Matrix Models: a Way to Quantum Moduli Spaces

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Abstract

We give the description of discretized moduli spaces (d.m.s.) \( \overline{\mathcal{M}}_{g,n}^{\text{disc}} \) introduced in [1] in terms of a discrete de Rham cohomologies for each moduli space \( \mathcal{M}_{g,n} \) of a genus \( g \), \( n \) being the number of punctures. We demonstrate that intersection indices (cohomological classes) calculated for d.m.s. coincide with the ones for the continuum moduli space \( \overline{\mathcal{M}}_{g,n} \) compactified by Deligne and Mumford procedure. To show it we use a matrix model technique. The Kontsevich matrix model is a generating function for these indices in the continuum case, and the matrix model with the potential \( N \alpha \text{tr} \left( -\frac{1}{2}AXAX + \log(1 - X) + X \right) \) is the one for d.m.s. In the last case the effects of reductions become relevant, but we use the stratification procedure in order to express integrals over open spaces \( \overline{\mathcal{M}}_{g,n}^{\text{disc}} \) in terms of intersection indices which are to be calculated on compactified spaces. The coincidence of the cohomological classes for both continuum and d.m.s. models enables us to propose the existence of a quantum group structure on d.m.s. Then d.m.s. are nothing but cyclic (exceptional) representations of a quantum group related to a moduli space \( \overline{\mathcal{M}}_{g,n} \). Considering the explicit expressions for integrals of Chern classes over \( \overline{\mathcal{M}}_{g,n} \) and \( \overline{\mathcal{M}}_{g,n}^{\text{disc}} \), we conjecture that each moduli space \( \overline{\mathcal{M}}_{g,n} \) in the Kontsevich parametrization can be presented as a coset \( \overline{\mathcal{M}}_{g,n} = T^{d}/G \), \( d = 3g - 3 + n \), where \( T^{d} \) is some \( d \)-dimensional complex torus and \( G \) is a finite order symmetry group of \( T^{d} \).

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1 Introduction.

In our recent papers [1], [2] we have proposed and developed an approach to discretization of arbitrary moduli space of algebraic curve. The connection of these spaces to a matrix model was established and also it was demonstrated explicitly that in the limit when discretization parameter became small this matrix model goes to the Kontsevich one [3]. This model, in its turn, is a generating function for intersection indices or integrals of first Chern classes on the corresponding moduli space. It was a proposal by Witten [4] that these integrals yield correlation functions for the two-dimensional gravity coupled to the matter. Still the notion of the discretization of the moduli space was not very certain. In the present paper we ensure this object with some explicit description and, using the results of [2], we prove the main identity:

\[ \langle \tau_{d_1} \ldots \tau_{d_n} \rangle_g = \langle \tau_{d_1} \ldots \tau_{d_n} \rangle_g, \]  

where in both cases \( \langle \tau_{d_1} \ldots \tau_{d_n} \rangle_g \) are someway determined integrals of products of Chern classes \( \omega_i \) and \( \tilde{\omega}_i \) over moduli spaces. (We shall denote as tilde-variables the discrete counterparts of the continuum objects.) In the continuum case

\[ \langle \tau_{d_1} \ldots \tau_{d_n} \rangle_g = \int_{M_{g,n}} \prod_{i=1}^n \omega_i^{d_i}. \]  

On the l.h.s. of (1.1) \( \langle \tau_{d_1} \ldots \tau_{d_n} \rangle_g \) can be presented in a form similar to (1.2) but with all quantities being related to the discretized moduli space (d.m.s.). In what follows we shall provide a proof to (1.1) using a matrix model technique.

We shall use a fat graph technique in order to introduce coordinates on the moduli spaces. The coordinatization means that we assign lengths \( l_i \) to all edges of the fat graph and the number of punctures, \( n \), is the number of faces of the graph. We call this space \( \mathcal{M}_{g,n}^{\text{comb}} \). The discretization of \( \mathcal{M}_{g,n}^{\text{comb}} \) is rather simple – we assume that all these lengths are to be integer numbers (probably zeros). When taking all these possibilities we get the points of the discretized moduli space \( \overline{\mathcal{M}}_{g,n}^{\text{disc}} \). For a general oriented graph of the genus \( g \) and the number of faces \( n \) the total number of edges (for trivalent vertices of the general position) is \( 6g - 6 + 3n \) which exceeds the dimension of \( \mathcal{M}_{g,n} \) by \( n \). So there are \( n \) extra parameters which are not related to the coordinates on the original moduli space itself. Namely, they are perimeters of the faces of the graph. In the continuum case we have due to Strebel theorem [5] an isomorphism \( \mathcal{M}_{g,n} \otimes \mathbb{R}_+^n \simeq \mathcal{M}_{g,n}^{\text{comb}} \) and we define a projection \( \pi : \mathcal{M}_{g,n}^{\text{comb}} \to \mathbb{R}_+^n \) to the space of perimeters. The fibers \( \pi^{-1}(p) \) of the inverse map are isomorphic to the initial moduli space \( \mathcal{M}_{g,n} \) and hence they all are isomorphic to each other. We are able now also to define another projection \( \tilde{\pi} : \mathcal{M}_{g,n}^{\text{disc}} \to \mathbb{Z}_+^{n} | \sum_{p_i \in \mathbb{Z}_+} \) where all perimeters are strictly positive integers with even total sum and consider its fibers \( \tilde{\pi}^{-1}(p) \). They are, generally speaking, finite sets of points belonging to the initial moduli space \( \overline{\mathcal{M}}_{g,n} \). These sets are no more isomorphic to each other. Moreover, among these points there are always points which correspond exactly to reduced surfaces (“infinity points”). We assume that the space of reduced surfaces is \( \partial \mathcal{M}_{g,n} = \overline{\mathcal{M}}_{g,n} - \mathcal{M}_{g,n} \). In the
usual Teichmüller picture all these points lie at the infinity, but in what follows we should include them into the game. We are able to introduce an analogue of De Rham complex on these spaces using finite difference structures instead of differential ones. There are the spaces we call discretized moduli spaces (d.m.s.). Also instead of $U(1)$–bundles for continuum case we shall consider “$\mathbb{Z}_p$–bundles” over these spaces. Thus we can define cohomological classes for d.m.s. as well

$$\langle \langle \tau_{d_1} \ldots \tau_{d_n} \rangle \rangle = \int_{\pi^{-1}(p_\ast)} \prod_{i=1}^n \omega_i^{d_i}. \quad (1.3)$$

There is an unique (up to isomorphisms) closed moduli space $\overline{\mathcal{M}}_{g,n} = \pi^{-1}(p_\ast)[\mathcal{M}^{\text{comb}}_{g,n}]$ and an infinite series of nonisomorphic $\pi^{-1}(p_\ast)[\mathcal{M}^{\text{disc}}_{g,n}]$, but for all of them the relation (1.1) holds true. We shall present matrix model arguments in favour of this statement, but right now we want to discuss consequences of the relation (1.1).

From the one hand, it means that there exists an invariant which doesn’t depend on which space from the set $\{\pi^{-1}(p_\ast), \pi^{-1}(p_\ast)\}$ we choose. Since we replace the differential structure for $\pi^{-1}(p_\ast)$ by a discrete one for $\pi^{-1}(p_\ast)$, it is natural to look for an analogy with the Quantum Group (QG) description for quantum deformations. Then for any $\pi^{-1}(p_\ast)$ we expect that the space of functions $f[\pi^{-1}(p_\ast)]$ is a representation of the Quantum Group $Q_{g,n}$ related to the initial moduli space $\overline{\mathcal{M}}_{g,n}$. This representation has a feature that it does not contain highest weight vectors, hence it should be a cyclic (exceptional) representation of QG [3, 7]. These representations exist only for specific values of quantum deformation parameters $q_j = e^{2\pi im_j/s_j}$, $(m_j, s_j \in \mathbb{Z})$, the values of $n_j$ being related to the set of perimeters $\{p_j\}$. In particular, for $\overline{\mathcal{M}}_{g,n}$ with $n = 1$ (one puncture) $s_j = p_j/2$. Then the intersection indices themselves are values of some invariant traces of quantum operators acting on these representations. It is worth to note that due to orbifold structure of the moduli space $\overline{\mathcal{M}}_{g,n}$ we can view it as a coset of some finite covering $\mathcal{T}_{g,n}$ of $\overline{\mathcal{M}}_{g,n}$ over a symmetry group $G$: $\overline{\mathcal{M}}_{g,n} = \mathcal{T}_{g,n}/G$. It means that these representations themselves possess this internal symmetry group.

From the other hand, we shall show that in the Kontsevich’s parametrization the very evaluation of the integrals over $\pi^{-1}(p_\ast)$ and $\pi^{-1}(p_\ast)$ can be reduced to a calculation of integrals over volume forms on abovementioned finite coverings $\mathcal{T}_{g,n}$. The discretization on this language means that we introduce an equidistant lattice on $\mathcal{T}_{g,n}$ and while calculating the volume we merely count a total number of sites in this lattice and divide it by some product of $p_1^{a_1}, p_1^{2a_1} \ldots p_n^{a_n}$ where $\sum_{i=1}^n a_i = d = 3g - 3 + n$ – the total dimension of $\overline{\mathcal{M}}_{g,n}$. Note also that all $\mathcal{T}_{g,n}$ are compact spaces without boundaries. One may imagine that when doing the sum over all points of the lattice they contribute to this sum together with unit cubes therefore calculating the volume of $\mathcal{T}_{g,n}$. But it is true only if all these points are nonsingular points, i.e. of zero curvature! Because of a huge variety of sets $\pi^{-1}(p_\ast)$ we know that for each orbifold point there exists a set $\{p_1, \ldots, p_n\}$ which includes this point. Were there exists a singularity of metric in this point, then the equality (1.1) is broken, but it contradicts to our proof of its validity in which we use only matrix model arguments without any reference to an underlying geometrical structure. It means that
there is no points of nonzero internal curvature in the space $T_{g,n}$, hence $T_{g,n}$ should be a torus of the complex dimension $d$, and in the Kontsevich parametrization:

$$\overline{\mathcal{M}}_{g,n} = T^d/G,$$

where $T^d$ is a $d$–dimensional complex torus and $G$ is a finite order symmetry group of $T^d$. In particular, the order of this group is a common denominator for all intersection indices on this moduli space.

In order to find a connection between moduli spaces $\overline{\mathcal{M}}_{g,n}$ and d.m.s. we shall use a matrix model technique. Matrix models recently revealed a lot of applications in various branches of mathematical physics: two–dimensional quantum field theory, intersection theory on the moduli space of Riemann surfaces, etc.

An old days concept for usual 1–matrix hermitian model was the following: in a “fat graph” technique starting with each graph we can construct the dual one corresponding to some Riemann surface with singularities of curvature concentrated in vertices of this dual graph. Then faces of this graph correspond to vertices in the matrix model graph and vice versa. If the initial potential contains only three valent vertices we can speak about “triangulation” of the Riemann surface. In what follows we shall deal with potentials of an arbitrary order, but we use the same term “triangulation”. The model with an arbitrary potential was solved exactly in [8] in the double scaling limit when the number of triangles tends to infinity and these singular metrics approximate “random metrics” on the surface. These model was presented by a hermitian $N \times N$ one–matrix model

$$\int \exp(\, \text{tr} \, P(X)) \, DX,$$

where $P(X) = \sum_n T_n \, \text{tr} \, X^n$, $T_n$ being times for the one–matrix model. For such system discrete Toda chain equations holds with an additional Virasoro symmetry imposed [8]. In the limit $N \to \infty$ the Korteveg–de–Vries equation arises. The partition function of the two–dimensional gravity for this approach is a series in an infinite number of variables and coincides with the logarithm of some $\tau$–function for KdV hierarchy.

