CAUCHY PROBLEM FOR HYPERBOLIC OPERATORS WITH TRIPLE EFFECTIVE CHARACTERISTICS ON THE INITIAL PLANE

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ABSTRACT. We study effectively hyperbolic operators $P$ with triple characteristics points lying on $t = 0$. Under some conditions on the principal symbol of $P$ one proves that the Cauchy problem for $P$ in $[0,T] \times U$ is well posed for every choice of lower order terms. Our results improves those in [11] since we don’t assume the condition (E) of [11] satisfied.

1. INTRODUCTION

In this paper we study the Cauchy problem for a differential operator

$$P(t, x, D_t, D_x) = \sum_{k+|\alpha|\leq 3} c_{k,\alpha}(t, x) D_t^k D_x^\alpha , \quad D_t = -i\partial_t, \quad D_{x_j} = -i\partial_{x_j}$$

of order 3 with smooth coefficients $c_{k,\alpha}(t, x)$, $t \in \mathbb{R}$, $x \in \Omega$, $c_{3,0} \equiv 1$. Denote by

$$p(t, x, \tau, \xi) = \sum_{k+|\alpha|=3} c_{k,\alpha}(t, x) \tau^k \xi^\alpha = \tau^3 + q_1(t, x, \xi)\tau^2 + q_2(t, x, \xi)\tau + q_3(t, x, \xi)$$

the principal symbol of $P$. With a real symbol $\varphi \in S^0_{1,0}$ one can write

$$P = (D_t - \text{Op}(\varphi)(D))^3 + \text{Op}(a)(D)(D_t - \text{Op}(\varphi)(D))^2 - \text{Op}(b)(D)^2(D_t - \text{Op}(\varphi)(D))$$

$$+ \text{Op}(c)(D)^3 - \sum_{j=0}^{2} \text{Op}(\tilde{b}_{1j})(D)^j(D_t - \text{Op}(\varphi)(D))^{2-j} \quad (1.1)$$

which is a differential operator in $t$. Here the symbols $a$, $b$, $c \in S^0_{1,0}$ coincide with

$$q_1(\xi)^{-1} + 3\varphi, \quad -(q_2(\xi)^{-2} + 2\varphi q_1(\xi)^{-1} + 3\varphi^2), \quad q_3(\xi)^{-3} + \varphi q_2(\xi)^{-2} + \varphi^2(\xi)^{-1} + \varphi^3,$$

respectively, $\tilde{b}_{1j} \in S^0_{0,1}$, $j = 0, 1, 2$ (see [3]), and $\langle D \rangle$ has symbol $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$. Throughout the paper we work with symbols $s(t, x, \xi)$ which depend smoothly on $t \in [0,T]$ and we use the Weyl quantization

$$s(t, x, D)u = (\text{Op}_w(s)u)(x) = (2\pi)^{-n} \int \int e^{i(x-y)\xi} \hat{s}\left(t, \frac{x+y}{2}, \xi\right) u(y) dy d\xi.$$

We will use the notation $S^m_{0,1}$ for the class of symbols (see [3]) and we abbreviate $S^m_{1,0}$ to $S^m$ and $\text{Op}_w(s)$ to $\text{Op}(s)$. First we assume that the principal symbol

$$p(t, x, \tau, \xi) = (\tau - \varphi(\xi))^3 + a(\xi)(\tau - \varphi(\xi))^2 - b(\xi)^2(\tau - \varphi(\xi)) + c(\xi)^3 \quad (1.2)$$

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is hyperbolic, that is the roots of equation $p = 0$ with respect to $\tau$ are real for $(t, x, \xi) \in [0, T] \times \Omega \times \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^a$ is an open set. Second we assume also that $p$ has triple characteristic points only if $t = 0$ and $P$ is effectively hyperbolic (see [4], [11]) at every triple characteristic points $\rho = (0, x, \tau, \xi)$ which is equivalent (see [4]) to the condition

$$\frac{\partial^2 p}{\partial t \partial \tau}(\rho) < 0.$$  

Recall that an operator is effectively hyperbolic if the fundamental matrix $F_p(z)$ of the principal symbol $p$ has two non-vanishing eigenvalues $\pm \mu(z)$ at every point $z$ where $dp(z) = 0$. An effectively hyperbolic operator may have triple characteristics only for $t = 0$ or $t = T$ (see [4]). Consequently, at a triple characteristic point $\rho_0 = (0, x_0, 0, \xi_0)$, assuming $\varphi(0, x_0, \xi_0) = 0$, we have $b_1(0, x_0, \xi_0) > 0$. Moreover, at $\rho_0$ we have $a(0, x_0, \xi_0) = b(0, x_0, \xi_0) = c(0, x_0, \xi_0) = 0$.

Our purpose is to study the Cauchy problem for $P$ and to prove that under some conditions on $p$ this problem is well posed for every choice of lower order terms. This property is called strong hyperbolicity and the effective hyperbolicity of $P$ is a necessary condition for it ([11]). For operators having only double characteristics every effectively hyperbolic operator is strongly hyperbolic and we refer to [9] for the related works. The conjecture is that effectively hyperbolic operators with triple characteristic points on $t = 0$ are strongly hyperbolic (see [4], [1], [11]). On the other hand, for some class of hyperbolic operators with triple characteristics the above conjecture has been proved in [6], [1], [11], but the general case is still an open problem.

In [11] the strong hyperbolicity was established under the condition (E) saying that for some $\delta > 0$ we have the lower bound

$$\frac{\Delta}{\langle \xi \rangle^6} \geq \delta t \left( \frac{\Delta_0}{\langle \xi \rangle^2} \right)^2.$$  

Here $\Delta \in S^6$ is the discriminant of the equation $p = 0$ with respect to $\tau$, while $\Delta_0 \in S^2$ is the discriminant of the equation $\frac{\partial p}{\partial \tau} = 0$ with respect to $\tau$. In [11] it was introduced also a weaker condition (H) saying that with some constant $\delta > 0$ we have

$$\frac{\Delta}{\langle \xi \rangle^6} \geq \delta^2 \frac{\Delta_0}{\langle \xi \rangle^2}.$$  

We can consider a microlocal version of the conditions (E) and (H) assuming the above inequalities hold for $(t, x, \xi)$ in a small conic neighborhood $W_0$ of every triple characteristic point $(0, x_0, \xi_0)$. The main goal of this paper will be stated in Theorem 4.1 and Corollary 4.5 which improve the results in [11] and show that we have a strong hyperbolicity for some operators for which (E) is not satisfied, but (H) holds. In particular, we cover the case of operators whose principal symbol $p$ admits a microlocal factorization with one smooth root under the condition that there are no double characteristic points of $p$ converging to a triple characteristic point $(0, x, 0, \xi)$ (see Example 1.1).

Concerning the symbols $a(t, x, \xi)$, $b(t, x, \xi)$, $c(t, x, \xi)$, we assume the existence of $\delta_1 > 0$ such that

$$b(t, x, \xi) \geq \delta_1 t, \quad c = O(b^2), \quad \langle \xi \rangle^\alpha \partial_\xi^\alpha \partial_x^3 c = O(b), \quad |\alpha + \beta| = 1, \quad \langle \xi \rangle^\alpha \partial_\xi^\alpha \partial_x^3 c = O(\sqrt{b}), \quad |\alpha + \beta| = 2, \quad (1.3)$$

$$\partial_{\xi} c = O(b), \quad \langle \xi \rangle^\alpha \partial_\xi^\alpha \partial_x^3(ac) = O(\sqrt{b}), \quad |\alpha + \beta| = 3.$$
It is clear that the condition (1.3) are satisfied if
\[ b(t, x, \xi) \geq \delta_1 t, \quad \langle \xi \rangle^{\alpha} \partial_t \partial_{\xi}^2 c = O(b^{2-|\alpha+\beta|/2-|\gamma|}) \quad \text{for} \quad |\alpha + \beta + \gamma| \leq 3, \quad \gamma = 0, 1. \] (1.4)
In fact, we assume a slightly weaker microlocal conditions formulated in (3.11) and Theorem 4.1.

Below we present two examples of operators with triple characteristics on \( t = 0 \) satisfying the above assumptions.

**Example 1.1.** Assume \( c \equiv 0 \). Then the symbol \( p \) becomes \( p = ((\tau - \varphi(\xi))^2 + a(\xi)(\tau - \varphi(\xi)) - b(\xi)^2)(\tau - \varphi(\xi)) \). Let \( p = (0, x_0, \varphi(0, x_0, \xi_0)), \xi_0, \) be a triple characteristic point. For small \( t > 0 \) we have \( b(t, x_0, \xi_0) > 0 \). If for some \( (y, \eta) \) sufficiently close to \( (x_0, \xi_0) \) we have \( b(0, y, \eta) < 0 \), then there exists \( z = (t^*, x^*, \xi^*) \) with \( t^* > 0 \) such that \( b(z) = 0 \) and the equation \( (\tau - \varphi(\xi))^2 + a(\xi)(\tau - \varphi(\xi)) - b(\xi)^2 = 0 \) has a root \( \varphi(z)/\varphi(\xi^*) \) for \( z \). This implies the existence of a double characteristic point \((t^*, x^*, \varphi(z)/\varphi(\xi^*)), \xi^*)\) of \( p \). We exclude this possibility, assuming \( b(0, x, \xi) \geq 0 \) for \((x, \xi)\) close to \((x_0, \xi_0)\).

