BTZ black hole with KdV-type boundary conditions: Thermodynamics revisited

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ABSTRACT: The thermodynamic properties of the Bañados-Teitelboim-Zanelli (BTZ) black hole endowed with Korteweg-de Vries (KdV)-type boundary conditions are considered. This family of boundary conditions for General Relativity on AdS3 is labeled by a non-negative integer \( n \), and gives rise to a dual theory which possesses anisotropic Lifshitz scaling invariance with dynamical exponent \( z = 2n + 1 \). We show that from the scale invariance of the action for stationary and circularly symmetric spacetimes, an anisotropic version of the Smarr relation arises, and we prove that it is totally consistent with the previously reported anisotropic Cardy formula. The set of KdV-type boundary conditions defines an unconventional thermodynamic ensemble, which leads to a generalized description of the thermal stability of the system. Finally, we show that at the self-dual temperature \( T_s = \frac{1}{8z(z+1)^{z+1}} \), there is a Hawking-Page phase transition between the BTZ black hole and thermal AdS3 spacetime.
1 Introduction

In the pursuit of a better understanding of quantum gravity, in the past two decades, a lot of interest has been put into the so called Gauge/Gravity correspondence, whose most celebrated example is the AdS/CFT duality [1, 2]. In this context, AdS$_3$/CFT$_2$ correspondence has played an important role. One of the first main results was the renowned article from Brown and Henneaux [3], where they showed that the asymptotic symmetries of General Relativity in three dimensions with negative cosmological constant correspond to the conformal algebra in two dimensions with a classical central extension. This result naturally suggest that a quantum theory of gravity in three dimensions could be described by a CFT at the boundary. Based on this result, Strominger [4] proved that the entropy of the Bañados-Teitelboim-Zanelli (BTZ) black hole [5, 6] can be recovered by a microscopic counting of states by means of the Cardy formula [7]. This simple example gave rise to an active field of research regarding the thermodynamic properties of lower dimensional black holes and how they could be holographically related to a dual field theory that describes the much sought after quantum theory of gravity.

In the static case, the thermodynamic stability of the BTZ black hole has qualitatively a different behaviour than its higher dimensional counterparts. In fact, in four dimensions, the canonical ensemble for the Schwarzschild solution is not well-defined [8]. This difficulty is avoided by the presence of the negative cosmological constant [9], but nevertheless, since the specific heat of Schwarzschild-AdS$_4$ black hole presents discontinuities, the system can not reach thermal equilibrium with a thermal bath at any temperature. In the case of three dimensions, none of the above arguments hold [10–12], since the specific heat of the
BTZ is a monotonically increasing positive function of the temperature, as the case of any Chern-Simons black hole in odd dimensions [13–15].

Several efforts have been made to generalize the Gauge/Gravity proposal for non AdS asymptotics (see e.g. [16–19]). In this scenario, a lot of attention has been placed on gravity dual theories with anisotropic scaling properties, which are found in the context of non-relativistic condense matter physics (see references in [20]). The main work on this subject has been done along the lines of Lifshitz holography, where the gravity counterparts are given by asymptotically Lifshitz geometries (see e.g. [21] and references therein). However, this class of spacetimes are not free of controversies. In particular, the Lifshitz spacetime (which would play the role of ground state in the thermodynamic description) suffer from divergent tidal forces. Additionally, asymptotically Lifshitz black holes are not vacuum solutions to General Relativity, and it is mandatory to include extra matter fields as in the case of Proca fields [21], p-form gauge fields [22, 23], to name a couple of examples.

In the present work, we will adopt a different approach to this holographic realization, where the anisotropic scaling properties of the boundary field theory instead emerge from a very special choice of boundary conditions for General Relativity on AdS$_3$. This new set of boundary conditions is labeled by a nonnegative integer $n$, and is related with the Korteweg-de Vries (KdV) hierarchy of integrable systems [24]. Allowing that at the asymptotic region the Lagrange multipliers could depend on the global charges, it was shown that the reduced phase space of Einstein field equations is given precisely by two copies of the $n$-th member of the KdV hierarchy. It is worth to emphasize that, although these boundary conditions describe asymptotically locally AdS$_3$ spacetimes, the associated dual field theory possesses an anisotropic scaling of Lifshitz type,

$$t \to \lambda^z t, \quad \phi \to \lambda \phi,$$

(1.1)

where the dynamical exponent is given by $z = 2n + 1$. In the context of black hole thermodynamics, KdV-type boundary conditions define an unconventional thermodynamic ensemble, which leads to a generalized thermodynamic description of the BTZ black hole. Remarkably, this thermodynamic description shows very similar features with the ones found in the study of Lifshitz black holes [27–31], and has a deep relationship with the work of Hardy and Ramanujan on the counting of partitions of an integer into $z$-th powers [32]. In this work, we will focus on the main characteristics that this thermodynamic ensemble implies, and its differences with the standard analysis.

