Horizon Dynamics of a BTZ Black Hole

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Abstract

It has been suggested in the literature that, given a black hole spacetime, a relativistic membrane can provide an effective description of the horizon dynamics. In this paper, we explore such a framework in the context of a 2+1-dimensional BTZ black hole. Following this membrane prescription, we are able to translate the horizon dynamics (now described by a string) into the convenient form of a 1+1-dimensional Klein-Gordon equation. We proceed to quantize the solutions and construct a thermodynamic partition function. Ultimately, we are able to extract the quantum-corrected entropy, which is shown to comply with the BTZ form of the Bekenstein-Hawking area law. We also substantiate that the leading-order correction is proportional to the logarithm of the area.

I. INTRODUCTION

Many explorations into quantum gravity have centered in the realm of black hole thermodynamics [1]. In this regard, a particularly important open question is the origin of the Bekenstein-Hawking entropy [2,3],

\[ S_{BH} = \frac{A}{4G}. \]  

(1)

It is commonly believed that a derivation of the Bekenstein-Hawking area law from first principles will be a significant step towards realizing the fundamental theory of quantum gravity [4]. Meanwhile, although this law is well established at the level of semiclassical thermodynamics, the statistical origin of the entropy still remains as enigmatic as ever.

We should point out that there has been undeniable success in calculating \( S_{BH} \) by way of state-counting procedures [5]. Nevertheless, it is not at all evident that the states being

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\(^1\)Here, \( A \) is the area of the black hole horizon (or the analogue of area when the dimensionality of the spacetime differs from four), \( G \) is the gravitational coupling constant, and all other fundamental constants have been set to unity.
counted have any real physical significance [6]. This dilemma may be viewed as a manifestation of our ignorance of physics below the Planck scale. That is to say, without the resolution of “subplanckian” distances, it seems unlikely that theorists will be able to identify, never mind count, the microscopic degrees of freedom that (presumably) underlie the black hole entropy.

In view of this ignorance, it is perhaps beneficial to “take a step back” and see what we can learn about black holes when the subplanckian degrees of freedom have been a priori suppressed. To this end, a very elegant framework has been proposed by Maggiore [7]. This proposal will serve as the focal point of our current analysis, so let us proceed with a pertinent discussion.

We begin by considering a fiducial observer; that is, a static observer who remains eternally outside of the black hole. It immediately follows from the “no-hair” theorem of black holes [8] that, as far as this observer is concerned, only the degrees of freedom outside of the horizon are of relevance. Therefore, we should limit considerations to the region of spacetime that spans from the horizon surface \( r = r_+ \) to spatial infinity \( r \rightarrow \infty \). In terms of path integral formalism [9], this implies that the relevant partition function is expressible as follows:

\[
Z = \int_{M_{ext}} D[g_{\mu\nu}] e^{iI_g[g_{\mu\nu}]}. \tag{2}
\]

Here, \( M_{ext} \) denotes the exterior manifold, \( g_{\mu\nu} \) is the metric for the background (black hole) spacetime, \( I_g \) is the appropriate gravitational action, and \( D \) indicates a suitable measure.

An immediate problem with the above picture is our ignorance with regard to the position of the horizon. (This is, of course, essentially the same ignorance that has been alluded to above.) At the classical level, we can, given \( g_{\mu\nu} = g^{cl}_{\mu\nu} \), pinpoint the horizon precisely; however, at the quantum level, the metric is fluctuating and, therefore, so is the position of the horizon. Let us assume that the fluctuations have a maximal spatial extent of \( \epsilon \), which is presumably on the order of a few Planck lengths. It is then natural to separate the exterior spacetime into a pair of submanifolds that are defined by \( r_+ < r < r_+ + \epsilon \) and \( r > r_+ + \epsilon \).\(^2\) In the spirit of Wilson’s renormalization group [11], one can then integrate out the degrees of freedom in the near-horizon shell \( (M_\epsilon) \) and employ the classical metric on the outside. Following this prescription, we can schematically re-express the partition function (2) as follows:

\[
Z = \left[ e^{iI_g[g_{\mu\nu}]} \int_{M_\epsilon} D[\xi] e^{iI_\xi[\xi]} \right]_{g_{\mu\nu} = g^{cl}_{\mu\nu}}. \tag{3}
\]

Here, \( \xi \) collectively represents any physically relevant variables that remain after integrating the “fast variables” out of \( M_\epsilon \), and \( I_\xi[\xi] \) represents whatever effective action has been induced by this coarse-graining process.

