Research Article

Hartman-Type and Lyapunov-Type Inequalities for a Fractional Differential Equation with Fractional Boundary Conditions

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We prove Hartman-type and Lyapunov-type inequalities for a class of Riemann–Liouville fractional boundary value problems with fractional boundary conditions. Some applications including a lower bound for the corresponding eigenvalue problem are obtained.

1. Introduction

In [1], Lyapunov established the following striking inequality:

**Theorem 1.** Let $q \in C([a,b], \mathbb{R})$. Assume that the problem
\[
\begin{aligned}
\omega'' + q(x)\omega &= 0, & x \in (a,b), \\
\omega(a) &= \omega(b) = 0,
\end{aligned}
\]
has a solution $\omega \in C([a,b], \mathbb{R})$ such that $\omega(x) \neq 0$ for $x \in (a,b)$. Then,
\[
(b-a) \int_a^b |q(z)|dz > 4,
\]
and constant 4 is the best possible largest number.

It has been shown that this result serves as a good tool in the study of several properties of solutions of differential equations (such as eigenvalue problems and eigenvalue inequalities) (see, for example, [2–5] and the references therein). Many authors have worked on generalizations of classical inequalities (see, for instance, [4–16] and the references therein).

In [11], the authors use the Hahn integral operator to prove a description of new generalization of Minkowski’s inequality.

In [5], the authors improve inequality in (2) by proving the following Hartman–Winter inequality:
\[
\int_a^b (b-z)(z-a)q^+(z)dz > b-a,
\]
where $q^+(z) = \max(q(z), 0)$ is the nonnegative part of $q(z)$.

Inequality (3) is also known as the best Lyapunov inequality.

In [17], Ferreira considered the following fractional differential problem:
\[
\begin{aligned}
D_+^\alpha a \omega + q(x)\omega &= 0, & a < x < b, 1 < \alpha \leq 2, \\
\omega(a) &= \omega(b) = 0,
\end{aligned}
\]
where $q \in C([a,b], \mathbb{R})$ and $D_+^\alpha$ denotes the Riemann–Liouville fractional derivative of order $\alpha$ (see Definition 2 in the following).

The author established the following Lyapunov-type inequality for problem (4).

**Theorem 2** (see [17]). Assume that problem (4) has a solution $\omega \in C([a,b], \mathbb{R})$ such that $\omega(x) \neq 0$ for $x \in (a,b)$. Then,
\[
\frac{1}{\Gamma(\alpha)} \int_a^b |q(z)|dz > \left(\frac{4}{b-a}\right)^{\alpha-1}.
\]
Remark 1. Note that if we let \( \alpha = 2 \) in (5), one obtains Lyapunov’s classical inequality (2).

For the convenience of the reader, we recall the concept of fractional integral and derivative of order \( \gamma \geq 0 \).

Definition 1 (see [18, 19]). The Riemann–Liouville fractional integral of order \( \gamma \geq 0 \) for a real-valued function \( \omega \) is defined by 
\[
(I_a^\gamma \omega)(x) = \frac{1}{\Gamma(\gamma)} \int_a^x (x-z)^{\gamma-1} \omega(z)dz, \quad \gamma > 0, x \in [a,b],
\]
where \( \Gamma(\gamma) \) is the Euler gamma function.

Definition 2 (see [18, 19]). The Riemann–Liouville fractional derivative of order \( \gamma \geq 0 \) for function \( \omega \) is defined by 
\[
(D_a^\gamma \omega)(x) = \left( \frac{d}{dx} \right)^n (I_{a+}^{\gamma-n} \omega)(x), \quad \gamma > 0,
\]
where \( n = [\gamma] + 1 \) with \( [\gamma] \) the integer part of \( \gamma \).

The new development in fractional calculus has attracted the attention of researchers of various disciplines. Different mathematical procedures have been considered by several authors through different research-oriented aspects of fractional differential equations (see, for instance, [20–22] and the references therein).

Our goal in this paper is to establish Hartman-type and Lyapunov-type inequalities for the following problem:
\[
\begin{aligned}
D_a^\alpha \omega + q(x)\omega &= 0, \quad x \in (a,b), \\
\omega(a) &= D_a^{\alpha-3} \omega(a) = D_a^{\alpha-2} \omega(a) = \omega''(b) = 0,
\end{aligned}
\]
where \( \alpha \in (3, 4] \) and \( q \in \mathcal{C}([a,b], \mathbb{R}) \). Some applications are given to illustrate our result.

