ON THE ATKIN AND SWINNERTON-DYER TYPE CONGRUENCES FOR SOME TRUNCATED HYPERGEOMETRIC $1F_0$ SERIES

YONG ZHANG AND HAO PAN

ABSTRACT. Let $p$ be an odd prime and let $n$ be a positive integer. For any positive integer $\alpha$ and $m \in \{1, 2, 3\}$, we have

$$p^{\alpha n - 1} \sum_{k=0}^{p^{\alpha n - 1} \left( \frac{1}{2} \right)_k \cdot (-4)^k \equiv \left( \frac{m(m-4)}{p} \right)^{\alpha n - 1} \sum_{k=0}^{p^{\alpha n - 1} \left( \frac{1}{2} \right)_k \cdot (-4)^k \pmod{p^{2\alpha}}},$$

where $(x)_k = x(x+1)\cdots(x+k-1)$ and $(\cdot)$ denotes the Legendre symbol. Also, when $m = 4$,

$$p^{\alpha n - 1} \sum_{k=0}^{p^{\alpha n - 1} (-1)^k \cdot \left( \frac{1}{2} \right)_k \equiv p^{\alpha n - 1} \sum_{k=0}^{p^{\alpha n - 1} (-1)^k \cdot \left( \frac{1}{2} \right)_k \pmod{p^{2\alpha}}},$$

1. Introduction

In [2], Aktin and Swinnerton-Dyer systematically investigated the arithmetic properties of the Fourier coefficients of noncongruence modular forms. They observed that if $\Gamma$ is a noncongruence subgroup of $SL_2(\mathbb{Z})$ with a finite index and $k \geq 2$ is even, then for some good primes $p$, there exists a basis $\{f_i\}_{1\leq i \leq d}$ of $S_k(\Gamma)$, where $d = \dim S_k(\Gamma)$, such that for each $1 \leq i \leq d$ and $\alpha \geq 1$,

$$a_{np^n}(f_i) - \lambda_{p,i} \cdot a_{np^{\alpha-1}}(f_i) + p^{k-1} a_{np^{\alpha-2}}(f_i) \pmod{p^{(k-1)\alpha}}, \quad \forall n \geq 1,$$

where $\lambda_{p,i}$ is an algebraic integer with $|\lambda_{p,i}| \leq 2p^{-\frac{1}{3}}$, $a_n(f)$ denotes the $n$-th coefficients in the Fourier expansion of $f(z)$ and $a_x(f) = 0$ if $x \notin \mathbb{Z}$. Subsequently, the work of Aktin and Swinnerton-Dyer was greatly developed by Scholl in [9].

Nowadays, for a sequence $\{a_n\}_{n \geq 0}$ of integers, the congruence of the form

$$a_{np^n} \equiv \lambda_p \cdot a_{np^{\alpha-1}} \pmod{p^{k^{\alpha}}}, \quad \forall n \geq 1,$$

is also often called Atkin and Swinnerton-Dyer type congruence, where $p$ is a prime and $k, r \geq 1$. The Atkin and Swinnerton-Dyer type congruences have been established for many combinatorial sequences. For examples, Beukers [5] proved that
the Apéry number
\[ A_n := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \]
which was used to prove the irrationality of \( \zeta(3) = \sum_{n \geq 1} n^{-3} \) by Apéry, satisfies the Atkin and Swinnerton-Dyer type congruence
\[ A_{np^{\alpha-1}} \equiv A_{np^{\alpha-1} - 1} \pmod{p^{3\alpha}}, \quad \forall n \geq 1, \quad (1.1) \]
where \( p \geq 5 \) is prime and \( \alpha \geq 1 \). Another example due to Coster and Hamme is concerning the Legendre polynomial
\[ P_n(z) := \sum_{k=0}^{n} \binom{n}{k}(-n-1)_k \cdot \left( \frac{1-z}{2} \right)^k. \]
Coster and Hamme \[4\] proved that if the elliptic curve \( y^2 = x(x^2 + Ax + B) \) has the complex multiplication, then the sequence \( \{P_n(z)\}_{n \geq 0} \), where \( z = (1 - A/\sqrt{A^2 - 4B})/2 \), obeys some Atkin and Swinnerton-Dyer type congruences. In \[6\], Li and Long gave a nice survey on the Atkin and Swinnerton-Dyer congruences. For more related results, the reader may refer to \[10, 8\]. In particular, recently Sun \[13\] proposed many conjectured Atkin and Swinnerton-Dyer type congruences.

