Completion of the proof of the Geometrization Conjecture

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This paper builds upon and is an extension of [13]. In this paper, we complete a proof of the following:

Geometrization Conjecture: Any closed, orientable, prime 3-manifold $M$ contains a disjoint union of embedded 2-tori and Klein bottles such that each connected component of the complement admits a locally homogeneous Riemannian metric of finite volume.

Recall that a Riemannian manifold is homogeneous if its isometry group acts transitively on the underlying manifold; a locally homogeneous Riemannian manifold is the quotient of a homogeneous Riemannian manifold by a discrete group of isometries acting freely. Recall also that a prime 3-manifold is one which is not diffeomorphic to $S^3$ and which is not a connected sum of two manifolds neither of which is diffeomorphic to $S^3$. It is a classic result in 3-manifold topology, see [12] that every 3-manifold is a connected sum of a finite number of prime 3-manifolds, and this decomposition is unique up to the order of the factors.

The main part of this paper is devoted to giving a proof of Theorem 7.4 stated in [19] on locally volume collapsed 3-manifolds with curvature bounded from below which is the last step in the proof of the Geometrization Conjecture. In the introduction we will summarize the major results on Ricci flow contained in [13] and also briefly discuss results on Ricci flow in dimension 3 beyond those contained in [13] that are needed in proving the Geometrization conjecture.

The Geometrization Conjecture was proposed by W. Thurston in early 1980’s. It includes the Poincaré conjecture as a special case. Thurston himself solved this conjecture for a large class of 3-manifolds, namely those containing an incompressible surface; i.e., an embedded surface of genus $\geq 1$ whose fundamental group injects into the fundamental group of the 3-manifold.

Thurston’s Geometrization Conjecture suggests the existence of especially nice metrics on 3-manifolds and consequently, a more analytic approach to the problem of classifying 3-manifolds. Richard Hamilton formalized one such approach in [7] by introducing the Ricci flow equation on the space of Riemannian metrics:

$$\frac{\partial g(t)}{\partial t} = -2\text{Ric}(g(t)),$$

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where $\text{Ric}(g(t))$ is the Ricci curvature of the metric $g(t)$. In dimension 3, the fixed points (up to rescaling) of this equation are the Riemannian metrics of constant Ricci curvature. Beginning with any Riemannian manifold $(M, g_0)$, there is a solution $g(t)$ of this Ricci flow on $M$ for $t$ in some interval such that $g(0) = g_0$. It is hoped that if $M$ is a closed 3-manifold, then $g(t)$ exists for all $t > 0$ after appropriate scaling and converges to a nice metric outside a part with well-understood topology.

In [8], R. Hamilton showed that if the Ricci flow exists for all time and if there is an appropriate curvature bound together with another geometric bound, then as $t \to \infty$, after rescaling to have a fixed diameter, the metric converges to a metric of constant negative curvature. However, the general situation is much more complicated to formulate and much more difficult to establish. There are many technical issues with this program: One knows that in general the Ricci flow will develop singularities in finite time, and thus a method for analyzing these singularities and continuing the flow past them must be found. Furthermore, even if the flow continues for all time, there remain complicated issues about the nature of the limiting object at time $t = \infty$. For instance, if the topology of $M$ is sufficiently complicated, say it is a non-trivial connected sum, then no matter what the initial metric is one must encounter finite-time singularities, forced by the topology. More seriously, even if $M$ has simple topology, beginning with an arbitrary metric, one expects to (and cannot rule out the possibility that one will) encounter finite-time singularities in the Ricci flow. These singularities may occur along proper subsets of the manifold, not the entire manifold. Thus, one is led to study a more general evolution process called Ricci flow with surgery, first introduced by Hamilton in the context of four-manifolds, [9]. This evolution process is still parameterized by an interval in time, so that for each $t$ in the interval of definition there is a compact Riemannian 3-manifold $M_t$. But there is a discrete set of times at which the manifolds and metrics undergo topological and metric discontinuities (surgeries). In each of the complementary intervals to the singular times, the evolution is the usual Ricci flow, though, because of the surgeries, the topological type of the manifold $M_t$ changes as $t$ moves from one complementary interval to the next. From an analytic point of view, the surgeries at the discontinuity times are introduced in order to ‘cut away’ a neighborhood of the singularities as they develop and insert by hand, in place of the ‘cut away’ regions, geometrically nice regions. This allows one to continue the Ricci flow (or more precisely, restart the Ricci flow with the new metric constructed at the discontinuity time). Of course, the surgery process also changes the topology. To be able to say anything useful topologically about such a process, one needs results about Ricci flow, and one also needs to control both the topology and the geometry of the surgery process at the singular times. For example, it is crucial for the topological applications that we do surgery along 2-spheres rather than surfaces of higher genus. Surgery along 2-spheres produces the connected sum decomposition, which is well-understood topologically, while, for example, Dehn surgeries along tori can completely destroy the topology, changing any 3-manifold into any other.

The change in topology turns out to be completely understandable and amazingly, the surgery processes produce exactly the topological operations needed to cut the
manifold into pieces on which the Ricci flow can produce the metrics sufficiently controlled so that the topology can be recognized.

Following Perelman in [19], we gave a detailed proof in [13] of the following long-time existence result for Ricci flow with surgery:

**Theorem 0.1.** Let \((M, g_0)\) be a closed Riemannian 3-manifold. Suppose that there is no embedded, locally separating \(P^2\) contained\(^1\) in \(M\). Then there is a Ricci flow with surgery, say \((M_t, g(t))\), defined for all \(t \in [0, \infty)\) with initial metric \((M, g_0)\).

The set of discontinuity times for this Ricci flow with surgery is a discrete subset of \([0, \infty)\). The topological change in the 3-manifold as one crosses a surgery time is a connected sum decomposition together with removal of connected components, each of which is diffeomorphic to one of \(S^2 \times S^1\), \(RP^3#RP^3\), the non-orientable 2-sphere bundle over \(S^1\), or a manifold admitting a metric of constant positive curvature. Furthermore, there are two decreasing functions \(r(t) > 0\) and \(\kappa(t) > 0\) such that (1) \((M_t, g(t))\) is \(\kappa(t)\)-non-collapsed (see [13] Definition 9.1) and (2) any point \(x \in M_t\) with \(R(g(t)) \geq r(t)^{-2}\) satisfies the so called strong canonical neighborhood assumption (see [13] Definition 9.78 and Theorem 15.9).

Theorem 0.1 is central for all applications of Ricci flow to the topology of three-dimensional manifolds. The book [13] dealt with the case that \(M_t = \emptyset\) for \(t\) sufficiently large, that is, the case when the Ricci flow with surgery becomes extinct at finite time. Then it follows from the above theorem that the initial manifold \(M\) is diffeomorphic to a connected sum decomposition together with removal of connected components, each of which is diffeomorphic to one of \(S^2 \times S^1\), \(RP^3#RP^3\), the non-orientable 2-sphere bundle over \(S^1\), or a manifold admitting a metric of constant positive curvature. It was shown in [13] that if \(M\) is a simply-connected 3-manifold, then for any initial metric \(g_0\) the corresponding Ricci flow with surgery becomes extinct at finite time, see also ([20] and [5]). Consequently, \(M\) is diffeomorphic to \(S^3\), thus proving the Poincaré conjecture.

If \(g_0\) has positive scalar curvature, then the Ricci flow with surgery \(g(t)\) becomes extinct in time at most \(\frac{3}{2a}\) where \(a\) is a positive lower bound of the scalar curvature of \(g_0\). This follows from the maximum principle and the induced scalar curvature evolution equation for Ricci flow. By the above theorem, we see that \(M\) in this case is diffeomorphic to a connected sum of copies of 2-sphere bundles over \(S^1\) and metric quotients of the round \(S^3\). If the scalar curvature is only nonnegative, then by the strong maximum principle it instantly becomes positive unless the metric is (Ricci-)flat; thus in this case, \(M\) is a flat manifold.

However, if the scalar curvature is negative somewhere, then one needs to analyze the asymptotic behavior of \(g(t)\) as \(t\) goes to \(\infty\). For this purpose, one first examines when the sectional curvature can be bounded at \(t = \infty\). This was given in Section 6 of Perelman’s second paper [19] and more details can be found in [11], [14].

Roughly speaking, Perelman showed that for any \(w > 0\), there are \(\tau = \tau(w) > 0\), \(K = K(w) < \infty\), \(\eta = \eta(w) > 0\) and \(\theta(w) > 0\) such that if the ball \(B(x, t, r)\) of \((M_t, g(t))\) centered at \(x\) and of radius \(r\) has its sectional curvature bounded from

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\(^1\)That is, no embedded \(P^2\) in \(M\) with trivial normal bundle. Clearly, all orientable manifolds satisfy this condition.
below by $-r^{-2}$, where $\theta^{-1}(w)h(t/2) \leq r \leq \tau \sqrt{t}$ (Here $h(s)$ is the surgery scale at time $s$, see §4.4 of [19] or §15.3 of [13],) and its volume is bounded from below by $wr^3$, then its scalar curvature is bounded by $Kr^{-2}$ on $B(x,t,r/4) \times [t - \tau r^2, t]$ (cf. Corollary 6.8, [19]). Using this and the strong canonical neighborhood assumption one can conclude that given $w > 0$ for all $t$ sufficiently large if $B(x,t,r)$ has volume $\geq wr^3$ and sectional curvatures bounded below by $-r^{-2}$, then its scalar curvature is bounded by $Kr^{-2}$ on $B(x,t,r/4) \times [t - \tau r^2, t]$.

By the above discussion, one may assume that our initial manifold does not admit a metric with nonnegative scalar curvature, and that once we get a component with nonnegative scalar curvature, it is immediately removed. Hence, we can assume that the scalar curvature of $g(t)$ is negative somewhere on each component and each time $t$. To see what the limit can be as $t \to \infty$, using the above curvature estimates Perelman adapted the arguments of R. Hamilton in [9] to the Ricci flow with surgery $g(t)$. For the readers’ convenience, we outline the arguments following [19] (For more details, see [11], [14]).

Recall that for the Ricci flow with surgery $g(t)$, one still has the evolution equation on its scalar curvature $\mathcal{R}$

$$\frac{d\mathcal{R}}{dt} = \Delta \mathcal{R} + 2|\text{Ric}^0|^2 + \frac{2}{3} \mathcal{R}^2,$$  

where $\text{Ric}^0$ is the trace-free part of $\text{Ric}$. Let $R_{\min}(t)$ be the minimum of the scalar curvature $\mathcal{R}(g(t))$ of $g(t)$. Then by the usual (scalar) maximum principle we have

$$\frac{dR_{\min}}{dt} \geq \frac{2}{3} R_{\min}^2.$$  

It follows that

$$R_{\min}(t) \geq -\frac{3}{2} \left( \frac{1}{t + 1/4} \right).$$  

Let $V$ be the volume, then

$$\frac{dV}{dt} \leq -R_{\min} V.$$  

It follows that the function $V(t)(t + 1/4)^{-\frac{3}{2}}$ is non-increasing in $t$. Denote by $\mathcal{V}$ its limit as $t \to \infty$. Put $\hat{R} = R_{\min} V^{\frac{1}{2}}$. It is a scale invariant and satisfies

$$\frac{d\hat{R}}{dt} \geq \frac{2}{3} \hat{R} V^{-1} \int (R_{\min} - R) dV.$$  

The right-handed side is nonnegative since $R_{\min} \leq 0$ on each component. So $\hat{R}(t)$ has a unique limit, say $\hat{R}$, as $t \to \infty$.

If $\mathcal{V} > 0$, then it follows from Equations (0.3) and (0.4) that $R_{\min}(t)$ is asymptotic to $-3/2t$, that is, $\hat{R} \mathcal{V}^{-\frac{3}{2}} = -\frac{3}{2}$. Now Inequality (0.5) implies that whenever we have
a sequence of parabolic balls \( P(x_a, t_a, r\sqrt{t_a}, -r^2 t_a) \) for some fixed small \( r > 0 \), such that the scalings of \( g(t) \) by factor \( t_a^{-1} \) converge to some limit flow in the smooth topology, defined in a certain parabolic ball \( P(\tau, 1, r, -r^2) \), then the scalar curvature of this limit flow is independent of the space variables and equals \( -\frac{\Delta t}{t} \) at time \( t \in [1 - r^2, 1] \). Moreover, applying the strong maximum principle to Equation (1.1), one can easily show that the sectional curvature of the limit at time \( t \) is constant and equals \( -\frac{1}{\Delta t} \). If \( \nabla = 0 \), then no such a sequence of parabolic balls can exist, so the above conclusion is automatically valid. Furthermore, using curvature estimates from Section 6 in [19], Perelman showed a more effective estimate on how close \( g(t) \) is to a hyperbolic metric: Given \( w > 0 \), \( r > 0 \), \( \xi > 0 \), one can find \( T = T(w, r, \xi) < \infty \) such that if the ball \( B(x, t, r\sqrt{t}) \) at some time \( t \geq T \) has volume at least \( wr^3t^{3/2} \) and sectional curvature at least \( -r^{-2}t^{-1} \), then Ricci curvature satisfies

\[
|2t \text{Ric}(x, t) + g(x, t)|_{g(x, t)} < \xi. \tag{0.6}
\]

If one allows \( T \) also to depend on \( A \in (1, \infty) \), one can even have the above inequality for all points in the parabolic ball \( P(x, t, Ar\sqrt{t}, -Ar^2t) \).

Now one can introduce the thick-thin decomposition of \( M_t \) for \( t \) sufficiently large. Let \( \rho(x, t) \) denote the radius \( \rho \) of the ball \( B(x, t, \rho) \) where \( \inf \text{Rm} = -\rho^{-2} \). Fix \( w > 0 \) a small positive constant. Let \( M_-(w; t) \) denote the thin part of \( M_t \) consisting of all \( x \in M_t \) satisfying:

\[ \text{Vol}(B(x_0, t, \rho(x, t))) \leq w\rho(x, t)^3. \]

Let \( M_+(w; t) \) be its complement.

As Perelman pointed out in Section 7.3 of [19], using the curvature pinching inequality along \( g(t) \) and curvature estimates from Section 6 in [19], one can show: For any \( w > 0 \), there are \( \tau = \tau(w) \) and \( \tau = \tau(w) \) such that if \( t \geq \tau \) and \( \rho(x, t) < \tau \sqrt{t} \), then

\[ \text{vol}(B(x, t, \rho(x, t))) < w\rho(x, t)^3. \tag{0.7} \]

It follows that for any \( w > 0 \), if \( x \in M_+(w; t) \) and \( t \) is sufficiently large, then

\[ \rho(x, t) \geq \tau \sqrt{t}. \tag{0.8} \]

Then for any given \( w \) and \( \xi \), for \( t \) sufficiently large, every point \( (x, t) \in M_+(w; t) \) satisfies the estimates for curvature in Inequality (0.6). Using Inequalities (0.6) and (0.8), one can show that if \( \{x_a \in M_+(w; t_0)\} \) is a sequence of points with \( t_0 \to \infty \), then based manifolds \((M_{t_0}, t_0^{-1}g(t_0), x_a)\) converge, along a subsequence of \( a \to \infty \), to a complete hyperbolic manifold of finite volume. The limit may depend on choices of \((x_a, t_0)\). If one of these limits is closed, then \( M_{t_0} \) is diffeomorphic to this limit when \( a \) is sufficiently large and \( t_0^{-1}g(t_0) \) converges to a hyperbolic metric as \( a \to \infty \). Then using the rigidity of hyperbolic metrics, one can further show that \((M_t, t^{-1}g(t))\) converges to the same hyperbolic manifold. So one may assume that none of the limits is closed, let \( H_1 \) is such a limit with the least number of cusps. Define \( H_1(w') \)

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\( P(x_0, t_0, r_0, -\Delta t) \) means the product of the ball \( B(x_0, t_0, r_0) \) of radius \( r_0 \) centered at \( x_0 \) in the metric \( g(t_0) \) with the time interval \([t_0 - \Delta t, t_0]\). Implicitly, we assume that this ball exists for all times \( t \in [t_0 - \Delta t, t_0] \).
to be the set of points in $H_1$ where the injectivity radius is not less than $w'$. Then by using an argument in [9] based on hyperbolic rigidity, Perelman showed that for all sufficiently small $w$ and for $t$ large enough, $M_+(w/2, t)$ contains an almost isometric copy of $\varphi_t: H_1(w) \to M_+(w/2, t)$, which in turn contains a component of $M_+(w, t)$; Moreover, this embedded copy $\varphi_t(H_1(w))$ moves by isotopy as $t \to \infty$. If for some $w > 0$ the complement $M_+(w, t) \backslash \varphi_t(H_1(w))$ is not empty for a sequence of $t \to \infty$, then one can repeat the above arguments and get other complete hyperbolic manifolds $H_2, \ldots, H_k$. Since each hyperbolic 3-manifold with finite volume has a uniform lower bound, one can conclude that for $k$ sufficiently large and each sufficiently small $w > 0$, the images of $\varphi_t(H_1(w)), \ldots, \varphi_t(H_k(w))$ in $M_t$ cover $M_+(w, t)$ for all sufficiently large $t$. Clearly, each boundary component of $H_j(w)$ is a torus. In fact, those boundary tori of $\varphi_t(H_j(w))$ are incompressible in $M_t$. The proof is identical to that of R. Hamilton in [9] and is a minimal surface argument, using a result of Meeks and Yau. An alternative proof for this incompressibility was given in [11] by using volume comparison. This later proof is simpler and more elementary.

For $t$ sufficiently large, we define

$$\tilde{M}_-(w, t) = M_t \backslash \varphi_t(H_1(w)) \cup \cdots \cup \varphi_t(H_k(w)).$$

It is a compact, codimension-0 submanifold of $M_-(w, t)$ with incompressible tori as boundary components. Furthermore, if $w$ is sufficiently small and $t$ is sufficiently large, each boundary component of $(M_-(w, t), t^{-1} g(t))$ is convex, has diameter at most $w$ and has a (topologically trivial) collar of length one, where the sectional curvatures are between $-1/4 - \epsilon$ and $-1/4 + \epsilon$, where $\epsilon > 0$ can be as small as one wants so long as $t$ is large enough. For sufficiently small $w > 0$ and sufficiently large $t$, $\tilde{M}_-(w, t)$ is actually diffeomorphic to a graph manifold, and consequently satisfies the Geometrization Conjecture\(^3\) as does $M_t$. The fact that, for $t$ sufficiently large and $w$ sufficiently small, the $\tilde{M}_-(w, t)$ are graph manifolds is a consequence of the following theorem on collapsing with local lower curvature bound, applied to the $(M_-(w, t), t^{-1} g(t))$. The goal of this paper is to give a self-contained, complete proof of this theorem, which is a slight rewording\(^4\) of the result stated (without proof) as Theorem 7.4 of [20].

**Theorem 0.2.** Suppose that $(M_n, g_n)$ is a sequence of compact, oriented Riemannian 3-manifolds, closed or with convex boundary, and that $w_n$ is a sequence of positive numbers tending to zero as $n$ tends to $\infty$. Assume that:

1. For each point $x \in M_n$ there exists a radius $\rho = \rho_n(x)$ such that the ball $B_{g_n}(x, \rho)$ has volume at most $w_n \rho^3$ and all the sectional curvatures of the restriction of $g_n$ to this ball are all at least $-\rho^{-2}$;

2. Each component of the boundary of $M_n$ is an incompressible torus of diameter at most $w_n$ and with a topologically trivial collar containing the all points within

\(^3\)The definition of a graph manifold and a discussion of the fact that graph manifolds satisfy the Geometrization Conjecture are both given in the next section.

\(^4\)The difference is that we have not restricted $\rho(x)$ to be less than the diameter of the manifold. This is taken care of by the argument in Section 1.4.2.
distance 1 of the boundary on which the sectional curvatures are between $-5/16$ and $-3/16$;

3. For every $w' > 0$ there exist $\tau = \tau(w') > 0$ and constants $K_m = K_m(w') < \infty$ for $m = 0, 1, 2, \ldots$, such that for all $n$ sufficiently large, and any $0 < r \leq \tau$, if the ball $B_{g_n}(x, r)$ has volume at least $w'r^3$ and sectional curvatures at least $-r^{-2}$, then the curvature and its $m$th-order covariant derivatives at $x$, $m = 1, 2, \ldots$, are bounded by $K_0r^{-2}$ and $K_mr^{-m-2}$, respectively.

Then for every $n$ sufficiently large, $M_n$ is a graph manifold. If, in addition, $M_n$ is aspherical, there is a finite collection $T_n$ of disjoint copies of $T^2 \times I$ and twisted $I$-bundles over the Klein bottle in $M_n$ such that each component of $M_n \setminus \text{int} T_n$ is a Seifert fibered 3-manifold with incompressible boundary. It follows that each component of $M_n \setminus T_n$ admits a locally homogeneous geometry of finite volume.

From the above discussions we see that, given any $w > 0$ for all $t$ sufficiently large the manifolds $\tilde{M}_-(w, t)$ satisfy the hypotheses of Theorem 0.2. Thus, applying this theorem, we see that for all $w$ sufficiently small and for all $t$ sufficiently large, the $\tilde{M}_-(w, t)$ are graph manifolds with incompressible boundary. Thus, fixing $w$ sufficiently small and fixing $t$ sufficiently large, we have the following: There is a decomposition $M_t = M_+(w, t) \cup \tilde{M}_-(w, t)$ where $M_+(w, t)$ and $\tilde{M}_-(w, t)$ are compact codimension-0 submanifolds meeting along their boundaries, these boundaries being incompressible tori. Furthermore, the interior of $M_+(w, t)$ is diffeomorphic to a complete hyperbolic manifold of finite volume and $\tilde{M}_-(w, t)$ is a graph manifold with incompressible boundary. Since each connected component of $M_t$ is either prime or diffeomorphic to the 3-sphere, the same is true for $\tilde{M}_-(w, t)$. It is known that every connected, orientable, prime graph manifold with incompressible torus boundary either is diffeomorphic to one of $T^2 \times I$ or a twisted $I$-bundle over the Klein bottle, or can be decomposed by a finite collection of disjoint incompressible tori and Klein bottles into manifolds which admit complete, locally homogeneous Riemannian metrics. It follows immediately that the same statement is true for $M_t$.

This completes the outline of the proof of the Geometrization Conjecture.

At the beginning of this introduction we stated that in this paper we are extending and building upon the work of [13] in order to present a complete proof the Geometrization Conjecture. Let us clarify that statement. What we present here in detail is a proof of the theorem about locally volume collapsed 3-manifolds (Theorem 7.4 of [19]). In addition to that and the material in [13] one also needs the material in Section 6 and in the first part of section 7 of [19]. In this paper we have been content to outline the main results from this material. It is our plan to combine this manuscript with an exposition of the remaining material from [19] into a sequel to [13]. Together these two books will give an entirely self-contained proof of the Geometrization Conjecture using Ricci flow and Alexandrov spaces.

There are other approaches to the Geometrization Conjecture which use variations of Theorem 7.4. As was indicated above, if a 3-manifold $M$ admits an incompressible torus, then it falls into the class of 3-manifolds for which the Geometrization Conjecture had been established by Thurston himself. A detailed proof of the Ge-
ometrization Conjecture for those 3-manifolds was given in [16] and [17]. In view of this, it suffices to prove Theorem 7.4 for closed manifolds (again appealing to the Ricci flow results from [18] and the material in [19] preceding Theorem 7.4). A version of Theorem 7.4 for closed 3-manifolds has been proved in a series of papers of Shioya-Yamaguchi ([22], [23]). They did not make use of Assumption 3 on bounds on derivatives of curvature. So their result can be applied to 3-manifolds which do not necessarily arise from Ricci flow. However, because they are not relying on Assumption 3, to prove their result, Shioya-Yamaguchi need to use a hard stability theorem on Alexandrov spaces with generalized curvature bounded from below. This stability theorem was due to Perelman and its proof was given in an unpublished manuscript in 1993. Recently, V. Kapovitch posted a preprint, [10], which proposes a more readable proof for this stability theorem of Perelman. Putting all these together, one has a Perelman-Shioya-Yamaguchi-Kapovitch proof of Theorem 7.4 for closed manifolds without Assumption 3. As we have indicated, this proof requires a much deeper knowledge on Alexandrov spaces than the proof we present. Our presentation of the collapsing space theory is motivated by, and to a large extent follows, the Shioya-Yamaguchi paper.

There is another approach to the proof of the Geometrization Conjecture due to Besson etc. [2] which avoids using Theorem 7.4. Again, this result relies on Thurston’s theorem that 3-manifolds with incompressible surfaces satisfy the Geometrization Conjecture, so that, as in the previous approach, one only needs to consider the case when the entire closed 3-manifold is collapsed. Rather than appealing to the theory of Alexandrov spaces, this approach relies on other deep works on geometry and topology, e.g., results on the Gromov norms of 3-manifolds.

Thus, all other approaches rely on Thurston’s result on geometrization of 3-manifolds containing an incompressible surface. The proof of this result uses completely different techniques than Ricci flow and is highly non-trivial in its own right, relying as it does on delicate and powerful results from hyperbolic geometry. For this reason, we feel that it is worthwhile to have a self-contained argument based on Ricci flow with surgery not making use of Thurston’s results on hyperbolic manifolds and the hard stability theorem of Alexandrov spaces.

One can find other, related works on Ricci flow and the Geometrization Conjecture on www.claymath.org and in the long introduction of our previous book [13]. This paper will eventually be a part of our book project [14] on the Geometrization Conjecture.

1 The Collapsing Theorem: First remarks

From now on in this article 3-manifolds are implicitly assumed to be orientable. Recall that a Seifert fibration structure on a compact 3-manifold is a locally-free circle action on a 2-sheeted covering $\tilde{M}$ of $M$ such that, denoting the covering structure...
transformation on $\tilde{M}$ by $\tau$, we have $\tau(\zeta \cdot x) = \overline{\zeta} \cdot x$ for all $x \in \tilde{M}$ and all $\zeta \in S^1$. Seifert fibration structures are classified in terms of their base orbifolds, local Seifert invariants, and, when the base is closed, an ‘Euler class,’ see [21] or [15]. A compact 3-manifold is said to be Seifert fibered if it admits a Seifert fibration structure. A compact, connected, Seifert fibered 3-manifold is either diffeomorphic to a solid torus or has boundary consisting of incompressible tori. Furthermore, the interior of any compact, connected, Seifert fibered 3-manifold not diffeomorphic to a solid torus, to $T^2 \times I$, or to a twisted $I$-bundle over a Klein bottle admits a complete, locally homogeneous Riemann metric of finite volume. A graph manifold is a compact 3-manifold that is a connected sum of manifolds each of which is either diffeomorphic to the solid torus or can be cut apart along a finite collection of incompressible tori into Seifert fibered 3-manifolds. Thus, a graph manifold with incompressible boundary satisfies Thurston’s geometrization conjecture. One result we need is that the union along boundary tori of the total space of a locally trivial $S^1$-fibration and a collection of solid tori is a graph manifold, see [25]. Furthermore, if the fiber of the $S^1$-fibration is homotopically essential in each of the solid tori, then the result is a Seifert fibered 3-manifold.

For the rest of this paper we fix the constants $\tau(w')$ and the $K_m(w')$, $m = 0, 1, \ldots$ as in the statement of Theorem 0.2.

### 1.1 Outline of the Proof

According to Theorem 1.17 in Section 1.6 of [1], a closed, connected 3-manifold admitting a flat metric is Seifert fibered and hence is a graph manifold. If a closed, orientable 3-manifold has a metric of non-negative sectional curvature then by [7] it is diffeomorphic to one of the following:

1. a spherical 3-dimensional space-form,
2. a manifold double covered by $S^2 \times S^1$, or
3. a flat 3-manifold.

Thus, without loss of generality we can make the following assumption.

**Assumption 1.** For each $n$, no connected, closed component of $M_n$ admits a Riemann metric of non-negative sectional curvature.

The idea of the proof is to consider a sequence of balls of the form $B_{g_n'(x_n)}(x_n, 1) \subset M_n$, $n = 1, 2, \ldots$, where by definition $g_n'(x_n) = \rho_n(x_n)^{-2}g_n$. The hypotheses of the theorem and Assumption 1 imply that each of these balls is non-compact, but locally complete and of sectional curvature $\geq -1$. The general theory of Alexandrov spaces implies that given any such sequence there is a subsequence that converges in the sense of Gromov-Hausdorff to a ball of radius one in an Alexandrov space of curvature $\geq -1$ and of dimension at least 1 and at most 3. The volume condition implies that the limit is a 1- or 2-dimensional. We then use results on the structure of Alexandrov spaces of dimension 1 and 2 to deduce strong information about the structure of these balls in $M_n$ for all $n$ sufficiently large. These local structures can
then be pieced together to form a global result, proving the theorem stated above. In Section 2 we recall the general theory on Alexandrov spaces, and in Section 3 we give a more detailed analysis of 1- and 2-dimensional Alexandrov spaces that is necessary to prove this result. In this introduction we assume that these basic notions are understood and we formulate the precise structural results that will be proved. In Section 4 we deduce the local results, i.e., the possible structures of the balls $B_{\varrho/(x)}(x,1)$, and in Section 5 we piece the local results together proving Theorem 1.1 below.

1.2 The collapsing theorem

Let us now state the topological theorem that is established using the compactness of Alexandrov spaces of curvature $\geq -1$ and the volume collapsing hypotheses.

**Theorem 1.1.** Suppose that we have a sequence of compact 3-manifolds satisfying the hypothesis of Theorem 0.2 and satisfying Assumption 1. Then, for every $n$ sufficiently large there are compact, codimension-0, smooth submanifolds $V_{n,1} \subset M_n$ and $V_{n,2} \subset M_n$ with $\partial M_n \subset V_{n,1}$ satisfying the following.

1. Each connected component of $V_{n,1}$ is diffeomorphic to one of the following:
   - a $T^2$-bundle over $S^1$ or a union of two twisted $I$-bundles over the Klein bottle along their common boundary;
   - $T^2 \times I$ or $S^2 \times I$, where $I$ is a closed interval;
   - a compact 3-ball or the complement of an open 3-ball in $\mathbb{R}P^3$;
   - a twisted $I$-bundle over the Klein bottle; or a solid torus.

   In particular, every boundary component of $V_{n,1}$ is either a 2-sphere or a 2-torus.

2. $V_{n,2} \cap V_{n,1} = \partial V_{n,2} \cap \partial V_{n,1}$.

3. If $X_0$ is a 2-torus component of $\partial V_{n,1}$, then $X_0 \subset \partial V_{n,2}$ if and only if $X_0$ is not a boundary component of $M_n$.

4. If $X_0$ is a 2-sphere component of $\partial V_{n,1}$, then $X_0 \cap \partial V_{n,2}$ is diffeomorphic to an annulus.

5. $V_{n,2}$ is the total space of a locally trivial $S^1$-bundle and $\partial V_{n,1} \cap \partial V_{n,2}$ is saturated under this fibration.

6. $M_n \setminus \text{int} (V_{n,2} \cup V_{n,1})$ is a disjoint union of solid tori and solid cylinders, i.e., copies of $D^2 \times I$. The boundary of each solid torus is a boundary component of $V_{n,2}$, and each solid cylinder $D^2 \times I$ meets $V_{n,1}$ exactly in $D^2 \times \partial I$. 
1.3 Proof that Theorem 1.1 implies Theorem 0.2

In deducing Theorem 0.2 from Theorem 1.1 we shall introduce several topological simplifications in the decomposition given in the conclusion of Theorem 1.1. While the decomposition given in Theorem 1.1 is deduced from the collapsing theory (in particular, $V_{n,1}$ is the part of $M_n$ close to a 1-dimensional space and $V_{n,2}$ is the part close to a 2-dimensional space), as we modify the decomposition we work purely topologically and do not try to keep the connection with the collapsing geometry.

**Claim 1.2.** It suffices to establish Theorem 0.2 under the assumption that we have a decomposition as given in Theorem 1.1 that satisfies the following additional properties:

1. $V_{n,1}$ has no closed components.
2. Each 2-sphere component of $\partial V_{n,1}$ bounds a 3-ball component of $V_{n,1}$.
3. Each 2-torus component of $\partial V_{n,1}$ that is compressible in $M_n$ bounds a solid torus component of $V_{n,1}$.

**Proof.** By assumption, each closed component of $V_{n,1}$ can be decomposed along a single incompressible $T^2$ into Seifert fibered manifolds, and hence these satisfy the conclusion of Theorem 0.2. Thus, without loss of generality we can assume that there are no closed components of $V_{n,1}$. In the similar way, we can suppose that no component of $M_n$ is the union of two solid tori, the union of a solid torus and a twisted $I$-bundle over the Klein bottle, or the union of two twisted $I$-bundles over the Klein bottle along a common boundary torus, since manifolds of the first two types admit Riemannian metrics of non-negative sectional curvature and those of the third decompose along an incompressible torus into pieces that are Seifert fibered.

Let $C$ be a 2-sphere component of $\partial V_{n,1}$. If $C$ bounds a component $\hat{C}$ of $V_{n,1}$ diffeomorphic to $\mathbb{R}P^3 \setminus B^3$, then we remove $\hat{C}$ from $M_n$ and from $V_{n,1}$ and replace it in each with a 3-ball in each. This has the effect of removing a prime factor diffeomorphic to $\mathbb{R}P^3$ from $M_n$. This allows us to assume that there are no components of $V_{n,1}$ diffeomorphic to $\mathbb{R}P^3 \setminus B^3$ and hence that the only components of $V_{n,1}$ with boundary 2-spheres are either 3-balls or diffeomorphic to $S^2 \times I$.

Now let $C$ be a 2-sphere component of $\partial V_{n,1}$, but not bounding a 3-ball component of $V_{n,1}$. We cut $M_n$ open along $C$ and cap off the resulting two copies of $C$ with 3-balls. We add these balls to $V_{n,1}$ forming $V'_{n,1}$, and we leave $V_{n,2}$ unchanged. The resulting subsets $V'_{n,1}$ and $V_{n,2}$ satisfy all the conclusions of Theorem 1.1. If we can show that the result is a graph manifold, then the same is true for $M_n$. Induction then allows us to assume that every $S^2$-boundary component of $V_{n,1}$ bounds a 3-ball or diffeomorphic to $S^2 \times I$.

Next, we consider a 2-torus component $T$ of $\partial V_{n,1}$ that is a compressible 2-torus in $M_n$, but one that does not bound a solid torus component of $V_{n,1}$. By Dehn’s lemma there is an embedded disk in $M_n$ meeting $T$ only along its boundary, that intersection being homotopically non-trivial in $T$. First, suppose that $T$ separates $M_n$. We write $M_n = P \cup_T N$. A thickening of $T \cup D$ has a 2-sphere boundary
component $S$, which we can suppose lies in $P$. Let $R$ be the region between $T$ and $S$; it is diffeomorphic to the complement in a solid torus of a 3-ball. We form $A = P \cup_T F$ where $F$, is a solid torus, glued in such a way that $R \cup_T F$ is diffeomorphic to a 3-ball. We set $V_{n,2}(A) = V_{n,2} \cap P$ and $V_{n,1}(A) = (V_{n,1} \cap A) \cup F$. We also form $B = \tilde{R} \cup_T N$ where $\tilde{R}$ is the solid torus obtained from $R$ by attaching a 3-ball to its $S^2$-boundary. We set $V_{n,2}(B) = V_{n,2} \cap N$ and $V_{n,1}(B) = (V_{n,1} \cap N) \cup \tilde{R}$. It is easy to see that $M_n$ is diffeomorphic to $A \# B$ and that the given decompositions of $A$ and $B$ satisfy all the conclusions of Theorem 0.1 unless $T$ bounds a component of $V_{n,1}$ that is a twisted $I$-bundle over the Klein bottle. In this case, that component of $V_{n,1}$ is $N$ and $\tilde{R} \cup_T N$ is Seifert fibered, whereas the conclusions of Theorem 1.1 hold for $A$. Thus, by a straightforward induction argument, allows us to assume that every compressible 2-torus component of $\partial V_{n,1}$ that separates $M_n$ bounds a solid torus component of $V_{n,1}$. If $T$ does not separate $M_n$ we cut $M_n$ open along $T$, add a solid torus $F$ as before to the copy of $T$ bounding $R$ and add a copy of $\tilde{R}$ to the other copy of $T$. Then $M_n$ is diffeomorphic to the connected sum of the resulting manifold, $M'_n$, and $S^2 \times S^1$. Furthermore, adding $\tilde{R} \coprod F$ and to $V_{n,1}$ and leaving $V_{n,2}$ unchanged produces a new decomposition satisfying the hypotheses of Theorem 1.1. Again a simple induction argument shows that repeated application of this operation removes all non-separating compressing tori boundary components of $V_{n,1}$ without creating any new compressing tori boundary components that do not bound solid torus components of $V_{n,1}$. This completes the proof of the claim. \hfill $\Box$

With all these simplifying assumptions in place, we are ready to complete the proof that Theorem 0.1 implies Theorem 0.2. Let us consider the union, $X$, of the $D^2 \times I$ components of the closure of $M_n \setminus (V_{n,1} \cup V_{n,2})$ and the 3-ball components of $V_{n,1}$. Since every 2-sphere boundary component of $V_{n,1}$ bounds a 3-ball component of $V_{n,1}$, each $D^2 \times I$ meets the disjoint union of the 3-balls exactly in $D^2 \times \partial I$ and the boundary of each 3-ball contains exactly two disks in common with $\coprod D^2 \times \partial I$. It then follows from the fact that $M_n$ is orientable that $X$ is diffeomorphic to a disjoint union of a finite number of solid tori. Hence, the closure of $M_n \setminus V_{n,2}$ is a finite collection of solid tori, components diffeomorphic to $T^2 \times I$, and components diffeomorphic to twisted $I$-bundles over the Klein bottle. Furthermore, all boundary components of the $T^2 \times I$ and twisted $I$-bundles over the Klein bottle are incompressible in $M_n$. We remove from $M_n$ all components of $M_n \setminus V_{n,2}$ diffeomorphic to either $T^2 \times I$ or to a twisted $I$-bundle over the Klein bottle. The result, $W_n$, is a manifold that is the union of $V_{n,2}$ and a collection of solid tori glued in along boundary components. According to [25], since $V_{n,2}$ is an $S^1$-fibration, $W_n$ is a graph manifold. Since the tori boundary components that we cut along are incompressible, $\partial W_n$ consists of incompressible boundary tori. It follows that each prime factor of $W_n$ has the property that removing a disjoint union of submanifolds diffeomorphic to $T^2 \times I$ and twisted $I$-bundles over the Klein bottle results in an open manifold each component of which admits complete homogeneous metrics of finite volume. The same is then true of $M_n$.

If $M_n$ is aspherical, then it is not a non-trivial connected sum. Removing from $M_n$ copies of $T^2 \times I$ and twisted $I$-bundles over the Klein bottle yields a manifold
each component of which is aspherical. But an aspherical graph manifold with incompressible boundary has the property that taking out further copies of $T^2 \times I$ and twisted $I$-bundles over the Klein bottle results in a manifold each component of which is Seifert fibered with incompressible boundary. This completes the proof that Theorem 1.1 implies Theorem 0.2. The rest of this paper is devoted to the proof of Theorem 1.1.

1.4 First reductions in the proof of Theorem 1.1

1.4.1 A smooth limit result

As we have already indicated, the entire argument revolves around considering sequences \( \{x_n \in M_n\}_{n=1}^{\infty} \), rescaling the metrics \( g_n \), and, after passing to a subsequence, extracting a limit (usually a Gromov-Hausdorff limit) of the metric unit balls in the rescaled metrics. In general, a limit like this can be of dimension 1, 2, or 3 (although when we use \( \rho_n^{-2}(x_n) \) to rescale the limit, the volume collapsing hypothesis implies that the limit has dimension 1 or 2) and depending on which it is we get a different structure for balls. The easiest case to treat is when the limit is 3-dimensional. As the next theorem shows, because of the assumption on bounds on the curvature and its derivatives in the statement of Theorem 0.2 such limits are automatically smooth limits, rather than the more general Gromov-Hausdorff limits that occur in the other two cases.

