ON THE DISTRIBUTIONAL ROBUSTNESS OF FINITE RATIONAL INATTENTION MODELS

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ABSTRACT. In this paper we study a rational inattention model in environments where the decision maker faces uncertainty about the true prior distribution over states. The decision maker seeks to select a stochastic choice rule over a finite set of alternatives that is robust to prior ambiguity. We fully characterize the distributional robustness of the rational inattention model in terms of a tractable concave program. We establish necessary and sufficient conditions to construct robust consideration sets. Finally, we quantify the impact of prior uncertainty, by introducing the notion of Worst-Case Sensitivity.

Keywords: Rational Inattention, prior uncertainty, misspecified models, Robust Optimization, \(\phi\)-divergences, Shannon Entropy, Risk Measures.

JEL classification: D11, D81, D83
1. Introduction

Rational Inattention (RI) is now a prominent approach for modeling boundedly rational behavior in many areas of economics (Sims [2003, 2010]). Formally, the RI approach models a decision maker (DM) who is uncertain about the payoffs associated to the different alternatives that she faces. In this environment the DM can choose how much to study the different options and what to learn about them subject to information costs, which can involve pecuniary, time, or psychological costs. Given the costs of processing information, the DM can optimally choose to be imperfectly informed about some of the available options. One of the major advantages of the RI model is its flexibility as it does not impose any particular assumption on what agents learn or how they go about learning it. Intuitively, in an RI problem a DM derives the information structure from her utility-maximizing behavior under the constraint that information is costly to acquire and processes. Thus, a rationally inattentive DM process information that she finds useful and ignores information that is not worth the effort of acquiring and processing.

In a fundamental contribution, Matejka and Mckay [2015] show that the DM’s optimal information-processing strategy results in probabilistic choices that reflects both the actions’ true payoffs as well as the DM’s prior beliefs. In particular, they show that when information costs are modeled using the Shannon mutual information between actions and states, the resulting choice probabilities in the RI model take the form of a generalized multinomial logit (MNL) model, which depends both on the actions’ true payoffs as well as on prior beliefs. In deriving their result, Matejka and Mckay [2015] assume that the DM knows the correct prior distribution over the set of possible states. In other words, they assume that the DM faces uncertainty about payoffs realizations but not about the distribution that generate them. In the language of Hansen and Sargent [2001, 2008], Maccheroni et al. [2006] the DM does not face prior ambiguity or model misspecification. So even though the RI framework generalizes the problem faced by the DM in terms of incorporating informational constraints, we still end up in a scenario in which the DM is endowed with a sharp knowledge of the prior belief brought about in an ex machina fashion.

1A prior consists of a probability measure over the set of states of the world.

2The assumption that the DM has correct prior beliefs is relevant not only in theoretical work but also in empirical applications of the RI approach. For instance the econometric work by Dasgupta and Mondria [2018], Brown and Jeon [2021], Porcher [2020], Joo [Forthcoming], and DeDad et al. [2021] assume that the DM’s prior beliefs are correct or follow a determined known parametric distribution.
In this paper we ask the following question: what should a rational inattentive DM do if the prior probability distribution over states is unknown or cannot be trusted due to misspecification?

We answer this question by developing an RI framework which allows for prior ambiguity and/or model misspecification. Formally, we study the RI problem in an environment in which the DM has some best guess/estimate about the true unknown prior but does not fully trust it. Instead, the DM considers many other probability distributions to be feasible, with feasibility diminishing proportionally to their statistical distance from her best guess prior. We employ the notion of $\phi$-divergence to measure the distance between probability distributions (Liese and Vajda [1987]). Examples of $\phi$-divergence functions are the Shannon and Renyi entropy, respectively. This notion of distance allows us to introduce the concept of an uncertainty set, which represents the set of feasible priors that the DM can consider.

Endowed with the notion of an uncertainty set, we study the distributional robustness of the RI model. In particular, we introduce a robust optimization approach that allows us to relax the assumption that the DM fully knows and trusts the prior governing the state realizations. Formally, we exploit the fact that optimal rational inattentive behavior can be expressed in terms of a stochastic optimization program with respect to the (unconditional) choice probability vector (Matejka and Mckay [2015, Cor. 1]). We leverage this structure to jointly solve for the optimal choice probability vector and the worst-case prior. We denote the resulting model as the Distributionally Robust Rational Inattention (DRO-RI) model.

We make three main contributions. First, we characterize the distributional robustness of the RI model in terms of a concave optimization program (Theorem 1). Using this result we provide necessary and sufficient conditions to compute the optimal choice probability vector and the worst-case prior. Similar to Matejka and Mckay [2015], we show that the stochastic choice rule is given by a generalized MNL model. However, our result differs in that it incorporates the role of choosing the worst-case prior distribution. More generally, our results highlight the role of the $\phi$-divergence function choice and the nature of information acquisition. In particular, we show that the DRO-RI model can be interpreted through the lens of averse stochastic optimization and risk.

For instance, Joo [Forthcoming] introduces an approach that allows for a subjective prior which captures heterogeneity in consumer behavior consistent with market data. He shows that by introducing this flexible notion of priors, the RI model can be econometrically identified. However, his framework assumes that consumers do not face prior ambiguity.
measures (Ruszczyński and Shapiro [2021]). We discuss, how by choosing the appropriate $\phi$-divergence function, a robust DM selects the choice probability vector and the worst-case prior considering the risk associated to prior uncertainty as measured by the Average-Value-at-Risk (AVaR) and entropic risk, respectively. In addition, we study the class of Cressie-Read divergences. For this class, we show how that in order to protect against worst-case distributional shifts, a robust DM will optimize the tail performance of the worst-case prior. To the best of our knowledge, the connection between the DRO-RI model, risk measures, and the Cressie-Read family is new to the RI literature.

In our second contribution, we provide necessary and sufficient conditions for the formation of robust consideration sets in the DRO-RI model. In establishing this result, we employ Lagrangian techniques similar to Caplin et al. [2018]. However our characterization of consideration sets differs from theirs in at least two aspects. First, our characterization highlights the role of jointly choosing the choice probability vector and the worst-case prior. In fact, our result makes explicit how these two variables interact to add or exclude alternatives from the consideration set. Second, our characterization allows us to show that, given prior uncertainty, a robust DM will add more alternatives to the consideration set than an environment where the DM has correctly specified priors. We establish this property in a situation where the DM must choose between a number of goods, one of which is of high quality and the remainder alternatives are of low quality.

The paper’s third contribution is the characterization of the effect of prior uncertainty in terms of the DM’s utility loss. Following Gotoh et al. [2020], we introduce the notion of Worst-Case Sensitivity, which is defined as the worst-case rate of decrease in the expected utility of a DRO-RI problem when the size of the uncertainty set vanishes. We show that worst-case sensitivity is proportional to the standard deviation associated to the DM’s expected utility (under the best guess prior). More generally, we show that the DRO-RI model is essentially a mean-(worst-case) sensitivity problem when the degree of uncertainty is small. We provide a closed-form expression of worst-case sensitivity exploiting the smoothness of several well-known $\phi$-divergence functions.

1.1. Related literature. This paper belongs to the growing and active research on RI and stochastic choice models. As we mentioned earlier, the connection between costly information and the MNL model was first established by Matejka and Mckay [2015]. The paper by Fosgerau et al. [2020] extends this connection to a general class of discrete choice models. They use the notion of Bregman information costs. The paper by Caplin et al. [2018]
provides necessary and sufficient first order conditions for the formation of consideration sets. The work by [Armenter et al. 2021] introduces a geometric approach to study RI in the case of Shannon information costs. The recent paper by [Caplin et al. 2022] introduces three new classes of cost functions that generalizes the traditional RI model based on Shannon mutual entropy. Similarly, [Morris and Yang 2021] and [Hébert and Woodford 2021] introduce new information cost functions to the study of stochastic choice behavior with information frictions. All of these papers study rational inattentive behavior under the assumption that the DM’s prior is correct. In other words, none of these papers address the problem of robustness in the RI model.

The paper by [Hansen et al. 2022] is the closest work to ours. Similar to us, they do study the robustness of finite RI models, but following the macroeconomics literature on robustness ([Hansen and Sargent 2001, 2008]), they study prior uncertainty in terms of the Kullback-Leibler distance. They provide necessary and sufficient conditions for the robust solution and develop numerical methods to solve it. In spite of the overall coincidence of the main question that both papers tackle, there are at least four important differences between our work and theirs. First, our analysis is based on distributionally robust optimization techniques, where we formally exploit Lagrangian duality to express the DRO-RI model as a finite dimensional concave problem. Our approach is by no means nonessential, as we are able to study a large family of $\phi$-divergence functions where the the Kullback-Leibler distance ends up being a particular case. This fact implies that our characterization of robust consideration sets does not depend on the choice of the $\phi$-divergence function. Second, for a large class of $\phi$-divergences, we provide necessary and sufficient conditions to compute a solution to the DRO-RI model. For instance, our approach can be used to compute a solution when the $\phi$-divergence function is given by the Cressie-Read family, which includes the case of Kullback-Leibler and Renyi entropy as particular cases. Third, our analysis allows us to connect robust rationally inattentive behavior with the notion of risk measures. In fact, we actually discuss the way in which [Hansen et al. 2022]’s approach corresponds to the entropic risk measure case. Finally, in a fourth important difference with [Hansen et al. 2022], we do carry out sensitivity analysis with respect to the degree of robustness. In all, even though both research papers study the problem of robustness in finite RI models, they concurrently provide different and complementary results.

