A Monopole Solution in open String Theory

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Abstract

We investigate a solution of the Weyl invariance conditions in open string theory in 4 dimensions. In the closed string sector this solution is a combination of the SU(2) Wess–Zumino–Witten model and a Liouville theory. The investigation is carried out in the $\sigma$-model approach where we have coupled all massless modes (especially an abelian gauge field via the boundary) and tachyon fields. Neglecting all higher derivatives in the field strength we get an exact result which can be interpreted as a monopole configuration living in non-trivial space time. The masses of both tachyon fields are quantized by $c_{\text{wzw}}$ and vanish for $c_{\text{wzw}} = 1$.

1 Introduction

Solutions of the conformal invariance conditions in closed string theory have been widely discussed in the last time. First of all these are solutions corresponding to (gauged) Wess–Zumino–Witten (WZW) theories (see e.g. [1]). It is possible to obtain a non-linear $\sigma$ model by starting with a gauged WZW theory, choosing a gauge and integrating out the 2d gauge field. From this $\sigma$ model one can read off the metric, antisymmetric tensor and the dilaton. Usually one gets by this approach the background field only in an $\alpha'$ expansion. In order to get exact results one has to use algebraic Hamiltonian techniques [2] or to compute first the effective action of the gauged WZW model and then to eliminate the 2d gauge field [3]. Then the corresponding background fields solve the $\beta$ equations in all orders in $\alpha'$. Another class are the instanton, soliton and monopole solutions [4, 5] that one gets from the heterotic string theory with self dual non-abelian gauge field. They have been constructed in the lowest order in $\alpha'$ and are exact due to the non-renormalization theorem in extended worldsheet supersymmetric theories [6]. The soliton solution corresponds to a semi-wormhole in the space time and near the singularity this theory is approximately given by a combination of a SU(2) WZW model and a Liouville theory (after incorporation of a tachyon field). Therefore, in this limit it is also possible to construct an exact conformal theory in the bosonic case. The generalization of this closed string theory to an open string theory is the aim this paper.

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The $\sigma$ model containing the massless modes and tachyon field of the closed and open string is given by

$$Z = \int Dx \ e^{-S},$$

$$S = \frac{1}{4\pi\alpha'} \int_M d^2z \sqrt{g} \left( g^{ab} \partial_a x^\mu \partial_b x^\nu G_{\mu\nu} - i \epsilon^{ab} \partial_a x^\mu \partial_b x^\nu B_{\mu\nu} + \alpha' R^{(2)} \phi + \alpha' T_1 \right) +$$

$$+ \frac{1}{2\pi\alpha'} \int_{\partial M} (\alpha' k \phi + \alpha' T_2 + i e A_\mu \dot{x}^\mu).$$

Here $G_{\mu\nu}$ corresponds to the metric in the target space (space time), $B_{\mu\nu}$ is the antisymmetric tensor field, $\phi$ is the dilaton, $A_\mu$ is an abelian gauge field, $T_{1,2}$ the tachyon fields for closed or open string respectively. We have to decide between both tachyons because they carry different world sheet dimensions. $R^{(2)}$ is the worldsheet scalar curvature, $k$ the corresponding curvature of the boundary and $e$ an electric charge. This model is conformally invariant if the Weyl anomaly coefficients vanish. These coefficients are the $\beta$ functions which are mainly given by the renormalization group $\beta$ functions \[7\]. In general it is only possible to solve these equations in an $\alpha'$ expansion. But for an appropriate choice of the background fields it is possible to find exact solutions, e.g. as a WZW theory.

Starting with an exact solution in the closed string sector (sect. 2) we will show in sect. 3 that it is possible to find an exact solution for the $\beta$ equations in the open string sector too. Both sections are rather technical whereas in section 4 we summarize all results and give an interpretation and discussion of the results.

## 2 Closed string sector

We start with the investigation of the closed string sector and thereafter we study the generalization to open strings. So we neglect here the couplings via the boundary in (1). As we have already noted for special background fields and in four space time dimensions it is possible to rewrite the model (1) in a $SU(2)$ WZW model and a Liouville model \[3, 8, 9\]. Namely, if

$$G_{\mu\nu} = \frac{Q}{r^2} \delta_{\mu\nu}, \quad \mu, \nu = 0, 1, 2, 3,$$

$$H_{\mu\nu\lambda} \equiv \partial_{[\mu} B_{\nu\lambda]} = \pm \epsilon_{\mu\nu\lambda} \partial_{\sigma} \log \frac{\sqrt{Q}}{r}$$

$$\phi = \phi(r) \quad \text{and} \quad T_1 = T_1(r)$$

we can write (1) as

$$S = S_{WZW}(SU(2)) + \frac{1}{4\pi\alpha'} \int ((\partial u)^2 Q + \alpha' R^{(2)} \phi(u) + \alpha' T_1(u))$$

and

$$S_{WZW} = \frac{Q}{4\pi\alpha'} \int d^2z \ tr (\partial_a g^{-1} \partial_a g) + \frac{Q}{6\pi\alpha'} \int d^3z \ tr \ \epsilon^{abc} ((g^{-1} \partial_a g)(g^{-1} \partial_b g)(g^{-1} \partial_c g))$$

where $g = \frac{1}{2} (x^0 1 + ix^i \sigma^i)$, $\sigma^i$ are the Pauli matrices, $u = \log r$, $r^2 = (x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$ and $Q = \text{const}$. After introducing polar coordinates one finds immediately that the angle degrees of freedom are controlled by the WZW part and the second Liouville part

