THE SPACE OF DENSITY STATES IN GEOMETRICAL QUANTUM MECHANICS

JESÚS CLEMENTE-GALLARDO AND GIUSEPPE MARMO

ABSTRACT. We present a geometrical description of the space of density states of a quantum system of finite dimension. After presenting a brief summary of the geometrical formulation of Quantum Mechanics, we proceed to describe the space of density states $\mathcal{D}(\mathcal{H})$ from a geometrical perspective identifying the stratification associated to the natural $GL(\mathcal{H})$-action on $\mathcal{D}(\mathcal{H})$ and some of its properties. We apply this construction to the cases of quantum systems of two and three levels.

Keywords: Density states, projective space, geometric quantum mechanics
PACS: 03.65.-w, 03.65.Ta

1. INTRODUCTION

A comparison of the frameworks underlying classical and quantum mechanics shows that the two descriptions have several common mathematical structures. However, a striking difference emerges: the classical setting is geometrical and nonlinear while the quantum is algebraic and linear. The emphasis on the underlying linearity in quantum mechanics is usually attributed to the description of the interference phenomena \cite{10}. Therefore, the carrier space of quantum systems is required to be a Hilbert space $\mathcal{H}$ from the beginning. The Hermitian structure is required to describe the probabilistic interpretation of Quantum Mechanics. However, it is exactly this probabilistic interpretation which forces on us the identification of physical states not with the Hilbert space but rather with the space of rays, i.e. the complex projective space of $\mathcal{H}$, say $\mathcal{RH}$. Of course $\mathcal{RH}$ is a genuine nonlinear manifold and on it the Hermitian structure gives rise to a Kähler structure.

The appearance of this manifold in the quantum setting calls for a geometrical formulation of Quantum Mechanics. It is clear that in the manifold viewpoint we have to give up the usual “superposition of states” and the notion of operators, eigenvectors and eigenstates as usually presented. Nevertheless, due to their physical relevance and interpretation we must be able to recover these “attributes” for quantum systems also at the manifold level. The overall formulation must allow for nonlinear transformations and therefore only tensorial objects should be identified with physically relevant quantities. To fully exploit the geometrical picture, one prefers to work with real differential manifolds, i.e. one replaces the complex vector space $\mathcal{H}$ with its realification $\mathcal{RH}$. The Hermitian structure then splits into a complex structure, a symplectic structure and a Riemannian structure (compatible among them to define a Kähler structure) Hermitian operators are transformed into functions by replacing them by their expectation values. These functions project onto $\mathcal{RH}$. With the help of the Poisson tensor associated with the symplectic structure it is possible to give rise to a flow by integrating the Hamiltonian vector field...
associated with the expectation value function corresponding to a given Hermitian operator [1, 2, 3, 4, 5, 6, 7, 8, 9, 15, 23].

The symplectic structure appearing in Quantum Mechanics makes also possible to consider it as a “classical field theory” associated with a Lagrangian description with relativity group the Galilei group [21], since we deal with non-relativistic Quantum Mechanics. Thus, classical and quantum descriptions have in common a symplectic structure. However, this is the only quantum feature that has a direct classical analogue. Some characteristic features like the quantum uncertainties and state vector reduction in a measurement process are strictly related to the additional complex structure, available in Quantum Mechanics but not present in Classical Mechanics [20]. This additional structure lies at the heart of the difference between the mathematical structures underlying the two theories, much more than the linear structure.

Going back to the manifold viewpoint introduced in Quantum Mechanics, i.e. the identification of $\mathcal{RH}$ as the true manifold of physical states, we have to recover the notion of superposition of physical states [19]. This has been done and creates a deep relation with Pancharatnam connection, Bargmann invariants and geometric phases [22]. In addition we have to recover the notion of “eigenvector” and “eigenvalue”. As a matter of fact by considering the expectation value function associated with any operator we find that their critical points will correspond to eigenstates and their values at those critical points correspond to the eigenvalues.

If we consider the expectation value-functions of generic operators we get complex valued functions on $\mathcal{RH}$ and they provide us with a $\mathbb{C}^*$–algebra, thus paving the way for the geometrization of the $\mathbb{C}^*$–algebraic approach to quantum theories. In this latter approach usually the space of states is identified with the space of normalized positive linear functionals on the $\mathbb{C}^*$–algebra describing the quantum system. As a provisional geometrization of this approach we shall construct these spaces with the help of the momentum map associated with the symplectic action of the unitary group on the Kähler manifold $\mathcal{RH}$.

2. A BRIEF EXPOSITION OF GEOMETRICAL QUANTUM MECHANICS

The aim of this section is to present a brief summary of the set of the geometrical tools which characterize the description of Quantum Mechanics [13, 14, 18].

2.1. The states. The first step consists in replacing the usual complex vector space structure of the Hilbert space $\mathcal{H}$ of a quantum system by the corresponding realification of the vector space. We shall denote as $\mathcal{H}_R$ the resulting vector space. In this realification process the complex structure on $\mathcal{H}$ will be represented by a tensor $J$ on $\mathcal{H}_R$.

The natural identification is then provided by

$$\psi_R + i\psi_I = \psi \in \mathcal{H} \mapsto (\psi_R, \psi_I) \in \mathcal{H}_R.$$ 

Under this transformation, the Hermitian product becomes, for $\psi^1, \psi^2 \in \mathcal{H}$

$$\langle (\psi_R^1, \psi_I^1), (\psi_R^2, \psi_I^2) \rangle = \langle \psi_R^1, \psi_R^2 \rangle + \langle \psi_I^1, \psi_I^2 \rangle + i(\langle \psi_R^1, \psi_I^2 \rangle - \langle \psi_I^1, \psi_R^2 \rangle).$$

To consider $\mathcal{H}_R$ just as a real differential manifold, the algebraic structures available on $\mathcal{H}$ must be converted into tensor fields on $\mathcal{H}_R$. To this end we have to introduce the tangent bundle $T\mathcal{H}_R$ and its dual the cotangent bundle $T^*\mathcal{H}_R$. The
linear structure available in $\mathcal{H}_R$ is encoded in the vector field $\Delta$

$$\Delta : \mathcal{H}_R \to T\mathcal{H}_R \quad \psi \mapsto (\psi, \psi)$$

We can consider the Hermitian structure on $\mathcal{H}_R$ as an Hermitian tensor on $T\mathcal{H}_R$. With every vector we can associate a vector field

$$X_\psi : \phi \mapsto (\phi, \psi)$$

Therefore, the Hermitian tensor, denoted in the same way as the scalar product will be

$$\langle X_\psi_1, X_\psi_2 \rangle = \langle \psi_1, \psi_2 \rangle$$

The scalar product above is written as

$$\langle \psi_1, \psi_2 \rangle = g(X_{\psi_1}, X_{\psi_2}) + i\omega(X_{\psi_1}, X_{\psi_2}),$$

where $g$ is now a symmetric tensor and $\omega$ a skew-symmetric one. It is also possible to write them as a pull-back by means of the dilation vector field $\Delta$ as:

$$\langle \Delta^*(g + i\omega), \phi \rangle = \langle (\psi, \phi) \rangle_{\mathcal{H}}$$

The properties of the Hermitian product ensure that:

- the symmetric tensor is positive definite and non-degenerate, and hence defines a Riemannian structure on the real vector space.
- the skew-symmetric tensor is also non degenerate, and is closed with respect to the natural differential structure of the vector space. Hence, the tensor is a symplectic form.

