Integrability, conservation laws and solitons of a many-body dynamical system associated with the half-wave maps equation

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Abstract
We consider the half-wave maps (HWM) equation which is a continuum limit of the classical version of the Haldane-Shastry spin chain. In particular, we explore a many-body dynamical system arising from the HWM equation under the pole ansatz. The system is shown to be completely integrable by demonstrating that it exhibits a Lax pair and relevant conservation lows. Subsequently, the analytical multisoliton solutions of the HWM equation are constructed by means of the pole expansion method. The properties of the one- and two-soliton solutions are then investigated in detail as well as their pole dynamics. Last, an asymptotic analysis of the $N$-soliton solution reveals that no phase shifts appear after the collision of solitons. This intriguing feature is worth noting since it is the first example observed in the head-on collision of rational solitons. A number of problems remain open for the HWM equation, some of which are discussed in concluding remarks.
1. Introduction

The half-wave maps (HWM) equation arises from a continuum limit of a classical version of the Haldane-Shastry spin chain \([1-3]\). The latter is also known as the classical spin Calogero-Moser (CM) system whose complete integrability has been established \([4-6]\).

The HWM equation describes the time evolution of a spin density \(m(x, t) \in S^2\), where \(t\) and \(x\) are the temporal and spatial variables, respectively and \(S^2\) is the two-dimensional (2D) unit-sphere. The evolution equation for \(m\) is given by \([1-3]\)

\[
m_t = m \times H m_x,
\]

with the nonlocal operator \(H\) defined by

\[
H m(x, t) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{m(y, t)}{y - x} dy,
\]

being the Hilbert transform. The subscripts \(t\) and \(x\) appended to \(m\) denote partial derivative and the symbol "\( \times \)" is the vector product of 3D vectors in \(\mathbb{R}^3\) or \(\mathbb{C}^3\). Specifically, the latter is defined by \(a \times b = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)\) for 3D vectors \(a = (a_j)_{1 \leq j \leq 3}\) and \(b = (b_j)_{1 \leq j \leq 3}\). Throughout the paper, we restrict our consideration to the analysis on the real line.

Our main concern is the complete integrability of the HWM equation since it has been derived from a continuum limit of an integrable system. A recent study reveals that the HWM equation admits a Lax representation, as well as an infinite number of conservation laws \([7,8]\). As is well-known, the integrable soliton equations exhibit multisoliton solutions. The one-soliton (or traveling solitary wave) solution has been obtained \([1,2]\) together with numerical computation of the two- and three-soliton solutions \([1]\). Very recently, a procedure which is now known as the pole expansion method \([9]\) was applied to the HWM equation \([10]\). The method works well in obtaining the rational solutions of certain nonlinear evolution equations such as the Korteweg-de Vries (KdV) and Benjamin-Ono (BO) equations \([11-14]\) for which the equations of motion for the poles are governed by the finite-dimensional dynamical systems. The treatment of solutions expressed by hyperbolic functions becomes more sophisticated since the number of poles becomes infinite \([9]\). Although some numerical computations were performed to obtain soliton solutions of the HWM equation, the detail of the interaction process of solitons was not clarified \([10]\). One reason for this is that the explicit forms of the multisoliton solutions are not available yet even for the two-soliton case.

First, we summarize the main result given in \([10]\) for future uses (see Theorem 2.1 in \([10]\)):

We introduce the pole ansatz for solutions of Eq. (1.1)

\[
m(x, t) = m_0 + i \sum_{j=1}^{N} \frac{s_j(t)}{x - x_j(t)} - i \sum_{j=1}^{N} \frac{s_j(t)^*}{x - x_j(t)^*},
\]

(1.2)
where $\mathbf{m}_0$ is an arbitrary constant vector in $\mathbb{S}^2$ describing the boundary value $\mathbf{m}(\pm\infty, t)$ at spatial infinity, the complex functions $x_j(t)$ represent poles in the upper-half complex plane $\mathbb{C}_+$, $s_j$ are spin variables which take values in $\mathbb{C}^3$, the asterisk denotes complex conjugate and $N$ is an arbitrary positive integer characterizing the total number of poles (or solitons). It follows by taking the scalar product of Eq. (1.1) with $\mathbf{m}$ that $\mathbf{m} \cdot \mathbf{m}_t = 0$, and hence $\mathbf{m}^2$ is a constant independent of $t$, which is set to 1 hereafter. This implies that one can put $\mathbf{m}_0^2 = 1$ as well. Then, the spin field $\mathbf{m}$ from (1.2) solves Eq. (1.1) if $s_j$ and $x_j$ obey the system of nonlinear ordinary differential equations

$$
\dot{s}_j(t) = -2 \sum_{k \neq j}^N \frac{s_j(t) \times s_k(t)}{(x_j(t) - x_k(t))^2}, \quad (j = 1, 2, \ldots, N), \quad (1.3)
$$

$$
\ddot{x}_j(t) = 4 \sum_{k \neq j}^N \frac{s_j(t) \cdot s_k(t)}{(x_j(t) - x_k(t))^3}, \quad (j = 1, 2, \ldots, N), \quad (1.4)
$$

with the initial conditions $s_j(0) = s_{j,0}, x_j(0) = x_{j,0}$ and

$$
\dot{x}_j(0) = \frac{s_{j,0} \times s_{j,0}^*}{s_{j,0} \cdot s_{j,0}^*} \cdot \left( i \mathbf{m}_0 - \sum_{k \neq j}^N \frac{s_{k,0}}{x_{j,0} - x_{k,0}} + \sum_{k=1}^N \frac{s_{k,0}^*}{x_{j,0} - x_{k,0}^*} \right), \quad (j = 1, 2, \ldots, N). \quad (1.5)
$$

Furthermore, the $2N$ constraints are imposed on $s_{j,0}$ and $x_{j,0}$ such that

$$
s_{j,0}^2 = 0, \quad s_{j,0} \cdot \left( i \mathbf{m}_0 - \sum_{k \neq j}^N \frac{s_{k,0}}{x_{j,0} - x_{k,0}} + \sum_{k=1}^N \frac{s_{k,0}^*}{x_{j,0} - x_{k,0}^*} \right) = 0, \quad (j = 1, 2, \ldots, N). \quad (1.6)
$$

In the above expressions, the notation $\sum_{k \neq j}^N$ is short for the sum $\sum_{k=1}^N \sum_{(k \neq j)}$, and the dot appended to $s_j$ denotes the time derivative. The scalar product of 3D vectors $\mathbf{a} = (a_j)_{1 \leq j \leq 3}$ and $\mathbf{b} = (b_j)_{1 \leq j \leq 3}$ in $\mathbb{C}^3$ is defined by $\mathbf{a} \cdot \mathbf{b} = \sum_{j=1}^3 a_j b_j$. Recall that the spin variables $s_j$ and poles $x_j$ in Eqs. (1.3) and (1.4) evolve according to the dynamics of an exactly solvable spin CM system which has been explored extensively in [4, 6]. However, due to the constraints (1.6), the analysis of solutions becomes more complicated than that of the original spin CM system. It is important to remark that the constraints (1.6) hold for arbitrary $t > 0$ if they are satisfied at $t = 0$ [10]. In particular, taking the scalar product of $s_j$ with Eq. (1.3) leads to the relation $s_j \cdot \dot{s}_j = (1/2)(s_j^2)_t = 0$. Thus, $s_j^2(t) = 0, (j = 1, 2, \ldots, N)$ for arbitrary $t$. Note that since $s_j \in \mathbb{C}^3$, the relation $s_j^2 = s_j \cdot s_j = 0$ does not imply $s_j = 0$. Unlike the spin CM case, the permissible solutions are restricted by these conditions.

The purpose of the present paper is twofold. The first one is to explore a many-body dynamical system associated with the HWM equation which has been described above. In particular, we show that it admits a Lax pair as well as a number of conservation laws,
showing that the system is completely integrable. Although it has been pointed out in [10] that the dynamical system governed by the equations of motion (1.3) and (1.4) is identical with the rational spin CM system, the explicit form of the Lax pair has not been discovered yet. The second one is to provide the general $N$-soliton formula of the HWM equation under the pole ansatz (1.2).

The present paper is organized as follows. In Section 2, we present a Lax pair for equations (1.3) and (1.4). In Section 3, we derive the conservation laws of our system by using the Lax pair and exemplify some of them. We also clarify the Hamiltonian structure of the system, and subsequently, the integration of the system is performed by means of the standard procedure [15]. In Section 4, we develop an exact method of solution for constructing the multisoliton solutions of the HWM equation. We use a procedure employed in [16] to obtain the rational multisoliton solutions of the nonlocal nonlinear Schrödinger (NLS) equation. In Section 5, we give the explicit forms of the one- and two-soliton solutions. Specifically, the interaction process of two solitons is investigated in detail based on their pole dynamics. The feature of the $N$-soliton solution is discussed briefly by focusing on its asymptotic behavior. Section 6 is devoted to concluding remarks in which some open problems associated with the HWM equation are addressed. In Appendices A-D, the four propositions posed in Section 5 are proved.

2. Lax pair and integration of the system of equations

2.1. Lax pair

Here, we establish the following theorem.

**Theorem 1.** The system of equations (1.3) and (1.4) for $s_j$ and $x_j$ admits a Lax pair

$$\dot{L} = [B, L] \equiv BL - LB,$$

where $L$ and $B$ are $N \times N$ matrices whose elements are given respectively by

$$L = \begin{pmatrix} l_{jk} \end{pmatrix}_{1 \leq j, k \leq N}, \quad l_{jk} = \delta_{jk} \dot{x}_j + (1 - \delta_{jk}) \frac{\epsilon_{jk} \sqrt{2 s_j \cdot s_k}}{x_j - x_k},$$

$$B = \begin{pmatrix} b_{jk} \end{pmatrix}_{1 \leq j, k \leq N}, \quad b_{jk} = (1 - \delta_{jk}) \frac{\epsilon_{jk} \sqrt{2 s_j \cdot s_k}}{(x_j - x_k)^2}.$$ (2.2)

Here, $\delta_{jk}$ is Kronecker’s delta and $\epsilon_{jk}$ is an anti-symmetric symbol defined by

$$\epsilon_{jk} = -\epsilon_{kj}, \quad \epsilon_{jk}^2 = 1 - \delta_{jk}, \quad (j, k = 1, 2, ..., N).$$ (2.4)

**Proof.** First, we provide a key relation which will be used frequently in our analysis:

$$(s_j \times s_k) \cdot s_l = \epsilon_{jk} \epsilon_{kl} \epsilon_{lj} \sqrt{2 s_j \cdot s_k \sqrt{s_k \cdot s_l \sqrt{s_l \cdot s_j}}}.$$ (2.5)
To verify (2.5), we note the identity
\[
\{(s_j \times s_k) \cdot s_l\}^2 = 2(s_j \cdot s_k)(s_k \cdot s_l)(s_l \cdot s_j),
\] (2.6a)
which follows from the formula
\[
\{(s_j \times s_k) \cdot s_l\}^2 = \begin{vmatrix} s_j^2 & s_j \cdot s_l & s_k \cdot s_l \\ s_j \cdot s_l & s_j^2 & s_j \cdot s_k \\ s_k \cdot s_l & s_j \cdot s_k & s_k^2 \end{vmatrix},
\] (2.6b)
and the constraints \(s_j^2 = s_k^2 = s_l^2 = 0\). The square root of (2.6a) yields, after taking into account the properties of the scalar and vector products, (2.5).

