The Standard Model in the Alpha Gauge is Not Renormalizable

Hung Cheng\textsuperscript{1}
Department of Mathematics, Massachusetts Institute of Technology
Cambridge, MA 02139, U.S.A.

and

S.P. Li\textsuperscript{2}
Institute of Physics, Academia Sinica
Nankang, Taipei, Taiwan, Republic of China

PACS: 11.10.Gh; 11.15.-q; 12.20.-m
Keywords: Standard Model; Ward-Takahashi identities; Renormalizability
1. E-mail: huncheng@math.mit.edu
2. E-mail: spli@phys.sinica.edu.tw
Abstract

We study the Ward-Takahashi identities in the standard model with the gauge fixing terms given by (1.1) and (1.2). We find that the isolated singularities of the propagators for the unphysical particles are poles of even order, not the simple poles people have assumed them to be. Furthermore, the position of these poles are ultraviolet divergent. Thus the standard model in the alpha gauge in general, and the Feynman gauge in particular, is not renormalizable. We study also the case with the gauge fixing terms (1.3), and find that the propagators remain non-renormalizable. The only gauge without these difficulties is the Landau gauge. As emphasized by Bonneau[1], one must make a distinction between the renormalizability of the Green functions and that of the physical scattering amplitudes.
1. Introduction

In this paper, we shall explore some of the exact and non-perturbative consequences of the Ward-Takahashi identities in the standard model.

The organization of this paper will be as follows: in Sec. 2, we study the $2 \times 2$ mixing matrix of propagators for the longitudinal $W$ and the charged Higgs meson with the gauge fixing term of $W$ chosen to be

$$-\frac{1}{\alpha_W}(\partial^\mu W^\mu_\nu + i\alpha_W M_0 \phi^+)(\partial^\nu W^-_\nu - i\alpha_W M_0 \phi^-), \quad (1.1)$$

where $\phi^\pm$ are the charged Higgs fields, $M_0$ is the bare mass\[4\] of $W$, and $\alpha_W$ is the gauge parameter. This study has been carried out by a number of authors\[2,3\]. Indeed, we could have started where they left off, but we opt for making the presentation more self-contained.

In Sec. 3, we study the $3 \times 3$ mixing matrix of propagators for the longitudinal $A$, the longitudinal $Z$ and $\phi^0$, where $\phi^0$ is the unphysical neutral Higgs meson, (the imaginary part of the neutral Higgs meson) with the gauge fixing term chosen to be

$$-\frac{1}{2\alpha_Z}(\partial_\mu Z^\mu + \alpha_Z M_0 \phi^0)^2 - \frac{1}{2\alpha_A}(\partial_\mu A^\mu)^2, \quad (1.2)$$

where $M_0'$ is the bare mass of $Z$, and $\alpha_Z$ and $\alpha_A$ are gauge parameters.

In Sec. 4, we study these propagators when the gauge fixing terms are, instead of (1.1) and (1.2),

$$-\frac{1}{2\alpha_Z}(\partial_\mu Z^\mu)^2 - \frac{1}{\alpha_W}(\partial^\mu W^\mu_\nu)(\partial^\nu W^-_\nu) - \frac{1}{2\alpha_A}(\partial_\mu A^\mu)^2. \quad (1.3)$$

In Sec. 5, we discuss the meaning of our results.

In Appendix A, we list the BRST variations of the fields in the standard model, in the gauge of (1.1) and (1.2). In Appendix B, we present an alternative way to derive the Ward-Takahashi identities\[4\], and list the three Ward-Takahashi identities for propagators involving the longitudinal $W$, in the gauge of (1.1). In Appendix C, we list the nine Ward-Takahashi identities for propagators involving the longitudinal $A$ and the longitudinal $Z$, in the gauge of (1.2). In Appendix D, we present the derivations of the three relations satisfied

\[1\]The bare mass $M_0$ is equal to $\frac{1}{2}g_0 v_0$, where $v_0$ may be defined either as the classical vacuum value or the quantum vacuum value of the Higgs field. The Green functions differ with different definitions of $v_0$, but by gauge invariance, physical quantities remain the same with either definition.
by the 1PI amplitudes of the propagators in the $3 \times 3$ mixing matrix, in the gauge of (1.2). In Appendix E, we list the twelve Ward-Takahashi identities in the gauge of (1.3), and derive the relations satisfied by the 1PI self-energy amplitudes.

2. The Mixing of $W$ and the Charged Higgs Meson

In the standard model, a longitudinal $W$ meson may propagate into either a longitudinal $W$ or a Higgs meson with the same charge. Thus all of the following propagators

\[ <0|T W_\mu^+(x) W_\nu^-(0)|0>, \quad <0|T W_\mu^+(x) \phi^-(0)|0>, \]
\[ <0|T \phi^+(x) W_\mu^-(0)|0>, \quad <0|T \phi^+(x) \phi^-(0)|0>, \]

are none-zero and together they form a $2 \times 2$ mixing matrix. We shall denote the Fourier transform of such a propagator with the symbol $G$ and put

\[ G_{W^+ W^-}^{W^{\mu}}(k) \equiv -iD_{WW}^T(k^2)T_{\mu\nu} - i\alpha W D_{WW}(k^2)L_{\mu\nu}, \]
\[ G_{W^+ \phi}^{\phi^+(x)}(k) = G_{\phi^+ W^-(x)}^{\phi^-(x)}(k) \equiv \frac{i\alpha W}{M_0}D_{WW}(k^2), \]
\[ G_{\phi^+ \phi^-}^{\phi^+}(k) \equiv iD_{\phi^+ \phi^-}(k^2), \]

where

\[ T_{\mu\nu} \equiv g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}, \quad L_{\mu\nu} \equiv \frac{k_\mu k_\nu}{k^2}. \]

The $D$ functions are so defined that their unperturbed forms in the gauge of (1.1) are simply

\[ D_{WW}^{(0)}(k^2) = D_{\phi^+ \phi^-}^{(0)}(k^2) = \frac{1}{k^2 - \alpha_w M_0^2} \quad (2.1) \]

They are finite and non-zero at $\alpha_W = 0$. The unperturbed form for $D_{W\phi}$ is zero.

The charged Higgs meson mixes not with the transverse $W$ but with the longitudinal $W$, as is indicated by the factor $k_\mu$ in the expression for $G_{W^+ \phi}^{W^-}$. Thus, in the $2 \times 2$ mixing matrix under discussion, this factor $k_\mu$ can be replaced by $K$, where

\[ K \equiv \sqrt{k^2} \]

For the same reason, $k_\mu k_\nu / k^2$ can be replaced by unity. The $2 \times 2$ mixing matrix is therefore equal to

\[
\begin{bmatrix}
-i\alpha W D_{WW} & i\alpha W D_{W\phi} / M_0 \\
i\alpha W D_{W\phi} / M_0 & iD_{\phi^+ \phi^-}
\end{bmatrix}
\]

(2.2)
We shall express the propagators by their 1PI amplitudes.

In a field theory without mixing, let the unperturbed propagator for a particle be $i(k^2 - m^2)^{-1}$, and let the 1PI self-energy amplitude be $\Pi$, then this propagator is the inverse of $-i(k^2 - m^2 - \Pi)$. Now that the unperturbed propagators in the mixing matrix are given by (2.1), the mixing matrix (2.2) is equal to the inverse of

$$
\begin{bmatrix}
i(k^2 - M_0^2) - \alpha W\Pi_{W\phi}/\alpha W & iK\Pi_{W\phi}/M_0 \\
iK\Pi_{W\phi}/M_0 & -i(k^2 - \alpha W M_0^2 - \Pi_{\phi+\phi^-})
\end{bmatrix},
$$

(2.3)

where $\Pi_{W\phi}$, for example, is the 1PI amplitudes for $D_{W\phi}$. The 1PI amplitudes are functions of $k^2$ and $\alpha$, the dependence of which will be exhibited only when needed. Inverting the matrix in (2.3), we find that the mixing matrix in (2.2) is equal to

$$
\begin{bmatrix}
-i(k^2 - \alpha W M_0^2 - \Pi_{\phi+\phi^-})\alpha W & -iK\alpha W\Pi_{W\phi}/M_0 \\
-iK\alpha W\Pi_{W\phi}/M_0 & i(k^2 - \alpha W M_0^2 - \alpha W \Pi_{W\phi})
\end{bmatrix}.
$$

(1)

