Abstract
The entropy of the Ising model in the mean field approximation is derived by the Hamilton–Jacobi formalism. We consider a grand canonical ensemble with respect to the temperature and the external magnetic field. A cusp arises at the critical point, which shows a simple and new geometrical aspect of this model. In an educational sense, this curve with a cusp helps students acquire a more intuitive view of statistical phase transitions.

Keywords: phase transition, cusp singularity, Hamilton–Jacobi structure, mean field Ising model, Finsler geometry

1. Introduction
Phase transitions, critical phenomena, and the corresponding critical exponents are fundamental topics in statistical mechanics. Though they should have a close relationship with critical points of maps [1], catastrophe theory [2, 3], and singularity theory [4, 5] in mathematics, there is little application of these simple physical problems. In these geometrical standpoints, it is natural to expect critical points to be singularities on a certain surface or a curve. A critical phenomenon, accompanied by the Hamilton–Jacobi structure, is simply visualized in the present paper. We take the Ising model in the mean field approximation as an example.

The relationship between thermodynamics and Hamilton–Jacobi theory have been discussed for many years [6–10]. However, they are mostly considered under quasi-static
conditions. Among them, one notable proposal was offered by Suzuki [11]. The second law of thermodynamics can be considered as a variational principle that determines reversible or irreversible processes. He recognized a Finsler structure in the thermodynamics, and identified the equation of state, or the virial relation, as the constraint which inevitably arises in the Finsler–Lagrangian formulation [11–13], and derived a Hamilton–Jacobi structure. The idea was supplemented by one of the authors [14].

Following Suzuki’s method, we review the Finsler–Lagrangian formulation in the next section. In section 3, we apply this method to the mean field Ising model and show several graphs which reveal singularities at the critical points.

2. Finsler–Lagrangian formulation and Suzuki’s Hamilton–Jacobi thermodynamics

2.1. Review of Finsler–Lagrangian formulation

We start with the definition of the Finsler manifold. A Finsler manifold is a set of a manifold $M$ and a function $F$ on the tangent bundle $TM$. The function $F$ is called the Finsler metric and is defined so that it admits the properties (1) of the homogeneity of $F$:

$$F(x, \lambda dx) = \lambda F(x, dx), \quad \lambda > 0,$$

and (2) the domain of $F$:

$$F : D(F) \subset TM \to \mathbb{R},$$

where $D(F)$ is a subbundle of $TM$ on which $F$ and its derivative are well defined. In mathematics [17, 18], they additionally assume (3) positivity: $F > 0$ and (4) regularity: $\det (g_{\mu\nu}(x, dx)) \neq 0$ for $g_{\mu\nu}(x, dx) = \frac{1}{2} \partial^2F / \partial x^\mu \partial x^\nu$. Since these additional assumptions are not essential and can be obstacles for physical applications, we adopt conditions (1) and (2) alone.

Next, we review a Finsler geometrical formulation of the Lagrangian formalism, which we call the Finsler–Lagrangian formulation [12–16]. Let $Q = \{ (q^1, q^2, ..., q^n) \}$ be a configuration space and $L = L(q^i, \dot{q}^i, t)$ be a Lagrangian. It is well known that the Lagrangian $L$ constructs a Finsler metric $F$ on the extended configuration space $M := \mathbb{R} \times Q$ as

$$F = F(x^\mu, dx^\nu) := L\left(x^i, \frac{dx^i}{dx^0}, x^0\right) dx^0,$$

where we set $x^0 = t$, $x^i = \dot{q}^i$. This technique is known as the homogenization technique. The set $(M, F)$ becomes a $(n + 1)$-dimensional Finsler manifold. Though a configuration space $Q$ does not have a structure of metric geometry by itself, an extended configuration space $M$, which is a set of time and a configuration space, does have a metric structure. We also note that the homogenization technique does not change any physical properties that the original Lagrangian $L$ holds. Time evolutions of the system are represented by oriented curves $C = \{ c \}$ on the extended configuration space $M$. The action of the Lagrangian system is defined by a line integral of $F = F(x, dx)$ along an oriented curve $c \in C$