To establish (2.1), we compute the \((j,k)\) elements of both sides to obtain
\[
(\dot{L})_{jk} = \delta_{jk}\ddot{x} + (1 - \delta_{jk}) \left\{ -\frac{\epsilon_{jk}\sqrt{2}s_j \cdot s_k}{(x_j - x_k)^2}(\dot{x}_j - \dot{x}_k) + \frac{\epsilon_{jk}}{\sqrt{2}(x_j - x_k)} \frac{\dot{s}_j \cdot s_k + s_j \cdot \dot{s}_k}{\sqrt{s_j \cdot s_k}} \right\},
\] (2.7)
\[
([B, L])_{jk} = -(1 - \delta_{jk})\frac{\epsilon_{jk}\sqrt{2}s_j \cdot s_k}{(x_j - x_k)^2}(\dot{x}_j - \dot{x}_k) + 2 \sum_{l \neq j,k} \frac{\epsilon_j\epsilon_{lk}\sqrt{s_j \cdot s_l \sqrt{s_l \cdot s_k}}}{(x_j - x_l)^2(x_k - x_l)^2}(2x_l - x_j - x_k).
\] (2.8)

For \(j = k\), noting the relation \(\epsilon_{jl}\epsilon_{ij} = -1, (j \neq l)\), Eq. (2.1) with (2.7) and (2.8) yields (1.4) whereas for \(j \neq k\), it reduces to
\[
\frac{\epsilon_{jk}}{\sqrt{2}(x_j - x_k)} \frac{\dot{s}_j \cdot s_k + s_j \cdot \dot{s}_k}{\sqrt{s_j \cdot s_k}} = 2 \sum_{l \neq j,k} \frac{\epsilon_j\epsilon_{lk}\sqrt{s_j \cdot s_l \sqrt{s_l \cdot s_k}}}{(x_j - x_l)^2(x_k - x_l)^2}(2x_l - x_j - x_k).
\] (2.9)

Multiplying \(\epsilon_{jk}\sqrt{2}(x_j - x_k)\sqrt{s_j \cdot s_k}\) on both sides of (2.9) and using the relations (2.5) and \(\epsilon_{jk}^2 = 1, (j \neq k)\), Eq. (2.9) can be recast in the form
\[
\left\{ \dot{s}_j + 2 \sum_{l \neq j} \frac{s_j \times s_l}{(x_j - x_l)^2} \right\} \cdot s_k + \left\{ \dot{s}_k + 2 \sum_{l \neq k} \frac{s_k \times s_l}{(x_k - x_l)^2} \right\} \cdot s_j = 0.
\] (2.10)

Since \(s_j\) and \(s_k\) satisfy Eq. (1.3), the left-hand side of (2.10) becomes zero, which proves the Lax Eq. (2.1). □

We note that the system of equations (1.3) for \(s_j\) has a Lax representation
\[
\dot{S} = [B, S],
\] (2.11)
where \(S\) is an \(N \times N\) matrix with elements
\[
S = (s_{jk})_{1 \leq j,k \leq N}, \quad s_{jk} = \epsilon_{jk}\sqrt{2}s_j \cdot s_k.
\] (2.12)

The proof of (2.11) can be carried out in the same way as that of Eq. (2.1). Actually, Eq. (2.11) reduces to (2.10) and hence it holds identically by virtue of Eq. (1.3). The Lax
pair (2.1) is formally identical with that of the spin CM system given in [4, 5]. However, missing the relation (2.5), its explicit form has not been found yet and presented here for the first time.

2.2. Integration of the system of equations

To integrate the system of equations (1.3) and (1.4), we provide the following proposition:

**Proposition 1.** Let $X$ be an $N \times N$ matrix with elements

$$X = (x_{jk})_{1 \leq j, k \leq N}, \quad x_{jk} = \delta_{jk} x_j. \tag{2.13}$$

Then, $X$ evolves according to the equation

$$\dot{X} = L + [B, X]. \tag{2.14}$$

**Proof.** The proof can be done by a direct computation using (2.2) and (2.3). □

Let us now solve the system of equations (1.3) and (1.4). First, we introduce the quantity $J = J(t) = U^{-1}(t)X(t)U(t)$, where $U$ satisfies the equation $\dot{U} = BU$ subjected to the initial condition $U(0) = I$ ($I : N \times N$ unit matrix). It then follows from (2.1) and (2.14) that

$$\dot{J} = U^{-1}LU, \quad \ddot{J} = 0. \tag{2.15a}$$

Eq. (2.15b) can be integrated twice with respect to $t$, giving

$$J(t) = \dot{J}|_{t=0}t + J(0) = L(0)t + X(0). \tag{2.16}$$

The matrix $U(t)$ determines the time evolution of $L$ and $S$ in accordance with the relations

$$L(t) = U(t)L(0)U(t)^{-1}, \tag{2.17}$$

$$S(t) = U(t)S(0)U(t)^{-1}, \tag{2.18}$$

thus providing a complete set of solutions to the system of equations (1.3) and (1.4). See also [4, 5] for more detailed discussion on the integration of the system under consideration.

To proceed, we introduce the tau-function $f_N$ which plays a central role in constructing soliton solutions:

$$f_N = \prod_{j=1}^{N} (x - x_j) = |Ix - X|. \tag{2.19}$$
Referring to the relation $X = U J U^{-1}$ with (2.16), we find that
\[ f_N = |U(Ix - L(0)t - X(0))U| = |Ix - L(0)t - X(0)|. \]  
(2.20)

The expressions of the poles $x_j$ can be obtained from (2.20) by solving the algebraic equation $f_N = 0$ of the $N$th degree in $x$. However, in general, it is impossible to find their explicit analytical solutions. In Section 4, we show that one needs only the fundamental symmetric polynomials of $x_j$ in constructing soliton solutions. They follow immediately from (2.19) and (2.20) by means of a purely algebraic procedure. For latter use, we write the explicit forms of $f_1$ and $f_2$:
\[ f_1 = x - \dot{x}_{1,0}t - x_{1,0}, \]  
(2.21a)
\[ f_2 = x^2 - \{(\dot{x}_{1,0} + \dot{x}_{2,0})t + x_{1,0} + x_{2,0}\}x + \left(\dot{x}_{1,0}x_{2,0} - \frac{2s_{1,0} \cdot s_{2,0}}{(x_{1,0} - x_{2,0})^2}\right)t^2 \]
\[ + (\dot{x}_{1,0}x_{2,0} + x_{1,0}\dot{x}_{2,0})t + x_{1,0}x_{2,0}. \]  
(2.21b)

The construction of the solutions for $s_j$, on the other hand, is not addressed here. It will be considered in Section 4 where we develop a new method of solution based on an elementary theory of linear algebra.

**Remark 1.** In the case of the periodic solutions, the pole ansatz may be expressed in the form [10]
\[ m(x, t) = m_0 + i \sum_{j=1}^{N} s_j(t) \kappa \cot \kappa (x - x_j(t)) - i \sum_{j=1}^{N} s^*_j(t) \kappa \cot \kappa (x - x^*_j(t)), \]  
(2.22)
where $\kappa$ is a positive parameter. Then, the evolution equations of $s_j$ and $x_j$ read [10]
\[ \dot{s}_j(t) = -2 \sum_{k \neq j}^{N} s_j(t) \times s_k(t) \frac{\kappa^2}{\sin^2[\kappa(x_j(t) - x_k(t))]}, \quad (j = 1, 2, ..., N), \]  
(2.23)
\[ \ddot{x}_j(t) = 4 \sum_{k \neq j}^{N} s_j(t) \cdot s_k(t) \frac{\kappa^3 \cos[\kappa(x_j(t) - x_k(t))] \sin^3[\kappa(x_j(t) - x_k(t))]}{\sin^3[\kappa(x_j(t) - x_k(t))]}, \quad (j = 1, 2, ..., N). \]  
(2.24)

The Lax pair for the above system of equations takes the same form as (2.1) except that the elements of the matrices $L$ and $B$ are given respectively by
\[ l_{jk} = \delta_{jk} \dot{x}_j + (1 - \delta_{jk}) \epsilon_{jk} \sqrt{2 s_j \cdot s_k} \frac{\kappa}{\sin \kappa(x_j - x_k)}, \quad (1 \leq j, k \leq N), \]  
(2.25)
\[ b_{jk} = (1 - \delta_{jk}) \epsilon_{jk} \sqrt{2 s_j \cdot s_k} \frac{\kappa^2 \cos \kappa(x_j - x_k)}{\sin^2 \kappa(x_j - x_k)}, \quad (1 \leq j, k \leq N). \]  
(2.26)
As confirmed easily, in the limit of $\kappa \to 0$, the expressions (2.22)-(2.26) reduce to the corresponding ones for the soliton solutions. The analysis of the periodic solutions is more involved than that of the soliton solutions. Recall that the similar periodic problem has been exploited in [16] for the nonlocal NLS equation. This interesting issue will be addressed elsewhere.

3. Conservation laws and Hamiltonian formulation

The Lax pairs (2.1) and (2.11) for $L$ and $S$ allow us to obtain the conservation laws for the dynamical system described by the equations of motion (1.3) and (1.4). Here, we derive several independent conservation laws. We also show that our system of equations can be written as a Hamiltonian system under appropriate Poisson brackets.

3.1. Conservation laws

The direct consequence of (2.1) and (2.11) is given by the following proposition:

**Proposition 2.** The quantities

$$\mathcal{H} = \frac{1}{n} \text{Tr}(L + \mu S)^n,$$

(3.1)

are the constants of motion, where $n$ is an arbitrary positive integer and $\mu$ is an expansion parameter.

