ANOSOV AUTOMORPHISMS OF NILPOTENT LIE ALGEBRAS

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Abstract. Each matrix $A$ in $GL_n(\mathbb{Z})$ naturally defines an automorphism $f$ of the free $r$-step nilpotent Lie algebra $f_{n,r}$. We study the relationship between the matrix $A$ and the eigenvalues and rational invariant subspaces for $f$. We give applications to the study of Anosov automorphisms.

1. Introduction

1.1. Anosov maps. Anosov maps are fundamental objects in the field of dynamical systems. A $C^1$ diffeomorphism $f$ of a compact Riemannian manifold $M$ is called Anosov if there exist constants $\lambda$ in $(0, 1)$ and $c > 0$ along with a $df$-invariant splitting $TM = E^s \oplus E^u$ of the tangent bundle of $M$ so that for all $n \geq 0$,

$$\|df^n_x v\| \leq c \lambda^n \|v\| \quad \text{for all } v \text{ in } E^s(x), \text{ and}$$

$$\|df^{-n}_x v\| \leq c \lambda^n \|v\| \quad \text{for all } v \text{ in } E^u(x).$$

The standard example of an Anosov map is a toral map defined by a unimodular hyperbolic automorphism of $\mathbb{R}^n$ that preserves an integer lattice.

The only other known examples of Anosov maps arise from automorphisms of nilpotent groups. A hyperbolic automorphism of a simply connected nilpotent Lie group $N$ that fixes a torsion-free lattice $\Gamma < N$ descends to an Anosov diffeomorphism of the compact nilmanifold $N/\Gamma$. It is also possible that such an $N/\Gamma$ has a finite quotient, called an infranilmanifold, and that the Anosov map on $N/\Gamma$ finitely covers an Anosov map of the infranilmanifold.

An automorphism of a Lie algebra that descends to an Anosov map of a compact quotient of the corresponding Lie group is called an Anosov automorphism. Nilpotent Lie algebras are the only Lie algebras that admit Anosov automorphisms. A Lie algebra $\mathfrak{n}$ is called Anosov if there exists a basis $\mathcal{B}$ for $\mathfrak{n}$ with rational structure constants, and there exists a hyperbolic automorphism $f$ of $\mathfrak{n}$ with respect to which $f$ is represented relative to $\mathcal{B}$ by a matrix in $GL_n(\mathbb{Z})$. A simply connected Lie group admits a hyperbolic automorphism preserving a lattice if and only if its Lie algebra is Anosov ([AS70]).
In this paper we study the properties of Anosov automorphisms and Anosov Lie algebras. There has already been some progress in this area. S. G. Dani showed that the free $r$-step nilpotent Lie algebra $f_{n,r}$ on $n$ generators admits an Anosov automorphism when $r < n$ (Dan78). Dani and Mainkar considered when two-step nilpotent Lie algebras defined by graphs admit Anosov automorphisms (DM05). Real Anosov Lie algebras and all their rational forms have been classified in dimension eight and less (Ito89, LW). Lauret observed that the classification problem for Anosov Lie algebras contains within it the problem of classifying all Lie algebras admitting $\mathbb{Z}^+$ derivations (Lau03b, Lau03a).

Auslander and Scheuneman established the correspondence between Anosov automorphisms of nilpotent Lie algebras and semisimple hyperbolic automorphisms of free nilpotent Lie algebras preserving ideals of a certain type (AS70). A matrix $A$ in $GL_n(\mathbb{Z})$, together with a rational basis $B$ of $f_{n,r}$ induces an automorphism $f_A$ of $f_{n,r}$. Suppose that $i$ is an ideal of $f_{n,r}$ such that

1. $i$ is invariant under $f_A$,
2. the restriction of $f_A$ to $i$ is unimodular,
3. $i$ has a basis that consists of $\mathbb{Z}$-linear combinations of elements of $B$, and
4. all eigenspaces for $f_A$ for eigenvalues with modulus one are contained in $i$.

If we let $n = f_{n,r}/i$ and let $p : f_{n,r} \rightarrow n$ be the projection map, there is an Anosov automorphism $\overline{f} : n \rightarrow n$ such that $\overline{f}p = pf_A$. We will call the four conditions the Auslander-Scheuneman conditions. Auslander and Scheuneman showed that any semisimple Anosov automorphism $f$ of an $r$-step nilpotent Lie algebra $n$ may be represented in the manner just described, relative to a rational basis $B$ of a free nilpotent Lie algebra $f_{n,r}$, a semisimple matrix $A$ in $GL_n(\mathbb{Z})$, and an ideal $i$ in $f_{n,r}$ satisfying the four conditions. We will always assume without loss of generality that $i < [f_{n,r},f_{n,r}]$.

In order to understand general properties of Anosov Lie algebras, one must understand the kinds of ideals of free nilpotent Lie algebras that satisfy the Auslander-Scheuneman conditions for some automorphism $f_A$ defined by a matrix $A \in GL_n(\mathbb{Z})$. The dynamical properties of a toral Anosov automorphism of $\mathbb{R}^n/\mathbb{Z}^n$ are closely related to the algebraic properties of the characteristic polynomial $p$ of the matrix $A$ in $GL_n(\mathbb{Z})$ used to define the automorphism (See EW99). We show in this work that, similarly, the algebraic properties of the characteristic polynomial $p$ of the matrix $A$ defining the automorphism $f_A$ of a free nilpotent Lie algebra determine the structure of the ideals that satisfy the Auslander-Scheuneman conditions for $f_A$.

1.2. Summary of results. Now we summarize the main ideas of the paper. We associate to any automorphism $f_A, A \in GL_n(\mathbb{Z})$, of a free $r$-step nilpotent Lie algebra $f_{n,r}$ an $r$-tuple of polynomials $(p_1,p_2,\ldots,p_r)$. For $i = 1,\ldots,r$, the polynomial $p_i$ is the characteristic polynomial of a matrix representing the automorphism $f_A$ on an $i$th step $V_i$ of $f_{n,r}$. Let $K$ denote the splitting field of $p_1$ over $\mathbb{Q}$, and let $G$ denote the Galois group for $K$ over $\mathbb{Q}$. We associate to the automorphism the action of the finite
group $G$ on $\mathfrak{f}_{n,r}(K)$, the free nilpotent Lie algebra over $K$. We show in Theorem 5.1 that $G$ orbits in $\mathfrak{f}_{n,r}(K)$ correspond to rational invariant subspaces for $f^A$, and the characteristic polynomial for the restriction of $f^A$ to such an invariant subspace is a power of an irreducible polynomial.

We analyze Anosov Lie algebras using the following general approach. We fix a free $r$-step nilpotent Lie algebra $\mathfrak{f}_{n,r}$. We consider the class of automorphisms of $\mathfrak{f}_{n,r}$ whose associated polynomial $p_1$ has Galois group $G$, where $G$ is isomorphic to a subgroup of the symmetric group $S_n$. We let $(p_1, p_2, \ldots, p_r)$ be the $r$-tuple of polynomials associated to such a polynomial $p_1$. Our goal is to determine the factorizations of the polynomials $p_1, \ldots, p_r$; this will tell us what rational invariant subspaces for $f$ are. Such subspaces generate any ideal satisfying the Auslander-Scheuneman conditions. First we analyze the factorizations of $p_2, \ldots, p_r$ into powers of irreducibles by understanding orbits of the action of the Galois group of $p_1$ on $\mathfrak{f}_{n,r}(K)$. Then we determine whether the corresponding rational invariant subspaces are minimal and whether there are eigenvalues of modulus one using ideas from number theory (See Proposition 3.6 and Lemma 3.8).

We extend the classification of Anosov Lie algebras to some new classes of two-step Lie algebras.

**Theorem 1.1.** Suppose that $n$ is a two-step Anosov Lie algebra of type $(n_1, n_2)$ with associated polynomials $(p_1, p_2)$. Let $G$ denote the Galois group of $p_1$.

1. If $n_1 = 3, 4$ or $5$, then $n$ is one of the Anosov Lie algebras listed in Table 3.
2. If $p_1$ is irreducible and the action of $G$ on the roots of $p_1$ is doubly transitive, then $n$ is isomorphic to the free nilpotent Lie algebra $\mathfrak{f}_{n,2}$.

We can also classify Anosov Lie algebras admitting automorphisms whose polynomials $p_1$ have certain specified Galois groups.

**Theorem 1.2.** Let $\mathcal{T}$ be a semisimple Anosov automorphism of an $r$-step Anosov Lie algebra. Let $(p_1, \ldots, p_r)$ be the $r$-tuple of polynomials associated to the automorphism $f$ of the free nilpotent Lie algebra $\mathfrak{f}_{n,r}$ induced by $\mathcal{T}$. Suppose that $p_1$ is irreducible.

1. If the polynomial $p_1$ is of prime degree with cyclic Galois group, then $n$ is one of the Lie algebras of type $C_n$ defined over $\mathbb{R}$ as in Definition 4.5. Conversely, if $n$ is prime, and $i$ is an ideal of $\mathfrak{f}_{n,r}$ of cyclic type defined over $\mathbb{R}$ containing the ideal $\mathfrak{j}_{n,r}$ defined in Definition 3.3, then the Lie algebra $n = \mathfrak{f}_{n,r}/i$ is Anosov.
2. If the Galois group of $p_1$ is symmetric, then
   a. If $r = 2$, then $n$ is isomorphic to $\mathfrak{f}_{n,2}$,
   b. If $r = 3$, then $n$ is isomorphic to one of the following five Lie algebras: $\mathfrak{f}_{n,3}$, $\mathfrak{f}_{n,3}/F_1$, $\mathfrak{f}_{n,3}/F_2$, $\mathfrak{f}_{n,3}/(F_1 \oplus F_2)$, and $\mathfrak{f}_{n,3}/F_{2a}$, where the ideals $F_1$ and $F_2$ are as defined in Equation (1) of Section 3.2, and $F_{2a}$ is as in Proposition 5.5.

Matrices in $GL_n(\mathbb{Z})$ having characteristic polynomial with symmetric Galois group are dense in the sense of thick and thin sets ([Ser92]); hence, the second part of
the previous theorem describes Anosov automorphisms of two- and three-step Lie algebras that are generic in this sense.

We investigate some general properties of Anosov automorphisms. We can describe the dimensions of minimal nontrivial rational invariant subspaces. Out of such analyses we obtain the following special case.

**Theorem 1.3.** Suppose that \( n \) is an Anosov Lie algebra of type \((n_1, \ldots, n_r)\). If \( n_1 = 3 \), then \( n_i \) is a multiple of 3 for all \( i = 2, \ldots, r \), and if \( n_1 = 4 \), then \( n_i \) is even for all \( i = 2, \ldots, r \). If \( n_1 \) is prime and the polynomial \( p_1 \) is irreducible, then \( n_1 \) divides \( n_i \) for all \( i = 2, \ldots, r < n \).

The results of [Mai] give an alternate proof of this theorem.

One way to approach the classification problem is to fix the field in which the spectrum of an Anosov automorphism lies. The following theorem describes all Anosov automorphisms whose spectrum lies in a quadratic extension of \( \mathbb{Q} \).

**Theorem 1.4.** Let \( f \) be a semisimple Anosov automorphism of a two-step nilpotent Lie algebra \( n \). Let \( \Lambda \subset \mathbb{R} \) denote the spectrum of \( f \), and let \( K \) denote the finite extension \( \mathbb{Q}(\Lambda) \) of \( \mathbb{Q} \). If \( K \) is a quadratic extension of \( \mathbb{Q} \), then \( n \) is one of the Anosov Lie algebras defined in Definition 7.1.

The paper is organized as follows. In Section 2 we review background material on nilpotent Lie algebras, Anosov automorphisms and algebraic numbers, and we define the \( r \)-tuple of polynomials associated to an Anosov automorphism of a Lie algebra. In Section 3 we describe properties of the \( r \)-tuple of polynomials, such as their reducibility and their Galois groups. In Proposition 3.6 we consider the set of roots of an Anosov polynomial and describe multiplicative relationships among them; this number-theoretic result may be interesting in its own right. In Section 4 we associate to an automorphism \( f \) of a free nilpotent Lie algebra \( f_{n,r} \) the action of a Galois group \( G \), and in Theorem 5.1 we relate rational invariant subspaces of \( f_{n,r} \) to the orbits of \( G \). In Section 6 we consider Anosov Lie algebras for which the associated Galois group is symmetric or cyclic. Finally, in Section 7 we apply the results from previous sections to the problem of classification of Anosov Lie algebras whose associated polynomial \( p_1 \) has small degree. Although the theorems we have stated above follow from various results distributed throughout this work, for the sake of clarity, in Section 8 we provide self-contained proofs of the theorems.

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2. Preliminaries

In this section, we describe the structure of free nilpotent Lie algebras and their automorphisms, and we review some concepts from number theory that we will use later. We conclude with some examples to illustrate the concepts presented.

2.1. Nilpotent Lie algebras. Let \( \mathfrak{n} \) be a Lie algebra defined over field \( K \). The central descending series for \( \mathfrak{n} \) is defined by \( \mathfrak{n}^0 = \mathfrak{n} \), and \( \mathfrak{n}^i = [\mathfrak{n}, \mathfrak{n}^{i-1}] \) for \( i \geq 1 \). If \( \mathfrak{n}^r = 0 \) and \( \mathfrak{n}^{r-1} \neq 0 \), then \( \mathfrak{n} \) is said to be \( r \)-step nilpotent. When \( \mathfrak{n} \) is a nilpotent Lie algebra defined over field \( K \) and \( n_i \) is the dimension of the vector space \( \mathfrak{n}^i / \mathfrak{n}^{i-1} \) over \( K \), then \( (n_1, n_2, \ldots, n_r) \) is called the type of \( \mathfrak{n} \).

The free \( r \)-step nilpotent Lie algebra on \( n \) generators over the field \( K \), denoted \( \mathfrak{f}_{n,r}(K) \), is defined to be the quotient algebra \( \mathfrak{f}_n(K)/\mathfrak{f}_n^{r+1}(K) \), where \( \mathfrak{f}_n(K) \) is the free nilpotent Lie algebra on \( n \) generators over \( K \). Given a set \( \mathcal{B}_1 \) of \( n \) generators, the free nilpotent Lie algebra \( \mathfrak{f}_{n,r}(K) \) can be written as the direct sum \( V_1(K) \oplus \cdots \oplus V_r(K) \), where \( V_1(K) \) is defined to be the span of \( \mathcal{B}_1 \) over \( K \) and for \( i = 2, \ldots, r \), the subspace \( V_i(K) \) is defined to be the span over \( K \) of \( i \)-fold brackets of the generators. We will call the space \( V_i(K) \) the \( i \)th step of \( \mathfrak{f}_{n,r}(K) \) without always explicitly mentioning the dependence on \( \mathcal{B}_1 \). When the field \( K \) has characteristic zero, we identify the prime subfield of \( K \) with \( \mathbb{Q} \). For our purposes, fields that we consider will be intermediate to \( \mathbb{Q} \) and \( \mathbb{C} \): one of \( \mathbb{Q}, \mathbb{R}, \mathbb{C} \), or the splitting field for a polynomial in \( \mathbb{Z}[x] \). We will always assume that a generating set \( \mathcal{B}_1 \) for a free nilpotent Lie algebra \( \mathfrak{f}_{n,r}(K) \) has cardinality \( n \).

The most natural basis to use for a free nilpotent Lie algebra is a Hall basis. Let \( \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \) be \( n \) generators for \( \mathfrak{f}_{n,r}(K) \). We call these the standard monomials of degree one. Standard monomials of degree \( n \) are defined inductively: After the monomials of degree \( n - 1 \) and less have been defined, we define an order relation \( < \) on them so that if degree \( u < \) degree \( v \), then \( u < v \). Any linear combination of monomials of degree \( i \) will be said to be of degree \( i \). If \( u \) has degree \( i \) and \( v \) has degree \( j \), and \( i + j = k \), we define \([u,v]\) to be a standard monomial of degree \( k \) if \( u \) and \( v \) are standard monomials and \( u > v \), and if \( u = [x,y] \) is the form of the standard monomial \( u \), then \( v \geq y \). The standard monomials of degree \( r \) or less form a basis for \( \mathfrak{f}_{n,r}(K) \), called the Hall basis (Hal50). For \( i = 1, \ldots, r \), the subset \( \mathcal{B}_i = \mathcal{B} \cap V_i(K) \) of the basis \( \mathcal{B} \) is a basis for the \( i \)th step \( V_i(K) \) of \( \mathfrak{f}_{n,r}(K) \) consisting of elements of the Hall basis of degree \( i \). To each monomial of degree \( i \) we can also associate a Hall word of length \( i \) from a given alphabet \( \alpha_1, \ldots, \alpha_n \) of \( n \) letters; for example, \([\langle \mathbf{x}_3, \mathbf{x}_1 \rangle, \mathbf{x}_2] \) becomes the word \( \alpha_3 \alpha_1 \alpha_2 \).

