Diffraction by a quarter-plane.
Links between the functional equation, additive crossing
and Lamé functions

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Abstract

In our previous work (Assier & Shanin, QJMAM, 2019), we gave a new spectral formula-
tion in two complex variables associated with the problem of diffraction by a quarter-plane.
In particular, we showed that the unknown spectral function satisfies a condition of additive
crossing about its branch set. In this paper, we study a very similar class of spectral prob-
lem, and show how the additive crossing can be exploited in order to express its solution
in terms of Lamé functions. The solution obtained can be thought of as a tailored vertex
Green’s function whose behaviour in the near-field is directly related to the eigenvalues of the
Laplace–Beltrami operator. This is important since the correct near-field behaviour at the
tip of the quarter-plane had so far never been obtained via a Wiener–Hopf approach.

1 Introduction

We continue to address the problem of wave diffraction by a quarter-plane. This is a three-
dimensional scalar and stationary scattering problem governed by the Helmholtz equation. The
scatterer is a quarter-plane, i.e. it is an obstacle having zero thickness and the shape of a plane
angular sector with opening angle equal to $\pi/2$. Dirichlet boundary conditions are imposed on the
two faces of the scatterer.

A literature review dedicated to this problem can be found in our previous paper [4]. One
of the ways to tackle this problem, which we will continue to develop in the present work, is to
formulate it as a two-complex-variables functional equation of the Wiener–Hopf type. Various
(mostly unsuccessful) attempts (see e.g. [9]) to solve this functional equation were reviewed in [4].

One should note that the two-complex-variables Wiener–Hopf (2DWH) problem differs strongly
from its one-complex-variable analogue (1DWH). The unknown functions for the 1DWH problem
are some functions analytic in the upper or lower half-plane of a single complex spectral argument.
It is known that inverse Fourier transforms of such functions are equal to zero on the positive or
negative half-axis of the real physical coordinate variable. Such unknown functions can hence be
referred to as 1/2-based.

By analogy, for the 2DWH problem one obtains two unknown functions (of two complex spectral
arguments), one of which being 1/4-based, and another being 3/4-based. The 1/4-basedness means
that a 2D inverse Fourier transform of such a function is not zero only on a single quadrant of the physical coordinate plane. A 2D inverse Fourier transform of a 3/4-based function should be zero on one quadrant of the plane and non-zero on the remaining three quadrants.

While the criterion for 1/4-basedness is well-known (the function should be analytic in a product of two half-planes), there is no known criterion for 3/4-basedness. In [4] we introduced the concept of additive crossing of branch lines and showed that it was deeply connected with the 3/4-basedness of the unknown function. However, at this stage, it is not clear how the additive crossing property can be used in practice. With the present work, our aim is to bridge this gap and give a concrete example of usage of the additive crossing property for a simpler but related diffraction problem.

In [4] we considered the problem of diffraction of a plane wave by a quarter-plane. In the present paper we have in mind a slightly different wave propagation problem. The scatterer remains the same (this is a Dirichlet quarter-plane), but there is no incident field. Instead, the vertex condition imposed on the field is weakened: the field can grow as any power of the radius. Such field behaviour corresponds to an arbitrary configuration of sources located at the tip of the quarter-plane. This problem is considered in the spectral domain, i.e. a corresponding functional problem is formulated (it is called here a simplified functional problem to separate it from the functional problem derived for the problem of plane wave diffraction).

For the simplified problem, we show that the usage of the additive crossing property leads to the possibility of writing down the solution in an explicit form, namely in the form of a finite sum of products of Lamé functions. This form agrees perfectly with what can be expected from the point of view of the separation of variables method [10, 7].

The rest of the paper is organised as follows. We formulate the simplified spectral problem in Section 2, while Section 3 is dedicated to solving this problem. More precisely, in Section 3.1 we reformulate the simplified spectral problem using the so-called complex angular coordinates, and show using additive crossing that it can be recast as a stencil equation for a functions of two complex arguments in Section 3.2. The stencil equation is solved by separation of (complex) variables in Section 3.3. As a result, a 1D stencil equation (i.e. a difference equation) is obtained. The difference equation is reduced to an ordinary differential equation (ODE) in Section 3.4, and we show in Section 3.5 that this ODE can be reduced to the Lamé equation. Finally, in Section 4, the solution of the functional equation is transformed into a wave field via Fourier transform. The directivity of the field is studied, and expressed in terms of the same Lamé functions.

2 Problem formulation

2.1 Functional problem for the quarter-plane diffraction problem

As hinted in introduction, the present work is motivated by the canonical problem of diffraction of an incident plane wave \( u^\text{in} \) by a quarter-plane, which we will re-formulate here for completeness. The total field \( u^t \) satisfies the Helmholtz equation

\[
\Delta u^t + k^2 u^t = 0,
\]

in the three-dimensional space \((x_1, x_2, x_3) \in \mathbb{R}^3\), where \(\Delta\) is the three-dimensional Laplacian. The wavenumber parameter \(k\) is assumed to have a non-zero positive real part and a vanishingly small positive imaginary part. The imaginary part of \(k\) can be interpreted as the absorption of the medium. The scatterer is the quarter-plane \(QP \equiv \{(x_1, x_2, x_3), x_{1,2} > 0 \text{ and } x_3 = 0\}\). The total
field \( u^t \) obeys the Dirichlet boundary conditions \( u^t = 0 \) on the faces of the quarter-plane. The geometry of the problem is illustrated in figure 2.1.

The total field \( u^t \) is a sum of the incident field \( u^{in} \) and the scattered field \( u \):

\[
 u^t = u^{in} + u,
\]

and the incident field is a plane wave that can be expressed as

\[
 u^{in} = \exp \left\{ i(k_1 x_1 + k_2 x_2 - (k_2^2 - k_1^2 - k_2^2)^{1/2} x_3) \right\}.
\]

We assume that the wavenumber components of the incident wave are such that \( \text{Re}[k_{1,2}] > 0 \) and \( \text{Im}[k_{1,2}] > 0 \).

For the problem to be well-posed, the scattered field \( u \) should obey:

- the Helmholtz equation (2.1) in the free space,
- the inhomogeneous Dirichlet condition \( u = -u^{in} \) on QP,
- the radiation condition that can be formulated in the form of the limiting absorption principle,
- the edge conditions at the two edges of the quarter-plane: \( x_1 = x_3 = 0, x_2 > 0 \) and \( x_2 = x_3 = 0, x_1 > 0 \),
- the vertex condition at the tip of the quarter-plane.

The edge and vertex conditions take the form of Meixner conditions. They are equivalent to say that the energy-like combination \( |\nabla u^t|^2 + |u|^2 \) should be locally integrable near the edges and the vertex.

As often for diffraction problems, it is convenient to work in the Fourier space. We will consider \( x_{1,2,3} \in \mathbb{R} \) and \( \xi_{1,2} \in \mathbb{C} \) and will denote \( \xi = (\xi_1, \xi_2) \in \mathbb{C}^2 \) and \( x = (x_1, x_2) \in \mathbb{R}^2 \). Introduce the double Fourier transform \( \mathcal{F} \) and its inverse \( \mathcal{F}^{-1} \) defined by

\[
 \mathcal{F}[\phi](\xi, x_3) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x, x_3) e^{i\xi \cdot x} \, dx \quad \text{and} \quad \mathcal{F}^{-1} [\tilde{\Phi}](x) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{\Phi}(\xi) e^{-i\xi \cdot x} \, d\xi
\]

for any suitable physical function \( \phi(x, x_3) \) and spectral function \( \tilde{\Phi}(\xi) \).

In [4], we formulated a functional problem, which can be treated as a 2DWH problem. We introduced the unknown spectral functions \( \tilde{U} \) and \( \tilde{W} \) as

\[
 \tilde{U}(\xi) = \mathcal{F}[u](\xi, 0^+), \quad \tilde{W}(\xi) = \mathcal{F} \left[ \frac{\partial u}{\partial x_3} \right](\xi, 0^+),
\]

(2.2)
and showed that these functions obey the functional equation
\[ \tilde{K}(\xi)\tilde{W}(\xi) = i\tilde{U}(\xi), \] (2.3)
where the kernel \( \tilde{K} \) is defined by
\[ \tilde{K}(\xi) = (k^2 - \xi_1^2 - \xi_2^2)^{-1/2}. \]

Upon denoting \( k = (k_1, k_2) \), the incident plane wave takes the form \( u_{in}(x, x_3) = e^{i(k \cdot x - x_3/\tilde{K}(k))} \).

The unknown functions \( \tilde{U}(\xi) \) and \( \tilde{W}(\xi) \) are initially defined for real \( \xi \), but they can be analytically continued into much wider domains. This analytical continuation was the main subject of [4]. Thus, below, \( \xi \) is considered as a pair of complex variables.

Such functions of two complex variables can have singularities of polar and branching types. Such singularities are not isolated points like for one complex variable functions, but are located on analytic sets, which are surfaces of real dimension 2 embedded in the space \( (\xi_1, \xi_2) \in \mathbb{C}^2 \) having real dimension 4. For convenience, in [4], such sets have been called polar 2-lines and branch 2-lines. We will continue to use the same terminology here.

Let us introduce the domains \( \hat{H}^+ \) and \( \hat{H}^- \) as the upper and lower half-planes of the complex plane. Consider also the domains \( H^\pm \) that are the upper and lower half-planes cut along the cuts \( h^\pm \) as illustrated in figure 2.2. The cuts \( h^\pm \) are the images of the real axis under the mappings \( \xi \to \pm \sqrt{k^2 - \xi^2} \).

![Figure 2.2: The sets \( H^\pm \) and the cuts \( h^\pm \) within the \( \xi \) complex plane](image)

In [4] we introduced the important notion of additive crossing. The simplest possible definition of additive crossing is the following. Let \( D_1 \) and \( D_2 \) be some domains in the \( \xi_1 \) and \( \xi_2 \) complex planes respectively (see figure 2.3). Assume that these domains are cut along the cuts \( \chi_1, \chi_2 \) starting at the points \( d_{1,2} \) and denote the shores of the cuts by the symbols \( r \) (right) and \( \ell \) (left).

Consider \( \tilde{\Phi}(\xi_1, \xi_2) \) to be a function holomorphic in the domain \( (D_1 \setminus \chi_1) \times (D_2 \setminus \chi_2) \) and one-sided continuous on the shores of the cuts (here by cuts, we mean the sets \( \chi_1 \times D_2 \) and \( D_1 \times \chi_2 \)). Let \( \xi_1 \in \chi_1 \) and \( \xi_2 \in (D_2 \setminus \chi_2) \) and denote by \( \tilde{\Phi}(\xi_1^\ell, \xi_2^\ell) \) and \( \tilde{\Phi}(\xi_1^r, \xi_2^r) \) the values of \( \tilde{\Phi} \) on different shores of \( \chi_1 \). If these values are not equal then \( d_1 \times D_2 \) is a branch 2-line of \( \tilde{\Phi} \). Similarly, we can define \( \tilde{\Phi}(\xi_1^r, \xi_2^r) \) and \( \tilde{\Phi}(\xi_1^\ell, \xi_2^\ell) \) for \( \xi_1 \in (D_1 \setminus \chi_1) \) and \( \xi_2 \in \chi_2 \), and if these two quantities are not equal, then \( D_1 \times d_2 \) is also a branch 2-line of \( \tilde{\Phi} \). By continuity, it is hence possible to define the quantities \( \tilde{\Phi}(\xi_1^r, \xi_2^\ell, \xi_1^\ell, \xi_2^r) \).

**Definition 2.1.** We say that such a function \( \tilde{\Phi} \) has the additive crossing property about the branch 2-lines \( d_1 \times D_2 \) and \( D_1 \times d_2 \) if
\[ \tilde{\Phi}(\xi_1^\ell, \xi_2^\ell) + \tilde{\Phi}(\xi_1^r, \xi_2^r) = \tilde{\Phi}(\xi_1^r, \xi_2^\ell) + \tilde{\Phi}(\xi_1^\ell, \xi_2^r). \]
Theorem 2.2. Let $k_{1,2}$ be such that $\text{Re}[k_{1,2}] > 0$ and $\text{Im}[k_{1,2}] > 0$. For any function $\tilde{W}(\xi)$, consider the two associated functions $\tilde{U}(\xi)$ and $\tilde{U}'(\xi)$ defined by

$$\tilde{U}(\xi) = -i\tilde{K}(\xi)\tilde{W}(\xi) \quad \text{and} \quad \tilde{U}'(\xi) = \tilde{U}(\xi) - (\xi_1 + k_1)^{-1}(\xi_2 + k_2)^{-1}.$$  

If the function $\tilde{W}(\xi)$ and the associated function $\tilde{U}'(\xi)$ have the following properties:

FP1 $\tilde{W}$ is holomorphic in the domain $(\hat{H}^+ \times (\hat{H}^+ \cup \hat{H}^- \setminus \{-k_2\})) \cup ((\hat{H}^+ \cup \hat{H}^- \setminus \{-k_1\}) \times \hat{H}^+)$

FP2 $\tilde{W}(\xi)$ has poles (2-lines) at $\xi_1 = -k_1$ and $\xi_2 = -k_2$ with known residues

FP3 The associated function $\tilde{U}'(\xi)$ is holomorphic in the domain $(\hat{H}^- \setminus \{-k_1\}) \times (\hat{H}^- \setminus \{-k_2\})$

FP4 $\tilde{U}'(\xi)$ has the additive crossing property for the 2-lines $\xi_1 = -k$ and $\xi_2 = -k$ with cuts $\chi^-$ (it means that $D_{1,2} = \hat{H}^- \setminus \{-k_{1,2}\}$, $d_{1,2} = -k$ and $\chi_{1,2} = \chi^-$)

FP5 There exist some functions $E_1(\xi_1)$ and $E_2(\xi_2)$, defined for complex $\xi_1$ and $\xi_2$, such that

$$|\tilde{W}(\xi_1, \xi_2)| < E_1(\xi_1)|\xi_2|^{-1/2} \quad \text{as} \quad |\xi_2| \to \infty, \text{Im}[\xi_2] > 0$$

$$|\tilde{W}(\xi_1, \xi_2)| < E_2(\xi_2)|\xi_1|^{-1/2} \quad \text{as} \quad |\xi_1| \to \infty, \text{Im}[\xi_1] > 0$$

FP6 There exists a function $C(\beta, \psi_1, \psi_2)$, defined for $0 < \beta < \pi/2$ and $0 < \psi_1, \psi_2 < \pi$ such that for real $\Lambda$

$$|\tilde{W}(\xi_1, \xi_2)| < C(\beta, \psi_1, \psi_2)\Lambda^{-1-\mu} \quad \text{for some} \quad \mu > -1/2,$$

where $\xi_{1,2}$ are parametrised as follows for large real $\Lambda$

$$\xi_1 = \Lambda e^{i\psi_1} \cos(\beta) \quad \text{and} \quad \xi_2 = \Lambda e^{i\psi_2} \sin(\beta)$$

Then the field $u(\mathbf{x}, x_3)$ defined by $u(\mathbf{x}, x_3) = -i\mathcal{F}^{-1}[\tilde{K}\tilde{W}e^{i|x_3|/K}](\mathbf{x})$ is the sought-after solution to the quarter-plane problem.
From point FP1 it follows that the function $\tilde{W}$ is $1/4$-based (its inverse Fourier transform is non-zero only for $x_1 > 0$ and $x_2 > 0$). Point FP2 is responsible for the incident plane wave. Points FP3 and FP4 are nontrivial. From them it follows that $\tilde{U}'$ is $3/4$-based, i.e. its inverse Fourier transform is equal to zero for $x_1 > 0$ and $x_2 > 0$. Point FP5 is related to the edge conditions, while point FP6 is responsible for the vertex condition. The radiation condition should be fulfilled by construction.