4For a complete review of RI models, we refer the reader to [Maćkowiak et al. 2021].
Our paper is also related to the decision theory literature on ambiguity and model uncertainty. The seminal papers by Gilboa and Schmeidler [1989] and Maccheroni et al. [2006] provide axiomatic foundations to represent DM’s preferences in environments where she faces model ambiguity. Strzalecki [2011] studies from an axiomatic standpoint the connection between multiplier preferences and robustness. However, none of these papers study the problem of model uncertainty in the context of RI models.

Finally, our work is related to the active and fast growing literature on Distributionally Robust Optimization problems. Shapiro [2017a] and Kuhn et al. [2019] provide an up-to-date treatment of the subject. Applications vary from inventory management to regularization in machine learning. However, this literature has not been used to address the problem of robustness in RI models. Thus the results of this paper contribute to this literature by bringing a new class of models suitable to be analyzed using distributionally robust optimization techniques.

The rest of the paper is organized as follows. §2 presents the RI model and discusses the role of prior uncertainty. §3 presents the DRO-RI approach and characterizes the distributionally robustness of the RI model. §4 discusses the formation of consideration sets. §5 characterizes the worst-case sensitivity of the DRO-RI model. Finally, §6 concludes. Proofs and technical lemmas are gathered in Appendix A.

2. The RI model

In this section we introduce the RI model. A DM is presented with a discrete set of alternatives denoted by $A = \{1, \ldots, n\}$ from which he must choose one. There is a finite set of states denoted by $\Theta \subseteq \mathbb{R}^m$ with typical element $\theta = (\theta_1, \ldots, \theta_m) \in \Theta$. The DM is endowed with a utility function $u : A \times \Theta \mapsto \mathbb{R}$, where $u(a, \theta)$ denotes the utility associated to alternative $a$ when the state $\theta$ is realized. The state $\theta$ is unknown to the DM, and as consequence the payoff vector $u(\theta) = (u(a, \theta))_{a \in A}$ is also unknown to her. For instance in a consumer choice problem, $\theta$ can represent a vector of qualities associated to the goods in $A$.

The DM possesses some prior knowledge about the payoffs associated to the available options, given by a prior belief (probability measure) $\mu$ with support...
on \( \Theta \). In other words, the DM’s prior corresponds to \( \mu \in \Delta(\Theta) \). As we discussed in §1, the traditional RI approach assumes that the prior \( \mu \) is the correct model describing state realizations. In the language of [Hansen and Sargent 2001, 2008], and [Maccheroni et al. 2006] the DM does not face model uncertainty (or ambiguity). Assumption 1 below formalizes this condition.

**Assumption 1. [No model uncertainty]** The DM possesses some prior knowledge about the available options, given by a probability measure \( \mu \in \Delta(\Theta) \), where \( \mu(\theta) \triangleq P(\theta \text{ is realized}) > 0 \) for all \( \theta \in \Theta \).

The previous assumption makes explicit the fact that the DM only faces uncertainty about the realization of \( \theta \) but not about the distribution \( \mu \) governing such realizations.

Let \( p_a(\theta) \) denote the conditional choice probability of selecting \( a \) given the state \( \theta \). Accordingly, the conditional choice probability vector is denoted by \( p(\theta) = (p_a(\theta))_{a \in A} \). Let \( P = \{p(\theta)\}_{\theta \in \Theta} \) be the collection of conditional choice probabilities. Let \( \mathcal{P} \) denote the set of conditional choice probability collections. Given \( P \) and the prior \( \mu \), the unconditional choice probability vector is denoted by \( x = (x_a)_{a \in A} \) where \( x_a = \sum_{\theta \in \Theta} p_a(\theta) \mu(\theta) \) for all \( a \in A \).

In the RI paradigm the DM’s problem can be structured into two stages. First, she selects an information strategy (a mapping from state of the world to information signals) to refine her belief \( \mu \) about the state. In comparing how informative different information structures are, the DM must incur in a cost of processing information, where more precise information structures are more costly. In the second stage, the DM faces a standard decision problem under uncertainty with the beliefs generated in the first stage.

A fundamental result in [Matejka and Mckay 2015] shows that the optimization problem just described can be equivalently seen as directly choosing the set of conditional choice probabilities when Shannon entropy information-cost function is used. Formally, the Shannon cost function is defined as:

\[
\xi C(P, \mu) \triangleq \xi \left[ \sum_{a \in A} \sum_{\theta \in \Theta} p_a(\theta) \left( \log p_a(\theta) - \log x_a \right) \mu(\theta) \right],
\]

where the parameter \( \xi > 0 \) measures the unitary marginal cost of information.

Following [Matejka and Mckay 2015] and [Fosgerau et al. 2020], the problem of the rationally inattentive DM is to choose the optimal collection of conditional choice probabilities \( P \), balancing the expected payoff against the cost of
information. Formally, the DM solves the following optimization problem:

\[
(1) \max_{P \in \mathcal{P}} \left\{ \sum_{\theta \in \Theta} \left[ \sum_{a \in A} p_a(\theta) u(a, \theta) \right] \mu(\theta) - \xi C(P, \mu) \right\},
\]

subject to

\[
(2) \quad p_a(\theta) \geq 0 \text{ for all } a \in A \quad \text{and} \quad \sum_{a \in A} p_a(\theta) = 1 \quad \forall \theta \in \Theta.
\]

As in shown in [Matejka and Mckay 2015], a solution to this problem yields the optimal conditional choice probabilities given by:

\[
(3) \quad p_a(\theta) = \frac{x_a e^{u(a, \theta)/\xi}}{\sum_{b \in A} x_b e^{u(b, \theta)/\xi}} \text{ for all } a \in A.
\]

Expression (3) is the generalized MNL model in which the payoff vector \( u(\theta) \) is shifted by the term \( \log x_a \). This is a key result from [Matejka and Mckay 2015]. Remarkably, the influence of the prior information \( \mu \) is completely captured by this shift vector \( \log x_a \). In particular, DM’s prior knowledge and information-processing strategy are incorporated into the choice probabilities through the weights, \( \log x_a \). Intuitively these terms shift the choice probabilities toward those alternatives that are good choices a priori. More importantly, these weights are independent of the actual (contingent) payoffs of the alternatives. In particular, expression (3) makes explicit the fact that once we know the solution \( x \in \Delta_n \), we can recover the conditional choice probabilities. In other words, in order to construct the generalized MNL choice probabilities, one only needs to solve for the optimal choice probability vector \( x \in \Delta_n \). This intuition is formalized in the following result from [Matejka and Mckay 2015].

**Proposition 1** (Matejka and Mckay (2015)). Let Assumption \([\dagger]\) hold. Then the collection of conditional choice probabilities \( P = \{p(\theta)\}_{\theta \in \Theta} \) solves (1)-(3) if and only if the probabilities are given by the generalized MNL formula (3) with the (unconditional) choice probability vector \( x \) solving the concave optimization problem

\[
(4) \quad \max_{x \in \Delta_n} \mathbb{E}_\mu [H(x, \theta)]
\]

where \( H(x, \theta) \triangleq \xi \log(x^\top e^{u(\theta)/\xi}) \) and \( \Delta_n \triangleq \Delta(A) \) being the \( n \)-dimensional simplex.

The relevance of program (4) comes from the fact that it summarizes the role of the prior distribution \( \mu \) in shaping the optimal solution to the RI problem.
in terms of the unconditional choice probabilities $x$. In particular, different choices of $\mu$ lead to different solutions to (4). In addition, noting that Eq. (3) depends on the optimal solution to (4), it follows that the entire RI model depends on the fact that $\mu$ is correctly specified. In other words, Assumption 1 is key in the analysis of the RI model.

2.1. A robust approach to RI. Our main goal in this paper is to develop a framework that relaxes Assumption 1. In doing so, we develop a Distributionally Robust Optimization RI framework (henceforth DRO-RI) in which the DM is uncertain about the true prior distribution describing the realization of the random vectors $\theta$. We formalize this idea by replacing the program (4) by a distributionally robust optimization problem which allows for prior uncertainty. In particular, and in the spirit of Gilboa and Schmeidler [1989] and Maccheroni et al. [2006], the DM faces prior uncertainty and wants to be robust by solving the following max-min problem:

$$\max_{x \in \Delta_n} \min_{\nu \in B} \{E_\nu(H(x, \theta))\}$$

where $B \subseteq \Delta(\Theta)$ describes a set of priors from which the DM can choose from. In other words, in the DRO-RI model, the DM chooses a pair $(x, \nu)$ that solves the problem (5). It is worth pointing out that under Assumption 1 the program (4) is a particular case of (5) where the set $B$ boils down to $B = \{\mu\}$.

A key aspect of the DRO-RI model is related to the structure of the uncertainty set $B$. In §3 we define the set $B$ in terms of $\phi$-divergence functions. This latter concept enables us to use the notion of statistical distances between probability distributions. As we shall see, by doing this we are able to derive a simple characterization of (5) which connects the solutions to this problem with the degree of robustness that a rationally inattentive DM seeks. Furthermore, we shall show that despite its apparently complexity, program (5) is highly tractable, making it suitable for comparative statics and consideration set analysis.

3. The DRO-RI Model

In this section we formally introduce the DRO-RI model. In doing so, we exploit the concept of statistical distance between probability distributions using the idea of $\phi$-divergence functions [Ali and Silvey 1966, Csizár 1967].

