On the optimal paving over MASAs in von Neumann algebras

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Abstract

We prove that if $A$ is a singular MASA in a II$_1$ factor $M$ and $\omega$ is a free ultrafilter, then for any $x \in M \otimes A$, with $\|x\| \leq 1$, and any $n \geq 2$, there exists a partition of 1 with projections $p_1, p_2, \ldots, p_n \in A^\omega$ (i.e., a paving) such that $\left\| \sum_{i=1}^n p_i x p_i \right\| \leq 2\sqrt{n - 1}/n$, and give examples where this is sharp. Some open problems on optimal pavings are discussed.

1 Introduction

A famous problem formulated by R.V. Kadison and I.M. Singer in 1959 asked whether the diagonal MASA (maximal abelian $*$-subalgebra) $D$ of the algebra $B(\ell^2 N)$, of all linear bounded operators on the Hilbert space $\ell^2 N$, satisfies the paving property, requiring that for any contraction $x = x^* \in B(\ell^2 N)$ with 0 on the diagonal, and any $\varepsilon > 0$, there exists a partition of 1 with projections $p_1, \ldots, p_n \in D$, such that $\| \sum_{i=1}^n p_i x p_i \| \leq \varepsilon$. This problem has been settled in the affirmative by A. Marcus, D. Spielman and N. Srivastava in [MSS13], with an actual estimate $n \leq 12^4 \varepsilon^{-4}$ for the paving size, i.e., for the minimal number $n = n(x, \varepsilon)$ of such projections.

In a recent paper [PV14], we considered a notion of paving for an arbitrary MASA in a von Neumann algebra $A \subset M$, that we called so-paving, which requires that for any $x = x^* \in M$ and any $\varepsilon > 0$, there exist $n \geq 1$, a net of partitions of 1 with $n$ projections $p_{1,i}, \ldots, p_{n,i} \in A$ and projections $q_i \in M$ such that $\| q_i (\Sigma_{k=1}^n p_{k,i} x p_{k,i} - a_i) q_i \| \leq \varepsilon, \forall i$, and $q_i \rightarrow 1$ in the so-topology.

This property is in general weaker than the classic Kadison-Singer norm paving, but it coincides with it for the diagonal MASA $D \subset B(\ell^2 N)$. We conjectured in [PV14] that any MASA $A \subset M$ satisfies so-paving. We used the results in [MSS13] to check this conjecture for all MASAs in type I von Neumann algebras, and all Cartan MASAs in amenable von Neumann algebras and in group measure space factors arising from profinite actions, with the estimate $12^4 \varepsilon^{-4}$ for the so-paving size derived from [MSS13] as well.

We also showed in [PV14] that if $A$ is the range of a normal conditional expectation, $E : M \to A$, and $\omega$ is a free ultrafilter on $\mathbb{N}$, then so-paving for $A \subset M$ is equivalent to the usual Kadison-Singer paving for the ultrapower MASA $A^\omega \subset M^\omega$, with the norm paving size for $A^\omega \subset M^\omega$ coinciding with the so-paving size for $A \subset M$. In the case $A$ is a singular MASA in a II$_1$ factor $M$, norm-paving for the ultrapower inclusion $A^\omega \subset M^\omega$ has been established in [P13], with paving size $1250 \varepsilon^{-3}$. This estimate was improved to $< 16 \varepsilon^{-2} + 1$ in [PV14], while also shown to be $\geq \varepsilon^{-2}$ for arbitrary MASAs in II$_1$ factors.

In this paper we prove that the paving size for singular MASAs in II$_1$ factors is in fact $< 4 \varepsilon^{-2} + 1$, and that for certain singular MASAs this is sharp. More precisely, we prove that for any contraction $x \in M^\omega$ with 0 expectation onto $A^\omega$, and for any $n \geq 2$, there exists a partition of 1 with $n$ projections $p_i \in A^\omega$ such that $\| \Sigma_{i=1}^n p_i x p_i \| \leq 2\sqrt{n - 1}/n$. In fact, given any finite

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set of contractions $F \subset M^d \otimes A^d$, we can find a partition $p_1, ..., p_n \in A^d$ that satisfies this estimate for all $x \in F$, so even the multipaving size for singular MASAs is $< 4e^{-2} + 1$.