As the inner product is sesquilinear, it satisfies

$$\langle \psi_1, i\psi_2 \rangle = i\langle \psi_1, \psi_2 \rangle, \quad \langle i\psi_1, \psi_2 \rangle = -i\langle \psi_1, \psi_2 \rangle.$$  
This implies

$$g(X_{\psi_1}, X_{\psi_2}) = \omega(JX_{\psi_1}, X_{\psi_2}).$$
We also have that $J^2 = -\mathbb{I}$, and hence that the triple $(J, g, \omega)$ defines a Kähler structure. This implies, among other things, that the tensor $J$ generates both finite and infinitesimal transformations which are orthogonal and symplectic.

Linear transformations are converted into $(1,1)$–tensor fields by setting $A \to T_A$ where

$$T_A : T\mathcal{H}_R \to T\mathcal{H}_R \quad (\psi, \phi) \mapsto (\psi, A\phi).$$

The association $A \to T_A$ is an associative algebra isomorphism. It is possible to recover the Lie algebra of vector fields by setting $X_A = T_A(\Delta)$. Complex linear transformations will be represented by $(1,1)$–tensor fields commuting with $J$.

For finite dimensional Hilbert spaces it may be convenient to introduce adapted coordinates on $\mathcal{H}$ and $\mathcal{H}_R$. Fixing an orthonormal basis $\{|e_k\rangle\}$ of the Hilbert space allows us to identify this product with the canonical Hermitian product of $\mathbb{C}^n$:

$$\langle \psi_1, \psi_2 \rangle = \sum_k \langle \psi_1, e_k \rangle \langle e_k, \psi_2 \rangle$$

The group of unitary transformations on $\mathcal{H}$ becomes identified with the group $U(n, \mathbb{C})$, its Lie algebra $\mathfrak{u}(\mathcal{H})$ with $\mathfrak{u}(n, \mathbb{C})$ and so on.

The choice of the basis also allows us to introduce coordinates for the realified structure:

$$\langle e_k, \psi \rangle = (q_k + ip_k)(\psi),$$

and write the geometrical objects introduced above as:

$$J = \partial_{q_k} \otimes dq_k - \partial_{p_k} \otimes dp_k \quad g = dq_k \otimes dq_k + dp_k \otimes dp_k \quad \omega = dq_k \wedge dp_k$$
If we combine them in complex coordinates we can write the Hermitian structure in a simple way:

\[ h = \sum_k d\bar{z}_k \otimes dz_k \]

In an analogous way we can consider a contravariant version of these tensors. It is also possible to build it by using the isomorphism \( T\mathcal{H}_\mathbb{R} \leftrightarrow T^*\mathcal{H}_\mathbb{R} \) associated to the Riemannian tensor \( g \). The result in both cases is a Kähler structure for the dual vector space \( \mathcal{H}_\mathbb{R}^* \) with the dual complex structure \( J^* \), a Riemannian tensor \( G \) and a (symplectic) Poisson tensor \( \Omega \): The coordinate expressions with respect to the natural base are:

- the Riemannian structure \( G = \sum_{k=1}^n \left( \frac{\partial}{\partial q_k} \otimes \frac{\partial}{\partial q_k} + \frac{\partial}{\partial p_k} \otimes \frac{\partial}{\partial p_k} \right) \),
- the Poisson tensor \( \Omega = \sum_{k=1}^n \left( \frac{\partial}{\partial q_k} \wedge \frac{\partial}{\partial p_k} \right) \),
- while the complex structure has the form \( J = \sum_{k=1}^n \left( \frac{\partial}{\partial p_k} \otimes dq_k - \frac{\partial}{\partial q_k} \otimes dp_k \right) \).

2.2. The observables. The space of observables (i.e. of self-adjoint operators acting on \( \mathcal{H} \)) may be identified with the dual \( u^*(\mathcal{H}) \) of the real Lie algebra \( u(\mathcal{H}) \), according to the pairing between the unitary Lie algebra and its dual given by

\[ A(T) = \frac{i}{2} \text{Tr} AT \]

Under the previous isomorphism, \( u^*(\mathcal{H}) \) becomes a Lie algebra with product defined by

\[ i[A, B] = [A, B]_+ = (AB - BA) \]

We can also transfer the Jordan product:

\[ [A, B]_+ = 2A \circ B = AB + BA \]

Both structures are compatible. As a result, \( u^*(\mathcal{H}) \) becomes a Jordan-Lie algebra (see [11, 16]).

We can also define a suitable scalar product, given by:

\[ \langle A, B \rangle = \frac{1}{2} \text{Tr} AB \]

which turns the space into a real Hilbert space. This scalar product is the restriction of the one on \( \mathfrak{gl}(\mathcal{H}) \) defined as \( \langle M, N \rangle = \frac{1}{2} \text{Tr} M^1 N \).

Besides this scalar product is compatible with the Lie-Jordan structure in the following sense:

\[ \langle [A, \xi], B \rangle_{u^*(\mathcal{H})} = \langle A, [\xi, B] \rangle_{u^*(\mathcal{H})} \quad \langle [A, \xi]_+, B \rangle_{u^*(\mathcal{H})} = \langle A, [\xi, B]_+ \rangle_{u^*(\mathcal{H})} \]

These algebraic structures may be given a tensorial translation in terms of the association \( A \mapsto T_A \). However we can also associate complex valued functions with linear operators \( A \in \mathfrak{gl}(\mathcal{H}) \) by means of the scalar product

\[ \mathfrak{gl}(\mathcal{H}) \ni A \mapsto f_A = \frac{1}{2} \langle \psi, A\psi \rangle_{\mathcal{H}}. \]

In more intrinsic terms we may write

\[ f_A = \frac{1}{2} (g(\Delta, X_A) + i\omega(\Delta, X_A)). \]
Hermitian operators give rise thus to quadratic real valued functions.