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7In a recent paper, program (4) has been exploited by Armenter et al. [2021] to develop geometric methods to study finite-state RI problems. Their analysis assumes that Assumption 1 is satisfied.
and [Liese and Vajda 1987]). By employing the $\phi$-divergence concept we are able to formalize the notion of an uncertainty set, which accounts for the possibility of prior uncertainty on the DM’s behavior. In this section we show that the DRO-RI can be characterized as a tractable concave optimization program which accounts for the degree of ambiguity in the DM’s priors. In addition, we discuss how the DRO-RI connects with the notion of risk measures.

3.1. $\phi$-divergence functions. As we said before, a $\phi$-divergence is a function that measures the statistical distance between probability distributions. Formally, the notion of $\phi$-divergence function is defined as:

**Definition 1.** The $\phi$-divergence distance between two probability vectors $\nu$ and $\mu \in \Delta_m$ is defined as

\[
I_\phi(\nu, \mu) = \sum_{i=1}^{m} \mu_i \phi\left( \frac{\nu_i}{\mu_i} \right),
\]

where $\phi(t)$ is convex for $t \geq 0$, $\phi(1) = 0$, $0 \phi(a/0) \triangleq \lim_{t \to \infty} \phi(t)/t$ for $a > 0$, and $0 \phi(0/0) \triangleq 0$.

The function $\phi$ is referred as the $\phi$-divergence function. Throughout our analysis, we use the following technical assumption.

**Assumption 2** (Smooth $\phi$-divergences). The $\phi$-divergence function $\phi(t)$ is strictly convex, twice continuously differentiable in $t$, with $\phi(1) = 0$, $\phi'(1) = 0$ and $\phi''(1) > 0$.

The previous assumption is just a smoothness condition on $\phi$ and many $\phi$-divergence functions commonly used in the statistical literature satisfy it. For instance, when $\phi(t) = t \log t - t + 1$, expression (6) boils down to the well-known Kullback-Leibler distance: $\bar{I}_\phi(\nu, \mu) = \sum_{i=1}^{m} \nu_i \log(\nu_i / \mu_i)$. In this case straightforward algebra shows that $\phi''(1) = 1$.

Table 1 below displays several well-known $\phi$-divergence functions and their corresponding $I_\phi(\nu, \mu)$.

Another important notion in our analysis is the concept of convex conjugate. In the case of the $\phi$-divergence its convex conjugate is given by a function $\phi^* : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ defined as follows:

\[
\phi^*(s) = \sup_{t \geq 0} \{st - \phi(t)\}.
\]

8For a detailed treatment of the concept of $\phi$-divergence and its applications to information theory and statistical inference, we refer the reader to Liese and Vajda [2006], Pardo [2005], and Shapiro [2017b].

9We note that when the $\phi$-divergence is differentiable at $t = 1$, the function $\varphi(t) = \phi(t) - \phi'(t)(t - 1)$ also yields a $\phi$-divergence, satisfying for probability vectors $\bar{I}_\varphi(\nu, \mu) = \bar{I}_{\phi}(\nu, \mu)$, with $\varphi(1) = \varphi'(1) = 0$ and $\varphi(t) \geq 0$. For details we refer the reader to Pardo [2005].
It is worth noticing that in some cases $\phi^*$ may not exist in closed form. This is, for example, the case for the $J$-divergence. However, tractable reformulations can be derived using convex analysis results ([Rockafellar 1970]). Table 1 below displays several $\phi^*$ functions associated to commonly used $\phi$-divergences.

| Divergence          | $\phi(t)$                 | $\phi(t) \geq 0, t \geq 0$ | $I_\phi(\nu, \mu)$ | $\phi^*(s)$ |
|---------------------|---------------------------|-----------------------------|---------------------|-------------|
| Kullback-Leibler    | $\phi_{KL}(t)$            | $t \log t - t + 1$         | $\sum_{i=1}^{m} \nu_i \log (\nu_i/\mu_i)$ | $e^s - 1$   |
| Burg Entropy        | $\phi_{B}(t)$             | $- \log t + t - 1$         | $\sum_{i=1}^{m} \mu_i \log (\mu_i/\nu_i)$ | $-\log(1-s), s < 1$ |
| J-Divergence        | $\phi_j(t)$               | $(t-1) \log t$             | $\sum_{i=1}^{m} (\nu_i - \mu_i) \log \left( \frac{\nu_i}{\mu_i} \right)$ | No closed form |
| $\chi^2$-distance   | $\phi_e(t)$               | $\frac{1}{2}(t-1)^2$       | $\sum_{i=1}^{m} \frac{(\nu_i - \mu_i)^2}{\nu_i}$ | $2 - 2\sqrt{1 - s}$ |
| Modified $\chi^2$- distance | $\phi_{mc}(t)$           | $\frac{1}{2}(t-1)^2$       | $\sum_{i=1}^{m} \frac{(\nu_i - \mu_i)^2}{\nu_i}$ | $\begin{cases} -1 & s < -2 \\ s + s^2/4 & s \geq -2 \end{cases}$ |
| Hellinger distance  | $\phi_{H}(t)$             | $-(\sqrt{t} - 1)^2$        | $\sum_{i=1}^{m} (\sqrt{\nu_i} - \sqrt{\mu_i})^2$ | $\frac{1}{1-s}, s < 1$ |
| $\chi$-divergence of order $k > 1$ | $\phi_{\chi}(t)$          | $|t-1|^k$                   | $\sum_{i=1}^{m} \mu_i \left| 1 - \frac{\nu_i}{\mu_i} \right|^k$ | $s + (k-1) \left( \frac{k}{s} \right)^{k/(k-1)}$ |
| Variation Distance  | $\phi_v(t)$               | $|t-1|$                     | $\sum_{i=1}^{m} |\nu_i - \mu_i|$ | $\begin{cases} -1 & s \leq -1 \\ s & -1 \leq s \leq 1 \end{cases}$ |
| Cressie-Read        | $\phi^{k}_{CR}(t)$       | $\frac{t^k - k + k - 1}{k(k-1)}, k \in \mathbb{R}\{0, 1\}$ | $\frac{1}{\pi} \left( 1 - \sum_{i=1}^{m} \nu_i^k \mu_i^{1-k} \right)$ | $\frac{1}{\pi} \left[ ((k-1) s + 1)^{k^*} - 1 \right]$ |

### 3.2. Uncertainty sets.**

Equipped with the notion of $\phi$-divergence, we formalize the idea that a DM faces prior ambiguity by defining the **uncertainty set** as follows:

$$B_\phi(\mu) = \{ \nu \in \Delta(\Theta) : I_\phi(\nu, \mu) \leq \rho \}.$$  

Following [Hansen and Sargent 2001], the set (8) is interpreted as an environment in which the DM has some best guess (estimate) $\mu$ of the true unknown probability distribution over $\Theta$, but does not fully trust it. Instead, the DM considers many other probability distributions $\nu$ to be feasible, with feasibility diminishing proportionally to their “distance” from $\mu$. Following the robust optimization literature, we denote $\mu$ as the nominal distribution.

Intuitively, the set (8) considers all distributions $\nu$ whose $\phi$-divergence from the nominal distribution is at most $\rho$. Accordingly, the parameter $\rho$ can be seen as an **index of robustness**, since it determines the size of $B_\phi(\mu)$. We want $\rho$ to be small but not too small. For instance, setting $\rho = 0$ one obtains that $B_\phi(\mu)$ is just the nominal distribution $\mu$. On the other hand, when $\rho \rightarrow \infty$ the ambiguity set $B_\phi(\mu)$ admits all possible probability distributions on $\Theta$. In this latter case the DM is overly conservative with the goal of maximizing the worst outcome. Thus $\rho$ can be seen as a **risk-level parameter** which reflects the DM's perceived ambiguity (uncertainty) in her choice.
Given the structure of the uncertainty set $B_\phi(\mu)$, the DRO-RI problem can be stated as the following distributionally robust optimization program:

\[
\max_{x \in \Delta_n} \min_{\nu \in B_\phi(\mu)} \left\{ \mathbb{E}_\nu(H(x, \theta)) \right\}.
\]

Without further delay we establish the main result of this section

**Theorem 1.** The DRO-RI problem (9) is equivalent to solve the following optimization problem:

\[
\max_{x \in \Delta_n, \lambda \geq 0, \eta \in \mathbb{R}} \left\{ -\eta - \rho \lambda - \lambda \mathbb{E}_\mu \left( \phi^* \left( \frac{-H(x, \theta) - \eta}{\lambda} \right) \right) \right\},
\]

where $\lambda$ and $\eta$ are the Lagrange multipliers associated to the constraints given by $B_\phi(\mu)$ and $\sum_{\theta \in \Theta} \nu(\theta) = 1$, respectively. Furthermore, the problem (10) is concave in $x, \lambda,$ and $\eta$.

Some remarks are in order. First, the result in Theorem 1 establishes that the max-min problem (9) is equivalent to solve a concave optimization program in the variables $x, \lambda, \eta$. More importantly, program (10) is a typical stochastic optimization problem where the expectation is taken with respect to the nominal $\mu$. This latter fact makes problem (10) suitable to be studied using, for instance, stochastic optimization techniques such as Sample Average Approximation (SAA) (Shapiro [2021]).

A second observation is related to the role of the convex conjugate $\phi^*$ in problem (10). As we mentioned before, in some cases $\phi^*$ may not be available in closed form. Fortunately, for many commonly used $\phi$-divergence functions, the conjugate $\phi^*$ has a closed form expression (see Table 1). Furthermore, in cases where $\phi^*$ is not available in closed-form, we can exploit the concave structure of program (10) combined with some stochastic optimization algorithm (Shapiro [2021]).