To construct pavings satisfying this estimate, we first use Theorem 4.1(a) in [P13] to get a unitary $u \in A^d$ with $u^i = 1$, $\tau(u^k) = 0$, $1 \leq k \leq n - 1$, such that any word with alternating letters from $\{u^k \mid 1 \leq k \leq n - 1\}$ and $F \cup F^*$ has trace 0. This implies that for each $x \in F$ the set $X = \{u^{-i}xu^{-i+1} \mid i = 1, 2, ..., n\}$ satisfies the conditions $\tau(\Pi_{k=1}^n(x_k^2 - 1x_k)) = 0 = \tau(\Pi_{k=1}^n(x_k^2 - 1x_k^2))$, for all $m$ and all $x_k \in X$ with $x_k \neq x_{k+1}$ for all $k$. We call $L$-freeness this property of a subset of a II$_1$ factor. We then prove the general result, of independent interest, that any $L$-free set of contractions $\{x_1, ..., x_n\}$ satisfies the norm estimate $\|\sum_{i=1}^n x_i\| \leq 2\sqrt{n} - 1$. We do this by first “dilating” $\{x_1, ..., x_n\}$ to an $L$-free set of unitaries $\{U_1, ..., U_n\}$ in a larger II$_1$ factor, for which we deduce the Kesten-type estimate $\|\sum_{i=1}^n U_i\| = 2\sqrt{n} - 1$ from results in [AO74]. This implies the inequality for the $L$-free contractions as well. By applying this to $\{u^{-i}xu^{-i} \mid i = 1, ..., n\}$ and taking into account that $\frac{1}{n}\sum_{i=1}^n u^{-i}xu^{-i} = \sum_{i=1}^n p_ip_x$, where $p_1, ..., p_n$ are the minimal spectral projections of $u$, we get $\|\sum_{i=1}^n p_ip_x\| \leq 2\sqrt{n} - 1/n, \forall x \in F$.

We also notice that if $M$ is a II$_1$ factor, $A \subset M$ is a MASA and $v \in M$ a self-adjoint unitary of trace 0 which is free with respect to $A$, then $\|\sum_{i=1}^n p_ip_x\| \geq 2\sqrt{n} - 1/n$ for any partition of 1 with projections in $A^d$, with equality if and only if $\tau(p_i) = 1/n, \forall i$. A concrete example is when $M = L(Z \ast (Z/2Z))$, $A = L(Z)$ (which is a singular MASA in $M$ by [PS1]) and $v = v^* \in L(Z/2Z) \subset M$ denotes the canonical generator. This shows that the estimate $4e^{-2} + 1$ for the paving size is in this case optimal.

The constant $2\sqrt{n} - 1$ is known to coincide with the spectral radius of the $n$-regular tree, and with the first eigenvalue less than $n$ of $n$-regular Ramanujan graphs. Its occurrence in this context leads us to a more refined version of a conjecture formulated in [PV14], predicting that for any MASA $A \subset M$ which is range of a normal conditional expectation, any $n \geq 2$ and any contraction $x = x^* \in M$ with 0 expectation onto $A$, the infimum $\varepsilon(A \subset M; n, k)$ for all norms of pavings of $x$, $\|\sum_{i=1}^n p_ip_x\|$, with $n$ projections $p_1, ..., p_n$ in $A^d$, $\sum_{i=1}^n p_i = 1$, is bounded above by $2\sqrt{n} - 1/n$, and that in fact $\sup\{\varepsilon(A \subset M; n, x) \mid x = x^* \in M \subset A, \|x\| \leq 1\} = 2\sqrt{n} - 1/n$. Such an optimal estimate would be particularly interesting to establish for the diagonal MASA $D \subset B(\ell^2Z)$.

2 Preliminaries

A well known result of H. Kesten in [K58] shows that if $F_k$ denotes the free group with $k$ generators $h_1, ..., h_k$, and $A$ is the left regular representation of $F_k$ on $\ell^2F_k$, then the norm of the Laplacian operator $L = \sum_{i=1}^k (\lambda(h_i) + \lambda(h_i^{-1}))$ is equal to $2\sqrt{2k} - 1$. It was also shown in [K58] that, conversely, if $k$ elements $h_1, ..., h_k$ in a group $\Gamma$ satisfy $\|\sum_{i=1}^k (\lambda(h_i) + \lambda(h_i^{-1}))\| = 2\sqrt{2k} - 1$, then $h_1, ..., h_k$ are freely independent, generating a copy of $F_k$ inside $\Gamma$. The calculation of the norm of $L$ in [K58] uses the formalism of random walks on groups, but it really amounts to calculating the higher moments $\tau(L^{2n})$ and using the formula $\|L\| = \lim_m (\tau(L^{2n}))^{1/2m}$, where $\tau$ denotes the canonical (normal faithful) tracial state on the group von Neumann algebra $L(F_k)$.

Kesten’s result implies that whenever $u_1, ..., u_k$ are freely independent Haar unitaries in a type II$_1$ factor $M$ (i.e., $u_1, ..., u_k$ generate a copy of $L(F_k)$ inside $M$), then one has $\|\sum_{i=1}^k u_i + u_i^*\| = 2\sqrt{2k} - 1$. In particular, if $M$ is the free group factor $L(F_k)$ and $u_i = \lambda(h_i)$, where $h_1, ..., h_k \in F_k$ as above, then $\|\sum_{i=1}^k \alpha_i u_i + \overline{\alpha_i} u_i^*\| = 2\sqrt{2k} - 1$, for any scalars $\alpha_i \in \mathbb{C}$ with $|\alpha_i| = 1$.

