DIFFERENTIAL GRADED VERSUS SIMPLICIAL CATEGORIES

GONÇALO TABUADA

Abstract. We construct a zig-zag of Quillen adjunctions between the homotopy theories of differential graded and simplicial categories. In an intermediate step we generalize Shipley-Schwede’s work [21] on connective DG algebras by extending the Dold-Kan correspondence to a Quillen equivalence between categories enriched over positive graded chain complexes and simplicial $k$-modules. As an application we obtain a conceptual explanation of Simpson’s homotopy fiber construction [22].

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1. Introduction

A differential graded (=dg) category is a category enriched in the category of complexes of modules over some commutative base ring $k$. Dg categories provide a framework for ‘homological geometry’ and for ‘non commutative algebraic geometry’ in the sense of Drinfeld and Kontsevich [4] [5] [14] [15] [16]. In [23] the homotopy theory of dg categories was constructed. This theory was allowed several developments such as: the creation by Toën of a derived Morita theory [24]; the construction of a category of non commutative motives [23]; the first conceptual characterization of Quillen-Waldhausen’s $K$-theory [23]...

On the other hand a simplicial category is a category enriched over the category of simplicial sets. Simplicial categories (and their close cousins: quasi-categories) provide a framework for ‘homotopy theories’ and for ‘higher category theory’ in the sense of Joyal, Lurie, Rezk, Toën ... [11] [17] [20] [25]. In [11] Bergner constructed a...

Key words and phrases. Dg category, Simplicial category, Dold-Kan correspondence, Quillen model structure, Eilenberg-MacLane’s shuffle map.
homotopy theory of simplicial categories by fixing an error in a previous version of [6]. This theory can be considered as one of the four Quillen models for the theory of \((\infty, 1)\)-categories, see [2] for a survey.

We observe that the homotopy theories of differential graded and simplicial categories are formally similar and so a ‘bridge’ between the two should be developed. In this paper we establish the first connexion between these theories by constructing a zig-zag of Quillen adjunctions relating the two:

In first place, we construct a Quillen model structure on positive graded dg categories by ‘truncating’ the model structure of [23], see theorem 4.7.

Secondly we generalize Shipley-Schwede’s work [21] on connective DG algebras by extending the Dold-Kan correspondence to a Quillen equivalence between categories enriched over positive graded chain complexes and simplicial \(k\)-modules, see theorem 5.19.

Finally we extend the \(k\)-linearization functor to a Quillen adjunction between simplicial categories and simplicial \(k\)-linear categories.

As an application, the zig-zag of Quillen adjunctions obtained allow us to give a conceptual explanation of Simpson’s homotopy fiber construction [22] used in his nonabelian mixed Hodge theory.

2. Acknowledgments

I am deeply grateful to Gustavo Granja for several useful discussions and for his kindness.

3. Preliminaries

In what follows, \(k\) will denote a commutative ring with unit. The tensor product \(\otimes\) will denote the tensor product over \(k\). Let \(\mathbf{Ch}\) denote the category of complexes over \(k\) and \(\mathbf{Ch}_{\geq 0}\) the full subcategory of positive graded complexes. Throughout this article we consider homological notation (the differential decreases the degree).

Observe that \(\mathbf{Ch}_{\geq 0}\) is a full symmetric monoidal subcategory of \(\mathbf{Ch}\) and that the inclusion \(\mathbf{Ch}_{\geq 0} \hookrightarrow \mathbf{Ch}\) commutes with limits and colimits.

We denote by \(\mathbf{Ch}_{\geq 0}(\_, \_)\) the internal Hom-functor in \(\mathbf{Ch}_{\geq 0}\) with respect to \(\otimes\).

By a \textit{dg category}, resp. \textit{positive graded dg category}, we mean a category enriched over the symmetric monoidal category \(\mathbf{Ch}\), resp. \(\mathbf{Ch}_{\geq 0}\), see [1] [12] [13] [23]. We denote by \(\mathbf{dgcat}\), resp. \(\mathbf{dgcat}_{\geq 0}\), the category of small dg categories, resp. small positive graded dg categories.

Notice that \(\mathbf{dgcat}_{\geq 0}\) is a full subcategory of \(\mathbf{dgcat}\) and the inclusion \(\mathbf{dgcat}_{\geq 0} \hookrightarrow \mathbf{dgcat}\) commutes with limits and colimits.

Let \(\mathbf{sSet}\) be the symmetric monoidal category of simplicial sets and \(\mathbf{sMod}\) the category of simplicial \(k\)-modules. We denote by \(\wedge\) the levelwise tensor product of simplicial \(k\)-modules. The category \((\mathbf{sMod}, - \wedge -)\) is a closed symmetric monoidal category. We denote by \(\mathbf{sMod}(\_, \_)\) its internal Hom-functor.

By a \textit{simplicial category}, resp. \textit{simplicial \(k\)-linear category}, we mean a category enriched over \(\mathbf{sSet}\), resp. \(\mathbf{sMod}\), see [1].
We denote by $s\text{Set}\text{-Cat}$, resp. $s\text{Mod}\text{-Cat}$, the category of small simplicial categories, resp. simplicial $k$-linear categories.

Let $(\mathcal{C},\otimes,\mathbb{I}_C)$ and $(\mathcal{D},\wedge,\mathbb{I}_D)$ be two symmetric monoidal categories. A \textit{lax monoidal functor} is a functor $F: \mathcal{C} \to \mathcal{D}$ equipped with:

- a morphism $\eta: \mathbb{I}_D \to F(\mathbb{I}_C)$ and
- natural morphisms $\psi_{X,Y}: F(X) \wedge F(Y) \to F(X \otimes Y)$, $X,Y \in \mathcal{C}$

which are coherently associative and unital (see diagrams 6.27 and 6.28 in [3]).

A lax monoidal functor is \textit{strong monoidal} if the morphisms $\eta$ and $\psi_{X,Y}$ are isomorphisms.

Throughout this article the adjunctions are displayed vertically with the left, resp. right, adjoint on the left side, resp. right side.

4. Homotopy theory of positive graded DG categories

In this section we will construct a Quillen model structure on $\text{dgcat}_{\geq 0}$. For this we will adapt to our situation the Quillen model structure on $\text{dgcat}$ constructed in chapter 1 of [23].

Remark 4.1. Chapter 1 of [23] (and the whole thesis) is written using cohomological notation. Throughout this article we are always using homological notation.

We now define the weak equivalences in $\text{dgcat}_{\geq 0}$.

\textbf{Definition 4.2.} A dg functor $F: \mathcal{A} \to \mathcal{B}$ in $\text{dgcat}_{\geq 0}$ is a quasi-equivalence if:

(i) $F(x,y): \mathcal{A}(x,y) \to \mathcal{B}(x,y)$ is a quasi-isomorphism in $\text{Ch}_{\geq 0}$ for all objects $x,y \in \mathcal{A}$ and

(ii) The induced functor $H_0(F): H_0(\mathcal{A}) \to H_0(\mathcal{B})$ is essentially surjective.

\textbf{Notation 4.3.} We denote by $Q_{qe}$ the class of quasi-equivalences in $\text{dgcat}_{\geq 0}$.

Remark 4.4. Notice that the class $Q_{qe}$ consist exactly of those quasi-equivalences in $\text{dgcat}$, see [23, 1.6], which belong to $\text{dgcat}_{\geq 0}$.

In order to build a Quillen model structure on $\text{dgcat}_{\geq 0}$ we consider the generating (trivial) cofibrations in $\text{dgcat}$ which belong to $\text{dgcat}_{\geq 0}$ and introduce a new generating cofibration. Let us now recall these constructions, see section 1.3 in [23].

