Triangles on $\mathbb{Z}^2$ and sliding phenomenon

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Abstract

We show that for all $D > 0$ there exists an acute-angled triangle inscribes in $\mathbb{Z}^2$ with side-lengths at least $D$ and area $\sqrt{3}/4 \cdot D^2 + O(D^{4/5})$. As a byproduct, we confirm a conjecture from [1] claiming that there are only finitely many values of $D$ for which so-called sliding phenomenon in dense-packing configurations occurs.

1 Introduction

In the paper [1] authors study extremal Gibbs measures for the high-density hard-core model on the square lattice. It turns out that the structure of extremal Gibbs measures is very sensitive to arithmetic properties of the hard-core exclusion distance $D$. Namely, the question boils down to the study of the set $\mathcal{T}(D)$ of acute-angled triangles inscribed in $\mathbb{Z}^2$ with all sides at least $D$. Depending on the number and form of triangles from $\mathcal{T}(D)$ of smallest area, possible values of $D$ are split into three classes which then lead to different structure of Gibbs states. We refer to [1] for details on the model and the study of its Gibbs measures.

One of the classes is formed by values of $D$ which generate sliding. We say that sliding occurs if there exist two triangles from the set $\mathcal{S}(D)$ which share two vertices, see Definition 2.2 for a formal definition. It was conjectured in [1] that sliding occurs only for finitely many values of $D$.

Motivated by this question, we study the smallest possible area of a triangle from $\mathcal{T}(D)$. We show that this area is asymptotically $\sqrt{3}/2 \cdot D^2 + O(D^{4/5})$, whereas if sliding occurs then this area cannot be smaller than $\sqrt{3}/2 \cdot D^2 + cD$ for some small constant $c > 0$.

The note is organised as follows. In Section 2 we show an upper bound on the smallest possible area of a triangle from $\mathcal{T}(D)$ and in Section 3 we show an upper bound for this area when sliding occurs.

2 Smallest possible area.

The aim of this section is to show a general upper bound on the smallest possible area of an acute-angled triangles inscribed in $\mathbb{Z}^2$ with side-lengths at least $D$. We start with some definitions.
**Definition 2.1.** For a positive $D$ we define $\mathcal{T}(D)$ to be the set of all triangles with vertices in $\mathbb{Z}^2$, side-lengths at least $D$, and angles at most $\pi/2$. Following [1], we denote twice the smallest area of a triangle from $\mathcal{T}(D)$ by $S(D)$. We refer to triangles from $\mathcal{T}(D)$ of minimal area as $M$-triangles.

**Definition 2.2.** We say that for $D > 0$ sliding occurs if there exists two $M$-triangles OW $A$ and OW $B$ with $A$ and $B$ being on the same side of OW. We refer to OW as to the base of the triangle.

In the following, for a real $x$ we denote by $\{x\}$ the fractional part of $x$ and by $\|x\|$ the distance from $x$ to the closest integer. We also use the notation $O(f(n))$ to denote any function which is upper-bounded by $C \cdot f(n)$ for some absolute constant $C > 0$. We start from a reduction to a number theoretic problem.

**Lemma 2.1.** Let $D > 100$ and $\varepsilon < 0.1$ and suppose there exist integers $x$ and $y$ such that

$$
(1) \|\sqrt{3}x\| < \varepsilon \quad (2) \|\sqrt{3}y\| < \varepsilon \quad (3) 2\sqrt{x^2 + y^2} \in [D + \sqrt{2\varepsilon}, D + 2\varepsilon],
$$

Then $S(D) < \sqrt{3}/2 \cdot D^2 + 6D\varepsilon + 8\varepsilon^2$.

**Proof.** Note that points $A(0,0), B(2x, 2y) \text{ and } C(x - \sqrt{3}y, y + \sqrt{3}x)$ form an equilateral triangle with side length between $D + \sqrt{2\varepsilon}$ and $D + 2\varepsilon$. Let $z$ and $t$ be integers closest to $x - \sqrt{3}y$ and $y + \sqrt{3}x$ respectively. First two conditions ensure that $|z - (x - \sqrt{3}y)| < \varepsilon$ and $|t - (y + \sqrt{3}x)| < \varepsilon$, and so points $C'(z, t)$ and $C(x - \sqrt{3}y, y + \sqrt{3}x)$ are at most $\sqrt{2\varepsilon}$ apart. Thus, by triangle inequality, all sides of the triangle $ABC'$ are at least $D$.

