Partial embedding of the quantum mechanical analog of the nonlinear sigma model

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Abstract

We consider the quantum mechanical analog of the nonlinear sigma model. There are difficulties to completely embed this theory by directly using the Batalin, Fradkin, Fradkina, and Tyutin (BFFT) formalism. We show in this paper how the BFFT method can be conveniently adapted in order to achieve a gauge theory that partially embeds the model.

I. INTRODUCTION

The interest in embedding of systems with nonlinear constraints has been started with the works by Banerjee et al. \textsuperscript{1}. The general and systematic formalism for embedding was developed by Batalin, Fradkin, Fradkina, and Tyutin (BFFT) \textsuperscript{2,3} where systems with second class constraints \textsuperscript{4} are transformed into first class ones, i.e. they are embedded into more general (gauge) theories. This is achieved with the aid of auxiliary variables with the general rule such that there is one pair of canonical variables for each second class constraint to be transformed.

The BFFT method is quite elegant and the obtainment of first class constraints is done in an iterative way. The first correction to the constraints is linear in the auxiliary variables, the second one is quadratic, and so on. In the case of systems with just linear constraints, like chiral-bosons \textsuperscript{5}, one obtains that just linear corrections are enough to make them first class \textsuperscript{6,7}. Here, we mention that the method is equivalent to express the dynamic quantities by means of shifted coordinates \textsuperscript{8}.

However, for systems with nonlinear constraints, the iterative process may go beyond the first correction. This is a crucial point for the use of the method. This is so because the first iterative step may not give a unique solution and one does not know \textit{a priori} what should be the most convenient solution we have to choose for the second step. There are systems where this choice is very natural and it is feasible to carry out all the steps. We mention for example the massive Yang-Mills theory \textsuperscript{9}. However, for the nonlinear sigma-model (and \textit{CP} \textsuperscript{N-1}) not all the solutions of the first step lead to a solution in the second one \textsuperscript{10}. The same occurs from the second to the third step, and so on, making the method not feasible to be applied. More than that, in the case of the nonlinear sigma model one can not assure that these higher order solutions actually exist \textsuperscript{10}. It is important to emphasize that this is not a problem necessarily related to the method, what may happen is that there might be no gauge theory that completely embeds the nonlinear sigma-model.

We shall address to this problem in the present paper. We are going to study the quantum mechanical analog of the nonlinear sigma-model. The use of the BFFT method in this model also presents similar difficulties in providing a complete embedding. However, we show how the method can be conveniently adapted to partially embedding it.

Our paper is organized as follows. In Sec. II we make a brief review of the BFFT method and introduce the general lines of the partially embedding procedure. In Sec. III, for future comparisons, we discuss the constraint dynamics of the quantum mechanical analog of the nonlinear sigma-model, that corresponds to a particle constrained to move on a \textit{N}-dimensional sphere and show the difficulties we have for totally embedding it. We develop the partially embedding of this theory in Sec. IV and discuss the time evolution and the consequences of the gauge invariance of the model into Sec. V. We left Sec. VI for some concluding remarks.

II. BRIEF REVIEW OF THE BFFT FORMALISM

Let us take a system described by a Hamiltonian \( H_c \) in a phase-space with variables \((q_i, p_i)\) where \(i\) runs from 1 to \(N\). It is also supposed that there exist second class constraints only since this is the case that will be investigated. Denoting them by \( T_a \), with \(a = 1, \ldots, M < 2N\), we have

\[
\{ T_a, T_b \} = \Delta_{ab},
\]

where \(\det(\Delta_{ab}) \neq 0\).
The first objective is to transform these second-class constraints into first-class ones. Towards this goal auxiliary variables \( \eta^a \) are introduced, one for each second class constraint (the connection between the number of constraints and the new variables in a one-to-one correlation is to keep the same number of the physical degrees of freedom in the resulting extended theory), which satisfy a symplectic algebra

\[
\{ \eta^a, \eta^b \} = \omega^{ab}, \tag{2.2}
\]

where \( \omega^{ab} \) is a constant quantity with \( \det(\omega^{ab}) \neq 0 \). The first class constraints are now defined by

\[
\tilde{T}_a = \tilde{T}_a(q,p;\eta), \tag{2.3}
\]

and satisfy the boundary condition

\[
\tilde{T}_a(q,p;0) = T_a(q,p). \tag{2.4}
\]

A characteristic of these new constraints is that they are assumed to be strongly involutive, i.e.

