λ–Symmetries and integrability by quadratures

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Abstract

It is investigated how two (standard or generalized) λ–symmetries of a given second-order ordinary differential equation can be used to solve the equation by quadratures. The method is based on the construction of two commuting generalized symmetries for this equation by using both λ–symmetries. The functions used in that construction are related with integrating factors of the reduced and auxiliary equations associated to the λ–symmetries. These functions can also be used to derive a Jacobi last multiplier and two integrating factors for the given equation.

Some examples illustrate the method; one of them is included in the XXVII case of the Painlevé-Gambier classification. An explicit expression of its general solution in terms of two fundamental sets of solutions for two related second-order linear equations is also obtained.

Keywords: λ–symmetries, first integrals, integrating factors, Jacobi last multiplier.

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1. Introduction

A remarkable application of the Lie group theory to differential equations is that the general solution of an nth-order ordinary differential equation (ODE) that admits an nth-dimensional solvable symmetry algebra can be found by means of n successive quadratures \cite{1} \cite{2} \cite{3}. However, the existence of non-trivial point symmetries is not a necessary condition for the integrability by quadratures of ODEs. An example of a family of second-order ODEs integrable by quadratures whose point symmetry group is trivial was firstly provided in
Such example, and many others that appeared later in the literature, motivated the need of developing a more general context than the framework of Lie point symmetries to explain the integrability by quadratures of a given ODE.

The integrability of many of these equations can be explained by using $\lambda$-symmetries (also called $C^\infty$-symmetries). This concept was introduced in [8] and it is based on a prolongation formula that generalizes the usual prolongation of vector fields. $C^\infty$-symmetries can be used to reduce the order of ODEs as Lie point symmetries do and have been widely studied and generalized from very different points of view (see [10]–[13] and the references therein).

However, it seems that no studies have been made on the consequences of having two or more $C^\infty$-symmetries for a given ODE. In fact, the reduction process associated to one of the $\lambda$-symmetries is independent of the corresponding one for any of the remaining $C^\infty$-symmetries.

The integrability of 2nd-order ODEs admitting two $C^\infty$-symmetries is studied in this paper. Section 2 includes a review of the basic notions (limited to second-order ODEs) and the extension of the initial notion of $C^\infty$-symmetry introduced in [8] to consider $\lambda$-prolongations of generalized vector fields. In Sections 3 and 4 a systematic procedure to construct two commuting generalized symmetries from two given (standard or generalized) $C^\infty$-symmetries of a second-order ODE is provided. According to [14], two independent first integrals of the ODE can be found, not necessarily by quadrature, by means of a procedure that uses each $C^\infty$-symmetry separately. In Section 5 the two $C^\infty$-symmetries and the functions that appear in the construction of the generalized symmetries are used simultaneously to find such first integrals by quadrature.

Several significant objects in the analytical study of the ODEs arise as an immediate consequence of the described procedure. For instance, a well-known result on the relationships between Jacobi last multipliers and Lie point symmetries is that the knowledge of two independent symmetries provides an explicit formula for a Jacobi last multiplier. In Section 6 a new explicit formula for a Jacobi last multiplier of 2nd-order ODEs that involve two $C^\infty$-symmetries admitted by the equation is provided.

In Section 6 the known $\lambda$-symmetries and the functions that appear in the construction of the generalized symmetries are used to obtain two integrating factors for the second-order ODE. The cited functions are closely related to integrating factors of the reduced and auxiliary equations that appear in the reduction processes associated to the given generalized $C^\infty$-symmetries.

The results in this paper are applied to a subclass of equations in the XXVII case of the Painlevé-Gambier classification. In the general case the equa-
tions in this subclass do not admit Lie point symmetries, but any equation of the family admits two non-equivalent $\mathcal{C}^\infty$-symmetries, which have been recently found by the authors of [21]. Such $\mathcal{C}^\infty$-symmetries are used to illustrate the procedure of integration of the equations by quadratures. The study of that family of equations has been carried out through several examples to illustrate the different steps that appear in the method. As a consequence, the general solution of any of the equations in the family can be expressed in terms of two fundamental sets of solutions for two second-order linear equations. Explicit expressions for a Jacobi last multiplier and the integrating factors of the reduced and auxiliary equations are also provided.

In Section 7 a step-by-step description of the procedure to facilitate its practical application is presented. Several examples show how the method can be used to solve equations lacking Lie point symmetries or which admit just one Lie point symmetry.

2. Preliminaries

Throughout this paper $M$ will denote an open subset of the space of the independent and dependent variables $(x, u)$ of a given 2nd-order ODE:

$$u_{xx} = \phi(x, u, u_x),$$

where the subscript denotes derivation with respect to $x$.

The vector field on the jet space $M^{(1)}$ associated to equation (1) is defined by $A = \partial_x + u_x \partial_u + \phi(x, u, u_x) \partial_{u_x}$. The total derivative operator with respect to $x$ is defined by

$$D_x = \partial_x + u_x \partial_u + u_{xx} \partial_{u_x} + \cdots.$$

Recall [8] that if $v = \xi(x, u) \partial_x + \eta(x, u) \partial_u$ is a vector field on $M$ and $\lambda = \lambda(x, u, u_x)$ is a smooth function defined on $M^{(1)}$ then the first-order $\lambda$–prolongation of $v$ is the vector field $v^{(\lambda,(1))}$ on $M^{(1)}$ defined by

$$v^{(\lambda,(1))} = v + \left( (D_x + \lambda)(\eta) - (D_x + \lambda)(\xi) u_x \right) \partial_{u_x}. \quad (2)$$

The functions $\xi$ and $\eta$ will be called the infinitesimals of $v$. Observe that for $\lambda = 0$, (2) is the standard first-order prolongation $v^{(1)}$ of $v$ [3].

The pair $(v, \lambda)$ is a $\mathcal{C}^\infty$-symmetry (also called a $\lambda$–symmetry) of equation (1) if

$$[v^{(\lambda,(1))}, A] = \lambda v^{(\lambda,(1))} + \mu A, \quad (3)$$

where $\mu$ is a smooth function defined on $M^{(1)}$. The functions $\lambda$ and $\mu$ will be called the coefficients of the symmetry $(v, \lambda)$.
where $\mu = -(A + \lambda)(\xi)$. For $\lambda = 0$, if $v$ satisfies the condition \((3)\) then $v$ is a standard Lie point symmetry of \((1)\) \cite{3}.

When a $C^\infty$-symmetry $(v, \lambda)$ is known, a method to solve the given ODE proceeds as follows: if $y = y(x, u), w = w(x, u, u_x)$ are two invariants of $v^{[A,(1)]}$, then \((1)\) can be written as a reduced equation $\Delta(y, w, w_y) = 0$. Then the general solution of \((1)\) arises from the general solution $w(x, u, u_x) = H(y(x, u), C)$, where $C \in \mathbb{R}$ (see Theorem 3.2 in \cite{8} for details). This method generalizes the classical Lie method and has been successfully applied to integrate or reduce the order of many ODEs lacking Lie point symmetries \cite{8, 9}.

If $\xi = \xi(x, u, u_x)$ and $\eta = \eta(x, u, u_x)$ are two smooth functions defined on $M^{(1)}$ then
\[
v = \xi(x, u, u_x) \partial_x + \eta(x, u, u_x) \partial_u
\]
is a well-defined vector field on $M^{(1)}$. If $\lambda = \lambda(x, u, u_x)$ is also a smooth function on $M^{(1)}$ then by considering the vector field $A$, the vector field $v$ can be prolonged by using a formula similar to \((2)\) but changing $D_x$ by $A$: i.e.
\[
v^{[A,(1)]} = v + [(A + \lambda)(\eta) - (A + \lambda)(\xi) u_x] \partial_{u_x}, \tag{4}
\]
which is a well-defined vector field on $M^{(1)}$. The pair $(v, \lambda)$ will be called a generalized $C^\infty$-symmetry (or a generalized $\lambda$-symmetry) of equation \((1)\) if \((3)\) holds, where $v^{[A,(1)]}$ is defined by \((4)\). It is clear that if the pair $(v, \lambda)$ is a generalized $C^\infty$-symmetry of equation \((1)\) for the function $\lambda = 0$ then the vector field $v^{[0,(1)]} = v^{(1)}$ is a generalized symmetry of equation \((1)\) \cite{3}.

According to \((3)\), any given generalized $C^\infty$-symmetry $(v, \lambda)$ of \((1)\) defines a vector field $v^{[A,(1)]}$ such that the system \{$A, v^{[A,(1)]}\}$ is in involution. By Fröbenius Theorem \((1)\), the system \{$A, v^{[A,(1)]}\}$ is integrable and the integral submanifold is locally defined by a first integral, $I \in C^\infty(M^{(1)})$, such that
\[
A(I) = v^{[A,(1)]}(I) = 0.
\]
In this case, $I$ will be called a first integral of $A$ associated to $(v, \lambda)$.

A given first integral of $A$ can be associated to several generalized $C^\infty$-symmetries. This fact is related to the notion of equivalence of $C^\infty$-symmetries established in \cite{14, 22}. Next this concept is extended for generalized $C^\infty$-symmetries:

**Definition 1.** Two generalized $C^\infty$-symmetries $(v_1, A_1)$ and $(v_2, A_2)$ of the equation \((1)\) will be called $A$-equivalent if the set of vector fields \{$A, v_1^{[A_1,(1)]}, v_2^{[A_2,(1)]}\}$...
is dependent over $\mathcal{C}^\infty(M^{(1)})$. In this case the notation $(\mathbf{v}_1, \lambda_1) \sim^A (\mathbf{v}_2, \lambda_2)$ will be used.

It follows from Definition [1] that the first integrals of $A$ associated to equivalent generalized $\mathcal{C}^\infty-$symmetries are functionally dependent. Consequently, a first integral of $A$ associated to a generalized $\mathcal{C}^\infty-$symmetry $(\mathbf{v}, \lambda)$ can be found by using any element of its class of equivalence. A particularly simple element in the equivalence class of a given generalized $\mathcal{C}^\infty-$symmetry $(\mathbf{v}, \lambda)$ is the pair

$$(\partial_u, \lambda + A(Q)/Q),$$

where $Q = \eta - \xi u_x$ denotes the characteristic of $\mathbf{v} = \xi \partial_x + \eta \partial_u$. This pair is called the *canonical* representative of the equivalence class of $(\mathbf{v}, \lambda)$ [14].

