ON FINITE IMAGINARIES

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Abstract. We study finite imaginaries in certain valued fields, and prove a conjecture of Cluckers and Denef.

1. Introduction

In their beautiful paper [4], Cluckers and Denef study actions of linear algebraic groups on varieties over local fields of high residue characteristic. The orbits of these actions, known to be finite in number, can be viewed as imaginary elements of the theory $HF_0$ of Henselian fields of residue characteristic zero. Cluckers and Denef relate them to finite imaginaries of a certain extension, $T^{(d)}_\infty$ (cf. 2.2). They formulate a “tameness” conjecture (2.16) on the nature of such imaginaries, and show that it is true for imaginaries arising from algebraic group actions over number fields ([4] Theorem 1.1). From this they obtain consequences for orbital integrals (Theorem 1.2).

Sections 3 of this note contain a proof of Conjecture 2.16 in general (Theorem 2.3). This in turn is a special case of a more general description of imaginaries in this theory, and indeed in a wider class of theories of Henselian fields. However as Denef suggested finite imaginaries have a certain autonomy that permits their direct classification. The proof of the finite case follows the lines of the general (unpublished) proof, and has the merit of containing most the main ideas while avoiding technicalities. §2 contains some general observations on finite imaginaries, while §3 describes them for the theory $T^{(d)}_\infty$.

The rest of the paper contains two further comments on [4], from different points of view. Consider theories $T$ of Henselian valued fields of residue characteristic 0. The data of [4] consists of an algebraic group $G$ and a homogeneous space $V$ for $G$, in the sense of algebraic geometry. This means that an action of $G$ on $V$ is given, and over an algebraically closed field the action is transitive. But for a given field $L \models T$ the action need not be transitive, and the question concerns the space of orbits of $G(L)$ on $V(L)$. The goal is to show that $V(L)/G(L)$ reduce to imaginaries of the residue field and the value group; and indeed to special, “tame” imaginaries (see below.)

The data $G, V$ is geometric (i.e. quantifier-free) and group-theoretic. Both of these qualities are lost in the reductions of $G(F)/V(F)$ to the residual sorts given by one of the above methods. In §4 we describe another method that does not lose track of the group theory, and is geometric in the sense of being independent of a particular completion of the theory of Henselian fields. We use the theory ACVF; but we cannot simply interpret the implied quantifiers of $G$-conjugacy on $V$ in terms of the quantifier elimination of ACVF, since the resulting quotient would be trivial. Reformulating the question in terms of groupoids does allow us to work in ACVF without trivializing the problem. We illustrate this in the special case $V = G/T$, where $T$ is a torus.

In [4] the residue field is pseudo-finite, the value group essentially $\mathbb{Z}$-like, and tameness is defined in concrete terms; an imaginary is tame if in the Denef-Pas language it is definable over the sorts $k$ (the residue field) and $\Gamma/\eta\Gamma$ (where $\Gamma$ is the value group); but not $\Gamma$ itself. In the
final section we attempt to understand the role of tameness from a more abstract viewpoint. In [3], Cherlin, Van den Dries and Macintyre consider imaginaries coding the Galois groups of finite extensions of a field $F$. In the context of $HF_0$, we note that Galois imaginaries satisfy a strong (and more symmetric) form of tameness. We also show that $V(L)/G(L)$ is analyzable over the Galois sorts, and hence over the tame imaginaries. We leave open the question of whether analyzability can be replaced by internality. In an appendix, we discuss Galois sorts for theories more general than fields.

§1-3, §4, and §5 can each be read independently of the others, and of [4]; however §5 (like [4]) relies on [10]. In this paper, by “definable” we mean 0-definable, i.e. invisible parameters are not allowed.

2. Finite sorts

We will consider first order theories $T$ in many-sorted languages $L$. $T$ need not be complete but everything we do can easily be reduced to the complete case. Terms like “surjective” applied to definable relations and functions mean: provably in $T$. Thus a definable surjection $D \rightarrow D'$ means a definable $R \subseteq D \times D'$, such that in any model $M \models T$, $R(M)$ is the graph of a surjection $D(M) \rightarrow D'(M)$.

A sort $S$ of $L$ is called finite if $S(M)$ is finite for all $M \models T$.

A family $\mathcal{F}$ of sorts is said to be closed under products if the product of two sorts in $\mathcal{F}$ is definably isomorphic to a third.

Definition 2.1. If $S$ is a sort of $L$ and $\{S_j\}_{j \in J}$ is a family sorts, we say $S$ is dominated by $\{S_j\}_{j \in J}$ if there exists a definable $D \subseteq S_{j_1} \times \ldots \times S_{j_k}$ and a definable surjective map $f : D \rightarrow S$, with $j_1, \ldots, j_k \in J$.

Equivalently, in any $M \models T$, $S(M) \subseteq \text{dcl}(\cup_{j \in J} S_j(M))$.

In this language, Conjecture 2.16 can be stated as follows.

Definition 2.2 ([4]). Let $T^{(d)}_\infty$ be the theory of Henselian fields with pseudo-finite residue field of characteristic 0, and (dense) value group elementarily equivalent to $\langle (\frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{N}, (b, d) = 1) \leq \mathbb{Q}, <, + \rangle$

Theorem 2.3. Every finite imaginary sort of $T^{(d)}_\infty$ is dominated by the set of sorts consisting of the residue field $k$ and the finite value group quotient sorts $\Gamma/n\Gamma$.

This will be proved at the end of §3.

Definition 2.4. Let $S, F$ be sorts of $T$ with $F$ finite. Define the imaginary sort $S \subseteq F$ to be the sort of partial functions $F \rightarrow S$.

Explanation: by definition, for some $N \in \mathbb{N}, T \models |F| \leq N$. Then $S \subseteq F = \cup_{n \leq N} S \subseteq F_n$ where $S \subseteq F_n$ is the sort of partial functions $F \rightarrow S$ with $n$ element domain. By identifying a function with its graph, an $n$-element set of pairs, and then identifying a set of pairs with a tuple of pairs up to $\text{Sym}(n)$, we see that $S \subseteq F_n$ embeds naturally into the imaginary sort $(F \times S)^n/\text{Sym}(n)$. Observe in particular that $S \subseteq F$ is dominated by $S, F$.

1cf. §1 of [2] for an account and further references. In these references, the field $F$ is PAC, but this condition is irrelevant here.

2It follows that in the Denef-Pas language they are definable over the sorts $\Gamma/n\Gamma$ and $k$. Asides from this remark, we use the intrinsic valued field structure in this paper, and do not split RV.
We view an element of \( S^{\subseteq \mathcal{F}} \) as a tuple of elements of \( S \), indexed by a finite subset of \( F \) in place of a finite subset of \( N \). We thus refer to the sorts \( S^{\subseteq \mathcal{F}} \) as \((\text{Aut}(\mathcal{F})\cdot)\) twisted Cartesian powers of \( S \).

**Lemma 2.5.** Let \( T \) be a theory. Assume given a family \( S \) of sorts, and a family \( \mathcal{F} \) of finite sorts, closed under products. Let \( T' \) be obtained from \( T \) by naming the elements of each \( S \in \mathcal{F} \). I.e. \( T' = \text{Th}((M, c_j)_{j \in J}) \) for some \( M \models T \) and enumeration \((c_j : j \in J) \) of \( \mathcal{F} = \cup_{S \in \mathcal{F}} S(M) \). If \( T' \) eliminates imaginaries to the sorts \( S \), then \( T \) eliminates imaginaries to the sorts \( \{S^{\subseteq \mathcal{F}} : S \in S, F \in \mathcal{F}\} \)

**Proof.** Observe that \( (S_1 \times S_2)^{\subseteq \mathcal{F}} \) embeds naturally into \( S_1^{\subseteq \mathcal{F}} \times S_2^{\subseteq \mathcal{F}} \). Thus we may assume \( S \) closed under Cartesian products.

Let \( M \models T \) and let \( e \) be an imaginary element of \( M \). We have to find \( h \in S^{\subseteq \mathcal{F}} \) for some \( S, F \) with \( \text{dcl}(e) = \text{dcl}(h) \).

