Self-associated three-dimensional cones

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Abstract
For every proper convex cone $K \subset \mathbb{R}^3$ there exists a unique complete hyperbolic affine 2-sphere with mean curvature $-1$ which is asymptotic to the boundary of the cone. Two cones are associated if the corresponding affine spheres can be mapped to each other by an orientation-preserving isometry. This equivalence relation is generated by the groups $SL(3, \mathbb{R})$ and $S^1$, where the former acts by linear transformations of the ambient space, and the latter by multiplication of the cubic holomorphic differential of the affine sphere by unimodular complex constants. The action of $S^1$ generalizes conic duality, which acts by multiplication of the cubic differential by $-1$. We call a cone self-associated if it is linearly isomorphic to all its associated cones, in which case the action of $S^1$ induces (nonlinear) isometries of the corresponding affine sphere. We give a complete classification of the self-associated cones and compute isothermal parameterizations of the corresponding affine spheres. Their metrics can be expressed in terms of degenerate Painlevé III transcendent. The boundaries of generic self-associated cones can be represented as conic hulls of vector-valued solutions of a certain third-order linear ordinary differential equation with periodic coefficients, but there exist also self-associated cones with polyhedral boundary parts. The self-associated cones are the second family of non-trivial 3-dimensional cones for which the affine spheres can be computed explicitly, the first being the semi-homogeneous cones.

Keywords  Affine sphere · Associated family · Monge-Ampère equation · Painlevé transcendent

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1 Introduction

In this work we classify the self-associated convex cones in $\mathbb{R}^3$, by which we mean those cones that are linearly isomorphic to all its associated cones. The notion of associated cones has been introduced in the paper of Lin and Wang (2016) and is by virtue of the Calabi theorem derived from the notion of associated families of affine spheres (Simon and Wang 1993). The results in this paper in spirit partly resemble the work of Dumas and Wolf (2015), which were the first to establish explicit non-trivial relations between complete hyperbolic affine spheres and their asymptotic cones.

Self-associated cones are akin to semi-homogeneous cones, which have been introduced in Hildebrand (2014) and whose corresponding affine spheres’ isothermal parametrizations have been explicitly computed in Lin and Wang (2016). While the affine spheres of the latter possess a continuous isometry group which is generated by linear automorphisms of the cone, the affine spheres of the former have isometries which are of a nonlinear nature and correspond to a continuously parameterized symmetry of the cone generalizing the duality symmetry of self-dual cones.

1.1 Background

A proper, or regular, convex cone $K \subset \mathbb{R}^n$ is a closed convex cone with non-empty interior and containing no line. In the sequel we shall speak of cones for brevity, meaning always proper convex cones. The Calabi conjecture on affine spheres (Calabi 1972), proven by the efforts of several authors, states that for every cone there exists a unique complete hyperbolic affine sphere with mean curvature $-1$ which is asymptotic to the boundary of the cone (Cheng and Yau 1977; Sasaki 1980), and conversely, every complete hyperbolic affine sphere is asymptotic to the boundary of some cone (Cheng and Yau 1986; Li 1990, 1992). These affine spheres are equipped with a complete non-positively curved Riemannian metric, the affine metric $h$, and a totally symmetric trace-less $(0, 3)$-tensor, the cubic form $C$. The two objects are equivariant under the action of the group $SL(n, \mathbb{R})$ of unimodular transformations. Moreover, the affine sphere and the corresponding cone $K$ can be reconstructed from $(h, C)$ up to a unimodular transformation of the ambient space $\mathbb{R}^n$. For more on affine spheres see, e.g., Nomizu and Sasaki (1994), for a survey on the Calabi conjecture see (Li et al. 1993, Chap. 2).

Complete hyperbolic affine 2-spheres are non-compact simply connected Riemann surfaces, and their affine metrics have the form $h = e^u|dz|^2$ in an isothermal complex coordinate $z = x + iy \in M, M \subset \mathbb{C}$ being a simply connected domain and $u : M \to \mathbb{R}$ an analytic conformal factor. The cubic form can be represented as $C = 2Re(U(z)d\bar{z}^3)$, with $U : M \to \mathbb{C}$ a holomorphic function, the so-called cubic differential (more precisely, the cubic differential is $Udz^3$, but we shall refer to $U$ for brevity). It satisfies the compatibility condition (Wang 1991)

$$e^u = \frac{1}{2} \Delta u + 2|U|^2 e^{-2u}, \quad (1)$$
also called Wang’s equation. Here \( \Delta u = u_{xx} + u_{yy} = 4u_{zz} \) is the Laplacian of \( u \). In the sequel, when we speak of complete solutions \((u, U)\) of (1), we always assume that the metric \( h \) is complete and \( U \) is holomorphic.

The affine sphere is given by an embedding \( f : M \to \mathbb{R}^3 \), which is determined by the solution \((u, U)\) up to a unimodular transformation of the ambient space \( \mathbb{R}^3 \). We shall speak of the complete solution \((u, U)\) of Wang’s equation (1) as corresponding to the affine sphere \( f \) or the convex cone \( K \). In Wang (1991) deduced the moving frame equations whose integration allows to reconstruct the affine sphere \( f \) from a given solution \((u, U)\). Define the \( 3 \times 3 \) real matrix \( F = (e^{-u/2} f_x, e^{-u/2} f_y, f) \). The first two columns of \( F \) form an \( h \)-orthonormal basis of the tangent space to the embedding, while the third column is the position vector of the embedding. The condition that the mean curvature of the affine sphere equals \(-1\) is equivalent to unimodularity of \( F \).

From the structure equations Simon and Wang (1993)

\[
f_{zz} = u_z f_z - U e^{-u} f_{\bar{z}}, \quad f_{\bar{z}z} = \frac{1}{2} e^u f, \quad f_{zz} = -\bar{U} e^{-u} f_z + u_z f_{\bar{z}}
\]

we obtain the frame equations

\[
F_x = F \begin{pmatrix} -e^{-u} \text{Re} U & \frac{u_y}{2} + e^{-u} \text{Im} U & e^{u/2} \\ -\frac{u_x}{2} + e^{-u} \text{Im} U & 0 & 0 \\ e^{u/2} & 0 & 0 \end{pmatrix}, \quad F_y = F \begin{pmatrix} e^{-u} \text{Im} U & -\frac{u_y}{2} + e^{-u} \text{Re} U & 0 \\ \frac{u_x}{2} + e^{-u} \text{Re} U & 0 & e^{u/2} \\ 0 & -e^{-u} \text{Im} U & 0 \end{pmatrix}.
\]

Choosing an arbitrary point \( z_0 = x_0 + iy_0 \in M \) and an arbitrary initial value \( F(z_0) = F_0 \in SL(3, \mathbb{R}) \), we can after integrating (2) recover the embedding \( f \) from the third column of the matrix-valued function \( F(z) \).

The isothermal coordinate system is not unique, but defined up to conformal isomorphisms of the domain \( M \). Let \( \hat{M} \subset \mathbb{C} \) be the pre-image of the domain \( M \) under the biholomorphic map \( b : w \leftrightarrow z \), and consider the embedding \( f \circ b : \hat{M} \to \mathbb{R}^3 \). Then the embeddings \( f \circ b \) and \( f \) define the same affine sphere, considered as a surface in \( \mathbb{R}^3 \), but its parametrization is different. Let \((\hat{u}, \hat{U})\) be the solution of (1) corresponding to \( f \circ b \). Equivariance of the pair \((h, C)\) then yields the transformation law

\[
\hat{U}(w) = U(z) \left( \frac{dz}{dw} \right)^3, \quad \hat{u}(w) = u(z) + 2 \log \left| \frac{dz}{dw} \right|.
\]

Let us single out the following consequence for further reference.

**Fact 1** Equivalence classes of complete solutions \((u, U)\) of (1) under bi-holomorphisms between the domains of definition are in one-to-one correspondence with equivalence classes of regular convex cones \( K \subset \mathbb{R}^3 \) under the action of the group \( SL(3, \mathbb{R}) \).
Several questions arise:
1. Given a complete solution \((u, U)\) of (1), describe the corresponding cone \(K \subset \mathbb{R}^3\) and vice versa.
2. Given a holomorphic function \(U\) on a simply connected domain \(D \subset \mathbb{C}\), characterize all complete solutions \((u, U)\) of (1) on \(D\).
3. Given an analytic function \(u\) on a simply connected domain \(D \subset \mathbb{C}\), characterize all complete solutions \((u, U)\) of (1) on \(D\).

Results on the first question are scarce. Dumas and Wolf (2015) have shown that if \(U\) is a polynomial on \(\mathbb{C}\), then a complete solution \((u, U)\) corresponds to a polyhedral cone in \(\mathbb{R}^3\), with the degree of the polynomial being equal to the number of extreme rays of the cone less 3. On the other hand, a polyhedral cone \(K\) corresponds to a complete solution \((u, U)\) with \(U\) a polynomial on \(\mathbb{C}\).

In Lin and Wang (2016) the complete solutions \((u, U)\) have been computed for the semi-homogeneous cones.

The second question has been answered completely. Obviously it suffices to consider the unit disc \(\mathbb{D}\) and the complex plane \(\mathbb{C}\), as any other simply connected non-compact domain is conformally equivalent to one of these two. Using techniques from Wan and Au (1994), Li (2019) observed that if \(U\) is a cubic holomorphic differential on the unit disc, then (1) has a unique complete solution. He used this to prove that if \(U\) is a non-zero holomorphic cubic differential on \(\mathbb{C}\), then (1) has a unique complete solution (Li 2019, Theorem 3.1). We may summarize these results as follows.

**Fact 2** For every holomorphic \(U\) on a simply-connected domain \(M \subset \mathbb{C}\) except for the zero function on \(M = \mathbb{C}\) there exists a unique complete solution \((u, U)\) of (1).

Prior to this, existence and uniqueness results have been obtained for cubic differentials \(U\) induced on the universal covers of compact Riemann surfaces (Wang 1991; Loftin 2001; Labourie 2007), on the universal covers of punctured compact Riemann surfaces with poles at the punctures (Loftin 2004), for functions on \(\mathbb{D}\) satisfying certain bounded-ness conditions (Benoist and Hulin 2013, 2014), and for polynomials on \(\mathbb{C}\) (Dumas and Wolf 2015).

The third question has been answered in Simon and Wang (1993). A necessary and sufficient condition on the affine metric \(h\) induced by the analytic function \(u\) has been given to admit a solution \((u, U)\). For any two such solutions the holomorphic functions \(U\) differ by a multiplicative unimodular complex constant \(e^{i\varphi}\), and multiplying the function \(U\) of some solution by \(e^{i\varphi}\) yields another solution.

The last result gives rise to the concept of associated cones which is central for the present paper. If an affine 2-sphere with mean curvature \(-1\), affine metric \(h = e^u|dz|^2\), and cubic differential \(U\) is given, then the affine spheres constructed from the pairs \((u, e^{i\varphi}U)\), where \(\varphi\) runs through \([0, 2\pi)\), exhaust all affine 2-spheres with affine metric \(h\) and mean curvature \(-1\). The orbits of complete hyperbolic affine 2-spheres with respect to the action of \(SL(3, \mathbb{R})\) are hence arranged in 1-parametric associated families, on which the circle group \(S^1 \simeq \{c \in \mathbb{C} ||c| = 1\}\) acts by multiplication of the cubic differential \(U\) by the unimodular group element. Affine spheres belonging to orbits in the same family are called associated (Simon and Wang 1993).

By virtue of the Calabi theorem this notion can naturally be extended to cones in \(\mathbb{R}^3\). The action of the circle group \(S^1\) on the solutions of (1) induces an action on the set of
$SL(3, \mathbb{R})$-orbits of cones, and these $SL(3, \mathbb{R})$-orbits are also arranged in 1-parametric families. Cones belonging to orbits in the same family are called associated (Lin and Wang 2016).

The action of the group $S^1$ on a single associated family of $SL(3, \mathbb{R})$-orbits of cones does not need to be faithful. It may well be that two solutions $(u, U)$, $(u, cU)$ of Wang’s equation (1) for $c \neq 1$ lead to isomorphic affine spheres or, equivalently, to the same $SL(3, \mathbb{R})$-orbit of cones. In this contribution we characterize those cones $K \subset \mathbb{R}^3$ whose $SL(3, \mathbb{R})$-orbit is a fixed point of the action of $S^1$, and compute the corresponding solutions $(u, U)$ of Wang’s equation (1).

**Definition 1** Let $K \subset \mathbb{R}^3$ be a cone and let $(u, U)$ be a complete solution of (1) corresponding to $K$. For $\varphi \in [0, 2\pi)$, let $K_\varphi \subset \mathbb{R}^3$ be a cone corresponding to the solution $(u, e^{i\varphi} U)$ of (1). We call the cone $K$ self-associated if for every $\varphi \in [0, 2\pi)$ the cone $K_\varphi$ is linearly isomorphic to $K$.

Note that we did not restrict the linear isomorphisms between $K$ and $K_\varphi$ to be unimodular, and hence the condition in Definition 1 is a priori weaker than the condition that the $SL(3, \mathbb{R})$-orbit of $K$ is a fixed point of the action of $S^1$. Later we shall show that these two conditions are actually equivalent.

Explicit results on associated families of cones are scarce. In Lin and Wang (2016) the associated families have been computed for the semi-homogeneous cones. From the results of Dumas and Wolf (2015) it follows that any cone associated to a polyhedral cone is also polyhedral with the same number of extreme rays, and that the self-associated polyhedral cones are exactly the cones over the regular $n$-gons.

In (Loftin 1999, Corollary 4.0.4) Loftin observed that multiplying the cubic differential of an affine 2-sphere by $-1$ leads to the projectively dual affine 2-sphere. However, if an affine 2-sphere is asymptotic to the boundary of a convex cone $K \subset \mathbb{R}^3$, then the projectively dual affine 2-sphere is asymptotic to the boundary of the dual cone $K^* = \{y \in \mathbb{R}^3 \mid x^T y \geq 0 \ \forall \ x \in K\}$. Hence $K^*$ is always associated to $K$, and its $SL(3, \mathbb{R})$-orbit can be obtained from the $SL(3, \mathbb{R})$-orbit of $K$ by the action of the group element $e^{i\pi} \in S^1$. Any self-associated cone $K$ must hence be self-dual, in the sense that it is linearly isomorphic to its dual $K^*$. The property of being self-associated is therefore stronger than self-duality.

**1.2 Outline**

In this work we provide a full classification of self-associated cones and explicitly describe their boundaries as well as the affine spheres which are asymptotic to these boundaries. We now outline the contents of the paper, sketch the strategy of the proofs, and summarize the results.

In Sect. 2 we show that a cone $K \subset \mathbb{R}^3$ is self-associated if and only if its cubic differential $U : M \rightarrow \mathbb{C}$ satisfies condition (4), where $\psi$ is a vector field generating a flow of conformal automorphisms of the domain $M$. In other words, the flow multiplies the cubic differential by unimodular complex constants. Actually, we show that the flow consists of isometries of the affine metric, i.e., it preserves the conformal factor $u$, and $\psi$ is a Killing vector field.
In Sect. 3 we classify the solutions \((\psi, U)\) of (4) up to conformal isomorphisms of the domain \(M\). We distinguish two cases, apart from the trivial solution \(U \equiv 0\) on a hyperbolic domain, which corresponds to the Lorentz cone \(L_3\). In the first case the domain can be transformed to an open disc with radius \(R \in (0, +\infty]\), the generated conformal automorphisms are the rotations of the disc, and \(U(z) = z^k, k \in \mathbb{N}\). In the second case the domain can be transformed to a vertical strip \((a, b) + i\mathbb{R}\), where \(-\infty \leq a < b \leq +\infty\), the automorphisms are the vertical translations, and \(U(z) = e^z\). We shall refer to these cases as the rotational and the translational case, respectively.

In Sect. 4 we find for every pair \((\psi, U)\) the unique complete solution \(u\) of Wang’s equation. Its invariance with respect to the automorphisms generated by \(\psi\) allows to reduce (1) to the degenerate Painlevé III equation (26). For each \((\psi, U)\) we characterize the corresponding solution \(v(t)\) of (26), in particular, its asymptotics at the boundary of the interval of definition.

Other Painlevé III equations play a role in the description of constant or harmonic inverse mean curvature surfaces with radial symmetry (Bobenko and Its 1995; Bobenko et al. 1997), Amsler surfaces (Bobenko and Kitaev 1998), and Smyth surfaces and indefinite affine spheres with intersecting straight lines (Bobenko and Eitner 2000).

