Dimensional crossovers and Casimir forces for the Bose gas in anisotropic optical lattices

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We consider the Bose gas on a $d$-dimensional anisotropic lattice employing the imperfect (mean-field) gas as a prototype example. We study the dimensional crossover arising as a result of varying the dispersion relation at finite temperature $T$. We analyze in particular situations where one of the relevant effective dimensionalities is located at or below the lower critical dimension, so that the Bose-Einstein condensate becomes expelled from the system by anisotropically modifying the lattice parameters controlling the kinetic term in the Hamiltonian. We clarify the mechanism governing this phenomenon. Subsequently we study the thermodynamic Casimir effect occurring in this system. We compute the exact profile of the scaling function for the Casimir energy. As an effect of strongly anisotropic scale invariance, the Casimir force below or at the critical temperature $T_c$ may be repulsive even for periodic boundary conditions. The corresponding Casimir amplitude is universal only in a restricted sense, and the power law governing the decay of the Casimir interaction becomes modified. We also demonstrate that, under certain circumstances, the scaling function is constant for sufficiently large values of the scaling variable, and in consequence is not an analytical function. At $T > T_c$, the Casimir-like interactions reflect the structure of the correlation function, and, for certain orientations of the system relative to the confining walls, show exponentially damped oscillatory behavior so that the corresponding force is attractive or repulsive depending on the distance.

I. INTRODUCTION

Ultracold atomic gases in optical lattices have remained a topic of great interest over the last years from both theoretical and experimental points of view. The progressing experimental developments allowed for exploiting physical situations inaccessible in traditional condensed matter setups (and also in continuum cold gases) and stimulated enormous theoretical developments worldwide.

In this paper we investigate the physics emergent in optical-lattice Bose systems as a result of introducing strong spatial anisotropies resulting in the presence of (at least) two distinct lengths scales divergent at the transition to the condensed phase. First we analyze the possibility of controlling the Bose-Einstein condensation by anisotropically varying the hopping parameters (or, equivalently, the dispersion relation). By suitably tuning the lattice parameters one induces crossovers to physical situations characterized by fractional effective dimensionalities. This may lead in particular to configurations where some of the effective dimensionalities relevant for the system are at or below the lower critical dimension $d_c$, while others are above. This yields certain features of the phase diagram (in particular the crossover scales) not obvious and presumably sometimes hard to access within numerical approaches. Due to the mean-field character of the studied system, the present analysis may be carried out exactly. Nonetheless, we argue that many of the studied features are not necessarily restricted to mean-field models and could also be found in systems characterized by realistic interactions. The analyzed setup might be conceivable in future experiments in optical lattices, where the system parameters could be controlled with high precision.

The second issue of the present paper concerns the thermodynamic Casimir effect in anisotropic Bose systems, where the anisotropy is inherited from the lattice. As was indicated in a relatively recent work on the $O(N)$ models in the vicinity of the Lifshitz point, strong anisotropy, manifested by nontrivial scaling of two correlation lengths, leads to the remarkable effect of modifying the power law governing the decay of the Casimir force. As a consequence, the asymptotic expression for the Casimir energy (at or below the critical temperature $T_c$) for condensation) contains at least one length scale in addition to the system extension $D$ and the scaling function is universal only after the appropriate dimensionful coefficient is correctly identified and factored out. Equally remarkably, for certain orientations of the system relative to the confining walls, the Casimir force turns out to be repulsive even for the periodic boundary conditions (PBC). We confirm this picture within the present exact study of the imperfect Bose gas. If the temperature is fixed above $T_c$, the Casimir force is exponentially suppressed at distances larger than the correlation length. We find however, that its typically attractive character may be significantly modified by varying the orientation of the confining walls. The Casimir force then shows damped oscillatory behavior and its actual sign depends on the distance $D$ between the confining walls. The obtained behavior should be of relevance for the entire universality class of anisotropic $O(N)$-symmetric models in the limit $N \to \infty$.

The outline of the paper is as follows: In Sec. II we introduce the imperfect Bose gas model on an anisotropic lattice. Its bulk properties in the relevant regime of low temperatures are reviewed with particular focus on the effects caused by anisotropies. In Sec. III we present our results on the dimensional crossovers with emphasis on the the possibility of tuning the system continuously to a state characterized by the effective dimensionality at or below the lower critical dimension $d_c$ (Sec. III B). We give arguments suggesting that the results of this section are not necessarily restricted to mean-field models and may apply to a broad class of systems characterized by realistic microscopic interactions. In Sec. IV we present our derivation of the expression for the Casimir energy and extract its asymptotic behavior for large separations between the confining walls. We compute and discuss the scaling function for the Casimir force. The entire study is carried out by means of an exact analysis. Sec. V contains a
summary and outlook.

II. MODEL AND ITS BULK SOLUTION

We consider bosons on a lattice at a fixed temperature $T$, chemical potential $\mu$ and contained within the volume $V = L^d$. The system is governed by the Hamiltonian

$$\hat{H} = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} \hat{n}_{\mathbf{k}} + \frac{a}{2V} \hat{N}^2.$$

The particles are assumed spinless for simplicity, and we impose periodic boundary conditions. The dispersion relation $\epsilon_{\mathbf{k}}$ is controlled by the optical lattice parameters and we will specify to a hypercubic lattice later in the calculation. The wave vectors $\mathbf{k}$ are contained in the first Brillouin zone. The physical content of the repulsive mean-field interaction term $V_{mf} = \frac{a}{2V} \hat{N}^2$ ($a > 0$) is best understood by noting that it arises from the long-range repulsive part $v(r)$ of a 2-particle interaction potential upon performing the Kac scaling limit $\lim_{\gamma \to 0} \gamma^4 v(\gamma r)$, i.e. for vanishing interaction strength and diverging range. The presence of the $1/V$ factor in $V_{mf}$ assures extensivity of the system. The continuum version of the model in the bulk was studied in Refs. 12–15. The finite-size effects were addressed in Refs. 16–19. The setup involving a harmonic trap was considered in Ref. 20. Before proceeding, we invoke two results of high relevance to the present study: as was established in Ref. 17, the Bose–Einstein condensation in the (isotropic) imperfect Bose gas is controlled by the same critical exponents as the spherical model, which in turn belongs to the bulk universality class of the $O(N \to \infty)$ model. However, the scaling function for the excess surface free energy obtained in Ref. 17 turned out to differ from its counterpart in the spherical model [21] by a global factor of two. This issue was further analyzed in Ref. 18, which established an equivalence between the isotropic imperfect Bose gas and the $O(2N)$ model for $N \to \infty$ providing a resolution of the puzzle. The key features of the phase diagram and correlation functions of the imperfect Bose gas in presence of anisotropies were addressed in Ref. 6. The following part of the present section is a brief summary of some aspects of that study and Sec. III constitutes its extension accounting for the interplay between the different terms of $\epsilon_{\mathbf{k}}$ giving rise to the dimensional crossovers. We begin with the expression for the grand canonical partition function [6, 16]

$$\Xi(\mu, T) = -i \exp \left( \frac{\beta V}{2a} \mu^2 \right) \sqrt{\frac{V}{2\pi \beta a}} \int_{\beta a - i\omega}^{\beta a + i\omega} ds \exp[-V\varphi(s)],$$

where $\alpha < 0$ is arbitrary, $\beta^{-1} = k_b T$ and

$$\varphi(s) = \frac{1}{\beta a} \left( -\frac{s^2}{2} + s\beta \mu \right) - \frac{1}{V} \log \Xi(\frac{s}{\beta}, T).$$

