NEWTON–OKOUNKOV BODIES OVER DISCRETE VALUATION RINGS

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Abstract. The theory of Newton–Okounkov bodies attaches a convex body to a line bundle on a variety equipped with flag of subvarieties. This convex body encodes the asymptotic properties of sections of powers of the line bundle. We study Newton–Okounkov bodies for schemes defined over discrete valuation rings. We give the basic properties and then focus on the case of toric schemes and families of curves. We describe the Newton–Okounkov bodies for semistable families of curves in terms of the Baker–Norine theory of linear systems on graphs, making a connection between asymptotics of linear systems and tropical geometry.

1. Introduction

Let $\mathcal{O}$ be a discrete valuation ring with fraction field $K$ and residue field $k$. Let $\pi$ be a uniformizer of $\mathcal{O}$. For convenience, we shall call semistable any irreducible, regular scheme proper, flat, and of finite type over $\mathcal{O}$ whose generic fiber is smooth and whose closed fiber is a reduced normal crossings divisor. Let $\mathcal{X}$ be a projective regular semistable scheme over $\mathcal{O}$ with closed fiber $X$. Let $\mathcal{Y}$ denote a descending flag of subschemes

$$\mathcal{X} = Y_0 \supsetneq Y_1 \supsetneq \ldots \supsetneq Y_{d+1}$$

where each $Y_i$ is a codimension $i$ subscheme that is either a semistable scheme over $\mathcal{O}$ or a smooth subvariety of a component of the closed fiber $\mathcal{X} \times_{\mathcal{O}} k$. Let $\mathcal{D}$ be a divisor on $\mathcal{X}$ that surjects to Spec $\mathcal{O}$. Let $j$ be the index such that $Y_{j-1}$ is semistable over $\mathcal{O}$ and $Y_j$ is contained in the closed fiber. Then $Y_j$ is a component of $Y_{j-1} \times_{\mathcal{O}} k$. We will give names to two special cases: when $j = 1$, we are said to be in the Arakelovian case; when $j = d + 1$, we are in the tropical case. These names are explained below.

In this paper we will define a Newton–Okounkov body, $\Delta_{\mathcal{Y}}(\mathcal{D}) \subset \mathbb{R}^{d+1}$ by considering sections of $\mathcal{O}(\mathcal{D})$ evaluated at the valuations attached to the flag. In contrast to the classical case, this Newton–Okounkov body will be unbounded, albeit in a single direction. This unboundedness is a consequence of the fact that if $s \in H^0(\mathcal{X}, \mathcal{O}(\mathcal{D}))$, then $\pi^k s \in H^0(\mathcal{X}, \mathcal{O}(\mathcal{D}))$ for any $k \geq 0$.

We will investigate these Newton–Okounkov bodies, determining their basic properties. We have the following result establishing their boundedness except in a single direction:

**Theorem 1.1.** Let $p_\pi : \mathbb{R}^{d+1} \to \mathbb{R}^d$ be projection along the direction through the valuation vector of $\pi$. The image $\Delta = p_\pi(\Delta_{\mathcal{Y}}(\mathcal{D}))$ is compact.

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We will give a description of $\Delta_{\psi}(\mathcal{D})$ in the tropical case. Recall that for $\Delta \subset \mathbb{R}^d$, a convex body and a convex function $\psi : \Delta \to \mathbb{R}$, we may define the upper hull in $\mathbb{R}^{d+1}$ to be the set

$$\{(x,t) \mid x \in \Delta, \ t \geq \psi(x)\}.$$ 

If $\psi$ is piecewise linear, its domains of linearity give the Newton subdivision of $\Delta$ [5], an important notion of tropical geometry. A related construction occurs in the work of Yuan [14] and Witt-Nyström [13].

**Theorem 1.2.** In the tropical case, $\Delta_{\psi}(\mathcal{D})$ is given by the upper hull of a convex function $\psi : p_\pi(\Delta_{\psi}(\mathcal{D})) \to \mathbb{R}$.

We will give a description of Newton–Okounkov bodies in the case of toric schemes with respect to a toric flag. Here, it is analogous to the field case and involves a particular polyhedron $P_\mathcal{D}$ depending on $\mathcal{D}$, and $\phi_\mathcal{R}$: a certain linear map depending on $\mathcal{Y}$.

**Theorem 1.3.** Let $\mathcal{D}$ be a torus-invariant divisor on a toric scheme $\mathcal{X}$ that surjects onto Spec $\mathcal{O}$ with generic fiber $D$ such that $\mathcal{O}(D)$ is a big line bundle on $X$. Then, the Newton–Okounkov body of $\mathcal{O}(\mathcal{D})$ is given by

$$\Delta_{\psi}(\mathcal{D}) = \phi_\mathcal{R}(P_\mathcal{D}).$$

We will consider the case of semistable families of curves where we can give a fairly complete description of the Newton–Okounkov bodies. In fact, they can be described in terms of the Baker–Norine theory of linear systems on graphs [2]. As an intermediate object for describing the Newton–Okounkov bodies, we introduce the Newton–Okounkov linear system, $L^+(\rho(\mathcal{D}))$, which measures the vanishing orders on components on the special fiber of sections of $\mathcal{O}(m\mathcal{D})$ for $m \in \mathbb{Z}_{\geq 1}$. We prove that the Newton–Okounkov linear system is combinatorial by relating it to a combinatorially-defined effective linear system $L^+(\rho(\mathcal{D}))$ where $\rho(\mathcal{D})$ is the combinatorial specialization of the horizontal divisor $\mathcal{D}$ on the family $\mathcal{C}$:

**Theorem 1.4.** If the generic fiber of $\mathcal{D}$ has positive degree, then we have the equality between the Newton–Okounkov linear system and the combinatorial effective linear system:

$$L^+(\rho(\mathcal{D})) = L^+(\rho(\mathcal{D})).$$

After enriching the above theorem by considering vanishing orders of sections at a point in the closed fiber of $\mathcal{C}$, we are able to give a description for the Newton–Okounkov bodies in the tropical and Arakelovian cases in Theorem 6.16 and Theorem 6.17, respectively.

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2. **Notation and conventions**

For a scheme $\mathcal{X}$ over $\mathcal{O}$, we will use $\mathcal{X}_\mathbb{K}$ to denote its generic fiber. If $\mathcal{C}$ is a semistable curve over $\mathcal{O}$, irreducible divisors are either horizontal or vertical. Here, horizontal divisors
are those that surject onto $\text{Spec} \mathcal{O}$ while vertical divisors are contained in the closed fiber. Any divisor on $\mathcal{C}$ can be decomposed into a sum of horizontal and vertical divisors.

Recall that a convex cone in $\mathbb{R}^n$ is a convex set invariant under rescaling by elements of $\mathbb{R}_{\geq 0}$. For a set $S \subset \mathbb{R}^n$, we write $\text{cone}_{\mathbb{R}^{n+1}}(S)$ for the minimal convex cone containing $S$.

We will write $0$ for the empty divisor. For divisors $D, E$, we will write $D \leq E$ if $E - D$ is effective. If $D$ is a divisor on a smooth variety $X$, sections of $\mathcal{O}(D)$ can be interpreted in two ways: as a section of a line bundle or as a rational function $s$ whose principal divisor satisfies $(s) + D \geq 0$. Consequently if $D \leq E$, a section of $\mathcal{O}(D)$ can be interpreted as a section of $E$ although its zero locus will differ by $E - D$. We will point out the relevant interpretation by using the words “section” or “rational function.”

3. Construction of the Newton–Okounkov body

Let us recall the construction of Newton–Okounkov bodies in the classical setting.

