THE SPECTRUM OF THE AVERAGING OPERATOR ON A NETWORK (METRIC GRAPH)

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Abstract. A network is a countable, connected graph $X$ viewed as a one-complex, where each edge $[x,y] = [y,x]$ ($x, y \in X^0$, the vertex set) is a copy of the unit interval within the graph’s one-skeleton $X^1$ and is assigned a positive conductance $c(xy)$. A reference “Lebesgue” measure on $X^1$ is built up by using Lebesgue measure with total mass $c(xy)$ on each edge $[x,y]$. There are three natural operators on $X$: the transition operator $P$ acting on functions on $X^0$ (the reversible Markov chain associated with $c$), the averaging operator $A$ over spheres of radius 1 on $X^1$, and the Laplace operator $\Delta$ on $X^1$ (with Kirchhoff conditions weighted by $c$ at the vertices). The relation between the $\ell^2$-spectrum of $P$ and the $H^2$-spectrum of $\Delta$ was described by Cattaneo [4]. In this paper we describe the relation between the $\ell^2$-spectrum of $P$ and the $L^2$-spectrum of $A$.

1. Introduction

Let $X$ be a countable, connected graph with symmetric neighbourhood relation $\sim$ and without loops and multiple edges. We shall view it as a one-complex, where each edge is a (homeomorphic) copy of the unit interval and edges are glued together at common endpoints (vertices). We write $X^0$ for the vertex set and $X^1$ for the one-skeleton of $X$. Every point in $X^1$ is of the form $(xy, t)$, the point at distance $t$ from $x$ on the non-oriented edge $[x,y] = [y,x]$, where $0 \leq t \leq 1$, and $x, y \in X^0, \ x \sim y$. Thus, $(xy,0) = x$ and $(xy, t) = (yx, 1-t)$. In this way, the discrete graph metric $d(\cdot, \cdot)$ on the vertex set (minimal length = number of edges of a connecting path) has a natural extension to $X^1$.

We equip each edge $[x,y]$ with a positive conductance $c(xy) = c(yx)$. On $X^0$, we consider the discrete measure $m^0$, where $m^0(x) = \sum_{y:y\sim x} c(xy)$. Our basic assumption is that $m^0(x) < \infty$ for all $x \in X^0$. On $X^1$, we introduce the continuous weighted “Lebesgue” measure $m^1$ which at the point $(xy, t)$ is given by $c(xy) \cdot dt$, if $0 < t < 1$ (the vertex set has $m^1$-measure 0). The pair $(X, c)$, together with these measures, is called a network, or – in the recent literature – also metric graph or quantum graph.

Associated with a network, there are three natural operators.

The first is the transition operator $P$ acting on functions $g : X^0 \to \mathbb{C}$ by

\begin{equation}
Pg(x) = \frac{1}{m^0(x)} \sum_{y:y\sim x} c(xy) g(y).
\end{equation}
The second is the Laplace operator $\Delta$. It can be defined via Dirichlet form theory, or by considering the space of all continuous functions $F : X^1 \to \mathbb{C}$ which are twice differentiable in the interior of each edge and satisfy the Kirchhoff equations

$$\sum_{y : y \sim x} c(xy) F'(xy, 0+) = 0 \quad \text{for all } x \in X^0.$$  

We then have

$$\Delta F(xy, t) = F''(xy, t),$$

the 2nd derivative with respect to $t \in (0, 1)$, and $\Delta$ has to be closed suitably. See e.g. Cattaneo [4], Solomyak [18] or Eells and Fuglede [7] for precise details. (The paper [4] seems to have escaped the attention of most people working on metric graphs.)

The third operator is the averaging operator $A$ over balls of radius 1. It acts on locally integrable functions $F : X^1 \to \mathbb{C}$ by

$$AF(xy, t) = \frac{1}{m^0(x)} \sum_{u \sim x} c(xu) \int_{0}^{1-t} F(xu, s) \, ds + \frac{1}{m^0(y)} \sum_{v \sim y} c(yv) \int_{0}^{t} F(yv, s) \, ds.$$  

In the regular case, i.e., when $m^0(\cdot)$ is constant, this is just the $m^1$-average of $F$ over the ball with radius 1 centered at $(x, t)$.

Each of the three operators gives rise to a Markov process. For $P$, this is the random walk (reversible Markov chain) with discrete time and state space $X^0$ whose transition probabilities are $p(x, y) = c(xy)/m^0(x)$, if $y \sim x$, and $p(x, y) = 0$, otherwise.

The Laplace operator $\Delta$ is the infinitesimal generator of Brownian motion on the network.

The stochastic interpretation of $A$ is more similar to that of $P$. Namely, $A$ governs the random walk with discrete time and state space $X^1$, where at any time $n$, if the current position is $(x, t)$, the next step goes to a random point in $X^1$ at distance at most 1. The random choice depends on $t$, the conductance of $[x, y]$ and the edges incident with $[x, y]$.

These stochastic aspects are not at the heart of the present paper. What we are interested in here is relation between the spectra of the operators $A$ and $P$. Cattaneo [4] has given a complete description of the $H^2$-spectrum of $\Delta$ in terms of the $\ell^2$-spectrum of $P$. Our plan is to describe the $L^2$-spectrum of $A$ in terms of the $\ell^2$-spectrum of $P$.

This refers to the (complex) Hilbert spaces $L^2(X^1, m^1)$ and $\ell^2(X^0, m^0)$. The inner product of the latter is given by

$$\langle g_1, g_2 \rangle = \sum_{x \in X^0} g_1(x) \overline{g_2(x)} \, m^0(x),$$

and it is well known and easy to check that $P$ is self-adjoint with $\|P\| \leq 1$ on this space. Analogously, the inner product on $L^2(X^1, m^1)$ is

$$\langle F_1, F_2 \rangle = \frac{1}{2} \sum_{x \in X^0} \sum_{y : y \sim x} c(xy) \int_{0}^{1} F_1(xy, t) \overline{F_2(xy, t)} \, dt.$$  

The factor $\frac{1}{2}$ occurs because corresponding to each edge $[x, y]$, we get two equal terms on the right, namely $\int_{0}^{1} F_1(xy, t) \overline{F_2(xy, t)} \, dt$ and $\int_{0}^{1} F_1(yx, t) \overline{F_2(yx, t)} \, dt$. Again, it is straightforward to check that $A$ is self-adjoint with norm bounded by 1 on $L^2(X^1, m^1)$.
There is a large body of literature on the spectrum of transition (resp. adjacency and discrete Laplace) operators on finite graphs, see e.g. the books by Biggs [2], Cvetković, Doob and Sachs [6] and Chung [5]. Transition operators on infinite graphs are also very well studied objects, see e.g. the books by Soardi [17] and Woess [19]. A lot is known about the $\ell^2$-spectrum of transition operators on various classes of infinite graphs, see e.g. Mohar and Woess [14] for a general survey (up to 1989), and the many more recent papers, mostly embedded into the context of Markovian convolution operators on groups, of which we quote here only a few: de la Harpe, Robertson, and Vallette [10], Bartoluzzi and Woess [1], Cartwright [3], Grigorchuk and Zuk [8], [9], Bartholdi and Woess [1] and many more.

On the other hand, not much work has been done regarding the spectra of averaging operators on networks, whence it appears to be useful to have a recipe for translating the spectrum of $P$ into the spectrum of $A$. Our main result is the following.

(1.3) Theorem. The spectrum of $A$ is

$$\text{spec}(A) = \{0\} \cup \left\{ \frac{\sin \omega}{\omega} : \omega \in \mathbb{R} \setminus \{0\}, \cos \omega \in \text{spec}(P) \right\} \cup \{1 : 1 \in \text{spec}(P)\}.$$  

Here, by "$\cup \{1 : 1 \in \text{spec}(P)\}$" we mean that 1 is included in $\text{spec}(A)$ if and only if $1 \in \text{spec}(P)$. This theorem has the following obvious consequence.

(1.4) Corollary. Let $\rho = \rho(P)$ denote the spectral radius of $P$. Then the spectral radius of $A$ is

$$\rho(A) = \begin{cases} 1, & \text{if } \rho = 1, \\ \sqrt{1 - \rho^2} / \arccos(\rho), & \text{if } \rho < 1. \end{cases}$$

Let $\text{spec}_p(P)$ denote the point spectrum of $P$, i.e., the set of $\ell^2(X^0, m^0)$-eigenvalues of $P$.

(1.5) Theorem. We have

$$\text{spec}_p(A) \setminus \{0\} = \{1 : 1 \in \text{spec}_p(P)\} \cup \left\{ \frac{\sin \omega}{\omega} : \omega \in \mathbb{R} \setminus \{0\}, \cos \omega \in \text{spec}_p(P) \right\}.$$  

Moreover, $0 \in \text{spec}_p(A)$ unless $m^0(X^0) = \infty$ and $X$ is a tree with the property that after removal of any edge, at least one of the two connected components is recurrent.

For the precise meaning of this last condition, see Definition 3.8 and Proposition 3.9 below.

The structure of this paper is as follows. In §2, we set up the basic tools for relating $P$ and $A$. In §3, we study the contribution to the kernel of $A$ that comes from flows in the network. In §4, we prove the above two main theorems, and we also specify for finite graphs how one can obtain an orthonormal basis of $L^2(X^1, m^1)$ consisting of eigenvectors (-functions) of $A$. In §5, we exhibit several examples.
2. Interpolation of functions on the vertex set

For \( g \in \ell^2(X^0, m^0) \) and \( u \in L^2[0, 1] \), define a function \( F_{g,u} \) on \( X^1 \) by

\[
F_{g,u}(xy, t) = g(x)u(1 - t) + u(t)g(y).
\]

(2.1)

Recall that \((xy, t) = (yx, 1 - t)\); the definition of \( F_{g,u} \) is compatible with this parametrization. It is easy to check that \( F_{g,u} \in L^2(X^1, m^1) \). In fact, it is routine to calculate, for \( g_1, g_2 \in \ell^2(X^0, m^0) \) and \( u_1, u_2 \in L^2[0, 1] \), that

\[
\langle F_{g_1,u_1}, F_{g_2,u_2} \rangle = \langle g_1, g_2 \rangle \langle u_1, u_2 \rangle + \langle p_{g_1}, g_2 \rangle \langle u_1, Su_2 \rangle,
\]

(2.2)

where \( S \) is defined in (2.4) below, and \( \langle u_1, u_2 \rangle = \int_0^1 u_1(t)u_2(t) \, dt \) is the standard inner product on \( L^2[0, 1] \), while the inner products on \( L^2(X^1, m^1) \) and \( \ell^2(X^0, m^0) \) are those defined in the introduction.

(2.3) Lemma. The action of \( A \) on a function \( F_{g,u} \) is given by

\[
AF_{g,u} = F_{g,Jsu} + F_{pg,Jsu},
\]

where the operators \( S \) and \( J \) are given by

\[
Su(t) = u(1 - t) \quad \text{and} \quad Ju(t) = \int_0^t u(s) \, ds.
\]

(2.4)

(2.5) Definition. Denote by \( \mathcal{M}_0 \) the linear span of the functions \( F_{g,u} \), where \( g \in \ell^2(X^0, m^0) \), and where \( u \in L^2[0, 1] \). Let \( \mathcal{M} \) denote the closure of \( \mathcal{M}_0 \) in \( L^2(X^1, m^1) \).