The quantity $\Xi(\frac{s}{\beta}, T)$ is the grand canonical partition function of the noninteracting Bose gas [22] evaluated at chemical potential $\mu = \frac{s}{\beta}$ and temperature $T$. The presence of the volume factor in the term $\exp[-V\varphi(s)]$ in Eq. (2) implies that the saddle point treatment of Eq. (2) becomes exact in the thermodynamic limit. The saddle-point equation $\varphi'(s = s_0) = 0$ yields

$$-s_0 \frac{1}{a\beta} + \mu = \frac{1}{V} \sum_{n=1}^{\infty} e^{\beta n} \sum_{\mathbf{k} \neq 0} e^{-\beta \mu} + \frac{1}{V} \sum_{\mathbf{k} \neq 0} e^{\beta \mu} \tag{4}$$

It is crucial for exploiting the thermodynamics of the system and the whole analysis to follow (see Ref. 16 for detailed explanations in the isotropic case). The left-hand side of Eq. (4) represents a linear function of $s_0$. If the first term on the right-hand side is unbounded from above, there always exists a nonzero solution, and the last term (which is interpreted in terms of the condensate density) vanishes in the thermodynamic limit. This happens at low dimensionalities. In the opposite situation, a finite solution exists only for $\beta$ sufficiently small, while for $\beta$ large one finds $s_0 \to 0^-$ for $V \to \infty$, which signals the Bose–Einstein condensed phase [16].

As demonstrated in Ref. [3], the thermodynamic properties of the system in the vicinity of the critical temperature (and for $T$ low enough) are fully determined by the asymptotic form of the dispersion relation $\epsilon_{\mathbf{k}}$ at $|\mathbf{k}|$ small. The system displays a line of second order phase transitions $T_c(\mu)$ down to $T_c(\mu \to 0) \to 0$. Considering the hypercubic lattice as a specific example, we take

$$\epsilon_{\mathbf{k}} = \sum_{\mathbf{R}} 2\pi (1 - \cos(\mathbf{k} \cdot \mathbf{R})).$$

In a typical situation, expansion around $\mathbf{k} = 0$ leads to the following asymptotic form

$$\epsilon_{\mathbf{k}} \to \bar{\epsilon}_{\mathbf{k}} = \sum_{i=1}^{d} c_i (k_i A)^2,$$

where $c_i$ are numerical coefficients, and $A$ denotes the lattice constant. From the point of view of universal properties, the behavior of the system characterized by the dispersion of Eq. (6) is identical to that of the continuum imperfect Bose gas. By tuning the hopping parameters it is however possible to cancel one or more of the coefficients $c_i$ in which case, the corresponding leading order term in the $i$-th direction becomes quartic (or even higher order). Specific examples of such a tuning procedure are given in Ref. 4. In a general situation, the asymptotic form of the dispersion may be written as

$$\bar{\epsilon}_{\mathbf{k}} = \sum_{i=1}^{d} t_i |k_i|^\alpha_i, \quad t_i, \alpha_i > 0 \tag{7}$$

For the hypercubic lattice $\alpha_i$ are even natural numbers. The thermodynamics and correlations of the system characterized by Eq. (7) were thoroughly studied in Ref. 6 which pointed at the affinity to the isotropic system in an effective dimensionality

$$d_{eff} = \frac{2}{\psi} \tag{8}$$
where
\[ \frac{1}{\psi} = \frac{1}{\alpha_1} + \ldots + \frac{1}{\alpha_d}. \]

In particular, the system is above its lower critical dimension (and therefore hosts a Bose-Einstein condensed phase in its phase diagram at \( T > 0 \)) if \( \frac{1}{\psi} > 1 \). The upper critical dimension is on the other hand determined by the condition \( \frac{1}{\psi} = 2 \). We also remark that the asymptotic shape of the critical line in the phase diagram is given by the universal exponent \( \tilde{\tau} \) so that \( \mu_c(T) \sim T^{\tilde{\tau}} \) (for the continuum case one has \( \mu_c(T) \sim T^{d/2} \)). We point out that overall the role played in the continuum case by the spatial dimensionality \( d \) is for the optical lattice taken over by the parameter \( \tilde{\tau} \). This may in turn be experimentally tuned leading to crossovers between different effective dimensionalities involving also fractional values.

The analysis of such crossovers effects requires, however, going beyond the asymptotic form of \( \epsilon_k \) given by Eq. (7) and accounting for the next-to leading contributions. We present such an extension below.

III. DIMENSIONAL CROSSOVER

In the most standard setup, the dimensional crossover is realized by confining the system in one or more directions and manipulating the thermodynamic parameters so that the characteristic length scale becomes larger (or smaller) than the confining parameter [see e.g. 23–27]. Here we analyze a natural alternative avenue, already outlined in Sec. II, where the dimensional crossover between distinct effective dimensionalities is tuned by manipulating the hopping parameters. To this aim we now extend the expansion of the dispersion given by Eq. (7) and consider

\[ \epsilon_k = t_0(k_1A)^2 + t(k_3A)^{2m} + \tau(k_1A)^2 + \tau'(k_3A)^4, \]

where \( t_0, t, \tau, \tau' > 0 \). This allows us to analyze different physical situations depending, in particular, on the value of \( m \). On the other hand, the fixed signs of the kinetic couplings restricts to uniform ordered phases, ruling out the modulated states related to the Lifshitz points. [28] For the time being we specified to \( d = 3 \). We will restore the generality of \( d \) in the discussion of the Casimir effect (Sec. IV).

With \( \epsilon_k \) given above, the saddle-point equation may be written as

\[ -s_0 \frac{1}{a \beta} + \frac{\mu}{a} = \frac{1}{A^3} \frac{\Gamma(1 + \frac{1}{2m})}{2^{3/2} \pi^{5/2}} \frac{1}{\beta^{1 + \frac{1}{2m}} \sqrt{\tau \theta}} \sum_{n=1}^{\infty} n^{1 + \frac{1}{2m}} \frac{1}{n^{1 + \frac{1}{2m}}}, \]

where:

\[ f(x) = \frac{1}{\sqrt{\pi}} x^{3/2} K_{1/2}(x), \quad \theta = \frac{\beta r^2}{8 r'}. \]

and \( K_{\alpha}(x) \) is the Bessel function. As we show below, the dimensionless parameter \( \theta \) serves as the scaling variable controlling the dimensional crossover. Note that it may be varied between 0 and infinity either by manipulating the hopping parameters, or temperature. The function \( f(x) \) is monotonously increasing and bounded. Its asymptotic behavior is given by

\[ f(x) \sim \begin{cases} (1/4)^{1/4} & \text{as } x \to 0^+ \\ \sqrt{\tau} & \text{as } x \to \infty \end{cases} \]

The expression for the critical line is obtained [16] by dropping the last term in Eq. (11) and putting \( s_0 = 0 \). It reads:

\[ \mu_c(T) = \frac{1}{A^3} \left( \frac{1}{2^{3/2} \pi^{5/2}} \frac{1}{\beta^{1 + \frac{1}{2m}} \sqrt{\tau \theta}} \sum_{n=1}^{\infty} \frac{1}{n^{1 + \frac{1}{2m}}} \right) \]

The phase hosting the condensate is stable for \( \mu > \mu_c(T) \).

