Complexity Analysis of Stein Variational Gradient Descent
Under Talagrand’s Inequality T1

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Abstract

We study the complexity of Stein Variational Gradient Descent (SVGD), which is an algorithm to sample from \( \pi(x) \propto \exp(-F(x)) \) where \( F \) smooth and nonconvex. We provide a clean complexity bound for SVGD in the population limit in terms of the Stein Fisher Information (or squared Kernelized Stein Discrepancy), as a function of the dimension of the problem \( d \) and the desired accuracy \( \varepsilon \). Unlike existing work, we do not make any assumption on the trajectory of the algorithm. Instead, our key assumption is that the target distribution satisfies Talagrand’s inequality T1.

1 Introduction

Sampling from a given target distribution \( \pi \) is a fundamental task of many Machine Learning procedures. In Bayesian Machine Learning, the target distribution \( \pi \) is known up to a multiplicative factor and often takes the form

\[
\pi(x) \propto \exp(-F(x)),
\]

(1)

where \( F : \mathcal{X} \to \mathbb{R} \) is a smooth nonconvex function defined on \( \mathcal{X} := \mathbb{R}^d \) satisfying

\[
\int \exp(-F(x)) dx < \infty.
\]

As sampling algorithms are intended to be applied to large scale problems, it has become increasingly important to understand their theoretical properties, such as their complexity, as a function of the dimension of the problem \( d \), and the desired accuracy \( \varepsilon \). In this regard, most of the Machine Learning literature has concentrated on understanding the complexity (in terms of \( d \) and \( \varepsilon \)) of (variants of) the Langevin algorithm in various settings, see e.g. the works of [2019], [2018], [2018], [2018], [2020], [2018], [2017], [2017], [2017], [2019], [2019], [2017], [2019], [2019], [2018], [2018].

1.1 Stein Variational Gradient Descent (SVGD)

Stein Variational Gradient Descent (SVGD) [2016], [2017] is an alternative to the Langevin algorithm and has been applied in several contexts in machine learning, including Reinforcement Learning [2017], sequential decision making [2018, 2019], Generative Adversarial Networks [2019], Variational Auto Encoders [2017], and Federated Learning [2019].

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However, the theoretical understanding of SVGD is limited compared to that of Langevin algorithm [Lu et al., 2019, Duncan et al., 2019, Liu, 2017, Chewi et al., 2020, Nüsken and Renger, 2021]. In particular, the first complexity result of SVGD, due to [Korba et al., 2020, Corollary 6], appeared only recently, and relies on an assumption on the trajectory of the algorithm, which cannot be checked prior to running the algorithm. Basically, this assumption is that the first moment of the iterates remain uniformly bounded by a constant $C$ at every iteration. Moreover, their complexity bound [Korba et al., 2020, Corollary 6] does not make the dependence on the dimension $d$ of the problem explicit, precisely because this bound depends on $C$.

### 1.2 Related works

Most of the existing results on SVGD deal with the continuous time approximation of SVGD (in the population limit) [Lu et al., 2019, Duncan et al., 2019, Liu, 2017, Nüsken and Renger, 2021], or with an approximation of that continuous time dynamics Chewi et al. [2020]. In particular, Duncan et al. [2019] propose a Stein logarithmic Sobolev inequality that implies the linear convergence of the continuous time dynamics. However, it is not yet understood when this inequality holds. Besides, Chewi et al. [2020] showed that the Wasserstein gradient flow of the chi-squared divergence can be seen as an approximation of the continuous dynamics behind SVGD, and showed linear convergence of the Wasserstein gradient flow of the chi-squared under Poincaré inequality. Other results, such as those of [Lu et al., 2019, Liu, 2017, Nüsken and Renger, 2021], include asymptotic convergence properties of the continuous time SVGD, but do not include convergence rates.

To our knowledge, the only existing complexity result (i.e., convergence rate) for SVGD in discrete time is due to [Korba et al., 2020]. They showed that, in the population limit, $O\left(\frac{L\varepsilon}{\varepsilon}\right)$ iterations of SVGD algorithm are sufficient to achieve $\varepsilon$ accuracy in terms of the Stein Fisher Information. However, this complexity result does not express the dependence in the dimension $d$. More importantly, this complexity result is established under the assumption that $\sup_n I_{\text{Stein}}(\mu_n|\pi) < \infty$, and the quantity $C = \sup_n I_{\text{Stein}}(\mu_n|\pi)$ appears in the hidden constants. In particular, the dependence of $C$ in $d$ and other parameters is not known.

Finally, the ML literature on the complexity of sampling from a non logconcave target distribution has many focused on the Langevin algorithm. In particular, Vempala and Wibisono [2019] showed that $O\left(\frac{L^2\lambda_2}{\varepsilon^2}\right)$ iterations of Langevin algorithm are sufficient to achieve $\varepsilon$ accuracy in terms of the KL-divergence, if the target distribution satisfies LSI, which is stronger than T1; see Table[1].

### 1.3 Contributions

Our paper intends to provide a clean analysis of SVGD, without any assumptions on the trajectory, and with complexity bounds depending on $\varepsilon$ and $d$. Instead, our key assumption is that the target distribution $\pi$ satisfies T1, the mildest of the Talagrand’s inequalities, which is satisfied under mild assumptions on the tail of the distribution; see Villani [2008, Theorem 22.10]. Moreover, T1 is implied, for example, by the Logarithmic Sobolev Inequality (LSI) [Villani, 2008, Theorem 22.17].

Assuming that the T1 inequality holds, we provide

- a convergence rate for SVGD in the so-called population limit, without assumption on the trajectory of the algorithm,

- a complexity bound for SVGD in terms of the dimension $d$ and the desired accuracy $\varepsilon$,

- a generic weak convergence result for SVGD.

