Dirac Operator Zero-modes on a Torus

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Abstract

We study Dirac operator zero-modes on a torus for gauge background with uniform field strengths. Under the basic translations of the torus coordinates the wave functions are subject to twisted periodic conditions. In a suitable torus coordinates the zero-mode wave functions can be related to holomorphic functions of the complex torus coordinates. We construct the zero-mode wave functions that satisfy the twisted periodic conditions. The chirality and the degeneracy of the zero-modes are uniquely determined by the gauge background and are consistent with the index theorem.
1 Introduction

Toric geometry appears in periodic systems. In many interesting cases background gauge field is also introduced. We may then consider topologically nontrivial gauge fields. They affects the spectrum of the hamiltonian in an interesting way even if the charged particle does not touch the magnetic field. In Dirac theory on a compact space we have chiral zero-modes for gauge background with nonvanishing topological charge. This is well-known as the Atiyah-Singer index theorem [1]. It can be used as a mechanism of keeping fermions massless in Kaluza-Klein compactifications.

If the gauge fields carry topological information of the periodic system, it cannot be periodic but changes by a gauge transformation under translations by a period of the system. Similarly the boundary condition for wave functions of a charged particle becomes twisted from the naive periodic boundary condition. This makes the task of solving eigenvalue problems nontrivial. In this paper we investigate Dirac fields in an abelian gauge background of constant field strength on a torus and construct zero-mode solutions of the Dirac operator. They are interesting in gauge theories in connection to chiral anomaly [2, 3] and also in Kaluza-Klein scenario as a mechanism to have massless fermions. In Ref. [4] Sakamoto and Tanimura developed Fourier analysis for wave functions satisfying twisted boundary conditions on an arbitrary torus. They analyzed the eigenvalue problem of magnetic laplacian. Their arguments are based on the representation theory of magnetic translation group [5, 6, 7]. Though their approach is powerful and systematic, it is possible to find Dirac operator zero-modes in a way that is more accessible to physicists. Namely, we directly solve massless Dirac equation on a euclidean torus and find solutions satisfying twisted boundary conditions. We show that the zero-mode wave functions can be expressed as gaussian wave function times a function of complex torus coordinates. We find that the twisted boundary conditions determine the zero-mode wave functions up to an overall normalization.

This paper is organized as follows. In the next section we argue general features of gauge potentials on a torus. We also show periodicity conditions for wave functions. In Sect. we solve the massless Dirac equation on $T^2$. This system exhibits some characteristics of Dirac operator zero-modes on tori. We show that the zero-mode wave functions are related to holomorphic or anti-holomorphic functions satisfying periodicity conditions with a twist. In Sect. Dirac operator zero-modes in an arbitrary even dimensions are analyzed for field strengths of a standard form. Most general field strengths are treated in Sect. The zero-mode wave functions can be related to holomorphic functions. We derive the periodicity conditions for the holomorphic functions. In Sect. we solve the periodicity conditions and give a construction of zero-mode wave functions. Sect. is devoted to summary. A periodicity relation for general translations is derived in Appendix A. In the appendices and we give proofs of the key mathematical facts that are used in this paper. The normalization constant of the zero-mode wave functions is given in Appendix D.
2 Gauge Potential on a Torus

A $d$ dimensional torus $T^d$ can be regarded as a quotient space $\mathbb{R}^d/\Lambda$, where $\Lambda$ is some $d$ dimensional lattice. Or equivalently, it can be obtained from a $d$ dimensional parallelogram by identifying its opposite $d-1$ dimensional faces. In this paper we take a hypercubic regular lattice $\Lambda = \mathbb{Z}^d$ generated by the standard basis vectors $e_a$ ($a = 1, \ldots, d$) with $(e_a)^b = \delta^b_a$ and define $T^d = \mathbb{R}^d/\Lambda^d$ by identifying all the points that differ by a vector in $\Lambda$. In Ref. [4] a more general torus is considered by taking an arbitrary set of vectors in place of the orthonormal basis $e_a$. It is straightforward to carry out this kind of generalization in the following development.

Abelian gauge fields on $T^d$

\[ A = \sum_a A_a(x)dx^a \]  \hspace{1cm} (2.1)

are topologically classified by a set of integers $\phi_{ab}$ given by

\[ \phi_{ab} = \frac{1}{2\pi} \int_{D_{ab}} F, \]  \hspace{1cm} (2.2)

where $D_{ab}$ is an $(a,b)$-plane taken in $T^d$ and $F$ is the field strength 2-form

\[ F = \frac{1}{2} \sum_{a,b} F_{ab} dx^a \wedge dx^b = dA. \]  \hspace{1cm} (2.3)

We see that $2\pi \phi_{ab}$ is the flux through $D_{ab}$.

That $\phi_{ab}$ must be an integer can be understood by considering a parallel transport of a wave function along a closed curve $C$ taken in $D_{ab}$ as depicted in Fig. 1. The phase factor arising from the parallel transport along $C$ is given by

\[ \exp\left[-i \int_C A\right]. \]  \hspace{1cm} (2.4)

In applying the Stokes theorem there are essentially two different possibilities of choosing a two-dimensional surface with $C$ as its boundary. One is the region enclosed by $C$, the shaded region of Fig. 1(a). It is contractible to a point. The other is the shaded region of Fig. 1(b). This is also an allowed surface since the left side of the square is identified with the right and the upper side with the lower. These two must yield the same phase factor. This implies

\[ \exp\left[-i \int_{D_{ab}} F\right] = \exp[-2\pi i \phi_{ab}] = 1. \]  \hspace{1cm} (2.5)

We thus find that $\phi_{ab}$ must be an integer.
Figure 1: Two possible choices of two-dimensional surface with $C$ as its boundary. The square denotes the $(a,b)$-plane $D_{ab}$ taken in $T^d$.

The gauge potential $A$ cannot be single-valued on $T^d$ if it belongs to a topologically nontrivial sector. However, the field strength $F$ must be well-defined on $T^d$. In other words, it must satisfy the periodic boundary conditions

$$F_{ab}(x + e_c) = F_{ab}(x). \quad (2.6)$$

This implies that $A_a(x)$ and $A_a(x + e_b)$ can only differ by a gauge transformation as

$$A_a(x + e_b) = A_a(x) + \partial_a \lambda_b(x), \quad (2.7)$$

where $\lambda_b$ may be a function on $\mathbb{R}^d$. This also gives rise to the transformation property of a wave function $\psi$ under the translations as

$$\psi(x + e_a) = g_a(x)\psi(x), \quad g_a(x) = e^{i\lambda_a(x)}, \quad (2.8)$$

where we are assuming that the covariant derivative is given by

$$D_a\psi(x) = (\partial_a - iA_a(x))\psi(x). \quad (2.9)$$

In order for the translations $(2.8)$ to give a unique wave function under the combined translations $x \rightarrow x + e_a \rightarrow x + e_a + e_b$ and $x \rightarrow x + e_b \rightarrow x + e_a + e_b$, the $g_a(x)$ must satisfy the cocycle conditions

$$g_a(x + e_b)g_b(x) = g_b(x + e_a)g_a(x). \quad (2.10)$$

That these are fulfilled can be seen by considering the parallel transport of the wave function along the boundary of the square $D$ as depicted in Fig. 2. The flux through $D$ can be computed as

$$\int_D F = \int_{\partial D} A = \lambda_a(x + e_b) - \lambda_a(x) - \lambda_b(x + e_a) + \lambda_b(x). \quad (2.11)$$

*We include the coupling constant in the definition of gauge potential.*
In deriving this use has been made of (2.7). Since the flux through \( D \) is equal to \( 2\pi \) times an integer, we see that the cocycle conditions (2.10) are satisfied.

We now turn to gauge potential with uniform field strength. In this case \( F_{ab} \) can be written as

\[
F_{ab} = 2\pi \phi_{ab}. \tag{2.12}
\]

The gauge potential 1-form \( A \) in axial gauge is given by

\[
A = \sum_{a<b} F_{ab} x^a dx^b + \sum_b a_b dx^b. \tag{2.13}
\]

Since we can freely shift the constant part \( a_b \) by \( 2\pi \) by carrying out a gauge transformation on \( T^d \) given by \( g(x) = e^{2\pi i x^b} \), we can restrict \( a_b \) to be in the interval \( 0 \leq a_b < 2\pi \). Furthermore, if \( \det F_{ab} \neq 0 \), we can remove the constant part of \( A_a \) by a suitable shift of the coordinates \( x \). This is not obvious in axial gauge. We come back to this point soon in connection to symmetric gauge. Henceforth, we restrict ourselves to the case \( \det F_{ab} \neq 0 \) and choose the coordinates so as for the gauge potential \( A \) to be given by

\[
A = \sum_{a<b} F_{ab} x^a dx^b. \tag{2.14}
\]

This satisfies complete axial gauge conditions

\[
A_1(x_1, \cdots, x_d) = A_2(0, x_2, \cdots, x_d) = A_3(0, 0, x_3, \cdots, x_d) = \cdots = A_d(0, \cdots, 0, x_d) = 0.
\tag{2.15}
\]

The gauge potential (2.14) satisfies the periodicity

\[
A(x + e_a) = A(x) + d\lambda_a(x), \tag{2.16}
\]
where $\lambda_a(x)$ is given by

$$\lambda_a(x) = \sum_{b=a+1}^{d} F_{ab} x^b.$$  \hfill (2.17)

Due to (2.12) the gauge transformation $g_a(x)$ in (2.8) is periodic in $\mathbb{R}^d$, i.e.,

$$g_a(x + e_b) = g_a(x).$$  \hfill (2.18)

We can regard it as a gauge transformation on $T^d$. The wave function $\psi$ must satisfy the periodicity (2.8) with $\lambda_a$ given by (2.17).

