On the ramification of étale cohomology groups

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Abstract

Let $K$ be a complete discrete valuation field whose residue field is perfect and of positive characteristic, let $X$ be a connected, proper scheme over $\mathcal{O}_K$, and let $U$ be the complement in $X$ of a divisor with simple normal crossings.

Assume that the pair $(X,U)$ is strictly semi-stable over $\mathcal{O}_K$ of relative dimension one and $K$ is of equal characteristic. We prove that, for any smooth $\ell$-adic sheaf $G$ on $U$ of rank one, at most tamely ramified on the generic fiber, if the ramification of $G$ is bounded by $t+$ for the logarithmic upper ramification groups of Abbes-Saito at points of codimension one of $X$, then the ramification of the étale cohomology groups with compact support of $G$ is bounded by $t+$ in the same sense.

0 Introduction

Let $K$ be a complete discrete valuation field with perfect residue field $k$ of characteristic $p > 0$. Let $X$ be a connected, proper scheme over $\mathcal{O}_K$, $D$ a divisor with simple normal crossings on $X$, and $U = X - D$. Assume that the pair $(X,U)$ is strictly semi-stable over $\mathcal{O}_K$ of relative dimension $d$ (see Definition 1.2).

Let $\ell$ be a prime number different from $p$ and $\mathcal{G}$ be a smooth $\ell$-adic sheaf on $U$, by which we mean a smooth $\mathbb{Q}_\ell$-sheaf on $U$. Assume that $\mathcal{G}$ is at most tamely ramified on the generic fiber $X_K$. Write $D = \bigcup_{i=1}^{n} D_i$, where $D_i$ are the irreducible components of $D$. Let $\xi_i$ be the generic point of $D_i$, $\mathcal{O}_{M_i} = \mathcal{O}_{X,\xi_i}$, the henselization of the local ring at $\xi_i$, $M_i$ its field of fractions, and $\eta_i = \text{Spec} M_i$.

Let $G_{M_i}$ and $G_K$ denote the absolute Galois groups of $M_i$ and $K$, respectively, and $(G_{M_i,\log}^t)_{t \in \mathbb{Q} \geq 0}$, $(G_{K,\log}^t)_{t \in \mathbb{Q} \geq 0}$ the corresponding Abbes-Saito logarithmic upper ramification filtrations (see [1]). Put, for a real number $s \geq 0$, $G_{M_i,\log}^{s+} = \bigcup_{t \in \mathbb{Q}, t > s} G_{M_i,\log}^t$ and $G_{K,\log}^{s+} = \bigcup_{t \in \mathbb{Q}, t > s} G_{K,\log}^t$. Then we have the following conjecture:

Conjecture 1. Under the assumptions above, if $G_{M_i,\log}^{t+}$ acts trivially on $\mathcal{G}_{\mathcal{M}_i}$ for every $i$, then $G_{M_i,\log}^{t+}$ acts trivially on $H^j_c(U,\mathcal{G})$ for every $j$.

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Our main result is Theorem 4.1, in which we prove the conjecture in the special case where \( G \) is of rank 1, \( K \) has characteristic \( p \), and the relative dimension is \( d = 1 \).

The structure of this paper is as follows: in the first section, we shall briefly review some properties of the Abbes-Saito logarithmic upper ramification filtration and some notions on the ramification of characters. In the second section, we give a criterion for \( G_{+K, \log} \) to act trivially on an \( \ell \)-adic sheaf. In the third section, we provide an application of the Kato-Saito conductor formula. In the fourth section, we present and prove the main result.

1 Preliminary notions

1.1 The Abbes-Saito filtration

We shall briefly review some properties of the Abbes-Saito logarithmic upper ramification filtration. For a complete discrete valuation field \( K \), Abbes and Saito constructed a decreasing filtration \( (G_{tK, \log})_{t \in \mathbb{Q} > 0} \) of the absolute Galois group \( G_K \), extended by \( G_{0K, \log} = G_K \).

When the residue field of \( K \) is perfect, \( (G_{tK, \log})_{t \in \mathbb{Q} > 0} \) coincides with the classical upper ramification filtration. \( (G_{tK, \log})_{t \in \mathbb{Q} > 0} \) is stable under tame base change; more precisely, if \( L \) is a finite separable extension of \( K \) of ramification index \( e \) that is tamely ramified, we have \( G_{etL, \log} = G_{tK, \log} \). In general, for a finite separable extension \( L/K \) of ramification index \( e \), not necessarily tamely ramified, we have \( G_{etL, \log} \subset G_{tK, \log} \).

