Physical model for the electron spin correlation

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November 9, 2021

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Abstract

The quantum formula for the spin correlation of the bipartite singlet spin state, \( C_Q(a, b) \), is derived on the basis of a probability distribution \( \rho(\phi) \) that is generic, i.e., independent of \((a, b)\). In line with a previous result obtained within the framework of the quantum formalism, the probability space is partitioned according to the sign of the product \( A = \alpha \beta \) of the individual spin projections \( \alpha \) and \( \beta \) onto \( a \) and \( b \); this precludes the transfer of \( \alpha \) or \( \beta \) from \( C_Q(a, b) \) to \( C_Q(a, b') \), for \( b' \neq b \). A specific physical model that reproduces the quantum spin correlation serves to validate our approach.

1 Introduction

In a recent paper [1] a detailed analysis was made of the spin projection operator correlation function \( C_Q(a, b) = \langle (\hat{\sigma} \cdot a) (\hat{\sigma} \cdot b) \rangle \) for the bipartite singlet spin state. The analysis, conducted strictly within the framework of the quantum formalism, led to an unequivocal probabilistic reading. Specifically, the calculation of \( C_Q(a, b) \) was shown to entail a partitioning of the probability space, which is dependent on the directions \((a, b)\). This result is the outcome of a purely theoretical analysis; however, it can be readily translated to the laboratory language, meaning that the series of values \((\pm 1)\) obtained for the projections \( \alpha \) and \( \beta \) leading to the experimental correlation \( C(a, b) \), cannot be mixed or combined with those obtained for \( \alpha \) and \( \beta' \) and leading to \( C(a, b') \), if \( b' \neq b \).

In the present paper we take these results further and obtain the quantum formula for the spin correlation, constructed on the basis of a unique probability distribution \( \rho(\phi) \). The distribution function is independent of \((a, b)\); the dependence on the directions resides exclusively in the partitioning of the probability space. In other words, for a given pair \((a, b)\), the entire set of values for \( \phi \), \( \{\phi\} \), is formed by two complementary subsets \( \{\phi\}^\pm_{ab} \), leading respectively to \( A = \alpha \beta = \pm 1 \). This is illustrated with the help of a specific physical model that reproduces the spin correlation.
The paper is organized as follows. Section II contains a brief introduction to the quantum description of the bipartite singlet state, and an explanation of the disaggregation of the correlation $C_Q(a, b)$ on the basis of the spin projection eigenfunctions associated with the directions $(a, b)$, entailing a partitioning of the probability space dependent on $(a, b)$. In Section III a (local hidden-variable) distribution function is derived, and the quantum spin correlation for the entangled state is constructed on this basis. A simple geometric model for the spin orientations serves to give concrete meaning to the quantum result. The theoretical discussion is translated in Section IV to the experimental language, in order to make contact with Bell-type tests.

\section{Quantum description of the bipartite singlet spin correlation}

We consider a system of two $1/2$--spin particles in the (entangled) singlet state

$$|\Psi^0\rangle = \frac{1}{\sqrt{2}} (|+r\rangle |-, -r\rangle + |-, +r\rangle |+r\rangle), \quad (1)$$

in terms of the simplified (standard) notation $|\phi\rangle |\chi\rangle = |\phi\rangle \otimes |\chi\rangle$, with $|\phi\rangle$ a vector in the Hilbert space of particle 1, and $|\chi\rangle$ a vector in the Hilbert space of particle 2. The state vectors

$$|+r\rangle = \cos \frac{\theta_r}{2} |+z\rangle + e^{i\varphi_r} \sin \frac{\theta_r}{2} |-z\rangle, \quad (2a)$$

$$|-, -r\rangle = e^{-i\varphi_r} \sin \frac{\theta_r}{2} |+z\rangle + \cos \frac{\theta_r}{2} |-, -z\rangle, \quad (2b)$$

form an orthogonal basis, with $0 \leq \theta_r \leq \pi$ and $0 \leq \varphi_r \leq 2\pi$, $\theta_r$ and $\varphi_r$ being the zenithal and azimuthal angles that define the Bloch vector $r = \mathbf{i} \sin \theta_r \cos \varphi_r + \mathbf{j} \sin \theta_r \sin \varphi_r + \mathbf{k} \cos \theta_r$ (see, e. g., Ref. [2]).

