Confined Klein-Gordon oscillator in cosmic string spacetime with uniform cosmic screw dislocation; position-dependent mass and torsion effect

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Abstract: We study the Klein-Gordon (KG) oscillator in a cosmic string spacetime with a uniform cosmic screw dislocation (torsion). We start with a confined (in a Cornell-type potential) KG-oscillator and report the torsion effect on the exact energy levels. We observe shifts/dislocations of the energy levels along the torsion’s parameter δ-axis by δ = ℓ/kz; ℓ = 0, ±1, ±2, · · ·. Such energy levels shifts/dislocations manifestly yield energy levels crossings (i.e., occasional degeneracies). Moreover, we observe eminent energy levels clusterings when |δ| >> 1, for each value of the magnetic quantum number ℓ. To find out parallel systems that admit invariance and isospectrality with the confined KG-oscillator, we discuss the KG-oscillator in a deformed cosmic string spacetime background with a uniform cosmic screw dislocation. Such parallel systems are found to inherit the same effects as above. Yet, we suggest a new recipe for position-dependent mass (PDM) KG-oscillator using the PDM-momentum operator of Mustafa and Algadhi [38]. Two PDM illustrative examples are used, a power-law type PDM, and an exponentially growing PDM. For the exponentially growing PDM, we show that such a PDM introduces a Cornell-type confinement as its own byproduct. Hereby, we observe clustering of the energy levels, as the PDM parameter ξ grows up, but no energy levels crossing are found feasible for a fixed torsion parameter value.

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I. INTRODUCTION

In the early universe and as a consequence of vacuum symmetry breaking phase transitions, the grand unified theories have predicted possible topological defects [1–4] that have been investigated in many areas of physics. For example, in condensed matter physics [5], in gravitation [6–8] (where the linear defects are due dislocation (torsion) and curvature (disclinations)), in domain wall [2, 8], in cosmic string [9, 10], in global monopole [11], etc. However, in their work on Volterra distortions and cosmic defects, Puntigam and Soleng [6] have generalized the Volterra distortion to (3+1)-dimensions using differential geometric and gauge theoretical methods and introduced the concept of Volterra defected/distorted spacetime. They have constructed solutions of the Einstein-Cartan field equation that match with the Volterra defected spacetime, where the resulting matter distributions are interpreted as cosmic strings and cosmic dislocations. Hereby, distortions are line-like defects characterized by a delta-function-valued curvature (classified as disclination) and torsion (classified as dislocation) distributions that result in rotational and translational holonomy.

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Dislocation may be in the form of a spiral-type or a screw-type. In the current study, we investigate the torsion effects on the Klein-Gordon (KG) oscillator in cosmic string spacetime with cosmic screw-type dislocation.

The KG-oscillator has been a subject of research studies in recent years ever since the introduction of the Dirac oscillator by Moshinsky and Szczepaniak. It has been studied in the Gödel and Gödel-type spacetime background, in cosmic string spacetime and Kaluza-Klein theory backgrounds, in Som-Raychaudhuri, in the (2+1)-dimensional Gürses spacetime backgrounds. The concept of position-dependent effective mass (PDM in short) settings of Mathews-Lakshmanan oscillator, on the other hand, has sparked research interest on PDM in both classical and quantum mechanics. Such a PDM concept is, in fact, a metaphorical manifestation of coordinate deformation/ transformation. The coordinate transformation, in effect, changes the form of the canonical momentum in classical and the momentum operator in quantum mechanics (e.g., and related references therein). In classical mechanics, for example, negative the gradient of the potential force field is no longer the time derivative of the canonical momentum but it is rather related to the time derivative of the pseudo-momentum (also called Noether momentum).

In quantum mechanics, however, the PDM momentum operator is constructed and used to find the PDM creation and annihilation operators for the PDM-Schrödinger oscillator. It would be interesting, therefore, to investigate the effects of torsion on the PDM KG-oscillator in the (3+1)-dimensional cosmic string spacetime background with cosmic screw-type dislocation.

Nevertheless, attempts were made to include PDM settings in the Dirac and/or KG relativistic equations through the assumption that where denotes the rest mass energy, PDM, the Lorentz scalar potential interaction, and denotes effective PDM in the relativistic wave equation at hand. However, in the current methodical proposal, we depart from such practice and adopt the fact that PDM is a manifestation of coordinate deformation/ transformation that yields, in turn, PDM-quantum mechanical operator (see below). Under such settings, we shall study torsion effects on PDM KG-oscillator in cosmic string spacetime with cosmic screw dislocation. The organization of this paper is in order.

