MILNOR OPERATIONS AND CLASSIFYING SPACES

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ABSTRACT. We give an example of a nonzero odd degree element of the classifying space of a connected Lie group such that all higher Milnor operations vanish on it. It is a counterexample of a conjecture of Kono and Yagita.

1. INTRODUCTION

For each prime number $p$, there are the mod $p$ and Brown-Peterson cohomology. For a compact connected Lie group $G$, the mod $p$ cohomology of the classifying space $BG$ has no nonzero odd degree element if the integral cohomology of $G$ has no $p$-torsion. So does the Brown-Peterson cohomology. On the one hand, if the integral homology of $G$ has $p$-torsion, the mod $p$ cohomology of $BG$ has a nonzero odd degree element. On the other hand, for the Brown-Peterson cohomology, Kono and Yagita conjectured the following:

**Conjecture 1.1** (Kono and Yagita, (1) in Conjecture 4 in [KY93]). There is no nonzero odd degree element in the Brown-Peterson cohomology of the classifying space of a compact Lie group.

Conjecture 1.1 is interesting in conjunction with Totaro’s conjecture on the cycle map from the Chow ring of the classifying space of a complex linear algebraic group $G$ to its Brown-Peterson cohomology. In [Tot97], Totaro showed that the cycle map from the Chow ring of a complex smooth algebraic variety to its ordinary cohomology factors through the Brown-Peterson cohomology after localized at $p$. In [Tot99], he defined the Chow ring $CH^*(BG)$ of a linear algebraic group $G$ and conjectured the following.

**Conjecture 1.2** (Totaro, p.250 in [Tot99]). For a complex linear algebraic group $G$, if there is no nonzero odd degree element in the Brown-Peterson cohomology $BP^*(BG)$, the cycle map

$$CH^i(BG) \rightarrow (\mathbb{Z}(p) \otimes_{BP^*} BP^*(BG))^{2i}$$

is an isomorphism.

With Conjectures 1.1 and 1.2 we expect a close connection between the Chow ring in algebraic geometry and the Brown-Peterson cohomology in algebraic topology. In [KY93], Kono and Yagita confirmed Conjecture 1.1 for some compact connected Lie groups with $p$-torsion by computing the Atiyah-Hirzebruch spectral sequences. However, the non-triviality of Milnor operations on odd degree elements yields non-trivial differentials sending odd degree elements to non-zero elements, so odd degree elements do not survive to the $E_\infty$-term. With their computational

This work was supported by JSPS KAKENHI Grant Number JP17K05263.
results on the Brown-Peterson cohomology of classifying spaces, Kono and Yagita conjectured the following:

**Conjecture 1.3** (Kono and Yagita, Conjecture 5 in [KY93]). For each nonzero odd degree element $x$ of the mod $p$ cohomology of the classifying space of a compact connected Lie group, there exists an integer $i$ such that for $m \geq i$,

$$Q_m x \neq 0.$$  

Conjecture 1.3 is interesting in the cohomology theory of classifying spaces of non-simply connected Lie groups. In [VV05], Vavpetič and Viruel showed that if $p$ is an odd prime, Conjecture 1.3 holds for the projective unitary group $PU(p)$. Moreover, the cohomology of classifying spaces of non-simply connected Lie groups has recently enjoyed renewed interest. Many mathematicians have studied it in various contexts. Antieau, Gu and Williams ([AW14], [Gu19], [Gu20], [GZZZ22]) studied it for the topological period-index problem. Antieau, the author and Tripathy ([Ant16], [Kam15], [Kam17], [Tri16]) studied it for integral Hodge conjecture modulo torsion. Furthermore, the Atiyah-Hirzebruch spectral sequence is used in theoretical physics to study anomalies, cf. García-Etxebarria and Montero [GEM19].

In this paper, we give a counterexample for Conjecture 1.3 in the case $p = 2$. Our result is as follows: Let $\mathbb{H}$ be the quaternions. Let $Sp(1) \subset \mathbb{H}$ be the symplectic group consisting of unit quaternions. Let $G$ be the quotient of the 3-fold product $Sp(1)^3$ of the symplectic groups $Sp(1)$ by the subgroup $\Gamma_2$ generated by $(-1, -1, 1)$ and $(-1, 1, -1)$.

