Abstract

Since the first attempts to quantize Gauge Theories and Gravity in the loop representation, the problem of the determination of the corresponding classical actions has been raised. Here we propose a general procedure to determine these actions and we explicitly apply it in the case of electromagnetism. Going to the lattice we show that the electromagnetic action in terms of loops is equivalent to the Wilson action, allowing to do Montecarlo calculations in a gauge invariant way. In the continuum these actions need to be regularized and they are the natural candidates to describe the theory in a “confining phase”.

D. Armand Ugon, R. Gambini, J. Griego and L. Setaro
Instituto de Física, Facultad de Ciencias
Tristán Narvaja 1674
Montevideo, Uruguay.

July 1993
Loop space provides a common scenario for a nonlocal description of Gauge Theories [1], [2], [3] and Quantum Gravity [4], [5]. Loops form a group, and all the kinematical properties of gauge theories are imbedded in this group. This is the remarkable and fundamental property of the group of loops. Gauge theories arise by considering different representations of this group [6], [7]. The Loop representation [8], [3], [4] is usually constructed by means of the non canonical algebra of a complete set of gauge invariant operators. These operators act on a state space of loop wavefunctions $\psi(\gamma)$. Once the complete set of invariant operators are realized in the space of loops, the action of any other gauge invariant operator (like the hamiltonian) can be obtained from them. In the abelian case, the noncanonical algebra is given in terms of the holonomy $H_A(\gamma)$ and the conjugate electric field $E^a(x)$.

The non canonical character of the algebra of the fundamental invariant operators shadows the classical counterpart of the theory. We do not have at our disposal a pair of canonical variables whose commutator can be related with a classical Poisson bracket. As a consequence, we are not able to make a Legendre transform from the hamiltonian to the classical action of the theory. Classical actions in terms of loops would be interesting in its own right because they are the natural candidates to describe the theory in a “confining phase”. They would also allow to do Montecarlo calculations in Lattice Gauge Theories in a gauge invariant way, combining the power of Montecarlo methods with the advantage of the geometrical character of loop space [3]. The classical action may be also useful to obtain semiclassical approximations to the gauge theory under consideration or to General Relativity in terms of the Ashtekar’s variables.

Recently, an extension of the loop space with the structure of an infinite dimensional Lie group has been proposed [9]. The usual group of loops is a subgroup of this extended loop group and generalized holonomies can be defined in the extended space. In this letter, we show that the introduction of a new representation of gauge theories in terms of this extended structure allows to obtain their corresponding classical actions in terms of loops. We shall consider in detail the electromagnetic case and sketch the general method for an arbitrary gauge theory. In a forthcoming paper the extended representation of the Yang Mills theories and Quantum Gravity will be introduced and the corresponding classical action will be explicitly calculated. These representations correspond to the quantized version of the classical theories whose actions are given in terms of the elements of the extended group. Once the extended classical theory is given, it is straightforward to specialize them to loops, giving rise to a classical description of the theory in the confining phase.

The consistency of the loop dependent action for quantum electromagnetism will be studied, showing that the loop dependent classical action of electromagnetism in the lattice leads to the usual Kogut-Susskind hamiltonian.

We start with a brief review of the extended loop group and the extended loop representation. A more complete treatment of these subjects can be found in references [9].
The holonomy of a nonabelian connection $A_{ax}$ can be written in the following way

$$H_A(\gamma) = 1 + \sum_{n=1}^{\infty} \int dx_1^3 \ldots dx_n^3 A_{a_1}(x_1) \ldots A_{a_n}(x_n) X^{a_1 \ldots a_n}(x_1, \ldots, x_n, \gamma), \quad (1)$$

where

$$X^{a_1 \ldots a_n}(x_1, \ldots, x_n, \gamma) = \oint dy_n^{a_n} \ldots \oint dy_1^{a_1} \delta(x_n - y_n) \ldots \delta(x_1 - y_1) \Theta_{\gamma}(0, y_1, \ldots, y_n) \quad (2)$$

and $\Theta_{\gamma}(0, y_1, \ldots, y_n)$ orders the points along the contour starting at the origin of the loop. These relations define the multitangent field of rank $n$ associated to the loop $\gamma$. They behave as multivector densities under general coordinate transformations. No more information from the loop is needed in order to compute the holonomy than what is present in the multitangents of all orders.