\textbf{Definition 4.5.} Following Drinfeld [4, 3.7.1] we define $K$ to be the dg category that has two objects 1, 2 and whose morphisms are generated by $f \in K(1,2)_0$, $g \in K(2,1)_0$, $r_1 \in K(1,1)_1$, $r_2 \in K(2,2)_1$ and $r_{12} \in K(1,2)_2$ subject to the relations $d(f) = d(g) = 0$, $d(r_1) = gf - 1_1$, $d(r_2) = fg - 1_2$ and $d(r_{12}) = fr_1 - r_2f$.

\[ \begin{array}{c}
1 \\
\downarrow \quad \quad \downarrow \\
2 \\
\quad r_{12} \\
\quad r_1 \\
\quad g \\
\quad r_2
\end{array} \]

Let $\bar{k}$ be the dg category with one object 3, such that $\bar{k}(3,3) = k$. Let $F$ be the dg functor from $\bar{k}$ to $K$ that sends 3 to 1. Let $\mathcal{B}$ be the dg category with two objects 4 and 5 such that $\mathcal{B}(4,4) = k$, $\mathcal{B}(5,5) = k$, $\mathcal{B}(4,5) = 0$ and $\mathcal{B}(5,4) = 0$. Let $n \geq 1$, $S^{n-1}$ the complex $k[n-1]$ and let $D^n$ be the mapping cone on the identity
of $S^{n-1}$. We denote by $P(n)$ the dg category with two objects 6 and 7 such that
$P(n)(6, 6) = k$, $P(n)(7, 7) = k$, $P(n)(7, 6) = 0$, $P(n)(6, 7) = D^n$ and whose composition given by multiplication.
Let $R(n)$ be the dg functor from $B$ to $P(n)$ that sends 4 to 6 and 5 to 7. Let $C(n)$ be the dg category with
two objects 8 and 9 such that $C(n)(8, 8) = k$, $C(n)(9, 9) = k$, $C(n)(9, 8) = 0$, $C(n)(8, 9) = S^{n-1}$ and whose
composition given by multiplication. Let $S(n)$ be the dg functor from $C(n)$ to $P(n)$ that sends 8 to 6, 9 to 7 and
$S^{n-1}$ to $D^n$ by the identity on $k$ in degree $n - 1$. Let $Q$ be the dg functor from the empty dg category $\emptyset$,
which is the initial object in $\mathsf{dgcat}_{\geq 0}$, to $\emptyset$. Finally let $N$ be the dg functor from $B$ to $C(1)$ that
sends 4 to 8 and 5 to 9.

Let us now recall the following standard recognition theorem:

**Theorem 4.6.** [8, 2.1.19] Let $\mathcal{M}$ be a complete and cocomplete category, $W$ a class
of maps in $\mathcal{M}$ and $I$ and $J$ sets of maps in $\mathcal{M}$ such that:

1. The class $W$ satisfies the two out of three axiom and is stable under retracts.
2. The domains of the elements of $I$ are small relative to $I$-cell.
3. The domains of the elements of $J$ are small relative to $J$-cell.
4. $J - \text{cell} \subseteq W \cap I - \text{cof}.$
5. $I - \text{inj} \subseteq W \cap J - \text{inj}.$
6. $W \cap I - \text{cof} \subseteq J - \text{cof}$ or $W \cap J - \text{inj} \subseteq I - \text{inj}.$

Then there is a cofibrantly generated model category structure on $\mathcal{M}$ in which $W$ is
the class of weak equivalences, $I$ is a set of generating cofibrations, and $J$ is a set
of generating trivial cofibrations.

**Theorem 4.7.** If we let $\mathcal{M}$ be the category $\mathsf{dgcat}_{\geq 0}$, $W$ be the class $\mathcal{Q}_{qe}$, $J$ be
the set of dg functors $F$ and $R(n), n \geq 1$, and $I$ the set of dg functors $Q, N$ and
$S(n), n \geq 1$, then the conditions of the recognition theorem 4.6 are satisfied. Thus,
the category $\mathsf{dgcat}_{\geq 0}$ admits a cofibrantly generated Quillen model structure whose
weak equivalences are the quasi-equivalences.

4.1. **Proof of Theorem 4.7.** We start by observing that the category $\mathsf{dgcat}_{\geq 0}$ is
complete and cocomplete and that the class $\mathcal{Q}_{qe}$ satisfies the two out of three
axiom and that it is stable under retracts. We observe also that the domains and
codomains of the morphisms in $I$ and $J$ are small in the category $\mathsf{dgcat}_{\geq 0}$. This
implies that the first three conditions of the recognition theorem 4.6 are verified.

**Lemma 4.8.** $J - \text{cell} \subseteq \mathcal{Q}_{qe}$.

**Proof.** Since the inclusion

$$\mathsf{dgcat}_{\geq 0} \hookrightarrow \mathsf{dgcat}$$

preserves colimits and the class $\mathcal{Q}_{qe}$ consist exactly of those quasi-equivalences in
$\mathsf{dgcat}$ which belong to $\mathsf{dgcat}_{\geq 0}$, the proof follows from lemma 1.10 in [23].

We now prove that $J - \text{inj} \cap \mathcal{Q}_{qe} = I - \text{inj}$. For this we introduce the following
auxiliary class of dg functors:

**Definition 4.9.** Let $\mathsf{Surj}_{\geq 0}$ be the class of dg functors $G : H \to I$ in $\mathsf{dgcat}_{\geq 0}$ such
that:

- $G(x, y) : H(x, y) \to I(Gx, Gy)$ is a surjective quasi-isomorphism for all
  objects $x, y \in H$ and
- $G$ induces a surjective map on objects.
Remark 4.10. Notice that the class \( \text{Surj}_{\geq 0} \) consist exactly of those dg functors in \( \text{Surj} \), see section 1.3.1 in [23], which belong to \( \text{dgcat}_{\geq 0} \).

**Lemma 4.11.** \( I \rightarrow \text{inj} = \text{Surj}_{\geq 0} \).

**Proof.** We prove first the inclusion \( \supseteq \). Let \( G : \mathcal{H} \rightarrow \mathcal{I} \) be a dg functor in \( \text{Surj}_{\geq 0} \). By remark 4.10 \( G \) belongs to \( \text{Surj} \) and so lemma 1.11 in [23] implies that \( G \) has the right lifting property with respect to the dg functors \( Q \) and \( S(n), n \geq 1 \). Since the morphism of complexes \( G(x, y) : \mathcal{H}(x, y) \rightarrow \mathcal{I}(Gx, Gy), \ x, y \in \mathcal{H} \) is surjective on the degree zero component, the dg functor \( G \) also has the right lifting property with respect to \( N \). This proves the inclusion \( \supseteq \).

We now prove the inclusion \( \subseteq \). Let \( R : \mathcal{C} \rightarrow \mathcal{D} \) be a dg functor in \( I \rightarrow \text{inj} \). Lemma 1.11 in [23] implies that:
- \( R \) induces a surjective map on objects and
- for all objects \( x, y \in \mathcal{C} \):
  - \( R(x, y) : \mathcal{C}(x, y) \rightarrow \mathcal{D}(Rx, Ry) \) is a surjective quasi-isomorphism for \( n \geq 1 \) and
  - \( H_0R(x, y) : H_0\mathcal{C}(x, y) \rightarrow H_0\mathcal{D}(Rx, Ry) \) is an injective map.

Since \( R \) belongs to \( I \rightarrow \text{inj} \) it has the right lifting property with respect to \( N \) and so the morphism of complexes \( R(x, y) \) is also surjective on the degree zero component. This clearly implies that \( R \) belongs to \( \text{Surj}_{\geq 0} \) and proves the inclusion \( \subseteq \).

We now consider the following ‘diagram chasing’ lemma:

**Lemma 4.12.** Let \( f : M_\bullet \rightarrow N_\bullet \) be a morphism in \( \text{Ch}_{\geq 0} \) such that:
- \( f_n : M_n \rightarrow N_n \) is surjective map for \( n \geq 1 \) and
- \( H_n(M_\bullet) \rightarrow H_n(N_\bullet) \) is an isomorphism for \( n \geq 0 \).

Then \( f_0 : M_0 \rightarrow N_0 \) is also a surjective map.

**Proof.** It’s a simple diagram chasing argument.

**Lemma 4.13.** \( J \rightarrow \text{inj} \cap \mathcal{Q}_{\text{qe}} = \text{Surj}_{\geq 0} \).

**Proof.** The inclusion \( \supseteq \) follows from remark 4.10 and from the inclusion \( \supseteq \) in lemma 1.12 of [23]. We now prove the inclusion \( \subseteq \). Let \( R : \mathcal{C} \rightarrow \mathcal{D} \) be a dg functor in \( J \rightarrow \text{inj} \cap \mathcal{Q}_{\text{qe}} \). Since \( R \) belongs to \( \mathcal{Q}_{\text{qe}} \) and it has the right lifting property with respect to the dg functors \( R(n), n \geq 1 \) the morphism of complexes \( R(x, y) : \mathcal{C}(x, y) \rightarrow \mathcal{D}(Rx, Ry), \ x, y \in \mathcal{C} \) satisfies the conditions of lemma 4.12 and so \( R(x, y) \) is a surjective quasi-isomorphism. Finally the fact that \( R \) induces a surjective map on objects follows from lemma 1.12 in [23].

This proves the lemma.