The conditions of Theorem 2.2 form what we will refer to as the functional problem (FP) for $\tilde{W}(\xi)$. By this we mean that the functional problem for $\tilde{W}$ is the following:

$$\text{FP} : \text{Find a function } \tilde{W}(\xi) \text{ obeying the points } \text{FP1–FP6}$$

(2.4)

2.2 A simplified functional problem

The purpose of the present work is to illustrate the significance and practical implications of the additive crossing property. Hence, for simplicity, let us now consider a modified version of the spectral formulation (FP) by making the following simplifications. We will disregard the polar singularities due to $k_{1,2}$ corresponding to the incident wave (affecting points FP1, FP2 and FP3, making the conditions imposed on $\tilde{W}$ stronger). In addition, we will weaken the vertex growth conditions (affecting point FP6 of Theorem 2.2) imposed on $\tilde{W}$, allowing an arbitrary power growth (the parameter $\mu$ should now be considered as arbitrary). Finally, let us abandon the edge growth conditions (point FP5 of Theorem 2.2). Surprisingly, we will see that with the weakened vertex condition, the edge conditions need to be formulated in a slightly different form. This is why we do not consider the edge conditions now, and will return to them later, ultimately using them for selecting the right solutions in Section 3.4.

This results in the following simplified functional problem (SFP):

$$\text{SFP} : \text{Find a function } \tilde{W}(\xi) \text{ obeying the points } \text{SFP1–SFP4},$$

(2.5)

where the four properties are

SFP1 $\tilde{W}$ is analytic in the domain $(\hat{H}^+ \times (\hat{H}^+ \cup H^-)) \cup ((\hat{H}^+ \cup H^-) \times \hat{H}^+)$

SFP2 The function $\tilde{U}(\xi) = -i\tilde{K}(\xi)\tilde{W}(\xi)$ is analytic in the domain $H^- \times H^-$

SFP3 The function $\tilde{U}(\xi)$ has the additive crossing property for the 2-lines $\xi_1 = -k$ and $\xi_2 = -k$ with associated cuts $h^-$

SFP4 There exists a real parameter $\mu$ such that for any $\xi_{1,0}$, $\xi_{2,0}$

$$|\tilde{W}(\Lambda\xi_{1,0}, \Lambda\xi_{2,0})| < \Lambda^{-1-\mu}$$

for large enough $\Lambda > 0$, where the points $\xi_{1,0}$ and $\xi_{2,0}$ are chosen such that $(\Lambda\xi_{1,0}, \Lambda\xi_{2,0})$ remains within the domain of analyticity of $\tilde{W}$.

The present paper is dedicated to the resolution of this simplified functional problem SFP. Below we show that solutions of this functional problem correspond to wave fields generated by some source configurations at the vertex of the quarter-plane. The connection between the solution of the simplified functional problem FP and the simplified functional problem SFP is not completely clear at this stage and is beyond the scope of this work.
2.3 On the function $\tilde{W}(\xi)$ and its associated wave field

A solution $\tilde{W}(\xi)$ of the simplified functional problem SFP corresponds to a wave field $u(x, x_3)$ defined by

$$u(x, x_3) = -\frac{i}{4\pi^2} \int_{\Gamma_\xi} \int_{\Gamma_\xi} \tilde{K}(\xi_1, \xi_2) \tilde{W}(\xi_1, \xi_2) e^{ix_3 K^{-1} (\xi_1, \xi_2)} e^{-i(\xi_1 x_1 + \xi_2 x_2)} d\xi_1 d\xi_2,$$  \hspace{1cm} (2.6)

for $x_3 > 0$, where $\Gamma_\xi$ is just the real segment $(-\infty, \infty)$. This representation is inherited from the definition (2.2). The properties of the integral (2.6) should be considered carefully.

The integrand has no singularities on the surface of integration, i.e. on the real plane, since $k$ has a small positive imaginary part. Formally, the convergence of the integral can be questionable, since $\tilde{W}$ can grow at infinity as an arbitrary power of $|\xi|$. However, note that if $x_3 > 0$ then the integral converges exponentially since $\text{Im}[K^{-1}] > 0$. When $x_3 = 0$, in order for it to remain exponentially convergent, it is necessary to regularise the integral by deforming the contour ever so slightly. This point is addressed in Appendix A.3.

The exponential convergence of the regularised integral (2.6) enables one to differentiate it with respect to the variables $x_{1,2,3}$, which play the role of parameters. One can easily show that $u(x_1, x_2, x_3)$ obeys the Helmholtz equation (2.1) for $x_3 > 0$. This is supported by the elementary observation that (2.6) has the structure of a plane wave decomposition.

Using the methods presented in [4], one can prove that $u(x_1, x_2, 0) = 0$ for $x_1 > 0$ and $x_2 > 0$. This follows from point SFP1 of the simplified functional problem (implying the $1/4$-basedness of $\tilde{W}$). Besides, using point SFP2, one can prove that $\frac{\partial u}{\partial x_3}(x_1, x_2, 0) = 0$ for $x_1 < 0$ or $x_2 < 0$.

By applying the multidimensional saddle-point method [6] (which is not elementary in this case) one can prove that $u(x, x_3)$ obeys the radiation condition for $x_3 > 0$. Intuitively this is clear, since the plane wave decomposition (2.6) contains only waves that are outgoing and decaying for $x_3 \to \infty$. Also it is possible to show that in the area $x_3 = 0$, $\sqrt{x_1^2 + x_2^2} \to \infty$ the field $u$ contains only outgoing waves.

Since we ultimately wish to let $\text{Im}[k] \to 0$, one should be careful to indent the contour of integration of (2.6) properly in order to bypass the singularity defined by the equation $\xi_1^2 + \xi_2^2 = k^2$. Since the contour is a two-dimensional surface embedded in a four-dimensional space, this is non-trivial. In order to facilitate this task, we will make use of the bridge and arrow notation, to which the Appendix B is dedicated.

A consequence of the application of the multidimensional saddle-point method is that the values of $\tilde{W}(\xi_1, \xi_2)$ for the circle $\xi_1^2 + \xi_2^2 = k^2$ play an important role: the directivity pattern of the field (also known as diffraction coefficient) is proportional to the function $\tilde{W}$ taken at these values (see [1], where a similar result is used).

We shall see later that the function $\tilde{W}$ is multivalued. The values corresponding to the directivity (i.e. belonging to the surface of integration within this circle) will be referred to as belonging to the physical sheet of the Riemann manifold of $\tilde{W}$.

Finally, according to general properties of the Fourier transform, the condition SFP4 guarantees that the field has a singularity at the origin no stronger than of power type.

We can hence conclude that, if $\tilde{W}$ satisfies the simplified functional problem SFP formulated above, it corresponds to a mixed homogeneous boundary-value problem on a quarter-plane. The solution has a weakened vertex condition (comparatively to the classical diffraction problem) and abandoned edge conditions (we plan to impose the edge conditions later). Such solution contains only outgoing wave components.
3 Solution of the simplified functional problem

3.1 The simplified functional problem in the angular coordinates

Let us introduce the so-called angular coordinates $\alpha_{1,2}$, linked to the Fourier coordinates $\xi_{1,2}$ by

$$\alpha_{1,2} = \arcsin(\xi_{1,2}/k) \quad \text{and} \quad \xi_{1,2} = k \sin(\alpha_{1,2}). \quad (3.1)$$

Here of course, $\alpha_{1,2}$ are understood to be complex. Let us list the properties of the mapping $\xi \to \alpha$.

The points $\pm k$ located on the physical sheet are mapped to the points $\pm \pi/2$. Indeed, the points with affixes (coordinates) $\xi = \pm k$ located on other sheets of the Riemann surface of the analytical continuation of $\tilde{W}$ are mapped to $\pm \pi/2 + \pi n$.

The shores of the cut $h^+$ are mapped to the curve $m^+$ shown in figure 3.1. The left shore of the cut is mapped to the part of $m^+$ with $\text{Im}[\alpha] > 0$, while the right shore is mapped to the part of $m^+$ with $\text{Im}[\alpha] < 0$. Similarly, the shores of $h^-$ are mapped to the curve $m^-$. This mapping from $m^\pm$ to $h^\pm$ is illustrated on figure 3.1 and summarised in table 1.

| set in $\alpha$ plane | lower part of $m^-$ | upper part of $m^-$ | lower part of $m^+$ | upper part of $m^+$ |
|-----------------------|---------------------|---------------------|---------------------|---------------------|
| set in $\xi$ plane    | left shore of $h^-$ | right shore of $h^-$ | right shore of $h^+$ | left shore of $h^+$ |

Table 1: Summary of which part of $h^\pm$ are mapped to which part of $m^\pm$ by (3.1)

One should also note that the sets $m^\pm$ have the inherent symmetry property that the upper part of $m^\pm$ (such that $\text{Im}[\alpha] \geq 0$) and the lower part of $m^\pm$ (such that $\text{Im}[\alpha] \leq 0$) are the image of each other via the mapping $\alpha \to \pm \pi - \alpha$, as summarised in table 2.

| set in $\alpha$ plane | $m^-$ | $m^+$ |
|-----------------------|-------|-------|
| symmetry mapping linking upper and lower part | $-\pi - \alpha$ | $\pi - \alpha$ |

Table 2: Inherent symmetry of the sets $m^\pm$

As is also shown in figure 3.1, we introduce the contour $\Gamma_\alpha$ as the image of the contour $\Gamma_\xi$ by the mapping (3.1). It is clear that we have $\Gamma_\alpha = m^- + \frac{\pi}{2} = m^+ - \frac{\pi}{2}$. Both $\Gamma_\xi$ and $\Gamma_\alpha$ are shown by a red line in the figure. We can also introduce the curved strips $M^\pm$ as it is shown in the figure. Following from the properties of conformal mappings, it is clear that the domains $H^\pm$ are mapped to $M^\pm$ by (3.1).

In what follows we will use notations like $\pi - M^+$. This particular notation correspond to the set of values of all $\alpha = \pi - \alpha'$ where $\alpha' \in M^+$.

For any function $\tilde{\Phi}(\xi_1, \xi_2)$, we can define a new function $\hat{\Phi}(\alpha_1, \alpha_2)$ by

$$\hat{\Phi}(\alpha_1, \alpha_2) = \tilde{\Phi}(k \sin(\alpha_1), k \sin(\alpha_2)).$$

Using this rule, we define the functions $\hat{W}$, $\hat{K}$ and $\hat{U}$ of $\alpha = (\alpha_1, \alpha_2)$. We can reformulate the simplified functional problem $\text{SFP}$ using the angular coordinates. This leads to a new problem referred to as the angular simplified formulation (ASF), defined as follows:

$$\text{ASF} : \text{Find a function } \hat{W}(\alpha) \text{ obeying the points ASF1–ASF5} \quad (3.2)$$

The following proposition lists the conditions of this new problem.
Proposition 3.1. Let \( \hat{W}(\xi) \) be a function that solves the simplified functional problem \( \text{SFP} \). Then its associated function \( \hat{W}(\alpha) \) has the following properties (they compose the angular simplified formulation \( \text{ASF} \)):

ASF1 \( \hat{W}(\alpha) \) is analytic in the domain \( (M^+ \times (M^+ \cup M^-)) \cup ((M^+ \cup M^-) \times M^+) \)

ASF2 We have \( \hat{W}(\alpha_1, \alpha_2) = \hat{W}(\pi - \alpha_1, \alpha_2) \) and \( \hat{W}(\alpha_1, \alpha_2) = \hat{W}(\alpha_1, \pi - \alpha_2) \) on \( m^+ \times m^+ \).

ASF3 The function \( \hat{U}(\alpha) = -i\hat{K}(\alpha)\hat{W}(\alpha) \) is analytic on \( M^- \times M^- \)

ASF4 The function \( \hat{U}(\alpha) \) satisfies the following equation on \( m^- \times m^- \)

\[
\hat{U}(\alpha_1, \alpha_2) + \hat{U}(-\pi - \alpha_1, -\pi - \alpha_2) = \hat{U}(\alpha_1, -\pi - \alpha_2) + \hat{U}(-\pi - \alpha_1, \alpha_2) \quad (3.3)
\]

ASF5 There exists a constant \( \mu \) such that for real \( \alpha_{1,0}, \alpha_{2,0}, \alpha'_{1,0}, \alpha'_{2,0} \)

\[
\hat{W}(\alpha_{1,0}, \alpha_{2,0}, \alpha'_{1,0}, \alpha'_{2,0}) < \exp\{\mu\Lambda\}
\]

for large enough \( \Lambda \). Parameters \( \alpha_{1,0}, \alpha_{2,0}, \alpha'_{1,0}, \alpha'_{2,0} \) are chosen such that the points \((\alpha_{1,0} + \Lambda\alpha'_{1,0}, \alpha_{2,0} + \Lambda\alpha'_{2,0})\) fall into the domains of analyticity defined in points ASF1 and ASF3, and \((\alpha'_{1,0})^2 + (\alpha'_{2,0})^2 = 1 \).

Reciprocally, if \( \hat{W}(\alpha) \) satisfies these conditions, then the associated function \( \hat{W}(\xi) \) satisfies the simplified functional formulation \( \text{SFP} \).

Proof. The points ASF1, ASF3 and ASF5 are straightforwardly equivalent to the points SFP1, SFP2 and SFP4 respectively, since the sets \( H^\pm \) and the cuts \( h^\pm \) are sent to \( M^\pm \) and \( m^\pm \) via the change of variables \((3.1)\). The point ASF4 comes from the additive property SFP3 satisfied by \( \hat{U}(\xi) \) and from the mapping and symmetry results summarised in tables 1–2. Indeed, let us consider \( \xi^\dagger \) to belong to the left shore of \( h^- \), and let \( \alpha \) be its image by the mapping \((3.1)\) then it is clear that \( \alpha \) belongs to the lower part of \( m^- \). Now the symmetry of \( m^- \) implies that \(-\alpha - \pi \) belongs to the upper part of \( m^- \) and is hence mapped back by \((3.1)\) to the point \( \xi' \) belonging to the right shore of \( h^- \). Hence \( \xi^\dagger_{1,2} \leftrightarrow \alpha_{1,2} \) and \( \xi^\dagger_{1,2} \leftrightarrow -\pi - \alpha_{1,2} \) and the additive crossing property SFP3 of \( \hat{U}(\xi) \) is equivalent to the point ASF4 for \( \hat{U}(\alpha) \).
The point ASF2 can be proved similarly from the analyticity property SFP1 of \( \hat{W} \). Indeed, let us consider \((\xi_1, \xi_2) \in h^+ \times \hat{H}^+\) and refer to \(\xi'_i\) and \(\xi'_1^i\) as being on the left and right shores of \(h^+\) respectively. The function \(\tilde{W}\) satisfies \(\tilde{W}(\xi'_1, \xi_2) = \hat{W}(\xi'_1, \xi_2)\), since it is analytic on \(h^+\) (as a function of \(\xi'_1\)). Let \(\alpha_1 \in M^+\) and \(\alpha_2 \in M^+\) be the image of \(\xi'_1\) and \(\xi_2\) by the mapping (3.1), it is clear from tables 1–2 that \(\alpha_1\) belongs to the upper part of \(m^+\), and that \(\pi - \alpha_1\) is mapped back to \(\xi'_1\). We hence have that \(\hat{W}(\alpha_1, \alpha_2) = \hat{W}(\alpha_1, \pi - \alpha_2)\) on \(M^+ \times M^+\). And hence by continuity, both equalities are simultaneously valid on \(m^+ \times m^+\), as required. 