In order to simplify the notation, it is convenient to use the following conventions

$$X^{\mu_1 \ldots \mu_n}(\gamma) = X^{a_1 \ldots a_n} x_1 \ldots x_n(\gamma) = X^{a_1 \ldots a_n}(x_1, \ldots, x_n, \gamma), \quad (3)$$

with $\mu_i \equiv (a_i x_i)$. The $X$'s are not independent quantities, they obey two kinds of constraints: the algebraic and differential constraints.

The algebraic constraints stem from the relations satisfied by the generalized Heaviside function and have the general form

$$\sum P_{k} X^{P_{k} \mu_1 \ldots \mu_n} = X^{\mu_1 \ldots \mu_k} X^{\mu_{k+1} \ldots \mu_n} \quad (4)$$

where the sum goes over all the permutations of the $\mu$ variables which preserve the ordering of the $\mu_1, \ldots, \mu_k$ and the $\mu_{k+1}, \ldots, \mu_n$ between themselves.

The differential constraints ensure that the holonomy has the correct transformation properties under gauge transformations, and can be readily derived from Eq. (1). They are given by,

$$\frac{\partial}{\partial x_i^{a_i}} X^{a_1 x_1 \ldots a_i x_i \ldots a_n x_n} = \left( \delta(x_i - x_{i-1}) - \delta(x_i - x_{i+1}) \right) X^{a_1 x_1 \ldots a_{i-1} x_{i-1} a_{i+1} x_{i+1} \ldots a_n x_n} \quad (5)$$

In this expression, points $x_0$ and $x_{n+1}$ are to be understood as the basepoint of the loop.

An important property of the constraints is that any multitensor density $X^{a_1 x_1 \ldots a_n x_n}$ that satisfies them can be used in Eq. (1) and the resulting object is a gauge covariant quantity. When restricted to the multitangents $X(\gamma)$ associated with loops, the resulting object is the holonomy. It is this property that allows to extend the loops to a more general structure. One can in general deal with arbitrary multitensor densities $X$ (not necessarily related with loops) and construct gauge invariant objects, by taking the trace. The multitensor densities need not be distributional functions as
the multitangents associated with a loop. They could be perfectly smooth functions on the manifold.

With this construction in hand, one could go further and forget loops and holonomies altogether. Since one can represent any gauge covariant object using the X’s, one can represent a gauge theory entirely in terms of X’s. The underlying mathematical structure of this extended representation will be called the “extended group of loops” which has the structure of an infinite dimensional Lie group.

The extended group can be introduced in the following way: given a set of arbitrary (unconstrained) multitensor densities of any rank $E^{\mu_1...\mu_n}$ we define the object $E$ as

$$E = (E, E^\mu_1, ..., E^{\mu_1...\mu_n}, ...) \equiv (E, \vec{E})$$

(6)

where $E$ is a real number. It can be readily checked that the set $\{E\}$ has the structure of a vector space (denoted as $E$) with the usual composition laws of addition and multiplication.

We will now introduce a product law in $E$ in the following way: given two vectors $E_1$ and $E_2$, we define $E_1 \times E_2$ as the vector with components

$$E_1 \times E_2 = (E_1 E_2, E_1 \vec{E}_2 + \vec{E}_1 E_2 + \vec{E}_1 \times \vec{E}_2)$$

(7)

where $\vec{E}_1 \times \vec{E}_2$ is given by,

$$(\vec{E}_1 \times \vec{E}_2)^{\mu_1...\mu_n} = \sum_{i=1}^{n-1} E_1^{\mu_1...\mu_i} E_2^{\mu_{i+1}...\mu_n}.$$  

(8)

The product law is associative and distributive with respect to the addition of vectors. It has a null element (the null vector) and a identity element, given by

$$I = (1, 0, \ldots, 0, \ldots).$$

(9)

An inverse element exists for all vectors with nonvanishing zero rank component. It is given by

$$E^{-1} = E^{-1} I + \sum_{i=1}^{\infty} (-1)^i E^{-i-1}(E - EI)^i$$

(10)

such that

$$E \times E^{-1} = E^{-1} \times E = I.$$  

(11)

The set of all vectors with nonvanishing zero rank component (notice the $E^{-1}$ role in Eq. (10)) forms a group with the $\times$-product law.