**Theorem 1.4.** In the mod 2 cohomology of the classifying space of the compact connected Lie group $G$ above, there exists a nonzero element $x_{13}$ of degree 13 such that

$$Q_m x_{13} = 0$$  

for $m \geq 1$.

This paper is organized as follows. In Section 2, we describe the action of Milnor operations on the mod 2 cohomology of $BSO(3)$. In Section 3, we prove Theorem 1.4 as Proposition 3.5.

The author would like to thank the anonymous referee for suggestions to improve the exposition of this paper.

### 2. Milnor operations

In this section, we recall Milnor operations

$$Q_m : H^i(X; \mathbb{Z}/2) \to H^{i+2^m+1}(X; \mathbb{Z}/2)$$

and the mod 2 cohomology of the classifying space $BSO(3)$. Milnor operations $Q_m$ are defined by

$$Q_0 = Sq^1, \quad Q_m = Sq^{2^m} Q_{m-1} + Q_{m-1} Sq^{2^m} \quad (m \geq 1).$$

They have the following properties:

$$Q_m Q_n = Q_n Q_m,$$

$$Q_m^2 = 0,$$

and

$$Q_m(x \cdot y) = (Q_m x) \cdot y + x \cdot (Q_m y).$$
These formulae are essential in our proofs Propositions 2.2 and 3.5. The mod 2 cohomology of \(BSO(3)\) is a polynomial ring
\[ H^*(BSO(3); \mathbb{Z}/2) = \mathbb{Z}/2[w_2, w_3] \]
generated by two elements \(w_2, w_3\) of degree 2, 3, respectively. The action of Steenrod squares on these elements is well-known as the Wu formula. In particular, we have
\[ \begin{align*}
\text{Sq}^1 w_2 & = w_3, & \text{Sq}^2 w_2 & = w_2^2, \\
\text{Sq}^1 w_3 & = 0, & \text{Sq}^2 w_3 & = w_2 w_3.
\end{align*} \]
By the Wu formula and by the definition and elementary properties of Milnor operations stated above, it is easy to obtain
\[ \begin{align*}
Q_0 w_2 & = w_3, & Q_1 w_2 & = w_2 w_3, & Q_0 Q_1 w_2 & = w_3^2, \\
Q_0 w_3 & = 0, & Q_1 w_3 & = w_3^2, & Q_0 Q_1 w_3 & = 0.
\end{align*} \]
This section aims to prove the following lemma on the action of Milnor operations on the mod 2 cohomology of \(BSO(3)\).

**Lemma 2.1.** For \(m \geq 2\), there exists a polynomial \(g_m\) in \(w_2^2\) and \(w_2^3\) such that we have
\[ Q_m Q_1 w_2 = g_m w_3^4.\]
in the mod 2 cohomology of \(BSO(3)\).

To prove Lemma 2.1, we recall the relation between Dickson invariants and Milnor operations as Proposition 2.2. The connection between Dickson invariants and Milnor operations is an exciting subject in algebraic topology. Thus, we refer the reader to the classical work of Adams and Wilkerson ([AW80], [Wil83]) for more detail on the background of this section. However, to make this paper self-contained as far as possible, we give detailed proof for Lemma 2.1 without mentioning Dickson invariants and the above background.

Let \((\mathbb{Z}/2)^2 = \mathbb{Z}/2 \times \mathbb{Z}/2\) be the elementary abelian 2-subgroup of \(SO(3)\) generated by diagonal matrices
\[ \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \]
We denote by \(\iota: (\mathbb{Z}/2)^2 \to SO(3)\) the inclusion map. The induced homomorphism
\[ Bi^*: H^*(BSO(3); \mathbb{Z}/2) \to H^*(B(\mathbb{Z}/2)^2; \mathbb{Z}/2) \]
is injective, and its image is the subring generated by the following elements.
\[ \begin{align*}
Bi^*(w_2) & = s_1^2 + s_1 s_2 + s_2^2, \\
Bi^*(w_3) & = s_1^2 s_2 + s_2^3.
\end{align*} \]