As discussed in [4], and as could be anticipated from the definition of \(\tilde{K}(\xi)\) for which it is a singular set, the set

\[
\gamma^* = \{ \xi \in C^2 \text{ such that } \xi^2_1 + \xi^2_2 = k^2 \},
\]

that we will refer to as the complexified circle is very important to us. Its image by the mapping (3.1), and hence the singular set of \(\tilde{K}(\alpha)\), is the set of 2-lines \(\sigma^\pm_n\) defined for any \(n \in \mathbb{Z}\) by

\[
\sigma^\pm_n = \{ \alpha \in C^2 \text{ such that } \alpha_1 \pm \alpha_2 = \pi/2 + n\pi \},
\]

(3.4)
corresponding to the \(\alpha_{1,2}\) such that \(\cos^2(\alpha_1) = \sin^2(\alpha_2)\). In order to help with the visualisation of such sets, we will often look at their real trace. The real trace of a singular set \(\gamma\) is a set of real curves describing \(\gamma \cap \mathbb{R}^2\). In Figure 3.2, we show the real traces of \(\gamma^*\) (left) and of some of the \(\sigma^\pm_n\) (right) for real \(k\), i.e. this is the limiting case \(\text{Im}[k] \to 0\).

![Real traces of \(\gamma^*\) (left) and \(\sigma^\pm_n\) (right)](image)

Note that the region \(\xi^2_1 + \xi^2_2 < k^2\) encircled by the real trace of \(\gamma^*\), corresponding to the directivity of the field \(u\), is mapped onto the square \(|\alpha_1| + |\alpha_2| < \pi/2\) shown in figure 3.2, right.

The point ASF2 of Proposition 3.1 is important since it will allow us to analytically continue the function \(\hat{W}(\alpha)\) to a wider domain, as summarised in Proposition 3.2 below.

**Proposition 3.2.** If \(\hat{W}(\alpha)\) satisfies the angular simplified formulation ASF of Proposition 3.1, it can be analytically continued to the domain \(\Omega \cup \Omega_{+-} \cup \Omega_{-+} \cup \Omega_{--}\), where

\[
\Omega = (M^+ \cup M^-) \times (M^+ \cup M^-) \quad \Omega_{+-} = (M^+ \cup M^-) \times (\pi - (M^+ \cup M^-)) \\
\Omega_{-+} = (\pi - (M^+ \cup M^-)) \times (M^+ \cup M^-) \quad \Omega_{--} = (\pi - (M^+ \cup M^-)) \times (\pi - (M^+ \cup M^-))
\]
Note that \( \hat{W}(\alpha) \) will have branch lines within this domain at \( \sigma_1^+, \sigma_2^+, \sigma_1^- \) and \( \sigma_2^- \), where it can be represented as a regular function multiplied by a square root. In particular, we can show that the function

\[
[(\alpha_1 + \alpha_2 + \pi/2)(\alpha_1 + \alpha_2 - 5\pi/2)(\alpha_1 - \alpha_2 - 3\pi/2)(\alpha_1 - \alpha_2 + 3\pi/2)]^{-1/2}\hat{W}(\alpha_1, \alpha_2)
\]

is analytic on \( \Omega \cup \Omega_+ \cup \Omega_- \cup \Omega_{--} \).

**Proof.** Points ASF1 and ASF3 of Proposition 3.1 provide some information on the behaviour of \( \hat{W} \) in the domain \( \Omega \). Namely, ASF1 states that \( \hat{W} \) is holomorphic in \((M^+ \times (M^+ \cup M^-)) \cup ((M^+ \cup M^-) \times M^+)\), while ASF3 states that \( \hat{W} \) has a branch 2-line in \( M^- \times M^- \), and this 2-line is \( \sigma_1^+ \).

The reasons for this being that on \( M^- \times M^- \), we have \( \hat{W} = i\hat{U}/\hat{K}, \hat{U} \) being analytic, and \( \sigma_1^+ \) being the only part of the singularity set of \( \hat{K}(\alpha) \) that belongs to \( M^- \times M^- \).

Another way to see this is that according to ASF3 and the definition of \( \hat{K} \), the product

\[
(\alpha_1 + \alpha_2 + \pi/2)^{-1/2}\hat{W}(\alpha_1, \alpha_2)
\]

is holomorphic in \( M^- \times M^- \). Hence, allowing for this branch 2-line, we can analytically continue \( \hat{W} \) onto \( M^- \times M^- \). This means that overall we have an analytic continuation on

\[
(M^+ \times (M^+ \cup M^-)) \cup ((M^+ \cup M^-) \times M^+)) \cup (M^- \times M^-),
\]

which can easily be seen to be equal to \( \Omega \).

The point ASF2 can now be used to continue \( \hat{W} \) into the domain \( \Omega_{--} \). In order to do so just pick a point in \( \Omega_{--} \), and construct the analytic continuation of \( \hat{W} \) to this point as follows. Start from within \( \Omega \), at \((0, 0)\) say, move to \( m^+ \times m^+ \), and use the first formula to analytically continue \( \hat{W} \) past the \( m^+ \) border of the \( \alpha_1 \) plane until reaching the sought-after point in \( \pi - (M^+ \cup M^-) \). Analytical continuation to \( \Omega_{--} \) can be done similarly by using the second formula of ASF2, while for the analytical continuation to \( \Omega_- \), both formulae in ASF2 should be used simultaneously. The corresponding reflections of the branch 2-line \( \sigma_1^- \) are \( \sigma_1^- \), \( \sigma_2^- \), and \( \sigma_2^+ \). They are also branch 2-lines of the analytical continuation of \( \hat{W} \) and are of the same type as \( \sigma_1^+ \). \( \blacksquare \)

### 3.2 The stencil equation

The function \( \hat{K}(\alpha_1, \alpha_2) \) is branching in the domain of two complex variables \((\alpha_1, \alpha_2) \in \mathbb{C}^2\), moreover, its branch 2-lines \( \sigma_n^\pm \) have one dimensional real traces on the real plane \((\alpha_1, \alpha_2) \in \mathbb{R}^2\) as can be seen in figure 3.2. Let us fix the value of this function on the real plane. In order to do this, we will use the *bridge and arrow* notations defined and developed in Appendix B.

The starting point of fixing the value of \( \hat{K} \) is to find its value at \((\alpha_1, \alpha_2) = (0, 0)\), or, the same, the value \( \hat{K}(0, 0) \). As it is clear from the form of (2.6), one should choose

\[
\hat{K}(0, 0) = k^{-1}
\]

(3.5)

to get the plane waves outgoing for \( x_3 \to \infty \).

The integration surface in the field reconstruction formula (2.6) is the real plane \((\xi_1, \xi_2) \in \mathbb{R}^2\). For \( \text{Im}[k] \to 0 \) the real square \( \xi \in [-k, k]^2 \) is mapped onto the square \( \alpha \in [-\pi/2, \pi/2]^2 \) by (3.1). This square in the real \((\alpha_1, \alpha_2) \) plane is denoted by \( \Gamma' \) and shown in grey in figures 3.3 and B.4. Let us fix the way in which the integration surface bypasses the singularities. The proper bypasses are shown in figure B.4.
The bypass symbol introduced in this way enables one to define the value of \( \hat{K}(-\frac{\pi}{2}, -\frac{\pi}{2}) \). Namely, this value is equal to \(-i k^{-1}\). Hence, remembering that \(( -\frac{\pi}{2}, -\frac{\pi}{2} ) \in m^- \times m^-\), the values of \( \hat{K} \) can be found on the whole set \( m^- \times m^- \) by continuity. Thus, the link \( \hat{U} = \hat{K}\hat{W} \) becomes clarified on \( m^- \times m^- \).

Because of the inherent symmetry of \( m^- \) given in table 2 and the definition of \( \hat{K} \), the following identities are valid on \( m^- \times m^- \):

\[
\hat{K}(\alpha_1, \alpha_2) = \hat{K}(-\pi - \alpha_1, \alpha_2) = \hat{K}(\alpha_1, -\pi - \alpha_2) = \hat{K}(-\pi - \alpha_1, -\pi - \alpha_2). \tag{3.6}
\]

Thus, from point ASF4 of Proposition 3.1 and the definition of \( \hat{U} \), it follows that, on \( m^- \times m^- \), we have

\[
\hat{W}(\alpha_1, \alpha_2) + \hat{W}(-\pi - \alpha_1, -\pi - \alpha_2) = \hat{W}(\alpha_1, -\pi - \alpha_2) + \hat{W}(-\pi - \alpha_1, \alpha_2). \tag{3.7}
\]

Now, remembering that ASF2 was used in order to continue the function \( \hat{W} \) analytically, thus it remains valid for the continued function \( \hat{W} \). Namely, the condition

\[
\hat{W}(\alpha_1, \alpha_2) = \hat{W}(\pi - \alpha_1, \alpha_2) \tag{3.8}
\]

links the values of \( \hat{W} \) on \( m^- \times m^- \) with the values on \((2\pi + m^-) \times m^-\). Similarly, the condition

\[
\hat{W}(\alpha_1, \alpha_2) = \hat{W}(\alpha_1, \pi - \alpha_2) \tag{3.9}
\]

links the values on \( m^- \times m^- \) with the values on \( m^- \times (2\pi + m^-) \). Finally, using these two relations, one can obtain the relation

\[
\hat{W}(\alpha_1, \alpha_2) = \hat{W}(\pi - \alpha_1, \pi - \alpha_2) \tag{3.10}
\]

linking \( m^- \times m^- \) with \((2\pi + m^-) \times (2\pi + m^-) \). However, these three links should be clarified, since the function \( \hat{W} \) is branching.

Note that the function \( \hat{W} \) is single-valued on the sets \( m^- \times m^- \), \((2\pi + m^-) \times m^- \), \( m^- \times (2\pi + m^-) \), \((2\pi + m^-) \times (2\pi + m^-) \), since these do not intersect the branch 2-lines. Thus, one should find four reference points linked by the aforementioned relations, and then continue the relations by continuity. These points can be easily found by taking into account that the point \(( \frac{\pi}{2}, \frac{\pi}{2} ) \) belongs to the physical sheet of \( \hat{W} \). They are the points \((-\frac{\pi}{2}, -\frac{\pi}{2}) \), \((-\frac{\pi}{2}, \frac{3\pi}{2}) \), \((\frac{3\pi}{2}, -\frac{\pi}{2}) \), \((\frac{3\pi}{2}, \frac{3\pi}{2}) \) shown in figure 3.3 and belonging respectively to \( m^- \times m^- \), \((2\pi + m^-) \times m^- \), \( m^- \times (2\pi + m^-) \) and \((2\pi + m^-) \times (2\pi + m^-) \). The figure 3.3 shows the paths along which one can reach the reference points from the point \(( \frac{\pi}{2}, \frac{\pi}{2} ) \) belonging to the physical sheet.

Let us now combine the additive branching relation (3.7) with the analyticity relations (3.8), (3.9), (3.10). As a result, we obtain the so-called stencil equation

\[
\hat{W}(\alpha_1, \alpha_2) + \hat{W}(2\pi + \alpha_1, 2\pi + \alpha_2) - \hat{W}(\alpha_1, 2\pi + \alpha_2) - \hat{W}(2\pi + \alpha_1, \alpha_2) = 0, \tag{3.11}
\]

valid for \((\alpha_1, \alpha_2) \in m^- \times m^- \).

The values of \( \hat{W} \) in (3.11) are chosen by continuity from the reference points shown in figure 3.3. Therefore, we obtain the stencil formulation (StF) of the angular functional formulation ASF, as summarised in following proposition:

**Proposition 3.3.** If \( \hat{W}(\alpha) \) is a function that satisfies the angular formulation ASF of Proposition 3.1, then it has the following properties (they compose the stencil formulation StF):
3.3 Separation of variables for the stencil equation

Let us now focus on finding some solutions obeying the stencil functional problem \( \text{StF} \) formulated above in Proposition 3.3. Note that we are not trying to build a general solution of the stencil functional problem. Instead, we are building a basis of partial solutions, and later on, by comparison with the standard separation of variables for a quarter-plane, the way of constructing a general solution will become clear.

We will start by concentrating solely on items \( \text{StF1} \) and \( \text{StF2} \), that is the domain of analyticity of \( \hat{W}(\alpha) \) and the stencil equation (3.11). Then we will prove that the obtained solutions obey naturally the symmetry conditions of the third item.

Let us take a somewhat similar approach to that of separation of variables. Since the singularities of \( \hat{W} \) are located along the lines \( \sigma_{\pi}^{\pm} \), it seems natural to change the variables from \( \alpha = (\alpha_1, \alpha_2) \) to \( \alpha = (\alpha_1 + \alpha_2, \alpha_1 - \alpha_2) \).
to $\bm{\beta} = (\beta_1, \beta_2)$ defined by

$$\beta_1 \equiv \alpha_1 + \alpha_2 \quad \text{and} \quad \beta_2 \equiv \alpha_1 - \alpha_2. \quad (3.12)$$

The possible singularities are now given by the lines

$$\sigma^+_n : \beta_1 = \pi/2 + \pi n \quad \text{and} \quad \sigma^-_n : \beta_2 = \pi/2 + \pi n,$$

and we can consider the new unknown function $W^\dagger(\bm{\beta})$ defined by

$$W^\dagger(\beta_1, \beta_2) = \hat{W}((\beta_1 + \beta_2)/2, (\beta_1 - \beta_2)/2).$$

Using this change of variables, the stencil equation (3.11) becomes

$$W^\dagger(\beta_1, \beta_2) + W^\dagger(\beta_1 + 4\pi, \beta_2) - W^\dagger(\beta_1 + 2\pi, \beta_2 - 2\pi) - W^\dagger(\beta_1 + 2\pi, \beta_2 + 2\pi) = 0, \quad (3.13)$$

where $\bm{\beta}$ belongs to the image of $m^- \times m^-$ by (3.12) and the point connections and singularities bypasses can be visualised in a similar way as in figure 3.3, but this time in the real $\beta_1, \beta_2$ plane, as can be seen in figure 3.4. An important step of our consideration will be to assume that (3.13) can be continued to all $(\beta_1, \beta_2)$ belonging to the Riemann manifold over $\mathbb{C}^2$ associated to $W^\dagger$.

