ON HECKE EIGENVALUES OF SIEGEL MODULAR FORMS IN THE MAASS SPACE

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ABSTRACT. In this article, we prove an omega-result for the Hecke eigenvalues $\lambda_F(n)$ of Maass forms $F$ which are Hecke eigenforms in the space of Siegel modular forms of weight $k$, genus two for the Siegel modular group $Sp_2(\mathbb{Z})$. In particular, we prove

$$\lambda_F(n) = \Omega(n^{k-1} \exp(c \sqrt{\frac{\log n}{\log \log n}})),$$

when $c > 0$ is an absolute constant. This improves the earlier result

$$\lambda_F(n) = \Omega(n^{k-1} \frac{\sqrt{\log n}}{\log \log n})$$

of Das and the third author. We also show that for any $n \geq 3$, one has

$$\lambda_F(n) \leq n^{k-1} \exp(c_1 \sqrt{\frac{\log n}{\log \log n}}),$$

where $c_1 > 0$ is an absolute constant. This improves an earlier result of Pitale and Schmidt. Further, we investigate the limit points of the sequence $\{\lambda_F(n)/n^{(k-1)/2}\}_{n \in \mathbb{N}}$ and show that it has infinitely many limit points. Finally, we show that $\lambda_F(n) > 0$ for all $n$, a result earlier proved by Breulmann by a different technique.

1. INTRODUCTION

For $g \geq 1$, let $\Gamma_g := Sp_g(\mathbb{Z})$ be the Siegel modular group of genus $g$ and $S^0_k$ be the space of cuspidal Siegel modular forms of weight $k$ and genus $g$ for $\Gamma_g$. One of the interesting problems in the theory of modular forms is to understand arithmetic nature of eigenvalues of the Hecke operators acting on the space $S^0_k$. Let $f$ be a normalised Hecke eigenform of weight $k$ and genus $g = 1$ with the Hecke eigenvalues $\lambda_f(n)$. By a celebrated result of Deligne, one has

$$|\lambda_f(n)| \leq d(n) n^{(k-1)/2},$$

where $d(n)$ is the number of divisors of $n$. One would like to know the optimality of the above result, i.e. an omega result for the sequence $\{\lambda_f(n)/n^{(k-1)/2}\}_{n \in \mathbb{N}}$. In 1973, Rankin [10] showed that

$$\limsup_n \frac{\lambda_f(n)}{n^{(k-1)/2}} = +\infty.$$
In 1983, Ram Murty \cite{6} showed that
\[ |\lambda_f(n)| = \Omega\left(n^{(k-1)/2}\exp(c\log n / \log \log n)\right), \]
where \(c > 0\) is an absolute constant.

It is natural to investigate similar questions for higher genus. In this direction, the generalised Ramanujan-Petersson conjecture \cite{8} implies that for any prime \(p\) and \(\epsilon > 0\), one has
\[ \lambda_F(p) \ll_{g,\epsilon} p^{g(k/2-g(g+1)/4+\epsilon)}. \]
However, it is known that when \(g = 2\), the elements of the Maass space in \(S^2_k\) are precisely the ones which fail to satisfy equation (1). R. Weissauer \cite{12} showed that Ramanujan-Petersson conjecture is true when \(F\) does not belong to the Maass space in \(S^2_k\).

From now on, we concentrate on the space of Maass cusp forms denoted by \(S^*_k\) in the space \(S^2_k\). In a recent work \cite{4}, Das along with the third author studied the question of omega result for Hecke eigenvalues of the Hecke operators acting on \(S^*_k\). But there is a considerably large gap between the known upper bound
\[ \lambda_F(n) \ll_{\epsilon} n^{k-1+\epsilon}, \quad \text{for any } \epsilon > 0 \]
and the known omega result.

In this article, we investigate arithmetic behaviour of Hecke eigenvalues of Maas forms in \(S^*_k\) and also study the limit points of the sequence \(\{\lambda_F(n)/n^{k-1}\}_{n \in \mathbb{N}}\). More precisely, we prove the following theorems.