Let \( M' \) be an expansion of \( M \) to a model of \( T' \). By assumption, in \( M', \text{dcl}_{M'}(e) = \text{dcl}_{M'}(g) \) for some tuple \( g \) from the sorts \( S \); so we may assume \( g \in S(M), S \in S \). It follows that there exists \( F \in \mathcal{F} \) and \( d \in F(M) \) such that \( \text{dcl}(e, d) = \text{dcl}(g, d) \) in the sense of \( M \). So \( g = H(e, d) \) and \( e = G(g, d) \) for some definable functions \( G, H \). We can restrict the domains of \( H, G \) to any given definable set containing \((e, d)\) (respectively \((g, d)\).) So we may assume that \( G(H(x, y), y) = x \).

Let \( H_e(y) = H(e, y) \). So \( H_e \) is a function with nonempty domain contained in \( F \), and range in \( S \). So \( H_e \) is coded by some \( e \)-definable element \( h \) of \( S^{\subseteq \mathcal{F}} \). On the other hand \( e \) is determined by \( H_e \), in fact \( e = G(H_e(y), y) \) for any \( y \in \text{dom}(H_e) \). So \( \text{dcl}(e) = \text{dcl}(h) \). \( \square \)

**Lemma 2.6.** Suppose \( \mathcal{F} \) is a collection of finite cyclic groups; and the relation: “there exists a definable surjective homomorphism \( A \rightarrow B \)” is a directed partial ordering on \( \mathcal{F} \), i.e any two groups in \( \mathcal{F} \) are definable morphomorphic images of a third. Then the condition of being closed under products in Lemma 2.5 can be dispensed with.

**Proof.** Let \( B_1, \ldots, B_n \in \mathcal{F} \). There exists \( B \in \mathcal{F} \) and definable surjective maps \( h_i : B \rightarrow B_i \). For any \( S \in \mathcal{S} \), this gives rise to a surjective \( h : B^\mathbb{N} \rightarrow \Pi_{i=1}^n B_i \) and hence a definable injection \( S^{B_1 \times \cdots \times B_n} \rightarrow S^{B^\mathbb{N}}, y \mapsto y \circ h \). Thus a twisted powers of a set \( S \) by a product of sorts of \( \mathcal{F} \) embeds into a twisted power \( X^{B^\mathbb{N}} \) by a power of single such \( B \).

Now we use the fact that \( B \) is cyclic, say of order \( d \). Then the map \( B \times [1, \ldots, d]^k \rightarrow B^k, (b, (a_1, \ldots, a_k)) \mapsto (b^{a_1}, \ldots, b^{a_k}) \), is a surjective map. Let \( N = d^k \) and fix a bijection \( \{1, \ldots, N\} \rightarrow [1, \ldots, d]^k \). Then as above \( S^{B^k} \) embeds definably into \( S^{B \times [1, \ldots, d]^k} = (S^{B})^N \), and similarly for partial functions. So the sorts \( S^B, B \in \mathcal{F} \) dominate the sorts \( S^{B_1 \times \cdots \times B_n}, B_1, \ldots, B_n \in \mathcal{F} \). \( \square \)

**Lemma 2.7.** Let \( T_1, T_2 \) be two theories, and let \( T = T_1 \times T_2 \) be their Feferman-Vaught product, so that a model of \( T \) is a product of a model of \( T_1 \) with one of \( T_2 \), and a relation is a Boolean combination of products of relations. Assume \( T_1, T_2 \) eliminate imaginaries and that every sort of \( T_2 \) is linearly ordered. Then \( T \) eliminates imaginaries.

**Proof.** Let \( M \models T \) and let \( R \) be an \( M \)-definable relation on \( M \). Say \( e \) is a canonical code for \( R \), possibly imaginary. Then \( M = M_1 \times M_2, M_i \models T_i, \) and \( R \) is a finite disjoint union of products of a definable set of \( M_1 \) with a definable set of \( M_2 \). Let \( B_2 \) be the Boolean algebra of definable subsets of \( M_2 \) generated by the sections \( R(a) = \{y : (a, y) \in R\} \). This algebra is finite; let \( \{R_i : i = 1, \ldots, k\} \) be an enumeration of the atoms (without repetitions.) Let \( e_2^i \) be a canonical parameter for \( R_i \). Then the set \( \{e_2^i\} \) is \( e \)-definable. Since we assumed each sort of \( T_2 \) is linearly ordered, each \( e_2^i \) is actually \( e \)-definable. Let \( R_1^0 = \{x \times R_2^0 \subseteq R\} \). Then \( R_1^0 \) is clearly \( e \)-definable, and \( R = \cup_i R_1^i \times R_2^0 \). Let \( e_1^i \) be a canonical code for \( R_1^i \). Then \( \text{dcl}(e) = \text{dcl}(e_1^i, e_2^i) \). So \( e \) is coded by a real element. \( \square \)
2.8. Elimination of finite imaginaries. Let $T$ be a theory in a many-sorted language $L$, with sorts $S_i$ ($i \in I$). Let $I^- \subseteq I$, and let $L^-$ be the language consisting of the sorts $S_i$ ($i \in I^-$) and the relations among them; let $T^- = T|L^-$. If $M \models T$, $M^-$ will denote the restriction of $M$ to the sorts $S_i$ ($i \in I^-$).

If $C \subseteq C' \subseteq M \models T$, we say $C'$ is stationary over $C$ if $\text{dcl}^q(C') \cap \text{acl}^q(C) = \text{dcl}^q(C)$, i.e. every imaginary element that is definable over $C'$ and algebraic over $C$ is definable over $C$. It is clear that if $C''$ is stationary over $C'$, and $C'$ over $C$, then so is $C''$ over $C$. A type $p = \text{tp}(c/C)$ is called stationary if $C \cup \{c\}$ is stationary over $C$.

**Lemma 2.9.** (1) A type $p$ over $C$ is stationary if and only if for any $C$-definable equivalence relation $E$ on a $C$-definable set $D$ with finitely many classes, if $p(x) \models D(x)$ then $p$ chooses one of the classes of $E$, i.e. $p(x) \cup p(y) \models xEy$.

(2) If there exists a $C$-definable type $\bar{p}$ over $M$ extending $p$, then $p$ is stationary.

**Proof.** (1) If $p = \text{tp}(c/C)$ is not stationary, then for some $C$-definable function $f$ (possibly with imaginary values), $f(c)$ is algebraic over $C$ but not definable. Define $xEy \iff f(x) = f(y)$. Then $E$ divides the solutions of $p$ to finitely many classes, but more than one. Conversely if a $C$-definable equivalence relation $E$ divides $p$ into finitely many classes (but more than one), then $c/E$ is an imaginary element in $\text{acl}^q(C) \setminus \text{dcl}^q(C)$.

(2) Let $E$ be a $C$-definable equivalence relation with finitely many classes. Since $\bar{p}$ is a complete type over $M$ there is a unique class $D$ of $E$ such that $(xEa) \in \bar{p}$ iff $a \in D$. By definability of the type, there exists a formula $\theta(y)$ over $C$ such that $(xEa) \in \bar{p}$ iff $\theta(a)$. Thus $D$ is $C$-definable, by $\theta$; and $p \models D(x)$. \hfill $\square$

**Lemma 2.10.** Assume

(1) $T$ admits elimination of $S_i$-quantifiers for each $i \in I \setminus I^-$. $T^-$ is stably embedded in $T$.

(2) Let $M \models T$ be a countable model. Then there exists $C$ containing $M^-$ and stationary over $M^-$, such that $\text{acl}(C) \prec M$.

(3) For $A \subseteq M \models T$, let $T_A = \text{Th}(M,a)_{a \in A}$. If $F$ is a finite $T_A$-definable set then there exists a finite $T_A$-definable set $F'$ of $M^-$ and a definable bijection $g : F \to F'$.

Then every finite imaginary sort of $T$ is definably isomorphic to one of $T^-$.

**Proof.** Quantifier-elimination will be used in the background, to avoid disagreement about the notion of a definable set between $T$ and $T^-$. Let $S = D/E$ be a finite imaginary sort, where $D$ is a definable set in $T$; let $\pi : D \to S$ be the canonical map. We may assume all elements of $S$ realize the same type. Let $M \models T$. View $M^-$ as a subset of $M$, and let $C$ be as in (2), and $N = \text{acl}(C)$. Then by (2), $N \prec M$. Since $S$ is finite, $S(N) = S(M)$, so each class of $E$ has a representative in $N$. Thus there exists $e \in C$ and a finite $e$-definable set $H_e \subseteq D$, meeting every $E$-class. Using (3), let $W$ be a finite $e$-definable subset of $M^-$ and $h_e : W \to H_e$ a definable bijection.