Let $X$ be a smooth irreducible projective variety of dimension $n$ over a field $K$. Given a divisor $D$ on $X$, we want to construct a convex compact subset of $\mathbb{R}^n$ called the Newton–Okounkov body of $D$. A flag of smooth irreducible subvarieties

$$Y_\bullet : X = Y_0 \supset Y_1 \supset \cdots \supset Y_{n-1} \supset Y_n = \{pt\}$$

is called full and admissible if $\text{codim} Y_i = i$ and $Y_n$ is a smooth point on each $Y_i$. For every non-zero section $s$ of $\mathcal{O}(D)$, if $s_0 := s$, for $i = 1, \ldots, n$ define

$$(3.1) \quad \nu_i(s) := \text{ord}_{Y_i}(s_{i-1}), \quad s_i := \frac{s_{i-1}}{g_i^{\nu_i(s)}_{Y_i}},$$

where $g_i$ is the local equation of $Y_i$ in $Y_{i-1}$ near $Y_n$. Here, $s_0$ is considered as a section of $L_0 = \mathcal{O}(D)$ while $s_i$ is considered as a section on $Y_i$ of $L_i = L_{i-1}|_{Y_i} \otimes \mathcal{O}_{Y_i}(-\nu_i(\varphi)Y_i)$. We obtain the vector $\nu(s) = (\nu_1(s), \ldots, \nu_n(s)) \in \mathbb{Z}^n$. Define the semi-group of valuation vectors as

$$\Gamma_{Y_\bullet}(D) := \{(\nu(s), m) \in \mathbb{Z}^n \times \mathbb{N} \mid s \in H^0(X, \mathcal{O}_X(mD))\}$$

and the Newton–Okounkov body of $D$ as

$$\Delta_{Y_\bullet}(D) := \text{cone}_{\mathbb{R}^{n+1}}(\Gamma_{Y_\bullet}(D)) \cap (\mathbb{R}^n \times \{1\}).$$

We write $\Gamma_{Y_\bullet}(D)_n$ for $\Gamma_{Y_\bullet}(D) \cap (\mathbb{Z}^n \times \{n\})$. The same construction can be performed for non-complete graded linear series as well, see [9].

3.1. The case of curves. Let us consider a curve $C$ of positive genus $g$ and a divisor $D$ of degree $d > 0$. Let $Y_\bullet = \{C \supset p\}$, with $p$ a general point. Then the Newton–Okounkov body is the segment $[0, d]$. This follows from the Riemann-Roch theorem [12]. Here, the Okounkov body only captures numerical data.
3.2. The case of surfaces. In the case of surfaces, the Zariski decomposition of big divisors can be used to show that the Newton–Okounkov body lies between the graphs of two functions on an interval \([1,2]\). This description provides all the information about possible shapes of Okounkov bodies of surfaces.

Any pseudoeffective divisor \(D\) (that is, a divisor in the closure of the effective cone in the Neron–Severi group) can be written as \(D = P_D + N_D\), where \(P_D\) is nef, \(P_D \cdot N_D = 0\), and \(N_D\) is effective with a negative definite intersection matrix.

Let us consider the rank two valuation induced by a general flag \(\mathcal{Y} = \{ X \supseteq C \supseteq p \}\) such that \(p \notin \text{supp}(N_D)\). Let \(\alpha(D) = \text{ord}_p(N_D)\), \(\beta(D) = \alpha(D) + C \cdot P_D\), and \(\mu := \sup\{x| D - xC \text{ is big} \} \). Then, we have

\[
\Delta_{\mathcal{Y}}(D) = \{(x,y) \in \mathbb{R}^2 | 0 \leq x \leq \mu \text{ and } \alpha(D - xC) \leq y \leq \beta(D - xC) \}.
\]

**Example 3.2.** Let \(X\) be the blow up of \(\mathbb{P}^2\) at two points with exceptional divisors \(E_1, E_2\) and consider the flag \(\mathcal{Y} = \{ X \supseteq l \supseteq p \}\) given by a general line and a general point on it. Let \(H\) denote the pullback of the class of a line in \(\mathbb{P}^2\).

Let \(D = 2H - E_1 - E_2\). In this case we have \(\mu = 1\) and the Zariski decomposition of \(D - xH\) is the sum \((1-x)(2H - E_1 - E_2) + x(H - E_1 - E_2)\), obtaining the following body:

![Newton–Okounkov body](image)

Newton–Okounkov bodies in higher dimensions can be much more complicated and there is no general strategy for writing them down. We will see that for toric schemes with respect to a torus-invariant flag, the Okounkov body is determined by combinatorics.

4. Newton–Okounkov bodies over discrete valuation rings

4.1. Definition for schemes over discrete valuation rings. We now describe the case of Newton–Okounkov bodies on schemes over discrete valuation rings. Let \(\mathcal{O}\) be a discrete valuation ring with fraction field \(K\), residue field \(k\), and valuation \(\text{val}\). Let \(\pi\) be a uniformizer of \(\mathcal{O}\). Let \(\mathcal{X}\) be an \(n\)-dimensional semistable scheme over \(\mathcal{O}\). We will write \(X = \mathcal{X} \times_{\mathcal{O}} K\) for the generic fiber of \(\mathcal{X}\). Let \(\mathcal{Y}\) denote a descending flag of proper subschemes

\[
\mathcal{X} = Y_0 \supseteq Y_1 \supseteq ... \supseteq Y_{n+1}
\]

where each \(Y_i\) is a codimension \(i\) subscheme that is either a semistable scheme over \(\mathcal{O}\) or a proper smooth subvariety of the closed fiber \(\mathcal{X} \times_{\mathcal{O}} k\). Let \(\mathcal{D}\) be a divisor on \(\mathcal{X}\) flat over \(\mathcal{O}\).

Let \(j\) be the index such that \(Y_{j-1}\) is semistable over \(\mathcal{O}\) and \(Y_j\) is a closed subvariety of \(\mathcal{X} \times_{\mathcal{O}} k\). Then \(Y_j\) is a component of \(Y_{j-1} \times_{\mathcal{O}} k\). We will give names to special cases: when \(j = 1\), we are said to be in the Arakelovian case; when \(j = d + 1\), we are in the tropical case.
Here, the name “Arakelovian” is motivated by some constructions by Yuan \cite{Yuan} of which we describe the function field analogue. The name “tropical” is motivated by a fundamental notion in tropical geometry, the Newton subdivision \cite{Tropical} to which our construction specializes in the toric case. So in a certain sense, our work interpolates between tropical geometry and function field Arakelov theory. Our work is closely related to Okounkov bodies of filtered linear series as in the work of Boucksom–Chen \cite{BC}.

The Newton–Okounkov body $\Delta_{\mathcal{D}}(\mathcal{D})$ is defined as a convex body in $\mathbb{R}^{n+1}$ exactly as above by considering sections of $\mathcal{O}(m\mathcal{D})$ over $\mathcal{D}$ for $m \in \mathbb{N}$ evaluated at the valuations attached to the flag.

### 4.2. Boundedness.

In contrast to the field case, Newton–Okounkov bodies over discrete valuation rings are not bounded. However, the failure of boundedness can be precisely described.

**Lemma 4.1.** Let $\nu(\pi) \in \mathbb{R}^{n+1}$ be the valuation of the uniformizer $\pi \in \mathcal{O}$, viewed as a rational function on $\mathcal{D}$. The Newton–Okounkov body $\Delta_{\mathcal{D}}(\mathcal{D})$ is closed under positive translations in the $\nu(\pi)$-direction.

