A FAVARD TYPE THEOREM FOR HURWITZ POLYNOMIALS

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Dedicated to Prof. Juan J. Nieto on the occasion of his 60th birthday

Abstract. A Favard type theorem for Hurwitz polynomials is proposed. This result is a sufficient condition for a sequence of polynomials of increasing degree to be a sequence of Hurwitz polynomials. As in the Favard celebrated theorem, the three-term recurrence relation is used. Some examples of Hurwitz sequences are also presented. Additionally, a characterization of constructing a family of orthogonal polynomials on \([0, \infty)\) by two couples of numerical sequences \((A_{1,j}, B_{1,j})\) and \((A_{2,j}, B_{2,j})\) is stated.

1. Introduction. The basic differential-difference equation [3, page 416]

\[
\dot{x} = -\frac{\pi}{4} x - \frac{\pi}{3} x(t-1)
\]

has a characteristic function \(g(z) = (\frac{\pi}{4} + z) e^z + \frac{\pi}{3}\). The entire function \(g\) has roots with a negative real part [18, Theorem A.5]. On the other hand, the sequence of polynomials \((g_n)_{n=1}^{\infty}\) given by \(g_n(z) = \left(\frac{\pi}{4} + z\right) \left(1 + \frac{\pi}{3}\right)^n + \frac{\pi}{3}\) uniformly converges to \(g\). Moreover, for \(n = 1, 2, 3\) and 4 the polynomials \(g_n\) are Hurwitz polynomials. Recall that a real polynomial \(f_n(z)\) of degree \(n\)

\[
f_n(z) := a_0 z^n + a_1 n z^{n-1} + \cdots + a_{n-1} n z + a_n
\]

is called a Hurwitz polynomial if all its roots are located in the open left half complex plane.

In this paper we present a certain sufficient condition for a family of polynomials \((f_n)_{n=1}^{\infty}\) to be Hurwitz polynomials for every \(n\). We do not consider the convergence of a sequence \((f_n)_{n=1}^{\infty}\) as \(n \to +\infty\).

Definition 1.1. The sequence of polynomials \((f_j)_{j=1}^{\infty}\) (resp. \((f_j)_{j=1}^{n}\)) is called a Hurwitz sequence (resp. finite Hurwitz sequence) if \(\deg f_j = j\) and every \(f_j\) is a Hurwitz polynomial.

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Our first motivation is to develop the connection between the Hurwitz polynomials [21, 25] and orthogonal polynomials [4, 27], which was started in [6]; see also [7] and [8]. For this purpose, let us recall the well-known Favard theorem, which basically states that if a sequence of polynomials \((p_n(x))_{n=0}^{\infty}\), each of them of degree \(n\), satisfies a three-term recurrence relation, then \((p_n(x))_{n=0}^{\infty}\) is a sequence of orthogonal polynomials with respect to some distribution function; see e.g. [12, 10] and references therein. In [14] a sequence of Hurwitz polynomials is studied from the algebraic point of view. A three term relation is employed that differs from the proposed analysis in this work.

Our second motivation is related with the question whether a sequence of Hurwitz polynomials converge uniformly to a stable entire function. Such functions are important in the study of the stability of linear delay differential equations. For more details on stable entire functions and their approximation, see [17], [22], [26], [2] and references therein. In [14] a sequence of Hurwitz polynomials is studied from the form of a continued fraction.

We present two main results in this paper. The first one is a sufficient condition for an infinite or finite monic sequence of polynomials \((f_n(z))_{n=1}^{\infty}\) to be a sequence of Hurwitz polynomials; see Theorem 1.2 and Remark 2. Here and hereafter, the superscript \(\Xi\) denotes \(\infty\) or a nonnegative integer. The second main result is Theorem 3.2 where a recursive relationship between the sequences \((A_{1,j}, B_{1,j})_{j=0}^{\infty}\) and \((A_{2,j}, B_{2,j})_{j=0}^{\infty}\) is given. Remark 5 allows the construction of a couple of finite or infinite sequences \((A_{2,j}, B_{2,j})_{j=0}^{\infty}\) from a given couple of sequences \((A_{1,j}, B_{1,j})_{j=0}^{\infty}\) in the form of a continued fraction.

A characterization of constructing a family of orthogonal polynomials on \([0, \infty)\) by two couples of sequences \(A_{1,j}, B_{1,j}\) and \(A_{2,j}, B_{2,j}\) is stated in Theorem 3.4. This result does not require the boundedness of the corresponding Jacobi operator constructed by the three-term recurrence coefficients \(A_{1,j}, B_{1,j}\) and \(A_{2,j}, B_{2,j}\). The mentioned characterization could complete the well-known approach of constructing a family of orthogonal polynomials on \([0, \infty)\) by a positive distribution function on \([0, \infty)\), as well as a family of positive Stieltjes sequences; see [4] and [24, Page 260], respectively.

In the proof of the main result, we decisively use a finite continued expansion of the quotient \(\frac{a_n}{h_n}\), where \(h_n\) and \(g_n\) are related to \(f_n\) as follows:

\[
 f_n(z) = h_n(z^2) + zg_n(z^2), \tag{2}
\]

where

\[
 h_n(z) := \begin{cases} 
 a_{0n} z^m + a_{2n} z^{m-1} + \cdots + a_{n-2} z + a_{nn}, & n = 2m, \\
 a_{1n} z^m + a_{3n} z^{m-1} + \cdots + a_{n-2} z + a_{nn}, & n = 2m + 1,
\end{cases} \tag{3}
\]

\[
 g_n(z) := \begin{cases} 
 a_{1n} z^{m-1} + a_{3n} z^{m-2} + \cdots + a_{n-3} z + a_{n-1n}, & n = 2m, \\
 a_{0n} z^m + a_{2n} z^{m-1} + \cdots + a_{n-3} z + a_{n-1n}, & n = 2m + 1.
\end{cases} \tag{4}
\]