Estimates of norms of linear combinations of elements satisfying more general free independence
relations in group $\Pi_1$ factors $L(\Gamma)$ have later been obtained in [L73], [B74], [AO74]. These estimates involve elements in $L(\Gamma)$ (viewed as convolvers on $\ell^2(\Gamma)$) that are supported on a subset $\{g_1, \ldots, g_n\} \subset \Gamma$ satisfying the following weaker freeness condition, introduced in [L73]: whenever $k \geq 1$ and $i_s \neq j_s$, $j_s \neq i_{s+1}$ for all $s$, we have that

$$g_{i_1}g_{j_1}^{-1} \cdots g_{i_k}g_{j_k}^{-1} \neq e.$$ 

In [B74] and [AO74], this is called the Leinert property and it is proved to be equivalent with $\{g_1^{-1}g_2, \ldots, g_n^{-1}g_n\}$ freely generating a copy of $F_{n-1}$. The most general calculation of norms of elements $x = \sum c_i \lambda(g_i) \in L(\Gamma)$, supported on a Leinert set $\{g_i\}$, with arbitrary coefficients $c_i \in \mathbb{C}$, was obtained by Akemann and Ostrand in [AO74]. The calculation shows in particular that if $\{g_1, \ldots, g_n\}$ satisfies Leinert’s freeness condition then $\|\sum_{i=1}^n \lambda(g_i)\| = 2\sqrt{n-1}$. Since $h_1, \ldots, h_k \in \Gamma$ freely independent implies $\{h_i, h_i^{-1} \mid 1 \leq i \leq k\}$ is a Leinert set, the result in [AO74] does recover Kesten’s theorem as well. Like in [K58], the norm of an element of the form $L = \sum_{i=1}^n c_i \lambda(g_i)$ in [AO74] is calculated by evaluating $\lim n \tau((L^*L)^n)^{1/2n}$ (by computing the generating function of the moments of $L^*L$).

An argument similar to [K58] was used in [Le96] to prove that, conversely, if some elements $g_1, \ldots, g_n$ in a group $\Gamma$ satisfy $\|\sum_{i=1}^n \lambda(g_i)\| = 2\sqrt{n-1}$, then $g_1, \ldots, g_n$ is a Leinert set. On the other hand, note that if $g_1, \ldots, g_n$ are arbitrary elements in an arbitrary group $\Gamma$ and we denote $L = \sum_{i=1}^n \lambda(g_i)$ the corresponding Laplacian, then the $n$th moment $\tau((L^*L)^n)$ is bounded from below by the $n$th moment of the Laplacian obtained by taking $g_i$ to be the generators of $F_n$. Thus, we always have $\|\sum_{i=1}^n \lambda(g_i)\| \geq 2\sqrt{n-1}$. More generally, if $v_1, \ldots, v_n$ are unitaries in a von Neumann algebra $M$ with normal faithful trace state $\tau$, such that any word $v_{i_1}^* v_{j_1} v_{i_2} v_{j_2}^* \cdots v_{i_m} v_{j_m}^*$, $\forall m \geq 1$, $\forall i_k, j_k \leq n$, has trace with non-negative real part, then $\|\sum_{i=1}^n \tau(v_i)\| \geq 2\sqrt{n-1}$. In particular, for any unitaries $u_1, \ldots, u_n \in M$ one has $\|\sum_{i=1}^n u_i \otimes \overline{u_i}\| \geq 2\sqrt{n-1}$.

For convenience, we state below some norm calculations from [AO74], formulated in the form that will be used in the sequel:

**Proposition 2.1 ([AO74]).** If $v_1, v_2, \ldots, v_{n-1} \in M$ are freely independent Haar unitaries, then

$$\|1 + \sum_{i=1}^{n-1} v_i\| = 2\sqrt{n-1}. \tag{2.1}$$

Also, if $\alpha_0, \ldots, \alpha_{n-1} \in \mathbb{C}$, $\sum_i |\alpha_i|^2 = 1$, then

$$\|\alpha_0 1 + \sum_{i=1}^{n-1} \alpha_i v_i\| \leq 2\sqrt{1 - 1/n}. \tag{2.2}$$

