PROJECTIVE VARIETIES WITH BAD SEMI-STABLE REDUCTION AT 3 ONLY

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Abstract. Suppose $F = W(k)[1/p]$ where $W(k)$ is the ring of Witt vectors with coefficients in algebraically closed field $k$ of characteristic $p \neq 2$. We construct integral theory of $p$-adic semi-stable representations of the absolute Galois group of $F$ with Hodge-Tate weights from $[0, p)$. This modification of Breuil’s theory results in the following application in the spirit of Shafarevich’s Conjecture.

If $Y$ is a projective algebraic variety over $\mathbb{Q}$ with good reduction modulo all primes $l \neq 3$ and semi-stable reduction modulo 3 then for the Hodge numbers of $Y_C = Y \otimes_{\mathbb{Q}} \mathbb{C}$, it holds $h^2(Y_C) = h^{1,1}(Y_C)$.

Introduction

Everywhere in the paper $p$ is a fixed prime number, $p \neq 2$, $k$ is algebraically closed field of characteristic $p$, $F$ is the fraction field of the ring of Witt vectors $W(k)$, $\bar{F}$ is a fixed algebraic closure of $F$ and $\Gamma_F = \text{Gal}(\bar{F}/F)$ is the absolute Galois group of $F$.

Suppose $Y$ is a projective algebraic variety over $\mathbb{Q}$. Denote by $Y_C$ the corresponding complex variety $Y \otimes_{\mathbb{Q}} \mathbb{C}$. For integers $n, m \geq 0$, set $h^n(Y_C) = \dim_{\mathbb{C}} H^n(Y_C, \mathbb{C})$ and $h^{n,m}(Y_C) = \dim_{\mathbb{C}} H^n(\Omega^m_{Y_C})$.

The main result of this paper can be stated as follows.

Theorem 0.1. If $Y$ has semi-stable reduction modulo 3 and good reduction modulo all primes $l \neq 3$ then $h^2(Y_C) = h^{1,1}(Y_C)$.

Remind that a generalization of the Shafarevich Conjecture about the non-existence of non-trivial abelian varieties over $\mathbb{Q}$ with everywhere good reduction was proved by Fontaine [12] and the author [2], and states that

$$(0.1) \quad h^1(Y_C) = h^3(Y_C) = 0, \quad h^2(Y_C) = h^{1,1}(Y_C)$$

if $Y$ has everywhere good reduction. (The Shafarevich Conjecture appears then as the equality $h^1(Y_C) = 0$.) This result became possible due to the following two important achievements of Fontaine’s theory of $p$-adic crystalline representations:

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the Fontaine-Messing theorem relating etale and de Rham cohomology of smooth proper schemes over $W(k)$ in dimensions $[0, p)$, \[8\] (which was later proved by Faltings in full generality, \[8\]);

— the Fontaine-Laffaille integral theory of crystalline representations of $\Gamma_F$ with Hodge-Tate weights from $[0, p - 2]$, \[9\].

Note that the Fontaine-Laffaille theory works essentially for Hodge-Tate weights from $[0, p)$ but does not give all Galois invariant lattices in the corresponding crystalline representations. Nevertheless, this theory admits improvement developed by the author in \[1\]. As a result, there was obtained a suitable integral theory for Hodge-Tate weights from $[0, p)$, which allowed us to prove some extras to statements (0.1), in particular, that modulo the Generalized Riemann Hypothesis it holds $h_4(Y_C) = h_{2,2}(Y_C)$.

Since that time there was a huge progress in the study of semi-stable $p$-adic representations. Tsuji \[19\] proved a semi-stable case of the relation between etale and crystalline cohomology and Breuil \[5, 6\] developed an analogue of the Fontaine-Laffaille theory in the context of semi-stable representations (even for ramified basic fields). The papers \[4\] and \[17\] studied the problem of the existence of abelian varieties over $\mathbb{Q}$ with only one prime of bad semi-stable reduction. Note that the progress in this direction is quite restrictive because our knowledge of algebraic number fields with prescribed ramification at a given prime number $p$ (and unramified outside $p$) is very far from to be complete.

Theorem 0.1 represents an exceptional situation where the standard tools: the Odlyzko estimates of the minimal discriminants of algebraic number fields and the modern computing facilities (SAGE) are enough to resolve upcoming problems. In addition, the proof of this theorem requires a modification of Breuil’s theory to work with semi-stable representations of $\Gamma_F$ with Hodge-Tate weights from $[0, p)$.

The structure of this paper can be described as follows.

In Section 1 we introduce the category $\mathcal{L}^\ast$ of filtered $(\varphi, N)$-modules over $\mathcal{W}_1 := k[[u]]$. This is a special pre-abelian category, that is an additive category with kernels, cokernels and sufficiently nice behaving short exact sequences. Note that such categories play quite appreciable role in all our constructions. In Section 2 we construct the functor $\mathcal{V}^\ast$ from $\mathcal{L}^\ast$ to the category $\mathcal{M}_{\mathcal{G}}$ of $\mathbb{F}_p[\Gamma_F]$-modules. The version $\mathcal{CV}^\ast$ of $\mathcal{V}^\ast$ gives a fully faithful functor from $\mathcal{L}^\ast$ to the category of cofiltered $\Gamma_F$-modules $\mathcal{CM}_{\mathcal{G}}$. In Section 3 we give an interpretation of Breuil’s theory in terms of $\mathcal{W} := W(h)[[u]]$-modules (Breuil worked with modules over the divided powers envelope of $\mathcal{W}$) by introducing the category of filtered $(\varphi, N)$-modules $\mathcal{L}^{\mathcal{ft}}$ over $\mathcal{W}$. The advantage of this construction is that the objects of this category appear as strict subquotients of $p$-divisible groups in suitable pre-abelian category. This allows us to use devissage despite that all involved categories are not abelian. We also introduce the subcategories $\mathcal{L}^{u,\mathcal{ft}}$ and, resp., $\mathcal{L}^{m,\mathcal{ft}}$ of
unipotent and, resp., multiplicative objects in $\mathcal{L}^{\text{fl}}$ and prove that any $\mathcal{L} \in \mathcal{L}^{\text{fl}}$ is a canonical extension

$$(0.2) \quad 0 \longrightarrow \mathcal{L}^u \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}^m \longrightarrow 0$$

of a multiplicative object $\mathcal{L}^m$ by a unipotent object $\mathcal{L}^u$. In Section 4 we study Breuil’s functor $\mathcal{V}^{\text{fl}} : \mathcal{L}^{\text{fl}} \longrightarrow \mathcal{M}_F^\Gamma$ in the situation of Hodge-Tate weights from $[0, p)$. We show that on the subcategory $\mathcal{L}^{u, \text{fl}}$ this functor is still fully faithful by proving that on the subcategory of killed by $p$ unipotent filtered modules the functors $\mathcal{V}^{\text{fl}}$ and $\mathcal{V}^*$ coincide. Then we show that for any killed by $p$ object $\mathcal{L}$ of $\mathcal{L}^{\text{fl}}$, the functor $\mathcal{V}^{\text{fl}}$ transforms the standard short exact sequence $(0.2)$ into a short exact sequence in $\mathcal{M}_F^\Gamma$, which admits a functorial splitting. This splitting is used then to construct a modified version $\widehat{\mathcal{V}}^{\text{fl}} : \mathcal{L}^{\text{fl}} \longrightarrow \mathcal{C}_\mathcal{M}_F^\Gamma$ of $\mathcal{V}^{\text{fl}}$, which is already fully faithful. This gives us an efficient control on all Galois invariant lattices of semi-stable representations with weights from $[0, p)$. Especially, we have an explicit description of all killed by $p$ subquotients of such lattices and the corresponding ramification estimates. Finally, in Section 5 we give a proof of Theorem 0.1 following the strategy from [2].

Essentially, we obtain the following result: if $V$ is a 3-adic representation of $\Gamma_\mathbb{Q} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which is unramified outside 3 and is semi-stable at 3 then there is a $\Gamma_\mathbb{Q}$-equivariant filtration by $\mathbb{Q}_3$-subspaces $V = V_0 \supset V_1 \supset V_2 \supset V_3 = 0$ such that for $0 \leq i \leq 2$, the $\Gamma_\mathbb{Q}$-module $V_i/V_{i+1}$ is isomorphic to the product of finitely many copies of the Tate twist $\mathbb{Q}_3(i)$. If $V = H^2_{\text{et}}(Y_F, \mathbb{Q}_3)$ then looking at the eigenvalues of the Frobenius morphisms of reductions modulo $l \neq 3$, we obtain that $V = V_1$ and $V_2 = 0$, and this implies that $h^2(Y_{\overline{\mathbb{C}}}) = h^{1,1}(Y_{\overline{\mathbb{C}}})$.

Note that our construction of the modification of Breuil’s functor gives automatically the modification of the Fontaine-Laffaille functor, which is very close to its modification constructed in [1]. It is worth mentioning that switching from Breuil’s $S$-modules to $W$-modules means moving in the direction of Kisin’s approach [14] and recent approach to integral theory of $p$-adic representations by Liu [15, 16]. Finally, mention quite surprising matching of the ramification estimates for semi-stable representations and the Leopoldt conjecture for the field $\mathbb{Q}(\sqrt[3]{3}, \zeta_9)$, cf. Section 5.

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1. The categories $\tilde{L}^*$, $\tilde{L}^*_0$, $L^*$, $L^*_0$

Remind that $k$ is algebraically closed field of characteristic $p > 2$. Let $W = W(k)[[u]]$, where $W(k)$ is the ring of Witt vectors with coefficients in $k$ and $u$ is an indeterminate. Denote by $\sigma$ the automorphism of $W(k)$ induced by the $p$-th power map on $k$ and agree to use the same symbol for its continuous extension of $\sigma$ to $W$ such that $\sigma(u) = u^p$. Denote by $N : W \rightarrow W$ the continuous $W(k)$-linear derivation such that $N(u) = -u$.

Let $W_1 = W/pW$ with induced $\sigma$, $\varphi$ and $N$. We shall often use the following statement.

**Lemma 1.1.** Suppose $L$ is a module of finite rank over $W_1$ and $A$ is a $\sigma$-linear operator on $L$. Then the operator $\text{id} - A$ is epimorphic. If, in addition, $A$ is nilpotent then $\text{id} - A$ is bijective.

**Proof.** Part b) is obvious. In order to prove a) notice first that we can replace $L$ by $L/uL$ and assume that $L$ is a finite dimensional vector space over $k$. Then $L = L_1 \oplus L_2$, where $A$ is invertible on $L_1$ and nilpotent on $L_2$. It remains to note that $L_1 = L_0 \otimes_{\mathbb{F}_p} k$, where $L_0$ is a vector space over $\mathbb{F}_p$ such that $A|_{L_0} = \text{id}$. $\square$

1.1. Definitions and general properties.

**Definition.** The objects of the category $\tilde{L}^*_0$ are the triples $L = (L, F(L), \varphi)$, where

- $L$ and $F(L)$ are $W_1$-modules such that $L \supset F(L)$;
- $\varphi : F(L) \rightarrow L$ is a $\sigma$-linear morphism of $W_1$-modules; (Note that $\varphi(F(L)) = \sigma(W_1)$-submodule in $L$.)

If $L_1 = (L_1, F(L_1), \varphi)$ is also an object of $\tilde{L}^*_0$ then the morphisms $f \in \text{Hom}_{\tilde{L}^*_0}(L_1, L)$ are given by $W_1$-linear maps $f : L_1 \rightarrow L$ such that $f(F(L_1)) \subset F(L)$ and $f \varphi = \varphi f$.

**Definition.** The objects of the category $\tilde{L}^*$ are the quadruples $L = (L, F(L), \varphi, N)$, where

- $(L, F(L), \varphi)$ is an object of the category $\tilde{L}^*_0$;
- $N : L \rightarrow L/u^2L$ is a $W_1$-differentiation, i.e. for all $w \in W_1$ and $l \in L$, $N(wl) = N(w)(l \mod u^2L) + wN(l)$;
- if $L_1 = (L_1, F(L_1), \varphi, N)$ is another object of $\tilde{L}^*_0$ then morphisms $\text{Hom}_{\tilde{L}^*_0}(L_1, L)$ are given by $f : (L_1, F(L_1), \varphi) \rightarrow (L, F(L), \varphi)$ from $\tilde{L}^*_0$ such that $fN = Nf$. (We use the same notation $f$ for the reduction of $f$ modulo $u^2L$.)

The categories $\tilde{L}^*$ and $\tilde{L}^*_0$ are additive.

**Definition.** The category $L^*_0$ is a full subcategory of $\tilde{L}^*_0$ consisting of the objects $L = (L, F(L), \varphi)$ such that
• $L$ is a free $\mathcal{W}_1$-module of finite rank;
• $F(L) \supset u^{p-1}L$;
• the natural embedding $\varphi(F(L)) \subset L$ induces the identification $\varphi(F(L)) \otimes_{\sigma(\mathcal{W}_1)} \mathcal{W}_1 = L$.

**Definition.** The category $\mathcal{L}^*$ is a full subcategory of $\tilde{\mathcal{L}}^*$ consisting of the objects $\mathcal{L} = (L, F(L), \varphi, N)$ such that

- $(L, F(L), \varphi) \in \mathcal{L}^*$;
- for all $l \in F(L)$, $uN(l) \in F(L) \mod u^{2p}L$ and $N(\varphi(l)) = \varphi(uN(l))$ (we use the same notation $\varphi$ for the morphism $\varphi \mod u^{2p}L$).

The categories $\mathcal{L}^*$ and $\tilde{\mathcal{L}}^*$ are additive.

In the case of objects $(\tilde{L}, F(L), \varphi, N)$ of $\tilde{\mathcal{L}}^*$ the morphism $N$ can be uniquely recovered from the $\mathcal{W}_1$-differentiation $N_1 = N \mod u^{p}L$ due to the following property.

**Proposition 1.2.** Suppose $N_1 : L \mapsto L/u^p$ is a $\mathcal{W}_1$-differentiation such that for any $m \in F(L)$, $uN_1(m) \in F(L) \mod u^{2p}L$ and $N_1(\varphi(l)) = \varphi(uN_1(l))$. Then there is a unique $\mathcal{W}_1$-differentiation $N : L \mapsto L/u^{2p}$ such that $N \mod u^p = N_1$ and for any $m \in F(L)$, $N(\varphi(m)) = \varphi(uN(m))$.

**Proof.** Choose a $\mathcal{W}_1$-basis $m_1, \ldots, m_s \in F(L)$. Then $l_1 = \varphi(m_1), \ldots, l_s = \varphi(m_s)$ is a $\mathcal{W}_1$-basis of $L$ and a $\sigma(\mathcal{W}_1)$-basis of $\varphi(F(L))$.

Let $N(l_i) = \varphi(uN_1(m_i)) \in L/u^{2p}$, where $N_1(m_i)$ are some lifts of $N_1(m_i)$ to $L/u^{2p}$. Clearly, the elements $N(l_i) \in \varphi(F(L)) \subset L/u^{2p}$ are well-defined (use that $\varphi(u^{p+1}L) \subset u^{2p}L$).

For any $l = \sum w_i l_i \in L$, let $N(l) = \sum N(w_i) l_i + \sum w_i N(l_i)$. Then $N : L \mapsto L/u^{2p}$ is a $\mathcal{W}_1$-differentiation and $N \mod u^p = N_1$. Clearly, $N$ is the only candidate to satisfy the requirements of our Proposition.

Now suppose $m = \sum w_i m_i \in F(L)$ with all $w_i \in \mathcal{W}_1$. Then $N(\varphi(m)) = \sum \sigma(w_i) l_i \mod u^{2p}$. On the other hand, $\varphi(uN(m))$ equals

$$\sum \sigma(N(w_i)) l_i + \sum \varphi(w_i uN(m_i)) = \sum \sigma(w_i) l_i \mod u^{2p}$$

because all $\sigma(N(w_i)) \in u^p \sigma(\mathcal{W}_1)$.

The proposition is proved. $\square$

**Proposition 1.3.** $\mathcal{L}^*$ and $\tilde{\mathcal{L}}^*$ are pre-abelian categories, cf. Appendix.

**Proof.** Suppose $\mathcal{S}$ is additive category and $f \in \text{Hom}_S(A, B)$. Then $i \in \text{Hom}_S(K, A)$ is a kernel of $f$ if for any $D \in \mathcal{S}$, the sequence of abelian groups

$$0 \rightarrow \text{Hom}_S(D, K) \xrightarrow{i} \text{Hom}_S(D, A) \xrightarrow{f} \text{Hom}_S(D, B)$$

is exact. Similarly, $j \in \text{Hom}_S(B, C)$ is a cokernel of $f$ if for any $D \in \mathcal{S}$, the sequence

$$0 \rightarrow \text{Hom}_S(C, D) \xrightarrow{j} \text{Hom}_S(B, D) \xrightarrow{f} \text{Hom}_S(A, D)$$
is exact.

Let \( FW_1 \) be the category of free \( W_1 \)-modules with filtration. This category is pre-abelian. More precisely, consider two its objects \( L = (L, F(L)), M = (M, F(M)) \) and \( f \in \text{Hom}_{FW_1}(L, M) \).

Then \( \text{Ker}_{FW_1} f \) appears as a natural embedding \( i_L : K = (K, F(K)) \rightarrow L \), where \( K = \text{Ker}(f : L \rightarrow M) \) and \( F(K) = K \cap F(L) \). Then \( \text{Im}_{FW_1} f = \text{Coker}_{FW_1}(\text{Ker}_{FW_1} f) \) appears as a natural projection \( j_L : L \rightarrow L' = (L', F(L')) \), where \( L' = f(L) \) and \( F(L') = f(F(L)) \).

Similarly, \( \text{Coker} f \) appears as a natural projection \( j_M : M \rightarrow C = (C, F(C)) \), where \( C = (M/L')/(M/L')_{tor} \) and \( F(C) = j_M(F(M)) \). Then \( \text{Coim}_{FW_1} f = \text{Ker}_{FW_1}(\text{Coker}_{FW_1} f) \) is a natural embedding \( M' = (M', F(M')) \rightarrow M \), where \( M' = M \cap \text{Ker} f \) and \( F(M') = F(M) \cap M' \).

As usually, we have a natural map \( L' \rightarrow M' \) induced by \( L' \subset M' \). Note that \( M/M' \) is free and \( M'/L' \) is torsion \( W_1 \)-modules and these properties completely characterize \( M' \) as a \( W_1 \)-submodule of \( M \).

Now suppose \( L = (L, F(L), \varphi), M = (M, F(M), \varphi) \) are objects of \( L_0^* \) and \( f \in \text{Hom}_{L_0^*}(L, M) \). Use the obvious forgetful functor \( L_0^* \rightarrow FW_1 \) and the same notation for the corresponding images of \( L, M \) and \( f \). Show that \( K = \text{Ker}_{FW_1} f \) and \( C = \text{Coker}_{FW_1} f \) have natural structures of objects of \( L_0^* \) and with respect to this structure they become the kernel and, resp, cokernel of \( f \) in \( L_0^* \). Indeed,

\[
w^{-1}K = w^{-1}L \cap K \subset F(L) \cap K = F(K) = \text{Ker}(f : F(L) \rightarrow F(M)).
\]

Therefore, \( \varphi(F(K)) \subset K \cap \varphi(F(L)) \) and there is a natural embedding \( i : \varphi(F(K)) \otimes_{W_1} W_1 \subset K \). On the one hand,

\[
\text{rk}_{W_1}(\varphi(F(K))) = \text{rk}_{W_1}(F(K)) = \text{rk}_{W_1}(K).
\]

On the other hand, \( L/K \) has no \( W_1 \)-torsion. This implies that the quotient \( \varphi(F(L))/\varphi(F(K)) \) has no \( \sigma W_1 \)-torsion and the factor of \( L = \varphi(F(L))/\varphi(F(K)) \otimes_{W_1} W_1 \) by \( \varphi(F(K)) \otimes_{W_1} W_1 \) also has no \( W_1 \)-torsion. So, the above embedding \( i \) becomes the equality \( \varphi(F(K)) \otimes_{W_1} W_1 = K \) and \( K = (K, F(K), \varphi) = \text{Ker}_{L_0^*} f \).

The above description of \( \text{Ker}_{L_0^*} \) implies that \( w^{-1}L' \subset F(L'), \varphi F(L') = \varphi(F(M))/\varphi(F(K)) \) and \( L' = \varphi(F(L')) \otimes_{W_1} W_1 \). In other words, \( L' = (L', F(L'), \varphi) \in L_0^* \).

Now note that for \( M' = (M', F(M')) \) we have

\[
w^{-1}M' = (w^{-1}M) \cap M' \subset F(M) \cap M' = F(M')
\]

and, therefore, \( F(M')/F(L') \) is torsion \( W_1 \)-module and

- \((\varphi(F(M')) \otimes_{W_1} W_1)/L' \) is torsion \( W_1 \)-module;

On the other hand, \( F(M)/F(M') \) is torsion free \( W_1 \)-module implies that \( \varphi(F(M))/\varphi(F(M')) \) is torsion free \( \sigma W_1 \)-module and, therefore,

- \( M/(\varphi(F(M')) \otimes_{W_1} W_1) \) is torsion free \( W_1 \)-module.
Lemma 1.4. \( \varphi(F(L')) \supset u^p \varphi(F(M')) \) (and, therefore, \( L' \supset u^p M' \)).

Proof of Lemma. Otherwise, there is an \( n \) such that \( u^n F(L') \) and \( u^n F(M') \) together with their natural embedding \( u^n F(L') \to u^n F(M') \). As a matter of fact these two objects of \( L'_n \) do not differ very much.

\[ L'/u^p L' \leftarrow L'/u^p M' \to M'/u^p M' \to M'/u^p M \]

to deduce that \( N(K) \subset \text{Ker}(L/u^p L \to L'/u^p L') \) and

\[ N(K) \text{ mod } u^p L \subset \text{Ker}(L/u^p L \to L'/u^p L') = K/u^p K. \]

Therefore, by Proposition 1.2, \( N \) (as a unique lift of \( N_1 = N \text{ mod } u^p \)) maps \( K \to K/u^p K \) and \( (K, F(K), \varphi, N) \in L^* \).

The above property of \( \text{Ker}_F f \) implies that \( N(L') \subset L'/u^p L' \). Now use that \( u^p M' \subset L' \), \( u^p L' \subset u^p M' \), and \( N(u^p M') \subset u^p M/u^p M \) to deduce that

\[ N(u^p M') \subset L'/u^p M' \cap u^p M/u^p M = u^p M'/u^p M'. \]

So, \( N \text{ mod } u^p \) maps \( M' \) to \( M'/u^p M' \) and again by Proposition 1.2, \( N(M') \subset M'/u^p M' \). This means that the kernel of the above constructed \( \text{Coker}_F f : (M, F(M), \varphi) \to (C, F(C), \varphi) \) is provided with the structure of object of the category \( L^* \). Therefore, \( N \) induces the map \( N : C \to C/u^p C \) and \( (C, F(C), \varphi, N) \in L^* \). The proposition is proved.

Remark. The above proof shows that the kernels and cokernels in the category \( L^* \) appear on the level of filtered modules as the kernel and cokernel of the corresponding map of filtered modules \( (L_1, F(L_1)) \) to
(L, F(L)) in the category of filtered $\mathcal{W}_1$-modules. Therefore, the category $\mathcal{L}_r^*$ is special, cf. Appendix A, and we can apply the corresponding formalism of short exact sequences. In particular, if we take another object $\mathcal{L}_1 = (L_1, F(L_1), \varphi, N) \in \mathcal{L}_r^*$ then

- $i \in \text{Hom}_{\mathcal{L}_r^*}(\mathcal{L}_1, \mathcal{L})$ is strict monomorphism iff $i : L_1 \rightarrow L$ is injective and $i(L_1) \cap F(L) = i(F(L_1));$

- $j \in \text{Hom}_{\mathcal{L}_r^*}(\mathcal{L}, \mathcal{L}_2)$ is strict epimorphism iff $j : L \rightarrow L_2$ is epimorphic and $j(F(L)) = F(L_2).$

As usually, cf. Appendix A, if $i$ is strict monomorphism then $j = \text{Coker } i$ is strict epimorphism and if $j$ is strict epimorphism then $i = \text{Ker } j$ is strict monomorphism and under these assumptions $0 \rightarrow \mathcal{L}_1 \xrightarrow{i} \mathcal{L} \xrightarrow{j} \mathcal{L}_2 \rightarrow 0$ is a short exact sequence.

1.2. The category $\mathcal{L}_r^*$.

**Proposition 1.5.** Suppose $\mathcal{L} = (L, F(L), \varphi, N) \in \mathcal{L}_r^*$. Then the following conditions are equivalent:

(a) $N(F(L)) \subset F(L) \mod u^{2p}L$;

(b) $N(\varphi(F(L))) \subset u^pL \mod u^{2p}L$.

**Proof.** (a) $\Rightarrow$ (b): if for any $l \in F(L)$, $N(l) \in F(L) \mod u^{2p}L$ then $N(\varphi(l)) = \varphi(uN(l)) = u^p\varphi(N(l)) \in u^pL \mod u^{2p}L$.

(b) $\Rightarrow$ (a): for any $l \in F(L)$, $\varphi(uN(l)) = N(\varphi(l)) \in u^pL \mod u^{2p}L$; now use that $\varphi$ induces an embedding of $F(L)/uF(L)$ into $L/u^pL$ to deduce that $uN(l) \in uF(L) \mod u^{2p}L$, i.e. $N(l) \in F(L) \mod u^{2p}L$ (use that $u^{p-1}L \subset F(L)$). \hfill $\Box$

**Definition.** The category $\mathcal{L}_r^*$ is a full subcategory of $\mathcal{L}_r^*$ consisting of $(L, F(L), \varphi, N)$ such that $N : L \rightarrow L$ satisfies the equivalent conditions from Proposition 1.5.

**Remark.** If $\mathcal{L} = (L, F(L), \varphi, N) \in \mathcal{L}_r^*$ then $N_1 = N \mod u^p$ is the unique $\mathcal{W}_1$-differentiation $N_1 : L \rightarrow L/u^p$ such that its restriction to $\varphi(F(L))$ is the zero map.

**Proposition 1.6.** Suppose $\mathcal{L} = (L, F(L), \varphi, N) \in \mathcal{L}_r^*$. Then there is a $\sigma(\mathcal{W}_1)$-basis $l_1, \ldots, l_s$ of $\varphi(F(L))$ and integers $0 \leq c_i < p$, where $1 \leq i \leq s$, such that $u^{c_i}l_1, \ldots, u^{c_i}l_s$ is a $\mathcal{W}_1$-basis of $F(L)$.

**Proof.** Choose a $\mathcal{W}_1$-basis $m_1, \ldots, m_s$ of $L$ such that for suitable integers $c_1, \ldots, c_s$, the elements $u^{c_1}m_1, \ldots, u^{c_s}m_s$ form a $\mathcal{W}_1$-basis of $F(L)$. Clearly all $0 \leq c_i < p$.

For $1 \leq i \leq s$ and $j \geq 0$, let $l_{ij} \in \varphi(F(L))$ be such that $m_i = \sum_{j \geq 0} u^j l_{ij}$. Note that $\{l_{0i} \mid 1 \leq i \leq s\}$ is a $\sigma(\mathcal{W}_1)$-basis of $\varphi(F(L))$ and it will be sufficient to prove that all $u^{c_i}l_{0i} \in F(L)$ because then the elements $l_i := l_{0i}$ will satisfy the requirements of our proposition.
For all $1 \leq i \leq s$, the element
\[
N(u^c m_i) = - \sum_j (j + c_i)u^{j+c_i}(l_{ij} \mod u^{2p} L) + \sum_j u^{j+c_i}N(l_{ij})
\]
belongs to $F(L) \mod u^{2p} L$ if and only if $\sum_j (j + c_i)u^{j+c_i}l_{ij} \in F(L)$. (Use that $u^pL \subset uF(L)$.) This implies that for all integers $k \geq 0$, $\sum_j (j + c_i)k^u^{j+c_i}l_{ij} \in F(L)$. Therefore, for any $\alpha \in \mathbb{Z}/p\mathbb{Z}$,
\[
\sum_{(j+c_i)\mod p = \alpha} u^{j+c_i}l_{ij} \in F(L).
\]
In particular, taking $\alpha = c_i \mod p$ and using that $u^p l_{ij} \in F(L)$, we obtain that $u^{c_i}l_{i0} \in F(L)$.

Consider the category of filtered Fontaine-Laffaille modules $\text{MF}_p$. The objects of this category are finite dimensional $k$-vector spaces $M$ with decreasing filtration of length $p$ by subspaces $M = M^0 \supset M^1 \supset \cdots \supset M^{p-1} \supset M^p = 0$ and $\sigma$-linear maps $\varphi_i : M^i \rightarrow M$ such that $\text{Ker} \varphi_i \supset M^{i+1}$, where $0 \leq i < p$, and $\sum_i \text{Im} \varphi_i = M$. The morphisms in $\text{MF}_p$ are the morphisms of filtered vector spaces which commute with the corresponding morphisms $\varphi_i$, $0 \leq i < p$.

Define the functor $\mathcal{M}$ from $\text{MF}_p$ to $\mathcal{L}^*$ as follows. Suppose $M$ is an object of $\text{MF}_p$ and $l_1, \ldots, l_s$ is a $k$-basis of $M$ which is compatible with the filtration $\{M^j\}_{0 \leq j < p}$. In other words, if for $0 \leq i < p$, the index $j(i)$ is such that $l_i \in M^{j(i)} \setminus M^{j(i)+1}$ then the elements of the set $\{l_i \mod M^{j+1} \mid j(i) = j\}$ form a $k$-basis of $M^j/M^{j+1}$. Then $\mathcal{M}(M) = (L, F(L), \varphi, N)$ is such that
- $L = M \otimes_k W_1$;
- $F(L) = \sum_i W_1 u^{p-1-j(i)} l_i$;
- $\varphi$ is a unique $\sigma$-linear map such that for all $0 \leq i < p$, it holds $\varphi(u^{p-1-j(i)} l_i) := \varphi_j(l_i)$;
- $N$ is uniquely recovered from the condition $N|_{\varphi(F(L))} = 0 \mod u^p$.

Clearly, the above correspondence $M \mapsto (L, F(L), \varphi, N)$ can be naturally extended to the functor $\mathcal{F}$ from the category $\text{MF}_p$ to the category $\mathcal{L}^*_\sigma$.

**Proposition 1.7.** The functor $\mathcal{F}$ is surjective on objects.

**Proof.** Suppose $\mathcal{L} = (L, F(L), \varphi, N) \in \mathcal{L}^*_\sigma$. By Proposition 1.6 there are a $\sigma(W_1)$-basis $l_1, \ldots, l_s$ of $\varphi(F(L))$, integers $0 \leq c_1, \ldots, c_s < p$, a matrix $A \in \text{GL}_s(k)$ and an $s$-vector $a$ with coordinates in $u^p \varphi(F(L))$ such that
\[
(\varphi(u^{c_1} l_1), \ldots, \varphi(u^{c_s} l_s)) = (l_1, \ldots, l_s) A + a.
\]
Suppose $l = (l_1, \ldots, l_s)$ and $l' = (l'_1, \ldots, l'_s) = l + b$, where $b = (b_1, \ldots, b_s)$ is a vector with coordinates in $u^p \varphi(F(L))$. It will be sufficient to prove that $b$ can be chosen in such a way that
\[
(\varphi(u^{c_1} l'_1), \ldots, \varphi(u^{c_s} l'_s)) = l' A.
\]
Equivalently, $b$ must satisfy the relation
\[ b - (w^{pc_1} \varphi(b_1), \ldots, w^{pc_s} \varphi(b_s))B = \tilde{a}, \]
where $B = A^{-1}$ and $\tilde{a} = aB$. Therefore, the existence of $b$ follows from Lemma 1.1 because the correspondence
\[ (X_1, \ldots, X_s) \mapsto (w^{pc_1} \varphi(X_1), \ldots, w^{pc_s} \varphi(X_s))B \]
is a $\sigma$-linear endomorphism of the $\sigma(W_1)$-module $u^p \varphi(F(L))^s$. \hfill $\square$

**Remark.** The above functor $\mathcal{F}$ is not equivalence of categories. For example, $\mathcal{MF}_{p-1}$ is abelian category but $\mathcal{L}_{\mathbb{F}_p}$ is not.