The association of operators with quadratic functions allows also to recover the algebraic structures on \( u(\mathcal{H}) \) and \( u^*(\mathcal{H}) \) by means of appropriate \((0,2)\)-tensors on \( \mathcal{H}_R \). By using the contravariant form of the Hermitian tensor \( G + i\Omega \) given by:

\[
G + i\Omega = 4 \frac{\partial}{\partial z_k} \otimes \frac{\partial}{\partial \bar{z}_k} = \frac{\partial}{\partial q_k} \otimes \frac{\partial}{\partial \bar{q}_k} + \frac{\partial}{\partial p_k} \otimes \frac{\partial}{\partial \bar{p}_k} + i \frac{\partial}{\partial q_k} \wedge \frac{\partial}{\partial p_k},
\]

it is possible to define a bracket

\[
\{f, h\}_H = \{f, h\}_g + i\{f, h\}_\omega
\]

In particular, for quadratic real valued functions we have

\[
\{f_A, f_B\}_g = f_{AB + BA} = 2f_{AB} \quad \{f_A, f_B\}_\omega = -if_{AB - BA}
\]

The imaginary part, i.e., \( \{\cdot, \cdot\}_\omega \), defines a Poisson bracket on the space of functions. Both brackets allow us to define a tensorial version of the Lie-Jordan algebra of the set of operators.

For Hermitian operators we recover previously defined vector fields:

\[
\text{grad} f_A = \tilde{A}; \quad \text{Ham} f_A = i\tilde{A}
\]

where the vector fields associated with operators, we recall, are defined by:

\[
\tilde{A} : \mathcal{H}_R \rightarrow T\mathcal{H}_R \quad \psi \mapsto (\psi, A\psi)
\]

\[
i\tilde{A} : \mathcal{H}_R \rightarrow T\mathcal{H}_R \quad \psi \mapsto (\psi, JA\psi)
\]

We can also consider the algebraic structure associated to the full bracket \( \{\cdot, \cdot\}_H \), as we associated above the Jordan product and the commutator of operators to the brackets \( \{\cdot, \cdot\}_g \) and \( \{\cdot, \cdot\}_\omega \) respectively. It is simple to see that it corresponds to the associative product of the set of operators, i.e.

\[
\{f_A, f_B\}_H = \{f_A, f_B\}_g + i\{f_A, f_B\}_\omega = f_{AB + BA} + i f_{AB - BA} = 2f_{AB}
\]

This particular bilinear product on quadratic functions may be written also as a star product

\[
\{f_A, f_B\}_H = 2f_{AB} = \langle df_A, df_B \rangle_{H^*} = f_A * f_B
\]

The set of quadratic functions endowed with such a structure turns out to be a \( \mathbb{C}^* \)-algebra.

We see then that we can reconstruct all the information of the algebra of operators starting only with real-valued functions defined on \( \mathcal{H}_R \). We have thus

**Proposition 1.** The Hamiltonian vector field \( X_f \) (defined as \( X_f = \hat{\Omega}(df) \)) is a Killing vector field for the Riemannian tensor \( G \) if and only if \( f \) is a quadratic function associated with an Hermitian operator \( A \), i.e. there exists \( A = A^\dagger \) such that \( f = f_A \).

Finally, we can consider the problem of how to recover the eigenvalues and eigenvectors of the operators at the level of the functions of \( \mathcal{H}_R \). It is simple to see that

- eigenvectors correspond to the critical points of functions \( f_A \), i.e.

\[
df_A(\psi_*) = 0 \text{ iff } \psi_* \text{ is an eigenvector of } A
\]
the corresponding eigenvalue is recovered by the value

\[
\frac{f_A(\psi_*)}{\langle \psi_*, \psi_* \rangle}
\]

Thus we can conclude that the Kähler manifold \((\mathcal{H}_R, J, \omega, g)\) contains all the information of the usual formulation of Quantum Mechanics on a complex Hilbert space.

Up to now we have concentrated our attention on states and observables. If we consider observables as generators of transformations, i.e. we consider the Hamiltonian flow associated to the corresponding functions, the invariance of the tensor \(G\) implies that the evolution is actually unitary. It is, therefore, natural, to consider the action of the unitary group on the realification of the complex vector space.

3. The momentum map: geometrical structures on \(g^*\)

The unitary action of \(U(\mathcal{H})\) on \(\mathcal{H}\) induces a symplectic action on the symplectic manifold \((\mathcal{H}_R, \omega)\). By using the association

\[
F : \mathcal{H}_R \times \mathfrak{u}(\mathcal{H}) \to \mathbb{R} \quad (\psi, A) \mapsto \frac{1}{2} \langle \psi, A\psi \rangle = f_A(\psi),
\]

we find, with \(F_A = f_A : \mathcal{H}_R \to \mathbb{R}\), that

\[
\{F(A), F(B)\}_\omega = iF([A, B]).
\]

Thus if we fix \(\psi\), we have a mapping \(F(\psi) : \mathfrak{u}(\mathcal{H}) \to \mathbb{R}\). Thus with any element \(\psi \in \mathcal{H}\) we have an element in \(\mathfrak{u}^*(\mathcal{H})\). Hence it defines a momentum map

\[\mu : \mathcal{H} \to \mathfrak{u}^*(\mathcal{H}),\]

which provides us with a symplectic realization of the natural Poisson manifold structure available in \(\mathfrak{u}^*(\mathcal{H})\). We can write the momentum map from \(\mathcal{H}_R\) to \(\mathfrak{u}^*(\mathcal{H})\) as

\[\mu(\psi) = |\psi\rangle \langle \psi|\]

If we make the convention that the dual \(\mathfrak{u}^*(\mathcal{H})\) of the (real) Lie algebra \(\mathfrak{u}(\mathcal{H})\) is identified with Hermitian operators by means of a scalar product, the product pairing between Hermitian operators \(A \in \mathfrak{u}^*(\mathcal{H})\) and the anti-Hermitian element \(T \in \mathfrak{u}(\mathcal{H})\) will be given by

\[A(T) = \frac{i}{2} \text{Tr}(AT)\]

If we denote the linear function on \(\mathfrak{u}^*(\mathcal{H})\) associated with the element \(iA \in \mathfrak{u}(\mathcal{H})\) by \(\hat{A}\), we have

\[\mu^*(\hat{A}) = f_A\]

The pullback of linear functions on \(\mathfrak{u}^*(\mathcal{H})\) is given by the quadratic functions on \(\mathcal{H}_R\) associated with the corresponding Hermitian operators.

It is possible to show that the contravariant tensor fields on \(\mathcal{H}_R\) associated with the Hermitian structure are \(\mu\)-related with a complex tensor on \(\mathfrak{u}^*(\mathcal{H})\):

\[\mu_*(G + i\Omega) = R + i\Lambda;\]

where the two new tensors \(R\) and \(\Lambda\) are defined by

\[R(\xi)(\hat{A}, \hat{B}) = \langle \xi, [A, B]_+ \rangle_{\mathfrak{u}^*} = \frac{1}{2} \text{Tr}(\xi(AB + BA))\]

and

\[\Lambda(\xi)(\hat{A}) = \frac{i}{2} \text{Tr}(\xi[A, B])\]
and
\[ \Lambda(\xi)(\hat{A},\hat{B}) = \langle \xi, [A, B]_\pm \rangle_{u^*} = \frac{1}{2i} \text{Tr}(\xi(AB - BA)) \]

Clearly,
\[ G(\mu^* \hat{A}, \mu^* \hat{B}) + i\Omega(\mu^* \hat{A}, \mu^* \hat{B}) = \mu^*(R(\hat{A}, \hat{B}) + i\Lambda(\hat{A}, \hat{B})). \]

As we know that \( u^*(\mathcal{H}) \) is foliated by symplectic manifolds, we wish to consider more closely the map from \( \mathcal{H}_R \) to the minimal symplectic orbit on \( u^*(\mathcal{H}) \).