In Sect. 5 we consider the asymptotic behaviour of the moving frame near the boundary of the domain \(M\) and compute pieces of the boundary \(\partial K\) of the self-associated cones. In order to capture the asymptotics of the moving frame we shall employ the technique of osculation maps which has been introduced in Dumas and Wolf (2015). We compare the moving frame \(F\) with an explicit diverging unimodular matrix \(V\) such that the ratio \(G = FV^{-1}\) has a finite limit as the argument tends to the boundary of \(M\). In the limit the dynamics of the moving frame along the boundary in general reduces to a linear third-order ordinary differential equation (ODE) with periodic coefficients. In these cases the corresponding boundary piece of \(K\) can be described as the conic hull of a vector-valued solution of the ODE. In the remaining cases the boundary piece consists of planar faces. The latter situation has been encountered by Dumas and Wolf (2015).

In Sect. 6 we assemble the boundary pieces to obtain the whole boundary of the cone. The main tool to accomplish this task are the linear automorphisms \(\Sigma, T\) of the cone that are generated by complex conjugation of the domain and by its rotation by an angle \(\frac{2\pi}{k+3}\) or translation by \(2\pi i\), respectively. These symmetries generate a subgroup of automorphisms which is isomorphic to a finite dihedral group in the rotational case (Lemma 14) and to the infinite dihedral group in the translational case (Lemma 15).

According to the whether the trace of \(T\) is smaller, equal, or greater than 3 we distinguish cones of elliptic, parabolic, and hyperbolic type (Definition 2). The first arises in the rotational case, the second in the translational case with \(a = -\infty\), and the last in the translational case with \(a\) finite.

In Sect. 7 we summarize our classification in Theorems 3, 4, and 5, describing the three types of cones, respectively.

In Sect. 8 we pose some open questions and suggest directions for further research.
2 Killing vectors and self-associated cones

In this section we characterize affine spheres which are asymptotic to the boundary of self-associated cones by the existence of a 1-parameter subgroup of automorphisms of the domain which non-trivially multiplies the cubic differential by unimodular complex constants. The infinitesimal generator of the subgroup turns out to be a Killing vector field (i.e., generator of isometries) for the affine metric.

This will be accomplished in two steps. First we show that if all associated cones \( K_\varphi \) of a cone \( K \subset \mathbb{R}^3 \) lie in the same \( GL(3, \mathbb{R}) \)-orbit as \( K \), then they must also lie in the same \( SL(3, \mathbb{R}) \)-orbit. In a second step we show the existence of the Killing vector field.

2.1 Isomorphisms with negative determinant

Suppose two cones \( K, \tilde{K} \) are linearly isomorphic, but not in the same \( SL(3, \mathbb{R}) \)-orbit. Then there exists a linear map with determinant \(-1\) taking \( K \) to \( \tilde{K} \). The next result shows how the corresponding solutions of (1) are related to each other.

**Lemma 1** Let \( K \subset \mathbb{R}^3 \) be a cone and let \( A \in GL(3, \mathbb{R}) \) be a linear map with determinant \(-1\). Let \((u, U)\) be the complete solution of (1) corresponding to \( K \), defined on a simply connected domain \( M \subset \mathbb{C} \). Then a complete solution \((\tilde{u}, \tilde{U})\) of (1) corresponding to the cone \( A[K] \) is given by \( \tilde{u}(z) = u(\overline{z}) \) and \( \tilde{U}(z) = \overline{U(\overline{z})} \) on the complex conjugate domain \( \overline{M} \).

**Proof** The domain \( \overline{M} \) is simply connected, \( \tilde{U} \) is holomorphic, and the pair \((\tilde{u}, \tilde{U})\) satisfies Wang’s equation (1). Moreover, the metric \( \tilde{h} = e^{\tilde{u}}|dz|^2 \) is complete on \( \overline{M} \), because complex conjugation is an isometry between \( M \) and \( \overline{M} \). Hence \((\tilde{u}, \tilde{U})\) is a solution of (1).

Let \( F : M \to SL(3, \mathbb{R}) \) be a solution of the frame equations (2) such that the embedding \( f : M \to \mathbb{R}^3 \) given by the third column of \( F \) is asymptotic to \( \partial K \). It is not hard to see that equations (2) are invariant under the substitution \((x, y, u, U, f) \mapsto (x, -y, u, \tilde{U}, -f)\). Note that \(-A \in SL(3, \mathbb{R})\). The embedding \( \tilde{f} : \overline{M} \to \mathbb{R}^3 \) defined by \( \tilde{f}(z) = (-A)(-f(\overline{z})) = Af(\overline{z}) \) is then asymptotic to \( \partial A[K] \) and gives rise to a solution \( \tilde{F} : \overline{M} \to SL(3, \mathbb{R}) \) of the frame equations corresponding to \((\tilde{u}, \tilde{U})\). Therefore the solution \((\tilde{u}, \tilde{U})\) corresponds to the cone \( A[K] \). \( \square \)

We now show that for a self-associated cone, the isomorphisms between the associated cones can be assumed to have positive determinant.

**Lemma 2** A cone \( K \subset \mathbb{R}^3 \) is self-associated if and only if all cones which are associated to \( K \) are in the \( SL(3, \mathbb{R}) \)-orbit of \( K \).

**Proof** The “if” direction follows from the definition of self-associated cones. Let us prove the “only if” direction.

Let \( K \subset \mathbb{R}^3 \) be self-associated. Then every associated cone \( K_\varphi \) is in the \( SL(3, \mathbb{R}) \)-orbit of \( K \) or in the \( SL(3, \mathbb{R}) \)-orbit of \(-K\). If \( K, -K \) are in the same \( SL(3, \mathbb{R}) \)-orbit, then the claim of the lemma follows. Let us therefore suppose that the \( SL(3, \mathbb{R}) \)-orbits of the cones \( K, -K \) are distinct. Define the disjoint complementary subsets \( S_+, S_- \subset S^1 \) such that \( e^{i\varphi} \in S_\pm \) if \( K_\varphi \) is in the \( SL(3, \mathbb{R}) \)-orbit of \( \pm K \), respectively.
2.2 Existence of the Killing vector field

We now proceed to the second step, showing that domains with solutions of (1) corresponding to self-associated cones carry a Killing vector field with the properties claimed at the beginning of Sect. 2. Note that in the complex isothermal coordinate isomorphisms taking the solution \((u, U)\), \((\tilde{u}, \tilde{U})\) of Wang’s equation are conformally equivalent, then for all \(\varphi\) the solutions \((u, e^{i\varphi}U)\), \((\tilde{u}, e^{i\varphi}\tilde{U})\) are also conformally equivalent, by the same isomorphism. Therefore if \(e^{i\varphi_1}, e^{i\varphi_2}\) are in the same subset \(S_+\) or \(S_-\), then \(e^{i(\varphi_1+\varphi)}, e^{i(\varphi_2+\varphi)}\) are also in the same subset for every \(\varphi\). In particular, \(1\) and \(e^{i\varphi}\) are in the same subset if and only if \(e^{-i\varphi}\) and \(1\) are in the same subset. It follows that \(e^{i\varphi}, e^{-i\varphi}\) are in the same subset. But then \(e^{2i\varphi}, 1\) are in the same subset, for every \(\varphi\). Since \(1 \in S_+\), we obtain \(S_- = \emptyset\). This completes the proof. □

Let \(K \subset \mathbb{R}^3\) be a self-associated cone, and let \((u, U)\) be a corresponding complete solution of Wang’s equation (1) on a domain \(M \subset \mathbb{C}\). Then there exists a holomorphic function \(\psi\) on \(M\) which generates a 1-parameter subgroup of conformal automorphisms of \(M\) and satisfies the relations

\[
U(z) + U'(z)\psi(z) + 3U(z)\psi'(z) = 0, \tag{4}
\]

\[
\text{Re}(u'(z)\psi(z) + \psi'(z)) = 0, \tag{5}
\]

where the prime denotes the derivative with respect to the complex coordinate \(z\).

**Proof** Assume the conditions of the lemma.

If \(U \equiv 0\), then \(\psi(z) \equiv 0\) satisfies the requirements. Let us henceforth assume that \(U \not\equiv 0\).

By Fact 1 and Lemma 2 all solutions \((u, e^{i\varphi}U)\) of (1) on \(M\) are mutually conformally equivalent. Let \(\text{Aut}\ M\) be the finite-dimensional Lie group of biholomorphic automorphisms of \(M\). For every \(e^{i\varphi} \in S^1\), let \(G_{\varphi} \subset \text{Aut}\ M\) be the set of all conformal isomorphisms taking the solution \((u, e^{i\varphi}U)\) of (1) to \((u, U)\). Since \(e^{i\varphi}U \not\equiv e^{i\varphi'}U\) and hence \(G_{\varphi} \cap G_{\varphi'} = \emptyset\) whenever \(e^{i\varphi} \not\equiv e^{i\varphi'}\), we may define a surjective map \(\alpha : G = \bigcup_{e^{i\varphi} \in S^1} G_{\varphi} \to S^1\) by sending \(G_{\varphi}\) to \(e^{i\varphi}\).

Since the action of the automorphism group on the solution \((u, U)\) is continuous in the group element, the subgroups \(G, G_0\) are closed in \(\text{Aut}\ M\), and hence \(G_0\) is closed in the subspace topology of \(G\). It follows that \(G_0, G\) are Lie subgroups of \(\text{Aut}\ M\), and \(G_0\) is a Lie subgroup of \(G\). Since \(G_0\) is the subgroup leaving \((u, U)\) and hence \((u, e^{i\varphi}U)\) invariant for every \(\varphi\), the map \(\alpha\) is a group homomorphism with kernel \(G_0\).

If a sequence \(b_k : w \mapsto z_k\) of automorphisms \(b_k \in \text{Aut}\ M\) tends to the identity map, then for every fixed \(w \in M\) we have \(b_k(w) \to w\) and \(b_k'(w) \to 1\). Therefore \(\alpha\) is continuous by (3). Thus \(\alpha\) is a Lie group homomorphism and defines a group isomorphism between \(S^1\) and the quotient \(G/G_0\).

Let \(g, g_0\) be the Lie algebras of the Lie groups \(G, G_0\), respectively. Then \(\alpha\) defines a Lie algebra homomorphism \(\alpha : g \to \mathbb{R}\) of \(g\) into the Lie algebra \(\mathbb{R}\) of \(S^1\), such

\[ \square \]
that \(g_0 \subset \ker a\). For the sake of contradiction, let us assume that \(\ker a = g\). Then the
connection component of the identity in \(G\) is mapped to \(1 \in S^1\) and is hence a subset
of \(G_0\). It follows that \(G_0\) is open in \(G\), and the quotient topology of \(G/G_0\) is discrete.
However, \(G/G_0\) is an uncountable set, because \(S^1\) is uncountable. Hence \(G\) is not
second-countable, contradicting the property that it is a finite-dimensional Lie group.

Thus \(\ker a\) must be a strict Lie subalgebra of \(g\), and \(a\) is surjective.

Let \(b \in g\) be such that \(a(b) = -1\). Let \(\{g_t\}_{t \in \mathbb{R}} \subset G\), \(g_t : w \mapsto z_t\), be the one-
parameter subgroup generated by the Lie algebra element \(b\). Note that \(z_t\) is holomorphic
for all \(t\), because \(g_t\) are conformal. We have \(\alpha(g_t) = e^{-it}\), and from (3) we get

\[
e^{-it} U(w) = U(z_t) \left( \frac{dz_t}{dw} \right)^3, \quad u(w) = u(z_t) + 2 \log \left| \frac{dz_t}{dw} \right| \tag{6}
\]

for every \(t \in \mathbb{R}\).

Let \(\psi(w) = \left. \frac{dz_t(w)}{dt} \right|_{t=0}\) be the velocity field of the flow generated by \(b\), represented
as a complex-valued function on \(M\). Since the \(z_t\) are holomorphic, the vector field \(\psi\)
is also holomorphic. By virtue of the relations

\[
z_0(w) \equiv w, \quad \left. \frac{dz_t}{dw} \right|_{t=0} \equiv 1, \quad \left[ \frac{d}{dt} \frac{dz_t}{dw} \right]_{t=0} = \psi'(w)
\]

differentiation of (6) with respect to \(t\) at \(t = 0\) yields

\[
-iU(w) = U'(w)\psi(w) + 3U(w)\psi'(w), \quad 0 = 2Re (u'(w)\psi(w)) + 2Re \psi'(w).
\]

The claim of the lemma readily follows. \(\Box\)

The opposite implication is also true.

**Lemma 4** Let \(K \subset \mathbb{R}^3\) be a cone, and let \((u, U)\) be a complete solution of Wang’s
equation (1) corresponding to \(K\), defined on a domain \(M \subset \mathbb{C}\). Suppose there exists
a vector field on \(M\), represented by a holomorphic function \(\psi\) satisfying (4) and
generating a 1-parameter group \(\{g_t\}_{t \in \mathbb{R}}\) of conformal automorphisms of \(M\). Then the
cone \(K\) is self-associated.

**Proof** Integrating (4) along the trajectories of the flow \(\{g_t\}\) we obtain the first relation in
(6). This means that the automorphism \(g_t\) transforms the holomorphic cubic differential
\(e^{-it}U\) to \(U\).

Fix \(t \in \mathbb{R}\). Since equation (1) is left invariant if \(U\) is multiplied by a constant
phase factor, the pair \((u, e^{-it}U)\) is also a complete solution of (1) on \(M\). Applying the
automorphism \(g_t\) to this solution, we obtain another complete solution \((u_t, U)\), where
\(u_t\) is the image of \(u\) under \(g_t\). However, by the uniqueness property in Fact 2 \(u\) and \(u_t\)
must coincide.

Since the complete solutions \((u, e^{-it}U)\) and \((u, U)\) of (1) on \(M\) are related by the
automorphism \(g_t\), they correspond by Fact 1 to the same \(SL(3, \mathbb{R})\)-orbit of cones. This
holds for all \(t\), and thus \(K\) is self-associated. \(\Box\)
In this section we expressed the property of a cone $K \subset \mathbb{R}^3$ to be self-associated by an analytic condition on the corresponding cubic differential $U : M \to \mathbb{C}$. Namely, an infinitesimal generator $\psi$ of a 1-parameter subgroup of automorphisms of the domain $M$ has to exist which satisfies (4). In the next section we shall classify all such pairs of holomorphic functions $(\psi, U)$ together with their domains $M$.

### 3 Classification of cubic holomorphic differentials

In this section we classify, up to biholomorphic isomorphisms, all pairs $(\psi, U)$ of holomorphic functions on $M$ satisfying condition (4), where $\psi$ represents a vector field generating a 1-parameter subgroup of biholomorphic automorphisms of $M$. At first we shall single out the following special case, however.

**Case 0**: $U \equiv 0$. This condition implies that the cubic form $C$ of the affine sphere vanishes identically. The Theorem of Pick and Berwald (Nomizu and Sasaki 1994, Theorem 4.5, p. 53) states that this happens if and only if the affine sphere is a quadric. A quadric can be asymptotic to the boundary of a cone in $\mathbb{R}^3$ only if this cone is ellipsoidal, i.e., linearly isomorphic to the Lorentz cone

$$L_3 = \left\{ x = (x_0, x_1, x_2)^T \mid x_0 \geq \sqrt{x_1^2 + x_2^2} \right\}.$$

The affine sphere is then one sheet of a two-sheeted hyperboloid and is isometric to a hyperbolic space form. The domain of the solution can be transformed to the unit disc $\mathbb{D}$ by a conformal isomorphism.

In the rest of this section we assume that $U \neq 0$. Then we also have $\psi \neq 0$. The simply connected domains $M \subset \mathbb{C}$ admitting a 1-parameter group of biholomorphic automorphisms have been classified up to conformal isomorphisms and can be reduced to the following five cases (Dorfmeister and Ma 2016a, Theorem 3.2, (2)), the domains in the other items of this theorem being either compact or not simply connected:

1. $M = \mathbb{C}, \psi(z) \equiv \mu$;
2. $M = \mathbb{C}, \psi(z) = i\mu z$;
3. $M = \mathbb{D}, \psi(z) = i\mu z$;
4. $M = \{z \mid \text{Im} z > 0\}, \psi(z) \equiv \mu$;
5. $M = \{z \mid 0 < \text{Im} z < \pi\}, \psi(z) \equiv \mu$.