$$2$$
contains the dependence on the radius. Background fields like (2) have been investigated in
the context of semi–wormholes or solitons in [4] and its cosmological relevance is discussed in [8]. The WZW theory is well defined and conformally invariant if [10]: $\frac{c}{\sigma} \equiv k = 1, 2, 3, ...$ (the level of the WZW theory). The corresponding central charge for the $SU(2)$
is given by: $c_{wzw} = \frac{3k}{k+2}$. In addition, the conformal invariance requires that the tachyon and dilaton have to fulfill the $\tilde{\beta}$ equations of the Liouville part in (3)

$$
\phi''(u) = 0
$$

$$
1 + c_{wzw} - 26 + \frac{6}{k} (\phi')^2 = 0
$$

$$
-\frac{1}{2k} T'_1 - 2T_1 + \frac{1}{k} \phi T'_1 = 0
$$

(26 is the contribution of the ghosts to the central charge and $\phi'(u) \equiv \frac{d}{du} \phi(u)$). From the first equation it follows that the dilaton is at most linear in $u$, the second equation ensures the vanishing of the total central charge and the last one defines the tachyon field. As solution one finds

$$
\phi(u) = \phi_0 + qu \quad \text{with}: \quad q^2 = \frac{k}{6} \left(25 - \frac{3k}{k+2}\right) , \quad u = \log r .
$$

$$
T_1^{(1)}(u) \sim e^{qu} \sqrt{\left(\frac{k}{6} \left(\frac{3k}{k+2} - 1\right)\right)}^{-1} \sin \left(\frac{k}{6}(\frac{3k}{k+2} - 1)u\right) ,
$$

$$
T_1^{(2)}(u) \sim e^{qu} \cos \left(\frac{k}{6}(\frac{3k}{k+2} - 1)u\right) .
$$

These exact results have been obtained by rewriting the original $\sigma$ model in the known WZW model and the Liouville theory. Since this procedure is not practicable in open string theory we want now to consider this result from the $\sigma$ model point of view. This means we have to show that the background fields (2) and (5) correspond to zeros of the $\tilde{\beta}$ functions and therewith to a conformal field theory. These functions are given by [7]

$$
\tilde{\beta}^G_{\mu\nu} = \beta^G_{\mu\nu} + D_{(\mu} M_{\nu)} , \quad M_{\nu} = 2\alpha' \partial_{\nu} \phi + W_{\nu} ,
$$

$$
\tilde{\beta}^B_{\mu\nu} = \beta^B_{\mu\nu} + H_{\mu\nu\lambda} M^{\lambda} + \partial_{[\mu} K_{\nu]} ,
$$

$$
\tilde{\beta}^\phi = \beta^\phi + \frac{1}{2} M^{\nu} \partial_{\nu} \phi ,
$$

$$
\tilde{\beta}^T = \beta^T_{1} - 2T_1 + \frac{1}{2} M^{\nu} \partial_{\nu} T_1 .
$$

Up to the second order in $\alpha'$ one gets for the renormalization group $\beta$ functions

$$
\beta^{T_1} = -\frac{1}{2} \alpha' D^2 T_1 - \alpha'^2 \frac{1}{8} (H^2)_{\mu\nu} D_{\mu} \partial_{\nu} T_1 ,
$$

$$
\beta^G_{\mu\nu} = \alpha' \hat{R}_{[\mu|\nu]} + \frac{1}{2} \alpha'^2 \left(\hat{R}^{\alpha\beta\lambda}_{\rho\sigma} R_{\rho\sigma\alpha\beta\lambda} - \frac{3}{2} \hat{R}^{\beta\lambda\alpha}_{\nu} \hat{R}_{\mu\alpha\beta\lambda} + \frac{1}{2} \hat{R}_{\lambda[\mu\nu]} \beta (H^2)_{\lambda\beta}\right) ,
$$

$$
\beta^B_{\mu\nu} = \alpha' \hat{R}_{[\mu|\nu]} + \frac{1}{2} \alpha'^2 \left(\hat{R}^{\alpha\beta\lambda}_{\rho\sigma} \hat{R}_{\rho\sigma\alpha\beta\lambda} - \frac{3}{2} \hat{R}^{\beta\lambda\alpha}_{[\nu} \hat{R}_{\mu]\alpha\beta\lambda} + \frac{1}{2} \hat{R}_{\lambda[\mu\nu]} \beta (H^2)_{\lambda\beta}\right) ,
$$

$$
\beta^\phi = \frac{1}{6} (d - 26) - \frac{1}{2} \alpha' D^2 \phi - \frac{1}{8} \alpha'^2 (H^2)_{\mu\nu} D_{\mu} D_{\nu} \phi +
$$

$$
\quad + \frac{1}{16} \alpha'^2 \left(R_{\mu\nu\alpha\lambda} \frac{11}{2} R H H + \frac{5}{24} H^4 + \frac{3}{8} (H^2)_{\mu\nu}^2 + \frac{4}{3} D H \cdot D H \right) .
$$

$^a$We neglect in our consideration all non-perturbative contributions, e.g. a tachyon potential.
and $W_\mu = -(\alpha'^2/24)\partial_\mu H^2$, $K_\mu = \mathcal{O}(\alpha'^3)$, $H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda}$, $D_H \cdot D_H \equiv D_\mu H_{\nu\lambda\beta} D^\mu H^{\nu\lambda\beta}$ and $\hat{R}_{\mu\nu\lambda\beta}$ is the generalized curvature tensor computed in terms of the connection $\hat{\Gamma}^\mu_{\nu\lambda} = \Gamma^\mu_{\nu\lambda} - \frac{1}{2}H^\mu_{\nu\lambda}$.

