Recovering Lorentz Invariance of DLCQ *

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Abstract
We propose a way to recover Lorentz invariance of the perturbative S matrix in the Discrete Light-Cone Quantization (DLCQ) in the continuum limit without spoiling the trivial vacuum.

1. INTRODUCTION

The Discrete Light-Cone Quantization (DLCQ) was first introduced by Maskawa and Yamawaki (MY) [1] in 1976 based on the canonical quantization for the constrained system due to Dirac and was also considered by Casher [2] slightly later in 1976 and by Pauli and Brodsky [3] in 1985 in different contexts. In the original paper of MY [1], the light-like coordinate $x^\pm$ was compactified $-L \leq x^\pm \leq L$ with periodic boundary condition thus discretizing the conjugate momentum, $p^\pm = (n\pi)/L \quad (n = 0, \pm 1, \pm 2, \ldots)$, in order to isolate the zero mode ($n = 0$), $\phi_0 \equiv \frac{1}{2L} \int_{-L}^{L} dx^- \phi$, from the non-zero mode, $\varphi \equiv \phi - \phi_0$.

The most important finding of MY [1] is the discovery of the Zero-Mode Constraint (ZMC):

\[ \Phi_3 \equiv \frac{1}{2L} \int_{-L}^{L} dx^- \left[ (m^2 - \partial_+^2)\phi + \frac{\partial V(\phi)}{\partial \phi} \right] = \left[ \frac{\partial H}{\partial \phi} \right]_0 = 0 \quad (1) \]

in the 4-dimensional scalar theory with mass $m$ and self-interaction $V(\phi)$, where $H = \frac{1}{2}\phi(m^2 - \partial_+^2)\phi + V(\phi)$ is the light-cone Hamiltonian density and $[O]_0 \equiv \frac{1}{2L} \int_{-L}^{L} dx^- O$ is the zero mode of the operator $O$. Thanks to ZMC, MY succeeded to compute the well-defined Dirac bracket at $x^+ = y^+$:

\[ \{ \phi(\vec{x}), \phi(\vec{y}) \}_D \equiv \{ \phi(\vec{x}), \phi(\vec{y}) \} - \int d\vec{z}d\vec{z}' \{ \phi(\vec{x}), \Phi_i(\vec{z}) \} C^{-1}_{ij}(\vec{z}, \vec{z}') \{ \Phi_j(\vec{z}'), \phi(\vec{y}) \} \quad (2) \]

where for convenience the primary second-class constraint $\Phi \equiv \pi - \partial_\phi \phi$ may be divided into the non-zero mode $\Phi_1 \equiv \pi_0 - \partial_\phi \phi$ and the zero mode $\Phi_2 \equiv [\Phi_0]_0 = \pi_0$, and $C^{-1}$ is the inverse of $C_{ij}(\vec{x}, \vec{y}) \equiv \{ \Phi_i(\vec{x}), \Phi_j(\vec{y}) \}$, with $\vec{x} \equiv (x^-, x^\perp)$ and $(i, j) = 1, 2, 3$. Without ZMC, $\Phi_3$, which yields $C_{i, 3} = -C_{3, i} \neq 0 \quad (i = 1, 2)$, we would have trouble to get the inverse $C^{-1}$ and hence the Dirac bracket, since $C_{1, 2} = C_{2, 1} = C_{2, 2} = 0$. Thus the ZMC is vital to a well-defined canonical light-cone commutator $[\phi(x), \phi(y)]_{x^+ = y^+}$ which is given as $i\hbar$ times the Dirac bracket (up to operator ordering).

From the ZMC [1] and the Dirac bracket [2] (hence the canonical commutator), MY obtained two important physical consequences [1]:

- **Proof of the Trivial Vacuum**
  Since the translation invariance, $[\phi, P^+] = i\partial_\phi \phi$ is manifest in DLCQ even for finite $L$, the vacuum is defined independently of the dynamics (thus “trivial”) as the lowest $p^+$ state ($p^+ = 0$).
  This trivial vacuum proved a unique $p^+ = 0$ state and hence the true vacuum, since the zero mode is not an independent degree of freedom, written in terms of the non-zero modes, $\phi_0 = \phi_0(\varphi)$, by solving the ZMC (at least in perturbation), and hence can be removed from the physical Fock space.