It should be mentioned that the representation (2), as well as, the method of Markov parameters (as in Subsection 1.1), were considered by M.G. Krein/M. Naimark [23], F.R. Gantmacher [13, Chapter XV] and other authors. The interlacing property of the roots of the polynomials \(p_n\) and \(q_n\) was employed to establish the Hurwitness of the given polynomial \(f_n\) [25]. In this paper, we crucially use the explicit relation between a Hurwitz polynomial and a member of a family of orthogonal polynomials and its second kind polynomial. This relation, to the knowledge of the authors, was first presented in [6]. The reason why the latter relation was not noted by previous
authors lies mainly in two facts. First, instead of the Laurent expansion (12), the following expansion was used [13, page 233]:
\[
\frac{g_{2m+1}(-z)}{h_{2m+1}(-z)} = \hat{s}_{-1} - \frac{s_0}{z} - \frac{s_1}{z^2} - \cdots - \frac{s_{2m-1}}{z^{2m}} - \cdots,
\]
where \(\hat{s}_{-1}\) is a positive number and \((\hat{s}_j)_{j=0}^\infty\) are the corresponding Markov parameters. Second, the relational functions \(\frac{g_{2m}(-z)}{h_{2m}(-z)}\) and \(\frac{h_{2m+1}(-z)}{(-z)g_{2m+1}(-z)}\) happen to be the so-called extremal solutions of the truncated Stieltjes moment problem; see (11) and (12). For more details, see [11].

Now we state the first main result of the present work.

**Theorem 1.2.** Let \(f_0(z) := 1\), and let \((f_j)_{j=1}^\infty\) be a sequence of monic polynomials with \(\deg f_j = j\). Furthermore, let \(A_{k,j}, B_{k,j}\) be positive numbers for \(j \geq 0, k = 1, 2\) with \(A_{2,0} = A_{1,0} + \frac{B_{1,1}}{A_{1,0}}\). Moreover, let \(s_0 > 0\). If
\[
f_1(z) = z + s_0, \tag{6}
\]
\[- A_{1,0}f_0(z) + f_2(z) = z^2f_0(z) + s_0z, \tag{7}
\]
\[- A_{2,0}f_1(z) + f_3(z) = z^2f_1(z) - A_{1,0}s_0, \tag{8}
\]
and for \(j \geq 1\),
\[
B_{1,j-1}f_{2j-1}(z) - A_{1,j}f_{2j}(z) + f_{2j+1}(z) = z^2f_{2j}(z), \tag{9}
\]
\[
B_{2,j-1}f_{2j-1}(z) - A_{2,j}f_{2j+1}(z) + f_{2j+3}(z) = z^2f_{2j+1}(z), \tag{10}
\]
then the sequence \((f_j)_{j=1}^\infty\) is a Hurwitz sequence.

As a consequence, we obtain the second main result of our work: a recursive relation between coefficients \((A_{1,j}, B_{1,j})_{j=0}^\infty\) and \((A_{2,j}, B_{2,j})_{j=0}^\infty\) that appears in (7)-(10). See Theorem 3.2 and Remark 5.

In the subsequent two subsections, we recall important results concerning the Markov parameters, orthogonal polynomials on \([0, +\infty)\) and Hurwitz polynomials.

In Section 2 the proof of Theorem 1.2 is given. Relations between the three-term recurrence relation coefficients \(A_{1,j}, B_{1,j}\) and \(A_{2,j}, B_{2,j}\) are presented in Section 3.

In Section 4 the first polynomials of a sequence of Hurwitz polynomials \((f_j)_{j=0}^\infty\) in terms of the Markov parameters, the three-term recurrence relation coefficients, the Stieltjes parameters and the Schur complements are given.

Finally, in Section 5 some explicit examples of Hurwitz sequences of polynomials are presented.

1.1. Markov parameters. The Laurent expansions of the rational functions \(\frac{g_{2m}(-z)}{h_{2m}(-z)}\) and \(\frac{g_{2m+1}(-z)}{h_{2m+1}(-z)}\),
\[
\frac{g_{2m}(-z)}{h_{2m}(-z)} = -\frac{s_0}{z} - \frac{s_1}{z^2} - \cdots - \frac{s_{2m-1}}{z^{2m}} - \cdots, \tag{11}
\]
\[
\frac{h_{2m+1}(-z)}{(-z)g_{2m+1}(-z)} = -\frac{s_0}{z} - \frac{s_1}{z^2} - \cdots - \frac{s_{2m}}{z^{2m+1}} + \cdots, \tag{12}
\]
play an important role in the interrelation between the Hurwitz polynomials and orthogonal polynomials. These expansions allow us to incorporate the Markov
The sequence \((p_1, p_2, \ldots, p_n)\) is a necessary and sufficient condition of the Hurwitzness of a polynomial \(s\). Further development of the Markov parameter approach, which in turn is related to Hankel matrices

\[
H_{1,j} := \begin{pmatrix}
    s_0 & s_1 & \cdots & s_j \\
    s_1 & s_2 & \cdots & s_{j+1} \\
    \vdots & \vdots & \ddots & \vdots \\
    s_j & s_{j+1} & \cdots & s_{2j}
\end{pmatrix},
\]

\[
H_{2,j} := \begin{pmatrix}
    s_1 & s_2 & \cdots & s_{j+1} \\
    s_2 & s_3 & \cdots & s_{j+2} \\
    \vdots & \vdots & \ddots & \vdots \\
    s_{j+1} & s_{j+2} & \cdots & s_{2j+1}
\end{pmatrix},
\]

(13)

The connection between the latter Hankel matrices and Hurwitz polynomials was first studied by Hurwitz in [21, Equalities (17) and (20)] and extensively studied in Chapter XV of Gantmacher’s book, [13]. Further development of the Markov parameter was considered by [9, 19, 20], and references therein.

In [8, Theorem 3.4], it is proved that the positiveness of both Hankel matrices (13) is a necessary and is a sufficient condition of the Hurwitzness of a polynomial \(f_n\). To prove this condition, the following notion on the finite sequences \((s_j)_{j=0}^{2m}\) (resp. \((s_j)_{j=0}^{2m+1}\)) was employed in [8].

**Definition 1.3.** The sequence \((s_j)_{j=0}^{2m}\) (resp. \((s_j)_{j=0}^{2m+1}\)) is called a Stieltjes positive sequence if \(H_{1,m}\) and \(H_{2,m-1}\) (resp. \(H_{1,m}\) and \(H_{2,m}\)) are positive definite matrices.