**Proof.** Using (2.1) and (2.11) with the trace identity $\text{Tr}(AB) = \text{Tr}(BA)$

$$\mathcal{H} = \text{Tr} \left\{ (\hat{L} + \mu \hat{S})(L + \mu S)^{n-1} \right\}$$

$$= \text{Tr} \left\{ [B, L + \mu S](L + \mu S)^{n-1} \right\}$$

$$= \text{Tr} \left\{ B(L + \mu S)^n - (L + \mu S)B(L + \mu S)^{n-1} \right\}$$

$$= \text{Tr} \left\{ B(L + \mu S)^n - B(L + \mu S)^n \right\}$$

$$= 0.$$  

(3.2)

Expanding (3.1) in powers of $\mu$, one can see that the coefficients at different powers of $\mu$ are also conserved. In particular, the quantities

$$\mathcal{H}_{m,n} = \frac{1}{m + n} \text{Tr}(L^m S^n),$$

(3.3)

are the constants of motion. Below, we present some explicit examples:

$$\mathcal{H}_{2,0} = \frac{1}{2} \sum_{j=1}^N x_j^2 + \sum_{j \neq k}^N \frac{s_j \cdot s_k}{(x_j - x_k)^2},$$

(3.4a)
\[ \mathcal{H}_{3,0} = \frac{1}{3} \sum_{j=1}^{N} \ddot{x}_j^3 + 2 \sum_{j \neq k}^{N} \frac{\dot{x}_j \mathbf{s}_j \cdot \mathbf{s}_k}{(x_j - x_k)^2} + \frac{2}{3} \sum_{j \neq k \neq l}^{N} \frac{\mathbf{s}_j \cdot (\mathbf{s}_k \times \mathbf{s}_l)}{(x_j - x_k)(x_k - x_l)(x_l - x_j)}, \]  
\[ \mathcal{H}_{1,1} = -\sum_{j \neq k}^{N} \frac{\mathbf{s}_j \cdot \mathbf{s}_k}{x_j - x_k}, \]  
\[ \mathcal{H}_{1,2} = -\frac{1}{3} \sum_{j \neq k}^{N} \dot{x}_j \mathbf{s}_j \cdot \mathbf{s}_k + \frac{2\sqrt{2}}{3} \sum_{j \neq k \neq l}^{N} \frac{\mathbf{s}_j \cdot (\mathbf{s}_k \times \mathbf{s}_l)}{x_j - x_k}. \]  

As in the case of the CM system, we have additional constants of motion [17]. To show this, we define the quantity \( \mathcal{I}_n = \text{Tr}(XL^{n-1}) \). Then, we establish

**Proposition 3.**

\[ \dot{\mathcal{I}}_n = \text{Tr} L^n. \]  

**Proof.** Using (2.1) and (2.14), one can perform a sequence of computations to obtain

\[ \dot{\mathcal{I}}_n = \text{Tr} \left( \dot{X}L^{n-1} + X \sum_{j=2}^{n} L^{n-j} \dot{L}L^{j-2} \right) \]
\[ = \text{Tr} \left( (L + [B, X])L^{n-1} + X \sum_{j=2}^{n} L^{n-j} [B, L]L^{j-2} \right) \]
\[ = \text{Tr} \left( L^n + BL^{n-1} - XBL^{n-1} + X(BL^{n-1} - L^{n-1}B) \right) \]
\[ = \text{Tr} L^n. \]  

\[ \square \]

Integration of (3.5) gives

\[ \mathcal{I}_n = t \text{Tr} L^n(0) + \text{Tr} L^n(0) \]
\[ = nt \mathcal{H}_{n,0} + \text{Tr} L^n, \]  

where \( \text{Tr} L^n \equiv \text{Tr} L^n(t) = \text{Tr} L^n(0) \). It follows from (3.7) that the quantities

\[ \mathcal{J}_{m,n} = m \mathcal{H}_{m,0} - n \mathcal{H}_{n,0} \]
\[ = \text{Tr}(XL^{n-1})\text{Tr} L^n - \text{Tr}(XL^{m-1})\text{Tr} L^n, \]  

are the constants of motion. In particular, for \( m = 1 \), (3.8) reduces to

\[ \mathcal{J}_{1,n} = \text{Tr}(XL^{n-1})\text{Tr} L - \text{Tr} X \text{Tr} L^n. \]  

\[ 9 \]
The information of the conservation laws about the dynamical system under consideration does not provide directly that of the conservation laws of the HWM equation itself. However, a few conservation laws are available for the HWM equation which are associated with the global symmetries of the equation \[1\]. For instance, the integral
\[
\mathcal{J} = \frac{1}{2} \int_{-\infty}^{\infty} \mathbf{m} \cdot \mathbf{H} \mathbf{m} dx
\]
corresponding to energy (or Hamiltonian) is preserved in \( t \). Substituting \( \mathbf{m} \) from (1.2) into this expression, we obtain
\[
\mathcal{J} = 2\pi N \sum_{j,k=1}^{N} s_{jk} \cdot s_{jk}^{*} \left( x_{j} - x_{k}^{*} \right)^{2},
\]
As will be expected from the existence of a Lax pair \[7\], an infinite number of conservation laws may exist for the HWM equation which would establish under appropriate conditions the complete integrability of the equation. This interesting issue deserves a future investigation.

3.2. Hamiltonian formulation

We start from the Hamiltonian from (3.4a)
\[
\mathcal{H}_{2,0} = \frac{1}{2} \sum_{j=1}^{N} p_{j}^{2} + \frac{1}{2} \sum_{j \neq k}^{N} \frac{s_{jk}^{2}}{(x_{j} - x_{k})^{2}},
\]
where \( p_{j} = \dot{x}_{j} \) are the momentum variables and \( s_{jk} \) are given by (2.12). The system described by the Hamiltonian (3.11) has been introduced in \[4, 5\]. The number of the independent variables is found to be \( 2N + N(N - 1)/2 = N(N + 3)/2 \) by taking into account the antisymmetric property of the variable \( s_{jk} \). In accordance with \[5\], we define the Poisson brackets
\[
\{f, g\}_{p} = \sum_{j=1}^{N} \left( \frac{\partial f}{\partial x_{j}} \frac{\partial g}{\partial p_{j}} - \frac{\partial g}{\partial x_{j}} \frac{\partial f}{\partial p_{j}} \right),
\]
\[
\{s_{ij}, s_{kl}\}_{s} = \delta_{il} s_{kj} + \delta_{ik} s_{jl} + \delta_{jl} s_{ik} + \delta_{jk} s_{li}.
\]
Then, the equations of motion for \( x_{j} \) and \( p_{j} \) can be written as
\[
\dot{x}_{j} = \{x_{j}, \mathcal{H}_{2,0}\}_{p} = p_{j},
\]
\[
\dot{p}_{j} = \{p_{j}, \mathcal{H}_{2,0}\}_{p} = \sum_{k \neq j}^{N} \frac{2s_{jk}^{2}}{(x_{j} - x_{k})^{3}},
\]
wheres those of \( s_{jk} \) are given by
\[
\dot{s}_{jk} = \{s_{jk}, \mathcal{H}_{2,0}\}_{s} = -\sum_{l \neq j,k}^{N} \left( \frac{s_{jl}s_{lk}}{(x_{j} - x_{l})^{2}} - \frac{s_{jl}s_{lk}}{(x_{k} - x_{l})^{2}} \right).
\]
The expressions (3.15) with (3.14) coincide with (1.4), and (3.16) are found to reduce to (2.10) after a few manipulations. This result exhibits the Hamiltonian structure of the system. Although the independence of the various conserved quantities derived in Section 3.1 under the above Poisson brackets is an interesting issue, we do not discuss it here and instead refer to [4].

4. Construction of the \( N \)-soliton solution

4.1. \( N \)-soliton solution

Here, we provide an explicit formula for the rational \( N \)-soliton solution of the HWM equation. It has a form of the pole representation given by (1.2). To this end, we make a few preparations. First, we write the tau-function \( f_N \) from (2.19) in the form

\[ f_N = \sum_{j=0}^{N} (-1)^j \sigma_j x^{N-j}, \]  

(4.1a)

where \( \sigma_j \) are fundamental symmetric polynomials of \( x_j \) given by

\[ \sigma_0 = 1, \quad \sigma_1 = \sum_{j=1}^{N} x_j, \quad \sigma_2 = \sum_{j<k} x_j x_k, ..., \quad \sigma_N = \prod_{j=1}^{N} x_j. \]  

(4.1b)

We define the polynomials \( \sigma_{n,j} \) by the relation

\[ (x-x_1)...(x-x_{j-1})(x-x_{j+1})...(x-x_N) = \sum_{n=0}^{N-1} (-1)^n \sigma_{n,j} x^{N-n-1}, \quad (j = 1, 2, ..., N). \]  

(4.2)

Multiplying \( x - x_j \) by (4.2) and comparing the coefficients of \( x^{N-n+l} \) on both sides, we obtain the recursion relation \( \sigma_{n-l,j} + x_j \sigma_{n-l-1,j} = \sigma_{n-l} \). If we multiply \( (-1)^l x_j^l \) by this and add the resultant expression from \( l = 0 \) to \( l = n - 1 \), we arrive at the relation

\[ \sigma_{n,j} = \sum_{l=0}^{n} (-1)^l x_j^l \sigma_{n-l}. \]  

(4.3)

Furthermore, we introduce the vector quantities

\[ J_n = \sum_{j=1}^{N} s_j x_j^n, \quad (n = 0, 1, 2, ...). \]  

(4.4)

Then, we establish our main result:

**Theorem 2.** The solution of the HWM equation under the pole ansatz (1.2) is represented explicitly in terms of \( \sigma_j \) and \( J_l \) as

\[ m = m_0 + i \left( \frac{\sum_{n=0}^{N-1} \sum_{l=0}^{n} (-1)^{n+l} J_l \sigma_{n-l} x^{N-n-1}}{\sum_{j=0}^{N} (-1)^j \sigma_j x^{N-j}} - c.c. \right), \]  

(4.5)
where the notation c.c. stands for the complex conjugate expression of the preceding expression.

**Proof.** We modify \( m \) from (1.2) by using (4.2) to obtain

\[
m = m_0 + i \left( \frac{1}{N} \sum_{j=1}^{N} \sum_{n=0}^{N-1} (-1)^n \sigma_{n,j} s_j x^{N-n-1} - c.c. \right).
\]

The \( N \)-soliton formula (4.5) follows simply by inserting (4.1), (4.3) and (4.4) into the above expression. \( \square \)

**Remark 2.** Before proceeding, we make a few comments. First, the expressions (2.19) and (2.20) imply that \( \sigma_n \) is an \( n \)th-order polynomial of \( t \) whose coefficients depend only on the initial conditions. As evidenced by (4.5), the solutions \( x_j \) themselves are not necessary, but one needs the polynomials \( \sigma_j \) to obtain \( m \). This observation is quite important since the explicit analytical expression of \( x_j \) are not available in general. Second, the solutions \( s_j \) are expressed in terms of \( x_j (j = 1, 2, ..., N) \) and \( J_n (n = 1, 2, ..., N-1) \) by solving the system of linear algebraic equations (4.4) for \( s_j \). The explicit formula for \( s_j \) will be presented in Section 4. See (4.30). We recall that an analogous formula has been given for the rational \( N \)-soliton solution of the nonlocal NLS equation [16].

