DECAY RATES OF BOUND STATES AT THE SPECTRAL 
THRESHOLD OF MULTI-PARTICLE SCHRÖDINGER OPERATORS 

SIMON BARTH AND ANDREAS BITTER 

ABSTRACT. We consider $N$-body Schrödinger operators with $N \geq 3$ particles in di-
mension $d \geq 3$ in the critical case when the lowest eigenvalue coincides with the bottom 
of the essential spectrum of the operator. We give the asymptotic behaviour of the cor-
responding eigenfunction in dependence of the dimension and the number of particles 
of the system. 

1. INTRODUCTION 

It is well known that if a Schrödinger operator has an eigenvalue lying below its essen-
tial spectrum, then the corresponding eigenfunctions decay exponentially [1]. However, 
the situation changes completely at the threshold of the essential spectrum. In the work 
at hand we consider a multi-particle Schrödinger operator

$$H_N(\lambda) = -\sum_{i=1}^{N} \frac{1}{2m_i} \Delta x_i + \lambda \sum_{1 \leq i < j \leq N} V_{ij}(x_{ij})$$ (1.1) 

with a coupling constant $\lambda > 0$. After the separation of the center of mass motion this 
operator can be written as

$$H_0(\lambda) = -\Delta_0 + \lambda \sum_{1 \leq i < j \leq N} V_{ij}(x_{ij}),$$ (1.2) 

where $\Delta_0$ is the Laplace-Beltrami operator on the space of relative motion of the system. 

We study the case when $\lambda$ takes its critical value, i.e. for $0 < \lambda \leq \lambda_0$ the spectrum of the 
Hamiltonian $H_0(\lambda)$ coincides with the half-line $[0, \infty)$ and for $\lambda > \lambda_0$ sufficiently close 
to $\lambda_0$ the operator $H_0(\lambda)$ has negative eigenvalues. The decay properties of solutions 
of the Schrödinger equation corresponding to the critical coupling constant $\lambda_0$ play an 
important role in different physical phenomena, see for example [4] and [5]. They are 
in particular crucial for the existence and non-existence of the so-called Efimov effect 
[8]. In a recent work [3] it was shown that in case of short-range potentials for $\lambda = \lambda_0$ 
the operator $H_0(\lambda)$ has an non-degenerate eigenvalue at zero, where the corresponding 
eigenfunction $\varphi_0$ satisfies

$$(1 + |x|)^{\alpha} \varphi_0 \in L^2 \quad \text{for any} \quad \alpha < \frac{d(N - 2)}{2} - 2. \quad (1.3)$$ 

Here $N \geq 3$ is the number of particles and $d \geq 3$ is the dimension. However, the question 
remains open whether this estimate from below on the decay rate of $\varphi_0$ is close to the 
optimal one.
In this work we answer this question and give an explicit formula for the asymptotics of the eigenfunction $\varphi_0$ for large distances to the center of mass.

2. Notation and Main Result

We consider a system of $N \geq 3$ particles in dimension $d \geq 3$ with masses $m_i > 0$ and position vectors $x_i \in \mathbb{R}^d$, $i = 1, \ldots, N$. Such a system is described by the Hamiltonian (1.1), where $\lambda > 0$ is a real parameter, $x_{ij} = x_i - x_j$ and the potentials $V_{ij}$ describe the pair interactions. We assume that $V_{ij}$ satisfy

$$|V_{ij}(x_{ij})| \leq c|x_{ij}|^{-2-\nu}, \quad x_{ij} \in \mathbb{R}^d, \quad |x_{ij}| \geq A$$  \hspace{1cm} (2.1)

for some constants $c, \nu, A > 0$ and

$$\begin{cases} V_{ij} \in L^2_{\text{loc}}(\mathbb{R}^d), & \text{if } d = 3, \\ V_{ij} \in L^p_{\text{loc}}(\mathbb{R}^d) \text{ for some } p > 2, & \text{if } d = 4, \\ V_{ij} \in L^\infty_{\text{loc}}(\mathbb{R}^d), & \text{if } d \geq 5. \end{cases}$$  \hspace{1cm} (2.2)