A cautionary comment is in order. It is implicit in this procedure - for which only a small fraction of the spacetime is subject to quantization - that the degrees of freedom of a

\(^2\)Such a near-horizon cutoff in a black hole spacetime is philosophically similar to the brick-wall model proposed by ’t Hooft [10].
black hole spacetime are mostly localized within a small region near the horizon. Although
this extreme degree of localization may be intuitively unsettling, just such a notion has,
in fact, been frequently advocated in the literature (e.g., [10,12]). Heuristically speaking,
this localization follows from the immense gravitational blue-shifting of any energy in the
near-horizon vicinity.

In practice, it would be extremely difficult to obtain an explicit formulation of the effective
action, \( I_\epsilon[\xi] \). Nevertheless, Maggiore has argued on the grounds of invariance principles
that, at least to the lowest order, the renormalization group procedure should induce the
action of a relativistic bosonic membrane [7]. That is,

\[
I_\epsilon = -\mathcal{T} \int d^n \xi \sqrt{-h},
\]

where \( n+1 \) is the dimensionality of the black hole spacetime, the \( \xi \)-variables now parametrize
the \( n \)-dimensional world-volume of the membrane, \( h \) is the determinant of a suitably defined
induced metric, and \( \mathcal{T} \) is the tension of the membrane.

The above formalism suggests an intriguing picture: the dynamics of the black hole
horizon can effectively be described by a membrane whose equilibrium position is at a
distance \( \epsilon \) from the horizon. This membrane position can, in fact, be identified with the
so-called “stretched horizon” (see below) of the black hole. Meanwhile, our ignorance of
subplanckian physics is now encapsulated in the arbitrariness of the parameter \( \mathcal{T} \); that is,
the membrane tension. Presumably, this lost information can be retrieved from a more
fundamental theory, but this is not necessary for semiclassical considerations.

It is worth noting that the above framework also follows intuitively from the viewpoint
of the membrane paradigm [13]; which stresses that, for a fiducial observer, the black hole
horizon behaves as if it were a real membrane that is endowed with physical properties. (A
useful definition of these properties necessitates that the membrane is moved out a small
distance, which effectively describes the location of the stretched horizon.) On the other
hand, a free-falling observer would see no membrane at all; however, this apparent paradox
has been nicely resolved by the principle of black hole complementarity [14].

That this membrane picture leads to a self-consistent description of horizon dynamics
was amply demonstrated through the cited work of Maggiore [7]. In a related study [15],
Lousto applied the membrane description in a novel way and demonstrated that, for a
“conventional” 3+1-dimensional theory, the membrane fluctuations could be described by
a 2+1-dimensional Klein-Gordon equation. This was followed by a procedure of quanti-
zation and then a thermodynamic analysis. Most notably, the leading-order entropy was
found to comply with the Bekenstein-Hawking area law. Higher-order corrections were also
considered.

Lousto’s verification of the area law can be viewed as highly non-trivial, inasmuch as
a Klein-Gordon description of the horizon fluctuations could not have been \( a \ priori \) antici-
pated. It should, therefore, be of considerable interest to see if the basic outcomes persist for
more exotic black hole scenarios. The purpose of the current paper is to consider just such
a scenario; in particular, the BTZ model [16], which describes solutions of 2+1-dimensional
anti-de Sitter gravity that have all the properties of black holes. Our choice is motivated,
in part, by a subsequent paper by Maggiore which demonstrated that the general philoso-
phy can indeed be translated into a BTZ context [17]. (Note, however, that the relativistic
membrane is, in this case, a string.) Furthermore, the BTZ black hole, although essentially
a toy model, has generated substantial interest in various aspects of gravitational theory. For instance, the BTZ solution is dually related to certain stringy black holes [18,19], has played a featured role in microscopic entropy calculations [20,21], and has served as a useful "laboratory" for studying one-loop thermodynamics [22–27].