### 1.3. Simple objects in $\mathcal{L}^*$.

**Definition.** An object $\mathcal{L}$ of $\mathcal{L}^*$ is simple if any strict monomorphism $i : \mathcal{L}_1 \rightarrow \mathcal{L}$ in $\mathcal{L}^*$ is either isomorphism or the zero morphism. Equivalently, $\mathcal{L}$ is simple iff any strict epimorphism $j : \mathcal{L} \rightarrow \mathcal{L}_2$ is either isomorphism or the zero morphism.

All simple objects in $\mathcal{L}^*$ can be described as follows.

Let $[0, 1]_p = \{ r \in \mathbb{Q} | 0 \leq r \leq 1, v_p(r) = 0 \}$, where $v_p$ is a $p$-adic valuation. Then any $r \in [0, 1]_p$ can be uniquely written as $r = \sum_{i \geq 1} a_i p^{-i}$, where the digits $0 \leq a_i = a_i(r) < p$ form a periodic sequence. The minimal positive period of this sequence will be denoted by $s(r)$.

Let $\tilde{r} = 1 - r$. Then $\tilde{r} \in [0, 1]_p$ and $\tilde{r} = \sum_{i \geq 1} \tilde{a}_i p^{-i}$, where for all $i \geq 1$, the digits $\tilde{a}_i = a_i(\tilde{r})$ are such that $a_i + \tilde{a}_i = p - 1$.

**Definition.** For $r \in [0, 1]_p$, let $\mathcal{L}(r) = (L(r), F(L(r)), \varphi, N)$ be the following object of the category $\mathcal{L}^*$:

- $L(r) = \bigoplus_{i \in \mathbb{Z}/s(r)} W_1 l_i$;
- $F(L(r)) = \bigoplus_{i \in \mathbb{Z}/s(r)} W_1 u^{\tilde{a}_i} l_i$;
- for $i \in \mathbb{Z}/s(r)$, $\varphi(u^{\tilde{a}_i} l_i) = l_{i+1}$;
- $N$ is uniquely recovered from the condition $N|_{\varphi(F(L))} = 0 \pmod{u^p}$.

If $n \in \mathbb{N}$ and $r \in [0, 1]_p$ set $r(n) = \sum_{i \geq 1} a_{i+n}(r) p^{-i}$. Extend this definition to any $n \in \mathbb{Z}$ by setting $r(n) := r(n+Ns(r))$ for a sufficiently large $N \in \mathbb{N}$. Then we have the following properties.

**Proposition 1.8.** a) If $r \in [0, 1]_p$ then $\mathcal{L}(r)$ is simple;

b) if $r_1, r_2 \in [0, 1]_p$ then $\mathcal{L}(r_1) \simeq \mathcal{L}(r_2)$ if and only if there is an $n \in \mathbb{Z}$ such that $r_1 = r_2(n)$;

c) if $L$ is a simple object of the category $\mathcal{L}^*$ then there is an $r \in [0, 1]_p$ such that $\mathcal{L} \simeq \mathcal{L}(r)$.

**Proof.** The proof of a) and b) is straightforward. Then c) can be proved along the following lines. Suppose $\mathcal{L} = (L, F(L), \varphi, N) \in \mathcal{L}^*$. Take a non-zero $l \in L \setminus (aL)$ and form the sequence $l_n \in \mathcal{L}^*$ as follows. Set $l_0 = l$ and by induction on $n \geq 0$, let $a_n \in \mathbb{Z}_{\geq 0}$ be such that
Proof. one can easily see that the elements \( a_n \) is periodic for \( n \gg 0 \). Then (using that \( k \) is algebraically closed) one can choose an \( \alpha \in k \) such that the sequence \( \alpha_n \) is periodic for \( n \gg 0 \). Then one can easily see that the elements \( N(l'_n) \) give either a similar sequence or \( N(l'_n) \in u^n L \) mod \( u^{2^n} L \) for \( n \gg 0 \). Because \( N \text{mod} p \) is nilpotent on \( \varphi(F(L)) \), we can assume the second alternative and in this case for \( n \gg 0 \), \( \mathcal{L}(r(n)) \supset \mathcal{L} \). Clearly, this embedding is strict and, therefore, \( \mathcal{L} = \mathcal{L}(r(n)) \). \( \square \)

1.4. Extensions in \( \mathcal{L}_s' \). Suppose \( r_1, r_2 \in [0,1]_p \). Choose an \( s \in \mathbb{N} \) which is divisible by \( s(r_1) \) and \( s(r_2) \) and introduce the objects \( \mathcal{L}_1 = (L_1, F(L_1), \varphi, N) \) and \( \mathcal{L}_2 = (L_2, F(L_2), \varphi, N) \) of the category \( \mathcal{L}_s' \) as follows:

- \( L_1 = \oplus_{i \in \mathbb{Z}/s} W_1 t_i^{(1)} \), \( F(L_1) = \oplus_{i \in \mathbb{Z}/s} W_1 u^i t_i^{(1)} \), where \( r_1 = \sum_{i \geq 1} a_i p^{-i} \) with the digits \( 0 \leq a_i < p \), \( a_i = (p - 1) - a_i \) and for all \( i \in \mathbb{Z}/s \), \( \varphi(u^i t_i^{(1)}) = t_i^{(1)} \).

- \( L_2 = \oplus_{j \in \mathbb{Z}/s} W_1 t_j^{(2)} \), \( F(L_2) = \oplus_{j \in \mathbb{Z}/s} W_1 b_j t_j^{(2)} \), where \( r_2 = \sum_{j \geq 1} b_j p^{-j} \) with the digits \( 0 \leq b_j < p \), \( b_j = (p - 1) - b_j \) and for all \( j \in \mathbb{Z}/s \), \( \varphi(b_j t_j^{(2)}) = t_j^{(2)} \).

Note that for \( i = 1,2 \), \( \mathcal{L}_i \) is isomorphic to the product of \( s/s(r_i) \) copies of the simple object \( \mathcal{L}(r_i) \).

Suppose \( \mathcal{L} = (L, F(L), \varphi, N) \in \text{Ext}_{\mathcal{L}_s'}(\mathcal{L}_2, \mathcal{L}_1) \). Consider a \( \sigma(W_1) \)-linear section \( S : t_j^{(2)} \mapsto l_j \), \( j \in \mathbb{Z}/s \), of the corresponding epimorphic map \( \varphi(F(L)) \rightarrow \varphi(F(L_2)) \). Then:

- \( L = L_1 \oplus (\oplus_{j \in \mathbb{Z}/s} W_1 l_j) \);

- for all indices \( j \in \mathbb{Z}/s \), there are unique elements \( v_j \in L_1 \), such that \( F(L) = F(L_1) + \sum_{j \in \mathbb{Z}/s} W_1 (b_j l_j + v_j) \) and \( \varphi(b_j l_j + v_j) = l_{j+1} \);

- \( F(L) \supset u^{p-1} L \) if and only if for all \( j \in \mathbb{Z}/s \), \( u^j v_j \in F(L_1) \);

- if \( S' : t_j^{(2)} \mapsto l_j' = l_j + \varphi(w_{j-1}) \), where \( j \in \mathbb{Z}/s \) and \( w_{j-1} \in F(L_1) \), is another section of the epimorphism \( \varphi(F(L)) \rightarrow \varphi(F(L_2)) \) then for the corresponding elements \( v_j' \in L_1 \), it holds \( v_j' - v_j = w_j - u^j \varphi(w_{j-1}) \).

Proposition 1.9. For a given \( \mathcal{L} \in \text{Ext}_{\mathcal{L}_s'}(\mathcal{L}_2, \mathcal{L}_1) \), there is a section \( S \) such that the corresponding system of factors \( \{v_j \in L_1 \mid j \in \mathbb{Z}/s \} \), satisfies the following normalization condition:

(C1) if \( v_j = \sum_{i,t} \gamma_{ijt} u^i t_i^{(1)} \) with all \( \gamma_{ijt} \in k \), then \( \gamma_{ijt} = 0 \) if \( t = \tilde{b}_j \).

Proof. Choose a section \( S \) of the projection \( \varphi(F(L)) \rightarrow \varphi(F(L_2)) \) such that the number of elements in the set \( \gamma(S) = \{ \gamma_{ijt} \neq 0 \mid t = \tilde{b}_j \} \) is minimal. Suppose \( \gamma(S) \neq \emptyset \) (otherwise, the proposition is proved).

Let \( \{v_j \mid j \in \mathbb{Z}/s \} \) be the corresponding system of factors.
Suppose \( i_0, j_0 \in \mathbb{Z}/s \) are such that \( \gamma = \gamma_{i_0 j_0 b_{j_0}} \neq 0 \). Replace \( \{v_j \mid j \in \mathbb{Z}/s\} \) by an equivalent system \( \{v'_j \mid j \in \mathbb{Z}/s\} \) via the elements \( \{w_j \in F(L_1) \mid j \in \mathbb{Z}/s\} \) such that \( w_j = 0 \) if \( j \neq j_0 - 1 \) and \( w_{j_0 - 1} = \sigma^{-1}(\gamma)u_{\tilde{a}_{i_0 - 1}l_{i_0}^{(1)}} \). If \( v'_j = \sum_{i,t} \gamma'_{ijt}u_{l_i}^{(1)} \) then \( \gamma'_{i_0 j_0 b_{j_0}} = 0 \) and (because of the above minimality condition for \( S \)) we must have \( \tilde{a}_{i_0 - 1} = \tilde{b}_{j_0 - 1} \) and \( \gamma'_{i_0 - 1,j_0 - 1,\tilde{b}_{j_0 - 1}} = \sigma^{-1}(\gamma) \). In particular, the new section \( S' \) again satisfies the minimality condition.

Repeating the above procedure we obtain for all \( n \in \mathbb{Z}/s \), that \( \tilde{a}_{i_0 - n} = \tilde{b}_{j_0 - n} \), that is \( \tilde{r}_1(i_0) = \tilde{r}_2(j_0) \). In addition, for all \( m \in \mathbb{Z}/s \), it holds \( \gamma_{i_0 + m,j_0 + m,b_{j_0 + m}} = \sigma^m(\gamma) \neq 0 \).

Choose \( \beta \in K \) such that \( \beta - \sigma^s(\beta) = \gamma \) and consider \( w_j \in F(L_1) \) such that for all \( n \in \mathbb{Z}/s \), \( w_{j_0 + n} = \sigma^n(\beta)u_{\tilde{b}_{j_0 + n}l_{i_0 + n}^{(1)}} \). Then for the corresponding new system of factors \( \{v'_j \mid j \in \mathbb{Z}/s\} \), where

\[
v'_j = v_j + w_j - u_{\tilde{b}_j} \varphi(w_{j-1}) = \sum_{i,t} \gamma'_{ijt}u_{l_i}^{(1)},
\]

it holds \( \gamma'_{i_0 j_0 b_{j_0}} = 0 \), and \( \gamma_{ijt} = \gamma'_{ijt} \) if \( (i,j,t) \neq (i_0,j_0,\tilde{b}_{j_0}) \). This contradicts to our original assumption that the number of elements in \( \gamma(S) \) is minimal. \( \square \)

**Proposition 1.10.** For a given \( \mathcal{L} \in \text{Ext}_{\mathbb{L}}^s(\mathcal{L}_2, \mathcal{L}_1) \) there is a section \( S \) such that the corresponding system \( \{v_j \mid j \in \mathbb{Z}/s\} \), satisfies the above condition (C1) and the following normalisation condition:

(C2) the coefficients \( \gamma_{ijt} = 0 \) if \( t > \tilde{a}_i \).

*Proof.* Suppose \( v^{(0)} = \{v_j \mid j \in \mathbb{Z}/s\} \) is such that \( v_{j_0} = \gamma u_{\tilde{a}_i l_{i_0}^{(1)}} \) with \( \gamma \in k, \tilde{a}_0 > \tilde{a}_i \) and for \( j \neq j_0 \), \( v_j = 0 \). It will be sufficient to prove that any such system of factors is equivalent to the trivial system of factors.

Take \( u^{(0)} = \{u_j^{(0)} \mid j \in \mathbb{Z}/s\} \) such that \( u_{j_0}^{(0)} = -\gamma u_{\tilde{a}_i l_{i_0}^{(1)}} \) and \( u_j^{(0)} = 0 \) if \( j \neq j_0 \). Then the corresponding equivalent system \( \{v_j^{(1)} \mid j \in \mathbb{Z}/s\} \) is such that \( v_{j_0}^{(1)} = 0 \) if \( j \neq j_0 + 1 \), and \( v_{j_0 + 1}^{(1)} = \gamma p u_{\tilde{a}_i l_{i_0}^{(1)} + 1} \), where \( t_1 = \tilde{b}_{j_0 + 1} + (t_0 - \tilde{a}_i)p \). This implies that \( t_1 > p > \tilde{a}_i + 1, t_0 - \tilde{a}_i + 1 > t_0 - \tilde{a}_i \), and \( t_1 - \tilde{a}_i + 1 > t_0 - \tilde{a}_i \) unless \( \tilde{b}_{j_0 + 1} = 0, t_1 = p \) and \( \tilde{a}_i + 1 = p - 1 \).

Therefore, we can repeat this procedure to get for all \( n \geq 0 \), the systems of factors \( w^{(n)} = \{w_j^{(n)} \mid j \in \mathbb{Z}/s\} \) and the corresponding equivalent systems of factors \( v^{(n)} = \{v_j^{(n)} \mid j \in \mathbb{Z}/s\} \) such that \( v_{j_0}^{(n)} = 0 \) if \( j \neq j_0 + n \), and \( v_{j_0 + n}^{(n)} = \gamma p^n u_{\tilde{a}_i l_{i_0}^{(1)} + n} \).

If \( (\tilde{r}_1, t_0) \neq (0, 1, p) \) then \( t_n \to \infty \) and we can use the system \( w = \{\sum_{n \geq 0} w_j^{(n)} \mid j \in \mathbb{Z}/s\} \) to trivialize our original system of factors \( v^{(0)} \).
If \((\tilde{r}_2, \tilde{r}_1, t_0) = (0, 1, p)\), one can trivialize \(v^{(0)}\) via \(w = \{w_j \mid j \in \mathbb{Z}/s\}\), where for \(0 \leq n < s\), \(w_{j_0+n} = \kappa^{p^n} u^{j_0} l_{j_0+n}^{(1)}\) and \(k \in E\) is such that \(\kappa - \kappa^{p^n} = \gamma\).

Suppose, as earlier, that \(\mathcal{L} \in \text{Ext}_{\mathcal{L}^*}(\mathcal{L}_2, \mathcal{L}_1)\) and assume that it is given via the system of factors \(\{v_j \mid j \in \mathbb{Z}/s\}\), where all \(v_j\) satisfy the normalization condition \((C1)\) from Proposition 1.9.

**Proposition 1.11.**

(a) If all \(v_j \in F(L_1)\) then there is a crystalline \(\mathcal{L}' \in \text{Ext}_{\mathcal{L}^*}(\mathcal{L}_2, \mathcal{L}_1) \subseteq \text{Ext}_{\mathcal{L}^*}(\mathcal{L}_2, \mathcal{L}_1)\) with the same system of factors.

(b) If \(\mathcal{L} \in \mathcal{L}^*\) then all \(v_j \in F(L_1)\).

**Proof.**

(a) If all \(v_j \in F(L_1)\) then \(N(v_j) \in F(L_1)\) mod \(u^{2p} L_1\) and \(u^{b_j} l_j \in F(L)\) mod \(u^{2p} L\). Therefore, the congruences

\[
N(\varphi(m)) \equiv \varphi(uN(m)) \mod u^p L,
\]

for all \(m \in F(L)\), are equivalent to the congruences

\[
N(l_{j+1}) \equiv \varphi(u^{b_j+1} N(l_j)) \mod u^p L.
\]

Therefore, by Proposition 1.2 the zero map \(\varphi(F(L)) \rightarrow L/\mathcal{L}\) can be extended to a unique \(\mathcal{W}_1\)-differentiation \(N' : L \rightarrow L/\mathcal{L}\) such that the quadruple \((L, F(L), \varphi, N') \in \mathcal{L}^*\).

(b) Suppose that \(\mathcal{L} \in \mathcal{L}^*\) and all \(v_j = \sum_{i,t} \gamma_{ijt} u^t l_i^{(1)}\), where \(\gamma_{ijb} = 0\).

Consider the congruence (use that \(-u^{b_j} l_j \equiv v_j \mod F(L)\))

\[
(1.1) \quad N(u^{b_j} l_j + v_j) \equiv \sum_{i,t} \gamma_{ijt}(\tilde{b}_j - t) u^t l_i^{(1)} + u^{b_j} N(l_j) \mod F(L).
\]

The condition \(\mathcal{L} \in \mathcal{L}^*\) implies that \(N(u^{b_j} l_j + v_j) \in F(L)\) mod \(u^{2p} L\) and \(N(l_j) \in u^p L\) mod \(u^{2p} L \subseteq F(L)\) mod \(u^{2p} L\). This means that all \((\tilde{b}_j - t) \gamma_{ijt} u^t l_i^{(1)} \in F(L_1)\). Therefore, for \(t \neq \tilde{b}_j\), \(\gamma_{ijt} u^t l_i^{(1)} \in F(L_1)\), and \(v_j \in F(L_1)\). The proposition is proved.

**Definition.** A pair \((i_0, j_0) \in (\mathbb{Z}/s)^2\) is \((r_1, r_2)_{cr}\)-admissible if \(\tilde{a}_{i_0} \neq \tilde{b}_{j_0}\) and there is an \(m_0 \in \mathbb{N}\) such that \(\tilde{a}_{i_0+m_0} > \tilde{b}_{j_0+m_0}\) and for \(0 < m < m_0\), \(\tilde{a}_{i_0+m} = \tilde{b}_{j_0+m}\).

**Remark.** Clearly, for any \((r_1, r_2)_{cr}\)-admissible pair of indices \((i_0, j_0)\), it holds \(r_1(i_0) > r_2(j_0)\).

**Definition.** For \((i_0, j_0) \in (\mathbb{Z}/s)^2\) and \(\gamma \in k\), denote by \(E_{cr}(i_0, j_0, \gamma)\) the extension \(\mathcal{L} \in \text{Ext}_{\mathcal{L}^*}(\mathcal{L}_2, \mathcal{L}_1)\) given by the system \(\{v_j \mid j \in \mathbb{Z}/s\}\) such that \(v_{j_0} = \gamma a_{i_0} \circ i_0 l_i^{(1)}\) and \(v_j = 0\) if \(j \neq j_0\).

**Proposition 1.12.** Any element \(\mathcal{L} \in \text{Ext}_{\mathcal{L}^*}(\mathcal{L}_2, \mathcal{L}_1)\) can be obtained in a unique way as a sum of \(E_{cr}(i, j, \gamma_{ij})\), where \((i, j) \in (\mathbb{Z}/s)^2\) runs over the set of all \((r_1, r_2)_{cr}\)-admissible pairs and all \(\gamma_{ij} \in k\).
Proof. Propositions 1.9-1.11 imply that any $\mathcal{L} = (L, F(L), \varphi, N)$ from the group $\text{Ext}_r^{1}(\mathcal{L}_2, \mathcal{L}_1)$ can be presented as a sum of extensions $E_{cr}(i,j,\gamma_{ij})$, where $i, j \in \mathbb{Z} / s$ are such that $\tilde{a}_i \neq \tilde{b}_j$, and all $\gamma_{ij} \in k$.

If $m_0 \in \mathbb{N}$ is such that $\tilde{a}_{i+m_0} < \tilde{b}_{j+m_0}$ and for $0 < m < m_0$, it holds $\tilde{a}_{i+m} = \tilde{b}_{j+m}$, then the extension $E_{cr}(i,j,\gamma_{ij})$ is trivial, cf. the proof of Proposition 1.10. Therefore, any $\mathcal{L}$ can be obtained as a sum of $E_{cr}(i,j,\gamma_{ij})$, where all $\gamma_{ij} \in k$ and the pairs $(i,j)$ are $(r_1,r_2)_{cr}$-admissible.

Prove the uniqueness of such presentation. Suppose $\mathcal{L}$ is given via the system of factors $v = \{v_j \mid j \in \mathbb{Z} / s\}$ such that for all $j \in \mathbb{Z} / s$, $v_j = \sum_i \gamma_{ij}u^{\tilde{a}_i}l^{(1)}_i$, where the coefficient $\gamma_{ij} = 0$ if $(i,j)$ is not $(r_1,r_2)_{cr}$-admissible. Suppose there is a system $w = \{w_j \in F(L) \mid j \in \mathbb{Z} / s\}$ such that $v_j = w_j - u^{\tilde{b}_j} \varphi(w_{j-1})$.

For some coefficients $\kappa_{ij} \in k$, $w_j = \sum_i \kappa_{ij}u^{\tilde{a}_i}l^{(1)}_i \mod uF(L)$ and, therefore,

$$v_j = \sum_i \left(\kappa_{ij}u^{\tilde{a}_i} - \kappa_{ij}^{p-1}u^{\tilde{b}_j}\right)l^{(1)}_i \mod uF(L).$$

It will be sufficient to prove that all $\kappa_{ij} = 0$.

Suppose $i_1,j_1 \in \mathbb{Z} / s$.

Choose the indices $i_0, j_0$ and $m_0 \geq 1$ such that

- $\tilde{a}_{i_0} \neq \tilde{b}_{j_0}$;
- $\tilde{a}_{i_0+m_0} \neq \tilde{b}_{j_0+m_0}$, but for all $0 < m < m_0$, $\tilde{a}_{i_0+m} = \tilde{b}_{j_0+m}$;
- $(i_1,j_1) \in \{(i_0 + m, j_0 + m) \mid 0 \leq m < m_0\}$.

Suppose $\tilde{b}_{j_0+m_0} < \tilde{a}_{i_0+m_0}$, that is $(i_0,j_0)$ is $(r_1,r_2)_{cr}$-admissible. Then relation (1.2) together with relations $\gamma_{i_0+m,j_0+m,b_{j_0+m}} = 0$, where $m \in \{m_0, m_0 - 1, \ldots, 1\}$, imply that $\kappa_{i_0+m_0-1,j_0+m_0-1} = \cdots = \kappa_{i_0,j_0} = 0$.

Suppose $\tilde{b}_{j_0+m_0} > \tilde{a}_{i_0+m_0}$, i.e. $(i_0,j_0)$ is not $(r_1,r_2)_{cr}$-admissible. Then relation (1.2) together with relations $\gamma_{i_0+m,j_0+m,a_{i_0+m}} = 0$, where $m \in \{0,1,\ldots,m_0-1\}$, imply that $\kappa_{i_0,j_0} = \cdots = \kappa_{i_0+m_0-1,j_0+m_0-1} = 0$.

In both cases we have $\kappa_{i_1,j_1} = 0$.

Proposition 1.12 gives a description of the subgroup $\text{Ext}_{cr}^{1}(\mathcal{L}_2, \mathcal{L}_1)$ of $\text{Ext}_{cr}^{1}(\mathcal{L}_2, \mathcal{L}_1)$. In particular, working modulo this subgroup we can describe the extensions in the whole category $\mathcal{L}^*$ via the systems of factors $\{v_j \mid j \in \mathbb{Z} / s\}$ such that the elements $v_j = \sum_{i,t} \gamma_{ijt}u^{t}_i$ satisfy the normalization conditions (C1) and

(C3) the coefficients $\gamma_{ijt} = 0$ if $t \geq \tilde{a}_i$.

**Proposition 1.13.** Suppose the system of factors $\{v_j \mid j \in \mathbb{Z} / s\}$ satisfies the conditions (C1) and (C3). Then it determines an element $\mathcal{L} = (L, F(L), \varphi, N)$ of the group $\text{Ext}_{cr}^{1}(\mathcal{L}_2, \mathcal{L}_1)$ if and only if the coefficients $\gamma_{ijk} \in k$ satisfy the following conditions:
a) if \( t = \tilde{a}_i - 1 \) and either \( \tilde{b}_j = p - 1 \) or \( \tilde{a}_i = 0 \) then \( \gamma_{ijt} = 0 \);

b) \( \gamma_{ijt} = 0 \) if \( t < \tilde{a}_i - 1 \);

c) if \( t = \tilde{a}_i - 1 \) and there is an \( m_0 \in \mathbb{N} \) such that \( \tilde{a}_{i+m_0} - 1 > \tilde{b}_{j+m_0} \) and for all \( 1 \leq m < m_0 \), \( \tilde{a}_{i+m} - 1 = \tilde{b}_{j+m} \), then \( \gamma_{ijt} = 0 \);

d) if \( t = \tilde{a}_i - 1 \) and for all \( m \in \mathbb{Z}/s \), \( \tilde{a}_{i+m} - 1 = \tilde{b}_{j+m} \) then \( \gamma_{ijt} = 0 \).

\[ \text{Proof. Suppose } v_j = \sum_i \gamma_{ijt} u^t l_i. \]

By our assumptions \( \gamma_{ijt} = 0 \) if \( t \geq \tilde{a}_i \) or \( t = \tilde{b}_j \). This immediately implies a) (for \( \tilde{b}_j = p - 1 \) use that \( u^{p-1}l_j \in F(L) \) implies that \( v_j \in F(L_1) \) should be zero by C(3)).

Then the relation \( uN(u^{\tilde{b}_j}l_j + v_j) \in F(L) \) implies that all \( \gamma_{ijt} = 0 \) if \( t < \tilde{a}_i - 1 \) (use congruence (1.1)) and b) is proved.

Now we can set for all indices \( i, j, \gamma_{ij} := \gamma_{i,j,\tilde{a}_i-1}. \)

Let \( \kappa_{ij} \in k \) be such that \( \kappa_{ij} \equiv \sum_i \kappa_{ij} l_i^{(1)} \mod u^pL \) and suppose \( \gamma_{ij} \neq 0 \) (this implies that \( \tilde{b}_j \neq \tilde{a}_i - 1 \)). For \( m \geq 0 \), consider the relations

\[ (1.3) \quad N(l_{j_{m+1}}) = \varphi(uN(u^{\tilde{b}_{j+m}}l_{j+m} + v_{j+m})). \]

If \( m = 0 \) then it implies \( \kappa_{i+1,j+1} = \gamma_{ij}^{(p)}(\tilde{b}_j - \tilde{a}_i + 1) \). Suppose that there is an \( m_0 \geq 0 \) such that for all \( 1 \leq m < m_0 \), \( \tilde{a}_{i+m} - 1 = \tilde{b}_{j+m} \) and \( \tilde{a}_{i+m_0} - 1 \neq \tilde{b}_{j+m_0} \). Then (1.3) together with congruence (1.1) (where \( j \) is replaced by \( j + m \)) imply that for \( 1 \leq m < m_0 \),

\[ \kappa_{i+m+1,j+m+1} = \kappa_{i+m,j+m}^{(p)}(\tilde{b}_j - \tilde{a}_i + 1). \]

In particular, \( N(l_{j+m_0}) \mod u^pL \) contains \( l_{i+m_0}^{(1)} \) with the coefficient \( \gamma_{ij}^{(p_m)}(\tilde{b}_j - \tilde{a}_i + 1) \). Therefore, \( uN(u^{\tilde{b}_{j+m_0}}l_{j+m_0} + v_{j+m_0}) \mod u^pL \) contains \( l_{j+m_0}^{(1)} \) with the coefficient \( u^{\tilde{b}_{j+m_0}+1} \gamma_{ij}^{(p_m)}(\tilde{b}_j - \tilde{a}_i + 1) \). But this monomial must belong to \( F(L_1) \). This proves that \( \tilde{b}_{j+m_0} + 1 > \tilde{a}_{i+m_0} \).

Finally, suppose that for all \( m \geq 1 \), \( \tilde{a}_{i+m} - 1 = \tilde{b}_{j+m} \). Then \( \tilde{a}_i - 1 = \tilde{a}_{i+s} - 1 = \tilde{b}_{j+s} = \tilde{b}_j \) and \( \gamma_{ij} = \gamma_{i,j,\tilde{a}_i-1} = \gamma_{i,j,\tilde{b}_j} = 0. \)

\[ \square \]

\[ \text{Definition. A pair } (i_0, j_0) \in (\mathbb{Z}/s)^2 \text{ is } (r_1, r_2)_{st}-\text{admissible if:} \]
\[ \bullet \tilde{b}_{j_0} \neq p - 1 \text{ and } \tilde{a}_{i_0} \neq 0; \]
\[ \bullet \tilde{a}_{i_0} - 1 \neq \tilde{b}_{j_0}; \]
\[ \bullet \text{there is an } m_0 = m_0(i_0, j_0) \in \mathbb{N} \text{ such that } \tilde{a}_{i_0+m_0} - 1 < \tilde{b}_{j_0+m_0} \text{ and for } 1 \leq m < m_0, \tilde{a}_{i_0+m} - 1 = \tilde{b}_{j_0+m}. \]

\[ \text{Definition. A pair } (i_0, j_0) \in (\mathbb{Z}/s)^2 \text{ is } (r_1, r_2)_{sp}-\text{admissible if } i_0 = 0 \text{ and for all } m \in \mathbb{Z}/s, \tilde{a}_m - 1 = \tilde{b}_{j_0+m}. \]

\[ \text{Proposition 1.14. a) If } (i_0, j_0) \text{ is an } (r_1, r_2)_{st}-\text{admissible pair then } \]
\[ r_1(i_0) + 1/(p - 1) > r_2(j_0); \]
\[ \text{b) if } (0, j_0) \text{ is an } (r_1, r_2)_{sp}-\text{admissible pair then } r_1 + 1/(p - 1) = r_2(j_0). \]
Proof. a) Here for $1 \leq m < m_0$, $a_{i_0 + m} + 1 = b_{j_0 + m}$ and $a_{i_0 + m_0} \geq b_{j_0 + m_0}$. Therefore,

$$r_1(i_0) + 1/(p-1) > \sum_{1 \leq m \leq m_0} (a_{i_0 + m} + 1)p^{-m} > \sum_{1 \leq m \leq m_0} b_{j_0 + m}p^{-m} + \sum_{m > m_0} (p - 1)p^{-m} \geq r_2(j_0).$$

The part b) can be obtained similarly. □

Proposition 1.13 implies the following statements.

**Proposition 1.15.** Suppose $(i_0, j_0) \in (\mathbb{Z}/s)^2$ is $(r_1, r_2)_{st}$-admissible and $\gamma \in k$. Then there is a unique $L \in \text{Ext}_{\mathcal{L}}(\mathcal{L}_2, \mathcal{L}_1)$ given by the system of factors $\{v_j \mid j \in \mathbb{Z}/s\}$ such that $v_{j_0} = \gamma u^{\tilde{a}_{i_0} - l^{(1)}_{i_0}}$ and $v_j = 0$ if $j \neq j_0$, and the map $N$ is uniquely determined by the condition:

- if $j \in \{j_0 + 1, \ldots, j_0 + m_0\}$ then $N(l_j) \equiv \gamma p^m (b_j - \tilde{a}_i + 1)l^{(1)}_{i + m} \mod u^p L$
- and, otherwise, $N(l_j) \equiv 0 \mod u^p L$.

**Proposition 1.16.** Suppose $(0, j_0) \in (\mathbb{Z}/s)^2$ is $(r_1, r_2)_{sp}$-admissible and $\gamma \in \mathbb{F}_q$, $q = p^s$. Then there is a unique $L \in \text{Ext}_{\mathcal{L}}(\mathcal{L}_2, \mathcal{L}_1)$ given by the zero system of factors and the map $N$ is uniquely determined by the condition:

- $N(l_{j_0 + m}) \equiv \gamma p^m l^{(1)}_m \mod (u^p L)$, where $m \in \mathbb{Z}/s$.

**Definition.** Set $E_{st}(i_0, j_0, \gamma) := L$, if $L$ is the extension from Proposition 1.15 and set $E_{sp}(j_0, \gamma) := L$ if $L$ is the extension from Proposition 1.16.

**Proposition 1.17.** Modulo the subgroup $\text{Ext}_{\mathcal{L}}(\mathcal{L}_2, \mathcal{L}_1)$ any element $L \in \text{Ext}_{\mathcal{L}}(\mathcal{L}_2, \mathcal{L}_1)$ can be obtained uniquely as a sum of the extensions $E_{st}(i, j, \gamma_{ij})$ and $E_{sp}(j, \gamma_j)$, where $(i, j)$, resp. $(0, j)$, runs over the set of all $(r_1, r_2)_{st}$-admissible, resp. $(r_1, r_2)_{sp}$-admissible, pairs from $(\mathbb{Z}/s)^2$, all $\gamma_{ij} \in k$ and all $\gamma_j \in \mathbb{F}_q$.