Suppose \( \mathfrak{g} \) is a Lie algebra defined over a field \( K \) of characteristic zero. Suppose that \( \mathcal{B} \) is a basis of \( \mathfrak{g} \) having rational structure constants. The basis \( \mathcal{B} \) determines a rational structure on \( \mathfrak{g} \). A subspace \( E \) of \( \mathfrak{g} \) spanned by \( \mathbb{Q} \)-linear combinations of elements of \( \mathcal{B} \) is called a rational subspace for this rational structure. Since the structure constants for the free nilpotent Lie algebra \( \mathfrak{f}_{n,r}(K) \) relative to a Hall basis \( \mathcal{B} \) are rational, a Hall basis \( \mathcal{B} \) for \( \mathfrak{f}_{n,r}(K) \) defines a rational structure on \( \mathfrak{f}_{n,r}(K) \).
Example 2.1. Let $C_1 = \{z_i\}_{i=1}^n$ be a set of $n$ generators for the free $r$-step nilpotent Lie algebra $f_{n,r}(K)$ on $n$ generators over a field $K$ and let $C = \cup_{i=1}^r C_i$ be the Hall basis determined by $C_1$. Elements of $C_2$, where $r \geq 2$, are of the form $[z_i, z_j]$ with $i > j$, hence the dimension of $V_2(K)$ over $K$ is $\left(\begin{array}{c} n \\ 2 \end{array}\right)$. When $r \geq 3$, from the definition of Hall monomial, elements in the set $C_3$ for $f_{n,r}(K)$ are of the form $[[z_i, z_j], z_i]$ or $[[z_i, z_j], z_k]$ with $i > j$ or, if $n \geq 3$, of the form $[[z_i, z_j], z_k]$ with $i, j, k$ distinct and $i$ and $k$ greater than $j$. There are $n(n-1)$ standard Hall monomials of the first type, and when $n \geq 3$, there are $2\left(\begin{array}{c} n \\ 3 \end{array}\right)$ standard Hall monomials of the second type, for a dimensional total of $\frac{1}{3}(n+1)n(n-1)$ for the third step $V_3(K)$ of $f_{n,r}(K)$. We let $C'_3$ denote the set of standard Hall monomials of the first type, and let $C''_3$ denote the set of standard Hall monomials of the second type:

\[ C'_3 = \cup_{1 \leq j < i \leq n} \{[[z_i, z_j], z_i], [[z_i, z_j], z_k]\}, \quad \text{and} \quad C''_3 = \cup_{1 \leq j < i < k \leq n} \{[[z_i, z_j], z_k], [[z_i, z_j], z_k]\}. \]

Define subspaces $F_1(K)$ and $F_2(K)$ of $f_{n,r}(K)$ by

(1) \[ F_1(K) = \text{span}_K C'_3, \quad \text{and} \quad F_2(K) = \text{span}_K C''_3. \]

The subspace $V_3(K)$ is the direct sum of $F_1(K)$ and $F_2(K)$ since $C_3$ spans $V_3(K)$ and is the disjoint union of $C'_3$ and $C''_3$.

2.2. Anosov automorphisms. As we discussed previously, every Anosov automorphism can be represented in terms of a matrix $A$ in $GL_n(\mathbb{Z})$, an automorphism $f^A$ of $f_{n,r}$ induced by $A$ and an ideal $i < f_{n,r}$ satisfying the four Auslander-Scheuneman conditions. In this section we spell out some of the details involved in such a representation.

Let $f_{n,r}(K) = \bigoplus_{i=1}^r V_i(K)$ be the free $r$-step nilpotent Lie algebra over field $K$ with a set $B_1$ of $n$ generators. Let $B = \cup_{i=1}^r B_i$ be the Hall basis determined by $B_1$. Let $A$ be a matrix in $GL_n(\mathbb{Z})$ having no eigenvalues of modulus one. Together the matrix $A$ and the basis $B_i$ define a linear map $f_i : V_i(K) \to V_i(K)$. The map $f_i$ induces an automorphism $f^K_i$ of $f_{n,r}(K)$ that when restricted to $V_i(K)$ equals $f_i$. For all $i = 1, \ldots, r$, the restriction $f_i$ of $f^K_i$ to $V_i(K)$ can be represented with respect to the basis $B_i$ of $V_i(K)$ by a matrix $A_i$ having integer entries that are independent of the field $K$.

For $i = 1, \ldots, r$, let $p_i$ denote the characteristic polynomial of $A_i$. We define the $r$-tuple of polynomials associated to $f$ to be $(p_1, \ldots, p_r)$. Note that all of the polynomials are monic with integer coefficients, and there is no dependence on $K$ in defining the polynomials: either the matrix $A$ or the polynomial $p_1$ alone is enough to uniquely define the $r$-tuple $(p_1, \ldots, p_r)$.

A Lie algebra admits an Anosov automorphism if and only if it admits a semisimple Anosov automorphism ([AS70]). Assume that the linear map $f_1 : V_1(L) \to V_1(L)$ defined by $A \in GL_n(\mathbb{Z})$ and rational basis $B$ is diagonalizable over the field $L$ (where
char $L = 0$). The vector space $V_1(L)$ can be decomposed into the direct sum of minimal nontrivial rational $f_1$-invariant subspaces $E_1, \ldots, E_s$. For each rational invariant subspace $E_j, j = 1, \ldots, s$, the restriction of $f_1$ to $E_j$ is diagonalizable over $L$. Hence there is a basis $C_1 = \{z_1, \ldots, z_n\}$ of $V_1(L)$ consisting of eigenvectors of $f_1$ with each eigenvector properly contained in one of the subspaces $E_1, \ldots, E_s$. Let $C$ be the Hall basis of $f_{n,r}(L)$ determined by $C_1$. We will call such an eigenvector basis for an automorphism $f$ of $f_{n,r}(L)$ compatible with the rational structure, and in the future, when we use eigenvector bases for free nilpotent Lie algebras we will always choose them to be compatible with the rational structure determined by a fixed Hall basis.

**Notation 2.2.** We shall use $B$ to denote the Hall basis of a free nilpotent Lie algebra $f_{n,r}(K)$ that determines the rational structure and that with a matrix in $GL_n(Z)$ defines the Anosov automorphism, while we will use $C$ to denote a Hall basis that diagonalizes the Anosov automorphism.

Suppose that $K$ has characteristic zero and $B$ is a fixed Hall basis of $f_{n,r}(K)$, and identify the prime subfield of $K$ with $\mathbb{Q}$. We will use $f_{n,r}(\mathbb{Q})$ to denote the subset of $f_{n,r}(K)$ that is the $\mathbb{Q}$-span of $B$ in $f_{n,r}(K)$.

At times we will move between free nilpotent Lie algebras $f_{n,r}(K)$ and $f_{n,r}(L)$ defined over different field extensions $K$ and $L$ of $\mathbb{Q}$. We define a correspondence between rational $f_{K}^A$-invariant subspaces of $f_{n,r}(K)$ and rational $f_{L}^A$-invariant subspaces of $f_{n,r}(L)$:

**Definition 2.3.** Let $K$ and $L$ be fields with characteristic zero. Let $B_1(K)$ and $B_1(L)$ be generating sets, both of cardinality $n$, for free nilpotent Lie algebras $f_{n,r}(K)$ and $f_{n,r}(L)$ respectively, and let $B(K)$ and $B(L)$ be the Hall bases defined by $B_1(K)$ and $B_1(L)$ respectively. A bijection $i_1 : B_1(K) \rightarrow B_1(L)$ of the generating sets naturally induces a bijection $i : B(K) \rightarrow B(L)$ of the Hall bases, and this in turn defines an isomorphism $\tilde{i}$ from $f_{n,r}(\mathbb{Q}) < f_{n,r}(K)$ to $f_{n,r}(\mathbb{Q}) < f_{n,r}(L)$, where $f_{n,r}(\mathbb{Q})$ denotes the $\mathbb{Q}$-span of the fixed Hall basis.

Endow $f_{n,r}(K)$ and $f_{n,r}(L)$ with the rational structures defined by $B(K)$ and $B(L)$ respectively. Given a matrix $A \in GL_n(\mathbb{Z})$, let maps $f_K^A \in \text{Aut}(f_{n,r}(K))$ and $f_L^A \in \text{Aut}(f_{n,r}(L))$ be defined by $A$ and $B_1(K)$ and $B_1(L)$ respectively. Observe that $[f_K^A]|_{B(K)} = [f_L^A]|_{B(L)} \in M_n(\mathbb{Z})$, where $N = \dim f_{n,r}$.

Let $E$ be a rational $f_K^A$-invariant subspace of $f_{n,r}(K)$ spanned by vectors $v_1, \ldots, v_m$ in $f_{n,r}(K)$; i.e. coordinates of $v_1, \ldots, v_m$ with respect to $B(K)$ are in $\mathbb{Q}$. Define the subspace $E_L$ of $f_{n,r}(L)$ to be the $L$-span of the vectors $\tilde{i}(v_1), \ldots, \tilde{i}(v_m)$ in $f_{n,r}(L)$. Clearly $E_L$ is rational and $f_L^A$-invariant subspace of $f_{n,r}(K)$.

**Remark 2.4.** Observe that $i$ satisfies the Auslander-Scheuneman conditions for a semisimple automorphism $f$ of a free nilpotent Lie algebra, then it satisfies the conditions for $f^2$. Therefore, when seeking ideals of a free nilpotent Lie algebra satisfying the four conditions for an automorphism $f$, by moving to $f^2$ if necessary, we may
assume that the eigenvalues of the automorphism have product 1, and that all the real eigenvalues are positive.

The next example clarifies some of our definitions and notation.

**Example 2.5.** Let \( f_{3,2}(\mathbb{R}) = V_1(\mathbb{R}) \oplus V_2(\mathbb{R}) \) be the free two-step nilpotent Lie algebra on three generators \( x_1, x_2, \) and \( x_3 \). These three generators span the subspace \( V_1(\mathbb{R}) \). The Hall words of length two are \( x'_1 = [x_3, x_2], x'_2 = [x_3, x_1], \) and \( x'_3 = [x_2, x_1] \); they span \( V_2(\mathbb{R}) \). The union \( B \) of \( B_1 = \{ x_1, x_2, x_3 \} \) and \( B_2 = \{ x'_1, x'_2, x'_3 \} \) is the Hall basis determined by \( x_1, x_2, x_3 \).

Now let \( A = A_1 \) be a \( 3 \times 3 \) matrix in \( SL_3(\mathbb{Z}) \) that has eigenvalues \( \alpha_1, \alpha_2, \alpha_3 \), none of which has modulus one. The matrix \( A \) and the basis \( B_1 \) define the linear map \( f_1 : V_1 \to V_1 \). The linear map \( f_1 \) induces an automorphism \( f^A \) of \( f_{3,2}(\mathbb{R}) \). Let \( A_2 \) denote the matrix representing the restriction of \( f^A \) to \( V_2(\mathbb{R}) \) with respect to the basis \( B_2 \).

The matrix \( A_1 \) has characteristic polynomial
\[
p_1(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3).
\]
A short calculation shows that \( A_2 \) is similar to \( A_1^{-1} \) and has characteristic polynomial
\[
p_2(x) = (x - \alpha_2 \alpha_3)(x - \alpha_1 \alpha_3)(x - \alpha_1 \alpha_2) = (x - \alpha_1^{-1})(x - \alpha_2^{-1})(x - \alpha_3^{-1}).
\]
Neither \( A_1 \) or \( A_2 \) has any eigenvalues of modulus one, so \( f^A \) is an Anosov automorphism of \( f_{3,2}(\mathbb{R}) \).

**2.3. Polynomials and algebraic numbers.** We will call a monic polynomial Anosov if has integer coefficients, it has constant term \( \pm 1 \), and it has no roots with modulus one. The roots of an Anosov polynomial are algebraic units.

We can identify each monic polynomial \( p \) in \( \mathbb{Z}[x] \) of degree \( n \) with an automorphism of the free nilpotent Lie algebra \( f_{n,r}(\mathbb{R}) = \oplus_{i=1}^r V_i(\mathbb{R}) \) with generating set \( B_1 = \{ x_1, \ldots, x_n \} \). Suppose \( p = q_1 q_2 \cdots q_s \) is a factorization of \( p \) into irreducibles. Let \( A_i \) be the companion matrix for \( q_i \), for \( i = 1, \ldots, s \), and define the matrix \( A_p \) to be block diagonal with the matrices \( A_1, \ldots, A_s \) down the diagonal. As already described, the matrix \( A_p \) and the basis \( B_1 \) together define an automorphism of the free \( r \)-step nilpotent Lie algebra on \( n \) generators.

If \( E \) is a nontrivial rational invariant subspace for an Anosov automorphism \( f \) of an Anosov Lie algebra, we will let \( p_E \) denote the characteristic polynomial for the restriction of \( f \) to \( E \). If \( p \) and \( q \) are polynomials in \( \mathbb{Z}[x] \), we define the polynomial \( p \wedge q \) to be the characteristic polynomial of the matrix \( A_p \wedge A_q \).

Next we illustrate how an Anosov polynomial determines a class of Anosov automorphisms.

**Example 2.6.** Let \( p \) be an Anosov polynomial of degree \( n \) that is a product of two irreducible factors \( r_1 \) and \( r_2 \) of degrees \( d_1 \) and \( d_2 \) respectively. The companion matrices
$B_1$ and $B_2$ to the polynomials $r_1$ and $r_2$ are in $GL_{d_1}(\mathbb{Z})$ and $GL_{d_2}(\mathbb{Z})$ respectively. Putting these matrices together in a block diagonal matrix gives a matrix

$$A_p = A_1 = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$$

in $GL_n(\mathbb{Z})$ with characteristic polynomial $p$.

Let $f_{n,2}(\mathbb{R}) = V_1(\mathbb{R}) \oplus V_2(\mathbb{R})$ be the real free two-step nilpotent Lie algebra on $n$ generators with generating set $B_1$. The matrix $A_1$ and the basis $B_1$ of $V_1(\mathbb{R})$ define a linear map $f_1 : V_1(\mathbb{R}) \to V_1(\mathbb{R})$ which induces an automorphism $f^A$ of $f_{n,2}(\mathbb{R})$. Let $f_2 = f^A|_{V_2(\mathbb{R})}$. The map $f_2$ may be represented by a matrix $A_2$ that is block diagonal with matrices $B_1 \wedge B_1, B_1 \wedge B_2$ and $B_2 \wedge B_2$ along the diagonal. Let $\alpha_1, \ldots, \alpha_{d_1}$ denote the roots of $r_1$ and let $\beta_1, \ldots, \beta_{d_2}$ denote the roots of $r_2$. It can be shown that the matrix $A_2$ has characteristic polynomial

$$p_2 = (r_1 \wedge r_1)(r_1 \wedge r_2)(r_2 \wedge r_2),$$

where

$$(r_1 \wedge r_1)(x) = \prod_{1 \leq i < j \leq d_1} (x - \alpha_i \alpha_j),$$

$$(r_1 \wedge r_2)(x) = \prod_{1 \leq i \leq d_1} (x - \alpha_i \beta_j), \quad \text{and}$$

$$(r_2 \wedge r_2)(x) = \prod_{1 \leq i < j \leq d_2} (x - \beta_i \beta_j).$$

As long as none of the roots of $p_2$ have modulus one, the map $f^A$ is Anosov.

Later we will need to know when polynomials in $\mathbb{Z}[x]$ have roots of modulus one and will use the following observation.

**Remark 2.7.** Suppose that an irreducible polynomial $p$ in $\mathbb{Z}[x]$ of degree $n$ has a root $\alpha$ with modulus one. Then the complex conjugate $\bar{\alpha} = \alpha^{-1}$ is also a root of $p$, so $p$ is self-reciprocal and $n$ is even. If $q$ is the minimal polynomial of $\alpha + \alpha^{-1}$, then $p(x) = x^{n/2}q(x + 1/x)$. Hence, the Galois group for $p$ is a wreath product of $C_2$ and the Galois group for the polynomial $q$.

### 3. Polynomials associated to automorphisms

#### 3.1. Properties of the $r$-tuple of characteristic polynomials.

Now we present some properties of the tuple of polynomials associated to an automorphism of a free nilpotent Lie algebra.

**Proposition 3.1.** Let $f^A$ be a semisimple automorphism of the free nilpotent Lie algebra $f_{n,r}(\mathbb{R}) = \bigoplus_{i=1}^r V_i(\mathbb{R})$ defined by a matrix $A$ in $GL_n(\mathbb{Z})$ and the Hall basis $\mathcal{B}$ defined by generating set $\mathcal{B}_1 = \{ x_i \}_{i=1}^n$. Let $(p_1, p_2, \ldots, p_r)$ be the $r$-tuple of polynomials
associated to \( f \), let \( \alpha_1, \ldots, \alpha_n \) denote the roots of \( p_1 \), and let \( K \) denote the splitting field for \( p_1 \). Let \( \mathcal{C}_1 = \{ z_i \}_{i=1}^n \) be a \( f_K^n \)-eigenvector basis of \( V_1(K) < f_{n,r}(K) \) compatible with the rational structure defined by \( \mathcal{B} \) and let \( \mathcal{C} = \bigcup_{i=1}^{r} \mathcal{C}_i \) be the Hall basis of \( f_{n,r}(K) \) associated to \( \mathcal{C}_1 \).

1. Each standard Hall monomial of degree \( i \) on \( z_1, \ldots, z_n \) in the set \( \mathcal{C}_i \) is an eigenvector for \( f_{K|V_i(K)}^A \) whose eigenvalue is the corresponding Hall word in \( \alpha_1, \ldots, \alpha_n \).

2. For \( i = 1, \ldots, r \), let \( p_i = r_{i,1} \cdots r_{i,d_i} \) be a factorization of \( p_i \) into \( d_i \) irreducible monic polynomials in \( \mathbb{Z}[x] \), and let \( V_i(\mathbb{R}) = \bigoplus_{j=1}^{e_i} E_{i,j} \) be a decomposition of \( V_i(\mathbb{R}) \) into \( e_i \) minimal nontrivial rational \( f_A^n \)-invariant subspaces. For all \( i = 1, \ldots, r \), \( d_i = e_i \) and the map that sends \( E_{i,j} \) to the characteristic polynomial of \( f|_{E_{i,j}} \) is a one-to-one correspondence between the set of rational subspaces \( \{ E_{i,j} \}_{j=1}^{d_i} \) and the set of factors \( \{ r_{i,j} : j = 1, \ldots, e_i \} \) of \( p_i \).

It follows from Part (1) of the proposition that if the matrix \( A \) is diagonalizable over \( \mathbb{C} \), then the automorphism \( f_A^n \) is semisimple. In particular, if the polynomial \( p_1 \) is separable over \( \mathbb{Q} \), then \( f_A^n \) is semisimple.

**Remark 3.2.** As a consequence of the third part of the proposition, if \( f \) is unimodular, the characteristic polynomial for the restriction of \( f \) to a rational invariant subspace \( E \) has a unit constant term, hence the restriction of \( f \) to any rational invariant subspace \( E \) is unimodular. Therefore, the second of the four Auslander-Scheuneman conditions is automatic.

It is well known that there are no Anosov Lie algebras of type \( (n_1, \ldots, n_r) \), where \( n_1 = 2 \) and \( r > 1 \). This follows from Part (1) of the proposition. Henceforth we shall only consider nilpotent Lie algebras where \( n_1 \geq 3 \) and \( r \geq 2 \).