The paper is organized as follows. In Section 2 we provide a brief review of KdV-type boundary conditions in the context of Chern-Simons description of General Relativity with negative cosmological constant in three dimensions. In Section 3 it is shown that an anisotropic Smarr formula emerges from the radially conserved charge associated with the scale invariance of the reduced Einstein-Hilbert action endowed with KdV-type boundary conditions. Section 4, is devoted to the anisotropic Cardy formula and the relation with its Smarr counterpart. Finally, the thermal stability of BTZ black hole with KdV-type

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**Other examples of this relationship between 2D integrable systems and gravity in 2+1, have been also made for the cases of “flat” and “soft hairy” boundary conditions in [25] and [26], respectively.**
boundary conditions is deeply analyzed in Section 5. We conclude with some comments in Section 6.

2 General Relativity on AdS\(_3\) and the KdV-type boundary conditions

General Relativity with negative cosmological constant in three dimensional spacetimes can be formulated as the difference of two Chern-Simons actions for gauge fields \(A^\pm\), evaluated on two independent copies of the \(sl(2,\mathbb{R})\) algebra [33, 34],

\[
I_{EH} = I_{CS}[A^+] - I_{CS}[A^-],
\]

(2.1)

The above action corresponds precisely to General Relativity on AdS\(_3\) only if both Chern-Simons levels are given by \(k = \ell/4G\), where \(\ell\) is the AdS radius and \(G\) the Newton constant.

In order to describe the asymptotic form of the fields, it is convenient to do the analysis using auxiliary fields \(a^\pm\), which are defined by a precise gauge transformation on \(A^\pm\) [35],

\[
A^\pm = b^\pm_1 (a^\pm + d) b^\pm_2,
\]

(2.2)

with \(b^\pm = e^{\pm \log(r/\ell)L^\pm_0}\). So that, the radial component of \(a^\pm\) vanishes, while the remain ones only depend on time and the angular coordinate. Proceeding as in [36, 37], the non-vanishing components of the auxiliary fields are given by

\[
a^\pm_0 = L^\pm_0 - \frac{1}{4} L^\pm_1 L^\mp_0, \quad a^\pm_t = \pm \mu^\pm L^\pm_0 - \partial_\phi \mu^\pm L^\pm_0 \pm \frac{1}{2} \left( \partial_\phi^2 \mu^\pm - \frac{1}{2} \mu^\pm L^\pm_0 \right) L^\pm_0,
\]

(2.3)

where \(L^\pm(t,\phi)\) stand for the dynamical fields, and \(\mu^\pm(t,\phi)\) correspond to the values of the Lagrange multipliers at infinity. In the asymptotic region, the field equations reduce to

\[
\partial^\pm_t L^\pm = \pm D^\pm \mu^\pm, \quad D^\pm := (\partial_\phi L^\pm) + 2 L^\pm_0 \partial_\phi - 2 \partial_\phi^3.
\]

(2.4)

It is worth highlighting that the boundary conditions are fully specified once a precise form of the Lagrange multipliers at infinity is provided. In the standard approach of Brown and Henneaux [3], the Lagrange multipliers are set as \(\mu^\pm = 1\). Going one step further, one can generalize this analysis by choosing arbitrary functions of the coordinates, \(\mu^\pm = \mu^\pm(t,\phi)\), which, in order to have a well-defined action principle, are held fixed at the boundary (\(\delta \mu^\pm = 0\)) [36, 37]. However, even beyond that, one can still guarantees the integrability of the boundary term in the action, if one allow that the Lagrange multipliers may depend on the dynamical fields and their spatial derivatives, giving rise to a complete new set of boundary conditions. Here, we will focus on the family of KdV-type boundary conditions, introduced in [24], which are labeled by a non negative integer \(n\). In this context, the Lagrange multipliers are chosen to be given by the \(n\)-th Gelfand-Dikii polynomial evaluated on \(L^\pm\) and can be obtained by the functional derivative with respect to \(L^\pm\) of the \(n\)-th Hamiltonian of the KdV hierarchy, i.e.,

\[
\mu^\pm(n)[L^\pm] = \frac{\delta H^\pm(n)}{\delta L^\pm},
\]

(2.5)

\(\delta\)The representation that we use is the same as in [36].
where the following recursion relation is satisfied
\[
\partial_\phi \mu^{(n+1)}_\pm = \frac{n+1}{2n+1} D^\pm \mu^{(n)}_\pm .
\] (2.6)

Thus, for the case \( n = 0 \), one recovers the Brown-Henneaux boundary conditions \( \mu^{(0)}_\pm = 1 \), and in consequence, according to (2.4), the dynamical fields are chiral. In the case \( n = 1 \), the Lagrange multipliers are given by \( \mu^{(1)}_\pm = L_\pm \), and then the field equations reduce to two copies of the KdV equation, while for the remaining cases \( (n > 1) \) the field equations are given by the corresponding \( n \)-th member of the KdV hierarchy.

