Manifestation of exciton Bose condensation in induced two-phonon emission and Raman scattering

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The unusual two-photon emission by Bose-condensed excitons caused by simultaneous recombination of two excitons with opposite momenta leaving the occupation numbers of excitonic states with momenta \( p \neq 0 \) unchanged (below coherent two-exciton recombination) is investigated. Raman scattering accompanied by the analogous two-exciton recombination (or creation) is also analyzed. The excess momentum equal to the change of the electromagnetic field momentum in these processes can be transferred to phonons or impurities. The processes under consideration take place if there is Bose condensation in the interacting exciton system, and, therefore, can be used as a new method to reveal exciton Bose condensation. If the frequency of the incident light \( \omega < 2\Omega \) (\( \Omega \) is the frequency corresponding to the recombination of an exciton with \( p=0 \)), the coherent two-exciton recombination with the excess momentum elastically transferred to impurities leads to the appearance of the spectral line \( 2\Omega - \omega \) corresponding to the induced two-photon emission. In this case the anti-Stokes line on frequency \( \omega + 2\Omega \) also appears in the Raman spectrum. If \( \omega > 2\Omega \), there are both Stokes and anti-Stokes lines on frequencies \( \omega \pm 2\Omega \) in the Raman spectrum. The induced two-photon emission is impossible in this case. The spectral lines mentioned above have phonon replicas on frequencies \( |\omega \pm (2\Omega - n\omega_0^s)| \) corresponding to the transmission of the excess momentum (partially or as a whole) to optical phonons of frequency \( \omega_0^s \) (\( n \) is an integer number). The quantitative estimation shows that the light corresponding to the coherent two-exciton recombination can be experimentally observed in \( \mathrm{Cu}_2\mathrm{O} \).

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I. INTRODUCTION

One of the most interesting collective properties of excitons is their possible Bose condensation and superfluidity. There are interesting reports about the experimental observation of Bose condensation and superfluidity of excitons in Cu$_2$O based on the detected changes in the exciton luminescence spectrum and the ballistic exciton transport (see also the discussion in Refs. 1-4). There are also the reports about the observation of condensation of indirect excitons in coupled quantum well structures in strong magnetic field (see also theoretical investigations of superfluidity in coupled quantum wells in Refs. 5-7; strong magnetic field effects are considered in Ref. 8). In view of the strong interest to the search for exciton Bose condensation the new methods of the detection of Bose condensation of excitons seems to be a very important problem.

If excitons are Bose-condensed, the average of the destruction (or creation) operator of an exciton with momentum $p = 0$ is not zero:

$$\langle N - 1 | Q_0 | N \rangle = \langle N + 1 | Q_0^+ | N \rangle = \sqrt{N_0}. \quad (1)$$

Here $|N\rangle$ is the ground state of the exciton system corresponding to average number of excitons $N$, $Q_0$ is the destruction operator of an exciton with $p = 0$, $N_0$ is the number of condensate excitons.

From averages (1) it is easy to see that after recombination or creation of an exciton with $p = 0$ the Bose-condensed exciton system, being in the ground state, also appears in the ground state which differs from the initial one only by the average number of excitons with $p = 0$. The condensate peak in the exciton luminescence spectrum on frequency $\Omega = [E_0(N) - E_0(N - 1)]/\hbar$ corresponds to the recombination of excitons with $p = 0$ (here $E_0(N)$ is the energy of the ground state of the exciton system).

If exciton-exciton interaction is present, not only averages (1) but also the product of two destruction (or creation) operators of excitons with opposite momenta averaged on the ground state of the exciton system is not zero:

$$\langle N - 2 | Q_{-p} Q_p | N \rangle \neq 0;$$
$$\langle N + 2 | Q_{+p} Q_p^+ | N \rangle \neq 0. \quad (2)$$

The unusual optical properties of excitons in Bose-condensed state due to nonzero averages (2) are investigated in this paper. It is shown that the interaction of Bose-condensed excitons with electromagnetic field leads to the possibility of the simultaneous recombination (or creation) of two excitons with opposite momenta corresponding to averages (2). These processes leave the occupation numbers of excitons with $p \neq 0$ unchanged. The only difference between initial and final states of the exciton system is the average number of excitons with $p = 0$. Below the recombination (creation) of two excitons with opposite momenta corresponding to averages (2) will be called coherent two-exciton recombination (creation).

The coherent two-exciton recombination can accompany the induced two-photon emission and Raman scattering. Raman scattering can also be accompanied by the coherent two-exciton creation. In these processes the momentum of the exciton-photon system is not conserved. In fact, the momentum of the exciton system remains unchanged, while the momentum of the electromagnetic field is changed. The excess momentum equal to the change of the electromagnetic field momentum in these processes is transferred to phonons or impurities.

If the frequency of the incident light $\omega < 2\Omega$, the spectral line on frequency $2\Omega - \omega$ corresponding to the induced impurity-assisted two-photon emission and anti-Stokes line on frequency $\omega + 2\Omega$...
corresponding to impurity-assisted Raman scattering appear. Both lines correspond to the coherent two-exciton recombination: the energy of the initial state of exciton system exceeds the energy of the final state by $2\hbar\Omega$, where $\Omega$ is the frequency corresponding to the recombination of an exciton with $p = 0$. The transmission of the excess momentum to impurities is supposed to be elastic. If $\omega > 2\Omega$, there are anti-Stokes line $\omega + 2\Omega$ corresponding to the coherent two-exciton recombination and Stokes line $\omega - 2\Omega$ corresponding to the coherent two-exciton creation in the Raman spectrum. The induced two-photon emission is not possible in this case. The appearance of the lines on frequencies $|\omega \pm 2\Omega|$ is possible only if excitons are Bose-condensed. If exciton system is in the normal state, these lines are absent.

Analogous effects take place if the excess momentum is taken by photons (partially or as a whole). If phonons are optical, there are phonon replicas of spectral lines $|\omega \pm 2\Omega|$ on frequencies $|\omega \pm (2\Omega - n\omega_0^*)|$ ($\omega_0^*$ is the frequency of the optical phonons, $n$ is an integer number).

The quantitative estimation for excitons in Cu$_2$O shows that the experimental observation of the spectral line on frequency $2(\Omega - \omega_0^*) - \omega$ corresponding to the coherent two-exciton recombination is possible and, therefore, can be used to detect exciton Bose condensation.

**II. THE IMPURITY-ASSISTED TWO-PHOTON EMISSION**

Effective Hamiltonian $\hat{H}_X$ responsible for the optical impurity-assisted exciton recombination can be represented in the following form (see Appendix A):

$$\hat{H}_X = \hat{X} + \hat{X}',$$

$$\hat{X} = \sum_{j,pq} \left(X_{pq}^j Q_p c_q^+ + h.c.\right),$$

$$\hat{X}' = \sum_{j,pq} \left(X_{pq}^{j'} Q_p c_q + h.c.\right),$$

where $X_{pq}^j = i\sqrt{2\pi\omega_q}(e^*d_{pq}^j)$, $X_{pq}^{j'} = -i\sqrt{2\pi\omega_q}(ed_{pq}^{j})$. Here $Q_p$ is the destruction operator of an exciton with momentum $p$, $c_q$ is the destruction operator of a photon with momentum $q$ ($\omega_q$ and $e$ are the frequency and the polarization vector of the photon). Matrix elements $d_{pq}^j(d_{pq}^{j'})$ are responsible for the optical recombination of an exciton with momentum $p$ accompanied by the transmission of excess momentum $p \mp q$ to impurity $j$ (see Appendix A). Of course, in the first order on $\hat{X}'$ the exciton recombination is impossible due to the energy conservation.

Let us consider Bose-condensed excitons at $T = 0$, i.e. in vacuum state $|i\rangle_{exc} = |0\rangle_{exc}$ with respect to quasiparticles (the case $T \neq 0$ will be considered in another paper). Because the excitons are in the coherent state, the recombination (creation) of two excitons with equal and opposite momenta accompanied by the excitation of no quasiparticles is possible (see averages (2)). Therefore, two-photon processes (two-photon emission and Raman scattering) accompanied by the coherent two-exciton recombination (creation) can take place. These processes correspond to the transition of the exciton system to the final state $|f\rangle_{exc} = |0\rangle_{exc}$ which differs from $|i\rangle_{exc}$ only by the average number of excitons with $p = 0$.

In this section we shall consider the induced two-photon emission accompanied by the coherent two-exciton recombination with the excess momentum transferred to impurities. The case of the transmission of the excess momentum to phonons is considered in Sec III. Raman scattering with coherent two-exciton recombination (creation) is discussed in Sec IV.
If the scattering of excitons on impurities is elastic, the energy conservation law for the induced two-photon emission accompanied by the coherent two-exciton recombination is

\[ 2\Omega = \omega + \omega', \tag{4} \]

where \( \omega \) and \( \omega' \) are the energies of the emitted photons (here and further \( \hbar = 1 \) if it is not specially mentioned), \( \Omega \) is the energy corresponding to the recombination of an exciton with \( p = 0 \). In case of the induced two-photon emission one of the frequencies (\( \omega \)) in Eq.(4) coincides with the frequency of the incident light. Thus, the spectral line on frequency \( 2\Omega - \omega \) corresponding to the induced two-photon emission by excitons appears at the temperature below the Bose-Einstein condensation point.

The matrix element for the two-photon emission under consideration is

\[ \langle \hat{H}_X \rangle_{fi} = \sum_{\nu} \left( \frac{\langle \hat{X}_{f\nu}(\hat{X}_{\nu} \rangle}{\omega_{\nu} - \omega + i\delta} + \frac{\langle \hat{X}_{f\nu}(\hat{X}_{\nu} \rangle}{\omega_{\nu} - \omega' + i\delta} \right) \tag{5} \]

where \( |i\rangle = |0\rangle_{exc}|0\rangle_{phot} \) and \( |f\rangle = |0\rangle_{exc}|1_k, 1_{k'}\rangle_{phot} \) are initial and final states of the exciton-photon system, \( k \) and \( k' \) are the momenta of the emitted photons, \( \omega_{\nu} \) is the energy difference between initial and intermediate states of the exciton system. Frequencies \( \omega \) and \( \omega' \) are related by energy conservation law (4).