**Proof.** Proceed along the lines from the proof of Proposition 1.12. □

1.5. **Standard exact sequences.** Suppose $L = (\mathcal{L}, F(L), \varphi, N) \in \mathcal{L}^*$. Introduce a $\sigma$-linear map $\phi : L \longrightarrow L$ by the correspondence $\phi : l \mapsto \varphi(u^{p-1}l)$.

**Definition.** The module $L$ is etale (resp., connected) if $\phi \mod u$ is invertible (resp., nilpotent) on $L/uL$.

Denote by $\mathcal{L}^*_{et}$ (resp., $\mathcal{L}^*_{c}$) the full subcategory of $\mathcal{L}^*$ consisting of etale (resp. connected) objects. One can verify the following properties:

- $\mathcal{L} \in \mathcal{L}^*_{et}$ iff any its (strict) simple subquotient is isomorphic to $\mathcal{L}(0)$;
- $\mathcal{L} \in \mathcal{L}^*_{c}$ iff any its (strict) simple subquotient is isomorphic to $\mathcal{L}(r)$, where $r \in [0, 1/p] \setminus \{0\}$. 
Proposition 1.18. Any $\mathcal{L} \in \mathcal{L}^*$ contains a unique maximal etale subobject $(\mathcal{L}^e, i^e)$ and a unique maximal connected quotient object $(\mathcal{L}^c, j^c)$ and the sequence $0 \to \mathcal{L}^e \xrightarrow{i^e} \mathcal{L} \xrightarrow{j^e} \mathcal{L}^c \to 0$ is short exact.

Proof. Use that for any $r \in [0, 1]_p \setminus \{0\}$, it holds $\text{Hom}_{\mathcal{L}^*}(\mathcal{L}(0), \mathcal{L}(r)) = \text{Ext}_{\mathcal{L}^*}(\mathcal{L}(0), \mathcal{L}(r)) = 0$. □

Suppose $\mathcal{L} = (L, F(L), \varphi, N) \in \mathcal{L}^*$. Then $\varphi(F(L))$ is a $\sigma(W_1)$-module and $L = \varphi(F(L)) \otimes_{\sigma(W_1)} W_1$. If $l \in L$ and for $0 \leq i < p$, $l^{(i)} \in F(L)$ are such that $l = \sum_{0 \leq i < p} \varphi(l^{(i)}) \otimes u^i$ then set $V(l) = l^{(0)}$. Then $V \text{mod } u$ is a $\sigma^{-1}$-linear endomorphism of the $k$-vector space $L/u$.

Definition. The module $\mathcal{L}$ is multiplicative (resp., unipotent) if $V \text{mod } u$ is invertible (resp., nilpotent) on $L/uL$.

Denote by $\mathcal{L}^{*m}$ (resp., $\mathcal{L}^{*u}$) the full subcategory of $\mathcal{L}^*$ consisting of multiplicative (resp., unipotent) objects. One can verify the following properties:

• $\mathcal{L} \in \mathcal{L}^{*m}$ iff any its simple subquotient is isomorphic to $\mathcal{L}(1)$;
• $\mathcal{L} \in \mathcal{L}^{*u}$ iff any its simple subquotient is isomorphic to $\mathcal{L}(r)$, where $r \in [0, 1]_p \setminus \{1\}$.

Proposition 1.19. Any $\mathcal{L} \in \mathcal{L}^*$ contains a unique maximal multiplicative quotient object $(\mathcal{L}^m, j^m)$ and a unique maximal unipotent subobject $(\mathcal{L}^u, i^u)$ and the sequence $0 \to \mathcal{L}^u \xrightarrow{i^u} \mathcal{L} \xrightarrow{j^m} \mathcal{L}^m \to 0$ is short exact.

Proof. Use that for any $r \in [0, 1]_p \setminus \{1\}$, it holds $\text{Hom}_{\mathcal{L}^*}(\mathcal{L}(r), \mathcal{L}(1)) = \text{Ext}_{\mathcal{L}^*}(\mathcal{L}(r), \mathcal{L}(1)) = 0$. □
2. The functor $\mathcal{CV}^* : \mathcal{L}^* \rightarrow \text{CM}_{\mathcal{F}}$

2.1. $\mathcal{F}$-module $\mathcal{R}_{st}^0 \in \mathcal{L}^*$. Let $R = \lim_n (\hat{O}/p)_n$ be Fontaine’s ring; it has a natural structure of $k$-algebra via the map $k \rightarrow R$ given by $\alpha \mapsto \lim_n (\sigma^{-n}\alpha \mod p)$, where for any $\gamma \in k$, $[\gamma] \in W(k) \subset \hat{O}$ is the Teichmüller representative of $\gamma$.

Choose $x_0 = (x_0^{(n)} \mod p)_{n \geq 0} \in R$ and $\varepsilon = (\varepsilon^{(n)} \mod p)_{n \geq 0}$ such that for all $n \geq 0$, $x_0^{(n+1)} = x_0^{(n)}$ and $\varepsilon^{(n+1)} = \varepsilon^{(n)}$ with $x_0^{(0)} = -p$, $\varepsilon^{(0)} = 1$ but $\varepsilon^{(1)} \neq 1$. Set $R^0 = R/x^2p$.

Let $Y$ be an indeterminate.

Denote by $R^0(Y)$ the divided power envelope of $R^0[Y]$ with respect to the ideal $(Y)$. If for $j \geq 0$, $\gamma_j(Y)$ is the $j$-th divided power of $Y$ then $R^0(Y) = \oplus_{j \geq 0} R^0 \gamma_j(Y)$. Denote by $R_{st}^0$ its completion $\prod_{j \geq 0} R^0 \gamma_j(Y)$ and set, $\text{Fil}^p R_{st}^0 = \prod_{j \geq 1} R^0 \gamma_j(Y)$. Define the $\sigma$-linear morphism of the $R$-algebra $R^0(Y)$ by the correspondence $Y \mapsto x^p_0 Y$. We shall denote it below by the same symbol $\sigma$.

Introduce a $\mathcal{W}_1$-module structure on $R_{st}^0$ by the $k$-algebra morphism $\mathcal{W}_1 \rightarrow R_{st}^0$ such that $u \mapsto \iota(u) := x_0 \exp(-Y)$. Set $F(R_{st}^0) = \bigoplus_{0 \leq i < p} R^0 \gamma_i(Y) + \text{Fil}^p R_{st}^0$. Define the continuous $\sigma$-linear morphism of $R^0$-modules $\varphi : F(R_{st}^0) \rightarrow R_{st}^0$ by setting for $0 \leq i < p$, $\varphi(x_0^{p-1-i} \gamma_i(Y)) = \gamma_i(Y)(1 - (i/2)x_0^p Y)$, and for $i \geq p$, $\varphi(\gamma_i(Y)) = 0$. One can easily see that for any $a \in R_{st}^0$ and $b \in F(R_{st}^0)$, $\varphi(ab) = \sigma(a)\varphi(b)$.

Let $N$ be a unique $R$-differentiation of $R_{st}^0$ such that $N(Y) = 1$.

**Proposition 2.1.** a) $\varphi$ is a $\sigma$-linear morphism of $\mathcal{W}_1$-modules;

b) for any $b \in R_{st}^0$ and $w \in \mathcal{W}_1$, $N(wb) = N(w)b + wN(b)$;

c) for any $l \in F(R_{st}^0)$, $uN(l) \in F(R_{st}^0)$ and $N(\varphi(l)) = \varphi(uN(l))$.

**Proof.** a) Use that the multiplication by $\sigma(u) = u^p$ comes as the multiplication by $\iota(u)^p = x_0^p = x_0^p \exp(-x_0^p Y) = \sigma(\iota(u))$.

b) Use that $N(\iota(u)) = -\iota(u)$.

c) It will be enough to check the identity for $l = x_0^{p-1-i} \gamma_i(Y)$ with $1 \leq i < p$. Then $N(\varphi(l)) = \gamma_{i-1}(Y)(1 - (1/2)(i+1)x_0^p Y)$. On the other hand, $uN(l) = x_0^{p-1-(i-1)} \gamma_{i-1}(Y) \exp(-Y)$ and $\varphi(uN(l))$ is equal to $\gamma_{i-1}(Y)(1 - (1/2)(i-1)x_0^p Y) \exp(-x_0^p Y) = \gamma_{i-1}(Y)(1 - (1/2)(i+1)x_0^p Y)$.

\[\Box\]

**Corollary 2.2.** $\mathcal{R}_{st}^0 := (R_{st}^0, F(R_{st}^0), \varphi, N) \in \mathcal{L}^*$

2.2. The functor $\mathcal{V}^*$. If $\mathcal{L} = (L, F(L), \varphi, N) \in \mathcal{L}^*$ then the triple $(L, F(L), \varphi)$ is an object of $\mathcal{L}^*$ which will be denoted below by the same symbol $\mathcal{L}$. 

Definition. Let $\mathcal{R}^0 = (R^0, F(R^0), \varphi) \in \widehat{\mathcal{C}}^*$, where the $\mathcal{W}_1$-module structure on $R^0$ is given via $u \mapsto x_0$, $F(R^0) = x_0^{-1}R^0$ and for any $r \in F(R^0)$, $\varphi(r) = (\hat{r}/x_0^{p-1})^p \mod x_0^p$ with $\hat{r} \in R$ such that $\hat{r} \mod x_0^p = r$.

For any $\mathcal{L} = (L, F(L), \varphi, N) \in \mathcal{L}^*$, consider the abelian group $W(\mathcal{L}) = \text{Hom}_{\mathcal{L}^*}(\mathcal{L}, \mathcal{R}_0^0)$. If $f \in W(\mathcal{L})$ and $i \geq 0$, introduce the $k$-linear morphisms $f_i : L \rightarrow R^0$ such that for any $l \in L$, $f(l) = \sum_{i \geq 0} f_i(l)\gamma_i(Y)$. Then $f_0 \in W_0(L) := \text{Hom}_{\mathcal{L}^0}(\mathcal{L}, \mathcal{R}^0)$ and the correspondence $f \mapsto f_0$ gives the homomorphism of abelian groups $\text{pr}_0 : W(\mathcal{L}) \rightarrow W_0(\mathcal{L})$.

Proposition 2.3. $\text{pr}_0$ is isomorphism.

Proof. Suppose $f \in \text{Ker} \text{pr}_0$. Then for all $i \geq 0$ and $l \in L$, $f_i(l) = f_0(N^i(l))) = 0$, i.e. $f = 0$.

Suppose $g \in \text{Hom}_{\mathcal{L}^0}(\mathcal{L}, \mathcal{R}^0)$. This means that $g : L \rightarrow R^0$ is $\sigma$-linear morphism of $\mathcal{W}_1$-modules, $g(F(L)) \subset F(R^0)$ and for any $l \in F(L)$, $g(\varphi(l)) = (g(l)/x_0^{p-1})^p$.

Set for any $l \in L$, $f(l) = g(l) + g(Nl)\gamma_1(Y) + \cdots + g(N^il)\gamma_i(Y) + \cdots$.

Clearly, for any $l \in L$, $f(N(l)) = N(f(l))$ and our Proposition is implied by the following Lemma.

Lemma 2.4. a) for any $l \in L$, $f(ul) = x_0 \exp(-Y)f(l)$;

b) for any $l \in F(L)$, $\varphi(f(l)) = f(\varphi(l))$;

Proof of Lemma. a) For any $l \in L$, $f(ul) = \sum_{i \geq 0} g(N^i(ul))\gamma_i(Y) = x_0 \sum_{i, s} (-1)^s \binom{i}{s} g(N^s l)\gamma_s(Y)\gamma_j(Y) = x_0 \sum_{j, s} (-1)^s g(N^s l)\gamma_s(Y)\gamma_j(Y) = x_0 \exp(-Y)f(l)$.

b) Let $l \in L$. Prove by induction on $i \geq 1$ that $N^i(\varphi(l)) = \varphi((uN)^i(l)) = -\frac{i(i-1)}{2} u^p \varphi(u^{i-1}N^{i-1}(l)) + \varphi(u^iN^i(l))$.

This implies that $g(N^i(\varphi(l))) = -\frac{i(i-1)}{2} x_0^p \left( \frac{g(u^{i-1}N^{i-1}(l))}{x_0^{p-1}} \right)^p + \left( \frac{g(u^iN^i(l))}{x_0^{p-1}} \right)^p$.

Therefore, $f(\varphi(l))$ is equal to $\sum_{i \geq 0} g(N^i(\varphi(l)))\gamma_i(Y) = \sum_{i \geq 0} \left( \frac{g(N^i l)}{x_0^{p-1}} \right)^p \left( \gamma_i(Y) - \frac{i(i+1)}{2} x_0^p \gamma_{i+1}(Y) \right) = \varphi(f(l))$. 

Corollary 2.5. a) If $\text{rk}_{\mathcal{W}_1} L = s$ then $|W(\mathcal{L})| = p^s$;

b) the correspondence $\mathcal{L} \mapsto W(\mathcal{L})$ induces exact functor from $\mathcal{L}^*$ to the category of abelian groups.
Proof. a) Proceed as in [11, 3]. Suppose the structure of the filtered \( \varphi \)-module \( L \) is given by a choice of a \( \mathcal{W}_1 \)-basis \( m_1, \ldots, m_s \) of \( F(L) \) and a non-degenerate matrix \( A \in M_s(\mathcal{W}_1) \) such that \( (m_1, \ldots, m_s) = (\varphi(m_1), \ldots, \varphi(m_s)) \). Let \( X = (X_1, \ldots, X_s) \) be a vector with \( s \) independent variables. Let \( R_0 = \text{Frac} R \). Consider the quotient \( A_L \) of the polynomial ring \( R_0[\hat{X}] \) by the ideal generated by the coordinates of the vector \( (X_A)^p - x_0^{p(p-1)} \hat{X} \). Then \( A_L \) is etale \( R_0 \)-algebra of rank \( p^s \) (use that \( (u^{p-1}I_s)A^{-1} \in M_s(\mathcal{W}_1) \)) and all its \( R_0 \)-points give rise to the elements of \( W_0(L) \).

b) This follows from a) because the functor \( L \mapsto W_0(L) \) is left exact.

\[ \square \]

Introduce the ideal \( J^0 = \sum_{0 \leq i < p} x_0^{p+1-i} \gamma_i(Y) + \text{Fil}^0 R_{st}^0 \) of \( R_{st}^0 \). The quotient \( R_{st}^0/J^0 \) is provided with a continuous \( \Gamma_F \)-action as follows.

For any \( \tau \in \Gamma_F \), let \( k(\tau) \in \mathbb{Z} \) be such that \( \tau(x_0) = \varepsilon^k(\tau)x_0 \) and let \( \log(1 + X) = X - X^2/2 + \cdots - X^{p-1}/(p - 1) \) be the truncated logarithm. Then the cocycle relation \( \varepsilon^{k(\tau_1)}(\tau_1 \varepsilon)^k(\tau) = \varepsilon^{k(\tau_1 \tau)} \), where \( \tau_1, \tau \in \Gamma_F \), implies the cocycle relation

\[ k(\tau_1) \tilde{\log} \varepsilon + k(\tau) \tilde{\log}(\tau_1 \varepsilon) \equiv k(\tau_1 \tau) \tilde{\log} \varepsilon \mod x_0^{p^2/(p-1)}. \]

(Use that \( \tilde{\log} : (1 + \mathbb{F}_p[[X,Y]])^\times \to \mathbb{F}_p[[X,Y]] \) mod deg \( p \) is group homomorphism and \( \varepsilon \equiv 1 \mod x_0^{p/(p-1)} \).) Then we extend the natural \( \Gamma_F \)-action on \( R^0 \) to \( R_{st}^0/J^0 \) by setting for \( \tau \in \Gamma_F \), \( \tau(Y) := Y + k(\tau) \tilde{\log} \varepsilon \).

**Proposition 2.6.** a) The projection \( R_{st}^0 \to R_{st}^0/J^0 \) induces the embedding \( \iota_L : W(L) \to \text{Hom}_{L^0}(L, R_{st}^0/J^0) \), where \( R_{st}^0/J_0 \) denotes the \( \mathcal{W}_1 \)-module \( R_{st}^0/J^0 \) with the structure of an object of the category \( L^0 \) induced by the natural projection from \( R_{st}^0 \).

b) if \( g \in \iota_L(W(L)) \), \( \tau \in \Gamma_F \) and \( \tau^*(g) \in \text{Hom}_{L^0}(L, R_{st}^0/J^0) \) is such that for any \( l \in L, \tau^*(g)(l) = \tau(g(l)) \), then \( \tau^*(g) \in \iota_L(W(L)) \).

**Proof.** Part a) follows because \( \varphi \) is zero on \( J^0 \).

For part b), suppose that our filtered \( \varphi \)-module \( L \) is given in terms of vectors \( \tilde{m}, \tilde{l} = (\varphi(m_1), \ldots, \varphi(m_s)) \) and matrix \( A \) from the proof of Corollary 2.5. Notice that \( x_0 \exp(-Y) \equiv x_0 \exp(-Y) \mod J^0 \), where \( \exp \) is the truncated exponential. Therefore, the elements \( f \in W(L) \) can be uniquely characterized by the congruences \( \varphi(f(\tilde{m})) \equiv f(\tilde{l}) \mod J^0 \) and \( f(\tilde{m}) \equiv f(\tilde{l})A \mod J^0 \). Thus our Proposition is implied by the following Lemma.

**Lemma 2.7.** \( x_0 \exp(-Y) \mod J^0 \in (R_{st}^0/J^0)^{\Gamma_F} \).

**Proof.** We must prove that for any \( m \in \mathbb{Z} \),

\[ x_0 e^m \exp(-Y) \equiv x_0 \exp(-Y + m \tilde{\log} \varepsilon) \mod J^0. \]
Following the coefficients for \(\gamma_{i_0}(Y)\), \(0 \leq i_0 < p\), reduce this to proving of the congruence
\[
\varepsilon^m \equiv \sum_{0 \leq j < p-i_0} \gamma_j(m \log \varepsilon) \mod x_0^{n(i_0)}
\]
where \(n(i_0) \geq p - i_0\). Notice that
\[
P_{i_0} = \sum_{0 \leq j < p-i_0} \gamma_j(m \log(1 + X)) \in \mathbb{Z}_p[X]
\]
is congruent modulo \(X^{p-i_0}\) to the \((p - i_0 - 1)\)-th partial sum of the formal series \((1 + X)^m = \exp(m \log(1 + X))\). So, \((1 + X)^m - P_{i_0}\) belongs to \(X^{p-i_0}\mathbb{Z}_p[[X]]\). This implies that congruence (2.1) holds with \(n(i_0) = (p - i_0)/p/(p - 1) > p - i_0\) because \(\varepsilon \equiv 1 \mod(x_0^{p/(p-1)})\). \(\Box\)

**Definition.** The \(\Gamma_F\)-module \(\mathcal{V}^*(\mathcal{L})\) is the abelian group \(W(\mathcal{L})\) with the action of \(\Gamma_F\) defined via the embedding \(i_{\mathcal{L}}\). Then \(\mathcal{V}^* : \mathcal{L}^u \to M\Gamma_F\) is the functor induced by the correspondence \(\mathcal{L} \mapsto \mathcal{V}^*(\mathcal{L})\), where \(\mathcal{L} \in \mathcal{L}^u\).

For future references point out the following characterization of the \(\Gamma_F\)-modules \(\mathcal{V}^*(\mathcal{L})\) with \(\mathcal{L} \in \mathcal{L}^u\), which comes from the identification \(\text{Hom}_{\mathcal{L}^u}(\mathcal{L}, \mathcal{R}^0) = \text{Hom}_{\mathcal{L}^u}(\mathcal{L}, \mathcal{R}_{st}^0)\) of Proposition 2.3.

**Corollary 2.8.**

\[
\mathcal{V}^*(\mathcal{L}) = \left\{ \sum_{0 \leq i < p} N^i(f_0)\gamma_i(Y) \mod J^0 \mid f_0 \in \text{Hom}_{\mathcal{L}^u}(\mathcal{L}, \mathcal{R}^0) \right\}
\]

We also can use \(\tilde{R}^0 = (R/x_0^p m_R, x_0^{p-1} R/x_0^p m_R, \varphi) \in \mathcal{L}^u\), where \(\varphi(r) = (r/x_0^{p-1})^p\), and the ideal \(\tilde{J}^0 = \sum_{0 \leq i < p} x_0^{p-i} m_R \gamma_i(Y) + \text{Fil}^p R_{st}^0\) to state the following analogue of the above Corollary.

**Corollary 2.9.**

\[
\mathcal{V}^*(\mathcal{L}) = \left\{ \sum_{0 \leq i < p} N^i(f_0)\gamma_i(Y) \mod \tilde{J}^0 \mid f_0 \in \text{Hom}_{\mathcal{L}^u}(\mathcal{L}, \tilde{R}^0) \right\}
\]

**Remark.** If \(\mathcal{L} \in \mathcal{L}^u\) then in the above Corollary we can replace \(\tilde{R}^0\) and \(\tilde{J}^0\) by, respectively, \(\tilde{R}^u = (R/x_0^p, x_0^{p-1} R/x_0^p, \varphi) \in \mathcal{L}^u\) and the ideal \(\tilde{J}^u = \sum_{0 \leq i < p} R x_0^{p-i} \gamma_i(Y) + \text{Fil}^p R_{st}^0\).

2.3. The category \(\text{CM}^{\Gamma_F}\) and the functor \(CV^*\).

**Definition.** The objects of the category \(\text{CM}^{\Gamma_F}\) are the triples \(\mathcal{H} = (H, H^0, j)\), where \(H, H^0\) are finite \(\mathbb{Z}_p[\Gamma_F]\)-modules, \(\Gamma_F\) acts trivially on \(H^0\) and \(j : H \to H_0\) is an epimorphic map of \(\mathbb{Z}_p[\Gamma_F]\)-modules. If \(\mathcal{H}_1 = (H_1, H_1^0, j_1) \in \text{CM}^{\Gamma_F}\) then \(\text{Hom}_{\text{CM}^{\Gamma_F}}(\mathcal{H}_1, \mathcal{H})\) consists of the couples \((f, f^0)\), where \(f : H_1 \to H\) and \(f^0 : H_1^0 \to H^0\) are morphisms of \(\Gamma_F\)-modules such that \(j_1 f^0 = f j\).
The category $\mathbf{CMΓF}$ is pre-abelian, cf. Appendix A, and its objects have a natural group structure. In particular, with notation from the above definition, Ker$(f,f^0) = (\text{Ker}f,j_1(\text{Ker}f))$ together with natural embedding to $\mathcal{H}_1$. Similarly, Coker$(f,f^0) = (H/f(H_1),H^0/j(f(H_1)))$. For example, the map $(\text{id},0) : (H,H) \rightarrow (H,0)$ has the trivial kernel and cokernel. In addition, the monomorphism $(f_1,f^0_1) : \mathcal{H}_1 \rightarrow \mathcal{H}$ is strict if and only if $f_1(\text{Ker}j_1) = f_1(H_1) \cap \text{Ker}i$. Suppose $\mathcal{H}_2 = (H_2,H^0_2,j_2)$ and $(f_2,f^0_2) : \mathcal{H} \rightarrow \mathcal{H}_2$ is an epimorphism. Then it is strict if and only if $f^0_2$ induces an epimorphic map of Ker$j$ to Ker$j_2$. We can use the formalism of short exact sequences and the corresponding 6-term Hom$_{\mathbf{CMΓF}}$ - Ext$_{\mathbf{CMΓF}}$ exact sequences, cf. Appendix A.

**Definition.** Suppose $\mathcal{L} \in \mathcal{L}^*$ and $i^\ast : \mathcal{L}^\ast \rightarrow \mathcal{L}$ is the embedding of the maximal etale subobject. Then $\mathcal{CV}^* : \mathcal{L}^* \rightarrow \mathbf{CMΓF}$ is the functor such that $\mathcal{CV}^*(\mathcal{L}) = (\mathcal{V}^*(\mathcal{L}),\mathcal{V}^*(\mathcal{L}^\ast),\mathcal{V}^*(i^\ast))$.

The simple objects in $\mathbf{CMΓF}$ are of the form either $(H,0,0)$, where $H$ is a simple $\mathbb{Z}_p[\Gamma_F]$-module, or $(\mathbb{F}_p,\mathbb{F}_p,\text{id})$, where $\mathbb{F}_p$ is provided with the trivial $\Gamma_F$-action. In this context it will be very convenient to use the following formalism.

For $s \in \mathbb{N}$, introduce the continuous characters $\chi_s : \Gamma_F \rightarrow k^*$ such that for $\tau \in \Gamma_F$, $\chi_s(\tau) = (\tau x_s)/x_s$ mod $x^p_0$, where $x_s \in R$ is such that $x^p_s - 1 = x_0$. If $\chi$ is any continuous character of $\Gamma_F$ then there are $s,m \in \mathbb{N}$ such that $0 < m \leq p^s - 1$ and $\chi = \chi_s^m$. Set $r(\chi) = m/(p^s - 1)$. Then $r(\chi)$ depends only on $\chi$ and the correspondence $\chi \mapsto r(\chi)$ gives a bijection of the set of all continuous characters of $\Gamma_F$ (with the values in $k^*$) and $[0,1][p] \setminus \{0\}$.

For $r \in [0,1][p]$, $r \neq 0$, introduce the $\Gamma_F$-module $\mathbb{F}(r)$ such that $\mathbb{F}(r) = \mathbb{F}_{p(r)}$, where $s(r)$ is the period of the $p$-digit expansion of $r$, cf. Subsection 1.2 with the $\Gamma_F$-action given by the character $\chi$ such that $r(\chi) = r$. We have:

- all $\mathbb{F}(r)$ are simple $\mathbb{Z}_p[\Gamma_F]$-modules;
- $\Gamma_F$-modules $\mathbb{F}(r_1)$ and $\mathbb{F}(r_2)$ are isomorphic if and only if there is an $n \in \mathbb{Z}$ such that $r_1 = r_2(n)$;
- any simple $\mathbb{Z}_p[\Gamma_F]$-module is isomorphic to some $\mathbb{F}(r)$.

It will be natural to set $\mathcal{F}(r) := (\mathbb{F}(r),0,0)$ for all $r \in (0,1][p]$, and to set separately $\mathcal{F}(0) := (\mathbb{F}_p,\mathbb{F}_p,\text{id})$.

With above notation we have the following property, where the objects $\mathcal{L}(r)$ were introduced in Subsection 1.3.

**Proposition 2.10.** For any $r \in [0,1][p]$, $\mathcal{CV}^*(\mathcal{L}(r)) = \mathcal{F}(r)$.

2.4. **A criterion.** Suppose $\mathcal{L}_1,\mathcal{L}_2$ are given in notation of Subsection 1.4 and $q = p^s$. Then for $i = 1,2$, $\mathcal{V}^*(\mathcal{L}_i) = V_i$ are 1-dimensional vector spaces over $\mathbb{F}_q$ with $\Gamma_F$-action given by the character $\chi_i : \Gamma_F \rightarrow k^*$ such that $r(\chi_i) = r_i$. (Note that $(q - 1)r_i \in \mathbb{Z}$ and, therefore, $\chi_i(\Gamma_F) \subset \mathbb{F}_q^\ast$.) Choose $\pi_s \in \bar{F}$ such that $\pi^{q^s-1} = -p$. Then $\bar{F}_s = F(\pi_s)$ is a tamely
ramified extension of $F$ of degree $q - 1$ and all points of $V_i$ are defined over $F_s$. We can identify $V_i$ with the $F_p[\Gamma_F]$-module $\mathbb{F}_q \tilde{\pi}^{(q-1)r_i} \subset \hat{O}/p\hat{O}$, where $\tilde{\pi} = \pi_s \mod p$. These identifications allow us to fix the points $h^0_i := \tilde{\pi}^{(q-1)r_i} \in V_i$ and we have $V_i = \{\alpha h^0_i \mid \alpha \in \mathbb{F}_q\}$.

Suppose $h_1 \in V_1$. Define the homomorphism

$$F_{h_1} : \text{Ext}_{F_p[\Gamma_F]}(V_1, V_2) \longrightarrow Z^1(\Gamma_F, \mathbb{F}_q) = \text{Hom}(\Gamma_F, \mathbb{F}_q),$$

where $\Gamma_F = \text{Gal}(\overline{F}/F_s)$, as follows. If $V \in \text{Ext}_{F_p[\Gamma_F]}(V_1, V_2)$ and $h \in V$ is a lift of $h_1$ then $F_{h_1}(V) = \{a \in \mathbb{F}_q \mid \tau \in \Gamma_F\}$ such that $\tau h - h = a_s h^0_i$.

Obviously, we have the following criterion.

**Proposition 2.11.** $V$ is the trivial extension if and only if for all $h_1 \in V_1$, it holds $F_{h_1}(V) = 0$.

### 2.5. Galois modules $\mathcal{V}^*(E_{cr}(i_0, j_0, \gamma))$.

Suppose we have an object $\mathcal{L} = (L, F(L), \varphi, N)$ of the category $\mathcal{L}_c^*$. Then there is a special $\sigma$($\mathcal{V}_1$)-basis $l_1, \ldots, l_s$ of $\varphi(F(L))$ such that for some integers $0 \leq c_1, \ldots, c_s < p$ and a matrix $A \in \text{GL}_c(k)$, the elements $u^{c_1}l_1, \ldots, u^{c_s}l_s$ form a $\mathcal{V}_1$-basis of $F(L)$ and

$$(\varphi(u^{c_1}l_1), \ldots, \varphi(u^{c_s}l_s)) = (l_1, \ldots, l_s)A.$$  

For $1 \leq i \leq s$, set $\tilde{c}_i = (p - 1) - c_i$. The following Proposition is a special case of Corollary 2.9.

**Proposition 2.12.** With above notation, $\mathcal{V}^*(\mathcal{L})$ is the $F_p[\Gamma_F]$-module of all $(\theta_1, \ldots, \theta_s) \mod x_0^m \mathbb{F}_R \in (R/x_0^m \mathbb{F}_R)^s$ such that

$$(\theta_1^p/x_0^{\tilde{c}_1}, \ldots, \theta_s^p/x_0^{\tilde{c}_s}) = (\theta_1, \ldots, \theta_s)A.$$  

Remark. In [1] [2] it was proved that the category of $F_p[\Gamma_F]$-modules $\mathcal{V}^*(\mathcal{L})$, where $\mathcal{L} \in \mathcal{L}_c^*$, coincides with the category of all killed by $p$ subquotients of crystalline representations of $\Gamma_F$ with weights from $[0, p)$. This result also reappears in Subsection 4 where we establish that the category of $F_p[\Gamma_F]$-modules $\mathcal{V}^*(\mathcal{L})$, where $\mathcal{L} \in \mathcal{L}_c^*$, coincides with the category of all killed by $p$ subquotients of semi-stable representations of $\Gamma_F$ with weights from $[0, p)$.

For an $(r_1, r_2)_{cr}$-admissible pair $(i_0, j_0) \in (\mathbb{Z}/s)^2$ and $\gamma \in k$, use the description of $E_{cr}(i_0, j_0, \gamma)$ from Subsection 1.4. Then by Corollary 2.9, $V = \mathcal{V}^*(E_{cr}(i_0, j_0, \gamma))$ is identified with the additive group of all solutions in $R$ of the following system of equations

$$X_i^{(1)p}/x_0^{p\alpha_i} = X_i^{(1)}, \quad X_i^{(1)}, \quad \text{for all } i \in \mathbb{Z}/s;$$

$$X_j^{p}/x_0^{pb_j} = X_{j+1} - \delta_{j0} \gamma^p X_{i_0+1}^{(1)}, \quad \text{for all } j \in \mathbb{Z}/s$$

Note that the first group of equations describes $V_1 = \mathcal{V}^*(\mathcal{L}_1)$ and the correspondences $X_i^{(1)} \mapsto 0$ and $X_j \mapsto X_j^{(2)}$ with $i, j \in \mathbb{Z}/s$, define the map $V \rightarrow V_2$, where $V_2 = \mathcal{V}^*(\mathcal{L}_2)$ is associated with all solutions in $R$ of the equations $X_j^{(2)p}/x_0^{pb_j} = X_j^{(2)}$, $j \in \mathbb{Z}/s$. As it was noted
in Subsection 2.2 the correspondent $\Gamma_F$-action on $V$, $V_1$ and $V_2$ comes from the natural $\Gamma_F$-action on $R/x_0^m R$.