4. The complex projective space

As we have already remarked, the association of states of the quantum system with vectors in the Hilbert space needs further qualifications because of the probabilistic interpretation required in Quantum Mechanics. More specifically, states should be identified with rays in the Hilbert space, i.e. equivalence classes of vectors, orbits of non-null vectors under the action of \( C_0 = \mathbb{C} - \{0\} \). The equivalence class of the vector \( \psi \in \mathcal{H} \) will be denoted then as
\[ [\psi] = \{ \lambda \psi, \lambda \in C_0 \} \]

As the infinitesimal generators of the real and imaginary parts of the action of \( C_0 \) on \( \mathcal{H}_R \) are given by the dilation vector field \( \Delta \) and the vector field \( J(\Delta) \) respectively, it is clear that we have to undertake the projection of the relevant tensors on \( \mathcal{H}_R \) to the complex projective space or ray space \( \mathcal{R} \mathcal{H} \).

Without entering in too many details, we find that we have to modify \( G \) and \( \Omega \) by a conformal factor to turn them into projectable tensors. Specifically we have
\[ \tilde{G} = g(\Delta, \Delta)G, \quad \tilde{\Lambda} = g(\Delta, \Delta)\Lambda. \]

These tensors are projectable onto non-degenerate contravariant tensors on \( \mathcal{R} \mathcal{H} \) and gives rise to a Lie-Jordan algebra structure on the space of real valued functions whose Hamiltonian vector fields are also Killing vector fields for the projection \( \tilde{G} \).

As a matter of fact a theorem by Wigner allows us to state that these functions are necessarily projections of expectation values of Hermitian operators
\[ e_A(\psi) = \frac{\langle \psi, A\psi \rangle}{\langle \psi, \psi \rangle}. \]

The action of the unitary group may also be projected and gives rise to a symplectic action on \( \mathcal{R} \mathcal{H} \). The momentum map from \( \mathcal{H}_R \) projects onto the momentum map from \( \mathcal{R} \mathcal{H} \) because it is equivariant with respect to the action of \( \Delta_R \) on \( \mathcal{H}_R \) and the action of \( \Delta_{u^*} \) on \( u^*(\mathcal{H}) \).

From \( \mu^*(\hat{A}) = f_A \) we find
\[ \Delta_R \mu^*(\hat{A}) = 2\mu^*(\Delta_{u^*} \hat{A}) = 2\mu^*(\hat{A}) \]

The momentum map for the projected action may be written in the form
\[ \mu([\psi]) = \frac{|\psi\rangle \langle \psi|}{\langle \psi, \psi \rangle} = \rho_\psi. \]

This map identifies \( \mathcal{R} \mathcal{H} \) with the Hermitian operators in \( u^*(\mathcal{H}) \) which are of rank one and are projectors, i.e.
\[ \rho_\psi \rho_\psi = \rho_\psi, \quad \text{Tr} \rho_\psi = 1. \]

As the ray space is a principal bundle with base manifold \( \mathcal{R} \mathcal{H} \) and structure group \( C_0 \), we may look for a connection one form.
The connection one-form $\theta$ is given by

$$\theta(\psi) = \left< \psi, d\psi \right> \left< \psi, \psi \right>,$$

with associated curvature form $\omega = d\theta$, because the structure group is Abelian. This curvature form coincides with the symplectic structure on $\mathcal{R}\mathcal{H}$ arising from the projection of $\tilde{\Lambda}$ (conformally related to $\Lambda$).

It is also possible to write the Hermitian tensor which coincides with the Hermitian tensor on $\mathcal{R}\mathcal{H}$ when evaluated on horizontal vector fields. We thus have

$$\frac{\left< d\psi, d\psi \right>}{\left< \psi, \psi \right>} - \frac{\left< \psi, d\psi \right> \left< d\psi, \psi \right>}{\left< \psi, \psi \right>^2}.$$

It is not difficult to see that both $\Delta$ and $J(\Delta)$ are annihilated by this tensor.

The embedding of $\mathcal{R}\mathcal{H}$ into $\mathfrak{u}^*(\mathcal{H})$ by means of the momentum map allows us to consider convex combinations of the image $\mu(\mathcal{R}\mathcal{H}) \subset \mathfrak{u}^*(\mathcal{H})$. The convex combinations will generate the space of density states, i.e. normalized positive linear functionals on the Lie-Jordan algebra of observables. This convex body inherits some structures from those existing on $\mathfrak{u}^*(\mathcal{H})$, which are particularly important and useful when we are interested in describing evolutions of states which are not unitary. In particular they inherits a Poisson structure and a Jordan structure. In the next section we shall study more closely the space of density states.

5. THE SPACE OF DENSITY STATES

As we have already remarked the space of density states is the space of positive normalized linear functionals on the real linear space of observables. A theorem by Gleason [12] asserts that these functionals may be represented by suitable operators when the trace is used as a bilinear pairing. By using this theorem we can start by considering states directly as appropriate operators.

Let us introduce first the space of all non-negatively defined operators, i.e. the space of all those $\rho \in \mathfrak{gl}(\mathcal{H})$ which can be written in the form

$$\rho = T^\dagger T \quad T \in \mathfrak{gl}(\mathcal{H}).$$

We shall denote by $\mathcal{P}\mathcal{H}$ this space of operators, which is a convex cone in $\mathfrak{u}^*(\mathcal{H})$. By imposing the condition $T^\dagger T = 1$ we select in $\mathcal{P}\mathcal{H}$ the convex body of density states which we denote by $\mathcal{D}(\mathcal{H})$. We shall also consider non-negative Hermitian operators and density states of rank $k$, and denote as $\mathcal{P}^k(\mathcal{H})$ and $\mathcal{D}^k(\mathcal{H})$ respectively the corresponding spaces.

The complex projective space is in one-to-one correspondence with $\mathcal{D}^1(\mathcal{H})$. Indeed, any state in $\mathcal{D}(\mathcal{H})$ can be written as a convex combination of distinct states $\rho = p_1 \rho_1 + (1 - p_1) \rho_2$, with $0 \leq p_1 \leq 1$. We shall call extremal states those which can not be written in this form (i.e. as convex combination of two $\rho_1$ and $\rho_2$). The extremal states are thus given by $\mathcal{D}^1(\mathcal{H})$.

