Decoherence of Fock States Leads to a Maximally Quantum State

Andrew C. McClung, Tyler E. Keating, Adam T. C. Steege, and Arjendu K. Pattanayak

Department of Physics and Astronomy, Carleton College,
One North College Street, Northfield, MN 55057

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We consider the Wigner function evolution of Fock states \(|n\rangle\) linearly coupled to a Markovian bath of oscillators. In the absence of environmental coupling, apparent “quantumness” increases with \(n\), but the presence of any environmental interaction causes high-\(n\) states to lose their quantum features more rapidly than low-\(n\) states. Using the negative volume of the Wigner function as a metric \([1]\), we observe a time-dependent quantumness peak across the eigenstates.

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The macroscopic world is constructed of fundamentally quantum objects, but the observation of quantum effects requires either very small or very cold systems. That is, quantum effects are most easily visible in systems sufficiently isolated from their environments to prevent decoherence. It is generally more difficult to adequately protect larger systems from decoherence; this difficulty gives rise to the classical behavior of the macroscopic scale \([2]\). Contemporary investigations of the transition from quantum to classical are motivated by fundamental considerations as well as practical issues of control and engineering. The transition is being investigated in a wide array of systems, including nanomechanical systems \([3, 4]\), mesoscopic systems such as Josephson junction devices \([5]\), as well as cavity-QED systems \([6]\).

This quantum-classical transition is understood to be a multi-parameter transition, controlled by internal system parameters and behaviors as well as (a) the relative size of \(\hbar\) compared to the characteristic action of the system, (b) the strength of the system-environment interaction, and (c) the temperature and other characteristics of the environment, leading to a rich landscape of possible behaviors.

In this paper we argue that the interplay between the relative effects of decoherence and increasing quantum number leads to nonmonotonicity in the “quantumness” of a unipartite system. We illustrate this in the behavior of Fock states \(|n\rangle\) for a harmonic oscillator coupled to a system modeled as a Markovian bath of oscillators and demonstrate the existence of a time-dependent quantumness peak. Signatures of this effect have been seen in recent experimental work mapping the decoherence of Fock states \([7, 8]\), although the significance had not been previously realized.

In the interaction picture, the time evolution of the density matrix \(\varrho\) for a quantum harmonic oscillator coupled to a continuum of oscillators in the Markovian approximation and with appropriate assumptions about the spectrum of the environment is governed by the master equation \([9]\)

\[
\dot{\varrho} = -\frac{\gamma}{2} \left( \mathcal{N} L[a^\dagger] \varrho + (\mathcal{N} + 1) L[a] \varrho \right).
\]

Here the dot represents the time derivative, the Lindblad superoperator is defined as \(L[O] \varrho \equiv 2O \varrho O^\dagger - \varrho O^\dagger O - [O, \varrho]\), and \(a\) and \(a^\dagger\) are the raising and lowering operators for the harmonic oscillator, respectively, \(\gamma\) represents the degree of coupling of the oscillator to its environment, and \(\mathcal{N}\) corresponds to the mean number of thermal photons in the bath.

In our analysis below, we consider the Wigner function \(W(\alpha)\), a phase space representation of the state of the system, which is defined most easily (if circuitously) via the Fourier transform of \(\chi(\lambda)\), the Wigner characteristic function \([9]\):

\[
W(\alpha) = \int d^2 \lambda \frac{\pi}{\alpha^2} \chi(\lambda) \exp(\lambda^* \alpha - \lambda \alpha^*),
\]

where \(\alpha = (x + i p)/\sqrt{2}\), in terms of the phase space quadratures \(x\) and \(p\); and \(\lambda = (x + i p)/\sqrt{2}\) is defined on the Fourier transform of phase space. The function \(\chi(\lambda)\) is written in terms of the density matrix as \(\chi(\lambda) = \text{Tr} [\varrho \exp(\lambda a^\dagger - \lambda^* a)]\). The master equation \([10]\) can then be rewritten \([10]\) as a diffusion equation for the characteristic function, \(\chi\):

\[
\dot{\chi} = -\frac{\gamma}{2} \left[ \frac{\partial^2 \chi}{\partial x^2} + \frac{\partial^2 \chi}{\partial p^2} + \left(1 + \mathcal{N}\right) (\dot{x}^2 + \dot{p}^2) \chi \right].
\]

Restricting our attention to systems initialized in the \(n\)th Fock state, the solution to Eq. \([3]\) can be written \([11]\)

\[
\chi_n(\lambda, t) = L_n(|\lambda|^2 e^{-\gamma t}) e^{\frac{1}{2} |\lambda|^2 (1 - 2N (e^{-\gamma t} - 1))},
\]

where \(L_n(x) = \sum_{m=0}^n \frac{(-x)^n}{m!} \frac{n^m}{m!}\) is the Laguerre polynomial of order \(n\).