Lemma 2.3 shows that \( \mathcal{M}_0 \) is invariant under \( A \), and therefore \( \mathcal{M} \) is too.

(2.6) Lemma. The orthogonal complement of \( \mathcal{M} \) in \( L^2(X^1, m^1) \) consists of the (equivalence classes of) square integrable functions \( F : X^1 \to \mathbb{C} \) such that for each \( x \in X^0 \),

\[
\sum_{y : y \sim x} c(xy) F(xy, t) = 0 \quad \text{for almost all} \quad t \in [0, 1].
\]

(2.7)

Proof. Suppose that \( F \in \mathcal{M}^\perp \). Then in particular, \( \langle F_{g,u}, F \rangle = 0 \) for each \( u \in L^2[0, 1] \) and for \( g = \delta_x \), and for this \( g \), one calculates that

\[
\langle F_{g,u}, F \rangle = \int_0^1 u(1 - t) \sum_{y : y \sim x} c(xy) F(xy, t) \, dt
\]

Since \( u \in L^2[0, 1] \) is arbitrary, (2.7) holds.

Conversely if (2.7) holds, then \( \langle F_{g,u}, F \rangle = 0 \) for all \( g \in \ell^2(X^0, m^0) \) of the form \( g = \delta_x \). By linearity, \( \langle F_{g,u}, F \rangle = 0 \) if \( g \) is finitely supported, and using finite approximations, this implies that \( \langle F_{g,u}, F \rangle = 0 \) for all \( g \in \ell^2(X^0, m^0) \) and \( u \in L^2[0, 1] \).

(2.8) Corollary. The operator \( A \) leaves \( \mathcal{M} \) and \( \mathcal{M}^\perp \) invariant, and is identically zero on \( \mathcal{M}^\perp \).

Proof. If \( F \in \mathcal{M}^\perp \), then \( AF = 0 \), as is immediate from the definition of \( A \) and Lemma 2.6.

\( \square \)
In the next Section 3, we shall give a complete description of $\mathcal{M}^\perp$ in terms of flows in the network, characterising, in particular, those networks for which $\mathcal{M}^\perp = \{0\}$.

Let us record some elementary properties of the operators $J$ and $S$ arising in (2.4).

**Lemma.** The operator $S$ satisfies $S^* = S$ and $S^2 = I$. If $v, w \in L^2[0,1]$, with $Sv = v$ (in which case we say that $v$ is even) and $Sw = -w$ (in which case $w$ is called odd), then $\langle v, w \rangle = 0$. For any $u \in L^2[0,1]$, we can write
\begin{equation}
(2.10) \quad u = v + w, \quad \text{where} \quad v = \frac{u + Su}{2} \quad \text{and} \quad w = \frac{u - Su}{2},
\end{equation}
and then $Sv = v$, $Sw = -w$ (and so $\langle v, w \rangle = 0$).

**Lemma.** The operator $J$ is compact (in fact, Hilbert-Schmidt), but not normal. Moreover, $J^* = SJS$, so that $JS$ and $SJ$ are self-adjoint operators on $L^2[0,1]$. For any $u \in L^2[0,1]$ we have
\begin{equation}
(2.12) \quad JSu + SJu = \langle u, 1 \rangle 1,
\end{equation}
where $1$ is the function taking the constant value $1$ on $[0,1]$.

**Proof.** For the compactness of $J$, see Meise and Vogt [13, Proposition 16.12 and Lemma 16.7(1)] or Pedersen [15, Proposition 3.4.16 and Lemma 3.4.5]. The other assertions are easily checked. $\square$

**Lemma.** Let $-1 \leq \lambda \leq 1$. For $u, v \in L^2[0,1]$, write
\begin{equation}
\langle u, v \rangle_\lambda = \langle u, v \rangle + \lambda \langle u, Sv \rangle.
\end{equation}
Then
\begin{equation}
(2.14) \quad (1 - |\lambda|) \langle u, u \rangle \leq \langle u, u \rangle_\lambda \leq (1 + |\lambda|) \langle u, u \rangle.
\end{equation}
If $-1 < \lambda < 1$ then $\langle \cdot, \cdot \rangle_\lambda$ is an inner product on $L^2[0,1]$.

In the degenerate cases $\lambda = \pm 1$, we have $\langle u, u \rangle_1 = 0 \iff Su = -u$, and $\langle u, u \rangle_{-1} = 0 \iff Su = u$, respectively.

**Proof.** This is routine, using $\langle u, Sv \rangle = \langle Su, v \rangle$. $\square$

For $-1 < \lambda < 1$, we shall denote by $L^2_\lambda$ the space $L^2[0,1]$ endowed with the inner product $\langle \cdot, \cdot \rangle_\lambda$. By (2.14), it is a Hilbert space.

**Lemma.** Let $-1 < \lambda < 1$. Then the operator $J_\lambda = JS + \lambda J$ is compact and self-adjoint on the Hilbert space $L^2_\lambda$.

**Proof.** If $u, v \in L^2[0,1]$, then using $J^* = SJS$ and $(JS)^* = JS$,\begin{align}
\langle J_\lambda u, v \rangle_\lambda &= \langle (JS + \lambda J)u, v \rangle + \lambda \langle (JS + \lambda J)u, Sv \rangle \\
&= \langle JSu, v \rangle + \lambda \left( \langle Ju, v \rangle + \langle JSu, Sv \rangle \right) + \lambda^2 \langle Ju, Sv \rangle \\
&= \langle u, JSv \rangle + \lambda \left( \langle u, SJSv \rangle + \langle u, Jv \rangle \right) + \lambda^2 \langle u, SJv \rangle \\
&= \langle u, J_\lambda v \rangle_\lambda.
\end{align}
The compactness of $J_\lambda$ follows from the compactness of $J$ on $L^2[0, 1]$, plus the fact that the norms of $L^2[0, 1]$ and $L^2_\lambda$ are equivalent. □

It follows from [13, Proposition 16.2] or [15, Theorem 3.3.8] that $L^2_\lambda$ has an orthonormal basis consisting of eigenfunctions for $J_\lambda$. More explicitly:

(2.16) Lemma. Let $-1 < \lambda < 1$ and set $\omega = \arccos \lambda \in (0, \pi)$. Then the functions

$$u_{\lambda,n}(t) = \frac{\sqrt{2}}{\sin \omega} \sin((\omega + 2\pi n) t), \quad n \in \mathbb{Z},$$

form a complete orthonormal basis of $L^2_\lambda$ consisting of eigenvectors of $J_\lambda$. In fact, $u_{\lambda,n}$ is an eigenfunction for the eigenvalue

$$\mu_{\lambda,n} = \frac{\sin \omega}{\omega + 2\pi n}.$$

Proof. Setting $u(t) = \sin(\vartheta t)$, where $\vartheta \neq 0$, we compute

$$J_\lambda u(t) = \int_0^t \sin(\vartheta(1 - s)) \, ds + \lambda \int_0^t \sin(\vartheta s) \, ds$$

$$= \sin \vartheta \sin(\vartheta t) + \frac{\lambda - \cos \vartheta}{\vartheta} (1 - \cos(\vartheta t)).$$

So if $\vartheta$ is such that $\cos \vartheta = \lambda$, then $J_\lambda u = \mu u$, where $\mu = \sin \vartheta / \vartheta$. Taking $\vartheta = \omega + 2\pi n$, we see that $u_{\lambda,n}$ is an eigenfunction for the eigenvalue $\mu_{\lambda,n}$. Since these eigenvalues are distinct for distinct $n$’s, the $u_{\lambda,n}$’s are orthogonal in $L^2_\lambda$. It is routine to check that they are in fact orthonormal.

Suppose that $u \in L^2_\lambda$ and that $\langle u, u_{\lambda,n} \rangle = 0$ for all $n \in \mathbb{Z}$. We claim that $u = 0$. Taking $v(t) = \sin(\vartheta t)$, where $\cos \vartheta = \cos \omega$, we find that $(v + (\cos \omega) S v)(t) = \sin(\vartheta t) \cos(\vartheta(1 - t))$.

So from $\langle u, u_{\lambda,n} + \lambda S u_{\lambda,n} \rangle = 0$ for all $n$ we find that

$$\int_0^1 u(1 - t) \cos((\omega + 2\pi n)t) \, dt = 0$$

(2.18) for all $n$. Adding and subtracting (2.18), and (2.18) with $n$ replaced by $-n$, we find that for all $n \in \mathbb{Z}$,

$$\int_0^1 \sin(2\pi nt) \sin(\omega t) u(1 - t) \, dt = 0$$

and

$$\int_0^1 \cos(2\pi nt) \cos(\omega t) u(1 - t) \, dt = 0.$$

The first of these conditions implies that $v(t) = \sin(\omega t) u(1 - t)$ satisfies $v(1 - t) = v(t)$ for almost all $t$, and the second condition implies that $w(t) = \cos(\omega t) u(1 - t)$ satisfies $w(1 - t) = -w(t)$ for almost all $t$. That is, for almost all $t$,

$$\sin(\omega(1 - t)) u(t) = \sin(\omega t) u(1 - t) \quad \text{and} \quad \cos(\omega(1 - t)) u(t) = -\cos(\omega t) u(1 - t).$$

Multiplying the first of these equations by $\cos(\omega t)$ and the second by $\sin(\omega t)$ and adding, we find that $u(t) = 0$ almost everywhere. Hence the family $\{u_{\lambda,n} : n \in \mathbb{Z}\}$ is a complete orthonormal basis for $L^2_\lambda$. □
3. Flows, and the space $\mathcal{M}^\perp$

We now study in detail the space $\mathcal{M}^\perp$ defined in Lemma 2.6; recall the defining relation (2.7).

Given our graph $X$, we consider the edge set $E = E(X)$ to be the set of ordered pairs $xy$, where $x, y \in X^0$ and $x \sim y$. We set $r(xy) = 1/c(xy)$, the resistance of the edge $xy$. Let $\ell^2(E, r)$ be the Hilbert space of all functions $\Phi : E(X) \to \mathbb{C}$ for which $\langle \Phi, \Phi \rangle < \infty$, where the inner product is

$$\langle \Phi, \Psi \rangle = \frac{1}{2} \sum_{x,y : x \sim y} \Phi(xy) \overline{\Psi(xy)} r(xy).$$

(3.1) Definition. A flow on the network $(X, c)$ is a function $\Phi \in \ell^2(E, r)$ such that

$$\sum_{y : y \sim x} \Phi(xy) = 0 \quad \text{for all } x \in X.$$ (3.2)

The flow is called odd, if $\Phi(xy) = -\Phi(yx)$, and it is called even, if $\Phi(xy) = \Phi(yx)$.

