Tunneling between fermionic vacua and the overlap formalism.

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Abstract

The probability amplitude for tunneling between the Dirac vacua corresponding to different signs of a parity breaking fermionic mass $M$ in $2 + 1$ dimensions is studied, making contact with the continuum overlap formulation for chiral determinants. It is shown that the transition probability in the limit when $M \rightarrow \infty$ corresponds, via the overlap formalism, to the squared modulus of a chiral determinant in two Euclidean dimensions. The transition probabilities corresponding to two particular examples: fermions on a torus with twisted boundary conditions, and fermions on a disk in the presence of an external constant magnetic field are evaluated.

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1 Introduction.

Quantum mechanical tunneling between different classical vacua is a phenomenon whose importance in Quantum Field Theory should not need to be stressed. A paradigmatic example of this are instanton effects and the related tunneling between vacua in Yang-Mills theories, essential to the understanding of the vacuum structure of such models [1]. No less relevant to tunneling effects is the role of fermions. Under certain conditions their contribution to the transition probability can alter dramatically the results obtained from the other fields involved. This is particularly true of models with massless fermions couple to a gauge field with non-zero Pontryagin index in 4 dimensions, where the presence of fermionic zero modes suppress the transition probability [2].

In this letter we study the probability amplitude for fermionic transition between vacua in 2+1 dimensions, the different vacua being characterized by different signs of the fermion mass. This situation is concretely realised by introducing a real pseudoscalar field with a Yukawa coupling to the fermions, and having a spontaneous symmetry breaking potential with two minima, one for each sign of the mass. The fermions are also coupled to an external gauge field $A_\mu$ with $A_0 = 0$ and $A_j = A_j(\vec{x})$. A question that arises naturally is which is the dependence of the transition probability on the external gauge field configuration and the parameters of the pseudoscalar field potential.

The relevance of this problem to the understanding of Quantum Electrodynamics in 2 + 1 dimensions stems from the fact that different signs of the mass correspond to different parity violating configurations. The existence of fermionic tunneling between them should imply that the true vacuum is a symmetric configuration of the two, with no net parity violation coming from the fermionic sector. The possibility of having different signs for the fermion mass does not necessarily come from the existence of a spontaneous symmetry breaking potential. It may also come, for example, from the fact that one is dealing with the massless theory, whose infrared divergences render it ill-defined. The mass could then be seen as an infrared regulator, whose sign determines the properties of the vacuum.

When defining the probability amplitude, one is faced with an object which has an striking similarity with the squared modulus of the so-called ‘overlap’, the basic construct in the overlap formalism [3]. This is a proposal, based on an earlier idea of Kaplan [4], to define fermionic chiral determinants.
When implemented on the lattice, it seems to overcome the kinematical constraint stated by the Nielsen-Ninomiya theorem [5], and thus could provide a suitable framework to study non-perturbative phenomena in models containing chiral fermions.

In this method, the determinant of a massless chiral Dirac operator in $2d$ dimensions is defined as an overlap between the Dirac vacuum states of two auxiliary Hamiltonians acting on massive Dirac fermions in $2d + 1$ dimensions. This implies that the probability amplitude for tunneling between vacua corresponds, when the modulus of $M$ tends to infinity, to the squared modulus of a chiral determinant in $2d$ dimensions. The presence of the modulus guarantees the gauge invariance of the results.

This paper is organized as follows: In section 2 the kind of model we are studying is introduced and the corresponding transition probability defined. The formula for the transition probability is evaluated for the case of twisted boundary conditions on a torus in section 3, and for a constant magnetic field in section 4. In section 5 we discuss our results and present our conclusions.
2 Transition probability.