Here $\mu$ is an arbitrary non-zero real constant determining the parametrization of the automorphism subgroup. Cases (a),(d),(e) correspond to real translations, cases (b),(c) to rotations about the origin. For each case we first find all solutions $U$ of (4). In a second step we transform the domain by a conformal isomorphism to obtain a standardized form for $U$.

**Case T**: If the automorphism group consists of translations, $\psi \equiv \mu$, then the solutions of (4) are given by $U(z) = \gamma e^{-i\pi/\mu}$, where $\gamma$ is an arbitrary non-zero complex number. After application of the inverse of the conformal isomorphism $w \mapsto z = \alpha w + \beta$, where $\alpha = i\mu, \beta = -\mu(\frac{\pi}{2} + i \log(\gamma \mu^3))$, we obtain by (3) the solution $(\psi, U) = (-i, e^w)$. 

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The applied map transforms the domain $M$ to the complex plane $\mathbb{C}$ in case (a), to the right half-plane $\{w \mid \text{Re} w > \log |\gamma \mu^3|\}$ in case (b) when $\mu > 0$, to the left half-plane $\{w \mid \text{Re} w < \log |\gamma \mu^3|\}$ in case (b) when $\mu < 0$, to the vertical strip $\{w \mid \log |\gamma \mu^3| < \text{Re} w < \log |\gamma \mu^3| + \frac{\pi i}{\mu}\}$ in case (c) if $\mu > 0$, and to the vertical strip $\{w \mid \log |\gamma \mu^3| + \frac{\pi i}{\mu} < \text{Re} w < \log |\gamma \mu^3|\}$ in case (c) when $\mu < 0$.

The solution of (4) can hence be brought to the form $(\psi, U) = (-i, e^w)$ on the domain $M_{(a,b)} = \{w \mid a < \text{Re} w < b\}$, which is a vertical strip of finite or infinite width, parameterized by $-\infty \leq a < b \leq +\infty$ or equivalently by the set of non-empty open intervals $(a, b) \subset \mathbb{R}$.

Case R: If the automorphism group consists of rotations, $\psi = i \mu z$, then the solutions are locally given by $U(z) = \gamma z^{-(3\mu+1)/\mu}$, where $\gamma$ is an arbitrary non-zero complex number. This solution is defined in a neighbourhood of $z = 0$ if and only if $-\frac{3\mu+1}{\mu}$ is a natural number, i.e., $\mu = -\frac{1}{k+3}$ for some $k \in \mathbb{N}$. After application of the inverse of the conformal isomorphism $w \mapsto z = \gamma^\mu w$ we obtain by (3) the solution $(\psi, U) = (-i \frac{w}{k+3}, w^k)$. The applied maps transforms the domain $M$ to the complex plane $\mathbb{C}$ in case (b), and to the open disc $B_R = \{w \mid |w| < R\}$ of radius $R = |\gamma|^{-\mu} \in (0, +\infty)$ in case (c).

The solution of (4) can therefore be brought to the form $(\psi, U) = (-i \frac{w}{k+3}, w^k)$, where $k \in \mathbb{N}$ is a discrete parameter, on the domain $B_R = \{w \mid |w| < R\}$, which is an open disc of radius $R \in (0, +\infty]$.

Lemma 5 Let $K \subset \mathbb{R}^3$ be a self-associated cone. Then the cubic differential $U$ from any solution of (1) corresponding to $K$ can be transformed by a conformal isomorphism to exactly one of the following cases:

0: $U \equiv 0$ on $M = \mathbb{D}$;
R: $U = z^k$, $k \in \mathbb{N}$, on $M = B_R = \{z \mid |z| < R\}$, $R \in (0, +\infty]$;
T: $U = e^z$ on $M = M_{(a,b)} = \{z \mid a < \text{Re} z < b\}$, $-\infty \leq a < b \leq +\infty$.

Every listed case corresponds to an $SL(3, \mathbb{R})$-orbit of self-associated cones, and the corresponding solution $u$ of (1) satisfies condition (5) with $\psi = -i \frac{z}{k+3}$ in Case R, and $\psi = -i$ in Case T.

Proof Let $K \subset \mathbb{R}^3$ be a self-associated cone, and $(u, U)$ a complete solution of (1) corresponding to $K$. By Lemma 3 there exists a holomorphic function $\psi$ satisfying (4) and generating a 1-parameter group of conformal automorphisms of the domain $M$ of the solution. By the classification provided in this section the cubic differential together with its domain can be transformed to exactly one of the cases in the lemma by a biholomorphic isomorphism of the domain. This proves the first assertion.

Let now $U : M \to \mathbb{C}$ be one of the holomorphic functions listed in the lemma, with $\psi$ the corresponding vector field. Then $U$ satisfies (4), and the corresponding function $\psi$ generates a 1-parameter group of automorphisms of $M$. Let $u$ be the unique complete solution of (1), which exists by Fact 2. By Fact 1 this solution corresponds to an $SL(3, \mathbb{R})$-orbit of cones. By Lemma 4 these cones are self-associated. Finally, by Lemma 3 the function $u$ satisfies (5). \qed

Thus the zero case corresponds to a single $SL(3, \mathbb{R})$-orbit, in the rotational case we obtain a countably infinite number of 1-parametric families of $SL(3, \mathbb{R})$-orbits.
while in the translational case we obtain a 2-parametric family of $SL(3, \mathbb{R})$-orbits of self-associated cones.

In this section we classified the cubic differentials $U$ together with their domain $M$ of definition which correspond to self-associated cones, up to conformal isomorphism. In the next section we find the corresponding solutions $u$ of (1).

### 4 Solving Wang’s equation

In the previous section we have classified all solutions $(\psi, U)$ of equation (4), which in the case $U \not\equiv 0$ led to the canonical forms $z^k$ and $e^z$ for the holomorphic function $U$ on families of domains $M \subset \mathbb{C}$. In this section we obtain the corresponding real-valued function $u$ by solving (1). We use condition (5), which holds by Lemma 5, to reduce Wang’s equation to an ODE. This ODE turns out to be equivalent to the Painlevé III equation, which is considered in more detail in the Appendix. The boundary conditions for the solution of the ODE are obtained from the completeness condition on the metric $h = e^u |dz|^2$ on $M$. Existence and uniqueness of the solution are guaranteed by Fact 2.

We shall consider Cases R and T from Lemma 5 separately.

**Case R:** Plugging $\psi = -iz^k$ into (5), we obtain that $u$ is invariant under rotations of the domain $B_R$ about the origin, i.e., $u(z) = \chi(r)$ for some function $\chi: [0, R) \to \mathbb{R}$ which can be analytically extended to an even function on $(-R, R)$. Plugging $u(z) = \chi(r)$ into equation (1), we obtain the second-order ODE

$$\frac{d^2 \chi}{dr^2} = 2e^\chi - \frac{1}{r} \frac{d \chi}{dr} - 4r^2 e^{-2\chi}. \tag{7}$$

Set $t_R = \sqrt{\frac{32R^{k+3}}{(k+3)^3}} \in (0, +\infty]$. Making the substitution $e^\chi = \sqrt{\frac{k+3}{2}} v r^{(k-1)/2}$, $t = \sqrt{\frac{32}{(k+3)^3}} r^{(k+3)/2}$ in (7), we obtain a solution $v(t)$ of Painlevé equation (26) which is positive on $(0, t_R)$. It hence equals one of the functions $v_{s,c}(t)$, $(s, c) \in [-1, 3] \times \mathbb{R}$, from Proposition 6 in the Appendix. We now turn to the boundary conditions.

It is checked by straightforward calculation that the analyticity condition for $\chi$ at $r = 0$ is equivalent to the condition that the corresponding Painlevé transcendent is of the form $v_{s,c}(t) = t^{-(k-1)/(k+3)} \kappa(t^{4/(k+3)})$, where $\kappa$ is an analytic function in some neighbourhood of the origin satisfying

$$\kappa(0) = e^{u(0)} 2^{(3k-1)/(k+3)} (k+3)^{-2k/(k+3)} > 0.$$
\[ \kappa(0) = \begin{cases} 
  e^c, & k = 0; \\
  \lambda_k^28^{1-2\lambda_k}e^{3\lambda_k c/2} \frac{\Gamma(1-\frac{\lambda_k}{2})\Gamma(1-\lambda_k)}{\Gamma(1+\frac{\lambda_k}{2})\Gamma(1+\lambda_k)}, & k > 0.
\end{cases} \]

Here \( \lambda_k = \frac{2}{k+3} \).

Completeness of the metric \( h = e^u |dz|^2 \) on \( B_R \) implies that the metric \( e^x dr^2 \) is complete on \((-R, R)\). In particular, in the case \( R < \infty \) we must have \( \lim_{r \to R} \chi(r) = +\infty \), which implies \( \lim_{t \to t_R} v(t) = +\infty \). But then \( v(t) \) must have a double pole with expansion (28) at \( t = t_R \).

**Lemma 6** Let \( R \in (0, \infty), k \in \mathbb{N}, U(z) = z^k \) on \( B_R \). Assume above definitions. Then the corresponding unique complete solution \( u \) of (1) on \( B_R \) is rotationally symmetric and determined by the unique positive solution \( v_{s_k,c}(t) \) of Painlevé III equation (26) which is pole-free on \((0, t_R)\) and has a double pole at \( t = t_R \) for \( R \) finite, or is pole-free on the whole positive real axis for \( R = +\infty \). If \( R \) runs through the interval \((0, +\infty)\), then \( c \) runs through \([0, +\infty)\) in the opposite direction.

**Proof** Existence and uniqueness of \( u \) follows from Fact 2, its rotational symmetry and expression through \( v_{s_k,c} \) from the above considerations.

For \( R = +\infty \) the parameter \( c \) must equal zero, because only for \( c = 0 \) the solutions \( v_{s_k,c} \) are pole-free on a neighbourhood of \(+\infty\) on the real axis. In particular, it follows that \( v_{s_k,0} \) is pole-free and positive on \( \mathbb{R}_+ \), because it is the Painlevé transcendent which generates the complete solution \( u \) on \( \mathbb{C} \).

Let \( R \) be finite. Note that \( s_k \in (-1, 3) \), and hence by Corollary 3 there can exist at most one function \( v_{s_k,c} \) which is positive on \((0, t_R)\) and has a double pole at \( t = t_R \). Thus there exists a unique such function, namely that generating the solution \( u \) on \( B_R \).

Let now \( c > 0 \) be arbitrary. By Lemma 20 the solution \( v_{s_k,c} \) is positive on \((0, \tau_c)\), where \( \tau_c \) is the location of its left-most pole. But then by the preceding \( v_{s_k,c} \) generates the solution \( u \) of (1) on \( B_R \), where \( R \) is chosen such that \( t_R = \tau_c \). For \( c < 0 \) the solution \( v_{s_k,c} \) cannot be positive up to its left-most pole by Lemma 20, and hence does not generate a complete solution \( u \).

Hence the parameters \( c \in [0, +\infty) \) and \( R \in (0, +\infty) \) are in bijective correspondence. By Lemma 20 \( c \) is a decreasing function of \( R \), which proves the last assertion.

**Case T:** Plugging \( \psi = -i \) into (5), we obtain that the solution \( u \) is invariant with respect to vertical translations, i.e., of the form \( u(z) = \chi(x) \) for some analytic function \( \chi : (a, b) \to \mathbb{R} \). Plugging \( u(z) = \chi(x) \) into equation (1), we obtain the second-order ODE

\[ \frac{d^2\chi}{dx^2} = 2e^x - 4e^{2x}e^{-2x}. \]
Set $t_a = \sqrt{32}e^{a/2} \in [0, +\infty)$, $t_b = \sqrt{32}e^{b/2} \in (0, +\infty]$. Making the substitution

$$e^x = \frac{1}{8}vt, \quad t = \sqrt{32}e^{x/2},$$

we again obtain a solution $v(t)$ of Painlevé equation (26).

Completeness of the metric $h = e^{u}|dz|^2$ on $M$ is equivalent to completeness of the metric $e^x dx^2$ on $(a, b)$.

We shall treat the cases $a, b$ finite or infinite separately.

Consider first the case of finite $a$ ($b$). Then we must have $\lim_{x \to a} x(x) = +\infty$ (or $\lim_{x \to b} x(x) = +\infty$). This condition is equivalent to $\lim_{t \to t_a} v(t) = +\infty$ (or $\lim_{t \to t_b} v(t) = +\infty$). Hence $v(t)$ must have a double pole with expansion (28) at $t = t_a$ ($t = t_b$).

Let us now consider the case $a = -\infty$. Then the solution $v$ is pole-free on $(0, t_b)$ and hence as in Case $R$ of the form $v_{x,c}$ for some $(s, c) \in [-1, 3] \times \mathbb{R}$. At the left end of the interval $(-\infty, b)$ the completeness condition is equivalent to the divergence of the integral $\int_{-\infty}^x e^{x(s)/2} ds = \int_0^t \sqrt{\frac{v(s)}{2s}} ds$. Comparing with the asymptotic expressions in Proposition 6 we see that this integral diverges if and only if $\lambda = 0$, or equivalently $s = 3$.

In the case $b = +\infty$ the solution $v(t)$ is pole-free on the interval $(t_a, +\infty)$, and as in Case $R$ above the parameter $c$ of the solution $v_{x,c}$ must equal zero.

**Lemma 7** Let $-\infty \leq a < b \leq +\infty$, and let $U(z) = e^z$ be defined on $M_{(a,b)} = (a, b) + i\mathbb{R}$. Assume above definitions. Then the corresponding unique complete solution $u$ of (1) on $M_{(a,b)}$ is invariant with respect to vertical translations and determined by the unique positive solution $v(t)$ of Painlevé equation (26) which is pole-free on $(t_a, t_b)$ and

- of the form $v_{3,c}$ with a double pole at $t = t_b$ for $a = -\infty$, $b$ finite, here if $b$ runs through $\mathbb{R}$, then $c$ runs through $\mathbb{R}_+$ in the opposite direction;
- of the form $v_{3,0}$ with a double pole at $t = t_a$ for a finite, $b = +\infty$, here if $a$ runs through $\mathbb{R}$, then $s$ runs through $(3, +\infty)$ in the same direction;
- with double poles at $t = t_a$, $t = t_b$ for $a, b$ finite;
- equal to $v_{3,0}$ for $-a = b = +\infty$.

**Proof** Existence of $u$ follows from Fact 2, its translational symmetry and expression through solutions of (26) with the specified properties from the above considerations.

The uniqueness of these solutions $v(t)$ follows from the fact that their asymptotics at $t = t_a, t = t_b$ implies completeness of the solution $u$ on $M_{(a,b)}$ constructed from them. But this complete solution is unique by Fact 2.

Let us prove the assertions on the dependence of the parameters $s, c$ on $a, b$. Note that the solution $v_{3,0}$ is positive on $\mathbb{R}_+$, because it generates the complete solution $u$ of (1) on $\mathbb{C}$.

By Lemma 19, for every $s > 3$ the solution $v_{s,0}$ is positive on $(\tau_s, +\infty)$, where $\tau_s$ is the location of its right-most pole. Hence it generates the complete solution $u$ of (1) on $M_{(a, +\infty)}$, where $a$ is chosen such that $t_a = \tau_s$. For $s < 3$ the solution $v_{s,0}$ cannot have a pole on $\mathbb{R}$ and be positive up to the right-most pole by Lemma 19, and hence

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does not generate a complete solution $u$. Therefore $s \in (3, +\infty)$ and $a \in \mathbb{R}$ are in bijective correspondence. By Lemma 19 $s$ is an increasing function of $a$.

Likewise, by Lemma 20, for every $c > 0$ the solution $v_{3,c}$ is positive on $(0, \tau_3)$, where $\tau_3$ is the location of its left-most pole. Hence it generates the complete solution $u$ of (1) on $M(-\infty, b)$, where $b$ is chosen such that $t_b = \tau_3$. For $c < 0$ the solution $v_{3,c}$ cannot have a pole on $\mathbb{R}$ and be positive up to the left-most pole by Lemma 20, and hence does not generate a complete solution $u$. Therefore $c \in (0, +\infty)$ and $b \in \mathbb{R}$ are in bijective correspondence. By Lemma 20 $c$ is a decreasing function of $b$. \(\square\)

In this section we have reduced Wang’s equation (1) to Painlevé III equation (26) and singled out those solutions of (26) which generate the sought complete solutions of (1). We are now ready to obtain the corresponding self-associated cones in the next sections.

