A HYBRID EULER-HADAMARD PRODUCT FORMULA FOR THE Riemann Zeta Function

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Abstract. We use a smoothed version of the explicit formula to find an approximation to the Riemann zeta function as a product over its nontrivial zeros multiplied by a product over the primes. We model the first product by characteristic polynomials of random matrices. This provides a statistical model of the zeta function that involves the primes in a natural way. We then employ the model in a heuristic calculation of the moments of the modulus of the zeta function on the critical line. This calculation illuminates recent conjectures for these moments based on connections with random matrix theory.

1. Introduction

An important theme in the study of the Riemann zeta function, \( \zeta(s) \), has been the estimation of the mean values (or moments)

\[
I_k(T) = \frac{1}{T} \int_0^T \left| \zeta\left( \frac{1}{2} + it \right) \right|^{2k} \, dt .
\]

These have applications to bounding the order of \( \zeta(s) \) in the critical strip as well as to estimating the possible number of zeros of the zeta function off the critical line. Moreover, the techniques developed in these problems, in addition to being interesting in their own right, have been used to estimate mean values of other important functions in analytic number theory, such as Dirichlet polynomials.

In 1918 Hardy and Littlewood [8] proved that

\[
I_1(T) \sim \log T
\]
as \( T \to \infty \). Eight years later, in 1926, Ingham [10] showed that

\[
I_2(T) \sim \frac{1}{2\pi^2} (\log T)^4 .
\]

There are no proven asymptotic results for \( I_k \) when \( k > 2 \), although it has long been conjectured that

\[
I_k(T) \sim c_k (\log T)^{k^2}
\]
for some positive constant \( c_k \). Conrey and Ghosh (unpublished) cast this in a more precise form, namely,

\[
I_k(T) \sim \frac{a(k)g(k)}{\Gamma(k^2 + 1)} (\log T)^{k^2} ,
\]

where

\[
a(k) = \prod_p \left( 1 - \frac{1}{p} \right)^{k^2} \sum_{m=0}^{\infty} \left( \frac{\Gamma(m + k)}{m! \Gamma(k)} \right)^2 p^{-m} ,
\]

(1)

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the product being taken over all prime numbers, and \( g(k) \) is an integer when \( k \) is an integer. The results of Hardy–Littlewood and Ingham give \( g(1) = 1 \) and \( g(2) = 2 \), respectively. However, until recently, no one had formed a plausible conjecture for \( g(k) \) when \( k > 2 \). Then, in the early 1990’s, Conrey and Ghosh \[4\] conjectured that \( g(3) = 42 \). Later, Conrey and Gonek \[5\] conjectured that \( g(4) = 24024 \). The method employed by the last two authors reproduced the previous values of \( g(k) \) as well, but it did not produce a value for \( g(k) \) when \( k > 4 \).

It was recently suggested by Keating and Snaith \[13\] that the characteristic polynomial of a large random unitary matrix can be used to model the value distribution of the Riemann zeta function near a large height \( T \). Their idea was that because the zeta function is analytic away from the point \( s = 1 \), it can be approximated at \( s = \frac{1}{2} + it \) by polynomials whose zeros are the same as the zeros of \( \zeta(s) \) close to \( t \). These zeros (suitably renormalized) are believed to be distributed like the eigenangles of unitary matrices chosen with Haar measure, so they used the characteristic polynomial

\[
Z_N(U, \theta) = \prod_{n=1}^{N} (1 - e^{i(\theta_n - \theta)}),
\]

where the \( \theta_n \) are the eigenangles of a random \( N \times N \) unitary matrix \( U \), to model \( \zeta(s) \). For scaling reasons they used matrices of size \( N = \log T \) to model \( \zeta(\frac{1}{2} + it) \) when \( t \) is near \( T \). They then calculated the moments of \( |Z_N(U, \theta)| \) and found that

\[
E_N \left[ |Z_N(U, \theta)|^{2k} \right] \sim \frac{G^2(k + 1)}{G(2k + 1)} N^{k^2},
\]

where \( E_N \) denotes expectation with respect to Haar measure, and \( G(z) \) is Barnes’ \( G \)-function. When \( k = 1, 2, 3, 4 \) they observed that

\[
\frac{G^2(k + 1)}{G(2k + 1)} = \frac{g(k)}{\Gamma(k^2 + 1)},
\]

where \( g(k) \) is the same as in the results of Hardy–Littlewood and Ingham, and in the conjectures of Conrey–Ghosh and Conrey–Gonek given above. They then conjectured that this holds in general. That is, they asserted

**Conjecture 1** (Keating and Snaith). *For \( k \) fixed with \( \text{Re} \ k > -1/2 \),

\[
\frac{1}{T} \int_{T}^{2T} |\zeta(\frac{1}{2} + it)|^{2k} \, dt \sim a(k) \frac{G^2(k + 1)}{G(2k + 1)} (\log T)^{k^2},
\]

as \( T \to \infty \), where \( a(k) \) is given by \[1\] and \( G \) is the Barnes \( G \)-function.*

The characteristic polynomial approach has been successful in providing insight into other important and previously intractable problems in number theory as well (see, for example, \[16\] for a survey of recent results). However, the model has the drawback that it contains no arithmetical information—the prime numbers never appear. Indeed, they must be inserted in an \textit{ad hoc} manner. This is reflected, for example, by the absence of the arithmetical factor \( a(k) \) in equation \[3\]. Fortunately, in the moment problem it was only the factor \( g(k) \), and not \( a(k) \), that proved elusive. A realistic model for the zeta function (and other \( L \)-functions) clearly should include the primes.

In this paper we present a new model for the zeta function that overcomes this difficulty in a natural way. Our starting point is an explicit formula connecting the zeros and the primes from which we deduce a representation of the zeta function as a partial Euler product times a partial Hadamard product. Making certain assumptions about how these products behave, we then reproduce Conjecture \[1\]. Our model is based on the following representation of the zeta function.
Theorem 1. Let \( s = \sigma + it \) with \( \sigma \geq 0 \) and \( |t| \geq 2 \), let \( X \geq 2 \) be a real parameter, and let \( K \) be any fixed positive integer. Let \( u(x) \) be a nonnegative \( C^\infty \) function of mass 1, supported on \([e^{1-1/X}, e]\), and set

\[
U(z) = \int_0^\infty u(x)E_1(z \log x) \, dx, \tag{4}
\]

where \( E_1(z) \) is the exponential integral \( \int_z^\infty e^{-w/w} \, dw \). Then

\[
\zeta(s) = P_X(s)Z_X(s) \left( 1 + O \left( \frac{X^{K+2}}{(|s| \log X)^K} \right) + O(X^{-\sigma \log X}) \right), \tag{5}
\]

where

\[
P_X(s) = \exp \left( \sum_{n \leq X} \frac{\Lambda(n)}{n^s \log n} \right), \tag{6}
\]

\( \Lambda(n) \) is von Mangoldt’s function, and

\[
Z_X(s) = \exp \left( - \sum_{\rho_n} U((s - \rho_n) \log X) \right). \tag{7}
\]

The constants implied by the \( O \) terms depend only on \( u \) and \( K \).

We remark that Theorem 1 is unconditional—it does not depend on the assumption of any unproved hypothesis. Moreover, it can easily be modified to accommodate weight functions \( u \) supported on the larger interval \([1, e]\). Finally, as will be apparent from the proof, the second error term can be deleted if we replace \( P_X(s) \) by

\[
\tilde{P}_X(s) = \exp \left( \sum_{n \leq X} \frac{\Lambda(n)}{n^s \log n} v(e^{\log n/\log X}) \right),
\]

where \( v(t) = \int_t^\infty u(x) \, dx \).

To clarify (5), we temporarily assume the Riemann Hypothesis (RH) and take \( s = \frac{1}{2} + it \). We shall denote the nontrivial zeros of \( \zeta(s) \) by \( \rho_n = \frac{1}{2} + i\gamma_n \), ordered by their height above the real axis, with \( \gamma_{-n} = -\gamma_n \). Since the support of \( u \) is concentrated near \( e \), \( U(z) \) is roughly \( E_1(z) \), which is asymptotic to \(-\gamma - \log z\) as \( z \to 0 \). Here \( \gamma = 0.5772\ldots \) is Euler’s constant. Thus, for those ordinates \( \gamma_n \) close enough to \( t \), we see that

\[
\exp \left( - U(i(t - \gamma_n) \log X) \right) \approx 1(t - \gamma_n) e^{\gamma \log X}.
\]

We expect the ordinates farther away not to contribute substantially to the exponential defining \( Z_X(s) \). Now, \( P_X(s) \approx \prod_{\rho \leq X} (1 - p^{-s})^{-1} \), hence our formula looks roughly like

\[
\zeta \left( \frac{1}{2} + it \right) \approx \prod_{\rho \leq X} (1 - p^{-\frac{1}{2} - it})^{-1} \prod_{|t - \gamma_n| < 1/\log X} (1(t - \gamma_n) e^{\gamma \log X}). \tag{8}
\]

This formula is a “hybrid” consisting of a truncated Euler product and (essentially) a truncated Hadamard product, with the parameter \( X \) mediating between them. Near height \( T \) we are approximating part of the zeta function by a polynomial of degree about \( \log T/\log X \). The rest of the zeta function, which comes from the zeros we have neglected, is approximated by the finite Euler product. Formally, when we take \( X \) large, we reduce the number of zeros used to approximate zeta, but make up for it with more primes; and when we take \( X \) small, we approach the previous model (2). Note however, that in order for the error terms in (5) to be smaller than the main term, it is necessary to work in an intermediate regime, where both the zeros and the primes contribute.
To see how to use the model, and as a test case, we heuristically calculate $I_k(T)$. The new model is more elaborate than the original one, so more work is required. Nevertheless, the idea is straightforward. The $2k$th moment of $|ζ \left( \frac{1}{2} + it \right)|$ is asymptotic to the $2k$th moment of $|P_X(\frac{1}{2} + it)|$. We argue that when $X$ is not too large relative to $T$, the $2k$th moment of this product splits as the product of the moments. We call this the “Splitting Conjecture”.

**Conjecture 2. (Splitting Conjecture.)** Let $X$ and $T \to \infty$ with $X = O((\log T)^{2-\epsilon})$. Then for $k > -1/2$ we have

$$\frac{1}{T} \int_T^{2T} |ζ(\frac{1}{2} + it)|^{2k} dt \sim \left( \frac{1}{T} \int_T^{2T} |P_X(\frac{1}{2} + it)|^{2k} dt \right) \times \left( \frac{1}{T} \int_T^{2T} |Z_X(\frac{1}{2} + it)|^{2k} dt \right).$$

In Section 3 we calculate the moments of $P$ rigorously and establish the following theorem.

**Theorem 2.** Let $1/2 \leq c < 1$, $\epsilon > 0$, and let $k$ be any real number. Suppose that $X$ and $T \to \infty$ and $X = O\left((\log T)^{1/(1-c-\epsilon)}\right)$. Then we have

$$\frac{1}{T} \int_T^{2T} |P_X(\sigma + it)|^{2k} dt = a(k, \sigma) F_X(k, \sigma) \left( 1 + O_k \left( \frac{1}{\log X} \right) \right)$$

uniformly for $c \leq \sigma \leq 1$, where

$$a(k, \sigma) = \prod_p \left\{ \left( 1 - \frac{1}{p^{2\sigma}} \right)^{k^2} \sum_{m=0}^{\infty} \frac{d_k(p^m)^2}{p^{2m\sigma}} \right\}$$

and

$$F_X(k, \sigma) = \begin{cases} \zeta(2\sigma)^k e^{-k^2 E_1((2\sigma-1) \log X)} & \text{if } \sigma > 1/2, \\ (e^\gamma \log X)^k & \text{if } \sigma = 1/2. \end{cases}$$

Here $E_1$ is the exponential integral, and $\gamma = 0.5772\ldots$ is Euler’s constant.

Note that $a(k, \frac{1}{2})$ is the same as $a(k)$ in (1).

In Section 4 we conjecture an asymptotic estimate for $\int_T^{2T} |Z_X(\frac{1}{2} + it)|^{2k} dt$ using random matrix theory. We introduce random matrix theory in the following way. The statistical distribution of the ordinates $γ_n$ is conjectured to coincide with that of the eigenangles $θ_n$ of $N \times N$ random unitary matrices chosen with Haar measure for some $N$ (see for example [17], [15] and [15]). The choice of $N$ requires consideration. The numbers $γ_n$ are spaced $2\pi/\log T$ apart on average, whereas the average spacing of the $θ_n$ is $2\pi/N$, and so we take $N$ to be the greatest integer less than or equal to $\log T$. We therefore conjecture that the $2k$th moment of $|Z_X(\frac{1}{2} + it)|$, when averaged over $t$ around $T$, is asymptotically the same as $|Z_X(\frac{1}{2} + it)|^{2k}$ when the $γ_n$ are replaced by $θ_n$ and averaged over all unitary matrices with $N$ as specified above. We perform this random matrix calculation in section 3 and so obtain the following conjecture:

**Conjecture 3.** Suppose $X, T \to \infty$ with $X = O((\log T)^{2-\epsilon})$. Then for any fixed $k > -1/2$, we have

$$\frac{1}{T} \int_T^{2T} |Z_X(\frac{1}{2} + it)|^{2k} dt \sim \frac{G^2(k+1)}{G(2k+1)} \left( \frac{\log T}{e^\gamma \log X} \right)^k.$$

We actually expect conjecture 3 to hold for a much larger range of $X$, but the correct bound on the size of $X$ with respect to $T$ is unclear.

We note that this asymptotic formula coincides with that in (3) when there $N$ is taken to be on the order of $\log T/e^\gamma \log X$. This is consistent with the fact that the polynomial in (3) is of about this degree. Alternatively, the mean density of eigenvalues is $N$ divided by $2\pi$, and this is
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comparable to the mean density of the ordinates of the zeros when multiplied by $e^\gamma \log X$, as they are in \(3\).