In this paper we shall make use of the following definition:

**Definition 1.1.** For a real number \( s \geq 0 \), define \( G^{s+}_{K, \log} = \bigcup_{t \in \mathbb{Q}, t > s} G_{tK, \log} \).

We shall need the following property of this filtration:

**Lemma 1.1** ([2], Lemma 5.2). Let \( K \) be a complete discrete valuation field with residue field \( k \) of characteristic \( p \). Assume that there is a map of complete discrete valuation fields \( K \to L \) inducing a local homomorphism \( \mathcal{O}_K \to \mathcal{O}_L \), that the ramification index is prime to \( p \), and that the induced extension of residue fields is separable. Then, for \( t \in \mathbb{Q} > 0 \), the map \( G_L \to G_K \) induces a surjection \( G_{etL, \log} \to G_{tK, \log} \).

As a consequence, we also have surjections \( G_{etL, \log}^{s+} \to G_{tK, \log}^{s+} \).

1.2 Ramification of characters

In this subsection, assume that the residue field \( k \) of \( K \) has characteristic \( p > 0 \) and is not necessarily perfect.

We recall the definition of the \( k \)-vector space \( \Omega_k(\log) \). There exists a canonical map \( d \log : K^\times \to \Omega_k \), and \( \Omega_k(\log) \) is the amalgamate sum of the differential module \( \Omega_k \) with \( k \otimes_{\mathbb{Z}} K^\times \) over \( k \otimes_{\mathbb{Z}} \mathcal{O}_K^\times \) with respect to \( d \log : \mathcal{O}_K^\times \to \Omega_k \) and \( \mathcal{O}_K^\times \hookrightarrow K^\times \). There is a residue map \( \text{res} : \Omega_k(\log) \to k \) induced by the valuation map of \( K \) and an exact sequence
In \[5\], Kato constructs an increasing filtration \((F_r H^1(K, \mathbb{Q}/\mathbb{Z}))_{r \in \mathbb{N}}\) and defines, putting \(Gr_r H^1(K, \mathbb{Q}/\mathbb{Z}) = F_r H^1(K, \mathbb{Q}/\mathbb{Z})/F_{r-1} H^1(K, \mathbb{Q}/\mathbb{Z})\) for \(r \geq 1\), an injection

\[
\text{rsw}_{r,K} : Gr_r H^1(K, \mathbb{Q}/\mathbb{Z}) \to \text{Hom}_k(\mathfrak{m}_K^r/\mathfrak{m}_K^{r+1}, \Omega_k(\log)),
\]

where \(\mathfrak{m}_K\) denotes the maximal ideal of \(\mathcal{O}_K\). For \(\chi \in F_r H^1(K, \mathbb{Q}/\mathbb{Z})\setminus F_{r-1} H^1(K, \mathbb{Q}/\mathbb{Z})\), the injection

\[
\text{rsw}_{r,K}(\chi) : \mathfrak{m}_K^r/\mathfrak{m}_K^{r+1} \to \Omega_k(\log)
\]

is denoted by \(\text{rsw}_K(\chi)\) and called the refined Swan conductor of \(\chi\).

In \[3\], Corollary 9.12, Abbes and Saito relate Kato’s construction to the upper ramification groups defined in \[1\]. More specifically, they prove that, when \(K\) is of equal characteristic, \(\chi \in F_r H^1(K, \mathbb{Q}/\mathbb{Z})\) if and only if \(\chi\) kills \(G^{r+}_{r,K, \log}\).

