**Generalized Bell States and Quantum Teleportation**

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**Abstract**

We make a brief comment on measurement of quantum operators with degenerate eigenstates and apply to quantum teleportation. We also try extending the quantum teleportation by Bennett et al [5] to more general situation by making use of generalized Bell states.

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1 Introduction

Quantum Teleportation is one of main subjects in Quantum Information Theory. See for example [1], [2], [3], [4] or [9] on quantum information theory.

This concept was first proposed by Bennett et al. [5] and has been studying actively. Alice send (teleport) a quantum state localized near her to Bob making use of some manipulations. The significant feature of this is to use the so-called Bell states ([6], [7]) which are maximally entangled.

In the process Alice must measure some physical operator which eigenstates are just the Bell states to know the final state through the reduction of state. Here much attention should be required. If a physical operator which Alice will use has degenerate eigenstates, then we cannot in general get one of Bell states explicitly by the principle of Quantum Mechanics. This has been pointed out by [8]. Alice must choose a physical operator with simple eigenvalues which eigenstates are the Bell states.

By the way the generalized Bell states were defined by [10] making use of generalized coherent states and have been calculated by [11]. Therefore it is very natural to try applying our generalized Bell states to quantum teleportation. Such a generalization is not so difficult (see Sect. 3). This is our main result.

2 Review on Quantum Teleportation

We in this section revisit the quantum teleportation by Bennett et al. [5].

The famous Bell states ([6], [7]) in the case of spin $\frac{1}{2}$ are:

$$|\phi_1\rangle_{12} = \frac{1}{\sqrt{2}}(|0\rangle_1 \otimes |0\rangle_2 + |1\rangle_1 \otimes |1\rangle_2),$$

$$|\phi_2\rangle_{12} = \frac{1}{\sqrt{2}}(|0\rangle_1 \otimes |0\rangle_2 - |1\rangle_1 \otimes |1\rangle_2),$$

$$|\phi_3\rangle_{12} = \frac{1}{\sqrt{2}}(|0\rangle_1 \otimes |1\rangle_2 + |1\rangle_1 \otimes |0\rangle_2).$$
\[ |\phi_4\rangle_{12} = \frac{1}{\sqrt{2}}(|0\rangle_1 \otimes |1\rangle_2 - |1\rangle_1 \otimes |0\rangle_2). \]  

(1)

These are a basis in \( C^2 \otimes C^2 \). Conversely we have

\[ |0\rangle_1 \otimes |0\rangle_2 = \frac{1}{\sqrt{2}}(|\phi_1\rangle_{12} + |\phi_2\rangle_{12}), \quad |1\rangle_1 \otimes |1\rangle_2 = \frac{1}{\sqrt{2}}(|\phi_1\rangle_{12} - |\phi_2\rangle_{12}), \]

\[ |0\rangle_1 \otimes |1\rangle_2 = \frac{1}{\sqrt{2}}(|\phi_3\rangle_{12} + |\phi_4\rangle_{12}), \quad |1\rangle_1 \otimes |0\rangle_2 = \frac{1}{\sqrt{2}}(|\phi_3\rangle_{12} - |\phi_4\rangle_{12}). \]  

(2)

The quantum teleportation by Bennett et al \[5\] is as follows: Alice and Bob share beforehand a system of two particles \( A \) and \( B \) and they can constitute the EPR pair

\[ \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B). \]  

(3)

Next Alice wants to send (transport) a state

\[ \alpha|0\rangle_1 + \beta|1\rangle_1 \]  

where \( \alpha, \beta \in C \)

to Bob. For that she manipulates a system of three particles 1, \( A \) and \( B \) as follows:

**Fundamental Formula I \([5]\)**

\[ (\alpha|0\rangle_1 + \beta|1\rangle_1) \otimes \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B) \]

\[ = \frac{1}{2} |\phi_1\rangle_{1A} \otimes (\alpha|1\rangle_B - \beta|0\rangle_B) + \frac{1}{2} |\phi_2\rangle_{1A} \otimes (\alpha|1\rangle_B + \beta|0\rangle_B) \]

\[ + \frac{1}{2} |\phi_3\rangle_{1A} \otimes (-\alpha|0\rangle_B + \beta|1\rangle_B) + \frac{1}{2} |\phi_4\rangle_{1A} \otimes (-\alpha|0\rangle_B - \beta|1\rangle_B). \]  

(5)

The proof is very easy making use of \(2\).