The following theorem is crucial in the subsequent analysis:

**Theorem 3.** The quantity \( J_n \) defined by (4.4) is an \( n \)th-order polynomial of \( t \). More precisely, it can be expressed in the form

\[
J_n = \sum_{k=0}^{n} \frac{c_{n,k}}{k!} t^k,
\]

(4.6a)

with

\[
c_{n,k} = \frac{d^k J_n}{dt^k} \bigg|_{t=0} = \sum_{j=1}^{N} \frac{d^k (s_j x_j^n)}{dt^k} \bigg|_{t=0}.
\]

(4.6b)

Note that the coefficients \( c_{n,k} \) are determined by the initial conditions for \( x_j, \dot{x}_j, s_j \) and \( \dot{s}_j (j = 1, 2, ..., N) \). The expression (4.6) implies that \( d^{n+1} J_n/dt^{n+1} = 0 \) \( (n = 0, 1, ..., N-1) \). A direct verification of these relations using (1.3) and (1.4) is possible for the first few \( n \)'s. However, the amount of computations becomes formidable as \( n \) increases. Therefore, we employ an alternative approach.

Theorem 3 is now proved in several steps, which we shall now demonstrate by establishing a sequence of propositions.

**Proposition 4.** Let \( Y = (I - \epsilon X)^{-1} \). Then,

\[
\dot{Y} = BY - YB + \epsilon YLY,
\]

(4.7)
where $\epsilon$ is an expansion parameter.

**Proposition 5.** Let $K = YL$. Then,

$$\dot{K} = BK - KB + \epsilon K^2.$$  \hfill (4.8)

**Proposition 6.** Let $\mathcal{P}_n = \text{Tr}(S^2K^nY)$. Then,

$$\dot{\mathcal{P}}_n = \epsilon(n + 1) \mathcal{P}_{n+1}.$$  \hfill (4.9)

**Proposition 7.** Let $\mathcal{Q}_n = \text{Tr}(S^2X^n)$. Then,

$$\sum_{l=n}^{\infty} \epsilon^l \frac{d^n \mathcal{Q}_l}{dt^n} = \epsilon^n n! \mathcal{P}_n.$$ \hfill (4.10)

Proposition 7 comes from Propositions 4-6 which are proved in Appendices A-D.

Now, we are ready for the proof of Theorem 3.

**Proof of Theorem 3.** We rewrite $\mathcal{Q}_n$ defined in (4.10) by using (2.12) and (2.13), giving

$$\mathcal{Q}_n = -2 \left( \sum_{j=1}^{N} s_j x_j^n \right) \cdot \left( \sum_{j=1}^{N} s_j \right) = -2J_n \cdot J_0.$$ \hfill (4.11)

We expand $\mathcal{P}_n$ in powers of $\epsilon$ and compare the coefficient of $\epsilon^n$ on both sides of (4.10). This leads to the relation

$$\frac{d^n \mathcal{Q}_n}{dt^n} = n! \text{Tr}(S^2L^n).$$ \hfill (4.12)

Differentiating (4.12) with respect to $t$ and taking into account the constant of motion (3.3), we obtain

$$\frac{d^{n+1} \mathcal{Q}_n}{dt^{n+1}} = n!(n + 2) \frac{d \mathcal{H}_{n,2}}{dt} = 0.$$ \hfill (4.13)

If we substitute (4.13) into (4.11), we find

$$\frac{d^{n+1} J_n}{dt^{n+1}} \cdot J_0 = 0.$$ \hfill (4.14)

Since $J_0 = \sum_{j=1}^{N} s_j$ is an arbitrary constant vector specified by the initial conditions, one concludes that $d^{n+1} J_n / dt^{n+1} = 0$, and hence $J_n$ is an $n$th-order polynomial of $t$. This completes the proof of Theorem 3. \hfill \square
4.2. Solution of the constraints

For the complete description of the $N$-soliton solution, we must specify the initial conditions for $s_j$, $x_j$ and $\dot{x}_j$. This problem becomes complicated because of the constraints (1.6) imposed on $s_{j,0}$. We seek the solutions of (1.6) of the form [10]

$$s_{j,0} = s_j(n_{j,1} + in_{j,2}), \quad (j = 1, 2, \ldots, N),$$  \hspace{1cm} (4.15)

where $n_{j,1}$ and $n_{j,2} \in \mathbb{S}^2$ are unit vectors satisfying the conditions $n_{j,1} \cdot n_{j,2} = 0 (j = 1, 2, \ldots, N)$ and $s_j \in \mathbb{C}$ are unknown parameters to be determined later. In addition, we define the vectors

$$n_{j,+} = n_{j,1} + in_{j,2}, \quad n_j = n_{j,1} \times n_{j,2}, \quad (j = 1, 2, \ldots, N),$$  \hspace{1cm} (4.16)

as well as the parameters

$$\kappa_{jk} = n_{j,+} \cdot n_{k,+}, \quad \nu_{jk} = n_{j,+} \cdot n_{k,+}^*, \quad \mu_j = n_{j,+} \cdot m_0, \quad (j, k = 1, 2, \ldots, N).$$  \hspace{1cm} (4.17)

Note that $n_j$ is a unit vector orthogonal to the vectors $n_{j,1}$ and $n_{j,2}$. The first constraints in (1.6) are fulfilled due to the orthogonality relations $n_{j,1} \cdot n_{j,2} = 0 (j = 1, 2, \ldots, N)$. The second constraints, on the other hand, can be written in terms of the parameters defined by (4.17) as

$$i\mu_j + \frac{2s_j^*}{x_{j,0} - x_j} - \sum_{k \neq j} \frac{\kappa_{jk} s_k^*}{x_{j,0} - x_{k,0}} + \sum_{k \neq j} \frac{\nu_{jk} s_k^*}{x_{j,0} - x_{k,0}^*} = 0, \quad (j = 1, 2, \ldots, N).$$  \hspace{1cm} (4.18)

The complex conjugate expression of (4.18) reads

$$-i\mu_j^* + \frac{2s_j}{x_{j,0} - x_j} - \sum_{k \neq j} \frac{\kappa_{jk}^* s_k}{x_{j,0} - x_{k,0}^*} + \sum_{k \neq j} \frac{\nu_{jk}^* s_k}{x_{j,0}^* - x_{k,0}} = 0, \quad (j = 1, 2, \ldots, N).$$  \hspace{1cm} (4.19)

If we introduce the matrices $F$ and $G$ together with the vectors $s$ and $\mu$ whose elements are given by

$$F = (f_{jk})_{1 \leq j, k \leq N}, \quad f_{jk} = \frac{2\delta_{jk}}{x_{j,0} - x_{k,0}} + (1 - \delta_{jk}) \frac{\nu_{jk}}{x_{j,0} - x_{k,0}^*},$$  \hspace{1cm} (4.20)

$$G = (g_{jk})_{1 \leq j, k \leq N}, \quad g_{jk} = -(1 - \delta_{jk}) \frac{\kappa_{jk}}{x_{j,0} - x_{k,0}},$$  \hspace{1cm} (4.21)

$$s = (s_1, s_2, \ldots, s_N)^T, \quad s^* = (s_1^*, s_2^*, \ldots, s_N^*)^T,$$  \hspace{1cm} (4.22)

$$\mu = (\mu_1, \mu_2, \ldots, \mu_N)^T, \quad \mu^* = (\mu_1^*, \mu_2^*, \ldots, \mu_N^*)^T,$$  \hspace{1cm} (4.23)

then, the system of linear algebraic equations (4.18) and (4.19) for $s_j$ and $s_j^*$ can be put into the compact form as

$$Gs + Fs^* = -i\mu.$$  \hspace{1cm} (4.24a)
\[ F^* \mathbf{s} + G^* \mathbf{s}^* = i \mu^*. \] (4.24b)

If the conditions \(|F| \neq 0\) and \(|F^* - G^* F^{-1} G| \neq 0\) are satisfied, then Eqs. (4.24) can be solved uniquely for \( \mathbf{s} \) to give

\[ \mathbf{s} = i (F^* - G^* F^{-1} G)^{-1} (G^* F^{-1} \mu + \mu^*). \] (4.25)

In the case of the two-soliton solution which will be described in detail in Section 5, (4.25) is expressed in the form

\[ s_1 = \frac{\Delta_1}{\Delta}, \quad s_2 = \frac{\Delta_2}{\Delta}, \] (4.26a)

where

\[
\Delta = 1 + \frac{\nu_1^2 \nu_2}{4} \left( \frac{x_{1,0} - x_{1,0}^*}{|x_{1,0} - x_{2,0}|^2} \right) \left( x_{2,0} - x_{2,0}^* \right) - \frac{\kappa_1^2 \kappa_2}{4} \left( \frac{x_{1,0} - x_{1,0}^*}{|x_{1,0} - x_{2,0}|^2} \right) \left( x_{2,0} - x_{2,0}^* \right), \tag{4.26b}
\]

\[
\Delta_1 = -\frac{i}{4} \left( x_{1,0} - x_{1,0}^* \right) \left( x_{2,0} - x_{2,0}^* \right) \left( \frac{\nu_1^2 \mu_2}{x_{1,0} - x_{2,0}} - \frac{\kappa_1^2 \mu_2}{x_{1,0}^* - x_{2,0}^*} \right) - \frac{1}{2} (x_{1,0} - x_{1,0}^*) \mu_1^*, \tag{4.26c}
\]

\[
\Delta_2 = -\frac{i}{4} \left( x_{1,0} - x_{1,0}^* \right) \left( x_{2,0} - x_{2,0}^* \right) \left( \frac{\nu_1^2 \mu_1}{x_{2,0} - x_{1,0}} - \frac{\kappa_1^2 \mu_1}{x_{2,0}^* - x_{1,0}^*} \right) - \frac{1}{2} (x_{2,0} - x_{2,0}^*) \mu_2^*. \tag{4.26d}
\]

Note that \( \Delta \in \mathbb{R} \) and \( \kappa_{21} = \kappa_{12}, \nu_{21} = \nu_{12} \).