Another approach to the two–dimensional gravity is to do the integral over all classes of conformally equivalent metrics on Riemann surfaces. It may be presented as an integral over the finite–dimensional space of conformal structures. This integral has a cohomological description as an intersection theory on the compactified moduli space of complex curves. Edward Witten presented compelling evidence for a relationship between random surfaces and the algebraic topology of moduli space [1], [3]. In fact, he suggested that these expressions coincide since both satisfy the same equations of KdV hierarchy. It was Maxim Kontsevich who proved this assumption [3]. Surprisingly, he explicitly presented a new matrix model defining exactly the values of intersection indices or, on the language of 2D gravity, correlation functions of observables $\mathcal{O}_n$ of the type

$$< \mathcal{O}_{n_1} \ldots \mathcal{O}_{n_s} >_g,$$
where $< \ldots >_g$ denotes the expectation value on a Riemann surface with $g$ handles. Then the string partition function $\tau(t)$ has an asymptotic expansion of the form

$$\tau(t) = \exp \sum_{g=0}^{\infty} \left\langle \exp \sum_n t_n O_n \right\rangle_g,$$

and it is certainly a tau–function of the KdV hierarchy taken at a point of Grassmannian where it is invariant under the action of the set of the Virasoro constraints: $L_n \tau(t) = 0, \ n \geq -1$ \[1\], \[2\], \[3\], \[4\]. One might say that the Kontsevich model is used to triangulate moduli space, whereas the original models triangulated Riemann surfaces (see e.g. \[5\]).

The generalization of the Kontsevich model — so-called Generalized Kontsevich Model (GKM) \[6\] is related to the two–dimensional Toda lattice hierarchy and it originated from the external field problem defined by the integral

$$Z[\Lambda; N] = \int DX \exp \left\{ N \text{tr} (\Lambda X - V_0(X)) \right\},$$

where $V_0(X) = \sum_n t_n \text{tr} X^n$ is some potential, $t_n$ are related to times of the hierarchy. This model is equivalent to the Kontsevich integral for $V_0(X) \sim \text{tr} X^3$. To solve the integral (1.8) one may use the Schwinger–Dyson equation technique \[7\] written in terms of eigenvalues of $\Lambda$. The Kontsevich model was solved in the genus expansion in the papers \[8\], \[9\] for genus zero (planar diagrams) and in \[10\] for higher genera.

Recently, the Kontsevich–Penner model was introduced \[11\]. The Lagrangian of this model has the following form:

$$Z[\Lambda] = \int DX \exp \left( N \text{tr} \left\{ -\frac{1}{2} \Lambda X \Lambda X + \alpha[\log(1+X) - X] \right\} \right), \quad \Lambda = \text{diag}(\Lambda_1, \ldots, \Lambda_N).$$

This model may be readily reduced to (1.8) with $V_0(X) = -X^2/2 + \alpha \log X$. It was solved in genus expansion in \[12\], \[13\]. It appears (see \[14\], \[15\]) that it is in fact equivalent to the one–matrix hermitean model (1.5) with the general potential

$$P(X) = \sum_{n=0}^{\infty} T_n \text{tr} X^n,$$

which times are defined by the kind of Miwa transform ($\eta = \Lambda - \alpha \Lambda^{-1}$):

$$T_n = \frac{1}{n} \text{tr} \eta^{-n} - \frac{N}{2} \delta_{n2} \quad \text{for} \ n \geq 1 \quad \text{and} \quad T_0 = \text{tr} \log \eta^{-1}.$$

It was demonstrate in \[11\], that this new model describes in a natural way the intersection indices for the case of d.m.s. The only complication is that this model does not present generating function for the indices (1.3) straightforwardly because of contribution from reductions. Indeed, any matrix model can deal with only open strata of a moduli space. It was not essential for the case of the Kontsevich model since there the integration went over cells of the highest dimension in the simplicial complex partition of the moduli space $\mathcal{M}_{g,n}$. 


All singular points are simplices of lower dimensions in $\mathcal{M}_{g,n}$ and give no contribution to the integral. But in the case of d.m.s. integrals over simplices of all dimensions are relevant due to the total discretization, so the integrals over reduced surfaces give non-zero contribution which we should exclude in order to compare with the matrix model. The way to do it is to use a stratification procedure $[24]$ which permits to express open moduli space $\mathcal{M}_{g,n}$ via $\mathcal{M}_{g,n}$ and moduli spaces of lower genera.

All this reveals one new side of seems–to–be–well–known standard one–matrix model. This model appears to have the direct relation to 2D–gravity not only in the d.s.l. but also beyond it. On this language d.s.l. means that we introduce using the Laplace transform a small parameter $\varepsilon$ and while $\varepsilon$ goes to zero the leading contribution to the sum over d.m.s. originates from d.m.s. with large values of $p_i$, i.e. from d.m.s. with more and more dense distribution of points of d.m.s. inside the original moduli space $\mathcal{M}_{g,n}^{\text{disc}}$. Note that on the language of quantum description $\varepsilon$ plays the role of the Plank constant $\eta$.

In the paper $[2]$ the explicit solution to the Kontsevich–Penner, or, equivalently, to the general one–matrix model was found in genus expansion. The key role in this consideration was played by so-called “momenta” of the potential resembling in many details “momenta” which appeared in genus expansion solution to the Kontsevich model $[19]$. We shall use some proper reexpansion of these momenta in terms of new quantities which stand just by the intersection indices (1.2), (1.3). These new variables are quantum analogues of the Kontsevich “times” $T_n = \text{tr} \Lambda^{-2n-1}$. Summing up we stress that the usual one–matrix model appears to be still rather interesting even beyond the double scaling limit, and it may provide a way to quantization of the moduli spaces.

The paper is organized as follows: The short review of geometric approach to the Kontsevich model is given in Section 2. The Penner–Kontsevich model as well as the definition of the discretized moduli spaces are presented in Section 3. In Section 5 the matrix model technique is actively used in order to compare both models and it is shown by this comparison that the intersection indices in both models coincide. The proposal for the orbifold structure of modular spaces in the Kontsevich parametrization is the matter of Section 5. Also we discuss there the possibility to introduce Quantum Group structures on these discrete moduli spaces. Then the short Conclusion section is followed by Appendix in which we present an explicit solution for the modular space $\mathcal{M}_{2,1}$ of genus 2.

## 2 The geometric approach to the Kontsevich model.

In his original paper $[3]$ Kontsevich proved that

$$
\sum_{d_1, \ldots, d_n = 0}^{\infty} <\tau_{d_1}, \tau_{d_2}, \ldots, \tau_{d_n}> = \prod_{i=1}^{n}(2d_i - 1)!!\lambda_i^{-(2d_i+1)} = \sum_{\Gamma} \frac{2^{-\#X_0}}{\# \text{ Aut} (\Gamma)} \prod_{\{ij\}} \frac{2}{\lambda_i + \lambda_j}.
$$

where the objects standing in angular brackets on the left–hand side are (rational) numbers describing intersection indices, and on the right–hand side the sum runs over all...
oriented connected trivalent “fatgraphs” $\Gamma$ with $n$ labeled boundary components, regardless of the genus, $#X_0$ is the number of vertices of $\Gamma$, the product runs over all the edges in the graph and $#\text{Aut}$ is the volume of discrete symmetry group of the graph $\Gamma$.

The amazing result by Kontsevich is that the quantity on the right hand side of (2.1) is equal to a free energy in the following matrix model:

$$e^{F_N(\Lambda)} = \frac{\int dX \exp \left(-\frac{1}{2} \text{tr} \Lambda X^2 + \frac{1}{6} \text{tr} X^3 \right) \int dX \exp \left(-\frac{1}{2} \text{tr} \Lambda X^2 \right)}{\int dX \exp \left(-\frac{1}{2} \text{tr} \Lambda X^2 \right)},$$

where $X$ is an $N \times N$ hermitian matrix and $\Lambda = \text{diag} (\lambda_1, \ldots, \lambda_N)$. The distinct feature of the expression (2.1) is that in spite of the fact that each selected diagram has quantities $(\lambda_i + \lambda_j)$ in the denominator, when taking a sum over all diagrams of the same genus and the same number of boundary components all these quantities are cancelled with the ones from nominator.

Feynman rules for the Kontsevich matrix model are the following: as in the usual matrix models, we deal with so-called “fat graphs” or “ribbon graphs” with propagators having two sides, each carries corresponding index. The Kontsevich model varies from the standard one–matrix hermitian model since there appear additional variables $\lambda_i$ associated with index loops in the diagram, the propagator being equal to $2/(\lambda_i + \lambda_j)$, where $\lambda_i$ and $\lambda_j$ are variables of two cycles (perhaps the same cycle) which the two sides of propagator belong to. Also there are trivalent vertices presenting the cell decomposition of the moduli space.

It is instructive to consider the simplest example of genus zero and three boundary components which we symbolically label $\lambda_1$, $\lambda_2$ and $\lambda_3$. There are two kinds of diagrams giving the contribution in this order (Fig.1). The contribution to the free energy arising from this sum is

$$\frac{1}{6(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)} + \frac{1}{3} \left\{ \frac{1}{4\lambda_1(\lambda_2 + \lambda_1)(\lambda_3 + \lambda_1)} \right.$$ $$+ \left. (1 \rightarrow 2, 2 \rightarrow 3, 3 \rightarrow 1) + (1 \rightarrow 3, 3 \rightarrow 2, 2 \rightarrow 1) \right\}$$

$$= \frac{2\lambda_1\lambda_2\lambda_3 + \lambda_2\lambda_3(\lambda_2 + \lambda_3) + \lambda_1\lambda_3(\lambda_1 + \lambda_3) + \lambda_1\lambda_2(\lambda_1 + \lambda_2)}{12\lambda_1\lambda_2\lambda_3(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)}$$

$$= \frac{1}{12\lambda_1\lambda_2\lambda_3}. \quad (2.3)$$

This example demonstrates the cancellations of $(\lambda_i + \lambda_j)$–terms in the denominator above mentioned.
2.1 Geometry of fiber bundles on $\overline{M}_{g,n}$.

Now the sketch of Kontsevich’s proof is in order. Let us associate with each edge $e_i$ of a fat graph its length $l_i > 0$. We consider the orbispace $\mathcal{M}_{g,n}^{\text{comb}}$ of fat graphs with all possible lengths of edges and arbitrary valencies of vertices. Two graphs are equivalent if an isomorphism between them exists. Let us introduce an important object — the space of $(2,0)$–meromorphic differentials $\omega(z)dz^2$ on a Riemann surface with $g$ handles and $n$ punctures, the only poles of $\omega(z)$ are $n$ double poles placed in the points of punctures with strictly positive quadratic residues $p_i^2 > 0, \ (i = 1, \ldots, n)$. It is Strebel’s theorem which claims that the natural mapping from $\mathcal{M}_{g,n}^{\text{comb}}$ to the moduli space $\mathcal{M}_{g,n} \otimes \mathbb{R}^n_+$, where $\mathbb{R}^n_+$ is the space of residues, $p_i > 0$ being perimeters of cycles, is homeomorphism. Thus, varying $l_j$ and taking the composition of all graphs we span the whole space $\mathcal{M}_{g,n} \otimes \mathbb{R}^n_+$.

