The critical equation of state
of the 2D Ising model

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Abstract

We compute the $2n$-point coupling constants in the high-temperature phase of the 2d Ising model by using transfer-matrix techniques. This provides the first few terms of the expansion of the effective potential (Helmholtz free energy) and of the equation of state in terms of the renormalized magnetization. By means of a suitable parametric representation, we determine an analytic extension of these expansions, providing the equation of state in the whole critical region in the $t,h$ plane.
1 Introduction

Despite its apparent simplicity and the fact that since more than fifty years the exact expression of the free energy along the $h = 0$ axis is exactly known, the two-dimensional Ising model still provides many interesting issues. Since the original Onsager solution [1], several other exact results have been obtained for this model. In particular, closed expressions for the two-point correlation function at $h = 0$ and an S-matrix solution on the $t = 0, h \neq 0$ axis [2] exist (for a review, see [3]). However, little is known for generic values of $t$ and $h$ and in particular, there is no exact expression for the free energy and for the critical equation of state in the whole $(t, h)$ plane.

In order to determine the Helmholtz free energy (also called effective potential) and the equation of state, we begin by computing the first terms of their expansion in powers of the magnetization in the high-temperature (HT) phase. The coefficients of this expansion are directly related to the $n$-point zero-momentum renormalized couplings $g_n$ that are also interesting in themselves, since they summarize relevant (zero-momentum) physical properties of the quantum field theory that describes the Ising model in the vicinity of the critical point. For this purpose we combine transfer matrix (TM) techniques, conformal field theory (CFT) methods, and general renormalization-group (RG) properties of critical systems. Recent works [4, 5] have shown that a combination of these approaches can lead to very accurate estimates of universal ratios in the 2d Ising model. In this respect this paper is the natural continuation of [5] in which the first nontrivial coupling $g_4$ was determined with the same techniques used here.

Starting from the expansion of the free energy in powers of the magnetization in the HT phase, we determine approximate representations of the equation of state that are valid in the critical regime in the whole $(t, h)$ plane. This requires an analytic continuation in the complex $t$-plane [6, 7], extending the expansion valid for $t > 0$ to the low-temperature phase $t < 0$. For this purpose, we use parametric representations [8–10], which implement in a rather simple way the known analytic properties of the equation of state (Griffiths’ analyticity). We construct a systematic approximation scheme based on polynomial parametric representations [7] and on a global stationarity condition [11]. This approach was successfully applied to the three-dimensional Ising model leading to an accurate determination of the critical equation of state and of the universal amplitude ratios that can be obtained from it [7, 11].

The paper is organized as follows: In Sec. 2 we set up the formalism and define the quantities that parametrize the free energy for small values of the magnetization. These coefficients are evaluated numerically in Sec. 3 by using TM techniques. In Sec. 4 we use a parametric representation to analytically extend this expansion to the whole critical region, obtaining approximate, but still quite precise, expressions of the critical equation of state in the whole $(t, h)$ plane. Finally in Sec. 5 we draw our conclusions. In App. A we report our notations for the universal amplitude ratios, in App. B we present a detailed discussion of the finite-size behaviour of the free energy and of its derivatives, and in App. C we explicitly compute $e_h$, one of the coefficients appearing in the expansion of the nonlinear scaling field associated with the magnetic field.
2 Small-field expansion of the effective potential in the high-temperature phase

In the theory of critical phenomena continuous phase transitions can be classified into universality classes determined only by a few basic properties characterizing the system, such as space dimensionality, range of interaction, number of components and symmetry of the order parameter. RG theory predicts that, within a given universality class, the critical exponents and the scaling functions are the same for all systems. Here we consider the two-dimensional Ising universality class, which is characterized by a real order parameter and effective short-range interactions. A representative of this universality class is the standard square-lattice Ising model defined by the partition function

\[ Z = \sum_{\sigma_i = \pm 1} e^{\beta \sum_{\langle n,m \rangle} \sigma_n \sigma_m + h \sum_n \sigma_n}, \]  

where the field variable \( \sigma_n \) takes the values \( \{ \pm 1 \} \), \( n \equiv (n_0, n_1) \) labels the sites of a square lattice of size \( L_0 \times L_1 \), and \( \langle n,m \rangle \) denotes a lattice link connecting two nearest-neighbour sites. In our calculations we treat asymmetrically the two directions. We denote by \( n_0 \) the “time” coordinate and by \( n_1 \) the “space” one. We indicate by \( N \equiv L_0 L_1 \) the number of sites of the lattice and define the reduced temperature

\[ t \equiv \frac{\beta_c - \beta}{\beta_c}, \]  

where \( \beta_c = \log (\sqrt{2} + 1)/2 \approx 0.4406868 \ldots \) is the critical point.

We introduce the Gibbs free-energy density

\[ F(t, h) \equiv \frac{1}{N} \log(Z(t, h)), \]  

and the related Helmholtz free-energy density—in the field-theoretical framework usually called effective potential—

\[ \mathcal{F}(t, M) = M h - F(t, h), \]  

where \( M \) is the magnetization per site. For \( t > 0 \), \( F(t, h) \) and \( \mathcal{F}(t, M) \) have regular expansions in even powers of \( h \) and \( M \) respectively. Explicitly

\[ F(t, h) - F(t, 0) = \sum_{n=1}^{\infty} \frac{1}{(2n)!} \chi_{2n}(t) h^{2n}, \]  

\[ \mathcal{F}(t, M) - \mathcal{F}(t, 0) = \sum_{n=1}^{\infty} \frac{1}{(2n)!} \chi_{2n}^{\text{PI}}(t) M^{2n}, \]  

where \( \chi_{2n}(t) \) is the zero-momentum \( n \)-point function and \( \chi_{2n}^{\text{PI}}(t) \) is its one-particle irreducible counterpart. Note that \( \chi_2(t) = \chi(t) \), where \( \chi(t) \) is the magnetic susceptibility.

Expansion (2.6) can also be written in the equivalent forms

\[ \mathcal{F}(t, M) - \mathcal{F}(t, 0) = \xi_{2n}^{-1} \left( \frac{1}{2} z^2 + \sum_{n=2}^{\infty} \frac{1}{(2n)!} g_{2n} z^{2n} \right), \]  

\[ = -\frac{\chi^2}{\chi_4} \left( \frac{1}{2} z^2 + \frac{1}{4!} z^4 + \sum_{n=3}^{\infty} \frac{1}{(2n)!} r_{2n} z^{2n} \right), \]
where
\[ \varphi^2 = \frac{\xi_{\text{2nd}}(t)^2}{\chi(t)} M^2, \quad z^2 = \frac{\chi_4(t)}{\chi(t)^3} M^2, \]
and \( \xi_{\text{2nd}} \) is the second-moment correlation length
\[ \xi_{\text{2nd}} = \frac{1}{4\chi} \sum_x x^2 \langle s(0)s(x) \rangle. \] (2.10)

The coefficients \( g_{2n} \) correspond to the \( 2n \)-point renormalized coupling constants and \( r_{2n} = g_{2n}/g_4^{n-1} \). The couplings \( g_{2n} \) and the related \( r_{2n} \) can be computed in terms of \( \chi_{2n} \) and \( \xi_{\text{2nd}} \). The four-point coupling is given by
\[ g_4(t) = -\frac{\chi_4}{\chi^2 \xi_{\text{2nd}}}. \] (2.11)

An explicit expression of \( r_{2n} \) in terms of \( \chi_{2n} \) is given at the beginning of Sec. 3.

The advantage of the expansions (2.7) and (2.8) is the fact that the coefficients have a finite limit—in the following we indicate it by the same symbol—for \( t \to 0 \). Notice that for \( t \to 0 \), \( z \sim Mt^{-\beta} \) and \( \chi^2/\chi_4 \sim t^{2\nu} \), so that Eq. (2.8) defines the HT expansion of the scaling part of the free energy \( F(t, M) \sim t^{2\nu} F_{\text{scal}}(Mt^{-\beta}) \).

### 3 Calculation of \( r_{2n} \) with transfer-matrix techniques

In this section we wish to report the calculation of the first few coefficients \( r_{2n} \) at the critical point \( t = 0 \). It is easy to express them in terms of the critical amplitudes of the \( 2n \)-point functions \( \chi_{2n}(t) \). Defining
\[ C_n^+ = \lim_{t \to 0} \chi_n(t) t^{15n/8-2}, \] (3.1)
it is possible to show by direct calculation that
\[ r_6 = 10 - \frac{C_6^+ C_2^+}{(C_4^+)^2}, \] (3.2)
\[ r_8 = 280 - 56 \frac{C_6^+ C_2^+}{(C_4^+)^2} + \frac{C_8^+ (C_2^+)^2}{(C_4^+)^3}, \] (3.3)
\[ r_{10} = 15400 - 4620 \frac{C_6^+ C_2^+}{(C_4^+)^2} + 120 \frac{C_8^+ (C_2^+)^2}{(C_4^+)^3} \]
\[ + 126 \frac{(C_6^+)^2 (C_2^+)^2}{(C_4^+)^4} - \frac{C_{10}^+ (C_2^+)^3}{(C_4^+)^4}, \] (3.4)
\[ r_{12} = 1401400 - 560560 \frac{C_6^+ C_2^+}{(C_4^+)^2} + 17160 \frac{C_8^+ (C_2^+)^2}{(C_4^+)^3} + 36036 \frac{(C_6^+)^2 (C_2^+)^3}{(C_4^+)^4} \]
\[ - 220 \frac{C_{10}^+ (C_2^+)^3}{(C_4^+)^4} - 792 \frac{C_8^+ C_6^+ (C_2^+)^3}{(C_4^+)^5} + \frac{C_{12}^+ (C_2^+)^4}{(C_4^+)^5}. \] (3.5)

The aim of this section is to estimate the constants \( C_{2n}^+ \), by using TM techniques and the exact knowledge of several terms of the small-\( t \) expansion of \( \chi_{2n}(t) \). Using the equations reported above, we will then obtain our estimates of \( r_{2n} \).
3.1 The transfer-matrix approach

We obtained our numerical estimates of the $2n$-point functions $\chi_{2n}(t)$ following a two-step procedure. First, by using TM techniques, we obtained estimates of these quantities on lattices of finite transverse size $L_1$, with $L_1 \leq 24$. Second, we extrapolated these results to the thermodynamic limit by using the exact knowledge of their Finite Size Scaling (FSS in the following) behaviour. To perform this second part of the analysis we have elaborated a new extrapolation scheme which is rather interesting in itself. We shall discuss it in detail in the last part of this section.