As a consequence of the reduction of Einstein field equations to the KdV hierarchy\(^3\), the “boundary gravitons” (global gravitational excitations) possess an anisotropic scaling of Lifshitz type
\[
t \to \lambda^z t , \quad \phi \to \lambda \phi , \quad \mathcal{L}_\pm \to \lambda^{-2} \mathcal{L}_\pm ,
\] (2.7)
where the dynamical exponent \( z \) is related to the KdV label \( n \) by \( z = 2n + 1 \). It must be remarked that, although the solutions of Einstein field equations are locally AdS\(_3\), they inherit an anisotropic scaling from the choice of KdV-type boundary conditions. For this reason, the thermodynamic properties of black holes will also carry a \( z \)-dependence.

According to the canonical approach [40], the variation of the generators of the asymptotic symmetries are readily found to be given by \( \delta Q = \delta Q_+ [\varepsilon_+] - \delta Q_- [\varepsilon_-] \), where
\[
\delta Q_\pm [\varepsilon_\pm] = - \frac{\ell}{32 \pi G} \int d\phi \varepsilon_\pm \delta \mathcal{L}_\pm .
\] (2.8)

In particular, when the gauge parameters are related with the asymptotic Killing vectors \( \partial_\phi \) and \( \partial_t \), one can integrate (2.8) directly. Indeed, the angular momentum is given by
\[
Q [\partial_\phi] = \frac{\ell}{32 \pi G} \int d\phi (\mathcal{L}_+ - \mathcal{L}_-) ,
\] (2.9)
while for time translations,
\[
\delta Q [\partial_t] = \frac{\ell}{32 \pi G} \int d\phi (\mu_+ \delta \mathcal{L}_+ + \mu_- \delta \mathcal{L}_-) ,
\] (2.10)
which by virtue of (2.5), the energy integrate as
\[
Q [\partial_t] = \frac{\ell}{32 \pi G} \left( H_+^{(k)} + H_-^{(k)} \right) .
\] (2.11)

2.1 The BTZ black hole with KdV-type boundary conditions

For each allowed choice of \( n \) (or equivalently \( z \)), the spectrum of solutions is quite different. Nonetheless, BTZ black hole [5, 6] fits within every choice of boundary conditions in (2.5).

\(^3\)As shown in [38], by performing the Hamiltonian reduction of KdV-type boundary conditions, the equations (2.4) actually corresponds to the conservation law of the energy-momentum tensor of the corresponding theory at the boundary. For the particular case \( n = 0 \), the field equations are equivalent to the aforementioned conservation law. See also [39], for a recent related result, in the case of “near horizon” boundary conditions.
Indeed, this class of configurations is described by constant $L_\pm$, which trivially solves (2.4) for all possible values of $n$. In this case, according to the normalization choice in (2.6), it is possible to show that the Lagrange multipliers generically acquire a remarkably simple form, $\mu_\pm^{(n)} = L_\pm^n N_\pm$, where $N_\pm$ is assumed to be fixed without variation at the boundary ($\delta N_\pm = 0$). Note that $\mu_\pm^{(0)} = N_\pm$, so in that special case, the Lagrange multipliers at infinity are held constants but given by arbitrary values, and the standard Brown-Henneaux analysis [3] is recovered by setting $N_\pm = 1$. In this scenario, along the lines of [37], the Lagrange multipliers are allowed to depend on the dynamical fields, which amounts to a different fixing of the “chemical potentials” at the boundary, implying that we are dealing with the same black hole configuration but in a different thermodynamic ensemble (see e.g., [41]). In what follows, we will use the dynamical exponent $z$, instead of the KdV-label $n$, so by using $z = 2n + 1$, we can rewrite the KdV-type Lagrange multipliers as\footnote{In the context of AdS/CFT holography, the relationship between the chemical potentials and conserved charges is known as “multi-trace deformations” of the dual theory [42–45].},

$$\mu_\pm = L_\pm^{\frac{z+1}{2}} N_\pm . \tag{2.12}$$

The energies of the left and right movers also take a simple form for a generic choice of $n$, namely $E_\pm = \ell \frac{1}{8G} L_\pm^{\frac{z+1}{2}}$. Therefore, in terms of the dynamical exponent we can rewrite them as

$$E_\pm = \ell \frac{1}{8G} L_\pm^{\frac{z+1}{2}} . \tag{2.13}$$

From the gravitational perspective, according to (2.11), the energy of the BTZ black hole is determined by $E = E_+ + E_-$. In the next section we will show that an anisotropic version of Smarr formula naturally emerges as the consequence of the scale invariance of the reduced Einstein-Hilbert action, as long as we consider the KdV-type boundary conditions.

3 The anisotropic Smarr formula as a radial conservation law

In [46], the authors showed that the reduced Einstein-Hilbert action coupled to a scalar field in AdS$_3$ spacetimes is invariant under a set of scale transformations which leads to a radial conservation law by using the Noether theorem. When this conserved quantity is evaluated in a particular solution of the theory, namely, a black hole solution, one obtains a Smarr relation [47]. This method has been successfully applied to several cases in the literature [48–55] for different theories. By following this procedure, we show that it is possible as well, to obtain a generalization of the Smarr formula for the BTZ black hole endowed with KdV-type boundary conditions. As a consequence, the entropy as a bilinear form of the global charges of the black hole, manifestly depends on the dynamical exponent.

