GRASSMANNIANS AND CONFORMAL STRUCTURE ON ABSOLUTES

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Abstract. We study grassmannians associated with a linear space with a nondegenerate hermitian form. The geometry of these grassmannians allows us to explain the relation between a (pseudo-)riemannian projective geometry and the conformal structure on its ideal boundary (absolute). Such relation encompasses, for instance, the usual conformal structure on the absolute of real hyperbolic space, the usual conformal structure on the absolute of de Sitter space, the conformal contact structure on the absolute of complex hyperbolic space, and the causal structure on the absolute of anti-de Sitter space.

Dedicated to the memory of Waldyr Rodrigues Jr.

1. Introduction

We study the geometry of a nondegenerate hermitian form on an \( \mathbb{R} \)- or \( \mathbb{C} \)-vector space \( V \) from the perspective of the grassmannian \( \text{Gr}(k, V) \). The grassmannian consists of two complementary parts: \( \text{Gr}^0(k, V) \), formed by nondegenerate subspaces, and the absolute.

The generic part \( \text{Gr}^0(k, V) \) of the grassmannian has several components, each corresponding to the \( k \)-dimensional subspaces of \( V \) of a certain nondegenerate signature. A tangent vector at a point \( p \in \text{Gr}^0(k, V) \) is a linear map \( t \in \text{Lin}(p, p^\perp) \) and the trace \( \text{tr}(t_1^* t_2) \) provides an hermitian metric on \( \text{Gr}^0(k, V) \). In particular, each component of \( \text{Gr}^0(k, V) \) is endowed with the pseudo-riemannian metric \( \text{Re} \text{tr}(t_1^* t_2) \).

There exists, however, a more adequate way to deal with the geometry of grassmannians. It involves observations and the product. Extending by zero, the tangent vector \( t \in \text{Lin}(p, p^\perp) \) at \( p \in \text{Gr}^0(k, V) \) can be viewed as a linear map \( t \in \text{Lin}(V, V) \). Conversely, a linear map \( t \in \text{Lin}(V, V) \) gives rise to the tangent vector

\[
  t_p := \pi[p] t \pi'[p]
\]

at \( p \in \text{Gr}^0(k, V) \), where \( \pi'[p] : V \to p \) and \( \pi[p] : V \to p^\perp \) stand for the orthogonal projections onto \( p \) and \( p^\perp \), respectively. The tangent vector \( t_p \) is called the observation of \( t \) at \( p \). Observations produce smooth vector fields on \( \text{Gr}^0(k, V) \) that play, in the context of grassmannian geometries, a role similar to that played by left invariant vector fields in the theory of Lie groups. Such vector fields are already important in the projective case \( k = 1 \) [AGR1].

In the expression \( \text{tr}(t_1^* t_2) \) of the hermitian metric on \( \text{Gr}^0(k, V) \), one can take tangent vectors \( t_1, t_2 \) observed at distinct points. Moreover, the characteristic polynomial \( \text{char}(t_1^* t_2) \), or simply the product \( t_1^* t_2 \) itself, can be taken instead of the hermitian metric. The coefficients of \( \text{char}(t_1^* t_2) \) provide not only the hermitian metric but also other geometric characteristics such as the invariant \( \frac{\det(t^* t)}{\text{tr}^k(t^* t)} \) of
the geodesic in $\text{Gr}^0(k, V)$ determined by a tangent vector $t$ [AGr2]. Not infrequently, a fact involving the hermitian/pseudo-riemannian metric is just a simple consequence of some property of the product. For example, the $m$-Plücker embedding
\[ E^m : \text{Gr}^0(k, V) \to \text{Gr}^0 \left( \binom{k}{m}, \bigwedge^m V \right), \quad p \mapsto \bigwedge^m p \]
is a (minimal) isometric embedding [AGr2]; this a trivial consequence of the functorial behaviour of $E^m$ with respect to the product: $(E^m t_1)^*E^m t_2 = E^m (t_1^* t_2)$. In other words, $E^m$ is an ‘isometric embedding’ in the sense of the product.

