Two Mutual Exclusion Algorithms for Shared Memory

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Abstract

In this paper, we introduce two algorithms that solve the mutual exclusion problem for concurrent processes that communicate through shared variables, [2]. Our algorithms guarantee that any process trying to enter the critical section, eventually, does enter it. They are formally proven to be correct.

The first algorithm uses a special coordinator process in order to ensure equal chances to processes waiting for the critical section. In the second algorithm, with no coordinator, the process exiting the critical section is in charge to fairly elect the following one. In the case that no process is waiting, the turn is marked free and will be determined by future waiting processes. The type of shared variables used are a turn variable, readable and writable by all processes; and a flag array, readable by all with flag[i] writable only by process i. There is a version of the first algorithm where no writable by all variable is used.

The bibliography reviewed for this paper is [3] and [4], all the rest is original work.
1 Asymmetric Algorithm

This algorithm uses a coordinator process in charge of assigning the right to access the critical section to processes trying to enter it. The algorithms are written using the notation described in [1].

Shared variables

- Processes $\in \{p_0, ..., p_{N-1}\}$, plus the coordinator.
- $turn \in \{0, ..., N-1, \text{THINKING}\}$, initially THINKING, readable and writable by all.
- $flag[i] \in \{\text{REMAINDER, WAITING}\}, i \in \{0, ..., N-1\}$. Initially, $(\forall i | 0 \leq i \leq N-1 : flag[i] = \text{REMAINDER})$. $flag[i]$ is readable by all and writable by $p_i$.

A process $p_i$ willing to enter the critical section announces this by setting $flag[i]$ to WAITING, and then waits until $turn$ is $i$. On exit, it reverts $flag[i]$ to REMAINDER and sets $turn$ to THINKING, so the coordinator knows the process is done and turn must be updated.

Algorithm for process $p_i$

\[
\begin{align*}
flag[i] & \leftarrow \text{WAITING} \\
\text{if} & \hspace{1em} turn \neq i \\
\text{CRITICAL SECTION} \\
flag[i] & \leftarrow \text{REMAINDER} \\
\text{turn} & \leftarrow \text{THINKING}
\end{align*}
\]

The coordinator performs an infinite loop where it serially inspects the state of processes. When a process is found waiting for the critical section, the coordinator grants the access to it; and then waits until that process leaves the section —signaled by setting $turn$ to THINKING.
Algorithm for the coordinator process

\[ p \leftarrow 0 \]

\[ \begin{align*}
\infty & \quad flag[p] = WAITING \implies \\
\implies turn & \leftarrow p \\
\implies turn & \neq THINKING \\
p & \leftarrow (p + 1) \mod N
\end{align*} \]

Although the above algorithms are written mathematically, problems on concurrency are difficult to reason about. It is much more suitable to define algorithms by means of finite state automata with edges representing events – transitions between two states. An event may have a precondition, written “precondition \(\implies\)”; or an effect, written “\(\implies\) effect”. This is a practical simplification of the formalism in [4].

Automaton for process \(p_i\).

\[ \begin{align*}
1 & \quad \implies flag[i] = WAITING \\
2 & \quad \implies turn = i \\
\text{critical section} & \quad \implies flag[i] = REMAINDER \\
3 & \quad \implies turn = THINKING \\
4 & \quad \implies turn = THINKING
\end{align*} \]
Theorem 1.1 Mutual exclusion.

No two processes can be at the same time in the critical section state.

Proof of theorem 1.1

Suppose, for a contradiction, that there exists an execution where process $b$ is the first one to enter the critical section while another one, $a$, is still inside. This timeline shows the situation.

If $b$ has to enter the critical section while $a$ is in, there must be another process, say $c$, to change $\text{turn}$ to THINKING before $a$ does (within the lapse $[\text{CS}_a, 4_a]$), so the coordinator escapes from 2.2 and can reach 2.2 again to let $b$ in. But that means that $c$ is in the critical section overlapped with $a$ before $b$ overlaps with $a$. 
This is a contradiction, as we assumed that \( b \) is the first one breaching the critical section. \( \square \)

**Lemma 1.2** If the turn is assigned to a process, then, it will enter the critical section.

**Proof of lemma 1.2**

If process \( a \) is in a state 2 and turn is assigned to it –as the coordinator reaches 2.2; only \( a \) may enter the critical section (by the condition \( \text{turn} = i \Rightarrow \) ) since the coordinator will be in state 2.2 while \( \text{turn} \neq \text{THINKING} \). In this point, there is no other process in the critical section (or \( a \) could violate mutual exclusion) able to change \( \text{turn} \) to THINKING.