Substituting the matrix elements in (2.4) into the Ward identity (B.4a), we get

$$
(M_0^2 + \Pi_{WW})(M_0^2 - M_0^2\Pi_{\phi+\phi^-}/k^2) = (M_0^2 + \Pi_{W\phi})^2,
$$

(2.5)

As a side remark, (2.5) provides a subtraction condition. By setting $k^2$ to zero in (2.5), we find that

$$
\Pi_{\phi+\phi^-}(0) = 0.
$$

(2.6)

This ensures the divergence of $\Pi_{\phi+\phi^-}$ to be a logarithmic one. If we express the real part of the neutral Higgs field as $(v_0 + H)$, and choose $v_0$ to be the quantum value of the Higgs field, then the Higgs potential has a term linear in $H$. This term acts as a counter term which cancels all contributions of tadpole diagrams. With the absence of the contributions of the tadpole diagrams, the self-energy amplitude $\Pi_{W\phi}$ is only logarithmically divergent, not quadratically divergent. By (2.5), so is $\Pi_{WW}$. We also note that, with $v_0$ the quantum vacuum value, the Higgs potential has a mass term for $\phi^\pm$. When we calculate $\Pi_{\phi+\phi^-}$, we include the contributions of this mass term. But to incorporate the condition (2.6), we
make a subtraction for $\Pi_{\phi^+ \phi^-}$ at $k^2 = 0$. The contributions of the mass term vanish after subtraction.

The denominator in (2.4) is simplified if we take advantage of (2.5). We may, for example, use (2.5) to eliminate $\Pi_{\phi^+ \phi^-}$ from the denominator, which then becomes

$$\frac{J_W^2}{1 + \Pi_{WW} M_0^{-2}},$$

where

$$J_W \equiv k^2 - \alpha W M_0^2 - \alpha W \Pi_{WW} + \frac{k^2}{M_0^2} \Pi_{w\phi}. \tag{2.7}$$

Thus we have

$$D_{WW} = \frac{(k^2 - \alpha W M_0^2 - \Pi_{\phi^+ \phi^-})(1 + \Pi_{WW} M_0^{-2})}{(J_W)^2}, \tag{2.9a}$$

$$D_{\phi^+ \phi^-} = \frac{(k^2 - \alpha W M_0^2 - \alpha_w \Pi_{WW})(1 + \Pi_{WW} M_0^{-2})}{(J_W)^2}, \tag{2.9b}$$

and

$$D_{W\phi} = -\frac{\Pi_{W\phi}(1 + \Pi_{WW} M_0^{-2})}{(J_W)^2}. \tag{2.9c}$$

Note that the denominator in the expressions in (2.9) is the square of $J_W$. Since the poles of these propagators come from the zeroes of $J_W$, and since $J_W$ is a linear superposition of 1PI self-energy amplitudes, which are analytic functions of $k^2$, the order of the poles of the propagators are always even.

Propagators in quantum field theories generally have simple poles. It is therefore easy to be lulled into believing that poles of propagators are simple poles. We have found that this is often not true when two fields mix under the auspices of the Ward-Takahashi identities. In particular, the charged $\phi$ and the longitudinal $W$ have the same unperturbed mass. Thus the poles of the unperturbed propagators for these fields are both at $k^2 = \alpha W M_0^2$. As the interactions are turned on and the propagators form a mixing matrix, the positions of the poles change but the Ward-Takahashi identities force them to remain to be at the same point. Thus the two simple poles merge to form a double pole.

The function $J_W$ contains divergent integrals. Since no subtraction conditions are available, we expect that the location of the zero of $J_w$ be ultraviolet divergent. This is confirmed
by a perturbative calculation. We find that $J_w$ has a zero at

$$k^2 = \alpha_W M_0^2 \left\{ 1 + \frac{I}{\mu_0^2} + \frac{g_0^2}{8\pi^2\epsilon} \left[ 3 - \frac{s^2\alpha_A}{4} - \frac{\alpha_W}{2} - \frac{s^4\alpha_Z}{4c^2} \right] \right\},$$

(2.10)

where $\mu_0^2$ is the bare mass squared of the physical Higgs and is equal to $\lambda_0 v_0^2/2$, $\lambda_0$ and $v_0$ are the bare four point coupling and the vacuum expectation value of the $\phi$-field respectively. $s$ and $c$ are the sine and cosine of the Weinberg angle. $I$ is the contribution from the tadpole and $\epsilon = 4 - D$, with $D$ the space-time dimension. $I$ is given by

$$I = \left[ \frac{3\lambda}{2} + \frac{g_0^2}{2} \left( 1 + \frac{1}{2c^2} \right) \right] I_2$$

$$+ \frac{v_0^2}{64\pi^2\epsilon} \left[ 3\lambda^2 + \alpha_W \lambda g_0^2 + c \lambda g_0^2 + 3g_0^4 \left( 1 + \frac{1}{2c^4} \right) \right].$$

$I_2$ is the quadratic divergent term and is given by

$$I_2 = \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2}.$$

For comparison, the pole of the transverse part of the W is located at

$$k^2 = M_0^2 \left\{ 1 + \frac{2I}{\mu_0^2} + \frac{g_0^2}{8\pi^2\epsilon} \left[ \frac{17}{3} - \frac{3}{4c^2} - \frac{\alpha_W}{2} - \frac{\alpha_Z}{2} \left( c^2 + \frac{1}{2c^2} \right) \right] \right\}.$$

In (2.10), we keep only the terms which are ultraviolet divergent and we have not included the contributions of quark or lepton loops. Since the ratio of (2.10) and the transverse mass pole in the above expression is infinite, the double pole in (2.10) cannot be made finite.

Let the Fourier transform of $<0|T\eta^+(x)\xi^-(0)|0>$ be denoted as

$$G^{\eta^+\xi^-} \equiv iD_{\eta^+\xi^-},$$

where $\eta^+$ and $\xi^-$ are hermitian ghost fields associated with $W$. The Ward-Takahashi identity (B.4b) gives

$$D_{\eta^+\xi^-} = \frac{1 + \Pi_{W\phi} M_0^{-2}}{J_W (1 + g_0 E)},$$

(2.11)

where $E$ is defined in (B.5). The ghost propagator has only one factor of $J_W$ in the denominator.
Since the pole of a function cannot be removed by multiplying the function by a constant, the propagators cannot be made finite by wavefunction renormalizations. The standard model in the alpha gauge in general, and the Feynman gauge in particular, is not renormalizable.

Let us set $\alpha_W = 0$. In this limit, (2.8) becomes

$$J_W = k^2(1 + \Pi_{W\phi}M_0^{-2}).$$

(2.12)

Thus we have, in the Landau gauge,

$$D_{\phi^+\phi^-} = \frac{1}{k^2 - \Pi_{\phi^+\phi^-}},$$

(2.13a)

and

$$D_{\eta^+\xi^-} = \frac{1}{k^2(1 + g_0E)},$$

(2.13b)

with the propagators $G_{W^+\phi^-}$ and $G_{L^+W^-}$ vanishing at $\alpha_W = 0$. Because of (2.6), both propagators in (2.13) have a simple pole at $k = 0$. These propagators are renormalizable by wavefunction renormalizations.

### 3. The Mixing of $A$, $Z$ and $\phi^0$

In this section we discuss the $A$, $Z$, and $\phi^0$ mixing with the gauge fixing terms given by (1.2).