The organization of the paper is as follows. In Section 2, we derive the explicit expression of the Green function corresponding to problem (8) and we establish some properties on it. This allows us to prove Hartman-type and Lyapunov-type inequalities for problem (8). In Section 3, we present some applications including a lower bound for the corresponding eigenvalue problem.

## 2. Main Results

### 2.1. Green’s Function

First, we recall the following well-known properties (see, for example, [18, 19]).

Lemma 1. Let \( \alpha \in (3, 4) \) and \( \omega \in \mathcal{C}((a,b), \mathbb{R}) \cap L^1((a,b)) \). Then,

(i) For \( 0 < \gamma < \alpha \), \( D_a^\gamma (I_a^\gamma \omega) = I_a^{\gamma-\gamma} \omega \) and \( D_a^\gamma (I_a^\gamma \omega) = \omega \).

(ii) Assume that \( D_a^\gamma \omega \in \mathcal{C}((a,b), \mathbb{R}) \cap L^1((a,b)) \), then
\[
(I_a^\gamma (D_a^\gamma \omega))(x) = \omega(x) + \sum_{i=1}^{\alpha-1} c_i (x-a)^{\alpha-i},
\]
where \( c_i \in \mathbb{R} \), for \( i \in \{1, 2, 3, 4\} \).

Lemma 2. Let \( \omega \in \mathcal{C}([a,b]) \) be a solution of problem (8). Then,
\[
\omega(x) = \int_a^b G_a(x,y)q(y)\omega(y)dy,
\]
where \( G_a(x, y) \) is Green’s function of problem (8) given by
\[
G_a(x, y) = \begin{cases} 
\frac{b-y}{b-a} (x-a)^{\alpha-1} - I_a^\alpha (q\omega)(x), & a \leq y \leq x \leq b, \\
\frac{b-y}{b-a} (x-a)^{\alpha-1}, & a \leq x \leq y \leq b.
\end{cases}
\]

Proof. Let \( \omega \) be such solution. By Lemma 1, we have
\[
\omega(x) = \sum_{i=1}^{\alpha-1} c_i (x-a)^{\alpha-i} - I_a^\alpha (q\omega)(x).
\]
Using the fact that \( \omega(a) = D_a^{\alpha-3} \omega(a) = D_a^{\alpha-2} \omega(a) = \omega''(b) = 0 \), we obtain \( c_1 = c_2 = c_4 = 0 \) and \( (b-a)^{\alpha-3} I(a) c_3 = \int_a^b (b-y)^{\alpha-3} q(y) \omega(y)dy \).

Therefore,
\[
\omega(x) = \frac{1}{\Gamma(\alpha)} \int_a^b \left( \frac{b-y}{b-a} \right)^{\alpha-3} (x-a)^{\alpha-1} q(y)\omega(y)dy
\]
\[
- \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} q(y)\omega(y)dy
\]
\[
= \int_a^b G_a(x, y)q(y)\omega(y)dy.
\]
This ends the proof.

To get a quick perspective, in Figure 1, we have the representation of Green’s function $G_{7/2}(x,y)$ with the contours and some projections.

One can see from Figure 1 that Green’s function $G_{7/2}(x,y)\geq 0$ and it is nondecreasing with respect to the first variable. This important observation will be proved for $G_a(x,y)$ with $a \in (3,4]$.

**Definition 3.** Let $f,g:[a,b] \times [a,b] \longrightarrow \mathbb{R}$ with $f,g \geq 0$. We say that

$$f(x,y) \leq g(x,y) \text{ on } [a,b] \times [a,b],$$

(14)

if there exists $c>0$ such that $(1/c)g(x,y) \leq f(x,y) \leq cg(x,y)$ for all $(x,y) \in [a,b] \times [a,b]$.

**Remark 2.** Let $\tau > 0$ and $x,z \in [0,1]$. Then,

$$\min(1,\tau)(1-zx) \leq 1-zx^z \leq \max(1,\tau)(1-zx).$$

(15)

Next, we establish some properties on Green’s function $G_a(x,y)$ given by (11).