Figure 3.4: Points connection and singularities bypasses for the $\bm{\beta}$ stencil equation (3.13)

Now, upon dividing (3.13) through by $W^\dagger(\beta_1 + 2\pi, \beta_2)$, we get

$$\frac{W^\dagger(\beta_1, \beta_2) + W^\dagger(\beta_1 + 4\pi, \beta_2)}{W^\dagger(\beta_1 + 2\pi, \beta_2)} = \frac{W^\dagger(\beta_1 + 2\pi, \beta_2 - 2\pi) + W^\dagger(\beta_1 + 2\pi, \beta_2 + 2\pi)}{W^\dagger(\beta_1 + 2\pi, \beta_2)}, \quad (3.14)$$
and looking for separated solutions of the form

\[ W^\dagger(\beta_1, \beta_2) = \Theta(\beta_1)\Psi(\beta_2), \]  

(3.15)
a standard separation of variables argument implies that

\[ \Theta(\beta_1) + \Theta(\beta_1 + 4\pi) - \lambda\Theta(\beta_1 + 2\pi) = 0, \]

(3.16)
\[ \Psi(\beta_2 - 2\pi) + \Psi(\beta_2 + 2\pi) - \lambda\Psi(\beta_2) = 0, \]

(3.17)
where \( \lambda \) is a separation constant. We can hence look for solutions \( W^\dagger(\beta_1, \beta_2; \lambda) = \Theta(\beta_1; \lambda)\Psi(\beta_2; \lambda) \), where \( \Theta(\beta_1; \lambda) \) and \( \Psi(\beta_2; \lambda) \) satisfy (3.16) and (3.17) respectively.

Remember that this has been obtained solely from considering the items StF1 and StF2 of the stencil equation formulation of Proposition 3.3. Let us now focus on the item StF3. Under the \( \beta \) formulation, this item can be rewritten as

\[ W^\dagger(\beta_2 - 2\pi, \beta_1 - 2\pi; \lambda) = W^\dagger(\beta_1, \beta_2; \lambda), \]

(3.18)
\[ W^\dagger(\beta_2 + \pi, \beta_1 - \pi; \lambda) = W^\dagger(\beta_1, \beta_2; \lambda), \]

(3.19)
and should be valid on the image of \( m^- \times m^- \) by (3.12). Conveniently, these symmetry conditions arise naturally from the separated form of our solution when making certain assumptions summarised in the following lemma.

**Lemma 3.4.** The symmetry equations (3.18) and (3.19) are automatically satisfied provided that

\[ \Theta(\beta + \pi; \lambda) = \Psi(\beta; \lambda), \]

(3.20)
\[ \Psi(-\beta; \lambda) = \Psi(\beta; \lambda). \]

(3.21)
Moreover, these two conditions imply that

\[ \Theta(\beta; \lambda) = \Theta(2\pi - \beta; \lambda). \]

(3.22)

**Proof.** In this proof, for brevity, we will drop the \( \lambda \) dependency as it will be assumed throughout. Let us start by proving that (3.18) is satisfied, by first noticing that according to (3.15), we have

\[ W^\dagger(\beta_1, \beta_2) = \Theta(\beta_1)\Psi(\beta_2), \]

(3.23)
\[ W^\dagger(\pi - \beta_2, \pi - \beta_1) = \Theta(\pi - \beta_2)\Psi(\pi - \beta_1). \]

(3.24)
All we need to do now is to prove that the RHS of (3.23) and (3.24) are equal by using the conditions (3.20) and (3.21):

\[ \Theta(\pi - \beta_2)\Psi(\pi - \beta_1) = \Psi(-\beta_2)\Psi(\pi - \beta_1) \]

(3.20)
\[ \Psi(\beta_2)\Psi(\beta_1 - \pi) = \Psi(\beta_2)\Theta(\beta_1), \]

(3.21)
as required. In order to get (3.19), notice that by (3.15), we have

\[ W^\dagger(\beta_2 + \pi, \beta_1 - \pi) = \Theta(\beta_2 + \pi)\Psi(\beta_1 - \pi), \]

(3.25)
hence all we need to do is to show that the RHS of (3.23) and (3.25) are equal to each other by using to conditions (3.20) and (3.21):

\[ \Theta(\beta_2 + \pi)\Psi(\beta_1 - \pi) = \Psi(\beta_2)\Theta(\beta_1), \]

(3.20)
as required. The resulting condition (3.22) at the end of the lemma can also be obtained as follows:

\[ \Theta(\beta) = \Psi(\beta - \pi) = \Psi(\pi - \beta) = \Theta(2\pi - \beta), \]

(3.20)
as required.
Using the application of the separation of variables performed above and the previous lemma, we can hence make further progress in constructing some solutions to the stencil formulation problem $\text{StF}$ as summarised in the following proposition. This constitutes a new functional problem, the one-dimensional stencil formulation $1\text{DSt}$.

**Proposition 3.5.** Let us assume that for some constant $\lambda$ there exists a function $T(\beta) = T(\beta; \lambda)$ satisfying the following properties:

1DSt1 the combination

$$\left( \beta + \pi/2 \right)^{-1/2} \left( \beta - 5\pi/2 \right)^{-1/2} T(\beta)$$

is analytic in the strip $-\pi \leq \text{Re}[\beta] \leq 3\pi$;

1DSt2 $T(\beta)$ obeys the functional equation

$$T(\beta) + T(\beta + 4\pi) - \lambda T(\beta + 2\pi) = 0; \quad (3.27)$$

1DSt3 $T(\beta)$ obeys a growth restriction of the form

$$|T(\beta)| < C \exp\{\kappa |\text{Im}[\beta]|\} \quad (3.28)$$

for some real constants $C$ and $\kappa$;

1DSt4 $T$ obeys the symmetry condition

$$T(\beta) = T(2\pi - \beta). \quad (3.29)$$

Then the function $\hat{W}(\alpha)$ defined by

$$\hat{W}(\alpha_1, \alpha_2) = T(\alpha_1 + \alpha_2; \lambda)T(\alpha_1 - \alpha_2 + \pi; \lambda) \quad (3.30)$$

obeys the stencil formulation $\text{StF}$ of Proposition 3.3.

Any finite linear combination of such $\hat{W}(\alpha_1, \alpha_2)$ taken for different $\lambda$ also obeys the stencil formulation $\text{StF}$.

Here $T(\beta; \lambda)$ is to be understood as playing the role of $\Theta(\beta; \lambda) = \Psi(\beta - \pi; \lambda)$ introduced above. The stencil equation for $T$, the positions of branch points, and the symmetry conditions are coming directly from those for $\Theta$.

A stencil equation for a branching function should be clarified by indicating the positions of the points linked by this equation. To make this clarification, we show in figure 3.5 the reference points $(-\pi, \pi, 3\pi)$ and the paths connecting them in the $\beta$-plane. The other triplets $(\beta, \beta + 2\pi, \beta + 4\pi)$ can be obtained from the reference triplet by continuity.

Naturally, the growth condition (3.28) guarantees the fulfillment of the condition $\text{ASF5}$ of Proposition 3.3.
3.4 Solving one-dimensional stencil equations with ODEs

Solutions of the functional problem $1DSt$ formulated in Proposition 3.5 can be obtained by solving an ordinary differential equation (ODE) of Fuchsian type. This link is summarised in the following theorem, the proof of which can be found in Appendix C.1.

**Theorem 3.6.** Let $T(\beta) = T(\beta; \lambda)$ be a function obeying the conditions $1DSt1$-$1DSt4$ of Proposition 3.5. Then $T(\beta)$ is a solution of an ODE of the form

$$\left[ \frac{d^2}{d\beta^2} + f(\beta) \frac{d}{d\beta} + g(\beta) \right] T(\beta) = 0,$$

(3.31)

where $f(\beta)$ and $g(\beta)$ are rational functions (i.e. ratios of polynomials) of $r = e^{i\beta}$. The coefficients $f$ and $g$ obey the symmetry relations

$$f(-\beta) = -f(\beta), \quad g(-\beta) = g(\beta),$$

(3.32)

and are bounded as $|\text{Im}[\beta]| \to \infty$.

Theorem 3.6 embeds the 1D stencil equations (3.16) and (3.17) into the rich context of Fuchsian equations. Fuchsian ODEs are linear ODEs whose coefficients are rational functions and whose singular points (including infinity) are all regular singular points. Indeed, using the change of variables $\beta \to r = e^{i\beta}$, the equation (3.31) becomes a Fuchsian ODE of the second order. We are not planning to use this equation, but, for completeness, we write its form here:

$$\left[ \frac{d^2}{dx^2} + f^*(x) \frac{d}{dx} + g^*(x) \right] T^*(x) = 0,$$

(3.33)

$$T^*(x) = T(\beta(x)), \quad f^*(x) = \frac{1 - i f(\beta(x))}{x}, \quad g^*(x) = -\frac{g(\beta(x))}{x^2}, \quad \beta(x) = -i \log(x).$$

Such a Fuchsian equation possesses (regular) singular points that are the (polar) singularities of the coefficients of (3.33). These singular points can be divided into two sorts: they may be false or strong (see [13]). A strong singular point is a point at which not only the coefficients $f^*$ and $g^*$ have singularities, but also the solutions (at least one of the two linearly independent solutions) have singularities. At false singular points, conversely, both linearly independent solutions are regular. The removing of false singular points by isomonodromic transformation is the subject of [13].
The only possible strong singular points of equation (3.33) can be found from the behaviour of the functions $T(\beta)$ and $R(\beta) = T(\beta + 2\pi)$. They are $\tau = i, -i, 0, \infty$. As expected, all singular points are regular (in the usual sense). The pairs of exponents at the points $\pm i$ are $(0, 1/2)$ (see the condition 1DSt1 of Proposition 3.5). The exponents of the points $0$ and $\infty$ are unknown a priori.

Let us call the equation (3.33) minimal if it has only strong singular points and let us also call the equation (3.31) minimal if the equation (3.33) resulting from it is minimal. Below we only study the properties of minimal equations (3.31). In particular, we will prove that the wave field $u$ obtained from a minimal equation by the procedure

$$T(\beta) \to \hat{W}(\alpha_1, \alpha_2) \to \hat{W}(\xi_1, \xi_2) \to u(x_1, x_2, x_3)$$

obeys the edge conditions. Moreover, a more detailed study, which falls beyond the scope of the present work, shows that if the equation is not minimal then the resulting wave field $u$ does not obey the edge conditions.

Note that an ODE with four strong regular singular points is Heun’s equation. Unfortunately, no analytical representation for its solution or at least its monodromy matrix is known.

The form of a minimal equation (3.31) is given by the following proposition, the proof of which can be found in Appendix C.2.

**Proposition 3.7.** For the equation (3.31) to be minimal and for its solutions to have the correct behaviour at the points $\beta = \pm \pi/2$, the coefficients $f$ and $g$ have to be

$$f(\beta) = -\frac{1}{2} \tan \beta, \quad g(\beta) = \frac{a}{\cos \beta} + b$$

(3.34)

for some constants $a$ and $b$.

Hence if the equation (3.31) is minimal, it takes the form

$$\left[ \frac{d^2}{d\beta^2} - \frac{\tan \beta}{2} \frac{d}{d\beta} + \frac{a}{\cos \beta} + b \right] T(\beta) = 0,$$  

(3.35)

for some arbitrary constant parameters $a$ and $b$ that should be found from some additional conditions.

Let us now consider the points $\beta = -\pi/2, \pi/2, 3\pi/2$ in more details. All these points are regular singular points of (3.35) with a pair of exponents $(0, 1/2)$. From the theory of ODEs (see e.g. [8]), it is known that at each point there exists a basis of two linearly independent solutions of (3.35), such that the first component of this basis is regular at the corresponding singular point, and the second solution is a regular function multiplied by a square root singularity. Let us write these basis functions in the form

$$B_{-\pi/2}(\beta) = \left( \begin{array}{c} \psi_{1,1}(\beta + \pi/2) \\ (\beta + \pi/2)^{1/2}\psi_{2,1}(\beta + \pi/2) \end{array} \right),$$

$$B_{\pi/2}(\beta) = \left( \begin{array}{c} \psi_{1,2}(\beta - \pi/2) \\ (\beta - \pi/2)^{1/2}\psi_{2,2}(\beta - \pi/2) \end{array} \right),$$

$$B_{3\pi/2}(\beta) = \left( \begin{array}{c} \psi_{1,3}(\beta - 3\pi/2) \\ (\beta - 3\pi/2)^{1/2}\psi_{2,3}(\beta - 3\pi/2) \end{array} \right),$$
where $\psi_{m,n}$ are functions holomorphic in some neighbourhood of zero. Since there can only be two linearly independent solutions of (3.35), the bases can be linearly expressed in terms of each other by connection matrices:

$$B_{\pi/2} = MB_{-\pi/2} = NB_{3\pi/2}$$

for some constant $2 \times 2$ matrices $M = (m_{j,\ell})$ and $N = (n_{j,\ell})$.

The following proposition, the proof of which can be found in Appendix C.3, formulates the restrictions imposed on the solutions of (3.35). These conditions should be satisfied by choosing appropriate values of $a$ and $b$.

**Proposition 3.8.** Let the constants $a$ and $b$ introduced in (3.34) be chosen in such a way that the connection matrices defined in (3.36) for the equation (3.35) have the following properties:

$$m_{1,1} = 0$$

and

$$n_{1,2} = 0.$$  

Then (3.35) has a solution $T(\beta)$ obeying the conditions of Proposition 3.5.

### 3.5 Link between (3.35) and the Lamé equation

The main aim of this section is to highlight the strong link between the stencil equation (that is a result of the additive crossing property) and Lamé equation. The aim being to make a connection between the separated solution of the physical field (treated for example in [7]) and our separation technique in the Fourier space. In order to do so, let us introduce the change of variable

$$\chi = \chi(\beta) = \arccos(\sqrt{2} \cos(\beta/2)), \quad \text{with} \quad \chi \in [\pi/2, 3\pi/2],$$

which maps the segment $[\pi/2, 3\pi/2]$ of $\beta$ onto the segment $[0, \pi]$ of $\chi$. Upon introducing the function $T^L(\chi)$ defined by

$$T^L(\chi(\beta)) = T(\beta),$$

the ODE (3.35) can be written in terms of the $\chi$ variable and becomes

$$\left[ \frac{d^2}{d\chi^2} + \frac{\cos \chi \sin \chi}{2 - \cos^2 \chi} \frac{d}{d\chi} + \frac{\nu(\nu + 1) \sin^2 \chi}{2 - \cos^2 \chi} - \frac{4a}{2 - \cos^2 \chi} \right] T^L(\chi) = 0,$$

where $\nu$ is a constant parameter linked to $b$ by the relations

$$\nu(\nu + 1) = 4b, \quad \nu = \frac{-1 + \sqrt{1 + 16b}}{2}. \quad (3.40)$$

In a very similar way, let us now introduce the change of variable

$$\tau = \tau(\beta) = \arccos(\sqrt{2} \sin(\beta/2)),$$

mapping the segment $[-\pi/2, \pi/2]$ of $\beta$ onto the segment $[0, \pi]$ of $\tau$. Upon introducing the function $T^R(\tau)$ defined by

$$T^R(\tau(\beta)) = T(\beta),$$

the equation (3.35) transforms into

$$\left[ \frac{d^2}{d\tau^2} + \frac{\cos \tau \sin \tau}{2 - \cos^2 \tau} \frac{d}{d\tau} + \frac{\nu(\nu + 1) \sin^2 \tau}{2 - \cos^2 \tau} + \frac{4a}{2 - \cos^2 \tau} \right] T^R(\tau) = 0.$$  

(3.43)

It is, at this stage, important to realise that equations (3.39) and (3.43) are both Lamé differential equations written in their trigonometric forms (see [5]).