Note that (2.1) above shows in particular that if $p, q \in M$ are projections with $\tau(p) = 1/2$ and $\tau(q) = 1/n$, for some $n \geq 3$, and they are freely independent, then $\|pq\| = 1/2 + \sqrt{n-1/n}$. Indeed, any two such projections can be thought of as embedded into $L(F_2)$ with $p$ and $q$ lying in the MASAs of the two generators, $p \in A_1$, respectively $q \in A_2$. Denote $v = 2p - 1$. Let $g_1 = q_1, q_2, \ldots, q_n \in A_2$ be mutually orthogonal projections of trace $1/n$ and denote $u = \sum_{j=1}^n \lambda^{-1} j$, where $\lambda = 2\exp(2\pi i/n)$. It is then easy to see that the elements $v_k = vu^k v u^{-k}$, $k = 1, 2, \ldots, n-1$ are freely independent Haar unitaries. By (2.1) we thus have

$$\|\sum_{k=0}^{n-1} u^k v u^{-k}\| = \|1 + \sum_{k=1}^{n-1} vu^k v u^{-k}\| = 2\sqrt{n-1}.$$ But $\sum_{k=0}^{n-1} u^k v u^{-k} = n(\sum_{j=1}^n g_j v q_j)$, implying that.

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3See also the more “rough” norm estimates for elements in $L(F_n)$ obtained by R. Powers in 1967 in relation to another problem of Kadison, but published several years later in [Po75], and which motivated in part the work in [AO74].
\[ \|qvq\| = \|q(2p - 1)q\| = 2\sqrt{n - 1}/n = 2\sqrt{\tau(q)(1 - \tau(q))} \]

or equivalently
\[ \|qpq\| = 1/2 + \sqrt{n - 1}/n = \tau(p) + \sqrt{\tau(q)(1 - \tau(q))}. \]

The computation of the norm of the product of freely independent projections \( q, p \) of arbitrary trace in \( M \) (in fact, of the whole spectral distribution of \( qpq \)) was obtained by Voiculescu in [Vo86], as one of the first applications of his multiplicative free convolution (which later became a powerful tool in free probability). We recall here these norm estimates, which in particular show that the first of the above norm calculations holds true for projections \( q \) of arbitrary trace (see also [ABH87] for the case \( \tau(q) = 1/n, \tau(p) = 1/m \), for integers \( n \geq m \geq 2 \):

**Proposition 2.2 (Vo86).** If \( p, q \in M \) are freely independent projections with \( \tau(q) \leq \tau(p) \leq 1/2 \), then
\[ \|qpq\| = \tau(p) + \tau(q) - 2\tau(p)\tau(q) + 2\sqrt{\tau(p)\tau(1 - p)\tau(q)\tau(1 - q)}. \]  

If in addition \( \tau(p) = 1/2 \) and we denote \( v = 2p - 1 \), then
\[ \|qvq\| = 2\sqrt{\tau(q)\tau(1 - q)}. \]

3 \( L \)-free sets of contractions and their dilation

Recall from [P13] that two selfadjoint sets \( X, Y \subset M \ominus C1 \) of a tracial von Neumann algebra \( M \) are called freely independent set\(^4\) if the trace of any word with letters alternating from \( X \) and \( Y \) is equal to 0. Also, a subalgebra \( B \subset M \) is called freely independent of a set \( X \), if \( X \) and \( B \ominus C1 \) are freely independent as sets. Several results were obtained in [P13] about constructing a “large subalgebra” \( B \) inside a given subalgebra \( Q \subset M \) that is freely independent of a given countable set \( X \). Motivated by a condition appearing in one such result, namely [P13] Theorem 4.1, and by a terminology used in [AO74], we consider in this paper the following free independence condition for arbitrary elements in tracial algebras:

**Definition 3.1.** Let \( (M, \tau) \) be a von Neumann algebra with a normal faithful tracial state. A subset \( X \subset M \) is called \( L \)-free\(^5\) if
\[ \tau(x_1x_2^*x_2^* \cdots x_{2k-1}x_{2k}^*) = 0 \quad \text{and} \quad \tau(x_1^*x_2 \cdots x_{2k-1}x_{2k}^*) = 0, \]

whenever \( k \geq 1, x_1, \ldots, x_{2k} \in X \) and \( x_i \neq x_{i+1} \) for all \( i = 1, \ldots, 2k - 1 \).

Note that if the subset \( X \) in the above definition is taken to be contained in the set of canonical unitaries \( \{u_g \mid g \in \Gamma\} \) of a group von Neumann algebra \( M = L(\Gamma) \), i.e. \( X = \{u_g \mid g \in F\} \) for some subset \( F \subset \Gamma \), then \( L \)-freeness of \( X \) amounts to \( F \) being a Leinert set. But the key example of an \( L \)-free set that is important for us here occurs from a diffuse algebra \( B \) that is free independent from a set \( Y = Y^* \subset M \ominus C1 \); given any \( y_1, \ldots, y_n \in Y \) and any unitary element \( u \in U(B) \) with \( \tau(u^k) = 0, 1 \leq k \leq n - 1 \), the set \( \{u^{k-1}y_ku^{-k+1} \mid 1 \leq k \leq n\} \) is \( L \)-free.