All these results hold without assuming $F$ to be convex. Our main complexity result is summarized in Table[1].
Table 1: Comparison of known complexity results for Langevin and our complexity results for SVGD. Recall that the Logarithmic Sobolev Inequality (LSI) implies T1 with the same constant \( \lambda \); see Villani [2008, Theorem 22.17].

| Algorithm | Reference | Language | SVGD |
|-----------|-----------|----------|------|
|           | Vempala and Wibisono [2019, Theorem 1] | This paper | Theorem1 Corollary3 |

| Assumptions | LSI with constant \( \lambda \) \( F \) is \( L \)-smooth | T1 with constant \( \lambda \) \( F \) is \( L \)-smooth |
| Criterion | \( \text{KL}(\mu|\pi) \) | \( J_{\text{Stein}}(\mu|\pi) \) |
| Complexity | \( \tilde{O}\left(\frac{L^2d}{\lambda^2\varepsilon}\right) \) | \( \tilde{O}\left(\frac{L^d\varepsilon^2}{\lambda_1\varepsilon}\right) \) |

1.4 Paper structure

The remainder of the paper is organized as follows. In Section 2 we introduce the necessary mathematical and notational background on optimal transport, reproducing kernel Hilbert spaces and SVGD in order to be able to describe and explain our results. Section 3 is devoted to the development of our theory. Since this is a theoretical work, we include all key proofs in the paper. Finally, in Section 4 we formulate three corollaries of our key result, capturing weak convergence and complexity estimates for SVGD. We include the proofs of the first two corollaries here, and only leave the proof of Corollary 3 to the appendix.

2 Background and Notation

For any Hilbert space \( H \), we denote by \( \langle \cdot, \cdot \rangle_H \) the inner product of \( H \) and by \( \| \cdot \|_H \) its norm. Moreover, \( \| \cdot \|_{\text{op}} \) denotes the operator norm on the set of matrices.

2.1 Optimal transport

Consider \( p \geq 1 \). We denote by \( \mathcal{P}_p(\mathcal{X}) \) the set of Borel probability measures \( \mu \) over \( \mathcal{X} \) with finite \( p \)-th moment: \( \int \|x\|^p d\mu(x) < \infty \). We denote by \( L^p(\mu) \) the set of measurable functions \( f : \mathcal{X} \rightarrow \mathcal{X} \) such that \( \int \|f\|^p d\mu < \infty \). Note that the identity map \( I \) of \( \mathcal{X} \) satisfies \( I \in L^p(\mu) \) if \( \mu \in \mathcal{P}_p(\mathcal{X}) \). Moreover, denoting the image (or pushforward) measure of \( \mu \) by a map \( T \) as \( T\#\mu \), we have that if \( \mu \in \mathcal{P}_p(\mathcal{X}) \) and \( T \in L^p(\mu) \) then \( T\#\mu \in \mathcal{P}_p(\mathcal{X}) \) using the transfer lemma.

For every \( \mu, \nu \in \mathcal{P}_p(\mathcal{X}) \), the \( p \)-Wasserstein distance between \( \mu \) and \( \nu \) is defined by

\[
W_p^p(\mu, \nu) = \inf_{s \in S(\mu, \nu)} \int \|x - y\|^p ds(x, y),
\]

where \( S(\mu, \nu) \) is the set of couplings between \( \mu \) and \( \nu \), i.e., the set of nonnegative measures over \( \mathcal{X}^2 \) such that \( P\#s = \mu \) (resp. \( Q\#s = \nu \)) where \( P : (x, y) \mapsto x \) (resp. \( Q : (x, y) \mapsto y \)) denotes the projection onto the first (resp. the second) component. The \( p \)-Wasserstein distance is a metric over \( \mathcal{P}_p(\mathcal{X}) \). The metric space \( (\mathcal{P}_p(\mathcal{X}), W_2) \) is called the Wasserstein space.

In this paper, we consider a target probability distribution \( \pi \) proportional to \( \exp(-F) \), where \( F \) satisfies the following.

**Assumption 1.** The Hessian \( H_F \) is well-defined and \( \exists L \geq 0 \) such that \( \|H_F\|_{\text{op}} \leq L \).
To specify the dependence in the dimension $d$ of our complexity bounds, we need the following proposition.

**Proposition 1.** Under Assumptions [7], there exists $x_\ast \in \mathcal{X}$ for which $\nabla F(x_\ast) = 0$, i.e., $F$ admits a stationary point.

The task of sampling from $\pi$ can be formalized as an optimization problem. Indeed, define the Kullback-Leibler (KL) divergence as

$$KL(\mu | \pi) = \int \log \left( \frac{d\mu}{d\pi}(x) \right) d\mu(x),$$

if $\mu$ admits the density $\frac{d\mu}{d\pi}$ with respect to $\pi$, and $KL(\mu | \pi) = +\infty$ else. Then, $KL(\mu | \pi) \geq 0$ and $KL(\mu | \pi) = 0$ if and only if $\mu = \pi$. Therefore, assuming $\pi \in \mathcal{P}_2(\mathcal{X})$, the optimization problem

$$\min_{\mu \in \mathcal{P}_2(\mathcal{X})} F(\mu) := KL(\mu | \pi),$$

admits a unique solution, the distribution $\pi$. We will see in Section [3] that SVGD can be seen as an optimization algorithm to solve (4).

Indeed, the Wasserstein space can be endowed with a differential structure. In particular, when it is well defined, the Wasserstein gradient of the functional $F$ denoted by $\nabla_W F(\mu)$ is an element of $L^2(\mu)$ and satisfies $\nabla_W F(\mu) = \nabla \log \left( \frac{d\mu}{d\pi} \right)$.