To further motivate our choice to study, in particular, the BTZ black hole, let us take note of a relevant paper by Horowitz and Welch [28]. These authors made the important observation that the BTZ black hole is essentially equivalent, under an appropriate duality, to a three-dimensional black string solution.\(^3\) (Especially pertinent to this observation: the 2+1-dimensional anti-de Sitter metric is the natural choice for formulating the SL(2,R) projection of the Weiss-Zumino-Witten model. Significantly, this WZW model uses a conformal field theory to describe string propagation.) It is quite feasible that the string description of the current paper is some sort of semiclassical manifestation of the string in the WZW model. If this relationship could be rigorously established, it would provide an intriguing physical motivation for the Horowitz-Welch duality. We, perhaps boldly, suggest that the current treatment can be viewed as a modest step in this direction. (Note that other string theoretical descriptions have been advocated for the BTZ black hole [21], and it remains an open question as to how any of the various interpretations might be related.)

The rest of the paper is organized as follows. In Section 2, we consider the action of a relativistic string embedded in the background of a BTZ black hole spacetime. Keeping in mind that the string serves as an effective description of horizon dynamics, we are able to express the first-order field equations in the form of a 1+1-dimensional Klein-Gordon equation. In Section 3, we quantize the relevant solutions, which can be identified with fluctuations in the string’s radial position, and obtain a discrete energy spectrum. We then go on to construct a thermodynamic partition function, from which the free energy, internal energy and entropy are extracted. The resulting expression for the entropy is discussed in detail. Section 4 ends with a brief summary.

II. EFFECTIVE ACTION AND FIELD EQUATIONS

On the basis of our preceding discussion (also see [7,17]), we will assume that the horizon dynamics of a BTZ black hole can be effectively described by the action of a relativistic (bosonic) string. That is,

\[
I = -\mathcal{T} \int d^2\xi \sqrt{-h}. \tag{5}
\]

Here, \(\xi^i = \{\tau, \sigma\}\) are the (1+1-dimensional) world-volume coordinates and \(h\) is the determinant of the following induced metric:

\[
h_{ij} = g_{\mu\nu} \frac{\partial X^\mu}{\partial \xi_i} \frac{\partial X^\nu}{\partial \xi_j}, \tag{6}
\]

3. It should be stressed that no such duality is apparent in the case of four dimensions of spacetime. Hence, the current considerations are indigenous to theories of gravity that can be cast, at least locally, into a three-dimensional framework.
where $X^\mu = X^\mu(\tau, \sigma)$ describes the embedding of the string in a 2+1-dimensional spacetime and $g_{\mu\nu}$ is the target-space metric.

The above Nambu-Goto action [29] is known to be equivalent to

$$I = -\frac{T}{2} \int d^2\xi \sqrt{-\eta^{ij} g_{\mu\nu} \partial_i X^\mu \partial_j X^\nu}, \quad (7)$$

where $\partial_i = \partial/\partial \xi^i$. Varying this form with respect to $X^\mu$, we obtain the following field equation:

$$\partial_i \left[ \sqrt{-\eta^{ij} g_{\mu\nu} \partial_j X^\nu} \right] - \frac{1}{2} \sqrt{-\eta^{ij} \partial_i X^\nu \partial_j X^\rho \partial_\mu g_{\nu\rho}} = 0, \quad (8)$$

where $\partial_\mu = \partial/\partial X^\mu$.

For a target-space metric, we now specialize to the curved background of a static BTZ black hole, and so [16]

$$ds^2 = -U(r) dt^2 + \frac{1}{U(r)} dr^2 + r^2 d\phi^2, \quad (9)$$

such that

$$U(r) = \frac{r^2}{l^2} - 8GM = \frac{r^2}{l^2} - \frac{r_{+}^2}{l^2}. \quad (10)$$

Here, $G$ is the 2+1-dimensional Newton constant (i.e., $G \sim l_p$, a Planck length), $M$ is the ADM black hole mass, $l$ is the curvature radius (i.e., $\Lambda = -l^{-2}$ is the cosmological constant), and $r_{+} = \sqrt{8GMl^2}$ is the radius of the black hole horizon. Note that the coordinate $\phi$ is identified with a period of $2\pi$. Also note that we assume a semiclassical regime; meaning $M$ is large enough so that $r_{+} >> l_p$.