The variations involving observations and the product are endless. In order to measure distance in a riemannian component $C$ of $\text{Gr}^0(k, V)$, for example, we do not really need the metric, only observations. Indeed, let $p_1, p_2 \in C$ and let $t \in \text{Lin}(p_1, p_1^\perp) \subset \text{Lin}(V)$ be a nonnull tangent vector at $p_1$. We observe $t$ at $p_2$ and then observe the result back at $p_1$ thus obtaining a new tangent vector $t'$ at $p_1$. The change suffered by $t'$ reflects how $p_1, p_2$, and $t$ are related and allows us to infer the distance $\text{dist}(p_1, p_2)$. If, say, $k = 1$ and $t$ is tangent to the projective line joining $p_1$ and $p_2$, then $t' = \text{ta}^2(p_1, p_2) t$, where
\[ \text{ta}(p_1, p_2) := \frac{\langle p_1, p_2 \rangle \langle p_2, p_1 \rangle}{\langle p_1, p_1 \rangle \langle p_2, p_2 \rangle} \]
and $\text{dist}(p_1, p_2) = \arccosh \sqrt{\text{ta}(p_1, p_2)}$ when the form on the subspace $\mathbb{R} p_1 + \mathbb{R} p_2 \subset V$ is indefinite or $\text{dist}(p_1, p_2) = \arccos \sqrt{\text{ta}(p_1, p_2)}$ when the mentioned form is definite. Analogously, given points $p_1, p_2, p_3 \in C$ and a tangent vector $t$ at $p_1$, a similar procedure of successive observations leads to a new tangent vector $t'$ at $p_1$. Again, $t'$ reflects the relation between $p_1, p_2, p_3$ and $t$. If $p_1, p_2, p_3 \in C$ lie in a complex projective line, i.e., the triangle $\Delta(p_1, p_2, p_3)$ is $\mathbb{C}$-plane, and $t$ is tangent to this line, then $t' = t_{12}^2 t_{23}^2 t_{31}^2 \exp \left( 2i \text{Area} \Delta(p_1, p_2, p_3) \right) t$, where $t_{ij} := \text{ta}(p_i, p_j)$.

The above process of successive observations may be performed even when the metric is not riemannian or when the points do not belong to a same component of $\text{Gr}^0(k, V)$. In fact, in the global picture composed of all the pseudo-riemannian pieces of $\text{Gr}^0(k, V)$, observations and the product tie everything together. Tangent vectors to a point in one piece are observable from the points in the other pieces and we can take the product of tangent vectors at points in different pieces.

But what about the geometry on the absolute?

Our main objective in this paper is to show that observations and the product partially survive at a degenerate point $p \in \text{Gr}(k, V)$. Roughly speaking, let $q$ be the kernel of the hermitian form on $p$ and take the quotient $V_q := q^\perp / q$. The points on the absolute of $\text{Gr}(k, V)$ with the same $q$ form a fibre of a certain bundle and observations and the product are defined for tangent vectors to such fibre because $V_q$ is naturally equipped with a nondegenerate hermitian form. In particular, we obtain a hermitian metric on the fibre.

When we consider the generic part of the absolute of $\text{Gr}(k, V)$, that is, the points $p \in \text{Gr}(k, V)$ such that the kernel of the hermitian form on $p$ is one-dimensional, the bundle in question fibres over the absolute of a projective geometry $\mathbb{P} V$. Surprisingly, in this case, the hermitian metric on the fibres provides the conformal (or conformal contact) structure on the absolute of $\mathbb{P} V$. In other words, the conformal structure is exactly what remains from the metric when we arrive at the absolute.

The basic geometrical objects in projective geometries, such as geodesics, totally geodesic subspaces, equidistant loci, etc., have linear nature (see [AGr1] and references therein). In other words, grassmannians constitute a natural habitat for important geometrical objects in classical spaces. Now, we can see that the geometry of grassmannians also accounts for the relation between the (pseudo-)riemannian metric of a projective geometry and the conformal structure on its absolute. Particular cases of conformal structures that are amenable to this approach include the usual conformal structure on the absolute.
of real hyperbolic space, the usual conformal structure on the absolute of de Sitter space, the conformal contact structure on the absolute of complex hyperbolic space, and the causal structure on the absolute of anti-de Sitter space (see Section 3).

