Global Well-Posedness for NLS with a Class of $H^8$-Supercritical Data

Jinsheng Han

Department of Mathematics, Sichuan Normal University, Chengdu 610068, P. R. China.

Baoxiang Wang

LMAM, School of Mathematics, Peking University, Beijing 100871, P. R. China.

(Emails: pkuhan@math.scnu.edu.cn, wbx@pku.edu.cn)

Abstract. We study the Cauchy problem for NLS with a class of $H^s$-super-critical data

$$i u_t + \Delta u + \lambda |u|^{2\kappa} u = 0, \quad u(0) = u_0$$

(0.1)

and show that (0.1) is globally well-posed and scattering in $\alpha$-modulation spaces $M^{\kappa}_{2,1} (\alpha \in [0, 1), \ s > d\alpha/2 - \alpha/\kappa$, $\kappa \in \mathbb{N}$ and $\kappa \geq 2/d$) for the sufficiently small data. Moreover, NLS is ill-posed in $M^{\kappa}_{2,1}$ if $s < d\alpha/2 - \alpha/\kappa$. In particular, we obtain a class of initial data $u_0$ satisfying for any $M \gg 1$,

$$\|u_0\|_2 \sim M^{1/\kappa - d/2}, \ \|u_0\|_\infty = \infty, \ \|u_0\|_{M^{\kappa}_{2,1}} \geq M^{(1-\alpha)/\kappa}, \ \|u_0\|_{B^{s}_{2,\infty}} = \infty$$

such that NLS is globally well-posed in $M^{\kappa}_{2,1}$ if $\kappa > 2/d$, $\alpha \in [0, 1)$, $d\alpha/2 - \alpha/\kappa < s < s(\kappa) := d/2 - 1/\kappa$. Such a kind of data are super-critical in $H^s(\kappa)$ and have infinite amplitude.

Key words. $\alpha$-modulation space, NLS, Global well-posedness, super-critical data in $H^s$.

AMS subject classifications. 42 B35, 35 Q55

1 Introduction

In this paper we study the Cauchy problem for the nonlinear Schrödinger equation (NLS):

$$i u_t + \Delta u + \lambda |u|^{2\kappa} u = 0, \quad u(0) = u_0,$$

(1.1)

where $u(t,x)$ is a complex valued function of $(t,x) \in \mathbb{R} \times \mathbb{R}^d$, $i = \sqrt{-1}$, $u_t = \partial/\partial t$, $\Delta = \partial^2/\partial^2 x_1 + \ldots + \partial^2/\partial^2 x_d$, $\kappa \in \mathbb{N}$, $\lambda \in \mathbb{R}$, $u_0$ is a complex valued functions of $x \in \mathbb{R}^n$. $\lambda < 0$ ($\lambda > 0$) is said to be the defocusing (focusing) case for NLS. The solutions of NLS satisfy the conservations of mass and energy:

$$M(u(t)) = \|u(t)\|_2^2 = M(u_0),$$

(1.2)

$$E(u(t)) = \|\nabla u(t)\|_2^2 - \frac{2\lambda}{1+\kappa} \|u(t)\|_{2+2\kappa}^{2+2\kappa} = E(u_0).$$

(1.3)

It is well known that (1.1) is invariant under the scaling

$$u_\sigma(t,x) := \sigma^{1/\kappa} u(\sigma^2 t, \sigma x), \quad \sigma > 0,$$

(1.4)
which means that if \( u \) solves (1.1), so does \( u_\sigma \) with initial data \( \sigma^{1/\kappa} u_0(\sigma \cdot) \). Denote \( s(\kappa) = d/2 - 1/\kappa \). \( \dot{H}^{s(\kappa)} \) is a scaling invariant Sobolev space for NLS, namely, \( \| u_\sigma \|_{\dot{H}^s} = \| u \|_{\dot{H}^s} \) for all \( \sigma > 0 \) if \( s = s(\kappa) \), which is said to be a critical Sobolev space for NLS. If \( s > s(\kappa) \) (\( s < s(\kappa) \)), then \( \dot{H}^s \) is said to be subcritical (supercritical) Sobolev spaces. It is known that NLS is local well posed in all subcritical spaces \( \dot{H}^s \) with \( s > s(\kappa) \), and globally well posed in critical space \( \dot{H}^{s(\kappa)} \) if initial data are sufficiently small; cf. Cazenave and Weissler [9] and Nakamura and Ozawa [31] for exponential growth nonlinearity. Moreover, NLS is ill posed in all super-critical spaces \( \dot{H}^s \) with \( s < s(\kappa) \), cf. [2].

In the \( \dot{H}^1 \)-subcritical and \( \dot{H}^1 \)-critical cases, a large amount of work has been devoted to the study of the global well-posedness and the existence of scattering operators of NLS (cf. [8, 10, 14, 32, 34, 35, 36] and references therein), where the conservations of mass and energy play important roles. However, for the \( \dot{H}^1 \)-supercritical cases, the nonlinearity is out of the control of the energy space and up to now, it is not very clear for the mechanism of the well/ill posedness of NLS. For the defocusing energy-supercritical NLS (\( \lambda < 0, \kappa > 2/(d-2) \)), the global well-posedness for large data has been open for many years. In the general \( \dot{H}^s \) super-critical cases, we have a more delicate question:

**Question 1.1.** Are there any initial data out of the critical Sobolev spaces \( \dot{H}^{s(\kappa)} \) (or more general critical Besov spaces \( \dot{B}^{s(\kappa)}_{2,\infty} \)) so that NLS is still local and global well posed in suitable function spaces?

The above question is also standing for the other dispersive equations. In [30], Krieger and Schlag considered a class of energy-supercritical NLW in 3D (higher dimensional cases are similar to 3D)

\[
\begin{align*}
    u_{tt} - \Delta u \pm u^7 &= 0, \\
    u(0, x) &= u_0(x), \\
    u_t(0, x) &= u_1(x)
\end{align*}
\]

and they obtained the following result (notice that \( \dot{H}^{7/6} \times \dot{H}^{1/6} \) is a critical space):

**Theorem 1.1.** Let \( d = 3 \). There exist \((u_0, u_1) \in C^\infty \times C^\infty\) with

\[
\|(u_0, u_1)\|_{\dot{H}^{7/6} \times \dot{H}^{1/6}} = \infty, \quad \|(u_0, u_1)\|_{\dot{H}^s \times \dot{H}^{s-1}} < \infty, \quad s > 7/6
\]

such that there exists globally in forward time as a \( C^\infty \)-smooth solution. These solutions are stable under a certain class of perturbations.

Moreover, Krieger and Schlag considered the global existence for some initial data out of \( \dot{H}^{7/6} \times \dot{H}^{1/6} \) and large enough in \( \dot{B}^{7/6}_{2,\infty} \times \dot{B}^{1/6}_{2,\infty} \) and \( L^\infty(\{|x| \geq 1\}) \) in the defocusing case.

On the other hand, in the focusing case \( \lambda > 0 \), it is known that NLS generates blow up solutions at finite time for a class of initial data with negative energy. Let us recall Glassey’s result (cf. [19]): Let \( \lambda > 0, \kappa > 2/d \) and \( \kappa \in \mathbb{N} \), \( u_0 \in \mathcal{S} \) satisfying

\[
E(u_0) < 0, \quad \exists \int x u_0 \nabla u_0 < 0,
\]

...
then the solution of (1.1) blows up at some finite time $T_0 > 0$ in the sense
\[ \lim_{t \to T_0} \| \nabla u(t) \|_2 = \infty, \quad \lim_{t \to T_0} \| u(t) \|_{\infty} = \infty. \tag{1.7} \]

Recently Du, Wu and Zhang (cf. [13]) obtained for the focusing NLS, if $\kappa \geq 2/(d-2)$ and $\kappa \in \mathbb{N}$. $u_0 \in H^s$, $s > s(\kappa)$ and

\[ E(u_0)\|u_0\|^2_2 < \left| \int u_0 \nabla u_0 \right|^2, \tag{1.8} \]

then there exists $T_0 \leq \infty$ such that the solution of (1.1) blows up at $T_0$ in the sense that
\[ T_0 < \infty, \quad \lim_{t \to T_0} \| u(t) \|_{H^s} = \infty, \tag{1.9} \]
\[ T_0 = \infty, \quad \lim_{t \to T_0} \| u(t) \|_p = \infty, \quad \forall p > 2\kappa + 2. \tag{1.10} \]

In view of Glassey’s and Du, Wu and Zhang’s results, it is impossible to set up the global well-posedness for the focusing NLS without any conditions on initial data. In fact, for the energy critical focusing NLS, Kenig and Merle [25] obtained some optimal global well-posedness and scattering results in the radial case by using the profile decomposition together with the concentration compactness techniques. A sharp condition for scattering of the radial 3D cubic focusing NLS was obtained by Holmer and Roudenko [24].

In this paper, we mainly study the global well posedness of (1.1) with small initial data in $\alpha$-modulation spaces $M^{s,\alpha}_{2,1}$ and show the global well posedness of NLS in an $\alpha$-modulation spaces $M^{s,\alpha}_{2,1}$ and as an application of the well-posedness in $M^{s,\alpha}_{2,1}$, we have a positive answer to Question 1.1. $\alpha$-modulation spaces $M^{s,\alpha}_{p,q}$ were introduced by Gröbner [17], which can be regarded as intermediate function spaces to connect modulation and Besov spaces. In [22], we studied some standard properties including the dual space, embedding, scaling and algebraic structure of $\alpha$-modulation spaces and in current work we give their applications to NLS. Now we recall the definition of $M^{s,\alpha}_{2,1}$. Let $Q^\alpha_k := \langle k \rangle^{\frac{\alpha}{1-\alpha}}(k + [-C, C]^d)$ and
\[ \| f \|_{M^{s,\alpha}_{2,1}} = \sum_{k \in \mathbb{Z}^d} \langle k \rangle^{\frac{s}{1-\alpha}} \| \hat{f} \|_{L^2(Q^\alpha_k)}. \tag{1.11} \]

In the case $\alpha = 0$, $M^{s}_{2,1} := M^{s,0}_{2,1}$ is said to be a modulation space, cf. Feichtinger [15]. $\alpha$-modulation spaces have been applied in pseudo-differential operators and related PDE in recent years, cf. [1, 3, 4, 12, 16, 17, 29, 37, 38, 39, 40, 42, 43, 45].

Recall that the NLS has the following equivalent form:
\[ u(t) = S(t)u_0 + i\mathcal{A}(\|u\|^{2\kappa}u), \tag{1.12} \]
\[ S(t) = e^{it\Delta}, \quad \mathcal{A} = \int_0^t S(t-\tau) \cdot d\tau. \tag{1.13} \]

We have the following results.
Theorem 1.2. Let \( d \geq 1, \kappa \geq 2/d \) and \( \kappa \in \mathbb{N}, s_\kappa = d\alpha/2 - \alpha/\kappa \) and \( s > s_\kappa \). Assume that \( u_0 \in M_{2,1}^{s,\alpha} \). There exists \( \varepsilon > 0 \) such that if \( \|u_0\|_{M_{2,1}^{s,\alpha}} \leq \varepsilon \), then (1.12) has a unique solution \( u \in C(\mathbb{R}_+, M_{2,1}^{s,\alpha}) \cap X_\Delta^{s,\alpha} \) and

\[
\|u\|_{C(\mathbb{R}_+, M_{2,1}^{s,\alpha}) \cap X_\Delta^{s,\alpha}} \leq \varepsilon.
\]

If \( s < s_\kappa \), then NLS is illposed in the sense that the solution map \( u_0 \to u(t) \) is not \( C^{2\kappa+1} \) in \( M_{2,1}^{s,\alpha} \). Moreover, in the case \( \alpha = 0 \), the solution map \( u_0 \to u(t) \) is discontinuous at zero in \( M_{2,1}^{s,\alpha} \) if \( s < s_k = 0 \).

We point out that the argument of Theorem 1.2 implies the scattering operators of NLS carries a neighbourhood of zero in \( M_{2,1}^{s,\alpha} \) into \( M_{2,1}^{s,\alpha} \) if the conditions of Theorem 1.2 are satisfied.

Let us recall the definition of Besov spaces. Let \( s \in \mathbb{R} \), set \( R_0 := \{ \xi : |\xi| \leq 1 \}, R_j := \{ \xi : 2^{j-1} \leq |\xi| < 2^j \} \). Recall that the norm on Besov spaces \( B_{2,q}^s (1 \leq q \leq \infty) \) can be defined as

\[
\|f\|_{B_{2,q}^s} := \left( \sum_{j \in \mathbb{Z}_+} 2^{jsq} \|\hat{f}\|_{L^2(R_j)}^q \right)^{1/q} < \infty
\]

with usual modification for \( q = \infty \).

Theorem 1.3. Let \( d \geq 1, \kappa > 2/d \) and \( \kappa \in \mathbb{N}, \alpha \in (0,1), d\alpha/2 - \alpha/\kappa < s < s(\kappa), M \gg 1 \). Then there exists \( u_0 \in M_{2,1}^{s,\alpha} \) satisfying

\[
\|u_0\|_2 \sim M^{1/\kappa - d/2}, \quad \|u_0\|_{M_{2,1}^{s,\alpha}} \geq M^{(1-\alpha)/\kappa};
\]

\[
\|u_0\|_{\infty} = \infty, \quad \|u_0\|_{B_{2,\infty}^{s(\kappa)}} = \infty
\]

such that (1.12) has a unique solution \( u \in C(\mathbb{R}, M_{2,1}^{s,\alpha}) \cap X_\Delta^{s,\alpha} \).

Comparing the above result with Krieger and Schlag’s Theorem 1.1, we see that the initial data \( u_0 \in M_{2,1}^{s,\alpha} \) cannot be \( C^\infty \) functions. However, noticing that \( \|P_N u_0 - u_0\|_{M_{2,1}^{s,\alpha}} \to 0 \) as \( N \to \infty \), where \( P_N = \mathcal{F}^{-1} \chi_{|\xi| \leq N} \mathcal{F} \), we can consider \( P_N u_0 \in C^\infty \) for \( N \gg 1 \) as the initial value so that the solution is \( C^\infty \). We easily see that \( P_N u_0 \) satisfies (5.14) and by carefully choosing \( u_0 \), we have

\[
\|u_0\|_{\infty} \gg 1, \quad \|u_0\|_{B_{2,\infty}^{s(\kappa)}} \gg 1,
\]

see Section 5 for details. In the case \( \alpha = 0 \), we can show that there exist a class of initial data satisfying for any \( M \gg 1 \),

\[
\|u_0\|_2 \sim M^{1/\kappa - d/2}, \quad \|u_0\|_{M_{2,1}^{s,\alpha}} \geq M^{1/\kappa};
\]

\[
M^{1/\kappa} \leq \|u_0\|_{\infty} < \infty, \quad \|u_0\|_{B_{2,\infty}^{s(\kappa)}} = \infty
\]

\(*X_\Delta^{s,\alpha} \) will be defined in Section 2.3
such that (1.12) has a unique solution \( u \in C(\mathbb{R}, M^s_{2,1} \cap X^{s,0}_\Delta) \) for \( 0 < s \ll 1, \ d \geq 1, \ \kappa > 2/d \).

The results in the case \( \alpha = 0 \) were essentially obtained in [44].

Theorem 1.3 is particularly interesting to the focusing NLS with the energy super-critical power \( \kappa > 2/(d - 2) \), which implies that there exist a class of initial data out of the critical Besov spaces \( B^{s,\infty}_{2,\infty} \) so that NLS is globally well posed. Moreover, such a kind of data have neither small amplitudes nor small \( \alpha \)-modulation norm in \( M^{s,\alpha}_{2,1} \) for \( s > s_k, \)

In [23], we announced a result to showed that NLS is globally well posed in \( \alpha \)-modulation spaces \( M^{s,\alpha}_{2,1} \) if \( s > \tilde{s}_k, \)

\[ \tilde{s}_k = \frac{d\alpha}{2} - \frac{\alpha}{\kappa} + \frac{\alpha(1 - \alpha)(n\kappa + 2)}{2\kappa[(1 + \alpha)\kappa + 1 - \alpha]} \]

\( \tilde{s}_k \) is not optimal if \( \alpha \neq 0, 1 \). Our Theorem 1.2 has improved the result in [23] and except for the end point case \( s = s_\kappa, \) our result is sharp. The proof of Theorem 1.2 relies upon \( U_p \) and \( V_p \) spaces first applied in [21, 26], together with the bilinear and Strichartz’ estimates and variant \( \alpha \)-decompositions.

Throughout this paper, \( A \lesssim B \) stands for \( A \leq CB, \) and \( A \sim B \) denote \( A \lesssim B \) and \( B \lesssim A, \) where \( C \) is a positive constant which can be different at different places. We write \( a \vee b = \max(a, b), \) \( a \wedge b = \min(a, b). \) We denote by \( s+ = s + \varepsilon, \ 0 < \varepsilon \ll 1. \) For any \( p \in [1, \infty], \) \( p' \) will stand for the dual number of \( p, \) i.e., \( 1/p + 1/p' = 1. \)

The paper is organized as follows. In Section 2 we introduce some function spaces and study their properties, such as duality, embedding, which are useful in the whole paper. In Section 3, we get some bilinear estimates with respect to the \( \alpha \)-decompositions. Global well-posedness for NLS in \( \alpha \)-modulation space is obtained in Sections 4 and 5. In the last section, we show the ill-posedness for NLS in \( \alpha \)-modulation space.