**Proof.** It suffices to show $\Gamma_{\mathcal{D}}(\mathcal{D}) + k(\nu(\pi),0) \subset \Gamma_{\mathcal{D}}(\mathcal{D})$ for any $k \in \mathbb{Z}_{\geq 0}$. Any point of $\Gamma_{\mathcal{D}}(\mathcal{D})$ is of the form $(\nu(s),m)$ for $s \in H^0(\mathcal{D},\mathcal{O}_X(m\mathcal{D}))$. Now, $\pi^k s \in H^0(\mathcal{D},\mathcal{O}_X(m\mathcal{D}))$ and $\nu(\pi^k s) = \nu(s) + k\nu(\pi)$. \hfill $\square$

The Newton–Okounkov body is bounded in other directions. Let $p_{\pi}: \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}/(\mathbb{R}\nu(\pi))$ be the projection along the $\nu(\pi)$-direction.

**Lemma 4.2.** The image under projection, $p_{\pi}(\Delta_{\mathcal{D}}(\mathcal{D}))$ is bounded.

**Proof.** Choose an ample divisor $H$ on $X$ and an ample divisor $h$ on $Y_j$, which is a component of the projective variety $Y_{j-1} \times \mathcal{O}$. We will follow \cite{BC} Lemma 1.10, whose methods come from \cite{Tropical}. We begin with the following observation: give a line bundle $L$ and a prime divisor $Y$ on some scheme $Z$, there exists an integer $b$ such that for any section $s_m$ of $L^m$, the vanishing order of $s_m$ on $Y$ is at most $mb$. Indeed, choose $b$ sufficiently large such that $(L - bY) \cdot H^{d-1} < 0$. Multiplying this inequality by $m$, we see that $mL - mbY$ cannot have any regular sections.

We claim that there exists positive integers $b_1,\ldots, b_{j-1}$ such that for any $s \in H^0(X,\mathcal{O}(m\mathcal{D}))$, $\nu_i(s) \leq mb_i$ for $i = 1,\ldots, j-1$. Given $s \in H^0(\mathcal{D},\mathcal{O}(m\mathcal{D}))$, restriction to the generic fiber gives $s \in H^0(X,\mathcal{O}(m\mathcal{D}_K))$. We choose $b_1$ as in the above paragraph. Then, $\nu_2$ is given by the vanishing order at $Y_2$ of the restriction of $s$ to $Y_1$ considered as a section of $L_{1,a} = \mathcal{O}(D)|_{Y_1} \otimes \mathcal{O}_{Y_1}(-aY_1)$ for some $a$ with $0 \leq a \leq b_1$. Choose $b_2$ to be the max of the $b$’s produced in the above paragraph for $L_{1,a}$ for $0 \leq a \leq b_1$. We continue by defining $b_i$’s inductively.

Now there are finitely many line bundles on $Y_{j-1}$ whose sections we will restrict to $Y_j$. We will take the maximum of the $b$’s chosen for all line bundles. Suppose we have one line bundle on $Y_{j-1}$. We may restrict our attention to $Y'_{j-1}$, and so it suffices to prove the result for $j = 1$. Let us consider elements of $H^0(X,\mathcal{O}(D))$. These are exactly rational sections of
$O(m\mathcal{D})$ over $\mathcal{X}$ that are allowed to have poles along components of the closed fiber. By multiplying such sections by a suitable power of $\pi$, we can ensure that the section is regular. Therefore, the Newton–Okounkov body associated to $H^0(X, O(D))$ with respect to the flag $Y_1 \supseteq \ldots \supseteq Y_d$ is exactly the Minkowski sum of the Newton–Okounkov body of $H^0(\mathcal{X}, O(\mathcal{D}))$ and the line $\mathbb{R} \nu(\pi)$. Since we're only interested in the projection of that Newton–Okounkov body along the line through $\nu(\pi)$, it suffices to consider the complementary slice given by elements of $H^0(X, O(D))$ that have neither poles nor zeroes generically along $Y_1$.

Now, the sections under consideration restrict to sections on $Y_1$. We may now apply the above argument with $H$ replaced by $h$ to obtain $b_1, \ldots, b_d$.

Remark 4.3. In the tropical case, $Y_{d+1}$ is a smooth point of the central fiber, $p_\pi$ is the projection along the $(n+1)$-st component.

4.3. Newton subdivision in the tropical case. We now consider the tropical case where the admissible flag $\mathcal{Y}_\bullet$ is given by semistable schemes $Y_1, \ldots, Y_d$ in $\mathcal{X}$ and a point $Y_{d+1}$ in the closed fiber $\mathcal{X}_k$. Let $\mathcal{D}$ be a divisor on $\mathcal{X}$ flat over $\text{Spec} \mathcal{O}$ whose generic fiber is $D \subset X$. Let $\mathcal{Y}_\bullet$ denote the flag on $X$ given by $Y_1, \ldots, Y_d$.

Theorem 4.4. We have a surjection of Newton–Okounkov bodies,

$$p_\pi : \Delta_{\mathcal{Y}_\bullet}(\mathcal{D}) \rightarrow \Delta_{Y_\bullet}(D).$$

Moreover, $\Delta_{\mathcal{Y}_\bullet}$ is given as the upper hull of a convex function

$$\psi : \Delta_{Y_\bullet}(D) \rightarrow \mathbb{R}.$$ 

Proof. Because any section $s \in H^0(X, O(D))$ has some multiple by $\pi$ satisfying $\pi^k s \in H^0(\mathcal{X}, O(\mathcal{D}))$, we have that the projection $p_\pi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ maps $\Delta_{\mathcal{Y}_\bullet}(\mathcal{D})$ surjectively to $\Delta_{Y_\bullet}(D)$.

Now, $\Delta_{\mathcal{Y}_\bullet}(\mathcal{D})$ is closed under positive translation by $e_{n+1} = \nu(\pi)$. From the convexity of Newton–Okounkov bodies, it follows that $\Delta_{\mathcal{Y}_\bullet}(\mathcal{D})$ is the set of points in $\mathbb{R}^{n+1}$ lying above the graph of a convex function $\psi : \Delta_{Y_\bullet}(D) \rightarrow \mathbb{R}$. □

In circumstances where the function $\psi$ is piecewise linear, we can consider the domains of linearity of $\psi$ as giving a subdivision of $\Delta_{Y_\bullet}(D)$. This is of interest in tropical geometry [6].

We note that Newton–Okounkov bodies of schemes over discrete valuation rings need not be polyhedral. Indeed, one may take a semistable model of a variety with a flag over $K$ that already has a non-polyhedral Newton–Okounkov body [11] and extend the flag to a tropical one.