Combining the result of Theorem \(2\) with the formula in Conjecture \(3\) and using the Splitting Conjecture, we recover precisely the conjecture put forward by Keating and Snaith. Note that, as must be the case, all $X$-dependent terms cancel out.

In Section 5, we prove Theorem \(3\).

Let $\epsilon > 0$ and let $X$ and $T \to \infty$ with $X = O((\log T)^{2-\epsilon})$. Then for $k = 1$ and $k = 2$ we have

$$
\frac{1}{T} \int_T^{2T} |\zeta(1/2 + it)| P_X(1/2 + it)^{-1} |2k\ dt \sim \frac{G^2(k+1)}{G(2k+1)} \left( \frac{\log T}{e^\gamma \log X} \right)^{k^2}.
$$

Since $\zeta(1/2 + it) P_X(1/2 + it)^{-1} = Z_X(1/2 + it) (1 + o(1))$ for $t \in [T, 2T]$, it follows from this that Conjecture \(3\) holds when $k = 1$ and $k = 2$. Moreover, combining Theorem \(3\) with our estimate for

$$
\frac{1}{T} \int_T^{2T} |P_X(1/2 + it)|^{2k} \ dt
$$

from Theorem \(2\) we also see that Conjecture \(2\) holds for $k = 1$ and $k = 2$. Thus, we obtain the

Corollary. Conjectures \(2\) and \(3\) are true for $k = 1$ and $k = 2$.

Clearly our model can be adapted straightforwardly to other $L$-functions (see \(14\)). It can also be used to reproduce other moment results and conjectures, such as those given by Gonek \(10\) and by Hughes, Keating and O’Connell \(19\) concerning derivatives of the Riemann zeta function at the zeros of the zeta function. We also expect it to provide further insight into the connection between prime numbers and the zeros of the zeta function. It would be particularly interesting to determine whether the model can be extended to capture lower order terms in the asymptotic expansions of the moments of $\zeta(1/2 + it)$ and other $L$-functions, c.f. \(3\).

2. THE PROOF OF THEOREM 1

We begin the proof by stating a smoothed form of the explicit formula due to Bombieri and Hejhal \(2\).

Lemma 1. Let $u(x)$ be a real, nonnegative, $C^\infty$ function with compact support in $[1, e]$, and let $u$ be normalized so that if

$$
v(t) = \int_t^\infty u(x) \ dx
$$

then $v(0) = 1$. Let

$$
\tilde{u}(z) = \int_0^\infty u(x)x^{z-1}dx
$$

be the Mellin transform of $u$. Then for $s$ not a zero or pole of the zeta function, we have

$$
-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=2}^\infty \frac{\Lambda(n)}{n^s} v(e^{\log n}/\log X) - \sum_{\rho} \frac{\tilde{u}(1 - (s - \rho) \log X)}{s - \rho} + \frac{\tilde{u}(1 - (s - 1) \log X)}{s - 1} + \sum_{m=1}^\infty \frac{\tilde{u}(1 - (s + 2m) \log X)}{s + 2m},
$$

where the sum over $\rho$ runs over all the nontrivial zeros of the zeta function.
This lemma is proved in a familiar way, beginning with the integral
\[ \frac{1}{2\pi i} \int_{(c)} \frac{\zeta'(z + s)}{\zeta(z)} \bar{u}(1 + z \log X) \frac{dz}{z}, \]
where the integral is over the vertical line \( \text{Re} \, z = c = \max\{2, 2 - \text{Re} \, s\}. \)

The support condition on \( u \) implies that \( v(e^{\log n/\log X}) = 0 \) when \( n > X \), so the sum over \( n \) is finite. Furthermore, if \( |\text{Im} \, z| > 2 \), say, then integrating \( \bar{u} \) by parts \( K \) times, we see that
\[
|\bar{u}(z)| \leq \max_x |u^{(K)}(x)| \left| \frac{\Gamma(z)}{\Gamma(z + K)} \right| \left( e^{\text{Re} \, z + K} + 1 \right)
\]
for any positive integer \( K \). Thus, the sums over \( \rho \) and \( m \) on the right-hand side of (10) converge absolutely so long as \( s \neq \rho \) and \( s \neq -2m \). This, in fact, is the reason we require smoothing.

Next we integrate (10) along the horizontal line from \( s_0 = \sigma_0 + it_0 \) to \( +\infty \), where \( \sigma_0 \geq 0 \) and \( |t_0| \geq 2 \). If the line does not pass through a zero, then on the left-hand side we obtain \( -\log \zeta(s_0) \).
We choose the branch of the logarithm here so that \( \lim_{|z| \to \infty} \log \zeta(s_0) = 0 \). If the line of integration does pass through a zero, we define \( \log \zeta(\sigma + it) = \lim_{\epsilon \to 0^+} \frac{1}{2} (\log \zeta(\sigma + i(t + \epsilon) + \log \zeta(\sigma + i(t - \epsilon)) \).

Recalling the definition of \( U(z) \) in (4), we see that
\[
\int_{s_0}^{\infty} \bar{u}(1 - (s - z) \log X) \frac{ds}{s - z} = \int_0^{\infty} u(x) E_1((s_0 - z) \log X \log x) \, dx \]
\[ = U((s_0 - z) \log X), \tag{12} \]
provided that \( s_0 - z \) is not real and negative (so as to avoid the branch cut of \( E_1 \)). If it is, we use the convention that \( U((s_0 - z) \log X) = \lim_{\epsilon \to 0^+} \left( U((s_0 - z) \log X + \epsilon) + U((s_0 - z) \log X - \epsilon) \right). \)
Note that the logarithms in (12) are both positive since the support of \( u \) is in \([1, e]\) and \( X \geq 2 \). It therefore follows from (10) that
\[
\log \zeta(s_0) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^{s_0} \log n} v(e^{\log n/\log X}) - \sum_{\rho} U((s_0 - \rho) \log X)
\]
\[ + U((s_0 - 1) \log X) - \sum_{m=1}^{\infty} U((s_0 + 2m) \log X). \tag{13} \]
The interchange of summation and integration in the sums is justified by absolute convergence. This representation holds for all points in \( \text{Re} \, s \geq 0 \) not equal to the pole or one of the zeros of the zeta function.

We next suppose that the support of \( u \) is contained in \([e^{1-1/X}, e]\) with the same \( X \) as in (13). It is easy to see that there is a smooth nonnegative function \( f \) with support in \([0, 1]\) and total mass one such that \( u(x) = X f(X \log(x/e) + 1)/x \). Since \( \max_x |f^{(K)}(x)| \) is bounded and independent of \( X \), we see that \( \max_x |u^{(K)}(x)| \ll_X X^{K+1} \). It therefore follows from (11) that
\[
\bar{u}(s) \ll_X e^{\max(\sigma, 0)} X^{K+1} (1 + |s|)^{-K}. \]
From this and (12), and since $|t_0| \geq 2$, we find that if $r$ is real, then

$$U((s_0 - r) \log X) = \int_{s_0}^{\infty} \frac{\tilde{u}(1 - (s - r) \log X)}{s - r} \, ds$$

$$\ll K \frac{X^{K+1}}{(|s_0| \log X)^K} \int_{s_0}^{\infty} \left| \frac{X_{\max\{r-\sigma,0\}}}{(\log X)^K} \right| ds$$

$$\ll K \frac{X^{K+1+\max\{r_0-\sigma,0\}}}{(|s_0 - r| \log X)^K}.$$  

In particular, for any fixed positive integer $K$ we have that

$$U((s_0 - 1) \log X) \ll K \frac{X^{K+1+\max\{1-\sigma_0,0\}}}{(|s_0| \log X)^K},$$

and, since $\sigma_0 \geq 0$, that

$$\sum_{m=1}^{\infty} U((s_0 + 2m) \log X) \ll K \frac{X^{K+1}}{(|s_0 + 2m| \log X)^K}.$$

Inserting these estimates into (13) and replacing $s_0$ by $s$, we find that

$$\log \zeta(s) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s \log n} \nu(e^{\log n/\log X}) - \sum_{\rho} U((s - \rho) \log X) + O\left( \frac{X^{K+2}}{(|s| \log X)^K} \right)$$

for $\sigma \geq 0$, $|t| \geq 2$, and $K$ any fixed positive integer. Exponentiating both sides, we obtain

$$\zeta(s) = \tilde{P}_X(s)Z_X(s) \left( 1 + O\left( \frac{X^{K+2}}{(|s| \log X)^K} \right) \right), \quad (14)$$

where

$$\tilde{P}_X(s) = \exp \left( \sum_{n \leq X} \frac{\Lambda(n)}{n^s \log n} \nu(e^{\log n/\log X}) \right)$$

and

$$Z_X(s) = \exp \left( - \sum_{\rho} U((s - \rho) \log X) \right).$$

We now wish to show that replacing $\tilde{P}_X(s)$ by

$$P_X(s) = \exp \left( \sum_{n \leq X} \frac{\Lambda(n)}{n^s \log n} \right)$$
only introduces a small error term into (14). To see this, note that 
\(v((e^{\log n/\log X}) = 1 \) for \( n \leq X^{1-1/X} \) because the support of \( u(x) \) is in \([e^{1-1/X}, e]\). Therefore,

\[
\frac{\tilde{P}_X(s)}{P_X(s)} = \exp \left( \sum_{\substack{X^{1-1/X} \leq n \leq X}} \frac{\Lambda(n)}{n^s \log n} \left( v(e^{\log X/\log n}) - 1 \right) \right) 
\ll \exp \left( \sum_{\substack{X^{1-1/X} \leq n \leq X}} \frac{1}{n^\sigma} \right) 
\ll \exp \left( X^{-\sigma \log X} \right).
\]

This completes the proof of Theorem 1 provided that \( s \) is not a nontrivial zero of the zeta function. To remove this restriction, we recall the formula

\[
E_1(z) = -\log z - \gamma - \sum_{m=1}^{\infty} \frac{(-1)^m z^m}{m! m},
\]

where \(|\arg z| < \pi\), \(\log z\) denotes the principal branch of the logarithm, and \(\gamma\) is Euler’s constant. From this and (4) we observe that we may interpret \(\exp(-U(z))\) to be asymptotic to \(Cz\) for some constant \(C\) as \(z \to 0\). Thus, both sides of (5) vanish at the zeros.

3. The Proof of Theorem 2

We begin with several lemmas.

**Lemma 2.** Let \(X \geq 2\) and set

\[
P_X^*(s) = \prod_{p \leq X} \left(1 - \frac{1}{p^s}\right)^{-1} \prod_{\sqrt{X} < p \leq X} \left(1 + \frac{1}{2 p^s}\right)^{-1}.
\]

Then for \(k\) any real number we have

\[
P_X(s)^k = P_X^*(s)^k \left(1 + O_k \left(\frac{1}{\log X}\right)\right)
\]

uniformly for \(\sigma \geq 1/2\).

**Proof.** By (3) we have

\[
P_X(s)^k = \exp \left( k \sum_{n \leq X} \frac{\Lambda(n)}{n^s \log n} \right) = \prod_{p \leq X} \exp \left( k \sum_{1 \leq j \leq N_p} \frac{1}{jp^js} \right),
\]

where \(N_p = \lfloor \log X/\log p \rfloor\), the integer part of \(\log X/\log p\). Therefore

\[
P_X(s)^k P_X^*(s)^{-k} = \exp \left( -k \sum_{p \leq X} \sum_{j > N_p} \frac{1}{jp^js} - k \sum_{\sqrt{X} < p \leq X} \sum_{j=1}^{\infty} \frac{(-1)^j}{jp^{2js}} \right).
\]

The primes \(\sqrt{X} < p \leq X\) have \(N_p = 1\), and we note that the \(j = 2\) term for these primes in the first double sum exactly cancels the \(j = 1\) term in the second. Hence the argument in the exponent is

\[
\ll |k| \left( \sum_{p \leq \sqrt{X}} \frac{1}{p^{\sigma(N_p+1)}} + \sum_{\sqrt{X} < p \leq X} \frac{1}{p^{\sigma}} \right).
\]
Now \( p^{N_p+1} > X \) since \( N_p + 1 > \log X/\log p \), so, for \( \sigma \geq 1/2 \), this is
\[
\ll |k| \left( \frac{X^{-1/2}}{} \sum_{p \leq \sqrt{X}} 1 + \sum_{\sqrt{X} < p \leq X} \frac{1}{p^{3/2}} \right)
\ll |k| \left( \frac{1}{\log X} + \frac{1}{X^{1/4} \log X} \right) \ll \frac{|k|}{\log X}.
\]
It follows that
\[
 PX(s)^k P_X^*(s)^{-k} = 1 + O_k \left( \frac{1}{\log X} \right),
\]
as required.

Lemma 3. Let \( k \) be a real number. Let \( 1/2 \leq c < 1 \) be arbitrary but fixed, and suppose that \( 2 \leq X \ll (\log T)^{1/(1-c+\epsilon)} \), where \( \epsilon > 0 \) is also fixed. Then
\[
\frac{1}{T} \int_T^{2T} |P_X^*(\sigma + it)|^{2k} \, dt = a(k, \sigma) \prod_{p \leq X} \left( 1 - \frac{1}{p^{2\sigma}} \right)^{-k^2} \left( 1 + O_k \left( \frac{X^{1/2-2\sigma}}{\log X} \right) \right)
\]
uniformly for \( c \leq \sigma \leq 1 \), where \( a(k, \sigma) \) is given by (9).

