t- Intuitionistic Fuzzy Subalgebra of $BG$-Algebras

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Abstract. The aim of this paper is to introduced the notion of t-intuitionistic fuzzy subalgebra and t-intuitionistic fuzzy normal subalgebra of $BG$-algebras. We state and prove some theorems in t-intuitionistic fuzzy subalgebra and t-intuitionistic fuzzy normal subalgebra in $BG$-algebras. The homomorphic image and inverse image are investigated in both t-intuitionistic fuzzy subalgebra and normal subalgebras.

Introduction

In 1966, Imai and Iseki [6] introduced the two classes of abstract algebras, viz., $BCK$-algebras and $BCI$-algebras. It is known that the class of $BCK$-algebra is a proper subclass of the class of $BCI$-algebras. Neggers and Kim [8] introduced a new concept, called $B$-algebras, which are related to several classes of algebras such as $BCI/BCK$-algebras. Kim and Kim [7] introduced the notion of $BG$-algebra which is a generalization of $B$-algebra. The concept of intuitionistic fuzzy subset (IFS) was introduced by Atanassov [5] in 1983, which is a generalization of the notion of fuzzy sets. The concept of fuzzy subalgebras of $BG$-algebras was introduced by Ahn and Lee in [1]. The study of intuitionistic fuzzification of subalgebras and ideals of $BG$-algebras is done by Senapati et. al in [9]. The idea of t-intuitionistic fuzzy sets in fuzzy subgroups and fuzzy subrings is introduced by Sharma in [10, 11]. Here in this paper, we introduced the notion of t-intuitionistic fuzzy sets in fuzzy subalgebra and fuzzy normal subalgebras of $BG$-algebras and study their properties.

Preliminaries

Definition 0.1 ([1]) A $BG$-algebra is a non-empty set $X$ with a constant ‘0’ and a binary operation ‘*$’ satisfying the following axioms:

(i) $x * x = 0$,

(ii) $x * 0 = x$,

(iii) $(x * y) * (0 * y) = x, \forall x, y \in X$.

For brevity, we also call $X$ a $BG$-algebra. We can define a partial ordering “$\leq$” on $X$ by $x \leq y$ iff $x * y = 0$.

Definition 0.2 ([1]) A non-empty subset $S$ of a $BG$-algebra $X$ is called a subalgebra of $X$ if $x * y \in S$, for all $x, y \in S$. 

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Definition 0.3 Let $X$ and $Y$ be two nonempty sets and $f : X \rightarrow Y$ be a mapping. Let $A$ and $B$ be IFS's of $X$ and $Y$ respectively. Then the image of $A$ under the map $f$ is denoted by $f(A)$ and is defined by $f(A)(y) = (\mu_{f(A)}(y), \nu_{f(A)}(y))$, where $\mu_{f(A)}(y) = \bigwedge \{\mu_A(x) : x \in f^{-1}(y)\}$ and $\nu_{f(A)}(y) = \bigvee \{\mu_A(x) : x \in f^{-1}(y)\}$

Also pre image of $B$ under $f$ is denoted by $f^{-1}(B)$ and is defined as $f^{-1}(B)(x) = (\mu_{f^{-1}(B)}(x), \nu_{f^{-1}(B)}(x)) = (\mu_B(f(x)), \nu_B(f(x)))) \forall x \in X$.

Remark Note that $\mu_A(x) \leq \mu_{f(A)}(f(x))$ and $\nu_A(x) \geq \nu_{f(A)}(f(x)) \forall x \in X$ however equality hold when the map $f$ is bijective.

Definition 0.4 ([2, 3]) An intuitionistic fuzzy set (IFS) $A$ in a nonempty set $X$ is an object of the form $A = \{< x, \mu_A(x), \nu_A(x) > | x \in X \}$ where $\mu_A : X \rightarrow [0, 1]$ and $\nu_A : X \rightarrow [0, 1]$ with the condition $0 \leq \mu_A(x) + \nu_A(x) \leq 1, \forall x \in X$. The numbers $\mu_A(x)$ and $\nu_A(x)$ denote respectively the degree of membership and the degree of non-membership of the element $x$ in the set $A$. For the sake of simplicity we shall use the symbol $A = (\mu_A, \nu_A)$ for the intuitionistic fuzzy set $A = \{< x, \mu_A(x), \nu_A(x) > | x \in X \}$. The function $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ for all $x \in X$. is called the degree of uncertainty of $x \in A$. The class of IFSs on a universe $X$ is denoted by $IFS(X)$.