Since (1) is a 2nd-order equation, its general solution can be obtained by considering two first integrals $I_1$ and $I_2$ of $A$ which are respectively associated to two known non-equivalent generalized $\mathcal{C}^\infty-$symmetries $(\mathbf{v}_1, \lambda_1)$ and $(\mathbf{v}_2, \lambda_2)$ of (1). In the following sections a procedure to compute such first integrals by quadratures will be described.

3. **Commuting generalized $\mathcal{C}^\infty-$symmetries**

Assume that equation (1) admits two non-equivalent generalized $\mathcal{C}^\infty-$symmetries $(\mathbf{v}_1, \lambda_1)$ and $(\mathbf{v}_2, \lambda_2)$ and let $\mathcal{A}_1$ and $\mathcal{A}_2$ be their respective $A$-equivalence classes. By using their canonical representatives it can be considered, without loss of generality, that such $\mathcal{C}^\infty-$symmetries are of the form $(\partial_u, \lambda_1) \in \mathcal{A}_1$ and $(\partial_u, \lambda_2) \in \mathcal{A}_2$, where $\lambda_1 \neq \lambda_2$ because $\mathcal{A}_1 \neq \mathcal{A}_2$. In this case the functions $\lambda_1$ and $\lambda_2$ are particular solutions of the determining equation (see equation (7) in [14]):

$$\lambda_x + \lambda u_x + \lambda u_x \phi + \lambda^2 = \phi u + \lambda \phi u_x.$$  \hspace{1cm} (5)

In order to simplify the notation, for $i = 1, 2$, $(\partial_u)^{[\lambda_i,(1)]}$ will be denoted by $X_i$:

$$X_i = (\partial_u)^{[\lambda_i,(1)]} = \partial_u + \lambda \partial_u x, \quad (i = 1, 2).$$ \hspace{1cm} (6)

As a direct consequence of former definitions, it can be checked that

$$[X_1, A] = \lambda_1 X_1, \quad [X_2, A] = \lambda_2 X_2, \quad [X_1, X_2] = \rho(X_1 - X_2),$$ \hspace{1cm} (7)

where the function $\rho = \rho(x, u, u_x)$ is given by

$$\rho = \frac{X_1(\lambda_2) - X_2(\lambda_1)}{\lambda_1 - \lambda_2}.$$ \hspace{1cm} (8)
Since it is assumed that $\lambda_1 \neq \lambda_2$, then $X_1 - X_2 \neq 0$ and therefore the vector fields $X_1$ and $X_2$ commute if and only if $\rho = 0$. In this case, by applying the Jacobi identity to the vectors fields $\{A, X_1, X_2\}$ and by using (7),

$$0 = [X_1, [X_2, A]] + [X_2, [A, X_1]] + [A, [X_1, X_2]]$$
$$= [X_1, \lambda_2 X_2] - [X_2, \lambda_1 X_1] + [A, 0]$$
$$= X_1(\lambda_2) \cdot X_2 - X_2(\lambda_1) \cdot X_1. \quad (9)$$

Since the vector fields $X_1, X_2$ are not proportional, because $\lambda_1 \neq \lambda_2$, (9) implies that

$$X_1(\lambda_2) = X_2(\lambda_1) = 0. \quad (10)$$

In other words, the vector fields $X_1$ and $X_2$ commute if and only if (10) holds.

The following lemma will be used to prove the existence of two $\mathcal{C}^\infty$-symmetries which are $A$–equivalent to $(\partial_{u}, \lambda_1)$ and $(\partial_{u}, \lambda_2)$ respectively and are such that the corresponding first-order $\lambda$–prolongations commute:

**Lemma 1.** With the above notations, if $f_1 = f_1(x, u, u_x)$ and $f_2 = f_2(x, u, u_x)$ are two functions such that

$$\frac{X_1(f_2)}{f_2} = \frac{X_2(f_1)}{f_1} = \rho,$$ \quad (11)

where $\rho$ is given by (8), then

$$[f_1 X_1, A] = \rho_1(f_1 X_1), \quad [f_2 X_2, A] = \rho_2(f_2 X_2), \quad [f_1 X_1, f_2 X_2] = 0, \quad (12)$$

where

$$\rho_i = \lambda_i - \frac{A(f_i)}{f_i}, \quad \text{for} \ i = 1, 2. \quad (13)$$

**Proof.** By the properties of the Lie bracket, for any functions $f_1$ and $f_2$ the relations

$$[f_1 X_1, f_2 X_2] = f_1 f_2 [X_1, X_2] + f_1 X_1(f_2) \cdot X_2 - f_2 X_2(f_1) \cdot X_1$$
$$= f_1 f_2 \rho(X_1 - X_2) + f_1 X_1(f_2) \cdot X_2 - f_2 X_2(f_1) \cdot X_1$$
$$= f_2 \left( \rho - \frac{X_2(f_1)}{f_1} \right) \cdot f_1 X_1 - f_1 \left( \rho - \frac{X_1(f_2)}{f_2} \right) \cdot f_2 X_2$$

hold. If $f_1, f_2$ satisfy (11), then $[f_1 X_1, f_2 X_2] = 0$.

The first two relations in (12) can be proved similarly. \qed
In order to simplify the notations, if \( f_1 \) and \( f_2 \) satisfy (11) then the vector fields \( f_1X_1 \) and \( f_2X_2 \) will be denoted by \( Y_1 \) and \( Y_2 \) respectively: \( Y_i = f_iX_i \), for \( i = 1, 2 \). With these notations, equations in (12) can be written as

\[
[Y_1,A] = \rho_1 Y_1, \quad [Y_2,A] = \rho_2 Y_2, \quad [Y_1,Y_2] = 0, \tag{14}
\]

where \( \rho_1 \) and \( \rho_2 \) are given by (13).

By (3), the first two relations in (14) show that

\[
Y_i = f_iX_i = (f_i\partial_u)^{[\rho_i,(1)]}, \quad \text{for} \quad i = 1, 2. \tag{15}
\]

According to Definition [1], the pairs \(( f_1\partial_u, \rho_1) \) and \(( f_2\partial_u, \rho_2) \) are two generalized \( \mathcal{C}_\infty \)–symmetries of (1) which are \( \mathcal{A} \)–equivalent to \(( \partial_u, \lambda_1) \) and \(( \partial_u, \lambda_2) \) respectively.

Previous discussion shows that if \(( \partial_u, \lambda_1) \in \mathcal{A}_1 \) and \(( \partial_u, \lambda_2) \in \mathcal{A}_2 \) are two non-equivalent \( \mathcal{C}_\infty \)–symmetries of (1) then \(( f_1\partial_u, \rho_1) \in \mathcal{A}_1 \) and \(( f_2\partial_u, \rho_2) \in \mathcal{A}_2 \) and by (15) their first-order \( \lambda \)–prolongations commute.

For further reference, the next theorem collects the main aspects of former discussion.

**Theorem 2.** Let \(( \partial_u, \lambda_1) \) and \(( \partial_u, \lambda_2) \) be the canonical representatives of two non-equivalent generalized \( \mathcal{C}_\infty \)–symmetries of (1) and let \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) be their respective equivalence classes. Denote \( X_i = (\partial_u)^{[\lambda_i,(1)]}, \) for \( i = 1, 2 \). Let \( \rho \) be the function defined by (8) and let \( f_1, f_2 \) be two functions satisfying (11). Then:

1. \(( f_1\partial_u, \rho_1) \in \mathcal{A}_1 \) and \(( f_2\partial_u, \rho_2) \in \mathcal{A}_2 \), where \( \rho_1, \rho_2 \) are given by (13).
2. By denoting \( Y_1 = ( f_1\partial_u)^{[\rho_1,(1)]} \) and \( Y_2 = ( f_2\partial_u)^{[\rho_2,(1)]} \), the relations (14) hold. In particular, \( Y_1, Y_2 \) commute.

The above-described method is illustrated in the next example by constructing two commuting generalized \( \mathcal{C}_\infty \)–symmetries from two known \( \mathcal{C}_\infty \)–symmetries of an equation.

**Example 3.1.** Let us consider the family of 2nd-order equations

\[
u_{xx} - \frac{1}{2u}u_x^2 + 2uu_x + \frac{1}{2}u^3 - F(x)u + \frac{1}{2u} = 0, \tag{16}
\]

which is a particular case of the XXVII equation in the Painlevé-Gambier classification [20].
For an arbitrary function $F = F(x)$, equation (16) does not admit Lie point symmetries. Nevertheless, it has been shown in [21] that two non-equivalent $C^\infty$-symmetries of (16), $(\partial_u, \lambda_1)$ and $(\partial_u, \lambda_2)$, are defined by:

$$\lambda_1 = \frac{u_x}{u} - u + \frac{1}{u}, \quad \lambda_2 = \frac{u_x}{u} - u - \frac{1}{u}.$$  

(17)

It can be checked that these two functions are particular solutions of the corresponding determining equation [5]. For this case, the vector fields (6) satisfy:

$$[X_1, X_2] = \frac{2}{u} (X_1 - X_2).$$

Since $\rho = \frac{2}{u}$ does not depend on $u_x$, two functions $f_1$ and $f_2$ satisfying (11) can be easily computed:

$$f_1(x, u, u_x) = f_2(x, u, u_x) = u^2.$$  

(18)

The vector fields $Y_1 = u^2X_1$ and $Y_2 = u^2X_2$ become

$$Y_1 = u^2\partial_u + (u_xu - u^3 + u)\partial_{u_x},$$

$$Y_2 = u^2\partial_u + (u_xu - u^3 - u)\partial_{u_x}$$  

(19)

and satisfy $[Y_1, Y_2] = 0$. According to (15), it can be written $Y_1 = (u^2\partial_u)^{[\rho_1,(1)]}$ and $Y_2 = (u^2\partial_u)^{[\rho_2,(1)]}$ for the functions

$$\rho_1 = \lambda_1 - \frac{A(u^2)}{u^2} = -\frac{u_x}{u} - u + \frac{1}{u} \quad \text{and} \quad \rho_2 = \lambda_2 - \frac{A(u^2)}{u^2} = -\frac{u_x}{u} - u - \frac{1}{u}.$$