**Remark 1.1.** For the operator in Example 1.1, the discriminant of the equation \( p = 0 \) has the form \( \Delta = b^2(a^2 + 4b)(\xi)^6 \), while \( \Delta_0 = 4(a^2 + 3b)(\xi)^2 \). Therefore the condition (E) is reduced to
\[ b^2(a^2 + 4b) \geq \delta t(a^2 + 3b)^2. \]
If \( b = O(t) \), this inequality yields \( b^2a^2 + 4b^3 \geq \delta ta^4 \) and hence \( a^2 \leq O(t^2)/\delta t = O(t) \) which is not satisfied in any small neighborhood of a triple characteristic point \((0, x_0, \varphi(0, x_0, \xi_0)), \xi_0, \) unless \( a(0, x, \xi) = 0 \) for all \((0, x, \xi)\) close to the point \((0, x_0, \xi_0)\). On the other hand, the inequality
\[ b^2(a^2 + 4b) \geq \delta t^2(a^2 + 3b) \]
obviously holds \((b \geq \delta t \text{ is assumed})\), hence \((H) \) is satisfied.

The Example 1.1 covers the case when the principal symbol \( p \) admits a factorization
\[ p = (\tau^2 + 2d(t, x, \xi))\tau + f(t, x, \xi)(\tau - \lambda(t, x, \xi)) \]
with \( C^\infty \) smooth real root \( \lambda(t, x, \xi) \) and \( p \) has not double characteristic points in a neighborhood of \((0, x_0, \xi_0)\). In fact, we may write
\[ p = ((\tau - \lambda)^2 + 2(\lambda + d)(\tau - \lambda) + \lambda^2 + 2d\lambda + f)(\tau - \lambda) \]
and taking \( \varphi = \lambda(\xi)^{-1} \) we reduce the symbol to Example 1.1. Notice that effectively hyperbolic operators with principal symbols admitting above factorization have been studied by V. Ivrii in [6] who proved the strong hyperbolicity constructing parametrix. Here we present another proof based on energy estimates with weight \( t^{-N} \), assuming \( P \) strictly hyperbolic for small \( t > 0 \).

**Example 1.2.** Consider the operator with principal symbol
\[ p = \xi^3 - (t + \alpha(x, \xi))(\xi)^2\tau - (t^2b_2 + tb_1 + b_0)(\xi)^3, \]
where \( \alpha, b_0, b_1, b_2 \) are zero order pseudo-differential operators and \( \alpha \geq 0 \). This class of operators has been studied in [11] under the condition (E). We write \( p \) as follows
\[ p = (\tau + b_1(\xi))^3 - 3b_1(\xi)(\tau + b_1(\xi))^2 - (t + \alpha - 3b_1^2)(\xi)^2(\tau + b_1(\xi)) \]
\[ - \left(t^2b_2 + b_0 - b_1\alpha + b_1^3\right)(\xi)^3. \]
Choosing \( \varphi = -b_1(t, x, \xi) \) one reduces the symbol \( p \) to the form (1.2) with \( a = -3b_1, \ b = t + \alpha - 3b_1^2, \ c = -(t^2b_2 + b_0 - b_1\alpha + b_1^3) \). If \( \alpha \geq 3b_1^2, \ b_0 = b_1\alpha - b_1^3, \) the condition (1.4) is satisfied, while for
the problem to the one for first order pseudo-differential systems. In Section 2 we construct a symmetrizer for operators with multiple characteristics (see [4], [1], [11]). We follow the approach in [11] reducing the problem to the one for first order pseudo-differential system. This phenomenon is typical for effectively hyperbolic operators with multiple characteristics (see [4], [1], [11]). We follow the approach in [11] reducing the problem to the one for first order pseudo-differential system. In Section 2 we construct a symmetrizer S for the principal symbol of the system following a general result (see Lemma 2.1) which has independent interest. Moreover, det S = \( \frac{1}{2\pi} \Delta \) and under our assumptions one shows that det S \( \geq \delta b^2(a^2 + 4b) \), \( \delta > 0 \). Therefore \( \Delta \geq \varepsilon t^2(a^2 + 4b) \), \( \varepsilon > 0 \), and in general the condition (E) is not satisfied. This leads to difficulties in Section 3, where a more fine analysis of the matrix pseudo-differential operators is needed. In Section 4 we show that the microlocal conditions (1.3) are sufficient for the energy estimates in Theorems 4.1 and 4.2.

2. Symmetrizer

First we recall a general result concerning the existence of a symmetrizer. Let \( p(\zeta) = \zeta^m + a_1 \zeta^{m-1} + \cdots + a_m \) be a monic hyperbolic polynomial of degree \( m \) and let \( q(\zeta) = p'(\zeta) \). Here \( a_j(t, x, \xi) \) depend on \( (t, x, \xi) \) but we omit this in the notations below. Let

\[
h_{p, q}(\zeta, \tilde{\zeta}) = \frac{p(\zeta)q(\tilde{\zeta}) - p(\tilde{\zeta})q(\zeta)}{\zeta - \tilde{\zeta}} = \sum_{i, j=1}^{m} h_{ij} \zeta^i \tilde{\zeta}^{j-1}
\]

be the Bézout form of \( p \) and \( q \). It is well known that the matrix \( H = (h_{ij}) \) is nonnegative definite (see for example [5]).

Consider the Sylvester matrix \( A_p \) corresponding to \( p(\zeta) \) which has the form

\[
A_p = \begin{pmatrix}
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 \\
-a_m & -a_{m-1} & \cdots & -a_1
\end{pmatrix}
\]

One has the following result [10] and for the sake of completeness we present the proof.

**Lemma 2.1.** ([10] Lemma 2.3.1) \( H \) symmetrizes \( A_p \) and \( \det H = \Delta^2 \) where \( \Delta \) is the difference-product of the roots of \( p(\tau) = 0 \).

**Proof.** We first treat the case when \( p(\zeta) \) is a strictly hyperbolic polynomial. Let \( \lambda_j, j = 1, \ldots, m \) be the different roots of the equation \( p(\zeta) = 0 \). Write \( p(\zeta) = \prod_{j=1}^{m} (\zeta - \lambda_j) \) and set

\[
\sigma_{\ell, k} = \prod_{1 \leq j_1 < \cdots < j_{k} \leq m, j_p \neq k} \lambda_{j_1} \cdots \lambda_{j_k}.
\]

Since \( p'(\zeta) = \sum_{k=1}^{m} \prod_{j=1, j \neq k}^{m} (\zeta - \lambda_j) = \sum_{i=1}^{m} (-1)^{m-i} \sigma_{m-i, k} \zeta^{i-1} \) it is easy to see

\[
h_{ij} = \sum_{k=1}^{m} (-1)^{i+j} \sigma_{m-i, k} \sigma_{j-1, k}.
\]
Denote by $R$ the Vandermonde’s matrix having the form

$$R = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_m \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{m-1} & \lambda_2^{m-1} & \cdots & \lambda_m^{m-1} \end{pmatrix}.$$ 

Since $\lambda_i \neq \lambda_j$, $i \neq j$, the matrix $R$ is invertible and $|\det R| = |\Delta|$. It is clear that

$$A_pR = R \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix}.$$ 

Denote by $\text{co}R = (r_{ij})$ the cofactor matrix of $R$ and by $\Delta(\lambda_1, \ldots, \lambda_k)$ the difference-product of $\lambda_1, \ldots, \lambda_k$. It is easily seen that $r_{ij}$ is divisible by $\Delta_i = \Delta(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_m)$, hence

$$r_{ij} = c_{ij}(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_m)\Delta_i. \quad (2.1)$$

Since $r_{ij}$ and $\Delta_i$ are alternating polynomials in $(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_m)$ of degree $m(m-1)/2 - j + 1$ and $(m-1)(m-2)/2$ respectively, then $c_{ij}$ is a symmetric polynomial of degree

$$m - j = m(m-1)/2 - j + 1 - (m-1)(m-2)/2.$$ 

Therefore $c_{ij}$ is a polynomial in fundamental symmetric polynomials of $(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_m)$. Noting that $\Delta_i$ is of degree $m - 2$ and $r_{ij}$ $(j \neq m)$ is of degree $m - 1$ respectively with respect to $\lambda_\ell$ $(\ell \neq i)$, one concludes that $c_{ij}$ is of degree 1 with respect to $\lambda_\ell$ $(\ell \neq i)$ which proves that

$$c_{ij} = (-1)^{i+j} \sigma_{m-j,i}. \quad (2.2)$$

Thus denoting $C = (c_{ij})$ we have $tCC = (h_{ij}) = H$. In particular, this shows that the symmetric matrix $H$ is nonnegative definite as it was mentioned above.

Set $D = \text{diag}(\Delta_1, \ldots, \Delta_m)$ and note that $D$ is invertible. Moreover it follows from $(2.1)$ that

$$C = D^{-1}(\text{co}R) = (\det R)D^{-1}R^{-1}$$

and hence

$$CA_pC^{-1} = D^{-1}(R^{-1}A_pR)D.$$ 

It is clear that $CA_pC^{-1}$ is a diagonal matrix because both $R^{-1}A_pR$ and $D$ are diagonal matrices. Then $CA_pC^{-1} = tC^{-1}tA_p^tC$ yields $tCCA_p = tA_p^tCC$ which proves that $HA_p$ is symmetric. From $C = (\det R)D^{-1}R^{-1}$ it follows that

$$C = \text{diag}\left(\pm \prod_{k \neq 1} (\lambda_i - \lambda_k), \pm \prod_{k \neq 2} (\lambda_i - \lambda_k), \ldots, \pm \prod_{k \neq m} (\lambda_i - \lambda_k)\right)R^{-1}$$

and hence $|\det C| = |\prod_{j=1}^m \prod_{k \neq j} (\lambda_k - \lambda_j)|/|\Delta| = |\Delta|$. Consequently, $\det H = \Delta^2$ and this completes the proof for strictly hyperbolic polynomial $p(\zeta)$.

