Minimal complete arcs in $PG(2,q)$, $q \leq 32$

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Abstract. In this paper it has been verified, by a computer-based proof, that the smallest size of a complete arc is 14 in $PG(2,31)$ and in $PG(2,32)$. Some examples of such arcs are also described.

1 Introduction

In the projective plane $PG(2,q)$ over the Galois field $GF(q)$ an $n$-arc is a set of $n$ points no 3 of which are collinear. An $n$-arc is called complete if it is not contained in an $(n+1)$-arc of the same projective plane. For a detailed description of the most important properties of these geometric structures, we refer the reader to [8]. In [10] the close relationship between the theory of complete $n$-arcs, coding theory and mathematical statistics is presented. In particular arcs and linear maximum distance separable codes (MDS codes) are equivalent objects (see [19], [20], [21]). Partly because of this fact, in recent years, the problem of determining the spectrum of values $n$ for which a complete arc exists has been intensively investigated. For recent results on the sizes of complete arcs in projective planes see [5]. The full classification of complete $n$-arcs is known for $q \leq 29$, see [4] and the references therein. This paper concerns the minimal complete arcs in $PG(2,q)$ for $q \leq 32$. The minimal size of a complete $n$-arc of $PG(2,q)$ is indicated by $t(2,q)$. General lower bounds on $t(2,q)$ are given in the following table:

| $q$ | $t(2,q)$ | References |
|-----|----------|------------|
| $q$ | $\sqrt{2q+1}$ | [18] |
| $q = p^h$, $p$ prime, $h = 1,2,3$ | $\sqrt{3q+1/2}$ | [2], [3], [16] |

Lower bounds for $t(2,q)$

The values of $t(2,q), q \leq 29$ are stated in the following table:
In this paper it is demonstrated by a computer-based proof that
t_2(2,31) = 14 and t_2(2,32) = 14.

This result has been obtained by an exhaustive computer search. The search
has been feasible because projective equivalence properties among arcs have
been exploited and a simple parallelization technique has been used.

We also performed a partial classification of the smallest complete arcs in
PG(2,31) and in PG(2,32) obtaining 3391 and 9300 non-equivalent examples
respectively. Equivalence up to PGL(3,31) has been considered for PG(2,31),
while equivalence up to PΓL(3,32) has been considered for PG(2,31). The aim
of this search has been to look for examples with large automorphism group,
but the maximum order of the automorphism group of the examined examples
is 6 and almost all the examples have trivial automorphism group.

In Section 2 the computation of the values t_2(2,31) and t_2(2,32) is described;
examples of the smallest complete arcs in PG(2,31) and in PG(2,32) are
presented in Section 3.

### 2 The determination of t_2(2,31), t_2(2,32)

The results presented in this paper have been obtained by an exhaustive
computer search. The exhaustive search has been feasible because projective
properties among arcs have been exploited to avoid obtaining too many iso-
morphic copies of the same solution arc and to avoid searching through parts
of the search space isomorphic to previously searched portions.

Also a simple parallelization technique has been used to divide the load of
the computation in a multiprocessor computer (a Quad-Core Linux computer
with 4 processors).

The used algorithm starts constructing a tree structure containing a rep-
resentative of each class of non-equivalent arcs of size less than or equal to a
fixed threshold $h$. If the threshold $h$ were equal to the actual size of the sought
arcs, the algorithm would be orderly, that is capable of constructing each goal
configuration exactly once [17]. However, in the present case, the construction
of the tree with the threshold $h$ equal to the size of the sought arcs would have
been too space and time consuming. For this reason a hybrid approach has
been adopted. The tree representing the non-equivalent arcs of size less than
or equal to eight has been constructed and then every non-equivalent 8−arc
has been extended using a backtracking algorithm trying to obtain complete
arcs of the desired size. In the backtracking phase, the information obtained
during the classification of the arcs has been further exploited to prune the
search tree. In fact the points that would have given arcs equivalent to already
obtained ones have been excluded from the backtracking steps. The algorithm
is described in detail in [13].

Each 8−arc can be extended in independent way. To distribute the load
of computation, a certain number of 8-arcs has been assigned to each of the 4
processors to be extended with the backtracking algorithm.