A. Case \( \epsilon_k = t_0(k_1A)^2 + t_0(k_3A)^2 + \tau(k_1A)^2 + \tau'(k_3A)^4 \)

Here we consider \( m = 1 \) and analyze the crossover in the effective dimensionality realized by changing \( \theta \). For example we may vary the parameter \( \tau > 0 \) towards zero, gradually giving way to the subdominant term proportional to \( k_3^4 \) in the dispersion along the 3rd direction. The relevant values of \( 1/\psi \) are \( 1/\psi = 3/2 \) (for \( \tau > 0 \)) and \( 1/\psi = 5/4 \) (for \( \tau = 0 \)). The series in Eq. (14) is convergent for any \( \theta > 0 \). At fixed \( \tau \) and \( \theta' \) the variable \( \theta \) may be tuned between the asymptotic regimes \( \theta \gg 1 \) and \( \theta \ll 1 \) by varying temperature \( T \). Alternatively, at given \( T \) (and \( \tau' \)), one may use \( \tau \) as the control parameter. In the asymptotic regime \( \theta \gg 1 \) we may replace \( f(n\theta) \) by its limiting form for large arguments. This leads to

\[ \mu_c^{\infty}(T) \approx \frac{1}{8 \pi^{5/2}} \left( \frac{\zeta(3/2)}{2} \right) \frac{1}{A^3} \frac{1}{t_0} \frac{1}{\sqrt{\tau}} (k_3 T)^{3/2}, \]

where \( \zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \) (with \( z > 1 \)) is the Riemann zeta function. The expression of Eq. (15) coincides with the formula for the critical point derived in Ref. [6].

In the opposite limit \( \theta \ll 1 \) the series is dominated by terms with \( n\theta \ll 1 \) and one may use asymptotic form of the function \( f \) for small arguments. This leads to the expression

\[ \mu_c^\infty(T) \approx \frac{1}{16 \pi^{5/2}} \left( \frac{\zeta(5/2)}{4} \right) \frac{1}{A^3} \frac{1}{t_0} \sqrt{\tau} (k_3 T)^{5/4}, \]

which also agrees with the predictions of Ref. [6]. Estimating the crossover temperature \( T_{\text{cross}} \) by the condition \( \mu_c^{\infty}(T_{\text{cross}}) = \mu_c^0(T_{\text{cross}}) \), we find

\[ k_3 T_{\text{cross}} = \frac{2^2 \left( \frac{\Gamma(1/4) \zeta(3/2)}{2^{3/4}} \right)^4}{\tau' \left( \frac{\zeta(3/2)}{2} \right)^4}. \]

The emergent picture is illustrated in Fig. 1.
FIG. 1: The critical line as computed from Eq. (14) with $m = 1$. The plot parameters are $t_0 = t = 1, \tau = 0.2, \tau' = 2.43$. The plotted quantities are scaled by $\eta = \frac{1}{\sqrt{T}}$. One finds the crossover between the scaling behavior with $\psi = 2/3$ (low $T$) and $\psi = 4/5$ (high $T$). The crossover temperature (red cross) precisely agrees with Eq. (17). The inset presents the critical line in linear scale for $\tau = 0$ (and therefore $\theta = 0$) and $\tau = 1$, where the scaling with only one value of $\psi (2/3 \text{ or } 4/5)$ occurs in the entire temperature range.

The kinetic term is therefore used as the con-
tive lower critical dimension $d^c$. We consider the setup where the behavior of the critical line at $\theta \gg 1$ corresponds to a value such that (at $\theta > 0$ fixed) the first term in Eq. (13) constitutes a valid approximation to the entire series. The last term of Eq. (13) is bounded from above by:

$$\sum_{N(\theta) + 1}^{\infty} \frac{1}{n^5/4} f(n\theta) < \frac{\pi}{2} \sqrt{\frac{4}{\sqrt{N(\theta)}}}.$$  

The upper bound obviously vanishes for $N(\theta) \to \infty$. The first term on the right-hand side (RHS) of Eq. (18) may be estimated using the Euler formula

$$\sum_{N(\theta) + 1}^{\infty} \frac{1}{n^5/4} f(n\theta) \approx \log N(\theta) + \gamma + \frac{1}{2N(\theta)} + O\left(\frac{1}{N(\theta)^4}\right)$$

with $\gamma$ denoting the Euler-Mascheroni constant. Truncating this expansion at the leading term and using Eq. (18), the asymptotic form of Eq. (14) becomes

$$\mu^0(T) \approx \frac{\Gamma(1/4)^2}{32\pi^{3/2}} \frac{a}{A^4} \frac{1}{\sqrt{\theta(t^4/4)}} \log \left(\frac{1}{\theta}\right)(k_B T).$$

Note that the unspecified constant $a$ [relating $\theta$ and $N(\theta)$] as well as the constant $\gamma$ influence only the subdominant contribution to $\mu^0(T)$, which is also linear in $T$, but does not involve the log-divergent coefficient $\sim \log(\frac{1}{\theta})$. The above calculation reveals a somewhat subtle behavior of the critical line $T_c(\mu)$. As we have shown, at fixed $\theta$ the dependence $T_c(\mu)$ is linear for $\mu$ small (up to log corrections) and, at $\mu$ larger, it crosses over to the power law behavior with the exponent $\psi = 4/5$. When the parameter $\tau$ is then tuned towards zero (implying vanishing $\theta$), the coefficient governing the low-$\mu$ (low $T_c$) linear behavior vanishes logarithmically, thus suppressing the critical temperature towards zero. This is accompanied by shifting the scale corresponding to the onset of the power-law regime towards infinite chemical potentials. This picture clarifies the mechanism leading to continuously depleting the condensate from the system for $\tau \to 0$. The corresponding illustrations are presented in Fig. 2.

Concerning the situation with a value of $m$, where the effective dimensionality $d_{\text{eff}}$ is below $d_t$ for $\tau = 0$, one may show that the picture is similar to the one extracted above for $m = 2$. The critical line is linear in the low-$T$ regime [$T_c(\mu) \approx A\mu$] and crosses over to a power-law with an exponent $\psi < 1$ at $T$ higher. The role of $m$ reveals itself in the way the coefficient $\tilde{A}$ governing this linear behavior vanishes for $\tau \to 0$. Instead of the behavior $\tilde{A} \sim -1/\log(\tau)$ obtained above for $m = 2$, one finds a power-law dependence $\tilde{A} \sim \tau^\epsilon(m)$. For $m = 3$ we obtain $\kappa(3) = 1/6$.

We finally point out an observation concerning the relation to a general situation with short-ranged interaction potentials and suggesting that the obtained picture may be valid also for non-mean-field models. As we already remarked, the
We now move on to discuss the thermodynamic Casimir effect in the system. We consider a general situation where the $d$-dimensional system is enclosed in volume $V = L^{d-1}D$, where $L \gg D \gg l_{\text{mic}}$ and $l_{\text{mic}}$ denotes all the microscopic length scales present in the system. The quantity $D$ measures the system extension in the $d$-th direction. We will separately consider the situation with the confining walls perpendicular to $k_1$ in Sec. IVC. We analyze the case of periodic boundary conditions. The dispersion displays $\sim k^2$ behavior in $m < d$ directions, and the usual $\sim k^2$ behavior in the remaining $d - m$ directions:

$$
\epsilon_k = \tilde{\epsilon}_k = \sum_{i=1}^{d-m} t_0(k_iA)^2 + \sum_{i=d-m+1}^{d} t(k_iA)^4. \tag{23}
$$

