QUASI-COARSE SPACES, HOMOTOPY AND HOMOLOGY

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Abstract. We introduce the notion of a quasi-coarse structure on a set, which enables one to endow non-trivial ‘coarse-like’ structures to compact metric spaces, something which is impossible in coarse geometry. We begin the study of homotopy in this context, and we then construct homology groups which are invariant under quasi-coarse homotopy equivalence. We further show that any undirected graph \( G = (V, E) \) induces a quasi-coarse structure on its set of vertices \( V_G \), and that the respective quasi-coarse homology is isomorphic to the Vietoris-Rips homology. This, in turn, leads to a homotopy invariance theorem for the Vietoris-Rips homology of undirected graphs.

CONTENTS

1. Introduction 2
2. Quasi-Coarse Spaces 3
2.1. Fundamental Concepts and Examples 3
2.2. Subspaces 11
2.3. Product Quasi-Coarse Spaces 13
2.4. Quotient Spaces 18
2.5. Coarse Space Induced by a Quasi-Coarse Space 20
3. Homotopy 23
3.1. Homotopy and Homotopy Equivalence for Quasi-Coarse Spaces 23
3.2. Homotopy Groups 26
3.3. Connectedness 30
3.4. The Quasi-Coarse Fundamental Group of Cyclic Graphs 32
3.5. Long Exact Sequence in Homotopy 36
4. Homology 39
4.1. Simplicial Homology 39
4.2. Graphs and Quasi-Coarse Spaces 42
Acknowledgements 44
Conflict of Interest Statement 44
References 44

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1. Introduction

Coarse geometry [6] is often referred to as ‘geometry in the large’, or ‘large scale geometry’, for the reason that, on a metric space endowed with the natural coarse structure, all bounded phenomena are trivial from the coarse point of view. The study of coarse geometry has led to a number of important advances, in particular in group theory, where it has enabled many questions about groups to be addressed geometrically via the analysis of their Cayley graphs (see, for instance, [3] for a recent book-length discussion of this approach). Nonetheless, the large-scale nature of coarse geometry forces all finite-diameter spaces to be coarsely-equivalent to a point. It is not difficult, however, to imagine scenarios where one might like to ‘coarsen’ a space up to a certain scale, but where one does not want to erase all bounded phenomena at every scale. Many problems of topological data analysis, for instance, in which one would like to deduce the topological invariants of a space from algebraic invariants built from a finite subset of the space, present themselves naturally as problems of ‘medium-scale’ coarsening. Several possibilities for a formal context for ‘coarsening’ a space up to a scale were developed by the first author in [4] and [5], where he studied the problem from the point of view of Čech closure spaces and semi-uniform spaces, respectively. The connection to coarse geometry in these earlier works, however, remained unclear, and this question forms the primary motivation for the work presented here.

Whereas the closure structures and semi-uniform structures studied in [4] and [5] are generalizations of topological structures and uniform structures, respectively, in this article, we focus instead on a generalization of coarse structures, modifying the axioms of coarse geometry [6] to allow ‘coarsenings’ only up to a preferred size. Technically, this is achieved by eliminating the product axiom in the definition of the coarse structure, which, indeed, has a similar flavor to the elimination of the idempotence axiom in the passage from Kuratowski (topological) closure structures to more general Čech closure structures, or to the elimination of the product axiom in the passage from uniform spaces to semi-uniform spaces. While removing the product axiom effectively destroys the notion of coarse equivalence, we find that coarse equivalence can be profitably generalized to a useful notion of homotopy equivalence in this setting. We call the resulting structures and spaces quasi-coarse structures and quasi-coarse spaces, respectively. Many examples of quasi-coarse spaces exist. The examples of most interest to the topological data analysis community may be built from pseudo-metric spaces and a preferred scale \( r > 0 \), but many others exist as well, in particular those constructed from semi-uniform spaces.

The outline of this article is as follows. In Section 2, we give the basic definitions, point-set properties, and principal examples of quasi-coarse spaces. We will also show in that section how to obtain a coarse space from a quasi-coarse space through a certain limiting process. In Section 3, we begin the study of homotopy invariants in the quasi-coarse category. Unfortunately, the interval does not appear to have a natural non-trivial quasi-coarse structure, significantly complicating the construction. We are able circumvent this shortcoming by adapting the homotopy construction from [1], using finite-length subsequences of \( \mathbb{Z} \) in place of the interval to construct cylinders. While technically delicate, this nonetheless allows us to define homotopy groups, prove a long-exact sequence of pairs, and compute the fundamental group of a ‘quasi-coarse circle’ with four points. Finally, in Section 4, we construct homology groups for quasi-coarse spaces, inspired by the Vietoris-Rips
construction now commonly used in topological data analysis. We give a construction for any quasi-coarse space, demonstrate the invariance of the homology groups with respect to the homotopy introduced in Section 3 and show that, for a countable quasi-coarse space, this homology is exactly the Vietoris-Rips homology of an associated graph.

Many of the results in this paper were first presented in the Master’s thesis of the second author [7], written under the supervision of the first author, where quasi-coarse spaces and structures were called pseudo-coarse.

2. Quasi-Coarse Spaces

In this section, we define quasi-coarse spaces, bornologous functions, and we give examples of quasi-coarse structures constructed from a metric space and a scale parameter $r > 0$. We then define quasi-coarse quotients, disjoint unions, and products, and we show how to build an induced coarse structure from a quasi-coarse structure.

2.1. Fundamental Concepts and Examples. We begin by setting some notation which we will use throughout the article.

**Definition 2.1.1.** Let $X$ be a set. We denote by $\mathcal{P}(X)$ the collection of all subsets of $X$, and

1. $X^n$ will denote $X \times X \times \cdots \times X$.
2. $\Delta_X := \{(x, x) \in X \times X : x \in X\}$ will be called the diagonal of $X$.
3. For $V \in \mathcal{P}(X \times X)$, we define
   $$V^{-1} := \{(y, x) \in X \times X : (x, y) \in V\},$$
   which we call the inverse of $V$.
4. For $V, W \in \mathcal{P}(X \times X)$, we define
   $$V \circ W := \{(x, y) \in X \times X : \exists z \in X, (x, z) \in V \land (z, y) \in W\},$$
   which will be called the set product of $V$ and $W$.
5. Let $X$ and $Y$ be sets, $f : X \to Y$ a set function, and $V \in \mathcal{P}(X \times X)$. Then
   $$(f \times f)(V) := \{(f(x), f(x')) \in Y \times Y : (x, x') \in V\},$$
   and we call $(f \times f)(V)$ the image of $V$ under $f \times f$.

We now define quasi-coarse spaces, our principal object of interest.

**Definition 2.1.2 (Quasi-coarse space).** Let $X$ be a set, and let $\mathcal{V} \subset \mathcal{P}(X \times X)$ be a collection of subsets of $X \times X$ which satisfies

1. $\Delta_X \in \mathcal{V}$,
2. If $B \in \mathcal{V}$ and $A \subset B$, then $A \in \mathcal{V}$,
3. If $A, B \in \mathcal{V}$, then $A \cup B \in \mathcal{V}$,
4. If $A \in \mathcal{V}$, then $A^{-1} \in \mathcal{V}$.

We call $\mathcal{V}$ a quasi-coarse structure on $X$, and we say that the pair $(X, \mathcal{V})$ is a quasi-coarse space.

If, in addition, $\mathcal{V}$ satisfies
(qc5) If \( A, B \in \mathcal{V} \), then \( A \circ B \in \mathcal{V} \).

Then \( \mathcal{V} \) will be called a coarse structure, and \((X, \mathcal{V})\) will be called a coarse space, as in [6].

The elements of \( \mathcal{V} \) will be called controlled sets. Moreover, if there exist \( a, b \) such that \( \{(a, b)\} \in \mathcal{V} \) we will say that \( a \) and \( b \) are adjacent. If \( \mathcal{V} \) and \( \mathcal{V}' \) are quasi-coarse structures on \( X \) such that \( \mathcal{V} \subset \mathcal{V}' \), then we say that \( \mathcal{V}' \) is finer than \( \mathcal{V} \) and \( \mathcal{V} \) is coarser than \( \mathcal{V}' \). Finally, when the structure \( \mathcal{V} \) is unambiguous, we will sometimes refer to the quasi-coarse space only by \( X \).

The functions of interest between quasi-coarse spaces will be those which preserve the quasi-coarse structure.

**Definition 2.1.3.** We will say that \( f : X \to Y \) is a \((\mathcal{V}, \mathcal{W})\)-bornologous function, or simply bornologous, if \( f \times f \) maps each controlled set \( V \in \mathcal{V} \) to a controlled set \((f \times f)(V) \in \mathcal{W}\).

Since the composition of set maps is associative, the composition of bornologous maps is bornologous, and the identity is bornologous for every quasi-coarse space \((X, \mathcal{V})\), we have

**Theorem 2.1.4.** Quasi-coarse spaces and bornologous functions form a category.

We denote the category of quasi-coarse spaces and bornologous functions by \( \text{QCoarse} \).

For our first collection of examples, we show how undirected graphs induce quasi-coarse structures. We begin with a pair of lemmas.

**Lemma 2.1.5.** Let \( X \) be a set, and let \( W \subset X \times X \) such that

(1) \( \Delta_X \subset W \),
(2) \( W = W^{-1} \).

Then \((X, \mathcal{P}(W))\) is a quasi-coarse space.

**Proof.** We verify directly that the axioms \([\text{qc1}] - [\text{qc4}]\) from Definition 2.1.2 are satisfied.

(\text{qc1}) \( \Delta_X \subset W \), so \( \Delta_X \in \mathcal{P}(W) \).
(\text{qc2}) \( \mathcal{P}(W) \) is closed under taking subsets by definition.
(\text{qc3}) Since any two sets \( A, B \in \mathcal{P}(W) \) are subsets of \( W \), the union \( A \cup B \subset W \), and therefore \( A \cup B \in \mathcal{P}(W) \).
(\text{qc4}) Suppose \( A \in \mathcal{P}(W) \). Then \( A \subset W \). However, by hypothesis on \( W \), if \( (a, b) \in W \) then \( (b, a) \in W \). Therefore \( A^{-1} \subset W \), so \( A^{-1} \in \mathcal{P}(W) \) as well. It follows from the above that \((X, \mathcal{P}(W))\) is a quasi-coarse space, as desired. \( \square \)

It will sometimes be convenient to use this lemma in the following alternative form.

**Lemma 2.1.6.** Let \( X \) be a set and suppose that \( U \subset \mathcal{P}(X \times X) \) is a collection of subsets of \( X \times X \) such that

(1) \( U \in U \implies U^{-1} \in U \), and
(2) \( \Delta_X \in U \).

Let \( W := \bigcup_{U \in U} U \). Then \((X, \mathcal{P}(W))\) is a quasi-coarse space.

**Proof.** By construction, \( W \) satisfies the hypotheses of Lemma 2.1.5 The conclusion follows. \( \square \)
We now use these lemmas to give several important examples of quasi-coarse spaces.

**Example 2.1.6.1.** Let $G = (V, E)$ be an undirected graph (i.e. $(u, v) \in E \iff (v, u) \in E$). Define $\mathcal{V} \subset \mathcal{P}(V \times V)$ to be

$$\mathcal{V}_G := \mathcal{P}(E \cup \Delta_V),$$

where $\Delta_V$ is the diagonal of $V \times V$. By Lemma 2.1.6 above, the pair $(V, \mathcal{V}_G)$ is a quasi-coarse space.

**Definition 2.1.7.** Given a graph $G = (V, E)$, we say that the quasi-coarse space $(V, \mathcal{V}_G)$ constructed in Example 2.1.6.1 is the quasi-coarse space generated by the graph $G$.

Another important class of examples may be constructed from metric spaces combined with a positive scale parameter $r > 0$.

**Example 2.1.7.1.** (a) Let $(X, d)$ be a metric space. Let $r > 0$ be a positive real number and define

$$U_r := \{(x, x') \in X \times X \mid d(x, x') \leq r\},$$

Then $(X, \mathcal{P}(U_r))$ is a quasi-coarse space by Lemma 2.1.5.

(b) Similarly, defining $U_r^< := \{(x, x') \in X \times X \mid d(x, x') < r\}$,

Lemma 2.1.5 gives that $(X, \mathcal{P}(U_r^<))$ is a quasi-coarse space.

For the next example, which generalizes the ones in Example 2.1.7.1 above, we introduce semi-pseudometric spaces.

**Definition 2.1.8** (Semi-Pseudometric; [2], 18 A.1.). Let $X$ be a set and $d : X \times X \to \mathbb{R}$ be a function, we will say $d$ is a semi-pseudometric on $X$ if they satisfies the next conditions

(m1) For each $x \in X$, $d(x, x) = 0$.

(m2) For every $x, y \in X$, $d(x, y) = d(y, x) \geq 0$.

A semi-pseudometric on $X$ is a pseudometric on $X$ if also

(m3) For each $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$, i.e. $d$ satisfies the triangle inequality.

A semi-pseudometric will be called a semi-metric, if it also satisfies

(m4) $d(x, y) = 0$ implies $x = y$.

Finally, a semi-pseudometric $d$ will be called a metric iff it satisfies (m1)–(m4) that is, if $d$ is both a semi-metric and a pseudometric.

A semi-pseudometric space is an ordered pair $(X, d)$ where $X$ is a set and $d$ is a semi-pseudometric on $X$. Similarly, $(X, d)$ is a semi-metric, pseudometric, or metric space when $d$ is a semi-metric, pseudometric, or metric, respectively.

We will also need the following semi-pseudometrics constructed from a metric and a pre-determined scale $r > 0$. 
Definition 2.1.9. Let \((X, d)\) be a metric space and let \(r \geq 0\) be a non-negative real number. We will define the functions \(d_r, d_r^c, d_r^\infty : X \times X \to [0, \infty)\) by
\[
d_r^c(x, y) = \begin{cases} 
0 & \text{if } d(x, y) \leq r, \\
1 & \text{if } d(x, y) > r,
\end{cases}
\]
\[
d_r^\infty(x, y) = \begin{cases} 
0 & \text{if } d(x, y) < r, \\
1 & \text{if } d(x, y) \geq r.
\end{cases}
\]
\[d_r(x, y) = \max\{0, d(x, y) - r\}\]

Remark 2.1.10. For \(r \geq 0\), the function \(d_r\) satisfies \(d_r(x, x) = 0\) and \(d_r(x, y) = d_r(y, x)\), by the symmetry of \(d\). Therefore, \(d_r\) is a semi-pseudometric on \(X\) for any \(r \geq 0\). Similarly, the functions \(d_r^c\) and \(d_r^\infty\) are semi-pseudometrics on \(X\) for \(r \leq 0\) and \(r > 0\), respectively.

Example 2.1.10.1. (a) Let \(d : X \times X \to [0, \infty)\) be a semi-pseudometric, and define \(U := \{(x, x') \in X \times X \mid d(x, x') = 0\}\). Then \((X, \mathcal{P}(U))\) is a quasi-coarse space by Lemma 2.1.5. The semi-pseudometrics \(d_r^c\) and \(d_r^\infty\) from Definition 2.1.9 give the quasi-coarse structures in Examples 2.1.7.1(a) and 2.1.7.1(b) respectively.

The following examples are independent of the Lemmas 2.1.5 and 2.1.6 and give examples of quasi-coarse spaces which are not coarse.

Example 2.1.10.2. (a) Let \((X, d)\) be a metric space which is at most countably infinite. Define \(V \subset \mathcal{P}(X \times X)\) by
\[V \in \mathcal{V} \iff \{|V| < \infty \mid \forall (x, x') \in V, d(x, x') < 1\}.
\]
Then \((X, \mathcal{V})\) is a quasi-coarse space. If \(X = \mathbb{Q}\) with the Euclidean metric, then \((\mathbb{Q}, \mathcal{V})\) is a quasi-coarse space which is not coarse.

(b) Let \((X, c)\) be an uncountable metric space. Define \(V \subset \mathcal{P}(X \times X)\) by
\[V \in \mathcal{V} \iff \{|V| \leq \aleph_0 \mid \forall (x, x') \in V, d(x, x') < 1\}.
\]
Then \((X, \mathcal{V})\) is a quasi-coarse space. If \(X = \mathbb{R}\) with the Euclidean metric, then \((\mathbb{R}, \mathcal{V})\) is a quasi-coarse space which is not coarse.

Finally, we show how to generate a quasi-coarse structure from a semi-uniform space, generalizing Examples 2.1.6.1, 2.1.7.1 and 2.1.10.1 above. Semi-uniform spaces (described in detail in [2], Chapter 23) are a generalization of uniform spaces which are no longer necessarily topological. After the following preliminary definition, we recall the definition of semi-uniform spaces and give several examples. We then show how to construct a quasi-coarse space from a semi-uniform space.

Definition 2.1.11. A filter \(\mathcal{U}\) on a set \(X\) is a non-empty collection of subsets of \(X\) such that

\((f1)\) \(\emptyset \notin \mathcal{U}\),
\((f2)\) If \(A, A' \in \mathcal{U}\), then \(A \cap A' \in \mathcal{U}\),
\((f3)\) If \(A \in \mathcal{U}\) and \(A \subset A'\), then \(A' \in \mathcal{U}\).

A subcollection \(\mathcal{U}_0\) of \(\mathcal{U}\) is a filter base of \(\mathcal{U}\) iff each element of \(\mathcal{U}\) contains some element of \(\mathcal{U}_0\).

Remark 2.1.12. A filter is sometimes defined in the literature without \((f1)\) above, in which case a filter which also satisfies \((f1)\) is called a proper filter. We will not
make this distinction in the present article, and we assume that a filter always satisfies (f1).

**Definition 2.1.13** (Semi-Uniform Space; [2], 23 A.3.). Let $X$ be a set and $\mathcal{U}$ be a filter on $X \times X$. We call $\mathcal{U}$ a **semi-uniform structure on** $X$ and the pair $(X, \mathcal{U})$ a **semi-uniform space** iff

(su1) Each element $U \in \mathcal{U}$ contains the diagonal $\Delta$ of $X$,

(su2) If $A \in \mathcal{U}$, then $A^{-1}$ contains an element of $\mathcal{U}$.

Since $\mathcal{U}$ is a filter, as noted in [2], Axiom (su2) may be replaced by

(su2’) If $A \in \mathcal{U}$, then $A^{-1} \in \mathcal{U}$.

A semi-uniform space $(X, \mathcal{U})$ is called a **uniform space** iff, in addition to the above,

(su3) For every $U \in \mathcal{U}$, there exists a $V \in \mathcal{U}$ such that $V \circ V \subset U$.

When context allows it, we will just represent the semi-uniform space by its set $X$.

Let $(X, \mathcal{U})$ and $(Y, \mathcal{T})$ be semi-uniform spaces. We will say that $f : (X, \mathcal{U}) \to (Y, \mathcal{T})$ is uniformly continuous iff for each $B \in \mathcal{T}$ there exists a $A \in \mathcal{U}$ such that $(f \times f)(A) \subset (B)$.

**Definition 2.1.14.** We denote by $\text{SUnif}$ the category of semi-uniform spaces and uniformly continuous maps.

Following [2], Example 23.A.7, we have the following construction of a semi-uniform space from a semi-pseudometric space.

**Example 2.1.14.1.** (1) Let $(X, d)$ be a semi-pseudometric space, and let $\mathcal{U}_d$ be the semi-uniform structure on $X \times X$ generated by the collection of sets $\{B_q\}_{q > 0}$, where $B_q = \{(x, y) : d(x, y) < q\}$.

(2) Given a metric space $(X, d)$ and a positive real number $r > 0$, the construction in item (1) above applied to the semi-pseudometrics $d_r$, $d_r^\infty$, and $d_r^\gamma$ from Definition 2.1.9 gives examples of semi-uniform spaces which are not uniform.

**Definition 2.1.15.** Given a semi-pseudometric space $(X, d)$, We call $\mathcal{U}_d$ the **semi-uniform structure induced by** the semi-pseudometric $d$. If, in addition, $r \geq 0$ is a non-negative real number, then we write $\mathcal{U}_r$ for the semi-uniform structure induced by the semi-pseudometric $d_r$ from Definition 2.1.9.

We now show how to build a semi-uniform structure from a quasi-coarse structure and vice-versa. We first state the following simple lemma.

**Lemma 2.1.16.** Let $X$ be a set and suppose that $A, B \in \mathcal{P}(X \times X)$. Then

(i) $A = (A^{-1})^{-1}$, and

(ii) If $A \subset B$, then $A^{-1} \subset B^{-1}$.

**Proof.** (i) By definition, we have that

$$(A^{-1})^{-1} = \{(x, y) : (y, x) \in A^{-1}\} = \{(x, y) : (x, y) \in A\} = A.$$ 

(ii) Let $(x, y) \in A^{-1}$, then $(y, x) \in A$, so that $(y, x) \in B$. Therefore, $(x, y) \in B^{-1}$. \qed
**Proposition 2.1.17** (Quasi-Coarse Space from a Semi-Uniform Space). Let \((X, \mathcal{U})\) be a semi-uniform space, and define the collection \(\mathcal{U}^\downarrow \subset \mathcal{P}(X \times X)\) by

\[
\mathcal{U}^\downarrow := \left\{ B \subset X \times X : B \subset \bigcap_{A \in \mathcal{U}} A \right\}.
\]

Then \((X, \mathcal{U}^\downarrow)\) is a quasi-coarse space.

**Proof.** Let \((X, \mathcal{U})\) and \(\mathcal{U}^\downarrow\) be as in the statement of the Proposition. We check that the axioms for quasi-coarse structures are satisfied by \(\mathcal{U}^\downarrow\).

(qc1) By definition of a semi-uniform space, the diagonal \(\Delta_X \subset U\) for each \(U \in \mathcal{U}\). Therefore, \(\Delta_X \in \mathcal{U}^\downarrow\).

(qc2) Let \(B \in \mathcal{U}^\downarrow\) and \(B' \subset B\). Then \(B' \subset B \subset A\) for each \(A \in \mathcal{U}\). So \(B' \in \mathcal{U}^\downarrow\).

(qc3) Let \(B, B' \in \mathcal{U}^\downarrow\). By definition, we have that \(B, B' \subset A\) for every \(A \in \mathcal{U}\), and therefore \(B \cup B' \subset A\) for every \(A \in \mathcal{U}\) as well. Therefore, \(B \cup B' \in \mathcal{U}^\downarrow\).

(qc4) Let \(B \in \mathcal{U}^\downarrow\), then \(B \subset A\) for each \(A \in \mathcal{U}\). By Lemma 2.1.16, we have that \(B^{-1} \subset A^{-1}\) for each \(A \in \mathcal{U}\), and since \(A \in \mathcal{U}\) implies \(A^{-1} \in \mathcal{U}\), further implies that \(B^{-1} \subset A\) for each \(A \in \mathcal{U}\). Therefore, \(B^{-1} \in \mathcal{U}^\downarrow\).

It now follows that \((X, \mathcal{U}^\downarrow)\) is a quasi-coarse space. \(\square\)

**Definition 2.1.18.** The quasi-coarse structure \(\mathcal{U}^\downarrow\) in Proposition 2.1.17 is called the quasi-coarse structure induced by the semi-uniform structure \(\mathcal{U}\).

We now construct a semi-uniform structure \(\mathcal{V}^\uparrow\) from a quasi-coarse space \((X, \mathcal{V})\).

**Proposition 2.1.19.** Let \((X, \mathcal{V})\) be a quasi-coarse space, and define the collection \(\mathcal{V}^\uparrow \subset \mathcal{P}(X \times X)\) by

\[
\mathcal{V}^\uparrow := \left\{ U \subset X \times X : \left( \bigcup_{V \in \mathcal{V}} V \right) \subset U \right\}.
\]

Then \((X, \mathcal{V}^\uparrow)\) is a semi-uniform space.