Our definition requires, in particular, that $\langle \Phi, \Phi \rangle < \infty$. The latter number is often called the energy – or, more appropriately, the power – of the flow $\Phi$. In the literature, the term flow usually applies to what we call an odd flow here. In this case, one may imagine each edge $[x, y]$ as a tube with unit length and cross-section $c(xy)$, the tubes are connected at the vertices of $X$, and the network of tubes is filled with liquid. Then $\Phi(xy)$ is the amount of liquid per time unit that flows from $x$ to $y$, whence $-\Phi(xy) = \Phi(yx)$ flows in the reverse direction. The condition (3.2) is Kirchhoff’s law: the amount of liquid per time unit that enters at any vertex coincides with the amount that exits. In particular, our flows have no source or sink – they are “passive flows”. In the above definition, even flows do not have such a nice physical interpretation. We shall write $\mathcal{J}^e$ and $\mathcal{J}^o$ for the (closed and orthogonal) subspaces of $\ell^2(E, r)$ consisting of all even and odd flows on the network $(X, c)$, respectively.

(3.3) Remark. A graph is called bipartite if we can partition its vertex set $X^0$ in two classes $C_1, C_2$ such that every edge has one endpoint in $C_1$ and the other in $C_2$. Equivalently, this means that $X$ has no odd cycles (as defined below). On a bipartite network, there is an obvious one-to-one correspondence between odd and even flows:

$$\Phi \in \mathcal{J}^o \iff \tilde{\Phi} \in \mathcal{J}^e,$$

where $\tilde{\Phi}(xy) = (-1)^i \Phi(xy)$, if $x \in C_i (1 = 1, 2)$.

Returning to $L^2(X^1, m^1)$, a function $F$ in that space is called even if $F(xy, 1-t) = F(xy, t)$, and odd if $F(xy, 1-t) = -F(xy, t)$, for all $t \in [0, 1]$ and each $xy \in E(X)$. Each $F$ has an orthogonal decomposition as a sum of its even and odd part.

It is straightforward to verify the following lemma.

(3.5) Lemma. (a) If $F \in \mathcal{M}^\perp$ is even (respectively, odd), and $u \in L^2[0, 1]$, then

$$\tilde{\Phi}(xy) = \Phi F_u(xy) = c(xy) \int_0^1 F(xy, t) u(t) \, dt$$

defines an even (respectively, odd) flow with $\langle \Phi, \Phi \rangle \leq \langle F, F \rangle \langle u, u \rangle$. 

(b) If \( \Phi \) is an even flow and \( u \in L^2[0,1] \) is even (respectively, if \( \Phi \) is an odd flow and \( u \in L^2[0,1] \) is odd), then

\[
F(xy,t) = \Phi(xy)u(t)/c(xy)
\]
defines an even (respectively, odd) function in \( \mathcal{M}^\perp \) with \( \langle F,F \rangle = \langle u,u \rangle \langle \Phi,\Phi \rangle \).

The simple proof is left to the reader. Regarding (a), note that when one of \( F \in \mathcal{M}^\perp \) and \( u \in L^2[0,1] \) is even and the other is odd, then \( \Phi_{F,u} \equiv 0 \). Thus, we may restrict to even \( u \) when \( F \) is even and to odd \( u \) when \( F \) is odd. We set \( d^e = \dim \mathcal{J}^e \) and \( d^o = \dim \mathcal{J}^o \) \((\leq \infty)\). In view of Lemma 3.5, the following is now the consequence of basic Fourier analysis.

\[\textbf{(3.6) Proposition.} \text{ Let } \{\Phi_m^e : 0 \leq m < d^e \} \text{ and } \{\Phi_m^o : 0 \leq m < d^o \} \text{ be orthonormal bases of the spaces } \mathcal{J}^e \text{ and } \mathcal{J}^o, \text{ respectively. Then an orthonormal basis of the subspace } \mathcal{M}^\perp \text{ of } L^2(X^1,m^1) \text{ defined by } (2.7) \text{ is given by the set of all functions:} \]

\[
C_{m,0}^e(xy,t) = \frac{\Phi_m^e(xy)}{c(xy)}, \quad C_{m,n}^e(xy,t) = \frac{\sqrt{2} \Phi_m^e(xy) \cos(2\pi nt)}{c(xy)}, \quad \text{and} \quad C_{m,n}^o(xy,t) = \frac{\sqrt{2} \Phi_m^o(xy) \sin(2\pi nt)}{c(xy)},
\]

where \( n \in \mathbb{N} = \{1,2,\ldots\} \), and \( 0 \leq m < d^e \) or \( 0 \leq m < d^o \), respectively.

A cycle in \( X \) is a sequence \( c = [x_0, \ldots, x_n] \) \((n \geq 3)\) of vertices such that \( x_0, \ldots, x_{n-1} \) are distinct, \( x_i x_{i+1} \in E(X) \) for \( i = 0, \ldots, n-1 \), and \( x_n = x_0 \). Associated with \( c \), there is the natural flow \( \Phi_c \in \mathcal{J}^o \) defined by

\[
\Phi_c(x_0x_{i+1}) = 1, \quad \Phi_c(x_ix_{i+1}) = -1 \quad (i = 0, \ldots, n-1),
\]

\[
\Phi_c(xy) = 0, \quad \text{if } xy \text{ is not an edge on } c.
\]

We remark that our cycle \( c = [x_0, x_1, \ldots, x_n] \) has an orientation, and that \( \Phi_{c^*} = -\Phi_c \) when \( c^* = [x_n, x_{n-1}, \ldots, x_0] \).

We now want to characterise those networks for which \( \mathcal{M}^\perp = \{0\} \). For this purpose we recall the following.

\[\textbf{(3.8) Definition.} \text{ The network } (X, \mathcal{C}) \text{ is called transient, if } \sum_{n \geq 0} \langle P^n \delta_x, \delta_y \rangle < \infty \text{ for some (equivalently, for all) } x, y \in X. \text{ Otherwise, the network is called recurrent.} \]

For the significance of this probabilistic notion, see e.g. \[19\] or \[17\].

\[\textbf{(3.9) Proposition.} \text{ One has } \mathcal{M}^\perp = \{0\} \text{ if and only if } (i) \text{ } X \text{ is a tree and (ii) after removal of any edge, at least one of the two connected components is recurrent as a subnetwork.} \]

\[\text{Proof.} \text{ Suppose that } X \text{ has a cycle } c. \text{ Then by Lemma 3.5(b) we can use the odd flow } \Phi = \Phi_c \text{ to construct a non-zero function in } \mathcal{M}^\perp. \]

Thus, \( X \) has to be a tree if \( \mathcal{M} = L^2(X^1,m^1) \). Now suppose that \( X \) is a tree.

It follows from the flow criterion for transience of networks that on the tree \( X \) there is a non-zero odd flow with finite power if and only if there is an edge that disconnects \( X \).
into two transient subtrees, see e.g. [17], Theorems 3.33 and 4.20. Thus, what is left is to show that on a tree, $\mathcal{M}^\perp \neq \{0\}$ if and only if there is a non-zero $\Phi \in \mathcal{J}^\circ$.

Suppose that $\Phi$ is such a flow. Then Lemma 3.3(b) shows how one can construct a non-zero, odd function in $\mathcal{M}^\perp$. Conversely, suppose that $F \in \mathcal{M}^\perp$ is non-zero. If the odd (respectively, even) part $F^o$ (respectively, $F^e$) of $F$ is non-zero then there must be an odd (respectively, even) function $u \in L^2[0, 1]$ such that the odd flow $\Phi_{F^o, u}$ (respectively, even flow $\Phi_{F^e, u}$) defined in Lemma 3.3(a) is non-zero. By Remark 3.3, when $F^e \neq 0$, the even flow $\Phi_{F^e, u}$ can be transformed into a non-zero odd flow, since every tree is bipartite. □

Our final goal in this section is to describe how one finds (orthonormal) bases of $\mathcal{J}^e$ and $\mathcal{J}^o$, when $X$ is a finite graph, in which case the flow spaces do not depend on the specific conductances assigned to the edges. A spanning tree of the graph $X$ is a subtree $Y$ of $X$ which contains all vertices of $X$. It defines a subnetwork $(Y, c_Y)$ whose conductance function $c_Y$ is the restriction of $c$ to $Y$. Recall that $E(X)$ consists of ordered pairs of adjacent vertices, that is, we have associated with each unoriented edge $[x, y]$ two oppositely oriented edges $xy$ and $yx$. It will be convenient to choose for each unoriented edge of $X$ one of its endpoints as the initial and the other as the terminal point. We write $\vec{E}(X)$ for the resulting set of oriented edges, so that $E(X)$ is the disjoint union of the sets of ordered pairs $\{xy : xy \in \vec{E}(X)\}$ and $\{xy : yx \in \vec{E}(X)\}$. Also, we set $\vec{E}(Y) = \vec{E}(X) \cap (Y \times Y)$.

Consider an edge $xy \in \vec{E}(X) \setminus \vec{E}(Y)$. Adding this edge to the tree, the new graph has precisely one cycle $c_{xy} = [x_0, \ldots, x_k]$ ($k \geq 3$) which is oriented such that $x = x_i$ and $y = x_{i+1}$ for some $i$. We define

$$\text{Cyc}(X : Y) = \{c_{xy} : xy \in \vec{E}(X) \setminus \vec{E}(Y)\}.$$  

The following is well known.

(3.11) Lemma. Let $Y$ be a spanning tree of the finite graph $X$. Then the set of flows $\{\Phi_c : c \in \text{Cyc}(X : Y)\}$ is a basis of $\mathcal{J}^o$. Every odd flow $\Phi$ in $X$ has the unique decomposition

$$\Phi = \sum_{xy \in \vec{E}(X) \setminus \vec{E}(Y)} \Phi(xy) \cdot \Phi_{c_{xy}}.$$  

Proof. The function

$$\Psi = \Phi - \sum_{xy \in \vec{E}(X) \setminus \vec{E}(Y)} \Phi(xy) \cdot \Phi_{c_{xy}}$$

vanishes on all edges in $E(X) \setminus \vec{E}(Y)$. Thus, $\Psi$ defines an odd flow in the finite tree $Y$, whence $\Psi \equiv 0$. Linear independence of the $\Phi_c$, $c \in \text{Cyc}(X : Y)$, is immediate. □

If $X$ is finite and bipartite, then all cycles are even (have even length), and (3.4) implies that the set of even flows

$$\{\bar{\Phi}_c : c \in \text{Cyc}(X : Y)\}$$

is a basis of $\mathcal{J}^e$. In general, the situation is slightly more complicated. We decompose

$$\text{Cyc}(X : Y) = \text{Cyc}^e(X : Y) \cup \text{Cyc}^o(X : Y),$$
where $\text{Cyc}^c(X : Y)$ consists of all even and $\text{Cyc}^o(X : Y)$ consists of all odd cycles in $\text{Cyc}(X : Y)$. If $\text{Cyc}^o(X : Y) \neq \emptyset$, then we can choose an (oriented) edge $x_0 y_0 \in \tilde{E}(X) \setminus \tilde{E}(Y)$ such that $c_0 = c_{x_0 y_0}$ is an odd cycle. Now let $xy \in \tilde{E}(X) \setminus \tilde{E}(Y)$ be any other edge such that $c = c_{xy}$ is odd. We can define an associated even flow $\Phi_{c_0,c}$ with $\Phi_{c_0,c}(xy) = 1$ by distinguishing the following two cases: (i) if $c$ and $c_0$ intersect, we can define $\Phi_{c_0,c}$ by suitably alternating the values 1 and $-1$ on all edges of $c_0 \cup c$ whose endpoints do not lie on both $c$ and $c_0$; (ii) If $c$ and $c_0$ do not intersect, we can define $\Phi_{c_0,c}$ by suitably alternating the values 1 and $-1$ on all edges of $c_0 \cup c$, and by suitably alternating the values 2 and $-2$ on all the edges of the unique path in the tree $Y$ that connects $c$ with $c_0$. The simple details are best understood by drawing a few figures.

