Dynamics of the 2D two-component plasma near the Kosterlitz-Thouless transition

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Abstract. We study the dynamics of a classical, two-component plasma in two dimensions, in the vicinity of the Kosterlitz-Thouless (KT) transition where the system passes from a dielectric low-temperature phase (consisting of bound pairs) to a conducting phase. We use two “complementary” analytical approaches and compare to simulations. The conventional, “intuitive” approach is built on the KT picture of independently relaxing, bound pairs. A more formal approach, working with Mori projected dynamic correlation functions, avoids to assume the pair picture from the start. We discuss successes and failures of both approaches, and suggest a way to combine the advantages of both.

1. INTRODUCTION

The two-component plasma in two dimensions (2D) is the generic model for topological excitations in 2D systems with a U(1) order parameter symmetry, usually called vortices [1]. Prominent examples of such systems are the “2D superfluids”, superconducting or $^4$He films and Josephson junction arrays (JJAs), others are the XY model of 2D planar magnets, and — with some modifications — 2D melting [2]. The 2D plasma undergoes the well-known Kosterlitz-Thouless (KT) transition at a finite temperature $T_{KT}$.

The dynamic behavior close to this transition is of both principal and practical interest. It is conveniently described by a frequency dependent, complex dielectric function $\epsilon(\omega)$, related to the dynamic correlation function $\Phi_\rho(\mathbf{k}, \omega)$ of the charge density $\rho(\mathbf{r}, t)$ of the plasma:

$$\frac{1}{\epsilon(\omega)} = 1 - \frac{2\pi}{k_B T k^2} \left[ q^2 \bar{n} S_{\rho}(\mathbf{k}) + i \omega \Phi_\rho(\mathbf{k}, \omega) \right] \bigg|_{k \to 0},$$

$$\Phi_\rho(\mathbf{k}, \omega) = \int_0^\infty dt e^{i \omega t} \int d^2 r e^{-i \mathbf{k} \cdot \mathbf{r}} \langle \rho(\mathbf{r}, t) \rho(0, 0) \rangle.$$  (1)

Here $\bar{n}$ is the mean number density of the particles with charge $\pm q$, and $S_{\rho}(\mathbf{k}) = \Phi_\rho(\mathbf{k}, t=0)/q^2 \bar{n}$ is the static charge structure factor, determining the zero frequency limit of $\epsilon(\omega)$. As usual, $\epsilon(\omega)$ is related to the dynamic conductivity $\sigma(\omega)$ by $\epsilon(\omega) = 1 - 2\pi \sigma(\omega)/i \omega$.

Recent numerical simulations by Jonsson and Minnhagen (JM) [3] of the 2D XY model with Ginzburg-Landau dynamics have shed new light onto the critical dynamics of the associated vortex plasma. Using an appropriate definition for the vortex density $\rho(\mathbf{r})$, JM have computed $\Phi_\rho$ and extracted the complex dielectric function $\epsilon(\omega)$ using (1). It shows the following, interesting features:

- $1/\epsilon(\omega)$ approximately has a scaling form with a characteristic frequency $\omega_{ch}$ going to zero as $T \to T_{KT}$ (“critical slowing down”) from both sides.
- For low frequencies, $\text{Re}[1/\epsilon(\omega) - 1/\epsilon(0)] \propto \omega$ over a relatively large $\omega$ interval, contrary to a simple “Drude” behavior ($\propto \omega^2$). This corresponds to a non-analyticity $\sigma(\omega) \propto \ln |\omega|$ at low $\omega$.

JM obtain very good fits of their data by a phenomenological expression proposed earlier [4] by Minnhagen. Experiments on JJAs [3] seem indeed to confirm the non-analytic frequency dependence...
of \(\epsilon(\omega)\), and various theoretical attempts have been made \cite{6} in order to explain this finding. The critical slowing down has, to our knowledge, not been theoretically addressed so far.

2. AHNS PAIR RELAXATION PICTURE

The “standard” dynamic theory of the KT transition has been developed in 1978 by Ambegaokar et al. (AHNS) \cite{6}, who built up the dielectric function \(\epsilon(\omega)\) additively from contributions of pairs of any size and of free particles. We use a slight modification of this idea here, defining by

\[
\frac{1}{\epsilon_b(\omega)} = 1 + \int_0^\xi \frac{d[1/\tilde{\epsilon}(r)]}{dr} \frac{1}{1 - i\omega\tau(r)}, \quad \tau(r) \approx \frac{r^2\tilde{\epsilon}(r)\Gamma}{3.5q^2} \tag{3}
\]