\(^4\)We specifically consider this condition for subsets \( X, Y \subset M \ominus C1 \), not to be confused with the freeness of the von Neumann algebras generated by \( X \) and \( Y \).

\(^5\)Note that this notion is not the same as (and should not be confused with) the notion of \( L \)-sets used in [P92].
Note that we do not need to impose both conditions on the traces being zero in Definition 3.1, because we cannot deduce $\tau(x_i^* x_2 x_3 x_1) = 0$ from $\tau(y_1 y_2^* y_3 y_1^*) = 0$ for all $y_i \in X$ with $y_1 \neq y_2$, $y_2 \neq y_3$, $y_3 \neq y_4$. However, if $X \subset U(M)$ consists of unitaries, then only one set of conditions is sufficient. We in fact have:

**Lemma 3.2.** Let $X = \{u_1, \ldots, u_n\} \subset U(M)$. Then the following conditions are equivalent:

(a) $X$ is an L-free set.
(b) $\tau(u_{i_1}^* u_{j_1}^* \cdots u_{i_k}^* u_{j_k}) = 0$ whenever $k \geq 1$ and $i_s \neq j_s$, $j_s \neq i_{s+1}$ for all $s$.
(c) $u_1^* u_2, \ldots, u_n^* u_n$ are free generators of a copy of $L(\mathbb{F}_{n-1})$.

**Proof.** This is a trivial verification. $\square$

**Corollary 3.3.** If $\{u_1, \ldots, u_n\}$ is an L-free set of unitaries in $U(M)$, then $\|\Sigma_{i=1}^n u_i\| = 2\sqrt{n-1}$. Moreover, if $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ with $\Sigma_{i=1}^n |\alpha_i|^2 \leq 1$, then

$$\left\|\sum_{i=1}^n \alpha_i u_i\right\| \leq 2\sqrt{1-1/n}.$$

**Proof.** Since $\|\Sigma_{i=1}^n \alpha_i u_i\| = \|\alpha_1 + \Sigma_{i=2}^n \alpha_i u_i^* u_i\|$, the statement follows by applying (2.2) to the freely independent Haar unitaries $v_j = u_i^* u_j$, $2 \leq j \leq n$. $\square$

**Proposition 3.4.** Let $M$ be a finite von Neumann algebra with a faithful tracial state $\tau$. If $\{x_1, \ldots, x_n\} \subset M$ is an L-free set with $\|x_i\| \leq 1$ for all $i$, then there exists a tracial von Neumann algebra $(M, \tau)$, a trace preserving unital embedding $M \subset M$ and an L-free set of unitaries $\{U_1, \ldots, U_n\} \subset U(M)$ with $M = M_{n+1}(\mathbb{C}) \otimes M$ so that, denoting by $(e_{ij})_{i,j=0, \ldots, n}$ the matrix units of $M_{n+1}(\mathbb{C})$, we have $e_{00}U_ie_{00} = x_i$ for all $i$.

**Proof.** Define $\mathcal{M} = M * L(\mathbb{F}_{n(n-1)})$ and denote by $u_{i,j}$, $i \neq j$, free generators of $L(\mathbb{F}_{n(n-1)})$. For every $i \in \{1, \ldots, n\}$, define

$$c_i = \sqrt{1-x_i^* x_i} \quad \text{and} \quad d_i = -\sqrt{1-x_i^* x_i}.$$

Put $\tilde{M} = M_{n+1}(\mathbb{C}) \otimes \mathcal{M}$ and define the unitary elements $U_i \in U(\tilde{M})$ given by

$$U_i = (e_{00} \otimes x_i) + (e_{ii} \otimes x_i^*) + (e_{0i} \otimes c_i) + (e_{i0} \otimes d_i) + \sum_{j \neq i} (e_{jj} \otimes u_{i,j}).$$

Note that $U_i$ is the direct sum of the unitary

$$\begin{pmatrix} x_i & c_i \\ d_i & x_i^* \end{pmatrix}$$

in positions 0 and $i$, and the unitary $\bigoplus_{j \neq i} u_{i,j}$ in the positions $j \neq i$.

By construction, we have that $e_{00}U_i e_{00} = x_i e_{00}$. So, it remains to prove that $\{U_1, \ldots, U_n\}$ is L-free.