\[
\{ T_a^{(0)}, T_b^{(0)} \}_{(q,p)} + \{ T_a^{(1)}, T_b^{(1)} \}_{(\eta)} = 0, \tag{2.8}
\]

\[
\{ T_a^{(0)}, T_b^{(1)} \}_{(q,p)} + \{ T_a^{(1)}, T_b^{(0)} \}_{(q,p)} + \{ T_a^{(1)}, T_b^{(2)} \}_{(\eta)} + \{ T_b^{(2)}, T_a^{(1)} \}_{(\eta)} = 0, \tag{2.9}
\]

\[
\{ T_a^{(0)}, T_b^{(2)} \}_{(q,p)} + \{ T_a^{(1)}, T_b^{(1)} \}_{(q,p)} + \{ T_a^{(2)}, T_b^{(0)} \}_{(q,p)} + \{ T_a^{(1)}, T_b^{(3)} \}_{(\eta)} + \{ T_a^{(3)}, T_b^{(1)} \}_{(\eta)} = 0. \tag{2.10}
\]

The notations \( \{ , \}_{(q,p)} \) and \( \{ , \}_{(\eta)} \) represent the parts of the Poisson bracket \( \{ , \} \) relative to the variables \( (q,p) \) and \( (\eta) \).

The above equations are used iteratively to obtain the corrections \( T_a^{(n)} (n \geq 1) \). Equation (2.8) shall give \( T_a^{(1)} \). With this result and (2.3), one calculates \( T_a^{(2)} \), and so on. Since \( T_a^{(1)} \) is linear in \( \eta \) we may write

\[
T_a^{(1)} = X_{ab}(q,p) \eta^b. \tag{2.11}
\]

Introducing this expression into (2.8) and using the boundary condition (2.4), as well as (2.3) and (2.4), we get

\[
\Delta_{ab} + X_{ac} \omega^{cd} X_{bd} = 0. \tag{2.12}
\]

We notice that this equation contains two unknowns \( X_{ab} \) and \( \omega^{ab} \). Usually, first of all \( \omega^{ab} \) is chosen in such a way that the new variables are unconstrained. It is opportune to mention that it is not always possible to make such a choice. In consequence, the consistency of the method requires the introduction of other new variables in order to transform these constraints also into first-class. This may lead to an endless process. However, it is important to emphasize that \( \omega^{ab} \) can be fixed anyway.

After fixing \( \omega^{ab} \), we pass to consider the coefficients \( X_{ab} \). They cannot be obtained unambiguously since, even after fixing \( \omega^{ab} \), expression (2.13) leads to less equations than variables. The choice of \( X \)'s has therefore to be done in a convenient way.

The knowledge of \( X_{ab} \) permits us to obtain \( T_a^{(1)} \). If \( X_{ab} \) does not depend on \( (q,p) \), it is easily seen that \( T_a + T_a^{(1)} \) is already strongly involutive and we succeed in obtaining \( \tilde{T}_a \). This is what happens for systems with linear constraints. For nonlinear constraints, on the other hand, \( X_{ab} \) becomes variable dependent which necessitates the analysis to be pursued beyond the first iterative step. All the subsequent corrections must be explicitly computed, the knowledge of \( T_a^{(n)} (n = 0, 1, 2, \ldots n) \) leading to the evaluation of \( T_a^{(n+1)} \) from the recursive relations. Once again the importance of choosing the proper solution for \( X_{ab} \) becomes apparent otherwise the series of corrections cannot be put in a closed form and the expression for the involutive constraints becomes unintelligible and uninteresting.
Another point in the Hamiltonian formalism is that any dynamic function \( A(q, p) \) (for instance, the Hamiltonian) has also to be properly modified in order to be strongly involutive with the first-class constraints \( \tilde{T}_a \). Denoting the modified quantity by \( \tilde{A}(q, p; \eta) \), we then have

\[
\{ \tilde{T}_a, \tilde{A} \} = 0. 
\] (2.13)

In addition, \( \tilde{A} \) has also to satisfy the boundary condition

\[
\tilde{A}(q, p; 0) = A(q, p). 
\] (2.14)

To obtain \( \tilde{A} \) an expansion analogous to (2.4) is considered,

\[
\hat{A} = \sum_{n=0}^{\infty} A^{(n)} ,
\] (2.15)

where \( A^{(n)} \) is also a term of order \( n \) in \( \eta \)'s. Consequently, compatibility with (2.14) requires that

\[
A^{(0)} = A .
\] (2.16)