By stable embeddedness (1), $W$ is actually defined over some $e' \in M^-$. Write $W = H'_{e'}$. We have an induced $e'$-definable surjection $\psi_{e'} : H'_{e'} \to S$. But there are only finitely many maps $H'_{e'} \to S$, hence all are algebraic over $M^-$; by the stationarity assumption (2), since $\psi_{e'}$ is $C$-definable it is also $M^-$-definable. By enlarging $e'$ we may assume it is $e'$-definable.

Let $H'$ be a definable set of $T^-$, and $\psi : H' \to D$ a definable map, such that $H'_{e'} = \{x : (x,e') \in H', \psi_{e'}(x) = \psi(x,e')\}$. Then the composition $\pi \circ \psi : H' \to S$ is surjective. Let $E'$ be the kernel, i.e. define $E'(x,y) \iff \pi \circ \psi(x) = \pi \circ \psi(y)$. Then we have a definable bijection $H'/E' \to S$. By stable embeddedness again, $E'$ is $T^-$-definable and $H'/E'$ is an imaginary sort of $T^-$. \hfill $\square$
3. Finite sorts of $T^{(d)}$

Let $PF$ be the theory of pseudo-finite fields, $PF_0 = PF + \"characteristic 0\"$. Let $F^Gal_n$ be a finite imaginary sort whose elements code the elements of the Galois group of the unique field extension of order $n$. Let $F^Gal_n = \{ F_2, F_3, F_4, \ldots \}$. Let $PF'$ be the theory obtained from $PF$ by naming the elements of $F^Gal_n$, for each $n$. It was shown in [1] that $PF'$ eliminates imaginaries. Since $F^Gal_n$ surjects canonically onto $F^Gal_m$ when $m$ divides $n$, Lemma 2.6 applies. Hence, by Lemma 2.5 we have:

**Example 3.1.** $PF$ eliminates imaginaries to the level of $F^Gal_n$-twisted powers of the field sort.

The finite group $\mathbb{Z}/n\mathbb{Z}$ admits elimination of imaginaries to the level of subsets of $\mathbb{Z}/n\mathbb{Z}$. To see this, it suffices to note that any subgroup $H$ of the automorphism group $G$ of $\mathbb{Z}/n\mathbb{Z}$ has the form $H = \{ g \in G : gY = Y \}$ for some $Y \subseteq \mathbb{Z}/n\mathbb{Z}$. Indeed, we have $G = (\mathbb{Z}/n\mathbb{Z})^*$ so that $H \subseteq G \subset \mathbb{Z}/n\mathbb{Z}$, and we can let $Y = H$.

**Proof.** The theory $T(d)$ admits elimination of quantifiers, and it is easy to classify the definable subsets of $\Gamma$ and the definable functions $\Gamma \rightarrow \Gamma$ (in one variable) and see explicitly that they are coded. This suffices in general, cf. [6], and shows that $T(d)$ eliminates imaginaries.

It follows from Lemma 2.6 that $Th(\mathbb{Z}^{(d)})$ eliminates imaginaries to the level of twisted products $\Gamma^{\mathbb{Z}/n_1} \times \ldots \times \Gamma^{\mathbb{Z}/n_2}$. This reduces to $\Gamma^{(\Gamma/n)^k}$ and again, by Lemma 2.6 to the sorts $\Gamma/\Gamma$. Now a function $\Gamma/\Gamma \rightarrow \Gamma$ carries the same information as a subset of $\Gamma$ of size $\leq n$ (the image), together with a partial ordering of a subset $\Gamma/\Gamma$. As remarked above this reduces to subsets of $\Gamma/\Gamma$.

**Corollary 3.3.** Let $T$ be the model-theoretic disjoint sum of $PF_0$ and $Th(\mathbb{Z}^{(d)})$: $T$ has two sorts $k, \Gamma$, with relations $+, \cdot, 0, 1$ on $k$, $+, <, 0$ on $\Gamma$; such that $k \models PF_0$ and $\Gamma \models Th(\mathbb{Z}^{(d)})$. Then $T$ eliminates imaginaries to the sorts $\Gamma$ together with the sorts $S(n_1, n_2)$ for $n_1, n_2 \in \mathbb{N}$, where $S(n_1, n_2) = k^{F_1} \times F_2$, with $F_1 = F^Gal_{n_1}, F_2 = \Gamma/n_2$.

**Proof.** Let $S$ be the set of Cartesian products of these sorts.

**Claim** $S$ is closed under twisted powers by $\Gamma/\Gamma$.

**Proof.** Using the linear ordering, a function $\Gamma/\Gamma \rightarrow \Gamma$ can be coded by an $n$-tuple of elements of $\Gamma$ together with a partial ordering on $\Gamma/\Gamma$, namely the pullback of the linear ordering on $\Gamma$. In turn this partial ordering can be coded by a function from $\Gamma/\Gamma$ to a subset of the prime field of $k$. This shows that functions $\Gamma/\Gamma \rightarrow \Gamma$ are coded in $S$. On the other hand a function $\Gamma/\Gamma \rightarrow Y^{\Gamma/m}$ (with $Y = k^{F_1}$) can be viewed as a function $\Gamma/\Gamma \times \Gamma/m \rightarrow Y$, and handled using Lemma 2.6.

Thus by Lemma 2.6 it suffices to prove that the theory $T'$ obtained by naming the elements of each $\Gamma/\Gamma$ eliminates imaginaries to the sorts $\Gamma$ together with $k^{F^Gal_{n_1}}$. But this follows from Lemma 2.7 together with Example 3.1 and the first part of Example 3.2. □
3.4. Finite imaginaries in $T_{\infty}^{(d)}$.

We take the theory $T_{\infty}^{(d)}$ with the sorts VF, k, $\Gamma$, as well as the sorts $\Gamma/2, \Gamma/3, \ldots$ and all twisted product sorts $k^{\mathbb{C}_{\omega_1}} \times \Gamma \times \mathbb{Z}$. The sort of these last two kinds will be called $I_{\Gamma}$. The language includes the field structure on VF, a valuation map $VF \setminus \{0\} \to \Gamma$, predicates for the valuation ring $0$ and its maximal ideal $M$, the residue homomorphism $0/M \to k$, and finally a group isomorphism $VF^*/(1+M) \to (k^* \times \Gamma)$ splitting the canonical maps $k^* \to VF^*/(1+M) \to \Gamma$. The projection to $\Gamma$ is thus canonical, while the projection to $k^*$ is the Denef-Pas “angular component” map.

Let $I^- = \{\Gamma\} \cup I_{\Gamma}, I = I^- \cup \{VF\}$ where VF is the valued field sort. We will now show that the hypotheses (1-3) of Lemma 2.10 are valid for $T_{\infty}^{(d)}$.

Lemma 2.10 (1) follows from Denef-Pas elimination of quantifiers, [5]. Stable embeddedness is clear from the form of the quantifier-elimination; see [6] for an identical proof in the case of ACVF.

Lemma 3.5. Let $C \subseteq M \models T_{\infty}^{(d)}$. Assume $M^- \subseteq C$ and the maps val, ac restricted to C are onto the value group and residue field of M. Then acl($C$) $\langle M$.

Proof. Let $N = acl(C) \cap VF(M)$. Then $N$ is Henselian, so by the Ax-Kochen, Ershov principle for the Denef-Pas language applied to $T_{\infty}^{(d)}$, we see that $N, VF(M)$ are elementarily equivalent, and by model completeness, $N \subseteq VF_M$. It follows immediately from the surjectiveness that acl($C$) $\langle M$.

Now to prove Lemma 2.10 (2), let $M \models T = T_{\infty}^{(d)}$. Let RV = VF/(1+M), rv : VF $\to$ RV the quotient map. Let $\alpha - = \{\text{rv}(m) : m \in M\}$. From the point of view of $M$, rv($m$) is equidefinable with the pair (val($m$), ac($m$)), and so $\alpha -$ can be identified with val($M$) $\times$ $k^*(M) \subseteq acl(M^-)$. But $\alpha -$ can also be considered as a substructure of a model of ACVF, in the language considered in [7].