(3.13) Lemma. Let $Y$ be a spanning tree of the finite graph $X$. Then the set of flows

$$
\{\tilde{\Phi}_c : c \in \text{Cyc}^c(X : Y)\} \cup \{\Phi_{c_0,c} : c \in \text{Cyc}^o(X : Y), c \neq c_0\}
$$

(where $c_0 \in \text{Cyc}^o(X : Y)$, if the latter set is non-empty) is a basis of $\mathcal{J}^e$. Every even flow $\Phi$ in $X$ has the unique decomposition

$$
\Phi = \sum_{xy \in \tilde{E}(X) \setminus \tilde{E}(Y), c_{xy} \text{ even}} \Phi(xy) \cdot \tilde{\Phi}_{c_{xy}} + \sum_{xy \in \tilde{E}(X) \setminus \tilde{E}(Y), c_{xy} \text{ odd}, xy \neq x_0 y_0} \Phi(xy) \cdot \Phi_{x_0,c_{xy}}.
$$

Proof. If $\Psi$ is the difference between $\Phi$ and the sum on the right hand side, then $\Psi$ is an even flow on the graph obtained from the tree $Y$ by adding the edge $x_0 y_0$ (if $X$ has odd cycles) or just on the tree $Y$ itself (if $X$ is bipartite). If $xy$ is an edge of that graph such that $y$ is the only neighbour of $x$, (3.2) implies that $\Psi(xy) = 0$, so that $\Psi$ also is an even flow on the graph that remains after deleting $x$ and the edge $xy$. Thus, after repeatedly “chopping off” finitely many edges where $\Psi = 0$, we are left with an even flow on the odd cycle $c_{x_0 y_0}$, which must vanish on each edge. Thus, $\Psi \equiv 0$. Once more, linear independence of the proposed basis is immediate. $\square$

If $X$ is finite, the last two lemmas provide a simple algorithm for finding bases of the (finite dimensional) spaces $\mathcal{J}^o$ and $\mathcal{J}^e$, which can orthonormalized by the Gram-Schmidt method. Then Proposition 3.6 leads to an orthonormal basis of the space $\mathcal{M}^+ \subset \ker A$.

4. The spectral measure, and proof of the main results

Recall the Spectral Theorem for a normal operator $T$ on a Hilbert space $\mathcal{H}$ (see [13, Chapter 18] or [15, §§ 4.4, 4.5], for example). Let $\mathcal{B}(\mathcal{H})$ denote the $C^*$-algebra of bounded linear operators on $\mathcal{H}$. Let $C^*(T)$ denote the closure in $\mathcal{B}(\mathcal{H})$ of the space of polynomials in $T$ and $T^*$. Then there is an isometric $*$-isomorphism $\Phi : f \mapsto f(T)$ from the $C^*$-algebra $\mathcal{C}(\text{spec}(T))$ of continuous functions on $\text{spec}(T)$ onto $C^*(T)$. This isomorphism maps the function $f(\lambda) \equiv \lambda^n$ to $T^n$ for $n = 0, 1, \ldots$. Now let $\mathcal{F}_\infty(K)$ denote the $C^*$-algebra of bounded Borel measurable functions on the compact set $K \subset \mathbb{C}$. Then there is a $*$-homomorphism $\Psi : \mathcal{F}_\infty(\text{spec}(T)) \to \mathcal{B}(\mathcal{H})$, also written $f \mapsto f(T)$, which extends $\Phi$, and which is continuous in the following sense: if $(f_n)$ is a uniformly bounded sequence of measurable functions on $\text{spec}(T)$ converging pointwise to a function $f$ on $\text{spec}(T)$, then $\langle f_n(T)x, y \rangle \to \langle f(T)x, y \rangle$ for each $x, y \in \mathcal{H}$. 
For each Borel set \( B \subset \text{spec}(T) \), denote by \( E(B) \) the operator \( \Psi(\mathbf{1}_B) \), where \( \mathbf{1}_B \) is the indicator function of \( B \). Each \( E(B) \) is a self-adjoint projection, and the map \( B \mapsto E(B) \) is called the spectral measure of \( T \). For each \( x, y \in H \), the map \( \mu_{x,y} : B \mapsto \langle E(B)x, y \rangle \) is a regular Borel complex measure, and

\[
\langle f(T)x, y \rangle = \int_{\text{spec}(T)} f(\lambda) \, d\mu_{x,y}(\lambda)
\]

for each \( f \in \mathcal{F}_\infty(\text{spec}(T)) \). It is convenient to also write \( \int_{\text{spec}(T)} f(\lambda) \, d\langle E_\lambda x, y \rangle \) for the integral on the right, interpreting the latter as a Lebesgue-Stieltjes integral with respect to the function \( \lambda \mapsto \langle E_\lambda x, y \rangle = \langle E((-\infty, \lambda])x, y \rangle \).

Now suppose that \( T \) is self-adjoint, so that \( \text{spec}(T) \subset \mathbb{R} \). If \( f \) is a bounded Borel measurable function defined on a Borel set of \( \mathbb{R} \) containing \( \text{spec}(T) \), then \( f(T) \) is by definition \( f(\text{spec}(T))(T) \). For example, in Lemma 18 below, we apply the above spectral theory to the operators \( A \) and \( J_\lambda = JS + \lambda J \), which are self-adjoint, and of norm at most 1, so that their spectra are contained in \([-1, 1]\). So if \( f \in \mathcal{F}_\infty([-1, 1]) \) we can form the operators \( f(A) \) and \( f(J_\lambda) \), acting on \( L^2(X^1, m^1) \) and \( L_\lambda^2 \), respectively, whenever \(|\lambda| < 1\).

Similarly, it is convenient to define \( E(B) = E(B \cap \text{spec}(T)) \) for any Borel subset \( B \) of \( \mathbb{R} \). With this notation, if \( \lambda \in \mathbb{R} \), then \( \lambda \in \text{spec}(T) \) if and only if the operator \( E((\lambda - \epsilon, \lambda + \epsilon)) \) is non-zero for each \( \epsilon > 0 \). Equivalently, \( \lambda \notin \text{spec}(T) \) if and only if \( f(A) = 0 \), the zero operator, for every continuous function \( f \) supported in \((\lambda - \epsilon, \lambda + \epsilon)\). Also, \( \lambda \) is an eigenvalue of \( T \) if and only if \( E(\{\lambda\}) \neq 0 \) (see [13] Lemma 18.5(3) and Proposition 18.14 or [13] Proposition 4.5.10).

In the sequel, \( E \) will always be the spectral measure of the operator \( P \).

The following Perron-Frobenius-type proposition concerning \( P \) can be found in the literature in a few places, mostly under the assumption that \( m^0(\cdot) \) is bounded away from 0 on \( X^0 \) (in which case it becomes easier). Since it appears not to be as well known as it should be, we include its proof, whose first part is extrapolated from Kersting [12] Lemma 3.1.

**Proposition (4.1)**. The operator \( P \) has eigenfunctions in \( \ell^2(X^0, m^0) \) for the eigenvalue 1 if and only if \( m^0(X^0) < \infty \). In this case, the 1-eigenspace \( \mathcal{H}_1 \) consists of the constant functions on \( X^0 \).

Furthermore, it has eigenfunctions in \( \ell^2(X^0, m^0) \) for the eigenvalue \(-1 \) if and only if \( m^0(X^0) < \infty \) and the graph \( X \) is bipartite. In this case, the \(-1\)-eigenspace \( \mathcal{H}_{-1} \) is spanned by the single function \( 1_{C_1} - 1_{C_2} \), where \( C_1 \) and \( C_2 \) are the two bipartite classes.

**Proof**. If \( m^0(X^0) < \infty \), then the constant functions are in \( \ell^2(X^0, m^0) \) and are eigenfunctions of \( P \) for the eigenvalue 1. If \( m^0(X^0) < \infty \) and \( X \) is bipartite, then \( 1_{C_1} - 1_{C_2} \) is an eigenfunction of \( P \) for the eigenvalue \(-1 \).

Conversely, suppose that \( g \in \ell^2(X^0, m^0) \) is nonzero, and \( Pg = \lambda_0 g \), where \( \lambda_0 \in \{-1, 1\} \).

We first show that \((X, c)\) must be recurrent (see Definition). For \( \lambda \in \mathbb{R} \), the operator \( E(\{\lambda\}) \) is the orthogonal projection of \( \ell^2(X^0, m^0) \) onto the \( \lambda \)-eigenspace of \( P \) (non-trivial
if and only if $\lambda \in \text{spec}_p(P)$. Then for each fixed $g_1, g_2 \in \ell^2(X^0, m^0)$,

$$
\langle P^n g_1, g_2 \rangle = \int_{[-1,1]} \lambda^n d\langle E_\lambda g_1, g_2 \rangle
= \langle E(\{1\})g_1, g_2 \rangle + (-1)^n \langle E(\{-1\})g_1, g_2 \rangle + o(1) \quad \text{as } n \to \infty
$$

by the Bounded Convergence Theorem. By hypothesis, $E(\{\lambda_0\})g = g \neq 0$, and there must be an $x \in X$ so that $E(\{\lambda_0\})\delta_x \neq 0$. So

$$
\langle P^{2n} \delta_x, \delta_x \rangle = \langle E(\{1\})\delta_x, \delta_x \rangle + \langle E(\{-1\})\delta_x, \delta_x \rangle + o(1)
= \|E(\{1\})\delta_x\|^2 + \|E(\{-1\})\delta_x\|^2 + o(1)
$$

tends to a nonzero limit as $n \to \infty$. Hence $\sum_{k=0}^\infty \langle P^k \delta_x, \delta_x \rangle = \infty$, and $(X, c)$ is recurrent.