**Remark 1.** As the referee pointed out, the comparison between Kato’s filtration and the Abbes-Saito logarithmic upper ramification groups remains open in the mixed characteristic case. This is the only reason we assume that \(K\) is of characteristic \(p > 0\) in sections 3 and 5 of this paper. All results of this paper are valid whenever \(K\) is a complete discrete valuation field having the property that, for all \(\chi \in H^1(K, \mathbb{Q}/\mathbb{Z})\) and \(r \in \mathbb{N}\),

\[
\chi \in F_r H^1(K, \mathbb{Q}/\mathbb{Z}) \text{ if and only if } \chi : G_K \to \mathbb{Q}/\mathbb{Z} \text{ kills } G^{r+}_{r,K, \log}.
\]

Consider now the following case. Let \(S = \text{Spec} \mathcal{O}_K\) and \(X\) be a regular flat separated scheme over \(S\). Let \(D = \bigcup_{i=1}^n D_i\) be a divisor with simple normal crossings, where \(D_i\) denotes the irreducible components of \(D\). For each \(i\) let \(\xi_i\) be a generic point for \(D_i\), \(\mathcal{O}_{M_i} = \mathcal{O}_{X,\xi_i}^h\) the henselization of the local ring at \(\xi_i\), \(M_i\) its field of fractions, and \(k_i\) the residue field of \(M_i\). Let \(U = X - D\) and \(\chi \in H^1(U, \mathbb{Q}/\mathbb{Z})\). For each \(i\), denote by \(\chi_i \in H^1(M_i, \mathbb{Q}/\mathbb{Z})\) the restriction of \(\chi\), and by \(r_i\) the Swan conductor \(\text{Sw}_{M_i, \chi_i}\). Define the Swan divisor

\[
D_{\chi} = \sum_i r_i D_i
\]

and let

\[
E = \sum_{r_i > 0} D_i
\]

be the support of \(D_{\chi}\). It’s shown by \[5\], (7.3), that there exists an injection

\[
\text{rsw}_{\chi} : \mathcal{O}_X(-D_{\chi}) \otimes_{\mathcal{O}_X} \mathcal{O}_E \to \Omega^1_{X/S}(\log D) \otimes_{\mathcal{O}_X} \mathcal{O}_E
\]

inducing \(\text{rsw}_{M_i}(\chi_i)\) at \(\xi_i\). We say that \(\chi\) is clean if \(\text{rsw}_{\chi}\) is a locally splitting injection.
1.3 Semi-stable pairs

In this subsection, we let $K$ be a complete discrete valuation field with perfect residue field $k$ of characteristic $p > 0$, $X$ a proper scheme of finite presentation over $\mathcal{O}_K$, and $U$ an open and dense subscheme of $X$. We recall the definition of a semi-stable pair (9, Definition 1.6):

**Definition 1.2.** The pair $(X, U)$ is said to be semi-stable over $\mathcal{O}_K$ of relative dimension $d$ if, étale locally on $X$, $X$ is étale over $\text{Spec} \, \mathcal{O}_K[T_0, \ldots, T_d]/(T_0 \cdots T_r - \pi)$ and $U$ is the inverse image of $\text{Spec} \, \mathcal{O}_K[T_0, \ldots, T_d, T_0^{-1}, \ldots, T_m^{-1}]/(T_0 \cdots T_r - \pi)$ for some $0 \leq r \leq m \leq d$ and prime $\pi$ of $K$.

When $(X, X_K)$ is semi-stable over $\mathcal{O}_K$, we say that $X$ is semi-stable over $\mathcal{O}_K$.

If we substitute the condition “étale locally” by “Zariski locally”, the pair $(X, U)$ is then said to be strictly semi-stable.

We shall need the following property of semi-stable pairs, which is a consequence of Theorem 2.9 in 9:

**Theorem 1.1.** Let $(X, U)$ be a strictly semi-stable pair over $\mathcal{O}_K$ and $L$ be a finite separable extension of $K$. Then there exists a proper birational morphism $X' \to X_{\mathcal{O}_L}$ inducing an isomorphism $U' \to U_{\mathcal{O}_L}$, where $U'$ is the inverse image of $U_{\mathcal{O}_L}$, and such that $(X', U')$ form a strictly semi-stable pair over $\mathcal{O}_L$.

2 The action of $G_{K, \text{log}}^{t+}$

In this section, we let $K$ be a complete discrete valuation field of equal characteristic with perfect residue field $k$ of characteristic $p > 0$, $\ell$ be a prime different than $p$, and $M, N$ be finite-dimensional representations of $G_K$ over $\overline{\mathbb{Q}}_{\ell}$ which come from finite-dimensional continuous representations of $G_K$ over a finite extension of $\mathbb{Q}_\ell$ contained in $\overline{\mathbb{Q}}_{\ell}$. We shall provide a criterion for $G_{K, \text{log}}^{t+}$ to act trivially on $M$.