We start, in section 2, with a confined (in a Cornell-type potential, commonly used in heavy quarkonium spectroscopy) KG-oscillator in a cosmic string spacetime background with a uniform cosmic screw-type dislocation and report the torsion effect on the exact energy levels. We found that torsion results in energy levels shifts/dislocations along the torsion parameter axis and consequently energy levels crossings are unavoidable in the process. Energy levels crossings is a phenomenon responsible for electron transfer in protein, it underlies stability analysis in mechanical engineering, and appears in algebraic geometry (e.g., and references cited therein). Moreover, clusterings of energy levels are found eminent when where denotes torsion parameter. To find out parallel systems that admit invariance and isospectrality with the confined KG-oscillator, we discuss (in section 3) the KG-oscillator in a deformed cosmic spacetime background with a uniform cosmic screw dislocation. Such parallel systems are found to inherit the same effects as above. Moreover, we suggest (in section 4) a new recipe for PDM KG-oscillator. Hereby, we use the PDM-momentum operator constructed by Mustafa and Alqadhi and discuss the effects of torsion and PDM settings on the KG-oscillator in a cosmic spacetime with a uniform cosmic screw-type dislocation. Two PDM illustrative examples are used. A power-law type PDM is found to have similar trend of behavior as that for the confined KG-oscillator of section 2. However, for an exponentially growing PDM, we found that the KG-oscillator in confined in its own PDM-byproducted Cornell-type confinement. Yet, an obvious energy levels clustering is observed.
as the PDM parameter, $\xi \geq 0$, grows up from zero, but no energy levels crossings are found feasible for a fixed value of the torsion parameter $\delta$. However, the effect of the torsion parameter $\delta$ on the energy levels, for a fixed PDM parameter $\xi$, is found to maintain the same trend of behavior as that in section 2. Our concluding remarks are given in section 5. To the best of our knowledge, such a methodical proposal has never been reported elsewhere.

II. CONFINED KG-OSCILLATOR IN COSMIC STRING SPACETIME WITH UNIFORM COSMIC SCREW DISLOCATION

The cosmic string spacetime metric that characterizes a Volterra screw dislocation of order three [6, 54] (in $\hbar = c = 1$ units) is given as

$$ds^2 = -dt^2 + dr^2 + r^2 d\varphi^2 + (dz + \delta d\varphi)^2,$$

where $\delta$ denotes a uniform screw-type torsion (i.e., dislocation) parameter and is related to Burger’s vector $b$ through $\delta = b/2\pi$. The covariant and contravariant metric tensors in this case, respectively, read

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (r^2 + \delta^2) & \delta \\ 0 & 0 & \delta & 1 \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & -\frac{\delta}{r^2} \\ 0 & 0 & -\frac{\delta}{r^2} & \left(1 + \frac{\delta^2}{r^2}\right) \end{pmatrix}; \quad \det (g) = -r^2.$$

On the other hand, the Klein-Gordon equation (KG), with a scalar confinement $S(r)$ (i.e., $m \rightarrow m + S(r)$), is given by

$$\frac{1}{\sqrt{-g}} \partial_{\mu} \left( \sqrt{-g} g^{\mu\nu} \partial_{\nu} \Psi \right) = (m + S(r))^2 \Psi.$$  

Moreover, we may now couple the KG-oscillator using Mirza et al.’s recipe [55] and allow the momentum operator to indulge the oscillator through

$$p_{\mu} \rightarrow p_{\mu} + i\eta \chi_{\mu},$$

with $\eta$ denoting the frequency of the oscillator and $\chi_{\mu} = (0, r, 0, 0)$. This would, in effect, transform KG equation into

$$\frac{1}{\sqrt{-g}} \left( \partial_{\mu} + \eta \chi_{\mu} \right) \left[ \sqrt{-g} g^{\mu\nu} \left( \partial_{\nu} - \eta \chi_{\nu} \right) \Psi \right] = (m + S(r))^2 \Psi,$$

and yields

$$\left\{ -\partial_t^2 + \left( \partial_r^2 + \frac{1}{r} \partial_r \right) + \frac{1}{r^2} \partial_{\varphi}^2 + \left( 1 + \frac{\delta^2}{r^2} \right) \partial_z^2 - \frac{2\delta}{r^2} \partial_r \partial_z - \eta^2 r^2 - 2\eta - (m + S(r))^2 \right\} \Psi = 0.$$