**Lemma 4.14.** \( J \rightarrow \text{cell} \subseteq I \rightarrow \text{cof} \).

**Proof.** Observe that the morphisms in \( J \rightarrow \text{cell} \) have the left lifting property with respect to the class \( J \rightarrow \text{inj} \). By lemmas 4.11 and 4.13 \( I \rightarrow \text{inj} = J \rightarrow \text{inj} \cap \mathcal{Q}_{\text{qe}} \) and so the morphisms in \( J \rightarrow \text{cell} \) have also the left lifting property with respect to the class \( I \rightarrow \text{inj} \), i.e. \( J \rightarrow \text{cell} \subseteq I \rightarrow \text{cof} \).
We have shown that $J = \text{cell} \subseteq Q_{qc} \cap I = \text{cof}$ (lemmas 4.8 and 4.14) and that $I = \text{inj} = J = \text{inj} \cap Q_{qc}$ (lemmas 4.11 and 4.13). This implies that the last three conditions of the recognition theorem 4.6 are satisfied. This finishes the proof of theorem 4.7.

**Remark 4.15.** Since every object in $\text{dgcat}$ is fibrant, see remark 1.14 in [23], and the set $J$ of generating trivial cofibrations in $\text{dgcat}_{\geq 0}$ is a subset of the generating trivial cofibrations in $\text{dgcat}$ we conclude that every object in $\text{dgcat}_{\geq 0}$ is also fibrant.

### 4.2. The truncation functor.

In this subsection we construct a functorial path object in the Quillen model category $\text{dgcat}_{\geq 0}$.

Consider the following adjunction:

$$
\begin{array}{ccc}
\text{Ch} & \overset{\tau_{\geq 0}}{\downarrow} & \text{Ch}_{\geq 0} \\
\mathbb{I} & \overset{i}{\downarrow} & \mathbb{I} \\
\end{array}
$$

where $\tau_{\geq 0}$ denotes the ‘intelligent’ truncation functor: to a complex

$$M_\bullet : \cdots \leftarrow M_{-2} \overset{d_{-1}}{\leftarrow} M_{-1} \overset{d_0}{\leftarrow} M_0 \overset{d_1}{\leftarrow} M_1 \leftarrow \cdots$$

it associates the complex

$$\tau_{\geq 0}M_\bullet : \cdots \leftarrow 0 \leftarrow 0 \leftarrow \text{Ker}(d_0) \overset{d_1}{\leftarrow} M_1 \leftarrow \cdots .$$

The truncation functor $\tau_{\geq 0}$ is a lax monoidal functor. In particular we have natural morphisms

$$\tau_{\geq 0}M_\bullet \otimes \tau_{\geq 0}N_\bullet \longrightarrow \tau_{\geq 0}(M_\bullet \otimes N_\bullet), \quad M_\bullet, N_\bullet \in \text{Ch}$$

which satisfy the associativity conditions. Observe that the truncation functor $\tau_{\geq 0}$ preserve the unit

$$\cdots \leftarrow 0 \leftarrow k \leftarrow 0 \leftarrow \cdots$$

of both symmetric monoidal structures.

**Definition 4.16.** Let $\mathcal{A}$ be a small dg category. The truncation $\tau_{\geq 0}\mathcal{A}$ of $\mathcal{A}$ is the positive graded dg category with the same objects as $\mathcal{A}$ and whose complexes of morphisms are defined as

$$\tau_{\geq 0}\mathcal{A}(x, y) := \tau_{\geq 0}\mathcal{A}(x, y), \quad x, y \in \mathcal{A}.$$ 

For $x, y$ and $z$ objects in $\tau_{\geq 0}\mathcal{A}$ the composition is defined as

$$\tau_{\geq 0}\mathcal{A}(x, y) \otimes \tau_{\geq 0}\mathcal{A}(y, z) \longrightarrow \tau_{\geq 0}(\mathcal{A}(x, y) \otimes \mathcal{A}(y, z)) \overset{\tau_{\geq 0}(c)}{\longrightarrow} \tau_{\geq 0}\mathcal{A}(x, z),$$

where $c$ denotes the composition operation in $\mathcal{A}$. The units in $\tau_{\geq 0}\mathcal{A}$ are the same as those of $\mathcal{A}$.

Observe that we have a natural adjunction

$$
\begin{array}{ccc}
\text{dgcat} & \overset{\tau_{\geq 0}}{\downarrow} & \text{dgcat}_{\geq 0} \\
\mathbb{I} & \overset{i}{\downarrow} & \mathbb{I} \\
\end{array}
$$

**Remark 4.17.** Notice that both functors $i$ and $\tau_{\geq 0}$ preserve quasi-equivalences.
Proposition 4.18. The adjunction \((i, \tau_{\geq 0})\) is a Quillen adjunction.

Proof. Clearly, by remark 4.4 the functor \(i\) preserves weak equivalences. We now show that it also preserves cofibrations. The Quillen model structure of theorem 4.7 is cofibrantly generated and so by proposition 11.2.1 in [7] the class of cofibrations equals the class of retracts of relative \(I\)-cell complexes. Since the functor \(i\) preserves colimits it is then enough to prove that it sends the generating cofibrations in \(\text{dgcat}_{\geq 0}\) to cofibrations in \(\text{dgcat}\). This is clear, by definition, for the generating cofibrations \(Q\) and \(S(n), n \geq 1\). We now observe that \(i(N) = N\) is also a cofibration in \(\text{dgcat}\). In fact \(N\) can be obtained by the following push-out

\[
\begin{array}{ccc}
C(0) & \xrightarrow{P} & B \\
S(0) \downarrow \downarrow & \tau_{\geq 0} & \downarrow N \\
P(0) & \xrightarrow{\sim} & C(1),
\end{array}
\]

where \(S(0)\) is a generating cofibration in \(\text{dgcat}\), see section 1.3 in [23], and \(P\) sends 8 to 4 and 9 to 5. This proves the lemma.

Remark 4.19. Recall from [23, 4.1] the construction of a path object \(P(A)\) for each dg category \(A \in \text{dgcat}\).

Lemma 4.20. Let \(A\) be a positive graded dg category. Then \(\tau_{\geq 0}P(A)\) is a path object of \(A\) in \(\text{dgcat}_{\geq 0}\).

Proof. Consider the diagonal dg functor

\[
\begin{array}{ccc}
A \xrightarrow{\Delta} A \times A \\
\delta \downarrow & \sim & \downarrow P \\
P(A), & & \end{array}
\]

in \(\text{dgcat}\). We have, as in [23, 4.1], a factorization

\[
\begin{array}{ccc}
A \xrightarrow{\Delta} A \times A \\
\tau_{\geq 0}(I) \sim & \tau_{\geq 0}P(A), & \tau_{\geq 0}(P)
\end{array}
\]

where \(I\) is a quasi-equivalence and \(P\) a fibration. By remark 4.17 and lemma 4.18 the functor \(\tau_{\geq 0}\) preserves quasi-equivalences and fibrations. Since the functor \(\tau_{\geq 0}\) also preserves limits we obtain the following factorization

\[
\begin{array}{ccc}
A \xrightarrow{\Delta} A \times A \\
\tau_{\geq 0}(I) \sim & \tau_{\geq 0}P(A) & \tau_{\geq 0}(P)
\end{array}
\]

This proves the lemma.

5. Extended Dold-Kan equivalence

In this section we will first construct a Quillen model structure on \(s\text{Mod-Cat}\) and then show that it is Quillen equivalent to the model structure on \(\text{dgcat}_{\geq 0}\) of theorem 4.7.
Recall from [9, III-2.3] the Dold-Kan equivalence between simplicial \(k\)-modules and positive graded complexes

\[
\begin{array}{ccc}
s \text{Mod} & \xrightarrow{\Gamma} & N \\
\downarrow & & \downarrow \\
\text{Ch}_{\geq 0} & \xrightarrow{\nabla} & N(A \land B), \quad A, B \in s \text{Mod}
\end{array}
\]

where \(N\) is the normalization functor and \(\Gamma\) its inverse. The normalization functor \(N\) is a lax monoidal functor, see [21, 2.3], via the Eilenberg-MacLane shuffle map, see [18, VIII-8.8]

\[
\nabla : NA \otimes NB \longrightarrow N(A \land B), \quad A, B \in s \text{Mod}.
\]

Observe that the normalization functor \(N\) preserves the unit of the two symmetric monoidal structures.

As it is shown in [21, 2.3] the lax monoidal structure on \(N\), given by the shuffle map \(\nabla\), induces a lax comonoidal structure on \(\Gamma\):

\[
\tilde{\psi} : \Gamma(M \otimes M') \longrightarrow \Gamma(M) \land \Gamma(M'), \quad M, M' \in \text{Ch}_{\geq 0}.
\]

Now, let \(I\) be a set.