We shall consider a fermionic field interacting with an external (i.e., non-dynamical) gauge field $A\mu$ and a dynamical pseudoscalar field $\varphi$ in $2 + 1$ dimensions. The system is described by the action

$$S = \int d^3x \mathcal{L} \quad \mathcal{L} = \mathcal{L}_F + \mathcal{L}_\varphi$$

$$\mathcal{L}_F = \bar{\psi}(i \nabla - e A - g \varphi)\psi \quad \mathcal{L}_\varphi = \frac{1}{2}(\partial \varphi)^2 - \frac{\lambda}{2}(\varphi^2 - \nu_0^2)^2$$

where $\lambda > 0$. The form we have chosen for the pseudoscalar field Lagrangian is such that spontaneous symmetry breaking does occur, the two classical vacua for this field being simply $\varphi = \pm \varphi_0$. To study the vacuum structure of the full system, we need to put the fermions in their vacua as well. As the fermionic Lagrangian is sensitive to the sign chosen for the vacuum value of $\varphi$, the form it adopts for each of the $\varphi$ vacua has to be studied separately. This yields

$$\mathcal{L}_F^\pm = \bar{\psi}(i \nabla - e A \mp M)\psi$$

where $M = g\varphi_0$. The fermionic vacua will simply be the two possible Dirac vacua corresponding to $\mathcal{L}_F^+$ and $\mathcal{L}_F^-$. We shall deal with gauge field configurations such that $A_0 = 0$ and $A_j$ is static (in order to have a vacuum state). Dirac vacua are constructed by filling all the negative energy states. The second-quantized Hamiltonians corresponding to (2):

$$H^\pm(A) = \int d^2x \Psi^\dagger(x)\mathcal{H}_\pm(A)\Psi(x)$$

where $\Psi(x)$ is the fermionic field operator, and $\mathcal{H}_\pm$ are the two 1-body Dirac Hamiltonians

$$\mathcal{H}_\pm(A) = \bar{\alpha} \cdot (-i \nabla - e A) \pm M \beta .$$

We define the eigenstates of $\mathcal{H}_\pm(A)$

$$\mathcal{H}_\pm(A) \ u_\pm(\lambda|A,x) = \omega(\lambda|A,x) \ u_\pm(\lambda|A,x)$$

$$\mathcal{H}_\pm(A) \ v_\pm(\lambda|A,x) = \ -\omega(\lambda|A,x) \ v_\pm(\lambda|A,x)$$

where $u_\pm(\lambda|A,x)$ and $v_\pm(\lambda|A,x)$ are the positive and negative energy eigenstates, respectively. $\lambda$ is an index which labels the eigenstates, and is assumed
to be discrete for the sake of simplicity, but a continuous spectrum can also be dealt with in a completely analogous manner. The $x$ and $A$ dependence of the eigenspinors, as well as the $A$-dependence of the energy $\omega$ have been made explicit. Note that the above objects do also depend on the value of the mass $M$. We assume the orthonormality relations

\begin{align*}
( u_{\pm}(\lambda|A,x), u_{\pm}(\lambda'|A,x) ) &= \delta_{\lambda,\lambda'} \\
( v_{\pm}(\lambda|A,x), v_{\pm}(\lambda'|A,x) ) &= \delta_{\lambda,\lambda'} \\
( v_{\pm}(\lambda|A,x), u_{\pm}(\lambda'|A,x) ) &= 0
\end{align*}

where the scalar products mean inner product in spinorial two-component space as well as integration over the coordinates. In terms of the eigenspinors $\{\bar{u}\}$, the fermionic field may be expanded using either sign of the mass, \begin{equation}
\Psi(x) = \sum_\lambda \left[ b_{\pm}(\lambda) u_{\pm}(\lambda|A,x) + d_{\pm}^\dagger(\lambda) v_{\pm}(\lambda|A,x) \right]
\end{equation}
\begin{equation}
\Psi^\dagger(x) = \sum_\lambda \left[ b_{\pm}^\dagger(\lambda) u_{\pm}^\dagger(\lambda|A,x) + d_{\pm}(\lambda) v_{\pm}^\dagger(\lambda|A,x) \right].
\end{equation}

Whence the canonical anticommutation relations become

\begin{align*}
\{\Psi_\alpha(x), \Psi_\beta(y)\} &= 0 \\
\{\Psi_\alpha(x), \Psi_\beta^\dagger(y)\} &= \delta_{\alpha\beta} \delta^{(2)}(x - y).
\end{align*}