The longitudinal $Z$ meson mixes with $\phi^0$ as well as with the longitudinal $A$, where $\phi^0$ is the imaginary part of the neutral Higgs meson and is unphysical. Thus the propagators for these fields form a $3 \times 3$ matrix. Let the propagators be denoted as

$$G_{\mu\nu}^{AA}(k) \equiv -iD_{TAA}(k^2)T_{\mu\nu} - i\alpha_AD_{AA}(k^2)L_{\mu\nu},$$

$$G_{\mu\nu}^{ZZ}(k) \equiv -iD_{TZZ}(k^2)T_{\mu\nu} - i\alpha_ZD_{ZZ}(k^2)L_{\mu\nu},$$

$$G_{\mu\nu}^{AZ}(k) = G_{\mu\nu}^{ZA}(k) \equiv -iD_{TAZ}(k^2)T_{\mu\nu} - i\alpha_A\alpha_ZD_{AZ}(k^2)L_{\mu\nu},$$

$$G_{\mu}^{A\phi}(k) = \quad -G_{\mu}^{\phi A}(k) \equiv \alpha_A\frac{k\mu}{M_0^2}D_{A\phi}(k^2),$$

$$G_{\mu}^{Z\phi}(k) = \quad -G_{\mu}^{\phi Z}(k) \equiv \alpha_Z\frac{k\mu}{M_0^2}D_{Z\phi}(k^2),$$

$$G_{\mu}^{\phi\phi}(k) \equiv \quad iD_{\phi\phi\phi}(k^2),$$
where \( M'_0 \) is the bare mass of \( Z \). In the above, the function \( G_{\mu\nu}^{AZ} \), for example, is the Fourier transform of

\[
< 0 | T A_\mu(x) Z_\nu(0) | 0 > .
\]

These propagators form the mixing matrix

\[
\begin{bmatrix}
-i\alpha_A D_{AA} & -i\alpha_A \alpha_Z D_{AZ} & K\alpha_A D_{A\phi}/M'_0 \\
-i\alpha_A \alpha_Z D_{AZ} & -i\alpha_Z D_{ZZ} & K\alpha_Z D_{Z\phi}/M'_0 \\
-K\alpha_A D_{A\phi}/M'_0 & -K\alpha_Z D_{Z\phi}/M'_0 & iD_{\phi\phi}\phi\phi
\end{bmatrix}
\]

(3.1)

The unperturbed forms for these propagators are

\[
D^{(0)}_{AA}(k^2) = \frac{1}{k^2},
\]

\[
D^{(0)}_{ZZ}(k^2) = D^{(0)}_{\phi\phi\phi}(k^2) = \frac{1}{k^2 - \alpha_Z M'_0^2},
\]

with all other unperturbed forms vanishing. Thus the matrix in (3.1) is the inverse of

\[
\begin{bmatrix}
  i(k^2 - \alpha_A \Pi_{AA}) / \alpha_A & -i\Pi_{AZ} & K\Pi_{A\phi}/M'_0 \\
  -i\Pi_{AZ} & i(k^2 - \alpha_Z M'_0^2 - \alpha_Z \Pi_{ZZ}) / \alpha_Z & K\Pi_{Z\phi}/M'_0 \\
  -K\Pi_{A\phi}/M'_0 & -K\Pi_{Z\phi}/M'_0 & -i(k^2 - \alpha_Z M'_0^2 - \Pi_{\phi\phi}\phi\phi)
\end{bmatrix}
\]

(3.2)

where \( \Pi_{AA} \), for example, is the 1PI amplitude for \( D_{AA} \). There are nine Ward-Takahashi identities for these propagators, the derivation of which is given in Appendix C. Three of these identities give the following three relations among the 1PI amplitudes,[2,3]:

\[
(M'_0^2 + \Pi_{ZZ})(M'_0^2 - M'_0^2 \Pi_{\phi\phi\phi}/k^2) = (M'_0^2 + \Pi_{Z\phi})^2,
\]

(3.3a)

\[
(M'_0^2 + \Pi_{ZZ})\Pi_{AA} = \Pi_{AZ}^2,
\]

(3.3b)

and

\[
(M'_0^2 + \Pi_{Z\phi})\Pi_{AZ} = (M'_0^2 + \Pi_{ZZ})\Pi_{A\phi}.
\]

(3.3c)

The derivation of these relations is presented in Appendix D. Note the resemblance of (2.5) with (3.3a), indeed with (3.3b) and (3.3c) as well, if one takes into account that the photon is massless.
Next we calculate the inverse of (3.2) and equate it to the matrix in (3.1). We defer the
details to Appendix D and give only the results here:

\[ D_{AA}(k^2) = \frac{1}{k^2}, \quad (3.4a) \]

\[ D_{ZZ}(k^2) = \frac{k^2(k^2 - \alpha_Z M_0^2) + \alpha_A \alpha_Z M_0^2 \Pi_{AA} - k^2 \Pi_{\phi^0,\phi^0} \left[ 1 + \frac{\Pi_{ZZ}}{M_0^2} \right]}{k^2 J_Z^2}, \quad (3.4b) \]

\[ D_{AZ}(k^2) = -D_{A\phi^0}(k^2) = \frac{\Pi_{AZ}}{k^2 J_Z}, \quad (3.4c) \]

\[ D_{Z\phi}(k^2) = -\frac{(\alpha_A M_0^2 \Pi_{AA} + \Pi_{ZZ} k^2) \left( 1 + \frac{\Pi_{ZZ}}{M_0^2} \right)}{k^2 J_Z^2}, \quad (3.4d) \]

and

\[ D_{\phi^0,\phi^0}(k^2) = \frac{(k^2 - \alpha_Z M_0^2 - \alpha_Z \Pi_{ZZ} - \alpha_A \Pi_{AA}) \left( 1 + \frac{\Pi_{ZZ}}{M_0^2} \right)}{J_Z^2}, \quad (3.4e) \]

where

\[ J_Z = k^2 - \alpha_Z M_0^2 - \alpha_Z \Pi_{ZZ} + k^2 \frac{\Pi_{Z\phi}}{M_0^2}. \quad (3.5) \]

Note that all the propagators above except \( D_{AA} \) have a double pole at the simple zero
of \( J_Z \). The existence of this double pole is again easy to understand. The unperturbed
longitudinal \( Z \) and the unperturbed \( \phi^0 \) have the same mass, while the unperturbed \( A \) has
zero mass. Thus \( D_{ZZ} \) and \( D_{\phi^0,\phi^0} \) have simple poles at the same position, while \( D_{AA} \) has
a simple pole at \( k^2 = 0 \). As the interactions are turned on, the Ward-Takahashi identities
require that the position of the former two poles remain to be the same, while that of the
last pole remains to be zero. Thus the propagators in (3.4) have a double pole as well as a
simple pole at \( k^2 = 0 \).

A perturbative calculation shows that the zero of \( J_Z \) is located at

\[ k^2 = \alpha_Z M_0^2 \left\{ 1 + \frac{I}{\mu_0^2} + \frac{g_0^2}{8\pi^2} \left[ \frac{3c^2}{2} - \frac{\alpha_Z}{4c^2} - \frac{s^2}{2} \right] \right\}. \quad (3.6) \]

These propagators cannot be made finite by wavefunction renormalizations. One can easily
check that this double pole is different from the pole of the transverse part of the photon-Z
mixing sector.

From the Ward-Takahashi identities (C.4)–(C.7), we get

\[ D_{\eta_A \xi_A}(k^2) = \frac{(1 + g_0 c F_Z) J_Z + \alpha_Z c_0 F_Z \Pi_{AZ}}{k^2 J_Z(1 + c_0 F_A + g_0 c F_Z)}, \quad (3.7a) \]
\[
D_{\eta_1\xi_2}(k^2) = -\frac{\alpha_Z(1 + e_0 F_A)\Pi_{AZ} + g_0 c F_A J_Z}{k^2 J_Z(1 + e_0 F_A + g_0 c F_Z)}, \quad (3.7b)
\]

\[
D_{\eta_2\xi_1}(k^2) = -\frac{e_0 F_Z(1 + \Pi_{Z\phi}/M_0^2)}{J_Z(1 + e_0 F_A + g_0 c F_Z)}, \quad (3.7c)
\]

\[
D_{\eta_2\xi_2}(k^2) = \frac{(1 + e_0 F_A)(1 + \Pi_{Z\phi}/M_0^2)}{J_Z(1 + e_0 F_A + g_0 c F_Z)}, \quad (3.7d)
\]

which express the ghost propagators in terms of the 1PI self-energy amplitudes as well as the three-point functions \( F_A \) and \( F_Z \) defined by the equations following (C.4).

Note that the ghost propagators in (3.7) have poles at \( k^2 = 0 \) as well as at a zero of \( J_Z \). This is because the unperturbed propagators of the ghosts have a simple pole at \( k^2 = 0 \) and a simple pole at \( k^2 = \alpha_Z M_0^2 \), same as the position of the unperturbed \( D^{(0)}_{\phi^0}\phi^0 \). As interactions are turned on, the Ward-Takahashi identities require that the former remains to be at \( k^2 = 0 \), while the latter remains to be at the same position as the pole of \( D_{\phi^0}\phi^0 \).

Finally, we go to the Landau gauge by setting all alphas to zero. Then the only non-zero \( G \) for the unphysical mesons is

\[
G^{\phi^0\phi^0} = \frac{i}{k^2 - \Pi_{\phi^0}\phi^0}, \quad (3.8)
\]

which is logarithmically divergent and is renormalizable by a wave function renormalization of \( \phi^0 \).