For the sake of completeness the recall the [8, Theorem 3.4]:

**Remark 1.** Let \(n \geq 2\) and \(a_{0n} > 0\). The polynomial \(f_{2m+1}\) (resp. \(f_{2m}\)) is a Hurwitz polynomial if and only if \((s_j)_{j=0}^{2m}\) (resp. \((s_j)_{j=0}^{2m-1}\)) is a Stieltjes positive sequence.

In [7] the Kharitonov’s theorem for interval polynomials is given in terms of orthogonal polynomials on \([0, +\infty)\) and their second kind polynomials. Additionally, a family of robust stabilizing controls for the canonical system is proposed.

1.2. **Relation between orthogonal and Hurwitz polynomials.** Recently an explicit relation between orthogonal polynomials, their second kind polynomials and Hurwitz polynomials was found by the author in [6].

**Definition 1.4.** Let \(\sigma(t)\) be a positive distribution function on \([0, \infty)\) such that all moments \(s_j := \int_0^\infty t^j d\sigma(t)\) are finite for \(j \in \mathbb{N} \cup \{0\}\). The sequence of polynomials \((p_j)_{j=0}^{\infty}\) with \(p_j(x) = x^j + \cdots, j \geq 0\) and

\[
\int_0^\infty p_j(t)p_k(t)d\sigma(t) = \begin{cases} 0, & j \neq k, \\
c_j, & j = k, \end{cases}
\]

is called the sequence of monic orthogonal polynomials on \([0, \infty)\) with respect to \(d\sigma(t)\). Here \(c_j\) is a positive number.

In [6], it was proved that every Hurwitz polynomial can be written in terms of a member of a family of orthogonal polynomials \((p_{k,j})_{j=0}^{\infty}\), \(k = 1, 2\) on \([0, \infty)\) and their second kind polynomials \((q_{k,j})_{j=0}^{\infty}\), \(k = 1, 2\):

\[
f_n(z) = \begin{cases} (-1)^n (p_{1,m}(-z^2) - z q_{1,m}(-z^2)), & n = 2m, \\
(-1)^n (q_{2,m}(-z^2) + z p_{2,m}(-z^2)), & n = 2m + 1. \end{cases}
\]

(14)

More precisely, the sequence \((p_{1,j})_{j=0}^{\infty}\) (resp. \((p_{2,j})_{j=0}^{\infty}\)) is orthogonal with respect to a given positive distribution function \(\sigma(t)\) (resp. \(t\sigma(t)\)) on \([0, \infty)\), while \(q_{1,j}, q_{2,j}\)
are defined as follows:

\[ q_{1,j}(x) := \int_0^\infty \frac{p_{1,j}(x) - p_{1,j}(t)}{x - t} \, d\sigma(t), \]

\[ q_{2,j}(x) := \int_0^\infty \frac{x p_{2,j}(x) - t p_{2,j}(t)}{x - t} \, d\sigma(t), \]

for \( j \geq 0 \). See [5, Remark E.4] and [5, Lemma E.11].

Clearly, by using a given sequence of infinite or finite polynomials \((p_{1,j}, q_{1,j})_{j=0}^\infty\) or \((p_{2,j}, q_{2,j})_{j=0}^\infty\), one can generate a sequence of Hurwitz polynomials. A reciprocal question arises: Under what conditions is a sequence of polynomials \((f_j)_{j=1}^\infty\) a sequence of Hurwitz polynomials? In Theorem 1.2, a sufficient condition is given. To prove this assertion, we use a continued fraction approach, in contrast to [6] where the Markov parameter approach was employed to prove (14).

2. Proof of the Favard type theorem for Hurwitz polynomials.

In this section, we present the proof of Theorem 1.2.

Proof. Step 1. By using (2), (3) and (4), we rewrite equalities (6)-(10) in terms of \( h_j \) and \( g_j \) and set \( z = \omega \) for \( \omega \in \mathbb{R} \). Extend the real and imaginary parts of the resulting relations to the complex plane. We denote again the resulting expressions as \( h_j(z) \) and \( g_j(z) \). These functions can be expressed as follows:

\[ h_2(z) = z + A_{1,0}, \quad g_2(z) = s_0, \]

\[ h_{2(j+1)}(z) = (z + A_{1,j}) h_{2j}(z) + B_{1,j-1} h_{2(j-1)}(z), \]

\[ g_{2(j+1)}(z) = (z + A_{1,j}) g_{2j}(z) + B_{1,j-1} g_{2(j-1)}(z), \]

\[ h_1(z) = s_0, \quad g_1(z) = 1, \]

\[ h_{3}(z) = (z + A_{2,0}) h_{1}(z) - A_{1,0} s_0, \quad g_{3}(z) = (z + A_{2,0}) g_{1}(z), \]

\[ h_{2j+3}(z) = (z + A_{2,j}) h_{2j+1}(z) + B_{1,j-1} h_{2j-1}(z), \]

\[ g_{2j+3}(z) = (z + A_{2,j}) g_{2j+1}(z) + B_{2,j-1} g_{2j-1}(z). \]

Step 2. Let

\[ m_0 := s_0^{-1}, \quad l_0 := A_{1,0}^{-1} s_0, \quad m_{j+1} := \frac{1}{B_{1,j} l_j m_j}, \quad \text{and} \quad l_{j+1} := \frac{1}{B_{2,j} l_j m_j^2}, \]

for \( j \geq 0 \). The quantities \( m_j \) and \( l_j \) are called Stieltjes parameters [24, page 198]. Write \( \frac{g_{2m}}{h_{2m}} \) and \( \frac{g_{2m+1}}{h_{2m+1}} \) through the sequences \((m_j)_{j=0}^{m-1}\) and \((l_j)_{j=0}^{m-1}\) (resp. \((m_j)_{j=0}^m\) and \((l_j)_{j=0}^m\)) [6, Theorem 5.4]:

\[ g_{2m}(-z) \]

\[ h_{2m}(-z) \]

\[ = m_0 + \frac{1}{z l_0 + \frac{1}{\ddots + \frac{1}{z l_{m-1} + m_{m-1}}}}, \]

\[ g_{2m+1}(z) \]

\[ h_{2m+1}(z) \]

\[ = m_0 + \frac{1}{z l_0 + \frac{1}{\ddots + \frac{1}{z l_{m-1} + m_{m-1}}}}. \]
Furthermore, [13, Theorem 16, Chapter XV] ensures that the polynomial (2) is a Hurwitz polynomial if and only if (22)-(23) holds with \( m_j > 0 \) and \( l_j > 0 \). Consequently, we attain that the sequence \( (f_j)_{j=1}^\infty \) is a Hurwitz sequence. \( \square \)