Passing to the general case, introduce the polynomial

$$p_\varepsilon(\zeta) = \left(1 + \varepsilon \frac{\partial}{\partial \zeta}\right)^{m-1} p(\zeta), \; \varepsilon \neq 0.$$ 

According to [12], $p_\varepsilon(\zeta)$ is strictly hyperbolic and let $H_\varepsilon = tC_\varepsilon C_\varepsilon$ be the symmetrizer for $A_{p_\varepsilon}$ constructed above. Obviously, as $\varepsilon \to 0$, we have $A_{p_\varepsilon} \to A_p$ since the coefficients of $p_\varepsilon(\zeta)$ go to the ones of $p(\zeta)$. The roots of $p(\zeta)$ depend continuously on the coefficients and this yields $\lambda_{j,\varepsilon} \to \lambda_j, \lambda_{j,\varepsilon}$ being the roots of $p_\varepsilon(\zeta) = 0$. The equalities $(2.2)$ imply $C_\varepsilon \to C$ and passing to the limit $\varepsilon \to 0$, we obtain the result. 

\qed
Note that $H$ is different from the Leray’s symmetrizer\(^\text{[7]}\) since if $B$ is the Leray’s symmetrizer, then $\det B = \Delta^2 (m-1)$. Now consider

$$
\tilde{A}_p = \begin{pmatrix}
-a_1 & -a_2 & \ldots & -a_m \\
1 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ldots & 1 & 0
\end{pmatrix}.
$$

**Corollary 2.1.** Let $J = (\delta_{i,m+1-j})$, where $\delta_{ij}$ is the Kronecker’s delta. Then $\tilde{H} = JH^tJ$ symmetrizes $\tilde{A}_p$ and $\det \tilde{H} = \Delta^2$.

**Proof.** Since $\tilde{A}_p = JA_p^tJ$ and $^tJJ = I$ the proof is immediate. \(\square\)

With $U = ^t((D_t - \text{Op}(\varphi)(D))^2u, (D)(D_t - \text{Op}(\varphi)(D))u, (D)^2u)$ the equation $Pu = f$ is reduced

$$
D_tU = \text{Op}(\varphi)(D)U + (\text{Op}(A)(D) + \text{Op}(B))U + F,
$$

(2.3)

where $F = ^t(f, 0, 0)$ and

$$
A(t, x, \xi) = \begin{pmatrix}
-a & b & -c \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}, \quad B(t, x, \xi) = \begin{pmatrix}
b_{10} & b_{11} & b_{12} \\
0 & b_{21} & 0 \\
0 & 0 & b_{32}
\end{pmatrix},
$$

where $b_{ij} \in S^0_{1,0}$.

Introduce

$$
S(t, x, \xi) = \frac{1}{3} \begin{pmatrix}
3 & 2a & -b \\
2a & 2(a^2 + b) & -ab - 3c \\
-b & -ab - 3c & b^2 - 2ac
\end{pmatrix},
$$

which is a representation matrix (conjugated by $J$ in Corollary 2.1) of the Bézout form of $p(\tau) = \tau^3 + a\tau^2 - b\tau + c$ and $p'(\tau)$ (see for example [3], [5]). Therefore $S$ symmetrizes $A$ so that

$$
S(t, x, \xi)A(t, x, \xi) = \frac{1}{3} \begin{pmatrix}
-a & 2b & -3c \\
2b & ab - 3c & -2ac \\
-3c & -2ac & bc
\end{pmatrix}.
$$

(2.4)

Note that when $c = 0$ one has

$$
S_0(t, x, \xi) = \frac{1}{3} \begin{pmatrix}
3 & 2a & -b \\
2a & 2(a^2 + b) & -ab \\
-b & -ab & b^2
\end{pmatrix}
$$

and hence

$$
\det S_0(t, x, \xi) = \frac{1}{27} b^2 (a^2 + 4b).
$$

**Lemma 2.2.** There exist $\bar{\varepsilon} > 0$ and $\delta > 0$ such that

$$
\det S \geq \delta b^2 (a^2 + b)
$$

if $|ac| \leq \bar{\varepsilon} b^2$ and $|c| \leq \bar{\varepsilon} b^{3/2}$.\(\quad \)
Lemma 2.4. Assume \( \varepsilon \) indeed since there exist \( \varepsilon \geq \varepsilon b^2 a^2 \), \( |abc| \leq \varepsilon b^3 \), \( |c^2| \leq \varepsilon^2 b^3 \) choosing \( \varepsilon = 1/50 \) for instance, the assertion is clear.

**Proof.** Note that
\[ \det S = \det S_0 + \frac{1}{27} \left\{ -4a^3c - 18abc - 27c^2 \right\}. \]

Since
\[ |a^3c| \leq \varepsilon b^2 a^2, \quad |abc| \leq \varepsilon b^3, \quad |c^2| \leq \varepsilon^2 b^3 \]
choosing \( \varepsilon = 1/50 \) for instance, the assertion is clear.

**Lemma 2.3.** There exist \( \varepsilon > 0 \) and \( \varepsilon_1 > 0 \) such that
\[ S(t, x, \xi) \gg \varepsilon_1 t \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b \end{pmatrix} = \varepsilon_1 t J, \]
provided \( |ac| \leq \varepsilon b^2 \) and \( |c| \leq \varepsilon b^{3/2} \).

**Proof.** Since
\[ 3S - \varepsilon_1 t J = \begin{pmatrix} 3 - \varepsilon_1 t & 2a & -b \\ 2a & 2a^2 + 2b - \varepsilon_1 t & -ab - 3c \\ -b & -ab - 3c & b^2 - \varepsilon_1 tb - 2ac \end{pmatrix}, \]
one obtains
\[ \det (3S - \varepsilon_1 t J) = \det 3S + \varepsilon_1 O(b^2 (b + a^2)). \]

Indeed
\[ (3 - \varepsilon_1 t)(2a^2 + 2b - \varepsilon_1 t)(b^2 - \varepsilon_1 tb - 2ac) = 3(2a^2 + 2b)(b^2 - 2ac) + \varepsilon_1 O(t(b + a^2)), \]
\[ b^2(2a^2 + 2b - \varepsilon_1 t) = b^2(2a^2 + 2b) + \varepsilon_1 O(t(b + a^2)), \]
\[ 4a^2(b^2 - \varepsilon_1 tb - 2ac) = 4a^2(b^2 - 2ac) + \varepsilon_1 O(t(ba^2)), \]
\[ (3 - \varepsilon_1 t)(ab + 3c)^2 = 3(ab + 3c)^2 + \varepsilon_1 O(t(b^2)). \]

Noting \( b \geq \delta t \), one gets the above representation and we deduce \( \det(3S - \varepsilon_1 t J) \geq 0 \) for small \( \varepsilon_1 \).

In the same way one treats the principal minors of order 2. For example
\[ (3 - \varepsilon_1 t)(2a^2 + 2b - \varepsilon_1 t) - 4a^2 = 2a^2 + 6b - \varepsilon_1 t(2a^2 + 2b) + \varepsilon_1^2 t^2 \geq 2(a^2 + b)(1 - \varepsilon_1 t) \geq 0, \]
\[ (3 - \varepsilon_1 t)(b^2 - \varepsilon_1 tb - 2ac) - b^2 = 2b^2 - 6ac - \varepsilon_1 t(b^2 - 2ac + 3b) + \varepsilon_1^2 t^2 b \]
\[ \geq b^2 - 4ac - 3\varepsilon_1 tb + (b^2 - 2ac)(1 - \varepsilon_1 t) \]
\[ \geq (1 - 4\varepsilon)b^2 - 3\varepsilon_1 tb + (1 - 2\varepsilon)(1 - \varepsilon_1 t)b^2 \geq 0, \]
\[ (2a^2 + 2b - \varepsilon_1 t)(b^2 - \varepsilon_1 tb - 2ac) - (ab + 3c)^2 \geq a^2b^2 + 2b^3 - 10abc - 9c^2 - 4a^3c \]
\[ -3\varepsilon_1 tb^2 + 2\varepsilon_1 tac - 2\varepsilon_1 tba^2 \]
\[ \geq (1 - 4\varepsilon)a^2b^2 + (2 - 10\varepsilon - 9\varepsilon^2)b^3 - (3\varepsilon_1 + 2\varepsilon_1 \varepsilon)tb^2 - 2\varepsilon_1 tba^2 \geq 0 \]
since all terms involving \( \varepsilon_1 t \) can be compensated by \( a^2 b^2 + 2b^3 \).

**Lemma 2.4.** Assume \( (\xi) \alpha c^{(\alpha)}_{(\beta)} = O(\sqrt{b}) \) for \( |\alpha + \beta| = 2 \) and \( (\xi)^{\alpha}(ac)^{(\alpha)}_{(\beta)} = O(\sqrt{b}) \) for \( |\alpha + \beta| = 3 \).

There exists \( C > 0 \) such that for \( U \in C^\infty(\mathbb{R}^t : C^\infty(\mathbb{R}^n)) \) we have
\[ \text{Re}(\text{Op}(S)U, U) \geq \varepsilon_1 t \left( \sum_{j=1}^2 \|U_j\|^2 + (\text{Op}(b)U_3, U_3) \right) - Ct^{-1}\|D\|^{-1}U\|^2. \]
Proof. We will follow the argument of [11, Section 3] and we use the notation \( \partial^\alpha \partial_x^\beta Q = Q^{(\alpha)}_{(\beta)} \).

Recall that we have the representation
\[
Q_F - \text{Op}(Q) = \text{Op}\left( \sum_{2\leq |\alpha + \beta| \leq 3} \psi_{\alpha,\beta}(\xi)Q^{(\alpha)}_{(\beta)} \right) + \text{Op}(R)
\] (2.5)
with \( R \in S^{-2}_{1/2,0} \) and real symbols \( \psi_{\alpha,\beta} \in S^{(|\alpha| - |\beta|)/2} \), where \( Q_F \) is the Freidrichs part of \( Q \) (see [11, Appendix], [2]) and hence \((Q_F U, U) \geq 0\).

