Notes on endomorphisms, local cohomology and completion

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Abstract. Let \( M \) denote a finitely generated module over a Noetherian ring \( R \). For an ideal \( I \subset R \) there is a study of the endomorphisms of the local cohomology module \( H^i_I(M) \), \( g = \text{grade}(I, M) \), and related results. Another subject is the study of left derived functors of the \( I \)-adic completion \( \hat{\Lambda}_I^i(H_f^g(M)) \), motivated by a characterization of Gorenstein rings given in [25]. This provides another Cohen-Macaulay criterion. The results are illustrated by several examples. There is also an extension to the case of homomorphisms of two different local cohomology modules.

In honour of Sylvia and Roger Wiegand for their contributions in Commutative Algebra.

1. Introduction

Let \( I \) denote an ideal of a Noetherian ring \( R \). Let \( M \) be a finitely generated \( R \)-module, and let \( H^i_I(M), i \in \mathbb{Z}, \) denote the local cohomology modules of \( M \) with respect to \( I \) (see [9] or [3] for definitions). In the case of \( (R, m) \), a local ring, we denote by \( E_R([k]) \) the injective hull of the residue field \( R/m = \kappa \) and by \( \text{Hom}_R(\cdot, E_R([k])) = D(\cdot) \) the Matlis Duality functor.

In their paper (see [14]) Hellus and Stückrad investigated the endomorphism ring of \( H_f^g(R) \) and \( D(H_f^g(R)) \) in the case of a complete local ring \( R \) and an ideal satisfying \( H_f^i(R) = 0 \) for all \( i \neq g \). In fact, they proved the isomorphisms of endomorphism rings

\[
R \cong \text{Hom}_R(H_f^g(R), H_f^g(R)) \cong \text{Hom}_R(D(H_f^g(R)), D(H_f^g(R))).
\]

A characterization of ideals such that \( H_f^i(R) = 0 \) for all \( i \neq g \), so-called cohomologically complete intersections, is described by Hellus and the author (see [13]). Some of these results were generalized to finitely generated modules by Zargar (see [26]) under the name relative Cohen-Macaulay modules. In the case of a Gorenstein ring \( (R, m) \) weaker conditions than \( H_f^i(R) = 0 \) for \( i \neq \text{grade}(I, R) \) are sufficient for the previous isomorphisms (see [24]).

In the following we shall extend some of the previous results to the situation of an \( R \)-module \( M \) and extend the result by Hellus and Stückrad (see 4.4). Namely we will investigate the local cohomology module \( H_f^g(M), g = \text{grade}(I, M) \). To this end we denote by \( \hat{\cdot}^I \) the \( I \)-adic completion functor. For the maximal ideal \( I = m \) we write \( \hat{\cdot} \). As main results we shall prove the following:

Theorem 1.1. Let \( I \) denote an ideal of a local ring \( (R, m) \). Let \( M \) be a finitely generated \( R \)-module and \( g = \text{grade}(I, M) \).

(a) \( \text{Hom}_k(H_f^g(\hat{M}), H_f^g(\hat{M})) \cong \text{Hom}_R(D(H_f^g(M)), D(H_f^g(M))) \) and therefore

\[
\hat{R} \cong \text{Hom}_k(H_f^g(\hat{M}), H_f^g(\hat{M})) \text{ if and only if } \hat{R} \cong \text{Hom}_R(D(H_f^g(M)), D(H_f^g(M))).
\]
(b) If \(H^i_I(M) = 0\) for all \(i \neq g\), then
\[
\text{Hom}_R(M,M) \otimes_R \hat{R}^i \cong \text{Hom}_R(H^i_I(M), H^i_I(M)) \quad \text{and} \quad \text{Ext}^j_R(M,M) \otimes_R \hat{R}^i \cong \text{Ext}^{g+j}_R(H^g_I(M), M) \quad \text{for all} \ j.
\]

Extensions of the isomorphisms in 1.1 (b) for two different modules is given in Section 6. Moreover, there is also another characterization of the grade of a module. For our further results we denote by \(\Lambda^i_I(\cdot)\) the left derived functor of the completion \(\cdot^I\). We refer to \([25]\) for definitions and basic results. In \([25, 10.5.9]\) it is shown that \(R\) is a \(d\)-dimensional Gorenstein ring if and only if \(\Lambda^d_I(E_R(\mathfrak{k})) = 0\) for all \(i \neq d\) and \(\Lambda^d_I(E_R(\mathfrak{k})) \cong \hat{R}\). As a generalization we prove the following result:

**THEOREM 1.2.** Let \(I \subset R\) be an ideal. Let \(M\) denote a finitely generated \(R\)-module and \(g = \text{grade}(I,M)\).

(a) There is a natural homomorphism \(\Lambda^I_I(H^I_I(M)) \xrightarrow{\tau} \hat{M}^I\).

(b) Suppose \(H^I_I(M) = 0\) for all \(i \neq g\), then \(\tau\) is an isomorphism and \(\Lambda^I_I(H^I_I(M)) = 0\) for all \(i \neq g\).

As an application it yields a further Cohen-Macaulay criterion: \(M\) is a \(d\)-dimensional Cohen-Macaulay module if and only if \(\Lambda^m_d(H^m_m(M)) \cong \hat{M}^m\) and \(\Lambda^m_d(H^d_m(M)) = 0\) for all \(i \neq d\) (see 5.3). Furthermore, there is an intrinsic characterization of canonically Cohen-Macaulay modules in the sense of \([21]\). A few more results about modules satisfying the assumption in 1.1 (b) are shown.

