Baryons and Confining Strings

Oliver Jahn\textsuperscript{a}\textsuperscript{*} and Philippe de Forcrand\textsuperscript{ab}

\textsuperscript{a}Institute for Theoretical Physics, ETH Zürich, CH-8093 Zürich, Switzerland
\textsuperscript{b}CERN, Theory Division, CH-1211 Genève 23, Switzerland

The subleading term of the heavy quark potential (the analogue of the Lüscher term) is computed in a string model for the case of three quarks. It turns out to be positive in 2+1 dimensions, making the potential non-concave as a function of the scale for fixed geometry. The results are compared to numerical simulations of the lattice gauge theory.

1. Motivation

The potential of three heavy quarks has recently been the object of detailed numerical studies \cite{1,2}, which lend support to the so-called Y law, inspired by a string picture of heavy baryons. Here, we study the leading effect of string fluctuations on the potential, the analogue of the Lüscher term \cite{3,4,5} in the quark-antiquark potential. Understanding such corrections to the Y law is particularly important since the transition from ∆ to Y law occurs at large quark separations \cite{1}.

2. Setup

We study three static quarks in a D-dimensional Euclidean space with periodic Euclidean time extent $T$. The potential is obtained from the $T \to \infty$ limit,

$$V_{qqq} = - \lim_{T \to \infty} \frac{1}{T} \ln Z_{qqq}.$$  \hfill (1)

In the string picture, the classical ground state of the three-quark system is given by three strings meeting at a junction, whose position is determined by the requirement of minimal total length of the strings. The balance of tensions implies angles of $\frac{2\pi}{3}$ between the strings. The classical potential is proportional to the total length of the strings in this configuration,

$$V_{cl} = \sigma(L_1 + L_2 + L_3) + m = \sigma L_Y + m,$$  \hfill (2)

where $\sigma$ is the same string tension that appears in the mesonic potential. $m$ represents the self-energy of the junction.

Following the mesonic case \cite{4}, we shall expand the action to second order in the transverse fluctuations $\xi_a(t, s)$ of the string world sheets $a = 1, 2, 3$ around the classical configuration. Transversality means $\xi_a^D = 0$ and $\vec{e}_a \cdot \vec{\xi}_a = 0$, where $\vec{e}_a$ is a spatial unit vector in the direction of string $a$. With $\vec{\phi}(t)$ denoting the position of the fluctuating junction, the boundary conditions are

$$\xi_a(t, 0) = \vec{e}_a, \quad \xi_a(t, L_a + \vec{e}_a \cdot \vec{\phi}(t)) = \vec{\phi}_{\perp a}(t),$$  \hfill (3)

where $\vec{\phi}_{\perp a} \equiv \vec{\phi} - \vec{e}_a(\vec{e}_a \cdot \vec{\phi})$, and periodic in $t$.

Invariance under Euclidean transformations inside the plane of the sheet and perpendicular to it fixes the leading term in a derivative expansion of the bulk string action to

$$S_{\text{bulk}}^{\text{fluct}} = \frac{\mu}{2} \sum_{a=1}^3 \int_{\Gamma_a} \partial \vec{\xi}_a \cdot \partial \vec{\xi}_a,$$

where $\Gamma_a$ is the domain of $\xi_a$ implied by the boundary conditions \cite{6}. We also include a boundary term,

$$S_{\text{bound}}^{\text{fluct}} = \frac{\mu}{2} \int dt |\vec{\phi}|^2.$$

The change of area caused by fluctuations of the junction in the plane of a given sheet cancels in the sum over sheets. The three-quark potential including leading fluctuation effects can thus be extracted from the partition function

$$Z_{qqq} = \int \mathcal{D}\vec{\phi} \int \prod_{a=1}^3 \mathcal{D}\vec{\xi}_a e^{-TV_{cl} - S_{\text{bulk}}^{\text{fluct}} - S_{\text{bound}}^{\text{fluct}}}.$$  \hfill (4)
3. Calculation