Notice that \( b \) is real, hence \((\text{Op}(b)U_3, U_3) = \text{Re}(\text{Op}(b)U_3, U_3)\). Setting \( Q = S - 2\varepsilon_1 tJ \), we have

\[
\text{Re}(\text{Op}(S)U, U) = \text{Re}(\text{Op}(Q)U, U) + 2\varepsilon_t \left( \sum_{j=1}^{2} \|U_j\|^2 + (\text{Op}(b)U_3, U_3) \right),
\]
and it is enough to prove

\[
|\text{Re}(\text{Op}\left( \sum_{2\leq |\alpha + \beta| \leq 3} \psi_{\alpha,\beta}Q^{(\alpha)}_{(\beta)} \right)U, U)| \leq \varepsilon_1 t \left( \sum_{j=1}^{2} \|U_j\|^2 + (\text{Op}(b)U_3, U_3) \right) + C\varepsilon_1^{-1}t^{-1}\|D\|^{-1}U\|^2. \quad (2.6)
\]

Indeed if this is true, then we have

\[
\text{Re}(\text{Op}(Q)U, U) \geq (Q_F U, U) - \varepsilon_1 t \left( \sum_{j=1}^{2} \|U_j\|^2 + (\text{Op}(b)U_3, U_3) \right) - C\varepsilon_1^{-1}t^{-1}\|D\|^{-1}U\|^2 - C\|D\|^{-1}U\|^2
\]
\[
\geq -\varepsilon_1 t \left( \sum_{j=1}^{2} \|U_j\|^2 + (\text{Op}(b)U_3, U_3) \right) - C\varepsilon_1^{-1}t^{-1}\|D\|^{-1}U\|^2.
\]

Thus we conclude the assertion.

To prove (2.6), consider \( \text{Re}(\text{Op}(\psi_{\alpha,\beta}Q^{(\alpha)}_{(\beta)}U, U) \) with \(|\alpha + \beta| = 2\). Setting \( g = b^2 - \varepsilon tb - 2ac \), one has

\[
Q^{(\alpha)}_{(\beta)} = \begin{pmatrix}
0 & S^{-|\alpha|} & S^{-|\alpha|} \\
S^{-|\alpha|} & S^{-|\alpha|} & S^{-|\alpha|} \\
S^{-|\alpha|} & S^{-|\alpha|} & g^{(\alpha)}_{(\beta)}
\end{pmatrix}.
\]

Here and below \( S^m \) denotes some symbol in the class \( S^m \). This yields

\[
\psi_{\alpha,\beta}Q^{(\alpha)}_{(\beta)} = \begin{pmatrix}
0 & S^{-1} & S^{-1} \\
S^{-1} & S^{-1} & S^{-1} \\
S^{-1} & S^{-1} & \psi_{\alpha,\beta}g^{(\alpha)}_{(\beta)}
\end{pmatrix}
\]
and hence

\[
|(\text{Op}(\psi_{\alpha,\beta}Q^{(\alpha)}_{(\beta)}U, U)| \leq \varepsilon_1 t \sum_{j=1}^{2} \|U_j\|^2 + C\varepsilon_1^{-1}t^{-1}\|D\|^{-1}U\|^2 + |\text{Re}(\text{Op}(\psi_{\alpha,\beta}g^{(\alpha)}_{(\beta)}U_3, U_3)|.
\]
Let $T = \psi_{\alpha\beta}g^{(\alpha)}_{(\beta)}\langle \xi \rangle$. Then $\psi_{\alpha\beta}g^{(\alpha)}_{(\beta)} = \text{Re}(T\#\langle \xi \rangle^{-1}) + S^{-2}$ and
\[\text{Re}(\text{Op}(\psi_{\alpha\beta}g^{(\alpha)}_{(\beta)})U_3, U_3) \leq \varepsilon_1 t\|\text{Op}(T)U_3\|^2 + C\varepsilon_1^{-1}t^{-1}\|\langle D \rangle^{-1}U_3\|^2.\]

Note that $\|\text{Op}(T)U_3\|^2 = (\text{Op}(T\#T)U_3, U_3)$ and $T\#T = T^2 + S^{-2}$. Therefore there exists $C > 0$ such that
\[T^2 \leq Cb\]
because $\langle \xi \rangle^{\beta} c^{(\alpha)}_{(\beta)} = O(\sqrt{b})$ and $\langle \xi \rangle^{\alpha} (b(b - \varepsilon_1 t))^{(\alpha)}_{(\beta)} = O(\sqrt{b})$ and $b \geq \delta t$. Applying the Fefferman-Phong inequality for the operator with symbol $Cb - T^2$, one proves the assertion.

For the case $|\alpha + \beta| = 3$ with $T_1 = \psi_{\alpha\beta}g^{(\alpha)}_{(\beta)}\langle \xi \rangle^{3/2}$ we have the inequality
\[T_1^2 \leq Cb\]
with some $C > 0$. Indeed, $\langle \xi \rangle^{\alpha}(ac)^{\alpha}_{(\beta)} = O(\sqrt{b})$ and $\langle \xi \rangle^{\alpha} (b(b - \varepsilon_1 t))^{(\alpha)}_{(\beta)} = O(\sqrt{b})$. Repeating the above argument, we complete the proof. \qed

**Corollary 2.2.** Let $\tilde{S} = S + \lambda t^{-1}\langle \xi \rangle^{-2}I$. Then there exists $\lambda_0 > 0$ such that for $\lambda \geq \lambda_0$ we have
\[\text{Re}(\text{Op}(\tilde{S})U, U) = \text{Re}(\text{Op}(S)U, U) + \lambda t^{-1}\|\langle D \rangle^{-1}U\|^2\]
\[\geq \varepsilon_1 t\left(\sum_{j=1}^{2}\|U_j\|^2 + (\text{Op}(b)U_3, U_3)\right) + (\lambda/2)t^{-1}\|\langle D \rangle^{-1}U\|^2.\]

**Corollary 2.3.** There exist $\delta_2 > 0$ and $\lambda_0 > 0$ such that
\[\text{Re}(\text{Op}(\tilde{S})U, U) \geq \delta_2 t^2\|U\|^2 + (\lambda/2)t^{-1}\|\langle D \rangle^{-1}U\|^2, \quad \lambda \geq \lambda_0.\]

**Proof.** Since there exists $\delta_1 > 0$ such that $b \geq \delta_1 t$ from the Fefferman-Phong inequality for the scalar symbol $b - \delta_1 t$ one deduces
\[\text{Op}(b)U_3, U_3 \geq \delta_1 t\|U_3\|^2 - C\|\langle D \rangle^{-1}U_3\|^2\]
which proves the assertion thanks to Corollary 2.2. \qed

### 3. Energy estimates

Consider the energy $(t^{-N}e^{-\gamma t}\text{Op}(\tilde{S})U, U)$, where $(\cdot, \cdot)$ is the $L^2(\mathbb{R}^n)$ inner product and $N > 0$, $\gamma > 0$ are positive parameters. Then one has
\[\partial_t(t^{-N}e^{-\gamma t}\text{Op}(\tilde{S})U, U) = -N(t^{-N-1}e^{-\gamma t}\text{Op}(\tilde{S})U, U) - \gamma(t^{-N}e^{-\gamma t}\text{Op}(\tilde{S})U, U)\]
\[+(t^{-N}e^{-\gamma t}\text{Op}(\partial_t S)U, U) - \lambda(N + 1)t^{-N-2}e^{-\gamma t}\|\langle D \rangle^{-1}U\|^2 - \lambda\gamma t^{-N-1}e^{-\gamma t}\|\langle D \rangle^{-1}U\|^2 \quad (3.1)\]
\[= 2\text{Im}(t^{-N}e^{-\gamma t}\text{Op}(\tilde{S})\langle \varphi \rangle\langle D \rangle + \text{Op}(A)\langle D \rangle + \text{Op}(B)\langle U, U \rangle) - 2\text{Im}(t^{-N}e^{-\gamma t}\text{Op}(\tilde{S})F, U).\]

Consider $S\#A\#\langle \xi \rangle = \langle \xi \rangle\#A^*\#S$. Note that
\[S\#A = SA + \sum_{|\alpha + \beta| = 1} \frac{(-1)^{|\beta|}}{2i} S^{(\alpha)}_{(\beta)} A^{(\beta)}_{(\alpha)} + \sum_{|\alpha + \beta| = 2} \cdots + S^{-3}.\]
Writing $S = (s_{ij})$ one has

$$
\sum_{|\alpha + \beta| = 2} \cdots = \sum_{|\alpha + \beta| = 2} \cdots \left( s_{ij}^{(\alpha)} \right) \begin{pmatrix}
-a_{ij}^{(\beta)} & b_{ij}^{(\beta)} & -c_{ij}^{(\beta)} \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
S^{-2} & S^{-2} & O(\sqrt{\beta})S^{-2} \\
S^{-2} & S^{-2} & O(\sqrt{\beta})S^{-2} \\
S^{-2} & S^{-2} & O(\sqrt{\beta})S^{-2}
\end{pmatrix},
$$

because $c_{ij}^{(\beta)} = O(\sqrt{\beta})$ for $|\alpha + \beta| = 2$. Then

$$(S\#A)\langle \xi \rangle = (SA)\langle \xi \rangle + \left( \sum_{|\alpha + \beta| = 1} \cdots \right)\langle \xi \rangle + \begin{pmatrix}
S^{-1} & S^{-1} & O(\sqrt{\beta})S^{-1} \\
S^{-1} & S^{-1} & O(\sqrt{\beta})S^{-1} \\
S^{-1} & S^{-1} & O(\sqrt{\beta})S^{-1}
\end{pmatrix} + S^{-2}.
$$