**Proposition 2.2.** Suppose that \(m \geq 2\). For an element \(x\) of the mod 2 cohomology of \(B(\mathbb{Z}/2)^2\), let
\[ D_m x = Q_m x + Bi^*(w_2^{m-1}) Q_{m-1} x + Bi^*(w_3^{m-1}) Q_{m-2} x. \]
Then, we have
\[ D_m x = 0. \]
Proof. Here, in the proof of Proposition 2.2 by the mod 2 cohomology ring, we mean the mod 2 cohomology ring of $B(\mathbb{Z}/2)^2$ unless otherwise stated explicitly. Recall that
\[ Q_m(x \cdot y) = (Q_m x) \cdot y + x \cdot (Q_m y), \]
for $x, y \in H^*(X; \mathbb{Z}/2)$. For $i = 1, 2$, we have
\[
D_m s_i = Q_m s_i + B_i^* (w_2^{2m-1}) Q_{m-1} s_i + B_i^* (w_3^{2m-1}) Q_{m-2} s_i
\]
\[
= s_i^{2m+1} + (s_1^2 + s_1 s_2 + s_2^2) s_i^{2m-1} + (s_1^2 s_2 + s_1 s_2^2) s_i^{2m-1}
\]
\[
= (s_1^4 + (s_1^2 + s_1 s_2 + s_2^2) s_1^2 + (s_1^2 s_2 + s_1 s_2^2) s_1) s_i^{2m-1}
\]
\[
= 0.
\]
Thus, for elements $x, y$ in the mod 2 cohomology ring, we have
\[
D_m (x \cdot y) = D_m x \cdot y + x \cdot D_m y.
\]
Therefore, since the mod 2 cohomology ring is generated by $s_1, s_2$, the fact that $D_m s_i = 0$ for $i = 1, 2$ implies that $D_m x = 0$ for each element $x$ in the mod 2 cohomology ring. \hfill \Box

Now, for $m \geq 2$, we describe the action of the Milnor operation $Q_m$ in terms of certain polynomials $f_{m,0}, f_{m,1}$ in $w_2^2$ and $w_3^2$ and Milnor operations $Q_0, Q_1$. Since the induced homomorphism
\[
B_i^* : H^*(BSO(3); \mathbb{Z}/2) \rightarrow H^*(B(\mathbb{Z}/2)^2; \mathbb{Z}/2)
\]
is injective, by Proposition 2.2 for each $x$ in the mod 2 cohomology of $BSO(3)$, we have
\[
Q_m x = w_2^{2m-1} Q_{m-1} x + w_3^{2m-1} Q_{m-2} x.
\]
We may write it in the following form.
\[
\begin{pmatrix}
Q_m x \\
Q_{m-1} x
\end{pmatrix} =
\begin{pmatrix}
w_2^{2m-1} & w_3^{2m-1} \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
Q_{m-1} x \\
Q_{m-2} x
\end{pmatrix}.
\]
Let us define a matrix $A_m$ whose coefficients are polynomials in $w_2^2, w_3^2$ as follows:
\[
A_m =
\begin{pmatrix}
w_2^{2m-1} & w_3^{2m-1} \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
w_2^{2m-2} & w_3^{2m-2} \\
1 & 0
\end{pmatrix}
\cdots
\begin{pmatrix}
w_2^4 & w_3^4 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
w_2^2 & w_3^2 \\
1 & 0
\end{pmatrix}.
\]
Furthermore, let us define polynomials $f_{m,0}, f_{m,1}$ by
\[
(f_{m,1} \quad f_{m,0}) = (1 \quad 0) A_m.
\]
Then, for $x$ in the mod 2 cohomology of $BSO(3)$, we have
\[
Q_m x = (1 \quad 0) \begin{pmatrix}
Q_m x \\
Q_{m-1} x
\end{pmatrix} = (1 \quad 0) A_m \begin{pmatrix}
Q_1 x \\
Q_0 x
\end{pmatrix} = f_{m,1} Q_1 x + f_{m,0} Q_0 x.
\]