The combination of (2.6), (2.13) and (2.15) gives

\[
\begin{align*}
\{ T_a^{(0)}, A^{(0)} \}_{(q,p)} + \{ T_a^{(1)}, A^{(1)} \}_{(\eta)} &= 0 , \\
\{ T_a^{(0)}, A^{(1)} \}_{(q,p)} + \{ T_a^{(1)}, A^{(0)} \}_{(q,p)} + \{ T_a^{(1)}, A^{(2)} \}_{(q,p)} + \{ T_a^{(2)}, A^{(1)} \}_{(\eta)} &= 0 , \\
\{ T_a^{(0)}, A^{(2)} \}_{(q,p)} + \{ T_a^{(1)}, A^{(1)} \}_{(q,p)} + \{ T_a^{(2)}, A^{(0)} \}_{(q,p)} + \{ T_a^{(1)}, A^{(3)} \}_{(\eta)} + \{ T_a^{(2)}, A^{(2)} \}_{(\eta)} + \{ T_a^{(3)}, A^{(1)} \}_{(\eta)} &= 0 ,
\end{align*}
\] (2.17)

which correspond to the coefficients of the powers \( \eta^0, \eta^1, \eta^2, \) etc., respectively. The expression (2.17) above gives us \( A^{(1)} \)

\[
A^{(1)} = -\eta^a \omega_{ab} X^{bc} \{ T_c, A \} ,
\] (2.18)

where \( \omega_{ab} \) and \( X^{ab} \) are the inverses of \( \omega^{ab} \) and \( X_{ab} \).

It was earlier seen that \( T_a + T_a^{(1)} \) was strongly involutive if the coefficients \( X_{ab} \) do not depend on \( (q, p) \). However, the same argument does not necessarily apply in this case. Usually we have to calculate other corrections to obtain the final \( \hat{A} \). Let us discuss how this can be systematically done. The correction \( A^{(2)} \) comes from equation (2.18), that we conveniently rewrite as

\[
\{ T_a^{(1)}, A^{(2)} \}_{(\eta)} = -G^{(1)} ,
\] (2.19)

where

\[
G^{(1)} = \{ T_a, A^{(1)} \}_{(q,p)} + \{ T_a^{(1)}, A \}_{(q,p)} + \{ T_a^{(2)}, A^{(1)} \}_{(\eta)} .
\] (2.20)

Thus

\[
A^{(2)} = -\frac{1}{2} \eta^a \omega_{ab} X^{bc} G^{(1)} .
\] (2.21)

In the same way, other terms can be obtained. The final general expression reads

\[
A^{(n+1)} = -\frac{1}{n+1} \eta^a \omega_{ab} X^{bc} G^{(n)} ,
\] (2.22)

where

\[
G^{(n)} = \sum_{m=0}^{n} \{ T_a^{(n-m)}, A^{(m)} \}_{(q,p)} + \sum_{m=0}^{n-2} \{ T_a^{(n-m)}, A^{(m+2)} \}_{(\eta)} + \{ T_a^{(n+1)}, A^{(1)} \}_{(\eta)} .
\] (2.23)
The partially embedding procedure we are going to apply consists in transforming into first-class just part of the constraints. Let us suppose we take $M'$ among the $M$ second-class constraints to be converted into first-class. There are two general steps to be done. The first one is to achieve the strong involutive algebra for these constraints, namely,

$$\{\hat{T}_{a'}, \hat{T}_{b'}\} = 0 \quad a', b' = 1, \ldots, M' \quad (2.26)$$

by introducing $M'$ variables $\eta^a$ of auxiliary variables and the same steps of the BFFT formalism. Consequently, in the partially embedding formalism we work with less auxiliary variables than in the full procedure. Of course, the choice of what constraints we intend to convert into first-class may be a crucial point for the success of the method.

The fact of having achieved a strong involutive algebra for some of the constraints does not necessarily means that these constraints are first-class. We have also to convert the remaining $M - M'$ ones in order to have involutive algebras with all the $\hat{T}_{a'}$. This is the second general step we have talked above and it is achieved by treating the remaining constraints in a similar way of the quantization. Consequently, the second-class constraints by $\hat{T}_{a'}^*$, with $a^* = 1, \ldots M - M'$, we would have to obtain $\hat{T}_{a'}(q, p, \eta')$ in the same lines that $A$ was done in the BFFT formalism. In this way

$$\{\hat{T}_{a'}, \hat{T}_{a'}^*\} = 0 , \quad (2.27)$$

but, in general, the matrix $\Delta_{a^*b'^*} = \{\hat{T}_{a'^*}, \hat{T}_{b'^*}\}$ will be nonsingular, which means that the $\hat{T}_{a'^*}$'s remain second-class.

