Optimal Bubble Riding:
A Mean Field Game with Varying Entry Times

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June 27, 2022
1. Introduction

2. Bubble Riding Model
   - N-Player Game
   - Mean Field Game (Fixed Entry)

3. Results and Numerics
   - Existence and Approximation
   - Numerical Results
Bubbles

(a) Tech Bubble  
(b) Meme Stocks  
(c) Cryptocurrencies
Kindleberger-Minsky: 5 Phases of Bubble

1. **Displacement**: New technologies, paradigm shifts, or shocks.
2. **Boom**: Entry, media coverage, the hype-building phase.
3. **Euphoria**: Skyrocketing price, herding, "greater-fool".
4. **Profit-Taking**: Optimal stopping/execution, coordinated attacks.
5. **Panic**: Endogenous or exogenous burst, asset price plummets.

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1 (Kindleberger and Bernstein 2000)
Outline

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$N$ players trade on the same asset with finite horizon $T$. The bubble starts at time $t = 0$.

Player $i$ begins trading at $t^i \in [0, \eta]$ with empirical CDF $F^{N}_{T}(t)$

-"Awareness window".\(^2\)
- Different risk preferences.

Independent initial inventory $X_{t^i}^i = \iota^i \overset{d}{\sim} \lambda_0$.

Control: Trading rate $\alpha^i = (\alpha^i_t)_{t^i \leq t \leq T}$, A valued process. Assume $A \subset \mathbb{R}$ is a compact interval. $\alpha^i_t = 0$ on $t < t^i$.

Inventory trajectory is given by

$$dX^i_t = \alpha^i_t dt + \sigma dW^i_t, \quad X^i_t = 0 \text{ on } t < t^i, \quad X^i_{t^i} = \iota^i.$$ 

\(^2\)(Abreu and Brunnermeier 2003)
Number of players in the game by time $t$ is $N_{in}(t) = \sum_{i=1}^{N} 1\{t^i \leq t\}$. We have the average trading rate:

$$\bar{\theta}_t := \begin{cases} 0 & \text{if } N_{in}(t) = 0 \\ \frac{1}{N_{in}(t)} \sum_{i=1}^{N} \alpha_t^i = \frac{1}{F_T^{N}(t)} \int_A a \theta_t^N(da) & \text{if } N_{in}(t) \neq 0 \end{cases}$$

Average inventory:

$$\bar{\mu}_t := \begin{cases} 0 & \text{if } N_{in}(t) = 0 \\ \frac{1}{N_{in}(t)} \sum_{i=1}^{N} X_t^i = \frac{1}{F_T^{N}(t)} \int_{\mathbb{R}} x \mu_t^N(dx) & \text{if } N_{in}(t) \neq 0 \end{cases}$$

where $\theta_t^N = \frac{1}{N} \sum_{i=1}^{N} \delta \alpha_t^i$ and $\mu_t^N = \frac{1}{N} \sum_{i=1}^{N} \delta X_t^i$. 
Pre-burst Price Dynamics

\[ dP_t^+ = b(t, F_T^N(t))dt + \left[ g(\theta_t^N)dt + \sigma_0 dW_t^0 \right], \quad P_0^+ = P_0, \]

- **Bubble component** \( \gamma_t \):
  \[ d\gamma_t = b(t, F_T^N(t))dt, \quad \gamma_0 = 0 \]

- **Fundamental price component** \( Q_t \): the N-player extension of the price impact model\(^1\). \( g \): permanent impact function (e.g. linear).
  \[ dQ_t = g(\theta_t^N)dt + \sigma_0 dW_t^0, \quad Q_0 = P_0 \]

Similar role to the reference price\(^2\), or the fundamental value process\(^3\).

\(^1\)(Almgren and Chriss 2001; Carlin, Lobo, and Viswanathan 2007)
\(^2\)(Johansen, Sornette, and Ledoit 1999)
\(^3\)(Abreu and Brunnermeier 2003)
An *endogenous* burst refers to the collapse caused by the cumulative selling pressure (within the system).

An *exogenous* burst, sometimes called a sunspot event, refers to a commonly observable event that triggers the cascading price adjustment from outside the system.

**Definition (N-Player endogenous burst time)**

\[
\bar{\tau}^N(\mu^N) := \inf \left\{ t > \min_{i \in \{1,2,\ldots,N\}} t^i : \inf_{s \in [0,t]} \bar{\mu}_s^N \leq \zeta_0 + \zeta(t) \right\} \wedge T
\]

**Definition (Bubble Burst Time)**

The actual burst time is

\[
\tau^* := \bar{\tau}^N(\mu^N) \wedge \tau.
\]
The crash consumes a fraction\(^1\) \(\beta_t \in [0, 1]\) of the bubble component. At burst time, price drops by \(\beta_{\tau^*} \gamma_{\tau^*}\), and player \(i\) loses

\[
X^i_{\tau^*} \beta_{\tau^*} \gamma_{\tau^*} = X^i_{\tau^*} \beta_{\tau^*} \int_0^{\tau^*} b(t, F_N(t)) \, dt.
\]