A second case is when process \( a \) leaves the critical section. Because \( a \) sets \( \text{flag}[i] \) to REMAINDER before signaling the coordinator to choose the following process for the critical section, with \( \text{turn} = \text{THINKING} \), \( a \) can’t reach state 4 with the turn reassigned to it. \( \square \)

**Theorem 1.3** A process waiting in state 2 will enter the critical section.

**Proof of theorem 1.3**

The coordinator assigns turns in a round-robin fashion between processes with \( \text{flag}[i] = \text{WAITING} \) –in the loop 2, 2.1, 2.2, 3 using counter \( p \).

Once a process \( a \) is waiting in 2, processes with number \([a + 1, ..., N - 1, 0, ..., a - 1]\) could enter the critical section before \( a \). If one such process is not waiting, \( p \) increases towards \( a \); if it is waiting, it will enter the critical section and, at exit, \( p \) is equally increased towards \( a \). Finally, \( p = a \) and \( a \) will enter the critical section. \( \square \)
2 Asymmetric Algorithm with one-writer/multiple-reader variables

A new version of our previous algorithm can be devised in order to have the turn variable to be writable only by the coordinator process. This is at the expense of making the exiting process, \( p_i \), to wait until the coordinator changes turn to something different from \( i \). Otherwise, \( p_i \) would be able to re-enter the critical section using its previous turn. And worse, the coordinator could change turn to a different waiting process and the mutual exclusion would be broken.

Shared variables

- Processes \( \in \{ p_0, ..., p_{N-1} \} \), plus the coordinator.
- \( \text{turn} \in \{ 0, ..., N-1, \text{THINKING} \} \), initially THINKING, writable only by the coordinator process, readable by all.
- \( \text{flag}[i] \in \{ \text{REMAINDER}, \text{WAITING} \} \), \( i \in \{ 0, ..., N-1 \} \). Initially, \((\forall i | 0 \leq i \leq N - 1 : \text{flag}[i] = \text{REMAINDER})\). \( \text{flag}[i] \) is readable by all and writable by \( p_i \).

| Algorithm for process \( p_i \) |
|----------------------------------|
| \( \text{flag}[i] \leftarrow \text{WAITING} \) |
| \( \quad \text{turn} \neq i \) |
| CRITICAL SECTION |
| \( \quad \text{flag}[i] \leftarrow \text{REMAINDER} \) |
| \( \quad \text{turn} \neq i \) |
Algorithm for the coordinator process

\[ p \leftarrow 0 \]

\[
\text{flag}[p] = \text{WAITING} \Rightarrow
\]

\[ \text{turn} \leftarrow p \]

\[ \text{flag}[p] \neq \text{REMAINDER} \]

\[ \text{turn} \leftarrow \text{THINKING} \]

\[ p \leftarrow (p + 1) \text{ mod } N \]

3 Symmetric Algorithm

This algorithm has no special coordinator process, but, in some way, it integrates its task. When a process exits the critical section, it fairly decides which waiting process is granted the next access—setting variable \( \text{turn} \). If no one is waiting, \( \text{turn} \) is set to FREE, same as at the begining. If the critical section is found free, incoming processes will elect one of them to enter the critical section.

Shared variables

- Processes \( \in \{p_0, ..., p_{N-1}\} \)

- \( \text{turn} \in \{0, ..., N - 1, \text{THINKING}, \text{FREE}\} \), initially FREE, readable and writable by all.

- \( \text{flag}[i] \in \{\text{REMAINDER}, \text{WAITING}, \text{CANDIDATE}\}, i \in \{0, ..., N - 1\} \). Initially, \( (\forall i | 0 \leq i \leq N - 1 : \text{flag}[i] = \text{REMAINDER}) \). \( \text{flag}[i] \) is readable by all and writable by \( p_i \).
Algorithm for process $p_i$, entry part

$flag[i] \leftarrow$ WAITING

$\implies \quad \text{turn} = \text{THINKING}$

$\text{turn} = \text{FREE} \implies$

$\quad flag[i] \leftarrow$ CANDIDATE

$\implies \quad \text{turn} = \text{FREE} \land flag[i] = \text{CANDIDATE}$

$n\text{Candidates} \leftarrow 0$

$\min\text{Candidate} \leftarrow -1$

$N - 1 \implies j \rightarrow 0$

$\quad flag[j] = \text{CANDIDATE} \implies$

$\quad \quad n\text{Candidates} \leftarrow n\text{Candidates} + 1$

$\quad \quad \min\text{Candidate} \leftarrow j$

$\text{turn} = \text{FREE} \land n\text{Candidates} = 1 \implies$

$\quad \text{turn} \leftarrow i$

$\text{turn} \neq \text{FREE} \lor \min\text{Candidate} < i \implies$

$\quad flag[i] \leftarrow$ WAITING

$\quad flag[i] \leftarrow$ WAITING

$\implies \quad \text{turn} \neq i$

CRITICAL SECTION
Algorithm for process $p_i$, exit part

CRITICAL SECTION

turn $\leftarrow$ THINKING
flag[$i$] $\leftarrow$ REMAINDER
nextTurn $\leftarrow$ THINKING