- **Violation of the Lorentz Invariance**
  Based on the canonical commutator mentioned above, the Lorentz algebra $[\phi(x), M_{\mu\nu}]$ was explicitly computed: for the Lorentz generators $M^{\mu+}, M^{\mu-}$ which change the quantization plane (light-front), the Lorentz invariance is violated at operator level due to the unwanted surface term at $x^- = \pm L$ which does not vanish even in the continuum limit ($L \to \infty$).

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Thus the incompatibility between the trivial vacuum and the Lorentz invariance was the very nature of DLCQ from the beginning back in 1976 and has been a long-standing problem. (See Ref. [4].)

One might consider that the trivial vacuum could be fake, namely it could not be reconciled with the nonperturbative phenomena such as the confinement and the spontaneous symmetry breaking (SSB) which are attributed to the complicated vacuum structure in the usual quantization. However, it was demonstrated [5,6] that the operator solution of ZMC together with the trivial vacuum can in fact describe the SSB in four dimensions for continuous symmetry in a way to regularize the zero mode of the Nambu-Goldstone boson through explicit symmetry breaking which is taken to zero at the end of all calculations. (See also footnote[7]) What about the Lorentz invariance?

In this talk we shall demonstrate that

- The Lorentz invariance is violated at S matrix level as well as at operator level due to lack of the zero-mode loop in the perturbative DLCQ in the continuum limit [7].
- Lorentz invariance at perturbative S matrix level can be recovered by modifying the naive DLCQ action into the one with additional operator arising from the zero-mode loop in such a way that the trivial vacuum remains intact by this modification [8].

2. DLCQ PERTURBATION

To be definite we confine ourselves to two dimensional scalar theory with $V(\phi) = \frac{\lambda}{4!} \phi^4$. Extension will be discussed in the end. The ZMC [1] in two dimensions reads $m^2 \phi_{0} + \frac{\lambda}{2} ([\varphi^2]_0 + 3(\varphi^2)_0 \phi_{0}) = 0$. This is a cubic equation for $\phi_{0}$ once $[\varphi^3]_0$ and $[\varphi^2]_0$ are given, and hence has three solutions (up to operator ordering): [8]

$$
\phi_{0}^{(0)} = G_{-} - G_{+}, \quad \phi_{0}^{(\pm \nu)} = \pm \left[ e^{\frac{4\pi \nu}{m \lambda}} G_{+} - e^{\frac{2\pi \nu}{m \lambda}} G_{-} \right],
$$

where

$$
G_{\mp} \equiv \left( \pm \frac{1}{2} [\varphi^3]_0 \pm \frac{1}{2} \sqrt{\left([\varphi^3]_0)^2 + 4 \left([\varphi^2]_0 + \frac{2m^2}{\lambda}\right)^3} \right)^{\frac{1}{3}}.
$$

In the case of $m^2 > 0$, only $\phi_{0}^{(0)}$ is a real solution where its Taylor expansion for $\lambda \ll 1$ coincides with the perturbative solution [6] (see Fig. 1):

$$
\phi_{0}^{(0)}(x^+) = -i \lambda \int d^2y \Delta_0(x-y)[\varphi^3(y)] \frac{3!}{3!}
$$

$$
+(-i\lambda)^2 \int d^2y_1 \int d^2y_2 \Delta_0(x-y_1)[\varphi^2(y_1)] \frac{3!}{2!} \Delta_0(y_1-y_2)[\varphi^3(y_2)] + \cdots,
$$

with $\int d^2x \equiv \int dx^+ \int_{-L}^{L} dx^-$, where $\Delta_0(x-y) = \frac{1}{2L \cdot m^4} \delta(x^+ - y^+)$ is the zero-mode propagator (actually not “propagating” due to instantaneous delta function $\delta(x^+ - y^+)$, reflecting the fact that the zero mode is not an independent degree of freedom). Note that the ZMC solution produces only the tree zero-mode graph.