**Notation 5.1.** We denote by \(\text{Ch}_{\geq 0}^I\)-Gr, resp. by \(\text{Ch}_{\geq 0}^I\)-Cat, the category of \(\text{Ch}_{\geq 0}\)-graphs with a fixed set of objects \(I\), resp. the category of categories enriched over \(\text{Ch}_{\geq 0}\) which have a fixed set of objects \(I\). The morphisms in \(\text{Ch}_{\geq 0}^I\)-Gr and \(\text{Ch}_{\geq 0}^I\)-Cat induce the identity map on the objects.

We have a natural adjunction

\[
\begin{array}{ccc}
\text{Ch}_{\geq 0}^I\text{-Cat} & \xrightarrow{T_I} & \text{Ch}_{\geq 0}^I\text{-Gr} \\
\downarrow U & & \downarrow \\
\text{Ch}_{\geq 0}\text{-Gr} & \xrightarrow{T_I} & \text{Ch}_{\geq 0}\text{-Cat}
\end{array}
\]

where \(U\) is the forgetful functor and \(T_I\) is defined as

\[
T_I(A)(x, y) := \begin{cases} 
  k \oplus \bigoplus_{x, x_1, \ldots, x_n, y} A(x, x_1) \otimes \ldots \otimes A(x_n, y) & \text{if } x = y \\
  \bigoplus_{x, x_1, \ldots, x_n, y} A(x, x_1) \otimes \ldots \otimes A(x_n, y) & \text{if } x \neq y
\end{cases}
\]

Composition is given by concatenation and the unit corresponds to \(1 \in k\).

**Remark 5.2.**

- Notice that the categories \(\text{Ch}_{\geq 0}^I\)-Gr and \(\text{Ch}_{\geq 0}^I\)-Cat admit standard Quillen model structures whose weak equivalences (resp. fibrations) are the morphisms \(F : A \to B\) such that

\[
F(x, y) : A(x, y) \longrightarrow B(x, y), \quad x, y \in I
\]

is a weak equivalence (resp. fibration) in \(\text{Ch}_{\geq 0}\). In fact the projective Quillen model structure on \(\text{Ch}_{\geq 0}\), see [9, III-2], naturally induces a model structure on \(\text{Ch}_{\geq 0}^I\)-Gr which can be lifted along the functor \(T_I\) using theorem 11.3.2 in [7].

- If the set \(I\) has a unique element, then the previous adjunction corresponds to the (Quillen) adjunction between connective dg algebras and positive graded complexes, see [10].
Notation 5.3. We denote by $s\text{Mod}^I$-$\text{Gr}$, resp. by $s\text{Mod}^I$-$\text{Cat}$, the category of $s\text{Mod}$-graphs with a fixed set of objects $I$, resp. the category of categories enriched over $s\text{Mod}$ which have a fixed set of objects $I$. The morphisms in $s\text{Mod}^I$-$\text{Gr}$ and $s\text{Mod}^I$-$\text{Cat}$ induce the identity map on the objects.

In an analogous way we have an adjunction

$$
\begin{array}{ccc}
s\text{Mod}^I$-$\text{Cat} & \overset{T_I}{\rightarrow} & s\text{Mod}^I$-$\text{Gr} \\
\downarrow U & & \downarrow \\
s\text{Mod}^I$-$\text{Cat} & \overset{T_I}{\rightarrow} & s\text{Mod}^I$-$\text{Gr},
\end{array}
$$

where $U$ is the forgetful functor and $T_I$ is defined as

$$
T_I(B)(x,y) := \begin{cases} 
     k\Delta_0 \oplus \bigoplus_{x,x_1,...,x_n,y} B(x,x_1) \land \cdots \land B(x_n,y) & \text{if } x = y \\
     \bigoplus_{x,x_1,...,x_n,y} B(x,x_1) \land \cdots \land B(x_n,y) & \text{if } x \neq y
\end{cases}
$$

Composition is given by concatenation and the unit corresponds to $1 \in k\Delta_0$.

Remark 5.4. If the set $I$ has an unique element, then the previous adjunction corresponds to the classical adjunction between simplicial $k$-algebras and simplicial $k$-modules, see [9, II-5.2].

Clearly the Dold-Kan equivalence induces an equivalence of categories

$$
\begin{array}{ccc}
s\text{Mod}^I$-$\text{Gr} & \overset{N}{\rightarrow} & \text{Ch}_{\geq 0}$-$\text{Gr} \\
\downarrow \Gamma & & \downarrow \\
s\text{Mod}^I$-$\text{Cat} & \overset{N_I}{\rightarrow} & \text{Ch}_{\geq 0}$-$\text{Cat},
\end{array}
$$

that we still denote by $N$ and $\Gamma$.

Since the functor $N : s\text{Mod} \rightarrow \text{Ch}_{\geq 0}$ is lax monoidal it induces, as in [21, 3.3], a normalization functor

$$
\begin{array}{ccc}
s\text{Mod}^I$-$\text{Cat} & \overset{N_I}{\rightarrow} & \text{Ch}^I_{\geq 0}$-$\text{Cat}.
\end{array}
$$

In fact, let $A \in s\text{Mod}^I$-$\text{Cat}$ and $x,y$ and $z$ objects in $A$. Then $N_I(A)$ has the same objects as $A$, the complexes of morphisms are given by

$$
N_I(A)(x,y) := NA(x,y), \quad x,y \in A
$$

and the composition is defined by

$$
NA(x,y) \otimes NA(y,z) \xrightarrow{\nabla} N(A(x,y) \land A(y,z)) \xrightarrow{N(c)} NA(x,z),
$$

where $c$ denotes the composition operation in $A$. The units in $N_I(A)$ are induced by those of $A$ under the normalization functor $N$.

As it is shown in section 3.3 of [21] the functor $N_I$ admits a left adjoint $L_I$. 

Let $\mathcal{A} \in \text{Ch}^I_{\geq 0}$-$\text{Cat}$. The value of the left adjoint $L_I$ on $\mathcal{A}$ is defined as the coequalizer of two morphisms in $\text{sMod}^I$-$\text{Cat}$

$$T_I\Gamma U T_I U(A) \xrightarrow{\psi_1} T_I\Gamma U(A) \xrightarrow{\psi_2} L_I(A).$$

The morphism $\psi_1$ is obtained from the unit of the adjunction $T_I U \rightarrow A$ by applying the composite functor $T_I \Gamma U$; the morphism $\psi_2$ is the unique morphism in $\text{sMod}^I$-$\text{Cat}$ induced by the $\text{sMod}^I$-$\text{Gr}$ morphism $\Gamma U T_I U(A) \rightarrow UT_I \Gamma U(A)$ whose value at $\Gamma U T_I U(A)(x, y)$, $x, y \in I$ is

$$\bigoplus_{x, x_1, \ldots, x_n, y} \Gamma(A(x, x_1) \otimes \ldots \otimes A(x_n, y)) \xrightarrow{\tilde{\psi}} \bigoplus_{x, x_1, \ldots, x_n, y} \Gamma A(x, x_1) \wedge \ldots \wedge \Gamma A(x_n, y),$$

where $\tilde{\psi}$ is the lax comonoidal structure on $\Gamma$ induced by the lax monoidal structure on $N$, see section 3.3 of [21].

5.1. **Left adjoint.** Notice that the normalization functor $N_I : \text{sMod}^I$-$\text{Cat} \rightarrow \text{Ch}^I_{\geq 0}$-$\text{Cat}$, of the previous subsection, can be naturally defined for every set $I$ and so it induces a ‘global’ normalization functor

$$\begin{array}{ccc}
\text{sMod}$-$\text{Cat}$ & \xrightarrow{N} & \text{dgcat}_{\geq 0} \\
\downarrow & & \\
\text{dgc} & & \\
\end{array}$$

In this subsection we will construct the left adjoint of $N$.

Let $\mathcal{A} \in \text{dgcat}_{\geq 0}$ and denote by $I$ its set of objects. Define $L(I)$ as the simplicial $k$-linear category $L_I(A)$.