There is a canonical slope decomposition (see [7], Proposition 1.1, or [4], Lemma 6.4)

$$M = \bigoplus_{r \in \mathbb{Q}_{\geq 0}} M^{(r)}$$

characterized by the following properties: if $P$ is the wild inertia subgroup of $G_K$, then $M^P = M^{(0)}$. Further, for all $r > 0$,

$$(M^{(r)})^{G_{K, \text{log}}} = 0$$

and

$$(M^{(r)})^{G_{K, \text{log}}^{t+}} = M^{(r)}.$$}

We have $M^{(r)} = 0$ except for finitely many $r$. The values of $r$ for which $M^{(r)} \neq 0$ are called slopes of $M$.

**Definition 2.1.** We say that $M$ is isoclinic if it has only one slope.
The following proposition gives our criterion:

**Proposition 2.1.** Let $t$ be a nonnegative real number. Assume that, for any totally tamely ramified extension $L/K$ of degree $e$ prime to $p$, we have the following: if $M_L$ denotes the representation of $G_L$ induced by $M$, then, for any character $\chi : G_L \to \mathbb{Q}_\ell^\times$ for which $Sw_L(\chi) > et$, we have

$$Sw_L(M_L \otimes \chi) = \text{rk}(M_L)Sw_L(\chi).$$

Then $G_K^{t+}$ acts trivially on $M$.

The proof will be presented shortly. The general strategy is the following:

- We first show that the behavior of the tensor product of isoclinic $M$ and $N$ is similar to that of the tensor product of characters;
- Next, we use the previous result to understand the slope decomposition of the tensor product $M \otimes \chi$ and prove the proposition.

We start with the lemma:

**Lemma 2.1.** If $M$ is isoclinic of slope $r$ and $N$ is isoclinic of slope $s$, where $r > s$, then $M \otimes N$ is isoclinic of slope $r$.

**Proof.** We have

$$M_{G_K^{r-}} = 0,$$

$$M_{G_K^{r+}} = M,$$

$$N_{G_K} = 0,$$

and

$$N_{G_K^{s+}} = N.$$

Since $r > s$, $(M \otimes N)_{G_K^{r+}} = M \otimes N$. On the other hand, $G_K^r$ acts trivially on $N$ and $M_{G_K^{r-}} = 0$, so $(M \otimes N)_{G_K^r} = 0$. Hence $M \otimes N$ is isoclinic of slope $r$. \qed

**Proof of Proposition 2.1.** We need to show that, if $r > t$, then $M^{(r)} = 0$. Let $R$ be the maximum slope of $M$. Assume, by contradiction, that $R > t$. Let $m, e$ be positive integers such that:

(i) $e$ is prime to $p$,

(ii) $\frac{m}{e} < R$,

(iii) $\frac{m}{e}$ is strictly greater than any other slope of $M$,

(iv) $\frac{m}{e} > t$.

Let $L$ be a totally tamely ramified extension of degree $e$ of $K$. By [1], Proposition 3.15, $G_{K,\log}^{s} = G_{L,\log}^{s}$ for any $s \in \mathbb{Q}_{\geq 0}$, so the slopes of $M_L$ are of the form $er$, where $r$ is a slope of $M$. Take $\chi$ with $Sw_L(\chi) = m$. Then, by assumption,

$$Sw_L(M_L \otimes \chi) = \text{rk}(M_L)Sw_L(\chi) = \text{rk}(M_L)m.$$
By Lemma 2.1 for all $r < m$ we have that $M^{(r)}_L \otimes \chi$ is isoclinic of slope $m$, while $M^{(eR)}_L \otimes \chi$ is isoclinic of slope $eR$. It follows that

$$Sw_L(M_L \otimes \chi) = \sum_{r \in \mathbb{Q}_{\geq 0}} Sw_L(M^{(r)}_L \otimes \chi) = \sum_{r \in \mathbb{Q}_{\geq 0}, r < m} \text{rk}(M^{(r)}_L)m + \text{rk}(M^{(eR)}_L)eR.$$  

Combining the two expressions we get

$$\text{rk}(M^{(eR)}_L)eR = \text{rk}(M^{(eR)}_L)m,$$

which is a contradiction, since, by assumption, $m < eR$ and $M^{(eR)}_L \neq 0$. \hfill \square