**Proposition 1.3.** Let \((M_n, g_n)\) and \(w_n\) be as in the statement of Theorem 0.2. Suppose that we have a sequence of points \( x_n \in M_n \) such that \( B_n = B_{g_n}(x_n, \rho_n(x_n)) \) is disjoint from \( \partial M_n \) and a sequence of constants \( \lambda_n \) with a Gromov-Hausdorff limit of a subsequence of \((B_n, \lambda_n g_n, x_n)\), which is a 3-dimensional Alexandrov space. Then, passing to a further subsequence, there is a smooth limit of the \((B_n, \lambda_n g_n, x_n)\), which is a complete, non-compact manifold of non-negative curvature.

**Proof.** First step:

**Claim 1.4.** If \((B_n, \lambda_n g_n, x_n)\) converges to a 3-dimensional Alexandrov space, then there is a sequence of points \( y_n \in M_n \) converging to a point \( y \) in the limit and constants \( r > 0 \) and \( \kappa > 0 \) such that for all \( n \) sufficiently large \( \text{Vol} B_{\lambda_n g_n}(y_n, r) \geq \kappa r^3 \).

**Proof.** Fix \( \delta > 0 \) sufficiently small. Let \( X \) be the limiting 3-dimensional Alexandrov space. By Corollary 6.7 of [3] the subset \( R_\delta(X) \) consisting of points with a \((3, \delta)\)-strainer is dense. Choose \( y \in R_\delta(X) \) and let \( y_n \in M_n \) be a sequence converging to \( y \). Then there is a \((3, \delta)\)-strainer \( \{a_1, b_1, a_2, b_2, a_3, b_3\} \) at \( y \). Let \( d \) be the size of this strainer. Hence for all \( n \) sufficiently large, there is a \((3, \delta)\)-strainer of size \( d/2 \) at \( y_n \) in \( \lambda_n B_n \). According to Proposition 2.25 this means that for some \( r << d/2 \), but depending only on \( d \), there is an almost bilipschitz homeomorphism from \( B_{\lambda_n g_n}(y_n, r) \) to the ball of radius \( r \) in Euclidean space, where the error estimate goes to zero with \( \delta \). Hence, for any \( \epsilon > 0 \) sufficiently small, the cardinality of a maximal \( \epsilon \)-net in \( B_{\lambda_n g_n}(y_n, r) \) is at least \( \alpha \epsilon^{-3} r^3 \) for a universal constant \( \alpha > 0 \). If
we choose $\epsilon > 0$ sufficiently small then the volume in $\lambda_n g_n$ of any ball of radius $\epsilon/2$ centered at a point of $B_{\lambda_n g}(y_n, r)$ is at least $\omega_0(\epsilon/2)^3$ where $\omega_0$ is the volume of the unit ball in Euclidean 3-space. Hence, $\text{Vol} B_{\lambda_n g_n}(y_n, (r + \epsilon)) \geq \alpha \omega r^3/8$. Taking the limit as $\epsilon \to 0$ gives the uniform lower bound to the volume of the ball of radius $B_{\lambda_n g_n}(y_n, r)$.

**Second Step:** Suppose that $y_n \in B_n$ is as in the previous claim. Then, we see that the $(B_n, \lambda_n g_n)$ are uniformly volume non-collapsed at $y_n$. That is to say for some $r > 0$ and $w' > 0$, for all $n$ the volume of $B_{\lambda_n g_n}(y_n, r)$ is at least $w'r^3$. Since there is $\rho(y_n)$ such that the ball $B(y_n, \rho(y_n))$ has volume is at most $w_n \rho(y_n)^3$ where $w_n \to 0$ as $n \to \infty$, it follows from Bishop-Gromov volume comparison that

$$\rho(y_n) \sqrt{\lambda_n} \to \infty$$

as $n$ tends to infinity. Hence, for any $A < \infty$, for all $n$ sufficiently large, we have $A < \rho(y_n) \sqrt{\lambda_n}$. Thus, by our assumption, for all $n$ sufficiently large, the curvature of $\lambda_n g_n$ on $B_{\lambda_n g_n}(y_n, A) \geq -\lambda_n^{-1} \rho(y_n)^{-2} > -A^{-2}$. Also, for all $n$ sufficiently large, $A/\sqrt{\lambda_n} < \tau(w')$. Hence, for any $A < \infty$, for all $n$ sufficiently large, we have uniform bounds on the curvature and all its derivatives in $B_{\lambda_n g_n}(y_n, A)$. Since we also have uniform volume non-collapsing at the base point, we can pass to a subsequence, which has a smooth complete limit. Since $\rho(y_n)^{-2} \lambda_n^{-1}$ tends to zero, the curvature of the limit manifold is $\geq 0$. \hfill \square

This result about the 3-dimensional limits will be important as we study the 1- and 2-dimensional limits.

### 1.4.2 Adjusting $\rho_n$

**Lemma 1.5.** Let $M_n$, $w_n$ and $\rho_n$ satisfy the hypotheses of Theorem 0.2 and suppose that the $M_n$ satisfy Assumption 1. After passing to a subsequence, there are constants $w'_n$ and functions $\rho_n : M_n \to (0, \infty)$ satisfying the hypothesis of Theorem 0.2 such that in addition the following hold:

1. For any connected component $M_n^0$ of $M_n$ and for any $x \in M_n^0$ we have

   $$\rho_n(x) \leq \frac{1}{2} \text{diam } M_n^0,$$

   and

2. if $y \in B(x, \rho_n(x)/2)$ then $\rho_n(y)/2 \leq \rho_n(x) \leq 2 \rho_n(x)$.

**Proof.** Without loss of generality we can assume that $M_n$ is connected. If $M_n$ is closed, then by assumption it is not the case that $\text{Rm} \geq 0$ on all of $M_n$. If $M_n$ has non-empty boundary, then also by assumption $\text{Rm}$ is not everywhere positive. Thus, for each $x \in M_n$, there is a maximum $r = r_n(x) \geq \rho_n(x)$ such that the $\text{Rm} \geq -r^{-2}$ on $B(x, r)$. Furthermore, by volume comparison (the Bishop-Gromov theorem)

$$\text{vol } B(x, r) \leq \frac{V_{\text{hyp}}(1)}{V_{\text{Eucl}}(1)} w_n r^3,$$
where $V_{\text{hyp}}(1)$, resp. $V_{\text{Eucl}}(1)$, is the volume of the unit ball in hyperbolic, resp. Euclidean, 3-space. Thus, at the expense of changing the $w_n$ by a factor independent of $n$, we can define the function $\rho_n$ so that $\rho_n(x)$ is this maximum $r(x)$. It follows immediately that if $y \in B(x, \rho_n(x)/2)$ then

$$\frac{1}{2} \rho_n(x) \leq \rho_n(y) \leq 2 \rho_n(x).$$

If, for all $n$, we have $\rho_n(x) \leq \text{diam } M_n$ for all $x \in M_n$, then we simply replace $\rho_n(x)$ by $\rho_n(x)/2$ and $w_n$ by $8w_n$ and we have established the claim in this case.

Now suppose (after passing to a subsequence) that for each $n$ there is $x_n \in M_n$ with $\rho_n(x_n) > \text{diam } M_n$. This implies that $\text{Rm}(x) \geq -(\text{diam } M_n)^{-2}$ for all $x \in M_n$ and hence that $\rho_n$ is a constant function; we denote its value by $\rho_n$. Passing to a subsequence we can assume that $\text{vol}(M_n)/(\text{diam } M_n)^3$ tends to a limit (possibly $+\infty$) as $n \to \infty$. First, we consider the case when this limit is non-zero. The fact that the volume divided by the cube of the diameter is bounded away from zero and the volume inequality assumed in Theorem 0.2 imply that $\text{diam } M_n/\rho_n$ tends to 0 as $n \to \infty$. By the hypothesis about the boundary of $M_n$, this implies that $M_n$ is closed. Rescaling $M_n$ to make its diameter 1 yields a manifold whose sectional curvatures are bounded below by $-(\text{diam } M_n)^2/\rho_n^2$ and whose volume is bounded away from zero. By Proposition 1.3 we see that passing to a subsequence there is a smooth limit which has non-negative sectional curvature. This is contrary to Assumption 1. Thus, we can suppose that $\text{vol}(M_n)/(\text{diam } M_n)^3$ tends to zero as $n$ goes to infinity. We take $w'_n = 8\text{vol}(M_n)/(\text{diam } M_n)^3$ and we take $\rho_n$ to be the constant $\text{diam } M_n/2$.

Assumption 2 and notation: Now we fix the constants $w_n$ and the functions $\rho_n : M_n \to (0, \infty)$ satisfying Lemma 1.5. For any $n$ and any $x \in M_n$ we denote by $g'_n(x)$ the metric $\rho_n(x)^{-2}g_n$. Thus, $B_{g_n}(x, \rho_n(x)) = B_{g'_n(x)}(x, 1)$.

## 2 Gromov-Hausdorff convergence of Alexandrov spaces

### 2.1 Basics about Alexandrov Spaces

Let $X$ be locally compact metric space. Then $X$ is a *local length space* if $X$ is covered by open subsets $U_i$ such that for each $i$ and for every two points $x, y$ in $U_i$ there is a closed interval $I \subset \mathbb{R}$ and an isometric embedding $I \to X$ whose endpoints are $x$ and $y$. In particular, the image of $I$ is rectifiable and its length is the distance between its endpoints. Fix a number $k$. We denote by $H_k$ the complete, simply connected surface of constant curvature $k$. Given three points $p, q, r$ in a metric space and a real number $k$, then a *comparison triangle* $\overrightarrow{pqr}$ in $H_k$ is a triangle whose side lengths are equal to the distances between the corresponding points of $X$, e.g., $d_{H_k}(\overrightarrow{pqr}) = d_X(p, q)$. Such a comparison triangle exists and is unique up to isometries of $H_k$, provided only that if $k > 0$, then the sum of the three pair-wise distances is at most $2\pi/\sqrt{k}$. We define the $k$-comparison angle (or the comparison angle if $k$ is clear from the context) $\angle \overrightarrow{pqr}$ to be the angle $\angle \overrightarrow{pqr}$ in $H_k$. By definition,
a local Alexandrov space with curvature bounded below by $k$ is a locally compact, local length space with the property that for every $p \in X$ there is a neighborhood $U \subset X$ of $p$ such that for any three points $q, r, s$ in $U$ the $k$-comparison angles satisfy

$$\tilde{\angle} qpr + \tilde{\angle} rps + \tilde{\angle} spr \leq 2\pi,$$

(2.1)

see [3]. In addition, if the dimension of $X$ is one and $k > 0$, then we require the diameter of $X$ to be at most $\pi/\sqrt{k}$. A local Alexandrov space is a local Alexandrov space of curvature bounded below by some $k$. If $X$ is a local Alexandrov space with curvature $\geq k$ and if $\lambda > 0$ then the metric space, denoted $\lambda X$, obtained by multiplying the given metric by $\lambda^2$ is a local Alexandrov space with curvature bounded below by $-\lambda k$. An Alexandrov space is a complete metric space which is a length space in the sense that any two points are jointed by an isometric embedding of an arc, and that is also a local Alexandrov space. In such spaces Inequality 2.1 holds globally, i.e., for all 4-tuples of points in the space. (See §3 of [3].)

Let $A, B$ be disjoint compact subsets of a local Alexandrov space $X$. By a geodesic from $A$ to $B$ we mean an isometric embedding of an interval into $X$ with one endpoint contained in $A$ and the other contained in $B$ and such that the length of this interval is equal to the distance from $A$ to $B$. By a geodesic we mean a geodesic between the compact sets which are its endpoints. It is an easy exercise to show that if $\gamma$ is a geodesic from $A$ to $B$ and $q$ is an interior point of this geodesic, then the sub-interval of $\gamma$ from $A$ to $q$ is the unique geodesic from $A$ to $q$.

An elementary application of Toponogov [24] theory shows the following.

**Example 2.1.** Let $M$ be a Riemannian manifold with locally convex boundary of with all sectional curvatures $\geq k$. Then $M$ is a local Alexandrov space of curvature bounded below by $k$. If $M$ is complete, then it is an Alexandrov space.

Some of the basic properties of Alexandrov spaces $X$ that follow from this definition are:

**Proposition 2.2.** Let $X$ be a local Alexandrov space with curvature bounded below by $k$ and let $U \subset X$ be an open subset with the property that every pair of points in $U$ is connected by a geodesic and with the property that Inequality 2.1 holds for all quadruples of points in $U$.

1. Let $p, q, r$ be three points in $U$ and let $\tilde{\triangle} qpr$ be a comparison triangle in $H_k$. Let $\gamma$ be a geodesic in $U$ with endpoints $q$ and $r$ and let $x \in \gamma$ be at distance $s$ from $q$. Let $\bar{x} \in \tilde{\triangle} qpr$ be a point on the corresponding geodesic in $H_k$ at distance $s$ from $\bar{q}$. Then $d(p, x) \leq d(p, x)$.

2. Let $\gamma$ and $\mu$ be geodesics in $U$ emanating from $p$. Set $a$, respectively $b$, equal to the length of $\gamma$, respectively $\mu$. For $0 < s \leq a$ and $0 < t \leq b$ let $\gamma(s)$ be the point on $\gamma$ at distance $s$ from $p$ and let $\mu(t)$ be the point on $\mu$ at distance $t$ from $p$. Define the function

$$f(s, t) = \tilde{\angle} \gamma(s)p\mu(t).$$
Then \( f(s,t) \) is a monotone non-increasing function of either variable \( s \) and \( t \) when the other is held fixed. In particular, the limit as \( s \) and \( t \) both go to zero \( f(s,t) \) exists and is denoted \( \angle qpr \).

In particular, if \( X \) is complete, then all these properties hold with \( U = X \).

These are all proved in \( \S 3 \) for \( [3] \).

An important result is the following splitting theorem for complete Alexandrov spaces.

**Proposition 2.3.** Suppose that \( X \) is an Alexandrov space of curvature \( \geq 0 \) and that \( X \) contains a geodesic line \( L \subset X \) (i.e., a copy of \( \mathbb{R} \) isometrically embedded in \( X \)) parameterized as \( \zeta(s) \). Then there is an Alexandrov space \( Y \) and an isometry \( \mathbb{R} \times Y \to X \) so that \( L \) is the image of \( \mathbb{R} \times \{y\} \) for some \( y \in Y \). The parallel copies of \( Y \) in the product are the level sets of the function \( f = \lim_{s \to -\infty} (d(\zeta(s), \cdot) - s) \).

This leads immediately by induction to:

**Corollary 2.4.** Suppose that \( X \) is an Alexandrov space of curvature \( \geq 0 \) containing an isometric copy of \( \mathbb{R}^m \) for some \( m > 0 \). Then there is an Alexandrov space \( Y \) and an isometric product decomposition \( X = \mathbb{R}^m \times Y \) with the property that the given copy of \( \mathbb{R}^m \) is identified with \( \mathbb{R}^m \times \{y\} \) for some \( y \in Y \).

The proofs given in \( [24] \) in the case of smooth manifolds work mutatis-mutandis for Alexandrov spaces.

**Definition 2.5.** The *dimension* of a local Alexandrov space is its Hausdorff dimension. Later, we shall see there is a much more precise statement about dimension using strainers (or burst points).

**2.2 Gromov-Hausdorff convergence**

Hausdorff convergence is defined for metric spaces. The Hausdorff distance between two metric spaces \( X \) and \( Y \) is less \( \epsilon \) if there is a metric on \( X \bigsqcup Y \) extending the given metrics on \( X \) and \( Y \) with \( X \) contained in the \( \epsilon \)-neighborhood of \( Y \) and \( Y \) contained in the \( \epsilon \)-neighborhood of \( X \).

**Definition 2.6.** Let \( X \) be a metric space. For any compact subset \( A \subset X \) and any \( r > 0 \) we define the metric ball \( B(A, r) = \{x \in X \mid d(x, A) < r\} \) and we denote the metric sphere \( S(A, r) = \{x \in X \mid d(A, x) = r\} \). Notice that balls are open subsets.

**Definition 2.7.** Let \( (X_n, p_n) \) be a sequence of pointed, locally compact metric spaces and that \( (X, p) \) is a complete pointed metric space. We say that the \( (X_n, p_n) \) converge in the Gromov-Hausdorff sense to a ball \( (X, p) \) if for every \( R < \infty \) the balls \( B(p_n, R) \) converge in the Hausdorff topology to \( B(p, R) \).

A crucial property (and indeed one of the main reasons for introducing Alexandrov spaces) is the following compactness result (see \( \S 8.5 \) of \( [3] \)).
Proposition 2.8. Fix an integer $N$ and a real number $k$. The collection of complete, pointed Alexandrov spaces of dimension $\leq N$ with curvature bounded below by $k$ is sequentially compact in the Gromov-Hausdorff sense, meaning that if $(X_n,p_n)$ is a sequence of such Alexandrov spaces, then there is a subsequence converging in the Gromov-Hausdorff sense to a complete pointed Alexandrov space $(X,p)$ which itself has dimension at most $N$ and curvature bounded below by $k$.

It is important to recognize that even if the $X_n$ all have the same dimension $N$, it may well be the case that the limit $X$ has lower dimension.

We shall also need a local version of convergence.

Definition 2.9. An Alexandrov ball is a local Alexandrov space $B = B(p,r)$ that is a metric ball and has the property for any $y,z \in B$ with $d(y,z)+d(p,y)+d(p,z) < 2r$ are joined by a geodesic in $B$.

The local version that we need is the following.

Proposition 2.10. Suppose that $B_n = B(p_n,r_n)$ are Alexandrov balls of dimension $N$ and of curvature bounded below by $k$. Suppose that the $r_n \to r$ as $n \to \infty$ with $0 < r \leq \infty$. Then, after passing to a subsequence, there is an Alexandrov ball $(B,p)$ of dimension at most $N$ and of radius $r$ such that for every $r' < r$ the balls $B(p_n,r')$ converge in the Gromov-Hausdorff topology to $B(p,r')$.

The essential point here is that for every $s < r$ for all $n$ sufficiently large there are balls of radius $(r-s)/3$ centered at each point of $B(p_n,s)$ on which the Alexandrov property holds. Given this uniformity, the argument in the complete case adapts to establish this result.

If spaces $X_n$ converge in the Gromov-Hausdorff sense to $X$, then we say that a sequence $x_n \in X_n$ converges to $x \in X$ (or that $x$ is the limit of the $x_n$) if, setting $d_n$ equal to the distance between $x_n$ and $x$ in the given metric on $X_n \coprod X$, the $d_n(x_n, x)$ tend to zero as $n$ tends to infinity. Suppose that $(X_n,p_n)$ is a sequence of pointed local Alexandrov spaces of dimension $\leq N$ with curvature bounded below by $k$ which are either complete or are Alexandrov balls of radius $r_n \to r > 0$. If the $X_n$ are complete, then we set $r = \infty$. Suppose that the $(X_n,p_n)$ are of dimension $\leq N$ and that they converge in the Gromov-Hausdorff sense to $(X,p)$. Then all of the following are immediate consequences of the definitions and usual compactness arguments.

1. Suppose that $s < r$ and that for each $n$ we have a point $x_n \in B(p_n,s)$. Then, after passing to a subsequence, there is a limit point $x \in B(p,r) \subset X$ for the sequence.

2. Suppose that we have a sequence of geodesics $\gamma_n$ in $X_n$ whose lengths converge to a non-zero (but possibly infinite) limit as $n$ tends to $\infty$ and suppose that the initial points of the $\gamma_n$ converge to a point of $X$. Then after passing to a further subsequence the geodesics converge, uniformly on compact sets, to a geodesic in $X$. 

(3) Suppose that we have distinct points \( q, r, s \) in \( X \) and sequences \( q_n, r_n, s_n \) in \( X_n \) such that \( \lim q_n = q, \lim r_n = r \) and \( \lim s_n = s \). Then
\[
\lim_{n \to \infty} \zeta q_n r_n s_n = \zeta qrs.
\]

(4) Suppose that for each \( n \) we have geodesics \( \gamma_n \) and \( \mu_n \) in \( X_n \) both emanating from a point \( p_n \in X_n \) with \( \lim \gamma_n = \gamma, \lim \mu_n = \mu \). Then
\[
\lim \inf_{n \to \infty} \zeta \gamma_n \mu_n \geq \zeta \gamma \mu.
\]

### 2.2.1 Limits that are products

**Proposition 2.11.** Fix \( r > 0 \). Let \( \lambda_n \to \infty \) and \( \delta_n \to 0 \) as \( n \to \infty \). Suppose that \( (X_n, x_n) \) is a sequence of local Alexandrov spaces of dimension \( N \) and curvature \( \geq -1 \). Suppose that for each \( n \) there are compact sets \( \{ A_n^+, A_n^- \} \) with \( d(x_n, A_n^+) \geq r \). We suppose that for each point \( z_n \) in the ball \( B(x_n, r) \) there are geodesics from \( z_n \) to \( A_n^\pm \). We also suppose that the comparison angle \( \angle A_n^- x_n A_n^+ > \pi - \delta_n \). Suppose that the \( (\lambda_n X_n, x_n) \) converge to \( (X, x) \). Then there is a based Alexandrov space \( (Y, y) \) of dimension \( \leq N - 1 \) and isometry \( (X, x) \cong (Y, y) \times (\mathbb{R}, 0) \) with the property that for any sequence of points \( z_n \in X_n \) converging to a point \( z \in X \) and geodesics \( \gamma_n^\pm \) from \( x_n \) to \( A_n^\pm \), the \( \gamma_n^\pm \) converge (uniformly on compact sets containing \( x_n \)) to the geodesic rays from \( z \) in the positive and negative \( \mathbb{R} \)-directions in the product.

**Proof.** Denote by \( g_n \) the metrics on \( X_n \); the rescaled metrics are \( \lambda_n^2 g_n \). Let \( \zeta_n^\pm \) be geodesics from \( x_n \) to \( A_n^\pm \). Since the comparison angle \( \angle A_n^- x_n A_n^+ \) is greater than \( \pi - \delta_n \), by monotonicity for any points \( u_n^\pm \) on \( \zeta_n^\pm \) the comparison angle \( \angle u_n^- x_n u_n^+ \) is greater than \( \pi - \delta_n \). Hence, rescaling by the \( \lambda_n \) and taking limits we see that for points \( u_n^\pm \) on the limiting geodesic rays \( \zeta^\pm \) the comparison angle \( \angle u^- x u^+ = 0 \), meaning that \( \zeta = \zeta^- \cup \zeta^+ \) is a geodesic line. Since the \( X_n \) have curvature \( \geq -1 \) and the \( \lambda_n \to \infty \), the limit \( X \) has curvature \( \geq 0 \). Hence, by Proposition 2.3 it splits as a product \( Y \times \mathbb{R} \) in such a way that \( \zeta \) is the factor in the \( \mathbb{R} \)-direction through the base point. Furthermore, it also follows from this proposition that, letting \( f_n \) be the function \( d_{\lambda_n^2 g_n}(A_n^-, \cdot) - d_{\lambda_n^2 g_n}(A_n^+, x_n) \), the \( f_n \) converge to a function \( f: X \to \mathbb{R} \) whose level sets are the parallel copies of \( Y \) in the product structure. Let \( z_n \in B(x_n, r) \) be a sequence of points converging to \( z \in X \), and let \( \gamma^\pm \) be a geodesic from \( z_n \) to \( A_n^\pm \). The directional derivatives of the \( f_n \) at the \( x_n \) in the directions of the \( \gamma_n^\pm \) converge to 1 as \( n \) goes to infinity. Hence, the \( \gamma_n^+ \) converge to rays in the positive \( \mathbb{R} \)-direction. It follows that the \( \gamma_n^- \) converge to rays in the negative \( \mathbb{R} \)-direction.

### 2.3 Local geometry of Alexandrov spaces

In this section we introduce the notion of the tangent cone for an Alexandrov space. Using this we define directional derivatives for Lipschitz functions and also the boundary of an Alexandrov space.
2.3.1 The space of directions and the tangent cone

Let $X$ be a local Alexandrov space of dimension $\leq N$, and let $p \in X$ be a point. Let $\Sigma'_p$ be the set of equivalences classes of geodesics with $p$ as one endpoint where, by definition, two geodesics are equivalent if their intersection contains a geodesic of positive length emanating from $p$. The set of equivalence classes has a metric: $d([\gamma_1], [\gamma_2])$ is equal to the angle at $p$ of representatives $\gamma_1$ and $\gamma_2$ of the equivalence classes, which is clearly independent of the choice of representatives. The metric completion of $\Sigma'_p$ is the space of directions at $p$, denoted $\Sigma_p$. The dense subset $\Sigma'_p$ in $\Sigma_p$ is called the set of directions realizable by geodesics.

**Proposition 2.12.** The space $\Sigma_p$ is a compact Alexandrov space of dimension one less than the dimension of $X$. If the dimension of $X$ is at least three, then $\Sigma_p$ is an Alexandrov space of curvature $\geq 1$. If $X$ has dimension 2, then either $\Sigma_p$ is isometric to a circle or an interval and has diameter at most $\pi$.

For a proof of this result see Section 7 of [3].

Notice that the length of a metric circle is twice its diameter.

**Definition 2.13.** The tangent cone $T_pX$ is the cone over $\Sigma_p$ to the point $p$. It is an Alexandrov space of curvature $\geq 0$ of the same dimension as $X$.

**Lemma 2.14.** Let $\lambda_n$ be a sequence of positive real numbers tending $+\infty$. Then the based local Alexandrov spaces $(\lambda_n X, p)$ converge in the Gromov-Hausdorff topology to $(T_pX, \{p\})$.

For a proof, see Theorem 7.8.1 of [3].

2.3.2 The boundary of an Alexandrov space

**Definition 2.15.** The boundary of a local Alexandrov space is defined inductively on dimension. Let $X$ be a one-dimensional local Alexandrov space. Then it is either isometric to either an interval or a circle. Its boundary as an Alexandrov space is its topological boundary. More generally, we define the boundary of a higher dimensional local Alexandrov space by induction. For $X$ an $n$-dimensional local Alexandrov space, we define $\partial X$ to be the subset of $X$ consisting of points $p$ for which $\Sigma_p$ is an $(n-1)$-dimensional Alexandrov space with non-empty boundary. Then $\partial X$ is a closed subset. Its complement is denoted int $X$.

2.4 Regular functions

For the material in this section see Section 11 of [3].

**Definition 2.16.** Let $X$ be a local Alexandrov space. We say that a Lipschitz function $f: X \to \mathbb{R}$ has a directional derivative at $q \in X$ if there is a continuous function $f'_q: \Sigma_q \to \mathbb{R}$ such that for any geodesic $\gamma$ starting at $q$ and parameterized as $\gamma(s)$ where $s$ is the distance from $q$, the function $s \mapsto f(\gamma(s))$ has a one-sided derivative at $s = 0$, and this derivative is $f'_q([\gamma])$. We say that a Lipschitz function $f$
is regular at \( q \) if it has a directional derivative at \( q \) and if the directional derivative in some direction at \( q \) is positive. We say that a 1-Lipschitz function \( f \) is \( a \)-strongly regular at \( q \), if \( f \) has a directional derivative and if there is a direction \( \tau \) such that \( f'_q(\tau) > a \). The function \( f \) is strongly regular if it is \( a \)-strongly regular for every \( a < 1 \).

**Definition 2.17.** Let \( A \) be a compact subset of a local Alexandrov space \( X \) and let \( q \) be a point of \( X \setminus A \). We denote by \( A' \subset \Sigma_q \) the set of directions at \( q \) to all geodesics \( \gamma \) from \( A \) to \( q \). It is a closed subset of \( \Sigma_q \).

**Lemma 2.18.** Let \( A \subset X \) be a compact subset, and set \( f = d(A, \cdot) \). Then at any point \( q \in X \setminus A \) for which there is at least one geodesic from \( A \) to every point in a neighborhood of \( q \), the function \( f \) is a 1-Lipschitz function and \( f \) has a directional derivative at \( q \). The derivative \( f'_q \) is given by

\[
f'_q(\xi) = -\cos(|\xi A'|),
\]

where \( |\xi A'| \) denotes the distance in \( \Sigma_q \) from \( \xi \) to the closed set \( A' \) of all tangent directions at \( p \) to geodesics from \( A \) to \( p \). In particular, \( f \) is \( a \)-strongly regular if and only if there is a direction \( \tau \) such that \( d(A', \tau) > \cos^{-1}(-a) \).

**Proof.** See §11.4 of [3].

The following is an elementary consequence of this lemma.

**Corollary 2.19.** Let \( A \) a compact subset of \( X \) and let \( V \subset X \setminus A \) be an open subset. Suppose that there is a geodesic from \( A \) to each point of \( V \). Then the subset of \( V \) where \( d(A, \cdot) \) is regular, respectively \( a \)-strongly regular, is an open set of \( V \) and includes any point \( q \in V \) with the property that there is a geodesic from \( A \) to a point \( w \neq q \) and passing through \( q \).

Similarly, one shows:

**Corollary 2.20.** Suppose that we have a sequence of pointed local Alexandrov spaces \((X_n, p_n)\) with curvature \( \geq k \) converging in the Gromov-Hausdorff topology to a limit \((X, p)\). Suppose that there are compact subsets \( A_n \subset X_n \) converging to \( A \subset X \) and open subsets \( V_n \) converging to \( V \). Suppose that there is a geodesic from \( A_n \) to each point of \( V_n \), and suppose that \( q \in V \) and \( q_n \in V_n \) is a sequence converging to \( q \). Then if \( d(A, \cdot) \) is regular at \( q \), resp. \( a \)-strongly regular at \( q \), then for all \( n \) sufficiently large, \( d(A_n, \cdot) \) is regular at \( q_n \), resp. \( a \)-strongly regular at \( q_n \).

**Lemma 2.21.** Suppose that \( f : X \to \mathbb{R} \) is a Lipschitz function with directional derivatives and that \( q_n \in f^{-1}(f(q)) \) is a sequence converging to \( q \). Let \( \gamma_n \) be a geodesic from \( q \) to \( q_n \). Suppose that the unit tangent vectors to the \( \gamma_n \) at \( q \) converge to a tangent direction \( \tau \). Then \( f'_q(\tau) = 0 \).

**Proof.** This is elementary from the comparison results, see §11.3 of [3].
2.4.1 Regular functions on smooth manifolds

We shall need information about level sets of regular functions on smooth manifolds.

**Lemma 2.22.** Suppose that $X$ is a locally complete Riemannian manifold and that $f$ is the distance function from a compact set $A$ and that $f$ is regular (in the Alexandrov sense) at $q_0 \in X \setminus A$. Then there is a neighborhood $U$ of $q_0$ and a smooth unit vector field $\tau$ on $U$ with the property that $f'_q(\tau) > 0$ for all $q \in U$. Furthermore, there is an open interval $J$, an open subset $U'$ of $\mathbb{R}^{n-1}$, and a bi-Lipschitz homeomorphism $U \cong U' \times J$ with the property that the level sets of $f|_U$ are identified with the subsets $U' \times \{j\}$ for $j \in J$. In particular, the level sets of $f$ are topologically locally flat, codimension-1 submanifolds near $q$.

**Proof.** Consider the subset of the unit tangent bundle of $X$ consisting of directions $\chi_q \in T_qX$ with the property that $f'_q(\chi_q) > 0$ as $q$ varies over an open neighborhood $U$ of $q_0$. Arguments similar to the above show that this is an open subset of $TX$. If we take $U$ small enough, the fiber over every $q \in U$ is non-empty. Hence, there is a smooth vector field $\tau$ defined in a neighborhood $U$ of $q$ and $\alpha > 0$ such that $f'_q(\tau(q)) \geq \alpha$ for all $q \in U$. Now we integrate $\tau$ to define a smooth local coordinate system $(x^1, \ldots, x^n)$ near $q_0$ such that $\tau = \partial/\partial x^1$. We replace $U$ be a smaller open set which is the product of an open ball in $(x^2, \ldots, x^n)$-space with an interval in the $x^1$-direction. Since $f'((\partial/\partial x^1)) > 0$ everywhere, we see that the level sets of $f$ meet each interval in the $x^1$-direction in at most one point. That is to say, near $q_0$ these level sets are given by the graphs of functions $x^1 = \varphi(x^2, \ldots, x^n)$. Elementary arguments show that the map $(x^1, \ldots, x^n) \mapsto (f(x^1, \ldots, x^n), x^2, \ldots, x^n)$ is the required bi-Lipschitz homeomorphism. \hfill $\Box$

We also need a fairly restricted version of an analogous result for maps to the plane. The following is an elementary lemma.

**Lemma 2.23.** Given $\epsilon' > 0$, the following holds for all $\epsilon > 0$ sufficiently small. Let $B(0, \epsilon^{-1})$ be the ball of radius $\epsilon^{-1}$ in the Euclidean plane centered at the origin. We denote by $(x, y)$ the Euclidean coordinates on this ball and by $\theta$ the usual coordinate along the circle. Let $g$ be a Riemann metric on $U = B(0, \epsilon^{-1}) \times S^1$ that is within $\epsilon$ in the $C^N$-topology (where $N = [\epsilon^{-1}]$) of the product of the usual Euclidean metric on $B(0, \epsilon^{-1})$ and the Riemannian metric of length 1 on the circle. Suppose that $F = (f_1, f_2): U \rightarrow \mathbb{R}^2$ is a map with the property that $f_1$ and $f_2$ are 1-Lipschitz with respect to $g$ with directional derivatives at all points of $U$. Suppose further that the directional derivatives of $f_i$ with respect to $g$ satisfy:

\[
|f'_1(\partial_x) - 1| < \epsilon \\
|f'_2(\partial_y) - 1| < \epsilon \\
\max(|f'_1(\pm \partial_y)|, |f'_2(\pm \partial_x)|, |f'_1(\pm \partial_x)|, |f'_2(\pm \partial_y)|) < \epsilon.
\]

Then any fiber $F^{-1}(p)$ that meets $B(0, \epsilon^{-1}/2)$ is a circle that is $\epsilon'$-orthogonal to the family of horizontal spaces $B(0, \epsilon^{-1}) \times \{0\}$ in the sense that, fixing $a \in F^{-1}(p)$, as
\(b \in F^{-1}(p)\) approaches a the angle (measured with respect to product metric) of the geodesic (in the product metric) from \(a\) to \(b\) with the horizontal space through \(a\) is within \(\epsilon'\) of \(\pi/2\). Furthermore, any fiber \(F^{-1}(p)\) that meets \(B(0, \epsilon^{-1}/2)\) intersects each horizontal space \(\{\theta\} \times B(0, \epsilon^{-1})\) in a single point.

### 2.5 Almost manifold regions in Alexandrov spaces

We introduce an open dense set of ‘good’ points in an \(n\)-dimensional Alexandrov space and show that these points have neighborhoods that are \((1 + \epsilon)\)-Lipschitz equivalent to \(\mathbb{R}^n\).

#### 2.5.1 \((m, \delta)\)-strainers

Let \(X\) be an \(n\)-dimensional local Alexandrov space of curvature bounded below by \(k\). Fix \(x \in X\) and let \(U \subset X\) be a neighborhood of \(x\) in which the Alexandrov property holds. A \((m, \delta)\)-strainer at \(x \in X\) is a set of \(2m\) points \(\{a_1, b_1, \ldots, a_m, b_m\}\) in \(U\) such that:

1. \(\tilde{\angle}a_ipb_i > \pi - \delta\) for all \(i = 1, \ldots, m,\)
2. \(\tilde{\angle}a_ipa_j > \pi/2 - \delta\) for all \(i \neq j,\)
3. \(\tilde{\angle}b_ipb_j > \pi/2 - \delta\) for all \(i \neq j.\)

Notice that it follows from the defining property of an Alexandrov space that \(\tilde{\angle}a_ipa_j < \pi/2 + 2\delta\) for every \(i \neq j,\) and similarly for the \(b_i, b_j\). The size of an \((m, \delta)\)-strainer is defined to be the minimum of the \(2m\) distances \(\{d(p, a_i), d(p, b_i)\}_{i=1}^m\). For any local Alexandrov space \(X\) and any \(m \geq 0\) and any \(\delta > 0,\) the subset of points \(x \in X\) that have a \((m, \delta)\)-strainer is an open subset.

**Definition 2.24.** Let \(X\) be an \(n\)-dimensional local Alexandrov space. Then for any \(\delta > 0\) denote by \(R_\delta(X)\), the \(\delta\)-regular set, the subset of \(X\) consisting of points with an \((n, \delta)\)-strainer. According to Section 6 of \([3]\) \(R_\delta(X)\) is an open dense subset of \(X\).

**Proposition 2.25.** Fix \(n > 0\). Given \(\epsilon > 0\) there is \(\delta > 0\) such that the following holds. Let \(X\) an \(n\)-dimensional local Alexandrov space and suppose that \(x \in R_\delta(X)\), and suppose that the Alexandrov property holds on the ball of radius \(r\) centered at \(x\). Then there is a neighborhood \(U\) of \(x\) and a \((1 + \epsilon)\)-bilipschitz homeomorphism from \(U\) to an open subset of \(\mathbb{R}^n\). The open set \(U\) contains a metric ball about \(x\) whose radius depends only on the size of the \((n, \delta)\)-strainer at \(x\) contained in \(B(x, r)\).

**Proof.** Let \(\{a_1, b_1, \ldots, a_n, b_n\}\) be a \(\delta\)-strainer of size \(s\) at \(x\) contained in \(B(x, r)\). We define a map \(X \rightarrow \mathbb{R}^n\) by \(y \mapsto (d(y, a_1), \ldots, d(y, a_n))\). According to Theorem 9.4 of \([3]\) this map has the required properties in a ball about \(x\) whose radius depends only on \(s\) and \(n.\)
Any compact, connected 1-dimensional Alexandrov space is isometric either to a closed interval or to a circle. In each case, the only invariant up to isometry is the total length of the space. Furthermore, for any $\delta > 0$ the subset of $\delta$-regular points is the interior of the interval or the entire circle. The size of a $(1, \delta)$-strainer is the distance from the boundary in the case of the interval, or one-quarter the length in the case of the circle.

### 2.6 A blow-up argument

We need a special result about rescaling Alexandrov spaces so as to construct higher dimensional limits. We need this result in order to handle sequences of points $x \in M_n$ converging to a singular point of a 2-dimensional limit. The following is a reformulation in our context of Lemma 3.6 of [22].

**Proposition 2.26.** For any $\delta > 0$, the following holds for all $\mu > 0$ sufficiently small. Fix $r > 0$. Suppose that $B_n = B(x_n, 1)$ is a sequence of Riemannian balls of radius 1 with curvature $\geq -1$ in a complete Riemannian manifolds with convex boundary. Suppose that the $B_n$ are non-compact and converge to an Alexandrov ball $(X, x)$. Suppose that $\dim X$ is either 1 or 2. Suppose also that rescaling $B = B(x, r)$ by $r^{-2}$ produces a ball that is within $\mu$ in the Gromov-Hausdorff distance of a flat cone. If $B$ is 1-dimensional, then we require that this flat cone is the cone on a single point. If $B$ is 2-dimensional we require that the flat cone is the cone either on a circle or on an interval and the diameter of the base of the cone is most $\pi - \delta$. Then, after passing to a subsequence, there are points $\hat{x}_n \in X_n$ with $d(x_n, \hat{x}_n) \to 0$ as $n \to \infty$ such that one of the following holds:

1. $d(\hat{x}_n, \cdot)$ has no critical points in $B(\hat{x}_n, r) \setminus \{\hat{x}_n\}$. In this case $B(\hat{x}_n, r')$ is diffeomorphic to a ball for every $0 < r' < r$.