Proof. We write
\[
\sum_{n=1}^{\infty} \frac{\alpha_k(n)}{n^s} = P_X^*(s)^k = \prod_{p \leq X} (1 - p^{-s})^{-k} \prod_{\sqrt{X} < p \leq X} (1 + \frac{1}{2} p^{-2s})^{-k}.
\]
Let \( S(X) \) denote the set of \( X \)-smooth numbers, that is, \( S(X) = \{ n : p \mid n \implies p \leq X \} \). Then \( \alpha_k(n) = d_k(n) \), the \( k \)th divisor function, if \( n \in S(\sqrt{X}) \); \( \alpha_k(p) = d_k(p) \) for all \( p \leq X \); and \( \alpha_k(n) = 0 \) if \( n \not\in S(X) \). It is also easy to see that
\[
(1 - p^{-s})^{-k} (1 + \frac{1}{2} p^{-2s})^{-k} = \exp \left( k \left( \frac{1}{p^s} + \frac{1}{3p^{3s}} + \frac{1 + \left( \frac{1}{7} \right)^1}{4p^{4s}} + \frac{1}{5p^{5s}} + \frac{1 - \left( \frac{1}{7} \right)^2}{6p^{6s}} + \cdots \right) \right).
\]
Comparing this with
\[
(1 - p^{-s})^{-k} = \exp \left( k \left( \frac{1}{p^s} + \frac{1}{2p^{2s}} + \frac{1}{3p^{3s}} + \frac{1}{4p^{4s}} + \cdots \right) \right),
\]
we find that for \( k \geq 0, \sqrt{X} < p \leq X \), and \( j = 1, 2, \ldots \),
\[
0 \leq \alpha_k(p^j) \leq d_{3k/2}(p^j),
\]
while for \( k < 0 \)
\[
|\alpha_k(p^j)| \leq d_{|k|/2}(p^j).
\]

We now truncate the sum in (15) at \( T^\theta \), where \( \theta \) is a small positive number to be chosen later, and obtain
\[
\sum_{n \in S(X) \atop n \leq T^\theta} \frac{\alpha_k(n)}{n^s} + O \left( \sum_{n \in S(X) \atop n > T^\theta} \frac{|\alpha_k(n)|}{n^\sigma} \right).
\]
For \( \varepsilon > 0 \) fixed and \( \sigma \geq c \), the sum in the \( O \)-term is

\[
\ll \sum_{n \geq T^\theta \atop n \in S(X)} \left( \frac{n}{T^\theta} \right)^\varepsilon \frac{d_3|k|/2(n)}{n^\sigma} \leq T^{-\varepsilon \theta} \sum_{n \in S(X)} \frac{d_3|k|/2(n)}{n^{\sigma-\varepsilon}}
\]

\[
= T^{-\varepsilon \theta} \prod_{p \leq X} (1 - p^{\varepsilon - \varepsilon})^{-3|k|/2} = T^{-\varepsilon \theta} \exp \left( O \left( \frac{|k| \sum_{p \leq X} p^{\varepsilon - \varepsilon}}{(1 - c + \varepsilon) \log X} \right) \right)
\]

\[
\ll T^{-\varepsilon \theta} \exp \left( O \left( \frac{|k| X^{1-c+\varepsilon}}{(1 - c + \varepsilon) \log X} \right) \right).
\]

Now suppose that \( 2 < X \ll (\log T)^{1/(1-c+\varepsilon)} \) with the same \( \varepsilon \). Then this is

\[
\ll T^{-\varepsilon \theta} \exp \left( O \left( \frac{|k| \log T}{\log \log T} \right) \right) \ll k^{-\varepsilon \theta/2}.
\]

Thus, we find that

\[
P_X^*(s)^k = \sum_{n \in S(X) \atop n \leq T^\theta} \frac{\alpha_k(n)}{n^s} + O_k \left( T^{-\varepsilon \theta/2} \right). \tag{16}
\]

Next we calculate \( \frac{1}{T} \int_T^{2T} |P_X^*(s)|^{2k} \, dt \). By Montgomery and Vaughan’s mean value theorem for Dirichlet polynomials \([18]\), we have

\[
\int_T^{2T} \left| \sum_{n \leq T^\theta \atop n \in S(X)} \frac{\alpha_k(n)}{n^{\sigma+it}} \right|^2 \, dt = \left( T + O(T^\theta) \right) \sum_{n \leq T^\theta \atop n \in S(X)} \frac{\alpha_k(n)^2}{n^{2\sigma}}.
\]

Using the method above, we may extend the sum on the right to infinity with an error again no larger than \( O_k \left( T^{-\varepsilon \theta/2} \right) \). Thus, taking \( \theta = 1/2 \), say, we find that

\[
\frac{1}{T} \int_T^{2T} \left| \sum_{n \leq T^{1/2} \atop n \in S(X)} \frac{\alpha_k(n)}{n^{\sigma+it}} \right|^2 \, dt = \sum_{n \in S(X)} \frac{\alpha_k(n)^2}{n^{2\sigma}} \left( 1 + O_k(T^{-\varepsilon/4}) \right). \tag{17}
\]

We next note that if \( A_i = \frac{1}{T} \int_T^{2T} |a_i(t)|^2 \, dt \), \( i = 1, 2 \), and \( A_1 \neq 0 \), then

\[
\frac{1}{T} \int_T^{2T} |a_1(t) + a_2(t)|^2 \, dt = A_1 \left( 1 + O \left( \left( A_2/A_1 \right)^{1/2} \right) \right) + A_2,
\]

the \( O \)-term arising from the Cauchy-Schwarz inequality applied to the “cross term”. We use this with \( a_1(t) \) the sum on the right-hand side of \([16]\) and \( a_2(t) \) the error term (with \( \theta = 1/2 \)). Since \( \alpha_k(1) = 1 \), we see from \([17]\) that \( A_1 \gg 1 \). It therefore follows that

\[
\frac{1}{T} \int_T^{2T} |P_X^*(\sigma + it)|^{2k} \, dt = \left( 1 + O_k(T^{-\varepsilon/4}) \right) \sum_{n \in S(X)} \frac{\alpha_k(n)^2}{n^{2\sigma}}. \tag{18}
\]
Since \( \alpha_k(n) = d_k(n) \) for \( n \in S(\sqrt{X}) \), and \( \alpha_k(p) = d_k(p) \) for \( \sqrt{X} < p \leq X \), we may write the sum as

\[
\sum_{n \in S(X)} \frac{\alpha_k(n)^2}{n^{2\sigma}} = \prod_{p \leq X} \left( \sum_{m=0}^{\infty} \frac{\alpha_k(p^m)^2}{p^{2m\sigma}} \right)
= \prod_{p \leq \sqrt{X}} \left( \sum_{m=0}^{\infty} \frac{d_k(p^m)^2}{p^{2m\sigma}} \right) \prod_{X < p \leq X} \left( 1 + \frac{d_k(p)^2}{p^{2\sigma}} + \sum_{m=2}^{\infty} \frac{\alpha_k(p^m)^2}{p^{2m\sigma}} \right)
= \prod_{p \leq X} \left( \sum_{m=0}^{\infty} \frac{d_k(p^m)^2}{p^{2m\sigma}} \right) \prod_{X < p \leq X} \left\{ \left( 1 + \frac{d_k(p)^2}{p^{2\sigma}} + \sum_{m=2}^{\infty} \frac{\alpha_k(p^m)^2}{p^{2m\sigma}} \right) / \sum_{m=0}^{\infty} \frac{d_k(p^m)^2}{p^{2m\sigma}} \right\}.
\]

Factoring \( 1 + d_k(p)^2/p^{2\sigma} \) (which is at least 1) out of the numerator and denominator of the last product, we see that the product equals

\[
\prod_{\sqrt{X} < p \leq X} \left( 1 + O_k \left( \frac{1}{p^{2\sigma}} \right) \right) = \exp \left( O_k \left( \frac{X^{\frac{1}{2}-2\sigma}}{\log X} \right) \right) = 1 + O_k \left( \frac{X^{\frac{1}{2}-2\sigma}}{\log X} \right).
\]

Hence,

\[
\sum_{n \in S(X)} \frac{\alpha_k(n)^2}{n^{2\sigma}} = \prod_{p \leq X} \left( \sum_{m=0}^{\infty} \frac{d_k(p^m)^2}{p^{2m\sigma}} \right) \left( 1 + O_k \left( \frac{X^{\frac{1}{2}-2\sigma}}{\log X} \right) \right).
\]

Writing the product here as

\[
\prod_{p \leq X} \left( \left( 1 - \frac{1}{p^{2\sigma}} \right)^k \sum_{m=0}^{\infty} \frac{d_k(p^m)^2}{p^{2m\sigma}} \right) \prod_{p \leq X} \left( 1 - \frac{1}{p^{2\sigma}} \right)^{-k^2},
\]

we note that the first of the two factors may be extended over all the primes, because

\[
\prod_{p > X} \left( \left( 1 - \frac{1}{p^{2\sigma}} \right)^k \sum_{m=0}^{\infty} \frac{d_k(p^m)^2}{p^{2m\sigma}} \right) = \prod_{p > X} \left( 1 + O_k \left( \frac{1}{p^{4\sigma}} \right) \right)
= 1 + O_k \left( \frac{X^{1-4\sigma}}{\log X} \right).
\]

Thus, by the definition of \( a(k, \sigma) \) in [18], we find that

\[
\sum_{n \in S(X)} \frac{\alpha_k(n)^2}{n^{2\sigma}} = a(k, \sigma) \prod_{p \leq X} \left( 1 - \frac{1}{p^{2\sigma}} \right)^{-k^2} \left( 1 + O_k \left( \frac{X^{\frac{1}{2}-2\sigma}}{\log X} \right) \right).
\]

The lemma follows from this and [18].

**Lemma 4.** If \( k \) is a real number, then

\[
\prod_{p \leq X} \left( 1 - \frac{1}{p^{2\sigma}} \right)^{-k^2} = F_X(k, \sigma) \left( 1 + O_k \left( \frac{1}{\log X} \right) \right)
\]

uniformly for \( \sigma \geq 1/2 \), where

\[
F_X(k, \sigma) = \begin{cases} 
\zeta(2\sigma)k^2 e^{-k^2 \gamma / 2} \log X & \text{if } \sigma > 1/2, \\
(e^{-\gamma} \log X)^k & \text{if } \sigma = 1/2,
\end{cases}
\]

and \( E_1 \) is the exponential integral.
Proof. Mertens’ theorem asserts that
\[
\prod_{p \leq X} \left( 1 - \frac{1}{p} \right)^{-1} = e^\gamma \log X \left( 1 + O\left( \frac{1}{\log X} \right) \right).
\]
Raising both sides to the \(k^2\) power establishes the result when \(\sigma = 1/2\). When \(\sigma > 1/2\), we see that
\[
\prod_{p \leq X} \left( 1 - \frac{1}{p^{2\sigma}} \right)^{-1} = \zeta(2\sigma) \exp \left( \sum_{p > X} \log \left( 1 - \frac{1}{p^{2\sigma}} \right) \right).
\]
By the prime number theorem in the form \(\psi(x) = \sum_{n \leq x} \Lambda(n) = x + O(x/(\log x)^A)\), we find that
\[
\sum_{p > X} \log \left( 1 - \frac{1}{p^{2\sigma}} \right) = -\sum_{p > X} \left( \frac{1}{p^{2\sigma}} + O\left( \frac{1}{p^{4\sigma}} \right) \right)
\]
\[
= -\int_X^\infty \left( \frac{1}{u^{2\sigma}} + O\left( \frac{1}{u^{4\sigma}} \right) \right) \frac{du}{\log u} + O\left( \frac{1}{(\log X)^A} \right)
\]
\[
= -E_1((2\sigma - 1) \log X) + O\left( \frac{1}{(\log X)^A} \right).
\]
Hence,
\[
\prod_{p \leq X} \left( 1 - \frac{1}{p^{2\sigma}} \right)^{-k^2} = \zeta(2\sigma)^{k^2} \exp \left( -k^2 E_1((2\sigma - 1) \log X) \right) \left( 1 + O_k\left( \frac{1}{\log X} \right) \right),
\]
as asserted.

The proof of Theorem 2 now follows immediately from Lemmas 2, 3 and 4.

4. Support for Conjecture 3

In this section we give heuristic arguments supporting Conjecture 3 which we restate as
\[
\frac{1}{T} \int_{T}^{2T} \left| Z_X\left( \frac{1}{2} + it \right) \right|^{2k} dt \sim \frac{G^2(k+1)}{G(2k+1)} \left( \frac{\log T}{e^\gamma \log X} \right)^{k^2}
\]
as \(T \to \infty\), where \(Z_X(s)\) is given by (7).