Consequently, two new $C^\infty$-symmetries of equation (16) have been constructed, $(u^2\partial_u, \rho_1)$ and $(u^2\partial_u, \rho_2)$, which are A-equivalent to $(\partial_u, \lambda_1)$ and $(\partial_u, \lambda_2)$ respectively and satisfy

$$[Y_1, Y_2] = \left[ (u^2\partial_u)^{[\rho_1,(1)]}, (u^2\partial_u)^{[\rho_2,(1)]} \right] = 0.$$

\[\square\]

An important advantage of the pairs $(f_1\partial_u, \rho_1) \in \mathcal{A}_1$ and $(f_2\partial_u, \rho_2) \in \mathcal{A}_2$ constructed in Theorem 2 is that $Y_1 = (u^2\partial_u)^{[\rho_1,(1)]}$ and $Y_2 = (u^2\partial_u)^{[\rho_2,(1)]}$ can be simultaneously straightened by using quadratures alone. The next objective is to search for two independent functions $w_1 = w_1(x, u, u_x)$ and $w_2 = w_2(x, u, u_x)$ such that in the local system of coordinates $\{x, w_1, w_2\}$ of $M^{(1)}$ the vector fields $Y_1$ and $Y_2$ can be written as

$$Y_1 = \partial_{w_2}, \quad Y_2 = \partial_{w_1}.$$  

(20)
The conditions (20) would imply
\[ Y_1(w_2) = 1, \quad Y_2(w_2) = 0, \] (21)
and
\[ Y_1(w_1) = 0, \quad Y_2(w_1) = 1. \] (22)
Since \( Y_1 = f_1 \partial_u + f_1 \lambda_1 \partial_{u_x} \) and \( Y_2 = f_2 \partial_u + f_2 \lambda_2 \partial_{u_x} \), equations (21) would imply:
\[ Y_1(w_2) = (w_2)_u Y_1(u) + (w_2)_{u_x} Y_1(u_x) = f_1(w_2)_u + \lambda_1 f_1(w_2)_{u_x} = 1, \]
\[ Y_2(w_2) = (w_2)_u Y_2(u) + (w_2)_{u_x} Y_2(u_x) = f_2(w_2)_u + \lambda_2 f_2(w_2)_{u_x} = 0. \]
and then \( (w_2)_u \) and \( (w_2)_{u_x} \) would be defined by
\[ (w_2)_u = \frac{\lambda_2}{f_1(\lambda_2 - \lambda_1)}, \quad (w_2)_{u_x} = -\frac{1}{f_1(\lambda_2 - \lambda_1)}. \] (23)

The local existence of such function \( w_2 \) is warranted because the mixed partials coincide:
\[ \left( \frac{\lambda_2}{f_1(\lambda_2 - \lambda_1)} \right)_{u_x} = -\left( \frac{1}{f_1(\lambda_2 - \lambda_1)} \right)_u; \]
this can be checked by using that \( X_2(f_1) = \rho_1 f_1 \) and a straightforward calculation. It is clear that the conditions (23) imply that \( w_2 \) can be constructed by quadratures if \( f_1 \) is known.

A similar reasoning can be followed to prove the local existence of a function \( w_1 \) satisfying (20), by using system (22) instead of (21); \( w_1 \) satisfies
\[ (w_1)_u = \frac{\lambda_1}{f_2(\lambda_1 - \lambda_2)}, \quad (w_1)_{u_x} = -\frac{1}{f_2(\lambda_1 - \lambda_2)}. \] (24)

For further reference, the following proposition collects some properties of the considered coordinate system \( \{x, w_1, w_2\} \).

**Proposition 1.** Let \( (\partial_u, \lambda_1) \) and \( (\partial_u, \lambda_2) \) be the canonical representatives of two non-equivalent generalized \( C^\infty \)-symmetries of (1). Consider the vector fields \( Y_1 \) and \( Y_2 \) given in Theorem 2. A local system of coordinates \( \{x, w_1, w_2\} \) on \( M^{(1)} \) in which \( Y_1 = \partial w_2 \) and \( Y_2 = \partial w_1 \) can be constructed by quadratures by using the infinitesimals of \( Y_1 \) and \( Y_2 \).
The functions \( f_1 \) and \( f_2 \) that satisfy \( (11) \) let the construction by quadratures of the invariants \( w_1, w_2 \) of \( Y_1, Y_2 \), respectively. Since \( Y_1 = f_1 X_1 \) and \( Y_2 = f_2 X_2 \), the functions \( w_1, w_2 \) are also invariants of \( X_1 \) and \( X_2 \), respectively. These invariants can be used to construct reduced equations associated to each \( C^\infty \)-symmetry as follows.

The first equation in \( (14) \) shows that if \( w_1 \) is an invariant of \( Y_1 \) then \( \phi_1 = A(w_1) \) is also an invariant of \( Y_1 \) and \( \phi_1 \) can be expressed in terms of \( \{x, w_1\} \). Similarly, \( \phi_2 = A(w_2) \) is an invariant of \( Y_2 \) which can be expressed in terms of \( \{x, w_2\} \). Therefore, in terms of the local system of coordinates \( \{x, w_1, w_2\} \), the vector field \( A \) becomes

\[
A = \partial_x + \phi_1(x, w_1) \partial_{w_1} + \phi_2(x, w_2) \partial_{w_2}.
\] (25)

In consequence

\[
(w_1)_x = \phi_1(x, w_1) \quad \text{and} \quad (w_2)_x = \phi_2(x, w_2)
\] (26)

are two reduced equations associated to \( (\partial_x, \lambda_1) \) and \( (\partial_x, \lambda_2) \), respectively. The respective vector fields associated to the reduced equations \( (26) \) are

\[
A_1 = \partial_x + \phi_1(x, w_1) \partial_{w_1} \quad \text{and} \quad A_2 = \partial_x + \phi_2(x, w_2) \partial_{w_2}.
\] (27)

In the next example it is illustrated the procedure given in the proof of the Proposition \( 1 \) to compute invariants of the vector fields \( (19) \) by using \( (18) \) and quadratures alone. These invariants will be used to compute the reduced equations associated to the \( C^\infty \)-symmetries defined by \( (17) \).

**Example 3.2.** By proceeding with the study of equation \( (16) \) made in Example \( 3.1 \) let \( Y_1 \) and \( Y_2 \) be the vector fields given in \( (19) \). The functions \( \lambda_1 \) and \( \lambda_2 \) given in \( (17) \) and the functions \( f_1 = f_2 = u^2 \) given in \( (18) \) will be used to construct the systems corresponding to \( (24) \) and \( (23) \):

\[
(w_1)_u = \frac{u_x - u^2 + 1}{2u^2}, \quad (w_1)_u = -\frac{1}{2u}, \\
(w_2)_u = -\frac{u_x - u^2 - 1}{2u^2}, \quad (w_2)_u = \frac{1}{2u}.
\]

Both systems can be easily solved by quadratures and

\[
w_1 = -\frac{u_x + u^2 + 1}{2u} \quad \text{and} \quad w_2 = \frac{u_x + u^2 - 1}{2u}
\] (28)

are, respectively, particular solutions.
It can be checked that, for this example, the vector fields (19) become \( Y_1 = \partial_{w_2} \) and \( Y_2 = \partial_{w_1} \) in variables \( \{x, w_1, w_2\} \). In these variables the vector field \( A \) associated to equation (16) is

\[
A = \partial_x + \left( w_1^2 - \frac{1}{2} (F(x) + 1) \right) \partial_{w_1} - \left( w_2^2 - \frac{1}{2} (F(x) - 1) \right) \partial_{w_2},
\]

which according to (26) provides the following reduced equations

\[
(w_1)_x = w_1^2 - \frac{1}{2} (F(x) + 1), \quad (w_2)_x = -w_2^2 + \frac{1}{2} (F(x) - 1).
\]

(29)

These equations were also obtained by the authors of [21] by following a different procedure.

Thus far, the relations (7) have been simplified by changing the initial generalized \( C^\infty \)-symmetries by other equivalent ones that define the vector fields \( Y_1 \) and \( Y_2 \) given in (15). As a consequence, reduced equations associated to the \( C^\infty \)-symmetries can be constructed by quadratures.

In the next section it is shown how to construct two standard generalized symmetries that are equivalent to the generalized \( C^\infty \)-symmetries that define \( Y_1 \) and \( Y_2 \). Once these generalized symmetries are known, the initial equation (1) can be completely integrated by quadratures.

4. Commuting generalized symmetries from generalized \( C^\infty \)-symmetries

In this section it is investigated the existence of two non identically zero functions \( g_1, g_2 \in C^\infty (M^{(1)}) \) such that

\[
[g_1 Y_1, A] = [g_2 Y_2, A] = [g_1 Y_1, g_2 Y_2] = 0,
\]

(30)

where \( Y_1 \) and \( Y_2 \) are the vector fields constructed in Theorem 2 from two known \( C^\infty \)-symmetries of equation (1). In case of existence of such functions \( g_1, g_2 \), equations (14) and the properties of the Lie bracket would imply that \( g_1, g_2 \) satisfy

\[
\left\{ \begin{array}{l}
A(g_1) = \rho_1 g_1, \\
Y_2(g_1) = 0,
\end{array} \right. \quad \left\{ \begin{array}{l}
A(g_2) = \rho_2 g_2, \\
Y_1(g_2) = 0,
\end{array} \right.
\]

(31)

where the functions \( \rho_1 \) and \( \rho_2 \) are defined by (13). Conversely, it can be checked that if \( g_1, g_2 \in C^\infty (M^{(1)}) \) are respectively solutions of the systems in (31) then \( g_1, g_2 \) satisfy (30).

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Observe that each system in (31) is a coupled system of two first-order partial differential equations with three independent variables whose compatibility is, a priori, not obvious. The local system of coordinates \( \{x, w_1, w_2\} \) obtained in Proposition 1 will be used to reduce simultaneously each system in (31) to a single partial differential equation with two independent variables, and the compatibility of the systems in (31) will be straightforward.

**Lemma 3.** There exist two functions \( g_1 = g_1(x, u, u_x) \) and \( g_2 = g_2(x, u, u_x) \) which satisfy the corresponding system in (31).