Next, let us utilize the gauge symmetry of the system and fix the coordinates as appropriate for a fiducial (static, external) observer. This choice immediately implies that $t, r$ and $\phi$ can be identified with $X^0$, $X^1$ and $X^2$ (respectively). Moreover, the static nature and axial symmetry of the spacetime naturally leads to the following gauge-fixing conditions:

$$t = X^0(\tau, \sigma) = \tau, \quad (11)$$

$$\phi = X^2(\tau, \sigma) = \sigma. \quad (12)$$

Thus, all of the dynamics of the system are contained within the yet-to-be-determined radial function, $r = X^1(\tau, \sigma)$.

As discussed in Section 1, this effective description follows from the premise of a fluctuating horizon with quantum fluctuations on the order of a few Planck lengths. Moreover, the string should maintain an equilibrium position at the order of unity (in Planck units) from the actual horizon, given that short-distance effects have already been accounted for via an implied coarse-graining procedure. It is, therefore, appropriate to write

$$r = X^1(\tau, \sigma) = r_{+} + \epsilon + \delta r(\tau, \sigma)$$

$$= r_e + \delta r(\tau, \sigma), \quad (13)$$

where $X^\mu = X^\mu(\tau, \sigma)$ describes the embedding of the string in a 2+1-dimensional spacetime and $g_{\mu\nu}$ is the target-space metric.
where $r_e$ is the equilibrium position of the string, while $\epsilon$ (a constant “cutoff” length) and $\delta r(\tau, \sigma)$ (a quantum fluctuation) are both on the order of a few Planck lengths. Alternatively, under our semiclassical assumption, both $\epsilon$ and $\delta r << r_e + \epsilon$.

Incorporating the above formalism into Eq.(6) for the induced metric, we find (up to the first order in $\delta r$

$$ds^2_{\hbar} = -[U(r_e) + U'(r_e)\delta r] d\tau^2 + \left[r_e^2 + 2r_e\delta r\right] d\sigma^2,$$

(14)

where a prime denotes differentiation with respect to $r$.

The above form of the induced metric enables an explicit evaluation of the field equation (8). Doing so, we obtain for the $r$, $t$ and $\phi$ components (respectively)

$$-\frac{1}{U(r_e)} \partial^2_r (\delta r) + \frac{1}{r_e^2} \partial^2_\sigma (\delta r) - \left[U'(r_e) - \frac{U(r_e)}{r_e^2} + \frac{1}{2} U''(r_e)\right] \delta r$$

$$= \frac{U(r_e)}{r_e} + \frac{U'(r_e)}{2} + O[(\delta r)^2],$$

(15)

$$\left[\frac{U(r_e)}{r_e} + \frac{U'(r_e)}{2}\right] \partial_\tau (\delta r) + O[(\delta r)^2] = 0,$$

(16)

$$\left[\frac{U(r_e)}{r_e} + \frac{U'(r_e)}{2}\right] \partial_\sigma (\delta r) + O[(\delta r)^2] = 0.$$

(17)

From the last pair of equations, it is quite evident that, for a non-trivial solution of $\delta r$, the quantity inside of the square brackets (in either equation) must vanish. Imposing this constraint on the remaining field equation (15), we are left with

$$-\left[\frac{1}{U(r_e)} \partial^2_r - \frac{1}{r_e^2} \partial^2_\sigma + \mu^2\right] (\delta r) = 0,$$

(18)

where:

$$\mu^2 = \frac{U'(r_e)}{r_e} - \frac{U(r_e)}{r_e^2} + \frac{1}{2} U''(r_e).$$

(19)

It is interesting that the above (18) is simply a two-dimensional Klein-Gordon equation, with the background metric corresponding to the classical limit of the induced metric; cf, Eq.(14). In this way, we can identify $\mu^2$ with the effective mass (squared) of the first-order fluctuations, $\delta r$.