5 Asymptotics of the moving frame

In order to describe the cone $K$ corresponding to a solution $(u, U)$ of Wang’s equation on a domain $M \subset \mathbb{C}$ we have to compute how the moving frame behaves at the boundary of $M$. Recall that the affine sphere $f : M \rightarrow \mathbb{R}^3$ which is asymptotic to $\partial K$ is given by the third column of the unimodular moving frame $F : M \rightarrow SL(3, \mathbb{R})$. If a sequence of points $z_k \in M$ tends to the boundary of $M$, including infinity, and the ray through $f(z_k)$ tends to some limit ray, then this limit ray belongs to $\partial K$. Instead of a sequence of points, it will be more convenient to consider the limit of the ray through $f(z)$ when $z$ moves along a line towards the boundary. A continuous family of such lines will yield a family of limit rays which allows to construct a piece of the boundary $\partial K$.

In order to capture the asymptotics of $F$ at the boundary $\partial M$ we employ the technique of osculation maps introduced in Dumas and Wolf (2015). We compare $F$ to an explicit model frame $V : M \rightarrow SL(3, \mathbb{R})$ (in Dumas and Wolf (2015) the role of $V$ is played by the frame $F_T$ of a Titeica surface), such that the ratio $G = FV^{-1}$ (in Dumas and Wolf (2015) the role of $G$ is played by the osculation map $\hat{F}$) remains finite at the boundary of the domain $M$. We can then read off $\partial K$ from the limit $G_0$ of $G$. Different pieces of the boundary necessitate different model frames, and in this section we compute only these individual pieces.

In order to treat Case R in the same framework as Case T, we apply the coordinate transformation

$$re^{i\varphi} = z \mapsto -3 \log(k + 3) + (k + 3) \log z = w = x + iy \quad (10)$$

which transforms the cubic differential $U = z^k$ into $\tilde{U} = e^w$, the solution $\chi$ of (7) into the solution $\tilde{\chi} = \chi - \frac{2k}{k+3} \log(k + 3) + \frac{2x}{k+3}$ of (8), the independent variable $r$ of (7) into the independent variable $x = -3 \log(k + 3) + (k + 3) \log r$ of (8), and the punctured domain $B_R \setminus \{0\}$ into the (semi-)infinite horizontal strip $(-\infty, b) + i(-k+3)\pi, (k+3)\pi]$ with $b = -3 \log(k + 3) + (k + 3) \log R \in (-\infty, +\infty]$. It is more convenient, however, to work with the universal cover of $B_R \setminus \{0\}$, which is
mapped to the left half-plane \((-\infty, b) + i\mathbb{R}\). We then have to keep in mind that the points \((x, y)\) and \((x, y + 2(k + 3)\pi)\) in this domain correspond to the same point on the affine sphere. In particular, the moving frame satisfies the periodicity relation

\[
F(x, y + 2(k + 3)\pi) = F(x, y)
\]  

(11)

for all \(x < b, y \in \mathbb{R}\). Since we are interested only in the behaviour of \(F\) near the boundary of the domain \(B_R\), the loss of the central point \(z = 0\) is not relevant for the analysis.

In both Cases R and T the domain \(M\) of definition of the solution \((u, U)\) is now given by \((a, b) + i\mathbb{R}\), where \(-\infty \leq a < b \leq +\infty\), the cubic differential by \(U = e^z\), and the conformal factor by \(u(x, y) = \chi(x)\). The solution \(\chi(x)\) of ODE (8) on \((a, b)\) relates by transformation (9) to the solution \(\nu(t)\) of the Painlevé equation (26) which is described in Lemmas 6, 7.

The frame equations (2) take the form

\[
\begin{align*}
F_x &= F \begin{pmatrix}
-e^{-x} e^x \cos y & e^{-x} e^x \sin y & e^{x^2/2} \\
e^{-x} e^x \sin y & e^{-x} e^x \cos y & 0 \\
e^{x^2/2} & 0 & 0
\end{pmatrix} =: FA, \\
F_y &= F \begin{pmatrix}
e^{-x} e^x \sin y & \frac{x'}{2} + e^{-x} e^x \cos y & 0 \\
\frac{x'}{2} + e^{-x} e^x \cos y & -e^{-x} e^x \sin y & e^{x^2/2} \\
0 & e^{x^2/2} & 0
\end{pmatrix} =: FB.
\end{align*}
\]

(12)

We shall now study the asymptotic behaviour of the frame \(F(x, y)\) as \(x\) tends to the limiting values \(a\) or \(b\) while \(y\) remains fixed. For each value \(y \in \mathbb{R}\), the last column \(f(x, y)\) of \(F(x, y)\) is asymptotic to some boundary ray of the cone \(K\). We shall see that as \(y\) changes, the limiting rays either continuously sweep a boundary piece of the cone, or cluster at discrete values in case the boundary piece consists of just one ray or is polyhedral. The latter behaviour has been observed in Dumas and Wolf (2015). In case the boundary piece is smooth we shall describe it by a vector-valued ODE. Different initial conditions for this ODE lead to solutions which are related by a linear map. Thus an ODE is a convenient way to describe an \(SL(3, \mathbb{R})\)-orbit of conic boundaries rather than the boundary of a single cone.

We shall treat the cases of finite and infinite limiting values separately, which yields 4 cases in total to consider.

**Case** \(x \to b = +\infty\) turns out to be directly covered by the results developed in Dumas and Wolf (2015) and corresponds to polyhedral boundary pieces.

In order to apply the apparatus developed (Dumas and Wolf 2015, Sect. 5) we have to perform a coordinate transformation. Consider a horizontal strip \(S = (a, +\infty) + i(y_0, y_0 + 3\pi) \subset M\) of width \(3\pi\). Applying the coordinate transformation \(z \mapsto w = 2^{1/6} \cdot 3 \cdot e^{z/3}\), we map \(U = e^z\) to \(\tilde{U} = \frac{u}{2}\), the strip \(S\) to some half-plane in \(\mathbb{C}\) minus possibly a compact disc centered at the origin, vertical segments \(x + i \cdot (y_0, y_0 + 3\pi)\) to semi-circles with radius \(r = 2^{1/6} \cdot 3 \cdot e^{z/3}\), and \(u(z) = \chi(x)\) to a radial function \(\tilde{u}(w) = \tilde{\chi}(r) = \chi(x) - \frac{1}{3} \log 2 - \frac{2\pi}{3}\).
Recall that $\chi(x)$ is related by (9) to a solution $v_{x,0}(t)$ of the Painlevé equation (26). From its asymptotics in the Appendix we obtain that

$$e^\chi \sim 2^{1/3} e^{2x/3} + 2^{-1/2} 3^{-1/4} \pi^{-1/2} s e^{x/2} e^{-2/3} \frac{3^2}{2} e^{x/3}$$

as $x \to +\infty$ and hence

$$e^{\tilde{\chi}} \sim 1 + 2^{-3/4} 3^{1/4} \pi^{-1/2} s \tau^{-1/2} e^{-\sqrt{6} \tau}$$

as $r \to \infty$. This is precisely the asymptotics obtained in Sect. 5 and needed for the results in Sect. 6 of Dumas and Wolf (2015) to be applicable. After the reverse transformation back to the coordinates $x, y$ on $M$ these results translate into the following limiting behaviour of the affine sphere $f$.

**Lemma 8** Let $a \in [\infty, +\infty)$, $M = (a, +\infty) + i \mathbb{R}$, and let the cubic differential $U = e^z, z = x + i y$, be defined on $M$. Let $(u, U), u(z) = \chi(x)$, be the solution of Wang’s equation (1) on $M$ which in Case R is obtained by transformation (10) from the (punctured) complete solution $(u, z^k)$ on $B_{+\infty} = \mathbb{C}$ (in which case $a = -\infty$), and in Case T is the complete solution on $M$. Let $f : M \to \mathbb{R}^3$ be a corresponding affine sphere, obtained by integration of the frame equations (12).

Then for $\frac{\sqrt{\tau}}{2\pi}$ not an integer, the limit as $x \to +\infty$ of the ray through $f(x, y)$ depends only on the integer part of $\frac{\sqrt{\tau}}{2\pi}$. The limits of two neighboring strips of width $2\pi$ are distinct. The ray through $f(x, y)$ with $\frac{\sqrt{\tau}}{2\pi}$ an integer tends for $x \to +\infty$ to a proper convex combination of the limiting rays of the neighboring strips. The other convex combinations of these limiting rays can be obtained as the limit of the rays through $f(x, y(x))$ along a curved path $y = y(x)$ on $M$ as $x \to +\infty$.

**Corollary 1** Assume the notations of Lemma 8. The cone $K$ to whose boundary $f$ is asymptotic has a polyhedral boundary piece. The image under $f$ of the horizontal strip $(a, +\infty) + 2\pi i \cdot [n, n + 1], n \in \mathbb{Z}$, is a surface which is asymptotic to a conic wedge $W_n$, consisting of an extreme ray $\rho_n$ of $K$ and parts of the neighbouring two faces of $K$.

In Case R the union of the wedges $W_n, n \in \mathbb{Z}$, is the whole boundary $\partial K, K$ is polyhedral and has $k + 3$ extreme rays. In Case T the boundary piece consisting of the wedges $W_n$ for all $n \in \mathbb{Z}$ is non-self-intersecting and has infinitely many extreme rays.

**Proof** The first part of the Corollary is a consequence of Lemma 8. Let us prove the second part.

By virtue of (11), in Case R we have $f(x, y) = f(x, y + 2\pi (k + 3))$ for all $(x, y) \in M$. Hence $W_n = W_{n+k+3}$ for all $n \in \mathbb{Z}$, and there are only $k + 3$ distinct wedges. Moreover, the first and the last wedge in a sequence of $k + 3$ consecutive wedges intersect in a boundary ray, and the boundary piece formed of these $k + 3$ wedges closes in on itself. Hence the cone has $k + 3$ extreme rays in total, and the boundary piece is already the whole boundary $\partial K$. That $K$ is polyhedral with $k + 3$ extreme rays also follows (Dumas and Wolf 2015, Theorem 6.3).
In Case T the rays through \( f(x, y) \) for different points \((x, y) \in M\) are distinct, and the boundary piece \( \bigcup_{n \in \mathbb{Z}} W_n \) is non-self-intersecting. It is composed of infinitely many wedges, and hence it contains infinitely many extreme rays. \( \square \)

**Case** \( x \to a > -\infty \): The solution \( v(t) \) of the Painlevé equation (26) corresponding to \( \chi(x) \) has a double pole at \( t_0 = \sqrt{32} e^{a/2} \) with expansion (28). Plugging this expansion into \( e^x = \frac{v(t)}{8} \) and writing shorthand \( \delta = x - a \), or equivalently, \( t - t_0 = \sqrt{32} e^{a/2} (e^{b/2} - 1) \), we obtain the expansions

\[
e^{x} = \delta^{-2} + \tilde{a} + O(\delta^2), \quad e^{x/2} = \delta^{-1} + \frac{\tilde{a}}{2} \delta + O(\delta^3),
\]

\[
e^{-x+\delta} = \frac{1}{2} \delta^2 + \delta^3 + \left( \frac{1}{2} - \tilde{a} \right) \delta^4 + \left( \frac{1}{6} - \tilde{a} \right) \delta^5 + O(\delta^6),
\]

\[
\chi' = -\delta^{-1} + \tilde{a} \delta + O(\delta^3),
\]

where \( \tilde{a} = \frac{ae^{a/2}}{\sqrt{2}} - \frac{1}{48} \), and \( a \) is the parameter from (28).

Consider the matrix variable \( G = FV^{-1} \), where \( V = \begin{pmatrix} 0 & 0 & \delta \\ 0 & 1 & 0 \\ -\delta^{-1} & 0 & \delta^{-1} \end{pmatrix} \). For further reference we note that with \( J = \text{diag}(1, -1, -1) \) we have

\[
\lim_{\delta \to 0} V(\delta) J V^T(\delta) = \lim_{\delta \to 0} \begin{pmatrix} -\delta^2 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.
\] (13)

Since \( V \) is unimodular, we have \( G \in SL(3, \mathbb{R}) \). The frame equations (12) become

\[
G_x = GVAV^{-1} - GV_x V^{-1}, \quad G_y = GV BV^{-1}.
\]

Inserting the above expansions, we obtain

\[
G_x = G \begin{pmatrix} \frac{\tilde{a}}{2} \delta + O(\delta^3) & 0 & -\delta - \frac{\tilde{a}}{2} \delta^3 + O(\delta^5) \\ e^a (\delta + \frac{\delta^3}{2}) \sin y + O(\delta^5) & e^a (\delta^2 + \frac{\delta^3}{2}) \cos y + O(\delta^4) & -e^a (\delta^3 + \frac{\delta^3}{2}) \sin y + O(\delta^4) \\ e^a (1 + \delta) \cos y + O(\delta^2) & -e^a (\delta + \frac{\delta^3}{2}) \sin y + O(\delta^3) & -\frac{\delta}{2} e^a \delta^2 \cos y + O(\delta^3) \end{pmatrix},
\]

\[
G_y = G \begin{pmatrix} 0 & 0 & 1 + \frac{\delta}{2} \delta^2 + O(\delta^4) \\ \frac{3\delta}{2} + e^a \delta \cos y + O(\delta^2) & -e^a (\delta^2 + \frac{\delta^3}{2}) \sin y + O(\delta^4) & 1 - \tilde{a} \delta^2 + O(\delta^3) \\ -e^a (1 + \delta) \sin y + O(\delta^2) & \frac{3\delta}{2} e^a \delta \cos y + O(\delta^2) & e^a (\delta^2 + \frac{\delta^3}{2}) \sin y + O(\delta^4) \end{pmatrix}.
\]

The limit of the expression \( G^{-1} \nabla G \) as \( \delta \to 0 \) (equivalently, \( x \to a \)) hence exists and the convergence is uniform in \( y \). Therefore the function \( G(x, y) \), which as a function of \( x \) for every fixed \( y \) is the solution of a linear ODE, extends analytically to some unimodular matrix-valued function \( G_0(y) =: G(a, y) \). This function obeys the linear
ODE

\[
\frac{dG_0}{dy} = G_0 \cdot \begin{pmatrix} 0 & 1 & 0 \\ \frac{3\tilde{\alpha}}{2} & 0 & 1 \\ -e^a \sin y & \frac{3\tilde{\alpha}}{2} & 0 \end{pmatrix}
\]

(14)

with the $2\pi$-periodic coefficient matrix being the limit of $G^{-1}G_y$ as $\delta \to 0$.

Denote the third column of $G_0$ by $f_0$. Equation (14) allows to express the derivatives of $f_0$ as a function of $G_0$. In particular, we get

\[
(f_0'', f_0', f_0) = G_0 \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{3\tilde{\alpha}}{2} & 0 & 1 \end{pmatrix}
\]

(15)

and hence

\[
\det(f_0'', f_0', f_0) = 1.
\]

(16)

Therefore the vector $f_0(y)$ never vanishes. Moreover, from (14) it follows that it obeys the linear third-order ODE

\[
\frac{d^3 f_0}{dy^3} - 3\tilde{\alpha} \cdot \frac{df_0}{dy} + e^a \sin y \cdot f_0 = 0
\]

(17)

with $2\pi$-periodic coefficients. We are now able to prove the following result.

**Lemma 9** Let $a \in \mathbb{R}$, $b \in (a, +\infty]$, $M = (a, b) \cup i\mathbb{R}$, and let the cubic differential $U = e^z$, $z = x + iy$, be defined on $M$. Let $(u, U)$, $u(z) = \chi(x)$, be the complete solution of Wang’s equation (1) on $M$. Let $f : M \to \mathbb{R}^3$ be a corresponding affine sphere, obtained by integration of frame equations (12), and let $K \subset \mathbb{R}^3$ be the cone to whose boundary $f$ is asymptotic.

Then for fixed $y \in \mathbb{R}$ the limit as $x \to a$ of the ray through $f(x, y)$ is a boundary ray $\rho_y$ of $K$. The union $\bigcup_{y \in \mathbb{R}} \rho_y$ of these boundary rays forms a non-self-intersecting boundary piece of $K$ which is analytic everywhere except the tip of the cone. It can be obtained as the conic hull of a vector-valued solution $f_0$ of ODE (17) on $\mathbb{R}$ satisfying (16).

**Proof** Assume above notations. Fix $y \in \mathbb{R}$. The vector $f$ equals $G$ times the last column of $V$. Hence $\delta \cdot f(x, y) = G(x, y) \cdot ((x - a)^2, 0, 1)^T$, and this product tends to $f_0(y)$ for $x \to a$. Therefore the ray through $f(x, y)$ tends to the ray through $f_0(y)$ as $x \to a$. Since $f$ is asymptotic to $\partial K$, this limit $\rho_y$ is a boundary ray of $K$. By definition $f_0$ satisfies (17) and (16).