**Proof.** Recall that in the coordinates \( \{x, w_1, w_2\} \) given by Proposition 1, the vector field \( Y_1 \) (resp. \( Y_2 \)) can be written as \( Y_1 = \partial_{w_2} \) (resp. \( Y_2 = \partial_{w_1} \)). Since \( g_1, g_2 \) must satisfy \( Y_2(g_1) = Y_1(g_2) = 0 \), the functions \( g_1 \) and \( g_2 \) in coordinates \( \{x, w_1, w_2\} \) must be of the form \( g_1 = g_1(x, w_2) \) and \( g_2 = g_2(x, w_1) \), respectively. Since \( g_1 \) (resp. \( g_2 \)) must satisfy \( A(g_1) = \rho_1 g_1 \) (resp. \( A(g_2) = \rho_2 g_2 \)), the expression of \( A \) in coordinates \( \{x, w_1, w_2\} \) given in (25), suggests that \( g_1 = g_1(x, w_2) \) (resp. \( g_2 = g_2(x, w_1) \)) could be any particular solution of the first-order partial differential equation

\[
(g_1)_x + (g_1)_{w_2}\phi_2 = g_1(\phi_2)_{w_2} \quad \text{(resp. } (g_2)_x + (g_2)_{w_1}\phi_1 = g_2(\phi_1)_{w_1}). \tag{32}
\]

It can be checked that if \( \tilde{g}_1(x, w_1) \) (resp. \( \tilde{g}_2(x, w_2) \)) is a solution for (32), then, by writing \( w_1 \) and \( w_2 \) in terms of \( \{x, u, u_x\} \), the functions \( g_1 = \tilde{g}_1(x, w_2(x, u, u_x)) \) and \( g_2 = \tilde{g}_2(x, w_1(x, u, u_x)) \) are solutions of the corresponding system in (31). \( \square \)

As a consequence of the previous results, the main theorem in this section can now be proved:

**Theorem 4.** Let \( (\partial_u, \lambda_1) \) and \( (\partial_u, \lambda_2) \) be the canonical representatives of two non-equivalent generalized \( \mathcal{C}^\infty \)-symmetries of equation (1) and let \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) be their respective equivalence classes. Let \( f_1, f_2 \in \mathcal{C}^\infty(M^{(1)}) \) be two functions satisfying (11) and assume that \( g_1, g_2 \in \mathcal{C}^\infty(M^{(1)}) \) satisfy (31). Let \( h_1, h_2 \in \mathcal{C}^\infty(M^{(1)}) \) be the functions defined by

\[
h_i = f_i g_i, \quad (i = 1, 2), \tag{33}
\]

and denote \( Z_i = h_i \partial_u + \Lambda(h_i)\partial_{u_x} \), for \( i = 1, 2 \). The following relations hold:

1. \( (h_1 \partial_u, 0) \in \mathcal{A}_1 \) and \( (h_2 \partial_u, 0) \in \mathcal{A}_2 \).
2. The vector fields \( Z_1 \) and \( Z_2 \) are generalized symmetries of (1).
3. The system $\{Z_1, Z_2\}$ is a system of commuting generalized symmetries of $A$:

$$[Z_1, A] = [Z_2, A] = [Z_1, Z_2] = 0. \tag{34}$$

**Proof.** With the previous notations, by using (13) and (31),

$$\lambda_i - \frac{A(h_i)}{h_i} = \lambda_i - \frac{A(f_i g_i)}{g_i} = \lambda_i - \frac{A(f_i)}{f_i} - \frac{A(g_i)}{g_i} = \rho_i - \frac{A(g_i)}{g_i} = 0. \tag{35}$$

This proves that, for $i = 1, 2$, the pair $(h_i \partial_u, 0)$ is a generalized $C^\infty$-symmetry of (1), which is $A$-equivalent to $(\partial_u, \lambda_i)$, according to Definition 1. In fact, by denoting $Z_i = h_i \partial_u + A(h_i) \partial_{u_i}$, for $i = 1, 2$, the vector fields $Z_1$ and $Z_2$ are generalized symmetries of (1). Relations (35) imply

$$Z_i = h_i \partial_u + A(h_i) \partial_{u_i} = g_i Y_i = h_i X_i, \tag{36}$$

for $i = 1, 2$.

By using (7), (35), (36) and the properties of the Lie bracket, it follows

$$[Z_i, A] = [h_i X_i, A] = h_i [X_i, A] - A(h_i) X_i = h_i \lambda_i X_i - A(h_i) X_i = 0,$$

for $i = 1, 2$. Finally, by (14), (31) and (36)

$$[Z_1, Z_2] = [g_1 Y_1, g_2 Y_2] = g_1 g_2 [Y_1, Y_2] + g_1 Y_1 (g_2) \cdot Y_2 - g_2 Y_2 (g_1) \cdot Y_1 = 0.$$

The results obtained in this section will now be used in the following example to construct two commuting generalized symmetries from the $C^\infty$-symmetries (17) of equation (16).

**Example 4.1.** The studies of equation (16) made in Examples 3.1 and 3.2 will be continued in this example. Lemma 3 will be used to construct two functions $g_1$ and $g_2$ satisfying (31). In terms of the variables $\{x, w_1, w_2\}$ where $w_1$ and $w_2$ are given by (28), the required functions $g_1 = g_1(x, w_2)$ and $g_2 = g_2(x, w_1)$ are solutions of the equations (32):

$$(g_1)_x - \left( w_2^2 - \frac{1}{2} (F(x) - 1) \right) (g_1)_{w_2} = -2 w_2 g_1, $$

$$(g_2)_x + \left( w_1^2 - \frac{1}{2} (F(x) + 1) \right) (g_2)_{w_1} = 2 w_1 g_2. \tag{37}$$
In what follows, explicit expressions of some solutions of (37) will be obtained in terms of solutions of the 2nd-order linear equations

\[ \psi_{xx} = \frac{1}{2} \left( F(x) + 1 \right) \psi, \quad \theta_{xx} = \frac{1}{2} \left( F(x) - 1 \right) \theta. \]  

(38)

These are the linear equations obtained from the Riccati-type equations (29) by means of the standard transformations \( w_1 = -\psi' / \psi \) and \( w_2 = \theta' / \theta \), respectively. Let the pairs \( \psi_1, \psi_2 \) and \( \theta_1, \theta_2 \) be linearly independent solutions of the respective equations given in (38). Their corresponding Wronskians will be denoted by \( W_1 = W(\psi_1, \psi_2) = \psi_1 \psi_2' - \psi_2 \psi_1' \) and \( W_2 = W(\theta_1, \theta_2) = \theta_1 \theta_2' - \theta_2 \theta_1' \). It can be checked that the functions

\[ g_1(x, w_2) = \frac{1}{W_2} \left( \theta_2 w_2 - \theta_2' \right)^2 \quad \text{and} \quad g_2(x, w_1) = \frac{1}{W_1} \left( \psi_2 w_1 + \psi_2' \right)^2 \]  

(39)

satisfy the corresponding equation in (37). By using (28) and the functions \( f_1 \) and \( f_2 \) obtained in (18) the functions \( h_1 \) and \( h_2 \) given by (33) become

\[ h_1 = \frac{1}{4W_2} \left( (u_x + u^2 - 1) \theta_2 - 2u \theta_2' \right)^2, \quad h_2 = \frac{1}{4W_1} \left( (u_x + u^2 + 1) \psi_2 - 2u \psi_2' \right)^2. \]  

(40)

Finally, according to (36), two commuting generalized symmetries for equation (16) are given by

\[ Z_1 = \frac{1}{4W_2} \left( (u_x + u^2 - 1) \theta_2 - 2u \theta_2' \right)^2 \left( \partial_u + \left( \frac{u_x}{u} - u + \frac{1}{u} \right) \partial_{u_x} \right), \]

\[ Z_2 = \frac{1}{4W_1} \left( (u_x + u^2 + 1) \psi_2 - 2u \psi_2' \right)^2 \left( \partial_u + \left( \frac{u_x}{u} - u - \frac{1}{u} \right) \partial_{u_x} \right). \]

In the next section it will be shown that the system \( \{Z_1, Z_2\} \) of commuting symmetries constructed from two non-equivalent \( C^\infty \)-symmetries of equation (1) permits the integration of the equation by using quadratures alone.

5. Generalized \( C^\infty \)-symmetries and integrability by quadratures

As a consequence of Theorem 4, two non-equivalent generalized \( C^\infty \)-symmetries of equation (1) can be used to construct a system of commuting symmetries of \( A \). Next, this system is used to compute by quadratures first integrals of the equation (1) associated to the generalized \( C^\infty \)-symmetries. The
Theorem 5. Let $\partial u, \lambda_1$ and $\partial u, \lambda_2$ be the canonical representatives of two non-equivalent generalized $C^\infty$-symmetries of equation (1). Let $f_1, f_2$ (resp. $g_1, g_2$) be some pairs of functions in $C^\infty(M^{(1)})$ satisfying (11) (resp. (31)). Two functionally independent first integrals $I_1$ and $I_2$ of $A$, associated to $\partial u, \lambda_1$ and $\partial u, \lambda_2$, respectively, can be found by quadratures from (42) and (43).
Example 5.1. This example is a continuation of Examples 3.1, 3.2 and 4.1. Theorem 5 will be applied to integrate by quadratures equation (16). The functions $h_1 = f_1 g_1$ and $h_2 = f_2 g_2$ obtained in (40) let the construction of systems (42)-(43). Two first integrals $I_1$ and $I_2$ for this equation can be obtained by quadratures by using (42) and (43) respectively:

\[ I_1 = \frac{(u^2 + u_x + 1) \psi_1 - 2u \psi'_1}{(u^2 + u_x + 1) \psi_2 - 2u \psi'_2} \quad \text{and} \quad I_2 = \frac{(u^2 + u_x - 1) \theta_1 - 2u \theta'_1}{(u^2 + u_x - 1) \theta_2 - 2u \theta'_2}. \]  

(44)

These first integrals provide the general solution of equation (16) in terms of the solutions of the corresponding linear equations (38):

\[ u(x) = \left( \frac{C_1 \psi_2 - \psi'_1}{C_1 \psi_2 - \psi'_1} - \frac{C_2 \theta'_2 - \theta'_1}{C_2 \theta_2 - \theta'_1} \right)^{-1} \quad (C_1, C_2 \in \mathbb{R}). \]  

(45)

In [20] the general solution of equation (16) is expressed in terms of the solutions of a third-order nonlinear ODE that becomes a fourth-order linear ODE under differentiation. Observe that in this paper the general solution (45) has been obtained in terms of two fundamental sets of solutions for two second-order linear ODEs, although equation (16) is not linearisable.