**Proof of Proposition 3.1** The first part is elementary. Hall words of degree \( i \) in \( z_i \) span \( V_i(K) \), for \( i = 1, \ldots, r \). Because \( f_K^n \) is an automorphism of \( f_{n,r}(K) \), a Hall word in \( z_1, \ldots, z_n \) is an eigenvector for \( f_K^n \) whose eigenvalue is the same Hall word in \( \alpha_1, \ldots, \alpha_n \).

The last part of the proposition follows from the existence of the elementary divisors. Rational canonical forms for matrices. The fact that the matrix is semisimple implies that the elementary divisors are irreducible.

3.2. The polynomials \( p_2 \) and \( p_3 \). Let \( A \) be a semisimple matrix in \( GL_n(\mathbb{Z}) \), and let \( K \) be the splitting field for the characteristic polynomial \( p_1 \) of \( A \). Let \( f_A^n \) be the semisimple automorphism of \( f_{n,r}(\mathbb{R}) = \bigoplus_{i=1}^{r} V_i(\mathbb{R}) \), where \( n \geq 3 \), induced by \( A \) and a basis \( \mathcal{B}_1 \) for \( V_1(\mathbb{R}) \). Let \( \mathcal{C}_1 = \{ z_i \}_{i=1}^n \) be an eigenvector basis for \( V_1(K) < f_{n,r}(K) \) compatible with the rational structure determined by \( \mathcal{B}_1 \), where \( f_A^n(z_j) = \alpha_j z_j \), for \( j = 1, \ldots, n \); and let \( \mathcal{C} = \bigcup_{i=1}^{r} \mathcal{C}_i \) be the Hall basis of \( f_{n,r}(K) \) determined by \( \mathcal{C}_1 \).
Proposition 3.3. The polynomial $p_2(x) = \prod_{1 \leq i < j \leq n} (x - \alpha_i \alpha_j)$. Let $C'_3$ and $C''_3$ be as defined in Example 2.1. By Proposition 3.1, an element $[[z_j, z_i], z_j]$ of $C'_3$ is an eigenvector for $f'_R$ with eigenvalue $\alpha_i \alpha_j^2$, and an element $[[z_i, z_j], z_k]$ of $C''_3$ is an eigenvector for $f''_R$ with eigenvalue $\alpha_i \alpha_j \alpha_k$. Define the polynomials $q_1$ and $q_2$ by

$$q_1(x) = \prod_{1 \leq i,j \leq n, i \neq j} (x - \alpha_i \alpha_j^2), \quad \text{and} \quad q_2(x) = \prod_{1 \leq i < j < k \leq n} (x - \alpha_i \alpha_j \alpha_k).$$

Because they are invariant under the action of $G$, $q_1$ and $q_2$ have integral coefficients. The polynomial $p_3 = q_1 q_2^2$ is the characteristic polynomial for the restriction of the automorphism $f''_R$ to $V_3(L)$ for any extension $L$ of $\mathbb{Q}$.

3.3. Anosov polynomials and their roots. In this section, we discuss Anosov polynomials and their properties.

**Proposition 3.3.** Let $p_1$ be an Anosov polynomial in $\mathbb{Z}[x]$ of degree $n \geq 3$. Let $(p_1, \ldots, p_r)$ be the associated $r$-tuple of polynomials.

1. If $p_1$ has constant term one, then its reciprocal polynomial $(p_1)_R$ is a factor of $p_{n-1}$ and $(p_2)_R$ is a factor of $p_{n-2}$. If the constant term of $p_1$ is $-1$, then $(p_1)_R(-x)$ is a factor of $p_{n-1}$ and $(p_2)_R(-x)$ is a factor of $p_{n-2}$.
2. The polynomial $p_1$ has at least one root of modulus greater than or equal to one and at least one root of modulus less than one for all $i \geq 2$.
3. If the roots $\alpha_1, \ldots, \alpha_n$ of $p_1$ are viewed as indeterminates, then the constant term of $p_i$ is $(\alpha_1 \cdots \alpha_n)^{D(i)}$, with the exponent $D(i)$ given by

$$D(i) = \frac{1}{ni} \sum_{d|i} \mu(d)n^{i/d},$$

where $\mu$ is the Möbius function.

**Proof.** Let $\alpha_1, \ldots, \alpha_n$ denote the roots of $p_1$. Suppose that the constant term of $p_1$ is $(-1)^n$, so $\alpha_1 \cdots \alpha_n = 1$. The reciprocal of $\alpha_j$, for $j = 1, \ldots, n$, is $\alpha_1 \cdots \hat{\alpha}_j \cdots \alpha_n$, the Hall word $\alpha_n \cdots \hat{\alpha}_j \cdots \alpha_1$ of length $n - 1$. By Proposition 3.1 Part (1), this number is a root of $p_{n-1}$. Thus $(p_1)_R$ is a factor of $p_{n-1}$. Similarly, the reciprocal of the root $\alpha_j \alpha_{j'}$ of $(p_2)_R^n$, where $1 \leq j_1 < j_2 \leq n$, is $\alpha_1 \cdots \hat{\alpha}_{j_1} \cdots \hat{\alpha}_{j_2} \cdots \alpha_n$, which is a permutation of a Hall word on $n - 2$ letters, and therefore is a root of $p_{n-2}$. Thus $(p_2)_R$ is a factor of $p_{n-2}$. If the constant term of $p_1$ is $(-1)^{n+1}$, then $\alpha_1 \cdots \alpha_n = -1$ and the same argument shows that whenever $\alpha$ is a root of $p_1$, $-\alpha^{-1}$ is a root of $p_{n-1}$, and when $\alpha$ is a root of $p_2$, then $-\alpha^{-1}$ is a root of $p_{n-2}$.

Now we prove Part (2). Fix $i \geq 2$. By hypothesis, $\alpha_1 \cdots \alpha_n = \pm 1$ and not all of the roots have modulus one. Therefore, some root, say $\alpha_1$, has modulus less than
one. Then the modulus of the product of the \( n - 1 \) other roots is \( 1/|\alpha_1| > 1 \), so the modulus of at least one of the other roots, say \( \alpha_2 \), must satisfy \( |\alpha_2| \geq 1/\sqrt[n-1]{|\alpha_1|} \). The root \( \alpha_1^{-1}\alpha_2 \) of \( p_i \) has modulus equal to \( |\alpha_1|^{-1}\frac{1}{n-1} \). The exponent \( i - 1 - \frac{1}{n-1} \) is positive, so \( \alpha_1^{-1}\alpha_2 \) has modulus less than one. By the same reasoning, there exists a root of \( p_i \) with modulus greater than one.

The dimension of \( V_i(\mathbb{R}) \) is a Dedekind number \( \frac{1}{2} \sum_{d|i} \mu(d)n^{i/d} \), where \( \mu \) is the Möbius function (Corollary 4.14, [Reu93]). Each of \( \alpha_1, \ldots, \alpha_n \) must occur the same number of times in the constant term of \( p_1 \) by Lemma 1 of [AS70]. Therefore, the constant term of \( p_i \), for \( 1 \leq i \leq r \), is \( \alpha_1 \cdots \alpha_n \) to the power \( 1/n \cdot \dim(V_i(\mathbb{R})) \), as claimed. \( \Box \)

The next lemma helps identify roots of modulus one for automorphisms whose first polynomial \( p_1 \) has Galois group of odd order.

**Lemma 3.4.** Suppose that the characteristic polynomial \( p_1 \) of a semisimple hyperbolic matrix \( A \) in \( GL_n(\mathbb{Z}) \), where \( n \geq 3 \), has Galois group of odd order. Let \( f^A : f_{n,r}(\mathbb{R}) \to f_{n,r}(\mathbb{R}) \) be the automorphism of the free \( r \)-step nilpotent Lie algebra on \( n \) generators induced by \( A \). If \( \lambda \) is an eigenvalue of \( f^A \) with modulus one, then \( \lambda = 1 \) or \( \lambda = -1 \).

**Proof.** Let \( p_1 \) and \( f^A \) be as in the statement of the lemma. Let \( (p_1, \ldots, p_r) \) be the \( r \)-tuple of polynomials associated to \( f^A \).

If a monic irreducible nonlinear polynomial in \( \mathbb{Z}[x] \) has a root of modulus one, by Remark 2.7 its Galois group \( G \) has even order.

By Proposition 3.1, \( p_1 \) is an eigenvalue of \( f^A \), it is a root of a reducible factor \( q \) of \( p_i \) for some \( i = 1, \ldots, r \). Since the splitting field for \( q \) is a subfield of the splitting field for \( p_1 \), the Galois group \( H \) for \( q \) is the quotient of \( G \) by a normal subgroup; hence if \( q \) is nonlinear, \( H \) has odd order. Then, either \( q \) is linear and \( \alpha = \pm 1 \), or \( q \) is nonlinear and \( |\alpha| \neq 1 \). \( \Box \)

### 3.4. The full rank condition

Suppose that \( p_1 \) is an Anosov polynomial in \( \mathbb{Z}[x] \) with roots \( \alpha_1, \ldots, \alpha_n \). We will want to know when the equation

\[
\alpha_1^{d_1} \alpha_2^{d_2} \cdots \alpha_n^{d_n} = 1
\]

has integer solutions \( d_1, \ldots, d_n \). Note that if \( p_1 \) has constant term \((-1)^n\), then \( \alpha_1 \cdots \alpha_n = 1 \), and \( d_1 = \cdots = d_n = d \) is a solution for any integer \( d \).

**Definition 3.5.** Let \( \Lambda = \{\alpha_1, \ldots, \alpha_n\} \) be the set of roots of a polynomial \( p \in \mathbb{Z}[x] \) with constant term \((-1)^n\) and degree \( n \geq 2 \). The set \( \Lambda \) is said to be of full rank if the only integral solutions to Equation (1) are of form \( d_1 = d_2 = \cdots = d_n \).

The next proposition describes how multiplicative relationships among the roots of some polynomials in \( \mathbb{Z}[x] \) depend on their Galois groups.

**Proposition 3.6.** Suppose that \( \alpha_1, \ldots, \alpha_n \) are roots of a degree \( n \) irreducible monic polynomial \( p \in \mathbb{Z}[x] \) with constant term \((-1)^n\), and suppose that none of \( \alpha_1, \ldots, \alpha_n \) are roots of unity. Let \( G \) denote the Galois group for \( p \).

The set \( \{\alpha_1, \ldots, \alpha_n\} \) is of full rank in the following situations.
(1) When the permutation representation of $G$ on $\mathbb{Q}^n$ is the sum of the principal representation and a representation that is irreducible over $\mathbb{Q}$.
(2) When the action of $G$ on the set of roots of $p$ is doubly transitive.
(3) When $p$ is Anosov, and precisely one of its roots $\alpha_1$ has modulus greater than one.

An algebraic number $\alpha_1$ as in Part (3) of the proposition is a P.-V. number. Properties of P.-V. numbers were first investigated by Pisot and Vijayaraghavan. (See [Mey72] and [BDGGH+92] for background on P.-V. numbers.) The proof of Part (3) of the proposition is due to Bell and Hare ([BH]); we repeat it here for the sake of completeness.

The action of the Galois group $G$ of $p_1 \in \mathbb{Z}[x]$ on the set $\{\alpha_1, \ldots, \alpha_n\}$ of enumerated roots of $p_1$ gives an identification of $G$ with a subgroup of $S_n$, and we can define a permutation representation $\rho$ of $G$ on $\mathbb{Q}^n$, with

\[
\rho(g)(\beta_1, \ldots, \beta_n) = (\beta_{g(1)}, \ldots, \beta_{g(n)})
\]

for $g \in G$ and $(\beta_1, \ldots, \beta_n) \in \mathbb{Q}^n$.

Proof. Fix $\alpha_1, \ldots, \alpha_n$ as in the statement of the theorem. If $(d_1, \ldots, d_n) \in \mathbb{Q}^n$ is a solution to Equation (4), and $\sigma$ is in $G$, then

\[
\sigma(\alpha_1)^{d_1} \sigma(\alpha_2)^{d_2} \cdots \sigma(\alpha_n)^{d_n} = 1,
\]

which may be alternately expressed as

\[
\alpha_1^{d_{\sigma^{-1}(1)}} \alpha_2^{d_{\sigma^{-1}(2)}} \cdots \alpha_n^{d_{\sigma^{-1}(n)}} = 1.
\]

Therefore, the set of integral solutions to Equation (4) is invariant for the permutation representation $\rho$ : For all $\sigma$ in $G$, $(d_1, \ldots, d_n) \in \mathbb{Q}^n$ is a solution to Equation (4) if and only if $(d_{\sigma^{-1}(1)}, \ldots, d_{\sigma^{-1}(n)})$ is a solution to Equation (4). It is easy to see that the set $S$ of solutions to Equation (4) in $\mathbb{Z}^n$ is closed under addition and subtraction. Therefore, if $(d_1, \ldots, d_n)$ is an integral solution to Equation (4), then any vector in $\text{span}_{\mathbb{Z}}\{\rho(\sigma)(d_1, \ldots, d_n) : \sigma \in G\}$ is also a solution to Equation (4).

Suppose that $\rho$ decomposes as the sum of the trivial representation on $\mathbb{Q}(1, 1, \ldots, 1)$ and an irreducible representation on $W = (1, 1, \ldots, 1)^\perp$. We will show by contradiction that the set $\{\alpha_1, \ldots, \alpha_n\}$ has full rank. Suppose that $(d_1, \ldots, d_n)$ is an $n$-tuple of integers so that Equation (4) holds and that $(d_1, \ldots, d_n)$ is not a scalar multiple of $(1, 1, \ldots, 1)$. After subtracting the appropriate multiple of $(1, 1, \ldots, 1)$, we may assume that the solution $(d_1, \ldots, d_n)$ is a nontrivial vector in $W$. The representation of $G$ on $W$ is irreducible over $\mathbb{Q}$, and $(d_1, \ldots, d_n)$ is a nontrivial element of $W$, so the invariant subspace $\text{span}_{\mathbb{Q}}\{\rho(\sigma)(d_1, \ldots, d_n) : \sigma \in G\}$ is all of $W$. Then $\text{span}_{\mathbb{Q}} S = \mathbb{Q}^n$, implying that $(1, 0, \ldots, 0)$ is a $\mathbb{Q}$-linear combination of solutions $(d_1, \ldots, d_n)$ to Equation (4). But then there exists an integer $N$ such that $\alpha_1^N = 1$, a contradiction. Hence, every solution to Equation (4) is a scalar multiple of $(1, 1, \ldots, 1)$.
If the action of $G$ is two-transitive, then the permutation representation of $G$ on $\mathbb{C}^n$ is the sum of the trivial representation on $\mathbb{C}(1,1,\ldots,1)$ and a representation on the orthogonal complement that is irreducible over $\mathbb{C}$ ([Ser77], Exercise 2.6). Then the permutation representation of $G$ on $\mathbb{Q}^n$ is the sum of the trivial representation on $\mathbb{Q}(1,1,\ldots,1)$ and a representation on the orthogonal complement that is irreducible over $\mathbb{Q}$, and the set of roots is of full rank, by Part (1). Therefore, if $G$ is two-transitive, then the set of roots is of full rank.

Now assume that $p$ is irreducible, and without loss of generality that $\alpha_1 > 0$. Suppose that the roots of $p$ satisfy the condition that

$$\alpha_1 > 1 > |\alpha_2| \geq \cdots \geq |\alpha_n|,$$

and that Equation (4) holds for $d_1, d_2, \ldots, d_n$. Let $m$ be the index so that $d_m$ achieves the minimum of the set $\{d_i : i = 1, \ldots, n\}$. Since $(d_m, \ldots, d_m)$ is a solution to Equation (4), the $n$-tuple

$$(e_1, \ldots, e_n) = (d_1 - d_m, d_2 - d_m, \ldots, d_n - d_m)$$

is a solution to Equation (4) with $e_m = 0$ and $e_i \geq 0$ for all $i = 1, \ldots, m$.

Because $p$ is irreducible, there exists a permutation $\sigma$ in $G$ such that $\sigma(\alpha_1) = \alpha_m$, or if we identify $G$ with a subgroup of $S_n$ in the natural way, $\sigma(1) = m$. We then have

$$\rho(\sigma)(e_1, \ldots, e_n) = (e_{\sigma(1)}, e_{\sigma(2)}, \ldots, e_{\sigma(n)}) = (0, e_{\sigma(2)}, \ldots, e_{\sigma(n)})$$

is also a solution to the equation, so

$$\alpha_2 \alpha_3 \cdots \alpha_n = 1.$$

But $|\alpha_i| < 1$ for $i = 2, \ldots, n$, and all the exponents are nonnegative, hence all the exponents $e_i$ must be zero. Then $d_1 = d_2 = \cdots = d_n$ as desired, and Part (3) holds. 

The next lemma shows that when the set of roots of an Anosov polynomial $p_1$ is of full rank, there are strong restrictions on the Galois groups for irreducible factors of polynomials $p_2, p_3, \ldots$ associated to $p_1$.

**Lemma 3.7.** Let $p_1$ be an Anosov polynomial of degree $n$ with constant term $(-1)^n$. Suppose that the set of roots $\{\alpha_1, \ldots, \alpha_n\}$ of $p_1$ has full rank. Let $G$ denote the Galois group of $p_1$, and let $(p_1, \ldots, p_r)$ be the $r$-tuple of polynomials associated to $p_1$ for some $r > 1$.

Fix $p_i$ for some $i = 2, \ldots, r$, and suppose that $q$ is an irreducible nonlinear factor of $p_i$ over $\mathbb{Z}$ with root $\beta = \alpha_1^{d_1} \cdots \alpha_s^{d_s}$ for $s \leq n - 1$. The Galois group $G$ acts on the splitting field $\mathbb{Q}(p_1)$ for $p_1$, and the splitting field $\mathbb{Q}(q)$ for $q$ is a subfield of $\mathbb{Q}(p_1)$. Let $H < G$ be the stabilizer of $\mathbb{Q}(q)$. Any element $\sigma$ in $H$ has the properties

1. $\sigma$ permutes the set $\{\alpha_{s+1}, \ldots, \alpha_n\}$.
2. For $j, k = 1, \ldots, s$, $\sigma(\alpha_j) = \alpha_k$ only if $d_j = d_k$. That is, $\sigma$ permutes the sets of roots having the same exponent in the expression for $\beta$ in terms of $\alpha_1, \ldots, \alpha_n$. 