On the other hand, define the truncated hypergeometric function
\[ _{m+1}F_m \left[ \begin{array}{c} a_0 \ a_1 \ \ldots \ a_m \\ b_1 \ \ldots \ b_m \end{array} \right] z_n := \sum_{k=0}^{n} \frac{(a_0)_k(a_1)_k \ldots (a_m)_k}{(b_1)_k \ldots (b_m)_k} \cdot \frac{z^k}{k!}, \]
where
\[ (a)_k = \begin{cases} a(a+1) \cdots (a+k-1), & \text{if } k \geq 1, \\ 1, & \text{if } k = 0. \end{cases} \]
Clearly the truncated hypergeometric function is just a finite analogue of the original hypergeometric function. Recently the arithmetic properties of the truncated hypergeometric functions are widely studied. In this paper, we shall consider the simplest truncated hypergeometric function
\[ _1F_0 \left[ \begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array} \right] z_n = \sum_{k=0}^{n} \frac{(\frac{1}{2})_k}{k!} \cdot z^k. \]
For each non-zero integer \( m \), as a consequence of \( \), for each odd prime \( p \) we have
\[ _1F_0 \left[ \begin{array}{c} \frac{1}{2} \\ - \frac{4}{m} \end{array} \right] \equiv \left( \frac{m(m-4)}{2} \right) \pmod{p}, \quad (1.2) \]
where \((\cdot)\) denotes the Legendre symbol. In fact, (1.2) also easily follows from that
\[
\begin{align*}
1F_0 \left[ \frac{1}{2} \right] - \frac{4}{m} \right]_{p-1} & \equiv 1F_0 \left[ \frac{1-p}{2} \right] - \frac{4}{m} \right]_{p-1} = \sum_{k=0}^{p-1} \left( \frac{p-1}{2} \right) k \cdot \left( -\frac{4}{m} \right)^k \\
& = \left( 1 - \frac{4}{m} \right)^{p-1} \equiv \left( \frac{m(m-4)}{p} \right) \pmod{p}.
\end{align*}
\]

In [11], Sun extended (1.2) to
\[
\begin{align*}
1F_0 \left[ \frac{1}{2} \right] - \frac{4}{m} \right]_{p-1} & \equiv \left( \frac{m(m-4)}{p} \right) + u_{p-\left(\frac{m(m-4)}{p}\right)}(m-2,1) \pmod{p^2}, \tag{1.3}
\end{align*}
\]
where the Lucas sequence \(\{u_n(A, B)\}_{n \geq 0}\) is given by
\[
u_0(A, B) = 0, \quad u_1(A, B) = 1, \quad u_n(A, B) = Au_{n-1}(A, B) - Bu_{n-2}(A, B), \quad \forall n \geq 2.
\]
Recently, Sun [13] also obtained an Atkin and Swinnerton-Dyer type generalization of (1.3):
\[
\begin{align*}
1F_0 \left[ \frac{1}{2} \right] - \frac{4}{m} \right]_{np^\alpha-1} & \equiv \left( \frac{m(m-4)}{p} \right) + u_{p-\left(\frac{m(m-4)}{p}\right)}(m-2,1) \pmod{p^{\alpha+1}}. \tag{1.4}
\end{align*}
\]
Clearly (1.3) easily follows from (1.4) by substituting \(\alpha = 1\) and \(n = 1\).