In our terminology we follow the textbooks \([3]\) and \([15]\). As a reference for Homological Algebra we refer to \([5]\). Particular results about local duality and completion as well as its derived functors are found in \([25]\). By ”\(\sim\)” we denote quasi-isomorphisms in the sense of \([25, 1.1.3]\). In a final section we illustrate the results by several examples. In particular, \(\text{Hom}_R(H^g_I(M), H^g_I(M), g = \text{grade}(I,M)\), is not a finitely generated \(R\)-module in general (see 7.4).

## 2. Preliminaries and the truncation complex

**NOTATION 2.1.** (A) In the following let \(R\) denote a commutative \(d\)-dimensional ring. If \((R,m)\) is local, let \(\mathfrak{k} = R/m\) be its residue field. In this case we denote by \(E = E_R(\mathfrak{k})\) the injective hull of the residue field and by \(D(\cdot) = \text{Hom}_R(\cdot;E)\) the Matlis Duality functor. Let \(I \subset R\) be an ideal and \(M\) be an \(R\)-module. For a prime ideal \(p \in \text{Spec} R\) let \(E_R(R/p)\) denote the injective hull of \(R/p\) as \(R\)-module.

(B) Let \(M\) denote an \(R\)-module. Let \(E_R^i(M) : 0 \rightarrow M \rightarrow E_R^0(M) \rightarrow E_R^1(M) \rightarrow \cdots \rightarrow E_R^i(M) \rightarrow \cdots\) be a minimal injective resolution. By Matlis’ Structure theory it follows that \(E_R^i(M) \cong \bigoplus_{p \in \text{Supp}_{\mathfrak{m}}M} E_R(R/p)^{\mu_i(p,M)}\), where \(\mu_i(p,M) = \dim_{k(p)} \text{Ext}_R^i(k(p),M_p)\). For the details we refer to \([2]\) and \([5]\).

(C) For a finitely generated \(R\)-module \(M\) we define (following D. Rees)
\[
\text{grade}(I,M) = \text{min}\{i \in \mathbb{N} | \text{Ext}^i_R(R/I, M) \neq 0\}.
\]

For basic properties of \(\text{grade}(I,M)\) we refer to \([4, 1.2]\), in particular \(\text{grade}(I,M) = \inf\{\text{depth}M_p|p \in V(I)\}\) (see e.g. \([4, 1.2.10]\)).

(D) With the previous notation let \(H^i_I(M), i \in \mathbb{N}\), denote the local cohomology modules of \(M\) with respect to \(I\). We refer to \([9]\) or to \([25]\) for some basic results. Note that \(\text{Hom}_R(H^I_I(M), H^I_I(M), g = \text{grade}(I,M)\), is not a finitely generated \(R\)-module in general (see 7.4).

In the following we shall modify the construction of the truncation complex as introduced by the author (see \([19]\) and \([21]\)) and used several times.

**DEFINITION 2.2.** Truncation complex. With the previous notation let \(g = \text{grade}(I,M)\). We apply the section functor \(\Gamma_I(\cdot)\) to the minimal injective resolution \(E_R(M)\). By view of 2.1 (B) it follows that \(\Gamma_I(E_R^i(M)) = 0\) for all \(i < g\). So there is an injection \(0 \rightarrow H^g_I(M)[-g] \rightarrow \Gamma_I(E_R^g(M))\), where \(H^g_I(M)[-g]\) denotes the module \(H^g_I(M)\) considered as a complex sitting in cohomological degree \(g\). The complex
$C_M(I)$ obtained as the cokernel is denoted the truncation complex. So there is a short exact sequence of complexes

$$0 \to H^g_I(M)[-g] \to \Gamma_I(E_R(M)) \to C_M(I) \to 0$$

and $H^i(C_M(I)) \cong H^i_I(M)$ for $i > g$ and $H^i(C_M(I)) = 0$ for $i \leq g$.

Next we shall prove some homological results related to modules with the property that their support is contained in $V(I)$.

**Lemma 2.3.** Let $I \subset R$ denote an ideal. Let $M, N$ denote two $R$-module. Suppose that $\text{Supp}_R M \subseteq V(I)$. There are the following natural isomorphisms:

(a) $\text{Hom}_R(M, \Gamma_I(N)) \cong \text{Hom}_R(M, N)$,

(b) $M \otimes_R \text{Hom}_R(\Gamma_I(N), E_R(k)) \cong M \otimes_R \text{Hom}_R(N, E_R(k))$.

**Proof.** The proof in (a) is easy to see. For the proof of (b) assume at first that $M$ is a finitely generated $R$-module. Then $\text{Hom}_R(\text{Hom}_R(M, \cdot), E_R(k)) \cong M \otimes_R \text{Hom}_R(\cdot, E_R(k))$ and the claim follows by (a) and adjointness. The general case turns out because any $R$-module can be written as a direct limit of finitely generated submodules and tensor products commute with direct limits. \hfill \Box

**Lemma 2.4.** Let $X, M$ be $R$-modules and $\text{Supp}_R X \subseteq V(I)$, where $I \subset R$ denotes an ideal with $g = \text{grade}(I, M)$.

(a) $\text{Ext}_R^i(X, M) = 0$ for $i < g$ and $\text{Ext}_R^g(X, M) \cong \text{Hom}_R(X, H^g_I(M))$.

(b) $\text{Tor}_R^g(X, D(M)) = 0$ for $i < g$ and $\text{Tor}_R^g(X, D(M)) \cong X \otimes_R D(H^g_I(M))$.

**Proof.** As above let $M \xrightarrow{\sim} E_R(M)$ denote a minimal injective resolution. By view of Lemma 2.3 there is an isomorphism of complexes

$$\text{Hom}_R(X, \Gamma_I(E_R(M))) \cong \text{Hom}_R(X, E_R(M))$$

since $\text{Supp}X \subseteq V(I)$. By 2.2 there is an exact sequence $0 \to H_I^g(M) \to \Gamma(E_M)^g \to \Gamma(E_M)^g$.