6. Integrating factors and Jacobi last multipliers without additional integration

In this section, it is shown how several classical objects (as Jacobi last multipliers and integrating factors) associated to the given ODE, can be considered as by-products of the procedure of integration by quadratures described in the previous section.

6.1. Integrating factors and a Jacobi last multiplier for equation (1)

Some relations between first integrals of 2nd-order ODEs and $\mathcal{C}^\infty$-symmetries have been established in [14, 22, 23]: if $I \in \mathcal{C}(\mathcal{M}^{(1)})$ is a first integral of $\mathbf{A}$ associated to a $\mathcal{C}^\infty$-symmetry whose canonical representative is $(\partial_u, \lambda)$, then $\mu = I_{ux}$ is an integrating factor of equation (1) such that $\mu(u_{xx} - \phi) = D_x(I)$ and the following identities hold:

\[ I_x = \mu(\lambda u_x - \phi), \quad I_u = -\lambda \mu. \]  

(46)
Suppose that two non-equivalent \( \mathcal{C}^\infty \)−symmetries \( (\partial_u, \lambda_1) \) and \( (\partial_u, \lambda_2) \) of equation (1) are known, and let \( f_1, f_2 \) (resp. \( g_1, g_2 \)) be some pairs of functions in \( \mathcal{C}^\infty(M^{(1)}) \) satisfying (11) (resp. (31)). By (42)–(43),

\[
\mu_1 = \frac{1}{f_2 g_2 (\lambda_2 - \lambda_1)}, \quad \mu_2 = \frac{1}{f_1 g_1 (\lambda_1 - \lambda_2)} \tag{47}
\]

are two integrating factors of (1). Observe that the two first equations in (42) (resp. (43)) are the equations (46) for \( \mu = \mu_1 \) (resp. \( \mu = \mu_2 \)) given in (47).

Another consequence of Theorem 4 is that the system of commuting symmetries can be used to derive an explicit expression of a Jacobi last multiplier of the equation. According to [15] and with the same notation used in the previous sections, the reciprocal of the determinant

\[
\begin{vmatrix}
0 & f_1 g_1 & A(f_1 g_1) \\ 0 & f_2 g_2 & A(f_2 g_2) \\ 1 & u_x & \phi
\end{vmatrix}
= \begin{vmatrix}
0 & f_1 g_1 & \lambda_1 f_1 g_1 \\ 0 & f_2 g_2 & \lambda_2 f_2 g_2 \\ 1 & u_x & \phi
\end{vmatrix} = f_1 g_1 f_2 g_2 (\lambda_2 - \lambda_1)
\]

is a Jacobi last multiplier of equation (1). The earlier discussion proves the next theorem:

**Theorem 6.** Let \( (\partial_u, \lambda_1) \) and \( (\partial_u, \lambda_2) \) be the canonical representatives of two non-equivalent generalized \( \mathcal{C}^\infty \)−symmetries of equation (1). Let \( f_1, f_2 \) (resp. \( g_1, g_2 \)) be some pairs of functions in \( \mathcal{C}^\infty(M^{(1)}) \) satisfying (11) (resp. (31)). Then:

1. The function \( \mu_1 = \mu_1(x, u, u_x) \) (resp. \( \mu_2 = \mu_2(x, u, u_x) \)) given in (47) is an integrating factor of the equation (1). A first integral \( I_1 \) (resp. \( I_2 \)) of \( A \) associated to \( (\partial_u, \lambda_1) \) (resp. \( (\partial_u, \lambda_2) \)) can be obtained by quadratures from (42) (resp. (43)).
2. The function \( M = M(x, u, u_x) \) defined by

\[
M = \frac{1}{f_1 g_1 f_2 g_2 (\lambda_2 - \lambda_1)} \tag{48}
\]

is a Jacobi last multiplier of the equation (1).

**Example 6.1.** By continuing the Examples [3.1–5.1] Theorem 6 implies that two integrating factors of equation (16) can be calculated as in (47) by using the functions given in (17), (18) and (39):

\[
\mu_1 = -\frac{2 u W_1}{(u^2 + u_x + 1) \psi_2 - 2 u \psi'_2} \quad \mu_2 = -\frac{2 u W_2}{(u^2 + u_x - 1) \theta_2 - 2 u \theta'_2} \tag{49}
\]
expressed in terms of the solutions of the corresponding linear equations (38). A Jacobi last multiplier of the equation (16) can be obtained by using (48):

\[ M = \frac{8uW_1W_2}{((u^2 + u_x + 1)\psi_2 - 2u\psi'_2)^2((u^2 + u_x - 1)\theta_2 - 2u\theta'_2)^2} \]

\[ \square \]

6.2. Integrating factors of the reduced equations associated to the \(C^\infty\)-symmetries

Theorem 5 provides a novel procedure that simultaneously uses two known generalized \(C^\infty\)-symmetries of (1) to construct, by quadratures, two independent first integrals, \(I_1\) and \(I_2\) of \(A\). In this section it is shown how any pair of functions \(g_1, g_2\) in \(C^\infty(M^{(1)})\) satisfying (31) can be used to provide integrating factors of the reduced equations associated to the given generalized \(C^\infty\)-symmetries.

Consider the system of coordinates \(\{x, w_1, w_2\}\) constructed in Proposition 1 and the reduced equations (26) associated to the given \(C^\infty\)-symmetries. Recall that a function \(\mu = \mu(x, u)\) is an integrating factor of a first-order ODE \(u_x = \varphi(x, u)\) if \(\mu_x + (\varphi\mu)u = 0\) [24, 25]; in other words, \(\mu\) is an integrating factor of that equation if

\[ A_0(\mu) = -\mu\varphi_u, \] (50)

where \(A_0 = \partial_x + \varphi\partial_u\) denotes the corresponding vector field. According to (32) and taking (27) into account, \(A_1(g_2) = g_2(\phi_1)w_1\) (resp. \(A_2(g_1) = g_1(\phi_2)w_2\)). Therefore

\[ A_1\left(\frac{1}{g_2}\right) = -\frac{g_2(\phi_1)w_1}{(g_2)^2} = -\frac{(\phi_1)w_1}{g_2}, \text{ and } A_2\left(\frac{1}{g_1}\right) = -\frac{(\phi_2)w_2}{g_1}. \]

This implies, by using (50), that the reciprocal of the functions \(g_1\) and \(g_2\) provide integrating factors of the reduced equations (26):

**Theorem 7.** Let \((\partial_x, \lambda_1)\) and \((\partial_u, \lambda_2)\) be the canonical representatives of two non-equivalent generalized \(C^\infty\)-symmetries of equation (1). Let \(\{x, w_1, w_2\}\) be the local system of coordinates of \(M^{(1)}\) given in Proposition 1 and let \(g_1 = g_1(x, w_2)\) and \(g_2 = g_2(x, w_1)\) be particular solutions of the corresponding equation in (32). Then

\[ v_1(x, w_1) = \frac{1}{g_2(x, w_1)} \text{ and } v_2(x, w_2) = \frac{1}{g_1(x, w_2)} \] (51)

are respectively integrating factors of the two reduced equations that appear in (26).
6.3. Integrating factors of the auxiliary equations associated to the \( \mathcal{C}^\infty \)–symmetries

Suppose that \( \hat{I}_1 = \hat{I}_1(x, w_1) \) and \( \hat{I}_2 = \hat{I}_2(x, w_2) \) denote the respective first integrals of the reduced equations \( \text{(26)} \) associated to the integrating factors \( \text{(51)} \). Then
\[
\hat{I}_i(x, w_i) = C_i, \quad (i = 1, 2) \tag{52}
\]
provide the general solutions of the respective reduced equations in \( \text{(26)} \), where \( C_1, C_2 \in \mathbb{R} \). The corresponding auxiliary equations are obtained by writing \( w_i \) in terms of \( (x, u, u_x) \) in \( \text{(52)} \):
\[
\hat{I}_i(x, w_i(x, u, u_x)) = C_i, \quad (i = 1, 2). \tag{53}
\]
Locally \( \text{(53)} \) can be written in the form
\[
u(x, w_i(x, u, C_i)) = 1 \quad (i = 1, 2).
\]

Now it is shown that some of the previous results can be used to obtain integrating factors of the equations in \( \text{(54)} \). To this end, recall that according to Corollary 6 in \( \text{(26)} \), any given Jacobi last multiplier of \( \text{(1)} \) provides an integrating factor of the auxiliary equation that appears in the reduction process associated to a given \( \lambda \)–symmetry. In the following theorem the Jacobi last multiplier \( \text{(48)} \) is used to find integrating factors of the auxiliary equations \( \text{(54)} \).

**Theorem 8.** Let \( (\partial_u, \lambda_1) \) and \( (\partial_u, \lambda_2) \) be the canonical representatives of two non-equivalent generalized \( \mathcal{C}^\infty \)–symmetries of equation \( \text{(1)} \). Let \( f_1, f_2 \) (resp. \( g_1, g_2 \)) be some functions in \( \mathcal{C}^\infty(M^{(1)}) \) satisfying \( \text{(11)} \) (resp. \( \text{(31)} \)). Then the function
\[
\tilde{\nu}(x, u) = \frac{1}{f_i(x, u, H_i(x, u, C_i)) g_i(x, u, H_i(x, u, C_i))}, \quad (i = 1, 2) \tag{55}
\]
is an integrating factor of the corresponding auxiliary equation \( \text{(54)} \).