Take $x_s \in R$ such that $x_s^{p-1} = x_0$ and $x_s \mapsto \pi_s \mod p$ under the natural identification $R/x_0^m \cong \mathcal{O}/p\mathcal{O}$. (This identification is given by the correspondence $r = \lim(r_n \mod p) \mapsto r^{(1)} := \lim r_n^{p^m}$. For $i, j \in \mathbb{Z}/s$, set $x_0^{r_1(i)} := x_s^{(q-1)r_1(i)}$ and $x_0^{r_2(j)} := x_s^{(q-1)r_2(j)}$, and introduce the variables $Z_i^{(1)} = x_0^{-pr_1(i)} X_i^{(1)}$, $Z_j = x_0^{-pr_2(j)} X_j$, $Z_j^{(2)} = x_0^{-pr_2(j)} X_j^{(2)}$. Then the elements of $V$ appear as the solutions in $R_0 := \text{Frac}(R)$ of the following system of equations

$$Z_i^{(1)p} = Z_i^{(1)}, \quad Z_j^q = Z_j, \quad Z_{j_0+1}^q - Z_{j_0+1}^{(1)p} = \gamma p Z_{i_0+1}^{(1)} x_0^{(r_1(i_0)-r_2(j_0))}$$

Note that for the points $h_1^0 \in V_1$ and $h_2^0 \in V_2$ chosen in Subsection 2.4 it holds $Z_i^{(1)}(h_0) = Z_i^{(2)}(h_0) = 1$, where $i \in \mathbb{Z}/s$.

Suppose $\alpha \in \mathbb{F}_q$ and $h_1 = \alpha h_0^1 \in V_1$.

Let $\mathcal{F}_s = k((x_s)) \subset R_0 = \text{Frac} R$. Then the field-of-norms functor gives a natural embedding of the absolute Galois group $\Gamma_{\mathcal{F}_s}$ of $\mathcal{F}_s$ into $\Gamma_{\mathcal{F}_s}$, where $F_s = F(\pi_s)$. Then the restriction $F_{h_1}(V)|_{\Gamma_{\mathcal{F}_s}}$ of the cocycle

$$F_{h_1}(V) = \{ A_{\tau,\alpha}(i_0, j_0, \gamma) \in \mathbb{F}_q \mid \tau \in \Gamma_{\mathcal{F}_s} \}$$

from Subsection 2.4 can be described as follows.

Let $U \in R_0$ be such that $U - U^q = \gamma x_0^{r_1(i_0)-r_2(j_0)}$. Then for any $\tau \in \Gamma_{\mathcal{F}_s}$, $\sigma^{j_0}(A_{\tau,\alpha}(i_0, j_0, \gamma)) = \sigma^{j_0}(\alpha)(\tau(U) - U)$ and therefore

$$A_{\tau,\alpha}(i_0, j_0, \gamma) = \sigma^{j_0}(\alpha)\sigma^{-j_0}(\tau U - U).$$

The following Lemma is an immediate consequence of the definition of $(r_1, r_2)_{cr}$-admissible pairs.

**Lemma 2.13.** With above notation let $C = -(q-1)(r_1(i_0) - r_2(j_0))$. Then $C$ is a prime to $p$ integer and $1 \leq C \leq q - 1$.

2.6. **Galois modules** $\mathcal{V}^*(E_{\mathcal{F}}(i_0, j_0, \gamma))$. For an $(r_1, r_2)_{cr}$-admissible pair $(i_0, j_0) \in (\mathbb{Z}/s)^2$ and $\gamma \in k$, use the description of $E_{\mathcal{F}}(i_0, j_0, \gamma)$ from Subsection 1.4.

By Subsection 2.2 $V = \mathcal{V}^*(E_{\mathcal{F}}(i_0, j_0, \gamma))$ is identified (as an abelian group) with the solutions $

$$\left( \{ X_i^{(1)} \mid i \in \mathbb{Z}/s \}, \{ X_j \mid j \in \mathbb{Z}/s \} \right) \in \mathbb{R}^{2s}$$

of the following system of equations

$$X_i^{(1)p}/x_0^{p a_i} = X_{i+1}^{(1)}, \text{ for all } i \in \mathbb{Z}/s; \quad X_j^p/x_0^{p b_j} + \delta_{j_0}(p X_i^{(1)} + x_0^{p a_i} + p = X_{j+1}, \text{ for all } j \in \mathbb{Z}/s.$$
the natural $\Gamma_F$-action on $\tilde{R}^0_{st}$, and the embedding of $V$ into $(R^0_{st})^{2s}$
given by the following correspondences:

$-\text{ if } i \in \mathbb{Z}/s \text{ then } X^{(1)}_i \mapsto X^{(1)}_i \mod x^2_0$;

$-\text{ if } j \notin \{j_0 + 1, \ldots, j_0 + m_0\} \text{ then } X_j \mapsto X_j \mod x^2_0$;

$-\text{ for } 1 \leq m \leq m_0, X^{j_0+m} \mapsto X^{j_0+m} + \gamma^m (\tilde{b} \tilde{a}_0) X^{(1)}_{j_0+m} \mod x^2_0$.

Similarly to Subsection 2.5, introduce the new variables by the re-

lations $Z^{(1)}_i = x^{-pr(i)}_0 X^{(1)}_i$, $Z_i = x^{-pr_i(i)}_0 X_i$ and $Z^{(2)}_i = x^{-pr_2(i)}_0 X^{(2)}_i$,

$i \in \mathbb{Z}/s$, and rewrite system of equations (2.2) in the following form:

$Z^{(1)}_i = Z_{i+1}$, for all $i \in \mathbb{Z}/s$;

$Z^{(2)}_j = Z_{j+1}$, for all $j \neq j_0 + 1$;

$Z_{j_0+1} - Z^{(2)}_{j_0+1} = \gamma^p Z^{(1)}_{j_0+1} x^2_0 (r_1(i_0) - r_2(j_0))$.

If $\alpha \in \mathbb{F}_q$ and $h_1 = \alpha^h_1 \in V_1$, then the restriction to $\Gamma_{F_2}$ of the
cocycle

$F_{h_1}(V) = \{A_{\tau,\alpha}(i_0, j_0, \gamma) \mid \tau \in \Gamma_{F_2}\}$
can be described as follows. Let $U \in R_0$ be such that

$U - U^q = \gamma x_0^{r_1(i_0) - r_2(j_0)}$.

Then for any $\tau \in \Gamma_{F_2}$, $\sigma^{j_0}(A_{\tau,\alpha}(i_0, j_0, \gamma)) = \sigma^{i_0}(\alpha)(\tau U - U$).

Thus

$A_{\tau,\alpha}(i_0, j_0, \gamma) = \sigma^{i_0 - j_0}(\alpha) \sigma^{j_0}(\tau U - U)$.

The following Lemma is a direct consequence of the definition of

$(r_1, r_2)_{st}$-admissible pairs, cf. also Proposition 1.14

Lemma 2.14. Let $C = -(q - 1)(r_1(i_0) - r_2(j_0) - 1)$. Then $C$ is a
prime to $p$ integer such that $1 \leq C < (q - 1)(1 + 1/(p - 1))$.

2.7. Galois modules $E_{sp}(j_0, \gamma)$. In this subsection $(0, j_0)$ is an $(r_1, r_2)_{sp}$-
admissible pair (i.e. $r_1 + 1/(p - 1) = r_0(j_0)$) and $\gamma \in \mathbb{F}_q$. Then

$V = V^s(E_{sp}(j_0, \gamma))$ is identified as an abelian group with the solutions

$(\{X^{(1)}_i \mid i \in \mathbb{Z}/s\}, \{X^{(2)}_j \mid j \in \mathbb{Z}/s\}) \in R^{2s}$
of the following system of equations

$X^{(1)}_i/p x_0^{pa_i} = X^{(1)}_{i+1}$, for all $i \in \mathbb{Z}/s$;

$X^{(2)}_j/p x_0^{pb_j} = X^{(2)}_{j+1}$, for all $j \in \mathbb{Z}/s$.

The corresponding $\Gamma_F$-action comes from the natural $\Gamma_F$-action on $\tilde{R}^0_{st}$
and the embedding of $V$ into $(R^0_{st})^{2s}$ given by the following correspondences:

$-\text{ if } i \in \mathbb{Z}/s \text{ then } X^{(1)}_i \mapsto X^{(1)}_i \mod x^2_0$;

$-\text{ if } m \in \mathbb{Z}/s \text{ then } X^{(2)}_{j_0+m} \mapsto X^{(2)}_{j_0+m} + \gamma^m X^{(1)}_m \mod x^2_0$.

If $\alpha \in \mathbb{F}_q$ and $h_1 = \alpha^h_1 \in V_1$ then the cocycle

$F_{h_1}(V) = \{A_{\tau,\alpha}^p(j_0, \gamma) \mid \tau \in \Gamma_{F_2}\}$.
can be described as follows. Note that the point \( h_1 \) corresponds to the collection \( \{ \sigma^i(\alpha)x_0^{pr(i)} \mid i \in \mathbb{Z}/s \}, \{ \sigma^i-j_0(\alpha\gamma)x_0^{pr(i-j_0)}Y \mid i \in \mathbb{Z}/s \} \). Then for \( \tau \in \Gamma_{F_s} \), \( \tau(h_1) \) corresponds to the collection
\[
\{ \sigma^i(\alpha)x_0^{pr(i)} \mid i \in \mathbb{Z}/s \}, \{ \sigma^i-j_0(\alpha\gamma)x_0^{pr(i-j_0)}(Y + k(\tau)\log \varepsilon) \mid i \in \mathbb{Z}/s \}.
\]
Therefore, \( \tau(h_1) - h_1 \) corresponds to the collection
\[
\{ 0 \mid i \in \mathbb{Z}/s \}, \{ \sigma^i-j_0(\alpha\gamma)x_0^{pr(i)}k(\tau) \mid i \in \mathbb{Z}/s \},
\]
which corresponds to \( \sigma^{-j_0}(\alpha\gamma)h_1^0 \). Therefore, \( A_{r,\alpha}^{sp}(j_0, \gamma) = \sigma^{-j_0}(\alpha\gamma)k(\tau) \).

Notice that for any \( \tau \in \Gamma_{F_s} \subset \Gamma_{F_1} \), \( A_{r,\alpha}^{sp}(j_0, \gamma) = 0 \).

2.8. Fully faithfulness of \( CV^* \).

In this subsection we prove the following important property.

**Proposition 2.15.** The functor \( CV^* \) is fully faithful.

**Proof.** We must prove that for all \( L_1, L_2 \in \mathcal{L}^* \), the functor \( CV^* \) induces a bijective map
\[
\Pi(L_1, L_2) : \text{Hom}_{\mathcal{L}}(L_2, L_1) \rightarrow \text{Hom}_{CM,\mathfrak{p}}(CV^*(L_1), CV^*(L_2)).
\]

By induction on lengths of composition series for \( L_1 \) and \( L_2 \) it will be sufficient to verify that for any two simple objects \( L_1 \) and \( L_2 \):

- \( \Pi(L_1, L_2) \) is bijective;
- the functor \( CV^* \) induces an injective map
\[
\Pi(L_1, L_2) : \text{Ext}_{\mathcal{L}}^*(L_2, L_1) \rightarrow \text{Ext}_{CM,\mathfrak{p}}(CV^*(L_1), CV^*(L_2)).
\]

The first fact has been already checked in Subsection 2.3.

In order to verify the second property, notice that for any two objects \( L_1, L_2 \in \mathcal{L}^* \), the natural map
\[
\text{Ext}_{CM,\mathfrak{p}}(CV^*(L_1), CV^*(L_2)) \rightarrow \text{Ext}_{\mathfrak{p}}(V^*(L_1), V^*(L_2))
\]
is injective. Therefore, we can prove the injectivity of \( \Pi(L_1, L_2) \) on the level of functor \( V^* \). In addition, for \( n_1, n_2 \in \mathbb{N} \), \( \text{Ext}_{\mathcal{L}}^*(L_1^{n_1}, L_1^{n_2}) = \text{Ext}_{\mathcal{L}}^*(L_2, L_1)^{n_1n_2} \) (the formation of Ext is compatible with direct sums). Therefore, we can assume that \( L_1, L_2 \) are products of copies of simple objects introduced in Subsection 2.3.

By Propositions 1.12 and 1.17 any element of \( \text{Ext}_{\mathcal{L}}^*(L_2, L_1) \) appears as a sum of standard extensions of the form \( E_{sr}(i, j, \gamma_{ij}) \), \( E_{st}(i, j, \gamma_{ij}) \) and \( E^{sp}(j, \gamma^{sp}_j) \), where \( (i, j) \) is either \( (r_1, r_2)_{sr} \)-admissible or \( (r_1, r_2)_{st} \)-admissible, all \( \gamma_{ij} \in k, j \in \mathbb{Z}/s \) is such that \( (0, j) \) is \( (r_1, r_2)_{sp} \)-admissible and \( \gamma^{sp} \in \mathbb{F}_q \). (Note that \( (i, j) = (0, j) \) can be \( (r_1, r_2)_{sr} \)-admissible and \( (r_1, r_2)_{sp} \)-admissible at the same time.) By results of Subsections 2.5, 2.7, we can attach to these standard extensions the 1-cocycles \( A_{r,\alpha}(i, j, \gamma_{ij}) \) and \( A_{r,\alpha}^{sp}(j, \gamma^{sp}_j) \), where \( \tau \in \Gamma_{F_s} \). It remains to prove that
the sum of these cocycles is trivial only if all corresponding coefficients 
\( \gamma_{ij} \) and \( \gamma_{isp} \) are equal to 0.

First, we need the following lemma.

**Lemma 2.16.** Suppose for all \((i,j) \in (\mathbb{Z}/s)^2\), the elements \( U_{ij} \in R_0 = \text{Frac} R \) are such that \( U_{ij} - U_{ij}^p = \gamma_{ij} x_{s,ij} \), where all \( \gamma_{ij} \in k \) and all \( C_{ij} \) are prime to \( p \) natural numbers. For \( \tau \in \Gamma_{F_s} \), let \( B_\tau(i,j,\gamma_{ij}) = \tau(U_{ij}) - U_{ij} \in F_q \). If for all \( \alpha \in F_q \) and all \( \tau \in \Gamma_{F_s} \),

\[
(2.3) \quad \sum_{i,j \in \mathbb{Z}/s} \sigma^{i-j}(\alpha) \sigma^{-j}B_\tau(i,j,\gamma_{ij}) = 0
\]

then all \( \gamma_{ij} = 0 \).

**Proof of Lemma.** For different prime to \( p \) natural numbers \( C_{ij} \) the extensions \( F_s(U_{ij}) \) behave independently. Therefore, we can assume that all \( C_{ij} = C \) are the same.

Let \( j_0 = j_0(j) \) be such that \( 0 \leq j_0 < s \) and \( j_0 \equiv -j \bmod s \). Then \((2.3)\) means that for any \( \alpha \in F_q \),

\[
B_\alpha := \sum_{i,j \in \mathbb{Z}/s} \sigma^{i-j}(\alpha) \sigma^{j_0}(U_{ij}) \in F_s.
\]

Then

\[
B_\alpha - B_\alpha^p = \sum_{j \in \mathbb{Z}/s} \left( \sum_{i \in \mathbb{Z}/s} \sigma^{i-j}(\alpha) \gamma_{ij}^{p^{-j}} \right) x_{s,-p^{j_0}C}.
\]

Looking at the Laurent series of \( B_\alpha \in F_s \) we conclude that all \( B_\alpha \in F_q \). This means that for all \( j \in \mathbb{Z}/s \) and \( \alpha \in F_q \), \( \sum_{i \in \mathbb{Z}/s} \sigma^i(\alpha) \gamma_{ij} = 0 \) and, therefore, all \( \gamma_{ij} = 0 \). The lemma is proved \( \square \)

Now suppose that for all \( \alpha \in F_q \) and \( \tau \in \Gamma_{F_s} \), the sum of cocycles \( A_{\tau,\alpha}(i,j,\gamma_{ij}) \) and \( A_{\tau,\alpha}^{sp}(j,\gamma_{jsp}) \) is zero. Restrict this sum to the subgroup \( \Gamma_{F_s} \). Then all \( sp \)-terms will disappear and by above Lemma 2.16 all \( \gamma_{ij} = 0 \). So, for all \( \tau \in \Gamma_{F_s} \) and \( \alpha \in F_q \), \( \sum_{j \in \mathbb{Z}/s} \sigma^{-j}(\alpha \gamma_{jsp}) = 0 \), and this implies that all \( \gamma_{js} = 0 \). \( \square \)

**Corollary 2.17.** The functor \( \mathcal{V}^* \) is fully faithful on the subcategories of unipotent objects \( \mathcal{L}^{*u} \) and of connected objects \( \mathcal{L}^{*c} \).

**Proof.** Indeed, on both categories the map \( \Pi(\mathcal{L}_1, \mathcal{L}_2) \) is already bijective on the level of functor \( \mathcal{V}^* \). \( \square \)

2.9. **Ramification estimates.** Suppose \( \mathcal{L} \in \mathcal{L}^* \) and \( H = \mathcal{V}^*(\mathcal{L}) \). For any rational number \( v \geq 0 \), denote by \( \Gamma_{F}^{(v)} \) the ramification subgroup of \( \Gamma_{F} \) in upper numbering, \( \mathbb{L}^* \).

**Proposition 2.18.** If \( v > 2 - \frac{1}{p} \) then \( \Gamma_{F}^{(v)} \) acts trivially on \( H \).
A proof can be obtained along the lines of the paper [13] (which adjusts Fontaine’s approach from [10]). Alternatively, one can apply author’s method from [3]: if $\tau \in \Gamma^{(v)}$ with $v > 2 - 1/p$ then there is an automorphism $\psi$ of $R$ such that $\psi(x_0) = \tau(x_0)$ and $\psi$ induces the trivial action on $H$; therefore we can assume that $\tau$ comes from the absolute Galois group of $k((x_0))$ and the characteristic $p$ approach from [3] gives the ramification estimate which coincides with the required by the theory of field-of-norms.

**Corollary 2.19.** If $\tilde{F}$ is the common field-of-definition of points of $\mathbb{F}_p[\Gamma_F]$-modules $\mathcal{V}(\mathcal{L})$ for all $\mathcal{L} \in \mathcal{L}^*$, then $v_p(D(\tilde{F}/F)) < 3 - \frac{1}{p}$, where $D(\tilde{F}/F)$ is the different of the field extension $\tilde{F}/F$. 
3. Semistable representations with weights from \([0, p]\) and filtered \(W\)-modules

3.1. The ring \(S\). Let \(v = u + p \in W\) and let \(S\) be the \(p\)-adic closure of the divided power envelope of \(W\) with respect to the ideal generated by \(v\). Use the same symbols \(\sigma\) and \(N\) for natural continuous extensions of \(\sigma\) and \(N\) from \(W\) to \(S\). For \(i \geq 0\), denote by \(\text{Fil}^i S\) the \(i\)-th divided power of the ideal \((v)\) in \(S\). Then for \(0 \leq i < p\), there are \(\sigma\)-linear morphisms \(\phi_i = \sigma/p^i : \text{Fil}^i S \to S\). Note that \(\phi_0 = \sigma\) and agree to use the notation \(\varphi\) for \(\phi_{p-1}\). One can see also that \(S\) is the \(p\)-adic closure of \(W(k)[v_0, v_1, \ldots, v_n, \ldots]\), where \(v_0 = v\) and for all \(n \geq 0\), \(v_{n+1}/p = v_n\).

Consider the ideals \(m_S = (p, v, v_1, \ldots, v_n, \ldots)\), \(I = (p, v_1, v_2, \ldots)\) and \(J = (p, v_1 v, v_2, \ldots, v_n, \ldots)\) of \(S\). Then
- \(m_S\) is the maximal ideal in \(S\);
- \(I = \text{Fil}^0 S + pS \supset J\);
- \(\varphi(I) \subset S\) and \(\varphi(J) \subset pS\);
- \(\varphi(v^{p-1}) \equiv 1 - v_1 \pmod{J}\) and \(\varphi(v^i) \equiv 1 \pmod{J}\).

3.2. The ring of semi-stable periods \(\hat{A}_{\text{st}}\). Let \(R\) be Fontaine’s ring and let \(x_0, \varepsilon \in R\) be the elements chosen in Subsection 2.1.

Denote by \(A_{\text{cr}}\) the Fontaine crystalline ring. It is the \(p\)-adic closure of the divided power envelope of \(W(R)\) with respect to the ideal \(([x_0] + p)\) of \(W(R)\), where \([x_0] \in W(R)\) is the Teichmüller representative of \(x_0\). Then for \(i \geq 0\), \(\text{Fil}^i A_{\text{cr}}\) is the \(i\)-th divided power of the ideal \((x_0 + p)\) in \(A_{\text{cr}}\). Denote by \(\sigma : A_{\text{cr}} \to A_{\text{cr}}\) the natural morphism induced by the \(p\)-th power on \(R\). Then for \(0 \leq i < p\), there are \(\sigma\)-linear maps \(\phi_i = \sigma/p^i : \text{Fil}^i A_{\text{cr}} \to A_{\text{cr}}\). We shall often use the simpler notation \(\varphi = \phi_{p-1}\) and \(F(A_{\text{cr}}) = \text{Fil}^{p-1} A_{\text{cr}}\). Notice that \(A_{\text{cr}}\) is provided with the natural continuous \(\Gamma_F\)-action.

Let \(X\) be an indeterminate. Then \(\hat{A}_{\text{st}}\) is the \(p\)-adic closure of the ring \(A_{\text{cr}}[\gamma_i(X) \mid i \geq 0] \subset A_{\text{cr}}[X] \otimes_{\Z_p} \Q_p\), where for all \(i \geq 0\), \(\gamma_i(X) = X^i/i!\).

The ring \(\hat{A}_{\text{st}}\) has the following additional structures:

- the \(S\)-module structure given by the natural \(W(k)\)-algebra structure and the correspondence \(u \mapsto [x_0]/(1 + X)\);
- the ring endomorphism \(\sigma\), which is the extension of the above defined endomorphism \(\sigma\) of \(A_{\text{cr}}\) via the condition \(\sigma(X) = (1 + X)^p - 1\);
- the continuous \(A_{\text{cr}}\)-derivation \(N : \hat{A}_{\text{st}} \to \hat{A}_{\text{st}}\) such that \(N(X) = X + 1\);
- for any \(i \geq 0\), the ideal \(\text{Fil}^i \hat{A}_{\text{st}}\), which is the closure of the ideal \(\sum_{i_1 + i_2 \geq 1} (\text{Fil}^{i_1} A_{\text{cr}}) \gamma_{i_2}(X)\);
- the action of \(\Gamma_F\), which is the extension of the \(\Gamma_F\)-action on \(A_{\text{cr}}\) such that for all \(\tau \in \Gamma_F\), \(\tau(X) = \varepsilon[k(\tau)](X + 1) - 1\). Here all \(k(\tau) \in \Z_p\) are such that \(\tau(x_0) = \varepsilon[k(\tau)]x_0\).
Note that for $0 \leq m < p$, $\sigma(\text{Fil}^m \hat{A}_st) \subset \text{Fil}^m \hat{A}_st$ and, as earlier, we can set $\phi_m = \text{Fil}^m \sigma |_{\text{Fil}^m \hat{A}_st}$ and introduce the simpler notation $\varphi = \phi_{p-1}$ and $F(\hat{A}_st) = \text{Fil}^{p-1} \hat{A}_st$.

3.3. Construction of semi-stable representations of $\Gamma_F$ with weights from $[0, p)$. For $0 \leq m < p$, consider the category $\tilde{S}_m$ of quadruples $\mathcal{M} = (M, \text{Fil}^m M, \phi_m, N)$, where $\text{Fil}^m M \subset M$ are $S$-modules, $\phi_m : \text{Fil}^m M \rightarrow M$ is a $\sigma$-linear map and $N : M \rightarrow M$ is a $W(k)$-linear endomorphism such that for any $s \in S$ and $m \in M$, $N(sx) = N(s)x + sN(x)$ The morphisms of the category $\tilde{S}_m$ are $S$-linear morphisms of filtered modules commuting with the corresponding morphisms $\phi_m$ and $N$. Notice that for $0 \leq m < p$, $\hat{A}_st$ has a natural structure of the object of the category $\tilde{S}_m$. As earlier, we shall use the simpler notation $\varphi = \phi_{p-1}$ and $F(M) = \text{Fil}^{p-1} M$.

For $0 \leq m < p$, the Breuil category $S_m$ of strongly divisible $S$-modules of weight $\leq m$ is a full subcategory of $\tilde{S}_m$ consisting of the objects $\mathcal{M} = (M, \text{Fil}^m M, \phi_m, N)$ such that

1. $M$ is a free $S$-module of finite rank;
2. $(\text{Fil}^m S)M \subset \text{Fil}^m M$;
3. $(\text{Fil}^m M) \cap pM = p\text{Fil}^m M$;
4. $\phi_m(\text{Fil}^m M)$ spans $M$ over $S$;
5. $N\phi_m = p\phi_m N$;
6. $(\text{Fil}^1 S)N(\text{Fil}^m M) \subset \text{Fil}^m M$

For $\mathcal{M} \in S_m$, let $T^*_st(\mathcal{M})$ be the $\Gamma_F$-module of all $S$-linear and commuting with corresponding morphisms $\phi_m$ and $N$, maps $f : M \rightarrow \hat{A}_st$ such that $f(\text{Fil}^m M) \subset \text{Fil}^m \hat{A}_st$. Then one has the following two basic facts:

- $T^*_st(\mathcal{M})$ is a continuous $\mathbb{Z}_p[\Gamma_F]$-module without $p$-torsion, its $\mathbb{Z}_p$-rank equals $\text{rk}_S M$, and $V^*_st(\mathcal{M}) = T^*_st(\mathcal{M}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is semi-stable $\Gamma_F$-module with Hodge-Tate weights from $[0, m]$;

- any semi-stable representation of $\Gamma_F$ with Hodge-Tate weights from $[0, m]$, $0 \leq m < p$, appears in the form $V^*_st(\mathcal{M})$ for a suitable $\mathcal{M} \in S_m$.

By Theorem 1.3 [6] these facts follow from the existence of strongly divisible lattices in $S \otimes_W F$-modules associated with weakly admissible $(\phi_0, N)$-modules with filtration of length $m$. Breuil proved this for all $m \leq p - 2$ but his method can be easily extended to cover the case $m = p - 1$ as well, cf. also [7].
3.4. The category $\mathcal{L}^f$. In this section we introduce $\mathcal{W}$-analogues of Breuil’s $S$-modules from the category $\mathcal{S}_{p-1}$ and prove that they can be also used to construct semi-stable representations of $\Gamma_F$ with Hodge-Tate weights from $[0, p]$.

**Definition.** Let $\tilde{\mathcal{L}}$ be the category of $\mathcal{L} = (L, F(L), \varphi, N_S)$, where $L \supset F(L)$ are $\mathcal{W}$-modules, $\varphi : F(L) \rightarrow L$ is a $\sigma$-linear morphism of $\mathcal{W}$-modules and $N_S : L \rightarrow L_S := L \otimes_{\mathcal{W}} S$ is such that for all $w \in \mathcal{W}$ and $l \in L$, $N_S(wl) = N(w)l + (w \otimes 1)N_S(l)$. For $\mathcal{L}_1 = (L_1, F(L_1), \varphi, N_S) \in \tilde{\mathcal{L}}$, the morphisms $\text{Hom}_{\mathcal{L}}(\mathcal{L}, \mathcal{L}_1)$ are $\mathcal{W}$-linear $f : L \rightarrow L_1$ such that $f(F(L)) \subset F(L_1)$, $f\varphi = \varphi f$ and $fN_S = N_S(f \otimes 1)$.

Let $A_{st} = (\hat{A}_{st}, F(\hat{A}_{st}), \varphi, N_S)$, where $N_S = N \otimes 1$. Then $A_{st}$ is an object of the category $\tilde{\mathcal{L}}$.

Suppose $\mathcal{L} = (L, F(L), \varphi, N_S) \in \tilde{\mathcal{L}}$. Set $L_S \coloneqq L \otimes_{\mathcal{W}} S$, $F(L_S) = (F(L) \otimes 1)S + (L \otimes 1)\text{Fil}^pS$, and $\varphi_S : F(L_S) \rightarrow F(L_S)$ is a unique $\sigma$-linear map such that $\varphi_S|_{F(L) \otimes 1} = \varphi \otimes 1$ and for any $s \in \text{Fil}^pS$ and $l \in L$, $\varphi_S(l \otimes s) = (\varphi(v^{-1}l) \otimes 1)\varphi(s)/\varphi(v^{-1})$.

**Definition.** Denote by $\mathcal{L}^f$ the full subcategory in $\tilde{\mathcal{L}}$ consisting of the quadruples $\mathcal{L} = (L, F(L), \varphi, N_S)$ such that
- $L$ is a free $\mathcal{W}$-module of finite rank;
- $v^{-1}L \subset F(L)$, $F(L) \cap pL = pF(L)$ and $L = \varphi(F(L)) \otimes_{\sigma\mathcal{W}} \mathcal{W}$;
- for any $l \in F(L)$, $vN_S(l) \in F(L_S)$ and $\varphi_S(vN_S(l)) = cN_S(\varphi(l))$, where $c = 1 + u^p/p$.

It can be easily seen that for $\mathcal{L} = (L, F(L), \varphi, N_S) \in \mathcal{L}^f$ and the map $N = N_S \otimes 1 : L_S \rightarrow L_S$, the quadruple $\mathcal{L}_S = (L_S, F(L_S), \varphi_S, N)$ is the object of the category $\mathcal{S}_{p-1}$.

The main result of this Subsection is the following statement.

**Proposition 3.1.** For any $\mathcal{M} = (M, F(M), \varphi, N) \in \mathcal{S}_{p-1}$, there is an $\mathcal{L} = (L, F(L), \varphi, N_S) \in \mathcal{L}^f$ such that $\mathcal{M} = \mathcal{L}_S$.

**Corollary 3.2.** a) If $\mathcal{L} \in \mathcal{L}^f$ and $T_{st}(\mathcal{L}) = \text{Hom}_{\mathcal{L}}(\mathcal{L}, \hat{A}_{st})$ with the induced structure of $\mathbb{Z}_p[\Gamma_F]$-module then $V_{st}^*(\mathcal{L}) = T_{st}(\mathcal{L}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a semi-stable $\mathbb{Q}_p[\Gamma_F]$-module with Hodge-Tate weights from $[0, p]$ and $\dim_{\mathbb{Q}_p} V_{st}^*(\mathcal{L}) = \text{rk}_{\mathcal{W}} L$.

b) For any semi-stable $\mathbb{Q}_p[\Gamma_F]$-module $V_{st}^*$ with Hodge-Tate weights from $[0, p)$, there is an $\mathcal{L} \in \mathcal{L}^f$ such that $V_{st}^* \simeq V_{st}^*(\mathcal{L})$.

**Proof of Proposition 3.1.** Let $d$ be a rank of $M$ over $S$. If $L \subset M$ is a free $\mathcal{W}$-submodule of rank $d$ in $M$ we say that $L$ is $\mathcal{W}$-structural (with respect to $M$).

Let $F(L) = F(M) \cap L$. 


Lemma 3.3. If $L$ is $\mathcal{W}$-structural for $M$ then

a) $F(L) \supset v^{p-1}L$;

b) $F(L) \cap pL = pF(L)$;

c) $F(L)$ is a free $\mathcal{W}$-module of rank $d$.

Proof. a) $v^{p-1}L \subset (\text{Fil}^{p-1}S)M \cap L \subset F(M) \cap L = F(L)$.

b) $F(L) \cap pL = L \cap F(M) \cap pL = F(M) \cap pM \cap pL = pF(M) \cap pL = pF(L)$.

c) $F(L)$ has no $p$-torsion. Therefore, it will be sufficient to prove that $\frac{F(L)}{pF(L)}$ is a free $k[[u]]$-module of rank $d$. Consider the following natural embeddings of $k[[v]]$-modules

$$L/pL \supset F(L)/pF(L) \supset v^{p-1}L/pv^{p-1}L \simeq L/pL$$

(Use b) and that $pL \cap v^{p-1}L = pv^{p-1}L$.) It remains to note that $L/pL$ is free of rank $d$ over $k[[v]]$.