The two-photon emission under consideration takes place through the intermediate states of two types with respect to the exciton occupation numbers:

I. After the recombination of an exciton with momentum \( p = 0 \) the exciton system transfers into the intermediate state containing no quasiparticles. This intermediate state differs from the initial one only by the average number of excitons with \( p = 0 \). During the following transition into the final state other exciton with \( p = 0 \) recombines.

II. After the recombination of an exciton with momentum \( p \neq 0 \) the exciton system transfers into the excited intermediate state containing a quasiparticle with momentum \(-p\). During the following transition of the exciton system into the final state an exciton with the opposite momentum recombines and the quasiparticle disappears. Thus, there are no quasiparticles in the final state of the exciton system.

The amplitudes of the intermediate states of the exciton system and the matrix elements for the corresponding transitions are

\[ |\nu\rangle_{exc} = [\delta_p + (1 - \delta_p)O_p^+]|0\rangle_{exc}, \]
\[ (\hat{X})_{\nu} = X^i_{pk} [\sqrt{N_0} \delta_p + (1 - \delta_p)u_p], \]
\[ (\hat{X})_{f\nu} = X^j_{pk'} [\sqrt{N_0} \delta_p + (1 - \delta_p)u_p], \tag{6} \]

where \( \delta_p = 1 \) at \( p = 0 \); \( \delta_p = 0 \) at \( p \neq 0 \). Here \( O_p^+ \) is the creation operator of a quasiparticle in the exciton system defined by the Bogoliubov transformation (we consider the dilute exciton system):

\[ O_p^+ = u_pQ_p^+ - v_pQ_{-p}. \tag{7} \]

In process of the two-photon emission the change of the momentum of the electromagnetic field is \( k' + k \), where \( k \) and \( k' \) are the momenta of the emitted photons. Since the momentum of the exciton system remains unchanged in this process, the excess momentum \( \delta k = -(k + k') \) is transferred to impurities.
At first we shall consider the case when two different impurities take the excess momentum corresponding to the induced two-photon emission accompanied by the coherent two-exciton recombination. Let impurities \( i \) and \( j \) take momenta \( \mathbf{p} - \mathbf{k} \) and \(- (\mathbf{k}' + \mathbf{p})\) correspondingly. There can be only quasiparticles with momenta \( \pm \mathbf{p} \) and \( \pm (\mathbf{p} - \mathbf{k} + \mathbf{k}') \) in the exciton system in this case. Diagrams corresponding to such two-photon emission (see Appendix E) are shown in Fig.1.

The matrix element for the two-photon emission with the excess momentum transferred to impurities \( i \) and \( j \) has the following form

\[
(H^ij_X)_{ji} = \left[ \frac{X^j_{pk}X^i_{pk}}{\Omega - \epsilon_p - \omega + i\Gamma_p/2} + \frac{X^i_{pk}X^j_{pk}}{\Omega - \epsilon_p - \omega' + i\Gamma_p/2} \right] \times
\]

\[
\times (N_0\delta_p + (1 - \delta_p)u_pv_p) +
\]

\[
+ \left[ \frac{X^i_{p-q,k'}X^j_{p+q,k}}{\Omega - \epsilon_p - \omega + i\Gamma_p/2} + \frac{X^j_{p-q,k}X^i_{p+q,k'}}{\Omega - \epsilon_p - \omega' + i\Gamma_p/2} \right] \times
\]

\[
\times (N_0\delta_{p-q} + (1 - \delta_{p-q})u_{p-q}v_{p-q}),
\]  

(8)

where \( \Gamma_p \) is the width of the energy level of the exciton system corresponding to the quasiparticle with momentum \( \mathbf{p} \); \( \mathbf{q} = \mathbf{k} - \mathbf{k}' \). In Eq. (8) we have taken into account that the difference between the energies of the exciton system in initial and intermediate states is \( \omega_{iv} = \Omega - \epsilon_p \). At \( p \neq 0 \) the quantity \( \epsilon_p \) is the energy of the quasiparticle in the intermediate state of the exciton system; \( \epsilon_p = 0 \) at \( p = 0 \).

Using energy conservation law (4), matrix element (8) can be expressed in terms of the anomalous Green function of Bose-condensed excitons:

\[
(H^ij_X)_{ji} = -\left[ 2\pi iN_0(\Omega - \omega)\delta_p - (1 - \delta_p)\hat{G}_p(\Omega - \omega) \right] X^j_{pk}X^i_{pk} -
\]

\[
- \left[ 2\pi iN_0(\Omega - \omega)\delta_{p-q} - (1 - \delta_{p-q})\hat{G}_{p-q}(\Omega - \omega) \right] X^j_{p-q,k}X^i_{p+q,k},
\]  

(9)

where

\[
\hat{G}_p(\omega) = \frac{u_pv_p}{\omega - \epsilon_p + i\Gamma_p/2} - \frac{u_pv_p}{\omega + \epsilon_p - i\Gamma_p/2}
\]  

(10)

is the anomalous Green function of the dilute Bose-condensed exciton system at \( T = 0 \) (see Appendix E),

\[
N_0(\omega) = \frac{N_0}{\pi} \frac{\Gamma_0/2}{\omega^2 + \Gamma_0^2/4}.
\]  

(11)

For the differential cross-section of the induced two-photon emission with the excess momentum transferred to two different impurities one has

\[
d\sigma_X^j = \frac{\omega(2\Omega - \omega)^3}{c^4} \sum_{(ij)} \sum_p |(s^ij_p)_nm\epsilon^*_m\epsilon^*_n|_p^2 d\Omega'.
\]  

(12)

where

\[
(s^ij_p)_nm = \left[ 2\pi iN_0(\Omega - \omega)\delta_p - (1 - \delta_p)\hat{G}_p(\Omega - \omega) \right] (d^i_{-pk})_nm(d^j_{pk})_m +
\]

\[
+ \left[ 2\pi iN_0(\Omega - \omega)\delta_{p-q} - (1 - \delta_{p-q})\hat{G}_{p-q}(\Omega - \omega) \right] (d^i_{p-q,k})_m(d^j_{p+q,k})_m.
\]  

(13)

At \( |\Omega - \omega| \gg \Gamma_0 \) the terms \( \sim N_0 \) give a negligibly small contribution to the cross-section (12). The cross-section is proportional to the chemical potential of excitons in this case, because \( \hat{G}_p(\omega) \sim \mu \). Therefore, if \( |\Omega - \omega| \gg \Gamma_0 \), cross-section (12) does not directly depend on factor \( N_0 \) of macroscopic
filing of the state with \( p = 0 \). It is defined by anomalous averages \( \langle \rangle \) caused by the existence of Bose-condensate in the system of interacting excitons. The chemical potential of three-dimensional ideal Bose gas is zero below Bose-Einstein condensation point. Thus, at \(|\Omega - \omega| \gg \Gamma_0\) the induced two-photon emission accompanied by the coherent two-exciton recombination with the excess momentum transferred to two different impurities is possible only in the nonideal Bose gas of excitons.

For weakly interacting Bose gas of excitons at \( T = 0 \) the anomalous Green function is \( \hat{G}_p(\omega) = -\mu / [\omega^2 - (\epsilon_p - i \Gamma_p / 2)^2] \), where \( \epsilon_p = p^2 / (2m) + \mu; \mu \) is the chemical potential of the excitons and \( m \) is the exciton mass. If the condition \(|\Omega - \omega| = \epsilon_p\) is fulfilled at \( p \gg q \), we can suppose that \( \epsilon_{p-q} \approx \epsilon_p \) in tensor \( \langle \rangle \).

Replacing the summation on \( p \) by \( p \)-integration in Eq. (12) yields

\[
\sum_p |\langle s^{ij}_p \rangle|_{nm} e_n^* e_m^*|^2 = \mu^2 \int \frac{d^3p}{(2\pi)^3} \frac{(e^{*i}_{d^i_{-pk}})(e^j_{d^j_{pk}}) + (e^{*i}_{d^i_{-pk}})(e^j_{d^j_{pk}})^*}{(\Omega - \omega - i \Gamma_p / 2)^2 - \epsilon_p^2}. \tag{14}
\]

Integral (14) diverges if \( \Gamma_p \to 0 \). Thus, if \(|\Omega - \omega| \gg \Gamma_p\), the main contribution to the cross-section of the induced two-photon emission is given by the resonant levels corresponding to quasiparticles with energies \( \epsilon_p \sim |\Omega - \omega| \) in the Bose-condensed exciton system. In this case matrix elements \( d^i_{-pk} \cdot d^j_{pk} \) can be replaced by their values at momentum \( p_X \) defined by the condition \( \epsilon(p_X) = |\Omega - \omega| \). Replacing the \( p \)-integration by the integration on \( t = \xi_p / \mu \) yields

\[
\sum_p |\langle s^{ij}_p \rangle|_{nm} e_n^* e_m^*|^2 = \frac{2^{1/2} m^{3/2} \mu^{-1/2}}{2\pi^2} ||d^i_{n}(\omega_X)d^j_{m}(\omega_X) + d^i_{n}(\omega'_X)d^j_{m}(\omega_X)|| e_n^* e_m^*|^2 \times 
\int_1^\infty \frac{dt}{(\alpha^2_{X+} + 1 - t^2)(\alpha^2_{X-} + 1 - t^2)}, \tag{15}
\]

where \( \alpha_{X\pm} = \alpha_X \pm i \gamma_p, \alpha_X = |\Omega - \omega| / \mu, \gamma_p = \Gamma_p / (2\mu) \). Here \( d^i(\omega_X) = \int d^i(p_X, k)d_{0p_X} / (4\pi) \) and \( d^i(\omega'_X) = \int d^i(p_X, k')d_{0p_X} / (4\pi) \) are the matrix elements averaged over \( p_X \) directions.

