An estimation of the eigenfunctions of periodic Sturm-Liouville problems

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ABSTRACT
In this paper, the numerical estimations for the eigenfunctions corresponding to the eigenvalues of Sturm-Liouville problem with periodic and semi-periodic boundary conditions are considered. Eigenfunctions are obtained by using the finite-difference method and shown to be matched with previous asymptotic studies.

1. Introduction and preliminary facts
Let \( P(q) \) and \( S(q) \) be the operators generated in \( L_2[0, \pi] \) by the differential expression
\[
-y''(x) + q(x)y(x)
\]
with the periodic
\[
y(\pi) = y(0), \quad y'(\pi) = y'(0)
\]
and semi-periodic
\[
y(\pi) = -y(0), \quad y'(\pi) = -y'(0)
\]
boundary conditions, where \( q \) is a real periodic function with period \( \pi \). The spectrum of the operators \( P(q) \) and \( S(q) \) consists of the eigenvalues called as periodic and semiperiodic eigenvalues respectively (Eastham, 1973).

This paper gives the estimations for the eigenfunctions corresponding to the periodic and semi-periodic eigenvalues which have been calculated in Dinibutun and Veliev (2013) when the real periodic potential \( q \) belongs to the Sobolev space \( W_1^1[0, \pi] \) with \( l > 1 \). These assumptions on the potential \( q \) imply that
\[
q(x) = \sum_{n \in \mathbb{Z}} q_n e^{2nx}, \quad q_{-n} = \overline{q}_n, \quad |q_n| \leq \frac{r}{(2n)^l}, \quad (4)
\]
where
\[
q_n = (q.e^{2nx}) = \int_0^\pi q(x)e^{-2nx}dx, \quad r = \int_0^\pi |q^{(l)}(x)|dx.
\]
Without loss of generality, it is assumed that \( q_0 = 0 \).

The consideration of the eigenfunction of the Sturm-Liouville problem arises directly as mathematical models of motion according to Newton’s law, but more often as a result of using the method of separation of variables to solve the classical partial differential equations of physics, such as Laplace’s equation, the heat equation, and the wave equation (Al-Gwaiz, 2008). Sturm-Liouville Problems with periodic boundary conditions are also common in physics, such as quantum physics or frequency and vibration theory, etc. Moreover, the periodic case describes the motion of a particle in the bulk matter. Here the obtained result about eigenfunctions shows that, under perturbation \( g \), the plane waves \( e^{2nx} \) and \( e^{-2nx} \) interface each other and standing wave \( \cos 2nx \) is a result of the interference between two waves \( e^{2nx} \) and \( e^{-2nx} \) traveling in the opposite directions.

In literature, there are many studies about numerical estimation for the eigenvalues and eigenfunctions of Sturm-Liouville problems. John D. Pryce explained many numerical methods about general Sturm-Liouville problems in his book (Pryce, 1993). The operators \( P(q) \) and \( S(q) \) are the most commonly studied ones among the Sturm-Liouville operators. Moreover, many different methods, such as the finite difference method, the finite element method, Prüfer transformations, and the shooting method, have been used for the investigations of the small eigenvalues of these operators. For instance, Andrew considered the computations of the eigenvalues by using the finite element method (Andrew, 1988) and the finite difference method (Andrew, 1989). Then
these results were extended by Condon (1999) and Vanden Berghe, Van Daele, and De Meyer (1995). Ghelardoni found some approximations of Sturm-Liouville eigenvalues using boundary value methods (Ghelardoni, 1997). Chein-Shan Liu calculated eigenvalues and eigenfunctions of Sturm-Liouville problems by using the Lie-group shooting method (Liu, 2008). Ji and Wong (1991) used Prüfer method, Ji (1994) used shooting algorithm, Celik and Gokmen (2005) used Chebysev collocation method for periodic and semiperiodic Sturm-Liouville eigenvalue problems. Malathi, Suleiman, and Taib (1998) used expansions technique. Yuan, Sun, and Zettl (2017) extended the classical results for periodic Sturm-Liouville problems. Yucel (2015) computed the eigenvalues of periodic Sturm-Liouville problems. Vanden Berghe, Van Daele, and De Meyer (1995). Chein-Shan Liu calculated eigenvalues using boundary value methods (Ghelardoni, 1997). Chein-Shan Liu calculated eigenvalues using boundary value methods (Ghelardoni, 1997). Chein-Shan Liu calculated eigenvalues using boundary value methods (Ghelardoni, 1997). Chein-Shan Liu calculated eigenvalues using boundary value methods (Ghelardoni, 1997). Chein-Shan Liu calculated eigenvalues using boundary value methods (Ghelardoni, 1997).