Let us now see both these steps in more detail.

3.1.1 The TM computation

This part of the procedure was already discussed in Ref. [5]. We report here the main features of the algorithm for completeness and refer to [5] for a more detailed discussion.

The main idea is to use TM techniques to extract the $h$-dependence of the magnetization at fixed $t$. To this end, we computed numerically the magnetization $M(t, h, L_1)$ on lattices of size $\infty \times L_1$, for $L_1 \leq 24$. In this work we use again the data obtained in Ref. [5]. In addition, we generated new data for $\beta = 0.335$ and 0.345 and added $L_1 = 24$ results for $\beta = 0.35$ and 0.355. All computations were performed with double-precision floating-point arithmetic on commercial workstations. Typically we obtained $M$ with 15 correct digits.

Then, for each given value of $L_1$ and $\beta$, we determined the coefficients of the series

$$h = b_1 M + b_3 M^3 + b_5 M^5 \ldots$$

(3.6)

which we truncated at order $M^{15}$. Using the numerical results of $M$ for 8 values of $h$, we computed the coefficients $b_1, b_3, \ldots, b_{15}$. The optimal choice for these eight values of $h$ is the one for which the errors due to the truncation of the series and those due to numerical rounding are of the same magnitude. This optimal range changes as a function of $\beta$ and $L_1$. For instance, just to give an idea of the magnitude of the magnetic fields which we studied, for $\beta = 0.36$ and $L_1 = 24$ the best choice is $h = 0.002 j$ with $j = 1 \ldots 8$. The accuracy of the $b_i$ obtained in this way is decreasing with increasing order. For example, for $\beta = 0.37$, we obtain $b_1$ with 14 significant digits and, e.g., $b_{11}$ with only 3 significant digits.

In principle we could have also used the expansion

$$M = a_1 h + a_3 h^3 + a_5 h^5 \ldots$$

(3.7)

However, in practice, the use of (3.7) leads to results with significantly larger truncation errors, since in contrast to what happens for $b_i$, the sign of the coefficients $a_i$ alternates.

Then, for each $\beta$ and $L_1$, we constructed the $\chi_{2n}$ functions out of the $b_i$ constants. These are the inputs of the FSS analysis discussed in the next section. The relations between the $b_i$ and the $\chi_{2n}$ can be easily obtained by a straightforward calculation. We report them here for completeness:

$$\chi_4 = -3! \frac{b_3}{b_1^4},$$

(3.8)

$$\chi_6 = -5! \left( \frac{b_5}{b_1^6} - 3 \frac{b_3^2}{b_1^4} \right),$$

(3.9)
\[ \chi_s = -7! \left( \frac{b_7}{b_1^8} + 12 \frac{b_3^3}{b_1^{10}} - 8 \frac{b_3 b_5}{b_1^9} \right), \quad (3.10) \]

\[ \chi_{10} = -9! \left( \frac{b_9}{b_1^{10}} - 55 \frac{b_3^4}{b_1^{13}} - 10 \frac{b_3 b_7}{b_1^{11}} + 55 \frac{b_3^2 b_5}{b_1^{12}} - 5 \frac{b_3^3}{b_1^{11}} \right), \quad (3.11) \]

\[ \chi_{12} = -11! \left( \frac{b_{11}}{b_1^{12}} + \frac{273 b_3^5}{b_1^{16}} - \frac{364 b_3^3 b_5}{b_1^{15}} + \frac{78 b_3 b_7}{b_1^{14}} + \frac{78 b_3^2 b_5}{b_1^{14}} - \frac{12 b_3^2 b_7}{b_1^{13}} - \frac{12 b_3 b_9}{b_1^{12}} \right). \quad (3.12) \]

This last step is the only difference in the present calculation with respect to the analogous one reported in \[\text{[5]}\] where we directly studied the thermodynamic limit of the \(b_i\) coefficients. The reason is that as \(n\) increases the \(\chi_{2n}\) show a much better and smoother FSS behaviour than the \(b_i\).

### 3.1.2 The thermodynamic limit

In the second step of the analysis we extrapolate our results to the thermodynamic limit. We know on general grounds that, since \(L_1 \gg \xi\), the convergence towards the thermodynamic limit of the \(\chi_{2n}\) functions is exponentially fast, the decay rates being related to the spectrum of the theory. This result holds for any lattice model. However, in the case of the two-dimensional Ising model, thanks to the fact that the model can be solved exactly also on finite lattices \[\text{[12]}\] (or, in the language of S-matrix theory, thanks to the fact that the S-matrix of the model is very simple), much more informations can be obtained on the FSS properties of the free energy and of its derivatives. In particular, one can explicitly compute the functional form of the FSS behaviour of \(\chi_{2n}\) (see App. \[\text{[3]}\]). It turns out to be

\[ \chi_{2n}(L_1) - \chi_{2n}(\infty) = L_1^{2n-3/2} (g_{2n,1}(1/L_1)e^{-mL_1} + g_{2n,2}(1/L_1)e^{-2mL_1} + g_{2n,3}(1/L_1)e^{-3mL_1} + ...) \quad (3.13) \]

where \(g_{2n,i}(1/L_1)\) are functions of \(1/L_1\), which can be expanded in (positive) powers of \(1/L_1\).

By fitting our values of \(\chi_{2n}(L_1)\) with this law (by expanding the functions \(g_{2n,i}(L_1)\) and using as free parameters the coefficients of the first few terms of the expansion) it is possible to obtain very precise estimates for the thermodynamic limit \(\chi_{2n}(\infty)\). However, this is not the most efficient strategy, since much of the information contained in the \(\chi_{2n}(L_1)\) is lost in the determination of the coefficients of the Taylor expansion of the \(g_{2n,i}(L_1)\) functions. Thus, we have elaborated an alternative iterative procedure which is much simpler to perform than the above multiple fits and allows to reach higher precision (up to one additional accurate digit). The idea behind this Iterative Algorithm (IA, in the following) is first to absorb the pre-exponential factors of Eq. \[\text{[3.13]}\] in the masses by allowing them to depend on \(L_1\). Then, the resulting exponential corrections are eliminated by iteratively solving the system of equations

\[ \begin{align*}
\chi_{2n}^{(i)}(L_1 - 2) &= c \exp(-x(L_1 - 2)) + \chi_{2n}^{(i+1)}(L_1) \\
\chi_{2n}^{(i)}(L_1 - 1) &= c \exp(-x(L_1 - 1)) + \chi_{2n}^{(i+1)}(L_1) \\
\chi_{2n}^{(i)}(L_1) &= c \exp(-xL_1) + \chi_{2n}^{(i+1)}(L_1) 
\end{align*} \quad (3.14) \]

with respect to \(\chi_{2n}^{(i+1)}(L_1)\), \(c\) and \(x\).
The index \( i \) in \( \chi_{2n}^{(i)} \) denotes the order of the iteration, in particular the \( \chi_{2n}^{(0)} \) are the input data of the algorithm obtained as discussed in the previous section. As \( i \) increases the \( L_1 \) dependence of the \( \chi_{2n}^{(i)} \) becomes smaller. After a certain number of steps the extrapolation becomes unstable since rounding errors accumulate. Typically, the final estimate was obtained by iterating 1, 2, 3 or 4 times, depending on \( \beta \) and on \( n \). The residual dependence on \( L_1 \) is used to estimate the error in the best estimate of \( \chi_{2n}(\infty) \). The results obtained in this way are in perfect agreement with those obtained by directly fitting Eq. (3.13) but, as mentioned above, turn out to be more precise. By using the IA we were able to obtain reliable estimates of \( \chi_n \) for \( n \leq 12 \) up to \( \beta = 0.37 \). Our results are summarized in Tables 1 and 2.

In order to verify if our method of estimating the errors with the IA method (quoted in Tables 1 and 2) is reliable we made the following test. We studied with the IA a sample of data at a very low value of \( \beta \) where rather large values of \( L_1/\xi \) could be reached, so that the results for the \( \chi_{2n}(L_1) \) on our largest lattices were already good estimates of \( \chi_{2n}(\infty) \) with no need of further manipulations. Then, we performed our analysis using only the data at small \( L_1 \). In this way we could explicitly check that our extrapolation scheme gives accurate results for the thermodynamic limit starting from data such that \( L_1/\xi \approx 7 \), which is what we reached at our largest value of \( \beta, \beta = 0.37 \). Moreover in App. B.3 we report a further test of IA. We applied it to a test function of the type (3.13), finding again support to the reliability of the method.

In order to give a feeling of the performances of the IA for \( \beta = 0.37 \)—this is the value of \( \beta \) that is nearer to the critical point among those we studied and hence the worst one from the point of view of FSS—we have reported in Table 3 the results of the first four iterations for \( \chi_4 \) at \( \beta = 0.37 \). From these numbers we estimate the thermodynamic limit of \( \chi_4 \) to be \( \chi_4(\infty) = -4.052238(1) \) to be compared with the result obtained by directly fitting Eq. (3.13) which is \( \chi_4(\infty) = -4.052242(10) \).

This same IA was also used in our previous paper [4], the only difference being that in that case it was applied to extract the thermodynamic limit of the \( b_i \) constants. Let us stress, as a final remark, that this algorithm is very general and can be used even when no exact information is available on the FSS behaviour of the quantity of interest except from the fact that it is dominated by an exponential decay.

### 3.2 Small-\( t \) expansion of \( \chi_n(t) \)

As it is well known, the TM approach gives reliable results only for rather small values of \( \beta \). In order to perform the extrapolation \( \beta \to \beta_c \), it is thus mandatory to have a good control of the scaling corrections of \( \chi_n(t) \). In this section we address this problem in detail. Our goal will be to obtain for each \( \chi_{2n} \) the exact form of the scaling function up to \( O(t^4) \) and the spectrum of the possible scaling dimension (i.e. the exponents of the corresponding contributions in the scaling function) up to \( O(t^5) \)).