A substitution in the form of

$$\Psi(t, r, \varphi, z) = \exp \left( i [\ell \varphi + k_z z - Et] \right) \psi(r) = \exp \left( i [\ell \varphi + k_z z - Et] \right) \frac{R(r)}{\sqrt{r}}$$

would result in

$$R''(r) + \left[ \lambda - \frac{(\ell^2 - 1/4)}{r^2} - \eta^2 r^2 - 2mS(r) - S(r)^2 \right] R(r) = 0,$$

where

$$\lambda = \frac{(\ell^2 - 1/4)}{r^2}.$$
\[ \lambda = E^2 - k_z^2 - 2\eta - m^2; \quad \ell^2 = (\ell - k_z\delta)^2. \] (9)

Obviously, equation (8) represents, with \( S(r) = 0 \), the 2-dimensional radial harmonic oscillator with an effective oscillation frequency \( \eta \) and consequently inherits its textbook eigenvalues

\[ \lambda = 2\eta \left( 2n_r + |\ell| + 1 \right) \iff E^2 = 2\eta \left( 2n_r + |\ell - k_z\delta| + 2 \right) + k_z^2 + m^2 \] (10)

and radial eigenfunctions

\[ R(r) \sim r^{|\ell - k_z\delta|+1/2} \exp \left( -\frac{\eta r^2}{2} \right) L^{(\ell - k_z\delta)} (\eta r^2) \iff \psi(r) \sim r^{(\ell - k_z\delta)} \exp \left( -\frac{\eta r^2}{2} \right) L^{(\ell - k_z\delta)} (\eta r^2). \] (11)

Let us now consider the above KG-oscillator confined in a Cornell type potential

\[ S(r) = ar + \frac{b}{r}, \] (12)

to imply that equation (8) reads

\[ R''(r) + \left[ \hat{\lambda} - \frac{\hat{\gamma}^2 - 1/4}{r^2} - \hat{\omega}^2 r^2 - 2mar - \frac{2mb}{r} \right] R(r) = 0, \] (13)

where

\[ \hat{\lambda} = E^2 - k_z^2 - 2\eta - m^2 - 2ab; \quad \hat{\gamma}^2 = (\ell - k_z\delta)^2 + b^2; \quad \hat{\omega}^2 = \eta^2 + a^2. \] (14)

Equation (14) admits a finite/bounded solution in the form of biconfluent Heun functions in the form of

\[ R(r) \sim r^{|\gamma|+1/2} \exp \left( -\frac{\hat{\omega}^2 r^2 + 2ar}{2\hat{\omega}} \right) H_B \left( 2|\gamma|, \frac{2ma}{\hat{\omega}^{3/2}}, \frac{a^2m^2 + \hat{\lambda} \hat{\omega}^2}{\hat{\omega}^3}, 4mb \sqrt{\hat{\omega}} \sqrt{\hat{\omega} r} \right), \] (15)

where \( H_B (a, \beta, \gamma, \delta, r) \) in the biconfluent Heun polynomial of degree \( 2n_r \geq 0 \). This would immediately suggest that \( \gamma = 2(2n_r + 1) + \alpha \) (this choice is manifested by the fact that when \( a = b = 0 \) the energies in (10) should naturally be recovered), hence

\[ \frac{a^2m^2 + \hat{\lambda} \hat{\omega}^2}{\hat{\omega}^3} = 2 (2n_r + |\gamma| + 1) \iff \hat{\lambda} = 2\hat{\omega} (2n_r + |\gamma| + 1) - \frac{m^2a^2}{\hat{\omega}^2}. \] (16)

In this case, we get the relation for the energy eigenvalues as

\[ E^2 = 2 \left( \sqrt{\eta^2 + a^2} \right) \left( 2n_r + \sqrt{(\ell - k_z\delta)^2 + b^2} + 1 \right) - \frac{m^2a^2}{\eta^2 + a^2} + 2\eta + k_z^2 + m^2 + 2ab. \] (17)

In Figures 1 and 2 we show the effect of the torsion related parameter \( \delta \) on the energy levels of a confined KG-oscillator in a cosmic string spacetime with a uniform cosmic screw dislocation. We clearly observe that the first term under the square root determines the shifts/dislocations in the energy levels at \( \delta = \ell/k_z \), on the \( \delta \)-axis. That is, for negative \( \ell \) values the shifts/dislocations will be in the negative \( \delta \) region, whereas for positive \( \ell \) values the shifts will be in the positive \( \delta \) region. This would, in effect, manifestly yield energy levels crossings (i.e., occasional degeneracies, as shown in figures 1(a), 1(b), and 1(c), with the Cornell confinement). Moreover, in Figures 2(a), 2(b), and 2(c), we observe eminent energy levels clusterings when \( |\delta| > 1 \), for each value of the magnetic quantum number \( \ell = 0, \pm 1, \pm 2, \ldots \).