In recent studies, we can see different asymptotical and numerical methods which are mostly about computing eigenvalues for different types of Sturm-Liouville problems. Yucel (2015) computed the eigenvalues using the Chebyshev polynomial expansions technique. Yuan, Sun, and Zettl (2017) identified the eigenvalues of periodic Sturm-Liouville problems with complex boundary conditions. Wang and Zettl (2017) extended the classical results for discontinuous boundary conditions. Gao, Li, and Zhang (2018) obtained some results about the eigenvalues of discrete Sturm-Liouville problems with nonlinear eigenparameter. Ao and Yang (2019) investigated the eigenvalues of Sturm-Liouville problems with distribution potentials. Mukhtarov and Yucel (2020) computed the eigenvalues and corresponding eigenfunctions of singular Sturm–Liouville problem using a new algorithm based on Adomian decomposition method.

We can also see many recent studies in vibration and solid mechanics journals. For example, in the study (Hei & Zheng, 2020), an approximate analytical solution of the oil film force for the supporting bearing of the rotor is proposed based on the method of separation of variables and Sturm–Liouville theory. In another study (Kutsenko, Shuvalov, Poncelet, & Norris, 2013), solutions of the equation satisfy a quasi-periodic boundary condition which yields the Floquet parameter K.

There are many papers about numerical estimation of the periodic eigenvalues. However, in Dinibutun and Veliev (2013), the matrix form of the operator $Q(q)$ has been considered and obtained very sharp estimation for the small eigenvalues. The approximation of order $10^{-18}, 10^{-15}$ and $10^{-12}$ for the first 201 eigenvalues has been calculated and proved with the examples. This paper can be considered as a continuation of the paper (Dinibutun & Veliev, 2013).

Since the eigenfunctions are obtained by using the finite-difference method, let us briefly recall it.

The finite-difference method is a common method to solve boundary-value problems. The method replaces each of the derivatives in the differential equation with a different-quotient approximation.

The finite-difference method for the linear second order boundary-value problem:

\[ y''(x) = p(x) y'(x) + q(x) y(x) + r(x), \quad a \leq x \leq b, \quad y(a) = \alpha, \quad y(b) = \beta \]

requires that difference-quotient approximations to approximate $y'$ and $y''$ (Burden, 2001).

For an integer $N > 0$, let's divide the interval $[a, b]$ into $N + 1$ equal subintervals. The endpoints will be the mesh points $x_i = a + ih$ for $i = 0, 1, 2, \ldots, N + 1$ where $h = \frac{b - a}{N+1}$. For these mesh points, the differential equation in (5) will turn into:

\[ y''(x_i) = p(x_i) y'(x_i) + q(x_i) y(x_i) + r(x_i). \]

For $y \in C^2[x_{i-1}, x_{i+1}]$, the centered-difference formulae for $y'$ and $y''$ can be found as follows:

\[ y'(x_i) = \frac{y(x_{i+1}) - y(x_{i-1})}{2h} - \frac{h^2}{6} [y''(\eta_i)] \]

\[ y''(x_i) = \frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} - \frac{h^2}{12} [y''''(\zeta_i)] \]

for some $\eta_i \in (x_{i-1}, x_{i+1})$ and $\zeta_i \in (x_{i-1}, x_{i+1})$ (Burden, 2001).