The analysis of sampling algorithm in the case where $F$ is nonconvex often goes through functional inequalities.

**Definition 1** (Logarithmic Sobolev Inequality (LSI)). The distribution $\pi$ satisfies the Logarithmic Sobolev Inequality if there exists $\lambda > 0$ such that for all $\mu \in \mathcal{P}_2(\mathcal{X})$,

$$F(\mu) \leq \frac{2}{\lambda} \left\| \nabla_W F(\mu) \right\|^2_{L^2(\mu)}.$$

LSI is the key assumption in the analysis of Langevin algorithm in the case when $F$ is not convex [Vempala and Wibisono [2019]].

**Definition 2** (Talagrand’s Inequality $T_p$). Let $p \geq 1$. The distribution $\pi$ satisfies the Talagrand’s Inequality $T_p$ if there exists $\lambda > 0$ such that for all $\mu \in \mathcal{P}_p(\mathcal{X})$, we have $W_p(\mu, \pi) \leq \sqrt{\frac{2p(\mu)}{\lambda}}$.

We now claim that T1 is milder than LSI. Indeed, using $W_1(\mu, \pi) \leq W_2(\mu, \pi)$, T2 implies T1 with the same constant $\lambda$. Moreover, using [Villani, 2008, Theorem 22.17], LSI implies T2 with the same constant $\lambda$. In conclusion, LSI $\Rightarrow$ T2 $\Rightarrow$ T1, with the same constant $\lambda$.

Our key assumption on $\pi$ is that it satisfies the Talagrand’s inequality T1 [Villani, 2003, Definition 22.1].

**Assumption 2.** The target distribution $\pi$ satisfies T1.

The target distribution $\pi$ satisfies T1 if and only if there exist $a \in \mathcal{X}$ and $\beta > 0$ such that $\int \exp(\beta \|x - a\|^2) d\pi(x) < \infty$, see [Villani, 2008, Theorem 22.10]. Therefore, Assumption 2 is essentially an assumption on the tails of $\pi$. In particular, $\pi \in \mathcal{P}_2(\mathcal{X})$.

Somehow, T1 allows $F$ to be "more nonconvex" than LSI. Moreover, if $F$ is $\lambda$-strongly convex, then LSI and T2 hold with constant $\lambda$. 


2.2 Reproducing Kernel Hilbert Space

We consider a kernel $k$ associated to a Reproducing Kernel Hilbert Space (RKHS) denoted by $\mathcal{H}_0$. We denote the so-called feature map $x \mapsto k(x, \cdot)$ by $\Phi(x) := k(\cdot, x) \in \mathcal{H}_0$, for every $x \in \mathcal{X}$. The Hilbert space $\mathcal{H}_0^d$ is denoted by $\mathcal{H} := \mathcal{H}_0^d$. We make the following assumption on the kernel $k$.

**Assumption 3.** There exists $B > 0$ such that the inequalities

$$\|\Phi(x)\|_{\mathcal{H}_0} \leq B \quad \text{and} \quad \|\nabla \Phi(x)\|_{\mathcal{H}} = \left(\sum_{i=1}^{d} \|\partial_i \Phi(x)\|_{\mathcal{H}_0}^2\right)^{\frac{1}{2}} \leq B$$

hold for all $x \in \mathcal{X}$.

Under Assumption 3, $\mathcal{H} \subset L^2(\mu)$ for every probability distribution on $\mathcal{X}$, and the inclusion map $\iota_\mu : \mathcal{H} \to L^2(\mu)$ is continuous. We denote by $P_\mu$ its adjoint defined by the relation: for every $f \in L^2(\mu)$, $g \in \mathcal{H}$,

$$\langle f, \iota_\mu g \rangle_{L^2(\mu)} = \langle P_\mu f, g \rangle_{\mathcal{H}}.$$  \hspace{1cm} (5)

Then, $P_\mu$ can be expressed as a convolution with $k$ [Carmeli et al., 2010, Proposition 3]:

$$P_\mu f(x) = \int k(x, y) f(y) d\mu(y), \quad \text{or} \quad P_\mu f = \int \Phi(y) f(y) d\mu(y) \hspace{1cm} (6)$$

where the integral converges in norm.

2.3 Stein Variational Gradient Descent

Stein Variational Gradient Descent (SVGD) is an algorithm to sample from $\pi \propto \exp(-F)$. SVGD proceeds by maintaining a set of particles over $\mathbb{R}^d$, whose empirical distribution $\mu_n$ at time $n$ aims to approximate $\pi$ as $n \to \infty$, see Liu and Wang [2016]. In this paper, we analyze SVGD in the so-called population limit, where the number of particles is infinite. In this limit, the distribution $\mu_n$ follows the dynamics

$$\mu_{n+1} = (I - \gamma h_{\mu_n}) \# \mu_n, \hspace{1cm} (7)$$

where

$$h_{\mu}(x) := \int k(x, y) \nabla F(y) - \nabla_y k(x, y) d\mu(y)$$

or

$$h_{\mu} := \int \nabla F(y) \Phi(y) - \nabla \Phi(y) d\mu(y).$$

**Our point of view on SVGD.** We now provide the intuition behind our results on SVGD.

In the population limit, SVGD can be seen as a Riemannian gradient descent, thanks to the following two reasons.