Take $k \geq 1$ and indices $i_s, j_s$ such that $i_s \neq j_s$, $j_s \neq i_{s+1}$ for all $s$. We must prove that

$$\tau(U_{i_1}^* U_{j_1}^* \cdots U_{i_k}^* U_{j_k}^*) = 0.$$  \hspace{1cm} (3.1)

Consider $V := U_{i_1}^* U_{j_1}^* \cdots U_{i_k}^* U_{j_k}^*$ as a matrix with entries in $\mathcal{M}$. Every entry of this matrix is a sum of “words” with letters

$$\{x_i, x_i^*, c_i, d_i \mid i = 1, \ldots, n\} \cup \{u_{i,j}, u_{i,j}^* \mid i \neq j\}.$$

We prove that every word that appears in a diagonal entry $V_{ii}$ of $V$ has zero trace. The following types of words appear.
1° Words without any of the letters \( u_{a,b} \) or \( u_{a,b}^* \). These words only appear as follows:

- in the entry \( V_{00} \) as \( x_{i_1} x_{j_1}^* \cdots x_{i_k} x_{j_k}^* \), which has zero trace;
- if \( i_1 = j_k = i \), in the entry \( V_{ii} \) as \( w = d_i x_{j_1}^* x_{i_2} \cdots x_{i_{k-1}} x_{j_{k-1}} x_{i_k} d_i^* \). Then we have
  \[
  \tau(w) = \tau(x_{j_1}^* x_{i_2} \cdots x_{j_{k-1}} x_{i_k} d_i^* d_i) \\
  = \tau(x_{j_1}^* x_{i_2} \cdots x_{j_{k-1}} x_{i_k} x_i) - \tau(x_{j_1}^* x_{i_2} \cdots x_{j_{k-1}} x_{i_k} x_i) \\
  = 0 - \tau(x_{i_1} x_{j_1}^* \cdots x_{i_k} x_{j_k}^*) = 0 ,
  \]
  because \( i = i_1 \) and \( i = j_k \).

2° Words with exactly one letter of the type \( u_{a,b} \) or \( u_{a,b}^* \). These words have zero trace because \( \tau(Mu_{a,b}M) = \{0\} \).

3° Words \( w \) with two or more letters of the type \( u_{a,b} \) or \( u_{a,b}^* \). Consider two consecutive such letters in \( w \), i.e. a subword of \( w \) of the form

\[
u_{i,j}^* w_0 \nu_{i',j'}^*
\]

with \( \varepsilon, \varepsilon' = \pm 1 \) and where \( w_0 \) is a word with letters from \( \{x_i, x_i^*, c_i, d_i \mid i = 1, \ldots, n\} \). We distinguish three cases.

- \( (\varepsilon', i', j') \neq (-\varepsilon, i, j) \).
- \( u_{i,j} w_0 u_{i,j}^* \).
- \( u_{i,j}^* w_0 u_{i,j} \).

To prove that \( \tau(w) = 0 \), it suffices to prove that in the last two cases, we have that \( \tau(w_0) = 0 \). A subword of the form \( u_{i,j} w_0 u_{i,j}^* \) can only arise from the \( jj \)-entry of

\[
U_s U_{j_s}^* \cdots U_{i_t} U_{j_t}^* \quad \text{with} \quad i_s = j_t = i , \quad j_s = j_t = j
\]

(and thus, \( t \geq s + 2 \)). In that case,

\[
w_0 = c_j^* x_{i_{s+1}} x_{j_{s+1}}^* \cdots x_{i_{t-1}} x_{j_{t-1}}^* c_j .
\]

Thus,

\[
\tau(w_0) = \tau(x_{i_{s+1}} x_{j_{s+1}}^* \cdots x_{i_{t-1}} x_{j_{t-1}}^* c_j c_j^*) \\
= \tau(x_{i_{s+1}} x_{j_{s+1}}^* \cdots x_{i_{t-1}} x_{j_{t-1}}^* ) - \tau(x_{i_{s+1}} x_{j_{s+1}}^* \cdots x_{i_{t-1}} x_{j_{t-1}}^* x_j x_j^* ) \\
= 0 - \tau(x_j x_{j_{s+1}}^* \cdots x_{j_{t-1}} x_{j_{t-1}} x_i) = 0 ,
\]

because \( j = j_s \) and \( j = i_t \).

Finally, a subword of the form \( u_{i,j}^* w_0 u_{i,j} \) can only arise from the \( jj \)-entry of

\[
U_{j_s}^* U_{j_s} \cdots U_{j_{t-1}} U_{j_t} \quad \text{with} \quad j_{s-1} = i_t = i , \quad i_s = j_{t-1} = j
\]

(and thus, \( t \geq s + 2 \)). In that case,

\[
w_0 = d_j x_{j_{s+1}} x_{i_{s+1}}^* \cdots x_{j_{t-2}} x_{i_{t-1}}^* d_j^* .
\]

As above, it follows that \( \tau(w_0) = 0 \).
So, we have proved that every word that appears in a diagonal entry \( V_{ii} \) of \( V \) has trace zero. Then also \( \tau(V) = 0 \) and it follows that \( \{U_1, \ldots, U_n\} \) is an L-free set of unitaries.

