Non-Markovian quantum jumps from measurements in bipartite Markovian dynamics

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The quantum jump approach allows to characterize the stochastic dynamics associated to an open quantum system submitted to a continuous measurement action. In this paper we show that this formalism can consistently be extended to non-Markovian system dynamics. The results rely in studying a measurement process performed on a bipartite arrangement characterized by a Markovian Lindblad evolution. Both a renewal and non-renewal extensions are found. The general structure of non-local master equations that admit an unravelling in terms of the corresponding non-Markovian trajectories are also found. Studying a two-level system dynamics, it is demonstrated that non-Markovian effects such as an environment-to-system flow of information may be present in the ensemble dynamics.

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I. INTRODUCTION

One of the central achievement of the theory of open quantum systems is the possibility of assigning to a given master equation an ensemble of stochastic realizations. They can be put in one-to-one correspondence with a well defined continuous-in-time measurement process performed over the system of interest. When the measurement apparatus is sensible to (detect) transitions between the system’s levels \[1\,5\], the realizations consist in a sequence of disruptive instantaneous changes, associated to the measurement recording events, while in the intermediate time regime the ensemble dynamics is smooth, being defined by a non-unitary dynamics. These basic ingredients, which define the quantum jump approach (QJA) \[6\,8\], are well understood for Markovian dynamics, that is, those where the evolution of the system density matrix is local in time.

In the last ten years, an ever increasing interest has been paid to the development of a consistent non-Markovian generalization of the standard (Markovian) open quantum system theory \[8\]. In the generalized scheme the system density matrix evolution is characterized by (time-convoluted) memory contributions \[9\,21\]. Both a theoretical interest as well as a wide range of physical applications motivates this line of research.

Relevant achievements in the study of non-Markovian master equations were formulated on the basis of stochastic phenomenological approaches \[10\,13\] and related concepts \[14\,21\]. On the other hand, much less progress was achieved in the formulation of stochastic processes that can be read as the result of a continuous measurement action performed over a system characterized by a non-local in time (non-Markovian) evolution. In fact, while there exist different stochastic dynamics that in average recover a non-Markovian density matrix evolution, its reading in terms of a continuous measurement process is problematic. Remarkable examples are the non-Markovian quantum state diffusion model \[22\] and the unravelling of local in time master equations characterized by negative transition rates \[23\]. The realizations associated to these approaches can only be read in the context of hidden-like variables models \[22\,23\].

The main goal of this paper is to demonstrate that it is possible to formulate a consistent generalization of the QJA such that in average the ensemble of measurement realizations recover a non-local non-Markovian density matrix evolution. The basic idea of our analysis is to study the QJA in a bipartite Markovian arrange. Then, we search for the conditions (interaction symmetries) that allows to formulate a closed stochastic dynamics for the system of interest. The coupling with the second or auxiliary system introduce the memory effects. In contrast with previous approaches \[22\,23\], the reading of the stochastic realizations in terms of a continuous-in-time measurement process is guaranteed by construction.

We show that a renewal non-Markovian measurement process can be obtained from the bipartite dynamics. Renewal means that the interval statistics between successive events is always the same being defined by a probability distribution called waiting time distribution \[2\]. A non-renewal dynamics is also defined. As in the standard Markovian formalism, the arising of each case depends on the properties of the resetting state \[5\] associated to each measurement event. The structure of the corresponding non-Markovian master equations are also found.

We remark that there exist previous studies where the QJA is formulated for a system that interacts with extra unobserved “classical” degrees of freedom \[24\,26\]. While our approach relies on a similar underlying dynamic (strictly, here not any classicality condition is imposed), we demonstrate that over a similar basis it is possible to get a consistent non-Markovian generalization of the QJA. In fact, in contrast with previous contributions \[24\,26\], we focus the analysis on the possibility of establishing a closed stochastic system dynamics, that is,
without involving “explicitly” the degrees of freedom of the auxiliary system.

The paper is outlined as follows. In Sec. II, in order to introduce the notation as well as basic results over which our analysis rely, we provide a resume of the standard Markovian QJA. In Sec. III we demonstrate that the basic structure of the standard QJA can be embedded in a bipartite Markovian dynamics, providing in this way the theoretical background for its non-Markovian generalization. Possible (bipartite) interactions that lead to a closed system dynamics are found. The non-Markovian density matrix evolution is determined for both renewal and non-renewal measurement processes. In Sec. IV we study a particular example that explicitly shows the consistence of the present proposal. Furthermore, it demonstrate that non-Markovian features such as an environment-to-system flow of information may be present in the ensemble dynamics. The conclusions are presented in Sec. V. In Appendix A we provide a derivation of the statistics of the measurement events in the standard case. In Appendix B we work out an alternative derivation of the non-Markovian system density matrix evolution based on the measurement statistics.

II. MARKOVIAN QUANTUM JUMPS

The standard QJA allows to define the (stochastic) dynamics of an open quantum system when it is subjected to a measurement process. The basic ingredients of the formalism are the system density matrix evolution, the definition of the apparatus measurement action, the conditional dynamic between detections events and their statistical characterization. Below, we review these elements.

We write the evolution of the system density matrix $\rho_t$ as

$$\frac{d}{dt}\rho_t = (\hat{L}_0 + \sum_{\alpha} \gamma_\alpha \hat{C}[V_\alpha]) \rho_t,$$  \hspace{1cm} (1)

where $\hat{L}_0$ is an arbitrary superoperator that may include Hamiltonian as well as dissipative (Lindblad) superoperators. The second contribution in (1) is defined by an addition of Lindblad channels

$$\hat{C}[V] \rho = V \rho V^\dagger - \frac{1}{2} \{ V^\dagger V, \rho \}_+,$$  \hspace{1cm} (2)

each one characterized by the operator $V_\alpha$ and the transition rate $\gamma_\alpha$. With $\{.,.\}_+$ we denotes an anticommutation operation.

We assume that the system is monitored by only one measurement apparatus, which is sensible to all Lindblad transitions channels $\hat{C}[V_\alpha]$. Hence, the master equation (1) is rewritten as

$$\frac{d}{dt}\rho_t = (\hat{\mathcal{D}} + \hat{J}) \rho_t.$$  \hspace{1cm} (3)

The superoperator $\hat{J}$ reads

$$\hat{J} \rho = \sum_{\alpha} \gamma_\alpha V_\alpha \rho V_\alpha^\dagger.$$  \hspace{1cm} (4)

It defines the system transformation after a measurement event. In fact, when a recording event happens, consistently with a quantum measurement theory, the system density matrix suffer the disruptive transformation $\rho \rightarrow \hat{M} \rho$ (jump or state collapse),

$$\hat{M} \rho = \frac{\hat{J} \rho}{Tr_s[\hat{J} \rho]} = \frac{\sum_{\alpha} \gamma_\alpha V_\alpha \rho V_\alpha^\dagger}{\left\{ \sum_{\alpha} \gamma_\alpha Tr_s[V_\alpha^\dagger V_\alpha \rho] \right\}},$$  \hspace{1cm} (5)

where $Tr_s[\cdot]$ denotes a trace operation. On the other hand, in Eq. (3) the superoperator $\hat{D}$ is defined as

$$\hat{D} \rho = \hat{L}_0 \rho - \frac{1}{2} \sum_{\alpha} \gamma_\alpha \{ V_\alpha^\dagger V_\alpha, \rho \}_+.$$  \hspace{1cm} (6)

In the QJA, this superoperator defines the system dynamics between detection events. In fact, given that in the interval $(\tau, t)$ not any detection event happens, the system dynamic is defined by the (conditional) normalized propagator

$$\hat{T}_c(t - \tau) \rho = \frac{\hat{T}(t - \tau) \rho}{Tr_s[\hat{T}(t - \tau)]},$$  \hspace{1cm} (7)

The superoperator $\hat{D}$ generates the dynamics of the unnormalized propagator $\hat{T}(t - \tau)$, which reads

$$\hat{T}(t - \tau) \rho = \exp[(t - \tau) \hat{D}] \rho.$$  \hspace{1cm} (8)

In this way, the trajectories associated to the measurement process are a piecewise deterministic process which combine a deterministic time-evolution [Eq. (7)] with jump process [Eq. (5)].