The solution of the constraints makes it possible to solve the initial value problem of the system of equations (1.3) and (1.4), which we shall summarize. For a given \( \mathbf{m}_0 \), prepare the initial pole positions \( x_{j,0} = x_j(0) \in \mathbb{C}_+ (j = 1, 2, ..., N) \) and the directions \( \mathbf{n}_j \in \mathbb{S}^2 \) of the initial spins \( \mathbf{s}_{j,0} (j = 1, 2, ..., N) \) which are given by (4.16). The initial conditions for the spin variables are computed in accordance with (4.15) and (4.25). The initial conditions (1.5) for \( \dot{x}_{j,0} \in \mathbb{C} \) then reduce, after introducing (4.15), to the transparent forms

\[ \dot{x}_{j,0} = -i \mathbf{n}_j \cdot \left( i \mathbf{m}_0 - \sum_{k \neq j} \frac{\mathbf{s}_{k,0}}{x_{j,0} - x_{k,0}} + \sum_{k=1}^N \frac{\mathbf{s}_{k,0}^*}{x_{j,0} - x_{k,0}^*} \right), \quad (j = 1, 2, ..., N). \] (4.27)

### 4.3. Consistency of an overdetermined system

As already shown by (4.5), the spin variables themselves are not required for the purpose of constructing the \( N \)-soliton solution. However, since these are used to evaluate the asymptotic behavior of the solutions, we derive their explicit expressions. First, we note that the system of equations for \( \mathbf{s}_j \) is overdetermined so that one must verify its consistency. Specifically, we show that it has a solution. To this end, let \( \mathbf{J}_{N+m} = \sum_{j=0}^{N-1} c_j \mathbf{J}_j \) \((m \geq 0)\) with \( c_j \) being unknown constants to be determined. Invoking the definition of \( \mathbf{J}_j \), this expression can be written as \( \sum_{k=1}^N \mathbf{s}_k x_k^{N+m} = \sum_{k=1}^N \mathbf{s}_k \sum_{j=0}^{N-1} c_j x_k^j. \)
Equating the coefficients of $s_k$ on both sides, one obtains the system of linear algebraic equations for $c_j$: \[ \sum_{j=0}^{N-1} c_j x_k^j = x_k^{N+m} \quad (k = 1, 2, \ldots, N) \] This system is solved simply to give the solution

\[
c_j = (-1)^{N-j-1} \begin{vmatrix}
R_0 & R_{-1} & \cdots & R_{-N+1} \\
R_1 & R_0 & \cdots & R_{-N+2} \\
\vdots & \vdots & \ddots & \vdots \\
R_{j-1} & R_{j-2} & \cdots & R_{-N+j} \\
R_{j+1} & R_j & \cdots & R_{-N+j+2} \\
\vdots & \vdots & \ddots & \vdots \\
R_{N-1} & R_{N-2} & \cdots & R_0 \\
R_{N+m} & R_{N+m-1} & \cdots & R_{m+1}
\end{vmatrix}, \quad (j = 0, 1, 2, \ldots, N-1), \quad (4.28a)
\]

where $R_n$ are defined by the relation

\[
\prod_{j=1}^{N} (1 - \varepsilon x_j)^{-1} = 1 + \sum_{n=1}^{\infty} \varepsilon^n R_n, \quad (4.28b)
\]

with $R_0 = 1$ and $R_n = 0 \quad (n < 0)$. It follows from (4.28b) that the quantities $R_n$ are expressed in terms of the fundamental symmetric polynomials $\sigma_j \quad (j = 1, 2, \ldots, N)$ defined by (4.1b). In the case of $N = 3$ and $m = 0, 1, 2$, for instance, the resulting expressions of $J_3, J_4$ and $J_5$ are given respectively by

\[
J_3 = \sigma_1 J_2 - \sigma_2 J_1 + \sigma_3 J_0, \quad (4.29a)
\]

\[
J_4 = (\sigma_1^2 - \sigma_2) J_2 - (\sigma_1 \sigma_2 - \sigma_3) J_1 + \sigma_1 \sigma_3 J_0. \quad (4.29b)
\]

\[
J_5 = (\sigma_1^3 - 2 \sigma_1 \sigma_2 + \sigma_3) J_2 - (\sigma_1^2 \sigma_2 - \sigma_2^2 - \sigma_1 \sigma_3 + \sigma_4) J_1 + (\sigma_1 \sigma_3^2 - \sigma_2 \sigma_3 - \sigma_1 \sigma_4 + \sigma_5) J_0. \quad (4.29c)
\]

The above discussion reveals that $N$ equations are independent among (4.4) and the other ones are redundant. In accordance with this observation, we apply Cramer’s rule to the first $N$ equations and obtain the solution

\[
s_j = \begin{vmatrix}
1 & \cdots & 1 & J_0 & 1 & \cdots & 1 \\
x_1 & \cdots & x_{j-1} & J_1 & x_{j+1} & \cdots & x_N \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
x_1^{N-1} & \cdots & x_1^{N-1} & J_{N-1} & x_{N-1}^{N-1} & \cdots & x_N^{N-1}
\end{vmatrix} / V_N, \quad (j = 1, 2, \ldots, N), \quad (4.30a)
\]

where

\[
V_N = |(x_k^{j-1})|_{1 \leq j, k \leq N} = \prod_{1 \leq k < j \leq N} (x_j - x_k), \quad (4.30b)
\]

is the Vandermonde determinant. If we expand the determinant in (4.30a) with respect to the $j$th column, then we can express $s_j$ as a linear combination of $J_k \quad (k = 0, 1, \ldots, N-1)$. 

5. Soliton solutions

In this section, we present the explicit examples of soliton solutions. In particular, we explore the properties of the one- and two-soliton solutions which are most fundamental constituents among soliton solutions. The \( N \)-soliton solution will be described briefly focusing on its asymptotic behavior at large time.

5.1. One-soliton solution

The one-soliton solution is given by (4.5) with \( N = 1 \). It reads

\[
\mathbf{m} = \mathbf{m}_0 + i \left( \frac{s_1(t)}{x - x_1(t)} - \frac{s_1^*(t)}{x - x_1^*(t)} \right), \tag{5.1a}
\]

with

\[
x_1(t) = x_{1,0} t + x_{1,0}, \quad s_1(t) = s_{1,0}, \tag{5.1b}
\]

where (5.1b) follows from (2.21a) and (4.6) with \( n = 0 \). In accordance with (4.27), the initial condition of \( \dot{x}_1 \) is

\[
\dot{x}_1,0 = -i n_1 \cdot \left( i \mathbf{m}_0 + \frac{s_{1,0}^*}{x_{1,0} - x_{1,0}^*} \right), \tag{5.2a}
\]

and the constraint for \( s_{1,0} \) comes from (1.6) with \( N = 1 \), giving

\[
s_{1,0} \cdot \left( i \mathbf{m}_0 + \frac{s_{1,0}^*}{x_{1,0} - x_{1,0}^*} \right) = 0. \tag{5.2b}
\]

If we substitute \( s_{1,0} \) from (4.15) into (5.2b), we can determine \( s_1 \). Consequently,

\[
s_{1,0} = \text{Im} x_{1,0} (n_{1,+}^* \cdot \mathbf{m}_0)n_{1,+}. \tag{5.3}
\]

Introducing (5.3) into (5.2a) and taking into account the relation \( n_{1,+}^* \cdot n_1 = 0 \), we obtain \( \dot{x}_{1,0} = n_1 \cdot \mathbf{m}_0 \equiv v \) \( (|v| < 1) \). It turns out from (5.1b) that \( x_1(t) = vt + x_{1,0} \). With these results, the one-soliton solution (5.1) is expressed in the form

\[
\mathbf{m} = \mathbf{m}_0 + \text{Im} x_{1,0} \left\{ \frac{n_{1,+}^* \cdot \mathbf{m}_0}{x - vt - x_{1,0}} n_{1,+} - \frac{n_{1,+} \cdot \mathbf{m}_0}{x - vt - x_{1,0}^*} n_{1,+}^* \right\}. \tag{5.4}
\]

Let \( \mathbf{e}_1 = (1, 0, 0), \mathbf{e}_2 = (0, 1, 0), \mathbf{e}_3 = (0, 0, 1) \) be the normal orthogonal bases in \( \mathbb{R}^3 \). We then put \( n_{1,1} = \mathbf{e}_1, n_{1,2} = \mathbf{e}_2, n_1 = \mathbf{e}_3 \) and

\[
\mathbf{m}_0 = \cos \theta \mathbf{e}_1 + \sin \theta \mathbf{e}_3, \tag{5.5}
\]
Fig. 1. A profile of the one-soliton solution as function of the traveling wave coordinate $X = x - vt$ with the parameters $m_0 = (1/2, 0, \sqrt{3}/2), x_{1,0} = i, s_{1,0} = (1/4, i/4, 0), n_{1,1} = (1, 0, 0), n_{1,2} = (0, 1, 0), \theta = \pi/3 (v = \sqrt{3}/2)$.

where $\theta$ is a real parameter. Note in this setting that $n_{1,+} = e_1 + i e_2$. Using the relations $v = n_1 \cdot m_0 = \sin \theta, n_{1,+} \cdot m_0 = \cos \theta$, the expression of $m$ from (5.4) is rewritten in the form

$$m = m_0 - \frac{2 \text{Im} x_{1,0} \cos \theta}{(x - vt - \text{Re} x_{1,0})^2 + (\text{Im} x_{1,0})^2} \left\{ \text{Im} x_{1,0} e_1 + (x - vt - \text{Re} x_{1,0}) e_2 \right\}. \quad (5.6)$$

Introducing the function $B(z)$

$$B(z) = \frac{z - x_{1,0}}{z - x_{1,0}^*} = 1 - \frac{2(\text{Im} x_{1,0})^2}{(z - \text{Re} x_{1,0})^2 + (\text{Im} x_{1,0})^2} - 2i \frac{\text{Im} x_{1,0} (z - \text{Re} x_{1,0})}{(z - \text{Re} x_{1,0})^2 + (\text{Im} x_{1,0})^2}, \quad (5.7)$$

and taking into account (5.5), we can recast (5.6) into the form

$$m = \cos \theta \text{Re} B(x - vt) e_1 + \cos \theta \text{Im} B(x - vt) e_2 + \sin \theta e_3. \quad (5.8)$$

This recovers the traveling wave solution of the HWM equation discovered in [1, 2].