The following proposition is valid:
Proposition 3.9. If the values of $a$ and $b$ are chosen such that the condition of Proposition 3.8 are satisfied, then there exist solutions $T^L(\chi)$ and $T^R(\tau)$ of equations (3.39) and (3.43), respectively, such that

\begin{align*}
T^L(\chi) &= T^L(-\chi), & T^L(\chi) &= T^L(2\pi - \chi), \\
T^R(\tau) &= T^R(-\tau), & T^R(\tau) &= -T^R(2\pi - \tau).
\end{align*}

The inverse statement is also valid.

Proof. Let $\chi_0 \in (0, \pi)$ and consider a direct path $P_{\chi_0}$ in the $\chi$ complex plane joining $\chi_0$ to $-\chi_0$. Then the image of this path in the $\beta$ plane, $2\arccos(\sqrt{\frac{1}{2}} \cos(P_{\chi_0}))$, is a closed path starting and ending at $\beta(\chi_0)$ and encircling the point $\beta = \frac{\pi}{2}$ once. Since this point is not a branch point of $T(\beta)$, the value of $T$ is the same at the start and the end of this path, implying that $T^L(\chi_0) = T^L(-\chi_0)$. Similarly, the image in the $\beta$ plane of a path joining $\chi_0$ to $2\pi - \chi_0$ is a closed path encircling $\beta = \frac{3\pi}{2}$, which is not a branch point of $T(\beta)$, implying that $T^L(\chi_0) = T^L(2\pi - \chi_0)$. The second equation can be proven similarly by considering the image of a path joining $\tau_0$ to $2\pi - \tau_0$ encircles the point $\beta = \frac{\pi}{2}$, which is a branch point of $T(\beta)$ with square root behaviour. Hence $T(\beta)$ changes its sign along this path, which leads to $T^R(\tau_0) = -T^R(2\pi - \tau_0)$.

It hence transpires that, by comparison$^1$ to [7], the values $a$, $b$ and the functions $T^L$, $T^R$ obey the Sturm–Liouville problem derived for a flat Dirichlet cone in the sphero-conal coordinates.

4 Interpretation of the solution

4.1 Imposing the edge conditions

Let us reconstruct the wave field using the integral representation (2.6), and consider the singularities of the field on the edges of the quarter-plane, namely the lines $x_1 = 0, x_2 > 0, x_3 = 0$ and $x_1 > 0, x_2 = 0, x_3 = 0$. A usual formulation of a diffraction problem includes Meixner conditions at the edges. These conditions have been skipped so far, and we now return to them.

In the case of a half-plane, the Meixner condition is equivalent to continuity of the field near the edge. If $\rho$ is the distance between the edge and the observation point, which is close to the edge, the field allowed by the Meixner condition behaves as $\sim \rho^{1/2}$, while the first prohibited term of the same symmetry type is $\sim \rho^{-1/2}$.

Thus, in order for the edge condition to be satisfied, it is enough for the integral (2.6) to be convergent at the edges.

Let us prove this for the edge $x_1 > 0, x_2 = 0$, the other edge can be dealt with similarly. The integral for the field at the edge $x_1 > 0, x_2 = x_3 = 0$ takes the form

\begin{equation}
\int_{\Gamma_\xi} \int_{\Gamma_\xi} K(\xi_1, \xi_2) \tilde{W}(\xi_1, \xi_2) e^{-i \xi_1 x_1} d\xi_1 d\xi_2,
\end{equation}

where we recall that $\Gamma_\xi$ is just the real axis. For the edge condition to be satisfied, we demand that this integral converges.

To see this, fix $\xi_2 \in \Gamma_\xi$ and consider the integral over $\xi_1$. Deform the contour of integration in the $\xi_1$ plane to the contour $\zeta$ as shown in figure 4.1. We shift the contour into the lower half-plane

$^1$In particular see Eq. (3.5) and (3.6) of [7] when, in their notations, $k = k' = 1/\sqrt{2}$. 

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since the term $e^{-i\xi_1 x_1}$ decays exponentially there if $x_1 > 0$. The integration over $\xi_2$ is held over the real axis $\Gamma_{\xi}$.

Figure 4.1: Deformation of the integration contour in the $\xi_1$-plane

In figure 4.1, the integration contour $\zeta$ is shown at some distance from $h^-$ for clarity. In fact, we are letting this distance tend to zero so that for each point of $h^-$ there is a small portion of $\zeta$ going from $-i\infty$ to $-k$, and a small portion of $\zeta$ going in the opposite direction.

So, the integral becomes rewritten in the form

$$u(x_1, 0, 0) = -\frac{i}{4\pi^2} \int_{\Gamma_{\xi}} \int_{\zeta} \tilde{K}(\xi_1, \xi_2) \tilde{W}(\xi_1, \xi_2) e^{-i\xi_1 x_1} d\xi_1 d\xi_2. \quad (4.2)$$

Because the function $\tilde{W}(\xi_1, \xi_2)$ grows algebraically, the integral in the $\xi_1$-plane is always convergent. Our aim is to study the convergence of the external integral over $\xi_2$.

Because, as $|\xi_2| \to \infty$, we have $\tilde{K}(\xi_1, \xi_2) = O(1/|\xi_2|)$, one can see that the integral is convergent if

$$\tilde{W}_L(\xi_1, \xi_2) - \tilde{W}_R(\xi_1, \xi_2) = o(1) \text{ as } |\xi_2| \to \infty, \quad (4.3)$$

where $\tilde{W}_L$ and $\tilde{W}_R$ are the values taken, respectively, on the left and and the right shore of the contour $\zeta$ for some $\xi_1 \in h^-$.  

In the angular coordinates $(\alpha_1, \alpha_2)$ introduced in Section 3.1, the integration contours have the shape shown in figure 4.2. Namely, the integration in the $\alpha_1$-plane is held over the contour $m^- + \epsilon$, where $\epsilon$ is a small positive real number, and the limit $\epsilon \to 0$ corresponds to the contour $\zeta$ falling onto $h^-$. The integration in the $\alpha_2$-plane is held over the contour $\Gamma_\alpha = m^- + \pi/2$, which is the image of the real axis under the mapping $\xi \to \alpha$, as discussed in Section 3.1. The contour $m^- + \epsilon$ possesses a symmetry $\alpha \to -\pi + 2\epsilon - \alpha$. In the limit $\epsilon \to 0$, this symmetry corresponds to the formation of a pair of points belonging to the right and the left shore of the cut $h^-$, as summarised in tables 1 and 2.

In the variables $(\alpha_1, \alpha_2)$, the condition (4.3) can be rewritten as:

$$\lim_{\epsilon \to 0} \left[ \tilde{W}(\alpha_1, \alpha_2) - \tilde{W}(-\pi + 2\epsilon - \alpha_1, \alpha_2) \right] = o(1), \quad (4.4)$$

for $\alpha_1 \in m^- + \epsilon$ and $\alpha_2 \in m^- + \pi/2$, as $|\text{Im}[\alpha_2]| \to \infty$.

Now let us show that the condition (4.4) is satisfied for the solution $\hat{W}$ found above in Section 3.4. Using the ODE (3.35) and the restrictions on $a$ and $b$ formulated in Section 3.4, find a value
of $\lambda$ and a corresponding function $T(\beta) = T(\beta; \lambda)$. Remembering that $T(\beta) = \Theta(\beta) = \Psi(\beta - \pi)$, construct $\hat{W}$ as follows:

$$\hat{W}(\alpha_1, \alpha_2) = T(\alpha_1 + \alpha_2)T(\pi - \alpha_1 + \alpha_2).$$

(4.5)

The function $T(\beta)$ has branch points on the real axis at $\beta = -\pi/2$ and $\beta = 5\pi/2$. Cut the complex $\beta$-plane along the intervals $(-\infty, -\pi/2], [5\pi/2, \infty)$ belonging to the real axis and note that $T(\beta)$ is single-valued over this cut plane, where it obeys the symmetry condition (3.29). It is also worth noting that the arguments $\alpha_1 + \alpha_2$ and $\pi - \alpha_1 + \alpha_2$ of the function $T$ in (4.5) belong to the cut plane if $\alpha_1 \in m^- + \epsilon$, $\alpha_2 \in m^- + \pi/2$.

Upon introducing

$$J(\alpha_1, \alpha_2; \epsilon) \equiv \hat{W}(\alpha_1, \alpha_2) - \hat{W}(-\pi + 2\epsilon - \alpha_1, \alpha_2),$$

we obtain

$$J(\alpha_1, \alpha_2; \epsilon) = T(\alpha_1 + \alpha_2)T(\pi - \alpha_1 + \alpha_2) - T(\alpha_1 + \alpha_2 + 2\pi - 2\epsilon)T(-\pi - \alpha_1 + \alpha_2 + 2\epsilon),$$

(4.6)

and we demand that

$$\lim_{\epsilon \to 0} J(\alpha_1, \alpha_2; \epsilon) = o(1) \text{ as } |\text{Im}[\alpha_2]| \to \infty.$$  

(4.7)

Let us reconsider the ODE (3.35) obeyed by $T(\beta)$, and focus on its behaviour as $\beta \to +i\infty$. As we established in Appendix C.3, $T$ can be spanned over a basis composed of two functions $F_{1,2}$ such that

$$F_j(\beta) \sim e^{\kappa_j \beta},$$

where $\kappa_{1,2}$ are given in (C.16). It means that there exist two constants $q_{1,2}$ such that

$$T(\beta) = q_1 F_1(\beta) + q_2 F_2(\beta).$$

(4.8)

Due to the symmetry (3.29), in the lower half-plane of $\beta$ (here we mean the plane cut over the cuts introduced above) we have

$$T(\beta) = q_1 F_1(2\pi - \beta) + q_2 F_2(2\pi - \beta)$$

(4.9)

with the same $q_1$ and $q_2$. 

Figure 4.2: Integration contours in the $\alpha$-planes
Let us now fix $\alpha_1 \in m^- + \epsilon$ and take $\alpha_2 \in m^- + \pi/2$ such that $\text{Im}[\alpha_2] > 0$. After noticing that due to the ansatz (C.14)

$$F_j(\alpha_1 + \alpha_2)F_j(\pi - \alpha_1 + \alpha_2) - F_j(\alpha_1 + \alpha_2 + 2\pi)F_j(-\pi - \alpha_1 + \alpha_2) = 0, \quad j = 1, 2,$$ (4.10)

substitute (4.8) into (4.7) to conclude that

$$\lim_{\epsilon \to 0} J(\alpha_1, \alpha_2; \epsilon) = q_1q_2[F_1(\alpha_1 + \alpha_2)F_2(\pi - \alpha_1 + \alpha_2) + F_2(\alpha_1 + \alpha_2)F_1(\pi - \alpha_1 + \alpha_2)]$$

$$- q_1q_2[F_1(\alpha_1 + \alpha_2 + 2\pi)F_2(-\pi - \alpha_1 + \alpha_2) - F_2(\alpha_1 + \alpha_2 + 2\pi)F_1(-\pi - \alpha_1 + \alpha_2)].$$ (4.11)

Hence, according to (C.14) and (C.16), this expression grows as

$$\lim_{\epsilon \to 0} J(\alpha_1, \alpha_2; \epsilon) \sim \exp\{(\kappa_1 + \kappa_2)\alpha_2\} = \exp\{i\alpha_2/2\}.$$ (4.11)

In the upper half-plane, i.e. $\text{Im}[\alpha_2] \to +\infty$, this is a decay.

Due to the symmetry $\beta \to 2\pi - \beta$ of $T$ the function (4.11) decays as well in the lower half-plane and hence the integral (4.2) is convergent. It is important to note that this consideration is based on the fact that $\kappa_1 + \kappa_2 = i/2$ (see (C.16)) and that this fact only holds if the equation (3.31) is taken to be minimal.

### 4.2 Wave field $u(x, x_3)$ as the solution in the spherico-conal variables

The following theorem describes the link between the solution built in the paper and the classical solution arising from separation of variables in the spherico-conal coordinates.

**Theorem 4.1.** Let $T(\beta) = T(\beta; \lambda)$ be a solution of the ODE inverse monodromy problem formulated in Proposition 3.8 for some triplet $(\lambda, a, b)$, and let the field $u(x, x_3), x_3 > 0$, be defined by (2.6). Then

$$u(x_1, x_2, x_3) = -\frac{i\epsilon^{\nu\pi}}{\sqrt{2\pi}} \frac{1}{\sqrt{kr}} H_{\nu+1/2}^{(1)}(kr) T^R(\tau) T^L(\chi),$$ (4.12)

where $\nu$ is related to $b$ by (3.40), $H_{\nu+1/2}^{(1)}(kr)$ is the Hankel function of the first kind of order $\nu + 1/2$ and $T^R(\tau)$ and $T^L(\chi)$ are related to $T$ by

$$T^R(\tau) = T(2\arcsin(\frac{1}{\sqrt{2}} \cos \chi)) \text{ and } T^L(\chi) = T(2\arccos(\frac{1}{\sqrt{2}} \cos \chi)).$$

The coordinates $(r, \chi, \tau)$ are defined for $0 < \chi < \pi$ and $0 < \tau < \pi$ by

$$x_1 = x_1(r, \chi, \tau) = r \left[ \frac{\cos \chi}{\sqrt{2}} \sqrt{1 - \frac{\cos^2 \tau}{2}} - \frac{\cos \tau}{\sqrt{2}} \sqrt{1 - \frac{\cos^2 \chi}{2}} \right],$$ (4.13)

$$x_2 = x_2(r, \chi, \tau) = -r \left[ \frac{\cos \chi}{\sqrt{2}} \sqrt{1 - \frac{\cos^2 \tau}{2}} + \frac{\cos \tau}{\sqrt{2}} \sqrt{1 - \frac{\cos^2 \chi}{2}} \right],$$ (4.14)

$$x_3 = x_3(r, \chi, \tau) = r \sin \chi \sin \tau.$$ (4.15)
Proof. The proof of this theorem is based on the differential form notations introduced in Appendix A and on the statements proven within this appendix. In particular, we have already shown in Appendix A.1 that the field \( u(x, x_3) \) defined by (2.6) does satisfy the Helmholtz equation. Hence, by rewriting the Laplace operator in spherical coordinates, we obtain

\[
\left[ \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \tilde{\Delta}_\nu \right] u(\nu, r) = 0,
\]

where \( \tilde{\Delta}_\nu \) is the physical Laplace-Beltrami operator defined in Appendix A.2 and \( \nu \) can be considered as a point on the unit sphere (see (A.1)). As discussed in Appendix A the field \( u \) defined by (2.6) can be rewritten as per (A.6) as

\[
u
\]

where \( \tilde{\Gamma} \) is an integration surface that can be slightly deformed in order to ensure the exponential convergence of the integral, as discussed in Appendix A.3. The point \( \omega \) defined in (A.1) belongs to the spectral complexified sphere \( S \) defined in (A.2). The function \( w: S \to \mathbb{C} \), the differential 2-form \( \psi_\omega \) and plane wave function \( p \) are defined in (A.3), (A.4) and (A.5) respectively.