Definition 0.5 ([2, 3]) If $A = \{< x, \mu_A(x), \nu_A(x) > | x \in X \}$ and $B = \{< x, \mu_B(x), \nu_B(x) > | x \in X \}$ be any two IFS of a set $X$ then

$A \subseteq B$ iff for all $x \in X, \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$

$A = B$ iff for all $x \in X, \mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$

$A \cap B = \{< x, (\mu_A \cap \mu_B)(x), (\nu_A \cup \nu_B)(x) > | x \in X \}$ where

$\mu_A \cap \mu_B(x) = \min\{\mu_A(x), \mu_B(x)\}$ and $\nu_A \cup \nu_B(x) = \max\{\nu_A(x), \nu_B(x)\}$

$A \cup B = \{< x, (\mu_A \cup \mu_B)(x), (\nu_A \cap \nu_B)(x) > | x \in X \}$ where

$\mu_A \cup \mu_B(x) = \max\{\mu_A(x), \mu_B(x)\}$ and $\nu_A \cup \nu_B(x) = \min\{\nu_A(x), \nu_B(x)\}$

Definition 0.6 ([4]) For any IFS $A = \{< x, \mu_A(x), \nu_A(x) > | x \in X \}$ of $X$ and $\alpha \in [0, 1]$, the operator $\square : IFS(X) \rightarrow IFS(X), \Diamond : IFS(X) \rightarrow IFS(X), D_\alpha : IFS(X) \rightarrow IFS(X)$ are defined as

(i) $\square(A) = \{< x, \mu_A(x), 1 - \mu_A(x) > | x \in X \}$ is called necessity operator

(ii) $\Diamond(A) = \{< x, 1 - \nu_A(x), \nu_A(x) > | x \in X \}$ is called possibility operator

(iii) $D_\alpha(A) = \{< x, \mu_A(x) + \alpha \pi_A(x), \nu_A(x) + (1 - \alpha) \pi_A(x) > | x \in X \}$ is called $\alpha$-Model operator.

Clearly $\square(A) \subseteq A \subseteq \Diamond(A)$ and the equality hold, when $A$ is a fuzzy set also $D_0(A) = \square(A)$ and $D_1(A) = \Diamond(A)$. Therefore the $\alpha$-Model operator $D_\alpha(A)$ is an extension of necessity operator $\square(A)$ and possibility operator $\Diamond(A)$.

Definition 0.7 ([4]) For any IFS $A = \{< x, \mu_A(x), \nu_A(x) > | x \in X \}$ of $X$ and for any $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$, the $(\alpha, \beta)$-model operator $F_{\alpha, \beta} : IFS(X) \rightarrow IFS(X)$ is defined as $F_{\alpha, \beta}(A) = \{< x, \mu_A(x) + \alpha \pi_A(x), \nu_A(x) + \beta \pi_A(x) > | x \in X \}$, where $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ for all $x \in X$. Therefore we can write

$F_{\alpha, \beta}(A) = (\mu_{F_{\alpha, \beta}}(A), \nu_{F_{\alpha, \beta}}(A))$

where $\mu_{F_{\alpha, \beta}}(x) = \mu_A(x) + \alpha \pi_A(x)$ and $\nu_{F_{\alpha, \beta}}(A)(x) = \nu_A(x) + \beta \pi_A(x)$.

Clearly, $F_{0, 0}(A) = \square(A), F_{1, 0}(A) = \Diamond(A)$ and $F_{1, 1}(A) = D_\alpha(A)$

Definition 0.8 ([9]) An intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ of a $BG$-algebra $X$ is said to be an intuitionistic fuzzy subalgebra of $X$ if

(i) $\mu_A(x \ast y) \geq \min\{\mu_A(x), \mu_A(y)\}$

(ii) $\nu_A(x \ast y) \leq \max\{\nu_A(x), \nu_A(y)\} \ \forall x, y \in X.$
Definition 0.9 ([7]) An IFS \( A \) of a BG-algebra \( X \) is said to be an IF normal subalgebra of \( X \) if

(i) \( \mu_A((x * a) * (y * b)) \geq \min\{\mu_A(x * y), \mu_A(a * b)\} \),

(ii) \( \nu_A((x * a) * (y * b)) \leq \max\{\nu_A(x * y), \nu_A(a * b)\}, \forall x, y \in X \).