If we plug (7) and (8) into (6), we find:

\[ y(x_{i+1}) - 2y(x_i) + y(x_{i-1}) \]

\[ = p(x_i) y(x_{i+1}) - y(x_{i-1}) + q(x_i) y(x_i) + r(x_i) \]

\[ = \frac{h^2}{2h} \left[ p(x_i) y''(x_i) + q(x_i) y(x_i) + r(x_i) \right] \]

\[ = -\frac{h^2}{12} \left[ 2p(x_i) y''(x_i) - y''''(x_i) \right]. \]

So, if we write the boundary conditions $y(a) = \alpha$ and $y(b) = \beta$ as

\[ w_0 = \alpha, w_{N+1} = \beta, \]

Equation (9) can be written as

\[ \frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} + p(x_i) \frac{w_{i+1} - w_{i-1}}{2h} + q(x_i) w_i = -r(x_i) \]

with truncation error of order $O(h^2)$. If we arrange (11), we get

\[ -\frac{h^2}{12} \left[ 2p(x_i) y''(x_i) - y''''(x_i) \right] \]

\[ = -r(x_i) \]

and for $i = 1, 2, \ldots, N$, we can write the system
\[-(1 - \frac{h}{2} p(x_1)) w_2 + (2 + h^2 q(x_1)) w_3 - (1 + \frac{h}{2} p(x_1)) w_0 = -h^2 r(x_1)\]
\[-(1 - \frac{h}{2} p(x_2)) w_3 + (2 + h^2 q(x_2)) w_4 - (1 + \frac{h}{2} p(x_2)) w_1 = -h^2 r(x_2)\]
\[-(1 - \frac{h}{2} p(x_3)) w_4 + (2 + h^2 q(x_3)) w_5 - (1 + \frac{h}{2} p(x_3)) w_2 = -h^2 r(x_3)\]
\[-(1 - \frac{h}{2} p(x_N)) w_{N+1} + (2 + h^2 q(x_N)) w_N - (1 + \frac{h}{2} p(x_N)) w_{N-1} = -h^2 r(x_N).\]  

(13)

This system can be expressed as a \(N \times N\) matrix which forms the system \(Aw = b\) where (Burden, 2001)

\[
A = \begin{bmatrix}
(2 + h^2 q(x_1)) & (1 - \frac{h}{2} p(x_1)) & 0 & 0 & \ldots \\
-(1 + \frac{h}{2} p(x_2)) & (2 + h^2 q(x_2)) & (1 - \frac{h}{2} p(x_2)) & 0 & \ldots \\
0 & -(1 + \frac{h}{2} p(x_3)) & (2 + h^2 q(x_3)) & (1 - \frac{h}{2} p(x_3)) & \ldots \\
& & & \ddots & \ddots \\
0 & 0 & 0 & -(1 + \frac{h}{2} p(x_N)) & (2 + h^2 q(x_N))
\end{bmatrix},
\]

(14)

\[
w = \begin{bmatrix}
w_1 \\
w_2 \\
w_3 \\
\vdots \\
w_{N-1} \\
w_N
\end{bmatrix}, \quad \text{and} \quad b = \begin{bmatrix}
-h^2 r(x_1) + \left(1 + \frac{h}{2} p(x_1)\right) w_0 \\
-h^2 r(x_2) \\
-h^2 r(x_3) \\
\vdots \\
-h^2 r(x_{N-1}) \\
-h^2 r(x_N) + \left(1 - \frac{h}{2} p(x_N)\right) w_{N+1}
\end{bmatrix}
\]

(15)

2. Estimation of eigenfunctions

The finite-difference method can be used to find approximate value of the eigenfunctions of the periodic boundary-value problem

\[-y'' + (q(x) - \lambda) y = 0\]
\[y(\pi) = y(0), y'(\pi) = y'(0).\]  

(16)

By using the method explained in the previous chapter, for \(x_i = ih\) and \(h = \frac{\pi}{N}\) with the centered-difference formulae \(y = w_i\) and \(y'' = \frac{-w_{i+1} + 2w_i - w_{i-1}}{h^2}\) (16) can be written as

\[-w_{i+1} + 2w_i - w_{i-1} + (q(x) - \lambda) w_i = 0,\]  

(17)

and to continue with the matrix form, (17) is written as

\[-\frac{1}{h^2} w_{i-1} + \left(\frac{2}{h^2} + q(x) - \lambda\right) w_i - \frac{1}{h^2} w_{i+1} = 0\]  

(18)

Here, without loss of generality, it is assumed that \(w_0 = w_N = 1\), and from (18), we obtain the system

\[\frac{1}{h^2} w_{i-1} + \left(\frac{2}{h^2} + q(x) - \lambda\right) w_i - \frac{1}{h^2} w_{i+1} = 0\]  

(19)

for \(i = 1, \ldots, N - 1\).