Integral in Eq. (15) can be represented as the sum of two integrals each of which does not diverge at \( \gamma_p \to 0 \):

\[
\int_1^\infty \frac{dt}{(\alpha^2_{X+} + 1 - t^2)(\alpha^2_{X-} + 1 - t^2)} = \frac{1}{\beta^2_+ - \beta^2_-} \left[ \int_1^\infty \frac{dt}{t^2 - \beta^2_+} - \int_1^\infty \frac{dt}{t^2 - \beta^2_-} \right], \tag{16}
\]

where \( \beta^2_\pm = \alpha^2_{X\pm} + 1 \). Thus, the integration in the right side of Eq. (16) can be done supposing \( \beta^2_\pm = \beta^2 \pm i \delta, \delta = 0+ \). One has

\[
\sum_p |\langle s^{ij}_p \rangle|_{nm} e_n^* e_m^*|^2 = \frac{2^{1/2} m^{3/2} \mu^{-1/2}}{8\pi \alpha_X \gamma_X} \left( \frac{\sqrt{\alpha^2_{X+} + 1} - 1}{\sqrt{\alpha^2_{X+} + 1}} \right)^{1/2} \times 
\times ||d^i_{n}(\omega'_X)d^j_{m}(\omega_X) + d^i_{n}(\omega_X)d^j_{m}(\omega_X)|| e_n^* e_m^*|^2, \tag{17}
\]

where \( \gamma_X = \Gamma_X / (2\mu) \). Here \( \Gamma_X \) is the inverse lifetime of a quasiparticle with energy \( \epsilon(p_X) = |\Omega - \omega| \) in the Bose-condensed exciton system.

Inserting the obtained expression in Eq. (12) the summation over impurities can be done. Supposing all the impurities to be identical yields

\[
d\sigma_N^X = \omega(2\Omega - \omega) \frac{2^{1/2} m^{3/2} \mu^{-1/2}(\sqrt{\alpha^2_{X+} + 1} - 1)^{1/2}}{\alpha_X \gamma_X \sqrt{\alpha^2_{X+} + 1}} N(N - 1)|d^i_{n}(\omega'_X)d^j_{m}(\omega_X)|| e_n^* e_m^*|^2 d\Omega', \tag{18}
\]
where \( N \) is the number of the impurities per unit volume.

If the excitons and the impurities are in the isotropic medium and the exciton scattering on the impurities is also isotropic, one has \( d_{nm}^2(\omega_X) = d^2(\omega_X)/3 \). If the incident light is polarized along some direction, the equality \( |\epsilon_n^*d_m(\omega_X)|^2 = d^2(\omega_X)/3 \) is valid, because in case of the induced two-photon emission photon \( \omega \) is identical to the incident one. Summing over the polarization of photon \( \omega' \) and integrating over the direction of its momentum yields the total cross-section of the induced two-photon emission with the coherent two-exciton recombination and the excess momentum transferred to two different impurities

\[
\sigma_2^X = \frac{\omega(2\Omega - \omega)^3a^2/2m_0^3/2}{9c^4\alpha_X\Gamma_X} \left( \sqrt{\alpha^2_X + 1} - 1 \right)^{1/2} N(N - 1)d^2(\omega_X)d^2(\omega'_X).
\]

(19)

Now we shall consider the induced two-photon emission with the coherent two-exciton recombination accompanied by the transmission of excess momentum \( \delta \mathbf{k} \) to one of impurities (for example, to impurity \( j \)) as a whole (see also Ref.\cite{2}). In this process the states of the other impurities remain unchanged, that is why the momentum of a quasiparticle in the intermediate state of the exciton system does not depend on the final state of impurity \( j \). Therefore, the matrix element for such two-photon emission is given by the expression

\[
(H^j_X)_{fi} = \sum_p \left( \frac{X^j_{-pk'}X^j_{pk}}{\Omega - \epsilon_p - \omega + i\Gamma_p/2} + \frac{X^j_{-pk}X^j_{pk'}}{\Omega - \epsilon_p - \omega' + i\Gamma_p/2} \right) \times
\]

\[
\times [N_0\delta_p + (1 - \delta_p)u_p v_p],
\]

(20)
in which the summation is performed over all possible momenta \( p \) of the quasiparticle in the intermediate state of the exciton system. Diagrams corresponding to the two-photon emission under consideration are shown in Fig.2. The sum of the diagrams with all possible momenta \( p \) corresponds to matrix element \( \sigma^j \) (see Appendix B).

For the differential cross-section of the induced two-photon emission with coherent two-exciton recombination and the excess momentum transferred to one of the impurities as a whole one has

\[
d\sigma_2^X = \frac{\omega(2\Omega - \omega)^3}{c^4} \sum_j |(s^{ij})_{nm}\epsilon_n^*\epsilon_m^*|^2 d',
\]

(21)

where tensor \( (s^{ij})_{nm} \) is given by the relation

\[
(s^{ij})_{nm} = \sum_p \left[ 2\pi iN_0(\Omega - \omega)\delta_p - (1 - \delta_p)\hat{G}_p(\Omega - \omega) \right] (d^j_{-pk'})_n(d^j_{pk})_m.
\]

(22)

Here \( \hat{G}_p(\omega) \) is the anomalous Green function of Bose-condensed excitons (see Eq.(10)), \( N_0(\omega) \) is defined by Eq.(11).

If the frequency of the incident light satisfies the condition \( |\Omega - \omega| \gg \Gamma_0 \), the terms \( \sim N_0 \) give a negligibly small contribution in tensor \( (s^{ij}) \). Cross-section \( \sigma(\Omega) \) as well as cross-section \( \sigma(\omega) \) (see above) does not directly depend on factor \( N_0 \) of macroscopic filling of the exciton state with \( p = 0 \) in this case. Therefore, the induced two-photon emission by the ideal Bose gas of excitons with the coherent two-exciton recombination and transmission of the excess momentum to one of the impurities as a whole is impossible if \( |\Omega - \omega| \ll \Gamma_0 \).

Replacing the summation on \( p \) by \( p \)-integration yields

\[
(s^{ij})_{nm} = \mu \int_0^\infty \frac{d^3p}{(2\pi)^3} \frac{(d^j_{-pk'})_n(d^j_{pk})_m}{(\Omega - \omega + i\Gamma_p/2 - \epsilon)(\Omega - \omega - i\Gamma_p/2 + \epsilon)}.
\]

(23)
If $|\Omega - \omega| \gg \Gamma_p$, the main contribution to the tensor of the two-photon emission with transmission of the excess momentum to one of the impurities as a whole is given by the intermediate states corresponding to quasiparticles with energies $\epsilon_p \sim |\Omega - \omega|$. In this case $d^j_{\mathbf{p} k}$; $d^2_{\mathbf{p} k}$. can be replaced by $d^j(\omega_X)$ and $d^j(\omega'_X)$. The definition of $d^j(\omega_X)$ and $d^j(\omega'_X)$ is after Eq.(15). At the same time, integral (23) does not diverge even at $\Gamma_p \to 0$, contrary to the case when two different impurities take the excess momentum. Thus, $\Gamma_p$ can be replaced by $\delta = 0+$ if $|\Omega - \omega| \gg \Gamma_p$. Replacing the $\mathbf{p}$-integration by the integration on $\xi_p$ we have:

$$ (s^{jj})_{nm} = -d^j_{n} (\omega'_{X}) d^m_{n} (\omega_{X}) \frac{2}{2\pi^2} \int_{\mu}^{\infty} \frac{d\xi_p \sqrt{\xi_p - \mu}}{(\Omega - \omega)^2 - \epsilon^2_p + i\delta}. \quad (24) $$

After the substitutions $t = \xi_p / \mu$ and $u = t - 1$ the integration can be easily done. The tensor of the two-photon emission with the coherent two-exciton recombination and the transmission of the excess momentum to impurity $j$ is

$$ (s^{jj})_{nm} = \frac{2^{1/2}m^{3/2}\mu^{1/2}}{4\pi \sqrt{\alpha_X} + 1} \left[ (\sqrt{\alpha^2_X + 1} + 1)^{1/2} + i(\sqrt{\alpha^2_X + 1} - 1)^{1/2} \right] d^j_{n}(\omega'_{X}) d^m_{n}(\omega_{X}). \quad (25) $$

where $\alpha_X = |\Omega - \omega| / \mu$.

Inserting tensor (22) in Eq.(21) the summation on the impurities can be done. Integrating on the polarization and the direction of photon $\omega'_{X}$ yields the overall cross-section of the induced two-photon emission with transmission of the excess momentum to one of the impurities as a whole

$$ \sigma^X_{1} = \frac{\omega(2\Omega - \omega)^3}{c^4} \frac{2m^3\mu N}{9\pi \sqrt{\alpha_X} + 1} d^2(\omega_{X}) d^2(\omega'_{X}). \quad (26) $$

Therefore, the cross-section of the induced impurity-assisted two-photon emission accompanied by the coherent two-exciton recombination $\sigma^X$ is the sum of two terms:

$$ \sigma^X(\omega) = \sigma^X_{1}(\omega) + \sigma^X_{2}(\omega), \quad (27) $$

where $\sigma^X_{1}(\omega)$ and $\sigma^X_{2}(\omega)$ are the cross-sections of the induced two-photon scattering with the excess momentum transferred to one of the impurities as a whole and to two different impurities correspondingly (see Eqs.(20),(19)). Despite $\sigma^X_{1}/\sigma^X_{2} \sim 1/N$; $N \gg 1$, the ratio between these cross-sections can not be estimated in general case, because it essentially depends on other parameters involved in the problem:

$$ \frac{\sigma^X_{1}}{\sigma^X_{2}} = \frac{m^{3/2}\mu^{1/2}}{2^{3/2}\pi \hbar^2 N \tau^X} \frac{\alpha_X}{(\sqrt{\alpha^2_X + 1} - 1)^{1/2}}. \quad (28) $$

where $\tau^X$ is the lifetime of quasiparticle with energy $\epsilon(p_X) = |\Omega - \omega|$ in Bose-condensed exciton system. Eq.(28) is written in ordinary units.