Proof of Lemma 2.7. We have the following congruence.
\[
A_m \equiv
\begin{pmatrix}
w_2^{2m-1} & 0 \\
1 & 0
\end{pmatrix}
\cdots
\begin{pmatrix}
w_2^2 & 0 \\
1 & 0
\end{pmatrix}
\equiv
\begin{pmatrix}
w_2^{2m-2} & 0 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
w_2^{m-1-2} & 0 \\
1 & 0
\end{pmatrix}
\mod (w_3^2).
\]
Hence, we have $f_{m,0} \equiv 0 \mod (w_2^2)$. Therefore, there exists a polynomial $g_m$ in $w_2^2$ and $w_3^2$ such that
\[
f_{m,0} = g_m w_3^2.
\]
Recall the fact that $Q_0Q_1w_2 = w_3^3$ and $Q_1Q_1 = 0$. Then, we have

$Q_mQ_1w_2 = f_{m,1}Q_1w_2 + f_{m,0}Q_0Q_1w_2$

$= f_{m,0}Q_0Q_1w_2$

$= g_mw_3^4$. □

**Example 2.3.** For $m = 2, 3, 4$, elements $Q_mx$ and polynomials $g_m$ in Lemma 2.1 are as follows:

$Q_2x = w_1^2Q_1x + w_3^2Q_0x$, $g_2 = 1$,

$Q_3x = (w_2^6 + w_3^4)Q_1x + w_2^4w_3^2Q_0x$, $g_3 = w_4^3$,

$Q_4x = (w_2^{14} + w_2^9w_3^1 + w_2^2w_3^8)Q_1x + (w_2^{12} + w_3^8)w_3^2Q_0x$, $g_4 = w_2^{12} + w_3^8$.

3. **The Nonzero Odd Degree Element**

In this section, we prove Theorem 1.4 as Proposition 3.5.

We begin with recalling the definition of the connected Lie group $G$ in Section 1 and set up notations. Let us consider the 3-fold product of symplectic groups $Sp(1) \subset \mathbb{H}$ consisting of unit quaternions. Let

$\Gamma_3 = \{(\pm 1, \pm 1, \pm 1)\}$

be the center of $Sp(1)^3$. Let $\Gamma_2$ be its subgroup generated by $(-1, -1, -1), (1, -1, -1)$ and

$G = Sp(1)^3/\Gamma_2$.

Let $\mathbb{Z}/2 = \{(\pm 1, 1, 1)\} \subset \Gamma_3$. Then, $\mathbb{Z}/2$ and $\Gamma_2$ generate $\Gamma_3$. Moreover, we have

$Sp(1)^3/\Gamma_3 = SO(3)^3$.

Therefore, we have the following fiber sequence:

$B\mathbb{Z}/2 \to BG \to BSO(3)^3$.

We denote by

$\{E_r^{p,q}, d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}\}$

the Leray-Serre spectral sequence associated with this fiber sequence. Let us denote its $E_r$-term by

$E_r = \bigoplus_{p,q} E_r^{p,q}$.

We compute the mod 2 cohomology of $BG$ using the above Leray-Serre spectral sequence. Although it is easy, we quickly review it. See [Kam19] for more detail.

We describe the $E_2$-term and compute the first non-trivial differential $d_2$. Let

$B\pi_i : BSO(3)^3 \to BSO(3)$

be the map induced by the projection to the $i$th factor for $i = 1, 2, 3$. We denote by $w_i'$, $w_i''$, $w_i'''$ the cohomology classes $B\pi_1^*(w_i)$, $B\pi_2^*(w_i)$, $B\pi_3^*(w_i)$, respectively. Let $u_1$ be the generator of the mod 2 cohomology $H^1(B\mathbb{Z}/2; \mathbb{Z}/2) \cong \mathbb{Z}/2$ of the fibre $B\mathbb{Z}/2$. The $E_2$-term is given by

$E_2 = \mathbb{Z}/2[w_1', w_2', w_3', w_2'', w_3'', w_2''', w_3'''] \otimes \mathbb{Z}/2[u_1]$.