**Theorem 1.** Let \(F \in S^*_k\) be a non-zero Hecke eigenform. Then there exists an absolute constant \(c > 0\) such that
\[ \lambda_F(n) = \Omega(n^{k-1}\exp(c\sqrt{\log n / \log \log n})) \]

Our next theorem shows that the above omega result is not too far from an upper bound one can derive. In particular, we have

**Theorem 2.** Let \(F \in S^*_k\) be a non-zero Hecke eigenform. Then there exists an absolute constant \(c_1 > 0\) such that
\[ \lambda_F(n) \leq n^{k-1}\exp\left(c_1\sqrt{\log n / \log \log n}\right) \]
for all \(n \in \mathbb{N}\) with \(n \geq 3\).

**Remark 1.1.** Theorem 2 improves an earlier result of Pitale and Schmidt (see page 101 of \cite{9}).

We also have the following lower bound.
Theorem 3. Let \( F \in S^*_k \) be a non-zero Hecke eigenform. Then there exist absolute constants \( c_2, c_3 > 0 \) such that

\[
\lambda_F(n) \geq c_2 n^{k-1} \exp \left( -c_3 \sqrt{\frac{\log n}{\log \log n}} \right)
\]

for all \( n \in \mathbb{N} \) with \( n \geq 3 \).

As a corollary, we now get the following result of Breulmann [2] whose proof is different from ours.

Corollary 1. If \( F \in S^*_k \) is a non-zero Hecke eigenform with Hecke eigenvalues \( \lambda_F(n) \), then \( \lambda_F(n) > 0 \).

Since \( \frac{\lambda_F(n)}{n^{k-1}} > 0 \), one can ask whether this result is optimal. Our next theorem shows that the answer is positive.

Theorem 4. Let \( F \in S^*_k \) be a non-zero Hecke eigenform. Then

\[
\liminf_n \frac{\lambda_F(n)}{n^{k-1}} = 0
\]

Finally, we investigate the limit points of the sequence \( \{\lambda_F(n)/n^{k-1}\}_{n \in \mathbb{N}} \). In this direction, we have the following result.

Theorem 5. Infinitely many limit points of the sequence \( \{\frac{\lambda_F(n)}{n^{k-1}}\}_{n \in \mathbb{N}} \) are greater than 1 and infinitely many of them are less than 1.

In order to prove our results, we rely on an idea of Rankin [10] and some standard analytic techniques. We manage to avoid the use of Sato-Tate conjecture which is now a theorem due to Barnet-Lamb, Geraghty, Harris and Taylor [3].

2. NOTATION AND PRELIMINARIES

Throughout the article, let \( \mathcal{P} \) denote the set of all rational prime numbers. Also we use the notation \( q := e^{2\pi iz} \), where \( z \in \mathbb{H} \), the complex upper half-plane. We say that \( f(x) = \Omega(g(x)) \) to indicate that \( \limsup_{x \to \infty} |f(x)/g(x)| > 0 \). Moreover, we shall write \( f(x) \sim g(x) \) when \( \lim_{x \to \infty} f(x)/g(x) = 1 \). A subset \( A \subset \mathcal{P} \) is said to have the lower natural density \( \alpha(A) \) if

\[
\alpha(A) = \liminf_{x \to \infty} \frac{\#\{p \leq x : p \in A\}}{\#\{p \leq x\}}
\]

We now recall the following lemma which we will use to prove our results.

Lemma 6. ([4, lemma 3.1]) Let \( f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S^*_k \) be a normalised Hecke eigenform. Then there exists an absolute constant \( 0 < \beta < 2 \) such that the set

\[
\{ p \in \mathcal{P} : a(p) > \beta \cdot p^{(k-1)/2} \}
\]

has positive lower density.
One can use the Sato-Tate conjecture to get the above result but proof of the lemma avoids that.

3. **Proof of Theorem**

Let $F \in S_k^*$ be a nonzero Hecke eigenform with eigenvalues $\lambda_F(n)$. Then there exists a normalised Hecke eigenform $f$ of weight $2k - 2$ for the full modular group $SL_2(\mathbb{Z})$ such that $F$ is the Saito-Kurokawa lift of $f$. We know that for any prime $p$, one has (see [2] for details)

$$\lambda_F(p) = p^{k-1}(1 + \frac{1}{p} + \frac{a(p)}{p^{k-1}}).$$

Note that by lemma 6 there exists an absolute constant $0 < \beta < 2$ such that the set $A := \{p : a(p) > \beta \cdot p^{k-3/2}\}$ has positive lower density. For any $x > 0$, let