Finally we mention that the strong involutive Hamiltonian are obtained in the same way as $\hat{T}_{a'}$. Further details will be displayed when we use the formalism in the example we are going to consider.

### III. THE MODEL AND DIFFICULTIES FOR TOTALLY EMBEDDING

The motion of a particle on a $N$-dimensional sphere of radius 1 is described by the Lagrangian

$$L = \frac{1}{2} \dot{q}^2 + \frac{1}{2} \lambda (q_i q_i - 1) . \quad (3.1)$$

As can be easily verified, the Euler-lagrange equations for $\lambda$ and $q_i$ are respectively given by

$$q_i^2 - 1 = 0 , \quad \ddot{q}_i - \lambda q_i = 0 . \quad (3.2)$$

In the first equation above we are using a short notation. We do it from now on where there is no misunderstanding. Derivating the first of the equations above twice with respect to the evolution parameter and using the second equation, we see that $\lambda = -\frac{\dot{q}_i^2}{q_i^2}$, and so we get the expected equation of motion for a particle moving on a $N$-dimensional sphere:

$$\ddot{q}_i = -\frac{\dot{q}_i^2}{q_i^2} q_i . \quad (3.3)$$

Let us now consider this theory in the canonical formalism. The canonical Hamiltonian corresponding to (3.1) reads

$$H_c = \frac{1}{2} p_i^2 - \frac{1}{2} \lambda (q_i^2 - 1) , \quad (3.4)$$

where $p_i$ is the canonical conjugated momentum to $q_i$. Using the Dirac constraint formalism we obtain that the constraints of this theory are

$$T_1 = q_i^2 - 1 , \quad T_2 = q_i p_i , \quad T_3 = p_i \lambda , \quad T_4 = \lambda q_i^2 + p_i^2 , \quad (3.5)$$

where $p_\lambda$ is the canonical conjugated momentum to $\lambda$. These constraints are second-class. In fact, for the antisymmetric quantities $\Delta_{ab}$ given by (2.13), we have

$$\Delta_{12} = 2q_i^2 , \quad \Delta_{14} = 4q_i p_i , \quad \Delta_{24} = -2\lambda q_i^2 + 2p_i^2 , \quad \Delta_{34} = -q_i^2 , \quad (3.6)$$

and one can verify that the matrix $\Delta = (\Delta_{ab})$ is regular.

The Dirac brackets between any two quantities $A$ and $B$ is constructed in the usual way,

$$\{A, B\}_D = \{A, B\} - \{A, T_a\} \Delta_{ab}^{-1} \{T_b, B\} \quad (3.7)$$

and since any quantity has null Dirac brackets with any one of the $T$’s, the time evolution generated by $H_c$, or by any of its extensions by adding to it terms such as $\lambda T_a$, gives the same result. Actually, by considering the form of the constraints, we note that (3.4) can be written as

$$H_c = -\lambda T_1 + \frac{1}{2} T_4 - \frac{1}{2} \lambda , \quad (3.8)$$
and so, under Dirac brackets, the dynamics of the system is generated just by $-\frac{1}{2} \dot{\lambda}$. As can be verified,

$$\dot{q}_i = -\frac{1}{2} \{q_i, \lambda \}_D = \frac{1}{q^2} p_i,$$
$$\dot{p}_i = -\frac{1}{2} \{p_i, \lambda \}_D = -\frac{\lambda}{q^2} q_i,$$
$$\dot{\lambda} = -\frac{1}{2} \{\lambda, \lambda \}_D = 0,$$  \hspace{1cm} (3.9)

which give, on the constraint surface, the same dynamics as the one generated by the Euler-Lagrange equations \(2.12\).