Now by virtue of (16) the derivative $\frac{df_0}{dy}$ and the vector $f_0$ are linearly independent for every $y$. Hence locally for different $y$ the limit rays $\rho_y$ are distinct, the solution curve $f_0$ is analytic, and the boundary piece $\bigcup_{y \in \mathbb{R}} \rho_y$ generated by this curve is also analytic everywhere except at the origin. The rays through $f(x, y)$ for different points $(x, y) \in M$ are distinct, and hence the boundary piece is non-self-intersecting. \qed
Case \( x \to b < +\infty \): Replacing \( a \) by \( b \) we obtain the same expansions and can perform the same constructions as in the previous case. However, since now \( x \to b \) from the left, we have that \( \delta = x - b \) is negative. Therefore the product \( \delta \cdot f(x, y) \) will tend to a point on \( -\partial K \) rather than on \( \partial K \). In order to correct this sign change we shall multiply \( f_0 \) by \( -1 \), i.e., we define \( f_0(y) \) as minus the third column of the matrix \( G_0(y) \). Instead of (16) we then get

\[
\det(f''_0, f'_0, f_0) = -1.
\]

However, the vector-valued function \( f_0(y) \) is still a solution of the linear third-order ODE

\[
d^3f_0 \over dy^3 - 3\tilde{\alpha} \cdot df_0 \over dy + e^b \sin y \cdot f_0 = 0
\]

with \( \tilde{\alpha} = {a e^{b/2} \over \sqrt{2}} - {1 \over 48} \), \( \alpha \) being the parameter from expansion (28) at \( t_0 = \sqrt{32}e^{b/2} \).

We get the following result.

Lemma 10 Let \( b \in \mathbb{R} \), \( a \in [\infty, b) \), \( M = (a, b) + i\mathbb{R} \), and let the cubic differential \( U = e^z \), \( z = x + iy \), be defined on \( M \). Let \( (u, U) \), \( u(z) = \chi(x) \), be the solution of Wang’s equation (1) on \( M \) which in Case \( R \) is obtained by transformation (10) from the (punctured) complete solution \( (u, z^k) \) on \( B_R \), where \( b = -3\log(k+3) + (k+3) \log R \) (in which case \( a = -\infty \)), and in Case \( T \) all rays are different for different \( y \), and the solution \( f_0 \) is 2\((k+3)\)-periodic. In Case \( T \) all rays \( \rho_y \) are different, and the boundary piece is non-self-intersecting.

Proof Repeating the arguments in the proof of Lemma 9, we obtain that for fixed \( y \in \mathbb{R} \) the ray through \( f(x, y) \) tends to the ray through \( f_0(y) \) as \( x \to b \), and this limit ray is a boundary ray of \( K \), denote it by \( \rho_y \). Here \( f_0 \) is a vector-valued solution of ODE (19) satisfying (18). Linear independence of \( f_0, f'_0 \) implies that locally the rays \( \rho_y \) are different for different \( y \), and the solution curve \( f_0 \) as well as the boundary piece generated by it are analytic except at the origin.

By virtue of (11), in Case \( R \) we have \( f(x, y) = f(x, y + 2\pi(k+3)) \) for all \((x, y) \in M \). But then \( f_0(y) = f_0(y + 2\pi(k+3)) \), and the solution \( f_0 \) is 2\((k+3)\)-periodic. Hence the boundary piece generated by \( f_0 \) closes in on itself, and makes up the whole boundary \( \partial K \).

In Case \( T \) the rays through \( f(x, y) \) are distinct for different points \((x, y) \), and the boundary piece generated by \( f_0 \) cannot close in on itself. \( \square \)
Case $x \to a = -\infty$: Recall that $\chi(x)$ relates by (9) to a solution $v_{3,c}(t)$ of the Painlevé equation (26). From Lemma 22 we obtain the expansions

$$e^x = \frac{1}{(-x + \alpha)^2} + O(e^{2x}x^2), \quad e^{-x+\alpha} = (-x + \alpha)^2 e^x + O(e^{3x}x^6),$$

$$e^{x/2} = \frac{1}{-x + \alpha} + O(e^{2x}x^3), \quad \frac{\chi'}{2} = \frac{1}{-x + \alpha} + O(e^{2x}x^3)$$

as $x \to -\infty$, where $\alpha = \log 2 - 3\gamma - \frac{3}{2}c$, and $\gamma$ is the Euler–Mascheroni constant.

Set $\delta = \frac{1}{-x+\alpha}$ and define the matrix-valued function $G = FV^{-1}$ with $V = \left( \begin{array}{ccc} \frac{\delta}{2} & 0 & \frac{\delta}{2} \\ 0 & 1 & 0 \\ -\delta^{-1} & 0 & \delta^{-1} \end{array} \right)$. Again $V$ is unimodular and hence $G \in SL(3, \mathbb{R})$. By virtue of the above expansions and the relation $\frac{d\delta}{dx} = \delta^2$ the frame equations become

$$G_x = GVAV^{-1} - GV_xV^{-1} = G \left( \begin{array}{ccc} O(\delta^{-2}e^x) & O(\delta^{-1}e^x) & O(e^x) \\ O(\delta^{-3}e^x) & O(\delta^{-2}e^x) & O(\delta^{-1}e^x) \\ O(\delta^{-4}e^x) & O(\delta^{-3}e^x) & O(\delta^{-2}e^x) \end{array} \right),$$

$$G_y = GVBV^{-1} = G \left( \begin{array}{ccc} 2 + O(\delta^{-3}e^x) & O(\delta^{-2}e^x) & O(\delta^{-1}e^x) \\ O(\delta^{-4}e^x) & 2 + O(\delta^{-3}e^x) & O(\delta^{-2}e^x) \end{array} \right).$$

Here the constants in the $O$ terms are uniformly bounded over $y \in \mathbb{R}$. It follows that $G(x, y)$ converges to some function $G_0(y)$ as $x \to -\infty$, and this function obeys

$$\frac{dG_0}{dy} = G_0 \cdot \left( \begin{array}{ccc} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{array} \right) \quad \Rightarrow \quad G_0(y) = G_0(0) \cdot \left( \begin{array}{ccc} 1 & 0 & 0 \\ 2y & 1 & 0 \\ 2y^2 & 2y & 1 \end{array} \right). \quad (20)$$

**Lemma 11** Let $b \in (-\infty, +\infty]$, $M = (-\infty, b) + i\mathbb{R}$, and let the cubic differential $U = e^z$, $z = x + iy$, be defined on $M$. Let $(u, U)$, $u(z) = \chi(x)$, be the complete solution of Wang’s equation (1) on $M$. Let $f : M \to \mathbb{R}^3$ be a corresponding affine sphere, obtained by integration of frame equations (12), and let $K \subset \mathbb{R}^3$ be the cone to whose boundary $f$ is asymptotic.

Then for fixed $y \in \mathbb{R}$ the limit as $x \to -\infty$ of the ray through $f(x, y)$ is a boundary ray $\hat{\rho}$ of $K$ which is independent of $y$.

**Proof** For fixed $y \in \mathbb{R}$ the product $\delta \cdot f(x, y) = G(x, y) \cdot (\frac{\delta^2}{2}, 0, 1)^T$ tends to the last column of $G_0(y)$ for $x \to -\infty$, or equivalently $\delta \to 0$. By (20) the latter equals the last column of $G_0(0)$ and is hence a fixed vector $f_0$, independent of $y$. It follows that the ray through $f(x, y)$ tends to the ray through $f_0$, which is hence a boundary ray $\hat{\rho}$ of the cone $K$. \qed

In this section we described the boundary pieces of $K$ to which the affine sphere $f(x, y)$ is asymptotic as $x \to a$ or $x \to b$ for bounded $y$. In the cases $x \to \pm \infty$
we obtain a polyhedral boundary piece and a single ray, respectively, while for finite limits \( x \to a, x \to b \) the boundary piece is the conic hull of a vector-valued solution of ODE (17), (19), respectively. In the next section we assemble these pieces in order to construct the whole cones.

### 6 Computing the self-associated cones

The most important tool to assemble the boundary pieces obtained in the previous section are automorphisms of \( K \) which arise from certain invariants of the moving frame.

**Lemma 12** Let \(-\infty \leq a < b \leq +\infty\), let \( U = e^z \) be defined on the domain \( M = (a, b) + i\mathbb{R}, \) let \( u(z) = \chi(x) \) be a solution of (1), where \( z = x + iy, \) and let \( F(x, y) \) be a solution of frame equations (12). Set \( D = \text{diag}(1, -1, 1) \) and \( J = \text{diag}(1, -1, -1). \)

Then there exist linear maps \( T, \Sigma : \mathbb{R}^3 \to \mathbb{R}^3 \) with determinant \( \pm 1, \) respectively, and a unimodular quadratic form \( \Omega \) on \( \mathbb{R}^3 \) such that

\[
F(x, y + 2\pi) = TF(x, y), \quad F(x, -y) = \Sigma F(x, y)D, \quad \Omega^{-1}
\]

for all \( z = x + iy \in M. \) In particular, the third column \( f \) of \( F \) obeys

\[
f(x, y + 2\pi) = Tf(x, y), \quad f(x, -y) = \Sigma f(x, y)
\]

(21) for all \( z = x + iy \in M, \) and \( T, \Sigma \) are linear automorphisms of the surface given by \( f : M \to \mathbb{R}^3. \)

The products \( T^n \Sigma \) are similar to \( D, \) the products \( T^n \Omega^{-1} \) are symmetric with signature \((+−−)\) for all \( n \in \mathbb{Z}. \) The objects \( T, \Sigma, \Omega \) obey the relations

\[
\Sigma^2 = I, \quad T^{-1} = \Sigma T \Sigma^{-1}, \quad T \Omega^{-1} = \Sigma \Omega^{-1} \Sigma^T.
\]

**Proof** The coefficient matrices \( A, B \) in frame equations (12) are invariant with respect to the translation \((x, y) \mapsto (x, y+2\pi). \) Hence the gradient of the expression \( F(x, y + 2\pi)F^{-1}(x, y) \) vanishes, and this product equals a fixed unimodular matrix \( T. \)

Further, \( A(x, -y) = DA(x, y)D, \) \( B(x, -y) = -DB(x, y)D \) for all \( x + iy \in M. \) Hence the gradient of the expression \( F(x, -y)DF^{-1}(x, y) \) vanishes, and this product equals a fixed matrix \( \Sigma \) with determinant equal to \(-1. \)

Likewise, \( A(x, y - \pi)J = -JA^T(-y), \) \( B(x, y - \pi)J = JB^T(-y) \) for all \( x + iy \in M. \) Hence the gradient of the expression \( F(x, y - \pi)JF^T(x, -y) \) vanishes, and this product equals the inverse of a fixed unimodular matrix \( \Omega. \)

For an arbitrary \( x \in (a, b) \) and \( n \in \mathbb{Z} \) we obtain

\[
T^n \Sigma = (F(x, -n\pi + 2n\pi)F^{-1}(x, -n\pi))(F(x, -n\pi)DF^{-1}(x, n\pi)) \]

\[
= F(x, n\pi)DF^{-1}(x, n\pi),
\]

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and $T^n \Sigma$ is similar to $D$. In particular, $\Sigma^2 = (T \Sigma)^2 = I$, which yields $T^{-1} = \Sigma T \Sigma^{-1}$. In the same spirit,

$$T^n \Omega^{-1} = \left( F\left(x, -\left(n + \frac{1}{2}\right) \pi + 2n \pi\right) F^{-1}\left(x, -\left(n + \frac{1}{2}\right) \pi\right) \right) \cdot \left( F\left(x, -\left(n - \frac{1}{2}\right) \pi - \pi\right) JF^T\left(x, \left(n - \frac{1}{2}\right) \pi\right) \right)$$

$$= F\left(x, \left(n - \frac{1}{2}\right) \pi\right) JF^T\left(x, \left(n - \frac{1}{2}\right) \pi\right),$$

is symmetric and has the same signature as $J$. Finally,

$$\Sigma \Omega^{-1} \Sigma^T = \Sigma F(x, y - \pi) JF^T(x, -y) \Sigma^T = F(x, -y + \pi) DJDF^T(x, y)$$

$$= TF(x, -y - \pi) JF^T(x, y) = T \Omega^{-1}. $$

This completes the proof. □

Lemma 12 applies in particular to the affine spheres $f$ which are asymptotic to the boundary of a self-associated cone $K$. In this case $T, \Sigma$ are linear automorphisms of $K$. Since $T$ and $T^{-1}$ are similar, $T$ must have spectrum $\{1, \lambda, \lambda^{-1}\}$ with $\lambda$ on the real line or on the unit circle.

**Definition 2** We shall call the self-associated cone $K$ of

- **elliptic** type if its automorphism $T$ has spectrum $\{1, e^{i\varphi}, e^{-i\varphi}\}$ with $\varphi$ not a multiple of $\pi$;
- **parabolic** type if $T$ has spectrum in $\{-1, +1\}$;
- **hyperbolic** type if $T$ has spectrum $\{1, \lambda, \lambda^{-1}\}$ with $|\lambda| > 1$.

Let us investigate how the automorphisms $T, \Sigma$ act on the boundary pieces which have been obtained in the previous section.

**Lemma 13** Assume the notations of Lemma 8 and Corollary 1. Then the automorphism $T$ of $K$ acts on the polyhedral boundary piece described in Corollary 1 by a shift $W_n \mapsto W_{n+1}$, $\rho_n \mapsto \rho_{n+1}$ of the sequences of conic wedges and extreme rays. In particular, in Case R the automorphism permutes the extreme rays of $K$ cyclically, while in Case T the infinite sequence of extreme rays is shifted by one. The automorphism $\Sigma$ acts by maps $W_n \mapsto W_{-(n+1)}$, $\rho_n \mapsto \rho_{-(n+1)}$. In particular, it reverses the order of planar faces and extreme rays in the polyhedral boundary piece, mapping the face formed by the wedges $W_{-1}$, $W_0$ to itself.

Assume the notations of Lemma 9 (Lemma 10). Then the automorphisms $T, \Sigma$ of $K$ act on the solution $f_0$ of ODE (17) (ODE (19)) by $f_0(y + 2\pi) = Tf_0(y)$, $f_0(-y) = \Sigma f_0(y)$ for all $y \in \mathbb{R}$. In particular, the analytic boundary piece defined by $f_0$ is mapped to itself by both automorphisms.

Assume the notations of Lemma 11. Then the boundary ray $\hat{\rho}$ of $K$ is left invariant by both $T$ and $\Sigma$. 

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Proof The first (third) part of the lemma follows from Corollary 1 (Lemma 11) and relations (21).

Assume the notations of Lemma 9 (Lemma 10). By construction, for fixed $y \in \mathbb{R}$ the ray through $f(x, y)$ tends to the ray through $f_0(y)$ as $x \rightarrow a$ ($x \rightarrow b$). Relations (21) then imply that $T$ maps the ray through $f_0(y)$ to the ray through $f_0(y+2\pi)$, i.e., $Tf_0(y) = \mu(y)f_0(y+2\pi)$ for some analytic positive scalar function $\mu$. Differentiating this relation and inserting the derivatives into (16) (into (18)) yields $\mu = 1$. In the same way one obtains the relation $f_0(-y) = \Sigma f_0(y)$. \qed

We now consider the action of $T$, $\Sigma$ on the cones corresponding to Cases R and T from Lemma 5.

Lemma 14 In Case R the subgroup of automorphisms of $K$ generated by $T$ and $\Sigma$ is isomorphic to the dihedral group $D_{k+3}$. The cones $K$ are of elliptic type.

Proof It is more convenient to pass back to the domain $M = B_R$, with the cubic differential given by $U = z^k$. Let $F(z)$ be the moving frame of the corresponding affine sphere. Relations (21) transform to $f(e^{2\pi i/(k+3)}z) = Tf(z)$, $f(\bar{z}) = \Sigma f(z)$, which by continuity hold also for $z = 0$.