Though the axial gauge is useful in our analysis, it lacks the rotational covariance. The shortcoming can be remedied by using the gauge potential in symmetric gauge given by

$$\tilde{A}_a(x) = \frac{1}{2} \sum_{b=1}^{d} F_{ba} x^b.$$  \hfill (2.19)

It is related to the potential in axial gauge by a gauge transformation in $\mathbb{R}^d$

$$\tilde{A}_a = A_a + \partial_a \Lambda, \quad \Lambda = -\frac{1}{2} \sum_{a<b} x^a x^b F_{ab}.$$  \hfill (2.20)

The wave function in symmetric gauge is related to the wave function in axial gauge by

$$\tilde{\psi}(x) = e^{i \Lambda(x)} \psi(x).$$  \hfill (2.21)

The gauge potential (2.19) is not of a most general form. We can add an arbitrary constant term to the potential without affecting the field strength. It can be removed, however, by a suitable shift of the coordinates $x$ if the field strength satisfies $\det F_{ab} \neq 0$.

3 Massless Dirac Equation on $T^2$

To solve massless Dirac equation in a background gauge field with constant field strength we need several machinery. In two dimensions, however, the problem is rather simple but its solution is illuminating to understand higher dimensional cases. In this section we give a construction of Dirac operator zero-modes on $T^2$ to illustrate some points that also apply in higher dimensional cases.

The Dirac operator in two dimensions is given by

$$\slashed{D} = \sum_{k=1,2} \sigma^k D_k = \begin{pmatrix} 0 & \partial_x - i \partial_y - i (A_x - i A_y) \\ \partial_x + i \partial_y - i (A_x + i A_y) & 0 \end{pmatrix}$$  \hfill (3.1)
where $\sigma$’s are the Pauli matrices and the gauge potential for a constant field strength $F_{12} = B$ in axial gauge is given by

$$A_x = 0, \quad A_y = Bx. \quad (3.2)$$

We can write $B = 2\pi N\nu$, where $\nu$ is a positive integer and $N = \text{sgn}B$. In two dimensions the distinction between $B$ and $2\pi\nu$ is not essential and the introduction of $\nu$ and $N$ might be considered redundant. As we will see in later sections, the distinction becomes crucial in the case of general uniform field strength in higher dimensions.

The zero-mode of the Dirac operator is a solution to the euclidean massless Dirac equation

$$\not{D} \psi = 0. \quad (3.3)$$

In terms of components

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \quad (3.4)$$

the Dirac equation (3.3) is given by

$$(\partial_x + is\partial_y + sBx)\psi_s = 0. \quad (3.5)$$

The $s = \pm$ label the chiral components of $\psi_s$ if we define the chiral matrix by $\Gamma_3 = -i\sigma_1\sigma_2 = \sigma_3$. We see from (3.5) that $\psi_s$ can be written as

$$\psi_s(x, y) = e^{-\frac{i}{2}sBx^2}f_s(x + isy), \quad (3.6)$$

where $f_s$ is a function of $x + isy$ to be determined shortly.

Under the translations $x \rightarrow x + e_a$ ($a = 1, 2$), the gauge potential (3.2) is changed by a gauge transformation $\lambda_a(x) = By\delta_{a1}$. This gives the periodicity conditions on $\psi$ as

$$\psi(x + 1, y) = e^{iBy}\psi(x, y), \quad \psi(x, y + 1) = \psi(x, y) \quad (3.7)$$

These can be transcribed into the conditions for $f_s(x + isy)$ as

$$f_s(x + sN + isy) = e^{2\pi\nu(x + isy + \frac{1}{2}sN)}f_s(x + isy),$$

$$f_s(x + is(y + N)) = f_s(x + isy). \quad (3.8)$$

Since $f_s(x + isy)$ are periodic under the translation $y \rightarrow y + N$, we can Fourier expand them as

$$f_s(x + isy) = \sum_n c_n^s e^{2\pi n(x + isy)}. \quad (3.9)$$
The Fourier coefficients $c^s_n$ can be determined from the first condition of (3.8). They must satisfy the recursion relation

$$c^s_n = e^{-\pi sN(2n-\nu)}c^s_{n-\nu}. \quad (3.10)$$

By putting $n = \nu k + r$ with $k \in \mathbb{Z}$ and $r = 0, 1, \cdots, \nu - 1$, these can be solved as

$$c^s_{\nu k + r} = c^s_r e^{-\pi sN(\nu k^2 + 2rk)}, \quad (3.11)$$

where $c^s_r$ are arbitrary constants. The $f_s(x + isy)$ is now given by

$$f_s(x + isy) = \sum_{r=0}^{\nu-1} \sum_{k=-\infty}^{+\infty} e^{-\pi sN(\nu k^2 + 2rk) + 2\pi(\nu k + r)(x + isy)}. \quad (3.12)$$

Since $B = 2\pi \nu N$, the sum defining $f_-$ does not converge if $B > 0$. This implies that $c^-_r$ and, hence, $f_-$ must vanish. There are only $\nu$ independent $f_+$ corresponding to an arbitrary choice of $c^+_r$. We thus obtain $\nu$ independent zero-modes with positive chirality. Conversely, $f_+$ does not exists for $B < 0$ and there are $\nu$ independent $f_-$, giving zero-modes with negative chirality.

The absence of zero-mode of $D_1 - iD_2$ for $B > 0$ can be shown without solving the Dirac equation. Using

$$(D_1 + iD_2)(D_1 - iD_2) = D_1^2 + D_2^2 - B, \quad (3.13)$$

we get

$$\int_{T^2} d^2x |(D_1 - iD_2)\psi_-|^2 = \int_{T^2} d^2x (|D_1\psi_-|^2 + |D_2\psi_-|^2 + B|\psi_-|^2). \quad (3.14)$$

The rhs of this expression cannot vanish if $\psi_- \neq 0$. This implies that $D_1 - iD_2$ has no nontrivial zero-modes for $B > 0$. Conversely, $D_1 + iD_2$ has no zero-mode if $B < 0$.

Combining (3.6) and (3.12), we finally obtain the normalized zero-modes

$$\psi_{r,R} = u_+ \psi_{r,+}(x, y), \quad (B > 0)$$

$$\psi_{r,L} = u_- \psi_{r,-}(x, y), \quad (B < 0) \quad (3.15)$$

where we have introduced the basis of two-component chiral spinors $u_\pm$ given by

$$u_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (3.16)$$

The $\psi_{r,s}(x, y)$ are defined by

$$\psi_{r,s}(x, y) = (2\nu)^{\frac{3}{4}} \sum_{k=-\infty}^{+\infty} e^{-\frac{\pi sN}{\nu}(sN x - k - r)^2 + 2\pi is\nu k + ry}. \quad (3.17)$$

$$= (2\nu)^{\frac{3}{4}} \sum_{n \equiv r \pmod{\nu}} e^{-\frac{\pi sN}{\nu}(sN x - n)^2 + 2\pi isy}. \quad (3.17)$$
where the last sum is taken over integers of the form \( n = \nu k + r \) \((k \in \mathbb{Z})\). The overall factor is so chosen that the wave functions satisfy the orthonormality
\[
\int_{T^2} d^2 x \psi^*_{r,s} \psi_{r',s} = \delta_{r,r'}.
\] (3.18)

The \( \psi_{rR} \) and \( \psi_{rL} \) are eigenstates of the chiral matrix \( \Gamma_3 \). If we denote the number of positive and negative chirality zero-modes by \( n_R \) and \( n_L \) and define the index of \( \mathcal{D} \) by
\[
\text{index } \mathcal{D} = n_R - n_L,
\] (3.19)
we find \( \text{index } \mathcal{D} = \nu N = B/2\pi \) irrespective of the sign of \( B \). This is completely consistent with the index theorem.
\[
\text{index } \mathcal{D} = \frac{B}{2\pi} = \frac{1}{2\pi} \int_{T^2} F.
\] (3.20)

### 4 Extension to Higher Dimensions

The analysis given in the previous section can be easily extended to arbitrary even dimensions \( d = 2n \) in the case that the field strength 2-form takes a special form
\[
F = \sum_{k=1}^{n} B_k dx^{2k-1} \wedge dx^{2k},
\] (4.1)
where \( B_k \) is a flux density through \((2k-1, 2k)-\)plane and \( \nu_k = |B_k|/2\pi \) is a positive integer. Most general uniform field strength can be made into the form (4.1) by a suitable coordinate rotation. In the rotated coordinates, however, the periodicity conditions become complicated. We postpone the analysis of the Dirac operator zero-modes for the most general uniform field strength to later sections. In this section we argue the Dirac operator zero-modes for the special gauge background given by (4.1) and develop some machinery needed in later sections.

