OPTIMIZING LINEAR EXTENSIONS

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Abstract. The minimum number of elements needed for a poset to have exactly \( n \) linear extensions is at most \( 2\sqrt{n} \). In a special case, the bound can be improved to \( \sqrt{n} \).

1. Introduction and definitions

A partially ordered set, or poset, \( P = (X, \preceq) \) consists of a set \( X \) together with a partial ordering \( \preceq \) on \( X \). For background on these structures, the reader is encouraged to review [3] and [4].

One statistic that can hint at how much information is missing in a partial ordering is based on the following definition.

Definition 1.1. A linear extension of a poset \( P = (X, \preceq) \) is a total ordering of the elements of \( X \) that is compatible with \( \preceq \). The number of linear extensions of \( P \) is denoted \( e(P) \).

As suggested in [3], the number of linear extensions of a poset gives an indication of the intricacy of the original partial ordering. Thus understanding the function \( e \) can provide some insight into the complexity of the structure of partial orderings.

Another poset statistic, the number of order ideals in a poset, is considered in [1], and a bound is given for the minimal number of elements needed to have a particular number of order ideals. Here, the analogous question is answered for the function \( e \).

Definition 1.2. The size of a poset \( P = (X, \preceq) \), denoted \( |P| \), is the cardinality of \( |X| \).

Definition 1.3. For any integer \( n \geq 1 \), set \( \lambda(n) = \min\{|P| : e(P) = n\} \).

The main result of this work, Theorem 3.2, is the bound

\[ \lambda(n) \leq 2\sqrt{n}. \]

In a certain case, as discussed in Section 4, this bound can be improved further to \( \sqrt{n} \). As displayed in Table 1, there are values of \( n \) for which \( \lambda(n) \) equals \( 2\sqrt{n} \).

In the next section, the values of \( \lambda(n) \) for small \( n \) are given, together with examples of the posets that obtain them. Furthermore, the poset operations that give the primary tools for proving Theorem 3.2 are stated. Section 3 consists of the main result, and a special case is treated in the last section.

2. Examples and arithmetic of poset operations

Before describing how basic poset operations affect the function \( \lambda \), it is instructive to calculate \( \lambda(n) \) for some small values of \( n \), and to view the posets that give these values. These examples appear in Table 1 and as sequence A160371 in [2].

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Two elementary operations on posets are the direct sum and the ordinal sum. Note that a poset which can be constructed entirely by these two operations is called series-parallel.

**Definition 2.1.** Let $P$ and $Q$ be posets on the sets $X_P$ and $X_Q$, respectively, with order relations $\preceq_P$ and $\preceq_Q$, respectively. The direct sum $P + Q$ is the poset defined on $X_P \cup X_Q$, with order relations $\preceq_P \cup \preceq_Q$. The ordinal sum $P \oplus Q$ is the poset defined on $X_P \cup X_Q$, with order relations $\preceq_P \cup \preceq_Q \cup \{x_P \preceq x_Q : x_P \in X_P, x_Q \in X_Q\}$.

The next lemma follows immediately from the definitions.

**Lemma 2.2.** For posets $P$ and $Q$,

$$e(P + Q) = \binom{|P| + |Q|}{|P|} e(P)e(Q)$$

and

$$e(P \oplus Q) = e(P)e(Q).$$

**Definition 2.3.** For any $\ell \geq 0$, let the poset $C_\ell$ be the chain of $\ell$ elements, where $C_0 = \emptyset$.

Certainly the poset $C_\ell$ is already a total ordering, so $\lambda(C_\ell) = 1$ for all $\ell$. Moreover, it follows from the identities of Lemma 2.2 that

$$e(P + C_\ell) = \binom{|P| + \ell}{|P|} e(P)$$

and

$$e(P \oplus C_\ell) = e(P)$$

for all $\ell \geq 0$. Equation (1) implies that a poset with $n$ linear extensions can have arbitrarily large size. Perhaps unexpectedly, equation (1) will be very helpful in bounding $\lambda(n)$. The key is to employ it as in the following result.

**Proposition 2.4.** For all $\ell \geq 0$, $e((P \oplus C_\ell) + C_1) = (|P| + \ell + 1)e(P)$.

Proposition 2.4 gives the following initial result for all $n$.

**Corollary 2.5.** If $n = ab$ for $a, b \in \mathbb{Z}^+$ with $a < b$, then $\lambda(n) \leq b$.

**Proof.** First note that

$$\lambda(n) \leq n$$

for all $n \in \mathbb{Z}^+$, by considering the $n$-element poset $C_{n-1} + C_1$, which has $n$ linear extensions.
Let $P$ be a poset of size $\lambda(a)$, with $e(P) = a$. Since $a < b$, equation (2) implies $\lambda(a) < b$, and so $b - 1 - |P| \geq 0$. Set $Q = (P \oplus C_{b-1-|P|}) + C_1$. Then $|Q| = |P| + b - 1 - |P| + 1 = b$, and $e(Q) = (|P| + b - 1 - |P| + 1)e(P) = ab = n$. \hfill \square

3. Bounds

The proof of the main result, Theorem 3.2, begins with an analysis of the following $m$-element poset $Q_{i,j,m}$, where $1 \leq i < j \leq m - 2$. Note that $Q_{i,j,m}$ is not series-parallel.