In Eq. (1) the direction of $r$ is arbitrary, since the singlet state is spherically symmetric. The projection of the first spin operator along an arbitrary direction $\mathbf{a}$ is described by $(\hat{\mathbf{\sigma}} \cdot \mathbf{a}) \otimes \mathbb{I}$, and the projection of the second spin operator along $\mathbf{b}$ is described by $\mathbb{I} \otimes (\hat{\mathbf{\sigma}} \cdot \mathbf{b})$. With the purpose of carrying out a detailed calculation of the correlation

$$C_Q(a, b) = \langle \Psi^0 | (\hat{\mathbf{\sigma}} \cdot \mathbf{a}) \otimes (\hat{\mathbf{\sigma}} \cdot \mathbf{b}) |\Psi^0\rangle, \quad (3)$$

we use Eqs. (2) to obtain

$$\langle \pm r | \hat{\mathbf{\sigma}} \cdot \mathbf{a} |\pm r\rangle = \pm r \cdot \mathbf{a} = \pm \cos \theta_{ra} \quad (4a)$$

and

$$\langle -r | \hat{\mathbf{\sigma}} \cdot \mathbf{a} |+r\rangle = \langle +r | \hat{\mathbf{\sigma}} \cdot \mathbf{a} |-, -r\rangle^* = e^{i\varphi} (\theta + i\varphi) \cdot \mathbf{a},$$

whence

$$|\langle \mp r | \hat{\mathbf{\sigma}} \cdot \mathbf{a} |\pm r\rangle| = |r \times \mathbf{a}|. \quad (4b)$$
In terms of the complete set of vectors in the composite Hilbert space,

\[ |\Psi_1\rangle = |+_r\rangle |-_r\rangle, \quad |\Psi_2\rangle = |-_r\rangle |+_r\rangle, \]
\[ |\Psi_3\rangle = |+_r\rangle |+_r\rangle, \quad |\Psi_4\rangle = |-_r\rangle |-_r\rangle, \]

we get, with the help of Eqs. (4),

\[ C_Q(a, b) = \langle \Psi_0 | (\hat{\sigma} \cdot a) \left( \sum_{k=1}^{4} |\Psi_k\rangle \langle \Psi_k| \right) (\hat{\sigma} \cdot b) |\Psi_0\rangle = \sum_{k=1}^{4} F_k, \]

with

\[ F_1 = -\frac{1}{2} (r \cdot a)(r \cdot b) = F_2, \]
\[ F_3 = \frac{1}{2} [(r \times a) \cdot (r \times b) - ir \cdot (a \times b)] = F_4^* . \]

These equations are greatly simplified by making \( r \) lie on the plane formed by \( a \) and \( b \), i.e., \( \varphi_r = \varphi_a = \varphi_b = 0 \); with \( \theta_{ra} = \theta_r - \theta_a \) and \( \theta_{rb} = \theta_r - \theta_b \) they become

\[ F_1 = F_2 = -\frac{1}{2} \cos \theta_{ra} \cos \theta_{rb}, \]
\[ F_3 = F_4 = -\frac{1}{2} \sin \theta_{ra} \sin \theta_{rb} . \]

The sum of the four terms gives of course \( C_Q(a, b) = -a \cdot b \). The fact that this depends only on the angle formed by \( a \) and \( b \) is due to the spherical symmetry of the singlet spin state. Looking at the terms separately, however, we observe that \( F_1 + F_2 \), involving intermediate states (\( |\Psi_1\rangle \) and \( |\Psi_2\rangle \)) of antiparallel spins (along the arbitrary direction \( r \)), gives the product of the projections of \( a \) and \( b \) onto \( r \), whilst \( F_3 + F_4 \), involving intermediate states (\( |\Psi_3\rangle \) and \( |\Psi_4\rangle \)) of parallel spins, contains their vector products. In other words, the two spin projection operators \( \hat{\sigma} \cdot a, \hat{\sigma} \cdot b \) establish a correlation not just through the intermediate states representing antiparallel spins—as one might naively suppose for the entangled spin-zero state—but also through the intermediate states of parallel spins, \( |+_r\rangle |+_r\rangle \) and \( |-_r\rangle |-_r\rangle \).