### 3 The Kato-Saito conductor formula

Let $K$ be a complete discrete valuation field with perfect residue field $k$ of characteristic $p > 0$. Let $\ell$ be a prime number different from $p$, $U$ be a smooth separated scheme of finite type over $K$, and $\mathcal{F}$ be a smooth $\ell$-adic sheaf of constant rank on $U$. In [6], Kato and Saito defined the Swan class $Sw_U\mathcal{F}$, a 0-cycle class with coefficients in $\mathbb{Q}$ supported on the special fiber of a compactification of $U$ over $\mathcal{O}_K$, and proved the conductor formula

$$Sw_KR\Gamma_c(U_{\overline{K}}, \mathcal{F}) = \text{deg} Sw_U\mathcal{F} + \text{rk}(\mathcal{F})Sw_KR\Gamma_c(U_{\overline{K}}, \mathbb{Q}_\ell),$$

where $Sw_KR\Gamma_c(U_{\overline{K}}, \mathcal{F})$ denotes the alternating sum $\sum_j (-1)^j Sw_KH^j_c(U_{\overline{K}}, \mathcal{F})$.

In this section, assume that $X$ is a regular flat separated scheme of finite type over $S = \text{Spec} \mathcal{O}_K$. Let $D \subset X$ be a divisor with simple normal crossings and write $D = \bigcup_{i=1}^n D_i$, where $D_i$ are the irreducible components of $D$. Put $U = X - D$ and consider a smooth $\ell$-adic sheaf $\mathcal{F}$ of rank 1 on $U$, at most tamely ramified on $X_K$ and with clean ramification with respect to $X$.

The Swan 0-cycle class $c_\mathcal{F}$ of $\mathcal{F}$ is defined as follows. Let $E$ be the support of the Swan divisor $D_\mathcal{F} = \sum r_i D_i$. Then define $c_\mathcal{F} \in CH_0(E)$ as

$$c_\mathcal{F} = \{c(\Omega^1_{X/S}(\log D) \otimes_{\mathcal{O}_X} \mathcal{O}_E)^* \cap (1 + D_\mathcal{F})^{-1} \cap D_\mathcal{F}\}_{\text{dim } 0}.$$ 

Under the assumption that $\dim U_K \leq 1$, by Corollary 8.3.8 of [6], the Kato-Saito conductor formula becomes simply

$$Sw_KR\Gamma_c(U_{\overline{K}}, \mathcal{F}) = \text{deg} c_\mathcal{F} + Sw_KR\Gamma_c(U_{\overline{K}}, \mathbb{Q}_\ell).$$

The following proposition is an application of this formula that will be useful in the next section:

**Proposition 3.1.** Let $X$, $S$ and $U = X - D$ be as above. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two smooth $\ell$-adic sheaves on $U$ of rank one, $\mathcal{F}_2$ having clean ramification with respect to $X$. Write $D_{\mathcal{F}_1} = \sum r_i D_i$ and $D_{\mathcal{F}_2} = \sum s_i D_i$. Assume that $r_i < s_i$ for every $i$. Then $\mathcal{F}_1 \otimes \mathcal{F}_2$ has clean ramification and

$$c_{\mathcal{F}_1 \otimes \mathcal{F}_2} = c_{\mathcal{F}_2}.$$
Proof. Since \( r_i < s_i \) for every \( i \), we have \( D_{\mathcal{F}_1 \otimes \mathcal{F}_2} = D_{\mathcal{F}_2} \) and the refined Swan conductors of \( \mathcal{F}_1 \otimes \mathcal{F}_2 \) and \( \mathcal{F}_2 \) coincide. Denote by \( E_i \) the support of \( D_{\mathcal{F}_i} \) and by \( E \) be the support of \( D_{\mathcal{F}_1 \otimes \mathcal{F}_2} \). We have \( E = E_2 \), so

\[
c_{\mathcal{F}_1 \otimes \mathcal{F}_2} = \{ c(\Omega^1_{X/S}(\log D) \otimes O_X \otimes E)^* \cap (1 + D_{\mathcal{F}_1 \otimes \mathcal{F}_2})^{-1} \cap D_{\mathcal{F}_1 \otimes \mathcal{F}_2} \}_\dim 0
\]

\[
= c(\Omega^1_{X/S}(\log D) \otimes O_{E_2})^* \cap (1 + D_{\mathcal{F}_2})^{-1} \cap D_{\mathcal{F}_2}
\]

\[= c_{\mathcal{F}_2}. \]

\[\square\]