The propagator $\hat{T}(t)$ completely define the statistics of the measurement process. In fact, it allows to calculate the survival probability between measurement events. Given that at time $\tau$ the state of the system is $\rho_\tau$, the probability $P_0(t - \tau | \rho_\tau)$ of not happening any detection event in the interval $(\tau, t)$ is

$$P_0(t - \tau | \rho_\tau) = Tr_s[\hat{T}(t - \tau) \rho_\tau].$$  \hspace{1cm} (9)

The probability distribution $w(t - \tau | \rho_\tau)$ of the interval $(t - \tau)$ follows as $w(t - \tau | \rho_\tau) = -(d/dt) P_0(t - \tau | \rho_\tau)$, delivering

$$w(t - \tau | \rho_\tau) = -Tr_s[\hat{D} \hat{T}(t - \tau) \rho_\tau].$$  \hspace{1cm} (10)

By using that $(d/dt) Tr_s[\rho_t] = 0$, Eq. (8) implies that $-Tr_s[\hat{D} \rho_t] = Tr_s[\hat{J} \rho_t]$, leading to the equivalent expression $w(t - \tau | \rho_\tau) = Tr_s[\hat{J} \hat{T}(t - \tau) \rho_\tau]$. On the other hand, notice that $P_0(t - \tau | \rho_\tau)$, or equivalently $w(t - \tau | \rho_\tau)$, depends explicitly on the state $\rho_\tau$. 

From the previous statistical objects it is possible to define the “conditional distribution” \[3\]

\[ w_c(t - \tau | \rho_{\tau}) = \frac{w(t - \tau | \rho_{\tau})}{P_0(t - \tau | \rho_{\tau})}. \] (11)

It defines the probability density for recording a detection event at time \( t \), given no counts are recorded in the interval \((\tau, t)\), and that the last one was recorded at time \( \tau \). Therefore, \( w_c(t - \tau | \rho_{\tau}) \) gives the probability density for a jump at time \( t \) given that we know that no event occurred up to the present time since the last one. Trivially, from Eqs. (9) and (10) it can be written as

\[ w_c(t - \tau | \rho_{\tau}) = -\frac{\text{Tr}_s[\hat{D}\hat{T}(t - \tau)|\rho_{\tau}]}{\text{Tr}_s[\hat{T}(t - \tau)|\rho_{\tau}]} . \] (12)

## A. Stochastic dynamics

With the previous elements, it is possible to define the dynamics of a stochastic density matrix \( \rho_s^a(t) \) such that its average over realizations, denoted by an overbar, recovers the system state

\[ \rho_s^a = \bar{\rho}_s^a(t). \] (13)

Each realization corresponds to a given recording realization of the measurement apparatus. Its structure can be established by studying the counting statistics of the measurement process (see Appendix A).

Given the initial state \( \rho_0^a \), we can evaluate \( P_0(t - 0 | \rho_0^a) \). The time \( t_1 \) of the first detection event follows by solving the equation \( P_0(t_1 - 0 | \rho_0^a) = r \), where \( r \) is a random number in the interval \((0, 1)\). The dynamic of \( \rho_s^a(t) \) in the interval \((0, t_1)\) is defined by Eq. (7). At \( t = t_1 \) the disruptive transformation \( \rho_s^a(t_1) \rightarrow \hat{M}\rho_s^a(t_1) \) is applied. The subsequent dynamics is the same. In fact, after the \( n_u \)-measurement event at time \( t_n \), \( \rho_s^a(t_n) \rightarrow \hat{M}\rho_s^a(t_n) \), the time \( t_{n+1} \) for the next detection event follows from

\[ P_0(t_{n+1} - t_n | \hat{M}\rho_s^a(t_n)), \] (14)
equated to \( r \), where again \( r \) is a random number in the interval \((0, 1)\). The dynamic in the interval \((t_n, t_{n+1})\) is defined by the conditional propagator (7).

The previous algorithm determine the realizations over finite time intervals \[4\]. It is also possible to obtain the evolution over infinitesimal intervals. Its structure remains the same [Eqs. (5) and (7)]. Nevertheless, instead of Eq. (14), the jump statistic is determined from

\[ w_c(t - \tau | \rho_{\tau}) \text{, Eq. (11)} . \] Given that the last event happened at time \( \tau \) and that not any detection was detected in the interval \((\tau, t)\), the probability \( \Delta P \) of having a detection event in the infinitesimal interval \((t, t + dt)\) is (by definition) \[8\]

\[ \Delta P = w^c(t - \tau | \rho_s^a(\tau)) \, dt. \] (15)

From Eqs. (6), (7), and (12), we can write

\[ \Delta P = -dt\text{Tr}_s[\hat{D}\hat{T}(t - \tau)|\rho_{\tau}] = dt \sum_{\alpha} \gamma_{\alpha} \text{Tr}_s[V_{\alpha}^aV_{\alpha}^a(\tau)]. \] (16)

The happening or not of a detection follows by comparing \( \Delta P \) with a random number in the interval \((0, 1)\). This alternative algorithm generate the same realizations than the previous one \[4\]. Nevertheless, in this last scheme the Markovian property of the underlying master equation is self-evident in the expression of \( \Delta P \). In fact, \( \Delta P \) does not depends on the “history” of \( \rho_s^a(t) \) in the interval \((\tau, t)\)

## B. Renewal and non-renewal measurement processes

An extra understanding of the QJA is achieved by specifying the operators \( \{V_{\alpha}\} \) that determine the measurement transformation Eq. (5).

When the system state after a measurement event (resetting state) is always the same, the statistics of the time interval between events is defined by a unique probability distribution (waiting time distribution). In this case, the measurement process is a renewal one. This situation arises when the measurement apparatus is sensible to all transitions \( \{u \rightarrow r_{\alpha}\} \) between a given system state \( |u\rangle \) and a set of alternative states \( \{|r_{\alpha}\rangle\} \). Therefore, the operators \( \{V_{\alpha}\} \) have the structure

\[ V_{\alpha} = |r_{\alpha}\rangle \langle u|, \] (17)

which in turn, from Eq. (5), imply the measurement transformation

\[ \hat{M}\rho = \bar{\rho}_{s} \equiv \sum_{\alpha} p_{\alpha} |r_{\alpha}\rangle \langle r_{\alpha}|, \quad p_{\alpha} = \frac{\gamma_{\alpha}}{\sum_{\alpha} \gamma_{\alpha}}. \] (18)

Hence, the conditional dynamics [Eq. (7)] always start in the same resetting state \( \bar{\rho}_s \). Furthermore, the survival probability and waiting time distribution [Eqs. (9) and (10) respectively], after the first event \( |r_{\tau}\rangle \rightarrow \hat{M}\rho = \bar{\rho}_{s} \) are always the same, being defined as

\[ P_0(t) = \text{Tr}_s[\hat{F}(t)|\bar{\rho}_s], \quad w(t) = -\text{Tr}_s[\hat{D}\hat{T}(t)|\bar{\rho}_s]. \] (19)

In consequence, the interval statistics does not depends explicitly on the time \( \tau \) of the last events and it is always the same. The operators (18) arise for example in optical systems such as two-level fluorescent systems, where \( \bar{\rho}_s \) is a pure state, and three-level \( \Lambda \) configurations \[3\], \[4\].