The procedure of quantum teleportation goes as follows:

(i) Alice measures a physical operator \( \hat{Q} \) related to two particles system \( \{1, A\} \) which eigenstates are \(4\) and after that she obtains a state (one of \(4\)) by the reduction of state.

(ii) Alice informs this outcome to Bob by some classical means of communication (there is no inconsistency to the theory of special relativity).
(iii) Bob knows by this what the state corresponding to a particle $B$ is

$$\alpha \langle 1 |_B - \beta \langle 0 |_B, \quad \alpha \langle 1 |_B + \beta \langle 0 |_B, \quad -\alpha \langle 0 |_B + \beta \langle 1 |_B, \quad -\alpha \langle 0 |_B - \beta \langle 1 |_B.$$  

(iv) Bob operates some operators to get the final result

\[ i\sigma_2 (\alpha \langle 1 |_B - \beta \langle 0 |_B) = \alpha \langle 0 |_B + \beta \langle 1 |_B, \]
\[ \sigma_1 (\alpha \langle 1 |_B + \beta \langle 0 |_B) = \alpha \langle 0 |_B + \beta \langle 1 |_B, \]
\[ -\sigma_3 (-\alpha \langle 0 |_B + \beta \langle 1 |_B) = \alpha \langle 0 |_B + \beta \langle 1 |_B, \]
\[ -\frac{1}{2} (\alpha \langle 0 |_B - \beta \langle 1 |_B) = \alpha \langle 0 |_B + \beta \langle 1 |_B. \]\n
(6)

Here we have something to worry. In (i) Alice measures a physical operator $\hat{Q}$.

**What a kind of physical operator does she measure?**

If we measure physical operators like

$$\hat{Q} = \sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2$$  

(7)

or

$$\hat{Q} = \sigma_1 \otimes \sigma_1 + \sigma_3 \otimes \sigma_3$$  

(8)

then we meet some troubles. This has been pointed out by Adenier [8]. For example let us consider the operator (8) which eigenvalues are $\{2, 0, -2\}$ and corresponding eigenstates are $|\phi_1\rangle, \{|\phi_2\rangle, |\phi_3\rangle\}, |\phi_1\rangle$ in this order. Namely 0 is an eigenvalue with multiplicity 2.

Let us here assume that we get 0–eigenvalue when measuring $\hat{Q}$. What is the state we get after the reduction of state? Since

$$|0\rangle \in \text{Vect}_C \{ |\phi_2\rangle, |\phi_3\rangle \},$$

we have some

$$\alpha |\phi_2\rangle + \beta |\phi_3\rangle \quad \text{where} \quad \alpha, \beta \in \mathbb{C}$$  

(9)

\[^{1}\text{I don’t agree to his assertion in spite of his good pointout}\]
by the principle of Quantum Mechanics. That is, it is dangerous for us to use a physical
operator with degenerate eigenstates in the process of measurement.
Therefore we must use a physical operator $\hat{Q}$ with simple eigenvalues which corresponding
eigenstates are just (10). Let us consider an operator
\[
\hat{Q} = a|\phi_1\rangle\langle \phi_1| + b|\phi_2\rangle\langle \phi_2| + c|\phi_3\rangle\langle \phi_3| + d|\phi_4\rangle\langle \phi_4|
\]
where $a, b, c, d$ are mutually distinct real numbers. Then it is clear that the eigenvalues
of $\hat{Q}$ are $\{a, b, c, d\}$ and corresponding eigenstates are $\{|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle, |\phi_4\rangle\}$ in this order.
We want to rewrite (11) making use of
\[
|0\rangle\langle 0| = \frac{1}{2}(1_2 + \sigma_3), \quad |1\rangle\langle 1| = \frac{1}{2}(1_2 - \sigma_3),
\]
\[
|0\rangle\langle 1| = \frac{1}{2}(\sigma_1 + i\sigma_2), \quad |1\rangle\langle 0| = \frac{1}{2}(\sigma_1 - i\sigma_2).
\]
The result reads
\[
\hat{Q} = \frac{a+b+c+d}{4}1_2 \otimes 1_2 + \frac{a-b+c-d}{4}\sigma_1 \otimes \sigma_1
\]
\[
+ \frac{-a+b+c-d}{4}\sigma_2 \otimes \sigma_2 + \frac{a+b-c-d}{4}\sigma_3 \otimes \sigma_3.
\]
The proof is easy.
For simplicity we set $a = 3, \ b = 1, \ c = -1, \ d = -3$ to get a simple form
\[
\hat{Q} = \sigma_1 \otimes \sigma_1 + 2\sigma_3 \otimes \sigma_3.
\]
That is, Alice should measure this operator in place of (8) or (7).
3 More on Quantum Teleportation

