Fields in the Vicinity of a Superconducting Cosmic String

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Abstract

Superconducting cosmic strings may be viewed as wires of thickness 1/Λ with Λ = 10^{16} TeV. We show that the weak interactions will spread out the current to distances \( r = (1/M_Z) \ln(I/M_Z) \), where \( I \) is the magnitude of the current in the string. Consequences for the scattering of light by these strings is presented.

Symmetry breaking at scales of Λ = 10^{16} TeV, or higher, may induce strings that behave as superconducting wires carrying currents of the order of \( I = 10^{20} \) A. The radius of these “wires” is governed by the masses of the Higgs particles responsible for the symmetry breaking and will be of the order of 1/Λ. It has been noted that, due to other interactions, various instabilities will develop at larger radii. Hadronic chiral symmetry breaking will screen fields for \( r \leq I/f_{\pi}m_{\pi} \); \( r \) is the distance from the wire. At distances of the order of \( r = I/M_{W}^{2} \) the anomalous magnetic moment of the \( W \) boson induces a condensation of

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the fields corresponding to this particle. Although the screening discussed in Ref. [2] extends to the largest distances considered so far, the mechanism is suspect in that it relies on a low energy effective model rather than a fundamental theory. In this work we shall show that at even smaller distances, \( r \leq (1/M_Z) \ln(I/M_Z) \), the magnetic field is partially screened by the weak interactions. The definition of an electric current in the presence of a non Abelian gauge theory with the electromagnetic field being one of the adjoint fields is problematic as a local gauge transformation can rotate the current to an other direction in group space. We shall return to this point shortly.

As in the SU(2) \times U(1) theory of the weak interactions the electromagnetic field is a combination of an SU(2) and a U(1) field we shall, for pedagogical reasons, start with the Georgi-Glashow O(3) model [3] and then return to the full Weinberg-Salam theory. The fields of the O(3) model are \( W^{(1)}_\mu \), \( W^{(2)}_\mu \) and \( A_\mu = W^{(3)}_\mu \). In addition there is a triplet of Higgs fields \( \phi^{(i)} \), \( i = 1, 2, 3 \). Again, for simplicity we shall work in the nonlinear limit where \( \phi^2 \) is a constant and scaled to equal one. The Lagrangian for these fields coupled to an external electric current \( j_\mu \) is

\[
L = -\frac{1}{4} F^{(i)}_{\mu\nu} F^{(i)\mu\nu} + \frac{v^2}{2} (\partial_\mu \phi^{(i)} + ge^{ijk} W^{(j)}_\mu \phi^{(k)})(\partial^\mu \phi^{(i)} + ge^{ijk} W^{(j)}_\mu \phi^{(k)}) + j_\mu A_\mu; \tag{1}
\]

\( g \) is the coupling constant and \( v \) is the vacuum expectation value of the Higgs field. The current \( j \) will be taken as that due to a wire at the origin and extending along the z direction, \( j_z = g I \delta(r) \), with \( r \) the spatial vector transverse to the current direction. We may perform a gauge rotation, so that the Higgs field points everywhere in the 3 direction,

\[
L = -\frac{1}{4} F^{(i)}_{\mu\nu} F^{(i)\mu\nu} + \frac{M^2}{2} \left(W^{(1)}_\mu W^{(1)}_\mu + W^{(2)}_\mu W^{(2)}_\mu\right) - g I \delta(r) \left[ O^3 \phi^{(i)} + \frac{1}{g} e^{ijk} \partial_z O^j O^{3k} \right]; \tag{2}
\]

\( M = gv \) is the mass of the charged vector mesons.

As we shall be interested in the static energies of configurations that are solutions of the equations of motion derived from Eq. (2), it is the Hamiltonian we need;

\[
\mathcal{H} = \int d^2r \frac{1}{4} \left[ \partial_\mu A_\mu - \partial_\mu A_\mu + g(W^{(1)}_a W^{(2)}_b - W^{(1)}_b W^{(2)}_a) \right]^2
\]
\[
+ \frac{1}{4} \left[ \partial_a W^{(i)}_b - \partial_b W^{(i)}_a + g e^{ij} (W^{(j)}_a A_b - W^{(j)}_b A_a) \right]^2 \\
+ g I \delta(r) \left[ O^{3i} W^{(i)}_z + \frac{1}{g} e^{ijk} \partial_z O^{3j} O^{3k} \right]; \quad (3)
\]