**Proof.** Let \((X, \mathcal{V})\) and \(\mathcal{V}^\uparrow\) be as in the statement of the proposition. We first show that \(\mathcal{V}^\uparrow\) is a filter.

(f1) Since \(\Delta_X \in \mathcal{V}\), \(\Delta_X \subset (\bigcup_{V \in \mathcal{V}} V) \subset A \cap A'\), and, in particular, \(\emptyset \neq A \cap A'\).

(f2) Suppose that \(A, A' \in \mathcal{V}^\uparrow\). Then \((\bigcup_{V \in \mathcal{V}} V) \subset A \cap A'\) by construction, so \(A \cap A' \in \mathcal{V}^\uparrow\).

(f3) Suppose that \(A \in \mathcal{V}^\uparrow\) and \(A' \subset X \times X\) such that \(A \subset A'\). Then \((\bigcup_{V \in \mathcal{V}} V) \subset A \subset A'\), and therefore \(A' \in \mathcal{V}^\uparrow\) as well.

We now prove that \(\mathcal{V}^\uparrow\) satisfies the axioms of a semi-uniform structure.

(su1) Since \(\Delta_X \in \mathcal{V}\), then \(\Delta_X \subset (\bigcup_{V \in \mathcal{V}} V) \subset U\) for each \(U \in \mathcal{V}^\uparrow\).

(su2') Since \(V \in \mathcal{V} \iff V^{-1} \in \mathcal{V}\), it follows that \(\bigcup_{V \in \mathcal{V}} V = \bigcup_{V \in \mathcal{V}} V^{-1}\). Now suppose that \(A \in \mathcal{V}^\uparrow\). Then \(\bigcup_{V \in \mathcal{V}} V = \bigcup_{V \in \mathcal{V}} V^{-1} \subset A^{-1}\), so \(A^{-1} \in \mathcal{V}^\uparrow\).

It follows that \((X, \mathcal{V}^\uparrow)\) is a semi-uniform space. \(\square\)

**Definition 2.1.20.** We call the semi-uniform structure \(\mathcal{V}^\uparrow\) in Proposition 2.1.19 the Semi-uniform structure induced by the quasi-coarse space \((X, \mathcal{V})\).

Propositions 2.1.17 and 2.1.19 motivate the following definitions.
Definition 2.1.21. (Roof) Let \((X, \mathcal{V})\) be a quasi-coarse space. We call \(\mathfrak{R}(\mathcal{V}) := \bigcup_{V \in \mathcal{V}} V\) the roof of \(\mathcal{V}\), and we say that the quasi-coarse space \((X, \mathcal{V})\) is roofed iff \(\mathfrak{R}(\mathcal{V}) \in \mathcal{V}\). Otherwise, we say that \((X, \mathcal{V})\) is non-roofed.

Example 2.1.21.1. The quasi-coarse spaces in Lemma 2.1.6 are roofed with roof \(W\).

Definition 2.1.22. Let \((X, \mathcal{U})\) be a semi-uniform space. We call \(\mathfrak{F}(\mathcal{U}) := \bigcap_{U \in \mathcal{U}} U\) the foundation of the semi-uniform structure \(\mathcal{U}\). We say that \((X, \mathcal{U})\) is cemented (from the Spanish cimentado) iff \(\mathfrak{F}(\mathcal{U}) \in \mathcal{U}\). Otherwise, we say that \((X, \mathcal{U})\) is non-cemented.

Remark 2.1.23. Note that the roof is the maximal in the quasi-coarse structure with order relation \(A < B\) iff \(A \subset B\), while the foundation is the minimal in the semi-uniform structure with the same order.

Proposition 2.1.24. (1) Let \(U \subset X \times X\) such that \(\Delta_X \subset U\) and \(U = U^{-1}\), and denote by \([U]\) the filter generated by \(U\). Then \((X, [U])\) is a cemented semi-uniform space with foundation \(U\).

(2) Let \(\mathcal{U}\) and \(\mathcal{T}\) be semi-uniform structures on \(X\). If \(\mathfrak{F}(\mathcal{U}) = \mathfrak{F}(\mathcal{T})\) and \((X, \mathcal{U})\) is cemented, then \(\mathcal{T} \subset \mathcal{U}\). It follows that, for each foundation, there exists a unique cemented semi-uniform space with that foundation.

Proof. (1) Let \(U \subset X \times X\) such that \(\Delta_X \subset U\) and \(U = U^{-1}\), and denote by \([U]\) the filter generated by \(U\). Since \(\Delta_X \subset U\) and \(U = U^{-1}\), then \(\Delta_X \subset A\) for all \(A \in [U]\). Furthermore, if \(A \in [U]\), then \(U \subset A\) and \(U = U^{-1} \subset A^{-1}\), from which it follows that \(A^{-1} \in [U]\). Thus, \((X, [U])\) is a semi-uniform space.

Moreover, since \(U \subset A\) for all \(A \in [U]\), then

\[
U \subset \bigcap_{A \in [U]} A \subset U
\]

Thus \((X, [U])\) is a cemented semi-uniform space with foundation \([U]\).

(2) Let \(\mathcal{U}\) and \(\mathcal{T}\) be semi-uniform structures on \(X\) such that \(\mathfrak{F}(\mathcal{U}) = \mathfrak{F}(\mathcal{T})\), and suppose that \((X, \mathcal{U})\) is cemented. Therefore, for all \(B \in \mathcal{T}\), \(\mathfrak{F}(\mathcal{U}) \subset B\), and since \(\mathfrak{F}(\mathcal{U})\), therefore \(B \in \mathcal{U}\). Thus, \(\mathcal{T} \subset \mathcal{U}\) and the result follows. \(\square\)

Proposition 2.1.25. Let \(\mathcal{V}\) and \(\mathcal{T}\) be a quasi-coarse structures on a set \(X\). Suppose that \(\mathcal{V}\) is roofed, and that \(\mathfrak{R}(\mathcal{V}) = \mathfrak{R}(\mathcal{T})\). Then \(\mathcal{T} \subset \mathcal{V}\), and for each roof, there exists a unique roofed quasi-coarse structure with that roof.

Proof. For all \(B \in \mathcal{T}\), \(B \subset \mathfrak{R}(\mathcal{T}) = \mathfrak{R}(\mathcal{V}) \in \mathcal{V}\), and therefore \(B \in \mathcal{V}\). The result follows. \(\square\)

Proposition 2.1.26. Let \((X, \mathcal{V})\) and \((Y, \mathcal{W})\) be quasi-coarse spaces, and suppose that \((X, \mathcal{V})\) is roofed. A map \(f : X \to Y\) is bornologous iff \((f \times f)(\mathfrak{R}(\mathcal{V})) \in \mathcal{W}\).

Proof. Any set \(V \in \mathcal{V}\) is a subset of \(\mathfrak{R}(\mathcal{V})\) by definition. Therefore, if \((f \times f)(V) \subset (f \times f)(\mathfrak{R}(\mathcal{V})) \in \mathcal{W}\), then \((f \times f)(V) \in \mathcal{W}\), so \(f\) is bornologous.

Conversely, suppose that \(f\) is bornologous. Then \((f \times f)(\mathfrak{R}(\mathcal{V})) \in \mathcal{W}\). \(\square\)

Similarly, for cemented semi-uniform spaces, we have
Proposition 2.1.27. Let \((X, \mathcal{U})\) and \((Y, \mathcal{T})\) be semi-uniform spaces and suppose that \((Y, \mathcal{T})\) is cemented. A map \(f : X \to Y\) is uniformly continuous iff \((f \times f)(\mathcal{U}) \subset \mathcal{T}\).

Proof. Let \((X, \mathcal{U})\) and \((Y, \mathcal{T})\) be cemented semi-uniform spaces, and let \(f : (X, \mathcal{U}) \to (Y, \mathcal{T})\) be a uniformly continuous function. Then, for each \(B \in \mathcal{T}\) there exists a set \(A_B \in \mathcal{U}\) such that \((f \times f)(A_B) \subset B\). Therefore

\[
(f \times f)(\mathcal{U}) = (f \times f) \left( \bigcap_{A \in \mathcal{U}} A \right) \subset \bigcap_{B \in \mathcal{T}} (f \times f)(A) \subset \bigcap_{B \in \mathcal{T}} B = \mathcal{T}.
\]

Conversely, assume that \((f \times f)(\mathcal{U}) \subset \mathcal{T}\). If \(B \in \mathcal{T}\), then \(\mathcal{T} \subset B\), and therefore \((f \times f)(\mathcal{U}) \subset B\) as well. However, since \((X, \mathcal{U})\) is cemented, we have that \(\mathcal{U} \in \mathcal{T}\), from which it follows that \(f\) is uniformly continuous. \(\Box\)

Proposition 2.1.28. Let \(X\) be a set, and suppose that \(\mathcal{V}\) and \(\mathcal{U}\) are quasi-coarse and semi-uniform structures on \(X\), respectively. Then

(i) \((X, \mathcal{U}^\dagger)\) is a roofed quasi-coarse space.

(ii) \((X, \mathcal{V}^\dagger)\) is a cemented semi-uniform space.

(iii) If \((X, \mathcal{U})\) is a cemented semi-uniform space, then \((\mathcal{U}^\dagger)^\dagger = \mathcal{U}\).

(iv) If \((X, \mathcal{V})\) is a roofed quasi-coarse space, then \((\mathcal{V}^\dagger)^\dagger = \mathcal{V}\).

Proof. Let \(X, \mathcal{V}\), and \(\mathcal{U}\) be as in the hypothesis of the proposition.

(i) First note that, by definition, \(\mathcal{U} \in \mathcal{U}^\dagger\). Now suppose that \(A \in \mathcal{U}^\dagger\). By construction, \(A \subset B\) for every \(B \in \mathcal{U}\), and therefore \(A \subset \mathcal{U}\). Therefore, \(\mathcal{U}^\dagger = \mathcal{U} \in \mathcal{U}^\dagger\), and \((X, \mathcal{U}^\dagger)\) is a roofed quasi-coarse space.

(ii) As before, note that \(\mathcal{V} \in \mathcal{V}^\dagger\) by definition. Now let \(B \in \mathcal{V}^\dagger\). By construction, \(A \subset B\) for every \(A \in \mathcal{V}\), and therefore \(\mathcal{V} \subset B\). It follows that \(\mathcal{V}^\dagger = \mathcal{V} \in \mathcal{V}^\dagger\), so \((X, \mathcal{V}^\dagger)\) is a cemented semi-uniform space.

(iii) By points (i) and (ii) above, \(\mathcal{U} = \mathcal{U}^\dagger = \mathcal{U}^\dagger^\dagger\). Therefore, \(\mathcal{U}^\dagger = (\mathcal{U}^\dagger)^\dagger\) by Proposition 2.1.24.

(iv) By points (i) and (ii) above, \(\mathcal{V} = \mathcal{V}^\dagger = \mathcal{V}^\dagger^\dagger\). Therefore, \(\mathcal{V}^\dagger = (\mathcal{V}^\dagger)^\dagger\) by Proposition 2.1.27. \(\Box\)

We now show that the constructions above are functorial. Let \(\text{RQCoarse}\) and \(\text{CSUnif}\) denote the full subcategories of roofed quasicoarse spaces and cemented semi-uniform spaces, respectively.

Proposition 2.1.29. Let \(\Phi : \text{QCoarse} \to \text{SUnif}\) be the map \(\Phi(X, \mathcal{V}) = (X, \mathcal{V}^\dagger)\) on objects and \(\Phi(f : X \to Y) = f\) on morphisms, and let \(\Psi : \text{SUnif} \to \text{QCoarse}\) be the map \(\Psi(X, \mathcal{U}) = (X, \mathcal{U}^\dagger)\) on objects and \(\Psi(f : X \to Y) = f\) on morphisms.

Then \(\Phi\) and \(\Psi\) are functors, \(\Phi(\text{QCoarse}) = \text{CSUnif}\), and \(\Psi(\text{CSUnif}) = \text{RQCoarse}\). Moreover, we have \(\Phi|_{\text{RQCoarse}} \circ \Psi|_{\text{CSUnif}} = 1_{\text{CSUnif}}\), and \(\Psi|_{\text{CSUnif}} \circ \Phi|_{\text{RQCoarse}} = 1_{\text{RQCoarse}}\).
Proof. It suffices to prove that $\Phi$ maps bornologous function to uniformly continuous functions and that $\Psi$ maps uniformly continuous functions to bornologous functions. The rest of the proposition is a direct consequence of Proposition 2.1.28.

Let $f : (X, V) \to (Y, W)$ be a bornologous function. Then $(f \times f)(A) \in W$ for every $A \in V$. We therefore have that

$$(f \times f) \left( \bigcup_{A \in V} A \right) \subset \bigcup_{B \in W} B.$$ 

Thus, the foundation of $W^\uparrow$ contains the image of the foundation of $V^\uparrow$ under $f \times f$, i.e., $(f \times f)(\mathfrak{F}(V^\uparrow)) \subset U$ for every $U \in W^\uparrow$. Therefore $f : (X, U^\uparrow) \to (Y, W^\uparrow)$ is uniformly continuous by Proposition 2.1.27.

Now let $f : (X, U) \to (Y, T)$ be a uniformly continuous function. By definition, for every $T \in T$, there exists a $U_T \in U$ such that $(f \times f)(U_T) \subset T$. We therefore have that

$$(f \times f)(\mathfrak{T}(U)) = (f \times f) \left( \bigcap_{U \in U} U \right) \subset \bigcap_{T \in T} (f \times f)(U_T) \subset \bigcap_{T \in T} T$$

$$= \mathfrak{T}(f \times f)(U).$$

Therefore, the roof of $T^\uparrow$ contains the image of the roof of $U$ under $f \times f$. It follows that $(f \times f)(A) \subset rf(Y, T^\uparrow)$, or, equivalently, $(f \times f)(A) \in T^\uparrow$ for every $A \in U^\uparrow$. Therefore, $f : (X, U^\uparrow) \to (Y, T^\uparrow)$ is a bornologous function. $\square$

In addition to the above, the next result shows that, when restricted to cemented semi-uniform spaces and roofed quasi-coarse spaces, the functors in Proposition 2.1.29 are adjoints of each other.

2.2. Subspaces. We will now proceed to build new quasi-coarse spaces from existing ones, in particular, constructing subspaces, products, and quotient spaces in the quasi-coarse category. Our first step, in this section, will be to construct quasi-coarse subspaces of a quasi-coarse space $(X, V)$.

We first require the following lemma.

Lemma 2.2.1. Let $X$ be a set, and let $\{A_\lambda\}_{\lambda \in \Lambda}$ a collection of elements of $\mathcal{P}(X \times X)$, where $\Lambda$ is an index set. Then

1. $\bigcup_{\lambda \in \Lambda} (A_\lambda)^{-1} = \left( \bigcup_{\lambda \in \Lambda} A_\lambda \right)^{-1}$,
2. $\bigcap_{\lambda \in \Lambda} (A_\lambda)^{-1} = \left( \bigcap_{\lambda \in \Lambda} A_\lambda \right)^{-1}$.

Proof. Let $X$ be a set, let $\Lambda$ be a index set, and let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a collection of elements of $\mathcal{P}(X \times X)$. 

(i) We see that
\[(x, y) \in \bigcup_{\lambda \in \Lambda} (A_\lambda)^{-1} \iff \exists \lambda_0 \in \Lambda \text{ such that } (x, y) \in (A_{\lambda_0})^{-1} \iff (y, x) \in A_{\lambda_0}\]
\[\iff (y, x) \in \bigcup_{\lambda \in \Lambda} A_\lambda \iff (x, y) \in \left(\bigcup_{\lambda \in \Lambda} A_\lambda\right)^{-1}.\]
Therefore \(\bigcup_{\lambda \in \Lambda} (A_\lambda)^{-1} = \left(\bigcup_{\lambda \in \Lambda} A_\lambda\right)^{-1}\)

(ii) Now note that
\[(x, y) \in \bigcap_{\lambda \in \Lambda} (A_\lambda)^{-1} \iff \forall \lambda \in \Lambda, (x, y) \in (A_\lambda)^{-1} \iff (y, x) \in A_{\lambda} \forall \lambda \in \Lambda\]
\[\iff (y, x) \in \bigcap_{\lambda \in \Lambda} A_\lambda \iff (x, y) \in \left(\bigcap_{\lambda \in \Lambda} A_\lambda\right)^{-1}.\]
Therefore, \(\bigcap_{\lambda \in \Lambda} (A_\lambda)^{-1} = \left(\bigcap_{\lambda \in \Lambda} A_\lambda\right)^{-1}\) as desired.

**Proposition 2.2.2.** Let \((X, V)\) be a quasi-coarse space and suppose that \(Y \subset X\). Define the collection \(V_Y \subset P(Y \times Y)\) by
\[V_Y := \{V \cap (Y \times Y) \mid V \in V\}\]
Then the pair \((Y, V_Y)\) is a quasi-coarse space.

**Proof.** Let \((X, V), Y \subset X\), and \(V_Y\) be as in the statement of the proposition. We check that the axioms for a quasi-coarse space are satisfied by \((Y, V_Y)\).

(qc1) We observe that \(\Delta_Y = \Delta_X \cap (Y \times Y)\). Therefore \(\Delta_Y \in V_Y\).

(qc2) Let \(A \in V_Y\) and suppose that \(A' \subset A\). By definition, there is a \(B \in V\) such that \(A = B \cap (Y \times Y)\). We define \(B' := A' \cap B\) and we observe that \(B' \in V\)
\[\text{since } B' \subset B \in V.\]
However, \(B' \cap (Y \times Y) = A' \cap B \cap (Y \times Y) = A' \cap A = A'\), so \(A' \in V_Y\).

(qc3) If \(A, B \in V_Y\), then there are \(A', B' \in V\) such that \(A = A' \cap (Y \times Y)\) and \(B = B' \cap (Y \times Y)\). Therefore
\[A \cup B = (A' \cap (Y \times Y)) \cup (B' \cap Y \times Y) = (A' \cup B') \cap (Y \times Y)\]
Since \(A' \cup B' \in V\), it follows that \(A \cup B \in V_Y\).

(qc4) If \(A \in V_Y\), then there is \(A' \in V_Y\) such that \(A = A' \cap (Y \times Y)\), then by **Lemma 2.2.1**, we get \(A^{-1} = (A \cap (Y \times Y))^{-1} = A'^{-1} \cap (Y \times Y) = A'^{-1} \cap (Y \times Y)\). Therefore \(A^{-1} \in V_Y\).
It follows that \((Y, V_Y)\) is a quasi-coarse space, as desired.

**Definition 2.2.3** (Quasi-Coarse Subspace). Let \((X, V)\) be a quasi-coarse space and let \(Y \subset X\). The ordered pair \((Y, V_Y)\) from **Proposition 2.2.2** will be called a quasi-coarse subspace of \(X\). When the structure \(V_Y\) is clear from the context, we will simply refer to the subspace \((Y, V_Y)\) as \(Y\).

The following proposition gives a useful criterion for checking whether a function is bornologous on a quasi-coarse space \((X, V)\).
**Proposition 2.2.4.** Let \((X, \mathcal{V})\) and \((Y, \mathcal{W})\) be quasi-coarse spaces, and suppose that \((X_i, \mathcal{V}_i) \subseteq (X, \mathcal{V}), \ i \in \{1, \ldots , n\}\), are subspaces of \((X, \mathcal{V})\) such that \(\bigcup_{i=1}^{n} X_i = X\) and every set \(V \in \mathcal{V}\) may be written in the form

\[ V = \bigcup_{i=n}^{n} V_i \]

where each \(V_i \in \mathcal{V}_i\). Now suppose that \(f : X \to Y\) is a map such that the restrictions \(f_{|X_i} : (X_i, \mathcal{V}_i) \to (Y, \mathcal{W})\) are bornologous for all \(i \in \{1, \ldots , n\}\). Then \(f : (X, \mathcal{V}) \to (Y, \mathcal{W})\) is bornologous.

**Proof.** Let \(V \in \mathcal{V}\). Then, by hypothesis, \(V \in \mathcal{V}_i\) for some \(i \in \{1, \ldots , n\}\), and since \(f_{|X_i}\) is bornologous, we have that \((f \times f)(V) = (f_{|X_i} \times f_{|X_i})(V) \in \mathcal{W}\). Since \(V \in \mathcal{V}\) is arbitrary, it follows that \(f\) is bornologous. \(\Box\)

### 2.3. Product Quasi-Coarse Spaces

In this section, we construct product quasi-coarse structures on the product of sets. We begin with the following definition and several preliminary results.

**Definition 2.3.1.** Let \(X\) and \(Y\) be sets, and suppose that \(V \in \mathcal{P}(X \times X)\) and \(W \in \mathcal{P}(Y \times Y)\). We define

\[ V \boxtimes W := \{((a, b), (c, d)) \in (X \times Y) \times (X \times Y) : (a, c) \in V, (b, d) \in W\} \]

which we call the **Cartesian cross product** of \(V\) and \(W\).

**Lemma 2.3.2.** The Cartesian cross product is associative.

**Proof.** Let \(X, Y,\) and \(Z\) be sets, and suppose that \(U \in \mathcal{P}(X \times X), V \in \mathcal{P}(Y \times Y),\) and \(W \in \mathcal{P}(X \times X)\). We wish to show that \((V \boxtimes W) \boxtimes U \cong V \boxtimes (W \boxtimes U)\). By definition, we have that

\[
(U \boxtimes V) \boxtimes W = \{((a, c), e, (b, d), f)) \in ((X \times Y) \times Z) \times ((X \times Y) \times Z)
\]

\[
\mid (a, b) \in (X \times X), (c, d) \in (Y \times Y),\) and \((e, f) \in (Z \times Z)\}, \text{ and}
\]

\[
U \boxtimes (V \boxtimes W) = \{((a, (c, e)), (b, (d, f))) \in (X \times (Y \times Z)) \times (X \times (Y \times Z))
\]

\[
\mid (a, b) \in (X \times X), (c, d) \in (Y \times Y),\) and \((e, f) \in (Z \times Z)\}.
\]

Since the sets \((X \times Y) \times Z \cong X \times (Y \times Z)\) are isomorphic, this proves the result. \(\Box\)

**Lemma 2.3.3.** Let \(X\) and \(Y\) be sets, \(A \subseteq X \times X, B \subseteq Y \times Y, \{A_\lambda\}_{\lambda \in \Lambda}\) a collection of subsets of \(X \times X\) indexed by \(\Lambda\) and \(\{B_\gamma\}_{\gamma \in \Gamma}\) a collection of subsets of \(Y \times Y\) indexed by \(\Gamma\). Then

\[
(i) \quad (A \boxtimes B)^{-1} = A^{-1} \boxtimes B^{-1}.
\]

\[
(ii) \quad \bigcup_{\lambda \in \Lambda} (A_\lambda \boxtimes B) = \left( \bigcup_{\lambda \in \Lambda} A_\lambda \right) \boxtimes B.
\]

\[
(iii) \quad \bigcup_{\gamma \in \Gamma} (A \boxtimes B_\gamma) = A \boxtimes \left( \bigcup_{\gamma \in \Gamma} B_\gamma \right).
\]

**Proof.** Let \(X, Y, A \subseteq X \times X, B \subseteq Y \times Y, \{A_\lambda\}_{\lambda \in \Lambda}, \text{ and } \{B_\gamma\}_{\gamma \in \Gamma}\) be as in the statement of the proposition.
(i) We observe that
\[(A \boxtimes B)^{-1} = \{(x', y'), (x, y) : (x, x') \in A, (y, y') \in B\} \]
\[= \{(x', y'), (x, y) : (x', x) \in A^{-1}, (y, y') \in B^{-1}\} \]
\[= A^{-1} \boxtimes B^{-1}. \]

(ii) If \((a, c), (b, d) \in \bigcup_{\lambda \in \Lambda} (A_{\lambda} \boxtimes B)\), then there exists a \(\lambda_0 \in \Lambda\) such that \((a, b), (c, d) \in A_{\lambda_0} \boxtimes B\), so \((a, b) \in A_{\lambda_0}\) and \((c, d) \in B\). This implies that \((a, b) \in \bigcup_{\lambda \in \Lambda} A_{\lambda}\) and \((b, d) \in B\), from which we conclude that \((a, c), (b, d) \in \left(\bigcup_{\lambda \in \Lambda} A_{\lambda}\right) \boxtimes B\).