$$n_x := \prod_{5 \leq p \leq x, \ p \in A} p$$

with the convention that an empty product is $1$. Then for sufficiently large $x \in \mathbb{R}^+$, we have

$$\frac{\lambda_F(n_x)}{n_x^{k-1}} = \prod_{5 \leq p \leq x, \ p \in A} \left(1 + \frac{1}{p} + \frac{a(p)}{p^{k-1}}\right) \geq \prod_{5 \leq p \leq x, \ p \in A} \left(1 + \frac{a(p)}{p^{k-1}}\right) \geq \exp \left[ \sum_{5 \leq p \leq x, \ p \in A} \log(1 + \frac{\beta}{p^{1/2}}) \right] \geq \exp \left( c_4 \sum_{5 \leq p \leq x, \ p \in A} \frac{1}{p^{1/2}} \right),$$

where $c_4 > 0$ is an absolute constant. Since the set $A$ has positive lower density, by partial summation formula, it can be seen that

$$\sum_{5 \leq p \leq x, \ p \in A} \frac{1}{p^{1/2}} \gg \frac{\sqrt{x}}{\log x},$$

where the implied constant is absolute. Further note that for any $x \in \mathbb{R}^+$, we have

$$\log(n_x) = \sum_{5 \leq p \leq x, \ p \in A} \log p \ll x$$

with the convention that an empty sum is zero. Note that $\frac{\sqrt{x}}{\log x}$ is an increasing function for $x \geq 8$. Thus for sufficiently large $x$, we have

$$\frac{\lambda_F(n_x)}{n_x^{k-1}} \geq \exp \left( c_5 \frac{\sqrt{x}}{\log x} \right) \geq \exp \left( c_6 \frac{\log n_x}{\log \log n_x} \right),$$
where $c, c_5 > 0$ are absolute constants. This shows that given any natural number $M$, there exists a natural number $n$ with $n > M$ such that

$$\frac{\lambda_F(n)}{n^{k-1}} \geq \exp\left( c \frac{\sqrt{\log n}}{\log \log n} \right).$$

This completes the proof of Theorem 1.

4. PROOF OF THEOREM 2

In this section, we keep the notations as in the section 2 and section 3. It can be deduced from [2] that for all $m \in \mathbb{N}$ and any $p \in \mathcal{P}$, we have

$$\frac{\lambda_F(p^m)}{p^{m(k-1)}} = 1 + \frac{1}{p} + \left(1 + \frac{1}{p}\right) \sum_{\ell=1}^{m-1} \frac{a(p^\ell)}{p^{\ell(k-1)}} + \frac{a(p^m)}{p^{m(k-1)}}$$

with the convention that an empty sum is zero. For any $|\lambda| < 1$, the series

$$\sum_{n=2}^{\infty} (n+1)\lambda^n = \sum_{n=3}^{\infty} n\lambda^{n-3} = \frac{3\lambda^2 - 2\lambda^3}{(1 - \lambda)^2}.$$

This can be seen by considering the power series

$$f(Y) = \sum_{n \geq 3} Y^n = \frac{1}{1-Y} - 1 - Y - Y^2$$

and noting that

$$f'(Y) = \frac{3Y^2 - 2Y^3}{(1-Y)^2},$$

where $f'$ is the derivative of $f$. For any $p \in \mathcal{P}$, let us set

$$\alpha_p := \sum_{n=2}^{\infty} \frac{n+1}{p^{n/2}} = \frac{3p^{1/2} - 2}{p^{1/2}(p^{1/2} - 1)^2}.$$

By the work of Deligne, one knows that

$$\frac{a(n)}{n^{k-3/2}} \leq d(n),$$

where $d(n)$ denotes the number of divisors of $n$. This shows that for any $p \in \mathcal{P}$ and $m \in \mathbb{N}$ with $m \geq 2$, we have

$$\frac{\lambda_F(p^m)}{p^{m(k-1)}} \leq 1 + \frac{1}{p} + \left(1 + \frac{1}{p}\right) \frac{2}{p^{1/2}} + \left(1 + \frac{1}{p}\right) \alpha_p.$$