It is natural to ask whether in (1.4) modulo \(p^{\alpha+1}\) can be replaced by \(p^{2\alpha}\). Unfortunately, seemingly it is not easy to get such an extension for general \(m\). However, in this paper, for \(m = 1, 2, 3\), we shall prove that

**Theorem 1.1.** Let \(p\) be an odd prime and \(n\) be a positive integer. If \(m \in \{1, 2, 3\}\), then for any positive integer \(\alpha\),
\[
1F_0 \left[ \frac{1}{2} \right] - \frac{4}{m} \right]_{np^\alpha-1} \equiv \left( \frac{m(m-4)}{p} \right) 1F_0 \left[ \frac{1}{2} \right] - \frac{4}{m} \right]_{np^\alpha-1} \pmod{p^{2\alpha}}. \tag{1.5}
\]
Furthermore, when \(m = 4\),
\[
1F_0 \left[ \frac{1}{2} \right] - 1 \right]_{np^\alpha-1} \equiv p \cdot 1F_0 \left[ \frac{1}{2} \right] - 1 \right]_{np^\alpha-1} \pmod{p^{2\alpha}}. \tag{1.6}
\]

We mention that the special case \(m = \alpha = 1\) of (1.5) was also conjectured by Apagodu and Zeilberger[11] and proved by Liu [7].

Let us give an explanation on (1.5) from the viewpoint of convergent series. We know that
\[
\sum_{k=0}^{n} \frac{(\frac{1}{2})^k}{k!} \cdot z^k = \sqrt{1-z} \tag{1.7}
\]
for any \( z \in \mathbb{C} \) with \( |z| \leq 1 \). However, since \( (\frac{1}{2})_k/k! \) is not divisible by \( p \) for infinitely many \( k \), the series \((1.7)\) can't be convergent in the sense of \( p \)-adic norm. Let

\[
S_n = \sum_{k=0}^{n-1} \frac{\left(\frac{1}{2}\right)_k}{k!} \cdot \frac{(-4)^k}{m^k}.
\]

Then \((1.5)\) says that for each \( n \geq 1 \), both \( \{S_{np^2}\}_{\alpha \geq 0} \) and \( \{S_{np^{2\alpha-1}}\}_{\alpha \geq 1} \) are rapidly convergent subsequences of \( \{S_m\}_{m \geq 0} \) in the sense of \( p \)-adic norm.

Throughout this paper, we will show several lemmas in Sections 2. Theorem 1.1 will be proved in Sections 3.

2. SOME LEMMAS

**Lemma 2.1.** For any nonnegative integer \( k, n \) and \( \alpha \), we have

(i) If \( p \mid k \), then

\[
\binom{p^\alpha n}{k} \equiv \binom{p^{\alpha-1} n}{k/p} \pmod{p^{2\alpha}}.
\]

(ii) If \( p \nmid k \), then

\[
\binom{p^\alpha n}{k} \equiv \frac{p^\alpha n}{k} \binom{p^{\alpha-1} n - 1}{\left\lfloor \frac{k-1}{p} \right\rfloor} (-1)^{k-1-\left\lfloor \frac{k-1}{p} \right\rfloor} \pmod{p^{2\alpha}}.
\]

(iii)

\[
\binom{p^\alpha n - 1}{k} \equiv \binom{p^{\alpha-1} n - 1}{\left\lfloor k/p \right\rfloor} (-1)^{k-\left\lfloor k/p \right\rfloor} \pmod{p^\alpha},
\]

here \((2.3)\) is the Lemma2(i) in F. Beukers’ paper\[3\]. The following curious identity is due to Sun and Taurso:

**Lemma 2.2 (\[14\] (2.1)).** For any nonzero integer \( m \) and positive integer \( n \), we have

\[
m^{n-1} \sum_{k=0}^{n-1} \frac{1}{m^k} \binom{2k}{k} = \sum_{k=0}^{n-1} \binom{2n}{k} u_{n-k}(m-2, 1).
\]

**Lemma 2.3.** Let \( p > 2 \) be a prime. For any nonnegative integer \( \alpha \), \( s \) with \( \alpha \geq s \), then we have

\[
\frac{(mp^\alpha - p^{\alpha-1} - 1)}{2p^\alpha} \equiv \frac{(m^s - p^{s-1} - 1)}{4p^s} \pmod{p^s}.
\]
Proof.