The most important information which is needed to perform this analysis is the spectrum of the irrelevant operators of the theory. This problem was recently addressed numerically in Refs. [13, 15] where it was shown that, for rotationally invariant quantities like the free energy and its derivatives, the first correction due to the irrelevant operators in the square-lattice Ising model appears at order \( t^4 \) (hence no correction of order \( t^2 \) is present) and that
| $\beta$ | $4$ | $6$ |
|-------|-----|-----|
| 0.20  | $-3.2111149829(1)$ | $61.94504968(2)$ |
| 0.25  | $-3.4721394468(1)$ | $74.5437488(1)$ |
| 0.28  | $-3.622540033(1)$  | $82.255117(3)$ |
| 0.30  | $-3.720514859(2)$  | $87.459547(5)$ |
| 0.31  | $-3.768883189(5)$  | $90.081633(5)$ |
| 0.32  | $-3.81687386(2)$   | $92.71755(1)$ |
| 0.33  | $-3.86451567(5)$   | $95.36788(5)$ |
| 0.335 | $-3.88821530(5)$   | $96.6986(1)$ |
| 0.34  | $-3.91183945(10)$  | $98.0332(1)$ |
| 0.345 | $-3.9353922(1)$    | $99.3718(1)$ |
| 0.35  | $-3.9588780(5)$    | $100.7142(2)$ |
| 0.355 | $-3.9823005(10)$   | $102.0610(2)$ |
| 0.36  | $-4.0056653(10)$   | $103.4119(2)$ |
| 0.365 | $-4.028975(1)$     | $104.7673(2)$ |
| 0.37  | $-4.052238(1)$     | $106.127(1)$ |

Table 1: Thermodynamic-limit results for $\chi_n(t) t^{15n/8-2}$, for $n = 4, 6$. The quoted error bars are estimates of the systematic error of the extrapolation.

| $\beta$ | $8$ | $10$ | $12$ |
|-------|-----|-----|-----|
| 0.20  | $-2985.5224(2)$ | $267883.(1.)$ | $-23211300.(1000.)$ |
| 0.25  | $-4002.0604(2)$ | $400060.(1.)$ | $-43255500.(1000.)$ |
| 0.28  | $-4672.309(2)$  | $494203.(5.)$ | $-59550000.(10000.)$ |
| 0.30  | $-5144.20(2)$   | $563455.(10.)$ | $-72400000.(10000.)$ |
| 0.31  | $-5387.715(10)$ | $600090.(20.)$ | $-79445000.(30000.)$ |
| 0.32  | $-5636.31(2)$   | $638110.(50.)$ | $-86910000.(100000.)$ |
| 0.33  | $-5890.10(5)$   | $677600.(100.)$ | $-94830000.(100000.)$ |
| 0.335 | $-6018.9(1)$    | $697800.(100.)$ | $-98950000.(200000.)$ |
| 0.34  | $-6149.0(1)$    | $718300.(100.)$ | $-103190000.(200000.)$ |
| 0.345 | $-6280.5(2)$    | $739000.(500.)$ | $-107560000.(200000.)$ |
| 0.35  | $-6413.1(3)$    | $760600.(200.)$ | $-112000000.(500000.)$ |
| 0.355 | $-6547.1(3)$    | $782400.(200.)$ | $-116600000.(500000.)$ |
| 0.36  | $-6682.3(5)$    | $804400.(300.)$ | $-121300000.(1000000.)$ |
| 0.365 | $-6819.3(3)$    | $827100.(500.)$ | $-126100000.(1000000.)$ |
| 0.37  | $-6958.1(5)$    | $850000.(500.)$ | $-131000000.(1000000.)$ |

Table 2: Thermodynamic-limit results for $\chi_n(t) t^{15n/8-2}$ for $n = 8, 10, 12$. The quoted error bars are estimates of the systematic error of the extrapolation.
| $L_1$ | $F_4^{(0)}$ | $F_4^{(1)}$ | $F_4^{(2)}$ | $F_4^{(3)}$ | $F_4^{(4)}$ |
|-------|----------|----------|----------|----------|----------|
| 15    | −3.72609418989519 | −4.07455655 |          |          |          |
| 16    | −3.80380147054199 | −4.06509498 |          |          |          |
| 17    | −3.86341299292473 | −4.05978684 | −4.05300296 |          |          |
| 18    | −3.90897624823899 | −4.05675321 | −4.05300296 | −4.05243012 |          |
| 19    | −3.94370300243699 | −4.05498669 | −4.05229750 | −4.05223018 |          |
| 20    | −3.99015681870086 | −4.05315480 | −4.05229750 | −4.05223018 | −4.05223671 |
| 21    | −4.00535064954957 | −4.05293008 | −4.05229750 | −4.05223018 | −4.05223671 | −4.05223829 |
| 22    | −4.01685278847850 | −4.05268980 | −4.05229750 | −4.05223018 | −4.05223671 | −4.05223829 |
| 23    | −4.02551193514433 | −4.05253744 | −4.05229750 | −4.05223018 | −4.05223671 | −4.05223829 |

Table 3: Results of the IA analysis for $F_4 \equiv \chi_4 t^{-11/2}$ at $\beta = 0.37$.

the next irrelevant operator contributes only at order $t^6$. This result has been shown for the susceptibility in zero field, but a standard application of RG ideas implies that it should also apply to the free energy as a function of $t$ and $h$.\footnote{This pattern agrees with an independent analysis, in the framework of CFT, based on the spectrum of the quasi-primary operators of the critical Ising model. This result was already anticipated in Ref. \[4\] and will be discussed in full detail in a forthcoming publication.} This is also confirmed by the less precise numerical results of Ref. \[4\] for the free energy on the critical isotherm and by the analytic study of the two-point function at large distances \[17\].

This result allows us to determine the scaling corrections to $\chi_2 n(t)$ up to order $t^4$ by using standard RG techniques (see \[5, 17, 18\]).

As a first step, let us write the free energy of the model in terms of nonlinear scaling fields \[19\]:

$$F(t, h) = F_b(t, h) + |u_t|^{d/ny} f_{\text{sing}} \left( \frac{u_h}{|u_t|^y}, \left\{ \frac{u_j}{|u_t|^y} \right\} \right)$$

$$+ |u_t|^{d/ny} \log |u_t| f_{\text{sing}} \left( \frac{u_h}{|u_t|^y}, \left\{ \frac{u_j}{|u_t|^y} \right\} \right). \quad (3.15)$$

Here $F_b(t, h)$ is a regular function of $t$ and $h^2$, $u_t$, $u_h$, $\{u_j\}$ are the nonlinear scaling fields associated respectively to the temperature, the magnetic field and the irrelevant operators, and $y_t$, $y_h$, $\{y_j\}$ are the corresponding dimensions. For the Ising model $y_t = 1$ and $y_h = 15/8$.

Notice the presence of the logarithmic term, that is related to a “resonance” between the thermal and the identity operator.\footnote{In principle, other logarithmic terms may arise from additional resonances due to the fact that $y_j$ are integers or differ by integers from $y_h$, and indeed they have been observed in a high-precision analysis of the asymptotic behaviour of the susceptibility \[15\]. They will not be considered here since these contributions either are subleading with respect to those we are interested in or have a form that is already included.} Since all numerical data indicate that for $t \to 0$ all zero-momentum correlation functions diverge as a power of $t$ without logarithms—our results provide additional evidence for this cancellation—we shall assume in the following (as in
Ref. [17]) that $\tilde{f}_{\text{sing}}$ does not depend on its first argument $u_h/|u_l|^n/w$. There is also some evidence that the leading contribution due to the irrelevant operators is absent. Indeed, for the susceptibility one would expect a correction of order $t^4 \log |t|$ which is not found in the high-precision study of the susceptibility reported in [15]. In the following we will be conservative and we will report results with and without corrections of order $t^4 \log |t|$. Our final estimates assume however that such a term is absent.

The scaling fields are analytic functions of $t$ and $h$ that respect the $\mathbb{Z}_2$ parity of $h$. Let us write their Taylor expansion, keeping only those terms that are needed for our analysis (we use the notations of Ref. [17]):

\begin{equation}
\begin{aligned}
    u_h &= h \left[ 1 + c_h t + d_h t^2 + e_h h^2 + f_h t^3 + O(t^4, t h^2) \right], \\
    u_t &= t + b_t h^2 + c_t t^2 + d_t t^3 + e_t t h^2 + f_t t^4 + O(t^5, t^2 h^2, h^4).
\end{aligned}
\end{equation}

All these coefficients (except $e_h$) have been determined exactly or numerically with very high precision, see [5, 14, 18]. We list them here for completeness:

\begin{equation}
\begin{aligned}
    c_h &= \frac{\beta_c}{\sqrt{2}}, & d_h &= \frac{23 \beta_c^2}{16}, & f_h &= \frac{191 \beta_c^3}{48 \sqrt{2}}, \\
    c_t &= \frac{\beta_c}{\sqrt{2}}, & d_t &= \frac{7 \beta_c^2}{6}, & f_t &= \frac{17 \beta_c^3}{6 \sqrt{2}}, \\
    e_t &= b_t \beta_c \sqrt{2}, & b_t &= -\frac{E_0 \pi}{16 \beta_c^2}.
\end{aligned}
\end{equation}

where [20]

\begin{equation}
    E_0 = 0.0403255003...
\end{equation}

is the coefficient of the contribution proportional to $t \log |t|$ in the susceptibility.