## 2 Function spaces

Let \( \mathcal{S}(\mathbb{R}^d) \) be the Schwartz space and \( \mathcal{S}'(\mathbb{R}^d) \) be its dual space. Sobolev spaces \( H^s(\mathbb{R}^d) = (I - \Delta)^{-s/2}L^2, \ \tilde{H}^s(\mathbb{R}^d) = (-\Delta)^{-s/2}L^2. \) \( L^{p'}_tL^p_x(\mathbb{R}^{d+1}) \) is the space-time Lebesgue space equipped with norm

\[ \|f\|_{L^{p'}_tL^p_x(\mathbb{R}^{d+1})} := \|\|f\|_{L^p_x(\mathbb{R}^d)}\|_{L^{p'}_t(\mathbb{R})}. \]

### 2.1 Decompositions to frequency spaces

Let \( \varphi \) be a smooth radial bump function supported in the ball \( B(0, 2) = \{ \xi \in \mathbb{R}^d : |\xi| < 2 \} \) and \( \varphi(\xi) = 1 \) in the unit ball \( B(0, 1). \) Put

\[
\begin{align*}
\varphi_0(\xi) &= \varphi(\xi), \\
\varphi_j(\xi) &= \varphi(2^{-j}\xi) - \varphi(2^{-j+1}\xi), \quad j \in \mathbb{N},
\end{align*}
\]

(2.1)
and
\[ \Delta_j := \mathcal{F}^{-1} \varphi_j \mathcal{F}, \quad j \in \mathbb{N} \cup \{0\}, \]
which are said to be dyadic decomposition operators. Put
\[ \varphi^\alpha_k(\xi) := \varphi \left( \frac{\xi - \langle k \rangle^{\frac{\alpha}{1-\alpha}}}{C(k)^{\frac{\alpha}{1-\alpha}}} \right), \]
and denote
\[ \eta^\alpha_k(\xi) := \varphi^\alpha_k(\xi) \left( \sum_{l \in \mathbb{Z}^d} \varphi^\alpha_l(\xi) \right)^{-1}. \]

We define an operator sequence by
\[ \square^\alpha_k := \mathcal{F}^{-1} \eta^\alpha_k \mathcal{F}, \quad k \in \mathbb{Z}^d, \]
which are said to be the $\alpha$-decomposition operators. Formally, we have
\[ \sum_{j \in \mathbb{Z}^d} \Delta_j = \text{Id}, \quad \sum_{k \in \mathbb{Z}^d} \square^\alpha_k = \text{Id}. \]

The following are the embedding results between $\alpha$-modulation and Besov spaces. One can refer to [22].

**Proposition 2.1** (Embeddings). There hold the following sharp embeddings.

(i) If $0 \leq \alpha < 1, s_1 \geq s_2$, then $M_{s_1,\alpha} \subset B_{s_2,1}^{s_2,\alpha}$.

(ii) If $0 \leq \alpha < 1, s_1 \geq s_2 + d(1-\alpha)/2$, then $B_{2,1}^{s_1} \subset M_{2,1}^{s_2,\alpha}$.

### 2.2 $U^p$ and $V^p$ type spaces

$U^p$ and $V^p$, as a development of Bourgain’s spaces [5, 6] were first applied by Koch and Tataru in the study of NLS, cf. [26, 27, 28]. Let $Z$ be the set of finite partitions $-\infty = t_0 < t_1 < ... < t_{K-1} < t_K = \infty$. Let $1 \leq p < \infty$. For any \( \{t_k\}_{k=0}^K \subset Z \) and \( \{\phi_k\}_{k=0}^{K-1} \in L^2 \) with \( \sum_{k=0}^{K-1} \|\phi_k\|_p^p = 1, \phi_0 = 0 \). A step function \( a: \mathbb{R} \rightarrow L^2 \) given by
\[ a = \sum_{k=1}^K \chi_{[t_{k-1},t_k]} \phi_{k-1} \]
is said to be a $U^p$-atom. All of the $U^p$ atoms is denoted by $\mathcal{A}(U^p)$. The $U^p$ space is
\[ U^p = \left\{ u = \sum_{j=1}^\infty c_j a_j : a_j \in \mathcal{A}(U^p), \ c_j \in \mathbb{C}, \ \sum_{j=1}^\infty |c_j| < \infty \right\} \]
for which the norm is given by

\[
\|u\|_{U^p} = \inf \left\{ \sum_{j=1}^{\infty} |c_j| : u = \sum_{j=1}^{\infty} c_j a_j, \ a_j \in A(U^p), \ c_j \in \mathbb{C} \right\}.
\]

We define \( V^p \) as the normed space of all functions \( v : \mathbb{R} \to L^2 \) such that \( \lim_{t \to \pm \infty} v(t) \) exist and for which the norm

\[
\|v\|_{V^p} := \sup_{\{t_k\}_{k=0}^{K} \in \mathbb{Z}} \left( \sum_{k=1}^{K} \|v(t_k) - v(t_{k-1})\|_{L^2}^p \right)^{1/p}
\]

is finite, where we use the convention that \( v(-\infty) = \lim_{t \to -\infty} v(t) \) and \( v(\infty) = 0 \) (here \( v(\infty) \) and \( \lim_{t \to \infty} v(t) \) are different notations). Likewise, we denote by \( V^p \) the subspace of all \( v \in V^p \) so that \( v(-\infty) = 0 \). Moreover, we define the closed subspace \( V^p_{rc,\Delta} (V^p_{-r, \Delta}) \) as all of the right continuous functions in \( V^p (V^p) \).

We define

\[
U^p_{\Delta} := e^{-i\Delta U^p}, \quad \|u\|_{U^p_{\Delta}} = \|e^{-i\Delta} u\|_{U^p}.
\]

\[
V^p_{\Delta} := e^{-i\Delta V^p}, \quad \|u\|_{V^p_{\Delta}} = \|e^{-i\Delta} u\|_{V^p}.
\]

Similarly for the definition of \( V^p_{rc,\Delta} (V^p_{-r, \Delta}) \). We list some known results in \( U^p \) and \( V^p \) (cf. 26 27 28 21).

**Proposition 2.2.** (Embedding) Let \( 1 \leq p < q < \infty \). We have the following results.

(i) \( U^p \) and \( V^p \), \( V^p_{rc} \), \( V^p_{-r} \), \( V^p_{rc, -} \) are Banach spaces.

(ii) \( U^p \subset V^q_{rc, -} \subset U^q \subset L^\infty(\mathbb{R}, L^2) \). Every \( u \in U^p \) is right continuous on \( t \in \mathbb{R} \)

(iii) \( V^p \subset V^q \), \( V^p_{rc} \subset V^q \), \( V^p_{rc} \subset V^q \), \( V^p_{rc, -} \subset V^q_{rc, -} \).

(iv) \( \dot{X}^{0,1/2,1} \subset U^p_{\Delta} \subset V^2_{rc, -} \subset \dot{X}^{0,1/2,\infty} \), where Besov type Bourgain’s spaces \( \dot{X}^{s,b,q} \) are defined by

\[
\|u\|_{\dot{X}^{s,b,q}} := \left\| \int^{\infty}_{\tau = \tau_0} e^{i \tau \xi^2} \int^{\infty}_{\tau = \tau_0} e^{i \tau \xi^2} |\xi|^s |\tau + \xi^2|^{b} \eta(\tau, \xi) \right\|_{L^2_{\tau} L^{q}_{\xi}}.
\]

The following transference principle due to Hadac, Herr and Koch [21] will be frequently used in the later.

**Proposition 2.3** (Transference principle). Suppose that the linear operator \( T : L^2(\mathbb{R}^d) \times \cdots \times L^2(\mathbb{R}^d) \to L^2_{\ell} L^2_{\ell}(\mathbb{R}^{d+1}) \) satisfies for some \( 1 \leq q, r \leq \infty \),

\[
\|T((e^{it\Delta} \phi^{(\ell)})_{\ell=1}^{n})\|_{L^2_{\ell} L^2_{\ell}(\mathbb{R}^{d+1})} \leq C_1 \prod_{\ell=1}^{n} \|\phi^{(\ell)}\|_{L^2},
\]

(2.5)
then we have
\[ \|T((u(t)_{\ell=1}^n))\|_{L^p_t L^q_x(\mathbb{R}^{d+1})} \leq C_1 \prod_{\ell=1}^n \|u(t)\|_{U^\alpha_k}. \] (2.6)

Furthermore, if for some \( p > q \), there holds
\[ \|T((u(t)_{\ell=1}^n))\|_{L^p_t L^q_x(\mathbb{R}^{d+1})} \leq C_2 \prod_{\ell=1}^n \|u(t)\|_{U^\alpha_k}, \] (2.7)
then we conclude that
\[ \|T((u(t)_{\ell=1}^n))\|_{L^p_t L^q_x(\mathbb{R}^{d+1})} \approx C_1 \left( \ln \frac{C_2}{C_1} + 1 \right) \prod_{\ell=1}^n \|u(t)\|_{U^\alpha_k}. \] (2.8)

### 2.3 \( X^{s,\alpha}_\Delta \) and \( Y^{s,\alpha}_\Delta \)

In this subsection we introduce the working spaces \( X^{s,\alpha}_\Delta \) and \( Y^{s,\alpha}_\Delta \) and establish their duality. Moreover, we will give some embedding results between \( X^{s_1,\alpha_1}_\Delta \) and \( X^{s_2,\alpha_2}_\Delta \) (\( Y^{s_1,\alpha_1}_\Delta \) and \( Y^{s_2,\alpha_2}_\Delta \)), which are necessary for us to make nonlinear estimates.

**Definition 2.1.** Let \( 0 \leq \alpha < 1 \), \( s \in \mathbb{R} \). We define the following spaces:

\[
X^{s,\alpha}_\Delta = \{ u \in \mathcal{S}'(\mathbb{R}^{d+1}) : \|u\|_{X^{s,\alpha}_\Delta} := \sum_{k \in \mathbb{Z}^d} \langle k \rangle^{\frac{\alpha}{1-\alpha}} \|\Box_k u\|_{L^2} < \infty \}, \quad (2.9)
\]

\[
Y^{s,\alpha}_\Delta = \{ u \in \mathcal{S}'(\mathbb{R}^{d+1}) : \|v\|_{Y^{s,\alpha}_\Delta} := \sup_{k \in \mathbb{Z}^d} \langle k \rangle^{\frac{\alpha}{1-\alpha}} \|\Box_k v\|_{L^2} < \infty \}, \quad (2.10)
\]

\[
X^{s,1}_\Delta = \{ u \in \mathcal{S}'(\mathbb{R}^{d+1}) : \|u\|_{X^{s,1}_\Delta} := \sum_{j=0}^{\infty} 2^{js} \|\Delta_j u\|_{L^2} < \infty \}, \quad (2.11)
\]

\[
Y^{s,1}_\Delta = \{ u \in \mathcal{S}'(\mathbb{R}^{d+1}) : \|v\|_{Y^{s,1}_\Delta} := \sup_{j \in \mathbb{Z}^+} 2^{js} \|\Delta_j v\|_{L^2} < \infty \}, \quad (2.12)
\]

\[
\|u\|_{X^{s,\beta}_\Delta} = \|S(-t)u\|_{X^{s,\beta}_\Delta}, \quad \|u\|_{Y^{s,\beta}_\Delta} = \|S(-t)u\|_{Y^{s,\beta}_\Delta}, \quad \beta \in [0, 1]. \quad (2.13)
\]

For any time interval \( I \subset \mathbb{R} \), we denote
\[
\|u\|_{X^{s,\alpha}_\Delta(I)} = \inf \{ \|\tilde{u}\|_{X^{s,\alpha}_\Delta} : \tilde{u} \in X^{s,\alpha}_\Delta, \tilde{u}(t) = u(t), \quad \forall \ t \in I \}. \quad (2.14)
\]

**Proposition 2.4.** (Duality) Let \( 1 \leq p < \infty \), \( 1/p + 1/p' = 1 \). Then \((U_p)^* = V^{p'}\) in the sense that
\[
T : V^{p'} \to (U_p)^* ; \quad T(v) = B(\cdot, v), \quad (2.15)
\]
is an isometric mapping. The bilinear form \( B : U^p \times V^{p'} \) is defined in the following way: For a partition \( t := \{ t_k \}_{k=0}^K \in \mathcal{Z} \), we define
\[
B_t(u, v) = \sum_{k=1}^K (u(t_{k-1}), v(t_k) - v(t_{k-1})). \quad (2.16)
\]

8
Here \( \langle \cdot , \cdot \rangle \) denotes the inner product on \( L^2 \). For any \( u \in U^p \), \( v \in V'^p \), there exists a unique number \( B(u,v) \) satisfying the following property. For any \( \varepsilon > 0 \), there exists a partition \( t \) such that

\[
|B(u,v) - B_t(u,v)| < \varepsilon, \quad \forall \ t' \supset t.
\]

Moreover,

\[
|B(u,v)| \leq \|u\|_{U^p} \|v\|_{V'^p}.
\]

In particular, let \( u \in V^1 \) be absolutely continuous on compact interval, then for any \( v \in V'^p \),

\[
B(u,v) = \int \langle u'(t), v(t) \rangle dt.
\]

The duality of \( U^p \) and \( V'^p \) obtained by Hadac, Herr and Koch [21] is of importance for us to make nonlinear estimates for the dispersive equations. We need further consider its localized version with \( \alpha \)-decomposition, namely, the duality of \( X^{s,\alpha} \) and \( Y^{-s,\alpha} \).

### 2.4 Duality of \( X^{s,\alpha} \) and \( Y^{-s,\alpha} \)

**Proposition 2.5 (Duality).** Let \( 1 \leq p, q < \infty \), \( \alpha \in [0,1) \). Then we have

\[
(X^{s,\alpha})^* = Y^{-s,\alpha}; \quad (X^{s,1})^* = Y^{-s,1}.
\]  

in the sense that

\[
T: Y^{-s,\alpha} \to (X^{s,\alpha})^*; \quad T(v) = B(\cdot, v),
\]

is an isometric mapping, where the bilinear form \( B(\cdot , \cdot) \) is defined in Proposition 2.4. Moreover, we have

\[
|B(u,v)| \leq \|u\|_{X^{s,\alpha}} \|v\|_{Y^{-s,\alpha}}.
\]

**Proof.** By the almost orthogonality, we see that

\[
B_t(\square_k^0 u, \square_l^0 v) = 0, \quad |k - l| \geq C,
\]

For any \( v \in Y^{-s,\alpha} \), by Proposition 2.4 and Hölder's inequality, we have

\[
|B(u,v)| = \left| \sum_{|l| \leq C} \sum_{k \in \mathbb{Z}^d} B(\square_k^0 u, \square_k^0 v) \right| \\
\leq \sum_{k \in \mathbb{Z}^d} \|\square_k^0 u\|_{U^2} \|\square_k^0 v\|_{V^2} \lesssim \|u\|_{X^{s,\alpha}} \|v\|_{Y^{-s,\alpha}}.
\]

It follows that \( Y^{-s,\alpha} \subset (X^{s,\alpha})^* \) and \( \|v\|_{(X^{s,\alpha})^*} \lesssim \|v\|_{Y^{-s,\alpha}} \).

Conversely, considering the map

\[
X^{s,\alpha} \ni f \to \{\square_k^0 f\} \in \ell^s(\mathbb{Z}^d; U^2),
\]
where
\[ \ell_s^*(\mathbb{Z}^d; U^2) := \left\{ \{f_k\}_{k \in \mathbb{Z}^d} : \|\{f_k\}\|_{\ell_s^*(\mathbb{Z}^d; U^2)} := \left\| \{\langle k \rangle^{s/(1-\alpha)} \|f_k\|_{U^2} \} \right\|_{\ell_q} < \infty \right\}, \]
we see that it is an isometric mapping from \( X^{s,\alpha} \) into a subspace of \( \ell_1^s(\mathbb{Z}^d; U^2) \). So, \( v \in (X^{s,\alpha})^* \) can be regarded as a continuous functional in a subspace of \( \ell_1^s(\mathbb{Z}^d; U^2) \). In view of Hahn-Banach Theorem, it can be extended onto \( \ell_1^s(\mathbb{Z}^d; U^2) \) (the extension is written as \( \tilde{v} \)) and its norm will be preserved. In view of the well-known duality \((\ell_1^s(\mathbb{Z}; X))^* = \ell_{\infty}^{-s}(\mathbb{Z}; X^*)\), we have
\[ (\ell_s^*(\mathbb{Z}^d; U^2))^* = \ell_{\infty}^{-s}(\mathbb{Z}^d; V^2), \]
and there exists \( \{v_k\}_{k \in \mathbb{Z}} \in \ell_{\infty}^{-s}(\mathbb{Z}; V^2) \) such that
\[ \langle \tilde{v}, \{f_k\} \rangle = \sum_{k \in \mathbb{Z}^d} B(f_k, v_k), \quad \forall \{f_k\} \in \ell_1^s(\mathbb{Z}^d; U^2). \]
Moreover, \( \|v\|_{(X^{s,\alpha})^*} = \|\{v_k\}\|_{\ell_{-s,\alpha}(\mathbb{Z}; V^2)} \). Hence, for any \( u \in X^{s,\alpha} \),
\[ \langle v, u \rangle = \langle \tilde{v}, \{\Box_k^\alpha u\} \rangle = \sum_{k \in \mathbb{Z}^d} B(\Box_k^\alpha u, v_k). \]
From \( B(\Box_k^\alpha u, v) = B_t(u, \Box_k^\alpha v) \) we see that \( B(\Box_k^\alpha u, v) = B(u, \Box_k^\alpha v) \). It follows that
\[ v = \sum_{k \in \mathbb{Z}^d} \Box_k^\alpha v_k. \]
Obviously, we have
\[ \|v\|_{Y^{-s,\alpha}} \leq \|\{v_k\}\|_{\ell_{-s,\alpha}(\mathbb{Z}^d; V^2)} = \|v\|_{(X^{s,\alpha})^*}. \]
This proves \( (X^{s,\alpha})^* \subset Y^{-s,\alpha} \).