2.1. Finite sequence of monic Hurwitz polynomials. Now we are interested in considering a finite sequence of monic polynomials with \( \deg f_j = j \). Denote

\[
J_{k,m} := \begin{pmatrix}
-A_{k,0} & 1 & 0 & \cdots & 0 & 0 \\
B_{k,0} & -A_{k,1} & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & -A_{k,m-1} & 1 \\
0 & 0 & \cdots & \cdots & B_{k,m-1} & -A_{k,m}
\end{pmatrix}, \quad \text{for } k = 1, 2,
\]

and

\[
f_{[0,2m]}(z) := (f_0(z), f_1(z), \ldots, f_{2m}(z))^* \]

and

\[
f_{[1,2m+1]}(z) := (f_1(z), f_2(z), \ldots, f_{2m+1}(z))^*.
\]

Here \(^*\) denotes the conjugate transpose operator.

The following remark can be proved by using the procedure as in Theorem 1.2.

**Remark 2.** Let \( f_0(z) = 1 \), and let \( (f_j)_{j=0}^{2m+2} \) (resp. \( (f_j)_{j=0}^{2m+3} \)), a finite sequence of monic polynomials with \( \deg f_j = j \), be satisfied. Furthermore, let \( s_0, A_{k,j}, B_{k,j} \) for \( k = 1, 2 \) be as in Theorem 1.2:

\[
J_{1,m} f_{[0,2m]}(z) = z^2 f_{[0,2m]}(z) + (s_0 \bar{z}, 0, \ldots, 0, -f_{2m+2}(\bar{z}))^*,
\]

and

\[
J_{2,m} f_{[1,2m+1]}(z) = z f_{[1,2m+1]}(z) + (-A_{1,0} s_0, 0, \ldots, 0, -f_{2m+3}(\bar{z}))^*
\]

hold, then the sequences \( (f_j)_{j=1}^{2m+2} \) (resp. \( (f_j)_{j=1}^{2m+3} \)) are sequences of Hurwitz polynomials.

3. Relation between \( A_{1,j}, B_{1,j} \) and \( A_{2,j}, B_{2,j} \). In this section, we attain a recurrent relation between \( A_{1,j}, B_{1,j} \) and \( A_{2,j}, B_{2,j} \) appearing in Theorem 1.2. Equivalently, this relation is also valid for families of orthogonal polynomials. For this purpose, one uses explicit relations between \( h_n \) and \( g_n \) of Hurwitz polynomial \( f_n \): 

\[
h_{2m}(z) = (-1)^m p_{1,m}(-z), \quad g_{2m+1}(z) = (-1)^m p_{2,m}(-z),
\]

\[
g_{2m}(z) = (-1)^{m+1} q_{1,m}(-z), \quad h_{2m+1}(z) = (-1)^m q_{2,m}(-z),
\]

for \( m \geq 0 \). See [6, Pages 78 and 79].

Let us recall [5, Theorem 9.3, Part b)]. See also [10, Equation (1.5)].

**Lemma 3.1.** Let \( (s_j)_{j=0}^{2m-1} \) (resp. \( (s_j)_{j=0}^{2m} \)) be a Stieltjes positive sequence. Furthermore, let for \( k = 1, 2 \) and \( j \geq 1 \)

\[
\tilde{H}_{k,0} := s_{k-1}, \quad \tilde{H}_{k,j} := \frac{\det H_{k,j}}{\det H_{k,j-1}}.
\]
If $A_{k,j}$ and $B_{k,j}$ for $k = 1, 2$ and $j \geq 0$ satisfy the recurrence relation (15)-(17) and (18)-(21), then

$$A_{1,0} = \hat{H}_{2,0}\hat{H}_{1,0}^{-1}, \quad A_{2,0} = s_2\hat{H}_{2,0}^{-1},$$

$$A_{1,j} = \hat{H}_{2,j}\hat{H}_{1,j}^{-1} + \hat{H}_{1,j}\hat{H}_{2,j-1}, \quad j \geq 1,$$

$$A_{2,j} = \hat{H}_{1,j+1}\hat{H}_{2,j}^{-1} + \hat{H}_{2,j}\hat{H}_{1,j}, \quad j \geq 0,$$

$$B_{k,j} = \hat{H}_{k,j}\hat{H}_{k,j+1}.$$

(25)  
(26)  
(27)  
(28)

The following remark allows us to obtain the sequence $(s_j)_{j=0}^n$ from the Stieltjes parameters $m_j$ and $l_j$.

**Remark 3.** [6, Proposition 4.9] Let the sequence $(s_j)_{j=0}^{2m}$ (resp. $(s_j)_{j=0}^{2m+1}$) be recursively defined by

$$s_{2j} := \begin{cases} m_0^{-1}, & \text{if } j = 0, \\ Y_{1,j}^{-1}H_{1,j-1}^{-1}Y_{1,j} + \left(\prod_{k=0}^{j-2} m_k l_k\right)^{-1}m_j^{-1} & \text{if } j \geq 1, \end{cases}$$

and

$$s_{2j+1} := \begin{cases} (m_0l_0)^{-2}, & \text{if } j = 0, \\ Y_{2,j}^{-1}H_{2,j-1}^{-1}Y_{2,j} + \left(\prod_{k=0}^{j-2} m_k l_k\right)^{-2}l_j & \text{if } j \geq 1, \end{cases}$$

where $Y_{1,j} = (s_1, \ldots, s_{2j-1})$ and $Y_{2,j} = (s_{j+1}, \ldots, s_{2j})$. Thus, the sequence $(s_j)_{j=0}^{2m}$ (resp. $(s_j)_{j=0}^{2m+1}$) is a Stieltjes positive sequence.