Now, let $F : \mathcal{A} \rightarrow \mathcal{A}'$ be a dg functor. We denote by $I'$ the set of objects of $\mathcal{A}'$. The dg functor $F$ induces the following diagram in $\text{sMod}$-$\text{Cat}$:

$$\begin{array}{ccc}
T_I\Gamma U T_I U(A) & \xrightarrow{\psi_1} & T_I\Gamma U(A) \xrightarrow{\psi_2} L_I(A) =: L(A) \\
\downarrow & & \downarrow \\
T_I\Gamma U T_I U(A') & \xrightarrow{\psi_1} & T_I\Gamma U(A') \xrightarrow{\psi_2} L_{I'}(A') =: L(A'). \\
\end{array}$$

Notice that the square whose horizontal arrows are $\psi_1$ (resp. $\psi_2$) is commutative. Since the inclusions

$$\text{sMod}^I$-$\text{Cat} \hookrightarrow \text{sMod}$-$\text{Cat} \text{ and sMod}^{I'}$-$\text{Cat} \hookrightarrow \text{sMod}$-$\text{Cat}$$

...
clearly preserve coequalizers the previous diagram in \(s\text{Mod}\)-Cat induces a simplicial \(k\)-linear functor
\[
L(F) : L(A) \longrightarrow L(A').
\]
We have constructed a functor
\[
L : dgcat_{\geq 0} \longrightarrow s\text{Mod}\text{-Cat}.
\]

**Proposition 5.5.** The functor \(L\) is left adjoint to \(N\).

**Proof.** Let \(A \in dgcat_{\geq 0}\) and \(B \in s\text{Mod}\text{-Cat}\). Let us denote by \(I\) the set of objects of \(A\). We will construct two natural maps
\[
s\text{Mod}\text{-Cat}(L(A), B) \xrightarrow{\phi} dgcat_{\geq 0}(A, N(B))
\]
and then show that they are inverse of each other.

Let \(G : L(A) \rightarrow B\) be a simplicial \(k\)-linear functor. We denote by \(B'\) the full subcategory of \(B\) whose objects are those which belong to the image of \(G\). We have a natural factorization
\[
\begin{array}{ccc}
L(A) & \xrightarrow{G} & B \\
\downarrow{G'} & & \downarrow{\eta} \\
\end{array}
\]
Now, let \(\tilde{B}\) be the simplicial \(k\)-linear category whose set of objects is
\[
obj(\tilde{B}) := \{(a, b) | a \in L(A), b \in B' and G'(a) = b\}
\]
and whose simplicial \(k\)-module of morphisms is defined as
\[
\tilde{B}((a, b), (a', b')) := B'(b, b').
\]
The composition is given by the composition in \(B'\). Now consider the simplicial \(k\)-linear functor
\[
\tilde{G} : L(A) \longrightarrow \tilde{B}
\]
which maps \(a\) to \((a, G'(a))\) and the simplicial \(k\)-linear functor
\[
P : \tilde{B} \longrightarrow B'
\]
which maps \((a, b)\) to \(b\).

The above constructions allow us to factor \(G\) as the following composition
\[
\begin{array}{ccc}
L(A) & \xrightarrow{\tilde{G}} & \tilde{B} \\
\downarrow{\tilde{G}'} & & \downarrow{P} \\
B' & \longrightarrow & B.
\end{array}
\]
Notice that \(\tilde{G}\) induces a bijection on objects and so it belongs to \(s\text{Mod}^I\text{-Cat}\). Finally define \(\phi(G)\) as the following composition
\[
\begin{array}{ccc}
A & \xrightarrow{\tilde{G}^t} & NB \\
\downarrow{NP} & & \downarrow{NP'} \\
NB' & \longrightarrow & NB,
\end{array}
\]
where \(\tilde{G}^t\) denotes the morphism in \(Ch^I_{\geq 0}\text{-Cat}\) which corresponds to \(\tilde{G}\) under the adjunction \((I_1, N_I)\).
We now construct in a similar way the map $\eta$. Let $F : A \to NB$ be a dg functor and $(NB)'$ be the full subcategory of $NB$ whose objects are those which belong to the image of $F$. We have a natural factorization

$$\begin{array}{ccc}
A & \xrightarrow{F} & NB \\
| & & \\
| & \searrow & \\
| & & (NB)'
\end{array}$$

Now, let $\widehat{NB}$ be the positive graded dg category whose set of objects is

$$\text{obj}(\widehat{NB}) := \{(a,b) | a \in A, b \in NB' \text{ and } F'(a) = b\}$$

and whose positive graded complex of morphisms is defined as

$$\widehat{NB}((a,b), (a', b')) := (NB)'(b, b').$$

The composition is given by the composition in $(NB)'$. Consider the dg functor

$$\widetilde{F} : A \to \widehat{NB}$$

which maps $a$ to $(a, F'(a))$ and the dg functor

$$P : \widehat{NB} \to (NB)'$$

which maps $(a, b)$ to $b$.

The above constructions allow us to factor $F$ as the following composition

$$\begin{array}{ccc}
A & \xrightarrow{\widetilde{F}} & \widehat{NB} \\
| & \searrow & \\
| & \downarrow & \\
| & P & \\
NB & \xrightarrow{(NB)'} & NB.
\end{array}$$

Notice that $\widetilde{F}$ induces a bijection on objects and so belongs to $\text{Ch}_{\geq 0}^I\text{-Cat}$. Since the normalization functor $N$ preserve the set of objects, the above construction

$$\begin{array}{ccc}
\text{dgcat}_{\geq 0} & \xrightarrow{N} & s\text{Mod}\text{-Cat} \\
| & \searrow & \\
\widehat{NB} & \xrightarrow{P} & (NB)'
\end{array}$$



We can now define $\eta(F)$ as the following composition

$$\eta(F) : L(A) \xrightarrow{\tilde{F}^g} \tilde{B} \xrightarrow{\overline{\pi}} B' \xrightarrow{\iota} B,$$

where $\tilde{F}^g$ denotes the morphism in $s\text{Mod}^I\text{-Cat}$, which corresponds to $\tilde{F}$ under the adjunction $(L_I, N_I)$.

The maps $\eta$ and $\phi$ are clearly inverse of each other and so the proposition is proven.
5.2. **Path object.** In this subsection we lift the Quillen model structure on \( \text{dgcat}_{\geq 0} \), see theorem 4.7, along the adjunction

\[
\begin{array}{ccc}
\text{sMod-Cat} & \xrightarrow{N} & \text{dgcat}_{\geq 0} \\
\downarrow & & \downarrow \\
\text{dgcat}_{\geq 0} & \xleftarrow{L} & \text{sMod-Cat}
\end{array}
\]

of the previous subsection. For this we will use theorem 5.12 and proposition 5.13 of [23].

**Definition 5.6.** A simplicial \( k \)-linear functor \( G : A \to B \) is:
- a weak equivalence if \( NG \) is a quasi-equivalence in \( \text{dgcat}_{\geq 0} \).
- a fibration if \( NG \) is a fibration in \( \text{dgcat}_{\geq 0} \).
- a cofibration if it has the left lifting property with respect to all trivial fibrations in \( \text{sMod-Cat} \).

**Definition 5.7.** Let \( A \) be a small simplicial \( k \)-linear category. The homotopy category \( \pi_0(A) \) of \( A \) is the category which has the same objects as \( A \) and whose morphisms are defined as

\[
\pi_0(A)(x,y) := \pi_0(A(x,y)), \quad x,y \in A.
\]

**Lemma 5.8.** Let \( x \xrightarrow{f} y \) be a 0-simplex morphism in \( A \). Then \( \pi_0(f) \) is invertible in \( \pi_0(A) \) iff \( H_0(Nf) \) is invertible in \( H_0(NA) \).

**Proof.** We start by observing that if we restrict ourselves to the 0-simplex morphisms in \( A \) and to the degree zero morphisms in \( NA \) we have the same category. In fact the degree zero component of the shuffle map \( \nabla \), used in the definition of \( NA \), is the identity map, see [18, VIII-8.8].

Now suppose that \( \pi_0(f) \) is invertible. Then there exists a 0-simplex morphism \( g : y \to x \) and 1-simplex morphisms \( h_1 \in A(x,x) \) and \( h_2 \in A(y,y) \) such that \( d_0(h_1) = 1_x \), \( d_1(h_1) = gf \), \( d_0(h_2) = 1_y \) and \( d_1(h_2) = fg \). Observe that the image of \( h_1 \), resp. \( h_2 \), by the normalization functor \( N \) is a degree 1 morphism in \( NA(x,x) \), resp. in \( NA(y,y) \), whose differential is \( gf - 1_x \) (resp. \( fg - 1_y \)). This implies that \( H_0(Nf) \) is also invertible in \( H_0(NA) \).

To prove the converse we consider an analogous argument. \( \square \)

**Proposition 5.9.** A simplicial \( k \)-linear functor \( G : A \to B \) is a weak equivalence iff:

1. \( G(x,y) : A(x,y) \to B(Gx,Gy) \) induces an isomorphism on \( \pi_i \) for all \( i \geq 0 \) and for all objects \( x,y \in A \), and
2. \( \pi_0(G) : \pi_0(A) \to \pi_0(B) \) is essentially surjective.