### III. PHONON REPLICAS

The excess momentum corresponding to the two-photon emission accompanied by the coherent two-exciton recombination can be transferred not only to impurities but also to phonons. Thus, two more types of two-photon emission accompanied by the coherent two-exciton recombination are possible: 1) The excess momentum is transferred to phonons as a whole; 2) An impurity takes some
part of the excess momentum. The rest of it is transferred to a phonon. It is easy to see that in both cases the momentum of the quasiparticle in the intermediate state of the exciton system depends on the final state of the phonon subsystem of the crystal. Therefore, the two-photon emission with the excess momentum transferred to phonons (or to both a phonon and an impurity) is analogous to that accompanied by the transmission of the excess momentum to two different impurities.

In general, the optical recombination of an exciton can be assisted by an arbitrary number of phonons. We shall restrict ourselves to the consideration of an exciton assisted by one phonon. In this case the effective Hamiltonian responsible for the optical recombination (creation) of excitons assisted by a phonon can be represented in the form analogous to that of Hamiltonian (3) in Sec.II (see Appendix A):

\[ \hat{H}_L = \hat{L} + \hat{L}', \]

\[ \hat{L} = \sum_{pq} \left[ L_{pq}^> Q_p c_q^+ b_{p-q}^+ + L_{pq}^< Q_p c_q^+ b_{p-q} + h.c. \right], \]

\[ \hat{L}' = \sum_{pq} \left[ L_{pq}^> Q_p c_q b_{p+q} + L_{pq}^< Q_p c_q b_{p-q}^+ + h.c. \right], \]

where \( L_{pq}^> = i \sqrt{2 \pi \omega_p (e^{f_p^+} \langle < \rangle)}, \) \( L_{pq}^< = -i \sqrt{2 \pi \omega_p (e^{f_p^+} \langle < \rangle)}. \) Here \( b_p \) is the destruction operator of a phonon. Effective matrix elements \( f_p^+ \langle < \rangle \) and \( f_p^+ \langle < \rangle \) are responsible for the optical recombination of an exciton with momentum \( p \) accompanied by simultaneous emission or absorption of a phonon. The other designations are defined in the text after Hamiltonian (3) in Sec.I.

Let us consider the two-photon emission accompanied by the coherent two-exciton recombination and the emission of two phonons. These phonons take away the excess momentum. Diagrams corresponding to such two-photon emission are analogous to those shown in Fig.1. Only the lines corresponding to the interaction with impurities should be replaced by the phonon lines responsible for the exciton-phonon interaction.

For the two-photon emission under consideration the energy conservation law is

\[ 2\omega = \omega + \omega' + \omega_{p-k'} + \omega_{p-k}, \]

where \( \omega_{p-k} \) and \( \omega_{p-k'} \) are the energies of the emitted phonons. The total momentum of these phonons equals to the excess momentum \( \delta k = -k - k'. \)

In designations of Hamiltonian (29) the matrix element for the two-photon emission under consideration is given by the formula analogous to Eq.(8):

\[
(H_L)_{fi} = \left[ \frac{L_{-pk'}^> L_{-pk}^>}{\Omega - \omega_{p-k} - \epsilon_p - \omega + i\Gamma_p/2} + \frac{L_{pk}^> L_{-pk'}^>}{\Omega - \omega_{p-k'} - \epsilon_p - \omega' + i\Gamma_p/2} \right] \times
\]

\[
(\sum_{pq} \left[ N_0 \delta_p + (1 - \delta_p) u_{p-q} \right] \left[ \frac{L_{p-q,k}^> L_{p-q,k}^>}{\Omega - \omega_{p-q,k} - \epsilon_{p-q} - \omega + i\Gamma_{p-q}/2} + \frac{L_{p+q,k}^> L_{p+q,k}^>}{\Omega - \omega_{p+q,k} - \epsilon_{p+q} - \omega' + i\Gamma_{p-q}/2} \right] \times
\]

\[
(\sum_{pq} \left[ N_0 \delta_{p-q} + (1 - \delta_{p-q}) u_{p-q} \right] \left[ \right] \times
\]

where \( |i\rangle = |0\rangle_{exc} |0\rangle_{phot} |0\rangle_{phon}; |f\rangle = |0\rangle_{exc} |1_k, 1_{k'}\rangle_{phot} |1_{p-k}, 1_{p-k'}\rangle_{phon}. \)

As in case of impurity-assisted two-photon emission, using energy conservation law (30) yields matrix element (31) expressed in terms of the anomalous Green functions of Bose condensed excitons \( \hat{G}_p(\omega) \) (see Eq.(4)) and the function \( N_0(\omega) \) (see Eq.(11)).
\[
(H_L)_{fi} = - \left[ 2\pi i N_0(\Omega - \omega^*_p - \omega)\delta_p - (1 - \delta_p)\hat{G}_p(\Omega - \omega^*_p - \omega) \right] L^<_{pk} L^>_{pk} - \\
- \left[ 2\pi i N_0(\Omega - \omega^*_p - \omega)\delta_p - (1 - \delta_p)\hat{G}_p(\Omega - \omega^*_p - \omega) \right] L^>_{p,q,k} L^>_{p+q,k}.
\]

We shall limit ourselves to the consideration of the induced two-photon emission accompanied by the coherent two-exciton recombination and the emission of two optical phonons with negligibly small dispersion ($\omega^*_q = \omega^*_0$). The induced two-photon emission of this type leads to the appearance of the spectral line on frequency $2(\Omega - \omega^*_0) - \omega$, where $\omega$ is the frequency of the incident light. This line is the phonon replica of spectral line $2\Omega - \omega$ corresponding to the induced impurity-assisted two-photon emission with the coherent two-exciton recombination.

For the differential cross-section of the induced two-photon emission accompanied by the coherent two-exciton recombination and the emission of two optical phonons one has

\[
\frac{d\sigma^L}{d\omega} = \frac{\omega(2\Omega - \omega)^3}{2e^4} \sum_p |(s_p)_{nm}e^*_n e^*_{m}|^2 d\omega',
\]

where

\[
(s_p)_{nm} = \left[ 2\pi i N_0(\Omega_\omega - \omega)\delta_p - (1 - \delta_p)\hat{G}_p(\Omega_\omega - \omega) \right] (f^>_{pk})_n (f^<_{pk})_m + \\
+ \left[ 2\pi i N_0(\Omega_\omega - \omega)\delta_p - (1 - \delta_p)\hat{G}_p(\Omega_\omega - \omega) \right] (f^>_{p+q,k})_n (f^>_{p+q,k})_m.
\]

Here and further $\Omega_\omega = \Omega - \omega^*_0$. The summation over all possible $p$ takes into account the emission of two phonons with momenta $p - k$ and $-p - k'$ twice: $(s_p)_{nm} = (s_{-p+q})_{nm}$. That is why factor ”2” appears in the denominator of Eq. (33).

Further calculations are analogous to those for the induced two-photon emission with the excess momentum transferred to two different impurities. The overall cross-section of the induced two-photon emission with the coherent two-exciton recombination assisted by two emitted optical phonons is given by the formula

\[
\sigma^L = \frac{\omega(2\Omega_\omega - \omega)^3}{9e^4\alpha_L \Gamma_L} \left( \frac{\sqrt{\alpha_L^2 + 1} - 1}{\sqrt{\alpha_L^2 + 1}} \right) f^2(\omega_L) f^2(\omega_L')
\]

where $f(\omega_L) = \int f^>(p_L,k) dp_L/(4\pi)$; $f(\omega_L') = \int f^>(p_L,k') dp_L/(4\pi)$; momentum $p_L$ is defined by the condition $\epsilon(p_L) = |\Omega_\omega - \omega|$. Here $\alpha_L = |\Omega_\omega - \omega|/\mu$, $\Gamma_L$ is the inverse lifetime of the quasiparticle with energy $\epsilon(p_L)$ in the Bose-condensed exciton system.

The induced two-photon emission accompanied by the coherent two-exciton recombination and the transmission of the excess momentum to both a phonon and an impurity can be considered analogously. The energy conservation law for such two-photon emission is

\[
2\Omega = \omega + \omega' + \omega^*_p - \omega^*_q,
\]

where $\omega^*_p - \omega^*_q$ is the energy of the emitted phonon with momentum $-p - k'$. The rest of the excess momentum $p - k$ is transmitted to an impurity.

The matrix element for the two-photon emission under consideration is

\[
(H_X + H_L)_{fi} = \left[ \frac{L^>_{pk} X^i_{pk}}{\Omega - \epsilon_p - \omega + i\Gamma_p/2} + \frac{X^i_{pk} L^>_{pk'}}{\Omega - \omega^*_p - \omega^*_q - \epsilon_p - \omega' + i\Gamma_p/2} \right] \times \\
\times (N_0\delta_p + (1 - \delta_p)u_{p+1p}) +
\]
emitted optical phonons \((\omega_{\text{phon}} = \omega_0^p)\), the induced two-photon emission under consideration leads to the appearance of the spectral line on frequency \(2\Omega - \omega_0^p - \omega\) (see energy conservation law \((34)\)). Like the line on frequency \(2\Omega - \omega\), this line is the phonon replica of the spectral line \(2\Omega - \omega\).