Importantly, the dispersion parameters ($t_0$ and $t$) are assumed to be the same (i.e. independent of $i$) for each of the two classes of spatial directions. Relaxing this symmetry may lead to additional effects not addressed here. We keep only the dominant contributions in each of the directions, leaving the crossover effects aside. We also introduce $\tilde{\epsilon}_1 = t_0(k_1A)^2$ and $\tilde{\epsilon}_2 = t(k_2A)^4$. We are interested in the excess grand-canonical free energy density

$$
\omega_j(D, \mu, T) = \lim_{L \to \infty} \left[ \Omega(L, D, T, \mu) - D\omega_0(T, \mu) \right] \tag{24}
$$

which is related to the Casimir force $F(D, \mu, T)$ by $F(D, \mu, T) = -\partial \omega_j(D, \mu, T)/\partial \mu$. The grand-canonical free energy is given by $\Omega(L, D, T, \mu) = -\beta^{-1}\ln \Xi(T, L, D, \mu)$ and the bulk free energy density $\omega_j(T, \mu)$ follows from $\omega_j(T, \mu) = \lim_{L \to \infty} \frac{1}{L} \Omega(L, D = L, T, \mu)$. The excess contribution to the grand potential can be written as

$$
\omega_j(D, \mu, T) = \lim_{L \to \infty} \beta^{-1} D \left[ \varphi(\bar{s}) - \varphi(\bar{s}_0) \right], \tag{25}
$$

where

$$
\varphi(\bar{s}) = -\frac{s^2}{2a\beta} + \frac{\mu \bar{s}}{a}
$$

$$
\frac{1}{V} \left[ \sum_{\mathbf{k}} \sum_{\mathbf{k}_{\perp}} \sum_{r=1}^{\infty} \frac{1}{r} e^{r(\bar{s}-\beta\epsilon_{k_r})} - \sum_{k_r} \log \left( 1 - e^{r(\bar{s}-\beta\epsilon_{k_r})} \right) \right]. \tag{26}
$$

$\bar{s}$ represents the solution to the saddle-point equation $\varphi'(\bar{s}) = 0$, $\bar{s}_0$ corresponds to $\bar{s}$ in the bulk case (i.e. when $D = L$ and $L \to \infty$) and $\varphi(\bar{s}) = \lim_{L \to \infty} \varphi(s)$. In essence, our present goal amounts to solving the saddle-point equation at finite $D$ and evaluating Eq. (25). The following content is relatively technical due to the rich analytical structure of the problem. It is possible to read the summary section (Sec. V) before becoming acquainted with the part exposed below.

We identify two distinct thermal length scales

$$
\lambda_1 = 2A \sqrt{\beta t_0} \quad \lambda_2 = A \frac{\pi}{\Gamma(5/4)} (\beta t)^{1/4}, \tag{27}
$$
assumed large as compared to the lattice scale $A$. The bulk saddle-point equation can be written as

$$-s_0 \frac{1}{a\beta} + \frac{\mu}{a} = \frac{1}{\lambda_1^{d-m} \lambda_2^m} g_\psi(e^{\mu x}) + \frac{1}{V} e^{\mu x}$$  \hspace{0.5cm} (28)$$

with

$$\frac{1}{\psi} = \frac{d}{2} \frac{m}{4}$$

and the Bose function $g_\psi(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^m}$. We read off the expression for the critical line

$$\mu_c(T) = \frac{a}{\lambda_1^{d-m} \lambda_2^m} \left( \frac{1}{\psi} \right)$$

recovery the previously studied behavior $\mu_c(T) \sim T^{1/\gamma}$. Introducing the dimensionless parameter

$$\epsilon = \frac{\mu - \mu_c}{\mu_c}$$

and expanding the Bose function for $|s_0| \ll 1$ according to

$$g_\psi(e^{\mu x}) - \zeta(1/\psi) \approx \left( \Gamma(1 - \frac{1}{\psi})s_0^{1/\psi - 1}, \frac{1}{\psi} < 2 \right.$$  

$$-\zeta(1/\psi), \frac{1}{\psi} > 2 \right.$$

we may solve Eq. (28) for $|\epsilon| \ll 1$. The analysis of the bulk limit then proceeds along the line of Refs. 6 and 17.

A. Saddle-point equation

We now analyze the situation, where the system remains finite in one of the directions so that $L \to \infty$, but $D$ (i.e. the system extension in the $d$-th direction) is kept finite. We shall study the complementary situation where the confining walls are placed perpendicular to the 1st direction in Sec. IVC. The saddle-point equation is first cast in the form

$$\zeta(1/\psi) \left( -\frac{\bar{s}}{\mu_c} + \epsilon \right) = -\zeta(1/\psi) + \frac{A_2}{D} \sum_{r=1}^{\infty} \frac{e^{\pi r/\psi}}{r^{3/2}} \sum_{k_d} e^{-\beta \epsilon_{k_d}}$$

$$-\frac{A_1^{d-m} A_2^m}{V} \sum_{k_d} \frac{1}{1 - e^{\beta \epsilon_{k_d}}}.$$  \hspace{0.5cm} (33)$$

The sum occurring in the second term on the RHS of the equation can be transformed using the Poisson formula

$$\sum_{m=-\infty}^{\infty} f(m) = \sum_{n=-\infty}^{\infty} \hat{f}(n),$$

where $\hat{f}(n) = \int_{-\infty}^{\infty} dx e^{-\beta \epsilon_{k_d}} f(x)$. We obtain

$$\sum_{k_d} e^{-\beta \epsilon_{k_d}} = \frac{D}{2\Gamma(5/4) A_2^{1/4}} \sum_{m=-\infty}^{\infty} \psi(1/\psi) \frac{\pi m}{\Gamma(5/4) A_2} \frac{1}{r^{1/4}},$$

where

$$\phi(k) = \int_{-\infty}^{\infty} dx e^{ikx} e^{-\epsilon x}. \hspace{0.5cm} (36)$$

The properties of the function $\phi(k)$ are crucial for the results to follow. In particular $\lim_{k \to 0} \phi(k) = 2\Gamma(5/4)$, while the asymptotic behavior at $k$ large is described by

$$\phi(k) \sim 2^{7/6} \sqrt{\pi} \frac{1}{k^{4/3}} \exp \left( -\frac{3}{16} \frac{2^{1/3} k^{4/3}}{\pi} \right) \cos \left( \frac{3^{1/2} 2^{1/3}}{16} k^{4/3} - \frac{\pi}{6} \right)$$  \hspace{0.5cm} (37)

so that it exhibits exponentially damped oscillations. This gives rise to substantial differences as compared to the usual case with quadratic $\epsilon_{k_d}$, where the corresponding expression is a monotonously decreasing Gaussian function. An illustrative plot of $\phi(k)$ is given in Fig. 3 (left panel). We now consider

FIG. 3: The functions $\phi(x)$ and $G(x)$, see the main text.

$D \gg A_2$ replace the summation over $r$ in Eq. (33) with an integral in accord with the Euler-Maclaurin formula. We introduce the following notation

$$\sigma = \frac{\pi}{\Gamma(5/4) A_2} |\bar{s}|^{1/4} \quad F_\sigma(x) = \int_{0}^{\infty} dp e^{-p} \phi(x/p^{1/4}) \hspace{0.5cm} (38)$$

and transform the saddle-point equation to the following form

$$\zeta(1/\psi) \left( -\frac{\bar{s}}{\mu_c} + \epsilon \right) = g_\psi(e^{\epsilon}) - \zeta(1/\psi) +$$

$$\frac{\Gamma(5/4)^{2/5} (A_2)^{1/5} \sigma^{1/4}}{\pi^{1/4}} \sum_{n=1}^{\infty} F_\sigma(n \sigma) \frac{1}{\Gamma(5/4)^{1/4}} \frac{1}{k_d} \sum_{k_d} \frac{1}{1 - e^{\beta \epsilon_{k_d}}}.$$  \hspace{0.5cm} (39)$$