Now we apply the duality to the norm calculation to the inhomogeneous part of the solution of NLS in \( X^{s,\alpha}_\Delta \). By Propositions \([2.4]\) and \([2.5]\) we see that
\[ \|\mathcal{A}(f)\|_{X^{s,\alpha}_\Delta} = \sup \left\{ \left| B \left( \int_0^t e^{-i\Delta s} f(s) ds, v \right) \right| : \|v\|_{Y^{-s,\alpha}} \leq 1 \right\} \]
\[ \leq \sup_{\|v\|_{Y^{-s,\alpha}} \leq 1} \left| \int \langle f(s), e^{i\Delta s} v(s) \rangle ds \right| \]
\[ \leq \sup_{\|v\|_{Y^{-s,\alpha}} \leq 1} \left| \int \langle f(s), v(s) \rangle ds \right| . \tag{2.19} \]

**Corollary 2.1.** Let \( 0 \leq \alpha < 1 \). We have
\[ \|\mathcal{A}f\|_{X^{s,\alpha}_\Delta} = \sup_{\|v\|_{Y^{-s,\alpha}} \leq 1} \left| \int_{\mathbb{R}} \langle f(t), v(t) \rangle dt \right| , \tag{2.20} \]
\[ \|\mathcal{A}f\|_{X^{1,\alpha}_\Delta} = \sup_{\|v\|_{Y^{-1,\alpha}} \leq 1} \left| \int_{\mathbb{R}} \langle f(t), v(t) \rangle dt \right| . \tag{2.21} \]
For the purpose of later use, we need:

**Theorem 2.1 (Embeddings).** The following embeddings hold true.

(i) If $0 \leq \alpha_1, \alpha_2 < 1, s_1 \geq s_2 + 0 \lor d(\alpha_1 - \alpha_2)/2$, then $X^{s_1,\alpha_1} \subset X^{s_2,\alpha_2}$.

(ii) If $0 \leq \alpha < 1, s_1 \geq s_2 + d(1 - \alpha)/2$, then $X^{s_1,1} \subset X^{s_2,\alpha}$.

(iii) If $s_1 \geq s_2 + 0 \lor d(\alpha_2 - \alpha_1)/2$, then $Y^{s_1,\alpha_1} \subset Y^{s_2,\alpha_2}$.

(iv) If $0 \leq \alpha < 1, s_1 \geq s_2 + d(1 - \alpha)/2$, then $Y^{s_1,\alpha_1} \subset Y^{s_2,1}$.

**Proof.** For any $(l,\alpha_2) \in \mathbb{Z}^d \times [0,1)$, we define

$$
\Lambda^{\alpha_1}[(l,\alpha_2)] := \{ k \in \mathbb{Z}^d : \square_l^{\alpha_1} \square_l^{\alpha_2} f \neq 0, \forall f \in \mathcal{S}'(\mathbb{R}^d) \}.
$$

One can check that

$$
\#\Lambda^{\alpha_1}[(l,\alpha_2)] \lesssim 1 \lor \langle l \rangle \frac{d(\alpha_2 - \alpha_1)}{1 - \alpha_2}; \text{ and } \langle k \rangle \sim \langle l \rangle \frac{1 - \alpha_1}{1 - \alpha_2}
$$

provided that $k \in \Lambda^{\alpha_1}[(l,\alpha_2)]$. Let $\{t_i\}_{i=0}^K \in \mathbb{Z}$, Plancherel’s identity yields that

$$
\left( \sum_{i=1}^K \| \square_l^{\alpha_2} [u(t_i) - u(t_{i-1})] \|_2^2 \right)^{1/2} \leq \left( \sum_{k \in \Lambda^{\alpha_1}[(l,\alpha_2)]} \sum_{i=1}^K \| \square_l^{\alpha_1} \square_l^{\alpha_2} [u(t_i) - u(t_{i-1})] \|_2^2 \right)^{1/2},
$$

where the $k$ on the right-hand side of (2.22) belongs to $\Lambda^{\alpha_1}[(l,\alpha_2)]$. Therefore,

$$
\sup_{l \in \mathbb{Z}^d} \langle l \rangle \frac{s_2}{1 - \alpha_2} \sup_{\{t_i\}_{i=0}^K \in \mathbb{Z}} \left( \sum_{i=1}^K \| \square_l^{\alpha_2} [u(t_i) - u(t_{i-1})] \|_2^2 \right)^{1/2} \lesssim \sup_{k \in \mathbb{Z}^d} \langle k \rangle \frac{1}{1 - \alpha_1} [s_2 + \frac{d}{2}(\alpha_2 - \alpha_1)] \sup_{\{t_i\}_{i=0}^K \in \mathbb{Z}} \left( \sum_{i=1}^K \| \square_l^{\alpha_1} [u(t_i) - u(t_{i-1})] \|_2^2 \right)^{1/2},
$$

which implies the result of (iii). As a consequence of duality, we can obtain the result as desired. For instance, we prove (i). If $\alpha_1 \geq \alpha_2$ and $s_1 \geq s_2 + d(\alpha_1 - \alpha_2)/2$, then we have $Y^{-s_2,\alpha_2} \subset Y^{-s_1,\alpha_1}$. It follows from the duality that

$$
\|u\|_{X^{s_2,\alpha_2}} = \sup_{v \in (X^{s_2,\alpha_2})^* \setminus \{0\}} \frac{B(u,v)}{\|v\|_{(X^{s_2,\alpha_2})^*}} = \sup_{v \in Y^{-s_2,\alpha_2} \setminus \{0\}} \frac{B(u,v)}{\|v\|_{Y^{-s_2,\alpha_2}}} \leq C \sup_{v \in Y^{-s_1,\alpha_1} \setminus \{0\}} \frac{B(u,v)}{\|v\|_{Y^{-s_1,\alpha_1}}} = C\|u\|_{X^{s_1,\alpha_1}}.
$$

(2.23)
If \( \alpha_1 < \alpha_2 \) and \( s_1 \geq s_2 \), we can prove (i) in a similar way. \( \square \)

### 3 Bilinear Estimates

Using standard dual arguments, one has the following Strichartz estimates, which is useful to obtain the bilinear estimates, cf. \([\Pi]\).

**Proposition 3.1** (Strichartz). \((q, r)\) is said to be a admissible pair if \(q, r \geq 2, (q, r, d) \neq (2, \infty, 2)\) and

\[
\frac{1}{q} = \frac{1}{r} \cdot \frac{2}{dq} \geq 0. \tag{3.1}
\]

Let \((q, r)\) and \((\tilde{q}, \tilde{r})\) be two admissible pairs, then we have

\[
\|\Box_k e^{it\Delta} u_0\|_{L^q_t L^r_x} \lesssim \langle k \rangle^{\frac{d}{2d} - \frac{2}{rq}} \|\Box_k u_0\|_{L^q_t L^r_x}. \tag{3.2a}
\]

Applying the transference principle, we immediately have

\[
\|\Box_k a \|_{L^q_t L^r_x} \lesssim \langle k \rangle^{\frac{d}{2d} - \frac{2}{rq}} \|\Box_k a\|_{L^q_{t'} L^r_{x'}}. \tag{3.3}
\]

**Lemma 3.1** (General bilinear estimates). Let \(D_1\) and \(D_2\) are compact subsets of \(\mathbb{R}^d\). Assume that \(\text{supp} \hat{\varphi}_i \subset D_i, i = 1, 2\) and

\[
\lambda = \inf \{\|\xi^{(1)}_i - \xi^{(2)}_i\| : \xi^{(i)}(\xi) \in D_i, \xi^{(i)}(\xi) = (\xi^{(i)}_1, ..., \xi^{(i)}_d), i = 1, 2\} > 0 \tag{3.4}
\]

Denote \(\overline{D} = \{\xi : \exists \xi_1, (\xi_1, \xi) \in D\}\). Then we have

\[
\|e^{it\Delta} \varphi_1 e^{it\Delta} \varphi_2\|_{L^2_{x,t}} \lesssim \lambda^{-1/2} (|D_1| \wedge |D_2|)^{1/2} \|\varphi_1\|_2 \|\varphi_2\|_2. \tag{3.5}
\]

Lemma 3.1 is known in the literatures, see for instance, \([20]\) in 1D and \([7, 33]\) in higher dimensions in the dyadic version. However, the current version shows that the decay \(\lambda^{-1/2}\) in the bilinear estimate only depends on the distance of the supports of \(\varphi_1\) and \(\varphi_2\) in one direction.

**Proof.** We have

\[
\|e^{it\Delta} \varphi_1 e^{it\Delta} \varphi_2\|_{L^2_{x,t}} = \sup \|g\|_{L^2_{x,t}} \int_{\mathbb{R}^{d+1}} e^{it\Delta} \varphi_1 e^{it\Delta} \varphi_2 g(x, t) dx dt. \tag{3.6}
\]

By Plancherel's identity,

\[
\int_{\mathbb{R}^{d+1}} e^{it\Delta} \varphi_1 e^{it\Delta} \varphi_2 g(x, t) dx dt = \int_{\mathbb{R}^{2d+1}} (e^{-it|\xi|^2} \tilde{\varphi}_1) * (e^{-it|\xi|^2} \tilde{\varphi}_2) \tilde{g}(\xi, t) d\xi dt
\]

\[
= \int_{\mathbb{R}^{2d+1}} e^{-it|\xi|^2} \varphi_1(\xi_1) e^{-it|\xi|^2} \varphi_2(\xi_2) \tilde{g}(\xi_1 + \xi_2, t) d\xi_1 d\xi_2 dt
\]

\[
= \int_{\mathbb{R}^{2d}} \varphi_1(\xi_1) \varphi_2(\xi_2) \tilde{g}(\xi_1 + \xi_2, t) d\xi_1 d\xi_2
\]

\[
\leq \|\varphi\|_2 \|\varphi_2\|_2 \|\chi_{D_1}(\xi_1) \chi_{D_2}(\xi_2) \tilde{g}(\xi_1 + \xi_2, t) d\xi_1 d\xi_2\|
\]

\[
\leq \|\varphi\|_2 \|\varphi_2\|_2 \|\chi_{D_1}(\xi_1) \chi_{D_2}(\xi_2) \tilde{g}(\xi_1 + \xi_2, t) d\xi_1 d\xi_2\|. \tag{3.7}
\]

12
By Change of variables $\xi = \xi_1 + \xi_2$, $\tau = -|\xi_1|^2 - |\xi_2|^2$, we have
\[
\|\chi D_1(\xi) \chi D_2(\xi) \hat{g}(\xi_1 + \xi_2, -|\xi_1|^2 - |\xi_2|^2)\|_{L^2_{\xi_1, \xi_2}} \leq \lambda^{-1} |D_2| \tag{3.8}
\]
Since $D_1$ has the same role as $D_2$, we have the result, as desired. \(\square\)

Since we also need to handle the bilinear version $w\overline{v}$, similar to Lemma 3.1, we have

**Lemma 3.2.** Assume that the conditions of Lemma 3.1 are satisfied. Then we have
\[
\|e^{it\Delta} \varphi_1 e^{it\Delta} \varphi_2\|_{L^2_t} \lesssim \lambda^{-1/2} (|D_1| \wedge |D_2|)^{1/2} \|\varphi_1\|_2 \|\varphi_2\|_2. \tag{3.9}
\]

**Proof.** In the same way as in the proof of Lemma 3.1, noticing that
\[
\int_{\mathbb{R}^{d+1}} e^{it\Delta} \varphi_1 e^{it\Delta} \varphi_2 \, g(x, t) dx dt = \int_{\mathbb{R}^{2d}} \hat{\varphi}_1(\xi_1) \hat{\varphi}_2(\xi_2) \hat{g}(\xi_1 - \xi_2, |\xi_1|^2 - |\xi_2|^2) d\xi_1 d\xi_2, \tag{3.10}
\]
we can repeat the argument as in the proof of Lemma 3.1 to have the result, as desired. \(\square\)

**Corollary 3.1** ($\square_{\overline{k}}^\alpha$-Bilinear estimates I). Let $0 \leq \alpha < 1$, $l \in \mathbb{Z}^d$, $|l| \gg 1$. $|k| \leq |l|$. Suppose that
\[
|\langle k \rangle^{\alpha/(1-\alpha)} k - \langle l \rangle^{\alpha/(1-\alpha)} l | \gtrsim \langle l \rangle^{1/(1-\alpha)}. \tag{3.11}
\]
Then we have
\[
\|\square_{\overline{k}}^\alpha u \square_{\overline{l}}^\alpha v\|_{L^2_{t,x}} \lesssim \langle k \rangle^{(d-1)\alpha/2(1-\alpha)} \langle l \rangle^{-1/2(1-\alpha)} \ln \langle l \rangle \|\square_{\overline{k}}^\alpha u\|_{V_{\alpha}^d} \|\square_{\overline{l}}^\alpha v\|_{V_{\alpha}^d}; \tag{3.12}
\]
\[
\|\square_{\overline{k}}^\alpha \square_{\overline{l}}^\alpha v\|_{L^2_{t,x}} \lesssim \langle k \rangle^{(d-1)\alpha/2(1-\alpha)} \langle l \rangle^{-1/2(1-\alpha)} \ln \langle l \rangle \|\square_{\overline{k}}^\alpha u\|_{V_{\alpha}^d} \|\square_{\overline{l}}^\alpha v\|_{V_{\alpha}^d}; \tag{3.13}
\]
\[
\|\square_{\overline{k}}^\alpha u \square_{\overline{l}}^\alpha v\|_{L^2_{t,x}} \lesssim \langle k \rangle^{(d-1)\alpha/2(1-\alpha)} \langle l \rangle^{-1/2(1-\alpha)} \ln \langle l \rangle \|\square_{\overline{k}}^\alpha u\|_{V_{\alpha}^d} \|\square_{\overline{l}}^\alpha v\|_{V_{\alpha}^d}; \tag{3.14}
\]
\[
\|\square_{\overline{k}}^\alpha \square_{\overline{l}}^\alpha v\|_{L^2_{t,x}} \lesssim \langle k \rangle^{(d-1)\alpha/2(1-\alpha)} \langle l \rangle^{-1/2(1-\alpha)} \ln \langle l \rangle \|\square_{\overline{k}}^\alpha u\|_{V_{\alpha}^d} \|\square_{\overline{l}}^\alpha v\|_{V_{\alpha}^d}. \tag{3.15}
\]
Moreover, if $2^j \ll \langle l \rangle^{1/(1-\alpha)}$, then we have
\[
\|\Delta_j u \square_{\overline{l}}^\alpha v\|_{L^2_{t,x}} \lesssim 2^{j-d-1/2} \langle l \rangle^{-1/2(1-\alpha)} \ln \langle l \rangle \|\Delta_j u\|_{V_{\alpha}^d} \|\square_{\overline{l}}^\alpha v\|_{V_{\alpha}^d}. \tag{3.16}
\]
Notice that for $k \in \mathbb{Z}^d$, $|k| = |l|$, or $l_1 k_1 < 0$ with $|l_1| = \max_{1 \leq j \leq d} |l_j|$, (3.11) holds.