Define jump processes for \(\tau^*\): \(D^*_t := \mathbb{1}_{\{\tau^* \leq t\}}\). Then the price dynamics is

\[
P_t = (1 - D^*_t) P^+_t + D^*_t Q_t + (1 - \beta^*_{\tau}) \gamma_{\tau^*} D^*_t.
\]

\(^1\)“Bubble size” in Abreu and Brunnermeier 2003, “Loss Amplitude” in LPPL models. See e.g. Sornette et al. 2013.
Objective: Total Cost

Given realization of entry times $t = (t^1, \ldots, t^N) \in \mathcal{T}^N$ and $\alpha = (\alpha^1, \ldots, \alpha^N) \in \mathbb{A}^N(t)$, player $i$ minimizes

$J^{N,i}(\alpha, t) := \mathbb{E} \left[ -(V_T^i - V_{t^i}^i) + \int_{t^i}^T \phi(X_t^i)^2 dt + c(X_T^i)^2 \right]$

$= \mathbb{E} \left[ \int_{t^i}^T f(t, X_t^i, F_T^N(t), \mu_t^N, \theta_t^N, \alpha_t^i) dt + X_T^i \beta_T^* \gamma_T^* + c(X_T^i)^2 \right]$

where the running cost function $f$ is

$f(t, x, r, \mu^N, \theta, a) = \kappa a^2 + \phi x^2 - xg(\bar{\theta}) - xb(t, r) \mathbb{1}_{\{\tau^*(\mu^N) > t\}}$.

Notice that we also impose a temporary price impact (slippage cost) with parameter $\kappa > 0$. 
Mean Field Game (Fixed Entry): $N \to \infty$

For a given flow of probability measures on controls and states $(\theta, \mu)$:

**Definition (Burst Time)**

We define the endogenous burst time for the mean field game as

$$\bar{\tau}(\mu) := \inf \left\{ t > 0 : \inf_{s \in [0, t]} \bar{\mu}_s \leq \zeta_0 + \zeta(t) \right\} \wedge T.$$ 

The actual burst time of the mean field game is

$$\tau^* := \bar{\tau}(\mu) \wedge \tau.$$ 

Over *admissible* strategies, the representative player minimizes

$$J^{\theta, \mu}(\alpha) = \mathbb{E} \left[ X_{T^*} \beta_{T^*} \gamma_{T^*} + cX_T^2 + \int_0^T f(t, X_t, \mu, \theta_t, \alpha_t) dt \right]$$

where $X$ follows the forward inventory trajectory that corresponds to $\alpha$. 
Market information + Observing the Exogenous Burst.
Market information + Observing the Exogenous Burst.
Let $\mathcal{F}$ be the Brownian filtration. Enlarge to $\mathcal{G}$ with process $\mathbf{1}_{\{\tau \leq t\}}$. 

$\text{Driftless state}$

$X_t = \iota + \sigma W_t$.

For each $\alpha \in A$, define $P_\alpha$ by

$dP_\alpha = E_{R_{t \geq 0}} \sigma^{-1} \alpha s dW_s$.

$(X, P_\alpha, W_\alpha)$ solves inventory equation weakly, where $W_\alpha t = W_t - R_{t \geq 0} \sigma^{-1} \alpha s dW_s$.

Coupled FBSDE reduces to a BSDE with reduced Hamiltonian as generator.
Market information + Observing the Exogenous Burst. Let $\mathbb{F}$ be the Brownian filtration. Enlarge to $\mathbb{G}$ with process $1\{\tau \leq t\}$.

1. Progressive Enlargement of Filtration (Jeulin and Yor 1985)
2. Credit Risk (Elliott, Jeanblanc, and Yor 2000)
Market information + Observing the Exogenous Burst.
Let $\mathbb{F}$ be the Brownian filtration. Enlarge to $\mathbb{G}$ with process $1\{\tau \leq t\}$.

1. Progressive Enlargement of Filtration (Jeulin and Yor 1985)
2. Credit Risk (Elliott, Jeanblanc, and Yor 2000)
3. Weak Formulation of Mean Field Games (Carmona and Lacker 2015; Possamaï and Tangpi 2021)
   - Driftless state $X_t = \iota + \sigma W_t$.
   - For each $\alpha \in \mathbb{A}$, define $\mathbb{P}^\alpha$ by $\frac{d\mathbb{P}^\alpha}{d\mathbb{P}} = \mathcal{E}\left(\int_0^t \sigma^{-1} \alpha_s dW_s\right)_T$.
   - $(X, \mathbb{P}^\alpha, W^\alpha)$ solves inventory equation weakly, where $W^\alpha_t := W_t - \int_0^t \sigma^{-1} \alpha_s dW_s$
   - Coupled FBSDE reduces to a BSDE with reduced Hamiltonian as generator.
Then the objective is can be expressed as

\[ J^{\theta, \mu}(\alpha) := \mathbb{E}^{\mathbb{P}_\alpha} \left[ X_{\tau^*} \beta_{\tau^*} \gamma_{\tau^*} + cX_T^2 + \int_0^T f(s, X, \theta_s, \tau^*, \alpha_s) \, ds \right] \]