$$A[c] = \int_c F := \int_{t_0}^{t_1} F\left(x^i(\tau), \frac{dx^i}{d\tau}(\tau)\right) d\tau,$$

where $\tau$ is an arbitrary parameter of $c$. By homogeneity condition (1), $A[c]$ does not depend on the choice of the parametrization. The variational principle
leads to a covariant Euler–Lagrange equation

\[ 0 = \frac{\partial F}{\partial x^\mu} - \frac{d}{d\tau}\left( \frac{\partial F}{\partial \dot{x}^\mu} \right), \quad (\mu = 0, 1, 2, \ldots, n), \]  

(2.6)

where \( \frac{\partial F}{\partial x^\mu}(x(\tau), \dot{x}(\tau)) \) and \( \frac{\partial F}{\partial \dot{x}^\mu}(x(\tau), \dot{x}(\tau)) \) are functions of \( x^\mu(\tau) \) and \( \dot{x}^\mu(\tau) \). The important fact is that the covariant Euler–Lagrange equation (2.6) is parametrization invariant. Comparing it to the standard Euler–Lagrange equations, we additionally have an equation for \( m = 0 \), which has a strong relationship with the energy conservation law, due to the parametrization invariance.

Through Euler’s homogeneous function theorem, or differentiating the homogeneity condition (1) with respect to \( \lambda \) and set \( \lambda = 1 \), we have

\[ F = \frac{\partial F}{\partial x^\mu} dx^\mu. \]  

(2.7)

Differentiating it again with respect to \( dx^\nu \) on both sides, we have

\[ \frac{\partial F}{\partial x^\nu} = \frac{\partial^2 F}{\partial x^\mu \partial x^\nu} dx^\mu + \frac{\partial F}{\partial x^\nu}. \]  

(2.8)

The terms \( \frac{\partial F}{\partial x^\nu} \) cancel each other out, and it follows that

\[ 0 = \frac{\partial^2 F}{\partial x^\mu \partial x^\nu} dx^\mu = \frac{\partial p_\mu}{\partial x^\nu} = \frac{\partial p_\mu}{\partial x^\mu} dx^\mu, \]  

(2.9)

where we define a covariant conjugate momentum

\[ p_\mu \equiv \frac{\partial F}{\partial x^\mu}. \]  

(2.10)

The relation (2.9) indicates that the matrix \( \left( \frac{\partial p_\mu}{\partial x^\nu} \right) \) does not have the inverse matrix, that is, \( \det \left( \frac{\partial p_\mu}{\partial x^\nu} \right) = 0 \). The implicit function theorem for the Legendre transformation \( (x^\mu, dx^\nu) \rightarrow (x^\nu, p_\mu) \) promises that there exists at least one constraint

\[ G(x, p) = 0, \]  

(2.11)

among the variables \( (x^\mu, p_\mu) \).

Now, we show two simple examples. When we consider a relativistic free particle on an \( (n + 1) \) dimensional Lorentzian manifold \((M, g)\), we can take a Finsler metric on \( M \) as

\[ F = mc \sqrt{g_{\mu\nu}(x)dx^\mu dx^\nu}, \quad (\mu, \nu = 0, 1, 2, \ldots, n). \]  

(2.12)

With an arbitrary time parameter \( \tau \), the Euler–Lagrange equations are

\[ 0 = \frac{mc}{2\sqrt{g_{\alpha\beta}\ddot{x}^\alpha\ddot{x}^\beta}} \frac{\partial g_{\alpha\beta}}{\partial x^\mu} \dot{x}^\alpha \dot{x}^\beta - \frac{d}{d\tau}\left( mcg_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta \right), \]  

(2.13)

where \( \dot{x}^\mu = \frac{dx^\mu}{d\tau} \). If \( g_{\mu\nu} \) does not depend on \( x^0 \), the equation for \( \mu = 0 \) represents the energy conservation law.
The conjugate momenta are
\[
p_\mu = \frac{\partial F}{\partial x^\mu} = \frac{mc}{\sqrt{g_{\mu\nu}dx^\nu dx^\sigma g_{\sigma\alpha}dx^\alpha}}.
\]  
(2.15)