Introducing supplementary conditions related to the constraints, it is possible to define several base pointed subgroups of this general group. Among these we find the Extended Loop group, defined as the set $D_o$ of elements $X = (X, \vec{X})$ with $X$ a nonzero real number and where the multivector components $X^{\mu_1...\mu_n}$ satisfy the differential constraint (5) for any rank $n$. 


The group of loops $\mathcal{L}_o$ is a subgroup of the group $\mathcal{D}_o$ since any multitangent field $X(\gamma) \in \mathcal{D}_o$ and
\[ X(\gamma_1 \circ \gamma_2) = X(\gamma_1) \times X(\gamma_2) \] (12)
therefore $\mathcal{L}_o \subset \mathcal{D}_o$.

Matrix representations of these groups may be generated through a natural extension of the holonomy $[\mathbb{I}]$. The extended holonomies associated with a general connection $A_{ax}$ are defined by
\[ H_A(X) = \sum_{n=0}^{\infty} i^n A_{\mu_1 \ldots \mu_n} X^{\mu_1 \ldots \mu_n} \] (13)
where a generalized Einstein convention was assumed and $X^{\mu_1 \ldots \mu_0} = X$. It is straightforward to see that the generalized holonomies satisfy
\[ H_A(X_1) H_A(X_2) = H_A(X_1 \times X_2) \] (14)

Let us now recall how the loop representation is usually derived in the simplest case of the electromagnetic theory.

One starts by considering the non canonical algebra of a complete set of gauge invariant operators. In the Maxwell case $[\mathbb{I}], [\mathbb{II}]$ this algebra is defined in terms of the gauge invariant holonomy
\[ \hat{H}(\gamma) = \exp i \oint_\gamma \hat{A}_a(y) dy^a = \exp i \int d^3 x A_a(x) X^{ax}(\gamma) \] (15)
where
\[ X^{ax}(\gamma) = \oint_\gamma dy^a \delta(x - y) \] (16)
and the conjugate electric field $\hat{E}^a(x)$. These operators satisfy the following commutation relation
\[ [\hat{E}^a(x), \hat{H}(\gamma)] \equiv X^{ax}(\gamma) \hat{H}(\gamma) \] (17)
In the abelian case the theory is completely described in terms of the the rank one component of the multitangent fields $[\mathbb{I}]$. The gauge invariant operators act on a state space of abelian loop functions $\psi(\gamma)$ that may be related with the states in the connection representation by the loop transform
\[ \psi(\gamma) = \int d_{[A]} [A] \psi[A] H_A(\gamma) \] (18)
By means of the loop transform it is immediate to calculate the explicit action of the fundamental gauge invariant operators
\[ \hat{H}(\gamma_0) \psi(\gamma) = \psi(\gamma_0 \circ \gamma) \] (19)
\[ \hat{E}^a(x) \psi(\gamma) = \oint_\gamma \delta(x - y) dy^a \psi(\gamma) \] (20)

---

1This is true if one considers the restriction of the theory to the subgroup $X_o \subset \mathcal{D}_o$, where the elements of $X_o$ satisfy the differential and the algebraic constraints.
The action of any other gauge invariant operator in the abelian loop representation may be deduced from the above expressions. For instance the hamiltonian takes the form

$$\hat{H}\psi(\gamma) = \int d^3x \Delta_{ij}(x) \Delta_{ij}(x) + \oint \gamma dy^a \oint \gamma dy'^a \delta^3(y - y') \psi(\gamma)$$ (21)

where $\Delta_{ij}(x)$ is the loop derivative $^3$. The electric term of the hamiltonian is singular and needs to be regularized. One usually introduces a regularization of the $\delta$-function, for instance

$$f_\epsilon(x - y) = (\pi \epsilon)^{-3/2} \exp(- (x - y)^2 / \epsilon)$$ (22)

Similar regularizations are required for the loop representations of the nonabelian gauge theories and quantum gravity, and they are also required for other descriptions in terms of loops in the continuum. We shall see that the extended loop representation is free from singularities of this kind.