Because of the previous remark it induces a natural commutative diagram with exact rows

$$
\begin{array}{ccc}
0 & \to & \text{Hom}_R(X, H^g_I(M)) \\
\downarrow & & \downarrow \\
0 & \to & \text{Ext}_R^g(X, M)
\end{array}
\begin{array}{ccc}
& & \to \\
\downarrow & & \downarrow \\
& & \to \\
\end{array}
\begin{array}{ccc}
\text{Hom}_R(X, \Gamma_I(E_M))^g & \to & \text{Hom}_R(X, \Gamma_I(E_M))^g \\
\downarrow & & \downarrow \\
\text{Hom}_R(X, E(M))^g & \to & \text{Hom}_R(X, E(M))^g
\end{array}
\begin{array}{ccc}
& & \to \\
\downarrow & & \downarrow \\
& & \to \\
\end{array}
\begin{array}{ccc}
\text{Hom}_R(X, \Gamma_I(E_M))^{g+1} & \to & \text{Hom}_R(X, \Gamma_I(E_M))^{g+1} \\
\downarrow & & \downarrow \\
\text{Hom}_R(X, E(M))^{g+1} & \to & \text{Hom}_R(X, E(M))^{g+1}
\end{array}
$$

The last two vertical homomorphisms are isomorphisms by Lemma 2.3. Therefore the first vertical map is an isomorphism too. This finishes the proof of (a).

For the proof of (b) we first assume that $X$ is a finitely generated $R$-module. By the Ext–Tor duality (see e.g. [25, 1.4.1]) there are canonical isomorphisms

$$\text{Hom}_R(\text{Ext}_R^i(X, M), E_R(k)) \cong \text{Tor}_R^i(X, \text{Hom}_R(M, E_R(k)))$$

for all $i$. This proves the statement (b) for a finitely generated $R$-module $X$. In general let $X = \lim X_\alpha$ be the direct limit of $X$ by finitely generated submodules $X_\alpha \subseteq X$. Because the tensor product and $\text{Tor}$ commute with direct limits this completes the proof. \hfill \Box

**Remark 2.5.** The previous result 2.4 (b) provides another characterization of $\text{grade}(I, M)$ for a finitely generated module $M$ over a local ring. Namely $\text{grade}(I, M) = \min \{ i \in \mathbb{N} | \text{Tor}_R^i(R/I, D(M)) \neq 0 \}$. This is true because of $\text{Tor}_R^i(R/I, D(M)) \cong D(\text{Ext}_R^i(R/I, M))$ for all $i$. The dual notion of the grade is the width$(I, M)$ as introduced by Frankild (see [7]). It is related to the minimal non-vanishing of local homology. See also [25, 5.1.2], where it is denoted by Tor-codepth.
3. Endomorphisms of Local cohomology

The statements of Lemma 2.4 provide some information about the endomorphisms of the local cohomology module $H^g_I(M)$. If $g = \text{grade}(I,M)$, then in the following we will denote by $\hat{\cdot}$ the $I$-adic completion functor. In the case of a local ring $(R,m)$ we write $(\hat{\cdot})$ for the $m$-adic completion functor.

**Theorem 3.1.** Let $I$ denote an ideal of a local ring $(R,m)$. Let $M$ be a finitely generated $R$-module with $g = \text{grade}(I,M)$. Then there is a natural isomorphism

$$\text{Hom}_R(H^g_I(\hat{M}), H^g_{IR}(\hat{M})) \cong \text{Hom}_R(D(H^g_I(M)), D(H^g_I(M))).$$

**Proof.** By adjunction formula we get the natural isomorphism

$$\text{Hom}_R(D(H^g_I(M)), D(H^g_I(M))) \cong D(H^g_I(M) \otimes_R D(H^g_I(M))).$$

By virtue of 2.4 (b) there are the following isomorphisms to the second module above

$$D(\text{Tor}^g(I,M), D(M)) \cong \text{Ext}^g_R(H^g_I(M), \hat{M}) \cong \text{Ext}^g_R(H^g_{IR}(\hat{M}), \hat{M}).$$

For the first recall the Ext—Tor duality (see e.g. [25, 1.4.1]) and $D(D(M)) \cong \hat{M}$. For the second we refer to [23, Lemma 2.1] and recall that $H^g_I(M) \otimes_R \hat{M} \cong H^g_{IR}(\hat{M})$. Then the statement follows by 2.4 (a) applied to $\hat{R}$ and $\hat{M}$.

For an $R$-module $X$ there is a natural $R$-homomorphism $\phi : R \to \text{Hom}_R(X,X), r \mapsto rx$, the multiplication map by $x$ on $X$. Clearly $\ker \phi = \text{Ann}_R X$. In the following we shall have a look for the situation of $X = H^g_I(M)$. To this end we look also at the natural map $X \otimes_R \text{Hom}_R(X,Y), x \otimes \psi \mapsto \psi(x)$, where $X,Y$ denote $R$-modules.

**Corollary 3.2.** With the previous notation the following conditions about the corresponding natural $R$-homomorphisms are equivalent:

(i) $H^g_I(M) \otimes D(H^g_I(M)) \to E_R(\mathfrak{m})$ is an isomorphism.

(ii) $\hat{R} \to \text{Hom}_R(H^g_{IR}(\hat{M}), H^g_{IR}(\hat{M}))$ is an isomorphism.

(iii) $\hat{R} \to \text{Hom}_R(D(H^g_I(M)), D(H^g_I(M)))$ is an isomorphism.

**Proof.** The equivalence of (ii) and (iii) follows by Theorem 3.1. The natural homomorphism in (i) is an isomorphism if and only if $D(E_R(\mathfrak{m})) \to D(H^g_I(M) \otimes D(H^g_I(M)))$ is an isomorphism. The last module is isomorphic to $\text{Hom}_R(D(H^g_I(M)), D(H^g_I(M)))$. It is easy to see that this is the natural map.