As $\Lambda$ and $\tilde{\Lambda}$ are not invertible in $\mathfrak{u}^*(\mathcal{H})$, it is convenient to use the pairing between $\mathfrak{u}(\mathcal{H})$ and $\mathfrak{u}^*(\mathcal{H})$ defined by the trace, to introduce two tensor fields on $\mathfrak{u}^*(\mathcal{H})$. We set then

$$\tilde{J}, R : Tu^*(\mathcal{H}) \to Tu^*(\mathcal{H})$$
defined as
\[ \tilde{J}_\xi(X_A) = (\xi, [A, \xi]) = \Lambda(\xi)(d\hat{A}) \]
\[ R_\xi(X_A) = (\xi, [A, \xi]) = R(\xi)(d\hat{A}) \]

The image of \( \tilde{J} \) is the Hamiltonian involutive distribution associated with linear Hamiltonian functions, and we shall denote it as \( D_\Lambda \). The image of \( R \) is also a distribution, which we shall denote as \( D_R \), but in this case it is not involutive.

It is possible to see that combining \( D_R \) and \( D_\Lambda \) we can define two distributions \( D_0 = D_R \cap D_A \) and \( D_1 = D_R + D_\Lambda \) which are indeed involutive.

We notice that the tensors \( \tilde{J} \) and \( R \) commute, i.e. \( \tilde{J} \circ R = R \circ \tilde{J} \). More specifically we have
\[ \tilde{J}(\xi) \circ R(\xi)(X_A) = R(\xi) \circ \tilde{J}(\xi)(X_A) = [A, \xi] \]

As a result, we find that the distribution \( D_0 \) becomes:
\[ D_0(\xi) = \{ [A, \xi]; A \in u^*(\mathcal{H}) \} \]

On \( \mathcal{R}\mathcal{H} \), \( D_0 \) coincides with \( D_\Lambda \).

The distribution \( D_1 \) is involutive and the leaves are related to orbits of the following \( GL(\mathcal{H}) \)-action:
\[ GL(\mathcal{H}) \times u^*(\mathcal{H}) \to u^*(\mathcal{H}) \quad (T, \xi) \mapsto T\xi T^\dagger \]

We obtain some interesting results [13, 14]:

(1) The Hermitian operators \( \rho \) and \( \rho' \) belong to the same \( GL \)-orbit if and only if they have the same number \( K_+ \) of positive eigenvalues and the same number \( K_- \) of negative eigenvalues (counted with multiplicities).

(2) Any \( GL \)-orbit intersecting the positive cone \( \mathcal{P}\mathcal{H} \) lies entirely in \( \mathcal{P}\mathcal{H} \); so that \( \mathcal{P}\mathcal{H} \) is stratified by the \( GL \)-orbits. These \( GL \)-orbits in \( \mathcal{P}\mathcal{H} \) are determined by the rank of the operator, i.e. they are exactly \( \mathcal{P}^k(\mathcal{H}) \).

When we restrict to the space of density states by imposing the condition \( \text{Tr} \rho = 1 \), this \( GL \)-action will not preserve the states. It is however possible to define a new action that maps \( D(\mathcal{H}) \) into itself by setting
\[ GL(\mathcal{H}) \times D(\mathcal{H}) \to D(\mathcal{H}) \quad (T, \rho) \mapsto \frac{T\rho T^\dagger}{\text{Tr}(T\rho T^\dagger)} \]

This action does preserve the rank of \( \rho \) and then the following proposition holds true:

**Proposition 2.** The decomposition of the convex body of density states \( D(\mathcal{H}) \) into orbits of the \( GL(\mathcal{H}) \)-action \( \rho \mapsto \frac{T\rho T^\dagger}{\text{Tr}(T\rho T^\dagger)} \) is exactly the stratification
\[ D(\mathcal{H}) = \bigcup_{k=1}^{n} D^k(\mathcal{H}), \]
into states of a given rank.

The boundary of the convex body of density states consists of states of rank lower than \( n \), i.e. \( \partial D(\mathcal{H}) = \bigcup_{k=1}^{n-1} D^k(\mathcal{H}) \), and each stratum is a smooth submanifold in \( u^*(\mathcal{H}) \). However, the boundary \( \partial D(\mathcal{H}) \) is not smooth (for \( n > 2 \)). We have the following theorem:
Theorem 1. Every smooth curve $\gamma : \mathbb{R} \to \mathfrak{u}^*(\mathcal{H})$ through the convex body of density states is tangent, at every point, to the stratum to which it belongs, i.e.

$$\gamma(t) \in \mathcal{D}^k(\mathcal{H}) \Rightarrow T\gamma(t) \in T_{\gamma(t)}\mathcal{D}^k(\mathcal{H}).$$

One may gain further insight on the “location” of the boundary by using the notion of “face”

Definition 1. A non-empty closed convex subset $K_0$ of a closed convex set $K$ is called a face of $K$ if any closed segment in $K$ with an interior point in $K_0$ lies entirely in $K_0$.

Thus, for any $\rho \in \mathcal{D}(\mathcal{H})$ we may consider the decomposition $\mathcal{H} = \text{Im} \rho + \ker \rho$ into the kernel and the image of $\rho$. We have:

Proposition 3. The face of $\mathcal{D}(\mathcal{H})$ through $\rho \in \mathcal{D}^k(\mathcal{H})$ consists of states $A$ which are “projectable” with respect to the projection defined by $\ker \rho$, i.e. $\ker A \subseteq \ker \rho$. The face through $\rho$ is then equivalent to $\mathcal{D}(\text{Im} \rho)$.

The inner product defined by the trace allows to define a probability transition function

$$p(\rho_1, \rho_2) = \text{Tr} \rho_1 \rho_2,$$

when $\rho_1$ and $\rho_2$ belong to the boundary $\partial_e \mathcal{D}(\mathcal{H})$, the space of extremal states.

This function satisfies

$$0 \leq p(\rho_1, \rho_2) \leq 1 \quad p(\rho_1, \rho_2) = p(\rho_2, \rho_1).$$

Moreover, $p(\rho_1, \rho_2) = 1$ if and only if $\rho_1 = \rho_2$.

It is not difficult to show that the Hamiltonian vector fields which leave $\mathcal{H}$ invariant will preserve also the probability transition functions. This result is related to a theorem by Wigner and may be used to recover the space of density states starting with a Poisson space carrying a compatible probability transition function. Further details and a full treatment of Poisson spaces with a transition probability function have been considered by Landsman (see [17]).