The coefficient $e_h$ is the only term that was not reported in [4]. In App. C we show that it is possible to determine it by analyzing the scaling corrections to the free energy on the critical isotherm. Using the precise data of Ref. [4], we obtain

\begin{equation}
    e_h = -0.00727(15).
\end{equation}

Using Eq. (3.15) and the expansions of the scaling fields reported above, we can compute the leading terms in the asymptotic expansion of $\chi_n(t)$ for $t \to 0$. We obtain

\begin{equation}
    \chi_n(t) = t^{2-15n/8} a_n(t) + t^{19/4-15n/8} b_n(t) + \ldots,
\end{equation}

where

\begin{equation}
\begin{aligned}
    a_n(t) &= C_n^+(1 + \alpha_1 \beta_c t + \alpha_2 \beta_c^2 t^2 + \alpha_3 \beta_c^3 t^3 + O(t^4)), \\
    b_n(t) &= C_{n-2}^+(\zeta_0 + \zeta_1 t + O(t^2)).
\end{aligned}
\end{equation}

The coefficients $\alpha_i$ and $\zeta_i$ are given by:

\begin{equation}
\begin{aligned}
    \alpha_1 &= -\frac{7n - 16}{8 \sqrt{2}},
\end{aligned}
\end{equation}
\[ \alpha_2 = \frac{147 n^2 - 1080 n + 2176}{768}, \]  
\[ \alpha_3 = -\frac{334 n^3 - 5208 n^2 + 26240 n - 49152}{6144\sqrt{2}}, \]  
\[ \zeta_0 = \frac{E_0 n (n - 1)(15 n - 46)}{128\beta_c^2}, \]  
\[ \zeta_1 = -\frac{E_0 n (n - 1)(7n - 38)(15n - 46)}{1024\sqrt{2}\beta_c} + n(n - 2)(n - 1)e_h. \]  

Plugging these coefficients into Eq. (3.23) we obtain the exact form of the scaling function up to the contribution of the first irrelevant field, i.e. up to \( O(t^4) \). The simplest way to use the exact knowledge of these terms of the scaling function to extract the amplitudes \( C_n^+ \) is to construct the quantity:

\[ M^{(n)}(t) = \chi_n(t) t^{15n/8 - 2 - b_n(t) t^{15n/8 - 19/4}} / a_n(t) \]  

which has the following expansion for \( t \to 0 \):

\[ M^{(n)}(t) = C_n^+ (1 + p_1 t^4 + \tilde{p}_1 t^4 \log t + p_2 t^{4.75} + O(t^5)). \]  

The terms proportional to \( t^4 \) and to \( t^{4.75} \) are due to the first unknown coefficients of \( a_n(t) \) and \( b_n(t) \) respectively. They also take into account the possible presence of irrelevant operators contributing to order \( t^4 \). The term proportional to \( t^4 \log t \) is the only remnant of the \( \tilde{f}_{\text{sing}} \) term in Eq. (3.15) and it is due to the irrelevant operators in \( \tilde{f}_{\text{sing}} \).

The constants \( C_n^+ \) are determined in sequence. We start by using \( C_2^+ \), which is known to very high precision, \( C_2^+ = 0.9625817323087721140443 \ldots \) \([3, 14, 15]\), to estimate \( C_4^+ \) with a best-fit analysis of Eq. (3.32) with \( n = 4 \). The value of \( C_4^+ \) obtained in this way is then used as input to construct the function \( b_6(t) \) which appears in Eq. (3.31), thus allowing to estimate \( M^{(6)}(t) \). At this point, by using again Eq. (3.32), we obtain the best-fit value for \( C_6^+ \) and repeat the whole procedure for the next value of \( n \).

As a consequence, in the determination of \( M^{(n)}(t) \) there are three sources of uncertainty:

a] The uncertainty in the TM estimates of \( \chi_n(t) \) which we use as input of our analysis.

b] The uncertainty in the estimate of \( e_h \) and \( E_0 \).

c] The uncertainty in our best estimate of \( C_{n-2}^+ \).

These uncertainties must be treated in different ways. [a] can be straightforwardly propagated to \( M^{(n)}(t) \). On the contrary [b] and [c] appear as uncertainties of the whole fitting function. To deal with them we followed the simplest (and most conservative) strategy. Let us study as an example the uncertainty due to the error on \( e_h \). We constructed two sets of data, \( M^{(n)}_+(t) \) and \( M^{(n)}_-(t) \), obtained by using \( e_h + \delta e_h \) and \( e_h - \delta e_h \) respectively, and then, for each set, we determined the best-fit values of the parameters. The difference between these results is a conservative estimate of the error induced by the uncertainty on \( e_h \). In order to give an idea of the size of these uncertainties, we have reported in Table \([\ldots]\) those induced by \( e_h \) in the best-fit estimates of \( C_6^+ \).
It turns out that the errors induced by the uncertainties on $C_{n-2}^+$ and $E_0$ are always negligible with respect to that due to $e_h$ and, what is more important, that this last one is in any case negligible with respect to the uncertainty in the best-fit estimate, i.e. with respect to the uncertainty due to the lack of knowledge of the higher-order correction terms. For this reason we shall neglect the errors of type [b] and [c] in our final results.

3.3 The fitting procedure

To analyze the data we followed a procedure similar to that presented in Ref. [5]. For each set of data we performed five different fits of $M^{(n)}$ using Eq. (3.32),

f1] keeping $C_n^+$ and $p_1$ as free parameters, and setting $\tilde{p}_1 = p_2 = 0$;
f2] keeping $C_n^+$ and $\tilde{p}_1$ as free parameters, and setting $p_1 = p_2 = 0$;
f3] keeping $C_n^+, p_1$, and $\tilde{p}_1$ as free parameters, and setting $p_2 = 0$;
f4] keeping $C_n^+, p_1$, and $p_2$ as free parameters, and setting $\tilde{p}_1 = 0$;
f5] keeping $C_n^+, p_1, \tilde{p}_1, \text{ and } p_2$ as free parameters.

In order to estimate the systematic errors involved in the determination of $C_n^+$, we performed for all the fitting functions several independent fits, trying first to fit all the existing data (reported in Tables 1 and 2) and then eliminating the data one by one, starting from the farthest from the critical point. Among the set of estimates of the critical amplitudes we selected only those fulfilling the following requirements:

1] The reduced $\chi^2$ of the fit must be of order unity. In order to fix precisely a threshold, we required the fit to have a confidence level larger than 30%.

2] For all the subleading terms included in the fitting function, the amplitude estimated from the fit must be larger than the corresponding error.

3] If the fit contains $k$ free parameters besides $C_n^+$, then at least $2^k$ degrees of freedom must be used in the fit. This means that in fits of type f1 and f2 at least four data must be used ($k = 1$: two parameters plus two degrees of freedom) in those of type f3 and f4 at least 7 data must be used ($k = 2$: three parameters plus four degrees of freedom). In f5 at least 12 data must be used ($k = 3$: four parameters plus eight degrees of freedom).

This last constraint is a generalization to higher values of $n$ of the one that we proposed in [5]. It seems to encode very well the real statistical significance of the data.

Finally, among all the estimates of the critical amplitude $C_n^+$ fulfilling these requirements we selected the smallest and the largest ones as lower and upper bounds. We consider the mean of these two values as our best prediction for $C_n^+$. The values are reported in Table 4. The errors quoted in Table 4 are half of the difference between the upper and lower bounds. They seem to give a reliable estimate of the uncertainty of our results.

Let us briefly comment on these results:
Table 4: Best-fit estimates of the amplitudes $C_n^+$. 

|     | $\tilde{p}_1 \neq 0$       | $\tilde{p}_1 = 0$       |
|-----|----------------------------|-------------------------|
| $C_4^+$ | $-4.379095(8)$            | $-4.379094(6)$          |
| $C_6^+$ | $125.9330(11)$            | $125.9332(6)$           |
| $C_8^+$ | $-9066.5(9)$              | $-9066.4(7)$            |
| $C_{10}^+$ | $1216.33(80) \times 10^3$ | $1216.34(60) \times 10^3$ |
| $C_{12}^+$ | $-26260(150) \times 10^4$ | $-26175(60) \times 10^4$ |

a] In Table 4 we have reported in two separate columns the best-fit results with and without the log-type contribution $\tilde{p}_1 t^4 \log t$. The reason of this choice is that there are strong numerical indications [13] that $\tilde{p}_1 = 0$ in the Ising model. To keep our analysis as general as possible, we report also the result with $\tilde{p}_1 \neq 0$. It is interesting to note that the estimates of $C_n^+$ ($n \leq 10$) do not depend essentially on this choice, while the error decreases by a factor 1.5–2 if we assume $\tilde{p}_1 = 0$. This seems to indicate that the fitting procedure is stable and reliable.

b] In the first line of Table 4 we report our best-fit estimate of $C_4^+$, which we had already estimated in [3] with the same techniques used in the present paper. The only improvement with respect to [3] is that we now also know exactly the value of $\epsilon_h$ which was the dominant (of order $t^{3.75}$) unknown correction in [3]. It is instructive to compare the two estimates. In [3] we obtained, keeping also into account the log-type corrections, $C_4^+ = -4.379101(9)$, while the present value is $C_4^+ = -4.379095(8)$. The two results are, as expected, fully compatible. The new one is only slightly more precise than the previous one. The fact that the enhancement in precision is so small is due to the fact that, by using the exact value of $\epsilon_h$, we only improve the known small-$t$ expansion of $\chi_n(t)$ from $O(t^{3.75})$ to $O(t^4)$.

c] It is instructive to look in more detail to the results of the fits in one particular case. Let us consider, for instance, $C_6^+$. The results are reported in Table 5, where, for each type of fit, we quote the two fits which correspond to the highest and lowest values of $C_6^+$ (in quoting the uncertainties in the best-fit estimates we also report those induced by $\epsilon_h$, to give an idea of their size).

The most impressive feature of this set of fits is that with only the $t^4$ contribution besides $C_6^+$ (fit f1) one can fit all the data up to $\beta = 0.31$.

Looking at Table 4, we obtain our best estimate

$$C_6^+ = 125.9330(11).$$

(3.33)

If we additionally assume that $\tilde{p}_1 = 0$, i.e. we keep into account only the results of the fits of type f1 and f4, then we find

$$C_6^+ = 125.9332(6).$$

(3.34)

The estimates (3.33) and (3.34) are those reported in Table 4.
Table 5: Estimates of \( C_6^+ \). In the first column we report the best-fit results for the critical amplitude (the first error in parenthesis is that induced by the systematic errors of the input data while the second is due to the error on \( e_h \)), in the second column the number of degrees of freedom (i.e. the number of data used in the fit minus the number of free parameters) and in the last column the type of fit.

| \( C_6^+ \)          | d.o.f. | fit type |
|-----------------------|--------|----------|
| 125.93274(12)(0)      | 2      | f1       |
| 125.93298(5)(0)       | 9      | f1       |
| 125.93301(14)(2)      | 2      | f2       |
| 125.93389(16)(7)      | 7      | f2       |
| 125.93223(30)(1)      | 5      | f3       |
| 125.93341(15)(2)      | 8      | f3       |
| 125.93290(14)(0)      | 7      | f4       |
| 125.93369(12)(1)      | 9      | f4       |
| 125.93253(20)(2)      | 8      | f5       |
| 125.93258(7)(2)       | 10     | f5       |

3.4 Summary of the results

Using the estimates of \( C_2^{+n} \) obtained in the previous section, we can estimate \( r_{2n} \) by using Eqs. (3.2)-(3.5). We finally obtain

\[
g_4 = 14.697323(20), \quad (3.35) \\
r_6 = 3.67866(3)(2), \quad (3.36) \\
r_8 = 26.041(8)(3), \quad (3.37) \\
r_{10} = 284.5(1.4)(1.0), \quad (3.38) \\
r_{12} = 4200(320)(420). \quad (3.39)
\]

For each \( r_{2n} \) we report two errors: the first one is due to the error on \( C_2^{+n} \), while the second one expresses the uncertainty due to the error on all \( C_2^{+k} \), \( k < n \). Note that the high precision reached in the estimates of \( C_2^{+n} \) is partially lost in the estimates of \( r_{2n} \) because of the cancellations among the various terms in the sums. This effect is particularly important for \( r_{10} \) and \( r_{12} \).