As the backtracking algorithm exploits the information obtained during the
classification phase, the extension time of the 8−arcs is not equal. The 8−arcs
are extended following a certain order; when we extend an 8−arc we can avoid
to consider some possibilities because we know that we should obtain solutions
equivalent to solutions obtained extending 8−arcs already considered. It means
that the extension time of the first 8−arcs is much longer of the extension time
of the following 8−arcs. Therefore, to balance the computational load among
the 4 processors, we divided the number of 8−arcs to extend according to the
following proportions: 10%, 20%, 30%, 40%.

When studying the value of $t_2(2,31)$, during the classification, up to
$PGL(3,31)$, of the arcs of $PG(2,31)$ of size less than or equal to 8, we have
found 11 non-equivalent arcs of size five, 905 non-equivalent arcs of size six,
66,272 non-equivalent arcs of size seven and 3,768,298 non-equivalent arcs of
size eight. Each 8-arc has been extended trying to obtain complete arcs of size
less than or equal to 13. No examples have been found, so $t_2(2,31) = 14$.

When studying the value of $t_2(2,32)$, during the classification, up to
$PTL(3,32)$, of the arcs of $PG(2,32)$ of size less than or equal to 8, we have
found 3 non-equivalent arcs of size five, 213 non-equivalent arcs of size six,
16,593 non-equivalent arcs of size seven and 1,031,750 non-equivalent arcs of
size eight.
Each 8-arc has been extended trying to obtain complete arcs of size less than or equal to 13. No examples have been found, so $t_2(2,32) = 14$.

The search for the 13-arc in $PG(2,31)$ lasted about 197 days of total CPU time, while the search for the 13-arc in $PG(2,32)$ lasted about 100 days of total CPU time. The reason because the search in the bigger plane has been quicker is that $PGL(3,32)$ is much bigger than $PGL(3,31)$, so the consideration about isomorphism properties have reduced the search space in $PG(2,32)$, as we can see by the reduced number of classes to extend.

3 Examples of the smallest complete arcs in $PG(2,31)$ and in $PG(2,32)$

After having investigated the values of $t_2(2,31)$ and $t_2(2,32)$, we performed a partial search for examples of 14-arcs using the same algorithm.

We stopped the search for 14-arcs in $PG(2,31)$ after having found 500,000 examples. We performed a partial classification of them using MAGMA, a system for symbolic computation developed at the University of Sydney.

We obtained 3286 non-equivalent examples with trivial stabilizer group, 97 non-equivalent examples with stabilizer group of order two, 3 non-equivalent examples with stabilizer group isomorphic to $Z_4$, 4 non-equivalent examples with stabilizer group isomorphic to $Z_2 \times Z_2$ and one example with stabilizer group isomorphic to $S_3$.

We have stopped the search for 14-arcs in $PG(2,32)$ after having found 20,000 examples.

After a partial classification using MAGMA, we obtained 8759 non-equivalent examples with trivial stabilizer group and 541 non-equivalent examples with stabilizer group of order two, one example with stabilizer group isomorphic to $Z_4$ and one example with stabilizer group isomorphic to $Z_5$.

The field $GF(32)$ has been constructed using the primitive polynomial $\xi^3 + 2\xi^2 + 1$. Let $R = \{(0,0,1),(0,1,0),(1,0,0),(1,1,1)\}$.

The 14-arc in $PG(2,31)$ with stabilizer group $S_3$ is:

$$K_1 = R \cup \{(1,3,10),(1,5,11),(1,9,29),(1,12,19),(1,13,6),(1,14,3),\ (1,16,9),(1,20,26)(1,21,15),(1,22,16)\}.$$ 

The 14-arc in $PG(2,32)$ with stabilizer group $Z_4$ is:

$$K_2 = R \cup \{(1,\xi^{17},\xi^2),(1,\xi^{27},\xi^4),(1,\xi^{26},\xi^{13}),(1,\xi^2,\xi),(1,\xi,\xi^{28}),(1,\xi^{18},\xi^{19}), (1,\xi_{15},\xi^9),(1,\xi^{20},\xi^{10}),(1,\xi^{22},\xi^{28}),(1,\xi^{10},\xi^{12})\}.$$ 

The 14-arc in $PG(2,32)$ with stabilizer group $Z_5$ is:

$$K_3 = R \cup \{(1,\xi^{24},\xi),(1,\xi^7,\xi^{14}),(1,\xi,\xi^{28}),(1,\xi^8,\xi^{18}),(1,\xi^{28},\xi^{10}),(1,\xi^{22},\xi^{7}), (1,\xi^2,\xi^{25}),(1,\xi^{23},\xi^{29}),(1,\xi^{30},\xi^{13}),(1,\xi^{25},\xi^{23})\}.$$
References

[1] A.H. Ali, Classification of arcs in Galois plane of order thirteen, Ph.D. thesis, University of Sussex, 1993.