The \( \gamma \)-matrices \( \Gamma^a_n \) \((a = 1, \ldots, d)\) in \( d = 2n \) dimensions can be obtained recursively by the relations
\[
\Gamma^1_n = \sigma^1 \otimes 1_{2^{n-1}}, \quad \Gamma^2_n = \sigma^2 \otimes 1_{2^{n-1}}, \quad \Gamma^a_n = \sigma^a \otimes \Gamma^a_{n-1}, \quad (a = 3, \ldots, d),
\] (4.2)
where \( 1_N \) is the \( N \times N \) unit matrix and \( \Gamma^a_n \) are \( 2^n \times 2^n \) matrices. Explicitly they are given by
\[
\Gamma^{2k+1}_n = \underbrace{\otimes \cdots \otimes}_{k \text{ times}} \sigma^3 \otimes \underbrace{\sigma^1 \otimes 1_2 \otimes \cdots \otimes 1_2}_{n-k-1 \text{ times}},
\]
\[
\Gamma^{2k+2}_n = \underbrace{\otimes \cdots \otimes}_{k \text{ times}} \sigma^3 \otimes \underbrace{\sigma^2 \otimes 1_2 \otimes \cdots \otimes 1_2}_{n-k-1 \text{ times}}.
\] (4.3)
The chiral matrix $\Gamma_{d+1}$ is given by
\[
\Gamma_{d+1} = (-i)^n \Gamma_n^1 \cdots \Gamma_n^d = \underbrace{\sigma^3 \otimes \cdots \otimes \sigma^3}_{n \text{ times}}
\] (4.4)

The gauge potential that gives rise to the field strength (4.1) is a simple extension of (3.2) and is given by
\[
A_{2k-1} = 0, \quad A_{2k} = B_k x^{2k-1},
\] (4.5)

The Dirac operator is then given by
\[
\not{D} = \Gamma_n^a (\partial_a - i A_a) = \sum_{k=0}^{n-1} \sigma^3 \otimes \cdots \otimes \sigma^3 \otimes (\sigma^1 D_{2k-1} + \sigma^2 D_{2k}) \otimes 1_2 \otimes \cdots \otimes 1_2.
\] (4.6)

The components of the Dirac spinor can be defined by introducing a basis of the representation (4.3) defined as direct products of two-component spinors $u_s (s = \pm)$ given by (3.16). They satisfy
\[
\sigma^3 u_s = su_s, \quad \sigma^+ u_- = u_+, \quad \sigma^- u_+ = u_-
\] (4.7)

with $\sigma^\pm = \frac{1}{2} (\sigma^1 \pm i\sigma^2)$. The basis of spinors in $d = 2n$ dimensions is then given by
\[
u_{s_1} \otimes \cdots \otimes u_{s_n}.
\] (4.8)

Then an arbitrary spinor $\psi$ can be expanded as
\[
\psi(x) = \sum_{\{s\}} u_{s_1} \otimes \cdots \otimes u_{s_n} \psi_{s_1 \cdots s_n}(x),
\] (4.9)

where $\psi_{s_1 \cdots s_n}$ stand for the components of $\psi(x)$. Using $\sigma^1 u_s = u_{-s}$ and $\sigma^2 u_s = i s u_{-s}$, we can write the Dirac equation in component form as
\[
\sum_{k=1}^n s_1 \cdots s_{k-1} (D_{2k-1} - is_k D_{2k}) \psi_{s_1 \cdots s_{k-1} - s_k s_{k+1} \cdots s_n}(x) = 0.
\] (4.10)

These are the generalization of (3.5) to $d = 2n$ dimensions. Since each equation contains $n$ different components of $\psi(x)$, one might think it difficult to solve them. As we will see soon, this is not the case. We show that each component $\psi_{s_1 \cdots s_n}$ must satisfy uncoupled first order differential equations similar to (3.5).

To show this we first compute the square of the Dirac operator. By noting the commutation relations
\[
[D_{2k-1}, D_{2l-1}] = [D_{2k}, D_{2l}] = 0, \quad [D_{2k-1}, D_{2l}] = -iB_k \delta_{kl}, \quad (k, l = 1, \cdots, n)
\] (4.11)
we obtain

\[ \bar{\mathcal{D}}^2 = D^2 - i \sum_{k=1}^{n} \Gamma_n^{2k-1} \Gamma_n^{2k} B_k, \quad (4.12) \]

where we have introduced the Laplacian \( D^2 = \sum_a D_a^2 \). In the present representation of \( \Gamma^a \) we have

\[ \Gamma_n^{2k-1} \Gamma_n^{2k} = \underbrace{1_2 \otimes \cdots \otimes 1_2}_{k-1 \text{ times}} \otimes \underbrace{i\sigma^3 \otimes 1_2 \otimes \cdots \otimes 1_2}_{n-k \text{ times}}. \quad (4.13) \]

This gives

\[ \bar{\mathcal{D}}^2 \psi = \sum_{\{s\}} u_{s_1} \otimes \cdots \otimes u_{s_n} \left( D^2 + \sum_{k=1}^{n} s_k B_k \right) \psi_{s_1 \cdots s_n}. \quad (4.14) \]

We thus find that each component of the Dirac operator zero-modes must satisfy

\[ \left( D^2 + \sum_{k=1}^{n} s_k B_k \right) \psi_{s_1 \cdots s_n} = 0. \quad (4.15) \]

By multiplying \( \psi_{s_1 \cdots s_n}^* \) to this expression and then integrating over \( T^d \), we obtain

\[ \int_{T^d} d^d x \sum_k \left| (D_{2k-1} + i s_k D_{2k}) \psi_{s_1 \cdots s_n} \right|^2 = 0, \quad (4.16) \]

where use has been made of the relation

\[ (D_{2k-1} - i s_k D_{2k})(D_{2k-1} + i s_k D_{2k}) = D_{2k-1}^2 + D_{2k}^2 + s_k B_k. \quad (4.17) \]

We thus find that \( \psi_{s_1 \cdots s_n} \) must satisfy

\[ (D_{2k-1} + i s_k D_{2k}) \psi_{s_1 \cdots s_n} = 0. \quad (4.18) \]

These are essentially the same with the two-dimensional equations (4.5). As we have shown in the previous section, \( D_{2k-1} + i s_k D_{2k} \) has no zero-modes if \( \text{sgn} B_k < 0 \). This implies that \( \psi_{s_1 \cdots s_n} = 0 \) unless \( s_1 = \text{sgn} B_1, \cdots, s_n = \text{sgn} B_n \). The zero-modes of \( \bar{\mathcal{D}} \) is then given by a tensor product as

\[ \psi = \psi_1(x_1, x_2) \otimes \psi_2(x_3, x_4) \otimes \cdots \otimes \psi_n(x_{d-1}, x_d), \quad (4.19) \]

where \( \psi_k \) is a zero-mode of \( \sigma^1 D_{2k-1} + \sigma^2 D_{2k} \). There are \( \nu_k \) zero-modes of \( \sigma^1 D_{2k-1} + \sigma^2 D_{2k} \) with the two-dimensional chirality \( \text{sgn} B_k \). The chirality of \( \psi \) is the product of each chirality of the two-dimensional spinor \( \psi_k \) and coincides with \( \text{sgn} B_1 \cdots B_n \). Since the number of independent zero-modes of \( \bar{\mathcal{D}} \) is given by \( \nu_1 \nu_2 \cdots \nu_n \), we obtain the index relation

\[ \text{index} \bar{\mathcal{D}} = \prod_k \frac{B_k}{2\pi} = \frac{1}{(2\pi)^n n!} \int_{T^d} F^n. \quad (4.20) \]
5 Extension to Arbitrary Field Strengths

In the previous section we have considered rather special gauge field that can be treated by the method developed for the two dimensional system described in Sect. 3. A general uniform field strength can be more complicated. The flux through \((i,j)-\text{plane}\) may be nonvanishing for any \(i \neq j\). Fortunately, an arbitrary uniform field strength \(F_{ab}\) can be made into a standard form by an \(\text{SO}(d)\) rotation as

\[
F_{ab} = \sum_{c,d} \zeta_c^a \zeta_d^b B_{cd}, \quad B = (B_{ab}) = \begin{pmatrix}
0 & B_1 \\
-B_1 & 0 \\
& & 0 & B_2 \\
& & -B_2 & 0 \\
& & & & & & \ddots \\
& & & & & & & 0 & B_n \\
& & & & & & & -B_n & 0
\end{pmatrix}, \tag{5.1}
\]

where \(R = (\zeta_a^b)\) is a real orthogonal matrix satisfying \(RR^T = R^T R = 1\) and \(\det R = 1\). Since we are assuming \(\det F_{ab} \neq 0\), any eigenvalue \(B_k\) is nonvanishing. This ensures that the Dirac operator has a nonvanishing index. The essential difference between (4.1) and (5.1) is that the eigenvalue \(B_k\) in the latter is not necessarily a \(2\pi\) multiple of an integer. We also note that the field strengths \(F_{ab}\) can be expressed as

\[
F_{ab} = \sum_k B_k (\zeta_a^{2k-1} \zeta_b^{2k} - \zeta_b^{2k} \zeta_a^{2k}). \tag{5.2}
\]

We will frequently use the relation in the following development.

The goal of this paper is to find the solutions to the Dirac equation coupled to the gauge potential in axial gauge (2.14). It is given by

\[
\sum_a \Gamma_a \left( \partial_a + i \sum_{b < a} F_{ab} x^b \right) \psi(x) = 0. \tag{5.3}
\]

What makes the task nontrivial is the periodicity conditions (2.8). In the present case they are given by

\[
\psi(x + e_a) = e^{i \sum_{b < a} F_{ab} x^b} \psi(x). \tag{5.4}
\]

In Ref. [4] these are referred to as twisted periodicity conditions. We will see that they essentially determine the solution.

In the axial gauge, however, the rotational symmetry is not manifest. So, we carry out the gauge transformation (2.20) and move in symmetric gauge. The Dirac equation (5.3) then becomes

\[
\sum_a \Gamma_a \left( \partial_a + i \sum_b F_{ab} x^b \right) \tilde{\psi}(x) = 0. \tag{5.5}
\]
We now introduce new orthogonal coordinates $y^a$ by

$$y^a = \sum_b x^b \zeta^a_b.$$  \hfill (5.6)

The Dirac equation (5.5) can be converted to the form

$$\sum_a \Gamma^a \left( \frac{\partial}{\partial y^a} + i \frac{1}{2} \sum_b B_{ab} y^b \right) \tilde{\Psi}(y) = 0,$$  \hfill (5.7)

where $\tilde{\Psi}(y) = S\tilde{\psi}(x)$ with $S^\dagger \Gamma^a S = \sum_b \Gamma^b \zeta^a_b$. The $S$ stands for the spinor representation of the SO($d$) rotation $R$.