The causal structure on the absolute of anti-de Sitter space plays an important role in general relativity and the theory of black holes (see, for instance, [Car], [HoP], and references therein). Another known application is possibly of a more unexpected nature: the absolute of anti-de Sitter 4-space can be identified with the lagrangian grassmannian $\Lambda(2)$, the space of lagrangian planes in a 4-dimensional symplectic $\mathbb{R}$-linear space. In other words, $\Lambda(2)$ is naturally equipped with an extra structure arising from causal geometry (see [Cal] and Subsection 3.7). This phenomenon may be seen as a geometrical manifestation of the exceptional isomorphism $\text{Spin}(2, 3) = \text{Sp}(4, \mathbb{R})$.

Acknowledgements. It was with great sadness that we heard of the passing of Professor Waldyr Rodrigues Jr. He was a dear friend and an exceptional scientist with whom we had the pleasure and privilege to discuss many subjects related to mathematics and physics. Waldyr was very fond of natural and coordinate-free methods in geometry, a point of view that gave rise to this paper.

We are very grateful to Alexei L. Gorodentsev and Nikolay A. Tyurin for their stimulating interest in this work and to the referees whose suggestions have greatly contributed to the exposition.

This paper was partially developed while the first author was enjoying the hospitality of the IHES and, the third author, that of the MPIM.

2. Stratification and fibre bundle

Let $V$ be an $n$-dimensional $\mathbb{K}$-vector space equipped with a nondegenerate hermitian form $\langle -, - \rangle$ of arbitrary signature, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. The grassmannian $\text{Gr}_k(k, V)$ of $k$-dimensional $\mathbb{K}$-vector subspaces in $V$ can be described as follows. Fix a $\mathbb{K}$-vector space $P$ such that $\dim_\mathbb{K} P = k$. Denote by

$$M := \{ p \in \text{Lin}_k(P, V) \mid \ker p = 0 \}$$

the open subset of all monomorphisms in the $\mathbb{K}$-vector space $\text{Lin}_k(P, V)$. The group $\text{GL}_k P$ acts from the right on $\text{Lin}_k(P, V)$ and on $M$. The grassmannian $\text{Gr}_k(k, V)$ is simply the quotient space

$$\text{Gr}_k(k, V) := M/\text{GL}_k P, \quad \pi : M \to M/\text{GL}_k P.$$

We will not distinguish between the notation of points in $\text{Gr}_k(k, V)$ and of their representatives in $M$. Moreover, we will frequently write $p$ in place of the image $pP$ and $p^\perp$, in place of the orthogonal $(pP)^\perp$ to that image. For example, $V/p$ will denote $V/pP$.

The tangent space $T_p M$ is usually identified with $\text{Lin}_k(P, V)$ as follows. Given $\varphi : P \to V$, the curve $c(\varepsilon) := p + \varepsilon \varphi$ lives in $M$ for small $\varepsilon$ and $\varphi$ is identified with $\dot{c}(0)$. We use a slightly different identification $T_p M = \text{Lin}_k(p, V)$ where $p$ takes the place of $P$. In this way, $\overline{t} \in \text{Lin}_k(p, V)$ is interpreted as the tangent vector $\dot{c}(0)$, where $c(\varepsilon) := (1 + \varepsilon \overline{t})p$ and $\overline{t} \in \text{Lin}_k(V, V)$ extends $\overline{t}$. Note that $\overline{t} \in \text{Lin}_k(p, V)$ is tangent at $p$ to the orbit $p \text{GL}_k P$ if and only if $\overline{t}p \subset p$ since in this case $(1 + \varepsilon \overline{t})p = pg$ for small $\varepsilon$ and suitable $g \in \text{GL}_k P$. Therefore, $T_p \text{Gr}_k(k, V) = \text{Lin}_k(p, V/p)$.

2.1. Remark. Let $w \subset V$ be a linear subspace and let $\text{Gr}_k(k, w, V) \subset \text{Gr}_k(k, V)$ stand for the space of all $k$-dimensional linear subspaces in $V$ that are included in $w$. Then $T_p \text{Gr}_k(k, w, V) = \text{Lin}_k(p, w/p)$ for all $p \in \text{Gr}_k(k, w, V)$.