\[
\hat{R}_{\mu\nu\lambda\gamma} = R_{\mu\nu\lambda\gamma} + \frac{1}{2} (D_\gamma H_{\mu\nu\lambda} - D_\lambda H_{\mu\nu\gamma}) + \frac{1}{4} \left( H_{\mu\gamma\rho} H^{\rho}_{\nu\lambda} - H_{\mu\lambda\rho} H^{\rho}_{\nu\gamma} \right). \tag{8}
\]

It is now easy to prove that $\hat{R}$ vanishes for the metric and the torsion (2), i.e. the space time is parallelizable. The vanishing of $\hat{R}$ has the remarkable consequence that the $\beta$ functions of the metric and of the antisymmetric tensor vanish identically in all order in $\alpha'$. Furthermore, the torsion and the Riemann curvature are covariantly constant: $D_\lambda H_{\mu\nu\gamma} \equiv 0$, $D_\lambda R_{\mu\nu\rho\gamma} \equiv 0$. Let us show that $W_\nu$ and $K_\nu$ do not contribute to the $\beta$ functions. Both expressions are functions of the curvature and torsion only. Since $\hat{R} = 0$, we can substitute all curvature dependence by additional torsion terms. These terms are covariantly constant and thus: $D_\mu W_\nu = D_\mu K_\nu = 0$. So, they do not contribute to $\beta^{G,B}$. In addition, $W_\nu$ contributes to $\beta^{\phi,T}$ in the form of $W_\nu \partial_\nu \phi \sim W_\nu x^\nu$ which vanish too because $H_{\lambda\mu\nu} x^\nu = 0$. The tachyon term has the same structure because it depends on the radius only. Hence we can always set $W_\nu = K_\nu = 0$ or $M_\nu = 2\alpha' \partial_\nu \phi$. The whole $\beta$ functions of the metric and the antisymmetric tensor are thus given by

\[
\beta^{G,B}_{\mu\nu} \sim D_\mu D_\nu \phi, \quad \beta^{B}_{\mu\nu} \sim H_{\mu\nu} \partial_\lambda \phi \tag{9}
\]

and we find that for the dilaton (5) and the torsion (2) both expressions vanish. The Curci–Paffuti theorem has now the consequence that the dilaton $\beta$ function has to be constant, namely, the central charge. Because the dilaton $\beta$ function is only perturbatively known one can determine the parameter $q$ in the $\sigma$ model approach only order by order in $\alpha'$. Khuri has shown that one just gets the correct expansion in $\alpha'/q = 1/k$. If the tachyon depends on the radius only it is also possible to obtain an exact solution for the tachyon $\beta$ equation. Although one does not know $\beta^{T_1}$ in all order of the $\alpha'$ expansion the general structure is known

\[
\beta^{T_1} = P^{\mu\nu} D_\mu \partial_\nu T_1 - 2T_1 + \alpha' \partial^\mu \phi \partial_\mu T_1. \tag{10}
\]

Where $P^{\mu\nu}$ is a general operator depending on the curvature, the torsion, the covariant derivative and the metric. In the flat limit where all curvature and torsion terms drop out $P^{\mu\nu}$ is given by: $P^{\mu\nu} = -\frac{1}{2} \alpha' G^{\mu\nu}$. Since $\hat{R}_{\mu\nu\lambda\sigma} = 0$ and all covariant derivatives of $\hat{R}$ and $H$ vanish $P^{\mu\nu}$ can be written as a function of $H$ and covariant derivatives acting on the tachyon. Crucial for this argumentation is that: $D_\mu \partial_\nu T_1(r) \sim \frac{\partial^\mu \hat{T}_1}{\hat{T}_1}$ and that $\hat{\epsilon}^\mu_{\nu\tau}$ is a covariantly constant vector which can be commutated with the covariant derivatives. Since furthermore $H_{\mu\nu\lambda} x^\lambda = 0$ we find that all curvature and torsion terms drop out of $P^{\mu\nu}$ and we get the same structure as in the flat limit

\[
\beta^{T_1} = -\frac{1}{2} \alpha' D^2 T_1 - 2T_1 + \alpha' \partial^\mu \phi \partial_\mu T_1. \tag{11}
\]

It is easy to verify that for our solution (5) this expression vanish.
Similar to the WZW approach one obtains the quantization of \( k = \frac{Q}{\alpha'} \) only via a topological consistency condition. Namely, if \( g \) is an element of the SU(2) \((g : S^3 \rightarrow S^3)\) parametrized by \( g = \frac{1}{i}(x^0 1 + ix^i \sigma^i) \) we get:

\[
\frac{1}{2\pi \alpha'} \int_{S^3} d\Sigma^{\mu\nu\lambda} H_{\mu\nu\lambda} = \frac{Q}{2\pi \alpha'} \int_{S^3} \text{tr} e^{abc} ((g^{-1} \partial_a g)(g^{-1} \partial_b g)(g^{-1} \partial_c g)) = 2\pi \frac{Q}{\alpha'} \tag{12}
\]

and \( \frac{Q}{\alpha'} \) has to be an integer \([15]\), the topological charge. The reason is the following. There are two possibilities to continue the antisymmetric tensor contribution to a torsion contribution, either to the outer side or to the inner side of the world sheet. Because physically there is no difference we have the upper consistency relation.

