LOW REGULARITY BLOWUP SOLUTIONS FOR THE
MASS-CRITICAL NLS IN HIGHER DIMENSIONS

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ABSTRACT. In this paper, we study the $H^s$-stability of the log-log blowup regime (which has been completely described in a series of recent works by Merle and Raphael) for the focusing mass-critical nonlinear Schrödinger equations $i\partial_t u + \Delta u + |u|^4 u = 0$ in $\mathbb{R}^d$ with $d \geq 3$. We aim to extend the result in [Colliander and Raphael, Rough blowup solutions to the $L^2$ critical NLS, Math. Ann., 345(2009), 307-366.] for dimension two to the higher dimensions cases $d \geq 3$, where we use the bootstrap argument in the above paper and the commutator estimates in [M. Visan and X. Zhang, On the blowup for the $L^2$-critical focusing nonlinear Schrödinger equation in higher dimensions below the energy class. SIAM J. Math. Anal., 39(2007), 34-56.].

Key Words: nonlinear Schrödinger equation; blow up, low regularity.
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1. INTRODUCTION

We study the initial-value problem for focusing nonlinear Schrödinger equations of the form

\[
\begin{cases}
(i\partial_t + \Delta) u = -|u|^{\frac{4}{d}} u, & (t, x) \in \mathbb{R} \times \mathbb{R}^d, \\
u(0, x) = u_0(x),
\end{cases}
\]

(1.1)

where $u : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{C}$.

Equation (1.1) admits a number of symmetries in $H^1(\mathbb{R}^d)$, explicitly:

- **Space-time translation invariance**: if $u(t, x)$ solves (1.1), then so does $u(t + t_0, x + x_0)$, $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^d$;

- **Phase invariance**: if $u(t, x)$ solves (1.1), then so does $e^{i\gamma} u(t, x)$, $\gamma \in \mathbb{R}$;

- **Galilean invariance**: if $u(t, x)$ solves (1.1), then for $\beta \in \mathbb{R}^d$, so does $e^{i\frac{\beta}{2} \cdot (x - \beta t)} u(t, x - \beta t)$;

- **Scaling invariance**: if $u(t, x)$ solves (1.1), then so does $u_\lambda(t, x)$ defined by

\[
u_\lambda(t, x) = \lambda^{\frac{d}{2}} u(\lambda^2 t, \lambda x), \quad \lambda > 0.
\]

(1.2)

This scaling defines a notion of criticality for (1.1). In particular, one can check that the only homogeneous $L^2_x$-based Sobolev space that is left invariant under (1.2) is $L^2_x(\mathbb{R}^d)$, and we call problem (1.1) as mass-critical problem.

In above, we know that equation (1.1) admits a number of symmetries in the energy space $H^1$: if $u(t, x)$ solves (1.1), then for any $(\lambda_0, t_0, x_0, \beta_0, \gamma_0) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$, so does

\[
v(t, x) = \lambda_0^{\frac{d}{2}} e^{i\gamma_0} e^{i\frac{\lambda_0}{2} (x - \frac{x_0}{\lambda_0}) t} u(\lambda_0^2 t + t_0, \lambda_0 x + x_0 - \beta_0 t).
\]

(1.3)
From the Ehrenfest law or direct computation, these symmetries induce invariances in the energy space, namely

**Mass:** \( M(u) = \int_{\mathbb{R}^d} |u(t, x)|^2 \, dx = M(u_0); \) \hspace{1cm} (1.4)

**Energy:** \( E(u) = \int_{\mathbb{R}^d} \left( \frac{1}{2} |\nabla u(t, x)|^2 - \frac{d}{2(d + 2)} |u(t, x)|^\frac{2(d + 2)}{d} \right) \, dx = E(u_0); \) \hspace{1cm} (1.5)

**Momentum:** \( P(u) = \text{Im} \int_{\mathbb{R}^d} \nabla u \bar{\nabla} \, dx = P(u_0). \) \hspace{1cm} (1.6)

In the focusing nonlinear Schrödinger equation, the special solutions play an important role. They are the so-called solitary waves and are of the form \( u(t, x) = e^{i\omega t} Q_w(x) \) (which is a global solution but not scatters), where \( Q_w \) solves

\[
\begin{aligned}
&\Delta Q_w + Q_w |Q_w|^{\frac{4}{d}} = w Q_w, \\
&Q_w \in H^1(\mathbb{R}^d) \setminus \{0\}. 
\end{aligned}
\] (1.7)

Equation (1.7) is a standard nonlinear elliptic equation. It is known that if \( w \leq 0 \), then (1.7) does not have any solution. Therefore, we assume that \( w > 0 \).

In dimension \( d = 1 \), there exists a unique solution in \( H^1 \) up to translation to (1.7) and infinitely many with growing \( L^2 \)-norm for \( d \geq 2 \). Nevertheless, from \( [1, 8, 14] \), there is a unique positive solution up to translation \( Q_w(x) \). \( Q_w \) is in addition radially symmetric. Letting \( Q = Q_{w=1} \), then \( Q_w(x) = w^{1/2} Q(u^{1/2}x) \) from scaling property, i.e. \( Q \) solves

\[
\Delta Q + |Q|^{p-1} = Q. \] (1.8)

Using the Strichartz estimate and a standard fixed point argument, see Gindi, Bre and Velo [9] and Cazenave and Weissler [2], we derive that (1.1) is locally well-posed in \( H^s \) for \( 0 \leq s \leq 1 \) and the Cauchy problem is subcritical in \( H^s \) for \( s > 0 \): for \( u_0 \in H^s \), \( s > 0 \), there exists \( 0 < T \leq +\infty \) such that \( u \in C([0, T), H^s) \) and either \( T = +\infty \) and we say the solution is global, or \( T < +\infty \) and then

\[
\limsup_{t \to T} \|u(t)\|_{H^s} = +\infty
\]

and we say the solution blows up in finite time.

In the case \( \|u_0\|_{L^2} < \|Q\|_{L^2} \), from classical variational arguments, one can obtain the global well-posedness in \( H^1(\mathbb{R}^d) \). Indeed, this follows from the conservation of the energy, the mass and Gagliardo-Nirenberg inequality as exhibited by Weinstein in [28]:

\[
\forall \ u \in H^1: \ E(u) \geq \frac{1}{2} \left( \int |\nabla u|^2 \right) \left( 1 - \left( \frac{\int |u|^2}{\int Q^2} \right) \right). \] (1.9)

Moreover, the scattering result in \( L^2(\mathbb{R}^d) \) is obtained by Killip, Tao, Visan and Zhang [12] [13] for radial initial data and Dodson [10] for nonradial initial data.

In the case \( \|u_0\|_{L^2} = \|Q\|_{L^2} \), the pseudo-conformal transformation applied to the stationary solution \( e^{i\mu t} Q \) yields an explicit solution

\[
S(t, x) = \frac{1}{|t|^{d/2}} Q \left( \frac{x}{t} \right) e^{-i \frac{|x|^2}{4t} + \mu t}, \quad \|S(t)\|_{L^2} = \|Q\|_{L^2} \tag{1.10}
\]

which scatters as \( t \to -\infty \) in the sense that there exists \( \psi_\infty \in L^2(\mathbb{R}^d) \) such that

\[
\lim_{t \to -\infty} \|S(t, x) - e^{i\mu t} \psi_\infty\|_{L^2(\mathbb{R}^d)} = 0.
\]
And $S(t, x)$ blows up at $T = 0$ at the speed
\[ \| \nabla S(t) \|_{L^2} \sim \frac{1}{|t|}. \]

An essential feature of (1.10) is compact up to the symmetries of the flow, meaning that all the mass goes into the singularity formation
\[ |S(t)|^2 \to \|Q\|_{L^2}^2 \delta_{x=0} \text{ as } t \to 0. \quad (1.11) \]

It turns out that $S(t)$ is the unique minimal mass blow-up solution in $H^1$ in the following sense: let $u(-1) \in H^1$ with $\|u(-1)\|_{L^2} = \|Q\|_{L^2}$, and assume that $u(t)$ blows up at $T = 0$, then $u(t) = S(t)$ up to the symmetries of the equation, see Merle [15] (radial) and [16] (general case). Note that from direct computation
\[ E(S(t, x)) > 0, \quad \text{and} \quad \| \nabla S(t) \|_{L^2} = \frac{C}{|t|}. \]

The general intuition is that such a behavior is exceptional in the sense that such minimal elements can be classified.

The situation $\|u_0\|_{L^2} > \|Q\|_{L^2}$ has been clarified by Merle and Raphaël in the series of papers [17, 18, 19, 20, 21, 25]. Let us define the differential operator
\[ \Lambda := \frac{d}{2} + y \cdot \nabla, \]

which will be of constant use. Then we introduce the following property:

**Spectral property.** Let $d \geq 1$. Consider the two real Schrödinger operators
\[ \mathcal{L}_1 = -\Delta + \frac{2}{d} \left( \frac{4}{d} + 1 \right) Q^{\frac{4}{d} - 1} y \cdot \nabla Q, \quad \mathcal{L}_2 = -\Delta + \frac{2}{d} Q^{\frac{4}{d} - 1} y \cdot \nabla Q, \quad (1.12) \]

and the real quadratic form for $\varepsilon = \varepsilon_1 + i\varepsilon_2 \in H^1$:
\[ H(\varepsilon, \varepsilon) = (\mathcal{L}_1 \varepsilon_1, \varepsilon_1) + (\mathcal{L}_2 \varepsilon_2, \varepsilon_2). \]

Then there exists a universal constant $\tilde{\delta}_1 > 0$ such that for all $\varepsilon \in H^1$, if
\[ (\varepsilon_1, Q) = (\varepsilon_1, Q_1) = (\varepsilon_1, yQ) = (\varepsilon_2, Q_1) = (\varepsilon_2, Q_2) = (\varepsilon_2, \nabla Q) = 0, \]

then
\[ H(\varepsilon, \varepsilon) \geq \tilde{\delta}_1 \int \left( |\nabla \varepsilon|^2 + |\varepsilon|^2 e^{-(2-\epsilon)|y|} \right) \]

where $Q_1 = \Lambda Q$ and $Q_2 = \Lambda Q_1$, and $2_- = 2 - \epsilon$ with $0 < \epsilon \ll 1$.

**Remark 1.1.** We remark that this spectral property has been proved rigorously in [17] for dimension $d = 1$, since the ground state $Q$ is explicit in dimension one and the spectral property could be deduced from some known properties of the second-order differential operators [26]. For the dimensions $d \in \{2, 3, 4\}$, Fibich, Merle and Raphaël [7] gave a numerically-assisted proof of the above spectral property by using the numerical representation of the ground state $Q$. Recently, Yang, Roudenko and Zhao [30] showed the spectral property in dimensions $5 \leq d \leq 10$ (general case) and $d \in \{11, 12\}$ (radial).

Based on the works [17, 18, 19, 20, 21, 25], we now have:
Theorem 1.2 (Dynamics of NLS). Let \( d \geq 1 \) and assume the spectral property holds true. Then there exists \( \alpha^* > 0 \) and a universal constant \( C^* > 0 \) such that the following is true. Let \( u_0 \in B_{\alpha^*} \) with

\[
B_{\alpha^*} := \{ u_0 \in H^1(\mathbb{R}^d) : \int Q^2 \leq \int |u_0|^2 < \int Q^2 + \alpha^* \},
\]

let \( u(t) \) be the corresponding solution to (1.1) with \([0,T]\) its maximum time interval with existence on the right in \( H^1 \).

(i) Estimates on the blow-up speed: assume \( u(t) \) blows up in finite time i.e., \( 0 < T < +\infty \), for \( t \) close enough to \( T \), we have either

\[
\lim_{t \to T} \frac{\|\nabla u\|_{L^2}^2}{\|Q\|_{L^2}^2 \left( \frac{T - t}{\ln |\ln(T - t)|} \right)} = \frac{1}{\sqrt{2\pi}} \tag{1.13}
\]

or

\[
\|\nabla u(t)\|_{L^2} \geq \frac{C^*}{(T - t)\sqrt{E^G(u_0)}}, \tag{1.14}
\]

with

\[
E^G(u) := E(u) - \frac{1}{2} \frac{P(u)^2}{\|u\|_{L^2}^2}.
\]

(ii) Description of the singularity: assume \( u(t) \) blows up in finite time, then there exist parameters \((\lambda(t), x(t), \gamma(t)) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}\) and an asymptotic profile \( u^* \in L^2(\mathbb{R}^d) \) such that

\[
u(t) - \frac{1}{\lambda(t)^{d/2}} Q\left(\frac{x - x(t)}{\lambda(t)}\right)e^{i\gamma(t)} \to u^* \text{ in } L^2(\mathbb{R}^d) \text{ as } t \to T. \tag{1.15}\]

Moreover, the blow up point is finite in the sense that

\[x(t) \to x(T) \in \mathbb{R}^d, \text{ as } t \to T. \]

Moreover, assume \( u(t) \) satisfies (1.13), \( x(T) \) be its blow up point. Set

\[
\lambda_0(t) = \sqrt{2\pi} \left( \frac{T - t}{\ln |\ln(T - t)|} \right)^{1/2}
\]

then there exists a phase parameter \( \gamma_0(t) \in \mathbb{R} \) such that:

\[
u(t) - \frac{1}{\lambda_0(t)^{d/2}} Q\left(\frac{x - x_0(T)}{\lambda(t)}\right)e^{i\gamma_0(t)} \to u^* \text{ in } L^2(\mathbb{R}^d) \text{ as } t \to T. \tag{1.17}\]

(iii) Universality of blow up profile in \( \dot{H}^1 \): if we assume that \( u(t) \) blows up in finite time with (1.13), then there exist parameters \( \lambda_0(t) = \frac{\|\nabla Q\|_{L^2}^2}{\|\nabla u(t)\|_{L^2}^2} \), \( x_0(t) \in \mathbb{R}^d \) and \( \gamma_0(t) \in \mathbb{R} \) such that

\[
e^{i\gamma_0(t)} \lambda_0(t)^{d/2} u(t, \lambda_0(t)x + x_0(t)) \to Q \text{ in } \dot{H}^1, \text{ as } t \to T. \tag{1.18}\]

If \( u(t) \) satisfies (1.14), then asymptotic stability (1.18) holds on a sequence \( t_n \to T. \)

(iv) Sufficient condition for log-log blow-up: if

\[
E^G(u) := E(u) - \frac{1}{2} \frac{P(u)^2}{\|u\|_{L^2}^2} < 0, \tag{1.19}
\]

then \( u(t) \) blows up in finite time with the log-log speed (1.13). More generally, the set of initial data \( u_0 \in B_{\alpha^*} \) such that the corresponding solution \( u(t) \) to (1.1) blows
up in finite time $0 < T < +\infty$ with the log-log speed \((1.13)\) is open in $H^1$ ($H^1$ stability of the log-log regime).

(v) Asymptotic of $u^*$ on the singularity: assume $T < +\infty$; if $u(t)$ satisfies \((1.13)\), then for $R > 0$ small,

$$\frac{1}{C^*(\ln|\ln(R)|)^2} \leq \int_{|x-x(T)|<R} |u^*(x)|^2 \, dx \leq \frac{C^*}{(\ln |\ln(R)|)^2}$$

(1.20)

which implies $u^* \notin H^1$ and $u^* \notin L^p$ with $p > 2$. If $u(t)$ satisfies \((1.14)\), then

$$\int_{|x-x(T)|\leq R} |u^*(x)|^2 \, dx \leq C^* E_0 R^2, \quad \text{and} \quad u^* \in H^1.$$  

(1.21)

**Remark 1.3.** The above theorem asserts the existence and the stability of a log-log blowup regime, and gives sufficient conditions to ensure its occurrence. It also asserts that the log-log blowup regime is open in $H^1$. In [19], Colliander and Raphael proved that the log-log blowup dynamics described by Theorem \((1.2)\) are stable under small $H^s$ perturbations with $0 < s \leq 1$ in dimension $d = 2$. In this paper, we extend their result to higher dimensional cases $d \geq 3$.

Now, we state our main result.

**Theorem 1.4** ($H^s$-stability of the log-log regime). Let $d \geq 3$ and $\frac{1}{1+\min\{1, 4\}} < s \leq 1$, and assume the spectral property holds true. Then, the log-log blowup dynamics described by Theorem \((1.2)\) are stable under small $H^s$ perturbations. In other words, let $u_0 \in H^1$ evolve into a log-log blowup solution given by Theorem \((1.2)\). Then, there exists $\varepsilon = \varepsilon(s, u_0)$ such that for any $v_0 \in H^s$ with

$$\|v_0 - u_0\|_{H^s(\mathbb{R}^d)} < \varepsilon,$$

then, the corresponding solution $v(t)$ to \((1.1)\) blows up in finite time $0 < T < +\infty$ with the following blowup dynamics: there exist geometrical parameters $(\lambda(t), x(t), \gamma(t)) \in (0, +\infty) \times \mathbb{R}^d \times \mathbb{R}$ and an asymptotic residual profile $v^* \in L^2$, with $v^* \notin L^p(p > 2)$, such that as $t \to T$

$$v(t) - \frac{1}{\lambda(t)^\frac{4}{\pi}} Q\left(\frac{x-x(t)}{\lambda(t)}\right) e^{\gamma(t)} \to u^* \quad \text{in} \quad L^2,$$

(1.22)

$$x(t) \to x(T) \in \mathbb{R}^d,$$

(1.23)

$$\lambda(t) \sqrt{\frac{\log |\log(T-t)|}{T-t}} \to \sqrt{2\pi}.$$  

(1.24)

**Remark 1.5.** The restriction $\frac{1}{1+\min\{1, 4\}} < s$ follows from the commutator estimation in Lemma \(2.3\).

1.1. **Outline of proof of Theorem 1.4.** First, we recall from [18, 19] the existence of a one parameter family of localized self-similar profiles close to the ground state solution $Q$.

**Proposition 1.6** (Localized self-similar profiles, \[18\] \[19\]). There exist universal constants $C > 0$, $\eta^* > 0$ such that the following holds true. For all $0 < \eta < \eta^*$, there exist constants $\varepsilon^*(\eta) > 0$, $b^*(\eta) > 0$ going to zero as $\eta \to 0$ such that for all
\(|b| < b^*(\eta)|\), there exists a unique radial solution \(\tilde{Q}_b\) to
\[
\begin{cases}
\Delta \tilde{Q}_b - \tilde{Q}_b + ib\Lambda \tilde{Q}_b+
\tilde{Q}_b(\tilde{Q}_b)^{\frac{4}{3}} = 0, \\
P_b = \tilde{Q}_b e^{ib\Lambda} > 0 \text{ in } B_{R_b}, \\
(\tilde{Q}_b - Q(0)) \leq \varepsilon^*(\eta), \quad Q_b(R_b) = 0,
\end{cases}
\]
with \(R_b = \frac{2}{|b|}\sqrt{1 - \eta}\), \(B_{R_b} = \{y \in \mathbb{R}^d, \ |y| \leq R_b\}\). Let \(\phi_b(y)\) be a regular radially symmetric cut-off function satisfying
\[
\phi_b(y) = \begin{cases}
1 & \text{if } |x| \leq R_b^- = \sqrt{1 - \eta}R_b, \\
0 & \text{if } |x| \geq R_b,
\end{cases}
\]
and \(\|\nabla \phi_b\|_{L^\infty} + \|\Delta \phi_b\|_{L^\infty} \to 0\) as \(|b| \to 0\). Moreover, let
\[
Q_b(r) = \tilde{Q}_b(r)\phi_b(r),
\]
then
\[
\left\|e^{C_r}(Q_b - Q)\right\|_{H^{\mu} \cap C^2} \to 0 \quad \text{as } |b| \to 0,
\]
\[
\left\|e^{C_r}\left(\frac{Q_b}{b} + \frac{|b|^2}{4}Q\right)\right\|_{C^2} \to 0 \quad \text{as } |b| \to 0,
\]
\[
|E(Q_b)| \leq e^{-C|b|},
\]
and \(Q_b\) has super-critical mass:
\[
0 < \frac{d}{d(b^2)}\left(\int |Q_b|^2\right)_{b=0} = d_0 < +\infty.
\]

**Remark 1.7.** The profiles \(Q_b\) are not exact self-similar solutions and we define the error term \(\Psi_b\) by:
\[
\Delta Q_b - Q_b + ib\Lambda Q_b + Q_b(\tilde{Q}_b)^{\frac{4}{3}} = -\Psi_b,
\]
Indeed,
\[
-\Psi_b = 2\nabla \tilde{Q}_b \cdot \nabla \phi_b + \tilde{Q}_b(\Delta \phi_b) + ib\tilde{Q}_b(\Lambda \phi_b) + (\phi_b^{\frac{1}{3}} - \phi_b)(\tilde{Q}_b)^{\frac{4}{3}} \tilde{Q}_b.
\]

Next, we introduce the outgoing radiation escaping the soliton core according to the following Lemma.