4 Main results

In this section, we let \( K \) be a complete discrete valuation field with perfect residue field \( k \) of characteristic \( p > 0 \) and of equal characteristic, \( S = \text{Spec} \mathcal{O}_K \), and \( s = \text{Spec} k \). We will denote by \( X \) a proper, connected scheme of finite presentation over \( \mathcal{O}_K \), and \( U \) an open and dense subscheme of \( X \). We assume that \( D = X - U \) is a divisor with simple normal crossings and write \( \bigcup_{i=1}^n D_i \), where \( D_i \) are the irreducible components. We also assume that the pair \((X, U)\) is strictly semi-stable over \( \mathcal{O}_K \) of relative dimension 1, and that \( G \) is a smooth \( \ell \)-adic sheaf on \( U \), where \( \ell \) is a prime number different from \( p \). Further, we assume that \( G \) is of rank 1 and at most tamely ramified on the generic fiber \( X_K \). Denote by \( \xi_i \) the generic point of \( D_i \), \( O_{M_i} = O_{X,\xi_i} \) the henselization of the local ring at \( \xi_i \), \( M_i \) its field of fractions, \( k_i \) the residue field of \( M_i \), and \( \eta_i = \text{Spec} M_i \).

We shall prove the following theorem:

**Theorem 4.1.** Conjecture 1 is true when \( G \) is of rank 1, the relative dimension is 1, and \( K \) is of equal characteristic.

**Remark 2.** When the relative dimension is greater than 1, one should still be able to prove Conjecture 1 using the same methods used in this paper, as long as it is true that

\[
\text{Sw}_K R\Gamma_c(U_{\overline{\mathcal{F}}}, \mathcal{F}) = \deg c_{\mathcal{F}} + \text{Sw}_K R\Gamma_c(U_{\overline{\mathcal{F}}}, \mathbb{Q}_\ell)
\]

for smooth \( \ell \)-adic sheaves \( \mathcal{F} \) of rank 1 on \( U \), at most tamely ramified with clean ramification with respect to \( X \).

The proof is divided in two cases. First observe that, since the total constant field of \( X_K \) is a finite unramified extension of \( K \), we may assume that \( K \) is the total constant field of \( X_K \). Then there is an exact sequence of fundamental groups

\[
1 \longrightarrow \pi_1(U_{\overline{\mathcal{F}}}) \longrightarrow \pi_1(U) \longrightarrow G_K \longrightarrow 1.
\]

Let \( M \) be the function field of \( X \) and \( \eta = \text{Spec} M \). We first consider the case in which the action of \( \pi_1(U_{\overline{\mathcal{F}}}) \) is trivial on \( \mathcal{F}_\eta \), and then the case in which it is non-trivial.

To prove the first case, we shall need the following lemma:

**Lemma 4.1.** In addition to the assumptions of Theorem 4.1, assume that \( \mathcal{F}_\eta \) is the pullback of some \( \ell \)-adic representation \( \mathcal{H} \) of \( G_K \). If \( G_{M_i, \log}^{+} \) acts trivially on \( \mathcal{F}_\eta \), then \( G_{K}^{+} \) acts trivially on \( \mathcal{H} \).
We shall now prove Theorem 4.1 for the case in which we have that Lemma 4.1, the result follows.

By Lemma 2.1, and the fact that $H^i_c(U^\mathcal{R}, \mathcal{G})$ is at most tamely ramified ([8, Corollary 2]), we have that the slope decomposition of $H^i_c(U^\mathcal{R}, \mathcal{G})$ coincides with that of $\mathcal{H}$, in the following sense:

$$(H^i_c(U^\mathcal{R}, \mathcal{G}))^{(r)} = H^i_c(U^\mathcal{R}, \mathcal{Q}_L) \otimes \mathcal{H}^{(r)}.$$ 

It follows that $G_{K, \log}^i$ acts trivially on $H^i_c(U^\mathcal{R}, \mathcal{G})$ if and only if it acts trivially on $\mathcal{H}$. By Lemma 4.1, the result follows.