To compute the differential $d_2$, we consider the Leray-Serre spectral sequence

$\{\bar{E}_r^{p,q}, \bar{d}_r : \bar{E}_r^{p,q} \to \bar{E}_r^{p+r,q-r+1}\}$.
associated with the fiber sequence

$$
B\mathbb{Z}/2 \to BSp(1) \to BSO(3).
$$

Recall that its $E_2$-term is given as follows:

$$
\bar{E}_2 = \mathbb{Z}/2[w_2, w_3] \otimes \mathbb{Z}/2[u_1].
$$

and its first nontrivial differential $\bar{d}_2$ is given by

$$
\bar{d}_2(u_1) = w_2.
$$

Let

$$
B\iota_i: BSp(1) \to BG
$$

be the map induced by the inclusion map $\iota_i$ of $Sp(1)$ for $i = 1, 2, 3$ such that

$$
\iota_1(g) = (g, 1, 1), \quad \iota_2(g) = (1, g, 1), \quad \iota_3(g) = (1, 1, g).
$$

Then we have the following commutative diagram,

$$
\begin{array}{ccc}
B\mathbb{Z}/2 & \longrightarrow & BSp(1) \\
\downarrow & & \downarrow_{B\iota_i} \\
\mathbb{Z}/2 & \longrightarrow & BG
\end{array}
\quad
\begin{array}{ccc}
\longrightarrow & \longrightarrow & \longrightarrow \\
& B\iota_i & \\
& \longrightarrow & \\
BSO(3) & \longrightarrow & BSO(3)^3.
\end{array}
$$

Furthermore, we have

$$
B\iota_1^*(w'_2) = w_2, \quad B\iota_1^*(w''_2) = 0, \quad B\iota_1^*(w'''_2) = 0,
$$

$$
B\iota_2^*(w'_2) = 0, \quad B\iota_2^*(w''_2) = w_2, \quad B\iota_2^*(w'''_2) = 0,
$$

$$
B\iota_3^*(w'_2) = 0, \quad B\iota_3^*(w''_2) = 0, \quad B\iota_3^*(w'''_2) = w_2.
$$

Now, we are ready to compute the differential $d_2$. Suppose that the first nontrivial differential $d_2$ is given as follows:

$$
d_2(u_1) = \alpha_1 w'_2 + \alpha_2 w''_2 + \alpha_3 w'''_2,
$$

where $\alpha_1, \alpha_2, \alpha_3$ are in $\mathbb{Z}/2$. Since

$$
B\iota_i^*(d_2(u)) = \alpha_i w_2,
$$

and

$$
\bar{d}_2(u_1) = w_2,
$$

we obtain

$$
\alpha_i = 1
$$

for $i = 1, 2, 3$. Thus, the first nontrivial differential $d_2: E^{0,1}_2 \to E^{2,0}_2$ is given as follows:

$$
d_2(u_1) = w'_2 + w''_2 + w'''_2.
$$

Let us recall the relation between the transgression and Steenrod squares. For $r \geq 2$, the transgression

$$
d_r: E^{0,r-1}_r \to E^{r,0}_r
$$

commutes with Steenrod squares $Sq^i$. In other words, if $d_r(x) = y$ then we may have an element $Sq^i x \in E^{0,r-1+i}_r$ for $r \leq s$, an element $Sq^i y \in E^{r+i,0}_{r+i}$, and there hold that $d_s(Sq^i x) = 0$ for $r \leq s < r + i$ and that $d_{r+i}(Sq^i x) = Sq^i y$. 

Starting with the above $E_2$ and $d_2$, since $u_1^2 = Sq^1 u_1$, $u_1^4 = Sq^2 u_1^3$, and $u_1^6 = Sq^4 u_1^5$, we have the following $E_r$-terms and differentials up to $r \leq 9$.

\[
\begin{align*}
E_3 &= \mathbb{Z}/2[w'_2, w'_3, w''_2, w''_3, w'''_2, w'''_3] \otimes \mathbb{Z}/2[u_1^7], \\
d_3(u_1^2) &= Sq^1(w'_2 + w''_2 + w'''_2) \\
&= w'_3 + w''_3 + w'''_3, \\
E_4 &= \mathbb{Z}/2[w'_2, w'_3, w''_2, w''_3] \otimes \mathbb{Z}/2[u_1^4], \\
d_4(u_1^4) &= 0, \\
E_5 &= E_4, \\
d_5(u_1^4) &= Sq^2(w'_2 + w''_2 + w'''_2) \\
&= w'_2 w'_3 + w''_2 w''_3 + w'''_2 w'''_3, \\
E_6 &= \mathbb{Z}/2[w'_2, w'_3, w''_2, w''_3]/(w'_2 w''_3 + w''_2 w'_3) \otimes \mathbb{Z}/2[u_1^8], \\
d_6(u_1^8) &= 0, \\
d_7(u_1^8) &= 0, \\
d_8(u_1^8) &= 0, \\
E_9 &= E_6.
\end{align*}
\]