Dually, let $\text{Gr}_k(k, V, q) \subset \text{Gr}_k(k, V)$ denote the space of all $k$-dimensional linear subspaces in $V$ containing $q$, where $q \subset V$ is a given $d$-dimensional subspace. Then $T_p \text{Gr}_k(k, V, q) = \text{Lin}_k(p/q, V/p)$ for all $p \in \text{Gr}_k(k, V, q)$.
The degree of degeneracy of a point \( p \in \text{Gr}_K(k, V) \) is the dimension of the kernel of the hermitian form restricted to \( p \), that is, the dimension of the subspace \( p \cap p^\perp \subset V \). The grassmannian \( \text{Gr}_K(k, V) \) is stratified according to such degree:

\[
\text{Gr}_K(k, V) = \bigsqcup_d \text{Gr}^d_k(k, V), \quad \text{Gr}^d_k(k, V) := \{ p \in \text{Gr}_K(k, V) \mid \dim_K(p \cap p^\perp) = d \}.
\]

Subspaces of \( V \) of a given signature form an \( UV \)-orbit. Therefore, each stratum is the disjoint union of a finite number of such orbits, hence, a manifold (note that a stratum is not necessarily connected, so we use the word ‘stratification’ with a meaning slightly different from the usual one). There are finitely many strata and the closure of the stratum \( \text{Gr}^d_k(k, V) \) is the disjoint union of strata \( \bigsqcup_{d \geq d_0} \text{Gr}^d_k(k, V) \).

The generic part \( \text{Gr}^0_k(k, V) \) corresponding to the nondegenerate subspaces of \( V \) is open in \( \text{Gr}_K(k, V) \).

Associating to each \( p \in \text{Gr}_K(k, V) \) the kernel of the hermitian form on \( p \), we get the \( UV \)-equivariant fibre bundle

\[
\pi_d : \text{Gr}^d_k(k, V) \to \text{Gr}^d_k(d, V), \quad \pi_d : p \mapsto p \cap p^\perp.
\]

The fibre \( \pi_d^{-1}(q) \) can be naturally identified with \( \text{Gr}^0_k(k - d, V_q) \), where \( V_q := q^\perp/q \) is equipped with a natural nondegenerate hermitian form and \( \dim_K V_q = n - 2d \).

We are particularly interested in the case \( d = 1 \) corresponding to the generic part of the absolute of \( \text{Gr}_K(k, V) \). This is the case that will be applied, in the next section, to the study of conformal structures on the absolute of projective geometries: when \( d = 1 \), the bundle (2.2) fibres over \( \text{Gr}^1_k(1, V) \) which is the absolute, denoted by \( SV \), of \( \mathbb{P}_K V = \text{Gr}_K(1, V) \). Fibres are easy to visualize, as they correspond to subspaces ‘rotating’ about their common one-dimensional kernel, hence, forming a dense open part of the grassmannian \( \text{Gr}_K(k - 1, n - 2) \). The bundle itself is also simple to describe, as the next couple of propositions show.

### 2.3. Proposition

Let \( K = \mathbb{R} \). The bundle \( \pi_1 : \text{Gr}^1_k(k, V) \to \text{Gr}^1_k(1, V) \) is the nondegenerate part of a grassmannization of the tangent bundle of the absolute \( SV \) of the projective geometry \( \mathbb{P}_K V \).

**Proof.** Take \( q \in \text{Gr}^1_k(1, V) = SV \) and define \( V_q := q^\perp/q \). Since the fibre \( \pi_1^{-1}(q) \) is naturally identified with the nondegenerate part \( \text{Gr}^0_k(k - 1, V_q) \) of the grassmannian \( \text{Gr}_K(k - 1, V_q) \), and the subspace \( q \subset V \) is one-dimensional, it suffices to prove that the tangent space to \( SV \) at \( q \) has the form \( T_q SV = \mathbb{P}_K V(q, V_q) \).

Take \( u \in V \) such that \( SV \subset \mathbb{P}_K V \) is locally given by the equation \( f(x) = 0 \) in a neighbourhood of \( q \), where \( f(x) := \frac{\langle x, x \rangle}{\langle x, u \rangle \langle u, x \rangle} \).