\[
\text{nextTurn} = \text{THINKING} \land i + 1 \rightarrow j \rightarrow N - 1, 0 \rightarrow j \rightarrow i - 1
\]

\[
\text{flag}[j] \neq \text{REMAINDER} \Rightarrow
\text{nextTurn} \leftarrow j
\]

nextTurn = THINKING $\Rightarrow$

\[
\text{turn} \leftarrow \text{FREE}
\]

nextTurn $\neq$ THINKING $\Rightarrow$

\[
\text{turn} \leftarrow \text{nextTurn}
\]
This is the automaton version of the algorithm.

**Entry part**

1. $\Rightarrow \text{flag}[i] = \text{WAITING}$

2. $\Rightarrow \text{turn} \neq \text{THINKING}$

3. $\Rightarrow \text{turn} = \text{FREE}$
   - $\Rightarrow \text{flag}[i] = \text{CANDIDATE}$
   - $\Rightarrow \text{turn} \neq \text{FREE} \lor \text{flag}[i] \neq \text{CANDIDATE}$

4. $\Rightarrow \text{turn} \neq \text{FREE}$
   - $\Rightarrow \text{flag}[i] = \text{WAITING}$

5. $\Rightarrow \text{turn} = i$

**Critical section**

$\Rightarrow \text{flag}[i] = \text{WAITING}$
Exit part

\[ \Rightarrow \text{turn} = \text{THINKING} \]

\[ \Rightarrow \text{flag}[i] = \text{REMAINDER} \land j = (i + 1) \mod N \]

\[ \Rightarrow \text{inc}(j) \mod N \]

\[ j \neq i \Rightarrow \]

\[ \Rightarrow \text{flag}[j] = \text{REMAINDER} \Rightarrow \]

\[ \Rightarrow \text{flag}[j] \neq \text{REMAINDER} \Rightarrow \]

\[ \Rightarrow \text{turn} = \text{FREE} \]

\[ \Rightarrow \text{turn} = j \]
Lemma 3.1 No two processes are in state 3.5.1 simultaneously.

Proof of lemma 3.1 Suppose for a contradiction that two processes \(a\) and \(b\) are in state 3.5.1, being \(a\) the first to arrive.

The number of processes with \(flag[i] = \text{CANDIDATE}\) is computed in the loop 3.4, 3.4.1, 3.4.2. When \(a\) arrives at 3.5.1, \(n\text{Candidates} = 1\). That means that \(b\) has not reached state 3.2 (its flag is not \(\text{CANDIDATE}\)) when \(a\) inspects \(flag[b]\).

Now, by assumption, \(a\) remains in 3.5.1 until \(b\) arrives. That implies that \(flag[a] = \text{CANDIDATE}\) during that lapse. Eventually, \(b\) passes 3.2, marking itself as candidate and, when it ends the 3.4 loop, \(n\text{Candidates}\) for \(b\) must be at least 2 (itself and \(a\)).

Thus, \(b\) can’t enter 3.5.1 as the condition to do so fails. This contradicts the assumption that both \(a\) and \(b\) can be in 3.5.1 simultaneously. \(\Box\)

We say that a turn assignment (like the one when state 3.5.2 is reached) is an enabling turn assignment iff that assignment is the latest before the process granted by it enters the critical section.

The following timeline shows an enabling turn assignment for process \(a\). In the marked lapse, turn is not changed.
**Theorem 3.2** *Mutual exclusion.*

No two processes can be at the same time in the extended critical section – states in \([\text{critical section}, 8]\).

**Corollary 3.3** *(of theorem 3.2)*

Two processes can’t be simultaneously in the critical section.

**Proof of theorem 3.2**

Suppose, for a contradiction, that there exists an execution where process \(b\) is the first one to enter the extended critical section (states \([\text{critical section}, ... , 8]\) while another one, say \(a\), is still in. This is:

\[
\begin{array}{c}
\text{a} \\
\hline
\text{critical section} \quad \text{6.3 or 7} \\
\hline
\text{b}
\end{array}
\]

Now, let’s consider the cases where the enabling turn assignments of \(a\)’s and \(b\)’s can happen.