We now propose an alternative calculation, by resorting to the eigenvalue equations

\[ \hat{\sigma} \cdot a |\pm_a\rangle = \alpha |\pm_a\rangle, \quad \alpha = \pm 1, \]
\[ \hat{\sigma} \cdot b |\pm_b\rangle = \beta |\pm_b\rangle, \quad \beta = \pm 1, \]

(9)

to construct a new orthonormal basis for the bipartite system:

\[ |\phi^1\rangle = |+_a\rangle |-_b\rangle, \quad |\phi^2\rangle = |-_a\rangle |+_b\rangle, \]
\[ |\phi^3\rangle = |+_a\rangle |+_b\rangle, \quad |\phi^4\rangle = |-_a\rangle |-_b\rangle , \]

(10)
and write as before

\[ C_Q(a, b) = \langle \Psi^0| (\hat{\sigma} \cdot a) \left( \sum_{k=1}^{4} |\phi^k\rangle \langle \phi^k| \right) (\hat{\sigma} \cdot b) |\Psi^0\rangle. \]  

(11)

In view of (9) and (10), the terms that contribute to \( C_Q \) are

\[ (\hat{\sigma} \cdot a) \otimes I |\phi^k\rangle \langle \phi^k| I \otimes (\hat{\sigma} \cdot b) = A_k |\phi^k\rangle \langle \phi^k|, \]

(12)

with

\[ A_k = \alpha_k \beta_k, \]

(13)

where \( \alpha_k, \beta_k \) are the individual eigenvalues corresponding to the bipartite state \( |\phi^k\rangle \). Thus from Eqs. (11) and (12) we get

\[ C_Q(a, b) = \sum_{k=1}^{4} A_k(a, b) C_k(a, b), \]

(14)

with

\[ C_k = |\langle \phi^k|\Psi^0\rangle|^2. \]

(15)

In Eq. (14), \( A_k (k = 1, 2, 3, 4) \) are the eigenvalues of \((\hat{\sigma} \cdot a \otimes \hat{\sigma} \cdot b)\) corresponding to the bipartite states \( |\phi^k\rangle \), given according to Eqs. (9) and (10) by

\[ A_1 = A_2 = -1 \equiv A^-, \quad A_3 = A_4 = +1 \equiv A^+, \]

(16)

and \( C_k \) stands for the relative weight of eigenvalue \( A_k \). It is clear from (15) that the coefficients \( C_k \) are nonnegative and add to give

\[ \sum_k C_k = \sum_k \langle \Psi^0|\phi^k\rangle \langle \phi^k|\Psi^0\rangle = 1. \]

(17)

These coefficients represent thus the joint probabilities associated with their corresponding \( A_k \).

3 Probability distribution for the bipartite singlet spin state

We now proceed to calculate the coefficients \( C_k \) given by (15). To simplify the calculation we may, without loss of generality, select the vector \( r \) on the plane defined by the directions \( a \) and \( b \), so that Eqs. (2) reduce to

\[ |+_r\rangle = \cos \frac{\theta_r}{2} |+_z\rangle + \sin \frac{\theta_r}{2} |-_z\rangle, \quad |-_r\rangle = -\sin \frac{\theta_r}{2} |+_z\rangle + \cos \frac{\theta_r}{2} |-_z\rangle. \]

(18)

One thus obtains, using Eqs. (11) and (10),

\[ C_1(a, b) = C_2(a, b) = \frac{1}{2} \cos^2 \frac{\theta_{ab}}{2}, \]

(19a)
\( C_3(a, b) = C_4(a, b) = \frac{1}{2} \sin^2 \frac{\theta_{ab}}{2}, \)  

(19b) with \( \theta_{ab} = \theta_a - \theta_b \). These are, then, the relative weights of the eigenvalues \( A_k \) given by (16). Inserted into Eq. (14) they reproduce the quantum result

\[ C_Q(a, b) = -\cos \theta_{ab}, \]  

