Back Reaction to the Spectrum of Magnetic Field in the Kinetic Dynamo Theory — Modified Kulsrud and Anderson Equation —

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Abstract

We take account of the lowest order back reaction on the fluid and modify the Kulsrud and Anderson equation $\partial_t \mathcal{E}_M = 2\gamma \mathcal{E}_M$ obtained in the kinetic dynamo theory, where $\mathcal{E}_M$ is the energy density of the magnetic field. Furthermore, we apply our present result to some astrophysical stages where the magnetic field is expected to be amplified by the dynamo mechanism.

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1 Introduction

The magnetic fields have been observed in various astrophysical scales [1]. The origin is one of the important problems in cosmology [2]. Although an attractive mechanism in protogalaxy was proposed by Kulsrud et al [3] for the galactic magnetic field, it cannot explain how the magnetic field is made in intergalactic and intercluster regions [2]. Thus, it is worth investigating the generation and the evolution of the primordial magnetic field. Since the strength of these fields strength is too small we expect that the amplification of these fields occurs due to the dynamo mechanism. It is well known that the mean magnetic field can be amplified enough to explain the present observation in the kinetic dynamo theory [4]. However, as Kulsrud and Anderson showed [5], the growth rate of the fluctuation around the mean magnetic fields is much larger than that of the mean field in interstellar mediums. This means that the kinetic dynamo theory breaks down. Hence, one must investigate the effect of the back-reaction to the kinetic theory. So far the back-reaction on the mean field has been considered [3]. Setting apart the problem of the kinetic dynamo theory in interstellar mediums, it is obvious that the kinetic theory cannot hold near the equipartition state in general.

In this paper, we consider the back-reaction on the fluctuation and derive the evolutional equation of energy of the magnetic field, that is, modified Kulsrud and Anderson equation. Then we apply it to some examples.

The rest of the paper is organized as follows. In Sec. II, we derive the modified Kulsrud and Anderson equation with the lowest order back-reaction under a phenomenological assumption. In Sec. III and IV, we give applications and remarks, respectively.

2 Modified Kulrsrud and Anderson Equation

In the Fourier space the basic equations of the incompressible MHD are

\[ \partial_t v_i(k, t) = -iP_{ijk}(k) \int \frac{d^3q}{(2\pi)^3} \left[ v_j(k - q, t)v_k(q, t) - b_j(k - q, t)b_k(q, t) \right] \]  

and

\[ \partial_t b_i(k, t) = ik_j \int \frac{d^3q}{(2\pi)^3} \left[ v_i(k - q, t)b_j(q, t) - v_j(k - q, t)b_i(q, t) \right]. \]
where $b_i(k,t) := B_i(k,t) / \sqrt{4\pi \rho}$, $\rho$ is the energy density of the fluid and $P_{ijk}(k) = k_j P_{ik}(k) = k_j (\delta_{ik} - k_i k_k / |k|^2)$. For simplicity, we neglected the diffusion terms in the above equations. This simplification can be justified by the fact that almost of astrophysical systems have high magnetic Reynolds number.

Following Kulsrud and Anderson\cite{3} and considering a small time step as the parameter of expansion, we evaluate the time evolution of the magnetic field by iterations;

$$b_i(k,t) = b_i(k,0) + b_i^{(1)}(k,t) + b_i^{(2)}(k,t) + \cdots$$

(3)

where $b_i^{(0)}(k)$ is the initial field. For the fluid velocity, we take account of the back-reaction from the magnetic field (Lorentz force) as follows;

$$v_i(k,t) = v_i^{(1)}(k,t) + \delta v_i(k,t),$$

(4)

where $v_i^{(1)}(k,t)$ is statistically homogeneous and isotropic component and satisfy

$$\langle v_i^{(1)*}(k,t') v_j^{(1)}(q,t) \rangle = (2\pi)^3 \left[ J_1(k) P_{ij}(k) + i J_2(k) \epsilon_{ikj} k_k \right] \delta^3(k - q) \delta(t - t')$$

(5)

This statics holds in the region where is far from the boundary. $J_1(k)$ and $J_2(k)$ denote the velocity dispersion and the mean helicity of the fluid,

$$\langle v^{(1)}(x,t) \cdot v^{(1)}(x,t) \rangle = 2 \int \frac{d^3k}{(2\pi)^3} J_1(k) \delta(0)$$

and

$$\langle v^{(1)}(x,t) \cdot \nabla \times v^{(1)}(x,t) \rangle = -2 \int \frac{d^3k}{(2\pi)^3} k^2 J_2(k) \delta(0).$$

(6)

The second term in the right-hand side of the eq.(4), $\delta v_i(k,t)$, is determined by eq. (1) and this corresponds to the back-reaction term from the magnetic field.