As a third observation, we note that Theorem 1 follows from applying the arguments developed in the seminal paper by Ben-Tal et al. [2013]. Thus, our result can be seen as a connection between the RI and Robust optimization literature. We also remark that Theorem 1 can be extended to the case when

\[\text{max} \quad \sum_{t=1}^T \left\{ -\eta - \rho \lambda - \lambda \mathbb{E}_\mu \left( \phi^* \left( \frac{-H(x, \theta^t) - \eta}{\lambda} \right) \right) \right\},\]

\footnote{For instance, to apply the SAA, we first generate an independent and identically distributed (i.i.d.) sample $\theta^t = (\theta^t_1, \ldots, \theta^t_m)$ for $t = 1, \ldots, T$ from the distribution $\mu$, and then solve following optimization problem:}
\( \Theta \) is a continuous subset of \( \mathbb{R}^m \). Appendix B provides the details of this extension.

As we mentioned in the literature review section, our paper is related to the recent work by Hansen et al. [2022] which focuses on the case of \( \phi_{kl} \). Theorem 1 is a fundamental difference between their work and ours. In particular, our result establishes that the robustness of the RI model can be studied far beyond the case Kullback-Leibler divergence. Furthermore, we note that given the concave structure of problem (10), we can exploit the necessary and sufficient first order conditions to compute the worst-case probability vector \( \nu \). The next proposition formalizes this fact.

**Proposition 2.** Let \( s^*(\theta) \triangleq -\frac{H(x^*,\theta)+\eta^*}{\lambda^*} \). Then, for an optimal solution \( x^*, \lambda^*, \eta^* \) to problem (10) with \( \lambda^* > 0 \), the worst-case prior \( \nu \) satisfies the following equations:

\[
\begin{align*}
\nu(\theta) & \in \partial \phi^*(s^*(\theta)), \\
\sum_{\theta \in \Theta} \mu(\theta) \phi \left( \frac{\nu(\theta)}{\mu(\theta)} \right) & = \rho, \\
\sum_{\theta \in \Theta} \nu(\theta) & = 1.
\end{align*}
\]

The result in Proposition 2 is just a rearrangement of the first order conditions associated to problem (10). An important aspect of Eq. (11) is the presence of the convex conjugate \( \phi^* \). This structure gives us a method for recovering the worst-case distribution \( \nu \) from the dual problem. It uses Fenchel equality which establishes that \( \frac{\nu(\theta)}{\mu(\theta)} \in \partial \phi^*(s^*(\theta)) \) if and only if \( s^*(\theta) \in \partial \phi \left( \frac{\nu(\theta)}{\mu(\theta)} \right) \).

As it is inferred from Table 1, for many \( \phi \)-divergence functions the convex conjugate \( \phi^* \) is differentiable, implying that condition (11) can be expressed as \( \nu(\theta) = \mu(\theta) \phi'^*(s^*(\theta)) \). This latter condition is immediately satisfied if we impose Assumption 2 at the beginning of statement of Proposition 2.

It is worth noticing that expression (12) corresponds to the constraint associated to the uncertainty set. It is written as an equality because with \( \lambda^* > 0 \), the complementary slackness dictates that this constraint must be active. Similarly, expression (13) corresponds to the constraint associated to the worst-case probability \( \nu \).

\textsuperscript{11}We note that because \( \phi \) is a proper closed convex function, so is \( \phi^* \) ([Rockafellar, 1970, Thm. 12.2]). Hence, \( \phi^* \) is subdifferentiable on the relative interior of its domain ([Rockafellar, 1970, Thm. 23.4]).
In order to gain some intuition about how Proposition 2 works, the next example analyzes the well-known case of the Kullback-Leibler divergence.

**Example 2.** Let \( \phi_{kl} \) denote Kullback-Leibler divergence. Accordingly, we write

\[
I_{\phi_{kl}}(\nu, \mu) = \sum_{\theta \in \Theta} \mu(\theta) \log \left( \frac{\nu(\theta)}{\mu(\theta)} \right).
\]

In this case we have that \( \phi^* \) is differentiable where \( \phi^*(s^*(\theta)) = e^{s^*(\theta)} \) with \( s^*(\theta) \) defined as in Proposition 2. From (12) we know that for \( \lambda^* > 0 \), then condition (11) can be written as:

\[
\nu(\theta) = \mu(\theta) e^{-(H(x,\theta) + \eta)/\lambda} \quad \text{for all } \theta \in \Theta.
\]

Now we express the previous expression as:

\[
\nu(\theta) = \mu(\theta) e^{-H(x,\theta)/\lambda} e^{-\eta/\lambda},
\]

\[
\sum_{\theta \in \Theta} \nu(\theta) = e^{-\eta/\lambda} \sum_{\theta \in \Theta} e^{-H(x,\theta)/\lambda},
\]

\[
1 = e^{-\eta/\lambda} \sum_{\theta \in \Theta} \mu(\theta) e^{-H(x,\theta)/\lambda} \quad \text{(using condition (13)),}
\]

\[
e^{\eta/\lambda} = \sum_{\theta \in \Theta} \mu(\theta) e^{-H(x,\theta)/\lambda}.
\]

The last expression implies that:

\[
(14) \quad \nu(\theta) = \frac{\mu(\theta) e^{-H(x,\theta)/\lambda}}{\sum_{\theta' \in \Theta} \mu(\theta') e^{-H(x,\theta')/\lambda}} \quad \forall \theta \in \Theta.
\]

Thus, for the case of the Kullback-Leibler distance, we have a closed-form expression for the worst-case probability measure \( \nu(\theta) \).

It is worth mentioning that when \( \rho \) is small, we can find \( \nu \) in an explicit form for a general class of smooth \( \phi \)-divergences. In particular, Lemma 1 in §3.3 shows how to approximate the worst-case distribution \( \nu \) when \( \rho \) is small enough.

### 3.3. DRO-RI and risk measures.

In economic terms, the DRO-RI has a close connection with the theory of risk measures [Ruszczyński and Shapiro 2021]. In order to see this, we first discuss the case known as Average Value-at-Risk (AVaR). In doing so, for \( \alpha \in (0, 1] \) we define the function \( \phi \) as follows:

\[
\phi(t) = \begin{cases} 0, & 0 \leq t \leq 1/\alpha, \\ +\infty, & \text{otherwise.} \end{cases}
\]
The associate uncertainty set is given by:

\[
B_\phi(\mu) = \left\{ \nu \in \Delta(\Theta) : \frac{\nu(\theta)}{\mu(\theta)} \in [0, \alpha^{-1}] \right\}.
\]

In this case the conjugate of \( \phi \) is given by \( \phi^*(s) = \max \{0, \alpha^{-1} s\} \cong [\alpha^{-1} s]_+ \).

Furthermore, it is easy to see that for any \( \lambda > 0, \lambda \phi = \phi \).

Thus for a fixed \( x \in \Delta_n \), we define the functional \( R(x) \) as follows:

\[
\begin{align*}
R(x) &= \inf_{\eta \in \mathbb{R}, \lambda \geq 0} \left\{ \lambda \rho + \eta + \alpha^{-1} \mathbb{E}_\mu [\tilde{H}(x, \theta) - \eta]_+ \right\} \\
&= \inf_{\eta \in \mathbb{R}} \left\{ \eta + \alpha^{-1} \mathbb{E}_\mu [\tilde{H}(x, \theta) - \eta]_+ \right\}
\end{align*}
\]

where \( \tilde{H}(x, \theta) \cong -H(x, \theta) \) is a convex function on \( x \).

For a fixed \( x \), expression (16) is known as the AVaR risk measure\(^{12}\) The parameter \( \alpha \) is the confidence level that the loss \( \tilde{H}(x, \theta) \) will not exceed a determined amount. Accordingly, Theorem 1 can be rewritten in terms of optimization problem (17):

\[
\begin{align*}
\max_{x \in \Delta_n} \{-R(x)\}.
\end{align*}
\]

In this case, the DRO-RI model can be interpreted as an environment where a rational inattentive agent chooses a vector \( x \in \Delta_n \) that minimizes the associated VaR.

A second important example is the case of \( \phi_{kl}(t) = t \ln t - t + 1, t \geq 0 \). As we mentioned earlier, \( \phi_{kl}(t) \) defines the Kullback-Leibler distance. In this case we use the fact that the conjugate of \( \lambda \phi \) is given by \( (\lambda \phi)^*(s) = \lambda (e^{s/\lambda} - 1) \).

Then, for a fixed \( x \), and using Theorem 1, we define the functional \( R_\rho(x) \) as:

\[
R_\rho(x) = \inf_{\lambda \geq 0, \eta \in \mathbb{R}} \left\{ \lambda \rho + \eta + \alpha^{-1} \mathbb{E}_\mu \left[ e^{\tilde{H}(x, \theta)/\lambda} \right] - \lambda \right\}.
\]

Minimizing with respect to \( \eta \) the right hand of (18) yields \( \eta^* = \lambda \mathbb{E}_\mu [e^{\tilde{H}(x, \theta)/\lambda}] \).

By substituting this latter expression into (18), we obtain

\[
R_\rho(x) = \inf_{\lambda > 0} \left\{ \lambda \rho + \lambda \log \mathbb{E}_\mu \left[ e^{\tilde{H}(x, \theta)/\lambda} \right] \right\}.
\]

For \( \rho = 0 \) the functional \( R_0 \) is given by the minimum of the entropic risk measure \( \lambda \log \mathbb{E}_\mu \left[ e^{\tilde{H}(x, \theta)/\lambda} \right] \) (Ruszczyński and Shapiro [2021]). In this case the

\(^{12}\)In the financial literature the AVaR is also known as Conditional Value-at-Risk (CVaR), Expected Shortfall and Expected Tail Loss. For details we refer the reader to Rockafellar and Uryasev [2000], Rockafellar and Uryasev [2002], and Ruszczyński and Shapiro [2021].
optimization problem in Theorem 1 can be expressed as problem (20):

\[
\max_{x \in \Delta_n} \{-\mathcal{R}_0(x)\}.
\]

Intuitively, the previous program captures a situation where a DM wants to choose a robust \( x \in \Delta_n \) by minimizing the entropic risk measure \( \mathcal{R}_0 \). Noting that the Shannon mutual information is used in defining the cost \( \xi C(p, \mu) \) and the uncertainty set \( \mathcal{B}_\phi \), we can refer this instance of the DRO-RI as the double Shannon model.