Note that $\alpha_p \asymp \frac{1}{p}$. Hence there exists an absolute constant $c_7 > 0$ such that

$$\frac{\lambda_F(p^m)}{p^{m(k-1)}} \leq 1 + \frac{c_7}{p^{1/2}}.$$
for all \( m \in \mathbb{N} \). Let \( n \geq 3 \) be an arbitrary natural number and let \( t = \nu(n) \) be its number of distinct prime divisors. Then we can write \( n \) as

\[
n = p_1^{m_1} \cdots p_t^{m_t}
\]

where \( p_1 < \cdots < p_t \) and \( m_i > 0 \) for \( 1 \leq i \leq t \). Thus we have

\[
\frac{\lambda_F(n)}{n^{k-1}} \leq \prod_{1 \leq i \leq t} \left( 1 + \frac{c_7}{p_i^{1/2}} \right) = \exp \left( \sum_{1 \leq i \leq t} \log \left( 1 + \frac{c_7}{p_i^{1/2}} \right) \right) \leq \exp \left( c_7 \sum_{1 \leq i \leq t} \frac{1}{i^{1/2}} \right).
\]

Here we have used the fact that \( \log(1 + x) \leq x \) for any \( x > 0 \). Since \( i < p_i \), we have

\[
\frac{\lambda_F(n)}{n^{k-1}} \leq \exp \left( c_7 \sum_{1 \leq i \leq t} \frac{1}{i^{1/2}} \right) \leq \exp \left( c_8 i^{1/2} \right),
\]

where \( c_8 > 0 \) is an absolute constant. Note that \( t = \nu(n) \ll \log \log n \) for \( n \gg 1 \) (see [11], page 83 for details). Thus for any \( n \geq 3 \), we have

\[
\frac{\lambda_F(n)}{n^{k-1}} \leq \exp \left( c_1 \sqrt{\frac{\log n}{\log \log n}} \right),
\]

where \( c_1 > 0 \) is an absolute constant. This completes the proof of the theorem.

5. PROOF OF THEOREM 3

As earlier, we keep the notations as in the previous sections. We know that for any \( p \in \mathcal{P} \), one has

\[
\frac{\lambda_F(p^m)}{p^{m(k-1)}} = 1 + \frac{1}{p} + \left( 1 + \frac{1}{p} \right) \sum_{\ell=1}^{m-1} \frac{a(p^\ell)}{\ell^{k-1}} - \frac{a(p^m)}{p^{m(k-1)}}
\]

with the convention that an empty sum is zero. Proceeding as in section 4 for any \( p \in \mathcal{P} \) and \( m \in \mathbb{N} \) with \( m \geq 2 \), we see that

\[
\frac{\lambda_F(p^m)}{p^{m(k-1)}} \geq 1 + \frac{1}{p} + \left( 1 + \frac{1}{p} \right) \frac{a(p)}{p^{k-1}} - \left( 1 + \frac{1}{p} \right) \alpha_p.
\]

Since for any prime \( p \geq 11 \), one has \( \alpha_p < \frac{6}{p} \) and hence

\[
\frac{\lambda_F(p^m)}{p^{m(k-1)}} \geq 1 - \frac{1}{p^{1/2}} \left( 2 + \frac{5}{p^{1/2}} + \frac{2}{p} + \frac{6}{p^{3/2}} \right).
\]

Thus except for finitely many primes \( p \), there exists an absolute constant \( c_{10} > 0 \) such that for all \( m \in \mathbb{N} \),

\[
\frac{\lambda_F(p^m)}{p^{m(k-1)}} \geq 1 - \frac{c_{10}}{p^{1/2}} \text{ with } \frac{c_{10}}{p^{1/2}} < 1.
\]
It is easy to see that one can choose $c_{10} = 3.5$ so that the inequality (8) happens for any prime $p \geq 17$. Let 

$$T := \{ p \in \mathcal{P} : \text{the inequality (8) holds} \}$$

and $n \in \mathbb{N}$ be any natural number whose prime divisors are in $T$. As before, writing 

$$n = \prod_{1 \leq i \leq t} p_i^{m_i}$$

with $m_i > 0$ and $p_1 < \cdots < p_t$, we have 

$$\frac{\lambda_F(n)}{n^{k-1}} \geq \prod_{1 \leq i \leq t} \left( 1 - \frac{c_{10}}{p_i^{1/2}} \right) = \exp \left( \sum_{1 \leq i \leq t} \log \left( 1 - \frac{c_{10}}{p_i^{1/2}} \right) \right) \geq \exp \left( -c_{11} \sum_{1 \leq i \leq t} \frac{1}{p_i^{1/2}} \right) \geq \exp \left( -c_{11} \sum_{1 \leq i \leq t} \frac{1}{i^{1/2}} \right) \geq \exp \left( -c_{12} t^{1/2} \right),$$