**Lemma 1.8 (Linear outgoing radiation, [13](Lemma 15)).** There exist universal constants \(C > 0\) and \(\eta^* > 0\) such that \(\forall \ 0 < \eta < \eta^*\), there exists \(b^* > 0\) such that for any \(|b| < b^*\), the following holds true: with \(\Psi_b\) given by [13], there exists a unique solution \(\zeta_b\) to
\[
\begin{cases}
\Delta \zeta_b - \zeta_b + ib\Lambda \zeta_b = \Psi_b, \\
\int |\nabla \zeta_b|^2 < +\infty.
\end{cases}
\]
Moreover, let
\[
\Gamma_b := \lim_{|y| \to +\infty} |y|^d|\zeta_b(y)|^2,
\]
then,
\[
e^{-4(1+C\eta)|\cdot|^2} \leq \Gamma_b \leq e^{-4(1-C\eta)|\cdot|^2}.
\]
More precisely, it follows that
\[
\left\|\left(|\zeta_b| + |y||\nabla \zeta_b|\right)\right\|_{L^\infty(|y| \geq R_b)} \leq \Gamma_b^{\frac{1}{2} - C\eta}, \quad \int |\nabla \zeta_b|^2 \leq \Gamma_b^{1-C\eta}.
\]
For $|y|$ small, we have: For any $\sigma \in (0, 5)$, there exists $\eta^{**}(\sigma)$, such that for any $0 < \eta < \eta^{**}$, there exists $b^{**}(\eta)$ such that for any $0 < |b| < b^{**}(\eta)$, it follows that

$$\|\xi_b e^{-\frac{\sigma b |y|}{|b|}}\|_{C^2(|y| \leq \frac{R_b}{b})} \leq \Gamma_b^{\frac{1}{2} - C}.$$ 

Last, $\zeta_b$ is differentiable with respect to $b$ with estimate

$$\|\partial_b \zeta_b\|_{C^1} \leq \Gamma_b^{\frac{1}{2} - C \eta}.$$ 

Next, let us introduce some notations in the I-method, which consists in smoothing out the $H^s$-initial data with $0 < s < 1$ in order to access a good local theory available at the $H^1$-regularity. To do it, for $N \gg 1$, we define the Fourier multiplier $I_N$ by

$$\hat{I}_N u(\xi) := m_N(\xi) \hat{u}(\xi),$$

where $m_N(\xi)$ is a smooth radial decreasing cut off function such that

$$m_N(\xi) = \begin{cases} 1, & |\xi| \leq N, \\ \left(\frac{|\xi|}{N}\right)^{s-1}, & |\xi| \geq 2N. \end{cases}$$

Thus, $I_N$ is the identity operator on frequencies $|\xi| \leq N$ and behaves like a fractional integral operator of order $1 - s$ on higher frequencies. It is easy to show that the operator $I_N$ maps $H^s$ to $H^1$. Moreover, we have

$$\|u\|_{H^s} \lesssim \|I_N u\|_{H^1} \lesssim N^{1-s} \|u\|_{H^s}. \quad (1.36)$$

Let $\delta_\lambda f(x) = f(x/\lambda)$. Then, by a simple computation, we have

$$(I_N \delta_\lambda f)(x) = (\delta_\lambda I_N f)(x). \quad (1.37)$$

By a simple argument as in [5] and perturbation theory, we can reduce Theorem 1.4 to the following proposition.

**Proposition 1.9** (Explicit description of the blowup set). Let $s > s(d)$ and consider an initial data

$$u_0 = G(0) + H(0), \quad G(0) \in H^1, \quad H(0) \in H^s \quad (1.38)$$

such that $G(0)$ admits a geometrical decomposition:

$$G(0, x) = \frac{1}{\lambda(0)^{\frac{1}{2}}} (Q_b(0) + g(0)) \left(\frac{x - x(0)}{\lambda(0)}\right) e^{-i\gamma(0)}, \quad (1.39)$$

with the following controls:

(i) **Control of the scaling parameter:**

$$0 < b(0) \ll 1, \quad 0 < \lambda(0) \leq e^{-e^{-\frac{2}{\sqrt{\pi} b(0)}}}, \quad (1.40)$$

(ii) **$L^2$-control of the excess of mass:**

$$\|g(0)\|_{L^2} + \|H(0)\|_{L^2} \ll 1, \quad (1.41)$$

(iii) **$H^s$-control of the rough excess of mass:**

$$\|H(0)\|_{H^s} \leq \lambda(0)^{10}, \quad (1.42)$$

(iv) **$H^1$-smallness of $g(0)$:**

$$\int |\nabla g(0)|^2 + \int |g(0)|^2 e^{-|y|} \leq \Gamma_{b(0)}^{\frac{3}{10}}, \quad (1.43)$$
Control of the conservation laws for the $H^1$-part:

\[ |E(G(0))| \leq \frac{1}{\sqrt{\lambda(0)}}, \]  
\[ |P(G(0))| \leq \frac{1}{\sqrt{\lambda(0)}}. \]  

Then, the corresponding $H^s$ solution $u(t)$ to (1.1) blows up in finite time $0 < T < +\infty$ in the log-log regime and the conclusions of Theorem 1.4 hold true.

Hence, our goal is to prove Proposition 1.9. Let $u_0 \in H^s$ satisfy the hypotheses of Proposition 1.9. We can rewrite the decomposition (1.38) as

\[ u(0, x) = \frac{1}{\lambda(0)^{\frac{1}{2}}} (Q_{b(0)} + \varepsilon(0)) \left( \frac{x - x(0)}{\lambda(0)} \right) e^{-i\gamma(0)} \]  
(1.46)

with $\varepsilon(0) = g(0) + h(0)$ and

\[ H(0, x) = \frac{1}{\lambda(0)^{\frac{1}{2}}} h \left( 0, \frac{x - x(0)}{\lambda(0)} \right) e^{-i\gamma(0)}. \]  
(1.47)

We have by (1.42)

\[ \|h(0)\|_{H^s} = \|H(0)\|_{H^s} \leq \lambda(0)^{10+s}. \]  
(1.48)

This together with (1.41) and (1.43) yields that

\[ \|\varepsilon(0)\|_{H^s} \leq \|g(0)\|_{H^s} + \|h(0)\|_{H^s} \ll 1. \]  
(1.49)

Next, we derive a frequency localized version of (1.43) for $\varepsilon(0)$. Set

\[ N(0) = \left( \frac{1}{\lambda(0)} \right)^{\frac{1}{2}}, \]  
(1.50)

with $\beta$ given in Remark 1.10 then,

\[ 1 \ll \left( \frac{1}{\lambda(0)} \right)^{\frac{1}{1-s}} = N(0)\lambda(0). \]  
(1.51)

And so, using (1.34), (1.36), (1.40) and (1.48), we get

\[ \int |I_{N(0)\lambda(0)} \nabla h(0)|^2 \lesssim \langle N(0)\lambda(0) \rangle^{2(1-s)} \|h(0)\|^2_{H^s} \lesssim \lambda(0)^{10} \leq \Gamma_{b(0)}^{10}. \]

This together with (1.42) and (1.43) implies

\[ \int |I_{N(0)\lambda(0)} \nabla \varepsilon(0)|^2 + \int |\varepsilon(0)|^2 e^{-|y|} \leq \Gamma_{b(0)}^{\frac{3}{2}}. \]  
(1.52)

**Remark 1.10.** The restriction on $\beta$ stems from two sides. One comes from Corollary 2.10 below:

\[ \frac{4s}{\min\{4, d\} s - 4(1-s)} < \frac{1}{\beta}. \]

Another comes from Lemma 3.11 below

\[ \frac{4s}{\min\{4, d\} s^2 - (1-s)} > \frac{4s}{\min\{4, d\} s - 4(1-s) \min\{4, d\} s^2 - (1-s)} \]

to guarantee the convergence of the summation. Hence, we will take $\beta \in (0, 1)$ such that

\[ \frac{1}{\beta} > \min \left\{ \frac{4s}{\min\{4, d\} s - 4(1-s)}, \frac{4s}{\min\{4, d\} s - 4(1-s) \min\{4, d\} s^2 - (1-s)} \right\}. \]
Lemma 1.11 (Nonlinear modulation theory, [17, 18, 20]). There exists $\alpha_2 > 0$ such that for $\alpha_0 < \alpha_2$, there exist some functions $(\lambda, \gamma, x, b): [0, T) \to (0, +\infty) \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$ such that

$$\varepsilon(t, y) = e^{i\gamma(t)}\lambda(t)^{\frac{d}{2}}u(t, \lambda(t)y + x(t)) - Q_{b(t)}(y)$$

(1.53)

satisfies the following orthogonality conditions:

$$(\varepsilon_1(t), |y|^2\Sigma) + (\varepsilon_2(t), |y|^2\Theta) = 0,$$  

(1.54)

$$(\varepsilon_1(t), y\Sigma) + (\varepsilon_2(t), y\Theta) = 0,$$  

(1.55)

$$-(\varepsilon_1(t), \Lambda\Theta) + (\varepsilon_2(t), \Lambda\Sigma) = 0,$$  

(1.56)

$$-(\varepsilon_1(t), \Lambda^2\Theta) + (\varepsilon_2(t), \Lambda^2\Sigma) = 0,$$  

(1.57)

where $\varepsilon = \varepsilon_1 + i\varepsilon_2$ and $Q_{b(t)} = \Sigma + i\Theta$ in terms of real and imaginary part.

Remark 1.12. The existence of such a decomposition (1.53) requires only the smallness of the local $L^2$-norm of $\varepsilon$ due to the regularity of $Q_b$ and its fast decay in space. We note that (1.52) ensures that the deformed parameters ensuring the orthogonality conditions at time $t = 0$ are exponentially small in $b(0)$ compared to (1.40). We shall thus abuse notations at time $t = 0$ and identify these two decompositions which satisfy the initialized control of Proposition 1.9.

Our main claim now is that the controls of Proposition 1.9 determine a trapped dynamical region. In other words, we claim the following bootstrapped estimates. Consider a time interval $[0, T^+]$ such that the solution $u(t)$ admits a decomposition

$$u(t, x) = \frac{1}{\lambda(t)^{\frac{d}{2}}} (Q_{b(t)}(\cdot) + \varepsilon(t, \cdot)) \left(\frac{x - x(t)}{\lambda(t)}\right) e^{-i\gamma(t)}, \ t \in [0, T^+]$$

(1.58)

satisfying the orthogonality conditions (1.54)-(1.57). Now, let us assume the following uniform controls on $[0, T^+]$:

(i) Control of $b(t)$ and the $L^2$ mass:

$$b(t) > 0 \quad \text{and} \quad b(t) + \|\varepsilon(t)\|_{L^2} \leq 10\{b(0) + \|\varepsilon(0)\|_{L^2}\};$$

(1.59)

(ii) Control and monotonicity of the scaling parameter:

$$\lambda(t) \leq e^{-e^{\text{doubling}}}$$

(1.60)

and almost monotonicity:

$$\forall 0 < t_1 \leq t_2 \leq T^+, \ \lambda(t_2) \leq \frac{3}{2} \lambda(t_1).$$

(1.61)

Let $k_0 \leq k^+$ integers such that

$$\frac{1}{2k_0} \leq \lambda(0) \leq \frac{1}{2k_0-1}, \quad \frac{1}{2k^+} \leq \lambda(T^+) \leq \frac{1}{2k^+-1},$$

(1.62)

and for $k_0 \leq k \leq k^+$, let $t_k$ be a time such that

$$\lambda(t_k) = \frac{1}{2k},$$

(1.63)

then, we assume the control of the doubling time interval:

$$t_{k+1} - t_k \leq k\lambda(t_k)^2.$$

(1.64)
(iii) **Frequency localized control of the excess of mass**: let

\[ N(t) = \left( \frac{1}{\lambda(t)} \right)^{\frac{1}{s}}, \]

then,

\[ \int |I_{N(t)} \lambda(t) \nabla \varepsilon(t)|^2 + \int |\varepsilon(t)|^2 e^{-|y|} \leq \Gamma_{b(t)}^{\frac{1}{s}}. \]

We then claim the following Lemma which is the main step of the proof of Proposition 1.19 and states that all above estimates may be improved:

**Lemma 1.13** (Bootstrap lemma). There holds the following uniform control on \([0,T^+]:\)

\[ b(t) > 0 \text{ and } b(t) + \| \varepsilon(t) \|_{L^2} \leq 5 \left( b(0) + \| \varepsilon(0) \|_{L^2} \right), \]

\[ \lambda(t) \leq e^{-e^{\frac{1}{2} \lambda(t)}}, \]

\[ \forall 0 < t_1 \leq t_2 \leq T^+, \lambda(t_2) \leq \frac{5}{4} \lambda(t_1), \]

\[ t_{k+1} - t_k \leq \sqrt{k\lambda(t_k)^2}, \]

\[ \int |I_{N(t)} \lambda(t) \nabla \varepsilon(t)|^2 + \int |\varepsilon(t)|^2 e^{-|y|} \leq \Gamma_{b(t)}^{\frac{1}{s}}. \]

**Remark 1.14.** By (1.36), and the bootstrapped estimates (1.59), (1.65), (1.66), we obtain

\[ \| \varepsilon(t) \|_{H^s} \lesssim \| I_{N(t)} \lambda(t) \varepsilon(t) \|_{H^s} \ll 1. \]

This together with the geometrical decomposition (1.58), (1.27) yields that

\[ \| u(t) \|_{H^s} = \frac{\| Q_{b(t)} + \varepsilon(t) \|_{H^s}}{\lambda(t)^s} \sim \frac{1}{\lambda(t)^s}. \]

**2. Notation and Almost conservation law**

**2.1. Some notation.** For nonnegative quantities \(X\) and \(Y\), we will write \(X \lesssim Y\) to denote the estimate \(X \leq CY\) for some \(C > 0\). If \(X \lesssim Y \lesssim X\), we will write \(X \sim Y\). Dependence of implicit constants on the power \(p\) or the dimension will be suppressed; dependence on additional parameters will be indicated by subscripts. For example, \(X \lesssim_u Y\) indicates \(X \leq CY\) for some \(C = C(u)\). We denote \(a_\pm\) as \(a \pm \varepsilon\) with \(0 < \varepsilon \ll 1\).

For a spacetime slab \(I \times \mathbb{R}^d\), we write \(L^q_x L^r_t(I \times \mathbb{R}^d)\) for the Banach space of functions \(u : I \times \mathbb{R}^d \to \mathbb{C}\) equipped with the norm

\[ \| u \|_{L^q_x L^r_t(I \times \mathbb{R}^d)} := \left( \int_I \| u(t) \|_{L^r_t(\mathbb{R}^d)}^{\frac{1}{r}} \right)^{\frac{1}{q}}, \]

with the usual adjustments when \(q\) or \(r\) is infinity. When \(q = r\), we abbreviate \(L^q_x L^r_t = L^q_{x,t}\). We will also often abbreviate \(\| f \|_{L^r_x(\mathbb{R}^d)}\) to \(\| f \|_{L^r_x}\). For \(1 \leq r \leq \infty\), we use \(r'\) to denote the dual exponent to \(r\), i.e. the solution to \(\frac{1}{r} + \frac{1}{r'} = 1\).

The Fourier transform on \(\mathbb{R}^d\) is defined by

\[ \hat{f}(\xi) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx, \]
giving rise to the fractional differentiation operators $|\nabla|^s$ and $\langle \nabla \rangle^s$, defined by

$$|\nabla|^sf(\xi) := |\xi|^sf(\xi), \quad \langle \nabla \rangle^sf(\xi) := \langle \xi \rangle^sf(\xi),$$

where $\langle \xi \rangle := 1 + |\xi|$. This helps us to define the homogeneous and inhomogeneous Sobolev norms

$$\|f\|_{H^s} := \|\langle \xi \rangle^s f\|_{L^2_x}, \quad \|f\|_{H^s} := \|\langle \xi \rangle^s f\|_{L^2_x}.$$  

We will also need the Littlewood-Paley projection operators. Specifically, let $\varphi(\xi)$ be a smooth bump function adapted to the ball $|\xi| \leq 2$ which equals $1$ on the ball $|\xi| \leq 1$. For each dyadic number $N \in 2^\mathbb{Z}$, we define the Littlewood-Paley operators

$$\widetilde{P}_{\leq N}f(\xi) := \varphi(\frac{\xi}{N})\hat{f}(\xi),$$

$$\widetilde{P}_{> N}f(\xi) := \left(1 - \varphi(\frac{\xi}{N})\right)\hat{f}(\xi),$$

$$\widetilde{P}_Nf(\xi) := \left(\varphi(\frac{\xi}{N}) - \varphi(\frac{2\xi}{N})\right)\hat{f}(\xi).$$

Similarly we can define $P_{< N}$, $P_{\geq N}$, and $P_{M < N} = P_{< N} - P_{\leq M}$, whenever $M$ and $N$ are dyadic numbers. We will frequently write $f_{\leq N}$ for $P_{\leq N}f$ and similarly for the other operators.

The Littlewood-Paley operators commute with derivative operators, the free propagator, and the conjugation operation. They are self-adjoint and bounded on every $L^p_x$ and $\dot{H}^s_x$ space for $1 \leq p \leq \infty$ and $s \geq 0$, moreover, they also obey the following Bernstein estimates

$$\|P_{\geq N}f\|_{L^p} \lesssim N^{-s}\|\nabla|^sP_{\geq N}f\|_{L^p},$$

$$\|\nabla|^sP_{\leq N}f\|_{L^p} \lesssim N^s\|P_{\leq N}f\|_{L^p},$$

$$\|\nabla|^sP_Nf\|_{L^p} \sim N^s\|P_Nf\|_{L^p},$$

$$\|P_{\leq N}f\|_{L^q} \lesssim N^{\frac{s}{q}}\|P_{\leq N}f\|_{L^p},$$

$$\|P_Nf\|_{L^q} \lesssim N^{\frac{s}{q}}\|P_Nf\|_{L^p},$$

where $s \geq 0$ and $1 \leq p \leq q \leq \infty$.

We will also use the following basic inequalities.

**Lemma 2.1 (\cite{17}).** For any $z \in \mathbb{C}$ with $z = z_1 + iz_2$, there holds

$$|(1 + z_1)(1 + z)^\frac{4}{d} - 1 - (\frac{4}{d} + 1)z_1 + i\bar{z}_2((1 + z)^\frac{4}{d} - 1)|$$

$$\leq \begin{cases}  C|z|^{1 + \frac{4}{d}} + |z|^2 & \text{if } d = 3 \\  C|z|^2 & \text{if } d \geq 4, \end{cases} \quad (2.1)$$

and

$$|(1 + z_1)(1 + z)^\frac{3}{d} - 1 - (\frac{3}{d} + 1)z_1 - \frac{3}{d}(\frac{4}{d} + 1)\bar{z}_1^2 - \frac{2}{d}\bar{z}_2^2|$$

$$\leq \begin{cases}  C|z|^{3} & \text{if } d = 3 \\  C|z|^{2 + \frac{4}{d}} & \text{if } d \geq 4, \end{cases} \quad (2.2)$$
and

\[
\|1 + \frac{z^{2+\frac{d}{2}}}{2} - 1 - \left(\frac{d}{2} + 2\right)z_1 - \left(\frac{d}{2} + 1\right)\left(\frac{d}{2} + 1\right)z_1^2 - \left(\frac{d}{2} + 1\right)z_2^2\| \\
\leq \begin{cases} 
C|z^{2+\frac{d}{2}}| + |z|^d & \text{if } d = 3 \\
C|z|^d & \text{if } d \geq 4.
\end{cases}
\]  

(2.3)

2. Nonlinear estimate. For \(N > 1\), we define the Fourier multiplier \(I_N\) given by

\[I_N u(\xi) := m_N(\xi) \hat{u}(\xi),\]

where \(m_N(\xi)\) is a smooth radial decreasing cut off function by \([1, 3] \). Let us collect basic properties of \(I_N\).

**Lemma 2.2** \([27] \). Let \(1 < p < \infty\) and \(0 \leq s \leq s < 1\). Then,

\[\|I_N f\|_{L^p} \lesssim \|f\|_{L^p},\]

\[\|\nabla|^s P_N f\|_{L^p} \lesssim N^{s-1}\|\nabla I_N f\|_{L^p},\]

\[\|f\|_{H^s} \lesssim \|I_N f\|_{L^p} \lesssim N^{1-s}\|f\|_{H^s}.\]

(2.6)

We will need the following fractional calculus estimates from \([3]\).