da dielectric function of bound pairs, and setting \(\epsilon(\omega) = \epsilon_b(\omega) - 2\pi q^2 n_f / i\omega \Gamma\). The scale dependent dielectric constant \(\tilde{\epsilon}(r)\) is determined by the KT flow equations \cite{1, 2}, and the factor \(1/[1 - i\omega\tau(r)]\) in the integral describes the relaxation of individual pairs of size \(r\). \(r_0\) is a minimum length of the order of the vortex core size, \(\xi\) is the KT screening length, and \(n_f \approx \xi^{-2}\) the density of free particles. To determine \(\tilde{\epsilon}(r)\), we use Minnhagen’s \cite{1} version of the KT flow equations for the scale dependent “reduced temperature” \(\tau(r) = 1 - q^2/4\tilde{\epsilon}(r)\kB T\) and fugacity \(y(r)\). Their solutions are known analytically, in the regions \(T < T_{KT}\) and \(T > T_{KT}\) separately \cite{1}. The integral (3) is done numerically.

We find that the frequency at which \(\text{Im}[1/\epsilon(\omega)]\) has its maximum is more or less temperature independent below \(T_{KT}\), whereas above it varies rapidly like \(\xi(T)^{-2}\). Both above and sufficiently far below \(T_{KT}\), the low-\(\omega\) behavior of \(\text{Re}[1/\epsilon]\) is closer to linear than a simple Drude form. However, above \(T_{KT}\) there finally is a crossover to “Drude” \(\omega^2\) behavior as \(\omega \to 0\), and below \(T_{KT}\) both real and imaginary part of \(1/\epsilon(\omega) - 1/\epsilon(0)\) behave as \(\omega^\alpha\) at low \(\omega\), with an exponent \(\alpha = \frac{q^2}{2\epsilon(0)kB T} - 2\) which goes to zero as \(T \to T_{KT}\). If normalized by a factor of \(\tilde{\epsilon}(\xi)\), our data above \(T_{KT}\) display an almost perfect scaling behavior with a characteristic frequency \(\omega_{ch}(\xi) \approx 10q^2/\xi^2\tilde{\epsilon}(\xi)\Gamma\), over an extremely wide range in \(\xi\) (Fig. 1L). We find no comparable scaling below \(T_{KT}\), but if \(\omega_{ch}\) is defined here by the condition \(\text{Re}[1/\epsilon(\omega_{ch}) - 1/\epsilon(0)] = -\frac{q^2}{2\epsilon(0)kB T} \text{Im}[1/\epsilon(\omega_{ch})]\) (as done by JM), it decreases rapidly as \(T \to T_{KT}\) from both sides. In Fig. 1R we have plotted the corresponding temperature dependences of \(\omega_{ch}\) both above and below \(T_{KT}\), which is in good qualitative agreement with Fig. 6 of JM \cite{6}.
Figure 2: Left: Real and imaginary parts of the reduced charge friction function \( \hat{\gamma}(\hat{\omega}) \), calculated from Eq. (7) with \( A = 2 \). Right: The resulting \( 1/\epsilon(\omega) \) (full curves) compared to the phenomenology of JM (dashed) and to a Drude Form (dotted), with maxima of \(-\text{Im}\) at the same frequency \( (\hat{\omega} \approx 0.22) \) and amplitudes adjusted.

3. CALCULATIONS USING MORI’S TECHNIQUE

Mori’s technique [7] for evaluating dynamic correlation functions yields for the charge density correlator at small wave numbers \( k \):

\[
\Phi_{\rho}(k, \omega) \sim \frac{q^2 \bar{n} S_{\rho}(k)}{-1\omega + \frac{k_B T k^2}{S_{\rho}(k) \gamma_{\rho}(\omega)}}. \tag{4}
\]

The “memory function” (or “random force” correlation function) \( \gamma_{\rho}(\omega) \) obeys

\[
\gamma_{\rho}(\omega) \approx \Gamma + \frac{q^2}{2k_B T \bar{n} V^3} \sum_{k l} |k\cdot l| U_k U_l \int_0^\infty dt e^{i\omega t} \langle \delta n_{-k}(t) \rho_{k}(t) \delta n_{l}(0) \rho_{-l}(0) \rangle. \tag{5}
\]

where \( \delta n_k(t) \) denotes the deviation of the local number density from its mean value, and \( V \) is the system volume. \( \gamma_{\rho}(\omega) \) generalizes the bare friction constant \( \Gamma \), adding a contribution due to the particle interaction.