Conversely, if \((a, c), (b, d) \in \left(\bigcup_{\lambda \in \Lambda} A_{\lambda}\right) \boxtimes B\), then \((a, b) \in \bigcup_{\lambda \in \Lambda} A_{\lambda}\) and \((c, d) \in B\). This implies that there is \(\lambda_0 \in \Lambda\) such that \((a, b) \in A_{\lambda_0}\), which gives that \((a, c), (b, d) \in A_{\lambda_0} \boxtimes B\). This implies that \((a, c), (b, d) \in \bigcup_{\lambda \in \Lambda} (A_{\lambda} \boxtimes B)\), as desired.

(iii) The proof of this point is analogous to the proof of part [iii] above. \(\square\)

In the next proposition, we construct the product of two quasi-coarse spaces.

**Proposition 2.3.4.** Let \((X, V)\) and \((Y, W)\) be quasi-coarse spaces, and let \(V \times W\) be the collection of all subsets of finite unions of sets of the form \(V \boxtimes W\), where \(V \in V\) and \(W \in W\), i.e.
\[V \times W := \left\{ U \in \mathcal{P}((X \times Y) \times (X \times Y)) \mid \exists n \in \mathbb{N} \text{ such that } U \subset \bigcup_{i=1}^{n} V_i \boxtimes W_i, \right\}\]
where \(V_i \in V, W_i \in W \forall i \in \{1, \ldots, n\}\).

Then \((X \times Y, V \times W)\) is a quasi-coarse space.

**Proof.** Let \((X, V)\) and \((Y, W)\) be quasi-coarse spaces and let the collection \(V \times W\) be as in the statement of the proposition. We check that \((X \times Y, V \times W)\) satisfies the axioms for a quasi-coarse space.

(qc1) We observe that \(\Delta_X \boxtimes \Delta_Y = \{(x, y), (x, y) : (x, y) \in (X \times Y) \times (X \times Y) \mid x \in X, y \in Y\} = \Delta_{X \times Y}\).

(qc2) Let \(A \in V \times W\) and \(B \subset A\). Then there are \(n \in \mathbb{N}\), \(\{V_1, \ldots, V_n\} \subset V\) and \(\{W_1, \ldots, W_n\} \subset W\) such that \(B \subset A \subset \bigcup_{1 \leq k \leq n} (V_k \boxtimes W_k)\), and therefore \(B \in V \times W\).

(qc3) If \(A, B \in V \times W\). Then there are \(m, n \in \mathbb{N}\), \(\{V_1, \ldots, V_m\}, \{V'_1, \ldots, V'_n\} \subset V\) and \(\{W_1, \ldots, W_m\}, \{W'_1, \ldots, W'_n\} \subset W\) such that
\[A \subset \bigcup_{1 \leq k \leq m} (V_k \boxtimes W_k),\]
\[B \subset \bigcup_{1 \leq k \leq n} (V'_k \boxtimes W'_k).\]
It follows that \(A \cup B \subset \left(\bigcup_{1 \leq k \leq m} (V_j \boxtimes W_j)\right) \cup \left(\bigcup_{1 \leq k \leq n} (V'_k \boxtimes W'_k)\right)\), which implies that \(A \cup B \in V \times W\).

(qc4) If \(A \in V \times W\), then there is an \(n \in \mathbb{N}\), \(\{V_1, \ldots, V_n\} \subset V\), and \(\{W_1, \ldots, W_n\} \subset W\) such that \(A \subset \bigcup_{1 \leq k \leq n} (V_k \boxtimes W_k)\), so \(A^{-1} \subset \bigcup_{1 \leq k \leq n} (V_k^{-1} \boxtimes W_k^{-1})\) by [Lemma 2.2.1] and [Lemma 2.3.3] and therefore \(A^{-1} \in V \times W\).
We conclude that $\mathcal{V} \times \mathcal{W}$ is a quasi-coarse structure on the set $V \times W$. \hfill \Box

**Definition 2.3.5** (Product Quasi-Coarse Space). Let $(X, \mathcal{V})$ and $(Y, \mathcal{W})$ be quasi-coarse spaces, and let $\mathcal{V} \times \mathcal{W}$ be the quasi-coarse structure on $X \times Y$ constructed in [Proposition 2.3.3]. We call the ordered pair $(X \times Y, \mathcal{V} \times \mathcal{W})$ the **product quasi-coarse space of $(X, \mathcal{V})$ and $(Y, \mathcal{W})$**.

It is important to observe that this product is associative, which is established in the next proposition.

**Proposition 2.3.6.** Let $(X, \mathcal{V})$, $(Y, \mathcal{W})$ and $(Z, \mathcal{Z})$ be quasi-coarse spaces. Then $(\mathcal{V} \times \mathcal{W}) \times \mathcal{Z} = \mathcal{V} \times (\mathcal{W} \times \mathcal{Z})$.

*Proof.* Let $(X, \mathcal{V})$, $(Y, \mathcal{W})$ and $(Z, \mathcal{Z})$ be quasi-coarse spaces. Consider $A \in (\mathcal{V} \times \mathcal{W}) \times \mathcal{Z}$. Then there is a natural number $n$ and sets $\{D_k\}_{k=1}^n \subset \mathcal{V} \times \mathcal{W}$ and $\{E_k\}_{k=1}^n \subset \mathcal{Z}$ such that $A \subset \bigcup_{k=1}^n (D_k \boxtimes E_k)$. It follows that, for each $k \in \{1, \ldots, n\}$, there are natural numbers $\{m_k\}_{i=1}^{m_k} \subset \mathbb{N}$ and sets $\{B_i\}_{i=1}^{m_k} \subset \mathcal{V}$ and $\{C_i\}_{i=1}^{m_k} \subset \mathcal{W}$ such that $D_k \subset \bigcup_{i=1}^{m_k} (B_i \boxtimes C_i)$. This gives that $A \subset \bigcup_{k=1}^n \bigcup_{i=1}^{m_k} (B_i \boxtimes C_i \boxtimes E_k)$, by Lemma 2.3.2 and Lemma 2.3.3. Since this is a finite union, and, moreover, since $(C_i \boxtimes E_k) \in \mathcal{W} \times \mathcal{Z}$ and $B_i \in \mathcal{V}$ for each $i \in \{1, \ldots, m_k\}$, $k \in \{1, \ldots, n\}$, we have that $A \in \mathcal{V} \times (\mathcal{W} \times \mathcal{Z})$.

The proof that $(\mathcal{V} \times \mathcal{W}) \times \mathcal{Z} \supset \mathcal{V} \times (\mathcal{W} \times \mathcal{Z})$ is analogous. Thus, the two structures are identical. \hfill \Box

The following alternative characterization of the product structure will be useful for characterizing bornologous functions.

**Proposition 2.3.7.** Let $(X, \mathcal{V})$ and $(Y, \mathcal{W})$ be quasi-coarse spaces. The product structure $\mathcal{V} \times \mathcal{W}$ is the collection of subsets of sets of the form $V \boxtimes W$, where $V \in \mathcal{V}$ and $W \in \mathcal{W}$.

*Proof.* Let $(X, \mathcal{V})$ and $(Y, \mathcal{W})$ be quasi-coarse spaces, and let $\mathcal{V} \boxtimes \mathcal{W}$ denote the collection of subsets of sets of the form $V \boxtimes W$, where $V \in \mathcal{V}$, $W \in \mathcal{W}$.

By definition, it is clear that $\mathcal{V} \boxtimes \mathcal{W} \subset \mathcal{V} \times \mathcal{W}$. On the other hand, if $A \in \mathcal{V} \times \mathcal{W}$, then there is a natural number $n \in \mathbb{N}$ and sets $\{V_1, \ldots, V_n\} \in \mathcal{V}$ and $\{W_1, \ldots, W_n\} \in \mathcal{W}$ such that $A = \bigcup_{k=1}^n (V_k \boxtimes W_k)$. However, we also have that

$$\bigcup_{k=1}^n V_k \in \mathcal{V}, \quad \bigcup_{k=1}^n W_k \in \mathcal{W}, \quad \text{and} \quad A \subset \bigcup_{k=1}^n (V_k \boxtimes W_k) \subset \left(\bigcup_{k=1}^n V_k\right) \boxtimes \left(\bigcup_{k=1}^n W_k\right).$$

Therefore $A \in \mathcal{V} \boxtimes \mathcal{W}$, and the result follows. \hfill \Box

The following corollaries are now an immediate consequence of the above proposition.

**Corollary 2.3.8.** Let $(X, \mathcal{V})$, $(Y, \mathcal{W})$, and $(Z, \mathcal{A})$ be quasi-coarse spaces. Then $f : (X \times Y, \mathcal{V} \times \mathcal{W}) \to (Z, \mathcal{A})$ is a bornologous function iff $f(B \boxtimes C) \in \mathcal{A}$ for each $B \in \mathcal{V}$ and $C \in \mathcal{W}$. 
Corollary 2.3.9. Let \((X, \mathcal{V}), (Y, \mathcal{W}), (Z, \mathcal{A})\) be quasi-coarse spaces, and suppose that \((Y, \mathcal{W})\) is roofed. Then \(f : (X \times Y, \mathcal{V} \times \mathcal{W}) \to (Z, \mathcal{A})\) is a bornologous function iff \((f \times f)(V \times \mathcal{W}) \in \mathcal{A}\) for all \(V \in \mathcal{V}\).

Proof. If \((f \times f)(V \times \mathcal{W}) \in \mathcal{A}\) for all \(V \in \mathcal{V}\), then for any \(W \in \mathcal{W}\), \(W \in \mathcal{R}(\mathcal{V})\), so \((f \times f)(V \times W) \in (f \times f)(V \times \mathcal{W}) \in \mathcal{A}\). Therefore, \((f \times f)(V \times W) \in \mathcal{A}\) and \(f\) is bornologous by Proposition 2.3.7.

Conversely, suppose that \(f\) is bornologous. Then \((f \times f)(V \times \mathcal{W}) \in \mathcal{A}\). \(\square\)

We now discuss infinite products of quasi-coarse spaces. The next two propositions give quasi-coarse analogues of the box product and Tychonoff product of topological spaces.

Proposition 2.3.10. Let \(\Lambda\) be an index set and \(\{(X_\lambda, \mathcal{V}_\lambda)\}\) be a collection of quasi-coarse spaces. Let \(\prod_{\lambda \in \Lambda} \mathcal{V}_\lambda\) denote the collection of subsets of \((\prod_{\lambda \in \Lambda} X_\lambda)^2\) that are contained in some set of the form

\[
\left\{(x, y) \in \left(\prod_{\lambda \in \Lambda} X_\lambda\right)^2 : (\pi_\lambda(x), \pi_\lambda(y)) \in A_\lambda \text{ for each } \lambda \in \Lambda \right\}
\]

with \(A_\lambda \in \mathcal{V}_\lambda\).

Then \(\prod_{\lambda \in \Lambda} \mathcal{V}_\lambda\) is a quasi-coarse structure on the set \(\prod_{\lambda \in \Lambda} X_\lambda\).

Proof. Let \(\Lambda, \{(X_\lambda, \mathcal{V}_\lambda)\}_{\lambda \in \Lambda}\), and \(\prod_{\lambda \in \Lambda} \mathcal{V}_\lambda\) be as in the statement of the proposition. We check that \(\prod_{\lambda \in \Lambda} \mathcal{V}_\lambda\) satisfies the axioms of a quasi-coarse structure. We have

\[
\begin{align*}
(\text{qc1}) & \quad \Delta_{\prod_{\lambda \in \Lambda} X_\lambda} = \prod_{\lambda \in \Lambda} \Delta_{X_\lambda} \in \prod_{\lambda \in \Lambda} \mathcal{V}_\lambda, \\
(\text{qc2}) & \quad \text{If } V \in \prod_{\lambda \in \Lambda} \mathcal{V}_\lambda \text{ and } W \text{ is subset of } V, \text{ then there are sets } \{A_\lambda \in \mathcal{V}_\lambda\}_{\lambda \in \Lambda} \text{ such that } W \subset V \subset \prod_{\lambda \in \Lambda} A_\lambda, \text{ and therefore } W \in \prod_{\lambda \in \Lambda} \mathcal{V}_\lambda. \\
(\text{qc3}) & \quad \text{If } V, W \in \prod_{\lambda \in \Lambda} \mathcal{V}_\lambda, \text{ then there are } \{A_\lambda, B_\lambda \in \mathcal{V}_\lambda\}_{\lambda \in \Lambda} \text{ such that } V \subset \prod_{\lambda \in \Lambda} A_\lambda \text{ and } W \subset \prod_{\lambda \in \Lambda} B_\lambda. \text{ Therefore, } V \cup W \subset \prod_{\lambda \in \Lambda} (A_\lambda \cup B_\lambda), \text{ and we conclude that } V \cup W \in \prod_{\lambda \in \Lambda} \mathcal{V}_\lambda. \\
(\text{qc4}) & \quad \text{If } V \in \prod_{\lambda \in \Lambda} \mathcal{V}_\lambda, \text{ then there are } \{A_\lambda \in \mathcal{V}_\lambda\}_{\lambda \in \Lambda} \text{ such that } V \subset \prod_{\lambda \in \Lambda} A_\lambda, \text{ and therefore } V^{-1} \subset \prod_{\lambda \in \Lambda} (A_\lambda)^{-1}, \text{ which gives that } V^{-1} \in \prod_{\lambda \in \Lambda} \mathcal{V}_\lambda. \square
\end{align*}
\]

Definition 2.3.11. The quasi-coarse structure \(\prod_{\lambda \in \Lambda} \mathcal{V}_\lambda\) from Proposition 2.3.10 will be called the box product quasi-coarse structure on \((\prod_{\lambda \in \Lambda} X_\lambda)\). We call the pair \((\prod_{\lambda \in \Lambda} X_\lambda, \prod_{\lambda \in \Lambda} \mathcal{V}_\lambda)\) the box product of the family of quasi-coarse spaces \(\{(X_\lambda, \mathcal{V}_\lambda)\}\).

Proposition 2.3.12. Let \(\prod_{\lambda \in \Lambda} \mathcal{V}_\lambda\) be the collection of subsets of \((\prod_{\lambda \in \Lambda} X_\lambda)^2\) such they are contained in a set of the form

\[
\left\{(x, y) \in \left(\prod_{\lambda \in \Lambda} X_\lambda\right)^2 \mid (\pi_\lambda(x), \pi_\lambda(y)) \in A_\lambda \text{ if } \lambda \in F, \quad (\pi_\lambda(x), \pi_\lambda(y)) \in \Delta_{X_\lambda} \text{ if } \lambda \in \Lambda \setminus F \right\},
\]

where \(F \subset \Lambda\) is a finite set and \(A_\lambda \in \mathcal{V}_\lambda\) for all \(\lambda \in \Lambda\).

Then \(\prod_{\lambda \in \Lambda} \mathcal{V}_\lambda\) is a quasi-coarse structure on \(\prod_{\lambda \in \Lambda} X_\lambda\).
Proof. Let \( \prod_{\lambda \in \Lambda} V_\lambda \) be as in the statement of the proposition. We check that \( \prod_{\lambda \in \Lambda} V_\lambda \) the axioms for quasi-coarse structure.

(qc1) \( \Delta(\prod_{\lambda \in \Lambda} x_\lambda) = \prod_{\lambda \in \Lambda} \Delta x_\lambda \in \prod_{\lambda \in \Lambda} V_\lambda. \)

(qc2) If \( V \in \prod_{\lambda \in \Lambda} V_\lambda \) and \( W \) subset of \( V \), then there are sets \( \{ V_\lambda \in V_\lambda \}_{\lambda \in \Lambda} \) such that a finite number of \( V_\lambda \neq \Delta x_\lambda \) and \( W \subset V \subset \prod_{\lambda \in \Lambda} V_\lambda \), from which it follows that \( W \in \prod_{\lambda \in \Lambda} V_\lambda. \)

(qc3) If \( V, W \in \prod_{\lambda \in \Lambda} V_\lambda \), then there are sets \( \{ V_\lambda \in V_\lambda \}_{\lambda \in \Lambda}, \{ W_\lambda \in V_\lambda \}_{\lambda \in \Lambda} \) such that a finite number \( n \in \mathbb{N} \) of the \( V_\lambda \) satisfy \( V_\lambda \neq \Delta x_\lambda \), and a finite number \( m \) of the \( W_\lambda \) satisfy \( W_\lambda \neq \Delta x_\lambda \), and \( V \subset \prod_{\lambda \in \Lambda} V_\lambda, W \subset \prod_{\lambda \in \Lambda} W_\lambda. \)Therefore, \( V \cup W \subset \prod_{\lambda \in \Lambda} (V_\lambda \cup W_\lambda) \) and there are at most a finite number \( m + n \) of unions \( V_\lambda \cup W_\lambda \) which are not equal to the corresponding diagonal \( \Delta x_\lambda \). It follows that \( V \cup W \in \prod_{\lambda \in \Lambda} V_\lambda. \)

(qc4) If \( V \in \prod_{\lambda \in \Lambda} V_\lambda \), then there are sets \( \{ V_\lambda \in V_\lambda \}_{\lambda \in \Lambda} \) such that for a finite number of them, \( V_\lambda \neq \Delta x_\lambda \), and we have \( V \subset \prod_{\lambda \in \Lambda} V_\lambda \). It follows that \( V^{-1} \subset \prod_{\lambda \in \Lambda} (V_\lambda)^{-1} \), which implies that \( V^{-1} \in \prod_{\lambda \in \Lambda} V_\lambda. \)

\[\Box\]

**Definition 2.3.13.** We call the quasi-coarse structure \( \prod_{\lambda \in \Lambda} V_\lambda \) from Proposition 2.3.12 the Tychonoff product of the family \( \{ V_\lambda \}_{\lambda \in \Lambda} \), and we call the pair \( (\prod_{\lambda \in \Lambda} X_\lambda, \prod_{\lambda \in \Lambda} V_\lambda) \) the Tychonoff product of the family of quasi-coarse spaces \( \{(X_\lambda, V_\lambda)\}_{\lambda \in \Lambda} \).

The following proposition shows us that the box product is the preferred product for products of infinite families of quasi-coarse spaces. Naturally, when \( |\Lambda| < \infty \), the box product and the Tychonoff product coincide.

**Proposition 2.3.14.** Let \( (Y, W) \) be a quasi-coarse space, and let \( \{(X_\lambda, V_\lambda)\}_{\lambda \in \Lambda} \) be a collection of quasi-coarse spaces indexed by the set \( \Lambda \). A map \( f : (Y, W) \to (\prod_{\lambda \in \Lambda} X_\lambda, \prod_{\lambda \in \Lambda} V_\lambda) \) is bornologous iff \( \pi_\lambda \circ f : (Y, W) \to (X_\lambda, V_\lambda) \) is bornologous for each \( \lambda \in \Lambda \).

*Proof.* First, suppose that \( f \) is bornologous. Since \( \pi_\lambda : (\prod_{\lambda \in \Lambda} X_\lambda, \prod_{\lambda \in \Lambda} V_\lambda) \) is bornologous for any \( \lambda \in \Lambda \), it follows that \( \pi_\lambda \circ f \) is bornologous for each \( \lambda \in \Lambda \), since the composition of bornologous functions is bornologous.

Now suppose that \( \pi_\lambda \circ f \) is bornologous for each \( \lambda \in \Lambda \), and let \( V \in W \). Then \( \pi_\lambda \circ f(V) \in V_\lambda \). So \( \prod_{\lambda \in \Lambda} (\pi_\lambda \circ f(V)) \in \prod_{\lambda \in \Lambda} V_\lambda \). Since \( f(V) \subset \prod_{\lambda \in \Lambda} (\pi_\lambda \circ f(V)) \), \( f \) is a bornologous function. \( \Box \)

Our final discussion in this section will be about the relation between products and the functors sending quasi-coarse spaces to semi-uniform spaces and vice-versa. We first recall the definition of the Cartesian product of semi-uniform spaces.

**Definition 2.3.15 (Product Semi-Uniform Space; [2], 23.D.10.).** The product of a family \( \{(X_a, U_a) : a \in A\} \) of semi-uniform spaces, denoted by \( \prod_{a \in A} (X_a, U_a) \) is defined to be the semi-uniform space \( (X, U) \) where \( X \) is the cartesian product of the family \( \{X_a\} \), and \( U \), called the product semi-uniformity, is the collection of subsets of \( X \times X \) containing a set of the form

\[\{(x, y) \in X \times X : a \in F \Rightarrow (\pi_a(x), \pi_a(y)) \in U_a\}\]

where \( F \) is a finite subset of \( A \) and \( U_a \in U_a \). Sets of the above form are called the canonical elements of the product semi-uniformity.
Theorem 2.3.16. Let \( \{(X_a, \mathcal{U}_a)\} \) be a collection of semi-uniform spaces indexed by a set \( A \), \( \mathcal{V}_a \) the quasi-coarse structure induced by \( \mathcal{U}_a \), \( \mathcal{U} \) the product semi-uniformity and \( \mathcal{V} \) the quasi-coarse structure induced by \( \mathcal{U} \). Then, \( \mathcal{V} = \prod_{a \in A} \mathcal{V}_a \), the box product quasi-coarse structure.

Proof. Let \( \{(X_a, \mathcal{U}_a)\} \) be a collection of semi-uniform spaces indexed by a set \( A \), \( \mathcal{V}_a \) the quasi-coarse structure induced by \( \mathcal{U}_a \), \( \mathcal{U} \) the product semi-uniformity and \( \mathcal{V} \) the quasi-coarse structure induced by \( \mathcal{U} \).

Suppose that \( V \in \mathcal{V} \). Then \( V \) is contained in every canonical element of the product semi-uniformity, which implies that \( (\pi_a \times \pi_a)(V) \in U_a \) for each \( U_a \in \mathcal{U}_a \), and therefore \( (\pi_a \times \pi_a)(V) \in \mathcal{V}_a \) for each \( a \in A \). We conclude that \( V \in \prod_{a \in A} \mathcal{V}_a \).

On the other hand, if \( V \in \prod_{a \in A} \mathcal{V}_a \), then \( (\pi_a \times \pi_a)(V) \subset U_a \) for each \( U_a \in \mathcal{U}_a \). In particular, for every finite subset \( F \) of \( A \) we have \( (\pi_a \times \pi_a)(V) \in U_a \) for each \( U_a \in \mathcal{U}_a \), \( a \in F \), and therefore \( U \) is contained in each canonical element of the product semi-uniformity. We conclude that \( V \in \mathcal{V} \), and the result follows. \( \square \)

2.4. Quotient Spaces. In this section, we study quotient spaces of quasi-coarse spaces, i.e. the quasi-coarse structures constructed on a set \( Y \) given a quasi-coarse space \( (X, \mathcal{V}) \) and a surjective map \( g : X \to Y \). We begin with the following definition.

Proposition 2.4.1. Let \( (X, \mathcal{V}) \) be a quasi-coarse space, let \( Y \) be a set, and let \( g : X \to Y \) be a surjective function. We define \( \mathcal{V}_g := \{(g \times g)(V) : V \in \mathcal{V}\} \).

Then \( (Y, \mathcal{V}_g) \) is a quasi-coarse space.

Moreover, \( \mathcal{V}_g \) is the coarsest quasi-coarse structure which makes the function \( g \) bornological.

Proof. Let \( (X, \mathcal{V}), Y, g : X \to Y \), and \( \mathcal{V}_g \) be as in the statement of the proposition. We verify that \( \mathcal{V}_g \) satisfies the axioms of a quasi-coarse structure.