**Lemma 2.3** (Fractional product rule \([3]\)). Let \(s \geq 0\) and \(1 < r, r_j, q_j < \infty\) satisfy \(\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}\) for \(i = 1, 2\). Then

\[\left\|\nabla|^a(fg)\right\|_{L^r(\mathbb{R}^d)} \lesssim \left\|f\right\|_{L_{r_1}^\infty(\mathbb{R}^d)} \left\|\nabla|^a g\right\|_{L_{r_2}^\infty(\mathbb{R}^d)} + \left\|\nabla|^a f\right\|_{L_{r_1}^\infty(\mathbb{R}^d)} \|g\|_{L_{r_2}^\infty(\mathbb{R}^d)} + \left\|\nabla|^a f\right\|_{L_{r_1}^\infty(\mathbb{R}^d)} \|g\|_{L_{r_2}^\infty(\mathbb{R}^d)} \lesssim \left\|f\right\|_{L^r(\mathbb{R}^d)} \left\|\nabla|^a g\right\|_{L^r(\mathbb{R}^d)}.\]

(2.7)

From \([3]\), we have

\[\left\|I_N \nabla (fg)\right\|_{L^r(\mathbb{R}^d)} \lesssim \left\|f\right\|_{L_{r_1}^{\frac{2d}{d+1}}(\mathbb{R}^d)} \left\|I_N \nabla g\right\|_{L_{r_2}^{\frac{2d}{d+1}}(\mathbb{R}^d)} + \left\|I_N \nabla f\right\|_{L_{r_1}^{\frac{2d}{d+1}}(\mathbb{R}^d)} \|g\|_{L_{r_2}^{\frac{2d}{d+1}}(\mathbb{R}^d)}.\]

(2.8)

**Lemma 2.4** (Fractional chain rule \([3]\)). Let \(G \in C^1(C), s \in (0, 1),\) and \(1 < r, r_1, r_2 < +\infty\) satisfy \(\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}\). Then

\[\left\|\nabla|^a (G(u))\right\|_{r} \lesssim \|G'(u)\|_{r_1} \|\nabla|^a u\|_{r_2}.\]

(2.9)

As noted in the introduction, one needs to estimate the commutator \(|u|^p \partial u - I(|u|^p u)|\) in the increment of modified energy \(E(I_u)(t)\). When \(p\) is an even integer, one can use multilinear analysis to expand this commutator into a product of Fourier transforms of \(u\) and \(I_u\), and carefully measure frequency interactions to derive an estimate (see for example \([1]\)). However, this is not possible when \(p\) in not an even integer. Instead, Visan and Zhang in \([27]\) established the following rougher (weaker, but more robust) estimate:

**Lemma 2.5** (Commutator estimate, \([27]\)). Let \(1 < r, r_1, r_2 < \infty\) be such that \(\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}\) and let \(0 < \nu < s\). Then,

\[\left\|I_N (f g) - (I_N f) g\right\|_{L^r} \lesssim N^{1-s+\nu} \left\|I_N f\right\|_{L^{r_1}} \left\|\nabla^{1-s+\nu} g\right\|_{L^{r_2}}.\]

(2.10)

Furthermore, let \(I\) be a time interval and let \(\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}\) be such that \(s < 1\), then we have

\[\left\|\nabla I_N (|u|^\frac{2d}{d+1} u) - (I_N \nabla u)|u|^\frac{2d}{d+1}\right\|_{L^r(I \times \mathbb{R}^d)} \lesssim N^{1-\min(1, \frac{2d}{d+1})} \left\|\nabla I_N u\right\|_{L^r(I)}^{1+\frac{d}{d+1}},\]

(2.11)

\[\left\|\nabla I_N (|u|^\frac{2d}{d+1} u) - (|u|^\frac{2d}{d+1} u)\right\|_{L^r(I \times \mathbb{R}^d)} \lesssim \left\|\nabla I_N u\right\|_{L^r(I)}^{1+\frac{d}{d+1}},\]

(2.12)

\[\left\|\nabla I_N (|u|^\frac{2d}{d+1} u) - (|u|^\frac{2d}{d+1} u)\right\|_{L^r(I \times \mathbb{R}^d)} \lesssim \left\|\nabla I_N u\right\|_{L^r(I)}^{\frac{d}{d+1}}.\]

(2.13)
where $S^0(I)$ and $N^0(I)$ is defined in Definition 2.7 below.

**Remark 2.6.** It is easy to check that for $(q, r) \in \Lambda_0$,

$$\| (\nabla)^s u \|_{L_t^q L_x^r} \lesssim \| (\nabla) I_N u \|_{L_t^q L_x^r},$$

where $\Lambda_0$ is defined in Definition 2.7 below.

2.3. **Strichartz estimates and local well-posedness.** In this subsection, we consider the Cauchy problem

$$\begin{cases}
i u_t + \Delta u - f(u) = 0, \\
u(0) = u_0.
\end{cases}$$

The integral equation for the Cauchy problem (2.15) can be written as

$$u(t, x) = e^{i(t-t_0)\Delta} u(t_0) - i \int_{t_0}^t e^{i(t-s)\Delta} f(u(s)) ds.$$  

(2.16)

Now we recall the dispersive estimate for the free Schrödinger operator $U(t) = e^{it\Delta}$. From the explicit formula

$$e^{it\Delta} f(x) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{i(x-y)^2/2t} f(y) dy,$$

it is easy to get the standard dispersive inequality

$$\| e^{it\Delta} f \|_{L_t^\infty L_x^q(\mathbb{R}^d)} \lesssim |t|^{-\frac{d}{2}} \| f \|_{L_t^1 L_x^\infty(\mathbb{R}^d)}$$

(2.17)

for all $t \neq 0$. On the other hand, since the free operator conserves the $L_x^2(\mathbb{R}^d)$-norm, we obtain by interpolation

$$\| e^{it\Delta} f \|_{L_t^q L_x^r(\mathbb{R}^d)} \leq C |t|^{-d(\frac{1}{2} - \frac{1}{q})} \| f \|_{L_t^1 L_x^\infty(\mathbb{R}^d)}$$

(2.18)

for all $t \neq 0$ and $2 \leq q \leq +\infty$, $\frac{1}{q} + \frac{1}{r} = 1$.

The Strichartz estimates involve the following definitions:

**Definition 2.7.** A pair of Lebesgue space exponents $(q, r)$ are called Schrödinger admissible for $\mathbb{R}^{d+1}$, or denote by $(q, r) \in \Lambda_0$ when $q, r \geq 2$, $(q, r, d) \neq (2, \infty, 2)$, and

$$\frac{2}{q} = d \left( \frac{1}{2} - \frac{1}{r} \right).$$

(2.19)

For a fixed spacetime slab $I \times \mathbb{R}^d$, we define the Strichartz norm

$$\| u \|_{S^0(I)} := \sup_{(q, r) \in \Lambda_0} \| u \|_{L_t^q L_x^r(I \times \mathbb{R}^d)}, \quad d \geq 3.$$

We denote $S^0(I)$ to be the closure of all test functions under this norm and write $N^0(I)$ for the dual of $S^0(I)$.

According to the above dispersive estimate, the abstract duality and interpolation argument(see [11]), we have the following Strichartz estimates.

**Lemma 2.8** (Strichartz estimate, [10][11]). Let $s \geq 0$, and let $I$ be a compact time interval, and let $u : I \times \mathbb{R}^d \to \mathbb{C}$ be a solution of the Schrödinger equation

$$iu_t + \Delta u + h = 0.$$

Then, for all $t_0 \in I$

$$\| (\nabla)^s u \|_{S^0(I)} \leq C \| (\nabla)^s u(t_0) \|_{L_x^2(\mathbb{R}^d)} + \| (\nabla)^s h \|_{N^0(I)}.$$
By the fixed point argument, we have the following local well-posedness (LWP).

**Lemma 2.9 (H^s-LWP).** Let $s \in (0, 1]$, $u_0 \in H^s(\mathbb{R}^d)$ and

$$T_{LWP} = c \|u_0\|_{H^s}^{-\frac{2}{d}}$$

(2.20)

with $c$ small depending the constant in Strichartz estimate and Sobolev embedding. Then, there exists a unique solution $u(t)$ to (1.1) on $[0, T_{LWP}]$ and satisfying

$$\|u\|_{S^0([0,T_{LWP}])} \leq 2C\|u_0\|_{L^2_x}, \quad \|\langle \nabla \rangle^s u\|_{S^0([0,T_{LWP}])} \leq 2C\|u_0\|_{H^s}.$$  

(2.21)

**Proof.** We apply the Banach fixed point argument to prove this lemma. First, we define the map

$$\Phi(u) = e^{it\Delta}u_0 - i \int_0^t e^{i(t-s)\Delta}(|u|^{\frac{4}{d}}u)(s)\, ds$$

on the complete metric space $B$

$$B := \{ u \in C(I; H^s) : \|u\|_{S^0([0,T_{LWP}])} \leq 2C\|u_0\|_{L^2_x}, \quad \|\langle \nabla \rangle^s u\|_{S^0([0,T_{LWP}])} \leq 2C\|u_0\|_{H^s} \}$$

with the metric $d(u, v) = \|u - v\|_{L^\infty_{t,x}([0,T_{LWP}] \times \mathbb{R}^d)}$, where $C$ is the constant in Strichartz estimates.

It suffices to prove that the operator $\Phi(u)$ is a contraction map on $B$ for $[0, T_{LWP}]$. In fact, if $u \in B$, then by Strichartz estimate, Hölder’s inequality and Sobolev embedding, we have

$$\|\Phi(u)\|_{S^0([0,T_{LWP}])} \leq C\|u_0\|_{L^2_x} + C\|\langle |u|^{\frac{4}{d}}u\rangle\|_{L^\infty_{t,x}([0,T_{LWP}] \times \mathbb{R}^d)}$$

$$\leq C\|u_0\|_{L^2_x} + C\|u\|_{L^\infty_{t,x}([0,T_{LWP}] \times \mathbb{R}^d)}\|\langle |u|^{\frac{4}{d}}u\rangle\|_{L^\infty_{t,x}([0,T_{LWP}] \times \mathbb{R}^d)}$$

$$\leq C\|u_0\|_{L^2_x} + C\|u\|_{L^\infty_{t,x}([0,T_{LWP}] \times \mathbb{R}^d)}\left(T_{LWP}\|\langle \nabla \rangle^s u\|_{S^0([0,T_{LWP}])}\right)^{\frac{2}{d}}$$

$$\leq C\|u_0\|_{L^2_x} + 2^{\frac{4}{d}}C^{2+\frac{d}{2}}c^{\frac{d}{2}}\|u_0\|_{L^2_x}$$

$$\leq 2C\|u_0\|_{L^2_x}$$

provided that $2^{\frac{4}{d}}C^{2+\frac{d}{2}}c^{\frac{d}{2}} < 1$ with $T_{LWP} = c\|u_0\|_{H^s}^{-\frac{2}{d}}$. Similarly, we obtain

$$\|\langle \nabla \rangle^s \Phi(u)\|_{S^0([0,T_{LWP}])} \leq C\|u_0\|_{H^s} + C\|\langle \nabla \rangle^s (|u|^{\frac{4}{d}}u)\|_{L^\infty_{t,x}([0,T_{LWP}] \times \mathbb{R}^d)}$$

$$\leq C\|u_0\|_{H^s} + C\|\langle \nabla \rangle^s u\|_{L^\infty_{t,x}([0,T_{LWP}] \times \mathbb{R}^d)}\left(T_{LWP}\|\langle \nabla \rangle^s u\|_{S^0([0,T_{LWP}])}\right)^{\frac{2}{d}}$$

$$\leq C\|u_0\|_{H^s} + 2^{\frac{4}{d}}C^{2+\frac{d}{2}}c^{\frac{d}{2}}\|u_0\|_{H^s} \leq 2C\|u_0\|_{L^2_x}.$$

Hence, $\Phi(u) \in B$. 

On the other hand, for \( u, v \in B \), by Strichartz estimate, we obtain
\[
d(\Phi(u), \Phi(v)) = \|\Phi(u) - \Phi(v)\|_{L^2_{t,x}([0,T_{LWP}] \times \mathbb{R}^d)} \leq C \|u - v\|_{L^2_{t,x}([0,T_{LWP}] \times \mathbb{R}^d)}
\]
and so
\[
\|u - v\|_{L^2_{t,x}([0,T_{LWP}] \times \mathbb{R}^d)} \leq C \|u - v\|_{L^2_{t,x}([0,T_{LWP}] \times \mathbb{R}^d)} \leq C \|u - v\|_{L^2_{t,x}([0,T_{LWP}] \times \mathbb{R}^d)} \leq C \|u - v\|_{L^2_{t,x}([0,T_{LWP}] \times \mathbb{R}^d)} \leq C \|u - v\|_{L^2_{t,x}([0,T_{LWP}] \times \mathbb{R}^d)} \leq C \|u - v\|_{L^2_{t,x}([0,T_{LWP}] \times \mathbb{R}^d)} \leq C \|u - v\|_{L^2_{t,x}([0,T_{LWP}] \times \mathbb{R}^d)} \leq 2^\frac{s}{2} C^2 + \frac{s}{4} d(u, v)
\]
provided that \( 2^\frac{s}{2} C^2 + \frac{s}{4} d(u, v) < \frac{1}{2} \).

Therefore, applying the fixed point theorem gives a unique solution \( u \) of \((1.1)\) on \([0, T_{LWP}]\) which satisfies the bound \((2.21)\).

Therefore, applying the fixed point theorem gives a unique solution \( u \) of \((1.1)\) on \([0, T_{LWP}]\) which satisfies the bound \((2.21)\).

□

**Corollary 2.10** (‘Modified’ \(H^s\)-LWP). For \( T^* \geq T_{LWP} \), we denote
\[
\tilde{T}_{LWP} := c_0 \|\nabla\|_{S^0([0, T_{LWP})]} \frac{2}{2} u_0 \|_{L^2}
\]
with \( c_0 \) small. By \((2.12)\), we obtain
\[
\tilde{T}_{LWP} \leq c_0 \|u_0\|_{H^s}.
\]
This together with Lemma \((2.12)\) implies that \((1.1)\) is well-posed on \([0, \tilde{T}_{LWP}]\), and
\[
\|u\|_{S^0([0, \tilde{T}_{LWP})]} \leq 2\|u_0\|_{L^2}, \|\nabla\|^s u\|_{S^0([0, \tilde{T}_{LWP})]} \leq 2\|u_0\|_{H^s}.
\]
Moreover, if \( s < 1, \frac{1}{1+\min\{1, \frac{s}{4}\}} < s < 1, \frac{1}{\min\{4, d\} s - \frac{4}{s} - \frac{1}{s} < 1, then there holds
\[
\|\nabla\|_{S^0([0, \tilde{T}_{LWP})]} \lesssim \|u_0\|_{H^s}.
\]

**Proof.** First, we have by Strichartz estimate and \((2.12)\)
\[
\|\nabla\|_{S^0([0, \tilde{T}_{LWP})]} \leq C \|\nabla\|_{L^2_{t,x}([0, T_{LWP})]} + C \|\nabla\|_{L^2_{t,x}([0, T_{LWP})]} + C \|\nabla\|_{L^2_{t,x}([0, T_{LWP})]},
\]
for any \( \epsilon > 0 \) sufficiently small. Using \((1.65)\), monotonicity \((1.61)\), Remark \(1.14\) \((1.73)\) and \((2.10)\), we get
\[
N(T^*)^{-1} = \lambda(T^*)^\frac{s}{2} \lesssim \lambda(0)^{s} \lesssim \|u_0\|_{H^s}^{\frac{s}{2}}
\]
and so
\[
N(T^*)^{-1} \lesssim \|u_0\|_{H^s}^{\frac{s}{2}}.
\]
Lemma 2.13. Let
\[ N(T^*) - \min\left(1, \frac{s}{2}\right) + \epsilon = N(T^*) - \min\left(1, \frac{s}{2}\right) \cdot \frac{4(\beta + 1 - s)}{\beta} - \frac{4(\beta + 1 - s)}{\beta} \]
\[ \leq \|I_N(T^*)u_0\|_{H^s}^{\frac{4}{\beta}} \sim \tilde{T}_{\text{LWP}}^{\frac{4}{\beta}}. \]

Plugging this into (2.25) implies
\[ \|\langle\nabla\rangle I_N u\|_{S^0([0, \tilde{T}_{\text{LWP}}])} \leq C \|I_N u_0\|_{H^s} + C\tilde{T}_{\text{LWP}}^\frac{4}{\beta} \|\langle\nabla\rangle I_N u\|^{1 + \frac{4}{\beta}}_{S^0([0, \tilde{T}_{\text{LWP}}])}. \]

Therefore, (2.24) follows from standard continuous argument.

Remark 2.11. Here the restriction on \( s \) is different from [27].

As a direct application of \( H^s \)-LWP, we can control the number of LWP intervals covering the interval \([t_k, t_{k+1}]\) as follows.

Lemma 2.12. Let \( \frac{1}{1 + \min\left(1, \frac{s}{2}\right)} < s < 1, \frac{4s}{\min\left(4, d\right) s - k(1 - s)} < \frac{1}{\beta} \). Let \( \{t_k\}_{k_0 \leq k \leq k^+} \) be defined as in (1.63), and \( T^* \geq t_{k+1} \). We cover the interval \([t_k, t_{k+1}]\) by LWP time interval \( \{\tau_j^k\}_{1 \leq j \leq J_k} \) given by Corollary 2.10. Then, we have
\[ J_k \lesssim k N(T^*)^{\frac{2(1-s)}{2(1-s) - 1}}. \] (2.26)

Proof. First, it follows from (2.20) that
\[ \tau_{j+1}^k - \tau_j^k \sim \frac{1}{\|\langle\nabla\rangle I_N(T^*)u(\tau_j^k)\|_{L^2}} \gtrsim \left(\frac{1}{N(T^*)^{1-s} \|u(\tau_j^k)\|_{H^s}}\right)^{\frac{1}{s}}. \] (2.27)

This together with Remark 1.14 (1.73) and the almost monotonicity (1.61) implies
\[ \tau_{j+1}^k - \tau_j^k \gtrsim \frac{1}{N(T^*)^{\frac{2(1-s)}{2(1-s) - 1}}} \lambda(\tau_j^k)^2 \sim \frac{1}{N(T^*)^{\frac{2(1-s)}{2(1-s) - 1}}} \lambda(t_k)^2. \] (2.28)

And so (2.26) follows from the control of the blowup speed (1.64).