5. Toric schemes

In this section, we discuss toric schemes over discrete valuation rings. See [10] for a classical source or [6] for a rigid analytic perspective.
5.1. **Toric varieties.** We begin by reviewing Newton–Okounkov bodies for smooth projective toric varieties [12, Section 6.1]. Let \( N \) be an \( n \)-dimensional lattice. A toric variety \( X(\Delta) \) is specified by a complete rational fan \( \Delta \) in \( N_\mathbb{R} = N \otimes \mathbb{R} \cong \mathbb{R}^n \). The variety \( X(\Delta) \) is smooth if and only if the fan is unimodular, that is, the fan is simplicial and every cone is spanned by integer vectors forming a subset of a basis of \( N \). Let \( T = N \otimes \mathbb{K}^* \) denote the \( n \)-dimensional algebraic torus acting on \( X(\Delta) \). To each \( k \)-dimensional cone \( \sigma \) of \( \Delta \), there corresponds an orbit closure \( V(\sigma) \) which is a codimension \( k \) subvariety. Any torus-invariant divisor is given by

\[
D = \sum_{\sigma} a_{\sigma} V(\sigma)
\]

where the sum is over rays \( \sigma \) in \( \Delta \) and \( a_{\sigma} \in \mathbb{Z} \). Attached to \( D \) is a polyhedron \( P_D \subset M_\mathbb{R} \) where \( M \) is the dual lattice of \( N \) defined by

\[
P_D = \text{Conv}(\{ m \in M \mid \langle m, u_{\sigma} \rangle \geq -a_{\sigma} \})
\]

where \( u_{\sigma} \in N \) is the primitive integer vector (with respect to \( N \)) along \( \sigma \). This polyhedron arises by considering sections of \( \mathcal{O}(D) \): the vector space of sections \( H^0(X, \mathcal{O}(D)) \) has a decomposition into \( T \)-eigenspaces; the lattice points of \( P_D \) are exactly the characters of \( T \) that arise; for \( m \in P_D \), the character \( \chi^m \) on \( T \) extends to a section of \( \mathcal{O}(D) \) on \( X \). Indeed, the vanishing order of \( \chi^m \) (considered as a section of \( \mathcal{O}(D) \)) on the divisor \( V(\sigma) \) is \( \langle m, u_{\sigma} \rangle + a_{\sigma} \) so the inequalities defining \( P_D \) are exactly the conditions that \( \chi^m \) is regular at the generic point of the torus-invariant divisors. Consequently, \( \dim H^0(X, \mathcal{O}(D)) = |P_D \cap M| \). The line bundle \( \mathcal{O}(D) \) is big if and only if \( P_D \) is \( n \)-dimensional, and it is ample if and only if the normal fan to \( P_D \) is exactly \( \Delta \).

Because \( X(\Delta) \) is smooth, a \( T \)-invariant flag \( Y_1, Y_2, \ldots, Y_n \) can be written as \( Y_i = D_1 \cap \cdots \cap D_i \) for a choice of \( T \)-invariant divisors \( D_1, \ldots, D_n \) corresponding to rays \( \sigma_1, \ldots, \sigma_n \). Let \( u_1, \ldots, u_n \) be the primitive integer vectors along \( \sigma_1, \ldots, \sigma_n \). We define a linear map \( \phi : M_\mathbb{R} \to \mathbb{R}^n \) by \( \phi(v) = (\langle v, u_{\sigma_i} \rangle + a_{\sigma_i})_{1 \leq i \leq n} \). We have the following equality for big line bundles \( \mathcal{O}(D) \):

\[
\Delta_{Y_\bullet}(D) = \phi(P_D).
\]

5.2. **Toric schemes.** Complete toric schemes over a discrete valuation ring \( \mathcal{O} \) are described by complete fans in \( N_\mathbb{R} \times \mathbb{R}_{\geq 0} \) where \( N \cong \mathbb{Z}^d \) is a lattice. Given such a fan \( \Sigma \), there is a natural morphism of toric varieties \( X(\Sigma)_\mathbb{Z} \to X(\mathbb{R}_{\geq 0})_\mathbb{Z} = \mathbb{A}_\mathbb{Z}^1 \) and the toric scheme is given by \( \mathcal{X} = X(\Sigma) \times_{\mathbb{A}_\mathbb{Z}} \text{Spec}(\mathcal{O}) \). Here, we will map \( t \), the coordinate on \( \mathbb{A}_1 \), to the uniformizer of \( \mathcal{O} \). We will suppose that \( \Sigma \) is a unimodular fan and therefore that the total space \( \mathcal{X} \) is regular. If we set \( \Delta = \Sigma \cap (N_\mathbb{R} \times \{0\}) \), then the generic fiber of \( \mathcal{X} \) is the toric variety \( X = X(\Delta) \). The closed fiber of \( \mathcal{X} \) is a union of toric varieties described combinatorially by the polyhedral complex \( \Sigma_1 = \Sigma \cap (N_\mathbb{R} \times \{1\}) \) in \( N_\mathbb{R} \times \{1\} \). The components of the closed fiber are in bijective correspondence with the vertices of \( \Sigma_1 \). We will suppose that the vertices of \( \Sigma_1 \) are at points of \( N \times \{1\} \) which ensures that \( \mathcal{X} \) has reduced closed fiber and, therefore, is a semistable family. Let \( T = N \otimes \mathbb{K}^* \) denote the torus of \( X \).
A $T$-invariant divisor $D$ on $X$ has many extensions $\mathcal{D}$ to $X$. In particular, we may write $D = \sum_{\sigma} a_{\sigma} V(\sigma)$ where $a_{\sigma} \in \mathbb{Z}$ and $V(\sigma)$ is the divisor of $X$ corresponding to a ray $\sigma$ of $\Delta$. Any extension is of the form

$$\mathcal{D} = \sum_{\sigma} a_{\sigma} V(\sigma) + \sum_{v} a_{v} V(v)$$

where $a_{v} \in \mathbb{Z}$ and $V(v)$ are the divisors on $X$ corresponding to rays through the vertices of $\Sigma$. We construct the Newton–Okounkov body. Considering the total space $X(\Sigma)$ as an $(n+1)$-dimensional toric variety, we define a polyhedron $P_{\mathcal{D}} \subset M_{\mathbb{R}} \times \mathbb{R}_{\geq 0}$ by

$$P_{\mathcal{D}} = \text{Conv} \left( \{ (m, h) \in M_{\mathbb{R}} \times \mathbb{R}_{\geq 0} \mid \langle m, u_{\sigma} \rangle \geq -a_{\sigma}, \langle m, v \rangle + h \geq -a_{v} \} \right).$$

The second set of inequalities come from $v \in \Sigma$ corresponding to vertices $(v, 1) \in \Sigma$. Note that this projects onto $P_{\mathcal{D}}$ by $(m, u) \mapsto m$. We may define a piecewise-linear convex function on $P_{\mathcal{D}}$,

$$\psi(m) = \max(-a_{v} - \langle m, v \rangle)$$

where $v$ is taken over vertices of $\Sigma$. Then $P_{\mathcal{D}}$ is the set of points lying above the graph of $\psi$. This function $\psi$ (studied in [5]) is important in tropical geometry, and its domains of linearity give the Newton subdivision of $P_{\mathcal{D}}$.

Now, we will explain how the polyhedron $P_{\mathcal{D}}$ relates to the Newton–Okounkov body of $X$ with respect to a flag $\mathcal{F}$ of torus-fixed subschemes. Following [12], we may choose toric prime divisors $D_{1}, \ldots, D_{d+1}$ of $X(\Sigma)$ such that $Y_{i} = D_{1} \cap \cdots \cap D_{i}$. Suppose that $D_{i}$ corresponds to a ray in $N_{\mathbb{R}} \times \mathbb{R}$ whose primitive integer vector is $w_{i} \in N \times \mathbb{Z}$. Here, $\{w_{1}, \ldots, w_{d+1}\}$ is a basis for $N \times \mathbb{Z}$. Write

$$\mathcal{D} = \sum_{w} a_{w} V(\sigma_{w})$$

where $w$ runs over primitive integer vectors of rays $\sigma_{w}$ of $\Sigma$.