Let \( t \in \text{Lin}(q, V/q) \) be a tangent vector to \( \mathbb{P}_K V \) at \( q \) and let \( \tilde{t} \in \text{Lin}(V, V) \) be a lift of \( t \). Then \( t \) is tangent to \( SV \) if and only if \( tf = 0 \), i.e.,

\[
\left. \frac{d}{de} \right|_{e=0} \frac{\langle q + e\tilde{t}q, q + e\tilde{t}q \rangle}{\langle q, u \rangle \langle u, q + e\tilde{t}q \rangle} = \frac{2 \text{Re}(\tilde{t}q, q)}{\langle q, u \rangle \langle u, q \rangle} = \frac{2 \text{Re}(\tilde{t}q, q)}{\langle q, u \rangle \langle u, q \rangle} = 0.
\]

(In the formula, \( q \in V \) denotes an element representing the subspace \( q \).)

When \( K = \mathbb{C} \), a calculus analogous to the one presented in the proof of Proposition 2.3 shows that the tangent space to the absolute \( SV \) of the projective geometry \( \mathbb{P}_C V = \text{Gr}_C(1, V) \) at a point \( q \in SV \) is given by

\[
T_q SV = \{ t \in \text{Lin}_C(q, V/q) \mid \text{Re}(\tilde{t}q, q) = 0 \}.
\]

Hence, \( \text{Lin}_C(q, V_q) \subset T_q SV \) is not the entire tangent space at \( q \) as in the real case, but a maximal complex subspace of \( T_q SV \). In other words, the subspaces of the form \( \text{Lin}_C(q, V_q) \) give rise to a CR-distribution on \( SV \). Summarizing, we have the following proposition.
2.4. Proposition. Let $\mathbb{K} = \mathbb{C}$. The bundle $\pi_1 : \text{Gr}^1_{\mathbb{C}}(k, V) \to \text{Gr}^1_{\mathbb{C}}(1, V)$ is the nondegenerate part of a grassmannization of the CR-distribution in $TSV$ given by $\text{Lin}_{\mathbb{C}}(q, V_q) \subset T_qSV$, where $q \in SV$ and $V_q := q^\perp/q$.

The case $d > 1$ deals with a ‘degenerate’ part of the absolute of the grassmannian. The bundle $\pi_d : \text{Gr}^d_{\mathbb{C}}(k, V) \to \text{Gr}^d_{\mathbb{K}}(d, V)$ is no longer the grassmannization of the tangent bundle of $\text{Gr}^d_{\mathbb{K}}(d, V)$ (considering, say, the real case). Nevertheless, the bundle indicates a distinguished distribution whose geometric nature would be interesting to dwell on.

The fact that $\pi_d^{-1}(q) = \text{Gr}^0_{\mathbb{K}}(k - d, V_q)$, where $V_q := q^\perp/q$ is taken with its natural induced non-degenerate hermitian form, means that the fibres of the bundle (2.2) are naturally equipped with the geometric structure consisting of observations and of the product (this structure on the nondegenerate part of a grassmannian was discussed in the introduction).

Observations and the product on the fibre $\pi_d^{-1}(q)$ can be explicitly described as follows. By Remark 2.1,

$$T_p\pi_d^{-1}(q) = \text{Lin}_\mathbb{K}(p/q, q^\perp/p)$$

for all $p \in \pi_d^{-1}(q)$ because $\text{Gr}^0_{\mathbb{K}}(k - d, V_q)$ is open in $\text{Gr}_\mathbb{K}(k - d, V_q)$. Let $p \in \pi_d^{-1}(q)$. Clearly, $q = p \cap p^\perp$, $q^\perp = p + p^\perp$, and $V_q = p_0 \oplus p_0^\perp$, where $p_0 := p/q$ and $p_0^\perp = p^\perp/q$. We have $T_p \pi_d^{-1}(q) = \text{Lin}(p_0, p_0^\perp)$. Denote by $\pi'[p_0]$ and $\pi[p_0]$ the orthogonal projections onto $p_0$ and $p_0^\perp$, respectively. Then the linear map $t \in \text{Lin}_\mathbb{K}(V_q, V_q)$ gives rise to the tangent vector

$$t_{p_0} := \pi[p_0]t\pi'[p_0]$$

to $\pi_d^{-1}(q)$ at $p_0$ (the observation of $t$ at $p_0$). The product $t_1^*t_2$, where $t_1, t_2 \in \text{Lin}(V_q, V_q)$, gives rise to the hermitian metric $\text{tr}(t_1^*t_2)$ on the fibre when $t_1$ and $t_2$ are observed from a same point. In particular, the fibre $\pi_d^{-1}(q)$ consists of a collection of pseudo-Riemannian manifolds, each corresponding to the $(k - d)$-dimensional subspaces of $V_q$ of a given signature. Some particular examples will be considered in the next section.