Recall that in the absence of environmental interaction, the Wigner function for the \(n\)th Fock state is given by \([12]\)

\[
W_n(x, p) = \frac{(-1)^n}{\pi} e^{-\left(x^2 + p^2\right)} L_n(2x^2 + p^2).
\]

Given Eq. \([5]\), for zero temperature \((\mathcal{N} = 0)\), the time-dependent solution \([13]\) for an environmentally-interacting harmonic oscillator becomes a time-dependent superposition of non-interacting Wigner
functions $W_m(x, p)$, where $m \leq n$:

$$W_n(x, p, t) = e^{-\gamma t} \sum_{m=0}^{n} \binom{n}{m} \left( \frac{2}{(2\pi)^2} \right)^m \left( \frac{n}{m} \right) f_m(x, p),$$

(7)

That is, the Wigner function of a decohering Fock state is composed exclusively of diagonal elements.

To demonstrate Eq. (9), we note that substituting Eq. (7) into Eq. (2) yields

$$f_n - f_{n-1} = \sum_{m=0}^{n} \binom{n}{m} (-1)^{m-n} \binom{n}{m} f_m(x, p),$$

(9)

Proposition. We must now establish that

$$f_n(x, p) = (2\pi)^2 m! \sum_{k=0}^{m} (-1)^k \binom{m}{k} W_k(x, p).$$

(10)

Proof. We prove Eq. (10) by induction. First, we rearrange Eq. (7) to get a recursion relation $f_n(x, p)$, yielding

$$f_n(x, p) = (2\pi)^2 n! \left( \frac{-e^{-\gamma t}}{e^{-\gamma t}} \right)^n W_n(x, p, t)$$

$$- \sum_{m=0}^{n-1} \binom{n-1}{m} \frac{(-e^{-\gamma t})^m}{(2\pi)^2 m!} f_m(x, p),$$

(10)

for $n > 0$ and

$$f_0(x, p) = (2\pi)^2 W_0(x, p, t),$$

(11)

which is the base case ($n = 0$) of Eq. (10). To prove that Eq. (10) holds true in general, we now show that Eq. (10) holds for the $n+1$th case. We consider $f_{n+1}(x, p)$ by evaluating Eq. (10) at $t = 0$:

$$f_{n+1}(x, p) = \frac{(2\pi)^2 (n+1)!}{(-1)^{n+1}} W_{n+1}(x, p, t)$$

$$- \sum_{m=0}^{n} \frac{(-1)^m}{(2\pi)^2 m!} \binom{n+1}{m} f_m(x, p).$$

(12)

Because the sum only considers $f_m(x, p)$ where $m \leq n$, we may employ Eq. (9), giving

$$f_{n+1}(x, p) = (2\pi)^2 (n+1)! \left\{ (-1)^{n+1} W_{n+1}(x, p, t) \right\}$$

$$+ \sum_{m=0}^{n} \sum_{k=0}^{m} (-1)^{k+m+n} \binom{n+1}{m} \binom{m}{k} W_k(x, p).$$

(13)

The term in the square brackets is the $n+1$th term of the sum in Eq. (9); we must demonstrate is that the double sum in Eq. (13) constitutes the first $n$ terms. The double sum can be rewritten as

$$\sum_{m=0}^{n} \sum_{k=0}^{m} (-1)^{k+m+n} \binom{n+1}{m} \binom{m}{k} W_k(x, p)$$

$$+ \sum_{k=0}^{n} (-1)^k \binom{n+1}{k} W_k(x, p)$$

$$- (-1)^n W_{n+1}(x, p).$$

(14)

The second term in Eq. (14) constitutes the first $n$ terms of the sum in Eq. (9) evaluated for $n+1$. All that remains to be shown to establish Eq. (9) is that the first and third terms in expression (14) cancel exactly; that is, we must show

$$\sum_{m=0}^{n} \sum_{k=0}^{m} (-1)^{k+m+n} \binom{n+1}{m} \binom{m}{k} W_k(x, p) = W_{n+1}(x, p).$$

(15)

The LHS of Eq. (15) can be rewritten as

$$\sum_{m=0}^{n+1} \sum_{k=m}^{n+1} (-1)^{k+m} \binom{n+1}{k} \binom{k}{m};$$

(16)

but the second sum in Eq. (16) is the well-known combinatorial identity

$$\sum_{b=a}^{d} (-1)^{b-a} \binom{b}{a} = \delta_{a,b},$$

(17)

where $\delta_{a,b}$ is the Kronecker delta [11]; therefore the only surviving term in the LHS of Eq. (16) is $W_{n+1}(x, p)$, thereby establishing Eq. (9).}

Inserting Eq. (10) into Eq. (7) gives

$$W_n(x, p, t) = \sum_{m=0}^{n} \sum_{k=0}^{m} (-1)^{k+m} e^{-m \gamma t} \binom{n}{m} \binom{m}{k} W_k(x, p);$$

(18)

by collecting like $W_k(x, p)$, we can rewrite Eq. (18) as

$$W_n(x, p, t) = \sum_{m=0}^{n} W_m(x, p) \binom{n}{m} \binom{m}{k} W_k(x, p);$$

(19)

By changing the index on the second sum to $j = k - m$, Eq. (19) becomes

$$W_n(x, p, t) = \sum_{m=0}^{n} W_m(x, p) \binom{n}{m} \times e^{-\gamma t} \sum_{j=0}^{n-m} (-1)^j (\gamma t)^{n-m-j} \binom{n-m}{j};$$

(20)
but the second sum in Eq. 20 is simply the expansion of \( e^{\gamma t - 1} \)^{n-m}, and hence Eq. 6 is recovered.