A typical profile of the spin density $m = (m_1, m_2, m_3)$ as function of the traveling wave coordinate $X = x - vt$ is depicted in Fig. 1. In this example, it has the form

$$m(X) = \left( \frac{X^2 - 1}{2(X^2 + 1)}, \frac{X}{X^2 + 1}, \frac{\sqrt{3}}{2} \right). \quad (5.9)$$

An inspection of the component $m_1$ in (5.8) reveals that its amplitude measured from a constant level at spatial infinity becomes $2\sqrt{1 - v^2}/\text{Im} x_{1,0}$ and hence it decreases as the velocity increases. This unusual feature can be remedied if one observes the profile of $m_1$.
in the coordinate system moving to the right with a constant velocity \( v = 1 \), for instance. As a result, the amplitude becomes a monotonically increasing function of the velocity in agreement with the velocity-amplitude relation of the soliton.

5.2. Two-soliton solution

It follows from (4.5) with \( N = 2 \) that the two-soliton solution takes the form

\[
m = m_0 + i \sum_{j=1}^{2} \left( \frac{s_j(t)}{x - x_j(t)} - c.c. \right)
\]

\[
= m_0 + i \left[ \frac{1}{f_2} \{ J_0 \sigma_0 x - J_0 \sigma_1 + J_1 \sigma_0 \} - c.c. \right],
\]

where \( \sigma_0 = 1, \quad \sigma_1 = (\dot{x}_{1,0} + \dot{x}_{2,0})t + x_{1,0} + x_{2,0}, \quad J_0 = s_{1,0} + s_{2,0}, \quad J_1 = (\dot{x}_{1,0}s_{1,0} + \dot{x}_{2,0}s_{2,0} + x_{1,0}s_{1,0}^* + x_{2,0}s_{2,0}^*)t + x_{1,0}s_{1,0} + x_{2,0}s_{2,0} \).

\[
\dot{x}_{1,0} = -\text{im}_1 \cdot \left( \text{im}_0 - \frac{s_{2,0}}{x_{1,0} - x_{2,0}} + \frac{s_{1,0}^*}{x_{1,0} - x_{2,0}^*} + \frac{s_{2,0}^*}{x_{2,0} - x_{2,0}^*} \right),
\]

\[
\dot{x}_{2,0} = -\text{im}_2 \cdot \left( \text{im}_0 - \frac{s_{1,0}}{x_{2,0} - x_{1,0}} + \frac{s_{1,0}^*}{x_{2,0} - x_{1,0}^*} + \frac{s_{2,0}^*}{x_{2,0} - x_{2,0}^*} \right),
\]

\[
\dot{s}_{1,0} = -2 \frac{s_{1,0} \times s_{2,0}}{(x_{1,0} - x_{2,0})^2}, \quad \dot{s}_{2,0} = -2 \frac{s_{2,0} \times s_{1,0}}{(x_{2,0} - x_{1,0})^2}.
\]

Here, \( s_{j,0} (j = 1, 2) \) are given by (4.15) with (4.26).

The tau-function \( f_2 \) from (2.21b) may be written in a convenient form suitable for investigating the asymptotics of the two-soliton solution. To be more specific, we put

\[
f_2 = \begin{vmatrix} x - v_1 t + \alpha_1 & \beta_{1,2} \\ \beta_{2,1} & x - v_2 t + \alpha_2 \end{vmatrix}
\]

\[
= x^2 - \{ (v_1 + v_2)t - (\alpha_1 + \alpha_2) \} x + v_1 v_2 t^2 - (v_2 \alpha_1 + v_1 \alpha_2) t + \alpha_1 \alpha_2 - \beta_{1,2} \beta_{2,1},
\]

where \( v_1, v_2 \in \mathbb{R} \) and \( \alpha_1, \alpha_2, \beta_{1,2}, \beta_{2,1} \in \mathbb{C} \) are unknown parameters. To determine these unknowns, we compare (2.21b) with (5.11) and obtain the following system of algebraic equations:

\[
v_1 + v_2 = \dot{x}_{1,0} + \dot{x}_{2,0},
\]

\[
\alpha_1 + \alpha_2 = -(x_{1,0} + x_{2,0}),
\]

\[
v_1 v_2 = \dot{x}_{1,0} \dot{x}_{2,0} - \frac{2 s_{1,0} \cdot s_{2,0}}{(x_{1,0} - x_{2,0})^2},
\]

\[
v_2 \alpha_1 + v_1 \alpha_2 = -(\dot{x}_{1,0} x_{2,0} + x_{1,0} \dot{x}_{2,0}),
\]

\[
\alpha_1 \alpha_2 - \beta_{1,2} \beta_{2,1} = x_{1,0} x_{2,0}.
\]
If we take into account (5.12a) and (5.12c), we see that \( v_1 \) and \( v_2 \) follow by solving the quadratic equation for \( v \)

\[
v^2 - (\dot{x}_{1,0} + \dot{x}_{2,0})v + \dot{x}_{1,0}\dot{x}_{2,0} - \frac{2 s_{1,0} \cdot s_{2,0}}{(x_{1,0} - x_{2,0})^2} = 0, \quad (5.13a)
\]

which gives rise to the solutions

\[
v_{1,2} = \frac{1}{2} \left\{ \dot{x}_{1,0} + \dot{x}_{2,0} \mp \sqrt{(\dot{x}_{1,0} - \dot{x}_{2,0})^2 + \frac{8 s_{1,0} \cdot s_{2,0}}{(x_{1,0} - x_{2,0})^2}} \right\}, \quad (5.13b)
\]

where the plus (minus) sign corresponds to \( v_2(v_1) \). The reality of the velocities \( v_1 \) and \( v_2 \) will be shown in remark 3. On the other hand, \( \alpha_1 \) and \( \alpha_2 \) are found from (5.12b) and (5.12d) to be

\[
\alpha_1 = \frac{1}{v_1 - v_2} \left\{ -(x_{1,0} + x_{2,0})v_1 + \dot{x}_{1,0}x_{2,0} + x_{1,0}\dot{x}_{2,0} \right\}, \quad (5.14a)
\]

\[
\alpha_2 = \frac{1}{v_2 - v_1} \left\{ -(x_{1,0} + x_{2,0})v_2 + \dot{x}_{1,0}x_{2,0} + x_{1,0}\dot{x}_{2,0} \right\}. \quad (5.14b)
\]

Introducing (5.14) with (5.13) into (5.12e), we obtain, after a few computations, the following simple relation

\[
\beta_{1,2} = \frac{2 s_{1,0} \cdot s_{2,0}}{(v_1 - v_2)^2}. \quad (5.15)
\]

Consequently, we can put

\[
\beta_{1,2} = \frac{\sqrt{2 s_{1,0} \cdot s_{2,0}}}{|v_1 - v_2|}. \quad (5.16)
\]

The new parameterization of the tau-function \( f_2 \) allows us to explore the properties of the two-soliton solution. A typical example of the two-soliton solution is depicted in Fig. 2 in which the form of \( m \) at \( t = 0 \) is given by

\[
m(x, 0) = \left( -\frac{4x(x^2 - 1)}{(x^2 + 1)(x^2 + 4)}, \frac{8x^2}{(x^2 + 1)(x^2 + 4)}, \frac{x^2 - 4}{x^2 + 4} \right). \quad (5.17)
\]

The figure displays the head-on collision of two solitons. The solitonic nature of the solution is seen obviously, i.e., each soliton appears without changing its profile after the collision.

The detailed information of the interaction process of solitons may be extracted from the motion of poles in the complex plane. This purpose can be attained in principle by solving the equations of motion (1.4) for \( x_1 \) and \( x_2 \). When only solitons are present, however, an alternative way is to find the pole positions by solving the quadratic equation \( f_2 = 0 \). Actually, using (5.11) with (5.16), we find that

\[
x_{1,2} = \frac{1}{2} \left\{ (v_1 + v_2)t - \alpha_1 - \alpha_2 \pm \sqrt{D} \right\}, \quad (5.18a)
\]
Fig. 2. Profiles of the two-soliton solution at three different times: left panel $t = -50$; middle panel $t = 0$; right panel $t = 50$. The parameters are $m_0 = (0, 0, 1), x_{1,0} = i, x_{2,0} = 2i, s_{1,0} = (-4i/3, 4/3, 0), s_{2,0} = (10i/3, -8/3, 2), n_{1,1} = (0, -1, 0), n_{1,2} = (1, 0, 0), n_{2,1} = (0, 5/4, -3/5), n_{2,2} = (-1, 0, 0), v_1 = -0.854, v_2 = 0.521.

where

$$D = \left\{ (v_1 - v_2)t - (\alpha_1 - \alpha_2) \right\}^2 + \frac{8 s_{1,0} \cdot s_{2,0}}{(v_1 - v_2)^2}. \quad (5.18b)$$

A simple analysis reveals that the asymptotic forms of $x_1$ and $x_2$ from (5.18) as $t \to \pm \infty$ take the same forms and given respectively by

$$x_1 \sim v_1 t - \alpha_1 + \frac{2 s_{1,0} \cdot s_{2,0}}{(v_1 - v_2)^2} \frac{1}{(v_1 - v_2)t - (\alpha_1 - \alpha_2)}, \quad (5.19a)$$

$$x_2 \sim v_2 t - \alpha_2 - \frac{2 s_{1,0} \cdot s_{2,0}}{(v_1 - v_2)^2} \frac{1}{(v_1 - v_2)t - (\alpha_1 - \alpha_2)}. \quad (5.19b)$$

On the other-hand, it follows from (4.30) with $N = 2$ that

$$s_1 = \frac{1}{x_2 - x_1} (J_0 x_2 - J_1), \quad (5.20a)$$

$$s_2 = \frac{1}{x_1 - x_2} (J_0 x_1 - J_1). \quad (5.20b)$$

Taking into account (5.10c) and (5.19), one can see that both $s_1$ and $s_2$ approach the constant values as $t \to \pm \infty$. Actually, the leading order asymptotics of $s_1$ and $s_2$ are
Fig. 3. The time evolution of the poles \( x_1 \) (solid line) and \( x_2 \) (dotted line) in the time interval \(-10 \leq t \leq 10\), where the parameters are the same as those used in Fig. 2. The arrows indicate the direction of motion, and the black dots mark the positions of the poles at \( t = 0 \). In this example, \( \text{Im} x_1(\pm \infty) = 0.409, \text{Im} x_2(\pm \infty) = 2.59 \).