Let $C$ be a maximal subset of VF($M$) such that $C \sqsubseteq M^-$ is stationary over $M^-$. Let $A$ be the definable closure of $\alpha - \cup C$ in the sense of ACVF.

We have to show that acl($C \cup M^-$) $\langle M$, and by Lemma 3.5 it suffices to show that (val, ac)|$C$ is surjective. In other words given $m \in M$, to find $c \in C$ with val($c$) = val($m$), ac($c$) = ac($m$). Let $\beta = rv(m), B_\beta = rv^{-1}(\beta)$. We have to find $c \in B_\beta(C) := B_\beta \cap C$.

Claim 1. $B_\beta$ is not transitive in ACVF$_A$. In other words some proper sub-ball of $B_\beta$ is ACVF$_A$-definable.

Proof. Suppose for contradiction that $B_\beta$ is not transitive in ACVF$_A$. In this case, for any polynomial $F$, $rv\circ F$ is constant on $B_\beta$. ([7] Lemma 3.47.) So ac($F$) and val($F$) are constant. By Denef-Pas quantifier elimination, we see that $B_\beta$ is also transitive in $(T_{\infty}^{(d)}(\Gamma \cup C))$. (Compare [7] Lemma 12.1) Let $p$ be the unique type over $M^- \cup C$ of elements of $B_\beta$.

Moreover there exists a $\beta$-definable type $p_\beta(x)$ over $M$ concentrating on elements of $B_\beta$. Namely, first pick a val($m$)-definable type $r(t)$ of elements of $\Gamma$, concentrating on intervals (val($m$), val($m$) + $\epsilon$) $\subseteq \Gamma$. Over $M$, the type $r$ can be

$$r(t) = \{t \in \Gamma, t > \text{val}(m)\} \cup \{t < s : s \in \Gamma(M), s > \text{val}(m)\} \cup \{(3t')(nt' = t) : n = 1, 2, 3, \ldots\}$$

Then let $p_\beta(x) = \{x \in B_\beta\} \cup \\{\text{val}(x-u)\models r : u \in B_\beta(M)\} \cup \{\text{ac}(x-u) = 1 : u \in B_\beta(M)\}$

It is clear that this is a complete, consistent type over $M$ and is (ac($m$), val($m$))-definable.
By Lemma 2.10, \( p = p_B(M^- \cup \{C\}) \) is stationary. Choose any \( c \in B_\beta(M^- \cup \{C\}) \). Then \( \text{tp}(c/M^- \cup \{C\}) = p \). So \( C \cup \{c\} \cup M^- \) is stationary over \( M^- \), contradicting the maximality of \( C \). \( \Box \)

Hence \( B_\beta \) contains a proper ACVF-definable closed ball. In this case by [7] Lemma 3.39 \( B_\beta \) contains an ACVF-definable point \( d \). So \( d \in \text{ddl}(M^- \cup \{C\}) \) and hence \( d \in C \).

This finishes the proof of (2).

(3) We may take \( F \subset V F^n \). \( T_\infty^{(d)} \) is algebraically bounded in the sense of [12], so \( F \) is contained in a finite ACVF-definable quantifier-free definable set \( F' \). This reduces us to the same lemma for ACVF; for a proof, see for example [7] Lemma 3.9.

**Corollary 3.6.** Every finite definable imaginary set of \( T_\infty^{(d)} \) can be definably embedded into some power of the twisted product sorts \( k_{|n \Gamma|/n^2} \).

**Proof.** Lemma 2.10 reduces this to imaginary sorts of \( (T_\infty^{(d)})^- \); so by Corollary 3.3 it suffices to show this for a finite definable \( D \subset \Gamma^m \times P \), where \( P \) is a power of twisted product sorts. We can use induction on the cardinality, so we may assume \( D \) is not the union of two proper definable subsets. Since \( \Gamma \) is linearly ordered, it follows that the projection \( D \to \Gamma^m \) has a one-point image. Thus \( D \) projects injectively to a product of twisted product sorts. \( \Box \)

By the remark below Definition 2.4, the twisted product sorts \( k_{|n \Gamma|/n^2} \) are dominated by the sorts \( k, \Gamma/2, \Gamma/3, \ldots \), and \( F_{\text{Gal}}^\Gamma \). These Galois sorts are dominated by \( k \); hence every finite definable imaginary set of \( T_\infty^{(d)} \) is dominated by the tame sorts \( k \) and \( \Gamma/2, \Gamma/3, \ldots \).

This proves Theorem 2.3.

4. **Groupoids**

A groupoid is a category \( \Gamma = (\text{Ob}_\Gamma, \text{Mor}_\Gamma) \) in which every morphism is invertible. We will consider definable groupoids with a single isomorphism type. See [8], though the use of definable groupoids there is different. A morphism between groupoids is a quantifier-free definable functor.

Given a groupoid \( \Gamma \) defined without quantifiers in a theory \( T \), one obtains an equivalence relation \( E_\Gamma \), defined uniformly over \( T = Th(L) \) for any definably closed subset \( L \) of a model of \( T \). Namely the equivalence relation of \( \Gamma \)-isomorphism on the objects of \( \Gamma \): for \( a, b \in \text{Ob}_\Gamma(L), aE_\Gamma b \) iff \( \text{Mor}(a,b)(L) \neq \emptyset \). Let \( \text{Mor}(\Gamma; L) \) be the quotient \( \text{Ob}_\Gamma(L)/E_\Gamma, \) i.e. the set of isomorphism classes of \( \Gamma(L) \).

Let \( HF_0 \) be the theory of Henselian fields with residue field of characteristic 0. This can be viewed as the theory of definably closed substructures of models of ACVF \( \mathbb{Q} \) (with trivially valued \( \mathbb{Q} \)). Fix \( L \models HF_0 \). We will use only quantifier-free formulas, and notions such as dcl will refer to \( ACVF_L \).

We wish to reduce a given quantifier-free definable groupoid \( \Gamma \) over ACVF to a groupoid \( \Gamma' \) over RV, in a way that yields a reduction of the imaginaries \( \text{Iso}(\Gamma; L) \) to imaginaries \( \text{Iso}(\Gamma'; L) \), uniformly over Henselian valued fields \( L \) with various theories.

Note that a morphism \( f : \Gamma \to \Gamma' \) yields, for any \( L \models HF_0 \), a map \( f_* : \text{Iso}(\Gamma; L) \to \text{Iso}(\Gamma'; L) \). We say that \( f \) is an elementary reduction of \( \Gamma \) to \( \Gamma' \) (respectively, of \( \Gamma' \) to \( \Gamma \)) if \( f_* \) is injective (resp., surjective.)  A reduction is a finite sequence of elementary reductions.

Let \( G \) be a definable group acting on a definable set \( V \). Define a groupoid \( \Gamma = \Gamma(G, V) \) whose objects are the points of \( V \). The morphisms \( v \to v' \) are defined to be: \( \text{Mor}_\Gamma(v, v') = \).

\(^3\) This connection between groupoids and imaginaries is different from the one considered in [8]. The approach in this section is apparently in the spirit of stacks.
\( \{ g \in G : gv = v' \} \). Composition is multiplication in \( G \). Then \( \text{Iso}(\Gamma, L) \) is precisely the orbit space \( V(L)/G(L) \).

Let \( \Gamma \) be a groupoid, with one isomorphism class. Then all isomorphism groups \( G_a := \text{Mor}(a, a) \) are isomorphic to each other, non-canonically: given \( a, b \in \text{Ob} \Gamma \), choose \( f \in \text{Mor}(a, b) \); then \( g \mapsto f \circ g \circ f^{-1} \) is an isomorphism \( G_a \to G_b \). The isomorphism \( G_a \to G_b \) is defined up to conjugation; if the \( G_a \) are Abelian, this isomorphism is canonical, so all \( G_a \) are canonically isomorphic to a fixed group \( H \). (This is not essential to the discussion that follows, but simplifies it.) Let \( N \) be a normal subgroup of \( H \). We define a quotient groupoid \( \Gamma/N \). It has the same objects as \( \Gamma \), but the morphism set is \( \text{Mor}_{\Gamma/N}(a, b) = \text{Mor}_{\Gamma}(a, b)/N \). There is a natural morphism \( \Gamma \to \Gamma/N \).