\[
\frac{(m^{p^s-p^s-1} - 1)}{2p^s} = \sum_{k=1}^{\lfloor \frac{m}{p} \rfloor} \binom{p^s-1}{k} (m^{p^s-p^s-1} - 1)^k = \frac{(m^{p^s-p^s-1} - 1)}{2p^s}
\]

\[
\frac{(m^{p^s-p^s-1} - 1)}{2p^s} \left( \sum_{k=2}^{p^s-1} \binom{p^s-1}{k} (m^{p^s-p^s-1} - 1)^{k-1} + \sum_{k=1}^{p^s-1} \binom{p^s-1}{p^s-k} (m^{p^s-p^s-1} - 1)^{p^s-k-1} \right)
\]

\[
\equiv \frac{(m^{p^s-p^s-1} - 1)}{2p^s} \pmod{p^s}.
\]

\[\square\]

**Lemma 2.4.** Let \( p > 2 \) be a prime. For any nonnegative integer \( n, l, \alpha, s \) and \( \alpha \geq s \). If \( m = 1, 2, 3 \), then we have

\[
\sum_{|k/p^s| = l} \binom{p^s-1}{k} u_{p^s n-k} (m-2, 1) \equiv \frac{(m(m-4))^{s}(-m^{p^s-p^s-1} + 1)(-1)^l}{2p^s} (u_{p^s n-l} (m-2, 1) + u_{p^s n-l-1} (m-2, 1)) \pmod{p^s},
\]

here \( \sum_{|k/p^s| = l} \) denotes the sum of \( k \) with \( p \nmid k \).

**Proof.** Let \( m \) be an integer. We first assume that the following congruence is right.

\[
\sum_{k=1}^{p^s-1} \frac{(-1)^k u_{p^s n-k} (m-2, 1)}{k} \equiv \frac{(m(m-4))^{s}(-m^{p^s-p^s-1} + 1)}{2p^s} (u_{p^s n-l} (m-2, 1) + u_{p^s n-l-1} (m-2, 1)) \pmod{p^s}.
\]

Note that \( u_{-k} (m-2, 1) = -u_k (m-2, 1) \), then

\[
\sum_{|k/p^s| = l} \frac{(-1)^k u_{p^s n-k} (m-2, 1)}{k} = \sum_{k=1}^{p^s-1} \frac{(-1)^{p^s l + k} u_{p^s n-p^s l-k} (m-2, 1)}{p^s l + k}
\]

\[
\equiv (-1)^{p^s l} \sum_{k=1}^{p^s-1} \frac{(-1)^k u_{p^s n-k-p^s l} (m-2, 1)}{k} \pmod{p^s}
\]

\[
= (-1)^{p^s l+1} \sum_{k=1}^{p^s-1} \frac{(-1)^k u_{p^s (l+1-p^s n)-k} (m-2, 1)}{k}.
\]
Here we take \( s, (l + 1 - p^s - n) \) instead of \( \alpha, n \) in (2.5), then (2.4) is done.

\[
\sum_{k=1}^{p^s-1} \binom{p^s}{k} u_{p^s(n-k)}(m-2,1) = \sum_{k=0}^{p^s-1} \binom{p^s}{k} u_{p^s(n-k)}(m-2,1) - \sum_{l=0}^{p^s-1} \binom{p^s}{pl} u_{p^s-1(n-l)}(m-2,1),
\]

(2.6)

Next we will prove (2.5). On the one hand, we split the sum into a sum with \( p \nmid k \) and one with \( k = lp \),

\[
\sum_{k=1}^{p^s-1} \binom{p^s}{k} u_{p^s(n-k)}(m-2,1) = \sum_{k=0}^{p^s-1} \binom{p^s}{k} u_{p^s(n-k)}(m-2,1) - \sum_{l=0}^{p^s-1} \binom{p^s}{pl} u_{p^s-1(n-l)}(m-2,1),
\]

with the help of Lemma 2.1(i) and \( u_{pl}(m-2,1) = \left(\frac{m(m-4)}{p}\right) u_{l}(m-2,1) \), then

\[
\sum_{l=0}^{p^s-1} \binom{p^s}{pl} u_{p^s-1(n-l)}(m-2,1) \equiv \left(\frac{m(m-4)}{p}\right) \sum_{k=0}^{p^s-1} \binom{p^s-1}{k} u_{p^s-1(n-k)}(m-2,1) \pmod{p^{2s}}.
\]