Tables 6 and 7 compare our results with the existing data from other approaches. Our estimates of \( r_{2n} \) perfectly agree with those obtained analyzing HT expansions. On the contrary, the Monte Carlo results for \( r_{2n} \) of Ref. [25] are systematically larger. This could be an indication that the finite-size scaling curves obtained in Ref. [25] are still affected by large scaling corrections.

4 The critical equation of state
Table 6: Estimates of $g_4$. We also report the existing results from the form-factor approach (FF), high-temperature expansions (HT), Monte Carlo simulations (MC), field theory (FT) based on the $\epsilon$-expansion and the fixed-dimension $d = 2$ $g$-expansion, and a method based on a dimensional expansion around $d = 0$ ($d$-expansion).

| Method            | Ref.  | $g_4$     |
|-------------------|-------|-----------|
| TM+CFT            | [this work] | 14.697323(20) |
| FF                | [21]  | 14.6975(1) |
| HT                | [22]  | 14.694(2)  |
| HT                | [23]  | 14.693(4)  |
| HT                | [24]  | 14.700(17) |
| MC                | [21]  | 14.69(2)   |
| MC                | [25]  | 14.7(2)    |
| FT $\epsilon$-expansion | [26] | 14.7(4)   |
| FT $g$-expansion  | [27]  | 15.5(8)    |
| FT $g$-expansion  | [28]  | 15.4(3)    |
| $d$-expansion     | [29]  | 14.88(17)  |

Table 7: Estimates of $r_{2n}$. We also report the existing results from high-temperature expansions (HT), Monte Carlo simulations (MC), and field theory (FT) based on the $\epsilon$-expansion and the fixed-dimension $d = 2$ $g$-expansion.

|       | TM+CFT       | HT | HT | MC | FT $\epsilon$-exp. | FT $g$-exp. |
|-------|--------------|----|----|----|---------------------|-------------|
| $r_6$ | 3.67866(5)   | 3.678(2) | 3.679(8) | 3.93(12) | 3.69(4)             | 3.68         |
| $r_8$ | 26.041(11)   | 26.0(2)  | 28.0(1.6) | 26.0(2)  | 26.4(1.0)           |             |
| $r_{10}$ | 284.5(2.4) | 275(15) |             |         |                     |             |
| $r_{12}$ | 4200(740)   |         |             |         |                     |             |

4.1 General features

The basic result of the RG theory is that asymptotically close to the critical point the equation of state may be written in the scaling form [32]

$$ h = \frac{\partial F}{\partial M} \propto M|M|^{\delta-1}f(x), \quad x \propto t|M|^{-1/\beta}, \quad (4.1) $$

where $f(x)$ is a universal scaling function normalized in such a way that $f(-1) = 0$ and $f(0) = 1$. The value $x = -1$ corresponds to the coexistence curve, and $x = 0$ to the critical point $t = 0$. The function $h(M,t)$, representing the external field in the critical equation of state, satisfies Griffiths’ analyticity, i.e. it is regular at $t = 0$ for $M > 0$ fixed and at $M = 0$ for $t > 0$ fixed. This implies that $f(x)$ is analytic at $x = 0$, and it has a regular expansion...
for large-$x$ of the form
\[ f(x) = x^\gamma \sum_{n=0}^{\infty} f_n x^{-2n\beta}. \] (4.2)

As already mentioned in the introduction, many things are exactly known for the two-dimensional Ising model. However, there is no exact expression for the free energy and for the critical equation of state in the whole $(t, h)$ plane. In Table 8 we report a summary of the known results for the two-dimensional Ising model (there we consider only infinite-volume quantities). Many of them are known exactly, for the others we report their best estimate. The results that have not been derived in this paper have been taken from Refs. [5, 33–36].

In the following we will determine the equation of state, starting from its expansion for small magnetization in the HT phase. It is therefore useful to introduce a different representation that is analytic for $M \to 0$. Using the results of Secs. 2 and 3.2, in particular Eq. (3.15) and the discussion following it, one may write the Helmholtz free energy as
\[ F_{\text{sing}}(t, M) = a t^2 V(z) + \frac{A}{2} t^2 \log |t| \] (4.3)

where
\[ z = b |M|^t, \quad V(z) = \frac{z^2}{2} + \frac{z^4}{4!} + O(z^6), \] (4.4)
\[ a = -(C^+)/C_4^+, \quad b = \left[ -\frac{C_4^+}{(C^+)^3} \right]^{1/2}. \]

The constant $A$ is related to the amplitudes of the specific heat for $h \to 0$ defined in Eq. (A.1). Indeed the analyticity of the free energy for $t = 0, h \neq 0$ implies $A^+ = A^- \equiv A$, and thus $U_0 \equiv A^+/A^- = 1$. The presence of the logarithmic term gives rise to logarithms in the expansions of $V(z)$ for $z \to \infty$. Indeed, the analyticity of $F_{\text{sing}}(t, M)$ for $t = 0, |M| \neq 0$ implies, for large $z$,
\[ V(z) = z^{16} \sum_{k=0} c_k z^{-8k} + c_{\log} \log z, \] (4.5)
The constant $c_{\log}$ is easily expressed in terms of invariant amplitude ratios:
\[ c_{\log} = \frac{4A}{a} = 4Q^+ g_4^+. \] (4.6)

For the equation of state we have
\[ h = \frac{\partial F}{\partial M} = a b t^{15/8} \frac{\partial V(z)}{\partial z} \equiv a b t^{15/8} B(z) \] (4.7)

where, using Eq. (2.8),
\[ B(z) = z + \frac{1}{6} z^3 + \sum_{j=3} \frac{T_{2j}}{(2j-1)!} z^{2j-1}. \] (4.8)

For large $z$, using (4.5) we obtain the expansion
\[ B(z) = z^{15} \sum_{k=0} B_k^\infty z^{-8k}. \] (4.9)
Table 8: Critical exponents and universal amplitude ratios for the two-dimensional Ising universality class. See App.  \[\] for the definitions.
The constant $B_0^\infty$ can be expressed in terms of invariant amplitude ratios:

$$B_0^\infty = R_\chi(R_4^+)^{(1-\delta)/2} = 0.592357(6) \times 10^{-5}, \quad (4.10)$$

where we have used the numerical results of Table 8. Moreover, by using Eq. (4.13), we obtain

$$B_2^\infty = c_{\log} = 4Q^+g_4 = 9.349087(13). \quad (4.11)$$

To reach the coexistence curve, corresponding to $t < 0$ and $h = 0$, one should perform an analytic continuation in the complex $t$-plane [3,7]. The spontaneous magnetization is related to the complex zero $z_0 = |z_0|e^{-i\pi\beta}$ of $B(z)$ [7], where

$$|z_0|^2 = R_1^+ \equiv -C_4^+B^2/(C_2^+)^3 = 7.336774(10). \quad (4.12)$$

Therefore, the description of the coexistence curve is related to the behaviour of $B(z)$ in the neighbourhood of $z_0$.

The functions $B(z)$ and $f(x)$ give equivalent representations of the equation of state. Indeed, they are simply related by

$$z^{-\delta}B(z) = B_0^\infty f(x), \quad z = |z_0|x^{-\beta}. \quad (4.13)$$

### 4.2 Parametric representations

In order to obtain a representation of the critical equation of state that is valid in the whole critical region, we need to extend analytically the expansion (4.8) to the low-temperature region $t < 0$. For this purpose, one may use parametric representations, which implement in a simple way all scaling and analytic properties [8–10]. One may parametrize $M$ and $t$ in terms of $R$ and $\theta$ according to

$$\begin{align*}
M &= m_0 R^\delta \theta, \\
t &= R(1-\theta^2), \\
h &= h_0 R^{\delta h}(\theta),
\end{align*} \quad (4.14)$$

where $h_0$ and $m_0$ are normalization constants. The variable $R$ is nonnegative and measures the distance from the critical point in the $(t, h)$ plane; the critical behaviour is obtained for $R \to 0$. The variable $\theta$ parametrizes the displacements along the lines of constant $R$. The line $\theta = 0$ corresponds to the HT phase $t > 0$ and $h = 0$, the line $\theta = 1$ to the critical isotherm $t = 0$, and $\theta = \theta_0$, where $\theta_0$ is the smallest positive zero of $h(\theta)$ to the coexistence curve $T < T_c$ and $h \to 0$. Of course, one should have $\theta_0 > 1$. The regularity properties of the critical equation of state require $h(\theta)$ to be analytic in the physical domain $0 \leq \theta < \theta_0$. This is at variance with what happens for the scaling functions $f(x)$ and $B(z)$, that are nonanalytic for $x \to \infty$ and $z \to \infty$ respectively. This fact is important from a practical point of view. Indeed, in order to obtain approximate expressions of the equation of state, one can approximate $h(\theta)$ with analytic functions. The structure of the parametric representation automatically ensures the correct analytic properties of the equation of state.

Note that the mapping (4.14) is invertible only in the region $\theta < \theta_i$ where

$$\theta_i^2 = \frac{1}{1-2\beta} = \frac{4}{3}. \quad (4.15)$$
Thus, the physically relevant interval $0 \leq \theta \leq \theta_0$ must be contained in the region $\theta < \theta_l$, and thus we should have $\theta_0 < \theta_l$. In practice, since $\theta_l$ is a singular point of the mapping, it is important that $\theta_l - \theta_0$ is not too small. As we shall see, all our approximations satisfy this condition.