Definition 0.10 ([11]) Let \( A = (\mu_A, \nu_A) \) be an intuitionistic fuzzy set of BG-algebra \( X \). Let \( t \in [0, 1] \), then the intuitionistic fuzzy set \( A^t \) of \( X \) is called \( t \)-intuitionistic fuzzy subset (\( t \)-IFS) of \( X \) w.r.t \( A \) and is defined by \( A^t = \{< x, \mu_{A^t}(x), \nu_{A^t}(x) > | x \in X \} =< \mu_{A^t}, \nu_{A^t} > \) where \( \mu_{A^t}(x) = \min\{\mu_A(x), t\} \) and \( \nu_{A^t} = \max\{\nu_A(x), 1-t\}\forall x \in X \)

Remark 0.11 ([11]) Let \( A^t =< \mu_{A^t}, \nu_{A^t} > \) and \( B^t =< \mu_{B^t}, \nu_{B^t} > \) be two \( t \)-intuitionistic fuzzy subsets of BG-algebra \( X \), then
\( (A \cap B)^t = A^t \cap B^t \)

Remark 0.12 ([11]) Let \( f : X \rightarrow Y \) be a mapping. Let \( A \) and \( B \) are two IFS of \( X \) and \( Y \) respectively, then
\( (i) f^{-1}(B^t) = (f^{-1}(B))^t \) (ii) \( f(A^t) = (f(A))^t \) \( \forall t \in [0, 1] \)

Definition 0.13 Let \( A^t =< \mu_{A^t}, \nu_{A^t} > \) and \( B^t =< \mu_{B^t}, \nu_{B^t} > \) be two \( t \)-intuitionistic fuzzy subsets of BG-algebra \( X \). Then their cartesian product \( A^t \times B^t =< \mu_{A^t \times B^t}, \nu_{A^t \times B^t} > \) is defined by
\( \mu_{A^t \times B^t}(x, y) = \min\{\mu_{A^t}(x), \mu_{B^t}(y)\} \)
\( \nu_{A^t \times B^t}(x, y) = \max\{\nu_{A^t}(x), \nu_{B^t}(y)\} \) \( \forall x, y \in X \).

**t-Intuitionistic Fuzzy Subalgebra BG-algebra**

Now onwards, let \( X \) denote a BG-algebra unless otherwise stated.

Definition 0.14 Let \( A = (\mu_A, \nu_A) \) be an intuitionistic fuzzy set of BG-algebra \( X \). Let \( t \in [0, 1] \) then \( A \) is called \( t \)-intuitionistic fuzzy subalgebra (\( t \)-IFSA) of \( X \) if \( A^t \) is IFSA of \( X \) i.e. if \( A^t \) satisfies following conditions:
\( \mu_{A^t}(x * y) \geq \min\{\mu_{A^t}(x), \mu_{A^t}(y)\} \)
\( \nu_{A^t}(x * y) \leq \max\{\nu_{A^t}(x), \nu_{A^t}(y)\} \)

Theorem 0.15 If \( A = (\mu_A, \nu_A) \) is an intuitionistic fuzzy subalgebra BG-algebra \( X \), then \( A \) is also \( t \)-intuitionistic fuzzy subalgebra of \( X \).

**Proof.** Since \( A = (\mu_A, \nu_A) \) is an intuitionistic fuzzy subalgebra BG-algebra \( X \), therefore
\( \mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\} \)
\( \nu_A(x * y) \leq \max\{\nu_A(x), \nu_A(y)\}, \forall x, y \in X. \)

Now, \( \mu_{A^t}(x * y) = \min\{\mu_A(x * y), t\} \geq \min\{\min\{\mu_A(x), \mu_A(y)\}, t\} = \min\{\min(\mu_A(x), t), \min(\mu_A(y), t)\} = \min\{\mu_{A^t}(x), \mu_{A^t}(y)\} \)
\( \Rightarrow \mu_{A^t}(x * y) \geq \min\{\mu_{A^t}(x), \mu_{A^t}(y)\} \)
Similarly we can show
\( \nu_{A^t}(x * y) \leq \max\{\nu_{A^t}(x), \nu_{A^t}(y)\} \)
Hence \( A \) is also \( t \)-intuitionistic fuzzy subalgebra BG-algebra \( X \).
Remark 0.16 The converse of above Theorem is not true.