The differential cross-section of the induced two-photon emission with the coherent two-exciton recombination assisted by both a phonon and an impurity is

\[
d\sigma^{XL} = \frac{\omega(2\Omega - \omega_0^p - \omega)^3}{c^4} \sum_i \sum_p |(s^i_p)_{nm}\epsilon^*_{n}\epsilon^*_{m}|^2 d\omega',
\]

(39)

where

\[
(s^i_p)_{nm} = \left[2\pi i N_0(\Omega - \omega)\delta_p + (1 - \delta_p)\hat{G}_p(\Omega - \omega)\right] (f_{p,k}^i)_{n} (d_{p,k}^i)_{m} + \left[2\pi i N_0(\Omega - \omega)\delta_{p,q} + (1 - \delta_{p,q})\hat{G}_{p,q}(\Omega - \omega)\right] (d_{p,q,k}^i)_{n} (f_{p,q,k}^i)_{m}.
\]

(40)

The overall cross-section is

\[
\sigma^{XL} = \frac{\omega(2\Omega - \omega_0^p - \omega)^3}{9c^4} \left[ \frac{(\sqrt{\alpha_X} + 1 - 1)^{1/2}}{\alpha_X \Gamma_X \sqrt{\alpha^2 + 1}} N d^2(\omega_X) f^2(\omega_X) + \frac{(\sqrt{\alpha^2} + 1 - 1)^{1/2}}{\alpha_L \Gamma_L \sqrt{\alpha^2 + 1}} N d^2(\omega_L) f^2(\omega_L) \right],
\]

(41)

where \(f(\omega_X) = \int f^>(p_X,k')dp_{p_X}/(4\pi)\); \(d(\omega_L') = \int d(p_L,k')dp_{p_L}/(4\pi)\).

**IV. RAMAN SCATTERING**

Not only two-photon emission but also Raman scattering can be accompanied by the coherent two-exciton recombination. Initial and final states of the exciton-photon system are \(|i\rangle = |0\rangle_{\text{exc}}|1_k\rangle_{\text{phot}}\); \(|f\rangle = |0\rangle_{\text{exc}}|1_{k'}\rangle_{\text{phot}}\) in this case. The energy conservation law for the Raman scattering is

\[
\omega + (2\Omega - n\omega_0^p) = \omega',
\]

(42)

where \(n\) is an integer number. The excess momentum corresponding to the scattering is taken by impurities \((n = 0)\), or by both the emitted optical phonon and an impurity \((n = 1)\), or by two emitted optical phonons \((n = 2)\). In general, an arbitrary number of photons \(n\) can be emitted. The
Raman scattering with the coherent two-exciton recombination leads to the appearance of anti-Stokes components on frequencies defined by Eq. (12).

Raman scattering can be accompanied not only by the coherent two-exciton recombination but also by the coherent two-exciton \textit{creation}. For the last type of Raman scattering the energy conservation law is

$$\omega - (2\Omega - n\omega_0^s) = \omega'.$$

It is easy to see that Raman scattering with the coherent two-exciton creation and the transmission of the excess momentum to impurities \((n = 0)\) is possible only if \(\omega > 2\Omega\). Analogously, if the excess momentum is taken by both the emitted optical phonon and an impurity \((n = 1)\), or by two emitted optical phonons \((n = 2)\), Raman light scattering with coherent two-exciton creation takes place if \(\omega > 2\Omega_\omega - \omega_0^s\) and \(\omega > 2\Omega_{\omega -}\) correspondingly.

The cross-sections of Raman scattering are analogous to that of the induced two-photon emission. The cross-sections of Raman scattering accompanied by processes of the coherent two-exciton recombination or creation can be obtained from the formulae of Sec. II, III by the appropriate substitutions. To shorten the text, we only point out these substitutions below.

1) The cross-section of the impurity-assisted Raman scattering accompanied by the coherent two-exciton recombination can be obtained from Eq. (19) (two different impurities take the excess momentum) and Eq. (23) (the excess momentum is transferred to one of the impurities as a whole) by replacing \(d(\omega_X) \rightarrow d'(\tilde{\omega}_X); d'(\tilde{\omega}_X) \rightarrow d'(\tilde{\omega}_X); \omega \rightarrow -\omega, \alpha_X \rightarrow \tilde{\alpha}_X, \Gamma_X \rightarrow \tilde{\Gamma}_X\). Here \(\tilde{\alpha}_X = (\Omega + \omega)/\mu; \tilde{\Gamma}_X\) is the inverse lifetime of the quasiparticle with energy \(\epsilon(\tilde{p}_X) = \Omega + \omega\) in the Bose-condensed exciton system, \(d'(\tilde{\omega}_X) = \int d'(\tilde{p}_X, k)d\sigma_{p_{\tilde{L}_X}}/(4\pi); d'(\tilde{\omega}_X) = \int d'(\tilde{p}_X, k')d\sigma_{p_{\tilde{L}_X}}/(4\pi)\).

2) The cross-section of Raman scattering with the coherent two-exciton recombination assisted by two optical phonons can be obtained from Eq. (12) by replacing \(f(\omega_L) \rightarrow f'(\tilde{\omega}_L); f'(\tilde{\omega}_L) \rightarrow f'(\tilde{\omega}_L); \omega \rightarrow -\omega, \alpha_L \rightarrow \tilde{\alpha}_L, \Gamma_L \rightarrow \tilde{\Gamma}_L\). Here \(\tilde{\alpha}_L = (\Omega_{\omega -} + \omega)/\mu; \tilde{\Gamma}_L\) is the inverse lifetime of the quasiparticle with the energy \(\epsilon(\tilde{p}_L) = \Omega_{\omega -} + \omega\) in Bose-condensed exciton system, \(f'(\tilde{\omega}_L) = \int f'(\tilde{p}_L, k)d\sigma_{p_{\tilde{L}_L}}/(4\pi); f'(\tilde{\omega}_L) = \int f'(\tilde{p}_L, k')d\sigma_{p_{\tilde{L}_L}}/(4\pi)\). The cross-section of Raman scattering with coherent two-exciton recombination and the transmission of the excess momentum to the emitted optical phonon and an impurity can be obtained from Eq. (11) by replacing \(d(\omega_X) \rightarrow d'(\tilde{\omega}_X); f(\omega_X') \rightarrow f'(\tilde{\omega}_X'); f(\omega_X') \rightarrow f'(\tilde{\omega}_X'); d(\omega_L') \rightarrow d'(\tilde{\omega}_L'); \omega \rightarrow -\omega; \alpha_{X(L)} \rightarrow \tilde{\alpha}_{X(L)}; \Gamma_{X(L)} \rightarrow \tilde{\Gamma}_{X(L)}\). Here \(f'(\tilde{\omega}_X') = \int f'(\tilde{p}_X, k')d\sigma_{p_{\tilde{L}_X}}/(4\pi); d'(\tilde{\omega}_L') = \int d'(\tilde{p}_L, k')d\sigma_{p_{\tilde{L}_L}}/(4\pi)\).

3) The cross-section of the impurity-assisted Raman scattering with the coherent two-exciton creation \((\omega > 2\Omega)\) can be obtained from Eq. (10) (two different impurities take the excess momentum) and from Eq. (20) (the excess momentum is transferred to one of the impurities as a whole) by replacing \(d(\omega_X) \rightarrow d'(\tilde{\omega}_X).\)

4) The cross-section of Raman scattering accompanied by the coherent two-exciton creation and the emission of two optical phonons \((\omega > 2\Omega_{\omega -})\) can be obtained from Eq. (24) by replacing \(f(\omega_L) \rightarrow f'(\omega_L).\) The cross-section of Raman scattering with the coherent two-exciton creation and the transmission of the excess momentum to the emitted optical phonon and an impurity \((\omega > 2\Omega_{\omega -} - \omega_0^s)\) can be obtained from Eq. (11) by replacing \(d(\omega_X) \rightarrow d'(\omega_X); f(\omega_L) \rightarrow f'(\omega_L).\)
V. ON POSSIBILITY OF EXPERIMENTAL OBSERVATION OF TWO-PHOTON PROCESSES ACCOMPANIED BY COHERENT TWO-EXCITON RECOMBINATION (CREATION)

In this section we shall analyze the possibility of the experimental observation of the induced two-photon emission and Raman scattering accompanied by the coherent two-exciton recombination or creation. As an example, the induced two-photon emission assisted by optical phonons will be considered.

The intensity of the light $I_L(2\Omega_- - \omega)$ on frequency $2\Omega_- - \omega$ corresponding to the induced two-photon emission under consideration is given by the expression

$$I_L(2\Omega_- - \omega) = \frac{2\Omega_- - \omega}{\omega} \sigma^L(\omega) I(\omega), \quad (44)$$

where $I(\omega) \text{[W/cm}^2\text{]}$ is the intensity of the incident light on frequency $\omega$, $\sigma^L(\omega)$ is the cross-section of the induced two-photon emission assisted by optical phonons (see Eq.(35)).

If the frequency of the incident light $\omega < \Omega_-$, the spectral line corresponding to the induced two-photon emission under consideration will be observed on frequency $2\Omega_- - \omega > \Omega_-$. At $T = 0$ this line is out of the spectral interval of the luminescence of the Bose-condensed excitons assisted by optical phonons which takes place on frequencies $\omega' < \Omega_-$ at zero temperature (see Appendix C). If $\omega > \Omega_-$, more detailed analyses is needed, because the light corresponding to the induced two-photon emission under consideration will be observed against the background of the luminescence of the Bose-condensed excitons assisted by optical phonons. Intensity (44) can be represented as a sum of two terms

$$I^L(2\Omega_- - \omega) = \Delta I^L(2\Omega_- - \omega) + I^L_r(2\Omega_- - \omega), \quad (45)$$

where $I^L(2\Omega_- - \omega)$ is the intensity of the induced two-stage two-photon emission consisting of two successive processes each of which satisfies the energy conservation law. If the frequency of the incident light $\omega > \Omega_-$, the induced two-stage two-photon emission is the result of

I. The spontaneous recombination of an exciton with momentum $p_L$ accompanied by appearance of the quasiparticle with energy $\epsilon(p_L) = \omega - \Omega_-$ in the exciton system and spontaneous emission of a photon on frequency $2\Omega_- - \omega$.