The function $h(\theta)$ is odd in $\theta$, and is normalized so that $h(\theta) = \theta + O(\theta^3)$. Since $M = C^+_t h^{-\gamma}$ for $M \to 0, t > 0$, this condition implies $m_0 = C^+_t h_0$. Following Ref. [4], we then replace $h_0$ by a single normalization constant $\rho$ in such a way that we can write

$$z = \rho \theta \left(1 - \theta^2\right)^{-\beta},$$

$$B(z(\theta)) = \rho \left(1 - \theta^2\right)^{-\beta\delta} h(\theta).$$

In the exact parametric equation the value of $\rho$ may be chosen arbitrarily: clearly the physical function $B(z)$ does not depend on it. However, if we adopt an approximation for $h(\theta)$, as we will do, the dependence of $B(z)$ on $\rho$ is not eliminated. One may then choose $\rho$ to obtain an optimal approximation.

From the function $h(\theta)$ one may calculate the scaling functions $f(x)$, using the relations

$$x = \frac{1 - \theta^2}{\theta_0^2 - 1} \left(\frac{\theta}{\theta_0}\right)^{-1/\beta},$$

$$f(x) = \theta^{-\delta} \frac{h(\theta)}{h(1)},$$

and all universal amplitude ratios involving zero-momentum quantities, such as the $n$-point susceptibilities (see, e.g., Ref. [11] for a list of formulae).

### 4.3 Polynomial approximations for $h(\theta)$

In order to construct approximate parametric representations, we consider a systematic approximation scheme based on polynomial approximations of $h(\theta)$ [4], i.e.

$$h^{(k)}(\rho, \theta) = \theta + \sum_{i=1}^{k-1} h_{2i+1}(\rho) \theta^{2i+1}. \quad (4.19)$$

The coefficients $h_{2i+1}(\rho)$ are functions of $\rho, \gamma, \beta$, and are obtained by matching the small-$z$ expansion of $B(z)$ to $O(z^{2k-1})$, cf. Eq. (4.17). This kind of approximation turned out to be effective for the determination of the critical equation of state of three-dimensional Ising-like systems [3] [11], and was generalized to models with Goldstone singularities [37]. In order to optimize $\rho$ for a given truncation $h^{(k)}(\rho, \theta)$, we use a procedure based on the physical requirement of minimal dependence on $\rho$ of the resulting universal function

$$B^{(k)}(\rho, z) \equiv \rho \left(1 - \theta^2\right)^{-\beta\delta} h^{(k)}(\rho, \theta).$$

One may indeed prove [11] that for any truncation $k$ there exists a solution $\rho_k$ independent of $z$ that satisfies a global stationarity condition, i.e.

$$\frac{\partial B^{(k)}(\rho, z)}{\partial \rho} \bigg|_{\rho=\rho_k} = 0,$$
Table 9: Universal amplitude ratios obtained by taking different approximations of the parametric function \( h(\theta) \). The reported “errors” are only related to the uncertainty of the corresponding input parameters. Numbers marked with an asterisk are inputs, not predictions.

As input parameters we use the coefficients of the small-\( z \) expansion of \( B(z) \), i.e. the estimates of coefficients \( r_{2i} \) obtained by TM+CFT and reported in Table 7.

In Table 9 we report the universal amplitude ratios derived from truncations corresponding to \( k = 3, 4, 5 \). We will not report the results for \( k = 6 \) obtained using the available estimate of \( r_{12} \), because the relatively large uncertainty on \( r_{12} \) induces a very large error in the results of the \( k = 6 \) truncation. We only mention that results with \( k = 6 \) are perfectly consistent with those obtained from the \( k = 5 \) truncation. This can be inferred from the fact that the estimate of \( r_{12} \) obtained using \( h(5)(\theta) \), \( r_{12} \simeq 4215 \), is very close to the central value of the TM+CFT estimate, i.e. \( r_{12} = 4200(740) \). As we shall see, we will obtain a much better estimate of \( r_{12} \) in Sec. 4.4.

The results of Table 9 are not stable as \( k \) increases, showing a systematic drift up to \( k = 6 \), where the large uncertainty does not allow a meaningful comparison. We observe that the results for the universal amplitude ratios \( B^\infty_0, R^+_1, R_\chi, U_2 \) and \( v_3 \) effectively converge towards their precise estimates reported in Table 8. It is also reassuring that the difference between the exact value and the estimate obtained using \( h(5) \) is of the order of the variation of the estimates with changing \( k \).

### 4.4 Improved approximations from constrained polynomials

Although the polynomial approximations we presented in the previous Section are substantially consistent with the known results for the amplitude ratios, they do not provide an accurate approximation of the equation of state. The convergence appears rather slow, probably requiring the knowledge of a considerably larger number of coefficients \( r_{2j} \) to substantially improve the results. Here, we will present an improved approximation scheme that
Table 10: Polynomial approximations of $h(\theta)$ using the global stationarity condition for various values of the truncation parameter $k$, cf. Eq. (4.23). The reported expressions correspond to the central values of the input parameters.

| $k$ | $\rho_k$ | $\theta_0^2$ | $\bar{h}^{(k)}(\theta)/[\theta(1 - \theta^2/\theta_0^2)]$ |
|-----|-----------|--------------|-------------------------------------------------|
| 2   | 2.01116   | 1.15278      | $1 - 0.208408\theta^2$                         |
| 3   | 2.00770   | 1.15940      | $1 - 0.215675\theta^2 - 0.039403\theta^4$      |
| 4   | 2.00770   | 1.16441      | $1 - 0.219388\theta^2 - 0.041791\theta^4 - 0.013488\theta^6$ |
| 5   | 2.00881   | 1.16951      | $1 - 0.222389\theta^2 - 0.043547\theta^4 - 0.014809\theta^6 - 0.007168\theta^8$ |

significantly increases the precision of the results.

The approximation scheme can be improved by constructing constrained polynomial approximations of $h(\theta)$ that take into account the large-$z$ asymptotic behaviour of $B(z)$:

$$B(z) = B_0^\infty z^\delta \left[1 + O(z^{-1/\beta})\right],$$  
(4.22)

where the value of $B_0^\infty$ is reported in Eq. (4.10). We consider constrained polynomial approximations of the form

$$\bar{h}^{(k)}(\rho, \theta) = \theta + \sum_{i=1}^{k-1} \bar{h}_{2i+1}(\rho)\theta^{2i+1} + \bar{h}_{2k+1}(\rho)\theta^{2k+1},$$  
(4.23)

where the coefficients $\bar{h}_{2i+1}(\rho)$ with $i < k$ are determined as before, by matching the small-$z$ expansion of $B(z)$ to $O(z^{2k-1})$, while $\bar{h}_{2k+1}(\rho)$ is fixed by requiring that

$$B_0^\infty = \rho^{1-\delta}\bar{h}^{(k)}(\rho, 1) = 0.592357(6) \times 10^{-5}. \quad (4.24)$$

It follows

$$\bar{h}_{2k+1}(\rho) = \rho^{\delta-1}B_0^\infty - 1 - \sum_{i=1}^{k-1} \bar{h}_{2i+1}(\rho). \quad (4.25)$$

In this approximation scheme the free parameter $\rho$ can be still determined by requiring the global stationarity condition (4.22). This nontrivial property is essentially due to the fact that the constraint (4.24) is linear in the coefficients $\bar{h}_{2i+1}$. It can be proved by using arguments similar to those employed in App. C of Ref. [11] to show the global stationarity condition (4.21) for the approximation scheme (4.19). In Table 10, for $k = 2, 3, 4, 5$, we report the polynomials $\bar{h}^{(k)}(\theta)$ obtained by using the global stationarity condition to fix $\rho$, and the central values of the input parameters $F_0^\infty, \rho_6, \rho_8, \rho_{10}$. Note the stability of the coefficients of the polynomials with $k$ and that the size of the coefficients decreases with the order of the polynomial. The results for some universal quantities are reported in Table 11. They are in much better agreement with the exact results than those obtained without constraint.

In Fig. 1 we show the scaling function $B(z)$ obtained from $\bar{h}^{(k)}(\rho, \theta)$ for $k = 2, 3, 4, 5$. The convergence is good, indeed their differences are not visible in Fig. 1. This allows us to determine $B(z)$ for all real $z > 0$ with a relative precision of at least a few per mille (the least precision is found around $z \simeq |z_0| \simeq 2.71$). This fact is not trivial since the small-$z$ expansion has a finite convergence radius which is expected to be $|z_0| = (R_4^+)^{1/2} \simeq 2.71$. Therefore, the
Table 11: Universal amplitude ratios obtained from the constrained polynomial approximations (4.23) of the parametric function \( h(\theta) \). The reported “errors” are only related to the uncertainty of the corresponding input parameters (they are reported only if they are larger than the last figure shown). Numbers marked with an asterisk are inputs, not predictions.

determination of \( B(z) \) on the whole positive real axis from its small-\( z \) expansion requires an analytic continuation, which is effectively performed by the approximate parametric representations we have considered. We recall that the large-\( z \) limit corresponds to the critical isotherm \( t = 0 \), so that positive real values of \( z \) describe the HT phase up to \( t = 0 \). Note also the good agreement of the results for \( B_{\infty}^2 \) (see Table 11), i.e. the next-next-to-leading coefficient of the large-\( z \) expansion of \( B(z) \), with the precise estimate given in Eq. (4.11).

The convergence of the polynomial representations at the coexistence curve is slower. This can be seen by looking at the estimates reported in Table 11 for universal amplitude ratios involving quantities related to the coexistence curve, such as \( R^+_4 \), \( R^+_\chi \), and \( U_2 \), and comparing them with the corresponding known results reported in Tables 8. They appear to (monotonically) converge toward the correct results. The rate of convergence worsens when quantities with more and more derivatives with respect to \( h \) are involved in the amplitude ratio, as it can be already seen by comparing the results for \( R^+_4 \) and \( U_2 \).

Fig. 2 shows the scaling function \( f(x) \) as obtained by the truncations \( k = 2, 3, 4, 5 \). The accuracy of the determination of \( f(x) \) can be inferred from the convergence of the curves with increasing \( k \) and the comparison with the known behavior for \( x \to -1 \) and \( x \to +\infty \).