Thus, \( H \) is isomorphic to a subgroup of the direct product
\[
S_{k_1} \times S_{k_2} \cdots \times S_{k_{m-1}} \times S_{n-s}, \quad (k_1 + \cdots + k_{m-1} = s)
\]
of \( m \geq 2 \) symmetric groups.

**Proof.** Let \( \beta = \alpha_1^{d_1} \cdots \alpha_s^{d_s} \) be as in the statement of the theorem, and let \( \sigma \) be in the stabilizer \( H \) of \( \mathbb{Q}(q) \). Then \( \sigma(\beta) = \beta \) implies that
\[
\alpha_1^{d_1-1(1)} \alpha_2^{d_2-1(2)} \cdots \alpha_s^{d_s-1(s)} \alpha_n^0 = \alpha_1^{d_1} \cdots \alpha_s^{d_s},
\]
so then
\[
\alpha_1^{d_1-1(1)} - 1 \alpha_2^{d_2-1(2)} - d_2 \cdots \alpha_s^{d_s-1(s)} - d_s \alpha_n^0 = 1.
\]
By definition of full rank,
\[
d_{\sigma^{-1}(1)} - 1 = d_{\sigma^{-1}(2)} - d_2 = \cdots = d_{\sigma^{-1}(s)} - d_s = 0.
\]
Therefore, \( d_{\sigma^{-1}(i)} = d_i \) for all \( i = 1, \ldots, s \); in other words, if \( \sigma(i) = j \), then \( d_i = d_j \). Hence \( \sigma \) permutes each set of \( \alpha_i \)’s for which the exponents \( d_i \) agree, including the nonempty set \( \{\alpha_{s+1}, \ldots, \alpha_n\} \) where \( d_i = 0 \).

The following lemma describes how the polynomials in the \( r \)-tuple of polynomials \( (p_1, p_2, \ldots, p_r) \) can factor, in terms of the properties of \( p_1 \).

**Lemma 3.8.** Suppose \( p_1 \) is a monic polynomial in \( \mathbb{Z}[x] \) of degree \( n \geq 3 \) with constant term \((-1)^n\). Let \( G \) denote the Galois group for \( p_1 \). Let \( (p_1, p_2, \ldots, p_r) \) be an \( r \)-tuple of polynomials associated to \( p_1 \). Let \( q_1 \) and \( q_2 \) be as defined in Equation \( 3 \).

(1) Assume that \( p_1 \) is separable. If \( p_2 \) or \( q_1 \) factors over \( \mathbb{Z} \) as a power of an irreducible polynomial \( r \), then the degree of \( r \) is \( n - 1 \) or more. If \( q_2 \) factors over \( \mathbb{Z} \) as a power of an irreducible polynomial \( r \), then the degree of \( r \) is \( n - 2 \) or more.

(2) Suppose that the set of roots of \( p_1 \) is of full rank.
(a) For \( i = 2, \ldots, r \), the degree \( k \) of any factor of \( p_1 \), satisfies \( k = d(n/i) \) for some positive integer \( d \leq D(i) \), where \( D(i) \) is as defined in Proposition \( 3.3 \).
Therefore, if \( \gcd(n, i) = r \), then \( n/r \) divides \( k \).
(b) If \( q \) is a nonlinear irreducible factor of \( p_1 \) over \( \mathbb{Z} \) for some \( i = 2, \ldots, r \), then the normal subgroup \( N \) of \( G \) of automorphisms fixing \( \mathbb{Q}(q) \) does not act transitively on the roots of \( p_1 \).
(c) If \( p_2 = r^s \) or \( q_1 = r^s \) for an irreducible monic polynomial \( r \in \mathbb{Z}[x] \), then \( s = 1 \) when \( n \geq 3 \); and when \( n \geq 4 \), if \( q_2 = r^s \) for an irreducible \( r \), then \( s = 1 \).

**Proof.** Let \( \alpha_1, \ldots, \alpha_n \) denote the roots of \( p_1 \). If a polynomial \( r \) is irreducible with \( m \) distinct roots, then \( r^s \) has \( m \) distinct roots each of multiplicity \( s \). Thus, to prove the first part, we simply count the number of roots of each polynomial \( p_2, q_1, \) and \( q_2 \) that are guaranteed to be distinct, and the degree of \( r \) is necessarily greater or equal to that number if \( p_2, q_1, \) or \( q_2 \) is of form \( r^s \). For \( p_2 \), roots of form \( \alpha_1 \alpha_j, j = 2, \ldots, n \)
are distinct; for \( q_1 \), roots of form \( \alpha_1\alpha_2^2 \), \( j = 2, \ldots, n \) are distinct; and for \( p_2 \), roots of form \( \alpha_1\alpha_2\alpha_j \), \( j = 3, \ldots, n \) are distinct. This proves Part (1).

Now suppose that the set of roots of \( p_1 \) has full rank. Let \( q \) be a degree \( k \) factor of \( p_i \) over \( \mathbb{Z}[x] \) for some \( i = 2, \ldots, n \). The constant term of \( q \) is \( \pm 1 \), and by the full rank property, of form \( (\alpha_1\alpha_2 \cdots \alpha_n)^d \) for some positive integer \( d \). The constant term of \( p_i \) is \( (\alpha_1 \cdots \alpha_n)^{D(i)} \), by Proposition 3.3, so \( d \leq D(i) \). On the other hand, roots of \( p_i \) are Hall words of degree \( i \) in \( \alpha_1, \alpha_2, \ldots, \alpha_n \), so the constant term of \( q \) is the product of \( k \) \( i \)-letter words in \( \alpha_1, \alpha_2, \ldots, \alpha_n \). Thus, \( ki = nd \), and the degree \( k \) is an integral multiple of \( n/i \) as desired. This proves Part (2a).

Now suppose that \( q \) is nonlinear and irreducible. Let \( \beta \) be a root of \( q \). Using the identity \( \alpha_1 \cdots \alpha_n = 1 \), we may write \( \beta \) in the form \( \alpha_1^{d_1} \cdots \alpha_n^{d_n} \) where no \( \alpha_n \) appears, and Lemma 3.7 applies. Then by the lemma, the action of \( N \) on \( \{\alpha_1 \cdots \alpha_n\} \) is not transitive.

Finally, to show irreducibility of \( p_2, q_1 \) and \( q_2 \), use the full rank condition to show that the roots of \( p_2 \) and \( q_1 \) are distinct when \( n \geq 3 \), and the roots of \( q_2 \) are distinct when \( n \geq 4 \). □

We obtain a corollary that describes the dimensions of the steps of certain sorts of Anosov Lie algebras.

**Corollary 3.9.** Let \( n \) be an \( r \)-step nilpotent Lie algebra of type \( (n_1, \ldots, n_r) \), where \( n_1 \) is prime and \( 1 < r < n_1 \), that admits an Anosov automorphism \( f \). Let \( (p_1, p_2, \ldots, p_n) \) be the \( r \)-tuple of polynomials associated to an automorphism \( f \) of \( f_{n,r}(\mathbb{R}) \) that has \( f \) as a quotient. Suppose that the polynomial \( p_1 \) is irreducible. Then \( n_1 \) divides \( n_i \) for all \( i = 2, \ldots, r \).

**Proof.** Let \( G \) denote the Galois group of the polynomial \( p_1 \) associated to the Anosov automorphism \( f \). Because \( p_1 \) is irreducible, its roots are distinct, and \( f \) is semisimple. The number \( n_1 \) is prime, hence it divides the order of \( G \), and by Lagrange’s Theorem, there is a subgroup of \( G \) isomorphic to \( C_{n_1} \). The permutation representation of \( G \) on \( \mathbb{Q}^n \) is then the sum of the principal representation and a representation that is irreducible over \( \mathbb{Q} \). Hence the set of roots of \( p_1 \) has full rank by Proposition 3.6.

Now let \( q \) be the characteristic polynomial of a rational \( f \)-invariant subspace \( E \) contained in \( V_i(\mathbb{R}) \) for some \( i < n \). Because \( n_1 \) is prime, and \( i < n_1 \), the numbers \( i \) and \( n_1 \) are coprime. Therefore, the dimension \( k \) of a rational invariant subspace for \( f \) is an integral multiple of \( n_1 \), by Part (2a) of Lemma 3.8. Therefore, the dimension \( n_i \) of the \( i \)th step of \( n \) is a multiple of \( n_1 \) for all \( i = 2, \ldots, r \). □

### 3.5. The existence of Anosov polynomials with given Galois group.

In the next proposition, we summarize some results on the existence of Anosov polynomials with certain properties.

**Proposition 3.10.** There exist irreducible Anosov polynomials in \( \mathbb{Z}[x] \) satisfying the following conditions.
For all \( n \geq 2 \), for all \( r = 1, \ldots, n-1 \), there exists an irreducible Anosov polynomial \( p \) of degree \( n \) such that precisely \( r \) of the roots have modulus larger than one.

(2) For all \( n \geq 2 \), there exists an irreducible Anosov polynomial of degree \( n \) with Galois group \( S_n \).

(3) For all prime \( n \geq 2 \), there exists an irreducible Anosov polynomial of degree \( n \) with Galois group \( C_n \).

(4) Suppose that the group \( G \) acts transitively on some set of cardinality 2, 3, 4 or 5 (hence is the Galois group of an irreducible polynomial in \( \mathbb{Z}[x] \) of degree \( n \leq 5 \)), and \( G \) is not isomorphic the alternating group \( A_5 \). Then there exists an Anosov polynomial of degree \( n \) having Galois group \( G \).

D. Fried showed how to use the geometric version of the Dirichlet Unit Theorem and the results from [AS70] to construct Anosov automorphisms with given spectral properties ([Fri81]); these methods can also be used to prove the first part of the proposition. We provide an alternate proof that shows the actual polynomials defining the automorphisms.

**Proof.** The polynomial \( p(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + 1 \) has \( r \) roots greater than one in modulus and \( n-r \) roots less than one in modulus when

\[
|a_r| > 2 + |a_1| + \cdots + |a_{r-1}| + |a_{r+1}| + \cdots + |a_{n-1}|
\]

(May91, see [Taj21]). By letting \( a_r = 3 \) and \( a_i = 0 \) for \( i \neq r \) in \( \{1, \ldots, n-1\} \), we get a polynomial of degree \( n \) with precisely \( r \) roots greater than one in modulus that is irreducible by Eisenstein’s Criterion. By Remark 2.7, if \( n \neq 2r \), the polynomial can not have any roots of modulus one. If \( n = 2r \), then the polynomial is \((x^r)^2 + 3x^r + 1\); one can check by hand that neither of these polynomials have roots of modulus one. This proves Part (1).

The irreducible polynomial \( x^n - x - 1 \) has Galois group \( S_n \) (Ser92). As it is not self-reciprocal, by Remark 2.7, it has no roots of modulus one.

Now suppose that \( K \) is a Galois extension of \( \mathbb{Q} \) with Galois group \( C_n \) with \( n \geq 3 \) prime. It is well known that such a field exists (See the Kronecker-Weber Theorem, in [JLY92]). Let \( \eta \) be a Dirichlet fundamental unit for \( K \), and let \( p \) denote its minimal polynomial. Since \( \eta \) is a unit, the constant term of \( p \) is \pm 1. Because \( \eta \notin \mathbb{Q} \), the degree \( m \) of \( \eta \) is greater than one. Because \( m \) divides \( n \), and \( n \) is prime, \( m \) must equal \( n \). Thus, the minimal polynomial \( p \) for \( \eta \) has degree \( n \) and splitting field \( K \). The degree of \( p \) is odd, so by Lemma 3.3, \( p \) has no roots of modulus one. Thus we have shown that \( p \) is an Anosov polynomial.

Now we prove Part (3). It is simplest to list examples of Anosov polynomials of each kind: See Table 1. By Remark 2.7, the only polynomial in the table that could possibly have roots of modulus one is the self-reciprocal polynomial with Galois group \( V_4 \). An easy calculation shows that it does not have roots of modulus one.

\[\square\]
| Degree | Galois group | Anosov polynomial |
|--------|-------------|-------------------|
| 2      | $C_2$       | $p(x) = x^2 - x - 1$ |
| 3      | $C_3$       | $p(x) = x^3 - 3x - 1$ |
| 3      | $S_3$       | $p(x) = x^3 - x - 1$ |
| 4      | $C_4$       | $p(x) = x^4 + x^3 - 4x^2 - 4x + 1$ |
| 4      | $V_4$       | $p(x) = x^4 + 3x^2 + 1$ |
| 4      | $D_8$       | $p(x) = x^4 - x^3 - x^2 + x + 1$ |
| 4      | $A_4$       | $p(x) = x^4 + 2x^3 + 3x^2 - 3x + 1$ |
| 4      | $S_4$       | $p(x) = x^4 - x - 1$ |
| 5      | $C_5$       | $p(x) = x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1$ |
| 5      | $D_{10}$    | $p(x) = x^5 - x^3 - 2x^2 - 2x - 1$ |
| 5      | $F_{20}$    | $p(x) = x^5 + x^4 + 2x^3 + 4x^2 + x + 1$ |
| 5      | $A_5$       | Q: What is an example with small coefficients? |
| 5      | $S_5$       | $p(x) = x^5 - x - 1$ |

**Table 1.** The inverse Galois problem for Anosov polynomials of low degree

Many of the examples in Table 1 were taken from the appendix of [MM99]; the reader may find there a great many more examples of Anosov polynomials of degree $n \geq 6$ with a variety of Galois groups. To our knowledge, it is known whether the existence of a polynomial in $\mathbb{Z}[x]$ with Galois group $G$ guarantees the existence of a polynomial in $\mathbb{Z}[x]$ with constant term $\pm 1$ and Galois group $G$. In addition, we do not know of an example of an Anosov polynomial with Galois group $A_5$. 
4. Actions of the Galois group

4.1. Definitions of actions. In this section, we associate the \( \mathbb{Q} \)-linear action of a finite group to an automorphism of a free nilpotent Lie algebra that preserves the rational structure defined by a Hall basis.

**Definition 4.1.** Let \( A \) be a matrix in \( GL_n(\mathbb{Z}) \), let \( p_1 \) be the characteristic polynomial of \( A \) and let \( K \) be the splitting field for \( p_1 \). Suppose that \( f \) is the automorphism of \( \mathfrak{f}_{n,r}(K) \) determined by \( A \) and a set \( B_1 = \{ x_i \}_{i=1}^n \) of generators for \( \mathfrak{f}_{n,r}(K) \). Let \( B = \cup_{i=1}^r B_i \) be the Hall basis determined by \( B_1 \). Write \( \mathfrak{f}_{n,r}(K) = \oplus_{i=1}^r V_i(K) \), and for \( i = 1, \ldots, r \), let \( n_i = \text{dim}_K V_i(K) \).

Let \( G \) denote the Galois group for the field \( K \). Let \( x_1^i, \ldots, x_{n_i}^i \) denote the \( n_i \) elements of the basis \( B_i \) of \( V_i(K) \); they determine an identification \( V_i(K) \cong K^{n_i} \). For \( i = 1, \ldots, r \), the \( G \) action on \( K \) extends to a diagonal \( G \) action on \( V_i(K) \cong K^{n_i} \). In particular, if \( w = \sum_{j=1}^{n_i} \beta_j x_j^i \) is an element in \( V_i(K) \), where \( \beta_1, \ldots, \beta_{n_i} \in K \), and \( g \in G \), then \( g \cdot w \) is defined by

\[
g \cdot w = \sum_{j=1}^{n_i} (g \cdot \beta_j) x_j^i \in V_i(K).
\]

A \( G \) action on the free nilpotent Lie algebra \( \mathfrak{f}_{n,r}(K) = \oplus_{i=1}^r V_i(K) \) is defined by extending each of the \( G \) actions on \( V_i(K) \), for \( i = 1, \ldots, r \).

Next we describe properties of the action.

**Proposition 4.2.** Let \( A \) be a semisimple matrix in \( GL_n(\mathbb{Z}) \), let \( p_1 \) be the characteristic polynomial of \( A \), and let \( K \) be the splitting field for \( p_1 \) over \( \mathbb{Q} \). Let \( B_1 \) be a generating set for the free nilpotent Lie algebra \( \mathfrak{f}_{n,r} \) and let \( B \) be the Hall basis that it determines. Let \( f \) be the automorphism of \( \mathfrak{f}_{n,r}(K) \), induced by Hall basis \( B \) and the matrix \( A \). Let \( G \) be the Galois group for \( K \), and let \( G \cdot \mathfrak{f}_{n,r}(K) \to \mathfrak{f}_{n,r}(K) \) be the action defined in Definition 4.1. Then

1. The \( G \) action is \( \mathbb{Q} \)-linear, preserving \( \mathfrak{f}_{n,r}(\mathbb{Q}) < \mathfrak{f}_{n,r}(K) \), and it preserves the decomposition \( \mathfrak{f}_{n,r}(K) = \oplus_{i=1}^r V_i(K) \) of \( \mathfrak{f}_{n,r}(K) \) into steps.
2. The \( G \) action on \( \mathfrak{f}_{n,r}(K) \) commutes with the Lie bracket.
3. The function \( f : \mathfrak{f}_{n,r}(K) \to \mathfrak{f}_{n,r}(K) \) is \( G \)-equivariant.
4. The \( G \) action permutes the eigenspaces of \( f \): an element \( g \in G \) sends the \( \alpha \) eigenspace for \( f \) to the \( g \cdot \alpha \) eigenspace for \( f \).

Note also that because it fixes \( \mathfrak{f}_{n,r}(\mathbb{R}) \) stepwise, the \( G \) action on \( \mathfrak{f}_{n,r}(K) = \oplus_{i=1}^r V_i(K) \) commutes with the grading automorphism \( B \) defined by \( B(w) = e^t w \) for \( w \in V_i \).