We shall compute $Z_{qqq}$ in two steps. For fixed $\vec{\phi}$, each of the string partition functions,

$$Z_a(\vec{\phi}) = \int \mathcal{D}\xi \exp \left\{ -\frac{1}{2} \int |\partial\xi|^2 \right\} ,$$

splits into a minimal-area and a fluctuation part,

$$Z_a(\vec{\phi}) = e^{-\frac{1}{2} \int |\partial\xi_{a,\text{min}}|^2} \left| \text{det}_{\alpha} (-\Delta) \right|^{-\frac{D-2}{2}} ,$$

where $\xi_{a,\text{min}}$ is harmonic and satisfies (W) and the determinant is computed with Dirichlet boundary conditions on the domain $\Gamma_a = \{(t,s) \mid 0 \leq s \leq L_a + \vec{e}_a \cdot \vec{\phi}(t)\}$.

To leading order in $\vec{\xi}$,

$$\vec{\xi}_{a,\text{min}} = \frac{1}{\sqrt{T}} \sum_{\omega} \hat{\phi}_\omega \frac{\sinh(\omega s)}{\sinh(\omega L_a)} e^{i\omega t} + O(\phi^2) ,$$

where $\hat{\phi}_\omega$ are the Fourier components of $\vec{\phi}(t)$. This implies

$$\int |\partial\xi_{a,\text{min}}|^2 = \sum_\omega \omega \coth(\omega L_a) |\hat{\phi}_\omega|^2 + O(\phi^3) ,$$

which is the change in minimal area due to $\vec{\xi}_{a,\min}$.

The determinant induces a renormalisation of both $\sigma$ and $m$. After applying a Pauli–Villars regularisation, the determinant, which can be expressed in terms of the heat kernel of $\Delta$, can be computed by mapping the domain $\Gamma_a$ conformally to a rectangle $L'_a \times T$. Since the conformal map cannot change the ratio $L'_a/T$ (the modular parameter of the cylinder), one has to choose $L'_a = L_a + \frac{1}{T} \int \vec{e}_a \cdot \vec{\phi} \, dt$. To leading order in $\vec{\phi}$, the conformal map is

$$f(z) = z + \frac{1}{\sqrt{T}} \sum_{\omega \neq 0} \frac{\vec{e}_a \cdot \vec{\phi}_\omega}{\sinh(\omega L_a)} e^{\omega z} + O(\phi^2) .$$

Following Ref. [3], the determinant on $\Gamma_a$ can be related to that on the rectangle, and one finds

$$\ln \text{det}_{\alpha} (-\Delta) = 2 \ln \eta \left( \frac{iT}{2L'_a} \right) - \frac{1}{12\pi} \sum_\omega \omega^3 \coth(\omega L_a) |\vec{e}_a \cdot \vec{\phi}_\omega|^2 + O(\phi^3) ,$$

where $\eta$ is Dedekind’s function.

The Gaussian integral over $\vec{\phi}$ in (4) can now be performed and $V_{qqq}$ be extracted from the limit $T \to \infty$, cf. (4). One obtains:

$$V_{qqq} = V_{cl}^{\text{ren}} + V_{1/L}^{\parallel} + (D-3)V_{1/L}^{\perp} + O(L^{-2}) ,$$

$$V_{1/L}^{\parallel} = -\frac{\pi}{24} \sum_a \frac{1}{L_a}$$

$$+ \int_0^\infty \frac{d\omega}{2\pi} \ln \left[ \frac{1}{3} \sum_{a < b} \coth(\omega L_a) \coth(\omega L_b) \right] ,$$

$$V_{1/L}^{\perp} = -\frac{\pi}{24} \sum_a \frac{1}{L_a}$$

$$+ \int_0^\infty \frac{d\omega}{2\pi} \ln \left[ \frac{1}{4} \sum_a \coth(\omega L_a) \right] .$$

Here, we have separated contributions from fluctuations in the plane of the three quarks, $V_{1/L}^{\parallel}$, and perpendicular to it, $V_{1/L}^{\perp}$. Note that both expressions are homogeneous in $L_a$, representing exactly the $O(L^{-1})$ term.