Lemma 2.13. Let \( \{\tau_j^k, \tau_{j+1}^k\} \) be a LWP time interval as given by Corollary 2.10. Then, there holds:
\[ \|I_N(\tau_j^k)u - I_N(\tau_{j+1}^k)u\|_{L^2 L^2_{x,t}([\tau_j^k, \tau_{j+1}^k] \times \mathbb{R}^d)} \lesssim \lambda(t_k)^{\frac{\beta}{2}} \min\left(1, \frac{s}{2}\right) \|\langle\nabla\rangle I_N u\|_{L^\infty L^2_{x,t}([\tau_j^k, \tau_{j+1}^k] \times \mathbb{R}^d)}. \] (2.29)

Proof. We have by triangle inequality
\[ \|I_N u\|_{L^2 L^2_{x,t}([\tau_j^k, \tau_{j+1}^k] \times \mathbb{R}^d)} \lesssim \|I_N u\|_{L^2 L^2_{x,t}([\tau_j^k, \tau_{j+1}^k] \times \mathbb{R}^d)} + \|I_N u - I_N u\|_{L^2 L^2_{x,t}([\tau_j^k, \tau_{j+1}^k] \times \mathbb{R}^d)}. \]
Using (2.10) with \( \nu = \min\{1, \frac{4s}{d}\} s - (1 - s) \) and (2.13), we estimate

\[
\left\| (I_N u)\right\|_{L_t^2 L_x^s \mathbb{R}^d} \lesssim N(t_k)^{-\min\{1, \frac{4s}{d}\} s} \left\| I_N u \right\|_{L_t^2 L_x^s \mathbb{R}^d} \left\| |\nabla|^{\min\{1, \frac{4s}{d}\} s} - (|u|^{\frac{4}{d}}) \right\|_{L_t^\infty L_x^\infty \mathbb{R}^d}
\]

\[
\leq N(t_k)^{-\min\{1, \frac{4s}{d}\} s} \left\| I_N u \right\|_{L_t^\infty L_x^\infty \mathbb{R}^d} \left\| |\nabla|^{\min\{1, \frac{4s}{d}\} s} - (|u|^{\frac{4}{d}}) \right\|_{L_t^\infty L_x^\infty \mathbb{R}^d}
\]

Similarly,

\[
\left\| (I_N u)\right\|_{L_t^2 L_x^s \mathbb{R}^d} \lesssim \left\| I_N u \right\|_{L_t^\infty L_x^\infty \mathbb{R}^d} \left\| |\nabla|^{\min\{1, \frac{4s}{d}\} s} - (|u|^{\frac{4}{d}}) \right\|_{L_t^\infty L_x^\infty \mathbb{R}^d}
\]

Corollary 2.10. Then, for any \( T > \lambda(\tau_k^{s-}) \), the modified energy has a slow increment over the LWP time interval:

\[
|E(I_N(T)) u(\tau_k^{s+}) - E(I_N(T)) u(\tau_k^{s-})| \leq CN(T)^{-\min\{1, \frac{4s}{d}\} s} \left( \left\| I_N(T) u(\tau_k^{s-}) \right\|_{H^\frac{d}{2}}^{2 + \frac{d}{2}} + \left\| I_N(T) u(\tau_k^{s+}) \right\|_{H^\frac{d}{2}}^{2 + \frac{d}{2}} \right) \tag{2.30}
\]

Here, the constant \( C \) depends only on \( s \). The modified momentum has a slow increment over the LWP time interval:

\[
|P(I_N(T)) u(\tau_k^{s+}) - P(I_N(T)) u(\tau_k^{s-})| \leq CN(T)^{-\min\{1, \frac{4s}{d}\} s} \left\| I_N(T) u(\tau_k^{s+}) \right\|_{H^\frac{d}{2}}^{2 + \frac{d}{2}} \tag{2.31}
\]

Moreover,

\[
\left\| I_N(T) u(\tau_k^{s+}) \right\|_{H^\frac{d}{2}} \leq \frac{N(T)^{\frac{4s}{d}}}{N(\tau_k^{s-})} \frac{1}{\lambda(\tau_k^{s-})} \tag{2.32}
\]

**Proof.** First, (2.30) follows from Lemma 4.2 in [27] and Corollary 2.10. And (2.31) follows from (3.20) in [5]. Thus, we only need to show (2.31). We write \( N = N(T^*) \) in this proof. Note that

\[
\text{Re} \int \nabla \Delta I_N u \overline{I_N u} = 0 \quad \text{and} \quad \text{Re} \int \nabla (|I_N u|^{\frac{4}{d}} I_N u) \overline{I_N u} = 0,
\]
we derive that
\[
\frac{d}{dt} P(I_N u) = \text{Im} \int \nabla \partial_t I_N u \overline{I_N u} + \text{Im} \int \nabla I_N u \overline{\partial_t I_N u}
\]
\[
= \text{Re} \int (\nabla \Delta I_N u + \nabla I_N (|u|^4 \overline{u})) \overline{I_N u} - \text{Re} \int \nabla I_N u (\Delta \overline{I_N u} + I_N (|u|^4 \overline{u}))
\]
\[
= 2 \text{Re} \int \nabla I_N (|u|^4 \overline{u}) \overline{I_N u}
\]
\[
= 2 \text{Re} \int \nabla (I_N (|u|^4 \overline{u}) - |I_N u|^4 I_N u) \overline{I_N u}.
\]

Hence,
\[
|P(I_N(T^+) u(\tau_k^{j+1})) - P(I_N(T^+) u(\tau_k^j))| 
\leq \left| \int_{\tau_k^j}^{\tau_k^{j+1}} \frac{d}{dt} P(I_N u) \ dt \right|
\leq \int_{\tau_k^j}^{\tau_k^{j+1}} \int_{\mathbb{R}^4} |\nabla (I_N (|u|^4 \overline{u}) - |I_N u|^4 I_N u)\overline{I_N u}| \ dx \ dt
\leq \|\nabla [I_N (|u|^4 \overline{u}) - |I_N u|^4 I_N u]\|_{L^2_x L^{\frac{4n}{n+4}}((\tau_k^j, \tau_k^{j+1}) \times \mathbb{R}^4)} \|I_N u\|_{L^4_x L^{\frac{8n}{n+4}}((\tau_k^j, \tau_k^{j+1}) \times \mathbb{R}^4)}.
\]

It follows from (4.5) in [27] that
\[
\|\nabla [I_N (|u|^4 \overline{u}) - |I_N u|^4 I_N u]\|_{L^2_x L^{\frac{4n}{n+4}}((\tau_k^j, \tau_k^{j+1}) \times \mathbb{R}^4)} \lesssim N^{-\min\{1, \frac{4}{n}\}s+} \|\langle \nabla \rangle I_N u\|_{S^0((\tau_k^j, \tau_k^{j+1}))}^{1+\frac{4}{n}},
\]
this together with Hölder’s inequality implies that
\[
|P(I_N(T^+) u(\tau_k^{j+1})) - P(I_N(T^+) u(\tau_k^j))| 
\lesssim N^{-\min\{1, \frac{4}{n}\}s+} \|\langle \nabla \rangle I_N u\|_{S^0((\tau_k^j, \tau_k^{j+1}))}^{\frac{2n}{n+4}} |\tau_k^j - \tau_k^{j+1}|^{\frac{4}{n}}
\]
\[
\lesssim N^{-\min\{1, \frac{4}{n}\}s+} \|\langle \nabla \rangle I_N u(\tau_k^j)\|_{L^2}^{\frac{2n}{n+4}} |\tau_k^j - \tau_k^{j+1}|^{\frac{4}{n}}.
\]

Next, we consider the initial data.

**Lemma 2.15.** Let
\[
\Xi(t) := \frac{\lambda(t)^2}{2} \int (|\nabla G(0)|^2 - |\nabla I_N(t) G(0)|^2) \ dx.
\]
Then, we have for \(t \in [0, T^+]\),
\[
\left| E(I_N(t) u(0)) + \frac{\Xi(t)}{\lambda(t)^2} \right| \lesssim N(t)^{2(1-s)} + \frac{1}{\lambda(t)^2 - \frac{1}{2n} + \frac{s^2}{s+1}},
\]
\[
\left| P(I_N(t) u(0)) \right| \lesssim N(t)^{1-s} + \frac{1}{\lambda(t)^{1-\frac{1}{2n}}},
\]
\[
\left| E(I_N(t) u(0)) + \frac{\Xi(t)}{\lambda(t)^2} \right| \lesssim \left| E(I_N(t) G(0)) + \frac{\Xi(t)}{\lambda(t)^2} \right| + \left| E(I_N(t) (G(0) + H(0)) - E(I_N(t) G(0)) \right|.}
\]

**Proof.** Note that \(u_0 = G(0) + H(0)\), we have
\[
\left| E(I_N(t) u(0)) + \frac{\Xi(t)}{\lambda(t)^2} \right| \lesssim \left| E(I_N(t) G(0)) + \frac{\Xi(t)}{\lambda(t)^2} \right| + \left| E(I_N(t) (G(0) + H(0)) - E(I_N(t) G(0)) \right|.
The estimate of (2.36): A simple computation shows
\[
E(I_{N(t)}u(0)) + \frac{\Xi(t)}{\lambda(t)^2} = E(G(0)) + \frac{d}{2(d+2)} \int \left( |G(0)|^{\frac{2(d+2)}{d}} - |I_{N(t)}G(0)|^{\frac{2(d+2)}{d}} \right) dx.
\] (2.38)

Hence, by \( G(0) \in H^1 \) and \( ||G(0)||_{H^s} \sim \frac{1}{\lambda(0)^s} \), (1.34), (1.61) and (1.65), we get
\[
\left| E(I_{N(t)}G(0)) + \frac{\Xi(t)}{\lambda(t)^2} \right| \leq |E(G(0))| + \frac{d}{2(d+2)} \int \left| \left( |G(0)|^{\frac{2(d+2)}{d}} - |I_{N(t)}G(0)|^{\frac{2(d+2)}{d}} \right) dx \right|
\]
\[
\leq \frac{1}{\sqrt{\lambda(0)}} + \left| \left( |I_{N(t)} - Id)G(0)|^{\frac{2(d+2)}{d}} \right) \right|_{H^s}
\]
\[
\leq \frac{1}{\sqrt{\lambda(0)}} + \left| \left( |I_{N(t)} - Id)G(0)|^{\frac{2(d+2)}{d}} \right) \right|_{H^s}^{\sqrt{\frac{d}{d+2}}}
\]
\[
\leq \frac{1}{\lambda(0)^2} \left( \lambda(t)^\frac{3}{4} + \lambda(t)^{\frac{3}{4} + \frac{s}{d+2}} \right)
\]
\[
\leq \frac{1}{\lambda(0)^2} \left( \lambda(t)^{\frac{3}{4} - \frac{s}{d+2}} \right)
\] (2.39)

The estimate of (2.37): Observe that
\[
|E(u + v) - E(u)| \lesssim \|u\|_{L^2} \|v\|_{L^2} + \|\nabla v\|_{L^2}^2 + \int \left( |u|^{\frac{d+4}{d}} + |v|^{\frac{d+4}{d}} \right) |v| \ dx
\]
\[
\lesssim \|\nabla v\|_{L^2}^2 \left( 1 + \|v\|_{L^2}^2 \right) + \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} + \|\nabla u\|_{L^2}^{\frac{d+4}{d+2}} \|\nabla v\|_{L^2}^{\frac{d+4}{d+2}} \|u\|_{L^2} \|v\|_{L^2}^{\frac{d+4}{d+2}}.
\]

Using (1.36), (1.39), (1.42) and (1.43), we obtain
\[
\|\nabla I_{N(t)}G(0)\|_{L^2} \leq \|\nabla G(0)\|_{L^2} \lesssim \frac{1}{\lambda(0)},
\] (2.40)
\[
\|\nabla I_{N(t)}H(0)\|_{L^2} \leq N(t)^{1-s} \|H(0)\|_{H^s} \lesssim \lambda(0)^{10} N(t)^{1-s}.
\] (2.41)

This together with the uniform \( L^2 \) control (1.41) implies
\[
|E(I_{N(t)}(G(0) + H(0))) - E(I_{N(t)}G(0))| \lesssim N(t)^{2(1-s)}.
\] (2.42)

And so we obtain (2.34).

Next, we prove (2.35). This part is independent of nonlinear term, so this term is as in [2]. In fact, by (1.36), we have
\[
|P(I_{N(t)}u(0))| \leq |P(I_{N(t)}G(0) + H(0)) - P(I_{N(t)}G(0))| + |P(I_{N(t)}G(0) - P(G(0))| + |P(G(0))|
\]
\[
\lesssim |P(I_{N(t)}G(0) + H(0)) - P(I_{N(t)}G(0))| + |P(I_{N(t)}G(0) - P(G(0))| + \frac{1}{\sqrt{\lambda(0)}}
\]

A simple computation shows that for \( u, v \in \dot{H}^{\frac{d}{2}} \),
\[
|P(u + v) - P(u)| \lesssim \|v\|_{\dot{H}^{\frac{d}{2}}} \left( \|u\|_{\dot{H}^{\frac{d}{2}}} + \|v\|_{\dot{H}^{\frac{d}{2}}} \right).
\]
Combining this with (2.40) and (2.41), we derive that
\[
|P(I_{N(t)}(G(0) + H(0)) - P(I_{N(t)}G(0))| 
\lesssim \|I_{N(t)}H(0)\|_{\dot{H}^{\frac{1}{2}}} \left( \frac{1}{\lambda(0)} + \|I_{N(t)}H(0)\|_{\dot{H}^{\frac{1}{2}}} \right) \lesssim N(t)^{1-s}. 
\]

On the other hand,
\[
|P(I_{N(t)}G(0)) - P(G(0))| \lesssim \|(Id - I_{N(t)})G(0)\|_{\dot{H}^{\frac{1}{2}}} \|G(0)\|_{\dot{H}^{\frac{1}{2}}} 
\lesssim \frac{1}{\sqrt{\lambda(0)}} \left( \frac{1}{N(t)\lambda(0)} \right)^{\frac{1}{2}} 
\lesssim \frac{1}{\sqrt{\lambda(t)}} \left( \frac{1}{N(t)\lambda(t)} \right)^{\frac{1}{2}} 
\stackrel{\lambda(t)}{\lesssim} \frac{1}{\lambda(t)^{1-\frac{\alpha}{2}}}.
\]

\[\square\]

**Proposition 2.16 (Almost conservation laws).** Let \( \frac{1}{1+\min\{1,\frac{4}{3}\}} < s < 1, \quad \min(1,s) \approx s(1-s) < \frac{1}{\beta} \). There holds the following control of the modified energy and momentum on \([0,T^+]:\)
\[
|E(I_{N(t)}u(t)) + \Xi(t) - \frac{1}{\lambda(t)^{2(1-\alpha_1)}}| \leq \frac{1}{\lambda(t)^{2(1-\alpha_1)}}, \quad (2.43)
\]
\[
|P(I_{N(t)}u(t))| \leq \frac{1}{\lambda(t)^{1-\alpha_1}}, \quad (2.44)
\]
for some \( \alpha_1 = \frac{1-s}{4} \). In other words,
\[
|\lambda(t)^2 E(I_{N(t)}u(t)) + \Xi(t)| \leq \lambda(t)^2 \alpha_1 \leq \Gamma_0^{10}, \quad (2.45)
\]
\[
\lambda(t)|P(I_{N(t)}u(t))| \leq \lambda(t)^{\alpha_1} \leq \Gamma_0^{10}. \quad (2.46)
\]

**Proof.** Without loss of generality, we may assume \( t = T^+ \). By (2.30), (2.26), (2.34), (2.32), we have
\[
|E(I_{N(T^+)}u(T^+)) + \Xi(T^+) - \frac{1}{\lambda(T^+)^{2-\frac{1}{2} - \frac{1}{2}}} T^+| 
\lesssim |E(I_{N(T^+)}u(0)) + \Xi(T^+) - \frac{1}{\lambda(T^+)^{2-\frac{1}{2} - \frac{1}{2}}} T^+| 
\lesssim N(T^+)^{2(1-s) - \frac{1}{2}} + \frac{1}{\lambda(T^+)^2 - \frac{1}{2} - \frac{1}{2} - \frac{1}{2}} 
\]
\[
+ \sum_{k=k_0}^{k^+} k N(T^+) \sum_{j=1}^{J_k} \left( \frac{N(T^+)}{N(t_k)} \right)^{\frac{1}{2}} \left( \frac{1}{\lambda(t_k)^{2\frac{1}{2} - \frac{1}{2}}} \right) 
\lesssim \left( \frac{1}{\lambda(T^+)} \right)^{\frac{2(1-s)}{s}} + \frac{1}{\lambda(T^+)^2 - \frac{1}{2} - \frac{1}{2}} 
\]
\[
+ \sum_{k=k_0}^{k^+} k N(T^+) \sum_{j=1}^{J_k} \left( \frac{N(T^+)}{N(t_k)} \right)^{\frac{1}{2}} \left( \frac{1}{\lambda(t_k)^{2\frac{1}{2} - \frac{1}{2}}} \right) 
\lesssim \left( \frac{1}{\lambda(T^+)} \right)^{\frac{2(1-s)}{s}} + \frac{1}{\lambda(T^+)^2 - \frac{1}{2} - \frac{1}{2}} 
\]

We now sum up the geometric series from \(1.63\) to get
\[
\sum_{k=k_0}^{k^+} k N(T^+) \frac{2(1-s)}{s} N(T^+) - \min(1, \frac{2}{\beta}) s_+ \left[ \left( \frac{N(T^+)}{N(t_k)} \right)^{1-s} \frac{1}{\lambda(t_k)} \right]^{2+\frac{4}{s}}
\]
\[
\lesssim N(T^+) (1-s)(2+\frac{2}{\beta}+\frac{1}{s}) - \min(1, \frac{2}{\beta}) s_+ \sum_{k=k_0}^{k^+} k N(t_k) (2+\frac{2}{\beta})(1-s)
\]
\[
\lesssim k^+ N(T^+) (2+\frac{2}{\beta}) + \frac{2(1-s)}{s} - \min(1, \frac{2}{\beta}) s_+
\]
\[
\lesssim |\log(\lambda(T^+))| \left( \frac{1}{\lambda(T^+)} \right)^{\frac{1}{\beta}(2+\frac{2}{\beta}) - \frac{2(1-s)}{s} - \min(1, \frac{2}{\beta}) s_+}
\]
\[
\lesssim \left( \frac{1}{\lambda(T^+)} \right)^{\frac{1}{\beta} + 1 - \frac{1}{\beta} - \frac{1}{s}}.
\]

Similarly, using \(2.31\), \(2.20\), \(2.35\) and \(2.32\), we estimate
\[
|P(I_N(T^+)u(T^+))|
\]
\[
\leq |P(I_N(T^+)u(0))| + \sum_{k=k_0}^{k^+} \sum_{j=1}^{J_k} |P(I_N(T^+)u(\tau^j_k)) - P(I_N(T^+)u(\tau^j_k))|
\]
\[
\lesssim N(T^+) (1-s) + \frac{N(T^+)^{2(1-s)}}{\lambda(T^+)^{1-\frac{1}{s}}} + \sum_{k=k_0}^{k^+} k N(T^+) \left( \frac{N(T^+)}{N(t_k)} \right)^{1-s} \frac{1}{\lambda(t_k)} \right]^{2+\frac{4}{s}} - \frac{1}{\lambda(T^+)}\left( \frac{1}{\lambda(T^+)} \right)^{\frac{1}{\beta} + 1 - \frac{1}{\beta} - \frac{1}{s}}.
\]

\[
\delta \geq \left( \frac{1}{\lambda(T^+)} \right)^{\frac{1}{\beta} + 1 - \frac{1}{\beta} - \frac{1}{s}} + \left( \frac{1}{\lambda(T^+)} \right)^{\frac{1}{\beta} + 1 - \frac{1}{\beta} - \frac{1}{s}} + \left( \frac{1}{\lambda(T^+)} \right)^{\frac{1}{\beta} + 1 - \frac{1}{\beta} - \frac{1}{s}}.
\]

\[
3. \text{ PROOF OF PROPOSITION 1.9}
\]

In this section, we will show Proposition 1.9 and then we conclude the proof of our main Theorem 1.4.