2. There is a sequence of positive constants $\delta_n \to 0$ as $n \to \infty$ such that:

   (a) Every critical point of $d(\hat{x}_n, \cdot)$ in $B(\hat{x}_n, r)$ is within distance $\delta_n$ of $\hat{x}_n$, and

   (b) there is a critical point $q_n$ for $d(\hat{x}_n, \cdot)$ at distance $\delta_n$ from $\hat{x}_n$.

In this case, passing to a subsequence there is a limit of the $\frac{1}{\delta_n}B(\hat{x}_n, r)$. This limit is a complete Alexandrov space of curvature $\geq 0$ and of dimension at least one more that the dimension of $X$.

### 2.7 Gromov-Hausdorff limits of balls in the $M_n$

Now we turn from generalities about Alexandrov spaces to special properties of Gromov-Hausdorff limits of balls in the $M_n$. Recall that we have a sequence of constants $w_n \to 0$ as $n \to \infty$ and functions $\rho_n : M_n \to [0, \infty)$ with the property that $\rho_n(x) \leq \text{diam}(M_n^0)/2$ for every $n$ and every $x$ in the connected component $M_n^0$ of $M_n$. Thus, for every $n$ and every $x \in M_n$, the ball $B_{\rho_n}(x, \rho_n(x))$ is non-compact. Since $M_n$ is itself compact, it follows that for every $0 < r < \rho_n(x)$, the ball $B_{\rho_n}(x, r)$ has compact closure in $B(x, \rho_n(x))$. It then follows that the $B_{\rho_n}(x, \rho_n(x))$
are non-compact Alexandrov balls. Rescaling the metric by $\rho_n(x)^{-2}$, that is to say replacing the metric $g_n$ on this ball by the metric $g'_n(x) = \rho_n(x)^{-2}g_n$ we obtain non-compact Alexandrov balls $B_{g'_n}(x,1)$ of radius 1 with the property that their sectional curvatures are bounded below by $-1$, and their volumes are bounded above by $w_n$. Since $w_n \to 0$ as $n \to \infty$, the following is then immediate from Lemma 2.10 and Proposition 2.25.

**Lemma 2.27.** Let $x_n \in M_n$ be given for every $n \geq 1$. Then, after passing to a subsequence, the $B_{g'_n}(x_n,1)$ converge to a non-compact Alexandrov ball $B = B(x,1)$ of curvature $\geq -1$ and of dimension 1 or 2. The limiting ball contains points at every distance $< 1$ from $x$.

This leads immediately to the following corollary.

**Corollary 2.28.** There is a decreasing sequence of constants $\epsilon_n > 0$ tending to zero as $n \to \infty$ such that for every $n$ and for any $x \in M_n$ there is a non-compact Alexandrov ball $B$ of radius 1, of curvature $\geq -1$, and of dimension 1 or 2, such that $B_{g'_n}(x,1)$ is within $\epsilon_n$ in the Gromov-Hausdorff distance of $B$.

**Assumption 3:** We fix a sequence of $\epsilon_n \to 0$ as in the corollary.

## 3 2-DIMENSIONAL ALEXANDROV SPACES

In order get enough information about the structure of balls in the $M_n$ limiting (after rescaling) to a 2-dimensional Alexandrov space, we need fairly delicate information about 2-dimensional Alexandrov balls. We will cover these 2-dimensional spaces by four types of neighborhoods – those near flat balls in $\mathbb{R}^2$, those near flat circular cones on a circle of length $< 2\pi$, those near flat cones in $\mathbb{R}^2$ of angle $< \pi$, and those near flat boundary points. Establishing this is the subject of this section.

### 3.1 Basics

**Claim 3.1.** A 2-dimensional local Alexandrov space $X$ is a topological 2-manifold, possibly with boundary. The topological boundary of $X$ is Alexandrov boundary $\partial X$.

For a proof, see §12.9.3 of [3].

Let $X$ be a 2-dimensional local Alexandrov space. We define the cone angle at any point $p \in X$ to be the total length of $\Sigma_p$. It follows from the Alexandrov space axioms that if $p \in \text{int} X$ then the cone angle at $p$ is at most $2\pi$ and the tangent cone is a flat circular cone of this cone angle. If $p \in \partial X$, then the cone angle at $p$ is at most $\pi$, and the tangent cone is a subcone of $\mathbb{R}^2$ of this cone angle.

**Lemma 3.2.** Suppose that $(X_n, x_n)$ is a sequence of 2-dimensional Alexandrov balls converging to a 2-dimensional local Alexandrov ball $(X,x)$. Suppose that sequence $y_n \in X_n$ converges to $y \in X$. Then:

1. If $y_n \in \partial X_n$ for all $n$ then $y \in \partial X$. 
2. Conversely, if \( y \in \partial X \), then there is a sequence \( y_n \in \partial X_n \) converging to \( y \).

**Proof.** Let us suppose that \( y_n \in \partial X_n \) for all \( n \) and show that \( y \in \partial X \). Suppose to the contrary that \( y \in \text{int} X \). Let \( d_n \) be the Gromov-Hausdorff distance from \( (X_n, y_n) \) to \( (X, y) \). Choose constants \( \lambda_n \to \infty \) such that \( \lambda_n d_n \to 0 \). Then the Gromov-Hausdorff distance from \( (\lambda_n X_n, y_n) \) to \( (\lambda_n X, y) \) goes to zero and the \( (\lambda_n X, y) \) converge to the tangent cone to \( X \) at \( y \). This allows us to assume that the \( (X_n, y_n) \) converge to a flat circular cone \( C \) with cone point \( y \). For any compact subset \( K \subset C \setminus \{y\} \), for any sequence \( z_n \in X_n \) converging to a point in \( K \), and for any \( \delta > 0 \), for all \( n \) sufficiently large, there is a \((2, \delta)\) strainer at \( z_n \). Hence, \( z_n \in \text{int} X_n \). In particular, the boundary component of \( X_n \) containing \( y_n \) converges to the cone point \( y \), and thus the diameter of the boundary component converges to 0 as \( n \to \infty \). We take a piecewise geodesic approximation \( \gamma_1, \ldots, \gamma_k \) to the metric sphere \( S(y,1) \), with the endpoints of \( \gamma_i \) being \( z_i \) and \( z_{i+1} \) in \( S(y,1) \). Let \( z_{n,i} \) be points in \( X_n \) converging to \( z_i \) and let \( \gamma_{n,i} \) be a geodesic with endpoints \( z_{n,i} \) and \( z_{n,i+1} \). Then for all \( n \) sufficiently large, the union of the \( \gamma_{n,i} \) is a simple closed curve in \( \text{int} X_n \) separating \( X_n \) into two pieces, one of which, \( X_n' \), contains \( y_n \) and hence the entire component of \( \partial X_n \) containing \( y_n \). This component has convex boundary and hence is a compact 2-dimensional Alexandrov space \( X_n' \). It has at least two boundary components – the boundary component \( \partial_0 \) of \( X_n \) containing \( y_n \) and the component \( \partial_1 \) that is the union of the \( \gamma_{n,i} \). There is an infinite cyclic \( \tilde{X}_n' \) covering of \( X_n' \) that unwraps both \( \partial_0 \) and \( \partial_1 \). We give \( \tilde{X}_n' \) the length space metric with the property that the projection mapping to \( X_n' \) is a local isometry. This metric makes the total space of the infinite cyclic covering a complete local Alexandrov space and hence by Theorem 3.2 of [3], an Alexandrov space. Since the diameter of the boundary component of \( X_n' \) containing \( y_n \) is going to zero as \( n \to \infty \), the generator of the covering group of \( \tilde{X}_n' \) moves any point in the preimage of \( \partial_0 \) a distance \( d_n \) that goes to zero as \( n \to \infty \). Fix a metric ball \( B \) in the \( \delta \)-regular subset of the cone which is a topological ball. For all \( n \) sufficiently large, there are metric balls \( B_n \subset X_n' \) converging to \( B \). For \( n \) sufficiently large the \( B_n \) are topological balls. The area of \( B_n \) is bounded away from 0 as \( n \to \infty \) and the distance from \( B_n \) to \( y_n \) is also bounded, say by \( \epsilon \), as \( n \to \infty \). The preimage of \( B_n \) in the covering \( \tilde{X}_n' \) is a disjoint union of balls freely permuted by the infinite cyclic covering group. Fix a point \( \tilde{y}_n \in \tilde{X}_n' \) covering \( y_n \). Given any \( N \), there are distinct pre-images \( y_n \) all within distance \( N d_n \) of \( \tilde{y}_n \) and hence \( N d_n \) distinct preimages of \( B_n \) within \( \epsilon + N d_n \) of \( \tilde{y}_n \). Thus, for every \( n \) there are \( \lfloor \epsilon / d_n \rfloor \) distinct preimages of \( B_n \) within \( 2 \epsilon \) of \( \tilde{y}_n \). Since the areas of these preimages are bounded away from zero and the \( d_n \to 0 \) as \( n \to \infty \), and since the \( \tilde{X}_n' \) all have curvature \( \geq -1 \), for \( n \) sufficiently large this violates the Bishop-Gromov inequality. This proves that \( y \in \partial X \).

Conversely, suppose that \( y \in \partial X \) and let \( d_n \) be the distance from \( y_n \) to \( \partial X_n \). (We interpret \( d_n = \infty \) if \( \partial X_n = \emptyset \).) We suppose that the \( d_n \) are bounded away from 0. Then by Lemma 2.4 there is \( 0 < r < d_n \) for every \( n \) such that the distance function from \( y \) is \( \alpha \)-strongly regular on \( B(y, r) \setminus \{y\} \) and the corresponding level sets are arcs. Consequently, for all \( n \) sufficiently large, the distance function \( f_n \) from \( y_n \) is \( \alpha \)-strongly regular on \( B(y_n, r) \setminus B(y_n, r/2) \). This implies that this difference is a topological product of the level set \( f_n^{-1}(3r/4) \) with an interval. Furthermore,
furthermore $f_n^{-1}(3r/4)$ is a 1-manifold with boundary in $\partial X_n$. Because $d_n > r$, this level set is in fact unions of circles, and since $B(y, r) \setminus B(y, r/2)$ is connected, it follows that $f_n^{-1}(3r/4)$ is a single circle. But this is a contradiction. On the one hand, the $f_n^{-1}(3r/4)$ are converging in the Gromov-Hausdorff sense to the corresponding level set of the distance function from $y$ which is an arc, and on the other hand, a circle is not close in the Gromov-Hausdorff distance to an arc. This contradiction proves that $d_n \to 0$ as $n \to \infty$. Replacing $y_n$ by a nearest point on the boundary of $\partial X_n$ gives us a sequence of boundary points converging to $y$.

**Corollary 3.3.** Suppose that $X_n$ are 2-dimensional local Alexandrov spaces converging to a 2-dimensional local Alexandrov space $X$. Suppose that $x_n \in X_n$ converge to $x \in X$. Let $d_n$ be the distance from $x_n$ to $\partial X_n$ and let $d$ be the distance from $x$ to $\partial X$. Then $d = \lim_{n \to \infty} d_n$.

**Definition 3.4.** We say that a local Alexandrov space $(X, g)$ of dimension 1 or 2 is a **standard ball** if:

1. there is $x \in X$ such that $X = B_g(x, R)$ for some $0 < R \leq 1$ is an Alexandrov ball of radius $R$,

2. the curvature of $X$ is bounded below by $-1$, and

3. $X$ is non-compact, so that in particular, $X$ contains points at any distance $< R$ from $x$.

**3.2 The Interior**

We approximate interior points by cones, including flat cones.

**Definition 3.5.** Fix $\mu > 0$. Let $(X, g)$ be an 2-dimensional local Alexandrov space. Then $X$ is **interior $\mu$-good at a point** $y \in \text{int} X$ on scale $r$ and of angle $\alpha$ if $B_{r-2\mu}(y, 1)$ is a standard ball and is within $\mu$ in the Gromov-Hausdorff distance of the circular cone of cone angle $\alpha$. We say that $X$ is **interior $\mu$-flat at $y$ on scale** $s$ if $B_{s-2\mu}(y, 1)$ is a standard ball and is within $\mu$ in the Gromov-Hausdorff distance of the unit ball in $\mathbb{R}^2$. We say that $X$ is **flat at $y$ on scale** $s$ if for every $\mu > 0$ is $\mu$-flat at $y$ on scale $s$.

The first thing to notice is that being interior flat at one scale implies interior flatness at all smaller scales.

**Lemma 3.6.** Given $\mu > 0$ there is $\nu > 0$ such that the following holds. If $X = B(x, 1)$ is interior $\nu$-flat at $x$ on scale 1, then for any $0 < s \leq 1$, the ball $X$ is interior $\mu$-flat at $x$ on scale $s$.

**Proof.** Suppose that the result does not hold for some $\mu$. Then there is a sequence $\nu_n$ tending to 0 as $n \to \infty$ and be a $\nu_n$ counter-example, $X_n = B(x_n, 1)$, at scale $s_n$ with $0 < s_n \leq 1$. Passing to a subsequence we can suppose that the $X_n$ converge to $X = B(\bar{x}, 1)$ and that the $s_n$ converge to $0 \leq s \leq 1$. Since the $\nu_n \to 0$, the
limit $X$ is isometric to the unit ball in $\mathbb{R}^2$. Clearly, then $X$ is $\mu$-flat at $\gamma$ on any scale $s'$ with $0 < s' \leq 1$. Thus, if $s \neq 0$, rescaling the $X_n$ by $1/s_n$ we obtain a contradiction. Suppose now that $s = 0$. Now we rescale $(X_n, x_n)$ by $1/s_n$, and pass to a subsequence that has a limit. After rescaling there is a $(2, \delta_n)$-strainer at $x_n$ of size $1/s_n$, where $\delta_n$ depends only on $\nu_n$ and goes to zero as it does. It follows from Corollary 2.4 that the resulting limit is $\mathbb{R}^2$. This leads immediately to a contradiction as before.

Next, we see that interior good at a point implies interior flat in a nearby annular region where the constants depend on the area.

**Proposition 3.7.** Given $\mu > 0$ and $a > 0$ then for all $\mu' > 0$ sufficiently small there is $s_0 > 0$, depending only on $a$, such that the following holds. Suppose that $X = B(x, 1)$ is interior $\mu'$-good at $x$ on scale $1$, and suppose that the area of $X$ is $\geq a$. Then $X$ is interior $\mu$-flat at every point $y \in B(x, 7/8) \setminus B(x, 1/8)$ on all scales $\leq s_0$. Furthermore, for every $b \in (1/8, 7/8)$ the metric sphere $S(x, b)$ is a simple closed curve. (See Figure 1.)

**Proof.** Given $\mu > 0$, let $\nu > 0$ be as in Lemma 3.6. Without loss of generality we can assume that $a \leq \pi$. Then for any $a \leq a' \leq \pi$ there is a unique flat circular cone $C$ such that the unit ball about the cone point $p$ has area $a'$. Since every point of $C \setminus \{p\}$ is interior flat, there is $s_0 > 0$, depending only on $a$, such that every $y$ in the closure of if the annular sub-region $B(p, 7/8) \setminus B(p, 1/8)$ of $C$ is interior flat on all scales $\leq s_0$. The first statement follows immediately by taking limits and using Lemma 3.6.

Since $d(p, \cdot)$ is strongly regular on the annular region $B(p, 7/8) \setminus B(p, 1/8)$ in $C$, for any $\delta > 0$, provided that $\mu'$ sufficiently small, the distance $d(x, \cdot)$ is $(1 - \delta)$-strongly regular on $B(x, 7/8) \setminus B(x, 1/8)$. It then follows from §11 of [3] that $S(x, b)$ is a simple closed curve provided that $\mu'$ is sufficiently small.

**Definition 3.8.** If $B(x, 7/8) \setminus B(x, 1/8)$ satisfies the statement in the above proposition then we say that it is a $(\mu, s_0)$-good annular region.

### 3.3 The boundary

We turn to the analogues for the boundary of interior flatness and interior goodness.

**Definition 3.9.** Fix $\mu > 0$. Let $(X, g)$ be a 2-dimensional Alexandrov space and let $y \in \partial X$. We say that $X$ is boundary $\mu$-good on scale $r$ at $y$ of angle $\alpha$ if the rescaled ball $B_{s^{-2}g}(y, 1)$ is a standard ball and is within $\mu$ in the Gromov-Hausdorff distance of the unit ball with center the cone point in a flat 2-dimensional cone in $\mathbb{R}^2$ of cone angle $\alpha$. We say that $X$ is boundary $\mu$-good at $x$ on scale $r$, if it is boundary $\mu$-good on scale $r$ at $x$ of some angle $\alpha$. We say that $X$ is boundary $\mu$-flat at $y \in \partial X$ on scale $s$ if $B_{s^{-2}g}(y, 1)$ is within $\mu$ in the Gromov-Hausdorff distance to the unit ball centered at a boundary point of $\mathbb{R} \times [0, \infty)$, and we say that $X$ is boundary flat on scale $s$ at $y$ if it is boundary $\mu$-flat at $y$ on scale $s$ for every $\mu > 0$. 
The next observation is that boundary flatness at a point at one scale implies boundary flatness at that point at all smaller scales and interior flatness at nearby points.

**Lemma 3.10.** Given \( \mu > 0 \) for all \( \nu' > 0 \) sufficiently small the following holds. If \( X = B(x, 1) \) is boundary \( \nu' \)-flat at \( x \) on scale 1, then for any \( 0 < s \leq 1 \), the ball \( X \) is boundary \( \mu \)-flat at \( x \) on all scales \( s \), \( 0 < s \leq 1 \). Also, for any \( z \in \text{int} \cap B(x, 1/2) \) the ball \( X \) is interior \( \mu \)-flat at \( z \) on scales \( s \leq d/2 \) where \( d \) is the distance from \( z \) to \( \partial X \).

**Proof.** We begin the proof with a claim.

**Claim 3.11.** Fix \( \beta > 0 \). The following holds for all \( \nu' > 0 \) sufficiently small. Suppose \( X = B(x, 1) \) is boundary \( \nu' \)-flat at \( x \) on scale 1. Let \( y \in \partial X \cap B(x, 7/8) \) and let \( z_- \) and \( z_+ \) be points on \( \partial X \) at distance 1/8 from \( y \), lying on opposite sides of \( y \) in \( \partial X \). Then the comparison angle \( \tilde{\angle} z_+ yz_+ \) is at least \( \pi - \beta \). There is also a point \( w \in B(x, 1) \) at distance 1/8 from \( y \) such that the comparison angles \( \tilde{\angle} z_+ yw \) and \( \tilde{\angle} wyz_+ \) are both at least \( \pi/2 - \beta \).

**Proof.** These statements hold for \( \beta = 0 \) for the unit ball in \( \mathbb{R} \times [0, \infty) \) centered at a boundary point \( \bar{x} \). Thus, given \( \beta > 0 \), the result follows for all \( \nu' > 0 \) sufficiently small by taking limits.

Given this claim, the proof of the first statement of this result is analogous to the proof of Lemma \ref{lem:boundary-flatness} using Corollary \ref{cor:interior-flatness}.

Now we need the analogue of Proposition \ref{prop:good-annuli} producing good annular regions.

**Proposition 3.12.** Given \( \mu > 0 \) and \( a > 0 \) there is \( \mu'' > 0 \) such that the following holds for some positive constants \( s_1 \) and \( s_2 \) depending only on \( a \). Suppose that \( X = B(x, 1) \) is boundary \( \mu'' \)-good at \( x \) on scale 1 and of area \( \geq a \). Then at every point \( y' \in \partial X \cap (B(x, 7/8) \setminus B(x, 1/8)) \) the ball \( X \) is boundary \( \mu \)-flat on scale \( s_1 \), and for any \( b \in [1/8, 7/8] \) the metric sphere \( S(x, b) \) is an arc with endpoints in \( \partial X \). Furthermore, for any \( y \in B(x, 7/8) \setminus B(x, 1/8) \) one of the following holds.

1. \( X \) is interior \( \mu \)-flat at \( y \) on all scales \( \leq s_2 \).
2. There is \( y' \in \partial X \cap (B(x, 7/8) \setminus B(x, 1/8)) \) with \( y \in B(y', s_1/4) \), and the ball \( X \) is interior \( \mu \)-flat at \( y \) on all scales \( \leq d/2 \) where \( d \) is the distance of \( y \) to \( \partial X \).

(See Figure 2.)

**Proof.** Fix \( \mu > 0 \) and \( a > 0 \). Without loss of generality we can assume that \( a \leq \pi \). Let \( \nu' \) be the constants associated to \( \mu \) by Lemma \ref{lem:boundary-flatness} and Lemma \ref{lem:interior-flatness} respectively. First we show that for all \( \mu'' > 0 \) sufficiently small there is an \( s_1 \) so that at every point \( y' \in \partial X \cap (B(x, 7/8) \setminus B(x, 1/8)) \) the ball \( X \) is boundary \( \nu' \)-flat on all scales \( \leq s_1 \). Suppose not. Then there is a sequence of \( \mu''_k \rightarrow 0 \) and examples \( X_k = B(x_k, 1) \) of area \( \geq a \) that are boundary \( \mu''_k \)-good at \( x \) on scale 1, for which
there are points $y'_k \in \partial X_k \cap (B(x_k, 7/8) \setminus B(x_k, 1/8))$ at which $X_k$ is not boundary $\nu'$-flat on some scales $s'_k \to 0$. Passing to a subsequence, we can suppose that the $X_k$ converge to a limit $X = B(\mathbf{e}, 1)$ which is a flat cone of area $\geq a$. We can also assume that the $y'_k$ converge to $y'$ in the closure of $B(\mathbf{e}, 7/8) \setminus B(\mathbf{e}, 1/8)$, and by Lemma 3.2 we have $y' \in \partial X$. Thus, $X$ is boundary flat at $y'$ on a scale $\hat{s}_1 > 0$ that depends only on $a$, and hence for all $k$ sufficiently large, $X_k$ is boundary $\nu'$-flat at $y'_k$ on scale $\hat{s}_1$. This is a contradiction. Hence, there is $s_1 > 0$ such that assuming that $\mu''$ sufficiently small, $X$ is boundary $\nu'$-flat at every $y \in \partial X \cap (B(x, 7/8) \setminus B(x, 1/8))$ on scale $s_1$. By Lemma 3.10 this proves that for every such $y$, the ball $X$ is boundary $\mu$-flat on all scales $\leq s_1$ and for every point of $\partial X \cap (B(x, 7/8) \setminus B(x, 1/8))$ at distance $d \leq s_1/4$ of $\partial X$ is interior $\mu$-flat of all scales $\leq d/2$.

Now, provided that $\mu'' > 0$ is sufficiently small, we establish the existence of $s_2$ such that every point of $B(x, 7/8) \setminus B(x, 1/8)$ that is not within $s_1/4$ of $\partial X$ is interior $\nu$-flat on scale $s_2$. Suppose not. Then there are a sequence $\mu''_k \to 0$ and examples $X_k = B(x_k, 1)$ of area $\geq a$ that are boundary $\mu''_k$-good at $x_k$ on scale 1 and points $z_k \in B(x_k, 7/8) \setminus B(x_k, 1/8)$ at distance at least $s_1/4$ from $\partial X_k$ at which $X_k$ is not interior $\nu$-flat on some scale $s'_k \to 0$. Passing to a subsequence we have a limit $X$ which is a flat cone of area $\geq a$ and a limit $z \in X$ of the $z_k$. By Corollary 3.3 this is a point at distance at least $s_1/4$ from $\partial X$. Thus, by Lemma 3.10 $X$ is interior flat at $z$ on scale $s_1/8$. It follows that for all $k$ sufficiently large that $X_k$ is interior $\nu$-flat on scale $s_1/8$. This contradiction together with Lemma 3.6 proves the existence of $s_2$ as required.

Lastly, since the distance from the cone point in a flat cone is strongly regular on the corresponding annular region, given any $\delta > 0$, then assuming that $\mu''$ is sufficiently small, the distance from $x$ is $(1 - \delta)$-regular on $B(x, 7/8) \setminus B(x, 1/8)$. It follows from §11 of [3] that, provided that $\mu''$ is sufficiently small, for any $b \in [1/8, 7/8]$ the metric sphere $S(x, b)$ is an arc with endpoints in $\partial X$.

### 3.4 Geodesics approximating the boundary

It turns out that near the flat part of the boundary it is better to take neighborhoods centered around geodesics near the boundary rather than balls centered around boundary points. Here, we follow [23] closely.

**Definition 3.13.** Fix a 2-dimensional local Alexandrov space $X$ with curvature $\geq -1$. Suppose that $\gamma$ is an oriented geodesic in $X$ with initial point $e_-$ and final point $e_+$ and of length $\ell = \ell(\gamma)$. We define

$$f_\gamma = \frac{1}{2}(d(e_-, \cdot) - d(e_+, \cdot)) \quad \text{and} \quad h_\gamma = d(\gamma, \cdot).$$

These are 1-Lipschitz functions. Further, for any $\alpha > 0$ we define

$$\nu_\alpha(\gamma) = f_\gamma^{-1}([-\ell/4, \ell/4]) \cap h_\gamma^{-1}([0, \alpha\ell]),$$

and

$$\nu_\alpha(\gamma) = f_\gamma^{-1}([-\ell/4, \ell/4]) \cap h_\gamma^{-1}([0, \alpha\ell]).$$
We denote \( \nu_{\xi}^{0}(\gamma) = \nu_{\xi}(\gamma) \setminus \mathcal{P}_{\xi}^{2}(\gamma) \). The *ends* of \( \nu_{\alpha}(\gamma) \) are their intersections with \( f_{\gamma}^{-1}(\pm \ell/4) \), and the *side* of \( \mathcal{P}_{\alpha}(\gamma) \) is its intersection with \( h_{\gamma}^{-1}(\alpha \ell) \). For any \(-\ell/4 \leq a < b \leq \ell/4\) we set
\[
\nu_{\alpha,[a,b]}(\gamma) = f_{\gamma}^{-1}([a,b]) \cap h_{\gamma}^{-1}([0,\alpha \ell])
\]
and we denote by \( \mathcal{P}_{\alpha,[a,b]}(\gamma) \) its closure. As before, the boundary of \( \mathcal{P}_{\alpha,[a,b]}(\gamma) \) is made up of the side, given by \( h_{\gamma}^{-1}(\alpha \ell) \), and the two ends, given by \( f_{\gamma}^{-1}(a) \) and \( f_{\gamma}^{-1}(b) \). We say that \( \alpha \ell \) is the *width* of the neighborhood and \( (b-a)\ell \) is its *length*. The level set \( f_{\gamma}^{-1}(0) \) is the *center* of \( \nu_{\alpha}(\gamma) \).

**Lemma 3.14.** The following hold for any \( \xi > 0 \) sufficiently small and, given \( \xi \), for all \( \mu > 0 \) sufficiently small and for any \( s > 0 \). Suppose that \( X \) is a standard 2-dimensional ball and that \( \gamma \) is a geodesic of length between \( s/10 \) and \( s \) with endpoints in \( \partial X \). Then if there is a point \( x \in \partial X \) such that \( X \) is boundary \( \mu \)-flat at \( x \) on all scales \( \leq 5s \) and if \( \gamma \subset B(x,s) \) then the following hold.

1. The arcs \( \nu_{\xi}(\gamma) \cap \partial X \) and \( \gamma \cap \nu_{\xi}(\gamma) \) are within \( \xi^{2}\ell(\gamma) \) of each other in \( X \).

2. For each \( y \in \mathcal{P}_{\xi}(\gamma) \) the comparison angle \( \angle e_{-} ye_{+} \) is greater than \( \pi - 10\xi \).

3. For each point \( y \in \nu_{\xi}^{0}(\gamma) \) there is a geodesic \( \zeta \) from \( y \) to a point \( z \), with \( d(y,z) > 10\xi \ell \) such that for any \( w \in \zeta \) at distance at most \( 5\xi \ell \) from \( y \) the comparison angle \( \angle \gamma wz \geq \pi - \xi \), and the comparison angles \( \angle e_{\pm} wz \) are greater than \( \pi/2 - 10\xi \) and less than \( \pi/2 + 10\xi \).

4. For any level set \( L \) of \( f_{\gamma} \) in \( \mathcal{P}_{\xi}(\gamma) \) and for any \( c \in [\xi^{2},\xi] \) the distance from \( L \cap \gamma \to L \cap h_{\gamma}^{-1}(c\ell(\gamma)) \) is less than \( (1 + 2\xi)c\ell(\gamma) \).

(See Figure 3.)

**Proof.** Direct computation shows that the result holds for \( \xi > 0 \) sufficiently small, say \( \xi \leq \xi_{0} \) for some \( \xi_{0} > 0 \), for \( X \) being a ball of radius 1 centered at a boundary point of \( \mathbb{R} \times [0,\infty) \) and \( \gamma \) being a geodesic contained in \( \partial X \) of length between \( 1/50 \) and \( 1/5 \). Fix any \( 0 < \xi \leq \xi_{0} \). Now let \( B = B(x,5s) \) be a ball that is boundary \( \mu \)-flat at \( x \) on scale \( 5s \). Rescaling the metric by \( (1/5s)^{2} \), we can suppose that \( s = 1/5 \) and that \( B \) is a ball of radius 1. The result is now immediate by fixing \( \xi \) and taking limits as \( \mu \) tends to zero.

The exact same proof as in the above lemma shows the following result.

**Corollary 3.15.** The following holds for all \( \xi > 0 \) sufficiently small, and given \( \xi \) for all \( \mu > 0 \) sufficiently small. Let \( X \) be a standard 2-dimensional ball and suppose we have a geodesic \( \gamma \subset X \), a constant \( s \), and a point \( x \in X \) satisfying the hypotheses of the previous lemma. Suppose that \( \zeta \subset B(x,5s) \) is a geodesic of length between \( s/20 \) and \( 2s \) with endpoints in \( \partial X \). Fix a direction along \( \partial X \cap B(x,5s) \) and let endpoints of \( \gamma \) and \( \zeta \), denoted \( e_{-}(\gamma) \) and \( e_{+}(\zeta) \), be chosen so that in the given direction along \( \partial X \) we have \( e_{-}(\gamma) < e_{+}(\gamma) \) and \( e_{-}(\zeta) < e_{+}(\zeta) \). Suppose that there are constants \( c,c' \) with \( \xi \leq c,c' \leq 1 \). Then the following hold:
1. If $c\ell(\gamma) \leq (0.9)c\ell(\zeta)$, then the side of $\gamma_{\epsilon,\xi}(\zeta)$ is at distance at least $(0.05)c\ell(\zeta)$ from $\gamma_{\epsilon,\xi}(\gamma)$.

2. If $c\ell(\gamma) \geq (1.1)c\ell(\zeta)$, then the side of $\gamma_{\epsilon,\xi}(\gamma)$ is at distance at least $(0.05)c\ell(\gamma)$ from $\gamma_{\epsilon,\xi}(\zeta)$.

3. For any point $y \in \gamma_{\epsilon,\xi}(\gamma) \cap \gamma_{\epsilon,\xi}(\zeta)$, the comparison angles satisfy:
   
   $\bar{\zeta}e_{-}(\gamma)e_{+}(\zeta) > \pi - 10\xi$
   
   and
   
   $\bar{\zeta}e_{-}(\zeta)e_{+}(\gamma) > \pi - 10\xi$.

4. Suppose that a level set $L \subseteq \gamma_{\epsilon,\xi}(\zeta)$ for $f_{\zeta}$ meets $\nu_{\gamma}(\gamma)$. Then for any $y_1, y_2 \in L \cap \nu_{\gamma}(\gamma)$ we have
   
   $|f_{\gamma}(y_1) - f_{\gamma}(y_2)| < \xi^2\ell(\gamma)$.

(See Figure 3.)

**Definition 3.16.** We say that a geodesic $\gamma \subset X$ is a $\xi$-approximation to $\partial X$ on scale $s$ with $\mu$-control if $\gamma$ is a geodesic of length between $s/10$ and $s$ and if there is a point $x \in \partial X$ at which $X$ is boundary $\mu$-flat on scales $\leq 5s$ with $\gamma \subset B(x, s)$ such that:

1. the conclusions of Lemma 3.14 hold for $\xi$, and

2. the conclusions of Corollary 3.15 hold for $\xi$ and any geodesic $\zeta \subset B(x, 5s)$ of length between $s/20$ and $2s$.

The point $x$ is a $\mu$-control point for $\gamma$.

We must also compare flat regions near the boundary with balls around boundary points.

**Corollary 3.17.** The following holds for all $\xi > 0$ sufficiently small and, given $\xi$, for all $\mu > 0$ sufficiently small. Suppose that $X$ is a standard 2-dimensional ball that is boundary $\mu$-good at $y$ on scale $r$. Suppose that $\gamma \subset X$ is a $\xi$-approximation to $\partial X$ on scale $s \leq r/20$ that is contained in $B(y, 7r/8) \setminus B(y, r/8)$. We orient $\gamma$ so that $e_{+}$ is separated along $\partial X \cap B(y, r)$ by $e_{-}$ from $y$. Then for any $z \in \nu_{\gamma}(\gamma)$ the comparison angle $\bar{\zeta}yze_{+}$ is greater than $(.99)\pi$. Furthermore, for any level set $L$ of $d(y, \gamma)$ that meets $\nu_{\gamma}(\gamma)$ the intersection $L \cap \nu_{\gamma}(\gamma)$ is an interval with an endpoint in $\partial X$ and the other in the side of $\gamma_{\epsilon,\xi}(\gamma)$. Furthermore, the function $f_{\gamma}$ varies by at most $\xi^2\ell(\gamma)$ on this intersection. (See Figure 4.)

**Proof.** First we show the following.

**Claim 3.18.** There is $\omega$ depending only on $\mu$ and going to zero with $\mu$ such that the following holds. For any point $x \in \partial X \cap B(y, 15r/16)$ with $d(x, y) \geq r/16$ and any $w \in \partial X \cap B(y, r)$ that is separated from $y$ by $x$ along $\partial X \cap B(y, r)$, the comparison angle $\bar{\zeta}wxy > \pi - \omega$. 
3 2-DIMENSIONAL ALEXANDROV SPACES

Proof. First consider the point $w' \in \partial X \cap B(y, r)$ that is separated from $y$ by $x$ and lies at distance $\geq r/16$ from $x$. Since $B(y, r)$ is boundary $\mu$-good at $z$ on scale $r$, it follows that there is $\omega$ depending only on $\mu$ and going to zero as $\mu \to 0$ so that the comparison angle $\angle w'xy > \pi - \omega$. Next, we claim that for any $w \in \partial X \cap B(y, r)$ lying in the interior of the sub-interval of $\partial X$ with endpoints $x$ and $w'$ we have $\angle wxy \geq \angle w'xy$. To see this consider a geodesics $\alpha$ from $y$ to $w$ and $\beta$ from $x$ to $w'$. These must cross at exactly one point, say $u$. Also, let $\alpha'$ be a geodesic from $y$ to $x$ and $\beta'$ be a geodesic from $x$ to $w$. Then by the monotonicity of comparison angles $\angle wxy \geq \angle w'xy$. Also by monotonicity we have $\angle wxy + \angle w'xy$ is at most the sum of the angle between $\alpha'$ and $\beta$ at $x$ and the angle between $\beta$ and $\beta'$ at $x$. But since $x \in \partial X$, the sum of the angles between these geodesics is at most $\pi$. It follows that $\angle wxy + \angle w'xy \leq \pi$. Using this we see that $\angle yxw \geq \angle yxu + \angle uzw$. Thus, $\angle yxw \geq \angle yxw'$, completing the proof of the claim.

Let $z \in \nu_2(\gamma)$. Then $d(z, e_+) > \ell(\gamma)/4$ and $d(z, y) \geq r/4$. According to Lemma 3.14, $z$ is within $(1 + 2\xi)\ell(\gamma)$ of a point $q \in \gamma \cap \nu_2(\gamma)$ and every point of $\gamma \cap \nu_2(\gamma)$ is within $\xi^2\ell(\gamma)$ of a point in the boundary arc with endpoints $e_+$ and $e_-$. It follows that $z$ is within $(1 + 2\xi)\ell(\gamma)$ of a point $w$ in the arc with endpoints $e_+$ and $e_-$ of $\partial B(y, r)$. Since $\ell(\gamma) < r$, it follows from the law of cosines that there is $w'$ that goes to zero as $\omega$ and $\xi$ both go to zero, so that the comparison angle $\angle e_+zy$ is at least $\pi - \omega'$. This proves that given any $\omega > 0$, the angle $\angle yze_+ < \pi - \omega$ provided that $\xi$ and $\mu$ are sufficiently small.

With this estimate, the result now follows immediately by rescaling by to make $\ell(\gamma) = 1$ taking limits.

Corollary 3.19. Let $\xi > 0$ be given; fix $\mu > 0$ such that Lemma 3.14 and Corollaries 3.15 and 3.17 hold. Fix $a > 0$ and let $\mu'' > 0$ be such that Proposition 3.12 for these values of $\mu, a$ and $\mu''$. Then there are $s_1'$ and $s_2'$ such that the following holds. Suppose that $X = B(x, 1)$ is boundary $\mu''$-good on scale 1 at $x$ and of area $\geq a$. Then for any $b \in (1/8, 7/8)$, the metric sphere $S(x, b)$ is an interval with endpoints $y_1, y_2$ in $\partial X$. The space $X$ is boundary $\mu$-flat at $y_i$ on all scales $\leq s_1'$ and there are geodesics $\gamma_i$ that are $\xi$-approximations to $\partial X$ on scale $s_1'$ such that $S(x, b)$ is contained in the union of: (i) the open subset of points at which $X$ is interior $\mu$-flat on all scales $\leq s_2'$ and $\nu_{2, [\ell(\gamma_1)/8, \ell(\gamma_1)/8]}(\gamma_1) \cup \nu_{2, [\ell(\gamma_2)/8, \ell(\gamma_2)/8]}(\gamma_2).

Proof. Fixing $\xi, \mu, a, \mu''$ so that Proposition 3.12 holds, we take $s_1$ and $s_2$ as in Proposition 3.12. Then by §11 of 3, for any $b \in (1/8, 7/8)$ the metric sphere $S(x, b)$ is an interval with endpoints in $\partial X$. We denote these endpoints by $y_1$ and $y_2$. Let $\gamma_i$ be a geodesic of length $s_1$ with endpoints in $\partial X$ equidistance from $y_i$.

Claim 3.20. Every point of $S(x, b) \setminus (\nu_{2, [\ell(\gamma_1)/8, \ell(\gamma_1)/8]}(\gamma_1) \cup \nu_{2, [\ell(\gamma_2)/8, \ell(\gamma_2)/8]}(\gamma_2)$ is distance at least $\xi^2 s_1$ from $\partial X$.