4. Check on the mesonic string

As a check, we can (artificially) split the string connecting a quark and an antiquark into two sections of length $L_1$ and $L_2$, pretending there is a junction in between. The mass of the junction should not affect the large-$L$ behaviour, and we should recover the known result. In this case, there is no contribution $V_{1/L}^{\parallel}$ and the integral in $V_{1/L}^{\perp}$ becomes

$$\int_0^\infty \frac{d\omega}{2\pi} \ln \left[ \frac{1}{4} \left( \coth(\omega L_1) + \coth(\omega L_2) \right) \right]$$

$$= \frac{\pi}{24} \left( \frac{1}{L_1} + \frac{1}{L_2} - \frac{1}{L_1 + L_2} \right) ,$$

which just corrects the Lüscher terms of the two string sections into that of the full string.

5. Special cases

The following special cases reveal interesting consequences of our result. In the equilateral
case, \( L_a = L = L_3 \), it so happens that the fluctuations in the plane do not contribute. Those perpendicular to it yield
\[
V_{1/L} = V_{1/L}^\parallel + (D-3)V_{3/L}^\perp = -(D-3) \frac{\pi}{16L}.
\]
This means that there is no \( 1/L \) term in \( D = 3 \).

Expanding about the equilateral case, \( L_a = (1 + \varepsilon_a)L \) with \( \sum_a \varepsilon_a = 0 \), one finds
\[
V_{1/L} \approx \frac{\pi}{144L} \sum_a \varepsilon_a^2 - (D-3) \frac{\pi}{16L} \left( 1 + \frac{2}{9} \sum_a \varepsilon_a^2 \right),
\]
so in \( D = 3 \), the \( 1/L \) term is positive. This makes \( V_{qqq} \) non-concave as a function of the scale \( L \) for fixed angles between the quarks. This is actually true for all geometries, not just almost equilateral ones. Fig. 1 shows a density plot of \( V_{1/L} \), as given by Eq. (5), as a function of \( L_1 \) and \( L_2 \) for fixed \( L_3 = 1 \). In \( D \geq 4 \), the \( 1/L \) term is always negative, so \( V_{qqq} \) is concave.

**Figure 1.** \( \frac{24L}{\pi}V_{1/L}(L_1, L_2, 1) \) in \( D = 3 \).

**Figure 2.** \( V_{qqq} - \sigma_{q\bar{q}}L_Y - c_{q\bar{q}} \) from lattice simulations for rectangular, isosceles \( q\bar{q}q \) geometries with either the large side or the small sides of the triangle aligned with lattice axes. The curves are fits \( c - b/L_Y \). The \( Y \) law is approached from above.

6. Lattice gauge theory

Numerical simulations of lattice gauge theory were performed in \( D = 3 \). \( V_{qqq} \) was extracted from Polyakov loop correlators using the method of Ref. 5 on a \( 48^2 \times 32 \) lattice with lattice spacing \( a \approx 0.15 \text{ fm} \) (\( \beta = 11 \)), for one \( q\bar{q}q \) geometry. The deviation from a pure \( Y \) law (with the measured mesonic string tension) is shown in Fig. 2.

The asymptotic \( Y \) law is clearly approached from above as predicted, making the potential non-concave, but the magnitude of the \( 1/L \) term appears too large: \( V_{1/L} = 1.2(2) \frac{\pi}{24L_Y} \) compared to a coefficient of 0.39 as predicted by Eq. (5). The discrepancy may be due to the small quark separations: even for the largest geometries, the shortest string in the classical configuration is only 0.45 fm long, so the string picture is hardly applicable. In addition, it should be noted that the numerical result is very sensitive to the estimate of the mesonic string tension. Finally, a mandatory extrapolation to \( T \to \infty \) has not been performed yet, so that excited-state contributions may significantly affect the measured coefficient.

A more thorough check of the string predictions should also include other geometries. Finally, it would be very interesting to extend the calculation to larger \( SU(N) \) gauge groups and compare with predictions from \( 1/N \) expansions.

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