\(a, b\) are the spatial directions and \(e^{ij}\) is the two dimensional Levi-Civita symbol. We wish to minimize the energy with respect to the variables \(A_a, W^{(i)}_a, i = 1, 2\) and \(O^{ij}\). An obvious candidate is the solution corresponding to classical electromagnetism,

\[
A_z = \frac{g I}{2\pi} \ln \frac{r}{r_0}, \quad (4)
\]

and all other fields set equal to zero; \(r_0\) is a gauge parameter which may be set equal to the radius of the wire. The energy per unit length along the wire that is contained inside a cylinder of radius \(R\) is

\[
\mathcal{E} = \frac{g^2 I^2}{4\pi} \ln \frac{R}{r_0}. \quad (5)
\]

A configuration whose energy does not have the logarithmic dependence on \(R\) can be obtained by rotating the current into one of the massive directions, say we let it couple to \(W^{(1)}_z\); in this case only this field is excited and we find

\[
W^{(1)}_z = -\frac{I}{2\pi} K_0(Mr), \quad (6)
\]

and the corresponding energy per unit length is

\[
\mathcal{E} = -\frac{g^2 I^2}{4\pi} K_0(Mr_0), \quad (7)
\]

which is certainly lower than that of Eq. (3). This solution is however unacceptable as it is not what we would interpret as the fields resulting from an electric current.

We take as the definition of a configuration resulting from an electric current along the \(z\)-axis to be one where the magnetic field satisfies Ampère’s law at large distances from the wire

\[
\oint B \cdot dl = \frac{g I}{2\pi}, \quad (8)
\]
with the integral taken around a contour far from the wire. Certainly, the configuration described by Eq. (4) satisfies this criterion while that of Eq. (3) does not. There is however a configuration that does satisfy Eq. (8) and has a lower energy than that of Eq. (5).

As for the case of Eq. (6) we take the rotation to be such as to couple the current to $W_z(1)$ and look for configurations satisfying Eq. (8). We also find that only $A_z$, $W_z(1)$ and $W_{x,y}^{(2)}$ are excited. In order to simplify notation, we introduce

$$A_z = A,$$

$$W_z^{(1)} = W,$$

$$W_{x,y}^{(2)} = V_{x,y}.$$

In terms of these variables the Hamiltonian is

$$H = \int d^2r \left[ \frac{1}{2} (\partial_\rho A + gWV_a)^2 + \frac{1}{2} (\partial_\rho W - gAV_a)^2 + \frac{1}{2} (\partial_\rho V_b - \partial_\rho V_a)^2 + \frac{M^2}{2} (W^2 + V_a^2) + gWI\delta(r) \right].$$

As all these fields will depend only on the radial distance $r$ and the field $V_a$ will point in the radial direction, the term involving the derivatives of this field will vanish. Defining new scaled variables

$$\rho = g^2 Ir, \quad \mu = \frac{M}{g^2 I}, \quad a = \frac{A}{gI}, \quad w = \frac{W}{gI}, \quad v = \frac{V_r}{gI}.$$  \hspace{1cm} (11)

The energy is

$$H = g^2 I^2 \int d^2 \rho \left[ \frac{1}{2} (\partial_\rho a + wv)^2 + \frac{1}{2} (\partial_\rho w - av)^2 + \frac{\mu^2}{2} (w^2 + v^2) + w\delta(\rho) \right].$$  \hspace{1cm} (12)

As no kinetic energy terms appear for $v$ its equation of motion may be solved

$$v = \frac{w\partial_\rho a - a\partial_\rho w}{\mu^2 + w^2 + a^2}$$  \hspace{1cm} (13)

resulting in

$$H = g^2 I^2 \int d^2 \rho \left[ \frac{1}{2} (\partial_\rho a)^2 + \frac{1}{2} (\partial_\rho w)^2 + \frac{\mu^2}{2} w^2 - \frac{1}{2} \frac{(w\partial_\rho a - a\partial_\rho w)^2}{\mu^2 + w^2 + a^2} + w\delta(\rho) \right].$$  \hspace{1cm} (14)
The equations obtained from this Hamiltonian are far too non-linear in order to be able to obtain analytic solutions. We shall, instead, use the variational principle and show a solution with the property demanded by Eq. (8) and whose energy is lower than that of Eq. (5).