**Proof.** By Hölder’s inequality and (3.3),
\[
\|\square_{\overline{k}}^\alpha u \square_{\overline{l}}^\alpha v\|_{L^2_{t,x}} \leq \|\square_{\overline{k}}^\alpha u\|_{L^4_{t,x}} \|\square_{\overline{l}}^\alpha v\|_{L^4_{t,x}} \lesssim \langle k \rangle^{(d\alpha/4-\alpha/2)/(1-\alpha)} \langle l \rangle^{-d\alpha/4-\alpha/2/(1-\alpha)} \|\square_{\overline{k}}^\alpha u\|_{U_{\alpha}^d} \|\square_{\overline{l}}^\alpha v\|_{U_{\alpha}^d}. \tag{3.17}
\]
Using Lemma 3.1, we have
\[
\|\square_{\overline{k}}^\alpha u \square_{\overline{l}}^\alpha v\|_{L^2_{t,x}} \lesssim \langle k \rangle^{(d-1)\alpha/2(1-\alpha)} \langle l \rangle^{-1/2(1-\alpha)} \|\square_{\overline{k}}^\alpha u\|_{U_{\alpha}^d} \|\square_{\overline{l}}^\alpha v\|_{U_{\alpha}^d}. \tag{3.18}
\]
By transference principle, we have (3.12). Noticing that $|\square_{\overline{k}}^\alpha \square_{\overline{l}}^\alpha v| = |\square_{\overline{k}}^\alpha u \square_{\overline{l}}^\alpha v|$, we immediately have (3.13).
Corollary 3.2 ($\square_k^\alpha$-Bilinear estimate II). Let $0 \leq \alpha < 1$, $l \in \mathbb{Z}^d$, $|l| \gg 1$, $|k| \leq |l|$. Suppose that

$$|\langle k \rangle^{\alpha/(1-\alpha)} k + \langle l \rangle^{\alpha/(1-\alpha)} l| \gtrsim |l|^{1/(1-\alpha)}.$$  

Then we have

$$\|\square_k^\alpha u \square_l^\alpha v\|_{L^2_{t,x}} \lesssim |\langle k \rangle|^{\alpha/(2(1-\alpha))} |\langle l \rangle|^{-1/2(1-\alpha)} \ln(\|u\|_{Y^\alpha_{\Delta}^v} \|v\|_{Y^\alpha_{\Delta}^v});$$  

$$\|\square_k^\alpha u \square_l^\alpha v\|_{L^2_{t,x}} \lesssim |\langle k \rangle|^{\alpha/(2(1-\alpha))} |\langle l \rangle|^{-1/2(1-\alpha)} \ln(\|u\|_{Y^\alpha_{\Delta}^v} \|v\|_{Y^\alpha_{\Delta}^v});$$  

$$\|\square_k^\alpha u \square_l^\alpha v\|_{L^2_{t,x}} \lesssim |\langle k \rangle|^{\alpha/(2(1-\alpha))} |\langle l \rangle|^{-1/2(1-\alpha)} \ln(\|u\|_{Y^\alpha_{\Delta}^v} \|v\|_{Y^\alpha_{\Delta}^v});$$  

$$\|\square_k^\alpha u \square_l^\alpha v\|_{L^2_{t,x}} \lesssim |\langle k \rangle|^{\alpha/(2(1-\alpha))} |\langle l \rangle|^{-1/2(1-\alpha)} \ln(\|u\|_{Y^\alpha_{\Delta}^v} \|v\|_{Y^\alpha_{\Delta}^v}).$$

Moreover, if $2^j \ll |\langle l \rangle|^{1/(1-\alpha)}$, then we have

$$\|\Delta^j \square_k^\alpha v\|_{L^2_{t,x}} \lesssim 2^j \frac{d+1}{2} |\langle l \rangle|^{-1/2(1-\alpha)} \ln(\|u\|_{Y^\alpha_{\Delta}^v} \|v\|_{Y^\alpha_{\Delta}^v}).$$

Noticing that if $|k| \ll |l|$, or $k_1 l_1 > 0$ with $|l_1| = \max_{1 \leq j \leq d} |l_j|$, (3.19) holds.

4 Multi-linear Estimates

Let us write

$$\mathcal{L}(u, v) = \int_{\mathbb{R}^{d+1}} |u|^{2\kappa} u \overline{v} dxdt.$$  

Lemma 4.1. We have

$$|\mathcal{L}(u, v)| \lesssim \prod_{j=1}^{2\kappa+1} \|u\|_{X_{\Delta}^{\alpha}} \|v\|_{Y_{\Delta}^{-s,\alpha}}.$$  

For the proof of Lemma 4.1 we will only consider the case $s = s_k^+$, since the case of larger $s$ is easier to handle than that of $s = s_k$. For convenience, we write

$$u_1 = u_3 = ... = u_{2\kappa+1} = u, \quad u_2 = u_4 = ... = u_{2\kappa} = \overline{u}. \quad (4.3)$$

Applying the $\alpha$-decompositions,

$$\mathcal{L}(u, v) = \sum_{k^{(1)}, ..., k^{(2\kappa+1)} \in \mathbb{Z}^d} \int_{\mathbb{R}^{d+1}} \prod_{\ell=1}^{2\kappa+1} \square_k^{(\alpha)} u \overline{v} dxdt.$$  

We further denote for given $k^{(1)}, ..., k^{(2\kappa+1)}$,

$$K_j(k^{(1)}, ..., k^{(2\kappa+1)}) = \left| \sum_{\ell=1}^{2\kappa+1} (k^{(\ell)})^{\alpha/(1-\alpha)} k^{(\ell)}_j \right|, \quad j = 1, ..., 2\kappa+1,$$

$$K := K(k^{(1)}, ..., k^{(2\kappa+1)}) = \max_{j=1, ..., d} K_j(k^{(1)}, ..., k^{(2\kappa+1)}). \quad (4.6)$$
Put
\[
\Omega_0 = \left\{ (k^{(1)}, \ldots, k^{(2\kappa+1)}) \in \mathbb{Z}^{d(2\kappa+1)} : K \lesssim \sum_{\ell=1}^{2\kappa+1} \langle k^{(\ell)} \rangle^{\alpha/(1-\alpha)} \right\}, \tag{4.7}
\]
\[
\Omega_1 = \left\{ (k^{(1)}, \ldots, k^{(2\kappa+1)}) \in \mathbb{Z}^{d(2\kappa+1)} : K \sim \sum_{\ell=1}^{2\kappa+1} \langle k^{(\ell)} \rangle^{1/(1-\alpha)} \right\}, \tag{4.8}
\]
\[
\Omega_2 = \left\{ (k^{(1)}, \ldots, k^{(2\kappa+1)}) \in \mathbb{Z}^{d(2\kappa+1)} : \sum_{\ell=1}^{2\kappa+1} \langle k^{(\ell)} \rangle^{\alpha/(1-\alpha)} \ll K \ll \sum_{\ell=1}^{2\kappa+1} \langle k^{(\ell)} \rangle^{1/(1-\alpha)} \right\}. \tag{4.9}
\]

We can divide \( L(u, v) \) into the following three parts:
\[
L_i(u, v) = \sum_{(k^{(\ell)}) \in \Omega_i, \ k \in \mathbb{Z}^d} \int_{\mathbb{R}^{d+1}} \prod_{\ell=1}^{2\kappa+1} \square_{k^{(\ell)}}^\alpha u_t v dx dt. \tag{4.10}
\]

For convenience, we will denote \((k^{(\ell)}) := (k^{(\ell)})_{\ell=1}^{2\kappa+1} = (k^{(1)}, \ldots, k^{(2\kappa+1)})\) in the sequel.

### 4.1 Estimates of \( L_1(u, v) \)

Now we consider the estimate of \( L_1(u, v) \), which is easier than those of \( L_0(u, v) \) and \( L_2(u, v) \).

Using the \( \alpha \)-decomposition, we can rewrite \( L_1(u, v) \) as
\[
L_1(u, v) = \sum_{(k^{(\ell)}) \in \Omega_1, \ k \in \mathbb{Z}^d} \int_{\mathbb{R}^{d+1}} \prod_{\ell=1}^{2\kappa+1} \square_{k^{(\ell)}}^\alpha u_t v dx dt. \tag{4.11}
\]

So, one needs to control the right hand side of (4.11) and we prove that

**Lemma 4.2.**
\[
|L_1(u, v)| \lesssim \|v\|_{\tilde{Y}^{-s, \alpha}_{\Delta}} \|u\|_{X_{\Delta}^{\kappa+1}}^{2\kappa+1}. \tag{4.12}
\]

. First, we have

**Lemma 4.3.** If \( L_1(u, v) \) in (4.11) does not equal to zero, we must have
\[
\langle k \rangle^{\alpha/(1-\alpha)}(k_j - C) \leq \sum_{\ell=1}^{2\kappa+1} \langle k^{(\ell)} \rangle^{\alpha/(1-\alpha)}(k_j^{(\ell)} + C), \tag{4.13}
\]
\[
\langle k \rangle^{\alpha/(1-\alpha)}(k_j + C) \geq \sum_{\ell=1}^{2\kappa+1} \langle k^{(\ell)} \rangle^{\alpha/(1-\alpha)}(k_j^{(\ell)} - C). \tag{4.14}
\]

**Proof.** Using the fact that
\[
\int_{\mathbb{R}^{d+1}} \prod_{\ell=1}^{2\kappa+1} \square_{k^{(\ell)}}^\alpha u_t v dx dt = \int_{\mathbb{R}} \mathcal{F} \left( \prod_{\ell=1}^{2\kappa+1} \square_{k^{(\ell)}}^\alpha u_t v \right)(0, t) dt,
\]

15
we easily see the result. 

According to the duality argument, one needs to take $\ell^{\infty}$ norm on $k \in \mathbb{Z}^d$, we must remove the summation on $k \in \mathbb{Z}^d$ in the right hand side of (4.4). For convenience, we write

$$\Lambda((k^{(\ell)})_{\ell=1}^{2\kappa+1}) = \{k \in \mathbb{Z}^d : (4.13) \text{ and } (4.14) \text{ are satisfied}\}. \quad (4.15)$$

So, we can further rewrite $\mathcal{L}_1(u, v)$ as

$$\mathcal{L}_1(u, v) = \sum_{(k^{(\ell)})_{\ell=1}^{2\kappa+1} \in \Omega_1, \ k \in \Lambda((k^{(\ell)})_{\ell=1}^{2\kappa+1})} \int_{\mathbb{R}^{d+1}} \prod_{\ell=1}^{2\kappa+1} \alpha_k^{(\ell)} u_k^{(\ell)} \bar{v} dx dt. \quad (4.16)$$

For convenience, we denote by $\#A$ the number of the elements in $A$. It seems necessary to calculate $\#\Lambda((k^{(\ell)})_{\ell=1}^{2\kappa+1})$.

\textbf{Lemma 4.4.} Suppose that $(k^{(\ell)})_{\ell=1}^{2\kappa+1} \in \Omega_1$, $k \in \Lambda((k^{(\ell)})_{\ell=1}^{2\kappa+1})$. Then we have

$$K \sim \langle k \rangle^{1/(1-\alpha)}, \quad \#\Lambda((k^{(\ell)})_{\ell=1}^{2\kappa+1}) \lesssim 1. \quad (4.17)$$

\textbf{Proof.} It is a straightforward consequence of (4.13) and (4.14). \hfill $\square$

\textbf{Step 1.} We consider the estimates of $\mathcal{L}_1(u, v)$. We can assume that $K \gg 1$, since all of the summations are finite terms if $K \lesssim 1$. Since $k^{(1)}, k^{(3)}, \ldots, k^{(2\kappa+1)}$ have the equal positions, also $k^{(2)}, k^{(4)}, \ldots, k^{(2\kappa)}$ play the same roles, we can assume without loss of generality that

$$|k^{(1)}| \geq |k^{(3)}| \geq \ldots \geq |k^{(2\kappa+1)}|, \quad |k^{(2)}| \geq |k^{(4)}| \geq \ldots \geq |k^{(2\kappa)}|. \quad (4.18)$$

By (4.13) and (4.11), for $(k^{(1)}, \ldots, k^{(2\kappa+1)}) \in \Omega_1$, there exists $j \in \{1, \ldots, d\}$ such that

$$\langle k_j \rangle \sim \langle k \rangle \sim \langle k^{(1)} \rangle \sim \langle k^{(1)} \rangle, \quad \text{or } \langle k_j \rangle \sim \langle k \rangle \sim \langle k^{(2)} \rangle \sim \langle k^{(2)} \rangle. \quad (4.19)$$

\textbf{Case 1.} We consider the case

$$\langle k_j \rangle \sim \langle k \rangle \sim \langle k^{(1)} \rangle \sim \langle k^{(1)} \rangle. \quad (4.20)$$

We can assume that $j = 1$ in (4.20).

\textbf{Case 1.1.} We consider the case $|k^{(2)}|, |k^{(3)}| \ll |k^{(1)}|$. In $\mathcal{L}_1(u, v)$, using Hölder’s inequality,
bilinear and Strichartz’ estimates, we can bound
\[
\int_{\mathbb{R}^{d+1}} \prod_{\ell=1}^{2\kappa+1} \Box_{k^{(\ell)}}^\alpha u_{\ell} \Box_{k^{(\ell)}}^\alpha v dx dt \\
\leq \| \Box_{k^{(1)}}^\alpha v \|_{L^2_x(t)} \| \Box_{k^{(1)}}^\alpha u_1 \Box_{k^{(2)}}^\alpha u_2 \|_{L^2_x(t)} \prod_{\ell=4}^{2\kappa+1} \| \Box_{k^{(\ell)}}^\alpha u_{\ell} \|_{L^\infty_x(t)} \\
\lesssim \ln \langle k^{(1)} \rangle \langle k^{(1)} \rangle^{-1/(1-\alpha)} \langle k^{(2)} \rangle^{(d-1)\alpha/2/(1-\alpha)} \langle k^{(3)} \rangle^{(d-1)\alpha/2/(1-\alpha)} \| \Box_{k^{(\ell)}}^\alpha v \|_{V^2_\Delta} \| \Box_{k^{(\ell)}}^\alpha u_{\ell} \|_{U^2_\Delta} \\
\times \prod_{\ell=4}^{2\kappa+1} \langle k^{(\ell)} \rangle^s/(1-\alpha) \| \Box_{\pm k^{(\ell)}}^\alpha u_{\ell} \|_{U^2_\Delta} \\
\lesssim \ln \langle k^{(1)} \rangle \langle k^{(1)} \rangle^{-1/(1-\alpha)} \langle k^{(2)} \rangle^{(d-1)\alpha/2-s/(1-\alpha)} \langle k^{(3)} \rangle^{(d-1)\alpha/2-s/(1-\alpha)} \langle k \rangle^{-s/(1-\alpha)} \| \Box_{\pm k^{(\ell)}}^\alpha u_{\ell} \|_{U^2_\Delta} \\
\times \langle (k^{(4)}(\kappa-1)(d\alpha/2-s)/(1-\alpha) \rangle^{(d-1)\alpha/2-s)/(1-\alpha)} \prod_{\ell=1}^{2\kappa+1} \langle k^{(\ell)} \rangle^s/(1-\alpha) \| \Box_{\pm k^{(\ell)}}^\alpha u_{\ell} \|_{U^2_\Delta},
\] 
where we used (4.18). Further, by (4.18),
\[
\langle k^{(4)}(\kappa-1)(d\alpha/2-s)/(1-\alpha) \rangle^{(d-1)\alpha/2-s)/(1-\alpha)} \langle k^{(2)} \rangle^{(d-1)\alpha/2-s)/(1-\alpha)} \leq \langle k^{(2)} \rangle^{\kappa(d\alpha/2-s)-\alpha/2)/(1-\alpha)}.
\]
Notice that
\[
\kappa(d\alpha/2-s) - \alpha/2 \geq 0.
\]
By (4.29) we have \#\Lambda(k^{(1)}, \ldots, k^{(2\kappa+1)}) \lesssim 1 if \( (k^{(1)}, \ldots, k^{(2\kappa+1)}) \in \Omega_1 \). If \( |k^{(2)}|, |k^{(3)}| \ll |k^{(1)}| \) in the summation of \( \mathcal{L}_1(u, v) \), noticing that \(-(1+\alpha) + 2\kappa(d\alpha/2-s) < 0 \), we have
\[
\mathcal{L}_1(u, v) \lesssim \sum_{(k^{(\ell)})_{\ell=1}^{2\kappa+1} \in \Omega_1, \ k \in \Lambda((k^{(\ell)})_{\ell=1}^{2\kappa+1})} \ln \langle k^{(1)} \rangle \langle k^{(1)} \rangle^{-1/(1-\alpha)} \langle k^{(2)} \rangle^{(d-1)\alpha/2-s/(1-\alpha)} \langle k^{(3)} \rangle^{(d-1)\alpha/2-s/(1-\alpha)} \langle k \rangle^{-s/(1-\alpha)} \| \Box_{\pm k^{(\ell)}}^\alpha u_{\ell} \|_{U^2_\Delta} \\
\times \langle k \rangle^{-s/(1-\alpha)} \| \Box_{\pm k^{(\ell)}}^\alpha v \|_{V^2_\Delta} \prod_{\ell=1}^{2\kappa+1} \langle k^{(\ell)} \rangle^s/(1-\alpha) \| \Box_{\pm k^{(\ell)}}^\alpha u_{\ell} \|_{U^2_\Delta} \\
\lesssim \| v \|_{Y_{-s}^{\Delta,a}} \| u \|_{X_{-s}^{\Delta,2\kappa+1}}.
\]
\[
(4.23)
\]
**Case 1.2.** We consider the case \( |k^{(2)}| \ll |k^{(3)}| \sim |k^{(1)}| \) or \( |k^{(3)}| \ll |k^{(2)}| \sim |k^{(1)}| \). It suffices
to consider the former case. Using Hölder’s, bilinear and Strichartz estimates,

\[
\int_{\mathbb{R}^{d+1}} 2\kappa+1 \prod_{\ell=1}^{2\kappa+1} \Box_{k(\ell)}^\alpha u_{\ell} \Box_k v dx dt \\
\leq \|\Box_k^\alpha v\|_{L^2_{x,t}} \|\Box_{k(1)}^\alpha u_1\|_{L^4_{x,t}} \|\Box_{k(3)}^\alpha u_3\|_{L^4_{x,t}} \prod_{\ell=4}^{2\kappa+1} \|\Box_{k(\ell)}^\alpha u_{\ell}\|_{L^\infty_{x,t}} \\
\leq \ln\langle k(1) \rangle \langle k(2) \rangle \langle k(3) \rangle \langle k(4) \rangle \langle k(5) \rangle \langle k(6) \rangle \langle k(7) \rangle \langle k(8) \rangle \|\Box_k^\alpha v\|_{V^2_\Delta} \|\Box_{k(1)}^\alpha u\|_{U^2_\Delta} \\
\times \prod_{\ell=4}^{2\kappa+1} \langle k(\ell) \rangle \langle k(\ell) \rangle \langle k(\ell) \rangle \langle k(\ell) \rangle \langle k(\ell) \rangle \langle k(\ell) \rangle \langle k(\ell) \rangle \langle k(\ell) \rangle \|\Box_{k(\ell)}^\alpha u_{\ell}\|_{U^2_\Delta}.
\]