(2.7)

On the other hand, with the help of Lemma 2.1(iii), then we get

\[
\sum_{k=1}^{p^s-1} \binom{p^s}{k} u_{p^s(n-k)}(m-2,1) \equiv \sum_{k=1}^{p^s-1} \binom{p^s}{k} \left(\frac{p^s-1}{k} \right) u_{p^s(n-k)}(m-2,1)(-1)^{k-1-\left\lfloor \frac{m-4}{p} \right\rfloor}
\]

\[
= p^s \sum_{t=0}^{p^s-1} \binom{p^s-1}{t} (-1)^t \sum_{\left\lfloor \frac{m-4}{p} \right\rfloor = t} \frac{(-1)^{k-1}}{k} u_{p^s(n-k)}(m-2,1) \pmod{p^{2s}},
\]

(2.8)

we may assume \( s \geq 1 \) in Lemma 2.4. Clearly, we proceed by induction, that for \( s = 1, 2, \ldots, r-1 \), (2.4) is right.

\[
\frac{1}{p^s} \sum_{k=1}^{p^s-1} \binom{p^s}{k} u_{p^s(n-k)}(m-2,1) \equiv \sum_{t=0}^{p^s-1} \binom{p^s-1}{t} (-1)^t \left(\sum_{\left\lfloor \frac{k}{p} \right\rfloor = t} \frac{(-1)^{k-1}}{k} u_{p^s(n-k)}(m-2,1)
\]

\[- \left(\frac{m(m-4)}{p}\right) \left(-m^{p^s-p^s-1} + 1\right) \frac{(-1)^t}{2p^s} (u_{p^s-1(n-t)}(m-2,1) + u_{p^s-1(n-t-1)}(m-2,1))
\]

\[+ \left(\frac{m(m-4)}{p}\right) \left(-m^{p^s-p^s-1} + 1\right) p^{s-1} \sum_{k=0}^{p^s-1} \binom{p^s-1}{k} u_{p^s-1(n-k)}(m-2,1) \pmod{p^s}.
\]

(2.9)
We apply Lemma 2.1 with \( s - 1, 1 \) instead of \( \alpha, n, \) then

\[
\frac{1}{p^s} \sum_{k=1}^{p^s-1} \binom{p^s}{k} u_{\rho^n, k}(m - 2, 1) \equiv \sum_{n_1=0}^{p^s-2} \binom{p^s-2}{n_1} (-1)^{n_1} \left( \sum_{[k/p^s]=n_1} (-1)^{k-1} \frac{k}{p^s} u_{\rho^n, k} \right)
\]

\[
(2.9)
\]

\[
- \sum_{t=0}^{p-1} \left( \frac{m(m-4)}{p} \right) \frac{(-m^{p^s-p^s-1}+1)(-1)^m_{1+t}}{2p^s} (u_{\rho^n, 1-p, t}(m - 2, 1) + u_{\rho^n, 1-p, t-1}(m - 2, 1))
\]

\[
+ \left( \frac{m(m-4)}{p} \right) \frac{(-m^{p^s-p^s-1}+1)}{2p^s} \sum_{k=0}^{p^s-1} \frac{1}{k} f_{p^s-1-k}(m - 2, 1) \mod p^s,
\]

here

\[
\sum_{t=0}^{p-1} (-1)^{m_{1+t}} (u_{\rho^n, 1-p, t}(m - 2, 1) + u_{\rho^n, 1-p, t-1}(m - 2, 1))
\]

\[
= (-1)^{n_1} (u_{\rho^n, 1-p, 1}(m - 2, 1) + u_{\rho^n, 1-p, 1-1}(m - 2, 1))
\]

\[
= (-1)^{n_1} \left( \frac{m(m-4)}{p} \right) (u_{\rho^n, 1-p, 1}(m - 2, 1) + u_{\rho^n, 1-p, 1-1}(m - 2, 1)),
\]