**Proof.** The first assertion follows from the definition of the action.

The action of \( G \) commutes with the Lie bracket because structure constants for the Hall basis \( B \) are rational. Let \( y_1, \ldots, y_N \) be an enumeration of the Hall basis \( B \), and...
denote the structure constants relative to $\mathcal{B}$ by $\alpha_{ij}^k$. Consider the Lie bracket of two arbitrary vectors $\sum_{i=1}^N a_i y_i$ and $\sum_{j=1}^N b_j y_j$ in $f_{n,r}(K)$:
\[
g \cdot \left[ \sum_{i=1}^N a_i y_i, \sum_{j=1}^N b_j y_j \right] = g \cdot \sum_{k=1}^N \left( \sum_{i=1}^N \sum_{j=1}^N \alpha_{ij}^k a_i b_j \right) y_k = \sum_{k=1}^N \left( \sum_{j=1}^N \sum_{i=1}^N \alpha_{ij}^k (g \cdot a_i)(g \cdot b_j) \right) y_k = \left[ g \cdot \sum_{i=1}^N a_i y_i, g \cdot \sum_{j=1}^N b_j y_j \right].
\]

Now we show that the $G$ action on $f_{n,r}(K)$ commutes with the automorphism $f$. We can write $w$ in $f_{n,r}(K)$ as a linear combination $\sum_{j=1}^N \beta_j y_j$ of elements in the Hall basis $\mathcal{B}$ of $f_{n,r}(K)$. Take $g \in G$. Because $f$ is linear,
\[
f(g \cdot w) = f \left( \sum_{j=1}^N (g \cdot \beta_j) y_j \right) = \sum_{j=1}^N (g \cdot \beta_j) f(y_j).
\]

An integral matrix $(c_{ij})$ represents $f$ with respect to the basis $\mathcal{B}$. Therefore, for each $j = 1, \ldots, N$, the vector $f(y_j)$ is a $\mathbb{Z}$-linear combination $\sum_{i=1}^N c_{ij} y_i$ of $y_1, \ldots, y_N$. All of the integer entries of the matrix $(c_{ij})$ are fixed by the $G$ action. Hence, $f(g \cdot w)$ equals
\[
\sum_{j=1}^N (g \cdot \beta_j) f(y_j) = \sum_{j=1}^N (g \cdot \beta_j) \sum_{i=1}^N c_{ij} y_j = g \cdot \sum_{j=1}^N \sum_{i=1}^N \beta_j c_{ij} y_j = g \cdot f \left( \sum_{j=1}^N \beta_j y_j \right) = g \cdot (f w).
\]

Thus we have shown that $f(g \cdot w) = g \cdot (f w)$, so the $G$ action on $f_{n,r}(K)$ commutes with $f$ as asserted.

Consider an eigenvector $z$ for $f$ with eigenvalue $\alpha$. Because $z$ is in $\ker(f - \alpha \text{Id})$, and $\alpha \in K$, the vector $z$ is a $K$-linear combination of elements of the rational basis. For $g$ in $G$,
\[
f(g \cdot z) = g \cdot f(z) = g \cdot \alpha z = (g \cdot \alpha)(g \cdot z).
\]

Hence an automorphism $g$ in $G$ sends the $\alpha$-eigenspace to the $(g \cdot \alpha)$-eigenspace. Therefore, Part (4) of the proposition holds. \qed
Now we use the action defined in Definition 4.1 to describe certain important kinds of subspaces and ideals of Anosov Lie algebras.

**Definition 4.3.** Let $G$ be a finite group that acts on the free nilpotent Lie algebra $f_{n,r}(K)$ over field $K$, and let $z \in f_{n,r}(K)$. Define the subspace $E_G^K(z)$ of $f_{n,r}(K)$ to be the $K$-span of the $G$-orbit of $z$:

$$E_G^K(z) = \text{span}_K \{ g \cdot z : g \in G \}.$$

**Example 4.4.** Let $f_{n,2}(K) = V_1(K) \oplus V_2(K)$ be the free two-step Lie algebra on $n$ generators over field $K$. Let $C_1 = \{ z_i \}_{i=1}^n$ be a set of $n$ generators for $f_{n,2}(K)$. Let $G$ be the cyclic group of order $n$ that acts on $f_{n,2}(K)$ through the natural action of $G \cong C_n$ on $C_1$. For each $j = 2, \ldots, \lfloor n/2 \rfloor + 1$, define the ideal $i_j < V_2(K)$ by

$$i_j = E_G^K([z_j, z_1]) = \text{span}_K \{ [z_s, z_t] : s - t = j - 1 \mod n \}.$$

For example, when $n = 4$,

$$i_2 = \text{span}_K \{ [z_2, z_1], [z_3, z_2], [z_4, z_3], [z_1, z_4] \}, \quad \text{and} \quad i_3 = \text{span}_K \{ [z_3, z_1], [z_4, z_2] \}.$$

For distinct $j_1$ and $j_2$ in $\{1, \ldots, \lfloor n/2 \rfloor \}$, the subspaces $i_{j_1}$ and $i_{j_2}$ are independent, hence $V_2(K) = \bigoplus_{j=1}^{\lfloor n/2 \rfloor} i_j$. When $n = n_1$ is odd, there are $\frac{1}{2}(n - 1)$ such subspaces, each of dimension $n$. When $n = n_1$ is even, the subspaces $i_{j_i}, j = 1, \ldots, n/2 - 1$ are of dimension $n$, and the subspace $i_{n/2}$ is of dimension $n/2$.

For any proper subset $S$ of $\{1, \ldots, \lfloor n/2 \rfloor \}$, there is an ideal $i(S)$ of $f_{n,2}(K)$ defined by $\bigoplus_{j \in S} i_j$, and this ideal defines a two-step Lie algebra $n_S = f_{n,2}(K)/i_S$.

We define ideals of free nilpotent Lie algebras arising from group actions, as the Lie algebra $n_S$ in the previous example arises from the action of a cyclic group.

**Definition 4.5.** Let $G$ be a finite group that acts on the free $r$-step nilpotent Lie algebra $f_{n,r}(K) = \bigoplus_{i=1}^r V_i(K)$ over field $K$, where the field $K$ is an extension of $Q$. Let $L$ be another extension of $Q$, typically $R$ or $C$. Suppose that Hall bases $B \subset f_{n,r}(K)$ and $B' \subset f_{n,r}(L)$ define rational structures on $f_{n,r}(K)$ and $f_{n,r}(L)$ respectively, and let $E \to E^L$ be the correspondence of rational invariant subspaces defined in Definition 2.3 resulting from an identification of $B'$ and $B$.

A rational ideal of $f_{n,r}(L)$ generated by sets of the form $(E_G^L(w))$, where $E_G^L(w)$ is as defined in Definition 4.3 for $w \in f_{n,r}(K)$, is called an ideal of type $G$ defined over $L$. We use $i(G, w)$ denote the ideal of $f_{n,r}(L)$ generated by the subspace $(E_G^L(w))^L$.

A nilpotent Lie algebra of form $f_{n,r}(L)/i$, where $i$ is of type $G$ will be called a nilpotent Lie algebra of type $G$.

Now we describe ideals of symmetric type for two- and three-step free nilpotent Lie algebras.
Example 4.6. Suppose that \( f_{n,r}(K) \) has generating set \( C = \{ z_j \}_{j=1}^n \) and that the action of \( G \cong S_n \) on \( f_{n,r}(K) \) is defined by permuting elements \( z_1, \ldots, z_n \) of the generating set \( C \). Let \( C = \bigcup_{i=1}^r \mathcal{C}_i \) be the Hall basis determined by \( C \).

For any \( w = [z_i, z_j] \) in \( C_2 \), the subspace \( E^K_G(w) \) is all of \( V_2(K) \):

\[
E^K_G(w) = \text{span}_K \{ [z_j, z_i] \}_{1 \leq i < j \leq n} = V_2(K).
\]

Hence, an ideal \( i < f_{n,r}(K) \) of type \( S_n \) that intersects \( V_2(K) \) nontrivially must contain all of \( V_2(K) \).

When \( r \geq 3 \), there are two sets of form \( E^K_G(w) \) with \( w \in C_3 \), the subspaces \( F_1 \) and \( F_2 \) defined in Equation (1):

\[
F_1(K) = E^K_G([z_2, z_1], z_1]) \quad \text{and} \quad F_2(K) = E^K_G([z_2, z_1], z_3]).
\]

Therefore, for any ideal \( i \) of type \( S_n \), the subspace \( i \cap V_3(K) \) is one of the following: \{0\}, \( F_1(K) \), \( F_2(K) \), or \( V_3(K) \).

Next is an example showing nilpotent Lie algebras arising from dihedral groups.

Example 4.7. Let \( A \) be a semisimple matrix in \( GL_n(\mathbb{Z}) \) whose characteristic polynomial has splitting field \( K \) and Galois group \( G \) isomorphic to the dihedral group \( D_{2n} \) of order \( 2n \). Let \( f \) be the automorphism of \( f_{n,2}(K) \) induced by \( A \) and a Hall basis \( B \). Let \( z_1, \ldots, z_n \) denote a set of eigenvectors of \( f|_{V_1(K)} \) spanning \( V_1(K) \) compatible with the rational structure, and let \( \alpha_1, \ldots, \alpha_n \) denote corresponding eigenvalues.

The group \( D_{2n} \) is isomorphic to the group of symmetries of a regular \( n \)-gon. Enumerate the vertices of such an \( n \)-gon in counterclockwise order so that \( D_{2n} \cong \langle r, s \rangle \), where \( r \) is counterclockwise rotation by \( 2\pi/n \) and \( s \) is reflection through a line through center of the \( n \)-gon and the first vertex. Let \( X_n \) be the complete graph on \( n \) vertices obtained by adding edges connecting all distinct vertex pairs of the \( n \)-gon. Identify the roots of \( p_1 \) with the \( n \) vertices and the roots of \( p_2 \) with the \( \left( \frac{n}{2} \right) \) edges in such a way that the eigenvalue \( \alpha_i \alpha_j \) corresponds to the edge connecting vertices corresponding to eigenvalues \( \alpha_i \) and \( \alpha_j \). The \( G \) action on \( f_{n,2}(K) \) can then be visualized through the \( D_{2n} \) action on the graph \( X_n \).

For example, the \( G \) orbit of the eigenvector \([z_2, z_1]\) is given by

\[
\{ [z_2, z_1], [z_3, z_2], \ldots, [z_1, z_n] \},
\]

corresponding to the \( n \) "external" edges of the graph \( X_2 \). The subspace \( E^K_G([z_2, z_1]) \) defined by Definition 4.3 is the \( K \)-span of this set, an \( n \)-dimensional subspace of \( V_2(K) \). Other orbits depend on the value of \( n \) : if \( n = 3 \) there are no other orbits, and if \( n = 4 \) or \( 5 \), there is one more orbit coming from "interior" edges on the graph, yielding a subspace \( E^K_G([z_3, z_1]) \) of \( V_2(K) \) that is complementary to \( E^K_G([z_2, z_1]) \). When \( n \geq 6 \), there are two more orbits coming from interior edges.

5. Rational invariant subspaces

Given an automorphism of a free nilpotent Lie algebra, the next theorem describes how orbits of the \( G \) action on \( f_{n,r}(\mathbb{R}) \) relate to the factorization of the polynomials.
p_1, \ldots, p_r$. These restrictions on factorizations yield restrictions on the existence of Anosov quotients. Roughly speaking, when the Galois group of $p_1$ is highly transitive, the rational invariant subspaces for associated Anosov automorphisms tend to be big also, and when the group is small, the rational invariant subspaces are small. The larger rational invariant subspaces are, the fewer Anosov quotients there may be. The field $L$ in the theorem is typically $\mathbb{R}$ or $\mathbb{C}$.

**Theorem 5.1.** Let $A$ be a semisimple matrix in $GL_n(\mathbb{Z})$. Let $(p_1, \ldots, p_r)$ be the $r$-tuple of polynomials associated to $f$, let $K$ be the splitting field of $p_1$ and let $G$ be the Galois group for $K$. Let $f$ be the semisimple automorphism of $f_{n,r}(K)$ defined by Hall basis $B = \cup_{i=1}^r B_i$ of $f_{n,r}(K)$ and the matrix $A$. Let $C = \cup_{i=1}^r C_i$ be the Hall basis for $f_{n,r}(K)$ determined by a set of $C_1 = \{z_j\}_{j=1}^n$ of eigenvectors for $f|_{V_1(K)}$ that is compatible with the rational structure defined by $B$.

For all $i = 1, \ldots, r$, the vector subspace $V_i(K)$ of $f_{n,r}(K)$ decomposes as the direct sum of rational invariant subspaces of the form $E^K_G(z)$, where $z \in C_i$, and $E^K_G(z)$ is as defined in Definition 4.3. The characteristic polynomial $p_E$ for the restriction of $f$ to $E^K_G(z)$ is of form $p_E = r^s$, where $r$ is a polynomial that is irreducible over $\mathbb{Z}$.

Suppose that the field $L$ is an extension of $\mathbb{Q}$, and that $f$ is a semisimple automorphism of $f_{n,r}(L)$ defined by Hall basis $B' = \cup_{i=1}^r B'_i$ and the matrix $A$. Since the subspaces $E^K_G(z)$ of $f_{n,r}(K)$ are rational, for all $i = 1, \ldots, r$, there is a decomposition $V_i(L) = \oplus (E^K_G(z))^L$ of $V_i(L) < f_{n,r}(L)$ into rational $f$-invariant subspaces, through the correspondence defined in Definition 2.3 induced by the identification of the rational Hall bases $B$ and $B'$ of $f_{n,r}(K)$ and $f_{n,r}(L)$.

We illustrate the theorem by considering a special case of Example 4.7.

**Example 5.2.** Let $A$ be a semisimple hyperbolic matrix in $GL_5(\mathbb{Z})$ such that the splitting field $K$ for the characteristic polynomial $p_1$ for $A$ has Galois group $G$ isomorphic to the dihedral group $D_{10}$ of order 10. Let $f_K$ be the automorphism of $f_{5,2}(K) = V_1(K) \oplus V_2(K)$ induced by $A$ and a Hall basis $B$. Let $C_1 = \{z_1, \ldots, z_3\}$ be the set of eigenvectors of $f_K|_{V_1(K)}$ and let $\alpha_1, \ldots, \alpha_5$ denote the corresponding eigenvalues. Let $C = C_1 \cup C_2$ be the Hall basis of $f_{5,2}(K)$ determined by $C_1$.

We saw in Example 4.7 that the $D_{10}$ action on $f_{5,2}(K)$ has two orbits each of $K$-dimension five. By Theorem 5.1 these two orbits yield rational $f_K$-invariant subspaces

$$i_1(K) = E_G([z_2, z_1]), \quad \text{and} \quad i_2(K) = E_G([z_3, z_1])$$

and a decomposition $V_2(K) = i_1(K) \oplus i_2(K)$ of $V_2(K) < f_{5,2}(K)$ into rational invariant subspaces.

Let $f_{\mathbb{R}}$ be the automorphism of $f_{5,2}(\mathbb{R})$ induced by $A$ and Hall basis $B' = \cup_{i=1}^r B'_i$ of $f_{5,2}(\mathbb{R})$. Letting $L = \mathbb{R}$ in the Theorem 5.1 and using the correspondence between $f_{5,2}(K)$ and $f_{5,2}(\mathbb{R})$ as in Definition 2.3, we get a rational $f_{\mathbb{R}}$-invariant decomposition of $V_2(R) < f_{5,2}(\mathbb{R})$ into rational $f_{\mathbb{R}}$-invariant subspaces $i_1(\mathbb{R}) = (i_1(K))^\mathbb{R}$ and $i_2(\mathbb{R}) = (i_2(K))^\mathbb{R}$. 
Proof of Theorem 5.1. Fix an element \( D \) as in Definition 4.3. By Lemma 3.3, it is an eigenvector for \( K \) in \( G \). Since \( \text{dim} G = 5 \), we compute that \( \text{dim} V_1 = 1 \), \( \text{dim} V_2 = 3 \), \( \text{dim} V_3 = 1 \), \( \text{dim} V_4 = 1 \), \( \text{dim} V_5 = 1 \). The vector \( \text{dim} V_1 = 1 \) is preserved when moving to \( G_1 \), so that the quotient algebras \( f(\mathbb{R})/i_1 \) and \( f(\mathbb{R})/i_2 \) are irreducible. By \( \text{dim} G = 5 \), we have that \( \text{dim} V_2 = 3 \). (In particular, the \( \text{dim} V_1 = 1 \) coset in \( f(\mathbb{R})/i_1 \) is the unique element having a three-dimensional centralizer.)

In summary, in addition to \( f(\mathbb{R}) \), there are exactly two isomorphic two-step nilpotent quotients, both of type \((5,5)\) to which \( f \) descends as an Anosov map, and there are exactly two nonisomorphic Anosov Lie algebras of type \((5,5)\) with Anosov automorphisms yielding Galois group \( D_{10} \).

Proof of Theorem 5.1. Fix an element \( w \) in the basis \( C_i \) for \( V_i(K) < f_{n,r}(K) \). By Proposition 3.1, it is an eigenvector for \( f \); let \( \alpha \) denote its eigenvalue. When represented with respect to the rational basis \( B \), the vector \( w \) has coordinates in \( K^{n_i} \), where \( n_i = \text{dim} V_i \). Let \( E^K_G(w) \) be the subspace of \( V_i(K) \) generated by \( w \) and \( G \) as in Definition 4.3.

First we show that \( E^K_G(w) \) is invariant under \( f \). By Proposition 4.2, part (4), for all \( g \) in \( G \), the vector \( g \cdot w \) is an eigenvector with eigenvalue \( g \cdot \alpha \). An element \( u \) of
$V_i(K)$ is in $E^K_i(w)$ if and only if it is of the form $u = \sum_{g \in G} c_g (g \cdot w)$, where $c_g \in K$ for $g \in G$. Then

$$f(u) = \sum_{g \in G} c_g f(g \cdot w) = \sum_{g \in G} c_g (g \cdot \alpha) g \cdot w,$$

so $f(u)$ is also in $E^K_i(w)$.

The vector space $E^K_i(w)$ is spanned by vectors with coordinates in $K$, so there exists a polynomial function $\phi : V_i(\mathbb{R}) \cong \mathbb{R}^n \to \mathbb{R}$ with coefficients in $K$ such that $E^K_i(w)$ is the zero set of $\phi$. Because $E^K_i(w)$ is invariant under the $G$ action, for all $g \in G$, $E^K_i(w)$ is also the zero set of the function $\phi_g(x) = \phi(g \cdot x)$. Let $\tilde{\phi} = \prod_{g \in G} \phi_g$.

The function $\tilde{\phi}$ has rational coefficients, so $E^K_i(w)$ is a rational $G$-invariant subspace.