3. Conformal structures

The (pseudo-)riemannian metric on a space typically corresponds to some geometric structure on its ideal boundary (absolute). The correspondence is such that both the metric and the boundary structure have the same automorphisms. For example, the absolute of real hyperbolic space is a conformal sphere, the absolute of de Sitter space is a conformal sphere, the absolute of complex hyperbolic space is a CR-sphere, and the absolute of anti-de Sitter space has a causal structure.

We will see in this section that the bundle (2.2) allows us to explain these ‘gravity/conformal structure’ correspondences. At the end of the day, the hermitian form on $V$ is responsible not only for the metric, but also for the geometric structure on the absolute. The role of the bundle (2.2) is to provide a way for this hermitian form to ‘reach’ the tangent space of the absolute.

We call the bundle $\pi_1 : \text{Gr}^1_{\mathbb{K}}(k, V) \to \text{Gr}^1_{\mathbb{K}}(1, V)$ the conformal (conformal contact) when $\mathbb{K} = \mathbb{C}$ structure on the absolute $SV = \text{Gr}^1_{\mathbb{K}}(1, V)$ of the projective geometry $\mathbb{P}_\mathbb{K}V = \text{Gr}_{\mathbb{K}}(1, V)$. Let us analyze the conformal (conformal contact) structure for $k = 2$ in a few cases.

We will use the following observation in all the examples below. Let $p \in \text{Gr}^0_{\mathbb{K}}(1, V)$ and let $t_1, t_2 \in \text{Lin}(p, p^\perp)$ be tangent vectors at $p$. It is easy to see that, in these projective settings, the expression

$$\pm \text{tr}(t_1^*t_2) = \pm \frac{(t_1(p, t_2)p)}{(p, p)}$$

gives the (pseudo-)riemannian metrics in the components of $\text{Gr}^0_{\mathbb{K}}(1, V)$.
3.2. Real hyperbolic and de Sitter spaces. Let $\mathbb{K} = \mathbb{R}$ and assume that $V$ is equipped with an hermitian (i.e., bilinear symmetric) form of signature $+ \cdots + -$. Let $n := \dim_{\mathbb{R}} V$. Then $Gr_{\mathbb{K}}^{0}(1, n)$ has two components: one, called the real hyperbolic space $\mathbb{H}^{n-1}_{\mathbb{R}}$, consists of negative points; the other, called the de Sitter space $d\mathbb{S}^{n-1}$, consists of positive points. The absolute equals $Gr_{\mathbb{K}}^{1}(1, n)$ and is the sphere $\mathbb{S}^{n-2}$ of null points. In other words, the real hyperbolic space and the de Sitter space are glued along their absolutes; we call the projective space $Gr_{\mathbb{K}}^{1}(1, V)$ the extended real hyperbolic space.

Let $p \in Gr_{\mathbb{K}}^{0}(1, V)$ and let $t_{1}, t_{2} \in \text{Lin}_{\mathbb{R}}(p, p^{\perp})$ be tangent vectors at $p$. We introduce the metric $\langle t_{1}, t_{2} \rangle := -\text{tr}(t_{1}^{*}t_{2})$ on $Gr_{\mathbb{K}}^{0}(1, V)$. Due to (3.1) and to the signature of the form, this is a riemannian metric on $\mathbb{P}^{n-1}_{\mathbb{R}}$ and a lorentzian metric on $d\mathbb{S}^{n-1}$. In this way, we obtain the usual constant curvature metrics on the hyperbolic and de Sitter spaces [AGr1].

Let $q \in SV$. The fibre $\pi_{1}^{-1}(q)$ of the conformal structure $\pi_{1} : Gr_{\mathbb{K}}^{1}(2, V) \to Gr_{\mathbb{K}}^{1}(1, V)$ is naturally identified with $Gr_{\mathbb{K}}^{0}(1, V_{q}) = \mathbb{P}^{n-3}_{\mathbb{R}}$ because $k = 2$, $d = 1$, and $\dim_{\mathbb{R}} V_{q} = n - 2$, where $V_{q} := q^{\perp}/q$. The induced form on $V_{q}$ is positive-definite since the signature of the form on $q^{\perp}$ is $0 + \cdots +$. The form on $V_{q}$ therefore provides a riemannian metric on each fibre. The latter is a metric of constant curvature on $\mathbb{P}^{n-3}_{\mathbb{R}}$ [AGr1].