Applying the Laplace-Beltrami operator \( \tilde{\Delta}_\nu \) to the field \( u \), and using Proposition A.2, we obtain

\[
\tilde{\Delta}_\nu u = \int_{\tilde{\Gamma}} \tilde{\Delta}_\omega [p(kr, \nu, \omega)] w(\omega) \psi_\omega,
\]

which, using Proposition A.1 and the exponential convergence of the integral, leads to

\[
\tilde{\Delta}_\nu u = \int_{\tilde{\Gamma}} p(kr, \nu, \omega) \tilde{\Delta}_\omega [w(\omega)] \psi_\omega.
\]

In Section 3.5, we have demonstrated the link between the inverse monodromy problem for equation (3.35) and the Sturm-Liouville problem for equations (3.39) and (3.43). These two equations are the same as that derived in [7], where they emerged as the result of a separation of variables method applied to the Laplace–Beltrami operator and its associated eigenvalue problem, and are obeyed by \( T^R(\tau) \) and \( T^L(\chi) \). Hence, since \( w(\omega) = \frac{k}{4\pi^2} T^R(\tau) T^L(\chi) \), it is clear that \( w \) has to be an eigenfunction of the operator \( \tilde{\Delta}_\omega \). A comparison with [7] implies that the associated eigenvalue is \( -\nu(\nu + 1) \), i.e.

\[
\tilde{\Delta}_\omega w(\omega) = -\nu(\nu + 1)w(\omega),
\]

where \( \nu \) is linked to \( b \) by (3.40) and the separation constant is \( a \). Note that a detailed study of these eigenvalues can be found in [3].

Because of this and (4.18), it transpires that \( u(\nu, r) \) is itself an eigenfunction of the operator \( \tilde{\Delta}_\nu \) with the same eigenvalue, and hence, a separation of variables argument implies that

\[
\left[ r^2 \frac{\partial^2}{\partial r^2} + 2r \frac{\partial}{\partial r} + k^2 r^2 - \nu(\nu + 1) \right] u(\nu, r) = 0.
\]

This equation (the spherical Bessel equation) can be solved explicitly to show that

\[
u
\]

for some unknown functions \( A \) and \( B \). To find \( A \) and \( B \), one should consider the asymptotics of \( u \) as \( r \to \infty \). First of all, since the radiation condition should be satisfied, we must have \( B \equiv 0 \).
Moreover, by applying the multi-dimensional saddle-point method (see e.g. [6] or [1]) to (4.18), and considering solely the leading order, one obtains

$$u(\nu, r) = -2i\pi \frac{e^{ikr}}{kr} w(\nu) + o\left(\frac{1}{kr}\right).$$  

(4.22)

Using the large argument asymptotic formula for the Hankel function, (4.21) leads to

$$u(\nu, r) = -i\sqrt{\frac{2}{\pi}} e^{-\nu\pi} A(\nu) \frac{e^{ikr}}{kr} + o\left(\frac{1}{kr}\right),$$

(4.23)

and comparing (4.23) and (4.22), we obtain

$$A(\nu) = 2\sqrt{\frac{2}{\pi}} e^{\nu\pi} w(\nu).$$

(4.24)

Remembering that because of (A.3), we have

$$w(\nu) = \frac{k}{4\pi^2} T_R(\nu) T_L(\chi),$$

we can input (4.24) into (4.21) to obtain the expected formula (4.12).

\[ \square \]

**Remark 4.2.** As seen above, it is clear that our solution (4.12) satisfies the Helmholtz equation. Moreover, one should note that with the definition of the variables \((r, \chi, \tau)\), the quarter-plane is described by \(\tau = \pi\). Moreover, because of the second equation of (3.45), we have \(T_R(\pi) = 0\), which implies that our solution satisfies the Dirichlet boundary condition on the quarter-plane. Similarly, the other three quadrant of the \(x_3 = 0\) plane are represented by \(\chi = \pi, \tau = 0\) and \(\chi = 0\) respectively. Moreover, because of Proposition 3.9, we clearly have \(\frac{dT}{d\chi}(\pi) = \frac{dT}{d\chi}(0) = \frac{dT}{d\tau}(0) = 0\), which implies that our solution satisfies the Neumann boundary condition on the remaining three quadrants. By construction, this solution also satisfies the radiation condition and the edge condition. Hence, as expected, the solution of the type (4.12) can be thought of as a tailored vertex Green's function, i.e. a function resulting from a point source placed on the vertex of the quarter-plane, that satisfies all the correct boundary, radiation and edge conditions. Note that this solution, as expected, is singular at the origin and behaves like \(r^{-\nu-1}\) as \(r \to 0\) and that \(\nu\) is directly related to an eigenvalue of the Laplace–Beltrami operator. Such Green’s functions, with a source located at a geometric singularity of an obstacle, have proved very useful in diffraction theory. They indeed play a critical role in the derivation of the so-called embedding formulae, which, amongst other achievements, have led to some substantial progress in the quarter-plane diffraction problem (see e.g. [12, 2]).

5 Conclusion

This paper can in some way be considered as a proof of concept. It shows how important the study of functions of several variables in general, and the additive crossing property in particular, can be to diffraction theory.

More precisely, we started with the physical problem of diffraction of a plane wave by a quarter-plane and its associated functional problem \(FP\) (2.4) arising from our previous work [4]. We then considered a simplified functional problem \(SFP\) (2.5) that crucially retained the additive crossing property that existed in \(FP\), but relaxed some of the other assumptions. Using the angular coordinates (3.1), we showed that it resulted in the so-called Stencil equation (3.11). Thanks to the change of variables (3.12) and a separation of variables argument, we successfully embedded our problem within the rich context of Fuchsian ODEs. The property of these ODEs where exploited to construct explicit wave-fields whose Fourier Transforms are solution to the simplified functional
problem SFP. These resulting wave-fields are expressed in terms of some Lamé functions and some eigenvalues of the Laplace–Beltrami operator, and correspond to tailored vertex Green’s functions.

In order to carry out our arguments, it was necessary to introduce and develop the bridge and arrow notations (see Appendix B) that allow us to precisely describe the concept of indentation of contour integrations in \( \mathbb{C}^2 \). It was also necessary to make use of the differential form theory (see Appendix A), and we note that the latter can be extremely useful when trying to build the theory of Sommerfeld integrals in difficult situations (see e.g. [14]).

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A  On Fourier integrals in a 2D domain

A.1  Differential form notation

Upon denoting 
\[ r = \sqrt{x_1^2 + x_2^2 + x_3^2}, \]
let us define \( \nu \in \mathbb{R}^3 \) and \( \omega \in \mathbb{C}^3 \) by

\[
\nu = (\nu_1, \nu_2, \nu_3) = \left( \frac{x_1}{r}, \frac{x_2}{r}, \frac{x_3}{r} \right) \quad \text{and} \quad \omega = (\omega_1, \omega_2, \omega_3) = \left( \frac{\xi_1}{k}, \frac{\xi_2}{k}, \sqrt{\frac{k^2 - \xi_1^2 - \xi_2^2}{k^2}} \right). \tag{A.1}
\]

The points \( \nu \) are real, and they belong to the unit sphere. The points \( \omega \), however, are complex. We will say that they belong to the complexified unit sphere, i.e. to the manifold \( S \) defined by

\[
S = \{(\omega_1, \omega_2, \omega_3) \in \mathbb{C}^3 \mid \omega_1^2 + \omega_2^2 + \omega_3^2 = 1 \}. \tag{A.2}
\]

This manifold is analytic everywhere; it has complex dimension 2 and real dimension 4. If \( \omega_3 \neq 0 \), one can take \((\omega_1, \omega_2)\) as its local coordinates. Otherwise, one should take \((\omega_1, \omega_3)\) (if \( \omega_2 \neq 0 \)) or take \((\omega_2, \omega_3)\) (if \( \omega_1 \neq 0 \)).

One can consider the function \( \tilde{W}(\xi_1, -\xi) \) being defined on some subset of \( S \). More precisely, we will work with the function \( w : S \to \mathbb{C} \) defined by

\[
w(\omega) \equiv \frac{k\tilde{W}(\omega_1 k, -\omega_2 k)}{4\pi^2 i}. \tag{A.3}
\]

The sign of the arguments of \( \tilde{W} \) is chosen for convenience. Initially \( w \) is defined only for \((\omega_1 k, -\omega_2 k) \in \mathbb{R}^2 \), but we should note that the solution \( T \) of the ODE (3.35) is defined on the complex plane almost everywhere (continued along any contour not passing through the singular points). Thus, the same is valid for the combination (3.30). Therefore, \( w \) can be continued analytically onto \( S \) with some branching, i.e. \( \partial w = 0 \) almost everywhere on \( S \). Here we use the differential notation from [11].

Let us introduce the differential 2-form \( \psi_\omega \) on \( S \) by

\[
\psi_\omega \equiv \frac{d\omega_1 \wedge d\omega_2}{\omega_3}. \tag{A.4}
\]

Note that this form is analytic everywhere on \( S \). Indeed, it can be continued to the points of \( S \) with \( \omega_3 = 0 \) by the relations

\[
\frac{d\omega_1 \wedge d\omega_2}{\omega_3} = -\frac{d\omega_1 \wedge d\omega_3}{\omega_2} = \frac{d\omega_2 \wedge d\omega_3}{\omega_1}
\]
valid on \( S \).

Upon introducing the following notation for a plane wave:

\[
p(kr, \nu, \omega) \equiv \exp\{ikr \nu \cdot \omega \} = \exp\{ikr(\nu_1 \omega_1 + \nu_2 \omega_2 + \nu_3 \omega_3)\}, \tag{A.5}
\]

one can easily check that as a function of \( \omega \), \( p \) is analytic everywhere on \( S \). It is also possible to rewrite (2.6) as follows:

\[
\tilde{u}(\nu, r) = \int_{\Gamma} w(\omega) p(kr, \nu, \omega) \psi_\omega, \tag{A.6}
\]

where \( \Gamma \) is the integration surface on \( S \) having real dimension 2 and corresponding to the real plane \( \Gamma_\xi \times \Gamma_\xi \) in the initial \((\xi_1, \xi_2)\)-coordinates. The orientation of the integration surface is chosen
appropriately. Note that (A.6) is a common representation of a three-dimensional wave field, i.e. it is a general plane wave decomposition.

Since all factors in (A.6) are analytic on \( \mathcal{S} \), one can use Stokes’ theorem and deform \( \tilde{\Gamma} \) if necessary. Such a deformation should be a homotopy and it should not cross the singularities of \( w \). In this case, the value of the integral remains unchanged after the contour deformation (see e.g. [11]). This fact is the main benefit of using the differential form notations. We get more possibilities of changing the integration contour comparatively to considering the representation (2.6) as two repeated 1D integrals in \( \mathbb{C} \).

Note that \( p \) is a plane wave obeying the Helmholtz equation
\[
\Delta p + k^2 p = 0. \tag{A.7}
\]
Since \( p \) is the only part of the integrand of (A.6) depending on the physical variables \( (x_1, x_2, x_3) \), we can conclude that the field \( u \) defined by (2.6) also obeys the Helmholtz equation.

A.2 The Laplace–Beltrami operator on \( \mathcal{S} \)

Let us introduce the Laplace-Beltrami operator on \( \mathcal{S} \) by introducing the global complex coordinates \( (\theta_\omega, \varphi_\omega) \) on \( \mathcal{S} \). One possible choice is to use the formulae
\[
\theta_\omega = \arcsin(\sqrt{\omega_1^2 + \omega_2^2}), \quad \varphi_\omega = \arctan(\omega_2 / \omega_1). \tag{A.8}
\]
Indeed, one can take a pair \( (\omega_2, \omega_3) \) or a pair \( (\omega_3, \omega_1) \) instead of the pair \( (\omega_1, \omega_2) \). The change of variable (A.8) is not biholomorphic everywhere, however one can use the ambiguity shown above and take a neighbourhood small enough to make the change of variable locally biholomorphic.

Introduce the Laplace-Beltrami operator by the usual formula. For any function \( \phi(\omega) \) in the neighbourhood
\[
\tilde{\Delta}_\omega[\phi](\omega) = \frac{1}{\sin \theta_\omega} \frac{\partial}{\partial \theta_\omega} \left( \sin \theta_\omega \frac{\partial \phi}{\partial \theta_\omega} \right) + \frac{1}{\sin^2 \theta_\omega} \frac{\partial^2 \phi}{\partial \varphi_\omega^2}. \tag{A.9}
\]
One can show that this definition is coordinate invariant, i.e. the result does not depend on the ambiguity of the coordinate change discussed above.

In the coordinates \( (\theta_\omega, \varphi_\omega) \), the form \( \psi_\omega \) has the following representation:
\[
\psi_\omega = \sin \theta_\omega \, d\theta_\omega \wedge d\varphi_\omega. \tag{A.10}
\]
An important property of the Laplace-Beltrami operator on \( \mathcal{S} \), which is indeed a complexification of a corresponding property on a real sphere, is the following:

Proposition A.1. Let \( \Gamma \) be a two-dimensional integration manifold on \( \mathcal{S} \) with a boundary \( \partial \Gamma \) at infinity, and let \( \phi_1(\omega) \) and \( \phi_2(\omega) \) be some functions holomorphic at the points of \( \Gamma \). If \( \phi_1 \) and \( \phi_2 \) decay exponentially at infinity, then
\[
\int_\Gamma \phi_1(\omega) \tilde{\Delta}_\omega[\phi_2](\omega) \psi_\omega = \int_\Gamma \phi_2(\omega) \tilde{\Delta}_\omega[\phi_1](\omega) \psi_\omega \tag{A.11}
\]
Proof. Consider the 1-form \( \Omega \) in the coordinates \( (\theta_\omega, \varphi_\omega) \) defined by
\[
\Omega = \phi_1 \left[ \frac{\sin \theta_\omega}{\sin \theta_\omega} \frac{\partial \phi_2}{\partial \theta_\omega} d\varphi_\omega - \frac{1}{\sin \theta_\omega} \frac{\partial \phi_2}{\partial \varphi_\omega} d\theta_\omega \right] - \phi_2 \left[ \frac{\sin \theta_\omega}{\sin \theta_\omega} \frac{\partial \phi_1}{\partial \varphi_\omega} d\varphi_\omega - \frac{1}{\sin \theta_\omega} \frac{\partial \phi_1}{\partial \theta_\omega} d\theta_\omega \right]. \tag{A.12}
\]
A detailed consideration shows that this form is coordinate-independent in the sense discussed above. Note in particular that

\[ d\Omega = \phi_1(\omega)\tilde{\Delta}\omega[\phi_2](\omega) - \phi_2(\omega)\tilde{\Delta}\omega[\phi_1](\omega). \]

Applying Stokes’ theorem on manifolds (see e.g [11]), we obtain

\[ \int_{\partial\Gamma} \Omega = \int_{\Gamma} d\Omega, \quad (A.13) \]

and using the exponential decay at infinity, we get \( \int_{\partial\Gamma} \Omega = 0 \) and hence (A.11) is valid.