Using the inverse \( g^{\mu\nu} \) of \( g_{\mu\nu} \), we have
\[
g^{\mu\nu}p_\mu p_\nu = \frac{(mc)^2}{g_{\mu\nu}dx^\nu dx^\sigma g_{\sigma\alpha}dx^\alpha}g^{\alpha\beta}g_{\nu\beta}dx^\beta = \frac{(mc)^2}{g_{\mu\nu}dx^\nu dx^\sigma g_{\sigma\alpha}dx^\alpha}g^{\alpha\beta}g_{\nu\beta}dx^\beta = (mc)^2.
\]  
(2.16)

This gives a constraint
\[
G(x, p) = g^{\mu\nu}(x)p_\mu p_\nu - (mc)^2 = 0.
\]  
(2.17)

For the next example, we consider a non-relativistic particle under a potential force. The corresponding Finsler metric is
\[
F = \frac{mg_{ij}(x)dx^idx^j}{2dx^0} - V(x)dx^0, \quad (i, j = 1, 2, 3).
\]  
(2.18)

With an arbitrary time parameter \( \tau \), the Euler–Lagrange equations are
\[
0 = \frac{m\partial g_{ij}(x)dx^idx^j}{2dx^0} - \partial_0 Vx^0 - \frac{d}{d\tau}\left( \frac{mg_{ij}(x)dx^idx^j}{2(dx^0)^2} - V \right),
\]  
(2.19)

\[
0 = \frac{m\partial g_{ij}(x)dx^idx^j}{2dx^0} - \partial_0 Vx^0 - \frac{d}{d\tau}\left( \frac{mg_{ij}(x)dx^idx^j}{dx^0} \right),
\]  
(2.20)

where \( \ddot{x}^\mu = \frac{dx^\mu}{d\tau} \). If \( g_{ij} \) and \( V \) do not depend on \( x^0 \), the equation for \( \mu = 0 \) represents the energy conservation law
\[
\frac{d}{d\tau}\left( \frac{mg_{ij}(x)dx^idx^j}{dx^0} + V \right) = 0.
\]  
(2.21)

Taking \( \tau = x^0 \), we get the usual expression of the equations of motion and the energy conservation law.

The conjugate momenta are
\[
p_0 = \frac{\partial F}{\partial x^0} = \frac{mg_{ij}(x)dx^idx^j}{2(dx^0)^2} - V,
\]  
(2.22)

\[
p_i = \frac{\partial F}{\partial x^i} = \frac{mg_{ik}(x)dx^k}{dx^0}.
\]  
(2.23)

Using the inverse \( g^{ij} \) of \( g_{ij} \), we have
\[
g^{ij}p_ip_j = \frac{m^2g^{ij}g_{ik}dx^kdx^l}{(dx^0)^2} = \frac{m^2g_{ik}dx^kdx^l}{(dx^0)^2}.
\]  
(2.24)
Substituting the above relation into the expression \( p_0 \), we obtain
\[
p_0 = -\frac{1}{2m} g^{ij}(x)p_ip_j - V(x). \tag{2.25}
\]

\( p_0 \) is often written as \(-E\) with the energy \( E \) of the system, and we have a constraint
\[
G(x, p) = -E + \frac{1}{2m} g^{ij}(x)p_ip_j + V(x) = 0. \tag{2.26}
\]

Hamilton’s principal function \( W \) is defined as a line integral along a solution curve \( c \) of \((2.6)\),
\[
W(\xi) = \int_c F = \int_{\tau_0}^{\tau_1} F\left(x^\mu(\tau), \frac{dx^\mu}{d\tau}(\tau)\right) d\tau,
\]
where \( c \) is a solution curve connecting between \( x^\mu(\tau_0) \in M \) and \( \xi = (x^\mu(\tau)) \in M \).