In the metric formulation, the Einstein-Hilbert action has the following form

$$I_{EH} = \int d^3x \sqrt{-g} \left[ \frac{1}{2\kappa} (R - 2\Lambda) \right] , \tag{3.1}$$

where $\kappa = 8\pi G$, and the cosmological constant is related to the AdS radius by $\Lambda = -\ell^{-2}$.
By considering stationary and circularly symmetric spacetimes described by the following line element
\[ ds^2 = -N(r)^2 \mathcal{F}(r)^2 dt^2 + \frac{dr^2}{\mathcal{F}(r)^2} + r^2 \left( d\phi + N^\phi(r) dt \right)^2 , \] the reduced action principle in the canonical form is given by
\[ I = -2\pi (t_2 - t_1) \int dr \left( \mathcal{H}N + \mathcal{H}^\phi N^\phi \right) + B , \]
where the boundary term \( B \) must be added in order to have a well-defined variational principle. The surface deformation generators \( \mathcal{H}, \mathcal{H}^\phi \) acquire the following form
\[ \mathcal{H} = -\frac{r}{2\kappa} + 4\kappa r(\pi^{r\phi})^2 + \frac{\left(\mathcal{F}^2\right)'}{2\kappa} , \]
\[ \mathcal{H}^\phi = -2(r^2\pi^{r\phi})' , \]
where \( \mathcal{N} \) and \( \mathcal{N}^\phi \) stand for their corresponding Lagrange multipliers. The only nonvanishing component of the momenta \( \pi^{ij} \) is explicitly given by
\[ \pi^{r\phi} = -\frac{(\mathcal{N}^\phi)' r}{4\kappa N^\phi} , \]
where prime denotes derivative with respect to \( r \).

The above reduced action principle turns out to be invariant under the following set of scale transformations
\[ \tilde{r} = \xi r , \quad \tilde{N} = \xi^{-2} N , \quad \tilde{N}^\phi = \xi^{-2} N^\phi , \quad \tilde{\mathcal{F}}^2 = \xi^2 \mathcal{F}^2 , \]
where \( \xi \) is a positive constant. By applying the Noether theorem, we obtain a radially conserved charge associated with the aforementioned symmetries,
\[ C(r) = \frac{1}{4G} \left[ -N\mathcal{F}^2 + \frac{rN(\mathcal{F}^2)'}{2} - \frac{r^3 (N^\phi)' (\mathcal{N}^\phi)}{N} \right] , \]
which means that \( C' = 0 \) on-shell.

We will find a Smarr formula by exploiting the fact that this conserved charge must satisfy \( C(\infty) = C(r_+) \), where \( r_+ \) is the event horizon of the BTZ black hole solution with KdV-type boundary conditions.

### 3.1 Conserved charge at infinity

For the class of configurations considered here, the metric functions in “Schwarzschild” coordinates are given by
\[ \mathcal{N}(r) = \frac{\ell}{2} (\mu_+ + \mu_-) , \]
\[ \mathcal{N}^\phi(r) = \frac{1}{2} (\mu_+ - \mu_-) + \frac{\ell^2}{8r^2} (\mathcal{L}_+ - \mathcal{L}_-) (\mu_+ + \mu_-) , \]
\[ \mathcal{F}^2(r) = \frac{r^2}{\ell^2} - \frac{1}{2} (\mathcal{L}_+ + \mathcal{L}_-) + \frac{\ell^2}{16r^2} (\mathcal{L}_+ - \mathcal{L}_-)^2 . \]
where $\mu_\pm$ corresponds to the arbitrary values of the Lagrange multipliers at infinity, leading to a simple expression for the radially conserved charge at infinity,

$$C(\infty) = \frac{\ell}{8G} (\mu_+ L_+ + \mu_- L_-).$$

(3.10)

Thus in the case of KdV-type boundary conditions (2.12), can be written as

$$C(\infty) = \frac{\ell}{8G} \left( N_+ L_+^{z+1} + N_- L_-^{z+1} \right),$$

(3.11)

which in terms of the left and right energies (2.13), reads

$$C(\infty) = (z + 1) N_+ E_+ + (z + 1) N_- E_-.$$

(3.12)

### 3.2 Conserved charge at the event horizon

To evaluate the radial charge at the event horizon we must ensure that the Euclidean configuration is smooth around this point. The inner and outer horizons $r_{\pm}$ in these coordinates, are determined by $\mathcal{F}^2(r_{\pm}) = 0$, where $r_{\pm} = \frac{\ell}{2} \left( \sqrt{L_+} \pm \sqrt{L_-} \right)$. In consequence, it is clear that the regularity requirement in this gauge translates into,

$$\mathcal{N}(r_+)^2 r_+ = 4\pi, \quad \mathcal{N}^\phi(r_+) = 0,$$

(3.13)

which implies that the Euclidean metric becomes regular for $\mu_\pm = \frac{2\pi}{\sqrt{L_\pm}}$. Considering the regularity conditions (3.13) and the metric functions $\mathcal{F}^2$, $\mathcal{N}$ and $\mathcal{N}^\phi$, the value of the radial charge at the event horizon is

$$C(r_+) = \frac{\pi r_+}{2G} = S,$$

(3.14)

which corresponds to the entropy of the BTZ black hole.