It is worth remarking that the analysis in Hansen et al. [2022] can be interpreted in terms of the risk measure (19). Intuitively, Hansen et al. [2022]’s approach can be connected to the notion of entropic risk measure. We leave for future research an in-depth analysis of this connection.

3.4. DRO-RI and tail performance. Now we focus on the important case of the Cressie-Read divergence, which is denoted as \( \phi_{cr}^k \) (Cressie and Read [1984]). Our main goal here is to connect the program (10) with an explicit expression for the convex conjugate \( \phi_{cr}^k \). From Table 1 we know that the Cressie-Read family is parameterized by \( k \in (-\infty, \infty) \setminus \{0, 1\} \) and \( k^* = \frac{k}{k - 1} \) with

\[
\phi_{cr}^k(t) \triangleq \frac{t^k - k t + k - 1}{k(k - 1)} \quad \text{and} \quad \phi_{cr}^{k^*}(s) \triangleq \frac{1}{k} \left[ ((k - 1)s + 1)^{k^*} - 1 \right].
\]

For \( t < 0 \) we let \( \phi_{cr}^k(t) = +\infty \) and we define \( \phi_{cr}^0 \) and \( \phi_{cr}^1 \) as their respective limits as \( k \to 0, 1 \). The family of divergences (21) includes \( \chi^2 \)-divergence \((k = 2)\), empirical likelihood \( \phi_{cr}^0(t) = -\log t + t - 1 \), and KL-divergence \( \phi_{cr}^1(t) = t \log t - t + 1 \) as particular cases.

The following is the main result of this section.

**Proposition 3.** For any prior \( \mu \) on \( \Delta(\Theta) \), \( k \in (1, \infty) \), \( k^* = k/(k - 1) \), any \( \rho > 0 \), and \( m_k(\rho) := (1 + k(k - 1)\rho)\frac{k}{k - 1} \), we have for all \( x \in \Delta_n \),

\[
\mathcal{R}_k(x; \rho) = \inf_{\eta \in \mathbb{R}} \left\{ m_k(\rho) \mathbb{E}_\mu \left[ (\tilde{H}(x, \theta) - \eta)_{+}^{k^*} \right]^{\frac{1}{k^*}} - \eta \right\}.
\]

Furthermore, the solution to program (10) is giving by:

\[
\max_{x \in \Delta_n} \{-\mathcal{R}_k(x; \rho)\}.
\]

Some remarks are in order. First, the simplified dual form (22) shows that in the DRO-RI model, a DM protecting against worst-case distributional shifts optimizes the tail-performance of the worst-case distribution in \( \mathcal{B}_{\phi_{cr}}(\mu) \). In particular, for a given \( x \in \Delta_n \) the worst-case objective \( \mathcal{R}_k(x; \rho) \) only penalizes losses above the optimal dual variable \( \eta^*(x) \). Second, we point out that the
$L^k_{\star} (\mu)$-norm up-weights the tail values of $\tilde{H}(x, \theta)$, giving a worst-case objective that focuses on “hard” regions of the state space $\Theta$. Third, the simplified dual (22) also makes explicit the relationship between the behavior (growth) of $\phi^k_{cr}$ and the worst-case objective $R_k(x; \rho)$: as the growth of $\phi^k_{cr}(t)$ for large $t$ becomes steeper ($k \uparrow \infty$), the uncertainty set $B_{\phi^k_{cr}} (\mu)$ shrinks, and the risk measure $R_k(x; \rho)$ becomes less conservative (smaller). Since the dual form (22) quantifies this with the $L^k_{\star} (\mu)$-norm of the loss (utility) above the quantile $\eta$, we see that $\phi^k_{cr}$ with $k \in (1, \infty)$ is a possible choice if the loss has finite $k_{\star}$-moments under the nominal distribution $\mu$.

4. Worst-case comparative statics

In this section we study how the optimal value of the DRO-RI problem varies when $\rho \downarrow 0$. In particular, we want to quantify the impact of model uncertainty in terms of the DM’s utility loss. In our analysis we adapt the worst-case sensitivity approach proposed in Gotoh et al. [2020]. Accordingly, and recalling that $H(x, \theta) \triangleq \xi \log \sum_{a \in A} x_a e^{u(a, \theta)/\xi}$, we define the function $V_{\phi}(x; \rho)$ as:

$$
V_{\phi}(x; \rho) \triangleq \min_{\nu \in B_{\phi}(\mu)} \mathbb{E}_\nu[H(x, \theta)] \quad \forall x \in \Delta_n, \rho > 0.
$$

From expression (24) it follows then that:

$$
\max_{x \in \Delta_n} V_{\phi}(x; \rho) \triangleq \max_{x \in \Delta_n} \min_{\nu \in B_{\phi}(\mu)} \mathbb{E}_\nu[H(x, \theta)].
$$

Following Gotoh et al. [2020], we define the worst-case sensitivity with growth rate $g(\rho)$ as the right derivative of the value function at $\rho = 0$:

$$
S_{\mu}(x) = V'_{\phi}(x; 0^+) = \lim_{\rho \downarrow 0} \frac{\mathbb{E}_\mu(H(x, \theta)) - V(x; \rho)}{g(\rho)}.
$$

From an economic standpoint, $S_{\mu}(x)$ captures how small deviations from the nominal distribution $\mu$ can result in a large decrease in the expected value $\mathbb{E}_\mu(H(x, \theta))$; when that is the case, the decision of choosing $x$ is not robust. In order to provide a characterization of the worst-case sensitivity, we make use of the following technical lemma in Gotoh et al. [2020].

**Lemma 1.** Let Assumption \( \Xi \) hold. Then for $\rho \downarrow 0$ and for all $\theta \in \Theta$, the worst-case probability distribution is given by:

$$
\nu_\rho(\theta) = \mu(\theta) \left\{ 1 - \sqrt{\frac{2 \rho}{\mathbb{E}_\mu(H(x, \theta') \phi''(1)[H(x, \theta) - \mathbb{E}_\mu(H(x, \theta'))]}} \right\} + o(\sqrt{\rho}).
$$
The previous lemma provides a closed-form expression to the worst-case probability vector $\nu_\rho$ for the case of $\rho \to 0$. From expression (27) it is easy to see that $\nu_\rho \to \mu$ when $\rho \downarrow 0$. With this result in place, we are ready to establish the main contribution of this section:

**Theorem 3.** Let Assumption 2 hold. Then in the DRO-RI model for $g(\rho) = \sqrt{\rho}$ the following holds:

$$S_\mu(x) = \sqrt{\frac{2V_\mu(H(x, \theta))}{\phi''(1)}}$$

for all $x \in \Delta_n$.

Some remarks are in order. First, the previous result provides a simple formula to quantify the impact of prior uncertainty in the DM’s decisions. Intuitively, expression (28) can be interpreted as the *price of robustness*. In particular, the result in Theorem 3 allows one to interpret the DRO-RI model as one that captures the trade-off between mean and worst-case sensitivity. Formally, from Theorem 3, the following relationship holds:

$$\max_{x \in \Delta_n} V_\phi(x; \rho) \approx \max_{x \in \Delta_n} \{E_\mu(H(x, \theta)) - \sqrt{\rho S_\mu(x)} + o(\sqrt{\rho})\}.$$

Thus, for small values of $\rho$, the DRO-RI framework can be approximated through a mean-variance model.

Second, we point out that for many $\phi$-divergence functions we get $\phi''(1) = 1$, so that expression (28) simplifies to $S_\mu(x) = \sqrt{2\rho V_\mu(H(x, \theta))}$.

**Example 4.** Let us consider the modified $\chi^2$-divergence given by $\phi_{mc}(t) = \frac{1}{2}(t - 1)^2$. It is easy to see that $\phi''_{mc}(1) = 1$. From Lemma 1, the worst-case distribution is given by

$$\nu_\rho(\theta) = \mu(\theta) \left\{1 - \sqrt{\frac{2\rho}{V_\mu(H(x, \theta'))}}(H(x, \theta) - E_\mu(H(x, \theta')))\right\}.$$

This holds for all $\rho \geq 0$ as long as $\nu_\rho(\theta) \geq 0$ and not just when $\rho$ is small. Thus, it follows that

$$V_\phi(x; \rho) = E_\mu(H(x, \theta)) - \sqrt{\rho \sqrt{2V_\mu(H(x, \theta')}}).$$

From the previous expression we note that $V_\phi(x; 0) - V_\phi(x, \rho) \approx O(\sqrt{\rho})$, which implies that:

$$S_\mu(x) = \sqrt{2\rho V_\mu(H(x, \theta'))}.$$
5. Robust Consideration Sets

In this section we use Theorem 1 and Proposition 2 to characterize the construction of distributionally robust consideration sets (DRCS) in the context of the DRO-RI model.