Each cycle can be interpreted as a boundary components $I_i$ of the Riemann surface since in the Strebel metric it can be presented as half-infinite cylinder with the puncture point placed at infinity. The boundary of it consists of a finite number of intervals (edges). We consider a set of line bundles $L_i$ which fiber at a point $\Sigma \in \mathcal{M}_{g,n}$ is the cotangent space to the puncture point $x_i$ on the surface $\Sigma$. The first Chern class $c_1(L_i)$ of the line bundle $L_i$ admits a representation in terms of the lengths of the intervals $l_j$. The perimeter of the boundary component is $p_i = \sum_{l_{\alpha} \in I_i} l_{\alpha}$.

The first step in constructing $c_1(L_i)$ is to determine $\alpha_i$ which is $U(1)$–connection on the boundary component corresponding to $i$th puncture. In order to explicitly describe this construction it is convenient to introduce “polygon bundles” $BU(1)^{\text{comb}}_{(i)}$ in Kontsevich’s notations. These polygon bundles are sets of equivalent classes of all sequences of positive real numbers $l_1, \ldots, l_k$ modulo cyclic permutations.

$BU(1)^{\text{comb}}$ is the moduli (orbi)space of numbered ribbon graphs with metric whose underlying graphs are homeomorphic to the circle. There is an $S^1$–bundle over this orbispace whose total space $EU(1)^{\text{comb}}$ is an ordinary space. The fiber of the bundle over the equivalence class of sequences $l_1, \ldots, l_k$ is a union of intervals of lengths $l_1, \ldots, l_k$.

Figure 1. the $g=0, s=3$ contribution to Kontsevich’s model
with pairwise glued ends, i.e. a polygon. One may prove that the map \( \mathcal{M}_{g,n} \otimes \mathbb{R}^n_+ \to (BU(1)^{comb})^n \) extend continuously to \( \overline{\mathcal{M}}_{g,n} \times \mathbb{R}^n_+ \). The inverse images of \( S^1 \)–bundles are naturally isomorphic to the circle bundles associated with the complex line bundles \( L_i \).

Let us now compute the first Chern class of the circle bundle on \( BU(1)^{comb} \). The points of \( EU(1)^{comb} \) can be identified with pairs \( (p, S) \) where \( p \) is a perimeter and \( S \) is a nonempty finite subset (vertices) of the circle \( \mathbb{R}/p\mathbb{Z} \). Denote \( 0 \leq \phi_1 < \ldots < \phi_k < p \) representatives of points of \( S \).

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\[
l_i = \phi_{i+1} - \phi_i \quad (i = 1, \ldots, k - 1), \quad l_k = p + \phi_1 - \phi_k.
\]

Now we should choose some convenient form for \( S^1 \)–connections on these polygon bundles. Denote by \( \alpha \) the 1-form on \( EU(1)^{comb} \) which is equal to

\[
\alpha = \sum_{i=1}^{k} \frac{l_i}{p} \times d \left( \frac{\phi_i}{p} \right).
\]

Certainly \( \alpha \) is well–defined and the integral of it over each fiber of the universal bundle \( EU(1)^{comb} \to BU(1)^{comb} \) is equal \(-1\). The differential \( d\alpha \) is the pullback of a 2–form \( \omega \) on the base \( BU(1)^{comb} \),

\[
\omega = \sum_{1 \leq i < j \leq k-1} d \left( \frac{l_i}{p} \right) \wedge d \left( \frac{l_j}{p} \right).
\]

Extrapolating these results to the compactified moduli spaces we obtain that the pullback \( \omega_i \) of the form \( \omega \) under the \( i \)th map \( M_{g,n} \times \mathbb{R}^n_+ \to BU(1)^{comb} \) represents the class \( c_1(L_i) \).

Denote by \( \pi : \mathcal{M}_{g,n}^{comb} \to \mathbb{R}^n_+ \) the projection to the space of perimeters. Intersection indices are given by the formula:

\[
\langle \tau_{d_1} \cdots \tau_{d_n} \rangle = \int_{\pi^{-1}(p_*)} \prod_{i=1}^{n} \omega_i^{d_i},
\]

where \( p_* = (p_1, \ldots, p_n) \) is an arbitrary sequence of positive real numbers and the \( \pi^{-1}(p_*) \) is a fiber of \( \overline{\mathcal{M}}_{g,n} \) in \( \mathcal{M}_{g,n}^{comb} \).

### 2.2 Matrix Integral

We denote by \( \Omega \) the two–form on open strata of \( \mathcal{M}_{g,n}^{comb} \):

\[
\Omega = \sum_{i=1}^{n} p_i^2 c_1(L_i),
\]

which restriction to the fibers of \( \pi \) has constant coefficients in the coordinates \( (l(e)) \). Denote by \( d \) the complex dimension of \( \mathcal{M}_{g,n} \), \( d = 3g - 3 + n \). The volume of the fiber of \( \pi \) with respect to \( \Omega \) is

\[
\text{vol}(\pi^{-1}(p_1, \ldots, p_n)) = \int_{\pi^{-1}(p_*)} \frac{\Omega^d}{d!} = \frac{1}{d!} \int_{\pi^{-1}(p_*)} (p_1^2 c_1(L_1) + \ldots + p_n^2 c_1(L_n))^d =
\]

\[
= \sum_{d_i = d} \prod_{i=1}^{n} \frac{p_i^{2d_i}}{d_i!} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g.
\]
One important note is in order. It is a theorem by Kontsevich that these integrations extend continuously to the closure of the moduli space \( \overline{M}_{g,n} \) following the procedure by Deligne and Mumford \cite{Deligne}. (It means that we deal with a stable cohomological class of curves.)

In order to compare with a matrix model we should take the Laplace transform over variables \( p_i \) of volumes of fibers of \( \pi \):

\[
\int_0^\infty dp_i e^{-p_i \lambda_i} p_i^{2d_i} = (2d_i)! \lambda_i^{-2d_i-1}
\]

for the quantities standing on the right–hand side of (2.9). On the left–hand side we have

\[
\int_0^\infty \ldots \int_0^\infty dp_1 \wedge \ldots \wedge dp_n e^{-\sum p_i \lambda_i} \int_{\overline{M}_{g,n}} e^\Omega,
\]

and due to cancellations of all \( p_i^2 \) multipliers with \( p_i \)'s in denominators of the form \( \Omega \) we get:

\[
e^\Omega dp_1 \wedge \ldots \wedge dp_n = \rho \prod_{e \in X_1} dl_e.
\]

We use standard notations: \( X_q \) is a total number of \( q \)–dimensional cells of a simplicial complex. \( (X_1 \) is the number of edges, \( X_0 \) – the number of vertices, etc). \( \rho \) is a positive function defined on open cells, it is equal to the ratio of measures:

\[
\rho = \left( \prod_{i=1}^n |dp_i| \times \frac{Q^d}{d!} \right) : \prod_{e \in X_1} |dl(e)|.
\]

Surprisingly, the constant \( \rho \) does depend only on Euler characteristic of the graph \( \Gamma \),

\[
\rho = 2^{-\kappa}, \quad \rho = 2^{d+\#X_1-\#X_0}.
\]

The integral

\[
I_g(\lambda_*) := \int_{\mathcal{M}_{g,n}^{\text{comb}}} \exp(-\sum \lambda_i p_i) \prod_{e \in X_1} |dl(e)|
\]

is equal to the sum of integrals over all open strata in \( \mathcal{M}_{g,n}^{\text{comb}} \). These open strata are in correspondence with a complete set of three–valent graphs contributing to this order in \( g \) and \( n \). It is necessary also to take into account internal automorphisms of the graph (their number, in fact, counts how many replicas of moduli space one may find in this cell if one treat all \( l_e \) independently). The last step is to perform the sum \( \sum \lambda_i p_i \) in a form dependent on \( l_e \). By a standard procedure for simplicial complexes

\[
\sum_{i=1}^n \lambda_i p_i = \sum_{e \in X_1} l_e (\lambda_e^{(1)} + \lambda_e^{(2)}).
\]

Here \( \lambda_e^{(1)} \) and \( \lambda_e^{(2)} \) are variables of two cycles divided by \( e \)th edge. Performing now the Laplace transform we get the relation (2.1). The quantity standing in the r.h.s. is nothing
but a term from $1/N$ expansion of the Kontsevich matrix model and eventually we have:

$$\sum_{g=0}^{\infty} N^{2-2g} \alpha^{2g-n} \prod_{s_1+2s_2+\ldots+ks_k=d} \langle (\tau_0)^{s_0} \ldots (\tau_k)^{s_k} \rangle_g \frac{1}{s_0! \ldots s_k!} \prod_{i=1}^{n} \text{tr} \frac{(2d_i - 1)!!}{\Lambda^{2d_i + 1}}$$

Thus the Kontsevich matrix model is a generating function for intersection indices of the first Chern classes on moduli (orbi)spaces.

3. The Penner–Kontsevich model.

Now let us turn to the case of the Penner–Kontsevich model (PK model) [20], [22] with the partition function given by

$$Z[\Lambda] = \frac{\int DX \exp \left( \alpha N \text{tr} \left\{-\frac{1}{4} \Lambda X^3 X - \frac{1}{2} [\log(1 - X) + X] \right\} \right)}{\int DX \exp \left( \alpha N \text{tr} \left\{-\frac{1}{2} \Lambda X^3 X + \frac{1}{4} X^2 \right\} \right)}, \quad \Lambda = \text{diag}(\mu_1, \ldots, \mu_N).$$

(3.1)

It includes in variance with the Kontsevich model all powers of $X^n$ in the potential since it describes the partition of moduli space into cells of a simplicial complex, the sum running over all simplices with different dimensions. (On the language of the Kontsevich model the lower the dimension, the more and more edges of the fat graph are reduced).

We find the Feynman rules for the Kontsevich–Penner theory (1.9). First, as in the standard Penner model, we have vertices of all orders in $X$. Due to rotational symmetry, the factor $1/n$ standing with each $X^n$ cancels, and only symmetrical factor $1/#\text{Aut} \Gamma$ survives. Also there is a factor $(\alpha/2)$ standing with each vertex. As in the Kontsevich model, there are variables $\mu_i$ associated with each cycle. But the form of propagator changes — instead of $2/(\lambda_i + \lambda_j)$ we have $2/(\mu_i \mu_j + \alpha)$.

Let us consider the same case ($g = 0, n = 3$) as for Kontsevich model. One additional diagram resulting from vertex $X^4$ arises (Fig.2).