We again go back to axial gauge in the coordinate system $y$. This can be achieved by the inverse of the gauge transformation (2.21). The Dirac equation (5.3) finally becomes

$$\sum_a \Gamma^a \left( \frac{\partial}{\partial y^a} + i \sum_{b < a} B_{ab} y^b \right) \Psi(y) = 0,$$  \hfill (5.8)

where $\Psi$ is related to the original $\psi$ by

$$\Psi(y) = e^{i \frac{1}{2} \sum_{a < b} y^a y^b B_{ab} - \frac{i}{2} \sum_{a < b} x^a x^b F_{ab} S \psi(x)}.$$  \hfill (5.9)

The Dirac operator on the rhs of (5.8) is of the form given by (4.6). This implies that the Dirac equation can be reduced to the form

$$\left( \frac{\partial}{\partial y^{2k-1}} + i s_k \frac{\partial}{\partial y^{2k}} + s_k B_k y^{2k-1} \right) \Psi_{s_1 \cdots s_n} = 0. \quad (k = 1, \cdots, n)$$  \hfill (5.10)

We see that $\Psi_{s_1 \cdots s_n}$ can be expressed as

$$\Psi_{s_1 \cdots s_n}(y) = e^{i \frac{1}{2} \sum_k s_k B_k (y^{2k-1})^2} f_{s_1 \cdots s_n}(z),$$  \hfill (5.11)

where $f_{s_1 \cdots s_n}$ depends only on the $n$ complex variables $z^k = y^{2k-1} + i s_k y^{2k} (k = 1, \cdots, n)$.

We now turn to the periodicity conditions for the wave function $\Psi$. The translation $x \to x + e_a$ corresponds to a translation of $y$ by a vector $\zeta_a = (\zeta^b_a)$ as can be seen from

$$y^b = \sum_c x^c \zeta^b_c \to y'^b = \sum_c (x + e_a)^c \zeta^b_c = (y + \zeta_a)^b.$$  \hfill (5.12)

Using the periodicity (5.4) under the translation $x \to x + e_a$, we can find the periodicity of $\Psi$ under the translation (5.12) as

$$\Psi(y + \zeta_a) = e^{i \sum_{b < c} \zeta^b_a B_{bc} \left( y^c + \frac{1}{2} \zeta^b_a \right)} \Psi(y).$$  \hfill (5.13)

The overall factor of gauge transformation

$$g_a(y) = e^{i \sum_{b < c} \zeta^b_a B_{bc} \left( y^c + \frac{1}{2} \zeta^b_a \right)}$$  \hfill (5.14)
satisfies the cocycle conditions analogous to (2.10). This can be shown explicitly by noting the relation (5.2). The $g_a(y)$, however, is not single-valued on $T^d$. This is contrasted with the corresponding factor appearing in (5.4), where the gauge transformation is periodic under $x \rightarrow x + e_b$.

Under the translation (5.12) the $z^k$ are transformed as

$$z^k \rightarrow z^k + \Xi^k_a \quad \text{with} \quad \Xi^k_a = \zeta^k_a - 1 + i s_k \zeta^2_k.$$

(5.15)

The periodicity conditions (5.13) can be transcribed as the conditions for $f_{s_1 \cdots s_n}(z)$. They are given by

$$f_{s_1 \cdots s_n}(z + \Xi) = e^{\sum_k s_k B_k \zeta^{2k-1}(z^k + \frac{1}{2} \Xi^k_a)} f_{s_1 \cdots s_n}(z).$$

(5.16)

These ensure that the wave functions defined in $\mathbb{R}^d$ can be regarded as those on $T^d$. We will see that they imposes very tight constraints on the wave functions and are sufficient to determine $f_{s_1 \cdots s_n}$.

Before closing this section we generalize (5.16) to an arbitrary translation by a vector $\Xi = \sum_a N^a \Xi^a$, where $N^a$ $(a = 1, \cdots, 2n)$ are integers. This can be achieved by applying (5.16) repeatedly to $f_{s_1 \cdots s_n}(z + \Xi)$ until we arrive at $f_{s_1 \cdots s_n}(z)$. Since we will use a similar formula in later sections we give the periodicity conditions for an arbitrary translation in Appendix A. They are given by

$$f_{s_1 \cdots s_n}(z + \Xi) = e^{\sum_k s_k B_k \zeta^{2k-1}(z^k + \frac{1}{2} \Xi^k_a) - \frac{i}{2} \sum_a N^a N^b F_{ab} f_{s_1 \cdots s_n}(z)},$$

(5.17)

where $\zeta^{2k-1} = \sum_a N^a \zeta^{2k-1}_a$ is the real part of $\Xi^k_a$.

The periodicity under the translation $z \rightarrow z + \Xi$ becomes much simpler if we can find a set of integers $N^a$ for which $\zeta^{2k-1} (k = 1, \cdots, n)$ vanish. In the next section we show that we can find a set of translations under which $f_{s_1 \cdots s_n}(z)$ is periodic or anti-periodic.

6 Solving the Periodicity Conditions

In this section we give a construction of $f_{s_1 \cdots s_n}(z)$. To achieve this we search for a basis vectors other than $\{e_a\}$ for the lattice $\mathbb{Z}^{2n}$ and convert the periodicity conditions (5.16) into a tractable form.

We first note that the field strengths $F_{ab}$ can be made into a block-diagonal form

$$\nu_{ab} = \frac{1}{2\pi} L_a^c L_b^d F_{ab} = \begin{pmatrix} 0 & \nu_1 & 0 & \cdots & 0 & \nu_n \\ -\nu_1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \nu_2 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & -\nu_2 & 0 \\ -\nu_n & 0 & \cdots & 0 & \cdots & \nu_n \end{pmatrix},$$

(6.1)
where $L = (L_{ab})$ is a regular $2n \times 2n$ matrix with integral elements satisfying $|\det L| = 1$ and $\nu_p$ ($p = 1, \cdots, n$) are positive integers with the property that $\nu_p$ divides $\nu_q$ for $p < q$. As before, we are assuming a general field strengths $F_{ab}$ with $\det F_{ab} \neq 0$ and any $\nu_p$ is nonvanishing. For given $F_{ab}$ we can find $L$ in a systematic way. To illustrate this we give a proof of this lemma [8] in Appendix B.

We denote $L$ by $2n$ vectors $M_p = (M_p^a)$ and $N_p = (N_p^a)$ as

$$L = (M_1, N_1, \cdots, M_n, N_n). \quad (6.2)$$

Then the relation (6.1) can be expressed as

$$F(M_p, N_q) = \nu_p \delta_{pq}, \quad F(M_p, M_q) = F(N_p, N_q) = 0, \quad (6.3)$$

where we have introduced a skew symmetric bilinear form

$$F(\xi, \eta) = \frac{1}{2\pi} \sum_{a,b} F_{ab} \xi^a \eta^b. \quad (6.4)$$

It is integer-valued if both $\xi$ and $\eta$ are integral vectors.

As we show in Appendix B, any vector in $\Lambda = \mathbb{Z}^d$ can be uniquely expressed by a linear combination of the $2n$ vectors $M_p$ and $N_p$ with integral coefficients. This implies that $\Lambda$ can also be generated by the basis $\{M_p, N_p\}$. In defining $T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$, the $2n$ dimensional hypercubic region $0 \leq x^a \leq 1$ in $\mathbb{R}^{2n}$ can be replaced with the rectangular region $D$ given by

$$x^a = \sum_p (M_p^a \xi^p + N_p^a \eta^p). \quad (0 \leq \xi^p, \eta^p \leq 1) \quad (6.5)$$

Using (6.3), we can solve this with respect to $\xi^p$ and $\eta^p$ as

$$\xi^p = \frac{F(x, N_p)}{\nu_p}, \quad \eta^p = \frac{F(M_p, x)}{\nu_p}. \quad (6.6)$$

The volume of $D$ can be easily found as

$$\int_D d^{2n}x = \int_0^1 d^n \xi \int_0^1 d^n \eta |\det(M_p^a, N_p^a)| = 1, \quad (6.7)$$

where use has been made of $|\det L| = 1$. These imply that $\xi^p$ and $\eta^p$ can be regarded as the coordinates of the $2n$-torus $T^{2n}$. Later in this section we consider the translations

$$x^a \rightarrow x^a + M_p^a, \quad N_p^a. \quad (6.8)$$

These correspond to unit translations of the $(\xi, \eta)$-coordinates.
In Sect. 5, we have used the fact that the field strength can be made into a standard form \((5.1)\) by an \(SO(2n)\) ration \(R = (\zeta_b^a)\). The choice of \(R\), however, is not unique but we can replace \(\zeta_b^a\) by

\[
\zeta_b^a = \left( \begin{array}{c} \zeta_{2k-1}^a \cos \theta_k + \zeta_{2k}^a \sin \theta_k, \\ -\zeta_{2k-1}^a \sin \theta_k + \zeta_{2k}^a \cos \theta_k, \end{array} \right),
\]

where \(\theta_k\) is an arbitrary rotation angle. By utilizing this arbitrariness we can always choose \(\zeta_b^a\) to satisfy

\[
\sum_a N_p^a \zeta_{2k-1}^a = 0, \quad (k, p = 1, \cdots, n) \tag{6.10}
\]

where \(N_p = (N_p^a)\) are those vectors that constitute the \(L\) as given by \((6.2)\). In Appendix C, we give a proof of the existence of such a basis \(\zeta_b^a = (\zeta_b^a)\).