Now $Pg = \lambda_0 g$, and so $|g| = |Pg| \leq P|g|$. Let $f = P|g| - |g|$. Since $\|g\|_2 = \|g\|_2$,

$$
\left\| \sum_{k=0}^{n-1} P^k f \right\|_2 = \left\| P^n |g| - |g| \right\|_2 \leq 2\|g\|_2,
$$

whence we have for each $x \in X^0$

$$
\sum_{k=0}^{n-1} \langle P^k \delta_x, \delta_x \rangle f(x) \leq \left\langle \sum_{k=0}^{n-1} P^k f, \delta_x \right\rangle \leq 2\sqrt{m^0(x)} \|g\|_2
$$

for each $x \in X^0$ and each integer $n \geq 1$. So if $f(x) > 0$ for some $x \in X^0$, then

$$
\sum_{k=0}^\infty \langle P^k \delta_x, \delta_x \rangle \leq 2\sqrt{m^0(x)} \|g\|_2/f(x) < \infty,
$$

contradicting recurrence. So $f = 0$. Therefore $|g|$ is a nonnegative harmonic function, that is, $P|g| = |g|$, and so is constant by recurrence, see e.g. [19, Theorem 1.16]. Since the constant is nonzero, and $g \in \ell^2(X^0, m^0)$, we have $m^0(X^0) < \infty$. Now fix $x_0 \in X^0$. Multiplying $g$ by a scalar, we may assume that $g(x_0) = 1$. Then $|g(y)| = 1$ for all $y \in X$, and

$$
\sum_{y:y \sim x_0} \frac{c(x_0, y)}{m^0(x_0)} |g(y)| = 1 = g(x_0) = \lambda_0 \sum_{y:y \sim x_0} \frac{c(x_0, y)}{m^0(x_0)} g(y) = \left| \sum_{y:y \sim x_0} \frac{c(x_0, y)}{m^0(x_0)} \lambda_0 g(y) \right|.
$$

Hence equality holds in the triangle inequality, and therefore $\lambda_0 g(y) = 1$ for each $y \in X^0$ such that $y \sim x_0$. So if $\lambda_0 = 1$, the connectedness of $X$ implies that $g(y) = 1$ for all $y \in X^0$. If $\lambda_0 = -1$, connectedness of $X$ implies that $g(y) = (-1)^{\text{dist}(x_0, y)}$, and that $X^0$ is bipartite, with $C_1$ and $C_2$ the sets of vertices at even and at odd distance from $x_0$, respectively. \hfill \Box

It follows from (2.2) that if $g_1, g_2 \in \ell^2(X^0, m^0)$ are in two mutually orthogonal subspaces of $\ell^2(X^0, m^0)$ which are also $P$-invariant, then $\langle F_{g_1, u_1}, F_{g_2, u_2} \rangle = 0$ for any $u_1, u_2 \in L^2[0, 1]$. So if $\mathcal{H}'$ denotes the orthogonal complement in $\ell^2(X^0, m^0)$ of the sum of the eigenspaces $\mathcal{H}_1$ and $\mathcal{H}_{-1}$ (which are at most 1-dimensional), then the orthogonal decomposition

$$(4.2) \quad \ell^2(X^0, m^0) = \mathcal{H}_1 + \mathcal{H}_{-1} + \mathcal{H}'$$

gives rise to a corresponding orthogonal decomposition of $\mathcal{M}$:

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_{-1} + \mathcal{M}'.\]
where $\mathcal{M}'$ is the closure of the linear span of functions $F_{g,u}$, where $g \in \mathcal{H}'$ and $u \in L^2[0,1]$, and $\mathcal{M}_{\pm 1}$ are constructed analogously from $\mathcal{H}_{\pm 1}$.

**Lemma (4.3).** The subspaces $\mathcal{M}_1$, $\mathcal{M}_{-1}$ and $\mathcal{M}'$ are invariant under $A$. Let $u \in L^2[0,1]$. For $g \in \mathcal{H}_1$, $AF_{g,u} = F_{g,(u,1)}$. For $g \in \mathcal{H}_{-1}$, $AF_{g,u} = 0$.

**Proof.** The invariance of the subspaces is immediate from Lemma 2.3. If $g \in \mathcal{H}_1$, then write $u = v + w$ as in (2.10). Then $F_{g,w} = 0$ because $g$ is constant. Thus $F_{g,u} = F_{g,v}$, and $AF_{g,u}$ equals

$$AF_{g,v} = F_{g,JSv} + F_{Pg,Jv} = F_{g,JSv} + F_{g,Jv} = F_{g,JSv+Jv} = F_{g,(v,1)} = F_{g,(u,1)}$$

because $JSv + Jv = 2Jv$, and applying (2.10) to $2Jv$ in place of $u$, we see that the “$v$-component” of $2Jv$ is $Jv + SJv = JSv + SJv = \langle v, 1 \rangle$ by (2.12).

Similarly, if $g \in \mathcal{H}_{-1}$, then write $u = v + w$ as in (2.10). Then $F_{g,w} = 0$ because $g(y) = -g(x)$ for each edge $xy$. Thus $F_{g,u} = F_{g,w}$, and $AF_{g,u}$ equals

$$AF_{g,w} = F_{g,JSw} + F_{Pg,Jw} = F_{g,JSw} + F_{-g,Jw} = F_{g,JSw-Jw} = F_{g,-2Jw} = 0$$

because applying (2.10) to $2Jw$ in place of $u$, we see that the “$w$-component” of $2Jw$ is $Jw - SJw = -(JSw + SJw) = -\langle w, 1 \rangle$ = 0 by (2.12).

The following is one of our main tools for linking the spectra of $P$ and $A$.

**Proposition (4.4).** Suppose that $g_1, g_2 \in \ell^2(X^0, m^0)$ and $u_1, u_2 \in L^2[0,1]$. Then for $n = 0, 1, \ldots,$

$$\langle A^n F_{g_1,u_1}, F_{g_2,u_2} \rangle = \int_{\text{spec}(P)} \langle J^n_{\lambda} u_1, u_2 \rangle \lambda d\langle E_{\lambda} g_1, g_2 \rangle.$$

**Proof.** The proof is by induction. By the Spectral Theorem for $P$,

$$\langle P^k g_1, g_2 \rangle = \int_{\text{spec}(P)} \lambda^k d\langle E_{\lambda} g_1, g_2 \rangle \quad \text{for } k = 0, 1, \ldots$$

Case $n = 0$ of (4.3) is immediate from (2.2), together with the cases $k = 0, 1$ of (4.6).

Assume (4.5) holds for $n$. By Lemma 2.3, $A^{n+1} F_{g_1,u_1} = A^n (F_{g_1,JSu_1} + F_{Pg_1,Ju_1})$. This and the induction hypothesis yield

$$\langle A^{n+1} F_{g_1,u_1}, F_{g_2,u_2} \rangle = \langle A^n F_{g_1,JSu_1}, F_{g_2,u_2} \rangle + \langle A^n F_{Pg_1,Ju_1}, F_{g_2,u_2} \rangle$$

$$= \int_{\text{spec}(P)} \left( \langle J^n_{\lambda} JSu_1, u_2 \rangle + \langle J^n_{\lambda} Ju_1, u_2 \rangle \right) \lambda d\langle E_{\lambda} g_1, g_2 \rangle$$

$$= \int_{\text{spec}(P)} \langle J^{n+1}_{\lambda} u_1, u_2 \rangle \lambda d\langle E_{\lambda} g_1, g_2 \rangle$$

because $\int_{\text{spec}(P)} f(\lambda) d\langle E_{\lambda} Pg_1, g_2 \rangle = \int_{\text{spec}(P)} \lambda f(\lambda) d\langle E_{\lambda} g_1, g_2 \rangle$ for any continuous function $f(\lambda)$ on $[a,b]$, as on seen by first taking $f(\lambda) = \lambda^k$, for $k = 0, 1, \ldots,$ and using (4.6). This completes the induction step.

Let $u_1, u_2 \in L^2[0,1]$, and let $f : [-1, 1] \to \mathbb{C}$ be bounded and Borel measurable. If $-1 < \lambda < 1$ then the function $\bar{f} = \bar{f}_{u_1,u_2}$

$$\bar{f}(\lambda) = \langle f(J_{\lambda} u_1, u_2) \rangle$$
For any $\lambda \in \mathbb{R}$ and letting $H$ be a fixed Hilbert space on which $J$ is selfadjoint. When $\lambda = \pm 1$ then we cannot speak of $f(J\lambda)$, because $\langle \cdot, \cdot \rangle_{\pm 1}$ is degenerate and we have no Hilbert space on which $J\lambda$ is selfadjoint.

We set $\mathrm{spec}(P)' = \mathrm{spec}(P) \cap (-1, 1)$. Recall the definition of the subspace $\mathcal{H}'$ of $\ell^2(X, \mu)$ in (4.2).

**Lemma 4.8.** If $u_1, u_2 \in L^2[0, 1]$, and $f : [-1, 1] \to \mathbb{C}$ is continuous, then the function $\tilde{f}$ defined on $(-1, 1)$ by (4.7), is continuous, and

$$
\langle f(A)F_{g_1,u_1}, F_{g_2,u_2} \rangle = \int_{\mathrm{spec}(P)'} \tilde{f}(\lambda) \, d\langle E_\lambda g_1, g_2 \rangle
$$

(4.9)

for all $g_1 \in \mathcal{H}'$ and $g_2 \in L^2(X, \mu)$. If $f$ is the indicator function $1_{\{0\}}$ of a singleton, then $\tilde{f}$ is bounded and Borel measurable on $(-1, 1)$, and (4.9) holds.

**Proof of Lemma 4.8.** First of all, notice that when $g_1 \in \mathcal{H}'$, the integral in Proposition 4.4 can be regarded as over $\mathrm{spec}(P)'$. For $\mathcal{H}_1$, $\mathcal{H}_{-1}$ and $\mathcal{H}'$ are the images of $E(\{1\})$, $E(\{-1\})$ and $E((-1,1))$, respectively. Hence $E(B)g_1 = E(B)E((-1,1))g_1 = E(B \cap (-1,1))g_1 = 0$ for any $B \subset \{-1,1\}$.

Next, for arbitrary $h : (-1, 1) \to \mathbb{C}$ which is bounded and Borel measurable, we have

$$
|h(\lambda)| \leq \|h(J\lambda)\| \|u_1\| \|u_2\| \lambda \leq 2 \|h\|_{\infty} \|u_1\|_2 \|u_2\|_2 \quad \text{for all} \quad \lambda \in (-1, 1),
$$

(4.10)

where $\|h(J\lambda)\|_\lambda$ is the operator norm on $L_\lambda^2$.

Now, given $f$ (continuous), choose a sequence $(p_n)$ of polynomials such that $p_n \to f$ uniformly on $[-1, 1]$. Clearly, $\tilde{p}_n(\lambda)$ is a polynomial, and replacing $h$ with $f - p_n$ in (4.10), we see that

$$
|\tilde{f}(\lambda) - \tilde{p}_n(\lambda)| \leq 2 \|f - p_n\|_{\infty} \|u_1\|_2 \|u_2\|_2
$$

for all $\lambda \in (-1, 1)$. Consequently, $(p_n)$ converges uniformly to $\tilde{f}$, and so the latter is a continuous function on $(-1, 1)$. By (4.5), we have

$$
\langle p_n(A)F_{g_1,u_1}, F_{g_2,u_2} \rangle = \int_{\mathrm{spec}(P)'} \tilde{p}_n(\lambda) \, d\langle E_\lambda g_1, g_2 \rangle,
$$

and letting $n \to \infty$, we see that this also holds for $f$ and $\tilde{f}$ in the place of $p_n$ and $\tilde{p}_n$, respectively.