\( \text{(qc1) Since } g \text{ is surjective, } g(X) = Y \text{, and therefore } (g \times g)(\Delta_X) = \Delta_Y \text{, which gives that } \Delta_Y \in \mathcal{V}_g. \)

\( \text{(qc2) Let } B \in \mathcal{V}_g \text{ and suppose that } A \subset B. \text{ Then there is a set } W \in \mathcal{V} \text{ such that } (f \times f)(W) = B, \text{ and for each } (y, y') \in A, \text{ there is an } (x, x') \in W \text{ such that } g(x) = y \text{ and } g(x') = y'. \text{ Defining }\)

\[ A_g := \{(x, x') \in W : (g \times g)(x, x') \in A\}, \]

we have that \( A_g \subset W \) and \( (g \times g)(A_g) = A \). Since \( A_g \in \mathcal{V} \), it follows that \( A \in \mathcal{V}_g. \)

\( \text{(qc3) Let } A, B \in \mathcal{V}_g. \text{ Then there are } A', B' \in \mathcal{V} \text{ such that } (g \times g)(A') = A \text{ and } (g \times g)(B') = B. \text{ Now note that } A \cup B = (g \times g)(A') \cup (g \times g)(B') = (g \times g)(A' \cup B'), \text{ so } A \cup B \in \mathcal{V}_g. \)

\( \text{(qc4) Let } A \in \mathcal{V}_g, \text{ then there is } W \in \mathcal{V} \text{ such that } (g \times g)(A) = W. \text{ We observe that }\)

\[ (g \times g)(A^{-1}) = \{(x, y) \mid \exists (x', y') \in A^{-1}, (g \times g)(x', y') = (x, y)\} = \{(x, y) \mid \exists (x', y') \in A, (g \times g)(x', y') = (x, y)\} = \{(x, y) \mid \exists (x', y') \in A, (g \times g)(x', y') = (x, y)\} = ((g \times g)(A))^{-1}. \]

We conclude that \( \mathcal{V}_g \) is a quasi-coarse structure on \( Y \).
We now show that \( \mathcal{V}_g \) is the coarsest quasi-coarse structure making \( g \) bornologous. By definition, \( g \) is a bornologous function iff for each \( A \in \mathcal{V} \) we have that \( (g \times g)(A) \in \mathcal{V}_g \). However, by definition, \( \{(g \times g)(A) : A \in \mathcal{V}\} = \mathcal{V}_g \), so it follows that any quasi-coarse structure \( W \) making \( g \) bornologous must contain \( \mathcal{V}_g \), and therefore \( \mathcal{V}_g \) is the coarsest such structure.

\[ \square \]

**Definition 2.4.2.** The quasi-coarse structure \( \mathcal{V}_g \) constructed in Proposition 2.4.1 is called the quasi-coarse structure inductively generated by the function \( g : X \to Y \), and \( (Y, \mathcal{V}_g) \) is called the quasi-coarse space inductively generated by the function \( g : X \to Y \).

**Theorem 2.4.3.** Let \( (Y, \mathcal{V}_g) \) be a quotient quasi-coarse space inductively generated by the function \( g : X \to Y \) and let \( (Z, \mathcal{Z}) \) be a quasi-coarse space. A function \( f : (Y, \mathcal{V}_g) \to (Z, \mathcal{Z}) \) is bornologous iff \( f \circ g : X \to Z \) is bornologous.

**Proof.** Suppose that \( f : (Y, \mathcal{V}_g) \to (Z, \mathcal{Z}) \) is a bornologous function and let \( A \subset X \), \( A \in \mathcal{V} \). By definition, \( (g \times g)(A) \in \mathcal{V}_g \), so \( (f \circ g \times f \circ g)(A) = f \circ g \times f \circ g)(A) \in \mathcal{Z} \).

Now suppose that \( f \circ g : (X, \mathcal{V}) \to (Z, \mathcal{Z}) \) is a bornologous function, and let \( A \in \mathcal{V}_g \). Then there is a set \( W \in \mathcal{V} \) such that \( (g \times g)(W) = A \). It follows that \( (f \times f)(A) = (f \circ g \times f \circ g)(W) \in \mathcal{Z} \), and we conclude that \( f \) is bornologous. \( \square \)

An important example of an inductively generated quasi-coarse space is the quotient space generated by an equivalence relation.

**Example 2.4.3.1** (Quotient Quasi-Coarse Space). Let \( (X, \mathcal{V}) \) be a quasi-coarse space and let \( \sim \) be an equivalence relation on \( X \). Let \( p : X \to X/\sim \) be the map \( x \mapsto [x] \) sending each point \( x \in X \) to its equivalence class \([x] \in X/\sim \). Then the quasi-coarse space \((X/\sim, \mathcal{V}_p)\) is called the quotient space of \( X \) induced by the equivalence relation \( \sim \).

It will be useful in the following to have an alternate formulation of the quotient quasi-coarse structure induced by an equivalence relation, which we provide in the next proposition.

**Proposition 2.4.4.** Let \( (X, \mathcal{V}) \) be a quasi-coarse space and let \( \sim \) be an equivalence relation on \( X \). Furthermore, extend the relation \( \sim \) to \( X \times X \) by defining \( (x, y) \sim (x', y') \) iff \( x \sim x' \) and \( y \sim y' \). We denote by \([x, y]\) the equivalence class of \((x, y)\), and for a subset \( A \subset X \times X \), we define

\[ [A] := \{[x, y] \mid (x, y) \in A\}, \]

and we let \( \mathcal{V}/\sim \) denote the collection

\[ \mathcal{V}/\sim := \{[B] \mid B \in \mathcal{V}\}. \]

Finally, let \( p : X \to X/\sim \) be the map \( x \mapsto [x] \) sending each point \( x \in X \) to its equivalence class \([x] \in X/\sim \).

Then \( \mathcal{V}_p = \mathcal{V}/\sim \).

**Proof.** First, let \( A \in \mathcal{V}/\sim \). Then there is a set \( A' \in \mathcal{V} \) such that \( A = [A'] \). Furthermore, \( A = [A'] = (p \times p)(A') \), and we conclude that \( A \in \mathcal{V}_p \), so \( \mathcal{V}/\sim \subset \mathcal{V}_p \).

Now suppose that \( A \in \mathcal{V}_p \). Then there is a set \( A' \in \mathcal{V} \) such that \((p \times p)(A') = A \). However, by the definition of \( p \), \((p \times p)(A') = [A'] = A \), and therefore \( A \in \mathcal{V}/\sim \), so \( \mathcal{V}_p \subset \mathcal{V}/\sim \) as well. \( \square \)
2.5. Coarse Space Induced by a Quasi-Coarse Space. In this section, we discuss a method of generating a coarse space from a quasi-coarse space. In order to proceed with the construction, we will first discuss direct limits for quasi-coarse spaces. We begin with constructing the a quasi-coarse structure on a disjoint union of quasi-coarse spaces.

**Proposition 2.5.1.** Let \( \{(X_\lambda, V_\lambda)\}_{\lambda \in \Lambda} \) be a collection of quasi-coarse spaces indexed by the set \( \Lambda \), and let \( \sqcup_{\lambda \in \Lambda} V_\lambda \) be the collection of sets of the form

\[
\sqcup_{\lambda \in \Lambda} A_\lambda,
\]

where each \( A_\lambda \in V_\lambda \). (Note that any given \( A_\lambda \) may be the emptyset.) Then \( (\sqcup_{\lambda \in \Lambda} X_\lambda, \sqcup_{\lambda \in \Lambda} V_\lambda) \) is a quasi-coarse space.

**Proof.** We show that \( \sqcup_{\lambda \in \Lambda} V_\lambda \) satisfies the axioms for a quasi-coarse structure.

1. \( \Delta_{\sqcup_{\lambda \in \Lambda} X_\lambda} = \sqcup_{\lambda \in \Lambda} \Delta_\lambda \subseteq \sqcup_{\lambda \in \Lambda} V_\lambda \).
2. If \( A := \sqcup_{\lambda \in \Lambda} A_\lambda \in \sqcup_{\lambda \in \Lambda} V_\lambda \) and \( W \subseteq A \), then \( W = \sqcup_{\lambda \in \Lambda} W_\lambda \), where \( W_\lambda \subseteq A_\lambda \) for every \( \lambda \in \Lambda \). Therefore each \( W_\lambda \in V_\lambda \), and \( W \subseteq \sqcup_{\lambda \in \Lambda} V_\lambda \).
3. If \( A, W \subseteq \sqcup_{\lambda \in \Lambda} V_\lambda \), then there are \( \sqcup_{\lambda \in \Lambda} A_\lambda \) and \( \sqcup_{\lambda \in \Lambda} W_\lambda \) with \( A_\lambda, W_\lambda \in V_\lambda \) such that \( A = \sqcup_{\lambda \in \Lambda} A_\lambda \) and \( W = \sqcup_{\lambda \in \Lambda} W_\lambda \). So \( A \cup W \subseteq \sqcup_{\lambda \in \Lambda} (A_\lambda \cup W_\lambda) \), from which we conclude that \( A \cup W \subseteq \sqcup_{\lambda \in \Lambda} V_\lambda \).
4. If \( A \subseteq \sqcup_{\lambda \in \Lambda} V_\lambda \), then there is \( \sqcup_{\lambda \in \Lambda} A_\lambda \) with \( A_\lambda \in V_\lambda \) such that \( A = \sqcup_{\lambda \in \Lambda} A_\lambda \). So

\[
A^{-1} = (\sqcup_{\lambda \in \Lambda} A_\lambda)^{-1} = \sqcup_{\lambda \in \Lambda} (A_\lambda)^{-1},
\]

and therefore \( A^{-1} \in \sqcup_{\lambda \in \Lambda} V_\lambda \).

It follows that \( (\sqcup_{\lambda \in \Lambda} X_\lambda, \sqcup_{\lambda \in \Lambda} V_\lambda) \) is a quasi-coarse space. \( \square \)

**Definition 2.5.2.** We call the quasi-coarse structure \( \sqcup_{\lambda \in \Lambda} V_\lambda \) the **disjoint union quasi-coarse structure**, and we call the quasi-coarse space \( (\sqcup_{\lambda \in \Lambda} X_\lambda, \sqcup_{\lambda \in \Lambda} V_\lambda) \) the **disjoint union of the quasi-coarse spaces** \( \{(X_\lambda, V_\lambda)\}_{\lambda \in \Lambda} \).

We will now introduce the notion of a directed system in quasi-coarse category, which we will use to construct a coarse space from a quasi-coarse space.

**Definition 2.5.3.** Let \( \Lambda \) be a directed set. We will call \( \{(X_\alpha, V_\alpha), f^\beta_\alpha, \Lambda\} \) a **directed system of quasi-coarse spaces** if \( \{X_\alpha, f^\beta_\alpha, \Lambda\} \) is a directed system of sets, \( (X_\alpha, V_\alpha) \) are quasi-coarse spaces for each \( \alpha \in \Lambda \), and each \( f^\beta_\alpha \) is hornologous.

**Proposition 2.5.4.** Let \( \Lambda \) be a directed set, let \( \{(X_\alpha, V_\alpha), f^\beta_\alpha, \Lambda\} \) be a directed system of quasi-coarse spaces, and let \( \sim \) be the equivalence relation on \( \sqcup_{\lambda \in \Lambda} X_\lambda \) such that for \( x^\alpha \in X_\alpha \) and \( x^\beta \in X^\beta \), \( x^\alpha \sim x^\beta \) iff there is a \( \gamma \in \Lambda \) satisfying \( \alpha \leq \gamma \), \( \beta \leq \gamma \) and where \( f^\gamma_\alpha x^\alpha = f^\gamma_\beta x^\beta \). Then

\[
\lim_{\longrightarrow} \{(X_\alpha, V_\alpha), f^\beta_\alpha, \Lambda\} = (\sqcup_{\lambda \in \Lambda} X_\lambda)/\sim, (\sqcup_{\lambda \in \Lambda} V_\lambda)/\sim),
\]

where the left hand side of the above equation is the direct limit of the directed system, and the right hand side is the quotient quasi-coarse space from Example 2.4.3.1.
Proof. It is enough to show that, for any \((Y, W)\) and a collection of diagrams of solid arrows of the form
\[
\begin{array}{c}
\xymatrix{ (X_{\lambda}, V_{\lambda}) \ar[rr]^{f_{\alpha}} \ar[d]_{p_{\lambda}} & & (X_{\alpha}, V_{\alpha}) \ar[d]^{p_{\alpha}} \\
\left( \bigsqcup_{\lambda \in \Lambda} X_{\lambda} \right) / \sim \ar[d]_{\psi_{\lambda}} & & \left( \bigsqcup_{\lambda \in \Lambda} V_{\lambda} \right) / \sim \ar[d]_{\psi_{\alpha}} \\
(Y, W) & & (Y, W) \ar[u]_{g}
}\end{array}
\]
where \(\alpha, \lambda \in \Lambda\) satisfy \(\lambda \leq \alpha\) and the solid arrows are bornologous and commute, there exists a bornologous map
\[
g : \left( \bigsqcup_{\lambda \in \Lambda} X_{\lambda} \right) / \sim \rightarrow (Y, W)
\]
making the entire diagram commute for any choice of \(\alpha, \lambda \in \Lambda\). However, the existence of a set map \(g\) making the diagram commute is guaranteed by the fact that \(\bigsqcup_{\lambda \in \Lambda} X_{\lambda}\) is the direct limit of the directed system viewed as sets, and \(g\) is bornologous by Theorem 2.4.3. □

We now characterize the direct limit of some special direct systems which we will use to construct a coarse structure from a quasi-coarse structure on a set.

**Corollary 2.5.5.** Let \(X\) be a set, suppose that \(\Lambda\) is a totally ordered set, and let \(\{V_{\lambda}\}_{\lambda \in \Lambda}\) be a family of quasi-coarse structures on \(X\) satisfying \(V_{\lambda} \subset V_{\lambda'}\) if \(\lambda \leq \lambda'\). Denote by \(\{(X_{\alpha}, V_{\alpha}), i_{\alpha}, \Lambda\}\) the directed system such that \(X = X_{\alpha}\) for all \(\alpha \in \Lambda\) and the maps \(i_{\alpha}\) are all identity maps.

Then the direct limit of \(\{(X_{\alpha}, V_{\alpha}), i_{\alpha}, \Lambda\}\) is the space \((X, \bigcup_{\lambda \in \Lambda} V_{\lambda})\).

**Proof.** We note that \(X = (\bigsqcup_{\lambda \in \Lambda} X_{\lambda}) / \sim\) and \(\bigcup_{\lambda \in \Lambda} V_{\lambda} = (\bigsqcup_{\lambda \in \Lambda} V_{\lambda}) / \sim\). The result now follows from Proposition 2.5.4. □

**Corollary 2.5.6.** Let \(\{(X_{\alpha}, V_{\alpha}), i_{\alpha}, \Lambda\}\) be the directed system such that \(X = X_{\alpha}\) for all \(\alpha \in \Lambda\), the \(i_{\alpha}\) are the identity maps, and, in addition, suppose that the following condition holds.

\[
(1) \quad \forall \lambda_1, \lambda_2 \in \Lambda \exists \lambda_3 \in \Lambda \text{ such that } V \circ W \in V_{\lambda_3} \forall V \in V_{\lambda_1}, W \in V_{\lambda_2}.
\]

Then \(\lim_{\rightarrow} V_{\lambda}\) is a coarse structure on \(X\).

**Proof.** We must show that the quasi-coarse structure \((\bigcup_{\lambda \in \Lambda} V_{\lambda}) / \sim\) satisfies axiom (qc5) in Definition 2.1.2, but this follows immediately from Condition (1) in the statement of the corollary. □

Given a quasi-coarse space \((X, V)\), we will proceed to construct a coarse space by repeatedly adding sets of the form \(V \circ W\) to the quasi-coarse structure \(V\). We first show that adding \(\{V \circ W \mid V, W \in V\}\) to a quasi-coarse structure \(V\) gives another quasi-coarse structure. To do so, we will require the following lemma about properties of the set product.

**Lemma 2.5.7.** Let \(X\) and \(Y\) be sets and let \(A, B, C, D \in \mathcal{P}(X \times X)\). Then

\[
A \circ B = C \circ D. 
\]
(i) \((A \circ B)^{-1} = B^{-1} \circ A^{-1}\).
(ii) \((A \circ B) \cup (C \circ D) \subset (A \cup C) \circ (B \cup D)\).
(iii) Let \(f : X \to Y\) be a function of sets. Then
\[(f \times f)(A \circ B) \subset (f \times f)(A) \circ (f \times f)(B)\.

Proof. Let \(X\) and \(Y\) be sets and let \(A, B, C, D \in \mathcal{P}(X \times X)\). Then

(i) Let \((x, y) \in (A \circ B)^{-1}\), that is, \((y, x) \in A \circ B\). This is equivalent to the existence of a point \(z \in X\) such that \((y, z) \in A\) and \((z, x) \in B\). This, in turn, is true iff \((x, z) \in B^{-1}\) and \((y, z) \in A^{-1}\). By definition, we have \((x, y) \in B^{-1} \circ A^{-1}\).

(ii) Let \((x, y) \in (A \circ B) \cup (C \circ D)\). Then \((x, y) \in A \circ B\) or \((x, y) \in C \circ D\), and therefore there exists a point \(z \in X\) such that \((x, z) \in A\) and \((z, y) \in B\) or \((x, z) \in C\) and \((z, y) \in D\), which implies that \((x, z) \in A \cup C\) and \((z, y) \in B \cup D\). Thus, \((x, y) \in (A \cup C) \circ (B \cup D)\).

(iii) Suppose that \((a', b') \in (f \times f)(A \circ B)\). Then there exists an element pairs \((a, b) \in A \circ B\) such that \(f(a) = a'\) and \(f(b) = b'\), and therefore there exists \(x \in X\) with \((a, x) \in A\) and \((x, b) \in B\). It follows that \((a', f(x)) \in (f \times f)(A)\) and \((f(x), b') \in (f \times f)(B)\), which implies that \((a', b') \in (f \times f)(A) \circ (f \times f)(B)\). Therefore, \((f \times f)(A \circ B) \subset (f \times f)(A) \circ (f \times f)(B)\), as desired. \(\square\)

Remark 2.5.8. In item (ii) of the previous lemma, the other inclusion is not necessarily true: Let \(A\) be a non-empty set, \(D = A^{-1}\) and \(B = C = \emptyset\), then \(A \circ B = C \circ D = \emptyset\) and \((A \cup C) \circ (B \cup D) = A \circ A^{-1} \neq \emptyset\), so that \((A \circ B) \cup (C \circ D) \neq (A \cup C) \circ (B \cup D)\).

Proposition 2.5.9. Let \((X, \mathcal{V})\) be a quasi-coarse space, and define
\[\mathcal{V}^{PE} := \{C \subset X \times X \mid \exists A, B \in \mathcal{V} \text{ with } C \subset A \circ B\}\]
Then \((X, \mathcal{V}^{PE})\) is a quasi-coarse space, and \(\mathcal{V} \subset \mathcal{V}^{PE}\).

Proof. We first verify that \(\mathcal{V}^{PE}\) satisfies the axioms of a quasi-coarse structure.

(\text{qc1}) We observe that \(\Delta_X = \Delta_X \circ \Delta_X\). Thus \(\Delta_X \in \mathcal{V}^{PE}\).

(\text{qc2}) Let \(B \in \mathcal{V}^{PE}\) and suppose that \(A \subset B\). Then there are sets \(A', B' \in \mathcal{V}\) such that \(A \subset B' \circ A'\), so it follows that \(A \in \mathcal{V}^{PE}\).

(\text{qc3}) Let \(A, B \in \mathcal{V}^{PE}\). Then there are \(A', B', A'', B'' \in \mathcal{V}\) such that \(A \subset A' \circ A''\) and \(B \subset B' \circ B''\). Since \(A' \cup B', A'' \cup B'' \in \mathcal{V}\), we have that \((A' \cup B') \circ (A'' \cup B'') \in \mathcal{V}^{PE}\). By \text{Lemma 2.5.7} it follows that \(A \cup B \subset (A' \cup B') \circ (A'' \cup B'') \in \mathcal{V}^{PE}\). We conclude that \(A \cup B \in \mathcal{V}^{PE}\).

(\text{qc4}) Let \(C \in \mathcal{V}^{PE}\). Then there are \(A, B \in \mathcal{V}\) such that \(C \subset A \circ B\). Since \(A^{-1}, B^{-1} \in \mathcal{V}\), \text{Lemma 2.5.7} implies that \(C^{-1} \subset B^{-1} \circ A^{-1}\), and we conclude that \(C^{-1} \in \mathcal{V}^{PE}\).

Thus \((X, \mathcal{V}^{PE})\) is a quasi-coarse space.

Finally, for any \(A \in \mathcal{V}\), \(A \circ \Delta_X = A\), and therefore \(A \in \mathcal{V}^{PE}\). \(\square\)

Definition 2.5.10. Let \((X, \mathcal{V})\) be a quasi-coarse space. We call the structure \(\mathcal{V}^{PE}\) in \text{Proposition 2.5.9} the set product extension of \(\mathcal{V}\), and the ordered pair \((X, \mathcal{V}^{PE})\) will be called the set product extension of \((X, \mathcal{V})\). For any \(k \in \mathbb{N}\), we recursively define \(\mathcal{V}^{\text{PE}(k)}\) to be the set product extension of \(\mathcal{V}^{\text{PE}(k-1)}\).

Proposition 2.5.11. If \(f : (X, \mathcal{V}) \to (Y, \mathcal{W})\) is bornologous, then \(f : (X, \mathcal{V}^{PE}) \to (Y, \mathcal{W}^{PE})\) is bornologous.
Proof. If $A \in \mathcal{V}^{PE}$, then there exist sets $A', A'' \in \mathcal{V}$ such that $A \subseteq A' \circ A''$, so $(f \times f)(A'), (f \times f)(A'') \in \mathcal{W}$. By Lemma 2.5.7 item (iii) we have

$$(f \times f)(A') \subseteq (f \times f)(A' \circ A'') \subseteq (f \times f)(A') \circ (f \times f)(A'') \in \mathcal{V}^{PE}.$$ 

Therefore, $(f \times f)(A) \in \mathcal{V}^{PE}$ and $f : X \to Y$ is a $(\mathcal{V}^{EP}, \mathcal{W}^{EP})$-bornologous function. □

**Remark 2.5.12.** Note that Proposition 2.5.11 shows that the map

$$\Psi : \mathcal{QCoarse} \to \mathcal{QCoarse}$$ 

$$\Psi(X, \mathcal{V}) = (X, \mathcal{V}^{PE})$$ 

$$\Psi(f) = f$$

is a functor.

As a final observation, we note that the set product of roofed quasi-coarse spaces takes a special form.

**Remark 2.5.13.** Let $(X, \mathcal{V})$ be a roofed quasi-coarse space. Then, for every $A, B \in \mathcal{V}$, $A \circ B \subseteq rf(X, \mathcal{V}) \circ rf(X, \mathcal{V})$. Thus, $(X, \mathcal{V}^{EP})$ is a roofed quasi-coarse space and $rf(X, \mathcal{V}^{EP}) = rf(X, \mathcal{V}) \circ rf(X, \mathcal{V})$.

We are now ready to construct a coarse structure from a quasi-coarse space $(X, \mathcal{V})$.

**Theorem 2.5.14.** Let $(X, \mathcal{V})$ be a quasi-coarse space, and let $\{(X, \mathcal{V}^{kPE}), f_{ij}^k, N\}_{k=0}^\infty$ be the directed system of quasi-coarse structures such that all the $f_{ij}^k : (X, \mathcal{V}^{PE}) \to (X, \mathcal{V}^{jPE})$ are the identity map. Then $\lim_{\rightarrow} \{(X, \mathcal{V}^{kPE}), f_{ij}^k, N\}$ is a coarse space.