Note that 3.1 and 3.2 prove Theorem 1.1 (a). The following comparison result could be of some interest in order to relate different endomorphism rings to each other.

**Remark 3.3.** Let $J \subseteq I$ denote two ideals such that $\text{grade}(I,M) = \text{grade}(J,M) = g$. Then there is an exact sequence

$$0 \to H^g_I(M) \to H^g_J(M) \to H^g_{V(I) \cap V(J)}(\hat{M})$$

as easily seen. Here $\hat{M}$ denotes the associated sheaf of $M$ on $\text{Spec} R$. Let $X = \text{Coker} \psi$ then it induces an exact sequence

$$0 \to \text{Hom}_R(H^g_J(M), H^g_I(M)) \to \text{Hom}_R(H^g_I(M), H^g_I(M)) \to \text{Ext}^{g+1}_R(X,M)$$

as follows by applying $\text{Hom}_R(\cdot, M)$ and by view of 2.4. Note that $\text{Supp}_R X \subseteq V(J)$ and use 2.4 (a).

In the following we shall investigate when $\psi$ is an isomorphism.
LEMMA 3.4. Let \( J \subseteq I \) denote two ideals with \( \text{grade}(I, M) = \text{grade}(J, M) = g \). Suppose that 
\[
\text{depth}_{R_p} M_p \geq g + 1 \quad \text{for all} \quad p \in \text{Supp}_R M \cap (V(J) \setminus V(I)).
\]
Then the homomorphisms
\[
H^i_f(M) \to H^i_f(M) \quad \text{and} \quad \text{Hom}_R(H^i_f(M), H^i_f(M)) \to \text{Hom}_R(H^i_f(M), H^i_f(M))
\]
are isomorphisms.

PROOF. For the proof it will be enough to prove the first isomorphism. To this end recall that 
\(
\text{Supp}_R X \subseteq \text{Supp}_R M \cap (V(J) \setminus V(I))
\). By the assumption \( \text{depth}_{R_p} M_p \geq g + 1 \) for all \( p \in \text{Supp}_R M \cap (V(J) \setminus V(I)) \). Whence \( H^{g+1}_{\psi(V(J) \setminus V(I))}(\tilde{M}) = 0 \) and \( \psi \) is an isomorphism. \( \square \)

Note that \( \text{depth}_{R_p} M_p \geq g \) for all \( p \in \text{Supp}_R M \cap (V(J) \setminus V(I)) \) as follows by the characterization of 
\( \text{grade}(J, M) = g \). Moreover \( \text{Supp}_R M \cap (V(J) \setminus V(I)) = \text{Supp}_R M / JM : M \langle I \rangle \), where \( JM : M \langle I \rangle \) denotes the stable value of \( JM : M \langle I \rangle \) for \( n \gg 0 \).

4. Endomorphisms of modules

In this section we shall relate the endomorphism ring of an \( R \)-module \( M \) to that of a certain local 
cohomology module. An approach to the Ext-cohomology of \( H^i_E(M) \) is given in the following result.

THEOREM 4.1. With the above notation let \( M, N \) be a finitely generated \( R \)-modules and \( g = \text{grade}(I, M) \). Let \( E^i_R(M), E^i_R(N) \) be minimal injective resolutions of \( M \) and \( N \) resp. Then there is a 
long exact sequence
\[
\ldots \to H^i(\text{Hom}_R(C^i_M(I)), E^i_R(N)) \to \text{Ext}^i_R(M, N) \otimes_R \hat{R}^l \to \text{Ext}^{i+1}_R(H^i_f(M), N)
\]
\[
\ldots \to H^{i+1}(\text{Hom}_R(C^i_M(I)), E^i_R(N)) \to \ldots
\]

PROOF. We start with the short exact sequence of the truncation complex as given in 2.2. Then we 
apply the functor \( \text{Hom}_R(\cdot, E^i_R(N)) \). Therefore there is the following short exact sequence of complexes
\[
0 \to \text{Hom}_R(C^i_M(I), E^i_R(N)) \to \text{Hom}_R(\Gamma_I(E^i_R(M), E^i_R(N)) \to \text{Hom}_R(H^i_f(M), E^i_R(N))[g] \to 0
\]
because \( E^i_R(N) \) is a left bounded complex of injective modules. The \( i \)-th cohomology of the last complex is \( \text{Ext}^{i+1}_R(H^i_f(M), N) \). Next we inspect the cohomology of the complex in the middle. To this end let \( x = x_1, \ldots, x_t \) denote a generating set of the ideal \( I \) and let \( \tilde{L}_x \) denote the bounded free resolution of the \( \tilde{C}_x \) complex as constructed in [25, 6.2.2 and 6.2.3] or [20]. Note that \( M \to E^i_R(M) \) is a quasi-isomorphism 
and therefore \( \Gamma_I(E^i_R(M)) \simeq \tilde{L}_x \otimes_R M \). Because \( E^i_R(N) \) is a left bounded complex of injective modules the complex in the middle is quasi-isomorphic to
\[
\text{Hom}_R(\tilde{L}_x \otimes_R M, E^i_R(N)) \cong \text{Hom}_R(\tilde{L}_x, \text{Hom}_R(M, E^i_R(N))),
\]
(see also [25] for more details). Now it follows that
\[
(\ast) \quad H^i(\text{Hom}_R(\tilde{L}_x, \text{Hom}_R(M, E^i_R(N)))) \cong \Lambda^i(\text{Hom}_R(M, E^i_R(N))).
\]
Since the cohomology \( \text{Ext}^i_R(M, N) \), the cohomology of \( \text{Hom}_R(M, E^i_R(N)) \), is finitely generated it implies the isomorphisms
\[
H^i(\text{Hom}_R(\Gamma_I(E^i_R(M)), E^i_R(N))) \cong \text{Ext}^i_R(M, N) \otimes_R \hat{R}^l
\]
for all \( i \). By the long exact cohomology sequence it proves the claim. \( \square \)