Repeat this process \( s - 1 \) times as (2.9), then

\[
\frac{1}{p^s} \sum_{k=1}^{p^s-1} \binom{p^s}{k} u_{\rho^n, k}(m - 2, 1)
\]

\[
\equiv \ldots \equiv \sum_{n_1=0}^{p-1} \frac{(p-1)}{n_1} (-1)^{n_1} \left( \sum_{[k/p^s]=n_1} (-1)^{k-1} \frac{k}{p^s} u_{\rho^n, k}(m - 2, 1) \right)
\]

\[
- \left( \frac{m(m-4)}{p} \right)^{s-1} \frac{(-m^{p^s-p^s-1}+1)(-1)^{m_1}}{2p^s} (u_{\rho^n, s+1-n, 1}(m - 2, 1) + u_{\rho^n, s+1-n, 1-1}(m - 2, 1))
\]

\[
+ \left( \frac{m(m-4)}{p} \right) \frac{(-m^{p^s-p^s-1}+1)}{2p^s} \sum_{k=0}^{p^s-1} \frac{1}{k} u_{\rho^n, s-n}(m - 2, 1)
\]

\[
\equiv \sum_{k=1}^{p^s-1} \frac{(-1)^{k-1}}{k} u_{\rho^n, k}(m - 2, 1) - \left( \frac{m(m-4)}{p} \right)^{s} \frac{(-m^{p^s-p^s-1}+1)}{2p^s} (u_{\rho^n, s-n}(m - 2, 1)
\]

\[
+ u_{\rho^n, s-n-1}(m - 2, 1) + \left( \frac{m(m-4)}{p} \right) \frac{(-m^{p^s-p^s-1}+1)}{2p^s} \sum_{k=0}^{p^s-1} \frac{1}{k} u_{\rho^n, s-n}(m - 2, 1) \mod p^s,
\]

\[
(2.10)
\]
from (2.6), (2.7) and (2.10), then we only need to prove that

\[
\frac{1}{p^s} \sum_{k=0}^{p^s} \binom{p^s}{k} u_{p^s n-k} \equiv \frac{1}{p^s} \left( \frac{m(m-4)}{p} \right) \sum_{k=0}^{p^s-1} \binom{p^s-1}{k} u_{p^s n-k} \quad (\text{mod } p^s).
\]

Substitute \( m = 1 \) in (2.11). Then

\[
\sum_{k=0}^{p^s-1} \binom{p^s-1}{k} u_{p^s n-k}(-1, 1) = (1 + i)^{p^s} \nu_{p^s-1}(-1, 1) - (1 - i)^{p^s} \nu_{p^s-1}(-1, 1) = 0,
\]

we are done. Because

\[
\sum_{k=0}^{p^s-1} \binom{p^s-1}{k} u_{k-p^s n}(0, 1) = \frac{(1 + i)^{p^s} \nu_{p^s-1} - (1 - i)^{p^s} \nu_{p^s-1}}{2i}.
\]

Next we will take \( m = 2 \) in (2.11). By (2.12), then it suffices to show that

\[
\frac{2p^s - p^s - 1}{2p^s - 1} \equiv \left( \left( \frac{-1}{p} \right)^{2p^s - 1} \right) \times \left( \frac{u_{p^s - (\frac{1}{2} p)^{s-1}}(0, 1) + (\frac{1}{p})^{s-1} u_{p^s - (\frac{1}{2} p)^{s-1}}(0, 1)}{u_{p^s - (\frac{1}{2} p)^{s-1}}(0, 1) + (\frac{1}{p})^{s-1} u_{p^s - (\frac{1}{2} p)^{s-1}}(0, 1)} - 1 \right).
\]

Here

\[
(\frac{-1}{p})^{2p^s - 1} u_{p^s - (\frac{1}{2} p)^{s-1}}(0, 1) = u_{p^s - (\frac{1}{2} p)^{s-1}}(0, 1) = (\frac{-1}{p})^{2p^s - 1} u_{p^s - (\frac{1}{2} p)^{s-1}}(0, 1)
\]

and

\[
(\frac{-1}{p})^{2p^s - 1} u_{p^s - (\frac{1}{2} p)^{s-1}}(0, 1) = u_{p^s - (\frac{1}{2} p)^{s-1}}(0, 1) = (\frac{-1}{p})^{2p^s - 1} u_{p^s - (\frac{1}{2} p)^{s-1}}(0, 1).
\]
At last, with the help of Lemma 2.3 and the following congruence