Example 1. Consider a BG-algebra $X = \{0, 1, 2\}$ with the following Cayley table:

|   | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 2 |
| 1 | 1 | 0 | 1 |
| 2 | 2 | 2 | 0 |

The intuitionistic fuzzy subset $A = \{< x, \mu_A(x), \nu_A(x) > | x \in X \}$ given by $\mu_A(0) = 0.4$, $\mu_A(1) = 0.5, \mu_A(2) = 0.3$ and $\nu_A(0) = 0.5, \nu_A(1) = 0.4, \nu_A(2) = 0.6$. Since $\mu_A(0) = 0.4 \not\leq \min\{\mu_A(1), \mu_A(1)\}$. Therefore $A$ is not an intuitionistic fuzzy BG-subalgebra of $X$.

Take $t = 0.2$. Then $\mu_A(x) > t$ for all $x \in X$ and also $\nu_A(x) < 1 - t$ for all $x \in X$.

Therefore $\mu_A(x \cdot y) \geq \min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x \cdot y) \leq \max\{\nu_A(x), \nu_A(y)\}$ for all $x \in X$.

Hence $A$ is t-intuitionistic fuzzy subalgebra of $X$.

Theorem 0.17 If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy set of BG-algebra $X$ and let $t < \min\{p, 1 - q\}$, where $p = \min\{\mu_A(x) | x \in X \}$ and $q = \max\{\nu_A(x) | x \in X \}$ then $A$ is also t-intuitionistic fuzzy subalgebra BG-algebra $X$.

Proof. Since $t < \min\{p, 1 - q\}$

$t < \min\{p, 1 - q\}$

$\Rightarrow p > t \quad \text{and} \quad 1 - q > t$

$\Rightarrow p > t \quad \text{and} \quad q < 1 - t$

$\Rightarrow \min\{\mu_A(x) | x \in X \} > t \quad \text{and} \quad \max\{\nu_A(x) | x \in X \} < 1 - t$

$\Rightarrow \mu_A(x) > t, \forall \ x \in X \quad \text{and} \quad \nu_A(x) < 1 - t, \forall \ x \in X$

Therefore $\mu_A(x \cdot y) \geq \min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x \cdot y) \leq \max\{\nu_A(x), \nu_A(y)\}$ for all $x \in X$ hold. Hence $A$ is t-intuitionistic fuzzy subalgebra of $X$.

Theorem 0.18 Any IF set of BG-algebra $X$ can be realised as t-intuitionistic fuzzy subalgebra $X$.

Proof. It follows from Theorem 0.17 and Theorem 0.15.

Theorem 0.19 The intersection of two t-intuitionistic fuzzy subalgebra BG-algebra $X$ is also a t-intuitionistic fuzzy subalgebra of $X$. 

Proof. Let $x, y \in X$. Then
\[
\mu_{(A \land B)}(x * y) = \min \{\mu_{(A \land B)}(x * y), t\}
\geq \min \{\min \{\mu_A(x * y), \mu_A(x * y)\}, t\}
= \min \{\min \{\mu_A(x * y), t\}, \min \{\mu_B(x * y), t\}\}
= \min \{\mu_A(x * y), \mu_B(x * y)\}
\geq \min \{\min \{\mu_A(x), \mu_A(y)\}, \min \{\mu_B(x), \mu_B(y)\}\}
= \min \{\min \{\mu_A(x), \mu_B(x)\}, \min \{\mu_A(y), \mu_B(y)\}\}
= \min \{\mu_{(A \land B)}(x), \mu_{(A \land B)}(y)\}
\Rightarrow \mu_{(A \land B)}(x * y) \geq \min \{\mu_{(A \land B)}(x), \mu_{(A \land B)}(y)\}
\]
Similarly we can show that
\[
\nu_{(A \land B)}(x * y) \leq \max \{\nu_{(A \land B)}(x), \nu_{(A \land B)}(y)\}
\]

**Theorem 0.20** The intersection of any number of t-intuitionistic fuzzy subalgebra BG-algebra $X$ is also a t-intuitionistic fuzzy subalgebra of $X$.

**Theorem 0.21** For every t-intuitionistic fuzzy subalgebra $A^t$ of $X$, the following properties hold
(i) $\mu_{A^t}(0) = \mu_{A^t}(x)$
(ii) $\nu_{A^t}(0) \leq \nu_{A^t}(x), \forall x \in X$.

Proof. We have $\mu_{A^t}(0) = \mu_{A^t}(x * x) \geq \min \{\mu_{A^t}(x), \mu_{A^t}(x)\} = \mu_{A^t}(x)$
and $\nu_{A^t}(0) = \nu_{A^t}(x * x) \leq \max \{\nu_{A^t}(x), \nu_{A^t}(x)\} = \nu_{A^t}(x)$

**Theorem 0.22** If $A$ be IF subalgebra of BG-algebra $X$, then $\Box A$, $\Diamond A$ and $F_{\alpha, \beta}(A)$ are also t-intuitionistic fuzzy subalgebra of $X$.