Also, as we set \( \alpha_Z \) to zero, we find that

\[
D_{\eta_1\xi_2} = D_{\eta_2\xi_1},
\]

and the mixing matrix of the propagators of the neutral ghosts is symmetric. Such a matrix can be diagonalized by an orthogonal transformation. We may therefore renormalize these ghost propagators by renormalizing the rotated ghost fields obtained by diagonalization.

### 4. The Pure Alpha Gauge

We have shown that, if the alphas are not zero, the propagators in the preceding two sections have double poles with positions which are ultraviolet divergent. Consequently, the standard model with the gauge fixing terms of (1.1) and (1.2) are not renormalizable. We emphasize that the divergence of the double pole is sufficient but not necessary for the theory
of the unrenormalizable. An example is provided by the quantum theory of the standard model with the gauge fixing terms those in (1.3).

The unperturbed propagators of the fields in this gauge are given in (E.4). We see from the matrix which follows (E.4) that if we set $M'_0$ to zero, the off-diagonal propagators vanish while both diagonal propagators have a simple pole at $k^2 = 0$. In this limit, the charged Higgs meson is a Goldstone boson decoupled from the longitudinal $W$, the latter meson being also massless as a result of the gauge condition. Next we turn $M$ to a non-zero value. The charged Higgs meson remains to be a Goldstone boson but now it couples with $W$, which also remains massless because of the gauge condition. The two simple poles at $k^2 = 0$ merge and form a double pole at $k^2 = 0$, (The propagator $D_{WW}$ has only a simple pole at $k^2 = 0$ because of the gauge condition.) Similar considerations hold for the propagators of $A$, $Z$, and $\phi^0$.

To see what happens when the coupling constants are turned on, we first derive the twelve Ward-Takahashi identities satisfied by the two-point functions. These identities are listed in (E.3). They are somewhat different in forms from their counterparts in the gauge of (1.1) and (1.2), but they lead to the same relations among the 1PI self-energy amplitudes given by (2.5) and (3.3).

Using (2.5) and (3.3), we get the following expressions for the propagators:

\begin{align}
D_{WW} &= D_{AA} = D_{ZZ} = \frac{1}{k^2}, \\
D_{AZ} &= 0, \\
D_{\phi^+\phi^-} &= \frac{k^2 - \alpha_W M_0^2 - \alpha_W \Pi_{WW}}{(k^2)^2(1 - \Pi_{\phi^+\phi^-}/k^2)}, \\
D_{\phi^0\phi^0} &= \frac{k^2 - \alpha_Z M_0^2 - \alpha_Z \Pi_{ZZ} - \alpha_A \Pi_{AA}}{(k^2)^2(1 - \Pi_{\phi^0\phi^0}/k^2)}, \\
D_{W\phi} &= -\frac{M_0^2 + \Pi_{W\phi}}{(k^2)^2(1 - \Pi_{\phi^+\phi^-}/k^2)}, \\
D_{A\phi} &= -\frac{\Pi_{A\phi}}{(k^2)^2(1 - \Pi_{\phi^0\phi^0}/k^2)}, \\
D_{Z\phi} &= -\frac{M'_0^2 + \Pi_{Z\phi}}{(k^2)^2(1 - \Pi_{\phi^0\phi^0}/k^2)},
\end{align}
We see that the double pole for the $2 \times 2$ mixing matrix remains located at $k^2 = 0$. Thus the mixing of the unphysical Higgs mesons with the longitudinal gauge mesons does not change the massless nature of these particles as a result of the Ward identities.

Although the double pole is located at $k^2 = 0$, not infinity, the propagators still cannot be made finite by renormalizations. To see this, we first note that (4.1a) says that there are no radiative corrections to the propagators in (4.1a), which are already finite without being divided by wavefunction renormalization constants. Indeed, if we were to divide the longitudinal $A$ or the longitudinal $Z$ (or their rotated fields) by wavefunction renormalization constants which have ultraviolet divergences, the resulting renormalized propagators for these fields would be ultraviolet divergent. Thus the longitudinal $A$ and the longitudinal $Z$ need no renormalizations, and the only fields which we may renormalize are the Higgs fields. But it is not possible to make the propagators finite by doing so. To see this, let the wavefunction renormalization constant for $\phi^\pm$ be $Z_\phi$. Then $G^{W\phi}$ multiplied by $\sqrt{Z_\phi}$ and $G^{\phi^+\phi^-}$ multiplied by $Z_\phi$ are the renormalized propagators. If both renormalized propagators are finite, so is the ratio $M_0^2 D_{\phi^+\phi^-}/D_{W\phi}^2$. But this ratio is

$$
(k^2)^2 \left[ \frac{k^2}{M_0^2 + \Pi_{WW}(k^2)} - \alpha_W \right].
$$

(4.2)

Let us examine this expression in the limit of $k^2 \to 0$. As we have mentioned, the 1PI self-energy amplitude in this limit for the longitudinal vector meson is the same as that for the transverse vector. Thus the first term inside the bracket in (4.2) is equal to $k^2$ times the propagator of the tranverse $W$ at zero momentum, and is ultraviolet divergent. Thus the standard model in the gauge of (1.3) is not renormalizable. As before, the difficulty of renormalization disappears as we set all alphas to zero. Indeed, (4.1c) and (4.1d) are in the same forms as (2.13a) and (3.10) as we set all alphas to zero.

Finally, the Ward-Takahashi identities (E.2b), and (E.3c)–(E.3f) enable us to express the ghost propagators as

$$
D_{\eta^+\xi^-}(k^2) = \frac{1}{k^2(1 + g_0 E)},
$$

(4.3a)

$$
D_{\eta\xi A}(k^2) = \frac{1 + g_0 c F_{\rho A}}{k^2 1 + e_0 F_A + g_0 c F_{\rho}},
$$

(4.3b)

$$
D_{\eta\xi Z}(k^2) = -\frac{g_0 c F_A}{k^2 1 + e_0 F_A + g_0 c F_{\rho}},
$$

(4.3c)
\[ D_{ηZξ}(k^2) = \frac{1}{k^2} \frac{1 + e_0 F_A}{1 + e_0 F_A + g_0 c F_Z}, \quad (4.3d) \]
\[ D_{ηZξA}(k^2) = -\frac{1}{k^2} \frac{e_0 F_Z}{1 + e_0 F_A + g_0 c F_Z}. \quad (4.3e) \]

Note that the ghost propagators in (4.3) are identical in form with the corresponding ones in (3.7) if we set \( α_Z \) to zero, and can be renormalized for the same reason as before.

5. Conclusion

Recognizing that the Ward-Takahashi identities provide constraints on the divergences among amplitudes in the theory, people have accepted, ever since the appearance of the pioneering works of t’Hooft and Veltman [5,6], that the BRST invariance of the standard model guarantees the renormalizability of the quantum theory of the model in the alpha gauge.

In the standard model, there are twelve Ward-Takahashi identities for the two-point functions. Four of these identities lead to four relations satisfied by the nine 1PI self-energy amplitudes of the unphysical mesons, leaving five of the 1PI self-energy amplitudes independent. Five other Ward-Takahashi identities relate the five independent ghost propagators and three 3-point functions to these same 1PI self-energy amplitudes. The last three Ward-Takahashi identities set the mass scale of the physical vector mesons. For example, the Ward-Takahashi identity (B.6) leads to a relation between the \( W \)-mass and the quantum vacuum expectation of the Higgs meson, as will be discussed in more details in another paper. We shall only point out here that, in the gauge of (1.1) and (1.2), with no conditions of subtraction available for the position of the double pole in the unphysical propagators, there is nothing to prevent it from being infinite.

Over a year ago, we first realized that, in the Abelian gauge field theory with Higgs mesons, with a gauge fixing term of the form of (1.2), one of the Ward-Takahashi identities enforces the isolated singularity of the propagators in the \( 2 \times 2 \) mixing matrix in this theory to be a double pole. Explicit perturbative calculations on the position of this double pole verified that the position of this pole is ultraviolet divergent. Thus the Abelian gauge field theory with Higgs mesons is not renormalizable in the alpha gauge.
Since then we have shown that all these are also true in the standard model, both for the $2 \times 2$ mixing matrix and for the $3 \times 3$ mixing matrix for $\nu_0$ the classical vacuum value or the quantum vacuum value. We have also extended the treatment to the alpha gauge with the gauge fixing terms of (1.3). While the double pole in this case is at $k^2 = 0$, not infinity, the propagators cannot be rendered finite with wavefunction renormalizations.