Fig.2. $g=0, s=3$ contribution to Penner–Kontsevich’s model.
This contribution is (symmetrized over \(\mu_1, \mu_2\) and \(\mu_3\)):

\[
- \frac{1}{3} \left\{ \frac{2\alpha^{-1}}{2(\mu_1\mu_2 - 1)(\mu_1\mu_3 - 1) + \text{perm.}} \right\} + \frac{2\alpha^{-1}}{6(\mu_1\mu_2 - 1)(\mu_1\mu_3 - 1)(\mu_2\mu_3 - 1)} \\
+ \frac{1}{3} \left\{ \frac{2\alpha^{-1}}{2(\mu_1^2 - 1)(\mu_1\mu_2 - 1)(\mu_1\mu_3 - 1) + \text{perm.}} \right\}
\]

(3.2)

Again collecting all terms we get:

\[
\frac{2\alpha^{-1}}{6 \prod_{i<j}(\mu_i\mu_j - 1)} \left\{ \sum_{i<j} \mu_i\mu_j - 2 + \left( \frac{\mu_2\mu_3 - 1}{\mu_1^2 - 1} + \frac{\mu_1\mu_2 - 1}{\mu_2^2 - 1} + \frac{\mu_1\mu_3 - 1}{\mu_3^2 - 1} \right) \right\},
\]

(3.3)

and after a little algebra we obtain an answer:

\[
F_{0,3} = \alpha^{-1} \frac{\mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3 + 1}{3(\mu_1^2 - 1)(\mu_2^2 - 1)(\mu_3^2 - 1)}.
\]

(3.4)

We see that here, just as in standard Kontsevich model, the cancellation of intertwining terms in the denominator occurs that leads to factorization of the answer over \(1/(\mu_i^2 - 1)\)–terms. This simplest example shows that there should be some underlying geometric structures in this case as well.

Note that technically the reason why there is the dependence only on \(\text{tr} \Lambda^k\) \((k \leq 0)\) for the partition function (3.1) is because this model as well as the Kontsevich one belongs to the class of Generalized Kontsevich Models [16]. It means that after some simple transformation we can get from (3.1) the model with the potential \(\Lambda X + V(X)\) for which only the dependence on Miwa’s times exists.

### 3.1 The Discretized Moduli Space.

Let us now consider a discretization of the moduli spaces \(\overline{\mathcal{M}}_{g,n}\) and \(\mathcal{M}^{\text{comb}}_{g,n}\):

\[
l_i \in \mathbb{Z}_+ \cup \{0\}, \quad p_i \in \mathbb{Z}_+, \quad \sum_{i=1}^{n} p_i \in 2\mathbb{Z}_+.
\]

(3.5)

So all \(l_i\) and \(p_i\) are now integers, but some of \(l_i\) can be zeros while all perimeters are strictly positive, the sum of all perimeters must be even, because each edge contributes twice into this sum. We call this (combinatorial) space \(\overline{\mathcal{M}}_{g,n}^{\text{disc}}\). It is worth to note that now we explicitly include into play such points of the original \(\overline{\mathcal{M}}_{g,n}\) which are points of reductions. It is easy to see from (3.5). While keeping all \(p_i\) fixed in a general case we can put a number of \(l_j\) exactly equal zero. Some of these configurations belong to interior of \(\mathcal{M}_{g,n}\), but not all — it means that among the points of \(\overline{\mathcal{M}}_{g,n}^{\text{disc}}\) there are points which lie on the boundary \(\partial \mathcal{M}_{g,n}\) and they correspond to reductions of the algebraic curve. Also we shall use the notation \(\mathcal{M}_{g,n}^{\text{disc}}\) for such subset of \(\overline{\mathcal{M}}_{g,n}^{\text{disc}}\) where all points of reduction are excluded.

It appears that this choice for d.m.s. is rather natural since all the quantities (2.5–2.8) have corresponding counterplates in this discrete case.
First we need to define the action of the external derivative $d$ and the integration over these (orbi)spaces. We shall write its action on functions, the extrapolation to the space of skewsymmetric forms is obvious

$$df(l_1, \ldots, l_k) = \sum_{i=1}^{k} (f(l_1, \ldots, l_i + 1, \ldots, l_k) - f(l_1, \ldots, l_k)) dl_i$$ (3.6)

As for the integral over a domain $\Omega$, there is again a proper generalization of it to this discrete case:

$$\int_\Omega f(l_1, \ldots, l_k) dl_1 \ldots dl_k := \sum_{l_i \in \mathbb{Z}_p \cup \{0\}} \sum_{\{l_1, \ldots, l_k\} \in \Omega} f(l_1, \ldots, l_k).$$ (3.7)

Instead of $BU(1)^{comb}$ we have (orbi)space of equivalence classes of all sequences of non-negative integers $l_1, \ldots, l_k$ modulo cyclic permutations. An analog of $S^1$–bundle is now a kind of $\mathbb{Z}_p$–"bundle" over this new discrete orbispace whose total space $E\mathbb{Z}_p^{comb}$ is an ordinary rectangular lattice. The fiber of the bundle over the equivalence class of sequences $l_1, \ldots, l_k$ is again the polygon with integer lengths of edges $l_1 \ldots, l_k$.

We denote $\phi_i$ coordinates on $E\mathbb{Z}_p$ just as in (2.4):

$$l_i = \phi_{i+1} - \phi_i \quad (i = 1, \ldots, k - 1), \quad l_k = p + \phi_1 - \phi_k.$$ (3.8)

Due to linearity of (2.5) in $l_i$, and $\phi_j$ it can be straightforwardly generalized to our case. Denote by $\tilde{\alpha}$ the 1–form on $E\mathbb{Z}_p^{comb}$ which is equal to

$$\tilde{\alpha} = \sum_{i=1}^{k} \frac{l_i}{p} \times \frac{d\phi_i}{p}.$$ (3.9)

The integral of $\tilde{\alpha}$ over each fiber of the universal bundle $E\mathbb{Z}_p^{comb} \to BU(1)^{comb}$ is equal $-1$. The differential $d\tilde{\alpha}$ is the pullback of a 2–form $\tilde{\omega}$ on the base $B\mathbb{Z}_p^{comb}$,

$$\tilde{\omega} = \sum_{1 \leq i < j \leq k-1} \frac{dl_i}{p} \wedge \frac{dl_j}{p}.$$ (3.10)

Extrapolating these results to the whole discrete moduli space we obtain that the pullback $\tilde{\omega}_i$ of the form $\tilde{\omega}$ under the ith map $\overline{\mathcal{M}}_{g,n}^{disc} \to B\mathbb{Z}_p^{comb}$ represents the class $\tilde{c}_1(\mathcal{L}_i)$.

Denote by $\tilde{\pi} : \overline{\mathcal{M}}_{g,n}^{disc} \to [\mathbb{Z}_n]_{even}$ the projection to the space of perimeters with the restriction $\sum_i p_i \in 2 \cdot \mathbb{Z}_+$. Intersection indices are given again by the formula:

$$\langle \langle \tau_{d_1} \ldots \tau_{d_n} \rangle \rangle = \int_{\tilde{\pi}^{-1}(p_s)} \prod_{i=1}^{n} \tilde{\omega}_i^{d_i},$$ (3.11)

where $p_s = (p_1, \ldots, p_n)$ is an arbitrary sequence of positive integer numbers with even sum and the $\pi^{-1}(p_s)$ is an analogue of the fiber of $\overline{\mathcal{M}}_{g,n}$ in $\overline{\mathcal{M}}_{g,n}^{disc}$.

One important note is in order. These fibers $\tilde{\pi}^{-1}(p_s)$ contain finite number of points each and are not isomorphic to each other. But they are just the analogues of the initial
moduli space $\mathcal{M}_{g,n}$ labelled by different perimeters. This is a point why we call them discretized moduli spaces. It appears that without any reference how many points from the initial $\mathcal{M}_{g,n}$ participate in a fiber $\pi^{-1}(p_\ast)$ (it can be even only one point of reduction, as we shall see for $\mathcal{M}_{1,1}$), the relation (3.11) remains valid. Intuitive considerations which can convince us in it are the following: Values of these intersection indices are some rational numbers due to the orbifold nature of the initial moduli space $\mathcal{M}_{g,n}$. On the language of graphs this nature reveals itself as symmetries of the graphs. But these symmetry properties are the same whatever case – continuum or discrete, and whatever values of perimeters we choose. Thus preserving symmetry properties we preserve the values of cohomological classes on both continuum and discrete moduli spaces.

3.2 Matrix Integral for Discretized Moduli Space.

We denote by $\tilde{\Omega}$ the two–form on $\mathcal{M}^{\text{disc}}_{g,n}$:

$$\tilde{\Omega} = \sum_{i=1}^{n} p_i^2 \tilde{\omega}_i,$$

which restriction to the fibers of $\tilde{\pi}$ has constant coefficients in the coordinates $(l(e))$. $d$ is again the complex dimension of $\mathcal{M}_{g,n}$, $d = 3g - 3 + n$. The volume of the fiber of $\tilde{\pi}$ with respect to $\tilde{\Omega}$ is

$$\text{vol}(\tilde{\pi}^{-1}(p_1, \ldots, p_n)) = \int_{\tilde{\pi}^{-1}(p_\ast)} \frac{\tilde{\Omega}^d}{d!} = \frac{1}{d!} \int_{\tilde{\pi}^{-1}(p_\ast)} (p_1^2 \tilde{\omega}_1 + \ldots + p_n^2 \tilde{\omega}_n)^d =$$

$$= \sum_{d_i=d} \prod_{i=1}^{n} p_i^{2d_i} (\langle \tau_{d_1} \cdots \tau_{d_n} \rangle)_g. \quad (3.13)$$

Next step is to do a Laplace transform in both sides of (3.13). Of course, now we should replace continuum Laplace transform by the discrete one and also explicitly take into account that sum of all $p_i$ is even. On the R.H.S. we have:

$$\sum_{\sum p_i \in Z_+} e^{-\sum_i \lambda_i p_i^{2d_i} \cdots p_i^{2d_n}} = \prod_{i=1}^{n} \left( \frac{\partial}{\partial \lambda_i} \right)^{2d_i} \times \frac{1}{2} \left\{ \prod_{i=1}^{n} \frac{1}{e^{\lambda p_i} - 1} + (-1)^n \prod_{i=1}^{n} \frac{1}{e^{\lambda p_i} + 1} \right\}. \quad (3.14)$$

On the L.H.S. of (3.13) we again substitute

$$e^{\tilde{\Omega}} dp_1 \wedge \ldots dp_n \big|_{\sum p_i \in 2Z_+} = \tilde{\rho} \prod_{e \in X_1} dl_e. \quad (3.15)$$

Here the constant $\tilde{\rho}$ is the ratio of measures similar to (2.13) and we only need to take into account the restriction that the sum of all $p_i$ is even. It leads to renormalization of the $\rho$ for the case of d.m.s.:

$$\tilde{\rho} = \rho/2, \quad (3.16)$$

where $\rho$ is given by (2.14).
Now we want to find a matrix model description for these “new” intersection indices. Here we immediately encounter some troubles. Let us look which graphs correspond to different points of $\tilde{\pi}^{-1}(p_*)$. First, there are points of a general position for which all $l_i$ are greater than zero, which correspond to graph with only trivalent vertices. Second, there are such points of $\tilde{\pi}^{-1}(p_*)$ that some $l_i$ are zeros, but these points still do not correspond to reductions. For example, see Fig.3, where for the torus case Fig.3a represents a point of a general position, $l_i > 0$, $i = 1, 2, 3$, and Fig.3b gives an example of the graph for which one (and only one) of $l_i$ is zero. Such graphs by no means correspond to a reduction point, but rather belong to subspace of $\mathcal{M}_{g,n}$ (in Teichmüller parametrization) with modular parameter $\tau$ being purely imaginary. But certainly, if we want to include such graphs into consideration we should consider not only trivalent vertices, but vertices of arbitrary order. In the continuum limit we did not take into account such graphs since they correspond to subdomains of lower dimensions in the interior of the moduli space, the integration measure being continuous, and we may neglect them. Now the situation changed and we should take into account all such diagrams as well.