**Proposition 1**

(i) On $[a,b] \times [a,b]$,

$$G_a(x,y) \approx (x-a)^{\alpha-2}(b-y)^{\alpha-3}\min(x-a,y-a).$$

(16)

(ii) On $[a,b] \times [a,b]$,

$$\frac{\partial}{\partial x}G_a(x,y) = (x-a)^{\alpha-3}(b-y)^{\alpha-2}\min(x-a,y-a).$$

(17)

(iii) The function $G_a$ satisfies the following property:

$$0 \leq G_a(x,y) \leq G_a(b,y), \quad (x,y) \in [a,b] \times [a,b].$$

(18)

**Proof** (i) From Lemma 2, for $x,y \in (a,b)$, we have

$$G_a(x,y) = \frac{1}{\Gamma(a)} \left( \frac{b-y}{b-a} \right)^{\alpha-3} \left( x-a \right)^{\alpha-1} \left[ 1 - \left( \frac{b-y}{b-a} \right)^2 \left( \frac{(b-a)(x-y)^*}{(x-a)(b-y)} \right)^{\alpha-1} \right].$$

(19)

where $(x-y)^* = \max((x-y),0)$. Now, since $((b-a)(x-y)^*)/((x-a)(b-y)) \in [0,1]$, for $x,y \in (a,b)$, then by using Remark 2, with $\tau = a-1$ and $z = ((b-a)^{2}/((b-a)^2)) \in [0,1]$, we obtain

$$G_a(x,y) \approx (b-y)^{\alpha-2}(b-a)(x-a) - (b-y)(x-y)^*.$$

(20)

Hence, inequalities in (16) follow by observing that

$$(b-a)(x-a) - (b-y)(x-y)^* \approx \min((x-a),(y-a)).$$

(21)

(ii) We have

$$\frac{\partial}{\partial x}G_a(x,y) = \frac{(\alpha-1)}{\Gamma(a)} \left( \frac{b-y}{b-a} \right)^{\alpha-3} (x-a)^{\alpha-2} - (x-y)^{\alpha-2}, \quad a \leq y \leq x \leq b,$$

$$\frac{\partial}{\partial x}G_a(x,y) = \frac{(\alpha-1)}{\Gamma(a)} \left( \frac{b-y}{b-a} \right)^{\alpha-3} (x-a)^{\alpha-2}, \quad a \leq x \leq y \leq b.$$ 

(22)

Similar to case (i), by using the fact that

$$\frac{\partial}{\partial x}G_a(x,y) = \frac{(\alpha-1)}{\Gamma(a)} \left( \frac{b-y}{b-a} \right)^{\alpha-3} (x-a)^{\alpha-2} - \left[ \frac{(b-y)}{(b-a)} \left( \frac{(b-a)(x-y)^*}{(x-a)(b-y)} \right)^{\alpha-2} \right].$$

(23)

and applying Remark 2 with $\tau = \alpha-2$ and $z = (b-y)/(b-a) \in [0,1]$, we obtain the required result.

(iii) Let $y \in [a,b]$. Since the function $x \longrightarrow (\partial/\partial x)G_a(x,y)$ is nondecreasing on $[a,b]$, we deduce that
\(0 = G_a(a, y) \leq G_a(x, y) \leq G_a(b, y)\). This completes the proof.

2.2. Statements and Proofs of Main Results

**Theorem 3** (Hartman–Winter-type inequality)

Let \(q \in C([a, b], \mathbb{R})\). Assume that problem (8) has a solution \(\omega \in C([a, b], \mathbb{R})\) such that \(\omega(x) \neq 0\) for \(x \in (a, b)\). Then,

\[
\int_a^b (b - z)^{a-3} (z - a) (2b - a - z) q^+ (z) \, dz \geq \Gamma'(a),
\]

where \(q^+(z) = \max(q(z), 0)\).

**Proof.** From Lemma 2, we know that

\[
\omega(x) = \int_a^b G_a(x, z) q(z) \omega(z) \, dz, \quad x \in [a, b].
\]

Without loss of generality, we may assume that \(\omega(x) > 0\) for \(x \in (a, b)\).