**Proof.** This follows from Corollary 2.5.6 the definition of $\mathcal{V}^{kPE}$, and the fact that $\mathcal{V}^{iPE} \subseteq \mathcal{V}^{jPE}$ for any $i < j$. □

**Definition 2.5.15.** We call the coarse structure constructed in Theorem 2.5.14 the coarse structure induced by $\mathcal{V}$, and which we denote by $\mathcal{V}^{\infty}$.

3. Homotopy

In this section, we develop the basics of homotopy theory in quasi-coarse spaces. One of the key difficulties in doing so is that there is no natural quasi-coarse structure on the topological interval $[0, 1]$. In order to resolve this, we adapt a construction in [1] to the quasi-coarse category, using finite intervals in $\mathbb{Z}$ endowed with their ‘nearest neighbor’ quasi-coarse structure in place of the interval for homotopical constructions. In contrast to [1], however, the product which we use throughout is the categorical product for quasi-coarse spaces, which gives the resulting theory a more simplicial, rather than cubical, character.

### 3.1. Homotopy and Homotopy Equivalence for Quasi-Coarse Spaces.

Before we define homotopy, we define the quasi-coarse structure on $\mathbb{Z}$ which will be used throughout.

**Definition 3.1.1.** We call the quasi-coarse structure on $\mathbb{Z}$ generated by the graph $G = (\mathbb{Z}, E)$, where $E := \bigcup_{z \in \mathbb{Z}} \{(z, z - 1), (z, z), (z, z + 1)\}$, i.e.
the canonical quasi-coarse structure of \( \mathbb{Z} \), and we denote this structure by \( \mathcal{Z} \).

**Remark 3.1.2.** Note that the canonical quasi-coarse structure on \( \mathbb{Z} \) is the roofed quasi-coarse structure on \( \mathbb{Z} \) with roof \( E \).

With the quasi-coarse structure on \( Z \) in place, we now define homotopy between bornologous maps.

**Definition 3.1.3.** Let \((X, V)\) and \((Y, W)\) be quasi-coarse spaces and let \( f, g : (X, V) \to (Y, W) \) be bornologous functions. We will say that \( f \) is homotopic to \( g \), written \( f \simeq_{qc} g \), iff there is a bornologous function \( H : (X \times \mathbb{Z}, V \times \mathbb{Z}) \to (Y, W) \) and integers \( N, M \in \mathbb{Z} \) with \( M < N \), where \( H(x, n) = f(x) \) if \( n \leq M \) and \( H(x, n) = g(x) \) if \( n \geq N \).

We will say that two quasi-coarse spaces \((X, V)\) and \((Y, W)\) are homotopy equivalent when there are bornologous functions \( f : X \to Y \) and \( g : Y \to X \) such that \( g \circ f \simeq_{qc} id_X \) and \( f \circ g \simeq_{qc} id_Y \).

**Proposition 3.1.4.** Homotopy equivalence \( \simeq_{qc} \) is an equivalence relation on bornologous functions.

**Proof.** We check that \( \simeq_{qc} \) satisfies the axioms of an equivalence relation.

**Reflexivity:** Let \((X, V)\) and \((Y, W)\) be quasi-coarse spaces, let \( f : (X, V) \to (Y, W) \) be a bornologous function, and suppose that \( H : X \times \mathbb{Z} \to Y \) is a map such that \( H(x, z) = f(x) \) for each \( z \in \mathbb{Z} \). Since \( H(x, n) = x \) for all \( n \in \mathbb{Z} \), to see that \( f \simeq_{qc} f \), it is enough to prove that \( H \) is bornologous. From the definition of \( H \), we have

\[
(H \times H)(V \boxtimes \mathcal{R}(\mathbb{Z})) = (f \times f)(V) \in W,
\]

so that \( H \) is bornologous by [Corollary 2.3.9](#). It follows that \( f \simeq_{qc} f \).

**Symmetry:** Suppose that \( f \simeq_{qc} g \). Then there exists a bornologous function \( H : X \times \mathbb{Z} \to Y \) and integers \( M < N \) such that \( H(x, n) = f(x) \) for all \( n \leq M \) and \( H(x, n) = g(x) \) for all \( n \geq N \). Defining \( H'(x, n) = H(x, -n) \), we have that \( H'(x, n) = g(x) \) for each \( x \in X \) if \( n \leq -N \), and \( H'(x, n) = f(x) \) for each \( x \in X \) is \( n' \geq -M \). Moreover, the function \( h : (X \times \mathbb{Z}, V \times \mathbb{Z}) \to (X \times \mathbb{Z}, V \times \mathbb{Z}) \) given by \( h(x, n) \coloneqq (x, -n) \) is bornologous by [Corollary 2.3.9](#) since \((h \times h)(V \times \mathcal{R}) = V \times \mathcal{R} \in V \times \mathcal{R} \). Therefore, \( H' = H \circ h \) is a bornologous function, from which it follows that \( g \simeq_{qc} f \).

**Transitivity:** Suppose that \( f \simeq_{qc} g \) and \( g \simeq_{qc} h \). Then there are bornologous functions \( H_{fg} : (X \times \mathbb{Z}, V \times \mathbb{Z}) \to Y \) and \( H_{gh} : X \times \mathbb{Z} \to Y \) and pairs of integers \( M_f < N_g \) and \( M_g < N_h \) such that

\[
H_{fg}(x, n) = \begin{cases} 
  f(x) & n \leq M_f \\
  g(x) & n \geq N_g \end{cases}
\]

\[
H_{gh}(x, n) = \begin{cases} 
  g(x) & n \leq M_g \\
  h(x) & n \leq N_h \end{cases}
\]

Without loss of generality, we take \( M_f < -1 = N_g \) and \( M_g = 1 < N_h \).
Define \( H_{fh} : X \times \mathbb{Z} \to Y \) by

\[
H_{fh}(x, n) = \begin{cases} 
H_{fg}(x, n) & n \leq 0 \\
H_{gh}(x, n) & n > 0,
\end{cases}
\]

and observe that \( H_{fh}(x, n) = f(x) \) for all \( n \leq M_f \), \( H_{fh}(x, n) = h(x) \) for all \( n \geq N_h \), and \( H_{fh}(x, n) = g(x) = H_{fg}(x, n) = H_{gh}(x, n) \) for \(-1 = N_g \leq n \leq M_g = 1\). \( H_{fh} \) is bornologous by Proposition 2.2.3 where \( X_1 = X \times ((-\infty, 0] \times \mathbb{Z}) \) and \( X_2 = X \times ([0, \infty) \cap \mathbb{Z}) \), and \( X_1 \) and \( X_2 \) are endowed with the respective subspace quasi-coarse structures induced from \( V \times \mathbb{Z} \).

We conclude that \( f \simeq_{qc} h \). \( \square \)

The following proposition will be useful for determining whether proposed homotopies are bornologous.

**Proposition 3.1.5.** Let \((X, V)\) and \((Y, W)\) be quasi-coarse spaces, let \( f, g : (X, V) \to (Y, W) \) be bornologous maps, and suppose that \( H : X \times \mathbb{Z} \to Y \) is a (possibly non-bornologous) map such that there exist integers \( M_f < N_g \) with \( H(x, n) = f(x) \) for all \( n \leq M_f \) and \( H(x, n) = g(x) \) for all \( n \geq N_g \).

If \((H \times H)(V \boxtimes (n, n+1)), (H \times H)(V \boxtimes (n, n))\), and \((H \times H)(V \boxtimes (n, n-1))\) are controlled sets of \((Y, W)\) for each \( V \in \mathcal{V} \) and \( n \in \mathbb{Z} \), then \( H \) is a bornologous function.

**Proof.** Let \( \mathbb{Z}^2_f := \mathfrak{R}(\mathbb{Z}) \cap \{(a, b) \in \mathbb{Z}^2 \mid M_f \leq a \leq N_g\} \). Since \( H(x, n) = f(x) \) for \( n \leq M_f \) and \( H(x, n) = g(x) \) for \( n \geq N_g \), it follows that, for any \( V \in \mathcal{V} \),

\[
(H \times H)(V \boxtimes \mathfrak{R}(\mathbb{Z})) = (H \times H)(V \boxtimes \mathbb{Z}^2_f) = (H \times H) \left( \bigcup_{(a,b) \in \mathbb{Z}^2_f} (V \boxtimes (a, b)) \right).
\]

Since the finite union of sets in \( \mathcal{W} \) are in \( \mathcal{W} \), \( H \) is bornologous iff, for all \((a, b) \in \mathbb{Z}^2_f\), \((H \times H)(V \boxtimes (a, b)) \in \mathcal{W} \), which is true iff \((H \times H)(V \boxtimes (n, n+1)), (H \times H)(V \boxtimes (n, n))\), and \((H \times H)(V \boxtimes (n, n-1))\) are controlled sets of \((Y, W)\) for every \( n \in \mathbb{Z} \). \( \square \)

In the following, we define the homotopy of maps of pairs and triples for quasi-coarse spaces and show that they are also an equivalence relation.

**Definition 3.1.6** (Homotopy of Maps of Pairs and Triples). Let \((X, V)\) and \((Y, W)\) be quasi-coarse spaces, let \( B \subset A \subset X \) and \( C \subset D \subset Y \) be endowed with the subspace quasi-coarse structures, and let \( f, g : (X, A, B; V) \to (Y, C, D; W) \) be bornologous functions of a triple, i.e., such that \( f|_A \subset C \) and \( f|_B \subset D \). We say that \( f \) is relatively homotopic to \( g \) and write \( f \simeq_{qc} g \) iff there is a homotopy \( H : (X \times \mathbb{Z}, V \times \mathbb{Z}) \to (Y, W) \) such that \( H|_{A \times \mathbb{Z}} \subset C \) and \( H|_{B \times \mathbb{Z}} \subset D \).

We define a homotopy between maps of pairs \( f, g : (X, A; V) \to (Y, C; W) \) to be the homotopy between maps of a triple as above with \( B = A \) and \( C = D \).

**Lemma 3.1.7.** Let \((X, V)\) and \((Y, W)\) be quasi-coarse spaces, and \( B \subset A \subset X \), \( D \subset C \subset Y \) be endowed with the subspace quasi-coarse structures. Then relative homotopy of triples \( \simeq_{qc} \) is an equivalence relation.

**Proof.** The proof is nearly identical to the proof of Definition 3.1.3 with the addition that we need to observe that \( H_{fh}|_A \subset C \) (and \( H_{fh}|_D \subset B \)). However, this
follows the fact that $H_{fg}$ and $H_{gh}$ are homotopies of triples and from the definition of $H_{fh}$. \qed

We end this section with the definition of homotopy equivalence.

**Definition 3.1.8.** Two quasi-coarse spaces $(X, \mathcal{V})$ and $(Y, \mathcal{W})$ are said to be homotopy equivalent, written $(X, \mathcal{V}) \simeq (Y, \mathcal{W})$, iff there exist bornologous maps $f : (X, \mathcal{V}) \to (Y, \mathcal{W})$ and $g : (Y, \mathcal{W}) \to (X, \mathcal{V})$ such that $f \circ g \simeq_{qc} 1_Y$ and $g \circ f \simeq_{qc} 1_X$.

### 3.2. Homotopy Groups

As mentioned in Section 1 to define the homotopy groups for quasi-coarse spaces, we will largely follow the ideas in the construction of the $A$-homotopy groups from [1], with the difference that the product we use is the categorical product rather than a quasi-coarse version of the graph product. We begin in this section with the construction of the relative homotopy classes of maps of cubes in $\mathbb{Z}^n$ to a quasi-coarse space $(X, \mathcal{V})$ whose boundary is sent to a subset $B \subset X$. We set some notation in the next definition.

**Definition 3.2.1.** Let $n$ and $m$ be natural numbers. We denote by $(I_m, \mathcal{Z}_m)$ the quasi-coarse space where $I_m$ is the set $\{0, 1, \cdots, m\}$ and $\mathcal{Z}_m$ is the subspace quasi-coarse structure induced by the inclusion of $I_m$ in $(\mathbb{Z}, \mathcal{Z})$. Denote by $(I_m^n, \mathcal{Z}_m^n)$ the Cartesian product of $n$ copies of $(I_m, \mathcal{Z}_m)$. We write $a_i$ for the $i$-th coordinate of a point $a \in \mathbb{Z}_m^n$, where $i \in \{1, \cdots, n\}$, unless otherwise stated. Finally, we denote by $\partial I_m^n := I_m^n - I_m^{n-1}$, and we define $J_m^{n-1} \subset \partial I_m^n$ to be the set

$$J_m^{n-1} = \partial I_m^n - \{z \in \partial I_m^n \mid z_n = m\}.$$

To define the relative homotopy groups of a pointed quasi-coarse space pair $(X, A, \ast; \mathcal{V})$ (i.e. $\ast \in A \subset X$), we will consider maps of triples $f : (I_m^n, \partial I_m^n, J_m^{n-1}; \mathcal{Z}_m^n) \to (X, A, \ast; \mathcal{V})$, and we define $\pi_m(X, A, \ast; \mathcal{V})$ to be a certain direct limit which we construct below. First, however, we require the following definition and lemmas.

**Definition 3.2.2.** Let $m < m'$ be two positive integers. We define the retraction $\phi_{m'}^m : I_m^n \to I_m^n$ by

$$\langle \phi_{m'}^m(x) \rangle_k := \begin{cases} x_k & x_k \leq m, \\ m_k & m < x_k \leq m', \end{cases}$$

where $\langle \phi_{m'}^m(x) \rangle_k$ is the $k$-th coordinate of $\phi_{m'}^m(x)$. We will typically abuse notation and write $\phi_{m'}^m$ for the retraction $I_m^n \to I_m^n$ for any dimension $n$, as the dimension will usually be clear from context.

**Lemma 3.2.3.** The function $\phi_{m'}^m : (I_m^n, \mathcal{Z}_m^n) \to (I_m^n, \mathcal{Z}_m^n)$ is bornologous.

*Proof.* The result follows from Proposition 2.2.4 and the definition of the quasi-coarse structures $\mathcal{Z}_m^n$ and $\mathcal{Z}_m^n$. \qed

**Lemma 3.2.4.** Let $f : (I_m^n, \partial I_m^n, J_m^{n-1}; \mathcal{Z}_m^n) \to (X, A, \ast; \mathcal{V})$ be a bornologous function of triples. Then for any $m' \in \mathbb{Z}$ with $m' > m$, $f$ has a bornologous extension of triples $f' : (I_{m'}^n, \partial I_{m'}^n, J_{m'}^{n-1}; \mathcal{Z}_m^n) \to (X, A, \ast; \mathcal{V})$ given by $f'(z) = f(\phi_{m'}^m(z))$. 


Furthermore, if \( f \simeq_{qc} g \) as maps of triples \( (I^n_m, \partial I^n_m, J^{n-1}_m; \mathbb{Z}^n_m) \to (X, A, \ast; \mathcal{V}) \), then for the extensions of \( f' \) and \( g' \) of \( f \) and \( g \), respectively, to \( I^n_{m'} \), we have \( f' \simeq_{qc} g' \) as well.

**Proof.** The extension \( f' \) of \( f \) is bornologous since the composition of bornologous functions is bornologous.

If \( f \simeq_{qc} g \) rel \( \partial I^n_m \) then there exists a homotopy rel \( \partial I^n_m \) \( H : (I^n_m \times \mathbb{Z}, \mathbb{Z}^n_m \times \mathcal{Z}) \to (X, \mathcal{V}) \) between \( f \) and \( g \). We define \( H' : (I^n_{m'}, \mathbb{Z}, \mathcal{Z}^n_{m'} \times \mathcal{Z}) \to (X, \mathcal{V}) \) by

\[
H'(x, n) := H(\phi^n_{m'}(x), n).
\]

Since \( \phi^n_{m'} \times Id \) and \( H \) are bornologous, it follows that \( H' \) is bornologous, and therefore \( H' \) is a homotopy from \( f' \) to \( g' \) by definition. \[\square\]

The above implies that extending a map from \( I^n_m \) to \( I^n_{m'} \), as in [\text{Lemma } 3.2.4], induces a map on the relative homotopy classes

\[
i^m_m' : [(I^n_m, \partial I^n_m, J^{n-1}_m; \mathbb{Z}^n_m), (X, A, \ast; \mathcal{V})] \to [(I^n_{m'}, \partial I^n_{m'}, J^{n-1}_{m'}; \mathbb{Z}^n_{m'}), (X, A, \ast; \mathcal{V})]
\]

such that \( i^m_m = i^m_{m'} \circ i^m' \). The maps \( i^m_{m'} \) therefore make the homotopy classes into a directed system of sets \( \left( [(I^n_m, \partial I^n_m, \mathbb{Z}^n_m), (X, \ast; \mathcal{V})], i^m_{m'} \right) \). We define the classes \( \pi_n(X, A, \ast; \mathcal{V}) \) in the following manner.

**Definition 3.2.5.** Give the set \( \{1, \ast\} \) the diagonal quasi-coarse structure, which we denote \( D_{1,1} \). That is, \( D_{1,1} \) is the roofed quasi coarse structure such that \( \mathcal{R}(D_{1,1}) = \Delta_{1,1} = \{1, \ast\} \). We define the set \( \pi_0(X, A, \ast; \mathcal{V}) \) by

\[
\pi_0(X, A, \ast; \mathcal{V}) := \{\{1, \ast\}, \ast; D_{0,1}\}, (X, A, \ast; \mathcal{V})\}.
\]

For \( n \geq 1 \), we define \( \pi_n(X, A, \ast; \mathcal{V}) \) by

\[
\pi_n(X, A, \ast; \mathcal{V}) := \lim \left( [(I^n_m, \partial I^n_m, J^{n-1}_m; \mathbb{Z}^n_m), (X, A, \ast; \mathcal{V})], i^m_{m'} \right), \mathbb{N} \).
\]

For a homotopy class \( [f] \in [(I^n_m, \partial I^n_m, J^{n-1}_m; \mathbb{Z}^n_m), (X, A, \ast; \mathcal{V})]\), we write \( \langle f \rangle \) for its image in \( \pi_n(X, A, \ast; \mathcal{V}) \).

When \( A = \ast \), we write \( \pi_n(X, \ast; \mathcal{V}) \) as \( \pi_n(X; \mathcal{V}) \).

**Remark 3.2.6.** One may also construct the homotopy classes \( \pi_n(X, \ast; \mathcal{V}) \) as a direct limit of homotopy classes of maps of pairs \( f : (\mathbb{Z}^n, J^n_m, \mathbb{Z}^n) \to (X, \ast; \mathcal{V}) \), where \( J^n_m := \mathbb{Z} - J^{n-1}_m \). One sees that these formulations are equivalent by extending the maps \( f : (I^n_m, \partial I^n_m, \mathbb{Z}^n_m) \to (X, \ast; \mathcal{V}) \) to \( \mathbb{Z}^n \), where the extension \( f' \) sends all of \( J^n_m \) to \( \ast \in X \). The same argument as in [\text{Lemma } 3.2.4] shows that \( f' \) is bornologous, and it follows from the definitions that

\[
f \simeq_{qc} g \iff f' \simeq_{qc} g'.
\]

Therefore the homotopy classes \( \pi_n(X, \ast; \mathcal{V}) \) may be defined as

\[
\pi_n(X, \ast; \mathcal{V}) := \lim \left( [(\mathbb{Z}^n, E^n_m, \mathbb{Z}^n), (X, \ast; \mathcal{V})], i^m_{m'} \right), \mathbb{N} \)
\]

where, as before, the \( i^m_{m'} \) are the maps on the respective homotopy classes induced by the interpreting a map \( f : (\mathbb{Z}^n, E^n_m, \mathbb{Z}^n) \to (X, \ast; \mathcal{V}) \) as a map \( f : (\mathbb{Z}^n, E^n_m, \mathbb{Z}^n) \to (X, \ast; \mathcal{V}) \).

**Remark 3.2.7.** Let \( \langle f \rangle, \langle g \rangle \in \pi_n(X, A, \ast; \mathcal{V}) \). Observe that \( \langle f \rangle = \langle g \rangle \) iff there exist \( m \in \mathbb{N} \) and bornologous functions \( \tilde{f}, \tilde{g} : (I^n_m, \mathbb{Z}^n_m) \to (X, \mathcal{V}) \) such that \( \tilde{f} \in \langle f \rangle, \tilde{g} \in \langle g \rangle \) and \( \tilde{f} \simeq_{qc} \tilde{g} \).
We will now define an operation $*$ on the homotopy classes $\pi_n(X, \ast; \mathcal{V})$ and $\pi(X, A, \ast; \mathcal{V})$ and show this operation makes these classes into a group $n \geq 1$ and $n \geq 2$, respectively.

**Definition 3.2.8 (Operation $*$).** Let $X$ be a set, let $n \in \mathbb{N}$, $m$ and $m'$ be non-negative integers, and suppose that $f : I^n_m \to (X, \mathcal{V})$ and $g : I^{n'}_{m'} \to (X, \mathcal{V})$ are bornologous functions. We define the $*$-product $f \ast g : I^{n+m'}_{m+m'} \to X$ such that

$$f \ast g(\alpha) = \begin{cases} f(\phi^m_{m+m'}(\alpha + m'e_1)) & \text{if } 0 \leq \alpha_1 \leq m \\ g(\phi^{m'}_{m+m'}(\alpha + me_1)) & \text{if } m < \alpha_1 m + m' \end{cases}.$$

The following proposition tells us that the result of applying the $*$ operation to a pair of bornologous functions which agree on the boundary is bornologous.

**Proposition 3.2.9.** Let $n \in \mathbb{N}$, let $m$ and $m'$ be non-negative integers, and suppose that $f : (I^n_m, Z^n_m) \to (X, \mathcal{V})$ and $g : (I^{n'}_{m'}, Z^{n'}_{m'}) \to (X, \mathcal{V})$ are bornologous functions such that

$$f((m, \alpha_2, \ldots, \alpha_n)) = g((0, \alpha_2, \ldots, \alpha_n))$$

is satisfied for all $\alpha_2, \ldots, \alpha_n$ where both $f((m, \alpha_2, \ldots, \alpha_n))$ and $g((0, \alpha_2, \ldots, \alpha_n))$ are defined. Then $f \ast g$ is bornologous.

**Proof.** Let $n \in \mathbb{N}$, $m$ and $m'$ be non-negative integers, and suppose that $f : (I^n_m, Z^n_m) \to (X, \mathcal{V})$ and $g : (I^{n'}_{m'}, Z^{n'}_{m'}) \to (X, \mathcal{V})$ satisfy the hypothesis of the proposition. Let $K_1$ and $K_2$ be the sets

$$K_1 = \{ \alpha \in I^n_{m+m'} | 0 \leq \alpha_1 \leq m \}$$

$$K_2 = \{ \alpha \in I^n_{m+m'} | m \leq \alpha_1 \leq m' + m \}.$$

Then the restrictions $f \ast g|_{K_1}$ and $f \ast g|_{K_2}$ are bornologous, and $K_1$ and $K_2$ satisfy the hypothesis of Proposition 2.2.4. Therefore $f \ast g$ is bornologous by Proposition 2.2.4.

**Corollary 3.2.10.** Suppose that $f : (I^n_m, \partial I^n_m, J^{n-1}_m, Z^n_m) \to (X, A, \ast; \mathcal{V})$ and $g : (I^{n'}_{m'}, \partial I^{n'}_{m'}, J^{n'-1}_{m'}, Z^{n'}_{m'}) \to (X, A, \ast; \mathcal{V})$ are bornologous functions of triples. Then $f \ast g : (I^n_{m+m'}, \partial I^n_{m+m'}, J^{n-1}_{m+m'}, Z^n_{m+m'}) \to (X, \mathcal{V})$ is bornologous.

**Proof.** Since $f$ and $g$ are maps of triples, it follows by definition that

$$f((m, \alpha_2, \ldots, \alpha_n)) = \ast = g((-m', \alpha_2, \ldots, \alpha_n))$$

whenever both sides are defined, so the hypotheses of Proposition 3.2.9 are satisfied. The conclusion follows.