where $c_{11}, c_{12} > 0$ are absolute constants. Again since $t = \nu(n) \ll \frac{\log n}{\log \log n}$ for $n \gg 1$, we have for such $n \in \mathbb{N}$ with $n \geq 3$,

$$\lambda_F(n) \geq \exp \left( -c_3 \sqrt{\frac{\log n}{\log \log n}} \right),$$

where $c_3 > 0$ is an absolute constant. Note that (9) holds if all the prime divisors of $n$ are in the set $T$. Now if $n \in \mathbb{N}$ is such that $p|n \Rightarrow p \notin T$, then we use Hecke relation 

$$a(p^{n+1}) = a(p)a(p^n) - p^{2k-3}a(p^{n-1})$$

for $n \in \mathbb{N}$ and explicit calculations using Mathematica. In particular, we show that 

$$\lambda_F(n) \geq c_2,$$

where $c_2 > 0$ is an explicit constant. Combining (9) and (10), we now get 

$$\lambda_F(n) \geq c_2 n^{k-1} \exp \left( -c_3 \sqrt{\frac{\log n}{\log \log n}} \right)$$

for any natural number $n \in \mathbb{N}$ with $n \geq 3$.

5.1. **Proof of Corollary 1.** Since $\lambda_F$ is a non-zero multiplicative function (see [1] and [7]), we have $\lambda_F(1) = 1 > 0$. Also we know that 

$$\lambda_F(2) \geq \frac{3}{2} - \sqrt{2} > 0.$$

Now by applying Theorem 3, we have our corollary.
6. Proof of Theorem

Notations are as in the previous sections. For any prime \( p \), we get

\[
\lambda_F(p) = p^{k-1}(1 + \frac{1}{p} + \frac{a(p)}{p^{k-1}}).
\]

Write \( b(p) = a(p)/p^{k-3/2} \). For any absolute constant \( 0 < \tilde{\beta} < 2 \), consider the sums

\[
S(x) := \sum_{p \leq x} (b(p) + \tilde{\beta})(b(p) - 2) \quad \text{and} \quad S^+(x) := \sum_{\substack{p \leq x, \ b(p) < -\tilde{\beta}}} (b(p) + \tilde{\beta})(b(p) - 2).
\]

Note that

\[
S(x) \leq S^+(x) \leq 16 \# \{ p \in \mathcal{P} : b(p) < -\tilde{\beta} \} \log x.
\]

Then using the estimates (see pages 43 and 135 of [5] and Theorem 2 of [10])

\[
\sum_{p \leq x} b(p) \log p \ll x \exp(-\kappa \sqrt{\log x}), \quad \sum_{p \leq x} b^2(p) \log p \sim x \quad \text{and} \quad \sum_{p \leq x} \log p \ll x,
\]

where \( \kappa > 0 \) is an absolute constant and proceeding along the lines of the proof of lemma [6], one can show there exists an absolute constant \( 0 < \beta_1 < 2 \) such that

\[
B := \{ p : a(p) < -\beta_1 \cdot p^{k-3/2} \}
\]

has positive lower density. Let us take

\[
n_x = \prod_{x < p \leq 2x, \ p \in B} p,
\]

where \( x \) is sufficiently large so that \( \frac{2}{\sqrt{x}} < \beta_1 \). Then we have

\[
\frac{\lambda_F(n_x)}{n_x^{k-1}} = \prod_{x < p \leq 2x, \ p \in B} \left( 1 + \frac{1}{p} + \frac{a(p)}{p^{k-1}} \right) \leq \prod_{x < p \leq 2x, \ p \in B} \left( 1 + \frac{1}{p} + \frac{-\beta_1}{p^{1/2}} \right)
\]

\[
\leq \exp \left[ \sum_{x < p \leq 2x, \ p \in B} \log \left( 1 - \frac{\beta_1}{2p^{1/2}} \right) \right]
\]

\[
\leq \exp \left( -c_{13} \sum_{x < p \leq 2x, \ p \in B} \frac{1}{p^{1/2}} \right),
\]