We assume the Riemann Hypothesis. Since \(\text{Re} E_1(x) = -\text{Ci}(|x|)\) for \(x \in \mathbb{R}\), where
\[
\text{Ci}(z) = -\int_z^\infty \frac{\cos w}{w} dw,
\]
we find that
\[
\frac{1}{T} \int_{T}^{2T} \left| Z_X\left( \frac{1}{2} + it \right) \right|^{2k} dt = \frac{1}{T} \int_{T}^{2T} \prod_{\gamma_n} \exp \left( 2k \int_{1}^{\epsilon} u(y) \text{Ci}(|t - \gamma_n| \log y \log X) \ dy \right) dt,
\]
(19)
where \(u(y)\) is a smooth nonnegative function, supported on \([\epsilon^{1-1/X}, \epsilon]\) and of total mass 1. Since the terms in the exponent decay as \(|\gamma_n - t|\) increases, this product is effectively a local statistic. That is, the integrand depends only on those zeros close to \(t\). In recent years considerable evidence has been amassed suggesting that the zeros of the Riemann zeta function around height \(T\) are distributed like the eigenangles of unitary matrices of size \(\log T\) chosen with Haar measure (see, for example, the survey article [15]). We therefore model the right-hand side of (19) by replacing the
 ordinates $\gamma_n$ by the eigenvalues of an $N \times N$ unitary matrix and averaging over all such matrices with Haar measure, where $N = \lfloor \log T \rfloor$. Thus, the right-hand side of (14) should be asymptotic to

$$
\mathbb{E}_N \left[ \prod_{n=1}^{N} \exp \left( 2k \int_{1}^{e} u(y) \text{Ci}(\theta_n | \log y \log X) \, dy \right) \right],
$$

where the $\theta_n$ are the eigenvalues of the random matrix and $\mathbb{E}_N \left[ \cdot \right]$ denotes the expectation with respect to Haar measure. However, since the eigenvalues of a unitary matrix are naturally $2\pi$-periodic objects, it is convenient to periodicize our function, which we do by defining

$$
\phi(\theta) = \exp \left( 2k \int_{1}^{e} u(y) \left( \sum_{j=-\infty}^{\infty} \text{Ci}(\theta + 2\pi j | \log y \log X) \right) \, dy \right).
$$

It will follow from our proof of Lemma 6 that the terms with $j \neq 0$, which make the random matrix calculation much easier, only contribute $\ll k / \log X$ to $\phi(\theta)$ when $-\pi < \theta \leq \pi$. Hence they do not affect the accuracy of the model. Thus, we argue that

$$
\frac{1}{T} \int_{T}^{2T} |Z_X(\frac{1}{2} + it)|^{2k} \, dt \sim \mathbb{E}_N \left[ \prod_{n=1}^{N} \phi(\theta_n) \right].
$$

The remainder of this section is devoted to the proof of

**Theorem 4.** Let $\phi(\theta)$ be defined as in (20), then for fixed $k > -\frac{1}{2}$ and $X \geq 2$, we have as $N \to \infty$,

$$
\mathbb{E}_N \left[ \prod_{n=1}^{N} \phi(\theta_n) \right] \sim \frac{(G(k+1))^2}{G(2k+1)} \left( \frac{N}{e^\gamma \log X} \right)^k \left( 1 + O_k \left( \frac{1}{\log X} \right) \right).
$$

**Remark.** The random matrix model of Keating and Snaith [13] for the moments of the Riemann zeta function involved the characteristic polynomial (2). Note that if we set $M = N e^\gamma \log X$, then by (3) we have

$$
\mathbb{E}_M \left[ |Z_M(U, \theta)|^{2k} \right] \sim \frac{(G(k+1))^2}{G(2k+1)} \left( \frac{N}{e^\gamma \log X} \right)^k,
$$

which is the same answer we find in Theorem 4. This is easily explained by the fact that in our model the eigenvalues are multiplied by $e^\gamma \log X$ and so their mean density is $M/2\pi$. Given that for random matrices the mean density is the only parameter in the asymptotics of local eigenvalue statistics, it is natural that the result should be the same as for unitary matrices of dimension $M$, since their eigenvalues have precisely this mean density.

**Proof.** Heine’s identity [21] evaluates the expected value in (21) as a Toeplitz determinant

$$
\mathbb{E}_N \left[ \prod_{n=1}^{N} \phi(\theta_n) \right] = \det [\phi_{i-j}]_{1 \leq i, j \leq N},
$$

where

$$
\phi_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(\theta)e^{-i\theta} \, d\theta
$$

is the $n$th Fourier coefficient of $\phi(\theta)$. The Toeplitz symbol $\phi(\theta)$ is singular since it is zero when $\theta = 0$. Thus, the asymptotic evaluation of this determinant requires knowledge of the Fisher–Hartwig Conjecture in a form proved by Basor [1].

We factor out the singularity in $\phi(\theta)$ by writing

$$
\phi(\theta) = b(\theta)(2 - 2 \cos \theta)^k,
$$
where
\[
b(\theta) = \exp \left( -k \log(2 - 2 \cos \theta) + 2k \int_1^e u(y) \left( \sum_{j=-\infty}^{\infty} \text{Ci}(|\theta + 2\pi j| \log y \log X) \right) dy \right) .
\] (23)

As we will see in the proof of Lemma 6 below, the logarithmic singularities in the exponent on the right cancel. Thus \( b(\theta) \) never equals zero. The asymptotic behavior of the Toeplitz determinant with these symbols has been determined by Basor \[1\]. She showed that if \( k > -\frac{1}{2} \), then
\[
\det [\phi_{i-j}]_{1 \leq i,j \leq N} \sim E \exp \left( \frac{N}{2\pi} \int_{-\pi}^\pi \log b(\theta) \, d\theta \right) N^{k^2}
\] (24)
as \( N \to \infty \), where the constant \( E \) is given by
\[
E = \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{1}{2\pi} \int_{-\pi}^\pi \log b(\theta) e^{-in\theta} \, d\theta \right)^2 \right) b(0) \frac{G^2(k+1)}{G(2k+1)} .
\]

To evaluate \( E \) we need to know \( b(0) \) and the Fourier coefficients of \( \log b(\theta) \). These are given by the next two lemmas.

**Lemma 5.** Let \( b(\theta) \) be given by (23). Then
\[
\frac{1}{2\pi} \int_{-\pi}^\pi \log b(\theta) e^{-in\theta} \, d\theta = \begin{cases} 0 & \text{if } n = 0 , \\ \frac{1}{n} \log \left( e^{n/\log X} \right) & \text{if } n \geq 1 , \end{cases}
\]
where
\[
v(t) = \int_t^\infty u(y) \, dy .
\]

**Lemma 6.** Let \( b(\theta) \) be given by (23) and let \( u(x) \) have total mass one with support in \([e^{1-1/X}, e] \). Then
\[
b(0) = \exp \left( 2k (\log \log X + \gamma) \right) \left( 1 + O_k \left( \frac{1}{\log X} \right) \right) .
\]

Before proving the lemmas, we complete the proof of Theorem 4. Since \( u \) is a nonnegative function supported in \([e^{1-1/X}, e] \) of total mass one, we see that
\[
v(t) = \begin{cases} 1 & \text{if } t \leq e^{1-1/X} , \\ 0 & \text{if } t \geq e , \end{cases}
\]
and \( 0 \leq v(t) \leq 1 \) if \( t \in [e, e^{1-1/X}] \). Thus
\[
\sum_{n=1}^{\infty} \frac{1}{n} \left( \log \left( e^{n/\log X} \right) \right)^2 = \sum_{n \leq (1-1/X) \log X} \frac{1}{n} + O \left( \sum_{(1-1/X) \log X < n \leq \log X} \frac{1}{n} \right) .
\]
The first sum on the right equals \( \log \log X + \gamma + O \left( 1/\log X \right) \) and the second is \( O \left( X^{-1} \right) \). Hence, we find that
\[
\sum_{n=1}^{\infty} \frac{1}{n} \left( \log \left( e^{n/\log X} \right) \right)^2 = \log \log X + \gamma + O \left( \frac{1}{\log X} \right) .
\]
Using this and the value of \( b(0) \) given by Lemma 6, we obtain
\[
E = \exp \left( -k^2 (\log \log X + \gamma) \right) \frac{(G(k+1))^2}{G(2k+1)} \left( 1 + O_k \left( \frac{1}{\log X} \right) \right) .
\]
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The proof of Theorem 4 is completed by combining this, the case \( n = 0 \) of Lemma 5, and (21).

**Proof of Lemma 5.** We wish to evaluate
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \log b(\theta) e^{-in\theta} \, d\theta,
\]
where \( b(\theta) \) is given by (23). After some straightforward algebra we see that this equals
\[
\frac{-k}{\pi} \int_{0}^{\pi} \log(2 - 2\cos \theta) \cos n\theta \, d\theta + \frac{2k}{\pi} \int_{1}^{\infty} u(y) \left( \int_{0}^{\infty} \text{Ci}(\theta \log x) \cos n\theta \, d\theta \right) \, dy.
\]
(25)

When \( n = 0 \) the first integral vanishes by symmetry, and the second vanishes because
\[
\int_{0}^{\infty} \text{Ci}(\theta) \, d\theta = 0.
\]
This is a special case of the formula (see Gradshteyn and Ryzhik [7], p. 645)
\[
\int_{0}^{\infty} \text{Ci}(A\theta) \cos n\theta \, d\theta = \begin{cases} \frac{-\pi}{2n} & \text{if } A < n, \\ \frac{-\pi}{2n} & \text{if } A = n, \\ 0 & \text{otherwise} \end{cases}
\]
(26)
for \( A > 0 \), which we require below as well. Thus, both terms in (25) vanish and Lemma 5 holds in this case.

When \( n \) is a positive integer, the first term in (25) equals
\[
\frac{-k}{\pi} \int_{0}^{\pi} \log(2 - 2\cos \theta) \cos n\theta \, d\theta = \frac{-k}{\pi} \int_{0}^{\pi} \left( \log 4 + 2\log(\sin \frac{\theta}{2}) \right) \cos n\theta \, d\theta
\]
\[
= \frac{-4k}{\pi} \int_{0}^{\pi/2} \log(\sin \theta) \cos 2n\theta \, d\theta
\]
\[
= \frac{k}{n}
\]
(see Gradshteyn and Ryzhik [7], p. 584). The second term in (25) is, by (26),
\[
\frac{2k}{\pi} \int_{1}^{\infty} u(y) \left( \int_{0}^{\infty} \text{Ci}(\theta \log x) \cos n\theta \, d\theta \right) \, dy = \frac{-k}{n} \int_{1}^{e^{n/\log x}} u(y) \, dy
\]
\[
= \frac{k}{n} \left( v(e^{n/\log x}) - 1 \right).
\]
(28)

Inserting (28) and (27) into (25), we find that for \( n > 0 \) an integer,
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \log b(\theta) e^{-in\theta} \, d\theta = \frac{k}{n} v(e^{n/\log x}).
\]
This completes the proof of Lemma 5.

**Proof of Lemma 6.** We calculate \( b(0) \), where
\[
b(\theta) = \exp \left( -k \log(2 - 2\cos \theta) + 2k \int_{1}^{\infty} u(y) \left( \sum_{j=-\infty}^{\infty} \text{Ci}(\theta + 2\pi j \log x) \right) \, dy \right).
\]
(29)

Using the expansion
\[
\text{Ci}(x) = \gamma + \log x + O(x^2)
\]
for \( x > 0 \), we find that the first term in the exponent and the \( j = 0 \) term combined contribute

\[
-k \log(2 - 2 \cos \theta) + 2k \int_{1}^{e} u(y) \operatorname{Ci}(\theta \log y \log X) \, dy
\]

\[
= 2k \left\{ \log(|\theta|) + O(\theta^2) + \int_{1}^{e} u(y) \left( \log(|\theta| \log y \log X) + \gamma + O_X(\theta^2) \right) \, dy \right\}
\]

\[
= 2k \left\{ \gamma + \log \log X + \int_{1}^{e} u(y) \log \log y \, dy + O_X(\theta^2) \right\},
\]

since \( u(x) \) has total mass one. Moreover, \( u(x) \) is supported in \([e^{1/X}, e]\), so we have

\[
\int_{1}^{e} u(y) \log \log y \, dy \ll \frac{1}{X}.
\]

Therefore we find that

\[
\lim_{\theta \to 0} \left\{ -k \log(2 - 2 \cos \theta) + 2k \int_{1}^{e} u(y) \operatorname{Ci}(\theta \log y \log X) \, dy \right\} = 2k \left( \log \log X + \gamma \right) + O_k \left( \frac{1}{X} \right).
\]

Now consider the contribution of the terms with \( j \neq 0 \) in (29). An integration by parts shows that

\[
\operatorname{Ci}(x) = - \int_{x}^{\infty} \frac{\cos t}{t} \, dt = \frac{\sin x}{x} + O(\frac{1}{x^2})
\]

for \( x \) positive and \( \gg 1 \). Thus, since \((1 - 1/X) \log X \leq \log y \log X \leq \log X, X > 2, \) and \( \theta \in (-\pi, \pi] \), we see that

\[
\sum_{j=-\infty}^{\infty} \sum_{j \neq 0} \operatorname{Ci}(\theta + 2\pi j \log y \log X) = \frac{1}{\log y \log X} \sum_{j=-\infty}^{\infty} \frac{\sin (|\theta + 2\pi j| \log y \log X)}{|\theta + 2\pi j|} + O \left( \frac{1}{(\log X)^2} \right).
\]

In a standard way (via Abel partial summation), one can show that the series on the right is uniformly convergent for \( y \in [e^{1/X}, e] \), except possibly in the neighborhood of a finite number of points, and boundedly convergent over the whole interval. Moreover, the series may be bounded independently of \( \theta \in (-\pi, \pi] \). We may therefore multiply by the continuous function \( u(y) \) and integrate to find that

\[
\int_{1}^{e} u(y) \left( \sum_{j=-\infty}^{\infty} \sum_{j \neq 0} \operatorname{Ci}(\theta + 2\pi j \log y \log X) \right) \, dy \ll \frac{1}{\log X} \int_{1}^{e} \frac{u(y)}{\log y} \, dy + O \left( \frac{1}{(\log X)^2} \right) \ll \frac{1}{\log X}.
\]

uniformly for \( \theta \in (-\pi, \pi] \). Combining this and (30) with (29), we obtain

\[
b(0) = \exp \left( 2k (\log \log X + \gamma) + O_k \left( \frac{1}{\log X} \right) \right)
\]

\[
= \exp \left( 2k (\log \log X + \gamma) \right) \left( 1 + O_k \left( \frac{1}{\log X} \right) \right).
\]

This completes the proof of Lemma 6.
5. The Proof of Theorem 3

First we prove Theorem 3 when \( k = 1 \). In this case \( G^2(k + 1)/G(2k + 1) = G^2(2)/G(3) = 1 \), and by Lemma 2 we may replace \( P_X(\frac{1}{2} + it) \) by \( P_X^*(\frac{1}{2} + it) \). Thus, it suffices to show that for \( X \ll (\log T)^{2-\epsilon} \),

\[
\frac{1}{T} \int_T^{2T} \left| \zeta(\frac{1}{2} + it) P_X^*(\frac{1}{2} + it)^{-1} \right|^2 dt = \frac{\log T}{e^\gamma \log X} \left( 1 + O \left( \frac{1}{\log X} \right) \right).
\]

As in the proof of Lemma 3 we write \( S(X) = \{ n : p \mid n \implies p \leq X \} \) and

\[ P_X^*(\frac{1}{2} + it)^{-1} = \sum_{n \in S(X)} \frac{\alpha_1(n)}{n^{1/2+it}}, \]

where \( \alpha_1(n) = \mu(n) \), the Möbius function, if \( n \in S(\sqrt{X}) \); \( \alpha_1(p) = \mu(p) \) for all \( p \leq X \); and \( \alpha_1(n) \ll d_{3/2}(n) \ll d(n) \) for all \( n \in S(X) \). By (16), if the \( \epsilon \) above is sufficiently small, we find that

\[ P_X^*(\frac{1}{2} + it)^{-1} = \sum_{n \leq T^{\theta/9}} \frac{\alpha_1(n)}{n^{1/2+it}} + O \left( T^{-\theta/10} \right) \]