First, in a Riemannian interpretation of the Wasserstein space Villani [2008], for every $\mu \in \mathcal{P}_2(\mathcal{X})$, the map $\exp_\mu : \phi \mapsto (I + \phi) \# \mu$ can be seen as the exponential map at $\mu$. SVGD (7) can be rewritten as

$$\mu_{n+1} = \exp_{\mu_n}(-\gamma h_{\mu_n}).$$

Second, $-h_{\mu}$ can be seen as the negative gradient of $F$ at $\mu$ under a certain metric. Indeed, using integration by parts, $h_{\mu} = P_\mu \nabla_W F(\mu)$, see e.g. Korba et al. [2020], Duncan et al. [2019]. Therefore, for
every \( g \in \mathcal{H} \), \( \langle h_\mu, g \rangle_\mathcal{H} = \langle \nabla W \mathcal{F}(\mu), g \rangle_{L^2(\mu)} \), hence \( h_\mu \) can be seen as a Wasserstein gradient of \( \mathcal{F} \) under the inner product of \( \mathcal{H} \).

The Kernelized Stein Discrepancy (KSD) is a natural discrepancy between probability distributions that is used in the context of SVGD (to measure the convergence of the algorithm), see Liu and Wang [2016], Liu [2017]. The KSD is defined as the square root of the Stein Fisher Information [Duncan et al. 2019]

\[
I_{\text{Stein}}(\mu|\pi) := \|h_\mu\|_\mathcal{H}^2.
\] (8)

In this paper, we study the complexity of SVGD in terms of the Stein Fisher Information. Indeed, \( I_{\text{Stein}}(\mu|\pi) = 0 \) implies \( \mu = \pi \) if \( \mathcal{H} \) is rich enough [Liu et al. 2016, Chwialkowski et al. 2016, Oates et al. 2019], and, under further conditions on \( \nabla F \) and \( k \), we even have that \( I_{\text{Stein}}(\mu_n|\pi) \to 0 \) is equivalent to \( \mu_n \to \pi \) weakly [Gorham and Mackey, 2017, Theorem 8], see Section 4.

## 3 Analysis of SVGD

In this section, we analyze SVGD in the infinite number of particles regime. Recall that in this regime, SVGD is given by

\[
\mu_{n+1} = (I - \gamma h_\mu_n) \# \mu_n,
\]

where

\[
h_\mu := \int \nabla F(x) \Phi(x) - \nabla \Phi(x) d\mu(x).
\]

### 3.1 Notation

The Jacobian of a function \( \phi : \mathcal{X} \to \mathcal{X} \) is denoted by \( J\phi \). For every \( x \in \mathcal{X} \), \( J\phi(x) \) can be seen as a \( d \times d \) matrix. For any \( d \times d \) matrix \( A \), \( \|A\|_{\text{HS}} \) denotes the Hilbert Schmidt norm of \( A \) and by \( \|A\|_{\text{op}} \) the operator norm of \( A \) viewed as a linear operator \( A : \mathcal{X} \to \mathcal{X} \) (where \( \mathcal{X} \) is endowed with the standard Euclidean inner product). Finally, \( \delta_x \) is the Dirac measure at \( x \in \mathcal{X} \).

### 3.2 A fundamental inequality

We start by stating a fundamental inequality satisfied by \( \mathcal{F} \) for any update of the form

\[
\mu_{n+1} = (I - \gamma g) \# \mu_n,
\]

where \( g \in \mathcal{H} \).

**Proposition 2.** Let Assumptions 1 and 3 hold true. Let \( \alpha > 1 \) and choose \( \gamma > 0 \) such that \( \gamma \|g\|_\mathcal{H} \leq \frac{\alpha - 1}{\alpha B} \).

Then,

\[
\mathcal{F}(\mu_{n+1}) \leq \mathcal{F}(\mu_n) - \gamma \langle h_{\mu_n}, g \rangle_\mathcal{H} + \frac{\gamma^2 K}{2} \|g\|_\mathcal{H}^2,
\]

where \( K = (\alpha^2 + L)B \).

Inequality (10) is a property of the functional \( \mathcal{F} \), and not a property of the SVGD algorithm. Inequality (10) plays the role of a Taylor inequality, where \( h_{\mu_n} \) is the Wasserstein gradient of \( \mathcal{F} \) at \( \mu_n \) under the metric induced by \( \mathcal{H} \). Proposition 2 generalizes [Korba et al. 2020, Proposition 5]. Indeed, [Korba et al. 2020, Proposition 5] can be obtained from our Proposition 2 by taking \( g = h_{\mu_n} \), by assuming the uniform
We now identify each term. First, $\phi$ and $t$. Hence,

$$x \in X, \quad 0 \leq 1, \quad \alpha B \leq.$$  

Proof. Let $\phi_t = I - t g$ for $t \in [0, \gamma]$ and $\rho_t = (\phi_t) \# \mu_n$. Note that $\rho_0 = \mu_n$ and $\rho_1 = \mu_{n+1}$. First, for every $x \in X$,

$$\|g(x)\|^2 = \sum_{i=1}^{d} \langle k(x, .), g_i \rangle_{\mathcal{H}_0}^2 \leq \|k(x, .)\|^2 \|g\|^2_{\mathcal{H}} \leq B^2 \|g\|^2_{\mathcal{H}}$$  

and

$$\|Jg(x)\|_{\text{HS}}^2 = \sum_{i,j=1}^{d} \left| \frac{\partial g_i(x)}{\partial x_j} \right|^2 = \sum_{i,j=1}^{d} \langle \partial_{x_j} k(x, .), g_i \rangle_{\mathcal{H}_0}^2 \leq \sum_{i,j=1}^{d} \|\partial_{x_j} k(x, .)\|^2_{\mathcal{H}_0} \|g_i\|^2_{\mathcal{H}_0} = \|\nabla k(x, .)\|^2_{\mathcal{H}} \|g\|^2_{\mathcal{H}} \leq B^2 \|g\|^2_{\mathcal{H}}.$$  