When \( \xi_0 \) is fixed, \( W \) can be considered as a function on \( M, W: \xi \in M \Rightarrow W(\xi) \in \mathbb{R} \).

Let us introduce an infinitesimal transformation \( \epsilon \rightarrow \epsilon + \delta \epsilon, x^\mu \rightarrow x^\mu + \delta x^\mu \) which extends the solution curve infinitesimally from the endpoint \( x^\mu(\tau_1) \) toward the outside, while leaving the rest, including \( x^\mu(\tau_0) \), unchanged. This transformation changes the Hamilton’s principal function as
\[
\delta W = \int_{c+\delta c} F = \left[ \frac{\partial F}{\partial x^\mu} \delta x^\mu \right]_{\tau_0}^{\tau_1} + \int_{\tau_0}^{\tau_1} \left[ \frac{\partial F}{\partial x^\mu} - \frac{\partial}{\partial x^\nu} \left( \frac{\partial F}{\partial x^\mu} \right) \right] d\delta x^\mu = \frac{\partial F}{\partial x^\mu} \delta x^\mu. \tag{2.28}
\]

The last equality is derived from the fact that \( \delta x^\mu(\tau_0) = 0 \) and the on-shell condition \( \frac{\partial}{\partial x^\nu} \left( \frac{\partial F}{\partial x^\mu} \right) = 0 \). Since the conjugate momenta \( p_\nu \) are defined by \( \frac{\partial F}{\partial x^\nu} \), the above relation means
\[
dW = p_\mu d\delta x^\mu. \tag{2.29}
\]

The extended phase space is the hypersurface determined by the constraint \( G(x, p) = 0 \) in the cotangent bundle \( T^*M = \{ (x^\mu, p_\mu) \} \), and \( p_\mu = \frac{\partial W}{\partial x^\nu} \) should accordingly satisfy
\[
G\left(x^\mu, \frac{\partial W}{\partial x^\nu}\right) = 0. \tag{2.30}
\]

This is the covariant expression of the Hamilton–Jacobi equation, which is necessarily derived from the homogeneity condition \((1)\) in the Finsler–Lagrangian formulation. Taking a relativistic free particle and a non-relativistic particle under a potential as examples, we have
\[
g^{\mu\nu} \frac{\partial W}{\partial x^\mu} \frac{\partial W}{\partial x^\nu} = (mc)^2, \quad \frac{\partial W}{\partial x^\mu} \frac{\partial W}{\partial x^\nu} + \frac{1}{2m} g^{ij}(x) \frac{\partial W}{\partial x^i} \frac{\partial W}{\partial x^j} + V(x) = 0, \tag{2.31}
\]
which are exactly the same as the standard Hamilton–Jacobi equations.

### 2.2. Second law of thermodynamics as variational principle

Let \( \delta Q \) be the quantity of heat flowing into the system from the environment of temperature \( T^{\text{ex}} \) during an infinitesimal process, and \( dS \) be the difference in entropy between the initial and final equilibrium states. The second law of thermodynamics can be written as
\[
dS \geq \frac{\delta Q}{T^{\text{ex}}}. \tag{2.32}
\]

If the equality is satisfied, the thermal process is reversible. On the other hand, the inequality represents an irreversible process. Finsler geometry is historically introduced as a geometry of variational principle. The second law of thermodynamics, therefore, implies that there is a
Finsler structure. On that account, we will assume the right-hand side is supposed to be given by an integration of some Finsler metric defined on thermodynamic state space \( M = \{(U, V)\} \), where \( U \) is the energy and \( V \) the volume of the system [14]

\[
\int_{a \to b} \frac{\delta Q}{T} = \int_{\gamma^{a,b}} F(U, V, dU, dV),
\]

(2.33)

where \( \gamma^{a,b} \) is an oriented curve on \( M \) which represents the thermal process between the equilibrium states \( a \) and \( b \). Reversible processes are the paths which maximize the integral (2.33), and integral of \( \frac{\delta Q}{T} \) on a reversible process: \( a \to b \) becomes the entropy difference between \( a \) and \( b \)

\[
S(b) - S(a) = \int_{\gamma^{a,b}} F(U, V, dU, dV),
\]