- **Case 1.** The enabling assignments for both processes \(a\) and \(b\) occur at 3.5.2.
  
  By lemma 3.1, when \(a\) is in 3.5.1, \(b\) is not. From 3.5.1, \(a\) reaches the extended critical section, and, by assumption, it remains *alone* within it (until \(b\) also arrives at). Thus, during this lapse, *turn* is either \(a\) or THINKING, never FREE:

\[
\begin{array}{c}
\text{a} \\
\hline
1.5.1 \quad 1.5.2 \\
\hline
\text{CS} \quad \text{6.3 or 7} \\
\hline
\text{turn} \quad \text{(FREE)} \quad \text{a or THINKING}
\end{array}
\]

As a consequence, \(b\) can’t access state 3.5.1 (nor 3.5.2) to perform its enabling assignment, since *turn* \(\neq\) FREE. This contradicts this case.

- **Case 2.** No matter which is the enabling assignment for \(a\), the enabling assignment for process \(b\) is performed by a third process \(c\) going from state 6.3 to 8. This would be the situation.
By the contradicting assumption, $a$ must enter the critical section before $b$ (and remain there until it overlaps with $CS_b$). In this case, $CS_a$ must be before $8_c$ as well, because $turn = b$ is set at that moment and it is fixed until $CS_b$. This implies that $a$ must be overlapped in the critical section with process $c$. This contradicts the assumption that $a$ and $b$ are the first processes that break the mutual exclusion.

• Case 3 (last). The enabling assignment for process $a$ is done by a third process, say $c$; and the enabling assignment for $b$ occurs at state 3.5.2. This is the outline now:

Let’s consider when $b$ arrived at state 3.5.1. Because this needs $turn = FREE$, a process must assign $FREE$ to $turn$ as it reaches its state 8.

The assignment can’t be in $[CS_a, 3.5.2_b]$ (lapse 1), because that process would overlap its critical section with $a$’s. Neither in $[8_c, CS_a]$ because $turn = a$ is fixed for $a$ to enter the critical section. Neither in $[CS_c, 8_c]$ (lapse 2), because that overlaps with $c$’s critical section. Therefore, $b$ is in state 3.5.1 before $CS_c$. This is:
Now let’s consider how \textit{turn} is set to \textit{c} in \([3.5.1_b, \text{CS}_c]\). In this interval, by lemma \([3.1]\) \textit{c} cannot do a turn assignment to itself from state 3.5.1. Hence, a process, say \(j\), does this assignment in its state 8. When did \(j\) enter its critical section? If later than 3.5.1\(_b\), then again, another process in 8 should have performed \(j\)’s enabling assignment after 3.5.1\(_b\). But there may exist only a finite number of processes entering its critical section in \([3.5.1_b, \text{CS}_c]\) and enabling the following one until \(j\) and then \(c\). Let’s suppose it is process \(k\) the one which is in state 8 after 3.5.1\(_b\) but enters its critical section before. This is depicted in the following timeline.

Finally, we reach the desired contradiction: \textit{turn} can’t be changed to FREE in \([\text{CS}_k, 8_k]\) because there cannot exist another process in the critical section overlapped with \(k\) — \(a\) and \(b\) are the first ones to overlap.
Lemma 3.4 If the turn is assigned to a process, then, no other one will change this assignment before it enters the critical section.

Proof of lemma 3.4

If a process \( a \) assigns the turn to itself in state 3.5.2, the assignment won’t be changed (before \( a \) enters the critical section):

- Not at state 8. Otherwise, this would imply that a process is in \([\text{critical section}, 8]\) and \( a \) could use its turn to break theorem 3.2.

- Not at state 3.5.1, because, by lemma 3.1, it can’t be already in 3.5.1 and, as seen in the previous point, there is no one in the extended critical section to change turn to FREE.

If turn is assigned to \( a \) at state 8 by a process \( b \), again, no other one may change this assignment:

- Not at state 3.5.2. For a contradiction, suppose process \( c \) is the first one in doing so. \( c \) must be at state 3.5.1 (turn = FREE) before turn is set to \( b \), so the assumptions for this case hold. If this can happen, then, \( c \) could also reach the critical section in \([\text{CS}_b, 8_b]\), breaching theorem 3.2.
• Not at state 8. If, oppositely, a process $c$ is the first to do this, it must enter the critical section after $s_b$ (not to be overlapped with $b$). But this would require a change in $turn$ to $c$ before the supposed first change after $turn = a$.