**Corollary 3.5.** Let \((M, \tau)\) be a finite von Neumann algebra with a faithful normal tracial state. If \( \{x_1, \ldots, x_n\} \subset M \) is L-free with \( \|x_i\| \leq 1 \) for all \( i \), then

\[
\left\| \sum_{i=1}^{n} x_i \right\| \leq 2\sqrt{n - 1}.
\]

More generally, given any complex scalars \( \alpha_1, \ldots, \alpha_n \) with \( \sum_{i=1}^{n} |\alpha_i|^2 \leq 1 \), we have

\[
\left\| \sum_{i=1}^{n} \alpha_i x_i \right\| \leq 2\sqrt{1 - 1/n}.
\]

**Proof.** Assuming \( n \geq 2 \), with the notations from Proposition 3.4 and by using Corollary 3.3, we have

\[
\left\| \sum_{i=1}^{n} \alpha_i U_i \right\| \leq 2\sqrt{1 - 1/n}.
\]

Reducing with the projection \( e_{00} \), it follows that

\[
\left\| \sum_{i=1}^{n} \alpha_i x_i \right\| \leq 2\sqrt{1 - 1/n}.
\]

\[ \square \]

### 4 Applications to paving problems

Like in [P13], [PV14], if \( A \subset M \) is a MASA in a von Neumann algebra and \( x \in M \), then we denote by \( n(A \subset M; x, \varepsilon) \) the smallest \( n \) for which there exist projections \( p_1, \ldots, p_n \in A \) and \( a \in A \) such that \( \|a\| \leq \|x\| \), \( \sum_{i=1}^{n} p_i = 1 \) and \( \left\| \sum_{i=1}^{n} p_i x p_i - a \right\| \leq \varepsilon \|x\| \) (with the convention that \( n(A \subset M; x, \varepsilon) = \infty \) if no such finite partition exists), and call it the **paving size** of \( x \).

Recall also from [D54] that a MASA \( A \) in a von Neumann algebra \( M \) is called singular, if the only unitary elements in \( M \) that normalize \( A \) are the unitaries in \( A \).

**Theorem 4.1.** Let \( A_n \subset M_n \) be a sequence of singular MASAs in finite von Neumann algebras and \( \omega \) a free ultrafilter on \( \mathbb{N} \). Denote \( M = \prod_{\omega} M_n \) and \( A = \prod_{\omega} A_n \). Given any countable set of contractions \( X \subset M \ominus A \) and any integer \( n \geq 2 \), there exists a partition of 1 with projections \( p_1, \ldots, p_n \in A \) such that

\[
\left\| \sum_{j=1}^{n} p_j x p_j \right\| \leq 2\sqrt{n - 1/n}, \quad \text{for all } x \in X.
\]

In particular, the paving size of \( A \subset M \),

\[
n(A \subset M; \varepsilon) \overset{\text{def}}{=} \sup\{n(A \subset M; x, \varepsilon) \mid x = x^* \in M \ominus A\},
\]

is less than \( 4\varepsilon^{-2} + 1 \), for any \( \varepsilon > 0 \).

**Proof.** By Theorem 4.1(a) in [P13], there exists a diffuse abelian von Neumann subalgebra \( A_0 \subset A \) such that for any \( k \geq 1 \), any word with alternating letters \( x = x_0 \Pi_{i=1}^{k} (v_i x_i) \) with \( x_i \in X, 1 \leq i \leq k - 1, x_0, x_k \in X \cup \{1\}, v_i \in A_0 \ominus \mathbb{C}1 \), has trace equal to 0.
This implies that if $p_1, \ldots, p_n \in A$ are projections of trace $1/n$ summing up to 1 and we denote $u = \sum_{j=1}^n \lambda^j p_j$, where $\lambda = \exp(2\pi i/n)$, then for any $x \in X$ the set $\{u^{i-1} xu^{-i+1} \mid i = 1, 2, \ldots, n\}$ is L-free. Since $\frac{1}{n} \sum_{i=1}^n u^{i-1} xu^{-i} = \sum_{i=1}^n p_i x p_i$, where $p_1, \ldots, p_n$ are the minimal spectral projections of $u$, by Proposition 4.3 it follows that for all $x \in X$ we have

$$\|\sum_{i=1}^n p_i x p_i\| = \frac{1}{n} \|\sum_{i=1}^n u^{i-1} xu^{-i+1}\| \leq 2\sqrt{n} - 1/n.$$ 

To derive the last part, let $\varepsilon > 0$ and denote by $n$ the integer with the property that $2n^{-1/2} \leq \varepsilon < 2(n-1)^{-1/2}$. If $x \in M \ominus A$, $\|x\| \leq 1$, and $p_1, \ldots, p_n \in A$ are mutually orthogonal projections of trace $1/n$ that satisfy the free independence relation with $X = \{x\}$ as above, then $n < 4\varepsilon^{-2} + 1$ and we have

$$\|\sum_{i=1}^n p_i x p_i\| \leq 2\sqrt{n} - 1/n \leq \varepsilon,$$

showing that $n(A \subset M; x, \varepsilon) < 4\varepsilon^{-2} + 1$. \hfill \Box