**Proof.** A function \( g_1 \) (resp. \( g_2 \)) satisfying \( \text{(31)} \), written in terms of \( (x, w_2) \) (resp. \( (x, w_1) \)), is a solution of the first (resp. the second) equation in \( \text{(32)} \). By Theorem 7 the corresponding functions \( \text{(51)} \) are integrating factors of the respective equations in \( \text{(26)} \). The respective first integrals \( \hat{I}_1 = \hat{I}_1(x, w_1) \) and \( \hat{I}_2 = \hat{I}_2(x, w_2) \), satisfy
\[
(\hat{I}_1)_{w_1} = \frac{1}{g_2(x, w_2)}, \quad (\hat{I}_2)_{w_2} = \frac{1}{g_1(x, w_2)}. \tag{56}
\]
For \( i = 1, 2 \), let \( I_i = I_i(x, u, u_x) \) denote the function \( \hat{I}_i \) when \( w_i \) is expressed in terms of \( (x, u, u_x) \). Let \( M \) be the Jacobi last multiplier of equation (1) given in (48). According to Corollary 6 in [26], the restriction of the function

\[
\frac{M}{(I_1)_{u_x}} = \frac{1}{f_1 g_1 f_2 g_2 (\lambda_2 - \lambda_1) (I_1)_{u_x}}
\]

(57)
to \( \Delta_1 = \{(x, u, u_x) \in M^{(1)} : u_x = H_1(x, u, C_1)\} \) is an integrating function of the auxiliary equation \( u_x = H_1(x, u, C_1) \). By (24) and by taking (56) into account,

\[
(I_1)_{u_x} = -\frac{1}{f_2 g_2 (\lambda_1 - \lambda_2)}
\]

(58)
and (57) becomes

\[
\frac{M}{(I_1)_{u_x}} = \frac{1}{f_1 g_1}.
\]

(59)

Therefore the restriction of (59) to \( \Delta_1 \) gives us the integrating factor \( \tilde{\nu}_1 \) given in (55). A similar reasoning by using the Jacobi last multiplier \(-M\), where \( M \) is defined in (48), proves that the function \( \tilde{\nu}_2 \) given in (55) is an integrating factor of the auxiliary equation \( u_x = H_2(x, u, C_2), C_2 \in \mathbb{R} \).

Example 6.2. As a continuation of Example 6.1, by Theorem 7 the reciprocal of the functions (39), i.e. the functions

\[
\nu_1(x, w_1) = \frac{W_1}{(\psi_2 w_1 + \psi_1')^2} \quad \text{and} \quad \nu_2(x, w_2) = \frac{W_2}{(\theta_2 w_2 - \theta_1')^2},
\]

are integrating factors of the reduced equations (29). The associated first integrals become

\[
\hat{I}_1(x, w_1) = \frac{w_1 \psi_1 - \psi_1' w_1 \psi_2 - \psi_2'}{w_1 \psi_2 + \psi_2'} \quad \text{and} \quad \hat{I}_2(x, w_2) = \frac{w_2 \theta_1 - \theta_1' w_2 \theta_2 - \theta_2'}{w_2 \theta_2 - \theta_2'}.
\]

By using (28), the auxiliary equations (54) become:

\[
u_x = 2u \left( \frac{C_1 \psi_2 - \psi_1'}{C_1 \psi_2 - \psi_1} \right) - u^2 - 1 \quad \text{and} \quad u_x = 2u \left( \frac{C_2 \theta_2 - \theta_1'}{C_2 \theta_2 - \theta_1} \right) - u^2 + 1.
\]

(60)

By Theorem 8, the functions (55) constructed by using (18) and (39):

\[
\tilde{\nu}_1(x, u) = -\frac{C_1 \psi_2 - \psi_1'}{u^2 (C_1 \psi_2 - \psi_1')} \quad \text{and} \quad \tilde{\nu}_2(x, u) = \frac{C_2 \theta_2 - \theta_1}{u^2 (C_2 \theta_2 - \theta_1')}
\]

are integrating factors of the respective auxiliary equation in (60). The corresponding first integrals are the functions \( I_1 \) and \( I_2 \) derived in (49). \( \square \)
6.4. Scheme of the methods

Figure 1 presents a scheme of the alternative procedures that can be followed to integrate a second-order equation admitting two generalized \( C^\infty \)-symmetries. Their canonical representatives \((\partial_u, \lambda_1)\) and \((\partial_u, \lambda_2)\) will be used in the scheme.

1. In the central branches of the diagram, it is sketched the use of Theorem 5 to calculate two independent first integrals \( I_1, I_2 \) by quadratures through systems \((42)\) and \((43)\). The associated integrating factors are given in \((47)\).

2. The left branch of the figure solves the ODE by using the reduced and auxiliary equations associated to the first \( C^\infty \)-symmetry. The reduced equation can be solved by quadrature by using the integrating factor \( \nu_1 \) in Eq. \((51)\) (Theorem 7). The auxiliary equation can be solved by quadrature by using the integrating factor \( \tilde{\nu}_1 \) given in \((55)\) (Theorem 8). A similar process can be followed by using the second \( C^\infty \)-symmetry and the corresponding functions \( \nu_2 \) and \( \tilde{\nu}_2 \) (right branch of the scheme).

3. Finally, the functions \( \nu_1 \) and \( \nu_2 \) given in \((51)\) let the determination by quadrature of a first integral for each reduced equation (Theorem 8). The general solution of the ODE arises by following the dashed arrows in the diagram.
The lateral branches in Figure 1 correspond to the integration methods derived from the existence of two non-equivalent \( \mathcal{C}^\infty \)–symmetries [8]. The central branches of that figure summarize the main topic of this paper: a schematic description of the corresponding procedure will be shown in the next section.

### 7. Procedure for integration by quadratures and some examples

The following procedure describes the steps that can be followed to obtain the solution by quadratures, according to the method discussed in this paper.

1. Compute the function \( \rho = \frac{X_1(\lambda_2) - X_2(\lambda_1)}{\lambda_1 - \lambda_2} \), where \( X_i = \partial_u + \lambda_i \partial_{u_x} \), for \( i = 1, 2 \).
2. Calculate two functions \( f_1, f_2 \) such that \( \frac{X_1(f_2)}{f_2} = \frac{X_2(f_1)}{f_1} = \rho \).
3. Compute two functions \( w_1 = w_1(x, u, u_x) \) and \( w_2 = w_2(x, u, u_x) \), by solving by quadratures the systems

\[
(w_1)_u = \frac{\lambda_1}{f_2(\lambda_1 - \lambda_2)}, \quad (w_1)_{u_x} = -\frac{1}{f_2(\lambda_1 - \lambda_2)}.
\]
\[
(w_2)_u = \frac{\lambda_2}{f_1(\lambda_2 - \lambda_1)}, \quad (w_2)_{u_x} = -\frac{1}{f_1(\lambda_2 - \lambda_1)}.
\]

4. Calculate the function \( \phi_i(x, w_i) = \Lambda_i(w_i) \), for \( i = 1,2 \). Find a particular solution \( g_1 = g_1(x, w_2) \) (resp. \( g_2 = g_2(x, w_1) \)) to the first (resp. second) equation that follows:

\[
(g_1)_x + (g_1)_{u_x}\phi_2 = g_1(\phi_2)_{w_2} \quad \text{and} \quad (g_2)_x + (g_2)_{w_1}\phi_1 = g_2(\phi_1)_{w_1}.
\]

5. Use the functions \( f_1, f_2 \) of step 2 and the functions \( g_1, g_2 \) of step 4 (expressed in terms of \( \{x, u, u_x\} \)) to calculate two independent first integrals \( I_1 \) and \( I_2 \) by solving by quadratures the systems

\[
I_{1x} = \frac{\lambda_1 u_x - \phi}{f_2 g_2(\lambda_2 - \lambda_1)}, \quad I_{1u} = \frac{-\lambda_1}{f_2 g_2(\lambda_2 - \lambda_1)}, \quad I_{1u_x} = \frac{1}{f_2 g_2(\lambda_2 - \lambda_1)},
\]
\[
I_{2x} = \frac{\lambda_2 u_x - \phi}{f_1 g_1(\lambda_1 - \lambda_2)}, \quad I_{2u} = \frac{-\lambda_2}{f_1 g_1(\lambda_1 - \lambda_2)}, \quad I_{2u_x} = \frac{1}{f_1 g_1(\lambda_1 - \lambda_2)}.
\]

As an immediate consequence of the above-described procedure the following mathematical objects are obtained:

a) The general solution of the equation: \( I_1 = C_1, I_2 = C_2 \), where \( C_1, C_2 \in \mathbb{R} \).

b) The integrating factors of equation (1) given by

\[
\mu_1 = \frac{1}{f_2 g_2(\lambda_2 - \lambda_1)}, \quad \mu_2 = \frac{1}{f_1 g_1(\lambda_1 - \lambda_2)}.
\]

c) The Jacobi last multiplier of equation (1) defined by

\[
M = \frac{1}{f_1 g_1 f_2 g_2(\lambda_2 - \lambda_1)}.
\]

d) Two commuting generalized symmetries of equation (1):

\[
Z_i = (f_i g_i)\partial_u + A_i(f_i g_i)\partial_{u_x}, \quad (i = 1,2).
\]
Example 7.1. Although the general solution (45) of any equation of the form (16) has already been obtained through Examples 3.1–6.2, in this example it is considered the special case

\[ F(x) = 0, \]

for purposes of illustration of the procedure. In this case the computations appointed by the former procedure can be followed without the aid of a computer; they are relatively simple.

The three first steps of the method proceed as explained in Examples 3.1, 3.2 and 4.1 because the \( \lambda \)-symmetries defined by (17) do not depend on \( F(x) \):

Step 1 \( \rho(x, u, u_x) = \frac{2}{u} \).

Step 2 \( f_1(x, u, u_x) = f_2(x, u, u_x) = u^2 \).

Step 3 \( w_1 = -\frac{u_x + u^2 + 1}{2u} \) and \( w_2 = \frac{u_x + u^2 - 1}{2u} \).

Step 4 Since \( \phi_1 = A(w_1) = w_1^2 - \frac{1}{2} \) and \( \phi_2 = A(w_2) = -w_2^2 - \frac{1}{2} \), two particular solutions of

\[ (g_1)_x - \left( w_2^2 + \frac{1}{2} \right) (g_1)w_2 = -2w_2g_1 \quad \text{and} \quad (g_2)_x + \left( w_1^2 - \frac{1}{2} \right) (g_2)w_1 = 2w_1g_2 \]

can be easily found: \( g_1(x, w_2) = 2w_2^2 + 1 \) and \( g_2(x, w_1) = 2w_1^2 - 1 \).