(20)
as expected. Crucial is, however, to distinguish the contributions due to equal signs of \( \alpha_k \) and \( \beta_k \), leading to \( A_k = 1 \), from those due to opposite signs of \( \alpha_k \) and \( \beta_k \), leading to \( A_k = -1 \). This distinction implies a partitioning of the probability space, which we shall call \( \Phi \), into two separate, complementary probability spaces \( \Phi^{-ab}, \Phi^{+ab} \). In other words, if the associated probability distribution \( \rho(\phi) \) is a function of the continuous variable \( \phi \) spanning the entire probability space, such that \( \int_\phi \rho(\phi)d\phi = 1 \), the contributions to \( C_Q(a, b) \) stemming from the antiparallel and parallel spin projections are, respectively,

\[ \int_{\phi_{-ab}} \rho(\phi)d\phi = \cos^2 \frac{\theta_{ab}}{2}, \]  

(21a)

\[ \int_{\phi_{+ab}} \rho(\phi)d\phi = \sin^2 \frac{\theta_{ab}}{2}. \]  

(21b)

It is essential to note that the distribution function \( \rho(\phi) \) is the same regardless of the directions of the spin operators \( \hat{\sigma} \cdot a, \hat{\sigma} \cdot b \); only the partitioning of \( \Phi \) depends on the angle formed by \( a \) and \( b \).

### 3.1 General probability distribution function

As noted above, we are looking for a probability distribution function \( \rho(\phi) \) that complies with Eqs. (21) and reproduces the quantum correlation (20). Such a function can be readily found by observing that

\[ \cos^2 \frac{\theta_{ab}}{2} = \frac{1}{2}(1 + \cos \theta_{ab}) = \frac{1}{2} \int_{\theta_a}^{\pi} \sin \phi d\phi, \]  



\[ \sin^2 \frac{\theta_{ab}}{2} = \frac{1}{2}(1 - \cos \theta_{ab}) = \frac{1}{2} \int_{0}^{\theta_{ab}} \sin \phi d\phi. \]  

Therefore, the distribution function

\[ \rho(\phi) = \frac{1}{2} \sin \phi, \ 0 \leq \phi \leq \pi \]  

(23) is a general solution to our problem. With Eq. (23) the quantum correlation is given by

\[ C_Q(a, b) = \left( \int_{0}^{\theta_{ab}} A^+ + \int_{\theta_{ab}}^{\pi} A^- \right) \rho(\phi)d\phi = -\cos \theta_{ab} \]  

(24)
for any \((a,b)\), with \(A^+\) and \(A^-\) given by [16].

It is interesting to note that the same formula for the distribution, Eq. (22), has been previously obtained by Oaknin (3, see also [4]), based on the argument that giving up an assumption implicit in the proof of Bell’s inequalities, namely that there exists an absolute reference frame of angular coordinates with respect to which one can define the polarization properties of the hidden configurations of the pair of particles, it is possible to build a local and realistic model of hidden variables for Bell’s polarization states.

3.2 A physical model for the spin correlation

Given that we have found a general probability distribution and an appropriate separation of the probability space that accounts for the positive and negative outcomes contributing to the spin correlation, we now explore a possible geometric explanation for this result.

With this purpose in mind, let us take a pair of entangled spins and consider the situation in which the sign of the projection of spin 1 onto \(a\) has been determined, say \(\alpha = +1\); take the +z axis along this direction, and the x axis perpendicular to it. Because the bipartite system is in the singlet state, we know for sure that the projection of the second spin onto the +z axis would give -1. This means that the second spin lies in the lower half plane, and may form any angle \(\phi\) such that \(0 \leq \phi \leq \pi\), with the origin of \(\phi\) along the \(-x\) axis and \(\phi\) increasing counterclockwise. Conversely, if the sign of the projection of spin 1 is \(\alpha = -1\), the second spin lies in the upper half plane, and may form any angle \(\phi\) such that \(0 \leq \phi \leq \pi\), with the origin of \(\phi\) along the \(x\) axis. In both cases, \(A = -1\). (The argument is of course reversible, in the sense that the sign of the projection of spin 2 can be defined first, in which case the angle variable \(\phi\) refers to spin 1.)