Besides the Laplace–Beltrami operator \( \tilde{\Delta}\omega \), define a Laplace–Beltrami operator \( \tilde{\Delta}\nu \) on a real sphere. Such operator can be defined explicitly in the usual way using the usual physical spherical variables \( (\theta, \phi) \). The following statement can be checked explicitly in the angular coordinates:

**Proposition A.2.**

\[ \tilde{\Delta}\nu[p(kr, \nu, \omega)] = \tilde{\Delta}\omega[p(kr, \nu, \omega)]. \quad (A.14) \]

### A.3 Integration contours to compute the integral (2.6)

This appendix aims at explaining the slight contour deformations required to ensure that the field integral representation (2.6) remains exponentially convergent. Note first that if \( x_3 > 0 \) then the integral converges exponentially and no contour deformation is required. Thus, it is only necessary to regularise the integral for \( x_3 = 0 \) such that the resulting function is continuous (maybe except at the edges and the vertex of the quarter-plane \( x_3 = 0, x_{1,2} > 0 \)). The regularisation is as follows. Let us assume that \( x_{1,2} > 0 \) and consider the integral (2.6) for \( x_3 > 0 \). Change the integration contour so that the integral can be rewritten

\[ u(x, x_3) = -\frac{i}{4\pi^2} \int_{\gamma} \int_{\gamma} \tilde{K}(\xi_1, \xi_2)\tilde{W}(\xi_1, \xi_2)e^{ix_3\tilde{K}^{-1}(\xi_1, \xi_2)}e^{-i(\xi_1x_1 + \xi_2x_2)}d\xi_1d\xi_2, \quad (A.15) \]

where the contour \( \gamma \) is shown in figure A.1. Note that the tails of \( \gamma \) are slightly bent into the lower half-plane. Such a deformation of the contour can be performed first for \( \xi_1 \) and then for \( \xi_2 \). Cauchy’s theorem and the points SFP1-SFP2 of the simplified functional problem SFP guarantee that the value of the integral is not changing during these deformations. The integral (A.15) converges exponentially for \( x_3 \geq 0 \), thus providing the continuity. For other signs of \( x_{1,2} \) one should bend the tails of the contours (in the upper or lower half-plane) appropriately.

![Figure A.1: Deformation of the integration contour for regularisation of (2.6)](image-url)
B "Bridge and arrow" bypass symbols

B.1 Motivation

Consider the integral representation (2.6) of the wave field and consider the variables \((\alpha_1, \alpha_2)\) defined by (3.1). In these new variables, the resulting integral now reads as

\[
u(x, x_3) = -\frac{ik}{4\pi^2} \int_{\Gamma_\alpha \times \Gamma_\alpha} \exp \left\{ ik \left( -x_1 \sin \alpha_1 - x_2 \sin \alpha_2 + x_3 \sqrt{1 - \sin^2 \alpha_1 - \sin^2 \alpha_2} \right) \right\} \times \frac{\hat{W}(\alpha_1, \alpha_2) \cos \alpha_1 \cos \alpha_2}{\sqrt{1 - \sin^2 \alpha_1 - \sin^2 \alpha_2}} d\alpha_1 \wedge d\alpha_2 \tag{B.1}
\]

where \(\hat{W}\) is defined in Section 3.1 and the contour of integration is a product of two samples of \(\Gamma_\alpha\), also defined in Section 3.1 and illustrated in figure 3.1, right. For the purpose of this appendix, we adopt here the formalism of differential forms. The contour \(\Gamma_\alpha\) is understood to be oriented from left to right. Generally speaking, the integral (B.1) takes the form

\[
\int_{\Gamma} f(\alpha_1, \alpha_2) \, d\alpha_1 \wedge d\alpha_2 \tag{B.2}
\]

where \(\Gamma\) is a smooth (or a piece-wise smooth) oriented manifold of real dimension 2. The integrand \(f(\alpha_1, \alpha_2)\) is assumed to be a regular function of the arguments everywhere except at its singularities, i.e. at some sets \(\sigma_j\) defined for some analytic function \(g_j\) by:

\[
\sigma_j = \{ (\alpha_1, \alpha_2) \in \mathbb{C}^2 \text{ such that } g_j(\alpha_1, \alpha_2) = 0 \} . \tag{B.3}
\]

These sets are also manifolds of real dimension 2, and they can represent either polar sets or branch sets of \(f\). In our case, the branch sets are complexified lines that can be generically defined by

\[
g_j(\alpha_1, \alpha_2) \equiv a_j \alpha_1 + b_j \alpha_2 + c_j. \tag{B.4}
\]

In order for the integral (B.2) to be well-defined, \(\Gamma\) and the \(\sigma_j\) should not intersect:

\[
\Gamma \cap \sigma_j = \emptyset \text{ for all } j.
\]

Let us moreover assume that the “linear” functions introduced in (B.4) have real coefficients \(a_j, b_j\) and \(c_j\). It implies that the intersection of the real \((\alpha_1, \alpha_2)\)-plane and \(\sigma_j\) (which is what we referred to earlier as the real trace of \(\sigma_j\)) is a set of real dimension 1.

Let us consider that \(\Gamma\) is a manifold such that a part of it is close to the real \((\alpha_1, \alpha_2)\)-plane, in the sense that it can be parametrised by the values \(\text{Re}[\alpha_1]\) and \(\text{Re}[\alpha_2]\) by equations of the type

\[
\text{Im}[\alpha_1] = \eta(\text{Re}[\alpha_1], \text{Re}[\alpha_2]), \quad \text{Im}[\alpha_2] = \zeta(\text{Re}[\alpha_1], \text{Re}[\alpha_2]) \tag{B.5}
\]

where \(\eta\) and \(\zeta\) are small real continuous functions equal to zero everywhere except in the narrow vicinity of the branch sets \(\sigma_j\).

The detailed shape of the functions \(\eta\) and \(\zeta\) is not important due to the generalisation of the Cauchy theorem (or the Stokes theorem) in \(\mathbb{C}^2\) (see e.g [11] for details). However, it is important, whether \(\Gamma\) passes above or below the branch sets \(\sigma_j\). Of course, what is meant by above and below is not clear in \(\mathbb{C}^2\). The aim of this appendix is to introduce these concepts precisely.
We are here trying to describe the $\mathbb{C}^2$ equivalent of a situation that is very common in $\mathbb{C}$ in diffraction theory. Often when for example taking an inverse Fourier transform, one has to carefully make sure that the singularities of the integrand, located on the real axis, are not hit by the contour of integration. In order to do so, one should *indent* the contour either above or below the singularities (poles or branch points). The choice of indentation (above or below) is made based on physical considerations (typically linked to the radiation condition).

### B.2 The bridge and arrow notation

We will now introduce a diagrammatic notation, the “bridge and arrow” notation, in order to precisely illustrate how such manifold $\Gamma$ is located with respect to the singular sets $\sigma_j$. An example of such notation is shown in Fig. B.1.

![Diagram](image)

**Figure B.1:** Bridge and arrow notation. Left: bypass from above, right: bypass from below.

The plane of the figure is the real $(\alpha_1, \alpha_2)$ plane. The dashed line is the real trace of a singular complexified line $\sigma$. In the plane of the figure this is the straight line

$$a \operatorname{Re}[\alpha_1] + b \operatorname{Re}[\alpha_2] + c = 0$$

for some real coefficients $a$, $b$ and $c$.

Let us consider a point $\alpha^* = (\alpha_1^*, \alpha_2^*)$ belonging to the dashed line (i.e. to the real trace of $\sigma$) and introduce a local coordinate system $(z, t)$, with origin at this point defined for arbitrary real constants $\beta', \beta''$ and $\beta'''$ by

$$z = \beta'(a\alpha_1 + b\alpha_2 + c) \quad \text{and} \quad t = \beta''(-b\alpha_1 + a\alpha_2 + c^*) + \beta'''z \quad \text{with} \quad c^* = \frac{(a^2 + b^2)\alpha_1^* + ac}{b}. \quad (B.6)$$

Hence, in the real plane, the unit basis vectors $\hat{e}_z$ and $\hat{e}_t$ associated with these coordinates are given by $\hat{e}_z = \frac{\hat{e}_z}{|\hat{e}_z|}$ and $\hat{e}_t = \frac{\hat{e}_t}{|\hat{e}_t|}$, where $\hat{e}_z = \beta' \hat{n}$ and $\hat{e}_t = \beta'' \hat{t} + \beta''' \hat{n}$ with $\hat{n}$ and $\hat{t}$ being vectors normal and tangent to the dashed line respectively, given by $\hat{n} = (a, b)$ and $\hat{t} = (-b, a)$.

Let us now rewrite the surface parametrisation equations (B.5) in terms of the new variables:

$$\operatorname{Im}[z] = \eta'(\operatorname{Re}[z], \operatorname{Re}[t]), \quad \operatorname{Im}[t] = \zeta'(\operatorname{Re}[z], \operatorname{Re}[t]), \quad (B.7)$$

for some real and continuous functions $\eta'$ and $\zeta'$. For a fixed value of $\operatorname{Re}[t]$, it is clear that we must have

$$\eta'(0, \operatorname{Re}[t]) \neq 0. \quad (B.8)$$

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Otherwise, the point \((z, t) = (0, \text{Re}[t] + i \zeta'(0, \text{Re}[t]))\) would belong to \(\Gamma\), but this is impossible, since all points with \(z = 0\) belong to the singular set \(\sigma\).

For the value \(\text{Re}[t]\) fixed above, draw the graph of the function \(\text{Im}[z] = \eta'(\text{Re}[z], \text{Re}[t])\) as a function of \(\text{Re}[z]\). This function bypasses the point \((z, t) = (0, \text{Re}[t] + i \zeta'(0, \text{Re}[t]))\) (the origin of the complex \(z\)-plane) either from above \((\eta'(0, \text{Re}[t]) > 0)\) or below \((\eta'(0, \text{Re}[t]) < 0)\) as illustrated in Fig. B.1.

Now, if we let \(\text{Re}[t]\) vary continuously, the point \((z, t) = (0, \text{Re}[t] + i \zeta'(0, \text{Re}[t]))\) will move continuously along \(\sigma\). Since the function \(\eta'(0, \text{Re}[t])\) is continuous and non-zero, its sign will remain the same. Hence the graph of \(\text{Im}[z] = \eta'(\text{Re}[z], \text{Re}[t])\) considered as a function of \(\text{Re}[z]\) either bypasses \(\sigma\) from below for all \(\text{Re}[t]\), or from above for all \(\text{Re}[t]\).

We can hence introduce the bridge and arrow notation as follows. Pick a point on the real trace of \(\sigma\) and start by drawing the real \(z\) direction in the real \((\alpha_1, \alpha_2)\) plane in the neighbourhood of this point. This should locally resemble a straight line intersecting the real trace of \(\sigma\). Then, draw the imaginary axis of \(z\) normally to its real axis such that they intersect at the point picked earlier. The local fragment of the \(z\)-complex plane hence drawn does not belong to the real \((\alpha_1, \alpha_2)\) plane, but intersects it. In this fragment of the \(z\) complex plane, draw the function \(\text{Im}[z] = \eta'(\text{Re}[z], \text{Re}[t])\) for the fixed value \(\text{Re}[t]\) corresponding to the selected point of the real trace of \(\sigma\) using a thick curve. If this thick curve is above (below) the \(\text{Re}[z]\) axis, then we say that \(\Gamma\) bypasses \(\sigma\) from above (below).

### B.3 Usage rules

The symbols introduced above can be translated along \(\sigma\), rotated up to the directions parallel to \(\sigma\), and inverted without changing of the meaning of the symbol, i.e. without changing the value of the integral. Note that the limitation in the rotation comes from the fact that \(\text{Re}[z]\) cannot be aligned with \(\sigma\) without \(\Gamma\) and \(\sigma\) intersecting. Namely, all symbols in Fig. B.2 (left) correspond to the same bypass of \(\sigma\) by \(\Gamma\), while the symbols in Fig. B.2 (right) correspond to the other way of bypassing \(\sigma\). Translations and rotations should be clear from the figure. An inversion is obtained by changing \(z \rightarrow -z\) without changing of the “bridge”.

![Figure B.2: Possible translations, rotations, and inversions of the bridge and arrow symbol](image)

The proof of the possibility to translate and to rotate the symbol follows from the continuity. The inversion rule comes from the fact that changing \(z \rightarrow -z\) just corresponds to changing \(\beta''\) to
\(-\beta'\) in (B.6) and \(\eta'(\text{Re}[z], \text{Re}[t])\) to \(-\eta'(-\text{Re}[z], \text{Re}[t])\) in (B.7) and does not affect the location of \(\Gamma\) with respect to \(\sigma\).

A remark should be made about possible crossings of singular sets. Let two complexified lines \(\sigma_1\) and \(\sigma_2\) have an intersection at some point of the real \((\alpha_1, \alpha_2)\) plane. One should note that

i) the bypass symbols for \(\sigma_1\) and \(\sigma_2\) are independent;

ii) the bypass symbol for \(\sigma_1\) say is the same on the entire real trace of \(\sigma_1\) (the crossing cannot lead to a change of symbol).

An example of such situation is given in Fig. B.3. The reason behind these two properties is that the intersection point does not separate \(\sigma_1\) into two distinct parts. Hence the continuity argument remains valid.

An example of such situation is given in Fig. B.3. The reason behind these two properties is that the intersection point does not separate \(\sigma_1\) into two distinct parts. Hence the continuity argument remains valid.

Figure B.3: Bypass symbols in the vicinity of singular sets intersection

In the vicinity of the intersection of \(\sigma_1\) and \(\sigma_2\) it is possible to introduce the local coordinates \(z_1 = g_1(\alpha_1, \alpha_2), z_2 = g_2(\alpha_1, \alpha_2)\) such that in this neighbourhood, \(\Gamma\) is a product of two “bridges” corresponding to two bypass symbols.

### B.4 Bypass symbols for the considered problem

More specifically, in Section 3.1, because of the integral (B.1), we are interested in the behaviour of the function \(\bar{W}(\alpha_1, \alpha_2)\) in the close vicinity of the real square \(\Gamma' = [-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]\). On this real set we have shown in Proposition 3.2 that \(\bar{W}(\alpha_1, \alpha_2)\):

i) is singular on the real trace of the complexified line \(\sigma_{+1}^+ = \{\alpha \in \mathbb{C}^2, \alpha_1 + \alpha_2 = -\frac{\pi}{2}\}\), where it changes its sign (it behaves like a square root);

ii) is regular on the real trace of the complexified lines \(\sigma_0^- = \{\alpha \in \mathbb{C}^2, \alpha_1 - \alpha_2 = \frac{\pi}{2}\}\) and \(\sigma_0^+ = \{\alpha \in \mathbb{C}^2, \alpha_1 + \alpha_2 = \frac{\pi}{2}\}\) and \(\sigma_{-1}^- = \{\alpha \in \mathbb{C}^2, \alpha_1 - \alpha_2 = -\frac{\pi}{2}\}\);
iii) has no other singularities in the vicinity of $\Gamma'$.

The problematic part of the integrand in (B.1) involves the function

$$\hat{K}(\alpha_1, \alpha_2)\hat{W}(\alpha_1, \alpha_2) = \frac{\hat{W}(\alpha_1, \alpha_2)}{k\sqrt{1 - \sin^2(\alpha_1) - \sin^2(\alpha_2)}}.$$ 

Since this function has some singularities on $\Gamma'$, it is important to analyse the bypass symbols associated to the manifold $\Gamma$ when it is very close to the square $\Gamma'$. More precisely, we should specify the bypass symbols using the bridge and arrow notation for the complexified lines $\sigma_0^\pm$ and $\sigma_{-1}^\pm$. Using first a wavenumber $k$ with a small positive imaginary part, studying the position of resulting singularities close to $\Gamma'$ and letting $\text{Im}[k] \to 0$, one can show that the symbols described in figure B.4 should be used.