We now turn to the extended representation. The extended representation of an abelian gauge theory is based in the following extension of the holonomy

$$H_A(\gamma) \rightarrow H_A(\mathbf{X}) = \exp ig \int d^3x A_{ax} X^{ax}$$ (23)

where we have explicitly introduced the coupling constant $g$ and $X^{ax}$ is an arbitrary divergence free vector density field

$$\partial_{ax} X^{ax} = 0$$ (24)

that correspond to the first component of the multivector densities $\vec{X}$. Notice that now the fields $X^{ax}$ are not necessarily related with a loop, they may be smooth functions of the space coordinates. The new non canonical gauge invariant algebra

$$[\hat{E}^{ax}, \hat{H}_A(\mathbf{X})] = g X^{ax} \hat{H}_A(\mathbf{X})$$ (25)

can be realized on the linear space of extended abelian loop functions $\psi(X^{ax})$. The extended wavefunctions will be connected with the states in the connection representation by means of a generalized loop transform

$$\psi(\mathbf{X}) = \int d\mu[A] \psi[A] \exp(-ig \int d^3x A_a(x) X^{ax})$$ (26)

The operators $\hat{H}_A(\mathbf{X})$ and $\hat{E}^{ax}$ are realized in the extended space, as was done before in the loop representation. We get

$$\hat{H}_A(\mathbf{X}_0) \psi(\mathbf{X}) = \psi(\mathbf{X} + \mathbf{X}_0)$$ (27)

$$\hat{E}^{ax} \psi(\mathbf{X}) = g X^{ax} \psi(\mathbf{X})$$ (28)

The action of any other gauge invariant operator may be again deduced from them. The magnetic field operator takes the form

$$\hat{F}_{ab}(x) \psi(\mathbf{X}) = \frac{i}{g} \partial_{[a} \frac{\delta}{\delta X^{b]x}} \psi(\mathbf{X})$$ (29)
and the Hamiltonian operator is given by

\[ \hat{H} \psi(X) = \int d^3x \left[ \frac{g^2}{2} \hat{X}^a x \hat{X}^a x + \frac{1}{4g^2} (\partial_a \hat{P}_b x)^2 \right] \psi(X) \]  

(30)

where

\[ \hat{P}_{bx} = i \frac{\delta}{\delta X^{bx}} \]  

(31)

In the abelian case the extended representation is nothing but an electric field representation of electromagnetism. It is straightforward to see, that using (28) and (29) and observing that \( g \hat{A}_a(x) = \hat{P}_a(x) \), one reobtains the standard form of the Hamiltonian.

The transforms (26) and (26) allow to define a one to one correspondence between loop dependent solutions \( \psi(\gamma) = \psi[X(\gamma)] \) and the extended solutions \( \psi(X) \). The correspondence arises simply by substituting \( X(\gamma) \) by \( X \). For instance the vacuum of the electromagnetic theory in the loop representation takes the form

\[ \psi(\gamma) = \exp \left( -\frac{g^2}{2} \int_{\gamma} dy^a \int_{\gamma} dy'^a D_1(y - y') \right) \]  

(32)

where \( D_1 \) is the homogeneous symmetric propagator, while the corresponding state in the extended representation is

\[ \psi(X) = \exp \left( -\frac{g^2}{2} \int d^3x \int d^3y X^{ax} x \hat{X}^{ay} D_1(x - y) \right) \]  

(33)

where now \( X^{ax} \) is a general divergence free object. While the exponent of the loop dependent state is singular and needs to be regularized the wave function in the extended space is perfectly well defined when evaluated on smooth functions. An analogous situation appears in the non abelian gauge theories and quantum gravity where the use of the extended loop representation allows to get rid of ambiguities introduced by the regularization procedure. As shown by Ashtekar and Isham [12], the loop algebra (17) has several kinds of representations. In loop space, it is possible to introduce an inner product in terms of the \( X \) variables that allows to get a Fock representation of the theory. This representation gives a correct description of photons or gravitons in the linearized gravity case, which amounts to consider implicitly the extended representation. One can also consider another inequivalent representation which is defined in the space of functions of loops which are normalizable with respect to a discrete measure. This new Fock representation seems to describe the quantum states of a Type II superconductor and it should be considered as the most natural representation when one works with true loops.