For future reference, note that

$$\mu^2 = \frac{3}{l^2} + O[\epsilon],$$

(20)

where we have applied Eqs.(10) and (13). One can imagine generalizations of the 2+1-dimensional solution used here (for instance, a BTZ black hole with charge [30]). However, the precise form of the effective mass is irrelevant to later arguments, provided that $\mu^2$ remains well-defined in the limiting cases of interest; namely, $\epsilon \to 0$ and $r_+ \to \infty$. 
III. QUANTIZATION AND THERMODYNAMICS

To proceed, it is, of course, necessary to solve the above Klein-Gordon equation (18). For this purpose, let us first decompose the fluctuation field as follows:

$$\delta r(\tau, \sigma) = \sum_m e^{im\sigma} R_m(\tau),$$  \hspace{1cm} (21)

where the periodicity of \( \phi = \sigma \) imposes that \( m \) can only take on integral values. The above form allows us to separate the variables in Eq.(18) and eventually obtain

$$\ddot{R}_m(\tau) + \omega_m^2 R_m(\tau) = 0,$$  \hspace{1cm} (22)

where

$$\omega_m^2 = U(r_e) \left[ \mu^2 + \frac{m^2}{r_e^2} \right]$$  \hspace{1cm} (23)

and a dot denotes differentiation with respect to \( \tau \).

Eq.(22) can readily be identified with the equation of motion for a harmonic oscillator at frequency \( \omega_m \). Hence, the quantization of the modes, \( R_m \), yields the following energy spectrum:

$$E_{nm} = \left( n + \frac{1}{2} \right) \omega_m, \quad n = 0, 1, 2, \ldots$$  \hspace{1cm} (24)

It should be kept in mind that \( \omega_m \sim \sqrt{U(r_e)} \), which is just the Tolman red-shift factor at the stretched horizon of the BTZ black hole. As \( r_e \) approaches the true black hole horizon \( (r_+) \), this red shift goes to zero and Eq.(24) should then be regarded as an energy continuum. Hence, it is really our ignorance of physics below the Planck scale that necessitates a non-vanishing cutoff and, therefore, induces the discrete spacing between the energy levels. To take it a step further, if the spacetime is truly quantized below the Planck level, then Eq.(24) can also be viewed as a manifestation of this effect. A further point of interest is that the above energy levels can be interpreted as a discrete spectrum for the mass of a BTZ black hole. Significantly, this complies with Bekenstein’s notion of black hole spectroscopy [31].

Given the above outcomes, it is natural to construct a thermodynamic partition function in the following manner:

$$Z = \prod_{m=-\infty}^{+\infty} \sum_{n=0}^{\infty} e^{-\beta \left( n + \frac{1}{2} \right) \omega_m},$$  \hspace{1cm} (25)

where \( \beta \) is the inverse of the equilibrium temperature (discussed below). Identifying the sum over \( n \) as a geometric series, we have

$$Z = \prod_{m=-\infty}^{+\infty} \frac{e^{-\beta \omega_m/2}}{1 - e^{-\beta \omega_m}}.$$  \hspace{1cm} (26)

Alternatively, one can re-express this result in terms of the (Helmholtz) free energy:
\[ F = -\frac{1}{\beta} \ln Z = -\frac{1}{\beta} \sum_{m=-\infty}^{+\infty} \ln \left[ \frac{e^{-\beta \omega_m/2}}{1 - e^{-\beta \omega_m}} \right]. \quad (27) \]

The standard formula for the internal energy of a thermodynamic system gives us
\[ \mathcal{E} = \frac{\partial (\beta F)}{\partial \beta} = \sum_{m=-\infty}^{+\infty} \left[ \frac{\omega_m e^{\beta \omega}}{e^{\beta \omega} - 1} + \frac{\omega_m}{2} \right]. \quad (28) \]

The first term (on the right-hand side) is the anticipated Planckian or thermal spectrum, whereas the second term is a divergent contribution that would likely be removed upon a suitable process of renormalization.