In general, the operators may read

\[ V_{\alpha} = |r_{\alpha}\rangle \langle u_{\alpha}|, \] (20)

that is, the measurement apparatus is sensible to different transitions \( \{u_{\alpha}\} \rightarrow \{r_{\alpha}\} \). This case may happen when the natural frequencies of the different transitions are...
indistinguishable for the measurement apparatus; for example in cascade optical systems [5]. The measurement transformation

\[ \mathcal{M}\rho = \sum_\alpha \gamma_\alpha \langle u_\alpha | \rho | u_\alpha \rangle r_\alpha \langle r_\alpha | \rho | r_\alpha \rangle \left\{ \sum_\alpha \gamma_\alpha \langle u_\alpha | \rho | u_\alpha \rangle \right\}. \] (21)

delivers a state that depends on the pre-detection state. Hence, it is not possible to define a unique statistical object as in the previous case, that is, the survival probability and waiting time distribution correspond to the general expressions Eqs. (9) and (10) respectively.

### III. NON-MARKOVIAN QUANTUM JUMPS FROM BIPARTITE MARKOVIAN DYNAMICS

The previous elements and results that define the QJA, without introducing any new element, can also be established for bipartite dynamics. Here, in addition to the system of interest \( S \) we consider an auxiliary or ancilla system \( A \). Their joint dynamics is Markovian. Furthermore, we assume that the measurement apparatus is sensible to the same system transitions as before. Thus, we can define a stochastic density matrix \( \rho^a(t) \) such that its average over realizations recovers the bipartite density matrix \( \rho^a = \rho^a(t) \). The density matrix of \( S \) is recovered by a partial trace operation over the auxiliary system \( A \),

\[ \rho^s = \text{Tr}_a(\rho^a) = \text{Tr}_a(\rho^a(t)). \] (22)

Trivially, by introducing the stochastic matrix

\[ \rho^a(t) = \text{Tr}_a(\rho^a(t)), \] (23)

we recover Eq. (13), that is, \( \rho^s = \rho^s(t) \). At this point, we ask about the existence of different \( S - A \) interactions an evolutions under which it is possible to get a closed stochastic dynamics for \( \rho^a(t) \), that is, without involving explicitly the ancilla state. In addition to this constraint, here we search interaction structures that introduce a minimal modification of the standard approach, that is, it should be possible to define a measurement transformation [Eq. (5)], a conditional interevent dynamic [Eq. (7)], and a survival probability [Eq. (11)].

#### A. Bipartite Markovian embedding

Taking into account the evolution Eq. (11), we write the bipartite evolution as

\[ \frac{d}{dt} \rho_s^a = (\hat{L}_0 + \sum_{\alpha \lambda \mu} \gamma_{\alpha \lambda \mu} \hat{C}[V_{\alpha \lambda}]\rho_s^a). \] (24)

The operators \( V_{\alpha \lambda} \) are defined as

\[ V_{\alpha \lambda} = V_\alpha \otimes |a_\lambda \rangle \langle a_\lambda |. \] (25)

The set of states \( \{|a_\lambda \rangle\} \) provides an orthogonal and normalized basis of the ancilla Hilbert space. The system operators \( \{V_\alpha\} \) are the same as before. Notice that the diagonal contributions, defined by the operators \( V_{\alpha \alpha \alpha} = V_\alpha \otimes |a_\alpha \rangle \langle a_\alpha | \), correspond to system’s transitions that only happen when the ancilla system is in the state \( |a_\alpha \rangle \). The non-diagonal contributions \( V_{\alpha \lambda} = V_\alpha \otimes |a_\lambda \rangle \langle a_\alpha | \) correspond to system transitions that occur simultaneously with the ancilla transition \( |a_\alpha \rangle \leftrightarrow |a_\lambda \rangle \).

In Eq. (24), the superoperator \( \hat{M} \) not only includes the system evolution \( \{\hat{L}_0 \text{ in Eq. (11)}\} \) but also an arbitrary evolution for the ancilla system as well as the system-ancilla interaction. Over this last contribution, we only demand that it must not to include any interaction proportional to the transitions defined by the operators \( \{V_\alpha\} \). On the other hand, the measurement apparatus remains the same, that is, it only detects the system transitions. Therefore, we split the bipartite master equation (24) as

\[ \frac{d}{dt} \rho^a = (\hat{D} + \hat{J})\rho^a, \] (26)

where the superoperator \( \hat{J} \) reads

\[ \hat{J}\rho = \sum_{\alpha \lambda \mu} \gamma_{\alpha \lambda \mu} V_{\alpha \lambda} V_{\alpha \lambda}^\dagger. \] (27)

The measurement transformation [see Eq. (5)] in the bipartite Hilbert space becomes

\[ \hat{M}\rho = \frac{\hat{J}\rho}{\text{Tr}_{sa}[\hat{J}\rho]} = \left\{ \sum_{\alpha \lambda \mu} \gamma_{\alpha \lambda \mu} V_{\alpha \lambda} V_{\alpha \lambda}^\dagger \right\} \] (28)

The goal is to obtain a closed (stochastic) evolution for the system with almost the same elements than in the Markovian case. The free parameters are the rates \( \gamma_{\alpha \lambda \mu} \). In order to have the same measurement transformation than before [Eq. (5)], for arbitrary bipartite states \( \rho_{sa} \) one must to demand the condition

\[ \text{Tr}_a[\hat{M}\rho_{sa}] = \hat{M}[\rho_{sa}], \] (29)

where evidently \( \rho_s = \text{Tr}_a[\rho_{sa}] \). There exist different way of satisfying this condition. Here, for simplicity, we choose the constraint

\[ \hat{M}\rho_{sa} = \hat{M}[\rho_{sa}] \otimes \rho_a, \] (30)

where \( \rho_a \) is a particular ancilla density matrix. Notice that after a measurement event, the system and ancilla become uncorrelated. Trivially, this measurement transformation satisfy the previous condition Eq. (29).

The conditional system dynamics between collision events can be written as in Eq. (7), but now the unconditional propagator reads

\[ \hat{T}(t - \tau) = \text{Tr}_a[\exp(\hat{D}(t - \tau))]\rho_a]. \] (31)

It arises from the partial trace over the ancilla system of the bipartite conditional propagator \( \hat{T}(t - \tau) = \exp[\hat{D}(t - \tau)] \), and the condition (30). The superoperator \( \hat{D} \) is

\[ \hat{D}\rho = \hat{L}_0\rho - \frac{1}{2} \sum_{\alpha \lambda \mu} \gamma_{\alpha \lambda \mu} \left\{ V_{\alpha \lambda} V_{\alpha \lambda}^\dagger, \rho \right\}. \] (32)
As we have chosen the stronger separability condition \([30]\), the propagator defined by \(\mathcal{T}(t)\) [Eq. (31)] is not only completely positive but also its time evolution is given by an homogeneous equation. In fact, in a Laplace domain, \(f(z) \equiv \int_0^\infty dt e^{-zt} f(t)\), Eq. (31) becomes \(\mathcal{T}(z) = \text{Tr}_a[z^{-\hat{D}} \rho_a]\). This expression can be rewritten as \(\mathcal{T}(z) = \text{Tr}_a[(z - \hat{D})^{-1} \rho_a]^{-1} \times \{[\mathcal{T}(z)]^{-1}\}^{-1}\). Using in the curly brackets that \(M^{-1} \times N^{-1} = (N \times M)^{-1}\), where \(M\) and \(N\) are arbitrary matrices, it follows that \(\mathcal{T}(z) = (\mathcal{T}(z)^{-1} \rho_a - \text{Tr}_a[(z - \hat{D})^{-1} \rho_a])^{-1}\), which in turn leads to the expression \(\mathcal{T}(z) = [z - \hat{D}(z)]^{-1}\), where the system superoperator \(\hat{D}(z)\) is

\[
\hat{D}(z) = \left\{ \text{Tr}_a \left[ \frac{1}{z - \hat{D}} \rho_a \right] \right\}^{-1} \text{Tr}_a \left[ \frac{1}{z - \hat{D}} \rho_a \right]. \tag{33}
\]