Denoting the third term on the right-hand side by $K_2$, repeating the same arguments as before, it is easy to see

$$
|((Op(K_2) + Op(S^{-2}))U, U)| \leq C\left(\|D\|^{-1}U\|^2 + \sum_{j=1}^{2} \|U_j\|^2 + (Op(b)U_3, U_3)\right).
$$

(3.2)

Now we turn to the term with $|\alpha + \beta| = 1$. Note

$$
S_{ij}^{(\alpha)}A_{ij}^{(\beta)} = \left( s_{ij}^{(\alpha)} \right) \begin{pmatrix}
-a_{ij}^{(\beta)} & b_{ij}^{(\beta)} & -c_{ij}^{(\beta)} \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
S^{-1} & S^{-1} & O(\sqrt{\beta})S^{-1} \\
S^{-1} & S^{-1} & O(\sqrt{\beta})S^{-1} \\
O(\sqrt{\beta})S^{-1} & O(\sqrt{\beta})S^{-1} & O(\sqrt{\beta})S^{-1}
\end{pmatrix},
$$

since $c_{ij}^{(\alpha)} = O(\sqrt{\beta})$ and $b_{ij}^{(\alpha)} = O(\sqrt{\beta})$ for $|\alpha + \beta| = 1$ and hence

$$
\left( \sum_{|\alpha + \beta| = 1} \cdots \right)\langle \xi \rangle = \begin{pmatrix}
S^0 & S^0 & O(\sqrt{\beta})S^0 + S^{-1} \\
S^0 & S^0 & O(\sqrt{\beta})S^0 + S^{-1} \\
O(\sqrt{\beta})S^0 + S^{-1} & O(\sqrt{\beta})S^0 + S^{-1} & O(b)S^0 + O(\sqrt{\beta})S^{-1} + S^{-2}
\end{pmatrix} = K_1.
$$

The same arguments proves

$$
|((Op(K_1)U, U)| \leq C(\|D\|^{-1}U\|^2 + \sum_{j=1}^{2} \|U_j\|^2 + (Op(b)U_3, U_3)).
$$

Consider $A^*\#S$. We have the representation

$$
A^*\#S = A^*S + \sum_{|\alpha + \beta| = 1} \frac{(-1)^{|\beta|}}{2^r}(A^*)_{ij}^{(\alpha)}S_{ij}^{(\beta)} + \sum_{|\alpha + \beta| = 2} \cdots + S^{-3} = A^*S + \tilde{K}.
$$

Repeating similar arguments, one gets

$$
|((Op(\langle \xi \rangle \#\tilde{K})U, U)| \leq C(\|D\|^{-1}U\|^2 + \sum_{j=1}^{2} \|U_j\|^2 + (Op(b)U_3, U_3)).
$$

Since $A^*S = SA$, taking (2.3) into account, we see

$$
(SA)\langle \xi \rangle - \langle \xi \rangle \#(A^*S) = (SA)\langle \xi \rangle - \langle \xi \rangle \#(SA)
$$

$$
= \begin{pmatrix}
S^0 & S^0 & O(\sqrt{\beta})S^0 + S^{-1} \\
S^0 & S^0 & O(\sqrt{\beta})S^0 + S^{-1} \\
O(\sqrt{\beta})S^0 + S^{-1} & O(\sqrt{\beta})S^0 + S^{-1} & O(b)S^0 + O(\sqrt{\beta})S^{-1} + S^{-2}
\end{pmatrix}.
$$
Summarizing the above estimates, we obtain the following

**Lemma 3.5.** Assume \( \langle \xi \rangle^\alpha c_{(\beta)}^{(\alpha)} = O(\sqrt{b}) \) for \( |\alpha + \beta| \leq 2 \). There is \( C > 0 \) such that

\[
|(\text{Op}(S\#A\#\langle \xi \rangle - \langle \xi \rangle \#A^* \#S)U, U)| \leq C \left( \sum_{j=1}^{2} \|U_j\|^2 + (\text{Op}(b)U_3, U_3) + \|\langle D \rangle^{-1}U\|^2 \right).
\]

Consider \( S\#\varphi \#\langle \xi \rangle - \langle \xi \rangle \#\varphi \#S \), where \( \varphi \in S^0 \) is scalar. Recall

\[
S\#\varphi = \varphi S + \sum_{|\alpha + \beta| = 1} \frac{(-1)^{|\beta|}}{2i} S_{(\beta)}^{(\alpha)} \varphi_{(\alpha)} + \sum_{|\alpha + \beta| = 2} \cdots + S^{-3}.
\]

For \( |\alpha + \beta| = 2 \) one has

\[
S_{(\beta)}^{(\alpha)} \varphi_{(\alpha)} = \begin{pmatrix}
S^{-2} & S^{-2} & S^{-2} \\
S^{-1} & S^{-1} & S^{-1} \\
S^{-2} & S^{-1} & O(\sqrt{b})S^{-2}
\end{pmatrix}
\]

and hence

\[
(S\#\varphi)\#\langle \xi \rangle = (\varphi S)\#\langle \xi \rangle + \left( \sum_{|\alpha + \beta| = 1} \cdots \right)\#\langle \xi \rangle + \begin{pmatrix}
S^{-1} & S^{-1} & S^{-1} \\
S^{-1} & S^{-1} & S^{-1} \\
S^{-1} & S^{-1} & O(\sqrt{b})S^{-1} + S^{-2}
\end{pmatrix} + S^{-2}.
\]

Denoting the third term on the right-hand side by \( K_2 \), we have the same estimate as (3.2). Similarly one has

\[
\langle \xi \rangle \#(\varphi \#S) = \langle \xi \rangle \#(\varphi S) + \langle \xi \rangle \left( \sum_{|\alpha + \beta| = 1} \cdots \right) + \begin{pmatrix}
S^{-1} & S^{-1} & S^{-1} \\
S^{-1} & S^{-1} & S^{-1} \\
S^{-1} & S^{-1} & O(\sqrt{b})S^{-1} + S^{-2}
\end{pmatrix} + S^{-2}
\]

Consider the term with \( |\alpha + \beta| = 1 \) and observe that

\[
S_{(\beta)}^{(\alpha)} \varphi_{(\alpha)} = \begin{pmatrix}
S^{-1} & S^{-1} & O(\sqrt{b})S^{-1} \\
S^{-1} & S^{-1} & O(\sqrt{b})S^{-1} \\
O(\sqrt{b})S^{-1} & O(\sqrt{b})S^{-1} & g_{(\beta)}^{(\alpha)} \varphi_{(\alpha)}
\end{pmatrix}
\]

with \( g = b^2 - 2ac \). Therefore

\[
\langle \xi \rangle \#(S_{(\beta)}^{(\alpha)} \varphi_{(\alpha)}) = \begin{pmatrix}
S^0 & S^0 & O(\sqrt{b})S^0 + S^{-1} \\
S^0 & S^0 & O(\sqrt{b})S^0 + S^{-1} \\
O(\sqrt{b})S^0 + S^{-1} & O(\sqrt{b})S^0 + S^{-1} & O(b)S^0 + O(\sqrt{b})S^{-1} + S^{-2}
\end{pmatrix}
\]

(3.3)

because \( c_{(\beta)}^{(\alpha)} = O(b) \) for \( |\alpha + \beta| = 1 \) and then

\[
|(\text{Op}(\langle \xi \rangle \#(S_{(\beta)}^{(\alpha)} \varphi_{(\alpha)}))U, U)| \leq C \left( \sum_{j=1}^{2} \|U_j\|^2 + (\text{Op}(b)U_3, U_3) + \|\langle D \rangle^{-1}U\|^2 \right).
\]

Similar arguments are applied to \( |(\text{Op}(\varphi_{(\beta)}^{(\alpha)}S_{(\alpha)}^{(\beta)}U, U)| \). Finally, since

\[
\langle \xi \rangle \#(\varphi S) - (\varphi S)\#\langle \xi \rangle = \begin{pmatrix}
S^0 & S^0 & O(\sqrt{b})S^0 + S^{-1} \\
S^0 & S^0 & O(\sqrt{b})S^0 + S^{-1} \\
O(\sqrt{b})S^0 + S^{-1} & O(\sqrt{b})S^0 + S^{-1} & O(b)S^0 + O(\sqrt{b})S^{-1} + S^{-2}
\end{pmatrix},
\]
we obtain

**Lemma 3.6.** Assume $\langle \xi \rangle^\alpha c^{(\alpha)}_{(\beta)} = O(b)$ for $|\alpha + \beta| = 1$ and $\langle \xi \rangle^\alpha c^{(\alpha)}_{(\beta)} = O(\sqrt{b})$ for $|\alpha + \beta| = 2$. Then there exists $C > 0$ such that

$$
|\langle \text{Op}(S \# \varphi \# \langle \xi \rangle - \langle \xi \rangle \# \varphi \# S)U, U \rangle| \leq C \left( \sum_{j=1}^{2} \|U_j\|^2 + \langle \text{Op}(b)U_3, U_3 \rangle + \|D\|^{-1}U\|^2 \right).
$$

Combining Lemmas 3.5, 3.6 and Corollary 2.2, one concludes that for sufficiently large $N_1 > 0$ we have

$$
-N_1 \langle \text{Op}(\tilde{S})U, U \rangle - 2t \text{Im} \langle \text{Op}(\text{Op}(\varphi))D + \text{Op}(A)\rangle U, U \rangle \\
\leq (-N_1 \varepsilon_1 + 2C)t \left( \sum_{j=1}^{2} \|U_j\|^2 + \langle \text{Op}(b)U_3, U_3 \rangle \right) + (-N_1(\lambda/2)t^{-1} + 2Ct)\|D\|^{-1}U\|^2 \leq 0 \tag{3.4}
$$

Now we pass to the analysis of the term involving $\partial_t S$.