We recall that a consideration set consists of all actions that are chosen with positive probability. Formally, for a solution $x^*$ to problem (10) the associated robust consideration set is given by:

$$A(x^*) \triangleq \{ a \in A : x^*_a > 0 \}.$$ 

Given the concave structure of the DRO-RI model, we build upon the recent contribution by Caplin et al. [2018] which characterizes consideration sets in traditional RI models under Assumption 1. The next proposition establishes that their result can be extended to situations where the DM faces model uncertainty.

**Proposition 4** (Distributionally Robust Consideration Sets (DRCS)). Let $(x^*, \lambda^*, \eta^*)$ an optimal solution to problem (9) with $\lambda^* > 0$. Assume that $\phi^*(\cdot) > 0$ and let $\nu$ be the worst-case probability given by $\nu(\theta) \triangleq \mu(\theta) \times \phi^*(H(x^*, \theta) - \eta^*/\lambda^*)$ for all $\theta \in \Theta$. Then the solution $x^*$ is optimally robust if and only if

$$\sum_{\theta \in \Theta} \frac{e^{u(a,\theta)/\xi}\nu(\theta)}{\sum_{b \in A} x^*_b e^{u(b,\theta)/\xi}} \leq 1 \quad \forall a \in A,$$

with equality if $a \in A(x^*)$.

In proving this result, we combine the characterization in Theorem 1 and Proposition 2 with the Lagrangian techniques employed by Caplin et al. [2018]. However, a key difference is that our characterization of consideration sets replaces $\mu$ by the worst-case probability measure $\nu$. In other words, by exploiting the necessary and sufficient first order conditions of problem (10) we can jointly determine $x^*, \nu$, and the consideration set $A(x^*)$.

In order to understand how Proposition 4 can be applied to concrete $\phi$-divergence functions, the next corollary focus on the Kullback-Leibler case.

**Corollary 1.** Let $(x^*, \lambda^*, \eta^*)$ an optimal solution to problem (9) with $\lambda^* > 0$. Consider the Kullback-Leibler divergence $\phi_{KL}$. Then the solution $x^*$ is distributionally robust if and only if

$$\sum_{\theta \in \Theta} \left( \frac{e^{u(a,\theta)/\xi - H(x^*, \theta)/\lambda}}{\sum_{b \in A} x^*_b e^{u(b,\theta)/\xi - H(x^*, \theta)/\lambda}} \right) \mu(\theta) \leq 1 \quad \forall a \in A,$$
with equality if \(i \in A(x^*) \triangleq \{j \in A : x^*_j > 0\}\).

Intuitively, the previous corollary establishes that the DM will take into account the adjusted utility \(u(a, \theta)/\xi - H(x^*, \theta)/\lambda\) when adding or not option \(a\) to the consideration set \(A(x^*)\). Hansen et al. [2022] provide an in-depth discussion of the double Shannon model.

5.1. Application: Finding the best alternative. In order to see how Proposition 4 can be applied to concrete economic problems, we restudy a problem originally analyzed by Caplin et al. [2018]. We analyze a situation where the DM is faced with a range of possible goods identified as a set \(A = \{1, \ldots, N\}\). One of these options is good. The others are bad. The utilities of the good and bad options are \(u_G\) and \(u_B\) respectively, with \(u_G > u_B\). The DM has a prior on which of the available options is good. We define the state space to be the same as the action space, i.e., \(\Theta = A\), with the interpretation that state \(\theta_i\) is the state in which option \(i\) is good and all other alternatives are bad. Accordingly, the state contingent utilities are defined as:

\[
u(i, \theta_j) = \begin{cases} 
  u_G & i = j, \\
  u_B & i \neq j.
\end{cases}
\]

In this environment, \(\mu(\theta_i)\) is the prior probability that option \(a_i\) yields the good prize. Without loss of generality, we assume that the prior \(\mu\) satisfies:

\[
\mu_i \equiv \mu(\theta_i) \geq \mu(\theta_{i+1}) \equiv \mu_{i+1}.
\]

The DM can expend attentional effort to gain a better understanding of where the prize is located. The cost of improved understanding is defined by the Shannon model with parameter \(\xi > 0\).

To characterize the optimal strategy, we follow Caplin et al. [2018] using the following payoff transformation:

\[
\left(31\right) e^{u(i, \theta_j)/\xi} = \begin{cases} 
  e^{u_G} & i = j, \\
  e^{u_B} & i \neq j
\end{cases}
\]

with \(\pi, \delta > 0\).

The optimal policy will depend on \(\delta\) but not on \(\pi\). Increases in the utility differential \(u_G - u_B\) and reductions in the marginal cost \(\xi\) both affect the optimal policy through increases in \(\delta\).
In terms of $\phi$-divergences, we focus on the Kullback-Leibler case. From Example 2 we know that the worst-case probability is given by:

$$
\nu(\theta_i) = \frac{\mu(\theta_i) e^{-H(x, \theta_i)/\lambda}}{\sum_{j=1}^{N} \mu(\theta_j) e^{-H(x, \theta_j)/\lambda}} \quad i = 1 \ldots, N.
$$

Noting that $H(x, \theta_i) = \log \pi(1 + \delta x_i)$ and using the definition of $\nu$, the necessary and sufficient conditions for some action $i \in A(x^*)$ yields

$$
\frac{\delta \mu(\theta_i)}{(1 + \delta x_i^*)^\alpha} + \sum_{j \in A(x^*)} \frac{\mu(\theta_j)}{(1 + \delta x_j^*)^\alpha} + \sum_{k \in A \setminus A(x^*)} \mu(\theta_k) = 1.
$$

where $\alpha \triangleq 1/\lambda + 1$. Since the last two terms on the left-hand side are the same for all chosen actions, it follows that the optimal policy equalizes the first term:

$$
\frac{\mu(\theta_i)}{(1 + \delta x_i^*)^\alpha} \quad \forall i \in A(x^*).
$$

This equality implies that if the first $K$ actions are taken with positive probability, then the first order condition for the $K$-th action is given by:

$$
(\delta + K) \frac{\mu(\theta_K)}{(1 + \delta x_K^*)^\alpha} = \sum_{i \in A(x^*)} \mu(\theta_i).
$$

We use Eq. (32) to characterize the optimal policy $x^*$ and the consideration set $A(x^*)$. In doing so, we follow the arguments developed by Caplin et al. [2018]. We note that an optimal strategy $x_i^*$ must satisfy $x_i^* \geq 0$ for all $i \in A$, $x_i^* > 0$ for all $i \in A(x^*)$, and $\sum_{i \in A(x^*)} x_i^* = 1$. From Eq. (32), it follows that the optimal policy $x^*$ satisfies these requirements when for $x_K^*>0$ the following strict inequality holds:

$$
\mu(\theta_K) > \frac{\left(\sum_{a_i \in A(x^*)} (\mu(\theta_i))^{1/\alpha}\right)^\alpha}{\delta + K}
$$

and for all $j \in A \setminus A(x^*)$ the condition is $\mu(\theta_j) \leq \frac{\left(\sum_{a_i \in A(x^*)} (\mu(\theta_i))^{1/\alpha}\right)^\alpha}{(\delta + K)}$.

Based on this observation we obtain the following result.

**Proposition 5.** If $\mu(\theta_N) > \frac{1}{N+\delta}$, define $K = M$. If $\mu(\theta_N) < \frac{1}{N+\delta}$, define $K < N$ as the unique integer satisfying

$$
\mu(\theta_K) > \frac{\left(\sum_{j \in A(x^*)} (\mu(\theta_j))^{1/\alpha}\right)^\alpha}{K + \delta} \geq \mu(\theta_{K+1}).
$$

\[13\] We recall that without loss of generality we have assumed that $\mu(\theta_1) \geq \mu(\theta_2) \geq \ldots \geq \mu(\theta_N) > 0.$
Then the optimal attention strategy involves,

$$x_i^* = \frac{(\mu(\theta_i))^{1/\alpha}(K + \delta) - \sum_{j \in A(x^*)}(\mu(\theta_j))^{1/\alpha}}{\delta \sum_{j \in A(x^*)}(\mu(\theta_j))^{1/\alpha}} > 0 \quad \forall i \in A(x^*).$$

Furthermore, in the DRO-RI model, the following set inclusion relationship holds:

$$B(x) \subseteq A(x^*)$$

where $B(x)$ is the consideration set induced by the optimal policy $x$ under the assumption that $\mu$ is the correct model.

The first part of Proposition 5 extends Theorem 1 in Caplin et al. [2018]. The main difference between our result and theirs is that the characterization of the optimal policy $x^*$ depends on the parameter $\alpha$. In other words, the characterization (35) makes explicit that the DM faces ambiguity about the true model. In addition, it is worth pointing out that as $\lambda \to 0$, the Lagrange multiplier associated to the uncertainty set, the value of $1/\alpha$ gets closer to 1. This latter fact implies that when the constraint associated to the uncertainty set is not binding, i.e. when $\lambda = 0$, the solution of the DRO-RI model boils down to the characterization provided in Caplin and Dean [2013, Thm. 1]. Similarly, when $\lambda > 0$ our optimal policy (35) will differ from the one in Caplin and Dean [2013]. More importantly, using the fact that $\lambda$ represents the shadow price associated to the uncertainty constraint set, our characterization highlights the role of being robust in the DM’s optimal strategy.