By using the expansions defined by Eqs.(5-7), the Dirac vacua corresponding to the two possible signs of the mass become

\begin{equation}
|A \pm > = \left[ \prod_\lambda d_{\pm}(\lambda) \right] |0 >
\end{equation}

where $|0 >$ is the Fock vacuum, which satisfies $\Psi(x) |0 > = 0, \forall x$, and the product runs over all the possible values of the index $\lambda$. It goes without saying that the transition probability, normalized to 1 for $A_\mu = 0$ is

\begin{equation}
P_{\pm}(A) / P_{\pm}(0) = \frac{<A + |A- >}{< +| - >}^2
\end{equation}

where $|\pm > \equiv |0\pm >$. In [10] we recognize the object which appears in the overlap definition of the chiral determinant, in this case in two Euclidean dimensions. More precisely, in the overlap formalism one can obtain
the squared modulus of the normalized chiral determinant by means of the limit \[ \lim_{M \to \infty} \left| \frac{\det[(\theta + i e A)(\frac{1 + \gamma_5}{2})]}{\det[\theta(\frac{1 + \gamma_5}{2})]} \right|^2 \]

\[ = \frac{\lim_{M \to \infty} |<A + |A-|>|^2}{|<+|->|^2} \tag{11} \]

where the operators in (11) act on functions defined on two-dimensional Euclidean space. For example

\[ \theta = \gamma_1 \partial_1 + \gamma_2 \partial_2, \gamma_1 = \sigma_1, \gamma_2 = \sigma_2, \gamma_3 = \sigma_3. \tag{12} \]

The overlap between the two vacua may be given in terms of the negative energy eigenstates (5) of \( H_\pm(A) \). We first use (7) to write

\[ <A + |A-|> = <0|\prod_{\lambda} d_+^{\dagger}(\lambda)|\prod_{\lambda'} d_-^{\dagger}(\lambda')|0>. \tag{13} \]

We then use the relation between the creation and annihilation operators defined for both signs of the mass \( M \)

\[ b_+^{\dagger}(\lambda) = \sum_{\lambda'} \left[(u_+^{\dagger}(\lambda), u_-^{\dagger}(\lambda'))d_-^{\dagger}(\lambda') + (u_+^{\dagger}(\lambda), v_-^{\dagger}(\lambda'))d_-^{\dagger}(\lambda')\right] \]

\[ d_+^{\dagger}(\lambda) = \sum_{\lambda'} \left[(v_+^{\dagger}(\lambda), u_-^{\dagger}(\lambda'))d_-^{\dagger}(\lambda') + (v_+^{\dagger}(\lambda), v_-^{\dagger}(\lambda'))d_-^{\dagger}(\lambda')\right]. \tag{14} \]

By using the second line of (14) into (13), we may write each factor \( d_+^{\dagger} \) in terms of the full set of operators \( b_-^{\dagger}(\lambda) \) and \( d_-^{\dagger}(\lambda) \). The terms proportional to \( b_-^{\dagger}(\lambda) \) do not contribute because they can be anticommutated with all the \( d_-^{\dagger}(\lambda) \) operators, and they finally annihilate the Fock vacuum. Thus only the contributions depending on the scalar product between \( v_+^{\dagger}(\lambda) \) and \( v_-^{\dagger}(\lambda') \) survive, yielding

\[ <A + |A-|> = \det_{\lambda,\lambda'}(v_+^{\dagger}(\lambda), v_-^{\dagger}(\lambda')). \tag{15} \]

Hence the ratio between probabilities given in (10) yields

\[ \frac{P_\pm(A)}{P_\pm(0)} = \left| \frac{\det_{\lambda,\lambda'}(v_+^{\dagger}(A|\lambda), v_-^{\dagger}(A|\lambda'))}{\det_{\lambda,\lambda'}(v_+^{\dagger}(0|\lambda), v_-^{\dagger}(0|\lambda'))} \right|^2. \tag{16} \]

We shall now evaluate (16) in two situations of interest: (1) Fermions on a torus with constant metric and arbitrary twisting, (2) Fermions in the presence of an external magnetic field.
3 Fermions with twisted boundary conditions on a torus.

The two-dimensional torus is coordinatized by two real variables

\[ \sigma^1, \sigma^2, \quad 0 \leq \sigma^\mu \leq 1, \quad (17) \]

and is equipped with the Euclidean metric

\[ ds^2 = |d\sigma^1 + \tau d\sigma^2|^2 = g_{\mu\nu} d\sigma^\mu d\sigma^\nu, \]

\[ g_{\mu\nu} = \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}, \quad g^{\mu\nu} = \frac{1}{\tau_2^2} \begin{pmatrix} |\tau|^2 & -\tau_1 \\ -\tau_1 & 1 \end{pmatrix}, \quad (18) \]

where \( \tau = \tau_1 + i\tau_2 \) and \( \tau_2 > 0 \).