Under these assumptions on the potentials the operator $H_N$ is essentially self-adjoint on $L^2(\mathbb{R}^{dN})$. Following [7], we define the space $R_0$ of relative motion of the system as

$$R_0 = \left\{ x = (x_1, \ldots, x_N) \in \mathbb{R}^{dN} : \sum_{i=1}^N m_i x_i = 0 \right\}$$  \hspace{1cm} (2.3)

and the scalar product

$$\langle x, \bar{x} \rangle_1 = \sum_{i=1}^N m_i \langle x_i, \bar{x}_i \rangle, \quad |x|_1^2 = \langle x, x \rangle_1.$$  \hspace{1cm} (2.4)

For a fixed pair of particles $i \neq j$ we set

$$R_{ij} = \{ x = (x_1, \ldots, x_N) \in R_0 : m_i x_i + m_j x_j = 0 \} \text{ and } R_{ij}^\perp = R_0 \ominus R_{ij}.$$  \hspace{1cm} (2.5)

Let $P_{ij}$ and $P_{ij}^\perp$ be the projections in $R_0$ with respect to the scalar product $\langle \cdot, \cdot \rangle_1$ onto $R_{ij}$ and $R_{ij}^\perp$, respectively. For $x \in R_0$ we denote

$$q_{ij} = P_{ij} x \quad \text{ and } \quad \xi_{ij} = P_{ij}^\perp x.$$  \hspace{1cm} (2.6)

Note that for any $1 \leq i < j \leq N$ it holds

$$|q_{ij}|_1 = \frac{\sqrt{m_i m_j}}{\sqrt{m_i + m_j}} |x_{ij}|,$$  \hspace{1cm} (2.7)

which together with (2.1) implies

$$|V_{ij}(x_{ij})| \leq C|q_{ij}|_1^{-2-\nu} \quad \text{ for some } C > 0 \text{ and all } |x_{ij}| \geq A.$$  \hspace{1cm} (2.8)

In the following we denote

$$V(x) = \sum_{1 \leq i < j \leq N} V_{ij}(x_{ij}).$$  \hspace{1cm} (2.9)

By $\Delta_0$ we denote the Laplace Beltrami operator on $L^2(R_0)$. Then the Hamiltonian of the system after separation of the center of mass is given by

$$H_0(\lambda) = -\Delta_0 + \lambda V(x).$$  \hspace{1cm} (2.10)
In this work we consider systems for which the following important condition is fulfilled.

**Assumption:** We consider the case when \( \lambda \) takes its critical value \( \lambda_0 \), i.e. for \( \lambda \leq \lambda_0 \) the spectrum of \( H_0(\lambda) \) is \([0, \infty)\) and for \( \lambda > \lambda_0 \) the operator \( H_0(\lambda) \) has negative spectrum.

We assume that for all \( \lambda > \lambda_0 \) sufficiently close to \( \lambda_0 \) the essential spectrum of the operator \( H_0(\lambda) \) coincides with the half line \([0, \infty)\). For such \( \lambda \) the negative spectrum is purely discrete. Without loss of generality we set \( \lambda_0 = 1 \) and we write \( H_0 \) instead of \( H_0(\lambda_0) \). In [3] it was shown that under these assumptions in case of \( d \geq 3 \) and \( N \geq 3 \) zero is a non-degenerate eigenvalue of \( H_0 \). The following theorem gives the asymptotic behavior of the corresponding eigenfunction for large arguments.

**Theorem 2.1.** Assume that \( H_0 \) satisfies the conditions described above. Suppose that \( \varphi_0 \) is an eigenfunction of \( H_0 \) corresponding to the eigenvalue zero. Then the following assertions hold.