Proof. Let $z \in S(x, b) \setminus (\nu_{2, [\ell(\gamma_1)/8, \ell(\gamma_1)/8]}(\gamma_1) \cup \nu_{2, [\ell(\gamma_2)/8, \ell(\gamma_2)/8]}(\gamma_2)$ and suppose that $\zeta$ is a geodesic from $z$ to a point $w \in \partial X$ with the length of $\zeta$ being less than $\xi^2 s_1$. Then we have $|d(w, x) - b| < \xi^2 s_1$. By the first statement in Corollary 3.17 provided that $\xi$ is
sufficiently small, the value of the function \( d(x, \cdot) \) at the endpoints of \( \nu_{\xi}^2(\gamma_i) \) differs by at least \( 3s_1/8 \). Since the endpoints of \( \gamma_i \) are equidistance from \( y_i \), it follows that the distance of each of these the endpoints from \( b \) by at least \( s_1/8 \). Since by Claim 3.15, the distance from \( x \) is monotone along \( \partial X \cap (B(x, 7/8) \setminus B(x, 1/8)) \), it follows that on \( \partial X \setminus (\nu_{\xi}^2(\gamma_1) \cup \nu_{\xi}^2(\gamma_2)) \) the distance from \( x \) takes no value within \( s_1/8 \) of \( b \). Hence, the point \( w \in \nu_{\xi}^2(\gamma_1) \cup \nu_{\xi}^2(\gamma_2) \). By symmetry we can suppose that \( w \in \nu_{\xi}^2(\gamma_1) \). This argument shows that the distance from \( w \) to the endpoints of \( \partial X \setminus \nu_{\xi}^2(\gamma_1) \) is at least \( s_1/16 \). Since \( w \in \nu_{\xi}^2(\gamma_1) \), the geodesic \( \zeta \) must cross the frontier of \( \nu_{\xi}^2(\gamma_1) \). But the distance from \( x \) of any point in the ends of \( \nu_{\xi}^2(\gamma_1) \) is within \( 200\xi^2s_1 \) of the distance from \( x \) of the corresponding endpoint of \( \partial X \setminus \nu_{\xi}^2(\gamma_1) \). This means that the distance from \( w \) to the ends of \( \nu_{\xi}^2(\gamma_1) \) is greater than \( s_1/32 \). Provided that \( \xi \) is sufficiently small, this means that \( \zeta \) cannot cross the ends of \( \nu_{\xi}^2(\gamma_1) \), and hence it must cross the side of this neighborhood. It then also crosses the geodesic \( \gamma_1 \) and hence its length is at least \( \xi^2s_1 \), which is a contradiction. This contradiction proves the claim.

Now every point \( z \in S(x, b) \setminus (\nu_{\xi}^2(\gamma_1) \cup \nu_{\xi}^2(\gamma_2)) \) is either not within \( s_1/4 \) of \( \partial X \), in which case by Lemma 3.12 the space \( X \) is interior \( \mu \)-flat on all scales \( \leq s_2 \) at \( z \), or \( z \) is within \( s_1/4 \) of \( \partial X \), and \( X \) is interior \( \mu \)-flat at \( z \) on all scales less than or equal to \( d(z, \partial X)/2 \), and by the previous claim \( d(z, \partial X) \geq \xi^2s_1 \). Taking \( s'_1 = s_1 \) and \( s'_2 = \min(s_2, \xi^2s_1/2) \) then gives the result.

**Definition 3.21.** If \( B(x, 7/8) \setminus B(x, 1/8) \) satisfies the conclusions of Corollary 3.19 then we say that it is a \((\xi, \mu, s'_1, s'_2)\)-**good strip**.

### 3.5 The covering

Now we assemble all the local results to give a covering of a standard 2-dimensional ball.

**Theorem 3.22.** The following holds for all \( \xi > 0 \) sufficiently small, for all \( \mu > 0 \) less than a positive constant \( \mu_1(\xi) \), and for all \( a > 0 \). There are positive constants \( \delta \) and \( r_0 \), depending on \( \xi, \mu, a \), with \( r_0 \leq 10^{-3} \), such that for all \( s_1 > 0 \) less than a positive constant \( \tilde{s}_1(\xi, \mu, a, s_1) \leq r_0/20 \) and for all \( s_2 > 0 \) less than a positive constant \( \tilde{s}_2(\xi, \mu, a, s_1) \leq s_1 \), for any standard 2-dimensional ball \( X = B(x, 1) \) of area \( \geq a \), the ball \( B(x, 1/2) \) is covered by open subsets of the following four types:

1. The open subset of points \( y \in B(x, 1/2) \) with the property that \( X \) is interior \( \mu \)-flat on all scales \( \leq s_2 \) at \( y \).

2. The open subset of points that lie in the center of neighborhoods \( \nu_{\xi}^2(\gamma) \) where \( \gamma \) is a \( \xi \)-approximation to \( \partial X \) at scale \( s_1 \) with \( \mu \)-control.

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7 This means that \( \mu_1(\xi) \) is a constant that depends on \( \xi \). We shall use a similar convention throughout.
3. Open balls $B(y,r'/4)$ for some $r_0 \leq r' = r'(y) \leq 10^{-3}$ such that the open ball $B(y,r')$ is interior $\mu$-good at $y$ at scale $r'$ and angle $\leq 2\pi - \delta$ and

$$\frac{1}{r'} B(y,7r'/8) \setminus \frac{1}{r'} B(y,r'/8)$$

is a $(\mu,s_2)$-good annular region.

4. Open balls $B(y,r'/4)$, for some $r_0 \leq r' = r'(y) \leq 10^{-3}$, such that $X$ is boundary $\mu$-good at $y$ on scale $r'$ and of angle $\leq \pi - \delta$ and also $\frac{1}{r'} B(y,7r'/8) \setminus \frac{1}{r'} B(y,r'/8)$ is a $(\xi,\mu,s_1,s_2)$-good strip.

Proof. Fix $\xi > 0$ sufficiently small and let $\mu > 0$ be as in Corollary 3.19 for this value of $\xi$. Let $\eta$ and $\eta'$ be the values associated to $\mu$ by Lemma 3.6 and Lemma 3.10 respectively. Fix $a > 0$. Suppose that the result is not true. Then take decreasing sequences $\{k_i,r_{k_i},s_{1,k_i},s_{2,k_i}\}$ tending to zero, and suppose that there are examples $X_k = B(x_k,1)$, each satisfying the hypotheses for the given values of $\xi,\mu$, and $a$ and points $y_k \in B(x_k,1/2)$ not contained in an open set of any of the four types for any fixed values of the parameters $\delta_k, r_{0,k}, s_{1,k}, s_{2,k}$. Passing to a subsequence we can suppose that the $X_k$ converge to $X = B(\pi,1)$ and the $y_k$ converge to $y$ in the closure of $B(\pi,1/2)$. Since the $X_k$ all have area at least $a$, $X$ is a standard 2-dimensional ball of area at least $a$. By Lemma 2.14 given any $\epsilon > 0$, there is a constant $r = r(y) > 0$ such that $\frac{1}{r} B(y,r)$ is within $\epsilon$ in the Gromov-Hausdorff distance to the unit ball in a cone $C$. We can assume that $r \leq 10^{-3}$. Because the area of $X$ is $\geq a$, there is $a' > 0$, depending only on $a$, such that the area of $C$ is at least $a'$. The cone is either a flat cone in $\mathbb{R}^2$, if $y \in \partial X$, or a circular cone, if $y \in \text{int} X$. We consider the various cases.

**Case 1:** The cone is a circular cone of angle $2\pi$. Provided that we have chosen $\epsilon < \eta$, the ball $X$ is interior $\nu$-flat at $y$ on scale $s$ and hence for all $k$ sufficiently large $X_k$ is interior $\nu$-flat at $y_k$ on scale $s$ and hence by Lemma 3.6 for all $k$ sufficiently large $X_k$ is $\mu$-flat at $y_k$ on all scales $\leq s$. This is a contradiction since $s_{2,k} < s$ for all $k$ sufficiently large.

**Case 2:** The cone is a circular cone of angle $2\pi - \delta$ for some $\delta > 0$. In this case provided that we have chosen $\epsilon$ less than the $\mu'$ determined by $\mu$ and $a'$ by Proposition 3.7 it follows that $\frac{1}{r} B(y,r)$ is interior $\mu'$-good at $y$ on scale 1. The same is true for the $\frac{1}{r} B(y_k,r)$ for all $k$ sufficiently large. Hence, by Proposition 3.7 there is $s_0 = s_0(\alpha)$ such that for all $k$ sufficiently large, the region $\frac{1}{r} B(y_k,7r/8) \setminus \frac{1}{r} B(y_k,r/8)$ is a $(\mu,s_0(\alpha))$-good annular region. This is a contradiction since $r_{0,k} < r \leq 10^{-3}$ and $\delta_k < \delta$ and $s_{k,k} < s_0$ for all $k$ sufficiently large.

**Case 3:** The cone is a flat cone in $\mathbb{R}^2$ of cone angle $\pi$. This means that the limit is boundary flat at $y$ on scale $r$. Provided that we choose $\epsilon$ less than the $\nu'$ determined by $\mu$ by Lemma 3.10 it follows that for all $k$ sufficiently large, $X_k$ is boundary $\nu'$-flat at $y_k$ on scale $r$. Hence, by Lemma 3.10 for all $k$ sufficiently large $X_k$ is boundary $\mu$-flat at $y_k$ on all scales $\leq r$. Choose a geodesic $\gamma_k$ of length $r/20$ with endpoints on $\partial X_k$ equidistant from $y_k$. According to Lemma 3.14 this geodesic is a $\xi$-approximation to $\partial X_k$. Clearly, the limit of the $\gamma_k$ is a geodesic contained in $\partial X$ whose midpoint is $y$. In particular, $\nu_{\xi,2}(\gamma)$ contains an entire neighborhood of
y, and hence for all k sufficiently large the neighborhood $\nu_{\xi^2}(\gamma_k)$ contains $y_k$. Since the endpoints of $\gamma_k$ are equidistance for $y_k$, the point $y_k$ lies in the center of this neighborhood. This is a contradiction since for all $k$ sufficiently large $s_{1,k} < r/20$.

**Case 4: The cone is a flat cone in $\mathbb{R}^2$ of cone angle $\pi - \delta$ for some $\delta > 0$.** Provided that we have chosen $\epsilon$ less than the constant $\mu''$ associated to $\mu$ and $a'$ by Proposition 3.12, it follows that $X$ is boundary good at $y$ of angle $\pi - \delta$ on scale $r$. Hence, the same is true for $X_k$ at $y_k$ for all $k$ sufficiently large. It then follows from Corollary 3.19 that there are constants $s_1', s_2' > 0$ such that $\frac{1}{r}B(y_k, 7r/8) \setminus \frac{1}{r}B(y_k, r/8)$ is a $(\xi, \mu, s_1', s_2')$-good product region. This is a contradiction since $\delta_k < \delta$, $s_{1,k} < s_1'$ and $s_{2,k} < s_2'$ for all $k$ sufficiently large.

In all cases we have arrived at a contradiction, proving the result. □

### 3.6 Transition between the 2- and 1-dimensional part

We need to understand the passage between the 1- and 2-dimensional parts of the $M_n$. A 1-dimensional standard ball $B(x, 1)$ is either an open interval of length 2 or is a half-open interval of length $\ell$ with $1 \leq \ell \leq 2$.

**Lemma 3.23.** The following hold for all $\beta > 0$ and for all $a > 0$ less than a positive constant $a_2(\beta)$. Let $B(x, 1)$ be a standard 2-dimensional ball and suppose that there is a point $y \in B(x, 24/25)$ with the area of $B(y, 1/100)$ being at most $a$. Then $B(x, 1)$ is within $\beta$ in the Gromov-Hausdorff distance of a standard 1-dimensional ball $J$.

**Proof.** Fixing $\beta > 0$ suppose that the result does not hold for any $a > 0$. Take a sequence $B(x_k, 1)$ of standard 2-dimensional balls of area $a_k \to 0$ as $k \to \infty$ and points $y_k \in B(x_k, 24/25)$ for which the result does not hold. Passing to a subsequence we can extract a limit $\overline{B}$ with the $y_k$ converging to $\overline{y} \in \overline{B}$. Because of the area condition, the neighborhood $B(\overline{y}, 1/100)$ must be 1-dimensional, and hence $\overline{B}$ is a standard 1-dimensional ball. □

### 4 3-dimensional analogues

Now we discuss the structure of balls in a 3-dimensional Riemannian manifold that are close to the various 1- and 2-dimensional balls that we have been discussing. Since we shall need the results for 3-dimensional balls near 2-dimensional balls in our study of 3-dimensional balls near 1-dimensional balls, we start with the 2-dimensional case. Recall that for any $x \in M_n$ we denote by $g_n(x)$ the rescaled metric $\rho_n(x)^{-2}g_n$. Throughout this section we consider pairs consisting of a point $x_n \in M_n$ and a constant $\lambda \geq \rho_n(x_n)^{-2}$ with the property that $B_{\lambda g_n}(x_n, 1)$ is disjoint from $\partial M_n$. Of course, since $\lambda \geq \rho_n(x_n)^{-2}$ the sectional curvatures of these balls is bounded below by $-1$. Any time we refer to such $B_{\lambda g_n}(x_n, 1)$, unless we explicitly state the contrary, we are implicitly assuming that it is disjoint from the boundary.

#### 4.1 Generic interior points of 2-dimensional Alexandrov spaces

We begin with a description of the 3-dimensional part of a Riemannian 3-manifold $M$ that is near the ‘generic’ part of a 2-dimensional Alexandrov space.
Lemma 4.1. For all $\epsilon > 0$ and any $0 < s_2 \leq 1/2$, the following holds for all $\mu > 0$ less than a positive constant $\mu_2(\epsilon)$ and for all $\hat{c} > 0$ less than a positive constant $\hat{c}_0(s_2, \epsilon)$. Suppose that the ball $B_{\lambda g_n}(x_n, 1)$ is within $\epsilon$ of a 2-dimensional Alexandrov ball $B = B(\ol{x}, 1)$ which is interior $\mu$-flat at $\ol{x}$ on scale $s_2$. Then there exist an embedding $\varphi: S^1 \times B(0, \epsilon^{-1}) \to M_1$ with $x \in \varphi(S^1 \times \{0\})$ and a constant $\lambda' > \epsilon^{-2}\lambda$ such that the metric $\varphi^*\lambda g$ is within $\epsilon$ in the $C^{[1/\epsilon]}$-topology to the product of the metric of length 1 on the circle and the restriction of the standard Euclidean metric to $B(0, \epsilon^{-1})$. Lastly, there is a universal constant $C > 0$ such that, measured using $\lambda g_n$, the lengths of the circles in this product structure are less than $C\hat{c}$.

Proof. Let us first show that it suffices to prove the first conclusion for $s_2 = 1/2$. For, suppose that we have established the conclusion in this special case with constants $\mu_1(\epsilon)$ and $\hat{c}_0(\epsilon, 1/2)$, and let us consider the statement for another value $0 < s_2 \leq 1/2$. Then suppose for some $\mu < \mu_1(\epsilon)$ and $\hat{c} < 2s_2\hat{c}_0(\epsilon, 1/2)$ that we have $B_{\lambda g_n}(x_n, 1)$ within $\epsilon$ of $B(\ol{x}, 1)$, the latter being interior $\mu$-flat at $\ol{x}$ on scale $s_2$. Then $B_{(\lambda/2s_2^{n-1})g_n}(x_n, 1)$ is within $\hat{c}/(2s_2)$ of $\frac{1}{2s_2^2}B(\ol{x}, 2s_2)$, and the latter is $\mu$-flat at $\ol{x}$ on scale $1/2$. Since, by construction, $\hat{c}/(2s_2) < \hat{c}_0(\epsilon, 1/2)$, the result for $s_2 = 1/2$ implies the existence of a constant $\lambda'$ as required. (Of course, $\lambda' > (1/2s_2)\lambda$ since $B_{(1/4s_2^2)\lambda g_n}(x_n, 1)$ is close to a 2-dimensional ball whereas $B_{\lambda g_n}(x_n, 1)$ has 3-dimensional volume bounded away from zero.)

Thus, we can now assume that $s_2 = 1/2$. Fix $\epsilon > 0$ and suppose that the first conclusion does not hold for this constant. Then there is a sequence of $k \to \infty$ and $\epsilon_k \to 0$ both tending to zero as $k \to \infty$ such that for each $k$ there is an index $n(k)$ and a point $x_{n(k)} \in M_{n(k)}$ and constants $\lambda_k \geq \rho_{n(k)}(x_{n(k)})^{-2}$ so that the ball $B_{\lambda_k g_n(k)}(x_{n(k)}, 1)$ is within $\epsilon_k$ of a 2-dimensional Alexandrov ball $B_k = B(\ol{x}_k, 1)$ that is interior $\mu_k$-flat at $\ol{x}_k$ on scale $1/2$, yet no $x_{n(k)}$ satisfies the first conclusion of the lemma. Fix a sequence of positive constants $c_k \to \infty$ such that $c_k^2\epsilon_k \to 0$ as $k \to \infty$. The 2-dimensional balls $c_kB(\ol{x}_k, 1)$ are $\mu_k$-flat at $\ol{x}_k$ on scale $c_k/2$, and since $c_k \to \infty$, these rescaled balls converge in the Gromov-Hausdorff topology to $\mathbb{R}^2$. On the other hand, since $c_k\epsilon_k \to 0$, the balls $c_kB_{\lambda_k g_n(k)}(x_{n(k)}, 1)$ also converge to the same limit, $\mathbb{R}^2$, in the Gromov-Hausdorff topology. In particular, the volume of the unit balls centered at $x_{n(k)}$ in $c_kB_{\lambda_k g_n(k)}(x_{n(k)}, 1)$ tend to zero as $k \to \infty$.

Fix $\omega$ equal to one-half the volume of the 3-dimensional Euclidean ball of radius 1. We rescale, forming $\tilde{B}_k = c_kB_{\lambda_k g_n(k)}(x_{n(k)}, 1))$ in such a way that the volume of the unit ball in $\tilde{B}_k$ centered at $x_{n(k)}$ is $\omega$. These rescaling factors divided by $c_k$ tend to infinity, so that $\tilde{B}_k$ has a $(2, \mu_k)$ strainer at $x_{n(k)}$ of size that tends to infinity as $k \to \infty$. Let $(\tilde{B}, \ol{x})$ be the limit of a subsequence. By Proposition 2.3 this limit is a smooth, complete manifold of non-negative curvature and without boundary, and the convergence is a smooth. The existence of the $(2, \mu_k)$-strainers of size going to infinity in the sequence implies that there is an isometric copy of $\mathbb{R}^2$ through $\ol{x}$ in $\tilde{B}$. Hence, by Corollary 2.4 $\tilde{B}$ splits as a product of $\mathbb{R}^2$ with a complete, connected 1-manifold without boundary. This 1-manifold cannot be $\mathbb{R}^1$ because the volume of the unit ball in $\tilde{B}$ is one-half the volume of the unit ball in Euclidean space. Thus, $\tilde{B}$ is the product of a circle with $\mathbb{R}^2$. Rescaling again by a fixed constant, we can make the limit the product of the circle of length 1 with $\mathbb{R}^2$. The conclusion of the
lemma then holds for all $k$ sufficiently large by taking limits. This is a contradiction and proves the existence of the map $\varphi$ as required.

From this and the fact that $B_{\lambda g_n}(x_n, 1)$ is within $\hat{\epsilon}$ of a 2-dimensional ball, it is easy to see that the lengths of the fibers are at most $C\hat{\epsilon}$ for some universal constant $C$.

**Definition 4.2.** Anytime we have an embedding $\varphi: S^1 \times B(0, \epsilon^{-1}) \to M$ with $x \in \varphi(S^1 \times \{0\})$ that satisfies the conclusion of the previous lemma, we say that $x$ is the center of a $S^1$-product neighborhood with $\epsilon$-control. The horizontal spaces of an $S^1$-product neighborhood are the subspaces $\varphi(\{\theta\} \times B(0, \epsilon^{-1}))$ for $\theta \in S^1$.

We need a semi-local version of this result.

**Proposition 4.3.** Fix $\epsilon' > 0$ sufficiently small. Then there is $\epsilon_0(\epsilon') > 0$ such that the following hold for all $\epsilon < \epsilon_0$. Let $0 < \mu \leq \mu_2(\epsilon)$ as in Lemma 4.1. For any $d > 0$ there is $\tau(\epsilon, \mu, d) > 0$ such that the following holds for all $\hat{\epsilon} < \tau(\epsilon, \mu, d)$. Suppose that $B_{\lambda g_n}(x_n, 1)$ is within $\hat{\epsilon}$ of a standard 2-dimensional ball $B(\tau, 1)$. Suppose that $(a_1, a_2, b_1, b_2)$ is a $(2, \mu)$-strainer at a point $\overline{y} \in B(\tau, 1/2)$ of size $d$. Then there is a constant $d' > 0$ depending only on $d$ such that the following holds. Let $y_n \in B_{\lambda g_n}(x_n, 1/2)$ be a point within distance $\hat{\epsilon}$ of $\overline{y}$ and let $\tilde{a}_1, \tilde{a}_2$ be points of $B_{\lambda g_n}(x_n, 1)$ within $\hat{\epsilon}$ of $a_1, a_2$. Then there is an open subset $U_n$ with $B_{\lambda g_n}(y_n, d') \subset U_n \subset B_{\lambda g_n}(y_n, 2d')$ such that the function $F = (f_1, f_2)$, where $f_i = d(\tilde{a}_i, \cdot)$, determines a (topological) fibration of $U_n$ by circles over an open topological ball. Furthermore, for each $x \in U_n$ there is an $S^1$-product neighborhood with $\epsilon$-control centered at $x$, and the fiber through $x$ of this fibration makes an angle within $\epsilon'$ of $\pi/2$ to the horizontal spaces of this $S^1$-product structure. Lastly, this fiber is isotopic in that $S^1$-product neighborhood to the $S^1$-factor.

**Proof.** For any $\mu' > 0$, provided that $d' \ll d$ and $\hat{\epsilon}$ are sufficiently small, choosing points $b_1, b_2$ in $B_{\lambda g_n}(x_n, 1)$ within $\hat{\epsilon}$ of $b_1, b_2$, the quadruple $\{\tilde{a}_1, \tilde{a}_2, b_1, b_2\}$ is a $(2, \mu + \mu')$-strainer at any point $B_{\lambda g_n}(y_n, d')$ of size $d/2$. By Lemma 4.1 provided that $\mu + \mu'$ is sufficiently small, for all $\hat{\epsilon}$ sufficiently small (depending on $\mu + \mu'$ and $d'$) every point of $B_{\lambda g_n}(y_n, d')$ is the center of an $S^1$-product neighborhood with $\epsilon$-control. Clearly, the geodesics from each of the four points of the strainer to any point of this ball are almost horizontal in the $S^1$-product structure. It follows from Section 11 of [3] that the fibers of $F$ are circles and that they are almost orthogonal to all geodesics from the four points of the strainer, and hence they are almost orthogonal to the horizontal spaces. All the errors go to zero $\mu \to 0$, $d' \to 0$ and $\hat{\epsilon} \to 0$. Lastly, since the restriction of $F$ to any horizontal space of an $S^1$-product structure is a homeomorphism into, the fibers of the fibration structure on $U_n$ ‘go around’ the $S^1$-direction once and hence are isotopic to the $S^1$-factor.

This is a semi-local result: it is not small or the order of the fiber but it is small on the order of the base. But there is a truly global result obtained by piecing together the $S^1$-product structures to form a global $S^1$-fibration.

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8By angle we mean the limit as $q' \in S(q)$ approaches $q$ of angle between the direction of any geodesic from $q$ to $q'$ and the horizontal subspace at $q$. 
4.2 The global $S^1$-fibration

**Proposition 4.4.** Given $\epsilon' > 0$, the following holds for all $\epsilon > 0$ less than a positive constant $\epsilon_1(\epsilon')$. Suppose that $K \subset M$ is a compact subset and each $x \in K$ is the center of an $S^1$-product neighborhood with $\epsilon$-control. Then there is an open subset $V$ containing $K$ and a smooth $S^1$-fibration structure on $V$. Furthermore, if $U$ is an $S^1$-product structure with $\epsilon$-control that contains a fiber $F$ of the fibration on $V$, then $F$ is within $\epsilon'$ of vertical in $U$ and $F$ generates the fundamental group $U$. In particular, the diameter of $F$ is at most twice the length of any circle in the $S^1$-product structure centered at any point of $F$.

The proof of this proposition takes up this entire subsection. For $\epsilon > 0$ sufficiently small, we set $N = [1/\epsilon]$. Recall that an $S^1$-product neighborhood $U \subset M$ is the image $\varphi(S^1 \times B(0, \epsilon^{-1}))$ with the property that there is $\lambda_U > 0$ such that $\varphi^*(\lambda_U g)$ is within $\epsilon$ in the $C^N$-topology of $g_0$, the product of the Riemannian metric of length 1 on $S^1$ and the usual Euclidean metric on the ball $B(0, \epsilon^{-1})$ in the plane.

4.2.1 Comparing the standard metrics on the overlap

The first thing to do is to show that on the overlap of $S^1$-product neighborhoods the standard metrics are close.

**Claim 4.5.** Given $\epsilon' > 0$ there is $\epsilon > 0$ such that the following holds. Suppose that $U_1 = \varphi_1(S^1 \times B(0, \epsilon^{-1}))$ and $U_2 = \varphi_2(S^1 \times B(0, \epsilon^{-1}))$ are $S^1$-product neighborhoods with $\epsilon$-control in $M$. Suppose that there is a point

$$x \in \varphi_1(S^1 \times B(0, \epsilon^{-1}/2)) \cap \varphi_2(S^1 \times B(0, \epsilon^{-1}/2)).$$

Then for $i = 1, 2$ the circle factor $F_i$ though $x$ in the product structure on $U_i$ is within $\epsilon'$ of vertical in the product structure of $U_{3-i}$. The length of this fiber is between $1 - \epsilon'$ and $1 + \epsilon'$ times the length of any circle factor in the product structure of $U_{3-i}$ as is the ratio $\lambda_{U_i}/\lambda_{U_2}$. The homotopy class of $F_i$ generates $\pi_1(U_{3-i})$.

**Proof.** Without loss of generality we can assume that $\lambda_{U_2} \geq \lambda_{U_1}$. Let $\zeta$ be the $g$-shortest homotopically non-trivial loop through $x$ in $U_2$. Its $g$-length is close to $\lambda_{U_2}^{-1/2}$. Hence, it is contained in $U_1$ and its length with respect to the product metric $g_0$ on $U_1$ is close to $(\lambda_{U_1}/\lambda_{U_2})^{1/2} \leq 1$. Let us suppose that it is homotopically trivial in $U_1$. Then it bounds a disk contained in the $g$-neighborhood of size $2\lambda_{U_2}^{-1/2}$ of $x$. This disk is then contained in $U_2$, which is a contradiction. It follows that $\zeta$ is a homotopically non-trivial loop in $U_1$ through $x$. Since its length in the metric $g_0$ on $U_1$ is close $(\lambda_{U_1}/\lambda_{U_2})^{1/2} \leq 1$, the loop $\zeta$ generates the fundamental group of $U_1$. It follows that $\lambda_{U_1}/\lambda_{U_2}$ must be close to one. The errors in these estimates go to zero as $\epsilon$ tends to zero.

**Corollary 4.6.** We continue with the notation of the previous claim. Given $\epsilon' > 0$ if $\epsilon > 0$ is sufficiently small then the restrictions of $(\varphi_1^{-1})^*g_0$ and $(\varphi_2^{-1})^*g_0$ to $\varphi_1(S^1 \times B(0, \epsilon^{-1}/2)) \cap \varphi_2(S^1 \times B(0, \epsilon^{-1}/2))$ are within $\epsilon'$ in the $C^N$-topology.
4.2.2 Bounding the intersections

Now we turn to constructing a finite cover with a uniformly bounded number of neighborhoods meeting any given neighborhood.

**Claim 4.7.** Fix $R < \infty$ and $\epsilon' > 0$. Then for all $\epsilon > 0$ sufficiently small, there is a finite collection of $S^1$-product neighborhoods with $\epsilon$-control

$$\varphi_1(S^1 \times B(0, \epsilon^{-1})), \ldots, \varphi_T(S^1 \times B(0, \epsilon^{-1}))$$

such that the union of the images $U'_i = \varphi_i(S^1 \times B(0, R))$ cover $K$, and the $\varphi_i(S^1 \times B(0, R/3))$ are disjoint. Furthermore for every $i, j$, $(\varphi_i^{-1})^* g_0$ and $(\varphi_j^{-1})^* g_0$ are within $\epsilon'$ in the $C^N$-topology for Riemannian metrics

$$\varphi_i(S^1 \times B(0, \epsilon^{-1}/2)) \cap \varphi_j(S^1 \times B(0, \epsilon^{-1}/2)).$$

**Proof.** Fix $\epsilon > 0$ sufficiently small. If $\varphi_i(S^1 \times B(0, R/3)) \cap \varphi_j(S^1 \times B(0, R/3)) \neq \emptyset$, then, by the previous result, the standard metrics on the two images almost agree, and in particular, their union is contained in $\varphi_i(S^1 \times B(0, R))$. Take a collection $\{\tilde{U}_i = \varphi_i(S^1 \times B(0, \epsilon^{-1}))\}$ of $S^1$-product neighborhoods with $\epsilon$-control centered at points of $K$, maximal with respect to the property that the $\varphi_i(S^1 \times B(0, R/3))$ are disjoint. Then the $U'_i = \varphi_i(S^1 \times B(0, R))$ cover $K$. If we have chosen $\epsilon > 0$ sufficiently small, the last statement follows from the previous result. \qed

**Claim 4.8.** Given $R > 4$, there is an integer $C = C(R)$ such that the following holds for all $\epsilon > 0$ sufficiently small. Let $(M, g)$ be a Riemannian 3-manifold with curvature $\geq -1$. Suppose that we have a collection $\{\tilde{U}_i = \varphi_i(S^1 \times B(0, \epsilon^{-1}))\}_i$ of $S^1$-product neighborhoods with $\epsilon$-control. Let $U_i$ be the image of $\varphi_i(S^1 \times B(0, R + 1))$. Suppose also that $\varphi_i(S^1 \times B(0, R/3)) \cap \varphi_j(S^1 \times B(0, R/3)) = \emptyset$ for all $i \neq j$. Then for each $i$ the number of $j$ for which $U_i \cap U_j \neq \emptyset$ is at most $C$.

**Proof.** This is immediate from the fact that the standard metrics almost agree on the overlaps of the $U_i$. \qed

For $R < \epsilon^{-1}$ we define a reduced $S^1$-product structure with $\epsilon$-control of size $R$ to be an embedding $\varphi : S^1 \times B(0, R) \to M$ with the property that there is $\lambda > 0$ such that $\varphi^* \lambda g$ is within $\epsilon$ in the $C^N$-topology to the standard product metric $g_0$ on this product.

Fix $R$ and a covering $\{U_a\}_{a \in A}$ of $K$ as in Claim 4.7. It follows directly from Claim 4.8 that we can divide the open sets $\{U_a\}$ into $C$ groups $U_1, \ldots, U_C$ with the following properties:

1. Each $U_i$ consists of a disjoint union of finite number of the $U_a$, denoted $U_{i,1}, \ldots, U_{i,j_0(i)}$.
2. Each $U_a$ in the original collection occurs as exactly one of the $U_{i,j}$, so that in particular, setting $U'_i$ equal to the images $\varphi_{i,j}(S^1 \times B(0, R))$ for $1 \leq j \leq j_0(i)$, the union $\bigcup_{i=1}^C U'_i$ covers $K$.

**Definition 4.9.** For each $0 \leq D \leq 1$ we define $U_i^{[D]}$ to be the union of the images $\varphi_{i,j}(S^1 \times B(0, R + 1 - D))$. Notice that $U'_i = U_i^{[1]}$. 

4.2.3 The Gluing

Given a smooth fibration of a Riemannian manifold $M$ by circles, then on each fiber $F$ there is a unique measure $d\mu_F$ that is conformally equivalent to the measure induced by the restriction of the Riemannian metric to $F$ and in which $F$ has total length 1.

Suppose that we have an open subset $W \subset M$ that is the union of restrictions of $S^1$-product neighborhoods with $\alpha$-control to subsets $U_i = \varphi_i(S^1 \times B(0, R'))$ for some $R \leq R' \leq R + 1$, and suppose that the circle fibrations of the various $U_i$ are compatible so that they define a circle fibration on $W$. Suppose also that we have a reduced $S^1$-product structure with $\epsilon$ control $\varphi: S^1 \times B(0, R + 2) \to M$. Let $U = \varphi(S^1 \times B(0, R + 1))$. Assuming that $\alpha$ and $\epsilon$ are sufficiently small, let us define a map from the saturation of $U \cap W$ under the $S^1$-fibration on $W$ to $S^1 \times B(0, R + 2)$. For $\alpha$ and $\epsilon$ sufficiently small this saturation is contained in $\varphi(S^1 \times B(0, R + 2))$. Suppose that $p$ is a point of the saturation of $U \cap W$, say $p = \varphi(\theta, x)$. Let $F_p$ be the fiber of the fibration structure on $W$ through $p$. For each $q \in F_p$ we have $(\theta(q), x(q))$ defined by $\varphi^{-1}(q) = (\theta(q), x(q))$, so that $x: F_p \to B(0, R + 2)$. We form $\hat{x}(p) = \int_{F_p} x(q) d\mu_{F_p}$ and define the map

$$\psi(p) = (\theta(p), \hat{x}(p)).$$

The following is obvious from the definitions

**Claim 4.10.** If $F$ is an orbit of the $S^1$-fibration on $W$ passing through a point of $U$, then $\hat{x}: F \to B(0, R + 2)$ is constant.

Denote by $\text{Sat}(U \cap W)$ the saturation of $U \cap W$ under the fibration structure on $W$.

**Corollary 4.11.** Given $\epsilon_1 > 0$, then for all $\alpha, \epsilon > 0$ sufficiently small, the map $\hat{x}: \text{Sat}(U \cap W) \to B(0, R + 2)$ is within $\epsilon_1$ in the $C^{N+1}$-topology of the restriction to $\text{Sat}(U \cap W) \subset U$ of the composition of $\varphi^{-1}$ with the projection in product structure to $B(0, R + 2)$.

**Proof.** It follows immediately from Corollary 4.10 that the fibers of the $S^1$-fibration on $\text{Sat}(U \cap W)$ induced from the fibration on $W$ are geodesics in a metric that is $C^N$-close to the metric $g_0$ on $U$. From this we see that the map $p \mapsto \hat{x}(p)$ is $C^{N+1}$-close to the composition of $\varphi^{-1}$ with the projection to $B(0, R + 2)$ with the same error estimate.

It follows from Corollary 4.11 that given $\epsilon_1 > 0$, there is a constant $\alpha_0(\epsilon_1) > 0$ such that if $\alpha$ and $\epsilon$ are less than $\alpha_0(\epsilon_1)$, then we can define a map $\psi: \text{Sat}(U \cap W) \to S^1 \times B(0, R + 2)$ by sending $p = \varphi(\theta, x)$ to $\psi(p) = (\theta(p), \hat{x}(p))$. Again invoking Corollary 4.11 we see that:

**Corollary 4.12.** Provided that $\alpha$ and $\epsilon$ are less that $\alpha_0(\epsilon_1)$, the composition

$$\text{Sat}(U \cap W) \xrightarrow{\psi} S^1 \times B(0, R + 2) \xrightarrow{\varphi} \varphi(S^1 \times B(0, R + 2))$$

is within $\epsilon_1$ of the inclusion of $\text{Sat}(U \cap W) \subset \varphi(S^1 \times B(0, R + 2))$ in the $C^{N+1}$-topology.
Let $\beta : [0, R'] \to [0, 1]$ be a function that is identically 1 near $R'$ and identically zero on a neighborhood of $[0, R' - 1/C]$. We define $\beta_i : U_i \to [0, 1]$ by $\beta_i(\varphi_i(\theta, x)) = \beta_i(|x|)$. For all $i$ such that $U_i \cap U \neq \emptyset$, the gradients of the $\beta_i$ with respect to $\lambda_U g$ are bounded independent of $i$. (Recall that $\lambda_U g$ is the multiple of $g$ which is close to the standard product metric $g_0$ on $U$.) We set $\hat{\beta} : W \to [0, 1]$ equal to the product over the $i$ of the $\beta_i$. This function is identically 1 in the complement of $W$ and the restriction to $U$ of $\hat{\beta}$ has a gradient with respect to $g_0$ that is bounded depending only on $C$. Define $\Psi : U \to S^1 \times B(0, R + 2)$ by

$$\Psi(p) = \beta(p)\varphi^{-1}(p) + (1 - \beta(p))\psi(p).$$

Claim 4.13. Given $\epsilon_1$ there is $\alpha_1 = \alpha_1(\epsilon_1) > 0$ such that if $\alpha$ and $\epsilon$ are less than $\alpha_1$, then $\Psi$ is within $\epsilon_1$ of $\varphi^{-1}$ in the $C^N + 1$-topology using the metrics $\lambda_U g$ on the domain and $g_0$ on the range.

Proof. This follows immediately from Corollary 4.12.

We set $W' \subset W$ equal to $\beta^{-1}(0)$.

Claim 4.14. $W'$ is the union of $\varphi_i(S^1 \times B(0, R''))$ where $R'' = R' - 1/C$. In particular, $W'$ is saturated under the $S^1$-fibration structure on $W$. The image of $\Psi$ contains $S^1 \times B(0, R + 1 - 1/C)$. Setting $\varphi' : S^1 \times B(0, R + 1 - 1/C) \to M$ equal to the restriction of the inverse of $\Psi$, we have

1. $\varphi'$ is a reduced $S^1$-product neighborhood with $\epsilon'$-control of size $R + 1 - 1/C$.
2. If $\varphi'(0, x) \subset W'$, then $\varphi'(S^1 \times \{x\})$ is a fiber of the $S^1$-fibration on $W'$.
3. For any $T \leq R + 1$, the image $\varphi'(S^1 \times B(0, T))$ contains $\varphi(S^1 \times B(0, T - 1/C))$.

We denote the image $\varphi'(S^1 \times B(0, R + 1 - 1/C))$ by $U[1/C]$.

Corollary 4.15. The $S^1$-fibration structure on $U[1/C]$ coming from the $S^1$-product structure and the given $S^1$-fibration structure on $W'$ are compatible on the overlap $U[1/C] \cap W'$.

This claim shows that, at the expense of shrinking $W$ to $W'$ and at the expense of deforming $\varphi$ slightly to a reduced $S^1$-product structure with $\epsilon'$-control, $\varphi' : S^1 \times B(0, R + 1 - 1/C) \to M$, we can make the $S^1$-fibrations compatible on the overlap, so that together they define an $S^1$-fibration on the union $W' \cup U[1/C]$. One more remark is in order. If we have not a single reduced $S^1$-product neighborhood with $\epsilon$-control $U$, but rather a collection of them $U_{i_0, j,}$, $1 \leq j \leq j_0(i_0)$, whose images are disjoint, then we can perform this operation simultaneously on all of them, so as to deform them all to $S^1$-product neighborhoods with $\epsilon_1$-control compatible with the circle fibration on $W'$.

Now we are ready to apply this gluing argument by induction to the $U_1, \ldots, U_C$. We begin with $U_1$. In the inductive step, deforming and gluing in $U_{i_0}$, we cut down the $S^1$-product neighborhoods in the neighborhoods that make up the previous $U_i$. 
by $1/C$. The deformation of the maps $\varphi_{i_0,j}$ produces a reduced $S^1$-product neighborhood with $\epsilon_1$-control where the amount of the deformation and $\epsilon_1$ depend only on the control we have at the previous step. Thus, we can iterate this construction $C$ times keeping a fixed control, $\epsilon'$, on all the $S^1$-product neighborhoods and a given control on the size of the deformations, provided only that we arrange that the original control, $\epsilon$, is sufficiently small given $C$, $\epsilon'$, and the desired control on all deformations.

It follows from the second conclusion of Claim 4.14 that the $S^1$-fibrations induced by the product structures on the deformed $U_i$ are compatible and hence define a global $S^1$-fibration on the union. It follows from the third conclusion of Claim 4.14 that the union of the deformed $S^1$-product neighborhoods contains $K$. The last statement in the conclusion of Proposition 4.4 is immediate from the construction. This completes the proof of Proposition 4.4.