By Maschke’s Theorem, the subspace $V_i(K)$ may be written as the direct sum of subspaces of form $E^K_i(w)$.

Now we show that the characteristic polynomial for the restriction of $f$ to $E^K_i(w)$ is a power of an irreducible. Let $q$ denote the minimal polynomial for $\alpha$. By Proposition 3.1 the splitting field $\mathbb{Q}(q)$ is intermediate to $\mathbb{Q}$ and the splitting field $\mathbb{Q}(p_1)$ of $p_1$. The $G$-orbit of $\alpha$ is precisely the set of roots of $q$. Since $g \cdot \alpha$ is a root of $q$ for all $g \in G$, the space $E^K_i(w)$ is contained in the direct sum of the eigenspaces for eigenvalues $g \cdot \alpha, g \in G$. Therefore, the characteristic polynomial $p_E$ for the restriction of $f$ to $E^K_i(w)$ is a power of the irreducible polynomial $q$.

We will describe a some rational invariant subspaces that exist for any automorphism of a free nilpotent Lie algebra preserving a rational structure. First we need to make a definition.

**Definition 5.3.** Let $K$ be a field. Let $\mathcal{C}_1 = \{z_j\}_{j=1}^n$ be a generating set for the free $r$-step nilpotent Lie algebra $f_{n,r}(K)$, and let $\mathcal{C}$ be the associated Hall basis. Define the ideal $i_{n,r}$ of $f_{n,r}(K)$ to be the ideal generated by all elements $w$ of the Hall basis $\mathcal{C}$ having the property that there is a single number $k$ such that for all $j = 1, \ldots, n$, the letter $z_j$ occurs exactly $k$ times in the Hall word $w$.

For example, when $n = 3$, the ideal $i_{3,2} < f_{3,2}(K)$ is $\{0\}$, and the ideal $i_{3,3} < f_{3,3}$ given by

$$i_{3,3} = \text{span}_K \{[[z_2, z_1], z_3], [z_3, z_1], z_2]\} = F_2(K) < V_3(K),$$

where $F_2(K)$ is as defined in Equation (1), and the ideal $i_{4,3} < f_{4,3}(K)$ is given by

$$i_{4,3} = i_{3,3} \oplus [i_{3,3}, f_{4,3}],$$

where we map $i_{3,3}$ into $f_{4,3}$ in the natural way.

**Remark 5.4.** Since the product of the roots of an Anosov polynomial is always $\pm 1$, any ideal $i < f_{n,r}$ satisfying the Auslander-Scheuneman conditions for some $f$ must contain the ideal $i_{n,r}$ (defined relative to an eigenvector basis).
Proposition 5.5. Let $A$ be a semisimple matrix in $GL_n(\mathbb{Z})$ whose characteristic polynomial has splitting field $K$. Let $f_{n,r}(K) = \bigoplus_{i=1}^r V_i(K)$ be the free $r$-step nilpotent Lie algebra on $n \geq 3$ generators over $K$, endowed with the rational structure defined by a Hall basis $\mathcal{B}$. Let $f$ be the semisimple automorphism of $f_{n,r}(K)$ defined by the matrix $A$ and the Hall basis $\mathcal{B}$. Let $\mathcal{C}$ be the Hall basis of $f_{n,r}(K)$ determined by a set of eigenvectors $\mathcal{C}_1$ for $f|_{V_1(K)}$ that is compatible with the rational structure.

The ideal $\mathfrak{j}_{n,r}$ defined in Definition 5.3 is a rational invariant subspace; and when $r \geq 3$, the subspace $V_3(K)$ is the direct sum $F_1(K) \oplus F_2(K)$ of rational invariant subspaces where $F_1(K)$ and $F_2(K)$ are as in Equation (11), while $F_2(K)$ decomposes as the direct sum $F_2(K) = F_{2a}(K) \oplus F_{2b}(K)$ where $F_{2a}(K)$ and $F_{2b}(K)$ are rational invariant subspaces each of dimension $(\frac{n}{3})$. The characteristic polynomial for the restrictions of $f$ to $F_1(K)$ is $q_1$, and the characteristic polynomials for the restriction of $f$ to $F_{2a}(K)$ and $F_{2b}(K)$ are both $q_2$.

Proof. Let $G$ denote the Galois group of the polynomial $p_1$ associated to $f$. Suppose that $w$ is a $k$-fold bracket of elements in $\mathcal{C}_1 = \{z_j\}_{j=1}^n$.

Recall from Example 2.1 that $\mathcal{C}_3$ is the union of the set $\mathcal{C}_3'$ of standard Hall monomial of the first type and the set $\mathcal{C}_3''$ of standard Hall monomials of the second type. Let $g \in S_n$. It is easy to see that if $w \in \mathcal{C}_3'$ then $g \cdot w \in \mathcal{C}_3'$ or $-(g \cdot w) \in \mathcal{C}_3''$ and if $w \in \mathcal{C}_3''$ then $g \cdot w \in \mathcal{C}_3'$, $-(g \cdot w) \in \mathcal{C}_3''$, or $g \cdot w$ is a linear combination of elements of $\mathcal{C}_3''$ through the Jacobi Identity. The action of the group $G$ on $f_{n,r}(K)$ therefore preserves the subspaces $F_1$ and $F_2$, so that if $w$ in $\mathcal{C}_3$ is in $F_1(K)$ or $F_2(K)$, then $E_{n,K}^+(w) < F_1$ or $E_{n,K}^+(w) < F_2(K)$ respectively. The space $F_1(K)$ is the sum of the rational invariant spaces $E_{n,K}^+(w)$ as $w$ varies over elements of $\mathcal{C}_3'$, so is rational and invariant. By the same reasoning $F_2(K)$ is rational and $f$-invariant also.

The characteristic polynomial for the restriction of $f$ to $F_2(K)$ is $q_2^2$, where $q_2$ is as defined in Equation (3). The pair of elements of form $[z_i, z_j, z_k]$ and $[z_i, z_j, z_l]$, where $1 \leq j < i < k \leq n$, in $\mathcal{C}_3$ have the same eigenvalue, $\alpha_i \alpha_j \alpha_k$, where $\alpha_i$, $\alpha_j$, $\alpha_k$ are the eigenvalues of $z_i$, $z_j$, and $z_k$ respectively. These yield one basis vector for $F_{2a}$ and one basis vector for $F_{2b}$. Each factor $q_2$ in the characteristic polynomial $q_2^2$ for $F_2(K)$ yields one rational invariant subspace of $F_2(K)$ of dimension $\deg q_2 = (\frac{n}{3})$, each spanned by elements of $\mathcal{C}_3''$. Call these subspaces $F_{2a}(K)$ and $F_{2b}(K)$, so $F_2(K) = F_{2a}(K) \oplus F_{2b}(K)$.

Similarly, the set of $k$-fold brackets $w$ of $\mathcal{C}$ having the property that each element $z_i$ occurs the same number of times in $w$ is clearly invariant under the action of $G$, so the ideal that it generates, $\mathfrak{j}_{n,r}$, is $G$-invariant, hence rational. \hfill \Box

The following elementary proposition yields restrictions on possible dimensions of rational invariant subspaces for semisimple automorphisms of nilpotent Lie algebras.

Proposition 5.6. Let $f_{n,r}(\mathbb{R})$ be a free nilpotent Lie algebra, and let $f : f_{n,r}(\mathbb{R}) \rightarrow f_{n,r}(\mathbb{R})$ be a semisimple automorphism defined by a matrix $A$ in $GL_n(\mathbb{Z})$ and a Hall basis $\mathcal{B}$ of $f_{n,r}(\mathbb{R})$. Suppose that the characteristic polynomial $p_1$ of $A$ is irreducible with
Galois group \( G \). Let \( m \) be the dimension of a minimal nontrivial rational invariant subspace \( E \lt \mathfrak{g}_{n,r}(\mathbb{R}) \) for \( f \). Then \( G \) has a normal subgroup \( N \) such that \( G/N \) acts faithfully and transitively on a set of \( m \) elements.

Note that the subspace \( E \) is one-dimensional if and only if \( N = G \).

Proof. Suppose that \( E \) is a minimal nontrivial invariant subspace of dimension \( m \). The characteristic polynomial \( p_E \) for the restriction of \( f \) to \( E \) is irreducible. Since the roots of \( p_E \) are contained in the splitting field \( \mathbb{Q}(p_1) \), the Galois group for \( p_E \) is the quotient \( G/N \) of \( G \) by the normal subgroup of elements of \( G \) fixing \( \mathbb{Q}(p_E) \). The group \( G/N \) acts faithfully and transitively on the \( m \) roots of \( p_E \) since it the Galois group of an irreducible polynomial. □

The previous proposition can be used to find all possible dimensions of minimal nontrivial rational invariant subspaces for any Anosov automorphism whose associated Galois group is some fixed group \( G \). One simply needs to find all numbers \( m \) such that there exists a normal subgroup \( N \) of \( G \) and there exists a faithful transitive action of \( H = G/N \) on a set of \( m \) elements. Every faithful transitive action of a group \( H \) on a set \( X \) is conjugate to the action of \( H \) on the cosets \( X' = \{hK\}_{h \in H} \) of a subgroup \( K \) of \( H \) such that \( K \) contains no nontrivial normal subgroups of \( H \). To find all faithful transitive actions of a group \( H = G/N \), one must list all subgroups of \( H \) and eliminate any that contain nontrivial normal subgroups. The cardinalities \( |H|/|K| \) of the set \( \{hK\}_{h \in H} \) are admissible values for the cardinality \( m \) of a set \( X \) on which \( H \) acts faithfully and transitively. In our situation, where \( G \) is the Galois group of \( p_1 \), the number \( m \) could be the degree of a polynomial having Galois group \( G/N \), and \( m \) could be the dimension of a rational invariant subspace of the corresponding automorphism of \( \mathfrak{g}_{n,r}(\mathbb{R}) \).

In Table 2 we analyze the possible dimensions of rational invariant subspaces of Anosov Lie algebras \( n \) for which \( p_1 \) is irreducible and degree 3 or 4. The first two columns in the table give all possible Galois groups \( G \) for irreducible polynomials degrees three and four, grouped by degree. The fourth and fifth columns list the isomorphism class of proper normal subgroups \( N \) of each group \( G \), and the quotients \( G/N \). (Since \( n \) is Anosov, there are no one-dimensional rational invariant subspaces, and we omit the case \( N = G \).) The quotient groups \( G/N \) are potential Galois groups for characteristic polynomials of rational invariant subspaces of an Anosov Lie algebra with polynomial \( p_1 \) having Galois group \( G \). The last column gives the cardinalities of sets on which each \( G/N \) can act faithfully and transitively, found by the procedure described above. The numbers in the last column are listed in the same order as the subgroups in the second column, with the numbers for different subgroups separated by semicolons. In the third column we show when the roots of a polynomial with given Galois group must have full rank by Proposition 3.6. When the set of roots has full rank, some possibilities for the normal subgroup \( N \) may be prohibited by...
Table 2. Possible dimensions $m > 1$ for rational invariant subspaces $E$ for an Anosov automorphism of an $r$-step nilpotent Lie algebra $n = f_{n,r}(\mathbb{R})/i$ when $n = 3$ or 4 and $p_1$ is irreducible. An asterisk indicates that the marked values of $N, G/N$ and $\text{dim} E$ cannot occur by Lemma 3.8, Part 2b.

| $\deg p_1$ | $G$   | full rank? | $N \neq G$ | $G/N$            | dimension $m$ of $E$ |
|-----------|------|------------|------------|-----------------|-----------------|
| 3         | $S_3$ | yes        | $C_3^*, \{1\}$ | $C_2^*, S_3$    | $2^*; 3, 6$     |
| 3         | $C_3$ | yes        | $\{1\}$    | $C_3$           | 3               |
| 4         | $S_4$ | yes        | $A_4^*, V_4^*, \{1\}$ | $C_2^*, S_3^*, S_4$ | $2^*; 3^*, 6^*; 4, 6, 8, 12, 24$ |
| 4         | $A_4$ | yes        | $V_4^*, \{1\}$ | $C_3^*, A_4$    | $3^*; 4, 12$    |
| 4         | $D_8$ | no         | $C_4, C_2, \{1\}$ | $C_2, V_4, D_8$ | $2; 4; 4, 8$    |
| 4         | $C_4$ | no         | $C_2, \{1\}$ | $C_2, C_4$      | $2; 4$          |
| 4         | $V_4$ | no         | $C_2, \{1\}$ | $C_2, V_4$      | $2; 4$          |

Lemma 3.8: these subgroups and the corresponding dimensions are indicated in the table with asterisks.

From the table we obtain the following corollary to Theorem 5.1.

**Corollary 5.7.** Let $f$ be a semisimple automorphism of $f_{n,r}(\mathbb{R})$ induced by a hyperbolic matrix in $\text{GL}_n(\mathbb{Z})$ and a Hall basis $\mathcal{B}$. Let $(p_1, \ldots, p_r)$ be the $r$-tuple of polynomials associated to $f$. If $p_1$ is irreducible, then the dimension of any minimal nontrivial invariant subspace of $f_{n,r}(\mathbb{R})$ is 3 or 6 if $n = 3$ and is one of 2, 4, 6, 8, 12, 24 if $n = 4$.

6. Automorphisms with cyclic and symmetric Galois groups

In this section we use Theorem 5.1 to analyze the structure of Anosov Lie algebras whose associated polynomial $p_1$ has either a small Galois group, such as a cyclic group, or a large Galois group, such as a symmetric group. The following theorem describes Anosov automorphisms associated to Galois groups whose actions on the roots of $p_1$ is highly transitive.

**Theorem 6.1.** Suppose that $n$ is a real $r$-step Anosov Lie algebra admitting an Anosov automorphism defined by a semisimple matrix $A$ in $\text{GL}_n(\mathbb{Z})$, a Hall basis $\mathcal{B}$, and
an ideal \( i < f_{n,1}(\mathbb{R}) \) satisfying Auslander-Scheuneman conditions. Suppose that the polynomial \( p_1 \) associated to \( f \) is irreducible with Galois group \( G \). Let \( (p_1, \ldots, p_r) \) be the \( r \)-tuple of polynomials associated to \( p_1 \).

1. If the action of \( G \) on the roots of \( p_1 \) is two-transitive, then the polynomial \( p_2 \) is irreducible and Anosov, and if \( r = 2 \), then \( n \) is isomorphic to the free nilpotent algebra \( f_{n,2}(\mathbb{R}) \).

2. If the action of \( G \) on the roots of \( p_1 \) is three-transitive and \( r = 3 \), then \( n \) is isomorphic to \( f_{n,3}(\mathbb{R})/i \), where \( i \) is trivial or a sum of \( F_1(\mathbb{R}), F_{2a}(\mathbb{R}) \), and \( F_{2b}(\mathbb{R}) \), where \( F_1(\mathbb{R}) \) is as defined in Equation (11), and \( F_{2a}(\mathbb{R}) \) and \( F_{2b}(\mathbb{R}) \) are as in Proposition 5.5. If \( n = 3 \), then \( i \) contains \( F_2(\mathbb{R}) \).

3. A Lie algebra \( f_{n,r}(\mathbb{R})/i \) of type \( S_n \) is Anosov so long as the ideal \( i \) contains the ideal \( f_{n,r} \) as defined in Definition 5.3.

Note that when \( G \) is \( S_n \), the Anosov Lie algebra need not be of type \( S_n \) as in Definition 5.3: the Lie algebra \( f_{n,3}/F_{2a}(\mathbb{R}) \) is not of type \( S_n \) although it admits an Anosov automorphism with symmetric Galois group.

**Proof.** Let \( \alpha_1, \ldots, \alpha_n \) denote the roots of \( p_1 \), and let \( \mathcal{C}_1 = \{ z_1, \ldots, z_n \} \) be a set of corresponding eigenvectors of \( f|_{V_1(K)} \) that is compatible with the rational structure defined by \( B \). Let \( \mathcal{C} = \bigcup_{i=1}^r \mathcal{C}_i \) be the Hall basis of \( f_{n,r}(K) \) determined by \( \mathcal{C}_1 \).

Suppose that the action of the Galois group \( G \) on \( \{ \alpha_1, \ldots, \alpha_n \} \) is two-transitive. If the action of \( G \) on the roots of \( p_1 \) is two-transitive, then the set of roots of \( p_1 \) has full rank by Lemma 3.8. Since the action of the group \( G \) sends an eigenvector \( z_i \) to a multiple of another eigenvector \( z_j \), the \( G \) action sends an element of \( \mathcal{C}_2 = \{ [z_k, z_j] \}_{1 \leq j < k \leq n} \) to a scalar multiple of an element of \( \mathcal{C}_2 \). Because the action is doubly transitive, for any \( w \) in \( \mathcal{C}_2 \), the rational invariant subspace \( E^G_C(w) \) is all of \( V_2(\mathbb{R}) \). By Theorem 5.1, the characteristic polynomial \( p_2 \) for the restriction of \( f \) to \( V_2(\mathbb{R}) \) is a power of an irreducible polynomial. But actually, \( p_2 \) itself is irreducible by Part (2c) of Lemma 3.8. Thus, the only proper rational invariant subspace of \( V_2(\mathbb{R}) \) is trivial, and when \( r = 2 \), the only two-step Anosov quotient of \( f_{n,2}(\mathbb{R}) \) is itself. This proves the first part of the theorem.