3 Open string sector

Now we want to look for a solution for the corresponding \( \sigma \) model in open string sector. In this case there are two sets of \( \bar{\beta} \) functions which have to vanish in order to get a conformally invariant field theory \([16, 17]\). On one hand these are again the equations (6) corresponding to the background fields which couple at inner points of the world sheet. These fields correspond to excitations of the closed string. On the other hand we have in addition \( \bar{\beta} \) equations coming from fields which couple via the boundary of the world sheet and are the excitations of the open string theory: the gauge field \( A_\mu \) and a further tachyon field \( T_2 \). Crucial for the further investigation is that the gauge field \( A_\mu \) could only influence the \( \bar{\beta} \) functions (6) via higher genus contributions \([18]\) (see also the discussion), i.e. at the lowest genus the \( \bar{\beta} \) functions (6) are not changed by the boundary background fields. Therefore the solution (2),(5) remains valid in our consideration. At this point one has to note that the dilaton couples as well at inner points of the world sheet as at the boundary. Consequently, the dilaton \( \beta \) function on the boundary differs from the dilaton \( \beta \) function inside, e.g. by additional gauge field contributions. Later we will see that this does not matter for the solution (5). But first of all we should discuss the gauge symmetry of our theory (1). There are two gauge degrees of freedom: i) the usual gauge of \( A_\mu \) parameterized by a scalar field \( \omega \) and ii) the gauge of \( B_{\mu\nu} \) parameterized by a vector filed \( A_\mu \)

\[
B_{\mu\nu} \rightarrow B_{\mu\nu} + e \partial_\mu \Lambda_\nu
\]

\[
A_\mu \rightarrow A_\mu - \Lambda_\mu + \partial_\mu \omega \tag{13}
\]

The \( \bar{\beta} \) functions depend only on the gauge invariant field strength

\[
F_{\mu\nu} = B_{\mu\nu} + e f_{\mu\nu} \quad (f_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu) \tag{14}
\]

Before we turn to the concrete form of the \( \bar{\beta} \) functions let us do some remarks concerning the perturbation theory with respect to the gauge field coupling. Usually one considers the whole gauge field coupling as an interaction part of the theory. But it is possible to absorb the quadratic part of this interaction by a modification of the boundary condition of the propagator. This modified propagator is given by \([18]\)

\[
N_{\mu\nu} = -\frac{\alpha'}{2} \left( \eta_{\mu\nu} \log |z - z'|^2 + \left( \frac{\eta - F}{\eta + F} \right)_{\mu\nu} \log(z - \bar{z}') + \left( \frac{\eta + F}{\eta - F} \right)_{\mu\nu} \log(\bar{z} - z') \right) \tag{15}
\]
Using this propagator has the consequence that all gauge field vertices contain at least one derivative of the field strength $F$. If we neglect the higher derivative terms of $F$ it is possible to obtain an exact expression for the gauge field $\bar{\beta}$ function \[18\]. Let us now study the boundary $\bar{\beta}$ functions in detail. Neglecting all higher derivatives in $F$ they are given by \[18, 16\]

\[
\bar{\beta}_A^\mu = -\alpha' \left( \frac{1}{G - F^2} \right)^{\lambda \nu} D_\lambda F_{\nu \mu} + \frac{1}{2} M^\mu F_{\nu \mu} + \frac{1}{2} \left( \frac{F}{G - F^2} \right)^{\lambda \rho} H_{\lambda \rho \nu} F_{\nu \mu} - \frac{1}{2} e K_\mu ,
\]

\[
\bar{\beta}^\phi = \beta^\phi - \frac{1}{2} M^\mu \partial_\mu \phi ,
\]

\[
\bar{\beta}^{T_2} = \beta^{T_2} - T_2 + \frac{1}{2} M^\mu \partial_\mu T_2 .
\]

We start with the discussion of the dilaton $\bar{\beta}$ function. The Curci–Paffuti theorem \[13, 16\] which states that the dilaton $\bar{\beta}$ function is constant if the $\bar{\beta}$ functions of the metric, antisymmetric tensor and of the gauge field vanish is crucial for this investigation. Since the constant is the same for both dilaton $\bar{\beta}$ functions a vanishing inside causes a vanishing on the boundary. Therefore the dilaton (5) is a zero of both $\bar{\beta}$ functions. As in the closed string sector, $W_\mu$ and $K_\mu$ should not contribute in the open string sector. Both quantities enter the local Weyl anomaly as possible total derivatives and are absent in the global or integrated Weyl anomaly \[7, 16\]. Because these terms are not relevant for the inner $\bar{\beta}$ functions (see the discussion before (9)) we assume that they also not contribute to the boundary $\bar{\beta}$ functions. Hence we set again

\[
W_\mu = K_\mu = 0 \quad \text{or} \quad M_\mu \equiv 2\alpha' \partial_\mu \phi .
\]

Before we study the gauge field $\bar{\beta}$ function one has to note that in contrast to the closed string theory for open strings the potential for the torsion itself has physical meaning similar to the Aharanov–Bohm effect. In both theories only gauge invariant quantities enter the $\bar{\beta}$ functions. While in the $\sigma$ model for closed strings only $H$ is a gauge invariant quantity in open string theory there also appear the gauge invariant quantity $F$. By the way for the gauge invariant $F$ it is necessary to couple a gauge field $A$. This antisymmetric tensor field, which is also a possible potential for $H$, is entirely fixed by two equations

\[
\partial_{[\mu} F_{\nu \lambda]} \equiv \partial_{[\mu} B_{\nu \lambda]} = H_{\mu \nu \lambda} = \pm \epsilon_{\mu \nu \lambda \sigma} \partial^\sigma \log \frac{\sqrt{\bar{Q} r}}{r} , \quad \bar{\beta}_{\mu}^A = 0 .
\]