There have been many proofs [7,8,9,10] demonstrating that gauge field theories with symmetry breaking in general, and the standard model in particular, are renormalizable in the alpha gauge. Instead of addressing everyone of them in details, we would like to make the following comment. The considerations of renormalizability of quantum gauge field theories with symmetry breaking differ from those without symmetry breaking. In the former theories, the unphysical Higgs mesons mix with the longitudinal component of the associated gauge meson. It is untenable to argue that since the latter theories are renormalizable, so are the former theories.

To illustrate this point, let us consider the simple example of scalar QED in which the photon couples with a complex scalar field $\phi$ which has a real and positive bare mass $\mu_0$, and the vacuum symmetry is not broken. To quantize this theory, let us choose the gauge fixing term

$$-rac{1}{2\alpha}(\partial_\mu A^\mu - \alpha M\phi_2)^2$$

where $\phi_2$ is the imaginary part of $\phi$ and $M$ is an introduced parameter. The gauge fixing term for scalar QED is traditionally chosen to be the one in (5.1) with $M$ equal to zero, but there is nothing to forbid us from choosing a non-zero $M$ provided that we add the corresponding ghost terms to make the effective Lagrangian invariant under BRST variations. While the Green functions for a non-zero $M$ are different from those with $M$ equal to zero, the physical scattering amplitudes remain the same.

The gauge fixing term in (5.1) contains a term which mixes the longitudinal $A$ with $\phi$. This changes the tree amplitudes but not any of the 1PI amplitudes of the propagators. Thus we have

$$\Pi_{AA} = \Pi_{A\phi_2} = 0$$
and the $2 \times 2$ mixing matrix for the propagators of $A$ and $\phi_2$ is equal to the inverse of

\[
\begin{pmatrix}
\frac{ik^2}{\alpha} & KM \\
-KM & -i(k^2 - \mu_0^2 - \alpha M^2 - \Pi_{\phi\phi})
\end{pmatrix}.
\]

Thus this mixing matrix is equal to

\[
\begin{pmatrix}
-i\alpha(k^2 - \mu_0^2 - \alpha M^2 - \Pi_{\phi\phi}) & -\alpha KM \\
\alpha KM & ik^2
\end{pmatrix}
\]

\[
k^2(k^2 - \mu_0^2 - \Pi_{\phi\phi})
\]

We see from (5.2) that the propagators are not renormalizable for a finite and non-zero $M$. In order that the theory is renormalizable, a non-zero $M$ must be equal to a finite number times $1/\sqrt{Z_\phi}$, where $Z_\phi$ is the wavefunction renormalization constant for the $\phi_2$-field.

In the standard model there is no such freedom of choosing $M$, and the unphysical propagators are not renormalizable in the alpha gauge. Since graphs of unphysical propagators may appear as subgraphs in the graphs of other Green functions, Green functions in the standard model are generally not renormalizable.

The only gauge in which the difficulty of renormalization does not appear is the Landau gauge, which is obtained from the alpha gauge by setting alpha to zero. One of the reasons for this is that, as all alpha are set to zero, many propagators vanish and need no renormalization.

This does not necessarily mean that the standard model is renormalizable in Landau gauge. But if it is, and if physical (and on-shell) scattering amplitudes are gauge invariant, these amplitudes will be finite in the alpha gauge once they are finite in the Landau gauge. In a practical calculation of physical scattering amplitudes, the infinities from the Green functions cancel one another, provided that they are properly regularized and the procedures of subtractions and normalization as emphasized by Bonneau [11] are properly performed. On the other hand, off-shell Green functions are dependent on alpha, and are not renormalizable in the alpha gauge. In a gauge field theory with symmetry breaking, one must make a distinction between the renormalizability of the Green functions and that of the physical scattering amplitudes.
References

1. G. Bonneau, Nucl. Phys. B221 (1983) 178.

2. L. Baulieu and R. Coquereaux, Ann. Phys. 140 (1982) 163.

3. See also K. Aoki, Z. Hioki, R. Kawabe, M. Konuma, and T. Muta, Prog. Theor. Phys. Suppl. 73 (1982) 1, and the references quoted in this paper. A number of our results, e.g., (2.5), disagree with their counterparts in the paper of reference [2], but agree with their counterparts in this paper.

4. E.C. Tsai, (private communication, 1986).

5. G. t’Hooft, Nucl. Phys. B33 (1971) 173, ibid. B35 (1971) 167.

6. G. t’Hooft and M. Veltman, Nucl. Phys. B44 (1972) 189.

7. B.W. Lee and J. Zinn-Justin, Phys. Rev. D5 (1972) 3121, 3137, 3155.

8. J.C. Taylor, Nucl. Phys. B33 (1971) 436.

9. A.A. Slavnov, Theor. and Math. Phys. 10 (1972) 99.

10. K. Fujikawa, B.W. Lee, and A.I. Sanda, Phys. Rev. D6 (1972) 2923.

11. G. Bonneau, Int. J Mod. Phys. 5 (1990) 3831.
Appendix A

We list some of the BRST variations of fields in the standard model below:

\[
\delta W_\mu^\pm = \partial_\mu \xi^\pm \pm ig_0 (c Z_\mu + s A_\mu) \xi^\pm \mp ig_0 W_\mu^\pm (c \xi_Z + s \xi_A), \quad (A.1)
\]

\[
\delta Z_\mu = \partial_\mu \xi_Z + ig_0 c (W_\mu^+ \xi^- - W_\mu^- \xi^+), \quad (A.2)
\]

\[
\delta A_\mu = \partial_\mu \xi_A + ig_0 s (W_\mu^+ \xi^- - W_\mu^- \xi^+), \quad (A.3)
\]

\[
\delta \phi^\pm = \mp ig_0 \left[ \left( \frac{c^2 - s^2}{2c} \xi_Z + s \xi_A \right) \phi^\pm + \frac{1}{2} \xi^\pm (v_0 + H \pm i \phi^0) \right], \quad (A.4)
\]

\[
\delta H = -\frac{g_0}{2c} \xi_Z \phi - \frac{ig_0}{2} (\xi^- \phi^+ - \xi^+ \phi^-), \quad (A.5)
\]

\[
\delta \phi^0 = \frac{g_0}{2c} \xi_Z (v_0 + H) - \frac{g_0}{2} (\xi^- \phi^+ + \xi^+ \phi^-), \quad (A.6)
\]

\[
\delta i \eta^\pm = \frac{1}{\alpha_W} (\partial^\mu W_\mu^\pm \pm i \alpha_W M_0 \phi^\pm), \quad (A.7)
\]

\[
\delta i \eta_Z = \frac{1}{\alpha_Z} (\partial_\mu Z_\mu^\pm + \alpha M'_0 \phi^0), \quad (A.8)
\]

\[
\delta i \eta_A = \frac{1}{\alpha_A} \partial_\mu A_\mu. \quad (A.9)
\]

In the above,

\[
W^\pm \equiv \frac{W_1^\pm + i W_2^\pm}{\sqrt{2}},
\]

\[
M_0 \equiv \frac{1}{2} g_0 v_0,
\]

\[
M'_0 \equiv \frac{1}{c} M_0.
\]

\eta and \xi are the hermitian ghost fields, \( g_0 \) is the bare weak coupling constant, and \( \alpha \) is the gauge parameter. (We denote the gauge parameter for \( A \) as \( \alpha_A \).) Also, the Higgs field is given by

\[
\phi \equiv \begin{pmatrix} \phi^0 \\ (v_0 + H + i \phi^0)/\sqrt{2} \end{pmatrix}
\]

Appendix B

17
In a gauge field theory with an effective Lagrangian satisfying the BRST invariance, the vacuum state satisfies
\[ Q|0 >= 0. \tag{B.1} \]
In (B.1), \( Q \) is the BRST charge the commutator (anti-commutator) of which with a Bose (Grassmann) field is the BRST variation of the field.

The Ward-Takahashi identities can be derived directly from (B.1). We have, as a result of (B.1),
\[ <0|OQ|0 >= 0, \tag{B.2} \]
where \( O \) is any operator. Next we move \( Q \) in (B.2) to the left. Since
\[ <0|Q = 0, \]
we get
\[ <0|\delta O|0 >= 0, \tag{B.3} \]
where \( \delta O \) is the BRST variation of \( O \).