3.1. Control of the geometrical parameters. Recall the geometrical decomposition
\[
u(t, x) = \frac{1}{\lambda(t)^{\frac{1}{\beta}}}(Q_b(t)(\cdot) + \varepsilon(t, \cdot)) \left( \frac{x - x(t)}{\lambda(t)} \right) e^{-i \gamma(t)} \text{, } t \in [0, T^+].
\]

Let us introduce the rescaled time
\[
\frac{ds}{dt} = \frac{1}{\lambda(s)^2} \text{ with } s(0) = s_0 = e^{\delta \gamma_{\text{min}}}
\]
and \(y = \frac{x - x(t)}{\lambda(t)}\). Then, \(\varepsilon(s, y)\) satisfies on \([0, T^+]\) the equation:
\[
i \frac{\partial Q_b}{\partial s} b_s + i \partial_s \varepsilon + \Delta Q_b + \Delta \varepsilon + |Q_b + \varepsilon|^\frac{1}{\beta}(Q_b + \varepsilon)
\]
\[
= - \left( \gamma_s + i \frac{d}{2 \lambda} \right)(Q_b + \varepsilon) + i \left( \frac{x}{\lambda} + \frac{\lambda}{\lambda} y \right) \cdot \nabla (Q_b + \varepsilon).
\]
To simplify notations, we note
\[ \varepsilon = \varepsilon_1 + i \varepsilon_2, \quad Q_b = \Sigma + i \Theta \]
in terms of real and imaginary parts. We have by using Remark 1.7
\begin{align*}
\frac{\partial \Sigma}{\partial \gamma} + \partial_s \varepsilon_1 - M_-(\varepsilon) + b \Lambda \varepsilon_1 &= \left( \frac{\lambda_s}{\lambda} + b \right) \Lambda \Sigma + \tau_s \Theta + \frac{x_s}{\lambda} \cdot \nabla \Sigma + \\
&+ \left( \frac{\lambda_s}{\lambda} + b \right) \Lambda \varepsilon_1 + \tau_s \varepsilon_2 + \frac{x_s}{\lambda} \cdot \nabla \varepsilon_1 \\
&+ \text{Im}(\Psi_b) - R_2(\varepsilon) \\
&= \left( \frac{\lambda_s}{\lambda} + b \right) \Lambda \Theta + \tau_s \Sigma + \frac{x_s}{\lambda} \cdot \nabla \Theta + \\
&+ \left( \frac{\lambda_s}{\lambda} + b \right) \Lambda \varepsilon_2 - \tau_s \varepsilon_1 + \frac{x_s}{\lambda} \cdot \nabla \varepsilon_2 \\
&- \text{Re}(\Psi_b) + R_1(\varepsilon),
\end{align*}
\[(3.3)\]
with \( \tau(s) = -s - \gamma(s) \). The linear operator close to \( Q_b \) is now a deformation of the linear operator \( L \) close to \( Q \) and is \( M = (M_+, M_-) \) with
\begin{align*}
M_+(\varepsilon) &= -\Delta \varepsilon_1 + \varepsilon_1 - \left( \frac{4 \Sigma^2}{d \langle Q_b \rangle^2} + 1 \right) |Q_b|^\frac{d}{2} \varepsilon_1 - \left( \frac{4 \Sigma \Theta}{d \langle Q_b \rangle^2} |Q_b|^\frac{d}{2} \right) \varepsilon_2 \\
M_-(\varepsilon) &= -\Delta \varepsilon_2 + \varepsilon_2 - \left( \frac{4 \Theta^2}{d \langle Q_b \rangle^2} + 1 \right) |Q_b|^\frac{d}{2} \varepsilon_2 - \left( \frac{4 \Sigma \Theta}{d \langle Q_b \rangle^2} |Q_b|^\frac{d}{2} \right) \varepsilon_1.
\end{align*}
The formally quadratic in \( \varepsilon \) interaction terms are:
\begin{align*}
R_{1}(\varepsilon) &= (\varepsilon_1 + \Sigma)|\varepsilon + Q_b|^\frac{d}{2} - \Sigma|Q_b|^\frac{d}{2} - \left( \frac{4 \Sigma^2}{d \langle Q_b \rangle^2} + 1 \right) |Q_b|^\frac{d}{2} \varepsilon_1 - \left( \frac{4 \Sigma \Theta}{d \langle Q_b \rangle^2} |Q_b|^\frac{d}{2} \right) \varepsilon_2, \\
R_{2}(\varepsilon) &= (\varepsilon_2 + \Theta)|\varepsilon + Q_b|^\frac{d}{2} - \Theta|Q_b|^\frac{d}{2} - \left( \frac{4 \Theta^2}{d \langle Q_b \rangle^2} + 1 \right) |Q_b|^\frac{d}{2} \varepsilon_2 - \left( \frac{4 \Sigma \Theta}{d \langle Q_b \rangle^2} |Q_b|^\frac{d}{2} \right) \varepsilon_1.
\end{align*}
The formally cubic terms in \( \varepsilon \) are:
\begin{align*}
\bar{R}_1(\varepsilon) &= R_{1}(\varepsilon) - \varepsilon_1^2 |\varepsilon + Q_b|^\frac{d}{2} \\
&\quad \left[ \frac{2}{d} \left( 1 + \frac{4}{d} \right) \Sigma^3 + \frac{6}{d} \Sigma \Theta^2 \right] \\
&\quad - \varepsilon_2^2 |\varepsilon + Q_b|^\frac{d}{2} \\
&\quad \left[ \frac{2}{d} \Sigma^2 + \frac{2}{d} \left( \frac{4}{d} - 1 \right) \Theta \Sigma \right] \\
&\quad - \frac{4}{d} |\varepsilon + Q_b|^\frac{d}{2} \varepsilon_1 \varepsilon_2 \\
&\quad \left[ \frac{4}{d} - 1 \right] \Sigma \Theta + \Theta^3 \\
\bar{R}_2(\varepsilon) &= R_{2}(\varepsilon) - \varepsilon_1^2 |\varepsilon + Q_b|^\frac{d}{2} \\
&\quad \left[ \frac{2}{d} \left( 1 + \frac{4}{d} \right) \Theta^3 + \frac{6}{d} \Theta \Sigma^2 \right] \\
&\quad - \varepsilon_2^2 |\varepsilon + Q_b|^\frac{d}{2} \\
&\quad \left[ \frac{2}{d} \Theta^2 + \frac{2}{d} \left( \frac{4}{d} - 1 \right) \Sigma \Theta \right] \\
&\quad - \frac{4}{d} |\varepsilon + Q_b|^\frac{d}{2} \varepsilon_1 \varepsilon_2 \\
&\quad \left[ \frac{4}{d} - 1 \right] \Theta^2 \Sigma + \Sigma^3
\end{align*}

Remark 3.1. From Weinstein [29], the linearized operator \( L \) close to the ground state in dimension \( d \) can be explicitly written \( L = (L_+, L_-) \) with
\begin{align*}
L_+ &= -\Delta + 1 - \left( \frac{d}{2} + 1 \right) Q_b^\frac{d}{2}, \\
L_- &= -\Delta + 1 - Q_b^\frac{d}{2},
\end{align*}
and the following algebraic relations hold:

\[ L_+(\Lambda Q) = -2Q, \quad L_+(\nabla Q) = 0, \]
\[ L_-(Q) = 0, \quad L_-(yQ) = -2\nabla Q, \quad L_-(|y|^2Q) = -4\Lambda Q. \]

Recall

\[ \| e^{C_\gamma} (Q_b - Q) \|_{H^1 \cap C^2} \to 0 \quad \text{as} \quad |b| \to 0, \]

we can replace \( Q_{b(s)} \) by \( Q \) in the following with some loss such as \( \Gamma_{b(s)}^{10} \).

**Lemma 3.2** (Estimates induced by conservation laws). For all \( s \in [s_0, s^+] \) with \( s^+ = s(T^+) \), there holds:

\[
\left| 2(\varepsilon_1, \Sigma + b\Lambda \Theta - \text{Re}(\Psi_b)) + 2(\varepsilon_2, \Theta - b\Lambda \Sigma - \text{Im}(\Psi_b)) \right| - 2\Xi(s) - \int |I_{N(s,\lambda,\varepsilon)}| \leq \delta_0 \left( \int |I_{N(s,\lambda,\varepsilon)}| \right) \]
\[ + \int \left( \frac{C_\gamma}{d} + 1 \right) |Q| \frac{1}{2} (I_{N,\lambda,\varepsilon}^2) + \left| Q | \frac{1}{2} (I_{N,\lambda,\varepsilon})^2 \right| \]
\[ \leq \delta_0 \left( \int |I_{N(s,\lambda,\varepsilon)}| \right) \quad (3.5) \]

and

\[ \left| (\varepsilon_2, \nabla Q) \right| \leq \delta_0 \left( \int |I_{N(s,\lambda,\varepsilon)}| \right)^{1/2} + \Gamma_{b(s)}^{10}. \quad (3.6) \]

**Proof.** Note that

\[ I_{Nu}(t, x) = \frac{1}{\lambda^2} \left[ I_{\lambda,\varepsilon} (Q_b + \varepsilon) \right] \left( \frac{x - z(t)}{\lambda(t)} \right), \]

we have

\[ 2\lambda^2 E(I_{Nu}) = 2E(I_{N,\lambda}(Q_b + \varepsilon)) \]
\[ = 2E(I_{N,\lambda}(Q_b + \varepsilon)) + 2E(Q_b + I_{N,\lambda}) \]
\[ + 2E(Q_b + I_{N,\lambda}). \quad (3.7) \]

On the other hand, observe that

\[ \Delta Q_b - Q_b + ib\Delta Q_b + |Q_b|^{\frac{2d}{d+2}}Q_b = -\Psi_b, \]

we get

\[ 2E(Q_b + I_{N,\lambda}) = \int |\nabla (Q_b + I_{N,\lambda})|^2 - \frac{d}{d+2} \int |Q_b + I_{N,\lambda}|^{\frac{2(d+2)}{d}} \]
\[ = \int \left( |\nabla Q_b|^2 - 2\text{Re}(\Delta Q_b \cdot I_{N,\lambda}) + |\nabla I_{N,\lambda}|^2 \right) - \frac{d}{d+2} \int |Q_b + I_{N,\lambda}|^{\frac{2(d+2)}{d}} \]
\[ = 2E(Q_b) + \int |\nabla I_{N,\lambda}|^2 - 2\text{Re} \int (\Delta Q_b + |Q_b|^{\frac{2d}{d+2}}Q_b) I_{N,\lambda} \]
\[ - \frac{d}{d+2} \int \left( |Q_b + I_{N,\lambda}|^{\frac{2(d+2)}{d}} - |Q_b|^{\frac{2(d+2)}{d}} \right) - \frac{2(d+2)}{d} |Q_b|^{\frac{2d}{d+2}} \text{Re}(Q_b I_{N,\lambda}). \]
and
\[ 2\text{Re} \int (\Delta Q_b + |Q_b|^{\frac{2}{d}} Q_b) \overline{I_{N\lambda}} = 2\text{Re} \int (Q_b - ib\Lambda Q_b - \Psi_b) \overline{I_{N\lambda}} \]
\[ = 2\text{Re} \int (Q_b - ib\Lambda Q_b - \Psi_b) \overline{\varepsilon} + 2\text{Re} \int (I_{N\lambda} - \text{Id})(Q_b - ib\Lambda Q_b - \Psi_b) \overline{\varepsilon} \]
\[ = 2(\varepsilon_1, \Sigma + b\Lambda - \text{Re}(\Psi_b)) + 2(\varepsilon_2, \Theta - b\Lambda - \text{Im}(\Psi_b)) + 2\text{Re} \int (I_{N\lambda} - \text{Id})(Q_b - ib\Lambda Q_b - \Psi_b) \overline{\varepsilon}. \]

Hence,
\[ 2(\varepsilon_1, \Sigma + b\Lambda - \text{Re}(\Psi_b)) + 2(\varepsilon_2, \Theta - b\Lambda - \text{Im}(\Psi_b)) - 2\Xi(s) - \int |I_{N(s)\lambda(s)} \nabla \varepsilon(s)|^2 \]
\[ + \int \left( \frac{4\Sigma^2}{|Q_b|^2} + 1 \right) |Q_b|^{\frac{2}{d}} (I_{N\lambda} \varepsilon_1)^2 + \int \left( \frac{4\Theta^2}{|Q_b|^2} + 1 \right) |Q_b|^{\frac{2}{d}} (I_{N\lambda} \varepsilon_2)^2 \]
\[ = 2E(Q_b) - 2(\lambda^2 E(I_N u) + \Xi(s)) + 2E(I_N(Q_b + \varepsilon)) - 2E(Q_b + I_N \varepsilon) - 2\text{Re} \int (I_{N\lambda} - \text{Id})(Q_b - ib\Lambda Q_b - \Psi_b) \overline{\varepsilon} \]
\[ - 8 \int \frac{\Sigma \Theta}{d|Q_b|^2} |Q_b|^{\frac{2}{d}} (I_{N\lambda} \varepsilon_1 I_{N\lambda} \varepsilon_2) - \frac{d}{d+2} \int J(I_{N\lambda} \varepsilon), \]
(3.8)

where the cubic term \( J(\varepsilon) \) (is given in Appendix B in [20])
\[ J(\varepsilon) = \varepsilon + Q_b|^{2 + \frac{2}{d}} - |Q_b|^{2 + \frac{2}{d}} - \left( 2 + \frac{4}{d} \right) |Q_b|^{\frac{2}{d}} (\Sigma \varepsilon_1 + \Theta \varepsilon_2) \]
\[ - \frac{\varepsilon_1^2}{d} |Q_b|^{\frac{2}{d} - 2} \left[ (1 + \frac{2}{d})(1 + \frac{4}{d}) \Sigma^2 + \left( 1 + \frac{2}{d} \right) \Theta^2 \right] \]
\[ - \frac{\varepsilon_2^2}{d} |Q_b|^{\frac{2}{d} - 2} \left[ (1 + \frac{2}{d})(1 + \frac{4}{d}) \Theta^2 + \left( 1 + \frac{2}{d} \right) \Sigma^2 \right] \]
\[ - \varepsilon_1 \varepsilon_2 |Q_b|^{\frac{2}{d} - 2} \frac{8}{d} \left[ (1 + \frac{2}{d}) \Sigma \Theta \right]. \]

Indeed, \( J(\varepsilon) \) is the formal cubic term in \( \varepsilon \) in the expansion
\[ \varepsilon + Q_b|^{2 + \frac{2}{d}} = |Q_b|^{2 + \frac{2}{d}} \left[ 1 + \frac{2(\Sigma \varepsilon_1 + \Theta \varepsilon_2)}{|Q_b|^2} + \frac{|\varepsilon|^2}{|Q_b|^2} \right]. \]

In fact, \( J(\varepsilon) \) is not cubic in \( \varepsilon \) when \( d \geq 4 \), but from elementary inequality, \( J(\varepsilon) = O(|\varepsilon|^{2 + \frac{2}{d}}) \). This can give us the desired smallness in the sequel.

Then, (3.8) follows by (1.29), Proposition 2.16 and
\[ \| (I_{N(t)\lambda(t)} - \text{Id}) Q \|_{H^p} \lesssim \lambda(t)^C \lesssim \Gamma_{10}^{\tilde{b}(t)}. \]
(3.9)

Next, we turn to prove (3.6). Observe that
\[ \lambda P(I_N u) = P(I_{N\lambda}(Q_b + \varepsilon)) = (P(I_{N\lambda}(Q_b + \varepsilon)) - P(Q_b + I_{N\lambda} \varepsilon) + P(Q_b + I_{N\lambda} \varepsilon) \]
and
\[
P(Q_b + I_N \epsilon) = \text{Im} \int \nabla (Q_b + I_N \epsilon)(Q_b + I_N \epsilon) = -2 \int I_N \epsilon \nabla \Sigma - 2 \int \Theta \nabla \Sigma + 2 \int I_N \epsilon \nabla \Theta + 2 \int I_N \epsilon \nabla I_N \epsilon_2
\]
then, we obtain
\[
-2(\epsilon_2, \nabla Q) = \lambda P(I_N u) - \left( P(I_N (Q_b + \epsilon)) - P(Q_b + I_N \epsilon) \right) - 2\epsilon_2 \nabla (Q - \Sigma)) - 2(\epsilon_2, (\text{Id} - I_N \epsilon) \nabla \Sigma) + 2 \int \Theta \nabla \Sigma - 2 \int I_N \epsilon \nabla \Theta - 2 \int I_N \epsilon \nabla I_N \epsilon_2.
\]
Note that \((\Theta, \nabla \Sigma) = 0\) since \(Q_b\) is radial. Thus, we get (3.6) by (1.27) and Proposition 2.16. Therefore, we conclude the proof of Lemma 3.2.

Recall
\[
(Q_b)|_{b=0} = Q, \quad \left( \frac{\partial Q_b}{\partial b} \right)|_{b=0} = -\frac{1}{4} |y|^2 Q. \tag{3.10}
\]
It is easy to see that
\[
\nabla \Lambda f - \Lambda \nabla f = \nabla f, \quad (f, \Lambda g) = - (\Lambda f, g). \tag{3.11}
\]

**Lemma 3.3.** There holds

(i) \[
\left( \frac{\lambda \epsilon}{\alpha} + b \right) [\Lambda \Sigma, |y|^2 \Sigma] + (\Lambda \Theta_i, |y|^2 \Theta) \tag{3.12}
\]

(ii) \[
= (M_+(\epsilon), |y|^2 \Theta) - (M_-(\epsilon), |y|^2 \Sigma) - \frac{\lambda \epsilon}{\alpha} \left[ (\epsilon_1, \Lambda (|y|^2 \Sigma)) + (\epsilon_2, \Lambda (|y|^2 \Theta)) \right] + b_\epsilon \left[ (\frac{\partial}{\partial \theta}, |y|^2 \Sigma) + (\frac{\partial}{\partial \theta}, |y|^2 \Theta) - (\epsilon_1, \frac{\partial (|y|^2 \Sigma)}{\partial \theta}) - (\epsilon_2, \frac{\partial (|y|^2 \Theta)}{\partial \theta}) \right] + \gamma_\epsilon \left[ (\epsilon_1, |y|^2 \Theta) - (\epsilon_2, |y|^2 \Sigma) \right] + \frac{x_\epsilon}{\alpha} \left[ (\epsilon_1, |y|^2 \nabla \Sigma) + (\epsilon_2, |y|^2 \nabla \Theta) \right] - (R_1(\epsilon), |y|^2 \Theta) + (R_2(\epsilon), |y|^2 \Sigma) - (\text{Im}(\Psi_b), |y|^2 \Sigma) + (\text{Re}(\Psi_b), |y|^2 \Theta), \tag{3.13}
\]
Proof. Take the inner product of (3.3) with \( y \) and using the orthogonality condition (1.57), we obtain the following lemma.

**Lemma 3.4.** There holds

\[
(3.14)
\]

\[
(3.15)
\]

\[
(3.16)
\]

Note that

\[
\Lambda(fg) = g \Lambda f + f \Lambda g - \frac{d}{2} fg
\]

\[
\Delta(\Lambda f) = 2 \Delta f + \Lambda(\Delta f),
\]

and using the orthogonality condition (1.57), we obtain the following lemma.

**Lemma 3.4.** There holds

\[
- (M_+(\epsilon), \Lambda \Sigma) - (M_-(-\epsilon), \Lambda \Theta)
\]

\[
= 2(\epsilon_1, \Sigma + b \Lambda \Theta - \text{Re}(\Psi_b)) + 2(\epsilon_2, \Theta - b \Lambda \Theta - \text{Im}(\Psi_b))
\]

\[
- (\epsilon_1, \text{Re}(\Lambda \Psi_b)) - (\epsilon_2, \text{Im}(\Lambda \Psi_b)).
\]
Thus, this together with (3.12) and (2.2) yields that

\[
\begin{align*}
&\frac{1}{b_s} [(\partial_b \Theta, \Lambda \Sigma) - (\partial_b \Theta, \Lambda \Sigma) + (\epsilon_1, \partial_b \Lambda \Theta) - (\epsilon_2, \partial_b \Lambda \Sigma)] \\
&= 2 \Xi(s) + H(I_{N \Lambda \epsilon}, I_{N \Lambda \epsilon}) - (\epsilon_1, \Re(\Lambda \Psi_b)) - (\epsilon_2, \Im(\Lambda \Psi_b)) \\
&\quad - \frac{\gamma_s}{s} [(\epsilon_1, \Lambda \Sigma) + (\epsilon_2, \Lambda \Theta)] - \frac{\sigma_s}{s} \cdot \left[ (\epsilon_2, \nabla \Lambda \Sigma) - (\epsilon_1, \nabla \Lambda \Theta) \right] \\
&\quad + O(\delta_0 \left( \int |I_{N(s) \lambda(s)} \nabla \epsilon(s)|^2 + \int |\epsilon(s)|^2 e^{-|y|} \right) + O(\Gamma_{b(s)}^{1-C \eta}) + F(s),
\end{align*}
\]

with

\[
F(s) = (R_2(\epsilon) - R_2(I_{N \Lambda \epsilon}), \Lambda \Theta) + (R_1(\epsilon) - R_1(I_{N \Lambda \epsilon}), \Lambda \Sigma) \\
+ \tilde{H}_b(I_{N \Lambda \epsilon}, I_{N \Lambda \epsilon}) + (\tilde{R}_1(I_{N \Lambda \epsilon}), \Lambda \Sigma) + (\tilde{R}_2(I_{N \Lambda \epsilon}), \Lambda \Theta),
\]

\[
\tilde{H}_b(\epsilon, \epsilon) := \int |V_1(y)|i(\epsilon_1)^2 + \int |V_2(y)|i(\epsilon_2)^2 + \int |V_{12}(y)|i\epsilon_1\epsilon_2,
\]

\[
V_1(y) = \frac{2}{d} \left( 1 + \frac{4}{d} \right) \left[ \frac{|Q_{b}|^\frac{4}{d}}{|Q_b|^4} \Sigma^3 y \cdot \nabla \Sigma - Q^\frac{1}{d-1} y \nabla Q \\
+ \frac{|Q_{b}|^\frac{4}{d}}{|Q_b|^4} \Theta^2 \left[ 6 \Sigma \Lambda \Sigma - \left( \frac{4}{d} + 2 \right) \Sigma^2 - \Theta^2 \right] \\
+ \frac{2}{d} \frac{|Q_{b}|^\frac{4}{d}}{|Q_b|^4} \Theta \Lambda \Theta \left[ \Theta^2 + \left( \frac{4}{d} - 1 \right) \Sigma^2 \right], \right. \]

\[
V_2(y) = \frac{2}{d} \left[ \frac{|Q_{b}|^\frac{4}{d}}{|Q_b|^4} \Sigma^3 y \cdot \nabla \Sigma - Q^\frac{1}{d-1} y \cdot \nabla Q \\
+ \frac{|Q_{b}|^\frac{4}{d}}{|Q_b|^4} \Theta^2 \left[ \frac{2}{d} \left( \frac{4}{d} - 1 \right) \Sigma \Lambda \Sigma - \left( \frac{4}{d} + 2 \right) \Sigma^2 - \left( \frac{4}{d} + 1 \right) \Theta^2 \right] \\
+ \frac{2 \Theta \Lambda \Theta}{d} \frac{|Q_{b}|^\frac{4}{d}}{|Q_b|^4} \left[ 3 \Sigma^2 + \left( \frac{4}{d} + 1 \right) \Theta^2 \right], \right.
\]

\[
V_{12}(y) = \frac{4}{d} \frac{|Q_{b}|^\frac{4}{d}}{|Q_b|^4} \left[ \Theta \left( \Theta^2 \Lambda \Sigma + \left( \frac{4}{d} - 1 \right) \left( \Sigma^2 \Lambda \Sigma + \Sigma \Theta \Lambda \Theta \right) - 2 \Sigma |Q_b|^2 \right) + \Sigma^3 \Lambda \Theta \right].
\]

Moreover, the remainder F can be bounded by

\[
|F| \leq \delta(\alpha^*) \left( \int |\nabla I_{N \Lambda \epsilon}|^2 + \int |\epsilon|^2 e^{-|y|} \right) + \Gamma_{b(s)}^{1-C \eta}.
\]

**Remark 3.5.** By the estimation of F(s), we can absorb this term into \( \Gamma_{b(s)}^{1-C \eta} \). This is different from the dimension two case in \([5]\).