To compute higher terms and differentials, let us consider the ring homomorphism

\[
\phi: \mathbb{Z}/2[w'_2, w'_3, w''_2, w''_3] \to \mathbb{Z}/2[w'_2, w'_2, t_1]
\]

defined by

\[
\begin{align*}
\phi(w'_2) &= w'_2, \\
\phi(w'_3) &= t_1 w'_2, \\
\phi(w''_2) &= w''_2, \\
\phi(w''_3) &= t_1 w''_2.
\end{align*}
\]

We assign weight 0, 1, 0, 1 to $w'_2$, $w'_3$, $w''_2$, $w''_3$, respectively. We also assign weight 1 to $t_1$. Then, the ring homomorphism $\phi$ is weight-preserving.

Let

\[
M = \mathbb{Z}/2[w'_2, w'_3, w''_2, w''_3]/(w'_2 w''_3 + w''_2 w'_3).
\]

It is the bottom line of the $E_9$-term of the spectral sequence such that

\[
M = \bigoplus_p E_9^{p,0}.
\]

The ring homomorphism $\phi$ induces the weight-preserving ring homomorphism

\[
\tilde{\phi}: M \to \mathbb{Z}/2[w'_2, w'_2, t_1].
\]

It is clear that the ring homomorphism $\tilde{\phi}$ is injective. Thus, $M$ is isomorphic to the subring $\tilde{\phi}(M)$ of $\mathbb{Z}/2[w'_2, w'_2, t_1]$. Therefore, both $M$ and $\tilde{\phi}(M)$ are integral domains.
The next nontrivial differential is \( d_9 \). It is given by
\[
d_9(u_1^8) = \text{Sq}^4(u_2'w_3'' + w_2''w_3') \\
= w_2'^2w_2''w_3' + w_2'w_3'w_2''' + w_2''w_2'w_3' + w_3''w_2'w_2' \\
= w_2'w_2''(w_2'w_3'' + w_2''w_3') + w_2'w_3''w_2' + w_3''w_2'^2 \\
= w_2'w_3'' + w_3''w_2'.
\]
Since
\[
\bar{\phi}(w_3'w_3'^2 + w_3''w_3'') = t_3^1w_2'w_2''(w_2' + w_3''')
\]
is nonzero in \( \mathbb{Z}/2[w_2', w_2'', t_1] \), multiplication by \( w_3'w_3'^2 + w_3''w_3'' \) is injective on \( M \). Therefore, we have
\[
E_{10} = \mathbb{Z}/2[w_2', w_3', w_2'', w_3'']/(w_2'w_3'' + w_3'w_3'') \otimes \mathbb{Z}/2[u_1^{16}].
\]
We would like to point out that \( w_2'w_3'' + w_2''w_3' ', w_3'w_3'^2 + w_3''w_3' \) is a regular sequence in the polynomial ring \( \mathbb{Z}/2[w_2', w_2'', w_3', w_3''] \).

Finally, using the commutativity between the transgression and Steenrod squares again, we have
\[
d_r(u_1^{16}) = 0 \quad \text{for } 10 \leq r \leq 16 \text{ and }
\]
\[
d_{17}(u_1^{16}) = \text{Sq}^8(w_3'w_3'^2 + w_3''w_3'') \\
= w_3'w_3'w_3'w_3' + w_3''w_3''w_3''w_3'' \\
= (w_2'w_3' + w_2''w_3')w_3'w_3'^2 + w_2'w_3'(w_3' + w_3'')(w_3'w_3'^2 + w_3''w_3'') \\
= 0.
\]
Hence, we have \( d_r = 0 \) for \( r \geq 10 \) and \( E_\infty = E_{10} \).