COROLLARY 4.2. With the above notation we get the following:

(a) There is a natural homomorphism \( \text{Hom}_R(M, M) \otimes_R \hat{R}^l \to \text{Hom}_R(H^i_f(M), H^i_f(M)). \)

(b) Suppose that \( H^i_f(M) = 0 \) for all \( i \neq g \). Then the map in (a) is an isomorphism and 
\[
\text{Ext}^i_R(M, M) \otimes_R \hat{R}^l \cong \text{Ext}^{g+j}_R(H^i_f(M), M)
\]
for all \( j \).
PROOF. By 2.4 (b) there is an isomorphism \( \text{Hom}_R(H^g_i(M),H^g_i(M)) \cong \text{Ext}^g_i(H^g_i(M),M) \). Whence the statement in (a) follows 4.1.

Under the additional assumption in (b) \( C_M(I) \) is homologically trivial and \( \text{Hom}_R(C_M(I),E_R(M)) \) is exact. Therefore the statements follow by (a) and 4.1. \( \square \)

Note that 4.2 proves 1.1 (b). As a further application we get a result concerning a cohomological complete intersection, i.e. an ideal such that \( H^g_i(R) = 0 \) for all \( i \neq \text{grade}(I,R) \).

**Proposition 4.3.** Let \( I \subset R \) an ideal with \( H^g_i(R) = 0 \) for all \( i \neq \text{grade}(I,R) \). Let \( N \) denote an arbitrary \( R \)-module. Then there are isomorphisms

\[
\text{Ext}^{g-i}_R(H^g_i(R),N) \cong \Lambda^i(N)
\]

for all \( i \in \mathbb{Z} \).

**Proof.** We follow a slight modification of the proof of 4.1. By the short exact sequence at the beginning of the proof of 4.1 it follows that \( \text{Hom}_R(\Gamma_i(E_R(R)),E_R(N)) \xrightarrow{\sim} \text{Hom}_R(H^g_i(R),E_R(N)[g]) \) since \( C_R(I) \) is homologically trivial. For the isomorphisms (\( \star \)) in the proof of 4.1 with \( M = R \) it yields

\[
H_i(\text{Hom}_R(\check{L}_R, E_R(N))) \cong \Lambda^i(N).
\]

For its proof in 4.1 it is not necessary to assume that \( N \) is finitely generated. Because \( E_R(N) \) is an injective resolution we have \( \text{Hom}_R(\check{L}_R, E_R(N)) \cong \text{Hom}_R(\check{L}_R, N) \) and therefore \( H_i(\text{Hom}_R(\check{L}_R, N)) \cong \Lambda^i(N) \). Then the claim follows by the previous quasi-isomorphism (see also 5.1). \( \square \)

Another application is a result related to 3.2. This is an improvement of the result of Hellus and Stückrad (see [14]). For the proof we refer to 4.2 and 4.3. For further results on the endomorphism ring of \( H^g_i(R), g = \text{grade}(I,R) \), under additional assumptions on the ring we refer to [24].

**Corollary 4.4.** Let \( I \) denote an ideal of a Noetherian ring \( R \) and \( g = \text{grade}(I,R) \). Suppose that \( H^g_i(R) = 0 \) for all \( i \neq g \).

(a) There are isomorphisms \( \text{Ext}^i_{R}(H^g_i(R),H^g_i(R)) \cong \Lambda^i_{H^g_i(R)}(H^g_i(R)) \) for all \( i \in \mathbb{Z} \).

(b) The canonical map \( \check{R}^l \to \text{Hom}_R(H^g_i(R),H^g_i(R)) \) is an isomorphism, \( \text{Hom}_R(H^g_i(R),H^g_i(R)) \cong \Lambda^i_{H^g_i(R)}(H^g_i(R)) \) and \( \text{Ext}^i_{R}(H^g_i(R),R) = 0 \) for all \( i \neq g \).

The previous results have an interesting application on the annihilators of certain local cohomology modules.

**Remark 4.5.** Let \( I \) denote an ideal in a complete local ring \((R,m)\). Let \( M \) be a finitely generated \( R \)-module \( M \) such that \( H^g_i(M) = 0 \) for all \( i \neq \text{grade}(I,M) \). Then \( \text{Ann}_R M = \text{Ann}_R H^g_i(M) \). This follows since the natural homomorphism \( \Phi : R \to \text{Hom}_R(X,X) \) as in 3.2 is given by \( r \mapsto \phi(r) \) with \( \phi(r) : X \to X, x \mapsto rx \), the multiplication map and since \( \text{Ker} \Phi = \text{Ann}_R X \).

## 5. Relation to completions

**Notations 5.1.** (A) Let \( x = x_1, \ldots, x_l \) denote a system of elements in \((R,m)\) and \( I = (x) R \). The we recall the definition of the \( Č \)ech \( Č_x \) and its free resolution \( L_x \) as done in [25, 6.2.2 and 6.2.3] resp. [20].

(B) Let \( M \) be an \( R \)-module. In the case of a Noetherian ring it follows that \( \Lambda^i(M) \cong H_i(\text{Hom}_R(\check{L}_x,M)) \) (see [25] for more details and generalizations).

(C) Here \( \Lambda^i(M), i \geq 0 \), denote the \( i \)-th left derived functor of the \( I \)-adic completion with respect to \( M \).