Noticing that \(|k(1)| \geq |k(2)|, -\alpha/2 + \kappa(d\alpha/2 - s) > 0, -1/2 - 3\alpha/2 + 2\kappa(d\alpha/2 - s) < 0,

and using a similar way to Case 1.1, we can estimate \(L_1(u, v)\) by

\[
L_1(u, v) \lesssim \|v\|_{Y_{-\alpha, \alpha}} \|u\|_{X_{\alpha, \alpha}^{2\kappa+1}}.
\]

**Case 1.3.** We consider the case \(|k(2)| \sim |k(3)| \sim |k(1)|.** Using Hölder’s and Strichartz estimates,

\[
\int_{\mathbb{R}^{d+1}} 2\kappa+1 \prod_{\ell=1}^{2\kappa+1} \Box_{k(\ell)}^\alpha u_{\ell} \Box_k v dx dt \\
\leq \|\Box_k^\alpha v\|_{L^4_{x,t}} \|\Box_{k(1)}^\alpha u_1\|_{L^4_{x,t}} \|\Box_{k(3)}^\alpha u_3\|_{L^4_{x,t}} \prod_{\ell=4}^{2\kappa+1} \|\Box_{k(\ell)}^\alpha u_{\ell}\|_{L^\infty_{x,t}} \\
\leq \langle k(1) \rangle \langle k(2) \rangle \langle k(3) \rangle \langle k(4) \rangle \langle k(5) \rangle \langle k(6) \rangle \langle k(7) \rangle \langle k(8) \rangle \|\Box_k^\alpha v\|_{V^2_\Delta} \|\Box_{k(1)}^\alpha u\|_{U^2_\Delta} \\
\times \prod_{\ell=4}^{2\kappa+1} \langle k(\ell) \rangle \langle k(\ell) \rangle \langle k(\ell) \rangle \langle k(\ell) \rangle \langle k(\ell) \rangle \langle k(\ell) \rangle \langle k(\ell) \rangle \langle k(\ell) \rangle \|\Box_{k(\ell)}^\alpha u_{\ell}\|_{U^2_\Delta}.
\]

(4.26)
Similar to Case 1.1,

\[ L_1(u, v) \lesssim \sum_{\substack{\ell = 1 \text{ to } 2\kappa+1 \atop \{k^{(\ell)}\}_{\ell = 1}^{2\kappa+1} \in \Omega_1, \ k \in \Lambda(\{k^{(\ell)}\}_{\ell = 1}^{2\kappa+1})}} \langle k^{(\ell)} \rangle^{2\alpha/(1-\alpha) + 2\kappa(d\alpha/2 - s)/(1-\alpha)} \langle k^{(\ell)} \rangle^{-s/(1-\alpha)} \| \Delta^\alpha u \|_{V_{\Delta}^2} \]

\[ \lesssim \| v \|_{Y_{\Delta}^{-s}} \sum_{\substack{k^{(1)}, \ldots, k^{(2\kappa+1)} \in \mathbb{Z}^d}} \prod_{\ell = 1}^{2\kappa+1} \langle k^{(\ell)} \rangle^{s/(1-\alpha)} \| \Delta^\alpha u \|_{V_{\Delta}^2} \]

\[ \lesssim \| v \|_{Y_{\Delta}^{-s,\alpha}} \| u \|_{X_{\Delta}^{2\kappa+1,\alpha}}. \tag{4.27} \]

Case 2. If \( \langle k_j \rangle \sim \langle k \rangle \sim \langle k_j^{(2)} \rangle \), the arguments are similar to Case 1 and we omit the details.

4.2 Estimates of \( L_0(u, v) \)

When we estimate \( L_1(u, v) \), we see that \#\( \Lambda(\{k^{(\ell)}\}_{\ell = 1}^{2\kappa+1}) \sim 1 \), which leads to the summation over all \( k \in \Lambda(\{k^{(\ell)}\}_{\ell = 1}^{2\kappa+1}) \) has no contribution to the regularity index \( s > s_\kappa \). However, in the case \( \langle k^{(\ell)} \rangle_{\ell = 1}^{2\kappa+1} \in \Omega_0 \cup \Omega_2 \), \( \Lambda(\{k^{(\ell)}\}_{\ell = 1}^{2\kappa+1}) \) is a bit complicated and we have

Lemma 4.5. We have

(i) If \( K \lesssim \sum_{\ell = 1}^{2\kappa+1} \langle k^{(\ell)} \rangle^{\alpha/(1-\alpha)} \), then

\[ \# \Lambda(\{k^{(\ell)}\}_{\ell = 1}^{2\kappa+1}) \lesssim \sum_{\ell = 1}^{2\kappa+1} \langle k^{(\ell)} \rangle^{d\alpha}. \] \tag{4.28} \]

(ii) If \( K \gg \sum_{\ell = 1}^{2\kappa+1} \langle k^{(\ell)} \rangle^{\alpha/(1-\alpha)} \), then we have

\[ \# \Lambda(\{k^{(\ell)}\}_{\ell = 1}^{2\kappa+1}) \lesssim \sum_{\ell = 1}^{2\kappa+1} \langle k^{(\ell)} \rangle^{d\alpha/(1-\alpha) / K^{d\alpha}}. \] \tag{4.29} \]

Lemma 4.5 is a simple consequence of \( \langle 4.13 \rangle \) and \( \langle 4.14 \rangle \) and we omit the details for the proof. In order to estimate \( L_0(u, v) \), a straightforward idea is to follow the same way as in the estimates of \( L_1(u, v) \) and use Lemma 4.5 to calculate the number of \( \Lambda(\{k^{(\ell)}\}_{\ell = 1}^{2\kappa+1}) \). Unfortunately, the summation on \( k \in \Lambda(\{k^{(\ell)}\}_{\ell = 1}^{2\kappa+1}) \) will make troubles to the optimal regularity index \( s > s_\kappa = d\alpha/2 - \alpha/\kappa \). So, we need to use a different way to control \( L_0(u, v) \) and we use the dyadic decomposition on \( \bar{v} \). We can rewrite \( L_0(u, v) \) as

\[ L_0(u, v) = \sum_{\substack{(k^{(\ell)})_{\ell = 1}^{2\kappa+1} \in \Omega_0, \ j \in \mathbb{Z}^d \atop (k^{(\ell)})_{\ell = 1}^{2\kappa+1} \in \Omega_2}} \int_{\mathbb{R}^{d+1}} \prod_{\ell = 1}^{2\kappa+1} \Delta^\alpha u \Delta_j v dx dt. \] \tag{4.30} 

So, one needs to estimate the right hand side of \( \langle 4.30 \rangle \) and we prove that
Lemma 4.6.

\[ |\mathcal{L}_0(u, v)| \lesssim \|v\|_{Y_{-\kappa,\alpha}} \|u\|_{X_{-\kappa,\alpha}^\Lambda} \]  \hspace{1cm} (4.31)

Lemma 4.7. Let \((k^{(\ell)})_{\ell=1}^{2\kappa+1} \in \Omega_0\). If

\[ \int_{\mathbb{R}^{d+1}} \prod_{\ell=1}^{2\kappa+1} \Box_{k^{(\ell)}}^\alpha u_\ell \Delta_j v \, dx \, dt \neq 0, \]

then we have

\[ j \in \Lambda((k^{(\ell)})_{\ell=1}^{2\kappa+1}) := \left\{ j \in \mathbb{Z}_+ : 2^j \lesssim \bigvee_{\ell=1}^{2\kappa+1} (k^{(\ell)})^{\alpha/(1-\alpha)} \right\}. \]  \hspace{1cm} (4.32)

Note here and below in this subsection the notation \(\Lambda((k^{(\ell)})_{\ell=1}^{2\kappa+1})\) is different from (4.15) in the previous subsection. The proof of Lemma 4.7 proceeds in the same way as in Lemma 4.3 and we omit the details. Hence, we have

\[ \mathcal{L}_0(u, v) = \sum_{(k^{(\ell)})_{\ell=1}^{2\kappa+1} \in \Omega_0, j \in \Lambda((k^{(\ell)})_{\ell=1}^{2\kappa+1})} \int_{\mathbb{R}^{d+1}} \prod_{\ell=1}^{2\kappa+1} \Box_{k^{(\ell)}}^\alpha u_\ell \Delta_j v \, dx \, dt. \]  \hspace{1cm} (4.33)

One easily sees that

\[ \#\Lambda((k^{(\ell)})_{\ell=1}^{2\kappa+1}) \lesssim \max_{1 \leq \ell \leq 2\kappa+1} \ln \langle k^{(\ell)} \rangle, \]

it follows that the summation on \(j \in \Lambda((k^{(\ell)})_{\ell=1}^{2\kappa+1})\) in (4.33) is bounded by \(O(\max_{1 \leq \ell \leq 2\kappa+1} \ln \langle k^{(\ell)} \rangle)\), which is much less than that of any \(\alpha\)-decomposition. This is the main reason why we try to use the dyadic decomposition with respect to \(\pi\).

We can assume that \(\max_{1 \leq \ell \leq 2\kappa+1} |k^{(\ell)}| \gg 1\). In the opposite case all \(u_\ell\) and \(\pi\) have lower frequency, (4.31) can be easily obtained. As in (4.18), we can assume that

\[ |k^{(1)}| \geq |k^{(3)}| \geq \ldots \geq |k^{(2\kappa+1)}|, \quad |k^{(2)}| \geq |k^{(4)}| \geq \ldots \geq |k^{(2\kappa)}|. \]  \hspace{1cm} (4.34)

We divide the proof of Lemma 4.6 into the following three cases.

Case 1. We consider \(|k^{(1)}| = \max_{1 \leq \ell \leq 2\kappa+1} |k^{(\ell)}| \gg 1\). By (4.31) and \((k^{(1)}, \ldots, k^{(2\kappa+1)}) \in \Omega_0\), we have

\[ \langle k^{(1)} \rangle \sim \langle k^{(2)} \rangle, \quad \text{or} \quad \langle k^{(1)} \rangle \sim \langle k^{(3)} \rangle. \]  \hspace{1cm} (4.35)

If not, then \(|k^{(\ell)}| \ll |k^{(1)}|\) for all \(\ell = 2, \ldots, 2\kappa+1\). Assume that \(|k^{(i)}| \geq |k^{(1)}|\) for \(i = 2, \ldots, d\). Then

\[ 2^j \geq \langle k^{(1)} \rangle^{\alpha/(1-\alpha)} (|k^{(1)}| - C) - \sum_{\ell=2}^{2\kappa+1} \langle k^{(\ell)} \rangle^{\alpha/(1-\alpha)} (|k^{(1)}| + C) \gtrsim \langle k^{(1)} \rangle^{1/(1-\alpha)} \gg \bigvee_{\ell=1}^{2\kappa+1} \langle k^{(\ell)} \rangle^{\alpha/(1-\alpha)}. \]
A contradiction. The case $\langle k^{(1)} \rangle \sim \langle k^{(3)} \rangle$ is similar to the case $\langle k^{(1)} \rangle \sim \langle k^{(2)} \rangle$, it suffices to consider the case $\langle k^{(1)} \rangle \sim \langle k^{(2)} \rangle$.

**Case 1.1.** We consider the case $\langle k^{(1)} \rangle \sim \langle k^{(2)} \rangle \gg \langle k^{(\ell)} \rangle$, $\ell = 3, \ldots, 2\kappa + 1$. By Hölder’s inequality, bilinear estimates, we have

$$
\int_{\mathbb{R}^{d+1}} 2^{k+1} \prod_{\ell=1}^{2\kappa+1} \Box_{k(\ell)}^\alpha u_\ell \Delta_j v dx dt
\leq \|\Delta_j v \Box_{k(1)}^\alpha u_1\|_L^2_x \|\Box_{k(2)}^\alpha u_2 \Box_{k(3)}^\alpha u_3\|_L^2_x \prod_{\ell=4}^{2\kappa+1} \|\Box_{k(\ell)}^\alpha u_\ell\|_{L^\infty_x}
\lesssim \ln \langle k^{(1)} \rangle \langle k^{(1)} \rangle^{-1/(1-\alpha)} 2^{j(d-1)/2} \langle k^{(3)} \langle d-1 \rangle \alpha/2 \rangle \|\Delta_j v\|_{V^2_\Delta} \|\Box_{k(1)}^\alpha u\|_{U^2_\Delta} \|\Box_{k(2)}^\alpha u\|_{U^2_\Delta} \|\Box_{k(3)}^\alpha u\|_{U^2_\Delta}
\times \prod_{\ell=4}^{2\kappa+1} \langle k^{(\ell)} \rangle^{d/(1-\alpha)} \|\Box_{\pm k(\ell)}^\alpha u\|_{U^2_\Delta}
\lesssim \ln \langle k^{(1)} \rangle \langle k^{(1)} \rangle^{-(1+2s)/\alpha} 2^{j((d-1)/2+s+1(d-1)/2)} 2^{j(-s-d(1-\alpha)/2)} \|\Delta_j v\|_{V^2_\Delta}
\times \langle k^{(3)} \rangle \langle k^{(4)} \rangle \langle 2\kappa+1 \rangle \prod_{\ell=1}^{\kappa-1} \langle k^{(\ell)} \rangle^{s/(1-\alpha)} \|\Box_{\pm k(\ell)}^\alpha u\|_{U^2_\Delta}. 
$$

(4.36)

Since $\kappa(d\alpha/2-s)-\alpha/2 > 0$ and $|k^{(3)}|, |k^{(4)}| \leq |k^{(1)}|$ in the summation of $\mathcal{L}_0(u, v)$, noticing that

$$
\sum_{j \in \Lambda((k^{(\ell)})_{\ell=1}^{2\kappa+1})} 2^{j((d-1)/2+s+1(d-1)/2)} \lesssim \langle k^{(1)} \rangle \alpha((d-1)/2+s+1(d-1)/2)/(1-\alpha),
$$

(4.37)

we have

$$
\mathcal{L}_0(u, v) \lesssim \sum_{\langle (k^{(\ell)})_{\ell=1}^{2\kappa+1} \rangle \in \Omega_0, \ j \in \Lambda((k^{(\ell)})_{\ell=1}^{2\kappa+1})} \ln \langle k^{(1)} \rangle \langle k^{(1)} \rangle^{-1-2s-\alpha/2+(2\kappa-1)(d\alpha/2-s))}/(1-\alpha)
\times \prod_{\ell=1}^{2\kappa+1} \langle k^{(\ell)} \rangle^{s/(1-\alpha)} \|\Box_{\pm k(\ell)}^\alpha u\|_{U^2_\Delta} 2^{j((d-1)/2+s+1(d-1)/2)} \|v\|_{Y^s_{-s-d(1-\alpha)/2,1}}
\lesssim \|v\|_{Y^s_{-s-d(1-\alpha)/2,1}} \sum_{\langle k^{(\ell)} \rangle \in \mathbb{Z}^{d(2\kappa+1)}} \ln \langle k^{(1)} \rangle \langle k^{(1)} \rangle^{(2\kappa+1-\alpha)(d\alpha/2-s)/2-\alpha)}/(1-\alpha)
\times \prod_{\ell=1}^{2\kappa+1} \langle k^{(\ell)} \rangle^{s/(1-\alpha)} \|\Box_{\pm k(\ell)}^\alpha u\|_{U^2_\Delta}
\lesssim \|v\|_{Y^s_{-s-d(1-\alpha)/2,1}} \|u\|^{2\kappa+1}_{X^\alpha_{\Delta}}.
$$

(4.38)