To describe the \( E_\infty \)-term, let
\[
N = \mathbb{Z}/2[w_2', w_3', w_2'', w_3'']/(w_2'w_3'' + w_2''w_3' + w_3'w_3'^2 + w_3''w_3'').
\]
It is the bottom line of the \( E_\infty \)-term of the spectral sequence such that
\[
N = \bigoplus \limits_{p} E_{\infty}^{p,0}.
\]
It is also the subring of the mod 2 cohomology ring of \( BG \) generated by \( w_2', w_3', w_2'', w_3'' \). What we need is the fact that the induced homomorphism
\[
N \to H^*(BG; \mathbb{Z}/2)
\]
is injective, and \( N \) is closed under the action of Milnor operations \( Q_m \) for \( m \geq 0 \).

For a graded set \( \{x_1, x_2, \ldots\} \), we denote by \( \mathbb{Z}/2\{x_1, x_2, \ldots\} \) the graded \( \mathbb{Z}/2 \)-module spanned by \( \{x_1, x_2, \ldots\} \). Recall that we defined weight of \( w_2', w_3', w_2'', w_3'' \) as 0, 1, 0, 1, respectively. We have direct sum decompositions of \( M \) and \( N \) with respect to weight. Namely, \( M_k \), \( N_k \) are graded submodules of \( M \), \( N \) spanned by monomials of weight \( k \), respectively.

We will define the element \( x_{13} \) as an element in \( N_1 \). We also need the following Proposition 3.1 on the basis for \( N_1 \) to show that \( x_{13} \) is nonzero.
Proposition 3.1. For $N_0, N_1, N_2$, we have

\[ N_0 = \mathbb{Z}/2\{w_2^{m}w_3^n \mid m, n \geq 0\}, \]
\[ N_1 = \mathbb{Z}/2\{w_2^mw_3^3, w_2^mw_3^nw_3^m \mid m, n \geq 0\}, \]
\[ N_2 = \mathbb{Z}/2\{w_2^rw_3^2, w_2^mw_3^m, w_2^mw_3^n, w_2^mw_3^{n/2} \mid m, n \geq 0\}. \]

Proof. The weight of monomials in

\[ \tilde{\phi}(w_3^m + w_3^n) = t_1^3 w_2^m + t_1^3 w_2^n \]

is 3. Therefore, the ideal of $M$ generated by

\[ w_3^m + w_3^n \]

is spanned by monomials of weight greater than or equal to 3. Hence, we have.

$N_i = M_i$ for $i = 0, 1, 2$. It is clear that

\[ \tilde{\phi}(M_0) = \mathbb{Z}/2\{w_2^m w_3^m\}, \]
\[ \tilde{\phi}(M_1) = \mathbb{Z}/2\{t_1^m w_2^m \mid m + n \geq 1\}, \]
\[ \tilde{\phi}(M_2) = \mathbb{Z}/2\{t_1^m w_2^m \mid m + n \geq 2\} \]

and that

\[ \tilde{\phi}(\mathbb{Z}/2\{w_2^m w_3^m\}) = \mathbb{Z}/2\{w_2^m w_3^m\}, \]
\[ \tilde{\phi}(\mathbb{Z}/2\{t_1^m w_2^m \mid m + n \geq 1\}) = \mathbb{Z}/2\{t_1^m w_2^m \mid m + n \geq 2\}, \]
\[ \tilde{\phi}(\mathbb{Z}/2\{t_1^m w_2^m w_3^m\}) = \mathbb{Z}/2\{t_1^m w_2^m w_3^m \mid m + n \geq 2\}, \]

where $m, n$ range over the set of nonnegative integers. Since the ring homomorphism $\tilde{\phi}$ is injective, we obtain the desired results.

We need the following lemma on $N_k$ ($k \geq 3$) to show that $Q_{m}x_{13} = 0$ for $m \geq 2$.

Proposition 3.2. Suppose that $k \geq 3$. For $1 \leq i \leq k - 1$, $m \geq 0$, $n \geq 0$, we have

\[ w_2^m w_3^m w_3^i w_3^{mk-i} = w_2^m w_3^i w_3^{mk-1} \]

in $N_k$.