In the following we shall provide another application of the truncation complex.
THEOREM 5.2. Let \( I \subset R \) denote an ideal. Suppose that \( M \) is a finitely generated \( R \)-module and \( g = \text{grade}(I, M) \). Then there are an exact sequence and isomorphisms
\[
0 \to \Lambda^i_M(C_M(M)) \to \Lambda^i_M(H^g_I(M)) \xrightarrow{\tau} \hat{M}^i \to \Lambda^i_{g-1}(C_M(M)) \to \Lambda^i_{g-1}(H^g_I(M)) \to 0,
\]
\[
\Lambda^i_{g+1}(H^g_I(M)) \cong \Lambda^i_{g+1}(C_M(M)) \quad \text{for all } i \geq 1 \text{ and } i < -1.
\]

Suppose that \( H^i_I(M) = 0 \) for all \( i \neq g \) then \( \tau \) is an isomorphism and \( \Lambda^i_J(H^g_I(M)) = 0 \) for all \( j \neq g \).

PROOF. Let \( E_R(M) \) denote a minimal injective resolution of \( M \). Then we use the truncation complex (see (2.1)). We use the notions of the proof of 3.1 and we apply \( \text{Hom}_R({\hat{L}}_\omega, \cdot) \). Because \( \hat{L}_\omega \) is a bounded complex of free \( R \)-modules. Next we investigate the complex in the middle. By [25, Chapter 7] note the following quasi-isomorphisms
\[
\Gamma_1(E_R(M)) \simeq {\hat{L}}_\omega \otimes_R E_R(M) \simeq {\hat{L}}_\omega \otimes_R M.
\]

Therefore the complex in the middle is quasi-isomorphic to \( \text{Hom}_R({\hat{L}}_\omega, {\hat{L}}_\omega \otimes_R M) \simeq \text{Hom}_R({\hat{L}}_\omega, M) \), where the last quasi-isomorphism follows by view of [25, 6.5.4]. Now
\[
H_i(\text{Hom}_R({\hat{L}}_\omega, M)) \cong \Lambda^i_I(M) \quad \text{for all } i \geq 0
\]
(see 5.1 (B)). Because \( M \) is finitely generated it follows that \( \Lambda^i_I(M) \cong \hat{M}^i \) and the vanishing for \( i \neq 0 \). That is, the long exact homology sequence provides the first part of the statement. If \( H^i_I(M) = 0 \) for all \( i \neq g \) then \( C_M(I) \) and also \( \text{Hom}_R({\hat{L}}_\omega, C_M(M)) \) is exact. Therefore, by the long exact cohomology sequence it finishes the proof. \( \square \)

The previous Theorem proves 1.1 of the Introduction. In the following corollary we will discuss the situation of a Cohen-Macaulay module.

COROLLARY 5.3. Let \( M \) denote a \( d \)-dimensional module over a local ring \( (R, m) \). Then the following conditions are equivalent:

(i) \( M \) is a Cohen-Macaulay module.

(ii) \( \Lambda^m_I(H^d_m(M)) \cong \hat{M}^d \) and \( \Lambda^m_I(H^d_m(M)) = 0 \) for all \( i \neq d \).

PROOF. First prove \((i) \implies (ii)\). To this end recall that \( d = \text{grade}(m, M) = \text{depth}_R M \). Then the conclusion follows by 5.2.

For the proof of \((ii) \implies (i)\) we first prove that it will be enough to show the claim for \( M = \hat{M} \) the \( m \)-adic completion of \( M \) over the completed ring \( \hat{R} \). This follows because of \( \Lambda^m_I(H^d_m(\hat{M})) \cong \Lambda^m_I(H^d_m(M)) \) as easily seen (see e.g. [25, 9.8.3]). Because \( R \) is the quotient of a Gorenstein ring the canonical module \( K_M \) of \( M \) exists and \( H^d_m(M) \equiv D(K_M) \) (see [19] resp. [25, 10.3] for further details). Moreover
\[
\Lambda^m_I(H^d_m(M)) \equiv \Lambda^m_I(D(K_M)) \equiv D(H^d_m(K_M))
\]
for all \( i \) (see e.g. [25, 9.2.5]). By the assumptions it follows \( H^d_m(K_M) = 0 \) for all \( i \neq d \) and \( D(H^d_m(K_M)) = M \). Therefore \( K_M \) is a Cohen-Macaulay module as well as \( K_{K_M} \equiv D(H^d_m(K_M)) \equiv M \), which completes the proof since the canonical module of a Cohen-Macaulay module is Cohen-Macaulay. \( \square \)

In 5.3 (b) it is not enough to assume the vanishing of \( \Lambda^m_I(H^d_m(M)) \) for all \( i \neq d \). This implies (in the case of a complete local ring) that \( K_M \) is a Cohen-Macaulay module. A finitely generated \( R \)-module \( M \) such that \( K_M = \text{Hom}_R(H^d_m(M), E) \) is a Cohen-Macaulay \( \hat{R} \)-module is called canonically Cohen-Macaulay. See [21] for a study of canonically Cohen-Macaulay modules. The following Corollary 5.4 provides an intrinsic characterization of them. The proof is clear by the proof of 5.3.
COROLLARY 5.4. With the previous notation the following conditions are equivalent:
(i) $M$ is a canonically Cohen-Macaulay module.
(ii) $\Lambda_i^\infty(H_i^R(M)) = 0$ for all $i \neq d$.

6. A Generalization

At first we shall prove a slight improvement of an aspect of the so-called Greenlees-May duality. It was originally proved by Greenlees and May (see [8]) and by Lipman et al. (see [1]). It was generalized by Porta, Shaul and Yekutieli (see [16, Theorem 7.14] and the correction [17, Theorem 9]).