**Lemma 3.7.** Assume $\partial_t c = O(b)$. For $\varepsilon > 0$ sufficiently small we have

$$
S \gg \varepsilon \partial_t S.
$$

**Proof.** Since $\partial_t c = O(b)$, one has

$$
3S - \varepsilon \partial_t S = \begin{pmatrix}
3 & 2a + \varepsilon O(t) & -b + \varepsilon O(t) \\
2a + \varepsilon O(t) & 2a^2 + 2b + \varepsilon O(t) & -ab - 3c + \varepsilon O(at) + \varepsilon O(bt) \\
-b + \varepsilon O(t) & -ab - 3c + \varepsilon O(at) + \varepsilon O(bt) & b^2 - 2ac + \varepsilon O(bt)
\end{pmatrix}.
$$

It is not difficult to see that

$$
\det (3S - \varepsilon \partial_t S) = \det 3S + \varepsilon O(b^2(b + a^2))
$$

because $t = O(b)$. \qed

**Lemma 3.8.** Assume $\partial_t c = O(b)$, $\langle \xi \rangle^\alpha c^{(\alpha)}_{(\beta)} = O(\sqrt{b})$ for $|\alpha + \beta| = 2$ and $\langle \xi \rangle^\alpha c^{(\alpha)}_{(\beta)} = O(\sqrt{b})$ for $|\alpha + \beta| = 3$. There exist $\varepsilon > 0$ and $C > 0$ such that for $U \in C^\infty(\mathbb{R}_t : C_0^\infty(\mathbb{R}^n))$ we have

$$
\text{Re}(\text{Op}(S - \varepsilon t \partial_t S)U, U \rangle \geq -\varepsilon t \left( \sum_{j=1}^{2} \|U_j\|^2 + \langle \text{Op}(b)U_3, U_3 \rangle \right) - C \varepsilon t^{-1}\|D\|^{-1}U\|^2. \tag{3.5}
$$

**Proof.** Denoting $Q = S - 2\varepsilon t \partial_t S$, it suffices to prove

$$
|\text{Re}(\text{Op} \left( \sum_{2 \leq |\alpha + \beta| \leq 3} \psi_{\alpha\beta} Q^{(\alpha)}_{(\beta)} U, U \right) | \leq \varepsilon t \left( \sum_{j=1}^{2} \|U_j\|^2 + \langle \text{Op}(b)U_3, U_3 \rangle \right) + C \varepsilon t^{-1}\|D\|^{-1}U\|^2. \tag{3.6}
$$

Consider $\text{Re}(\text{Op}(\psi_{\alpha\beta} Q^{(\alpha)}_{(\beta)} U, U \rangle$ with $|\alpha + \beta| = 2$. Note that

$$
\psi_{\alpha\beta} Q^{(\alpha)}_{(\beta)} = \begin{pmatrix}
0 & S^{-1} & S^{-1} \\
S^{-1} & S^{-1} & S^{-1} \\
S^{-1} & S^{-1} & \psi_{\alpha\beta} \left( g^{(\alpha)}_{(\beta)} - \varepsilon t \langle \partial_t g \rangle^{(\alpha)}_{(\beta)} \right)
\end{pmatrix},
$$
where $g = b^2 - 2ac$. Consequently, one deduces
\[
|\langle \text{Op}(\psi_{\alpha,\beta} Q_{(\beta)}^{(\alpha)}) U, U \rangle| \leq \varepsilon t \sum_{j=1}^{2} \|U_j\|^2 + C\varepsilon^{-1}t^{-1}\|D\|^{-1} U\|^2 \\
+ |\text{Re}(\text{Op}(\psi_{\alpha,\beta}(g_{(\beta)}^{(\alpha)} - \varepsilon t(\partial_t g_{(\beta)}^{(\alpha)})) U, U)|.
\]
Setting
\[ T = \psi_{\alpha,\beta}(g_{(\beta)}^{(\alpha)} - \varepsilon t(\partial_t g_{(\beta)}^{(\alpha)})) \in S^0, \]
we obtain \( \text{Re}(\psi_{\alpha,\beta}(g_{(\beta)}^{(\alpha)} - \varepsilon t(\partial_t g_{(\beta)}^{(\alpha)}))) = T\#(\xi)^{-1} + S^{-2}. \)
Therefore
\[
\text{Re}(\text{Op}(\psi_{\alpha,\beta}(g_{(\beta)}^{(\alpha)} - \varepsilon t(\partial_t g_{(\beta)}^{(\alpha)})) U, U) \leq \varepsilon t\|\text{Op}(T) U\|^2 + C\varepsilon^{-1}t^{-1}\|D\|^{-1} U\|^2
\]
Note that \( \|\text{Op}(T) U\|^2 = (\text{Op}(T\#T) U, U) \) and \( T\#T = T^2 + S^{-2} \). There is \( C > 0 \) such that
\[ T^2 \leq Cb \]
because \( t = O(b) \) and \( \langle \xi \rangle^\alpha \ell_{(\beta)}^{(\alpha)} = O(\sqrt{b}) \) so that \( Cb - T^2 \geq 0 \). Then applying the Fefferman-Phong inequality, we prove the assertion. Let \( |\alpha + \beta| = 3 \) then with \( T_1 = (\psi_{\alpha,\beta}(g_{(\beta)}^{(\alpha)} - \varepsilon t(\partial_t g_{(\beta)}^{(\alpha)})) \#(\xi)^{3/2} \)
\[ T_1^2 \leq Cb \]
with some \( C > 0 \) since \( t = O(b) \) and \( \langle \xi \rangle^\alpha (ac)_{(\beta)}^{(\alpha)} = O(\sqrt{b}) \) and the proof is similar.

From \((3.3)\) setting \( N_2 = \varepsilon^{-1} \) and dividing by \( \varepsilon \), one deduces
\[
\text{Re}(\text{Op}(-N_2 S + t\partial_t S) U, U) \leq t \left( \sum_{j=1}^{2} \|U_j\|^2 + \|\text{Op}(b U, U)\| \right) + C t\varepsilon^{-2}\|D\|^{-1} U\|^2
\]
and applying Corollary \(2.2\), this implies
\[
\begin{align*}
- (N_2 + N_3) &\text{Re}(\text{Op}(\tilde{S}) U, U) + t \text{Re}(\text{Op}(\partial_t S) U, U) \\
&\leq (-N_3\varepsilon_1 + 1)t \left( \sum_{j=1}^{2} \|U_j\|^2 + \|\text{Op}(b U, U)\| \right) + t^{-1}(C\varepsilon^{-2} - N_3\lambda)\|D\|^{-1} U\|^2. \tag{3.7}
\end{align*}
\]
Fixing \( \varepsilon \) and \( N_2 \), we choose \( N_3 \) sufficiently large and we arrange the right hand side of the above inequality to be negative.

Next we turn to the analysis of \( 2\text{Im}(\text{Op}(\tilde{S}) \text{Op}(B) U, U) \). Recall that \( \langle \text{Op}(\tilde{S}) U, U \rangle \gg 0 \) by Corollary \(2.3\). Consequently,
\[
2|\langle \text{Op}(\tilde{S}) \text{Op}(B) U, U \rangle| \leq N^{-1/2}(t\text{Op}(\tilde{S}) \text{Op}(B) U, \text{Op}(B) U) + N^{1/2}(t^{-1}\text{Op}(\tilde{S}) U, U) \\
= N^{-1/2}(t\text{Op}(B^*)\text{Op}(\tilde{S}) \text{Op}(B) U, U) + N^{1/2}(t^{-1}\text{Op}(\tilde{S}) U, U) \tag{3.8}
\]
\[ \leq N^{-1/2}(t^{-1}\text{Op}(B^*)\text{Op}(\tilde{S}) \text{Op}(B) U, U) + N^{1/2}(t^{-1}\text{Op}(\tilde{S}) U, U) + C_2\lambda N^{-1/2}\|D\|^{-1} U\|^2.
\]

**Lemma 3.9.** There exists \( N_4 > 0 \) depending on \( T \) and \( B \) such that for \( 0 \leq t \leq T \) and any \( \varepsilon > 0 \) there exists \( D_\varepsilon > 0 \) such that
\[
\text{Re}(\text{Op}(N_4 S - t^2 B^* SB) U, U) \geq -\varepsilon t \left( \sum_{j=1}^{2} \|U_j\|^2 + \langle c U, U \rangle \right) - D_\varepsilon t^{-1}\|D\|^{-1} U\|^2.
\]
Proof. Recall
\[
3S - \varepsilon t^2 B^* SB = \begin{pmatrix}
3 + \varepsilon O(t^2) & 2a + \varepsilon O(t^2) & -b + \varepsilon O(t^2) \\
2a + \varepsilon O(t^2) & 2(a^2 + b) + \varepsilon O(t^2) & -ab - 3c + \varepsilon O(t^2) \\
-b + \varepsilon O(t^2) & -ab - 3c + \varepsilon O(t^2) & b^2 - 2ac + \varepsilon O(t^2)
\end{pmatrix}
\]
which proves \(3S - \varepsilon t^2 B^* SB \geq 0\) with some \(\varepsilon = \varepsilon(T) > 0\). To justify this, notice that the terms \(\varepsilon O(t^2 b), \varepsilon O(t^2 c), \varepsilon O(t^2 a^2), \varepsilon O(t^4 a)\) can be absorbed by \(\det S\) because \(b \geq \delta t\). For example,
\[
\varepsilon t^4 |a| \leq \frac{1}{2} \varepsilon (t^5 + t^3 a^2) \leq C\varepsilon t^2 (a^2 + b).
\]
Choosing \(\varepsilon(T)\) small enough, we obtain the result. Then the rest of the proof is just a repetition of the proof of Lemma 3.8.