**Theorem 3.2.** Let $s_0$, $A_{k,j}$, $B_{k,j}$ for $j \in \mathbb{N} \cup \{0\}$ and $k = 1, 2$ be as in Theorem 1.2, which satisfy relations (6)-(10) for a certain sequence of polynomials $(f_j)_{j=1}^\infty$ of increasing degree. Furthermore, let

$$\kappa_j := \frac{\prod_{\ell=0}^{j-1} B_{2,\ell}}{\prod_{\ell=0}^{j-1} B_{1,\ell}} A_{1,0}, \quad \text{with} \quad \kappa_0 := A_{1,0}$$

and

$$\xi_j := \frac{\prod_{\ell=0}^{j} B_{1,\ell} 1}{\prod_{\ell=0}^{j} B_{2,\ell} A_{1,0}}, \quad \text{with} \quad \xi_0 := \frac{B_{1,0}}{A_{1,0}}.$$

The following equalities then hold:

$$A_{2,j} = \kappa_j + \frac{B_{1,j}}{\kappa_j}, \quad j \geq 0$$

and

$$B_{2,0} = \xi_0 (A_{1,1} - \xi_0), \quad B_{2,j} = \xi_j (A_{1,j+1} - \xi_j), \quad j \geq 1.$$

(29)  
(30)

**Proof.** From Theorem 1.2, each member $f_n$ of the sequence $(f_n)_{n=1}^\infty$ is a Hurwitz polynomial. By Remark 1 the sequence of Markov parameters $(s_j)_{j=0}^{2m-1}$ (resp. $(s_j)_{j=0}^{2m}$) is a Stieltjes positive related to the polynomial $f_n$ for $n = 2m$ (resp. $n = 2m + 1$). By using the moments $(s_j)_{j=0}^{n-1}$, one constructs the Hankel matrices $H_{k,j}$ for $k = 1, 2$ and $j \geq 0$. Furthermore, we employ (24). For $j = 0$ Equality (29) follows by condition of Theorem 1.2. Let $j = 1$.

$$\kappa_1 + \frac{B_{1,1}}{\kappa_1} = \frac{B_{2,0}}{B_{1,0}} A_{1,0} + \frac{B_{1,0}B_{1,1}}{B_{2,0} A_{1,0}} = \frac{\hat{H}_{2,1}}{\hat{H}_{1,1}} + \frac{\hat{H}_{1,2}}{\hat{H}_{2,1}} = A_{2,1}.$$

In a similar manner, Equalities (29) for $j \geq 2$ and (30) can be proved. □

We following remark can be verified by direct calculations.
**Remark 4.** Let $s_0, A_{k,j}, B_{k,j}$ for $j \in \mathbb{N} \cup \{0\}$ and $k = 1, 2$ be as in Theorem 3.2. Thus, the following equalities hold:

$$
\kappa_j = A_{1,j} - \frac{B_{1,j-1}}{\kappa_{j-1}}, \quad j \geq 1,
$$

and

$$
\xi_j = \frac{B_{1,j}}{A_{1,j} - \xi_{j-1}}, \quad j \geq 1.
$$

**Remark 5.** The following continued fraction expressions for the sequence $(A_{2,j})_{j=0}^\infty$ are valid:

$$
A_{2,0} = A_{1,0} + \frac{B_{1,0}}{A_{1,0}},
$$

(31)

$$
A_{2,1} = A_{1,1} - \frac{B_{1,0}}{A_{1,0}} + \chi_1,
$$

(32)

$$
A_{2,n} = A_{1,n} - \chi_{n-1} + \chi_n,
$$

(33)

for $n \geq 2$, where $\chi_n := \frac{B_{1,n}}{A_{1,n} - \frac{B_{1,n-1}}{A_{1,n-1} - \frac{B_{1,n-2}}{\ddots \frac{B_{1,2}}{A_{1,1} - \frac{B_{1,0}}{A_{1,0}}}}}}, \quad n \geq 1.$

Furthermore, the following equalities for the sequence $(B_{2,j})_{j=0}^\infty$ are valid:

$$
B_{2,0} = \frac{B_{1,0}}{A_{1,0}} \left( A_{1,1} - \frac{B_{1,0}}{A_{1,0}} \right),
$$

(34)

$$
B_{2,1} = \chi_1 \left( A_{1,2} - \chi_1 \right),
$$

(35)

$$
B_{2,n} = \chi_n \left( A_{1,n+1} - \chi_n \right), \quad n > 1.
$$

(36)

The following remark guarantees the well-definiteness of the continued fractions of Remark 5.

**Definition 3.3.** A sequence of positive real numbers $s_0, (A_{k,j})_{j=0}^\infty$ and $(B_{k,j})_{j=0}^\infty$ for $k = 1, 2$ such that (6)-(10) are satisfied for a certain Hurwitz sequence is called a Favard-Hurwitz sequence.

Next we present a result concerning a characterization of the sequences $(A_{k,j})_{j=0}^\infty$ and $(B_{k,j})_{j=0}^\infty$.

**Theorem 3.4.** Let $F_{\infty}$ be a set of real positive numbers $s_0, (A_{k,j})_{j=0}^\infty$ and $(B_{k,j})_{j=0}^\infty$ for $k = 1, 2$. A necessary and sufficient condition for $F_{\infty}$ to be a Favard-Hurwitz sequence is that $F_{\infty}$ satisfies (31)-(33) and (34)-(36).

**Proof.** The necessary part of this theorem is proved in Theorem 3.2. To prove the sufficient part of this theorem, we apply Step 2 of the proof of Theorem 1.2. □

4. The Hurwitz polynomials via three-term recurrence relation coefficients, Markov parameters, Stieltjes parameters and Schur complements.

In this section we give four representations of a finite family of Hurwitz polynomials in terms of the three-term recurrent relation coefficients $A_{k,j}, B_{k,j}$, the Markov parameters $(s_j)_{j=0}^\infty$, the Stieltjes parameters $l_j, m_j$ and the Schur complements $\tilde{H}_{1,j}, \tilde{H}_{2,j}$. 
To this end, let us first recall the following remark. See [8, Lemma 3.1] or [7, Remark 1].