Fig. 4. The time evolution of the real part of poles \( \text{Re} x_1 \) (solid line) and \( \text{Re} x_2 \) (dotted line) in the time interval \(-10 \leq t \leq 10\) corresponding to Fig. 3.

found to be

\[
\begin{align*}
\mathbf{s}_1 & \sim \frac{1}{v_2 - v_1} \{(s_{1,0} + s_{2,0})v_2 - (\dot{x}_{1,0}s_{1,0} + \dot{x}_{2,0}s_{2,0} + x_{1,0}\dot{s}_{1,0} + x_{2,0}\dot{s}_{2,0})\}, \\
\mathbf{s}_2 & \sim \frac{1}{v_1 - v_2} \{(s_{1,0} + s_{2,0})v_1 - (\dot{x}_{1,0}s_{1,0} + \dot{x}_{2,0}s_{2,0} + x_{1,0}\dot{s}_{1,0} + x_{2,0}\dot{s}_{2,0})\}.
\end{align*}
\]

(5.21a)

(5.21b)

The asymptotic form of \( \mathbf{m} \) from (5.10) is obtained by introducing (5.19) and (5.21) into it, giving

\[
\mathbf{m} \sim \mathbf{m}_0 + i \sum_{j=1}^{2} \left( \frac{s_j(\infty)}{x - v_j t + \alpha_j} - \text{c.c.} \right),
\]

(5.22)

where \( s_j(\infty) (j = 1, 2) \) stand for the asymptotic values given by (5.21). The above expression shows clearly that the two-soliton solution is composed of a superposition of single solitons given by (5.4). Moreover, it exhibits no phase shifts after the collision. This remarkable feature is common to that of the rational (or algebraic) multisoliton solutions of the BO and nonlocal NLS equations [18-21]. However, to the best of our knowledge, it
is the first example observed in the head-on collision of rational solitons. Recall in this respect that the BO rational solitons exhibit no phase shifts after overtaking collisions [18].

Fig. 3 shows the time evolution of the poles $x_1$ and $x_2$ in the complex plane corresponding to the two-soliton solution depicted in Fig. 2. One can observe that the distance between two poles becomes minimum at $t = 0$ at which instant two solitons would collide. See the corresponding profiles of solitons in Fig. 2. A detailed inspection of the expressions of the poles $x_1$ and $x_2$ from (5.18) indicates that the imaginary part of each pole exhibits a discontinuity at $t = 0$ in such a way that the poles interchange their imaginary parts. This intriguing feature is sometimes called an exchange of identity. Fig. 4 depicts the time evolution of the real part of poles. While their trajectories are continuous during their motion, an exchange of identity occurs at the instant of the collision of poles.

5.3. N-soliton solution

The construction of the $N$-soliton solution can be done following a purely algebraic procedure developed in Section 4. Here, we discuss the property of the $N$-soliton solution focusing on its asymptotic behavior. Now, we assume the tau-function $f_N$ of the determinantal form

$$f_N = \left| \left( \delta_{jk}(x - v_j t + \alpha_j) + (1 - \delta_{jk}) \beta_{j,k} \right)_{1 \leq j,k \leq N} \right|, \quad (\beta_{j,k} = \beta_{k,j}), \quad (5.23)$$

which is a natural generalization of the tau-function of the two-soliton solution (5.11). We compare (2.20) with (5.23) and see that equating the coefficients of the terms $t^s x^j (j = 0, 1, ..., N-1; s = 0, 1, ..., N)$ yields the $N(N+3)/2$ equations for the unknowns $v_j, \alpha_j \ (j = 1, 2, ..., N)$ and $\beta_{j,k} \ (j, k = 1, 2, ..., N; j \neq k)$. Since the number of the unknowns is equal to that of the equations, we can determine these unknowns in principle. An explicit computation has been performed for the two-soliton solution, and as shown in (5.12), the five equations were derived for the same number of unknowns.

Let us now investigate the behavior of the solution. An asymptotic analysis using (5.23) reveals that the poles $x_j$ behave like

$$x_j \sim v_j t - \alpha_j, \quad t \to \pm \infty, \quad (j = 1, 2, ..., N). \quad (5.24)$$

Referring to (5.24), we find from (4.30) with (4.6) that the spin variables $s_j$ approach the constant vectors as $t \to \pm \infty$. The asymptotic form of $m$ then becomes

$$m \sim m_0 + i \sum_{j=1}^{N} \left( \frac{s_j(\infty)}{x - v_j t + \alpha_j} - c.c. \right). \quad (5.25)$$

This represents simply a superposition of $N$ single solitons, each of which undergoes no phase shifts after collisions.
Remark 3. The parameter $v_j$ in the asymptotic expression (5.24) is the velocity of the $j$th pole and hence it should be a real quantity. In the case of the two-soliton solution, it is given by (5.13b). Here, we provide a general proof that $v_j$ is real and satisfies the inequality $|v_j| < 1$. In addition, we show that $\text{Im} \alpha_j \neq 0$. We first recall that Eq. (1.5) is satisfied for $t > 0$ [10]. Taking the limit $t \to \infty$ for the expression of $\dot{x}_j(t)$ and then substituting the asymptotic form of $x_j$ from (5.24) under the assumptions $v_j \neq v_k$ for $j \neq k$ and $v_j, \alpha_j \in \mathbb{C}$, we obtain

$$v_j = i \frac{\mathbf{m}_0 \cdot (s_j(\infty) \times s_j^*(\infty))}{s_j(\infty) \cdot s_j^*(\infty)}, \quad (j = 1, 2, ..., N).$$

(5.26)

We see from this expression that $v_j^* = v_j$, implying that $v_j$ is real. If we use the formula (2.6b) subjected to the conditions $\mathbf{m}_0^2 = 1, s_j^2(\infty) = 0$, we can derive the relation

$$\{\mathbf{m}_0 \cdot (s_j(\infty) \times s_j^*(\infty))\}^2 = 2(\mathbf{m}_0 \cdot s_j(\infty))(\mathbf{m}_0 \cdot s_j^*(\infty))(s_j(\infty) \cdot s_j^*(\infty)) - (s_j(\infty) \cdot s_j^*(\infty))^2.

(5.27)

It follows from (5.26) and (5.27) that

$$v_j^2 = 1 - \frac{2(\mathbf{m}_0 \cdot s_j(\infty))(\mathbf{m}_0 \cdot s_j^*(\infty))}{s_j(\infty) \cdot s_j^*(\infty)}, \quad (j = 1, 2, ..., N).

(5.28)

Let $v_j^2 = 1 - \delta_j$ where $\delta_j$ stands for the second term on the right-hand side of (5.28). The inequality $0 < \delta_j \leq 1$ follows from (5.27) if $\mathbf{m}_0^2 = 1, s_j^2(\infty) = 0$. Plugging this inequality into (5.28), we finally arrive at the result $|v_j| < 1 \ (j = 1, 2, ..., N)$. A similar argument using the large time analog of (1.6) yields the relation

$$\text{Im} \alpha_j = -\frac{1}{2} \frac{s_j(\infty) \cdot s_j^*(\infty)}{\mathbf{m}_0 \cdot s_j(\infty)}, \quad (j = 1, 2, ..., N).

(5.29)

It turns out that $\text{Im} \alpha_j \neq 0$ and its sign is determined by the sign of the term $\mathbf{m}_0 \cdot s_j(\infty)$. The basic assumption in deriving the system of equations with constraints (1.3)-(1.6) is that $\text{Im} x_j(t) > 0 \ (j = 1, 2, ..., N)$ for $t > 0$. This implies that the tau-function $f_N(x, t)$ from (2.19) never becomes zero for real $x$. The proof of this fact has not been established as yet even though it can be checked numerically for $N = 2, 3$. However, the asymptotic form (5.24) of $x_j$ with $\text{Im} \alpha_j \neq 0$ strongly supports the above statement. This important problem should be addressed in a future study.

6. Concluding remarks

In this paper, we found an explicit Lax pair for a many-body dynamical system associated with the HWM equation. Although the dynamical system has been found to be equivalent to the spin CM system [10], its Lax pair was presented here for the first time, in which a key identity (2.5) played a central role. We also derived the conservation
laws of the system using the Lax pair following a standard procedure, and clarified the underlying Hamiltonian structure. We stress that analytical expressions of the multisoliton solutions of the HWM equation were also presented for the first time, even though a numerical scheme for obtaining them has been developed in [10]. A new parameterization of the $N$ soliton tau-function enables us to explore its large time asymptotics, showing that no phase shifts appear after the head-on collisions of solitons. The study of the HWM equation just begins and a number of problems remain unsolved. In conclusion, we discuss some of them.

1. One of the most important issues will be the analysis of the HWM equation by means of the inverse scattering transform method (IST). Although the IST has been used successfully to the local nonlinear evolution equations such as the KdV and NLS equations, its application to nonlocal nonlinear equations is far from satisfactory. See [22-25] for the BO and nonlocal NLS equations. The similar situation happens to the HWM equation for which the Lax pair takes a nonlocal form [7]. Nevertheless, the nonlocal Riemann-Hilbert approach may work effectively for these nonlocal equations. See, for instance, [26].

2. The HWM equation is formally shown to exhibit an infinite number of conservation laws including the mass, momentum and energy [1, 2]. However, the explicit forms of the higher conservation laws in terms of the spin densities have not been derived yet. In this respect, it should be noted that the information of the conservation laws of the spin CM system derived here has no direct relevance to that of the HWM equation.

3. The HWM equation is obtained formally from the classical Heisenberg ferromagnet equation $m_t = m \times m_{xx}$ if one replaces an $x$ derivative by the Hilbert transform. This formal derivation is just similar to the relation between the KdV and BO equations. A number of outcomes have been obtained for the Heisenberg ferromagnet equation [27]. Their analogs for the HWM equation will be worth studying. For instance, the Heisenberg ferromagnet equation is known to be gauge-equivalent to the NLS equation [28-30] and hence the HWM equation may be expected to have a gauge-equivalent nonlocal nonlinear equation.

4. There exist several exact methods of solutions for solving the soliton equations. Among them, the direct method provides a very powerful mean to construct multisoliton solutions [31, 32]. In fact, it has been applied to the BO and nonlocal NLS equations to obtain the explicit $N$-soliton formulas [18-21, 33]. On the other hand, the pole expansion method is rather sophisticated because it needs to solve the equations of motion for the corresponding dynamical system, as exemplified in this paper. This situation will be apparent if one compares the derivation of the $N$-soliton solution of the nonlocal NLS equation, for instance by means of the pole expansion method [16] and direct method [21]. An application of the direct method to the HWM equation is a challenging issue.

5. While we were concerned with soliton solutions on the real line, the construction of
periodic solutions is an important issue. Both the pole expansion method and direct method are amenable to this problem. An application of the former method to the nonlocal NLS equation has already been developed, whereby the quantities analogous to (4.4) were used effectively [16]. We will report the results associated with periodic solutions of the HWM equation in a subsequent paper.