6. Two Examples: $\mathfrak{g}_{u(2)}$ and $\mathfrak{g}_{u(3)}$

6.1. States of a two level system. We shall consider in some detail two examples. The first one is the two level system with carrier space $\mathcal{H} = \mathbb{C}^2$. We consider $\mathfrak{u}(2)$ and $\mathfrak{u}^*(2)$ and make a specific choice of basis

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We recall that

$$\sigma_1 \sigma_2 = i \sigma_3, \quad \sigma_2 \sigma_3 = i \sigma_1, \quad \sigma_3 \sigma_1 = i \sigma_2,$$

along with

$$\sigma_2 \sigma_1 = -i \sigma_3, \quad \sigma_3 \sigma_2 = -i \sigma_1, \quad \sigma_1 \sigma_3 = -i \sigma_2;$$

which may be obtained considering the conjugate-transpose of any product.

We define coordinate functions by writing

$$y_0(A) = \frac{1}{2} \text{Tr}(\sigma_0 A), \quad y_2(A) = \frac{1}{2} \text{Tr}(\sigma_2 A).$$
In these coordinates the corresponding Poisson brackets for the canonical Lie-Poisson structure on the dual of the Lie algebra read:

\[ \{ y_0, y_a \} = 0 \quad \{ y_a, y_b \} = 2 \epsilon_{abc} y_c. \]

The expression of the Poisson tensor thus becomes:

\[ \Lambda = 2 \left( y_1 \frac{\partial}{\partial y_2} \wedge \frac{\partial}{\partial y_3} + y_2 \frac{\partial}{\partial y_3} \wedge \frac{\partial}{\partial y_1} + y_3 \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y_2} \right) \]

It is also possible to construct the Riemann-Jordan tensor in the form:

\[ R = \frac{\partial}{\partial y_0} \otimes_s \left( y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2} + y_3 \frac{\partial}{\partial y_3} \right) + y_0 \left( \frac{\partial}{\partial y_0} \otimes \frac{\partial}{\partial y_0} + \frac{\partial}{\partial y_1} \otimes \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2} \otimes \frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_3} \otimes \frac{\partial}{\partial y_3} \right) \]

where \( \otimes_s \) means the symmetrized tensor product.

6.2. Distributions associated with \( \Lambda \) and \( R \). It is easy to see that the Hamiltonian distribution is generated by

\[ H_1 = y_3 \frac{\partial}{\partial y_2} - y_2 \frac{\partial}{\partial y_3}, \quad H_2 = y_1 \frac{\partial}{\partial y_3} - y_3 \frac{\partial}{\partial y_1}, \quad H_3 = y_2 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial y_2}, \]

while the distribution associated with the Riemann-Jordan tensor is

\[ X_0 = y^a \frac{\partial}{\partial y^a} + y^0 \frac{\partial}{\partial y^0}, \quad X_a = y^a \frac{\partial}{\partial y^0} + y^0 \frac{\partial}{\partial y^a} \]

It is clear that \( X_0 \) is central and \( \{ X_a \} \) are boosts of a four dimensional Lorentz group, therefore their commutator will provide us with the Lie algebra of the rotation group:

\[ [X_a, X_b] = y^a \frac{\partial}{\partial y^b} - y^b \frac{\partial}{\partial y^a}. \]

The intersection of the distribution associated with \( R \) and the Hamiltonian distribution associated will indeed be generated by the Hamiltonian vector fields and is involutive with leaves which are symplectic two dimensional spheres. The distribution generated by the union of the two distributions is the full Lorentz group centrally extended with the dilations. As the Lorentz group admits as a covering \( SL(2, \mathbb{C}) \) the central extension is isomorphic to \( GL(2, \mathbb{C}) \). This is a general property holding true in any dimension (see [14]).

We find that

**Lemma 1.** The rank of \( \Lambda \) is zero if \( y_1^2 + y_2^2 + y_3^2 = 0 \) and the rank is equal to 2 if \( y_1^2 + y_2^2 + y_3^2 > 0 \).

The situation is richer with \( R \):

**Lemma 2.** The rank of \( R \) is

- zero if \( y_0^2 + y_1^2 + y_2^2 + y_3^2 = 0 \)
- two if \( y_0 = 0 \) and \( y_1^2 + y_2^2 + y_3^2 > 0 \)
- three for \( y_0^2 = y_1^2 + y_2^2 + y_3^2 \)
- four if \( y_0^2 \neq y_1^2 + y_2^2 + y_3^2 \)
6.3. Density states. As we have already seen in the previous sections the set of states is identified with a subset of \( u^*(\mathcal{H}) \) satisfying a positivity condition and a normalization condition. In the specific situation we are considering, a generic Hermitian matrix \( A = y^0\sigma_0 + y^a\sigma_a \) will define a state if

\[
\text{Tr} A = 1 \quad 0 \leq y^0 + y^3 \leq 1, \quad 0 \leq y^0 - y^3 \leq 1, \quad \det A \geq 0.
\]

Explicitly we have

\[
y^0 = \frac{1}{2}, \quad \left( \frac{1}{2} + y^3 \right) \left( \frac{1}{2} - y^3 \right) - ((y^1)^2 + (y^2)^2) \geq 0,
\]

or

\[
(y^3)^2 + (y^2)^2 + (y^1)^2 \leq \frac{1}{4}.
\]

Thus in our parametrization states are determined by points in \( \mathbb{R}^4 \) on the hyperplane \( y^0 = \frac{1}{2} \), and on this three dimensional space are identified by the points in the ball of radius \( \frac{1}{2} \). When referring to states we replace \( A \) with \( \rho \) and write:

\[
\rho = \left( \begin{array}{cc}
\frac{1}{2} + y^3 & y^2 + iy^1 \\
\frac{1}{2} - y^3 & \frac{1}{2} - y^3
\end{array} \right).
\tag{1}
\]

The pure states corresponding to the vector \((z_1, z_2) \in \mathbb{C}^2\) with unit norm \(z_1\bar{z}_1 + z_2\bar{z}_2 = 1\) has a density state

\[
\rho = \left( \begin{array}{cc}
z_1 & \bar{z}_2 \\
\bar{z}_1 & z_2
\end{array} \right) = \left( \begin{array}{cc}
z_1\bar{z}_1 & \bar{z}_1\bar{z}_2 \\
z_2\bar{z}_1 & z_2\bar{z}_2
\end{array} \right).
\]

Within the previous parametrization we find

\[
y^3 = \frac{1}{2}(z_1\bar{z}_1 - z_2\bar{z}_2), \quad y^1 = \text{Im}(z_1\bar{z}_2), \quad y^2 = \text{Re}(z_1\bar{z}_2),
\]

and for these points the inequality is saturated thus implying that they lie on the surface of the ball of radius \( \frac{1}{2} \). We shall denote the set of density states by \( D \). This set is the convex hull of the sphere of pure states. For any \( \rho \in D \) there exist pure states \( \rho_1 \) and \( \rho_2 \) and a positive number \( p \) such that \( \rho = p\rho_1 + (1-p)\rho_2 \).