We extend the quantum teleportation by Bennett et al [3] to more general situation. In the following we treat the coherent representation of $u(3)$ based on $\frac{U(3)}{U(2)\times U(1)} \cong CP^2$ (see [11] and [12]) with $Q = 1$ only to avoid complicated situations. This is just the three dimensional representation.

Let $\{0, 1, 2\}$ be a basis of the representation space $V (\cong C^3)$. Namely

$$\sum_{j=0}^{2} |j\rangle\langle j| = 1_{Q=1} \quad \text{and} \quad \langle i|j\rangle = \delta_{ij}.$$ 

Let $\omega$ be an element in $C$ satisfying $\omega^3 = 1$. Then $1 + \omega + \omega^2 = 0$ and $\bar{\omega} = \omega^2$. Then generalized Bell states in this case ($n = 2$ and $Q = 1$) are given by [11]:

$$|\psi_1\rangle_{12} = \frac{1}{\sqrt{3}}(|0\rangle_1 \otimes |0\rangle_2 + |1\rangle_1 \otimes |1\rangle_2 + |2\rangle_1 \otimes |2\rangle_2),$$

$$|\psi_2\rangle_{12} = \frac{1}{\sqrt{3}}(|0\rangle_1 \otimes |0\rangle_2 + \omega|1\rangle_1 \otimes |1\rangle_2 + \omega^2|2\rangle_1 \otimes |2\rangle_2),$$

$$|\psi_3\rangle_{12} = \frac{1}{\sqrt{3}}(|0\rangle_1 \otimes |0\rangle_2 + \omega^2|1\rangle_1 \otimes |1\rangle_2 + \omega|2\rangle_1 \otimes |2\rangle_2),$$

$$|\psi_4\rangle_{12} = \frac{1}{\sqrt{3}}(|0\rangle_1 \otimes |1\rangle_2 + |1\rangle_1 \otimes |2\rangle_2 + |2\rangle_1 \otimes |0\rangle_2),$$

$$|\psi_5\rangle_{12} = \frac{1}{\sqrt{3}}(|0\rangle_1 \otimes |1\rangle_2 + \omega|1\rangle_1 \otimes |2\rangle_2 + \omega^2|2\rangle_1 \otimes |0\rangle_2),$$

$$|\psi_6\rangle_{12} = \frac{1}{\sqrt{3}}(|0\rangle_1 \otimes |1\rangle_2 + \omega^2|1\rangle_1 \otimes |2\rangle_2 + \omega|2\rangle_1 \otimes |0\rangle_2),$$

$$|\psi_7\rangle_{12} = \frac{1}{\sqrt{3}}(|0\rangle_1 \otimes |2\rangle_2 + |1\rangle_1 \otimes |0\rangle_2 + |2\rangle_1 \otimes |1\rangle_2),$$

$$|\psi_8\rangle_{12} = \frac{1}{\sqrt{3}}(|0\rangle_1 \otimes |2\rangle_2 + \omega|1\rangle_1 \otimes |0\rangle_2 + \omega^2|2\rangle_1 \otimes |1\rangle_2),$$

$$|\psi_9\rangle_{12} = \frac{1}{\sqrt{3}}(|0\rangle_1 \otimes |2\rangle_2 + \omega^2|1\rangle_1 \otimes |0\rangle_2 + \omega|2\rangle_1 \otimes |1\rangle_2). \quad (15)$$