(2.34)

and the maximal (stationary) integral of the RHS of (2.33) gives the Hamilton’s principal function \( W \). Therefore Hamilton’s principal function in thermodynamics is identical to the entropy function: \( W = S \). Thus, we get the relation

\[
dW = p_U dU + p_V dV = dS = \frac{1}{T} dU + \frac{p}{T} dV,
\]

(2.35)

where \( p_U = \frac{\partial F}{\partial U} \) and \( p_V = \frac{\partial F}{\partial V} \) are conjugate momenta of \( U \) and \( V \), and the third equality of (2.35) is the first law of thermodynamics. From the above equation (2.35), we can conclude the covariant conjugate momenta of \( (U, V) \) are

\[
(p_U, p_V) = \left( \frac{1}{T}, \frac{p}{T} \right).
\]

(2.36)

The constraint from the Finsler–Lagrangian formulation \( G(x, p) = 0 \) turns into an equation:

\[
G\left(U, V, \frac{1}{T}, \frac{p}{T}\right) = 0.
\]

(2.37)

In thermodynamics, a relation between \( (U, V, \frac{1}{T}, \frac{p}{T}) \) is called virial relation (equation of state), which is naturally identified with the above constraint. Taking the Hamilton–Jacobi formalism into account, we have the Hamilton–Jacobi equation

\[
G\left(U, V, \frac{\partial S}{\partial U}, \frac{\partial S}{\partial V}\right) = 0,
\]

(2.38)

as an equation to determine the entropy \( S \).

In the case of the ideal gas, it has the internal energy \( U = \frac{3}{2} N k T \) and the equation of state \( pV = N k T \), where \( N \) is the number of the gas particles, \( k \) the Boltzmann constant, \( T \) temperature, \( p \) pressure, \( V \) volume of the gas. Its virial relation is

\[
U = \frac{3}{2} pV.
\]

(2.39)

With (2.36), it becomes

\[
G(U, V, p_U, p_V) = p_U U - \frac{3}{2} p_V V = 0.
\]

(2.40)

From this virial equation and

\[
p_U = \frac{\partial S}{\partial U}, \quad p_V = \frac{\partial S}{\partial V},
\]

(2.41)
we can derive following Hamilton–Jacobi equation of the ideal gas:

\[ \frac{U}{\partial U} \frac{\partial S}{\partial U} - \frac{3V}{2} \frac{\partial S}{\partial V} = 0. \] (2.42)

The solution of the partial differential equation (2.42) gives the entropy of the ideal gas

\[ S = S(U, V) = \frac{3}{2} r \log U + r \log V + S_0, \] (2.43)

where \( r \) and \( S_0 \) are constants. Using (2.41), we also have

\[ p_U = \frac{1}{T} = \frac{3r}{2U}, \quad p_V = \frac{p}{V} = \frac{r}{V}, \] (2.44)

which reproduce the internal energy and state equation of the ideal gas. It is believed that the definition of the ideal gas needs both relations. However, the procedure of this section tells us that only the virial relation is needed, and through the Hamilton–Jacobi equation, the rest follows.

### 3. Ising model in mean field approximation

We apply the formulation reviewed in section 2 to a spin system. The Hamiltonian of the Ising model in the mean field approximation is expressed as

\[ H = \frac{N J z m^2}{2} - J z m \sum_{i=1}^{N} S_i, \] (3.1)

when there is no magnetic field. Here, \( N \) is the total site number, \( J \) the strength of the interaction, and \( z \) the coordination number. \( m \) is the expectation value of an Ising spin \( S_i = \pm 1 \), which admits the self-consistent equation. We start with the grand canonical ensemble

\[ \Xi(\beta, \xi) = \sum_{\text{configuration}} \exp \left[ -\beta \left( \frac{N J z m^2}{2} - J z m \sum_{i=1}^{N} S_i \right) - \xi \sum_{i=1}^{N} S_i \right]. \] (3.2)