The Lemma is proved. $\square$

Suppose $L$ is $\mathcal{W}$-structural for $M$.

Lemma 3.4. If $L$ is $\mathcal{W}$-structural then $\varphi(F(L))$ spans $M$ over $S$.

Proof. The equality $S = \mathcal{W} + \text{Fil}^pS$ implies that $M = L + (\text{Fil}^pS)L = L + (\text{Fil}^pS)M$. Therefore,

$$F(M) = F(M) \cap L + (\text{Fil}^pS)M = F(L) + (\text{Fil}^pS)L$$

(use that $F(M) \supset (\text{Fil}^pS)M$) and in notation of Subsection 3.1 it holds

$$F(M) = F(L) + v_1L + JM.$$ 

This implies that $\varphi(F(L)), \varphi(v_1L)$ and $\varphi(JM)$ span $M$ over $S$. But for any $t \in L$, $\varphi(v_1t) = \varphi(v_1)\varphi(v^{p-1}t)/\varphi(v^{p-1}) = (1 - v_1)^{-1}\varphi(v^{p-1}t) \equiv \varphi(v^{p-1}t) \bmod m_SM$. For similar reasons, $\varphi(JM) \subset pM \subset m_SM$. This means that $\varphi(F(L))$ spans $M$ modulo $m_SM$. The lemma is proved. $\square$

By above lemma it remains to prove the existence of a $\mathcal{W}$-structural $L$ for $M$ such that $\varphi(F(L)) \subset L$.

Let $\phi_0$ be a $\sigma$-linear endomorphism of the $S$-module $M \in S_{p-1}$ such that for all $m \in M$, $\phi_0(m) = \varphi(v^{p-1}m)/\varphi(v^{p-1})$. Clearly, $\phi_0(m_SM) \subset m_SM$ and, therefore, it induces a $\sigma$-linear endomorphism $\sigma_0$ of the $k$-vector space $M_k = M/m_SM$.

Lemma 3.5. Suppose $n \in \mathbb{Z}_{\geq 0}$, $L$ is $\mathcal{W}$-structural and $\varphi(F(L)) \subset L + p^nM$. Then there is a $\mathcal{W}$-structural $L'$ for $M$ such that $\varphi(F(L')) \subset L' + p^nJM$. 

Proof. Denote by $F(L)_k$ the image of $F(L)$ in the $k$-vector space $M/m_8M = L/(m_3 \cap W)L = L_k$. Let $s = \dim_k F(L)_k$, then $s \leq d = \dim_k L_k$. Choose a $W$-basis $e^{(1)}, \ldots, e^{(d)}$ of $L$ and a $W$-basis $f^{(1)}, \ldots, f^{(d)}$ of $F(L)$ such that

- for $1 \leq i \leq s$, $f^{(i)} = e^{(i)}$ and for $s < i \leq d$, $f^{(i)} \in vL$.

It will be convenient to use the following vector notation: $\bar{e} = (\bar{e}_1, \bar{e}_2)$, where $\bar{e}_1 = (e^{(1)}, \ldots, e^{(s)})$ and $\bar{e}_2 = (e^{(s+1)}, \ldots, e^{(d)})$, and $\bar{f} = (\bar{f}_1, \bar{f}_2)$, where $\bar{f}_1 = \bar{e}_1$ and $\bar{f}_2 = (f^{(s+1)}, \ldots, f^{(d)})$.

Then in obvious notation it holds $(\varphi(\bar{f}_1), \varphi(\bar{f}_2)) = (\bar{e}_1, \bar{e}_2)C$, where $C \in \text{GL}_d(S)$. Clearly, $C \equiv C_0 + p^n v_1 C_1 \mod p^n J$ with $C_0 \in \text{GL}_d(W)$ and $C_1 \in M_d(W)$. Clearly, $\varphi(F(L)) \subset L + p^n JM$ if $C_1 \equiv 0 \mod m_S$. Choose $\bar{g} = (\bar{g}_1, \bar{g}_2) \in L^d$ and set

$$
\bar{e}_1 = (e^{(1)}, \ldots, e^{(s)}) = \bar{e}_1 + p^n (v_1 - v^{p-1})\bar{g}_1
$$

$$
\bar{e}_2 = (e^{(s+1)}, \ldots, e^{(d)}) = \bar{e}_2 + p^n (v_1 - v^{p-1})\bar{g}_2
$$

Clearly, the coordinates of $\bar{e}' = (\bar{e}_1', \bar{e}_2')$ give an $S$-basis of $M$ and we can introduce the structural $W$-module $L' = \sum_i \text{We}^{(i)}$ for $M$.

Prove that the elements $e^{(i)}$, $1 \leq i \leq s$, and $f^{(i)}$, $s < i \leq d$, generate $F(L') \mod p^n JM$. Indeed, we have

$$(3.1) \quad L + p^n IM = L' + p^n IM$$

and this implies that the image $F(L)_k$ of $F(L)$ in $L_k$ coincides with its analogue $F(L')_k$. In addition, for $1 \leq i \leq s$,

$$e^{(i)} \in L' \cap (F(L) + p^n IM) \subset L' \cap F(M) = F(L').$$

Therefore, it would be sufficient to prove that $lvL' \cap F(L') \mod p^n JM$ is generated by the images of $ve^{(i)}$, $1 \leq i \leq s$, and $f^{(s+1)}, \ldots, f^{(d)}$. But relation $(3.1)$ implies that $lvL + p^n JM = lvL' + p^n JM$ and

$$(vL') \cap F(L') \mod p^n JM = (vL) \cap F(L) \mod p^n JM.$$ 

It remains to note that for $1 \leq i \leq s$, $ve^{(i)} \equiv ve^{(i)} \mod p^n JM$.

Therefore, we can define special bases for $L'$ and $F(L')$ by the relations $\bar{f}_1' = \bar{e}_1'$ and $\bar{f}_2' = \bar{f}_2$ and obtain that

$$(\varphi(\bar{f}_1'), \varphi(\bar{f}_2')) = (\varphi(\bar{f}_1), \varphi(\bar{f}_2)) + p^n v_1 (\sigma_0 \bar{g}_1, \bar{0}) \mod p^n JM$$

and

$$(\varphi(\bar{f}_1'), \varphi(\bar{f}_2')) \equiv (\bar{e}_1', \bar{e}_2')C_0 + p^n v^{p-1}(\bar{g}_1, \bar{g}_2)C_0 + p^n v_1 ((\bar{e}_1, \bar{e}_2)C_1 - (\bar{g}_1, \bar{g}_2)C_0 + (\sigma \bar{g}_1, \bar{0})) \mod p^n JM.$$

So, $\varphi(F(L')) \subset L' + p^n JM$ if and only if there is an $\bar{g} = (\bar{g}_1, \bar{g}_2) \in L^d$ such that $\sigma_0 \bar{g}_1, \bar{0}) \equiv (\bar{g}_1, \bar{g}_2)C_0 + \bar{h} \mod (m_S \cap W)L$, where $\bar{h} = (\bar{e}_1, \bar{e}_2)C_1 \in L$ and $C_0 \mod m_S \in \text{GL}_d(k)$. The existence of such vector $\bar{g}$ is implied by Lemma 3.6 below. □
Lemma 3.6. Suppose $V$ is a $d$-dimensional vector space over $k$ with a $\sigma$-linear endomorphism $\sigma_0 : V \to V$ and $\bar{a} = (\bar{a}_1, \bar{a}_2) \in V^d$, where $\bar{a}_1 \in V^s$ and $\bar{a}_2 \in V^{d-s}$. Then for any $C \in \text{GL}_d(k)$ there is an $\bar{g} = (\bar{g}_1, \bar{g}_2) \in V^d$ with $\bar{g}_1 \in V^s$ and $\bar{g}_2 \in V^{d-s}$ such that

\[ (\sigma_0 \bar{g}_1, \bar{0}) = \bar{g} C + \bar{a}. \]

Proof. Let $C^{-1} = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}$ with the block matrices of sizes $s \times s$, $(d - s) \times s$, $s \times (d - s)$ and $(d - s) \times (d - s)$.

Then the equality (3.2) can be rewritten as

\[ (\sigma_0 \bar{g}_1) D_{11} = \bar{g}_1 + \bar{a}_1', \quad (\sigma_0 \bar{g}_1) D_{21} = \bar{g}_2 + \bar{a}_2' \]

where $(\bar{a}_1', \bar{a}_2') = \bar{a} C^{-1}$. Clearly, it will be sufficient to solve the first equation in $\bar{g}_1$, but this is a special case of Lemma [1.1].

Lemma 3.7. Suppose $n \geq 0$ and $L$ is $\mathcal{W}$-structural for $M$ such that $\varphi(F(L)) \subset L + p^n J M$. Then there is a $\mathcal{W}$-structural $L'$ for $M$ such that $\varphi(F(L')) \subset L' + p^{n+1} M$.

Proof. Suppose the coordinates of $\bar{e} \in M^d$ form a $\mathcal{W}$-basis of $L$ and $D \in \mathcal{M}_d(\mathcal{W})$ is such that the coordinates of $\bar{f} = \bar{e} D$ form a $\mathcal{W}$-basis of $F(L)$. Then $\varphi(\bar{f}) = \bar{e} + p^n \bar{h}$, where $\bar{h} \equiv \bar{0}$ mod $JM$. Let $\bar{e}' = \bar{e} + p^n \bar{h}$ and let $L'$ be a $\mathcal{W}$-submodule in $M$ spanned by the coordinates of $\bar{e}'$. Clearly, $L'$ is $\mathcal{W}$-structural.

Prove that $F(L')$ is spanned by the coordinates of $\bar{e}' D$. Indeed, suppose $\bar{e}$ and $\bar{e}'$ have the coordinates $e^{(i)}$ and, resp., $e'^{(i)}$, $1 \leq i \leq s$. Then for all $i$, $e'^{(i)} = e^{(i)} + p^n h^{(i)}$, where $h^{(i)} \in JM \subset (\text{Fil}^p S) M$. This means that a $\mathcal{W}$-linear combination of $e^{(i)}$ belongs to $F(M)$ if and only if the same linear combination of $e'^{(i)}$ belongs to $F(M)$. This implies that $\bar{e}' D$ spans $F(L')$ over $\mathcal{W}$ because $\bar{e} D$ spans $F(L)$ over $\mathcal{W}$. Then $\varphi(F(L')) \subset L' + p^{n+1} M$ because $\varphi(\bar{h}) \in pM$ (use that $\varphi(J) \subset pS$) and

$\varphi(\bar{e}' D) = \varphi(\bar{e} D + p^n \bar{h} D) = \bar{e} + p^n \bar{h} + p^n \varphi(\bar{h}) \sigma(D) \equiv \bar{e}' \mod p^{n+1} M$.

It remains to notice that applying above Lemmas 3.6 and 3.7 one after another we shall obtain a sequence of $\mathcal{W}$-structural modules $L_n$ such that for all $n \geq 0$, $L_n + p^{n+1} M = L_{n+1} + p^{n+1} M$, where $L_0 \otimes \mathcal{W} S = M$. Therefore, $L = \varprojlim L_n/p^n$ is $\mathcal{W}$-structural and $\varphi(L) \subset L$.

The proposition is completely proved.

3.5. The categories $\mathcal{L}'$ and $\mathcal{L}^{H'}$.

Definition. $\mathcal{W}$-module $L$ is $p$-strict if it is isomorphic to $\oplus_{1 \leq i \leq s} \mathcal{W}/p^{n_i}$, where $n_1, \ldots, n_s \in \mathbb{N}$. 
In particular, if $L$ is $p$-strict and $pL = 0$ then $L$ is free $\mathcal{W}_1$-module. The $p$-strict modules can be efficiently studied via devissage due to the following property.

**Lemma 3.8.** $L$ is $p$-strict if and only if $pL$ and $L/pL$ are $p$-strict.

**Proof.** Specify Breuil’s proof of a similar statement but for more complicated ring $S = \mathcal{W}^{DP}$ from [6].

**Definition.** Denote by $\mathcal{L}'$ the full subcategory in $\tilde{\mathcal{L}}$ consisting of the quadruples $L = (L, F(L), \varphi, N_S)$ such that

- $L$ is $p$-strict;
- $\nu^{-1}L \subset F(L)$, $F(L) \cap pL = pF(L)$ and $L = \varphi(F(L)) \otimes_{\mathcal{W}} \mathcal{W}$;
- for any $l \in F(L)$, $\nu N_S(l) \in F(L_S)$ and $\varphi_S(\nu N(l)) = c N_S(\varphi(l))$, where $c = 1 + \nu^p/p$.

**Definition.** Denote by $\mathcal{L}'[1]$ the full subcategory in $\mathcal{L}'$, which consists of objects killed by $p$.

The category $\mathcal{L}'[1]$ is not very far from the category $\mathcal{L}'$ introduced in Section 1. Indeed, suppose $\mathcal{L} = (L, F(L), \varphi, N_S) \in \mathcal{L}'[1]$. Note that $N_S(L) \subset L_{S_1} := L \otimes_{\mathcal{W}_1} S_1 = L/u^pL + (L \otimes 1)Fil^pS_1$. With this notation we have the following property.

**Proposition 3.9.** There is a unique $N : L \rightarrow L/u^2p$ such that

- for any $l \in L$, $N(l) \otimes 1 = c N_S(l)$ in $L_{S_1}$, where $c = 1 + u^p/p \in S^*$;
- $(L, F(L), \varphi, N) \in \mathcal{L}'[1]$.

**Proof.** Let $N_1 := c N_S : L \rightarrow L_{S_1}$. Then for any $w \in \mathcal{W}_1$ and $l \in L$, it holds $N_1(wl) = N(wl) + w N_1(l)$ (use that $N(c) = 0$ in $S_1$) and there is a commutative diagram (use that $\sigma(c) = 1$ in $S_1$)

$$
\begin{array}{ccc}
F(L) & \xrightarrow{\varphi} & L \\
\downarrow uN_1 & & \downarrow N_1 \\
F(L)_S & \xrightarrow{\varphi} & L_{S_1}
\end{array}
$$

Prove that $N_1(\varphi(F(L))) \subset L/u^pL$ and, therefore, $N_1(L) \subset L/u^pL$.

Indeed, $(uN_1)(F(L)) \subset uN_1(L) \cap F(L)_S \subset (uL/u^pL \oplus (uL)Fil^pS_1) \cap (F(L)/u^pL + Fil^pS_1) \subset F(L)/u^pL + (uL)Fil^pS_1$. This implies that $N_1(\varphi(F(L))) \subset \varphi_S(uN_1(F(L))) \subset L/u^pL$ because $\varphi_S(uFil^pS_1) = 0$. So, by Proposition 1.3 there is a unique $N : L \rightarrow L/u^2p$ such that $N \mod u^p = N_1$ and $(L, F(L), \varphi, N) \in \mathcal{L}'[1]$. □

**Corollary 3.10.** With above notation the correspondence

$$(L, F(L), \varphi, N_S) \mapsto (L, F(L), \varphi, N)$$

induces the equivalence of categories $\Pi : \mathcal{L}'[1] \rightarrow \mathcal{L}'$.

**Corollary 3.11.** The category $\mathcal{L}'$ is preabelian.
Proof. Corollary\textsuperscript{5.10} and Proposition\textsuperscript{1.3} imply that \( \mathcal{L}^f[1] \) is pre-abelian. This can be extended then to the whole category \( \mathcal{L}^f \) by Breuil’s method from \cite{10} via above Lemma \textsuperscript{5.8}.

Note that if \( \mathcal{L} = (L, F(L), \varphi, N_S) \) and \( \mathcal{M} = (M, F(M), \varphi, N_S) \) are objects of \( \mathcal{L}^f \) and \( f \in \text{Hom}_\mathcal{L}(\mathcal{L}, \mathcal{M}) \) then:

- \( \text{Ker} f = (K, F(K), \varphi, N_S) \), where \( K = \text{Ker}(f : L \to M) \) and \( F(K) = F(L) \cap K \) with induced \( \varphi \) and \( N_S \);
- \( \text{Coker} f = (C, F(C), \varphi, N_S) \), where \( C = M/M' \), \( M' \) is equal to \((f(L) \otimes_W W[u^{-1}]) \cap M \) and \( F(C) = F(M)/(M' \cap F(M)) \) with induced \( \varphi \) and \( N_S \);
- \( f \) is strict monomorphic means that \( f : L \to M \) is monomorphism of \( \mathcal{W} \)-modules, \((f(L) \otimes_W W[u^{-1}]) \cap M = f(L) \) (or, equivalently, \( M/f(L) \) is \( p \)-strict) and \( f(F(L)) = L \cap F(M) \);
- \( f \) is strict epimorphic means that \( f \) is epimorphism of \( p \)-strict modules and \( f(F(L)) = F(M) \).

According to Appendix \textit{A}, we can use the concept of \( p \)-divisible group \( \{ \mathcal{L}^p(n), i_n \}_{n \geq 0} \) in \( \mathcal{L}^f \). In this case \( \mathcal{L}^p(n) = (L_n, F(L_n), N_S) \), where all \( L_n \) are free \( \mathcal{W}/p^n \)-modules of the same rank equal to the height of the \( p \)-divisible group \( \{ \mathcal{L}^p(n), i_n \} \). We have obvious equivalence of the category \( \mathcal{L}^f \) and the category of \( p \)-divisible groups of finite height in \( \mathcal{L}^f \).

**Definition.** Denote by \( \mathcal{L}^{f[1]} \) the full subcategory in \( \mathcal{L}^f \), which consists of strict subobjects of \( p \)-divisible groups in \( \mathcal{L}^f \). By \( \mathcal{L}^{f[1]} \) we denote the full subcategory in \( \mathcal{L}^{f[1]} \) consisting of all objects killed by \( p \).

It is easy to see that \( \mathcal{L}^{f[1]} \) contains all strict subquotients of the corresponding \( p \)-divisible groups. Contrary to the case of filtered modules coming from crystalline representations, the categories \( \mathcal{L}^{f[1]} \) and \( \mathcal{L}^f \) do not coincide but they have the same simple objects.

Note that the functor \( \Pi \) from Corollary\textsuperscript{5.10} identifies simple objects of the categories \( \mathcal{L}^f \) and \( \mathcal{L}^c \) and for any two objects \( \mathcal{L}_1, \mathcal{L}_2 \in \mathcal{L}^f[1] \), we have a natural isomorphism \( \text{Ext}^{\mathcal{L}^f[1]}(\mathcal{L}_1, \mathcal{L}_2) = \text{Ext}^{\mathcal{L}^c}(\Pi(\mathcal{L}_1), \Pi(\mathcal{L}_2)) \).

We can obviously extend the concepts of etale, connected, unipotent and multiplicative objects to the whole category \( \mathcal{L}^f \) and to obtain the following standard properties:

- for any \( \mathcal{L} \in \mathcal{L}^f \), there are a unique maximal etale subobject \( (\mathcal{L}^{et}, i^{et}) \) and a unique maximal connected quotient object \( (\mathcal{L}^c, j^c) \) in \( \mathcal{L}^f \) such that the sequence \( 0 \to \mathcal{L}^{et} \xrightarrow{i^{et}} \mathcal{L} \xrightarrow{j^c} \mathcal{L}^c \to 0 \) is exact and the correspondences \( \mathcal{L} \mapsto \mathcal{L}^{et} \) and \( \mathcal{L} \mapsto \mathcal{L}^c \) are functorial; if \( \mathcal{L} \in \mathcal{L}^{f[1]} \) then \( \mathcal{L}^{et} \) and \( \mathcal{L}^c \) are also objects of \( \mathcal{L}^{f[1]} \);
- for any \( \mathcal{L} \in \mathcal{L}^f \), there are a unique maximal unipotent subobject \( (\mathcal{L}^u, i^u) \) and a unique maximal multiplicative quotient object \( (\mathcal{L}^m, j^m) \)
in $L^t$ such that the sequence $0 \rightarrow L^u \xrightarrow{i_u} L \xrightarrow{j_m} L^m \rightarrow 0$ is exact and the correspondences $L \mapsto L^u$ and $L \mapsto L^m$ are functorial; if $L \in L^{ft}$ then $L^u$ and $L^m$ are also objects of $L^{ft}$.

Denote by $L^{et,t}$, $L^{c,t}$, $L^{u,t}$ and $L^{m,t}$ the full subcategories in $L^t$ consisting of, resp., etale, connected, unipotent and multiplicative objects. We have also the corresponding full subcategories $L^{et,ft}$, $L^{c,ft}$, $L^{u,ft}$ and $L^{m,ft}$ in $L^{ft}$.

The results of Subsection 1.5 and Appendix A imply that in the category $L^{ft}$:

- there is a unique etale $p$-divisible group $L^\infty(0) := \{L^{(n)}(0), i_n\}_{n \geq 0}$ of height 1 such that $L^{(1)}(0) = L(0)$;

- there is a unique multiplicative $p$-divisible group of height 1, $L^\infty(1) := \{L^{(n)}(1), i_n\}_{n \geq 0}$ such that $L^{(1)}(1) = L(1)$;

- for any $p$-divisible group $L^\infty$ there are functorial exact sequences of $p$-divisible groups

$$
0 \rightarrow L^{\infty,et} \rightarrow L^\infty \rightarrow L^{\infty,c} \rightarrow 0
$$

$$
0 \rightarrow L^{\infty,u} \rightarrow L^\infty \rightarrow L^{\infty,m} \rightarrow 0
$$

Here $L^{\infty,et}$ and $L^{\infty,m}$ are products of several copies of $L^\infty(0)$ and, resp., $L^\infty(1)$, and $L^{\infty,c}$ and $L^{\infty,u}$ are $p$-divisible groups in the categories $L^{c,ft}$ and, resp., $L^{u,ft}$. 
4. Semistable modular representations with weights $[0,p)$

In this section we prove that all killed by $p$ subquotients of Galois invariant lattices of semistable $\mathbb{Q}_p[\Gamma_F]$-modules with Hodge-Tate weights $[0,p)$ can be obtained via the functor $\mathcal{V}^*$ from Section 2.

4.1. The functor $\mathcal{V}^t : \mathcal{L}^t \to \mathcal{M}_F$. For $n \geq 1$, introduce the objects $A_{st,n} = (\hat{A}_{st,n}, F(\hat{A}_{st,n}), \varphi, N_S)$ of the category $\mathcal{L}$, with $\hat{A}_{st,n} = A_{st}/p^n A_{st}$, $\hat{F}(\hat{A}_{st,n}) = F(A_{st})/p^n F(A_{st})$ and induced $\varphi$ and $N_S$. Let $A_{st,\infty} = (A_{st,\infty}, F(A_{st,\infty}), \varphi, N_S)$ be the inductive limit of all $A_{st,n}$.

For $\mathcal{L} \in \mathcal{L}^t$, set $\mathcal{V}^t(\mathcal{L}) = \text{Hom}_F(\mathcal{L}, A_{st,\infty})$ with the induced structure of $\Gamma_F$-module. This gives the functor $\mathcal{V}^t : \mathcal{L}^t \to \mathcal{M}_F$. We shall use the same notation for its restriction to the category $\mathcal{L}^t$.

**Proposition 4.1.** Suppose $\mathcal{L} = (L, F(L), \varphi, N_S) \in \mathcal{L}^t$. Then $N_S|_{\varphi(F(L))}$ is nilpotent.

By devissage and Corollary 3.10 this is implied by the following statement for the objects of the category $\mathcal{L}^t$.

**Lemma 4.2.** If $\mathcal{L} = (L, F(L), \varphi, N) \in \mathcal{L}^t$ then $N^p(\varphi(F(L)) \subset u^p L$.

**Proof.** For any $l \in F(L)$, $N(\varphi(l)) = \varphi(uN(l))$. Use induction to prove that for $1 \leq m \leq p$, $N^m(\varphi(l)) \equiv \varphi(u^mN^m(l)) \mod u^p L$ and use then that $\varphi(u^pN^p(l)) \in \varphi(uF(L)) \subset u^p L$. $\square$

**Proposition 4.3.** For $n \geq 1$, $\bigoplus_{j \geq 0} A_{cr,n} \gamma_j(\log(1 + X))$ is the maximal $W(k)$-submodule of $\hat{A}_{st,n}$ where $N$ is nilpotent.

**Proof.** For any $j \geq 1$, it holds $N(\gamma_j(\log(1 + X))) = \gamma_{j-1}(\log(1 + X))$ and $N$ is nilpotent on $\bigoplus_{j \geq 0} A_{cr,n} \gamma_j(\log(1 + X))$. Therefore, it will be sufficient to prove that

$$\text{Ker} \left( N^p|_{A_{st,1}} \right) = \bigoplus_{0 \leq j < p} A_{cr,1} \gamma_j(\log(1 + X))$$

Let $C = \mathbb{F}_p(X)$ be the divided power envelope of $\mathbb{F}_p[X]$ with respect to the ideal $(X)$. Then $C = \mathbb{F}_p[X_0, X_1, \ldots, X_n, \ldots]_{<p}$ is the ring of polynomials in $X_i := \gamma_p(X)$, where for all $i \geq 0$, $X^p_i = 0$.

Let $m_C$ be the maximal ideal of $C$ and $Y = \log(1 + X) \in C$. Then $Y \equiv X_0 - X_1 \mod m_C^2$ and for all $j \geq 0$, $\gamma_{p^j}(Y) \equiv X_j - X_{j+1} \mod m_C^2$. This implies that with $Y_j = \gamma_j(Y)$ for all $j \geq 0$,

$$C = \mathbb{F}_p[X_0, Y_0, \ldots, Y_n, \ldots]_{<p} = \mathbb{F}_p(Y)[X]_{<p} = \bigoplus_{0 \leq i < p} \mathbb{F}_p(Y) \gamma_i(X).$$

So, $\hat{A}_{st,1} = \bigoplus_{0 \leq i < p} A_{cr,1} \gamma_i(X) \gamma_j(Y)$. Remind $N(X) = X + 1$ and for $j \geq 1$, $N(\gamma_j(Y)) = \gamma_{j-1}(Y)$. Using that $N^p$ is an $A_{cr,1}$-derivation we obtain that for any $P = \sum_{i,j} \alpha_{ij} X^i Y^j \in \mathbb{F}_p(Y)[X]_{<p}$ with $\alpha_{ij} \in \mathbb{F}_p$, it holds
(use that \( N^p(X) = X \) and \( N^p(\gamma_j + p(Y)) = \gamma_j(Y) \))

\[
N^p(P) = \sum_{i,j} \alpha_{ij}X^i\gamma_j(Y) + \sum_{i,j} \alpha_{i,j+p}X^i\gamma_j(Y).
\]

If \( P \in \text{Ker}N^p \) then for all involved indices \( i, j, \) \( i\alpha_{ij} + \alpha_{i,j+p} = 0. \) This implies that \( \alpha_{ij} = 0 \) if either \( i \neq 0 \) or \( j \geq p. \)

As earlier, introduce the category \( \tilde{\mathcal{L}}_0. \) Its objects are the triples \((L, F(L), \varphi), \) where \( L \supset F(L) \) are \( \mathcal{W}\)-modules and \( \varphi : F(L) \to L \) is a \( \sigma\)-linear morphism. For any object \( \mathcal{L} = (L, F(L), \varphi, N_S) \in \tilde{\mathcal{L}}, \) agree to use the same notation \( \mathcal{L} \) for the corresponding object \((L, F(L), \varphi) \in \tilde{\mathcal{L}}_0.\)

For all \( n \geq 0, \) set \( \mathcal{A}_{cr,n} = (A_{cr,n}, F(A_{cr,n}), \varphi) \in \tilde{\mathcal{L}}_0 \) with \( A_{cr,n} = A_{cr}/p^nA_{cr}, \) \( F(A_{cr,n}) = F(A_{cr})/p^nF(A_{cr}) \) and induced \( \varphi. \) Here the \( \mathcal{W}\)-module structure on \( A_{cr,n} \) is defined by the morphism of \( \mathcal{W}(k)\)-algebras \( \mathcal{W} \to A_{cr,n} \) such that \( u \mapsto [x_0]. \) Denote by \( \mathcal{A}_{cr,\infty} \) the inductive limit of all \( \mathcal{A}_{cr,n}. \)

Suppose \( \mathcal{L} \in \mathcal{L}^l \) and \( f \in \text{Hom}_{\mathcal{L}}(\mathcal{L}, \mathcal{A}_{st,n}). \) Then by Propositions \([4,1]\) and \([4,3]\)

\[
f(\varphi(F(L))) \subset \bigoplus_{j \geq 0} A_{cr,n}\gamma_j(\log(1 + X)).
\]

Consider the formal embedding of the algebra \( A_{st,n} \) into the completion \( \prod_{j \geq 0} A_{cr,n}\gamma_j(\log(1 + X)) \) of \( \bigoplus_{j \geq 0} A_{cr,n}\gamma_j(\log(1 + X)) \) such that \( X \mapsto \sum_{j \geq 1} \gamma_j(\log(1 + X)). \) Then any element of \( A_{st,n} \) can be uniquely written in the form \( \sum_{j \geq 0} a_j \gamma_j(\log(1 + X)), \) where all \( a_j \in A_{cr,n}. \) Note that the \( \mathcal{W}\)-module structure on \( A_{st,n} \) is given via the map

\[
u \mapsto [x_0]/(1 + X) = [x_0] \sum_{j \geq 0} (-1)^j \gamma_j(\log(1 + X)).
\]

For \( j \geq 0, \) introduce the \( \mathcal{W}(k)\)-linear maps \( f_j \in \text{Hom}(L, A_{cr,n}) \) such that for any \( l \in L, \) it holds \( f(l) = \sum_{j \geq 0} f_j(l)\gamma_j(\log(1 + X)). \) Then using methods from \([6]\) obtain the following property.

**Proposition 4.4.** a) The correspondence \( f \mapsto f_0 \) induces isomorphism of abelian groups \( \mathcal{V}^0(\mathcal{L}) = \text{Hom}_{\mathcal{L}_0}(\mathcal{L}, \mathcal{A}_{cr,n}); \)

b) for any \( j \geq 0 \) and \( l \in L, \) \( f_j(l) = f_0(N^j(l)). \)

**Corollary 4.5.** The functor \( \mathcal{V}^0 \) is exact.

**Proof.** Let \( \mathcal{L}^l \) be the full subcategory of \( \tilde{\mathcal{L}}_0 \) consisting of the triples \((L, F(L), \varphi) \) coming from all \( \mathcal{L} = (L, F(L), \varphi, N) \in \mathcal{L}^l. \) By Proposition \([4,1]\) it will be sufficient to prove that the functor \( \mathcal{V}^0 : \mathcal{L}^l \to (Ab), \) such that \( \mathcal{V}^0(\mathcal{L}) = \text{Hom}_{\mathcal{L}_0}(\mathcal{L}, \mathcal{A}_{cr,\infty}), \) is exact. The verification can be done by devissage along the lines of paper \([9]\). \]

**Remark.** One can simplify the verification of above corollary by replacing \( \mathcal{A}_{cr,1} \) by the corresponding object \( \widetilde{\mathcal{A}}_{cr,1} \) related to the module \( \tilde{\mathcal{A}}_{cr,1} = (R/x_0^p)T_1 \oplus (R/x_0^p) \) introduced in Subsection \([4,2]\) below.
Corollary 4.6. For $\mathcal{L} \in \mathcal{L}'$, let $\{\mathcal{L}^{(n)}, i_n\}_{n \geq 0}$ be the corresponding $p$-divisible group in the category $\mathcal{L}'^t$. Then in notation of Corollary 3.2 $T'_n(\mathcal{L}) = \lim_{\rightarrow n} \mathcal{V}^{e}(\mathcal{L}^{(n)})$.