These are a basis in $V \otimes V \cong C^3 \otimes C^3$. Conversely we have

$$|0\rangle_1 \otimes |0\rangle_2 = \frac{1}{\sqrt{3}}(|\psi_1\rangle_{12} + |\psi_2\rangle_{12} + |\psi_3\rangle_{12}),$$

$$|1\rangle_1 \otimes |1\rangle_2 = \frac{1}{\sqrt{3}}(|\psi_1\rangle_{12} + \omega^2|\psi_2\rangle_{12} + \omega|\psi_3\rangle_{12}),$$

$$|2\rangle_1 \otimes |2\rangle_2 = \frac{1}{\sqrt{3}}(|\psi_1\rangle_{12} + \omega|\psi_2\rangle_{12} + \omega^2|\psi_3\rangle_{12}).$$
|0⟩₁ ⊗ |1⟩₂ = \frac{1}{\sqrt{3}}(|ψ₁⟩₁₂ + |ψ₅⟩₁₂ + |ψ₆⟩₁₂),
|1⟩₁ ⊗ |2⟩₂ = \frac{1}{\sqrt{3}}(|ψ₄⟩₁₂ + ω²|ψ₅⟩₁₂ + ω|ψ₆⟩₁₂),
|2⟩₁ ⊗ |0⟩₂ = \frac{1}{\sqrt{3}}(|ψ₄⟩₁₂ + ω|ψ₅⟩₁₂ + ω²|ψ₆⟩₁₂),
|0⟩₁ ⊗ |2⟩₂ = \frac{1}{\sqrt{3}}(|ψ₇⟩₁₂ + |ψ₈⟩₁₂ + |ψ₉⟩₁₂),
|1⟩₁ ⊗ |0⟩₂ = \frac{1}{\sqrt{3}}(|ψ₇⟩₁₂ + ω²|ψ₈⟩₁₂ + ω|ψ₉⟩₁₂),
|2⟩₁ ⊗ |1⟩₂ = \frac{1}{\sqrt{3}}(|ψ₇⟩₁₂ + ω|ψ₈⟩₁₂ + ω²|ψ₉⟩₁₂).  \hspace{1cm} (16)

Alice and Bob share beforehand a system of two particles A and B and they can constitute the EPR–like pair
\[ \frac{1}{\sqrt{3}} (|0⟩ₐ ⊗ |1⟩ₐ + ω|1⟩ₐ ⊗ |2⟩ₐ + ω²|2⟩ₐ ⊗ |0⟩ₐ) \]  \hspace{1cm} (17)

Next Alice want to send (transport) a state
\[ α|0⟩₁ + β|1⟩₁ + γ|2⟩₁ \] where \( α, β, γ ∈ \mathbb{C} \)  \hspace{1cm} (18)
to Bob. For that she manipulates a system of three particles 1, A and B as follows:

**Fundamental Formula II**

\[
(α|0⟩₁ + β|1⟩₁ + γ|2⟩₁) ⊗ \frac{1}{\sqrt{3}} (|0⟩ₐ ⊗ |1⟩ₐ + ω|1⟩ₐ ⊗ |2⟩ₐ + ω²|2⟩ₐ ⊗ |0⟩ₐ)
\]

\[ = \frac{1}{3} \{ |ψ₁⟩₁ₐ ⊗ (α|1⟩ₐ + β|2⟩ₐ + γ|0⟩ₐ) \\
+ |ψ₂⟩₁ₐ ⊗ (α|1⟩ₐ + β|2⟩ₐ + γ|0⟩ₐ) \\
+ |ψ₃⟩₁ₐ ⊗ (α|1⟩ₐ + β|2⟩ₐ + γ|0⟩ₐ) \\
+ |ψ₄⟩₁ₐ ⊗ (α|1⟩ₐ + β|2⟩ₐ + γ|0⟩ₐ) \\
+ |ψ₅⟩₁ₐ ⊗ (α|1⟩ₐ + β|2⟩ₐ + γ|0⟩ₐ) \\
+ |ψ₆⟩₁ₐ ⊗ (α|1⟩ₐ + β|2⟩ₐ + γ|0⟩ₐ) \\
+ |ψ₇⟩₁ₐ ⊗ (α|1⟩ₐ + β|2⟩ₐ + γ|0⟩ₐ) \\
+ |ψ₈⟩₁ₐ ⊗ (α|1⟩ₐ + β|2⟩ₐ + γ|0⟩ₐ) \\
+ |ψ₉⟩₁ₐ ⊗ (α|1⟩ₐ + β|2⟩ₐ + γ|0⟩ₐ) \} . \hspace{1cm} (19)\]
The proof is straightforward. First expand the left hand side of (19) and next rearrange each terms making use of (18). We leave details to the readers.