Take again the case \(\alpha = +1\) for spin 1: if the second spin, lying in the lower half plane, is projected now onto the direction \(b\) forming an angle \(\theta_{ab}\) with the +z axis, \(A\) will still be negative for any angle \(\phi\) such that \(\theta_{ab} \leq \phi \leq \pi\), whilst it will become positive for \(0 \leq \phi \leq \theta_{ab}\). This gives a concrete meaning to Eq. (24).

What is it that determines in each instance the specific value of the variable \(\phi\) is unknown; we only know its probability distribution. When the direction \(b\) is changed to \(b'\), the probability space is subdivided accordingly:

\[
C_Q(a,b') = \left( \int_0^{\theta_{ab}} A^+ + \int_{\theta_{ab}}^{\pi} A^- \right) \rho(\phi) d\phi = -\cos \theta_{ab'} \tag{25}
\]

It is important to note that the subdivision depends only on the range of values of \(\phi\) for which the sign of the product \(A = \alpha \beta\) is either positive or negative, and not separately on the signs of \(\alpha\) and \(\beta\). This means that neither \(\alpha\) nor \(\beta\) may be transferred from (24) to (25); not even if the direction of \(a\) remains fixed. Precisely herein lies the essence of the correlation.
A similar reasoning can in fact be applied to the spin correlation for a single electron,

\[ C(a, b) = \langle \psi | (\hat{\sigma} \cdot a) (\hat{\sigma} \cdot b) | \psi \rangle. \]  

(26)

In this case, when the spin projection onto \(a\) (taken again along the +z axis) is +1, its projection onto \(b\) is +1 (i.e., \(A = +1\)) for any angle \(\phi\) such that \(\theta_{ab} \leq \phi \leq \pi\), whilst it is -1 (i.e., \(A = -1\)) for \(0 \leq \phi \leq \theta_{ab}\), and inversely if the spin projection onto \(a\) is negative. The two contributions taken together give the quantum result

\[ C(a, b) = \left( \int_0^{\theta_{ab}} A^- + \int_{\theta_{ab}}^{\pi} A^+ \right) \rho(\phi) d\phi = \cos \theta_{ab}, \]  

(27)

with \(\rho(\phi)\) given by (23).

4 Connecting with experiment

A large number of Bell tests have been conducted, mostly involving photons, and lately using also electron spins (see, e.g., [5] and references therein). Experimental loopholes are being brought to a closure with the improvement of experimental setups and detection devices, thereby leading to ever stronger confirmation of the quantum predictions and violation of Bell-type inequalities.

From the perspective provided by the present analysis, and according to the discussion of the previous section, the source of violation of Bell-type inequalities may be traced to a loophole in their derivation, namely the assumption that the values of the \(\alpha_k\) contained in \(A\) can be freely transferred from the integrals over the probability subspaces \(\Phi_{-ab}, \Phi_{+ab}\) involved in \(C_Q(a, b)\), Eq. (24), to the integrals over \(\Phi_{-ab'}, \Phi_{+ab'}\) involved in \(C_Q(a, b')\), Eq. (25), with \(\theta_{ab'} \neq \theta_{ab}\). As if the \(\alpha_k\) (\(\beta_k\)) were a function of the direction \(a\) (\(b\)) and carried with it this information from one experiment to another.

The experimental counterpart of (14) may be written as

\[ C_E(a, b) = \frac{1}{N_{ab}} \sum_{k=1}^{4} N_{k}^{ab} A_k, \]  

(28)

with \(N_{ab} = \sum_{k=1}^{4} N_{k}^{ab}\) the total number of coincidences, i.e., of pairs of entangled spins (1,2) projected onto (\(a, b\)), and \(A_k\) the product of their pairwise projections. like in (13). \(N_{k}^{ab}\) indicates the number of times the product \(A_k\) is obtained in this experiment, i.e., with \(a, b\) fixed; it represents the weight of the contribution of \(A_k\) to the total correlation. If one of the directions, say \(b\), is changed, a different series is obtained:

\[ C_E(a, b') = \frac{1}{N_{ab}} \sum_{k=1}^{4} N_{k}^{ab'} A_k. \]  

(29)
The individual results $A_k$ obtained in the first experiment leading to $C_E(a, b)$ may not be transferred to the second one; each series should be treated independently.

**Acknowledgment.** The author is grateful to David Oaknin for drawing her attention to Refs. [3] and [4].

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