Step 5 By using the functions \( f_1, f_2 \) of step 2 and the functions \( g_1, g_2 \) of step 4, written in variables \( (x, u, u_x) \) (by means of the expressions of \( w_1 \) and \( w_2 \) given in step 3), the systems that correspond to (42) and (43) are

\begin{align*}
I_{1,x} &= -\frac{1}{2}, & I_{1,u} &= -\frac{u^2 - u_x - 1}{(u^2 + u_x + 1)^2 - 2u^2}, & I_{1,u_x} &= -\frac{u}{(u^2 + u_x + 1)^2 - 2u^2}, \\
I_{2,x} &= \frac{1}{2}, & I_{2,u} &= \frac{u^2 - u_x + 1}{(u^2 + u_x - 1)^2 + 2u^2}, & I_{2,u_x} &= \frac{u}{(u^2 + u_x - 1)^2 + 2u^2}.
\end{align*}

These systems can be solved by quadratures and the solutions provide two independent first integrals for equation (61):

\begin{align*}
I_1 &= -\frac{1}{2} \left( x - \sqrt{2} \arctanh \left( \frac{u^2 + u_x + 1}{\sqrt{2}u} \right) \right), \\
I_2 &= \frac{1}{2} \left( x + \sqrt{2} \arctan \left( \frac{u^2 + u_x - 1}{\sqrt{2}u} \right) \right).
\end{align*}
Some consequences of the procedure are:

a) From $I_1 = C_1, I_2 = C_2$, where $C_1, C_2 \in \mathbb{R}$, the general solution of (61) can be (locally) written in explicit form as follows:

$$u(x) = \frac{\sqrt{2}}{\tanh\left(\frac{\sqrt{2}}{2} (x + 2C_1)\right) - \tan\left(\frac{\sqrt{2}}{2} (-x + 2C_2)\right)}, \quad C_1, C_2 \in \mathbb{R}. \quad (62)$$

b) Two integrating factors for (61) are:

$$\mu_1 = -\frac{u}{(u^2 + u_x + 1)^2 - 2u^2}, \quad \mu_2 = \frac{u}{(u^2 + u_x - 1)^2 + 2u^2}.$$

c) A Jacobi last multiplier for (61) is:

$$M = \frac{-2u}{(u^2 + u_x - 1)^2 + 2u^2}\left((u^2 + u_x + 1)^2 - 2u^2\right).$$

d) Two commuting generalized symmetries for (61) are:

$$Z_1 = \frac{1}{2}\left((u^2 + u_x - 1)^2 + u^2\right)\left(\partial_u + \frac{u_x - u^2 + 1}{u}\partial_u\right)$$

and

$$Z_2 = \frac{1}{2}\left((u^2 + u_x + 1)^2 - u^2\right)\left(\partial_u + \frac{u_x - u^2 - 1}{u}\partial_u\right).$$

Observe that equation (61) admits the Lie point symmetry $\partial_x$ and that this is the unique Lie point symmetry admitted by the equation. The classical Lie method of reduction provides the Abel equation

$$w_y = \frac{\left(\frac{u}{w^3}\right)^3 + 1}{2y} + 2w^2y - \frac{w}{2y}, \quad (63)$$

where $y = u$ and $w = 1/u_x$. The general solution of (63) can be expressed in the form $\Delta(y, w(y), C_1) = 0$, where $\Delta$ involves generalized hypergeometric functions. The general solution of (61) arises by solving the corresponding first-order equation $\Delta(u, 1/u_x, C_1) = 0$. Due to the involved expression of $\Delta$, to perform the quadrature needed for giving such explicit solution does not seem an easy task. Nevertheless, the presented procedure leads to the very compact form (62) for such general solution.
**Example 7.2.** The expressions obtained in Examples 3.1, 4.1 and 5.1 can be used to integrate any equation of the form (16), even if it does not admit Lie point symmetries. This is the case, for instance, of the equation

\[ u_{xx} - \frac{1}{2} u u_x^2 + 2 u u_x + \frac{1}{2} u^3 - (2 x + 1) u + \frac{1}{2} u = 0 \]  

which corresponds to \( F(x) = 2 x + 1 \).

For this example, the second-order linear equations (38) are the Airy equations

\[ \psi_{xx} = (1 + x) \psi, \quad \theta_{xx} = x \theta, \]  

(65)

Let \( \psi_1(x) = \text{Ai}(1 + x) \), \( \psi_2(x) = \text{Bi}(1 + x) \) and \( \theta_1(x) = \text{Ai}(x), \theta_2(x) = \text{Bi}(x) \) denote linearly independent solutions of the respective Airy equations (65).

Two independent first integrals of the equation (64) corresponding to the first integrals (44) become:

\[ I_1 = \frac{-2 \text{Ai}^{(1)}(x + 1) u + \text{Ai}(x + 1) \left( u^2 + u_x + 1 \right)}{-2 \text{Bi}^{(1)}(x + 1) u + \text{Bi}(x + 1) \left( u^2 + u_x + 1 \right)} \]

and

\[ I_2 = \frac{-2 \text{Ai}^{(1)}(x) u + \left( u^2 + u_x - 1 \right) \text{Ai}(x)}{-2 \text{Bi}^{(1)}(x) u + \left( u^2 + u_x - 1 \right) \text{Bi}(x)}. \]

These first integrals provide the general solution of equation (64) in terms of Airy functions:

\[ u(x) = \left( \frac{C_1 \text{Bi}^{(1)}(x + 1) - \text{Ai}^{(1)}(x + 1)}{C_1 \text{Bi}(x + 1) - \text{Ai}(x + 1)} - \frac{C_2 \text{Bi}^{(1)}(x) - \text{Ai}(x)}{C_2 \text{Bi}(x) - \text{Ai}(x)} \right)^{-1}, \]

where \( C_1, C_2 \in \mathbb{R} \).

Finally, observe that although equation (64) does not admit Lie point symmetries, Theorem 4 can be used to construct the following generalized symmetries for equation (64)

\[ Z_1 = \left( -2 \text{Bi}^{(1)}(x + 1) u + \text{Bi}(x + 1) \left( u^2 + u_x + 1 \right) \right)^2 \frac{4 \text{Ai}(x + 1) \text{Bi}^{(1)}(x + 1) - 4 \text{Bi}(x + 1) \text{Ai}^{(1)}(x + 1)}{4 \text{Ai}(x) \text{Bi}^{(1)}(x) - 4 \text{Ai}^{(1)}(x) \text{Bi}(x)} \left( \partial_u + \frac{u x - u^2 + 1}{u} \partial_{u_x} \right) \]

and

\[ Z_2 = \left( -2 \text{Bi}^{(1)}(x) u + \left( u^2 + u_x - 1 \right) \text{Bi}(x) \right)^2 \frac{4 \text{Bi}^{(1)}(x) \text{Ai}(x) - 4 \text{Ai}^{(1)}(x) \text{Bi}(x)}{4 \text{Bi}^{(1)}(x) \text{Ai}(x) - 4 \text{Ai}^{(1)}(x) \text{Bi}(x)} \left( \partial_u + \frac{u x - u^2 - 1}{u} \partial_{u_x} \right). \]
Example 7.3. In the application of the procedure some difficulties can appear; in this example it is shown how the choosing of an appropriate route in the diagram of Figure 1 may help to overcome these difficulties. The equation

$$u_{xx} + \frac{u_x}{u} + \frac{1}{u} + u = 0$$  \hspace{1cm} (66)

does only admit the Lie point symmetry $v = \partial_x$. This Lie point symmetry leads to the reduced equation

$$w_y = \left(\frac{y^2 + 1}{y}\right) \frac{w^3}{y} + \frac{w^2}{y},$$  \hspace{1cm} (67)

where $y = u$ and $w = 1/u_x$. Equation (67) is an Abel equation whose solution can be given in implicit form. By substituting $y = u$ and $w = 1/u_x$ into this general solution, the first-order equation

$$\sqrt{u^2 + (u_x + 1)^2} - \text{arctanh} \left(\frac{u_x + 1}{\sqrt{u^2 + (u_x + 1)^2}}\right) = C,$$  \hspace{1cm} (68)

where $C \in \mathbb{R}$, is obtained. If (68) is expressed in the form $u_x = H(u, C)$, then the general solution of (66) becomes

$$\int \frac{du}{H(u, C)} = x + K, \quad K \in \mathbb{R}.$$  

The obtained expression for this general solution requires a primitive of the function $1/H$. Since $u_x$ cannot be isolated from (68), to obtain the general solution of (66) in a closed form seems an impossible task.

The canonical representative of the equivalence class of the pair $(\partial_x, 0)$ is the $\lambda-$symmetry $(\partial_u, \lambda_1)$, for

$$\lambda_1 = \frac{A(Q)}{Q} = -\left(\frac{1}{u} + \frac{u^2 + 1}{u_x u}\right),$$

where $Q = -u_x$ is the characteristic of $\partial_x$ and $A$ denotes the vector field associated to (66). This function $\lambda_1$ solves the determining equation (5) for equation (66). If another particular solution is searched for such determining equation, it is not difficult to find a solution which is linear in $u_x$:

$$\lambda_2 = \frac{u_x + 1}{u}.$$  

Once two $\lambda-$symmetries are known, the algorithm can be started:
1. Construct the vector fields (6):

\[ X_1 = \partial_u - \left( \frac{1}{u} + \frac{u^2 + 1}{ux u} \right) \partial_{ux} \quad \text{and} \quad X_2 = \partial_u + \frac{ux + 1}{u} \partial_{ux} \]

and the function (8): \( \rho = \frac{ux + 1}{ux u} \).

2. A function \( f_1 \) satisfying (11) can be easily determined: \( f_1 = u_x \). However, the determination of a function \( f_2 \) such that \( X_1(f_2) = \rho f_2 \) seems quite complicated. The explicit expression of \( f_2 \) will be skipped, for a moment.

3. Since \( f_1 \) is known, system (23) can be constructed and integrated by quadrature to obtain the solution \( w_2 = \arctan \left( \frac{u}{1 + ux} \right) \).