![Fig.3a](image1) ![Fig.3b](image2) ![Fig.3c](image3)

**Fig.3.** Diagrams for different regions of $\overline{\mathcal{M}}_{1,1}$

But in each $\tilde{\pi}^{-1}(p_*)$ there are always (except the case $\mathcal{M}_{0,3}$) genuine points of reduction (see, for example Fig.3c when two of $l_i$ are zeros). We are not able to give a matrix model description to such points. At first sight it would mean that all the construction fails since we still not touch the question how to “exclude” such reduction points from $\tilde{\pi}^{-1}(p_*)$ modifying in a way the relation (3.11). We shall call $\mathcal{M}^{disc}_{g,n}$ such subset of $\overline{\mathcal{M}}^{disc}_{g,n}$ where all points of reduction are excluded. Thus we need to release somehow the integration over open $\mathcal{M}_{g,n}$ from the total integration over $\overline{\mathcal{M}}_{g,n}$. In order to do it we shall use a stratification procedure by Deligne and Mumford [24]. The idea is to present the open moduli space $\mathcal{M}_{g,n}$ as a combination of $\overline{\mathcal{M}}_{g,n}$ and $\overline{\mathcal{M}}_{g_j,n_j}$ of lower dimensions. The description of this procedure in the case of modulus one can find in [10], [15].

The geometrical meaning of the reduction procedure is that we subsequently pinch handles of the surface (Fig.4). One can see that there are two types of such reduction: in the first one when pinching a handle we result in the surface of genus lower by one and two additional punctures. Thus from the space $\overline{\mathcal{M}}_{g,n}$ we get after this reduction
type $\overline{M}_{g-1,n+2}$ (Fig.4a). In the second type pinching an intermediate cylinder we get two surfaces of the same total genus and two new punctures: one per each new component. It means that the initial moduli space $\overline{M}_{g,n}$ splits into the product $\overline{M}_{g_1,n_1+1} \otimes \overline{M}_{g_2,n_2+1}$, $g_1 + g_2 = g$, $n_1 + n_2 = n$ (Fig.4b).

\begin{equation}
\overline{M}_{g,n} = \sum_{\text{reductions } r_q=0}^{3g-3+n} (-1)^{r_q} \otimes \prod_{j=1}^{q} \overline{M}_{g_j,n_j+k_j},
\end{equation}

where sum runs over all $q$–component reductions, $r_q$ being the reduction degree and $k_j$ being the number of the additional punctures due to reductions. The dimension of $\overline{M}_{g_1,n_1+k_1}$ is $d_j = 3g_j - 3 + n_j + k_j$,

\begin{equation}
\sum_{j=1}^{q} n_j = n, \quad \sum_{j=1}^{q} d_j = d - r_q.
\end{equation}

Thus we have:

\[ \int_{\overline{M}_{g,n}^{\text{disc}}} e^\overline{\Omega} \times e^{\sum_i \lambda_i p_i} dp_1 \wedge \ldots \wedge dp_n = \frac{1}{d!} \int_{\overline{M}_{g,n}^{\text{disc}}} \left( \sum_{i=1}^{n} p_i^2 \tilde{\omega}_i \right)^d e^{\sum_i \lambda_i p_i} dp_1 \wedge \ldots \wedge dp_n + \]

Fig.4a. One–component type of the reduction.

Fig.4b. Two–component type of the reduction.
\[ \sum_{r_0=1}^{3g-3+n} (-1)^{r_0} \otimes_{j=1}^q \mathcal{M}_{g_j,n_j+k_j} \left( \sum_{a=1}^{n_j} p_a^2 \omega_a \right)^{d_j} e^{\sum_e \lambda_e p_1^e dp_1 \wedge \ldots \wedge dp_{n_j}}. \tag{3.19} \]

Now we can find using (3.13) a matrix model description for the L.H.S. of (3.19). Just as in the continuum case we have:

\[ \text{L.H.S.} = \int_{\mathcal{M}^{disc}} \exp \left\{ - \sum_{e \in X_1} l_e(\lambda^{(1)}_e + \lambda^{(2)}_e) \right\} \times \tilde{\rho} \times \prod_{e \in X_1} |dl(e)|, \tag{3.20} \]

where \( \tilde{\rho} = 2^{d+\#X_1-\#X_0-1} \).

This last expression can be presented as a sum over “fat graphs” \( \Gamma \) with vertices of all valencies which are possible for given genus and number of faces. We should again take into account the volume of the automorphism group for each graph which coincides with the number of copies of equivalent domains of the moduli space \( \mathcal{M}_{g,n} \) which constitute this cell of the combinatorial simplicial complex. The last step is to do the “integration” over each \( l(e) \) which is given by the sum over all positive integer values of \( l(e) \) (because we already took into account all zero values of \( l(e) \) doing the sum over all graphs). Eventually we have:

\[ \text{L.H.S.} = 2^{d-1} \sum_{\text{Graphs } \Gamma} \frac{1}{\#\text{Aut} (\Gamma)} \times 2^{-\#X_0} \times \prod_{e \in X_1} \frac{2}{e^{\lambda^{(1)}_e + \lambda^{(2)}_e} - 1}. \tag{3.21} \]

It is nothing but a term from the genus expansion of the matrix model (3.1) with

\[ \Lambda = \text{diag} \{ e^{\lambda_1}, \ldots, e^{\lambda_N} \}. \tag{3.22} \]

Then \( \log Z[\Lambda] \) has the following genus expansion:

\[ \log Z[\Lambda] = \sum_{g=0}^{\infty} \sum_{n=1}^{\infty} (N\alpha)^{2-2g} \alpha^{-n} w_g(\lambda_1, \ldots, \lambda_n). \tag{3.23} \]

Let us use the relations (3.14) in order to express the R.H.S. of (3.19) via intersection indices.

\[ w_g(\lambda_1, \ldots, \lambda_n) = \frac{1}{2^{d-1}} \sum_{q=1}^{\text{reductions}} \frac{1}{\text{q-component}} \left( \sum_{j=1}^{q} \prod_{d_l=3g_j-3+n_j+k_j} \frac{1}{n_j!^{d_j}} \frac{1}{\left( \tau_{d_1} \ldots \tau_{d_{n_j}} \tau_0 \ldots \tau_0 \right)_{g_j}} \times \text{tr} \prod_{k=1}^{n_j} \left( \frac{\partial}{\partial \lambda_k} \right)^{2d_k} \frac{1}{(d_k)!} \frac{1}{2} \left( \prod_{k=1}^{n_j} e^{\lambda_k} - 1 \right) + (-1)^{n_j} \prod_{k=1}^{n_j} e^{\lambda_k} + 1 \right) \right) \tag{3.24} \]

This formula is our main result. For practical reasons it is sometimes convenient to rewrite

\[ \sum_{d_l=3g_j-3+n_j+k_j} \frac{1}{n_j!^{s_j}} \left( \tau_{d_1} \ldots \tau_{d_{n_j}} \tau_0 \ldots \tau_0 \right)_{g_j} = \sum_{b_0+b_1+\ldots+b_k=n_j} \frac{1}{b_0! \ldots b_k!} \left( \tau_0 \right)^{b_0} \ldots \left( \tau_k \right)^{b_k} \left( \tau_0 \ldots \tau_0 \right)_{g_j}. \tag{3.25} \]
Taking this expression for the case $M_{0,3}$ (without reductions) and reminding that $\langle \tau_0^2 \rangle_0 = 1$ we immediately get the answer (3.4) after a substitution $\mu_i = e^{\lambda_i}$.

Since the matrix model (3.1) is equivalent to the hermitian one–matrix with an arbitrary potential, then the formulae (3.23)–(3.25) above give the solution to such models in geometric invariants of the d.m.s. We shall see in the next section that the relation (1.1) holds true, i.e. the intersection indices coincide for $\mathcal{M}_{g,n}$ and $\mathcal{M}_{g,n}^{disc}$ thus expressing the solution to one matrix model beyond the double scaling limit exactly in terms of variables which live just in the Kontsevich model or, equivalently, in the d.s.l.!

There is nevertheless some nontrivial point in the final relation (3.24). Namely, it is the “sum over reductions”. This sum appears to be rather involved by the following reasons: When we considered the orbits $\tilde{\pi}^{-1}(p_*)$ we assumed that they belonged to one copy of the moduli space $\mathcal{M}_{g,n}$. But when we deal with the cell decomposition it is much more convenient first to consider the total simplicial complex (which we shall denote $T_{g,n}$) and only afterward take into account internal automorphisms of $T_{g,n}$ which eventually produce $\mathcal{M}_{g,n}$ as a coset over a symmetry group $G_{g,n}$. One may consider instead of $\pi$ and $\tilde{\pi}$ mappings $\beta$ and $\tilde{\beta}$ correspondingly, $\beta : T_{g,n} \otimes \mathbb{R}_+$ $\rightarrow$ $\mathcal{M}_{g,n}^{comb}$ and $\tilde{\beta} : T_{g,n}^{disc}[p_*] \times [\mathbb{Z}_+^n]_{even} \rightarrow$ $\mathcal{M}_{g,n}^{disc}$, where $T_{g,n}^{disc}$ are again finite (nonisomorphic) sets of points of $T_{g,n}$ supplied with the discrete de Rham cohomology structure.

For these spaces $T_{g,n}$ an analogue of the formula (3.24) exists. The only difference is that we should multiply all indices $\langle \langle \tau_{d_1} \ldots \tau_{d_\nu} \rangle \rangle$ by the order of the symmetry group $G_{g,n}$. We shall present arguments in the next section that there is such choice of $T_{g,n}$ that all reductions $(T_{g,n},a_j+k_j)$ are counted integer number of times. But this number of copies multiplied by the order of the group $G_{g,n, a_j+k_j}$ does not necessarily divisible by the order of $G_{g,n}$. Thus when we write in the formula (3.24) “the sum over reductions” we should keep in mind that the coefficients in this sum are not necessarily integers! (See Section 5 for an example).

The next section is devoted to the comparison of the matrix integrals in the Kontsevich and matrix model for d.m.s. using exclusively matrix model tools, that permits to prove our basic relation (1.1).