These states, points on the surface, are in one-to-one correspondence with the unit rays in \( \mathbb{C}^2 \) and the map is given by the momentum map associated with the symplectic action of \( U(2) \) on \( \mathcal{RH} \sim \mathbb{C}P^1 \). The ball of the density states is foliated by symplectic leaves associated with the coadjoint action of \( U(2) \), which coincide also with the orbits of the \( SU(2) \) group.

The analysis of these orbits may also be done by considering the orbits passing through diagonal matrices, in other terms

\[
\rho = S \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} S^\dagger \quad a + b = 1 \quad a \geq 0, \quad b \geq 0.
\]

We visualize the situation with the help of the following diagram: the segment connecting \( (\frac{1}{2}, \frac{1}{2}) \) with \((1,0)\) parametrizes the family of two dimensional spheres. The point \((\frac{1}{2}, \frac{1}{2})\) coincides with the center of the ball and \((1,0)\) belongs to the outmost sphere of pure states.
What we have described is usually known as the Bloch sphere representation of one qubit. The decomposition of a density states $\rho$, a point in the ball, as a convex sum of two pure states $\rho_1 = |\psi_1\rangle \langle \psi_1|$ and $\rho_2 = |\psi_2\rangle \langle \psi_2|$, is given geometrically by drawing a straight line through $\rho$: the states $\rho_1$ and $\rho_2$ are the intersections of the line with the sphere. Evidently this decomposition may be done in a two parameter family of ways.

6.4. **States of a three level system.** Now $\mathcal{H} = \mathbb{C}^3$. The states are normalized positive $3 \times 3$ matrices inside $u^*(3)$. We first consider the geometrical tensors defined by means of the momentum map construction. We choose a basis for $u(3)$ given by the Gell-Mann matrices

$$
\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},
$$

$$
\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.
$$
They satisfy the scalar product relation
\[ \text{Tr} \lambda_\mu \lambda_\nu = 2 \delta_{\mu\nu} \]

Their commutation and anti-commutation relations are written in terms of the antisymmetric structure constants and symmetric d–symbols \(d_{\mu\nu\rho}\). We find
\[
[\lambda_\mu, \lambda_\nu] = 2iC_{\mu\nu\rho}\lambda_\rho \\
[\lambda_\mu, \lambda_\rho]_+ = 2\sqrt{\frac{2}{3}} \lambda_0 \delta_{\mu\nu} + 2d_{\mu\nu\rho}\lambda_\rho.
\]

The numerical values turn out to be
\[
C_{123} = 1, \quad C_{458} = C_{678} = \frac{\sqrt{3}}{2} \\
C_{147} = -C_{156} = C_{246} = C_{257} = C_{345} = -C_{367} = \frac{1}{2}
\]
The values of these symbols show the different embeddings of \(SU(2)\) into \(SU(3) \subset U(3)\). For the other coefficients we have
\[
d_{j0} = -d_{0j} = \sqrt{\frac{2}{3}} \quad j = 1, \ldots, 8 \\
-d_{888} = d_{8jj} = d_{j88} = d_{j8j} = \frac{1}{\sqrt{3}} \quad j = 1, 2, 3 \\
d_{8jj} = d_{jj8} = d_{j8j} = -\frac{1}{2\sqrt{3}} \quad j = 4, 5, 6, 7 \\
d_{jj3} = d_{j3j} = d_{j3j} = \frac{1}{2} \quad j = 4, 5 \\
d_{3jj} = d_{jj3} = d_{j3j} = -\frac{1}{2} \quad j = 6, 7 \\
d_{146} = d_{157} = d_{164} = d_{175} = -d_{247} = d_{256} = d_{265} = -d_{274} = \frac{1}{2} \\
d_{416} = -d_{427} = d_{461} = -d_{472} = d_{517} = d_{526} = d_{562} = d_{571} = \frac{1}{2} \\
d_{614} = d_{625} = d_{641} = d_{652} = d_{715} = -d_{724} = d_{751} = -d_{742} = \frac{1}{2}
\]
The indices appearing in the non-null structure constants are identifying the corresponding \(\lambda\)-matrices whose pairwise commutators define \(SU(2)\)-subgroups. It is now possible to introduce coordinate functions
\[ y^\mu(A) = \frac{1}{2} \text{Tr} \lambda_\mu A. \]
In these coordinates, a generic Hermitian matrix \(A\) can be written as
\[ A = y^0 \lambda_0 + y^r \lambda_r \]
The vector \((y^0, \vec{y}) \in \mathbb{R}^9\) plays a similar role to the one we saw on \(U(2)\). Under conjugation with \(S \in SU(3)\), any matrix \(A\) can be written as
\[ A = S \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} S^t. \]
The scalar product induced on vectors on $\mathbb{R}^8$ will be invariant under the action of $SO(8)$. It is now possible to write the Poisson tensor
\[ \Lambda = 2C_{\mu\nu\rho}y^\rho \frac{\partial}{\partial y^\mu} \wedge \frac{\partial}{\partial y^\nu} \]
and the Riemann-Jordan tensor
\[ R = \frac{\partial}{\partial y^0} \otimes_s y^\mu \frac{\partial}{\partial y^\mu} + y^0 \frac{\partial}{\partial y^0} \otimes \frac{\partial}{\partial y^0} + d_{\mu\nu\rho}y^\rho \frac{\partial}{\partial y^\nu} \otimes_s \frac{\partial}{\partial y^\mu}. \]

Now the analysis of the various distributions is more cumbersome, however it is easy to identify a few elements:
\[ R(dy^0) = y^\mu \frac{\partial}{\partial y^\mu}, \]
which is the dilation vector field on $\mathbb{R}^9$; while
\[ R(dy^r) = y^r \frac{\partial}{\partial y^0} + y^0 \frac{\partial}{\partial y^r} + d_{\mu\nu\rho}y^\rho \frac{\partial}{\partial y^\nu}, \]
where it is possible to identify a boost structure plus a correction due to the $d$–symbols. In any case the union of the Hamiltonian distribution and the Riemannian-Jordan distribution generates $GL(3, \mathbb{C})$.

The set of states will again be identified as the subset of the Hermitian matrices which are normalized and satisfy the positivity condition. If we set
\[ \rho = \begin{pmatrix} a & \bar{h} & g \\ h & b & \bar{f} \\ \bar{g} & f & c \end{pmatrix} \quad a, b, c \in \mathbb{R} \quad f, g, h \in \mathbb{C}. \]

The conditions for $\rho$ to be a state are:
- $a + b + c = 1$
- $a \geq 0, b \geq 0, c \geq 0$.
- $|f|^2 \leq bc, |g|^2 \leq ca, |h|^2 \leq ab$.
- $\det \rho = abc + 2\text{Re}(fgh) - (a|f|^2 + b|g|^2 + c|h|^2) \geq 0$

These matrices form a convex set of $\mathbb{R}^8$. The trace condition allows to identify this subset as a subset of the vector space of $\mathbb{R}^8$ corresponding to the dual space of the Lie algebra of $SU(3)$.