Hence $f(0) \in \mathbb{R}^3$ is an eigenvector of both $T$ and $\Sigma$ with eigenvalue 1. By Lemma 12 the tangent space to the surface $f(z)$ at $z = 0$ must be preserved by both $T$ and $\Sigma$. It can naturally be parameterized by the complex variable $z = x + iy \in \mathbb{C}$. By the above relations $T$ acts on this subspace by rotations $z \mapsto e^{2\pi i/(k+3)}z$ and $\Sigma$ by reflections $z \mapsto \bar{z}$. Hence the spectrum of $T$ is given by $\{1, e^{2\pi i/(k+3)}, e^{-2\pi i/(k+3)}\}$ and the claims in the lemma easily follow. \qed

Lemma 15 In Case T the subgroup of automorphisms of $K$ generated by $T$ and $\Sigma$ is isomorphic to the infinite dihedral group $D_{\infty}$. The spectrum of $T$ is real.

Proof Suppose for the sake of contradiction that $T$ has a complex eigenvalue $e^{i\varphi}$. Then $T$ is diagonalisable, and the sequence $T^n$, $n \in \mathbb{N}$, accumulates to the identity matrix. But then the sequence of vectors $f(x, y + 2\pi n)$ accumulates to $f(x, y)$, contradicting the fact that a complete hyperbolic affine sphere is an embedding. Hence the spectrum of $T$ is real.

The matrices $F(x, y + 2\pi n)$, $n \in \mathbb{Z}$, are mutually distinct. Hence $T^n$ are also mutually distinct. Thus the group generated by $T$, $\Sigma$ is isomorphic to $D_{\infty}$. \qed

Therefore in Case T the corresponding cones are of either parabolic or hyperbolic type. We now establish that this depends on whether $a = -\infty$ or $a$ finite.

Lemma 16 Let $-\infty < b \leq \infty$ and let $K$ be the cone corresponding to the complete solution of Wang’s equation on $M = (-\infty, b) + i\mathbb{R}$ with $U = e^z$. Then $K$ is of parabolic type, and $T$ has spectrum $\{1\}$ with a 1-dimensional eigenspace. The corresponding eigenvector generates a boundary ray $\hat{\rho}$ of $K$.

Proof In the previous section the moving frame was represented as a product $F(x, y) = G(x, y)V(x)$, where $G$, $V$ are unimodular matrix-valued functions.
relation $F(x, y + 2\pi) = TF(x, y)$ then yields $G(x, y + 2\pi) = TG(x, y)$. Passing to the limit $x \to -\infty$, we obtain $G_0(y + 2\pi) = TG_0(y)$. In particular,

$$T = G_0(2\pi)G_0(0)^{-1} = G_0(0)\begin{pmatrix} 1 & 0 & 0 \\ 4\pi & 1 & 0 \\ 8\pi^2 & 4\pi & 1 \end{pmatrix} G_0(0)^{-1}.$$ 

by virtue of (20). This shows that the only eigenvalue of $T$ is 1 and it has geometric multiplicity 1. The proof is completed by application of the third part of Lemma 13. □

**Corollary 2** Assume the conditions of Lemma 16.

If $b = +\infty$, then the boundary of $K$ is given by the closure of an infinite chain of 2-dimensional faces accumulating in both directions to $\hat{\rho}$.

If $b < +\infty$, then the boundary of $K$ is given by the closure of the conic hull of a vector-valued solution of ODE (19) satisfying (18) and tending to $\hat{\rho}$ both in the forward and the backward direction.

**Proof** Let $v \in \mathbb{R}^3$ be a non-zero vector. Then by Lemma 16 the sequence of rays generated by $T^nv$ tends to either $\hat{\rho}$ or $-\hat{\rho}$ as $n \to \pm \infty$. Clearly if $v$ lies on $\partial K$, this sequence also lies on $\partial K$ and hence tends to $\hat{\rho}$.

The proof is concluded by application of the first or second part of Lemma 13, setting $v$ equal to some non-zero vector in the conic wedge $W_0$ or to $f_0(0)$, respectively. □

Let us consider the case $b = +\infty$ in more detail. Lemma 16 states that the automorphism $T$ is similar to a full Jordan cell with eigenvalue 1. Hence in a convenient system of coordinates it can be written as $\begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$. We can still apply coordinate transformations generated by matrices $\begin{pmatrix} 1 & 2\alpha & \beta \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix}$, which leave $T$ invariant. The extreme ray $\rho_0$ of $K$ can by such a transformation be aligned with the basis vector $e_3$, otherwise all extreme rays $\rho_n = T^n\rho_0$ would lie in the linear subspace which is orthogonal to $e_3$. Then it easily follows that the extreme ray $\rho_n$ is generated by the vector $v_n = (n^2, -n, 1)^T$. In these coordinates the cone $K$ is therefore equal to the closed convex conic hull of the set $\{(n^2, n, 1)^T \mid n \in \mathbb{Z}\}$.

**Lemma 17** Let $-\infty < a < b \leq +\infty$ and let $K$ be the cone corresponding to the complete solution of Wang’s equation on $M = (a, b) + i\mathbb{R}$ with $U = e^z$. Then $K$ is of hyperbolic type, and $T$ has spectrum $\{1, \lambda, \lambda^{-1}\}$ with $\lambda > 1$. The boundary of $K$ is given by the closure of the union $\mathcal{Y}^- \cup \mathcal{Y}^+$ of two boundary pieces. Here $\mathcal{Y}^-$ is an analytic boundary piece given by the conic hull of a vector-valued solution $f_0^-(y)$ of ODE (17) satisfying (16). If $b < +\infty$, then $\mathcal{Y}^+$ is also analytic and given by the conic hull of a vector-valued solution $f_0^+(y)$ of ODE (19) satisfying (18). If $b = +\infty$, then $\mathcal{Y}^+$ is a two-sided infinite chain of planar faces of $K$.

**Proof** Let us assume first that $b$ is finite. By Lemmas 9, 10 the boundary $\partial K$ contains two analytic pieces $\mathcal{Y}^\pm$ which are the conic hulls of vector-valued solutions $f_0^\pm(y)$.
of ODEs (19), (17) satisfying (18), (16), respectively. The rays through \( f_0^+ (y) \) tend to some boundary rays \( \rho_+ \) of \( K \) as \( y \to \pm \infty \), respectively. In the same way, the rays through \( f_0^- (y) \) tend to boundary rays \( \rho_- \).

By the second part of Lemma 13 these limit rays must be generated by eigenvectors of \( T \) corresponding to positive eigenvalues \( \lambda_+ \), and \( \Sigma [\rho_{\pm}] = \rho_{\pm} \). Moreover, since the solutions \( f_0^\pm \) cannot tend to the origin as \( y \to \pm \infty \) due to (16), (18), we must have \( \lambda_+ \geq 1 \) and \( \lambda_- \leq 1 \).

If the four rays \( \rho_{\pm} \) are mutually distinct, then any three of them cannot be coplanar, otherwise at least one of the analytic boundary pieces has to lie in a flat face of \( K \), contradicting (16) or (18). It follows that \( T \) has an eigenspace of dimension 3 and must equal the identity, leading to a contradiction with \( f \) being an embedding.

Hence we have that \( \rho_+ = \rho_+ =: \rho_+ = \rho_- =: \rho_- \), because these two relations can hold only simultaneously by virtue of the symmetry \( \Sigma \). Since the two pieces \( \gamma_\pm \) link together at both ends, they together with the limit rays \( \rho_\pm \) make up the whole boundary \( \partial K \).

The product of the eigenvalues \( \lambda_\pm \) of \( T \) corresponding to \( \rho_\pm \) equals 1, because \( \Sigma \) conjugates \( T \) to \( T^{-1} \). Now suppose for the sake of contradiction that \( \lambda_+ = \lambda_- = 1 \). Then the whole 2-dimensional subspace spanned by \( \rho_{\pm} \) is left fixed by \( T \). However, this subspace intersects the interior of the cone \( K \), leading to a contradiction with \( f \) being an embedding. Hence \( \lambda_\pm = \lambda^{\pm} \) for some \( \lambda > 1 \), proving our claim.

In the case \( b = +\infty \) the argument is similar, with reference to Lemma 8 instead of Lemma 10 and with the analytic boundary piece \( \gamma^+ \) generated by the solution \( f_0^+ (y) \) replaced by an infinite chain of 2-dimensional faces. \( \square \)

For each of the pieces \( \gamma^{\pm} \) in Lemma 17 we have a unimodular initial condition \( f_0^{\pm} (0), f'_0 (0), f_0 (0)) \) to choose when integrating ODEs (17) and (19), respectively. Only one of these conditions can be chosen freely, corresponding to the choice of the representative in the \( SL(3, \mathbb{R}) \)-orbit of the cone. In order to choose the second initial condition correctly without having to integrate frame equations (2) we will make use of the automorphisms \( T, \Sigma \) and the invariant quadratic form \( \Omega \) which are common for both boundary pieces.

Suppose the boundary piece \( \gamma^- \) is defined by a solution \( f_0^- \) of (17). We have to show that the elements \( T, \Sigma, \Omega \) can be computed from this solution, and that these elements determine the initial conditions for the solution \( f_0^+ \) of (19) and hence the second boundary piece \( \gamma^+ \) uniquely. The second task is accomplished by the following lemma.

**Lemma 18** Let two linear automorphism \( T, \Sigma \) of \( \mathbb{R}^3 \) with determinant \( \pm 1 \), respectively, and a unimodular quadratic form \( \Omega \) on \( \mathbb{R}^3 \) be given such that the properties listed in the last paragraph of Lemma 12 are satisfied. Assume in addition that the spectrum of \( T \) equals \( \{1, \lambda, \lambda^{-1}\} \) for some \( \lambda > 1 \). Then there exists a unique non-trivial unimodular automorphism \( U \in SL(3, \mathbb{R}) \) which preserves \( T, \Sigma, \Omega \), namely the map negating the eigenspaces with eigenvalues \( \lambda, \lambda^{-1} \) of \( T \) and leaving the eigenspace with eigenvalue 1 fixed.

**Proof** Let \( U \in SL(3, \mathbb{R}) \) be a linear map preserving \( T, \Sigma, \Omega \). Since \( T \) has mutually distinct real eigenvalues, \( U \) has to leave each of its three eigenspaces invariant, multiplying the corresponding eigenvectors by constants \( \mu_1, \mu_\lambda, \mu_{\lambda^{-1}} \), respectively.
The map \( \Sigma \) conjugates \( T \) to \( T^{-1} \) and hence swaps the eigenspaces of \( T \) corresponding to the eigenvalues \( \lambda, \lambda^{-1} \). Since \( U \) preserves \( \Sigma \) it therefore has to satisfy the condition \( \mu_\lambda = \mu_{\lambda^{-1}} \).

The condition \( T \Omega^{-1} = \Omega^{-1}T^T \) implies \( T^T \Omega = \Omega T \), or equivalently \( \Omega(Tu, v) = \Omega(u, Tv) \) for all \( u, v \in \mathbb{R}^3 \). In particular, the eigenvectors of \( T \) are mutually \( \Omega \)-orthogonal, which in view of the non-singularity of \( \Omega \) implies that they have non-zero \( \Omega \)-length. But then \( U \) can multiply these eigenvectors only by constants \( \pm 1 \).

Finally, unimodularity of \( U \) implies \( \mu_1 \mu_\lambda \mu_{\lambda^{-1}} = \mu_1 \mu_\lambda^2 = \mu_1 = 1 \). Thus either \( U \) is the identity, or \( \mu_1 = 1, \mu_\lambda = \mu_{\lambda^{-1}} = -1 \).

Lemma 18 states that the elements \( T, \Sigma, \Omega \) determine the initial condition for \( f_0^+ \) up to the action of a single non-trivial map \( U \). However, this map multiplies the boundary rays \( \rho_\pm \) of the cone \( K \) by \( -1 \). Therefore only one of the two possible initial conditions for \( f_0^+ \) leads to a boundary piece \( \Upsilon^+ \) which matches the initial boundary piece \( \Upsilon^- \).

Let us now turn to the first task. Set \( \Phi_\pm(y) = (f_0^\pm(y), f_0^\pm(y), f_0^\pm(y)) \). By virtue of the second part of Lemma 13 we have \( \Phi_\pm(y + 2\pi) = T \Phi_\pm(y), \Phi_\pm(-y) = \Sigma \Phi_\pm(y) D \), and therefore

\[
T = \Phi_\pm(y + 2\pi) \Phi_\pm^{-1}(y), \quad \Sigma = \Phi_\pm(-y) D \Phi_\pm^{-1}(y)
\]

for arbitrary \( y \in \mathbb{R} \). In order to compute \( \Omega \) we use the definition of \( f_0^\pm \) together with relations (13),(15). Passing to the limit \( x \to a \) or \( x \to b \) in the expression \( \Omega^{-1} = F(x, y - \pi) J F^T(x, y) \) yields

\[
\begin{bmatrix}
\frac{3\tilde{\alpha}}{2} & 0 & 1 \\
0 & -1 & 0 \\
-1 & 0 & 0
\end{bmatrix}
\]

for arbitrary \( y \in \mathbb{R} \). Here \( \tilde{\alpha} \) is the coefficient from ODE (17) or (19), respectively.

If the boundary piece \( \Upsilon^+ \) is polyhedral, then we do not have the solution \( f_0^+ \) at our disposal and cannot assemble a corresponding matrix \( \Phi_+(y) \). Instead we shall use the following construction.

Assume the notations of Corollary 1. Let \( v_n, n \in \mathbb{Z} \), be a generator of the extreme ray \( \rho_n \) from \( \Upsilon^+ \). By the first part of Lemma 13 we may normalize \( v_n \) such that \( T v_n = v_{n+1}, \Sigma v_n = v_{-(n+1)} \) for all \( n \in \mathbb{Z} \). Multiplying all generators by a common factor we may further achieve \( \det \Phi_n = -1 \), where \( \Phi_n = (v_n, v_{n-1}, v_{n-2}) \) is a matrix composed of the generators of three consecutive extreme rays. We then have

\[
T = \Phi_{n+1} \Phi_n^{-1}, \quad \Sigma = \Phi_{-n+1} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \Phi_n^{-1}
\]
for arbitrary \( n \in \mathbb{Z} \). An involved calculation using the methods in (Dumas and Wolf 2015, Sect. 6) leads to the formula

\[
\Omega^{-1} = \Phi_n \cdot \begin{pmatrix} -1 & 0 & 0 \\ 0 & s & -1 \\ 0 & -1 & 0 \end{pmatrix} \cdot \Phi_{-n}^T, \tag{25}
\]

where \( n \in \mathbb{Z} \) is arbitrary and \( s = \text{tr} \ T > 3 \) turns out to be the parameter of the solution \( v_{x,0} \) of the Painlevé equation (26) determining the conformal factor \( u \).

In this section we described the boundaries and symmetries of the self-associated cones corresponding to the solutions \( (u, U) \) of Wang’s equation in Cases R and T of Lemma 5. In the next section we summarize our findings in formal theorems.

7 Results

The self-associated cones can be grouped into three types according to Definition 2, which are described in the theorems below. Besides these, the only self-associated cones are the ellipsoidal cones, which form the \( SL(3, \mathbb{R}) \)-orbit of the Lorentz cone \( L_3 \) described in Case 0 of Section 3.

**Theorem 3** The \( SL(3, \mathbb{R}) \)-orbits of self-associated cones of elliptic type are parameterized by a discrete parameter \( k \in \mathbb{N} \) and a continuous parameter \( R \in (0, +\infty) \). Every such cone possesses a subgroup of linear automorphisms which is isomorphic to the dihedral group \( D_{k+3} \). This subgroup is generated by an automorphism \( T \) with spectrum \( \{1, e^{2\pi i/(k+3)}, e^{-2\pi i/(k+3)}\} \) and a reflection \( \Sigma \). The cone possesses an interior ray which is left fixed by every automorphism in the subgroup.

The complete hyperbolic affine 2-sphere which is asymptotic to a cone of elliptic type corresponding to the parameters \( (k, R) \) possesses an isothermal parametrization with domain \( B_R \subset \mathbb{C} \), the open disc of radius \( R \), and cubic differential \( U = z^k \). The corresponding affine metric \( h = e^u |dz|^2 \) is given by the conformal factor \( e^{\mu(z)} = \sqrt{\frac{k+3}{2}} |z|^{(k-1)/2} v_{x_k,c}(t) \), where \( t = \sqrt{\frac{32}{(k+3)^2} |z|^{(k+3)/2}} \), \( s_k = 1 + 2 \cos \frac{2\pi}{k+3} \), and \( v_{x_k,c} \) is a solution of the Painlevé III equation (26). Here for every fixed \( k \) the parameter \( c \) is strictly monotonely decreasing in \( R \) such that \( c = 0 \) for \( R = +\infty \), and \( c \to +\infty \) for \( R \to 0 \). For finite \( R \) it is determined by the condition that \( v_{x_k,c} \) is positive on \( (0, t_0) \) and has a double pole at \( t_0 = \sqrt{\frac{32}{(k+3)^2} R^{(k+3)/2}} \) with expansion (28).