Let us now try to use the full BFFT method for the present theory in order to see the difficulties for embedding it. The use of the method requires the introduction of four coordinates $\eta^a$. We consider them such that $\eta^3$ and $\eta^4$ are the momenta conjugated to $\eta^1$ and $\eta^2$, respectively. Hence, the matrix $(\omega^{ab})$ given by \(2.2\) reads

$$(\omega^{ab}) = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}.$$  \hspace{1cm} (3.10)

The combination of \(2.13\), \(2.14\), \(3.3\), and \(3.10\) leads to the set of equations

$$X_{11}X_{23} + X_{12}X_{24} - X_{13}X_{21} - X_{14}X_{22} = -2q^2,$$
$$X_{11}X_{33} + X_{12}X_{34} - X_{13}X_{31} - X_{14}X_{32} = 0,$$
$$X_{11}X_{43} + X_{12}X_{44} - X_{13}X_{41} - X_{14}X_{42} = -4q,p,$$
$$X_{21}X_{33} + X_{22}X_{34} - X_{23}X_{31} - X_{24}X_{32} = 0,$$
$$X_{21}X_{43} + X_{22}X_{44} - X_{23}X_{41} - X_{24}X_{42} = 2\lambda q^2 - 2p^2,$$
$$X_{31}X_{43} + X_{32}X_{44} - X_{33}X_{41} - X_{34}X_{42} = q^2.$$  \hspace{1cm} (3.11)

The system above cannot be univocally solved. It contains sixteen variables in just six equations. In cases like this we examine the possibility of figuring out a solution where the first linear correction for the constrains could lead to a strongly involutive algebra. This is achieved if \(\lambda\) is considered a solution of \(2.11\), which leads to the following set of equations

$$\{T_a, X_{bc}\} + \{X_{ac}, T_b\} = 0,$$
$$\{X_{ac}, X_{bd}\} + \{X_{ad}, X_{bc}\} = 0.$$  \hspace{1cm} (3.12)

are also satisfied. Since the coefficients $X_{ab}$ depend on coordinates and momenta, there is no choice where this can be achieved.

In order to see the problem of going to the next steps of the method, let us make a choice that solves \(3.11\):

$$X_{11} = 0, \quad X_{12} = 0, \quad X_{13} = 2, \quad X_{14} = q.p,$$
$$X_{21} = q^2, \quad X_{22} = 0, \quad X_{23} = 0, \quad X_{24} = \frac{3}{2}p^2,$$
$$X_{31} = 0, \quad X_{32} = 0, \quad X_{33} = 0, \quad X_{34} = -\frac{1}{4}q^2,$$
$$X_{41} = 0, \quad X_{42} = 4, \quad X_{43} = 2\lambda, \quad X_{44} = 0.$$  \hspace{1cm} (3.13)

With this choice we have the following first-order correction for the constraints

$$T_1^{(1)} = 2 \eta^4 + q.p \eta^4,$$
$$T_2^{(1)} = q.q \eta^1 + \frac{1}{7} p.p \eta^4,$$
$$T_3^{(1)} = -\frac{1}{4} q.q \eta^4,$$
$$T_4^{(1)} = 4 \eta^2 + 2 \lambda \eta^3.$$  \hspace{1cm} (3.14)

We now have to consider these quantities into expression \(2.11\), which leads to the following set of equations

$$4 \frac{\partial T_2^{(2)}}{\partial \eta^1} + 2q.p \frac{\partial T_2^{(2)}}{\partial \eta^1} + q.q \frac{\partial T_2^{(2)}}{\partial \eta^1} - p.p \frac{\partial T_1^{(2)}}{\partial \eta^2} = 4q.p \eta^4,$$
$$8 \frac{\partial T_2^{(2)}}{\partial \eta^1} + 4q.p \frac{\partial T_2^{(2)}}{\partial \eta^1} + q.q \frac{\partial T_2^{(2)}}{\partial \eta^2} = 0,$$
$$2 \frac{\partial T_4^{(2)}}{\partial \eta^1} + p.q \frac{\partial T_4^{(2)}}{\partial \eta^1} + 4 \frac{\partial T_1^{(2)}}{\partial \eta^1} - 2\lambda \frac{\partial T_1^{(2)}}{\partial \eta^1} = 2\eta^4(p.p - q.q),$$
$$4q.q \frac{\partial T_4^{(2)}}{\partial \eta^1} - 2q.q \frac{\partial T_4^{(2)}}{\partial \eta^1} = -2q.q \eta^4,$$
$$2q.q \frac{\partial T_4^{(2)}}{\partial \eta^1} - p.q \frac{\partial T_4^{(2)}}{\partial \eta^1} - 8 \frac{\partial T_2^{(2)}}{\partial \eta^1} + 4\lambda \frac{\partial T_2^{(2)}}{\partial \eta^1} = 4q.p(\eta^4 - 2\eta^1),$$
$$q.q \frac{\partial T_4^{(2)}}{\partial \eta^1} - 16 \frac{\partial T_4^{(2)}}{\partial \eta^1} + 8\lambda \frac{\partial T_4^{(2)}}{\partial \eta^1} = 4(\eta^3 + q.p \eta^3).$$  \hspace{1cm} (3.15)
As one observes, this system may have many solutions. This can be verified if one writes $T^{(2)}_a$ as $X_{abc} \eta^b \eta^c$. So each $T^{(2)}_a$ has sixteen terms and the six equations (4.13) will involve ninety six quantities to be fixed. For any choice we make, this problem will be enlarged and enlarged at each step of the method.