Now, by making use of the equality $C(r_+) = C(\infty)$, the anisotropic Smarr formula is obtained$^5$,

$$S = (z + 1) N_+ E_+ + (z + 1) N_- E_-.$$

(3.15)

Identifying the Lagrange multipliers as the inverse of left and right temperatures $T_{\pm} = N_{\pm}^{-1}$, the above expression acquires the following form,

$$S = (z + 1) \frac{E_+}{T_+} + (z + 1) \frac{E_-}{T_-}.$$

(3.16)

Turning off the angular momentum, this expression reduces to$^6$

$$E = \frac{1}{(z + 1)} TS,$$

(3.17)

$^5$Resembling expressions for the entropy as a bilinear combination of the global charges times the chemical potentials have been previously found for three dimensional black holes and cosmological configurations in the context of higher spin gravity [36, 56, 57], hypergravity [58, 59] and extended supergravity [60]. The factor in front of each term corresponds to the conformal weight (spin) of the corresponding generator.

$^6$The left and right temperatures are related with the Hawking temperature through $T = \frac{2T_+ T_-}{(T_+ + T_-)}$, so in the absence of rotation $T_+ = T_- = T$. 

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which is in agreement with Smarr formula for Lifshitz black holes in three dimensions found in the literature (see e.g. [29, 30]). It is worth to point out that since the BTZ black hole with generic boundary conditions is asymptotically AdS$_3$, the contribution due to the rotation naturally appears in the anisotropic Smarr formula (3.15), despite the fact that, as far as the knowledge of the present authors, there is no a rotating Lifshitz black hole in three dimensions. It is also worth mentioning that in the limit $z \to 0$, (3.15) fits with the corresponding Smarr relation of soft hairy horizons in three spacetimes dimensions [61].

It is worth to remark that the scaling (3.7) is equivalent to the scaling of the Lifshitz type introduced in (2.7). By redefining the scale factor as $\lambda \to \xi^{-1}$ in (3.7), we obtain

$$t \to \xi^{-z} t, \quad r \to \xi^{-1} r, \quad \phi \to \xi^{-1} \phi, \quad L_{\pm} \to \xi^2 L_{\pm}.$$ (3.18)

Then, in the case of KdV-type boundary conditions (2.12), we can see that the Lagrange multipliers at infinity scales as $\bar{\mu}_\pm = \xi^{-1} \mu_\pm$, therefore from (3.9), we can deduce that $\bar{N}$ and $\bar{N}^\phi$ scales accordingly. Now, since the reduced Hamiltonian action does not depend on $t$ and $\phi$, before integrate them, we can absorb its scalings on the Lagrange multipliers, and in consequence, they must scale as $\bar{N} = \xi^{-2} N$ and $\bar{N}^\phi = \xi^{-2} N^\phi$, which is in full agreement with (3.7).

In the following chapter, we will show that by considering a two dimensional field theory defined on the torus, it is possible to recover the anisotropic Smarr formula by means of an anisotropic version of the standard S-modular invariance of the partition function.

4 The anisotropic Cardy formula

On this section, we will tackle the thermodynamical description of the BTZ black hole from an holographic perspective. The partition function of the 2D dual field theory is made up by the contribution of two non interacting left and right systems, each one with a corresponding temperature given by $T_+ = \beta^{-1}$ and $T_- = \beta^{-1}$. The modular parameter of the torus $\tau$, where the theory is defined, is related to the left and right periods of the thermal cycle, $\beta_+$ and $\beta_-$, by

$$\tau = \frac{i\beta_+}{2\pi}, \quad \bar{\tau} = \frac{-i\beta_-}{2\pi}.$$ (4.1)

As argued in [24] and [62], the partition function of the dual field theory at the boundary is invariant under the anisotropic S-duality transformation, given by

$$Z[\beta_{\pm}; z] = Z[(2\pi)^{1+\frac{1}{2}} \beta_\pm^{-\frac{1}{2}}; z^{-1}],$$ (4.2)

and, by assuming the existence of a gap in the energy spectrum it is possible to write the partition function as the contribution of two pieces

$$Z(\beta_+, \beta_-) = e^{-I(\beta_+, \beta_-)} + e^{-I_0(\beta_+, \beta_-)},$$ (4.3)

such that, at low temperatures, the contribution of the ground state $I_0$ dominates the partition function, so it can be approximated by

$$Z(\beta_+, \beta_-) \approx \exp \left(-\beta_+ E_0^0 - \beta_- E_-^0 \right),$$ (4.4)
and by virtue of the anisotropic S-duality, at high temperature regime the partition function acquires the following form

$$Z(\beta_+, \beta_-) \approx \exp \left( -\frac{(2\pi)^{1+\frac{1}{z}}}{\beta_+^{1+\frac{1}{z}}} E_+^0 \left[ z^{-1} \right] - \frac{(2\pi)^{1+\frac{1}{z}}}{\beta_-^{1+\frac{1}{z}}} E_-^0 \left[ z^{-1} \right] \right). \quad (4.5)$$