Proof. Here $A$ be IF subalgebra of BG-algebra $X$, By Theorem 0.15 $A$ is also t-intuitionistic fuzzy subalgebra of $X$.
\[
\mu_{A^t}(x * y) \geq \min \{\mu_{A^t}(x), \mu_{A^t}(y)\} \quad \text{(1)}
\nu_{A^t}(x * y) \leq \max \{\nu_{A^t}(x), \nu_{A^t}(y)\} \quad \text{\forall x, y \in X.} \quad \text{(2)}
\]

Now $\Box A^t = \{< x, \mu_{A^t}(x), 1 - \mu_{A^t}(x)|x \in X \} = \{< x, \mu_{A^t}(x), \overline{\mu_{A^t}(x)}|x \in X \}$
$\Diamond A^t = \{< x, 1 - \nu_{A^t}(x), \mu_{A^t}(x)|x \in X \} = \{< x, \overline{\nu_{A^t}(x)}, \mu_{A^t}(x)|x \in X \}$

Now
\[
\overline{\mu_{A^t}(x * y)} = 1 - \mu_{A^t}(x * y)
\leq 1 - \min \{\mu_{A^t}(x), \mu_{A^t}(y)\}\quad \text{By(1)}
= \max \{1 - \mu_{A^t}(x), 1 - \mu_{A^t}(y)\}
= \max \{\overline{\mu_{A^t}(x)}, \overline{\mu_{A^t}(y)}\}
\Rightarrow \overline{\mu_{A^t}(x * y)} \leq \max \{\overline{\mu_{A^t}(x)}, \overline{\mu_{A^t}(y)}\} \quad \text{(3)}
\]
Hence by Eq (1) and (3) $\Box A^t = \{< x, \mu_{A^t}(x), \overline{\mu_{A^t}(x)}|x \in X \}$ is t-intuitionistic fuzzy subalgebra of $X$.
Similarly we can show that
\[ A' = \{ x, \overline{A'}(x), \mu_{A'}(x)|x \in X \} \] is t-intuitionistic fuzzy subalgebra of X.

Again, we have
\[ F_{\alpha,\beta}(A) = \langle \mu_{F_{\alpha,\beta}(A)}, \nu_{F_{\alpha,\beta}(A)} \rangle > \text{let} \ x, y \in X, \text{then} \ F_{\alpha,\beta}(x \ast y) = (\mu_{F_{\alpha,\beta}(A)}(x \ast y), \nu_{F_{\alpha,\beta}(A)}(x \ast y)) \text{ where } \mu_{F_{\alpha,\beta}(A)}(x \ast y) = \mu_A(x \ast y) + \alpha \pi_A(x \ast y) \text{ and } \nu_{F_{\alpha,\beta}(A)}(x \ast y) = \nu_A(x \ast y) + \beta \pi_A(x \ast y) \]

\[ \mu_{F_{\alpha,\beta}A'}(x \ast y) = \mu_{A'}(x \ast y) + \alpha \pi_{A'}(x \ast y) \]
\[ = \mu_{A'}(x \ast y) + \alpha(1 - \mu_{A'}(x \ast y) - \nu_{A'}(x \ast y)) \]
\[ \geq \alpha + (1 - \alpha)\mu_{A'}(x \ast y) - \alpha \nu_{A'}(x \ast y) \]
\[ \geq \alpha + (1 - \alpha)\min(\mu_{A'}(x), \mu_{A'}(y)) - \alpha \max(\nu_{A'}(x), \nu_{A'}(y)) \quad \text{By (1)} \]
\[ \geq \alpha \min(1 - \nu_{A'}(x), 1 - \nu_{A'}(y)) + (1 - \alpha)\min(\mu_{A'}(x), \mu_{A'}(y)) \]
\[ \geq \min\{\alpha(1 - \nu_{A'}(x)) + (1 - \alpha)\mu_{A'}(x), \alpha(1 - \nu_{A'}(y)) + (1 - \alpha)\mu_{A'}(y)\} \]
\[ \geq \min\{\mu_{A'}(x) + \alpha(1 - \mu_{A'}(x) - \nu_{A'}(x)), \mu_{A'}(y) + \alpha(1 - \mu_{A'}(y) - \nu_{A'}(y))\} \]
\[ \geq \min\{\mu_{F_{\alpha,\beta}A'}(x), \mu_{F_{\alpha,\beta}A'}(y)\} \]

\[ \therefore \mu_{F_{\alpha,\beta}A'}(x \ast y) \geq \min\{\mu_{F_{\alpha,\beta}A'}(x), \mu_{F_{\alpha,\beta}A'}(y)\} \]