Let us next derive the Ward-Takahashi identities for the propagators in the mixing matrix for the longitudinal \( W \) and the charged \( \phi \).

By choosing \( T(i\eta^+\delta\eta^-) \) as \( O \) in (B.3), where \( T \) is the time-ordering operator, we get
\[ \left<0|T\left(\frac{1}{\alpha_W}\partial^\mu W_{\mu}^+ + iM_0\phi^+\right)\left(\frac{1}{\alpha_W}\partial^\nu W_{\nu}^- - iM_0\phi^-\right)|0\right> = 0 \]
or
\[ \frac{1}{\alpha_W} - \frac{k^2}{\alpha_W}D_{WW}(k^2) - 2k^2D_{W\phi}(k^2) + M_0^2D_{\phi^+\phi^-}(k^2) = 0. \tag{B.4a} \]
The Ward-Takahashi identity (B.4) relates the propagators in the \( 2 \times 2 \) mixing matrix.

From \( <0|\delta T i\eta^+ W_{\nu}^-|0 >= 0 \), we get
\[ \left<0|T\left(\frac{1}{\alpha_W}\partial^\mu W_{\mu}^+ + iM_0\phi^+\right)W_{\nu}^-|0\right> \]
\[ - \left<0|T i\eta^+ (\partial_{\nu}\xi^- - ig_0(cZ_{\nu} + sA_{\nu})\xi^- + ig_0W_{\nu}^-(c\xi_Z + s\xi_A))|0\right> = 0, \]
which leads to
\[ -D_{WW}(k^2) - \alpha_W D_{W\phi}(k^2) + D_{\eta^+\xi^-}(k^2)(1 + g_0E) = 0. \tag{B.4b} \]
where the Fourier transform of $i < 0|T i\eta^+[\{(c Z_\nu + s A_\nu)\xi^-- W^-_\nu(c \xi Z + s \xi A)]|0 >$ is defined to be

$$D_{\eta^+\xi^-}(k^2) E k^\nu.$$ (B.5)

The identity (B.5) relates the ghost propagator $D_{\eta^+\xi^-}$ to the propagators in the $2 \times 2$ mixing matrix.

From $< 0|\delta T i\eta^+ \phi^-|0 >= 0$, we get

$$\left< 0|T \left( \frac{1}{\alpha W} \partial^\mu W^\mu_\mu + i M_0 \phi^+ \right) \phi^-|0 \right> - ig_0 \left< 0|T i\eta^+ \left[ \left( \frac{c^2 - s^2}{2c} \xi Z + s \xi A \right) \phi^- + \frac{1}{2} \xi^- (v_0 + H - i \phi^0) \right]|0 \right> = 0,$$

which leads to

$$\frac{k^2}{M_0} D_{W\phi}(k^2) - M_0 D_{\phi^+\phi^-}(k^2) + g_0 D_{\eta^+\xi^-}(k^2) \left( \frac{< 0|v_0 + H|0 >}{2} + F \right) = 0,$$ (B.6)

where the Fourier transform of the connected part of

$$-i \left< 0|T i\eta^+ \left[ \left( \frac{c^2 - s^2}{2c} \xi Z + s \xi A \right) \phi^- + \frac{1}{2} \xi^- (H - i \phi^0) \right]|0 \right>$$

is defined to be

$$D_{\eta^+\xi^-}(k^2) F.$$ (B.7)

The identity (B.6) relates the propagators with the vacuum expectation value of the Higgs field.

**Appendix C**

In this Appendix, we derive the Ward-Takahashi identities involving the propagators in the $3 \times 3$ mixing matrix for the longitudinal $A$, the longitudinal $Z$, and $\phi^0$.

From $< 0|\delta T i\eta_A \partial_\mu A_\nu A^\nu|0 >= 0$, we get

$$\frac{1}{\alpha_A} < 0|T \partial_\mu A^\mu \partial_\nu A^\nu|0 >= 0,$$

which leads to

$$D_{A\lambda}(k^2) = \frac{1}{k^2}.$$ (C.1)
From \(0\mid \delta T i \eta_Z \left( \frac{1}{\alpha_Z} \partial_\nu Z^\nu + M'_0 \phi^0 \right) \mid 0 \rangle = 0\), we get
\[
\langle 0 \mid T \left( \frac{1}{\alpha_Z} \partial_\mu Z^\mu + M'_0 \phi^0 \right) \left( \frac{1}{\alpha_Z} \partial_\nu Z^\nu + M'_0 \phi^0 \right) \mid 0 \rangle = 0,
\]
which leads to
\[
\frac{1}{\alpha_Z} - \frac{k^2}{\alpha_Z} D_{ZZ} - 2k^2 D_{Z\phi} + M'_0^2 D_{\phi\phi^0} = 0. \tag{C.2}
\]
From \(0\mid \delta T i \eta_A \left( \frac{1}{\alpha_Z} \partial_\nu Z^\nu + M'_0 \phi^0 \right) \mid 0 \rangle = 0\), we get
\[
\langle 0 \mid T \frac{\partial_\mu A^\mu}{\alpha_A} \left( \frac{1}{\alpha_Z} \partial_\nu Z^\nu + M'_0 \phi^0 \right) \mid 0 \rangle = 0,
\]
which leads to
\[
D_{AZ} = -D_{A\phi}. \tag{C.3}
\]
The three Ward-Takahashi identities derived above are independent of the propagators for the ghost fields.

Next, from \(0\mid \delta T i \eta_A A_\nu \mid 0 \rangle = 0\), we get
\[
\frac{1}{\alpha_A} \langle 0 \mid T \partial_\mu A^\mu A_\nu \mid 0 \rangle - \langle 0 \mid T \eta_A \left[ \partial_\nu \xi_A + ie_0 (W_\nu^+ \xi^- - W_\nu^- \xi^+) \right] \mid 0 \rangle = 0,
\]
which leads to
\[
-\frac{1}{k^2} + D_{\eta_A \xi_A} (k^2) (1 + e_0 F_A) + D_{\eta_A \xi_Z} (k^2) e_0 F_Z = 0, \tag{C.4}
\]
where \(e_0\) is the bare electric charge and where
\[
\Gamma_{\eta_A W^+ \xi^-} - \Gamma_{\eta_A W^- \xi^+} \equiv k_\nu F_A(k^2),
\]
\[
\Gamma_{\eta_Z W^+ \xi^-} - \Gamma_{\eta_Z W^- \xi^+} \equiv k_\nu F_Z(k^2).
\]
In the above \(\Gamma_{\eta_A W^+ \xi^-}\) is the truncated 3-point function with the fields \(W_\nu^+\) and \(\xi^-\) joint at the same space-time point.

From \(0\mid \delta T i \eta_A Z_\nu \mid 0 \rangle = 0\), we get
\[
\frac{1}{\alpha_A} \langle 0 \mid T \partial_\mu A_\mu Z_\nu \mid 0 \rangle - \langle 0 \mid T \eta_A \left[ \partial_\nu \xi_Z + ig_0 c (W_\nu^+ \xi^- - W_\nu^- \xi^+) \right] \mid 0 \rangle = 0
\]
which leads to
\[
-\alpha_Z D_{AZ} (k^2) + D_{\eta_A \xi_Z} (k^2) (1 + g_0 c F_Z) + D_{\eta_A \xi_A} (k^2) g_0 c F_A = 0. \tag{C.5}
\]
From $<0|\delta T\eta Z A_\nu|0>=0$, we get
\[ \left< 0 | T \left( \frac{1}{\alpha_Z} \partial^\mu Z_\mu + M_0' \phi^0 \right) A_\nu |0 \right> - \left< 0 | T\eta Z [\partial_\nu \xi_A + i\epsilon_0 (W^+_\nu \xi^- - W^-_\nu \xi^+) ] |0 \right> = 0, \]
which leads to
\[ D_{\eta Z \xi_A} (k^2) (1 + \epsilon_0 F_A) + \epsilon_0 D_{\eta Z \xi_A} (k^2) F_Z = 0. \]  \hfill (C.6)
From $<0|\delta T\eta Z A_\nu|0>=0$, we get
\[ \left< 0 | T \left( \frac{1}{\alpha_Z} \partial^\mu Z_\mu + M_0' \phi^0 \right) Z_\nu |0 \right> - \left< 0 | T\eta Z [\partial_\nu \xi_Z + i\eta_0 c (W^+_\nu \xi^- - W^-_\nu \xi^+) ] |0 \right> = 0 \]
which leads to
\[ -D_{ZZ} (k^2) - \alpha_Z D_{\eta Z \phi} (k^2) + D_{\eta Z \xi_Z} (k^2) (1 + \eta_0 c F_Z) + \eta_0 c D_{\eta Z \xi_A} (k^2) F_A = 0. \]  \hfill (C.7)
The four Ward-Takahashi identities (C.4)–(C.7) relate the ghost propagators to the propagators in the mixing matrix.