We define

$$\phi_{\mathbb{R}} : M_{\mathbb{R}} \times \mathbb{R} \to \mathbb{R}^{d+1}, \quad \phi : (v, h) \mapsto \left( \langle (v, h), w_{i} \rangle + a_{i} \right)_{1 \leq i \leq d+1}$$

where the pairing is between $M_{\mathbb{R}} \times \mathbb{R}$ and $N_{\mathbb{R}} \times \mathbb{R}$. We have the following analogue of [12, Prop. 6.1]:

**Theorem 5.1.** Let $\mathcal{D}$ be a torus-invariant divisor on $X$ that surjects onto $\text{Spec} \mathcal{O}$ with generic fiber $D$ such that $\mathcal{O}(D)$ is a big line bundle on $X$. Then, the Newton–Okounkov body of $\mathcal{O}(\mathcal{D})$ is given by

$$\Delta_{\mathcal{F}}(\mathcal{O}(\mathcal{D})) = \phi_{\mathbb{R}}(P_{\mathcal{D}}).$$

**Proof.** By replacing $\mathcal{D}$ with a positive integer multiple, we may suppose that the vertices of $P_{\mathcal{D}}$ are points of $N \times \mathbb{Z}$.

Write the restriction of $s \in H^{0}(\mathcal{X}, \mathcal{O}(\mathcal{D}))$ to $X$ as

$$s = \sum_{m} c_{m} \chi^{m}$$
for $c_m \in K$ where the above is a finite sum over characters. The vanishing order of $s$ (considered as a rational function) on the divisor $D_w$ corresponding to a ray $\sigma_w$ of $\Sigma$ is

$$b_w = \min \left( \langle (m, \text{val}(c_m)), w \rangle \right)$$

and the pairing is the one between $M_R \times R$ and $N_R \times R$. Observe that in the above, if $\sigma_w$ is a ray of $\Delta$, then $\langle (m, \text{val}(c_m)), w \rangle = \langle m, u_\sigma \rangle_N$ where the second pairing is the pairing between $M_R$ and $N_R$. If $w = (v, 1)$ corresponds to a vertex of $\Sigma_1$, then $\langle (m, \text{val}(c_m)), w \rangle = \langle m, v \rangle_N + \text{val}(c_m)$. On $X$, we have that following formula for the principal divisor:

$$(s) = \sum b_w D_w.$$  

Note that the vanishing order of $s$, considered as a section of $\mathcal{O}(\mathcal{D})$, on $D_w$ is $b_w + a_w$.

Consequently, a sum of characters like the above corresponds to an element of $H^0((\mathcal{D}, \mathcal{O}(\mathcal{D}))$ if and only if $b_w \geq -a_w$ for all $w$. In fact, $\pi^h \chi^m \in H^0((\mathcal{D}, \mathcal{O}(\mathcal{D}))$ if and only if $(m, h) \in P_\mathcal{D}$.

Now, the valuation of such a section of $\mathcal{O}(\mathcal{D})$ is

$$\nu(s) = (b_{w_1} + a_{w_1}, \ldots, b_{w_{n+1}} + a_{w_{n+1}}).$$

It follows that $\nu(s) \in \phi_R(P_\mathcal{D})$. By considering sections of the form $\pi^h \chi^m$, we see that $\Delta_{\pi^h}(\mathcal{D})$ contains $\phi_R(P_\mathcal{D})$.  

\section{Newton–Okounkov bodies of curves}

We will relate the Newton–Okounkov bodies of curves over $\mathcal{O}$ to the Baker-Norine theory of linear systems on graphs. We will use $\mathbb{R}$-divisors on graphs whose theory is combinatorially much less rich than that of integral divisors.

\subsection{Review of linear systems on graphs}

We review some results on specialization of linear systems from curves to graphs due to Baker [1]. Let $\mathcal{C}$ be a semistable curve over $\text{Spec} \mathcal{O}$ such that the generic fiber $\mathcal{C}_K$ is smooth. The semistability condition ensures that the closed fiber $\mathcal{C}_0$ is reduced with only ordinary double points as singularities. A node in the closed fiber of $\mathcal{C}$ is formally locally parameterized in $\mathcal{C}$ by $\mathcal{O}[x, y]/(xy - \pi)$.

\textbf{Definition 6.1.} The \textit{dual graph} $\Sigma$ of a semistable curve $\mathcal{C}$ is a graph $\Sigma$ whose vertices $V(\Sigma)$ correspond to components of the normalization $\pi : \tilde{\mathcal{C}}_0 \to \mathcal{C}_0$ whose edges $E(\Sigma)$ correspond to nodes of $\mathcal{C}_0$. To each vertex $v$ is associated the component $C_v$ of $\tilde{\mathcal{C}}_0$.

Observe that each node of an irreducible component $C_v$ of $\mathcal{C}_0$ corresponds to a loop at $v$. We will denote the edges of $E(\Sigma)$ by $e = vw$ even though $\Sigma$ may not be a simple graph. Thus, when we sum over edges adjacent to $v$, we may need to sum over certain vertices more than once.

A \textit{divisor} on $\Sigma$ is an element of the free abelian group on $V(\Sigma)$. We write a divisor as $D = \sum_{v \in V(\Sigma)} a_v(v)$. The group of all divisors is denoted by $\text{Div}(\Sigma)$. We say a divisor $D$ is
effective and write $D \geq 0$ if $a_v \geq 0$ for all $v \in V(\Sigma)$. We write $D \geq D'$ if $D - D' \geq 0$. The degree of a divisor is given by
\[
\deg(D) = \sum_v a_v.
\]
We will study functions $\varphi : V(\Sigma) \to \mathbb{R}$. The Laplacian of $\varphi$, $\Delta(\varphi)$ is the divisor on $\Sigma$ given by
\[
\Delta(\varphi) = \sum_{v \in V(\Sigma)} \sum_{e \in E(\Sigma) | e = vw} (\varphi(v) - \varphi(w))(v).
\]
Note that $\Delta(\varphi)$ is of degree 0.

The specialization map $\rho : \text{Div}(\mathcal{C}) \to \text{Div}(\Sigma)$ is defined by, for $D \in \text{Div}(\mathcal{C})$,
\[
\rho(D) = \sum_{v \in \Gamma} \deg(\pi^*\mathcal{O}(D)|_{C_v})(v).
\]
The specialization of a vertical divisor $\sum_v \varphi(v)C_v$ satisfies
\[
\rho \left( \sum_v \varphi(v)C_v \right) = -\Delta(\varphi).
\]
For a divisor $H$ on $C_K$, we will write $\rho(H)$ to mean the specialization of its closure in $\mathcal{C}$. Observe that for $H$, horizontal and effective, we have $\rho(H) \geq 0$.

**Definition 6.2.** Let $\Lambda$ be a divisor on $\Sigma$. We define the linear system $L(\Lambda)$ to be the set of functions $\varphi : V(\Sigma) \to \mathbb{R}$ on $\Sigma$ with $\Delta(\varphi) + \Lambda \geq 0$. The effective linear system $L^+(\Lambda)$ is the subset of $L(\Lambda)$ consisting of non-negative functions $\varphi$.

Let $\mathcal{D}$ be a divisor on $\mathcal{C}$. Then we will interpret a global section of $\mathcal{O}(\mathcal{D})$ as a rational function $s$ on $\mathcal{C}$ such that $(s) + \mathcal{D} \geq 0$. If we write $(s) = \mathcal{H} + V$ where $\mathcal{H}$ is a horizontal divisor over $\mathcal{O}$ and $V$ is a vertical divisor contained in the closed fiber, we may decompose $V$ as
\[
V = \sum_v \varphi_s(v)C_v
\]
where we call $\varphi_s : V(\Sigma) \to \mathbb{Z}$ the vanishing function of $s$. For $s$, a rational function on $C$, we will abuse notation and take the vanishing function of $s$ to be vanishing function of the extension of $s$ to $\mathcal{C}$.