For the choice of \(\zeta_b^a\) satisfying \((6.10)\) the periodicity \((5.16)\) under the translations

\[
z \rightarrow z + \Omega_p \quad \text{with} \quad \Omega_k^a = i s_k \sum_a N_p^a \zeta_{2k}^a \tag{6.11}
\]

reduces to

\[
f_{s_1 \cdots s_n}(z + \Omega_p) = e^{-i \pi \epsilon_p} f_{s_1 \cdots s_n}(z), \tag{6.12}
\]

where \(\epsilon_p\) is defined by

\[
\epsilon_p = \frac{1}{2\pi} \sum_{a < b} N_p^a N_p^b F_{ab}. \tag{6.13}
\]

There are exactly \(n\) such translations corresponding \(N_p\) \((p = 1, \cdots, n)\). Since \(\epsilon_p\) is an integer, we see that \(f_{s_1 \cdots s_n}(z)\) is either periodic or anti-periodic under the translations \((6.11)\).

It is convenient to introduce new variables \(w^p\) by

\[
z^k = \sum_p \sum_a s_k N_p^a \zeta_{2k}^a w^p. \tag{6.14}
\]

By noting \((5.2)\) and \((6.10)\), this can be made into the form

\[
\sum_k s_k B_k \zeta_{2k-1}^a = 2\pi \sum_p s_{ap} w^p, \tag{6.15}
\]

where \(s_{ap}\) is an integer defined by

\[
s_{ap} = \frac{1}{2\pi} \sum_b F_{ab} N_p^b. \tag{6.16}
\]

From \((6.3)\) we see that \(s_{ap}\) satisfies the orthogonality relation

\[
\sum_a M_p^a s_{qa} = \nu_p \delta_{pq}. \tag{6.17}
\]
This enables us to solve (6.15) with respect to \( w_p \) as

\[
 w_p = \frac{1}{2\pi \nu_p} \sum_k \sum_a s_k B_k M_p^a \zeta_{sa}^{2k-1} z_k. \tag{6.18}
\]

The translation \( z \to z + \Omega_p \) corresponds to a unit imaginary translation in \( w \)

\[
 w \to w + ie_p, \tag{6.19}
\]

where \( e_p = (\delta_p^q) \) is a unit vector. This can be seen easily as

\[
 z^k \to z'^k = \sum_q \sum_a s_k N_q^a \zeta_{sa}^{2k} (w^q + i\delta_p^q) = z^k + \Omega_p^k. \tag{6.20}
\]

If we denote \( f_{s_1 \cdots s_n}(z) \) by \( g_{s_1 \cdots s_n}(w) \), the periodicity (6.12) can be written as

\[
 g_{s_1 \cdots s_n}(w + ie_p) = e^{-i\pi \epsilon_p} g_{s_1 \cdots s_n}(w). \tag{6.21}
\]

Since the change of variables (6.14) is nonsingular, the shift of \( w \to w + \Upsilon_a \) under the translation \( z \to z + \Xi_a \) is uniquely determined as

\[
 \Upsilon_a^p = \frac{1}{2\pi \nu_p} \sum_b \sum_k s_k B_k M_p^b \zeta_{sb}^{2k-1} \Xi_a^k. \tag{6.22}
\]

We thus obtain from (5.16) the periodicity relations of \( g_{s_1 \cdots s_n}(w) \) under \( w \to w + \Upsilon_a \) as

\[
 g_{s_1 \cdots s_n}(w + \Upsilon_a) = e^{2\pi \nu_p \sum_p \rho_p^q (w^p + \frac{1}{2} \Upsilon^p_a)} g_{s_1 \cdots s_n}(w), \tag{6.23}
\]

where use has been made of (6.15).

We can generalize (6.23) for arbitrary \( \Upsilon = \sum_a M^a \Upsilon_a \) as in (5.17). It is given by

\[
 g_{s_1 \cdots s_n}(w + \Upsilon) = e^{2\pi \nu_p \sum_p \sum_a M^a \Upsilon_a (w^p + \frac{1}{2} \Upsilon^p_a) + \frac{1}{2} \sum_a \sum_b M^a M^b F_{ab} g_{s_1 \cdots s_n}(w)}, \tag{6.24}
\]

where \( M^a \) are arbitrary integers. In particular under the translation

\[
 w \to w + \rho_p \quad \text{with} \quad \rho_p^q = \sum_a M_p^a \Upsilon_a^q, \tag{6.25}
\]

where \( M_p = (M_p^a) \) is a vector appearing in \( L \), the wave function is transformed as

\[
 g_{s_1 \cdots s_n}(w + \rho_p) = e^{2\pi \nu_p (w^p + \frac{1}{2} \rho_p^p) + i\tilde{\epsilon}_p \nu_p^p g_{s_1 \cdots s_n}(w),} \tag{6.26}
\]

In deriving this use has been made of (6.17). The \( \tilde{\epsilon}_p \) is defined by

\[
 \tilde{\epsilon}_p = \frac{1}{2\pi \nu_p} \sum_{a<b} M_p^a M_p^b F_{ab}. \tag{6.27}
\]

Note that only \( w_p \) appears in the exponent on the rhs of (6.26) under the translation (6.25).
We are now in a position to solve the periodicity conditions (6.21) and (6.26). From the periodicity (6.21) we can express $g_{s_1\cdots s_n}(w)$ in the Fourier expansion as

$$g_{s_1\cdots s_n}(w) = \sum_{\{n_p\}} c_{\{n_p\}}^{s_1\cdots s_n} e^{2\pi \sum_p (n_p - \frac{1}{2} s_p) w_p},$$

where $\{n_p\}$ stands for the abbreviation of $n_1\cdots n_n$.

The condition (6.26) can be translated into the recursion relations for the Fourier coefficients. They are given by

$$c_{\{n_q\}}^{s_1\cdots s_n} = c_{\{n_q - \nu_p \rho_{pq}\}}^{s_1\cdots s_n} e^{-\pi \sum_q \rho_p^a (2n_q - \epsilon_q) + \pi \nu_p \rho_p^a + i \pi \epsilon_p \nu_p}.$$  (6.29)

It is straightforward to obtain the general expression for $c_{\{n_q\}}^{s_1\cdots s_n}$ for $n_q = \nu_q k_q + r_q$ with $k_q \in \mathbb{Z}$ and $r_q = 0, 1, \cdots, \nu_q - 1$ is given by

$$c_{n_1\cdots n_n}^{s_1\cdots s_n} = c_{r_1\cdots r_n}^{s_1\cdots s_n} e^{-\pi \sum_{p,q} \rho_p^q \nu_q k_q - \pi \sum_{p,q} \rho_p^q \nu_q k_q - \pi \sum_{p,q} \rho_p^q (2r_q - \epsilon_q) + i \pi \sum_p \epsilon_p \nu_p k_p},$$

where $c_{r_1\cdots r_n}^{s_1\cdots s_n}$ are arbitrary constants to be determined later. In deriving this we have used the relation

$$\rho_p^q \nu_q - \rho_q^p \nu_p = -i F(M_p, M_q) = 0.$$  (6.31)

To each component of the wave function labeled by $\{s_k\}$ we have $\nu_1 \cdots \nu_n$ independent solutions to the recursion relations (6.30). They do not, however, necessarily correspond to nontrivial zero-mode solutions. The Fourier expansion (6.28) must converge. To establish the convergence of the rhs of (6.28), we must show that $c_{n_1\cdots n_n}^{s_1\cdots s_n} \to 0$ as $|n| \to \infty$, where $|n| = \sqrt{n_1^2 + \cdots + n_n^2}$. It suffices to verify the convergence of the most dominant part

$$|e^{-\pi \sum_{p,q} \rho_p^a \nu_q k_q k_p}|^2 = e^{-\sum_k s_k B_k \left(\sum_p \sum_a \epsilon_a^{2k-1} M_p^a \epsilon_k\right)^2}.$$  (6.32)

Since we have $\det \sum_a \epsilon_a^{2k-1} M_p^a \neq 0$, the rhs does not converge as $|n| \to \infty$ unless $s_k B_k > 0$ for $k = 1, \cdots, n$. This implies that the coefficients $c_{r_1\cdots r_n}^{s_1\cdots s_n}$ must vanish if $\{s_1, \cdots, s_n\} \neq \{\text{sgn} B_1, \cdots, \text{sgn} B_n\}$. We thus find that the only nontrivial components of the zero-modes is $\Psi_{s_1\cdots s_n}$ with $s_k = \text{sgn}(B_k)$. Hence the number of independent zero-modes is equal to the number of arbitrary constants $c_{r_1\cdots r_n}^{s_1\cdots s_n}$ and is given by

$$\nu_1 \cdots \nu_n = \sqrt{\det \left(\frac{1}{2\pi} F_{ab}\right)} = \frac{1}{(2\pi)^n \nu!} |\epsilon_{a_1 b_1 \cdots a_n b_n} F_{a_1 b_1} \cdots F_{a_n b_n}|,$$  (6.33)