\[ \frac{2^{p^s-p^s-1}}{2p^s} = 1 \frac{2^{p^s-p^s-1}}{2p^s} \left( -1 \frac{p^s}{4} - 1 \right)^2 + 2 \left( -1 \frac{p^s}{4} - 1 \right) \]

\[ \equiv 1 \frac{2^{p^s-p^s-1}}{2p^s} \left( -1 \frac{p^s}{4} - 1 \right) \pmod{p^s}, \]  \hspace{1cm} (2.14)

Lemma 2.4 with \( m = 2 \) is concluded because

\[ 2^{p^s-p^s-1} \equiv \left( \frac{2}{p} \right) = (-1)^{2^{s-1}} = (-1) \frac{p^s-1}{4} \pmod{p^s}. \]

When \( m = 3 \), (2.11) can be proved similarly, with the help of Lemma 2.3, then we have

\[ \left( -3 \right) \frac{3^{p^s-p^s-1}}{2p^s} = \left( \frac{3}{p} \right) - 1 \frac{3^{p^s-p^s-1}}{2p^s} \equiv 1 \pmod{p^s}, \]  \hspace{1cm} (2.15)

where

\[ \frac{\left( \frac{1+\sqrt{3}}{2} p^s - 1 \right)}{2} \left( \frac{1-\sqrt{3}}{2} p^s - 1 \right) \equiv 1 \pmod{p^s}. \]

(2.15) is proved when \( m = 3 \).

**Lemma 2.5.** Let \( a_k \in \mathbb{Z}_p (k = 0, 1, \ldots) \) be such that

\[ \sum_{|k/p^s|=l} a_k \equiv 0 \pmod{p^s}, \]

for any nonnegative integer \( m, n, \alpha \) and \( s \). Then

\[ \sum_{|k/p^s|=l} a_k \binom{mp^s n - 1}{k} (-1)^k \equiv 0 \pmod{p^\alpha}. \]  \hspace{1cm} (2.16)
Proof. We prove it by induction on $\alpha$. The above congruence is trivial when $\alpha = 0, 1$. Suppose that we have show it for $0, 1, \ldots, \alpha - 1$.

\[
\sum_{[k/p^\alpha] = l} a_k \binom{mp^\alpha n - 1}{k} (-1)^k \equiv \sum_{[k/p^\alpha] = l} a_k \binom{mp^{\alpha - 1}n - 1}{[k/p]} (-1)^{[k/p]} \quad (2.17)
\]

we now apply the induction hypothesis for $\alpha - 1$ with the new coefficients

\[
p\hat{a}_t = \sum_{[k/p] = t} a_k \equiv 0 \pmod{p},
\]

and

\[
\sum_{[t/p^{\alpha - 1}] = l} \hat{a}_t = \frac{1}{p} \sum_{[k/p^\alpha] = l} a_k \equiv 0 \pmod{p^{\alpha - 1}}.
\]

So we obtain

\[
\sum_{[k/p^\alpha] = l} a_k \binom{mp^\alpha n - 1}{k} (-1)^k \equiv p \sum_{[t/p^{\alpha - 1}] = l} a_t \binom{mp^{\alpha - 1}n - 1}{t} (-1)^t \equiv 0 \pmod{p^\alpha}.
\]

3. Proofs of Theorem 1.1

Proof. According to Lemma 2.1(i) and 2.2 so (1.5) can be rewritten as

\[
n \sum_{k=1}^{p^{\alpha - 1} - 1} \binom{2p^\alpha n - 1}{k - 1} u_{p^\alpha n - k} \binom{m - 2, 1}{k} \equiv \binom{m(m - 4)}{p} \binom{m(p^\alpha - p^{\alpha - 1})n - 1}{2p^\alpha} \sum_{k=0}^{p^{\alpha - 1} - 1} \binom{2p^{\alpha - 1}n}{k} u_{p^{\alpha - 1}n - k} \binom{m - 2, 1}{k} \pmod{p^\alpha}.
\]

(3.1)