4.2. The functor $\mathcal{V}[1]^*$. Note the following case of Proposition 4.4.

Proposition 4.7. Suppose $\mathcal{L} = (L, F(L), \varphi, N) \in \mathcal{L}'[1]$. Then there is an isomorphism of abelian groups $\mathcal{V}^t(\mathcal{L}) \simeq \text{Hom}^e(\mathcal{L}, \mathcal{A}_{cr,1})$ and $\Gamma_F$ acts on $\mathcal{V}^t(\mathcal{L})$ via its natural action on $\mathcal{A}_{st,1}$ and the identification

$$\iota_{\mathcal{L}} : \text{Hom}^e(\mathcal{L}, \mathcal{A}_{cr,1}) \longrightarrow \text{Hom}^e(\mathcal{L}, \mathcal{A}_{st,1}).$$

such that if $f_0 \in \text{Hom}^e(\mathcal{L}, \mathcal{A}_{cr,1})$ then for any $l \in L$,

$$\iota_{\mathcal{L}}(f_0)(l) = \sum_{j \geq 0} f_0(N^j(l)\gamma_j(\log(1 + X))).$$

Introduce the functor $\mathcal{V}[1]^* := \Pi^{-1} \circ \mathcal{V}^t|_{\mathcal{L}'[1]} : \mathcal{L}^* \longrightarrow \prod_{\Gamma_F}$, where $\Pi : \mathcal{L}'[1] \longrightarrow \mathcal{L}^*$ is the equivalence of categories from Corollary 3.10.

Proposition 4.8. On the subcategory of unipotent objects $\mathcal{L}^{su}$ of $\mathcal{L}^*$ the functors $\mathcal{V}[1]^*$ and $\mathcal{V}^*$ coincide.

Proof. The definition of $\mathcal{A}_{cr}$ implies that $\mathcal{A}_{cr,1} = (R/x_0^p)[T_1, T_2, \ldots]_{<p}$, where for all $i \geq 1$, $T_i$ comes from $\gamma_{p^i}(\frac{[x_0]}{p})$ and it holds $T_i^p = 0$. Set $F(\mathcal{A}_{cr,1}) = \text{Fil}^{p-1} \mathcal{A}_{cr,1} = (x_0^{p-1}R/x_0^pR) \oplus (R/x_0^p)I_1$, where the ideal $I_1$ is generated by all $T_i$. Then the corresponding map $\varphi : F(\mathcal{A}_{cr,1}) \longrightarrow \mathcal{A}_{cr,1}$ is uniquely determined by the conditions $\varphi(x_0^{p-1}) = 1 - T_1$, $\varphi(T_1) = 1$ and $\varphi(T_i) = 0$ if $i \geq 2$. In particular, $\varphi(\mathcal{A}_{cr,1}) \subset (R/x_0^p)T_1 \oplus (R/x_0^p)$.

Let $\tilde{\mathcal{A}}_{cr,1} = \mathcal{A}_{cr,1}/J_1$ with the induced structure of filtered $\varphi$-module $\tilde{\mathcal{A}}_{cr,1}$, where the ideal $J_1$ of $\mathcal{A}_{cr,1}$ is generated by the elements $T_1x_0^p$ and $T_i$ with $i \geq 2$. Then the projection $\mathcal{A}_{cr,1} \longrightarrow \tilde{\mathcal{A}}_{cr,1}$ induces for any object $\mathcal{L} = (L, F(L), \varphi, N)$ of the category $\mathcal{L}^*$, the identification (use that $\varphi|_J = 0$)

$$\text{Hom}^e(\mathcal{L}, \mathcal{A}_{cr,1}) = \text{Hom}^e(\mathcal{L}, \tilde{\mathcal{A}}_{cr,1}).$$

Introduce $a_0, a_{-1} \in \text{Hom}(L, R/x_0^p)$ such that for any $m \in L$, $f_0(m) = a_{-1}(m)T_1 + a_0(m)$. Note that $a_0$ and $a_{-1}$ are $\mathcal{W}_1$-linear, where the multiplication by $u$ on $L$ corresponds to the multiplication by $x_0$ in $R/x_0^p$.

Then for any $m \in F(L)$, the requirement $f_0(\varphi(m)) = \varphi(f_0(m))$ is equivalent to the conditions

\begin{equation}
\begin{aligned}
a_0(\varphi(m)) &= a_{-1}(m)^p + \frac{a_0(m)^p}{x_0^{p(p-1)}} \\
-1(\varphi(m)) &= -\frac{a_0(m)^p}{x_0^{p(p-1)}}
\end{aligned}
\end{equation}
Note that these conditions depend only on $\bar{m} = m \mod u^p L$.

Consider the operator $V : L \to L$ from Subsection 1.3. Clearly, $V(u^p L) \subset uF(L)$ and for $\tilde{L} := L / u^p L$, we obtain induced operator $\tilde{V} : \tilde{L} \to \tilde{L}$ (use that $F(L)/uF(L) \subset L / u^p L$).

For any $m \in \tilde{L}$, relations (1.3) can be rewritten as follows:

$$a_0(m) = \frac{a_0(\tilde{V} m)^p}{x_0^{p(p-1)}} + a_{-1}(\tilde{V} m)^p$$

$$a_{-1}(m) = -\frac{a_0(\tilde{V} m)^p}{x_0^{p(p-1)}}$$

Therefore, if $L$ is unipotent then for any $m \in \tilde{L}$,

$$a_{-1}(m) = -a_0(m) + a_{-1}(\tilde{V} m)^p = -a_0(m) + a_{-1}(\tilde{V}^2 m)^p = \cdots = -a_0(m).$$

This implies that for any $m \in F(L)$, $a_0(\varphi(m)) = a_0(m)^p / x_0^{p(p-1)}$. In other words, we have a natural identification

$$\text{Hom}_{\mathbb{Z}_p}(L, \tilde{R}^u) = \text{Hom}_{\mathbb{Z}_p}(L, \tilde{A}_{cr,1})$$

coming from the map of filtered $\varphi$-modules $\tilde{R}^u \to \tilde{A}_{cr,1}$ given by the $R$-linear map $R/x_0^p \to \tilde{A}_{cr,1} = (R/x_0^p)T_1 \oplus (R/x_0^p)$ such that for any $r \in R/x_0^p$, $r \mapsto (-rT_1, r)$.

This implies that for all unipotent $L \in \mathcal{L}^u$, $\mathcal{V}[1]^*(L) = \mathcal{V}^*(L)$.

Indeed, the above embedding $R/x_0^p \to \tilde{A}_{cr,1}$ can be extended to the embedding of $R_{st}/x_0^p R_{st}$ to $\tilde{A}_{st,1} = \prod_{j \geq 0} \tilde{A}_{cr,1} \gamma_j (\log(1 + X))$. This identifies the abelian groups $\mathcal{V}^*(L)$ and $\mathcal{V}(1)^*(L)$. Now notice that the ideal

$$J(\tilde{A}_{st,1}) = \sum_{1 \leq j < p} \text{Fil}_{p-j}(\tilde{A}_{cr,1}) \gamma_j (\log(1 + X)) + \text{Fil}^p(\tilde{A}_{st,1})$$

is $\Gamma_f$-invariant and we have the induced $\Gamma_f$-invariant embedding of $R_{st}^0/J^u$ into $\tilde{A}_{st,1}/J(\tilde{A}_{st,1})$. \hfill \Box

4.3. Splittings $\Theta$ and $\widetilde{\Theta}$. Suppose $L = (L, F(L), \varphi, N) \in \mathcal{L}^u$. Then there is a standard short exact sequence

$$0 \to \mathcal{L}^u \xrightarrow{i} \mathcal{L} \xrightarrow{j} \mathcal{L}^m \to 0,$$

where $(\mathcal{L}^u, i)$ is the maximal unipotent subobject and $(\mathcal{L}^m, j)$ is the maximal multiplicative quotient of $\mathcal{L}$.

If $\mathcal{L}^m = (L^m, F(L^m), \varphi, N)$ then $F(L^m) = L^m = L_0 \otimes_\mathbb{F}_p W_1$, where $L_0 = \{ l \in L^m \mid \varphi(l) = l \}$. Suppose $S : L^m \to F(L) \subset L$ is a $W_1$-linear section. Then for any $l_0 \in L_0$, $S(l_0) = \varphi(S(l_0)) + g(l_0)$, where $g \in \text{Hom}(L_0, L^u)$. If $S' : L^m \to F(L)$ is another $W_1$-linear section then for any $l_0 \in L_0$, $S'(l_0) = \varphi(S'(l_0)) + g'(l_0)$. Here $g' \in \text{Hom}(L_0, L^u)$ is such that for some $h \in \text{Hom}(L_0, L^u)$, it holds $g'(l_0) = h(l_0) - \varphi(h(l_0))$. \hfill \Box
Proposition 4.9. a) There is a section $S$ such that $g(L_0) \subset uL^u$.
   b) If $g(L_0), g'(L_0) \subset uL^u$ then $h(L_0) \subset uF(L^u)$.

Proof. a) It will be sufficient to prove that for any $l \in L^u$, there is an $h \in F(L^u)$ such that $l \equiv h - \varphi(h) \bmod uL^u$.

   Suppose $n_0 \geq 1$ is such that $V^{n_0}(L^u) \subset uF(L^u)$. Then for all $n \geq n_0, V^n(L^u) \subset uF(L^u)$. Let $h = -(Vl + V^2l + \cdots + V^{n_0}l)$. Then $h \in F(L^u)$ and $\varphi(h) \equiv -(l + Vl + \cdots + V^{n_0}l) \equiv -l + h \bmod uL^u$.

   b) We must prove that if $h \in F(L^u)$ and $h \neq \varphi(h) \in uL^u$ then $h \in uF(L^u)$.

   Indeed, we have $V(h) - h \in V(uL^u) \subset uF(L^u)$ and for all $n \geq 1, V^n(h) \equiv h \bmod uF(L^u)$ implies that $h \in uL^u$. Therefore, $\varphi(h) \in uL^u$ and $h \in uF(L^u)$.

Proposition 4.10. With above notation the short exact sequence

$$0 \rightarrow V[1]^*(L^u) \rightarrow V[1]^*(L) \rightarrow V[1]^*(L^u) \rightarrow 0$$

obtained from (4.2) by the functor $V[1]^*$, has a canonical functorial splittings $\Theta: V[1]^*(L^u) \rightarrow V[1]^*(L)$ and $\Theta: V[1]^*(L) \rightarrow V[1]^*(L^u)$ in the category $\text{MF}_F$.

Proof. It will be sufficient to prove the existence of a functorial splitting

$$\Theta: \text{Hom}_{L^u}(L^u, \tilde{A}_{cr,1}) \rightarrow \text{Hom}_{L^u}(L, \tilde{A}_{cr,1})$$

of the epimorphism $\text{Hom}_{L^u}(L^u, \tilde{A}_{cr,1}) \rightarrow \text{Hom}_{L^u}(L^u, \tilde{A}_{cr,1})$, obtained from exact sequence (4.2).

Suppose $f_0 = (a_{-1}, a_0): L^u \rightarrow (R/x_0^p)T_1 \oplus (R/x_0^p)$ belongs to $\text{Hom}_{L^u}(L^u, \tilde{A}_{cr,1}).$ Here $a_{-1}, a_0 \in \text{Hom}_{\mathcal{W}_1}(L^u, R/x_0^p)$ and for any $l \in L^u, a_{-1}(l) = -a_0(l)$, cf. Subsection 4.2.

Let $S: L^m \rightarrow L$ be a $\mathcal{W}_1$-linear section such that for any $l \in L_0, S(l_0) = \varphi(S(l_0)) + g(l_0)$, where $g \in \text{Hom}(L_0, uL^u)$.

Extend $f_0$ to $\Theta(f_0) = (a_{-1}, a_0): L \rightarrow (R/x_0^p)T_1 \oplus (R/x_0^p)$ by setting $a_0(S(l_0)) = -a_{-1}(S(l_0)) = \mathcal{X}$, where $\mathcal{X}$ is a unique element of $R/x_0^p$ such that $\mathcal{X} \cdot \mathcal{X}^p/x_0^{p(p-1)} = a_0(g(l_0))$. One can prove that $\Theta(f_0) \in \text{Hom}_{L^u}(L, \tilde{A}_{cr,1})$ by verifying relations (4.1) with $m = S(l_0)$.

4.4. A modification of Breuil’s functor. Remind that Breuil’s functor $\mathcal{V}^t: \mathcal{L}^t \rightarrow \text{MF}_F$ attaches to any $L \in \mathcal{L}^t$, the $\Gamma_F$-module $\mathcal{V}(L) = \text{Hom}_{\tilde{L}}(L, \tilde{A}_{st,\infty})$.

Proposition 4.11. The functor $\mathcal{V}^t$ is fully faithful on the subcategory of unipotent objects $\mathcal{L}^{t,u}$.

Proof. Indeed, by Subsection 2.3, $\mathcal{V}[1]^*$ is fully faithful. Then the exactitude of $\mathcal{V}^t$ together with Proposition 4.8 implies that $\mathcal{V}^t|_{\mathcal{L}^{t,u}}$ is fully faithful.
Proposition 4.10 implies that $V^t$ is very far from to be fully faithful on the whole $L^t$. Indeed, if $L \in L^t[1]$ and $0 \to L^u \to L \to L^m \to 0$ is the standard exact sequence then the corresponding sequence of $\Gamma_F$-modules admits a functorial splitting.

Introduce a modification $\tilde{V}^t : L^f \to \mathfrak{M}_F$ of Breuil’s functor.

Suppose $L \in L^f$. From the definition of the category $L^f$ in Subsection [3] it follows the existence of $L' \in L^f$ such that $pL' = L$. More precisely, there are a strict monomorphism $i_{L'} : L \to L'$ and a strict epimorphism $j_{L'} : L' \to L$ such that $p \text{id}_{L'} = j_{L'} \circ i_{L'}$. (Note that $i_{L'} \circ j_{L'} = p \text{id}_L$.)

Consider the following short exact sequences

\[(4.3) \quad 0 \to L \to L' \to \frac{C_p}{p} \to 0\]
\[(4.4) \quad 0 \to L' \to \frac{K_p}{p} \to L \to 0\]

and consider the corresponding sequence of $\Gamma_F$-modules and their morphisms

$$V^t(pL') \xrightarrow{\Theta} V^t(pL) \xrightarrow{V^t(C_p)} V^t(L') \xrightarrow{V^t(K_p)} V^t(\frac{K_p}{p}) \xrightarrow{\Theta} V^t(L^m).$$

As earlier, for any $L \in L^f$, $L^u$ is the maximal unipotent subobject and $L^m$ is the maximal multiplicative quotient object for $L$.

**Lemma 4.12.** $\text{Ker}(V^t(K_p) \circ \Theta) \supset \text{Im}(\Theta \circ V^t(C_p))$.

**Proof.** The section $\Theta$ depends functorially on objects of the category $L^t[1] \supset L^f[1]$. Therefore, we have the following commutative diagram

\[
\begin{array}{ccc}
V^t(pL') & \xrightarrow{V^t(C_p)} & V^t(L') \\
\downarrow{\Theta} & & \downarrow{\Theta} \\
V^t(pL) & \xrightarrow{V^t(K_p)} & V^t(L^m)
\end{array}
\]

and $\Theta \circ V^t(C_p) \circ V^t(K_p) \circ \Theta = V^t(K_p \circ C_p) \circ (\Theta \circ \Theta) = 0$ \hfill $\square$

**Definition.** Set $V^t_L(L) = \text{Ker}(V^t(K_p) \circ \Theta) / \text{Im}(\Theta \circ V^t(C_p))$.

**Proposition 4.13.** With above notation it holds:

a) $V^t_L(L) = \text{Coker} V^t(C_p) = V^t(L)$ if $L \in L^u,f$;

b) $V^t_L(L) = \text{Ker} V^t(K_p) = V^t(L)$ if $L \in L^m,f$;

c) for any $L \in L^f$, we have the induced exact sequence of $\Gamma_F$-modules

$$0 \to V^t(L^m) \to V^t_L(L) \to V^t(L^u) \to 0.$$ This sequence depends functorially on the pair $(L, L')$. 

Proof. The parts a) and b) are obtained directly from definitions. In order to prove c), note that $p\mathcal{L}' = \mathcal{L}$ implies that $p\mathcal{L}'' = \mathcal{L}'$ and $p\mathcal{L}''' = \mathcal{L}''$. This gives a functorial sequence

$$0 \longrightarrow V_{\mathcal{L}'''}(\mathcal{L}''') \longrightarrow V_{\mathcal{L}''}(\mathcal{L}) \longrightarrow V_{\mathcal{L}'}(\mathcal{L}') \longrightarrow 0.$$ 

Then a diagram chasing proves that this sequence is exact.

\[\Box\]

**Proposition 4.14.** Suppose for a given $\mathcal{L} \in \mathcal{L}^f$, the objects $\mathcal{L}', \mathcal{L}'' \in \mathcal{L}^f$ are such that $p\mathcal{L}' = p\mathcal{L}'' = \mathcal{L}$. Then there is a natural isomorphism $f(\mathcal{L}', \mathcal{L}'')$ of $\Gamma_F$-modules such that the following diagram is commutative

$$
\begin{array}{ccc}
0 & \longrightarrow & V^i(\mathcal{L}'') \\
\downarrow{id} & & \downarrow{f(\mathcal{L}', \mathcal{L}'')} \\
0 & \longrightarrow & V^i(\mathcal{L}') \\
\end{array}
\begin{array}{ccc}
0 & \longrightarrow & V^i(\mathcal{L}'') \\
\downarrow{id} & & \downarrow{id} \\
0 & \longrightarrow & V^i(\mathcal{L}') \\
\end{array}
$$

(The lines of this diagram are given by Prop 4.13)

**Proof.** By replacing $\mathcal{L}''$ by $\mathcal{L}' \big| \mathcal{L}''$ with respect to strict epimorphisms $j_{\mathcal{L}'}$ and $j_{\mathcal{L}''}$, we can assume that there is a map $f : \mathcal{L}'' \longrightarrow \mathcal{L}'$ which induces the identity map $p\mathcal{L}'' = \mathcal{L} \longrightarrow p\mathcal{L}' = \mathcal{L}$. Then the existence of $f(\mathcal{L}', \mathcal{L}'')$ follows from functoriality and the diagram chasing implies that it induces the identity maps on $V^i(\mathcal{L}''')$ and $V^i(\mathcal{L}'')$.

\[\Box\]

**Definition.** For $\mathcal{L}, \mathcal{L}' \in \mathcal{L}^{f}$ such that $p\mathcal{L}' = \mathcal{L}$, set $\widetilde{V}^f(\mathcal{L}) = V^f(\mathcal{L})$.

The correspondence $\mathcal{L} \longrightarrow \widetilde{V}^f(\mathcal{L})$ induces the additive exact functor $\widetilde{V}^f : \mathcal{L}^f \longrightarrow M^f$.

4.5. $\varphi$-filtered module $\widetilde{A}_{\mathcal{r}, 2} \in \mathcal{L}^f_0$. Let $\xi = [x_0] + p \in W(R) \subset A_{\mathcal{r}}$, and for $n \in \mathbb{N}$, $\gamma_n(\xi) = \xi^n / n!$

**Lemma 4.15.** If $n \geq 2p$ then $\varphi(\gamma_n(\xi)) \in p^2A_{\mathcal{r}}$.

**Proof.** We have $\varphi(\gamma_n(\xi)) = (p^{n-p+1}/n!)([x_0]p/p + 1)^n$. Therefore, it will be sufficient to verify that for $n \geq 2p$, $v_p(n!) + p + 1 \leq n$. Using the estimate $v_p(n!) < n/(p-1)$ we obtain that the required inequality holds for $p \geq 5$ if $n \geq p + 3$ and for $p = 3$ if $n \geq 8$. It remains to check that our inequality holds for $p = 3$ and $n = 6$ and 7.

Let $J_2$ be the closed ideal in $A_{\mathcal{r}}$ generated by $[x_0]^{p^n}p/p$ and all $\xi^n / n!$ with $n \geq 2p$. Then $J_2 \subset F(A_{\mathcal{r}})$ and $\varphi(J_2) \subset p^2A_{\mathcal{r}}$. Introduce $\widetilde{A}_{\mathcal{r}, 2} = A_{\mathcal{r}}/(J_2 + p^2A_{\mathcal{r}})$ and consider the corresponding induced filtered $\varphi$-module $\widetilde{A}_{\mathcal{r}, 2} = (\widetilde{A}_{\mathcal{r}, 2}, F(\widetilde{A}_{\mathcal{r}, 2}), \varphi) \in \mathcal{L}^f_0$. Clearly, for any $\mathcal{L} \in \mathcal{L}^f_0$, the natural projection $A_{\mathcal{r}, 2} \twoheadrightarrow \widetilde{A}_{\mathcal{r}, 2}$ induces the identification $\text{Hom}_{\mathcal{L}^f_0}(\mathcal{L}, A_{\mathcal{r}, 2}) \cong \text{Hom}_{\mathcal{L}^f_0}(\mathcal{L}, \widetilde{A}_{\mathcal{r}, 2})$.

Consider the structure of $\widetilde{A}_{\mathcal{r}, 2}$ more closely.

Let $T_1 = \xi^p / p$. With obvious notation the elements of $\widetilde{A}_{\mathcal{r}, 2}$ can be written uniquely modulo the subgroup $[x_0]^{p^n}R/T_1 + [x_0]^{p^n}p^{n+1}A_{\mathcal{r}} + \cdots$. Then $\varphi(T_1) = (\xi^p / p + 1)^n$, and $\varphi(T_1)^n / n! = (p^{n-p+1}/n!)([x_0]p/p + 1)^n$. Therefore, it will be sufficient to verify that for $n \geq 2p$, $v_p(n!) + p + 1 \leq n$. Using the estimate $v_p(n!) < n/(p-1)$ we obtain that the required inequality holds for $p \geq 5$ if $n \geq p + 3$ and for $p = 3$ if $n \geq 8$. It remains to check that our inequality holds for $p = 3$ and $n = 6$ and 7.

\[\Box\]
formally, we shall use that \( p^{2}W(R) \) in the form \([r-1]T_1 + [r_0] + p[r_1]\), where \( r_{-1}, r_0, r_1 \in R \). Informally, we shall use that \( r_{-1}, r_1 \in R/x_0^p \) and \( r_0 \in R/x_0^{2p} \). The \( W(R) \)-module structure on \( \tilde{A}_{cr,2} \) is induced by usual operations on Teichmüller’s representatives and the relation \( pT_1 \equiv [x_0]^p \mod p^2 W(R) \).

(Use that \( T_1 \equiv [x_0]^p/p + p[x_0]^{p-1} \mod p^2 W(R) \).)

The \( S \)-module structure on \( \tilde{A}_{cr,2} \) is induced by the \( W(k) \)-algebra morphism \( S \rightarrow W(R) \) such that \( u \mapsto [x_0] \). Then \( F(\tilde{A}_{cr,2}) \) is generated over \( W(R) \) by the images of \( T_1 \) and \( \xi^{p-1} \). Note that \( \xi^{p-1} \equiv [x_0]^{p-1} - p[x_0]^{p-2} \mod p^2 W(R) \).

The map \( \varphi : F(\tilde{A}_{cr,2}) \rightarrow \tilde{A}_{cr,2} \) is uniquely determined by the knowledge of \( \varphi(T_1) \) and \( \varphi(\xi^{p-1}) \). Note that

\[
\varphi(T_1) = \left( \frac{1 + [x_0]^p}{p} \right)^p \equiv 1 + [x_0]^p \mod (J + p^2 A_{cr,2} + p[m_R])
\]

\[
\varphi(\xi^{p-1}) = \left( 1 + \frac{[x_0]^p}{p} \right)^{p-1} \equiv 1 - T_1 \mod (J + p^2 A_{cr,2} + p[m_R])
\]

Suppose \( \mathcal{L} \in \mathcal{L}^{f^t}[1] \) and \( \mathcal{L}' \in \mathcal{L}^{f^t} \) is such that \( p\mathcal{L}' = \mathcal{L} \). Consider short exact sequences \( (4.3) \) and \( (4.4) \). Then the points \( f \in \mathcal{V}^t(p\mathcal{L}') \) and \( \mathcal{V}^t(C_p)(f) \in \mathcal{V}^t(\mathcal{L}') \) are related via the commutative diagram

\[
\begin{array}{ccc}
\mathcal{L}' & \xrightarrow{\mathcal{V}^t(C_p)(f)} & \tilde{A}_{cr,2} \\
\downarrow{c_p} & \downarrow{} & \downarrow{} \\
p\mathcal{L}' & \xrightarrow{f} & \tilde{A}_{cr,1}
\end{array}
\]

where the right vertical arrow is induced by the correspondence

\([r-1]T_1 + [r_0] + p[r_1] \mapsto [r_{-1}]T_1 + [r_0 \mod x_0^p] \).

Similarly, the points \( g \in \mathcal{V}^t(\mathcal{L}') \) and \( \mathcal{V}^t(K_p)(g) \in \mathcal{V}^t(\mathcal{L}_p') \) are related via the commutative diagram

\[
\begin{array}{ccc}
\mathcal{L}' & \xrightarrow{g} & \tilde{A}_{cr,2} \\
\downarrow{K_p} & \downarrow{} & \downarrow{} \\
\mathcal{L}_p' & \xrightarrow{\mathcal{V}^t(K_p)(g)} & \tilde{A}_{cr,1}
\end{array}
\]

where the right vertical arrow is induced by the correspondence

\([r_{-1}]T_1 + [r_0] \mapsto [r_{-1}x_0^0] + p[r_0] \).

4.6. **Filtered \( \varphi \)-modules** \( A_{cr,1}^0 \) and \( A_{cr,2}^0 \). Let \( A_{cr,2}^0 \) be the \( W(R) \)-submodule of \( \tilde{A}_{cr,2} \) consisting of elements \([r_{-1}]T_1 + [r_0] + p[r_1] \) such that \( r_{-1} = -r_0 \mod x_0^p \). Then \( F(A_{cr,2}^0) = F(\tilde{A}_{cr,2}) \cap A_{cr,2}^0 \) is generated over \( W(R) \) by \([x_0^{p-1}]^p T_1 + \xi^{p-1} \) and the congruence

\[
\varphi([x_0^{p-1}]^p T_1 + \xi^{p-1}) \equiv -T_1 + 1 \mod (J_2 + p^2 A_{cr,2} + p[m_R])
\]
implies that ϕ(F(A_{cr,2}^0) ⊂ A_{cr,2}^0 and $A_{cr,2}^0 = (A_{cr,2}^0, F(A_{cr,2}^0), ϕ) \in \mathcal{L}_0$.

Note that $pA_{cr,2}^0 = (pA_{cr,2}^0, pF(A_{cr,2}^0), ϕ) ∈ \mathcal{L}_0$. Then in notation from Subsection 4.4 it holds:

- $\text{Im} Θ = \text{Hom}_{\mathcal{L}_0}(p\mathcal{L}', pA_{cr,2}^0)$;
- $\mathcal{V}^{\ast}(C_p)(\text{Im} Θ) = \text{Hom}_{\mathcal{L}_0}(\mathcal{L}', A_{cr,2}^0)$;
- $\text{Ker} \tilde{Θ} = \text{Hom}_{\mathcal{L}_0}(\mathcal{L}', pA_{cr,2}^0)$;
- $\text{Ker}(\mathcal{V}^{\ast}(K_p) ∘ \tilde{Θ}) = \text{Hom}_{\mathcal{L}_0}(\mathcal{L}', pA_{cr,2}^0)$.

Therefore,

$$\tilde{V}^{\ast}(\mathcal{L}) = \mathcal{V}^{\ast}_C(\mathcal{L}) = \text{Hom}_{\mathcal{L}_0}(\mathcal{L}', A_{cr,2}^0/\mathcal{A}_{cr,1}^0) = \text{Hom}_{\mathcal{L}_0}(\mathcal{L}, \mathcal{A}_{cr,2}/\mathcal{A}_{cr,1}).$$

### 4.7. The functor $\tilde{\mathcal{V}}^{\ast}$. Let $\mathcal{L} ∈ \mathcal{L}_0^{\ast}$ and let $i^{et} : \mathcal{L}^{et} → \mathcal{L}$ be the maximal etale subobject of $\mathcal{L}$.

**Definition.** The functor $\tilde{\mathcal{V}}^{\ast} : \mathcal{L}_0^{\ast} → \text{CM}Γ_F$ is the functor induced by the correspondence $\mathcal{L} → \tilde{\mathcal{V}}^{\ast}(\mathcal{L}) = (\tilde{V}^{\ast}(\mathcal{L}), \tilde{V}^{\ast}(i^{et})).$

The functor $\tilde{\mathcal{V}}^{\ast}$ is not very far from Breuil’s functor $\mathcal{V}^{\ast}$ but it satisfies the following important property.

**Proposition 4.16.** The functor $\tilde{\mathcal{V}}^{\ast}$ is fully faithful.

**Proof.** By standard devissage it will be sufficient to verify this property for the restriction $\tilde{\mathcal{V}}^{\ast} |_{\mathcal{L}_0^{\ast}[1]}$. Due to Proposition 2.1 it will be sufficient to verify that the functor $\Pi^{-1} ∘ \tilde{V}^{\ast} |_{\mathcal{L}_0^{\ast}[1]}$ coincides with the functor $\mathcal{V}^{\ast}$ from Subsection 4.2. This can be proved similarly to the proof of the corresponding fact for unipotent objects in Subsection 4.2 as follows.

Let

$$A_{st,2}^0 = \prod_{j⩾0} A_{cr,2}^0 γ_j(\log(1 + X)) ⊂ \tilde{A}_{st,2} = \prod_{j⩾0} \tilde{A}_{cr,2}^0 γ_j(\log(1 + X))$$

with induced structures of the objects $A_{st,2}^0$ and $\tilde{A}_{st,2}$ of the category $\tilde{\mathcal{L}}$. Then from Subsection 4.6 it follows that

$$\mathcal{V}^{\ast}(\mathcal{L}) = \text{Hom}_{\tilde{\mathcal{L}}}(\mathcal{L}, A_{st,2}^0/pA_{st,2}^0).$$

One can see easily that the correspondence

$$[r_0 \mod x_0^0] T_1 + [r_0] + p[r_1] → (r_0 + x_0^0 r_1) \mod x_0^0 m_R$$

induces the morphism $A_{cr,2}^0/pA_{cr,2}^0 → \mathcal{R}^0$ in the category $\tilde{\mathcal{L}}_0$. This morphism induces a unique identification of the abelian groups $\mathcal{V}^{\ast}(\mathcal{L})$ and $\text{Hom}(\mathcal{L}, \mathcal{R}^0) = \mathcal{V}^{\ast}(\mathcal{L})$. Now going to a suitable factor of the object $A_{st,2}^0/pA_{st,2}^0$ we obtain that this identification is compatible with the $Γ_F$-actions on both abelian groups. □
Now we can describe all Galois invariant lattices of semi-stable $\mathbb{Q}_p[\Gamma_F]$-modules with weights from $[0, p)$.

**Corollary 4.17.** Suppose $V$ is a semi-stable representation of $\Gamma_F$ with weights from $[0, p)$, $\dim_{\mathbb{Q}_p} V = s$ and $T$ is a $\Gamma_F$-invariant lattice in $V$. Then there is a $p$-divisible group $\{\mathcal{L}^{(n)}, i_n\}_{n \geq 0}$ of height $s$ in $L^\text{fl}$ such that $\lim_{\leftarrow n} \mathcal{C}V^\text{fl}(\mathcal{L}^{(n)}) = (T, T^\text{et}, i^\text{et}) \in \text{CM} \Gamma_F$. 
5. Proof of Theorem 0.1

As earlier, \( p \) is a fixed prime number, \( p \neq 2 \). Starting Subsection 5.2 we assume \( p = 3 \).

5.1. For all prime numbers \( l \), choose an embedding of algebraic closures \( \overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_l \) and use it to identify the inertia groups \( I_l = \text{Gal}(\overline{\mathbb{Q}}_l/\mathbb{Q}_{l,\text{ur}}) \), where \( \mathbb{Q}_{l,\text{ur}} \) is the maximal unramified extension of \( \mathbb{Q}_l \), with subgroups in \( \Gamma_Q = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \).

Introduce the category \( \text{M}^t Q \). Its objects are the pairs \( H_Q = (H, \tilde{H}_{st}) \), where \( H \) is a finite \( \mathbb{Z}_p[\Gamma_Q] \)-module unramified outside \( p \) and \( \tilde{H}_{st} = (H_{st}, H_{st}^0, i) \in \text{CM}^t_F \), where \( H|_{l_p} = H_{st} \), \( F = W(\overline{\mathbb{F}}_p)[1/p] \) and \( \text{CM}^t_F \) is the image of the functor \( \tilde{\text{CV}}^{ft} \) from Subsection 4.7. The morphisms in \( \text{M}^t Q \) are compatible morphisms of Galois modules. Clearly, \( \text{M}^t Q \) is a special pre-abelian category, cf. Appendix A.