In each orders, the MHD equation becomes

$$\partial_t b_i^{(1)}(k,t) = 2ik_j \int \frac{d^3q}{(2\pi)^3} v_{ji}(k-q,t) b_j^{(0)}(q)$$

(6)
\[ \partial_t b_i^{(2)}(k, t) = 2ik_j \int \frac{d^3q}{(2\pi)^3} v_{ij}(k - q, t)b_j^{(1)}(q, t) \]  

and

\[ \partial_t v_i(k, t) = 2iP_{ijk}(k) \int \frac{d^3q}{(2\pi)^3} b_j^{(0)}(k - q)b_j^{(1)}(q, t). \]  

The last equation contains an effect of the lowest order back-reaction on the fluid and it gives an explicit expression of \( \delta v_i(k, t) \)

\[ \delta v_i(k, t) \simeq 2iP_{ijk}(k) \int_0^t dt' \int \frac{d^3q}{(2\pi)^3} b_j^{(0)}(k - q)b_j^{(1)}(q, t'). \]  

From the eqs. \((6) \sim (9)\), the time derivative of the energy becomes

\[ \partial_t \langle |b(k, t)|^2 \rangle = \langle b_i^{(1)*}(k, t)b_i^{(1)}(k, t) \rangle + \langle \delta^{(1)*}b_i^{(2)}(k, t) \rangle + \text{c.c.} \]

\[ = 4 \int_0^t dt' \int \frac{d^3q}{(2\pi)^3} \frac{d^3p}{(2\pi)^3} P_{ijk}b_j^{(0)*}(q)v_{ij}(k - q, t')v_{ij}(k - p, t)b_j^{(0)}(p) \]

\[ -4 \int_0^t dt' \int \frac{d^3q}{(2\pi)^3} \frac{d^3p}{(2\pi)^3} P_{ijk}b_j^{(0)*}(q)v_{ij}(q - k, t)v_{ij}(q - p, t')b_j^{(0)}(p) \]

\[ + \text{c.c.,} \]  

where

\[ \langle v_i^{(1)*}(k, t')v_j(k, t) \rangle \simeq \langle v_i^{(1)*}(k, t')v_j^{(1)}(q, t) \rangle \]

\[ + \langle v_i^{(1)*}(k, t')\delta v_j(q, t) \rangle + \langle \delta v_i^{(1)}(k, t')v_j^{(1)}(q, t) \rangle \]

\[ = (2\pi)^3 V_{ij}(k)\delta^{3}(k - q)\delta(t - t') \]

\[ -4 \int_0^t dt'' \int \frac{d^3p}{(2\pi)^3} P_{jkl}(q)p_{m}b_{m}^{(0)*}(p - k)V_{ij}(q)b_{k}^{(0)}(q - p) \]

\[ -4 \int_0^t dt'' \int \frac{d^3p}{(2\pi)^3} P_{jkl}(q)p_{m}b_{m}^{(0)*}(p - q)V_{ij}(q)b_{k}^{(0)*}(q - p) \]

\[ =: (2\pi)^3 V_{ij}(k)\delta^{3}(k - q)\delta(t - t') + \delta v_i^{(1)*}(k, t')v_j(k, t). \]  

The above eq. \((10)\) with \((11)\) is the formal equation with the effect of the back-reaction.