\[ a = \begin{cases} 
0 & \text{for } \rho < \rho_1 \\
\frac{1}{2\pi} \ln(\rho/\rho_1) & \text{for } \rho > \rho_1 
\end{cases} \]

\[ w = -\frac{1}{2\pi} K_0(\mu\rho) ; \]

\( \rho_1 \) is a parameter to be determined by minimizing the energy. The difference in the energies \( \delta(E) \) of the purely electromagnetic case, Eq. (5) and the one due to the above configuration is (for \( \mu\rho_1 > 1 \))

\[ \delta(E) = \frac{1}{4\pi} \ln(\mu\rho_1) + \pi \int_{\rho_1} \rho d\rho \frac{(w\partial_\rho a - a\partial_\rho w)^2}{\mu^2 + w^2 + a^2} . \]

\( \delta(E) \) is positive for all \( \rho_1 \)'s and thus we have found configurations satisfying Ampère’s law and whose energies are lower than that of the purely electromagnetic case.

It remains to find the optimal value of \( \rho_1 \). We are interested in large currents, therefore, for fixed vector meson mass \( M \) this means small \( \mu \). The full minimization of Eq. (16) cannot be done in an analytic form but an asymptotic one, valid for small \( \mu \), can be obtained

\[ \mu\rho_1 = -\ln(\mu) - \frac{1}{2} \ln \left[ -\frac{2\ln(\mu)}{\pi} \right] . \]

A comparison with a numerical minimization of Eq. (16) shows that this solutions valid for \( \mu \leq 0.5 \). We thus find that the magnetic field is screened for distances smaller than \( r_1 \) with

\[ r_1 = \frac{1}{M} F \left( \frac{g^2 I}{M} \right) . \]

For large currents \( F = \ln(g^2 I/M) \). This is the result presented at the beginning of this work.

In the full Weinberg-Salam model the electromagnetic field is a combination of the hypercharge field \( B \) and the third component of the weak isospin triplet \( W^{(3)} \),
\[ \mathcal{L} = \cdots + e j (\cos \theta_W B + \sin \theta_W W^{(3)}) . \] (19)

It is only the isospin part that can be rotated by an \( SU(2) \) transformation into a massive direction,

\[ \mathcal{L} = \cdots + e j \left[ \cos \theta_W B + \sin \theta_W (\cos \frac{\beta}{2} W^{(3)} + \sin \frac{\beta}{2} W^{(1)}) \right] . \] (20)

Re expressed in terms of the mass eigenstates \( A \) and \( Z \) this becomes

\[ \mathcal{L} = \cdots + e j \left\{ \sin \theta_W \sin \frac{\beta}{2} W^{(1)} \left[ (\cos \theta_W)^2 + (\sin \theta_W)^2 \cos \frac{\beta}{2} \right] A \\
+ \cos \theta_W \sin \theta_W (1 - \cos \frac{\beta}{2}) Z \right\} . \] (21)

The lowest energy configuration is obtained when the coefficient of the electromagnetic field is as small as possible. This is achieved for \( \beta = 2\pi \), which also eliminates the \( W^{(1)} \) term. Thus out to a distance

\[ r_1 = \frac{1}{M_Z} \ln \frac{I}{M_Z} \] (22)

\( \cos 2\theta_W \) of the electromagnetic field will penetrate and the \( Z \) field will appear with a coupling to the current of \( e \sin 2\theta_W \). Beyond \( r_1 \) the full electromagnetic field will be present.

These results have a consequence on the magnitude of the scattering cross section of light by cosmic strings. In Ref. [1] the cross section per unit length of string at a frequency \( \omega \) is

\[ \frac{d\sigma}{dz} = \frac{\pi}{2\omega \ln(\Lambda/\omega)} . \] (23)

This will be modified for the portion of the electromagnetic field that is screened.

\[ \frac{d\sigma}{dz} = \frac{\pi}{2\omega} \left[ \frac{\cos^2 2\theta_W}{\ln(\Lambda/\omega)} + \frac{1 - \cos^2 2\theta_W}{\ln(1/r_1 \omega)} \right] . \] (24)

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