**Corollary 3.2.11.** Let $f : (I^n_m, \partial I^n_m, Z^n_m) \to (X, \ast; \mathcal{V})$ and $g : (I^{n'}_{m'}, \partial I^{n'}_{m'}, Z^{n'}_{m'}) \to (X, \ast; \mathcal{V})$ be bornologous functions of triples. Then $f \ast g : (I^n_{m+m'}, \partial I^n_{m+m'}, Z^n_{m+m'}) \to (X, \mathcal{V})$ is bornologous.

**Proof.** The result follows from Corollary 3.2.10 with $A = \ast$.

The next lemma enables us to to extend the definition of $*$ from functions to homotopy classes.

**Lemma 3.2.12.** Let $n \geq 2$ and suppose that $(f), (g) \in \pi^{qc}_n(X, A, \ast)$, then the product $(f) \ast (g) := (f \ast g)$ is well-defined. Similarly, if $n \geq 1$ and $(f), (g) \in \pi^{qc}_n(X, \ast)$, then the product $(f) \ast (g) := (f \ast g)$ is well-defined.
Proof. Let $n \geq 2$, and suppose that $\langle f \rangle, \langle f' \rangle, \langle g \rangle, \langle g' \rangle \in \pi_n(X, A, *) \cup V)$ such that $\langle f \rangle = \langle f' \rangle$ and $\langle g \rangle = \langle g' \rangle$. Then, by Remark 3.2.7 there exist non-negative integers $m$ and $m'$ and functions $\tilde{f}, \tilde{f}' : (I^n_m, \partial I^n_m, J^{n-1}_m; Z^n_m) \rightarrow (X, A, *; V)$ and $\tilde{g}, \tilde{g}' : (I^{n'}_{m'}, \partial I^{n'}_{m'}, J^{n'-1}_{m'}; Z^{n'}_{m'}) \rightarrow (X, A, *; V)$ such that $\tilde{f} \simeq_{qc} \tilde{f}'$ and $\tilde{g} \simeq_{qc} \tilde{g}'$. We wish to show that $\tilde{f} \star \tilde{g} \simeq_{qc} \tilde{f}' \star \tilde{g}'$.

Let $H_1$ be a homotopy between $\tilde{f}$ and $\tilde{f}'$ and $H_2$ be a homotopy between $\tilde{g}$ and $\tilde{g}'$. Define a function

$$H : (I^{n+m'}_{m+m'} \times \mathbb{Z}, \partial I^{n+m'}_{m+m'} \times \mathbb{Z}, J^{n-1}_{m+m'} \times \mathbb{Z}, Z^{n}_{m+m'} \times \mathbb{Z}) \rightarrow (X, A, *; V)$$

by

$$H(\alpha, n) = \begin{cases} H_1(\alpha + m'e_1, n) & -m' - m \leq \alpha \leq -m' + m \\ H_2(\alpha + me_1, n) & -m' + m < \alpha \leq m' + m. \end{cases}$$

Then $H$ is a map of triples by definition, and $H$ is bornologous by Proposition 2.2.4 and $H$ is a homotopy between $\tilde{f} \star \tilde{g}$ and $\tilde{f}' \star \tilde{g}'$. By Remark 3.2.7 it follows that $\langle f \rangle \star \langle g \rangle = \langle f \star g \rangle$ is well-defined.

For $\pi_n(X, *; V)$, the above proof also applies for any $n \geq 1$, with the modification that all of the maps are bornologous maps of pairs instead of maps of triples. \qed

Given Lemma 3.2.12 we make the following definition.

Definition 3.2.13. For $n \geq 2$ and $\langle f \rangle, \langle g \rangle \in \pi_n(X, A, *; V)$, we define $\langle f \rangle \star \langle g \rangle := \langle f \star g \rangle$.

Similarly, for $n \geq 1$ and $\langle f \rangle, \langle g \rangle \in \pi_n(X, *; V)$, we define $\langle f \rangle \star \langle g \rangle := \langle f \star g \rangle$.

In parallel to topological spaces, we have the following theorem.

Theorem 3.2.14 (Quasi-coarse Homotopy Groups). Let $(X, V)$ be a quasi-coarse space, $A \subset X$, and $n \in \mathbb{N}$.

- If $n \geq 1$, then $(\pi^n_n(X, *), *)$ is a group.
- If $n \geq 2$, then $(\pi^n_n(X, *), *)$ is an abelian group.
- If $n \geq 2$, then $(\pi^n_n(X, A, *), *)$ is a group.
- If $n \geq 3$, then $(\pi^n_n(X, A, *), *)$ is an abelian group.

Proof. The proof is analogous to that in the topological case, replacing the $n$-disk $D^n$ with the blocks $I^n_m$. \qed

We now show that the homotopy groups of coarse spaces are trivial.

Definition 3.2.15. In what follows, we will write $\circ_{i=1}^N f_i$ to denote $f_N \circ \cdots \circ f_1$ in order to simplify notation.

Theorem 3.2.16. Let $(X, V)$ be a coarse space with $A \subset X$. Then for any $n \in \mathbb{N}$, we have $(\pi^n_n(X, A, *)) \cong \pi^n_n(X, *) \cong \{1\}$.

Proof. Let $(X, V)$ be a coarse space, $n \in \mathbb{N}$, and suppose that $h : I^n_m \rightarrow X$ is the constant map $h(\alpha) = *$ for all $\alpha \in I^n_m$.

Let $\langle f \rangle \in \pi^n_n(X, A, *)$. By definition, $\langle f(J^{n-1}) = \{*\}$, in particular $f(\alpha) = *$ when the first coordinate $a_1 = 0$. Note that, for all $\alpha \in I^n_m$ satisfies that $f(\alpha - ke_1)$ is adjacent to $f(\alpha - (k+1)e_1)$ for any $k \in \{0, \ldots, a_1 - 1\}$, that is $\{(f(\alpha), f(\alpha - e_1)), (f(\alpha - e_1), f(\alpha - 2e_1)), \ldots, (f(\alpha - (a_1 - 1)e_1), f(\alpha - a_1e_1))\} \in V$
because \( f \) is bornologous. Thereby, since \((X, \mathcal{V})\) is a coarse space and \(f(\alpha - \alpha_1e_1) = *\), we conclude that \(\{(f(\alpha),*)\} \in \mathcal{V}\) for every \(\alpha \in I^n\), and therefore

\[ f \simeq_{qc} *. \]

Therefore, \(\pi^n_{qc}(X, A, *) \cong \{1\}\). The proof for \(\pi^n_{qc}(X, *)\) is analogous. \(\square\)

While the above theorem says that quasi-coarse homotopy groups are trivial for coarse spaces, we expect there to be other quasi-coarse invariants, perhaps defined relative to infinity, which are non-trivial for both coarse and non-coarse quasi-coarse spaces. However, we leave this question open for future work.

3.3. Connectedness.

**Definition 3.3.1 (Quasi-Coarse \(n\)-Connected Space).** Let \((X, \mathcal{V})\) be a quasi-coarse space. We will say that \((X, \mathcal{V})\) is an \(n\)-connected (quasi-coarse) space iff \(\pi^n_{qc}(X, *) \cong \{0\}\). In particular, a 0-connected space will be called a connected (quasi-coarse) space.

In the next proposition we will observe that, when the quasi-coarse structure is coarse, then quasi-coarse connectedness is equivalent to coarse connectedness. We first recall the notion of coarse connectedness. See also [6] for more details on coarse connectedness.

**Definition 3.3.2 (Coarse connected space).** Let \((X, \mathcal{V})\) be a coarse space. We say that \((X, \mathcal{V})\) is coarsely connected iff every point in \(X \times X\) belongs to some controlled set \(V \in \mathcal{V}\).

**Proposition 3.3.3.** Let \((X, \mathcal{V})\) be a coarse space. Then \((X, \mathcal{V})\) is coarsely connected iff \(\pi_0(X, *; \mathcal{V}) \cong \{0\}\).

**Proof.** Let \((X, \mathcal{V})\) be a coarse space.

\((\Leftarrow)\) If \((X, \mathcal{V})\) is coarsely connected, then for each \((x, y) \in X \times X\) there exists \(E_{x,y} \in \mathcal{V}\) such that \((x, y) \in E_{x,y}\), that is, \(\{(x, y)\} \in \mathcal{V}\) for any \(x, y \in X\). Let \(f : \{(\ast, 1), *; D(\ast, 1)\} \to (X, *; \mathcal{V})\) and \(g : \{(\ast, 1), *; D(\ast, 1)\} \to (X, *; \mathcal{V})\) be bornologous maps. Note that, since \(D(\ast, 1)\) is the diagonal quasi-coarse structure, \(f(1)\) and \(g(1)\) may be arbitrary points of \(X\). Define \(H : \{\ast, 1\} \times \mathbb{Z} \to X\) such that \(H(\alpha, n) = f(\alpha)\) when \(n \leq 0\) and \(H(\alpha, n) = g(\alpha)\) when \(n \geq 1\). Since \(X\) is coarsely connected, we have that \(\{(f(1), g(1))\} \in \mathcal{V}\), and therefore \(H\) is bornologous. It follows that \(f \simeq_{qc} g\). However, \(f\) and \(g\) were arbitrary, and we conclude that \(\pi^n_0(X, *) \cong \{0\}\).

\((\Rightarrow)\) If \((X, \mathcal{V})\) is a coarse space with \(\pi_0(X, *; \mathcal{V}) = \{0\}\), then for each \(f : \{(\ast, 1), *\} \to (X, *)\) and \(g : \{(\ast, 1), *\} \to (X, *)\) there exist \(N, M \in \mathbb{Z}\) with \(N < M\) and a bornologous function \(H : \{\ast, 1\} \times \mathbb{Z} \to X\) such that \(H(e, z) = f(z)\) when \(z \leq N\) and \(H(e, z) = g(z)\) when \(z \geq M\). Given that \(H\) is bornologous, then \(H(1, z)\) is adjacent to \(H(1, z + 1)\) for each \(z \in \mathbb{Z}\). Therefore

\[ \{(f(1), H(1, N + 1))\}, \{(H(1, N + 1), H(1, N + 2))\}, \ldots \{(H(1, M - 1), g(1))\} \in \mathcal{V} \]

Then, \(\{(f(1), g(1))\} \in \mathcal{V}\). How \(f\) and \(g\) are arbitrary, \(f(1)\) and \(g(1)\) are any element of \(x\). Thus, for each \((x, y) \in X \times X\), \(\{(x, y)\} \in \mathcal{V}\). \(\square\)

The highlight in this section is being able to note that the base point does not matter when we have a connected space. We are going to prove a lemma which makes that easy.
Lemma 3.3.4. Let $(X, V)$ be a quasi-coarse space. If $x, y \in X$ are adjacent elements, then $\pi_n^{qc}(X, x) \cong \pi_n^{qc}(X, y)$ for $n \geq 1$.

Proof. Let $(X, V)$ be a quasi-coarse space, $n \geq 1$ and $x, y \in X$ be adjacent elements. Let $[(f)] : \langle I_n^m, \partial I_n^m \rangle \rightarrow (X, x)$, then there exists a non-negative integer $m$ such that $f' : \langle I_n^m, \partial I_n^m \rangle \rightarrow (X, x)$ and $f' \in (f)$. By definition, $(\phi_{m+2}^{m+2}) f' \in (f)$.

Let's define $B$ the block delimited by $A$ such that $a_i^+ = 1$ and $a_i^- = m - 1$ and

$$f_1(\beta) := \begin{cases} (\phi_{m+2}^{m+2}) f' (\beta - \sum_{i=1}^{n} e_i) & \text{if } \beta \in B + \sum_{i=1}^{n} e_i \\ x & \text{anywhere else} \end{cases}$$

We get that $(\phi_{m+2}^{m+2}) f'$ is bornologous and $(\phi_{m+2}^{m+2}) f' \simeq_{qc} f_1$. Let's define

$$f_2(\beta) := \begin{cases} y & \text{if } \beta \in \partial I_{m+2} \\ f_1(\beta) & \text{anywhere else} \end{cases}$$

which is bornologous because $x$ is adjacent to $y$.

Observe that we get a function $\Psi : \pi_n(X, x) \rightarrow \pi_n(X, y)$ which is well-defined by $(\phi_{m+2}^{m+2}) f' \simeq_{qc} f_1$. We need to prove that this is a group homomorphism.

Let $[(f)], [(g)] \in \pi_n^{qc}(X, x)$, then there exist non-negative integers $m, m'$ such that $f' : \langle I_m^m, \partial I_m^m \rangle \rightarrow (X, x)$, $g' : \langle I_{m'}^{m'}, \partial I_{m'}^{m'} \rangle \rightarrow (X, x)$, $f' \in (f)$ and $g' \in (g)$. We just need to prove that $(\Psi(f' \ast g')) \simeq_{qc} (\Psi(f') \ast \Psi(g'))$. Define $B(1)$ the block delimited by $A(1)$ such that

$$a_1^+ = m + m' + 2, \quad a_1^- = m + m' + 3$$

$$a_i^+ = 1, \quad a_i^- = m' + 1 \quad i \neq 1$$

and $B(2)$ the block delimited by $A(2)$ such that

$$a_1^+ = m + 1, \quad a_1^- = m' + 1$$

$$a_i^+ = 1, \quad a_i^- = m' + 1 \quad i \neq 1$$

Let's define $h_1 : I_{m+m'+4} \rightarrow X$ such that

$$h_1(\beta) := \begin{cases} (\phi_{m+m'+4}^{m+m'+4}) \Psi(f' \ast g')(\beta) & \text{if } \beta \in B(1) \\ x & \text{anywhere else} \end{cases}$$

Note that $h_1 \simeq_{qc} (\phi_{m+m'+4}^{m+m'+4}) \Psi(f' \ast g')$. Let's define

$$h_2(\beta) := \begin{cases} h_1(\beta - 2e_i) & \text{if } \beta \in B(2) + 2e_i \\ h_1(\beta - e_i) & \text{if } \beta \in (B(2) + e_i' + e_1) \\ h_1(\beta) & \text{anywhere else} \end{cases}$$

which is a bornologous function and $h_1 \simeq_{qc} h_2$. Let's define $B(3)$ and $B(4)$ delimited by $A(3)$ and $A(4)$ such that

$$A(3) := \{ \alpha \in I_{m+m'+4}^n : 1 \leq \alpha_i \leq m + 1 \}$$

$$A(4) := \{ \alpha \in I_{m+m'+4}^n : m + 3 \leq \alpha_1 \leq m + m' + 3, \ 1 \leq \alpha_i \leq m' + 1, \ i \neq 1 \}$$

Let's define

$$h_3(\beta) := \begin{cases} h_2(\beta) & \text{if } \beta \in B(3) \cup B(4) \\ y & \text{anywhere else} \end{cases}$$

which is bornologous and $h_3 \simeq_{qc} h_3$ because $x$ is adjacent to $y$. Finally, observe that $h_3 = \Psi(f' \ast g')$, getting what we want. Thus, $\Psi$ is a group homomorphism.
Under the same argument, we are able to define $\Theta : \pi_n^{qc}(X,y) \to \pi_n^{qc}(X,x)$. So, we just need to prove that $\Psi^{-1} = \Theta$ and we will get our isomorphism.

Let $(f_1) \in \pi_n^{qc}(X,x)$, then there exists a non-negative integer $m$ such that $f' : (I_m, \partial I_m) \to (X,x)$ and $f' \in (f)$. Since $x$ and $y$ are adjacent, we are able to replace every $y$ for $x$ in $\Theta \Psi (f')$ calling that function $h$, then $h$ is a bornologous function, $h \simeq_{qc} \Psi (f')$ and $h \in \langle f' \rangle = \langle f \rangle$. Thus, $\Theta \Psi([f]) = ([f])$, that is, $\Theta \Psi = 1_{\pi_n^{qc}(X,x)}$. Under the same argument, $\Psi \Theta = 1_{\pi_n^{qc}(X,y)}$, getting that $\Psi^{-1} = \Theta$.

**Theorem 3.3.5.** Let $(X,V)$ be a connected quasi-coarse space. Then, for every $x, y \in X$ we get $\pi_n^{qc}(X,x) \cong \pi_n^{qc}(X,y)$.

**Proof.** Let $(X,V)$ be a connected quasi-coarse space and $x, y \in X$. Then, there exist $N, M \in \mathbb{Z}$ such that $N < M$ and a bornologous function $H : \{*, 1\} \times \mathbb{Z} \to X$ such that $H(*, z) = *$ when $z \in \mathbb{Z}$, $H(1, z) = x$ when $z \leq N$ and $H(1, z) = y$ when $z \geq M$. Since $H$ is bornologous, then $\{(x, H(1, N + 1)), \{(H(1, N + 1), H(1, N + 2)), \ldots \{(H(1, M - 1), y) \in V$. Thus,

$$\pi_n^{qc}(X,x) \cong \pi_n^{qc}(X,H(1,N+1)) \cong \ldots \cong \pi_n^{qc}(X,H(1,M-1)) \cong \pi_n^{qc}(X,y).$$

Getting that $\pi_n^{qc}(X,x) \cong \pi_n^{qc}(X,y)$. \qed

### 3.4. The Quasi-Coarse Fundamental Group of Cyclic Graphs

In this section, we will compute the quasi-coarse fundamental group of cyclic graphs with different structures, and, in particular, this will provide a class of examples where quasi-coarse homotopy is non-trivial in dimensions greater than zero. (For the 0-homotopy class, we get that $\#(\pi_n^{qc}(X,*)) = \#X$, the number of quasi-coarse connected components of $X$.)

**Definition 3.4.1.** Let $C_n := \{0,1,2,\ldots,n-1\}$ and $m$ a positive integer. Let’s define $C_m^n$ as the graph with vertices $C_n$ and edges $(k,k-i), (k,k+i) \ mod(n)$ for every $k \in C_n$ and $i \in \{1,2,\ldots,m\}$.

For the remainder of this section, we denote by $C_{n,m}$ the quasi-coarse space $(C_n, \mathcal{C}_m^n)$, where $C_n$ is given the quasi-coarse structure induced by the graph $\mathcal{C}_m^n$. We denote by $c$ to the map from $\mathcal{C}_{n+1}$ onto $C_n$ such that $c(i) = i \ mod(n)$, and by $c^{-1}$ the map such that $c^{-1}(i) = c(n-i)$.

We will now define a class of functions which we can see as being endowed with an orientation. They will be particularly useful in the following.

**Definition 3.4.2.** We say that $f : I_k \to C_{n,m}$ is unidirectional if for every $i \in \{1,\ldots,k-2\}$ we have that $f(i)$ is not a neighbor of $f(i+2)$ (i.e. $(f(i), f(i+2)) \not\in \mathcal{C}_m^n$).

**Lemma 3.4.3.** For every bornologous function $f : I_k \to C_n$ there exists a bornologous function $f' : I_{k'} \to C_n$ such that $f \simeq_{qc} \langle f' \rangle$ and $f'$ is unidirectional.

**Proof.** Assume $f$ is not a unidirectional function, then there exists $i \in \{1,\ldots,k-2\}$ such that $f(i)$ is neighbor of $f(i+2)$. We are able to define $f_1 : I_k \to C_n$ such that $f_1(j) = f(j)$ with $j \neq i + 1$, $f_1(i+1) = f(i+2)$. Clearly $f_1 \simeq_{qc} f$.

We now define $f_1 : I_{k-1} \to C_n$ such that $f_1(j) = f_1(j)$ if $j \leq i$, $f_1(j) = f_1(j + 1)$ if $j > i$. It is not difficult to show that $\langle f \rangle \simeq_{qc} \langle f_1 \rangle$.

We repeat this process on $f_1$, and so on, until one is left with a unidirectional function. Observe this is a finite process since each iteration reduces the length of the domain of $f_1$ by 1. \qed
Remark 3.4.4. If \( \left\lceil \frac{n}{m} \right\rceil \leq 2 \), then no function \( f : I_k \to C_n \) is unidirectional for \( k > 0 \), because \( C_n^m \) is a complete graph.

The following result shows that we are able to reduce the map \( c \) to a standard form.

**Proposition 3.4.5.** Let \( \hat{c} : I_{\left\lceil \frac{n}{m} \right\rceil} \to C_n \) be the map \( \hat{c}(k) = (k - 1)m \) if \( 1 \leq i \leq \left\lceil \frac{n}{m} \right\rceil \), and \( \hat{c}(\left\lceil \frac{n}{m} \right\rceil) = 0 \). Then \( \hat{c} \simeq_{qc} c \).

**Proof.** Define \( \hat{c} : I_n \to C_n \) as follows:

\[
\hat{c}(i) := \begin{cases} 
0 & 0 \leq i \leq m - 1 \text{ or } i = n \\
m & m \leq i \leq 2m - 1 \\
2m & 2m \leq i \leq 3m - 1 \\
\vdots & \vdots \\
(\left\lceil \frac{n}{m} \right\rceil - 1)m & (\left\lceil \frac{n}{m} \right\rceil - 1)m \leq i \leq n - 1 
\end{cases}
\]

The \( C_n^m \) structure gives immediately that \( c \simeq_{qc} \hat{c} \). It now follows that \( \langle c \rangle \simeq_{qc} \langle \hat{c} \rangle \).

Our final proposition gives us that every unidirectional function has a standard form, up to homotopy.

**Proposition 3.4.6.** Every unidirectional function \( f : I_k \to C_n \) is homotopy equivalent to a function of one of the following forms: \( 1 \), \( c \ast c \ast \ldots \ast c \) or \( c^{-1} \ast \ldots \ast c^{-1} \).

**Proof.** We will assume that \( \frac{n}{m} > 2 \), otherwise we have complete graph which is homotopy equivalent a point, and the conclusion of the proposition is satisfied. We also assume that \( 1 \neq f \).

Let \( f : I_k \to C_n \) be unidirectional. Then \( f(0) = 0 \), and either \( f(1) \in \{1, \ldots, m\} \) or \( f(1) \in \{n - 1, \ldots, n - m\} \). We will work with the first case. The other case is analogous.

Let \( \hat{f}_1 : I_{k+f(1)} \to C_n \) be defined by

\[
\hat{f}_1(i) := \begin{cases} 
i & 1 \leq i \leq f(1) \\
f(i - f(1)) & f(1) < i \leq f(1) + k 
\end{cases}
\]

We observe that \( \langle f \rangle \simeq_{qc} \langle \hat{f}_1 \rangle \).

We call the transformation of \( f \) into \( \hat{f}_1 \) a flattening of \( f \) from \( i = 0 \) to \( i = 1 \). We now flatten \( \hat{f}_1 \) from \( f(1) \) to \( f(1) + 1 \), and we repeat this process until we arrive at a function \( \bar{f} \) such that \( \bar{f}(i) - \bar{f}(i + 1) = 1 \). Note that this takes a finite number of steps, and that \( \langle f \rangle \simeq_{qc} \langle \bar{f} \rangle \), which proves the result. \( \square \)

Combining the above two propositions, we get that the first homotopy group of \( (C_n, C_n^m) \) is isomorphic to the fundamental group of a different cyclic graph with a simpler structure, i.e.

**Corollary 3.4.7.** \( \pi_1^{qc}(C_n, C_n^m) \cong \pi_1^{qc}(C_{\left\lceil \frac{n}{m} \right\rceil}, C_{\left\lceil \frac{n}{m} \right\rceil}) \).

**Remark 3.4.8.** Let \( (C_n, C_n) \) be the \( n \)-cycle and \( n \in \{1, 2, 3\} \), then it follows that \( C_n \) is connected coarse structure, precisely \( C_n = \mathcal{P}(C_n \times C_n) \). So, their homotopy groups are trivial.

In the rest of this subsection, we compute the fundamental group of \( (C_n, C_n) \).