The function $F_\sigma(x)$ is characterized by oscillatory behavior inherited from $\phi(k)$. In the subsequent step we expand the Bose function for small $\bar{s}$ according to Eq. (32) and perform the limit $L \to \infty$. We also introduce the scaling variable $x$

$$x = \begin{cases} \frac{\epsilon}{\bar{s}}, & 1 < \frac{1}{\psi} < 2 \\ \frac{\epsilon}{\bar{s}}, & \frac{1}{\psi} > 2 \end{cases}, \hspace{0.5cm} (40)$$

which is positive below bulk $T_c$ and negative otherwise. This allows us to write Eq. (39) as a transparent relation between the variable $x$ and $\sigma = \sigma(x)$. As will turn out, the dependence
of the scaling function for the Casimir energy on $x$ can be absorbed into $\sigma$. For $1 < \frac{1}{\psi} < 2$ we obtain

$$
\zeta\left(\frac{1}{\psi}\right)x = \left(\frac{\Gamma(5/4)}{\pi}\right)^{\frac{1}{4}} \sigma^{\frac{1}{4}} \left[\Gamma\left(1 - \frac{1}{\psi}\right) + \frac{1}{\Gamma(5/4)} \sum_{n=1}^{\infty} F_{\frac{1}{\psi}}\right] 
+ \frac{\lambda_{d-m}}{V} \left(\frac{D}{\lambda_{2}}\right)^{\frac{1}{4}} \sum_{n} \frac{1}{\psi}^{|\psi_n|^{\frac{1}{2}} - 1},
$$

while for $\frac{1}{\psi} > 2$ we find

$$
\zeta\left(\frac{1}{\psi}\right)x = -\left(\frac{\Gamma(5/4)}{\pi}\right)^{\frac{1}{4}} \sigma^{\frac{1}{4}} \left[\zeta\left(\frac{1}{\psi}\right) + \zeta\left(\frac{1}{\psi} - 1\right)\right] + 
+ \frac{1}{\Gamma(5/4)} \left(\frac{D}{\lambda_{2}}\right)^{\frac{1}{4}} \sigma^{\frac{1}{4}} \sum_{n=1}^{\infty} F_{\frac{1}{\psi}}\right] + 
= \frac{\lambda_{d-m}}{V} \left(\frac{D}{\lambda_{2}}\right)^{\frac{1}{4}} \sum_{n} \frac{1}{\psi}^{|\psi_n|^{\frac{1}{2}} - 1}.
$$

In each of the cases, the dependence on $x$ occurs only on the left-hand side (LHS) of the equation, while the dependence on $\sigma$ on the corresponding RHS. We discuss the saddle-point solution in the two cases separately. We leave aside the case $\frac{1}{\psi} = 2$, where logarithmic corrections arise [see Eq. (42)].

1. Case $1 < \frac{1}{\psi} < 2$

Let us first concentrate on the low-$T$ phase ($x \geq 0$). One may show that the RHS of Eq. (41) is unbounded from above for $\frac{1}{\psi} \leq \frac{5}{4}$. This follows from the properties of the function $F_{\frac{1}{\psi}}(x)$. In such a situation for each $x \geq 0$ one finds a unique $\sigma(x) > 0$. In consequence, the corresponding value of $|\beta|$ is controlled by $D$ (i.e. vanished for $D \rightarrow \infty$) and the last term of Eq. (41) vanishes for $L \rightarrow \infty$. The situation is more complex for $\frac{1}{\psi} > \frac{5}{4}$. In this case the $L$-independent term on the RHS of Eq. (41) is bounded from above by its value at $\sigma \rightarrow 0^+$, which in turn may be expressed as

$$
G(\kappa) = \int_{0}^{\infty} dq \, q^{4 \kappa - 5} \phi(q).
$$

The properties of the above function are important for the analysis to follow. It is plotted in Fig. 3 (right panel) for illustration. We note in particular that

$$
F_{\kappa}(x) \approx \frac{1}{\frac{1}{4} x^{4 \kappa - 4}} G(\kappa)
$$

for $x \ll 1$ and $\kappa > 1$. The physical significance of the value $\frac{1}{\psi} = \frac{5}{4}$ is clear upon noticing that it corresponds to the lower critical dimension for condensation in a system with finite $D$ (i.e. after “excluding” the $d$-th direction in which the system is finite). One then finds a finite solution $\sigma(x) > 0$ for $x$ fulfilling the condition

$$
0 \leq x \leq x_{cr}\left(\frac{1}{\psi}\right) = \frac{1}{\zeta\left(\frac{1}{\psi}\right)} \left(\frac{\Gamma(5/4)}{\pi}\right)^{\frac{1}{4}} \frac{G\left(\frac{1}{\psi}\right)}{\zeta\left(\frac{1}{\psi}\right)}.
$$

In the opposite situation (for $x > x_{cr}$) the last term in Eq. (41) gives a finite contribution in the thermodynamic limit. This reflects the phase transition taking place (at finite) $D$ for

$$
\tilde{\mu}_{c}(T) = \mu_{c}(T) \left[ x_{cr}\left(\frac{1}{\psi}\right) \left(\frac{D}{\lambda_{2}}\right)^{\frac{1}{4}} + 1 \right].
$$

We then obtain $\sigma(x) = 0$ for $x > x_{cr}$. Inspection of the function $G\left(\frac{1}{\psi}\right)$ - see Fig. 4 reveals however that $G\left(\frac{1}{\psi}\right)$ has a zero at $\frac{1}{\psi} = \frac{5}{4}$. For $\frac{1}{\psi} > \frac{5}{4}$ we obtain $\sigma(x) = 0$ for all $x \leq 0$ in the limit $L \rightarrow \infty$. This behavior persists for $\frac{1}{\psi} > 2$, as discussed in the next subsection. Note however that in the “uniaxial” case $m = 1$ the value $\frac{1}{\psi} = \frac{5}{4}$ corresponds to the physical dimensionality $d = 4$, while for $m = 2$, $\frac{1}{\psi} = \frac{5}{4}$ implies and even higher value $d = \frac{9}{4}$. Obviously an experimentally meaningful value of $\frac{1}{\psi}$ is $\frac{5}{4}$.

As explained above, for $x > 0$ the behavior of $\sigma$ is controlled by either $D$ or the system volume. The situation is different for $x < 0$, where it is governed by the distance from the phase transition. Indeed, fixing $x < 0$ and passing to the limit $L \rightarrow \infty$, $D \rightarrow \infty$ we obtain a finite solution for $\sigma(x)$, which, for large $|x|$ (where we may replace $s$ with $x_0 < 0$) is given by the relation

$$
\zeta\left(\frac{1}{\psi}\right)x = \left(\frac{\Gamma(5/4)}{\pi}\right)^{\frac{1}{4}} \sigma^{\frac{1}{4}} \Gamma\left(1 - \frac{1}{\psi}\right) < 0.
$$

2. Case $\frac{1}{\psi} > 2$

For $\frac{1}{\psi} > 2$ the saddle-point equation [Eq. (42)] has a finite solution

$$
\sigma(x) = \frac{1}{\Gamma(5/4)} \left(\frac{1}{\mu_{c}} + \zeta\left(\frac{1}{\psi} - 1\right)\right)^{-1/4}
$$

for $x \leq 0$ in the limit $L \rightarrow \infty$, $D \rightarrow \infty$. For $x > 0$ inspection of the signs of the different terms in Eq. (42) leads directly to the conclusion that this equation is never fulfilled for $\sigma(x) > 0$. In consequence, the last term in Eq. (42) must give a finite contribution, which implies $\sigma(x) = 0$ for $x > 0$ and $L \rightarrow \infty$.

B. Excess free energy

We proceed to determine the excess grand canonical free energy given by Eq. (24). We again analyze the two cases distinguished by the value of $\frac{1}{\psi}$. 