(i) For all \( 1 \leq i < j \leq N \) we have
\[
V_{ij}(x_{ij})\varphi_0(x) \in L^1(R_0).
\] (2.11)

(ii) Let \( \beta = d(N-1) - 2 \) and denote by \( |S^{\beta-1}| \) the volume of the unit sphere in \( \mathbb{R}^\beta \). Further, let
\[
C_0 = \frac{1}{(\beta - 2)|S^{\beta-1}|} \int_{R_0} \sum_{1 \leq i < j \leq N} V_{ij}(x_{ij})\varphi_0(x) \, dx.
\] (2.12)

Then the function \( \varphi_0 \) has the following asymptotics
\[
\varphi_0(x) = \frac{C_0}{|x|^\beta} + g(x) \quad \text{as} \quad |x|_1 \to \infty,
\] (2.13)

where the remainder \( g \) belongs to \( L^p(R_0) \) for any \( p \) satisfying
\[
\frac{\beta + 2}{\beta + \frac{2}{1 + \gamma^*}} < p < \frac{\beta + 2}{\beta}
\quad \text{with} \quad \gamma^* = \min \left\{ \frac{d}{2} - 1, \nu \right\}.
\] (2.14)

**Remark.** (i) Note that \( \varphi_0 \) can be chosen to be strictly positive. If for all \( V_{ij} \) it holds \( V_{ij}(x) \leq 0 \), then we have \( C_0 \neq 0 \). In this case the leading term \( C_0|x|^\beta \chi_{\{|x|_1 > 0\}} \) belongs to \( L^q(R_0) \), only if \( q > \frac{\beta+2}{\beta} \).

(ii) (2.13) shows that the decay rate of \( \varphi_0 \) does not depend on the potentials as long as the pair potentials are short-range. At the same time, since \( |x|_1 = \sum_{i=1}^N m_i x_i \), the decay of \( \varphi_0 \) depends on the direction, if the masses are not equal.

**Proof of Theorem 2.1.** We will split the proof of the theorem into several propositions. The statement of the following proposition was proved in [3].

**Proposition 2.2.** The function \( \varphi_0 \) satisfies
\[
\nabla_0 (|x|^\alpha \varphi_0) \in L^2(R_0) \quad \text{for any} \quad 0 \leq \alpha < \frac{d(N-1) - 2}{2}.
\] (2.15)

The statement of assertion (i) of Theorem 2.1 is a special case of the following

**Proposition 2.3.** For all \( 1 \leq i < j \leq N \) and any \( 0 < \gamma < \gamma^* \) we have
\[
(1 + |x|_1)^\gamma V_{ij}(x_{ij})\varphi_0(x) \in L^1(R_0).
\] (2.16)
Proof of Proposition 2.3. By Proposition 2.2, together with $|\nabla q_{ij}| \leq |\nabla \varphi|$ and Hardy’s inequality in the space $H^1(R_{ij})$ we have

$$(1 + |q_{ij}|)^{-1} (1 + |x|)^\alpha \varphi_0 \in L^2(R_0).$$

(2.17)

Note that the potential $V_{ij}$ decays in the direction $x_{ij}$ but not necessarily in all directions. We will combine the decay property (2.1) of $V_{ij}$ and the a priori estimate (2.17) of $\varphi_0$ to get (2.11).

For any fixed $0 < \gamma < \gamma^*$ we write

$$(1 + |x|)^\gamma V_{ij}(x_{ij})\varphi_0(x) = (1 + |q_{ij}|)^{-1} (1 + |x|)^\alpha \varphi_0(x) \cdot f(x),$$

(2.18)

where

$$f(x) := (1 + |x|)^{-\alpha + \gamma} (1 + |q_{ij}|) V_{ij}(x_{ij}).$$

(2.19)

In view of (2.17) to prove Proposition 2.3 it suffices to show that $f$ belongs to $L^2(R_0)$. We decompose the function $f$ as

$$f(x) = f(x) \chi_{\{|x_{ij}| < A\}} + f(x) \chi_{\{|x_{ij}| \geq A\}}$$

(2.20)

and estimate the functions $f(x) \chi_{\{|x_{ij}| < A\}}$ and $f(x) \chi_{\{|x_{ij}| \geq A\}}$ separately, starting with the function $f(x) \chi_{\{|x_{ij}| < A\}}$. Note that $L^2(R_0) = L^2(R_{ij}) \otimes L^2(R_{ij}^\perp)$.