II. The induced recombination of an exciton with momentum $-p_L$ accompanied by disappearance of a quasiparticle with energy $\epsilon(p_L)$ and induced emission of photon $\omega$.

The number of quasiparticles with energy $\epsilon(p_L) = \omega - \Omega_-$ appearing in the exciton system per second due to the spontaneous exciton recombination is $I^L_s(2\Omega_- - \omega)/(2\Omega_- - \omega)$ (here $I^L_s(\omega)$ is the intensity of the luminescence, see Appendix C and also Ref. [24]). These quasiparticles disappear during their effective lifetime $\tau^L$. The disappearance of some part of the quasiparticles is accompanied by the induced emission on frequency $\omega$. Therefore, if $\omega > \Omega_-$, the intensity $I^L_r(2\Omega_- - \omega)$ can be found from the equation

$$I^L_r(2\Omega_- - \omega) = \frac{\tau^L}{\tau^L_L} I^L_s(2\Omega_- - \omega), \quad (46)$$

where $\tau^L_L$ is the lifetime of the quasiparticle $\epsilon(p_L)$ with respect to its disappearance accompanied by the induced emission on frequency $\omega$.

The ratio $\tau^L / \tau^L_L$ defines the portion of quasiparticles with energies $\epsilon(p_L)$, whose disappearance is accompanied by the induced emission on frequency $\omega$. Eq.(46) gives the part of the total intensity of
the luminescence on frequency \(2\Omega_+ - \omega\) corresponding to the induced two-stage two-photon emission. Thus, if the frequency of the incident light \(\omega > \Omega_+\), the intensity of the spectral line on frequency \(2\Omega_+ - \omega\) corresponding to the induced two-photon emission exceeds the intensity of the luminescence on this frequency by \(\Delta I^L(2\Omega_+ - \omega)\).

Using the Fermi’s golden rule yields

\[
\frac{1}{\tau^L} = \frac{(2\pi)^2}{3c} I^2(\omega_L) u^2_{p_L} I(\omega),
\]

where \(u_{p_L}\) is the coefficient of the Bogoliubov transformation corresponding to momentum \(p_L\).

Inserting Eqs. (47) and (48) (see Appendix A) into Eq. (46) one has \(I^L(2\Omega_+ - \omega) = I^L(2\Omega_+ - \omega)/2\). Therefore, the intensity of the spectral line on frequency \(2\Omega_+ - \omega\) corresponding to the induced two-photon emission exceeds the luminescence intensity on this frequency by the quantity

\[
\Delta I^L(2\Omega_+ - \omega) = \frac{(2\Omega_+ - \omega)^4 3^2/m^3/2 \mu \hbar^2}{9e^4 \alpha L^2} \left(\sqrt{\alpha^2_L + 1} - 1\right)^{1/2} L^2(\omega_L) I^2(\omega_L).
\]

Using Eq. (35) we shall estimate cross-section \(\sigma^L\) of the induced two-photon emission with coherent two-exciton recombination assisted by optical phonons. The cross-section expressed in ordinary units is

\[
\sigma^L = \frac{\tau^L V \omega (2\Omega_+ - \omega)^3 5/2 m^3/2 \mu \hbar^2}{9e^4 \hbar^4} \left(\sqrt{\alpha^2_L + 1} - 1\right)^{1/2} \frac{L^2(\omega_L) I^2(\omega_L)}{\alpha_L \sqrt{\alpha^2_L + 1}},
\]

where \(V\) is the volume of excitons interacting with the incident light, \(\tau^L\) is the lifetime of a quasiparticle with energy \(\hbar(\Omega_+ - \omega)\) in the Bose-condensed exciton system, \(\alpha_L = \hbar\Omega_+ / \mu\).

There are reports about the experimental observation of Bose condensation of excitons in \(Cu_2O\). To estimate the cross-section we consider the Bose-condensed excitons of density \(n = 10^{19}\,cm^{-3}\) in the \(Cu_2O\) at \(T = 0\). The exciton mass is equal to \(m = 2.7m_e\) in this crystal, the exciton radius is \(a = 7\,\angs\), the energy corresponding to the recombination of an exciton with \(p = 0\) is \(\hbar\Omega_+ \approx 2eV\). The optical recombination accompanied by the emission of an optical phonon with energy \(\hbar\omega_0 \approx 10meV\) is typical for excitons in \(Cu_2O\).

Chemical potential \(\mu\) of excitons can be roughly estimated from the formula for the chemical potential of weakly interacting Bose gas:

\[
\mu = \frac{4\pi \hbar^2}{m} na \approx 2(meV)
\]

at exciton density \(n = 10^{19}\,cm^{-3}\).

In experiments the excitons were pumped by powerful nanosecond laser pulses (the wavelength \(\lambda \approx 500\,nm\) focused in the spot of diameter \(d \approx 30\,\mu\text{m}\) on the crystal surface. Thus, the volume of Bose-condensed excitons can be estimated by the formula \(V = d^2l\), where \(l \approx 1\,\mu\text{m}\) is the penetration depth of the radiation with the wavelength 500\,nm.

If the frequency of the incident light \(\omega \to \Omega_+\) (\(\alpha_L \to 0\)), cross-section \(\sigma^L\) increases. We shall suppose that \(\hbar(\Omega_+ - \omega) = \mu\). In this case \(F(\omega_L) \approx f(\omega_L) \approx F(\omega_L)\), where \(F(\omega_L) = \int F^>(p_L,k)\,dp_L/(4\pi)\) is the matrix element of the optical recombination of an isolated exciton assisted by an optical phonon (see Appendix A). \(F(\omega_L)\) can be estimated from the expression

\[
\frac{1}{\tau_{exc}} = \frac{4\Omega^3}{3c^3\hbar} F^2(\omega_L),
\]
where $\tau_{exc}$ is the lifetime of an exciton with respect to the spontaneous recombination accompanied by the emission of a photon with energy $\hbar \Omega_-$ and the optical phonon with energy $\hbar \omega_0$. For paraexcitons in $Cu_2O$ the lifetime is $\tau_{exc} \sim 100\mu s$ (see Chap.13 in Ref.4).

The lifetime of the quasiparticle in the Bose-condensed exciton system $\tau^L$ is the subject of further investigation. It can be sufficiently smaller than the radiative lifetime of the excitons $\tau_{exc}$ even at $T = 0$ due to the possibility of the quasiparticle scattering with the emission of a phonon. Supposing $\tau^L$ is in the range $10^{-11} - 10^{-5}\text{sec}$ (the lower boundary corresponds to the condition $\Gamma_L = 10^{-1}\epsilon(p_L)$), the upper one is $10^{-1}\tau_{exc}$ yields the estimation $\sigma^L = 10^{-16} - 10^{-10}\text{cm}^2$.

The radiative lifetime of orthoexcitons in $Cu_2O$ is about $300\text{ns}$. Supposing $\tau^L$ is in the range $10^{-11} - 10^{-9}\text{sec}$ (the upper boundary is the lifetime of an orthoexciton with respect to ortho-to-para phonon-assisted conversion in this case) yields $\sigma^L = 10^{-11} - 10^{-9}\text{cm}^2$. So the induced two-photon emission accompanied by the coherent two-exciton recombination possibly can be observed in $Cu_2O$.

The cross-section of Raman scattering assisted by two emitted optical phonons is quadratic in the product of matrix elements $f'(\tilde{\omega}_L) = \int f'(\tilde{\omega}_L, k)d\sigma_{p\omega}/(4\pi)$ and $f(\tilde{\omega}_L') = \int f(\tilde{\omega}_L', k')d\sigma_{p\omega}/(4\pi)$, where $\tilde{\omega}_L$ is defined by the condition $\epsilon(\tilde{\omega}_L) = |\omega + \Omega_-|$ (see Sec.IV). Since the band gap in $Cu_2O$ is wide, the relation $\epsilon(\tilde{\omega}_L) \gg \omega_0$ is valid ($\Omega_- \sim 10^2\omega_0$ in $Cu_2O$). In this case $f(\tilde{\omega}_L')$ and $f'(\tilde{\omega}_L)$ are negligibly smaller than $f(\omega_L)$ involved in the cross-section of the induced two-phonon emission at $|\Omega_- - \omega| \sim \mu$ (see Eqs. (15),(16) in Appendix B). Besides, the cross-section of Raman scattering under consideration is proportional to the lifetime of a quasiparticle with energy $\epsilon(\tilde{\omega}_L) = |\omega + \Omega_-|$ which is supposed to be essentially smaller than $\tau^L$ of a quasiparticle with energy $\epsilon(p_L) = \mu$ involved in the cross-section (14) at $|\Omega_- - \omega| = \mu$. Therefore, the experimental observation of Raman scattering accompanied by the coherent two-exciton recombination is hardly possible in $Cu_2O$, contrary to the induced two-photon emission (see above). As for Raman scattering accompanied by the coherent two-exciton creation the situation is analogous.

VI. CONCLUSION

We have shown that the induced two-photon emission and Raman scattering accompanied by the recombination (or creation) of two excitons with opposite momenta leaving the exciton occupation numbers with $p \neq 0$ unchanged takes place in the interacting Bose-condensed exciton system. The excess momentum can be transferred to impurities or phonons involved in these processes. Both the induced two-photon emission and Raman scattering under consideration can be used to probe exciton Bose condensation, because they are absent if excitons are in normal state. The recombination (creation) of two excitons with opposite momenta leaving the exciton occupation numbers unchanged is called coherent two-exciton recombination (creation) in the paper.

If the frequency of the incident light $\omega < 2\Omega$ ($\Omega$ is the frequency corresponding to the recombination of an exciton with $p = 0$), there is the spectral line on frequency $2\Omega - \omega$ corresponding to the induced impurity-assisted two-photon emission accompanied by the coherent two-exciton recombination. The anti-Stokes line on frequency $\omega + 2\Omega$ corresponding to the impurity-assisted Raman scattering accompanied by the coherent two-exciton recombination also appears. If $\omega > 2\Omega$, there are both Stokes and anti-Stokes lines on frequencies $\omega \pm 2\Omega$ appear. The Stokes line corresponds to the coherent two-exciton creation. The induced two-photon emission is impossible in this case.