From $<0|\delta T\eta A \phi^0|0>=0$, we get
\[ \left< 0 | T \left( \frac{1}{\alpha_A} \partial^\mu A_\mu \phi^0 \right) \phi^0 |0 \right> - \left< 0 | T\eta A \left[ M_0' \xi_Z + \frac{\eta_0}{2c} \xi_Z H - \frac{\eta_0}{2} (\xi^- \phi^+ + \xi^+ \phi^-) \right] |0 \right> = 0 \]
which leads to
\[ -\frac{k^2}{M_0'} D_{\phi \phi} (k^2) - \frac{\eta_0}{2} D_{\eta A \xi_Z} (k^2) \left[ \frac{<0|\nu_0 + H|0>}{c} + \frac{1}{c} \Gamma_{\eta Z \xi_Z H} - \Gamma_{\eta Z \xi^- \phi^+} - \Gamma_{\eta Z \xi^+ \phi^-} \right] \]
\[ - \frac{\eta_0}{2} D_{\eta A \xi_Z} (k^2) \left[ \frac{1}{c} \Gamma_{\eta A \xi_Z H} - \Gamma_{\eta A \xi^- \phi^+} - \Gamma_{\eta A \xi^+ \phi^-} \right] = 0. \]  \hfill (C.8)
where $\Gamma_{\eta Z \xi_Z H}$, say, is the truncated 3-point Green function of the fields $\eta_Z$, $\xi_Z$ and $H$, with the latter two fields joined at the same space-time point.

From $<0|\delta T\eta Z \phi^0|0>=0$, we get
\[ \left< 0 | T \left( \frac{1}{\alpha_Z} \partial^\mu Z_\mu + M_0' \phi^0 \right) \phi^0 |0 \right> - \left< 0 | T\eta Z \left[ M_0' \xi_Z + \frac{\eta_0}{2c} \xi_Z H - \frac{\eta_0}{2} (\xi^- \phi^+ + \xi^+ \phi^-) \right] |0 \right> = 0, \]
which leads to
\[ -\frac{k^2}{M_0'} D_{Z \phi} (k^2) + M_0' D_{\phi \phi} (k^2) \]
\[ - \frac{\eta_0}{2} D_{\eta Z \xi_Z} (k^2) \left[ \frac{<0|\nu_0 + H|0>}{c} + \frac{1}{c} \Gamma_{\eta Z \xi_Z H} - \Gamma_{\eta Z \xi^- \phi^+} - \Gamma_{\eta Z \xi^+ \phi^-} \right] \]
\[ - \frac{\eta_0}{2} D_{\eta Z \xi_A} (k^2) \left[ \frac{1}{c} \Gamma_{\eta A \xi_Z H} - \Gamma_{\eta A \xi^- \phi^+} - \Gamma_{\eta A \xi^+ \phi^-} \right] = 0. \]  \hfill (C.9)
\[ 21 \]
The Ward-Takahashi identities (C.8) and (C.9) relate the propagators with the vacuum expectation value of the Higgs field.

**Appendix D**

Equating the matrix in (3.1) to the inverse of the matrix in (3.2), we express the propagators in (3.1) by their 1PI amplitudes. With the propagators expressed in such forms, we require them to satisfy the Ward-Takahashi identity (C.3). We get

\[
\Pi_{AZ}(k^2 - \alpha_Z M_0'^2 - \Pi_{\phi,\phi} - \alpha Z \Pi_{Z\phi}) = \Pi_{A\phi}(k^2 - \alpha_Z M_0'^2 - \alpha Z \Pi_{ZZ} + k^2 \Pi_{Z\phi}/M_0'^2). \quad (D.1)
\]

Similarly, by requiring (C.1) be satisfied, we get

\[
\Pi_{AA}(k^2 - \alpha_Z M_0'^2 - \Pi_{\phi,\phi} - \alpha Z \Pi_{Z\phi}) = \Pi_{A\phi}(-\alpha_Z \Pi_{AZ} + k^2 \Pi_{A\phi}/M_0'^2), \quad (D.2)
\]

where we have made use of (D.1). Also, imposing (C.2) gives

\[
(1 + \Pi_{ZZ}/M_0'^2)(1 - \frac{\Pi_{\phi\phi}}{k^2}) = (1 + \Pi_{Z\phi}/M_0'^2)^2, \quad (D.3)
\]

where we have made use of (D.1) and (D.2) to eliminate \(\Pi_{AA}\) and \(\Pi_{A\phi}\). We may reduce (D.1) further by making use of (D.3) to eliminate \(\Pi_{\phi\phi}\) and get

\[
\Pi_{AZ}(1 + \Pi_{Z\phi}/M_0'^2) = \Pi_{A\phi}(1 + \Pi_{ZZ}/M_0'^2). \quad (D.4)
\]

Similarly, we may eliminate \(\Pi_{\phi\phi}\) from (D.2) and get

\[
(1 + \Pi_{ZZ}/M_0'^2)(\Pi_{AA}/M_0'^2) = (\Pi_{AZ}/M_0'^2)^2. \quad (D.5)
\]

Note the resemblance of (D.3) and (D.5) with (2.5).

With (D.3)–(D.5), we may reduce the determinant of the 3 × 3 matrix in (3.2) into

\[
i k^2 J_Z^2/[\alpha A \alpha Z (1 + \Pi_{ZZ}/M_0'^2)] \quad (D.6)
\]

where

\[
J_Z = k^2 - \alpha_Z M_0'^2 - \alpha Z \Pi_{ZZ} + k^2 \Pi_{Z\phi}/M_0'^2. \quad (D.7)
\]

Equations (D.6) and (D.7) are obtained by eliminating \(\Pi_{AA}\) and \(\Pi_{A\phi}\) from the expression. Note the similarity between (D.7) and (2.8).
Appendix E

In this Appendix, we shall study the propagators in the pure alpha gauge defined by the gauge fixing terms (1.3). As in the preceding appendices, we shall use the Ward-Takahashi identities in this gauge to determine relations among the 1PI self-energy amplitudes. We then simplify the expressions for these propagators by the use of these relations.

In the pure alpha gauge, the BRST variations of the fields remain the same as the ones given in Appendix A with the following exceptions

\[
\delta i \eta^+ = \frac{1}{\alpha_W} \partial^\mu W^\mu_+,
\delta i \eta_Z = \frac{1}{\alpha_Z} \partial \delta^\mu Z_\mu.
\]

The Ward-Takahashi identities for the propagators of the longitudinal \( W \) and the unphysical charged \( \phi \) give

\[ D_{WW}(k^2) = \frac{1}{k^2}, \]
\[ D_{WW}(k^2) = D_{\eta^+\xi^-}(k^2)(1 + g_0 E), \]
where \( E \) is given by (B.6) and

\[ \frac{k^2}{M_0^2} D_{W\phi}(k^2) = -D_{\eta^+\xi^-}(k^2) \left( Z + \frac{F}{M_0} g_0 \right), \]

where \( F \) is given by (B.7) and

\[ Z \equiv \langle 0|v_0 + H|0 \rangle / v_0. \]