Similarly we can prove that
\[ \nu_{F_{\alpha,\beta}A'}(x \ast y) \leq \max\{\nu_{F_{\alpha,\beta}A'}(x), \nu_{F_{\alpha,\beta}A'}(y)\} \]

Hence \( F_{\alpha,\beta}(A) \) is t-intuitionistic fuzzy subalgebra of X.

**Theorem 0.23** Cartesian product of two t-intuitionistic fuzzy subalgebra of X is again a t-intuitionistic fuzzy subalgebra of \( X \times X \).

**Proof.** Let \( A' = \langle \mu_{A'}, \nu_{A'} \rangle > \text{and} \ B' = \langle \mu_{B'}, \nu_{B'} \rangle > \text{be two t-intuitionistic fuzzy subalgebra of BG-algebra}\ X \)

Then their cartesian product \( A' \times B' = \langle \mu_{A' \times B'}, \nu_{A' \times B'} \rangle > \), where
\[ \mu_{A' \times B'}(x, y) = \min\{\mu_{A'}(x), \mu_{B'}(y)\} \]
\[ \nu_{A' \times B'}(x, y) = \max\{\nu_{A'}(x), \nu_{B'}(y)\} \quad \forall x, y \in X. \]

Also
\[ \mu_{A'}(x \ast y) \geq \min\{\mu_{A'}(x), \mu_{A'}(y)\} \quad \text{(4)} \]
\[ \nu_{A'}(x \ast y) \leq \max\{\nu_{A'}(x), \nu_{A'}(y)\} \quad \forall x, y \in X. \quad \text{(5)} \]

\[ \mu_{A' \times B'}((x_1, y_1) \ast (x_2, y_2)) = \mu_{A' \times B'}(x_1 \ast x_2, y_1 \ast y_2) \]
\[ = \min\{\mu_{A'}(x_1 \ast x_2), \mu_{B'}(y_1 \ast y_2)\} \]
\[ \geq \min\{\min\{\mu_{A'}(x_1), \mu_{A'}(x_2)\}, \min\{\mu_{B'}(y_1), \mu_{B'}(y_2)\}\} \]
\[ = \min\{\min\{\mu_{A'}(x_1), \mu_{B'}(y_1)\}, \min\{\mu_{A'}(x_2), \mu_{B'}(y_2)\}\} \]
\[ = \min\{\mu_{A' \times B'}((x_1, y_1), \mu_{A' \times B'}((x_2, y_2)) \}

\[ \Rightarrow \mu_{A' \times B'}((x_1, y_1) \ast (x_2, y_2)) \geq \min\{\mu_{A' \times B'}((x_1, y_1), \mu_{A' \times B'}((x_2, y_2)) \}

Similarly we can show
\[ \nu_{A' \times B'}((x_1, y_1) \ast (x_2, y_2)) \leq \max\{\nu_{A' \times B'}((x_1, y_1), \nu_{A' \times B'}((x_2, y_2)) \} \]
Corollary 0.24 If $A^t =< \mu_{A^t}, \nu_{A^t} >$ and $B^t =< \mu_{B^t}, \nu_{B^t} >$ be two t-intuitionistic fuzzy subalgebra of BG-algebra $X$. Then $\Box(A^t \times B^t) \triangleleft (A^t \times B^t)$, $F_{\alpha,\beta}(A^t \times B^t)$ are also t-intuitionistic fuzzy subalgebra of $X \times X$.

Proof. Same as Corollary 0.24.

Theorem 0.25 If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy normal subalgebra BG-algebra $X$, then $A$ is also t-intuitionistic fuzzy normal subalgebra of $X$.