The first equation determines $F$ only up to a term $\sim \partial_{[\mu} \Omega_{\nu]}$ and the second equation fixes $\Omega_{\nu}$. The general solution of the first equation is given by

\[
F_{\mu \nu} = -i \frac{2}{\sqrt{2}} Q b_{[\mu} \partial_{\nu]} x + \partial_{[\mu} \Omega_{\nu]} \chi , \quad \text{where:} \quad b_\mu = J_{\mu} \lambda \partial_\lambda \log \frac{\sqrt{\bar{Q} r}}{r} , \quad \chi = \log \frac{Q r}{r} , \quad (J^2)^\mu_\nu = -G_{\mu}^\nu .
\]

\[\text{[19]}

\[\text{[19]}

Because the dilaton couples at inner points as well as at the boundary of the world sheet there are two different dilaton $\beta$ functions, e.g. only the boundary $\bar{\beta}$ function depends on the gauge field (at the lowest genus).
and  \( p \) is an arbitrary eigenvector of the matrix \( J \). There are six possible matrices \( J \):

\[
J_1^+ = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}, \quad J_1^- = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix},
\]

\[
J_2^+ = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad J_2^- = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix},
\]

\[
J_3^+ = \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}, \quad J_3^- = \begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix}.
\]

(20)

These matrices are discussed in [4] in the context of \((4,4)\) extended supersymmetric \( \sigma \) model and they are self dual \((J^+)\) or anti-self-dual \((J^-)\) respectively (see [20]). All \( J^+ \) commute with all \( J^- \) and both sets satisfy the algebra

\[
J_\pm^+ J_\pm^- = -\delta_{\pm s} + \epsilon_{rst} J_\pm^t.
\]

(21)

In order to determine the vector \( \Omega_\mu \) we insert \( F \) in \( \bar{\beta}_A \) and find as solution

\[
-\alpha' \left( \frac{G}{G - F^2} \right)^{\lambda \nu} D_\lambda F_{\nu \mu} + \alpha' \partial^\nu \phi F_{\nu \mu} + \alpha' \frac{1}{2} \left( \frac{F}{G - F^2} \right)^{\lambda \mu} H_{\lambda \rho}^{\nu} F_{\nu \mu} = 0
\]

if:

\[
\Omega_\mu = -i \frac{3}{4} Q b_\mu.
\]

(22)

Let us note some general properties of \( F \):

\[
(F^2)^2 = \frac{9}{4} F^2 \quad \rightarrow \quad (G - F^2)^{-1} = G - \frac{4}{5} F^2,
\]

\[
F \cdot x = F \cdot p = 0,
\]

\[
D_{(\mu} b_{\mu)} = 0,
\]

\[
F \wedge F = 0 \quad \text{(and also } R \wedge R = 0)\]

(23)

with \( b_\mu \) is given in (19) and \( p \) is an eigenvector of the matrix \( J \). The first projector property is useful in proving of (22) and the second one will be used for the tachyon \( \bar{\beta} \) function. From the third equation follows that in addition to the covariantly constant vector \( \partial_\mu \log r \) a further Killing vector is given by \( b_\mu \).

Finally we have yet to fix the boundary tachyon \( T_2 \). If we again assume that \( T_2 \) depends on the radius \( r \) only the corresponding \( \bar{\beta} \) function is given by

\[
\bar{\beta}^{T_2} = -\alpha' D^2 T_2 - T_2 + \alpha' \partial_\mu \phi \partial^\mu T_2 = 0.
\]

(24)

This function looks very similar to the \( \bar{\beta}^{T_1} \) function (11). The only differences are the factors in front of the first and second term. The first term has to be twice the term in (11)

\[\text{c The three Killing vectors } J_{\mu}^{(i)\nu} = J_{(i)\nu}^{(i)\mu} \partial_\nu \log r \text{ (e.g. with the set of matrices } J^+ \text{ in (20)) and the covariantly constant vector } \partial_\mu \log r \text{ are an orthogonal system of vectors.}\]
because the divergence on the boundary is twice than inside [14]. In the discussion of the $\beta^{T_1}$ function we pointed out that curvature and torsion terms have to drop out because the covariantly constant curvature and torsion are “transverse” to any covariant derivative of the tachyon. The same arguments are true for $T_2$. In addition to the curvature and torsion terms all $F_{\mu\nu}$ dependence drops out because $F_{\mu\nu} x^\nu = 0$ (see (23)) and $x'^\nu D_\lambda F_{\nu\mu} = 0$. In terms of the dilaton (5) $\phi = q \log \frac{r}{r_0}$ we find as solution

$$T_2 \sim r^{\alpha_\pm}, \quad \alpha_\pm = \frac{1}{2} \sqrt{\frac{k}{6}} \left( \sqrt{25 - \frac{3k}{k+2}} \pm \sqrt{1 - \frac{3k}{k+2}} \right).$$

Thus both tachyons have the same functional structure but: $\alpha_{\pm}^{\text{closed}} = 2\alpha_{\pm}^{\text{open}}$. Of course, in terms of sin or cos function it is possible to get two real solutions for all $k$ like (5).