**Remark 6.** Let \( f_n \) be a real polynomial of degree \( n \), and let \( h_n, g_n \) be as in (3) and (4). The Markov parameter sequence \((s_j)_{j=0}^{2m} \) (resp. \((s_j)_{j=0}^{2m-1} \)) from the relations (11) and (12) is determined by the following equalities:

\[
\begin{align*}
(s_0, s_1, \ldots, s_{2m-1})^* &= A_{2m}^{-1}(a_1, a_3, \ldots, a_{2m-1}, 0, \ldots, 0)^*, \quad n = 2m, \\
(s_0, s_1, \ldots, s_{2m})^* &= A_{2m+1}^{-1}(a_1, a_3, \ldots, a_{2m+1}, 0, \ldots, 0)^*, \quad n = 2m + 1,
\end{align*}
\]

where

\[
A_n := \begin{pmatrix}
a_0 & 0 & \ldots & 0 & 0 \\
a_2 & -a_0 & \ldots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
a_{2(n-1)} & -a_{2(n-2)} & \ldots & (-1)^n a_2 & (-1)^{n+1} a_0
\end{pmatrix},
\]

for \( n \geq 2 \) is the \( n \times n \) matrix with \( a_k = 0 \) for \( k > n \). Here we have omitted the second subindex in \( a_{k,n} \).

The following remark permits to write the sequence \((f_j)_{j=1}^5\) in terms of the coefficients \( A_{k,j} \) and \( B_{k,j} \) for \( k = 1, 2 \).

**Remark 7.** Let the pair sequences \((A_{k,j})_{j=1}^\infty, (B_{k,j})_{j=0}^\infty \) for \( k = 1, 2 \) be as in Theorem 1.2. By using (15)-(21), we can recursively compute the first elements of the sequence \((f_n)_{n=1}^\infty\):

\[
\begin{align*}
f_1(z) &= z + s_0, \\
f_2(z) &= z^2 + s_0 z + A_{1,0}, \\
f_3(z) &= z^3 + s_0 z^2 + A_{2,0} z + \frac{B_{1,0}}{A_{1,0}} s_0, \\
f_4(z) &= z^4 + s_0 z^3 + (A_{1,0} + A_{1,1}) z^2 + A_{1,1} s_0 z + A_{1,0} A_{1,1} - B_{1,0}, \\
f_5(z) &= z^5 + s_0 z^4 + (A_{2,0} + A_{2,1}) z^3 + s_0 \left( \frac{B_{10}}{A_{10}} + A_{21} \right) z^2 + (A_{20} A_{21} - B_{20}) z \\
&\quad + s_0 \left( \frac{A_{21} B_{10}}{A_{10}} - B_{20} \right).
\end{align*}
\]

By taking into account that all the coefficients of a Hurwitz polynomial are necessarily positive. Beginning from \( f_4 \), we have some new conditions for the coefficients \( A_{k,j} \) and \( B_{k,j} \). Since the construction of the Hurwitz polynomials \( f_n \) are based on the orthogonal polynomials, we get that the aforementioned conditions should be also satisfied by the three-term recurrence coefficients of any family of orthogonal polynomials generated by a positive measure on \([0, +\infty)\). Note that the positiveness of the coefficients of \( f_4 \) and \( f_5 \) can be verified by employing the representation given in Remark 9 or Remark 10.

Now by applying Remark 6, we express the sequence \((f_j)_{j=1}^5\) in terms of the moments \( s_j \).
Remark 8. In terms of the moments $s_j$, the Hurwitz polynomials have the following form:

\[
\begin{align*}
f_1(z) &= z + s_0, \\
f_2(z) &= z^2 + s_0z + \frac{s_1}{s_0}, \\
f_3(z) &= z^3 + s_0z^2 + \frac{s_2}{s_1}z - s_1 + \frac{s_0s_2}{s_1}, \\
f_4(z) &= z^4 + s_0z^3 + \frac{s_0s_3 - s_1s_2}{s_0s_2 - s_1^2}z^2 + \frac{s_1}{s_0s_2 - s_1^2}s_3 - \frac{s_2}{s_0s_2 - s_1^2}, \\
f_5(z) &= z^5 + s_0z^4 + \frac{s_2s_3 - s_1s_4}{s_2^2 - s_1s_3}z^3 + \frac{s_3}{s_2^2 - s_1s_3}\left(s_3 - \frac{s_2}{s_2^2 - s_1s_3}s_0s_4 + \frac{s_0s_2}{s_1s_3}\right)s_1 + \frac{s_0s_2s_3}{s_2^2 - s_1s_3}z^2 + \frac{s_2}{s_2^2 - s_1s_3}s_4 + \frac{3}{s_2^2 - s_1s_3}s_1.
\end{align*}
\]

For the convenience of the reader, we also express the sequence $\left(f_j^j\right)_{j=1}$ in terms of the Stieltjes parameters $m_j$ and $l_j$.

Remark 9. In terms of the Stieltjes parameters $l_j$ and $m_j$, the Hurwitz polynomials have the following form:

\[
\begin{align*}
f_1(z) &= z + \frac{1}{m_0}, \\
f_2(z) &= z^2 + \frac{1}{m_0}z + \frac{1}{l_0m_0}, \\
f_3(z) &= z^3 + \frac{1}{m_0}z^2 + \frac{m_0 + m_1}{l_0m_0m_1}z + \frac{1}{l_0m_0m_1}, \\
f_4(z) &= z^4 + \frac{1}{m_0}z^3 + \frac{l_0m_0 + l_1m_0 + l_1m_1}{l_0l_1m_0m_1}z^2 + \frac{l_0 + l_1}{l_0l_1m_0m_1}z + \frac{1}{l_0l_1m_0m_1}, \\
f_5(z) &= z^5 + \frac{1}{m_0}z^4 + \frac{l_0m_0m_1 + l_1m_0m_1 + l_0m_0m_2 + l_1m_0m_2}{l_0l_1m_0m_1m_2}z^3 + \frac{l_1m_1 + l_1m_2 + l_2}{l_0l_1m_0m_1m_2}z^2 + \frac{m_0 + m_1 + m_2}{l_0l_1m_0m_1m_2}z + \frac{1}{l_0l_1m_0m_1m_2}.
\end{align*}
\]

Now we also express of the sequence $\left(f_j^j\right)_{j=1}$ in terms of the Schur complements $\widehat{H}_{1,j}$ and $\widehat{H}_{2,j}$.