1. Case $1 < \frac{d}{\theta} < 2$

We treat the expression for $\varphi(\mathbf{r})$ given in Eq. (26) with a line of steps analogous to those applied above for the saddle-point equation. We employ the Poisson formula to the sum over $k_d$, replace the $r$-summation with an integral, and finally perform the expansion of the Bose function around $\bar{s} = 0$. It is here necessary to keep the two leading $\bar{s}$-dependent contributions, so that

$$g_{\bar{s}+1}(e^\ell) - \zeta \left( \frac{1}{\bar{s}} + 1 \right) = \Gamma \left( \frac{1}{\bar{s}} \right) \frac{1}{\bar{s}} - \zeta \left( \frac{1}{\bar{s}} \right) \frac{1}{\bar{s}} + \ldots . \quad (49)$$

As a result, in the limit $D \gg \lambda_2$, we obtain the following expression for $\omega_\lambda$:

$$\frac{\omega_\lambda}{k_BT} = -\chi^{d-m} \frac{\Delta^\perp(x)}{D^{\frac{d-1}{2}}} = -\chi^{d-m} \frac{\Delta^\perp(x)}{D^{2d-m-1}} , \quad (50)$$

where

$$\chi = \frac{\lambda_2^2}{\lambda_1} = A - \frac{\pi^{3/2}}{2} (\Gamma(5/4)^2 (\theta/\lambda)_0^{1/2} \quad (51)$$

is a temperature-independent microscopic length, while

$$\Delta^\perp(x) = \left( \frac{\Gamma(5/4)}{\pi} \right)^4 \left( \frac{1}{\bar{s}} \right)^{x/\bar{s}} \left( 1 + \frac{1}{\bar{s}} \right) \sum_{n=1}^{\infty} \frac{F_{\bar{s}+1}(\sigma(x))}{\sum_{n=1}^{\infty} F_{\bar{s}+1}(\sigma(x))} \quad (52)$$

represents the scaling function. The quantity $\sigma(x)$ must be determined from the saddle-point equation (41) as described in the previous sections. For $\frac{d}{\theta} > \frac{2}{3}$ and $x > x_c$, the scaling function is independent of $x$ and reads

$$\Delta^\perp(x) = \frac{4}{\Gamma(5/4)} \left( \frac{\Gamma(5/4)}{\pi} \right)^{\frac{d-2}{2}} \left[ G \left( \frac{1}{\bar{s}} + 1 \right) \zeta^\perp \left( \frac{1}{\bar{s}} \right) \right] . \quad (53)$$

We immediately note that the scaling function is constant for $x > x_c$, and monotonous for $x < x_c$. This implies that it is not an analytical function at $x = x_c$ [we do not exclude however the possibility that it is smooth (i.e. from the $C^\infty$ class)].

The corresponding plot is given in Fig. 4 for the physically interesting case $\frac{d}{\theta} = \frac{2}{3}$.

A few interesting facts are clear from Eq. (50). The power law governing the decay of the excess free energy is modified with respect to the standard case: the exponent $\xi_m = 2d - m - 1$ replaces the usual value $\xi_0 = d - 1$. Such an effect is accompanied by the appearance of a nonuniversal (dimensionful) scale factor $\chi^{d-m}$ multiplying the universal scaling function $\Delta^\perp(x)$. The obtained result for the exponent $\xi_m$ agrees with the general prediction of Ref. [11], which related $\xi_m$ to the anisotropy exponent $\theta_A$ such that

$$\xi_m = \frac{d - m}{\theta_A} + m - 1 , \quad (54)$$

where $\theta_A$ describes the ratio of the correlation-length exponents $\nu_\perp$ and $\nu_\parallel$ controlling the divergence of the correlation length in the two inequivalent directions (along say $k_1$ and $k_d$) at the critical temperature, so that $\xi_\perp \sim \xi_\parallel^{\theta_A}$. The detailed study of the correlation function of the anisotropic imperfect Bose gas (see Ref. [6]) shows that $\theta_A = 1/2$, which, after plugging into Eq. (50), yields $\xi_m = 2d - m - 1$ in agreement with Eq. (50). The profile of the scaling function $\Delta^\perp(x)$ obtained by solving Eq. (41) for $\sigma(x)$ and plugging into Eq. (52) is plotted in Fig. 4 for the experimentally meaningful case $\frac{d}{\theta} = \frac{2}{3}$ and $x \geq 0$. The negative sign of $\Delta^\perp(x)$ indicates repulsive character of the interaction in the low-temperature phase, in clear contrast to the usual situation with periodic boundary conditions. The asymptotic values $\Delta^\perp(0)$ and $\Delta^\perp(\infty)$ correspond to Casimir amplitudes at the transition and in the low-temperature phase, respectively. The negative sign of $\Delta^\perp(x)$ indicates repulsive character of the Casimir force. The difference between the values of $\Delta^\perp(0)$ and $\Delta^\perp(\infty)$ is tiny. The inset demonstrates the scaling function $\Delta^\perp(x)$ for $|x|$ sufficiently large so that $\bar{s}$ may be replaced with its bulk limit. The damped oscillatory behavior reflects the structure of the density-density correlation function and indicates that the sign of the exponentially suppressed interaction depends on the distance $D$.

FIG. 4: The scaling function $\Delta^\perp(x)$ in the low-temperature phase ($x \geq 0$) and $\bar{s} = \frac{2}{3}$. The asymptotic values at $x = 0$ and $x \to \infty$ correspond to Casimir amplitudes at the transition and in the low-temperature phase, respectively. The negative sign of $\Delta^\perp(x)$ indicates repulsive character of the Casimir force. The difference between the values of $\Delta^\perp(0)$ and $\Delta^\perp(\infty)$ is tiny. The inset demonstrates the scaling function $\Delta^\perp(x)$ for $|x|$ sufficiently large so that $\bar{s}$ may be replaced with its bulk limit. The damped oscillatory behavior reflects the structure of the density-density correlation function and indicates that the sign of the exponentially suppressed interaction depends on the distance $D$.

We now discuss the Casimir-like interaction in the high-temperature phase ($x < 0$), where the correlation lengths are finite and therefore the effective force is expected to decay exponentially for $D \gg \xi_\perp$. In this case we obtain an analytical expression for the scaling function in the regime $|x| \gg 1$, where we may replace $\bar{s}$ with its bulk value $\bar{s}_0$. The asymptotic behavior of $\omega_\lambda$ for $|x| \gg 1$ (and $x < 0$) is obtained as

$$\frac{\omega_\lambda}{k_BT} = -\chi^{d-m} \frac{\Delta^\perp(x)}{D^{\frac{d-1}{2}}} = -\chi^{d-m} \frac{\Delta^\perp(x)}{D^{2d-m-1}} , \quad (55)$$

where

$$\Delta^\perp(x) = \left( \frac{\Gamma(5/4)}{\pi} \right)^{\frac{d-2}{2}} \left( \frac{1}{\bar{s}} \right)^{\frac{d-2}{2}} \sum_{n=1}^{\infty} F_{\bar{s}+1}(\sigma(x)) . \quad (56)$$
The scaling function $\Delta^\gamma(x)$ displays exponentially damped oscillations deriving from the structure of the function $\phi(k)$ [Eq. (36)]. Its profile for $\frac{1}{\psi} = \frac{3}{4}$ is exhibited in the inset of Fig. 4.

As concerns the dependence of the scaling function ($\Delta^\gamma(x)$ first of all) on dimensionality $\frac{1}{\psi}$, it is clear from Eq. (53) that its sign and magnitude are controlled by the function $G$, whose sign may change depending on the argument. In fact, $\Delta^\gamma(x)$ features a complex and interesting structure as function of $\frac{1}{\psi}$ resulting in a change of sign of the scaling function (and, in consequence also the Casimir force). This is demonstrated in Fig. 5, where we plot $\Delta^\gamma(0)$ and $\Delta^\gamma(\infty)$ as function of $\frac{1}{\psi}$. Note however that for the physically most meaningful cases (such as $\frac{1}{\psi} = \frac{3}{4}$) - compare Sec.III) the force is repulsive.

![Image of a graph showing the dependence of the Casimir amplitudes $\Delta^\gamma(x = 0)$ and $\Delta^\gamma(x \to \infty)$ on $\frac{1}{\psi}$. The difference between the two quantities is nonzero up to $\frac{1}{\psi} = \frac{3}{4}$ (compare Fig. 4), but is not visible in the plot scale. Negative value of $\Delta^\gamma(x = 0)$ indicates a repulsive interaction.]