Hence, in the time domain we get

\[
\frac{d}{dt} \mathcal{T}(t) = \int_0^t dt' \mathcal{T}(t - t') \mathcal{T}(t'), \tag{34}
\]

where the memory superoperator \(\hat{D}(t)\) is defined by its Laplace transform [33]. We notice that in the Markovian case \(\mathcal{T}(t) = \exp[t\hat{D}]\) [see Eq. (33)] implying the local in time evolution \((d/dt)\mathcal{T}(t) = \hat{D}\mathcal{T}(t)\). Thus, in the present approach the conditional evolution between measurements events becomes non-local in time. This property also implies that in general, even for pure initial conditions \(|\Psi\rangle\), the conditional evolution cannot be decomposed in pure states [\(6, 7\)], that is,

\[
\mathcal{T}(t) (|\Psi\rangle \langle \Psi|) \neq |\Psi(t)\rangle \langle \Psi(t)|. \tag{35}
\]

Under the assumption Eq. (30), the previous analysis demonstrate that it is possible to obtain a closed evolution for the system dynamics. It remains to determine the statistics of the measurement events. As the bipartite dynamics is Markovian, here we also have a well defined survival probability [see Eq. (30)]. By using Eq. (30), it is possible to write

\[
P_0(t - \tau|\rho_\tau) = \text{Tr}_a[\exp[(t - \tau)\hat{D}] \rho_\tau \otimes \rho_a], \tag{36a}
\]

\[
= \text{Tr}_a[\mathcal{T}(t - \tau) \rho_\tau], \tag{36b}
\]

where \(\mathcal{T}(t)\) is given by Eq. (31). Notice that \(\rho_\tau\) is a system state. Furthermore, this expression has the same structure than Eq. (39). The definition of the conditional propagator \(\mathcal{T}(t)\) is the unique difference. The corresponding waiting time distribution [Eq. (10)] here reads

\[
w(t - \tau|\rho_\tau) = -\text{Tr}_a[\exp[(t - \tau)\hat{D}] \rho_\tau \otimes \rho_a]. \tag{37}
\]

From Eq. (37), it follows the equivalent expression

\[
w(t - \tau|\rho_\tau) = -\int_0^{t - \tau} dt' \text{Tr}_a[\hat{D}(t - \tau - t') \mathcal{T}(t') \rho_\tau], \tag{38}
\]

which leads to a natural non-Markovian generalization of Eq. (10). On the other hand, the conditional waiting time distribution, Eq. (11), here becomes

\[
w_c(t - \tau|\rho_\tau) = -\int_0^{t - \tau} dt' \text{Tr}_a[\hat{D}(t - \tau - t') \mathcal{T}(t') \rho_\tau] / \text{Tr}_a[\mathcal{T}(t - \tau) \rho_\tau]. \tag{39}
\]

B. Stochastic dynamics

As in the Markovian case, the previous objects [Eqs. (30), (31), and (33)] completely define the system realizations associated to the measurement process. Therefore, the algorithm associated to Eq. (14) remains exactly the same. The unique modification is the definition of the propagator \(\mathcal{T}(t)\), which in turn modify the conditional dynamics as well as the measurement events statistics.

On the other hand, the infinitesimal time step algorithm defined by Eq. (15) can also be applied. Nevertheless, in contrast to Eq. (10), here it is not possible to write a simple expression for \(\Delta P\) neither in terms of \(\rho_\alpha(t)\) or its history [see Eq. (39)]. Therefore, in this generalized non-Markovian approach the infinitesimal algorithm, while can be formally implemented, it does not provide an efficient numerical simulation method neither it has a simple physical interpretation.

C. Symmetries of the bipartite dynamics

It remains to demonstrate that in fact there exist different bipartite Lindblad equations that allow to fulfill the condition \([30]\), where the bipartite measurement transformation is given by Eq. (25). From Eq. (25), it can be written as

\[
\hat{M}\rho = \sum_{\alpha \beta \gamma \delta} \gamma_{\alpha \beta} \delta_{\gamma \delta} \rho_\alpha V_\beta \langle a_\delta | a_\beta \rangle V_\gamma^\dagger \langle a_\gamma | a_\alpha \rangle V_\delta^\dagger. \tag{40}
\]

The result of calculating \(\text{Tr}_a[\hat{M}\rho]\) can only be written in terms of \(\mathcal{M}\) [Eq. (5)] if \(\gamma_{\alpha \beta} = \gamma_{\alpha \beta} d_{\beta m}\), where \(d_{\beta m}\) is an arbitrary dimensionless coefficient. Eq. (40) becomes

\[
\hat{M}\rho = \sum_{\alpha \beta \gamma \delta} \gamma_{\alpha \beta} d_{\beta m} \langle a_\delta | a_\beta \rangle V_\gamma^\dagger \langle a_\gamma | a_\alpha \rangle V_\delta^\dagger. \tag{41}
\]

where \(\rho_\alpha^\dagger \equiv \sum_\gamma (\gamma_{\alpha \beta} / \gamma_{\alpha \delta}) \langle a_\gamma | a_\beta \rangle \langle a_\gamma | a_\alpha \rangle\), and \(\gamma_{\alpha \beta} \equiv \sum_\gamma \gamma_{\alpha \delta}\). With the operators definitions \([17, 20]\), Eq. (41) can satisfy the weaker condition \([29]\). Nevertheless, the resulting bipartite state is a classical correlated one (with vanishing discord). For satisfying the separability condition \([30]\), which leads to the homogeneous dynamics \([44]\), the states \(\rho_\alpha^\dagger\) must not to depend on index \(\alpha\). Hence, we demand \(\gamma_{\alpha \beta} = \gamma_{\alpha \beta} c_\beta\), where \(c_\beta\) is an also an arbitrary dimensionless coefficient. The rates \(\gamma_{\alpha \beta}\) become

\[
\gamma_{\alpha \beta} = \gamma_{\alpha \beta} c_\beta d_{\beta m}, \quad \sum_\beta c_\beta = 1, \tag{42}
\]
which from Eq. (40) leads to

\[ \hat{M}\rho = \hat{M}[\sum m d_m \langle a_m | \rho | a_m \rangle] \otimes \bar{\rho}_a. \]  

(43)

The ancilla resetting state \( \bar{\rho}_a \) is

\[ \bar{\rho}_a = \sum_t c_t |a_t\rangle \langle a_t|. \]  

(44)

For simplicity, we assumed \( \sum_t c_t = 1 \). If this condition is not met, it can always be satisfied by a renormalization of the measurement rates, \( \gamma_\alpha \rightarrow \gamma_\alpha / \sum_t c_t \).

The expression (42) can be read as a symmetry condition on the bipartite Lindblad evolution Eq. (24). It leads to Eq. (43), which does not recover explicitly Eq. (40). The fulfillment of this constraint can be achieved by choosing different set of values for the coefficients \( d_m \), which depend on the specific structure of \( \hat{M} \).

### 1. Renewal case

When the measurement transformation \( \hat{M} \) leads to a renewal process, Eqs. (17) and (18), independently of the coefficients \( d_m \) it follows \( \hat{M}[\sum m d_m \langle a_m | \rho | a_m \rangle] = \bar{\rho}_s \). Therefore, Eq. (43) leads to

\[ \hat{M}\rho = \bar{\rho}_s \otimes \bar{\rho}_a. \]  

(45)

Evidently this expression satisfies the condition (30). Furthermore, it says us that the stochastic dynamics developing in the bipartite \( S - A \) Hilbert space is also a renewal measurement process.

Similarly to the Markovian case, after the first detection event the statistic of the time interval between consecutive events is defined by a unique survival probability

\[ P_0(t) = \text{Tr}_s[\hat{T}(t)\bar{\rho}_s], \]  

(46)

or equivalently a unique waiting time distribution

\[ w(t) = -\int_0^t dt' \text{Tr}_s[\hat{D}(t-t')\hat{T}(t')\bar{\rho}_s]. \]  

(47)

These expressions follows from Eqs. (30) and (38) after introducing the resetting property defined by Eq. (44). They generalize the Markovian expressions (19).