**Case 1.2.** We consider the case $\langle k^{(1)} \rangle \sim \langle k^{(2)} \rangle \sim \langle k^{(3)} \rangle$, $\kappa \geq 2$. By Hölder’s inequality,
Using the bilinear and the Strichartz estimates we have
\[
\int_{\mathbb{R}^{d+1}} \prod_{\ell=1}^{2\kappa+1} \Box_{k(\ell)}^\alpha u_\ell \Delta_j v dx dt 
\leq \| \Delta_j v \|_{L^2_{x,t}} \| \Box_{k(1)}^\alpha u_1 \|_{L^4_{x,t}} \| \Box_{k(2)}^\alpha u_2 \|_{L^4_{x,t}} \| \Box_{k(3)}^\alpha u_3 \|_{L^4_{x,t}} \prod_{\ell=4}^{2\kappa+1} \| \Box_{k(\ell)}^\alpha u_\ell \|_{L^\infty_{x,t}} 
\lesssim \ln \langle k(1) \rangle \langle k(1) \rangle^{-1/2(1-\alpha)} 2^j(1/2) \langle k(2) \rangle (d\alpha/4-\alpha/2)/(1-\alpha) \langle k(3) \rangle (d\alpha/4-\alpha/2)/(1-\alpha) \| \Delta_j v \|_{Y_\Delta^2} \| \Box_{k(1)}^\alpha u \|_{U_\Delta^2} 
\times \| \Box_{-k(2)}^\alpha u \|_{U_\Delta^2} \| \Box_{k(3)}^\alpha u \|_{U_\Delta^2} \prod_{\ell=4}^{2\kappa+1} \langle k(\ell) \rangle^{d\alpha/(1-\alpha)} \| \Box_{\pm k(\ell)}^\alpha u_\ell \|_{U_\Delta^2} 
\lesssim 2^j ((1/2+s-d(1-\alpha)/2) \| \Delta_j v \|_{Y_\Delta^2} 
\times \ln \langle k(1) \rangle \langle k(1) \rangle^{-1/2-2s-\alpha+(2\kappa-1) (d\alpha/2-s)}/(1-\alpha) \prod_{\ell=1}^{2\kappa+1} \langle k(\ell) \rangle^{s/(1-\alpha)} \| \Box_{\pm k(\ell)}^\alpha u_\ell \|_{U_\Delta^2}.
\] (4.39)

In the case \(\kappa \geq 2\), noticing (4.37), we have we have
\[
\mathcal{L}_0(u, v) \lesssim \sum_{(\ell, k) \in \Omega_0, j \in \Lambda, \langle k(\ell) \rangle^{2\kappa + 1}} \ln \langle k(1) \rangle \langle k(1) \rangle^{-1-2s-\alpha+(2\kappa-1)(d\alpha/2-s)}/(1-\alpha) 
\times \prod_{\ell=1}^{2\kappa+1} \langle k(\ell) \rangle^{s/(1-\alpha)} \| \Box_{\pm k(\ell)}^\alpha u_\ell \|_{U_\Delta^2} 2^j ((1/2+s-d(1-\alpha)/2) \| v \|_{Y_\Delta^{-s-d(1-\alpha)/2, 1}} 
\lesssim \| v \|_{Y_\Delta^{-s-d(1-\alpha)/2, 1}} \sum_{(\ell, k) \in \Lambda, \langle k(\ell) \rangle^{2\kappa + 1}} \ln \langle k(1) \rangle \langle k(1) \rangle^{-1-2s-\alpha+(2\kappa-1)(d\alpha/2-s)-1/2-3\alpha/2}/(1-\alpha) 
\times \prod_{\ell=1}^{2\kappa+1} \langle k(\ell) \rangle^{s/(1-\alpha)} \| \Box_{\pm k(\ell)}^\alpha u_\ell \|_{U_\Delta^2} 
\lesssim \| v \|_{Y_\Delta^{-s-d(1-\alpha)/2, 1}} \| u \|_{X_\Delta^{2\kappa + 1, \alpha}}^\cdot. \] (4.40)

If \(\kappa = 1\), we will give the proof in the next Section.

Case 1.3. If \(\langle k(1) \rangle \sim \langle k(2) \rangle \sim \langle k(4) \rangle, \kappa \geq 2\), one can use the same way as Case 1.2 to show the result and we omit the details.

Case 2. We consider \(|k(2)| = \max_{1 \leq \ell \leq 2\kappa+1} |k(\ell)| \gg 1\). The argument is similar to Case 1.

4.3 Estimates of \(\mathcal{L}_2(u, v)\)

If \(\langle k(\ell) \rangle_{\ell=1}^{2\kappa+1} \in \Omega_2\), we see that there exists \(0 < \theta < 1\) verifying \(K = (\bigvee_{\ell=1}^{2\kappa+1} \langle k(\ell) \rangle^{-1})^{-1-\theta+\alpha/\theta}\). Put
\[
\tilde{\alpha} = \frac{\alpha}{1-\theta+\alpha \theta}.
\] (4.41)

Using the \(\tilde{\alpha}\)-decomposition to \(\mathcal{T}\), one can rewrite \(\mathcal{L}_2(u, v)\) as
\[
\mathcal{L}_2(u, v) = \sum_{k \in \mathbb{Z}^d, (k^{(\ell)}) \in \Omega_2} \int_{\mathbb{R}^{d+1}} \prod_{\ell=1}^{2\kappa+1} \Box_{k^{(\ell)}}^\alpha u_{k^{(\ell)}} v dxdt. \tag{4.42}
\]

So, one needs to control the right hand side of (4.42) and we prove that

**Lemma 4.8.**

\[
|\mathcal{L}_2(u, v)| \lesssim \|v\|_{\mathcal{M}^{s,a}_\Delta} \|u\|_{\mathcal{M}^{\kappa,a}_\Delta}^{2\kappa+1} \tag{4.43}
\]

First, we have

**Lemma 4.9.** If \(\mathcal{L}_2(u, v)\) in (4.42) does not equal to zero, we must have

\[
\langle k \rangle^{\alpha/(1-\tilde{\alpha})} (k_j - C) \leq \sum_{\ell=1}^{2\kappa+1} \langle k^{(\ell)} \rangle^{\alpha/(1-\alpha)} (k_j^{(\ell)} + C), \tag{4.44}
\]

\[
\langle k \rangle^{\alpha/(1-\tilde{\alpha})} (k_j + C) \geq \sum_{\ell=1}^{2\kappa+1} \langle k^{(\ell)} \rangle^{\alpha/(1-\alpha)} (k_j^{(\ell)} - C). \tag{4.45}
\]

**Proof.** See Lemma 4.3 \(\square\)

The idea to use the \(\tilde{\alpha}\)-decomposition is similar to the cases as in handling \(\mathcal{L}_0(u, v)\), one needs to remove the summation on \(k \in \mathbb{Z}^d\) in the right hand side of (4.42). For convenience, we write

\[
\Lambda((k^{(\ell)})_{\ell=1}^{2\kappa+1}) = \{k \in \mathbb{Z}^d : (4.44) \text{ and } (4.45) \text{ are satisfied}\}. \tag{4.46}
\]

Note here \(\Lambda((k^{(\ell)})_{\ell=1}^{2\kappa+1})\) has different meaning as in the previous two subsections. So, we can further rewrite \(\mathcal{L}_2(u, v)\) as

\[
\mathcal{L}_1(u, v) = \sum_{(k^{(\ell)})_{\ell=1}^{2\kappa+1} \in \Omega_2, \ k \in \Lambda((k^{(\ell)})_{\ell=1}^{2\kappa+1})} \int_{\mathbb{R}^{d+1}} \prod_{\ell=1}^{2\kappa+1} \Box_{k^{(\ell)}}^\alpha u_{k^{(\ell)}} v dxdt. \tag{4.47}
\]

**Lemma 4.10.** Let \((k^{(\ell)})_{\ell=1}^{2\kappa+1} \in \Omega_2\). We have \#\(\Lambda((k^{(\ell)})_{\ell=1}^{2\kappa+1})\) \(\lesssim 1\) and \(K \sim \langle k \rangle^{1/(1-\tilde{\alpha})}\) for any \(k \in \Lambda((k^{(\ell)})_{\ell=1}^{2\kappa+1})\).

**Proof.** First, we show that \(K \sim \langle k \rangle^{1/(1-\tilde{\alpha})}\) for any \(k \in \Lambda((k^{(\ell)})_{\ell=1}^{2\kappa+1})\). We can assume that \(K_1 = \max_{1 \leq i \leq 2\kappa+1} K_i = K\). Since \((k^{(\ell)})_{\ell=1}^{2\kappa+1} \in \Omega_2\), we see that

\[
\left| \sum_{\ell=1}^{2\kappa+1} \langle k^{(\ell)} \rangle^{\alpha/(1-\alpha)} (k_1^{(\ell)} \pm C) \right| \sim K.
\]

Using (4.44) and (4.45), we get \(K \sim \langle k \rangle^{1/(1-\tilde{\alpha})}\). Let \(k = (k_1, \overline{k}), \ k' = (k'_1, \overline{k}') \in \Lambda((k^{(\ell)})_{\ell=1}^{2\kappa+1})\). In view of mean value Theorem together with (4.44) and (4.45), one has that

\[
|k_1 - k'_1| \lesssim K^{-\tilde{\alpha}} \left( K^{\tilde{\alpha}} + \sum_{\ell=1}^{2\kappa+1} \langle k^{(\ell)} \rangle^{\alpha/(1-\alpha)} \right) \lesssim 1.
\]
We consider the estimates of $\mathcal{L}_2(u, v)$. As before, we can assume that

$$|k^{(1)}| \geq |k^{(3)}| \geq \ldots \geq |k^{(2\kappa+1)}|, \quad |k^{(2)}| \geq |k^{(4)}| \geq \ldots \geq |k^{(2\kappa)}|. \quad (4.48)$$

For $(k^{(1)}, \ldots, k^{(2\kappa+1)}) \in \Omega_2$, we see that $|k^{(1)}| \lor |k^{(2)}| = \max_{1 \leq \ell \leq 2\kappa+1} |k^{(\ell)}|$. 

**Case 1.** We assume that $|k^{(1)}| = \max_{1 \leq \ell \leq 2\kappa+1} |k^{(\ell)}|$. It is easy to see that

$$\langle k^{(1)} \rangle \sim \langle k^{(2)} \rangle, \text{ or } \langle k^{(1)} \rangle \sim \langle k^{(3)} \rangle. \quad (4.49)$$

We can further assume that $\langle k^{(1)} \rangle \sim \langle k^{(2)} \rangle$ and the case $\langle k^{(1)} \rangle \sim \langle k^{(3)} \rangle$ can be handled by using a similar way. There exists $j \in \{1, \ldots, d\}$ such that $\langle k^{(\ell)} \rangle \sim \langle k^{(j)} \rangle$. We can assume that $j = 1$. 

**Case 1.1.** We consider the case $|k^{(\ell)}| \ll |k^{(1)}|, \; \ell = 3, \ldots, 2\kappa + 1$. We have $\langle k^{(1)} \rangle^{1/(1-\alpha)} \gg K \sim \langle k^{1/(1-\bar{\alpha})} \rangle$. In $\mathcal{L}_2(u, v)$, using Hölder’s, bilinear and Strichartz’ estimates, we can bound

$$\int_{\mathbb{R}^{d+1}} \prod_{\ell=1}^{2\kappa+1} \Box^{\alpha}_{\epsilon, \ell} u \Box^{\alpha}_{\epsilon, \ell} v \; dx dt \leq \|\Box^{\alpha}_{\epsilon, 1} u\|_{L^2_{x,t}} \|\Box^{\alpha}_{\epsilon, 2} u_2 \Box^{\alpha}_{\epsilon, 3} u_3\|_{L^2_{x,t}} \prod_{\ell=4}^{2\kappa+1} \|\Box^{\alpha}_{\epsilon, \ell} u\|_{L^2_{x,t}}$$

$$\lesssim \ln \langle k^{(1)} \rangle \langle k^{(1)} \rangle^{-1/(1-\alpha)} \langle k^{(d-1)\bar{\alpha}/2(1-\bar{\alpha})} \rangle \langle k^{(3)} \rangle^{(d-1)\alpha/2(1-\alpha)} \|\Box^{\alpha}_{\epsilon, 2} v\|_{V^\alpha_{\Delta}} \|\Box^{\alpha}_{\epsilon, 1} u\|_{U^\alpha_{\Delta}} \times \prod_{\ell=4}^{2\kappa+1} \langle k^{(\ell)} \rangle^{d\alpha/(1-\alpha)} \|\Box^{\alpha}_{\epsilon, \ell} u\|_{U^\alpha_{\Delta}}$$

$$\lesssim \ln \langle k^{(1)} \rangle \langle k^{(1)} \rangle^{(1-2s)/(1-\alpha)} \langle k^{(d-1)\bar{\alpha}/2+s+d(\bar{\alpha}-\alpha)/2}/(1-\bar{\alpha}) \rangle \langle k^{(s-d(\bar{\alpha}-\alpha)/2)/(1-\bar{\alpha})} \rangle \langle k^{(4)} \rangle^{(\kappa-1)(d\alpha/2-s)/(1-\alpha)} \prod_{\ell=4}^{2\kappa+1} \langle k^{(\ell)} \rangle^{s/(1-\alpha)} \|\Box^{\alpha}_{\epsilon, \ell} u\|_{U^\alpha_{\Delta}}. \quad (4.50)$$

Notice that $\langle k^{1/(1-\bar{\alpha})} \rangle \lesssim \langle k^{(1)} \rangle^{(1-\theta+\alpha\theta)/(1-\alpha)}$ and

$$\kappa(d\alpha/2-s)-\alpha/2 \geq 0, \quad (4.51)$$

$$\langle k^{(d-1)\bar{\alpha}/2+s+d(\bar{\alpha}-\alpha)/2}/(1-\bar{\alpha}) \rangle \lesssim \langle k^{(1)} \rangle^{(d-1)\alpha/2+s+\theta(1-\alpha)(d\alpha/2-s)/(1-\alpha)} \quad (4.52)$$

$$\langle 2\kappa + \theta(1-\alpha) \rangle (d\alpha/2-s) < 1 + \alpha. \quad (4.53)$$

By Lemma 4.10 we have $\# \Lambda(k^{(1)}, \ldots, k^{(2\kappa+1)}) \lesssim 1$ if $(k^{(1)}, \ldots, k^{(2\kappa+1)}) \in \Omega_2$. We have from (4.50) to (4.53) that

$$\mathcal{L}_2(u, v) \lesssim \sum_{\substack{(k^{(\ell)}_{\ell=1}^{2\kappa+1} \in \Omega_2, \; k \in \Lambda((k^{(\ell)}_{\ell=1}^{2\kappa+1}))}} \ln \langle k^{(1)} \rangle \langle k^{(1)} \rangle^{-1/(1-\alpha+(2\kappa+\theta(1-\alpha))(d\alpha/2-s))/1-\alpha} \times \langle k^{(s-d(\bar{\alpha}-\alpha)/2)/(1-\bar{\alpha})} \rangle \langle k^{(4)} \rangle^{s/(1-\alpha)} \|\Box^{\alpha}_{\epsilon, \ell} u\|_{U^\alpha_{\Delta}}$$

$$\lesssim \|v\|_{V^\alpha_{\Delta-d(\bar{\alpha}-\alpha)/2, \bar{\alpha}}} \|u\|_{X^\alpha_{\Delta}}. \quad (4.54)$$
Case 1.2. We consider the case \(|k^{(1)}| \sim |k^{(2)}| \sim |k^{(3)}|\) and \(\kappa \geq 2\); or \(|k^{(1)}| \sim |k^{(2)}| \sim |k^{(4)}|\) and \(\kappa \geq 2\). It suffices to consider the case \(|k^{(1)}| \sim |k^{(2)}| \sim |k^{(3)}|\) and \(\kappa \geq 2\). We have \(\langle k^{(1)} \rangle^{1/\alpha} \gg K\).

In \(L_2(u, v)\), applying Hölder’s, bilinear and Strichartz’ estimates, we have

\[
\int_{\mathbb{R}^{d+1}} 2^{\kappa+1} \prod_{\ell=1}^{2\kappa+1} \|\Box_k u\|^2 \|\Box_k v\|^2 dxdt \\
\leq \|\Box_k u\|_{L_{x,t}^{2}} \|\Box_k v\|_{L_{x,t}^{2}} \prod_{\ell=4}^{2\kappa+1} \|\Box_k u\|_{L_{x,t}^{\infty}} \\
\lesssim \ln \langle k^{(1)} \rangle \langle k^{(1)} \rangle^{-1/2(1-\alpha)} \langle k^{(2)} \rangle^{(d-1)\tilde{\alpha}/2} \langle k^{(3)} \rangle^{(d\alpha/4-\alpha)/2} \langle h(\alpha/2) \rangle^{2(1-\alpha)} \langle k^{(3)} \rangle^{2(1-\alpha)} \|\Box_k u\|_{L_{x,t}^{2}} \\
\times \langle k^{(1)} \rangle \langle k^{(1)} \rangle^{(2\kappa-1)(d\alpha/2-s)-1/2-\alpha-2s)} \langle k^{(1)} \rangle^{s/(1-\alpha)} \|\Box_k u\|_{L_{x,t}^{2}} \\
\lesssim \ln \langle k^{(1)} \rangle \langle k^{(1)} \rangle^{-1/2(1-\alpha)} \langle k^{(2)} \rangle^{(d-1)\tilde{\alpha}/2} \langle k^{(3)} \rangle^{(d\alpha/4-\alpha)/2} \langle h(\alpha/2) \rangle^{2(1-\alpha)} \|\Box_k u\|_{L_{x,t}^{2}} \\
\times \langle k^{(1)} \rangle \langle k^{(1)} \rangle^{(2\kappa-1)(d\alpha/2-s)-1/2-\alpha-2s)} \langle k^{(1)} \rangle^{s/(1-\alpha)} \|\Box_k u\|_{L_{x,t}^{2}} \tag{4.55}
\]
A contradiction.