The Dirac Hamiltonian operators then become

\[ H_\pm(A) = \sigma^a e^\mu_a (\partial_\mu + iA_\mu) \pm M, \quad (19) \]

where the \( \sigma^a \)'s are the usual Pauli matrices

\[ \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (20) \]

and \( e^\mu_a \) are the zweibeins:

\[ g_{\mu\nu} e^\mu_a e^\nu_b = \delta_{ab}, \]

\[ e^\mu_1 = (1, 0) \quad e^\mu_2 = (-\frac{\tau_1}{\tau_2}, \frac{1}{\tau_2}). \quad (21) \]

Note that \( \tau_2 \) is the volume of the torus

\[ \tau_2 = \int d^2\sigma \sqrt{\det g_{\mu\nu}}. \quad (22) \]

We want to describe twisted fermions, i.e., the fermionic field has the boundary conditions

\[ \Psi(\sigma^1 + 1, \sigma^2) = -e^{2\pi i \varphi_1} \Psi(\sigma^1, \sigma^2) \]

\[ \Psi(\sigma^1, \sigma^2 + 1) = -e^{2\pi i \varphi_2} \Psi(\sigma^1, \sigma^2), \quad (23) \]
where $\varphi_1$ and $\varphi_2$ are real numbers such that $0 \leq \varphi_\mu \leq 1$ and $n_1, n_2$ run over all the half-integers. The ‘reference’ boundary condition about which we define the twistings is antiperiodicity in both directions, hence the minus signs on the rhs in (23).

We take advantage of the calculation of the overlap between the two Dirac vacua for this situation made in [7]. It follows from [7] that the ratio between probabilities (16) may be written as

$$\frac{P_\pm(\varphi)}{P_\pm(0)} = \prod_{n_1, n_2} \left[ \frac{(n + \varphi)^2 + \lambda^2}{(n + \varphi)^2} \frac{n^2}{(n^2 + \lambda^2)} \right]$$

where $\lambda = \frac{\tau M}{2\pi}$ and $n^2 \equiv g_{\mu\nu} n^\mu n^\nu$. It is not possible to give an exact analytic expression for (16), valid for all possible values of $M$. However, for $\lambda >> 1$, which means that the mass $M$ should be large as compared with the momentum spacing, we can use the leading contribution coming from the limit $\lambda \to \infty$, which is precisely one of the results given in [7]. Thus the probability ratio becomes equal to the squared modulus of the chiral determinant in 2 dimensions, which yields,

$$\frac{P_\pm(\varphi)}{P_\pm(0)} = \left| \vartheta(\alpha, \tau) \vartheta(0, \tau) \right|^2 e^{-2\pi \tau \varphi_1^2}$$

where $\alpha = \tau \varphi_1 - \varphi_2$ and

$$\vartheta(\alpha, \tau) = \sum_{n=-\infty}^{n=\infty} e^{i\pi \tau n^2 + 2\pi in\alpha}.$$  

This corresponds to the squared modulus of the corresponding chiral determinant, found by using conformal field theory methods in [8].

4 Constant magnetic field on a disc.

In this case, we assume there is an external constant magnetic field $B$, so that the gauge field verifies

$$e \epsilon_{jk}\partial_j A_k = B = \text{const}$$

which in the symmetric gauge is satisfied by the configuration

$$A_j = -\frac{B}{2e} \epsilon_{jk} x_k$$
and the Dirac Hamiltonians $\mathcal{H}_\pm (A)$ thus become

$$\begin{pmatrix}
    \pm M & 2\partial + \frac{B}{2} \bar{z} \\
    -2\partial + \frac{B}{2} z & \mp M
\end{pmatrix}$$  \hspace{1cm} (29)$$

where $z \equiv x_1 + ix_2$, $\bar{z} \equiv x_1 - ix_2$, and $\partial \equiv \frac{\partial}{\partial z}$.