**Proof:** The algebraic identity can be obtained directly from the proof of Lemma \([3,2]\). It is only a matter to estimate the remainder term F(s). Denote by

\[
F_1(s) = (R_2(\epsilon) - R_2(I_{N \Lambda \epsilon}), \Lambda \Theta) + (R_1(\epsilon) - R_1(I_{N \Lambda \epsilon}), \Lambda \Sigma),
\]

\[
F_2(s) = F(s) - F_1(s).
\]

Denote by

\[
\|I_{N \Lambda \epsilon}\|_{H_{1,exp}^{1}} := \int |I_{N \Lambda \epsilon}|^2 + \int |I_{N \Lambda \epsilon}|^2 e^{-|y|}.
\]
Estimate of $F_2$: From [18],
\[ |\tilde{H}_b(I_{N\lambda \epsilon}, I_{N\lambda \epsilon})| + |(R_1(I_{N\lambda \epsilon}), \Lambda \Sigma)| + |(\tilde{R}_2(I_{N\lambda \epsilon}), \Lambda \Theta)| \leq \delta_0 \|I_{N\lambda \epsilon}\|_{H^{1}_{exp}}^2.\]

Estimate of $F_1$: Note that $F_1$ can be written in the following form
\[ F_1(s) = \int |\epsilon|^2 (\text{Id} - I_{N\lambda}) \phi_1 + \int [I_{N\lambda \epsilon}^2 - I_{N\lambda}(|\epsilon|^2)] \phi_2 + O \left( \int \left( |\epsilon|^2 \chi_{d=3} + |\epsilon|^{2+\frac{2}{3}} \chi_{d\geq 4} \right) \phi_3 \right) \]
where $\phi_1, \phi_2, \phi_3$ are Schwartz functions built on $Q$ which decay exponentially as $r \to \infty$. We estimate
\[ \left| \int |\epsilon|^2 (\text{Id} - I_{N\lambda}) \phi_1 \right| \leq \|\epsilon\|_{L^2} \|\text{Id} - I_{N\lambda}\|_{L^\infty} \lesssim \frac{1}{N\lambda} \lesssim \Gamma^{10}_b, \]
and
\[ \left| \int [I_{N\lambda \epsilon}^2 - I_{N\lambda}(|\epsilon|^2)] \phi_2 \right| \lesssim \|I_{N\lambda \epsilon} - \epsilon\|_{L^2} \|\text{Id} - I_{N\lambda}\| \lesssim \frac{1}{N\lambda} \lesssim \Gamma^{10}_b. \]
To estimate the third term in $F_1$, we only deal with the case $d \geq 4$ here. Denote by $J_{N\lambda} = \text{Id} - I_{N\lambda}$, and from the assumption $s > \frac{1}{\Gamma^{10}_b} = \frac{d}{d+4}$, we know that $\frac{2d}{d-2s} > 2 + \frac{2}{3}$, we use Sobolev embedding to estimate
\[ \left| \int |\epsilon|^{2+\frac{2}{3}} \phi_3 \right| \leq C \int |I_{N\lambda \epsilon}|^{2+\frac{2}{3}} |\phi_3| + C \int |I_{N\lambda \epsilon}|^{2+\frac{2}{3}} |\phi_3|, \]
\[ \|I_{N\lambda \epsilon}\|_{L^{2+\frac{2}{3}}([\phi_3]|_{dy})} \leq \|I_{N\lambda \epsilon}\|_{L^{2+\frac{2}{3}}([\phi_3]|_{dy})} \|I_{N\lambda \epsilon}\|_{L^{2+\frac{2}{3}}([\phi_3]|_{dy})} \leq C \|I_{N\lambda \epsilon}\|_{L^2([\phi_3]|_{dy})} \|\nabla I_{N\lambda \epsilon}\|_{L^2} \leq C \|I_{N\lambda \epsilon}\|_{H^{1}_{exp}} \]
and
\[ \|J_{N\lambda \epsilon}\|_{L^{2+\frac{2}{3}}([\phi_3]|_{dy})} \leq \|J_{N\lambda \epsilon}\|_{L^{2+\frac{2}{3}}([\phi_3]|_{dy})} \|J_{N\lambda \epsilon}\|_{H^{1}_{exp}} \leq \|J_{N\lambda \epsilon}\|_{H^{1}_{exp}} \leq \frac{1}{(N\lambda)^{2\theta_2}} \|J_{N\lambda \epsilon}\|_{H^{\theta_2}}, \]
\[ \leq \frac{1}{(N\lambda)^{2\theta_2}} \|P_{\geq N\lambda}\|_{H^{\theta_2}} \leq \frac{1}{(N\lambda)^{2\theta_2}} \|\nabla I_{N\lambda \epsilon}\|_{L^2} \leq C \|I_{N\lambda \epsilon}\|_{H^{1}_{exp}}. \]
This completes the proof of Lemma 3.4.

\[ \square \]

View the four equalities (3.13), (3.14), (3.15), (3.17) as a linear system, which is invertible, we obtain
Lemma 3.6 (Control of the geometrical parameters). There holds:

\[
\left| \frac{\lambda_s}{\lambda} + b \right| + |b_s| \leq C \left( \Xi(s) + \int |\nabla I_{N(s)} \lambda(s) \varepsilon(s)|^2 + \int |\varepsilon(s)|^2 e^{-|y|} \right) \\
+ \Gamma_{b(s)}^{1-C \eta},
\]

\[
\left| \tilde{\gamma}_s - \left( \frac{\varepsilon_1}{L+\Lambda^2 Q} \right) \right| + \left| \tilde{\gamma}_s \right| \leq \delta_0 \left( \int |\varepsilon(s)|^2 e^{-|y|} \right)^{\frac{1}{2}} + \Gamma_{b(s)}^{1-C \eta} \\
+ C \left( \Xi(s) + \int |\nabla I_{N(s)} \lambda(s) \varepsilon(s)|^2 \right).
\]

(3.19)

(3.20)

3.2. Virial dispersion. In this subsection, we will derive two global virial dispersive estimates at the heart of the control of the log-log regime in [18, 20]. We begin with the global virial estimate first established in [17, 18].

Lemma 3.7 (Global virial estimate, [5]). We have

\[
b_s \geq c_0 \left( \Xi(s) + \int |\nabla I_{N(s)} \lambda(s) \varepsilon(s)|^2 + \int |\varepsilon(s)|^2 e^{-|y|} \right) - \Gamma_{b(s)}^{1-C \eta}.
\]

(3.21)

Next, we consider another dispersive control of a slightly different kind exhibited in [19, 20]. The main idea is that the profile \(Q_b + \zeta_b\) should be a better approximation of the solution. Let us introduce a cut-off parameter

\[
A(t) = e^{-\frac{2at}{L}}
\]

so that \(\Gamma_b^{-\frac{a}{2}} \leq A \leq \Gamma_b^{-\frac{4a}{7}}\)

(3.22)

for some small parameter \(0 < a \ll 1\) and

\[
\tilde{\zeta} = \chi \left( \frac{r}{L} \right) \zeta_b.
\]

where \(\chi(r)\) is a smooth cut-off function with

\[
\chi(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq \frac{3}{2} \\ 0 & \text{if } r \geq 2. \end{cases}
\]

We remark that \(\tilde{\zeta}\) is a small Schwartz function due to the A localization. We next consider the new variable

\[
\tilde{\varepsilon} = \varepsilon - \tilde{\zeta}.
\]

(3.23)

Then, by the same argument as Lemma 6 in [20] and the above, we obtain

Lemma 3.8 (Virial dispersion in the radiative regime). There holds for some universal constants \(c_1 > 0\) and \(s \in [s_0, s^+]\):

\[
\{ f_1(s) \} \geq c_1 \left( \Xi(s) + \int |\nabla I_{N(s)} \lambda(s) \tilde{\varepsilon}(s)|^2 + \int |\tilde{\varepsilon}(s)|^2 e^{-|y|} + \Gamma_b \right) \\
- \frac{1}{\delta_1} \int_A^{2A} \int |\tilde{\varepsilon}|^2,
\]

(3.24)

with

\[
f_1(s) = \frac{b}{4} \| yQ_b \|_{L^2}^2 + \frac{1}{2} \text{Im} \left( \int (y \cdot \nabla \tilde{\zeta}) \varepsilon \tilde{\zeta} \right) + \text{Re}(\varepsilon_2, \Lambda \tilde{\zeta}) - \text{Im}(\varepsilon_1, \Lambda \tilde{\zeta}).
\]

(3.25)
\[ \frac{df_1}{ds} = H(I_{N\lambda}e - \tilde{c}_b, I_{N\lambda}e - \tilde{c}_b) + (\epsilon_1 - \text{Re } \tilde{c}_b, \text{Re } \Lambda F) + (\epsilon_2 - \text{Im } \tilde{c}_b, \text{Im } \Lambda F) \]
\[ -2(\Lambda^2 E(I_{Nu}u + \Xi(s)) + b_s \left[ (\epsilon_2 - \text{Im } \tilde{c}_b, \partial_b \Lambda(\Sigma + \text{Re } \tilde{c}_b)) - (\epsilon_1 - \text{Re } \tilde{c}_b, \partial_b \Lambda(\Theta + \text{Im } \tilde{c}_b)) \right] \]
\[ - \frac{A_s}{\Lambda^2} \left[ (\epsilon_2 - \text{Im } \tilde{c}_b, \Lambda(y \cdot (\nabla \chi) \left( \frac{y}{A} \right) \text{Re } \tilde{c}_b)) - (\epsilon_1 - \text{Re } \tilde{c}_b, \Lambda(y \cdot (\nabla \chi) \left( \frac{y}{A} \right) \text{Im } \tilde{c}_b)) \right] \]
\[ - \left( \frac{\lambda_s}{\lambda} + b \right) \left[ (\epsilon_2 - \text{Im } \tilde{c}_b, \Lambda^2(\Sigma + \text{Re } \tilde{c}_b)) - (\epsilon_1 - \text{Re } \tilde{c}_b, \Lambda^2(\Theta + \text{Im } \tilde{c}_b)) \right] \]
\[ - \tilde{\gamma}_s \left[ (\epsilon_1 - \text{Re } \tilde{c}_b, \Lambda(\Sigma + \text{Re } \tilde{c}_b)) + (\epsilon_2 - \text{Im } \tilde{c}_b, \Lambda(\Theta + \text{Im } \tilde{c}_b)) \right] \]
\[ - \frac{x_s}{\lambda} \left[ (\epsilon_2 - \text{Im } \tilde{c}_b, \nabla \Lambda(\Sigma + \text{Re } \tilde{c}_b)) - (\epsilon_1 - \text{Re } \tilde{c}_b, \nabla \Lambda(\Theta + \text{Im } \tilde{c}_b)) \right] \]
\[ + (R_1(\epsilon), \Lambda \text{Re } \tilde{c}_b) + (R_2(\epsilon), \Lambda \text{Im } \tilde{c}_b) \]
\[ + (\epsilon_1 - \text{Re } \tilde{c}_b, \Lambda(1 + 4 \frac{d}{d|Q_b|^2}) Q^\frac{d}{2} \text{Im } \tilde{c}_b)) + (\epsilon_2 - \text{Im } \tilde{c}_b, \Lambda(\tilde{c}_b)) \]
\[ + (\epsilon_1, \tilde{L}) + (\epsilon_2, \tilde{K}) + \Xi_{N\lambda} + 2\Xi(s) + \text{Remainder} \]

with
\[ \tilde{L} = \left( \frac{4\Sigma^2}{d|Q_b|^2} + 1 \right) |Q_b|^\frac{d}{2} - \left( 1 + 4 \frac{d}{d|Q_b|^2} \right) Q^\frac{d}{2} \Lambda \text{Re } \tilde{c}_b + \frac{4\Sigma \Theta}{d|Q_b|^2} |Q_b|^\frac{d}{2} \Lambda \text{Im } \tilde{c}_b, \]
\[ \tilde{K} = \left( \frac{4\Theta^2}{d|Q_b|^2} + 1 \right) |Q_b|^\frac{d}{2} - Q^\frac{d}{2} \Lambda \text{Im } \tilde{c}_b + \frac{4\Sigma \Theta}{d|Q_b|^2} |Q_b|^\frac{d}{2} \Lambda \text{Re } \tilde{c}_b, \]

coming from the error term:

\[ (L_+ \epsilon_1 + b\lambda \epsilon_2, \Lambda \text{Re } \tilde{c}_b) + (L_- \epsilon_2 - b\lambda \epsilon_1, \Lambda \text{Im } \tilde{c}_b) \]
\[ -(M_+ \epsilon) + b\lambda \epsilon_2, \Lambda \text{Re } \tilde{c}_b) - (M_- \epsilon) - b\lambda \epsilon_1, \Lambda \text{Im } \tilde{c}_b). \]

Another error terms
\[ \Xi_{N\lambda} = ((I_{N\lambda} - \text{Id})\epsilon_1, \text{Re } \Psi_b) + ((I_{N\lambda} - \text{Id})\epsilon_2, \text{Im } \Lambda \Psi_b) \]
\[ + (L_+ (I_{N\lambda} - \text{Id}) \epsilon_1 + b\lambda (I_{N\lambda} - \text{Id}) \epsilon_2, \Lambda \text{Re } \tilde{c}_b) \]
\[ + (L_- (I_{N\lambda} - \text{Id}) \epsilon_2 - b\lambda (I_{N\lambda} - \text{Id}) \epsilon_1, \Lambda \text{Im } \tilde{c}_b) \]
\[ + ((I_{N\lambda} - \text{Id}) \epsilon_1, \Lambda(\text{Re } F + \left( 1 + 4 \frac{d}{d} \right) Q^\frac{d}{2} \text{Re } \tilde{c}_b)) \]
\[ + ((I_{N\lambda} - \text{Id}) \epsilon_2, \Lambda(\text{Im } F + Q^\frac{d}{2} \text{Im } \tilde{c}_b)) \]

and
\[ \text{Remainder} = \tilde{R}_b(I_{N\lambda} \epsilon, I_{N\lambda} \epsilon) + (R_1(\epsilon) - R_1(I_{N\lambda} \epsilon), \Lambda \Sigma) + (R_2(\epsilon) - R_2(I_{N\lambda} \epsilon), \Lambda \Theta) \]
\[ + (\tilde{R}_1(I_{N\lambda} \epsilon), \Lambda \Sigma) + (\tilde{R}_2(I_{N\lambda} \epsilon), \Lambda \Theta) - \frac{d}{d + 2} \int J(I_{N\lambda} \epsilon) \]
\[ + 2E(I_{N\lambda} Q_b + \epsilon)) - 2E(Q_b + I_{N\lambda} \epsilon) - 2 \text{Re } \int(I_{N\lambda} - \text{Id})(Q_b - ib\lambda Q_b - \Psi_b)\tau. \]
Step 2: Control of interaction terms.

Claim 1:
\[
\int |\text{Id} - I_{NL}\epsilon|^2 e^{-|y|} = O(\Gamma_1^{1+z_0}),
\]
\[
\int |\nabla I_{NL}\zeta|^2 + \int |\zeta|^2 e^{-|y|} = \int |\nabla (I_{NL}\epsilon - \zeta)|^2 + \int |I_{NL}\epsilon - \zeta|^2 e^{-|y|} + O(\Gamma_1^{1+z_0}),
\]
\[
\int |\epsilon|^2 e^{-|y|} \leq 2 \int |\zeta|^2 e^{-|y|} + \Gamma_1^{1+z_0},
\]
\[
\int |\nabla I_{NL}\epsilon|^2 \leq C \int |\nabla I_{NL}\zeta|^2 + \Gamma_1^{1-C}\eta
\]

Indeed, the first estimate comes from the boundness of $I_{NL}\epsilon$ in $H^1$ (thus the boundness of $\epsilon$ in $H^s$) as well as $(NL)^{-s} = O(\Gamma_1^{1+z_0})$ by our bootstrap assumption. Similarly, the second estimate comes from the smallness of $\zeta$:
\[
\|\zeta\|_{H^1} \leq \Gamma_1^{1-C}\eta
\]
and the error estimate $\|\text{Id} - I_{NL}\epsilon\|_{H^1 \to L^2} \leq \frac{1}{N\lambda}$. The third inequality comes from the property of $\zeta$:
\[
\|\zeta(y)e^{-\frac{\sigma d(y,\partial)}{2}}\|_{L^\infty(|y| \leq R_0)} \leq \Gamma_1^{1-\frac{\sigma}{2}}, \forall \sigma \in (0,5), \quad \|\zeta(y)|y|^{d/2}\|_{L^\infty(|y| > R_0)} \leq \Gamma_1^{1-C}\eta.
\]
\[
\int |\epsilon|^2 e^{-|y|} \leq 2 \int |\zeta|^2 e^{-|y|} + 2 \int |\tilde{\epsilon}|^2 e^{-|y|}
\]
\[
\leq 2 \int |\zeta|^2 e^{-|y|} + 2 \int \chi(y)^2 |\zeta|e^{-\frac{d(y,\partial)}{|y|}}
\]
\[
\leq 2 \int |\zeta|^2 e^{-|y|} + 2 \Gamma_1^{1-C}\eta R_0 + 2 \Gamma_1^{1-2C}\eta \log(2A)e^{-\frac{2L^2}{|y|}}
\]
\[
\leq 2 \int |\zeta|^2 e^{-|y|} + \Gamma_1^{1+z_0}.
\]

Claim 2:

1. For $d \geq 3$ and any $B \geq 2$,
\[
\int_{|y| \leq B} |\zeta|^2 \leq CB^2 \left( \int |\nabla I_{NL}\zeta|^2 + \int |\zeta|^2 e^{-|y|} \right) + \Gamma_1^{1+z_0}
\]

2. Second order interaction: for $R(\epsilon) = R_1(\epsilon)$ or $R_2(\epsilon)$,
\[
\int |R(\epsilon)| e^{-\frac{(1-C)\eta}{d(y,\partial)}} \leq C \left( \int |\nabla I_{NL}\zeta|^2 + \int |\zeta|^2 e^{-|y|} \right) + \Gamma_1^{1+z_0}.
\]

3. Small second-order interaction:
\[
\int |R(\epsilon)|(|\zeta| + |y \cdot \nabla \zeta|) \leq \delta(\alpha^*) \left( \int |\nabla I_{NL}\zeta|^2 + \int |\zeta|^2 e^{-|y|} \right) + \Gamma_1^{1+z_0}
\]

4. Small second-order scalar products: For any polynomial $P(y)$ and integers $0 \leq k \leq 2, 0 \leq l \leq 1$, there exists $C > 0$ such that
\[
\int |\zeta|^2 P(y) \left( |\nabla_y^k \tilde{\zeta} + \nabla_y^l \partial_y \tilde{\zeta}| \right) \leq C \left( \int |\nabla I_{NL}\zeta|^2 + \int |\zeta|^2 e^{-|y|} \right)^{\frac{1}{2}} + \Gamma_1^{1+z_0}
\]
(5) Cut-off \(\chi_A\) induced estimates

\[
\int |\epsilon|(|F| + |y \cdot F|) \leq C \Gamma_b^{\frac{3}{2}} \left( \int_{|y| \leq 2A} |\epsilon|^2 \right)^{\frac{1}{2}}.
\]

Proof:

(1) follows from \(\|\tilde{e}\|_{L^2(|y| \leq B)} \leq \|I_N \epsilon\|_{L^2(|y| \leq B)} + \|(Id - I_N \epsilon)\|_{L^2}\) and classical inequality (see [20])

\[
\int_{|y| \leq B} |v|^2 \leq C B^2 \left( \int |\nabla v|^2 + \int |v|^2 e^{-|y|} \right),
\]

combining with the first inequality in claim 1.