Case 1. We consider the case $|k_1^{(1)}| = \max_{1 \leq i \leq d} |k_i^{(1)}| \sim |k_1^{(2)}|$.

Case 1.1. $|k_1^{(1)}| = \max_{1 \leq i \leq d} |k_i^{(1)}| \sim |k_1^{(2)}| \gg |k_1^{(3)}|$. One can proceeds in the same way as in Case 1.1 in Section 4.2 to obtain the result, the details are omitted.

Case 1.2. $|k_1^{(1)}| = \max_{1 \leq i \leq d} |k_i^{(1)}| \sim |k_1^{(2)}| \sim |k_1^{(3)}|$. First, we conclude that $k_1^{(1)}$, $k_1^{(2)}$, $k_1^{(3)}$ cannot have the same signs, i.e., we must have one of the following cases:

$$
k^{(1)}_1 > 0, \quad k^{(2)}_1 < 0, \quad k^{(3)}_1 < 0; \quad \text{or} \quad k^{(1)}_1 < 0, \quad k^{(2)}_1 > 0, \quad k^{(3)}_1 > 0, \quad (4.58)
$$

$$
k^{(1)}_1 > 0, \quad k^{(2)}_1 < 0, \quad k^{(3)}_1 < 0; \quad \text{or} \quad k^{(1)}_1 < 0, \quad k^{(2)}_1 > 0, \quad k^{(3)}_1 > 0, \quad (4.59)
$$

$$
k^{(2)}_1 > 0, \quad k^{(1)}_1 < 0, \quad k^{(3)}_1 < 0; \quad \text{or} \quad k^{(2)}_1 < 0, \quad k^{(1)}_1 > 0, \quad k^{(3)}_1 > 0. \quad (4.60)
$$

If (4.58) holds, by Hölder’s inequality, we have

$$
\int_{\mathbb{R}^{d+1}} \prod_{\ell=1}^3 \Box_{k^{(1)}_\ell} u_\ell \Delta_j v dx dt \leq \left\| \Delta_j \Box \alpha_{k^{(1)}_\ell} u \right\|_{L^2_{x,t}} \left\| \Box \alpha_{k^{(1)}_\ell} u \right\|_{L^2_{x,t}} (4.61)
$$

To control the right hand side of (4.61), noticing that

$$\left| \langle k^{(1)}_1 \rangle_{\alpha/(1-\alpha)} k^{(1)}_1 - \langle k^{(2)}_1 \rangle_{\alpha/(1-\alpha)} k^{(3)}_1 \right| \geq \langle k^{(1)}_1 \rangle^{1/(1-\alpha)} \quad (4.62)
$$

and applying the bilinear estimates, we have

$$
\int_{\mathbb{R}^{d+1}} \prod_{\ell=1}^3 \Box_{k^{(1)}_\ell} u_\ell \Delta_j v dx dt \\
\lesssim \ln \langle k^{(1)} \rangle \langle k^{(1)} \rangle^{-1/(1-\alpha)} 2^{d(d-1)/2} \langle k^{(1)} \rangle^{-1/(1-\alpha)} \cdot \Delta_j v \| V^2_{\Delta} \left\| \Box \alpha_{k^{(1)}_\ell} u \right\|_{L^2_{x,t}} \left\| \Box \alpha_{k^{(2)}_\ell} u \right\|_{L^2_{x,t}} \left\| \Box \alpha_{k^{(3)}_\ell} u \right\|_{L^2_{x,t}} \\
\lesssim \ln \langle k^{(1)} \rangle \langle k^{(1)} \rangle^{-1/(1-\alpha)} 2^{d(d-1)/2} \langle k^{(1)} \rangle^{d/2} \langle k^{(1)} \rangle^{d/2} \Delta_j v \| V^2_{\Delta} \\
\times \langle k^{(2)} \rangle^{2(d-2-s)/s} \langle k^{(1)} \rangle^{s/(1-\alpha)} \prod_{\ell=1}^3 \langle k^{(1)} \rangle^{s/(1-\alpha)} \left\| \Box \alpha_{k^{(1)}_\ell} u \right\|_{V^2_{\Delta}}. \quad (4.63)
$$

This reduces to the estimate (4.36).

If (4.58) holds, $k^{(1)}$ and $k^{(3)}$ have equal positions, this case is the same as the above.

If (4.60) holds, we have

$$
\int_{\mathbb{R}^{d+1}} \prod_{\ell=1}^3 \Box_{k^{(1)}_\ell} u_\ell \Delta_j v dx dt \leq \left\| \Delta_j \Box \alpha_{k^{(1)}_\ell} u \right\|_{L^2_{x,t}} \left\| \Box \alpha_{k^{(1)}_\ell} u \right\|_{L^2_{x,t}} (4.64)
$$

By

$$\left| \langle k^{(1)} \rangle_{\alpha/(1-\alpha)} k^{(1)}_1 + \langle k^{(2)} \rangle_{\alpha/(1-\alpha)} k^{(2)}_1 \right| \geq \langle k^{(3)} \rangle^{1/(1-\alpha)} - K_1 \geq \langle k^{(1)} \rangle^{1/(1-\alpha)}
$$

and the bilinear estimates in Corollary 3.2 we still have (4.63). Repeating the procedures as above, we have the result, as desired.
Step 2. (Estimates of $L^2(u,v)$) Now let us connect the proof with Section 4.3 and we use the same notations as in Section 4.3. We need to consider the following case

$$(k^{(1)}, k^{(2)}, k^{(3)}) \in \Omega_2, \quad |k^{(1)}| \sim |k^{(2)}| \sim |k^{(3)}| \gg 1, \quad \kappa = 1.$$ 

Assume that $|k^{(1)}| = \max_{1 \leq i \leq d} k_i^{(1)}$. Since $K_1 \ll (k^{(1)})^{1/(1-\alpha)}$, we have

$$|k^{(1)}| \sim |k^{(2)}|, \quad \text{or} \quad |k^{(1)}| \sim |k^{(3)}|.$$

**Case 1.** We consider the case $|k^{(1)}| = \max_{1 \leq i \leq d} |k_i^{(1)}| \sim |k^{(2)}|$. 

**Case 1.1.** $|k^{(1)}| = \max_{1 \leq i \leq d} |k_i^{(1)}| \sim |k^{(2)}| \gg |k^{(3)}|$. One can proceeds in the same way as in Case 1.1 in Section 4.3 to obtain the result, the details are omitted.

**Case 1.2.** $|k^{(1)}| = \max_{1 \leq i \leq d} |k_i^{(1)}| \sim |k^{(2)}| \sim |k^{(3)}|$. First, we conclude that $k^{(1)}$, $k^{(2)}$, $k^{(3)}$ cannot have the same signs, i.e., we must have one of the alternative cases as in (4.58), (4.60) and (4.62).

If (4.58) is satisfied, by Hölder’s inequality, and (4.62), bilinear estimates, we have

$$\int_{\mathbb{R}^{d+1}} \prod_{\ell=1}^3 \Box_k^\alpha \Box_k^\alpha u \Box_k^\alpha v dx dt$$

$$\leq \| \Box_k^\alpha \Box_k^\alpha \Box_k^\alpha u \|_{L^2_{z,t}} \| \Box_k^\alpha \Box_k^\alpha \Box_k^\alpha u \|_{L^2_{z,t}}$$

$$\lesssim \ln \langle k^{(1)} \rangle \langle k^{(1)} \rangle^{1/(1-\alpha)} \langle k^{(3)} \rangle \langle k^{(3)} \rangle^{(d-1)/2(1-\alpha)} \| \Box_k^\alpha v \|_{Y_{\alpha}^2} \| \Box_k^\alpha \Box_k^\alpha u \|_{Y_{\alpha}^2} \| \Box_k^\alpha \Box_k^\alpha \Box_k^\alpha u \|_{Y_{\alpha}^2}$$

$$\lesssim \ln \langle k^{(1)} \rangle \langle k^{(1)} \rangle^{(1-2s)/(1-\alpha)} \langle k^{(3)} \rangle \langle k^{(3)} \rangle^{(d-1)/2(d-(\alpha-2)/2)} \| \Box_k^\alpha v \|_{Y_{\alpha}^2}$$

$$\times \langle k^{(3)} \rangle^{(d-2s)/2-(\alpha-2)/2} \prod_{\ell=1}^3 \langle k^{(1)} \rangle^{1/(1-\alpha)} \| \Box_k^\alpha \Box_k^\alpha \Box_k^\alpha u \|_{Y_{\alpha}^2}. \tag{4.65}$$

Using the same way as in (4.54), we have

$$L^2(u, v) \lesssim \| v \|_{Y_{\alpha}^{1-s-d(\alpha-2)/2, \alpha}} \| u \|^3_{X_{\alpha}^{s, \alpha}}. \tag{4.66}$$

### 4.5 Proof of Theorem 1.2

Now the proof of Theorem 1.2 seems quite standard after the multi-linear estimate is established. We consider the mapping

$$\mathcal{T} : u(t) \to S(t)u_0 + i\mathcal{A}(|u|^{2\kappa}u)$$

in the space

$$\mathcal{D} = \{ u \in X_{\alpha}^s((0, \infty)) : \| u \|_{X_{\alpha}^s((0, \infty))} \leq M \}.$$

Using the same way as in the proof of Theorem 1.1 in [21], we can show that $\mathcal{T}$ is a contraction mapping on $\mathcal{T}$ to obtain the global existence of solution in $X_{\alpha}^s((0, \infty))$. To prove the uniqueness, we need the following (see [21])

27
Lemma 4.11 (Right continuity). Let \( T > 0 \), \( u(0) = 0 \) and \( u \in X^s_\Delta([0,T]) \). For any \( \varepsilon > 0 \), there exists \( 0 < t_0 < T \) such that \( \|u\|_{X^s_\Delta([0,t_0])} \leq \varepsilon \).

Using Lemma 4.11 and following [21], we see that for two solutions \( u, v \in X^s_\Delta([0,\infty)) \) of NLS, we have

\[
u - v = i\mathcal{S}(|u|^{2\kappa} u - |v|^{2\kappa} v).
\]

Applying Lemmas 4.11 and 4.1, one has that

\[
\|u - v\|_{X^s_\Delta([0,t_0])} \lesssim (\|u\|^{2\kappa}_{X^s_\Delta([0,t_0])} + \|v\|^{2\kappa}_{X^s_\Delta([0,t_0])})\|u - v\|_{X^s_\Delta([0,t_0])} \lesssim \varepsilon^{2\kappa}\|u - v\|_{X^s_\Delta([0,t_0])}.
\]

This implies the uniqueness of \( u \in X^s_\Delta \).

5 Proof of Theorem 1.3

If \( \kappa > 2/d \) and \( 0 \leq \alpha < 1 \), then we have \( s_\kappa < s(\kappa) \). Let \( s = s_\kappa^+ \). Now we construct a suitable initial value. Choose a sequence \( \{k_j\} \) satisfying \( k_j \in \mathbb{N} \) and \( \langle k_j \rangle^\alpha/(1-\alpha)k_j \in [2^{j+1/4}, 2^{j+1/2}) \). Denote

\[
\mathbb{K}^d = \{(k_j, 0, ..., 0) : j \in J_\mathbb{N} \},
\]

where \( J \) will be chosen as below. Put for some small constant \( 0 < \varepsilon \ll 1 \),

\[
\tilde{u}_0 = \varepsilon \sum_{k \in \mathbb{K}^d} \frac{1}{\ln^2 |k|} \langle k \rangle^{-(s+\alpha/2)/(1-\alpha)} \mathcal{F}^{-1} \rho_k^\alpha,
\]

where \( \rho_k^\alpha = \varphi((\cdot - \langle k \rangle^{\alpha/(1-\alpha)}k)/c\langle k \rangle^{\alpha/(1-\alpha)}) \), \( \varphi \) is as in (2.3) and \( c = 1/8 \). We can write \( u_0 \) as

\[
u_0(x) = \varepsilon c^d \sum_{k \in \mathbb{K}^d} \frac{1}{\ln^2 |k|} \langle k \rangle^{-(s+\alpha/2)/(1-\alpha)} e^{ix\langle k \rangle^{\alpha/(1-\alpha)}} k \mathcal{F}^{-1} \varphi(c\langle k \rangle^{\alpha/(1-\alpha)} x).
\]

We may assume that \( (\mathcal{F}^{-1} \varphi)(0) = 1 \). We easily see that for \( \alpha > 0 \)

\[
u_0(0) = \varepsilon c^d \sum_{k \in \mathbb{K}^d} \frac{1}{\ln^2 |k|} \langle k \rangle^{-(s+\alpha/2)/(1-\alpha)} = \infty
\]

and for \( \alpha = 0 \)

\[
|u_0(x)| \leq \varepsilon c^d \sum_{k \in \mathbb{K}^d} \frac{1}{\ln^2 |k|} < \infty.
\]

We see that for \( \Lambda = \{ \ell \in \mathbb{Z}^d : |\ell| \leq C \}, \)

\[
\|u_0\|_{M^s_{2,1}} \lesssim \varepsilon \sum_{k \in \mathbb{K}^d + \Lambda} \frac{1}{\ln^2 |k|} \langle k \rangle^{-\alpha/2(1-\alpha)} \|\rho_k^\alpha\|_2 \sim \varepsilon \ll 1.
\]

Moreover, we see that for any \( s' > s \) and for \( k \in \mathbb{K}^d, |k| \gg 1, \)

\[
\|u_0\|_{B^{s'}_{2,\infty}} \gtrsim \frac{1}{\ln^2 |k|} \langle k \rangle^{(s'-s)/(1-\alpha)} \langle k \rangle^{-\alpha/2(1-\alpha)} \|\rho_k^\alpha\|_2 \gtrsim \frac{\varepsilon}{\ln^2 |k|} \langle k \rangle^{(s'-s)/(1-\alpha)}.
\]
It follows that
\[ \|u_0\|_{B_{2,\infty}^s} = \infty. \]
In view of Theorem 1.2, one sees that (1.12) has a global solution \( u \in C(\mathbb{R}, M_{2,1}^{s,\alpha}) \). Considering the scaling solution \( u_\sigma = \sigma^{1/\kappa} u(\sigma^2 t, \sigma x) \), which solves nonlinear Schrödinger equation (NLS):
\[ iv_t + \Delta v + \lambda |v|^{2\kappa} v = 0, \quad v(0) = \sigma^{1/\kappa} u_0(\sigma \cdot). \] 
(5.1)

We see that
\[ v(0, x) = \varepsilon c^{d/\alpha} \sum_{k \in \mathbb{k}^d} \frac{1}{\ln^2 |k|} \langle k \rangle^{(s-d\alpha)/2}/(1-\alpha) \langle k \rangle^{\alpha/(1-\alpha)} \langle \varphi \rangle_{\sigma \langle k \rangle^{\alpha/(1-\alpha)} x}. \]

One can rewrite \( v(0, \cdot) \) as
\[ \hat{v}(0, \xi) = \varepsilon c^{d/\alpha} \sum_{k \in \mathbb{k}^d} \frac{1}{\ln^2 |k|} \langle k \rangle^{(s-d\alpha)/2}/(1-\alpha) \varphi \left( \frac{\xi - \sigma \langle k \rangle^{\alpha/(1-\alpha)} k}{c \sigma \langle k \rangle^{\alpha/(1-\alpha)}} \right). \]

Let \( M \gg 1 \) and assume without loss of generality that \( \sigma = M/\varepsilon^{\kappa/(1-\alpha)} = 2^J \). We easily see that for \( \alpha = 0 \),
\[ |v(0, 0)| \geq c^d (M^{1/\kappa}/\varepsilon) |u_0(0)| \geq c^d M^{1-1/\kappa}, \] 
(5.2)
and for \( \alpha > 0 \),
\[ v(0, 0) = \infty. \] 
(5.3)

Denote
\[ \Lambda_k = \left\{ l : \eta_l^\alpha(\xi) \varphi \left( \frac{\xi - \sigma \langle k \rangle^{\alpha/(1-\alpha)} k}{c \sigma \langle k \rangle^{\alpha/(1-\alpha)}} \right) \neq 0 \right\}. \] 
(5.4)

It is easy to see that for \( k \in \mathbb{k}^d \),
\[ \#\Lambda_k \sim \sigma^d(1-\alpha); \quad \langle l \rangle \sim \sigma^{1-\alpha} \langle k \rangle \text{ for } l \in \Lambda_k. \] 
(5.5)