In any linear extension of $Q_{i,j,m}$, the elements $\{c_1, c_2, \ldots, c_{m-2}\}$ may appear in exactly one order. The element $a$ can appear anywhere after $c_i$, while the element $b$ can appear anywhere before $c_j$. The elements $a$ and $b$ are incomparable in $Q_{i,j,m}$, so they can appear in either order if they both appear between $c_k$ and $c_{k+1}$ in a linear extension. Thus

$$e(Q_{i,j,m}) = (m - 1 - i)j + (j - i) = (m - i)j - i,$$

and so

$$\lambda((m - i)j - i) \leq m.$$

**Proposition 3.1.** For all integers $n \geq 1$ and $d \geq 1$,

$$\lambda(n) \leq \lfloor n/d \rfloor + d.$$

**Proof.** This is proved by induction on $d$, where the case $d = 1$ follows from equation (2).

Now suppose that $d \geq 2$ and that the result holds for all $d' \in [1, d)$. The integer $n$ can be written as $n = qd - r$, where $r \in [0, d - 1]$. If $r \geq 1$ and $q + r - 2 \geq d$, then $Q_{r,d,q+r}$ is a poset having $n$ linear extensions and size

$$q + r \leq \lfloor n/d \rfloor + 1 + (d - 1) = \lfloor n/d \rfloor + d.$$

Thus it remains to consider when $r = 0$ or $q + r - 2 < d$.

If $r = 0$, then $n = qd$ and Lemma 2.2 implies that

$$\lambda(n) \leq \lambda(q) + \lambda(d) \leq q + d = \lfloor n/d \rfloor + d.$$
This leaves the case when \( r \in [1, d - 1] \) and \( q + r - 1 \leq d \). The few cases that remain when \( d \in \{2, 3\} \) are easy to check (in fact, they concern only \( n \leq 12 \), and so appear in Table 1). For the conclusion of the argument, suppose \( d \geq 4 \).

Rewrite \( n = n' q + r' \) where \( r' \in [0, d - 2] \). Because \( n = q(d - 1) + q + r \), the restrictions on \( q, r, \) and \( d \) imply that there is at most one extra factor of \( d - 1 \) in \( q + r \). That is, \( q' \in \{q, q + 1\} \). From the induction hypothesis for \( d' = d - 1 \), it follows that \( \lambda(n) \leq q' + d - 1 \leq q + d \), which completes the proof. \( \square \)

Although the bound in Proposition 3.1 is linear, the fact that it holds for all integers \( d \geq 1 \) indicates that it can be improved further.

**Theorem 3.2.** For all \( n \geq 1 \), \( \lambda(n) \leq 2\sqrt{n} \).

**Proof.** Apply Proposition 3.1 with \( d = \lceil \sqrt{n} \rceil \) and \( \varepsilon = \lceil \sqrt{n} \rceil - \sqrt{n} \), where \( \varepsilon \in [0, 1) \):

\[
\lambda(n) \leq \left\lceil \frac{n}{\sqrt{n}} \right\rceil + \lceil \sqrt{n} \rceil = \left\lceil \sqrt{n} - \varepsilon + \frac{\varepsilon^2}{\sqrt{n} + \varepsilon} \right\rceil + \sqrt{n} + \varepsilon. \tag{3}
\]

If \( \varepsilon = 0 \), then \( d = \sqrt{n} \), and the theorem holds. If \( \varepsilon \in (0, 5] \), then \( \varepsilon - 1 \leq -\varepsilon \), and

\[
\left\lceil \sqrt{n} - \varepsilon + \frac{\varepsilon^2}{\sqrt{n} + \varepsilon} \right\rceil \leq \lceil \sqrt{n} \rceil = \sqrt{n} + \varepsilon - 1 \leq \sqrt{n} - \varepsilon.
\]

On the other hand, if \( \varepsilon \in (5, 1) \), then \( \varepsilon - 2 < -\varepsilon \), and

\[
\left\lfloor \sqrt{n} - \varepsilon + \frac{\varepsilon^2}{\sqrt{n} + \varepsilon} \right\rfloor < \left\lfloor \sqrt{n} - \varepsilon + \frac{1}{2} \right\rfloor \leq \lceil \sqrt{n} \rceil.
\]

In other words, if \( \varepsilon \in (5, 1) \), then

\[
\left\lfloor \sqrt{n} - \varepsilon + \frac{\varepsilon^2}{\sqrt{n} + \varepsilon} \right\rfloor \leq \lceil \sqrt{n} \rceil - 1 = \sqrt{n} + \varepsilon - 2 < \sqrt{n} - \varepsilon.
\]

Therefore, for any \( \varepsilon \in [0, 1) \), it follows from inequality 3 that \( \lambda(n) \leq 2\sqrt{n} \). \( \square \)

4. A SPECIAL CASE

As suggested in Corollary 2.5, the number \( \lambda(n) \) is influenced by the factorization of \( n \). In particular, primality of \( n \) can be a challenge for the function \( \lambda \). On the other hand, if \( n \) factors in a particular way, then the bound on \( \lambda(n) \) can be further tightened along the lines of Corollary 2.5.

**Corollary 4.1.** If \( n = ab \) for \( a, b \in \mathbb{Z}^+ \) with \( 2\sqrt{b} \leq a \leq b \), then \( \lambda(n) \leq \sqrt{n} \).

**Proof.** Suppose that \( n = ab \), where \( 1 \leq a \leq b \leq (a/2)^2 \). Construct a poset \( P \) with \( e(P) = b \) and \( |P| = \lambda(b) \leq 2\sqrt{b} < a \). Let \( Q = (P \oplus C_{a-1-|P|}) + C_1 \). Note that \( e(Q) = ab = n \) and \( |Q| = a \). Since \( n = ab \) and \( a \leq b \), this implies that \( |Q| \leq \sqrt{n} \), and so \( \lambda(n) \leq \sqrt{n} \). \( \square \)

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