By Proposition 2.3, the bundle $\pi_{1} : Gr_{\mathbb{K}}^{1}(2, V) \to Gr_{\mathbb{K}}^{1}(1, V)$ is the projectivization of the tangent bundle of the absolute $\mathbb{S}^{n-2}$. Distances in the fibre are nothing but the angles of the standard $1$ conformal structure on the sphere $\mathbb{S}^{n-2}$. 

3.3. Anti-de Sitter space. Take $\mathbb{K} = \mathbb{R}$ and $V$ of signature $+ \cdots + -$. The anti-de Sitter space $ad\mathbb{S}^{n-1}$ is the negative part of $\mathbb{P}_{\mathbb{R}}V$. It is a lorentzian manifold. The fibre $\pi_{1}^{-1}(q)$ of the conformal structure $\pi_{1} : Gr_{\mathbb{K}}^{2}(2, V) \to Gr_{\mathbb{K}}^{1}(1, V)$ is naturally identified with $Gr_{\mathbb{K}}^{0}(1, V_{q})$ and is an open dense set in $\mathbb{P}^{n-3}_{\mathbb{R}}$. Since the signature of the form on $q^{\perp}$ is $0 - + \cdots +$, the signature of the form on $V_{q} = q^{\perp}/q$ is $- + \cdots +$ and the fibres carry the metric of the extended hyperbolic space of the previous example (each fibre is an extended hyperbolic space minus its absolute).

Such a conformal structure is known as a causal structure. It plays in anti-de Sitter geometry the same role played by the standard conformal structure in the absolute of the real hyperbolic space.

Curiously, the absolute of anti-de Sitter 4-space $ad\mathbb{S}^{4}$ can be naturally identified with the space $\Lambda(2)$ of lagrangian planes in a 4-dimensional symplectic $\mathbb{R}$-linear space (see Subsection 3.6). Hence, the lagrangian grassmannian $\Lambda(2)$ possesses a natural causal structure.

For $k > 2$, the geometry on the grassmannization $Gr_{\mathbb{K}}^{1}(k, V) \to Gr_{\mathbb{K}}^{1}(1, V)$ of the tangent bundle of the absolute (see Proposition 2.3) is related to the case $k = 2$ in the same way as are related the grassmannian and the projective geometries.

3.4. Complex hyperbolic space. Take $\mathbb{K} = \mathbb{C}$ and $V$ of signature $+ \cdots + -$. The complex hyperbolic space $H^{n-1}_{\mathbb{C}}$ is the negative part of $\mathbb{P}_{\mathbb{C}}V$ and its ideal boundary, the absolute, is the sphere $SV \simeq \mathbb{S}^{2n-3}$. Every fibre $\mathbb{P}_{\mathbb{C}}V_{q} \simeq \mathbb{P}^{n-3}_{\mathbb{C}}$ of the conformal contact structure carries the Fubini-Study metric. By Proposition 2.4, the conformal contact structure $\pi_{1} : Gr_{\mathbb{C}}^{2}(2, V) \to Gr_{\mathbb{C}}^{1}(1, V)$ is the projectivization of a CR-distribution in the tangent bundle of the absolute $\mathbb{P}^{n-3}_{\mathbb{C}}$ and the distances in a fibre are the angles between complex directions.

Note that any projective geometry can play the role of a (contact) conformal structure. In particular, the (contact) conformal structure can possess its own absolute and so on . . .

3.5. Comments and questions. Algebraic formulae that deal with geometrical quantities (the tance and the $\eta$-invariant are some simple examples, see [AGr1, Example 5.12]) work well for points in distinct pieces of $Gr_{\mathbb{R}}^{0}(k, V)$. Such formulae use to alter its geometrical sense when the points involved

\footnote{If one wishes to deal with angles varying in $[0, 2\pi]$, then the projectivization should be taken with respect to $\mathbb{R}^{+}$, from the very beginning.}
are taken in different components of $\text{Gr}_K^0(k, V)$. In this respect, it is interesting to understand if there is an explicit geometrical interpretation of observations and the product in the terms of the usual (pseudo-)riemannian concepts for $k > 2$.