Notice that once we have the quantum form of the Hamiltonian written in terms of the elements of the extended group (in this case in terms of the elements of the abelian extended group) it is straightforward to write a classical action associated
with the theory. The classical action associated with the hamiltonian (30) is

$$\mathcal{S} = \int dt \left\{ P_{ax} \dot{X}^{ax} - \left[ \frac{g^2}{2} X^{ax} X^{ax} + \frac{1}{4g^2} (\partial_0 P_{b|x})^2 \right] + \lambda_x X^{ax}_{,a} \right\}$$

(34)

that may be simply recognized as the usual action of electromagnetism if we impose

$$gX^{ax} = E^a(x) ; \quad \partial_0 E^a(x) = 0$$

(35)

and

$$P_{ax} = gA_a(x) ; \quad \lambda_x = A_0(x)$$

(36)

When this action is restricted to loops

$$X^{ax} = X^{ax}(\gamma) = \int_\gamma dy^a \delta(x - y)$$

(37)

one gets

$$\mathcal{S} = \int dt \left\{ \int_{\gamma_t} dy^a \dot{A}_a(y) + \frac{1}{2} \int d^3 x F_{ab}(x) F_{ab}(x) + g^2 \int_{\gamma_t} dy^a \int_{\gamma_t} dy'^a f_\epsilon(y - y') \right\}$$

(38)

where \(f_\epsilon\) is a regularization of the \(\delta\) function and the loops \(\gamma_t\) belongs to the surface \(t = \text{constant}\).

We may go to the second order form of the action. We get, ignoring the regularization

$$\mathcal{S} = \frac{g^2}{2} \int dt \int d^3 x \int d^3 y \left\{ -\dot{X}^{ax}(\gamma) \frac{1}{4\pi |x - y|} \dot{X}^{ay}(\gamma) - X^{ax}(\gamma) \delta(x - y) X^{ay}(\gamma) \right\}$$

(39)

where

$$\dot{X}^{ax}(\gamma) = \lim_{\delta t \to 0} \frac{X^{ax}(\gamma_{t+\delta t}) - X^{ax}(\gamma_t)}{\delta t}$$

(40)

As we have already noticed, this action is singular in the continuum and needs to be regularized. In the lattice we shall see that it is perfectly well defined and leads, via the transition matrix, to the usual loop representation of electromagnetism.

The extended representation for a nonabelian gauge theory may be obtained following a similar procedure as in the abelian case. In the nonabelian case the wavefunctions \(\psi(X)\) defined on the extended group that satisfies the differential constraint \(\mathcal{D}_0\) will depend on the multivector component \(\vec{X}\) of any rank. The hamiltonian may be written in terms of \(X^{\mu_1...\mu_n}\) and its conjugate momentum \(P_{\mu_1...\mu_n}\), given by

$$P_{\mu_1...\mu_n} = i \frac{\delta}{\delta X^{\mu_1...\mu_n}}$$

(41)

From the hamiltonian on can read the corresponding classical action written in terms of the canonical variables \(X^{\mu_1...\mu_n}, P_{\mu_1...\mu_n}\). The restriction of this extended action to multitangent fields

$$X^{\mu_1...\mu_n} = X^{\mu_1...\mu_n}(\gamma)$$

(42)
leads to an action in terms of loops for the Yang Mills theory. The physical meaning of this action is still unclear, but is natural to expect that it might be relevant to the semiclassical description of the physical excitations in the confining phase.

To prove the consistency of this approach we will now quantize the classical action (39) in the lattice using the transfer matrix formalism [13], [14]. We will first proceed to obtain a lattice version of (39). In a cubic lattice with \( N^3 \) sites the coordinates become

\[
x_i = a n_i
\]

where \( a \) is the lattice spacing and \(-\frac{N}{2} < n_i < \frac{N}{2}\).

The Laplacian operator can be easily inverted using the Fourier transform. The discrete Fourier transform of \( \dot{X}^a_x \) is

\[
\dot{X}^a_x(n) = \frac{1}{N^3 a^3} \sum_{\vec{q}} e^{-i\vec{q} \cdot \vec{n} a} \tilde{\dot{X}}^a_x(\vec{q})
\]

and its inverse

\[
\tilde{\dot{X}}^a_x(\vec{q}) = \sum_{\vec{n}} e^{i\vec{q} \cdot \vec{n} a} \dot{X}^a_x(n)
\]

where the \( \vec{q} \) components run over the first Brillouin zone \(-\frac{\pi}{a} < q_i \leq \frac{\pi}{a}\). Using the discrete form of the Laplacian operator we get

\[
\frac{1}{\nabla^2} \dot{X}^a_x(n) = -\sum_{\vec{n}'} f(\vec{n} - \vec{n}') \dot{X}^a_x(n')
\]

where

\[
f(\vec{n} - \vec{n}') = \frac{a^2}{2N^3} \sum_{\vec{q}} \sum_{i=1}^{3} \frac{e^{-i\vec{q} \cdot (\vec{n} - \vec{n}')}}{1 - \cos(q_i a)}
\]