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1In the case of double-well potential $m^2 < 0$, we can easily see [8] that other two solution $\phi_{0}^{(\pm \nu)}$ also become real for weak coupling $\lambda \ll 1$ and their Taylor expansion coincides with the perturbation around the non-zero value $\pm \nu$ ($v = \sqrt{-6m^2/\lambda}$) [5], where ±ν is nothing but the vacuum expectation value $\langle 0 \rangle : \phi_{0}^{(\pm \nu)} : |0\rangle$ on the trivial vacuum $|0\rangle$ ($\equiv$ is the normal ordering). This implies that the trivial vacuum is indeed the true vacuum even when SSB takes place. As was emphasized by Ref. [4], the information of SSB is carried by the operator but not by the vacuum which is always trivial in contrast to the usual quantization. The three operator solutions $\phi_{0}^{(0)}, \phi_{0}^{(\pm \nu)}$ are plugged into the original Hamiltonian, yielding three different effective Hamiltonians (without zero mode) $\mathcal{H}^{(0,\pm \nu)}(\varphi)$ whose vacuum energy is $\langle 0 \rangle : \mathcal{H}^{(0,\pm \nu)} : |0\rangle = 0$, $-\frac{2\nu}{2m}$ for the trivial vacuum $|0\rangle$. Thus $\mathcal{H}^{(- \nu)}$ or $\mathcal{H}^{(- \nu)}$, which is the operator non-invariant under $Z_2$, yields the physical (ground-state) solution, in perfect agreement with the result of the conventional SSB in a different language.
Plugging the solution \( \phi_0^{(0)} = \phi_0^{(0)}(\phi) \) into the original Hamiltonian, \( H = \frac{1}{2}(\phi_0^{(0)}(\phi_0^{(0)}) + \lambda_4)(\phi_0^{(0)}(\phi_0^{(0)}) + \lambda_4)^4 = \frac{1}{2}m^2\varphi^2 + H_{\text{int}}^{(0)}(\phi), \) we obtain the effective interaction Hamiltonian \( H_{\text{int}}^{(0)}(\phi) \) whose perturbative expansion reads \(6\):

\[
-i \int dx^+ H_{\text{int}}^{(0)}(\phi) = -i\lambda \int d^2 x \frac{1}{4!} \varphi^4(x) + \frac{1}{2!}(-i\lambda)^2 \int d^2 x \int d^2 y \left( \frac{\varphi^3(x)}{3!} \Delta_0(x-y) \frac{\varphi^3(y)}{3!} \right) + \frac{3}{3!}(-i\lambda)^3 \int d^2 x \int d^2 y \int d^2 z \left( \frac{\varphi^2(x)}{2!} \Delta_0(y-x) \frac{\varphi^2(z)}{2!} \Delta_0(x-z) \frac{\varphi^2(z)}{2!} \right) + O(\lambda^4),
\]

which has the zero-mode-induced terms (see Fig. 2) in addition to the original term (first term). Note again that zero-mode appears only in the tree internal line coupled to \([\varphi^2]_0 \) and \([\varphi^3]_0 \).

Now, the perturbative S matrix is given by the expansion of \( S = T \exp \left[ -i \int_{-\infty}^{\infty} dx^+ H_{\text{int}}^{(0)}(\phi) \right] \). It yields Feynman graphs which contain no closed single line of the zero-mode propagator, since there is no zero mode operator to be contracted in \( H_{\text{int}}(\phi) \), namely the zero-mode loop is absent.

### 3. NON-COVARIANT RESULT

Let us now show that as a simplest example the two-body scattering amplitude (forward scattering amplitude) with zero \( p^+ \) transfer in DLCQ disagrees with the covariant one \(7\). At one loop the covariant amplitude in this case is given by

\[
A_{\text{Cov}}^{\text{Forward}} = \frac{1}{2}(-\lambda)^2 \int \frac{dk^+ dk^-}{(2\pi)^2 i} \frac{1}{(m^2 - 2k^+ k^-)} \exp \left( -\frac{\xi m^2}{2k^+} \right) = \frac{(-\lambda)^2}{8\pi m^2} \neq 0,
\]