For (2), from the classical inequality (see [20])

\[
|R(\epsilon)| \leq C |\epsilon|^2 e^{-\left(\frac{1}{4} - 1\right)(1 - C \eta)^{\frac{\theta(b)(|y|)}{|b|}}} + |\epsilon|^{1 + \frac{4}{b}}, \quad d \leq 3,
\]

\[
|R(\epsilon)| \leq C \min \left(|\epsilon|^2 e^{-\left(\frac{1}{4} - 1\right)(1 - C \eta)^{\frac{\theta(b)(|y|)}{|b|}}}, |\epsilon|^{1 + \frac{4}{b}}\right), \quad d \geq 4,
\]

Using the classical inequality

\[
\int |v|^2 e^{-\kappa |y|} \leq C \left( \int |\nabla v|^2 + \int |v|^2 e^{-|y|} \right)
\]

and the third inequality in claim 1, for \(d \geq 4\), we have

\[
\int |\epsilon|^2 e^{-\left(\frac{1}{4} - 1\right)(1 - C \eta)^{\frac{\theta(b)(|y|)}{|b|}}} \leq \int |\tilde{e}|^2 e^{-\frac{3}{4}(1 - C \eta)^{\frac{\theta(b)(|y|)}{|b|}}} + \Gamma_b^{1 + z_0},
\]

we conclude by replacing \(\tilde{e}\) to \(I_N \epsilon\tilde{e}\) and an error term can be absorbed into \(\Gamma_b^{1 + z_0}\).

For \(d = 3\), we easily have

\[
\int |\epsilon|^2 e^{-\left(1 - C \eta\right)^{\frac{\theta(b)(|y|)}{|b|}}} \leq C \int |\tilde{e}|^2 e^{-\left(1 - C \eta\right)^{\frac{\theta(b)(|y|)}{|b|}}} + \Gamma_b^{1 + z_0}
\]

since \(\frac{7}{4} > 2\).

For (4), we do the case \(d \geq 4\) here. We estimate it by

\[
\int |\epsilon|^{1 + \frac{4}{b}}(|\tilde{G_a}| + |y \cdot \nabla \tilde{G_a}|)
\]

\[
\leq \Gamma_b^{-\frac{1}{2}}(1 - C \eta) \int_{|y| \leq 2A} |\epsilon|^{1 + \frac{4}{b}}
\]

\[
\leq C \Gamma_b^{\frac{1}{4}}(1 - C \eta) \int_{|y| \leq 2A} |\tilde{\epsilon}|^{1 + \frac{4}{b}} + \Gamma_b^{1 + z_0}
\]

\[
\leq C \Gamma_b^{\frac{1}{4}}(1 - C \eta) \int_{|y| \leq 2A} \left( |I_N \epsilon|^{1 + \frac{4}{b}} + |(Id - I_N \epsilon)\epsilon|^{1 + \frac{4}{b}} \right) + \Gamma_b^{1 + z_0}
\]

\[
\leq C \Gamma_b^{\frac{1}{4}}(1 - C \eta) A^C \left( \int_{|y| \leq 2A} |I_N \epsilon|^{\frac{2d+2}{2d}} \right)^{\frac{(d+2)(d-2)}{2d^2}} + \Gamma_b^{\frac{1}{2}}(1 - C \eta) A^{\frac{2d}{b}} \|\epsilon\|_{L^2}^{1 + \frac{4}{b}} + \Gamma_b^{1 + z_0}
\]

\[
\leq C \Gamma_b^{\frac{1}{4}}(1 - C \eta) A^C \|\nabla I_N \epsilon\|_{L^2}^{\frac{2d+4}{2d}} + \Gamma_b^{1 + z_0}
\]

\[
\leq \Gamma_b^C \|\nabla I_N \epsilon\|_{L^2}^2 + \Gamma_b^{1 + z_0},
\]

provided that \(a > 0, \eta > 0\) are small enough (\(A = e^{\frac{2d(a/2)}{b}}\)).
(4) follows from the classical pointwise bound
\[
P(y) \left( |\nabla_y \tilde{\zeta}_b| + |\nabla_y \partial_b \tilde{\zeta}_b| \right) \leq A^C \Gamma_b^{\frac{1}{2} - C^\eta}
\]
and (1).

(5) follows from the pointwise bound
\[
\|F\|_{L^\infty} + \|y \cdot \nabla F\|_{L^\infty} \leq \frac{C \Gamma_b^{\frac{1}{2}}}{A^{\frac{1}{2}}}
\]
and Cauchy-Schwartz inequality.

**Step 3:** Estimate of terms involving geometric parameters:
Denote by \( \tilde{\zeta}_1 = \zeta_1 - \text{Re} \tilde{\zeta}_b, \tilde{\zeta}_2 = \zeta_2 - \text{Re} \tilde{\zeta}_b \), the terms to be estimated are:

\[
\begin{align*}
\text{Term}_1 &= \Lambda_s \left[ (\tilde{\zeta}_2, \partial_b A(\Sigma + \text{Re} \tilde{\zeta}_b)) - (\tilde{\zeta}_1, \partial_b A(\Theta + \text{Im} \tilde{\zeta}_b)) \right], \\
\text{Term}_2 &= \left( \frac{\Lambda_s}{\Lambda} + b \right) \left[ (\tilde{\zeta}_2, \Lambda^2 (\Sigma + \text{Re} \tilde{\zeta}_b)) - (\tilde{\zeta}_1, \Lambda^2 (\Theta + \text{Im} \tilde{\zeta}_b)) \right], \\
\text{Term}_3 &= \frac{\Lambda_s}{\Lambda} \cdot \left[ (\tilde{\zeta}_2, \nabla A(\Sigma + \text{Re} \tilde{\zeta}_b)) - (\tilde{\zeta}_1, \nabla A(\Theta + \text{Im} \tilde{\zeta}_b)) \right], \\
\text{Term}_4 &= \frac{\gamma_s}{\Lambda} \left[ (\tilde{\zeta}_1, A(\Sigma + \text{Re} \tilde{\zeta}_b)) + (\tilde{\zeta}_2, A(\Theta + \text{Im} \tilde{\zeta}_b)) \right], \\
\text{Term}_5 &= \frac{A_s}{A} \left[ (\tilde{\zeta}_2, A(y \cdot \nabla \chi(\frac{y}{A}) \text{Re} \tilde{\zeta}_b)) - (\tilde{\zeta}_1, A(y \cdot \nabla \chi(\frac{y}{A}) \text{Im} \tilde{\zeta}_b)) \right]
\end{align*}
\]

Claim 3:

(1)
\[
|\text{Term}_1| + |\text{Term}_2| + |\text{Term}_3| \leq \delta(\alpha^*) \left( \Xi(s) + \int |I_N \lambda \tilde{e}|^2 + \int |\tilde{e}|^2 e^{-|y|} \right) + \Gamma_b^{1 + \varepsilon_0}.
\]

(2)
\[
|\text{Term}_4 - \frac{(\tilde{\zeta}_1, L + A^2 Q)(\tilde{\zeta}_1, A Q)}{\|A Q\|_{L^2}}| \leq \delta(\alpha^*) \left( \Xi(s) + \int |I_N \lambda \tilde{e}|^2 + \int |\tilde{e}|^2 e^{-|y|} \right) + \Gamma_b^{1 + \varepsilon_0}.
\]

(3)
\[
|\text{Term}_5| \leq \delta(\alpha^*) \left( \int |I_N \lambda \tilde{e}|^2 + \int |\tilde{e}|^2 e^{-|y|} \right) + \Gamma_b^{1 + \varepsilon_0}.
\]

**Proof:**
For (1), the three terms on the left hand side can be estimated in a similar way by using Lemma 3.6 and Claim 1.

\[
|\text{Term}_1| \leq |b_s| \left( \int |\tilde{e}|^2 e^{-\kappa |y|} \right)^\frac{1}{2}
\]
\[
\leq C \left( \Xi(s) + \int |\nabla I_N \lambda \tilde{e}|^2 + \int |\tilde{e}|^2 e^{-|y|} + \Gamma_b^{1 - C^\eta} \right) \left( \int |\tilde{e}|^2 e^{-|y|} \right)^\frac{1}{2}
\]
\[
\leq C \left( \Xi(s) + \int |\nabla I_N \lambda \tilde{e}|^2 + \int |\tilde{e}|^2 e^{-|y|} + \Gamma_b^{1 - C^\eta} \right) \left( \int |\tilde{e}|^2 e^{-|y|} \right)^\frac{1}{2}
\]
\[
\leq \delta(\alpha^*) \left( \Xi(s) + \int |\nabla I_N \lambda \tilde{e}|^2 + \int |\tilde{e}|^2 e^{-|y|} \right) + \Gamma_b^{1 + \varepsilon_0}
\]
where in the last step, we write the term
\[ \Gamma_b^{1-C_\eta} \left( \int |\bar{\xi}|^2 e^{-|y|} \right)^{\frac{1}{2}} = \Gamma_b^{1-C_\eta - \eta'} \left( \Gamma_b^{\eta'} \left( \int |\bar{\xi}|^2 e^{-|y|} \right)^{\frac{1}{2}} \right) \]
and use the elementary inequality \( XY \leq \frac{X^2 + Y^2}{2} \).

For (2), the left hand side can be estimated as
\[
\left| \gamma_s - \left( \frac{\mathcal{e}_1, L + \mathcal{L}^2 Q}{\|\mathcal{L}Q\|_{L^2}} \right) \right| \left( |\mathcal{e}_1, \Lambda(\Sigma + \Re \bar{\zeta})| + (\bar{\mathcal{e}}, \Lambda(\Theta + \Im \bar{\zeta})) \right) + \left| \left( \frac{\mathcal{e}_1, L + \mathcal{L}^2 Q}{\|\mathcal{L}Q\|_{L^2}^2} \right) \right| \left( |\mathcal{e}_1, \Lambda(\Sigma - Q + \Re \bar{\zeta})| + (\bar{\mathcal{e}}, \Lambda(\Theta + \Im \bar{\zeta})) \right).
\]

From Lemma 3.6,
\[
\left| \gamma_s - \left( \frac{\mathcal{e}_1, L + \mathcal{L}^2 Q}{\|\mathcal{L}Q\|_{L^2}^2} \right) \right| \leq \frac{\gamma_s - \left( \frac{\mathcal{e}_1, L + \mathcal{L}^2 Q}{\|\mathcal{L}Q\|_{L^2}^2} \right)}{\|\mathcal{L}Q\|_{L^2}^2} + \frac{|(\Re \bar{\zeta}, \mathcal{L}^2 Q)|}{\|\mathcal{L}Q\|_{L^2}^2} \leq \delta(\alpha^*) \left( \int |\nabla I_N \bar{\xi}|^2 + \int |\bar{\xi}|^2 e^{-|y|} \right)^{\frac{1}{2}} + \Gamma_b^{1+z_0} + C\Xi(s)
\]
since
\[
\int |\bar{\zeta}| |\mathcal{L}^2 Q| \leq C \int |\bar{\zeta}| e^{-|y|} \leq \Gamma_b^{1+z_0},
\]
thanks to the property of \( \zeta, \) (see [20]). Now (2) follows from (4) of Claim 2.

For (3), we note that \( \left| \frac{\mathcal{A}}{\|\mathcal{A}\|_{L^2}} \right| \leq C \left| \frac{\mathcal{A}}{\|\mathcal{A}\|_{L^2}} \right| \) and
\[
||\Lambda(y \cdot \nabla \left( \frac{y}{\|\mathcal{A}\|_{L^2}} \right) \zeta)|_{L^\infty} \leq A \Gamma_b^{1-C_\eta},
\]
thus
\[
|\text{Term}_5| \leq C \left| \frac{\mathcal{A}}{\|\mathcal{A}\|_{L^2}} \right| A \Gamma_b^{1-C_\eta} \left( \int |\bar{\xi}|^2 \right)^{\frac{1}{2}} \leq C \left| \frac{\mathcal{A}}{\|\mathcal{A}\|_{L^2}} \right| A \left( \int |\nabla I_N \bar{\xi}|^2 + \int |\bar{\xi}|^2 e^{-|y|} \right)^{\frac{1}{2}} + \Gamma_b^{1+z_0} \leq \delta(\alpha^*) \left( \int |\nabla I_N \bar{\xi}|^2 + \int |\bar{\xi}|^2 e^{-|y|} \right) + \Gamma_b^{1+z_0}
\]
by writing that
\[
||\bar{\xi}| |\mathcal{L}^2 Q|_{L^2(\{|y| \leq 2A\})} \leq ||I_N \bar{\xi}| |\mathcal{L}^2 Q|_{L^2(\{|y| \leq 2A\})} + ||(\Id - I_N \bar{\xi})||_{L^2}
\]
and using (1) of Claim 1.

**Step 4:** Estimate of degenerate scalar products.

**Claim 4:**
(1) \( (\mathcal{e}_1, Q)^2 \leq \delta(\alpha^*) \left( \int |\nabla I_N \bar{\xi}|^2 + \int |\bar{\xi}|^2 e^{-|y|} + \Xi(s) \right) + \Gamma_b^{1+z_0} \)

(2) \( (\bar{\mathcal{e}}, \bar{Q})^2 \leq \delta(\alpha^*) \left( \int |\nabla I_N \bar{\xi}|^2 + \int |\bar{\xi}|^2 e^{-|y|} \right) + \Gamma_b^{1+z_0} \).
\[
\begin{align*}
\sum_{\tilde{\epsilon}} (|y|^2 Q)^2 + (\tilde{\epsilon}, Q)^2 + (\tilde{\epsilon}, \Lambda Q)^2 + (\tilde{\epsilon}, 2Q)^2 \\
\leq \delta(\alpha^*) \left( \int |\nabla I_N \tilde{\epsilon}|^2 + \int |\tilde{\epsilon}|^2 e^{-|y|} \right) + \Gamma_b^{1+z_0}.
\end{align*}
\]

**Proof:**
We first indicate that \(\Xi(s) \leq \delta(\delta^*)\). Essentially,
\[
\Xi(s) = \frac{\lambda^2(s)}{2} \int |J_N(s) G(0, x)|^2 dx
\]
with \(J_N = \text{Id} - I_N\). Recall that
\[G(0, x) = \frac{1}{\lambda(0)^{\frac{1}{2}}} (Q_b(x) + g(0)) \left( \frac{x - x(0)}{\lambda(0)} \right) e^{-i\gamma(0)},\]
we estimate
\[
\frac{\lambda^2(s)}{2} \left\| J_N(s) G(0) \right\|^2_{L^2} \leq \frac{\lambda(s)}{\lambda(0)^2} \left\| \nabla g(0) \right\|^2_{L^2} + \left\| J_N(s) \lambda(0) (\nabla Q_b(0)) \right\|^2_{L^2}
\]
\[
\leq C\Gamma_b^{\frac{3}{2}z_0} + \frac{1}{\lambda(s)} \frac{\lambda(0)}{\lambda(s)}
\]
\[
\leq \delta(\alpha^*)
\]
thanks to the bootstrap assumption.

Now (1) follows from the following estimate induced by almost conservation of energy:
\[
(\tilde{\epsilon}, Q)^2 \leq \delta(\alpha^*) \left( \int |\nabla I_N \tilde{\epsilon}|^2 + \int |\tilde{\epsilon}|^2 e^{-|y|} + \Xi(s) \right) + \Gamma_b^{1+z_0} + (\lambda^2 E(I_N u))^2 (3.27)
\]
Indeed, we have already seen in Step 3 that
\[
\int |\tilde{\epsilon}|^2 e^{-\kappa|y|} \leq \Gamma_b^{\frac{1}{2}+z_0}.
\]
Thus we could replace the left hand side by \((\epsilon, Q)^2\). From the almost conservation of energy Lemma 3.2, we have
\[
2(\epsilon, Q) = -2(\epsilon, \Sigma - Q + b\Lambda \Theta) - 2(\epsilon, \Theta - b\Lambda \Sigma) + 2(\epsilon, \text{Re} \Psi) + 2(\epsilon, \text{Im} \Psi)
\]
\[
+ 2\Xi(s) + O \left( \int |\nabla I_N \epsilon|^2 + \int |\epsilon|^2 e^{-|y|} + \Gamma_b^{1-C\eta} \right)
\]
Then we conclude by the estimates
\[
|\text{Re}(\epsilon, Q_b - Q + ib\Lambda Q_b)| \leq \delta(\alpha^*) \left( \int |\nabla I_N \epsilon|^2 + \int |\epsilon|^2 e^{-|y|} \right)^{\frac{1}{2}} + \Gamma_b^{\frac{1}{2}+z_0},
\]
\[
\left| \int |\epsilon||\Psi| \right|^2 \leq \Gamma_b^{1-C\eta} \left( \int_{R_b \leq |y| \leq R_b} |\epsilon| \right)^2
\]
\[
\leq \Gamma_b^{1-C\eta} \left( \int |\nabla I_N \epsilon|^2 + \int |\epsilon|^2 e^{-|y|} \right)^{\frac{1}{2}} + \Gamma_b^{\frac{1}{2}+z_0}
\]
and the Claim 1.
(2) follows from the almost conservation of momentum

\[ -2(\varepsilon_2, \nabla Q) = \lambda P(I_{N\lambda}u) - \left( P(I_{N\lambda}(Q_b + \varepsilon)) - P(Q_b + I_{N\lambda}\varepsilon) \right) - 2(\varepsilon_2, \nabla(Q - \Sigma)) - 2(\varepsilon_2, (I_d - I_{N\lambda})\nabla\Sigma) - 2\int I_{N\lambda}\varepsilon_1\nabla\Theta - 2\int I_{N\lambda}\varepsilon_1\nabla I_{N\lambda}\varepsilon_2. \]

Again, we could replace the left hand side by \((\varepsilon_2, \nabla Q)^2\). The desired estimate follows from the same manipulation as for (1). Finally, (3) follows from orthogonality condition imposed by \(\varepsilon\) and the small deformation of \(Q\) to \(Q_b\).