Fig. 1(a) shows plots of the Wigner functions for zero-temperature harmonic oscillators initialized in the \( n = 1, 4 \) and 7 Fock states for three increasing values of \( \gamma t \). (Note that allowing the system to evolve for a longer time \( t \) at a given degree of coupling \( \gamma \) is equivalent to allowing the system to evolve for a shorter time at a higher degree of coupling—increasing \( \gamma t \) captures either of these behaviors.) When \( \gamma t = 0 \), the system initialized in the \( n = 7 \) Fock state exhibits strongly quantum features: its Wigner function has large-amplitude oscillations and contains large regions of negative quasi-probability, an obvious indicator of non-classical behavior. In these respects, the system initialized in the \( n = 4 \) Fock state is comparatively “less quantum” at \( \gamma t = 0 \), and the system initialized in the \( n = 1 \) Fock state even less so.

At \( \gamma t = 0.15 \), the oscillation amplitudes of the system initialized in the \( n = 7 \) Fock state are greatly reduced due to environmental interaction. The influence of the environment is present in the other two systems as well, but the effect is less pronounced. By \( \gamma t = 0.3 \), the systems initialized in the \( n = 4 \) and 7 Fock states resemble classical orbits. Though it began as the “least quantum” state, by \( \gamma t = 0.3 \), the Wigner function of the system initialized in the \( n = 1 \) Fock state has retained the comparatively “most quantum” oscillations and negative quasi-probability.

A numerical measure of quantumness for unipartite systems is necessary in this context to compare behavior. There are various ways of parameterizing nonclassicality—it is typical to define a coherent state with minimum uncertainty as the most classical system and to consider the quantumness of a state in question to be the minimum distance to the coherent state, using a distance metric based on the trace, the Hilbert-Schmidt distance, or a similar metric [12]. It is also possible to characterize quantum states based on the properties of their Wigner function representations, an attractive option in light of recent successes reconstructing the Wigner function experimentally [13, 14]. Naively, one might consider the purity, \( \mathcal{P} \), defined in phase space as

\[
\mathcal{P} = 2\pi \iint_{\mathbb{R}^2} dx dp \left| W(x, p) \right|^2,
\]

to be a good indicator of quantumness. However, at zero
of decoherence for larger $n$ states is higher. Fig. 2(a) shows the peak quantum state as a function of $\gamma t$. The existence of a quantumness peak can be understood precisely as resulting from these two competing trends: (a) the tendency for the fringing behavior (and therefore the negative volume) to increase with initial $n$, and (b) the robustness against decoherence to decrease with $n$. This behavior is arguably independent both of the measure of quantumness and of the system under consideration.

As mentioned above, the Wigner function for the interacting oscillator is composed exclusively of diagonal elements in the harmonic oscillator basis. Its density matrix representation can therefore be written $\rho_n = \sum_{k=0}^{\infty} p_k(t)|k\rangle\langle k|$, where $p_k(t)$ is the probability of observing the system in the $k$th eigenstate at time $t$. Since the density matrix is composed of mutually orthogonal states, we can use the phase space trace to determine the probabilities $p_k(t)$ [13]. For an oscillator initialized in the $n$th Fock state

$$p_k(t) = e^{-n\gamma t} \binom{n}{k} (e^{\gamma t} - 1)^{n-k}, \quad k \leq n. \quad (23)$$

Because the density matrix contains no off-diagonal elements, these probabilities are a complete description of the quantum state. The behavior of $p_k(t)$ for oscillators initialized in the $n = 1$ and $n = 3$ Fock states are shown in Fig. 2(b). The occupation probabilities for decohering Fock states up to $n = 15$ have been determined experimentally [7, 8]. The strong resemblance of the curves in Fig. 2(b) to experimental curves [16] suggests that evidence for a quantumness peak has already been observed.

In conclusion, the Wigner function representation of decohering zero-temperature Fock states can be used to compare the quantumness of systems, using the negative volume of the Wigner function as a quantumness metric. Though systems initialized in higher-$n$ Fock states start out as the “more quantum” states, they decohere more quickly than lower-$n$ states; the competing trends of quantum number and decoherence give rise to a time-dependent quantumness peak. The general nature of this argument suggests that this quantumness peak may be a universal feature of unipartite systems.

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\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2}
\caption{(a) Peak quantum state as a function of $\gamma t$. (b) (color online.) Time-dependent occupation probabilities for oscillators initialized in the $n = 1$ (left) and $n = 3$ (right) Fock states. The solid, dashed, dotted, and dot-dashed lines are the occupation probabilities for the $n = 0, 1, 2$ and 3 states, respectively. These curves strongly resemble those that have been determined experimentally [7, 8].}
\end{figure}

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