Now let $f$ be the indicator function $1_{\{0\}}$ of a singleton. Then there is clearly a uniformly bounded sequence $(f_n)$ of continuous functions converging pointwise to $f$ on $[-1, 1]$. Then for each fixed $\lambda \in (-1, 1),

$$
\tilde{f}_n(\lambda) = \langle f_n(J\lambda)u_1, u_2 \rangle \to \langle f(J\lambda)u_1, u_2 \rangle = \tilde{f}(\lambda)
$$

by the Spectral Theorem applied to $J\lambda$ on $L_\lambda^2$. Hence $\tilde{f}$ is Borel measurable. Moreover, estimating as in (4.10), $|\tilde{f}_n(\lambda)| \leq 2 \|f_n\|_{\infty} \|u_1\|_2 \|u_2\|_2$, and so the functions $\tilde{f}_n$ are uniformly bounded on $(-1, 1)$. Hence (4.9) holds for $f$ by the Bounded Convergence Theorem, and by the Spectral Theorem applied to $A$, because it holds for each $f_n$. \hfill \Box

**Remark.** The assertions in the lemma can be extended in two directions, neither of which we need in the sequel:
(a) In Lemma 1.8 if $f$ is continuous, then a little more work in the proof shows that $\hat{f}$ has a continuous extension to $[-1, 1]$, and (4.9) is in fact valid for any $g_1 \in \ell^2(X^0, m^0)$, with $\text{spec}(P)'$ replaced by $\text{spec}(P)$.

(b) Using [13] Lemma 18.1, it is easy to see that the last statement in Lemma 1.8 is valid for any bounded measurable function $f$.

**Proof of Theorem 1.8.** We must show that $\text{spec}(A)$ equals $\mathcal{S}$, where

$$\mathcal{S} = \{0\} \cup \{\mu_{\lambda,n} : \lambda \in \text{spec}(P)' \text{, } n \in \mathbb{Z}\} \cup \{1 : 1 \in \text{spec}(P)\}.$$  

First note that the set $\mathcal{S}$ is closed. For suppose that $\mu \in \mathbb{R}$ is the limit of a sequence of points $\mu_j = \mu_{\lambda_j,n_j}$ in $\mathcal{S} \setminus \{0, 1\}$. If $|n_j| \to \infty$, then $\mu_j \to 0$, and so $\mu = 0 \in \mathcal{S}$. So taking a subsequence, we may assume that there is an $n \in \mathbb{Z}$ so that $n_j = n$ for all $j$. Since $\text{spec}(P)$ is compact, taking a further subsequence, we may suppose that $\lambda_j \to \lambda \in \text{spec}(P)$. If $\lambda \neq \pm 1$, then $\mu = \lim_{j \to \infty} \mu_j = \mu_{\lambda,n} \in \mathcal{S}$. If $\lambda \in \{-1, 1\}$, then $\mu_j \to 0$ unless $\lambda = 1$ and $n = 0$, in which case $\mu_j \to 1 \in \mathcal{S}$.

To show that $\text{spec}(A) \subset \mathcal{S}$, suppose that $\mu_0 \in [-1, 1], \setminus \mathcal{S}$. Then $[\mu_0 - \epsilon, \mu_0 + \epsilon] \cap \mathcal{S} = \emptyset$ for some $\epsilon > 0$. If $\mu_0 \neq 1$, we may assume that $1 \notin [\mu_0 - \epsilon, \mu_0 + \epsilon]$. Let $f$ be any continuous function on $[-1, 1]$ supported in $[\mu_0 - \epsilon, \mu_0 + \epsilon]$. For any $\lambda \in \text{spec}(P)'$, $f(J_\lambda) = 0$ because $f(\mu) = 0$ for all $\mu \in \text{spec}(J_\lambda) \subset \mathcal{S}$, by Lemma 2.16. By Lemma 1.8 $\langle f(A)F_{g_1,u_1}, F_{g_2,u_2} \rangle = 0$ for all $g_1 \in \mathcal{H}$, $g_2 \in \ell^2(X^0, m^0)$ and $u_1, u_2 \in L^2[0, 1]$. Therefore $f(A) = 0$ on the $A$-invariant subspace $\mathcal{M}'$ of $L^2(X^1, m^1)$. By Corollary 2.8 and Lemma 1.3 $A = 0$ on $\mathcal{M}^\perp$ and $\mathcal{M}_{-1}$, and is an orthogonal projection on $\mathcal{M}_{1}$, and since $f(0) = 0$, we get $f(A) = 0$ on $\mathcal{M}_{-1} + \mathcal{M}^\perp$. If $\mu_0 \neq 1$, then $f(A) = 0$ on $\mathcal{M}_{1}$ because then $f(0) = f(1) = 0$, as arranged above. If $\mu_0 = 1$, then $1 \notin \text{spec}(P)$ and $\mathcal{M}_1 = \{0\}$. So in both cases, $f(A) = 0$ on the whole of $L^2(X^1, m^1)$ for every continuous $f$ supported in $[\mu_0 - \epsilon, \mu_0 + \epsilon]$. Therefore $\mu_0 \notin \text{spec}(A)$, and $\text{spec}(A) \subset \mathcal{S}$.

For the reverse inclusion, suppose that $\mu_0 \in \mathcal{S}$.

Consider first $\mu_0 \neq 0, 1$. Then $\mu_0 = \mu_{\lambda_0,n_0}$ for some $\lambda_0 \in \text{spec}(P)'$ and $n_0 \in \mathbb{Z}$. Suppose that $0 < \epsilon < 1 - |\lambda_0|$. By [13] Proposition 4.4.5, there is a $g \in \ell^2(X^0, m^0)$ so that $\|Pg - \lambda_0 g\| < \epsilon \|g\|$. Let $u = u_{\lambda_0,n_0}$. Then

$$AF_{g,u} - \mu_0 F_{g,u} = F_{g,Su} + F_{Pu} - \mu_0 F_{g,u} = F_{Pg \lambda_0 g, u},$$

and this has norm at most $2 \|Pg - \lambda_0 g\| \|u\| \leq 2\epsilon \|g\| \|u\|$, by (2.2). Again by (2.2),

$$\|F_{g,u}\|^2 \geq (1 - |\lambda_0| - \epsilon) \|g\|^2 \|u\|^2$$

because $|\langle Pg, g \rangle| \leq (|\lambda_0| + \epsilon) \|g\|^2$. Therefore

$$\|AF_{g,u} - \mu_0 F_{g,u}\|^2 \leq \frac{4\epsilon^2}{1 - |\lambda_0| - \epsilon} \|F_{g,u}\|^2,$$

and since $F_{g,u} \neq 0$, it follows that $\mu_0 \in \text{spec}(A)$.

Next, suppose that $\mu_0 = 1 \in \mathcal{S}$. This can only happen when $1 \in \text{spec}(P)$. If 1 is not an isolated point of $\text{spec}(P)$ then there is a sequence $(\lambda_n) \in \text{spec}(P)$ such that $\lambda_n \to 1$ from below. But then we just showed that $\mu_{\lambda_n,0} \in \text{spec}(A)$, and $\mu_{\lambda_n,0} \to 1$. Therefore, $1 \in \text{spec}(A)$. If 1 is an isolated point of $\text{spec}(P)$, then it must be an eigenvalue ([13] Proposition 18.14(3)) or [13] Proposition 4.4.5), and by Proposition 4.1 the constant
function $g \equiv 1$ is an associated eigenfunction in $\ell^2(X^0, m^0)$. But then the constant function $F \equiv 1$ on $X^1$ is in $L^2(X^1, m^1)$, whence $1 \in \text{spec}(A)$.

Finally, consider $\mu_0 = 0 \in \mathcal{S}$. If $\text{spec}(P)$ contains some $\lambda \in (-1, 1)$ then $\text{spec}(A) \ni \mu_{\lambda,n} \to 0$ as $|n| \to \infty$, whence $0 \in \text{spec}(A)$. Otherwise, $\text{spec}(P) \subset \{-1, 1\}$, so that by Proposition 4.1 the space $\ell^2(X^0, m^0) = \mathcal{M}_1 + \mathcal{M}_{-1}$ is at most 2-dimensional, which can happen only when $X$ has exactly two vertices and one edge. But in this case, $\text{spec}(A) = \{0, 1\}$, by Lemma 4.3.

**Proof of Theorem 1.5.** We first show that

\begin{equation}
\text{spec}_p(A) \setminus \{0, 1\} = \{\mu_{\lambda,n} : n \in \mathbb{Z}, \lambda \in \text{spec}_p(P) \setminus \{-1, 1\}\}.
\end{equation}

If $\lambda \in \text{spec}_p(P) \setminus \{-1, 1\}$ and $n \in \mathbb{Z}$, let $g \in \ell^2(X^0, m^0)$ be non-zero, and satisfy $Pg = \lambda g$. Let $u = u_{\lambda,n}$. Then $AF_{g,u} = \mu_{\lambda,n} F_{g,u}$ and $F_{g,u} \neq 0$. So $\mu_{\lambda,n} \in \text{spec}_p(A)$.

Since $\omega = \arccos \lambda \in (0, \pi)$, note that if $n \in \mathbb{Z}$ then,

$$0 < |\mu_{\lambda,n}| = \left| \frac{\sin \omega}{\omega + 2\pi n} \right| \leq \frac{\sin \omega}{\omega} < 1$$

So $\mu_{\lambda,n} \in \text{spec}_p(A) \setminus \{0, 1\}$.

On the other hand, let $\mu \in \text{spec}_p(A) \setminus \{0, 1\}$. Since $\mu \in \text{spec}(A)$, we can write $\mu = \sin \omega / \omega$ for some $\omega \in \mathbb{R}$ with $\lambda = \cos \omega \in \text{spec}(P)$, or equivalently, $\mu = \mu_{\lambda,n}$ for some $n \in \mathbb{Z}$. Since $\mu \neq 0$, a glance at the curve $\omega \mapsto \sin \omega / \omega$ shows that the number of solutions $\omega$ to $\sin \omega / \omega = \mu$ is finite. That is, the number of pairs $(\lambda, n) \in \text{spec}(P) \times \mathbb{Z}$ for which $\mu_{\lambda,n} = \mu$ is finite. Thus, the set $F_\mu$ of all $\lambda \in \text{spec}(P)$ such that $\mu = \mu_{\lambda,n}$ for some $n \in \mathbb{Z}$ is a finite set. Also, $F_\mu \subset \text{spec}(P)'$, since $\mu \notin \{0, 1\}$.