The function \( \Xi(\beta, \xi) \) is a grand partition function for parameters \( \beta \) and \( \xi \). We write the probability for a configuration \( r \) as \( p_r = \frac{1}{Z} \exp \left[ -U_r \right] \), where \( U_r = \frac{N J z m^2}{2} - J z m \sum_{i=1}^{N} (S_i)_r \), \( M_r = \sum_{i=1}^{N} (S_i)_r \), and the expected value of some physical quantity \( A \) is calculated as \( \langle A \rangle = \sum_r p_r A_r \). The parameters \( \beta \) and \( \xi \) correspond to \( \frac{1}{kT} \) and \( \frac{\hbar}{kT} \), respectively, which are confirmed for consistency with the thermodynamics at the end of this paragraph. \( \beta \) is related to the fluctuation of the internal energy \( \langle U \rangle \) and \( \xi \) is a parameter associated with the fluctuation of the total magnetization \( \langle M \rangle \). Throughout this section, we assume \( \xi \) to be non-zero, since an infinitesimally small magnetic field is necessary for the phase transition. The grand Massieu function \( \Psi(\beta, \xi) = k \log \Xi \) generates the magnetization \( \langle M \rangle \) and the internal energy \( \langle U \rangle \) as

\[ \langle M \rangle = -\frac{1}{\beta} \frac{\partial \Psi}{\partial \xi} = N \tanh \left( \frac{\beta J z \langle M \rangle}{N} - \xi \right). \] (3.3)

\[ \langle U \rangle = -\frac{1}{\beta} \frac{\partial \Psi}{\partial \beta} = -\frac{J z \langle M \rangle^2}{2N}. \] (3.4)
The equation (3.3) is the self-consistent equation and (3.4) gives a relation between \(\langle U \rangle\) and \(\langle M \rangle\) which should be kept all the time. The Shannon entropy is given by

\[
S = -k \sum_r p_r \log p_r \\
= -k \sum_r p_r (-\beta U_r - \xi M_r - \log \Xi) \\
= k \beta \sum_r p_r U_r + k \xi \sum_r p_r M_r + k \log \Xi \\
= \Psi - \beta \frac{\partial \Psi}{\partial \beta} - \xi \frac{\partial \Psi}{\partial \xi} = \Psi + k \beta \langle U \rangle + k \xi \langle M \rangle.
\]

(3.5)

Through this entropy, we make a connection with the thermodynamics. From now on, we abbreviate \(\langle U \rangle\) and \(\langle M \rangle\) to \(U\) and \(M\). The total derivative of the above entropy \(S\) is calculated as

\[
dS = d\Psi + kU d\beta + k\beta dU + kMd\xi + k\xi dM \\
= \frac{\partial \Psi}{\partial \beta} d\beta + \frac{\partial \Psi}{\partial \xi} d\xi + kU d\beta + k\beta dU + kMd\xi + k\xi dM \\
= -kU d\beta - kMd\xi + kU d\beta + k\beta dU + kMd\xi + k\xi dM \\
= k\beta dU + k\xi dM.
\]

(3.6)

It means the thermodynamic state space for this spin system is \(\{ (U, M) \}\) and the conjugate momenta \(p_U\) and \(p_M\) are

\[
p_U = \frac{\partial S}{\partial U} = k\beta, \quad p_M = \frac{\partial S}{\partial M} = k\xi.
\]

(3.7)

Additionally, (3.6) has information on the relation between the parameters \((\beta, \xi)\) and the temperature \(T\) and the magnetic field \(h\). From thermodynamic prediction, the energy change should be given by

\[
dU = TdS - hdM, \quad \text{or} \quad dS = \frac{1}{T}dU + \frac{h}{T}dM.
\]

(3.8)

Comparing it to the relation (3.6), we should take \(\beta = \frac{1}{T}\) and \(\xi = \frac{h}{T} = \beta h\).