It is well known that by taking the inverse Laplace transform of the partition function $(4.5)$ we can obtain the asymptotic growth of the density of states

$$\rho(E_+, E_-) = \frac{1}{(2\pi i)^2} \int_{-\infty}^{i\infty} d\beta_+ d\beta_- e^{\beta_+ E_+ + \beta_- E_-} Z(\beta_+, \beta_-) \approx \frac{1}{(2\pi i)^2} \int d\beta_+ d\beta_- e^{f_+ + f_-}, \quad (4.6)$$

where the function $f_\pm$ is defined as

$$f_\pm(\beta_\pm, E_\pm, E_0^\pm) := -\frac{(2\pi)^{1+\frac{1}{z}}}{\beta_\pm^{1+\frac{1}{z}}} E_\pm^0 \left[ z^{-1} \right] + \beta_\pm E_\pm. \quad (4.7)$$

We can evaluate this expression using the saddle-point approximation for fixed energies in the limit of $E_\pm \gg |E_0^\pm|$. Indeed, there is a critical point $\beta_\pm(E_\pm, E_0^\pm)$,

$$\beta_\pm = 2\pi \left( -\frac{E_\pm^0 \left[ z^{-1} \right]}{z E_\pm} \right)^{\frac{1}{1+\frac{1}{z}}}. \quad (4.8)$$

Then, the entropy of the system

$$S = \log \rho \approx f_+(\beta_+, E_+, E_0^+) + f_-(\beta_-, E_-, E_0^-), \quad (4.9)$$

will be given by two copies of the anisotropic Cardy formula\footnote{As explained in [24], for odd values of $n = (z-1)/2$, Euclidean BTZ with KdV-type boundary conditions is diffeomorphic to thermal $\text{AdS}_3$, but with reversed orientation, and in consequence, there is a opposite sign between Euclidean and Lorentzian left and right energies of the ground state. As it will be shown in the next section, this leads to a local thermodynamic instability of the system for odd values of $n$. So, it is mandatory to adopt $E_\pm^0 \left[ z^{-1} \right] \rightarrow -|E_\pm^0 \left[ z^{-1} \right]|$, in the Lorentzian ground state energies of the anisotropic Cardy formula.} [24], [62],

$$S = 2\pi (z + 1) \left( \left( \frac{|E_+^0 \left[ z^{-1} \right]|}{z} \right)^{\frac{1}{z+1}} \right)^{\frac{1}{1+\frac{1}{z}}} + 2\pi (z + 1) \left( \left( \frac{|E_-^0 \left[ z^{-1} \right]|}{z} \right)^{\frac{1}{z+1}} \right)^{\frac{1}{1+\frac{1}{z}}}. \quad (4.10)$$

Remarkably, if instead, we consider the ground state energies in terms of the inverse temperatures and the left and right energies,

$$|E_\pm^0 \left[ z^{-1} \right]| = z E_\pm \left[ z \right] \left( \frac{\beta_\pm}{2\pi} \right)^{\frac{1+\frac{1}{z}}{2}}, \quad (4.11)$$

in (4.9), we found that the entropy reduces to

$$S = (z + 1) E_+ \beta_+ + (z + 1) E_- \beta_-,$$
which exactly matches with the anisotropic Smarr formula previously obtained by considering the scale symmetry of the Einstein-Hilbert reduced action (3.16). This relationship between Cardy and Smarr formulas has been previously suggested in the literature [29]. Interestingly enough, the link between both expressions is the anisotropic version of the Stefan-Boltzmann law, which is nothing else than the relation between energy and temperature given by the critical point (4.11).

From equation (4.10), it is clear that the entropy written as a function of left and right energies scales as

$$ S (\lambda E_+, \lambda E_-) = \lambda \sigma S (E_+, E_-) ,$$

where

$$ \sigma = \frac{1}{(z + 1)} .$$

Hence, it is reassuring to prove that, by simply applying the Euler theorem for homogeneous functions, gives

$$ S (E_+, E_-) = (z + 1) (E_+ \beta_+ + E_- \beta_-) ,$$

in the same way than the original derivation found in [47].

The following section is devoted to the local and global thermal stability of the BTZ black hole endowed with KdV-type boundary conditions.