As shown in the preceding section Alice must measure a physical operator $\hat{Q}$ with non-degenerate eigenstates $\{|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_9\rangle\}$ such as

$$\hat{Q} = \sum_{j=1}^{9} a_j |\psi_j\rangle\langle\psi_j|$$

with mutually distinct real numbers $\{a_1, a_2, \ldots, a_9\}$. We want to rewrite (20) making use of

$$Z = |0\rangle\langle 0| + \omega|1\rangle\langle 1| + \omega^2|2\rangle\langle 2|,$$

$$X = |1\rangle\langle 0| + |2\rangle\langle 1| + |0\rangle\langle 2|,$$

$$Z^2 = |0\rangle\langle 0| + \omega|1\rangle\langle 1| + \omega|2\rangle\langle 2|,$$

$$X^2 = |2\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 2|,$$

$$Z^3 = 1_3, \quad X^3 = 1_3.$$  \hspace{1cm} \text{(21)}

For example

$$|0\rangle\langle 0| = \frac{1}{3} \left( 1_3 + Z + Z^2 \right),$$

$$|1\rangle\langle 1| = \frac{1}{3} \left( 1_3 + XZX + XZ^2X \right),$$

$$|2\rangle\langle 2| = \frac{1}{3} \left( 1_3 + X^2Z + Z^2X \right),$$

$$|0\rangle\langle 1| = \frac{1}{3} \left( X^2 + ZX^2 + Z^2X \right),$$

$$|1\rangle\langle 0| = \frac{1}{3} \left( X + XZ + XZ^2 \right),$$

$$|0\rangle\langle 2| = \frac{1}{3} \left( X + ZX + Z^2X \right),$$

$$|2\rangle\langle 0| = \frac{1}{3} \left( X^2 + X^2Z + X^2Z^2 \right),$$

$$|1\rangle\langle 2| = \frac{1}{3} \left( X^2 + XZ + XZ^2 \right),$$

$$|2\rangle\langle 1| = \frac{1}{3} \left( X + X^2Z + X^2Z^2 \right).$$ \hspace{1cm} \text{(22)}

We can with certainty rewrite (20) making use of (21) and (22) though the calculations are miserable. We leave it to the readers.
After receiving an information by Alice Bob operates some operator s to get the final result

\[
Z^2 X^2 \left( \alpha |1\rangle_B + \beta \omega |2\rangle_B + \gamma \omega^2 |0\rangle_B \right) = \alpha |0\rangle_B + \beta |1\rangle_B + \gamma |2\rangle_B,
\]

\[
X^2 (\alpha|1\rangle_B + \beta |2\rangle_B + \gamma |0\rangle_B) = \alpha |0\rangle_B + \beta |1\rangle_B + \gamma |2\rangle_B,
\]

\[
Z X^2 (\alpha|1\rangle_B + \beta \omega |2\rangle_B + \gamma \omega |0\rangle_B) = \alpha |0\rangle_B + \beta |1\rangle_B + \gamma |2\rangle_B,
\]

\[
\omega Z^2 X (\alpha \omega |2\rangle_B + \beta \omega^2 |0\rangle_B + \gamma |1\rangle_B) = \alpha |0\rangle_B + \beta |1\rangle_B + \gamma |2\rangle_B,
\]

\[
\omega^2 X (\alpha \omega |2\rangle_B + \beta \omega |0\rangle_B + \gamma \omega |1\rangle_B) = \alpha |0\rangle_B + \beta |1\rangle_B + \gamma |2\rangle_B,
\]

\[
\omega Z (\alpha \omega |2\rangle_B + \beta |0\rangle_B + \gamma \omega |1\rangle_B) = \alpha |0\rangle_B + \beta |1\rangle_B + \gamma |2\rangle_B
\]

This is an extended version of quantum teleportation by Bennett et al [5].

4 Discussion

In this paper we discussed the importance to use a physical operator with non–degenerate eigenstates (the Bell states) in the process of measurement and next applied our generalized Bell states to quantum teleportation.

The generalized Bell states including usual Bell states are deeply related to a (compact) complex geometry [11], [9], [13], so quantum teleportation should be in our opinion more geometrized.

We want to give a unified geometric approach to Quantum Information Theory including quantum computer, quantum teleportation, quantum cryptography.

In [14] such a trial will be given.
Acknowledgment. The author wishes to thank Shin’ichi Nojiri for useful discussions with him and Asher Peres for pointing out a misunderstanding on the Bell states.

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