Due to (2.7) and assumption (2.2) it holds $(1 + |q_{ij}|) V_{ij}(x_{ij}) \chi_{\{|x_{ij}| < A\}} \in L^2(R_{ij})$. Moreover, we can estimate $(1 + |x|)^{-1} \leq (1 + |x_{ij}|)^{-1}$. Since $\dim(R_{ij}^\perp) = d(N - 2)$, we have

$$(1 + |x_{ij}|)^{-\alpha + \gamma} \in L^2(R_{ij}^\perp)$$

if and only if $\alpha - \gamma > \frac{d(N - 2)}{2}$. (2.21)

Recall that $\gamma < \gamma^*$, which in particular implies that $\gamma < \frac{d}{2} - 1$. Therefore, the condition in (2.21) is satisfied if we choose $\alpha$ close enough to $\frac{d(N - 1)}{2}$. Hence, we have

$$(1 + |x_{ij}|)^{-\alpha + \gamma} \in L^2(R_{ij}^\perp)$$

and therefore

$$f(x) \chi_{\{|x_{ij}| < A\}} \in L^2(R_0).$$

(2.22)

In order to prove that the function $f(x) \chi_{\{|x_{ij}| \geq A\}}$ belongs to $L^2(R_0) = L^2(R_{ij}) \otimes L^2(R_{ij}^\perp)$, we show that it can be estimated as

$$|f(x) \chi_{\{|x_{ij}| \geq A\}}| \leq |f_1(q_{ij})| \cdot |f_2(\xi_{ij})|,$$

(2.23)

where $f_1 \in L^2(R_{ij})$ and $f_2 \in L^2(R_{ij}^\perp)$. Here, we will use the assumption that the potential $V_{ij}(x_{ij})$ decays faster than $|q_{ij}|^{-2}$ as $|x_{ij}| \to \infty$. Recall that $\dim(R_{ij}) = d$ and $\dim(R_{ij}^\perp) = d(N - 2)$, which implies that for any $0 < \varepsilon < \nu - \gamma$ we have

$$f_1(q_{ij}) := (1 + |q_{ij}|)^{-\frac{d}{2} - \varepsilon} \in L^2(R_{ij})$$

(2.24)

and

$$f_2(\xi_{ij}) := (1 + |\xi_{ij}|)^{-\alpha + \gamma - \nu + \varepsilon + \frac{d}{2} - 1} \in L^2(R_{ij}^\perp).$$

(2.25)

Note that we can always assume $\nu < \frac{d}{2} - 1$. By the use of $|q_{ij}|, |\xi_{ij}| \leq |x_{ij}|$ we get

$$(1 + |x|)^{-\alpha + \gamma} \leq (1 + |x_{ij}|)^{-\alpha + \gamma - \nu + \varepsilon + \frac{d}{2} - 1} (1 + |q_{ij}|)^{-\frac{d}{2} + \nu - \varepsilon}.$$`

(2.26)

This, together with (2.8) yields

$$|f(x) \chi_{\{|x_{ij}| \geq A\}}| \leq C|f_1(q_{ij})| \cdot |f_2(\xi_{ij})|$$

(2.27)
and therefore $f(x)\chi_{\{|x|\geq A\}} \in L^2(R_0)$, which completes the proof of Proposition 2.3. □