The effect of the torsion parameter on the energy levels of the confined KG-oscillator in a cosmic spacetime background with a uniform cosmic screw dislocation is clear, therefore.
FIG. 1: We plot the energy levels of (17) versus the torsion parameter $\delta$, for $m = k_z = \eta = 1$, $a = b = 2$ and for (a) $\ell = 0$, $n_r = 0, 1, 2, 3$, (b) $n_r = 1$, $\ell = 0, \pm 1, \pm 2$, and (c) $n_r = 3$, $\ell = 0, \pm 3, \pm 5$.

FIG. 2: We plot the energy levels of (17) versus the torsion parameter $\delta$, without the Cornell confinement, for $m = k_z = \eta = 1$, $a = b = 0$ and for (a) $\ell = 0$, $n_r = 0, 1, 2, 3$, (b) $\ell = 10$, $n_r = 0, 1, 2, 3$, and (c) $\ell = -10$, $n_r = 0, 1, 2, 3$.

III. KG-OSCILLATOR IN DEFORMED COSMIC STRING SPACETIME WITH UNIFORM COSMIC SCREW DISLOCATION: ISOSPECTRALITY AND INVARIANCE

Let the metric that describes the cosmic spacetime background with uniform cosmic screw dislocation be transformed/deformed in such a way that

$$ds^2 \rightarrow d\tilde{s}^2 = -d\tilde{t}^2 + d\tilde{r}^2 + \tilde{r}^2 d\tilde{\varphi}^2 + (d\tilde{z} + \delta d\tilde{\varphi})^2,$$

(18)

where

$$d\tilde{r} = \sqrt{m(r)} dr, \quad \tilde{r} = \sqrt{Q(r)} r, \quad d\tilde{\varphi} = d\varphi, \quad d\tilde{z} = dz \quad d\tilde{t} = dt,$$

(19)

and hence

$$\frac{d\tilde{r}}{dr} \Rightarrow \sqrt{m(r)} = \sqrt{Q(r)} \left[ 1 + \frac{Q'(r)}{2Q(r)} r \right],$$

(20)
to govern the correlation between \(m(r)\) and \(Q(r)\). Then the covariant and contravariant metric tensors (with \(f(r) = Q(r)\) \(r^2\) for economy of notations) in this case, respectively, read

\[
g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & m(r) & 0 & 0 \\ 0 & 0 & (f(r) + \delta^2) & \delta \\ 0 & 0 & \delta & 1 \end{pmatrix} \quad \leftrightarrow \quad g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{m(r)} & 0 & 0 \\ 0 & 0 & \frac{1}{f(r)} & -\frac{\delta}{f(r)} \\ 0 & 0 & -\frac{\delta}{f(r)} & (1 + \frac{\delta^2}{f(r)}) \end{pmatrix} ; \quad \det(g) = -m(r) f(r) . \quad (21)
\]

We now couple the PDM KG-oscillator using the momentum operator (4) of Mirza et al.’s recipe [55] and suggest that \(\chi_\mu = \left(0, \sqrt{m(r)f(r)}, 0 \right)\) to accommodate the new PDM-settings. This would, in effect, transform the KG-oscillator equation (5) into

\[
\left[ \frac{\partial_r}{\sqrt{m(r)f(r)}} \left( \frac{f(r)}{m(r)} \partial_r \right) U(r) \right] - \partial_r^2 + \frac{\partial_2}{f(r)} + \left[1 + \frac{\delta^2}{f(r)} \right] \partial_\varphi^2 - \frac{2 \delta \partial_\varphi \partial_\varphi}{f(r)} - 2\eta - \eta^2 f(r) - (m + S(r))^2 \right] \Psi = 0 . \quad (22)
\]