where $\epsilon_{a_1 b_1 \cdots a_n b_n}$ is the Levi-Civita symbol in $d = 2n$ dimensions. Furthermore, the chirality of these zero-modes coincides with $s_1 \cdots s_n$. These are perfectly consistent with the index theorem (4.20).
We now turn to the wave function (5.11). To write it in terms of the present coordinates let us denote \( w^p \) by its real and imaginary parts as
\[
\begin{align*}
  w^p &= u^p + iv^p. \quad (p = 1, \ldots, n)
\end{align*}
\]
Then we have from (6.18)
\[
\sum_k s_k B_k (y^{2k-1})^2 = 2\pi \sum_{p,q} g_{pq} u^p u^q,
\]
where \( g_{pq} \) is given by
\[
\begin{align*}
  g_{pq} &= \frac{1}{2\pi} \sum_{a,b} \sum_k s_k B_k \zeta_a^{2k} \zeta_b^{2k} N_a N_b.
\end{align*}
\]
It reminds us of a metric in geometry. To find the inverse of \( g_{pq} \) we note the relations
\[
\begin{align*}
  \sum_k \left( \sum_a s_k N_a \zeta_a^{2k} \right) \left( \sum_b s_k M_b \zeta_b^{2k-1} \right) &= 2\pi \nu_p \delta_{pq},
  \\
  \sum_p \left( \frac{1}{2\pi \nu_p} \sum_{a,b} s_k B_k M_p \zeta_a^{2k} \zeta_b^{2k-1} \right) \left( \sum_{a,b} s_l N_a \zeta_b^{2l} \right) &= \delta_{kl}.
\end{align*}
\]
These can be verified by using (5.2), (6.3) and (6.10). The inverse \( g^{pq} \) can be obtained from (6.36) and (6.38) as
\[
\begin{align*}
  g^{pq} &= \sum_{a,b} \sum_k \frac{s_k B_k}{2\pi \nu_p \nu_q} \zeta_a^{2k-1} \zeta_b^{2k-1} M_p M_q.
\end{align*}
\]
We see from (6.22) and (6.25) that \( g^{pq} \) is the real part of \( \rho^q_p / \nu_p \). We write \( \rho^q_p \) as
\[
\begin{align*}
  \rho^q_p &= \nu_p (g^{pq} + i\gamma^{pq}),
\end{align*}
\]
where \( \gamma^{pq} \) is defined by
\[
\begin{align*}
  \gamma^{pq} &= \frac{1}{4\pi \nu_p \nu_q} \sum_{a,b} \sum_k B_k (\zeta_a^{2k-1} \zeta_b^{2k} + \zeta_b^{2k-1} \zeta_a^{2k}) M_p M_q.
\end{align*}
\]
The wave function (5.11) for a given set of \( \{r_p\} \) can be written as
\[
\begin{align*}
  \Psi_{s_1, \ldots, s_n} &= \sum_{\pi_1, \ldots, \pi_n} e^{-i\pi \sum p,q \gamma^{pq} \pi_p \pi_q + i\pi \sum p \epsilon_p \pi_p - \pi \sum p,q g_{pq} (u^p + \sum r \rho^r \pi_r) (u^q + \sum s \rho^s \pi_s) + 2\pi i \sum p \pi_p v^p},
\end{align*}
\]
where the sum with respect to \( \pi_p \) \( (p = 1, \ldots, n) \) should be taken over the numbers of the form \( \pi_p = \nu_p k_p + r_p - \frac{1}{2} \epsilon_p \) \( (k_p = 0, \pm 1, \pm 2, \ldots) \). We have suppressed overall normalization constant in (6.42).
The wave function \([6.42]\) is not the final result. We must go back to the wave function \(\psi(x)\) in the original coordinates. As we have mentioned at the beginning of this section, \(\xi^p\) and \(\eta^p\) defined by \([5.5]\) are natural coordinates on \(T^{2n}\). The coordinates \(u^p\) and \(v^p\) are related to \(\xi^p\) and \(\eta^p\) by

\[
\begin{align*}
  u^p &= \sum_q g^{pq} v_q \xi^q, \\
  v^p &= \eta^p + \sum_q \gamma^{pq} v_q \xi^q.
\end{align*}
\tag{6.43}
\]

The translations \([6.19]\) and \([6.25]\) correspond to

\[
\begin{align*}
  \xi^q &\to \xi^q, \quad \eta^q &\to \eta^q + \delta_p^q \quad \text{under} \quad u^q &\to u^q + i \delta_p^q, \\
  \xi^q &\to \xi^q + \delta_p^q, \quad \eta^q &\to \eta^q \quad \text{under} \quad u^q &\to u^q + \rho_p^q.
\end{align*}
\tag{6.44}
\]

These can be understood by recalling the fact that the translation \([6.19]\) and \([6.25]\), respectively, correspond to \(x^a \to x^a + N_p^a\) and \(x^a \to x^a + M_p^a\). We thus find that \([6.19]\) and \([6.25]\) are the unit translations of the coordinates \(\xi^p\) and \(\eta^p\). These are just the basic moves on \(T^{2n}\). We can also directly show \([6.44]\) by making use of \([6.43]\).

It is now possible to write down the zero-mode wave function \(\psi(x)\) in terms of \(\xi^p\) and \(\eta^p\). Since the spinor part of \(\Psi\) is given by \(u_{s_1} \otimes \cdots \otimes u_{s_n}\), the normalized zero-mode for a given \(\{r_p\}\) can be obtained from \([5.9]\) and \([6.42]\) as

\[
\psi^{r_1\cdots r_n}(x) = \left[ \frac{2^n \nu_1 \cdots \nu_n}{\sqrt{\det g_{pq}}} \right]^\frac{1}{2} S^\dagger u_{s_1} \otimes \cdots \otimes u_{s_n} F_{s_1\cdots s_n}(x) e^{\frac{i}{2} \sum_{a<b} F_{ab} x^ax^b},
\tag{6.45}
\]

where \(F_{s_1\cdots s_n}(x)\) is given by

\[
\begin{align*}
F_{s_1\cdots s_n}(x) &= e^{-\frac{i}{2} \sum_{a<b} B_{ab} y^a y^b \psi^{r_1\cdots r_n}(y)} \quad \text{as} \quad \sum_{\pi_{s_1} \cdots \pi_{s_n}} e^{-\frac{i}{2} \sum_{p,q} (\gamma_{pq} + i \gamma_{pq} \nu_p \xi_q - \pi_q)(\nu_q \xi_p - \pi_p) - i \pi_p \xi_p^p \eta^p + i \pi_p \eta^p + i \pi_p \xi_p \pi_p}.
\end{align*}
\tag{6.46}
\]

In deriving this, use has been made of the relation

\[
\sum_{a<b} B_{ab} y^a y^b = 2\pi \sum_p \nu_p \xi^p \left( \sum_q \gamma_{pq} \nu_q \xi^q + \eta^p \right).
\tag{6.47}
\]

The orthogonality of \(\psi^{r_1\cdots r_n}\) with different set of \(\{r_p\}\) is obvious by the standard integral

\[
\int_0^1 d^n \eta e^{2\pi i \sum \pi_p \eta_p} e^{-2\pi i \sum \pi_q \eta_q} = \delta_{\pi_{s_1}, \pi_{r_1}} \cdots \delta_{\pi_{s_n}, \pi_{r_n}}.
\tag{6.48}
\]

The computation of the normalization constant is given in Appendix D.

The zero-mode wave function \([6.45]\) is the generalization of \([5.13]\) to arbitrary even dimensions. It is interesting to see how the periodicity conditions \([5.4]\), or their generalization for an arbitrary integral vector \(L = \sum_a L^a e_a\)

\[
\psi(x + L) = e^{i \sum_{a<b} F_{ab} L^a x^b} \psi(x),
\tag{6.49}
\]
are satisfied. These can be verified by applying (5.4) successively to the lhs of (6.49). Since the $2n$ vectors $M_p = (M^a_p)$ and $N_p = (N^a_p)$ span the entire lattice $\mathbb{Z}^{2n}$, it suffices to show (6.49) for $L = M_p, N_p$. In the form given by (6.45) it is completely straightforward to check these. For instance, under the translation $x \to x + M_p$ the overall exponential factor of $\psi(x)$ develops an extra phase factor as

$$e^{i \pi \nu_p \eta_p + i \pi \tilde{\epsilon}_p \nu_p} e^{i \sum_{a<b} M^a_p x^b F_{a,b} e^{i \sum_{a<b} x^a x^b F_{a,b}}}.$$  (6.50)

On the other hand $F_{r_1 \cdots r_n}(x)$ is transformed into

$$F_{s_1 \cdots s_n}^{r_1 \cdots r_n}(x + M_p) = e^{i \pi \nu_p \eta_p + i \pi \tilde{\epsilon}_p \nu_p} F_{s_1 \cdots s_n}^{r_1 \cdots r_n}(x).$$  (6.51)

Since $\tilde{\epsilon}_p \nu_p$ is an integer, the extra phase factors in the rhs’ of (6.50) and (6.51) cancel each other. Similar thing also happens for the translation $x \to x + N_p$.

We finally note that the transformation properties of $F_{r_1 \cdots r_n}(x)$ under (6.8) become symmetric as

$$F_{s_1 \cdots s_n}^{r_1 \cdots r_n}(x + M_p) = e^{i \sum_{a<b} F_{a,b}(M^a_p x^b - M^b_p x^a + M^a_p M^b_p) F_{s_1 \cdots s_n}^{r_1 \cdots r_n}(x)},$$

$$F_{s_1 \cdots s_n}^{r_1 \cdots r_n}(x + N_p) = e^{i \sum_{a<b} F_{a,b}(N^a_p x^b - N^b_p x^a - N^a_p N^b_p) F_{s_1 \cdots s_n}^{r_1 \cdots r_n}(x)}.$$  (6.52)

In deriving these we have used (6.5), (6.13) and (6.27).

## 7 Summary

We have investigated the Dirac operator zero-modes on a torus for a gauge background with constant field strength. The gauge potential possesses a characteristic transformation properties under the translations corresponding to the periods of the torus. This gives rise to nontrivial periodicity conditions for the wave function. In the coordinates where the field strength take a standard block-diagonal form the Dirac equation for the zero-modes can be reduced to first order differential equations for each spinor component. It give a kind of holomorphicity conditions and allows us to express the wave function by a set of holomorphic functions of the torus coordinates. We have shown that it is always possible to find a set of complex torus coordinates in which the holomorphic part of the wave function becomes either periodic or anti-periodic under unit translations along the imaginary axes. The periodicity conditions determine up to an overall normalization the holomorphic part of the wave function. For gauge backgrounds with uniform field strength only left-handed or right-handed chiral zero-modes are realized in a manner consistent with the index theorem.