which is non-vanishing result, where the integral \( dk^+ \) should be done before that of \( d\xi \), since otherwise \( d\xi \) first would give zero but \( dk^+ \) afterward would divergence and hence the integral would be ill-defined \( \propto 0 \). On the other hand, straightforward calculation via the DLCQ perturbation mentioned above yields for the same process the vanishing amplitude:

\[
A_{\text{DLCQ}}^{\text{Forward}} = \frac{1}{2}(-\lambda)^2 \frac{1}{2L} \sum_{l>0} \frac{1}{2(\pi l)^2} \int_{\xi}^{\infty} d\xi \delta(\xi) \exp \left( -\frac{\xi m^2}{2(\pi l)^2} \right) = 0
\]

independently of \( L \) and so does in the continuum limit \( L \to \infty \). Note that the DLCQ amplitude differs from \(7\) simply by the replacement \( 1/(2\pi) \int_{\xi}^{\infty} dk^+ \to 1/(2L) \sum_{l>0} \), where the zero mode loop is absent \( l \neq 0 \) in the DLCQ amplitude, and this time the integral \( d\xi \) can be done before \( \sum_{l>0} \), since \((2L)^{-1} \sum_{l>0} \[2(\pi l)^2]^{-1} = L/24 < \infty \) and hence the calculation is well-defined, (finite) \( \times 0 = 0 \).

The discrepancy is just the forward scattering amplitude with measure zero but has serious physical consequences. Actually, the same arguments can apply to the loop diagrams attached with arbitrary
number of sets of external lines with $p^+ = 0$ corresponding to $[\varphi^2]_0$ and $[\varphi^3]_0$. Vanishing of all these types of diagrams implies that the effective potential is zero, for example, as was emphasized in \[8\].

The absence of the zero-mode loop can be seen more transparently in the path integral formalism \[8\] where the second-class constraints $\Phi_1 \equiv \pi_\varphi - \partial_- \varphi, \Phi_2 \equiv \pi_0, \Phi_3 \equiv [\partial^2 \varphi]/[\partial^2 0]$ are incorporated in the standard way:

$$Z = \int [D\varphi D\pi D\varphi D\phi] \delta(\Phi_1) \delta(\Phi_2) \delta(\Phi_3) (Det C)^{1/2} \exp \left[ i \int (\pi \partial_+ \phi - \mathcal{H}) \right]$$

$$= \int [D\phi D\phi_0 DB DC D\bar{c}] \exp \left[ i \int \left\{ \mathcal{L} + \left( B [\partial \mathcal{H} / \partial \phi_0] + i \bar{c} \left[ \partial^2 \mathcal{H} / \partial \phi^2 \right]_0 c \right) \right\} \right], \quad (9)$$

where the integral $D\pi D\pi_0 D\delta(\Phi_1) \delta(\Phi_2) = \text{done trivially}$, and $\delta(\Phi_3) = \int [DB] \exp [i \int B[\partial \mathcal{H} / \partial \phi_0]]$ and $(Det C)^{1/2} = \text{Det} \left( [\partial^2 \mathcal{L} / \partial \phi^2]_0 \right) = \int [Dc D\bar{c}] \exp \left[ i \int i \bar{c} \left[ \partial^2 \mathcal{H} / \partial \phi^2 \right]_0 c \right], \text{with B, c, } \bar{c} \text{ being Nakanishi-Lautrup field, ghost and antighost, respectively.}^2$

Now, $\left( B [\partial \mathcal{H} / \partial \phi_0] + i \bar{c} \left[ \partial^2 \mathcal{H} / \partial \phi^2 \right]_0 c \right) \text{ is a BRS singlet, where the BRS transformation} \delta_{\text{BRS}} = \text{defined as } \delta_{\text{BRS}} c = i B, \delta_{\text{BRS}} \phi_0 = c, \delta_{\text{BRS}} B = \delta_{\text{BRS}} c = 0. \text{ Then } (\phi_0, B, \bar{c}, c) \text{ are BRS-quartet and hence the loop of } (\phi_0, B, \bar{c}, c) \text{ cancel each other in much the same way as in the gauge theories } \[10\]. \text{ We have explicitly checked this cancellation of the zero-mode loop by the } B, \bar{c}, c \text{ loops to two-loop order. In contrast, the covariant expression has no constraint and simply } Z = \int [D\phi] \exp \left[ i \int \mathcal{L} \right] \text{ where the loop effects of the zero-mode (though not clearly separated) are not canceled out.}$