(31)

(The exponent 1/10 in place of 1/2 is accounted for by the slight difference between the conditions \( X \ll (\log T)^{2-\epsilon} \) and \( X \ll (\log T)^{1/(1/2+\epsilon)} \).) Now for \( m \) and \( n \) coprime positive integers, we have the formula

\[
\int_T^{2T} \left| \zeta(\frac{1}{2} + it) \right|^2 \left( \frac{m}{n} \right)^{it} dt = \frac{T}{\sqrt{mn}} \left( \log \left( \frac{T}{2\pi mn} \right) + 2\gamma - 1 \right) + O \left( mnT^{8/9}(\log T)^6 \right).
\]

(For example, see Corollary 24.5 of [11].) Using this and the main term in (31) with \( \theta = 1/20 \), we find that

\[
\frac{1}{T} \int_T^{2T} \left| \zeta(\frac{1}{2} + it) \right|^2 \left| \sum_{n \leq T^{1/20}} \frac{\alpha_1(n)}{n^{1/2+it}} \right|^2 dt
\]

\[
= \sum_{m,n \leq T^{1/20}} \frac{\alpha_1(m)}{m} \frac{\alpha_1(n)}{n} (m,n) \left\{ \log \left( \frac{T(m,n)^2}{2\pi mn} \right) + 2\gamma - 1 + O \left( \frac{mn}{(m,n)^2} T^{-1/9}(\log T)^6 \right) \right\},
\]

(32)

where \( (m,n) \) denotes the greatest common divisor of \( m \) and \( n \). The \( O \)-term contributes

\[
\ll T^{-1/9}(\log T)^6 \left( \sum_{n \leq T^{1/20}} d(n) \right)^2 \ll T^{-1/90}(\log T)^8.
\]
Grouping together those $m$ and $n$ for which $(m,n) = g$, replacing $m$ by $gm$ and $n$ by $gn$, and then using the inequality $d(ab) \leq d(a)d(b)$, we find that

$$
\sum_{m,n \leq T^\theta \atop m,n \in \mathcal{S}(X)} \frac{\alpha_1(m)}{m} \frac{\alpha_1(n)}{n} (m,n) \left( \log \left( \frac{(m,n)^2}{2 \pi mn} \right) + 2\gamma - 1 \right)
\ll \sum_{g \in \mathcal{S}(X)} \frac{1}{g} \sum_{m,n \in \mathcal{S}(X)} ^{(m,n) = 1} d(gn)d(gn) \log mn \ll \sum_{g \in \mathcal{S}(X)} \frac{d(g)^2}{g} \left( \sum_{n \in \mathcal{S}(X)} \frac{d(n) \log n}{n} \right)^2.
$$

If we write $f(\sigma) = \sum_{n \in \mathcal{S}(X)} d(n)n^{-\sigma} = \prod_{p \leq X} (1 - p^{-\sigma})^{-2}$, then the sum over $n$ is $-f'(1)$, which, by logarithmic differentiation, is $2f(1)\sum_{p \leq X} \log p/(p - 1) \ll f(1)(\log X) \ll (\log X)^3$. We also have $\sum_{g \in \mathcal{S}(X)} d(g)^2g^{-1} \ll \prod_{p \leq X} (1 - p^{-1})^{-4} \ll (\log X)^4$, and so the expression above is $\ll (\log X)^{10}$.

Thus far then, we have

$$
\frac{1}{T} \int_T^{2T} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 \sum_{n \leq T/20} \frac{\alpha_1(n)}{n^{1/2 + it}} \, dt
= \log T \sum_{m,n \leq T^{1/20} \atop m,n \in \mathcal{S}(X)} \frac{\alpha_1(m)}{m} \frac{\alpha_1(n)}{n} (m,n) + O \left( (\log X)^{10} \right).
$$

Since $\sum_{g|\sigma} \phi(g) = n$, the remaining sum here is

$$
\sum_{m,n \leq T^{1/20} \atop m,n \in \mathcal{S}(X)} \frac{\alpha_1(m)}{m} \frac{\alpha_1(n)}{n} \left( \sum_{g|m \atop g \in \mathcal{S}(X)} \phi(g) \right) = \sum_{g \leq T^{1/20} \atop g \in \mathcal{S}(X)} \frac{\phi(g)}{g^2} \left( \sum_{n \leq T^{1/20}g^{-1} \atop n \in \mathcal{S}(X)} \frac{\alpha_1(gn)}{n} \right)^2.
$$

We wish to extend the sums on the right to all of $\mathcal{S}(X)$. For this we use several estimates. First,

$$
\sum_{n \in \mathcal{S}(X)} \frac{|\alpha_1(gn)|}{n} \ll d(g) \sum_{n \in \mathcal{S}(X)} \frac{d(n)}{n} = d(g) \prod_{p \leq X} \left( 1 - \frac{1}{p} \right)^{-2} \ll d(g)(\log X)^2.
$$

Second,

$$
\sum_{n > T^{1/20}g^{-1}} \frac{|\alpha_1(gn)|}{n} \ll d(g) \sum_{n > T^{1/20}g^{-1}} \frac{d(n)}{n} \ll d(g) \left( \frac{T^{1/20}}{g} \right)^{-1/4} \sum_{n \in \mathcal{S}(X)} \frac{d(n)}{n^{3/4}}
\ll d(g)g^{1/4}T^{-1/80} \prod_{p \leq X} \left( 1 - \frac{1}{p^{3/4}} \right)^{-2} \ll d(g)g^{1/4}T^{-1/80}e^{10X^{1/4}/\log X}
\ll d(g)g^{1/4}T^{-1/100},
$$

say. From these it follows that the square of the sum over $n$ in (34) is

$$
\left( \sum_{n \in \mathcal{S}(X)} \frac{\alpha_1(gn)}{n} \right)^2 + O \left( d(g)^2g^{1/2}T^{-1/100} \right).
$$
By arguments similar to those above we also find that
\[ \sum_{g \in S(X)} \frac{\phi(g)d(g)^2}{g^{3/2}} \ll T^{1/400} \quad \text{and} \quad \sum_{g \not\in S(X), \, g > T^{1/20}} \frac{\phi(g)d(g)^2}{g^2} \ll T^{-1/100}. \]

Using these and (35), we find that the right-hand side of (34) equals
\[ \left( \sum_{g \in S(X)} - \sum_{g \not\in S(X), \, g > T^{1/20}} \right) \frac{\phi(g)}{g^2} \left( \sum_{n \in S(X)} \frac{\alpha_1(gn)}{n} \right)^2 + O \left( T^{-1/200} \sum_{g \not\in S(X), \, g > T^{1/20}} \frac{\phi(g)d(g)^2}{g^{3/2}} \right) \]
\[ = \sum_{g \in S(X)} \frac{\phi(g)}{g^2} \left( \sum_{n \in S(X)} \frac{\alpha_1(gn)}{n} \right)^2 + O \left( T^{-1/200} \right). \]

Combining this with (35), we now have
\[ \frac{1}{T} \int_T^{2T} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 \sum_{n \leq T^{1/20}} \frac{\alpha_1(n)}{n^{1/2+it}} dt = \log T \left( \sum_{g \in S(X)} \frac{\phi(g)}{g^2} \left( \sum_{n \in S(X)} \frac{\alpha_1(gn)}{n} \right)^2 \right) + O \left( \left( \log X \right)^1 \right). \]

Since \( \alpha_1 \) and \( \phi \) are multiplicative functions, we may expand the entire sum into the Euler product
\[ \prod_{p \leq X} \left( \sum_{r, j, k} \frac{\varphi(p^r)\alpha_1(p^{j+r})\alpha_1(p^{k+r})}{p^{2r+j+k}} \right). \]

Recall that \( \alpha_1(n) = \mu(n) \), the Möbius function, if \( n \in S(\sqrt{X}) \); \( \alpha_1(p) = \mu(p) \) for all \( p \leq X \); and \( \alpha_1(n) \ll d_3/2(n) \ll d(n) \) for all \( n \in S(X) \). Thus, the product equals
\[ \prod_{p \leq \sqrt{X}} \left( 1 - \frac{1}{p} \right) \prod_{\sqrt{X} < p \leq X} \left( 1 - \frac{1}{p} + O \left( \frac{1}{p^2} \right) \right) = \prod_{p \leq X} \left( 1 - \frac{1}{p} \right) \prod_{\sqrt{X} < p \leq X} \left( 1 + O \left( \frac{1}{p^2} \right) \right) \]
\[ = \frac{1}{e^{\gamma} \log X} \left( 1 + O \left( \frac{1}{\log X} \right) \right). \]

Since \( \log X \ll \log \log T \), it now follows from (36) that
\[ \frac{1}{T} \int_T^{2T} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 \sum_{n \leq T^{1/20}} \frac{\alpha_1(n)}{n^{1/2+it}} dt = \log T \frac{\log X}{e^{\gamma} \log X} \left( 1 + O \left( \frac{1}{\log X} \right) \right). \]
Rewriting (31) (with \( \theta = 1/20 \)) as \( P_X^*(1/2 + it)^{-1} = \sum + O(T^{-\epsilon/200}) \), we see that
\[
\frac{1}{T} \int_T^{2T} |\zeta(1/2 + it)P_X(1/2 + it)^{-1}|^2 \, dt = \frac{1}{T} \int_T^{2T} |\zeta(1/2 + it)|^2 \left| \sum + O(T^{-\epsilon/200}) \right|^2 \, dt
\]
\[
= \frac{1}{T} \int_T^{2T} |\zeta(1/2 + it)|^2 \left| \sum \right|^2 \, dt + O \left( \frac{1}{T^{1+\epsilon/200}} \int_T^{2T} |\zeta(1/2 + it)|^2 \left| \sum \right| \, dt \right)
\]
\[
+ O \left( \frac{1}{T^{1+\epsilon/100}} \int_T^{2T} |\zeta(1/2 + it)|^2 \, dt \right).
\]

The final term is \( O(T^{-\epsilon/200}) \) since the second moment of the zeta function is \( O(T \log T) \). Also, by the Cauchy-Schwarz inequality and (37), the second term is
\[
\ll \frac{1}{T^{1+\epsilon/200}} \left( \int_T^{2T} |\zeta(1/2 + it) \sum|^2 \, dt \right)^{1/2} \int_T^{2T} |\zeta(1/2 + it)|^2 \, dt \ll \frac{1}{T^{1+\epsilon/200}} (T^{2 \log^2 T / \log T})^{1/2} \ll T^{-\epsilon/400}.
\]

From these estimates and (37), we may now conclude that
\[
\frac{1}{T} \int_T^{2T} |\zeta(1/2 + it)P_X^*(1/2 + it)^{-1}|^2 \, dt = \frac{\log T}{e^\gamma \log X} \left( 1 + O \left( \frac{1}{\log X} \right) \right)
\]
for \( X = O(\log T)^{2-\epsilon} \). This completes the proof of Theorem 3 in the case \( k = 1 \).

We now prove Theorem 3 for \( k = 2 \). By Lemma 2, we may again replace \( P_X(1/2 + it) \) by \( P_X^*(1/2 + it) \). Furthermore, \( C^2(3)/G(5) = 1/12 \), so it suffices to show that
\[
\frac{1}{T} \int_T^{2T} |\zeta(1/2 + it)^2 P_X(1/2 + it)^{-2}|^2 \, dt = \frac{1}{12} (1 + o(1)) \left( \frac{\log T}{e^\gamma \log X} \right)^4
\]
for \( X \ll (\log T)^{2-\epsilon} \). By (36) (see (31) also and the remark following it), we have
\[
P_X^*(1/2 + it)^{-2} = \sum_{n \in S(\mathcal{X})} \frac{\alpha_{-2}(n)}{n^{1/2 + it}} + O \left( T^{-\epsilon/10} \right),
\]
say, where \( \alpha_{-2}(p) = -2 \) for all \( p \leq X \), \( \alpha_{-2}(p^2) = 1 \) if \( p \leq \sqrt{X} \), \( \alpha_{-2}(p^2) = 2 \) if \( \sqrt{X} < p \leq X \), and \( \alpha_{-2}(p^2) = 0 \) otherwise. In particular, we note that \( |\alpha_{-2}(n)| \leq d(n) \).

In carrying out the proof of splitting for this case, we will gloss over some of the less important steps as these are handled analogously to those for the \( k = 1 \) case. In particular, by an argument similar to the one at the end of the proof of the case \( k = 1 \), one can show that
\[
\frac{1}{T} \int_T^{2T} |\zeta(1/2 + it)^2 P_X(1/2 + it)^{-2}|^2 \, dt
\]
\[
= \left( 1 + O \left( \frac{1}{\log X} \right) \right) \frac{1}{T} \int_T^{2T} |\zeta(1/2 + it)^2 \sum_{n \leq T^\theta} \frac{\alpha_{-2}(n)}{n^{1/2 + it}}|^2 \, dt,
\]
where \( Y = T^\theta \) and \( \theta > 0 \). Eventually we will take \( \theta \) very small.