Our reductions will use a sequence of canonical normal subgroups of a torus \( T \). First let \( T = G_{a \Gamma} \), a split torus. The valuation map induces a homomorphism \( T \to \Gamma^n \), with kernel \((\mathbb{O}^*)^\nu \) (where \( \mathbb{O} \) is the valuation ring.) Next, we have a reduction map \((\mathbb{O}^*)^\nu \to (\mathbb{k}^*)^\nu \), with kernel \((1 + M)^\nu \). Now if \( T \) is any torus defined over a valued field \( F \), by definition there exists a finite Galois extension \( L \) of \( F \), and an isomorphism \( f : T \to G_{a \Gamma}^\nu \) defined over \( L \). It is easy to see that \( N := f^{-1}(\mathbb{O}^*)^\nu \) and \( N^{-1} := f^{-1}(1 + M)^\nu \) do not depend on \( f \); so these are quantifier-free definable subgroups of \( T \). The quotient \( T/N \) is internal to \( \Gamma \), while \( N/N^{-1} \) is internal to the residue field (and \( N \) is generically metastable.) Note that \( N^{-1} \) is a uniquely divisible Abelian group.

We will assume \( L \) is not trivially valued, so that \( L^\text{alg} \models \text{ACVF} \). In particular all definable torsors have \( L^\text{alg} \)-definable points. The proof in the trivially valued case is left to the reader. \(^4\)

**Theorem 4.1.** Let \( G \) be a linear algebraic group, \( T \leq G \) a torus. Then \( \Gamma(G, G/T) \) reduces to a groupoid defined over \( \text{RV} \).

**Proof.** Let \( V = G/T, \Gamma = \Gamma(G, V) \). Consider first \( \Gamma/N \). Each automorphism group \( \text{Mor}(a, a) \) is a uniquely divisible Abelian group (isomorphic, over additional parameters, to \( \Gamma^n \).) Hence given a finite subset of \( \text{Mor}(a, b) \), it is possible to take the average, obtaining a unique point. In this way we can find for each \( a \in V \) a definable point \( c(a) \in \text{Mor}(1, a) \) (where \( 1 \) is the image in \( G/T \) of \( 1 \in G \).) Given \( a, b \in V \) let \( c(a, b) = c(b)c(a)^{-1} \in \text{Mor}(a, b) \). Then we have a subgroupoid of \( \Gamma/N \) with the same objects, and whose only morphisms are the \( c(a, b) \).

Let \( \Gamma_1 \) have the same objects as \( \Gamma \), and \( \text{Mor}_{\Gamma_1}(a, b) = \{ f \in \text{Mor}_{\Gamma}(a, b) : fN = c(a, b) \} \). The inclusion morphism \( \Gamma_1 \to \Gamma \) induces surjective maps \( \text{Iso}_{\Gamma_1}(L) \to \text{Iso}_{\Gamma}(L) \) for any \( L \), being surjective on objects. Thus \( \Gamma \) reduces to \( \Gamma_1 \).

Let \( \Gamma_2 = \Gamma_1/N^{-1} \). The automorphism groups of \( \Gamma_2 \) are isomorphic to \( t : N/N^{-1} \), a torus over \( \text{RV} \) (i.e. a group isomorphic, with parameters, to \( (\mathbb{k}^*)^\nu \).)

**Claim 1** Let \( L \models \text{HF}_0 \). Let \( Y \) be principal homogeneous space for \( N \) (defined in \( \text{ACVF}_L \). If \( Y/N^{-1} \) has a point in \( \text{dcl}(L) \), then so does \( Y \).

**Proof.** Let \( Y^- \) be an \( L \)-definable point of \( Y/N^{-1} \), i.e. an \( L \)-definable \( N^{-1} \)-subtorus of \( Y \). As above, since \( N^{-1} \) is uniquely divisible, \( Y^- \) has an \( L \)-definable point. \( \square \)

Applying this to \( \text{Mor}_{\Gamma_1}(a, b) \), we see that if \( a, b \) are \( \Gamma_2 \)-isomorphic then they are \( \Gamma_1 \)-isomorphic; so the natural morphism \( \Gamma_1 \to \Gamma_2 \) is injective on isomorphism classes over any \( L \models \text{HF}_0 \). Thus \( \Gamma_1 \) reduces to \( \Gamma_2 \).

Finally we reduce the objects. We have \( \text{Mor}_{\Gamma_2}(a, b) \subseteq \text{RV} \). By stable embeddedness of \( \text{RV} \) there exists a definable map \( j : \text{Ob}_{\Gamma_2} \to Y \subseteq \text{RV} \) such that \( \text{Mor}(1, a) \) is \( j(a) \)-definable. It

\(^4\) If \( F \) is trivially valued, then any element of \( \Gamma \) defined over \( F \) equals zero; the only definable subgroups of the additive group \( G_a \) are thus \( 0, \mathbb{O} \) and \( G_a \); the only definable \( \mathbb{O} \)-subtorus of \( G_a \) is \( \mathbb{O} \) (consider valuation of elements).
5.1. Definition of Galois sorts.

Let $F$ within $T$ in models have the form $(F, K, F, F)$, and the same morphism sets as $\Gamma_2$. The natural morphism $\Gamma_2 \to \Gamma_3$ (on objects, identity on morphisms) is bijective on $Iso$, since $\text{Mor}_{\Gamma_3}(j(a), j(b)) = \text{Mor}_{\Gamma_2}(a, b)$. Thus $\Gamma_2$ reduces to $\Gamma_3$. But $\Gamma_3$ is over $RV$.

Since we used only the quantifier elimination of ACVF, rather than HF, this method of investigation is not blocked in positive characteristic. Theorem 4.1 should go through for tori that split in a tamely ramified extension, replacing the unique divisibility argument for the additive groups by Hilbert 90 for the residue field, and an appropriate extension to $RV$. The right statement in the general case may give a lead with respect to motivic integration in positive characteristic.

5. Galois sorts

Let $F$ be any field, and consider the 2-sorted structure $(F, F_{\text{alg}}, +, \cdot)$. Working in the structure $(F, F_{\text{alg}})$ is convenient but harmless, since no new structure is induced on $F$. Let $T_0$ be a theory of fields (possibly with additional structure.) Let $T$ be the theory whose models have the form $(F, K, +, \cdot, \cdots)$, with $K$ an algebraically closed field and $F$ a distinguished subfield (possibly with additional structure) such that $F \models T$. We can restrict attention to $K = F_{\text{alg}}$ since $(F, F_{\text{alg}}) \prec (F, K)$. In this section, $T$ is fixed, and “definable” means: definable in $T$, imaginary sorts included. Let $F_0 = \text{dcl}(0)_T$.

5.1. Definition of Galois sorts. Let $E_n$ be the set of Galois extensions of $F$ of degree $n$, within $F_{\text{alg}}$, this is clearly a definable set of imaginaries of $(F, F_{\text{alg}})$. For $e \in E_n$ coding an extension $F_e$ of $F$, let $G_e$ be the Galois group $\text{Aut}(F_e/F)$. Let $\mathcal{S}_n$ be the disjoint union of the $G_e$; it comes with a map $\mathcal{S}_n \to E_n$. Let $\mathcal{S} = (\mathcal{S}_n : n \in \mathbb{N})$. See the Appendix for a definition at a greater level of generality, including some definitions for Galois cohomology.

5.2. A finiteness statement for $H^1$. Let $A$ an algebraic group defined over $F_0$, not necessarily commutative. We are interested in the first Galois cohomology set $H^1(F, A) = H^1(\text{Aut}(F_{\text{alg}}/F, A(F_{\text{alg}})))$, where $F \models T$.

To say that an object such as $H^1(F, A)$ is definable means that there exists a definable set $H$ of $T_{eq}$ and for any $F' \models T$, a canonical bijection $H(F') \to H^1(F', A)$. By standard methods of saturated models, such a definable set $H$, if it exists, is unique up to a definable bijection. Given a property $P$ of definable sets (invariant under definable bijections), we say that $H^1(F, A)$ has $P$ if $H$ has $P$.

Theorem 5.3. Let $F$ be a perfect field. If $A$ is a linear group, then $H^1(F, A)$ is definable, and $\mathcal{S}$-analyzable.

This will be proved as Proposition 5.10 below.

In case the Galois group of $L$ property $F$ in the sense of [10], (or bounded in the sense of [9]), Theorem 5.3 says simply that $H^1(F, A)$ (resp. the kernel $H^1(F, A) \to H^1(F, G)$) is finite. This is Theorem 5 of [10], Chapter 3, §4.