Throughout the paper we have assumed $\det F_{a,b} \neq 0$. If $\det F_{a,b} = 0$, then the index of the Dirac operator vanishes. We expect two possibilities. One is the occurrence of an equal number of positive and negative chirality zero-modes. Another is the absence of chiral
zero-modes. As noted in Sect. 2, we cannot remove some of the constant components of the gauge potential by shifting the coordinate origin if \( \det F_{ab} = 0 \). In general the presence of a constant gauge potential affects the spectrum of the Dirac operator and eliminates the chiral zero-modes. A complete analysis of the eigenvalue problem of the Dirac operator is interesting. We will argue this in a separate publication.

This work is supported in part by the Grant-in Aid for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology (No. 13135203).

A General Translations

We first consider a translation \( z \rightarrow z + n \Xi_a \) with an arbitrary integer \( n \). By a successive application of (5.16), we find

\[
f_{s_1 \cdots s_n}(z + n \Xi_a) = e^{\sum_k n s_k \zeta_a^{2k-1} B_k \left( z + \frac{1}{2} n \Xi_a \right)} f_{s_1 \cdots s_n}(z).
\]

We now consider the most general translation by

\[
\Xi^k = \sum_a N^a \Xi_a^k.
\]

By applying (A.1) to

\[
f_{s_1 \cdots s_n} \left( z + \sum_{a=1}^b N^a \Xi_a \right) = f_{s_1 \cdots s_n} \left( z + \sum_{a=1}^{b-1} N^a \Xi_a + N^b \Xi_b \right),
\]

we obtain a recursive relation

\[
f_{s_1 \cdots s_n} \left( z + \sum_{a=1}^b N^a \Xi_a \right) = e^{\sum_k N^b s_k \zeta_b^{2k-1} B_k \left( z + \sum_{a=1}^{b-1} N^a \Xi_a^b + \frac{1}{2} N^b \Xi_b^b \right)} f_{s_1 \cdots s_n} \left( z + \sum_{a=1}^{b-1} N^a \Xi_a \right).
\]

This gives

\[
f_{s_1 \cdots s_n}(z + \Xi) = \prod_{b=1}^{2n} \left( e^{\sum_k N^b s_k \zeta_b^{2k-1} B_k \left( z + \sum_{a=1}^{b-1} N^a \Xi_a^b + \frac{1}{2} N^b \Xi_b^b \right)} \right) f_{s_1 \cdots s_n}(z),
\]

where use has been made of \( \zeta^{2k-1} = \sum_a N^a \zeta_a^{2k-1} \). By noting the relation

\[
\sum_a (N^a)^2 \zeta_a^{2k-1} \Xi_a^k + 2 \sum_{a > b} N^a N^b \zeta_a^{2k-1} \Xi_a^b = \zeta^{2k-1} \Xi^k - i s_k \sum_{a < b} N^a N^b (\zeta_a^{2k-1} \zeta_b^{2k} - \zeta_a^{2k} \zeta_b^{2k-1})
\]

and using (5.2), we see that (A.3) can be written in the form (5.17).
B Proof of the Lemma

In this appendix we give a proof of the basic lemma mentioned in Sect. 6. It can be stated in the following form;

**Lemma:** Any nondegenerate antisymmetric matrix \( \phi \) with \( \phi_{ab} = -\phi_{ba} \in \mathbb{Z} \) can be converted to a block-diagonal form

\[
L_a^c L_b^d \phi_{cd} = \nu_{ab},
\]

with

\[
\nu = \begin{pmatrix}
0 & -\nu_1 \\
\nu_1 & 0 \\
0 & -\nu_2 \\
\nu_2 & 0 \\
& & \ddots \\
0 & -\nu_n \\
\nu_n & 0
\end{pmatrix},
\]

(B.1)

where \( L \) is a \( 2n \times 2n \) matrix with \( L_{ab} \in \mathbb{Z} \) and \(|\det L| = 1\). The \( \nu_1, \ldots, \nu_n \) are a sequence of positive integers with \( \nu_{p+1}/\nu_p \in \mathbb{Z} \).

To prove this let us consider a space \( \mathbb{Z}^{2n} \) spanned by the vectors

\[
\xi = \sum_a \xi^a e_a, \quad (\xi^a \in \mathbb{Z}).
\]

(B.2)

Then \( \phi \) naturally defines an integer-valued skew symmetric bilinear form

\[
\phi(\xi, \eta) = \sum_{a,b} \phi_{ab} \xi^a \eta^b, \quad (\xi, \eta \in \mathbb{Z}^{2n})
\]

(B.3)

Let \( M_1 \) and \( N_1 \) be such a pair of vectors that \( \nu_1 = \phi(M_1, N_1) \) becomes the smallest positive integer. Then we see that \( \phi(M_1, \xi) \) and \( \phi(N_1, \xi) \) for any integral vector \( \xi \) must be divided by \( \nu_1 \). This can be shown as follows: Let \( \xi \) be such an integral vector with \( \phi(M_1, \xi)/\nu_1 \notin \mathbb{Z} \). Then we have \( \phi(M_1, kN_1 + \xi) = k\nu_1 + \phi(M_1, \xi) \) for any integer \( k \). Since \( |\phi(M_1, \xi)| > \nu_1 \) by assumption, we can always find a suitable \( k \) such that \( 0 < \phi(M_1, kN_1 + \xi) < \nu_1 \) is satisfied. This contradicts the assumption that \( \nu_1 \) is the smallest positive integer taken by \( \phi \). Hence \( \phi(M_1, \xi) \) must be divided by \( \nu_1 \). This also applies to \( \phi(N_1, \xi) \). Using these properties, we can decompose any integral vector \( \xi \) in the form

\[
\xi = \xi' + \frac{\phi(\xi, N_1)}{\nu_1} M_1 + \frac{\phi(M_1, \xi)}{\nu_1} N_1,
\]

(B.4)

where \( \xi' \) satisfies

\[
\phi(M_1, \xi') = \phi(\xi', N_1) = 0.
\]

(B.5)

We next consider the space of integral vectors \( \xi' \) satisfying (B.5) and evaluate the bilinear form \( \phi \). Let \( M_2 \) and \( N_2 \) be such a pair of vectors that \( \nu_2 = \phi(M_2, N_2) \) becomes the smallest
positive integer on this restricted space. By assumption we have \( \nu_1 \leq \nu_2 \). To see that \( \nu_2/\nu_1 \in \mathbb{Z} \) we consider

\[
\phi(kM_1 + M_2, N_1 + N_2) = k\nu_1 + \nu_2, \quad (B.6)
\]

where \( k \) may be an arbitrary integer. Since \( k\nu_1 + \nu_2 \) cannot be a positive integer smaller than \( \nu_1 \) for any integer \( k \), we see that \( \nu_1 \) must divide \( \nu_2 \). We can also show that \( \phi(M_2, \xi) \) and \( \phi(N_2, \xi) \) for any vector \( \xi \in \mathbb{Z}^{2n} \) can be divided by \( \nu_2 \).

This procedure can be continued until we arrive at \( M_n \) and \( N_n \) with \( \nu_n = F(M_n, N_n) \). The matrix \( L_{ab} \) is then given by

\[
L_{2p-1}^a = N_p^a, \quad L_{2p}^a = M_p^a. \quad (p = 1, \cdots, n) \quad (B.7)
\]

In matrix notation this can be written as

\[
L = (N_1, M_1, \cdots, N_n, M_n). \quad (B.8)
\]

To show \( |\det L| = 1 \) we note that any vector \( \xi \in \mathbb{Z}^{2n} \) can be expanded uniquely as

\[
\xi = \sum_p (a_p N_p + b_p M_p) \quad \text{with} \quad a_p = \frac{1}{\nu_p} \phi(M_p, \xi), \quad b_p = \frac{1}{\nu_p} \phi(\xi, N_p), \quad (B.9)
\]

where the coefficients \( a_p \) and \( b_p \) are all integers. In particular we consider expansion of unit vectors \( e_a \) with \( (e_a)^b = \delta_a^b \) as

\[
e_a = \sum_p (A_{ap} N_p + B_{ap} M_p), \quad (B.10)
\]

where \( A_{ap} \) and \( B_{ap} \) are all integers. In terms of components we have

\[
\sum_p (A_{ap} N_p + B_{ap} M_p) = \delta_a^b. \quad (B.11)
\]

These can be expressed in a matrix form as

\[
\begin{pmatrix}
A_{11} & \cdots & A_{1n} & B_{11} & \cdots & B_{1n} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
A_{2n1} & \cdots & A_{2nn} & B_{2n1} & \cdots & B_{2nn}
\end{pmatrix}
\begin{pmatrix}
N_1^1 & \cdots & N_1^{2n} \\
\vdots & \vdots & \vdots \\
N_n^1 & \cdots & N_n^{2n} \\
M_1^1 & \cdots & M_1^{2n} \\
\vdots & \vdots & \vdots \\
M_n^1 & \cdots & M_n^{2n}
\end{pmatrix}
= 1. \quad (B.12)
\]

Taking the determinant of both sides of this expression, we find that

\[
|\det L| = |\det (N_1, \cdots, N_n, M_1, \cdots, M_n)| = 1. \quad (B.13)
\]

This completes the proof of the lemma.