**Theorem 3.4.9.** Let \( (C_n, C_n) \) be the quasi-coarse space induced by the \( n \)-cycle graph. Then, \( \pi_1^{qc}(C_n, C_n) \cong \mathbb{Z} \).
Before the proof, we are going to need proving the following three lemmas, for
which we define the function \( p : (\mathbb{Z}, \mathbb{Z}) \to (C_n, C_n) \) by \( p(k) := k \mod n \) This function
will be called projection of the integers onto \( n \)-cycle or simply the projection.
We note that \( p \) is a bornologous function.

**Lemma 3.4.10.** Let \( f : I_k \to C_n, n \geq 4, \) be a bornologous function with \( f(0) = 0. \)
Then, for each \( m \in p^{-1}(0) \) there exists a unique bornologous function \( \hat{f} : I_k \to \mathbb{Z} \)
such that \( \hat{f}(0) = m \) and \( f = p \circ \hat{f} \).

**Proof.** Let \( f : I_k \to C_n \) be a bornologous function with \( f(0) = 0, \) and let \( m \in p^{-1}(x_0) \).

We begin by proving that if there are two bornologous functions \( \hat{f}, \hat{g} : I_k \to (\mathbb{Z}, \mathbb{Z}) \) such that \( \hat{f}(0) = \hat{g}(0) = m \) and \( f = p \circ \hat{f} = p \circ \hat{g}, \) then \( \hat{f} = \hat{g} \). Given
that \( p \circ \hat{f} = p \circ \hat{g}, \) then \( \hat{f}(i) = \hat{g}(i) + n \cdot i \) for each \( i \in \{0, \ldots, k - 1\} \). Also,
by hypothesis, \( \hat{f}(0) = \hat{g}(0) \). Since both functions are bornologous, we have that \( \hat{f}(1) = \hat{f}(0) + i_0 \) and \( \hat{g}(1) = \hat{g}(0) + j_0 \) with \( i_0, j_0 \in \{ -1, 0, 1 \}, \) and \( i_0 \mod n = j_0 \mod n, \) which implies that \( i_0 = j_0 \). We conclude that \( \hat{f}(1) = \hat{g}(1) \). Inductively,
if \( \hat{f}(i - 1) = \hat{g}(i - 1), \) then \( \hat{f}(k) = \hat{g}(k) \) by the same argument made for \( i = 1. \)
Therefore, \( \hat{f}(i) = \hat{g}(i) \) for each \( i \in \{0, \ldots, m - 1\}, \) and \( \hat{f} = \hat{g} \) as desired.

We now construct a function \( \hat{f} : I_k \to \mathbb{Z} \) which satisfies \( \hat{f}(0) = m \) and \( f = p \circ \hat{f}. \)
First, define \( \hat{f}(0) = m, \) and \( i \in \{1, 2, \ldots, m - 1\} \) we define inductively \( \hat{f}(i) = \hat{f}(i - 1) + i_{i-1} \) where

\[
i_{i-1} = \begin{cases} 1 & \text{if } f(k) \equiv (f(k - 1) + 1) \mod n \\ 0 & \text{if } f(k) \equiv (f(k - 1)) \mod n \\ -1 & \text{if } f(k) \equiv (f(k - 1) - 1) \mod n \end{cases}
\]

So it follows by construction that \( \hat{f} \) is a bornologous function, that \( \hat{f}(0) = m \) and
that \( p \circ \hat{f} = f. \)

**Lemma 3.4.11.** Let \( f, g : I_k \to C_n \) be bornologous functions such that \( f(0) = g(0) = 0 \) and \( f(k) = g(k) = x_k. \) Assume we have a homotopy \( H : I_k \times \mathbb{Z} \to C_n, \)
between \( f \) and \( g \) such that \( H(0, z) = x_0 \) and \( H(k, z) = x_k \) for each \( z \in \mathbb{Z}. \) Then,
for each \( \hat{x_0} \in p^{-1}(0), \) there exists a unique lift \( \hat{H} : I_k \times \mathbb{Z} \to \mathbb{Z} \) of \( H \) such that \( p \circ \hat{H} = H \), \( \hat{H}(0, z) = \hat{x_0}, \) and \( \hat{H}(k, z) = \hat{x_0} \) for each \( z \in \mathbb{Z}. \)

**Proof.** Let \( f, g : I_k \to C_n \) be bornologous functions such that \( f(0) = g(0) = x_0 \) and \( f(k) = g(k) = x_k. \) Assume we have \( H : I_k \times \mathbb{Z} \to C_k, \) a homotopy between \( f \)
and \( g. \) By definition, there exist integers \( N, M \in \mathbb{Z}, \) \( N < M, \) such that \( H(m, i) = f(m) \)
for \( i \leq N \) and \( H(m, i) = g(m) \) for \( i \geq M. \) Therefore \( H(0, z) = x_0 \) and \( H(k, z) = x_k \)
for each \( z \in \mathbb{Z}. \) Consider a point \( \hat{x_0} \in p^{-1}(x_0). \)

For each \( z \in \mathbb{Z}, \) we define \( f_z : I_k \to C_n \) by \( f(x) := H(x, z) \) with \( x \in I_k. \) By
**Lemma 3.4.10** then there are unique bornologous functions \( \hat{f}_z : I_m \to \mathbb{Z} \) for each \( z \in \mathbb{Z} \)
such that \( p \circ \hat{f}_z = f_z \) and \( \hat{f}_z(0) = \hat{x_0}, \) so we can define \( \hat{H}(x, z) := \hat{f}_z(x) \)
for each \( z \in \mathbb{Z} \) and \( x \in I_k. \) Under this construction, it follows that \( p \circ \hat{H} = H, \)
\( \hat{H}(0, z) = \hat{x_0} \) for each \( z \in \mathbb{Z}, \) so it remains to show that \( H \) is bornologous.

Let’s observe first that by construction \( \hat{H}(x, z) = \hat{f}(x) \) if \( z \leq N, \) \( \hat{H}(x, z) = \hat{g}(x) \)
if \( z \geq M \) and \( \{ (\hat{H}(x, z), \hat{H}(x + 1, z)) \} \in \mathbb{Z} \) for each \( z \in \mathbb{Z} \) and \( x \in I_k. \) So, to prove
Remark 3.4.12. Let \( \hat{H} \) be bornologous is enough to show that
\[
\{(\hat{H}(x, z), \hat{H}(x, z+1)) \} \in Z, \{(\hat{H}(x+1, z), \hat{H}(x+1, z+1)) \} \in Z,
\]
for each \( x \in I_k \) and \( z \in \{N, N+1, \ldots, M-2, M-1\} \). Let \( x \in I_k \) and \( z \in \{N, N+1, \ldots, M-2, M-1\} \). For each of the pairs \((x, z), (x', z')\) considered, above, since \( H \) is bornologous, we have \( p \circ \hat{H}(x, z) = p \circ \hat{H}(x', z') \) where \( i_{x, x', z, z'} \in \{-1, 0, 1\} \). Then \( H(x, z) = H(x', z') + i_{x, x', z, z'} \) where \( \hat{i}_{x, x', z, z'} \in \{-1, 0, 1\} \mod n \). For \( x = 0 \), note that \( f_z(0) = \hat{f}_z(0) = \hat{x}_0 \), and therefore \( i_{0, z, 0, z'} = 0 \) for any \( z \in Z \) and \( z' \in \{z, z+1\} \). Since \( \hat{f}_z(x) = \hat{f}_z(x + 1) + i_{x, x+1, z} \) by construction, we also have that \( i_{x, x+1, z} = i_{x, x', z, z'} \in \{-1, 0, 1\} \) for all \( x \in I_k \). It follows that, for \( x, x' \in \{0, 1\} \), any \( z \), and \( z' \in \{z, z+1\} \), we have \( \hat{i}_{x, x', z, z'} = i_{x, x', z, z'} \in \{-1, 0, 1\} \) for any \( x, x' \in I_k \). Therefore \( \hat{H} \) is bornologous. \( \square \)

**Remark 3.4.14.** The homotopy is unique because \( \hat{f}_z \) is unique for each \( z \in Z \). If we change any element, the relations \( p \circ \hat{H} = H, f_z(0) = \hat{x}_0 \) will no longer be true.

**Lemma 3.4.13.** Let \( f : I_p \to Z \) and \( g : I_q \to Z \) be bornologous functions such that \( f(0) = g(0) \) and \( f(p) = g(q) \). Then \( f \simeq_{qc} q \).

**Proof.** Let \( f : I_p \to Z \) and \( g : I_q \to Z \) be bornologous functions such that \( f(0) = g(0) \) and \( f(p) = g(q) \).

Observe that there exist unidirectional functions \( f' : I_{p'} \to Z \) and \( g' : I_{q'} \to Z \) such that \( f(0) = g(0) \), \( f(p) = g(q) \), and \( m' = n' = 0 \), and the result follows.

Now assume that \( f(0) \neq f(p) \). Then \( f' \) must be the function \( f' : I_{|f(p) - f(0)|} \to Z \) such that \( f'(i) = f(0) + \text{sign}(f(m) - f(0))i \) for every \( i \in \{0, \ldots, |f(m) - f(0)|\} \), with \( \text{sign}() \) the sign function. Analogously for \( g' \). However, by hypothesis, \( f(p) = g(q) \). Thus \( f' = g' \). Therefore \( f \simeq_{qc} q \). \( \square \)

**Remark 3.4.14.** Let \( f, g : I_k \to X \) be bornologous functions such that \( f(k) = g(0) \). By [Proposition 3.2.3] \( f \star g \) is bornologous.

**Theorem 3.4.2.** Let \((C_n, C_n)\) be the quasi-coarse space induced by the \( n \)-cycle graph. We define \( \phi : \pi^{qc}_1(C_n, C_n) \) by \( \phi(z) := [c^z]\). Note that, by construction, \( c^z \) is a bornologous function for each \( z \in Z \). It is clear from the definition and the definition of the \( \star \) operation that \( \phi \) is a group homomorphism.

To prove that \( \phi \) is surjective, consider \( [\{f\}] \in \pi^{qc}_1(C_n, C_n) \) and, without loss of generality, assume that \( f \in \{f\} \) is a bornologous function \( f : I_k \to C_n \) with \( f(0) = f(k) = 0 \), and note that \( 0 \in \mathbb{Z} \). Thus, by [Lemma 3.4.10] there is a unique bornologous function \( \hat{f} \) such that \( \hat{f}(0) = 0 \) and \( f = p \circ \hat{f} \). Since \( (p \circ \hat{f})(k) = f(k) = 0 \), note that \( \hat{f}(k) = nq \) for some \( q \in \mathbb{Z} \). Thus, \( \hat{f} \simeq_{qc} c^n \) by [Lemma 3.4.13] and therefore
\[
f = p \circ \hat{f} \simeq_{qc} p \circ c^n \circ m = c^n\]
We conclude that \([\{f\}] = [\{c^n\}] = \phi(q)\), and therefore \( \phi \) is surjective.

To prove that \( \phi \) is injective, suppose that \( \phi(p) = \phi(q) \) for some \( p, q \in \mathbb{Z} \), so we have that \( c^p \) \( \simeq_{qc} c^q \). Let \( H : I_k \to C_n \) be a homotopy from \( c^p \) to \( c^q \), where \( k = |pqn| \), and, abusing notation, we understand \( c^p \) and \( c^q \) here to be the
extension of each map to $I_k$ by defining $c^*\rho(i) = 0$ if $i > |pn|$, and similarly for $c^*q$. Note that $H(0,i) = 0 \in C_n$ for each $i \in \mathbb{Z}$ by definition. We also note that $0 \in p^{-1}(\{0\}) = n\mathbb{Z}$.

By Lemma 3.4.11 there is a unique homotopy $\hat{H} : I_{pqn} \to C_n$ such that $H = p\circ \hat{H}$ and $H(0,n) = 0$ for each $n \in \mathbb{Z}$. By uniqueness of the homotopy, we have that $\hat{c^*\rho}(i) = \hat{H}(i,N)$ and $\hat{c^*q}(i) = \hat{H}(i,M)$. Furthermore, since $c^*\rho(|nq|) = c^*\rho(k) = H(k,N) = H(k,M) = c^*q(|nq|)$, we obtain $p = q$, and we conclude that the function is injective.

3.5. Long Exact Sequence in Homotopy. With the same goal, we will do the following definitions which will help us to prove the fundamental result: the long exact sequence in relative homotopy.

**Definition 3.5.1.** Let $(X, \mathcal{V})$ be a quasi-coarse space and $(A, \mathcal{V}_A)$ be a quasi-coarse subspace. Then,

- The bornologous function $r : X \to A$ is a retraction if satisfies that $r \circ i = id_A$, where $i : A \to X$ such that $i(a) = a$ for each $a \in A$.
- A bornologous function $F : (X \times \mathcal{Z}, \mathcal{V} \times \mathcal{Z}) \to (X, \mathcal{V})$ is a deformation retract of $X$ onto $A$ if there are $N < 0 < M$ such that, for each $x \in X$ and $a \in A$, $F(x,z) = x$ if $z \leq N$, $F(x,z) = a$ if $z \geq M$.
- If $F$ also satisfies that $F(a,z) = a$ for each $z \in \mathcal{Z}$, then $F$ is called strong deformation retraction.

**Lemma 3.5.2.** Let $f : I^n_m \to X$ be a bornologous function, $k \in I^n_m$ and $w \in \{1, \ldots, n\}$. Then $f \simeq g \simeq h$ such that

$$g(y) := \begin{cases} g(y) & \text{if } y_w \leq k_w \\ g(y - (y_w - k_w)e_w) & \text{if } y_w > k_w \end{cases}$$

and

$$h(y) := \begin{cases} h(y) & \text{if } y_w \geq k_w \\ h(y - (y_w - k_w)e_w) & \text{if } y_w < k_w \end{cases}$$

**Proof.** Let $f : I^n_m \to X$ be a bornologous function, $k \in I^n_m$, $w \in \{1, \ldots, n\}$,

$$g(y) := \begin{cases} g(y) & \text{if } y_w \leq k_w \\ g(y - (y_w - k_w)e_w) & \text{if } y_w > k_w \end{cases}$$

and

$$h(y) := \begin{cases} h(y) & \text{if } y_w \geq k_w \\ h(y - (y_w - k_w)e_w) & \text{if } y_w < k_w \end{cases}$$

We will just prove for $g$, the proof for $h$ is analogous. We are going to proceed by math induction:

**Base case:** Let $k$ such that $k_w = m - 1$, $f \simeq g$ such that

$$g(y) = \begin{cases} g(y) & \text{if } y_w \leq m - 1 \\ g(y - e_w) & \text{if } y_w = m \end{cases}$$

**Induction step:** Assume that $f \simeq g_i$ for every $i \in \{1, \ldots, k_w - 1\}$ where

$$g_i(y) := \begin{cases} g_i(y) & \text{if } y_w \leq i \\ g_i(y - (y_w - i)e_w) & \text{if } y_w > i \end{cases}$$
once again $f \simeq g$ such that
\[
g(y) := \begin{cases} 
  g(y) & \text{if } y_w \leq k_w \\
  g(y - (y_w - k_w)e_w) & \text{if } y_w > k_w
\end{cases}
\]

Observe that the last lemma makes us being able to replace either the plates “above” or “below” $(I^m_n)_{w=0}^{k-1}$. Thereby, the following lemma is a corollary. With the intention of making the proof clear, allow us call $g$ as $(f)^w_1$ and $h$ as $(f)^w_0$.

**Lemma 3.5.3.** Let $n$ be a natural number and $m$ be a non-negative integer number, then there is a strong deformation retraction $I^n_m$ onto $\{\ast\}$.

**Proof.** Let $n$ be natural number and $m$ be a non-negative integer number, let’s consider $\ast \in I^n_m$, then $\ast := k = (k_1, \ldots, k_n)$ where $k_i \in \{0, 1, \ldots, m\}$ fixed for each $i \in \{1, \ldots, n\}$. Denote $1^n_m$ the identity function, then $1^n_m \simeq (1^n_m)^{\uparrow} \simeq ((1^n_m)^{\uparrow})_{0}^{\downarrow} =: g_1$ by Lemma 3.5.2. Therefore, $g_i := ((g_{i-1})_{0}^{\downarrow})_{\ast} \simeq g_{i-1}$ for each $i \in \{2, \ldots, n\}$ by Lemma 3.5.2. Thus, $g_n \simeq f$. The strong deformation that we are looking for is precisely that homotopy.

The following lemma is a tool to prove the Compression Criterion, which one is our goal to finally get the long exact sequence in our homotopy.

**Lemma 3.5.4.** Let $f, g : (I^n_m, \partial I^n_m, J^{n-1}_m) \to (X, A, \ast)$ such that $f \simeq_{qc} g \rel(\partial I^n_m)$. Let $H$ the homotopy with $N < M$ and $N' > 0$, then $f \simeq_{qc} g \rel(\partial I^{n_n})$ with $h_{N'} : (I^n_{m+N'}, \partial I^n_{m+N'}, J^{n-1}_{m+N'}) \to (X, A, \ast)$ such that
\[
h_{N'}(x) := \begin{cases} 
  H(x - N' e_n, N + N') & x \in I^n_m + N' e_n \\
  H(x - k e_n, N + k) & x \in (I^n_m)_0 + k e_n \text{ and } k \in \{0, \ldots, N' - 1\} \\
  \ast & \text{otherwise}
\end{cases}
\]
where $(I^n_{m+n})_0 \subset I^n_m \subset I^n_{m+N'}$.

**Proof.** Let $f, g : (I^n_m, \partial I^n_m, J^{n-1}_m) \to (X, A, \ast)$ such that $f \simeq_{qc} g \rel(\partial I^n_m)$. Let $H$ the homotopy with $N < M$. We are going to do the proof by mathematical induction:

**Base case:** We have that $(f) \simeq_{qc} (g_1) \rel(\partial I^n_m)$ where $g_1 : (I^n_{m+1}, \partial I^n_{m+1}, J^{n-1}_{m+1}) \to (X, A, \ast)$
\[
g_1(x) = \begin{cases} 
  f(x - e_n) & x \in I^n_m + e_n \\
  f(x) & x \in (I^n_m)_0 \\
  \ast & \text{otherwise}
\end{cases}
\]

Since $H$ is a bornologous function, we are able to replace $f(x - e_n)$ by $H(x - e_n, N + 1)$, calling that function $h_1$, and we get $g \simeq_{qc} h_1 \rel(\partial I^n_{m+1})$. Note that $f(x) = H(x, N')$ when $x \in (I^n_m)_0$. So, $(f) \simeq_{qc} (h_1) \rel(\partial I^n_m)$.

**Induction step:** Suppose that for $w > 0$ we have $(f) \simeq_{qc} (h_w) \rel(\partial I^n_m)$ with $h_{N'} : (I^n_{m+w}, \partial I^n_{m+w}, J^{n-1}_{m+w}) \to (X, A, \ast)$ such that
\[
h_w(x) := \begin{cases} 
  H(x - w e_n, N + w) & x \in I^n_m + w e_n \\
  H(x - k e_n, N + k) & x \in (I^n_m)_0 + k e_n \text{ and } k \in \{0, \ldots, w - 1\} \\
  \ast & \text{otherwise}
\end{cases}
\]
We have that $\langle h_w \rangle \sim_{qc} \langle g_{w+1} \rangle \rel(\partial I^n)$ where $g_{w+1} : (I_{m+w+1}^n, \partial I_{m+w+1}^n, J_{m+w+1}^{n-1}) \to (X, A, \ast)$ such that

$$g_{w+1}(x) := \begin{cases} H(x - (w + 1)e_n, N + w) & x \in I_{m+w}^n + (w + 1)e_n \\ H(x - we_n, N + w) & x \in (I_{m+w}^n)_0 + we_n \\ H(x - ke_n, N + k) & x \in (I_{m+w}^n)_0 + ke_n \text{ and } k \in \{0, \ldots, w - 1\} \\ \ast & \text{otherwise} \end{cases}$$

Since $H$ is a bornologous function, we are able to replace $H(x - (w + 1)e_n, N + W)$ by $H(x - (w + 1), N + w + 1)$, calling that function $h_{w+1}$, and we get $g_{w+1} \sim_{qc} h_{w+1} \rel(\partial I_{m+w+1}^n)$.

**Lemma 3.5.5** (Compression Criterion). Let $(X, \mathcal{V})$ be a quasi-coarse space and $\ast \in A \subset X$. Then a function $f : (I^n, \partial I^n, J^{n-1}) \to (X, A, \ast)$ represents the zero in $\pi_n((X, \mathcal{V}), A, \ast)$ if, and only if, is relative homotopic in $\partial I^n$ to a function with its image contained in $A$.

**Proof.** Let $(X, \mathcal{V})$ be a quasi-coarse space, $\ast \in A \subset X$ and $f : (I^n, \partial I^n, J^{n-1}) \to (X, A, \ast)$.

$(\Rightarrow)$ If $[f] = 0$, then there exist $m \in \mathbb{N}$ with $f : (I_m^n, \partial I_m^n, J_m^{n-1}) \to (X, A, \ast)$ such that $f \sim_{qc} \ast$. Let $H$ be the homotopy with $N < M$, then by Lemma 3.5.4 we get $h_{M-N}$ as we write in such lemma which has its image contained in $A$ and $(f) \sim_{qc} (h_{M-N}) \rel(\partial I^n)$.

$(\Leftarrow)$ If $f$ is relative homotopic in $\partial I^n$ to a function with its image contained in $A$, let’s say $g$. As we do in Lemma 3.5.3 with $w = (0, \ldots, 0, m)$, then $g \sim_{qc} (g)_w^i = \ast$. Getting what we wanted.

**Definition 3.5.6.** Let $(X, \mathcal{V})$ and $(Y, \mathcal{W})$ be quasi-coarse spaces and $f : ((X, \mathcal{V}), A, \ast) \to ((Y, \mathcal{W}), B, \ast)$ be a Bornologous function. Then, we define $f_* : \pi_*^{qc}((X, \mathcal{V}), A, \ast) \to \pi_*^{qc}((Y, \mathcal{W}), B, \ast)$ as $f_*([h]) = ([f \circ h])$ for each $[h] \in \pi_*^{qc}((X, \mathcal{V}), A, \ast)$.

**Remark 3.5.7.** Let $(X, \mathcal{V})$ and $(Y, \mathcal{W})$ be quasi-coarse spaces and $f : ((X, \mathcal{V}), A, \ast) \to ((Y, \mathcal{W}), B, \ast)$ be a (based) Bornologous function. Then $f_*$ is well-defined and is an homomorphism. It is well-defined because, if we take $[g] \in [h]$ in $\pi_*^{qc}((X, \mathcal{V}), A, \ast)$, then there is a homotopy $H$ between them, so $f \circ H$ is a homotopy between $[f \circ h]$ and $(f \circ g)$. If $[h], [g] \in \pi_*^{qc}((X, \mathcal{V}), A, \ast)$. It is a homomorphism because $f \circ (g \ast h) = f \circ g(k_1, \ldots, k_n)$ if $0 \leq k_i \leq m$ for each $i \in \{1, \ldots, n\}$, $f \circ (g \ast h) = f \circ h(k_1, \ldots, k_n)$ if $m < k_1 \leq 2m 0 \leq k_i \leq m$ for each $i \in \{2, \ldots, n\}$, and $f \circ (g \ast h) = f(\ast)$ anywhere else, which is the same as $(f \circ g) \ast (f \circ h)$.