Hence,

$$\|t Jg(x)\|_{\text{op}} \leq \|t Jg(x)\|_{\text{HS}} \leq \gamma B \|g\|_{\mathcal{H}} \leq \frac{\alpha - 1}{\alpha} < 1,$$  

using our assumption on the step size $\gamma$. Inequality (13) proves that $\phi_t$ is a diffeomorphism for every $t \in [0, \gamma]$. Moreover,

$$\|(J \phi_t(x))^{-1}\|_{\text{op}} \leq \sum_{k=0}^{\infty} \|t Jg(x)\|^k_{\text{op}} \leq \sum_{k=0}^{\infty} \left( \frac{\alpha - 1}{\alpha} \right)^k = \alpha.$$  

Using [Villani, 2003, Theorem 5.34], the velocity field ruling the time evolution of $\rho_t$ is $w_t \in L^2(\rho_t)$ defined by $w_t(x) = -g(\phi_t^{-1}(x))$. Denote $\varphi(t) = \mathcal{F}(\rho_t)$. Using a Taylor expansion,

$$\varphi(\gamma) = \varphi(0) + \gamma \varphi'(0) + \int_0^\gamma (\gamma - t) \varphi''(t) dt.$$  

We now identify each term. First, $\varphi(0) = \mathcal{F}(\mu_n)$ and $\varphi(\gamma) = \mathcal{F}(\mu_{n+1})$. Then,

$$\varphi'(0) = -\langle h_{\mu_n}, g \rangle_{\mathcal{H}},$$  

and $\varphi''(t) = \psi_1(t) + \psi_2(t)$, where

$$\psi_1(t) = \mathbb{E}_{x \sim \rho_t}[\langle w_t(x), H_F(x)w_t(x) \rangle] \quad \text{and} \quad \psi_2(t) = \mathbb{E}_{x \sim \rho_t}[\|J w_t(x)\|^2_{\text{HS}}].$$  

1Note that assuming $\gamma C \frac{2}{\alpha} \leq \frac{\alpha - 1}{\alpha B}$ is more restrictive than assuming $\gamma \|h_{\mu_n}\|_{\mathcal{H}} \leq \frac{\alpha - 1}{\alpha B}$.

2Equations (16) and (17) are the equations that would deserve more explanations. We only sketch the proof, see [Korba et al., 2020, Lemma 11] for more details.
Recall that \( w_t = -g \circ (\phi_t)^{-1} \). The first term \( \psi_1(t) \) is bounded using the transfer lemma, Assumption I and Inequality (11):
\[
\psi_1(t) = \mathbb{E}_{x \sim \mu_n} [(g(x), H_V(\phi_t(x))g(x))] \leq L \|g\|_{L^2(\mu_n)}^2 \leq L B^2 \|g\|_{\mathcal{H}}^2.
\]
For the second term \( \psi_2(t) \), using the chain rule, \(-Jw_t \circ \phi_t = Jg(J\phi_t)^{-1}\). Therefore,
\[
\|Jw_t \circ \phi_t(x)\|_{\mathcal{H}}^2 \leq \|Jg(x)\|_{\mathcal{H}}^2 \|J(\phi_t)^{-1}(x)\|_{\mathcal{H}}^2 \leq \alpha^2 B^2 \|g\|_{\mathcal{H}}^2,
\]
using (12) and (14). Combining each of the quantity in the Taylor expansion (15) gives the desired result. □

### 3.3 Two lemmas

As mentioned in the last section, Proposition 2 can be applied to SVGD, i.e., the update
\[
\mu_{n+1} = (I - \gamma h_{\mu_n}) \# \mu_n,
\]
by setting \( g = h_{\mu_n} \in \mathcal{H} \). In this case, we obtain the following.

**Lemma 1.** Let Assumptions 1 and 3 hold true. Let \( \alpha > 1 \) and choose \( \gamma > 0 \) such that \( \gamma \|h_{\mu_n}\|_{\mathcal{H}} \leq \frac{\alpha-1}{\alpha B} \). Then,
\[
\mathcal{F}(\mu_{n+1}) \leq \mathcal{F}(\mu_n) - \gamma \left( 1 - \frac{\gamma K}{2} \right) \|h_{\mu_n}\|_{\mathcal{H}}, \tag{18}
\]
where \( K = (\alpha^2 + L)B \).

Contrary to Inequality (10), Inequality (18) is a property of the SVGD algorithm. In the language of the gradient descent algorithms, Inequality (18) is called a descent property.

**Lemma 2.** Let Assumptions 1, 2 and 3 hold true. Then, for every \( \mu \in \mathcal{P}_2(\mathcal{X}) \), we have
\[
\|h_{\mu}\|_{\mathcal{H}} \leq B \left( 1 + \|\nabla F(0)\| + L \int \|x\| d\pi(x) \right) + BL \sqrt{\frac{2F(\mu)}{\lambda}}
\]
and
\[
\|h_{\mu}\|_{\mathcal{H}} \leq B \left( 1 + L \sqrt{\frac{2F(\mu)}{\lambda}} + L \sqrt{\frac{2F(\mu)}{\lambda}} + L \int \|x - x\| d\mu_0(x) \right).
\]