### 4. RECOVERING LORENTZ INVARIANCE

We have seen that the violation of Lorentz invariance is a real effect in DLCQ as far as we use the naive discretization $\int_{-\infty}^{+\infty} dx^- \mathcal{L} \rightarrow \int_{-L}^{+L} dx^- \mathcal{L}$, with $\mathcal{L}$ in DLCQ being the same as that of the continuum theory. However we know in the lattice theory that the discretized action should be different in principle from that of the continuum theory: Lorentz covariant theory should be constructed by using the solution which has the second order phase transition and hence yields the sensible continuum limit.

In much the same spirit, here we propose the new DLCQ Lagrangian modified by adding extra operators $-\Delta H$ in such a way that the perturbative solution yields the covariant limit \[8\]. The extra term is generated by the zero-mode loop which can be explicitly estimated by the covariant perturbation theory. Important point of our method is that the trivial vacuum is not destroyed by the extra term.

At one loop order of the zero mode loop we have found the compact form of the extra operators in the Hamiltonian $\Delta H$:

$$\frac{\Delta H}{2L} = \frac{1}{2} \int \frac{dk^+ dk^-}{(2\pi)^2 i} \log \left[ m^2 + \frac{\lambda}{2} \left( [\varphi^2]_0 + (\phi_0^{(0,\pm \nu)})^2 \right) - k^+ k^- \right]$$

$$= - \sum_{n=1}^{\infty} \frac{\lambda}{2n} \frac{1}{2n} I_n \left( [\varphi^2]_0 + (\phi_0^{(0,\pm \nu)})^2 \right)^n, \quad (10)$$

where $\phi_0^{(0,\pm \nu)}$ is an operator solution of ZMC in \[5\] and $I_n (n \geq 1) = \int \frac{dk^+ dk^-}{(2\pi)^2 i} (m^2 - k^+ k^-)^{-n}$. $(n = 1$, the log divergence is regularized by Pauli-Villars regularization). This yields the correct Lorentz invariant scattering amplitude for any number of external legs with zero $p^+$ transfer, i.e., $[\varphi^2]_0$ and/or $[\varphi^3]_0$ (see Fig. 3). The term $n = 2$, for example, indeed yields the Lorentz-invariant two-body forward scattering amplitude with zero $p^+$ exchange in \[7\]. It is obvious that the extra term $\Delta H$ yields the correct effective potential by simply replacing $[\varphi^2]_0 + (\phi_0^{(0,\pm \nu)})^2$ by a c-number constant $\phi^2$.\[2\]

\[2\]If we perform the integral of the zero mode $[D\phi_0] \delta(\Phi_3) (Det C)^{1/2}$, we obtain $Z = \int [D\phi] \exp \left[ i \int \mathcal{L}^{(0)}(\varphi) \right]$, where $\mathcal{L}^{(0)}(\varphi)$ is the effective Lagrangian written only in terms of non-zero modes, corresponding to \[6\], which yields the same Feynman rule as in section 2. For the SSB solution $\phi_0^{(\pm \nu)}$, we make a simple replacement: $\mathcal{L}^{(0)}(\varphi) \rightarrow \mathcal{L}^{(\pm \nu)}(\varphi)$.\]
5. CONCLUSION

We have shown that the DLCQ with naive discretization violates Lorentz invariance in the perturbative S matrix due to the lack of zero mode loop, while DLCQ guarantees that the trivial vacuum is a true vacuum even when the SSB takes place. We have proposed to modify the DLCQ action by adding extra operators in such a way that the continuum limit S matrix recovers Lorentz invariance without spoiling the trivial vacuum. We have given such a modification explicitly at one loop level. Study at higher loops is in progress. Similar arguments will be applied to four dimensional theories but meet with substantial complexity due to the transverse degrees of freedom. This is also under investigation. Although our arguments are only within perturbation, we expect that our strategy to modify the DLCQ action based on the solution of the dynamics should be the right way to recover the Lorentz invariance without spoiling the trivial vacuum even in the non-perturbative approach.

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