Though the self-consistent equation (3.3) is a constraint, it should be rewritten in terms of \((U, V, p_U, p_M)\) to consider it as the one appearing in the Finsler–Lagrangian formulation. The first thing to do is to remove the statistical quantity \(N\) from the equation (3.3). From the relation (3.4), we have

\[
m = \frac{M}{N} = -\frac{2U}{JzM},
\]

(3.9)

which should take a value between \(\pm 1\). Substituting it into (3.3) to get rid of \(N\), we obtain our virial equation for the mean field Ising model:

\[
\beta \frac{2U}{M} + \xi = \tanh^{-1} \left( \frac{2U}{JzM} \right).
\]

(3.10)

It transforms into

\[
\frac{2U}{kM} \frac{\partial S}{\partial U} + \frac{1}{k} \frac{\partial S}{\partial M} = \tanh^{-1} \left( \frac{2U}{JzM} \right).
\]

(3.11)
after substituting the derivatives of \( S \) for \((\beta, \xi)\) in (3.10) using (3.7). This is the Hamilton–Jacobi equation for the Ising model in mean field approximation.

By solving the partial differential equation (3.11) directly, assuming homogeneity of \( S \) with respect to \((U, M)\), we find the general solution for the entropy as

\[
S = kM \tanh^{-1} \left( \frac{2U}{JzM} \right) + \frac{kJzM^2}{4U} \log \left( 1 - \left( \frac{2U}{JzM} \right)^2 \right) + aM^2U, \tag{3.12}
\]

where \( a \) is an arbitrary constant. The last term is set to be a linear function of \( M^2 \) because of the extensive property of the entropy. When \( U \) takes a value 0, this term pushes the entropy out to infinity, which makes it unphysical. Therefore we choose \( a = 0 \) for a physical solution.

After evaluating (3.7) and substituting (3.4), we obtain

\[
\beta = -\frac{JzM^2}{4U^2} \log \left( 1 - \frac{2U}{JzM} \right) = -\frac{1}{Jz^2m^2} \log (1 - m^2), \tag{3.13}
\]

\[
\xi = \tanh^{-1} \left( \frac{2U}{JzM} \right) + \frac{JzM}{2U} \log \left( 1 - \left( \frac{2U}{JzM} \right)^2 \right) = -\tanh^{-1}m - \frac{1}{m} \log (1 - m^2). \tag{3.14}
\]

By substituting the above relations, the entropy (3.12) becomes

\[
S = -kNm \tanh^{-1}m - \frac{km}{2} \log (1 - m^2), \tag{3.15}
\]

which is identical to what is derived from (3.5) except for a constant term. The expression (3.12), or (3.13)–(3.14), has much more information than (3.15), since \( \beta \) and \( \xi \) are parametrized by \( m \). It produces a curve in \((\beta, \xi, m)\) space as shown in figure 1, which resembles a partial curve of the famous cusp catastrophe surface. \((\beta, \xi)\) can be considered as control parameters which correspond to the splitting factor and the normal factor in catastrophe terminology, respectively. Figure 2 shows the projection of figure 1 onto the \((\beta, \xi, m)\) plane, and we observe a cusp exactly at the critical point \( kT_c = Jz \) and \( \delta h = 0 \) (\( \beta = 1 \) for \( k = Jz = 1 \)). Since the energy \( U \) is also considered a function of \( m \), we can exhibit the combination \((\beta, \xi, U)\) simultaneously (figure 3) to see a drastic change in the energy at the critical point. This fact gives a simple and new aspect of the critical phenomenon as the singularity theory.

Geometrically thinking, the critical exponents should be determined along the path to the critical point in this \((\beta, \xi, m)\) space. We verify that the curve in figure 1 is the path for the mean field approximation. The equations (3.13) and (3.14) expand as

\[
\beta = \frac{1}{Jz} + \frac{m^2}{2Jz} + \frac{m^4}{3Jz} + \cdots, \tag{3.16}
\]

\[
\xi = \frac{m^3}{6} + \frac{2m^5}{15} + \cdots. \tag{3.17}
\]

These expansions show \((\beta - \frac{1}{T})^3 \propto \xi^2\) around \( m \simeq 0 \), meaning a cusp catastrophe. From the second equation, we have \( \delta = 3 \) for \( m \propto |h|^{\frac{3}{2}} \). The reduced temperature \( t \) behaves as

\[
t = \frac{T - T_c}{T_c} = \frac{Jz\beta - 1}{Jz\beta} \simeq \frac{m^2}{2}, \tag{3.18}
\]
Figure 1. Solution curve in $(\beta, \xi, m)$.