**Proof:** Using Assumption 3,
\[
\|h_{\mu}\|_{\mathcal{H}} = \|E_{x \sim \mu} (\nabla F(x)\Phi(x) - \nabla \Phi(x))\|_{\mathcal{H}}
\leq E_{x \sim \mu} \|\nabla F(x)\Phi(x) - \nabla \Phi(x)\|_{\mathcal{H}}
\leq E_{x \sim \mu} \|\nabla F(x)\|_{\mathcal{H}} + E_{x \sim \mu} \|\nabla \Phi(x)\|_{\mathcal{H}}
= E_{x \sim \mu} \|\nabla F(x)\| \|\Phi(x)\|_{\mathcal{H}} + E_{x \sim \mu} \|\nabla \Phi(x)\|_{\mathcal{H}}
\leq B (E_{x \sim \mu} \|\nabla F(x)\| + 1).
\]

Using Assumption I, \( \|\nabla F(x)\| \leq \|\nabla F(0)\| + L \|x\| \). Therefore, using the triangle inequality for the metric \( W_1 \),
\[
\|h_{\mu}\|_{\mathcal{H}} \leq B \left( 1 + \|\nabla F(0)\| + L \int \|x\| d\mu(x) \right)
= B (1 + \|\nabla F(0)\| + LW_1(\mu, \delta_0))
\leq B (1 + \|\nabla F(0)\| + LW_1(\pi, \delta_0)) + BLW_1(\mu, \pi).
\]
We obtain the first inequality using Assumption 2:  
\[ W_1(\mu, \pi) \leq \sqrt{\frac{2F(\mu_0)}{\lambda}}. \]

To prove the second inequality, recall that \[ \|h_\mu\|_H \leq B \left( \mathbb{E}_{x \sim \mu} \|\nabla F(x)\| + 1 \right). \] Using Assumption 1 and Proposition 1, \[ \|\nabla F(x)\| = \|\nabla F(x) - \nabla F(x^*)\| \leq L\|x - x^*\|. \] Therefore, using the triangle inequality for the metric \( W_1 \),

\[
\int \|x - x^*\|d\mu(x) = W_1(\mu, \delta_{x^*}) \leq W_1(\mu, \pi) + W_1(\pi, \mu_0) + W_1(\mu_0, \delta_{x^*}) 
\]

Therefore,

\[
\|h_\mu\|_H \leq B \left( 1 + L \int \|x - x^*\|d\mu(x) \right) 
\]

\[
\leq B \left( 1 + L\sqrt{\frac{2F(\mu_0)}{\lambda}} + L\sqrt{\frac{2F(\mu)}{\lambda}} + LW_1(\mu_0, \delta_{x^*}) \right). \tag{19}
\]

\[ \]

3.4 Main result

Having established Proposition 2 and Lemmas 1 and 2, we are now ready to formulate and prove our main result.

**Theorem 1** (Descent lemma). *Let Assumptions 1, 2, and 3 hold true. Let \( \alpha > 1 \). If

\[
\gamma \leq (\alpha - 1) \left( \alpha B^2 \left( 1 + \|\nabla F(0)\| + L \int \|x\|d\pi(x) \right) + L\sqrt{\frac{2F(\mu_0)}{\lambda}} \right)^{-1}, \tag{20}
\]

or

\[
\gamma \leq (\alpha - 1) \left( \alpha B^2 \left( 1 + 2L\sqrt{\frac{2F(\mu_0)}{\lambda}} + L \int \|x - x^*\|d\mu_0(x) \right) \right)^{-1}, \tag{21}
\]

then

\[
F(\mu_{n+1}) \leq F(\mu_n) - \gamma \left( 1 - \gamma B(\alpha^2 + L) \right) I_{\text{Stein}}(\mu_n|\pi). \tag{22}
\]

Unlike in the work of [Korba et al. 2020], under Assumption 2 we prove that \( \gamma \) does not rely on the path information which makes our descent lemma self-contained.

**Proof.** We now prove by induction the first implication of Theorem 1 \((20) \Rightarrow (22)\). First, if \( \gamma > 0 \) satisfies \( (20) \), then, using Lemma 2, \( \gamma \|h_{\mu_0}\|_H \leq \frac{\gamma}{2}. \) Therefore, using Lemma 1

\[
F(\mu_1) \leq F(\mu_0) - \gamma \left( 1 - \frac{\gamma K}{2} \right) \|h_{\mu_0}\|_H^2,
\]

i.e., Inequality \( (22) \) holds with \( n = 0 \). Now, assume that the condition \( (20) \) implies Inequality \( (22) \) for every \( n \in \{0, \ldots, N - 1\} \) and let us prove it for \( n = N \). First, \( F(\mu_N) \leq F(\mu_0) \). Letting \( A := B \left( 1 + \|\nabla F(0)\| + L \int \|x\|d\pi(x) \right) \), this implies

\[
A + BL\sqrt{\frac{2F(\mu_N)}{\lambda}} \leq A + BL\sqrt{\frac{2F(\mu_0)}{\lambda}}.
\]
Therefore, if $\gamma > 0$ satisfies (20), then $\gamma \| h_{\mu_N} \|_H \leq \frac{\alpha - 1}{\alpha B}$. To see this, using Lemma 2 we obtain

$$\gamma \| h_{\mu_N} \|_H \leq \gamma \left( A + BL\lambda \sqrt{\mathcal{F}(\mu_N)} \right) \leq \gamma \left( A + BL\lambda \sqrt{\mathcal{F}(\mu_0)} \right) \leq \frac{\alpha - 1}{\alpha B}.$$ 

Therefore, using Lemma 1 the condition (20) implies Inequality (22) at step $n = N$:

$$\mathcal{F}(\mu_{N+1}) \leq \mathcal{F}(\mu_N) - \gamma \left( 1 - \frac{\gamma K}{2} \right) \| h_{\mu_N} \|_H^2.$$ 

Finally, it remains to recall that $\| h_{\mu_N} \|_H^2 = I_{\text{Stein}}(\mu_N | \pi)$. The proof of the second implication of Theorem 1, $(21) \Rightarrow (22)$, is similar. 

\[\square\]

### 4 Weak Convergence and Complexity

#### 4.1 Weak convergence

We now show that Theorem 1 implies weak convergence if $I_{\text{Stein}}(\mu_n | \pi) \to_{n \to +\infty} 0$ implies $\mu_n \to_{n \to +\infty} \pi$ weakly. This condition is studied in [Gorham and Mackey, 2017].