Let \( \text{M}^t Q[1] \) be the full subcategory of killed by \( p \) objects in \( \text{M}^t Q \). Denote by \( \mathcal{K}(p) \) an algebraic extension of \( \mathbb{Q} \) such that for any \( H_Q = (H, \tilde{H}_{st}) \in \text{M}^t Q[1] \), \( \Gamma_{\mathcal{K}(p)} \) acts trivially on \( H \). In other words, \( \mathcal{K}(p) \) can be taken as a common field-of-definition of points of all such \( \Gamma_Q \)-modules \( H \).

Now assume that

\[ (C) \text{ } \mathcal{K}(p) \text{ is totally ramified at } p. \]

Under this assumption identify \( \text{M}^t Q[1] \) with the full subcategory of \( \text{CM}^t_F \), consisting of \( (H_{st}, H_{st}^0, i) \) such that \( pH_{st} = 0 \) and all points of \( H_{st} \) are defined over \( \mathbb{Q} \). In other words, the objects of \( \text{M}^t Q[1] \) can be described via our local results about killed by \( p \) subquotients of semistable representations of \( \Gamma_F \).

Denote by \( \text{M}^{ft} Q[1] \) a full subcategory in \( \text{M}^t Q[1] \) which consists of killed by \( p \) subquotients of \( p \)-divisible groups in the category \( \text{M}^t Q \).

Under the above assumption \( (C) \), the maximal tamely ramified extension of \( F \in \mathcal{K}(p)F \) is \( F(\zeta_p) \), where \( \zeta_p \) is a primitive \( p \)-th root of unity. (Use that it comes from abelian extension of \( \mathbb{Q} \) unramified outside \( p \).) Therefore, all simple objects in \( \text{M}^t Q[1] \) are of the form \( \mathcal{F}(j) = (\mathbb{F}_p(j), 0, 0) \) if \( 1 \leq j < p \) and \( \mathcal{F}(0) = (\mathbb{F}_p(0), \mathbb{F}_p(0), \text{id}) \) if \( j = 0 \).

Let \( \mathcal{L}^{ft} Q[1] \) and \( \mathcal{L}^{ft} Q[1] \) be the full subcategories of \( \mathcal{L}^{ft}[1] \) mapped by the functor \( \tilde{\text{CV}}^{ft} \) to the objects of \( \text{M}^{ft} Q[1] \) and, resp., \( \text{M}^t Q[1] \). Clearly, \( \mathcal{L}^{ft} Q[1] \) is a full subcategory in \( \mathcal{L}^{ft} Q[1] \) and the only simple objects in these categories are \( \mathcal{L}(r) \), where \( r \in \{j/(p-1) \mid j = 0, 1, \ldots, p-1\} \).

Suppose \( H^{\infty} = \{H_Q^{(n)}, i_n\}_{n \geq 0} \) is a \( p \)-divisible group in the category \( \text{M}^{ft} Q \). Here all \( H_Q^{(n)} = (H^{(n)}, \tilde{H}_{st}^{(n)}) \) are objects of the category \( \text{M}^t Q \). Let \( \mathcal{L} \in \mathcal{L}^{ft} Q[1] \) be such that \( \tilde{\text{CV}}^{ft}(\mathcal{L}) = \tilde{H}_{st}^{(1)} \). Note that the maximal etale subobject \( \mathcal{L}^{et} \) of \( \mathcal{L} \) is isomorphic to \( \mathcal{L}(0)^{n_{et}} \), where
\( n_{\text{et}} = n_{\text{et}}(\mathcal{L}) \in \mathbb{Z}_{\geq 0} \), and \( \mathcal{L}/\mathcal{L}^\text{et} \) has no simple subquotients isomorphic to \( \mathcal{L}(0) \). Similarly, the corresponding maximal multiplicative quotient \( \mathcal{L}^m \) is isomorphic to \( \mathcal{L}(1)^{n_m} \), where \( n_m = n_m(\mathcal{L}) \in \mathbb{Z}_{\geq 0} \), and the kernel of the canonical projection \( \mathcal{L} \rightarrow \mathcal{L}^m \) has no simple subquotients isomorphic to \( \mathcal{L}(1) \). Therefore, for any \( \mathcal{M} \in \mathcal{L}_{L}^{f[1]} \),

\[
\text{Ext}_{\mathcal{L}_{L}^{f[1]}}(\mathcal{L}(0), \mathcal{M}) = \text{Ext}_{\mathcal{L}_{L}^{f[1]}}(\mathcal{M}, \mathcal{L}(1)) = 0.
\]

This implies that for any \( H \in \mathcal{M}_{\mathcal{L}^{f[1]}}[1] \),

\[
\text{Ext}_{\mathcal{M}_{\mathcal{L}^{f[1]}}[1]}(H, \mathcal{F}(0)) = \text{Ext}_{\mathcal{M}_{\mathcal{L}^{f[1]}}[1]}(\mathcal{F}(1), H) = 0.
\]

Therefore, by Theorem \((A.2)\) of Appendix A there is an embedding of \( p \)-divisible groups \( H_{\infty}^m \subset H_{\infty}^\text{et} \), where \( H_{\infty}^m = F_p(1)^{n_m} \), and there is a projection of \( p \)-divisible groups \( H_{\infty} \rightarrow H_{\infty}^\text{et} \), where \( H(1)^{\text{et}} = F_p(0)^{n_{\text{et}}} \).

Similarly, the corresponding maximal multiplicative quotient \( L_{\text{m}} \) is isomorphic to \( L(1) \), where \( n_{\text{m}} = n_{\text{m}}(L) \in \mathbb{Z}_{\geq 0} \), and the kernel of the canonical projection \( L \rightarrow L^m \) has no simple subquotients isomorphic to \( L(1) \). Therefore, for any \( L \in \mathcal{L}^{f[1]}[1] \),

\[
\text{Ext}_{\mathcal{L}_{L}^{f[1]}}(L(0), \mathcal{M}) = \text{Ext}_{\mathcal{L}_{L}^{f[1]}}(\mathcal{M}, L(1)) = 0.
\]

This implies that for any \( H \in \mathcal{M}_{\mathcal{L}^{f[1]}}[1] \),

\[
\text{Ext}_{\mathcal{M}_{\mathcal{L}^{f[1]}}[1]}(H, \mathcal{F}(0)) = \text{Ext}_{\mathcal{M}_{\mathcal{L}^{f[1]}}[1]}(\mathcal{F}(1), H) = 0.
\]

We state this result in the following form.

\textbf{Proposition 5.1.} Under assumption \((C)\) for any \( p \)-divisible group \( H_{\infty} \) in the category \( \mathcal{M}_{\mathcal{L}^{f[1]}} \) there is a filtration of \( p \)-divisible groups

\[
H_{\infty} \supset H_{1\infty}^m \supset H_{0\infty}^m
\]

such that \( H_{0\infty}^m = (\mathbb{Q}/\mathbb{Z})(p - 1)^{n_m} \), \( H_{\infty}/H_{1\infty}^m = (\mathbb{Q}/\mathbb{Z})^{n_{\text{et}}} \), and all simple subquotients in \( H_{1\infty}^m/H_{0\infty}^m \) come from the Tate twists \( F_p(j) \) with \( j = 1, \ldots, p - 2 \).

5.2. Assume that \( p = 3 \).

\textbf{Lemma 5.2.} \( K(3) = \mathbb{Q}(\sqrt[3]{3}, \zeta_9) \), where \( \zeta_9 \) is 9-th primitive root of 1.

This Lemma will be proved in Subsection \( 5.3 \) below.

So, \( K(3) \) satisfies the assumption \((C)\).

\textbf{Proposition 5.3.} If \( H_{\infty} \) is a 3-divisible group in \( \mathcal{M}_{\mathcal{L}^{f[1]}} \) then in its filtration from Proposition \( 5.1 \) the 3-divisible group \( \hat{H}_{\infty} = H_{1\infty}^m/H_{0\infty}^m \) is a product of finitely many trivial 3-divisible groups \( (\mathbb{Q}/\mathbb{Z})(1) \).

\textbf{Proof.} Let \( \mathcal{L}_{\mathbb{Q}} \) be the full subcategory of \( \mathcal{L}_{L}^{f[1]}[1] \) consisting of objects \( \mathcal{L} \) such that \( \mathcal{L}^m = \mathcal{L}^\text{et} = 0 \). This category has only one simple object \( \mathcal{L}(1/2) \). Let \( \mathcal{M}_{\mathbb{Q}} \) be the full subcategory in \( \mathcal{M}_{\mathcal{L}^{f[1]}}[1] \) consisting of the objects \( \mathcal{CV}^{f[1]}(\mathcal{L}) \), where \( \mathcal{L} \in \mathcal{L}_{\mathbb{Q}} \). Then \( \mathcal{L}_{\mathbb{Q}} \) and \( \mathcal{M}_{\mathbb{Q}} \) are antiequivalent categories and \( \hat{H}_{\infty} \) is a product of \( p \)-divisible groups \( (\mathbb{Q}/\mathbb{Z})(1) \). Clearly, our Proposition will be obtained by Theorems \((A.1)\) and \((A.2)\) from the following Proposition.\( \square \)
Proposition 5.4. $\text{Ext}_{\hat{\mathcal{L}}^Q} (\mathcal{L}(1/2), \mathcal{L}(1/2)) = 0$.

Proof. Consider the equivalence of the categories $\Pi : \mathcal{L}^{t} \rightarrow \mathcal{L}^{*}$ from Corollary [3.10]. This equivalence transforms the functor $\overline{\mathcal{C}V}$ into the functor $\mathcal{C}V^{*}$ from Section [2]. Therefore, the objects $\mathcal{L}$ of the category $\Pi(\mathcal{L}^{t}) := \mathcal{L}^{*}_{Q}$ are characterised by the condition that all points of $\mathcal{V}^{*}(\mathcal{L})$ are defined over the field $K(3)$. The objects $\mathcal{L}$ of the category $\Pi(\hat{\mathcal{L}}^{t}) := \hat{\mathcal{L}}^{*}_{Q}$ are characterised by the additional properties: they are all obtained by subsequent extensions via $\mathcal{L}(1/2)$ and $\mathcal{V}^{*}(\mathcal{L})$ appears as a subquotient of semi-stable representation of $\Gamma_{F}$ with Hodge-Tate weights from $[0,2]$.

Introduce the object $\mathcal{L}(1,1) = (L, F(L), \varphi, N)$ of the category $\mathcal{L}^{*}$ as follows:

- $L = W_{l}l \oplus W_{l}I_{l}$;
- $F(L)$ is spanned by $ul_{1}$ and $ul + l_{1}$;
- $\varphi(u_{l_{1}}) = l_{1}$, $\varphi(ul + l_{1}) = l$;
- $N(l_{1}) \equiv 0 \mod u^{3}L$, $N(l_{1}) \equiv l \mod u^{3}L$.

Clearly, $\mathcal{L}(1,1)$ has a natural structure of an element of the group $\text{Ext}_{\mathcal{L}^{*}}(\mathcal{L}(1), \mathcal{L}(1))$.

Lemma 5.5. a) $\mathcal{L}(1,1) \in \mathcal{L}^{*}_{Q}$;

b) $\text{Ext}_{\mathcal{L}^{*}}(\mathcal{L}(1), \mathcal{L}(1)) \simeq \mathbb{Z}/3$ and is generated by the class of $\mathcal{L}(1,1)$;

c) $\text{Ext}_{\mathcal{L}^{*}}(\mathcal{L}(1), \mathcal{L}(1)) = \text{Ext}_{\mathcal{L}^{*}}(\hat{\mathcal{L}}^{*}_{Q}, \mathcal{L}(1), \mathcal{L}(1)) = 0$.

This Lemma will be proved in Subsection [5.4] below.

Lemma [5.3] implies that $\text{Ext}_{\mathcal{L}^{*}}(\mathcal{L}(1), \mathcal{L}(1)) = 0$ and, therefore, any object $\mathcal{L}$ of $\in \mathcal{L}^{*}_{Q}$ is the product of several copies of $\mathcal{L}(1)$ and $\mathcal{L}(1,1)$.

Suppose $\mathcal{L} = L_{1} \times L(1,1) \in \hat{\mathcal{L}}^{*}_{Q}$. Then there is a 3-divisible group $H^{inf/tg}$ in $\overline{\mathcal{M}G}_{Q}$ such that $H^{(1)} = H' \times H(1,1)$, where $H'$ and $H(1,1) = \mathcal{C}V^{*}(\mathcal{L}(1,1))$ belong to $\overline{\mathcal{M}G}_{Q}$. Clearly, $\text{Ext}_{\overline{\mathcal{M}G}_{Q}[1]}(H', H(1,1)) = 0$ and applying Theorem [A.2] we obtain a 3-divisible group $H^{\infty}$ in $\overline{\mathcal{M}G}_{Q}$ such that $H^{(1)} = H(1,1)$. This implies the existence of 2-dimensional semi-stable (and non-crystalline) representation of $\Gamma_{F}$ with the only simple subquotient $\mathbb{F}_{3}(1)$, that is for any Galois invariant lattice $T$ of such representation, the $\Gamma_{F}$-module $T/3T$ has semi-simple envelope $\mathbb{F}_{3}(1) \times \mathbb{F}_{3}(1)$. But our situation appears as a very special case of Breuil’s classification of 2-dimensional semi-stable (and non-crystalline) representations. According to Theorem 6.1.1.2 of [5] the corresponding semi-simple envelope is either $\mathbb{F}_{3}(0) \times \mathbb{F}_{3}(1)$ or $\mathbb{F}_{3}(1) \times \mathbb{F}_{3}(2)$. The proposition is proved. \[\square\]

Now our main Theorem appears as the following Corollary.
Corollary 5.6. If $Y$ is a projective variety with semi-stable reduction modulo 3 and good reduction modulo all primes $l \neq 3$ then $h^2(Y_{\overline{\mathbb{C}}}) = h_{1,1}(Y_{\overline{\mathbb{C}}})$.

Proof. Indeed, let $V$ be the $\mathbb{Q}_3[\Gamma_F]$-module of 2-dimensional étale cohomology of $Y$. Then it is a semi-stable representation of $F$ and its $\Gamma_F$-invariant lattice determines a 3-divisible group in the category $\mathcal{M}_{\Gamma_F}$. By Proposition 5.3, this 3-divisible group can be built from the Tate twists $(\mathbb{Q}_3/\mathbb{Z}_3)(i)$, $i = 0, 1, 2$. Equivalently, all $\Gamma_F$-equivariant subquotients of $V$ are $\mathbb{Q}_3(i)$ with $i = 0, 1, 2$. Applying the Riemann Hypothesis (proved by Deligne) to the reductions $Y \mod l$ with $l \neq 3$, we obtain that $\mathbb{Q}(0)$ and $\mathbb{Q}(2)$ do not appear. Therefore, $V$ is the product of finitely many $\mathbb{Q}_3(1)$ and $h^2(Y_{\overline{\mathbb{C}}}) = h_{1,1}(Y_{\overline{\mathbb{C}}})$. \hfill \Box

5.3. Proof of Lemma 5.2. Use the ramification estimate from Subsection 4.4 to deduce that the normalized discriminant of $\mathcal{K}(3)$ over $\mathbb{Q}$ satisfies the inequality $|D(\mathcal{K}(3)/\mathbb{Q})|_{\mathcal{K}(3)/\mathbb{Q}}^{-1} < 3^{3-1/3} = 18.96236$. Then Odlyzko estimates imply that $[\mathcal{K}(3) : \mathbb{Q}] < 238$.

Let $K_0 = \mathbb{Q}(\zeta_9)$ and $K_1 = \mathbb{Q}(\sqrt[3]{3}, \zeta_9)$. Then $K_0$ is the maximal abelian extension of $\mathbb{Q}$ in $\mathcal{K}(3)$ and $K_1 \subset \mathcal{K}(3)$. We have also the inequality $[\mathcal{K}(3) : K_1] < 60$ and, therefore, $\text{Gal}(\mathcal{K}(3)/\mathbb{Q})$ is soluble.

Prove that $K_1 = \mathcal{K}(3)$.

Suppose the field $K_2$ is the maximal abelian extension of $K_1$ in $\mathcal{K}(3)$. One can apply the computer program SAGE to prove that the group of classes of $K_1$ is trivial. Therefore, $K_2$ is totally ramified at 3 and $\text{Gal}(K_2/\mathbb{Q})$ coincides with the Galois group of the corresponding 3-completions. In particular, $K_2/K_1$ is wildly ramified at 3. Therefore, there is an $\eta \in O_{K_1}^\ast$ such that $K_1(\sqrt[3]{\eta}) \subset K_2$. Then a routine computation shows that the normalized discriminant for $K_1(\sqrt[3]{\eta})$ over $\mathbb{Q}$ is less than $3^{3-1/3}$ if and only if $\eta \equiv 1 \mod O_{K_1}^\ast (1 + 3O_{K_1})^8$. We must verify that such $\eta \in O_{K_1}^\ast$. (This is equivalent to the Leopoldt Conjecture for the field $K_1$.) This was proved via a SAGE computer program written by R. Henderson (Summer-2009 Project at Durham University supported by Nuffield Foundation). This program, cf. Appendix B, constructed a basis $v_i \mod O_{K_1}^3$, $1 \leq i \leq 9$, of $O_{K_1}^3/O_{K_1}^3$, such that $18v_3(\varepsilon_i - 1)$ takes values in the set $\{1, 2, 4, 5, 7, 8, 10, 13, 16\}$. In other words, $v_3(\eta - 1) \geq 1 > 16/18$ implies that $\eta \in O_{K_1}^3$.

Lemma 5.2 is proved.

5.4. Proof of Lemma 5.5. a) Use the notation from the definition of the functor $V^i$ in Subsection 4.

If $f_0 \in V^i(\mathcal{L}(1, 1))$ then the correspondence $f_0 \mapsto (f_0(l_1), f_0(l_2))$ identifies $V^i(\mathcal{L}(1, 1))$ with the $\mathbb{F}_3$-module of couples $(X_{10}, X_0) \in (R/x^6_0)^2$. 


such that $X_0^3/3 = X_0$ and $(X_0^3 + X_1)/3 = X_0$. Then the $F_3[\Gamma_F]$-module $V^\prime(L(1,1))$ is identified with the module formed by the images of the couples $(X_1, X_0 + X_1) \in (R^0_{st})^2$ in the module $\bar{R}^0_{st} = R^0_{st}/(x_0^3m_R + x_0^3m_RY + x_0m_qY^2)$.

In particular, the corresponding $\Gamma_F$-action on $V^\prime(L(1,1))$ comes from the natural $\Gamma_F$-action on the residues of $X_1$ and $X_0$ modulo $x_0^3m_R$. Notice there is a natural $\Gamma_F$-equivariant identification

$$\iota : m_R/(x_0^3m_R) \longrightarrow m/3m,$$

where $m$ is the maximal ideal of the valuation ring of $\mathbb{Q}_3$. This isomorphism $\iota$ comes from the correspondence $r \mapsto \iota(r)$, where for $r = \lim r_n mod p$, $\iota(r) := \lim r_n^{p_n}$.

Then Hensel’s Lemma implies the existence of unique $Z_0, Z_0 \in m$ such that the following equalities hold $\iota(X_1 mod x_0^3m_R) = Z_0 mod 3m$, $\iota(X_0 mod x_0^3m_R) = Z_03m mod 3m$, $Z_0^3 + 3Z_0 = 0$ and $Z_0^3 + 3Z_0 = -Z_0$.

Clearly, $F(Z_0, Z_0) = F(\zeta_0)$. Therefore, if $\tau \in \Gamma_F$ is such that $\tau(\zeta_0) = \zeta_0$ then $\tau(X_1) = X_0$ and $\tau(X_0) = X_0$.

Finally, it follows directly from definitions that if $\tau(\sqrt{3}) = \sqrt{3}$ then $\tau$ acts as identity on the image of $Y$ in $\bar{R}^0_{st}$. The part a) of the Lemma is proved.

b) Suppose $\mathcal{L} = (L, F(L), \varphi, N) \in Ext\mathbb{Q}_3(\mathcal{L}(1), \mathcal{L}(1))$. Then $L = \mathcal{W}_1l \oplus \mathcal{W}_1l_1$, there is an $w \in \mathcal{W}_1$ such that $F(L)$ is spanned by $ul_1$ and $ul + w_l$ over $\mathcal{W}_1$, and it holds $\varphi(ul_1) = l_1$, $\varphi(ul + w_l) = l$, $N(l_1) \in w^3L$ and $N(l_1) \equiv w^3l \mod w^3L$. Notice that $\mathcal{L}$ splits in $\mathcal{L}^*$ iff $w \in u\mathcal{W}_1$.

Therefore, we can assume that $w = \alpha \in k$.

Then the field-of-definition of all points of $V^\prime(\mathcal{L})$ contains the field-of-definition of all solutions $(X_1, X) mod x_0^3m_R \in (R/x_0^3m_R)^2$ of the following congruences: $X_1^3/x_0^3 \equiv X_0 mod x_0^3m_R$ and $(X^3 + \alpha^3X_1)/x_0^3 \equiv X mod x_0^3m_R$.

Let $x_1 \in R$ be such that $x_1^3 = x_0$. Then we can take $X_1 = x_1^3$ and for $T = X/x_1^3$ one has the following Artin-Schreier-type congruence:

$$T^3 - T \equiv -\alpha^3/x_1^6 mod m_R.$$

Using calculations from above part a) we can conclude that $\mathcal{L} \in \mathcal{L}_Q^*$ if and only if the field-of-definition of $T mod m_R$ over $k((x_1))$ belongs to the field-of-definition of $T_0 mod m_R$ over $k((x_1))$, where $T_0 = x_0^6 mod m_R$. By Artin-Schreier theory this happens if and only if $\alpha \in F_3$ and therefore, $\mathcal{L} \cong \mathcal{L}(1,1)$.

c) Suppose $\mathcal{L} = (L, F(L), \varphi, N) \in Ext\mathbb{Q}_3(\mathcal{L}(1), \mathcal{L}(1))$.

Then we can assume that:

- $L = \mathcal{W}_1l \oplus \mathcal{W}_1l_1 \oplus \mathcal{W}_1m$;
- $F(L)$ is spanned over $\mathcal{W}_1$ by $ul_1$, $ul + l_1$ and $um + w + w_1l_1$ with $w, w_1 \in \mathcal{W}_1$;
- $\varphi(ul_1) = l_1$, $\varphi(ul + l_1) = l$ and $\varphi(um + w + w_1l_1) = m$.
Then the condition $u^2m \in F(L)$ implies that $wl_1 \in F(L)$, or $w \in uW_1$ and we can assume that $w = 0$. Then the submodule $W_1m + W_1l_1$ determines a subobject $\mathcal{L}'$ of $\mathcal{L}$, $\mathcal{L}' \in \mathcal{L}_{\mathbb{F}_q}^*$ and using calculations from b) we conclude that $w_1 \in \mathbb{F}_3 \mod uW_1$. Therefore, we can assume that $w_1 = \alpha \in \mathbb{F}_3$ and for $m' = m - \alpha l$ we have $m' \in F(L)$ and $\varphi(um') = m'$, i.e. $\mathcal{L}$ is a trivial extension.

Now suppose $\mathcal{L} = (L, F(L), \varphi, N) \in \text{Ext}_{\mathbb{L}}(\mathcal{L}(1, 1), \mathcal{L}(1))$.

Then we can assume that:

— $L = W_1m \oplus W_1m_1 \oplus W_1l$;
— $F(L)$ is spanned over $W_1$ by $ul$, $um_1 + wl$ and $um + m_1 + w_1l$ with $w, w_1 \in W_1$;
— $\varphi(ul) = l$, $\varphi(um_1 + wl) = m_1$ and $\varphi(um + m_1 + w_1l) = m$.

Again the condition $u^2m \in F(L)$ implies that $w \in uW_1$ and, therefore, we can assume that $w = 0$. Then the quotient module $L/W_1m_1$ is the quotient of $\mathcal{L}$ in the category $\mathcal{L}_{\mathbb{F}_q}^*$. This quotient must belong to the subcategory $\mathcal{L}_{\mathbb{F}_q}^*$. This implies that $w_1 \in \mathbb{F}_3 \mod uW_1$, and, as earlier, $\mathcal{L}$ becomes a trivial extension.

The Lemma is completely proved.
Appendix A. $p$-divisible groups in pre-abelian categories

A.1. Short exact sequences in pre-abelian categories.

A.1.1. Pre-abelian categories. Introduce the concept of a special pre-abelian category following mainly [24], cf. also [21, 22, 25]. Remind that $\mathcal{S}$ is a pre-abelian category if $\mathcal{S}$ is an additive category and for any its morphism $u \in \text{Hom}_\mathcal{S}(A, B)$, there exist $\text{Ker} u = (A_1, i)$ and $\text{Coker} u = (B_1, j)$, where $i \in \text{Hom}_\mathcal{S}(A_1, A)$ and $j \in \text{Hom}_\mathcal{S}(B, B_1)$. For any objects $A, B \in \mathcal{S}$, let $A \prod B$ and $A \coprod B$ be their product and coproduct, respectively. There is a canonical isomorphism $A \prod B \cong A \coprod B$ in $\mathcal{S}$. More generally, for given morphisms $\alpha \in \text{Hom}_\mathcal{S}(C, A)$, $\beta \in \text{Hom}_\mathcal{S}(C, B)$, there is a fibered coproduct $(A \coprod C B, i_A, i_B)$, with $i_A \in \text{Hom}_\mathcal{S}(A, A \coprod C B), i_B \in \text{Hom}_\mathcal{S}(B, A \coprod C B)$ which completes the diagram $A \leftarrow C \beta \rightarrow B$ to a cocartesian square;

- $f \in \text{Hom}_\mathcal{S}(A, C)$ and $g \in \text{Hom}_\mathcal{S}(B, C)$, there is a fibered product $(A \prod_C B, p_A, p_B)$, with $p_A \in \text{Hom}_\mathcal{S}(A \prod C B, A), p_B \in \text{Hom}_\mathcal{S}(A \prod C B, B)$, which completes the diagram $A \rightarrow f \leftarrow C \leftarrow g \rightarrow B$ to a cartesian square.

Similarly, suppose $i \in \text{Hom}_\mathcal{S}(A_1, A)$, $f \in \text{Hom}_\mathcal{S}(A_1, B)$ and $(B \coprod A_1, A, i_A, i_B)$ is their fibered coproduct. If $(A_2, j) = \text{Coker} i$ then there is a morphism $j_B : B \coprod A_1 A \rightarrow C$ such that the following diagram

is commutative (use the zero morphism from $B$ to $A_2$). A formal verification shows that $(A_2, j_B) = \text{Coker} i_B$.

Suppose $j \in \text{Hom}_\mathcal{S}(A, A_2), g \in \text{Hom}_\mathcal{S}(B, A_2)$ and $(B \coprod A_2, A, p_B, p_A)$ is their fibered product. If $(A_1, i) = \text{Ker} j$ then there is an $i_B : A_1 \rightarrow B \coprod A_2 A$ (use the zero map from $A_1$ to $B$) such that the following diagram

is commutative and $(A_1, i_B) = \text{Ker} p_B$.

A.1.2. Strict morphisms. A morphism $u \in \text{Hom}_\mathcal{S}(A, B)$ is strict if the canonical morphism $\text{Coim} u = \text{Coker}(\text{Ker} u) \rightarrow \text{Im} u = \text{Ker}(\text{Coker} u)$ is isomorphism. One can verify that always $\text{Ker} u = (A_1, i)$ is a strict monomorphism and $\text{Coker} u = (B_1, j)$ is a strict epimorphism. By
A pre-abelian category is special if it satisfies the following two axioms:

A1. if $\alpha : C \to A$ is strict monomorphism then $i_B : B \to \prod_C A$ is also strict monomorphism;

A2. if $f : A \to C$ is strict epimorphism then $p_B : \prod_C A \to B$ is also strict epimorphism.

A typical example of a pre-abelian special category is the category of modules with filtration.

Consider short exact sequence (A.1) in $\mathcal{S}$. If $f \in \text{Hom}_\mathcal{S}(A_1, B)$ then we have the following commutative diagram

\[
\begin{array}{ccccccccc}
0 & \to & A_1 & \xrightarrow{i} & A & \xrightarrow{j} & A_2 & \to & 0 \\
\downarrow{f} & & \downarrow{i_A} & & \downarrow{id} & & \downarrow{j_B} & & \downarrow{id} \\
0 & \to & B & \xrightarrow{i_B} & \prod_{A_1} A & \xrightarrow{j_B} & A_2 & \to & 0
\end{array}
\]

Then $j_B = \text{Coker} i_B$ is a strict epimorphism and by axiom A1, $i_B$ is a strict monomorphism. Then $\text{Ker} j_B = \text{Ker}(\text{Coker} i_B) = \text{Im} i_B = (B, i_B)$ and, therefore, the lower row of the above diagram is exact.

Dually, for any $g \in \text{Hom}_\mathcal{S}(B, A_2)$ there is a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \to & A_1 & \xrightarrow{i} & A & \xrightarrow{j} & A_2 & \to & 0 \\
\uparrow{id} & & \uparrow{p_A} & & \uparrow{g} & & \uparrow{p_B} & & \uparrow{0} \\
0 & \to & A_1 & \xrightarrow{i_B} & \prod_{A_1} A & \xrightarrow{p_B} & B & \to & 0
\end{array}
\]

where $i_B = \text{Ker} j_B$ is a strict monomorphism, by Axiom A2 $p_B$ is a strict epimorphism and the lower row of this diagram is exact.

A.1.3. Bifunctor Ext$_\mathcal{S}$. Notice that in special pre-abelian categories, the composition of strict monomorphisms (resp., epimorphisms) is again strict and in the following commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \to & A_1 & \xrightarrow{id} & A & \xrightarrow{id} & A_2 & \to & 0 \\
\downarrow{id} & & \downarrow{f} & & \downarrow{id} & & \downarrow{id} & & \downarrow{id} \\
0 & \to & A_1 & \xrightarrow{id} & A' & \xrightarrow{id} & A_2 & \to & 0
\end{array}
\]
the morphism $f$ is isomorphism. Therefore, one can introduce the set of equivalence classes of short exact sequences $\text{Ext}_S(A_2, A_1)$. This set is functorial in both arguments due to axioms A1 and A2.

Suppose the objects of $S$ are provided with commutative group structure respected by morphisms of $S$. Then for any $A, B \in S$, $\text{Ext}_S(A, B)$ has a natural group structure, where the class of split short exact sequences plays a role of neutral element. Remind that the sum $\varepsilon_1 + \varepsilon_2$ of two extensions $\varepsilon_1 : 0 \to A_1 \overset{i}{\to} A' \overset{j}{\to} A_2 \to 0$ and $\varepsilon_2 : 0 \to A_1 \overset{i''}{\to} A'' \overset{j''}{\to} A_2 \to 0$ is the lower line of the following commutative diagram relating the rows $l = \varepsilon_1 \oplus \varepsilon_2$, $\nabla^*(l)$ and $(+), \nabla^*(l)$,

\[
\begin{array}{ccccccccc}
l : 0 & \to & A_1 \prod A_1 & \overset{i'\Pi i''}{\to} & A' \prod A'' & \overset{j'\Pi j''}{\to} & A_2 \prod A_2 & \to & 0 \\
\nabla^*(l) : 0 & \to & A_1 \prod A_1 & \overset{i'\Pi i''}{\to} & A' \prod A'' & \overset{j'\Pi j''}{\to} & A_2 \prod A_2 & \to & 0 \\
(+), \nabla^*(l) : 0 & \to & A_1 & \overset{+}{\to} & A'' & \overset{id}{\to} & A_2 & \to & 0 \\
\end{array}
\]

Here $\nabla$ is the diagonal morphism, $+$ is the morphism of the group structure on $S$. For any $f \in \text{Hom}_S(A, B)$ and $g \in \text{Hom}_S(B, A)$ the corresponding morphisms $f_* : \text{Ext}_S(A_2, A_1) \to \text{Ext}_S(A_2, B)$ and $g^* : \text{Ext}_S(A_2, A_1) \to \text{Ext}_S(B, A_1)$ are homomorphisms of abelian groups. The proof is completely formal and goes along the lines of [23].