**Claim 5:**

\[ |(\varepsilon_1, \tilde{L})| + |(\varepsilon_2, \tilde{K})| + |(\varepsilon_1, \Lambda(Q^\dagger \Re \tilde{\zeta}_b))| + |(\varepsilon_2, \Lambda(Q^\dagger \Im \tilde{\zeta}_b))| \leq \delta(\alpha^*) \left( \int |\nabla I_{N\lambda}\tilde{\varepsilon}|^2 + \int |\tilde{\varepsilon}|^2e^{-|\tilde{y}|} \right) + \Gamma_b^{1+\varepsilon_0}. \]

**Proof:**

Using the expression of \(\tilde{L}\), the property of \(Q_b, \tilde{\zeta}_b\), we estimate

\[ |(\varepsilon_1, \tilde{L})| = \left| \int \varepsilon_1 \left[ \left( \frac{4\Sigma^2}{d|Q_b|^2} + 1 \right)|Q_b|^2 - \left( 1 + \frac{4}{d} \right) Q^\dagger \Re \tilde{\zeta}_b + \frac{4\Sigma\Theta}{d|Q_b|^2}|Q_b|^2 \Lambda \Im \tilde{\zeta}_b \right] \right| \]

\[ \leq C \left( \int |\varepsilon|^2e^{-\frac{2}{2}(1-C_\theta)\frac{A_{1/2}}{A_{1/6}}} \right)^{1/2} \Gamma_b^{1+\varepsilon_0 - C_\eta} \]

\[ \leq \delta(\alpha^*) \left( \int |\nabla I_{N\lambda}\tilde{\varepsilon}|^2 + \int |\tilde{\varepsilon}|^2e^{-|\tilde{y}|} \right) + \Gamma_b^{1+\varepsilon_0} \]

Similarly for \(|(\varepsilon_2, \tilde{K})|\). For the last two terms, we estimate, for example

\[ |(\varepsilon_1, \Lambda(Q^\dagger \Re \tilde{\zeta}_b))| \leq \int |\tilde{\varepsilon}|(|\tilde{\zeta}_b| + |\nabla \tilde{\zeta}_b|)e^{-\kappa|\tilde{y}|} \]

\[ \leq \left( \int |\tilde{\varepsilon}|^2e^{-\kappa|\tilde{y}|} \right)^{1/2} \Gamma_b^{1+\varepsilon_0} \]

\[ = \left( \Gamma_b^{1+\varepsilon_0} \right)^{1/2} \Gamma_b^{1+\varepsilon_0} \]

\[ \leq \delta(\alpha^*) \left( \int |\nabla I_{N\lambda}\tilde{\varepsilon}|^2 + \int |\tilde{\varepsilon}|^2e^{-|\tilde{y}|} \right) + \Gamma_b^{1+\varepsilon_0}. \]

**Step 5:** Estimate of principal terms.

**Claim 6:**

\[ H(I_{N\lambda}\epsilon - \tilde{\zeta}_b, I_{N\lambda}\epsilon - \tilde{\zeta}_b) - \left( \varepsilon_1, L + \Lambda^2(Q)\varepsilon_1, \Lambda Q \right) - \left[ (\Re \tilde{\zeta}_b, \Lambda \Re F) + (\Im \tilde{\zeta}_b, \Lambda \Im F) \right] \geq c_1 \left( \int |\nabla I_{N\lambda}\tilde{\varepsilon}|^2 + \int |\tilde{\varepsilon}|^2e^{-|\tilde{y}|} + \Gamma_b \right) - \delta(\alpha^*)\Xi(s) \]

\[ (3.28) \]

**Proof:**

Indeed, we may replace \(I_{N\lambda} - \tilde{\zeta}_b\) by \(I_{N\lambda}\tilde{\varepsilon}\) by adding an error \(\Gamma_b^{1+\varepsilon_0}\) which does not
change the type of the desired estimate. From the spectral property and Claim 4, we have
\[ H(I_{N\Lambda}, I_{N\Lambda}) - \frac{(\bar{c}_1, L_+\Lambda^2 Q)(\bar{c}_1, \Lambda Q)}{\|\Lambda Q\|^2_{L^2}} \geq c_0 \left( \int |I_{N\Lambda}|^2 + \int |I_{N\Lambda}|^2 e^{-|y|} \right) \]
- \frac{1}{c_0} \left[ (\bar{c}_1, Q)^2 + (\bar{c}_1, yQ)^2 + (\bar{c}_2, Q)^2 + (\bar{c}_2, \Lambda Q)^2 + (\bar{c}_2, \nabla Q)^2 \right]
\[ \geq c_1 \left( \int |I_{N\Lambda}|^2 + \int |\epsilon|^2 e^{-|y|} \right) - \delta(\alpha^*)\Xi(s) - \Gamma_{b}^{1+\varepsilon_0} \]

Finally, we conclude by the estimate proved in [20]:
\[ - \left[ (\text{Re} \bar{c}_b, \Lambda \text{Re} F) + (\text{Im} \bar{c}_b, \Lambda \text{Im} F) \right] > c\Gamma_{b}. \]

**Step 6:** Estimate of original reminder terms and conclusion.

The classical remainder term **Remainder** has been already estimated in the proof of Lemma 3.4, while the term \( \Upsilon_{N\Lambda} \) can be also bounded easily by \( \Gamma_{b}^{1+\varepsilon_0} \). Combining Step 1 to Step 5, we obtain the desired estimate.

\[ \square \]

Next, we need to control the \( L^2 \) type of term \( \int_A^{2A} |\epsilon|^2 \) in [3.24]. This is achieved by computing the flux of \( L^2 \) norm escaping the radiative zone. To do it, we introduce a radial nonnegative cutoff function \( \psi(r) \) such that
\[
\psi(r) = \begin{cases} 
0 & r \leq \frac{1}{2}, \\
1 & r \geq 3, \\
\frac{1}{4} \leq \psi'(r) \leq \frac{1}{2} & \text{for } 1 \leq r \leq 2, \quad \psi'(r) \geq 0.
\end{cases}
\]

Let
\[
\psi_A(s, r) = \psi\left(\frac{r}{A(s)}\right),
\]
with \( A(s) \) being given by [3.22], and so
\[
\begin{align*}
\psi_A(r) &= 0 \text{ for } 0 \leq r \leq \frac{1}{2}, \\
\frac{1}{2A} &\leq \psi'_A(r) \leq \frac{1}{2A} \text{ for } A \leq r \leq 2A, \\
\psi_A(r) &= 1 \text{ for } r \geq 3r, \\
\psi'_A(r) &\geq 0, \quad 0 \leq \psi_A(r) \leq 1.
\end{align*}
\]

Moreover, we restrict the freedom on the choice of the parameters \((\eta, a)\) by assuming \( a > C\eta \).

**Lemma 3.9** \((L^2 \text{ dispersion at infinity in space})\). *There holds for some universal constants \( C, c_3 > 0 \) and \( s \) large enough:*
\[
\left\{ \int_{\Omega} \psi_A |\epsilon|^2 \right\}_s \geq c_3 b \int_A^{2A} |\epsilon|^2 - \frac{C b^2}{\lambda^2} \Xi(s) - \Gamma_{b}^{1+C\eta} - \Gamma_{b}^{\frac{\eta}{\lambda}} \int |\nabla I_{N\lambda}(\lambda s)\epsilon|^2. \tag{3.29}
\]

**Proof.** Take the inner product of [3.3] with \( \psi_A \varepsilon_1 \) and of [3.4] with \( \psi_A \varepsilon_2 \) and integrate by parts. Note that the supports of \((Q_b, \Psi_b)\) and \( \psi_A \) are disjoint. Then, we obtain
\[
\begin{align*}
\frac{1}{2} \left\{ \int \psi_A |\epsilon|^2 \right\}_s = &\frac{1}{2} \int \frac{\partial \psi_A}{\partial s} |\epsilon|^2 + \frac{b}{2} \int y \cdot \psi_A |\epsilon|^2 + \text{Im} \left( \int \nabla \psi_A \cdot \nabla \varepsilon \right) \\
&- \frac{1}{2} \left( \frac{\lambda s}{\lambda^2} + b \right) \int y \cdot \nabla \psi_A |\epsilon|^2 - \frac{1}{2} \frac{x s}{\lambda} \int \nabla \psi_A |\epsilon|^2. \tag{3.30}
\end{align*}
\]
First observe from the choice of $\psi$:

\[ 10 \int \psi'\left(\frac{y}{A}\right)|\varepsilon|^2 \geq \frac{1}{A} \int y \cdot \nabla \psi\left(\frac{y}{A}\right)|\varepsilon|^2 \geq \frac{1}{10} \int \psi'\left(\frac{y}{A}\right)|\varepsilon|^2 \geq \frac{1}{40} \int_A^{2A} |\varepsilon|^2. \]  

(3.31)

The main term is

\[ \frac{b}{2} \int y \cdot \nabla_A |\varepsilon|^2 \geq \frac{1}{20} \int \psi'\left(\frac{y}{A}\right)|\varepsilon|^2. \]  

(3.32)

Using (3.20) and (3.19), we get

\[ \int \frac{\partial \psi_A}{\partial s} |\varepsilon|^2 = -\frac{A_s}{A^2} \int y \cdot \nabla \psi\left(\frac{y}{A}\right)|\varepsilon|^2 = 2a \frac{b}{4b} \int y \cdot \nabla \psi\left(\frac{y}{A}\right)|\varepsilon|^2 \]  

(3.33)

\[ \geq \frac{a}{4b^2} \left[ c_0 \left( \Xi(s) + \int |\nabla I_{N,\lambda}\varepsilon(s)|^2 + \int |\varepsilon(s)|^2 e^{-|y|} - \Gamma_{b(s)}^{1-C\eta} \right) \right] \int \psi'\left(\frac{y}{A}\right)|\varepsilon|^2. \]

Next, by Cauchy-Schwartz inequality, we have

\[
\left| \int \nabla \psi_A \cdot \nabla \varepsilon \right| \\
\leq \left| \int \nabla \psi_A \cdot \nabla (I_{N,\lambda}\varepsilon) \varepsilon \right| + \left| \int \nabla \psi_A \cdot \nabla ((I(I_{N,\lambda})\varepsilon) \varepsilon \right| \\
\leq \frac{b}{100} \int \psi'\left(\frac{y}{A}\right)|\varepsilon|^2 + \Gamma_b^{\frac{4}{5}} \int |\nabla I_{N,\lambda}\varepsilon|^2 + \left| \int \nabla \psi_A \cdot \nabla ((I(I_{N,\lambda})\varepsilon) \varepsilon \right|.
\]

On the other hand, by duality, Bernstein’s inequality, and interpolation, we estimate

\[
\left| \int \nabla \psi_A \cdot \nabla ((I(I_{N,\lambda})\varepsilon) \varepsilon \right| \leq \|\varepsilon(s)\|_{H^{s+1}} \|\nabla \psi_A \varepsilon\|_{H^{\frac{s+1}{s+2}}} \\
\lesssim \frac{(N\lambda)^{\frac{1}{2}-s}}{A} \|\varepsilon\|_{H^s} \|\varepsilon\|_{H^{\frac{s+1}{s+2}}} \\
\lesssim \Gamma_b^{\frac{4}{5}} \lambda(s) \|\varepsilon\|_{H^s} \|\varepsilon\|_{H^{\frac{s+1}{s+2}}} \|I_{N,\lambda}\varepsilon\|_{H^{\frac{s+1}{s+2}}} \\
\leq \Gamma_b^{10}.
\]

Using (3.20), we derive

\[
\left| \frac{x}{A} \int \nabla \psi_A |\varepsilon|^2 \right| \leq \frac{C}{A} \int \psi'\left(\frac{y}{A}\right)|\varepsilon|^2 \leq \frac{\Gamma_b^{\frac{4}{5}}}{A} \int \psi'\left(\frac{y}{A}\right)|\varepsilon|^2.
\]

(3.34)

By the same way, we have by (3.19)

\[
\left| \left( \frac{\lambda s}{A} + b \right) \int y \cdot \nabla \psi_A |\varepsilon|^2 \right| \\
\leq C \left( \Xi(s) + \int |\nabla I_{N,\lambda(s)}\varepsilon(s)|^2 + \int |\varepsilon(s)|^2 e^{-|y|} + \Gamma_{b(s)}^{1-C\eta} \right) \int \psi'\left(\frac{y}{A}\right)|\varepsilon|^2.
\]

(3.35)

\[\square\]

**Corollary 3.10 (Lyapunov functional).** For some universal constant $C > 0$ anf for $s$ large, the following holds:

\[
\{\mathcal{J}\}_s \leq -Cb \left( \Gamma_b + \Xi(s) + \int |\nabla I_{N,\lambda(s)}\varepsilon|^2 + \int |\varepsilon(s)|^2 e^{-|y|} \right) + \frac{C \Xi(s)}{b^2},
\]

(3.36)
with
\[
\mathcal{J}(s) = \left( \int |Q_b|^2 - \int Q^2 \right) + 2(\varepsilon_1, \Sigma) + 2(\varepsilon_2, \Theta) + \int (1 - \psi_A) |\varepsilon|^2 \\
- c \left( b \tilde{f}_1(b) - \int_0^b \tilde{f}_1(v) \, dv + b[\text{Re}(\varepsilon_2, \Lambda \tilde{c}) - \text{Im}(\varepsilon_1, \Lambda \tilde{c})] \right),
\]
(3.37)
where \( c > 0 \) denotes some small enough universal constant and
\[
\tilde{f}_1(b) = \frac{b}{4} \|y Q_b\|^2_{L^2} + \frac{1}{2} \text{Im} \left( \int (y \cdot \nabla \tilde{\zeta}) \tilde{\zeta} \right).
\]
(3.38)

Finally, by the same argument as Subsection 4.3 in [5], we conclude the proof of the bootstrap Lemma 1.13.

3.3. End proof of Proposition 1.9. Now, we are in position to proving Proposition 1.9. The proof is the same as in [3]. The main difference is the step 4: Strong \( L^2 \) convergence of excess mass outside the blowup point and Step 5: nonconcentration of the \( L^2 \) norm at the blowup point. We are reduced to show the following lemma.

**Lemma 3.11.** Let \( R > 0 \). Let \( x(T) = \lim_{t \to T} x(t) \). Then, there exists \( u^* \in L^2 \) such that
\[
u(t) \to u^* \text{ in } L^2(\mathbb{R}^d \setminus \{|x - x(T)| \leq R\}) \text{ as } t \to T.
\]
(3.39)
Furthermore, there holds
\[
\int |u|^2 = \lim_{i \to T} \int \chi_R(t)|I_N(t) u(t)|^2, \quad R(t) = 10A(t) \lambda(t),
\]
(3.40)
with
\[
\chi_R(x) = \chi\left( \frac{x - x(t)}{R} \right), \quad \chi(x) = \begin{cases} 0 & |x| \leq 1 \\ 1 & |x| \geq 2. \end{cases}
\]
(3.41)

**Proof.** First, we prove the claim. For \( R > 0 \), let \( w_R(t, x) = \chi_R(x) [I_N(t) u(t, x)] \) then, \( w_R \) solves
\[
i \partial_t w_R + \Delta w_R = i \chi_R \tilde{I}_N u + 2 \nabla \chi_R \cdot \nabla I_N u + \Delta \chi_R I_N u - \chi R I_N \left( |u|^2 u \right).
\]
(3.42)
where
\[
\tilde{I}_N u(\xi) = -\frac{N}{N} \hat{m}_N(\xi) \hat{I}_N u(\xi), \quad \hat{m}_N(\xi) = \frac{\xi}{N} \cdot \nabla \hat{m}_N(\xi) \cdot \frac{m_N(\xi)}{\hat{m}_N(\xi)}.
\]
(3.43)
It follows from [5] that
\[
\left\| \int_0^t e^{i(t-\tau)\Delta} (i \chi_R \tilde{I}_N u + 2 \nabla \chi_R \cdot \nabla I_N u + \Delta \chi_R I_N u) d\tau \right\|_{L^2_x([0, T] \times \mathbb{R}^d)} < +\infty.
\]
(3.44)

We only need to estimate the nonlinear term
\[
\left\| \int_0^t e^{i(t-\tau)\Delta} \chi_R I_N \left( |u|^2 u \right) d\tau \right\|_{L^2_x([0, T] \times \mathbb{R}^d)} < +\infty.
\]
(3.44)
And
\[
\left\| \int_0^t e^{i(t-\tau)\Delta} \chi_R I_N \left( |u|^2 u \right) d\tau \right\|_{L^\infty_x L^2([0, T] \times \mathbb{R}^d)} \leq C \left\| \chi_R I_N \left( |u|^2 u \right) \right\|_{L^\infty_x L^2([0, T] \times \mathbb{R}^d)}
\]
\[
\lesssim \left\| \chi_R \left( I_N u \right|^2 |I_N u \right\|_{L^\infty_x \tilde{L}^{\frac{2d}{d-2}}([0, T] \times \mathbb{R}^d)} + \left\| |I_N u|^2 I_N u - I_N (|u|^2 u) \right\|_{L^\infty_x \tilde{L}^{\frac{2d}{d-2}}([0, T] \times \mathbb{R}^d)}.
\]
On one hand,

\[
\| \chi_R([I_N u]^\frac{1}{2} I_N u)]^2 \|_{L^2_t L^\infty_x ([0, T] \times \mathbb{R}^d)} = \int_0^T \| \chi_R \frac{d}{dt} I_N u \|_{L^\infty_x}^{2(\frac{d+1}{d+2})} dt \\
\leq \int_0^T \| \chi_R I_N u \|_{L^\infty_x}^{2(\frac{d+1}{d+2})} dt \\
\leq \int_0^T \| \nabla (\chi_R I_N u) \|_{L^2}^2 < +\infty,
\]

where the last inequality follows from (4.49) in [5]. On the other hand, we have by Lemma 2.13 and (2.32)

\[
\| [I_N u]^\frac{1}{2} I_N u - I_N([u]^\frac{1}{2} u)]^2 \|_{L^2_t L^\infty_x ([0, T] \times \mathbb{R}^d)} \\
\leq \sum_{k=k_0}^{+\infty} \sum_{j=1}^{l_k} \| [I_N u]^\frac{1}{2} I_N u - I_N([u]^\frac{1}{2} u)]^2 \|_{L^2_t L^\infty_x ([\tau_k^j, \tau_k^j+1] \times \mathbb{R}^d)} \\
\leq \sum_{k=k_0}^{+\infty} \sum_{j=1}^{l_k} \lambda(t_k) \frac{2}{\min\{1, \frac{d}{d-1}\} s_{\frac{1}{2}-\frac{1}{d}}} \\
\leq \sum_{k=k_0}^{+\infty} k \lambda(t_k) \frac{2}{\min\{1, \frac{d}{d-1}\} s_{\frac{1}{2}-\frac{1}{d}}} \\
\leq \sum_{k=k_0}^{+\infty} k x \frac{2}{\min\{1, \frac{d}{d-1}\} s_{\frac{1}{2}-\frac{1}{d}}} < +\infty
\]

provided that \( \frac{1}{\beta} > \frac{d}{\min\{4, d\} s_{\frac{1}{2}}-\frac{1}{2}} \). Then, by the standard argument as in [5], we obtain (3.39).

Next, we turn to show (3.40). Observer that \( I_N u \) solves

\[
i \partial_t(I_N u) + \Delta(I_N u) = i \tilde{I}_N u - I_N([u]^\frac{1}{2} u).
\]

We then compute the flux of \( L^2 \)-norm:

\[
\frac{1}{2} \left\| \frac{d}{dt} \int \chi \left( \frac{x-x(\tau)}{R(t)} \right) |I_N u(\tau)|^2 \right\| dx \\
= \frac{1}{R(t)} \text{Im} \left( \int \nabla \chi \left( \frac{x-x(\tau)}{R(t)} \right) \cdot \nabla I_N u |I_N u|^2 \right) - \frac{x_t}{R(t)} \cdot \int \nabla \chi \left( \frac{x-x(\tau)}{R(t)} \right) |I_N u|^2 \\
+ \text{Re} \left( \int \chi \left( \frac{x-x(\tau)}{R(t)} \right) \tilde{I}_N u I_N u \right) + \text{Im} \left( \chi \left( \frac{x-x(\tau)}{R(t)} \right) (I_N u |I_N u|^2 - I_N(u|u|^\frac{1}{2}) |I_N u|^2) \right)
\]

and integrate from \( t \to T \). We obtain

\[
\left| \int \chi \left( \frac{x-x(T)}{R(t)} \right) |u|^2 - \int \chi \left( \frac{x-x(t)}{R(t)} \right) |I_N u|^2 \right| \\
\leq \frac{1}{A(t) \lambda(t)} \int_0^T \| \nabla I_N(u(\tau)) \|_{L^2} d\tau + \frac{1}{A(t) \lambda(t)} \int_0^T \left| \frac{x_t}{\lambda(\tau)} \right| d\tau + \frac{1}{A(t) \lambda(t)} \int_0^T \int \chi \left( \frac{x-x(\tau)}{R(t)} \right) \tilde{I}_N u |I_N u|^2 \\
+ \int_0^T \int (I_N u |I_N u|^\frac{1}{2} - I_N(u|u|^\frac{1}{2})) |I_N u|^2.
\]
By the same argument as in [5], we obtain
\[ \frac{1}{A(t)\lambda(t)} \int_0^T \|\nabla I_N u(t)\|_{L^2} \, dt + \frac{1}{A(t)\lambda(t)} \int_0^T \frac{t}{\lambda(t)} \, dt + \int_0^T \left\| \chi \left( \frac{x-x(\tau)}{R(\tau)} \right) I_N u \right\|_{L^2} \, dt \to 0 \quad \text{as} \quad t \to T. \]

Hence, we only need to show
\[ \int_0^T \left\| \left( I_N u \right| I_N u \right\| \frac{1}{\lambda(t)} \, dt \to 0 \quad \text{as} \quad t \to T. \]

Indeed, by (2.29), we obtain
\[ \int_0^T \left\| \left( I_N u \right| I_N u \right\| \frac{1}{\lambda(t)} \, dt \]
\[ \lesssim \sum_{k=k_t}^{+\infty} \sum_{j=1}^{L_k} \left\| I_N u \right\|_{L^2_t L^\infty_x \left( \tau_j, \tau_{j+1} \times \mathbb{R}^d \right)} \left\| I_N u \right\|_{L^2_t L^\infty_x \left( \tau_j, \tau_{j+1} \times \mathbb{R}^d \right)}
\[ \lesssim \sum_{k=k_t}^{+\infty} k \lambda(t_k) \left( \min(1, \frac{1}{2}) - \frac{1}{t_k} \right) - \min(1, \frac{1}{2})
\[ \lesssim \sum_{k=k_t}^{+\infty} k^2 \left( \frac{1}{2} \min(1, \frac{1}{2}) - \frac{1}{t_k} \right) - \min(1, \frac{1}{2}) \to 0 \quad \text{as} \quad k_t \to +\infty. \]

\[ \square \]

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