**Corollary 1 (Weak convergence).** Let Assumptions 1, 2 and 3 hold true. Assume moreover that $I_{\text{Stein}}(\mu_n | \pi) \to_{n \to +\infty} 0$ implies that $\mu_n \to_{n \to +\infty} \pi$ weakly. Let $\alpha > 1$. If $\gamma < \frac{2}{B(\alpha^2 + L)}$, and $\gamma$ further satisfies either (20) or (21), then $\mu_n \to_{n \to +\infty} \pi$ weakly.

**Proof.** Using Theorem 1 and iterating,

$$\mathcal{F}(\mu_n) \leq \mathcal{F}(\mu_0) - \gamma \left( 1 - \frac{\gamma B(\alpha^2 + L)}{2} \right) \sum_{k=0}^{n-1} I_{\text{Stein}}(\mu_k | \pi),$$

therefore, for every $n \geq 1$,

$$\gamma \left( 1 - \frac{\gamma B(\alpha^2 + L)}{2} \right) \sum_{k=0}^{n-1} I_{\text{Stein}}(\mu_k | \pi) \leq \mathcal{F}(\mu_0).$$

Consequently, $\sum_{n=0}^{+\infty} I_{\text{Stein}}(\mu_n | \pi) < \infty$. Therefore $I_{\text{Stein}}(\mu_n | \pi) \to_{n \to +\infty} 0$ and $\mu_n \to_{n \to +\infty} \pi$ weakly. 

\[\square\]

Conditions under which $I_{\text{Stein}}(\mu_n | \pi) \to_{n \to +\infty} 0 \Rightarrow \mu_n \to_{n \to +\infty} \pi$ can be found in [Gorham and Mackey, 2017, Theorem 8]. In particular, a sufficient condition is the combination the two following properties:

- $\pi$ being distant dissipative. For instance, $\pi$ is a finite Gaussian mixture with common covariance or $F$ is strongly convex outside a compact set (note that in this case, Assumption 2 is satisfied using [Villani, 2008, Theorem 22.10])

- $k$ is an inverse multiquadratic kernel, i.e., $k(x, y) = (c^2 + \|x - y\|^2)^\beta$ for some $c > 0$ and $\beta \in (-1, 0)$ (note that Assumption 3 is satisfied).
4.2 Complexity

Next, we provide an $O(1/n)$ convergence rate for the empirical mean of the squared Kernel Stein Discrepancy, obtained from our descent lemma (Theorem 1).

**Corollary 2 (Convergence rate).** Let Assumptions 1, 2, and 3 hold true. Let $\alpha > 1$. If $\gamma < \frac{2}{B(\alpha^2 + L)}$, and $\gamma$ further satisfies either (20) or (21), then

$$\frac{1}{n} \sum_{k=0}^{n-1} I_{\text{Stein}}(\mu_k | \pi) \leq \frac{2F(\mu_0)}{n\gamma}.$$  \hspace{1cm} (23)

**Proof.** Using Theorem 1, $F(\mu_{n+1}) \leq F(\mu_n) - \gamma I_{\text{Stein}}(\mu_n | \pi)$, and by iterating, we get

$$0 \leq F(\mu_n) \leq F(\mu_0) - \frac{\gamma}{2} \sum_{k=0}^{n-1} I_{\text{Stein}}(\mu_k | \pi).$$

We obtain the result by rearranging the terms.

Compared with the Langevin algorithm, whose complexity under LSI is $\tilde{O}(\frac{L^2d}{\lambda^2\varepsilon})$ in terms of KL, our complexity result $\tilde{O}(\frac{L^2d}{\lambda^2\varepsilon})$ for SVGD applies under T1 only, which is weaker than LSI, see Table 1.

**Corollary 3 (Complexity).** Let Assumptions 1, 2, and 3 hold true. Let $\alpha > 1$. If

$$\gamma \leq \min \left( (\alpha - 1) \left( \alpha B^2 \left( 1 + 2L \sqrt{\frac{2}{\lambda}} \sqrt{F(x_*) + \frac{d}{2} \log \left( \frac{L}{2\pi} \right) + \sqrt{Ld}} \right) \right)^{-1}, \frac{2}{B(\alpha^2 + L)} \right),$$

and if $\mu_0 = \mathcal{N}(x_*, \frac{1}{L}I)$, then

$$n = \tilde{\Theta} \left( \frac{Ld^{3/2}}{(\lambda/2\varepsilon)} \right)$$

iterations of SVGD are sufficient to output $\mu$ such that $I_{\text{Stein}}(\mu | \pi) \leq \varepsilon$.

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Appendix

A Proof of Proposition 1

First, we prove that $F$ is coercive, i.e., for every $C > 0$, the set $S = \{x \in X : F(x) \leq C\}$ is compact. Since $F$ is continuous, $S$ is closed. It remains to prove that $S$ is bounded. Assume, by contradiction, that $S$ is unbounded. Then, there exists a sequence $(x_n)$ of points in $X$ such that $F(x_n) \leq C$, $\|x_n\| \to +\infty$ and $B(x_n) \cap B(x_m) = \emptyset$ for every $n \neq m$, where $B(x)$ denotes the unit ball centered at $x$.

