Finite size mass shift formula for stable particles revisited

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Lüscher’s finite size mass shift formula in a periodic finite volume, involving forward scattering amplitudes in the infinite volume, is revisited for the two stable distinguishable particle system. The generalized mass shift formulae for the boson and fermion are derived in the boson-boson and fermion-boson systems, respectively. The nucleon mass shift is discussed in the nucleon-pion system.

1. Introduction

Nowadays the control of finite volume effects in lattice QCD simulations with dynamical fermions becomes a more important issue in order to determine the hadron spectrum precisely. Applications of chiral perturbation theory (ChPT) to the measured spectrum on the lattice have been done not only to achieve the chiral extrapolation, but also to find its finite volume dependence towards the thermodynamic limit.

In this context, Lüscher’s formula, relating the mass shift in finite volume with periodic boundary conditions to forward elastic scattering amplitudes in infinite volume, provides us with an elegant tool for such a purpose. Lüscher presented a rigorous proof of his formula for the case of a self-interacting bosonic theory. Among the applications of ChPT, the QCDSF-UKQCD collaboration estimated the finite volume effect on the nucleon mass from data in $N_f = 2$ lattice QCD, applying the mass shift formula derived within ChPT. Along the way, they found however, when expressing their formula in terms of $F_{N\pi}(\nu)$ in the same order of ChPT, that the factor of the pole term (the first term of Eq. (1) below) is twice larger than that of Lüscher’s in Ref. [11]. There seems to be no mistake in the formula in Ref. [10], at least within the infrared regularization scheme [11]. On the other hand, Lüscher’s formula being considered as general such that it can also be applicable to ChPT at any orders, this discrepancy poses a structural question.

In Ref. [11], we thus investigated the mass shift formula for the interacting two stable particle system along the lines of Lüscher’s proof for a self-interacting bosonic theory. In this report, we present the resulting formulae and an application to the nucleon mass shift in the $N\pi$ system.

2. Finite size mass shift formulae

The physical mass of a stable particle is given by the position of the pole of the propagator of an asymptotic field. In the framework of perturbation theory the pole is shifted from the bare one due to the self energy arising from virtual polarization effects. In finite volume, the expressions for the self energy involve sums over discrete spatial loop momenta, $\vec{q}(L) = 2\pi \vec{n}/L$ ($\vec{n} \in \mathbb{Z}^3$). By using the Poisson summation formula, such a summation can be rewritten as an integral with another summation over integer vectors $\vec{m} \in \mathbb{Z}^3$ and an exponential factor:

$$\frac{1}{L^3} \sum_{\vec{n} \in \mathbb{Z}^3} \int \frac{d\vec{q}_0}{2\pi} f(q_0, \vec{q}(L)) = \sum_{\vec{m} \in \mathbb{Z}^3} \int \frac{d^4q}{(2\pi)^4} e^{-iL\vec{m} \cdot \vec{q}} f(q_0, \vec{q}) , \quad (1)$$

where $f(q)$ is a function composed of propagators and vertex functions. Then, the difference of the self energies between the finite and infinite volumes, appearing in the definition of the mass shift, can be defined by the sum over $|\vec{m}| \neq 0$, since $|\vec{m}| = 0$ corresponds to the integral in infinite volume. The asymptotic formula at large $L$ is given by the contribution of $|\vec{m}| = 1$.

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Let $M(L)$ and $m$ be the masses in finite and infinite volumes, respectively. The boson mass shift formula for $\phi_A$ in the $\phi_A - \phi_B$ system, $\Delta m_A(L) = M_A(L) - m_A$, is found to be\(^2\):

$$
\Delta m_A(L) = -\frac{3}{8\pi m_A L} \left[ \frac{\lambda_{AAB}^2}{2\nu_B} e^{-\sqrt{m_B^2 - \nu_B^2}} + \int_{-\infty}^{\infty} \frac{dq_0}{2\pi} e^{-L\sqrt{m^2_{AB} + q^2}} F_{AB}(iq_0) + \int_{-\infty}^{\infty} \frac{dq_0}{2\pi} e^{-L\sqrt{m^2_{AB} + q^2}} F_{AB}(iq_0) \right] + O(e^{-Lm_A}). \tag{2}
$$

where $F_{AB}(\nu)$ and $F_{AA}(\nu)$ denote the forward scattering amplitudes of the processes $A + B \rightarrow A + B$ and $A + A \rightarrow A + A$ in the infinite volume, respectively. $\lambda_{AAB}$ is an effective renormalized coupling defined from the residue of $F_{AB}(\nu)$ at $\nu = \pm \nu_B = \pm m_B^2/(2m_A)$ as

$$
\lim_{\nu \rightarrow \pm \nu_B} (\nu^2 - \nu_B^2) F_{AB}(\nu) = \frac{\lambda_{AAB}^2}{2}. \tag{3}
$$

The error term is defined by $\bar{m} \geq \sqrt{2(m_B^2 - \nu_B^2)}$, which is due to the neglect of $|\bar{m}| \geq 2$ contributions. The formula is valid for $m_A \geq m_B$. However, if $m_A \gg m_B$, we may neglect the second term, since the contribution becomes smaller than the error term.