According to Lemma 3.9 and (3.8), one has
\[
2|\langle \text{Op}(\tilde{S})\text{Op}(B)U, U \rangle| \leq 2N_4^{-1/2}t^{1-1}(\text{Op}(\tilde{S})U, U) + \varepsilon t^2 \left( \sum_{j=1}^{2} \|U_j\|^2 + (\text{Op}(b)U_3, U_3) \right)
\]
\[
-\lambda N_4^{-1/2} t^{-1} \|D\|^{-1/2} t U_3 \|U\|^2 + C\lambda N_4^{-1/2} \|D\|^{-1} t U_3 \|U\|^2.
\]
Combining the estimates (3.4), (3.7), (3.9), it follows that
\[
\partial_t \text{Re}(t^{-N} e^{-\gamma t} \text{Op}(\tilde{S})U, U) \leq -2t^{N} e^{-\gamma t} \text{Op}(\tilde{S})F, F)
\]
\[
-(N - N_1 - N_2 - N_3 - 2N_4^{1/2})t^{-N-1} e^{-\gamma t} \text{Re}(\text{Op}(\tilde{S})U, U)
\]
\[
+ \left[ C\varepsilon - \lambda \left( N + 1 + N_4^{1/2} - \lambda C\varepsilon^{-1} \right) \right] t^{N-2} e^{-\gamma t} \|D\|^{-1} t U_3 \|U\|^2
\]
\[
+ \varepsilon t^{-N} e^{-\gamma t} \left( \sum_{j=1}^{2} \|U_j\|^2 + (\text{Op}(b)U_3, U_3) \right)
\]
\[
-(\gamma - D\varepsilon - C_1 \lambda - C t\lambda N_4^{-1/2}) t^{-N-1} e^{-\gamma t} \|D\|^{-1} t U_3 \|U\|^2.
\]
Note that
\[
2|t^{-N} e^{-\gamma t} \text{Op}(\tilde{S})F, U) \| \leq 2(t^{-N+1} e^{-\gamma t} \text{Op}(\tilde{S})F, F^{1/2} t^{-N-1} e^{-\gamma t} \text{Op}(\tilde{S})U, U^{1/2})
\]
\[
\leq (t^{-N+1} e^{-\gamma t} \text{Op}(\tilde{S})F, F) + (t^{-N-1} e^{-\gamma t} \text{Op}(\tilde{S})U, U).
\]
Denote \(N^* = N_1 + N_2 + N_3 + 2N_4^{1/2} + 2\) and we choose \(0 < \varepsilon \leq \varepsilon_1\). We fix \(\varepsilon\) and \(\lambda > 2C\varepsilon\). Next we fix \(N_4\) so that
\[
N_4^{1/2} > \lambda C\varepsilon^{-1} + 1.
\]
Then the term with \(t^{-N-2} e^{-\gamma t} \|D\|^{-1} t U_3 \|U\|^2\) is absorbed. Finally we choose \(N > N^*\) and \(\gamma\) such that \(\gamma - D\varepsilon - C_1 \lambda - C t\lambda N_4^{-1/2} T \geq 0\). Then we have
\[
\partial_t \text{Re}(t^{-N} e^{-\gamma t} \text{Op}(\tilde{S})U, U) \leq (t^{-N+1} e^{-\gamma t} \text{Op}(\tilde{S})F, F) - (N - N^*) \text{Re}(t^{-N-1} e^{-\gamma t} \text{Op}(\tilde{S})U, U).
\] (3.10)
Integrating (3.10) in \(\tau\) from \(\varepsilon > 0\) to \(t\) and taking Corollary 2.3 into account, one obtains
Proposition 3.1. Assume that

\[ b \geq \delta_1 t, \quad |ac| \leq \bar{\varepsilon} b^2, \quad |c| \leq \bar{\varepsilon} b^{3/2}, \]

\[ \langle \xi \rangle^{\alpha c}_{(\alpha)} = O(b) \text{ for } |\alpha + \beta| = 1, \quad \langle \xi \rangle^{\alpha c}_{(\beta)} = O(\sqrt{b}) \text{ for } |\alpha| = 2, \]

\[ \langle \xi \rangle^{\alpha (ac)}_{(\beta)} = O(\sqrt{b}), \quad |\alpha + \beta| = 3, \quad \partial_t c = O(b) \]

hold globally where \( \bar{\varepsilon} \) is given in Lemmas 2.2 and 2.3. Then there exist \( \delta_2 > 0, \gamma_0 > 0, N \in \mathbb{N} \) and \( C > 0 \) such that for \( \gamma \geq \gamma_0 \) and \( 0 < \varepsilon \leq t \leq T \) we have for any \( U \in C^\infty(\mathbb{R}_t \times C^0(\mathbb{R}^n)) \)

\[ \delta_2 t^{-N+2} e^{-\gamma t} \|U(t)\|^2 + \delta_2 (N - N^*) \int_{\bar{\varepsilon}}^t \tau^{-N+1} e^{-\gamma \tau} \|U(\tau)\|^2 d\tau \]

\[ \leq C\varepsilon^{-N-1} e^{-\gamma \varepsilon} \|U(\varepsilon)\|^2 + \int_{\bar{\varepsilon}}^t \tau^{-N+1} e^{-\gamma \tau} (\text{Op}(\tilde{S})F(\tau), F(\tau)) d\tau. \]

4. Microlocal energy estimates

First we prove the following

Lemma 4.10. Assume that (3.3) is satisfied in \([0, T] \times \tilde{W} \) where \( \tilde{W} \) is a conic neighborhood of \((x_0, \xi_0)\). Then there exist extensions \( \tilde{a}(t, x, \xi) \in S^0, \tilde{b}(t, x, \xi) \in S^0 \) and \( \tilde{c}(t, x, \xi) \in S^0 \) of \( a, b \) and \( c \) such that (3.11) holds globally.

Proof. Assume that (3.3) is satisfied in \([0, T] \times \tilde{W} \). Choose conic neighborhoods \( U, V, W \) of \((x_0, \xi_0)\) such that \( U \subset V \subset W \subset \tilde{W} \). Take \( 0 \leq \chi(x, \xi) \in S^0, 0 \leq \tilde{\chi}(x, \xi) \in S^0 \) such that \( \chi = 1 \) on \( V \) and \( \tilde{\chi} = 0 \) outside \( W \) and \( \chi = 0 \) on \( U \) and \( \tilde{\chi} = 1 \) outside \( V \). Choosing \( W \) and \( T \) small one can assume that \( \chi b \) is small as we please in \([0, T] \times \mathbb{R}^n\) because \( b(0, x_0, \xi_0) = 0 \). We define the extensions of \( a, b, c \) by

\[ \tilde{a} = \chi a, \quad \tilde{b} = \chi^2 b + M \tilde{\chi}, \quad \tilde{c} = \chi^3 c \]

where \( M > 0 \) is a positive constant which we will choose below. Note that

\[ |\tilde{a}\tilde{c}| = \chi^4 |ac| \leq C |a| \chi^2 b^2 \leq \varepsilon b^2, \]

\[ |\tilde{c}| = \chi^3 |c| \leq C \chi^3 b = \tilde{C} b^{3/2} \leq \varepsilon b^{3/2} \]

taking \( a(0, x_0, \xi_0) = 0, b(0, x_0, \xi_0) = 0 \) into account and choosing \( W \) small.

If \((x, \xi) \in V\) then \( \tilde{b}(t, x, \xi) = b + M \tilde{\chi} \geq \delta_1 t \) and if \((x, \xi)\) is outside \( V \) then \( \tilde{b}(t, x, \xi) = \chi b + M \geq \delta_1 t \) for \([0, T] \times \mathbb{R}^n\) choosing \( M \) so that \( M \geq \delta_1 T \). Thus we have

\[ \tilde{b}(t, x, \xi) \geq \delta_1 t \quad (t, x, \xi) \in [0, T] \times \mathbb{R}^n. \]

We turn to estimate derivatives of \( \tilde{c} \) and \( \tilde{a}\tilde{c} \). For \(|\alpha + \beta| = 1 \) it is clear that

\[ \langle \xi \rangle^{(\alpha)} \tilde{a}\tilde{c}_{(\alpha)} = \langle \xi \rangle^{(\alpha)} \langle \chi^3 c \rangle_{(\beta)} \leq C(\chi^2 b^2 + \chi^3 b) \leq C_1 \chi^2 b \leq C_1 \tilde{b}. \]

Similarly for \(|\alpha + \beta| = 2 \) one sees

\[ \langle \xi \rangle^{(\alpha)} \langle \chi^3 c \rangle_{(\beta)} \leq C(\chi^2 b^2 + \chi^2 b + \chi^3 \sqrt{b}) \leq C_1 \chi \sqrt{b} = C_1 (\chi^2 b)^{1/2} \leq C_1 \tilde{b}^{1/2}. \]
For $|\alpha + \beta| = 3$, taking $(\xi)^\alpha(a c)^{\alpha}_{(\beta)} = O(\sqrt{b})$ into account, one has

$$\langle \xi \rangle^{\alpha} |(ac)^{\alpha}_{(\beta)}| = \langle \xi \rangle^{\alpha} |(x^4 ac)^{\alpha}_{(\beta)}| \leq C(\chi b^2 + \chi^2 b + \chi^3 \sqrt{b} + \chi^4 \sqrt{b}) \leq C_1 \chi \sqrt{b} \leq C_1 \frac{b}{2}.$$ 

Since $|\partial_t e| = |\chi^3 \partial_t e| \leq C \chi^3 b \leq C b$ is obvious the proof is complete. \(\square\)

**Remark 4.2.** In the proof of Lemma 4.10 replacing $\tilde{b}$ by $\chi^2 b + M^2 \chi M' \chi_0(\xi)$ where $\chi_0(\xi) \in C_0^\infty(\mathbb{R}^n)$ which is 1 near $\xi = 0$ and $M > 0$ is a suitable positive constant it suffices to assume that (1.3) is satisfied in $[0, T] \times \overline{W}$ for $|\xi| \geq 1$.