[2] S. Ball, On sets of points in finite planes, Ph.D. thesis, University of Sussex, 1994.

[3] A. Blokhuis, Note on the size of a blocking set in $PG(2,p)$, Combinatorica 14 (1994), 111-114.

[4] K. Coolsaet and H. Sticker, A full classification of the complete k-arcs of $PG(2,27)$ and $PG(2,29)$, preprint 2009, [http://caagt.ugent.be/preprints/arcs2729-main.pdf](http://caagt.ugent.be/preprints/arcs2729-main.pdf).

[5] A.A. Davydov, G. Faina, S. Marcugini and F. Pambianco, On sizes of complete caps in projective spaces $PG(n,q)$ and arcs in planes $PG(2,q)$, Journal of Geometry 94 (2009), 31-58.

[6] G. Faina, S. Marcugini, A. Milani and F. Pambianco, The spectrum of the values $k$ for which there exists a complete $k$-arc in $PG(2,q)$ for $q \leq 23$, Ars Combinatoria 47 (1997), 3-11.

[7] C.E. Gordon, Orbits of arcs in $PG(N,K)$ under projectivities, Geom. Dedicata 42 (1992), 187-203.

[8] J.W.P. Hirschfeld, Projective geometries over finite fields, Clarendon Press, Oxford, 1998.

[9] J.W.P. Hirschfeld and A. Sadeh, The projective plane over the field of eleven elements, Mitt. Math. Sem. Giessen 164 (1984) 245-257.

[10] J.W.P. Hirschfeld and L. Storme, The packing problem in statistics, coding theory and finite geometry: update 2001, in Finite Geometries, Developments of Mathematics, (Proc. of the Fourth Isle of Thorns Conf., Chelwood Gate, July 16-21, 2000), pp. 201-246, Eds. A. Blokhuis, J.W.P. Hirschfeld, D. Jungnickel and J.A. Thas, Kluwer, 2001.

[11] P. Lisonek, Computer-assisted studies in algebraic combinatorics, Ph.D. thesis, Research Institute for Symbolic Computation, J. Kepler Univ. Linz, 1994, RISC-Linz Report Series No. 94-68.

[12] S. Marcugini, A. Milani and F. Pambianco, A computer search for orbits of complete arcs in $PG(2,q)$, $q = 8, 13, 16, 23$, Rapporto tecnico n. 5/98, Università degli Studi di Perugia (1998).
[13] S. Marcugini, A. Milani and F. Pambianco, Complete arcs in $\text{PG}(2,25)$: the spectrum of the sizes and the classification of the smallest complete arcs, *Discrete Math.* 307 (2007), 739-747.

[14] S. Marcugini, A. Milani and F. Pambianco, Minimal complete arcs in $\text{PG}(2,q)$, $q \geq 29$, *Journal of Combinatorial Mathematics and Combinatorial Computing* 47 (2003) 19-29.

[15] T. Penttila, G.F. Royle, Private Communication (1995).

[16] O. Polverino, Small blocking sets and complete $k$–arcs in $\text{PG}(2,p^3)$, *Discrete Math.* 208/209 (1999), 469-476.

[17] G.F. Royle, An orderly algorithm and some applications to finite geometry (1996), preprint.

[18] B. Segre, *Le geometrie di Galois, Ann. Mat. Pura Appl.* 48 (1959), 1-97.

[19] T. Szönyi, Arcs, caps, codes and 3-indipendent subsets, in *Giornate di Geometrie Combinatorie* (Proc. International Conference Univ. Perugia 1992) pp. 57-80, eds. G. Faina and G. Tallini, Università degli studi di Perugia, 1993.

[20] G. Tallini, Le geometrie di Galois e le loro applicazioni alla statistica e alla teoria delle informazioni, *Rend. Mat. e Appl.* 19 (1960), 379-400.

[21] J.A. Thas, M.D.S. codes and arcs in projective spaces. A survey, *Le Matematiche* 47 (1992), 315-328.