4 Comparison of two matrix models

This section is based on the results of the papers [21] and [2]. It was explicitly demonstrated in [22] [23] that the matrix model (3.1) is equivalent to the standard hermitian one–matrix model

$$Z[g, \tilde{N}] = \int_{\tilde{N} \times \tilde{N}} d\phi \exp(-\tilde{N} \text{tr} V(\phi)), \quad (4.1)$$

where the integration goes over hermitian $\tilde{N} \times \tilde{N}$ matrices and

$$V(\phi) = \sum_{j=1}^{\infty} \frac{g_j}{j} \phi^j \quad (4.2)$$
is a general potential. Then the following relation holds:

\[ Z[g, \tilde{N}(\alpha)] = e^{-N \operatorname{tr} \eta^2/2} Z_P[\eta, N], \quad \tilde{N}(\alpha) = -\alpha N. \]  (4.3)

Here the partition function \( Z_P[\eta, N] \) is

\[ Z_P[\eta, N] = \int_{N \times N} dX \exp \left[ N \operatorname{tr} \left( -\eta X - \frac{1}{2} X^2 - \alpha \log X \right) \right], \]  (4.4)

the integral being done over hermitian matrices of another dimension \( N \times N \) and the set of the coupling constants (4.2) being related to the matrix \( \eta \) by the Miwa transformation

\[ g_k = \frac{1}{N} \operatorname{tr} \eta^{-k} - \delta_{k,2} \text{ for } k \geq 1, \quad g_0 = \frac{1}{N} \operatorname{tr} \log \eta^{-1}. \]  (4.5)

Now after substitution

\[ \eta = \sqrt{\alpha}(\Lambda + \Lambda^{-1}) \]  (4.6)

and making the change of variables \( X \to (X - 1)\Lambda \sqrt{\alpha} \) we reconstruct the integral (3.1) (with \( \alpha \) multiplied by two).

Note that we can do a limiting procedure (which is a sort of the double scaling limit for the standard model (4.1)) resulting in the Kontsevich integral (2.2) starting from Kontsevich–Penner model (3.1). It looks even more natural in terms of this model than in terms of the one–matrix integral (4.1). Namely, let us take in (3.1)

\[ \Lambda = e^{\varepsilon \lambda}, \quad \alpha = \frac{1}{\varepsilon^3}. \]  (4.7)

Then after rescaling \( X \to \varepsilon X \) in the limit \( \varepsilon \to 0 \) we explicitly reproduce (2.2) from (3.1). During this procedure we can keep constant the size \( N \) of matrices of (3.1), but the size \( \tilde{N}(\alpha) \) of the matrices of hermitian model goes to infinity in the limit \( \varepsilon \to \infty \). There is a question whether this limit is a genuine d.s.l., because there exists another limiting procedure which generate a square of the Kontsevich model as the limit of (3.1), but for our present purposes we shall use this simplest scaling limit and we refer to it as d.s.l.

### 4.1 Review of the solutions to Kontsevich and KP models.

Since the models (3.1) and (4.1) are equivalent we can use the explicit answers for (4.1) found in [4] in order to check the validity of our formulae (3.24) and to compare the values of intersection indices in both Kontsevich model (2.2) and the model (3.1). Both these models were solved in genus expansion in terms of momenta. For the Kontsevich model this solution was presented in [13] and for (1.4) or, equivalently, (4.1) — in [2]. Here we present the results. (Throughout this section the expansion parameter \( \alpha \) should be replaced by \(-2\alpha\) in order to compare with the results of [2].

1. The solution to the Kontsevich model is

\[ \log Z_K[N, \Lambda] = \sum_{g=0}^{\infty} N^{2-2g} F^{Kont}_g. \]  (4.8)
For the genus expansion coefficients we have

\[
F_g^{Kont} = \sum_{\alpha_j > 1, \gamma} \langle \tau_{\alpha_1} \cdots \tau_{\alpha_n} \rangle_g \frac{I_{\alpha_1} \cdots I_{\alpha_n}}{(I_1 - 1)^{\alpha}} \quad \text{for } g \geq 1,
\]  

(4.9)

where \(\langle \cdot \rangle_g\) are just intersection indices and the moments \(I_k\)'s depending on an external field \(M\) are defined by

\[
I_k(M) = \frac{1}{(2k-1)!!} \frac{1}{N} \sum_{j=1}^N \frac{1}{(m_j^2 - 2u_0)^{k+1/2}} \quad k \geq 0,
\]  

(4.10)

and \(u_0(M)\) is determined from the equation

\[
u_0 = I_0(u_0, M).
\]  

(4.11)

2. The solution to the model (4.4) can be written as

\[
\log Z_P[N, \eta] = \sum_{g=0}^{\infty} N^{2-2g} F_g,
\]  

(4.12)

where

\[
F_g = \sum_{\alpha_j > 1, \beta_i > 1} \langle \alpha_1 \cdots \alpha_s; \beta_1 \cdots \beta_l | \alpha, \beta, \gamma \rangle_g \frac{M_{\alpha_1} \cdots M_{\alpha_s} J_{\beta_1} \cdots J_{\beta_l}}{M_1^s J_1^l d^r} \quad g > 1.
\]  

(4.13)

This solution was originated from the one-cut solution to the loop equations in the hermitian one–matrix model, \(x\) and \(y\) being endpoints of this cut, \(d = x - y\), and for momenta \(M_k, J_k\) we have

\[
M_k = \frac{1}{N} \sum_{j=1}^N \frac{1}{(\eta_j - x)^{k+1/2}(\eta_j - y)^{1/2}} - \delta_{k,1} \quad k \geq 0,
\]  

(4.14)

\[
J_k = \frac{1}{N} \sum_{j=1}^N \frac{1}{(\eta_j - x)^{1/2}(\eta_j - y)^{k+1/2}} - \delta_{k,1} \quad k \geq 0.
\]  

(4.15)

The brackets \(\langle \cdot \rangle_g\) denote rational numbers, the sum is finite in each order in \(g\), while the following restrictions are fulfilled: If we denote by \(N_M\) and \(N_J\) the total powers of \(M\)'s and \(J\)'s respectively, i.e.

\[
N_M = s - \alpha, \quad N_J = l - \beta,
\]  

(4.16)

then it holds that \(N_M \leq 0, N_J \leq 0\) and

\[
F_g : \quad N_M + N_J = 2 - 2g,
\]  

\[
F_g : \quad \sum_{i=1}^s (\alpha_i - 1) + \sum_{j=1}^l (\beta_j - 1) + \gamma = 4g - 4
\]  

(4.17)
We again have a nonlinear functional equations determining the positions of the end-points $x$ and $y$:

\[
\sum_{i=1}^{N} \frac{1}{\sqrt{(\eta_i - x)(\eta_i - y)}} - \frac{x + y}{2} = 0, \tag{4.18}
\]

\[
\sum_{i=1}^{N} \frac{\eta_i - \frac{x + y}{2}}{\sqrt{(\eta_i - x)(\eta_i - y)}} - \frac{(x - y)^2}{8} = -2\alpha + 1. \tag{4.19}
\]

The solutions to the first two genera have as usual some peculiarities. For $g = 1$ we have

\[
F_1 = -\frac{1}{24} \log M_1 J_1 d^4, \tag{4.20}
\]

and for zero genus we have after taking a double derivative in $\alpha$ in order to exclude divergent parts:

\[
\frac{d^2}{d\alpha^2} F_0 = 4 \log d. \tag{4.21}
\]

The last property of the expression (4.13) which we want to notice here is its symmetry under interchanging $x$ and $y$, or equivalently, $M_i$ and $J_i$:

\[
\langle \alpha_1 \ldots \alpha_s; \beta_1 \ldots \beta_l | \alpha, \beta, \gamma \rangle_g = (-1)^g \langle \beta_1 \ldots \beta_l; \alpha_1 \ldots \alpha_s | \beta, \alpha, \gamma \rangle_g. \tag{4.22}
\]

As we shall see in a moment this symmetrical relation is in a direct connection with the symmetrization $e^{\lambda} \rightarrow -e^{\lambda}$ in the formula (3.24).

3. In the d.s.l. $\varepsilon \rightarrow 0$ we may put

\[
y = -\frac{\sqrt{2}}{\varepsilon^{3/2}}, \quad x = \frac{\sqrt{2}}{\varepsilon^{3/2}} + \sqrt{2} u_0 + \ldots, \tag{4.23}
\]

and the equation (4.11) arises. The scaling behaviour of the momenta $M_k$, $J_k$ and $d$ is

\[
J_k \rightarrow -2^{-(3k/2+1)} \varepsilon^{(3k+1)/2} I_0 + \delta_{k1},
\]

\[
M_k \rightarrow -2^{(k-1)/2} \varepsilon^{-(k-1)/2} ((2k-1)!! I_k - \delta_{k1}),
\]

\[
d \rightarrow 2^{3/2} \varepsilon^{-3/2} \tag{4.24}
\]

Thus only the terms without $J_k$–dependence and only of the highest order in $\alpha_i$ survive in the d.s.l. in which the expression (4.13) is reduced to the answer for the Kontsevich model (4.9). Then the coefficients $\langle \alpha_1 \ldots \alpha_s; \{ \text{nothing} \} | \alpha, 0, \gamma \rangle_g$ coincide (up to some factorials and powers of two) with the Kontsevich intersection indices $\langle \tau_{\alpha_1} \ldots \tau_{\alpha_n} \rangle_g$. Having an explicit solution of the form (4.13) one may see it directly. In [2] an iterative procedure was proposed in order to find coefficients of the expansion (4.13) and all these coefficients were found in the genus 2 (for $g = 0, 1$ see [21]. It was proved there that coefficients of the highest order in $\alpha_k$ coincide in a proper normalization with the Kontsevich indices. It will permit us to prove the relation (1.1) for these intersection indices in the next subsection.
4.2 Relation between momenta and d.m.s. variables

Now our purpose is to study how one may reexpress the answers of the type (4.13) in terms of the quantities standing in the R.H.S. of (3.24). It is rather technical question, but important one if we want to deal with concrete expresions.

At first, let us expand both momenta $M_k$, $J_k$ and the restriction equations (4.18, 4.19) in terms of $\lambda$–variables, where $\eta = \sqrt{\alpha}(e^\lambda + e^{-\lambda})$. Then for the endpoints of the cut we have:

$$x = 2\sqrt{\alpha} + \xi, \quad y = -2\sqrt{\alpha} + \beta,$$

(4.25)

where $\xi$ and $\beta$ themselves are some polynomials in the higher momenta $M_i$ and $J_i$ with $i, j \geq 0$. Thus, after a little algebra we shall get for, say, the moment $M_k$:

$$M_k = \frac{1}{N} \text{tr} \frac{(e^\lambda)^{k+1}}{\eta - y} \left( (e^\lambda - 1)^2 - \frac{\xi}{\sqrt{\alpha}} e^\lambda \right)^{k+1/2} \left( (e^\lambda + 1)^2 - \frac{\beta}{\sqrt{\alpha}} e^\lambda \right)^{1/2} - \delta_{k,1}. \quad (4.26)$$

(For $J_k$ the expression is just the same with interchanging the powers $k + 1/2$ and $1/2$ for the two terms in the denominator.

It is convenient now to introduce new momenta:

$$\tilde{M}_k = \frac{1}{N} \text{tr} \frac{\sqrt{\eta - y}}{(\eta - x)^{k+1/2}},$$

$$\tilde{J}_k = \frac{1}{N} \text{tr} \frac{\sqrt{\eta - x}}{(\eta - y)^{k+1/2}}, \quad (4.27)$$

that are related to the initial ones by the following relations:

$$\tilde{M}_k = M_{k-1} + \delta_{k,2} + d(M_k + \delta_{k,1}),$$

$$\tilde{J}_k = J_{k-1} + \delta_{k,2} - d(J_k + \delta_{k,1}),$$

$$M_0 = J_0 = \frac{(\tilde{M}_0 - \tilde{J}_0)}{d}. \quad (4.28)$$

Then for these new $\tilde{M}_k$ we have

$$\tilde{M}_k = \frac{1}{N} \text{tr} \frac{1}{\sqrt{\alpha}^k} \left( \frac{(e^\lambda - 1)^2}{(e^\lambda + 1)^2} \left( 1 - \frac{\beta}{\sqrt{\alpha}} e^\lambda \right)^{1/2} \right)^{k+1/2}. \quad (4.29)$$

The expansion in (4.29) goes over the terms

$$H_{ab} = \frac{1}{N} \text{tr} \frac{(e^\lambda + 1) e^{a\lambda}}{(e^\lambda - 1)^{2a+1}} \frac{e^{b\lambda}}{(e^\lambda + 1)^{2b}}, \quad (4.30)$$

where $b \geq 0, a \geq k$.