Which upon the substitution

\[
\Psi(t, r, \varphi, z) = \exp \left( i [\ell \varphi + k_z z - Et] \right) U(r) , \quad (23)
\]
yields

\[
\frac{\partial_r}{\sqrt{m(r)f(r)}} \left( \frac{f(r)}{m(r)} \partial_r \right) U(r) + \left[ \lambda - \frac{\vec{r}^2}{f(r)} - \eta^2 f(r) - 2mS - S(r)^2 \right] U(r) . \quad (24)
\]

Obviously, the first term can be rewritten, with \(f(r) = Q(r)\) \(r^2 = \vec{r}^2\) and \(\partial_\varphi = \frac{1}{\sqrt{m(r)}} \partial_r\), as

\[
\frac{1}{f(r)} \frac{\partial_r}{\sqrt{m(r)}} \left( \frac{1}{\sqrt{m(r)}} \partial_r \right) U(r) = \frac{\partial}{\partial \vec{r}} \frac{\partial}{\partial \vec{r}} U(\vec{r}) = \left( \frac{\partial^2}{\partial \vec{r}^2} + \frac{1}{\vec{r}} \frac{\partial}{\partial \vec{r}} \right) U(\vec{r}) . \quad (25)
\]

To remove the first derivative we may define \(U(\vec{r}) = R(\vec{r}) / \sqrt{\vec{r}}\) to eventually imply

\[
\frac{d^2}{d\vec{r}^2} R(\vec{r}) + \left[ \lambda - V_{eff}(\vec{r}(r)) \right] R(\vec{r}) = 0 ; \quad V_{eff}(\vec{r}) = \frac{\vec{r}^2 - 1/4}{\vec{r}^2} + \eta^2 \vec{r}^2 + 2mS + S(r)^2 . \quad (26)
\]

This would in turn result, with \(R(\vec{r}) = R(\vec{r}(r)) = m(r)^{-1/4} \phi(r),\)

\[
\left( \frac{1}{\sqrt{m(r)}} \frac{d}{dr} \frac{1}{\sqrt{m(r)}} \frac{d}{dr} \right) m(r)^{-1/4} \phi(r) + \left[ \lambda - V_{eff} (\vec{r}(r)) \right] m(r)^{-1/4} \phi(r) = 0 . \quad (27)
\]

Now multiplying from the left by \(m(r)^{1/4}\) we get

\[
\left( m(r)^{-1/4} \frac{d}{dr} m(r)^{-1/2} \frac{d}{dr} m(r)^{-1/4} \right) \phi(r) + \left[ \lambda - V_{eff}(r) \right] \phi(r) = 0 , \quad (28)
\]

where

\[
V_{eff}(r) = \frac{\vec{r}^2 - 1/4}{Q(r) r^2} + \eta^2 Q(r) r^2 + 2mS(r) + S(r)^2 . \quad (29)
\]

One should be reminded that equation (28) is known in the literature as the von Roos PDM-Schrödinger equation [31], with the parametric ordering of Mustafa-Mazharimousavi [34, 37, 38]. Furthermore, it is obvious that for \(S(\vec{r}) = 0\)
equation (26) is isospectral and invariant with that in [8] and admits the same energies in [10]. Yet, for a confining potential in the form of a deformed Cornell type

$$S(\tilde{r}(r)) = a\sqrt{Q(r)r} + \frac{b}{\sqrt{Q(r)r}}.$$  

(30)

the deformed KG-oscillator in the deformed cosmic spacetime background with a uniform cosmic screw dislocation [13] would consequently inherit the eigenvalues [17] and eigen functions [15] of [13]. The two systems [13] and [26] are invariant and are isospectral, therefore.

IV. PDM KG-OSCILLATOR IN COSMIC STRING SPACETIME BACKGROUND WITH UNIFORM COSMIC SCREW DISLOCATION

Recently, Mustafa and Algadhi [38] have introduced, by construction, the PDM-momentum operator

$$\hat{p}(r) = -i\left(\nabla - \frac{\nabla m}{4m(r)}\right) \iff p_j = -i\left(\partial_j - \frac{\partial_j m}{4m(r)}\right).$$  

(31)