**Proof.** We show that condition (1), resp. condition (2), is equivalent to condition (i), resp. condition (ii), of definition 4.2. By the Dold-Kan equivalence, we have the following commutative diagram

\[
\begin{array}{ccc}
\pi_i A(x,y) & \xrightarrow{G} & \pi_i B(Gx,Gy) \\
\downarrow \sim & & \downarrow \sim \\
H_i NA(x,y) & \xrightarrow{NG} & H_i NB(Gx,Gy)
\end{array}
\]
where the vertical arrows are isomorphisms. This implies that condition (1) is equivalent to condition (i) of definition 4.2.

Concerning condition (2), we start by supposing that $\pi_0(G)$ is essentially surjective. Consider the functor

$$H_0(NG): H_0(NA) \to H_0(NB)$$

and let $z$ be an object in $H_0(NB)$. Since $\pi_0(B)$ and $H_0(NB)$ have the same objects we can consider $z$ as an object in $\pi_0(B)$. By hypothesis, $\pi_0(G)$ is essentially surjective and so there exists an object $w \in \pi_0(A)$ and a 0-simplex morphism

$$Gw \xrightarrow{f} z$$

which becomes invertible in $\pi_0(B)$. Now lemma 5.8 implies that $Nf$ is invertible in $H_0(NB)$ and so we conclude that the functor $H_0(NG)$ is essentially surjective. This shows that condition (2) implies condition (ii) of definition 4.2. To prove the converse we consider an analogous argument.

Theorem 5.10. The category $s\text{Mod}\text{-Cat}$ when endowed with the notions of weak equivalence, fibration and cofibration as in definition 5.6 becomes a cofibrantly generated Quillen model category and the adjunction $(L,N)$ becomes a Quillen adjunction.

The proof will consist on verifying the conditions of theorem 5.12 and proposition 5.13 in [23]. Since the Quillen model structure on $\text{dgcat}_{\geq 0}$ is cofibrantly generated, see theorem 4.14 every object in $\text{dgcat}_{\geq 0}$ is fibrant, see remark 4.15 and the functor $N$ clearly commutes with filtered colimits it is enough to show that:

- for each simplicial $k$-linear category $A$, we have a factorization

$$\begin{array}{ccc}
A & \xrightarrow{\Delta} & A \times A \\
I_A & \xrightarrow{\sim} & P(A), \\
& & P_0 \times P_1
\end{array}$$

with $I_A$ is a weak equivalence and $P_0 \times P_1$ is a fibration in $s\text{Mod}\text{-Cat}$.

For this we need a few lemmas. We start with the following definition.

Definition 5.11. Let us define $P(A)$ as the simplicial $k$-linear category whose objects are the 0-simplex morphisms $f: x \to y$ in $A$ which become invertible in $\pi_0(A)$. We define the simplicial $k$-module of morphisms

$$P(A)(x \xrightarrow{f} y, x' \xrightarrow{f'} y'), \quad f, f' \in P(A)$$

as the homotopy pull-back in $s\text{Mod}$ of the diagram

$$\begin{array}{ccc}
A(y, y') & \xrightarrow{f^*} & A(x, y') \\
\downarrow & & \downarrow \quad f_* \\
A(x, x') & \xrightarrow{f_*} & A(x, y')
\end{array}$$

by which we mean the simplicial $k$-module

$$A(x, x') \times_{A(x, y')} s\text{Mod}(k\Delta[1], A(x, y')) \times_{A(x, y')} A(y, y').$$
We denote the simplexes in $A(x, x')$ and $A(y, y')$ lateral morphisms and the simplexes in $sMod(k\Delta[1], A(x,y))$ homotopies. The composition operation

$$P(A)(f, f') \wedge P(A)(f', f'') \longrightarrow P(A)(f, f''), \quad f, f', f'' \in P(A)$$

decomposes on:

- a composition of lateral morphisms, which is induced by the composition on $A$ and
- a composition of homotopies, which is given by the map

$$sMod(k\Delta[1], A(x,y')) \wedge P(A)(f, f') \wedge sMod(k\Delta[0], A(x', x'')) \longrightarrow sMod(k\Delta[1], A(x,y')),$$

where the last map is induced by the diagonal map in $k\Delta[1]$.

**Remark 5.12.** Notice that a 0-simplex morphism $\alpha : f \to f'$ in $P(A)$ is of the form $\alpha = (m_x, h, m_y)$, with $m_x : x \to x'$ and $m_y : y \to y'$ 0-simplex morphisms in $A$ and $h$ is a 1-simplex morphism in $A(x,y')$ such that $d_0(h) = m_y f$ and $d_1(h) = f' m_x$.

We have a natural commutative diagram in $sMod\text{-Cat}$

\[
\begin{array}{ccc}
A & \Delta & A \times A \\
\downarrow t_A & & \downarrow p_0 \times p_1 \\
P(A) & & \\
\end{array}
\]

where $I_A$ is the simplicial $k$-linear functor that associates to an object $x \in A$ the 0-simplex morphism $x \xrightarrow{l_d} x$ and $P_0$, resp. $P_1$, is the simplicial $k$-linear functor that sends a morphism $x \xrightarrow{f} y$ in $P(A)$ to $x$, resp. $y$.

Notice that by applying the normalization functor $N$ to the above diagram and lemma 4.20 to the dg category $NA$ we obtain two factorizations

\[
\begin{array}{ccc}
NA & \Delta & NA \times NA \\
\downarrow \tau_{\geq 0}(I) & & \downarrow \tau_{\geq 0}(P) \\
NP(A) & & \\
\end{array}
\]

of the diagonal dg functor. By lemma 4.20, $\tau_{\geq 0}P(NA)$ is a path object of $NA$ in $\mathbf{dgcat}_{\geq 0}$. We will show in proposition 5.17 that $NP(A)$ is also a path object of $NA$.

**Lemma 5.13.** Let $A, B \in sMod$. The shuffle map $\triangledown$ induces a natural surjective chain homotopy equivalence

$$N(sMod(A, B)) \xrightarrow{\triangledown} Ch_{\geq 0}(NA, NB),$$
which has a natural section induced by the Alexander-Whitney map.

Proof. First note that if \((L, R)\) and \((L', R')\) are adjoint pairs of functors, a natural transformation \(\zeta : L \to L'\) induces a natural transformation \(\zeta^\#: R' \to R\) which is a natural equivalence iff \(\zeta\) is also.

Fixing a chain complex \(NA \in \mathsf{Ch}_{\geq 0}\) let \(L, L': \mathsf{Ch}_{\geq 0} \to \mathsf{Ch}_{\geq 0}\) be defined by

\[
L(C) := C \otimes NA, \quad L'(C) := N(\Gamma C \wedge A).
\]

Using the Dold-Kan equivalence in the case of \(L'\), we see that these functors have right adjoints

\[
R(C) = \mathsf{Ch}_{\geq 0}(NA, C), \quad R'(C) = N(\mathsf{Mod}(A, \Gamma C))
\]

respectively.

The shuffle map determines a natural inclusion \(\nabla : L \to L'\) which has a right inverse given by the Alexander-Whitney map \(\text{AW}\), see [21, 2.7]. It follows that \(\nabla^\#: R' \to R\) is a natural surjection with a section given by \(\text{AW}^\#\).

The fact that \(\nabla^\#\) is a natural transformation of bi-functors is clear. Since \(\nabla\) is a chain homotopy equivalence, in order to finish the proof it is now enough to show that the functors \(L, L', R, R'\) send chain homotopic maps to chain homotopic maps (for \((L, R)\) and \((L', R')\) will then induce adjunctions on the homotopy category \(\mathsf{Ho}(\mathsf{Ch}_{\geq 0})\) and \(\nabla : L \to L'\) will be a natural isomorphism between endo-functors of \(\mathsf{Ho}(\mathsf{Ch}_{\geq 0})\)).

The functors \(L\) and \(R\) clearly preserve the chain homotopy relation. For the same reason, \(L'\) and \(R'\) preserve the relation on \(\mathsf{Ch}_{\geq 0}(C, D)\) defined by the cylinder object

\[
\begin{array}{ccc}
\text{C} & \to & N(\Gamma C \wedge k\Delta[1]) \\
\downarrow & & \downarrow \\
\text{C} & \to & C
\end{array}
\]

Since the Alexander-Whitney and shuffle maps give maps between this cylinder object and the usual one, we see that this relation is the usual chain homotopy relation. This concludes the proof.