A second important aspect of Proposition 5 is the result that $B(x) \subseteq A(x^*)$. In order to gain some intuition about this relationship, we note that Caplin et al. [2018]'s condition to construct the consideration set $B(x)$ is given by

$$\mu(\theta_K) > \frac{\sum_{i \in A(x)} \mu(\theta_i)}{\delta + K},$$

which can be seen as a particular case of condition (33) for $\alpha = 1$. Noting that

$$\mu(\theta_K) > \frac{\sum_{i \in A(x^*)} \mu(\theta_i)}{\delta + K} > \left(\frac{\sum_{i \in A(x^*)}(\mu(\theta_i))^{1/\alpha}}{\delta + K}\right)^{\alpha},$$

it is easy to see that if $K$-th alternative is included for $\alpha = 1$ then that $K$-option is also added to $A(x^*)$. Thus we can conclude that $B(x) = A(x^*)$. In order to show the strict inclusion, we can consider the case where

$$\frac{\sum_{a_i \in A(x^*)} \mu(\theta_i)}{\delta + K} > \mu(\theta_K) > \left(\frac{\sum_{a_i \in A(x^*)}(\mu(\theta_i))^{1/\alpha}}{\delta + K}\right)^{\alpha}. $$
In this case clearly $B(x) \subset A(x^*)$. Thus we conclude that $B(x) \subseteq A(x^*)$. From an economic standpoint the relationship $B(x) \subset A(x^*)$ captures the fact that in the DRO-RI framework, a DM is more conservative in constructing the consideration set $A(x^*)$. The reason for this comes from the fact that given the ambiguity the DM is more conservative at the moment of adding alternatives to $A(x^*)$. In other words, in the DRO-RI model the DM is cautious and $A(x^*)$ contains more alternatives than the traditional consideration sets approach, which assumes that the prior $\mu$ is correct.

6. Final Remarks

In this paper, we study the RI model in a context where the DM faces prior uncertainty. Formally, we introduce the DRO-RI model. Exploiting the concave structure of the DRO-RI model, we provide necessary and sufficient conditions for the formation of distributionally robust consideration sets for a large class of $\phi$-divergence functions. In addition, we perform a worst-case sensitivity analysis and discuss how the DRO-RI model can be approximated in terms of a mean-variance optimization program. Finally, we show that our DRO-RI approach has an economic interpretation in terms of risk measures and tail performance.
A.1. **Proof of Theorem 1** We adapt the Lagrangian duality arguments in [Ben-Tal et al., 2013, Cor. 3]. We first study the inner minimization program. Using the fact that
\[
\min_{\nu \in B(\mu)} \phi(\mu) = -\max_{\nu \in B(\mu)} -\phi(\mu).
\]
The associated Lagrangian for \(\max_{\nu \in B(\mu)} -\phi(\mu)\) is given by
\[
L(\nu, \lambda, \eta) = -\phi(\mu) + \lambda \left( \rho - \sum_{\theta \in \Theta} \mu(\theta) \phi \left( \frac{\nu(\theta)}{\mu(\theta)} \right) \right) + \eta \left( 1 - \sum_{\theta \in \Theta} \nu(\theta) \right).
\]
For the Lagrangian (36), we define the dual objective function as
\[
(37) g(\lambda, \eta) = \max_{\nu \geq 0} L(\nu, \lambda, \eta).
\]
\[
g(\lambda, \eta) = \lambda \rho + \eta + \max_{\nu \geq 0} \sum_{\theta \in \Theta} \left( -\nu(\theta)H(x, \theta) - \lambda \mu(\theta) \phi \left( \frac{\nu(\theta)}{\mu(\theta)} \right) - \eta \nu(\theta) \right),
\]
\[
= \lambda \rho + \eta + \sum_{\theta \in \Theta} \max_{\nu(\theta) \geq 0} \left( -\nu(\theta)H(x, \theta) - \lambda \mu(\theta) \phi \left( \frac{\nu(\theta)}{\mu(\theta)} \right) - \eta \nu(\theta) \right),
\]
\[
= \lambda \rho + \eta + \sum_{\theta \in \Theta} \max_{\nu(\theta) \geq 0} \left( \nu(\theta)(-H(x, \theta) - \eta) - \lambda \mu(\theta) \phi \left( \frac{\nu(\theta)}{\mu(\theta)} \right) \right),
\]
\[
= \lambda \rho + \eta + \sum_{\theta \in \Theta} \mu(\theta) \max_{t \geq 0} (t(-H(x, \theta) - \eta) - \lambda \phi(t))
\]
\[
= \lambda \rho + \eta + \sum_{\theta \in \Theta} \mu(\theta)(\lambda \phi^*)(-H(x, \theta) - \eta)
\]
where in the equality (38) we define \(t \triangleq \frac{\nu(\theta)}{\mu(\theta)}\) for all \(\theta \in \Theta\). Using the definition of \(\phi^*\) and noting that \((\lambda \phi^*)(s) = \lambda \phi^*(s/\lambda)\) for \(\lambda \geq 0\), where we define \(0 \phi^*(s/0) \triangleq (0 \phi)^*(s)\), which equals 0 if \(s \leq 0\) and \(+\infty\) if \(s > 0\). Plugging in equality (38) in (9) we obtain
\[
(39) \max_{x \in D_n} \min_{\nu \in B(\mu)} \{ E_{\nu}(H(x, \theta)) \} = \max_{x \in D_n, \lambda \geq 0, \eta} \left\{ -\eta - \rho \lambda - \lambda \sum_{\theta \in \Theta} \mu(\theta) \phi^* \left( \frac{-H(x, \theta) - \eta}{\lambda} \right) \right\}.
\]
Noting that \(H(x, \theta)\) is concave on \(x \in D_n\), we conclude that the optimization problem (39) is concave on \(x, \lambda, \text{ and } \eta\). \qed
A.2. **Proof of Proposition 2** The proof follows from a simple rearrangement of the first order conditions used in the proof of Theorem 1.

The following lemma is useful in proving Proposition 3.

**Lemma 2.** For the Cressie-Read family the following holds:

\[(40) \quad \phi_{cr}^k(s) = \frac{1}{k}((k-1)s+1)^{k^*} - \frac{1}{k} \]

*Proof.* By definition we know that \(\phi_{cr}^k(s) = \sup_t \{st - \phi_{cr}^k(t)\}\). Then for \(t \geq 0\), the first order condition yields

\[\frac{\partial}{\partial t} [st - \phi_k(t)] = s - \frac{1}{k-1} (t^{k-1} - 1).\]

If \(s < 0\) then the above derivative is negative and the optimum is achieved at \(t = 0\). For \(s > -\frac{1}{k-1}\), we solve \(s - \frac{1}{k-1} (t^{k-1} - 1) = 0\). This latter expression implies that \(t = ((k-1)s+1)^{\frac{1}{k-1}}\). Plugging this expression into \(\phi_{cr}^k(t)\) we obtain

\[st - \phi_{cr}^k(t) = \frac{1}{k}((k-1)s+1)^{\frac{k}{k-1}} - \frac{1}{k}.\]

Finally the result follows from noting that \(1 - \frac{1}{k} = \frac{1}{k^*}\). □

A.3. **Proof of Proposition 3.** In proving this proposition we follow the proof of Lemma 1 in [Duchi and Namkoong 2021] and Lemma 2 above. From Theorem 1 we know the following:

\[\inf_{\lambda \geq 0, \eta \in \mathbb{R}} \left\{ \lambda \mathbb{E}_\mu \left( \phi_k^*(H(x, \theta) - \eta) \right) + \lambda \rho + \eta \right\} =\]

\[\inf_{\lambda \geq 0, \eta \in \mathbb{R}} \left\{ \frac{(k-1)^{k^*}}{k} \lambda^{1-k^*} \mathbb{E}_\mu \left( \frac{H(x, \theta) - \eta}{\lambda} \right)^{k^*} + \lambda \left( \rho - \frac{1}{k} \right) + \eta \right\} =\]

\[\inf_{\lambda \geq 0, \eta \in \mathbb{R}} \left\{ (k-1)^{k^*} \mathbb{E}_\mu (H(x, \theta) - \tilde{\eta})^{k^*} \lambda^{1-k^*} + \left( \rho + \frac{1}{k(k-1)} \right) \lambda + \tilde{\eta} \right\}\]

where in the last line followed by setting \(\tilde{\eta} := \eta - \frac{\lambda}{k-1}\). Taking derivatives with respect to \(\lambda\) to infimize the preceding expression, we have (noting that \((k^* - 1)/k^* = 1/k\))

\[\lambda = (k-1)(k(k-1)\rho + 1)^{-\frac{1}{k^*}} \left( \mathbb{E}_\mu (H(x, \theta) - \tilde{\eta})^{k^*} \right)^{\frac{1}{k^*}}\]

By substituting into the preceding expression, we find that the supremum is

\[\inf_{\tilde{\eta}} (k(k-1)\rho + 1)^{\frac{1}{k^*}} \left( \mathbb{E}_\mu (Z - \tilde{\eta})^{k^*} \right)^{1/k^*} + \tilde{\eta}.\]
Defining $m_k(\rho) = (1 + k(k - 1)\rho)^k$ the conclusion follows at once. \hfill \Box

A.4. **Proof of Lemma 1.** This follows from a direct application of Proposition 4.2 [Gotoh et al. 2020]. \hfill \Box

A.5. **Proof of Theorem 3.** We know that the optimal value of the program (10) is given by $E_{\nu}(H(x^*, \theta))$. Replacing $\nu$ by the approximated worst-case probability measure $\nu_\rho$, we find that

$$E_{\nu}(H(x^*, \theta)) = E_{\mu}(H(x^*, \theta)) - \sqrt{\rho} \sqrt{\frac{2V_{\mu}(U(x^*, \theta))}{\phi''(1)}} + o(\sqrt{\rho}),$$

$$E_{\mu}(H(x^*, \theta)) - E_{\nu}(H(x^*, \theta)) = \sqrt{\rho} \sqrt{\frac{2V_{\mu}(U(x^*, \theta))}{\phi''(1)}} + o(\sqrt{\rho}).$$