Let $V \Subset V_1 \Subset \Omega$ and $u \in C^\infty(\mathbb{R} t : C_0^\infty(V))$ Let $\{\chi_\alpha\}$ be a finite partition of unity with $\chi_\alpha(x, \xi) \in \mathcal{S}$ so that

$$\sum_\alpha \chi_\alpha^2(x, \xi) = 1,$$

where $\chi(x) = 1$ on $\overline{V}$ and supp $\chi \subset V_1$. We can suppose that supp $\chi_\alpha \subset V_1$. We repeat the argument in [11] Section 4], studying a system

$$D_t U_\alpha = (Op(\varphi)(D) + Op(A)(D) + Op(B))U_\alpha + F_\alpha$$

with $U_\alpha = \int \left[ (D_t - Op(\varphi)(D))^2 \chi_\alpha u, (D_t - Op(\varphi)(D)) \chi_\alpha u, (D_t - Op(\varphi)(D))^2 \chi_\alpha u \right] d\tau$. One extends the coefficients $a, b, c$ and $\varphi$ outside the support of $\chi_\alpha$ and one can assume that (3.11) are satisfied globally. Thus we obtain the following

**Theorem 4.1.** Let $Y \Subset \Omega$. Assume that for every point $(x_0, \xi_0) \in T^* \Omega \setminus \{0\}$ there exist a conic neighborhood $W \subset T^* \Omega \setminus \{0\}$ and $T(x_0, \xi_0) > 0$ such that the estimates (3.11) are satisfied for $0 \leq t \leq T(x_0, \xi_0)$ and $(x, \xi) \in W$. Then there exist $c > 0$, $T_0 > 0$, $\gamma_0 > 0$, $C > 0$ and $N \in \mathbb{N}$ such that for $\gamma \geq \gamma_0$, $0 < \varepsilon < t \leq T_0$ we have for any $U \in C^\infty(\mathbb{R} t : C_0^\infty(Y))$

$$ct^{-N+2} e^{-\gamma t} \|U(t)\|^2 + c \int_\varepsilon^t \tau^{-N+1} e^{-\gamma \tau} \|U(\tau)\|^2 d\tau \leq C \varepsilon^{-N+1} e^{-\gamma \varepsilon} \|U(\varepsilon)\|^2 + C \int_\varepsilon^t \tau^{-N+1} e^{-\gamma \tau} \|f(\tau)\|^2 d\tau.$$ 

(4.1)

**Corollary 4.4.** Let $Y \Subset \Omega$. Assume that for every point $(x_0, \xi_0) \in T^* \Omega \setminus \{0\}$ there exist a conic neighborhood $W \subset T^* \Omega \setminus \{0\}$ and $T(x_0, \xi_0) > 0$ such that the estimates (1.3) are satisfied for $0 \leq t \leq T(x_0, \xi_0)$ and $(x, \xi) \in W$. Then the same assertion as in Theorem 4.1 holds.

The same argument can be applied for the adjoint operator $P^*$. With

$$V = \int \left[ (D_t - Op(\varphi)(D))^2 v, (D_t - Op(\varphi)(D)) v, (D_t - Op(\varphi)(D))^2 v \right] d\tau$$

the equation $P^* v = g$ is reduced to

$$D_t V = Op(\varphi)(D) V + Op(A)(D) + Op(\tilde{B}) V + G,$$ 

(4.2)

with $G = \int \left[ (D_t - Op(\varphi)(D))^2 v, (D_t - Op(\varphi)(D)) v, (D_t - Op(\varphi)(D))^2 v \right] d\tau$. Here the principal symbol is the same, while the lower order terms change. To study the Cauchy problem for $P^*$ in $0 < t < T$ with initial data on $t = T$ one considers

$$-\partial_t (t^N e^{\gamma t} Op(\tilde{S}) V, V) = -N(t^{N-1} e^{\gamma t} Op(\tilde{S}) V, V) - \gamma (t^N e^{\gamma t} Op(\tilde{S}) V, V)$$

$$-t^N e^{\gamma t} Op(\partial_t S) V, V - \lambda (N-1)t^{N-2} e^{\gamma t} \|D^{-1}U\|^2 - \lambda \gamma t^{N-1} e^{\gamma t} \|D^{-1}U\|^2$$

$$+ 2 \Im \left( t^N e^{\gamma t} (Op(\tilde{S}) Op(\varphi)(D) + Op(A)(D) + Op(\tilde{B}) V, V) \right) + 2 \Im \left( t^N e^{\gamma t} Op(\tilde{S}) G, V \right).$$ 

(4.3)
Repeating the argument of Section 3, one obtains the following

**Theorem 4.2.** Let \( Y \Subset \Omega \). Assume that for every point \( (x_0, \xi_0) \in T^*\Omega \setminus \{0\} \) there exist a conic neighborhood \( W \subset T^*\Omega \setminus \{0\} \) and \( T(x_0, \xi_0) > 0 \) such that the estimates (3.11) are satisfied for \( 0 \leq t \leq T(x_0, \xi_0) \) and \( (x, \xi) \in W \). Then there exist \( c > 0, T_0 > 0, \gamma_0 > 0, C > 0 \) and \( N \in \mathbb{N} \) such that for \( \gamma \geq \gamma_0, 0 < \varepsilon < t \leq T_0 \) we have for any \( V \in C^\infty(\mathbb{R}_t : C^\infty(\mathbb{Y}')) \)

\[
ct^{N+2}e^{\gamma t}\|V(t)\|^2 + c \int_t^{T_0} \tau^{N+1}e^{\gamma \tau}\|V(\tau)\|^2\,d\tau \\
\leq CT_0^{N-1}e^{\gamma T_0}\|V(T_0)\|^2 + C \int_t^{T_0} \tau^{N+1}e^{\gamma \tau}\|g(\tau)\|^2\,d\tau.
\]

(4.4)

Following the argument in [11], we may absorb the weight \( \tau^{-N} \) and obtain energy estimates with a loss of derivatives. For the sake of completeness we recall this argument. Consider \( Pu = f \) for \( u \in C^\infty(\mathbb{R}_t : C^\infty(\mathbb{R}^n)) \). Assume \( u(\varepsilon, x) = u_t(\varepsilon, x) = u_{tt}(\varepsilon, x) = 0 \). Differentiating \( Pu = f \) with respect to \( t \), we determine the functions \( D_t^ju(\varepsilon, x) = u_j(\varepsilon, x) \in C^\infty(\mathbb{R}^n) \) and set

\[
u_M(t, x) = \sum_{j=0}^{M} \frac{1}{j!}u_j(x)(i(t - \varepsilon))^j, \quad 0 < \varepsilon < t \leq T_0.
\]

Therefore \( w = u - u_M \in C^\infty(\mathbb{R}_t : C^\infty(\mathbb{R}^n)) \) satisfies \( Pw = f_M \) with

\[
D_t^j f_M(\varepsilon, x) = 0, \quad j = 0, 1, \ldots, M - 3, \quad D_t^j w(\varepsilon, x) = 0, \quad j = 0, 1, \ldots, M.
\]

Consequently, from Theorem 4.1 one deduce the existence of \( N \in \mathbb{N} \) and \( C > 0 \) such that for \( \varepsilon > 0 \), \( Z \Subset \Omega \) and a solution of \( Pu = f \in C^\infty([0, T_0] \times Z) \) for \( 0 < \varepsilon < t \leq T_0, x \in Z \) with

\[
u(\varepsilon, x) = u_t(\varepsilon, x) = u_{tt}(\varepsilon, x) = 0
\]

we have

\[
\sum_{j+|\alpha| \leq 2} \int_{\varepsilon}^t \|\partial_t^j\partial_x^\alpha u(s, x)\|_{L^2(Z)}\,ds \leq C \int_{\varepsilon}^t \sum_{j+|\alpha| \leq N} \|\partial_t^j\partial_x^\alpha Pu(s, x)\|_{L^2(Z)}\,ds,
\]

(4.5)

where \( C \) is independent on \( \varepsilon \). We can obtain a similar estimates for higher order derivatives.

By applying the estimate (4.5) and the fact that under the assumptions of Theorem 4.1 the symbol \( p \) is strictly hyperbolic for \( 0 < t \leq T_0 \) one can obtain the existence of a solution of the Cauchy problem in \([0, T_0] \times Z\) repeating the argument in [3] Theorem 25.4.5. The fact that \( p \) is strictly hyperbolic for \( 0 < t \leq T_0 \), is equivalent to \( \Delta > 0 \) for \( 0 < t \leq T_0 \), \( \Delta \) being the discriminant of the equation \( p = 0 \) with respect to \( \tau \). On the other hand, \( \Delta = 27 \text{det } S \) (see also Corollary 2.1) and \( \text{det } S > 0 \) for \( t > 0 \) by Lemma 2.2. The local uniqueness of the solution of the Cauchy problem for \( P \) can be obtained taking into account Theorem 4.2 for the adjoint operator \( P^* \) and using the argument of [3] Theorem 25.4.5. We leave the details to the reader.

Finally, we deduce

**Corollary 4.5.** Under the assumptions of Theorem 4.1 the Cauchy problem for \( P \) is \( C^\infty \) well posed in \([0, T_0] \times Z\) for all lower order terms.
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