![Figure B.4: Bypass symbols associated with $\Gamma$ in the vicinity of the real square $\Gamma'$](image)

### C Proofs of theorems related to ODEs

#### C.1 Proof of Theorem 3.6

Since $T$ has a square root behaviour at $\beta = 5\pi/2$, it is clear that this point is a branch point of $T$. Let us start by changing the path connecting the reference points in figure 3.5 as it is shown in figure C.1. Since the branch point $\beta = 5\pi/2$ is bypassed in a different way now, one should change the sign of the corresponding term of the stencil equation. The new version of the stencil equation reads:

$$T(\beta + 4\pi) = T(\beta) - \lambda T(\beta + 2\pi).$$

(C.1)

Initially, we assume that this equation is valid for $\text{Re}[\beta] = -\pi$, and then we continue this equation onto the whole complex plane.

The aim of the change of path and, respectively, of the form of the stencil equation, is to connect the points $(\beta, \beta + 2\pi, \beta + 4\pi)$ in the simplest way at least in the upper half-plane. Namely, the function $T(\beta)$ is holomorphic in the half strip $-\pi < \text{Re}[\beta] < 3\pi$, $\text{Im}[\beta] > 0$, and the points $(\beta, \beta + 2\pi, \beta + 4\pi)$ for $\text{Re}[\beta] = -\pi$ are just points obtained from one another by simple translations.
Let us now define the function $R$ by

$$R(\beta) = T(\beta + 2\pi).$$  \hfill (C.2)

Here we will attempt to find some functions $g(\beta), f(\beta)$ such that the two equations

\[
\begin{bmatrix}
\frac{d^2}{d\beta^2} + f(\beta) \frac{d}{d\beta} + g(\beta)
\end{bmatrix} T(\beta) = 0, \\
\begin{bmatrix}
\frac{d^2}{d\beta^2} + f(\beta) \frac{d}{d\beta} + g(\beta)
\end{bmatrix} R(\beta) = 0
\]  \hfill (C.3)

are fulfilled. Note that the first equation is (3.31), thus, if the coefficients $f$ and $g$ posses the necessary properties, the statement of the theorem is proven.

The coefficients $f$ and $g$ can be found formally by solving (C.3) as a system of two linear algebraic equations:

$$f(\beta) = -\frac{D_{0,2}}{D_{0,1}}, \quad g(\beta) = \frac{D_{1,2}}{D_{0,1}},$$  \hfill (C.4)

where $D_{m,n}(\beta)$ are the generalised Wronsky determinants defined by

$$D_{m,n}(\beta) = \begin{vmatrix}
T^{(m)}(\beta) & T^{(n)}(\beta) \\
R^{(m)}(\beta) & R^{(n)}(\beta)
\end{vmatrix},$$  \hfill (C.5)

where $m$ and $n$ are non-negative integers and the subscript $^{(m)}$ say, corresponds to the $m$th derivative of a given function with respect to $\beta$. The determinants are defined in the strip $-\pi < \text{Re}[\beta] < \pi$. They are continuous on the edges of this strip, and are branching at the points $\beta = \pm \pi/2$. According to the properties of $T$ and $R$ at $\beta = \pm \pi/2$, each Wronsky determinant changes its sign when a point $\beta = \pm \pi/2$ is encircled.

Using the definition of $R$, is is possible to rewrite the stencil equation (C.1) in the vectorial form

$$\begin{pmatrix}
T(\beta + 2\pi) \\
R(\beta + 2\pi)
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
1 & -\lambda
\end{pmatrix} \begin{pmatrix}
T(\beta) \\
R(\beta)
\end{pmatrix},$$  \hfill (C.6)

for $\text{Re}[\beta] = -\pi$. By direct differentiation, the same relation is valid for any derivative of the vector $(\frac{T}{R})$, and hence, for any non-negative integers $m$ and $n$, we have

$$D_{m,n}(\beta + 2\pi) = \begin{vmatrix}
0 & 1 \\
1 & -\lambda
\end{vmatrix} D_{m,n}(\beta) = -D_{m,n}(\beta).$$  \hfill (C.7)

This relation enables one to perform the analytic continuation of $D_{m,n}(\beta)$ outside the strip $-\pi < \beta < \pi$. 

![Diagram](image-url)
Because \((\beta + \pi/2)^{-1/2}T(\beta)\) and \((\beta - \pi/2)^{-1/2}R(\beta)\) are analytic on the strip \(-\pi \leq \text{Re}[\beta] \leq \pi\), and because \(\cos(\beta)\) has simple zeros at \(\beta = \pm \pi/2\), one can show, by direct differentiation or otherwise, that the function

\[
P_{m,n}(\beta) = (\cos(\beta))^{\max(m,n)}D_{m,n}(\beta)/\sqrt{\cos(\beta)}
\]

is analytic on the strip and \(P_{m,n}(\pm \pi/2) \neq 0\). Since \(\cos(\beta)\) is \(2\pi\)-periodic, \((C.7)\) implies that \(P_{m,n}(2\pi + \beta) = -P_{m,n}(\beta)\). Hence the function \(\tilde{P}_{m,n}(\beta) = \sin(\beta/2)P_{m,n}(\beta)\) is analytic on the strip and satisfies \(\tilde{P}_{m,n}(\beta + 2\pi) = \tilde{P}_{m,n}(\beta)\).

The mapping \(r = e^{i\beta}\) maps the strip \(-\pi \leq \beta \leq \pi\) to the entire \(r\) plane, the left (resp. right) boundary of the \(\beta\) strip is sent to the bottom (resp. top) part of the negative real axis in the \(r\) plane. We can hence consider the function \(\tilde{p}_{m,n}(r) = \tilde{P}_{m,n}(\beta(r))\). It is clearly analytic everywhere in the \(r\) plane, with the possible exception of the negative real axis that can possibly be a branch cut. However, since \(\tilde{P}_{m,n}(\beta)\) is \(2\pi\)-periodic, its value on the right and left boundary is the same, and hence \(\tilde{p}_{m,n}(r)\) is continuous across the negative real axis, so there is no cut there and \(r = 0\) is hence not a branch point (but can still be a pole).

Moreover, the exponential growth\(^2\) of \(R\) and \(T\) ensures that \(D_{m,n}\) also has an exponential growth at infinity, and hence, it is clear that \(\tilde{P}_{m,n}(\beta(r))\) also has exponential growth. Noting that under the same \(r\) mapping, \(\beta = +i\infty\) (resp. \(\beta = -i\infty\)) is sent to 0 (resp. \(\infty\)). We conclude that \(\tilde{p}_{m,n}(r)\) grows no faster than a power of \(\tau\) at \(\infty\) and 0. Since \(r = 0\) is not a branch point, \(\tilde{p}_{m,n}(r)\) should grow like \(1/|r|^m\) for an integer power \(m\) say.

Hence the function \(r^m\tilde{p}_{m,n}(r)\) is entire and has power growth at infinity. According to the extended Liouville theorem, this has to be a polynomial. And hence \(\tilde{p}_{m,n}(r)\) is a rational function. Finally, we can conclude that the ratio of two generalised Wronsky determinants is a rational function. More precisely, we can show that

\[
f(\beta) = -\frac{D_{0,2}(\beta)}{D_{0,1}(\beta)} = \frac{-1}{\cos(\beta)} \frac{\tilde{P}_{0,2}(\beta)}{\tilde{P}_{0,1}(\beta)} = \frac{-2}{r + 1/r} \frac{\tilde{p}_{0,2}(r)}{\tilde{p}_{0,1}(r)},
\]

\[
g(\beta) = \frac{D_{1,2}(\beta)}{D_{0,1}(\beta)} = \frac{1}{\cos(\beta)} \frac{\tilde{P}_{1,2}(\beta)}{\tilde{P}_{0,1}(\beta)} = \frac{2}{r + 1/r} \frac{\tilde{p}_{1,2}(r)}{\tilde{p}_{0,1}(r)},
\]

are rational functions of \(r\). The boundedness of the coefficients \(f\) and \(g\) follows from the exponential behaviour of the solutions.

The reasoning above has been based on the fact that the Wronsky determinant \(D_{0,1}\) is not identically equal to zero. One can see that this happens if and only if \(T\) obeys a linear homogeneous ODE of the first order with a \(2\pi\)-periodic coefficient. Indeed, if \(T\) obeys such ODE, then so does \(R\), and then \(D_{0,1} = 0\). If \(D_{0,1} = 0\), then \(T\) obeys the ODE

\[
T' - \frac{R'}{R}T = 0, \tag{C.8}
\]

which is a homogeneous 1st order linear ODE with coefficient \(F(\beta) = -\frac{R'(\beta)}{R(\beta)}\). Using the 1D stencil equation, this coefficient can be shown to be \(2\pi\)-periodic:

\[
F(\beta + 2\pi) = -\frac{R'(\beta + 2\pi)}{R(\beta + 2\pi)} = \frac{T'(\beta) - \lambda R'(\beta)}{T(\beta) - \lambda R(\beta)} = -\frac{R'(\beta)}{R(\beta)}T(\beta) - \lambda R'(\beta)}{T(\beta) - \lambda R(\beta)} = F(\beta). \tag{C.9}
\]

\(^2\)To be rigorous here, one should impose the growth restriction of the type chosen not only on the function \(\tilde{W}\) but also on all its derivatives.
This cannot happen since in this case, we would get $T(2\pi + \beta) = \Upsilon T(\beta)$ for some constant $\Upsilon$, and this cannot be true because of the singularity structure of $T$.

Let us finish by proving the symmetry relations (3.32). The condition (3.29) implies that $R(\beta) = T(-\beta)$, and hence (3.32) follows from (C.4) and (C.5).

### C.2 Proof of Proposition 3.7

According to the periodicity of $f$ and $g$, their symmetry properties (3.32), the position of (regular) singular points, and the fact that no other singularities can occur due to the equation being minimal, the coefficients $f$ and $g$ can be rewritten as

$$f(\beta) = \frac{1}{\cos \beta} \sum_{n=1}^{N_f} a_n^{(f)} \sin(n\beta) = \frac{\sin \beta}{\cos \beta} \sum_{n=1}^{N_f} a_n^{(f)} U_{n-1}(\cos \beta),$$

$$g(\beta) = \frac{1}{\cos \beta} \sum_{n=0}^{N_g} a_n^{(g)} \cos(n\beta) = \frac{1}{\cos \beta} \sum_{n=0}^{N_g} a_n^{(g)} T_n(\cos \beta),$$

for some positive integers $N_{f,g}$ and constants $a_n^{(f,g)}$, where $T_n$ and $U_n$ are the Chebyshev polynomials of first and second kind respectively. Since $f$ and $g$ must remain bounded away from $\beta = \pm \pi/2$, (C.10) and (C.11) have to take the form

$$f(\beta) = \frac{c \sin \beta}{\cos \beta}, \quad g(\beta) = \frac{a + b \cos \beta}{\cos \beta},$$

for some constants $a, b$ and $c$. For the resulting equation (3.31) to have exponents $(0, 1/2)$ at the point $\beta = \pi/2$ (i.e. it will have a regular solution and a solution which is a product of a regular function and $\sqrt{\cos \beta}$ near this point), it is necessary that $c = -1/2$ (see e.g. [8]).

### C.3 Proof of Proposition 3.8

Let us pick $T(\beta) = \psi_{1,2}(\beta - \pi/2)$ and prove that it obeys all conditions of Proposition 3.5. Let us start by proving that $T$ obeys the symmetry condition 1DSt4. For this, consider the function $T^\text{h}(\beta) \equiv T(2\pi - \beta)$. The function $T^\text{h}(\beta)$ obeys equation (3.35) due to the symmetry of the coefficients of (3.35). It is regular at the point $\beta = \pi/2$ because $n_{1,2} = 0$, thus, its expansion in the basis $B_{\pi/2}$ does not contain the branching term. Thus, $T^\text{h}(\beta) = QT(\beta)$ for some constant $Q$.

Finally, taking $\beta = \pi$ we prove that $Q = 1$.

Because of this symmetry, the condition 1DSt1 is hence obeyed by construction. The condition 1DSt3 follows from the fact that 0 and $\infty$ are regular singular points of (3.33).

Let us finally prove that condition 1DSt2 of Proposition 3.5 is satisfied. For this, we consider $R(\beta) = T(\beta + 2\pi)$ and analyse the behaviour of the functions $T(\beta)$ and $R(\beta)$ as $\beta \to i\infty$ (i.e. in the upper half-plane far from the real axis). On the one hand, due to the $2\pi$-periodicity of the coefficients of (3.31) and by the construction of $R$, $R$ and $T$ are two independent solutions to our ODE, and $R(\beta + 2\pi)$ is also a solution. Hence $R(\beta + 2\pi)$ is a linear combination of $R$ and $T$, and we can write

$$\begin{pmatrix} T(\beta + 2\pi) \\ R(\beta + 2\pi) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ s & -\lambda \end{pmatrix} \begin{pmatrix} T(\beta) \\ R(\beta) \end{pmatrix}$$
for some constants $s$ and $\lambda$. The condition 1DSt2 will be proven if we establish that $s = 1$, i.e. we want (C.1) to be satisfied.

On the other hand, according to the theory of Fuchsian ODEs, in the upper half-plane the space of the solutions of (3.31) has a basis, whose terms can be expressed as formal series

$$F(\beta) = e^{\kappa \beta} \left(1 + \sum_{l=1}^{\infty} p_l e^{i\beta l}\right)$$

(C.14)

for some $p_l$. The parameter $\kappa$ can be defined from a characteristic equation obtained from (3.31) by taking the leading terms of the coefficients in the upper half-plane as $\beta \to +i\infty$:

$$\kappa^2 - \frac{i}{2} \kappa + b = 0.$$  

(C.15)

The roots of this equation are as follows:

$$\kappa_{1,2} = \frac{i}{4} \pm i\sqrt{b + 1/16}, \quad \kappa_1 + \kappa_2 = i/2.$$  

(C.16)

The two roots of this equation correspond to two functions $F_{1,2}(\beta)$ forming the solution basis.

Note that, for (C.1) to be satisfied, the values $\kappa$ should also obey the equation

$$e^{4\pi \kappa} + \lambda e^{2\pi \kappa} - 1 = 0.$$  

(C.17)

Note that this equation defines the values $\kappa$ only up to a term equal to $2\pi in$. Comparing (C.16) and (C.17) obtain

$$\lambda = -2i \cos(2\pi \sqrt{b + 1/16}).$$  

(C.18)

Due to (C.14), the transformation matrix for the basis $F_{1,2}$ is obviously diagonal:

$$\begin{pmatrix} F_1(\beta + 2\pi) \\ F_2(\beta + 2\pi) \end{pmatrix} = \begin{pmatrix} e^{2\pi \kappa_1} & 0 \\ 0 & e^{2\pi \kappa_2} \end{pmatrix} \begin{pmatrix} F_1(\beta) \\ F_2(\beta) \end{pmatrix}.$$  

(C.19)

Since the vector of unknowns in (C.19) is a linear transformation of the vector in (C.13), the determinants of the transformation matrices should be equal:

$$\det \begin{pmatrix} 0 & 1 \\ s & -\lambda \end{pmatrix} = \det \begin{pmatrix} e^{2\pi \kappa_1} & 0 \\ 0 & e^{2\pi \kappa_2} \end{pmatrix},$$

leading to

$$-s = \exp\{2\pi(\kappa_1 + \kappa_2)\} = \exp\{\pi i\} = -1.$$  

Thus, $s = 1$, equation (C.6) is valid for some $\lambda$ (given by (C.18)), and (C.1) is valid by construction of $R$. Finally, for the connection contours shown in figure 3.5 the functional equation of condition 1DSt2 is valid.