**Lemma 3.5** If the turn is assigned to a process, then, it will enter the critical section.

**Proof of lemma 3.5**
If a process sets $turn$ to itself in 3.5.2; it is able to reach state 4 – there is no condition against this.
A process in the extended critical section can’t choose itself for $turn$: it is not eligible in state 6.1 because its flag is REMAINDER.
If $turn$ is assigned to a process in 8, in case it is waiting at state 2, the assignment enables it to get to state 4.
Finally, since an assignment of $turn$ to a process doesn’t change (lemma 3.4) and it is able to arrive at state 4, it will eventually enter the critical section.

□
Lemma 3.6  A given process a in [5, 8] will elect another process b for turn, considering this order: a + 1, ..., N − 1, 0, 1, ... a − 1, such that b is smaller or equal than the minimum identifier of the processes which passed state 2 before process a arrived at 5.

Proof of lemma 3.6
The loop defined by the states 6, 6.1, 6.2 makes process a to look for a waiting process (flag[j] ≠ REMAINDER) with minimum identifier following this order: a + 1, ..., N − 1, 0, 1, ..., a − 1. Every process that has changed its flag to WAITING in state 2 before the loop begins (actually in state 5) will be eligible for turn in state 6.1.

There is the corner case that a process c becomes WAITING when the loop has begun. In this case, c will be considered only if the loop counter j < c. Note that c will remain in state 2 until the decision for turn is made, because turn is set to THINKING by a in 5. It will able to continue when a reaches state 8 and turn is set. □

Corollary 3.7 (of lemma 3.6)
When a process sets turn to FREE, there is no process in a state between [3, 4].

Proof of corollary 3.7 Any process in a state between [3, 4] has set its flag[i] to WAITING before turn is set to THINKING by the elector process in 5. Thus, it must be considered for turn in 6.1. This implies that turn cannot be set to FREE. □

Lemma 3.8 If turn is set to FREE, one of the waiting processes will be elected for turn in state 3.5.2.

Proof of lemma 3.8
If turn is set to FREE, by corollary 3.7 all waiting processes must be at state 2 before this assignment. The remaining processes have flag[i] = REMAINDER. The assignment to FREE allows processes waiting at 2 to continue (turn ≠ THINKING). At least one of them will reach state 3.2 (while turn = FREE), as well as at least one will reach state 3.3 (again while turn is not set in 3.5.2).

Let’s consider the set of processes, S, that reach state 3.3 from 3.1 since turn = FREE for the last time, and for which their flag[i] = CANDIDATE. S is not empty, and its size is bounded by N; this is, all the processes.
For a given execution, consider the instant \( t \) after which no more processes are added to \( S_t \) and let \( a \) be the process with the smallest identifier in \( S_t \).

If a process different from \( a \) is in state 3.5.1 then, by lemma 3.1, \( a \) won’t be able to reach 3.5.1 before that process sets \( \text{turn} \) to itself. And as a consequence, this is proves the theorem for this case.

If on the contrary, no process is already in 3.5.1, \( a \) will reach this state.

The loop defined by states 3.4, 3.4.1, 3.4.2 counts how many processes set \( \text{flag}[i] = \text{CANDIDATE} \) in 3.2 and which of them has the lowest identifier.

Any process \( b \in S_t, a < b \), will reach state 3.5 and because its \( \text{minCandidate} \) variable is less than \( b \), it will proceed to state 3.6 and, finally, it will set \( \text{flag}[b] \) to WAITING. This means that \( b \) is no more a candidate, it will not reach state 3.3 again in this loop, and that \( b \) is removed from \( S_t \). Eventually, only \( a \) will be in \( S_t \). Then, \( a \) will find that \( \text{turn} = \text{FREE} \) and \( n\text{Candidates} = 1 \) (being \( \text{minCandidate} = a \)). This allows \( a \) to assign \( \text{turn} \) to itself in state 3.5.2. \( \square \)

**Theorem 3.9** A waiting process (\( \text{flag}[i] \neq \text{REMAINDER} \)) will be elected for \( \text{turn} \) and it will enter the critical section.

**Proof of theorem 3.9**

By lemma 3.6 a process in the extended critical section fairly chooses the following process for \( \text{turn} \) among waiting processes, or sets it to FREE if no one is waiting when it performed the election.

In case \( \text{turn} \) is FREE, as initially, by lemma 3.8, \( \text{turn} \) is eventually granted to a process.

By lemma 3.4 once a process has the \( \text{turn} \), it is not changed before the process arrives at the critical section. By lemma 3.5 a process with the turn will reach the critical section.

Finally, a process that sets its \( \text{flag}[i] \) to WAITING, eventually arrives at state 4, and doesn’t change this setting, unless it reaches state 5 after the critical section.

Therefore, a waiting process will, eventually, enter the critical section. \( \square \)
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