5 Thermodynamic stability and phase transitions

We analyze the thermodynamic stability at fixed chemical potentials. Local stability condition can be determined by demanding a negative defined Hessian matrix of the free energy of the system (see e.g. [63]). Nonetheless, in this ensemble it can equivalently be performed by the analysis of the left and right specific heats with fixed chemical potential. From the anisotropic Stefan-Boltzmann law

$$ E_\pm [z] = \frac{1}{z^2} |E_\pm^0 [z^{-1}] | (2\pi)^{1+\frac{1}{z}} T_\pm^{1+\frac{1}{z}} ,$$

one finds that left and right specific heats are given by

$$ C_\pm [z] = \frac{\partial E_\pm}{\partial T_\pm} = \frac{z + 1}{z^2} |E_\pm^0 [z^{-1}] | (2\pi)^{1+\frac{1}{z}} T_\pm^{-\frac{1}{z}} .$$

We see that, for all possible values of $z$, the specific heats are continuous monotonically increasing functions of $T_\pm$, and always positive\(^8\), which means that the system is at least locally stable. It is important to remark that, as mentioned at the end of the last section, if we had not warned on the correct sign of the ground state energies for odd $n$, the sign of the specific heats would have depended on $z$, and in consequence, for odd values of $n$ the black hole would be thermodynamically unstable.

\(^8\)Strictly speaking, specific heats $C_\pm$ are always positive provided that $T_\pm > 0$. In terms of the temperature and angular velocity of the black hole, the above is equivalent to the non-extremality condition; $0 < T$, $-1 < \Omega < 1$. Since in the present paper we are not dealing with the extremal case, we will consider that this condition is always fulfilled.
Since the specific heats are finite and positive regardless of the value of $z$, the BTZ black hole with generic KdV-type boundary conditions can always reach local thermal equilibrium with the heat bath at any temperature.

Once local stability is assured, it makes sense to ask about the global stability of the system. Following the seminal paper of Hawking and Page [9], we use the free energies at fixed values of the chemical potentials of the two phases present in the spectrum (BTZ and thermal AdS$_3$), in order to realize which one is thermodynamically preferred.

In the semiclassical approximation, the on-shell Euclidean action is proportional to the free energy of the system. Taking into account the contributions of the left and right movers, the action acquires the following form

$$I = N_+ E_+ + N_- E_- - S.$$  \hspace{1cm} (5.3)

Assuming a non-degenerate ground state with zero entropy and whose left and right energies are equal and negative defined, $E_\pm \rightarrow -|E^0[z]|$, we see that the value of the action of the ground state is given by

$$I_0 = -|E^0[z]| \left( \frac{1}{T_+} + \frac{1}{T_-} \right).$$  \hspace{1cm} (5.4)

On the other hand, considering in (5.3) a system whose entropy is given by the formula (3.15), we obtain that

$$I = -z (N_+ E_+ + N_- E_-),$$  \hspace{1cm} (5.5)

hence, using the formulae for the energies in (5.1), the action then reads

$$I = -|E^0[z^{-1}]| \left( 2\pi \right)^{\frac{z+1}{z}} \left( T_+^{\frac{1}{z}} + T_-^{\frac{1}{z}} \right).$$  \hspace{1cm} (5.6)

Therefore, it is straightforward to see that, regardless of the value of $z$, the partition function $Z = e^{I + I_0}$, will be dominated by (5.4) at low temperatures, and by (5.6) at the high temperatures regime. It can also be shown that, consistently, we are able to found the same ground state action by making use of the anisotropic S-duality transformation (4.2) on (5.6).

In what follows, we will focus on the simplest case where the whole system is in equilibrium at a fixed temperature $T_\pm = T$. Then, the free energy of the system at high and low temperatures will respectively given by

$$F = -2|E^0[z^{-1}]| \left( 2\pi \right)^{1 + \frac{1}{z}} T^{1 + \frac{1}{z}}, \hspace{1cm} F_0 = -2|E^0[z]|,$$  \hspace{1cm} (5.7)

and comparing them, we can obtain the self-dual temperature, at where both free energies coincide,

$$T_s[z] = \frac{1}{2\pi} \left| \frac{E^0[z]}{E^0[z^{-1}]} \right|^{\frac{z}{z+1}},$$  \hspace{1cm} (5.8)

which manifestly depend on the dynamical exponent. An interesting remark is worth to be mentioned. The fact that the self-dual temperature $T_s$ depends on the specific choice of boundary conditions, is because the S-duality transformation involves an inversion of
the dynamical exponent between the high and low temperature regimes, namely, $z \to z^{-1}$. This is a highly non trivial detail in the calculation. If one does not take it into account, the self-dual temperature would be the same for all values of $z$.