Let us consider the simple example with the following initial condition

\[ b_i^{(0)}(x) = b_0\delta_{iz} \quad \text{or} \quad b_i^{(0)}(k) = b_0(2\pi)^3\delta^3(k)\delta_{iz}. \]
This condition holds approximately as long as the spatial scale of the magnetic field is much larger than the typical scale of eddies. In this case, the eq. (10) becomes

\[
\partial_t \langle |b(k,t)|^2 \rangle = 2(2\pi)^3 \delta(0) k_z^2 V_i(k) + 2 \int_0^t dt' k_z^2 \delta(v_i^*(k,t')v_i(k,t)) b_0^2 \\
= 4(2\pi)^3 k_z^2 J_1(k)(b_0^2) - 6(2\pi)^3 k_z^2 b_0^4(\Delta t)_k^2 J_1(k) \delta^3(0)
\]

where \((\Delta t)_k\) is the time scale of the eddy turnover and its expression will be given below. We assumed that the time integral should be estimated as \(\int_0^t dt'[\cdots] \sim (\Delta t)_k[\cdots]\) in the second line of the right-hand side of the eq. (13) because the back-reaction works only during the time scale of the eddy turnover. Here we assume Kolmogoroff spectrum for the inertial range \(k_0 < k < k_{\text{max}} \sim R^{3/4}k_0\), \(k_0\) is the wave number of the largest eddy and \(R\) is the Reynolds number. From the definition of the velocity dispersion

\[
\langle v^2 \rangle = 2 \int \frac{d^3 k}{(2\pi)^3} J_1(k) \delta(0) =: \int_{k_0}^{k_{\text{max}}} dk I(k),
\]

we obtain the relation

\[
I(k) = \frac{1}{\pi^2} k^2 J_1(k)(\Delta t)_k^{-1} \approx \frac{2}{3} v_0^2 k_0^{2/3} k^{-5/3},
\]

where \(v_0\) is the typical velocity \((v_0 \sim \sqrt{\langle v^2 \rangle})\) and we used \(\delta(0) \sim (\Delta t)_k^{-1}\). The expression of \((\Delta t)_k\) is given by the estimation of the order of the magnitude in the eq. (14), that is, \((1/k(\Delta t)_k)^2 \sim kI(k)\).

Integrating the above equation (13) over \(k\), we obtain the modified Kulsrud and Anderson equation

\[
\partial_t \rho_M = 2\gamma \rho_M - 2\zeta \rho_M^2,
\]

where

\[
\rho_M := \frac{\mathcal{E}_M}{4\pi \rho} := \frac{1}{V} \int \frac{d^3 k}{(2\pi)^3} \langle |b(k,t)|^2 \rangle
\]

\(\mathcal{E}_M\) is the magnetic energy density.

The inertial range is defined by the scale which is smaller than the largest eddy \((\sim k_0^{-1})\) and larger than a small scale \((\sim R^{-3/4}k_0^{-1})\) under where the viscosity term is dominant. In this range, the transfer of the energy works from large eddy to small one without the dissipation of the energy. This leads a sort of 'equilibrium state' with Kolmogoroff spectrum \(k^{5/3}\) (Kolmogoroff Theory).
\[ \gamma := 2 \int \frac{d^3k}{(2\pi)^3} k_z^2 J_1(k) \] (18)

and

\[ \zeta := 3 \int \frac{d^3k}{(2\pi)^3} k_z^4 J_1(k)(\Delta t)_k^2. \] (19)

In the above derivation, we used \( \delta^3(0) \sim V \), where \( V \) is the typical volume of the system.

Now we evaluate the coefficients \( \gamma \) and \( \zeta \). Results are given by

\[ \gamma \simeq \int_{k_0}^{k_{\text{max}}} dk k^2 I(k)(\Delta t)_k \simeq \int_{k_0}^{k_{\text{max}}} dk k I(k) \sim v_0 k_0^{1/3} k_{\text{max}}^{2/3} \sim R^{1/2} v_0 k_0 \] (20)

and

\[ \zeta \simeq \int_{k_0}^{k_{\text{max}}} dk k^4 I(k)(\Delta t)_k^3 \simeq \int_{k_0}^{k_{\text{max}}} dk k I(k)^{-1/2} \sim \frac{k_{\text{max}}^{4/3}}{v_0 k_0^{1/3}} \sim R \frac{k_0}{v_0}, \] (21)

respectively. Defining a dimensionless quantity \( \mu_M := \rho_M/v_0^2 \), we can see that the eq.(16) becomes

\[ \partial_t \mu_M = 2\gamma \mu_M - 2\zeta' \mu_M^2, \] (22)

where \( \zeta' = \zeta v_0^2 \sim Rk_0v_0 \). The second term in the right-hand side of the eq. (22) comes from the effect of the back-reaction effect. One can see easily from the above equation that the back-reaction gives an opposite effect to the original kinetic term and make the energy of the magnetic field balance with the energy of the fluid.