We now define a map \(\phi\) relating the \(\mathsf{Ch}_{\geq 0}\)-graphs associated with the dg categories \(NP(A)\) and \(\tau_{\geq 0}P(NA)\). Observe that:

- By lemma 4.20, \(NP(A)\) and \(\tau_{\geq 0}P(NA)\) have exactly the same objects and

- For each pair of objects \(x \xrightarrow{f} y, x' \xrightarrow{f'} y'\) in \(NP(A)\), the map of lemma 5.13 (with \(A = k\Delta[1]\)) induces a surjective quasi-isomorphism \(\phi_{f, f'}\)

\[
\begin{array}{ccc}
NA(x, x') & \times & N(\mathsf{Mod}(k\Delta[1], NA(x, y'))) \\
\otimes & & \otimes \\
\times & & \times \\
NA(x, y') & \times & NA(y, y')
\end{array}
\]

\[
\sim 1 \times \nabla \times 1
\]

\[
\begin{array}{ccc}
NA(x, x') & \times & \mathsf{Ch}_{\geq 0}(Nk\Delta[1], NA(x, y')) \\
\otimes & & \otimes \\
\times & & \times \\
NA(x, y') & \times & NA(y, y')
\end{array}
\]

in \(\mathsf{Ch}_{\geq 0}\).

Notation 5.14. We denote by

\[
\phi : NP(A) \to \tau_{\geq 0}P(NA)
\]
the map of $\text{Ch}_{\geq 0}$-graphs which is the identity on objects and $\phi_{f,f'}$ on the complexes of morphisms.

Remark 5.15. Notice that by definition of $P(A)$ and remark 5.12 the map $\phi$ preserve identities and the composition of degree zero morphisms.

We now establish a ‘homotopy equivalence lifting property’ of $\phi$.

Proposition 5.16. Let $\alpha$ be a degree zero morphism in $\tau_{\geq 0}P(NA)$ that becomes invertible in $H_0(\tau_{\geq 0}P(NA))$. Then there exists a degree zero morphism $\overline{\alpha}$ in $NP(A)$ which becomes invertible in $H_0(NP(A))$ and $\phi(\overline{\alpha}) = \alpha$.

Proof. Let $\alpha : (x \xrightarrow{f} y) \longrightarrow (x' \xrightarrow{f'} y')$ be a degree zero morphism in $\tau_{\geq 0}P(NA)$. Notice that $\alpha$ is of the form $(m_x, h, m_y)$ with $m_x : x \rightarrow x'$ and $m_y : y \rightarrow y'$ degree zero morphisms in $NA$ and $h : x \rightarrow y$ a degree 1 morphism in $NA$. Now, by definition of $P(A)$ we can choose a representative $\overline{h} \in A(x,y)_1$ of $h$ and so we obtain a degree zero morphism $\overline{\alpha} = (m_x, \overline{h}, m_y)$ in $NP(A)$ such that

$$\phi_{f,f'} : NP(A)(f,f') \rightarrow \tau_{\geq 0}P(NA)(f,f')$$

$$\overline{\alpha} = (m_x, \overline{h}, m_y) \mapsto (m_x, h, m_y).$$

Now suppose that $\alpha$ is invertible in $H_0(\tau_{\geq 0}P(NA))$. Then there exist morphisms $\beta$ of degree 0 and $r_1$ and $r_2$ of degree 1 such that $d(r_1) = \beta \alpha - 1$ and $d(r_2) = \alpha \beta - 1$. As above, we can lift $\beta$ to a morphism $\overline{\beta}$ in $NP(A)$. Since the map $\phi$ preserve the identities and the composition of degree zero morphisms it maps $\overline{\alpha} \overline{\beta}$ to $\alpha \beta$ and $\overline{\beta} \overline{\alpha}$ to $\beta \alpha$. Finally since the maps $\phi_{f,f'}$ are surjective quasi-isomorphisms we can lift $r_1$ to $\overline{r}_1$, resp. $r_2$ to $\overline{r}_2$, in $NP(A)$ by applying the lemma [8, 2.3.5] to the couple $(r_1, 1)$, resp. $(r_2, 1)$. This implies that $\overline{\alpha}$ is also invertible in $H_0(NP(A))$.

Proposition 5.17. In the following commutative diagram in $\text{dgcat}_{\geq 0}$

$$\begin{array}{ccc}
NA & \xrightarrow{\Delta} & NA \times NA \\
\downarrow{N(I_A)} & & \downarrow{N(P_0) \times N(P_1)} \\
NP(A) & \xrightarrow{N(P_0) \times N(P_1)} & N(P_0) \times N(P_1)
\end{array}$$

obtained by applying the normalization functor $N$ to the diagram [1] in $\text{sMod-Cat}$, the dg functor $N(I_A)$ is a quasi-equivalence and $N(P_0) \times N(P_1)$ is a fibration.

Proof. We first prove that $N(I_A)$ is a quasi-equivalence. By definition of $P(A)$ the dg functor $I_A$ clearly satisfies condition (1) of proposition 5.14. We now prove that $N(I_A)$ satisfies condition (ii) of definition 5.12. Let $f$ be an object in $NP(A)$. The dg categories $NP(A)$ and $\tau_{\geq 0}P(NA)$ have the same objects and so we can consider $f$ as an object in $\tau_{\geq 0}P(NA)$. Since the dg functor

$$\tau_{\geq 0}(I) : NA \longrightarrow \tau_{\geq 0}P(NA)$$

is a quasi-equivalence, see lemma 5.12, there exists an object $x$ in $NA$ and a homotopy equivalence $\alpha$ in $\tau_{\geq 0}P(NA)$ between $I(x)$ and $f$. By proposition 5.10 we can lift $\alpha$ to a homotopy equivalence $\overline{\alpha}$ in $NP(A)$ and so the dg functor

$$N(I_A) : NA \longrightarrow NP(A)$$
satisfies condition (ii) of definition 4.2. This proves that $N(I)$ is a quasi-equivalence.

We now prove that $N(P_0) \times N(P_1)$ is a fibration. By definition of $P(A)$ the dg functor $N(P_0) \times N(P_1)$ is clearly surjective on the complexes of morphisms. We now prove that it has the right lifting property with respect to the generating trivial cofibration $F$, see definition 4.5. Let $x \xrightarrow{f} y$ be an object in $NP(A)$ and $\gamma : (x, y) \to (x', y')$ a homotopy equivalence in $NA \times NA$. Since the dg functor

$$\tau_{\geq 0}(P) : \tau_{\geq 0}P(NA) \to NA \times NA$$

is a fibration there exists a homotopy equivalence $\alpha : f \to f'$ in $\tau_{\geq 0}P(NA)$ such that $\tau_{\geq 0}(P)(\alpha) = \gamma$. Now, by proposition 5.10, we can lift $\alpha$ to a homotopy equivalence $\overline{\alpha} : f \to f'$ in $N(PA)$ such that $N(P_0) \times N(P_1)(\overline{\alpha}) = \gamma$.

This proves the proposition. $\sqrt{}$

Remark 5.18. Since every object in $\text{dgcat}_{\geq 0}$ is fibrant, see remark 4.15, all simplicial $k$-linear categories will be fibrant with respect to this Quillen model structure.

5.3. Quillen equivalence. In this subsection we prove that the Quillen adjunction constructed in the previous subsection

$\text{sMod-Cat}$

$\downarrow \quad \downarrow N$

$\text{dgcat}_{\geq 0}$

is in fact a Quillen equivalence.

Theorem 5.19. The Quillen adjunction $(L, N)$ is a Quillen equivalence.

Proof. Let $A \in \text{dgcat}_{\geq 0}$ be a cofibrant dg category and $B$ a simplicial $k$-linear category. Recall from remark 5.18 that every object in $\text{sMod-Cat}$ is fibrant. We need to show that a simplicial $k$-linear functor

$$F : L(A) \to B$$

is a weak equivalence in $\text{sMod-Cat}$ iff the corresponding dg functor

$$F^\sharp : A \to NB$$

is a quasi-equivalence in $\text{dgcat}_{\geq 0}$.