FIG. 5: The dependence of the Casimir amplitudes $\Delta^\gamma(x = 0)$ and $\Delta^\gamma(x \to \infty)$ on $\frac{1}{\psi}$. The difference between the two quantities is nonzero up to $\frac{1}{\psi} = \frac{3}{4}$ (compare Fig. 4), but is not visible in the plot scale. Negative value of $\Delta^\gamma(x = 0)$ indicates a repulsive interaction.

2. Case $\frac{1}{\psi} > 2$

The analysis of this case proceeds along the same line as for $1 < \frac{1}{\psi} < 2$, but significantly simplifies due to vanishing of $\sigma(x)$ obtained from the solution of the saddle-point equation. For fixed $\frac{1}{\psi}$ and $x \geq 0$ one obtains a constant scaling function of value given by Eq. (53). The expression for $\Delta^\gamma(x)$ given in Eq. (50) remains valid also for the present case. The Casimir amplitude is plotted in Fig. 5 together with the results obtained for $\frac{1}{\psi} < 2$.

C. Walls perpendicular to $k_1$

We now analyze the complementary situation, where the confining walls are oriented perpendicular to $k_1$, and, as we show below, the scaling function for the Casimir energy has completely different properties as compared to the setup discussed above. In essence the computation proceeds along the same line, the difference being that the roles of $k_1$ and $k_d$ are interchanged and Eq. (35) becomes replaced by

$$\sum_{k_1} e^{-\psi_{k_1}} = \frac{D}{\lambda_1 \sqrt{\pi}} + 2 \frac{D}{\lambda_1 \sqrt{\pi}} \sum_{n=1}^{\infty} \exp\left(-\frac{\pi^2 n^2}{\lambda_1 r}ight), \quad (57)$$

where we again used the Poisson summation formula. The saddle point equation is then written as

$$\zeta\left(\frac{1}{\psi}\right) - \frac{\bar{s}}{\mu \beta} + \epsilon = g_\phi(e^\phi) - \zeta\left(\frac{1}{\psi}\right) \quad \zeta\left(\frac{1}{\psi}\right)$$

$$+ 2 \sum_{r=1}^{\infty} \sum_{n=1}^{\infty} \exp\left(-\frac{\pi^2 n^2}{\lambda_1 r^2}\right) - \frac{\lambda^{d-m} A^m_2}{V} \sum_{k_1} \frac{1}{e^{\psi_{k_1}} - 1} . \quad (58)$$

For $D \gg \lambda_1$ the $r$-summation can now be transformed into an integral by using the Euler-Maclaurin formula. The resulting integral is expressible via the Bessel function and the saddle-point equation takes the form

$$\zeta\left(\frac{1}{\psi}\right) - \frac{\bar{s}}{\mu \beta} + \epsilon = g_\phi(e^\phi) - \zeta\left(\frac{1}{\psi}\right)$$

$$+ 2 \sum_{n=1}^{\infty} \left(\frac{\lambda_1}{D}\right)^{\frac{1}{2}} \frac{1}{\pi^{\frac{1}{2}}} \exp\left(-\frac{\pi^2 n^2}{\lambda_1}\right) \sum_{k_1} \frac{1}{e^{\psi_{k_1}} - 1} . \quad (59)$$

where

$$\sigma' = 2 \sqrt{\pi \lambda_1 |\bar{s}|^{1/2}} . \quad (60)$$

Expanding the Bose function for $\bar{s} \ll 1$ we again encounter the different cases depending on the value of $\frac{1}{\psi}$. Introducing

$$x' = \begin{cases} \left(\frac{D}{\lambda_1}\right)^{1/2}, & 1 < \frac{1}{\psi} < 2 \\ \left(\frac{D}{\lambda_1}\right)^{1/2}, & \frac{1}{\psi} \geq 2 \end{cases} \quad (61)$$

the saddle-point equation is written as

$$\zeta\left(\frac{1}{\psi}\right) x'^{\frac{1}{2} - 1} = \frac{\Gamma(1 - \frac{1}{\psi})}{2^{\frac{1}{2} - 2}} \sigma'^{\frac{1}{2} - 2}$$

$$+ 2^{\frac{1}{2} - 1} \sigma'^{\frac{1}{2} - 1} \sum_{n=1}^{\infty} (n^{-1})^{\frac{1}{2} - 1} K_{\frac{1}{2} - 1}(n \sigma') \quad (62)$$

$$+ \pi^{\frac{1}{2} - 1} \left(\frac{\lambda_1}{D}\right)^{1/2} \frac{1}{A^m J^m} \frac{1}{V} \sum_{k_1} \frac{1}{e^{\psi_{k_1}} - 1}$$

for $\frac{1}{\psi} \in (1, 2)$ and

$$\zeta\left(\frac{1}{\psi}\right) x' = - \frac{1}{4 \pi} \sigma'^{2} \left(\zeta\left(\frac{1}{\psi} - 1\right) + \frac{1}{\mu \beta}\right)$$

$$+ 2^{\frac{1}{2} - 1} \left(\frac{\lambda_1}{D}\right)^{1/2} \sigma'^{\frac{1}{2} - 1} \sum_{n=1}^{\infty} (n^{-1})^{\frac{1}{2} - 1} K_{\frac{1}{2} - 1}(n \sigma') + \frac{D}{\lambda_1} \frac{1}{A^m J^m} \frac{1}{V} \sum_{k_1} \frac{1}{e^{\psi_{k_1}} - 1} \quad (63)$$
for $\frac{1}{\psi} > 2$, providing a relation between the quantities $x'$ and $\sigma'$, which is necessary to evaluate the excess free energy via Eq. (24). Focusing now on the case $\frac{1}{\psi} \in (1, 2)$ we note that the last term on the RHS of Eq. (62) can be neglected as long as $\sigma'$ (which solves Eq. (62)) is finite. For $\frac{1}{\psi} > \frac{1}{2}$ the RHS is in such a case bounded from above by

$$\lim_{\sigma' \to 0} 2^{3-\frac{1}{2}} \sigma'^{2-1} \left\{ \sum_{n=1}^{\infty} n^{1+\frac{1}{2}} K_{-1}(n\sigma') \right\} = 2\Gamma\left( \frac{1}{\psi} - 1 \right) \left( \frac{2}{\psi} - 2 \right).$$

This implies that for $x' > x'_c, \psi > 0$, where

$$x'_c(\psi) = \frac{2}{\zeta(\frac{1}{\psi})} \Gamma\left( \frac{1}{\psi} - 1 \right) \zeta\left( \frac{2}{\psi} - 2 \right)$$

$\sigma'$ vanishes. In consequence, the last term in Eq. (62) cannot be neglected. As a result, one obtains $\sigma' = 0$ for $x' > x'_c(\psi)$ (and $2 > \frac{1}{\psi} > \frac{1}{2}$). The scaling function is then constant (see below). For $\frac{1}{\psi} > 1$ one obtains $\sigma' = 0$ for any value of $x' > 0$, whereas for $x' < 0$ we obtain

$$\sigma' = \left( \frac{4\pi|x'|}{\pi \beta + \zeta\left( \frac{1}{\psi} - 1 \right)/\zeta\left( \frac{1}{\psi} \right)} \right)^{1/2}.$$