In the spirit of current practice in the theory of liquid dynamics [7] we factorize, as a first step, the combined correlator into a product of a number and a charge correlation function:

\[
\langle \delta n_{-k}(t) \rho_{k}(t) \delta n_{l}(0) \rho_{-l}(0) \rangle \approx V^2 \delta_{kl} \Phi_{n}(-k, t) \Phi_{\rho}(k, t). \tag{6}
\]

Using \( \Phi_{n}(k, \omega) \approx \bar{n}/(-i\omega + \frac{k_B T k^2}{\bar{n}}) \) and \( S_{\rho}(k) \approx k^2/(k^2 + b^{-2}) \) (the length \( b \) is a measure of the interparticle distance) we obtain the implicit equation

\[
\hat{\gamma}(\hat{\omega}) \approx 1 + \frac{A}{1 + i\omega} \ln \frac{1 + \hat{\gamma}(\hat{\omega})}{1 - i\hat{\omega}\hat{\gamma}(\hat{\omega})}, \tag{7}
\]

determining the function \( \hat{\gamma}(\hat{\omega}) = \gamma_{\rho}(\omega)/\Gamma \) which depends on \( \hat{\omega} = \omega/\omega_{\text{ch},1} \). The characteristic frequency \( \omega_{\text{ch},1} = k_B T/b^2 \Gamma \) reflects the spatial density of particles, and \( A = \pi q^4 n b^2 / 2(k_B T)^2 \) is a dimensionless parameter of order unity.

Fig. 2 L shows real and imaginary parts of the solution of (7) above \( T_{\text{KT}} \), where \( \sigma(\omega) \propto \gamma_{\rho}(\omega)^{-1} \). Re\( \hat{\gamma} \) shows the expected (see Sec. 1) logarithmic \( \omega \) dependence in some region, but has a finite limit for \( \omega \to 0 \). As seen in Fig. 2 R, our mode decoupling approach yields a dielectric function somewhat closer to the phenomenology of JM than a pure Drude form. However, the critical slowing down is
missing, since the factorization approximation \(6\) underestimates effects of charge binding into pairs on length scales \(\xi\).

A simple — and admittedly ad hoc — way to build pairing into the dynamic friction function consists in adding to the r.h.s. of (6) a slowly decaying term \(\propto e^{-t/\tau_{\text{esc}}}\), corresponding to a contribution \(\gamma_{\text{pairs}}(\omega) = \gamma_0/[\tau_{\text{esc}}^{-1} - i\omega]\) to \(\gamma_{\text{p}}(\omega)\) and describing the random force correlations of pairs that unbind after a typical time \(\tau_{\text{esc}}\). We interpret \(\tau_{\text{esc}}\) as the “thermal escape time” out of the logarithmic pair potential, screened at a distance \(\xi\):

\[
\tau_{\text{esc}}(\xi) \propto r_0^2 T\left(\frac{\xi}{r_0}\right)^z, \quad z = \frac{q^2}{k_B T} \tag{8}
\]

The dynamic exponent \(z \approx 4.7\) in the vicinity of the KT transition. Studying the correlations in an ensemble of independent pairs with average squared size \(\langle \Delta r^2 \rangle \approx 4b^2\), the amplitude of \(\gamma_{\text{pairs}}\) can be estimated as \(\gamma_0 \approx q^4/k_B T \langle \Delta r^2 \rangle\). The new term \(\gamma_{\text{pairs}}\) dominates \(\gamma_{\text{p}}\) for large \(\xi\) and low \(\omega\), and so the static limit of the solution diverges as \(\gamma_{\text{p}}(\omega=0) \propto \tau_{\text{esc}}(\xi) \propto \xi^z\). As a consequence, both the new characteristic frequency \(\omega_{\text{ch},2} = \tau_{\text{esc}}^{-1}\) and the static conductivity \(\sigma(0) = q^2 n/\gamma_{\text{p}}(0)\) of the plasma vanish smoothly as \(\xi(T)^{-z}\). The prediction \(z \approx 4.7\) is in striking contrast to the scaling with \(z = 2\) in Fig. 1 L obtained on the basis of AHNS theory, and to the usual expectation that \(\sigma(0) \approx q^2 n_f/\Gamma \sim \xi^{-2}\). We note, however, that recent dynamical scaling analyses \(8\) for different realizations of 2D superfluids actually do suggest anomalously large dynamical exponents in the range \(z \approx 4.5 \ldots 5.9\).

4. CONCLUSIONS

We have studied here the critical dynamics of a 2D, two-component plasma in the vicinity of the KT phase transition, using a modification of AHNS theory and a Mori type approach. The AHNS approach is quite successful as a convenient phenomenological interpretation of simulation data, but conceptually dissatisfactory in that pairs are put in “by hand”. The Mori approach is more fundamental, starting from microscopic equations of motion, but the correlations it predicts (within a mode decoupling approximation) are not strong enough to imply critical slowing down. We have proposed a way to cure this deficiency by introducing additional, long-lived random force correlations, describing temporary pairing. A virtue of this extension is that it contains two frequency scales, one finite and the other vanishing at \(T_{\text{KT}}\), in agreement with the numerical findings.

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