Spectral lines $|\omega \pm 2\Omega|$ have phonon replicas on frequencies $|\omega \pm (2\Omega - n\omega_0)|$ corresponding to the transmission of the excess momentum (partially or as a whole) to optical phonons of frequency $\omega_0$ ($n$ is
an integer number). The quantitative estimation shows that the spectral line on frequency \(2(\Omega - \omega_q') - \omega\) corresponding to the induced phonon-assisted two-photon emission can be experimentally observed in \(Cu_2O\).

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APPENDIX A: EFFECTIVE MATRIX ELEMENTS FOR THE OPTICAL EXCITON RECOMBINATION

Hamiltonian responsible for the interaction of excitons with impurities, phonons and electromagnetic field is

\[
\hat{V} = \hat{U} + \hat{W} + \hat{D},
\]

\[
\hat{U} = \sum_{jpq} U_{jpq}^\dagger Q^+_q Q_p,
\]

\[
\hat{W} = \sum_{pq} \left( W_{qp} Q^+_q Q_p b_{-p}^\dagger + W_{qp}^* Q^+_q Q_p^\dagger b_{-p}^\dagger \right),
\]

\[
\hat{D} = \sum_q \left( D_q Q^+_q c^+_q + D_q' Q^\dagger_{-q} c_q + h.c. \right), \tag{A1}
\]

where Hamiltonian \(\hat{U}\) corresponds to scattering of excitons on impurities, \(\hat{W}\) and \(\hat{D}\) are Hamiltonians responsible for exciton-phonon and exciton-photon interactions correspondingly; \(D_q = i\sqrt{2\pi\omega_q}(\mathbf{e}^\dagger \mathbf{d}_q), \quad D_q' = -i\sqrt{2\pi\omega_q}(\mathbf{e}^\dagger \mathbf{d}_q).\)

In the first order of perturbation series on \(\hat{V}\) only the direct optical recombination (creation) of an exciton is possible. In the second order of perturbation series on \(\hat{V}\) the indirect optical recombination (creation) of an exciton is also possible. For example, matrix element \(X_{pq}^j\) responsible for the optical recombination of an exciton with the transmission of the excess momentum to impurity \(j\) (see Hamiltonian \(\hat{U}\)) can be obtained from the expression

\[
X_{pq}^j \langle f | Q_p c^+_q | i \rangle = \frac{\langle f | D_q Q_p c^+_q | \nu_1 \rangle \langle \nu_1 | U_{jpq} Q^+_p Q_p | i \rangle}{E_i - E_{\nu_1}} + \frac{\langle f | U_{jpq} Q^+_p Q_p | \nu_2 \rangle \langle \nu_2 | D_q Q_p c^+_q | i \rangle}{E_i - E_{\nu_2}}, \tag{A2}
\]

where \(|i\rangle = |..m_p..\rangle_{exc}|..n_q..\rangle_{phot}\) and \(|f\rangle = |..m_p - 1..\rangle_{exc}|..n_q + 1..\rangle_{phot}\) are initial and final states of the exciton-photon system, \(|\nu_1\rangle = |..m_q + 1; m_p - 1..\rangle_{exc}|..n_q..\rangle_{phot}\) and \(|\nu_2\rangle = |..m_q - 1; m_p..\rangle_{exc}|..n_q + 1..\rangle_{phot}\) are its intermediate states, \(E_{\nu_1}\) and \(E_{\nu_2}\) are the energies of the intermediate states. Here \(m_p\) is the occupation number of excitons with momentum \(p\), \(n_q\) is the photon occupation number.

In case of the dilute Bose-condensed exciton gas at \(T = 0\), we have \(m_p = v_p^2; \quad E_i - E_{\nu_1} = -\epsilon_p - \epsilon_q; \quad E_i - E_{\nu_2} = \Omega - \omega_q - \epsilon_q\), where \(v_p\) is the coefficient of Bogoliubov transformation \(\mathbf{d}_p\), \(\epsilon_p\) is the energy of the Bogoliubov quasiparticle in the exciton system. Therefore,

\[
X_{pq}^j = i\sqrt{2\pi\omega_q}(\mathbf{e}^\dagger \mathbf{d}_pq), \quad \mathbf{d}_pq = -\mathbf{d}_p U_{jpq} \left[ \frac{\nu_p^2}{\epsilon_p + \epsilon_q} - \frac{\nu_q^2}{\Omega - \omega_q - \epsilon_q} \right]. \tag{A3}
\]
Matrix element $X^{ij}_{pq}$ (see Hamiltonian (3)) is given by the analogous expression:

$$X^{ij}_{pq} = -i \sqrt{2\pi} \omega \langle \mathbf{e} \mathbf{d}^{ij}_{pq} \rangle; \quad \mathbf{d}^{ij}_{pq} = -d_q U^{ij}_{qp} \left[ \frac{u^2_q}{\epsilon_p + \epsilon_q} - \frac{v^2_q}{\Omega + \omega_q - \epsilon_q} \right].$$  \hfill (A4)

Matrix element $L^>_{pq}$ corresponding to the optical phonon-assisted exciton recombination (see Hamiltonian (29)) can be obtained from the expression

$$L^>_{pq}(|f|Q_p c_{q}^+ b_{p-q}^+ |i\rangle) = \frac{\langle f\rangle D_q Q_q c_{q}^+ |\nu_1\rangle \langle \nu_1 | W^*_{pq} Q_q^+ Q_{p-q}^+ |\nu_2\rangle \langle \nu_2 | D_q Q_q c_{q}^+ |i\rangle}{E_i - E_{\nu_1}} + \frac{(|W^*_{pq} Q_q^+ Q_{p-q}^+ |\nu_2\rangle \langle \nu_2 | D_q Q_q c_{q}^+ |i\rangle)}{E_i - E_{\nu_2}},$$ \hfill (A5)

where $|i\rangle = |..m\ldots_{\text{exc}}|..s_{p-q}\ldots_{\text{phon}}|..n\ldots_{\text{phot}}\rangle$ and $|f\rangle = |..m\ldots_{\text{exc}}|..s_{p-q} + 1\ldots_{\text{phon}}|..n\ldots_{\text{phot}}\rangle$ are initial and final states of the system "excitons and phonons + electromagnetic field", $|\nu_1\rangle = |..m_q + 1; m\ldots_{\text{exc}}|..s_{p-q} + 1\ldots_{\text{phon}}|..n\ldots_{\text{phot}}\rangle$ and $|\nu_2\rangle = |..m_q - 1; m\ldots_{\text{exc}}|..s_{p-q} - 1\ldots_{\text{phon}}|..n\ldots_{\text{phot}}\rangle$ are the intermediate states of this system.

Supposing the phonons to be optical and neglecting their dispersion, for the dilute Bose-condensed excitons at $T = 0$ one has

$$L^>_{pq} = i \sqrt{2\pi} \omega q \langle e^* f^>_{pq} \rangle; \quad f^>_{pq} = -d_q W^*_{pq} \left[ \frac{u^2_q}{\epsilon_p + \epsilon_q + \omega_q^0} - \frac{v^2_q}{\Omega - \omega_q - \epsilon_q} \right].$$ \hfill (A6)

The analogous expressions for the other matrix elements in Hamiltonian (29) are

$$L^<_{pq} = i \sqrt{2\pi} \omega q \langle e^* f^<_{pq} \rangle; \quad f^<_{pq} = -d_q W^*_{pq} \left[ \frac{u^2_q}{\epsilon_p + \epsilon_q - \omega_q^0} - \frac{v^2_q}{\Omega - \omega_q - \epsilon_q} \right];$$

$$L^\prime_{pq} = -i \sqrt{2\pi} \omega q \langle e^* f^>_{pq} \rangle; \quad f^\prime_{pq} = -d_q W^*_{pq} \left[ \frac{u^2_q}{\epsilon_p + \epsilon_q + \omega_q^0} - \frac{v^2_q}{\Omega + \omega_q - \epsilon_q} \right];$$

$$L^\prime<_{pq} = -i \sqrt{2\pi} \omega q \langle e^* f^<_{pq} \rangle; \quad f^\prime<_{pq} = -d_q W^*_{pq} \left[ \frac{u^2_q}{\epsilon_p + \epsilon_q - \omega_q^0} - \frac{v^2_q}{\Omega + \omega_q - \epsilon_q} \right].$$ \hfill (A7)

If $\epsilon_p + \epsilon_q \ll \omega_q^0$ and $\Omega - \omega_q - \omega_q \ll \omega_q^0$, matrix element $f^>_{pq}$ is

$$f^>_{pq} = \frac{-d_q W^*_{pq}}{\omega_q^0}. \hfill (A8)$$

In this case $f^>_{pq}$ coincides with the matrix element for the optical recombination of an isolated exciton assisted by an optical phonon. In fact, using (A5) yields

$$L^>_{pq} = i \sqrt{2\pi} \omega q \langle e^* F^>_{pq} \rangle; \quad F^>_{pq} = \frac{d_q W^*_{pq}}{E_p - E_q - \omega_q^0}, \hfill (A9)$$

where $E_p = p^2/(2m)$. It is easy to see that $f^>_{pq} = F^>_{pq}$ if $E_p - E_q \ll \omega_q^0$.

**APPENDIX B: DIAGRAMS OF TWO-PHOTON PROCESSES ACCOMPANYING COHERENT TWO-EXCITON RECOMBINATION**

The induced two-photon emission and Raman scattering accompanied by the coherent two-exciton recombination (creation) can be considered in the formalism of Green functions of Bose-condensed
excitons. Diagrams shown in Figs.1-2 are responsible for the corresponding elements of $S$-matrix expressed in terms of anomalous Green functions of Bose-condensed excitons. Below it will be illustrated on the example of the impurity-assisted two-photon emission accompanied by the coherent two-exciton recombination.