Proof. For $i \geq 2$, we have

\[ w_3^i w_3^{mk-i} = w_3^i w_3^i \cdot w_3^{m-2} w_3^{mk-i-1} \]
\[ = w_3^i w_3^i \cdot w_3^{m-2} w_3^{mk-i-1} \quad (\because w_3^i w_3^i = w_3^i w_3^i) \]
\[ = w_3^i w_3^{mk-i}. \]

Iterating this process, we have $w_3^i w_3^{mk-i} = w_3^i w_3^{mk-1}$. For $m \geq 1$, we have

\begin{align*}
    w_2^m w_2^m w_3^i w_3^{mk-1} &= w_2^m w_2^m \cdot w_2^{m-1} w_3^i w_3^{mk-2} \\
    &= w_2^m w_2^m \cdot w_2^{m-1} w_3^i w_3^{mk-2} \quad (\because w_2^m w_2^m = w_2^m w_2^m) \\
    &= w_2^{m-1} w_2^{m+1} w_3^{mk-2} \\
    &= w_2^{m-1} w_2^{m+1} w_3^{mk-1} \quad (\because w_2^2 w_3^{k-2} = w_2^2 w_3^{mk-1}).
\end{align*}

Hence, we have the desired result $w_2^m w_2^m w_3^i w_3^{mk-1} = w_2^m w_3^i w_3^{mk-1}$.

Remark 3.3. With Proposition 3.2, it is easy to find a basis for $N_k$. And we have the following.

\[ N_k = \mathbb{Z}/2\{w_2^m w_3^i, w_2^m w_3^i w_3^{mk-1}, w_2^m w_3^{mk} \mid m, n \geq 0\}. \]
Remark 3.4. It is easy to compute the Poincaré series
\[
\frac{(1 - t^5)(1 - t^9)}{(1 - t^2)^2(1 - t^9)}
\]
of \(N\) since \(w'_2w''_3 + w''_3w'_4, w'_3w''_4 + w''_3w'_4\) is a regular sequence. To prove the linear independence of elements in Propositions 3.1 and 3.2, one may compute the Poincaré series of each \(N_k\) and add them up to obtain the Poincaré series of \(N\) above.

Proposition 3.5. Let us define an element \(x_{13}\) of degree 13 in the mod 2 cohomology of \(BG\) by
\[
x_{13} := B\pi^*(Q_1w_2)w''_2(w'^2_2 + w''_2).
\]
Then, \(x_{13}\) is nonzero and for \(m \geq 1\), we have
\[
Q_m x_{13} = 0.
\]

Proof. First, we verify that \(x_{13}\) is nonzero. Since \(B\pi^*(Q_1w_2) = w'_2w''_3\), we have
\[
x_{13} = w'^2_2w''_2w'_3 + w'_2w''_2w'_3
= w'_2w''_2w'_3 + w''_2w''_2w'_3
\neq 0
\]
in \(N_1\) by Proposition 3.1. Next, we compute \(Q_m x_{13}\). Since \(Q_m\) acts trivially on \(w''_2(w'^2_2 + w''_2)\),
\[
Q_m x_{13} = B\pi^*(Q_1w_2)w''_2(w'^2_2 + w''_2).
\]
For \(m = 1\), since \(Q_1Q_1 = 0\), we have \(Q_1 x_{13} = 0\). For \(m \geq 2\), by Lemma 2.1 we have
\[
B\pi^*(Q_1w_2)w''_2(w'^2_2 + w''_2) = B\pi^*(g_m w^3)w''_2(w'^2_2 + w''_2)
= B\pi^*(g_m)w''_2w''_2(w'^2_2 + w''_2).
\]
By Proposition 3.2 we obtain
\[
w'^4_2w''_2w'^3_2 = w'_2w''_2w'^3_2,
\]
\[
w'^4_2w''_2w''_2 = w''_2w'_2w'^3_2,
\]
hence, we have
\[
w'^4_2w''_2(w'^2_2 + w''_2) = 0.
\]
Therefore, we obtain \(Q_m x_{13} = 0\). \qed

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