As concerns the excess free energy, for the present case one finds

$$\varphi(3) = -\frac{2^2}{2a_0^2} - \frac{2^2}{2 \alpha_1^{2-m}} - \frac{1}{\lambda_1^{d-m} \lambda_2^{2-m}} g_{-1}(e^\delta) = -
\frac{1}{\lambda_1^{d-m} \lambda_2^{m}} \left( \frac{A_1}{D} \right)^{2-2m} \sigma'^2 \sum_{n=1}^{\infty} (n^{-1})^{2m} K_{-1}(n\sigma') +
\frac{1}{\pi} \sum_{k_1} \log \left( 1 - e^{-\beta\rho_{1,1}} \right),$$

and the last term always gives a vanishing contribution for $L \to \infty$. For $\frac{1}{\psi} \in (1, 2)$ and $x' \geq 0$ Eq. (24) can now be cast in the form

$$\omega = \frac{\omega_0}{k_B T} = \frac{1}{\chi^{m/2}} \frac{\Delta_c'(x)}{D^{m/2-1}} = \frac{1}{\chi^{m/2}} \frac{\Delta_c'(x')}{D^{m/2-1}}$$

where the scaling function is given by

$$\Delta_c'(x') = \frac{\zeta\left( \frac{1}{\psi} \right)}{4\pi} \sigma'(x')^2 + \frac{\Gamma\left( \frac{1}{\psi} \right)}{2\pi} \sigma'(x') \frac{2^2}{2 \pi^2} \sigma'(x') \sum_{n=1}^{\infty} (n^{-1})^{2m} K_{-1}(n\sigma'(x'))$$

and the relation $\sigma'(x')$ is determined from the solution of the saddle-point equation. We conclude that also for the present situation where the confining walls are oriented perpendicular to $k_1$, the power law governing the decay of the Casimir interaction is modified, which is accompanied by the appearance of a nonuniversal, dimensionful scale factor $\frac{1}{\sigma'}$. The obtained exponent $\zeta_m = d - m(1 - \theta_A) - 1$ again agrees with the form $\zeta_m = d - m(1 - \theta_A)$. The scaling function is monotonous and positive in each of the phases and points at attractive interaction, in contrast to the previous case of walls oriented perpendicular to $k_0$. The oscillations in the high-temperature phase are also absent. The profile of $\Delta_c'(x')$ is identical to that derived in Ref. (17) for isotropic continuum case upon identifying $\frac{1}{\psi} \to \frac{1}{2}$ so that the net effect of the anisotropy is the modification of the decay exponent $\zeta_m$ accompanied by the appearance of the scale factor $\frac{1}{\sigma'}$ [see Eq. (63)]. This holds true also for $\frac{1}{\psi} > 2$ as well as $2 > \frac{1}{\psi} > 1$ and $x' < 0$. We note however, that the existence of $x'_c(\frac{1}{\psi})$ and the fact that the scaling function is constant for $x' > x'_c(\frac{1}{\psi})$ for $\frac{1}{\psi} < \frac{1}{2}$ was not discussed in that study. This result implies that the scaling function $\Delta_c'(x')$ is not analytical at $x'_c(\frac{1}{\psi})$ which does not exclude the possibility that it remains within the $C^\infty$ class. This feature is related to the phase transition occurring at finite $D$ and should be shared by the entire $O(N \to \infty)$ universality class (also for the isotropic case) in dimensionality $3 < d < 4$. (40) We once again invoke here the recently established fact (18) that the critical behavior of the imperfect Bose gas with periodic boundary conditions maps exactly onto the corresponding classical $O(2N)$ model in the limit $N \to \infty$. The scaling functions of these two models are the same modulo a global factor of two. (17, 18, 21)

### V. Summary and Outlook

In this paper we addressed the Bose gas in optical lattices focusing on effects induced by spatial anisotropies which may be controlled by varying the lattice parameters. By suitably tuning the couplings, the system is driven into a strongly anisotropic setup, where condensation is characterized by two divergent length scales ($\xi_1$ and $\xi_2$) related by the anisotropy exponent $\theta_A$ (so that $\xi_1 \sim \xi_2^{\theta_A}$) and by fractional effective spatial dimensions. We addressed two aspects of the system induced by such anisotropies, the first one being related to dimensional crossovers in the bulk Bose-Einstein condensation (Sec. III), the other to Casimir interactions (Sec. IV). We employed the imperfect Bose gas as the prototypical model. As we argued at the end of Sec. III, we believe our major conclusions concerning the dimensional crossovers in the bulk should not be restricted to mean-field models. Indeed, the imperfect Bose gas is known to be closely related to the $O(N \to \infty)$ universality class, while realistic condensation corresponds to $N = 2$. It is however known that at least some of the features studied here (the universal asymptotics of the $T_c$-line in particular) are insensitive to the symmetry-breaking involved. Dimensional crossovers and the idea of tuning the system across the phase transition by varying dimensionality seems to be a problem of current experimental interest (25, 27) and we have provided an analytical understanding of these effects realized by tuning the lattice hoppings. Particularly interesting situations arise when one of the involved effective dimensionalities is located at or below the lower critical dimension ($d_{eff} = 2$) for condensation. The hoppings may then
be tuned to completely expel the condensate out of the system. We clarified the mechanism that governs this behavior.

Our results for the scaling function of the Casimir energy (Sec. IV) indicate a rather unusual behavior presumably generic for systems characterized by dispersions varying as $k^4$ (which however calls for further studies). Starting from a microscopic level and performing an exact analysis we have confirmed the picture of Ref. [11] concerning modifications of the decay exponent for the Casimir interaction. This is necessarily accompanied by the appearance of a nonuniversal dimensionful scale factor. We focused on periodic boundary conditions and addressed two configurations of the confining walls. In the first case, the walls are perpendicular to a direction characterized by a $\sim k^4$ dispersion; in the other setup the dispersion in the perpendicular direction is of the type $\sim k^2$. In the former situation and at physically most relevant effective dimensionalities, the obtained Casimir interaction turns out to be repulsive below and at the critical temperature $T \leq T_c$. In this regime we evaluated the entire profile of the universal scaling function, which turns out to be monotonous for positive values of the scaling variable $x$ (for $T \leq T_c$). In contrast, for $x < 0$ the scaling function shows exponentially damped oscillatory behavior and changes sign upon varying the distance $D$. In the present setup these effects are encoded in the rich structure of the function $b(k)$ (Fourier transform of the quartic Gaussian). By virtue of universality we expect similar behavior to apply to the entire $O(N \to \infty)$ universality class (up to proportionality factors understood in Ref. [18] for the isotropic case). An extension to finite $N$ is not an easy enterprise as is clear from Ref. [11]. For the standard case of isotropic $O(N)$ models the profile of the scaling function (but usually not its sign) may be different as compared to their $N \to \infty$ limiting shapes. [21] The possibility of obtaining repulsive Casimir forces was recently studied in a number of contexts. [42–51] In many situations, such scenarios are realized by varying the boundary conditions, which, to some extent may also be controlled experimentally by engineering the surface properties (see e.g. [52–54]). We point out, however, that the possibility of obtaining repulsive Casimir interaction for periodic boundary conditions is quite uncommon. Further theoretical verification of this possibility (at finite $N$ in particular) is an interesting direction for future studies. On the experimental side, besides the present context, anisotropic scale invariance is also present at the Lifshitz points as well in liquid crystals, to which (by virtue of universality) our results concerning the Casimir force may perhaps also apply.

As concerns the situation with $\sim k^2$ dispersion in the direction perpendicular to the confining walls, we have identified a similar effect of modifying the decay exponent for the Casimir energy, which is necessarily accompanied by emergence of a nonuniversal, dimensionful scale factor in the expression for the excess free energy. Once this is factored out, one recovers the universal scaling function identical to that obtained for the isotropic case, however in lower dimensionality $d_{eff} < 3$. The resulting Casimir interaction is then always attractive and shows no oscillations of the type observed in the case where the dispersion in the direction perpendicular to the walls is of the $\sim k^4$ type.

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