Let $n \geq 0$. Using the smoothness of $F$ (Assumption 1), for every $x \in B(x_n)$,

$$F(x) \leq F(x_n) + \langle \nabla F(x_n), x - x_n \rangle + \frac{L}{2}.$$ 

Denote by $V$ the volume of the unit ball centered at $x$, i.e., its Lebesgue measure. The positive number $V$ does not depend on $x$. Then

$$\int_{B(x_n)} \exp(-F(x)) dx \geq \int_{B(x_n)} \exp \left( -F(x_n) - \langle \nabla F(x_n), x - x_n \rangle - \frac{L}{2} \right) dx = V \exp \left( -F(x_n) - \frac{L}{2} \right) \int_{B(x_n)} \exp \left( \langle \nabla F(x_n), x_n - x \rangle \right) \frac{dx}{V} = V \exp \left( -F(x_n) - \frac{L}{2} \right) \int_{B(0)} \exp \left( \langle \nabla F(x_n), u \rangle \right) \frac{du}{V} \geq V \exp \left( -F(x_n) - \frac{L}{2} \right) \int_{B(0)} \exp \left( \langle \nabla F(x_n), u \rangle \right) \frac{du}{V} \geq V \exp \left( -C - \frac{L}{2} \right),$$

where we used Jensen’s inequality for the uniform distribution over $B(0)$, thanks to the convexity of $t \mapsto \exp(t)$. Finally,

$$\int \exp(-F(x)) dx \geq \sum_{n=0}^\infty \int_{B(x_n)} \exp(-F(x)) dx \geq \sum_{n=0}^\infty V \exp \left( -C - \frac{L}{2} \right) = +\infty,$$

which means that $\exp(-F)$ is not integrable. This contradicts the definition of $F$ and therefore, $S$ is bounded.

Next, since the set $S$ is compact, $F$ is coercive, and hence $F$ admits a stationary point. Indeed, $F$ is continuous over the compact set $\{x \in X : F(x) \leq 1\}$, and therefore, $F$ admits a minimizer, $x_*$, over this set. Moreover, this point $x_*$ is a stationary point i.e., $\nabla F(x_*) = 0$ (note that the point $x_*$ is actually a global minimizer of $F$).

B Proof of Corollary 3

Using Corollary 2 if

$$\gamma \leq \min \left( \alpha - 1, \alpha B^2 \left( 1 + 2L \sqrt{\frac{2F(\mu_0)}{\lambda}} + L \int \|x - x_*\| d\mu_0(x) \right) \right)^{-1}, \frac{2}{B(\alpha^2 + L)}$$,

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then, denoting \( p = \arg \min_{k \in \{0, \ldots, n-1\}} I_{\text{Stein}}(\mu_k | \pi) \),

\[
I_{\text{Stein}}(\mu_p | \pi) \leq \frac{1}{n} \sum_{k=0}^{n-1} I_{\text{Stein}}(\mu_k | \pi) \leq \frac{2F(\mu_0)}{n\gamma}.
\]

Using [Vempala and Wibisono, 2019, Lemma 1], \( F(\mu_0) \leq F(x_\star) + \frac{d}{2} \log \left( \frac{L}{2\pi} \right) \). Besides,

\[
\int \|x - x_\star\| d\mu_0(x) = \mathbb{E}_{X \sim \mu_0} \|X - x_\star\| = \frac{1}{\sqrt{L}} \mathbb{E}_{X \sim \mu_0} \|\sqrt{L}(X - x_\star)\|,
\]

and using the transfer lemma and Cauchy-Schwartz inequality,

\[
\int \|x - x_\star\| d\mu_0(x) = \frac{1}{\sqrt{L}} \mathbb{E}_{Y \sim N(0, I)} \|Y\| \leq \frac{1}{\sqrt{L}} (\mathbb{E}_{Y \sim N(0, I)} \|Y\|^2)^{1/2} = \sqrt{\frac{d}{L}}.
\]

Therefore,

\[
(\alpha - 1) \left( \alpha B^2 \left( 1 + 2L \sqrt{\frac{2F(\mu_0)}{\lambda}} + L \int \|x - x_\star\| d\mu_0(x) \right) \right)^{-1} \geq (\alpha - 1) \left( \alpha B^2 \left( 1 + 2L \sqrt{\frac{2}{\lambda} F(x_\star) + \frac{d}{2} \log \left( \frac{L}{2\pi} \right) + \sqrt{Ld}} \right) \right)^{-1} = \tilde{\Omega} \left( \frac{1}{\frac{L\sqrt{d}}{\sqrt{\lambda}} + \sqrt{Ld}} \right),
\]

and

\[
\gamma^{-1} = \tilde{O} \left( \frac{L\sqrt{d}}{\sqrt{\lambda}} + \sqrt{Ld} + L \right) = \tilde{O} \left( \frac{L\sqrt{d}}{\sqrt{\lambda}} \right).
\]

Since \( F(\mu_0) = \tilde{O}(d) \),

\[
\frac{F(\mu_0)}{\gamma} = \tilde{O} \left( \frac{Ld^{3/2}}{\sqrt{\lambda}} \right).
\]

Let \( \varepsilon > 0 \). To output \( \mu_p \) such that \( I_{\text{Stein}}(\mu_p | \pi) < \varepsilon \), it suffices to ensure that \( \frac{2F(\mu_0)}{n\gamma} < \varepsilon \). Therefore,

\[
\frac{2F(\mu_0)}{\gamma\varepsilon} = \tilde{\Omega} \left( \frac{Ld^{3/2}}{\varepsilon\sqrt{\lambda}} \right) \text{ iterations are sufficient.}
\]