To estimate the right-hand side we use an analogue of (32) due to Jose Gaggero [12]. Let \( A(s) = \sum_{n \leq T^\theta} a_n n^{-s} \), where the \( a_n \) are complex coefficients and \( Y = T^\theta \) with \( \theta < 1/150 \). Gaggero's
A HYBRID EULER-HADAMARD PRODUCT FORMULA FOR THE Riemann Zeta Function

Formula is

\[
\left( 1 + O\left( \frac{1}{(\log T)^2} \right) \right) \frac{1}{T} \int_T^{2T} \left| \zeta\left( \frac{1}{2} + it \right) A\left( \frac{1}{2} + it \right) \right|^2 dt
\]

\[= \sum_{k=1}^{4} \left\{ \sum_{m,n \leq Y} c_k(m, n) \frac{a_m a_n}{mn} (m, n) \left( \log \left( \frac{YT(m, n)}{2\pi mn} \right) + \log \left( \frac{T(m, n)}{2\pi Y} \right) \right) \right\}
- \sum_{m,n \leq Y} \frac{a_m a_n}{mn} \sum_{0 < d < Y/4} \frac{(m, d)(n, d)}{d} \left( \log \left( \frac{Y}{4d} \right) + O(1) \right) \sum_{v < V_1} \frac{1}{v} \sum_{u < U_1} \frac{1}{u}
- \sum_{m,n \leq Y} \frac{a_m a_n}{mn} \sum_{0 < d < mn/4Y} \frac{(m, d)(n, d)}{d} \left( \log \left( \frac{mn}{4dY} \right) + O(1) \right) \sum_{v < V_1} \frac{1}{v} \sum_{u < U_1} \frac{1}{u}.
\]

Here

\[U_1 = CYT/dn_d, \quad V_1 = CYT/dm_d, \quad (40)\]

\[U'_1 = CmnT/Ydn_d, \quad V'_1 = CmnT/Ydm_d, \quad C = 2/\pi, \quad B \text{ is an arbitrary positive number, and for integers } n \text{ and } d \text{ we write } n_d = n/(n, d). \]

Also, \(c_4(m, n) = (1/4\pi^2)\delta(m)\delta(n_m)\), where

\[\delta(n) = \prod_{p^r || n} \left( 1 + r \frac{(1 - 1/p)}{(1 + 1/p)} \right),\]

and \(c_j(m, n) \ll |c_4(m, n)|(\log \log 3mn)^{4-j} \) for \(j = 1, 2, 3\).

To estimate the right-hand side of (39), we take \(a_n = \alpha_{-2}(n)\) and \(Y = T^{\epsilon_1}\) in this and obtain

\[
\left( 1 + O\left( \frac{1}{(\log X)^2} \right) \right) \frac{1}{T} \int_T^{2T} \left| \zeta\left( \frac{1}{2} + it \right)^2 P_X\left( \frac{1}{2} + it \right)^{-2} \right|^2 dt
\]

\[= \left( \frac{1}{2\pi^2} + O(\epsilon_1) \right) \log^4 T \sum_{m,n \leq Y} \frac{\alpha_{-2}(m)\alpha_{-2}(n)\delta(m/(m, n))\delta(n/(m, n))}{mn} \sum_{m,n \leq Y} \frac{\alpha_{-2}(m)\alpha_{-2}(n)}{mn} \sum_{0 < d < Y/4} \frac{(m, d)(n, d)}{d} \left( \log \left( \frac{Y}{4d} \right) + O(1) \right) \sum_{v < V_1} \frac{1}{v} \sum_{u < U_1} \frac{1}{u} (41)
- \sum_{0 < d < mn/4Y} \frac{(m, d)(n, d)}{d} \left( \log \left( \frac{mn}{4dY} \right) + O(1) \right) \sum_{v < V'_1} \frac{1}{v} \sum_{u < U'_1} \frac{1}{u}
= T_1 - T_2 - T_3.
\]

say.
Let us denote the sum in $T_1$ by $S_1$. Grouping together those terms for which $(m, n) = g$ and then replacing $m$ by $mg$ and $n$ by $ng$, we obtain

$$S_1 = \sum_{g \leq Y \atop g \in S(X)} \frac{1}{g} \sum_{n \leq Y/g \atop n \in S(X)} \alpha_{-2}(gn)\delta(n) \left( \sum_{m \leq Y/g \atop (m, n) = 1} \alpha_{-2}(gm)\delta(m) \right). \quad (42)$$

Let $P = \prod_{p \leq X} p$. Since $\alpha_{-2}$ is supported on cube-free integers, the $g$’s we are summing over may be restricted to numbers of the form $g = g_1 g_2^2$, where $g_1 \mid P$, $g_2 \mid (P/g_1)$.

Note that this representation is unique and that $(g_1, g_2) = 1$. The summation over $g$ in (42) may therefore be replaced by the double sum

$$\sum_{g_1 \leq Y \atop g_1 \mid P} \sum_{g_2 \leq (Y/g_1)^{1/2} \atop g_2 \mid (P/g_1)} \cdots.$$

In the sum over $n$ we group terms together according to their greatest common divisor with $g = g_1 g_2^2$. Observe that we may assume that $(n, g_2) = 1$, for otherwise a cube divides $g_1 g_2^2 n$ and $\alpha_{-2}(gn)$ vanishes. If we then write $(n, g_1) = r$ and $n = rN$, we may replace the sum over $n$ in (42) by

$$\sum_{r \mid g_1 \atop N \leq (Y/r g_1 g_2^2)} \sum_{N \in S(X) \atop (N, (g_1/r) g_2) = 1} \cdots.$$

Ignoring the restriction $(m, n) = 1$ for the moment, we may similarly write the sum over $m$ in (42) as

$$\sum_{s \mid g_1 \atop M \leq (Y/s g_1 g_2^2)} \sum_{M \in S(X) \atop (M, (g_1/s) g_2) = 1} \cdots.$$  

Instead of $(m, n) = 1$ we now have $(sM, rN) = 1$ or, equivalently, $(M, N) = (r, s) = (N, s) = (M, r) = 1$. We may impose the condition $(r, s) = 1$ by replacing $s \mid g_1$ in (42) by $s \mid (g_1/r)$ since $g_1$ is square-free. Furthermore, since $(N, g_1/r) = 1$ and $s \mid (g_1/r)$, we automatically have $(N, s) = 1$. Thus, the coprimality conditions on $M$ are $(M, (g_1/s) g_2) = (M, N) = (M, r) = 1$. The first condition implies the third because $r \mid (g_1/s)$. Thus, we need only require that $(M, N(g_1/s) g_2) = 1$. The sum over $m$ may therefore be written

$$\sum_{s \mid g_1/r \atop M \leq (Y/s g_1 g_2^2)} \sum_{M \in S(X) \atop (M, N(g_1/s) g_2) = 1} \cdots.$$
We now have

\[ S_1 = \sum_{g_1 \leq Y / g_1 \mid P} \frac{1}{g_1} \sum_{g_2 \leq (Y / g_1)^{1/2}} \frac{1}{g_2} \sum_{r \mid g_1} \frac{1}{r} \sum_{s \mid (g_1 / r)} \frac{1}{s} \sum_{\alpha \leq (Y / r g_1 g_2^2) \in S(X) \cap \mathbb{N}} \frac{\alpha_2(r^2 g_2^2 N(g_1 / r)) \delta(r N)}{N} \times \sum_{M \leq (Y / s g_1 g_2^2) \in S(X) \cap \mathbb{N}} \frac{\alpha_2(s^2 g_2^2 M(g_1 / s)) \delta(s M)}{M} \].

Note that if \( N \) and \( r \) have a common factor, then \( \alpha_2(r^2 g_2^2 N(g_1 / r)) = 0 \), and similarly for \( M \) and \( s \). We may therefore replace the coprimality conditions in the sums over \( N \) and \( M \) by \( (N, g_1 g_2) = 1 \) and \( (M, N g_1 g_2) = 1 \), respectively. The new conditions then imply that \( \alpha_2(r^2 g_2^2 N(g_1 / r)) = \alpha_2(r^2) \alpha_2(g_2^2) \alpha_2(N) \alpha_2(g_1 / r), \delta(r N) = \delta(r) \delta(N), \) and similarly for \( \alpha_2(s^2 g_2^2 M(g_1 / s)) \) and \( \delta(s M) \). Hence

\[ S_1 = \sum_{g_1 \leq Y / g_1 \mid P} \frac{\alpha_2(g_1)^2}{g_1} \sum_{g_2 \leq (Y / g_1)^{1/2}} \frac{\alpha_2(g_2^2)^2}{g_2} \sum_{r \mid g_1} \frac{\alpha_2(r^2) \delta(r)}{\alpha_2(r) r} \sum_{s \mid (g_1 / r)} \frac{\alpha_2(s^2) \delta(s)}{\alpha_2(s) s} \sum_{N \leq (Y / r g_1 g_2^2) \in S(X) \cap \mathbb{N}} \frac{\alpha_2(N) \delta(N)}{N} \sum_{M \leq (Y / s g_1 g_2^2) \in S(X) \cap \mathbb{N}} \frac{\alpha_2(M) \delta(M)}{M} \].

We next extend each of the sums here to all of \( S(X) \). The error terms this introduces are handled as they were in the case \( k = 1 \), and they contribute at most “little \( o \)” of the main term. Observing also that \( M \) and \( N \) may be restricted to cube-free integers, we obtain

\[ S_1 = (1 + o(1)) \sum_{g_1 \mid P} \frac{\alpha_2(g_1)^2}{g_1} \sum_{g_2 \mid (P / g_1)^{1/2}} \frac{\alpha_2(g_2^2)^2}{g_2} \sum_{r \mid g_1} \frac{\alpha_2(r^2) \delta(r)}{\alpha_2(r) r} \sum_{s \mid (g_1 / r)} \frac{\alpha_2(s^2) \delta(s)}{\alpha_2(s) s} \sum_{N \mid (P / g_1 g_2)^2} \frac{\alpha_2(N) \delta(N)}{N} \sum_{M \mid (P / N g_1 g_2)^2} \frac{\alpha_2(M) \delta(M)}{M} \].

We now define the following multiplicative functions:

\[ A(n) = \prod_{d \mid n} \frac{\alpha_2(d) \delta(d)}{d} = \prod_{p^a \mid n} \left( 1 + \frac{\alpha_2(p) \delta(p)}{p} \cdots + \frac{\alpha_2(p^a) \delta(p^a)}{p^a} \right), \]

\[ B(n) = \prod_{d \mid n} \frac{\alpha_2(d) \delta(d) d A(d^2)}{d} = \prod_{p^a \mid n} \left( 1 + \frac{\alpha_2(p) \delta(p)}{p A(p^2)} + \cdots + \frac{\alpha_2(p^a) \delta(p^a)}{p^a A(p^{2a})} \right), \]

\[ C(n) = \prod_{d \mid (n, P)} \frac{\alpha_2(d^2) \delta(d)}{\alpha_2(d) d} = \prod_{p \mid (n, P)} \left( 1 + \frac{\alpha_2(p^2) \delta(p)}{\alpha_2(p^2) p} \right), \]

\[ D(n) = \prod_{d \mid (n, P)} \frac{\alpha_2(d^2) \delta(d) d A(d^2) C(d) d}{\alpha_2(d) d C(d) d} = \prod_{p \mid (n, P)} \left( 1 + \frac{\alpha_2(p^2) \delta(p)}{\alpha_2(p^2) C(p) p} \right), \]

\[ E(n) = \sum_{d \mid n} \frac{\alpha_2(d^2)^2}{A(d^2) B(d^2) d^2} = \prod_{p \mid l} \left( 1 + \frac{\alpha_2(p^2)^2}{A(p^2) B(p^2) p^2} \right), \]
and
\[
F(n) = \sum_{d \mid n} \frac{\alpha_2(d)^2 C(d) D(d)}{A(d^2) B(d^2) E(d)} = \prod_p \left( 1 + \frac{\alpha_2(p)^2 C(p) D(p)}{A(p^2) B(p^2) E(p)} \right).
\]

Using these definitions and working from the inside out in (13), we find first that the sum over \(M\) is \(A(P/N g_1 g_2)^2) = A(P^2)/A(N^2) A(g_1^2) A(g_2^2)\). The contribution of the sums over \(M\) and \(N\) together is then \((A(P^2)/A(g_1^2) A(g_2^2)) (B(P^2)/B(g_1^2) B(g_2^2))\). Thus, so far we have
\[
S_1 = (1 + o(1)) A(P^2) B(P^2) \sum_{g_1\mid P} \frac{\alpha_2(g_1)^2}{g_1 A(g_1^2) B(g_1^2)} \sum_{g_2\mid P/g_1} \frac{\alpha_2(g_2^2)^2}{g_2 A(g_2^2) B(g_2^2)} \sum_{r\mid g_1} \frac{\alpha_2(r^2) \delta(r)}{\alpha_2(r) r} \sum_{s\mid (g_1/r)} \frac{\alpha_2(s^2) \delta(s)}{\alpha_2(s) s}.
\]

The sums over \(r\) and \(s\) contribute \(C(g_1) D(g_1)\), and the sum over \(g_2\) is then \(E(P)/E(g_1)\). Thus, we see that
\[
S_1 = (1 + o(1)) A(P^2) B(P^2) E(P) \sum_{g_1\mid P} \frac{\alpha_2(g_1)^2 C(g_1) D(g_1)}{g_1 A(g_1^2) B(g_1^2) E(g_1)} = (1 + o(1)) A(P^2) B(P^2) E(P) F(P).
\]

Using the expression for \(F(P)\) as a product, we see that this is the same as
\[
S_1 = (1 + o(1)) \prod_p \left( A(p^2) B(p^2) E(p) + \frac{\alpha_2(p)^2 C(p) D(p)}{p} \right).
\]

By the definitions of \(C\) and \(D\) we see that
\[
C(p) D(p) = \left(1 + \frac{\alpha_2(p^2) \delta(p)}{\alpha_2(p) p}\right) = 1 - \frac{\alpha_2(p^2) \delta(p)}{p},
\]

since \(\alpha_2(p) = -2\) for \(p\) dividing \(P\). Similarly,
\[
A(p^2) B(p^2) E(p) = A(p^2) B(p^2) + \frac{\alpha_2(p^2)^2}{p^2}.
\]