### 2. Non-renewal case

When the measurement transformation \( \hat{M} \) does not lead to a renewal process [Eqs. (20) and (21)], the coefficients \( d_m \) can not be arbitrary. In fact, the only way of satisfying the condition (30) is by choosing \( d_m = 1 \) (after a rates renormalization we can also take \( d_m \) equal to an arbitrary real constant). As the states \( \{|a_m\}\) are a complete basis of the ancilla Hilbert space, for any bipartite state \( \rho_{sa} \) it follows \( \sum_m \langle a_m | \rho_{sa} | a_m \rangle = \text{Tr}_a[\rho_{sa}] = \rho_s \).

Thus, Eq. (43) recovers Eq. (39),

\[ \hat{M}\rho = \hat{M}[\rho_s] \otimes \bar{\rho}_a. \]  

(48)

Notice that this result is valid for both the non-renewal and renewal cases. Nevertheless, the condition \( d_m = 1 \) is only "necessary" in the former case. The symmetry condition on the rates \( \gamma_\alpha \) [Eq. (42)] then reads

\[ \gamma_\alpha t = \gamma_\alpha c_t, \quad \sum_t c_t = 1. \]  

(49)

In contrast to the renewal case, here the measurement statistics remains defined by the general expression Eqs. (30) and (38).

### D. Density matrix evolution

Under the symmetry conditions defined by Eqs. (42) and (49) the stochastic dynamics of \( \rho_{sa}^s(t) \) [Eq. (23)] has the same structure than in the Markovian case. For both renewal and non-renewal measurement processes, the main difference with the Markovian case is the conditional dynamics. It remains to calculate the time evolution of the system density matrix \( \rho_{sa}^s \), Eq. (22). In Appendix B we perform this calculus by averaging the realizations of \( \rho_{sa}^s(t) \), that is, from \( \rho_{sa}^s = \bar{\rho}_{sa}^s(t) \). Here, using an alternative procedure, the evolution of the system state is obtained from the bipartite dynamics (24) by using that \( \rho_{sa}^s = \text{Tr}_a[\rho_{sa}^s] \).

For simplicity, we take a separable bipartite initial condition

\[ \rho_{sa}^0 = \rho_0^a \otimes \bar{\rho}_a, \]  

(50)

where \( \rho_0^a \) is an arbitrary system state and \( \bar{\rho}_a \) is the ancilla resetting state defined by Eq. (44). The bipartite Lindblad evolution (24) can formally be integrated as

\[ \rho_{sa}^s = \exp[\hat{D}t]\rho_{sa}^0 + \int_0^t dt' \exp[\hat{D}(t-t')][\hat{M}\rho_{sa}^s]. \]  

(51)

The superoperators \( \hat{J} \) and \( \hat{D} \) were defined in Eqs. (27) and (32) respectively. By using the rates condition Eq. (45) and the operator definition (25), we get

\[ \hat{J}[\rho_{sa}^s] = \sum_\alpha \gamma_\alpha V_{\alpha}O[\rho_{sa}^s]\langle V_\alpha^\dagger \rangle \otimes \bar{\rho}_a. \]  

(52)

For shortening the notation we defined the superoperator

\[ O[\rho_{sa}^s] \equiv \sum_m d_m \langle a_m | \rho_{sa}^s | a_m \rangle. \]  

(53)

Taking the partial trace over the ancilla degrees of freedom, Eq. (51) leads to

\[ \rho_{sa}^s = \hat{T}(t)\rho_{sa}^0 + \int_0^t dt' \hat{T}(t-t') \sum_\alpha \gamma_\alpha V_{\alpha}O[\rho_{sa}^s]\langle V_\alpha^\dagger \rangle, \]  

(54)
which in turn, from Eq. (34), allows us to write
\[
\frac{d\rho_{\alpha}^a}{dt} = \int_0^t dt' \mathcal{D}(t-t')\rho_{\alpha}^a + \sum_{\alpha} \gamma_\alpha V_\alpha O[\rho^a_{\alpha}] V_\alpha^\dagger. \tag{55}
\]
If all \( d_m = 1 \), it follows \( O[\rho^a_{\alpha}] = \rho_{\alpha}^a \). Hence, from Eq. (55) we get the closed density matrix evolution
\[
\frac{d\rho_{\alpha}^a}{dt} = \int_0^t dt' \mathcal{D}(t-t')\rho_{\alpha}^a + \sum_{\alpha} \gamma_\alpha V_\alpha \rho_{\alpha}^a V_\alpha^\dagger. \tag{56}
\]
Notice that this evolution contains both convoluted as well as local in time contributions. It is valid for both, renewal and non-renewal measurement processes. On the other hand, in the case of renewal processes the coefficients \( d_m \) may be arbitrary and the previous expression does not apply. By using the specific form of the operators \( V_\alpha \) [Eq. (17)] it follows \( \sum_{\alpha} \gamma_\alpha V_\alpha O[\rho^a_{\alpha}] V_\alpha^\dagger = \tilde{\rho}_s \gamma \langle u | O[\rho^a_{\alpha}] | u \rangle \), where \( \gamma = \sum_{\alpha} \gamma_\alpha \) and the system resetting state \( \tilde{\rho}_s \) is defined by Eq. (18). By using in Eq. (55) that \( \langle d/dt \rangle Tr_s[\rho_{\alpha}^a] = 0 \), it follows \( \gamma \langle u | O[\rho^a_{\alpha}] | u \rangle = -\int_0^t dt' Tr_s[\mathcal{D}(t-t')\rho_{\alpha}^a] \), implying the closed density matrix evolution
\[
\frac{d\rho_{\alpha}^a}{dt} = \int_0^t dt' \mathcal{D}(t-t')\rho_{\alpha}^a - \tilde{\rho}_s \gamma \sum_{\alpha} \gamma_\alpha V_\alpha \rho_{\alpha}^a V_\alpha^\dagger. \tag{57}
\]
In the present approach, this expression correspond to the more general master equation consistent with a renewal measurement process. Notice that Eq. (57) is a particular case of this more general expression. By comparing both equations, we realize that it applies when \( \gamma \langle u | \rho_{\alpha}^a | u \rangle = -\int_0^t dt' Tr_s[\mathcal{D}(t-t')\rho_{\alpha}^a] \).

### E. Arbitrary master equations

Eqs. (56) and (57) are one of the central results of this section. They correspond to master equations that admit an unravelling in terms of an ensemble of trajectories associated to a continuous measurement action defined by the set of operators \( \{ V_\alpha \} \). Eq. (56) is valid for both renewal and non-renewal measurement processes [see Eqs. (14) and (20) respectively] while Eq. (57) is only valid for renewal processes [Eq. (17)]. Now we ask about which conditions an arbitrary non-Markovian master equation must to satisfy to admit the non-Markovian unravelling defined previously.

One condition is the possibility of rewriting the master equation with the structure defined by Eqs. (55) or (57). On the other hand, the ensemble representation can only be assigned if the memory superoperator \( \mathcal{D}(t) \) through the relation \( \langle d/dt \rangle \mathcal{T}(t) = \int_0^t dt' \mathcal{D}(t-t') \mathcal{T}(t') \) [Eq. (43)] defines a well behaved survival probability \( P_0(t-\tau | \rho) = Tr_s[\mathcal{T}(t-\tau | \rho)] \) [Eq. (30)] for “arbitrary” system states \( \rho \). A well behaved survival probability means that it is a decaying function, that is, for arbitrary times \( \tau < t_1 < t_2 \), it must to satisfy \( P_0(t_2 - \tau | \rho) \leq P_0(t_1 - \tau | \rho) \), implying
\[
Tr_s[\mathcal{T}(t_2 | \rho)] \leq Tr_s[\mathcal{T}(t_1 | \rho)], \quad t_1 < t_2. \tag{58}
\]

Taking into account that the realizations can be determine from \( P_0(t | \rho) \), the fulfillment of the previous inequality guarantees the possibility of assigning a non-Markovian unraveling to a master equation with the structure (55) or (57).