### Table 11: Universal amplitude ratios obtained from the constrained polynomial approximations (4.23) of the parametric function \( h(\theta) \). The reported “errors” are only related to the uncertainty of the corresponding input parameters (they are reported only if they are larger than the last figure shown). Numbers marked with an asterisk are inputs, not predictions.

|          | \( \tilde{h}^{(2)} \) | \( \tilde{h}^{(3)} \) | \( \tilde{h}^{(4)} \) | \( \tilde{h}^{(5)} \) |
|----------|----------------------|----------------------|----------------------|----------------------|
| \( \rho \) | 2.011                | 2.008                | 2.008                | 2.009(1)             |
| \( \theta_1^2 - \theta_0^2 \) | 0.181                | 0.174                | 0.169                | 0.164(2)             |
| \( r_6 \)  | 3.929                | *3.67866(5)          | *3.67866(5)          | *3.67866(5)          |
| \( r_8 \)  | 27.585               | 26.932               | *26.041(11)          | *26.041(11)          |
| \( r_{10} \) | 297.25               | 292.89               | 292.89               | *284.5(2.4)          |
| \( r_{12} \) | 4425.2               | 4385.6               | 4385.6               | 4443(16)             |
| \( r_{14} \) | 84387                | 84029                | 84029                | 84305(79)            |
| \( B(|z_0|/2) \) | 1.9798               | 1.9690               | 1.9675               | 1.9672               |
| \( B(|z_0|) \)  | 44.930               | 44.335               | 44.146(2)            | 44.05(3)             |
| \( B(3|z_0|/2) \) | 8442.2               | 8432.7               | 8429.4               | 8427.7(3)            |
| \( B(2|z_0|) \)  | 604619(6)            | 604548(6)            | 604524(6)            | 604511(7)            |
| \( B(3|z_0|) \)  | 2.63497(3)\times10^8 | 2.63496(3)\times10^8 | 2.63495(3)\times10^8 | 2.63495(3)\times10^8 |
| \( B_{\infty}^\infty \) | *0.592357(6)\times10^{-5} | *0.592357(6)\times10^{-5} | *0.592357(6)\times10^{-5} | *0.592357(6)\times10^{-5} |
| \( B_{\infty}^1 \)  | 0.021893             | 0.021375             | 0.021198(3)          | 0.02110(3)           |
| \( B_{\infty}^2 \)  | 9.3987               | 9.2611               | 9.2611               | 9.286(7)             |
| \( R^+_4 \)   | 7.458                | 7.396                | 7.371                | 7.355(5)             |
| \( R^+\chi \) | 7.602                | 7.172                | 7.002(2)             | 6.90(3)              |
| \( U_2 \)    | 45.918               | 42.358               | 40.76(2)             | 39.6(3)              |
| \( v_3 \)    | 28.328               | 29.201               | 29.837(9)            | 30.5(2)              |
Figure 1: The scaling function $B(z)$ versus $z$. We report the curves obtained from the constrained polynomial approximations (4.23) with $k = 2, 3, 4, 5$. Their differences are not visible.

and for $x \to +\infty$

$$f(x) = f_0^\infty x^\gamma + O \left( x^{\gamma-2\beta} \right),$$

$$f_0^\infty = R_\chi^{-1} = 0.14752994...$$

This shows that $f(x)$ is determined with a precision of a few per cent in the whole region. This happens also in the large-$x$ region, which corresponds to the HT phase, and therefore to $z \ll 1$ in $F(z)$, essentially because $f(x)$ is normalized at the coexistence curve, i.e. $x = -1$, where our approximation is worse. For $x > 0$, the error on $f(x)$ increases from 0 to 2%, the relative error on $R_\chi$.

Using the results of Table I we also obtain

$$r_{12} = 4.44(6) \times 10^3,$$

$$r_{14} = 8.43(3) \times 10^5,$$

$$B_1^\infty = 0.0211(2).$$

Note that the above-reported estimate of $r_{12}$ is perfectly consistent with the result obtained by TM+CFT, i.e. $r_{12} = 4.20(74) \times 10^3$, but much more precise.
constrained polynomial approximations (4.23) with $k = 2, 3, 4, 5$.

5 Conclusions

Let us briefly summarize the main results of this paper, which presents a determination of the equation of state of the two-dimensional Ising model in the whole $(t, h)$ plane. The starting point of the analysis was the determination of the $2n$-point couplings constants in the HT phase which are related to the expansion of the free energy in powers of the magnetization. These quantities were then used as input parameters for a systematic approximation scheme, which allowed us to obtain an accurate determination of the equation of state in the critical regime in the whole $(t, h)$ plane.

The determination of the coefficients $r_{2j}$ of the small-magnetization expansion of the Helmholtz free energy is in itself an interesting and nontrivial problem. We addressed this problem by using TM techniques. In order to overcome the typical problem of all TM calculations, i.e. the fact that they are constrained to rather small values of the lattice size in the transverse direction (in our case we could reach $L_1 = 24$ as maximum value) we used a two-step strategy:

1) In the first step we studied the model at very small values of $\beta$ so as to have small values for $\xi$ (whose maximum value was $\xi = 3.350288..$, which was reached for the largest value of $\beta$ that we studied: $\beta = 0.37$) and thus large values of the ratio $L_1/\xi$. We then extrapolated to the thermodynamic limit the TM results by using a new effective iterative algorithm. This algorithm is rather interesting in itself and could be of general utility for people working with TM methods.
Second, we used a combination of standard RG results and a set of high-precision numerical informations on the zero-field magnetic susceptibility to construct the scaling functions for the derivatives of the free energy involved in the construction of the $r_{2j}$ coefficients. With these scaling functions we could obtain the critical-limit values of the $r_{2j}$ coefficients with rather small errors even if the input data for this continuum-limit extrapolation correspond to rather small values of $\beta$.

The expansion of the free energy in powers of the magnetization in the HT phase was then used as a starting point to construct approximate representations of the equation of state that are valid in the critical regime of the whole $(t, h)$ plane. We considered a systematic approximation scheme based on polynomial parametric representations, devised to match the known terms of the small-magnetization expansion and the large-magnetization behaviour of the Helmholtz free energy. A global stationarity condition was used in order to optimize the polynomial approximation. This approximation scheme can be improved systematically by considering higher and higher order polynomials. It is only limited by the number of known terms in the small-magnetization expansion of the free energy. The knowledge of this expansion up to 10th order allowed us to obtain an accurate determination of the critical equation of state. We indeed obtained the scaling function $B(z)$, cf. Eq. (4.7), for all real $z > 0$ with a relative precision of at least a few per mille, and the scaling function $f(x)$, cf. Eq. (4.1), with a precision of a few per cent in the whole physical region $x \geq -1$. The approximation scheme is systematic, thus this precision can be improved by a more accurate knowledge of the small-magnetization expansion of the free energy in the HT phase.

The method that we used to reconstruct the equation of state from the small-magnetization expansion of the Helmholtz free energy is general and can be applied to other statistical models. We mention that similar methods have been successfully applied to the three-dimensional Ising and XY universality classes, leading to accurate determinations of the critical equation of state and of the universal ratios of amplitudes that can be extracted from it.

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A Notations for the amplitudes

Universal amplitude ratios characterize the critical behaviour of thermodynamic quantities that do not depend on the normalizations of the external (e.g. magnetic) field, order parameter (e.g. magnetization) and temperature. Amplitude ratios of zero-momentum quantities can be derived from the critical equation of state. Beside the amplitudes $C_n^+$ of the $n$-point functions $\chi_n(t)$ defined in Eq. (3.1) and the corresponding ones $C_n^-$ in the low-temperature phase, we consider amplitudes derived from the singular behaviour of the specific heat

$$C_H = A^\pm \ln(1/|t|),$$

(A.1)

the spontaneous magnetization on the coexistence curve

$$M = B(-t)^\beta.$$ (A.2)

We complete our list of amplitudes by considering the second-moment correlation length

$$\xi_{2nd} = f^{\pm} |t|^{-\nu},$$ (A.3)

and the true (on-shell) correlation length, describing the large-distance behaviour of the two-point function,

$$\xi_{gap} = f_{gap} |t|^{-\nu}.$$ (A.4)

One can also define amplitudes along the critical isotherm, e.g.,

$$\chi = C^c |h|^{-\gamma/\beta\delta},$$ (A.5)

$$\xi = f^c |h|^{-\nu/\beta\delta},$$ (A.6)

$$\xi_{gap} = f_{gap}^c |h|^{-\nu/\beta\delta}.$$ (A.7)

B Finite-size scaling of the free energy

Before starting the discussion on the FSS in the Ising model, let us stress that the analysis reported in this appendix is only a straightforward application of results which are already well-known in the literature. They can be found for instance in [39] and are based on the results of [40] and on the exact solution of the Ising model on a finite lattice obtained by Kaufman [12]. We decided to write it down in this appendix all the same so as to make this paper as self-contained as possible. We shall use the same notations as [39] to simplify the comparison. This means in particular that we shall use the letter $R$ to denote the finite size of the lattice in the transverse direction, which is denoted in the rest of the paper with $L_1$.

The aim of this appendix is to obtain the explicit form of the FSS of the free energy of the Ising model on a rectangular lattice of size $R \times \infty$, which is exactly the geometry we considered in our TM work. To obtain this result we shall work in the framework of the S-matrix approach to 2d integrable models and shall in particular use the so-called Thermodynamic Bethe Ansatz (TBA). In the TBA one looks at the theory defined on an infinitely long cylinder with circumference $R$. The only free parameter of the theory is $r \equiv mR$ where $m$ is the lowest mass of the model. The goal is to extract the behaviour of the free energy $E_0(R)$ and of the lowest mass $m(R)$ (and possibly, in models more complicated...
than the Ising one, of higher massive states) as a function of $R$. The geometry of the TBA is exactly the one which we have in our TM setting. The parameter $r$ is the ratio $L/\xi$ in our notations and the only thing that we must require to compare our findings with the TBA analysis is that we should be in the “scaling region,” i.e. that $\xi \gg 1$ so that we may neglect the lattice artifacts.