Proof. Since $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy normal subalgebra BG-algebra $X$, therefore

\[(i) \mu_A((x * a) * (y * b)) \geq \min\{\mu_A(x * y), \mu_A(a * b)\}\]
\[(ii) \nu_A((x * a) * (y * b)) \leq \max\{\nu_A(x * y), \nu_A(a * b)\}, \forall x, y \in X.\]

Now, $\mu_{A^t}((x * a) * (y * b)) = \min\{\mu_A((x * a) * (y * b)), t\}$
\[\geq \min\{\min\{\mu_A(x * y), \mu_A(a * b)\}, t\}\]
\[= \min\{\min\{\mu_A(x * y), t\}, \min\{\mu_A(a * b), t\}\}\]
\[= \min\{\mu_{A^t}(x * y), \mu_{A^t}(a * b)\}\]
\[\Rightarrow \mu_{A^t}((x * a) * (y * b)) \geq \min\{\mu_{A^t}(x * y), \mu_{A^t}(a * b)\}\]

Similarly we can show that $\nu_{A^t}((x * a) * (y * b)) \leq \max\{\nu_{A^t}(x * y), \nu_{A^t}(a * b)\}$

Hence $A$ is also t-intuitionistic fuzzy normal subalgebra BG-algebra $X$.

Remark 0.26 The converse of above Theorem is not true.

Theorem 0.27 If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy set of BG-algebra $X$ and let $t < \min\{p, 1-q\}$, where $p = \min\{\mu_A(x)|x \in X\}$ and $q = \max\{\nu_A(x)|x \in X\}$ then $A$ is also t-intuitionistic fuzzy normal subalgebra BG-algebra $X$.

Proof. Same as Theorem 0.17.

Theorem 0.28 The intersection of two t-intuitionistic fuzzy normal subalgebra BG-algebra $X$ is also a t-intuitionistic fuzzy normal subalgebra of $X$.

Proof. Same as Theorem 0.19.

Theorem 0.29 If $A$ be IF normal subalgebra of BG-algebra $X$, then $\Box A, \Diamond A$ and $F_{\alpha,\beta}(A)$ are also t-intuitionistic fuzzy normal subalgebra of $X$.

Proof. Same as Theorem 0.22.

Theorem 0.30 Cartesian product of two t-intuitionistic fuzzy normal subalgebra of $X$ is again a t-intuitionistic fuzzy normal subalgebra of $X \times X$.

Proof. Same as Theorem 0.23.

Corollary 0.31 If $A^t =< \mu_{A^t}, \nu_{A^t} >$ and $B^t =< \mu_{B^t}, \nu_{B^t} >$ be two t-intuitionistic fuzzy normal subalgebra of BG-algebra $X$. Then $\Box(A^t \times B^t), \Diamond(A^t \times B^t)$, $F_{\alpha,\beta}(A^t \times B^t)$ are also t-intuitionistic fuzzy normal subalgebra of $X \times X$.

Proof. Same as Corollary 0.24.
Homomorphism of t-intuitionistic fuzzy subalgebra $BG$-algebra

**Definition 0.32** Let $X$ and $Y$ be two $BG$-algebras, then a mapping $f : X \rightarrow Y$ is said to be homomorphism if $f(x * y) = f(x) * f(y)$, $\forall x, y \in X$.

**Theorem 0.33** Let $f : X \rightarrow Y$ be a homomorphism of $BG$-algebras, If $A$ be a t-intuitionistic fuzzy subalgebra of $Y$, then $f^{-1}(A)$ is t-intuitionistic fuzzy subalgebra $X$.

**Proof.** Let $A$ be a t-intuitionistic fuzzy subalgebra of $Y$. Let $x, y \in X$ be any elements, then $f^{-1}(A^t)(x * y) = (\mu_{f^{-1}(A^t)}(x * y), \nu_{f^{-1}(A^t)}(x * y))$

Now, $\mu_{f^{-1}(A^t)}(x * y)$

$= \mu_{A^t}(f(x) * f(y))$

$= \mu_{A^t}[f(x) * f(y)]$

$\geq \min \{\mu_{A^t}(f(x)), \mu_{A^t}(f(y))\}$ [Since $A$ is t-IF subalgebra of $Y$]

$= \min \{\mu_{f^{-1}(A^t)(x)}, \mu_{f^{-1}(A^t)(y)}\}$

Therefore $\mu_{f^{-1}(A^t)}(x * y) \geq \min \{\mu_{f^{-1}(A^t)(x)}, \mu_{f^{-1}(A^t)(y)}\}$

Similarly we can show that $\nu_{f^{-1}(A^t)}(x * y) \leq \max \{\nu_{f^{-1}(A^t)(x)}, \nu_{f^{-1}(A^t)(y)}\}$

Hence, $f^{-1}(A^t) = (f^{-1}(A))^t$ is t-intuitionistic fuzzy subalgebra $X$.