We have the following commutative diagram

$$\begin{array}{ccc}
A & \xrightarrow{F^\sharp} & NB \\
\eta \downarrow & & \downarrow NF \\
NL(A) & & \\
\end{array}$$

where $\eta$ is the counit of the adjunction $(L, N)$. Since, by definition, $F$ is a weak equivalence in $\text{sMod-Cat}$ iff $NF$ is a quasi-equivalence it is enough to show that $\eta$ is a quasi-equivalence. The dg functor $\eta$ is the identity map on objects and so it is enough to show that

$$\eta(x, y) : A(x, y) \to NL(A)(x, y), \quad x, y \in A$$
is a quasi-isomorphism. Now, let $I$ be the set of objects of $\mathcal{A}$. Since $\mathcal{A}$ is cofibrant in $\text{dgcat}_{\geq 0}$ it clearly stays cofibrant when considered as an object of the Quillen model structure on $\text{Ch}_{\geq 0}(-\text{Cat})$, see remark 5.2. By proposition 6.4 of [21] the adjunction morphism in $\text{Ch}_{\geq 0}(-\text{Gr})$

$$\Gamma U(\mathcal{A}) \longrightarrow L_I(\mathcal{A})$$

is such that

$$\Gamma U(\mathcal{A})(x, y) \longrightarrow L_I(\mathcal{A})(x, y)$$

induces an isomorphism in $\pi_i$ for $i \geq 0$ and for all objects $x, y \in \Gamma U(\mathcal{A})$. This implies by the Dold-Kan equivalence that

$$\mathcal{A}(x, y) = N(\Gamma U(\mathcal{A})(x, y)) \cong N(L_I(\mathcal{A})(x, y)), \quad x, y \in \mathcal{A}$$

is a quasi-isomorphism and so

$$\eta(x, y) : \mathcal{A}(x, y) \longrightarrow NL(\mathcal{A}(x, y)), \quad x, y \in \mathcal{A}$$

is a quasi-isomorphism. This proves the theorem. $

\Box$

Remark 5.20. Notice that the objects in $\text{dgcat}_{\geq 0}$, resp. in $\text{sMod-Cat}$, with only one object consist exactly on the connective dg algebras, see [21, 1.1], resp. simplicial $k$-algebras. We have the following commutative diagram

$$
\begin{array}{ccc}
\text{Alg} & \longrightarrow & \text{sMod-Cat} \\
L & N & L \\
\text{DGA}_{\geq 0} & \longrightarrow & \text{dgcat}_{\geq 0},
\end{array}
$$

where $\text{DGA}_{\geq 0}$ denotes the category of connective dg algebras and $\text{Alg}$ the category of simplicial $k$-algebras. Observe that if we restrict the Quillen model structures to these full subcategories we obtain Shipley-Schwede’s Quillen equivalence [21, 1.1]

$$
\begin{array}{ccc}
\text{Alg} & \longrightarrow & \text{sMod-Cat} \\
L & N & L \\
\text{DGA}_{\geq 0} & \longrightarrow & \text{dgcat}_{\geq 0}.
\end{array}
$$

We have then extended Shipley-Schwede’s work to a ‘several objects’ context: the notion of weak equivalence in $\text{sMod-Cat}$ and $\text{dgcat}_{\geq 0}$ (see definition 4.2 and proposition 5.6) is now a mixture between quasi-isomorphisms and categorical equivalences.

6. The Global Picture

Recall from [3, III] that we have an adjunction

$$
\begin{array}{ccc}
\text{sMod} & \longrightarrow & \text{sSet} \\
k(-) & U & \\
\text{sSet},
\end{array}
$$
where \( U \) is the forgetful functor and \( k(-) \) the \( k \)-linearization functor. The functor \( k(-) \) is lax strong monoidal and so we have the natural adjunction

\[
\begin{array}{c}
s\text{Mod-Cat} \\
\downarrow U \\
s\text{Set-Cat}
\end{array}
\]

Recall from [1, 1.1] that the category \( s\text{Set-Cat} \) is endowed with a Quillen model structure whose weak equivalences are the Dwyer-Kan (=DK) equivalences. Let us recall this notion.

**Definition 6.1.** A simplicial functor \( F : A \to B \) is a Dwyer-Kan equivalence if:
- for any objects \( x \) and \( y \) in \( A \), the map \( F(x,y) : A(x,y) \to B(Fx,Fy) \) is a weak equivalence of simplicial sets and
- the induced functor \( \pi_0(F) : \pi_0(A) \to \pi_0(B) \) is essentially surjective.

**Proposition 6.2.** The adjunction \( (k(-), U) \) is a Quillen adjunction, when we consider on \( s\text{Mod-Cat} \) the Quillen model structure of theorem 5.10.

**Proof.** We first observe that by proposition 5.9 the functor \( U : s\text{Mod-Cat} \to s\text{Set-Cat} \) preserves weak equivalences.

We now show that it also preserves fibrations. Let \( G : A \to B \) be a simplicial \( k \)-linear functor such that \( NG : NA \to NB \) is a fibration in \( \text{dgcat}_{\geq 0} \). We need to show that \( UG \) is a fibration in \( s\text{Set-Cat} \). Recall from [1] that \( UG \) is a fibration iff:

- for any object \( x \) and \( y \) in \( UA \), the map \( UG(x,y) : UA(x,y) \to UB(Gx,Gy) \) is a fibration in \( s\text{Set} \) and
- for any object \( x \in \text{UA} \), \( y \in UB \) and homotopy equivalence \( f : Gx \to y \) in \( UB \) (= \( f \) becomes invertible in \( \pi_0(UB) \)), there is an object \( z \in A \) and a homotopy equivalence \( h : x \to z \) in \( UA \) such that \( UG(h) = f \).

Since by hypothesis \( NG : NA \to NB \) is a fibration in \( \text{dgcat}_{\geq 0} \), the dg functors \( R(n), n \geq 1 \) (which belong to the set \( J \) of generating trivial cofibrations) allow us to conclude that the morphisms \( NG(x,y)_n : NA(x,y)_n \to NB(Gx,Gy)_n, \ x, y \in A \) are surjective for \( n \geq 1 \). Now, by [9] III-2.11, \( UG(x,y) \) is a fibration in \( s\text{Set} \) iff the morphisms \( NG(x,y)_n \) are surjective for \( n \geq 1 \). This implies that condition (F1) is verified.

Concerning condition (F2), let \( x \in UA, y \in UB \) and \( f : Gx \to y \) be a homotopy equivalence in \( UB \). This means that \( f \) is invertible in \( \pi_0(B) \) and so by lemma 5.8 \( N(f) \) is also invertible in \( H_0(NB) \). This data allow us to construct,
using proposition 1.7 in [23], the following (solid) commutative square

\[
\begin{array}{ccc}
  k & \rightarrow & NA \\
  \sim & \downarrow & \downarrow \\
  F \rightarrow & NG & \\
  \downarrow & \downarrow & \downarrow \\
  K & \rightarrow & NB.
\end{array}
\]

Since \( NG \) is a fibration in \( \text{dgcat}_{\geq 0} \) we can lift \( Nf \) to a morphism \( h : x \rightarrow z \) in \( NA \) which is invertible in \( H_0(NA) \). Since the 0-simplex morphisms in \( A \) and the degree zero morphisms in \( NA \) are exactly the same, lemma 5.8 implies that \( h : x \rightarrow z \), when considered as a morphism in \( UA \), satisfies condition (F2).

This proves the proposition.

We have obtained the following zig-zag of Quillen adjunctions relating the homotopy theories of differential graded and simplicial categories:

\[
\begin{array}{ccc}
  s\text{Set-Cat} & \rightarrow & s\text{Mod-Cat} \\
  \kappa(-) \downarrow & & \downarrow N \\
  \text{dgcat}_{\geq 0} \rightarrow & \tau_{\geq 0} & \rightarrow \\
  & \downarrow & \\
  & \text{dgcat}. &
\end{array}
\]

**Remark 6.3.** Since the adjunction \((L, N)\) is a Quillen equivalence

- the composed functor \( \mathbb{L}(N \circ k(-)) : \text{Ho}(s\text{Set-Cat}) \rightarrow \text{Ho}(\text{dgcat}) \) preserves homotopy colimits and

- the composed functor \( \mathbb{R}(U \circ L \circ \tau_{\geq 0}) : \text{Ho}(\text{dgcat}) \rightarrow \text{Ho}(s\text{Set-Cat}) \) preserves homotopy limits.

The following result was proved by Simpson in an adhoc way in [22, 5.1].

**Corollary 6.4.** Let \( F : A \rightarrow B \) be a dg functor and \( b \) an object of \( B \). Then the homotopy fiber of \( F \) over \( b \), denoted by \( \text{HFib}(F)/b \), is equivalent to the homotopy fiber of \( \mathbb{R}(U \circ L \circ \tau_{\geq 0})(F) \) over \( b \):

\[
\text{HFib}(F)/b \sim \text{HFib}(NF)/b.
\]

**Proof.** It follows from remark 6.3.

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