Question 5.4. Is $V(K)/G(K)$ in fact internal to $\mathcal{S}$?

Presumably it is not the case, in general, that $V(K)/G(K) \subseteq \text{dcl}(\mathcal{S})$, even over $\text{acl}(0)$; It would be good to have an example.

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\footnote{as one easily sees by an automorphism argument.}
If $L$ is a Galois extension of $F$, let $Z(L; A)$ be the set of maps $a : Aut(L/F) \to A(L)$ satisfying $a(st) = a(s)a(t)$. Two elements $a, a'$ are cohomologous if $a'(s) = b^{-1}a(s)b$ for some $b \in A(L)$. (Note $a'(1) = 1 = b^{-1}s(b)$ so $b \in Fix(s)$ whenever $a(s) = a'(s) = 1$.) The quotient of $Z(L, A)$ by this equivalence relation is denoted $H^1(L/F, A)$. If $L \leq L'$ we have a natural map $Z(L, A) \to Z(L', A)$, obtained by composing with the canonical quotient map $Aut(L'/F) \to Aut(L/F)$. This induces a map $H^1(L/F, A) \to H^1(L'/F, A)$, which is injective.

Let $Z(n; A)$ be the disjoint union of the sets $Z(L; A)$ over all Galois extensions $L$ of $F$ with $[L : F] | n$. Define an equivalence relation $E$ on $Z(n; A)$: if $a_i \in Z(L_i; A)$ for $i = 1, 2$, write $a_1 \sim a_2$ if $a_1, a_2$ are cohomologous in $L_1L_2$; equivalently, for some Galois extension $L$ of $F$ containing $L_1, L_2$, there exists $b \in A(L)$ such that for $s \in Aut(L/F)$, $a_1(s|L_1) = b^{-1}a_2(s|L_2)s(b)$. The second formulation shows that $\sim$ is an equivalence relation; the first shows that $\sim$ is definable. Definability of $Z(n; A)$ is clear. Denote the quotient $Z(n; A)/\sim = H(n; A)$. 

If $n|n'$ we obtain an injective map $H(n; A) \to H(n'; A)$. It is clearly definable. For any $F \models T$, $H(n; A)(F)$ is the set of elements of $H^1(F, A)$ in the image of $H^1(L/F, A)$ for some Galois extension $L$ of degree $n$.

More generally, if $\{A_n\}$ is a definable family of algebraic groups, for $b$ from $F$ the set $H(n; A_b)(F)$ of elements of $H^1(F, A_b)$ in the image of $H^1(L/F, A)$ for some Galois extension $L$ of degree $n$ is definable uniformly in the parameter $b$.

**Proposition 5.5.** Let $A$ be a linear algebraic group. Then $H^1(F, A)$ is definable; for some $n$, $H^1(F, A) = H(n; A)(F)$.

**Proof.** We have $A \leq GL_n$. By [11] Chapter X, Prop. 3, $H^1(F, GL_n) = 1$. The kernel of $H^1(F, A) \to H^1(F, GL_n)$ is canonically isomorphic to $GL_n(F)/A(F)$. Since $GL_n/F$ is definable, the kernel is as well, by standard compactness arguments, $\lim_n H(n; A) = \bigcup_n H((n + 1); A) \setminus H(n!; A)$ must also be definable, i.e. for large enough $n$ the set $H((n + 1)!; A) \setminus H(n!; A)$ must be empty.

**Corollary 5.6.** Let $A$ be a finite (linear) algebraic group. Then $H^1(F, A)$ is contained in $dcl(A, S)$.

**Proof.** A function from a finite set into the finite set $A$ is definable over the elements of $A$ and the elements of the domain. Thus $Z(n; A) \subseteq dcl(S_n, A)$.

We give a second proof, similar to the proof of (a) implies (b) in [10], 4.1, Proposition 8. This second proof goes through in a more general context, see the Appendix.

**Lemma 5.7.** Let $A$ be a finite definable group, of order $n$. Then $H^1(F, A) = H(n!^n; A)(F)$.

**Proof.** Let $L_0$ be the Galois extension of $F$ generated by the $n$ points of $A$. Since by a trivial estimate $|Aut(A)| \leq (n - 1)!$, we have $[L_0 : F] \leq (n - 1)!$. Let $L$ be any Galois extension of $F$, containing $L_0$, $G = Aut(L/F)$, and let $a \in Z(L; A)$. We have to show that the class of $a$ in $H^1(L/F, A)$ is in the image of some $H^1(L'/F, A)$ with $[L' : F] \leq (n!)^n$. In fact we will prove this even at the level of cocycles. The restriction of $a$ to $G_0 = Aut(L/L_0)$ is a homomorphism $a|G_0 : G_0 \to A$. Let $G_1$ be the kernel of $a|G_0$. Then $[G_0 : G_1] \leq n$, so $[G : G_1] \leq n!$. The number of conjugates of $G_1$ in $G$ is thus $\leq n!$; their intersection $G_2$ has index $\leq (n!)^n$. Let $L'$ be the fixed field of $G_2$. Then $a$ factors through a function $a'$ on $G/G_2$, so $a' \in Z(L'/F, A)$, as required.

**Corollary 5.8.** Let $A$ be a torus, i.e. $A$ becomes isomorphic to $G_n^m$ after base change to some Galois extension $F'$. Say $[F' : F] = m$, and let $B = \{a : A : a^m = 1\}$. Then $H^1(F, A) \subseteq dcl(B, S)$. If $F$ is perfect, then this holds for any connected solvable $A$ (for an appropriate finite group $B$).
Proof. The proof in [10], Chapter III, Theorem 4, goes through, showing that $H^1(F, A) \cong \hat{H}^1(F, B)$. \hfill \Box

Remark 5.9. We also have $H^1(F, A) \subseteq \dcl(F)$, since the Galois group $\text{Aut}(F^{\text{alg}}/F)$ acts trivially on $H^1(F, A)$.

Proposition 5.10. Assume $F$ is perfect. Let $A$ be a linear algebraic group. Then $H^1(F, A)$ is $\mathcal{S}$-analyzable.

Proof. The proof of [10], chapter III, Theorem 4 goes through. \hfill \Box

Remark 5.11. The proof of Proposition 5.2 shows that for any embedding of algebraic groups $A \to B$, the kernel of $H^1(F, A) \to H^1(F, B)$ is definable. It can be shown as in [10], chapter III, Theorem 5 that over a perfect field, this kernel is $\mathcal{S}$-analyzable.

Corollary 5.12. Let $L$ be a perfect field. Let $G$ be a linear algebraic group over $L$, and let $V$ be a homogeneous space for $V$ defined over $L$, i.e. $G$ acts on the variety $V$ and $G(L^{\text{alg}})$ acts transitively on $V(L^{\text{alg}})$. Then in the structure $(L, L^{\text{alg}}, +, \cdot)$, $V(L)/G(L)$ is $\mathcal{S}_L$-analyzable.

Proof. Immediate from Proposition 5.10 since after picking any point $c \in V(L)$ and letting $H = \{g \in G : gc = c\}$, we have a $c$-definable injective map $V(L)/G(L) \to H^1(F, H)$. \hfill \Box

5.13. Henselian fields. We now move to valued fields of residue characteristic zero.

Lemma 5.14. Let $K$ be a Henselian field with residue field of characteristic $0$. Let $k$ denote the residue field, $\mu = \cup_n \mu_n$, the roots of unity in $k^{\text{alg}}$, $\Gamma$ be value group. Then in $(K, K^{\text{alg}})$ we have $\mathcal{S}_K \subseteq \dcl(S_k, \mu, \Gamma/n\Gamma)$

Proof. Let $L$ be a finite Galois extension of $K$. We have to show that $\text{Aut}(L/K) \subseteq \dcl(S_k, \mu, \Gamma/n\Gamma)$ for some $n$.

We will use some standard valuation theory. Call $K'$ a ramified root extension of $K$ if it is a finite purely ramified extension obtained by adding roots to some elements of $K$.

Claim 1 There exists a finite unramified extension $L'$ of $L$, an unramified Galois extension $K_u \leq L'$ of $K$, and a ramified root extension $K_{rr} \leq L'$ of $K$, such that $L'/K_{rr}$ is unramified, and $\text{Aut}(L'/K) = \text{Aut}(L'/K_u)\text{Aut}(L'/K_{rr})$.