Dividing by $\sqrt{\rho}$ and taking limit when $\rho$ goes to 0 we conclude that

$$S_{\mu}(x^*) = \sqrt{\frac{2V_{\mu}(U(x^*, \theta))}{\phi''(1)}}.$$

\hfill \Box

A.6. **Proof of Proposition 4.** The proof of this result is a simple adaptation of [Caplin et al., 2018, Prop. 1] to the case of program (10). Accordingly, the optimization problem has the associated Lagrangian

$$L(x, \lambda, \eta, \varphi, \xi) = -\eta - \rho \lambda - \sum_{\theta \in \Theta} \mu(\theta) \phi^* \left( \frac{-H(x, \theta) - \eta}{\lambda} \right) + \varphi \left( 1 - \sum_{i \in A} x_i \right) + \sum_{i \in A} \xi_i x_i + \psi \lambda,$$

where $\varphi$ is the Lagrangian multiplier on the constraint that the unconditional probabilities $x = (x_1, \ldots, x_n)$ must sum to 1, and $\xi_i$ is the multiplier on the non-negativity constraint for $x_i$. Similarly, $\psi$ is the Lagrangian multiplier on the non-negativity constraint for $\lambda$

Defining $\nu(\theta) \triangleq \mu(\theta) \times \phi^* \left( \frac{-H(x, \theta) - \eta}{\lambda} \right) > 0$ for all $\theta \in \Theta$, the associated first order conditions with respect to $\eta, \lambda$, and $x$ are:

(41) \hspace{1cm} 1 - \sum_{\theta \in \Theta} \nu(\theta) = 0,

(42) \hspace{1cm} -\rho - \sum_{\theta \in \Theta} \mu(\theta) \phi^* \left( \frac{-H(x, \theta) - \eta}{\lambda} \right) - \frac{1}{\lambda} \sum_{\theta \in \Theta} \nu(\theta) (H(x, \theta) + \eta) + \psi = 0,

(43) \hspace{1cm} \sum_{\theta \in \Theta} \nu(\theta) \left( \frac{e^{\theta_i}}{\sum_{j \in A} x_j e^{\theta_j}} \right) - \varphi + \xi_i = 0 \text{ for all } i \in A.
Using the assumption that \( \phi^*(\cdot) > 0 \), it follows that \( \nu(\theta) > 0 \) for all \( \theta \in \Theta \). Combining this latter fact with (41), it follows that \( \nu \) is a well defined probability measure over the state space \( \Theta \).

The complementary slackness conditions with respect to the choice probabilities are \( \xi_i x_i = 0 \) and \( \xi_i \geq 0 \) for \( i = 1, \ldots, n \). If \( x_i > 0 \), then

\[
\sum_{\theta \in \Theta} \nu(\theta) \left( \frac{e^{\theta_i}}{\sum_{j \in A} x_j e^{\theta_j}} \right) = \varphi.
\]

The previous expression can be written as:

\[
\sum_{\theta \in \Theta} \hat{\mu}(\theta) \left( \frac{x_i e^{\theta_i}}{\sum_{j \in A} x_j e^{\theta_j}} \right) = x_i \varphi.
\]

Summing over \( i \) we get

\[ 1 = \varphi. \]

So if \( x_i > 0 \)

\[
\sum_{\theta \in \Theta} \nu(\theta) \left( \frac{e^{\theta_i}}{\sum_{j \in A} x_j e^{\theta_j}} \right) = 1.
\]

If \( x_i = 0 \) then \( \xi_i \geq 0 \) and

\[
\sum_{\theta \in \Theta} \nu(\theta) \left( \frac{e^{\theta_i}}{\sum_{j \in A} x_j e^{\theta_j}} \right) = 1 - \xi_i \leq 1.
\]

These are necessary and sufficient conditions for an optimal solution \( x \in \Delta_n \).

In (42) the complementary slackness condition on \( \lambda \) is given by \( \psi \lambda = 0 \) and \( \lambda \geq 0 \). For \( \lambda > 0 \) we have that (42) can be written as

\[
-\rho - \sum_{\theta \in \Theta} \mu(\theta) \phi^* \left( \frac{-H(x, \theta) - \eta}{\lambda} \right) = \frac{1}{\lambda} \sum_{\theta \in \Theta} \nu(\theta)(H(x, \theta) + \eta).
\]

The previous condition can be rewritten as:

\[
-\eta - \rho \lambda - \lambda \sum_{\theta \in \Theta} \mu(\theta) \phi^* \left( \frac{-H(x, \theta) - \eta}{\lambda} \right) = \sum_{\theta \in \Theta} \nu(\theta) H(x, \theta).
\]
A.7. Proof of Proposition 5. The derivation of inequality (34) and expression (35) follow the arguments developed in Caplin and Dean [2013]. The main difference is that in their derivation \( \alpha = 1 \). In particular, they used the inequality

\[
\mu(\theta_K) > \frac{\sum_{a_i \in A(x^*)} \mu(\theta_i)}{\delta + K}.
\]

Using the fact that \( \alpha > 0 \)

\[
\mu(\theta_K) > \frac{\sum_{a_i \in A(x^*)} \mu(\theta_i)}{\delta + K} > \left(\frac{\sum_{a_i \in A(x^*)} (\mu(\theta_i))^{1/\alpha}}{\delta + K}\right)^{\alpha}.
\]

Thus, we can apply the arguments used in proving Theorem 1 in Caplin and Dean [2013]. In order to show that \( A(x) \subseteq A(x^*) \), we first note that if in (44) we have \( \mu(\theta_N) > \frac{1}{K + \delta} \), then we set \( K = N \) and \( A(x) = A(x^*) \). Similarly, if in Eq. (44) we have that

\[
\sum_{a_i \in A(x^*)} \mu(\theta_i) > \mu(\theta_K) > \left(\frac{\sum_{a_i \in A(x^*)} (\mu(\theta_i))^{1/\alpha}}{\delta + K}\right)^{\alpha}.
\]

Then, it follows that \( A(x) \subset A(x^*) \). Thus we conclude that \( A(x) \subset A(x^*) \).

\[\square\]

Appendix B. DRO-RI with continuous distribution

In this Appendix we discuss how the Theorem 1 can be extended to the case of absolutely continuous distributions. As in the main text, we assume that the set of alternatives \( A \) is discrete. In particular, we assume \( A = \{1, \ldots, n\} \). We denote \( \Delta_n \) as the \( n \)-dimensional simplex over \( A \). The main difference comes from the definition of the state space \( \Theta \). Let \( \Theta \subseteq \mathbb{R}^m \) be the state space, \( \mu \) be the distribution on the measure space \((\Theta, A)\), \( \theta \) be a random element of \( \Theta \) and \( H : \Delta_n \times \Theta \rightarrow \mathbb{R} \) be the function \( H(x, \theta) = \xi \log \sum_{a \in A} e^{u(x, \theta)}/\xi \).

The rational inattentive agent solves the problem:

\[
\max_{x \in \Delta_n} \inf_{\nu \ll \mu} \{ \mathbb{E}_\mu[H(x, \theta)] : D_\phi(\nu \| \mu) \leq \rho \}
\]

where the \( \phi \)-divergence between \( \nu \) and \( \mu \) is given by:

\[
D_\phi(\nu \| \mu) \triangleq \int \phi \left( \frac{d\nu}{d\mu} \right) d\mu,
\]

where \( \phi : \mathbb{R} \rightarrow \mathbb{R}_+ = \mathbb{R}_+ \cup \{\infty\} \) is a convex function satisfying \( \phi(1) = 0 \) and \( \phi(t) = +\infty \) for any \( t < 0 \). Defining the uncertainty region

\[
B_\phi(\mu) \triangleq \{ \nu : D_\phi(\nu \| \mu) \leq \rho \}
\]
Using the likelihood ratio $L(\theta) \triangleq d\nu(\theta)/d\mu(\theta)$ to reformulate our distributionally robust problem (45) via

$$R_{\rho}(x; \mu) = \inf_{\nu} \{E_{\nu}(H(x, \theta)) : \nu \in \mathcal{B}_\phi(\mu)\}$$

(46)

$$= \inf_{L > 0} \left\{E_{\mu}[L(\theta)H(x; \theta)] \mid E_{\mu}[\phi(L(\theta))] \leq \rho, E_{\mu}[L(\theta)] = 1\right\}$$

where the infimum is over measurable functions. We now use a result from Ben-Tal et al. [2013] and Shapiro [2017a] to obtain a dual reformulation of the quantity (46), where $\phi^*(s) := \sup_t \{st - \phi(t)\}$ is the usual Fenchel conjugate.

**Proposition 6** (Shapiro [2017a]). Let $\mu$ be a probability measure on $(\Theta, \mathcal{A})$ and $\rho > 0$. Then

$$R_{\rho}(x; \mu) = \sup_{\lambda \geq 0, \eta \in \mathbb{R}} \left\{ -E_{\mu} \left[ \frac{-H(x; \theta) - \eta}{\lambda} \right] \right\}$$

for all $x$. Moreover, if the supremum on the left-hand side is finite, there are finite $\lambda(x) \geq 0$ and $\eta(x) \in \mathbb{R}$ attaining the infimum on the right-hand side.

It is worth remarking that the main appealing of Proposition 6 is the fact that transform an infinite dimensional problem into a finite dimensional concave optimization program. As a direct consequence of this result, solving the DRO-RI model corresponds to find a solution to:

$$\max_{x \in \Delta_n} R_{\phi}(x; \mu).$$

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