The temporal derivative of \( X^ax \) is given by

\[
\dot{X}^a_x(n) = \frac{1}{\tau} \left( X^a_x(n) - X^a_x(n) \right) = \frac{1}{\tau} X^a_{\gamma - \gamma}(\vec{n})
\]

where \( \gamma \) and \( \gamma' \) are loops in the spatial surfaces at the times \( t \) and \( t + \tau \), and \( \gamma' - \gamma = \gamma' \circ \vec{\gamma} \) where \( \vec{\gamma} \) is the loop \( \gamma \) with the opposite orientation. Notice that this result holds because the “loop coordinates” \( X^ax(\gamma) \) are linear functionals of loops in the abelian case.

On the lattice, the first term of the classical electromagnetic action reads

\[
\int d^3 x \, \dot{X}^ax_{\gamma} \frac{1}{\tau^2} \dot{X}^ax_{\gamma} \rightarrow \frac{a^3}{\tau^2} \sum_{\vec{n}} \sum_{\vec{n}'} f(\vec{n} - \vec{n}') X^a_{\gamma' - \gamma}(\vec{n}) X^a_{\gamma' - \gamma}(\vec{n}')
\]

with

\[
X^a_{\gamma}(\vec{n}) = \frac{1}{a^2} \sum_{\ell' \in \gamma} \vec{\theta}_{\ell \ell'}
\]
where \( \ell \) is the link with origin \( \vec{n} \) and orientation \( v \), and

\[
\delta_{\ell\ell'} = \begin{cases} 
1 & \text{for } \ell = \ell' \\
-1 & \text{for } \ell = \vec{n}' \\
0 & \text{otherwise}
\end{cases}
\]  

(51)

The second term takes the form

\[
\int d^3x X^a_\gamma X^{ax}_\gamma \rightarrow \frac{1}{a} \sum_{\ell \in \gamma} \sum_{\ell' \in \gamma} \delta_{\ell\ell'} = \frac{1}{a} \Lambda_\gamma
\]  

(52)

where \( \Lambda_\gamma \) is the quadratic length of the loop \( \gamma \). Doing a Wick rotation, we obtain for the euclidean lattice action the following expression

\[
S_E = K_\tau \sum_{n_0} D(\gamma' - \gamma) + K \sum_{n_0} \Lambda_\gamma
\]  

(53)

where \( n_0 \) is the discrete time variable, \( \gamma \) and \( \gamma' \) are loops in the spatial surfaces at the times \( n_0\tau \) and \( (n_0 + 1)\tau \), and

\[
K_\tau = \frac{g^2 a^3}{2\tau}, \quad K = \frac{g^2 \tau}{2a}
\]  

(54)

where \( g_\tau = g \) when \( \tau = a \) and with

\[
D(\gamma' - \gamma) = \sum_{\vec{n}} \sum_{\vec{n}'} f(\vec{n} - \vec{n}')X^a_\gamma X^a_{\gamma' - \gamma}(\vec{n})X^a_{\gamma' - \gamma}(\vec{n}')
\]  

(55)

Let us now introduce the partition function

\[
Z = \int_{-\infty}^{+\infty} [dX] e^{-S_E}
\]  

(56)

where the integral is done over all the configurations of the abelian loop coordinates \( X^{ax}(\gamma) \). Since the euclidean action in the lattice only involves the loop coordinates at \( n_0\tau \) and \( (n_0 + 1)\tau \), we can use the transfer matrix formalism and write the partition function as

\[
Z = \int_{-\infty}^{+\infty} [dX] \prod_{n_0} \langle X_{\gamma'} \mid \hat{T} \mid X_{\gamma} \rangle
\]  

(57)

where as before \( \gamma \) and \( \gamma' \) are loops in the spatial surfaces at the times \( n_0\tau \) and \( (n_0 + 1)\tau \) respectively.