Appendix A. Proof of Proposition 4

The proof of (4.7) can be performed by comparing the coefficients of $\epsilon^n$ on both sides. Explicitly, it yields

$$n\dot{X}X^{n-1} = BX^n - X^nB + \sum_{l=0}^{n-1} X^l LX^{n-l-1}. \tag{A.1}$$

The proof of (A.1) proceeds by the mathematical induction. For $n = 1$, (A.1) becomes

$$\dot{X} = BX - XB + L, \tag{A.2}$$

which is just (2.14). Assume that (A.1) holds for $n = m(\geq 2)$, which, multiplied by $X$ from the right, gives

$$m\dot{X}X^m = BX^{m+1} - X^{m}BX + \sum_{l=0}^{m-1} X^l LX^{m-l}. \tag{A.3}$$

If we introduce the term $BX$ from (A.2) into the second term on the right-hand side of (A.3), and using an obvious relation $XX = \dot{XX}$ which follows since $X$ is a diagonal matrix, we obtain

$$(m + 1)\dot{X}X^m = BX^{m+1} - X^{m+1}B + \sum_{l=0}^{m} X^l LX^{m-l}, \tag{A.4}$$

showing that (A.1) holds for $n = m + 1$. □

Appendix B. Proof of Proposition 5

It follows by differentiating $K = YL$ by $t$ and using (2.1) and (4.7) that

$$\dot{K} = (BY - YB + \epsilon YLY)L + Y(BL - LB)$$

$$= BYL + \epsilon YLYL - YLB$$

$$= BK - KB + \epsilon K^2.$$
Appendix C. Proof of Proposition 6

We differentiate $\mathcal{P}_n = \text{Tr}(S^2K^nY)$ by $t$ and then use the time derivative $\dot{S}$ from (2.11), $\dot{Y}$ from (4.7) with $K = YL$ and $\dot{K}$ from (4.8), respectively. This leads to

$$\dot{\mathcal{P}}_n = \text{Tr}\left[ (BS^2 - S^2B)K^nY + S^2 \sum_{l=0}^{n-1} K^l(BK - KB + \epsilon K^2)K^{n-l-1}Y \
+ S^2K^n(BY - YB + \epsilon YLY) \right]$$

$$= \text{Tr}\left[ S^2 \left\{ \left( \sum_{l=0}^{n} K^lBK^{n-l} - \sum_{l=-1}^{n-1} K^{l+1}BK^{n-l-1} \right) Y + \epsilon(n+1)K^{n+1}Y \right\} \right]$$

$$= \epsilon(n+1)\text{Tr}(S^2K^{n+1}Y)$$

$$= \epsilon(n+1)\mathcal{P}_{n+1}.$$

□

Appendix D. Proof of Proposition 7

The proof proceeds by the mathematical induction. For $n = 1$, (4.10) reduces to

$$\sum_{l=1}^{\infty} \epsilon^l \frac{d\mathcal{Q}_l}{dt} = \epsilon\text{Tr}(S^2KY), \quad (D.1)$$

which is verified as follows: First, referring to the definition $\mathcal{Q}_l = \text{Tr}(S^2X^l)$, one has

$$\sum_{l=1}^{\infty} \epsilon^l \mathcal{Q}_l = \text{Tr} \left\{ S^2(Y - I) \right\}. \quad (D.2)$$

Then, differentiating this expression by $t$ yields

$$\sum_{l=1}^{\infty} \epsilon^l \frac{d\mathcal{Q}_l}{dt} = \text{Tr} \left\{ (\dot{S}S + S\dot{S})(Y - I) + S^2\dot{Y} \right\}. \quad (D.3)$$

Last, substituting $\dot{S}$ from (2.11) and $\dot{Y}$ from (4.7) into (D.2), we deduce

$$\sum_{l=1}^{\infty} \epsilon^l \frac{d\mathcal{Q}_l}{dt} = \text{Tr} \left[ S^2 \{(Y - I)B - B(Y - I) + BY - YB + \epsilon YLY\} \right]$$

$$= \epsilon\text{Tr}(S^2KY). \quad (D.4)$$

which is (D.1). Assume that (4.10) holds for $n = m(>1)$, i.e.,

$$\sum_{l=m}^{\infty} \epsilon^l \frac{d_m\mathcal{Q}_l}{dt^m} = \epsilon^m m!\mathcal{P}_m. \quad (D.5)$$

27
We differentiate \( D.5 \) by \( t \) and use \((4.9)\) to obtain
\[
\sum_{l=m}^{\infty} \epsilon^l \frac{d^{m+1} Q_l}{dt^{m+1}} = \epsilon^{m+1} (m + 1)! H_{m+1}.
\] (D.6)

Comparison of the coefficients of \( \epsilon^m \) on both sides of (D.6) yields the relation \( d^{m+1} Q_m/dt^{m+1} = 0 \). It turns out that (D.6) reduces to
\[
\sum_{l=m+1}^{\infty} \epsilon^l \frac{d^{m+1} Q_l}{dt^{m+1}} = \epsilon^{m+1} (m + 1)! H_{m+1},
\] (D.7)
implying that (4.10) holds for \( n = m + 1 \). \( \square \).

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References

[1] T. Zhou and M. Stone, Solitons in a continuous classical Haldane-Shastry spin chain, Phys. Lett. A 379 (2015) 2817-2825.

[2] E. Lenzmann and A. Schikorra, On energy-critical half-wave maps into $S^2$, Inv. Math. 213 (2018) 1-82.

[3] E. Lenzmann and J. Sok, Derivation of the half-waves maps equation from Calogero-Moser spin system, arXiv. 2007.15323 (2020).

[4] J. Gibbons and T. Hermsen, A generalization of the Calogero-Moser system, Physica D11 (1984) 337-348.

[5] S. Wojciechowski, An integrable marriage of the Euler equations with the Calogero-Moser system, Phy. Lett. 111A (1985) 101-103.

[6] I. Krichever, O. Babelon, E. Billey and M. Talon, Spin generalization of the Calogero-Moser system and the matrix KP equation, in: Topics in Topology and Mathematical Physics, vol. 170 ed. S. Novikov, American Mathematical Society, Providence RI, 1995, 83-120.

[7] P. Gérard and E. Lenzmann, A Lax pair structure for the half-wave maps equation, Lett. Math. Phys. 108 (2018) 1635-1648.

[8] E. Lenzmann, A short primer on the half-wave maps equation, Journées Équations aux dérivées partielles (2018) 1-12.

[9] M. D. Kruskal, The Korteweg-de Vries equation and related evolution equations, Lect. Appl. Math. 15 (1974) 61-83.

[10] B. K. Berntson, R. Klabbers and E. Langmann, Multi-solitons of the half-wave maps equation and Calogero-Moser spin-pole dynamics, J. Phys. A: Math. Theor. 52 (2020) 505702 (32pp).

[11] H. Airault, H. P. Mckean and J. Moser, Rational and elliptic solutions of the Korteweg-de Vries equation and a related many-body problem, Comm. Pure and Appl. Math. 30 (1977) 95-148.

[12] D. V. Choodnovsky and G. V. Choodnovsky, Pole expansion of nonlinear partial differential equations, Il Nuovo Cimento B 40 (1977) 339-353

[13] K. M. Case, The $N$-soliton solution of the Benjamin-Ono equation, Proc. Natl. Acad. Sci. 75 (1978) 3562-3563.

[14] H. H. Chen, Y. C. Lee and N. R. Pereira, Algebraic internal wave solitons and integrable Calogero-Moser-Sutherland $N$-body problem, Phys. Fluids 22 (1979) 187-188.
[15] M. A. Olshanetsky and A. M. Perelomov, Classical integrable finite-dimensional systems related to Lie algebras, Phys. Rep. 71 (1981) 313-400.

[16] Y. Matsuno, Calogero-Moser-Sutherland dynamical systems associated with nonlocal nonlinear Schrödinger equation for envelope waves, J. Phys. Soc. Jpn. 71 (2002) 1415-1418.

[17] S. Wojciechowski, Superintegrability of the Calogero-Moser system, Phy. Lett. 95A (1983) 279-281.

[18] Y. Matsuno, Exact multi-soliton solution of the Benjamin-Ono equation, J. Phys. A: Math. Gen. 12 (1979) 619-621.

[19] Y. Matsuno, Interaction of the Benjamin-Ono solitons, J. Phys. A: Math. Gen. 13 (1980) 1519-1536.

[20] Y. Matsuno, Dynamics of interacting algebraic solitons, Int. J. Mod. Phys. B9 (1995) 1985-2081.

[21] Y. Matsuno, Multiperiodic and multisoliton solutions of a nonlocal nonlinear Schrödinger equation for envelope waves, Phys. Lett. A 278 (2000) 53-58.

[22] Y. Matsuno, Exactly solvable eigenvalue problems for a nonlocal nonlinear Schrödinger equation for envelope waves, Inverse Problems 18 (2002) 1101-1125.

[23] Y. Matsuno, A Cauchy problem for the nonlocal nonlinear Schrödinger equation, Inverse Problems 20 (2004) 437-445.

[24] Y. Matsuno, Recent topics on a class of nonlinear integrodifferential evolution equations of physical significance, in: Advances in Mathematics Research, vol. 4 ed. G. Oyibo, Nova Science, New York, 2003, 19-89.

[25] J.-C Saut, Benjamin-Ono and intermediate long wave equations: modeling, IST and PDE, in: Nonlinear Partial Differential Equations and Inverse Scattering, Fields Institute Communications vol. 83 eds. P. Miller, P. Perry, J.-C Saut and C. Sulem, Springer, New York, NY. 2019, 95-160.

[26] P. M. Santini, Integrable singular integral evolution equations, in: Important Developments in Soliton Theory, Springer series in Nonlinear Dynamics, eds. A. S. Fokas and V. E. Zakharov, Springer, New York, NY. 1993, 147-177.

[27] M. Lakshmanan, The fascinating world of the Landau-Lifshitz-Gilbert equation: an overview, Phil. Trans. R. Soc. A 369 (2011) 1280-1300.

[28] M. Lakshmanan, Continuous spin system as an exactly solvable dynamical system, Phys. Lett. 61A (1977) 53-54.
[29] L. A. Takhtajan, Integration of the continuous Heisenberg spin chain through the inverse scattering method, Phys. Lett. 64A (1977) 235-237.

[30] V. E. Zakharov and L. A. Takhtadzhyan, Equivalence of the nonlinear Schrödinger equation and the equation of a Heisenberg ferromagnet, Theor. Math. Phys. 38 (1979) 17-23.

[31] Y. Matsuno, Bilinear Transformation Method, Academic, New York, 1984.

[32] R. Hirota, The Direct Method in Soliton Theory, Cambridge University Press, Cambridge 2004.

[33] Y. Matsuno, A direct proof of the \( N \)-soliton solution of the Benjamin-Ono equation by means of Jacobi’s formula, J. Phys. Soc. Jpn. 57 (1988) 1924-1929.