For every \( k \in \mathbb{N} \) the self-associated cones corresponding to the parameters \( (k, +\infty) \) are linearly isomorphic to a cone over a regular \((k + 3)\)-gon.

The self-associated cones which correspond to the parameters \( (k, R) \) for finite \( R \) can be constructed as follows. Choose \( W \in SL(3, \mathbb{R}) \) arbitrarily and solve ODE (19) with initial values \( (f_0^\prime, f_0^\prime, f_0)|_{y=0} = -W \), where \( e^\mu = \frac{R^{k+3}}{(k+3)^3} \) and \( \tilde{\alpha} = \frac{\alpha f_0}{8} - \frac{1}{48} \). Then the solution \( f_0 : \mathbb{R} \to \mathbb{R}^3 \) traces a closed analytic \( 2(k + 3)\pi\)-periodic curve in \( \mathbb{R}^3 \). The boundary of the cone is analytic at every non-zero point and can be obtained as the conic hull of this curve. The \( SL(3, \mathbb{R}) \)-orbit of self-associated cones corresponding to \( (k, R) \) can be parameterized by the initial condition \( W \).
In Fig. 1 we present compact affine sections of some self-associated cones of elliptic type. Since each interior ray intersects the affine sphere inscribed in the cone in exactly one point, we may project the isothermal coordinate \( z \in B_R \) onto the interior of the affine section. A uniformly spaced polar coordinate grid on \( B_R \) projects on the sections as shown in the figure.

**Theorem 4** The \( SL(3, \mathbb{R}) \)-orbits of self-associated cones of parabolic type are parameterized by a continuous parameter \( b \in (-\infty, +\infty] \). Every such cone \( K \) possesses a subgroup of linear automorphisms which is isomorphic to the infinite dihedral group \( D_\infty \). This subgroup is generated by an automorphism \( T \) with spectrum \( \{1\} \) and eigenvalue of geometric multiplicity 1, and a reflection \( \Sigma \). There exists a boundary ray \( \hat{\rho} \) of \( K \) which is left fixed by every automorphism in the subgroup.

The complete hyperbolic affine 2-sphere which is asymptotic to a cone of parabolic type corresponding to the parameter \( b \) possesses an isothermal parametrization with domain \( (-\infty, b) + i \mathbb{R} \) and cubic differential \( U = e^z \). The corresponding affine metric \( h = e^u |dz|^2 \) is given by the conformal factor \( e^u(z) = \frac{v_{3,c}(t)}{8} \), where \( t = \sqrt{32} e^{Re z/2} \), and \( v_{3,c} \) is a solution of the Painlevé III equation (26). Here \( c \) is a strictly monotonely decreasing function of \( b \) such that \( c = 0 \) for \( b = +\infty \), and \( c \to +\infty \) for \( b \to -\infty \).

For finite \( b \) it is determined by the condition that \( v_{3,c} \) is positive on \((0, t_0)\) and has a double pole at \( t_0 = \sqrt{32} e^{b/2} \) with expansion (28).

Any cone corresponding to the parameter value \( b = +\infty \) is linearly isomorphic to the cone given by the closed convex conic hull of the infinite set of vectors \( v_n = (n^2, n, 1)^T \), \( n \in \mathbb{Z} \).
The self-associated cones which correspond to a finite parameter \( b \) can be constructed as follows. Choose \( W \in SL(3, \mathbb{R}) \) arbitrarily and solve vector-valued ODE (19) with initial values \((f''_0, f'_0, f_0)|_{y=0} = -W\), where \( \tilde{\alpha} = \frac{ae^{b/2}}{\sqrt{2}} - \frac{1}{48} \). Then the solution \( f_0 : \mathbb{R} \to \mathbb{R}^3 \) traces an analytic curve in \( \mathbb{R}^3 \). The conic hull of this curve is analytic at every non-zero point and meets itself at the ray \( \hat{\rho} \). The boundary of the cone can be obtained as the union of the conic hull of \( f_0 \) with the ray \( \hat{\rho} \). The \( SL(3, \mathbb{R}) \)-orbit of self-associated cones corresponding to the parameter \( b \) can be parameterized by the initial condition \( W \).

In Fig. 2 we present compact affine sections of self-associated cones of parabolic type for the parameter values \( b = -2, -1, 0, 1 \). As in Fig. 1, we project a uniformly spaced grid in the domain \( M \) on the section.

**Theorem 5** The \( SL(3, \mathbb{R}) \)-orbits of self-associated cones of hyperbolic type are parameterized by two continuous parameters \( -\infty < a < b \leq +\infty \). Every such
cone $K$ possesses a subgroup of linear automorphisms which is isomorphic to the infinite dihedral group $D_\infty$. This subgroup is generated by an automorphism $T$ with spectrum $\{1, \lambda, \lambda^{-1}\}$, $\lambda > 1$, and a reflection $\Sigma$. The cone possesses two boundary rays $\rho_\pm$ which are generated by eigenvectors of $T$ with eigenvalues $\lambda^{\pm1}$, respectively, and are mapped to each other by $\Sigma$.

The complete hyperbolic affine 2-sphere which is asymptotic to a cone of hyperbolic type corresponding to the parameters $(a, b)$ possesses an isothermal parametrization with domain $(a, b)+i\mathbb{R}$ and cubic differential $U = e^z$. The corresponding affine metric $h = e^u|dz|^2$ is given by the conformal factor $e^{u(z)} = \frac{t(u)}{8}$, where $t = \sqrt{32}e^{Rez/2}$, and $v$ is a solution of the Painlevé III equation (26). This solution is characterized by the condition that it is positive on $(t_a, t_b)$ with $t_a = \sqrt{32}e^{a/2}$, $t_b = \sqrt{32}e^{b/2}$, has a double pole at $t_a$ with expansion (28) featuring a constant $\alpha_a$, and for finite $b$ has a double pole at $t_b$ with expansion (28) featuring a constant $\alpha_b$. If $b = +\infty$, then $v$ is given by the solution $v_{s,0}$, where $s$ is a strictly monotonely increasing function of $a$ such that $s \to 3$ for $a \to -\infty$, and $s \to +\infty$ for $a \to +\infty$.

The boundary of $K$ is the closure of the union of two pieces $\Upsilon^\pm$ which join at the rays $\rho_\pm$.

The piece $\Upsilon^-$ can be constructed as follows. Choose $W \in SL(3, \mathbb{R})$ arbitrarily and solve vector-valued ODE (17) with initial values $\Phi_-(0) = (f_0^-\', f_0^-\,, f_0^-)_y |_{y=0} = W$, where $\tilde{\alpha} = \frac{a_\ell e^{a/2}}{\sqrt{2}} - \frac{1}{48}$. Then the solution $f^-_0: \mathbb{R} \to \mathbb{R}^3$ traces an analytic curve in $\mathbb{R}^3$, whose conic hull is analytic at every non-zero point and equals $\Upsilon^-$. The curve $f^-_0$ tends to $\rho_-$ for $y \to -\infty$. The automorphisms $T$, $\Sigma$ are given by formula (22), and formula (23) yields a quadratic form $\Omega$ on $\mathbb{R}^3$.

The piece $\Upsilon^+$ can be constructed as follows.

Case $b < +\infty$: Solve vector-valued ODE (19), where $\tilde{\alpha} = \frac{a_\ell e^{b/2}}{\sqrt{2}} - \frac{1}{48}$. The initial values $\Phi_+(0) = (f_0^+\', f_0^+\', f_0^+)_y |_{y=0}$ are chosen such that they produce the same objects $T$, $\Sigma$, $\Omega$ by relations (22), (23), and such that $f_0^+(y)$ tends to $\rho_\pm$ as $y \to \pm\infty$, respectively. Then the solution $f^+_0: \mathbb{R} \to \mathbb{R}^3$ traces an analytic curve in $\mathbb{R}^3$, whose conic hull is analytic at every non-zero point and equals $\Upsilon^+$.

Case $b = +\infty$: Set $s = \text{tr} T$ and let $v_0 \in \mathbb{R}^3$, $v_n = T^n v_0$, $\Phi_n = (v_n, v_{n-1}, v_{n-2})$, $n \in \mathbb{Z}$, be such that relations (24), (25) produce the same objects $T$, $\Sigma$, $\Omega$, and $v_n \to \rho_\pm$ for $n \to \pm\infty$, respectively. Then $\Upsilon^+$ is given by a two-sided infinite chain of planar faces $F_n$ spanned by the vectors $v_n$, $v_{n+1}$, $n \in \mathbb{Z}$.

The $SL(3, \mathbb{R})$-orbit of self-associated cones corresponding to $(a, b)$ can be parameterized by the initial condition $W$.

In Fig. 3 we present compact affine sections of self-associated cones of hyperbolic type for different parameter values $a$, $b$. As in the previous cases we project a uniformly spaced grid in the domain $M$ on the affine section.

8 Open problems

A cone $K \subset \mathbb{R}^3$ is self-associated if whenever there is another cone $\tilde{K} \subset \mathbb{R}^3$ such that the affine spheres which are asymptotic to the boundaries $\partial K$, $\partial \tilde{K}$ are isometric,
Fig. 3 Compact affine sections of self-associated cones of hyperbolic type with different parameter values \((a, b)\). The intervals \((a, b)\) are \((-3, 2), (-1, 0), (1, 2)\) in the first row, \((-4, -2), (-2, 0), (0, 2)\) in the second row, \((-6, 2), (-4, 0), (-2, 2)\) in the third row, and \((-12, -4), (-6, 2), (-14, 2)\) in the last row. The domain \(M = (a, b) + i\mathbb{R}\) with uniform grid in cartesian coordinates is projected onto the interior of the section. The step size in the horizontal direction equals \(\frac{1}{4}\), in the vertical direction \(\frac{7}{12}\)
the cones $K$, $\tilde{K}$ are linearly isomorphic. Generalization to higher dimension suggests the following problem.

**Problem 1** For $n > 3$, which cones $K \subset \mathbb{R}^n$ satisfy the following condition: whenever there exists another cone $\tilde{K} \subset \mathbb{R}^n$ such that the complete hyperbolic affine spheres with mean curvature $H = -1$ which are asymptotic to the boundaries $\partial K$, $\partial \tilde{K}$ are isometric, the cones $K$, $\tilde{K}$ are linearly isomorphic?

The affine spheres which are asymptotic to the boundary of self-associated cones possess a continuous group of isometries which multiply the cubic differential representing the cubic form by unimodular complex constants different from 1, and hence do not preserve the cubic form. We may then ask whether this is possible in higher dimensions.

**Problem 2** For $n > 3$, do there exist complete hyperbolic affine spheres $f : M \to \mathbb{R}^n$ with continuous groups of isometries which do not preserve the cubic form?

Any associated family of $SL(3, \mathbb{R})$-orbits of cones $K \subset \mathbb{R}^3$ admits a natural action of the circle group $S^1$, multiplying the holomorphic function $U$ representing the cubic differential by unimodular complex constants. For a self-associated cone its associated family consists of a single orbit. On the other hand, for a cone which is merely linearly isomorphic to its dual cone by a unimodular isomorphism we have that the action of the element $e^{\pi i} \in S^1$ leaves the orbits in the associated family fixed. This is a weaker condition than being self-associated. We may then consider the following intermediate notions.

**Problem 3** Let $k \geq 3$ be an integer. Find cones $K \subset \mathbb{R}^3$ (other than the self-associated cones) such that the action of the element $e^{2\pi i/k} \in S^1$ on its associated family of $SL(3, \mathbb{R})$-orbits of cones leaves the orbits fixed.

Affine spheres can be viewed as minimal Lagrangian manifolds in a certain para-Kähler space form (Hildebrand 2011). The affine spheres corresponding to self-associated cones can hence be represented as minimal Lagrangian surfaces with a continuous symmetry group. Similar surfaces in the space $\mathbb{C}P^2$ have been considered in Dorfmeister and Ma (2016a, b) with loop group methods (Dorfmeister and Eitner 2001). Surfaces with rotational and with translational symmetries have been distinguished and their loop group decompositions have been computed. Loop group methods are applicable to the case of definite affine 2-spheres as well (Liang and Ji 2010; Lin et al. 2017).

**Problem 4** Find the loop group decompositions of the affine spheres corresponding to the self-associated cones.

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The degenerate Painlevé equation

In this section we consider the Painlevé III equation

\[ tv'' = t(v')^2 - vv' + v^3 - t \]  
(26)

with parameters \((\alpha, \beta, \gamma, \delta) = (1, 0, 0, -1)\), corresponding to the degenerate case \(D_7\) in the classification of Sakai (2001). The ensemble of its solutions \(v_{s,c}(t)\) can be parameterized by two complex parameters \(s, c\) which encode monodromy data of an associated linear system of ODEs (Its and Novokshenov 1986). Equation (26) has no solution expressible in terms of algebraic and classical transcendental functions except \(v_{0,0}(t) = t^{1/3}\) (Ohyama et al. 2006). Equation (26) has been studied in Kitaev (1987). In particular, the asymptotics of the solutions in the neighbourhood of the singular points \(t = 0\) and \(t = +\infty\) have been computed. We summarize these results as follows.

**Proposition 6** The real solutions which are pole-free and positive in a neighbourhood of \(t = 0\) on the positive real axis are parameterized by \((s, c) \in [-1, 3] \times \mathbb{R}\), or equivalently \((\lambda, c) \in [0, 1] \times \mathbb{R}\), where \(s^{-1} = \cos(\lambda \pi)\), and have the following asymptotics. For \(s = 3\) or \(\lambda = 0\)

\[ v_{3,c}(t)t \sim \frac{2}{(\log t - 3 \log 2 + \frac{3}{2} \gamma + \frac{3}{4} c)^2}, \]

for \(0 < s < 3\) or \(\lambda \in (0, \frac{2}{3})\)

\[ v_{s,c}(t)t \sim \frac{2\lambda^2}{\sinh^2\left(\lambda \log t - 3 \log 2 + \frac{1}{2} \log \frac{\Gamma(1 - \frac{s}{2}) \Gamma(1 - \lambda)}{\Gamma(1 + \frac{s}{2}) \Gamma(1 + \lambda)} + \frac{3}{4} c\right)}, \]

for \(s = 0\) or \(\lambda = \frac{2}{3}\)

\[ v_{0,c}(t)t^{-1/3} \sim e^c \left(1 + \frac{9(e^c - e^{-2c})}{16} t^{4/3}\right), \]

for \(-1 < s < 0\) or \(\lambda \in (\frac{2}{3}, 1)\)

\[ v_{s,c}(t)t^{-1} \sim \frac{1}{2 - 2\lambda} \sinh\left(2(\lambda - 1) \log t + (6 - 6\lambda) \log 2 - \log \frac{-\Gamma(\frac{s}{2}) \Gamma(-1 + \lambda)}{\Gamma(1 - \frac{s}{2}) \Gamma(1 - \lambda)} + 3(1 - \lambda)c\right), \]

for \(s = -1\) or \(\lambda = 1\)

\[ v_{-1,c}(t)t^{-1} \sim -\log t + 2 \log 2 - \frac{3}{2} \gamma + \frac{3}{2} c, \]
where $\gamma$ is the Euler–Mascheroni constant.$^1$

The real solutions which are pole-free and positive in a neighbourhood of $t = +\infty$ on the positive real axis are parameterized by $s \in \mathbb{R}$, the second parameter $c$ being zero, and have the asymptotics

$$v_{s,0}(t) t^{-1/3} - 1 \sim 3^{-1/4} \pi^{-1/2} s t^{-1/3} e^{-3\sqrt{3} t^{2/3}/2}.$$ 

The solutions $v_{s,0}$ with $s \in [-1, 3]$ appear in both lists, and their asymptotics both at $t = 0$ and $t = +\infty$ are known.

For studying positive solutions $v(t)$ of (26) it is convenient to make the substitution $\tau = \log t$, $g = \log v - \frac{1}{2} \log t$. Then (26) is equivalent to the second-order ODE

$$\frac{d^2 g}{d\tau^2} = e^{4\tau/3} (e^g - e^{-2g})$$

(27)

on the function $g(\tau)$. By virtue of the strict monotonicity of the function $e^g - e^{-2g}$ we have $g'' > \tilde{g}''$ for two solutions $g, \tilde{g}$ whenever $g > \tilde{g}$.