The Ward-Takahashi identities for the propagators of the longitudinal \( A \), the longitudinal \( Z \), and the unphysical neutral \( \phi^0 \) are

\[ D_{AA}(k^2) = D_{ZZ}(k^2) = \frac{1}{k^2}, \]
\[ D_{AZ}(k^2) = 0, \]
\[ -\frac{1}{k^2} + D_{\eta_A \xi_A}(k^2)(1 + e_0 F_A) + D_{\eta_A \xi_Z}(k^2)e_0 F_Z = 0, \]
where \( F_A \) and \( F_Z \) are defined in the equations following (C.4),

\[ D_{\eta_A \xi_Z}(k^2)(1 + g_0 c F_Z) + D_{\eta_A \xi_A}(k^2)g_0 c F_A = 0, \]
We shall need the forms of the unperturbed propagators for the unphysical mesons:

\[ D_{\eta z}(k^2)(1 + e_0 F_A) + D_{\eta z}(k^2)e_0 F_Z = 0, \quad (E.3c) \]

\[ -\frac{1}{k^2} + D_{\eta z}(k^2)(1 + g_0 c F_Z) + D_{\eta z}(k^2)g_0 c F_A = 0, \quad (E.3f) \]

\[ -\frac{k^2}{M_0^2}D_{A\phi}(k^2) - \frac{g_0}{2}D_{\eta A\xi}(k^2)\left[ \frac{<0|v_0 + H|0>}{c} + \frac{1}{c} \Gamma_{\eta z\xi z} - \Gamma_{\eta z\xi^-} - \Gamma_{\eta z\xi^+} \right] \]

\[ -\frac{g_0}{2}D_{\eta A\xi}(k^2)\left[ \Gamma_{\eta A\xi z} - \Gamma_{\eta A\xi^-} - \Gamma_{\eta A\xi^+} \right] = 0, \quad (E.3g) \]

and

\[ -\frac{k^2}{M_0^2}D_{\zeta\phi}(k^2) - \frac{g_0}{2}D_{\eta z}(k^2)\left[ \frac{v_0 Z}{c} + \frac{1}{c} \Gamma_{\eta z\xi z} - \Gamma_{\eta z\xi^-} + \Gamma_{\eta z\xi^+} \right] \]

\[ -\frac{g_0}{2}D_{\eta z}(k^2)\left[ \frac{1}{c} \Gamma_{\eta A\xi z} - \Gamma_{\eta A\xi^-} + \Gamma_{\eta A\xi^+} \right] = 0. \quad (E.3h) \]

We shall need the forms of the unperturbed propagators for the unphysical mesons:

\[ D^{(0)}_{WW} = D^{(0)}_{AA} = D^{(0)}_{ZZ} = \frac{1}{k^2}, \]

\[ D^{(0)}_{A\zeta} = D^{(0)}_{A\phi} = 0, \]

\[ D^{(0)}_{W\phi} = -\frac{M_0^2}{(k^2)^2}, \]

\[ D^{(0)}_{Z\phi} = -\frac{M_0^2}{(k^2)^2}, \quad (E.4) \]

\[ D^{(0)}_{\phi^+\phi^-} = \frac{(k^2 - \alpha_W M_0^2)}{(k^2)^2}, \]

\[ D^{(0)}_{\phi^0\phi^0} = \frac{(k^2 - \alpha_Z M_0^2)}{(k^2)^2}. \]

Let us first deal with the \(2 \times 2\) mixing matrix given by (2.2) for the longitudinal \(W\) and the unphysical charged \(\phi\). Referring to (E.4), we find that the unperturbed form for this mixing matrix is

\[
\begin{bmatrix}
-\frac{i\alpha_W}{k^2} & -\frac{i\alpha_W K M_0}{(k^2)^2} \\
-\frac{i\alpha_W K M_0}{(k^2)^2} & \frac{i(k^2 - \alpha_W M_0^2)}{(k^2)^2}
\end{bmatrix}.
\]

The inverse of the matrix above is

\[
\begin{bmatrix}
\frac{i(k^2 - \alpha_W M_0^2)}{\alpha_W} & i K M_0 \\
K M_0 & -i k^2
\end{bmatrix}.
\]

(E.5)

Consequently, the \(2 \times 2\) mixing matrix is equal to the inverse of

\[
\begin{bmatrix}
\frac{i(k^2 - \alpha_W M_0^2 - \alpha_W \Pi_{WW})}{\alpha_W} & iK(M_0^2 + \Pi_{W\phi})/M_0 \\
iK(M_0^2 + \Pi_{W\phi})/M_0 & -i(k^2 - \Pi_{\phi^+\phi^-})
\end{bmatrix}.
\]

(E.6)
Let us impose (E.2a) on the first diagonal matrix element of the inverse of (E.6). We get, as in the alpha gauge of (1.1), the condition (2.5).

With (2.5), we reduce the denominator of the matrix in (E.5) into

\[ (k^2)^2(1 - \Pi_{\phi^+\phi^-}/k^2)/\alpha_W, \]  
\( (E.7) \)

Thus we have

\[ D_{\phi^+\phi^-}(k^2) = \frac{k^2 - \alpha_WM_0^2 - \alpha_W\Pi_{WW}}{(k^2)^2(1 - \Pi_{\phi^+\phi^-}/k^2)}, \]  
\( (E.8a) \)

and

\[ D_{W\phi}(k^2) = \frac{M_0^2 + \Pi_{W\phi}}{(k^2)^2(1 - \Pi_{\phi^+\phi^-}/k^2)}. \]  
\( (E.8b) \)

The Ward-Takahashi identity (E.2b) enables us to express the ghost propagator \( D_{\eta^+\xi^-} \) as

\[ D_{\eta^+\xi^-}(k^2) = \frac{1}{k^2(1 + g_0E)}. \]  
\( (E.9) \)

The Ward-Takahashi identity (E.2c) relates the 1PI amplitudes with the vacuum expectation value of the Higgs field.

Next we turn to the 3 \times 3 mixing matrix for the longitudinal \( A \), the longitudinal \( Z \), and the unphysical neutral Higgs meson \( \phi^0 \). Referring to (E.4), we find that the unperturbed form of this mixing matrix is

\[
\begin{pmatrix}
-i\alpha_A/k^2 & 0 & 0 \\
0 & -i\alpha_Z/k^2 & -\alpha_Z K M'_0/(k^2)^2 \\
0 & \alpha_Z K M'_0/(k^2)^2 & i(k^2 - \alpha_Z M'_0^2)/(k^2)^2
\end{pmatrix}.
\]

The inverse of the matrix above is

\[
\begin{pmatrix}
 i k^2/\alpha_A & 0 & 0 \\
0 & i(k^2 - \alpha_Z M'_0^2)/\alpha_Z & K M'_0 \\
0 & -K M'_0 & -i k^2
\end{pmatrix}.
\]

Thus the 3 \times 3 mixing matrix is equal to the inverse of

\[
\begin{pmatrix}
i (k^2 - \alpha_A \Pi_{AA})/\alpha_A & -i\Pi_{AZ} & K\Pi_{A\phi}/M'_0 \\
-i\Pi_{AZ} & i (k^2 - \alpha_Z M'_0^2 - \alpha_Z \Pi_{ZZ})/\alpha_Z & K(\Pi_{Z\phi} + M'_0^2)/M'_0 \\
-K\Pi_{A\phi}/M'_0 & -K(\Pi_{Z\phi} + M'_0^2)/M'_0 & -i(k^2 - \Pi_{\phi\phi^0})
\end{pmatrix}. \quad (E.10)
\]
Imposing (E.3b), we get

\[ \Pi_{AZ} \left( 1 - \frac{\Pi_{\phi^0\phi^0}}{k^2} \right) = \Pi_{A\phi} \left( 1 + \frac{\Pi_{Z\phi}}{M_0^2} \right). \]  \hspace{1cm} (E.11a)

By requiring \( D_{AA} = 1/k^2 \) and making use of (E.11a), we get

\[ \frac{\Pi_{AA}}{M_0^2} \left( 1 - \frac{\Pi_{\phi^0\phi^0}}{k^2} \right) = \left( \frac{\Pi_{A\phi}}{M_0^2} \right)^2, \]  \hspace{1cm} (E.11b)

By requiring \( D_{AA} = D_{ZZ} \), we get

\[ \left( 1 + \frac{\Pi_{ZZ}}{M_0^2} \right) \left( 1 - \frac{\Pi_{\phi^0\phi^0}}{k^2} \right) = \left( 1 + \frac{\Pi_{Z\phi}}{M_0^2} \right)^2. \]  \hspace{1cm} (E.11c)

The equations in (E.11) are the same as those in (3.3). Thus the relations among the 1PI amplitudes in the pure alpha gauge defined by the gauge fixing terms of (1.3) are the same as those in the alpha gauge defined by the gauge fixing terms of (1.2) and (1.3).

With (E.11), we find that the determinant of the matrix in (E.10) is equal to

\[ \frac{i(k^2)^3}{\alpha_A\alpha_Z} \left( 1 - \frac{\Pi_{\phi^0\phi^0}}{k^2} \right). \]  \hspace{1cm} (E.12)

The expressions for the propagators are listed in (4.2) and (4.3).