Suppose $\varepsilon \in \text{Ext}_S(A_2, A_1)$, then the extension $\varepsilon + (-\text{id})^*\varepsilon$ splits. We shall need below the following explicit description of this splitting.

Let $\varepsilon : 0 \to A_1 \overset{i}{\to} A \overset{j}{\to} A_2 \to 0$. Then $\varepsilon + (-\text{id})^*\varepsilon$ is the lower row in the following diagram

\[
\begin{array}{ccccccccc}
0 & \to & A_1 \prod A_1 & \overset{i'\Pi i}{\to} & A \prod A_0 & \overset{(i,j)}{\to} & A_2 & \to & 0 \\
\to & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \to \\
0 & \to & A_1 & \overset{\text{id}}{\to} & A_0 & \overset{\text{id}}{\to} & A_2 & \to & 0
\end{array}
\]

where the left vertical arrow is the cokernel of the diagonal embedding $\nabla : A_1 \to A_1 \prod A_1$. One can see that the epimorphic map $A_0 \to A_1$, which splits the lower exact sequence, is induced by the morphism $p_1 - p_2 : A \prod A_2 A \to A$.

Finally, one can apply Serre’s arguments [20] to obtain for any short exact sequence $0 \to A_1 \overset{i}{\to} A \overset{j}{\to} A_2 \to 0$ and any $B \in S$, the following standard 6-terms exact sequences of abelian groups

\[
\begin{align*}
0 & \to \text{Hom}_S(B, A_1) \overset{i_*}{\to} \text{Hom}_S(B, A) \overset{j_*}{\to} \text{Hom}_S(B, A_2) \\
& \overset{\delta}{\to} \text{Ext}_S(B, A_1) \overset{i_*}{\to} \text{Ext}_S(B, A) \overset{j_*}{\to} \text{Ext}_S(B, A_2)
\end{align*}
\]
Also, for all \(m \geq n \geq 0\), it holds

- \((C^{(m)}, i_{mn}) = \text{Ker}(p^m \text{id}_{C^{(n)}}), (C^{(m)}, j_{nm}) = \text{Coker}(p^{n-m} \text{id}_{C^{(n)}})\);
- \(i_{mn} = i_{m,m+1} \circ \ldots \circ i_{n-1,n}\) and \(j_{nm} = j_{n,n-1} \circ \ldots \circ j_{m+1,m}\).

The set of \(p\)-divisible groups in \(\mathcal{S}\) has a natural structure of category. This category is pre-abelian. In particular,

\[
0 \rightarrow (C_1^{(n)}, i_1^{(n)})_{n \geq 0} \xrightarrow{(\gamma_n)_{n \geq 0}} (C^{(n)}, i^{(n)})_{n \geq 0} \xrightarrow{(\delta_n)_{n \geq 0}} (C_2^{(n)}, i_2^{(n)})_{n \geq 0} \rightarrow 0
\]
is a short exact sequence of \( p \)-divisible groups iff for all \( n \geq 1 \), we have in \( \mathcal{S} \) the following commutative diagrams with short exact rows

\[
\begin{array}{c}
0 \to C(n) \to C(n) \to C(n) \to 0 \\
0 \to C(n+1) \to C(n+1) \to C(n+1) \to 0
\end{array}
\]

A.2.2. A property of uniqueness of \( p \)-divisible groups.

**Theorem A.1.** Let \( D \) be an object of \( \mathcal{S}_1 \) such that \( \text{Ext}_{\mathcal{S}_1}(D, D) = 0 \). If \( (C^{(n)}, i^{(n)})_{n \geq 0} \) and \( (C^{(n)}_1, i^{(n)}_1)_{n \geq 0} \) are \( p \)-divisible groups in \( \mathcal{S} \) such that \( C^{(1)} = C^{(1)}_1 \) then these \( p \)-divisible groups are isomorphic.

**Proof.** We must prove that for all \( n \geq 1 \), there are isomorphisms \( f_n : C^{(n)} \to C^{(n)}_1 \) such that \( f_n \circ i^{(n)}_1 = i^{(n)} \circ f_{n+1} \). Suppose \( n_0 \geq 1 \) and all such isomorphisms have been constructed for \( 1 \leq n \leq n_0 \). Therefore, we can assume for simplicity that \( C^{(n)} = C^{(n)}_1 \) for \( 1 \leq n \leq n_0 \). Consider the following commutative diagrams with exact rows:

\[
\begin{array}{c}
\varepsilon_{n_0+1} : & 0 \to C^{(1)} & \to C^{(n_0+1)} & \to C^{(n_0)} & \to 0 \\
\varepsilon_{n_0} : & 0 \to C^{(1)} & \to C^{(n_0)} & \to C^{(n_0-1)} & \to 0
\end{array}
\]

\[
\begin{array}{c}
\varepsilon'_{n_0+1} : & 0 \to C^{(1)} & \to C^{(n_0+1)} & \to C^{(n_0)} & \to 0 \\
\varepsilon'_{n_0} : & 0 \to C^{(1)} & \to C^{(n_0)} & \to C^{(n_0-1)} & \to 0
\end{array}
\]

Here in standard notation of section 1, \( i_1 = i_{1,n_0+1}, i'_1 = i'_{1,n_0+1}, i = i_{1,n_0}, j = j_{n_0,n_0-1}, j_1 = j_{n_0+1,n_0} \) and \( j'_1 = j'_{n_0+1,n_0} \) (dash means that the corresponding morphism is related to the second \( p \)-divisible group). We must construct an isomorphism \( f_{n_0+1} : C^{(n_0+1)} \to C^{(n_0+1)}_1 \) such that \( i^{(n_0)} \circ f_{n_0+1} = i^{(n_0)}_1 \). Consider the following commutative diagram obtained from above two diagrams.

\[
\begin{array}{c}
0 \to C^{(1)} \prod C^{(1)} & \to C^{(n_0+1)} \prod C^{(n_0)} & \to C^{(n_0)} & \to 0 \\
0 \to C^{(1)} \prod C^{(1)} & \to C^{(n_0)} \prod C^{(n_0)} & \to C^{(n_0-1)} & \to 0
\end{array}
\]
Notice that the morphisms of multiplication by $p$ in $C^{(n_0+1)}$ and $C_1^{(n_0+1)}$ can be factored as follows

\[
\begin{array}{ccc}
C^{(n_0+1)} & \xrightarrow{p} & C^{(n_0+1)} \\
\downarrow j_1 & & \downarrow j_1' \\
C^{(n_0)} & & C^{(n_0)}
\end{array} \quad \begin{array}{ccc}
C_1^{(n_0+1)} & \xrightarrow{p} & C_1^{(n_0+1)} \\
\downarrow j_1 & & \downarrow j_1' \\
C^{(n_0)} & & C^{(n_0)}
\end{array}
\]

Therefore, we obtain the following commutative diagram

\[(A.5) \quad \begin{array}{ccc}
C_{n_0+1}^{(n_0+1)} \prod_{C^{(n_0)}} C^{(n_0+1)} & \xrightarrow{p} & C_{1}^{(n_0+1)} \prod_{C^{(n_0)}} C^{(n_0+1)} \\
\downarrow (j_1', j_1) & & \downarrow (j_1', j_1) \\
C^{(n_0)} & \xrightarrow{\nabla} & C^{(n_0)} \prod_{C^{(n_0-1)}} C^{(n_0)}
\end{array}
\]

(here $\nabla$ is the diagonal morphism). Let $\alpha : C^{(1)} \prod_{C^{(1)}} C^{(1)} \to C^{(1)}$ be the cokernel of the diagonal morphism $\nabla : C^{(1)} \to C^{(1)} \prod C^{(1)}$. Clearly, $\nabla$ and $\alpha$ are, resp., strict monomorphism and strict epimorphism. Set $(D_{n_0+1}, \alpha_1) = \text{Coker} (\nabla \circ (i_1 \prod i_1'))$ and $(D_{n_0}, \alpha_0) = \text{Coker} (\nabla \circ (i \prod i))$.

Applying the morphism $\alpha_*$ to diagram (A.4) we obtain the two lower rows of the following diagram

\[(A.6) \quad \begin{array}{cccc}
0 & \xrightarrow{id} & C^{(1)} & \xrightarrow{D_0} & C^{(1)} & \xrightarrow{id} & 0 \\
\downarrow s & & \downarrow j_{n_0+1} & & \downarrow \ & & \downarrow \\
0 & \xrightarrow{id} & C^{(1)} & \xrightarrow{D_{n_0+1}} & C^{(n_0)} & \xrightarrow{id} & 0 \\
\downarrow u & & \downarrow i_{n_0-1,n_0} & & \downarrow & & \downarrow \\
0 & \xrightarrow{id} & C^{(1)} & \xrightarrow{D_{n_0}} & C^{(n_0-1)} & \xrightarrow{id} & 0
\end{array}
\]

Note that the middle line of this diagram equals $\varepsilon_{n_0+1} - \varepsilon'_{n_0+1} \in \text{Ext} (C^{(n_0)}, C^{(1)})$, and at the third row we have a trivial extension. This implies the existence of the first row of our diagram. As it was pointed out earlier, a splitting of the third line can be done via the morphism $f$ from the commutative diagram

\[(A.7) \quad \begin{array}{ccc}
C^{(n_0)} \prod_{C^{(n_0-1)}} C^{(n_0)} & \xrightarrow{p_1-p_2} & C^{(1)} \\
\downarrow \alpha_0 & & \downarrow f \\
D_{n_0} & & D_{n_0}
\end{array}
\]

(Notice that the morphism $s : D_{n_0+1} \to D_0$ is the cokernel of the composition $\text{Ker} f \to D_{n_0} \xrightarrow{u} D_{n_0+1}$.)

Above diagram (A.5) means that the morphism of multiplication by $p$ on $C_1^{(n_0+1)} \prod_{C^{(n_0)}} C^{(n_0+1)}$ factors through the diagonal embedding of $C^{(n_0)}$ into $C^{(n_0)} \prod_{C^{(n_0-1)}} C^{(n_0)}$. From diagram (A.7) it follows that $p \text{id}_{D_{n_0+1}}$ factors through the embedding $\text{Ker} f \to D_{n_0} \xrightarrow{u} D_{n_0+1}$.
Therefore, \( pD_0 = 0 \) i.e. the first line in diagram (A.6) is an element of the trivial group \( \text{Ext}_{S_1}(C^{(1)}, C^{(1)}) = 0 \). So, the second row in (A.6) is a trivial extension, i.e. the extensions \( \varepsilon_{n_0+1} \) and \( \varepsilon'_{n_0+1} \) from diagrams (A.2) and (A.3) are equivalent. This implies the existence of isomorphism \( j_{n_0+1} \). □

A.2.3. Splitting of extensions of \( p \)-divisible groups.

**Theorem A.2.** Suppose \( (C^{(n)}, i^{(n)})_{n \geq 0} \) is a \( p \)-divisible group in the category \( S \) and there are \( D_1, D_2 \in S_1 \) such that \( C^{(1)} \in \text{Ext}_{S_1}(D_2, D_1) \) and \( \text{Ext}_{S_1}(D_1, D_2) = 0 \). Then there is an exact sequence of \( p \)-divisible groups

\[
0 \rightarrow (C^{(n)}_1, i^{(n)}_1)_{n \geq 0} \rightarrow (C^{(n)}_1, i^{(n)}_1)_{n \geq 0} \rightarrow (C^{(n)}_2, i^{(n)}_2)_{n \geq 0} \rightarrow 0
\]

in \( S \) such that \( C^{(1)}_1 = D_1 \) and \( C^{(1)}_2 = D_2 \).

**Proof.** We have the exact sequence \( 0 \rightarrow D_1 \xrightarrow{i} C^{(1)} \xrightarrow{j} D_2 \rightarrow 0 \). Set \( C^{(1)}_1 = D_1 \) and \( \gamma_1 = i \). We must show for all \( n \geq 0 \), the existence of objects \( C^{(n)}_1 \), strict monomorphisms \( \gamma_n : C^{(n)}_1 \rightarrow C^{(n)} \) and \( i^{(n)}_1 : C^{(n)}_1 \rightarrow C^{(n+1)}_1 \) such that \( (C^{(n)}_1, i^{(n)}_1)_{n \geq 0} \) is a \( p \)-divisible group and the system \( (\gamma_n)_{n \geq 0} \) defines an embedding of this \( p \)-divisible group into the original \( p \)-divisible group \( (C^{(n)}, i^{(n)})_{n \geq 0} \). Agree to use for all \( 0 \leq m \leq n \), the notation \( i_{nm} \) and \( j_{nm} \) from Subsection A.2.1 for the original \( p \)-divisible group and set \( C^{(n)} = C^{(n)}_n \).

Illustrate the idea of proof by considering the case \( n = 2 \).

Consider the following commutative diagram with exact rows \( \varepsilon_2 \) and \( \varepsilon^{(1)}_2 = i \circ \varepsilon_2 \):

\[
\begin{array}{ccccccc}
\varepsilon_2 & : & 0 & \rightarrow & C_{10} & \xrightarrow{i_{12}} & C_{20} & \xrightarrow{j_{21}} & C_{10} & \rightarrow & 0 \\
\varepsilon^{(1)}_2 & : & 0 & \rightarrow & C_{10} & \xrightarrow{i^{(1)}_{12}} & C_{21} & \xrightarrow{j^{(1)}_{21}} & C_{11} & \rightarrow & 0 \\
\end{array}
\]

By axiom A1 from Subsection A.1.2 \( \gamma_2^{(1)} \) is a strict monomorphism and the equality \( p \text{id}_{C_{20}} = j_{21} \circ i_{12} \) implies that \( p \text{id}_{C_{21}} = j^{(1)}_{21} \circ (i \circ i^{(1)}_{12}) \). Then the morphism \( j_* : \text{Ext}_S(C_{11}, C_{10}) \rightarrow \text{Ext}_S(C_{11}, D_2) \) induces the following commutative diagram

\[
\begin{array}{ccccccc}
0 & \rightarrow & C_{10} & \xrightarrow{i^{(1)}_{12}} & C_{21} & \xrightarrow{j^{(1)}_{21}} & C_{11} & \rightarrow & 0 \\
0 & \rightarrow & D_2 & \xrightarrow{j} & D_{21} & \xrightarrow{f} & C_{11} & \rightarrow & 0 \\
\end{array}
\]

and \( (C_{11}, i \circ i^{(1)}_{12}) = \text{Ker} f \). From the above decomposition of \( p \text{id}_{C_{21}} \) it follows that it factors through the embedding of \( \text{Ker} f \), therefore,
that we constructed a segment of length 2 of the
Lemma A.3. Consider the general case.

\[ \begin{array}{c}
\text{id} \\
\gamma_2 \quad \gamma_2
\end{array} \]

Verify that one can set \( C_1^{(2)} = C_2 \) and \( \gamma_1^{(1)} = \gamma_2^{(2)} \). Indeed,
\[ p \text{id}_{C_2} \circ \gamma_2^{(2)} = \gamma_2^{(2)} \circ p \text{id}_{C_1} = (\gamma_2^{(2)} \circ j_2^{(1)}) \circ (i \circ \gamma_1^{(1)}) = j_2^{(2)} \circ \gamma_1^{(1)} \circ \gamma_2^{(2)} \]
and because \( \gamma_2^{(2)} \) is monomorphism, \( p \text{id}_{C_2} = j_2^{(2)} \circ \gamma_1^{(1)} \). This means that we constructed a segment of length 2 of the \( p \)-divisible group \((C_1^{(n)}, \gamma_1^{(n)})_{n \geq 0}\).

Consider the general case.

Lemma A.3. For \( k \geq 1 \) and \( 1 \leq t \leq k \), in the category \( \mathcal{S} \) there are the following commutative diagrams with exact lines (for second diagram \( \Omega_k^{(t)}, t \neq 1 \) and for forth diagram \( \Omega_k^{(t)}, t \neq k \)):

\[ \begin{array}{c}
E_k^{(1)} \\
E_k^{(t)} \\
\n\n\n\n\end{array} \]

(Here for all \( k \geq 0 \), \( C_0 = C^{(k)} \), \( i_{k, k+1} = i^{(k)} \), \( j_{k+1, k} = j^{(k)}_{k+1, k} \) and \( j_{k, k+1}^{(t)} = j^{(k)}_{k, k+1, t} \) are the morphisms from Subsection A.2.1, all \( i^{(t)}_{k,k+1} \) and \( \gamma^{(t)}_k \) are strict monomorphisms and all \( f^{(t)}_{k+1} \) and \( j^{(t)}_{k+1, k} \) are strict epimorphisms.)
Proof. Use diagram $E_1^1$ to set $C_{11} = D_1$, $\gamma_1^{(1)} = i$, $j_{11} = \text{id}_{C_{10}}$, $j_{11}^{(1)} = \text{id}_{C_{11}}$. Then for any $k \geq 2$, the upper row of $E_k^1$ is the short exact sequence $\varepsilon_k \in \text{Ext}_S(C_{10}, C_{k-1,0})$ from the original $p$-divisible group $(C_{k0}, i^{(k)})_{k \geq 0}$. Therefore, its lower row equals $i^* \varepsilon_k \in \text{Ext}_S(D_1, C_{k-1,0})$. This defines the objects $C_{k1}$, strict monomorphisms $i_{k-1,k}^{(1)}$, strict epimorphisms $j_{k1}^{(1)}$ and morphisms $\gamma_k^{(1)}$, which are strict monomorphisms (use axiom A1 and that $i$ is a strict monomorphism).

For any $k \geq 2$, the relation $(j_{k,k-1})_* (i^* \varepsilon_k) = i^* \varepsilon_{k-1}$. This gives the morphism $j_{k,k-1}^{(1)} : C_{k1} \rightarrow C_{k-1,1}$ such that $\Delta_k$ commutes. Because $j_{k-1,k-2}$ is a strict epimorphism so is the morphism $j_{k,k-1}^{(1)}$.

The upper row of diagram $\Omega_k^1$ is obtained from the middle column of diagram $E_k^1$ because $\text{Coker} \gamma_k^{(1)} \cong \text{Coker} i = (D_2, j)$. Similarly, the lower row of $\Omega_k^1$ is obtained from diagram $E_{k-1}^1$. The left square of $\Omega_k^1$ is commutative by the definition of $j_{k,k-1}^{(1)}$. The right square is commutative because $\Omega_k^1$ relates diagrams $E_k^1$ and $E_{k-1}^1$. For $k = 2$, the constructed morphism $j_{2,k-1}^{(1)}$ clearly coincides with the morphism $j_{k-1,k}^{(1)}$ from diagram $E_k^1$.

Suppose now we are given integers $k_0 \geq 2$ and $t_0 < k_0$ such that diagrams $E_{k_0}^1$, $\Delta_{k_0}$ and $\Omega_{k_0}$ have been already constructed for all $k < k_0$ and all relevant $t$ and for $k = k_0$ and all $1 \leq t \leq t_0$.

Constructing $E_{k_0}^{t_0+1}$. Consider the following diagram obtained by applying $(i_{k_0-1,k_0}^{(1)})_*$ to the lower row of $E_{k_0}^{t_0}$:

$$
\begin{array}{cccccc}
0 & \rightarrow & C_{k_0-1,t_0-1} & \xrightarrow{i_{k_0-1,k_0}^{(t_0)}} & C_{k_0,t_0} & \xrightarrow{j_{k_0}^{(t_0)}} & C_{11} & \rightarrow & 0 \\
& \downarrow{i_{k_0-1}^{(t_0)}} & & & \downarrow{j_{k_0}^{(t_0)}} & & & \\
0 & \rightarrow & D_2 & \rightarrow & D^* & \xrightarrow{id} & C_{11} & \rightarrow & 0
\end{array}
$$

Clearly, Ker$(C_{k_0t_0} \rightarrow D^*) = (C_{10} - C_{k_0-1,t_0}, \gamma_{10}^{(1)} \circ i_{k_0-1,k_0}^{(1)}(t_0))$. Consider the strict monomorphism $\gamma_{k_0t_0} := \gamma_{k_0t_0}^{(1)} \circ \gamma_{k_0-1,k_0}^{(1)}(t_0) : C_{k_0,t_0} \rightarrow C_{k_00}$ and an analogous morphism $\gamma_{k_0-1,t_0-1} : C_{k_0-1,t_0-1} \rightarrow C_{k_0-1,0}$. Because $t_0 \neq k_0$, one can obtain from diagrams $\Omega_{k_0}^1$ and $E_{k_0}^1$ the following commutative diagram:

$$
\begin{array}{cccccc}
C_{k_0,t_0} & \xrightarrow{\gamma_{k_0t_0}} & C_{k_00} \\
\downarrow{j_{k_0}^{(1)}} & & \downarrow{j_{k_0-1,k_0}^{(1)}} \\
C_{k_0-1,t_0} & \xrightarrow{\gamma_{k_0-1,t_0-1}} & C_{k_0-1,0} \\
\downarrow{i_{k_0-1,k_0}^{(1)}} & & \downarrow{i_{k_0-1,k_0}^{(1)}} \\
C_{k_0,t_0} & \xrightarrow{\gamma_{k_0t_0}} & C_{k_00}
\end{array}
$$
Then $\text{pid}_{C_{k_0}} = j_{k_0,k_0-1} \circ i_{k_0-1,k_0}$ implies that $\text{pid}_{C_{k_0}} = j_{k_0,k_0-1} \circ (\gamma_{k_0-1} \circ j_{k_0-1,k_0})$, i.e. $\text{pid}_{C_{k_0}}$ factors through $\text{Ker}(C_{k_0} \to D^*)$ and, therefore, $\text{pid}_{D^*} = 0$. Then $\text{Ext}_{S}(C_{11}, D_2) = 0$ implies that $(j_{k_0-1})_{*} \varepsilon_{k_0} = 0$, and we obtain from the exact sequence $\text{Hom}(\cdot, D_0) = 0$. Then $\text{Ext}_{S}(C_{11}, D_2) = 0$ implies that $(j_{k_0-1})_{*} \varepsilon_{k_0} = 0$, and we obtain from the exact sequence $\text{Hom}(\cdot, D_0) = 0$. Then $\text{Ext}_{S}(C_{11}, D_2) = 0$ implies that $(j_{k_0-1})_{*} \varepsilon_{k_0} = 0$, and we obtain from the exact sequence $\text{Hom}(\cdot, D_0) = 0$.

As we saw earlier, the commutativity of $(A.9)$ means that all diagrams $\Delta_{k_0}^{t_0+1}$ are commutative. Consider the short exact sequences from diagram $\Omega_{k_0-1}^{t_0}$. They give rise to the following exact sequences of abelian groups (where for $i = 1, 2$, $H_i := \text{Hom}(C_{11}, D_i)$ and $E_i = \text{Ext}(C_{11}, D_i)$).

$$(A.8)$$

$$
\begin{array}{cccc}
H_1 & \text{Ext}(C_{11}, C_{k_0-1,t_0}) & \gamma_{k_0-1}^{(t_0)} & \text{Ext}(C_{11}, C_{k_0-1,t_0-1}) & E_1 \\
\text{id} & j_{k_0-1,k_0-2}^{(t_0)} & \gamma_{k_0-2}^{(t_0)} & j_{k_0-1,k_0-2}^{(t_0)} & \text{id} \\
H_2 & \text{Ext}(C_{11}, C_{k_0-2,t_0}) & \gamma_{k_0-2}^{(t_0)} & \text{Ext}(C_{11}, C_{k_0-2,t_0-1}) & E_2
\end{array}
$$

As we saw earlier, the commutativity of $E_{k_0}^{t_0+1}$ is equivalent to the relation

$$(A.9)$$

$$(\gamma_{k_0-1}^{(t_0)})_{*} \varepsilon_{k_0}^{(t_0+1)} = \varepsilon_{k_0}^{(t_0)}$$

From $\Delta_{k_0}^{t_0}$ it follows that $\varepsilon_{k_0}^{(t_0)} = (j_{k_0-1,k_0-2})_{*} \varepsilon_{k_0}^{(t)}$, and from $E_{k_0-1}^{t_0+1}$ it follows that $(\gamma_{k_0-2}^{(t_0)})_{*} \varepsilon_{k_0-1}^{(t_0+1)} = \varepsilon_{k_0-1}^{(t_0)}$. Then $(A.8)$ implies that the extension $\varepsilon_{k_0}^{(t_0+1)}$ from relation $(A.9)$ can be chosen in such a way that $(j_{k_0-1,k_0-2})_{*} \varepsilon_{k_0-1}^{(t_0+1)} = \varepsilon_{k_0-1}^{(t_0)}$, and this gives $\Delta_{k_0}^{t_0+1}$.

Constructing $\Omega_{k_0}^{t_0+1}$. The above arguments imply that the left squares of diagrams $E_{k_0}^{t_0+1}$ and $E_{k_0-1}^{t_0+1}$ are related via the following commutative
From diagrams $\Omega_{t_0}$, $E_{t_0}$ and $E_{t_0}$ it follows that the induced map $\text{Coker} \gamma_{t_0} \to \text{Coker} \gamma_{t_0}$ is isomorphism. This is equivalent to the existence of $\Omega_{t_0+1}$. The lemma is proved.

For any $k \geq 1$, set $C_{kk} = C^{(k)}_1, i_{k-1,k}^{(k)} = i_1^{(k)}$. Then use diagrams $E_k$ to define the inductive system $(C^{(k)}_1, i_1^{(k)})_{k \geq 0}$. Denote by $\gamma_k$ the strict monomorphism $\gamma_k^{(k)} \circ \ldots \circ \gamma_1^{(1)} : C^{(k)}_1 \to C^{(k)}$. From diagrams $E_k$, $1 \leq t \leq k$, obtain the following commutative diagrams:

\[(A.10)\]

\[\begin{array}{cccccc}
0 & \longrightarrow & C^{(k-1)} & \xrightarrow{\gamma_{k-1}} & C^{(k)} & \xrightarrow{\gamma_k} & C^{(1)} & \longrightarrow & 0 \\
0 & \longrightarrow & C^{(k-1)} & \xrightarrow{i_1^{(k)}} & C^{(k)} & \xrightarrow{j_{k-1}} & C^{(1)} & \longrightarrow & 0
\end{array}\]

It remains only to prove that the inductive system $(C^{(n)}_1, i_1^{(n)})_{n \geq 0}$ is a $p$-divisible group in $\mathcal{S}$. From diagrams $E_k$ and $\Delta_{k-1}$ obtain the following
commutative diagrams with exact rows

\[
\begin{array}{ccc}
(A.11) & 0 & \longrightarrow C_{k-1,k-1} \\
& \downarrow \gamma_{k-1}^{(k-2)} & \downarrow \gamma_{k-1}^{(k-1)} \\
& C_{k-2,k-2} & \longrightarrow C_{k-1,k-1} \\
& \downarrow j_{k-2,k-1}^{(k-1)} & \downarrow j_{k-1,k-1}^{(k-1)} \\
& 0 & \longrightarrow 0 \\
\end{array}
\]

Thus, \( j_{k-2,k-1}^{(k-1)} \) and \( j_{k-1,k-1}^{(k-1)} \) are strict epimorphisms and are included in the following commutative diagrams

\[
\begin{array}{ccc}
(A.12) & C^{(k)} & \longrightarrow C^{(k-1)} \\
& \downarrow \gamma_k & \downarrow \gamma_{k-1} \\
& C_1^{(k)} & \longrightarrow C_1^{(k-1)} \\
\end{array}
\]

For \( 0 \leq m \leq n \), set \( j_{nm}' = j_{n,n-1}' \circ \ldots \circ j_{m+1,m}' \) and \( i_{nm}' = i_{m,m+1}' \circ \ldots \circ i_{n-1,n}' \). Composing diagrams (A.11) obtain the following commutative diagram with exact rows

\[
\begin{array}{ccc}
0 & \longrightarrow C_1^{(n-1)} & \longrightarrow C_1^{(n)} & \longrightarrow C_1^{(n)} & \longrightarrow 0 \\
& \downarrow j_{n-1,m-1}' & \downarrow j_{nm}' & \downarrow \text{id} \\
0 & \longrightarrow C_1^{(m-1)} & \longrightarrow C_1^{(m)} & \longrightarrow C_1^{(m)} & \longrightarrow 0 \\
\end{array}
\]

Thus, \( i_{n-1,n}' \) induces the isomorphism \( \text{Ker} j_{n-1,m-1}' \cong \text{Ker} j_{nm}' \). Therefore, \( \text{Ker} j_{nm}' = (C_1^{(n-m)}, i_{n-m,n}' \text{ if we prove that}

\[
(A.13) \quad \text{Ker} j_{k1}' = (C_1^{k-1}, i_{k-1,k}'). \]

As we noticed earlier, \( j_{k1}' = j_{k,k-1} \circ \ldots \circ j_{21} \). Therefore, diagrams (A.12) imply that \( \gamma_k \circ j_{k1} = j_{k1}' \circ \gamma_1 \). Now diagram (A.10) implies that \( j_{k1}^{(k)} \circ \gamma_1 = j_{k1}' \circ \gamma_1 \) and, therefore, \( j_{k1}^{(k)} = j_{k1}' \) because \( \gamma_1 \) is monomorphism. Hence equality (A.13) follows from diagram (A.10) and \( (C_1^{(n)}, i_1^{(n)})_{n \geq 0} \) satisfies the part a) of the definition of \( p \)-divisible groups.
From diagrams (A.10) and (A.12) one can easily obtain for all indices \(0 \leq m \leq n\), the commutativity of the following diagrams:

\[
\begin{array}{c}
\begin{array}{ccc}
C^{(n)} & \xrightarrow{j_{n,n-m}} & C^{(n-m)} & \xrightarrow{i_{n-m,n}} & C^{(n)} \\
\gamma_n & \downarrow & \gamma_{n-m} & \downarrow & \gamma_n \\
C^{(n)}_1 & \xrightarrow{j'_{n,n-m}} & C^{(n-m)}_1 & \xrightarrow{i'_{n-m,n}} & C^{(n)}_1
\end{array}
\end{array}
\]

Because \(\gamma_n\) is monomorphism, the equality \(j_{n,n-m} \circ i_{n-m,n} = p^m \text{id}_{C^{(n)}}\) implies the equality \(j'_{n,n-m} \circ i'_{n-m,n} = p^m \text{id}_{C^{(n)}_1}\). This gives the part b) of the definition of \(p\)-divisible groups for \((C^{(n)}_1, i^{(n)}_1)_{n \geq 0}\). The proposition is proved.

\[
\square
\]

**Appendix B. SAGE program**

This program finds the principal units \(f_1, f_2, \ldots, f_9\) in \(\mathbb{Q}(\zeta_9, \sqrt[3]{3})\) such that for the normalized 3-adic valuation \(v_3\) and all \(1 \leq i \leq 9\), the integers \(a_i = 18v_3(f_i - 1)\) are prime to 3 and \(1 \leq a_1 < a_2 < \cdots < a_9\). The result appears as the vector \(a f = (a_1, a_2, \ldots, a_9) = (1, 2, 4, 5, 7, 8, 10, 13, 16)\).

```python
p=3
L.<b>=NumberField(x^p-p);
R.<t>=L[]
M.<c>=L.extension(cyclotomic_polynomial(p^2));
L.<b>=NumberField(x^p-p);
R.<t>=L[]
M.<c>=L.extension(cyclotomic_polynomial(p^2));
X.<d>=M.absolute_field()
from_X, to_X=X.structure();
e=X.units()
e.append(X.zeta(p^2))

def n(x):
    for i in range(p):
        if valuation(norm(X(x-i)),p)!=0:
            break
    return norm(X(x-i))

def a_(x):
    return valuation(n(x),p)

p=3
a=[valuation(n(x),p) for x in e]
f=[e.pop(a.index(min(a)))]
```
h=[]
d=X.degree()

while len(e)!=0:
    a=[valuation(n(x),p) for x in e]
    i0=a.index(min(a))
    for j in range(len(f)):
        for k in range(d):
            s=0
            if a_(f[j]^p^k)>min(a):
                break
            if min(a)==a_(f[j]^p^k):
                s=1
                break
        if s==1:
            for i in range((p-1)^2+1):
                if min(a)<a_(e[i0]/(f[j]^(i*p^k))):
                    e[i0]=e[i0]/(f[j]^(i*p^k))
                    h.append((i,j,k))
                    break
            break
    if j+1==len(f) and s==0:
        f.append(e.pop(i0))
af=[valuation(n(x),p) for x in f]; af

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