Although we know from the procedure used here that the eq. (22) holds only in a small time step as \( \mu_M \ll 1 \), we try to extrapolate. As a result we find the solution

\[ \mu_M = \frac{\gamma}{\zeta'} \frac{1}{1 - \left(1 - \frac{1}{\mu_M(0) \zeta'}\right)e^{-2\gamma t}}. \] (23)

One can see easily that the magnetic ‘energy’ goes toward the terminal value \( \mu_M^* = \gamma/\zeta' \sim R^{-1/2} \) for a time scale \( \sim \gamma^{-1} \). This value corresponds to the saturation value, which is estimated naively on the assumption that the drain by the magnetic field is comparable to the turbulent power[3][5].
3 Applications

In this section we apply the eq. (22) to two examples which the magnetic field is amplified by the dynamo mechanism. First, we treat the time evolution of the magnetic field during the first order phase transition in the very early universe. We also consider briefly the amplification of the magnetic field in interstellar mediums.

3.1 Electroweak Plasma

There are attractive mechanisms of the generation of the primordial magnetic field in the course of cosmological phase transitions[8]. In these scenarios the strong magnetic field is expected to be amplified by MHD turbulence during the first order phase transition. The detail of the amplification has been discussed by using Kulsrud and Anderson equation in the ref. [9].

We reconsider the amplification of the magnetic field during the phase transition by using the modified Kulsrud and Anderson equation. The time scale for the equipartition is \( t_{\text{equi}} \sim \gamma^{-1} \sim R^{-1/2} v_0^{-1} k_0^{-1} \). Since the Reynolds number is \( R \sim 10^2 \) [9], we can see that it is the same order with the time scale of the phase transition. Thus, the magnetic field can be amplified enough and the final energy is given by

\[
\mathcal{E}_M^* \sim R^{-1/2} \mathcal{E}_v \sim 0.1 \times \mathcal{E}_v, \tag{24}
\]

where \( \mathcal{E}_v \) is the energy of the plasma fluid.

3.2 Interstellar Mediums

As we stated in Introduction, the kinetic dynamo theory breaks down in interstellar mediums[5]. For interstellar mediums, typical values of key quantities are \( 2\pi/k_0 \sim 100\text{pc} \), \( v_0 \sim 10^6\text{cm/s} \) and \( R \sim v_0/k_0 \nu \sim 10^8 \), where \( \nu \) denotes the kinetic ion viscosity; \( \nu \sim 10^{18}\text{cm}^2\text{s}^{-1} \). Then the typical time scale is given by \( t_{\text{ISM}} \sim \gamma^{-1} \sim 10^2\text{yr} \). Since the time scale of the mean field is \( \sim 10^{10}\text{yr} \), we realize again the mean field theory is meaningless in the present perturbative approach. The final energy of the magnetic field is given by \( \mathcal{E}_M^* \sim 10^{-4} \times \mathcal{E}_v \).
4 Concluding Remark

In this paper, we considered the lowest order back reaction to the kinetic dynamo theory and modified the equation for the energy of the magnetic field. As a result, we obtained the successful time evolution of the energy of the magnetic field. That is, the terminal value of the magnetic energy obtained from the eq. (23) equals to the previous qualitative estimation of the saturation energy\[3\][5]. We also presented the expression depending on \(k\) (eq. (13)), with which we can evaluate the evolution of the magnetic field for various scales. Since the present formalism is general, our equation is useful for other situations, for example, the fireball model for \(\gamma\)-ray bursts\[10\].

Finally, we should comment on our assumption for the initial condition (eq. (12)) and the extrapolation of the eq. (22). We choose the initial condition in order to obtain the simple result like the eq. (22). Although this assumption holds approximately in some cases, it may not be correct in general cases. We should also note that we considered only the effect of lowest order back-reaction. Properly speaking, if one wishes to analyse the vicinity of the equipartition, one must take account of effects of higher order back-reaction. The study near the equipartition might become clear by using something like the renormalization group approach. The study for more general initial condition and with higher order back reaction should be done in the future. At the same time, the spatial structure as the typical coherent length of the magnetic field also should be discussed.

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