We have
\[ \|v(0, \cdot)\|_{M_{2,1}^{s,\alpha}} \geq \varepsilon c^{1/\kappa - d} \ln^{-2} |k| \langle k \rangle^{-(s+d\alpha/2)/(1-\alpha)} \sum_{l \in \Lambda_k} \langle l \rangle^{s/(1-\alpha)} \left\| \eta_l^\alpha \varphi \left( \frac{\xi - \sigma \langle k \rangle^{\alpha/(1-\alpha)} k}{c \sigma \langle k \rangle^{\alpha/(1-\alpha)}} \right) \right\|_2 \]
\[ \geq \varepsilon c^{1/\kappa + s-d\alpha/2} \ln^{-2} |k| \geq \varepsilon \sigma^{(1-\alpha)+\kappa/\kappa} \ln^{-2} |k|. \] 
(5.6)

Taking \( |k|^{1/(1-\alpha)} \sim 2 \), we immediately have
\[ \|v(0, \cdot)\|_{M_{2,1}^{s,\alpha}} \geq M^{(1-\alpha)/\kappa}. \] 
(5.7)
6 Ill Posedness

Now let $u(\delta, t)$ satisfy

$$iu_t + \Delta u + \lambda |u|^{2\kappa} u = 0, \quad u(0) = \delta v_0. \quad (6.1)$$

One has that

$$\frac{\partial^{2\kappa+1}u(0, t)}{\partial \delta^{2\kappa+1}} = \int_0^t S(t - \tau)|S(\tau)v_0|^{2\kappa}S(\tau)v_0 d\tau. \quad (6.2)$$

It follows that

$$\frac{\partial^{2\kappa+1}u(0, t)}{\partial \delta^{2\kappa+1}} = e^{-it|\xi|^2} \int_{\mathbb{R}^{2\kappa d}} \frac{e^{-itP} - 1}{P} \hat{v}_0(\xi - \xi_1 - \ldots - \xi_{2\kappa}) \prod_{j=1}^{2\kappa} \hat{v}_0(\xi_j) d\xi_1 \ldots d\xi_{2\kappa}, \quad (6.3)$$

where

$$P = -\sum_{j=1}^{\kappa} |\xi_j|^2 - |\xi - \xi_1 - \ldots - \xi_{2\kappa}|^2 + \sum_{j=\kappa+1}^{2\kappa} |\xi_j|^2 + |\xi|^2.$$

Take $N \gg 1$, $k = (N, N, \ldots, N)$ and

$$\hat{v}_0(\xi) = \langle k \rangle^{-{(s+d\alpha/2)/(1-\alpha)}} (\varphi_k(\xi) + \varphi_{-k}(\xi)), \quad (6.4)$$

where we denote by $\varphi$ a smooth cut-off function adapted to the unit ball as in (2.1) and

$$\varphi_k(\xi) = \varphi \left( \frac{\xi - \langle k \rangle^{\alpha/(1-\alpha)} k}{c\langle k \rangle^{\alpha/(1-\alpha)}} \right).$$

By the definition of norm on $M_{2,1}^{s,\alpha}$, it is easy to see that

$$\|\delta v_0\|_{M_{2,1}^{s,\alpha}} \sim \delta.$$

Again, in view of the definition of norm on $M_{2,1}^{s,\alpha}$, we have

$$\left\| \frac{\partial^{2\kappa+1}u(0, t)}{\partial \delta^{2\kappa+1}} \right\|_{M_{2,1}^{s,\alpha}} \gtrsim \langle k \rangle^{(s -(2\kappa+1)(s+d\alpha/2)) /(1-\alpha)}$$

$$\times \left\| \varphi_k(\xi) \int_{\mathbb{R}^{2\kappa d}} \frac{e^{-itP} - 1}{P} \varphi_k(\xi - \xi_1 - \ldots - \xi_{2\kappa}) \prod_{j=1}^{\kappa} \varphi_{-k}(\xi_j) \prod_{j=\kappa+1}^{2\kappa} \varphi_k(\xi_j) d\xi_1 \ldots d\xi_{2\kappa} \right\|_2. \quad (6.5)$$

By change of variables

$$\xi_j + \langle k \rangle^{\alpha/(1-\alpha)} k = c\langle k \rangle^{\alpha/(1-\alpha)} \zeta_j, \quad j = 1, \ldots, \kappa,$$

$$\xi_j - \langle k \rangle^{\alpha/(1-\alpha)} k = c\langle k \rangle^{\alpha/(1-\alpha)} \zeta_j, \quad j = \kappa + 1, \ldots, 2\kappa,$$

$$\xi - \langle k \rangle^{\alpha/(1-\alpha)} k = c\langle k \rangle^{\alpha/(1-\alpha)} \zeta,$$
we see that
\[ P = \langle k \rangle^{2\alpha/(1-\alpha)} \left( -\sum_{j=1}^{\kappa} |\zeta_j|^2 + \sum_{j=\kappa+1}^{2\kappa} |\zeta_j|^2 - |\zeta - \zeta_1 - \ldots - \zeta_{2\kappa}|^2 + |\zeta|^2 \right). \]

By choosing \( t \sim \langle k \rangle^{-2\alpha/(1-\alpha)} \), from (6.5) it follows that
\[
\left\| \frac{\partial^{2\kappa+1} u(0, t)}{\partial \delta^{2\kappa+1}} \right\|_{M^{s,\alpha}_{2,1}} \geq \langle k \rangle^{(s-2\kappa+1)(s+\alpha/2)/(1-\alpha)} \langle k \rangle^{d\alpha/2(1-\alpha)} \langle k \rangle^{-2\alpha/(1-\alpha)}
\times \left\| \eta(\zeta) \int_{\mathbb{R}^{2\kappa}} \eta(\zeta - \zeta_1 - \ldots - \zeta_{2\kappa}) \prod_{j=1}^{2\kappa} \eta(\eta_j) d\zeta_1 \ldots d\zeta_{2\kappa} \right\|_2
\geq \langle k \rangle^{2\alpha(\alpha+2\kappa-s)-2\alpha/(1-\alpha)}.
\]

Hence, if \( s < d\alpha/2 - \alpha/\kappa \), we have for \( N \gg 1 \),
\[
\left\| \frac{\partial^{2\kappa+1} u(0, t)}{\partial \delta^{2\kappa+1}} \right\|_{M^{s,\alpha}_{2,1}} \geq N^\sigma, \quad \text{for some } \sigma > 0.
\]

In particular, in the case \( \alpha = 0 \), we have
\[
\left\| \frac{\partial^{2\kappa+1} u(0, t)}{\partial \delta^{2\kappa+1}} \right\|_{M^{s,\alpha}_{2,1}} \geq \langle k \rangle^{-2\kappa s}.
\]

Noticing that \( \|v_0\|_{M^{-s/2}_{2,1}} \sim N^{s/2} \ll 1 \) if \( s < s_\kappa = 0 \) (we can assume that \( s = 0+ \)), we have NLS is globally well-posed in \( C(\mathbb{R}, M^{-s/2}_{2,1}) \). So we have a Taylor expansion of \( u(\delta, t) \) at \( \delta = 0 \):
\[
u(\delta, t) = \delta S(t)v_0 + \sum_{j=2\kappa+1}^{\infty} \frac{\partial^j u(0, t)}{\partial \delta^j} \delta^j, \quad \text{in } C(\mathbb{R}, M^{-s/2}_{2,1}).
\]

Using Theorem 1.2, we have
\[
\left\| u(\delta, t) - \frac{\partial^{2\kappa+1} u(0, t)}{\partial \delta^{2\kappa+1}} \right\|_{M^{-s/2}_{2,1}} \leq \|u\|_{M^{-s/2}_{2,1}} + \left\| \frac{\partial^{2\kappa+1} u(0, t)}{\partial \delta^{2\kappa+1}} \right\|_{M^{-s/2}_{2,1}} \delta^{2\kappa+1}.
\]

Applying the multi-linear estimate,
\[
\left\| \frac{\partial^{2\kappa+1} u(0, t)}{\partial \delta^{2\kappa+1}} \right\|_{M^{-s/2}_{2,1}} \leq C\|S(t)v_0\|_{X^{-s/2}_{2,0}} \lesssim 1.
\]

However, we have for some fixed \( t \ll 1 \),
\[
\|u(\delta, t)\|_{M^{s,\alpha}_{2,1}} \geq \left\| \frac{\partial^{2\kappa+1} u(0, t)}{\partial \delta^{2\kappa+1}} \right\|_{M^{s,\alpha}_{2,1}} \delta^{2\kappa+1} - \left\| u(\delta, t) - \frac{\partial^{2\kappa+1} u(0, t)}{\partial \delta^{2\kappa+1}} \delta^{2\kappa+1} \right\|_{M^{-s/2}_{2,1}} \geq N^{-2\kappa s}\delta^{2\kappa+1} - C.
\]
Since $\delta$ is independent of $N$, we have $\|u(\delta, t)\|_{M_{2,1}^s} \gg 1$ by choosing $N^{-2\kappa s} \gg \delta^{-2\kappa+1}$. This implies that the map $\delta \to u(\delta, t)$ is discontinuous at $\delta = 0$.

Acknowledgement. B. Wang is supported in part by an NSFC grant 11771024. Part work of this paper has been talked in the “Workshop on nonlinear evolution equations” which was held in Kumamoto University during January 20-23, 2018. B. Wang is very grateful to Professor N. Hayashi for his invitation and support in the conference.

References

[1] Á. Bényi, K. Gröchenig, K. A. Okoudjou, L. G. Rogers, Unimodular Fourier multipliers for modulation spaces. J. Funct. Anal. 246 (2007), 366–384.

[2] B. Birnir, C. E. Kenig, G. Ponce, N. Svanstedt, L. Vega, On the ill-posedness of the IVP for the generalized Korteweg-de Vries and nonlinear Schrödinger equations. J. London Math. Soc. (2) 53 (1996), no. 3, 551–559.

[3] L. Borup, M. Nielsen, Boundedness for pseudodifferential operators on multivariate $\alpha$-modulation spaces, Ark. Mat. 44 (2006) 241–259.

[4] L. Borup, M. Nielsen, Nonlinear approximation in $\alpha$-modulation spaces, Math. Nachr. 279 (2006), 101–120.

[5] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations, Geom. Funct. Anal. 3 (1993), 107–156.

[6] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. II. The KdV-equation, Geom. Funct. Anal. 3 (1993), 209–262.

[7] J. Bourgain, Refinements of Strichartz inequality and applications to 2D-NLS with critical nonlinearity, Internat. Math. Res. Notices 5 (1998), 253–283.

[8] J. Bourgain, Global well-posedness of defocusing 3D critical NLS in the radial case, J. Amer. Math. Soc. 12 (1999), 145–171.

[9] T. Cazenave and F. B. Weissler, The Cauchy problem for the critical nonlinear Schrödinger equation in $H^s$, Nonlinear Anal. TMA, 14 (1990), 807–836.

[10] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in $R^3$, Ann. of Math. 167 (2008), 767–865.

[11] Q. Huang, J. C. Chen, Cauchy problem for dispersive equations in $\alpha$-modulation spaces. Electron. J. Differential Equations 2014, No. 158, 10 pp.

[12] S. Dahlke, M. Fornasier, H. Rauhut, G. Steidl, G. Teschke, Generalized coorbit theory, Banach frames, and the relation to $\alpha$-modulation spaces, Proc. Lond. Math. Soc., 96 (2008), no. 2, 464–506.

[13] D. P. Du, Y. F. Wu, K. J. Zhang, On blow-up criterion for the nonlinear Schrödinger equation. Discrete Contin. Dyn. Syst. 36 (2016), no. 7, 3639–3650.

[14] T. Duyckaerts, C. E. Kenig, F. Merle, Profiles for bounded solutions of dispersive equations, with applications to energy-critical wave and Schrödinger equations. Commun. Pure Appl. Anal. 14 (2015), 1275–1326.
[15] H. G. Feichtinger, *Modulation spaces on locally compact Abelian group*, Technical Report, University of Vienna, 1983.

[16] H. G. Feichtinger, C. Y. Huang, B. X. Wang, Trace operators for modulation, $\alpha$-modulation and Besov spaces, Appl. Comput. Harmon. Anal., 30 (2011), 110–127.

[17] P. Gröbner, *Banachräume Glatter Funktionen und Zerlegungsmethoden*, Doctoral thesis, University of Vienna, 1992.

[18] K. Gröchenig, *Foundations of Time-Frequency Analysis*, Birkhäuser Boston, MA, 2001.

[19] Glassey, R. T. On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations. J. Math. Phys. 18 (1977), no. 9, 1794–1797.

[20] A. Grünrock, Bi- and trilinear Schrödinger estimates in one space dimension with applications to cubic NLS and DNLS, IMRN, Int. Math. Res. Not. (2005), No. 41, 2525–2558.

[21] M. Hadac, S. Herr and H. Koch, Well-posedness and scattering for the KP-II equation in a critical space. Ann. Inst. H. Poincaré Anal. Non Linéaire 26 (2009), 917-941.

[22] J.S. Han and B.X. Wang, $\alpha$-modulation spaces (I) scaling, embedding and algebraic properties. J. Math. Soc. Japan 66 (2014), no. 4, 1315–1373.

[23] J. S. Han, B.X. Wang, $\alpha$-modulation spaces and the Cauchy problem for nonlinear Schrödinger equations. Harmonic analysis and nonlinear partial differential equations, 119–130, RIMS Kōkyūroku Bessatsu, B49, Res. Inst. Math. Sci. (RIMS), Kyoto, 2014.

[24] J. Holmer, S. Roudenko, A sharp condition for scattering of the radial 3D cubic nonlinear Schrödinger equation. Comm. Math. Phys. 282 (2008), no. 2, 435–467.

[25] C. E. Kenig, F. Merle, Global well-posedness, scattering and blow-up for the energy-critical, focusing, nonlinear Schrödinger equation in the radial case. Invent. Math. 166 (2006), no. 3, 645–675.

[26] H. Koch and D. Tataru, Dispersive estimates for principally normal pseudo-differential operators, Comm. Pure Appl. Math. 58 (2005), 217–284.

[27] H. Koch and D. Tataru, A priori bounds for the 1D cubic NLS in negative Sobolev spaces, Int. Math. Res. Not., 2007, no. 16, Art. ID rnm053, 36 pp.

[28] H. Koch and D. Tataru, Energy and local energy bounds for the 1D cubic NLS equation in $H^{1/4}$, Ann. Inst. H. Poincaré Anal. Non Linéaire 29 (2012), 955–988.

[29] M. Kobayashi, M. Sugimoto, N. Tomita, Trace ideals for pseudo-differential operators and their commutators with symbols in $\alpha$-modulation spaces. J. Anal. Math. 107 (2009), 141–160.

[30] J. Krieger and W. Schlag, Large global solutions for energy supercritical nonlinear wave equations on $\mathbb{R}^{3+1}$, arXiv:1403.2913v2.

[31] M. Nakamura and T. Ozawa, Nonlinear Schrödinger equations in the Sobolev space of critical order. J. Funct. Anal. 155 (1998), 364–380.

[32] Nakanishi, Kenji Energy scattering for nonlinear Klein–Gordon and Schrödinger equations in spatial dimensions 1 and 2. J. Funct. Anal. 169 (1999), 201–225.
[33] T. Ozawa and Y. Tsutsumi, Space-time estimates for null gauge forms and nonlinear Schrödinger equations, Differential Integral Equations 11 (1998), no. 2, 201–222.

[34] E. Ryckman and M. Visan, Global well-posedness and scattering for the defocusing energy-critical nonlinear Schrödinger equation in $\mathbb{R}^{1+4}$, Amer. J. Math. 129 (2007), 1–60.

[35] W. Strauss, Nonlinear scattering theory at low energy, J. Funct. Anal. 41 (1981), 110–133.

[36] Y. Tsutsumi, Scattering problem for nonlinear Schrödinger equations, Ann. Inst. H. Poincaré Phys. Théor. 43 (1985), 321–347.

[37] M. Sugimoto, N. Tomita, The dilation property of modulation space and their inclusion relation with Besov spaces, J. Funct. Anal., 248 (2007), 79–106.

[38] M. Sugimoto, B. X. Wang, R. R. Zhang, Local well-posedness for the Davey-Stewartson equation in a generalized Feichtinger algebra. J. Fourier Anal. Appl. 21 (2015), no. 5, 1105–1129.

[39] J. Toft, Continuity properties for modulation spaces, with applications to pseudo-differential calculus, I, J. Funct. Anal., 207 (2004), 399–429.

[40] J. Toft and P. Wahlberg, Embeddings of $\alpha$-modulation spaces, Arxiv: 1110.2681.

[41] H. Triebel, Theory of Function Spaces, Birkhäuser-Verlag, Basel, 1983.

[42] B. X. Wang, L. J. Han, C. Y. Huang, Global well-posedness and scattering for the derivative nonlinear Schrödinger equation with small rough data. Ann. Inst. H. Poincaré Anal. Non Linéaire 26 (2009), 2253–2281.

[43] B. X. Wang and C. Y. Huang, Frequency-uniform decomposition method for the generalized BO, KdV and NLS equations, J. Differential Equations, 239 (2007), 213–250.

[44] B. X. Wang and H. Hudzik, The global Cauchy problem for the NLS and NLKG with small rough data, J. Differential Equations, 232 (2007) 36–73.

[45] B. X. Wang L. Zhao, B. Guo, Isometric decomposition operators, function spaces $E^\lambda_p,q$ and applications to nonlinear evolution equations. J. Funct. Anal. 233 (2006), no. 1, 1–39.