The associated entropy can also be obtained via a standard relation, for which we find
\[ S = -\beta (F - \mathcal{E}) = \sum_{m=-\infty}^{+\infty} \left[ \frac{\beta \omega_m}{e^{\beta \omega} - 1} - \ln \left( 1 - e^{-\beta \omega_m} \right) \right]. \quad (29) \]

Considering that the string lives in the vicinity of the black hole horizon, we expect the spacing between adjacent energy levels to be correspondingly small; cf. Eq.(24) and the subsequent discussion. It thus follows that the above summation can be accurately evaluated as an integral over \( m \). That is,
\[ S = \frac{2 r_e}{\beta \sqrt{U(r_e)}} \int_{\beta \omega_0}^{\infty} dx \left[ \frac{x}{e^x - 1} - \ln \left( 1 - e^{-x} \right) \right], \quad (30) \]
where the integration variable has been changed for convenience and \( \omega_0 = \omega_{m=0} \).

So far, the equilibrium value of the temperature, \( \beta^{-1} \), has been left unspecified. However, it seems realistic that this value should be closely related to the Hawking temperature of the BTZ black hole: \( T_{BTZ} = r_+/2\pi l^2 \) \cite{16}. In fact, one would most naturally expect that \( \beta^{-1} = T_{BTZ} + \mathcal{O}[\epsilon] \), and we will assume that this is correct.

Applying the above, and also recalling that \( \omega_0 = \mu \sqrt{U(r_e)} \), \( \mu \sim l^{-1} \) and \( U(r_e) \sim \epsilon r_+/l^2 \) (cf. Eqs.(23,20,10,13)), we have
\[ \beta \omega_0 \sim \sqrt{\frac{\epsilon}{r_+}} \ll 1. \quad (31) \]

In light of this deduction, the above integral (30) can readily be evaluated to yield
\[ S = \frac{4 \zeta(2) r_e}{\beta \sqrt{U(r_e)}} + \mu r_e \ln \left[ \mu^2 \beta^2 U(r_e) \right] + \mathcal{O}[\beta \omega_0], \quad (32) \]
where we have discarded an irrelevant constant term.

Let us first focus on the leading-order term, which will be denoted by \( S_1 \). Applying the formalism of the last two paragraphs, we find that
\[ S_1 \approx \frac{\eta r_+^{3/2}}{\epsilon^{1/2}}, \quad (33) \]
where \( \eta \) represents a numerical factor of order unity.
To make sense of this result, it is necessary that the cutoff parameter, \( \epsilon \), be re-expressed in terms of an invariant, proper distance; say \( y \). More specifically,

\[
y = \int_{r_+}^{r_+ + \epsilon} \frac{dr}{\sqrt{U(r)}} = l\sqrt{\frac{2\epsilon}{r_+}} + O[\epsilon],
\]

and so

\[
S_1 \approx \eta \frac{r_+}{y}.
\]

By hypothesis, we have \( y \sim l_p \sim G \), so that \( S_1 \) is in agreement (up to a numerical factor of \( O[1] \)) with the Bekenstein-Hawking area law of a BTZ black hole [16],

\[
S_{BH} = \frac{A_+}{4G},
\]

where \( A_+ = 2\pi r_+ \) is the “area” of the horizon in 2+1 dimensions.

In view of the factorization of classical and quantum path integrals (cf. Eq.(3)), one would actually expect the total black hole entropy to be given by a sum: the tree-level area law (36) plus the entropy of the quantum fluctuations (32). This means that, to the leading quantum order, we can write

\[
S_{tot} = S_{BH} + S_1 = \frac{A_+}{4} \left[ \frac{1}{G} + \frac{\eta}{y} \right].
\]

It may appear, at a first glance, somewhat problematic that \( y \) can, in principle, be extrapolated to an infinitesimally small distance (i.e., \( y \ll l_p \sim G \)). For such an extrapolation, the (total) black hole entropy would apparently diverge; however, even in this event, the precise Bekenstein-Hawking formula can still effectively be preserved. This observation follows by virtue of the inverse gravitational coupling \((G^{-1}) \) always being uncertain up to a potentially infinite renormalization [32,33]. What is important, from our current perspective, is that \( S_1 \) does indeed comply with the area law, so that the leading-order effects can always be renormalized away.