PROPOSITION 6.1. Let $R$ denote a commutative ring (not necessarily Noetherian). Let $x = x_1, \ldots, x_m$ and $y = y_1, \ldots, y_n$ denote two weakly proregular system of elements and $I = (x)R$ and $J = (y)R$. Suppose that $\text{Rad} J \subseteq \text{Rad} I$. Then in the derived category there is an isomorphism

$$R\text{Hom}_R(R\Gamma_I(M), R\Gamma_J(N)) \cong \Lambda^I(R\text{Hom}_R(M, N))$$

for two $R$-modules $M, N$.

PROOF. Let $F : \longrightarrow M$ be a free resolution of $M$. Let $\check{L}_x$ and $\check{L}_y$ denote the bounded free resolutions of the Čech complexes $\check{C}_x$ and $\check{C}_y$ resp. (see [25, 6.2.2 and 6.2.3]). Then the following complex

$$\text{Hom}_R(\check{L}_x \otimes_R F, \check{L}_y \otimes_R N) \cong \text{Hom}_R(F, \text{Hom}_R(\check{L}_x, \check{L}_y \otimes_R N))$$

is a representative of the left complex in the statement. Now the natural morphism

$$\text{Hom}_R(\check{L}_x, \check{L}_y \otimes_R N) \rightarrow \text{Hom}_R(\check{L}_x, N)$$

is a quasi-isomorphism since $\text{Rad} yR \subseteq \text{Rad} xR$ (see [25, 6.5.4]). So there is a quasi-isomorphism

$$\text{Hom}_R(F, \text{Hom}_R(\check{L}_x, \check{L}_y \otimes_R N)) \rightarrow \text{Hom}_R(F, \text{Hom}_R(\check{L}_x, N)) \cong \text{Hom}_R(\check{L}_x, \text{Hom}_R(F, N))$$

Finally recall that the last complex in the previous sequence is a representative of the second complex in the statement (see again [25]). \qed

As an application of 6.1 there is an alternative proof of 4.2 and an extension to the relative situation.

COROLLARY 6.2. Let $I, J$ denote two ideals in a Noetherian ring $R$ with $\text{Rad} J \subseteq \text{Rad} I$. Let $M, N$ be two finitely generated $R$-modules. Suppose that $H_i^J(M) = 0$ for all $i \neq g$ and $H_j^J(N) = 0$ for all $j \neq h$. Then there are isomorphisms

$$\text{Ext}_{R}^{i+g-h}(H_i^J(M), H_j^J(N)) \cong \check{R}^l \otimes_R \text{Ext}_{R}^{i}(M, N)$$

for all $i \geq 0$.

PROOF. We apply the isomorphism of 6.1. Under the assumptions the $i$-th cohomology of the left complex is $\text{Ext}_{R}^{i+g-h}(H_i^J(M), H_j^J(N))$. Let $F : \longrightarrow M$ be a free resolution by finitely generated free $R$-modules. The complex at the right in 6.1 is represented by

$$\text{Hom}_R(\check{L}_x, \text{Hom}_R(F, N)) \cong \text{Hom}_R(F, \text{Hom}_R(\check{L}_x, N)) \cong \text{Hom}_R(F, R) \otimes_R \text{Hom}_R(\check{L}_x, N)$$

(see [25, 11.1.2]). Since $N$ is finitely generated $\text{Hom}_R(\check{L}_x, N)$ is quasi-isomorphic to $\check{R}^l \otimes_R N$. Therefore there is a quasi-isomorphism of the previous complexes to

$$\text{Hom}_R(F, R) \otimes_R N \otimes_R \check{R}^l \cong \text{Hom}_R(F, N) \otimes_R \check{R}^l.$$ 

Now note that $\check{R}^l$ is $R$-flat and taking cohomology provides the claim. \qed

For the isomorphism $\Lambda^I(R\text{Hom}_R(M, N)) \cong \check{R}^l \otimes_R R\text{Hom}_R(M, N)$ for finitely generated $R$-modules $M, N$ see also [7, 2.7] resp. [25]. Moreover, note that $\text{grade}(\text{Ann}_R M, N) = \inf\{i \in \mathbb{N} | \text{Ext}_R(M, N) \neq 0\}$ (see e.g. [4, 1.2.10]).
COROLLARY 6.3. Suppose that the ideals $I, J \subset R$ satisfy the assumptions of 6.2. Let $M, N$ denote two finitely generated $R$-modules with $c = \text{cd}(I, M)$ and $h = \text{grade}(J, N)$. Then there is an isomorphism $\text{Hom}_R(H^i_I(M), H^j_J(N)) \cong \hat{R}^i \otimes_R \text{Ext}^{c-i}_R(M, N)$.

**Proof.** We have $c = \sup\{i \in \mathbb{N} | H^i_I(M) \neq 0\}$ and $h = \inf\{H^j_J(N) \neq 0\}$. By view of [6, 2.1] it follows that $H^i(R \text{Hom}_R(\Gamma_I(M), \Gamma_J(N))) = 0$ for all $i < -c + h$ and

$$H^{-c+h}(R \text{Hom}_R(\Gamma_I(M), \Gamma_J(N))) \cong \text{Hom}_R(H^i_I(M), H^j_J(N)).$$

This finishes the proof as in 6.2. \qed

7. Examples

**Example 7.1.** (A) With the notation of 5.2 there is an exact sequence

$$0 \to H_1(\text{Hom}_R(\hat{L}_C, C_M(I))) \to \Lambda^I_\eta(\hat{H}^0_j(M)) \to \hat{M} \to H_0(\text{Hom}_R(\hat{L}_C, C_M(I))) \to \Lambda^I_{\eta -1}(\hat{H}^0_j(M)) \to 0.$$

It would be of some interest to have an intrinsic description of $\text{Ker} \tau$ and $\text{Coker} \tau$. Here $\chi = x_1, \ldots, x_t$ denotes a system of elements that generates the maximal ideal $m$ up to the radical, e.g. a system of parameters of $R$.