It is clear that \(A(p^2) = A(p^3) = \cdots\). Therefore
\[
B(p^2) = 1 + \frac{\alpha_2(p) \delta(p)}{p A(p^2)} + \frac{\alpha_2(p^2) \delta(p^2)}{p^2 A(p^2)} = 1 + \frac{1}{A(p^2)}(A(p^2) - 1) = 2 - \frac{1}{A(p^2)}
\]

and
\[
A(p^2) B(p^2) E(p) = 2 A(p^2) - 1 + \frac{\alpha_2(p^2)^2}{p^2}.
\]
We use this, (45), and \( \alpha_2(p) = -2 \), and obtain
\[
A(p^2)B(p^2)E(p) + \frac{\alpha_2(p)^2 C(p)D(p)}{p} = 2A(p^2) - 1 + \frac{\alpha_2(p^2)^2}{p^2} + 4 - \frac{4\alpha_2(p^2)^2\delta(p)}{p^2}
\]
\[
= 2\left(1 - \frac{2\delta(p)}{p^2} + \frac{\alpha_2(p^2)\delta(p^2)}{p^2}\right) - 1 + \frac{\alpha_2(p^2)^2}{p^2} + 4 - \frac{4\alpha_2(p^2)^2\delta(p)}{p^2}
\]
\[
= 1 + \frac{4 - 4\delta(p)}{p} + \frac{\alpha_2(p^2)\left(\alpha_2(p^2) - 4\alpha_2(p^2)\delta(p) + 2\delta(p^2)\right)}{p^2}
\].
Recall that \( \delta(p^r) = 1 + r \frac{1 - 1}{1 + 1/p^r} \), so that \( \delta(p) = 2/(1 + 1/p) \) and \( \delta(p^2) = 2\delta(p) - 1 \). Also recall that \( \alpha_2(p^2) = 1 \) if \( p \leq \sqrt{X} \). Thus, for \( p \leq \sqrt{X} \) the last line is
\[
= 1 + \frac{4 - 4\delta(p)}{p} + \frac{1 - 4\delta(p) + 2(2\delta(p) - 1)}{p^2} = 1 + \frac{4 - 4\delta(p)}{p} - 1
\]
\[
= 1 + \frac{4}{p} - \frac{8}{p + 1} - \frac{1}{1 + 1/p} = \frac{(1 - 1/p)^4}{1 - 1/p^2}.
\]
On the other hand, if \( \sqrt{X} < p \leq X \), then \( \alpha_2(p^2) = 2 \), and the last line is
\[
= \frac{(1 - 1/p)^4}{1 - 1/p^2} + O\left(1/p^2\right).
\]
Combining these results in (44), we find that
\[
S_1 = (1 + o(1)) \prod_{p \leq \sqrt{X}} \left(\frac{(1 - 1/p)^4}{1 - 1/p^2}\right) \prod_{\sqrt{X} < p \leq X} \left(\frac{(1 - 1/p)^4}{1 - 1/p^2} + O\left(1/p^2\right)\right)
\]
\[
= (1 + o(1)) \prod_{p \leq X} \left(\frac{(1 - 1/p)^4}{1 - 1/p^2}\right) \prod_{\sqrt{X} < p \leq X} \left(1 + O\left(1/p^2\right)\right)
\]
\[
= (1 + o(1)) \prod_{p \leq X} (1 - 1/p)^4 \prod_{p \leq X} (1 - 1/p^2)^{-1}
\]
\[
= (1 + o(1)) \frac{\pi^2}{6} (e^\gamma \log X)^{-4}.
\]
Since
\[
T_1 = S_1 \left(\frac{1}{2\pi^2} + O(\epsilon_1)\right) \log^4 T
\]
and \( \epsilon_1 > 0 \) may be taken as small as we like, we now see that
\[
T_1 = \left(\frac{1}{12} + o(1)\right) \left(\frac{\log T}{e^\gamma \log X}\right)^4.
\] (46)

To treat the second term on the right-hand side of (44), \( T_2 \), we require two lemmas.

**Lemma 7.** Suppose that \( a \) and \( b \) are positive integers with \( (a, b) = 1 \). Then for \( b \leq x \), we have
\[
\sum_{\substack{n \leq x \\ (an, b) = 1}} \frac{1}{n} = \frac{\phi(b)}{b} \log x + O(\log \log 2b).
\]
Proof. Since \((a, b) = 1\), the condition \((an, b) = 1\) is equivalent to \((n, b) = 1\). Thus, the sum is
\[
\sum_{n \leq x} \frac{1}{n} = \sum_{n \leq x} \frac{1}{n} \sum_{d \mid n} \mu(d) = \sum_{d \mid b} \frac{\mu(d)}{d} \sum_{m \leq x/d} \frac{1}{m} = \sum_{d \mid b} \frac{\mu(d)}{d} (\log x/d + O(1)).
\]

Now \(\sum_{d \mid b} \mu(d)/d = \phi(b)/b\) and
\[
\sum_{d \mid b} \frac{\mu(d) \log d}{d} = \frac{\phi(b)}{b} \sum_{p \mid b} \frac{\log p}{p-1} \ll \frac{\phi(b)}{b} \log 2b \ll \log 2b.
\]

Furthermore,
\[
\sum_{d \mid b} \frac{|\mu(d)|}{d} = \prod_{p \mid b} \left( 1 + \frac{1}{p} \right) \leq \frac{b}{\phi(b)} \ll \log 2b.
\]

Thus we find that
\[
\sum_{n \leq x} \frac{1}{n} = \frac{\phi(b)}{b} \log x + O(\log \log 2b).
\]

Lemma 8. Let \(\kappa(n) = \prod_{p \mid n} \left( 1 + \frac{1}{p} \right)^{-1}\) and let \(U, V\) be either \(U_1, V_1\) or \(U_1', V_1'\) as defined in (40). If \(m, n, d \ll Y \ll T^{1/150}\), and \((n_d, m_d) = 1\), then
\[
\sum_{v < V} \frac{1}{v} \sum_{u < U} \frac{1}{u} = \frac{6}{\pi^2} \kappa(m_d) \kappa(n_d) \log U \log V + O(\log T \log \log T).
\]

Proof. The conditions \((n_d u, m_d v) = 1\) and \((n_d, m_d) = 1\) are equivalent to \((v, n_d) = 1\) and \((u, m_d v) = 1\). Hence, by Lemma 7, the double sum equals
\[
\sum_{v < V} \frac{1}{v} \sum_{u < U} \frac{1}{u} = \log U \sum_{v < V} \frac{1}{v} \left( \frac{\phi(m_d v)}{m_d v} + O(\log \log(m_d V)) \right)
\]
\[
= \log U \sum_{v < V} \frac{1}{v} \left( \frac{\phi(m_d v)}{m_d v} \right) + O(\log T \log \log T).
\] (47)

Denoting the sum on the right by \(\sum\), we have
\[
\sum = \sum_{v < V} \frac{1}{v} \sum_{u < U} \frac{\mu(r)}{r} = \sum_{r < m_d V} \frac{\mu(r)}{r} \sum_{v < V} \frac{1}{v}.
\]

Now set \((m_d, r) = g\) and write \(r = gR\). Then \((m_d/g, R) = 1\) and we find that
\[
\sum = \sum_{g \mid m_d} \frac{1}{g} \sum_{R < m_d V/g} \frac{\mu(gR)}{R} \frac{1}{v}.
\]
If we set \( v = Rw \), then \( w < V/R \), and \((Rw, n_d) = 1\) is the same as the two conditions \((R, n_d) = 1\) and \((w, n_d) = 1\). Thus, using Lemma \( \mathfrak{1} \) and the observation that the inner sum vanishes unless \( R < V \), we obtain

\[
\sum = \sum_{g|m_d} \frac{1}{g} \sum_{R < V/g \atop (m_d/g, R) = 1 \atop (n_d, R) = 1} \mu(gR) \frac{1}{R^2} \sum_{w < V/R \atop (w, n_d) = 1} \frac{1}{w} = \sum_{g|m_d} \frac{1}{g} \sum_{R < V \atop (m_d/g, R) = 1 \atop (n_d, R) = 1} \mu(gR) \frac{1}{R^2} \left( \frac{\phi(n_d)}{n_d} \log \frac{V}{R} + O(\log \log 2n_d) \right).
\]

We may assume \((R, g) = 1\), for otherwise \(\mu(gR) = 0\). The coprimality conditions on the sum may then be written \((m_d n_d, R) = 1\), and we find that

\[
\sum = \sum_{g|m_d} \frac{\mu(g)}{g} \sum_{R < V \atop (R, m_d n_d) = 1} \frac{\mu(R)}{R^2} \left( \frac{\phi(n_d)}{n_d} \log \frac{V}{R} + O(\log \log 2n_d) \right)
\]

\[
= \frac{\phi(n_d)}{n_d} \log V \sum_{g|m_d} \frac{\mu(g)}{g} \sum_{R < V \atop (R, m_d n_d) = 1} \frac{\mu(R)}{R^2} + O \left( \log \log 2n_d \sum_{g|m_d} \frac{\mid \mu(g) \mid}{g} \sum_{R < V \atop (R, m_d n_d) = 1} \frac{\log R}{R^2} \right).
\]

Since \(\sum_{g|m_d} \mid \mu(g) \mid / g = \prod_{p|m_d} (1 + 1/p) \ll \log \log 2m_d\), the error term is \( \ll (\log \log 2m_d \log \log 2n_d)\). The main term is

\[
= \frac{\phi(n_d)}{n_d} \log V \sum_{g|m_d} \frac{\mu(g)}{g} \left( \sum_{R = 1}^{\infty \atop (R, m_d n_d) = 1} \frac{\mu(R)}{R^2} + O(V^{-1}) \right)
\]

\[
= \zeta(2)^{-1} \prod_{p|m_d n_d} \left( 1 - \frac{1}{p^2} \right)^{-1} \phi(n_d) \log V \sum_{g|m_d} \frac{\mu(g)}{g} + O \left( \frac{\log V}{V} \sum_{g|m_d} \frac{\mid \mu(g) \mid}{g} \right)
\]

\[
= \frac{6}{\pi^2} \prod_{p|m_d n_d} \left( 1 - \frac{1}{p^2} \right)^{-1} \phi(n_d) \log V + O \left( \frac{\log V \log \log 2m_d}{V} \right).
\]

By hypothesis, \((m_d, n_d) = 1\). Furthermore, \(\prod_{p|l} (1 - 1/p^2)^{-1} (\phi(l)/l) = \prod_{p|l} (1 + 1/p)^{-1} = \kappa(l)\). Thus, combining our estimates, we obtain

\[
\sum = \frac{6}{\pi^2} \kappa(m_d) \kappa(n_d) \log V + O(\log \log 2m_d \log \log 2n_d).
\]

Since \(m, n \ll T^{1/150}\), we obtain from this and (17) that

\[
\sum \frac{1}{v} \sum_{u < U \atop (n_d u, m_d v) = 1} \frac{1}{u} = \frac{6}{\pi^2} \kappa(m_d) \kappa(n_d) \log U \log V + O(\log T \log \log T).
\]

\(\square\)
Returning to $T_2$ in (41) and using Lemma 8, we have
\[ T_2 = \frac{6}{\pi^2} \sum_{m,n \leq Y \atop m,n \in S(X)} \frac{\alpha_{-2}(m)\alpha_{-2}(n)}{mn} \sum_{0 < d < Y/4 \atop d \in S(X)} \left( \frac{\log(Y/4d) + O(1)}{d} \right) \times \left( \kappa(m) \kappa(n) \log U_1 \log V_1 + O(\log T \log \log T) \right), \]
where $U_1 = CYT/dn_d, V_1 = CYT/dm_d$, and $Y = T'_1$. Interchanging the order of summation, we find that
\[ T_2 = \frac{6}{\pi^2} \sum_{0 < d < Y/4 \atop d \in S(X)} \frac{1}{d} \left( \frac{\log(Y/4d) + O(1)}{d} \right) \sum_{m,n \leq Y \atop m,n \in S(X)} \frac{\alpha_{-2}(m)\alpha_{-2}(n)(m,d)(n,d)}{mn} \times \left( \kappa(m) \kappa(n) \log U_1 \log V_1 + O(\log T \log \log T) \right). \]
Since $\kappa(n) \ll \log \log 3n$, the expression in the last parentheses is
\[ = \kappa(m) \kappa(n) \left( \log U_1 \log V_1 + O(\log T \log \log 3 T) \right) = (1 + O(\epsilon_1)) \kappa(m) \kappa(n) \log^2 T. \]
Thus,
\[ T_2 = \frac{6}{\pi^2} (1 + O(\epsilon_1)) \log^2 T \sum_{0 < d < Y/4 \atop d \in S(X)} \frac{\left( \frac{\log(Y/4d) + O(1)}{d} \right)\left( \sum_{n \in Y \atop n \in S(X)} \frac{\alpha_{-2}(n)(n,d)\kappa(n/(n,d))}{n} \right)^2}{\kappa(n/e)} = \left( 1 + o(1) \right) \sum_{e \mid d \mid P^2} \sum_{n \in S(X) \atop (n,d) = 1} \frac{\alpha_{-2}(n) \kappa(n/e)}{n}. \]
Denote the inner sum by $S(d)$. As on previous occasions, extending the sum to all of $S(X)$, we introduce an error term that is $o(1)$ times the main term. Thus, grouping together terms in $S(d)$ for which $(n,d) = e$, say, we obtain
\[ S(d) = \left( 1 + o(1) \right) \sum_{e \mid d \mid P^2} \sum_{n \in S(X) \atop (n,d) = 1} \frac{\alpha_{-2}(n) \kappa(n/e)}{n} = \left( 1 + o(1) \right) \sum_{e \mid d} \sum_{n \in S(X) \atop (N,d/e) = 1} \frac{\alpha_{-2}(e N) \kappa(N)}{N}. \]
Since $\alpha_{-2}$ is supported only on cube-free numbers in $S(X)$, we may assume that $e \mid P^2$. Therefore, $e \mid (d, P^2) = D$, say. Now $D$ may be written uniquely as $D = D_1 D_2$, where $D_1 \mid P$ and $D_2 \mid (P/D_1)$, so that, in particular, $(D_1, D_2) = 1$. Furthermore, we may write any divisor $e$ of $D$ as $e = e_1 e_2 e_3^2$, where $e_1 \mid D_1, e_2 \mid D_2$, and $e_3 \mid (D_2/e_2)$. Note that this means the $e_i$ are pairwise coprime. The condition $(N,d/e) = 1$ is now $(N, (D_1 e_1 e_2^2 / (e_1 e_2 e_3^2))) = 1$. Also, $\alpha_{-2}(eN) = \alpha_{-2}(e_1 e_2^2 e_3^2 N)$, so we may assume that $(N,e_3) = 1$ and, therefore, that $(N, (D_1 D_2^2 / (e_1 e_2))) = 1$. Observe, moreover, that $e_2 \mid D_2$ implies $e_2 \mid (D_2^2/e_2)$. Thus, $(N, (D_1 D_2^2 / e_2)) = 1$ is the same as $(N, (D_1 D_2 / e_1)) = 1$. It follows that $N$ and $e_1$ can have a common factor, but not $N$ and $e_2$ or $e_3$. We may therefore write $\alpha_{-2}(e_1 e_2^2 e_3^2 N) = \alpha_{-2}(e_1 N) \alpha_{-2}(e_2) \alpha_{-2}(e_3^2)$ and
\[ S(d) = \left( 1 + o(1) \right) \sum_{e_1 \mid D_1} \sum_{e_2 \mid (D_2/e_2)} \frac{\alpha_{-2}(e_2)}{e_3 \mid (D_2/e_2)} \sum_{N \in S(X) \atop (N,D_1 D_2 / (e_1 e_2)) = 1} \frac{\alpha_{-2}(e_1 N) \kappa(N)}{N} \sum_{e_3 \mid D_2} \frac{\alpha_{-2}(e_3^2)}{e_2 \mid (D_2/e_2)} = \left( 1 + o(1) \right) \sum_{e_1 \mid D_1} \sum_{N \in S(X) \atop (N,D_1 D_2 / (e_1 e_2)) = 1} \frac{\alpha_{-2}(e_1 N) \kappa(N)}{N} \sum_{e_3 \mid D_2} \frac{\alpha_{-2}(e_3^2)}{e_2 \mid (D_2/e_2)}. \]
The innermost sum is
\[ \sum_{e_2(D_2/e_3)} \alpha_{-2}(e_2) = \prod_{p|D_2/e_3} (1 + \alpha_{-2}(p)) = \prod_{p|D_2} (1 - 2) \]
\[ = \mu(D_2/e_3) = \mu(D_2)\mu(e_3). \]