We shall now prove Theorem 4.1 for the case in which $\pi_1(U^\mathcal{R})$ does not act trivially on $\mathcal{G}$. The core of strategy is the following: using the Kato-Saito conductor formula and the fact that $H^0(U^\mathcal{R}, \mathcal{G}) = H^0_c(U^\mathcal{R}, \mathcal{G}) = 0$, we show that $H^i_c(U^\mathcal{R}, \mathcal{G})$ satisfies the hypotheses of Proposition 2.1.

**Lemma 4.2.** Keep the assumptions of Theorem 4.1. Let $e$ be a natural number prime to $p$ and $L$ be a totally tamely ramified extension of $K$ of degree $e$. If $\chi : G_L \to \overline{\mathbb{Q}}_p^\times$ is a character such that $\text{Sw}_L(\chi) > et$, then

$$\text{Sw}_L(R\Gamma_c(U^\mathcal{R}, \mathcal{G}) \otimes \chi) = \text{rk}(R\Gamma_c(U^\mathcal{R}, \mathcal{G}))\text{Sw}_L(\chi).$$

**Proof.** First consider the following. By Theorem 4.1, there exists a proper birational morphism $X' \to X_{O_L}$ inducing an isomorphism $U' \to U_{O_L}$, where $U'$ is the inverse image of $U_{O_L}$ and such that $(X', U')$ is strictly semi-stable over $O_L$.

Let $D' = X' - U'$ and write $D' = \bigcup_{i=1}^{n'} D'_i$, where $D'_i$ are the irreducible components of $D'$. For each $1 \leq i \leq n'$ let $\xi'_i$ be the generic point of $D'_i$, $O_{M'_i} = O_{X', \xi'_i}$ the henselization of the local ring at $\xi'_i$, $M'_i$ its field of fractions, and $\eta'_i = \text{Spec } M'_i$.

There is a composition of blowups of closed points $\tilde{X} \to X$ and a point $\tilde{\xi}_i$ such that $O_{\tilde{X}, \tilde{\xi}_i} = O_{X', \xi'_i} \cap M$. Let $\tilde{M}_i$ be the field of fractions of $O_{\tilde{X}, \tilde{\xi}_i}^h$. Put $\tilde{\eta}_i = \text{Spec } \tilde{M}_i$. Denote by $e'_i$ and $\tilde{e}_i$ the ramification indices of $M'_i / \tilde{M}_i$ and $\tilde{M}_i / K$, respectively. We have $e = e'_i \tilde{e}_i$.

By [5], Theorem 8.1, and the fact that $G_{M'_i, \log}^i$ acts trivially on $\mathcal{G}_{\tilde{m}}$ for every $1 \leq i \leq n$, we have that $G_{M'_i, \log}^i$ acts trivially on $\mathcal{G}_{\tilde{m}}$ for every $1 \leq i \leq n'$. Further, since we have $G_{\tilde{M}_i, \log}^{e'_i \tilde{e}_i t} \subseteq G_{M'_i, \log}^{e'_i \tilde{e}_i t}$, we get that $G_{\tilde{M}_i, \log}^{e'_i \tilde{e}_i t}$ acts trivially on $\mathcal{G}_{\tilde{m}}$ for all $1 \leq i \leq n'$. Thus it is enough to prove that

$$\text{Sw}_K(R\Gamma_c(U^\mathcal{R}, \mathcal{G}) \otimes \chi) = \text{rk}(R\Gamma_c(U^\mathcal{R}, \mathcal{G}))\text{Sw}_K(\chi)$$

for $\chi : G_K \to \overline{\mathbb{Q}}_p^\times$ such that $\text{Sw}_K(\chi) > t$. 