Using (26), Proposition 1 (iii) and the fact that \(q(z) \leq q^+(z)\), we deduce that

\[
\omega(x) \leq \int_a^b G_a(x, z) q^+(z) \omega(z) \, dz \leq \int_a^b G_a(b, z) q^+(z) \omega(z) \, dz.
\]

Hence,

\[
\|\omega\| \leq \int_a^b G_a(b, z) q^+(z) \|\omega\| \, dz,
\]

or equivalently,

\[
1 \leq \int_a^b G_a(b, z) q^+(z) \, dz.
\]

Therefore,

\[
1 \leq \frac{1}{\Gamma'(a)} \int_a^b (b - z)^{a-3} (z - a) (2b - a - z) q^+(z) \, dz,
\]

from which inequality (25) follows.
Remark 3. Let \( q \in C([a,b], \mathbb{R}) \). Under the same conditions as in Theorem 3, we have
\[
\int_a^b (b - z)^{\alpha - 3}(z - a)(2b - a - z)q(z)dz \geq \Gamma(a). \tag{31}
\]
By applying the previous theorem with \( \alpha = 4 \), we obtain the following:

Corollary 1. Let \( q \in C([a,b], \mathbb{R}) \). Assume that the problems
\[
\begin{cases}
\omega^{(4)}(x) + q(x)\omega = 0, & x \in (a,b), \\
\omega(a) = \omega'(a) = \omega''(a) = \omega''(b) = 0,
\end{cases}
\]
admit a solution \( \omega \in C([a,b], \mathbb{R}) \) such that \( \omega(x) \neq 0 \) for \( x \in (a,b) \). Then,
\[
\int_a^b (z - a)(b - z)(2b - a - z)\omega'(z)dz \geq 6. \tag{33}
\]
In particular,
\[
\int_a^b (z - a)(b - z)\omega'(z)dz \geq \frac{3}{b - a} \tag{34}
\]
\[
\omega^{(4)} + q(x)\omega = 0, \quad x \in (a,b), \\
\omega(a) = \omega'(a) = \omega''(a) = \omega''(b) = 0,
\]
\[
\text{admit a solution } \omega \in C([a,b], \mathbb{R}) \text{ such that } \omega(x) \neq 0 \text{ for } x \in (a,b). \tag{40}
\]

3. Applications

3.1. Lower Bound for the Eigenvalues. Consider the following eigenvalue problem:
\[
\begin{cases}
\omega''(x) + \omega(x) = 0, & x \in (0,1), 3 < \alpha \leq 4, \\
\omega(0) = D_{0+}^{\alpha - 3}\omega(0) = D_{0+}^{-2}\omega(0) = \omega''(1) = 0.
\end{cases} \tag{42}
\]

Theorem 4. Assume that eigenvalue problem (42) has a solution \( \omega \in C([a,b], \mathbb{R}) \) such that \( \omega(x) \neq 0 \) for \( x \in (a,b) \). Then,
\[
|\lambda| \geq \frac{(\alpha - 2)\Gamma(\alpha + 1)}{2}. \tag{43}
\]

Proof. By Remark 3 with \( a = 0 \) and \( b = 1 \), we have
\[
|\lambda| \int_0^1 (1 - z)^{\alpha - 3}(2 - z)dz \geq \Gamma(a), \tag{44}
\]
from which inequality (43) follows by observing that
\[
\int_0^1 (1 - z)^{\alpha - 3}(2 - z)dz = \frac{2}{\alpha(\alpha - 2)}. \tag{45}
\]

3.2. Nonexistence Results. Consider the following problem:
\[
\begin{cases}
D_{0+}^{\alpha}\omega + q(x)\omega = 0, & x \in (0,1), 3 < \alpha \leq 4, \\
\omega(0) = D_{0+}^{\alpha - 3}\omega(0) = D_{0+}^{-2}\omega(0) = \omega''(1) = 0,
\end{cases} \tag{46}
\]
where \( q \in C([0,1], \mathbb{R}) \). Then, we have the following result.

Theorem 5. Assume that
\[
\int_0^1 q'(z)dz < \frac{\Gamma(\alpha - 1)^{\alpha - 2}}{2(\sqrt{\alpha - 1})(\alpha - 3)^{\alpha - 2}}. \tag{47}
\]

Then, problem (46) has no nontrivial solution.

Proof. The assertion follows from Corollary 2.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.
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