4.3 Balls centered at points of $\partial M_n$

The results about the generic behavior over interior points of the base is enough to establish what the neighborhoods of the boundary of the $M_n$ look like.

**Proposition 4.16.** Fix $\epsilon > 0$. For all $n$ sufficiently large, for any point $x \in \partial M_n$ the ball $B_{\rho_n^{-2}(x)g_n}(x, 1)$ is within $\epsilon$ of the interval of length 1, and $x$ is within $\epsilon$ of the endpoint 0.

**Proof.** Suppose that the result is not true. Then after passing to a subsequence (in $n$) we can suppose that for each $n$ we have $x_n \in \partial M_n$ for which the result does not hold. Let $T_n$ be the component of $\partial M_n$ containing $x_n$ and let $C_n$ be the topologically trivial collar containing the neighborhood of size 1 of $T_n$. Since $\partial M_n$ is convex and $\rho_n \leq \text{diam } M_n/2$, the balls $B_{\rho_n^{-2}(x_n)g_n}(x_n, 1)$ are Alexandrov balls. Because the curvature on the topologically trivial collar which includes the neighborhood of size 1 about $\partial M_n$, are bounded above by $-3/16$, it follows that $\rho_n(x_n) \leq \sqrt{16}/3$. Hence, $B_n = B_{\rho_n^{-2}(x_n)g_n}(x_n, 1/4)$ is contained in $C_n$. We shall show that, after passing to a subsequence the $B_{\rho_n^{-2}(x_n)g_n}(x_n, 1/4) \subset C_n$ converge to the interval $[0, 1/4)$ with the $x_n$ converging to the endpoint 0. Assuming this, it follows that the $B_{\rho_n^{-2}(x_n)g_n}(x_n, 1)$ also converge to a 1-dimensional Alexandrov space $\tilde{J}$ and that the $x_n$ converge to an endpoint of $\tilde{J}$. Since the diameter of $M_n$ is greater than $2\rho_n(x_n)$, it follows that $\tilde{J}$ has length 1.

We have already remarked that because of the convexity of $\partial M_n$, the $B_n$ are Alexandrov balls. Passing to a subsequence, there is a limiting Alexandrov space $J$ which is an Alexandrov ball of diameter 1/4 centered at $\overline{\pi} = \lim x_n$. Because of the volume collapsing condition on the $M_n$, it follows that $J$ is either of dimension 1 or 2. We rule out the possibility that $\dim J = 2$. Suppose to the contrary that the dimension of $J$ is 2. Fix $0 < \mu \leq \mu(\epsilon)$ from Lemma 4.1 and fix $0 < \alpha << 1/4$. Then there is a point $\overline{y}$ of $J$ within distance $\alpha$ of $\overline{\pi}$ that has a $(2, \mu)$-strainer of some size $d > 0$. Fix $d' > 0$ as in Proposition 4.3 for this value of $d$. For all $n$ sufficiently large $\epsilon_n < \overline{\pi}(\epsilon, \mu, d)$ from Proposition 4.3. By Proposition 4.3 this means that for all $n$ sufficiently large there is a point $y_n \in B_{\rho_n(x_n)}(x_n, 1/4)$ within $\epsilon_n$ of $\overline{\pi}$ with
a neighborhood $U_n$ with $B_{g_n'(y_n)}(y_n, d') \subset U_n \subset B_{g_n(y_n)}(y_n, 2d')$ that is fibered by circles over a topological ball. In particular, $\pi_1(U_n)$ is infinite cyclic and hence the image of $\pi_1(U_n)$ in $\pi_1(C_n)$ is either trivial or infinite cyclic.

We denote by $C_n$ the universal covering of $\tilde{C}_n$ with its inherited Riemannian metric. Fix lifts $\tilde{x}_n$ for $x_n$ and $\tilde{y}_n$ for $y_n$ that are within distance $\alpha$ of each other. Since $C_n$ is a topologically trivial collar of $T_n$, the group of covering transformations of $\tilde{C}_n$ over $C_n$ is a free abelian group of rank 2. We implicitly use the metric $\rho_n^{-2}(x_n)g_n$ on $C_n$ and the induced Riemannian metric on the covering. Since the diameter of $T_n$ in the metric $\rho_n^{-2}(x_n)g_n$ is at most $4w_n$, it follows that the fundamental group of $C_n$ is generated by elements which, acting as covering transformations on $\tilde{C}_n$, move $\tilde{x}_n$ a distance at most $8w_n$. In particular, we can choose an element $\gamma_n \in \pi_1(T_n)$ that moves $\tilde{x}_n$ a distance at most $8w_n$ and which is not of finite order in the quotient of $\pi_1(C_n)$ by the image of $\pi_1(U_n) \to \pi_1(C_n)$. Of course, $\gamma_n$ moves every point of $B(\tilde{x}_n, 4w_n)$ a distance at most $16w_n$. Since the translates of $B(\tilde{x}_n, 4w_n)$ by $\pi_1(T_n)$ cover all of $T_n$, and since the fundamental group is abelian, it follows that $\gamma_n$ moves every point of $\tilde{T}_n$ a distance at most $16w_n$ and consequently $\gamma_n^k$ moves $\tilde{x}_n$ a distance at most $16kw_n$. Since $U_n$ contains $B_{\rho_n^{-2}(x_n)}(y_n, d')$, it follows that each component of the preimage $\tilde{U}_n$ of $U_n$ contains the ball of radius $d'$ about each lift $\tilde{y}_n$ lying in that component. Since the group generated by $\gamma_n$ freely permutes the components of $\tilde{U}_n$, it follows that every power of $\gamma_n$ moves every preimage of $y_n$ a distance at least $2d'$.

The induced covering $\tilde{B}_n \subset \tilde{C}_n$ is a Riemannian manifold with convex boundary and hence is a local Alexandrov space. Furthermore, for $1 \leq k \leq \alpha/16w_n$ the image $\gamma_n^k\tilde{x}_n$ is within $\alpha$ of $\tilde{x}_n$. This implies that there are at least $m = \alpha/16w_n$ distinct translates of $\tilde{y}_n$, all within distance $2\alpha$ of $\tilde{x}_n$, and these translates are all at least distance $2d'$ apart. Thus, letting $w$ and $w'$ be any two such translates, the comparison angle $\angle w\tilde{x}_n w'$ is bounded away from zero. Since the distances are much smaller than 1/4, there are geodesics from $\tilde{x}_n$ to each of these $m$ translates of $\tilde{y}_n$. According to Remark 3.5 of [3] monotonicity of angles as in Part 2 of Proposition 2.2 holds in the region in which we are working. Thus, the angles that these geodesics make with each other at $\tilde{x}_n$ are bounded away from zero. As $n$ goes to infinity the number of these translates goes to infinity. But there is there is a fixed upper bound to the number of geodesics emanating from a point in a 3-manifold with the property that the angles between any two distinct ones is bounded below by a fixed positive constant. This contradiction proves that $J$ is 1-dimensional.

Take any point $y_n \in C_n$ at distance 1/2 from $T_n$ and join it to $T_n$ by a minimal geodesic $\gamma_n$. Let $x'_n$ be its other endpoint. This geodesic makes angle at most $\pi/2$ with any tangent vector at $x'_n$. Taking limits we see that there is a geodesic, $\gamma$ in the limit $J$ with one endpoint being the limit, $\pi$, of the $x_n$ such that $\gamma$ makes angle at most $\pi/2$ with any tangent direction at $\pi$. It follows that $\pi$ is an endpoint of $J$. $\Box$

4.4 The interior cone points

Proposition 4.17. For any $\epsilon > 0$ and $\alpha > 0$, the following holds for all $\mu > 0$ less than a positive constant $\mu_3(\epsilon, a)$, for any $0 < r_0 \leq 10^{-3}$, and for all $\dot{e} > 0$ less than
a positive constant $\hat{c}_1(\epsilon,a,r_0)$. Suppose that, for some $n$, there is a point $x_n \in M_n$ with the property that the ball $B_{\lambda_g}(x_n,1)$ is within $\hat{c}$ of a 2-dimensional Alexandrov ball $B(\overline{x},1)$ of area $\geq a$ that is interior $\mu$-good at $\overline{x}$ on scale $r'$, where $r_0 \leq r' \leq 10^{-3}$. Then there is a compact solid torus $S$ contained in $B_{\lambda_g}(x_n,3r'/4)$ and containing $B_{\lambda_g}(x_n,r'/2)$. Furthermore, every point of $U = B_{\lambda_g}(x_n,3r'/4) \setminus B_{\lambda_g}(x_n,r'/4)$ is the center of an $S^1$-product neighborhood with $\epsilon$-control.

Proof. First notice that it follows from the Bishop-Gromov inequality that there is $a' > 0$ depending only on $a$ such that if $B(\overline{x},1)$ is a standard 2-dimensional ball of area $\geq a$ then for any $0 < r \leq 1$, the area of $B(\overline{x},r)$ is at least $a'r^2$. First let us show that it suffices to prove the result when $r_0 = 10^{-3}$. For suppose that for every $\epsilon > 0$ and $a > 0$ we have positive constants $\mu_3(\epsilon,a)$ and $\lambda_1(\epsilon,a)$ so that the proposition holds for $r_0 = 10^{-3}$. Fix $\epsilon > 0$, $a > 0$, and $r_0 > 0$. Suppose we have $\mu < \mu_3(\epsilon,a')$ and $\lambda < (r_0/10^{-3})^2(\epsilon,a')$. Given balls $B_{\lambda_g}(x_n,1)$ and $B(\overline{x},1)$ as in the statement for these values of $\mu$ and $\lambda$ and $a$, and some $r'$ with $r_0 \leq r' \leq 10^{-3}$. Then $(10^{-3}/r')B(\overline{x},1)$ is interior $\mu$-good at scale $10^{-3}$ at $\overline{x}$. The unit subball centered at $\overline{x}$ has area $\geq a'$. On the other hand $B_{\lambda}(x_n,1)$ is within $(10^{-3}/r')^2\lambda g(\epsilon,a')$ of $(10^{-3}/r')B(\overline{x},1)$. By our assumption that the result holds in the special case when $r_0 = 10^{-3}$, we see that the conclusion holds for $B_{\lambda}(x_n,1)$ with $r'$ replaced by $10^{-3}$. Hence, rescaling it holds for $B_{\lambda}(x_n,1)$ with the given value of $r'$.

This allows us to assume that $r_0 = 10^{-3}$. Suppose that there are sequences $\mu_k \to 0$ and $\hat{c}_k \to 0$ as $k \to \infty$ and balls $B_{\lambda_k}(x_k,1)$ within $\hat{c}_k$ of standard 2-dimensional balls $B(\overline{x}_k,1)$ of area $\geq a$ that are interior $\mu_k$-good at $\overline{x}_k$ on scale $10^{-3}$ and yet the conclusion of the proposition does not hold for $r' = 10^{-3}$. Passing to a subsequence, we can suppose that the $B(\overline{x}_k,1)$ converge to a standard 2-dimensional ball $B(\overline{x},1)$. Because the $\mu_k \to 0$, it follows that $B(\overline{x},10^{-3})$ is a circular cone of some cone angle $\alpha \leq 2\pi$ which is bounded away from zero because $\alpha$ is greater than zero. Since the $\hat{c}_k \to 0$, the $B_{\lambda_k}(x_k,1)$ also converge to $B(\overline{x},1)$.

Let us first consider the case when $\alpha = 2\pi$ so that $B(\overline{x},10^{-3})$ is isometric to a ball in $\mathbb{R}^2$. It follows from Proposition 4.1 that there is $d' > 0$, a $(2,\mu)$-strainer $\{a_1, a_2, b_1, b_2\}$ for $x_n(k)$, and an open subset $U_n(k)$ containing $B_{\lambda_k}(x_k,d')$ and contained in $B_{\lambda_k}(x_n(k),2d')$ with the property that the function $F = (f_1, f_2)$ where $f_i = d(a_i, \cdot)$ determines a fibration of $U_n(k)$ by circles over a disk in the plane. Furthermore, by Lemma 4.1 for all $k$ sufficiently large, there is an $S^1$-product neighborhood $V$ with $\epsilon$ control centered at $x_n(k)$. Also, according to Proposition 4.1 the circle of the fibration structure on $U_n(k)$ passing through $x_n(k)$ is almost orthogonal to the horizontal spaces of the $S^1$-product structure centered at that point and this circle is isotopic in $V$ to the $S^1$-factor. This means that the closure of $V$ is a solid torus contained in $U_n(k)$ whose core is isotopic to the fiber of the fibration structure on $U_n(k)$. It follows that the inclusion of $V \subset U_n(k)$ induces an isomorphism on fundamental groups, both groups being isomorphic to $\mathbb{Z}$. Also, it follows that the region between $V$ and the closure of $U_n(k)$ is homeomorphic to $T^2 \times I$. We have inclusions $V \subset B_{\lambda_k}(x_n(k),d') \subset U_n(k) \subset B_{\lambda_k}(x_n(k),2d')$. For all $k$ sufficiently large, the distance function from $x_n(k)$ is regular on $B_{\lambda_k}(x_n(k),2d') \setminus B_{\lambda_k}(x_n(k),d')$, and consequently, the inclusion of the smaller ball into the larger induces an isomorphism


on the fundamental group. It then follows from the sequence of inclusions that the fundamental group of $B_{\lambda_k g_n(k)}(x_n(k), d')$ is isomorphic to $\mathbb{Z}$ and hence the metric sphere $S_{\lambda_k g_n(k)}(x_n(k), d')$ is a 2-torus. This 2-torus is contained in the complement of $V$ in the closure of $U_n(k)$ and separates the two boundary components of this region. Since we have already seen that this region is homeomorphic to a product $T^2 \times I$, it follows that $S_{\lambda_k g_n(k)}(x_n(k), d')$ is isotopic in the closure of $U_n(k)$ to the boundary of $U_n(k)$. Consequently, $B_{\lambda_k g_n(k)}(x_n(k), d')$ is a solid torus. Using the regularity of the distance function from $x_n(k)$ see that $B_{\lambda_k g_n(k)}(x_n(k), a)$ is a solid torus for every $a \in [d', 10^{-3}]$. The last statement in the proposition is immediate from Lemma [1.1].

This contradiction proves the result in the case when the limiting 2-dimensional space is flat.

Now we consider the case when the limiting cone angle $\alpha$ is less than $2\pi$. We rescale by $10^3$ so that $r'$ in effect becomes 1. In this case, according to Proposition [2.20] the following holds for all $k$ sufficiently large. There is $x_n(k) \in M_n(k)$ such that $d_{\lambda_k g_n(k)}(x_n(k), x_n'(k)) \to 0$ as $k \to \infty$ such that for each $k$ sufficiently large, one of the following two alternatives holds: for

1. the distance function from $x_n'(k)$ has no critical points on $B_{\lambda_k g_n(k)}(x_n'(k), 3/4) \setminus \{x_n'(k)\}$, or

2. there is $\delta_k \to 0$ such that the distance function from $x_n'(k)$ has no critical points in $B_{\lambda_k g_n(k)}(x_n'(k), 3/4) \setminus \overline{B}_{\lambda_k g_n(k)}(x_n'(k), \delta_k)$ and has a critical point at distance $\delta_k$ from $x_n'(k)$.

In Case 1 the level sets of the distance function are 2-spheres and the metric balls are topological 3-balls. Let us suppose that Case 2 holds. According to Proposition [2.20] after passing to a subsequence the rescaled balls $\overline{\delta_k^{-1} B_{\lambda_k g_n(k)}}(x_n'(k), 3/4)$ converge in the Gromov-Hausdorff topology to a complete 3-dimensional Alexandrov space of curvature $\geq 0$. By Proposition [1.3] the limit is actually a smooth, orientable Riemannian manifold of curvature $\geq 0$ and the convergence is $C^\infty$. Thus, the limit has a soul which is either a point, a circle, or a surface of non-negative curvature. We claim the soul is not a surface. For if the soul is a surface, then either the limiting 3-manifold or its double covering is a Riemannian product of that surface with $\mathbb{R}$. The limit cannot be the product of a surface with $\mathbb{R}$ because the complement of a small neighborhood about the soul is close to a connected 2-dimensional space and hence is connected. Thus, if the soul is a surface, the limiting 3-manifold is a non-orientable $\mathbb{R}$-bundle over that surface. It would then follow that given any $\beta > 0$ there is $R < \infty$ such that for all $k$ sufficiently large any triangle $ax_n'(k)b$ with $|ax_n'(k)| = |bx_n'(k)| = R$ has comparison angle less than $\beta$ at $x_n'(k)$. On the other hand, because the limit of the $B_{\lambda_k g_n(k)}(x_n'(k), 1)$ is 2-dimensional, there is $\beta_0 > 0$ such that for all $k$ sufficiently large there are geodesics from $x_n'(k)$ to points at a fixed positive distance that make a comparison angle at $x_n'(k)$ which is least $\beta_0$. This contradicts the monotonicity of the comparison angles.

This shows if Case 2 holds then the soul of the limiting manifold is either a circle or a point, and hence the level sets $d(x_n'(k), \cdot)^{-1}(a)$ are either 2-tori or 2-spheres for
every \( a \) with \( \delta_k < a \leq 3/4 \) and these bound either solid tori or 3-balls in the metric ball. In the first case, the level sets are topological 2-spheres and they bound 3-balls in the metric ball.

Next, we shall show that in either case, provided that \( \epsilon > 0 \) is sufficiently small, the level sets of the distance function from \( x'_{n(k)} \) must be 2-tori. Fix \( \epsilon' > 0 \) small and let \( \epsilon > 0 \) be such that Proposition 4.4 holds for these values of \( \epsilon' \) and \( \epsilon \). Consider the annular region \( A_k = d(x'_{n(k)}, \cdot)^{-1}([1/4, 3/4]) \). This is a compact subset and if \( k \) is sufficiently large, then every point of this compact set is within \( \epsilon \) of a point of \( B(\pi_k, 1) \) at which \( B(\pi_k, 1) \) is interior flat of some fixed scale \( s \). Having taking \( \epsilon \) sufficiently small, by Proposition \( 4.4 \) there is an open subset \( U_{n(k)} \subset M_{n(k)} \) containing \( A_k \) that is the total space of a circle fibration where the fibers of the fibration make angle at most \( \epsilon' \) with the horizontal spaces of the \( S^1 \)-product neighborhoods with \( \epsilon \)-control at every point of \( A_k \). Of course, there is a compact subsurface \( \Sigma_k \) contained in the base of the fibration with the property that the pre-image, \( W_k \), of \( \Sigma_k \) contains \( A_k \). Each component of \( \partial W_k \) is a torus. Thus, for every \( b \in (1/4, 3/4) \) the level set \( d(x'_{n(k)}, \cdot)^{-1}(b) \) separates two boundary components of \( W_k \). Since a 2-sphere in the total space of a circle bundle cannot separate boundary components of that circle bundle, it follows that these level sets are 2-tori.

This implies that for all \( k \) sufficiently large, Case 2 holds and the soul of the limiting 3-manifold is a circle. Thus, for every \( k \) sufficiently large, for every \( 0 < b \leq 3/4 \) the pre-image \( d(x'_{n(k)}, \cdot)^{-1}([0, b]) \) is a solid torus. We fix \( b \in (1/2, 3/4) \) and set the pre-image of \([0, b]\) equal to \( S \). Of course, provided that \( k \) is sufficiently large \( B(x_{n(k)}, 1/2) \subset S \subset B(x_{n(k)}, 3/4) \). This gives a contradiction and completes the proof of the result.

The argument above actually proves more.

**Corollary 4.18.** Fix \( \epsilon' > 0 \) sufficiently small and let \( 0 < \epsilon < \epsilon_1(\epsilon') \), where \( \epsilon_1(\epsilon') \) is as in Proposition \( 4.4 \). Under the hypothesis and notation of the previous proposition, suppose that we have an open subset \( \tilde{U} \) containing \( B_{\lambda g_n}(x_n, 3r'/4) \setminus \tilde{B}_{\lambda g_n}(x_n, r'/4) \) with \( \tilde{U} \) being the total space of an \( S^1 \)-fibration with fibers making angle within \( \epsilon' \) of \( \pi/2 \) with the horizontal spaces of the \( S^1 \)-product neighborhoods with \( \epsilon \)-control at every point of \( \tilde{U} \). Then there is a 2-torus in \( \tilde{U} \) that is invariant under the \( S^1 \)-fibration structure, which is contained in \( B_{\lambda g_n}(x_n, r'/2) \), and which bounds a solid torus in \( B_{\lambda g_n}(x_n, 3r'/4) \).

There is a further result that is not actually necessary for what follows but which makes the picture clearer and also simplifies somewhat several of the arguments.

**Proposition 4.19.** For \( \epsilon' > 0 \) sufficiently small and let \( 0 < \epsilon < \epsilon_1(\epsilon') \), where \( \epsilon_1(\epsilon') \) is as in Proposition \( 4.4 \). Under the hypothesis of the previous proposition, the \( S^1 \)-factors in the local \( S^1 \)-product structures with \( \epsilon \)-control contained in \( B_{\lambda g_n}(x_n, 3r'/4) \setminus B_{\lambda g_n}(x_n, r'/4) \) are homotopically non-trivial in \( B_{\lambda g_n}(x_n, 3r'/4) \).

**Proof.** Suppose that the result does not hold for any \( \epsilon' > 0 \). We take a sequence of \( \epsilon'_k \) tending to zero and \( \epsilon'_k \) counter-examples \( (B_k, x_k) \). After passing to a subsequence
these counter-examples converge to a 2-dimensional Alexandrov ball \((Z, z)\) with curvature \(\geq -1\) which is interior good at the limiting base point \(z\) on some scale \(r' \geq r_0\). The fundamental group \(\Gamma_k\) of \(\tilde{B}_k\) is infinite cyclic and the shortest homotopically non-trivial loop through \(x_k\) has a length that tends to zero as \(k \to \infty\). We consider the universal coverings \(\tilde{B}_k\) of the \(B_k\) and let \(\tilde{x}_k\) be a lifting of \(x_k\). For any fixed \(s > 0\) and any fixed \(N < \infty\) for all \(k\) sufficiently large there are at least \(N\) distinct preimages of \(x_k\) in \(B(\tilde{x}_k, s)\). On the other hand, suppose that the circles in the product structure contained in \(B_{\lambda g_k}(x_k, 3r'/4) \setminus B_{\lambda g_k}(x_k, r'/4)\) are homotopically trivial in \(B_{\lambda g_k}(x_k, 3r'/4)\). Then there is \(s > 0\) such that for all \(k\) sufficiently large and any point \(y_k \in (B_{\lambda g_k}(x_k, 3r'/4) \setminus B_{\lambda g_k}(x_k, r'/4))\) the preimage in \(\tilde{B}_k\) of the ball \(C_k\) of radius \(s\) centered at \(y_k\) is a disjoint union of components mapping homeomorphically onto the ball. Fix \(s \leq r'/8\). This means that each of these preimages contains the ball of radius \(s\) about the corresponding preimage of \(y_k\). Fix a pre-image \(\tilde{x}_k\) of \(x_k\), and a preimage \(\tilde{y}_k\) of \(y_k\) within distance \(3r'/4\) of \(\tilde{x}_k\). For \(k\) sufficiently large we have an arbitrarily large number of group elements of the fundamental group of \(B_k\) that move \(\tilde{x}_k\) a distance at most \(s\), but the balls of radius \(s\) about the corresponding translates of \(\tilde{y}_k\) are disjoint. Let \(G(k) \subset \pi_1(\tilde{B}_k, x_k)\) be the set of elements moving \(\tilde{x}_k\) a distance at most \(s\) and let \(N(k)\) be its cardinality. Since all of these points are contained in the ball of radius \(r'\) about \(\tilde{x}_k\), and the exponential mapping at the tangent space to \(\tilde{B}_k\) at \(\tilde{x}_k\) is defined out to distance at least \(2r'\). In particular, there are geodesics from \(\tilde{x}_k\) to each of the translates of \(\tilde{y}_k\) by elements of \(G(k)\), and consequently \(N(k)\) geodesics of length \(\leq r'\), all of whose endpoints are separated by distances at least \(2s\). Thus, the comparison angles at \(\tilde{x}_k\) for the triples of points consisting of \(\tilde{x}_k\) and two translates of \(\tilde{y}_k\) are bounded away from zero independent of \(k\). Since the exponential mapping is defined on the ball of radius \(2r'\) in the tangent space to \(\tilde{B}_k\) at \(\tilde{x}_k\), monotonicity holds for these triangles. This is a contradiction.

The topological import of this result about the fundamental group is the following:

**Corollary 4.20.** Under the notation and hypotheses of Corollary 4.18, the \(S^1\)-fibration structure on \(\tilde{U}\) extends to a Seifert fibration over \(\tilde{U} \cup B_{\lambda g_n}(x, 3r'/4)\) with one singular fiber.

**Definition 4.21.** \(B_{\lambda g_n}(x_n, r'/4)\) satisfying the conclusions of Propositions 4.17 and 4.19 and Corollary 4.18 is an \(\epsilon'\)-solid torus neighborhood near a 2-dimensional interior cone point.

**Remark 4.22.** In fact, a strengthening of this argument (see Theorem 0.2 and the material in Section 4 of [22]) proves that the order of the exceptional fiber is bounded above by \(2\pi/\alpha\) where \(\alpha\) is the cone angle of the nearby interior \(\mu\)-good ball at its central point. We shall not make use of this result.
4.5 Near almost flat boundary points

Now let us turn to the parts of the $M_\nu$ close to flat boundary points of a 2-dimensional Alexandrov ball.

**Proposition 4.23.** Given $\epsilon' > 0$, let $0 < \epsilon < \epsilon_1(\epsilon')$, where $\epsilon_1(\epsilon')$ is the constant given in Proposition 4.4. The following hold for all $\xi > 0$ less than a positive constant $\xi_0(\epsilon)$ and for all $\mu > 0$ less than a positive constant $\mu_4(\xi, \epsilon)$. For any $0 < s_1 \leq 1/4$ and, given $s_1$, for all $\epsilon > 0$ less than a positive constant $\epsilon_2(\epsilon, \mu, s_1)$, suppose that, for some $n$, there is a point $x_n \in M_\nu$ with the property that $B_{\lambda g_n}(x_n, 1)$ is within $\epsilon$ of a 2-dimensional Alexandrov ball $X = B(\pi, 1)$. Suppose that $\gamma$ is a $\xi$-approximation to $\partial X \cap B(\pi, 3/4)$ on scale $s_1$ with $\mu$-control. Suppose that $\tilde{\gamma}$ is a geodesic in $M_\nu$ within $\epsilon$ of $\gamma$. Then the subspace $\nu_\xi(\tilde{\gamma})$ is homeomorphic to $D^2 \times [0, 1]$ where the (closed) disks in this (topological) product structure are the level sets of $f_{\tilde{\gamma}}$. The subset $\nu_\xi(\tilde{\gamma})$ is homeomorphic to $S^1 \times [0, 1] \times (0, 1)$ where each circle factor is the intersection of a level set of $f_{\tilde{\gamma}}$ with a level set of $h_{\tilde{\gamma}}$. (These intersections are called level circles.)

**Proof.** We work with the metric $\lambda g_n$, so that in particular, $\ell(\tilde{\gamma})$ means the length of $\tilde{\gamma}$ with respect to this metric. Provided that $\tilde{\epsilon}$ is sufficiently small, it follows from Lemma 3.14 that $f_{\tilde{\gamma}}$ is $2\tilde{\epsilon}$-regular on $\nu_\xi(\tilde{\gamma})$ so that, provided that $\xi$ is sufficiently small, each level set $L$ is a Lipschitz surface and these level surfaces foliate $\nu_\xi(\tilde{\gamma})$. The conditions on the restrictions of $f_\gamma$ and $h_\gamma$ to $\nu_\xi(\tilde{\gamma})$ imply by arguments identical to the ones given in the proof of Lemma 4.3 that the map $F = (f_{\tilde{\gamma}}, h_{\tilde{\gamma}})$ determines a fibration of $\nu_\xi(\tilde{\gamma})$ with fibers circles. Hence, $\nu_\xi(\tilde{\gamma})$ is homeomorphic to $S^1 \times [0, 1] \times (0, 1)$ where the circles are the level circles.

We shall show that provided that $\xi > 0$ is sufficiently small, the level sets of $f_{\tilde{\gamma}}$ are homeomorphic to disks. From the immediately preceding discussion, it follows that the boundary of any level surface for $f_{\tilde{\gamma}}$ is a single circle. Since the level sets of $f_{\tilde{\gamma}}$ are connected, to show these level sets are homeomorphic to disks it suffices to show that they have virtually nilpotent fundamental groups and are orientable. The level sets are orientable since $M_\nu$ is and since they are the level sets of a regular Lipschitz function so that there is a neighborhood of the level set in $M_\nu$ that is homeomorphic to the product of the level set with $I$.

**Claim 4.24.** For $\xi > 0$ sufficiently small, the fundamental groups of the level sets of $f_\gamma$ are virtually abelian.

Of course, the image in the fundamental group of the manifold of the fundamental group of a level set is the same as the image of the fundamental group of the $\xi$-neighborhood of a point. The argument we give here is a simplification of the argument in the appendix of [3] proving a more general result.

**Proof.** We suppose that the claim does not hold. Then there is a sequence of $\xi_k$ and counter-examples $\nu_\xi(\tilde{\gamma}_k)$. Take as base points $p_k$ the midpoints of $\tilde{\gamma}_k$. According to what we just established every point of $\nu_\xi(\tilde{\gamma}_k)$ is the center of an $S^1$-product neighborhood with $\epsilon$-control. Thus, there are fixed constants $\epsilon > 0$ and $\delta > 0$, independent of $k$, such that the ball of radius $\epsilon$ about $p_k$ contains a ball of radius
Lemma 4.25. With notation and assumptions as in the previous proposition the following hold.

1. Each point of $\nu^0(\xi)$ is the center of an $S^1$-product neighborhood with $\epsilon$-control.

2. For any point $x \in \nu^0(\xi)$, the $S^1$-product structure with $\epsilon$-control centered at $x$, $\varphi: S^1 \times B(0, \epsilon^{-1}) \to M_n$, can be chosen so that the following hold for $R \leq \epsilon^{-1}/2$ and any point $q \in \varphi(S^1 \times B(0, R))$:

   (a) For any geodesic $\zeta$ from $\gamma$ to a point $q$, $\varphi^{-1}$ of intersection of $\xi$ with the neighborhood is within $\epsilon'$ of the straight line starting at $q$ in the negative $y$-direction in the $\mathbb{R}^2$ factor.
(b) For any geodesics $\zeta_\pm$ from $e_\pm(\gamma)$ to $q$, $\varphi^{-1}$ of the intersection of $\zeta_\pm$ with the neighborhood are within $\epsilon'$ of straight lines starting at $q$ in the $x_\pm$-directions.

3. For any $q \in \nu^0_{\xi}(\gamma)$ and for any $S^1$-product neighborhood with $\epsilon$-control containing $q$, the angle at $q$ between the level circle $S(q)$ through $q$ and the horizontal space of the $S^1$-product neighborhood is within $\epsilon'$ of $\pi/2$. Furthermore, if $q$ is contained in $\varphi(S^1 \times B(0, R))$, then $S(q)$ is isotopic in the $S^1$-product neighborhood to an $S^1$-factor.

Proof. For any point $x \in \nu^0_{\xi}(\gamma)$ the ball $B_{\lambda^0}(x, 1)$ is within $\epsilon$ of an Alexandrov ball $B(\bar{x}, 1)$ that is interior $\mu$-flat at $\bar{x}$ on scale $\xi^2 s_1/20$. It follows from Lemma 4.1 that for all $\epsilon$ sufficiently small, that every point of $\nu^0_{\xi}(\gamma)$ is the center of an $S^1$-product neighborhood with $\epsilon$-control. It then follows from the last statement in Lemma 4.1 that we can choose the Euclidean coordinates for this $S^1$-product structure so that any geodesics from $e_\pm$ is within $\epsilon'$ of the straight line in the horizontal space in the $x_\pm$-direction. Now consider any geodesic from $\tilde{\gamma}$ to a point of this product neighborhood. It makes angle near to $\pi/2$ with any geodesic from $e_\pm$ and also is almost horizontal since its length is much longer that the diameter of the circle factors. It follows that (possibly after reversing the $y$-coordinate) these geodesics are within $\epsilon'$ of the $y_-$-direction in the horizontal spaces. We consider $F = (f_{\tilde{\gamma}}, h_{\tilde{\gamma}})$ mapping this $S^1$-product neighborhood to $\mathbb{R}^2$. The restriction of $f_{\tilde{\gamma}}$ to any horizontal space is strictly increasing in the $x$-direction and in fact $|f_{\tilde{\gamma}}(x_0, y) - f_{\tilde{\gamma}}(x_1, y)| \geq (1 - \epsilon')|x_0 - x_1|$, whereas $|f_{\tilde{\gamma}}(x, y_0) - f_{\tilde{\gamma}}(x, y_1)| < \epsilon'|y_0 - y_1|$. The second coordinate function $h_{\tilde{\gamma}}$ satisfies analogous inequalities with the roles of $x$ and $y$ reversed. It follows that the restriction of $F$ to any horizontal space is one-to-one and hence the level sets of $F$ meet each the horizontal space in at most one point. Furthermore, the fact that the geodesics from $\tilde{\gamma}$ and from $e_\pm$ are nearly horizontal implies, by §11 of $[\mathbb{R}]$ that the level sets of $F$ are nearly orthogonal to the horizontal spaces in the sense described in the statement of the proposition. Choosing $\epsilon$ sufficiently small, we can arrange that these level sets make angle within $\epsilon'$ of $\pi/2$ with every horizontal space. It follows that any level set of $F$ that meets $\varphi \left(S^1 \times B(0, R)\right)$ is contained in $\varphi \left(S^1 \times B(0, \epsilon^{-1})\right)$ and is a circle meeting each horizontal space once. Such circles are isotopic in the neighborhood to the circle factors.

$\Box$

Definition 4.26. We call any neighborhood $\nu_{\xi}(\tilde{\gamma})$ for which there is a geodesic $\gamma$ in a 2-dimensional standard ball satisfying the hypotheses of Proposition 4.23 (and hence $\nu_{\xi}(\tilde{\gamma})$ satisfies the conclusions of the last two results) a $\epsilon'$-solid cylinder neighborhood at scale $s_1$ near a flat boundary, or simply an $\epsilon'$-solid cylinder neighborhood at scale $s_1$ for short.

For later use we need one final addendum about the $\nu_{\xi}(\tilde{\gamma})$. It follows directly from the corresponding statement in the 2-dimensional case (Lemma 3.14).

Lemma 4.27. With notation and assumptions as in the previous proposition the following holds. For any $c \in [\xi^2, \xi]$ and for any level surface $L$ of $f_{\tilde{\gamma}}$ the distance from any point of $L \cap h^{-1}_{\tilde{\gamma}}(c \cdot \ell(\tilde{\gamma}))$ to $L \cap \tilde{\gamma}$ is at most $(1 + 4\xi) c \cdot \ell(\tilde{\gamma})$. Also, for any
point \( y \in \nu^0_\tilde{\gamma} \) there is a geodesic \( \zeta \) of length \( 10\xi \) from \( y \) to a point \( z \) such that for any \( w \in \zeta \) at distance at most \( 5\xi \ell(\tilde{\gamma}) \) from \( y \) the comparison angle \( \tilde{\gamma}_wz \) is greater than \( \pi - 2\xi \). (Here all distances and \( \ell(\tilde{\gamma}) \) are measured with respect to \( \lambda g_n \).)

We shall also need smooth vector fields well-adapted to \( \nu_\xi(\tilde{\gamma}) \).

**Corollary 4.28.** There is a smooth unit vector field \( \tilde{\tau} \) on \( \nu_\xi(\tilde{\gamma}) \) such that, setting \( d_+ \) equal to the distance function from the endpoints \( e_\pm \) of \( \tilde{\gamma} \), we have \( d'_+(\tilde{\tau}) > 1 - 10\xi \), \( d'_-(\tilde{\tau}) > -1 + 10\xi \), and \( h_\tilde{\tau}(\tilde{\gamma}) < c\xi^2 \) for a universal constant \( c \). Provided that \( \xi \) is sufficiently small, for any points \( p,q \) on a flow line of the flow generated by \( \tilde{\tau} \) we have

\[
\left| \frac{h_\tilde{\tau}(p) - h_\tilde{\tau}(q)}{\tilde{f}_\tilde{\tau}(p) - \tilde{f}_\tilde{\tau}(q)} \right| < 2c\xi^2.
\]

In particular, for \( \xi > 0 \) sufficiently small, any maximal flow line of \( \tilde{\tau} \) that meets \( \nu_{3\xi/4}(\tilde{\gamma}) \) is an interval with endpoints in the ends of \( \nu_\xi(\tilde{\gamma}) \) and this interval meets each level set of \( f_\tilde{\tau} \) in a single point.

**Proof.** The existence of \( \tilde{\tau} \) as stated follows immediately from the definition of a \( \xi \)-approximation and Lemma 2.22. The last statements then follow easily.

**Definition 4.29.** The metric \( \lambda g_n \) that was used in the previous proposition is called the metric used to define the neighborhood \( \nu(\tilde{\gamma}) \). By \( \ell(\tilde{\gamma}) \) we always mean the length of the geodesic \( \tilde{\gamma} \) with respect to the metric used to define the neighborhood. By a *spanning disk* in an \( \varepsilon \)-solid cylinder we mean a 2-disk with boundary contained in the side of the solid cylinder that separates the ends of the solid cylinder.

### 4.5.1 Intersections of the \( \nu_\xi(\tilde{\gamma}) \)

It is important to have control over how the various \( \varepsilon \)-solid cylinder neighborhoods near a flat boundary intersect.

**Lemma 4.30.** The following hold for all \( \xi > 0 \) sufficiently small, for all \( \mu > 0 \) less than a positive constant \( \mu_5(\xi) \), for any \( 0 < s_1 \leq 1/4 \), and for all \( \varepsilon > 0 \) less than a positive constant \( \varepsilon_3(\xi, s_1) \). For \( i = 1, 2 \), let \( X_i = B(\overline{\gamma}_i, 1) \) be standard 2-dimensional balls and let \( \gamma_i \subset B(\overline{\gamma}_i, 3/4) \) be \( \xi \)-approximations to \( \partial X_i \) on scale \( s_1 \) with \( \mu \)-control with \( \overline{\gamma}_i \in B(\overline{\gamma}_i, 1) \) being the \( \mu \)-control point for \( \gamma_i \). Suppose we have points \( x_i \in M_n \) with the property that \( B_{\gamma_i}(x_i, 1) \) is within \( \varepsilon \) of \( B(\overline{\gamma}_i, 1) \) for \( i = 1, 2 \). Furthermore, suppose that \( \gamma_i \subset B_{\gamma_i}(x_i, 1) \) is within \( \varepsilon \) of \( \gamma_i \). Denote by \( \ell_i \) the length of \( \gamma_i \) as measured in the metric \( g_{\gamma_i}(x_i) \). Then the intersection of \( \gamma_1 \) with \( \nu_\xi(\overline{\gamma}_1) \) is contained in \( \nu_{\varepsilon_2}(\overline{\gamma}_1) \) and \( \overline{\gamma}_1 \) meets each level set of \( f_{\gamma_2} \) in at most one point. In particular, for any \( c \geq \xi \) the intersection of \( \gamma_2 \) with the boundary of \( \partial \nu_{c\varepsilon}(\overline{\gamma}_1) \) is contained in the ends of this neighborhood.