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Put \( r = \text{Sw}_K(\chi) \) and denote by \( \tilde{\chi} \) the pullback of \( \chi \) to \( U \). \( \tilde{\chi} \) has clean ramification because the following diagram

\[
\begin{array}{ccc}
\mathfrak{m}_K^r / \mathfrak{m}_K^{r+1} & \xrightarrow{\text{rsw}_K} & \Omega_k(\log) \\
\downarrow & & \downarrow \\
\mathfrak{m}_M^r / \mathfrak{m}_M^{r+1} & \xrightarrow{\text{rsw}_M} & \Omega_{k_i}(\log)
\end{array}
\]

is commutative. Indeed, since \( \chi \) is clean and \( \Omega_k(\log) \twoheadrightarrow \Omega_{k_i}(\log) \) is a splitting injection, \( \text{rsw} \tilde{\chi} \) is a locally splitting injection. Further, by Lemma 3.1, \( \text{Sw}_M(\tilde{\chi}) > t \) for every \( i \). From the Kato-Saito conductor formula, Proposition 3.1, and the fact that \((X, U)\) is semi-stable over \( \mathcal{O}_K \), we have that \( \mathcal{G} \otimes \tilde{\chi} \) is clean and \( \text{Sw}_K R\Gamma_c(U_K, \mathcal{G} \otimes \tilde{\chi}) = \text{deg} \ c_{\mathcal{G} \otimes \tilde{\chi}} = \text{deg} \ c_{\tilde{\chi}}. \)

Again by the Kato-Saito conductor formula,

\[
\text{Sw}_K R\Gamma_c(U_K, \tilde{\chi}) = \text{deg} \ c_{\tilde{\chi}}.
\]

Therefore, we have

\[
\text{Sw}_K R\Gamma_c(U_K, \mathcal{G} \otimes \tilde{\chi}) = \text{Sw}_K R\Gamma_c(U_K, \tilde{\chi}) = \text{Sw}_K R\Gamma_c(U_K, \mathcal{G} \otimes \chi).
\]

Since

\[
\text{Sw}_K R\Gamma_c(U_K, \mathcal{G} \otimes \tilde{\chi}) = \text{Sw}_K R\Gamma_c(U_K, \mathcal{G} \otimes \chi)
\]

and

\[
\text{Sw}_K (R\Gamma_c(U_K, \mathcal{G}) \otimes \chi) = \text{rk} (R\Gamma_c(U_K, \mathcal{G})) \text{Sw}_K(\chi) = \text{rk} (R\Gamma_c(U_K, \mathcal{G})) \text{Sw}_K(\chi),
\]

we conclude that

\[
\text{Sw}_K (R\Gamma_c(U_K, \mathcal{G}) \otimes \chi) = \text{rk} (R\Gamma_c(U_K, \mathcal{G})) \text{Sw}_K(\chi).
\]

**Lemma 4.3.** Let the assumptions be the same as in Lemma 4.2 and assume further that \( \pi_1(U_K) \) does not act trivially on \( \mathcal{G} \). Then

\[
H^j_c(U_K, \mathcal{G}) = 0
\]

for every \( j \neq 1 \).

**Proof.** By Poincaré duality and the fact that \( X \) is of dimension 2, it’s enough to show that \( H^0(U_T, \mathcal{G}) = 0 \). Since \( \pi_1(U_K) \) does not act trivially on \( \mathcal{G}_h \) and \( \text{rk}(\mathcal{G}) = 1 \), we get that \( H^0(U_T, \mathcal{G}) = \mathcal{G}_h^{\pi_1(U_K)} = 0 \).

**Proof of Theorem 4.1.** The theorem has already been proved in Proposition 4.1 for \( \mathcal{G} \) such that \( \pi_1(U_K) \) acts trivially on it, so we assume that \( \pi_1(U_K) \) does not act trivially. By Lemma 4.3, it’s enough to prove that \( \mathcal{G}_h^{\pi_1(U_K, \log)} \) acts trivially on \( H^1_c(U_K, \mathcal{G}) \).
From Lemmas 4.2 and 4.3 it follows that

$$\text{Sw}_L(H^1_c(U_{\mathcal{T}}, \mathcal{G}) \otimes \chi) = \text{rk}(H^1_c(U_{\mathcal{T}}, \mathcal{G})) \text{Sw}_L(\chi)$$

for any totally tamely ramified extension $L$ of $K$ of degree $e$ prime to $p$ and arbitrary character $\chi : G_L \to \overline{\mathbb{Q}}_\ell^\times$ satisfying $\text{Sw}_L(\chi) > et$.

From Proposition 2.1, we have that $G_{K, \log}^{t+}$ acts trivially on $H^1_c(U_{\mathcal{K}}, \mathcal{G})$. Hence $G_{K, \log}^{t+}$ acts trivially on $H^j_c(U_{\mathcal{K}}, \mathcal{G})$ for every $j$.

\[ \square \]

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