Figure 2. Graph of $(\beta, \xi)$. 
so that it gives the exponent $\beta = \frac{1}{2}$ for $m \propto |t|^2$. The differential $\frac{d\mu}{dm} = \frac{d\beta}{dm} + \beta \frac{dh}{dm}$ leads the magnetic susceptibility $\chi$ to

$$\chi = \frac{dm}{dh} = \frac{\beta m^2 (1 - m^2)}{m^2 + (1 - m^2) \log(1 - m^2)} \approx \frac{\beta}{1 - Jz \beta} = \frac{1}{kT - kT_c}. \quad (3.19)$$

Thus we have $\gamma = 1$ for $\chi \propto |t|^{-\gamma}$. The specific heat $C$ becomes

$$C = \frac{dU}{dT} = -\frac{JzN}{2} \frac{dm^2}{dT} - JzN \frac{d}{dT} \left( \frac{T_c}{T} \right) = kN. \quad (3.20)$$

by substituting (3.4) and (3.18), and we get $\alpha = 0$ for $C \propto |t|^{-\alpha}$. All these exponents are the same as the standard results.

Points on the solution curve appearing in three-space $\mathbb{R}^3 = \{ (\beta, \xi, m) \}$ in figure 1 satisfy a relation

$$f(u, v, m) = m^3 - um + v = 0, \quad u = 3(Jz\beta - 1), \quad v = 3\xi,$$

with the expansion $\tanh^{-1} m \approx m + \frac{m^3}{3}$. This equation is exactly the same as the equation which defines the cusp catastrophe surface. The surface $M_f$ which is defined by the above equation in $\{ (u, v, m) \}$ space is parametrized by two parameters $(u, m)$ with $v = -m^3 + um$. A cusp catastrophe map is defined as a map from $M_f$ to the control parameter space $\mathbb{R}^2 = (u, m)$ as $\chi_{j}$: $M_f \rightarrow \mathbb{R}^2$, $(u, v(u, m), m) \mapsto (u, v(u, m))$. Its Jacobi matrix is

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**Figure 3.** Graph of $(\beta, \xi, U)$. 

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and it is well known that the critical (singular) set for the map \( \chi_f \) on \( \mathcal{M}_f, u = 3m^2 \), forms a fold, which maps to a bifurcation set with a cusp. This bifurcation set corresponds to our curve in figure 2. The exact solution for the higher-dimensional Ising model with non-zero magnetic field can define an exotic surface as the cusp catastrophe surface in \( \{(\beta, \xi, m)\} \) space by considering the equation \( m = -\frac{1}{2N} \partial \phi \). However, it generally cannot define the unique exponents since there are infinite numbers of paths approaching the critical point. In contrast, the mean field Ising model gives a curve as seen in figure 1, so that we can define the unique exponents.

4. Discussion

We calculate the entropy of the Ising model in the mean field approximation as a Hamilton’s principal function on thermodynamic state space \( \{(U, M)\} \). Despite the fact that the Ising model is a statistical model, this formalism pushes the site number \( N \) away from the last results. It extracts the thermodynamical state, which has a clear singularity at the critical point. The critical exponents are uniquely determined along the solution curve. Standard calculations in various textbooks unknowingly assume this curve to derive the correct exponents.

Since our result shows a strong relationship with the catastrophe theory, we expect that there is a cusp catastrophe surface in the space \( \{(\beta, \xi, m)\} \) which includes the solution curve in figure 1, and it should be the exact solution of the Ising model. Our result suggests that physical critical phenomena can be studied by the singularity theory, and it also helps us acquire a more intuitive and simple geometric view.

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