Remark 10. In terms of the Schur complements $\widehat{H}_{1,j}$ and $\widehat{H}_{2,j}$, the Hurwitz polynomials have the following form:

\[
\begin{align*}
f_1(z) &= z + \widehat{H}_{1,0}, \\
f_2(z) &= z^2 + \widehat{H}_{1,0}z + \frac{\widehat{H}_{2,0}}{\widehat{H}_{1,0}}, \\
f_3(z) &= z^3 + \widehat{H}_{1,0}z^2 + \frac{\widehat{H}_{1,0}\widehat{H}_{1,1}}{\widehat{H}_{2,0}} + \frac{\widehat{H}_{1,1}}{\widehat{H}_{2,0}}z + \frac{\widehat{H}_{2,0}}{\widehat{H}_{1,0}}, \\
f_4(z) &= z^4 + \widehat{H}_{1,0}z^3 + \left(\frac{\widehat{H}_{1,1}}{\widehat{H}_{2,0}} + \frac{\widehat{H}_{2,0}}{\widehat{H}_{1,1}} + \frac{\widehat{H}_{2,1}}{\widehat{H}_{1,1}}\right)z^2 + \left(\frac{\widehat{H}_{1,0}\widehat{H}_{1,1}}{\widehat{H}_{2,0}} + \frac{\widehat{H}_{1,0}\widehat{H}_{2,1}}{\widehat{H}_{1,1}}\right)z.
\end{align*}
\]
Laguerre polynomials \([27]\), defined in terms of the hypergeometric series as Hurwitz sequence from Laguerre polynomials.

5. Examples.

5.1. Hurwitz sequence from Laguerre polynomials. Let us consider monic Laguerre polynomials \([27]\), defined in terms of the hypergeometric series as

\[
L_n^{(\alpha)}(x) = (-1)^n (\alpha + 1)_n F_1 \left( \frac{-n}{\alpha + 1} \left| x \right. \right),
\]

where \((A)_n = A(A+1) \cdots (A+n-1)\) denotes the Pochhammer symbol. For \(\alpha > -1\), these polynomials satisfy the orthogonality relation

\[
\int_0^\infty L_n^{(\alpha)}(x)L_m^{(\alpha)}(x)g^{(\alpha)}(x)dx = \Gamma(n + \alpha + 1) n! \delta_{n,m},
\]

where

\[
g^{(\alpha)}(x) = x^\alpha \exp(-x)
\]

and \(\delta_{n,m}\) stands for the Kronecker delta. If we consider \(p_{1,n}(x) = L_n^{(\alpha)}(x)\) orthogonal with respect to \(g^{(\alpha)}(x)\), then \((p_{2,n}(x))_{n=0}^\infty\) is orthogonal with respect to \(x^\alpha g^{(\alpha)}(x)\) i.e. \(p_{2,n}(x) = L_n^{(\alpha+1)}(x)\). The moments for this orthogonality weight function are given by

\[
s_n = \int_0^\infty x^n g^{(\alpha)}(x)dx = \Gamma(\alpha + n + 1), \quad \alpha > -1, \quad n \in \mathbb{N}.
\]

Therefore, the Hankel matrices \(H_{1,n}\) defined in (13) have determinants given by

\[
\det H_{1,n} = \varpi_n (\Gamma(\alpha + 1))^{n+1} \prod_{k=1}^n (\alpha + k)^{n+1-k},
\]

where \(\varpi_n\) are the superfactorials (the product of the first \(n\) factorials) defined as

\[
\varpi_0 = 1, \quad \varpi_1 = 1, \quad \varpi_n = n! \varpi_{n-1}.
\]

Similarly, the Hankel matrices \(H_{2,n}\) also defined in (13) have determinants given by

\[
\det H_{2,n} = \varpi_n (\Gamma(\alpha + 1))^{n+1} \prod_{k=1}^{n+1} (\alpha + k)^{n-k+2}.
\]

As a consequence, from (24) for \(j \geq 1\), we have

\[
\hat{H}_{1,j} = j! \Gamma(\alpha + 1) (\alpha + 1)_j, \\
\hat{H}_{2,j} = j! \Gamma(\alpha + 1) (\alpha + 1)_{j+1}.
\]
Therefore, from Lemma 3.1

\[ A_{1,0} = \alpha + 1, \]
\[ A_{2,0} = \alpha + 2, \]
\[ A_{1,j} = \alpha + 2j + 1, \quad j \geq 1, \]
\[ A_{2,j} = \alpha + 2j + 2, \quad j \geq 1, \]

and moreover (28) implies

\[ B_{1,j} = (j + 1)(\alpha + j + 1), \quad B_{2,j} = (j + 1)(\alpha + j + 2). \]

Let us compute the first \( q_{1,j} \)

\[
\begin{align*}
q_{1,1}(x) &= \Gamma(\alpha + 1), \\
q_{1,2}(x) &= (-\alpha + x - 3)\Gamma(\alpha + 1), \\
q_{1,3}(x) &= (\alpha^2 - 2(\alpha + 4)x + 6\alpha + x^2 + 11)\Gamma(\alpha + 1), \\
q_{1,4}(x) &= (-3(\alpha + 5)x^2 + \alpha(3\alpha + 25)x - \alpha(\alpha + 10) + 35) + x^3 + 58x - 50 \cdot \Gamma(\alpha + 1),
\end{align*}
\]

as well as the first \( q_{2,j} \)

\[
\begin{align*}
q_{2,1}(x) &= (x - 1)\Gamma(\alpha + 1), \\
q_{2,2}(x) &= (-\alpha + 5)x + x^2 + 2 \cdot \Gamma(\alpha + 1), \\
q_{2,3}(x) &= (x (\alpha^2 + \alpha(9 - 2x) + (x - 11)x + 26) - 6)\Gamma(\alpha + 1), \\
q_{2,4}(x) &= (-3(\alpha + 19)x^3 + 3(\alpha(\alpha + 11) + 34)x^2 - (\alpha + 7)(\alpha(\alpha + 7) + 22)x \\
&\quad + x^4 + 24)\Gamma(\alpha + 1).
\end{align*}
\]