where \( c_{13} > 0 \) is an absolute constant. Since the set \( B \) has positive lower density, as in section [3], we get

\[
\frac{\lambda_F(n_x)}{n_x^{k-1}} \leq \exp \left( -c_{15} \frac{\sqrt{x}}{\log x} \right) \leq \exp \left( -c_4 \frac{\sqrt{\log n_x}}{\log \log n_x} \right),
\]
where \( c_{15} > 0 \) is an absolute constant. Thus for given any natural number \( M \), there exists a natural number \( n \) with \( n > M \) such that

\[
\frac{\lambda_F(n)}{n^{k-1}} \leq \exp\left(-c_4\sqrt[4]{\log n} \frac{\log \log n}{\log n}\right).
\]

Hence we have the result.

7. PROOF OF THEOREM 5

Recall that for any \( m \in \mathbb{N} \) and any prime \( p \), one has (see equation (7))

\[
\frac{\lambda_F(p^m)}{p^{m(k-1)}} = 1 + \frac{1}{p} + \left(1 + \frac{1}{p}\right) \sum_{\ell=1}^{m-1} \frac{a(p^\ell)}{p^{(k-1)}(k-1)} + \frac{a(p^m)}{p^{m(k-1)}}
\]

with the convention that an empty sum is zero. Note that the series

\[
\sum_{\ell=1}^{\infty} \frac{a(p^\ell)}{p^{(k-1)}(k-1)}
\]

is absolutely convergent (see section 4 for details). This implies that the sequence

\[
\left\{\frac{\lambda_F(p^m)}{p^{m(k-1)}}\right\}_{m \in \mathbb{N}}
\]

is convergent. Further there exist absolute constants \( e_1, e_2 > 0 \) such that

(11) 

\[
1 + \frac{e_1}{p^{1/2}} \leq \frac{\lambda_F(p^m)}{p^{m(k-1)}} \leq 1 + \frac{e_2}{p^{1/2}}
\]

holds for all but finitely many primes \( p \in A \). Indeed, the upper bound is a consequence of section 4 whereas the lower bound follows from the fact that \( \alpha_p \leq 6/p \) for \( p \geq 11 \) and primes \( p \in A \) has the property that \( a(p) > \beta, p^{k-3/2} \) with absolute constant \( \beta \) (see section 3, section 4 and section 5).

Let us choose a prime \( p_1 \in A \) such that (11) holds. Since (11) is true for all but finitely many \( p \in A \), we can choose \( p_2 \in A \) such that \( p_2 > p_1 \) and

\[
1 + \frac{e_1}{p_2^{1/2}} \leq \frac{\lambda_F(p_2^m)}{p_2^{m(k-1)}} \leq 1 + \frac{e_2}{p_2^{1/2}} < 1 + \frac{e_1}{2p_1^{1/2}}.
\]

Proceeding in this way, we get a sequence \( \{p_n\}_{n \in \mathbb{N}} \) such that each

\[
\lim_{m \to \infty} \frac{\lambda_F(p_n^m)}{p_n^{m(k-1)}} > 1 \quad \text{and} \quad \lim_{m \to \infty} \frac{\lambda_F(p_i^m)}{p_i^{m(k-1)}} \neq \lim_{m \to \infty} \frac{\lambda_F(p_j^m)}{p_j^{m(k-1)}}
\]

for any \( i \neq j \). Thus there are infinitely many limit points of the sequence \( \{\frac{\lambda_F(n)}{n^{k-1}}\}_{n \in \mathbb{N}} \) which are > 1.
Considering the set $B$ (see section 6) and arguing as above, we can show there is a sequence 
$\{p_n\}_{n \in \mathbb{N}} \subset B$ for which

$$\lim_{m \to \infty} \frac{\lambda_F(p^m_n)}{p^m_n (k-1)} < 1 \quad \text{and} \quad \lim_{m \to \infty} \frac{\lambda_F(p^m_i)}{p^m_i (k-1)} \neq \lim_{m \to \infty} \frac{\lambda_F(p^m_j)}{p^m_j (k-1)}$$

for any $i \neq j$. This completes the proof.

**Acknowledgments:** The third author would like to thank the Institute of Mathematical Sciences for providing excellent working atmosphere where the work was done. The authors would like to thank Purusottam Rath for asking the possibility of infinitude of limit points of the sequence considered in Theorem 5 and going through an earlier version of the paper.

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