Extremal states are in one-to-one correspondence with the minimal symplectic orbit of the unitary group according to the coadjoint action and corresponds to $\mathbb{C}P^2$, the complex projective space of $\mathbb{C}^3$.

Pure states, rank one projectors, are given by vectors $(z_1, z_2, z_3) \in \mathbb{C}^3$ with the normalization condition $z_1 \bar{z}_1 + z_2 \bar{z}_2 + z_3 \bar{z}_3 = 1$ as
\[ \begin{pmatrix} z_1 \bar{z}_1 & z_1 \bar{z}_2 & z_1 \bar{z}_3 \\ z_2 \bar{z}_1 & z_2 \bar{z}_2 & z_2 \bar{z}_3 \\ z_3 \bar{z}_1 & z_3 \bar{z}_2 & z_3 \bar{z}_3 \end{pmatrix} \]

Previous inequalities are saturated by these states.

These extremal states may be written in terms of the $\lambda$-matrices
\[ \rho_{\psi} = \frac{\langle \psi | \langle \psi \rangle}{\langle \psi, \psi \rangle} = \frac{1}{3}(I + \sqrt{3}n^a \lambda_a), \]
with $n^a n_a = 1$ and $n \star n = n$, the star product being
\[ (a \star b)_l = \sqrt{3}d_{ijk} a_j b_k \]
By using the “radial-angular” parametrization of states, say

$$\rho = s \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} s^\dagger \quad a \geq 0, b \geq 0, c \geq 0, \quad s \in SU(3),$$

we may study the structure of this union of symplectic orbits by considering the family of diagonal matrices with the positivity condition (elements of a positive Weyl chamber in the Abelian Cartan subalgebra). The hyperplane $\text{Tr}\rho = 1$ identifies a triangle with the intersection with positive axes $(Oa, Ob, Oc)$; i.e. in the positive octant.

Each internal point of the triangle corresponds to a 6–dimensional symplectic orbit, out of which we may consider convex combinations. Due to the action of $SU(3)$ containing the action of the discrete Weyl group, the symplectic orbits are actually parametrized by the following smaller triangle.
When $a = b = c = \frac{1}{3}$ we have the “maximally mixed state” which play a crucial role when we consider composite systems and entangled states (the orbit passing through this point degenerates to a zero dimensional orbit). On the boundary of the bigger triangle the rank of $\rho$ is either 1 or 2. However the orbits passing through these points are diffeomorphic to $\mathbb{CP}^2$. For a generic point, the orbits are diffeomorphic to $SU(3)/U(1) \times U(1)$. It appears quite clearly that the set of states is a stratified manifold characterized by the rank of the state. We shall not indulge further on the geometrical analysis and refer to the literature for further details and applications.

**References**

[1] M.C. Abbati, R. Cirelli, P. Lanzavecchia and A. Maniá, *Pure states of general quantum mechanical systems as Kähler bundle*, Nuovo Cimento B 83, pp 43–60, 1984
[2] J.S. Anandan *A Geometric approach to Quantum Mechanics* Found.Phys. 21, pp 1265-1284, 1991
[3] A. Ashtekar and T.A. Schilling, *Geometrical formulation of Quantum Mechanics, on Einstein’s path*, pp 23-65, New York: Springer, 1999
[4] A.Benvegnu, N. Sansonetto and M. Spera *Remarks on geometric quantum mechanics* J.Geom.Phys. 51, pp 229-243, 2004
[5] A. Bloch, *An infinite-dimensional Hamiltonian system on a projective Hilbert space*, Trans. AMS 302, pp 787–796, 1987
[6] D. Brody and L.P. Hughston, *Geometric quantum mechanics*, J. Geom. Phys. 38, pp 19–53, 2001
[7] R. Cirelli and P. Lanzavecchia, *Hamiltonian vector fields in Quantum Mechanics*, Nuovo Cimento B, 79, pp 271–283, 1984
[8] R. Cirelli, A. Maniá and L. Pizzocchero, J.Math.Phys. 31, pp 2891-2903, 1990 (part I and II)
[9] D. Cruscinski and A. Jamiołkowski *Geometric phases in classical and Quantum Mechanics* Birkhauser,Boston,2004
[10] P.A.M. Dirac, *The Principles of Quantum Mechanics*, Clarendon Press, Oxford, 2nd edition, 1936.
[11] G.G. Emch, *Foundations of 20th century Physics*, North Holland, Amsterdam, 1984
[12] A.M. Gleason, Measures on the closed subspaces of a Hilbert space, J. Math. Mech. 6, pp 885-893, 1957
[13] J. Grabowski, M. Kuś and G. Marmo, Geometry of quantum systems: density states and entanglement, J. Phys. A: Math. Gen. 38, pp 10217-10244, 2005
[14] J. Grabowski, M. Kuś and G. Marmo, Symmetry, group actions and entanglement, Open sys. & Information dyn. 13, pp 343-362, 2006
[15] A. Heslot, Quantum mechanics as a classical theory, Phys Rev D 31, pp 1341–1348, 1985
[16] N.P. Landsman, Mathematical topic between Classical and Quantum Mechanics, Springer-Verlag, 1998
[17] N.P. Landsman, Poisson spaces with a transition probability, Rev Math Phys 9, pp 29-57, 1997
[18] V. I. Manko, G. Marmo, E.C.G. Sudarshan and F. Zaccaria, The geometry of density states, Rep. Math. Phys 55, pp 405-422, 2005
[19] V. I. Manko, G. Marmo, E.C.G. Sudarshan and F. Zaccaria, Interference and entanglement: an intrinsic approach, J. Phys A: Math Gen 35, pp 7137-7157, 2002
[20] G. Marmo, G. Scolarici, A. Simoni, F. Ventriglia, The Quantum-Classical Transition: The Fate of the Complex Structure Int. J. Mod. Geom. Meth. Phys. 2, pp 127-145, 2005
[21] G. Marmo and G. Vilasi, Symplectic Structures and Quantum Mechanics Mod. Phys. Letters B10, pp 545, 1996
[22] N. Mukunda, Arvind, E. Ercolessi, G. Marmo, G. Morandi, R. Simon, Bargmann invariants, null phase curves and a theory of geometric phase, Phys Rev A 67, 042114, 2003
[23] D.J. Rowe, A. Ryman and G. Rosensteel, Many body quantum mechanics as a symplectic dynamical system, Phys Rev A 22, pp 2362-2372, 1980

BIFI-Universidad de Zaragoza, Coruna de Aragon 42, 50009 Zaragoza-SPAIN

Dipartamento di Scienze Fisiche, Universita Federico II and INFN- Sezione Napoli, Via Cintia I-80126 Napoli-ITALY