The computation of function \( w_1 \) cannot be obtained by quadrature from system (24) without the expression of \( f_2 \). Nevertheless, observe that the left hand side of (68) defines a function \( \tilde{w}_1 \) that is a first integral of \( A \) and \( \partial_x \). Since \((\partial_x,0) \sim (\partial_u,\lambda_1)\), then \( \tilde{w}_1 \) is an invariant for \( X_1 \). In coordinates \( \{x, \tilde{w}_1, w_2\} \) the equation \( X_1(f_2) = \rho f_2 \) is simple and \( f_2(x, \tilde{w}_1, w_2) = \sin(w_2) \) is a particular solution. In the original variables such function becomes

\[ f_2(x, u, u_x) = \frac{u}{\sqrt{u^2 + (ux + 1)^2}}. \]

Now system (23) is known and can be integrated by quadrature to obtain the solution

\[ w_1 = \sqrt{u^2 + (ux + 1)^2} - \ln \left| \frac{\sqrt{u^2 + (ux + 1)^2 + ux + 1}}{u} \right|. \]

4. The functions \( \phi_1(x, w_2) = A(w_2) = 1 \) and \( \phi_1(x, u_1) = A(w_1) = 0 \) provide the first-order PDEs

\[ (g_1)_x + (g_1)_{w_2} = 0 \quad \text{and} \quad (g_2)_x = 0. \]

The particular solutions \( g_1(x, w_2) = 1 \) and \( g_2(x, w_1) = 1 \) arise immediately.

5. The functions \( f_1, f_2 \) of step 2 and 3 and the functions \( g_1, g_2 \) of step 4 (expressed in terms of \( \{x, u, u_x\} \)) is all it is needed to calculate two independent first integrals \( I_1 \) and \( I_2 \) by solving by quadratures systems (42)-(43):

\[ I_1 = \sqrt{u^2 + (ux + 1)^2} - \arctanh \left( \frac{ux + 1}{\sqrt{u^2 + (ux + 1)^2}} \right), \]

\[ I_2 = x - \arctan \left( \frac{u}{ux + 1} \right). \]
By using these first integrals the general solution of equation (66) can be locally given in explicit form:

\[ u(x) = \sin(C_2 - x) \left( C_1 - \arctanh(\cos(C_2 - x)) \right), \quad C_1, C_2 \in \mathbb{R}. \] (69)

As a remark, expression (69) can be used to derive the solution in parametric form of the Abel equation (67):

\[ y = \sin(C_2 - x) \left( C_1 - \arctanh(\cos(C_2 - x)) \right), \]

\[ w = \frac{1}{\cos(C_2 - x) \left( \arctanh(\cos(C_2 - x)) - C_1 \right) - 1}. \]

8. Concluding remarks

If two non-equivalent \( \mathcal{C}^\infty \)-symmetries for a given second-order ODE are known, then the independent use of these \( \mathcal{C}^\infty \)-symmetries gives two reduced equations whose integration, not necessarily by quadrature, can provide the general solution of the ODE. In this paper we provide a new method to obtain by quadratures two independent first integrals of the ODE by the combined use of both \( \mathcal{C}^\infty \)-symmetries.

This is done by constructing a system of two commuting generalized symmetries of the ODE. From these generalized symmetries two independent first integrals of the equation arise by quadratures.

A comparative study between this new method of integration of the ODE and the reduction methods associated to the given \( \mathcal{C}^\infty \)-symmetries is also provided. Some relationships between the functions involved in the new method and integrating factors of the reduced and auxiliary equations associated to the \( \mathcal{C}^\infty \)-symmetries have been established. Such functions and the \( \mathcal{C}^\infty \)-symmetries provide an explicit formula for a Jacobi last multiplier that generalizes (for \( n = 2 \)) the classical expression in terms of Lie point symmetries.

The results have been illustrated with a family of equations of the XXVII case in the Painlevé-Gambier classification. By using two \( \lambda \)-symmetries and two fundamental sets of solutions of two second-order linear ODEs, the equations can be integrated by quadratures. Explicit expressions for the first integrals, integrating factors, a Jacobi last multiplier and the general solution have also been provided as by-products of the quadrature process.
The method has been successfully applied to integrate by quadratures 2nd-order ODEs lacking Lie point symmetries or admitting just one Lie point symmetry.

The results presented in this paper could provide novel methods to find exact solutions of nonlinear ordinary and partial differential equations as well as establish new connections between analytical methods which are widely used in the contemporary literature [27][28][29].

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References

[1] P. Olver, Applications of Lie groups to differential equations, 2nd ed., Springer-Verlag, New York, 1993.

[2] N. H. Ibragimov, A practical course in differential equations and mathematical modelling, World Scientific, Beijing, 2010.

[3] H. Stephani, Differential equations: their solution using symmetries, Cambridge University Press, New York, 1989.

[4] A. González-López, Symmetry and integrability by quadratures of ordinary differential equations, Phys. Lett. A 133 (1988) 190–194. doi:10.1016/0375-9601(88)91015-8

[5] F. González-Gascón, A. González-López, Newtonian systems of differential equations, integrable via quadratures, with trivial group of point symmetries, Phys. Lett. A 129 (1988) 153–156. doi:10.1016/0375-9601(88)90134-X

[6] K. S. Govinder, P. G. L. Leach, A group-theoretic approach to a class of second-order ordinary differential equations not possessing Lie point symmetries, J. Phys. A: Math. Gen. 30 (1997) 2055–2068. doi:10.1088/0305-4470/30/6/026
[7] M. C. Nucci, Lie symmetries of a Painlevé-type equation without Lie symmetries, J. Nonlinear Math. Phys. 15 (2) (2008) 201–211. doi:10.2991/jnmp.2008.15.2.7.

[8] C. Muriel, J. L. Romero, New methods of reduction for ordinary differential equations, IMA J. Appl. Math. 66 (2001) 111–125. doi:10.1093/imamat/66.2.111.

[9] C. Muriel, J. L. Romero, $C^\infty$-Symmetries and reduction of equations without Lie point symmetries, J. Lie Theory 13 (2003) 167–188.

[10] G. Gaeta, Twisted symmetries of differential equations, J. Nonlinear Math. Phys. 16 (2009) 107–136. doi:10.1142/S1402925109000352.

[11] D. C. Ferraioli, P. Morando, Local and nonlocal solvable structures in the reduction of ODEs, J. Phys. A: Math. Theor. 42 (2009) 035210. doi:10.1088/1751-8113/42/3/035210.

[12] D. C. Ferraioli, P. Morando, Applications of solvable structures to the nonlocal symmetry-reduction of ODEs, J. Nonlinear Math. Phys. 16 (2009) 27–42. doi:10.1142/S1402925109000303.

[13] G. Gaeta, P. Morando, On the geometry of lambda-symmetries and PDE reduction, J. Phys. A: Math. Gener. 37 (27) (2004) 6955. doi:10.1088/0305-4470/37/27/007.

[14] C. Muriel, J. L. Romero, First integrals, integrating factors and $\lambda$-symmetries of second-order differential equations, J. Phys. A: Math. Theor. 42 (2009) 365207, 17. doi:10.1088/1751-8113/42/36/365207.

[15] E. Whittaker, W. McCrae, A Treatise on the Analytical Dynamics of Particles and Rigid Bodies, Cambridge Mathematical Library, Cambridge University Press, 1988.

[16] M. Nucci, Jacobi last multiplier and Lie symmetries: a novel application of an old relationship, J. Nonlinear Math. Phys. 12 (2005) 284–304. doi:10.2991/jnmp.2005.12.2.9.

[17] M. C. Nucci, P. G. L. Leach, An old method of Jacobi to find Lagrangians, J. Nonlinear Math. Phys. 16 (2009) 431–441. doi:10.1142/S1402925109000467.
[18] M. C. Nucci, P. G. L. Leach, Jacobi’s last multiplier and the complete symmetry group of the Euler-Poinsot system, J. Nonlinear Math. Phys. 9 (2002) 110–121. doi:10.2991/jnmp.2002.9.s2.10

[19] M. C. Nucci, P. G. L. Leach, Jacobi’s last multiplier and the complete symmetry group of the Ermakov-Pinney equation, J. Nonlinear Math. Phys. 12 (2005) 305–320. doi:10.2991/jnmp.2005.12.2.10

[20] E. Ince, Ordinary Differential Equations, Dover Books on Science, Dover Publications, New York, 1956.

[21] P. Guha, A. Choudhury, B. Khanra, $\lambda$–Symmetries, isochronicity and integrating factors of nonlinear ordinary differential equations, J. Eng. Math. 82 (2013) 85–99. doi:10.1007/s10665-012-9614-5

[22] C. Muriel, J. L. Romero, $\lambda$-Symmetries on the derivation of first integrals of ordinary differential equations, World Sci. Publ., Hackensack, NJ, 2010, pp. 303–308. doi:10.1142/9789814317429-0041

[23] C. Muriel, J. L. Romero, Integrating factors and $\lambda$-symmetries, J. Nonlinear Math. Phys. 15 (2008) 300–309. doi:10.2991/jnmp.2008.15.s3.29.

[24] G. W. Bluman, S. C. Anco, Symmetry and Integration Methods for Differential Equations, Springer-Verlag, New York, 2002.

[25] S. C. Anco, G. Bluman, Integrating factors and first integrals for ordinary differential equations, Euro. J. Appl. Math. 9 (1998) 245–259.

[26] C. Muriel, J. Romero, The $\lambda$-symmetry reduction method and Jacobi last multipliers, Commun. Nonlinear Sci. Numer. Simul. 19 (2014) 807–820. doi:10.1016/j.cnsns.2013.07.027

[27] R. Mohanasubha, V. K. Chandrasekar, M. Senthilvelan, M, Lakshmanan, Interplay of symmetries, null forms, Darboux polynomials, integrating factors and Jacobi multipliers in integrable second-order differential equations, P. Roy. Soc. A470 (2014) 20130656. doi:10.1098/rspa.2013.0656.

[28] R. Mohanasubha, V. K. Chandrasekar, M. Senthilvelan, M, Lakshmanan, On certain analytical methods in finding integrable systems and their interconnections, arXiv:1502.03914 (2015).
[29] A. Bhuvaneswari, R. A. Kraenkel, M. Senthilvelan, Application of the $\lambda$-symmetries approach and time independent integral of the modified Emden equation, Nonlinear Analysis: Real World Applications 3 (2012) 1102–1114. doi:10.1016/j.nonrwa.2011.08.030.