### 4 Discussion

Summarizing all results the background fields for which the Weyl anomaly vanish are given by

$$G_{\mu\nu} = \frac{Q}{r^2} \delta_{\mu\nu}, \quad H_{\mu\nu\lambda} = \pm \epsilon_{\mu\nu\lambda}^\sigma \partial_\sigma \log \frac{\sqrt{Q}}{r}, \quad \mu, \nu, \lambda = 0, 1, 2, 3$$

$$F_{\mu\nu} \equiv B_{\mu\nu} + e f_{\mu\nu} = -i \frac{3}{2} Q \left( b_{[\mu} \partial_{\nu]} \chi + \frac{1}{2} \partial_{[\mu} b_{\nu]} \right)$$

$$\phi = \phi_0 + q \log r$$

$$T_1 \sim r^{\alpha_\pm}, \quad T_2 \sim r^{\frac{\alpha_\pm}{2}}, \quad \alpha_\pm = \sqrt{\frac{k}{6}} \left( \sqrt{25 - \frac{3k}{k+2}} \pm \sqrt{1 - \frac{3k}{k+2}} \right)$$

where $b_{\mu}$ and $\chi$ are defined in (19). The metric could be understood as an approximation of a semi-wormhole near the singularity [8]. This semi-wormhole is given by a conformally flat metric with the conformal factor: $\sim C + \frac{Q}{r^2}$. Because the geometry of the space time in this limit is a cylinder with a three-sphere as cross-section

$$ds^2 = \frac{Q}{r^2} (dr^2 + r^2 d\Omega_3^2) = dt^2 + Q d\Omega_3^2, \quad t = \sqrt{Q} \log r$$

it is also possible to regard the space time as a Robertson–Walker universe with the world radius $Q$ [8, 9]. After this transformation we get a dilaton which depends linearly on the time with $q$ as a background charge. The tachyon fields differ only by a factor 2 in front of $\alpha_\pm$ and are in general oscillating fields of the time $t$. Only for $k = 1$ we get after the transformation (27) an exponential time-like tachyon. An additional spatial dependence of the tachyon for closed strings is discussed in [9].

Now we determine the gauge field $A_\mu$ explicitly. The $\beta$ functions define only $F_{\mu\nu} \equiv B_{\mu\nu} + e f_{\mu\nu} = -i \frac{3}{2} Q \left( b_{[\mu} \partial_{\nu]} \chi + \frac{1}{2} \partial_{[\mu} b_{\nu]} \right)$ but not the field strength $f_{\mu\nu} = \partial_{[\mu} A_{\nu]}$. An obvious choice would be: $A_\mu \sim b_\mu$. But $F$ can also be written as $F_{\mu\nu} \sim \chi \partial_{[\mu} b_{\nu]} + \partial_{[\mu} (1 - \chi) b_{\nu]}$ which would yield: $A_\mu \sim (1 - \chi) b_\mu$. Both gauge fields are equivalent from the $\sigma$ model point of view. They differ only by a special gauge transformation. In order to decide between
both fields we have to impose a gauge fixing condition. If we take, e.g., the Lorentz gauge condition for the more general gauge field $A_\mu = (c\chi + d)b_\mu$ ($c, d$ are some constants) we find
\[ D_\mu A^\mu = D_\mu(c\chi + d)b^\mu = cb^\mu \partial_\mu \chi = 0 \quad \leftrightarrow \quad c = 0 . \] (28)

Hence a possible gauge field and antisymmetric tensor are given by
\[ A_\mu = -i \frac{3}{4} Q b_\mu \quad \text{and} \quad B_{\mu\nu} = -e A_{[\mu} \partial_{\nu]} \chi . \] (29)

From (23) it follows that for this gauge fixing $A_\mu$ is just a Killing vector field.

We should remark that a priori $F_{\mu\nu}$ is not well defined everywhere. Because $H = dF$ and $\int H \neq 0$ we have to expect a Dirac singularity in $F_{\mu\nu}$. Indeed, performing the partial derivatives in $F$ yields
\[ F_{\mu\nu} = -i \frac{3}{2} Q \left( \frac{1}{r^2} J_{\mu\nu} + \frac{1}{x p} (b_\mu p_\nu - b_\nu p_\mu) \right) \] (30)
and we find a Dirac singularity for $|xp| = 0$ (note: the vectors $p_\mu$ are complex and we have to take the real part only). Therefore, we have to be careful in inserting the antisymmetric tensor in the $\sigma$ model (1). This can be done by seperating the manifold $\Sigma$ in two parts ($\Sigma^+, \Sigma^-$) and choosing for each part an eigenvector which does not cause a singularity in this area. Since $F_{\mu\nu}$ satisfies
\[ *d*F = 0 \] \[ *d F = j^M = Q d \log r . \] (31)
we see that $F_{\mu\nu}$ fulfills the equation of motion for a monopole field. To get the magnetic charge we have to perform the integration
\[ \int_{S^2} F = \int_{M_3} H = Q \] (32)
where $M_3$ is a spatial 3-manifold with the boundary $S_2$. The magnetic current is conserved ($D^\mu j^M_\mu = 0$) and the quantization of the magnetic charge $Q$ follows from (12). If we again take $t = \sqrt{Q} \log r$ as time, the spatial part of the torsion $H$ is given by: $H_{ijk} = Q \epsilon_{ijk}$. With $H_{ijk} = \partial_i F_{jk}$ one finds $\Box^M F_{ij} \sim \epsilon_{ijk} x^k$ and therefore we get the expected radial magnetic field: $B^i = \epsilon^{ijk} F_{jk} \sim x^i$.