Now suppose \( r = 3 \) and that the action of the Galois group \( G \) on \( \{ \alpha_1, \ldots, \alpha_n \} \) is three-transitive. Then \( G \) is two-transitive, and by the argument above, the only proper rational invariant subspace of \( V_2(\mathbb{R}) \) is \( \{0\} \). Recall from Proposition 5.5 that \( V_3(\mathbb{R}) \) is the direct sum \( V_3(\mathbb{R}) = F_1 \oplus F_{2a} \oplus F_{2b} \) of the rational invariant subspaces \( F_1, F_{2a}, \) and \( F_{2b} \) where \( F_2 = F_{2a} \oplus F_{2b} \), and the characteristic polynomial for the restrictions of \( f \) to \( F_1 \) is \( q_1 \) and the characteristic polynomials for the restriction of \( f \) to \( F_{2a} \) and \( F_{2b} \) are both \( q_2 \). By the three-fold transitivity of \( G \), the subspaces \( F_1 \) and \( F_2 \) are all single \( G \)-orbits; hence \( q_1 \) and \( q_2 \) are powers of irreducibles. But by Lemma 3.8 when \( n \geq 3 \), \( q_1 \) is irreducible, so \( F_1 \) has no nontrivial rational invariant subspaces, and when \( n > 3 \), the polynomial \( q_2 \) is irreducible; hence the subspaces \( F_{2a} \) and \( F_{2b} \) are minimal nontrivial invariant subspaces. If \( n = 3 \), then \( \alpha_1 \alpha_2 \alpha_3 = \pm 1 \), so \( q_2(x) = x \pm 1 \), and \( i \) must contain \( F_2 \). Therefore, in order for an ideal \( i \) of \( f_{n,3}(\mathbb{R}) \) to
satisfy the Auslander-Scheuneman conditions relative to \(f\), it is necessary for \(i\) to be a sum of \(\{0\}, F_1, F_2, F_2\). Thus, the second part of the theorem holds.

Now we consider the case that \(G\) is symmetric. Assume that \(i_0\) is an ideal of \(f_{n,r}(\mathbb{R})\) of type \(S_n\) relative to some Hall basis \(D\), and \(f_{n,r}(\mathbb{R})/i_0\) is a Lie algebra of type \(S_n\). Assume also that \(i_0\) contains \(j_{n,r}\). We need to show that \(i_0\) satisfies the Auslander-Scheuneman conditions relative to some automorphism \(f\) of \(f_{n,r}(\mathbb{R})\).

Let \(A\) be the companion matrix to an Anosov polynomial with Galois group \(S_n\), such as \(p_1(x) = x^3 - x - 1\) as in Proposition 3.10. Together, \(A\) and a set of generators \(B_1\) determine an automorphism \(f\) of \(f_{n,r}(\mathbb{R})\) that is rational relative to the Hall basis \(B\) determined by \(B_1\). Let \(C_1\) denote the set of eigenvectors for \(f\) in \(V_1(\mathbb{R})\), and let \(C\) be the corresponding Hall basis. There is an ideal \(i\) isomorphic to \(i_0\) that is the image of \(i_0\) under the isomorphism \(g : f_{n,r}(\mathbb{R}) \rightarrow f_{n,r}(\mathbb{R})\) defined by a bijection from \(D\) to \(C\).

The ideal \(i\) is invariant under \(f\) by Theorem 5.1. By Remark 3.2, the restriction of \(f\) to \(i\) is unimodular. The third of the Auslander-Scheuneman condition holds by the theory of rational canonical forms. The last condition is that all roots of modulus one or minus one are in \(i\) : this holds because the set of roots of \(p_1\) has full rank by Proposition 3.6 and \(j_{n,r} < i\). Thus, \(f\) descends to an Anosov automorphism of \(f_{n,r}(\mathbb{R})/i \cong f_{n,r}(\mathbb{R})/i_0\).

We can completely describe Anosov Lie algebras whose associated polynomials \(p_1\) are irreducible of prime degree \(n \geq 3\) with cyclic Galois group.

**Theorem 6.2.** Suppose that \(n = f_{n,r}(\mathbb{R})/i\) is an \(r\)-step Anosov Lie algebra of type \((n_1, \ldots, n_r)\) admitting an Anosov automorphism defined by a semisimple hyperbolic matrix in \(GL_n(\mathbb{Z})\), rational Hall basis \(B\), the resulting automorphism \(f\) in \(\text{Aut}(f_{n,r})\), and an ideal \(i\) satisfying Auslander-Scheuneman conditions. Let \((p_1, \ldots, p_r)\) be the associated \(r\)-tuple of polynomials. Suppose that \(p_1\) is irreducible of prime degree \(n = n_1 \geq 3\), and that \(p_1\) has cyclic Galois group \(G\). Then the ideal \(i\) is of cyclic type as in Definition 4.3 and \(i\) contains \(j_{n,r}\), where \(j_{n,r}\) is as defined in Definition 5.5. Furthermore, \(n = n_1\) divides \(n_i\) for all \(i = 2, \ldots, r\).

Conversely, for any prime \(n \geq 3\), a Lie algebra \(f_{n,r}(\mathbb{R})/i\) of cyclic type is Anosov, as long as the ideal \(i\) contains \(j_{n,r}\).

**Proof.** Let \(i, f_{n,r}\) and \((p_1, \ldots, p_r)\) be as in the statement of the theorem. By Remark 5.4, the ideal \(j_{n,r}\) is contained in \(i\). Recall that any irreducible polynomial in \(\mathbb{Z}[x]\) of prime degree \(n \geq 3\) and cyclic Galois group has totally real roots; hence \(f\) has real spectrum.

Let \(C_1\) be a basis of eigenvectors for \(f|_{V_1(\mathbb{R})}\) that is compatible with the rational structure, and let \(C = \bigcup_{i=1}^r C_i\) be the Hall basis defined by \(C_1\). By Theorem 5.1 for any \(w\) in \(C_i\), the orbit

\[E_G^K(w) = \text{span}_\mathbb{R}\{\sigma \cdot w : \sigma \in G\}\]

is a rational invariant subspace whose characteristic polynomial is a power \(r^s\) of an irreducible polynomial \(r\). The dimension \(d\) of \(E_G^K(w)\) is \(n\) or less because \(|G| = n\). The
dimension $d$ also satisfies $d = s \cdot \deg r$. Because the splitting field for $r$ is a subfield of $\mathbb{Q}(p_1)$ and the Galois group $G$ for $p_1$ is isomorphic to the simple group $C_n$, either $r$ is linear or $r$ is irreducible of degree $n$ and has Galois group $G$. Hence, either $i(G, w)$ is contained in $i$, or it is $n$-dimensional and its intersection with $i$ is trivial. The ideal $i$ is then a direct sum of subspaces of form $E_{n}^{r} (w)$, hence is of cyclic type. Since each step $V_i$ decomposes as the direct sum of $i \cap V_i$ and $n$-dimensional subspaces of the form $E_{n}^{r} (w)$, for $w \in C_i$, the dimension of the $i$th step of the quotient $n$ is divisible by $n$, for all $i = 2, \ldots , n$.

Now let $n$ be the quotient of $f_{n,r} (\mathbb{R}) = \oplus_{i=1}^{j} V_i$ by an ideal $i_0$ of cyclic type relative to some Hall basis $D = \cup_{i=1}^{j} D_i$, where $n \geq 3$ is prime, and suppose that $i_0 > j_{n,r}$ where $j_{n,r}$ is defined relative to $D$. We will show that $n$ is Anosov. By Proposition 3.10, there exists an Anosov polynomial $p_1$ whose Galois group is cyclic of order $n$. By using the companion matrix to $p_1$ and a Hall basis $B$ we can define an automorphism $f$ of $f_{n,r} (\mathbb{R})$. Let $C_1$ be an eigenvector basis for $V_1$ that is compatible with the rational structure defined by $B$, and let $C$ be the Hall basis defined by $C_1$. Then there is an ideal $i$ of $f_{n,r} (\mathbb{R})$ that is cyclic relative to the $G$ action on the Hall basis $C$ and is isomorphic to $i_0$.

The ideal $i$ is rational and invariant by Theorem 5.1 All we need to show is that the quotient map $\overline{f}$ on $f_{n,r} (\mathbb{R})/i$ has no roots of modulus one. But we have already shown in the first part of the proof that if a rational invariant subspace $i(w, G)$ is not in $i$, then the characteristic polynomial $r$ for the restriction of $f$ to $i(w, G)$ has odd degree $n$. By Remark 2.7, $r$ has no roots of modulus one. Hence $\overline{f}$ is Anosov.

7. Anosov automorphisms in low dimensions

In this section we describe Anosov automorphisms of some nilpotent Lie algebras that arise from Anosov polynomials of low degree.

7.1. When $p_1$ is a product of quadratics. We will analyze Anosov automorphisms for which the associated polynomial $p_1$ is a product of quadratic polynomials. To do this we need to define a family of two-step nilpotent Lie algebras.

Definition 7.1. Let $f_{n,2} (\mathbb{R}) = V_1 (\mathbb{R}) \oplus V_2 (\mathbb{R})$ be the free two-step Lie algebra on $2n$ generators, where $n \geq 2$. Let $C_1 = \{ z_1, \ldots , z_{2n} \}$ be a set of generating vectors spanning $V_1 (\mathbb{R})$, and let $C$ be the Hall basis determined by $C_1$. Let $S_1$ and $S_2$ be subsets of the set $\{ i, j \} : 1 \leq i < j \leq n$ of subsets of $\{ 1, 2, \ldots , n \}$ of cardinality two. To the subsets $S_1$ and $S_2$ associate the ideal $i(S_1, S_2)$ of $f_{n,2} (\mathbb{R})$ defined by

\begin{equation}
\begin{aligned}
i(S_1, S_2) &= \bigoplus_{i=1}^{n} \text{span}\{ [z_{2i-1}, z_{2i}] \} \oplus \bigoplus_{(i,j) \in S_1} \text{span}\{ [z_{2i-1}, z_{2j-1}], [z_{2i}, z_{2j}] \} \oplus \bigoplus_{(i,j) \in S_2} \text{span}\{ [z_{2i-1}, z_{2j}], [z_{2i}, z_{2j-1}] \}
\end{aligned}
\end{equation}
Define the two-step nilpotent Lie algebra \( n(S_1, S_2) \) to be \( f_{2n,2}(\mathbb{R})/i(S_1, S_2) \). A two-step Lie algebra of this form will be said to be of \textit{quadratic type}.

In the next theorem we classify two-step Anosov Lie algebras such that the polynomial \( p_1 \) is a product of quadratics.

**Theorem 7.2.** Suppose that the polynomial \( p_1 \) associated to a semisimple Anosov automorphism \( f \) of a two-step Anosov Lie algebra \( n \) is the product of quadratic polynomials. Then \( n \) is of quadratic type, as defined in Definition 7.1, and all the eigenvalues of \( f \) are real. Furthermore, every two-step nilpotent Lie algebra of quadratic type is Anosov.

**Proof.** Let \( f_{2n,2}(\mathbb{R}) = V_1(\mathbb{R}) \oplus V_2(\mathbb{R}) \) be the free two-step nilpotent Lie algebra on \( 2n \) generators. Let \( \mathcal{F} \) be an an Anosov automorphism of \( n = f_{2n,2}(\mathbb{R})/i \) defined by automorphism \( f \) of \( f_{2n,2}(\mathbb{R}) \), a Hall basis \( \mathcal{B} \) and an ideal \( i \) satisfying the Auslander-Scheuneman conditions. Without loss of generality, assume that \( i < V_2(\mathbb{R}) \). Let \((p_1, p_2)\) denote the pair of polynomials associated to \( f \), and assume that the polynomial \( p_1 \) of degree \( 2n \) is the product of \( n \) quadratic Anosov polynomials \( r_1, \ldots, r_n \).

By the quadratic equation, any roots of a quadratic Anosov polynomial are real. The subspace \( V_1(\mathbb{R}) \) decomposes as the direct sum \( \oplus_{i=1}^n E_i \) of rational invariant subspaces such that the characteristic polynomial for \( f \mid E_i \) is \( r_i \). For \( i = 1, \ldots, n \), let \( z_{2i-1} \) and \( z_{2i} \) denote eigenvectors in \( E_i \) with eigenvalues \( \alpha_{2i-1} \) and \( \alpha_{2i} \) respectively. We may assume without loss of generality that \( \alpha_{2i-1} > 1 > \alpha_{2i} = \alpha_{2i-1}^{-1} \).

As in Example 2.6, the polynomial \( p_2 \) may be written as
\[
p_2 = \prod_{i=1}^n (r_i \wedge r_i) \times \prod_{1 \leq i < j \leq n} (r_i \wedge r_j),
\]
and this factorization is corresponds to a decomposition \( V_2(\mathbb{R}) = \oplus_{1 \leq i < j \leq n}[E_i, E_j] \) of \( V_2(\mathbb{R}) \) into rational invariant subspaces.

For all \( i = 1, \ldots, n \), the polynomial \( r_i \wedge r_i \) is linear with root \( \alpha_{2i-1}\alpha_{2i} = 1 \), so \( (r_i \wedge r_i)(x) = x - 1 \). Therefore, the ideal \( i \) must contain the \( n \)-dimensional subspace \( \oplus_{i=1}^n [E_i, E_i] \) of \( V_2(\mathbb{R}) \). For \( i \neq j \), the polynomial \( r_i \wedge r_j \) is given by
\[
(r_i \wedge r_j)(x) = (x - \alpha_{2i-1}\alpha_{2j-1})(x - \alpha_{2i-1}\alpha^{-1}_{2j-1})(x - \alpha^{-1}_{2i-1}\alpha_{2j-1})(x - \alpha^{-1}_{2i-1}\alpha^{-1}_{2j-1}).
\]

Minimal nontrivial invariant subspaces of \([E_i, E_j]\) correspond to factorizations of \( r_i \vee r_j \) over \( \mathbb{Z} \).

If the splitting fields \( \mathbb{Q}(r_i) \) and \( \mathbb{Q}(r_j) \) do not coincide, then they are linearly disjoint, and \( \mathbb{Q}(r_i \vee r_j) = \mathbb{Q}(r_i)\mathbb{Q}(r_j) \) is a biquadratic extension of \( \mathbb{Q} \). Therefore, \( r_i \vee r_j \) is irreducible, and \([E_i, E_j]\) has no nontrivial rational invariant subspaces.

Now suppose that the splitting fields \( \mathbb{Q}(r_i) \) and \( \mathbb{Q}(r_j) \) are equal, for \( i \neq j \). Since the roots of Anosov quadratics are real, the field \( \mathbb{Q}(r_i) \) is a totally real quadratic extension of \( \mathbb{Q} \). By Dirichlet’s Fundamental Theorem, there are units \( \zeta = \pm 1 \), and
η, a fundamental unit in \( \mathbb{Q}(r_i) \), such that any unit \( \beta \) in \( \mathbb{Q}(r_i) \) can be expressed as \( \beta = \zeta^a \eta^b \), where \( a \in \{0, 1\} \) and \( b \in \mathbb{Z} \). We may choose \( \eta > 1 \). The Galois group for \( r_1 \) is generated by the automorphism of \( \mathbb{Q}(\eta) \) mapping \( \eta \) to \( \eta^{-1} \).

We can write

\[
\alpha_{2i-1} = \eta^{b_i}, \alpha_{2i} = \eta^{-b_i}, \alpha_{2j-1} = \eta^{b_j}, \text{ and } \alpha_{2j} = \eta^{-b_j},
\]

where \( b_i \) and \( b_j \) are in \( \mathbb{Z}^+ \). The four roots of \( (r_i \wedge r_j)(x) \) are then the numbers \( \eta^\pm b_i \pm b_j \).

Therefore, \( r_i \wedge r_j \) factors over \( \mathbb{Z}[x] \) as the product of two quadratics in \( \mathbb{Z}[x] \)

\[
(x - \eta^{b_i+b_j})(x - \eta^{-b_i-b_j}) = (x - \alpha_{2i-1}\alpha_{2j-1})(x - \alpha_{2i}\alpha_{2j}), \text{ and } (x - \eta^{b_i-b_j})(x - \eta^{-b_i+b_j}) = (x - \alpha_{2i-1}\alpha_{2j-1}^{-1})(x - \alpha_{2i-1}^{-1}\alpha_{2j-1}).
\]

The first polynomial is irreducible since \( b_i + b_j > 0 \), and \( \eta \) is not a root of unity. If \( \alpha_{2i-1} = \alpha_{2j-1} \), then \( b_i = b_j \) and the second polynomial is equal to \( (x - 1)^2 \).

This analysis shows that \( i \cap [E_i, E_j] \) must be one of the subspaces \( \{0\}, [E_i, E_j] \)

\[
E_{i,j} = \text{span}\{[z_{2i-1}, z_{2j-1}], [z_{2i}, z_{2j}]\} \text{ and } E'_{i,j} = \text{span}\{[z_{2i-1}, z_{2j}], [z_{2i}, z_{2j-1}]\}.
\]

Then \( i \) is the direct sum of \( \bigoplus_{i=1}^n [E_i, E_i] \), subspaces of the form \( E_{i,j} \) and subspaces of the form \( E'_{i,j} \). Therefore, \( n \) is of quadratic type.