**Proof.** Fix \( \xi > 0 \) and \( 0 < s_1 \leq 1/4 \). Let \( y_1 \in B_{\gamma_n_1}(x_1, 1) \) and \( y_2 \in B_{\gamma_n_2}(x_2, 1) \) be points within \( \varepsilon \) of \( \overline{\gamma}_1 \) and \( \overline{\gamma}_2 \) respectively. Since \( \nu_\xi(\overline{\gamma}_1) \cap \nu_\xi(\overline{\gamma}_2) \neq \emptyset \), it follows that \( \rho_n(x_1)/\rho_n(x_2) \) is between \( 1/2 \) and \( 2 \). The ball \( B_{\gamma_n_1}(y_1, s_1) \) contains \( \nu_\xi(\overline{\gamma}_1) \) and hence contains a point of \( \nu_\xi(\overline{\gamma}_2) \). The length of \( \gamma_2 \) with respect to \( g_{\gamma_n_2}(x_2) \) is
between $s_1/10$ and $s_1$, so its length with respect to $g'_n(x_1)$ is between $s_1/20$ and $2s_1$. If follows that $\tilde{\gamma}_2 \subset B_{g'_n(x_1)}(y_1, 4s_1)$. Similarly, $B_{g'_n(x_2)}(y_2, s_1) \subset B_{g'_n(x_1)}(y_1, 5s_1)$.

Now suppose that we have a sequence of $\mu_n \to 0$ and $\tilde{\epsilon}_n \to 0$ and counterexamples to the result for each of these constants. Denote the 2-dimensional balls associated to these counterexamples by $B(\tau_{n,1}, 1)$ and $B(\tau_{n,2}, 1)$ and denote the $\mu_n$-control points by $\gamma_{n,1}$ and $\gamma_{n,2}$. From the above we see that $B(\gamma_{n,2}, 1)$ is within $3\epsilon_n$ of a sub-ball of radius between $s_1/2$ and $2s_1$ in $B(\gamma_{n,1}, 5s_1)$. Passing to a subsequence so that limits $B(\gamma_{\infty,1}, 5s_1)$ and $B(\gamma_{\infty,2}, s_1)$ exist, we see that, taking the limit as $n$ tends to infinity, we see that $B(\gamma_{\infty,2}, s_1)$ is identified with a sub-ball of $B(\gamma_{\infty,1}, 5s_1)$. On the other hand since the $\mu_n \to 0$, both $B(\gamma_{\infty,1}, 5s_1)$ and $B(\gamma_{\infty,2}, s_1)$ are sub-balls of $[0, \infty) \times \mathbb{R}$ and the limiting geodesics $\gamma_{\infty,1}$ and $\gamma_{\infty,2}$ are geodesics in the boundary. Hence, the intersection of $\gamma_{\infty,2}$ with $\nu_\xi(\gamma_{\infty,1})$ is contained in $\gamma_{\infty,1}$. This is a contradiction, establishing the result.

We also need estimates about the vector fields from Lemma 4.28 and also about the distances between the sides of the neighborhoods.

**Lemma 4.31.** With notation and assumption as in the previous lemma the following hold, provided that $\xi > 0$ is sufficiently small.

1. For a unit vector field $\tilde{\tau}_1$ on $\nu_\xi(\gamma_1)$ satisfying Corollary 4.28 at any point of $\gamma_{\tilde{\tau}_1}(\gamma_1) \cap \tau_\xi(\gamma_2)$ we have
   $$\|f'(\tilde{\tau}_1)\| > 1 - 20\xi.$$

2. For any constants $c_1, c_2$ with $2\xi \leq c_i \leq 3/4$ and with
   $$c_1\ell_1\rho_n(x_1) < (0.9)c_2\ell_2\rho_n(x_2)$$
   each level set of $f_{\tilde{\tau}_2}$ in $\nu_{c_1\xi}(\gamma_2)$ that meets $\gamma_{\infty,1}$ meets $\nu_{c_1\xi}(\gamma_1)$ in a disk whose boundary is contained in the side of $\nu_{c_1\xi}(\gamma_1)$, a disk that separates the ends of $\nu_{c_1\xi}(\gamma_1)$.

**Proof.** The first statement is immediate from Corollary 3.19. It follows immediately from this that any level set of $f_{\tilde{\tau}_2}$ meets each flow line for $\tilde{\tau}_1$ in at most one point.

Now let us establish the second statement. Let $y$ be a point in
   $$\nu_\xi(\gamma_2) \cap \nu_{\xi,[-(24)\xi, (24)\xi]}(\gamma_1),$$
   and consider the level surface $L$ for $f_{\tilde{\tau}_2}$ through $y$. It follows from Corollary 3.19 that, provided that $\xi > 0$ is sufficiently small, the variation of $f_{\tilde{\tau}_1}$ on $L \cap \nu_\xi(\gamma_1)$ is less than $(0.001)\ell_1$. This implies that $L$ does not meet the ends of $\gamma_{\tilde{\tau}_1}(\gamma_1)$. Thus, under the given assumptions on $c_1$ and $c_2$ we see that $L \cap \left( h_{\tilde{\tau}_2}^{-1}([0, c_2\xi]) \right)$ crosses the side of $\nu_{c_1\xi}(\gamma_1)$.

Let us consider the intersection of $L$ with
   $$U = \nu_{c_1\xi}(\gamma_1) \setminus \nu_{c_2\xi}(\gamma_1).$$
On $U$ the functions $f_{\gamma_1}$ and $h_{\gamma_1}$ satisfy Lemma 2.23 and hence the intersection of the level sets of these functions are circles that are almost orthogonal to the horizontal spaces in $S^1 \times (0, 1)$ and is foliated by circles which are the intersections of $L$ with level sets of $h_{\gamma_1}$. Now we fix the circle $C = L \cap h_{\gamma_{n,1}}^{-1}(c_1 \ell_1/2)$ and let $D \subset L$ be the surface bounded by $C$. Let $z \in C$. We flow $D$ along the flow lines of $\tilde{\gamma}_1$ to a level set $L' = f_{\gamma_1}^{-1}(b)$ where $b$ is chosen so that $|b - f_{\gamma_1}(z)| = 4\xi^2 \ell_1$. By Corollary 3.19 for all $n$ sufficiently large, $f_{\gamma_1}$ varies by less than $2\xi^2 \ell_1$ on $L'$. By Corollary 3.19 $L' \cap L = \emptyset$ and the maximal difference between the values of $f_{\gamma_1}$ on $L$ and $L'$ is at most $4\xi^2 \ell_1$. Thus, any flow line for $\tau$ that starts on $D$ stays in $\nu_{c_1}(\tilde{\gamma}_1)$ as it flows from $D$ to $L'$. Thus, deforming along these flow lines gives a topological embedding of $D$ as a subsurface of $L'$. Since $\partial D = C$ is a single circle and since $L'$ is a disk, it follows that $D$ is a topological disk and hence $L$ is also a disk. Clearly, it separates the ends of $\nu_{c_1}(\tilde{\gamma}_1)$.

**Addendum 4.32.** In the previous two lemmas, we assumed the metrics were $g'_n(x_1)$ and $g'_n(x_2)$. The reason for this was that if $B_{g'_n(x_1)}(x_1, 1)$ have non-trivial intersection then the metrics are within a multiplicative factor of 2 of each other. We also have analogous results when we use a fixed multiple $\lambda g_n$ as the metric in two balls. The proofs are identical, since this time the metrics agree.

### 4.6 Boundary points of angle $\leq \pi - \delta$

**Proposition 4.33.** For all $a > 0$, given $a$ for all $\mu > 0$ sufficiently small, for all $0 < r \leq 10^{-3}$, and, given $a, \mu$, and $r$, for all $\epsilon > 0$ sufficiently small the following hold. Suppose that, for some $n$ there is a point $x_n \in M_n$ with the property that $B_{\lambda g_n}(x_n, 1)$ is within $\epsilon$ of a 2-dimensional Alexandrov space $X = B(\bar{\gamma}, 1)$ of area $\geq a$ that is boundary $\mu$-good at $\bar{\gamma}$ on scale $r$. Then the ball $B_{\lambda g_n}(x_n, 7r/8)$ is a topological 3-ball and the distance function, $d(x_n, \cdot)$, is $(1 - \beta)$-strongly regular on $B_{\lambda g_n}(x_n, 7r/8) \setminus B_{\lambda g_n}(x_n, r/8)$, where $\beta = \beta(\mu)$ limits to zero as $\mu$ tends to zero.

**Proof.** Fix $a > 0$. Suppose that there is a sequence $\mu_k \to 0$ as $k \to \infty$ for which the result does not hold, meaning that for each $k$ there is $0 < r_k \leq 10^{-3}$ for which there is no constant $\epsilon$ as required. This implies that for each $k$ there is a sequence of constants $\epsilon_k$ tending to zero and counter-examples $B_{\lambda g_n}(x_n, 1)$ with these values of the constants. The balls $B_{\lambda g_n}(x_n, 1)$ that are within $\epsilon_k$ of standard 2-dimensional balls $B(\bar{\tau}_k, \ell_1)$ of area at least $a$, balls that are boundary $\mu_k$-good at $\bar{\tau}_k$ on scale $r_k$. The ball $B(\lambda g_n(x_n, 1), (1/r_k)B(\bar{\tau}_k, \ell_1))$ is within $\epsilon_k$ of the unit ball centered at $\bar{\tau}_k$ in $(1/r_k)B(\bar{\tau}_k, \ell_1)$ and the latter is boundary $\mu_k$-good at $\bar{\tau}_k$ on scale 1. For each $k$ we choose $1/k$ sufficiently large such that $\epsilon_k (1/k) \to 0$. (We shall add another condition on how large $1/k$ must be later in the argument.) Re-index the constants by $k$ so that, for example, $\lambda_{k, \ell}(k)$ is denoted $\lambda_k$. Passing to a subsequence and taking a limit, the fact that the $\mu_k \to 0$ implies that the 2-dimensional unit balls converge to a flat cone $C$ in $\mathbb{R}^2$ of angle $\leq \pi$. The area of $C$ is bounded below by a positive constant $a'$ depending only on $a$. By our
In the first case, all the level sets for the distance function from \( \delta \) strictly between 0 and 1 are 2-spheres and the corresponding metric balls are homeomorphic to 3-balls. This is a contradiction, proving the result in this case.

Lastly, we also assume that for each \( k \) we have chosen \( \ell(k) \) sufficiently large so that \( \hat{\epsilon}_k, \ell(k) < \hat{\epsilon}_2(\epsilon, \mu_k, 1/4) \) for Proposition \([4.23]\). We consider first the case when the cone angle at the cone point of \( C \) is \( \pi \). In this case, \( C \) is isometric to a unit ball centered at a boundary point of \( \mathbb{R} \times [0, \infty) \). Since \( \mu_k \to 0 \) by Proposition \([4.23]\) and our choice of \( \ell(k) \), there is a constant \( \zeta > 0 \) such that for all \( k \) sufficiently large there neighborhood of \( x_k \) containing \( B(\lambda_k/(r_k)^2)g_{n(k)}(x_k, \zeta) \) and contained in \( B(\lambda_k/(r_k)^2)g_{n(k)}(x_k, 1) \) that is homeomorphic to \( D^2 \times I \). The boundary of this neighborhood, which is a 2-sphere, separates the level set for \( d_k \) at distance \( \zeta \) from the level set at distance 1. It follows that all the level sets are \( S^2 \)-spheres. Furthermore, since the level set at distance \( \zeta \) is a 2-sphere contained in a neighborhood of \( x_k \) homeomorphic to a 3-ball, this level set bounds a 3-ball in this neighborhood. It follows immediately that for all \( k \) sufficiently large, all the metric balls \( B(\lambda_k/(r_k)^2)g_{n(k)}(x_k, t) \) for \( \zeta \leq t \leq 1 \) are homeomorphic to 3-balls. This is a contradiction, proving the result in this case.

We now examine the case when the cone angle of \( C \) is strictly less than \( \pi \). For the rest of the argument we implicitly use the metric \( (\lambda_k/(r_k)^2)g_{n(k)} \) on \( M_{n(k)} \) and on all its subsets. According to the Proposition \([2.26]\) there is a sequence of points \( x'_{n(k)} \in M_{n(k)} \) with \( d(x'_{n(k)}; x_{n(k)}) \to 0 \) such that one of the two following cases holds:

1. the distance function from \( x'_{n(k)} \) has no critical points on \( B(x'_{n(k)}, 1) \setminus \{x'_{n(k)}\} \),

or

2. there is a sequence \( \delta_k \to 0 \) such that the distance function from \( x'_{n(k)} \) has no critical points at distances between \( \delta_k \) and 1 and has a critical point at distance \( \delta_k \).

In the first case, all the level sets for the distance function from \( x'_{n(k)} \) at distance strictly between 0 and 1 are 2-spheres and the corresponding metric balls are homeomorphic to 3-balls. In the second case, rescaling by \( \delta_k^{-2} \) we get a sequence of 3-manifolds with a subsequence converging to a 3-dimensional Alexandrov space of curvature \( \geq 0 \). By Proposition \([1.3]\) the convergence is in fact a smooth convergence and the limit is a smooth complete 3-manifold of non-negative curvature. It follows that for all \( k \) sufficiently large, one of these two possibilities holds for \( X_k \).

**Claim 4.34.** The level sets of the distance function \( f_k = d(x'_{n(k)}, \cdot) \) at distance between \( \delta_k \) and 1 are topological 2-spheres.
Let us assume this claim for a moment and complete the proof of the lemma. It follows from this claim that the end of the limiting manifold is homeomorphic to $S^2 \times [0, \infty)$. The limiting manifold has a soul which is a manifold of non-negative curvature. Because the neighborhood of infinity of the limit is diffeomorphic to $S^2 \times [0, \infty)$, the soul must be either a point or $\mathbb{R}P^2$. The second case is not possible, since in this case the original manifolds would converge to an interval not a 2-dimensional Alexandrov space of area $\geq a$. Since its soul is a point, the limiting manifold is diffeomorphic to $\mathbb{R}^3$. The result is then immediate.

It remains to prove the claim.

Proof. (of the claim) We know that

$$f_k: B(\lambda_k/(r_k)^2)g_{\alpha(k)}(x_{\alpha(k)}, 3/4) \setminus B(\lambda_k/(r_k)^2)g_{\alpha(k)}(x_{\alpha(k)}, 1/4) \to (1/4, 3/4)$$

is the projection mapping of a locally trivial fibration. Set $b^+ = 1/2$ and $b^- = (3/4)$. Then using the metric $(\lambda_k/(r_k)^2)g_{\alpha(k)}$, by our choice of constants and Proposition 4.23 there are good $\xi$-approximations $\nu_\xi(\gamma)$ and $\nu_\xi(\gamma')$ to the boundary of length $1/10$ centered at the two points of $\partial B$ at distance $(48)$ from $\mathfrak{F}$. Let $\nu_\xi(\gamma_k)$ and $\nu_\xi(\gamma_k')$ be corresponding neighborhoods in $M_{\alpha(k)}$. For $k$ sufficiently large we can choose these geodesics within $\epsilon'$ of $\gamma$ and $\gamma'$. Every point of the open subset $U$ which is the intersection of

$$B(\lambda_k/(r_k)^2)g_{\alpha(k)}(x_{\alpha(k)}, b^+) \setminus \overline{B(\lambda_k/(r_k)^2)g_{\alpha(k)}(x_{\alpha(k)}, b^-)}$$

with the complement of the closure of $\nu_\xi(\gamma_k) \cup \nu_\xi(\gamma_k')$ is the center of an $S^1$-product structure with $\epsilon$-control. Hence, this subset sits inside a larger open subset that is the total space of an $S^1$-fibration with fibers within $\epsilon'$ of orthogonal to the horizontal spaces of the $S^1$-product structures with $\epsilon$-control. This implies that there is an annulus in $U$ with boundary contained in $\nu_\xi(\gamma_k) \cup \nu_\xi(\gamma_k')$ that separates $f_k^{-1}(b^+) \cap U$ from $f_k^{-1}(b^-) \cap U$. Since the boundary loops of this annulus are homotopically trivial in $\nu_\xi(\gamma_k) \cup \nu_\xi(\gamma_k')$, it follows that there is a map of $S^2$ into $B(\lambda_kg_{\alpha(k)}(x_{\alpha(k)}, b^+) \setminus B(\lambda_kg_{\alpha(k)}(x_{\alpha(k)}, b)$ that is homologically non-trivial. The claim follows.

This argument actually proves more.

**Corollary 4.35.** Fix $\epsilon' > 0$ and let $\epsilon > 0$ be less than the constant $\epsilon_1(\epsilon')$ as in Proposition 1.21. For all $\xi > 0$ sufficiently small and every $a > 0$ the following holds for all $\mu$ less than a positive constant $\mu_0(\xi, a)$, for every $r > 0$ and for all $\eta$ less than a positive constant $\varepsilon_4(\epsilon, a, \mu, r)$. With the notation and assumptions of the previous proposition, fix $b \in (r'/8, 7r'/8)$. The level set $L_b = d(x, \cdot)^{-1}(b)$ is a topologically locally flat 2-sphere and the metric ball that it bounds is a topological 3-ball. Furthermore, there are two geodesics $\gamma_1$ and $\gamma_2$ within $\epsilon$ of geodesics $\gamma_1$ and $\gamma_2$ in $X$ that are $\xi$-approximations to $\partial X$ on scale $s_1$ with the property that every point of $L_b$ is not the center of an $S^1$-product neighborhood with $\epsilon$-control is contained in union $\nu_\xi(\gamma_1) \cup \nu_\xi(\gamma_2)$. 


Actually, we have more control over the intersections of the level sets with the \( \nu_\varepsilon(\gamma_i) \).

**Corollary 4.36.** With notation and assumptions as in the previous corollary, for any \( b \in (r'/8, 7r'/8) \), and any the level set \( L_b = d_{\lambda g_n}(x_n, \cdot)^{-1}(b) \) meets \( \pi_{\xi}(\gamma_i) \) in a spanning 2-disk in \( \pi_{\xi}(\gamma_i) \). Furthermore, for any \( c \in [\xi, 1] \) the level set \( h_{\gamma_i}^{-1}(c\xi(\gamma_i)) \) crosses \( L_b \) topologically transversally and the intersection is a circle bounding the disk \( L_b \cap \pi_{\xi}(\gamma_i) \).

**Proof.** Let \( f \) denote the distance function from \( x_n \). By Corollary 4.17 and the fact that \( B_{\lambda g_n}(x_n, 1) \) is within \( \varepsilon \) of a standard 2-dimensional ball \( B(\pi, 1) \) that is boundary \( \mu \)-good at \( \pi \), it follows that \( f \) is less than \( b \) on one end of \( \nu_\varepsilon(\gamma_i) \) and greater than \( b \) on the other end, so that \( L_b \cap \nu_\varepsilon(\gamma_i) \) separates the ends of \( \nu_\varepsilon(\gamma_i) \). Also, by the same lemma the function \( f \) is increasing on the flow lines of \( \tau \) is in Corollary 4.28 so that \( L_b \cap \nu_\varepsilon(\gamma_i) \) is transverse to these flow lines. Furthermore, it follows from Corollary 4.28 that any flow line from a point of \( L_b \cap \nu_\varepsilon(\gamma_i) \) remains in \( \nu_\varepsilon(\gamma_i) \) until it crosses the end of this region. Thus, flowing along these flow lines gives us an embedding of \( L_b \cap \nu_\varepsilon(\gamma_i) \) into a level disk for \( f_{\gamma_i} \) and hence embeds this surface as a subsurface \( \Sigma \) of a disk. It follows from §11 of [3] applied to the functions \( h_{\gamma_i} \) and \( d(x_0, \cdot) \) restricted to \( \nu_\varepsilon(\gamma_i) \) that the intersection \( L_b \cap \partial \nu_\varepsilon(\gamma_i) \) is homeomorphic to \( S^1 \times (0, 1) \) and the intersections of \( L_b \) with the level surfaces of \( h_{\gamma_i} \) foliate this region by circles. If follows that \( \Sigma \) has a single boundary component and hence is homeomorphic to a disk. From that it is immediate that for every \( c \in [\xi, 1] \) the intersection \( L_b \cap \nu_{\varepsilon, \xi}(\gamma_i) \) is a 2-disk. \( \square \)

**Definition 4.37.** We call any ball \( B_{\lambda g_n}(x_n, r'/4) \) satisfying the conclusions of Proposition 4.33, Lemma 4.35 and Corollary 4.36 a 3-ball near a 2-dimensional boundary corner.

### 4.7 Balls near open intervals

The following results describe the parts of \( M_n \) close to 1-dimensional Alexandrov balls.

**Lemma 4.38.** Given \( \epsilon' > 0 \) the following holds for all \( 0 < \epsilon \) less than a positive constant \( \epsilon_2(\epsilon') \). If \( B_{g_n}(x_n, 1) \) is within \( \epsilon \) of a standard 1-dimensional ball \( J \), then for any point \( y \in B_{g_n}(x_n, 24/25) \) whose distance from the endpoints of \( J \) (if any) is at least \( 1/25 \) there is an open set \( U = U(y) \), with \( B_{g_n}(x_n)(y, 1/50) \subset U \subset B_{g_n}(x_n)(y, 1/25) \), and an \( \epsilon' \)-approximation \( p_{x_n} : U \to J \), where \( J \) is an open interval of length \( 3/50 \) with central point \( p_{x_n}(y) \) such that the following hold:

1. There is a product structure on \( U \) such that \( p_{x_n} \) is the projection mapping onto the interval factor.
2. The fibers of \( p_{x_n} \) are homeomorphic to either 2-spheres or 2-tori.
3. There is a smooth unit vector unit field \( \chi \) on \( U \) such that for any (minimal) geodesic \( \gamma \) of length \( \geq 1/400 \), measured in the metric \( g_n(x_n) \), ending at a point \( z \in U \), the angle at \( z \) between \( \chi(z) \) and \( \gamma'(z) \) is within \( \epsilon' \) of either 0 or \( \pi \).
4. If \( z \in B_{g_n}(x_n)(y, 1/50) \) and if \( d_{g_n}(x_n)(w, z) \geq 1/400 \), then the level surface of the distance function \( d(w, \cdot) \) through \( z \) is contained in \( U \) and is isotopic in \( U \) to a fiber of \( p_{x_n} \).

Proof. Take a point \( u \) at distance 7/200 from \( x_n \) and define

\[
U = \left[ B_{g_n}(x_n)(u, 13/200) \setminus B_{g_n}(x_n)(u, 1/200) \right] \cap B_{g_n}(x_n)(x_n, 1/25)
\]

and let \( p_{x_n} : U \to (-3/100, 3/100) \) be \( d(u, \cdot) - 7/200 \). This is a \( (1 - \delta) \)-strongly regular Lipschitz function for some \( \delta \) that depends on \( \varepsilon \) and goes to zero as \( \varepsilon \) does. According to Lemma 2.22 there is a smooth unit vector field \( \chi \) on \( U \) with the property that \( p'_{x_n}(\chi) > 1 - \delta' \) for some \( \delta' \) that depends on \( \delta \) and goes to zero as \( \delta \) does. The integral curves of this vector field cross each level surface of \( p_{x_n} \) exactly once, and hence determine a product structure on \( U \) with \( p_{x_n} \) being the projection onto one factor. Now let \( \gamma \) be any geodesic of length at least 1/400 ending at \( z \in U \). By restricting \( \gamma \) to a possibly shorter geodesic we can suppose that \( \gamma \subset B_{g_n}(x_n)(x_n, 1/25) \). Denote by \( a \) its other endpoint. Suppose that we have points \( \underline{p}, \underline{z}, \overline{a} \in J \) such that under the \( \varepsilon \) approximation of between this ball and the open interval \( J \) that \( d(\underline{p}, u) < \varepsilon \), \( d(\underline{z}, z) < \varepsilon \) and \( d(\overline{a}, a) < \varepsilon \). Suppose that \( \overline{a} \) is separated from \( \overline{p} \) by \( \underline{z} \). Then the comparison angle \( \angle azu \) is close to \( \pi \), with an error that goes to zero as \( \varepsilon \) does. Hence, by monotonicity the actual angle that \( \gamma \) makes with any geodesic from \( u \) to \( z \) is close to \( \pi \) with an error going to zero as \( \varepsilon \to 0 \). Since the angle at \( z \) between \( \chi(z) \) and any geodesic from \( u \) to \( z \) is close to \( \pi \), it follows that the angle between \( \chi(z) \) and \( \gamma'(z) \) is less than an error term that goes to zero as \( \varepsilon \) goes to zero.

If \( \overline{p} \) lies on the same side of \( \overline{z} \) as \( \underline{p} \), then we choose \( u' \) at distance 7/200 from \( x_n \) but on the ‘other side’ of \( x_n \) (meaning the approximating points in the interval lie on the other side). The same argument then shows that the angle between any geodesic from \( u' \) to \( z \) and \( \gamma \) is close to \( \pi \), implying that the angle between \( \gamma \) and any geodesic from \( u \) to \( z \) is close to 0. Since the angle between \( \chi \) and any geodesics from \( u \) to \( z \) is close to \( \pi \), the angle between \( \chi \) and \( \gamma'(z) \) is close to \( \pi \), with an error that goes to zero as \( \varepsilon \) goes to zero.

Now let \( z \in B_{g_n}(x_n)(x_n, 1/50) \) and let \( w \in M_n \) be a point such that \( d(w, z) \geq 1/400 \). It follows easily from the result just established that if \( \varepsilon > 0 \) is sufficiently small, then the level set of \( d(w, \cdot) \) through \( z \) is contained in \( U \) and is transverse to \( \chi \) and hence isotopic in this open set to a fiber of \( p_{x_n} \).

It remains to show that, provided that \( \varepsilon > 0 \) is sufficiently small, the fibers of \( p_{x_n} \) are either 2-spheres or 2-tori. If not we take a sequence of \( \varepsilon_k \to 0 \) and examples \( p_k : U_k \to (-3/100, 3/100) \) with fibers \( L_k = p_k^{-1}(t_k) \) that are not 2-spheres or 2-tori. Fix points \( z_k \in L_k \), let \( d_k \) be the diameter of \( L_k \) and rescale, forming \( \frac{1}{d_k}(U_k, z_k) \), and, after passing to a subsequence take a limit. This limit is an Alexandrov space of dimension 2 or 3 and splits as a product \( \mathbb{R} \times Y \) where \( Y \) has diameter 1. If \( Y \) is 2-dimensional, then by Proposition 1.3 the convergence is smooth and \( Y \) is a surface of curvature \( \geq 0 \). Since \( Y \) is orientable, it follows in this case that \( Y \) and hence the fibers \( L_k \), for all \( k \) sufficiently large, are homeomorphic to either 2-spheres or 2-tori, which is a contradiction.
Suppose that \( Y \) is 1-dimensional. Then it is either a closed interval or circle, and there are rescalings \( \lambda_k \) such that \( \lambda_k U \) converges to the product \( \mathbb{R} \times Y \). If \( Y \) is a circle, we invoke Lemma 4.11 and Proposition 4.13 to see that for all \( k \) sufficiently large, any level set of \( p_k \) is contained in a compact subset \( V_k \subset \lambda_k U \) that is the total space of a circle fibration. We can take a slightly smaller compact fibration \( W_k \subset V_k \) still containing the level set. The boundary components of \( W_k \) are tori and at least one of them separates the two ends of \( \lambda_k U \). On the other hand, the level set \( L_k \) separates two of the boundary components of \( W_k \). These two facts together imply that for all \( k \) sufficiently large, \( L_k \) is a 2-torus, in contradiction to our assumption.

Lastly, suppose that \( Y \) is a closed interval. Then invoking Lemma 4.11, Proposition 4.14, and Proposition 4.23 we see that for all \( k \) sufficiently large every level set of \( p_k \) is contained in the union of the total space of an \( S^1 \)-fibration and two sets of the form \( \nu_k^2(\gamma_i) \) as in Proposition 4.23. Since the homotopy class of the fiber of the \( S^1 \)-fibration is trivial in \( \nu_k^2(\gamma_i) \), it follows that the level set of \( p_k \) is contained in an open subset of \( U_k \) whose fundamental group is the fundamental group of a connected surface with non-empty boundary; that is to say a free group. But the fundamental group of the level set maps isomorphically onto the fundamental group of \( U_k \) and is the group of a surface. This means that the level set is the 2-sphere.

**Definition 4.39.** A neighborhood \( U \), a point \( y \in U \), and a projection mapping \( p: U \to J \) satisfying the conclusions of the above lemma is called an interval product structure centered at \( y \) with \( \epsilon' \)-control. The content of the above lemma is that for \( \epsilon < e_2(\epsilon') \) if \( B_{g_n(x)}(x, 1) \) is within \( \epsilon \) of a standard 1-dimensional ball \( J \) and if \( y \in B_{g_n(x)}(x, 24/25) \) has distance at least 1/25 from the endpoints (if any) of \( J \), then there is an interval product structure centered at \( y \) with \( \epsilon' \)-control.

Now we need to understand what happens near the endpoints of the nearby interval. Unlike elsewhere in this section, here we do not assume that \( B_{g_n(x)}(x_n, 1) \) is disjoint from \( \partial M_n \).

**Lemma 4.40.** There is a \( a_1 > 0 \) such that the following holds for all \( \epsilon > 0 \) and for all \( \beta \) less than a positive constant \( \beta(\epsilon) \). Suppose that \( B_{g_n(x)}(x, 1) \) is within \( \beta \) of a standard 1-dimensional ball \( J \), that \( \overline{\pi} \) is an endpoint of \( J \) and that \( d(x, \overline{\pi}) < 1/25 \). One of the following two possibilities holds:

1. \( \overline{B}(x, 1/2) \) is diffeomorphic to \( T^2 \times [0, 1/2) \), to a solid torus, to a twisted I-bundle over the Klein bottle, to a 3-ball, or to \( \mathbb{R}P^3 \setminus B^3 \).

2. There is a \( \lambda \geq \epsilon^{-1} \rho_n(x)^{-2} \) such that \( B_{\lambda g_n}(x, 1) \) is within \( \epsilon \) of a standard 2-dimensional ball of area at least \( a_1 \), and \( \overline{B}_{g_n(x)}(x, 1/2) \setminus B_{g_n(x)}(x, 1/\lambda) \) is a topological product of a surface with an interval with \( d(x, \cdot) \) being the projection mapping to the interval of this product structure.

**Proof.** The first case to consider is when \( B(x, 1/2) \) meets the boundary of \( M_n \). Let \( x' \in B(x, 1/2) \cap \partial M_n \). According to Proposition 4.16 \( B(x', 1) \) is diffeomorphic to \( T^2 \times [0, 1] \) and the result follows easily in this case. Thus, we can suppose that \( B(x, 1) \) is disjoint from \( \partial M_n \). Suppose that there is no \( \beta \) as required. We take a sequence...
Lemma 3.23 and the limit is a punctured \( \mathbb{R} \) isotopic to the level sets of the distance function from \( x \) of the original \( \xi > \) level sets of the distance function for \( x \) of \( \gamma \) of \( \beta \) \( \epsilon \) tion 4.4, and this establishes by contradiction that Case 1 holds under these assumptions. We consider the case when the result is 3-dimensional. By Proposition 1.3 it is a complete 3-manifold of non-negative curvature, and as such it has a soul. If the soul is a point, then the limit is diffeomorphic to \( \mathbb{R}^3 \) and level sets of the distance function from \( \pi \) are 2-spheres. If the soul is a circle, then the limit is a solid torus and the level sets of the distance function from \( \pi \) are 2-tori. If the soul is a Klein bottle, then the level sets of the distance function from \( \pi \) are 2-tori. If the soul is \( \mathbb{R}P^2 \), then the limit is a punctured \( \mathbb{R}P^3 \) and the level sets are 2-spheres. Thus, in these cases the original \( B(x_k,1/2) \) is diffeomorphic to the limiting complete manifold and the level sets of the distance function for \( x_k \) away from the end point are topologically isotopic to the level sets of the distance function from \( x_k' \) at distances more than \( \delta_k ^2 \). This establishes by contradiction that Case 1 holds under these assumptions.

Suppose that the limit of the rescalings is 2-dimensional \((X, \pi)\). Consider points \( q_k \in B_{g_i(\pi, x_k)}' (x_k, 1/2) \) that converge to a point \( \bar{q} \in J \) at distance 1/4 from \( \pi \). The point \( x_k' \) is chosen as the unique local maximum for the distance function from \( q_k \) near the endpoint of \( J \). Let \( \gamma_k \) be a geodesic from \( x_k' \) to \( q_k \), and let \( q_k' \) be the point of \( \gamma_k \) at distance 2/\( \delta_k \) from \( x_k' \). Let \( p_k \) be the critical point for \( d(x_k', \cdot) \) at distance \( \delta_k \) from \( x_k' \). In the rescaled ball we have \( |x_k' q_k'| = 2 \) and \( x_k' p_k' = 1 \). The fact that \( x_k' \) is the unique local maximum for the distance function from \( q_k \) near the endpoint of \( J \), this implies that \( |x_k' q_k'| \geq |p_k, q_k'| \). Since \( p_k \) is a critical point for the distance function from \( x_k' \), the comparison angle \( x_k' p_k q_k' \) is at most \( \pi/2 \). This facts together imply that the area of the unit ball centered at \( \pi \) in \( X \) has area at least \( a_1 \) for some universal constant \( a_1 \). This shows that Case 2 holds under these assumptions, which is a contradiction. 

\[ \square \]

### 4.8 Determination of the Constants

We fix \( \epsilon' > 0 \) a universally small constant. Then \( \epsilon > 0 \) is chosen to be less than the minimum of the constants \( c_0(\epsilon') \) in Proposition 4.3 and \( \epsilon_2(\epsilon') \) in Lemma 4.38 and sufficiently small so that Lemma 2.23 holds. Then \( \beta \) is chosen less than \( \tilde{\beta}(\epsilon) \) in Lemma 4.40 and also less than \( \epsilon/2 \). Now we fix \( 0 < \xi \leq 10^{-3} \) with \( \xi \) sufficiently small so that Theorem 3.22, Lemma 4.30, Lemma 4.31 and Corollary 4.35 all hold. We also fix \( \xi > 0 \) less than the constant \( \xi_0(\epsilon) \) in Proposition 4.23. Next, we fix \( a > 0 \) less than the constants \( a_2(\beta/2) \) in Lemma 3.23 and \( a_1 \) in Lemma 4.40. We now fix \( \mu > 0 \) less than the minimum of \( \{\mu_1(\xi), \mu_2(\epsilon), \mu_3(\epsilon, a), \mu_4(\xi, \epsilon), \mu_5(\xi, \epsilon), \mu_6(\xi, a)\} \) where these are the constants given
Theorem 3.22, Lemma 4.1, Proposition 4.17, Proposition 4.23, Lemma 4.30, and Corollary 4.35. Now we fix $\delta, r_0$ positive constants as in Theorem 3.22 for the given values of $\xi, \mu$ and $a$. We fix $s_1 > 0$ less than $\tilde{s}_1(\xi, \mu, a)$ in Theorem 3.22. Then we choose $s_2 > 0$ less than the constants $\tilde{s}_2(\xi, \mu, a, s_1)$ in Theorem 3.22. With all of these constants determined, we are ready to fix $0 < \hat{\epsilon} < \beta/2$. We choose this constant less that the minimum of \{\hat{\epsilon}_0(\mu, s_2), \hat{\epsilon}_1(\epsilon, a, r_0), \hat{\epsilon}_2(\epsilon, \mu, s_1), \hat{\epsilon}_3(\xi, s_1), \hat{\epsilon}_4(\epsilon, a, \mu, r_0)\} as given in Lemma 4.1, Proposition 4.17, Proposition 4.23, Lemma 4.30, Corollary 4.35. We also choose $\hat{\epsilon} < 10^{-s_1}/C$ where $C$ is the constant in Lemma 4.1.

Now we pass to a subsequence of the $M_n$ so that $\hat{\epsilon}_n \leq \min(\hat{\epsilon}, \epsilon)$ for all $n$, and also so that Proposition 4.16 holds for all $n$.

5 The global result

At this point we have fixed all the constants appearing in the last two sections in such a way that the conclusions of all the results from these two sections hold. This gives us complete control over the local nature of the $(M_n, g_n)$, in the sense that we have complete control over the $B_{g_n}(x, 1/2) \subset B_{g_n}(x, 1)$ for every $x \in M_n$. The purpose of this section is to globalize these results establishing Theorem 1.1.

Definition 5.1. Given a ball $B_{\lambda g_n}(x, r)$ we say that $r$ is its rescaled radius and $r/\sqrt{\lambda}$ is its unrescaled radius.

5.1 Regions of $M_n$ close to open intervals

We begin the globalization by studying the generic “1-dimensional” regions of the $M_n$. We shall construct an open set $U'_{n,1} \subset M_n$ which is a first approximation to the submanifold $V_{n,1} \subset M_n$ referred to in Theorem 1.1. The manifold $U'_{n,1}$ will be an open submanifold. Eventually, when we define $V_{n,1}$ as follows: For each end of $U'_{n,1}$ either we truncate it by removing an open collar neighborhood of that end, or we extend it by adding compact external collar neighborhood. Also, we shall add disjoint compact 3-balls to $U'_{n,1}$ in creating $V_{n,1}$.

Proposition 5.2. Consider the subset $X_{n,1} \subset M_n$ consisting of all points $x_n \in M_n$ for which $B_{g_n}(x_n, 1) \subset B_{g_n}(x, 1)$ is within $\hat{\epsilon}$ of a standard 1-dimensional ball $J$ and the distance from $x_n$ to the endpoints (if any) of $J$ is at least $1/50$. Then there is an open subset $U_{n,1} \subset M_n$ containing $X_{n,1}$ with the following properties:

1. Each component of $U_{n,1}$ is either a 2-torus bundle over the circle, or diffeomorphic to a product of either $S^2$ or $T^2$ with an open interval.

2. For each non-compact end $E$ of $U_{n,1}$ there is a point $x_E \in X_{n,1}$, and an interval product structure centered at $x_E$ with $\epsilon'$-control, $p_{x_E} : U(x_E) \to J(x_E)$, where $J(x_E)$ is an interval of length $\geq 1/100$, with the property that $U(x_E)$ is a neighborhood of the end $E$. 
3. For distinct non-compact ends $\mathcal{E}$ and $\mathcal{E}'$ the neighborhoods $U(x_\mathcal{E})$ and $U(x_\mathcal{E}')$ are disjoint.

4. For each point $x \in X_{n,1}$, the ball $B_{g'_n(x)}(x,1/400)$ is contained in $U_{n,1}$.

Proof. Suppose that we have an open set $V \subset M_n$ satisfying the first three conclusions and a point $x \in X_{n,1}$ for which the fourth conclusion does not hold. Consider the open set $U(x)$ and projection $p_x: U(x) \rightarrow J'$ associated to $x$ by Lemma 4.38. Recall that $J'$ is of length $3/50$. Let $J'' \subset J'$ be an open interval of length $1/25$ centered at $p_x(x)$ and let $W = p^{-1}(J')$. If $W$ is disjoint from $V$ we replace $V$ by $V \cup W$. The result satisfies the first three conclusions. Suppose that $W$ meets a component of $V$. Since $W$ is close to an open interval, it has two ends. Suppose that there is a level set of $p$ near each end of $W$ that is contained in $V$. Then by Lemma 4.38 these level sets are isotopic to the fibers of the product structure of these components of $V$, and hence the union of $V \cup W$ still satisfies the first three conclusions of this proposition. Similarly, if one end of $W$ has such a level surface and the other end is disjoint from $V$, then the union $V \cup W$ satisfies the first three conclusions of this proposition.