Now we turn to the proof of assertion (ii) of Theorem 2.1. Since
\[ H_0\varphi_0 = (-\Delta_0 + V)\varphi_0 = 0 \tag{2.28} \]
and due to Proposition 2.3 $V\varphi_0 \in L^1(R_0)$, we can apply Theorem 6.21 in [6] to conclude
\[ \varphi_0(x) = \frac{-1}{(\beta - 2)|S^{\beta-1}|} \int_{R_0} |x - y|^{-\beta} V(y)\varphi_0(y) \, dy. \tag{2.29} \]
We derive the asymptotics (2.13) by studying the integral representation of $\varphi_0$ in (2.29). We will see that only certain regions contribute to the leading term in (2.13). We write
\[ \varphi_0(x) = \frac{-1}{(\beta - 2)|S^{\beta-1}|} (I_1(x) + I_2(x)), \tag{2.30} \]
where
\[
I_1(x) = \int_{\{|x-y| \leq 1\}} |x - y|^{-\beta} V(y)\varphi_0(y) \, dy, \\
I_2(x) = \int_{\{|x-y| > 1\}} |x - y|^{-\beta} V(y)\varphi_0(y) \, dy. \tag{2.31}
\]
At first, we prove that the function $I_1$ belongs to the remainder $g$ in (2.13), as we can see in the following

**Proposition 2.4.** The function $I_1$ is an element of $L^p(R_0)$ for all $1 \leq p < \frac{\beta + 2}{\beta}$.  

**Proof of Proposition 2.4.** Due to $\dim(R_0) = d(N-1)$ and $\beta = d(N-1) - 2$ we have
\[ |x|^{-\beta} \chi_{\{|x| \leq 1\}} \in L^p(R_0) \quad \text{for all} \quad 1 \leq p < \frac{\beta + 2}{\beta}. \tag{2.32} \]
By Proposition 2.3 we have $V\varphi_0 \in L^1(R_0)$, which together with Young's inequality yields the claim of Proposition 2.4. □

Now we will show that only a part of $I_2$ gives the leading term in (2.13). Let $\eta = \frac{1}{1+\gamma^*}$. For $x \in R_0$ we define
\[
\Omega_1(x) = \{y \in R_0 : |x - y|_1 > 1, \ |y|_1 > |x|_1^\gamma\}, \\
\Omega_2(x) = \{y \in R_0 : |x - y|_1 > 1, \ |y|_1 \leq |x|_1^\gamma\} \tag{2.33}
\]
and
\[ I_{2,k}(x) = \int_{\Omega_k(x)} |x - y|^{-\beta} V(y)\varphi_0(y) \, dy, \quad k = 1, 2. \tag{2.34} \]
We will show that only the function $I_{2,2}$ contributes to the leading term in (2.13). At first we consider the function $I_{2,1}$ and show that it belongs to the remainder in (2.13). Indeed, we have the following
Proposition 2.5. Let $I_{2,1}$ be given by (2.33) and (2.34). We have

$$I_{2,1} \in L^p(R_0) \quad \text{for all} \quad \frac{\beta + 2}{\beta + \frac{2}{1+\gamma}} < p < \frac{\beta + 2}{\beta}.$$  \hspace{1cm} (2.35)

Proof of Proposition 2.5. In the proof we will use Proposition 2.3. Let $\gamma < \gamma^*$. Using $|y|_1 > |x|_1^\theta$ for $y \in \Omega_1(x)$ we get

$$|I_{2,1}(x)| \leq \int_{\Omega_1(x)} |x - y|_1^{-\beta} |V(y)\varphi_0(y)| \, dy$$

$$\leq (1 + |x|_1^\theta)^{-\gamma} \int_{\Omega_1(x)} |x - y|_1^{-\beta}(1 + |y|_1)^\gamma |V(y)\varphi_0(y)| \, dy.$$ \hspace{1cm} (2.36)