In this section, we shall use such PDM-momentum operator and consider a PDM KG-oscillator in a cosmic spacetime background with cosmic screw dislocation of [11]. Therefore, the corresponding inverse metric tensor $g^{\mu\nu}$ is readily given in [2]. Moreover, we shall use the assumption that $m(r) = m(r)$ (i.e., the PDM function is only radially dependent). Under such settings, the momentum operator in [11] would inherit the PDM form so that

$$\hat{p}_\mu \rightarrow -i\partial_\mu + i\mathcal{F}_\mu; \mathcal{F}_\mu = \left(0, \eta r + \frac{m'(r)}{4m(r)}, 0, 0\right) \implies \mathcal{F}_r = \eta r + \frac{m'(r)}{4m(r)}.$$  

(32)

is used to construct the PDM KG-oscillator through

$$\frac{1}{\sqrt{-g}}(\partial_\mu + \mathcal{F}_\mu) \left[\sqrt{-g}g^{\mu\nu}(\partial_\nu - \mathcal{F}_\nu)\right] \Psi = (m + S(r))^2 \Psi.$$  

(33)

Then the PDM KG-oscillator equation (33), with the contravariant metric tensors in [2], would yield

$$\left[\frac{1}{r} \partial_r r \partial_r - \partial_t^2 + \frac{1}{r^2} \partial_\phi^2 + \left(1 + \frac{\delta^2}{r^2}\right) \partial_z^2 - \frac{2}{r^2} \partial_z \partial_r - \mathcal{F}_r - \frac{\mathcal{F}}{r} - \mathcal{F}_r - (m - S(r))^2\right] \Psi = 0.$$  

(34)

We may now use $\Psi(t, r, \phi, z)$ of [7] to obtain

$$R''(r) + \left[\lambda - \frac{2\ell^2 - 1/4}{r^2} - \eta^2 r^2 - 2mS(r) - S(r)^2 + M\right] R(r) = 0$$  

(35)

where $\lambda = E^2 - k_z^2 - 2\eta - m^2$, $\ell^2 = (\ell - k_z \delta)^2$ as in [2], and

$$M(r) = \frac{3}{16} \left(\frac{m'(r)}{m(r)}\right)^2 - \frac{1}{4} \frac{m''(r)}{m(r)} - \frac{m'(r)}{4m(r)} - \eta \frac{m'(r)}{2m(r)}.$$  

(36)

Obviously, for constant mass settings $m(r) = 1$, this equation collapses into that of [8] as should be. Yet, one should notice that when the KG-oscillator frequency $\eta = 0$, equation (33) would describe KG-particles in cosmic string spacetime background with cosmic screw dislocation, in general.

In connection with torsion effect on such PDM KG-oscillators, we provide two illustrative examples: a power-law PDM function $m(r) = Ar^\alpha$, and an exponentially growing PDM function $m(r) = B \exp(\xi r)$. 
A. Example 1: A power law PDM

A power-law PDM in the form of \( m(r) = Ar^\alpha \) would, through (36), imply

\[
M(r) = -\frac{\sigma^2}{16r^2} - \frac{\sigma}{2} \eta.
\]

(37)

Which, in turn, yields

\[
R''(r) + \left[ \mathcal{E} - \frac{(\zeta^2 - 1/4)}{r^2} - \eta^2 r^2 - 2mS(r) - S(r)^2 \right] R(r) = 0,
\]

where

\[
\mathcal{E} = E^2 - k_z^2 - 2\eta - m^2 - \frac{\sigma}{2} \eta; \quad \zeta^2 = (\ell - k_z \delta)^2 + \frac{\sigma^2}{16}.
\]

(38)

(39)

With \( S(r) = 0 \), this equation resembles that of the two dimensional radial Schrödinger oscillator discussed in section 2 (namely, equations (9), (10), and (11)) and admits eigenvalues

\[
\mathcal{E} = 2\eta (2n_r + |\zeta| + 1) \quad \iff \quad E^2 = 2\eta \left( 2n_r + \sqrt{(\ell - k_z \delta)^2 + \frac{\sigma^2}{16}} + 1 \right) + k_z^2 + m^2 + \frac{\sigma}{2} \eta,
\]

(40)

and eigenfunctions

\[
R(r) \sim r^{\zeta|+1/2} \exp \left( -\frac{\eta^2}{2} \right) L_{n_r}^{\zeta} (\eta^2) \quad \iff \quad \psi(r) \sim r^{\zeta} \exp \left( -\frac{\eta^2}{2} \right) L_{n_r}^{\zeta} (\eta^2).
\]

(41)