We obtain the following monotonicity results.

**Lemma 19** Let $t_a \geq 0$ and $s < s'$ be real numbers such that the solutions $v_{s,0}, v_{s',0}$ of (26) are positive on the interval $(t_a, +\infty)$. Let $g_s, g_{s'}$ be the corresponding solutions of (27). Then $g_{s'} > g_s, g_{s'}' < g_s'$ on $(\log t_a, +\infty)$ and $v_{s,0} < v_{s',0}$ on $(t_a, +\infty)$.

In particular, if the solution $v_{s,0}$ is positive on $\mathbb{R}_+$, then for $s' > s$ the solution $v_{s',0}$ is positive either on $\mathbb{R}_+$ or up to the right-most pole, and for $s' < s$ the solution $v_{s',0}$ must cease to be positive before it reaches its right-most pole.

**Proof** The asymptotics of $v_{s,0}$ for $t \to +\infty$ yields

$$g_s(\tau) \sim 3^{-1/4} \pi^{-1/2} s e^{-\tau/3 - 3\sqrt{3} \exp(2\tau/3)/2}$$

as $\tau \to +\infty$. Define the function $\delta = g_{s'} - g_s$ on $(\log t_a, +\infty)$. We get

$$\delta(\tau) \sim 3^{-1/4} \pi^{-1/2} (s' - s) e^{-\tau/3 - 3\sqrt{3} \exp(2\tau/3)/2}$$

as $\tau \to +\infty$. Moreover, $\delta'' > 0$ for all $\tau$ such that $\delta(\tau) > 0$.

Now we have $\delta(\tau) > 0$ for large enough $\tau$, and $\lim_{\tau \to +\infty} \delta(\tau) = 0$. Hence there exists a sequence $\tau_k \to +\infty$ such that $\delta(\tau_k) > 0, \delta'(\tau_k) \leq 0$ for all $k$. But then $\delta''(\tau) > 0, \delta'(\tau) < 0, \delta(\tau) > 0$ for all $\tau < \tau_k$, and hence for all $\tau \in (\log t_a, +\infty)$. The first claim of the lemma now easily follows.

Let us prove the second claim. For $s' > s$ we have that $v_{s',0}(t) > v_{s,0}(t) > 0$ as long as $v_{s',0} > 0$ on $(t, +\infty)$. Therefore the only way for $v_{s',0}$ to cease to be positive is to cross a pole. For $s' < s$ we have $v_{s',0}(t) < v_{s,0}(t)$ as long as $v_{s',0} > 0$ on $(t, +\infty)$. Hence $v_{s',0}$ must first hit the zero value before it can reach a pole. $\Box$

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$^1$ In the original paper Kitaev (1987) a factor of 2 is missing in the last expression.
Lemma 20  Let \( s \in [0, 3] \), \( t_b > 0 \), and \( c < c' \) be real numbers such that the solutions \( v_{s,c}, v_{s,c}' \) of (26) are positive on the interval \( (0, t_b) \). Let \( g_c, g_c' \) be the corresponding solutions of (27). Then \( g_c > g_c', g_c' > g_c' \) on \( (-\infty, \log t_b) \) and \( v_{s,c} < v_{s,c}' \) on \( (0, t_b) \).

In particular, if the solution \( v_{s,0} \) is positive on \( \mathbb{R}_+ \), then for \( c > 0 \) the solution \( v_{s,c} \) is positive up to the left-most pole, and for \( c < 0 \) the solution \( v_{s,c} \) cannot be positive up to its left-most pole.

Proof The asymptotics of \( v_{s,c} \) for \( t \to 0 \) from Proposition 6 yields

\[
g_c(t) \sim \begin{cases} -\frac{3}{4} t + 2 \log 2 - 2 \log(-\tau + 3 \log 2 - \frac{3}{4} \lambda), & s = 3; \\ (-\frac{3}{4} + 2\lambda) t + 2 \log \lambda + \frac{3}{2} \lambda \log (-2 \delta - 2 \lambda) \log 2 - \log \frac{\Gamma(1-\lambda/2)\Gamma(1+\lambda/2)}{\Gamma(1+\lambda/2)\Gamma(1-\lambda/2)} + \frac{3\lambda}{4} c, & s \in (0, 3); \\ c + \frac{9(e^{-2c} - e^{-2\lambda})}{16} e^{3\lambda t/3}, & s = 0 \end{cases}
\]

as \( t \to -\infty \). Here \( \frac{s-1}{2} = \cos(\pi \lambda) \). Define the function \( \delta = g_c' - g_c \) on \( (-\infty, \log t_b) \).

We get

\[
\delta(t) \sim \begin{cases} \frac{3(c' - c)}{2 t}, & s = 3; \\ \frac{3\lambda(c' - c)}{2}, & s \in (0, 3) \end{cases}
\]

as \( t \to -\infty \). Recall that \( \delta'' > 0 \) for all \( \tau \) such that \( \delta > 0 \).

It follows that \( \delta > 0, \delta'' > 0 \) for large enough \( |\tau| \), and \( \lim_{|\tau| \to \infty} \delta(t) < \infty \). But then also \( \delta' > 0 \) for large enough \( |\tau| \). Therefore \( \delta \) cannot become negative on the whole interval \( (-\infty, \log t_b) \), and the first claim of the lemma follows.

In order to prove the second claim, compare the solution \( v_{s,c} \) to the solution \( v_{s,0} \). For \( c > 0 \) we have that \( v_{s,c}(t) > v_{s,0}(t) \) as long as \( v_{s,c} > 0 \) on \( (0, t) \). Therefore the only way for \( v_{s,c} \) to cease to be positive is to cross a pole. For \( c < 0 \) we have \( v_{s,c}(t) < v_{s,0}(t) \) as long as \( v_{s,c} > 0 \) on \( (0, t) \). Therefore the only way for \( v_{s,c} \) to cease to be positive is to become zero. \( \square \)

By the Painlevé property the solutions of (26) can be extended to meromorphic functions on the universal cover of \( \mathbb{C} \setminus \{0\} \). If \( t = t_0 \neq 0 \) is a singularity of the solution, then inserting its Laurent expansion around \( t_0 \) into (26) easily yields

\[
v(t) = \frac{2t_0}{(t - t_0)^2} + \alpha - \frac{\alpha}{t_0} (t - t_0) + \frac{3\alpha^2 t_0 + 9\alpha}{10t_0^2} (t - t_0)^2 \\
\quad - \frac{3\alpha^2 t_0 + 4\alpha}{5t_0^3} (t - t_0)^3 + \cdots (28)
\]

for some \( \alpha \in \mathbb{R} \), i.e., \( t_0 \) is a double pole of the solution.

The family \( (v_{\alpha})_{\alpha \in \mathbb{R}} \) of solutions having a double pole at a fixed \( t_0 \) satisfies a similar monotonicity result.

Lemma 21  Let \( t_0 > 0 \), \( \alpha < \alpha' \) be real numbers, and let \( v_{\alpha}, v_{\alpha'} \) be the solutions of (26) with the corresponding expansions (28). Suppose that \( v_{\alpha}, v_{\alpha'} \) are positive on the interval \( (T, t_0) \) for some \( T \in (-\infty, t_0) \), and let \( g_{\alpha}, g_{\alpha'} \) be the corresponding solutions...
of (27) on \((T, \tau_0)\), where \(\tau_0 = \log t_0\). Then we have \(g_\alpha < g_{\alpha'}\), \(g'_\alpha > g'_\alpha\), \(g''_\alpha < g''_\alpha\) on \((T, \tau_0)\), and \(\nu_0 < \nu_{\alpha'}\) on \((T, \tau_0)\).

**Proof** Set \(\delta = g_{\alpha'} - g_\alpha\). Since the ratio \(\frac{\nu_{\alpha'}}{\nu_0}\) is analytic and non-zero in the neighbourhood of \(\tau_0\), its logarithm is also analytic and by virtue of (28) has the expansion

\[
\delta(\tau_0 + d) = \frac{\alpha' - \alpha}{2\tau_0} d^2 - \frac{\alpha' - \alpha}{2\tau_0^2} d^3 + \frac{t_0((\alpha')^2 - \alpha^2) + 18(\alpha' - \alpha)}{40\tau_0^3} d^4 - \cdots.
\]

It follows that for \(\tau < \tau_0\) close enough to \(\tau_0\) we have \(\delta(\tau) > 0\) and \(\delta'(\tau) < 0\). But \(\delta'' > 0\) as long as \(\delta > 0\), hence these relations are valid on the whole interval \((\tau_0, \log T)\). The claim of the lemma now easily follows. □

**Corollary 3** Let \(s, s' \in (-1, 3)\) and \(c, c' \in \mathbb{R}\) be numbers, and suppose that the corresponding solutions \(v_{s,c}, v'_{s',c'}\) of (26) are positive on \((0, \tau_0)\) and have a double pole at \(\tau_0\), such that the corresponding constants \(\alpha, \alpha'\) in their expansions (28) satisfy \(\alpha < \alpha'\). Then \(s < s'\).

**Proof** Let \(g_{s,c}, g'_{s',c'}\) be the corresponding solutions of (27), and set \(\delta = g'_{s',c'} - g_{s,c}\). By Lemma 21 we have \(\delta' < 0, \delta'' > 0\) on \((-\infty, \log \tau_0)\). From Proposition 6 we get after some calculations that the asymptotics of \(g_{s,c}, g'_{s',c'}\) in the limit \(\tau \to -\infty\) are given by

\[
g_{s,c} \sim \left(\frac{2\lambda - 4}{3}\right) \tau + C(s, c), \quad g'_{s',c'} \sim \left(\frac{2\lambda' - 4}{3}\right) \tau + C(s', c'),
\]

where \(\frac{\pi}{2} = \cos(\pi\lambda), \frac{\pi}{2} = \cos(\pi\lambda')\), and \(C(s, c), C(s', c')\) are constants. It follows that \(\delta \sim 2(\lambda' - \lambda)\tau + \text{const}\) as \(\tau \to -\infty\). But \(\delta\) is a strictly decreasing convex function, and therefore we must have \(\lambda' < \lambda\). This implies \(s' > s\). □

Finally, we shall estimate the error in the asymptotics of the solution \(v_{3,c}\) provided in Kitaev (1987). For given \(c \in \mathbb{R}\), set for brevity \(c' = 3 \log 2 - \frac{3}{4} \gamma - \frac{3}{4} c\).

**Lemma 22** The asymptotics of the solution \(v_{3,c}\) for \(t \to 0\) is given by

\[
\frac{2t^{-1}}{(-\log t + c')^2} = \frac{r^3(-\log t + c' + 1)^2}{32}.
\]

The asymptotics of its derivative is given by

\[
\frac{dv_{3,c}}{dt} \sim -\frac{2t^{-2}}{(-\log t + c')^2} + \frac{4t^{-2}}{(-\log t + c')^3} - \frac{3t^2(-\log t + c' + \frac{2}{3})^2}{32}.
\]

**Proof** By Proposition 6 the solution \(g(\tau)\) of (27) which corresponds to \(v_{3,c}\) has asymptotics

\[
g(\tau) \sim \log 2 - \frac{4}{3} \tau - 2 \log(-\tau + c') =: \tilde{g}(\tau).
\]
Note that the function \( \tilde{g} \) satisfies the ODE \( \tilde{g}'' = e^{4\tau/3} e^{\tilde{g}} \), and hence \( \delta = \tilde{g} - g \) satisfies

\[
\delta'' = e^{4\tau/3} (e^{\tilde{g}} - e^{g} + e^{-2g}) = 2 \left( -\tau + c' \right)^{-2} (1 - e^{-\delta}) + \mu(\tau),
\]

(29)

where \( \mu(\tau) = e^{4\tau/3} e^{-2g} \approx \frac{1}{4} e^{4\tau} (-\tau + c')^4 \).

Define the functions

\[
\mu(\tau) = \int_{-\infty}^{\tau} \int_{-\infty}^{s} \mu(r) \, dr \, ds, \quad \overline{\mu}(\tau) = \frac{1}{-\tau + c'} \int_{-\infty}^{\tau} (s - c')^2 \int_{-\infty}^{s} \frac{\mu(r)}{-r + c'} \, dr \, ds.
\]

Note that \( \overline{\mu''} = \frac{2\pi}{(\tau - c')^2} + \mu > \frac{2}{(\tau - c')^2} (1 - e^{-\overline{\mu}}) + \mu \), and both \( \mu \) and \( \overline{\mu} \) have the same asymptotics \( e^{\frac{4}{64} (\tau - c')^4} \) as \( \tau \to -\infty \). We shall now show that \( \mu < \delta < \overline{\mu} \).

Let \( \tau_0 < c' \) be such that \( g \) and hence also \( \mu \) and \( \delta \) are defined on \( (-\infty, \tau_0] \). Consider the integral operator \( \mathcal{I} \) taking a smooth function \( f : (-\infty, \tau_0] \to \mathbb{R} \) to \( \mathcal{I}f(\tau) = \int_{-\infty}^{\tau} \int_{-\infty}^{s} \frac{2}{(-r + c')^2} (1 - e^{-\overline{\mu}(r)}) + \mu(r) \, dr \, ds \). If \( f \) is such that \( 0 \leq f(\tau) \leq \overline{\mu}(\tau) \) for all \( \tau \in (-\infty, \tau_0] \), then

\[
0 \leq \mathcal{I}f(\tau) \leq \int_{-\infty}^{\tau} \int_{-\infty}^{s} \frac{2}{(-r + c')^2} (1 - e^{-\overline{\mu}(r)}) + \mu(r) \, dr \, ds
\]

\[
< \int_{-\infty}^{\tau} \int_{-\infty}^{s} \overline{\mu''}(r) \, dr \, ds = \overline{\mu}(\tau)
\]

for \( \tau \leq \tau_0 \). Hence \( \mathcal{I} \) is well-defined for every such function \( f \). Clearly if \( f < \tilde{f} \) everywhere on \( (-\infty, \tau_0] \), then also \( \mathcal{I}f < \mathcal{I}\tilde{f} \) everywhere on this interval.

Define now \( \delta_0 \equiv 0 \) and recursively \( \delta_{k+1} = \mathcal{I}\delta_k, k \geq 0 \). Then \( 0 = \delta_0 < \delta_1 = \mu \), and hence the sequence \( \delta_k(\tau) \) is strictly increasing for every \( \tau \leq \tau_0 \). On the other hand, this sequence is upper bounded by \( \overline{\mu}(\tau) \). Hence \( \delta_k \) converges point-wise to some limit function \( \delta^* \). This function is a fixed point of the operator \( \mathcal{I} \) and hence smooth and a solution of ODE (29). Thus it coincides with the sought function \( \delta \). By construction we obtain the desired bounds \( \mu < \delta < \overline{\mu} \) and therefore the asymptotics \( \delta \sim \frac{e^{4\tau} (\tau + c' + \frac{1}{2})^4}{64} \).

Switching back to the solution \( v(t) \) of (26) we get

\[
v_{3,c}(t) = t^{1/3} e^{\tilde{g} - \delta} \sim \frac{2t^{-1}}{(-\log t + c')^2} \left( 1 - \frac{t^4 (-\log t + c' + \frac{1}{2})^4}{64} \right),
\]

which yields the desired error term for \( v_{3,c} \).
Further, by (29) we have $\delta'' \sim \frac{e^{4\tau}(-\tau+c')^4}{4}$ and hence

$$\delta' (\tau) = \int_{-\infty}^{\tau} \delta''(s) \, ds \sim \frac{e^{4\tau}(-\tau+c'+\frac{1}{4})^4}{16}.$$ 

It follows that

$$g' (\tau) = \tilde{g}' - \delta' \sim -\frac{4}{3} + \frac{2}{\tau+c'} - \frac{e^{4\tau}(-\tau+c'+\frac{1}{4})^4}{16}.$$ 

Therefore by virtue of $\frac{d\tau}{dt} = t^{-1}$ we have

$$\frac{dv_{3,c}}{dt} = t^{-2/3} e^{\tilde{g}-\delta} \left( g' + \frac{1}{3} \right) \sim \frac{2t^{-2}}{(-\log t + c')^2} \left( 1 - \frac{t^4(-\log t + c'+\frac{1}{2})^4}{64} \right) \left( -1 + \frac{2}{-\log t + c'} - \frac{t^4(-\log t + c'+\frac{1}{4})^4}{16} \right),$$

which after a little calculus yields the desired error term. \( \square \)

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