The fermion mass shift formula for $\Psi_A$ in the $\Psi_A - \phi_B$ system ($m_A \geq m_B$) is found to be

$$
\Delta m_A(L) = -\frac{3}{8\pi m_A L} \left[ \frac{\lambda_{AAB}^2}{2\nu_B} e^{-\sqrt{m_B^2 - \nu_B^2}} + \int_{-\infty}^{\infty} \frac{dq_0}{2\pi} e^{-L\sqrt{m^2_{AB} + q^2}} F_{AB}(iq_0) + \int_{-\infty}^{\infty} \frac{dq_0}{2\pi} e^{-L\sqrt{m^2_{AB} + q^2}} F_{AB}(iq_0) \right] + O(e^{-Lm_A}). \tag{4}
$$

where three types of the forward scattering amplitudes contribute to the formula, $F_{AB}(\nu)$: $A + B \rightarrow A + B$, $F_{AA}(\nu)$: $A + A \rightarrow A + A$ and $F_{AB}(\nu)$: $A + A \rightarrow A + A$. Apart from the relative minus sign in front of the second term, which is due to Fermi statistics, the formula is almost the same as Eq. (2).

\(^2\)we assume that only $A$ particle carries a conserved charge.

3. The nucleon mass shift

As an application of Eq. (4), let us discuss the nucleon mass shift in the $N - \pi$ system. Since the formula is expected to hold nonperturbatively, it is interesting to estimate the mass shift by inserting the $N - \pi$ scattering amplitude which is known from experiment. Therefore, the following analysis can be regarded as an estimate of the finite volume effect on the nucleon mass when the realistic pion mass is achieved in lattice QCD simulations with dynamical fermions.

According to Höhler [10], the subthreshold expansion of the $N - \pi$ forward scattering amplitude around $\nu = 0$ is parametrized as

$$
D^+(\nu) = \frac{g^2}{m_N} \frac{\nu_B^2}{\nu^2} + d_0^+ m_\pi^{-1}
+ d_0^+ m_\pi^{-3} \nu^2 + d_2^+ m_\pi^{-4} \nu^4 + O(\nu^6). \tag{5}
$$

Here $m_N = 938$ MeV and $m_\pi = 140$ MeV are the masses of the nucleon and pion ($m_\pi/m_N = 0.149$), and $g^2/4\pi = 14.3$. The effect of isospin symmetry breaking is neglected. The first term is identified with the pseudovector nucleon Born term with $\nu_B = m_B^2/(2m_N) \approx 0.07m_\pi$. The coefficients of the other terms are given by $d_0^+ = -1.46(10)$, $d_1^+ = 1.12(2)$ and $d_2^+ = 0.200(5)$. We only take into account the mean of these values.

By sandwiching $D^+(\nu)$ between the nucleon spinors $\bar{u}$ and $u$ and taking into account the isospin factor, we can relate this to the amplitude $F_{AB}(\nu) = F_{N\pi}(\nu)$ in Eq. (3) by $F_{N\pi}(\nu) = 6m_N D^+(\nu)$. The effective coupling is then computed by using Eq. (3) as $\lambda_{N\pi}^2 = -12g^2\nu_B^2$. In this case, since $m_N \gg m_\pi$, the second term in Eq. (4) can be neglected. The mass shift formula, divided by the nucleon mass itself, is reduced to

$$
\delta(\xi = Lm_\pi) \equiv \Delta m_N/m_N
\approx \frac{9}{2\xi} \left( \frac{g^2}{4\pi} \right) \left( \frac{m_\pi}{m_N} \right)^3 e^{-\xi\sqrt{1 - \nu_B^2/m_\pi^2}}
- \frac{3}{16\pi^2\xi} \left( \frac{m_\pi}{m_N} \right)^2 \int_{-\infty}^{\infty} dy e^{-\xi\sqrt{1 + y^2}} F_{N\pi}(im_\pi y)
= \delta_P(\xi) + \delta_B(\xi) + \delta_R(\xi), \tag{6}
$$

where $\delta_P(\xi)$ is the pole term, and $\delta_B(\xi)$ and $\delta_R(\xi)$ correspond to the contributions of the pseudovec-
Figure 1. The nucleon mass shift as a function of $\xi = L m_\pi$.

We plot $\delta(\xi)$ in Fig. 1 where $\xi = 1$ corresponds to $L = 1.4$ fm. We find that $\delta(\xi)$ suffers strongly from higher order contributions of $\nu$ in the range $\xi \leq 1$. For instance, $\delta_B(\xi)$ causes the negative mass shift within the leading mass shift formula ($|\vec{m}| = 1$). In this range, the contribution from $|\vec{m}| > 1$ to the formula, of course, will not be negligible. On the other hand, $\delta(\xi)$ seems to be mostly described by $\delta_B(\xi)$ as $\xi$ increases. However, this is due to the cancellation between $\delta_B(\xi)$ and $\delta_R(\xi)$.

4. Summary

We have studied the general finite size mass shift formula for the two stable distinguishable particle system in a periodic finite volume along the lines of Lüscher's proof for an identical bosonic theory. The main results are Eqs. (2) and (4).

The overall error terms of the formulae can probably be reduced by taking into account the summation over integer vectors $\vec{m}$ in Eq. (1) without modifying the derivation, although a complete knowledge of the analyticity properties of the vertex functions is needed to control the final error term.

There are now valid finite size mass shift formulae for the two particle systems, in addition to that for the identical bosonic system [9]. For every case all these formulae are obtained in the same way by carefully analyzing the appropriate set of self-energy diagrams.

Acknowledgments

We are grateful to P. Weisz for introducing us to this interesting topic and also for numerous discussions during the course of the present work. We also appreciate useful comments from M. Lüscher and G. Colangelo. We are partially supported by the DFG Forschergruppe ’Lattice Hadron Phenomenology.’ M.K. is also supported by Alexander von Humboldt foundation, Germany.

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