The following lemma is standard and we include the proof only for completeness.

**Lemma 6.3.** Let $D$ be a divisor on $C_K$ whose closure $\mathcal{D}$ has specialization $\Lambda = \rho(\mathcal{D})$. For a rational function $s$ on $C$ corresponding to a section of $\mathcal{O}(D)$ with vanishing function $\varphi$, we have
\[
\Delta(\varphi) + \Lambda \geq 0
\]
or, in other words, $\varphi \in L(\Lambda)$.
Proof. Because $s$ is principal, we have $(s) \cdot C_v = 0$ for all components of the closed fiber. If we write $(s) = \mathcal{H} + \sum_v \varphi(v) C_v$, we have
\[
0 = \rho((s)) = \rho \left( \mathcal{H} + \sum_v \varphi(v) C_v \right) = \rho(\mathcal{H}) - \Delta(\varphi).
\]
Since $\mathcal{H} + D \geq 0$ is horizontal, we have
\[
0 \leq \rho(\mathcal{H}) + \rho(D) = \Delta(\varphi) + \rho(D).
\]
□

The linear system $L(\Delta)$ has a tropical semi-group structure as noted in [7]:

**Lemma 6.4.** For $\varphi_1, \varphi_2 \in L(\Lambda)$, let $\varphi: V(\Sigma) \to \mathbb{R}$ be the pointwise minimum of $\varphi_1, \varphi_2: V(\Sigma) \to \mathbb{R}$. Then $\varphi \in L(\Lambda)$.

**Proof.** Let $v \in V(\Sigma)$. Suppose without loss of generality that $\varphi(v) = \varphi_1(v)$. Then
\[
\Delta(\varphi) + \Lambda = \sum_{v \in V(\Sigma)} \sum_{e=uv} (\varphi_1(v) - \varphi(w)) + \Lambda \\
\geq \sum_{v \in V(\Sigma)} \sum_{e=uv} (\varphi_1(v) - \varphi_1(w)) + \Lambda \\
\geq 0.
\]
□

6.2. Classical and tropical linear systems.

**Definition 6.5.** Now, let $\mathcal{D}$ be a horizontal divisor on $\mathcal{C}$. Let $m \in \mathbb{N}$. There is a natural map
\[
\varrho_m: H^0(\mathcal{C}, m\mathcal{D}) \to L^+(\rho(\mathcal{D})) \\
s \mapsto \frac{1}{m} \varphi_s
\]
where $s \in H^0(\mathcal{C}, m\mathcal{D})$ is interpreted as a rational function $s$ with $(s) + m\mathcal{D} \geq 0$ and $\varphi_s$ is the vanishing function of $s$.

We define the Newton–Okounkov linear system $L^+_\Delta(\mathcal{D})$ to be the subset of $L^+(\rho(\mathcal{D}))$ given by the closure of the union of the convex hulls of the images of $\varrho_m$ for $m$ ranging over $\mathbb{Z}_{\geq 1}$. We may extend this definition to horizontal $\mathbb{Q}$-divisors by defining $L^+_\Delta(\mathcal{D})$ to be $\frac{1}{m} L^+_\Delta(m\mathcal{D})$ where $m$ is chosen arbitrarily divisible.

**Theorem 6.6.** If the generic fiber of $\mathcal{D}$ has positive degree, then we have the equality between the Newton–Okounkov linear system and the combinatorial effective linear system:
\[
L^+_\Delta(\mathcal{D}) = L^+(\rho(\mathcal{D})).
\]

Before proving the proposition, we need a preparatory lemma adapted from [8].
**Definition 6.7.** Let \( f : V(\Sigma) \to \mathbb{R} \) be a function. Set
\[
M(f) = \max_{S \subseteq V(\Sigma)} \left\{ \left| \sum_{v \in S} f(v) \right| \right\}.
\]

**Lemma 6.8.** Let \( \varphi : V(\Sigma) \to \mathbb{R} \) be a function. Then,
\[
\max \varphi - \min \varphi \leq M(\Delta(\varphi)) \cdot \text{diam}(\Sigma).
\]

*Proof.* It suffices to show that for any edge \( e = vw \) in \( \Sigma \), \( |\varphi(w) - \varphi(v)| \leq M(\Delta(\varphi)) \). Indeed, let \( v_0, v_1 \) be the vertices where the minimum and maximum of \( \varphi \) are achieved, respectively. By picking a path from \( v_0 \) to \( v_1 \) of length at most \( \text{diam}(\Sigma) \) and comparing the values of \( \varphi \) along that path, we achieve the desired conclusion.

Let \( t = \varphi(v_1) < \varphi(v_0) \). Set \( \Sigma \leq t \) be the subgraph of \( \Sigma \) induced by \( \varphi^{-1}([0,t]) \). Let \( O(\Sigma \leq t) \) be the set of outgoing edges, that is, the edges \( e = vw \in E(\Sigma) \) with \( v \in \Sigma \leq t \) and \( w \not\in \Sigma \leq t \). Observe that for such edges \( e = vw \), we have \( \varphi(w) - \varphi(v) > 0 \).

\[
M(\Delta(\varphi)) \geq \left| \sum_{v \in V(\Sigma \leq t)} \Delta(\varphi)(v) \right| = \left| \sum_{v \in V(\Sigma \leq t)} \left( \sum_{e=vw} (\varphi(v) - \varphi(w)) \right) \right| = \sum_{e=vw \in O(\Sigma \leq t)} (\varphi(w) - \varphi(v))
\]

because the contribution from edges contained in \( \Sigma \leq t \) cancel in pairs. From this we conclude that for any outgoing edge \( e = vw \), \( \varphi(w) - \varphi(v) \leq M(\Delta(\varphi)) \). \( \square \)

We have the following immediate corollary:

**Corollary 6.9.** Let \( \varphi, \vartheta : V(\Sigma) \to \mathbb{R}_{\geq 0} \) be functions such that
\[
\Delta(\varphi) - \Delta(\vartheta) = F - G
\]
where \( F \) and \( G \) are effective divisors of degree at most \( d \). Suppose that there are (not necessarily distinct) vertices \( v, w \) such that \( \varphi(v) = \vartheta(w) = 0 \). Then
\[
\max(|\varphi - \vartheta|) \leq d \cdot \text{diam}(\Sigma).
\]

*Proof.* By hypothesis, \( \min(\varphi - \vartheta) \leq \varphi(v) - \vartheta(v) \leq 0 \). We note that \( M(\Delta(\varphi - \vartheta)) \leq d \). Consequently,
\[
\max(\varphi - \vartheta) \leq \max(\varphi - \vartheta) - \min(\varphi - \vartheta) \leq d \cdot \text{diam}(\Sigma).
\]
Interchanging the roles of \( \varphi \) and \( \vartheta \), we get the conclusion. \( \square \)

We now prove Proposition 6.6.

*Proof.* Set \( \Lambda = \rho(\mathcal{D}) \). From Lemma 6.3, it follows that \( L_+^+(\mathcal{D}) \subseteq L_+^+(\Lambda) \).