Proof. The finite group $\Gamma(L)/\Gamma(K)$ is a direct sum of finite cyclic groups, $\oplus_{i=1}^k \mathbb{Z}/n_i\mathbb{Z}$: let $c_1, \ldots, c_k \in L$ be such that $\text{val}(c_1) + \Gamma(K), \ldots, \text{val}(c_k) + \Gamma(K)$ are generators for these cyclic groups. So $\text{val}(c_i^{n_i}) = \text{val}(d_i)$ for some $d_i \in K$. Let $e_i = c_i^{n_i}/d_i$, so that $\text{val}(e_i) = 0$. In a Henselian field of residue characteristic prime to $n$, if an element $f$ has $\text{val}(f) = 0$, and $\text{res}(f)$ has an $n$’th root, then by Hensel’s lemma so does $f$. Hence in some unramified Galois extension $L'$ of $L$, each $e_i$ has an $n_i$’th root. Clearly $d_i$ has an $n_i$’th root $r_i \in L'$. Let $K_{rr} = K(r_1, \ldots, r_k)$. Let $K_u$ be the maximal unramified subextension of $L'$. Then $L'$ has the same residue field over $K_u$; $L'/K_u$ is purely ramified. Since $\text{val}(L') = \text{val}(L) = \text{val}(K_{rr})$, $L'/K_{rr}$ is unramified. In particular $K_u \cap K_{rr} = K$, so $\text{Aut}(L'/K) = \text{Aut}(L'/K_u)\text{Aut}(L'/K_{rr})$ by Galois theory. \hfill \Box

Claim 2 $\text{Aut}(L'/K_{rr}) \subseteq \dcl(S_k)$.

Proof. The canonical homomorphism $\text{Aut}(L'/K_{rr}) \to \text{Aut}(k_{L'}/k)$ is an isomorphism, since $L'/K_{rr}$ is unramified. This homomorphism is definable, hence embeds the elements of $\text{Aut}(L'/K_{rr})$ into $\mathcal{S}_k$. \hfill \Box

Claim 3 $\text{Aut}(L'/K_u) \cong \text{Hom}(\Gamma(L)/\Gamma(K), \mu_n)$ (canonically and definably.)
Proof. Let \( E = \{ e \in L^* : e^n \in K^* \} \). Define a map \( b : \text{Aut}(L'/K_u) \times E \to \mu_n \) by \( b(\sigma, e) = \sigma(e)/e \). Clearly \( b \) is multiplicative in the second variable. If \( e \in E \) and \( \text{val}(e) = 0 \), then \( \text{res}(e^n) \) has an \( n \)'th root in \( \text{res}(L') \) and hence in \( \text{res}(K_u) \), since \( L'/K_u \) is purely ramified. So \( e^n \) has an \( n \)'th root in \( K_u \). Since all roots of unity in \( L' \) lie in \( K_u \), we have \( e \in K_u \); hence \( b(\sigma, e) = 1 \) for all \( \sigma \).

More generally if \( e \in E \) and \( \text{val}(e) \in \Gamma(K) \), then \( \text{val}(e/c) = 0 \) for some \( c \in K^* \), so \( b(\sigma, e/c) = 0 \) and hence \( b(\sigma, e) = 0 \) for all \( \sigma \). Thus \( b \) factors through \( b' : \text{Aut}(L'/K_u) \times \Gamma(L')/\Gamma(K) \to \mu_n \). We obtain a homomorphism \( \text{Aut}(L'/K_u) \to \text{Hom}(\Gamma(L')/\Gamma(K), \mu_n) \). It is injective since if \( b(\sigma, r_i) = 1 \) for each \( i \), then \( \sigma(r_i) = r_i \) so \( \sigma = 1 \). Surjectivity comes from cardinality considerations (but will not be needed).

\[ \Box \]

Claim 4 \( \text{Aut}(L'/K_u) \subseteq \text{dcl}(\mu_n, \Gamma/n\Gamma) \) where \( n = [L' : K_u] \).

Proof. By Claim 3, \( \text{Aut}(L'/K_u) \) is definably isomorphic to \( \text{Hom}(\Gamma(L)/\Gamma(K), \mu_n) \). Now \( \Gamma(L)/\Gamma(K) \) is a finite subgroup of \( (1/n)\Gamma(K)/\Gamma(K) \), hence isomorphic to a finite subgroup \( S \subseteq \Gamma(K)/n\Gamma(K) \). Each element of \( S \) lies in \( \Gamma/n\Gamma \), and so a homomorphism \( S \to \mu_n \) is definable over teh elements of \( S \) and the elements of \( \mu_n \), and the claim is proved.

The lemma follows from Claims 1, 2 and 4.

\[ \Box \]

Theorem 5.15. Let \( K \) be a Henselian field with residue field of characteristic 0. Let \( G \) be an algebraic group over \( K \), acting on a variety \( V \); assume \( G(K^{alg}) \) acts transitively on \( V(K^{alg}) \). Let \( V(K)/G(K) \) be the orbit space. Then for some \( n \), \( V(K)/G(K) \) is analyzable over sorts \( \Gamma/n\Gamma \) and the \( n \)'th Galois sort \( S_n(k) \) of the residue field.

Proof of Theorem 5.15. Given \( c \in V(K), H = \{ g : gc = c \} \), there is a canonical \( c \)-definable bijection between \( V(K)/G(K) \) and the kernel of \( H^1(K,H) \to H^1(K,G) \). Hence the theorem follows from Theorem 5.13 and Lemma 5.14.

\[ \Box \]

Remark 5.16. If we add to the structure the Denef-Pas splitting, (at least when e.g. \( \Gamma \) is a \( \mathbb{Z} \)-group, but probably in general), it follows that \( V(K)/G(K) \) is tame, i.e. \( V(K)/G(K) \subseteq \text{dcl}(k, \Gamma/n\Gamma) \). Indeed for every imaginary definable set \( D_0 \) of \( \Gamma \), either \( D_0 \subseteq \text{dcl}(\Gamma/n\Gamma) \) for some \( n \), or else \( D \subseteq \text{dcl}(D_0) \) for some infinite definable \( D \subseteq \Gamma \). But it is easy to see that \( D \) is not internal over \( k \cup \Gamma/m\Gamma \). Hence the definable sets internal over \( k \cup \Gamma/m\Gamma \) are contained in \( \text{dcl}(k \cup \Gamma/n\Gamma) \) for some \( n \), and by induction the same goes for analyzability.

6. Appendix

It may seem at first sight that Galois sorts are peculiar to fields; but in fact they can be defined for any theory eliminating imaginaries on its finite subsets, see below.

In particular, the Galois sorts of RV will be defined. This will permit the remark that, for a theory \( T \) of Henselian fields of residue characteristic 0, if \( T_{RV} \) is the induced theory on RV, we have \( S_T \cong S_{T_{RV}} \); which, along with Theorem 5.3 clarifies the reductions of [11] and the present paper.

Fix a language \( L \). Let \( \mathcal{I} \) be a theory admitting elimination of imaginaries with respect to finite sets of tuples. Thus for any finite product \( S \) of sorts of \( L \), and any \( m \in \mathbb{N} \), we have given a sort \( S[\leq m] \) and a function \( c_{S,m} : S^m \to S[\leq m] \), such that

\[ \mathcal{I} \models (c_{S,m}(x) = c_{S,m}(y) \iff \bigwedge_{\sigma \in \text{Sym}(m)} x^\sigma = y) \]

Let \( S[m] \) be the image of the distinct \( m \)-tuples.

For simplicity we assume also that \( \mathcal{I} \) eliminates quantifiers, and that any quantifier-free definable function of \( \mathcal{I} \) is given by a term (piecewise), so that substructures are definably
closed. Let $T_0$ be the theory of substructures of models of $T$. If $M \models T_0$, let $M^{alg}$ denote the algebraic closure of $M$ within some model of $T$. The Galois imaginaries, strictly speaking, belong to $T_0 = Th\{\{M, M^{alg} : M \models T_0\}\}$. We will also note some related imaginary sorts of $T_0$ itself. In practice it seems more convenient to use $T_0^g$, then note by considerations of stable embeddedness that definable sets on which the Galois group acts trivially belong to $T_0$. For instance this is the case with the cohomology sets $H^i(Aut(M^{alg}/M), A)$ considered below.