The TBA results depend on the entries of the S-matrix. They are in general very complicated. Thus it is usually impossible to find the exact FSS functions for any value of $r$ (one has usually to resort to perturbative expansions for small or large values of $r$). However, the thermal perturbation of the Ising model is so simple (the S matrix is simply $S = -1$ as a consequence of the fact that the model can be mapped to a free massive field theory) that in this case the explicit expression for any value of $r$ can be obtained.

**B.1 Free energy at $h = 0$**

The TBA prediction for the FSS behaviour of the free energy, in the HT phase of the 2d Ising model at $h = 0$, is

$$F(R) = F(\infty) - \frac{\pi c(r)}{6R^2}$$

(B.1)

where, following the standard TBA machinery (see p. 668 of Ref. [39]), $c(r)$ is given by

$$c(r) = \frac{6}{\pi^2} \int_0^\infty d\theta \, r \cosh(\theta) \ln(1 + e^{-r \cosh \theta}).$$

(B.2)

The integral can be exactly evaluated and gives:

$$c(r) = \frac{6r}{\pi^2} \sum_{k=1}^\infty \frac{(-1)^{k-1}}{k} K_1(k r),$$

(B.3)

where $K_1$ is the modified Bessel function of the second kind.

The normalization of Eq. (B.1) could seem strange, but it is chosen such that in the $r \to 0$ limit (i.e. at the critical point) the function $c(r)$ exactly becomes the central charge of the Ising model $c = 1/2$. The idea behind this choice is the so called $c$-theorem of Zamolodchikov which states that the FSS of any 2d integrable model can be parametrized by Eq.(B.1), where $c(r)$ is a “running central charge,” which in general interpolates between two critical points and, in this case, between the central charge of the Ising model and the value $c(\infty) = 0$ which is the appropriate one for a massive theory.

The limit in which we are interested is $r \gg 1$. In this limit, we can use the asymptotic expansion for the Bessel functions and Eqs. (B.1), (B.3) become:

$$F(R) = F(\infty) - \sqrt{\frac{m}{2\pi R^3}} \sum_{k=1}^\infty e^{-mkR} \frac{(-1)^{k-1}}{k^{1/2} k} (1 + \cdots)$$

(B.4)

where the dots stand for the $o(1)$ terms.

---

3Notice that in [39] the authors study, instead of the free energy $F(R)$, the quantity $E_0(R)$ that is related to $F(R)$ by $E_0(R) = R[F(R) - F(\infty)]$. 

26
B.2 Derivatives of the free energy at $h = 0$

Since we are interested in the derivatives with respect to $h$ of the free energy, we must try to extend Eq. (B.4) in the $h \neq 0$ plane. It is quite reasonable to assume that in (B.4) the $h$ dependence can only be hidden in the mass term. Moreover, we know from Refs. [41, 42] that this dependence (for small values of $h$) is well described by an analytic even function of $h$, i.e.

$$m(h) = m(1 + a \ h^2 + b \ h^4 + \ldots),$$  \hspace{1cm} (B.5)

with $a$ and $b$ unknown constants.

Inserting this expression in Eq. (B.4) and performing the derivatives we find

$$\chi_{2n}(R) - \chi_{2n}(\infty) = R^{2n-3/2} (g_{2n,1}(1/R)e^{-mR} + g_{2n,2}(1/R)e^{-2mR} + g_{2n,3}(1/R)e^{-3mR} + \ldots)$$  \hspace{1cm} (B.6)

where the $g_{2n,i}(1/R)$ may be expanded in positive powers of $1/R$ (whose coefficients could be in principle computed as functions of $a$, $b$, and $m$). This is the expression quoted in the text as Eq. (3.13).

B.3 A test of the iterative algorithm for the infinite-volume extrapolation

In order to further check the IA (3.14) for the infinite-size extrapolation, we apply it to a test function of the type (B.6). We consider the function

$$E(\xi, R) = 1 + \frac{\pi}{6R^2} \frac{\partial^4 c(r)}{\partial h^4} \bigg|_{h=0},$$  \hspace{1cm} (B.7)

where $c(r)$ is the function defined in Eq. (B.3) and

$$r = \frac{R}{\xi} \left(1 + h^2 + h^4 \right).$$  \hspace{1cm} (B.8)

Clearly, for $R \to \infty$, $E(\xi, \infty) = 1$. We compute $E(\xi, R)$ for finite values of $R$ in the typical range of our TM calculations, i.e. $3 \lesssim R/\xi \lesssim 7$, and apply the IA (3.14) to determine the $E(\xi, \infty)$. The comparison with the exact value gives an idea of the effectiveness of the method. Table 12 shows the results of the IA (3.14) for $\xi = 4$, analogously to Table 3 for $F_4$. We report the results for various iteration levels (up to four); the level zero corresponds to the original data. In order to obtain from Table 12 an estimate $E(\xi, \infty)$, we follow the same strategy used in Sec. 3.1.2, i.e. we consider the largest value of $R$ and four iterations of the IA. The error is estimated from the residual dependence on $R$ of the results obtained with four iterations of the algorithm. We obtain $E(\xi = 4, \infty) = 1.000000(1)$, which is in perfect agreement with the exact number. Analogous results are obtained for other values of $\xi$. This confirms the effectiveness of the procedure we used to perform the extrapolation to the thermodynamic limit, and the reliability of the uncertainty we considered.
\[ F(0, h) = f_b + A_{f,b} |h|^{16} \left( 1 + A_{f,1} |h|^{14} + A_{f,2} |h|^{12} + A_{f,3} |h|^{10} + \cdots \right). \] (C.1)

Each of these amplitudes has a precise physical meaning. Let us look at them in detail:

- \( f_b \) denotes the bulk contribution to the free energy and can be obtained from the exact solution of the Ising model on the lattice at the critical point. Explicitly \( f_b = \frac{2}{\pi}G + \frac{1}{\pi} \log 2 \), where \( G \) is Catalan’s constant.

- \( A_{f,b} \) is the amplitude of the singular part of the free energy. It can be evaluated exactly in the framework of the S-matrix approach to the model. Its value is \( A_{f,b} = 0.9927995\ldots \).

- \( A_{f,1} \) is due to the irrelevant operators in the Hamiltonian. This amplitude turns out to be compatible with zero.

- \( A_{f,2} \) is due to the \( b_i h^2 \) term in the Hamiltonian. This amplitude is compatible with zero.

Table 12: Results of the IA \((3.14)\) applied to the test function \( E(\xi = 4, R) \), cf. Eq. \((B.7)\).

| \( R \) | \( E^{(0)} \) | \( E^{(1)} \) | \( E^{(2)} \) | \( E^{(3)} \) | \( E^{(4)} \) |
|-------|-------|-------|-------|-------|-------|
| 12    | 1.002919604 |       |       |       |       |
| 13    | 1.003825616 |       |       |       |       |
| 14    | 1.004103404 | 1.0042408301 |       |       |       |
| 15    | 1.0040272017 |       | 1.0040304386 |       |       |
| 16    | 1.0040272017 |       | 1.0040304386 |       |       |
| 17    | 1.0040272017 |       | 1.0040304386 | 1.0040304386 |       |
| 18    | 1.0040272017 | 1.0040304386 | 1.0040304386 | 1.0040304386 |       |
| 19    | 1.0040272017 | 1.0040304386 | 1.0040304386 | 1.0040304386 | 1.0040304386 |
| 20    | 1.0040272017 | 1.0040304386 | 1.0040304386 | 1.0040304386 | 1.0040304386 |
| 21    | 1.0040272017 | 1.0040304386 | 1.0040304386 | 1.0040304386 | 1.0040304386 |
| 22    | 1.0040272017 | 1.0040304386 | 1.0040304386 | 1.0040304386 | 1.0040304386 |
| 23    | 1.0040272017 | 1.0040304386 | 1.0040304386 | 1.0040304386 | 1.0040304386 |
| 24    | 1.0040272017 | 1.0040304386 | 1.0040304386 | 1.0040304386 | 1.0040304386 |
| 25    | 1.0040272017 | 1.0040304386 | 1.0040304386 | 1.0040304386 | 1.0040304386 |
| 26    | 1.0040272017 | 1.0040304386 | 1.0040304386 | 1.0040304386 | 1.0040304386 |
| 27    | 1.0040272017 | 1.0040304386 | 1.0040304386 | 1.0040304386 | 1.0040304386 |
| 28    | 1.0040272017 | 1.0040304386 | 1.0040304386 | 1.0040304386 | 1.0040304386 |
$A_{f,3}^l$ is due to the $e_h h^2$ term in $u_h$.

In principle one could use the TM data to estimate all these constants. In practice this procedure works only for the first unknown amplitude. All the higher ones are then "shadowed" by the first.

The problem with $e_h$ is that it appears at a rather high level. In [4] we were able to fix exactly the amplitudes only up to $A_{f,1}^l$. The first unknown one was $A_{f,2}^l$ which was then estimated numerically, obtaining

$$0.020 < A_{f,2}^l < 0.022.$$  \hfill (C.2)

It was impossible to give any reliable estimate for $A_{f,3}^l$. The main progress of the present paper with respect to that analysis is that, thanks to the exact calculation of $b_t$ performed in [5, 14] we are now in the position to estimate exactly also $A_{f,2}^l$. A direct calculation gives

$$A_{f,2}^l = \frac{A_E^l \pi E_0}{A_f^l 8 \beta_c}, \hfill (C.3)$$

where $A_f^l$ was defined above, $E_0$ is given by Eq. (3.21), and $A_E^l$ is defined by the singular behaviour of the internal energy, i.e. $E_{\text{int}}(t = 0, h) = \frac{1}{2} \partial F/\partial \beta = E_{\text{bulk}} + A_E^l h^{8/15} + \ldots$

Numerically, $A_E^l = 0.58051\ldots$ [4, 14]. Substituting in Eq. (C.3), we find

$$A_{f,2}^l = 0.0210115\ldots \hfill (C.4)$$

in perfect agreement with the estimate of [4].

Substituting in Eq. (C.4), we may now estimate numerically the amplitude $A_{f,3}^l$, which is related to $e_h$ by:

$$A_{f,3}^l = \frac{16}{15} e_h. \hfill (C.5)$$

A standard application of the fitting procedure discussed in [4], using as input data those reported in Table 10 of [4] gives

$$e_h = -0.00727(15). \hfill (C.6)$$
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