On the other hand, computing the free energies of the BTZ black hole and thermal AdS$_3$ spacetime, we obtain

$$F_{\text{BTZ}} = -\frac{\ell}{4G} \frac{z}{z+1} (2\pi T)^{\frac{z+1}{2}}, \quad F_{\text{AdS}} = -\frac{\ell}{4G} \frac{1}{z+1},$$

and then, the self-dual temperature for which the two phases are equally likely is

$$T_s [z] = \frac{1}{2\pi} \left( \frac{1}{z} \right)^{\frac{1}{z+1}}.$$ (5.10)

which exactly matches with (5.8), if one identifies the ground state energy of the field theory with the one of the AdS$_3$ spacetime with KdV-type boundary conditions, i.e., $E^0 \to \frac{1}{2} E_{\text{AdS}} [z] = -\frac{\ell}{8G} \frac{1}{z+1}$.

As it shown in Figure 1, for an arbitrary temperature below the self-dual temperature ($T < T_s$), the thermal AdS$_3$ phase has less free energy than the BTZ, and therefore the former one is the most probable configuration, while if $T > T_s$, the black hole phase dominates the partition function and hence is the preferred one. Note that for higher values of $z$, the self-dual temperature becomes lower. The latter point entails to a remarkable result. In the case of Brown-Henneaux boundary conditions ($z = 1$), one can deduce that in order for the black hole reach the equilibrium with a thermal bath at the self-dual point $T_s$, the event horizon must be of the size of the AdS$_3$ radius, i.e., $r_+ = \ell$. Nonetheless, for a generic choice of $z$, the horizon size has to be

$$r_+ [z] = \ell \left( \frac{1}{z} \right)^{\frac{1}{z+1}}.$$ (5.11)
This means that the size of the black hole at the self-dual temperature decreases for higher values of $z$. In the same way, at $T_s$, the energy of the BTZ, $E_{BTZ} = E_+ + E_-$, endowed with generic KdV-type boundary conditions, acquires the following form

$$E_s[z] = \frac{\ell}{4G} \frac{1}{z(z+1)},$$

(5.12)

and when compared to the AdS$_3$ spacetime energy,

$$\Delta E = E_{BTZ} - E_{AdS} = \frac{\ell}{4G} \frac{1}{z},$$

(5.13)

we can observe that at the self-dual temperature there is an endothermic process, where the system absorbs energy from the surround thermal bath at a lower rate for higher values of $z$.

From these last points we can conclude that the global stability of the system is certainly sensitive to which KdV-type boundary condition is chosen, since the free energy of the possible phases of the system are explicitly $z$-dependent. Moreover, the temperature at which both phases have equal free energy, the size of the black hole horizon and the internal energy of the system at that temperature, decrease for higher values of $z$, giving rise to a qualitatively different behavior of the thermodynamic stability of the system, compared to the standard analysis defined by $z = 1$.

### 6 Outlook and ending remarks

The purpose of this work is twofold. On one hand, we have shown that the anisotropic Smarr relation for the BTZ black hole endowed with KdV-type boundary conditions can be obtained by following three different approaches. First, by means of the Noether theorem, we obtained a radial conserved quantity, which once evaluated in the BTZ solution naturally leads to an expression for the entropy as a $z$-dependent bilinear combination of the conserved charges times the chemical potentials at infinity. Secondly, we prove that the same formula can be obtained through the anisotropic S-duality of the partition function of a dual 2D field theory, and we show its close relationship with the corresponding Cardy-like formula. Finally, by considering the scaling properties of the entropy as a function of the charges, it was possible to recover the aforementioned Smarr relation from the Euler theorem for homogeneous functions.

The second aim of this work is devoted to the thermodynamical stability of the system. We have shown that, as it is expected for a black hole solution in a Chern-Simons theory, the specific heat of the BTZ black hole is a positive, monotonically increasing function of the temperature, independently of the choice of KdV-type boundary condition. In contrast, it was shown that the global stability of the system is sensitive to the specific choice of boundary conditions. There is Hawking-Page phase transition at an specific $z$-dependent self-dual temperature $T_s$, for which, at temperatures below this point, the preferred phase is the AdS$_3$ spacetime, and for higher temperatures, the BTZ black hole is the more stable phase. This self-dual temperature decreases for higher values of $z$, as does the size of the
black hole horizon and the energy that the system absorbs from the environment in order for the transition occurs.

Remarkably, the anisotropic scaling properties which are commonly realized in the context of Lifshitz holography, now take place in the General Relativity scenario. This is because the KdV-type boundary conditions induce these kind of scaling properties in the dual theory allowing to study Lifshitz holography in a simple setup. This fact leads to an interesting consequence, as it can be seen that rotation terms naturally appears in the anisotropic Smarr formula, despite that there is no a rotating Lifshitz black hole in three dimensions.

Along this work it has been assumed that the cosmological constant is a fixed constant without variation. However, is it possible to follow another point of view. If the energy of the black hole is no longer the mass but the thermodynamical enthalpy, the Smarr formula and a extended first law, can be found by considering a variable cosmological constant which can be related with pressure and volume terms (see e.g. [64–67]). In the literature, there is a standard mechanism described in [68], which explain how to promote the cosmological constant to a canonical variable. Nonetheless, at least in three dimensions, there is a superselection rule that forbids this possibility [69], in consequence, it cannot be rescaled.

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