Before we discuss the effective action corresponding to these background fields let us consider the $\bar{\beta}$ function once more. It is possible to express these functions in terms of

\[ ^e \text{For $J_1^+$ the eigenvectors are given by: (1, i, 0, 0), (0, 0, 1, i) and the complex conjugate vectors.} \]
Lie derivatives \[7, 13\]

\[\dot{\beta}^i = \beta^i + \sigma^i, \quad \sigma^G_{\mu\nu} = \mathcal{L}_M G_{\mu\nu}, \]

\[\sigma^B_{\mu\nu} = \mathcal{L}_M B_{\mu\nu} - \partial_{[\mu} (M \cdot B)_{\nu]}, \]

\[\sigma^A_{\mu} = \mathcal{L}_M A_{\mu} + \frac{1}{2\pi\alpha'} (M \cdot B)_{\mu} - \partial_{\mu} (M \cdot A), \]

\[\sigma^{T1} = \mathcal{L}_M T_1 - 2T_1, \]

\[\sigma^{T2} = \mathcal{L}_M T_2 - T_2, \]

\[\sigma^\phi = \mathcal{L}_M \phi \]

where the vector \(M_{\mu} = 2\alpha' \partial_{\mu} \phi\) yields the direction of the Lie derivative. If we insert the fields (26), (29) we find that the Lie derivative vanish for the metric, antisymmetric tensor and for the gauge field. Furthermore, one finds: \((M \cdot B)_{\mu} = (M \cdot A) = 0\) and hence \(\sigma^{G,B,A} = 0\). This has the consequence that for these background fields the renormalization group \(\beta\) function itself vanish and thus all counterterms. On the other side \(\sigma^{T1,2}\) and \(\sigma^\phi\) do not vanish for our solution, i.e. the tachyons as well as the dilaton have to be renormalized. Although the dilaton is a free field (after the transformation (27) it is linear in time) we have an additive renormalization of \(\phi\). The vanishing of the dilaton \(\dot{\beta}\) function is equivalent to the vanishing of the total central charge where the \(\sigma\) coefficient contains just the contribution of the background charge \(q\).

Let us shortly discuss the mass of the tachyons. We define the mass of the tachyon via the equation of motion, i.e. the \(\dot{\beta}\) equations (11) and (24). In order to read off the mass we have to decouple the tachyon field from the dilaton. This can be done in terms of the transformation \(T_1 = e^{\phi} \tilde{T}_1\) and we get after using the dilaton \(\dot{\beta}\) equation for \(\tilde{T}_1\)

\[-\frac{1}{2} \alpha' D^2 \tilde{T} - \frac{1}{12} \left( \frac{3k}{k+2} - 1 \right) \tilde{T} = 0. \]

For \(T_2\) one obtains a similar expression with one half of the mass. Thus the mass of both tachyons are proportional to: \((1 - \frac{3k}{k+2})\) and we get massless tachyons only if \(k = 1\) or \(c_{wzw} = 1 \)[9]. In this case: \(\alpha_+ = \alpha_- = q = 2\) and after the transformation (27) the massless solutions for \(T_{1,2}\) are given by

\[T^{(1)}_1 \sim e^{2t}, \quad T^{(2)}_1 \sim t e^{2t}, \]

\[T^{(1)}_2 \sim e^t, \quad T^{(2)}_2 \sim t e^t. \]

Here we should remark that all results we were computed in an \(\alpha'\) expansion. On the other hand in a weak field expansion for the tachyon one has to take into account additional divergencies, e.g. \(T^2\) terms. If one has an exponential tachyon as in the Liouville theory one can avoid divergencies like these by an appropriate continuation in the momentum like in the computation of Shapiro–Virasoro amplitudes. But this procedure does not work for oscillating tachyons. Applying this statement to our solution means that only massless tachyons (34) would yield an exact conformal field theory namely the Liouville
theory. Therefore the incorporation of non-perturbative divergencies would restrict us to \( k = 1 \).

Finally, we discuss the effective action yielding the fields (26). In the open string sector the gauge field \( \bar{\beta} \) function follows from the Born-Infeld action [21]

\[
S_{\text{open}} = -\kappa \int d^4x \sqrt{\det(G + F)} e^{-\phi} = -\kappa \int d^4x \sqrt{\det G} e^{-\phi} \sqrt{\det(1 + G^{-1}F)} = 
\]

\[
= -\kappa \int d^4x \sqrt{\det G} e^{-\phi} \sqrt{1 + \frac{1}{2}F^2 + \frac{1}{64}(F \wedge F)^2}
\]

where \( \kappa \) has to be positive in order to couple the gauge field to gravity with the right sign. In the closed string sector we have the known effective action [22]

\[
S_{\text{closed}} = \int \sqrt{\tilde{G}} e^{-2\phi} \left[ -\frac{2}{3}(26 - c) + \alpha'(R + 4(\partial \phi)^2 - \frac{1}{12}H^2) + O(\alpha''^2) \right]
\]

where \( c \) in our model is given by: \( c = 1 + \frac{3k}{k+2} \). In the flat limit \( (k \to \infty) \) we get \( c = 4 \), the number of space time dimensions. One should note here that the incorporation of the tachyons is not straightforward. The reason is the following. In our consideration we consider the \( \sigma \) model strictly in an \( \alpha' \) expansion. In this expansion the tachyons do not influence, e.g. the metric, because there are no counterterms to the metric. Only in a weak field expansion for the tachyon the metric \( \bar{\beta} \) function contains tachyon contributions [14]. This has the consequence that we can not simply incorporate the tachyon in the effective action in a covariant manner as a matter field. Therefore we want to neglect the tachyon fields in this consideration. The total effective action is then given by the sum [18]

\[
S_{\text{eff}} = S_{\text{closed}} + S_{\text{open}}
\]

In the equations of motion following from this effective action are additional terms, e.g. to the metric \( \bar{\beta} \) function (6) we have to add the contribution

\[
\sim \kappa e^\phi \sqrt{\det(1 + G^{-1}F)} \left( \frac{G + F^2}{G - F^2} \right)_{\mu\nu}
\]

Loop calculations show that terms like this can be interpreted as loop-correction to the metric \( \bar{\beta} \) function [18]. These contributions correspond to a gauge field couplings via small holes in the world sheet, i.e. higher genus contributions. Performing the integration of the radius from the hole causes this divergent term.