**Definition (Mean Field Equilibrium)**

A mean field equilibrium for fixed entry time is a triplet \((\hat{\alpha}, \theta, \mu)\) such that \(\hat{\alpha}\) is admissible and minimizes \(J^{\theta, \mu}\). Moreover, \(\mathbb{P}^{\hat{\alpha}} \circ X^{-1} = \mu\), and \(\mathbb{P}^{\hat{\alpha}} \circ \hat{\alpha}_t^{-1} = \theta_t\) for Lebesgue almost every \(t\).
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Theorem (Existence)

Under the assumption that $A$ is compact, along with mild continuity assumptions, there exists a MFG equilibrium $(\hat{\alpha}, \theta, \mu)$ for fixed entry time. Moreover, the optimal control can be decomposed as

$$\hat{\alpha}_t = \hat{\alpha}_t^+ 1\{t \leq \tau^*\} + \hat{\alpha}_t^-(\tau^*) 1\{t > \tau^*\}$$

where $\hat{\alpha}_t^+$ is $\mathbb{F}$-measurable and $\hat{\alpha}_t^-$ a Borel measurable function of $\tau^*$ that is also (jointly) $\mathbb{F}$-measurable.

Theorem (N-player Approximation)

The MFG equilibrium forms an $\varepsilon$-Nash equilibrium for the $N$-player game in the weak setup.
Proof Idea: BSDE and Decomposition

Given \((\theta, \mu)\), for each admissible \(\alpha\), let \((Y^\alpha, Z^\alpha)\) be a solution on \(\mathbb{G}\) to

\[
Y^\alpha_t = \xi_{\tau^*} + \int_t^T f(s, X, \theta_s, \tau^*, \alpha_s) + \sigma^{-1} \alpha_s Z_s^\alpha ds - \int_t^T Z_s^\alpha dW_s - \int_t^T U_s^\alpha dM_s
\]

Then the objective can be written as

\[
J^{\theta, \mu}(\alpha) = \mathbb{E}^\mathbb{P}^\alpha [Y^\alpha_0] = \mathbb{E}[Y^\alpha_0].
\]

1. Comparison principle: \(\hat{\alpha}_t = \hat{a}(t, x, z)\).
2. Brouwer-Schauder-Tychonov fixed point theorem.
3. Decomposition (Ankirchner, Blanchet-Scalliet, and Eyraud-Loisel 2010; Kharroubi and Lim 2014).
Equilibrium Trajectories

Default: $c = 10$, $\kappa = 0.5$, $\delta = 0.5$, $\phi = 0.1$, $\sigma = 1$, $\zeta = 2$, $k = 2$,
$\text{TotalCost} = -25.225$, Endo Burst: 51.9%

LowTransaction: $\kappa = 0.1$, $\delta = 0.3$,
$\text{TotalCost} = -115.49$, Endo Burst: 54.7%

FearExo: $k = 5$,
$\text{TotalCost} = 29.111$, Endo Burst: 23.6%

Figure 1

Figure 2

Figure 3
Average Reaction to Burst Times

Default

FearExo
\( k = 5 \)

NoBubble

LowTransaction
\( \kappa = 0.1, \delta = 0.3 \)

NoExo
\( k = 0.1 \)

BigBubble
Two Regimes on Priors

**Theta for Different Priors on Exogenous Burst**

- $k = 0.05$, $E[\tau] = 5.6$, Endo$\% = 98.6\%$
- $k = 0.1$, $E[\tau] = 3.96$, Endo$\% = 96.3\%$
- $k = 0.5$, $E[\tau] = 1.77$, Endo$\% = 83.4\%$
- $k = 1$, $E[\tau] = 1.25$, Endo$\% = 69.9\%$
- $k = 1.5$, $E[\tau] = 1.02$, Endo$\% = 60.6\%$
- $k = 2$, $E[\tau] = 0.89$, Endo$\% = 50.9\%$
- $k = 5$, $E[\tau] = 0.56$, Endo$\% = 25.3\%$
- $k = 7$, $E[\tau] = 0.47$, Endo$\% = 16.4\%$
- $k = 10$, $E[\tau] = 0.4$, Endo$\% = 5.5\%$
- $k = 20$, $E[\tau] = 0.28$, Endo$\% = 0.2\%$

**Mu for Different Priors on Exogenous Burst**

- $k = 0.05$, Cost = $-137.94$
- $k = 0.1$, Cost = $-133.33$
- $k = 0.5$, Cost = $-100.3$
- $k = 1$, Cost = $-68.69$
- $k = 1.5$, Cost = $-44.38$
- $k = 2$, Cost = $-25.33$
- $k = 5$, Cost = $29.19$
- $k = 7$, Cost = $42.89$
- $k = 10$, Cost = $53.78$
- $k = 20$, Cost = $68.09$
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Thank you!