In the Heisenberg representation Hamiltonian \( \hat{H}_X \) is

\[
\hat{H}_X(t) = \sum_{jpq} \left( X_{pq}^j c_q^+(t) c_p^j(t) + X_{pq}^{ij} c_q^+(t) c_p^{ij}(t) + h.c. \right),
\]

where \( c_q(t) = c_q e^{-i\omega_q t}; \quad Q_p(t) = Q_p e^{-i\epsilon_p t} \). The exciton energy is measured from the bottom of the exciton band: \( \epsilon(0) = 0 \).

We shall expand the evolution operator \( \hat{S}(t) = T \exp(-i \int_{-\infty}^{t} \hat{H}_X(t') dt') \) as a power series in \( \hat{H}_X \) up to the second order, inclusive. The element of \( S \)-matrix corresponding to the two-photon emission with the coherent two-exciton recombination and the transmission of momentum \( \mathbf{p} - \mathbf{k}' \) to impurity \( j \) is

\[
(S^{ij}_p)_{fi} = \frac{(-i)^2}{2!} \int \int_{-\infty}^{\infty} dt' dt'' e^{-i\Omega(t'+t'')} \times \left\{ \begin{array}{l}
\times \left\{ X_{pk}^i X_{-pk'}^{i'} \left[ N_0 \delta_p + i(1 - \delta_p) \tilde{G}_{-p}(t' - t'') \right] + \\
+ X_{p+q,k}^i X_{-p+q,k'}^{i'} \left[ N_0 \delta_{p-q} + i(1 - \delta_{p-q}) \tilde{G}_{p+q}(t' - t'') \right] \langle f| c_{k'}^{+}(t') c_{k}^{-}(t'') |i\rangle_{\text{phot}} + \\
+ \left[ X_{p-k}^{i} X_{-p+q}^{i'} \right] \left[ N_0 \delta_p + i(1 - \delta_p) \tilde{G}_{p}(t' - t'') \right] \langle f| c_{k'}^{+}(t') c_{k}^{-}(t'') |i\rangle_{\text{phot}} \right. \right. \\
\left. \left. + \left[ X_{p-k}^{i} X_{-p+q}^{i'} \right] \left[ N_0 \delta_{p-q} + i(1 - \delta_{p-q}) \tilde{G}_{-p+q}(t' - t'') \right] \langle f| c_{k'}^{+}(t') c_{k}^{-}(t'') |i\rangle_{\text{phot}} \right\}. \]
\]

where \( \tilde{G}_p(t' - t'') = -i \langle TQ_{-p}(t')Q_p(t'') \rangle \) is the anomalous Green function of the Bose-condensed excitons at \( T = 0 \) (see, for example, Ref.2).

The sum of diagrams shown in Fig.1 corresponds to the obtained element of \( S \)-matrix. The anomalous Green function of the Bose-condensed excitons is denoted by the thick line with oncoming arrows in this figure. If the momenta of this line are zero, this line denotes factor \( N_0 \) of macroscopic occupation of the exciton state with \( \mathbf{p} = 0 \). The wavy lines are responsible for the photon creation operators. The vertices on these diagrams correspond to the matrix elements \( X_{pk}^i \); \( X_{-pk}^{i'} \), where \( p \) and \( k \) are the momenta of the exciton and the photon lines coming out of the vertex.

The element of \( S \)-matrix corresponding to the two-photon emission accompanied by the coherent two-exciton recombination and the transmission of the excess momentum to one of the impurities as a whole can be obtained analogously:

\[
(S^{ij}_p)_{fi} = \frac{(-i)^2}{2!} \int \int_{-\infty}^{\infty} dt' dt'' e^{-i\Omega(t'+t'')} \times \\
\sum_p \left\{ X_{-pk}^i X_{pk'}^{i'} \left[ N_0 \delta_p + i(1 - \delta_p) \tilde{G}_p(t' - t'') \right] \langle f| c_{k'}^{+}(t') c_{k}^{-}(t'') |i\rangle_{\text{phot}} + \\
+ X_{-pk}^i X_{pk'}^{i'} \left[ N_0 \delta_{p} + i(1 - \delta_{p}) \tilde{G}_p(t' - t'') \right] \langle f| c_{k'}^{+}(t') c_{k}^{-}(t'') |i\rangle_{\text{phot}} \right\}. \]

The sum of diagrams with all possible momenta \( \mathbf{p} \) shown in Fig.2 corresponds to this element of \( S \)-matrix.

Integrating on \( t' - t'' \) and \( t'' \) in Eqs. (B2), (B3) yields the formulae (8), (20) for the matrix elements \( (\hat{H}_X^{ij})_{fi} \) and \( (\hat{H}_X^{ij})_{fi} \).
APPENDIX C: ONE-PHOTON EMISSION BY BOSE-CONDENSED EXCITONS AT T = 0

There are no quasiparticles in the Bose-condensed exciton system at zero temperature. Therefore, the recombination of an exciton with momentum \( \mathbf{p} \neq 0 \) is inevitably accompanied by the creation of a quasiparticle with momentum \(-\mathbf{p}\) in the exciton system at \( T = 0 \). The difference between the energies of the exciton system before and after the recombination is \( \Omega - \epsilon_p \) in this case, where \( \epsilon_p \) is the energy of the created quasiparticle. Hence, the one-photon emission by the Bose-condensed excitons with the excess momentum transferred to the impurities takes place on frequencies \( \omega < \Omega \) at \( T = 0 \). Analogously, the frequency of the one-photon emission with the excess momentum transferred to an optical phonon takes place at \( \omega < \Omega_{\omega} \).

At \( T = 0 \) the matrix element for the phonon-assisted exciton recombination with the emission of photon \( \omega \) is

\[
(\hat{L})_{fi} = i\sqrt{2\pi \omega_k (\epsilon^* f_{pL,k})} v_{pL} \sqrt{n_k} + 1;
\]

\[
v_{pL}^2 = \frac{\sqrt{\alpha_L^2 + 1 - \alpha_L}}{2\alpha_L},
\]

where \( v_{pL} \) is the coefficient of the Bogoliubov transformation corresponding to the energy \( \epsilon(p_L) = \Omega_{\omega} - \omega \), \( n_k \) is the photon occupation number in the initial state, \( \alpha_L = (\Omega_{\omega} - \omega)/\mu \). Here \( |i\rangle = |0\rangle_{exc} |0\rangle_{phon} |n_k\rangle_{phot} \) and \( |f\rangle = |1_{-p_L}\rangle_{exc} |1_{-p_L-k}\rangle_{phon} |n_k + 1\rangle_{phot} \), where \( |1_{-p_L}\rangle_{exc} \) is the state of the exciton system containing the quasiparticle with momentum \(-\mathbf{p}_L\), \( |0\rangle_{exc} \) is the vacuum state with respect to quasiparticles in the exciton system.

Using the Fermi’s golden rule yields the intensity of the one-photon emission by the Bose-condensed excitons with the excess momentum transferred to an optical phonon:

\[
I_{si}^L(\omega) = I_{s}^L(\omega) + I_{i}^L(\omega),
\]

\[
I_{s}^L(\omega) = \frac{\omega^5 2^{1/2} m^{3/2} \mu^{1/2}}{3\pi^2 \epsilon^3} \left( \frac{\sqrt{\alpha_L^2 + 1} - 1}{\sqrt{\alpha_L^2 + 1 + \alpha_L}} \right)^{1/2} f^2(\omega_L) \theta(\Omega_{\omega} - \omega) \theta(\Omega_{\omega} - \omega) I(\omega),
\]

\[
I_{i}^L(\omega) = \frac{\omega^5 2^{1/2} m^{3/2} \mu^{1/2}}{3\pi^2 \epsilon^3} \left( \frac{\sqrt{\alpha_L^2 + 1} - 1}{\sqrt{\alpha_L^2 + 1 + \alpha_L}} \right)^{1/2} f^2(\omega_L) \theta(\Omega_{\omega} - \omega) I(\omega),
\]

where \( \alpha_L = (\Omega_{\omega} - \omega)/\mu \); \( \theta(x) \) is the Heaviside unit-step function. Here \( I_{s}^L, I_{i}^L \) are the intensities of the spontaneous and the induced emission, \( I \) is the intensity of the incident light.

Analogous formulae for the intensity of the one-photon emission with the excess momentum elastically transferred to the impurity can be obtained from Eqs. (C2) by replacing \( f(\omega_L) \rightarrow N d(\omega_X) \); \( \Omega_{\omega} \rightarrow \Omega; \alpha_L \rightarrow \alpha_X \), where \( \alpha_X = (\Omega - \omega)/\mu \).

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Part of the results of the present paper concerning the induced two-photon emission and Raman scattering with the excess momentum transferred to impurities is presented in short communication.

We suppose that the exciton scattering on impurities does not essentially change the Green functions of excitons. It is possible if the concentration of impurities is sufficiently low.

Γ\(p\) is inversely proportional to the relaxation time of a quasiparticle with momentum \(p\) in the exciton system. Γ\(p\) can be essentially greater than Γ\(0\) for the condensate exciton.

The analogous process with the absorption of phonons is impossible at zero temperature, because phonons are absent at \(T = 0\).

In general, an arbitrary number of phonons can be involved in the induced two-photon emission accompanied by the coherent two-exciton recombination. Hence, the induced two-photon emission can lead to the appearance of spectral lines on frequencies \(2Ω - n\omega_0 - ω\), where \(n\) is an arbitrary integer number.

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Caption for Figures

**Fig.1.** Diagrams of the two-photon emission accompanied by the coherent two-exciton recombination with the excess momentum transferred to *two different* impurities.

**Fig.2.** Diagrams of the induced two-photon emission accompanied by the coherent two-exciton recombination with the excess momentum transferred to *one* of the impurities *as a whole*. 
Fig. 1

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Fig. 2.