Let $f = 1_{(\mu)}$. By assumption, $\mu \in \text{spec}_p(A)$. So $f(A)$, being the orthogonal projection onto the $\mu$-eigenspace of $A$, is non-zero. But $f(A)F = 0$ for all $F \in \mathcal{M}_1 \cup \mathcal{M}_{-1} \cup \mathcal{M}^\perp$, because $\mu \neq 0, 1$. So $f(A)$ cannot vanish on $\mathcal{M}'$. Thus there exist $g \in \mathcal{H}'$ and $u \in L^2[0, 1]$ so that $\langle f(A)F_{g,u}, F_{g,u} \rangle > 0$. Then by (4.9), we must have $\langle f(J_\lambda)u, u \rangle > 0$ for some $\lambda \in \text{spec}(P)'$. For any such $\lambda$, $f(J_\lambda) \neq 0$, and so $\mu = \mu_{\lambda,n}$ for some $n$. Therefore $\lambda \in F_\mu$. So the integrand in (4.9) is non-zero only for $\lambda \in F_\mu$, at most, and so the measure $B \mapsto \langle E(B)g, g \rangle$ must assign nonzero measure to $F_\mu$, and therefore to $\{\lambda\}$ for some $\lambda \in F_\mu$. This $\lambda$ must be in $\text{spec}_p(P)$, and $\mu = \mu_{\lambda,n}$ for some $n \in \mathbb{Z}$. This completes the proof of (4.12).

Now suppose that $1 \in \text{spec}_p(P)$. Then $0, 1 \in \text{spec}_p(A)$ because there are eigenfunctions of $A$ in $\mathcal{M}_1$ for both $0$ and $1$, by Lemma 4.3. If $1 \notin \text{spec}_p(P)$, then $\mathcal{H}' = \ell^2(X^0, m^0)$ and $\text{spec}(P)' = \text{spec}(P)$. So by Lemma 4.8 applied to $f = 1_{(0)}$ and $f = 1_{(1)}$, we see that $f(A)F = 0$ for any $F \in \mathcal{M}$, because $0$ and $1$ are not in the point spectrum of any $J_\lambda$, $\lambda \in (-1, 1)$. Of course $0 \in \text{spec}_p(A)$ if $\mathcal{M} \subsetneq L^2(X^1, m^1)$, since $A$ is zero on $\mathcal{M}^\perp$. This, together with Propositions 3.9 and 4.1, proves the last statement of Theorem 1.5.

Theorem 1.6 and its proof applies, in particular, to the case when the graph $X$ is finite. In this case, we also obtain an orthonormal basis of the operator $A$ by combining Theorem 1.6 with the following. In the proof of the next lemma, it is convenient to define $L^2_\lambda$ also when $\lambda \in \{-1, 1\}$. We define $L^2_1$ (respectively $L^2_{-1}$) to be the set of $u \in L^2[0, 1]$ such that $Su = u$ (respectively $Su = -u$). Note that for both these $\lambda$’s, $\langle u, v \rangle_\lambda = 2\langle u, v \rangle$ for $u, v \in L^2_\lambda$, so that $L^2_\lambda$ is a Hilbert space. Recall the definition 2.5 of $\mathcal{M}_0$. 

\[ \text{proposition} \]
(4.13) Lemma. Suppose that \( X \) is finite, and let \( g_1, \ldots, g_m \) be an orthonormal basis for \( \ell^2(X^0, m^0) \) consisting of eigenfunctions of \( P \), with \( Pg_j = \lambda_j g_j \) (with \(-1 \leq \lambda_j \leq 1\)) for each \( j \). Then \( \mathcal{M}_0 \) is closed, and consists of all functions

\[
F = F_{g_1,u_1} + \cdots + F_{g_m,u_m},
\]

where \( u_j \in L^2[0,1] \) for each \( j \), and where

\[
S u_j = u_j \text{ if } \lambda_j = 1, \quad \text{and} \quad S u_j = -u_j \text{ if } \lambda_j = -1.
\]

Moreover, the map \( F \mapsto (u_1, \ldots, u_m) \) is a linear isometry of \( \mathcal{M}_0 \) onto the orthogonal direct sum of the spaces \( L^2_{\lambda_j}, j = 1, \ldots, m \).

Proof. Each \( g \in \ell^2(X^0, m^0) \) is a linear combination of the \( g_j \)'s, and so each \( F_{g,u} \) is a sum of functions \( F_{g_j,u_j} \). If \( \lambda_j = 1 \), then \( g_j \) is constant on \( X^0 \), and so for each edge \( xy \), and any \( u \in L^2[0,1] \),

\[
F_{g_j,u}(xy,t) = u(1-t)g_j(x) + u(t)g_j(y) = (u(1-t) + u(t))g_j(x).
\]

Therefore, if \( Su = -u \), we have \( F_{g_j,u} \equiv 0 \). Applying (2.10) to \( u = u_j \), we see that \( F_{g_j,u} = F_{g_j,v} + F_{g_j,w} = F_{g_j,v} \). Similarly, if \( \lambda_j = -1 \), then \( F_{g_j,u} = F_{g_j,w} \), since \( g_j(y) = -g_j(x) \) for any edge \( xy \). Therefore each \( F_{g,u} \) can be written in the form (4.14), where (4.15) holds.

If \( F \) is as in (4.13), then \( \langle F_{g_j,u_j}, F_{g_k,u_k} \rangle = 0 \) if \( j \neq k \), and so

\[
\left\| \sum_{j=1}^{m} F_{g_j,u_j} \right\|^2 = \sum_{j=1}^{m} \left( \langle g_j, g_j \rangle \langle u_j, u_j \rangle + \lambda_j \langle g_j, g_j \rangle \langle u_j, Su_j \rangle \right) = \sum_{j=1}^{m} \langle u_j, u_j \rangle \lambda_j.
\]

It follows that \( \mathcal{M}_0 \) is isometric to the direct sum of the Hilbert spaces \( L^2_{\lambda_j} \), recalling the special definition of \( L^2_{\lambda} \) made above when \( \lambda = \pm 1 \). Therefore \( \mathcal{M}_0 \) is complete for its inner product, and so closed in \( L^2(X^1, m^1) \). \( \square \)

(4.16) Corollary. Suppose that \( X \) is finite, and let \( g_1, \ldots, g_N \) be an orthonormal basis for \( \ell^2(X^0, m^0) \) consisting of eigenfunctions of \( P \), with \( Pg_j = \lambda_j g_j \) (where \(-1 \leq \lambda_j \leq 1\)) for each \( j \). We assume that \( \lambda_1 = 1 \) and \( g_1 \) is constant, and when \( X \) is bipartite, that \( \lambda_N = -1 \) and \( g_N \) is constant on each bipartite class. Then

\[
\text{spec}(A) = \text{spec}_p(A) = \{0, 1\} \cup \{\mu_{\lambda_j,n} : |\lambda_j| < 1, \ n \in \mathbb{Z}\},
\]

with \( \mu_{\lambda_j,n} \) as in (2.17).

(i) An orthonormal basis of the subspace \( \mathcal{M} \) of \( L^2(X^1, m^1) \) consisting of eigenvectors of \( A \) is obtained as follows. (Note that \( m^1(X^1) = m^0(X^0)/2 \).)

(a) For the eigenvalue \( \mu = 1 \), the eigenspace is spanned by the function

\[
F^{(1,0)}(xy,t) = \frac{1}{\sqrt{m^1(X^1)}}.
\]

(b) For the eigenvalue \( \mu = 0 \), then the eigenspace is spanned by the functions

\[
F^{(1,n)}(xy,t) = \frac{\sqrt{2}}{\sqrt{m^1(X^1)}} \cos(2\pi nt), \quad n \in \mathbb{N},
\]
if $X$ is not bipartite. If $X$ is bipartite, then the eigenspace is spanned by the $F_{1,n}$ and the functions

$$F^{(-1,n)}(xy,t) = (-1)^{i(x)} \frac{\sqrt{2}}{\sqrt{m^1(X)}} \sin(2\pi nt), \quad n \in \mathbb{N},$$

where $i(x) = 1$ or $2$ according to whether $x$ lies in the bipartite class $C_1$ or $C_2$.

(c) For the eigenvalue $\mu$ with $0 < |\mu| < 1$, the eigenspace is spanned by the functions

$$F^{(j,n)}(xy,t) = \frac{\sqrt{2}}{\sin \omega_j} \left( g_j(x) \sin((\omega_j + 2\pi n)(1-t)) + g_j(y) \sin((\omega_j + 2\pi n)t) \right),$$

where $(j,n) \in \{1, \ldots, N\} \times \mathbb{Z}$ is such that $\mu_{\lambda_j,n} = \mu$ and $\omega_j = \arccos \lambda_j$.

(ii) An orthonormal basis of the subspace $\mathcal{M}^\perp \subset \ker A$ is obtained via Proposition 3.6 in combination with Lemmas 3.11 and 3.13.

5. Final remarks and examples

(5.1) Remark. One may ask why we call our operator $A$, defined in (1.2), the averaging operator, and not the one which takes the pure $m^1$-average over balls of radius 1. The latter is given by

$$\tilde{A}F(xy,t) = \frac{1}{(1-t)m^0(x) + t m^0(y)} \times \left( \sum_{u \sim x} c(xu) \int_0^{1-t} F(xu,s) \, ds + \sum_{v \sim y} c(yv) \int_0^t F(yv,s) \, ds \right).$$

The point is that unlike $A$, the latter operator does not enjoy a nice and natural compatibility with the transition operator $P$ and the Laplace operator on a network. Note, however, that $A = \tilde{A}$ when $m^0(\cdot)$ is constant. This occurs, in particular, when the graph $X$ is locally finite and regular, and $c(xy) = 1$ for each edge $xy$, in which case $P$ is called the simple random walk (SRW) operator.

We now give a few examples of locally finite, regular graphs with conductances $c(xy) = 1$ for each edge $[x,y]$, where the spectrum of $A$ can be computed via the (known) spectrum of $P$.

(5.2) Example. Equip the additive group $\mathbb{Z}$ of all integers with the typical graph structure, where the edges are between $x$ and $x+1$, $x \in \mathbb{Z}$. Then the SRW operator $P$, associated with conductances $\equiv 1$, is the convolution operator

$$P f = \varphi \ast f \quad \text{with} \quad \varphi = \frac{1}{2} (1_{\{1\}} + 1_{\{-1\}}).$$

Computing the Fourier transform $\hat{\varphi}(\omega) = \cos \omega$, $\omega \in [0, 2\pi]$, one finds the very well known fact that

$$\text{spec}(P) = [-1, 1] \quad \text{and} \quad \text{spec}_p(P) = \emptyset.$$
On the other hand, the one-skeleton of the network is the real line \( \mathbb{R} \) (with the integer points singled out as vertices), so that \( A \) is the convolution operator

\[
AF = \varphi * F \quad \text{with} \quad \varphi = \frac{1}{2} 1_{[-1,1]}.
\]

Since the Fourier transform of \( \varphi \) is \( \widehat{\varphi}(\omega) = (\sin \omega) / \omega \), \( \omega \in \mathbb{R} \), one finds that

\[
\text{spec}(A) = \{(\sin \omega^*) / \omega^*, 1\} \quad \text{and} \quad \text{spec}_p(A) = \emptyset,
\]

where \( \omega^* \) is the smallest positive solution of the equation \( \tan \omega = \omega \), so that \( (\sin \omega^*) / \omega^* = \cos \omega^* \). Numerical computation gives \( \omega^* = 4.493409 \ldots \) and \( \sin(\omega^*) / \omega^* = -0.217233 \ldots \).