**Theorem 3.5.8** (Long Exact Sequence in Homotopy). Let $(X, \mathcal{V})$ be a quasi-coarse space and $\ast \in A \subset X$. As well $i : (A, \ast) \to (X, \ast)$, $j : (X, \ast) \to (X, A)$ inclusions. Then there is a homomorphism $\partial_n : \pi_n^{qc}((X, \mathcal{V}), A) \to \pi_n^{qc}((A, \ast))$ such that the long sequence

$$\cdots \xrightarrow{j_*} \pi^{qc}_{n+1}(X, A) \xrightarrow{\partial_{n+1}} \pi^{qc}_n(A, \ast) \xrightarrow{i_*} \pi^{qc}_n(X, \ast) \xrightarrow{j_*} \cdots$$

$$\cdots \xrightarrow{\partial_2} \pi^{qc}_1(A, \ast) \xrightarrow{i_*} \pi^{qc}_1(X, \ast) \xrightarrow{j_*} \pi^{qc}_1(X, A)$$

is exact.
Proof. Let \((X, \mathcal{V})\) be a quasi-coarse space and \(* \in A \subset X\). As well \(i : (A, * ) \rightarrow (X, * )\), \(j : (X, * ) \rightarrow (X, A)\) inclusions. We are going to define \(\partial_n : \pi\text{qc}_n^q((X, \mathcal{V}), A) \rightarrow \pi\text{qc}_n^q(A, * )\) as \(f \mapsto f|_{I^{n-1}_M - \{0\}}\), which is well-defined and is a homomorphism.

Now we will prove that the long sequence is exact:

- \((\text{Im}(i_*) \subset \text{Ker}(j_*))\) If \(n \geq 1\) and \(f : (I^n, \partial I^n, J^{n-1}) \rightarrow (A, *, *)\) \(\in \pi\text{qc}_n^q(A, * )\), then \(j \circ i \circ f \in \pi\text{qc}_n^q(X, A)\), so by compression criterion we have that \([f] = ([*])\).

- \((\text{Im}(i_*) \supset \text{Ker}(j_*))\) If \(n \geq 1\) and \([f] \in \pi\text{qc}_n^q((X, \mathcal{V}), *)\) such that \(j_*[f] = ([*])\). Then by compression criterion, \(\langle f \rangle \simeq_{qc} \langle g \rangle\ \text{rel} (\partial I^n)\), where the image of \(g\) is contained in \(A\), thus \([g] = [f] \in \pi\text{qc}_n^q((X, \mathcal{V}), *)\) is in the image of \(i_*\).

- \((\text{Im}(j_*) \subset \text{Ker}(\partial))\) If \(n \geq 2\) and \([f] \in \pi\text{qc}_n^q((X, \mathcal{V}), *)\), then \(\partial j_*[f] = ([*])\). Then \(f|_{I^{n-1}_M - \{0\}} \in \pi\text{qc}_n^q(J^{n-1})\) is homotopic to \([*]\) by the compression theorem through a homotopy \(H : I^{n-1}_M - X \rightarrow A \text{rel} (\partial I^{n-1}_M)\), \(N < 0 < M\) such that \(H(x, z) = *\) if \(z \leq N\) and \(H(x, z) = f|_{I^{n-1}_M - \{0\}}(x, z) \) if \(z \geq M\). Let’s define \(g : (I_n^{n+M-N} \cap \partial I_n^{n+M-N}, J_n^{n+M-N}) \rightarrow ((X, \mathcal{V}), A, *)\) such that

\[g(k_1, \ldots, k_n) = H(k_1, \ldots, k_{n-1}, z = k_n)\]

with \(0 \leq k_i \leq m\) when \(i \in \{1, \ldots, n-1\}\) and \(0 \leq k_n \leq M - N\),

\[g(k_1, \ldots, k_n) = f(k_1, \ldots, k_{n-1}, k_n - M + N)\]

with \(0 \leq k_i \leq m\) when \(i \in \{1, \ldots, n-1\}\) and \(M - N + 1 \leq k_n \leq M - N + m\), and \(g(k_1, \ldots, k_n)\) anywhere else. So, we note that \(g(\partial I_n^{n+M-N}) = *\), that is, \([g] \in \text{Im}(j_*), \text{ and } [f] = ([*])\).

- \((\text{Im}(\partial) \subset \text{Ker}(i_*))\) If \(n \geq 2\) and \(f : (I_n, \partial I_n, J^{n-1}_m) \rightarrow ((X, \mathcal{V}), A, *)\).

Then, defining \(H : I^{n-1}_m - X \rightarrow (X, \mathcal{V})\) as \(H(k_1, \ldots, k_n) = f(k_1, \ldots, k_n)\) with \(0 \leq k_i \leq m\), \(H(k_1, \ldots, z) = f(k_1, \ldots, k_{n-1}, 0)\) if \(z \leq 0\), and \(H(k_1, \ldots, z) = f(k_1, \ldots, k_{n-1}, m)\) if \(z \geq m\), we get that \(f|_{I^{n-1}_m - \{0\}}\) is relative homotopic to \(*\) through \(H\). So, \(i_* \partial_[(f)] = ([*])\) by \underline{Lemma 3.5.5}\.

- \((\text{Im}(\partial) \supset \text{Ker}(i_*))\) If \(n \geq 2\) and \(f : (I_n, \partial I_n, J^{n-1}_m) \rightarrow (A, *, *)\) such that \(i_*([f]) = ([*])\), then we have a homotopy between \(f\) and \(*\), which gives us a function \(F : (I_n^{n+1} \cap \partial I_n^{n+1}, J_n^{n+1}) \rightarrow ((X, \mathcal{V}), A, *)\) such that \(\partial(F) = ([f])\). \(\square\)

4. Homology

In the last section, we will construct a Vietoris-Rips homology for quasi-coarse spaces and show that it is homotopy invariant. We finish with the observation that the homology defined here for a finite quasi-coarse space is isomorphic to the homology of the clique complex of a finite graph.

4.1. Simplicial Homology

Definition 4.1.1. Let \((X, \mathcal{V})\) be a quasi-coarse space and \(E\) be a controlled set, and let \(\mathcal{R}\) be a relation on \(X\) such that \(x \mathcal{R} y\) iff \((x, y) \in E\). We now define the following

- \(\Sigma_E^{(0)} := \{\{x\} : x \in X\},\)
We note that Remark 4.1.2. We note that \( \Sigma_E \) satisfies the definition of simplicial complex for any element \( E \in \mathcal{V} \) in the quasi-coarse structure \( X \). Definition 4.1.1, since \( \Sigma_E^{(0)} \) contains all sets with one vertex and every subset of a simplex is a simplex having all their elements related.

Definition 4.1.3. We define \( C_q(X, E) \) to be the free abelian groups generated by ordered simplicial chains, denoted \([v_0, v_1, \ldots, v_q]\), where \([v_0, v_1, \ldots, v_q] = 0\) if the vertices are not all pairwise different. We define the differential \( \partial_q \) by

\[
\partial_q[v_0, v_1, \ldots, v_q] := \sum_{i=0}^{q} (-1)^i [v_0, \ldots, \hat{v}_i, v_q]
\]

and we denote the chain complex by \( C_\ast(X, E) := \{C_q(X, E), \partial_q\} \) and the resulting homology groups by \( H_\ast(X, E) \).

The above definitions show how to construct homology groups from a single element of a quasi-coarse structure \( E \in \mathcal{V} \). The next lemma will allow us to construct a directed system from these homology groups.

Lemma 4.1.4. Let \((X, \mathcal{V})\) be a quasi-coarse space and \( E, E' \) controlled sets such that \( E \subset E' \). Then there exists a homomorphism \( i_* : H(X, E) \to H(X, E') \).

Proof. Let \((X, \mathcal{V})\) be a quasi-coarse space and let \( E, E' \) be controlled sets such that \( E \subset E' \). If \( i : E \to E' \) is the inclusion from \( E \) to \( E' \), then for a generator \( \sigma = [v_0, \ldots, v_q] \) of \( C_q(X, E) \), we define \((i_\#)_q : C_q(X, E) \to C_q(X, E')\) by

\[
(i_\#)_q[x_0, \ldots, x_q] := [i(x_0), \ldots, i(x_q)] = [x_0, \ldots, x_q] \in C_q(X, E').
\]

We then extend this by linearity. Note that the inclusion \( E \subset E' \) ensures that \( C_q(X, E) \subset C_q(X, E') \), since \( x \mathcal{R}_E y \implies x \mathcal{R}_{E'} y \) for any \( x, y \in X \). The inclusion \( i_\# \) also satisfies \( \partial_{E'} i_\# = i_\# \partial_E \), so \( i_\# \) is a chain map. The induced map \( i_* : H_\ast(X, E) \to H_\ast(X, E') \) such that \( i_*[\sigma] = [i_\#(\sigma)] = [\sigma] \in H_\ast(X, E') \) is the desired map in homology. \( \square \)

Definition 4.1.5. Let \((X, \mathcal{V})\) be a quasi-coarse space, and consider \( \mathcal{V} \) with the partial order given by inclusion of sets. We define

\[
H_\ast(X, \mathcal{V}) := \lim_{\rightarrow} \{H(X, E), \pi_E^\mathcal{V}\},
\]

where \( \pi_E^\mathcal{V} = i_* : H(X, E) \to H(X, E') \) from the last lemma. We call \( H_\ast(X, \mathcal{V}) \) the homology of the quasi-coarse space \((X, \mathcal{V})\). We will sometimes refer to \( H_\ast(X, \mathcal{V}) \) as the Vietoris-Rips homology of \((X, \mathcal{V})\).

We can note that the set of symmetric sets \( E \in \mathcal{V} \) is cofinal in \( \mathcal{V} \), i.e. for every \( E \subset \mathcal{V} \), we have \( E \subset E \cup E^{-1} \cup \Delta_X \in \mathcal{V} \). Denote by \( \mathcal{V}_S \) the collection \( \{E \in \mathcal{V} \mid E = E^{-1}\} \). By the cofinality We have that

\[
\lim_{\rightarrow} \{H(X, E), \pi_E^\mathcal{V}\} \cong \lim_{\rightarrow} \{H(X, E), \pi_E^\mathcal{V}, \mathcal{V}_S\},
\]

so it is enough to consider symmetric elements of \( \mathcal{V} \) when constructing the homology of \((X, \mathcal{V})\).
We will now show the quasi-coarse homotopy is a convariant functor. We will start by showing that homology is functorial for a fixed controlled set $E \in \mathcal{V}$.

**Lemma 4.1.6.** Let $(X, \mathcal{V})$ and $(Y, \mathcal{W})$ be quasi-coarse spaces, $E$ controlled by $\mathcal{V}$ and $f : (X, \mathcal{V}) \to (Y, \mathcal{W})$ be a bornologous function. Then,

(i) $f_* : C(X, E) \to C(Y, (f \times f)(E))$ where

$$ (f_*)_n[x_0, \cdots, x_n] = [f(x_0), \cdots, f(x_n)] $$

is a chain map.

(ii) $f_* : H(X, E) \to H(Y, (f \times f)(E))$ is a group homomorphism, where $(f_*)_n[\sigma^n] = [(f_*)_n\sigma^n]$.

**Proof.** Let $(X, \mathcal{V})$ and $(Y, \mathcal{W})$ be quasi-coarse spaces, $E$ controlled by $\mathcal{V}$ and $f : (X, \mathcal{V}) \to (Y, \mathcal{W})$ be a bornologous function. Since $f$ is bornologous, $\{(f(x), (y))\} \in (f \times f)(E) \in \mathcal{W}$ if $\{(x, y)\} \in \mathcal{V}$, so $f$ induces the simplicial map $f_*$ defined by

$$ (f_*)_n[x_0, \cdots, x_n] = [f(x_0), \cdots, f(x_n)]. $$

Since $f_*$ is a chain map, it induces the map $f_* : H_*(X, E) \to H_*(Y, (f \times f)(E))$ on homology by

$$ (f_*)_n[\sigma^n] = [(f_*)_n\sigma^n]. $$

**Remark 4.1.7.** The previous result is also true for $f_# : C(X, E) \to C(Y, A)$ and $f_* : H(X, E) \to H(Y, A)$ such that $(f \times f)(E) \subset A$ is controlled by $\mathcal{W}$. This follows from the previous lemma and [Lemma 4.1.4.]

**Theorem 4.1.8.** Quasi-coarse homology is a covariant functor $H_* : QCoarse \to \text{Ab}.$

**Proof.** Let $(X, \mathcal{V})$ and $(Y, \mathcal{W})$ be quasi-coarse spaces and let $f : (X, \mathcal{V}) \to (Y, \mathcal{W})$ be a bornologous function. Let $\{H(X, E), \pi^E_{E'}, \mathcal{V}\}$ and $\{H(Y, E), \pi^E_{E'}, \mathcal{W}\}$ be the directed systems of the homology groups, where the directed sets $\mathcal{V}$ and $\mathcal{W}$ are partially ordered by inclusion.

Note that, if $A \subset B \subset X \times X$, then $(f \times f)(A) \subset (f \times f)(B)$ so $(f \times f)$ preserves the preorders on $\mathcal{V}$ and $\mathcal{W}$. Therefore, for each $E, E' \in \mathcal{V}$ such that $E \subset E'$, the diagram

$$
\begin{array}{ccc}
H(X, E) & \xrightarrow{f_*} & H(Y, (f \times f)(E)) \\
\pi^E_{E'} \downarrow & & \downarrow \pi_{(f \times f)(E)^{E'}} \\
H(X, E') & \xrightarrow{f_*} & H(Y, (f \times f)(E'))
\end{array}
$$

commutes. It follows by definition that there exists a homomorphism $F : H_*(X, \mathcal{V}) \to H_*(X, \mathcal{W})$ between the directed limits, that is,

$$ F : \lim_{\rightarrow} \{H(X, E), \pi^E_{E'}, \mathcal{V}\} \to \lim_{\rightarrow} \{H(Y, E), \pi^E_{E'}, (f \times f)(\mathcal{V})\}. $$

**Theorem 4.1.9.** Let $(X, \mathcal{V})$ and $(Y, \mathcal{W})$ be quasi-coarse spaces. If $f, g : (X, \mathcal{V}) \to (Y, \mathcal{W})$ are homotopic functions, then the induced homomorphisms $f_*, g_* : H_*(X, \mathcal{V}) \to H_*(Y, \mathcal{W})$ are equal.
Proof. Let \((X, V)\) and \((Y, W)\) be quasi-coarse spaces. If \(f, g : (X, V) \to (Y, W)\) are homotopic functions, then there are a bornologous function
\[
H : (X \times \mathbb{Z}, V \times \mathbb{Z}) \to (Y, W)
\]
and \(N < 0 < M\) integer numbers such that \(H(x, z) = f(x)\) if \(z \leq N, x \in X\) and \(H(x, z) = g(x)\) if \(z \geq M, x \in X\). We will denote by \(h_z(x) := H(x, z)\) for \(x \in X\) and \(z \in \{N + 1, N + 2, \ldots, M - 2, M - 1\}\).

We will observe what happens with the induced homomorphisms by \(h\) and \(h_{N+1}\). Then, for each \(E\) controlled by \(V\), we define \((\Psi_E)_q : C_q(X, E) \to C(Y, H(E, \mathbb{Z}))\) such that
\[
(\Psi_E)_q[x_0, \ldots, x_q] := \sum_{i=0}^{q} (-1)^i [f(x_0), \ldots, f(x_i), h_{N+1}(x_i), \ldots, h_{N+1}(x_q)].
\]
A straightforward calculation shows that \(\Psi_E\) is a chain homotopy between the induced chain maps \(f_*\) and \((h_{N+1})_*\), so \(f_\# = (h_{N+1})_* : H(C(X, E)) \to H(C(Y, H(E, \mathbb{Z})))\).

Thereby, by similar arguments to the previous theorem, we get that
\[
f_* = (h_{N+1})_* : H(C(X, V)) \to H(C(Y, W)).
\]
We will repeat the same for \(h_i\) and \(h_{i+1}\) with \(i \in N + 1, \ldots, M - 1\), finally arriving at
\[
f_* = g_* : H(C(X, V)) \to H(C(Y, W)),
\]
as desired. \(\square\)

4.2. Graphs and Quasi-Coarse Spaces. Now that we have introduced the basic concepts of quasi-coarse homology, we recall every graph is a roofed quasi-coarse space, and, given the similarities between the constructions of the quasi-coarse homology and the Vietoris-Rips homology of a graph, is natural to ask whether they are isomorphic. We will answer this in the affirmative in this section, in addition to showing that the quasi-coarse homology only depends on the roof of the quasi-coarse structure. We begin with the following lemma.

Lemma 4.2.1. Let \((X, V)\) be a (possibly non-roofed) quasi-coarse space with roof \(\mathfrak{A}\). Then every finite subset \(A\) of \(\mathfrak{A}\) is an element of \(V\).

Proof. The lemma is immediate if \((X, V)\) is roofed. Recall that, by definition, \(\mathfrak{A} = \cup_{V \in V} V\). If \(A = \{a_0, \ldots, a_k\}\) is a finite subset of \(\mathfrak{A}\), then each element \(a_i\) of \(A\) is contained in some set \(V_i \in V\). By the axioms of a quasi-coarse structure, this implies that \(\{a_i\} \in V\) for each \(a_i \in A\), and therefore that \(A = \cup_{i=0}^k \{a_i\} \in V\). \(\square\)

We now use this to show that the quasi-coarse homology only depends on the roof of the quasi-coarse structure.

Theorem 4.2.2. Let \((X, V)\) be a quasi-coarse space. Then
\[
H(X, V) \cong H(X, r f(X, V)).
\]

Proof. Let \((X, V)\) be a roofed quasi-coarse space with roof \(\mathfrak{A}\). Then \(\{H(X, \mathfrak{A})\}\) is cofinal in the directed system \(\{H(X, E), \pi^E, V\}\), and the result follows.

Now assume that \(V\) is non-roofed with roof \(\mathfrak{A}\), and let \(W\) be the the roofed quasi-coarse structure with roof \(\mathfrak{A}\). Since \(V \subset W\), then there exists a homomorphism
\[
\pi^W : \lim_{\rightarrow} \{H(X, E), \pi^E, V\} \to \lim_{\rightarrow} \{H(X, E), \pi^E, W\}
\]
such that \( \pi^W_V(\sigma_E) = (\sigma_E)_W \).

**Surjectivity:** Let \( n \in \mathbb{N} \) and \( x \in \lim\{ H_n(X, E), \pi^E_E, W \} \), then there exists a set \( E \in W \) and \( y \in H_n(X, E) \) such that \( x = (y)_E \). If \( E \in V \), then \( \pi^W_V(y)_E = x \).

If \( E \notin V \), we need to do something else. As \( y \in H(X, E) \), then there exists a cycle \( z \in \text{Ker}(\partial^E_E) \) such that \( y = (z) \in H_n(X, E) \) and

\[
z = \sum_{i=1}^q \alpha_i \sigma_i, \quad \sigma_i = [z^i_0, \ldots, z^i_n], \alpha_i \in \mathbb{Z},
\]

where, the \( \sigma_i \) are the elements of the chain complex corresponding to the ordered simplices \([z^i_0, \ldots, z^i_n]\). By Lemma 4.2.1 each subset of vertices \( A_i := \{z^i_0, \ldots, z^i_n\} \in V \) forming each ordered simplex \( \sigma_i \) is in \( V \), and therefore \( A := \bigcup_i A_i \in V \) as well.

It now follows that \( z \in \text{Ker}(\partial^A_A) \), which the properties of the directed system imply that \( \pi^A_A \pi^A_E(\langle z \rangle) = \langle z \rangle_{AU} = \pi^A_A \pi^E_E(\langle z \rangle) \). and it follows that \( \langle z \rangle_A = \langle z \rangle \) represent the same element of \( H_*(X, W) \), i.e. \( \langle z \rangle_W = \langle z \rangle_W = x \), i.e. \( \langle z \rangle \). it now follows that \( \pi^W_V(\langle z \rangle_A) = x \). Thus, \( \pi^W_V \) is surjective.

**Injectivity:** In this part of the proof we will abuse notation and denote by \( \pi^W_U \) the homomorphism

\[
\pi^W_U: \lim\{ H(X, E), \pi^E_E, V \} \to H(X, \mathfrak{A})
\]

implicitly composing the original map with the isomorphism

\[
\lim\{ H(X, E), \pi^E_E, W \} \cong H(X, \mathfrak{A})
\]

Let \( n \in \mathbb{N} \) and \( x \in \lim\{ H_n(X, E), \pi^E_E, V \} \) such that \( \pi^W_U(x) = 0 \). Then there exists a set \( E \in V \) and \( z \in \text{Ker}(\partial^E_E) \) such that \( x = ([z])_E \) and \( \pi^W_U(z) = 0 \). Therefore, \( z \in \text{Im}(\partial^E_{n+1}) \), that is, there exists an element \( z' \in C^W_{n+1} \) such that \( \partial^E_{n+1}(z') = z \).

Since \( \pi^W_U \) is surjective, there is a set \( U \in V \) such that \( z' \in C^U_{n+1} \) and \( \partial^U_{n+1}(z') = z \in C^n_W \). This implies that \( [z]_U = 0_U \) and \( \pi^U_E([z])_E = 0_U \). Thus, \( [z]_E \) is injective. \( \pi^E_U \) is injective.

It now follows that \( \lim\{ H(X, E), \pi^E_E, V \} \cong H(X, \mathfrak{A}) \).

We now compare the quasi-coarse homology of the vertices of a graph with the quasi-coarse structure induced by the graph and the homology of the clique complex of a graph. We begin by recalling the definition of the clique complex.

**Definition 4.2.3.** A graph is called complete if each pair of vertices is adjacent. A \( k \)-clique in \( G \) is a complete subgraph of \( G \) with \( k \) vertices, and it is a maximal \( k \)-clique if it is not proper subgraph of another clique.

**Definition 4.2.4.** Given a graph \( G = (V, E) \) the clique complex \( \Sigma_G \) of \( G \) is the simplicial complex such that \( \Sigma^{(0)}_G = V \) and a finite set \( \sigma := \{v_0, \ldots, v_k\} \subset V \) is a \( k \)-simplex in \( \Sigma_G \) if the induced subgraph of \( G \) on the vertices in \( \sigma \) is a \( (k + 1) \)-clique. The Vietoris-Rips homology of a graph \( G \), denoted \( H^*_\mathcal{R}(G) \), is the simplicial homology of the simplicial complex \( \Sigma_G \).

**Theorem 4.2.5.** Let \( G = (V, E) \) be a graph and let \( (V, \mathcal{V}_G) \) be the quasi-coarse space induced by \( G \). Then \( H_*(V, \mathcal{V}_G) \cong H^*_\mathcal{R}(G) \).
Proof. Let $G = (V, E)$ be a graph and $(V, V_G)$ be quasi-coarse space associated to $G$. Note that $\bigcup_{E \in V} E = r f(V)$, and therefore

$$\lim_{\rightarrow} \{ H(V, E), \pi_E', \mathcal{V} \} = H \left( V, \bigcup_{E \in V} E \right),$$

by thereby we only have to examine the chain complex for the roof of $\mathcal{V}$.

Let $C(G)$ denote the ordered chain complex generated by ordered simplices of the clique complex of $G$, and define $\iota : C(X, \bigcup_{E \in V} E) \to C(G)$ such that $\iota_q(\{x_0, \ldots, x_q\}) = \{x_0, \ldots, x_q\}$ if all the elements are different and $\iota_q(\{x_0, \ldots, x_q\}) = 0$ if at least a pair of elements are equal. Note that $\iota$ is well-defined when all of elements are different because $x_i R x_j$ for each $i, j \in \{0, 1, \ldots, q\}$, thereby $\{x_i, x_j\} \in V$ when $i \neq j$, concluding $\{x_0, \ldots, x_q\}$ is a clique with $q + 1$ elements in $G$.

On the other hand, let’s define $\kappa : C(G) \to C(X, \bigcup_{E \in V} E)$ such that

$$\kappa_q(\{x_0, \ldots, x_q\}) = \{x_0, \ldots, x_q\}.$$  

By the argument in the previous paragraph, it is clear that $\kappa$ is well-defined. Moreover, $\iota \circ \kappa = 1_{C(X, \bigcup_{E \in V} E)}$ and $\kappa \circ \iota = 1_{C(G)}$, so $C(X, \bigcup_{E \in V} E) \cong C(G)$, thus $H(V(G), V) \cong H(G)$. \qed

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Conflict of Interest Statement

The authors declare that there is no conflict of interest.

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