Therefore, we must show that any \( \vartheta \in L_+^+(\Lambda) \) can be approximated by some element in the image of \( \rho_m \) for some positive integer \( m \). First, we may suppose that \( \vartheta \) takes rational values. Pick a sufficiently divisible \( m \) such that \( m\vartheta \) takes integer values. Because \( \varrho_m(\pi^k s) = \)

We have that $\Delta(m\vartheta)+m\Lambda \geq 0$. Pick an effective divisor $E$ on $\Sigma$ such that $\Delta(m\vartheta)+m\Lambda - E$ is effective of degree exactly $g$, the genus of $C$. Choose a horizontal divisor $\mathcal{E}$ such that $\rho(\mathcal{E}) = E$. Because $\deg(m\mathcal{D}_K - \mathcal{E}_K) = g$, the invertible sheaf $\mathcal{O}(m\mathcal{D}_K - \mathcal{E}_K)$ on $\mathcal{C}_K$ has a regular section by the Riemann-Roch theorem. Therefore, there is a rational function $s$ on $\mathcal{C}_K$ with

$$(s) + m\mathcal{D}_K - \mathcal{E}_K \geq 0.$$

From $s$ we will find a section whose image under $\rho_m$ approximates $\vartheta$. By multiplying $s$ by some power of $\pi$, we may ensure that $s$ (considered as a rational function on $\mathcal{C}$) is regular on the generic points of the components of the closed fiber and does not vanish identically on all of them. Consequently, $s$’s vanishing function $\varphi_s$ is nonnegative and takes the value 0 at some vertex $w$. Now,

$$\Delta(\varphi_s) - \Delta(m\vartheta) = (\Delta(\varphi_s) + m\Lambda - E) - (\Delta(m\vartheta) + m\Lambda - E)$$

is the difference of two effective degree $g$ divisors by Lemma 6.3. Consequently, by Corollary 6.9, we have

$$\left| \frac{\varphi_s}{m} - \vartheta \right| \leq \frac{g}{m} \text{diam}(\Sigma).$$

Because $\varrho_m(s) = \frac{s}{m}$, the conclusion follows by choosing large $m$. □

To handle the case where the divisor $D$ is of degree 0, we may employ the following result.

**Corollary 6.10.** Let $D$ be a horizontal divisor on $\mathcal{C}$ of nonnegative degree. Let $\mathcal{E}$ be an effective, non-empty horizontal divisor. Then, we have the equality

$$L^+(\rho(D)) = \bigcap_{\varepsilon > 0} L^+_{\Delta}(D + \varepsilon\mathcal{E}).$$

**Proof.** It suffices to prove that

$$L^+(\rho(D)) = \bigcap_{\varepsilon > 0} L^+(\rho(D + \varepsilon\mathcal{E})).$$

This follows immediately from definitions. □

The rank of a divisor on a graph as defined by Baker–Norine involves an additional lattice structure. Within the vector space set of functions $\varphi : V(\Sigma) \rightarrow \mathbb{R}$, there is a lattice $\varphi : V(\Sigma) \rightarrow \mathbb{Z}$. In [2], a divisor $\Lambda$ on a graph $\Sigma$ is said to have non-negative rank if there exists $\varphi : V(\Sigma) \rightarrow \mathbb{Z}$ such that $\Delta(\varphi) + \Lambda \geq 0$. From this concept, they build a Riemann–Roch theory for divisors on graphs. One may reformulate this theory in terms of lattice points in Newton–Okounkov linear systems. The authors do not know if this view leads to any new proofs of known results in the Baker–Norine theory.
6.3. Horizontal-Vertical decomposition. Now, we will define a decomposition of divisors on Σ analogous to the Zariski decomposition. We will use this in our description of Newton–Okounkov bodies of curves. Recall that the Zariski decomposition of a big $\mathbb{Q}$-divisor $D$ on a smooth projective surface $X$ is a particular decomposition of the linear equivalence class of $D$, $D = P + N$ where $P$ is nef and $N$ is effective. It has the property that for $m$ such that $mD$ and $mN$ are integral divisors, multiplication by $mN$ gives an isomorphism

$$H^0(X, mP) \rightarrow H^0(X, mD).$$

**Definition 6.11.** Let $\Lambda$ be a divisor on $\Sigma$ such that $L^+(\Lambda)$ is non-empty. The minimal element of $L^+(\Lambda)$, $\varpi: V(\Sigma) \rightarrow \mathbb{R}$ is defined by

$$\varpi(v) = \min(\varphi(v) | \varphi \in L^+(\Lambda)).$$

Here, we interpret $\varpi$ as a vertical divisor. We have the following straightforward lemma following from Lemma 6.4.

**Lemma 6.12.** Let $\varpi$ be the minimal element of $L^+(\Lambda)$. Addition of $\varpi$ gives an isomorphism

$$L^+(\Lambda - \Delta(\varpi)) \rightarrow L^+(\Lambda).$$

We can interpret $\Lambda = (\Lambda - \Delta(\varpi)) + \Delta(\varpi)$ as a sort of Zariski decomposition.

Moreover, if $L \subset L^+(\Lambda)$ is a sub-semigroup, we may define $\varpi_L$ to be the pointwise minimum of $\varphi \in L$.

6.4. Enriched Newton–Okounkov linear systems. We will connect the Newton–Okounkov linear systems with the Newton–Okounkov bodies of curves over discrete valuation rings. Such bodies must take into account the vanishing of sections along a flag $\mathcal{Y} = \{Y_1 \subset Y_2\}$ where $Y_2 = \{p\}$ is a smooth point of the closed fiber. Consequently, we will enrich the above theory by considering such vanishing. We will also need to consider elements of $H^0(C, mD)$ whose horizontal components do not have any components in common with a fixed horizontal divisor in order to gain control over the vanishing at $Y_1$ in the tropical case.

Let $D, F$ be horizontal divisors on $C$. Let $m \in \mathbb{Z}_{\geq 1}$.

**Definition 6.13.** The $F$-controlled linear system $H^0(C, mD)_{\mathcal{F}, \epsilon}$ is the set of all elements $s$ of $H^0(C, mD)$ which when considered as rational functions have the property that their principal divisors $(s)$ contains no component of $F$ with multiplicity greater than $m\epsilon$.

Let $p$ be a smooth point on a component $C_v$ of the closed fiber of $C$. Suppose that $p$ is chosen to be disjoint from $D$ and $F$. For a divisor $G$ on $C$, write $v_p(G)$ to be the multiplicity of $p$ in $G \cap C_v$ where $G$ is the horizontal part of $G$. We consider the natural map

$$\rho_{m,p}: H^0(C, mD)_{\mathcal{F}, \epsilon} \rightarrow L^+(\rho(D)) \times \mathbb{R}$$

$$s \mapsto \left(\frac{1}{m} v_p(s), \frac{1}{m} v_p((s) + mD)\right)$$

where $s \in H^0(C, mD)_{\mathcal{F}, \epsilon}$. Observe that the second component of $\rho_{m,p}(s)$ is the vanishing at $p$ of the horizontal component of the zero locus of $s$, considered as a section of $O(D)$. 

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Definition 6.14. The $p$-enriched Newton–Okounkov linear system $L^+_{\Delta,p}(\mathcal{D})(\mathcal{F},\epsilon)$ is the subset of $L^+(\Lambda) \times \mathbb{R}$ given by the closure of the union of the convex hulls of the images of $\mathcal{F}_m,\epsilon$ for $m \in \mathbb{Z}_{\geq 1}$. When the subscript $(\mathcal{F},\epsilon)$ is suppressed, this means that we consider the empty divisor.

For a divisor $\Lambda$ on $\Sigma$, let $L^+_p(\Lambda)$ be the subset of $L^+(\Lambda) \times \mathbb{R}$ given by

$$L^+_p(\Lambda) = \{ (\varphi, u) | \varphi \in L^+(\Lambda), \ 0 \leq u \leq \Delta(\varphi)(v) + \Lambda(v) \}.$$  

Observe that if $\Lambda = \rho(D)$ for a horizontal divisor $D$ on $C$, the quantity $\Delta(\varphi)(v) + \Lambda(v)$ is exactly the degree of the divisor $\sum_v \varphi(v)C_v + D$ restricted to $C_v$. So, the second component measures the possible multiplicities of $p$ in the horizontal part of $(s)$ varying from 0 to the maximal possible degree.