### IV. NON-MARKOVIAN RENEWAL TWO-LEVEL TRANSITIONS

Here, we work out an example that explicitly shows the consistency of the previous results. Both the system of interest and the ancilla are two-level systems. Their states are denoted \( |\pm\rangle \), and \( \{|1\rangle, |2\rangle\} \) respectively. The Markovian dynamic of the bipartite state \( \rho^a \) [Eq. (24)] here reads
\[
\frac{d}{dt} \rho^a = -i \frac{\hbar}{\gamma} \{ H_0 \rho^a + (\gamma C[\sigma_{11}] + \gamma' C[\sigma_{21}]) \rho^a. \tag{59}
\]
The bipartite Hamiltonian contribution is defined by the operator
\[
H_0 = \hbar \sigma_x \otimes \sigma_x, \tag{60}
\]
where \( \sigma_x \) is the \( x \)-Pauli matrix in the basis of each Hilbert space. The remaining Lindblad contributions [Eq. (2)] with rates \( \gamma \) and \( \gamma' \) are defined by the operators
\[
\sigma_{11} = \sigma \otimes |1\rangle \langle 1|, \quad \sigma_{12} = \sigma \otimes |1\rangle \langle 2|. \tag{61}
\]
The lowering system operator is defined as \( \sigma = |\rangle \langle +| \).

Notice that \( \sigma_{11} \) leads to system transitions between the upper and lower states \( |\rangle \sim |\rangle \) that can only happen when the ancilla is in the state \( |1\rangle \). In addition, \( \sigma_{12} \) leads to the same system transitions but in this case they simultaneously occur with the ancilla transition \( |2\rangle \sim |1\rangle \). Thus, the dissipative dynamic drives the system to its ground states. On the other hand, the unitary evolution can excite the system to its upper state. In consequence, the interplay between both contributions leads to successive system transitions \( |\rangle \sim |\rangle \). Each transition can be associated with a recording event in the measurement apparatus.

It is simple to check that Eq. (59) has the structure defined by Eqs. (24)- (25), and also fulfill the symmetry condition Eq. (12). Consistently with the previous analysis, the superoperator \( \mathcal{J} \) [Eq. (27)] is defined as
\[
\mathcal{J} \rho = \gamma \sigma_{11} \rho \sigma_{11}^\dagger + \gamma' \sigma_{12} \rho \sigma_{12}^\dagger, \tag{62}
\]
leading to the expression
\[
\mathcal{J} \rho = (\gamma \langle +1 | \rho | +1 \rangle + \gamma' \langle +2 | \rho | +2 \rangle) \langle -| \otimes |1\rangle \langle 1| \). \tag{63}
\]
From here, the measurement transformation [Eq. (28)] associated to each event reads
\[
\mathcal{M} \rho = \langle -| \otimes |1\rangle \langle 1|. \tag{63}
\]
Therefore, the state after a detection is independent of the previous bipartite state \( \rho \), which in turn implies that
the measurement process is a renewal one [see Eq. (15)]. The bipartite conditional dynamics between events is defined by the superoperator [Eq. (32)]

$$\tilde{\mathcal{D}}\rho = -i\frac{\hbar}{\mathbf{H}_0}\rho - \frac{1}{2}\{\mathbf{\gamma}\sigma_1^{\dagger}\sigma_{11} + \mathbf{\gamma}^{\dagger}\sigma_{12}\}, \rho\}^+.$$ (64)

In order to obtain simple analytical expressions from now on we analyze the case $\gamma' = \gamma$. Notice that it is also possible to take $\gamma' = 0$ with $\gamma > 0$, or $\gamma = 0$ with $\gamma' > 0$.

The conditional propagator $\tilde{T}(t)$ [Eq. (31)] can be defined when acting on an arbitrary initial condition $\rho$. By defining the state $\tilde{\rho}_k = \tilde{T}(t)\rho$, the time evolution of $\tilde{T}(t)$ [Eq. (31)] can be written in terms of the matrix elements

$$\tilde{\rho}_k^\pm = \langle \pm | \tilde{\rho}_k | \pm \rangle, \quad \tilde{c}_k^\pm = \langle \pm | \tilde{\rho}_k | \mp \rangle.$$ (65)

For the populations we get

$$\frac{d\tilde{\rho}_k^\pm}{dt} = -\int_0^t dt' k_{t-t'}^\pm \tilde{\rho}_{1-t'}^\pm + \int_0^t dt' k_{t-t'}^\mp \tilde{\rho}_{1-t'}^\mp,$$ (66a)

$$\frac{d\tilde{\rho}_k}{dt} = -\int_0^t dt' k_{t-t'} \tilde{\rho}_{1-t'} + (1 - \tilde{\delta}) \int_0^t dt' k_{t-t'} \tilde{\rho}_{1-t'}.$$ (66b)

Here, the constant $\tilde{\delta}$ must be taken as $\tilde{\delta} \rightarrow 1$. Thus, the last term does not contribute. The memory kernels are

$$k_t^\pm = \gamma'\delta(t) + \frac{\Omega^2}{2}e^{-t\gamma/2}, \quad k_t^{\mp} = \frac{\Omega^2}{2}e^{-t\gamma/2}.$$ (67)

The coherence evolve as

$$\frac{dc_k^\pm}{dt} = -\int_0^t dt' k_{t-t'}^\pm c_{1-t'}^\mp + \int_0^t dt' k_{t-t'}^\mp c_{1-t'}^\pm,$$ (68)

where the kernels $\tilde{k}_t$ and $\tilde{k}_t^\mp$ are

$$\tilde{k}_t = \gamma^2\delta(t) + \frac{\Omega^2}{4}(1 + e^{-t\gamma}), \quad \tilde{k}_t^{\mp} = \frac{\Omega^2}{4}(1 + e^{-t\gamma}).$$ (69)

Due to the symmetries of the problem, populations and coherences evolve independently of each of the others.

Eqs. (66) and (68) can be solved in a Laplace domain. The survival probability [Eq. (66)] reads

$$P_0(t|\rho) = Tr_s[\tilde{T}(t)\rho] = \tilde{\rho}_k^\pm + \tilde{\rho}_k^\mp.$$ (69)

We get

$$P_0(t|\rho) = Tr_s[\rho]e^{-\frac{\gamma^2}{2\nu^2}}\left\{\left(\frac{\gamma}{2\nu}\right)^2 \cosh(\nu t) - \left(\frac{\Omega}{\nu}\right)^2\right\},$$ (70)

where the “frequency” $\nu$ reads

$$\nu = \sqrt{(\gamma/2)^2 - \Omega^2}.$$ (71)

In Eq. (70) the dependence on the system state $\rho$ is given $Tr_s[\rho]$ and $Tr_s[\sigma_{z}\rho]$, where $\sigma_z$ is the $z$-Pauli Matrix. The normalization of $\rho$ it follows $Tr_s[\rho] = 1$, while $Tr_s[\sigma_{z}\rho] = \langle + | \rho | + \rangle - (\langle - | \rho | - \rangle).$ Therefore, $P_0(t|\rho)$ only depends on the populations of $\rho$.

In Fig. 1 we plotted $P_0(t|\rho)$ and its associated waiting time distribution $w(t|\rho) = -(d/dt)P_0(t|\rho)$ [Eq. (69)] for different initial states $\rho$. In Figs. (a) and (b) we took $\rho = |y\rangle\langle y|$, where $|y\rangle = (1/\sqrt{2})(|+\rangle - i |−\rangle)$. In Figs. (c) and (d), $\rho = |−\rangle\langle −|$, which correspond to the resetting state defined by Eq. (63). In all cases, the parameters satisfies $\Omega/\gamma = 4$.