In order to decouple the dilaton from the graviton it is common to perform the following Weyl rescalling [7]

\[
G_{\mu\nu} \to \tilde{G}_{\mu\nu} = e^{-2\phi} G_{\mu\nu}
\]

and one obtains the action [22]

\[
S = \int \sqrt{\tilde{G}} \left[ \alpha' \tilde{R} - \alpha' 2(\partial \phi)^2 - \alpha' \frac{1}{12}H^2 e^{-4\phi} + \frac{2}{3}(26 - c)e^{2\phi} - \right.
\]

\[
- \kappa e^{3\phi} \sqrt{1 + \frac{1}{2}\tilde{F}^2 + \frac{1}{64}(\tilde{F} \wedge \tilde{F})^2} + ... \right]
\]
with $\tilde{F}_{\mu\nu} = e^{-2\phi} F_{\mu\nu}$. The equation of motion e.g. for the metric is now given by

$$\tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{G}_{\mu\nu}\tilde{R} = T_{\mu\nu}^{\text{matter}}$$  \hspace{1cm} (42)$$

with the energy momentum tensor

$$T_{\mu\nu}^{\text{matter}} = \frac{1}{3\alpha'}(26 - c) e^{2\phi} \tilde{G}_{\mu\nu} + \frac{1}{4} \left(H_{\mu\nu}^2 - \frac{1}{6} \tilde{G}_{\mu\nu} H^2\right) e^{-2\phi} +$$

$$+ \left(\partial_{\mu}\phi \partial_{\nu}\phi - \frac{1}{2} (\partial\phi)^2 \tilde{G}_{\mu\nu}\right) - \kappa e^{3\phi} \sqrt{1 + \frac{1}{2} F^2 + \frac{1}{64} (F \wedge F)^2} \left(\tilde{G}/\tilde{F}^2\right)_{\mu\nu}.$$  \hspace{1cm} (43)$$

Because the field strength contribution correspond to a loop-correction to $\bar{\beta}G$ which we have neglected, our solution fulfills this metric equation of motion only for $\kappa = 0$. If we take our solution (26) and perform the Weyl transformation (40), we find for the metric

$$d\tilde{s}^2 = e^{-2\phi} \frac{Q}{\tau^2} (dt^2 + d\Omega_3^2)$$

$$= d\tau^2 + \frac{q^2 \tau^2}{[1 + \bar{r}^2 \tau^2]} ((dx^1)^2 + (dx^2)^2 + (dx^3)^2) \hspace{1cm} (44)$$

where: $q^2 = \frac{k}{6} (25 - \frac{3k}{k+2})$ and $\bar{r} = (x^1)^2 + (x^2)^2 + (x^3)^2$. This is just the Robertson–Walker metric for a linear expanding universe with the world radius: $K^2 = q^2 \tau^2$. The possible values of $q$ are restricted if we include non-perturbative contributions for the tachyons (see the discussion after (35)). In this case the tachyons have to be massless ($k = 1$) and therewith $q = 2$. Furthermore, if we insert our solution in (43) and after a time reparametrization we get for the energy momentum tensor (for $\kappa = 0$)

$$T_{\mu\nu}^{\text{matter}} = \left(\frac{1}{2q^2} + 2\right) \partial_{\mu}\log \tau \partial_{\nu}\log \tau - \frac{1}{2} \left(\frac{1}{2q^2} - 2\right) \frac{1}{\tau^2} \tilde{G}_{\mu\nu} \hspace{1cm} (45)$$

which correspond to an ideal liquid with the energy density: $\mu = (\frac{1}{4} + 2q^2) \frac{1}{q^2 \tau^2}$ and the isotropic pressure: $p = (q^2 - \frac{1}{4}) \frac{1}{q^2 \tau^2}$.  

5 Conclusion

In this paper we have firstly summarized the known results for a conformally exact $\sigma$ model in closed string theory [3, 8, 9]: a combination of the SU(2) WZW model with the Liouville theory. In addition, we have discussed this model from the $\sigma$ model point of view as solution of the $\bar{\beta}$ equations to all orders in $\alpha'$. In the second part we generalized these results to an open string theory where we coupled an abelian gauge field and a further tachyon via the boundary of the world sheet. Since we have neglected loop corrections (higher genus), the solution of the closed string model remained unchanged and we had yet to look for a solution for the gauge field and boundary tachyon $\bar{\beta}$ equation. Because we considered in this $\sigma$ model an antisymmetric tensor as well as a gauge field coupling, we had to take into account two different gauge transformations. A gauge invariant field strength was given by the sum of the antisymmetric tensor and the usual field strength. Assuming that the tachyon depends only on the radius and neglecting all contributions
which are quadratic or higher in the derivatives of the field strength we got for the open string sector an exact solution too (see (26)). This solution fulfills a monopole equation with a quantized magnetic charge and after a gauge fixing the corresponding gauge field was just given by a Killing vector field of this theory. The investigation of the tachyons showed that both tachyons have quantized masses. This quantization is given by the central charge of the SU(2) WZW model and both tachyons are massless if \( c_{wzw} = 1 \). In addition, only the massless tachyons are not oscillating in the time and yield a conformal field theory also after incorporation of non-perturbative contributions. Finally, we have discussed the effective action where the corresponding gauge field equation of motion follows from a Born-Infeld action. The space time can be interpreted as a linear expanding closed universe.

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