Remark 4.2. The above result suggests that an alternative way of measuring the so-paving size over a MASA in a von Neumann algebra $A \subset M$ admitting a normal conditional expectation, is by considering the quantity

$$\varepsilon(A \subset M; n) \overset{\text{def}}{=} \sup_{x \in (M_2^n \ominus A^*)_1} \left(\inf\{\|\sum_{i=1}^n p_i x p_i\| \mid p_i \in \mathcal{P}(A^\omega), \Sigma_i p_i = 1\}\right).$$

With this notation, the above theorem shows that for a singular MASA in a II$_1$ factor $A \subset M$, one has $\varepsilon(A \subset M; n) \leq 2\sqrt{n-1}/n$, $\forall n \geq 2$, a formulation that’s slightly more precise than the estimate $n_v(A \subset M; \varepsilon) = n(A^\omega \subset M^\omega; \varepsilon) < 4\varepsilon^{-2} + 1$. Also, the conjecture (2.8.2° in [PV14]) about the so-paving size can this way be made more precise, by asking whether $\varepsilon(A \subset M; n) \leq 2\sqrt{n-1}/n$, $\forall n$, for any MASA with a normal conditional expectation $A \subset M$. It seems particularly interesting to study this question in the classical Kadison-Singer case of the diagonal MASA $D \subset B = B(\ell^2(N))$, and more generally for Cartan MASAs $A \subset M$. So far, the solution to the Kadison-Singer paving problem in [MS13] shows that $\varepsilon(D \subset B; n) \leq 12n^{-1/4}$.

Also, while by [CEKP07] one has $n(D \subset B; \varepsilon) \geq \varepsilon^{-2}$ and by [PV14] one has $n_v(A \subset M; \varepsilon) \geq \varepsilon^{-2}$, for any MASA in a II$_1$ factor $A \subset M$, it would be interesting to decide whether $\varepsilon(D \subset B; n)$ and $\varepsilon(A \subset M; n)$ are in fact bounded from below by $2\sqrt{n-1}/n$, $\forall n$.

For a singular MASA in a II$_1$ factor, $A \subset M$, combining 4.1 with such a lower bound would show that $\varepsilon(A \subset M; n) = 2\sqrt{n-1}/n$, $\forall n$. While we could not prove this general fact, let us note here that for certain singular MASAs this equality holds indeed.

Proposition 4.3. 1° Let $M$ be a II$_1$ factor and $A \subset M$ a MASA. Assume $v \in M$ is a unitary element with $\tau(v) = 0$ such that $A$ is freely independent of the set $\{v, v^*\}$ (i.e., any alternating word in $A \oplus \mathbb{C}1$ and $\{v, v^*\}$ has trace 0). Then for any partition of 1 with projections $p_1, \ldots, p_n \in A^\omega$, we have $\|\sum_{i=1}^n p_i \tau(v) p_i\| \geq 2\sqrt{n-1}/n$, with equality iff all $p_i$ have trace 1/n. Also, $\varepsilon(A \subset M; n) \geq 2\sqrt{n-1}/n$, $\forall n$.

2° If $M = L(\mathbb{Z} \ast (\mathbb{Z}/2\mathbb{Z}))$, $A = L(\mathbb{Z})$ and $v = v^*$ denotes the canonical generator of $L(\mathbb{Z}/2\mathbb{Z})$, then $\varepsilon(A \subset M; v, n) = \varepsilon(A \subset M; n) = 2\sqrt{n-1}$, $\forall n$.

Proof. The free independence assumption in 1° implies that $A^\omega \ominus \mathbb{C} \{v, v^*\}$ are freely independent sets as well. This in turn implies that for each $i$, the projections $p_i$ and $\tau(v) p_i v^*$ are freely independent, and so by Proposition 2.2 one has $\|p_i \tau(v) p_i\| = \|p_i \tau(v) p_i v^*\| = 2\sqrt{(\tau(p_i))(1 - \tau(p_i))}$. 

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Thus, if one of the projections $p_i$ has trace $\tau(p_i) > 1/n$, then $\|\Sigma_j p_j v p_j\| \geq \|p_i v p_i\| > 2\sqrt{n-1}/n$, while if $\tau(p_i) = 1/n$, $\forall i$, then $\|\Sigma_j p_j v p_j\| = 2\sqrt{n-1}/n$.

By applying 1° to part 2°, then using [L1] and the fact that $A = L(\mathbb{Z})$ is singular in $M = L(\mathbb{Z} \ast \mathbb{Z})$ (cf. [P81]), proves the last part of the statement.

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