This system can be expressed as a \((N - 1) \times (N - 1)\) matrix which forms the system \(Aw = b\) where
\[ A = \begin{bmatrix}
(\frac{\lambda}{\pi} + q(x_1) - \lambda) & -\frac{1}{\pi} & 0 & 0 & \cdots \\
-\frac{1}{\pi} & (\frac{\lambda}{\pi} + q(x_2) - \lambda) & -\frac{1}{\pi} & 0 & \cdots \\
0 & -\frac{1}{\pi} & (\frac{\lambda}{\pi} + q(x_3) - \lambda) & -\frac{1}{\pi} & \cdots \\
& \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & -\frac{1}{\pi} & (\frac{\lambda}{\pi} + q(x_{N-1}) - \lambda)
\end{bmatrix}, \quad (20) \]

Figure 1. Graph of the eigenfunction corresponding to the eigenvalue \( \lambda = 1.859108072514 \).

Figure 2. Graph of the eigenfunction corresponding to the eigenvalue \( \lambda = 4.371300982731 \).
Therefore, the vector \( w \), which is obtained by the solution of the system \( Aw = b \), is the approximate value of the eigenfunction corresponding to the eigenvalue \( \lambda \) at the points \( x_i \).

\[
\begin{bmatrix}
  w_1 \\
  w_2 \\
  w_3 \\
  \vdots \\
  w_{N-1}
\end{bmatrix}, \quad \text{and} \quad b = 
\begin{bmatrix}
  \frac{1}{h^2} \\
  0 \\
  0 \\
  \vdots \\
  \frac{1}{h^2}
\end{bmatrix}.
\] (21)

3. Examples and conclusion

In this section, we illustrate the results of Sect. 2 for the following examples:

Let’s find the approximate values of the 1st, 2nd, 4th, and 20th eigenvalues of the following periodic boundary-value problem:

\[
-\gamma'' + q(x)y = \lambda y \\
y(\pi) = y(0), \quad y'(\pi) = y'(0)
\] (22)

where the potential \( q(x) = \sum_{k=1}^{1} e^{2ikx} = 2\cos 2x \).

The eigenvalue \( \lambda \) can be taken from Dinibutun and Veliev (2013) which gives high precision results for the calculation of the small eigenvalues.
So the eigenvalues we need are as follows:
for $i = 1$, $\lambda = 1.859108072514$,
for $i = 2$, $\lambda = 4.371300982731$,
for $i = 4$, $\lambda = 16.033832340360$,
for $i = 20$, $\lambda = 400.001253135326$.

Let’s divide the interval $[0, \pi]$ into $N = 500$ equal subintervals. If we create the system (19) and solve for $w$ in MATLAB, we obtain the graphs which can be seen in Figures 1–4.

Eastham (1973) shows us for the case $p(x) = s(x) = 1$, the eigenfunctions corresponding to the eigenvalues $\lambda_{2m+1}$ and $\mu_{2m}$ can be taken as $\psi_{2m+1}(x) = \cos x \sqrt{\lambda_{2m+1}}$, $\xi_{2m}(x) = \cos x \sqrt{\mu_{2m}}$ (see Eastham, 1973, pp. 39–40). So for large values of $\lambda$, eigenfunctions corresponding to eigenvalues have been proved to be very close to $\cos(\sqrt{\lambda}x)$ eigenfunction obtained when $q(x) = 0$. For the small value of $\lambda$, the graph in Figure 1 is not very close to $\cos(\sqrt{\lambda}x)$. However as seen in Figures 2–4, for the larger values of $\lambda$, the graph is very close to $\cos(\sqrt{\lambda}x)$.

Disclosure statement

The author declares that there is no conflict of interest.

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Data availability statement

The data which have been used for the example in Sect. 3 were taken from Dinibutun and Veliev (2013).

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