Now suppose that one of the ends of $W$ (say the end corresponding to $-1/50$) meets $V$ but no level surface near this end of $W$ is contained in $V$. Let $x'$ be the point as in the second item for the corresponding end of $V$. Then $\rho_n(x')$ and $\rho_n(x)$ are within a multiplicative factor of 2 of each other. We extend the end of $W$ by taking $\tilde{W} = p^{-1}(-3/100,1/50)$. According to Lemma 4.38 and the fact that $\rho_n(x') \leq 2\rho_n(x)$, there is a level surface of $\tilde{W}$ near the negative end of $\tilde{W}$ that is contained in $V$. Arguing as before shows that in all cases we can extend $V$ by taking its union with a set of the form $\tilde{W}$ in such a way that the first three conclusions still hold but also so that $B_{g'_n(x)}(x,1/400) \subset V \cup W$. Since $M_n$ is compact and $\rho_n$ is bounded below by a positive constant on $M_n$, it follows easily that after a finite number of such extensions we have arrived at a situation where the all four conclusions hold. \[\square\]

We fix $U_{n,1} \subset M_n$ as in the above proposition. For each non-compact end $\mathcal{E}$ of $U_{n,1}$ we fix a point $x_\mathcal{E}$ producing the neighborhood $U(x_\mathcal{E})$ of the end together with a projection mapping $p_{x_\mathcal{E}}: U(x_\mathcal{E}) \rightarrow J(x_\mathcal{E})$ as in Conclusion 2 of Proposition 5.2. In particular, $p_{x_\mathcal{E}}$ is an $\epsilon'$-approximation and $J(x_\mathcal{E})$ is an interval of length at least $1/50$ centered at $p_{x_\mathcal{E}}(x_\mathcal{E})$.

5.2 Balls close to half-open intervals

Now suppose that $x \in M_n \setminus U_{n,1}$ is in the closure of $U_{n,1}$. Since $\epsilon_n < \epsilon$ for all $n$, there are three possibilities for $B_{g'_n(x)}(x,1)$:

1. It is within $\epsilon$ of a standard 2-dimension ball $\overline{B}$ of area $\geq a$.
2. It is within $\epsilon$ of a 2-dimensional standard ball $\overline{B}$ of area $< a$.
3. It is within $\epsilon$ of a standard 1-dimensional ball $J$. 
In the second case, if follows from Lemma 3.23 and the fact $a < a_2(\beta/2)$ that $B$ is within $\beta/2$ of a standard 1-dimensional ball $J$ and since $\epsilon < \beta/2$, if follows that $B_{g'_n(x)}(x, 1)$ is within $\beta$ of $J$. Thus, in the second and third cases, $B_{g'_n(x)}(x, 1)$ is within $\beta$ of a standard 1-dimensional ball $J$. Suppose that either Case 2 or 3 above holds, and consider two further possibilities: (i) If the endpoints of $J$, if any, are at distance at least $1/25$ from $x$, and (ii) there is an endpoint of $J$ within distance $1/25$ of $x$. The first possibility contradicts the fact that $x \notin U_{n,1}$: Since we have chosen $\beta < \epsilon/2$ it follows from the definition if (i) holds then that $x \in X_{n,1} \subset U_{n,1}$.

This contradicts our assumption that $x \notin U_{n,1}$. Thus, we conclude that $x$ is within distance $1/25$ of an endpoint of $J$. Since $\beta < \beta(\epsilon)$ from Lemma 4.40, the conclusions of that lemma hold for $B = B_{g'_n(x)}(x, 1)$. That is to say: there exists an open subset $V = V(x) \subset B$ containing $B_{g'_n(x)}(x, 1/2)$ such that one of the following hold:

1. $V$ is diffeomorphic to $T^2 \times [0, 1)$ and contains a boundary component of $M_n$.

2. $V$ is an open 3-ball or is homeomorphic to (a) the complement of a closed 3-ball in $\mathbb{R}P^3$, (b) an open solid torus, or (c) an open twisted $I$-bundle over the Klein bottle.

3. There is a constant $\lambda > \epsilon^{-1}$ and a point $x' \in M_n$ such that $B' = B_{g'\lambda_n(x)}(x', 1)$ contains $B_{g'\lambda_n(x)}(x, 1/2)$ and is within $\epsilon$ of a 2-dimensional Alexandrov ball of radius 1 and area at least $a_0$.

Furthermore, in all cases the end of $V$ is contained in $U_{n,1}$ and there is a level set $\Sigma$ for the distance function from $x$ with $\Sigma \subset U_{n,1}$ and with $\Sigma$ isotopic in $V \cap U_{n,1}$ to a fiber of the fibration structure of $U_{n,1}$ (of course $\Sigma$ is either a 2-sphere or a 2-torus). Thus, in the first three cases the union of $V$ with the component of $U_{n,1}$ containing the end of $V$ is diffeomorphic to $V$. In the last case, the distance function from $x'$ has no critical points in $B \setminus \overline{B_{g'\lambda_n(x')}(x', 1)}$, and in particular, the region between $B'$ and $\Sigma$ is a topological product.

This completes the proof of the following:

**Lemma 5.3.** Let $A$ be a connected component of $M_n \setminus U_{n,1}$. Then one of the following holds.

1. For every point $x \in A$ the ball $B_{g'_n(x)}(x, 1)$ is within $\epsilon_n$ of a standard 2-dimensional ball $B$ of area $\geq a$.

2. There is a point $x \in A$ such that $A \subset B_{g'_n(x)}(x, 1/2)$ is diffeomorphic to $T^2 \times [0, 1]$, a solid torus, a twisted $I$-bundle over the Klein bottle, a closed 3-ball, or $\mathbb{R}P^3 \setminus B^3$. In all these cases the metric sphere $S_{g'_n(x)}(x, 1)$ is either a 2-torus or a 2-sphere and is isotopic in $U_{n,1}$ to a fiber in its fibration structure.

3. There is $\lambda > \epsilon^{-1}$ and a point $x \in A$ such that:

   (a) $A \subset B_{g'_n(x)}(x, 1/2)$.

   (b) $\overline{B_{g'_n(x)}(x, 1)} \setminus B_{g'_n(x)}(x, 1/\lambda)$ is a topological product with an interval and the distance function from $x$ is the projection mapping of this product structure.
(c) $B_{g'_n(x)}(x, 9/10) \setminus B_{g'_n(x)}(x, 1/10) \subset U_{n,1}$ and $S_{g'_n(x)}(x, 1/2)$ is isotopic in $U_{n,1}$ to a fiber of its fibration structure.

(d) $B_{\lambda g'_n(x)}(x, 1)$ is within $\epsilon$ of a standard 2-dimensional ball of area $\geq a$.

**Definition 5.4.** We call a component of $M_n \setminus U_{n,1}$ satisfying the Conclusion 3 above a component which is close to an interval but which expands to be close to a standard 2-dimensional ball and we call a component satisfying Conclusion 1 above a component close to a 2-dimensional space. For a component which is close to an interval but which expands to be close to a standard 2-dimensional ball, we use the metric $\lambda g'_n(x)$ as described in Part 3 of the previous lemma on the entire component. For components close to a 2-dimensional space we use varying metrics $g'_n(x_i)$ as described in Part 1 of the previous lemma.

At this point we add to $U_{n,1}$ every component of $M_n \setminus U_{n,1}$ of Type 2 in Lemma 5.3. Call the result $U'_{n,1}$. Some of the components of $U'_{n,1}$ are components of $U_{n,1}$. Let us consider the others. Fix a component $C'$ of $U'_{n,1}$ that is not a component of $U_{n,1}$. It contains a component $C$ of $U_{n,1}$. The component $C$ has at most two ends and $C'$ is the union of $C$ with either one or two neighboring components of $M_n \setminus C$ (neighboring in the sense that their closures meet $C$). Let $A$ be a component of $M_n \setminus C$ neighboring $C$ that is contained in $C'$. Then $A$ is diffeomorphic to one of the four manifolds listed in Conclusion 2 of Lemma 5.3. Furthermore, any fiber of the fibration structure on $C$, it divides $C'$ into two components one of which contains $A$ and is the union of $A$ and a collar neighborhood of the boundary of $A$. Hence, this closed complementary component is homeomorphic to $A$. If $C' = C \cup A$, then it follows that $C'$ is homeomorphic to $\int A$ and hence to the interior of one of the four manifolds listed in Conclusion 2 of Lemma 5.3. If $C' = C \cup A_1 \cup A_2$ for distinct components $A_1$ and $A_2$ of $M_n \setminus U_{n,1}$, then the same argument shows that $C'$ is the union of two manifolds homeomorphic to one of the four listed in Conclusion 2 of Lemma 5.3 along their common boundary. Any such manifold is a component of $M_n$, and every one of its prime factors is geometric. (The manifold is prime unless it is $S^3$ or $\mathbb{R}P^3 \# \mathbb{R}P^3$.)

Invoking the hypothesis that no closed component of $M_n$ admits a Riemannian metric of non-negative sectional curvature, allows us to conclude the following:

**Proposition 5.5.** The open subset $U'_{n,1} \subset M_n$ constructed in the previous paragraph satisfies the following:

1. Every component of $U'_{n,1}$ is diffeomorphic to one of the following:

   (a) a $T^2$-bundle or an $S^2$-bundle over either the circle or an interval with the fiber(s) over the endpoint(s) being boundary component(s) of $M_n$,

   (b) a twisted $I$-bundle over the Klein bottle whose boundary is a boundary component of $M_n$,

   (c) an open solid torus, an open twisted $I$-bundle over the Klein bottle, an open 3-ball, the complement of a closed 3-ball in $\mathbb{R}P^3$, or
(d) the union of two twisted \( I \)-bundles over the Klein bottle along their common boundary.

2. Each non-compact end of \( U'_{n,1} \) has a neighborhood that is a component of \( U_{n,1} \), and hence there is a non-compact end \( \mathcal{E} \) of \( U_{n,1} \) such that \( U(x_\mathcal{E}) \) is a neighborhood of this end.

3. Every complementary component \( M_n \setminus U'_{n,1} \) either is a component close to a 2-dimensional space or is a component which is close to an interval but which expands to be close to a standard 2-dimensional ball.

5.3 A decomposition into compact sets

For each non-compact end \( \mathcal{E} \) of \( U'_{n,1} \) we have the neighborhood \( U(x_\mathcal{E}) \) that fibers over an interval \( J(x_\mathcal{E}) \) by an \( \epsilon' \)-approximation. Denote by \( J^+(x_\mathcal{E}) \) the closed half-ray with endpoint the central point of \( J(x_\mathcal{E}) \) whose preimage is also a neighborhood in \( U'_{n,1} \) of the end \( \mathcal{E} \). Let \( \Sigma(\mathcal{E}) \subset U(x_\mathcal{E}) \) be the fiber over the central point of \( J(x_\mathcal{E}) \), and set \( U^+(\mathcal{E}) \) equal to the preimage of \( J^+(x_\mathcal{E}) \). We form the union \( W_{n,2} \) of \( M_n \setminus U'_{n,1} \) with the \( U^+(\mathcal{E}) \) as \( \mathcal{E} \) varies over the ends of \( U'_{n,1} \). Then \( W_{n,2} \) is compact and \( \partial W_{n,2} \) is a disjoint union of the \( \Sigma(\mathcal{E}) \) as \( \mathcal{E} \) varies over the non-compact ends of \( U'_{n,1} \). In particular \( \partial W_{n,2} \) consists of a disjoint union of 2-tori and 2-spheres. We set \( W_{n,1} \) equal to the complement in \( M_n \) of the interior of \( W_{n,2} \). It is the compact manifold with boundary obtained from \( U'_{n,1} \) by deleting the collar neighborhoods \( U^+(x_\mathcal{E}) \) as \( \mathcal{E} \) ranges over the non-compact ends of \( U'_{n,1} \). Its boundary consists of the boundary of \( W_{n,2} \) disjoint union the boundary of \( M_n \). Recall that the latter is a disjoint union of incompressible tori.

Let us recap our progress to date.

**Proposition 5.6.** We have a decomposition \( M_n = W_{n,1} \cup W_{n,2} \). The intersection \( W_{n,1} \cap W_{n,2} \) is the boundary of \( W_{n,2} \) and it is the union of the boundary components of \( W_{n,1} \) that are not boundary components of \( M_n \). For each end \( \mathcal{E} \) of \( U'_{n,1} \) there is one component of \( W_{n,1} \cap W_{n,2} \). This component is denoted \( \Sigma(\mathcal{E}) \). Each of these components is either a 2-torus or a 2-sphere, and each \( \Sigma(\mathcal{E}) \) is a fiber of the projection mapping \( p_{x_\mathcal{E}} : U(x_\mathcal{E}) \to J(x_\mathcal{E}) \). Each component of \( W_{n,1} \) is homeomorphic to one of the following:

1. a \( T^2 \)-bundle over either a circle or a compact interval,
2. an \( S^2 \)-bundle over a compact interval,
3. a twisted \( I \)-bundle over the Klein bottle,
4. a compact solid torus,
5. a compact 3-ball,
6. the complement in \( \mathbb{R}P^3 \) of an open 3-ball, or
7. the union of two twisted $I$-bundles over the Klein bottle along their common boundary.

Proof. This is immediate from Proposition 5.5 and the construction.

For each component $A$ of $M_n \setminus U'_{n,1}$, we set $\hat{A}$ equal to the union of $A$ together with $U^+(E)$ as $E$ varies over the ends of $U'_{n,1}$ whose closures meet $A$. Then $W_{n,2}$ is the disjoint union of the $\hat{A}$ as $A$ ranges over the components of $M_n \setminus U'_{n,1}$.

5.4 Covering of $W_{n,2}$

Next we must study the structure of components $\hat{A}$ of $W_{n,2}$. The crucial ingredient is to construct chains of $\epsilon'$-solid cylinder neighborhoods that together with $U_{2,\text{generic}}$, the $\epsilon'$-solid torus neighborhoods near interior cone points, and the 3-balls near a 2-dimensional boundary corner cover $W_{n,2}$.

The following two results are immediate consequences of Theorem 3.22 and the results of Section 4.

Lemma 5.7. Suppose that $A$ is a component of $M_n \setminus U'_{n,1}$ that is close to a 2-dimensional space. Then $\hat{A}$ is contained in the union of:

1. $U_{2,\text{generic}}$,
2. the open subset $U_{\text{cyl}}$ consisting of all points that are in the center of cores of $\epsilon'$-solid cylinders $\nu_{s_2}(\gamma)$ at scale $s_1$ near flat 2-dimensional boundary points, and a finite number of
3. $\epsilon'$-solid tori $B(z_i) = B_{g'_n(z_i)}(z_i, r(z_i)/4)$, for $i = 1, \ldots, N_t$, near interior cone points, and
4. 3-balls $B(x_i) = B_{g'_n(x_i)}(x_i, r(x_i)/4)$, for $i = 1, \ldots, N_c$, near 2-dimensional boundary corners.

For each 3-ball $B(x)$ near a 2-dimensional boundary corner, we denote by $\hat{B}(x)$ the ball $B_{g'_n(x)}(x, 7r(x)/8)$ and call it the expanded version of the ball.

Addendum 5.8. We can choose the neighborhoods in Lemma 5.7 so that in addition to the fact that they cover $\hat{A}$ we have:

(i) The 3-balls $B(x_i), 1 \leq i \leq N_c$, are disjoint.
(ii) Any 3-ball $B_{g'_n(x)}(x, r(x)/4)$ near a 2-dimensional boundary corner that meets one of the $B(x_i), 1 \leq i \leq N_c$, is contained in one of the expanded versions $\hat{B}(x_i)$.
(iii) The $\epsilon'$-solid tori $B(z_i), 1 \leq i \leq N_t$, are disjoint.
(iv) Each $\epsilon'$-solid torus in (3) of Lemma 5.7 is disjoint from each 3-ball in (4) of Lemma 5.7 and is also disjoint from $U_{\text{cyl}}$. 
Also, every 3-ball and every \(\epsilon\)'-solid torus in the collection above meets \(\hat{A}\).

Proof. We consider collections of disjoint 3-balls near 2-dimensional boundary corners. For any member of such a collection we have its unrescaled radius \(\rho_n(x_i)r(x_i)\).

If there is a disjoint 3-ball near a 2-dimensional boundary corner then we add it to the collection. If there is a 3-ball \(B\) near a 2-dimensional boundary corner that meets one of the 3-balls in the collection but is not contained in any of the expanded version of the 3-balls in the collection, then we add \(B\) to the collection and remove all the 3-balls in the collection that meet it. Since all these balls are contained in \(\hat{B}\) which itself is a union of \(B, U_{2,\text{gen}}\) and \(U_{\text{cyl}}\), removing these balls does not destroy the fact that we have a covering of \(\hat{A}\). In this case, it follows that the unrescaled radius of \(B\) is at least 1.1 times the unrescaled radius of each ball that we deleted.

Since \(\hat{A}\) is compact and thus the unrescaled radius of any ball is bounded above and below by positive constants, starting with the empty collection we can only repeat these two operations only finitely many times. When we can no longer repeat the operation we arrive at a collection of 3-balls near 2-dimensional boundary corners satisfying the first two conditions.

Consider the collection of \(\epsilon\)'-solid tori. If two of these meet, say \(B(z_i)\) and \(B(z_j)\), then without loss of generality we can suppose that

\[
\rho_n(z_i)r(z_i) \geq \rho_n(z_j)r(z_j).
\]

This implies that \(B(z_j) \subset B_{g_n(z_i)}(z_i, r(z_i)/2)\). Since

\[
B_{g_n(z_i)}(z_i, r(z_i)/2) \setminus B_{g_n(z_i)}(z_i, r(z_i)/4)
\]

is contained in \(U_{2,\text{gen}}\), we can remove \(B(z_j)\) from the collection and still have a covering. This allows us to make the \(\epsilon\)'-solid tori disjoint.

Now suppose that an \(\epsilon\)'-solid torus in the collection meets one of the 3-balls near a 2-dimensional boundary corner in our collection. If the unrescaled radius of the 3-ball is no greater than the unrescaled radius of the \(\epsilon\)'-solid torus, then the 3-ball is contained in the expanded version of the 3-solid torus, where, as before, in the expanded version we replace the radius \(r(x_i)/4\) by \(7r(x_i)/8\). But this is impossible, since the generic circle fibers in the 3-ball are isotopic in the solid torus to generic fibers of its Seifert fibration, but the circle fibers in the 3-ball are homotopically trivial in the 3-ball whereas the circle fibers in the solid torus are homotopically non-trivial in the solid torus. If the unrescaled radius of the 3-ball is greater than that of the solid torus, then the solid torus is contained in the expanded 3-ball and hence, by the same reasoning as above, it can be removed from the collection without destroying the fact that the collection covers \(\hat{A}\). This shows that we can make the \(\epsilon\)'-solid tori disjoint from the 3-balls near boundary corner points.

Lastly, suppose that an \(\epsilon\)'-solid torus in the collection meets \(U_{\text{cyl}}\). Then there is an \(\epsilon\)'-solid cylinder that is contained in the expanded version of the \(\epsilon\)'-solid torus. This is a contradiction for it implies that the generic fiber of the Seifert fibration on the expanded \(\epsilon\)'-solid torus is homotopically trivial in the \(\epsilon\)'-solid cylinder contained in the expanded \(\epsilon\)'-solid torus.
This completes the proof that there is a covering satisfying the listed properties. From this collection we simply remove any of the sets in the collection that does not meet $\hat{A}$. \qed

We have analogues of these results for components which are near intervals but that expand to be near 2-dimensional components.

**Lemma 5.9.** Suppose that $A$ is a component of $M_n \setminus U'_{n,1}$ that is close to an interval but that expands to be close to a standard 2-dimensional ball. Then for some $\lambda > \epsilon^{-1}$, using the metric $\lambda g_n(x)$ for an appropriate $x \in A$, we have that $\hat{A}$ is contained in the union of

1. $U_{2,\text{generic}}$,
2. the open subset $U_{\text{cyl}}$ of points in the center of cores of $\epsilon'$-solid cylinders $\nu_{2,\hat{\gamma}}$ at scale $s_1$ near flat 2-dimensional boundary points,

and a finite union of

3. $\epsilon'$-solid tori $B(z_i) = B_{\lambda g_n}(z_i, r(z_i)/4)$ near interior cone points, and
4. 3-ball components near 2-dimensional boundary corners

\[ B(x_i) = B_{\lambda g_n}(x_i, r(x_i)/4). \]

The same argument as in the proof of Addendum 5.8 shows:

**Addendum 5.10.** We can choose the neighborhoods in Lemma 5.9 so that in addition to the fact that they cover $\hat{A}$ we have

(i) The 3-balls $B(x_i)$ in (4) of Lemma 5.9 are disjoint.

(ii) Any 3-ball $B(x)$ that meets one of the $B(x_i)$ is contained in one of the expanded versions $\hat{B}(x_i)$.

(iii) The $\epsilon'$-solid tori $B(z_i)$ in (3) of Lemma 5.9 are disjoint.

(iv) Each $\epsilon'$-solid torus in (3) of Lemma 5.9 is disjoint from each 3-ball in (4) of Lemma 5.9 and from $U_{\text{cyl}}$.

Also, every 3-ball in (4) of Lemma 5.9 and every $\epsilon'$-solid torus in (3) of Lemma 5.9 meets $\hat{A}$.

**5.4.1 The circle fibration**

Above we constructed a covering of the compact submanifold $W_{n,2}$ by:

1. $U_{2,\text{gen}}$,
2. $U_{\text{cyl}}$,
3. a finite number of $\epsilon'$-solid tori $B(z_i)$, and
4. a finite number of 3-balls $B(x_i)$ near 2-dimensional boundary corners.

The constants $r(z_i)$ and $r(x_i)$ are bounded between $r_0$ and $10^{-3}$. The constants $\lambda_j$ or $\lambda_i$ multiplying the metric $g_0$ is the same for all the balls and all the solid tori that meet a given component of $W_{n,2}$ that is near to an interval but expands to be near to a 2-dimensional component. For balls and solid tori that meet any other component of $W_{n,2}$ the constant multiplying the metric $g_0$ is the value of $\rho_n^{-2}$ at the central point.

Proposition 4.4 and Lemma 4.1 imply that there is an open subset $U_2' \subset U_{2,\text{gen}}$ that contains the complement in $W_{n,2}$ of the union of the open sets in (3) and (4) in Lemmas 5.7 and 5.9 and the complement of $U_{\text{cyl}}$. Furthermore, $U_2'$ admits circle fibration whose fibers are $\epsilon'$-orthogonal to the $S^1$-product neighborhoods centered at each point of $U_2' \subset U_{2,\text{gen}}$ and also the fibers have length less than $C\epsilon$ where $C$ is the universal constant from Lemma 4.1.

5.4.2 Removing 3-balls from $W_{n,2}$

At this point we modify $W_{n,1}$ and $W_{n,2}$ by removing the 3-balls $B(x_i)$ near 2-dimensional boundary corners from $W_{n,2}$ and adding their closures as disjoint components of $W_{n,1}$. The results are denoted $W_{n,1}'$ and $W_{n,2}'$, respectively. A slight modification of Proposition 5.6 holds for these subsets.

**Corollary 5.11.** The conclusions of Proposition 5.6 hold for the compact submanifolds $W_{n,1}'$ and $W_{n,2}'$ with one change. The intersection $W_{n,1}' \cap W_{n,2}'$ is equal to the disjoint union of $W_{n,1}' \cap W_{n,2}'$ and the metric spheres $S(x_i, r(x_i)/4)$ that are the frontiers of the $B(x_i)$ are topological 2-spheres.

By doing this we have gained one thing: namely, $W_{n,2}'$ is covered by $U_2'$, $U_{\text{cyl}}$, and the $\epsilon'$-solid tori $B(z_i)$. The $\epsilon'$-solid tori do not meet $U_{\text{cyl}}$, and $\epsilon'$-solid tori are pairwise disjoint.

5.5 Deforming the splitting surfaces

At this point we have constructed a decomposition $M_n = W_{n,1}' \cup W_{n,2}'$ where the $W_{n,1}'$ are compact submanifolds meeting along their boundary. We must modify $W_{n,1}'$ and $W_{n,2}'$ in order to form $V_{n,1}$ and $V_{n,2}$ as required by Theorem 5.1. There are two steps in this modification. The first involves changing the boundary surfaces between $W_{n,1}'$ and $W_{n,2}'$ slightly so that they are well-positioned with respect to the circle fibration on $U'_2$. It is carried out in this section. The other involves removing $\epsilon'$-solid tori and chains of $\epsilon'$-solid cylinders from $W_{n,2}'$. It is carried out in the two subsections after this one.

5.5.1 Interface with the 3-balls near boundary corners

Let us deform the boundaries of the $B(x_i)$ slightly until they are the (overlapping) union of an annulus in $U_2'$ saturated under the $S^1$-fibration and two disks, each disk spanning an $\epsilon'$-solid cylinder.
Lemma 5.12. Let $B(x_i)$ be a 3-ball near a 2-dimensional boundary corner contained in the collection given in Lemma 5.4 or Lemma 5.9. Then there are disjoint $\epsilon'$-solid cylinders $\nu(1) = \nu(1, i)$ and $\nu(2) = \nu(2, i)$ of scale $s_1$ of length $s_1/2$ and of width $\xi s_1/2$ such that the centers of their cores meet the metric sphere $S(x_i, r(x_i)/4)$. Furthermore, there is a 2-sphere $S(x_i) \subset B(x_i, 3r(x_i)/8) \setminus \overline{B}(x_i, r(x_i)/4)$ that is the (overlapping) union of an annulus $A(x_i)$ and two 2-disks, $D_1$ spanning $\nu(1)$, and $D_2$ spanning in $\nu(2)$. These satisfy the following:

1. The annulus $A(x_i)$ is contained in $U_2'$ and is saturated under $S^1$-fibration on $U_2'$.  
2. One of the boundary circles of $A(x_i)$ is contained in $\nu(1)$ and the other is contained in $\nu(2)$.  
3. For $j = 1, 2$, the intersection of $D_j$ with $A(x_i)$ is an annulus which is a collar neighborhood in $D_j$ of $\partial D_j$ and is a collar neighborhood in $A(x_i)$ of one of its boundary components.  
4. Every point of $S(x_i) \setminus A(x_i)$ is contained in the sub-cylinder of one of the $\nu(i)$ of width $\xi s_1/8$.  
5. For $j = 1, 2$ and for any $t \in [3\xi s_1/8, \xi s_1/2]$ the intersection of $S(x_i)$ with $h_{\gamma_j}^{-1}(t)$ is a circle separating the ends of the level set $h_{\gamma_j}^{-1}(t)$ in $\nu_j$.

The 2-sphere $S(x_i)$ is isotopic in $B(x_i, 3r(x_i)/8)$ to the metric sphere $S(x_i, r(x_i)/4)$. In particular, the 2-sphere $S(x_i)$ separates the metric sphere $S(x_i, 3r(x_i)/8)$ from the metric sphere $S(x_i, r(x_i)/4)$. As a result it bounds a closed topological 3-ball $\overline{B}(x_i) \subset B(x_i, 3r(x_i)/8)$.

Proof. Since $B(x_i)$ is near a 2-dimensional boundary corner, there is a 2-dimensional Alexandrov ball $\overline{B}$ of radius 1 that is boundary $\mu$-good at some $\overline{\sigma}$ on scale $r(x_i)$ of angle $\leq \pi - \delta$ and $(B(x_i), 1, x_i)$ is within $\epsilon$ of $(\overline{B}, \overline{\sigma})$. The metric sphere $S(\overline{\sigma}, r(x_i)/4)$ is a topological interval that meets the boundary of $\overline{B}$ in its endpoints. Let $\gamma_1$ and $\gamma_2$ be geodesics of length $s_1$ in $\overline{B}$ whose endpoints lie in the boundary of $\overline{B}$ and whose central points lie in $S(\overline{\sigma}, r(x_i)/4)$ near to the two boundary points of this metric sphere. For $j = 1, 2$, let $\tilde{\gamma}_j$ be geodesics in $B(x_i, r(x_i))$ of length $s_1$ within $\epsilon$ of the $\gamma_j$. We can arrange that the central points of the $\gamma_j$ lie on the metric sphere $S(x_i, r(x_i)/4)$. For $j = 1, 2$ let $\nu(j) = \nu(j, i)$ be the $\epsilon'$-solid cylinders associated with the $\tilde{\gamma}_j$ of length $s_1/2$ and width $\xi s_1/2$. By construction the intersection of $S(x_i, r(x_i)/4)$ with $\nu(j)$ contains the central point of $\tilde{\gamma}_j$. Consider the saturated open subset $U_2(x_i)$ of $U_2'$ consisting of all fibers of the $S^1$-fibration on $U_2'$ that meet $E = B(x_i, .002)s_1 + r(x_i)/4) \setminus \overline{B}(x_i, .001)s_1 + r(x_i)/4)$. This open subset contains the complement in $E$ of the cores of $\nu(1)$ and $\nu(2)$ and is contained in $B(x_i, 3r(x_i)/8) \setminus \overline{B}(x_i, r(x_i)/4)$. For $j = 1, 2$ fix a point $y_j$ in the intersection of the level set $h_{\gamma_j}^{-1}((1.1)\xi s_1))$ and the central disk of $\nu(j)$. According to Lemma 4.23 there is a geodesic $\zeta_j$ from $y_j$ to a point $z$ at distance $(1.1)\xi s_1$ from $\tilde{\gamma}_j$ with the property that for any $w \in \zeta \cap h_{\gamma_j}^{-1}((.5)\xi s_1)$ the comparison angle $\hat{Z}_{\gamma_j}wz \geq \pi - 2\xi$. 
In particular, this geodesic meets each level set of \( h_{\gamma_j}^{-1}(t) \) for \((.11)\xi s_1 \leq t \leq \xi s_1/2\) in a single point. Of course, \( \zeta \subset U'_2(x_i) \). Let \( \hat{A}_j \) be the annulus that is the saturation of \( \zeta \cap \bar{h}_{\gamma_j}^{-1}([0, (.51)\xi s_1]) \) under the circle fibration on \( U'_2(x_i) \). This annulus crosses each level set of \( h_{\gamma_j}^{-1}(t) \) for \((.12)\xi s_1 \leq t \leq \xi s_1/2\) in \( \nu(j) \) in a single circle, a circle that separates the ends of that level set.

The base space of the circle fibration of \( U'_2(x_i) \) is a connected surface \( \Sigma \). Consider the saturated open subset \( V'_2(x_i) \subset U'_2(x_i) \) which is the union of all fibers that meet the complement of \( \nu(1) \cup \nu(2) \). It is also connected as is its quotient surface \( \Sigma' \subset \Sigma \). It follows that there is a saturated annulus \( \tilde{A}_0 \subset \Sigma' \) that connects orbits over the intersection of \( \zeta_j \cap h_{\gamma_j}^{-1}((.5)\xi s_1) \), for \( j = 1, 2 \) and which is disjoint from the union of the orbits over points of \( \zeta_j \cap h_{\gamma_j}^{-1}([0, (.51)\xi s_1]) \). The union \( \tilde{A}_1 \cup \tilde{A}_0 \cup \tilde{A}_2 \) is an annulus \( \tilde{A} \). Since in the rescaling of the metric giving the \( S^1 \)-product structure the fiber circles of the fibration structure lie within \( \epsilon' \) of an \( S^1 \)-factor in an \( \epsilon \)-product neighborhood, it follows immediately that the boundary circles of \( \tilde{A} \) are contained \( \cup_{j=1,2} h_{\gamma_j}^{-1}([0, (.12)\xi s_1]) \). For \( j = 1, 2 \), consider the level set \( L(j) = h_{\gamma_j}^{-1}((.12)\xi s_1) \) and the intersection of \( c_j = \tilde{A} \cap L(j) \). The circle \( c_j \) separates (in \( \tilde{A} \) the boundary component of \( \tilde{A} \) contained in \( \nu(j) \) from the intersection of \( \tilde{A} \) with the side of \( \nu(j) \). We define \( A'(x_i) \) as the subannulus of \( \tilde{A} \) bounded by \( c_1 \) and \( c_2 \). There is a disk \( D'_j \subset h_{\gamma_j}^{-1}([0, (.12)\xi s_1]) \) with boundary \( c_j \). We define \( D_j \) to be the union of \( D'_j \) and the intersection of \( A'(x_i) \cap \nu(j) \). We set \( A(x_i) \) equal to the sub-annulus of \( A'(x_i) \) bounded by the \( S^1 \)-fibers of \( U'_2 \) passing through the point on \( \zeta \cap h_{\gamma_j}^{-1}((.123)\xi s_1) \). Then \( A(x_i) \), the disks \( D_j \) and the union \( S(x_i) = D_1 \cup A(x_i) \cup D_2 \) are as required by the lemma.

5.5.2 Other interfaces

Lemma 5.13. Suppose that \( \lambda_n \geq \rho_n^{-2}(x_n) \) and that \( B_{\lambda_n, g_n}(x_n, 1) \) is within \( \hat{c} \) of a standard 2-dimensional Alexandrov ball of area \( \geq a \). Suppose also that for some \( \alpha \) with \( 1/100 \leq \alpha \leq 1/2 \) the function \( d(x_n, \cdot) \) is \((1-\epsilon)-regular on C(x_n, \alpha, \alpha +1/100) = B_{\lambda_n, g_n}(x_n, \alpha +1/100) \setminus \overline{B}_{\lambda_n, g_n}(x_n, \alpha) \) and determines a fibration of this open set over the interval \((\alpha, \alpha +1/100)\) with fiber either \( T^2 \) or \( S^2 \).

1. If the fibers of the restriction of \( d(x_n, \cdot) \) to \( C(x_n, \alpha, \alpha +1/100) \) are 2-tori, then there is a 2-torus \( T \subset U'_2 \cap A(x_n, \alpha, \alpha +1/100) \) that is saturated under the \( S^1 \)-fibration on \( U'_2 \) and that separates the metric spheres \( S_{\lambda_n, g_n}(x_n, \alpha) \) and \( S_{\lambda_n, g_n}(x_n, \alpha +1/100) \). This 2-torus is isotopic to the fibers of \( d(x_n, \cdot) \) on \( C(x_n, \alpha, \alpha +1/100) \).

2. If the fibers of the restriction of \( d(x_n, \cdot) \) to \( C(x_n, \alpha, \alpha +1/100) \) are 2-spheres, then there is a 2-sphere \( S \subset C(x_n, \alpha, \alpha +1/100) \) that is the union of an annulus \( A(x_n) \) in \( U'_2 \), an annulus saturated under the \( S^1 \)-fibration, and disks \( D_1 \) and \( D_2 \) in two \( \epsilon' \)-solid cylinders, \( \nu(1) \) and \( \nu(2) \). The \( D_j, A(x_n) \), and \( S(x_n) = D_1 \cup A(x_n) \cup D_2 \) satisfy Properties 1 – 5 listed in Lemma 5.12. The 2-sphere \( S(x_n) \) separates the metric spheres \( S_{\lambda_n, g_n}(x_n, \alpha) \) and \( S_{\lambda_n, g_n}(x_n, \alpha +1/100) \).
Proof. Let $W$ be the union of $C'(x_n) = C(x_n, \alpha + 10^{-3}, \alpha + (.0099))$ together with all $\epsilon'$-solid cylinders and 3-balls near 2-dimensional corner points that meet $C'(x_n)$. Then $W \subset C(x_n)$. First, we consider the case when the fibers of $d(x_n, \cdot)$ in $C(x_n) = C(x_n, \alpha, \alpha + 1/100)$ are 2-tori.

Claim 5.14. In this case, there is no 3-ball near a 2-dimensional boundary corner and no $\epsilon'$-solid cylinder that meets $C'(x_n)$.

Proof. Suppose that there is at least one $\epsilon'$-solid cylinder or 3-ball meeting $C'(x_n)$. Then by Van Kampen’s theorem, the fundamental group of $W$ is the quotient of the fundamental group of a Seifert fibration over a non-compact, connected surface when the class of the generic fiber is set equal to the trivial element. Hence, the fundamental group is a free product of cyclic groups. On the other hand, since $S(x_n, \alpha + 1/200) \subset W \subset C(x_n, \alpha, \alpha + 1/100)$, it follows that the fundamental group of $S(x_n, \alpha + 1/200)$ is identified with a subgroup of the fundamental group of $W$. Since $S(x_n, \alpha + 1/200)$ is a 2-torus, this is a contradiction.

This proves that $C'(x_n)$ is contained in the total space $X \subset U_2'$ of a Seifert fibration, and $X \subset C(x_n)$. In fact $X$ is the union of a saturated subset of $U_2'$ and a finite union of $\epsilon'$-solid tori $\tilde{\tau}(z_i)$. Since the complement of a compact subset of each $\tilde{\tau}(z_i)$ is contained in $U_2'$, it follows that the complement of a compact subset of $X$ is contained in $U_2'$ and hence there is a compact submanifold $X' \subset X$, containing $S(x_n, \alpha + 1/200)$ whose boundary is contained in $U_2'$ and is saturated under the $S^1$-fibration. Since $X'$ contains $S(x_n, \alpha + 1/200)$, if follows that $X'$ separates the ends of $C(x_n, \alpha, \alpha + 1/100)$ and hence one of the boundary components of $X'$ also separates these ends. This boundary component is a 2-torus contained in $U_2'$, saturated under the $S^1$-fibration. It separates the boundary components of $C(x_n, \alpha, \alpha + 1/100)$. Consequently, it is isotopic in $C(x_n, \alpha, \alpha + 1/100)$ to any fiber of $d(x_n, \cdot)$. This completes the proof in the case when $C(x_n)$ is fibered by 2-tori.

Next, we consider the case when the fiber $S(x_n, \alpha + 1/200)$ is a 2-sphere. The first thing to observe is that this 2-sphere is not isotopic in $X$ to a 2-sphere that is disjoint from the $\epsilon'$-solid cylinders and 3-ball neighborhoods. The reason is that any 2-sphere contained in a Seifert fibered 3-manifold with base a connected, non-compact surface is homotopically trivial in that manifold, but $S(x_n, \alpha + 1/200)$ is not homotopically trivial in $X$. Since $d(\pi, \cdot)$ is $(1-2\epsilon)$-regular on the metric annulus $\overline{C} = C(\pi, \alpha, \alpha+/100)$, it follows that the level sets of this function on $\overline{C}$ are intervals.

Any level set of $d(x_n, \cdot)$ meeting the center of an $\epsilon'$-solid cylinder, $C'(x_n)$, intersects the solid cylinder in a spanning 2-disk. We fix such two such $\epsilon'$-solid cylinders $\overline{\mathcal{P}}(1)$ and $\overline{\mathcal{P}}(2)$ with central geodesics $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ where the corresponding geodesics in $\overline{D}(\pi, 1)$ are near the two boundary components of $\overline{C}$. We construct a saturated annulus $\tilde{A}_j$ extending these by disks $D_j$ as in Lemma 5.12 so that the intersection of $\tilde{A}_j \cup D_j$ with $\overline{\mathcal{P}}(j)$ is a spanning disk. Since removing the 3-balls near 2-dimensional corner points and $\epsilon'$-solid tori from $W$ cannot disconnect it, it follows that there is a saturated annulus $A_0$ in $U_2' \cap C(x_n)$ connecting the outer boundaries of $\tilde{A}_1$ and $\tilde{A}_2$. The union $D_1 \cup A_0 \cup D_2$ is the 2-sphere as required. 

\qed
This leads immediately to the following two results.