From (B.1) and (B.13) we obtain \( \det \nu = \det \phi \). This immediately gives

\[
\nu_1 \cdots \nu_n = \frac{1}{n!} \epsilon_{a_1 b_1 \cdots a_n b_n} \phi_{a_1 b_1} \cdots \phi_{a_n b_n}. \quad (B.14)
\]
C Existence of a Standard Basis

Using the lemma given in Appendix B, we can always find $2n$ sets of integral vectors $N_p = (N_p^a)$ and $M_p = (M_p^a)$ satisfying

$$F(M_p, N_q) = \nu_p \delta_{pq}, \quad F(M_p, M_q) = F(N_p, N_q) = 0, \quad |\det(N_1, M_1, \cdots, N_n, M_n)| = 1,$$

where $\nu_1, \cdots, \nu_n$ are positive integers with the property that $\nu_p$ divides $\nu_q$ for $p < q$. To make the paper self-contained we give a proof of the existence of a set of orthonormal basis vectors $(\zeta^1, \cdots, \zeta^{2n})$ that converts the field strengths into a standard form (5.1) and satisfies the conditions (6.10). We basically follow Ref. [4].

We first note that an arbitrary vector $V \in \mathbb{R}^{2n}$ can be expanded in three ways as

$$V = \sum_a v^a e_a = \sum_a u^a L_a = \sum_a w^a \zeta^a,$$

where the bases $e_a, L_a = (N_1, M_1, \cdots, N_n, M_n)$ and $\zeta^a$ in $\mathbb{R}^{2n}$ are defined by

$$(e_a)^b = \delta_a^b, \quad (L_a)^b = L_a^b, \quad (\zeta^a)^b = \zeta^a_b.$$  

This gives

$$w^c = \sum_{a,b} u^a L_a^b \zeta^c_b.$$  

The transformation matrix

$$(LR)_a^b = \sum_c L_a^c \zeta_c^b$$

relates $B = (B_{ab})$ with $\nu = (\nu_{ab})$ as

$$\nu = LFL^T = LRB(LR)^T.$$  

We then define $n$ dimensional vector subspaces spanned by $\{N_1, \cdots, N_n\}$ and $\{M_1, \cdots, M_n\}$ as

$$M^- = \{V \in \mathbb{R}^{2n} | V = \sum_p v^p N_p, \quad v^p \in \mathbb{R}\},$$

$$M^+ = \{V \in \mathbb{R}^{2n} | V = \sum_p v^p M_p, \quad v^p \in \mathbb{R}\}.$$  

The field strength 2-form is degenerated on $M^+$ and $M^-$ in the sense

$$F(N_p, N_q) = F(M_p, M_q) = 0.$$
Furthermore, $M^+$ and $M^-$ are maximal degenerate subspaces of $\mathbb{R}^{2n}$. This implies $F(V, W) \neq 0$ for some $V \in M^+$ and $W \notin M^+$.

The $\mathbb{R}^{2n}$ can also be decomposed into $n$ two-dimensional orthogonal subspaces $W_1, \cdots, W_n$, where $W_k$ is spanned by the orthonormal basis vectors $\zeta^{2k-1} = (\zeta^{2k-1}_a)$ and $\zeta^{2k} = (\zeta^{2k}_a)$ satisfying

$$
\sum_{b,c} F_{ab} F_{bc} \zeta^{2k-1}_c = -B_k^2 \zeta^{2k-1}_a, \quad \zeta^{2k}_a = \frac{1}{B_k} \sum_b F_{ab} \zeta^{2k-1}_b. \quad (C.10)
$$

We note that the vector subspaces $W_k^\pm$ of $W_k$ defined by

$$
W_k^- = W_k \cap M^-, \quad W_k^+ = W_k \cap (M^-)^\perp \quad (C.11)
$$

must be one-dimensional. This can be seen as follows; If $\dim W_k^- = 2$, we have $W_k^- = W_k \subseteq M^-$. Since $F$ is degenerated on $M^-$, we obtain $F(V, V') = 0$ for arbitrary $V, V' \in W_k^-$. This contradicts the fact that the basis vectors $\zeta^{2k-1}$ and $\zeta^{2k}$ defined above satisfy

$$
F(\zeta^{2k-1}, \zeta^{2k}) = -\frac{B_k}{2\pi} \neq 0. \quad (C.12)
$$

On the other hand if $\dim W_k^+ = 2$, we see that $W_k^+ = W_k \subset (M^-)^\perp$ is orthogonal to $M^-$. Then we can expand $N_p$ as

$$
N_p = \sum_{l(\neq k)} (c^l_p \zeta^{2l-1} + d^l_p \zeta^{2l}). \quad (C.13)
$$

We now choose an arbitrary vector $V \in W_k^+$ and consider the $n+1$ dimensional subspace $\tilde{M}^- \supset M^-$ of $\mathbb{R}^{2n}$ spanned by $V$ and $\{N_1, \cdots, N_n\}$. Using (C.13), we immediately find $F(V, N_p) = 0$ for any $p$. This implies that $F$ is degenerated on $\tilde{M}^-$. This contradicts the fact that $M^-$ is a maximal degenerate subspace. We thus conclude that $\dim W_k^\pm = 1$.

We can always find an orthonormal basis $(\zeta^{2k-1}, \zeta^{2k})$ in $W_k$ to satisfy

$$
\zeta^{2k-1} \in W_k^-, \quad \zeta^{2k} \in W_k^+. \quad (C.14)
$$

Since $W_k^-$ is orthogonal to $M^-$ we obtain

$$
\sum_a N_p^a \zeta^{2k-1}_a = 0. \quad (C.15)
$$

This completes the proof of the existence of $\zeta^{b}_a$ satisfying (6.10).
D Computation of the Normalization Constant

In this appendix we compute the normalization constant of the wavefunction (6.45). Due to the orthogonality relations (6.48) we have only to compute the integral

\[
\int_D d^n x \left| \sum_{p_1, \ldots, p_n} e^{-\pi \sum_{p,q} (g^{pq} + i\gamma^{pq}) (\nu_p \xi^p - \pi_p) (\nu_q \xi^q - \pi_q) - i\pi \sum_p (\nu_p \xi^p - \pi_p) \eta^p + i\pi \sum_p \eta^p \xi^p \xi^p} \right|^2
\]

\[
= \int_0^1 d^n \xi \sum_{p_1, \ldots, p_n} \sum_{p_1', \ldots, p_n'} e^{-\pi \sum_{p,q} (g^{pq} + i\gamma^{pq}) (\nu_p \xi^p - \pi_p) (\nu_q \xi^q - \pi_q) - i\pi \sum_p (\nu_p \xi^p - \pi_p) \eta^p + i\pi \sum_p \eta^p \xi^p \xi^p} \times e^{-\pi \sum_{p,q} (g^{pq} - i\gamma^{pq}) (\nu_p \xi^p - \pi_p) (\nu_q \xi^q - \pi_q) + i\pi \sum_p (\nu_p \xi^p - \pi_p) \eta^p - i\pi \sum_p \eta^p \xi^p \xi^p} \int_0^1 d^n \eta e^{2i\pi \sum_p (\pi_p - \pi'_p) \eta^p}
\]

\[
= \int_0^1 d^n \xi \sum_{p_1, \ldots, p_n} e^{-2\pi \sum_{p,q} g^{pq} (\nu_p \xi^p - \pi_p) (\nu_q \xi^q - \pi_q)}
\]

\[
= \int_0^1 d^n \xi \sum_{k_1, \ldots, k_n = -\infty}^{+\infty} e^{-2\pi \sum_{p,q} g^{pq} (\nu_p \xi^p - \nu_p k_p - r_p + \frac{1}{2} \epsilon_p)(\nu_q \xi^q - \nu_q k_q - r_q + \frac{1}{2} \epsilon_q)}
\]

\[
= \sum_{k_1, \ldots, k_n = -\infty}^{+\infty} \int_{-k_n+1}^{-k_1+1} d\xi^1 \cdots \int_{-k_1}^{-k_n+1} d\xi^n e^{-2\pi \sum_{p,q} g^{pq} (\nu_p \xi^p - r_p + \frac{1}{2} \epsilon_p)(\nu_q \xi^q - r_q + \frac{1}{2} \epsilon_q)}
\]

\[
= \int_{-\infty}^{+\infty} d\xi^1 \cdots \int_{-\infty}^{+\infty} d\xi^n e^{-2\pi \sum_{p,q} g^{pq} \nu_p \nu_q \xi^p \xi^q}
\]

\[
= \frac{\sqrt{\det g_{pq}}}{2\pi^{\frac{n}{2}} \nu_1 \cdots \nu_n}.
\] (D.1)

This justifies the overall normalization of (6.45). In two dimensions we have $g_{11} = \nu$. The normalization constant appearing in (3.17) is consistent with (D.1).
References

[1] M. F. Atiyah and I. M. Singer, Bull. Amer. Math. Soc. 69 (1963) 422; Ann. Math. 87 (1968) 484, 546.

[2] S. L. Adler, Phys. Rev. 177 (1969) 2426.
   J. Bell and R. Jackiw, Nuovo Cim. 60A (1969) 47.

[3] H. B. Nielsen and M. Ninomiya, Phys. Lett. 130B (1983) 389.
   A. Manohar, Phys. Lett. 153B (1985) 153.
   T. Fujiwara and Y. Ohnuki, Prog. Theor. Phys. 76 (1986) 1182; 77 (1987) 1463.

[4] M. Sakamoto and S. Tanimura, J. Math. Phys. 44 (2003) 5042 [hep-th/0306006].

[5] E. Brown, Phys. Rev. 133 (1964) A1038.

[6] J. Zak, Phys. Rev. 134 (1964) A1602, A1607.

[7] S. Tanimura, J. Math. Phys. 43 (2002) 5926 [hep-th/0205053].

[8] J. Igusa, “Theta Functions,” Lemma 5, pp. 71-72, Springer-Verlag (Berlin) (1972).