Next, we now consider the PDM KG-oscillators confined in the Cornell-type potential of (12). This would, in effect, imply that equation (38) be rewritten as

\[
R''(r) + \left[ \mathcal{E} - \frac{(\bar{\zeta}^2 - 1/4)}{r^2} - \bar{\omega}^2 r^2 - 2ma\bar{\omega} - \frac{2mb}{r} \right] R(r) = 0,
\]

(42)

where

\[
\mathcal{E} = \mathcal{E} - 2ab; \quad \bar{\zeta}^2 = \zeta^2 + b^2; \quad \bar{\omega}^2 = \eta^2 + a^2.
\]

(43)

Under such settings, equation (12) admits a finite/bounded solution in the form of biconfluent Heun functions as

\[
R(r) \sim r^{\zeta|+1/2} \exp \left( -\frac{\bar{\omega}^2 r^2 + 2ma\bar{\omega}}{2\bar{\omega}} \right) H_B \left( 2 \zeta, 2ma \frac{a^2 m^2 + \mathcal{E} \bar{\omega}^2}{\bar{\omega}^3}, 4mb \sqrt{\bar{\omega}}, \sqrt{\bar{\omega}r} \right),
\]

(44)

where again \( H_B (\alpha, \beta, \gamma, \delta, r) \) is the biconfluent Heun polynomial of degree \( 2n_r \geq 0 \). This would immediately suggests that \( \gamma = 2 (2n_r + 1) + \alpha \) (this choice is again manifested by the fact that when \( a = b = 0 \) the energies in (10) should naturally be recovered), hence

\[
\frac{a^2 m^2 + \mathcal{E} \bar{\omega}^2}{\bar{\omega}^3} = 2 \left( 2n_r + |\zeta| + 1 \right) \quad \implies \quad \mathcal{E} = 2\bar{\omega} \left( 2n_r + |\zeta| + 1 \right) - \frac{m^2 a^2}{\bar{\omega}^2}.
\]

(45)

In this case, we get the relation for the energy eigenvalues as

\[
E^2 = 2 \left( \sqrt{\eta^2 + a^2} \right) \left( 2n_r + \sqrt{(\ell - k_z \delta)^2 + \frac{\sigma^2}{16} + b^2} + 1 \right) - \frac{m^2 a^2}{\eta^2 + a^2} + 2\eta + k_z^2 + m^2 + 2ab + \frac{\sigma}{2} \eta.
\]

(46)

Obviously, such energy levels inherit the behavior of those of (17) discussed in section 2. Namely, Figures 1 and 2. That is, one may rewrite this energy equation as

\[
E^2 = 2\tilde{\eta} \left( 2n_r + \sqrt{(\ell - k_z \delta)^2 + \tilde{b}^2} + 1 \right) - \frac{m^2 a^2}{\tilde{\eta}^2} + \left( 2 + \frac{\sigma}{2} \right) \eta + k_z^2 + m^2 + 2ab.
\]

(47)

where \( \tilde{\eta} = \sqrt{\eta^2 + a^2} \) and \( \tilde{b}^2 = b^2 + \sigma^2/16 \), to observe that similar trends of behavior are manifested herein.
FIG. 3: We plot the energy levels (51) of the exponentially growing PDM for \( m = k_z = \eta = 1 \). We show in (a) the effect of the PDM parameter \( \xi \) for \( n_r = \ell = 0 \), (b) the effect of the torsion parameter \( \delta \), for \( n_r = 2, \xi = 4, \ell = 0, \pm 1, \pm 2 \), and (c) the effect of the torsion parameter \( \delta \), for \( \ell = 2, \xi = 4, n_r = 0, 1, 2, 3, 4 \).

B. Example 2: A Cornell-type confinement as a byproduct of an exponentially growing PDM

An exponentially growing PDM function \( m(r) = B \exp(\xi r) \) would yield

\[
M(r) = -\frac{\xi^2}{16} - \frac{\xi}{4r} \frac{\xi}{2} \xi r.
\]