Incidentally, it can be (and has been [34]) argued that, for a calculation of this nature, the leading-order quantum term should be viewed as the principal source of black hole entropy rather than a “supplement” (as implied by Eq.(37)). However, thanks to the renormalizability of \( G \), these two viewpoints are operationally indistinguishable at the order of the area law.

Let us now cast our attention on the first-order correction to the area law, which will be denoted by \( S_2 \). On the basis of very general arguments (with origins in either state-counting [35] or thermodynamic principles [36]), the leading-order correction is expected to be directly proportional to the logarithm of the horizon area. Moreover, for a BTZ black hole in particular, this logarithmic correction appears to have a prefactor of \(-3/2 \) [35,36].

From an inspection of the second term in Eq.(32), it is clear that the inverted argument of the logarithm is indeed proportional to \( r_+ \sim A_+ \). More explicitly, we find (up to irrelevant constants, higher-order corrections and a \( \ln \epsilon \) term which will be commented on below) that

\[
S_2 \approx -\mu r_+ \ln(A_+).
\]
Substituting Eq.(20), we then obtain the following:

\[
S_2 \approx \left[ -\frac{\sqrt{3}}{l} r_+ + O(\epsilon) \right] \ln(A_+).
\]  

(39)

Here, we find that the prefactor coincides with the prescribed value of \(-3/2\) \[35,36\] only for the very special instance of \(r_+ = \sqrt{3l}/2\). We view this discrepancy as support for the notion that \(S_1\) is a supplementary rather than principal source of the black hole entropy (see the prior discussion). That is to say, if \(S_{BH}\) and \(S_1\) are fundamentally distinct quantities, then one would not \textit{a priori} expect their leading-order corrections to be in precise agreement.

Recall the importance, from a renormalization perspective, that \(S_1\) be in compliance with the area law. It is similarly important that the logarithmic prefactor in \(S_2\) remains finite as \(\epsilon \to 0\). The reason being that it is not at all clear that a term \(\propto \ln(A_+)\) can in any way be renormalized. In this regard, our above finding is quite reassuring. On the other hand, \(S_2\) also gives rise to a term \(\propto \ln(\epsilon)\), which is dangerously divergent. However, it is expected that such a term can indeed be renormalized away \[33,37\] and is, therefore, of no physical consequence.

\section*{IV. CONCLUSION}

In summary, we have been considering Maggiore’s “membrane model” \[7,17\] in the context of a 2+1-dimensional BTZ black hole \[16\]. The central idea is that a relativistic membrane (or string for this BTZ scenario) can effectively describe the horizon dynamics of a black hole. This, in turn, suggests that the elusive quantum degrees of freedom (in a black hole spacetime) can be identified with the fluctuations of a suitably defined membrane or string. In the current study, we have found that these fluctuations conform to a two-dimensional Klein-Gordon equation. Moreover, we have shown that the associated solutions can be readily quantized, thus leading to a discrete spectrum of energies. This formalism was then used to construct a thermodynamic partition function, from which the “quantum” black hole entropy could ultimately be extracted. At the leading order, we substantiated the Bekenstein-Hawking area law (for a BTZ black hole \[16\]), which indicates that the horizon dynamics effectively translate into a renormalization of the gravitational coupling \[32\]. We also verified a next-to-leading-order correction that is directly proportional to the logarithm of the horizon area. Although the logarithmic prefactor did not generally agree with some prior calculations \[35,36\], we have argued that our result is still consistent with any \textit{a priori} expectations.

It is worth re-emphasizing that the positive results of this analysis are highly non-trivial; insofar as our calculation of the black hole entropy followed from the analysis of a two-dimensional Klein-Gordon equation. Let us also remind the reader that similar outcomes were obtained by Lousto \[15\] in a 3+1-dimensional context. By generalizing this prior treatment, we have provided further support for the membrane interpretation of a black hole horizon \[13\]. Significantly, this membrane paradigm has already served as an antecedent for black hole complementarity \[14\] and the holographic principle \[38\]; both of which play a pivotal role in our current understanding of quantum gravity.
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