**Corollary 5.15.** Let \( A \) be a component of \( M_n \setminus U'_{n,1} \) that is close to a 2-dimensional space. Let \( \mathcal{E} \) be an end of \( U'_{n,1} \) neighboring \( A \). Suppose that the end \( U(x,\mathcal{E}) \) is fibered by 2-tori. Then there is a 2-torus \( T(\mathcal{E}) \subset U^+(x,\mathcal{E}) \cap U'_2 \) that is saturated under the \( S^1 \)-fibration and separates the ends of \( U(x,\mathcal{E}) \). In particular, \( T(\mathcal{E}) \) is isotopic in \( U^+(x,\mathcal{E}) \) to the boundary 2-torus \( \Sigma(\mathcal{E}) \) of \( U^+(x,\mathcal{E}) \).

**Corollary 5.16.** Suppose that \( A \) is a component of \( M_n \setminus U'_{n,1} \) that is close to a 2-dimensional space, and suppose that \( \mathcal{E} \) is an end of \( U'_{n,1} \) neighboring \( A \) and that \( U(x,\mathcal{E}) \) is fibered by 2-spheres. Then there is a 2-sphere \( S(\mathcal{E}) \subset U^+(x,\mathcal{E}) \) that is the union of an annulus \( E \subset U'_2 \) saturated under the \( S^1 \)-fibration and two 2-disks, \( D_1 \) and \( D_2 \), contained in two \( \epsilon' \)-solid cylinders. These satisfy the following:

1. Each boundary circle of \( E \) is contained in the interior of one of the \( \nu(j) \) and in fact lies in the sub solid cylinder of width \( \xi s_1/4 \).
2. \( S(\mathcal{E}) \) is isotopic in \( U^+(\mathcal{E}) \) to \( \Sigma(\mathcal{E}) \).

### 5.5.3 Case of components near 1-dimensional spaces but which expand to be near 2-dimensional spaces

Now we need to perform a similar construction for each component \( A \) of \( M_n \setminus \hat{U}'_{n,1} \) which is close to an interval but which expands to be near a 2-dimensional space. Then there is exactly one neighboring component of \( U'_{n,1} \) and the end \( \mathcal{E} \) of this component neighboring \( A \) is either fibered by 2-tori or fibered by 2-spheres. Furthermore, there is a point \( z \in \hat{A} \) and a constant \( \lambda > \rho_n^{-2}(x) \) such that \( B_{\lambda}(z,1) \) is close to a standard 2-dimensional ball \( B=B(\pi,1) \) and on the region between the metric sphere \( S_{\lambda}(x,1/2) \) and \( \Sigma(\mathcal{E}) \) distance function from \( x \) is regular. In particular, the region bounded by \( S_{\lambda}(x,1/2) \) and \( \Sigma(\mathcal{E}) \) is homeomorphic to a product of a closed surface (either a 2-sphere or a 2-torus) with a closed interval, and the distance function from \( x \) is regular on this region and gives this product structure.

Let us consider first the case when the annular region \( C_{\lambda_n}(x,1,2) \) is fibered by 2-tori. Applying Lemma 5.13 to the annular region \( C_{\lambda_n}(x,1/2,1) \) we see that in this case there is a 2-torus \( T(\mathcal{E}) \) in \( C(x,1/2,1) \cap U'_2 \) that is saturated under the circle fibration on \( U'_2 \) and separates the metric spheres \( S_{\lambda_n}(x,1/2) \) and \( S_{\lambda_n}(x,1) \). It follows that \( T(\mathcal{E}) \) is isotopic in \( C_{\lambda_n}(x,1/2,1) \) to either end and consequently the region between \( T(\mathcal{E}) \) and \( \Sigma(\mathcal{E}) \) is a product region.

Let us consider now the case when the annular region \( C_{\lambda_n}(x,1,2) \) is fibered by 2-spheres. Again applying Lemma 5.13 we see that there is a 2-sphere \( S(\mathcal{E}) \subset C_{\lambda_n}(x,1,2) \) that separates the ends and has the properties stated in the second part of that lemma. The region between \( S(\mathcal{E}) \) and \( \Sigma(\mathcal{E}) \) is a product region.

### 5.5.4 Redefinition of the boundary between \( W'_{n,1} \) and \( W'_{n,2} \)

Now we deform slightly \( W'_{n,1} \) and \( W'_{n,2} \) so as to replace the splitting surfaces \( W'_{n,1} \cap W'_{n,2} \) by the surfaces constructed in the previous sections.
For each end $E$ of $U'_{n,1}$ we have a surface either a 2-torus $T(E)$ that is contained in $U'_2$ and is saturated under the $S^1$-fibration or a 2-sphere $S(E)$ that is the union of an annulus in $U'_2$ saturated under the $S^1$-fibration and two spanning disks in $\epsilon'$-solid cylinders. In all cases this surface is contained in $W'_{n,2}$ and is parallel to the surface $\Sigma(E)$ which is a splitting surface for $W'_{n,1}$ and $W'_{n,2}$. For each end $E$ we remove the product region between either $T(E)$ or $S(E)$ and $\Sigma(E)$ from $W'_{n,2}$ and add it to $W'_{n,1}$. For each 3-ball component $B(x_i, r(x_i)/4)$ of $W'_{n,1}$ we have a surface $S(x_i)$ as constructed in Lemma 5.12. It is contained in $W'_{n,2}$ and the region between it and the metric sphere $S(x_i, r(x_i)/4)$ is a product. We remove this product region from $W'_{n,2}$ and add it to $W'_{n,1}$.

After making all these changes we relabel the results $W'_{n,1}$ and $W'_{n,2}$. What we have achieved is to make each component of the intersection either a 2-torus contained in $U'_2$ and saturated under the $S^1$-fibration or a 2-sphere that is the union of an annulus in $U'_2$ that is saturated under the $S^1$-fibration and two 2-disks spanning $\epsilon'$-solid cylinder neighborhoods. Also, each component of $W'_{n,1}$ satisfies the properties required of $V_{n,1}$ in Theorem 1.1. At this point we define $V_{n,1}$ to be $W'_{n,1}$.

Here is our progress to date.

**Corollary 5.17.** We have a decomposition $M_n = W'_{n,2} \cup V_{n,1}$. The intersection $W'_{n,2} \cap V_{n,1}$ is the boundary of $W'_{n,2}$ and is the union of all boundary components of $V_{n,1}$ that are not boundary components of $M_n$. Each component of $V_{n,1}$ is as listed in Theorem 1.1. Furthermore, the intersection $W'_{n,2} \cap V_{n,1}$ consists of 2-tori contained in $U'_2$ and saturated under the $S^1$-fibration structure on $U'_2$ and 2-spheres that are unions of annuli contained in $U'_2$ and saturated under the $S^1$-fibration and two spanning 2-disks in $\epsilon'$-solid cylinders.

**5.6 Overlaps of $\epsilon'$-solid cylinders**

Our next step is to show that we can arrange that the complement of the union of $U'_2$, the $\epsilon'$-solid tori near interior cone points and the 3-balls near 2-dimensional corner points referred to in Lemmas 5.7 and 5.9 is contained in the union of a finite set of cores of $\epsilon'$-solid cylinders and these cylinders have good intersections with each other and with the 3-balls and with the 2-spheres $S(E)$ associated to ends $E$. To do this we introduce the notion of chains of these neighborhoods.

**5.6.1 Chains of solid cylinders**

Let us begin with the local structure, namely, two solid cylinders meeting nicely. Let $y \in M_n$ and $\lambda \geq \rho_n(y)^{-2}$ be given. Suppose that $B_{\lambda g_n}(y, 1)$ is within $\epsilon$ of a standard 2-dimensional ball $X = B(\pi, 1)$ of area at least $a$, and suppose that $\gamma \subset B(\pi, 1/2)$ is a $\xi$-approximation to $\partial X$ on scale $s_1$ and that $\tilde{\gamma} \subset M_n$ is an $\hat{\epsilon}$-approximation to $\gamma$. We use the metric $\lambda g_n$ to measure things, so that in particular, $\ell = \ell(\tilde{\gamma})$ means the length of $\tilde{\gamma}$ in the metric $\lambda g_n$. Recall from Proposition 4.23 that for any constant $c \in [\xi^2, \xi]$ and any $-\ell/2 \leq a < b \leq \ell/2$, the region $\overline{\gamma}_{c, [a, b]}(\tilde{\gamma})$ is homeomorphic to
$D^2 \times I$ and is foliated by its intersections with the level sets of $f_{\tilde{\gamma}_i}$, each intersection being a 2-disk.

**Definition 5.18.** Suppose that we have points $y_1, y_2 \in M_n$ such that $B_{\rho_i}'(y_i)(y_i, 1)$ is within $\varepsilon$ of a standard 2-dimensional ball $B(\tau, 1)$. Suppose that we have geodesics $\gamma_i \subset B(\tau, 1/2)$ that are $\xi$-approximations to the boundary on scale $s_1$ and suppose that we have geodesics $\tilde{\gamma}_i$ that are $\varepsilon$ approximations to $\gamma_i$. We denote $\ell_i$ the length of $\tilde{\gamma}_i$ with respect to the metric $g_n(y_i)$. Suppose that we have constants $c_i \in [100\varepsilon^2, \xi]$, and intervals $[a_i, b_i] \subset [-\ell_i/4, \ell_i/4]$. We say that the $\tau(i) = \tau_{c_i, [a_i, b_i]}(\tilde{\gamma}_i)$ have good intersection if, possibly after reversing the directions either or both of the $\tilde{\gamma}_i$, the following hold:

1. The function $f_{\tilde{\gamma}_i}$ is an increasing function along $\tilde{\gamma}_2$ at any point of $\tilde{\gamma}_2 \cap \tau(\tilde{\gamma}_1)$.

2. There is a point in the negative end of $\tau_{c_2, [a_2, b_2]}(\tilde{\gamma}_2)$ that is contained in $f_{\tilde{\gamma}_2}^{-1}((b_1 - .02)\ell_1, b_1 - .01\ell_1)$ in $\tau_{c_1, [a_1, b_1]}(\tilde{\gamma}_1)$, and the positive end of $\tau(2)$ is disjoint from $\tau(1)$.

3. $c_1\ell_1\rho_n(y_1)$ is either at least $(1.1)c_2\ell_2\rho_n(y_2)$ or is at most $(0.9)c_2\ell_2\rho_n(y_2)$.

**Lemma 5.19.** With the notation above, suppose that for $i = 1, 2$ the sets $\tau(i) = \tau_{c_i, [a_i, b_i]}(\tilde{\gamma}_i)$ have good intersection. Then that intersection is homeomorphic to a 3-ball. If

$$c_1\ell_1\rho_n(y_1) < c_2\ell_2\rho_n(y_2),$$

then that 3-ball meets the boundary of $\tau(2)$ in a 2-disk contained in the negative end of $\tau(2)$ and the rest of the boundary consists of an annulus in the side of $\tau(1)$ together with the positive end of $\tau(1)$. If the reverse inequality holds in $\tau(2)$, the similar statements hold with the roles of $\tau(1)$ and $\tau(2)$ and ‘positive’ and ‘negative’ reversed.

**Proof.** We suppose that Inequality 5.1 holds. It follows from Lemma 5.16 that for all $n$ sufficiently large, the sides of $\tau(1)$ and of $\tau(2)$ do not intersect and in fact the side of $\tau(2)$ is disjoint from $\tau(1)$. Thus, the intersection of $\tau(1)$ and $\partial \tau(2)$ is contained in the negative end of $\tau(2)$. By Part 3 of Lemma 4.30 this intersection is a 2-disk. Hence, it cuts off a 3-ball in $\tau(1)$.

The other case is analogous. \[\]

**Corollary 5.20.** With notation and assumptions above, suppose that Inequality 5.1 holds. Then the boundary of $\tau(1) \cup \tau(2)$ consists of the union of two subsets: (i) the disjoint union of the negative end of $\tau(1)$ and the positive end of $\tau(2)$ and (ii) an annulus $A$. These two subsets are glued together along their boundaries. The annulus $A$ consists of the union of three annuli glued together along their boundaries. The first is the intersection of the side of $\tau(1)$ with the complement of the interior of $\tau(2)$. The second is the negative end of $\tau(2)$ minus its intersection with the interior of $\tau(1)$ and the third is the side of $\tau(2)$. If the opposite inequality to Inequality 5.1 holds, then there are similar statements with the roles of $\tau(1)$ and $\tau(2)$ and ‘positive’ and ‘negative’ reversed.
There are also an analogous definition and results when the balls are $B_{\lambda g_n}(y_1, 1)$ and $B_{\lambda g_n}(y_2, 1)$. (Notice that we are using the same multiple of the metric on the two balls.) Since the only place in these arguments where we used the fact that we were dealing with $g'_n(y_i)$ rather than arbitrary multiples of $g_n$ was when we compared $\rho_n(y_1)$ with $\rho_n(y_2)$. If we are using the same multiple of the metric for both balls, then the comparison factor is 1. We leave the explicit formulations to the reader.

### 5.6.2 Chains

Now suppose that we have a sequence of $\epsilon'$-solid cylinders $\{\overline{\mathcal{V}}(1), \ldots, \overline{\mathcal{V}}(k)\}$, with $\overline{\mathcal{V}}(i) = \overline{\mathcal{V}}_{c_i, [a_i, b_i]}(\overline{\gamma}_i)$ as in Proposition 4.23 with the geodesics $\overline{\gamma}_i$ oriented. We say that these form a linear chain of $\epsilon'$-solid cylinders with good intersections if:

1. For each $1 \leq i < k$ the open sets $\overline{\mathcal{V}}(i)$ and $\overline{\mathcal{V}}(i + 1)$ have a good intersection with the given orientations.

2. If $\overline{\mathcal{V}}(i) \cap \overline{\mathcal{V}}(j) \neq \emptyset$ for some $i \neq j$, then $|i - j| = 1$.

In addition to linear chains there are circular chains.

**Definition 5.21.** A circular chain of $\epsilon'$-solid cylinder neighborhoods with good intersections is a sequence $\{\overline{\mathcal{V}}(1), \ldots, \overline{\mathcal{V}}(k)\}$ of $\epsilon'$-solid cylinder neighborhoods, indexed by the integers modulo $k$, such that for each $i$, $1 \leq i \leq k$, the pair $\{\overline{\mathcal{V}}(i), \overline{\mathcal{V}}(i + 1)\}$ has good intersections and $\overline{\mathcal{V}}(i) \cap \overline{\mathcal{V}}(j) \neq \emptyset$ implies that $j \equiv i - 1, i$ or $i + 1$ (mod $k$).

**Lemma 5.22.** Suppose that $\{\overline{\mathcal{V}}(1), \ldots, \overline{\mathcal{V}}(k)\}$ is a linear chain of $\epsilon'$-solid cylinder neighborhoods with good intersections. Then $\overline{\mathcal{V}}(1) \cup \cdots \cup \overline{\mathcal{V}}(k)$ is homeomorphic to a 3-ball and its boundary is the union of the negative end of $\overline{\mathcal{V}}(1)$, the positive end of $\overline{\mathcal{V}}(k)$ and an annulus $A$.

**Proof.** This is proved easily by induction. \qed

The same arguments establish the analogue for circular chains.

**Lemma 5.23.** Let $\{\overline{\mathcal{V}}(1), \ldots, \overline{\mathcal{V}}(k)\}$ be a circular chain of $\epsilon'$-solid cylinder neighborhoods with good intersections contained in $M_n$. Then $\cup_i \overline{\mathcal{V}}(i)$ is a solid torus.

From now on a chain of $\epsilon'$-solid cylinders means a chain with good intersections.

**Definition 5.24.** By a complete chain of $\epsilon'$-solid cylinders we mean either:

1. a circular chain contained in $\text{int} W'_{n, 2}$, or

2. a linear chain with the property that each of the extremal solid cylinders in the chain meets a 2-sphere boundary component of $W'_{n, 2}$ in a spanning disk and all of the non-extremal solid cylinders in the chain are contained in $\text{int} W'_{n, 2}$. 


Proposition 5.25. Let $A$ be a component of $M_n \setminus U'_n,1$ and let $\tilde{A}$ be the associated compact submanifold as defined in Section 5.3. In the covering of $\tilde{A}$ given in either Lemma 5.7 or Lemma 5.9 we can replace the open subset $U_{cyl}$ by a finite set of cores of $\epsilon'$-solid cylinders. The $\epsilon'$-solid cylinders in this collection form a disjoint union of complete chains with good intersection. For each boundary component of $\tilde{A}$ one of the following holds.

1. The boundary component is a 2-torus and is disjoint from all the $\epsilon'$-solid cylinders.

2. The boundary component is a 2-sphere and there are exactly two $\epsilon'$-solid cylinders in the collection meet this boundary component. Each meets it in a spanning disk for the solid cylinder.

3. The width of each $\epsilon'$-solid cylinder in the collection is between $(.40)\xi s_1$ and $\xi s_1/2$.

Proof. For each 2-sphere boundary component $S$ of $W'_{n,2}$ there are two geodesics $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ of length $s_1$ and associated $\epsilon'$-solid cylinders $\mathcal{F}_\xi(\tilde{\gamma}_1(S))$ and $\mathcal{F}_\xi(\tilde{\gamma}_2(S))$ of length $s_1/2$ that meet $S$ in a spanning disk that itself meets the central 2-disk of the core of the $\epsilon'$-solid cylinder. The complement of the cores of these in $S(x_i)$, is contained in $U'_2$ and there is an annulus $A \subset S$ that is saturated under the $S^1$-fibration of $U'_2$ with the property that $S$ is the union of $A$ with the intersection of $S$ with these two $\epsilon'$-solid cylinders. As we run over all the boundary components $S$ of $W'_{n,2}$ these $\epsilon'$-solid cylinders are disjoint.

Suppose by induction that we have a disjoint collection $\mathcal{D}$ of such chains of $\epsilon'$-solid cylinders with good intersection containing all the solid cylinders constructed in the last paragraph. We also suppose that each extremal solid cylinder in each chain which is a member of $\mathcal{D}$ either has a free end, as defined below, or is one of the $\epsilon'$-solid cylinders meeting a boundary component of $W'_{n,2}$ constructed in the last paragraph. By a free end of an $\epsilon'$-solid cylinder we mean an end that is contained in a sub-cylinder $\mathcal{F}_{[a,b]}$ of length at least $s_1/6$ with the sub-cylinder being disjoint from all the other $\epsilon'$-solid cylinders in $\mathcal{D}$ and is also disjoint from the boundary components of $W'_{n,2}$. Lastly, we assume by induction that each cylinder in $\mathcal{D}$ is of the form $\mathcal{F}_{c,[a,b]}(\tilde{\gamma})$ where $(.40)\xi s_1 \leq c \leq \xi s_1/2$.

Suppose that the complement $Y$ in $\tilde{A}$ of the union of $U_{2,gen}$, the $\epsilon'$-solid tori $B(z_i)$ and the 3-ball neighborhoods $B(x_i)$ near 2-dimensional corners in the given collection is not contained in the union of the $\epsilon'$-solid cylinders in the family $\mathcal{D}$. First, we consider the possibility that one of the chains in $\mathcal{D}$ has a free end. Let $\mathcal{F}$ be an extremal member of a chain of $\mathcal{D}$ with a free end. We take a point $x \in M_n$ that is disjoint from all the chains of $\mathcal{D}$ and within $\hat{\epsilon}$ of the free end of the core of $\mathcal{F}$. We know that $B_{\hat{\epsilon}}(x,1)$ is within $\hat{\epsilon}$ of a 2-dimensional Alexandrov ball $\mathcal{B} = B(\mathcal{F},1)$. We examine the possibilities for $\mathcal{B}$ near $\mathcal{F}$.

Suppose that $\mathcal{B}$ was interior $\mu$-good at $\mathcal{F}$ on scale $r$. In this case $\mathcal{F}$ would be contained in a solid torus neighborhood near the interior cone point. The level circles of the end of $\mathcal{F}$ are almost orthogonal to the horizontal spaces, and hence
these level circles are homotopically non-trivial in the solid torus, which is absurd since they bound disks in $\gamma$ and $\gamma$ is contained in the solid torus.

Next, suppose that the free end of $\gamma$ is contained in a expanded version of a ball of the form $B_{y_1}(x_1, r(x_1)/3)$ for one of the 3-balls in our collection. In the annular region $B_{y_1}(x_1, r(x_1)/3) \setminus B_{y_2}(x_1, r(x_1)/3)$, we can form two chains of $\epsilon'$-solid cylinders with one end of each chain being one of the solid cylinders meeting $S_{y_1}(x_1, r(x_1)/4)$ constructed above. Then the free end of $\gamma$ must meet one of these chains. We can arrange that $\gamma$ has good intersection with this chain. In this way we extend the chain until it ends in the ball, keeping the intersections good.

Next suppose that $|\gamma|$ is boundary $\mu$-good at a point $\overline{y}$ on scale $r(\overline{y})$ with $r_0 \leq r(\overline{y}) \leq 10^{-5}$ and with angle $\leq \pi - \delta$ and with $\gamma \in B(\overline{y}, r(\overline{y})/4)$. Then there is $y \in M_n$ within $\epsilon$ of $y$ and the neighborhood $B_{y_1}(y, r(\overline{y}))$ that is a 3-ball containing the end of $\gamma$. But this 3-ball is contained in one of the extended balls in our collection, so we have already seen in this case how to extend the chain, with good intersections, until it meets one of the boundary components of $W_{n,2}$.

Suppose that there is a point within $\epsilon$ of the free end of $\gamma$ that is in the center of the core of an $\epsilon'$-solid cylinder near the flat boundary of a 2-dimensional Alexandrov ball. In this case we can take such a $\epsilon'$-solid cylinder, $\gamma'$ of width $\xi s_1/2$. By arranging its width (decreasing it by a factor of at most 1.1) and cutting it off appropriately on the end contained in $\gamma$, we can arrange it have good intersection with $\gamma$. One possibility is $\gamma'$ it meets no other $\epsilon'$-solid cylinder in the set we have constructed so far. In this case the other end of $\gamma'$ is a free end, which we can assume has length at least $s_1/6$. The other possibility is that $\gamma'$ meets some other $\epsilon'$-solid cylinder, $\gamma'(1)$, in the given set. Then, as we move along the geodesic $\gamma'$ away from $\gamma$, there is a first such $\epsilon'$-solid cylinder $\gamma'(1)$ that the geodesic meets. Again cutting off $\gamma'$ appropriately, and possibly decreasing its width by a factor of at most 1.1, we can arrange that the intersection of $\gamma'$ and $\gamma'(1)$ is good, without destroying the fact that the intersection of $\gamma$ and $\gamma'$ is good. In this case we have extended the chain so that it together with the other 3 sets contains a neighborhood of a fixed unrescaled size of the free end of $\gamma$. The width of $\gamma'$ is between $(.40)\xi s_1$ and $\xi s_1/2$.

This shows that in all cases we can extend the chain if it has a free end. Now let us consider the case when the chains that we have constructed have no free ends yet the cores of the chains do not cover the complement of the other three sets. Then we simply take a point not in the union of the other 3 open sets. It lies at the center of the core of an $\epsilon'$-solid cylinder of rescaled length $s_1/2$ near a flat boundary. We simply add this $\epsilon'$-solid cylinder to the collection. Since there are no free ends of the pre-existing chains, this solid cylinder is disjoint from all the previous ones.

After a finite number of repetitions of these two constructions, we have created chains of $\epsilon'$-cylinders as required that cover the complement of the other three open sets.

Since at each step we need only make the thickness of the $\epsilon'$-cylinder differ by a factor of 1.1 from two given numbers, we can arrange that all the intersections are good by taking widths of the $\epsilon'$-solid cylinders to be between $(.40)\xi s_1$ and $\xi s_1/2$. □
5.6.3 Refinements of chains

At this point we have covered $W_{n,2}'$ by $U_2'$, by $\epsilon'$-solid tori and by chains of $\epsilon'$-solid cylinders with good intersection. The union of $U_2'$ and the $\epsilon'$-solid tori has the structure of a Seifert fibration with the (possible) exceptional fibers along the cores of the $\epsilon'$-solid tori. A circular chain of $\epsilon'$-solid cylinder neighborhoods is homeomorphic to a solid torus, and a linear chain is homeomorphic to $D^2 \times I$ meeting the boundary of $W_{n,2}'$ in spanning disks contained in 2-sphere boundary components. The frontiers of these chains in $W_{n,2}'$ are contained in $U_2'$. The next step is to perturb these chains slightly to isotopic embeddings so that their frontiers in $W_{n,2}'$ are saturated under the $S^1$-fibration structure on $U_2'$. For this we first construct slightly smaller versions of these chains, called refinements.

**Definition 5.26.** Let $\mathcal{V}(i) = \mathcal{V}_{c_i,[a_i,b_i]}(\tilde{\gamma}_i)$, for $i = 1, \ldots, k$ be a chain of $\epsilon'$-solid cylinder neighborhoods with good intersection. Consider a consecutive pair $\nu(i), \nu(i+1)$. If Inequality 5.1 holds then we set

$$\mathcal{V}'(i) = \mathcal{V}_{(c_i/2),[a_i,b_i]}(\tilde{\gamma}_i)$$

and

$$\mathcal{V}'(i+1) = \mathcal{V}_{(c_{i+1}/2),[a_{i+1}+0.01\ell_i,b_{i+1}]}(\tilde{\gamma}_{i+1}).$$

If the opposite inequality holds then we set

$$\mathcal{V}'(i) = \mathcal{V}_{(c_i/2),[a_i,b_i-0.01\ell_i]}(\tilde{\gamma}_i)$$

and

$$\mathcal{V}'(i+1) = \mathcal{V}_{(c_{i+1}/2),[a_{i+1},b_{i+1}]}(\tilde{\gamma}_{i+1}).$$

Thus, we halve the width of both the solid cylinders and the truncate the end of the larger one. We perform an analogous operation for each pair of successive solid cylinders, so that it is possible that both ends of $\nu(i)$ are truncated, only one end is truncated, or neither end is truncated. In all cases the width of $\nu(i)$ is halved so as to become $c_i/2$. The result is called a refinement of the chain $\{\mathcal{V}(1), \ldots, \mathcal{V}(k)\}$. The boundary of $\mathcal{V}'(1) \cup \cdots \cup \mathcal{V}'(k)$ consists of the negative end of $\mathcal{V}'(1)$ union the positive end of $\mathcal{V}'(k)$ union an annulus $A'$ analogous to $A$.

It is easy to establish the following by induction.

**Lemma 5.27.** Suppose that $\{\mathcal{V}(1), \ldots, \mathcal{V}(k)\}$ is either a linear chain or a circular chain of $\epsilon'$-solid cylinder neighborhoods with good intersections where $\mathcal{V}(i) = \mathcal{V}_{c_i,[a_i,b_i]}(\tilde{\gamma}_i)$. Then there is a refinement $\{\mathcal{V}'(1), \ldots, \mathcal{V}'(k)\}$ of this chain. Furthermore:

1. If the chain is a linear chain, then $(\bigcup_{i=1}^k \mathcal{V}(i))$ is homeomorphic to a 3-ball and the 2-sphere $\partial (\bigcup_{i=1}^k \mathcal{V}(i))$ is made up of the negative end, $D_-(1)$, of $\mathcal{V}(1)$, the positive end, $D_+(k)$, of $\mathcal{V}(k)$ and an annulus $A$ with $A$ meeting each of $D_-(1)$ and $D_+(k)$ along its boundary circle. Similarly, $\partial (\bigcup_{i=1}^k \mathcal{V}'(i))$ is a 2-sphere consisting of the negative end $D'_-(1)$ of $\nu'_1$, the positive end $D'_+(k)$ of
\( \nu'(k) \) and an annulus \( A' \) meeting each of \( D'_-(1) \) and \( D'_+(k) \) along its boundary circle. Furthermore, there is a homeomorphism
\[
\left( \bigcup_{i=1}^{k} \overline{\nu(i)} \setminus \text{int} \left( \bigcup_{i=1}^{k} \nu'(i) \right), A, A' \right) \equiv (A \times I, A \times \{0\}, A \times \{1\}) .
\]

2. If \( \{\nu(1), \cdots, \nu(k)\} \) is a circular chain, then
\[
\partial \left( \bigcup_{i=1}^{k} \nu(i) \right)
\]
is homeomorphic to a 2-torus and
\[
\bigcup_{i=1}^{k} \overline{\nu(i)} \setminus \text{int} \left( \bigcup_{i=1}^{k} \nu'(i) \right)
\]
is homeomorphic to \( T^2 \times I \).

Lastly, for each \( i \) let \( g_{n,i} \) be the multiple of \( g_n \) that is used in defining \( \nu(i) \), let \( \tilde{\gamma}_i \) be the central geodesic of \( \nu(i) \) and let \( \ell_i \) be the length of \( \tilde{\gamma}_i \) in the metric \( g_{n,i} \). In each case the distance, measured in \( g_{n,i} \), from any point of \( A' \cap \nu(i) \) to \( A \) is at least \( \xi^2 \ell(\tilde{\gamma}_i) \).

5.7 Removing solid tori and solid cylinders from \( W'_{n,2} \)

At this point we have constructed the compact submanifold \( V_{n,1} \) and shown that it has all the properties required by Theorem 1.1 but we still must modify \( W'_{n,2} \) in order to produce \( V_{n,2} \). To produce \( V_{n,2} \) as required by Theorem 1.1 we shall remove solid tori and solid cylinders from \( W'_{n,2} \).

The compact set \( W'_{n,2} \) is covered by (i) \( U'_{2} \), (ii) a finite number of \( \epsilon' \)-solid tori \( B(z_i) = B(z_i, r(z_i)/4) \) near interior singular points, and (iii) a finite number of chains of refinements of \( \epsilon' \)-solid cylinders. Our first approximation to \( V_{n,2} \), we call it \( V'_{n,2} \), is to remove from \( W'_{n,2} \) the union of:

1. the interiors of all the refinements of \( \epsilon' \)-solid cylinders in the given chains and
2. the interiors of the \( \epsilon' \)-solid tori \( B(z_i) \) in the collection.

This does not produce \( V_{n,2} \) because, even though the frontiers of these components are contained in \( U'_{2} \), they are not saturated under the \( S^1 \)-fibration on \( U'_{2} \). In order to define \( V_{n,2} \), we must deform the solid tori and solid cylinders that we remove from \( W'_{n,2} \) slightly so as to arrange that their frontiers in \( W'_{n,2} \) are saturated under this action.

5.7.1 Removing solid tori near interior cone points

For the solid tori near interior cone points it is clear what to do. According to Corollary 4.18 for each \( \epsilon' \)-solid torus \( B(z_i) \) near an interior cone point the neighborhood of the boundary of this solid torus contains a 2-torus \( T(z_i) \subset U'_{2} \cap B(z_i, 3r(z_i)/8) \) that is saturated under the \( S^1 \)-fibration and bounds a solid torus \( \tilde{T}(z_i) \in B(z_i, 3r(z_i)/8) \).
Instead of removing $B(z_i, r(z_i)/4)$ from $W_{n, 2}'$ we remove the interior of $\widehat{T}(z_i)$. We are removing slightly larger solid tori, but this does not change the topological type since the region between $T(z_i)$ and the metric sphere $S(z_i, r(z_i)/4)$ is a product region, a region homeomorphic to $T^2 \times I$. The boundary component created by removing the interior of $\widehat{T}(z_i)$ is a 2-torus saturated under the $S^1$-fibration structure on $U_2'$.

### 5.7.2 Removing perturbations of chains of $\epsilon'$-solid cylinders

We wish to make an analogous removals of perturbations of the chains of $\epsilon'$-solid cylinders, perturbed so that their frontiers in $W_{n, 2}'$ are saturated under the $S^1$-fibration structure on $U_2'$. To do so requires more argument.

Let $C$ be a circular chain of $\epsilon'$-solid cylinders contained in $W_{n, 2}'$ and let $C'$ be the given refinement of it. Then by Lemma 5.9 the union $\widehat{T}_C$ of the solid cylinders in $C$ minus the union of the interiors $\text{int} \widehat{T}_{C'}$ of the solid cylinders in $C'$ is homeomorphic to $T^2 \times I$. Furthermore, $\widehat{T}_C \setminus \text{int} \widehat{T}_{C'}$ is contained in $U_2'$. We consider the union of all fibers of the $S^1$-fibration on $U_2'$ that either meet the complement of $\widehat{T}_C$ or are closer to this complement than they are to $\widehat{T}_{C'}$. This is an open subset $\Omega$ of $U_2'$ that is saturated under the $S^1$-fibration. Since $\Omega$ contains $\partial \widehat{T}_C$ and is disjoint from $\widehat{T}_{C'}$, it follows that we can find a compact 3-manifold $\Omega_0$ with boundary contained in $\Omega$ that is saturated under the $S^1$-fibration on $U_2'$. Then $\Omega_0$ contains $\partial \widehat{T}_C$ and of course is disjoint from $\widehat{T}_{C'}$. One of the boundary components, $T(C)$, of $\Omega_0$ must then separate $\partial \widehat{T}_C$ from $\partial \widehat{T}_{C'}$. Since this boundary component is fibered by circles and is orientable, it is diffeomorphic to a 2-torus. Since the region $\widehat{T}_C \setminus \text{int} \widehat{T}_{C'}$ is homeomorphic to $T^2 \times I$. Any 2-torus contained in this region that separates the boundary components, e.g., $T(C)$, is topologically isotopic in $\widehat{T}_C$ to either boundary component. It then follows that $T(C)$ bounds a solid torus $T(C)$ contained in $\widehat{T}_C$.

Now let us consider a linear chain $C$ of $\epsilon'$-solid cylinders. Let $\widehat{C}(C)$ be the union of the closed $\epsilon'$-solid cylinders in this chain. By Lemma 5.24 the submanifold $\widehat{C}(C)$ is homeomorphic to $D^2 \times I$. Let $\widehat{C}'(C)$ be the union of the solid cylinders in the refinement. Denote by $X(C)$ the complement $\widehat{C}(C) \setminus \text{int} \widehat{C}'(C)$ and by $E(X)$ the ends of $X$, i.e., the intersection of $X$ with the ends of $\widehat{C}(C)$. According to Lemma 5.24 the pair $(X, E(X))$ is homeomorphic to $(S^1 \times I \times I, S^1 \times \partial I \times I)$. Also, since the distance between any point $x \in \widehat{C}'(C) \cap \nu(i)$ and the complement of the $\widehat{C}(C)$, when measured in the multiple of the metric $g_n$ used to define $\nu(i)$, is at least $\ell(\widetilde{\gamma}_i) \xi^2 \geq s_1 \xi^2/10$, it follows from the fact that $\dot{\epsilon} < 10^{-3} \xi^2 s_1/C$, Lemma 4.1 and Proposition 4.4 that $\Omega$ contains $\partial \widehat{T}_C$ and is disjoint from $\widehat{T}_{C'}$. It then follows that we can find a compact 3-manifold $\Omega_0$ with boundary contained in $\Omega$ that is saturated under the $S^1$-fibration on $U_2'$. Then $\Omega_0$ contains $\partial \widehat{T}_C$ and of course is disjoint from $\widehat{T}_{C'}$. One of the boundary components, $T(C)$, of $\Omega_0$ must then separate $\partial \widehat{T}_C$ from $\partial \widehat{T}_{C'}$. Since this boundary component is fibered by circles and is orientable, it is diffeomorphic to a 2-torus. Since the region $\widehat{T}_C \setminus \text{int} \widehat{T}_{C'}$ is homeomorphic to $T^2 \times I$. Any 2-torus contained in this region that separates the boundary components, e.g., $T(C)$, is topologically isotopic in $\widehat{T}_C$ to either boundary component. It then follows that $T(C)$ bounds a solid torus $T(C)$ contained in $\widehat{T}_C$.
Each end of the chain crosses a 2-sphere boundary component of \( W'_{n,2} \). Let us denote these boundary components by \( S_\pm(C) \). By construction \( S_\pm(C) \) is the union of an annulus \( E_\pm(C) \) contained in \( U'_2 \) and saturated under the \( S^1 \)-fibration on \( U'_2 \) and two 2-disks contained in the of the extremal \( \epsilon \)-solid cylinders in the chain \( C \). Furthermore, the annulus \( E_\pm(C) \) contains all points of \( S_\pm(C) \) contained in the \( \hat{C}(C) \setminus \hat{C}'(C) \). Thus, the intersection of \( T \) with

\[
S_\pm(C) \cap \left( \hat{C}(C) \setminus \hat{C}'(C) \right)
\]

is a union of fibers of the \( S^1 \)-fibration on \( U'_2 \). Hence, there is an annulus \( P(C) \) in \( T \cap \left( \hat{C}(C) \setminus \hat{C}'(C) \right) \) that is saturated under the \( S^1 \)-fibration on \( U'_2 \) that has one boundary circle in \( E_\pm(C) \) and the other boundary circle in \( E_-(C) \) and is otherwise disjoint from the \( S_\pm(C) \). The intersection of \( P(C) \) with \( E_\pm(C) \) is a circle bounding a disk \( D_\pm \) in the \( \pm \) end of \( \hat{C}(C) \). The union of \( D_- \cup P(C) \cup D_+ \) is a 2-sphere contained in the interior of the 3-ball \( \hat{C}(C) \). As such, this union is the boundary of a 3-ball \( \Gamma(C) \) in \( \hat{C}(C) \). It follows that there is a homeomorphism

\[
(\Gamma(C), P(C)) \cong (D^2 \times I, \partial D^2 \times I).
\]

Now we are ready to define \( V_{n,2} \). We begin with the compact 3-manifold \( W'_{n,2} = M_n \setminus \text{int} \ V_{n,1} \). From this we remove the interiors of solid tori and of solid cylinders to form \( V_{n,2} \). For each \( \epsilon \)-solid torus \( B(z_i) \) near an interior cone point, we remove the interior of the solid torus \( \hat{C}(z_i) \) as described in above. For each circular chain of \( \epsilon \)-solid tori \( C \) we remove the interior of the solid torus \( \tau_C \) described above. For each linear chain of \( \epsilon \)-solid cylinders \( C \) we remove the sub region \( \Gamma(C) \) as described above. The boundary of \( V_{n,2} \) consists of 2-tori contained in \( U'_2 \) and saturated under the \( S^1 \)-fibration. They are of three types:

1. components that are disjoint from \( V_{n,1} \),

2. components that are unions of annuli meeting along their boundaries, annuli saturated under the \( S^1 \)-fibration on \( U'_2 \); the annuli alternate between the annuli \( P(C) \) associated to linear chains of \( \epsilon \)-solid cylinders whose interiors are disjoint from \( V_{n,1} \), and annuli \( E \) that are contained in the 2-spheres boundary components of \( W'_{n,2} \), and

3. components that are boundary components of \( V_{n,1} \).

Since \( V_{n,2} \subset U'_2 \) is a compact submanifold and its boundary is saturated under the \( S^1 \)-fibration on \( U'_2 \), it follows that \( V_{n,2} \) is a compact 3-manifold that is saturated under the \( S^1 \)-fibration on \( U'_2 \). Hence, \( V_{n,2} \) is the total space of a locally trivial circle bundle. Clearly from the construction the intersection of \( V_{n,2} \) with \( V_{n,1} \) consists of the union of (i) all the 2-torus boundary components of \( V_{n,1} \) that are not boundary components of \( M_n \) and (ii) an annulus in each 2-sphere boundary component of \( V_{n,1} \). Lastly, the complement \( M_n \setminus \text{int} \ (V_{n,1} \cup V_{n,2}) \) consists of a finite disjoint union of compact solid tori and of compact solid cylinders (manifolds homeomorphic to
$D^2 \times I$). The boundaries of the solid tori are boundary components of $V_{n,2}$, and the boundary of each solid cylinder meets $V_{n,2}$ in $\partial D^2 \times I$ and this intersection is saturated under the $S^1$-fibration on $V_{n,2}$. This completes the proof that the submanifolds $V_{n,1}$ and $V_{n,2}$ satisfy all the properties given in the conclusion of Theorem 1.1. This establishes Theorem 1.1 and as a consequence Theorem 0.2.

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