Below we abuse notation, identifying elements of $S[m]$ with $m$-element sets. Note however that if $M \models T$, then $S[m](M)$ corresponds to $m$-element subsets of $M^{alg}$, not of $M$.

We define the Galois sorts $\mathcal{S}$ of $T_0$. These are certain definable sorts of imaginaries (not quantifier-free.)

We have a relation $(\subseteq) \subset S[\leq m]^2$, corresponding to inclusion of finite sets. Let $S_{irr}[m]$ be the set of irreducible elements of $S[m]$:

$$x \in S_{irr}[m] \iff (x \in S[m] \land \bigwedge_{1 \leq k < m} \neg \exists y \in S[k](y \subseteq x))$$

Thus if $M \models T_0$, then $S_{irr}[m](M)$ corresponds to the set of orbits of $Aut(M^{alg}/M)$ on $M^{alg}$ of size $m$. If $k < m$, for $x \in S_{irr}[k]$, $y \in S_{irr}[m]$ we let $Mor(x, y)$ be the set of codes of functions $y \to x$ (viewed as subsets of $y \times x$.) This makes $\sqcup_{S,m} S_{irr}[m]$ into a category $\mathcal{E}$.

If $a \in S(M^{alg})$ has $m$ conjugates under $Aut(M^{alg}/M)$, let $s(a)$ be the (code for) the set of conjugates. Then $s(a) \in S_{irr}[m](M)$, and every element of $S_{irr}[m](M)$ arises in this way. A $\mathcal{E}$-morphism $s(b) \to s(a)$ exists iff $M(a) \leq M(b)$. In particular, $s(a), s(b)$ are $\mathcal{E}$-isomorphic iff $M(a) = M(b)$, i.e. $a, b$ generate the same substructure of $M^{alg}$ over $M$. Isomorphism classes of $\mathcal{E}$-objects correspond to finitely generated extensions of $M$ within $M^{alg}$.

If $M \models T_0$ and $a \in S_{irr}[k](M), b \in S_{irr}[m](M)$, then $Mor(a,b)(M)$ is a finite set, possibly empty. By irreducibility, any function coded in $Mor(a,b)(M)$ must be surjective. In particular if $k = m$ it must be bijective, so the subcategory with objects $S_{irr}[m](M)$ is a groupoid (every morphism is invertible.) For $s \in S_{irr}[m]$, let $H_s = Mor(s, s)$. Let $S_{gal}[m]$ be set of $s \in S_{irr}[m]$ such that $H_s$ acts regularly on $s$. Let $E_{S,m}$ be the set of $\mathcal{E}$-isomorphism classes of objects in $S_{gal}[m]$. If $s \in S_{gal}[m]$ codes a set $D_s$, let $Gal(s) = Aut(D_s; H_s)$ be the opposite group to $H_s$, i.e. the group of permutations of $D_s$ commuting with each element of $H_s$.

Let $E_s = Gal(s)^m/\sim$ be the set of $Gal(s)$-conjugacy classes on $Gal(s)^m$. (More canonically, we could look at $E^k_s = Gal(s)^k/\sim$ for all $k \in \mathbb{N}$, but all $E^k_s$ are definable from $E^m_s = E_s$.)

The definition of $Gal(s)$ requires knowledge of $D_s$, i.e. of a particular choice of an algebraic closure $M^{alg}$ of $M$. But $E_s$ is canonical and does not depend on this choice.

If $s, s'$ are isomorphic, i.e. $Mor(s, s') \neq \emptyset$, the choice of $f \in Mor(s, s')$ yields an isomorphism $Gal(s) \to Gal(s')$ (namely $\tau \mapsto f \circ \tau \circ f^{-1}$), and this isomorphism does not depend on the choice of $f$. Indeed let $M(s)$ be the substructure of $M^{alg}$ generated over $M$ by any element of $s$; the substructures $M(s), M(s')$ are equal, and $Gal(s) = Aut(M(s)/M)$. The induced bijection $E_s \to E_{s'}$ depends therefore neither on this choice nor on a choice of $M^{alg}$.

Let $\sim$ denote $\mathcal{E}$-isomorphism, and let $E_{S,m} = S_{gal}[m]/\sim$. then if $e = s/\sim \in S_{S,m}$ we may define $Gal(e) = Gal(s), M(e) = M(s), E_e = E_s$. Let $S_{S,m}$ be the disjoint union over $e \in E_{S,m}$ of the groups $Gal(e)$, and let $S_{S,m} \to E_{S,m}$ be the natural map. Similarly let $\tilde{S}$ be the direct limit over all $S, m$ of $E_{S,m}$.

Let $\tilde{S}$ denote the family of all sorts $\tilde{S}_{S,m}$ and $E_{S,m}$. These will be called the Galois sorts of $T_0$. Let $T_0^{gal}$ consist of all sorts interpretable over these sorts. (I.e. close under quotients by definable equivalence relations.) Note that the groups $Gal(e)$ are not themselves part of $\tilde{S}$, in general.
Remark 6.1. The Galois group $G = \text{Aut}(M^\text{alg}/M)$ is the projective limit over $E_{S,m}$ of all groups $G_{\sigma}(e)$. An element $e \in S_{S,m}(M)$ corresponds to a normal open subgroups $N(e)$ of $G$, and we have $G_{\sigma}(e) \cong G/N(e)$ canonically. Similarly for any $k$ the space $G^k/\text{ad}_G$ of conjugacy classes of $G$ on $G^k$ can be deduced from $S_{S,m}$ by an appropriate projective limit.

Remark 6.2. Let $M \models T$. Let $G = \text{Aut}(M^\text{alg}/M)$, $C$ the centralizer of $G$ in $\text{Aut}(M^\text{alg})$. Then $\text{Aut}(M/\hat{S}(M))$ is the image of $C$ in $\text{Aut}(M)$. Equivalently, $\sigma \in \text{Aut}(M/\hat{S}(M))$ if every extension $\tau$ of $\sigma$ to $\text{Aut}(M^\text{alg})$ normalizes $\text{Aut}(M^\text{alg}/M)$, and induces an inner automorphism of this group. It follows in particular that any such $\tau$ preserves each Galois extension of $M$.

Remark 6.3. Assume (PE): $L$ has a sort $S_{\text{basic}}$ for which the primitive element theorem is valid, i.e. any finite extension of $M \models T_0$ is generated by a single element of $S_{\text{basic}}$. Then only the Galois sorts $E_m = E_{S_{\text{basic}},m}$ and $\hat{S}_m = E_{S_{\text{basic}},m}$ need be considered; any Galois sort is in definable bijection with a definable subset of these.

6.4. Galois cohomology. Let $G$ be a profinite group, acting continuously on a discrete group $A$. We write $g^a$ for the action.

Recall the definition of $H^1(G,A)$ ([10] Chap. I, 5.1.) A 1-cocyle is a continuous map $a : G \to A$ satisfying $a(st) = a(s)a(t)$. The set of 1-cocycles is denoted $Z^1(G,A)$. Two cocycles $a,a'$ are cohomologous if $a'(s) = b^{-1}a(s)b$ for some $b \in A$. The quotient of $Z^1(G,A)$ by this equivalence relation is denoted $H^1(G,A)$. The action of $G$ on $Z^1(G,A)$, induced from the actions on $G$ and on $A$, satisfies: $g^a(t) = b^{-1}a(t)b$, where $b = a(g)$; hence $g^a$ is cohomologous to $a$, so $G$ acts trivially on $H^1(G,A)$.

When $A$ is a definable group of $\mathcal{T}$, for any $M \models T_0$ we have a profinite group $G_M := \text{Aut}(M^\text{alg}/M)$, and a continuous action of $G_M$ on $A(M^\text{alg})$. We write $G$ for the functor $M \mapsto G_M$.

Lemma 6.5. Assume (PE) (cf. [6]). Let $A$ be a definable finite group of $\mathcal{T}$. Let $G_M$ denote the automorphism group $\text{Aut}(M^\text{alg}/M)$. Then $H^1(G,A)$ is interpretable in the Galois sorts of $T_0$. In other words there exists a definable set $S$ of $T_0^\text{alg}$ and for any $M \models T_0$ a canonical bijection $S(M) \to H^1(G_M,A)$.

Proof. The proof of Lemma 6.4 goes through. \hfill \Box

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