The negative energy eigenstates (the only ones we need in order to evaluate the transition probability) are given by

$$v_\pm (n, l) = \sqrt{\omega_n \mp M \over 2 \omega_n} \begin{pmatrix}
    -\sqrt{2B} \\
    \omega_n \mp M \hat{a} f_{n,l}
\end{pmatrix}$$  \hspace{1cm} (30)$$

where $\omega_n = \sqrt{M^2 + sBn}$. $f_{n,l}$ are orthonormal eigenfunctions satisfying

$$(-2\partial + \frac{B}{2} \bar{z})(2\partial + \frac{B}{2} z)f_{n,l} = 2Bnf_{n,l}$$  \hspace{1cm} (31)$$

and we define creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$ by

$$\hat{a} = \frac{2\partial + \frac{B}{2} \bar{z}}{\sqrt{2B}} \quad \hat{a}^\dagger = \frac{2\partial + \frac{B}{2} z}{\sqrt{2B}} ,$$  \hspace{1cm} (32)$$

respectively. The functions $f_{n,l}$ may be built by successive application of the creation operator $\hat{a}^\dagger$ on the $n = 0$ state

$$f_{n,l} = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}} f_{0,l}$$

and

$$f_{0,l} = \left[\pi \left(\frac{2}{B}\right)^l + 1\right]^{-\frac{1}{2}} e^{-\frac{B}{2} \bar{z} z}$$  \hspace{1cm} (33)$$

The structure of the functions $f_{n,l}$ is quite similar to the ones one encounters when studying the eigenstates of a non-relativistic particle in the presence of an external magnetic field in two dimensions [9]. Indeed, in spite of the relativistic correction to the energies, the degeneracy of every Landau level remains the same as in the non-relativistic case, since this depends upon the structure of finite magnetic translations. The degeneracy of each level is thus $d_n = N_\phi = \frac{BL^2}{2\pi}$ where $N_\phi$ is the number of flux quanta piercing the disc, which is assumed to be of finite size.
The result of evaluating (16) can be put as
\[
P(B) = \prod_{n=1}^{\infty} \left( \frac{n}{n + x} \right)^{N_\phi}
\] (34)

where \(x \equiv \frac{M^2}{2B}\). To evaluate (34), it is convenient to take logarithms on both sides
\[
\log P(B) - \log P(0) = \frac{(ML)^2}{4\pi} \sum_{n=1}^{\infty} x \log \left( \frac{n}{n + x} \right).
\] (35)

To evaluate (35), we rewrite the logs by using Frullani’s identity:
\[
\log \left( \frac{a}{b} \right) = \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} \frac{ds}{s} \left( e^{-sb} - e^{-sa} \right)
\] (36)

which, after some elementary manipulations yields
\[
\frac{P(B)}{P(0)} = \exp \left[ \frac{(ML)^2}{2\pi} f(x) \right]
\] (37)

where
\[
f(x) = \int_{0}^{\infty} \frac{ds}{s} \frac{e^{-sx} - 1 + sx}{e^s - 1}.
\] (38)

5 Discussion.

We have obtained the transition probabilities between the two parity breaking vacua \((\pm M)\), as functions of the parameters of the model for two particular gauge field configurations (the twisted fermions can be equivalently regarded as untwisted but in the presence of an external constant gauge field \([7]\)).

It is worth remarking that there is a dependence of the probability ratio on the volume (area) of the system. In the case of the constant magnetic field, this dependence is such that when \(L \to \infty\), the ratio between probabilities tends to zero. This is a manifestation of the fact that when the volume tends to infinity, there appears a zero mode for the (two-dimensional) chiral determinant in the presence of a magnetic field. For the case of twisted fermions on the torus, the dependence on the area is through the parameter \(\tau_2\), which appears in a more involved way, but it is not difficult to show
that for large $\tau_2$ one gets a probability ratio that is equal to zero when the components $\varphi$ are nearly half-integer. As in the constant magnetic field case, this is explained by the fact that the corresponding chiral determinant develops a zero mode.

Finally, we note that there is a very important difference between the transition probability one gets for the fermions and the corresponding to the pseudoscalar field. For the fermions, the probability ratio is independent of the parameter $\lambda$ which appears in the symmetry breaking potential, and measures the height of the barrier. It only sees the magnitude and sign of the vacuum value of $\varphi$. Regarding the pseudoscalar field, if the volume is finite the transition probability may be non-zero, but it will vanish when the height of the barrier diverges, since tunneling configurations will cost an infinite action.
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