To upper-bound the area of the triangle $\mathcal{T}$ with vertices $A, B, C'$ note that $|AB| \leq D + 2\varepsilon$ and

$$
\rho(C', AB) \leq \rho(C, AB) + \sqrt{2\varepsilon} = \sqrt{3}/2 \cdot |AB| + \sqrt{2\varepsilon} \leq \sqrt{3}/2 \cdot (D + 2\varepsilon) + \sqrt{2\varepsilon} < \sqrt{3}/2 \cdot D + 4\varepsilon.
$$

Since $S(D)$ is at most the area of $\mathcal{T}$, we have

$$
S(D) \leq (D + 2\varepsilon) \cdot (\sqrt{3}/2 \cdot D + 4\varepsilon) < \sqrt{3}/2 \cdot D^2 + 6\varepsilon D + 8\varepsilon^2,
$$

which concludes the proof. \qed

**Remark 1.** Alternatively, to upper-bound the area of $\mathcal{T}$ one can note that all sides of $\mathcal{T}$ are at most $D + (2 + \sqrt{2})\varepsilon$.

In order to find a pair $(x, y) \in \mathbb{Z}^2$ satisfying (1) we need a simple lemma.

**Lemma 2.2.** There exists an absolute constant $C > 0$ such that for every $\varepsilon < 1$ and $x \in [0, 1]$ there exists an integer $s \leq C/\varepsilon$ such that $\{s\sqrt{3}\} \in [x, x + \varepsilon]$ which is probably standard.
Proof. It suffices to prove the statement for $\varepsilon_k := 10^{-k}$. We use induction on $k$. The base is trivial. Suppose we want to prove the statement for $k+1$ and some $x$. Using induction hypothesis we can find $s \leq C/\varepsilon_k$ such that \(\{s\sqrt{3}\} \in [x - \varepsilon_k, x]\). Then choose $v \leq 10/\varepsilon_{k+1}$ such that \(v\sqrt{3} < \varepsilon_{k+1}\) (for example, by solving Pell’s equation $3v^2 - 2 = u^2$). It is easy to see that we always have \(v\sqrt{3} > 1/(3v)\) and so by adding $v$ to $s$ at most $\varepsilon_k \cdot 3v \leq 300$ times we get $s'$ with \(\{s'\sqrt{3}\} \in [x, x+\varepsilon_{k+1}]\). We have $s' \leq 300v + s \leq 3000/\varepsilon_{k+1} + C/\varepsilon_k < C/\varepsilon_{k+1}$, provided $C$ is large enough.

Corollary 2.1. There exists an absolute constant $C > 0$ such that for every $\varepsilon < 1$ and every $n \in \mathbb{R}$ there exists an integer $k \in \lfloor n - C/\varepsilon \rfloor$ such that $\|k\sqrt{3}\| < \varepsilon$.

Proof. By increasing $C$ slightly, we may assume that $n$ is an integer. Let $x := \{n\sqrt{3}\}$. By Lemma 2.2 there exists an integer $s \leq C/\varepsilon$ such that \(\{s\sqrt{3}\}\) is $\varepsilon$-close to $x$ and then $k := n - s$ works.

We now construct a pair $(p, q)$ satisfying conditions similar to (1).

Lemma 2.3. There exists an absolute constant $C_1 > 0$ such that for any $N > 100$ and $\varepsilon < 1$ there exists $(x, y) \in \mathbb{Z}^2$ satisfying

\[
\begin{align*}
(1) \|\sqrt{3}x\| &< \varepsilon & (2) \|\sqrt{3}y\| &< \varepsilon & (3) x^2 + y^2 &\in [N, N + C_1(\varepsilon^{-2} + N^{1/4}\varepsilon^{-3/2})],
\end{align*}
\]

Proof. Let $C$ be the constant from Corollary 2.1. By Corollary 2.1 choose $x \in (\sqrt{N} - C/\varepsilon, \sqrt{N})$ such that $\|x\sqrt{3}\| < \varepsilon$. Then choose $y \in (\sqrt{N} - x^2, \sqrt{N} - x^2 + C/\varepsilon)$ with $\|y\sqrt{3}\| < \varepsilon$. We then have

\[
0 < N - x^2 < 2 \cdot (C/\varepsilon) \cdot \sqrt{N} = 2C \cdot N^{1/2} \varepsilon^{-1},
\]

and so

\[
0 < x^2 + y^2 - N < (C/\varepsilon)^2 + 2 \cdot (C/\varepsilon) \cdot \sqrt{N - x^2} < C^2 \varepsilon^{-2} + 2C^{3/2} N^{1/4} \varepsilon^{-3/2},
\]

and we can choose $C_1 = C^2$.