The operator \( \hat{T} \) acts in the space of loop coordinates and its matrix elements are

\[
\langle X_{\gamma'} \mid \hat{T} \mid X_{\gamma} \rangle = T(\gamma', \gamma)
\]

\[
= \exp \left( -K_\tau \sum_{\vec{n}} \sum_{\vec{n}'} f(\vec{n} - \vec{n}')X^a_{\gamma' - \gamma}(\vec{n})X^a_{\gamma' - \gamma}(\vec{n}') - K\Lambda_\gamma \right)
\]  

(58)
For a small temporal lattice spacing $\tau$ we have
\[ \hat{T} = e^{-\tau\hat{H}} \approx 1 - \tau\hat{H} \] (59)
where $a$ is held fixed and $\hat{H}$ is the Hamiltonian operator formulated on a spatial lattice.

Next we must determine how $K_{\tau}$ and $K$ should be adjusted so that the transfer matrix elements take the form (58). The diagonal elements of the matrix $T(\gamma', \gamma)$ are obtained when the loops at times $n_0 \tau$ and $(n_0 + 1)\tau$ are the same (that is to say, $\gamma' - \gamma$ is equivalent to the null path)
\[ T(\gamma, \gamma) = e^{-K\Lambda_\gamma} \approx 1 - \tau\hat{H} \mid_{\gamma,\gamma} \] (60)
As we shall see below, the largest contribution of the off-diagonal terms is obtained when the difference between the loops at times $n_0 \tau$ and $(n_0 + 1)\tau$ is a plaquette with positive or negative orientation ($\gamma' - \gamma = \Box$ or $\gamma' - \gamma = \Box$)
\[ T(\gamma \circ \Box, \gamma) = e^{-K_{\tau}D(\Box)}e^{-K\Lambda_\gamma} \approx -\tau\hat{H} \mid_{\gamma \circ \Box, \gamma} \] (61)
\[ \hat{T}(\gamma \circ \Box, \gamma) = e^{-K_{\tau}D(\Box)}e^{-K\Lambda_\gamma} \approx -\tau\hat{H} \mid_{\gamma \circ \Box, \gamma} \] (62)
For the case of $\gamma' - \gamma = \Box$ the sum (54) is reduced to
\[ D(\Box) = \frac{2(N + 1)^3}{3a^2N^3} \] (63)
and for $N \gg 1$ one can see (for the (2+1)-dimensional case) that
\[ D(\gamma' - \gamma) \approx m D(\Box) \] (64)
where $m$ is the number of plaquettes of the loop $\gamma' - \gamma$. In the (3+1)-dimensional case we get
\[ D(\gamma' - \gamma) \approx L \] (65)
where $L$ is the length of the loop $\gamma' - \gamma$. From Eqs. (41), (34) and (32) we see that the following conditions must be required
\[ K \approx \tau \]
\[ e^{-K_{\tau}D(\Box)} \approx \tau \] (66)
The others matrix elements do not survive since according to (34) or (33)
\[ T(\gamma', \gamma) = e^{-K_{\tau}D(\gamma' - \gamma)}e^{-K\Lambda_\gamma} \approx e^{-mK_{\tau}D(\Box)}e^{-K\Lambda_\gamma} = O(\tau^m) \] (67)
Then from Eq. (30) we must have
\[ K = \lambda e^{-K_{\tau}D(\Box)} \] (68)
If we identify the temporal lattice spacing as
\[ \tau = a e^{-K_D} \]
we find
\[ K = \frac{\lambda}{a} \tau \]
From Eqs. (61), (61) and (62) we conclude that
\[ \langle \gamma | \hat{H} | \gamma \rangle = \frac{\lambda}{a} \Lambda_\gamma \]
\[ \langle \gamma \circ \Box | \hat{H} | \gamma \rangle = -\frac{1}{a} \]
\[ \langle \gamma \circ \Box | \hat{H} | \gamma \rangle = -\frac{1}{a} \]
From the above results, the following expression for the hamiltonian operator in the lattice is obtained
\[ \hat{H} = -\frac{1}{a} \sum_{\Box} \left( W(\Box) + W^\dagger(\Box) \right) + \frac{\lambda}{a} \hat{E} \]
where the action of \( W(\gamma) \) and \( \hat{E} \) are defined by the following expressions
\[ W(\gamma') | \gamma \rangle = | \gamma' \circ \Box \rangle \]
\[ W^\dagger(\gamma') | \gamma \rangle = | \gamma' \circ \gamma \rangle \]
\[ \hat{E} | \gamma \rangle = \Lambda_\gamma | \gamma \rangle \]