(i) When $m = 1$, we need only to prove

\[
\sum_{k=1}^{p^{\alpha - 1} - 1} \binom{2p^\alpha n - 1}{k - 1} u_{p^\alpha n - k} \binom{-1, 1}{k} \equiv 0 \pmod{p^\alpha}.
\]

(3.2)

However

\[
\sum_{k=1}^{p^{\alpha - 1} - 1} \binom{2p^\alpha n - 1}{k - 1} u_{p^\alpha n - k} \binom{-1, 1}{k} \equiv \sum_{k=1}^{p^{\alpha - 1} - 1} \binom{2p^\alpha n - 1}{k} (-1)^k \binom{-1}{k} u_{p^\alpha n - k} \binom{-1, 1}{k} \pmod{p^\alpha}.
\]

We set $m = 2$ and $a_k = \binom{-1}{k} u_{p^\alpha n - k} \binom{-1, 1}{k}$ if $p \nmid k$, $a_k = 0$ otherwise in Lemma 2.5

Thus (1.5) with $m = 1$ immediately follows from Lemma 2.4.
(ii) Next we will prove it when $m = 2, 3$. It suffices to prove that

\[
\sum_{k=0}^{p^{\alpha-1}n-1} \left( \binom{p^{\alpha-1}n-1}{k} + \binom{2p^{\alpha-1}n-1}{k} \right) u_{p^{\alpha-1}n-k}(m-2,1) \equiv n \left( \frac{m(m-4)}{p} \right) \mod p^{\alpha},
\]

where

\[
\frac{m(p^{\alpha-p^{\alpha-1}}-1)}{2p^{\alpha}} = \frac{1}{2p^{\alpha}} \sum_{k=1}^{n} \left( \binom{n}{k} (m^{p^{\alpha}-p^{\alpha-1}}-1)^k \right) \equiv n \left( \frac{m^{p^{\alpha}-p^{\alpha-1}}-1}{2p^{\alpha}} \right) \mod p^{\alpha}.
\]

By Lemma 2.1, then

\[
\sum_{n_1=0}^{p^{\alpha-2}n-1} \left( \binom{2p^{\alpha-2}n-1}{n_1} \right) (-1)^{n_1} \left( \sum_{[k/p] = n_1}^{1} \binom{-1}{k} u_{k-p^{\alpha}n}(m-2,1) \right) \equiv 0 \mod p^{\alpha}.
\]

With the help of Lemma 2.4 with $s = 1$, we have

\[
\sum_{n_1=0}^{p^{\alpha-2}n-1} \left( \binom{2p^{\alpha-2}n-1}{n_1} \right) (-1)^{n_1} \left( \sum_{[k/p^2] = n_1}^{1} \binom{-1}{k} u_{k-p^{\alpha}n}(m-2,1) \right) \equiv 0 \mod p^{\alpha},
\]

then

\[
\sum_{n_1=0}^{p^{\alpha-2}n-1} \left( \binom{2p^{\alpha-2}n-1}{n_1} \right) (-1)^{n_1} \left( \sum_{[k/p^2] = n_1}^{1} \binom{-1}{k} u_{k-p^{\alpha}n}(m-2,1) \right) \equiv 0 \mod p^{\alpha},
\]
repeat this process $\alpha$ times, then we obtain
\[
\sum_{t=0}^{n-1} \binom{2n-1}{t} (-1)^t \left( \sum_{\lfloor k/p^\alpha \rfloor = 0}^{m-1} \frac{(-1)^{t+k} u_{k-p^\alpha} (m-1)}{k} - \left( \frac{m(m-4)}{p} \right)^{\alpha-1} \right) \equiv 0 \pmod{p^\alpha},
\]
where the last step we used Lemma 2.4 with $l = 0$.

\[\square\]

Acknowledgment. We are grateful to Professor Zhi-Wei Sun for his helpful discussions on this paper.

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E-mail address: yongzhang1982@163.com

Department of Mathematics and Physics, Nanjing Institute of Technology, Nanjing 211167, People's Republic of China

E-mail address: haopan79@zoho.com

School of Applied Mathematics, Nanjing University of Finance and Economics, Nanjing 210046, People’s Republic of China