The bundle $\pi_d : \text{Gr}_K^d(k, V) \to \text{Gr}_K^d(d, V)$ might admit a canonical connection. If so, what is its explicit description?

3.6. The lagrangian grassmannian. The material in this subsection is well-known and closely related to that in [Cal]. Our intention is simply to give it a complete description in the spirit of [AGr1] and of the present paper.

Let $V$ be a 4-dimensional $\mathbb{R}$-linear space. We fix $0 \neq \omega \in \bigwedge^4 V$. The wedge product endows $\bigwedge^2 V$ with the symmetric bilinear form

$$\langle -, - \rangle : \bigwedge^2 V \times \bigwedge^2 V \to \mathbb{R}, \quad a \wedge b = \langle a, b \rangle \omega.$$ 

The form $\langle -, - \rangle$ is nondegenerate of signature $- - - + +$. Indeed, if $b_1, b_2, b_3, b_4$ is a basis in $V$ such that $b_1 \wedge b_2 \wedge b_3 \wedge b_4 = \omega$, then the elements $b_1 \wedge b_2 \pm b_3 \wedge b_4$, $b_1 \wedge b_3 \pm b_2 \wedge b_4$, and $b_1 \wedge b_4 \pm b_2 \wedge b_3$ form an orthogonal basis in $\bigwedge^2 V$ from which the signature of the form can be easily inferred.

Given a 2-dimensional subspace $W \subseteq V$, the restriction of $\langle -, - \rangle$ to $\bigwedge^2 W \subseteq \bigwedge^2 V$ is null. In other words, the Plücker map

$$\text{Gr}_2(2, V) \to \text{Gr}_2(1, \bigwedge^2 V) = \mathbb{P} \bigwedge^2 V$$

embeds $\text{Gr}_2(2, V)$ into the absolute $S_V$ of the projective geometry $\mathbb{P} \bigwedge^2 V$. Hence, $\text{Gr}_2(2, V) \to S_V$ is a diffeomorphism, because $\text{Gr}_2(2, V)$ and $S_V$ are manifolds of the same dimension, $\text{Gr}_2(2, V)$ is compact without boundary, and $S_V$ is connected (alternatively, one may simply note that the conditions $0 \neq a \in \bigwedge^2 V$ and $a \wedge a = 0$ imply $a = v \wedge w$ for some linearly independent $v, w \in V$).

Let $0 \neq \eta \in \bigwedge^2 V$ be negative, that is, $\langle \eta, \eta \rangle < 0$. Then $V$ is equipped with the symplectic form

$$\langle -, - \rangle : V \times V \to \mathbb{R}, \quad (v, w) := \langle \eta, v \wedge w \rangle.$$ 

A 2-dimensional subspace $W \subseteq V$ is called lagrangian if the restriction of $\langle -, - \rangle$ to $W \times W$ is null, that is, if $\langle -, - \rangle|_{W \times W} = 0$. The subset $\Lambda(2) \subset \text{Gr}_2(2, V) = S_V$ corresponding to lagrangian planes is known as the lagrangian grassmannian. Clearly, $\Lambda(2) = \mathbb{P} \eta^+ \cap S_V$, where $\eta^+$ stands for the orthogonal complement to $\eta$ with respect to the symmetric bilinear form $\langle -, - \rangle$. It follows from the description of the tangent space to a point in $S_V$ in the proof of Proposition 2.3 and [AGG, Lemma 4.2.2] that the intersection of the hypersurfaces $\mathbb{P} \eta^+$ and $S_V$ in $\mathbb{P} \bigwedge^2 V$ is transversal. So, the lagrangian grassmannian is a 3-dimensional submanifold of $\text{Gr}_2(2, V)$.

The symmetric bilinear form restricted to $\eta^+$ has signature $- - + +$ and $\mathbb{P} \eta^+ \cap S_V$ is nothing but the absolute of the projective geometry $\mathbb{P} \eta^-$. In other words, the lagrangian grassmannian can be naturally identified with the absolute of anti-de Sitter 4-space. It is therefore endowed with a causal structure.

Causal structures in groups of symplectomorphisms/contactomorphisms constitute an important topic in the geometry of contact manifolds [EIP].

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