Conversely, we show that for any ideal \( i \) of quadratic type in \( \mathfrak{f}_{2n,2}(\mathbb{R}) \), the quotient \( n = \mathfrak{f}_{2n,2}(\mathbb{R})/i \) is an Anosov Lie algebra. Suppose that \( i(S_1, S_2) \) is an ideal as in the definition of quadratic type. Fix a fundamental unit \( \eta \) in a totally real quadratic extension of \( \mathbb{Q} \). Let \( b_1, \ldots , b_n \) be distinct positive integers. For each \( i = 1, \ldots , n \), define the polynomial \( r_i \) in \( \mathbb{Z}[x] \) by

\[
r_i(x) = (x - \eta^{b_i})(x - \eta^{-b_i})
\]

and let \( p_1 = r_1 \cdots r_n \). Then for all \( i \neq j \), the polynomial \( p_i \wedge p_j \) factors over \( \mathbb{Z} \) as the product of pairs of irreducible quadratic polynomials

\[
(x - \eta^{b_i+b_j})(x - \eta^{-b_i-b_j}) \text{ and } (x - \eta^{b_i-b_j})(x - \eta^{-b_i+b_j})
\]

in \( \mathbb{Z}[x] \), neither having roots of modulus one. The two factors give a two rational invariant subspace of form \( E_{ij} \) and \( E'_{ij} \). Therefore the ideal

\[
i = \bigoplus_{i=1}^n [E_i, E_i] \bigoplus \bigoplus_{(i,j) \in S_1} E_{ij} \bigoplus \bigoplus_{(i,j) \in S_2} E'_{ij}
\]

satisfies the four Auslander-Scheuneman conditions and is of the form \( i(S_1, S_2) \) with respect to the appropriate Hall basis of \( \mathfrak{f}_{2n,2}(\mathbb{R}) \). This completes the proof of the theorem.
$$n = f_{n,r}(\mathbb{R})/i$$

| $n = f_{n,r}(\mathbb{R})/i$ | type | ideal $i$ | reference for definition of $i$ |
|-----------------------------|------|----------|----------------------------------|
| $f_{3,2}$                   | (3, 3) | $\{0\}$ |                                   |
| $f_{4,2}$                   | (4, 6) | $\{0\}$ |                                   |
| $f_{4,2}/i$                 | (4, 4) | $i(V_4, [z_2, z_1])$ | Definition 4.5 |
| $f_{4,2}/i \cong h_3 \oplus h_3$ | (4, 2) | $i(C_4, [z_2, z_1])$ | Definition 4.5 |
| $f_{5,2}$                   | (5, 10) | $\{0\}$ |                                   |
| $f_{5,2}/i$                 | (5, 9) | $i_1$ | Definition 7.6 |
| $f_{5,2}/i$                 | (5, 6) | $i_1 \oplus i_2$ | Definition 7.6 |
| $f_{5,2}/i$                 | (5, 5) | $i(C_5, [z_2, z_1])$ | Definition 4.5 |
| $f_{5,2}/i$                 | (5, 5) | $i_2$ | Example 5.2 |
| $f_{5,2}/i \cong \mathbb{R}^2 \oplus f_{3,2}$ | (5, 3) | $i_1 \oplus i_3$ | Definition 7.6 |

Table 3. Two-step Anosov Lie algebras of type $(n_1, n_2)$ with $n_1 \leq 5$

7.2. When $p_1$ is a cubic. We can classify all Anosov Lie algebras of type $(3, \ldots, n_r)$ with $r = 2$ or $r = 3$.

**Theorem 7.3.** If $n$ is a two-step Anosov Lie algebra of type $(3, n_2)$, then $n \cong f_{3,2}$. If $n$ is three-step Anosov Lie algebra of type $(3, n_2, n_3)$, then $n$ is isomorphic to the Anosov Lie algebra $f_{3,3}/F_2$ of type $(3, 3, 6)$, where $i = F_2$ is as defined in Equation (1), or $i_1 = F_2 \oplus i(C_3, [[z_2, z_1], z_2])$ of type $(3, 3, 3)$, where $i(C_3, [[z_2, z_1], z_2])$ is as defined in Definition 4.5.

**Proof.** Suppose that $A$ is a semisimple hyperbolic matrix in $GL_n(\mathbb{Z})$ with associated triple of polynomials $(p_1, p_2, p_3)$. Let $\mathcal{B}_1$ be a generating set for $f_{3,3}$ and let $\mathcal{B}$ be the Hall basis determined by $\mathcal{B}$. Let $\alpha_1, \alpha_2, \alpha_3$ denote the roots of $p_1$. Since $p_1$ can not have 1 or -1 as a root, it is irreducible over $\mathbb{Z}$, and the Galois group $G$ of the splitting field of $p_1$ is either $C_3$ or $S_3$.

As demonstrated in Example 2.5, the cubic polynomial $p_2$ is irreducible and Anosov, so there are no Anosov Lie algebras of type $(3, n_2)$ other than $f_{3,2}$. 
Let $i$ be an ideal so that $f$ descends to an Anosov automorphism of a three-step nilpotent Lie algebra $f_{3,2}/i$. If $G$ is symmetric, then $i = F_2$ by Theorem 6.1. If $G$ is cyclic, by Theorem 6.2 $i$ is either $F_2$ or it is an ideal of form $F_2 \oplus (C_3, w)$, for $w \in C_3$.

Each Lie algebra that is listed may be realized by choosing appropriate Anosov polynomial from Table 1 and using its companion matrix $A$ to define an automorphism of $f_{3,2}$ or $f_{3,3}$. When $r = 3$, by Lemma 3.4, all vectors with eigenvalue $\pm 1$ will be in the kernel $F_2 = J_3$.

### 7.3. When $p_1$ is a quartic.

Now we consider the case that $p_1$ is a quartic Anosov polynomial. The next lemma is useful for understanding Anosov Lie algebras of type $(4, n_2)$.

**Lemma 7.4.** Let $(p_1, p_2)$ be the pair of polynomials associated to an irreducible Anosov polynomial $p_1$ of degree four. Let $G$ denote the Galois group of the splitting field for $p_1$. Then

1. $G \cong S_4$ or $G \cong A_4$ if and only if $p_2$ is irreducible.
2. $G \cong C_4$ or $G \cong D_8$ if and only if $p_2$ has an irreducible quartic factor.
3. $G \cong V_4$ if and only if $p_2$ has no irreducible factors of degree three or more.

Furthermore, roots of $p_2$ come in reciprocal pairs $\beta$ and $\pm \beta^{-1}$.

**Proof.** Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ denote the four distinct roots of $p_1$. Then the roots of $p_2$ are the six numbers $\alpha_i \alpha_j$, where $1 \leq i < j \leq 4$. Because $\alpha_1 \alpha_2 \alpha_3 \alpha_4 = \pm 1$, roots of $p_2$ come in pairs such as $\alpha_1 \alpha_2$ and $\alpha_3 \alpha_4 = \pm (\alpha_1 \alpha_2)^{-1}$.

The resolvent cubic $r$ for $p_1$ has roots

$$\beta_1 = \alpha_1 \alpha_2 + \alpha_3 \alpha_4, \quad \beta_2 = \alpha_1 \alpha_3 + \alpha_2 \alpha_4, \quad \text{and} \quad \beta_3 = \alpha_1 \alpha_4 + \alpha_2 \alpha_3.$$

Recall that one of three things must occur: (1) that none of $\beta_1, \beta_2, \beta_3$ lies in $\mathbb{Q}$, $r$ is irreducible, and $G \cong S_4$ or $G \cong A_4$, (2) exactly one of $\beta_1, \beta_2, \beta_3$ lies in $\mathbb{Q}$, $r$ is the product of an irreducible quadratic and a linear factor, and $G \cong C_4$ or $G \cong D_8$, and (3) $\beta_1, \beta_2, \beta_3$ all lie in $\mathbb{Q}$, $r$ splits over $\mathbb{Q}$, and $G \cong V_4$.

Since $\alpha_1 \alpha_2 \alpha_3 \alpha_4 = \pm 1$, $\beta_1 = \alpha_1 \alpha_2 + \alpha_3 \alpha_4$ is in $\mathbb{Q}$ if and only if $(x - \alpha_1 \alpha_2)(x - \alpha_3 \alpha_4)$ is a quadratic factor of $p_2$ over $\mathbb{Q}$. Factors that are the counterparts from $\beta_2$ and $\beta_3$,

$$r(x) = (x - \alpha_1 \alpha_3)(x - \alpha_2 \alpha_4) \quad \text{and} \quad (x - \alpha_1 \alpha_4)(x - \alpha_2 \alpha_3),$$

are in $\mathbb{Q}[x]$ depending on whether $\beta_1$ and $\beta_2$ are in $\mathbb{Q}$. Therefore, when all of $\beta_1, \beta_2, \beta_3$ lie in $\mathbb{Q}$, $p_2$ factors as the product of quadratics in $\mathbb{Q}[x]$, establishing the claim in Case (3).

In Case (1), $p_2$ is irreducible by Theorem 6.1. Part (1).

In the second case, when $G$ is $C_4$ or $D_8$, there is a $G$ orbit of cardinality four, something like $\{\alpha_1 \alpha_3, \alpha_1 \alpha_4, \alpha_2 \alpha_3, \alpha_2 \alpha_4\}$, that by Theorem 5.1 yields a factor

$$r(x) = (x - \alpha_1 \alpha_3)(x - \alpha_2 \alpha_4)(x - \alpha_1 \alpha_4)(x - \alpha_2 \alpha_3)$$

of $p_2$, and an orbit of cardinality two corresponding to a factor $(x - \alpha_1 \alpha_2)(x - \alpha_3 \alpha_4)$ of $p_2$. Then $\beta_1 \in \mathbb{Q}$ and $\beta_2, \beta_3 \notin \mathbb{Q}$. By Theorem 5.1 if $r$ were to factor, it would
be a power of an irreducible. Knowing that \( \beta_2, \beta_3 \not\in \mathbb{Q} \) rules out the polynomials in Equation (9) as factors of \( r \). The only other options for factors yield contradictions to the distinctness of roots of \( p_1 \). Thus, \( r \) is an irreducible quartic factor of \( p_2 \).

Now we are ready to classify two-step Anosov Lie algebras of type \((4, n_2)\) using the methods we have established.

**Theorem 7.5.** If \( \mathfrak{n} \) is an Anosov Lie algebra of type \((4, n_2)\), then \( \mathfrak{n} \) is one of the Anosov Lie algebras listed in Table 3.

**Proof.** Suppose that \( f \) is a semisimple automorphism of \( f_{4,2}(\mathbb{R}) = V_1(\mathbb{R}) \oplus V_2(\mathbb{R}) \) that projects to an Anosov automorphism of a two-step quotient \( \mathfrak{n} = f_{4,2}/i \). Let \((p_1, p_2)\) be the polynomials associated to \( f \), and let \( K \) be the splitting field for \( p_1 \). Let \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) be the roots of \( p_1 \) and let \( z_1, z_2, z_3, z_4 \) be corresponding eigenvectors for \( f_K \) in \( V_1(K) \).

If \( p_1 \) is reducible, since it is Anosov, it is a product of quadratics, and Theorem 7.2 describes the Anosov Lie algebras in that situation: \( i \) is one of \( i(V_4, [z_2, z_1]) \) and \( i(C_4, [z_2, z_1]) \).

Assume that \( p_1 \) is irreducible. The Galois group \( G \) of \( p_1 \) is then a transitive permutation group of degree four.

By Lemma 7.4 if \( G \) is \( S_4 \) or \( A_4 \), the polynomial \( p_2 \) is irreducible, so \( V_2(\mathbb{R}) \) has no nontrivial proper rational invariant subspaces; hence, \( i = \{0\} \), and \( \mathfrak{n} \) is free. If \( G \) is \( D_8 \) or \( C_4 \), then by considering the action of \( G \) on the set of roots it can be seen that \( f_{4,2}(K) \) is the direct sum of a four-dimensional rational \( f_K \)-invariant subspace \( i_1 \) and a two-dimensional rational \( f_K \)-invariant subspace \( i_2 \). If the roots are real, these may be represented as follows, renumbering the roots if necessary,

\[
i_1 = i(C_4, [z_2, z_1]) = i(D_8, [z_2, z_1]), \quad \text{and} \quad i_2 = i(C_4, [z_3, z_1]) = i(D_8, [z_3, z_1]).
\]

By Lemma 7.3 the minimal polynomial for \( i_1 \) is irreducible, so \( i \) is a minimal nontrivial invariant subspace. If there are complex roots, a short computation shows that the rational invariant subspaces \( i_1' \) and \( i_2' \) of \( f_{4,2}(K) \) yield rational invariant subspaces \( (i_1')^\mathbb{R} \) and \( (i_2')^\mathbb{R} \) of \( f_{4,2}(\mathbb{R}) \) isomorphic to \( i_1 \) and \( i_2 \). The characteristic polynomial for \( i_2 \) is either irreducible or it has roots \( \pm 1 \), in which case \( i_2 \) must be contained in the ideal \( i \). Thus, when \( G = C_4 \) or \( D_8 \), the ideal \( i \) is \( \{0\}, i_1 \) or \( i_2 \).

Finally, if \( G \) is the Klein four-group, by the same reasoning, \( V_2(K) \) is the direct sum of three two-dimensional ideals that are either minimal or must be contained in \( i \). Then \( \mathfrak{n} \) is of quadratic type.

Anosov automorphisms of all types \((4, n_2)\) may be realized by choosing an appropriate polynomial \( p_1 \) from Table 1. By Remark 7.1, the polynomial \( p_2 \) defined by such a \( p_1 \) can not have any nonreal roots of modulus one unless it is self-reciprocal. The only polynomials listed in Table 1 that is self-reciprocal is for \( V_4 \), so \( p_2 \) will not have roots of modulus one unless \( G = V_4 \), in which case the eigenspaces of those roots lie in the ideal \( i \).
7.4. When $p_1$ is a quintic. First we define some two-step nilpotent Lie algebras on five generators.

**Definition 7.6.** Let $f_{5,2} = V_1(\mathbb{R}) \oplus V_2(\mathbb{R})$ be a free Lie algebra on five generators \( \{z_i\}_{i=1}^{5} \). Define subspaces $E_1$ and $E_2$ of $V_1(\mathbb{R})$ by $E_1 = \text{span}_{\mathbb{R}}\{z_1, z_2\}$ and let $E_2 = \text{span}_{\mathbb{R}}\{z_3, z_4, z_5\}$, and define ideals of $f_{5,2}$ by

$$i_1 = [E_1, E_1], \quad i_2 = [E_1, E_2], \quad \text{and} \quad i_3 = [E_2, E_2].$$

Define two-step Lie algebras by

$$n_1 = f_{5,2}/i_1, \quad n_2 = f_{5,2}/(i_1 \oplus i_2), \quad \text{and} \quad n_3 = f_{5,2}/(i_1 \oplus i_3).$$

These are of types (5, 9), (5, 3) and (5, 6) respectively. Note that $n_2 \cong \mathbb{R}^2 \oplus f_{3,2}$.

**Theorem 7.7.** Suppose that $n$ is a two-step nilpotent Lie algebra of type $(5, n_2)$ admitting an Anosov automorphism $f$. Then $n$ is one of the Lie algebras listed in Table 3. Furthermore, all of the Lie algebras of type $(5, n_2)$ in Table 3 are Anosov.

**Proof.** Let $(p_1, p_2)$ be the pair of polynomials associated to an automorphism $f$ of $f_{5,2} = V_1(\mathbb{R}) \oplus V_2(\mathbb{R})$ that projects to an Anosov automorphism of an Anosov Lie algebra $n = f_{5,2}/i$, for some ideal $i$ of $f_{5,2}$ satisfying the Auslander-Scheuneman conditions. Without loss of generality we assume that the roots of $p_1$ have product 1.

First suppose that $p_1$ is irreducible, so that its Galois group $G$ is isomorphic to $S_5$, $A_5$, $D_{10}$, $C_5$ or the holomorph Hol($C_5$) of $C_5$. If $G$ is isomorphic to one of $S_5$, $A_5$, and Hol($C_5$), then the action of $G$ on the roots of $p_1$ is two-transitive, so by Theorem 6.1 $n$ is isomorphic to $f_{5,2}$. The case that the Galois group is $D_{10}$ was considered in Example 5.2 where it was found that either $n$ is free, $n$ is isomorphic to $n_1$ or $n_2$ in Example 5.2, both of type $(5, 5)$. The case of $C_5$ is covered by Theorem 6.2. Thus, in all possible cases, $n$ is one of the Lie algebras listed in Table 3. Each example may be realized by choosing a polynomial $p_1$ from Table 1 and using its companion matrix to define an automorphism of $f_{5,2}$. To get $n_1$, one needs to choose $p_1$ with all real roots, and to get $n_2$, one needs $p_2$ to have four nonreal roots. Examples of both kinds are in the table. The associated polynomials $p_2$ have no roots of modulus one by Lemma 3.4.

Now suppose that the Anosov polynomial $p_1$ is the product of a quadratic Anosov polynomial $r_1$ and a cubic Anosov polynomial $r_2$. Let $E_1$ and $E_2$ denote the rational invariant subspaces of $V_1(\mathbb{R})$ corresponding to $r_1$ and $r_2$ respectively. Because $r_1$ is quadratic, $r_1 \wedge r_1 = x \pm 1$, so $i_1 = [E_1, E_1]$ must be contained in $i$. As seen in Example 2.5 since $r_2$ is cubic, the polynomial $r_2 \wedge r_2$ is irreducible and Anosov. Therefore, the set $i_3 = [E_2, E_2]$ is a minimal nontrivial invariant subspace of $V_2(\mathbb{R})$.

Let $\alpha_1$ and $\alpha_2$ denote the roots of $r_1$, while $\alpha_3, \alpha_4, \alpha_5$ are the roots of $r_2$. Then $\alpha_1\alpha_3$ is a root of $r_1 \wedge r_2$. By standard arguments, \([\mathbb{Q}(\alpha_1\alpha_3) : \mathbb{Q}] = 6\), so $r_1 \wedge r_2$ is the minimal polynomial of $\alpha_1\alpha_3$. Therefore $r_1 \wedge r_2$ is irreducible and $i_2 = [E_1, E_2]$ is a minimal nontrivial rational invariant subspace. The subspace $V_2(\mathbb{R})$ decomposes as
the sum $i_1 \oplus i_2 \oplus i_3$ of minimal nontrivial invariant subspaces, where $i_1, i_2$ and $i_3$ are as in Definition 7.6, and the only possibilities for an ideal $i$ defining a two-step Anosov quotient are $i_1 \oplus i_2$ and $i_1 \oplus i_3$, as claimed.

Choosing $r_1$ and $r_2$ to be arbitrary Anosov polynomials of degree two and three respectively will yield an Anosov polynomial $p_1 = r_1 r_2$ such that the corresponding automorphism $f$ of $f_{5,2}$ admits quotients of all types listed in the table. By choosing $r_2$ so it has real roots, the roots of $p_2$ will be real, and $r_1 \wedge r_2$ will have no roots of modulus one.

8. PROOFS OF MAIN THEOREMS

Now we provide proofs for the theorems presented in Section 1.

Proof of Theorem 1.1. Suppose that $n$ is a two-step Anosov Lie algebra of type $(n_1, n_2)$ with associated polynomials $(p_1, p_2)$. If $n_1 = 3, 4,$ or $5$, then $n$ is one of the Lie algebras in Table 3, by Theorems 7.3, 7.5 and 7.7. Therefore, Part (1) of Theorem 1.1 holds.

The second part follows immediately from Part (1) of Theorem 6.1.

Proof of Theorem 1.2. The first part of the theorem follows immediately from Theorem 6.2. The second part is a consequence of Theorem 6.1.

Proof of Theorem 1.3. Corollaries 3.9 and 5.7 imply the theorem.

Proof of Theorem 1.4. Suppose that the spectrum of $f$ is in $\mathbb{Q}(\sqrt{b})$. Then the polynomial $p_1$ associated to $f$ is a product of quadratics, each of whose roots lie in $\mathbb{Q}(\sqrt{b})$. Theorem 7.2 implies that $n$ is one of the Lie algebras defined in Definition 7.1.

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