The hamiltonian (72) is nothing else that the familiar Kogut-Susskind hamiltonian for electromagnetism in the loop representation. therefore we have shown that the electromagnetic loop action when quantized in the lattice leads to the usual Kogut-Susskind.

We conclude with some general remarks. The principal objective of this letter was to show how a classical action in terms of loops can be derived for electromagnetism and implemented in the lattice. This loop action when quantize in the lattice leads to the usual Kogut-Susskind hamiltonian. We have now at our disposal a gauge invariant action, that may serve as the starting point of Montecarlo calculations of electromagnetism in terms of loops. One has the combined power of montecarlo methods with the geometric advantages of the loop representation. In particular, all the techniques developed in the hamiltonian loop representation of gauge theories [15] can now be applied in the statistical approach. In the case of 2+1 dimensions, the loop action has only two terms, one proportional to the length of the loop and the other to the difference area of the loops at time \( t \) and \( t + \tau \). Moreover, when one considers the regularized version of this action in the continuum, it does not lead to the usual Maxwell equations but to the equations for the electromagnetic interactions among confined electric or magnetic lines of flux in a Type II superconductor. We wish to thank, A. Aroca, C. Di Bartolo, H. Fort and J. Pullin for useful discussions and comments.
References

[1] S. Mandelstam, Ann. Phys. (NY) 19, 1 (1962).

[2] Yu. M. Makeenko, A. A. Migdal, Phys. Lett. B88, 135 (1979).

[3] R. Gambini and A. Trias, Nucl. Phys. B238, 436 (1986).

[4] C. Rovelli and L. Smolin, Phys. Rev. Lett. 61, 1155 (1988).
   C. Rovelli and L. Smolin, Nucl. Phys. B331, 80 (1990).

[5] R. Gambini, Phys. Lett. B 255, 180 (1991).
   R. Gambini, B. Brugmann and J. Pullin, Phys. Rev. Lett. 68, 431 (1992).
   B. Brugmann, R. Gambini and J. Pullin, Nucl. Phys. B385, 587 (1992).

[6] R. Gambini, A. Trias, Phys. Rev. D23, 553 (1981).

[7] J. Barrett, Int. J. Theor. Phys. 30, 1171 (1991).

[8] R. Gambini, A. Trias, Phys. Rev. D22, 1380 (1980).

[9] C. Di Bartolo, R. Gambini and J. Griego, The Extended Loop Group: an infinite
   dimensional manifold associated with loop space, to appear in Comm. Math. Phys.

[10] C. Di Bartolo, F. Nori, R. Gambini, A. Trias, Lett. Nuovo Cim 38, 497 (1983).

[11] A. Ashtekar, C. Rovelli, Class. Quant. Grav. 9, 1121 (1992).

[12] A. Ashtekar, C. Isham, Phys. Lett. B274, 393 (1992).

[13] J. B. Kogut, Rev. Mod. Phys. 51, 659 (1979).

[14] M. Creutz Quarks, gluons and lattices, Cambridge University Press, 1985.

[15] R. Gambini, L. Leal and A. Trias, Phys. Rev. D 39, 3127 (1989).
   C. Di Bartolo, R. Gambini and A. Trias, Phys. Rev. D 39, 3136 (1989).
   D. Arman-D-Ugon and R. Gambini Comp. Phys. Comm. 72, 29 (1992).
   H. Fort and R. Gambini, Phys. Rev. D 44, 1257 (1991).
   D. Arman-D-Ugon and Hugo Fort, Physics Letters B 282, 428 (1992).
   J. M. Aroca and H. Fort, Phys. Lett. B 299, 305 (1993).
   J. M. Aroca and H. Fort, Phys. Lett. B 299, 305 (1993).
   J. M. Aroca and H. Fort, preprint UAB-FT-301.