By (14), the first elements of this Hurwitz sequence are attained as

\[
\begin{align*}
f_1(z) &= \Gamma(\alpha + 1) + z, \\
f_2(z) &= z\Gamma(\alpha + 1) + \alpha + z^2 + 1, \\
f_3(z) &= z (\alpha + z^2 + 2) + (z^2 + 1)\Gamma(\alpha + 1), \\
f_4(z) &= \alpha^2 + 2(\alpha + 2)z^2 + z (\alpha + z^2 + 3)\Gamma(\alpha + 1) + 3\alpha + z^4 + 2, \\
f_5(z) &= 2(\alpha + 3)z^3 + ((\alpha + 5)z^2 + z^4 + 2)\Gamma(\alpha + 1) + (\alpha + 2)(\alpha + 3)z + z^5.
\end{align*}
\]

We have that each \( f_n(z) \) is a Hurwitz polynomial, i.e. all its roots are located in the open left-hand side of the complex plane. For instance, for \( \alpha = \sqrt{2} \) we have the following roots:

\[
\begin{align*}
f_1(z) : & \quad -1.25382, \\
f_2(z) : & \quad -0.626908 \pm 1.42169i, \\
f_3(z) : & \quad -0.408558, \quad -0.422629 \pm 1.70008i, \\
f_4(z) : & \quad -0.102035 \pm 2.22219i, \quad -0.524872 \pm 1.17906i, \\
f_5(z) : & \quad -0.180388, \quad -0.0420382 \pm 2.52333i, \quad -0.494676 \pm 1.39211i.
\end{align*}
\]

For this specific value of \( \alpha \), we have plotted the zeros of the first polynomials in different colors in Figure 1.
A Favard Type Theorem for Hurwitz Polynomials

1.2
1.0
0.8
0.6
0.4
0.2
-1.2
-1.0
-0.8
-0.6
-0.4
-0.2
-2
-1
1
2

Figure 1. Zeros of the polynomials $f_n(z)$ for $\alpha = \sqrt{2}$ and $n = 1, 2, 3, 4, 5$: $f_1(z)$ in black, $f_2(z)$ in blue, $f_3(z)$ in magenta, $f_4(z)$ in orange, and $f_5(z)$ in red.

5.2. Hurwitz sequence from a pair of sequences $(A_{1,j}, B_{1,j})_{j=0}^{\infty}$. Let the sequences $(A_{1,j})_{j=0}^{\infty}$, $(B_{1,j})_{j=0}^{\infty}$ with $A_{1,j} = 2j + 1$ and $B_{1,j} = (j + 1)^2$ be given. Let $s_0 = 1$ and $s_1 = 1$. By employing (29)-(30), we have $A_{2,j} = 2(j + 1)$ and $B_{2,j} = (j + 1)(j + 2)$ for $j \geq 0$. Finally, by employing (7)-(10), we attain

$$
\begin{align*}
  f_1(z) &= z + 1, \\
  f_2(z) &= z^2 + z + 1, \\
  f_3(z) &= z^3 + z^2 + 2z + 1, \\
  f_4(z) &= z^4 + z^3 + 4z^2 + 3z + 2, \\
  f_5(z) &= z^5 + z^4 + 6z^3 + 5z^2 + 6z + 2, \\
  f_6(z) &= z^6 + z^5 + 9z^4 + 8z^3 + 18z^2 + 11z + 6, \\
  f_7(z) &= z^7 + z^6 + 12z^5 + 11z^4 + 36z^3 + 26z^2 + 24z + 6.
\end{align*}
$$

The zeros of the first Hurwitz polynomials have been included in Figure 2.

5.3. Hurwitz sequence from exceptional Laguerre polynomials. In recent years, the so-called exceptional polynomials [15, 16] have attracted the interest of
many researchers. The exceptional polynomials do not exist for every degree, but they still constitute a complete orthogonal system. It is important to emphasize that the corresponding orthogonality relation is only true for the admissible degrees, and moreover, they are complete in an appropriate Hilbert space setting. In this example, we shall consider the $X_1$-exceptional Laguerre polynomials [15, Section 2.2], which are orthogonal with respect to

$$
\varrho(x; \alpha) = \frac{\exp(-x)x^\alpha}{(x + \alpha)^2}.
$$

This sequence of $X_1$-exceptional Laguerre polynomials starts with a polynomial of degree 1. The moments of the weight function can be computed for $\alpha > 0$ as

$$
s_n^{(\alpha)} = \int_0^\infty x^n \varrho(x) dx = (e^{\alpha(2\alpha + n)}E_{\alpha+n}(\alpha) - 1) \Gamma(\alpha + n), \quad n \geq 0,
$$

where $E_j(z)$ denotes the exponential integral function [1]. By using Remark 8, we get the following first elements of the new Hurwitz sequence from exceptional $X_1$-Laguerre polynomials that are detailed for the specific value $\alpha = 3$:

- $f_1(z) = z + 0.152522$,
- $f_2(z) = z^2 + 0.152522z + 3.05643$,
- $f_3(z) = (z + 0.033821) (z^2 + 0.118701z + 3.92213)$,
- $f_4(z) = (z^2 + 0.0384332z + 5.84973) (z^2 + 0.114089z + 2.10959)$,
- $f_5(z) = (z + 0.0122237) (z^2 + 0.0182556z + 6.96713) (z^2 + 0.122043z + 2.77288)$.

We have that each $f_n(z)$ is a Hurwitz polynomial, i.e. all its roots are located in the open left-hand side of the complex plane as shown in Figure 3:

- $f_1(z) : -0.152522$,
- $f_2(z) : -0.076261 \pm 1.7466 i$,
- $f_3(z) : -0.033821, -0.0593505 \pm 1.97955 i$,
- $f_4(z) : -0.0570444 \pm 1.45132 i - 0.0192166 \pm 2.41855 i$,
- $f_5(z) : -0.0122237, -0.0610213 \pm 1.66408 i, -0.0091278 \pm 2.63952 i$.

![Figure 3. Zeros of the polynomials $f_n(z)$ for $\alpha = 3$ and $n = 1, 2, 3, 4, 5$: $f_1(z)$ in black, $f_2(z)$ in blue, $f_3(z)$ in magenta, $f_4(z)$ in orange, and $f_5(z)$ in red.](image)
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