We then observe that it is not feasible to infer what is the general rule for the corrections and, consequently, this discard any possibility of obtaining a closed solution. We may conclude that the use of full BFET method to this problem is very tedious and uninteresting.

IV. USING THE PARTIALLY EMBEDDING FORMALISM

Considering again the set of constraints (3.3), let us just convert $T_1$ and $T_2$ into first class and let $T_3$ and $T_4$ as second class constraints. Then instead of two pair of canonical coordinated we introduce just one, that we simply denote by $\eta^1 = \eta$ and $\eta^2 = \pi$. From the solutions of the first step of the BFET method we make an analogous choice of Banerjee et al. [1] as second class constraints. Then instead of two pair of constraints $\tilde{T}_1$, $\tilde{T}_2$, $\tilde{T}_3$, and $\tilde{T}_4$ are first class. They also have $\{\tilde{T}_1, \tilde{T}_2\} = 0$. (4.1)

It is opportun to mention that this choice, which would correspond to $X_{11} = 2$ and $X_{23} = q.q$ of the full BFET method is not compatible with any solution for the set of equations given by (3.13).

Of course, the result given by (4.2) does not necessarily means that $\tilde{T}_1$ and $\tilde{T}_2$ are first class. They also have to have zero Poisson brackets, on the constraint surface, with the remaining constraints. We notice that this is actually true for the constraint $T_3$, but it is not for $T_4$. Let us then conveniently modify the constraint $T_4$ in order to have zero Poisson brackets with $\tilde{T}_1$ and $\tilde{T}_2$. As it was mentioned in Sec. II, this can be achieved in the same framework of the BFET formalism by taking $T_4$ as the quantity $A$ of Eq. (2.13). So, the general expression for the first correction for $T_4$ should be

$$T^{(1)}_4 = -\eta^a \omega_{abc} X^{bc} \{T_4\},$$

where the indices $a', b', c'$, $c'$ = 1, 2 just correspond to the constraints $\tilde{T}_1$ and $\tilde{T}_2$, and $\omega_{a'b'}$ and $X^{a'b'}$ are the inverse of $\omega^{a'b'}$ and $X_{a'b'}$ respectively. Considering expressions (3.4) and (3.5) we have

$$\omega_{a'b'} = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right), \quad X^{a'b'} = \left( \begin{array}{cc} \frac{1}{q^2} & 0 \\ 0 & -\frac{1}{q^2} \end{array} \right).$$

Hence, the first correction for $T_4$ reads

$$T^{(1)}_4 = 2 \left( \lambda - \frac{p^2}{q^2} \right) \eta - 2p.q \pi.$$ (4.5)

Using (2.21)–(2.25) we calculate other corrections for $T_4$. We list some of them below

$$T^{(2)}_4 = \left( \frac{2p_i}{q^2} \eta + q_i \pi \right)^2,$$

$$T^{(3)}_4 = -2 \eta \left( \frac{2p_i}{q^2} \eta + q_i \pi \right)^2,$$

$$T^{(4)}_4 = 4 \eta \left( \frac{2p_i}{q^2} \eta + q_i \pi \right)^2.$$ (4.6)

From these results we can infer that the general correction $T^{(n)}_4$ for $n \geq 2$, should be

$$T^{(n)}_4 = \left( -\eta \frac{2}{q^2} \right)^{n-2} \left( \frac{2p_i}{q^2} \eta + q_i \pi \right)^2.$$ (4.7)

We then see that the partially embedding procedure, contrarily to the use of the full BFET method, permitted us to infer the general rule for all the corrections. More than that, we can also show that the sum of all these terms to obtain $\tilde{T}_4$ can be cast in a closed form,

$$\tilde{T}_4 = \lambda q^2 + p^2 + 2 \lambda \eta - 2p \left( \frac{p_i}{q^2} \eta + q_i \pi \right)$$

$$+ \left( \frac{2p_i}{q^2} \eta + q_i \pi \right)^2 \sum_{n=0}^{\infty} \left( -\frac{2\eta}{q^2} \right)^n,$$

$$= \lambda q^2 + p^2 + 2 \left( \lambda - \frac{p^2}{q^2} \right) \eta - 2p.q \pi$$

$$+ \left( \frac{2p_i}{q^2} \eta + q_i \pi \right)^2 \left( 1 + \frac{2\eta}{q^2} \right)^{-1}.$$ (4.8)