**Example.** Equip the additive group \( \mathbb{Z}_N = \mathbb{Z}/(NZ) = \{0, \ldots, N - 1\} \) of integers modulo \( N \) with the structure of a cycle, where \( x \sim x + 1 \) (mod \( N \)) for \( x \in \mathbb{Z}_N \). If each edge is assigned conductance \( 1 \), then \( m^0 \) becomes twice the counting measure. An orthonormal basis of eigenvectors of the SRW operator \( P \) acting on \( \ell^2(\mathbb{Z}_N, m^0) \) with associated eigenvalues is given by

\[
g_j(x) = (2N)^{-1/2} \exp(2\pi i x j/N) \quad \text{and} \quad \lambda_j = \cos(2\pi j/N), \quad j = 0, \ldots, N - 1,
\]

so that \( \lambda_j = \lambda_{N-j} \) (1 \( \leq \) \( j \) \( \leq \) \( N - 1 \)) has multiplicity 2 unless \( j = N/2 \) for even \( N \).

The one-skeleton of \( \mathbb{Z}_N \) can be identified with the torus, interpreted as the compact additive group \( \mathbb{R}/(N\mathbb{R}) \) of real numbers modulo \( N \), parametrized by the interval \([0, N)\), or equivalently, \((-N/2, N/2)\), with Lebesgue measure. Thus, as in Example 5.2, the averaging operator \( A \) becomes the convolution operator \( AF = \varphi * F \) with \( \varphi = \frac{1}{2} 1_{[-1,1]} \), but this time modulo \( N \). The Fourier transform (on \( \mathbb{Z} \)) of \( \varphi \) is \( \widehat{\varphi}(n) = (\sin \varpi_n) / \varpi_n \), and

\[
\text{spec}(A) = \{(\sin \varpi_n) / \varpi_n : n \in \mathbb{Z}\}, \quad \text{where} \quad \varpi_n = 2\pi n / N.
\]

(We set \( (\sin 0)/0 = 1 \) by continuous extension.) The orthonormal basis of associated eigenfunctions is given by

\[
F_n(t) = N^{-1/2} \exp(i \varpi_n t), \quad t \in [0, N), \quad n \in \mathbb{Z}.
\]

Since the eigenvalues corresponding to \( \varpi_n \) and \( \varpi_n \) coincide, the eigenspace has dimension 2, unless \( n \) is a multiple of \( N \), or – when \( N \) is even – a multiple of \( N/2 \).

Relating this with Corollary 4.16, we get the following: for \( 1 \leq j < N/2 \), \( \mu_{\lambda_j, n} = \mu_{\lambda_{N-j}, n} = (\sin \varpi_{j+nN}) / \varpi_{j+nN} \), and elementary computations yield for the functions \( F^{(j, n)} \) and \( F^{(N-j, n)} \) of Corollary 4.16(c) that \( F^{(j, n)}(xy, t) = F_{j+nN}(x + t) \) and \( F^{(N-j, n)}(xy, t) = F_{-j-nN}(x + t) \), if \( x \in \{0, \ldots, N - 1\}, y = x + 1 \) (mod \( N \)), and \( t \in [0, 1] \).

The eigenspace associated with the eigenvector 1 consists of course of the constant functions.

Finally, \( \ker A \) is spanned by the functions \( F_{N, n}(t), n \in \mathbb{Z} \setminus \{0\} \), if \( N \) is odd, and by the functions \( F_{N/2, n}(t), n \in \mathbb{Z} \setminus \{0\} \), if \( N \) is even. Comparing with Corollary 4.16, we have to consider the decomposition \( \ker A = (\mathcal{M} \cap \ker A) \oplus \mathcal{M}^\perp \). If \( N \) is odd, then \( \mathcal{M} \cap \ker A \) is spanned by the functions \( F^{(1, n)}(xy, t) = \sqrt{2} \mathfrak{R}(F_{N, n}(x + t)) \), and – since \( \mathcal{J}^e = \{0\} \) and \( \dim \mathcal{J}^e = 1 \) – the space \( \mathcal{M}^\perp \) is spanned by the functions \( G_{0, n}(xy, t) = \sqrt{2} \mathfrak{R}(F_{N, n}(x + t)) \), \( n \in \mathbb{N} \), where \( \mathfrak{R}(\cdot) \) and \( \mathfrak{I}(\cdot) \) denote real and imaginary part.

When \( N \) is even, the situation is slightly more complicated. In this case, \( \mathcal{J}^e \) is one-dimensional and spanned by the even flow with value (-1)^e on the edge \([x, y]\) with \( y =
and maximum, respectively, of all numbers $|\omega|$. Indeed, in the limit, as $q \to \infty$, these functions are even discontinuous, whence they have to be expressed as Fourier series in terms of the functions $F_{n/2}$, $n \in \mathbb{Z}$.

(5.4) Example. Let $\mathbb{T}$ be the homogeneous tree with degree $q + 1$, where $q \geq 1$. It is well known that the spectrum of the SRW operator on $\mathbb{T}$ is

$$\text{spec}(P) = [-\rho, \rho], \quad \rho = \frac{2\sqrt{q}}{q+1}, \quad \text{and} \quad \text{spec}_p(P) = \emptyset,$$

see [14, §7.B] and the references given there. Removal of any edge splits $\mathbb{T}$ into two transient pieces. Therefore, in view of our theorems, $\text{spec}_p(A) = \{0\}$,

$$\text{spec}(A) = \left\{ \frac{\sin \omega}{\omega} : \omega > 0, \ |\cos \omega| \leq \rho \right\} \cup \{0\}, \quad \text{and} \quad \rho(A) = \frac{q-1}{q+1} / \arccos \frac{2\sqrt{q}}{q+1}.$$

The last formula for the spectral radius of $A$ was first found by SALLOFF-COSTE AND WOESS [16] by a completely different method, and indeed, the latter was the starting point for the present investigation.

A closer look at $\text{spec}(A)$ may be of interest. Let $\omega_{\pm\rho} = \arccos(\pm \rho)$, where $\rho = \rho(P)$, so that $0 < \omega_\rho < \pi/2$ and $\omega_{-\rho} = \pi - \omega_\rho$. For $n \geq 0$, let $m_n$ and $M_n$ be the minimum and maximum, respectively, of all numbers $|\sin \omega|/\omega$, where $n\pi + \omega_\rho \leq \omega \leq n\pi + \omega_{-\rho}$. In particular, $M_0 = \rho(A)$. Also, $M_n > M_{n+1}$ and $m_n > m_{n+1}$. Then

$$\text{spec}(A) = \{0\} \cup \bigcup_{n \geq 0} [m_{2n}, M_{2n}] \cup \bigcup_{n \geq 0} [-M_{2n+1}, -m_{2n+1}].$$

A closer analysis of these intervals shows that there is an $n_0$ such that $[m_n, M_n]$ and $[m_{n+2}, M_{n+2}]$ overlap for all $n \geq n_0$, whence $\text{spec}(A)$ is a union of finitely many intervals, one of which contains $[-M_{n_0}, M_{n_0+1}]$ if $n_0$ is odd, and $[-M_{n_0+1}, M_{n_0}]$ if $n_0$ is even.

When $q = 2$, one can verify that $n_0 = 0$, whence $\text{spec}(A) = [-M_1, M_0]$, but as $q$ increases, $\text{spec}(A)$ decreases and becomes a finite disjoint union of more and more intervals. Indeed, in the limit, as $q \to \infty$, $\text{spec}(A)$ tends to the set $\{0\} \cup \{(2n+1)/\pi : n \geq 0\}$.

Finally, we determine $-M_1 = \min\{\text{spec}(A)\}$. As in Example 5.2, let $\omega^* \in (\pi, \frac{3\pi}{2})$ be the smallest positive solution of $\tan \omega = \omega$. Then

$$-M_1 = \begin{cases} \frac{\sin \omega^*}{\omega^*} = \cos \omega^*, & \text{if } \pi + \omega_\rho < \omega^*, \\ \frac{\sin(\pi + \omega_\rho)}{\pi + \omega_\rho} = -\frac{q-1}{q+1} / \left( \pi + \arccos \frac{2\sqrt{q}}{q+1} \right), & \text{otherwise.} \end{cases}$$

The first of these two cases holds precisely when $\tan(\pi + \omega_\rho) < \pi + \omega_\rho$, or equivalently, when $\frac{q-1}{2\sqrt{q}} < \pi + \arccos \frac{2\sqrt{q}}{q+1}$, that is, for $q \leq 82$ by numerical computation.

It seems unlikely that these results could have been found without using the relation between the spectra of $P$ and $A$.

(5.5) Example. Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two homogeneous trees with degrees $q + 1$ and $r + 1$, respectively. In each of the trees, we choose a boundary point (end) and the associated
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Busemann (horocycle) function $h : \mathbb{T}_1 \to \mathbb{Z}$. The Diestel Leader graph $DL(q,r)$ is the horocyclic product of the two trees, i.e., the subgraph of their direct product,

$$DL(q,r) = \{ x_1x_2 \in \mathbb{T}_1 \times \mathbb{T}_2 : h(x_1) + h(x_2) = 0 \}.$$ 

See e.g. [1] for a detailed description and further references. In particular, in [1] it is shown that for the SRW operator $P$ on $DL = DL(q,r)$, the spectrum $\text{spec}(P) = [-\rho(P), \rho(P)]$ is pure point, i.e., it is the closure of the point spectrum, and there is an orthonormal basis of $\ell^2(DL)$ consisting of finitely supported eigenfunctions of $P$. One has

$$\text{spec}_p(P) = \left\{ \rho(P) \cos \frac{m}{n} \pi : n \geq 2, 1 \leq m \leq n-1 \right\}, \quad \rho(P) = \frac{2\sqrt{qr}}{q+r}.$$ 

By our theorems, we can compute the spectrum of $A$, which is also pure point and contains 0, since $DL$ is not a tree.

In the specific case when $r = q$, $DL(q,r)$ is a Cayley graph of the lamplighter group (wreath product) $\mathbb{Z}_q \wr \mathbb{Z}$, see again [1] for details. In that case, the spectrum of $P$ had been determined previously in [8], $\rho(P) = 1$, and the point spectrum of $A$ has the following particularly nice form:

$$\text{spec}_p(A) = \{0\} \cup \left\{ \frac{\sin \frac{m}{n} \pi}{(\frac{m}{n} + 2k)\pi} : n \geq 2, 1 \leq m \leq n-1, k \in \mathbb{Z} \right\} \quad (q = r).$$

In [1], an orthonormal basis of $\ell^2(DL^0)$ consisting of eigenvectors of $P$ is computed. One can of course adapt Corollary 4.16 in order to transfer the latter into an orthonormal basis of the subspace $\mathcal{M}$ of $L^2(DL^1)$ consisting of eigenvectors of $A$.

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