Consequently, the PDM KG-oscillator’s equation (35), with \( S(r) = 0 \), reads

\[
R''(r) + \left[ \Sigma - \left( \frac{\tilde{\ell}^2 - 1/4}{r^2} \right) - \eta^2 r^2 \right] - \frac{1}{2} \xi \frac{1}{2} \xi r - 2mS(r) - S(r)^2 \right] R(r) = 0, \tag{49}
\]

where \( \Sigma = E^2 - k_z^2 - 2\eta - m^2 - \xi^2/16 \), and \( \tilde{\ell}^2 = (\ell - k_z \delta)^2 \). It is clear that a Cornell-type confinement (i.e., \( \xi/4r + \xi \eta r/2 \)) is introduced as a byproduct of the PDM settings at hand. We, therefore, continue with \( S(r) = 0 \). This equation (49) admits a finite/bounded solution in the form of biconfluent Heun functions so that

\[
R(r) \sim r^{\frac{1}{2}} \exp \left( -\frac{1}{2} \eta r^2 - \frac{1}{2} \xi r \right) \text{HB} \left( 2 \to |\tilde{\ell}|, \frac{\xi^2 + 16\Sigma}{16\eta}, \frac{\xi}{2\sqrt{\eta}}, \frac{\xi}{2\sqrt{\eta}} \right). \tag{50}
\]

Hence, the corresponding energy levels are given by

\[
\frac{\xi^2 + 16\Sigma}{16\eta} = 2 \left( 2n_r + |\tilde{\ell}| + 1 \right) \Rightarrow E^2 = 2 \eta (2n_r + |\ell - k_z \delta| + 2) + m^2 + \frac{1}{4} \eta \xi^2. \tag{51}
\]

The energy levels are shown in Figure 3. In Figure 3(a), we show the energy levels against the PDM parameter \( \xi \geq 0 \) and observe eminent clustering of the energy levels as \( \xi \) grows up, but no energy levels crossing are found feasible. On the other hand, the torsion’s parameter \( \delta \) effect on the energy levels, for some fixed values of \( \xi \), maintains the same trend of behavior as that associated with (17) and discussed in section 2.

V. CONCLUDING REMARKS

In this work, we have studied the KG-oscillator in a cosmic string spacetime with a uniform cosmic screw dislocation (torsion). We have started with a confined (in a Cornell-type Lorentz scalar potential) KG-oscillator and reported the
torsion effect on the exact energy levels. This effect may very well be summarized as torsion yields shifts/dislocations of the energy levels along the torsion’s parameter $\delta$-axis by $\delta = \ell/k_z; \ell = 0, \pm 1, \pm 2, \cdots$ (documented in Figures 1(b), 1(c), 2(b), 2(c), 3(b), and 3(c)). That is, for negative $\ell$ values the shifts/dislocations will be in the negative $\delta$ region, whereas for positive $\ell$ values the shift will be in the positive $\delta$ region. This in turn manifestly resulted in energy levels crossings (i.e., occasional degeneracies, as shown in figures 1(b), 1(c), and 3(b)). Moreover, in Figures 2(a), 2(b), and 2(c), we have observed eminent energy levels clusterings when $|\delta| >> 1$, for each value of the magnetic quantum number $\ell = 0, \pm 1, \pm 2, \cdots$. In order to find parallel systems that admit invariance and isospectrality with the confined KG-oscillator, we have discussed (in section 3) KG-oscillator in a deformed cosmic string spacetime background with a uniform cosmic screw dislocation. Such parallel systems are found to inherit the same effects discussed above.

Yet, we have suggested (in section 4) a new recipe for the PDM KG-oscillator in cosmic string spacetime background with a uniform cosmic screw dislocation. We have used the PDM-momentum operator constructed by Mustafa and Al gadhi [38] and discussed the effects of PDM setting on the confined KG-oscillator, through two illustrative examples. For a power-law type PDM, the energy levels are shown to have similar trend of behavior as those of (17) discussed in section 2. Whereas, for the exponentially growing PDM, we found that such a PDM setting introduces a Cornell-like confinement. Such a KG-oscillator endowed with an exponentially growing PDM created its own byproducted confinement. Hereby, we have observed the effect of the exponential PDM parameter $\xi \geq 0$ on the energy levels. Obvious clustering of the energy levels are observed, as the PDM parameter $\xi$ grows up, but no energy levels crossing are found feasible for a fixed torsion parameter $\delta$ value (documented in figure 3(a)). Moreover, the effect of the torsion parameter $\delta$ on the energy levels, for a fixed PDM parameter $\xi$, is found to maintain the same trend of behavior as that associated with (17) and discussed in section 2.

Finally, the current methodical proposal may very well be extended to cover a more general case of PDM KG-particles in cosmic string background with uniform cosmic screw-type dislocation. To the best of our knowledge, such a PDM KG-oscillator methodical proposal has never been reported elsewhere.

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