This constraint can be further rewritten as

$$\tilde{T}_4 = \left( 1 + \frac{2\eta}{q^2} \right)^{-1} \left( p_i - \pi q_i \right)^2 + \lambda q^2 + 2\eta.$$ (4.9)

It is just a matter of algebraic work to check that $\tilde{T}_1$ and $\tilde{T}_2$ are actually first class, whereas $\tilde{T}_3 = T_3 = p_\lambda$ and $\tilde{T}_4$ are second class.

Using the partially embedding procedure in terms of the first-class constraints $\tilde{T}_1$ and $\tilde{T}_2$ we directly obtain
the partially embedding Hamiltonian $\tilde{H}_c$ that resembles the form of the Hamiltonian $H_c$ given by (5.8), namely
\begin{equation}
\tilde{H}_c = -\lambda \tilde{T}_1 + \frac{1}{2} \tilde{T}_4 - \frac{1}{2} \lambda .
\end{equation}

It is important to emphasize that it generates a consistent time evolution for each one of the constraints $\tilde{T}_a$.

V. TIME EVOLUTION AND GAUGE INVARiance

The first order Lagrangian for the theory described in the last section reads
\begin{equation}
\tilde{L} = p \cdot \dot{q} + \pi \dot{\lambda} + p \lambda - \tilde{H} ,
\end{equation}
where $\tilde{H}$ is the total Hamiltonian
\begin{equation}
\tilde{H} = \tilde{H}_c + \lambda a \tilde{T}_a .
\end{equation}

This theory must be invariant under the transformations generated by the first-class constraints. Considering that $y$ represents any one of the canonical coordinates of the system, we have
\begin{equation}
\delta y = a' \{ y, \tilde{T}_a \} D ,
\end{equation}
where $a'$ is the parameter characteristic of the gauge transformation generated by the first-class constraint $T_a$. The presence of the Dirac brackets here is to consistently eliminate the constraints $T_a$. We thus obtain
\begin{align}
\dot{q}_i &= \left( 2\tilde{T}_1 + 1 \right) \frac{q^2}{(q^2 + 2\eta)^2} \left( p_i - 2\pi q_i \right) - \lambda^2 q_i ,
\end{align}
\begin{align}
\dot{p}_i &= \pi \dot{q}_i + \pi q_i + \lambda^2 (p_i - \pi q_i) - (2\tilde{T}_1 + 1) \left( \frac{p_k + \pi q_k}{q^2 + 2\eta} \right)^2 q_i ,
\end{align}
\begin{align}
\dot{\lambda} &= 0 ,
\dot{p}_\lambda &= 0 ,
\end{align}
\begin{align}
\dot{\eta} &= - (2\tilde{T}_1 + 1) \frac{q^2}{(q^2 + 2\eta)^2} + \lambda^2 \frac{q^2 (p_i - \pi q_i)}{q^2 + 2\eta} \approx 0 ,
\end{align}
\begin{align}
\dot{\pi} &= 2\tilde{T}_1 + 1 \left( \frac{p_i - \pi q_i}{q^2 + 2\eta} \right)^2 q^2 - \lambda + 2(\lambda + \lambda_1) \approx 0 .
\end{align}

It is a matter of algebraic work to show that the above equations of motion are consistent with the gauge transformations defined in (5.3) in the sense that $\left[ \frac{d}{dt}, \delta y \right] = 0$ for any $y$. So gauge transformed variables satisfy the same equations of motion as the original ones.

The combination of the Eqs. (5.7) and (5.8) leads to
\begin{equation}
\frac{d}{dt} \left( q_i + \lambda^2 q_i \right) = - \left[ \lambda^2 + \frac{2\tilde{T}_1 (2\tilde{T}_1 + 1)}{(q^2 + 2\eta)^2} \right] (q_i + \lambda^2 q_i) - \frac{1}{q^2} (p_k + \lambda^2 q_k)^2 q_i .
\end{equation}

The gauge invariance of the corresponding action is then achieved if the Lagrange multipliers $\lambda a'$ transform as
\begin{equation}
\delta \lambda a' = - \epsilon a' .
\end{equation}

Let us finally consider the equations of motion generated by the total Hamiltonian (5.2). An important point regarding the embedding procedure is that the obtained theory, even though having more symmetries than the initial one, does not change its physics. In other words, the theory described by the total Hamiltonian $\tilde{H}$ must describe a particle moving on a sphere of radius one. If the partially embedding we have developed till now makes sense, this point has necessarily to be verified.