We also have
\[ \sum_{e_3}|D_2} \alpha_{-2}(e_3^2) = \prod_{p|D_2} (1 - \alpha_{-2}(p^2)) . \]

At this point it is convenient to define numbers
\[ P_1 = \prod_{p \leq \sqrt{X}} p \quad \text{and} \quad P_2 = \prod_{\sqrt{X} < p \leq X} p . \]

Notice that \( P = P_1 P_2 \). Since \( \alpha_{-2}(p^2) = 1 \) if \( p \mid P_1 \) and \( \alpha_{-2}(p^2) = 2 \) if \( p \mid P_2 \), the sum over \( e_3 \) equals 0 unless \( D_2 \mid P_2 \), in which case it equals \( \mu(D_2) \). Thus, if \( D_2 \) and \( P_1 \) have a common factor, \( S(D_1 D_2^2) = 0 \), whereas if \( D_2 \mid P_2 \), then
\[ S(d) = S(D_1 D_2^2) = (1 + o(1)) \sum_{e_1|D_1} \sum_{N \in \mathcal{S}(X)} \frac{\alpha_{-2}(e_1 N) \kappa(N)}{N} . \]

From this point on we shall therefore assume that \( D_2 \mid P_2 \).

Now set \( (N, e_1) = r \) and write \( N = r M \). Then we have
\[ S(D_1 D_2^2) = (1 + o(1)) \sum_{e_1|D_1} \sum_{r|e_1} \sum_{\substack{N \in \mathcal{S}(X) \\ (N, e_1) = r \\ (N, D_1 D_2/e_1) = 1}} \frac{\alpha_{-2}(e_1 N) \kappa(N)}{N} \]
\[ = (1 + o(1)) \sum_{e_1|D_1} \sum_{r|e_1} \sum_{\substack{M \in \mathcal{S}(X) \\ (M, e_1/r) = 1 \\ (r M, D_1 D_2/e_1) = 1}} \frac{\alpha_{-2}(r^2 M(e_1/r)) \kappa(r M)}{M} . \]

We may assume that \( (M, r) = 1 \) and \( (r, e_1/r) = 1 \), since otherwise \( \alpha_{-2}(r^2 M(e_1/r)) = 0 \). Actually, \( (r, e_1/r) = 1 \) is automatically satisfied because \( r \mid e_1 \) and \( e_1 \) is square-free. It follows that \( \kappa(r M) = \kappa(r) \kappa(M) \) and, since we also have \( (M, e_1/r) = 1 \), that \( \alpha_{-2}(r^2 M(e_1/r)) = \alpha_{-2}(r^2) \alpha_{-2}(M) \alpha_{-2}(e_1/r) \).

The coprimality conditions in the sum are now seen to be equivalent to the conditions \( (M, r) = (r, e_1/r) = (M, e_1/r) = (M, D_1 D_2/e_1) = (r, D_1 D_2/e_1) = 1 \). As we have already pointed out, the second of these is automatic. Similarly, so is the last. The remaining conditions are equivalent to \( (M, D_1 D_2) = 1 \), so we find that
\[ S(D_1 D_2^2) = (1 + o(1)) \sum_{e_1|D_1} \alpha_{-2}(e_1) \sum_{r|e_1} \frac{\alpha_{-2}(r^2) \kappa(r)}{r \alpha_{-2}(r)} \sum_{\substack{M \in \mathcal{S}(X) \\ (M, D_1 D_2) = 1}} \frac{\alpha_{-2}(M) \kappa(M)}{M} . \]

The sum over \( M \) equals
\[ \prod_{p|P/D_1 D_2} \left( 1 + \frac{\alpha_{-2}(p) \kappa(p)}{p} + \frac{\alpha_{-2}(p^2) \kappa(p^2)}{p^2} \right) = G(P/D_1 D_2) , \] (49)
say. Hence,

\[ S(D_1D_2^2) = (1 + o(1)) \frac{1}{G(P/D_1D_2)} \sum_{\alpha \in \mathcal{A}_{D_1}} \frac{\alpha \cdot \kappa(r)}{r} \sum_{\| \|} \frac{\alpha \cdot \kappa(r)}{r}. \]

The double sum equals

\[ \sum_{r \sim D_1} \frac{\alpha \cdot \kappa(r)}{r} \sum_{\eta \sim D_1} \alpha \cdot \kappa(r) = \mu(D_1) \sum_{r \sim D_1} \frac{\mu(r) \alpha \cdot \kappa(r)}{r}, \]

say. Thus,

\[ \mu(D_1) H(D_1), \]

say. Thus,

\[ S(d) = S(D_1D_2^2) = (1 + o(1)) \frac{1}{G(P/D_1D_2)} \mu(D_1) H(D_1), \]

provided \( D_2 | P \); otherwise \( S(d) = 0 \).

We use this in (45). Recall that for each \( d < Y/4 \) we set \( (d, P^2) = D_1D_2^2 \) with \( D_1 \) \( | \) \( P \) and \( D_2 \) \( \sim (P/D_1) \). Recall also that \( P = P_1P_2 \) and \( Y = T \epsilon_1 \). We therefore have that

\[ T_2 = \frac{6}{\pi^2} \epsilon_1 \log^2 T \sum_{0 < d < Y/4} \frac{(\log(Y/4d) + O(1))}{d} S(d)^2 \]

\[ = \frac{6}{\pi^2} \epsilon_1 \log^2 T \sum_{D_2 \mid P_2} \sum_{D_1 \mid (P/D_2)} \frac{S(D_1D_2^2)^2}{D_1} \sum_{0 < d < Y/4D_1D_2^2} \frac{(\log(Y/(4D_1D_2^2d)) + O(1))}{d}. \]

The coprimality condition in the last sum is equivalent to \( (\delta, P_1P_2/D_2) = 1 \). Thus, using (51), we find that

\[ T_2 = \frac{6}{\pi^2} \epsilon_1 \log^2 T \sum_{D_2 \mid P_2} \sum_{D_1 \mid (P/D_2)} \frac{1}{D_2^2 G(D_2)^2} \sum_{D_1 \mid (P/D_2)} \frac{H(D_1)^2}{D_1 G(D_1)^2} \]

\[ \times \sum_{0 < d < Y/4D_1D_2^2} \frac{(\log(Y/(4D_1D_2^2d)) + O(1))}{d}. \]

By Lemma 7 the sum over \( \delta \) is

\[ \ll \log Y \sum_{0 < \delta < Y/4D_1D_2^2} \frac{1}{\delta} \ll \frac{\phi(P)}{P} \frac{D_2}{\phi(D_2)} \log^2 Y. \]

Thus

\[ T_2 \ll \frac{G(P)^2 \phi(P)}{P} \log^2 T \log^2 Y \sum_{D_2 \mid P_2} \sum_{D_1 \mid (P/D_2)} \frac{1}{D_2 \phi(D_2) G(D_2)^2} \sum_{D_1 \mid (P/D_2)} \frac{H(D_1)^2}{D_1 G(D_1)^2}. \]

If we denote the innermost sum by \( I(P/D_2) \), then

\[ I(P/D_2) = \prod_{p \mid (P/D_2)} \left( 1 + \frac{H(p)^2}{p G(p)^2} \right), \]
and we find that

\[
\mathcal{T}_2 \ll G(P)^2 I(P) \frac{\phi(P)}{P} \log^2 T \log^2 Y \sum_{D_2 \mid P_2} \frac{1}{D_2 \phi(D_2) G(D_2)^2 I(D_2)}
\]

\[
\ll \epsilon_1^2 G(P)^2 I(P) \frac{\phi(P)}{P} \log^4 T \prod_{p \mid P_2} \left( 1 + \frac{1}{p \phi(p) G(p)^2 I(p)} \right)
\]

Now, by the definitions of \( G, H, \) and \( I \) in (19), (50), and (52), we have \( G(p) = 1 - \frac{\gamma}{p^2} + O\left(\frac{1}{p^2}\right) \), \( H(p) = 1 + O\left(\frac{1}{p}\right) \), and \( I(p) = 1 + \frac{1}{p} \left( (1 + O\left(\frac{1}{p}\right)) / (1 - \frac{2}{p} + O\left(\frac{1}{p^2}\right)) \right)^2 = 1 + \frac{1}{p} + O\left(\frac{1}{p^2}\right) \). From these estimates it is clear that the product over \( p \) dividing \( P_2 \) here is \( \prod_{\sqrt{X} \leq p \leq X} (1 + O(1/p^2)) \ll 1 \). Thus

\[
\mathcal{T}_2 \ll \epsilon_1^2 G(P)^2 I(P) \frac{\phi(P)}{P} \log^4 T \prod_{p \mid P_2} \left( 1 - \frac{1}{p} + O\left(\frac{1}{p^2}\right) \right) (1 + O(1/p^2))
\]

\[
\ll \epsilon_1^2 \log^4 T \prod_{p \mid P_2} \left( 1 - \frac{1}{p} \right)^4 \ll \epsilon_1 \left( \log \frac{T}{\log X} \right)^4.
\]

The treatment of \( \mathcal{T}_3 \) is almost identical and leads to the same bound. Thus, combining our estimates for \( \mathcal{T}_1 \) (see (46)), \( \mathcal{T}_2 \), and \( \mathcal{T}_3 \) with (53), and noting that we may take \( \epsilon_1 > 0 \) as small as we like, we obtain (38). This completes the proof of the case \( k = 2 \) of Theorem 3 and thus, also the proof of the theorem.

**Appendix A. Graphs**

To illustrate Theorem 4 in Figures 13 we have plotted \(|Z_X\left(\frac{1}{2} + it\right)|\) and \(|P_X\left(\frac{1}{2} + it\right)|\) for \( t \) near the 10^{12} th zero for two values of \( X \), and have compared their product with the Riemann zeta function. The values of \( X \) used are \( X = 26.31 \approx \log \gamma_{10^{12}} \) and \( X = 1000 \). Though the functions \( P_X \) and \( Z_X \) depend upon \( X \), when multiplied together the \( X \) dependence mostly cancels out, and we have an accurate pointwise approximation to the zeta function. The actual functions plotted are

\[
|P_X\left(\frac{1}{2} + i(x + t_0)\right)| = \exp \left( \sum_{n \leq X} \frac{\Lambda(n) \cos((x + t_0) \log n)}{\log n \sqrt{n}} \right)
\]

and

\[
|Z_X\left(\frac{1}{2} + i(x + t_0)\right)| = \prod_{n=N+1}^{N+100} \exp \left( \text{Ci}(|x + t_0 - \gamma_n| \log X) \right),
\]

where \( t_0 = \gamma_{10^{12} + 40} \). The values of the zeros of the zeta function came from Andrew Odlyzko’s tables [20]. The functions were plotted for \( x \) between 0 and 5, a range covering the zeros between \( \gamma_{10^{12} + 40} \) and \( \gamma_{10^{12} + 60} \). Note that the function \( Z_X \) we have plotted is an unsmoothed, truncated form of the function \( Z_X \) that appears in Theorem 4.
Figure 1. Graph of $|\zeta(\frac{1}{2} + i(x + t_0))|$ (solid) and $|P_X(\frac{1}{2} + i(x + t_0))Z_X(\frac{1}{2} + 1(x + t_0))|$, with $t_0 = \gamma_{10^{12}+40}$, with $X = \log t_0$ (dots) and $X = 1000$ (dash-dots).

Figure 2. Graph of $|P_X(\frac{1}{2} + i(x + t_0))|$, with $t_0 = \gamma_{10^{12}+40}$, with $X = \log t_0$ (dots) and $X = 1000$ (dash-dots).

Figure 3. Graph of $|Z_X(\frac{1}{2} + i(x + t_0))|$, with $t_0 = \gamma_{10^{12}+40}$, with $X = \log t_0$ (dots) and $X = 1000$ (dash-dots).
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