We have the following extension of Proposition 6.6.

Theorem 6.15. We have the equality of sets:

$$L^+_{\Delta,p}(\mathcal{D})(\mathcal{F},\epsilon) = L^+_p(\rho(D)).$$

This is proven by the same method as Proposition 6.6. We wish to find a section $s \in H^0(\mathcal{C}, mD)$ such that $\mathcal{F}_m(s)$ is close to some $(\varphi, u)$. We modify the proof to choose $\mathcal{F}$ to intersect $C_v$ with a multiplicity $m'$ close to $mu$ at $p$ and for $\mathcal{F}$ not to contain any component of $\mathcal{F}$. The section $s$ produced by the Riemann-Roch theorem has $(s) + mD - K$ equal to a degree $g$ divisor. Then, the horizontal component of $(s) + mD$ intersects $C_v$ with a multiplicity between $m'$ and $m' + g$. Moreover, the horizontal components of $(s) + mD$ supported on components of $\mathcal{F}$ are of degree at most $g$. By picking a sufficiently large $m$, we obtain a close approximation.

6.5. Newton–Okounkov bodies of curves. In this section, we give a combinatorial description of the Newton–Okounkov bodies of curves.

We first consider the tropical case of a flag $\mathcal{X} \supset Y_1 \supset Y_2$ where $Y_1$ is a horizontal divisor and $Y_2 = \{p\}$ is a smooth point of the closed fiber. Let $D$ be a horizontal divisor.

Theorem 6.16. Suppose that $Y_1$ is a horizontal divisor intersecting the closed fiber in smooth points and that $Y_2$ is a point on the component $C_v$. Moreover, suppose that $p$ is not contained in $D$. For $t \in \mathbb{R}$, let

$$L_t = L^+(\rho(D - tY_1)).$$

The Newton–Okounkov body is the set of points in $\mathbb{R}^2$ lying above the graph of

$$a: [0, \deg(D)/\deg(Y_1)] \to \mathbb{R}$$

$$t \mapsto \omega_{L_t}(v).$$

Proof. We first show that the Newton–Okounkov body lies above the graph of $a$. We observe that for $s \in H^0(\mathcal{C}, mD)$, $\nu_2(s)$ is always greater than or equal to the multiplicity of $C_v$ in
Proof. Observe that for $Y$ closed fiber and $t$ large enough such that $m\nu(s)$ is close to $(t,\varpi_{L_t}(v))$. Set $\mathcal{F} = Y_1$. For a small $\varepsilon > 0$, by Theorem 6.15 we may find $m$ large enough such that there exists $s \in H^0(\mathcal{C}, m(\mathcal{D}-t\mathcal{F}))(\mathcal{F},\varepsilon)$ (considered as a rational function) such that
\[
\frac{1}{m}v_2(s) = \frac{1}{m}(\varphi_s(v) + v_p((s) + m\mathcal{D} - \nu_1(s)\mathcal{F}))) > \varepsilon \quad \text{and} \quad \frac{1}{m}\varphi_s \text{ is within } \varepsilon \text{ of } \varpi_{L_t}.
\]
and follows that all but a small part of $v_2(s)$ comes from the vertical component of $(s)$ along $C_v$. Thus $v_2(s)$ can be made arbitrarily close to $\varpi_{L_t}(v)$.

Now, we consider the Arakelovian case.

**Theorem 6.17.** Let $\mathcal{D}$ be a horizontal divisor. Suppose that $Y_1$ is a component $C_v$ of the closed fiber and $Y_2 = \{p\}$ is a smooth point on $C_v$ not contained in $\mathcal{D}$. Let $L_t \subset L^+(\rho(D))$ be the sub-semigroup of elements $\varphi$ with $\varphi(v) = t$. Then the Newton–Okounkov body of $\mathcal{O}(D)$ is the set of points between the graphs of $a(t) = 0$ and
\[
b(t) = \rho(\mathcal{D})(v) + \max(\Delta(\varphi)(v) | \varphi \in L_t)
\]
for $t \geq 0$.

**Proof.** Observe that for $s \in H^0(\mathcal{C}, m\mathcal{D})$, $\nu_1(s) = \varphi_s(v)$ and
\[
v_2(s) = v_p((s) + m\mathcal{D}).
\]
The Newton–Okounkov body is the image of $L^+_\Delta(p)(\mathcal{D}) = L^+_p(\rho(\mathcal{D}))$ under the map $(\varphi, u) \mapsto (\varphi(v), u)$. The conclusion follows from Theorem 6.15.

**Example 6.18.** We conclude by giving an example of the Newton–Okounkov body for curves in the tropical and in the Arakelovian cases for the same linear system.

Let us consider the example in [II Section 4.4] of a smooth plane quartic curve of $X$ with genus $g(X) = 3$. This is a plane quartic degenerating into a conic $C$ and two lines $\ell_1, \ell_2$. To make the model semistable, one must blow up the intersection point of $\ell_1$ and $\ell_2$, introducing a new component $E$ of the degeneration. For this curve, the special fiber and the dual graph are given in the figure where the vertex $P$ corresponds to the conic, $Q_1, Q_2$ corresponds to the lines, and $P'$ corresponds to the curve $E$.

We will compute the Newton–Okounkov body for a general hyperplane section $\mathcal{D}$, whose specialization is given by $\rho(\mathcal{D}) = \Lambda = 2(P) + (Q_1) + (Q_2)$. Note that for any $\varphi \in L(\Lambda)$ we
have
\[
\Delta(\varphi) = (4\varphi(P) - 2\varphi(Q_1) - 2\varphi(Q_2))(P) \\
+ (3\varphi(Q_1) - \varphi(P') - 2\varphi(P))(Q_1) \\
+ (3\varphi(Q_2) - \varphi(P') - 2\varphi(P))(Q_2) \\
+ (2\varphi(P') - \varphi(Q_1) - \varphi(Q_2))(P').
\]

• **Tropical case:** we will pick as a flag $\mathcal{X} \supseteq Y_1 \supseteq Y_2$, with $Y_1$ a degree one horizontal divisor, intersecting the generic fiber $\mathcal{X}_K$ in a general point and intersecting the closed fiber in a generic point $Y_2$ of the conic $C$. It is straightforward to compute $\varpi_{L_t}$ for $t \in [0, 4]$ as follows:
  - for $t \in [0, 2]$, $\varpi_{L_t} = 0$,
  - for $t \in [2, 4]$, $\varpi_{L_t}(P) = (t - 2)/4$, $\varpi_{L_t}(Q_1) = \varpi_{L_t}(Q_2) = \varpi_{L_t}(P') = 0$.
This gives the following Newton–Okounkov body:

• **Arakelovian case:** in this let $Y_1$ be the conic $C$, and $Y_2$ a general point of $C$. The function $b(t)$ is achieved by the following choices for $\varphi$:
  - for $t \in [0, 1/2]$, $\varphi(P) = t$, $\varphi(Q_1) = \varphi(Q_2) = \varphi(P') = 0$,
  - for $t \in [1/2, \infty)$, $\varphi(P) = t$, $\varphi(Q_1) = \varphi(Q_2) = \varphi(P') = t - 1/2$.
Therefore, the Newton–Okounkov body is as follows:
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