The general expression for the time evolution of any canonical quantity, on the constraint surface, is
\begin{align}
\dot{y} &= \{ y, \tilde{H} \} D ,
\end{align}
\begin{align}
&= - \frac{1}{2} \{ y, \lambda \} D + (\lambda - \lambda_1) \{ \tilde{T}_1, y \} D - \lambda_2 \{ \tilde{T}_2, y \} D .
\end{align}

This leads, after some simplifications, to the equations of motion
As can be verified, the above equation is also consistent with the gauge transformations. Now, under the constraint surface, it reduces trivially to

$$
\frac{d}{dt}(q_i + \lambda^2 q_i) = -\lambda^2(q_i + \lambda^2 q_i) - \frac{1}{q^2}(q_k + \lambda^2 q_k)^2 q_i,
$$

(5.14)

which obviously reproduces (3.3) with the gauge choice \( \lambda^2 = 0 \), showing in this way that the partial embedding procedure introduces gauge degrees of freedom but keeps the same physical content, as it should be.

To conclude, let us recall the physical meaning of some embedding results found in literature by using the BFFT formalism. For the simplest case where the first class constraints of the embedding theory are linear in the new variables of the extended space, like chiral-bosons \([3,4]\) and Abelian massive vector theory \([3,4]\), the physical meaning is that the result is invariant for a shift of coordinates \([3]\). For the case where constraints are not linear in these variables, like the non-Abelian massive vector theory \([3,4]\), the embedding theory is equivalent to the generalized Stuckelberg formalism \([3,4]\). The natural question now is related to the physical meaning of the result expressed by the equation of motion given by (5.14). We can show that the result above and the initial one corresponding to a particle on a \(N\)-sphere described by coordinates \(q_i\), satisfying (3.3) are linked by a scale transformation. In fact, performing a scale transformation over the coordinates \(q_i\) in such a way that

$$
Q_i = e^\Lambda q_i,
$$

(5.15)

one directly verifies that (3.3) leads to

$$
\frac{d}{dt}(\dot{Q}_i - \dot{\Lambda}Q_i) = \dot{\Lambda}(\dot{Q}_i - \dot{\Lambda}Q_i) - \frac{1}{Q^2}(\dot{Q}_k - \dot{\Lambda}Q_k)^2 Q_i.
$$

(5.16)

By comparing (5.16) and (5.14) we see that \( \lambda^2 \) plays the role of minus the time derivative of the scale factor \( \Lambda \), when the \( q \)'s of (5.14) are interpreted as the \( Q \)'s of (5.16). This could also be directly seen from Eqs. (5.1) and (1.4), namely, \( \delta q_i = \varepsilon_2 q_i \) and \( \delta \lambda^2 = -\varepsilon_2 \). So we see that in some sense the effect of the embedding procedure on the Hamiltonian system describing the \( N \) dimensional rotor is related to a scale transformation. Fixing the gauge corresponds to make the scale factor constant over the time, as expected.

VI. CONCLUSION

In this work we have considered the quantum mechanical analog of the nonlinear sigma-model, corresponding to a particle constrained to move on a \(N\)-dimensional sphere of unit radius. The Hamiltonian treatment of this model generates four second class constraints. The embedding of this theory by using the BFFT formalism runs into difficulties. This is so because it is not natural the choice of solution in each step of the method and, consequently, one cannot infer the general rule for higher contributions. This makes the formalism uninteresting and not feasible to be applied, because we would have to get an infinite number of corrections to analyze the embedded theory. On the other hand, we have shown that the same BFFT method can be conveniently used in order to embed the theory in a partial way, where just two of the four constraints are converted into first-class. We have shown that, contrarily to the attempt of using the full method, all the steps of the corrections are naturally obtained and, more than that, can be cast in a closed form. Finally, we have discussed the dynamics and the physical meaning of the embedded theory. We have shown that it corresponds to a time dependent scale transformation of coordinates, suggesting us some equivalence with a geometrical conformal formulation.

Acknowledgment: This work is supported in part by Conselho Nacional de Desenvolvimento Científico e Tecnológico - CNPq, and Fundação Universitária José Bonifácio - FUJB (Brazilian Research Agencies).

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