We show that for any fixed $p$ satisfying (2.35) we find a constant $\gamma < \gamma^*$, such that the function on the r.h.s. of (2.36) belongs to $L^p(R_0)$. Note that $\frac{\gamma}{1+\gamma} = \eta\gamma^* \gamma$ sufficiently close to $\gamma^*$ it holds

$$p > \frac{\beta + 2}{\beta + \eta\gamma^*}.$$ \hspace{1cm} (2.37)

By Proposition 2.3 and Young’s inequality we have

$$I_{2,1}(x) := \int_{\Omega_1(x)} |x - y|_1^{-\beta}(1 + |y|_1)^\gamma |V(y)\varphi_0(y)| \, dy \in L^s(R_0), \quad s > \frac{d(N-1)}{d(N-1) - 2}. \hspace{1cm} (2.38)$$

Now we apply Hölder’s inequality to the r.h.s. of (2.36). For this purpose, we fix a constant $s > \frac{d(N-1)}{d(N-1) - 2}$ and define

$$t_1 = \frac{s}{s - p} \geq 1 \quad \text{and} \quad t_2 = \frac{p}{p} \geq 1 \quad \text{with} \quad \frac{1}{t_1} + \frac{1}{t_2} = 1. \hspace{1cm} (2.39)$$

Then we formally get

$$\int_{R_0} (1 + |x|_1^\theta)^{-\gamma} |I_{2,1}(x)|^p \, dx \leq \left( \int_{R_0} (1 + |x|_1^\theta)^{-\gamma t_1} \, dx \right)^{\frac{1}{t_1}} \left( \int_{R_0} |I_{2,1}(x)|^{pt_2} \, dx \right)^{\frac{1}{t_2}}. \hspace{1cm} (2.40)$$

Since $pt_2 = s$ and $I_{2,1} \in L^s(R_0)$, the second integral on the r.h.s of (2.40) is finite. Due to $\dim(R_0) = d(N-1)$, to prove the finiteness of the first integral on the r.h.s of (2.40) it suffices to show that $\eta\gamma pt_1 > d(N-1)$. By definition of $t_1$ this is equivalent to

$$\eta\gamma sp > d(N-1)(s - p) \quad \Leftrightarrow \quad p(\eta\gamma s + d(N-1)) > d(N-1)s$$

$$\Leftrightarrow \quad \frac{1}{p} < \frac{\eta\gamma s + d(N-1)}{sd(N-1)} = \frac{\eta\gamma}{d(N-1)} + \frac{1}{s}. \hspace{1cm} (2.41)$$

Since $p > \frac{d(N-1)}{d(N-1) - 2 + \eta}$, we see that the condition in (2.41) is fulfilled if $s$ is chosen sufficiently close to $\frac{d(N-1)}{d(N-1) - 2}$. Since $\beta = d(N - 1) - 2$, this completes the proof of Proposition 2.5. \hfill \Box

Now we finally show that the function $I_{2,2}$ yields the leading term in (2.13).
Proposition 2.6. Let $I_{2,2}$ be given by (2.34), then we have
\[ I_{2,2}(x) = |x|^{-\beta} \int_{\Omega_2(x)} V(y)\varphi_0(y) \, dy + h(x), \quad as \quad |x| \to \infty, \tag{2.42} \]
where
\[ h \in L^p(R_0) \quad \text{for all} \quad p > \frac{\beta + 2}{\beta + \frac{1}{1+\gamma}}. \tag{2.43} \]