Let us prove now that $H_{ab}$ can be presented as a linear sum of

$$L_a = \frac{1}{N} \text{tr} \frac{\partial^{2a}}{\partial \lambda^{2a}} \frac{1}{e^\lambda - 1},$$

$$R_b = \frac{1}{N} \text{tr} \frac{\partial^{2b}}{\partial \lambda^{2b}} \frac{1}{e^\lambda + 1}, \quad (4.31)$$
i.e. the sum goes over only even powers of derivatives in $\lambda$:

$$H_{ab} = \sum_{i=0}^{a} \alpha_{ab}^i L_i + \sum_{j=0}^{b-1} \beta_{ab}^j R_j. \quad (4.32)$$

We begin with considering the case $b = 0$. Then it is easy to see after a little algebra that

$$\frac{\partial^2}{\partial \lambda^2} H_{t0} = t^2 H_{t0} + 2(t+1)(2t+1)H_{t+1,0}. \quad (4.33)$$

It is enough to prove now that $H_{t1}$ can be presented in the form $\text{(4.32)}$. At first,

$$H_{01} = \frac{1}{N} \text{tr} \left( e^\lambda - 1 \right) \left( e^\lambda + 1 \right) = \frac{1}{2} (L_0 + R_0), \quad (4.34)$$

and it is trivial to see that

$$H_{t0} = H_{t-1,1} + 4H_{t,1}. \quad (4.35)$$

Thus $H_{t1}$ can be always written in the form $\text{(4.32)}$. Now by induction, multiplying both sides on $\frac{e^\lambda}{(e^\lambda + 1)^2}$ and using again the relation $\text{(4.32)}$ we shall prove it for an arbitrary $H_{tb}$.

Let us keep now only terms of zero and first orders in traces of $\lambda$ in the expressions for momenta. Then we get:

$$M_k \sim \frac{1}{\sqrt{\alpha^{k+1}}} \frac{1}{N} \text{tr} \frac{e^{\lambda(k+1)}}{(e^\lambda - 1)^{2k+1}(e^\lambda + 1)} + \delta_{k1},$$

$$J_k \sim \frac{1}{\sqrt{\alpha^{k+1}}} \frac{1}{N} \text{tr} \frac{e^{\lambda(k+1)}}{(e^\lambda - 1)(e^\lambda + 1)^{2k+1}} + \delta_{k1},$$

$$d \sim \sqrt{\alpha} \left\{ 4 - \frac{1}{\alpha} \frac{1}{N} \text{tr} \frac{2}{(e^\lambda - 1)(e^\lambda + 1)} \right\}. \quad (4.36)$$

If we pay an attention to the terms surviving in the d.s.l. then these terms are just the ones arising from the term without reductions on the L.H.S. of $\text{(3.24)}$. Then we eventually prove our basic relation $\text{(1.1)}$

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g = \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g. \quad (4.37)$$

Note that $L_a$ and $R_a$ are just analogues of the Kontsevich’s times $T_n = (2n-1)!!\Lambda^{2n+1}$. They are transforming into $T_n \cdot 2^n(n-1)!$ in the d.s.l. and in both cases there is no dependence on odd derivatives in $\Lambda$. As we have mentioned in Introduction there are two natural ways to do d.s.l. in the KP model. While doing the one we already considered $\text{(4.23-4.24)}$ only one set of times $\{L_n\}$ survived and it is going to the set $\{T_n\}$ in the limit $\varepsilon \to 0$. But if we choose $\Lambda$ to be symmetrical, $\Lambda = \text{diag}\{\lambda_1, \ldots, \lambda_{N/2}, -\lambda_1, \ldots, -\lambda_{N/2}\}$, then another limit is possible when $L_i = R_i$ and each of these sets generates $\{T_n\}$ thus giving a square of the integral $\text{(2.2)}$.

There is one question we want to discuss now. Let us consider the sum over multicomponent reductions on the L.H.S. of $\text{(3.24)}$. Using the matrix model technique we
have an opportunity to distinguish between different types of reductions mostly due to that remarkable fact that symmetrization $e^\lambda \rightarrow -e^\lambda$ goes in each component separately. Only this property makes $\lambda$–dependent terms different for, say, $\langle \tau_1(\tau_0)^3 \rangle_0 \cdot \langle \tau_2 \tau_0 \rangle_1$ and $\langle \tau_2 \tau_1(\tau_0)^4 \rangle_0$ (see Fig.5) — both these terms appear during the reduction procedure of the genus two surface with two punctures. But one of them is due to the two–component reduction and another – of the one–component type. Evidently, when fixing the number of punctures, $n$, only terms containing products of exactly $n$ traces of $\lambda$ contribute to the L.H.S. of (3.24) and it is a technical problem to pull these terms from the genus expansion in terms of momenta.

![Fig.5. Two examples of one– and two-component reduction for $\mathcal{M}_{2,2}$](image)

In the next section we apply all experience already obtained to the simplest case of $\overline{\mathcal{M}}_{1,1}$ – a torus with one puncture moduli space. This consideration enables us to make a conclusion about the structure of any modular space.

5 Orbifold structure of the moduli spaces.

In this section we want to shed a light on the question about the underlying structure of d.m.s. and the moduli spaces themselves.

5.1 The moduli space $\overline{\mathcal{M}}_{1,1}$.

Let us consider first an example of well–known–to–everybody modular space $\mathcal{M}_{1,1}$, i.e. the torus with one puncture. While thinking of it everyone immediately imagine the copy of this modular figure in Teichmuller upper half–plane, namely, a strip from $-1/2$ to $1/2$ along imaginary axis bounded from below by a segment of a semicircle of the radius 1 with the origin at zero point. In order to get the modular space itself we should identify both sides of the strip as well as two halves of this segment being correspondingly on the left and on the right of the imaginary axis $\Re z = 0$. From this picture it is clear that there are three peculiar points where the metric on the moduli space appears not to be
conformally flat, namely, \( z = i \) (square point), \( z = e^{i\pi/3} \) (or, the same, \( e^{2i\pi/3} \)) (triple point), and \( z = i\infty \) (infinity point). All these points also have a property that each of them is stable under the action of some operator from the modular transformation group. For triple point the subgroup of such operators has the third order, for square point – it is of order 2, and for the infinity point – of an infinite order.

This means that we can consider the modular space \( \mathcal{M}_{1,1} \) as an orbifold of the (open) upper half–plane. It was a way how Harer and Zagier \[25\] introduced virtual Euler characteristics for such spaces. And it was Penner \[26\] who had found a simple one–matrix hermitian model with the potential \( \log(1 + X) - X \) in order to generate these characteristics. We shall exploit the ideology of the Penner approach \[26\], \[27\]. The idea was to assign factor 1 to each edge (instead of an arbitrary length as it was in the Kontsevich case). Then for an arbitrary \( \mathcal{M}_{g,n} \) there is one–to–one correspondence between cells of the simplicial decomposition of the open moduli space \( \mathcal{M}_{g,n} \) and graphs of the Penner model, more, the symmetrical coefficients for each cell and corresponding graph coincide. Then the virtual Euler characteristic \( \kappa_{g,s} \) can be calculated by the formula:

\[
\kappa_{g,s} = \sum_{\text{cells}} \frac{(-1)^{n_G}}{\#\text{Aut } G},
\]

(5.1)

where \( n_G \) is the codimension of the cell in the simplicial complex.

In the case of \( \mathcal{M}_{1,1} \) the triple point graph corresponds to the higher dimensional cell, square point graph – to the cell of codimension 1. In the complex there is also an infinity point of the lowest dimension, but there is no graph corresponding to it. Thus, for the virtual Euler characteristic we get

\[
\kappa_{1,1} = \frac{1}{3} \cdot (-1)^0 + \frac{1}{2} \cdot (-1)^1 + \frac{1}{\infty} \cdot (-1)^2 = \frac{-1}{6}.
\]

(5.2)

Let us now consider the same case, but already in the Kontsevich parametrization. We know that there are three types of diagrams depicted on Fig.3a–c. The case of Fig.3a when all \( l_i \) are greater than zero corresponds to the cell of the higher dimension. In \( \mathcal{M}_{1,1} \) it is a domain where \( \sum_{i=1}^{3} l_i = p/2 \), i.e. it is an interior of the equilateral triangle. Note that due to two possibilities to choose an orientation there are two such congruent cells. The next case is when one of \( l_i \) is equal zero (Fig.3b). When putting equal zero various \( l_i \) we tend to the boundary of the previous case – they are open intervals lying on the boundary of the triangles. But it is not the whole boundary yet — it remains one point at the summit of the triangles and it corresponds to the last case, Fig.3c, where two of \( l_i \) are equal zero, that gives us a reduced case. The unique reduction of the torus with one puncture is the sphere with three punctures which modular space \( \mathcal{M}_{0,3} \) consists from only one point.

Now let us draw this simplicial complex \( \mathcal{T}_{1,1} \) graphically (Fig.6). We are to indentify the opposite edges of this parallelogram thus obtaining the torus. This torus complex consists from two open triangles (Fig.3a), three edges partiting these triangles (Fig.3b)
and the unique point of reduction – the vertex (Fig.3c). The centers of the triangles marked by small discs correspond to two copies of the triple point and centers of edges – to three copies of the square point (small circles). We have six copies of the original modular figure on this torus, one of them is coloured in black.

Fig. 6. A simplicial complex for $\mathcal{M}_{1,1}$ in the Kontsevich picture.

Thus we result in the conclusion that in the Kontsevich’s parametrization the modular space $\mathcal{M}_{1,1}$ is the orbifold of a torus $T^1$ with parameters $(1, e^{i\pi/3})$ which possesses an internal symmetry group $G_{1,1}$ of the sixth order. Thus

$$\mathcal{M}_{1,1} = T^1/G_{1,1}. \quad (5.3)$$

This torus is by all means totally flat compact space. And here is a point which is different from the Penner construction of orbifolds of the upper half–plane, because there we had infinity points of infinite order, and here the order of this point is obviously finite! In our case it is a point of sixth order. It means that for this case the formula $(5.2)$ will change and using $(5.1)$ we should add $1/6$ to $(5.2)$ thus obtaining zero for our new virtual Euler characteristic in the Kontsevich picture (as it should be for the torus).

To complete this geometric part, note that we can think about the torus $T^1$ as the fundamental domain of the subgroup $\Gamma_2$ of the modular group. This domain is depicted on Fig.7 and it again consists from six copies of the modular figure. Black discs mark the positions of triple points and small circles – the one of square points. If we identify left half–line with right half–circle and vice versa we shall obtain our torus (if we do not care about conformal properties of this transformation at the infinity point).