In Fig. 2, a realization of $p_\text{av}^\pm(t)$ is shown through the matrix elements $\langle + | p_\text{av}(t) | + \rangle$ [upper population, Fig. 2(a)] and $\langle + | p_\text{av}(t) | − \rangle$ [coherence, Fig. 2(b)]. The initial state is $p_\text{av}(0) = |y\rangle\langle y|$. In the behavior of $\langle + | p_\text{av}(t) | + \rangle$ is possible to observe the successive jumps, where the system state collapse to the resetting state $|−\rangle\langle −|$, or equivalently, $\langle + | p_\text{av}(t) | + \rangle \rightarrow 0$. The conditional interevent behavior is periodic.
FIG. 2: Realizations of the stochastic density matrix $\rho^s_0(t)$ and its ensemble average. In (a) and (b) are plotted the population ($+|\rho^s_0(t)|+$) and the imaginary part of the coherence ($+|\rho^s_0(t)|-$) respectively. In (c) and (d) are plotted an average over $2 \times 10^3$ realizations (noisy curves). The full lines correspond to the analytical solutions Eqs. (73) and (76). In all cases the initial system state is $\rho^s_0 = |y_+\rangle \langle y_-|$, while the characteristic parameters satisfy $\Omega/\gamma = 4$.

On the other hand, for the chosen initial condition the coherence $\langle +|\rho^s_0(t)|-\rangle$ does not have a real component. Hence, from Fig. 2(b) we conclude that after the first event is dyes out. This property follows from the replacement $\tilde{\rho}^c \to \rho^c$. Their explicit solution is

$$c^c_i = e^{-\frac{\Omega}{\gamma}t} \left[ a - b \cosh(\nu t) \right],$$

where the coefficients $a$ and $b$ read

$$a = (c^c_0 + c^c_0)\Omega^2/\nu^2,$$

$$b = (c^c_0 - c^c_0)\Omega^2/\nu^2.$$  (76)

In Figs. 2(c) and (d), the analytical expressions of both the populations and coherences, Eqs. (73) and (76) respectively, recover the ensemble average behavior. This result explicitly show the consistency of the proposed approach. On the other hand, Eqs. (73) and (76) lead to a diagonal stationary density matrix

$$\rho^s_0 = \lim_{t \to \infty} \rho^s_t = \text{diag}\left\{ \frac{\Omega^2}{\gamma^2 + 2\Omega^2}, \frac{\gamma^2 + \Omega^2}{\gamma^2 + 2\Omega^2} \right\}.$$  (77)

The evolution of the matrix elements (72) can also be rewritten in terms of the system density matrix $\rho^c$. From Eqs. (66) ($\tilde{\rho} \to 0$, $\tilde{\rho}^c \to \rho^c$, $\tilde{c}^c_i \to c^c_i$) and (68) we find

$$\frac{d\rho^c_t}{dt} = \gamma \tilde{C}[\sigma] \rho^c_t + \sum_{i=x,y,z} \int_0^t dt' k^c_{i-x} \tilde{C}[\sigma_i] \rho^c_t,$$  (78)

where the Lindblad channels are defined by Eq. (2), $\sigma_i$, $i = x, y, z$, are the Pauli matrices, while the memory functions are

$$k^c_x = \frac{\Omega^2}{8} (e^{-\gamma t/2} - 1)^2,$$

$$k^c_y = k^c_z = \frac{\Omega^2}{8} (e^{-\gamma t/2} - 1)^2.$$  (79)

As expected, the density matrix evolution (79) has the structure defined by Eq. (59), where the local in time contribution is directly associated to the system transitions recorded by the measurement apparatus.

**Genuine non-Markovian effects**

Quantum non-Markovian time convoluted master equations can always be rewritten in terms of local in time evolutions with time dependent rates [21]. If the rates are positive at all times, the measurement dynamics is still consistent with a standard QJA [23]. On the other hand, if the rates assume negative values, the dynamics develops “genuine” non-Markovian effects such an environment-to-system flow of information. This phenomenon can be detected through different measures [27], which in the Markovian case present a monotonous time
In both cases $\Omega / \gamma = 4$. For example, for $\rho^s_0 = |y\rangle \langle y|$, where $|y\rangle = (1/\sqrt{2})(|+\rangle - |-)$. In both cases $\Omega / \gamma = 4$.

In Fig. 3 the density matrix obey the evolution Eq. (79), whose solution is defined by Eqs. (79) and (77). The solid line corresponds to the initial condition and parameters values of Figs. 1 and 2. Evidently, the oscillatory behavior of $E(\rho^s_t|\rho^s_\infty)$ demonstrate that (79) cannot be rewritten in terms of a local in time evolution with (time-dependent) positive rates. The same property arises by choosing the initial conditions $\rho^s_0 = |\pm\rangle \langle \pm|$, in which case the system dynamics can be mapped with a classical two-level system. In general, the development of or not of the revivals strongly depends on the initial conditions. For example, for $\rho^s_0 = |x-\rangle \langle x-|$, where $|x-\rangle$ is an eigenvector of $\sigma_x$ with eigenvalue minus one, $|x-\rangle = (1/\sqrt{2})(|+\rangle - |-)$. $E(\rho^s_t|\rho^s_\infty)$ decay in a monotonous way (dotted line). This case can be understood in terms of the symmetries of the underlying bipartite dynamics, Eq. (59).

V. SUMMARY AND CONCLUSIONS

In this paper we established a non-Markovian generalization of the standard QJA. The underlying idea consist in embedding the system dynamics in a bipartite Markovian evolution [Eq. (24)]. Assuming that the measurement action is only performed on the system of interest, we demonstrated that there exist symmetries conditions on the Lindblad (bipartite) channels [Eqs. (12) and (19)] that lead to a closed system stochastic dynamics consistent with a quantum measurement theory.

For both, renewal and non-renewal measurement processes, the ensemble of realizations is similar to that of the standard case. At random times, the system state suffer a disruptive transformation, which is associated to each recording event. In the intermediate time intervals, the (conditional) system dynamics is smooth and non-unitary. The main difference with the standard approach is this last ingredient. Here, it is not defined by an exponential propagator [Eq. (41)]. In fact, it arises from a partial trace over the semigroup evolution associated to the Markovian bipartite dynamics [Eq. (31)]. Hence, in general, the stochastic dynamics does not admit an unravelling in terms of pure states [Eq. (55)].

As in the standard case, the jump statistics can be defined by a survival probability [Eq. (36)], which in general depends on the system state. In addition to the stochastic dynamics, we also characterized the system density matrix evolution. The structure of the corresponding non-Markovian quantum master equations is defined by Eqs. (50) and (57). Arbitrary master equations with this structure can be unravelled with the ensemble of trajectories if it is possible to assign a survival probability to the conditional dynamics, Eq. (55).

The consistence of the formalism was checked by studying the dynamics of a two level system whose non-Markovian dynamics lead to successive transition between the upper and lower levels. The simplicity of the model allowed us to obtain short analytical expressions for both the measurement statistics [Eq. (70)] as well as for the density matrix elements and the corresponding density matrix evolution [Eq. (79)]. The relevance of the example not only come from its simplicity. In fact, it also allowed us to demonstrate that the present generalization is consistent with a back flow of information from the environment to the system. This property follows from the non-monotonous decay of the relative entropy with respect to the stationary state (Fig. 3).

While the present formalism lead to a consistent non-Markovian generalization of the quantum jumps approach, it is clear that it can be extended in different directions. For example one may consider arbitrary initial bipartite states [Eq. (51)] or to introduce non-separable bipartite resetting states [Eq. (30)]. A less technical aspect should be to consider the case in which many different measurement apparatus are monitoring the system or to determine which kind of consistent non-Markovian generalization is not covered by a Markovian embedding.