Proof of Proposition 2.6. For $y \in \Omega_2(x)$ it holds (cf. [2])
\[ |x|^{-1}(1 - |x|^\eta) \leq |x - y|^{-1} \leq |x|^{-1}(1 + c|x|^\eta) \tag{2.44} \]
for some $c > 0$. We apply this inequality to the positive and the negative part of the integrand in the definition of $I_{2,2}$ separately. Let
\[ (V\varphi_0)_+(x) = \max \{V(x)\varphi_0(x), 0\} \quad \text{and} \quad (V\varphi_0)_- = -(V\varphi_0 - (V\varphi_0)_+), \tag{2.45} \]
then we have
\[ |x|^{-\beta}(1 - |x|^\eta) \int_{\Omega_2(x)} (V\varphi_0)_+(y) \, dy \leq \int_{\Omega_2(x)} \frac{(V\varphi_0)_+(y)}{|x - y|^\beta} \, dy \tag{2.46} \]
and
\[ \int_{\Omega_2(x)} \frac{(V\varphi_0)_+(y)}{|x - y|^\beta} \, dy \leq |x|^{-\beta}(1 + c|x|^\eta) \int_{\Omega_2(x)} (V\varphi_0)_+(y) \, dy. \tag{2.47} \]
Since $\dim(R_0) = d(N - 1)$ we see from (2.46) and (2.47) that there exist functions
\[ h_+ \in L^p(R_0), \quad p > \frac{d(N - 1)}{d(N - 1) - 2 + 1 - \eta}, \tag{2.48} \]
such that
\[ \int_{\Omega_2(x)} \frac{(V\varphi_0)_+(y)}{|x - y|^\beta} \, dy = |x|^{-\beta} \int_{\Omega_2(x)} (V\varphi_0)_+(y) \, dy + h_+(x) \tag{2.49} \]
for large $|x|$. Hence, we obtain
\[ I_{2,2}(x) = |x|^{-\beta} \int_{\Omega_2(x)} V(y)\varphi_0(y) \, dy + h(x) \quad \text{as} \quad |x| \to \infty, \tag{2.50} \]
where $h = h_+ - h_-$ belongs to $L^p(R_0)$ for $p$ given in (2.48). Due to $1 - \eta = \frac{\gamma}{1+\gamma}$ and $\beta = d(N - 1) - 2$ this concludes the proof of Proposition 2.6. \qed

By combining Propositions 2.4, 2.5 and 2.6 we get
\[ \varphi_0(x) = \frac{-|x|^{-\beta}}{(\beta - 2)|S^{\beta - 1}|} \int_{\Omega_2(x)} V(y)\varphi_0(y) \, dy + g(x) \quad \text{as} \quad |x| \to \infty \tag{2.51} \]
with
\[ g \in L^p(R_0) \quad \text{for} \quad \frac{\beta + 2}{\beta + \frac{1}{1+\gamma}} < p < \frac{\beta + 2}{\beta}. \tag{2.52} \]
Note that the integral on the r.h.s of (2.51) is over the set $\Omega_2(x)$, in contrast to (2.13), where the integral is over the whole space $R_0$. Therefore, to complete the proof of the theorem it remains to show that
\[
|x|_1^{-\beta} \int_{R_0 \setminus \Omega_2(x)} V(y) \varphi_0(y) \, dy
\]
does not contribute to the leading term in the asymptotic estimate of $\varphi_0$. Due to Proposition 2.3 it is easy to see that for any $\gamma < \gamma^*$ we have
\[
\left| \int_{R_0 \setminus \Omega_2(x)} V(y) \varphi_0(y) \, dy \right| \leq C (1 + |x|_1)^{-\eta \gamma}
\]
for $|x|_1$ sufficiently large. This implies
\[
|x|_1^{-\beta} \int_{R_0 \setminus \Omega_2(x)} V(y) \varphi_0(y) \, dy \in L^p(R_0) \quad \text{for} \quad p > \frac{\beta + 2}{\beta + \frac{2}{1+\gamma}}.
\]
Choosing $\gamma < \gamma^*$ sufficiently close to $\gamma^*$ and combining (2.51) and (2.55) completes the proof of the theorem. \hfill \Box

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\textbf{Simon Barth, Institut f"ur Analysis, Dynamik und Modellierung, Universit"at Stuttgart, Pfaffenwaldring 57, D-70569 Stuttgart, Germany}
\textit{E-mail address: simon.barth@mathematik.uni-stuttgart.de}