Fig.7. The fundamental domain for subgroup $\Gamma_2$ of the modular group.
Let us turn now to our basic formula (3.24). First, using diagram technique for the matrix model (3.1) it is easy to get the answer (after substitution \( \Lambda = e^\lambda \)). Combining all terms we obtain:

\[
F_{1, 1} = \alpha^{-1} \frac{3 e^{2\lambda} - 1}{6(e^{2\lambda} - 1)^3},
\]

and we need to express it in terms of derivatives (4.31). Note that the formula (5.4) one can obtain from the expansions (4.20) and (4.36) substituting \( \alpha \to -\alpha/2 \). After a little algebra we get an answer:

\[
F_{1, 1} = \frac{1}{48\alpha} \cdot \frac{\partial^2}{\partial \lambda^2} \left[ \frac{1}{e^\lambda - 1} - \frac{1}{e^\lambda + 1} \right] - \frac{1}{12\alpha} \left[ \frac{1}{e^\lambda - 1} - \frac{1}{e^\lambda + 1} \right].
\]

(5.5)

The first term gives us the proper value of \( \langle \tau_1 \rangle_1 = 1/24 \). As for the second term, it was originated from the “sum over reductions” and the only reduction of the torus is the sphere with three punctures for which \( \langle \tau_3 \rangle_0 = 1 \). We see that the sum over reductions gives an additional fractional factor 1/6, but now we know the nature of it. In the simplicial complex (Fig.6) there are six copies of the modular space \( \mathcal{M}_{1,1} \) and only one of the infinity point. So we see, that we just have “one sixth” of this point contributing to the expression (3.24) in this order in \( g \) and \( n \).

Now we are able to understand the structure of d.m.s. for \( \mathcal{M}_{1,1} \) and starting with this example we shall formulate a hypothesis about the structure of an arbitrary modular space \( \mathcal{M}_{g,n} \). Remind that in the Kontsevich parametrization we use the form \( \Omega \) (2.8) in order to evaluate the volume of the corresponding modular space. The fact that the intersection indices coincide for both continuum and discrete cases means that it does not matter how we calculate the total volume of the torus \( T^1 \): or by standard continuum integration, or by doing a sum over points of integer lattice, each taken with unit measure. For the torus with the perimeter equal \( p \) (which is always even) there are exactly \((p/2)^2\) points from d.m.s. lying in \( T_{1,1} = T^1 \). Thus the total volume per one copy of the initial moduli space is \((p/2)^2\) divided by number of copies, i.e. \( p^2/24 \) in our case.

### 5.2 General case of \( \mathcal{M}_{g,n} \).

The simplest example which has been considered in the previous subsection permits us to formulate the basic hypothesis about the structure of an arbitrary modular space \( \mathcal{M}_{g,n} \) in the Kontsevich parametrization. The main results we already have obtained are:

1. From the comparison of two matrix models (Section 4) we got the identity (4.37) — the indices for both continuum and d.m.s. coincide.

2. The calculation of all these indices can be reduced to evaluating the integrals of the volume forms over some finite coverings (simplicial complices) \( T_{g,n} \) corresponding to these modular spaces. In the discrete case values of these integrals are just the total numbers of integer points inside \( T_{g,n} \) for some fixed values of \( p \)-variables. It means that all such points contribute to the sum with the same coefficient — the volume of the unit cell. In
its turn it means that there should be no points of nonzero intrinsic curvature inside $\mathcal{T}_{g,n}$. In particular, all orbifold points are not singular points of metric in $\mathcal{T}_{g,n}$.

3. All continuum modular spaces are closed compact manifolds without boundaries after compactification by Deligne and Mumford.

Thus we conclude that in the Kontsevich parametrization for each modular space $\mathcal{M}_{g,n}$ we deal with the finite covering $\mathcal{T}_{g,n}$ which is totally conformally flat compact manifold without a boundary. The only possibility to ensure this property is to take $d$-dimensional complex torus $T^d$, $d = 3g - 3 + n$. It should possesses a rich discrete group of symmetries $G_{g,n}$. Thus we conclude that in the Kontsevich parametrization $\mathcal{M}_{g,n}$ is represented as a factor over this symmetry group:

$$\mathcal{M}_{g,n} = T^d/G_{g,n}. \quad (5.6)$$

It is this relation which permits us to make further assumptions about possibility to introduce quantum group structure on these modular spaces.

5.3 Proposal for Quantum Group Structure on $\mathcal{M}^{\text{disc}}_{g,n}$.

All our previous considerations went along the line of the matrix model theories. In this subsection we want to leave the frames of these models and to speculate about possible underlying algebraical structures which can exist on d.m.s. We suppose to deal with this question in a special paper and restrict ourselves here by making only few notes.

First, let us have a closer look on the structure of d.m.s. for $\mathcal{M}_{1,1}$. Here we again reproduce the explicit form of the torus $T^1/G_{1,1}$ for first few values of $p = 2, 4, 6, 8$ (Fig.8). We denote by small discs the points of $\tilde{\pi}^{-1}(p_*)[\mathcal{M}^{\text{disc}}_{1,1}]$.

![Fig. 8. Examples of orbits of $\tilde{\pi}^{-1}(p_*)[\mathcal{M}^{\text{disc}}_{1,1}]$.](image)

As $p \to \infty$ we reconstruct the space $\mathcal{M}_{1,1} = T^1/G_{1,1}$ for the continuum case. On the space $\mathcal{M}_{1,1}$ there exists Poisson structure due to Kontsevich [3]. We expect that there should be an analogue structure on each orbit $\tilde{\pi}^{-1}(p_*)[\mathcal{M}^{\text{disc}}_{1,1}]$. Here the following question arises: which structures except the trivial one (the commutative group of shifts along the axis directions) would exist for such spaces? We know that there are quantum group structures for arbitrary quantum tori [28] and we think that such the structures
should exist for the case under consideration as well. It is reasonable to propose that the parameter $p$ labels irreducible representations of some Quantum Group $Q_{1,1}$. Further, we claim that these representations are exceptional, i.e. without highest weight vectors (since our torus space is the compact space without the boundary). There is no such representations for the classical case, and in the quantum one they exist only for specific values of the quantum parameter $q = e^{2\pi i m/n}$. For such $q$ possible dimensions of these irreducible exceptional representations belong to the finite series. For $SU_q(N)$ they are $\{n^1, n^2, \ldots, n^{N(N-1)/2}\}$ — the highest power is the dimension of the space of the positive roots of the algebra. We see that it is compatible with our consideration if we assume $p = 2n$ and as a proper candidate for $Q_{1,1}$ we have $SU_q(3)$. Then it is worth to note that the torus $T^1$ we have is just the torus of the Weyl subgroup of $SU(3)$. The generators of $SU_q(3)$ should act on the space of functions $f[\tilde{\pi}^{-1}(p_*)]$ and these representations should also respect the additional symmetry transformation group $G_{1,1}$ which is a group of the sixth order in our case.

We propose a hypothesis that it is a general situation for every d.m.s. $\mathcal{M}^{\text{disc}}_{g,n}$. From the formula (5.6) it is clear that at least such construction is possible because it is valid for any such torus and the only question is whether such a representation exists which is invariant under the action of the symmetry group $G_{g,n}$. If it is a true, then the d.m.s. are nothing but spaces on which the exceptional representations of the Quantum Groups are living. We can also propose a hypothesis that for the spaces with one puncture, $\mathcal{M}^{\text{disc}}_{g,1}$, these representations are of the QG $SU_q(2d + 1)$, the torus $T^d$ being in this case a Weyl subgroup torus of this Lie group.

Then the intersection indices may have an interpretation as traces of some quantum operators which do not depend on the representation chosen and even coincide with the one for a classical case. This question obviously deserves a big field for an investigation.

6 Conclusions

In the conclusion we suppose to discuss briefly the perspectives of the present action as well as the connection with another approaches originated from the matrix model technique.

First, as we see from our basic formula (4.37), for each moduli space $\mathcal{M}_{g,n}$ there is a wide class of integrals which can be reduced to a sum over finite sets of points inside this space. It resembles a situation described in [29] where the hidden symmetry of functional integrals permits to reduce the procedure of doing the integral over the total space to taking the sum over singularity points. (We owe to A.Alexeyev by this note). So it is a reasonable task to search similar structures on the moduli spaces themselves. Second, it is interesting to investigate the very structure of moduli spaces using our main relation (3.24). (We present an example of such calculation in the Appendix for the modular space $\mathcal{M}_{2,1}$).
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8 Appendix. The explicit solution to $\overline{M}_{2,1}$

Here we shall find by the direct calculation the form of the formula (3.24) for the case of genus two moduli space with one puncture. In the paper [2] the explicit form of genus two partition function in terms of momenta was found:

$$F_2 = -\frac{119}{7680} J_2^2 d^4 - \frac{119}{7680} M_2^2 d^4 + \frac{181 J_2}{480} \frac{M_2}{J_2^2 d^3} - \frac{181 M_2}{480} \frac{J_2}{M_2^2 d^3}$$
$$+ \frac{3 J_2}{64 J_1^2 M_1^2 d^6} - \frac{3 M_2}{64 J_1^2 M_1^2 d^6} - \frac{11 J_2}{40 J_1^2 d^2} - \frac{11 M_2}{40 M_1^2 d^2}$$
$$+ \frac{21 J_2}{160 J_1^3 d} - \frac{21 M_2}{160 M_1^3 d} - \frac{29 J_2}{128 J_1^4 d} + \frac{29 M_2}{128 M_1^4 d}$$
$$+ \frac{35 J_4}{384 J_1^4 d} - \frac{35 M_4}{384 M_1^4 d}.$$  \(8.1\)

In order to investigate the modular space $\overline{M}_{2,1}$ it is enough to use the expansions (4.36), because we must keep only terms of the first order in traces. Then the only thing we need more is to express the quantities

$$p_k = \frac{e^{\lambda(k+1)}}{(e^\lambda - 1)^{2k+1}(e^\lambda + 1)}$$
$$q_k = \frac{e^{\lambda(k+1)}}{(e^\lambda - 1)(e^\lambda + 1)^{2k+1}}$$  \(8.2\)

via the derivatives $L_a$ and $R_a$ (4.31) using (4.33) and (4.35). We omit all lengthy calculations and present here only the final answer. After replacing $\alpha \to -\alpha/2$ we remain with

$$w_2(\lambda) = \frac{1}{1152} \left\{ \frac{L_4}{4!} \cdot \frac{1}{1152} - \frac{L_3}{3!} \cdot \frac{7}{1152} + L_2 \cdot \frac{1}{1152} \cdot \frac{1781}{30} - L_1 \cdot \frac{1}{1152} \cdot \frac{32581}{480} + L_0 \cdot \frac{119}{192} \cdot \frac{1}{40} + (L_a \to R_a) \right\}.  \(8.3\)$$

Here $\frac{1}{1152} = \langle \tau_4 \rangle_2$ and we see that the sum over reductions appear to give highly nontrivial coefficients. One should take into account that only for the first reduction degree there is
unique diagramm which correspond to, for all higher reductions there are various kinds of reductions, hence, the coefficients standing with the terms in the second line of (8.3) are themselves to be presented as sums of symmetrical coefficients, each corresponding to some selected type of the reduction.

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