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Appendix A: Quantum jumps statistics-Markovian case

Here we derive the statistical description of the ensemble of realizations associated to the Markovian QJA. The solution of Eq. (3) can formally be written as

$$\rho_t = \exp[\hat{D}t]\rho_0 + \int_0^t dt' \exp[\hat{D}(t - t')]\hat{J}[\rho_t'],$$  \hspace{1cm} (A1)

where $\rho_0$ is the initial system state. This expression can be iterated leading to the series expansion

$$\rho_t = \sum_{n=0}^{\infty} \rho_t^{(n)},$$  \hspace{1cm} (A2)

where each contribution satisfies the recursive relation

$$\rho_t^{(n)} = \int_0^t dt' \tilde{T}(t - t')\hat{J}\rho_t'^{(n-1)},$$  \hspace{1cm} (A3)

with $\rho_t^{(0)} = \tilde{T}(t)\rho_0$. Therefore, it follows ($n \geq 1$)

$$\rho_t^{(n)} = \int_0^t dt_n \cdots \int_0^{t_2} dt_1 \tilde{T}(t - t_n)\tilde{T}(t_2 - t_1)\hat{J}(t_1)\rho_0,$$

(A4)

The superoperators $\hat{J}$ and $\tilde{T}(t)$ are defined by Eqs. (A1) and (A2) respectively. Each contribution $\rho_t^{(n)}$ can be associated to trajectories with $n$-detection events. Its statistics can be obtained by writing the previous expression in terms of the measurement transformation $\hat{M}$ [Eq. (3)] and the normalized propagator $\tilde{T}(t)$ [Eq. (7)]. We get

$$\rho_t^{(n)} = \int_0^t dt_n \cdots \int_0^{t_2} dt_1 P_n[t, \{t_i\}_1^n]$$

$$\times \tilde{T}(t - t_n)\tilde{T}(t_2 - t_1)\hat{J}(t_1)\rho_0,$$

(A5)

(n $\geq 1$) and $\rho_t^{(0)} = P_0(t)\rho_0 \tilde{T}(t)\rho_0$. The function

$$P_n[t, \{t_i\}_1^n] = \text{Tr}_s[\hat{J}(t - t_n)\cdots\hat{J}(t_2 - t_1)\hat{J}(t_1)\rho_0],$$

(A6)

is the joint probability density for observing measurement events at times $\{t_i\}_1^n$. It completely characterize the statistic of the measurement process. By introducing the auxiliary states $\rho_{t_{i+1}} = \tilde{T}(t_{i+1} - t_i)\hat{M}\rho_{t_i}$, with $\rho_{t_i} = \tilde{T}(t_i, 0)\rho_0$, the previous object can be rewritten as

$$P_n[t, \{t_i\}_1^n] = P_0(t - t_n)\tilde{T}(t)\rho_0$$

$$\prod_{j=2}^n w(t_j - t_{j-1})\hat{M}\rho_{t_{j-1}}w(t_{j-1})\rho_0,$$

(A7)

where the survival probability $P_0(t)\rho_0$ and the waiting time distribution $w(t)\rho_0$ are defined by Eqs. (B4) and (B7) respectively.

The structure of both Eqs. (A5) and (A7) are consistent with the stochastic dynamics defined in Sec. II-A.

The second line of Eq. (A5) consists in successive applications of the measurement transformations $\hat{M}$ and intermediate evolution with the propagator $\tilde{T}(t)$. On the other hand, the weight of each realization, defined by Eq. (A7), have the same structure than a renewal process, that is, there exist a probability distribution (waiting time distribution) that define the statistic of the time interval between consecutive detection events. Nevertheless, here the distribution depends on the resetting state, that is, the state after a measurement event.

Appendix B: Non-Markovian master equations from the jumps statistics

We derived the non-Markovian extension of the QJA by studying the standard approach in a Markovian bipartite dynamics. Under the conditions obtained in Sec. III the system stochastic dynamics becomes closed, that is, it can be written without taking into account explicitly the ancilla dynamics. Here we derive the corresponding non-Markovian master equation [see Eqs. (S1) and (S7)] by averaging the ensemble of trajectories.

The full counting statistics can be derived from the Markovian evolution Eq. (26) and its formal solution (31). All calculation steps described in Appendix A can be extended, after a trivial change of notation ($\hat{J} \rightarrow \overline{J}$, $\tilde{T} \rightarrow \overline{T}$), to the bipartite evolution defined in terms of $\rho_t^{(n)}$. By performing a partial trace over the ancilla degrees of freedom on the corresponding expressions, by using the bipartite measurement transformation $\hat{M}$ and the initial bipartite state $\rho_0$, it is possible to demonstrate that Eqs. (A5) and (A7) are also valid for the non-Markovian system dynamics. Nevertheless, in the non-Markovian case, the propagator $\tilde{T}(t)$ is defined by Eq. (34) [or equivalently Eq. (33)] while the survival probability $P_0(t)\rho_0$ and waiting time distribution $w(t)\rho_0$ from Eqs. (B4) and (B7) respectively.

1. Renewal case

When the measurement process is a renewal one, we can write the joint probability density [Eq. (A7)] as

$$P_n[t, \{t_i\}_1^n] = P_0(t - t_n)\prod_{j=2}^n w(t_j - t_{j-1})w(t_1)\rho_0,$$

(B1)

where, in contrast to a Markovian renewal process, here the survival probability $P_0(t)$ and waiting time distribution $w(t)\rho_0$ are defined by Eqs. (B4) and (B7) respectively. From Eq. (A5) and by using the renewal property Eq. (B8), the previous expression for $P_n[t, \{t_i\}_1^n]$ allows us to write

$$\rho_t^{(n)} = \int_0^t dt' \tilde{T}(t - t')\rho_s f^{(n)}(t'),$$

(B2)
[\rho_i^{(0)} = \hat{T}(t)\rho_0^i] where the function \( f^{(n)}(t) \) is defined as
\[ f^{(n)}(t) = \int_0^t dt_1 \cdots \int_0^{t_2} dt_1 \prod_{j=2}^{n} w(t_j - t_{j-1})w(t_1|\rho_0^0). \]

From Eq. (B2) and the expression for the waiting time distribution \( w(t) \) [Eq. (A7)], we get the recursive relation
\[ \rho_i^{(n)}(t) = -\int_0^t dt' \hat{T}(t-t')\rho_s \int_0^{t'} d\tau \rho_s \{\hat{D}(t' - \tau)\hat{\rho}_i^{(n-1)}(\tau)\}. \]

By adding all these states [see Eq. (A2)] and by using the non-Markovian time evolution of the propagator \( \hat{T}(t) \) [Eq. (34)], after some calculations steps, the system density matrix evolution Eq. (57) is recovered.

2. Non-renewal case

By using the rate condition Eq. (49) correspondent to the non-renewal case, it is possible to demonstrate that the superoperator \( \hat{J} \) [Eq. (27)] satisfies the relation
\[ \hat{J} \rho = \hat{J} \{	ext{Tr}_a[\rho]\} \otimes \rho_a, \]
where the system superoperator \( \hat{J} \) is defined by Eq. (41) and the ancilla resetting state \( \tilde{\rho}_a \) follows from Eq. (44).

By writing Eq. (A6) in terms of bipartite objects \( (\hat{J} \to \hat{J}, \hat{J} \to \hat{T}) \), after introducing Eq. (B5), the joint probability distribution can be written as
\[ P_n[t, \{t_i\}^n_i] = \text{Tr}_s[\hat{T}(t - t_n)\hat{J} \cdots \hat{J} \hat{T}(t_2 - t_1)\hat{J} \hat{T}(t_1)\rho_0^s], \]

where \( \hat{T}(t) \) and \( \hat{J} \) follows from Eqs. (A1) and (1) respectively. Notice that in this case, the only difference with the Markovian case [Eq. (A6)] is the definition of \( \hat{T}(t) \).

In order to obtain the density matrix evolution we need a recursive relation for the states \( \rho_i^{(n)} \). Here, such kind of relation can be easily obtained from the recursive relation (A3) when applied to the bipartite dynamics. With the aid of Eq. (B5) we get
\[ \rho_i^{(n)} = \int_0^t dt' \hat{T}(t-t')\hat{J} \rho_i^{(n-1)}. \]

Consistently, the same relation arise from Eqs. (A3) and (B5). By adding all states \( \rho_i^{(n)} \), and by using the non-Markovian time evolution of the propagator \( \hat{T}(t) \) [Eq. (34)], we recover Eq. (59).
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