(B) Let $M$ denote a $d$-dimensional module with $d > g = \text{depth}_R M > 1$. Suppose that $H^0_d(M)$ is finitely generated for $i \neq d$, i.e. $M$ is a generalized Cohen-Macaulay module. Then $\Lambda^m_0(H^0_d(M)) = 0$ for all $i \neq 0$. For the homology of $\text{Hom}_R(\hat{L}_C, C_M(I))$ there is the spectral sequence

$$E^2_{i,j} = \Lambda^m_0(H^i_d(M)) \Longrightarrow E^\infty_{i-j} = \Lambda^m_{i-j}(C_M(m)).$$

Since $H^0_d(M)$ is finitely generated for $i \neq d$ it follows that $\Lambda^m_0(H^0_d(M)) = 0$ for $j \neq d$ and $i > 0$. Whence there is a partial degeneration to $\Lambda^m_0(C_M(m)) \cong \Lambda^m_d(H^0_d(M))$ at the stage $i - j = 0$ because $\text{depth}_R M > 1$. Therefore $M \cong \Lambda^m_0(H^0_d(M))$. Because $R$ is the quotient of a Gorenstein ring the canonical module $K_M$ of $M$ exists and $H^0_d(M) \cong D(K_M)$ (see [19] resp. [25, 10.3] for further details). Moreover $\Lambda^m_0(D(K_M)) \cong D(\Lambda^m_0(K_M)) \cong K_K$ for $M$. But this is well-known under the assumptions about $M$ (see [19]).

**Example 7.2.** Let $R = \mathbb{k}[x_1, x_2, x_3]$ the formal power series ring in three variables over the field $\mathbb{k}$. Let $I = (x_1, x_2, x_3) \cap (x_2, x_3) \cap (x_1, x_1) \supset J = (x_1, x_2, x_3 + x_2 x_2)$. Then $\text{Rad} I = \text{Rad} J$ as easily seen. Because $J$ is a complete intersection of height $2$ it follows $H^1_j(R) = 0$ for $i \neq 2$ and therefore $R \cong \text{Hom}_R(H^2_j(R), H^2_j(R)) \cong \text{Hom}_R(H^2_j(R), H^2_j(R))$.

**Example 7.3.** As a consequence of Hartshorne’s Second Vanishing Theorem (see [10]) one has the following: Let $C \subset \mathbb{P}^3_{\mathbb{k}}$ denote a connected curve. Then $H^1_I(R) = 0$ for $i \neq 2$, where $I \subset R = \mathbb{k}[[x_0, x_1, x_2, x_3]]$ is the saturated defining ideal of $C$. That is $R \cong \text{Hom}_R(H^2_I(R), H^2_I(R))$.

Note that the image $\mathbb{P}^1_{\mathbb{k}} \to \mathbb{P}^3_{\mathbb{k}}, (s : t) \to (s^4 : s^3 t : s t^3 : t^4)$, is set-theoretically a complete intersection for $\text{char}(\mathbb{k}) = p > 0$ (see [12] or [18]) while this is open in the case of $\text{char}(\mathbb{k}) = 0$. So, it seems that the endomorphism ring $\text{Hom}_R(H^2_I(R), H^2_I(R))$ is not sensitive enough in order to solve this problem.

Let $I \subset R$ denote an ideal of a local ring $(R, m)$ with $\text{grade}(I, R) = g$. In general the structure of the endomorphism ring $\text{Hom}_R(H^2_I(R), H^2_I(R))$ is difficult to understand. In general it is not a finitely generated $R$-module. The following example was suggested by R. Hartshorne (see [11]).

**Example 7.4.** (see [11, 3] and [24, Example 3.6]) Let $\mathbb{k}$ denote a field and $R = \mathbb{k}[[x, y, u, v]]/(xv - yu)$, where $\mathbb{k}[[x, y, u, v]]$ denotes the formal power series ring in four variables over $\mathbb{k}$. Let $I = (x, y) R$. Then $R$ is a Gorenstein ring with $\text{dim} R = 3, \text{dim} R/I = 2$ and $\text{grade}(I, R) = 1$. It follows that $H^0_J(R) = 0$ for $i \neq 1, 2$. Moreover $\text{Supp} H^0_J(R) \subset \{m\}$. We use the truncation complex (see 2.2). By the long exact sequence of the local cohomology it induces a short exact sequence

$$0 \to H^2_I(R) \to H^2_m(I) \to E \to 0.$$
(see [22, Lemma 2.2] for the details). R. Hartshorne (cf. [11, § 3]) has shown that the socle of $H^2_i(R)$ is not a finite dimensional $k$-vector space. Therefore, the socle of $H^2_m(H^1_i(R))$ is infinite dimensional. Moreover there are the following isomorphisms

$$\text{Hom}_R(H^1_i(R), H^1_i(R)) \cong \text{Ext}^1_R(H^1_i(R), R) \cong \text{Hom}_R(H^2_m(H^1_i(R)), E),$$

where the second isomorphism follows by the Local Duality Theorem (see e.g. [25, 10.4.2] for not necessarily finitely generated modules). By the Nakayama Lemma this means that $\text{Hom}_R(H^1_i(R), H^1_i(R))$ is not a finitely generated $R$-module.

A bit more is true: Let $S = k[[u, v, a]]$ and $R \rightarrow S$ the ring homomorphism induced by $x \mapsto au, y \mapsto av$ which is injective. In [24, Example 3.6] it is shown that $S \cong \text{Hom}_R(H^1_i(R), H^1_i(R))$. This proves that $\text{Hom}_R(H^1_i(R), H^1_i(R))$ is a Noetherian ring but not finitely generated over $R$. Also it yields another proof that the socle dimension of $H^2_i(R)$ is not finite since $S$ is not a finitely generated $R$-module.

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