**Theorem 0.34** Let $f : X \rightarrow Y$ be a homomorphism of $BG$-algebras, If $A$ be a t-intuitionistic fuzzy normal subalgebra of $Y$, then $f^{-1}(A)$ is t-intuitionistic fuzzy normal subalgebra $X$.

**Theorem 0.35** Let $f : X \rightarrow Y$ be a onto homomorphism of $BG$-algebras, If $A$ be t-intuitionistic fuzzy subalgebra $X$. Then $f(A)$ is t-intuitionistic fuzzy subalgebra of $Y$.

**Proof.** Let $y_1, y_2 \in Y$ Since $f$ is onto, therefore there exists $x_1, x_2 \in X$ such that $f(x_1) = y_1, f(x_2) = y_2$.

$f(A)(y_1 * y_2) = (\mu_f(A)(y_1 * y_2), \nu_f(A)(y_1 * y_2))$.

$\mu_f(A)(y_1 * y_2) = \mu_A(t)$ where $f(t) = y_1 * y_2 = f(x_1) * f(x_2) = f(x_1 * x_2)$

$\mu_f(A^t)(y_1 * y_2)$

$= \mu_f(A^t)(y_1 * y_2)$

$= \min \{\mu_f(A)(y_1 * y_2), t\}$

$= \min \{\mu_f(A)(f(x_1) * f(x_2)), t\}$

$= \min \{\mu_f(A)(f(x_1 * x_2)), t\}$

$= \min \{\mu_A(x_1 * x_2), t\}$

$= \mu_A(x_1 * x_2)$

$\geq \min \{\mu_A(x_1), \mu_A(x_2)\}$, for all $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_1) = y_1$

$= \min \{\min \{\mu_A(x_1), f(x_1) = y_1$ \} \ $; $ \min \{\mu_A(x_2), f(x_2) = y_2$ \} \}$

$= \min \{\mu_f(A^t)(y_1), \mu_f(A^t)(y_2)\}$

Therefore $\mu_f(A^t)(y_1 * y_2) \geq \min \{\mu_f(A^t)(y_1), \mu_f(A^t)(y_2)\}$

Similarly we can show that $\nu_f(A^t)(y_1 * y_2) \leq \max \{\nu_f(A^t)(y_1), \nu_f(A^t)(y_2)\}$

Hence $f(A)$ is t-intuitionistic fuzzy subalgebra of $Y$.
Theorem 0.36 Let $f : X \rightarrow Y$ be a onto homomorphism of BG-algebras, If $A$ be t-intuitionistic fuzzy normal subalgebra $X$, then $f(A)$ is t-intuitionistic fuzzy normal subalgebra of $Y$.

References

[1] S. S. Ahn and H. D. Lee, Fuzzy subalgebras of BG-algebras, Commun Korean Math.Soc 19(2) (2004) 243-251.

[2] K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy sets and Systems. vol.1(1986),87-96

[3] K. T. Atanassov, More on intuitionistic Fuzzy Sets, Fuzzy sets and systems, 33(1) (1989), 37-45.

[4] K. T. Atanassov , On Intuitionistic Fuzzy Sets Theory, Published by Springer-Verlag Berlin Heidelberg, 2012.

[5] K. T. Atanassov, Intuitionistic Fuzzy Sets, VII ITKR’s Session, Sofia,(Deposed in Central Sci. - Techn. Library of Bulg. Acad. of Sci., 1697/84) (June 1983)(in Bulg.)

[6] Y. Imai and K. Iseki, On Axiom systems of Propositional calculi XIV, Proc, Japan Academy, 42 (1966)19-22.

[7] C. B. Kim and H.S. Kim, on BG-algebras, Demonstratio Mathematica, 41(3) 497-505.

[8] J. Neggers and H. S Kim, On B-algebras, Math. Vensik, 54 (2002), 21-29.

[9] T. Senapati, M. Bhowmik, M. Pal, Intuitionistic fuzzifications of ideals in BG-algebras, Mathematical Aeterna 2 (9) (2012) 761-778.

[10] P. K. Sharma, t-Intuitionistic Fuzzy Quotient Group, Advances in Fuzzy Mathematics, 7(1) (2012), 1-9.

[11] P. K. Sharma, t-Intuitionistic Fuzzy Subrings, IJMS, 11(3-4) (2012), 265-275.

[12] L. A. Zadeh, Fuzzy sets, Information and Control (1965), 338-353.