Finally, we prove

Theorem 1. Asymptotically, for large $D$, one has $S(D) = \sqrt{3}/2 \cdot D^2 + O(D^{4/5})$.

Proof. Take $\varepsilon := C \cdot D^{-1/5}$ with large enough constant $C > 0$. Define $N := \left(\frac{D + \sqrt{2}\varepsilon}{2}\right)^2$.

By Lemma 2.3 we can find $(x, y) \in \mathbb{Z}^2$ with $\|\sqrt{3}x\| < \varepsilon$, $\|\sqrt{3}y\| < \varepsilon$ and $x^2 + y^2 \in [N, N + C_1(\varepsilon^{-2} + N^{1/4}\varepsilon^{-3/2})]$. We then use Lemma 2.1 to obtain $S(D) < \sqrt{3}/2 \cdot D^2 + 6D\varepsilon + 8\varepsilon^2 - \sqrt{3}/2 \cdot D^2 + O(D^{4/5})$. We now only need to check that the upper bound on $x^2 + y^2$ given by Lemma 2.3 matches the upper bound needed to use Lemma 2.1

\[
2\sqrt{x^2 + y^2} < 2 (N + C_1(\varepsilon^{-2} + N^{1/4}\varepsilon^{-3/2}))^{1/2} = 2\sqrt{N} + O(N^{-1/4}\varepsilon^{-3/2})
\]

\[
= D + \sqrt{2}\varepsilon + O(D^{-1/2}\varepsilon^{-3/2}) < D + 2\varepsilon,
\]

for $\varepsilon = C \cdot D^{-1/5}$ and an absolute constant $C$ large enough.
3 Sliding

In this section we give a lower bound for $S(D)$ in the presence of sliding.

**Theorem 2.** There exists an absolute constant $c > 0$ such that for all $D > 100$ for which sliding occurs one has $S(D) > \sqrt{3}/2 \cdot D^2 + c \cdot D$.

**Proof.** Set $\varepsilon := 0.1$. If at least one side of an M-triangle has length greater than $D + \varepsilon$ than the area $S(D)$ of this triangle is at least the area of a triangle with sides $D, D, D + \varepsilon$ which is at least $\sqrt{3}/2 \cdot D^2 + D/20$, so we are done. So from now on we assume that all sides of all M-triangles are in $[D, D + \varepsilon]$. It then follows that all angles of M-triangles are in $(\pi/3 - \pi/9, \pi/3 + \pi/9)$.

Let $OW$ be the base and $A$ and $B$ be remaining vertices of two M-triangles, see Definition 2.2. Since areas of all M-triangles is $S(D)$, line $AB$ is parallel to the base $OW$. By switching $A$ and $B$ if necessary, we may assume that points $O, A, B, W$ form a trapezoid in this order. Consider now triangle $OAB$. We have $\varphi := \angle OAB = \pi - \angle AOW > \pi - (\pi/3 + \pi/9) = 5\pi/9$. But then, using cosine theorem, we get

$$|OB| - |OA| = \frac{|OB|^2 - |OA|^2}{|OB| + |OA|} \geq \frac{|AB|^2 + 2 \cos (\pi - \varphi) |AB| \cdot |OA|}{2D + 2\varepsilon}.$$  

Since $|AB| \geq 1$, the numerator is at least $2 \cos (4\pi/9) \cdot D$, whereas the denominator is at most $3D$. Hence, $|OB| - |OA| \geq 2 \cos (4\pi/9)/3 > \varepsilon$ contradicting the fact that $|OB|, |OA| \in [D, D + \varepsilon]$.

Combining Theorem 1 and Theorem 2 we obtain

**Theorem 3.** For $D$ large enough sliding does not occur.

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**References**

[1] A. Mazel, I. Stuhl, and Y. Suhov, *High-density hard-core model on $\mathbb{Z}^2$ and norm equations in ring $\mathbb{Z}[\sqrt{3}]$*. 