A complete Lie symmetry classification of a class of 
(1+2)-dimensional reaction-diffusion-convection equations

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Abstract

A class of nonlinear reaction-diffusion-convection equations describing various processes in physics, biology, chemistry etc. is under study in the case of time and two space variables. The group of equivalence transformations is constructed, which is applied for deriving a Lie symmetry classification for the class of such equations by the well-known algorithm. It is proved that the algorithm leads to 32 reaction-diffusion-convection equations admitting nontrivial Lie symmetries. Furthermore a set of form-preserving transformations for this class is constructed in order to reduce this number of the equations and obtain a complete Lie symmetry classification. As a result, the so called canonical list of all inequivalent equations admitting nontrivial Lie symmetry (up to any point transformations) and their Lie symmetries are derived. The list consists of 22 equations and it is shown that any other reaction-diffusion-convection equation admitting a nontrivial Lie symmetry is reducible to one of these 22 equations. As a nontrivial example, the symmetries derived are applied for the reduction and finding exact solutions in the case of the porous-Fisher type equation with the Burgers term.

1 Introduction

Nowadays, it is generally accepted that a huge number of real processes arising in physics, biology, chemistry, material sciences, engineering, ecology, economics etc. can be adequately described only by nonlinear PDEs (or systems of such equations). The most widely used type of equations for modeling such processes are nonlinear reaction-diffusion-convection (advection) equations. In the 1970s several monographs were published, which are devoted to study and application of the nonlinear RDC equations in physics \cite{1,2,3}, biology \cite{4,5} and chemistry \cite{6,7}. In our opinion, these books had a great impact attracting many scholars to use reaction-diffusion-convection (RDC) for modeling real world processes and to study their properties. During the last two decades many new monographs appeared, especially for models related to the life sciences (see, \cite{8,9,10,11,12,13,14,15}).

The most general class of RDC equations occurring in applications reads as

\begin{equation}
\frac{u_t}{\mathbf{u} = \nabla \cdot (D(u) \nabla u) + K(u) \cdot \nabla u + R(u)}.
\end{equation}

where $u$ is the function of $t$ and $x_1, \ldots, x_n$, $\nabla = \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right)$, and $\cdot$ means the scalar product. Here he functions $D, K$ and $R$ are related to the three most common types of transport mechanisms occurring in real world processes. The diffusivity $D(u) > 0$ is the main characteristic of the diffusion (heat conductivity) process, the vector $K(u)$ typically means velocity, which can be positive and/or negative and describes the convective transport (in contrast to diffusion, one is not random) and the reaction term $R(u)$ describes the kinetics process (for example, this function

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presents interaction of the population \( u \) with the environment and its birth-death rate). Notably, any RDC equation with the constant vector \( K(u) = (c_1, \ldots, c_n) \) is reducible to the equation with the same structure but with \( K(u) = 0 \) via the known Galilei transformation

\[
x'_1 = x_1 + c_1 t, \ldots, x'_n = x_n + c_n t.
\]

The most common equations among the class of RDC equations (1) arising in applications are diffusion (heat) equations. Their typical form is

\[
u_t = \nabla \cdot (D(u)\nabla u).
\]

Another important case occurs if one takes into account the diffusion and the kinetics process, hence the so called reaction-diffusion (RD) equation

\[
u_t = \nabla \cdot (D(u)\nabla u) + R(u)
\]

is obtained.

Lie symmetries of RDC equations of the various forms were under study starting from the classical works S.Lie [16, 17], who calculated the maximal algebra of invariance of the linear heat equation in one-dimensional space, i.e. \( u_t = u_{xx} \). Much later, in the second half of 20th century, several papers were devoted to Lie symmetry classification (another common terminology is group classification) of different subclasses of (1) in the case \( n = 1 \), i.e. one space variable. The most general result was derived in [18, 19] (see also extended version in Chapter 2 of [20]). So, the problem of Lie symmetry classification (LSC) of the class of RDC equations (1) in the case \( n = 1 \) has been completely solved.

The LSC problem of (1) is still open in the multidimensional case, i.e. \( n > 1 \). Lie symmetries in the simplest case, when (1) is the multidimensional linear heat equation \( u_t = \Delta u \), were derived in papers [21] (case \( n = 2 \)) and [22] (case \( n = 3 \)). Obviously these results can be straightforwardly generalized on the arbitrary number of the space variables.

The first LSC of the class of nonlinear heat equations (2) with two space variables was derived in paper [23] (incidentally not cited so often as [24] published 13 years later). In fact, all generic extensions of Lie symmetry depending on the form of the diffusivity \( D(u) \) in equation (2) were identified in [23]. In particular, the author has shown that the diffusivity \( D(u) = u^{-1} \) leads to infinite-dimensional Lie algebra of invariance.

A complete LSC of the class of RD equations (3) was derived [24]. In particular, it was shown that the result is essentially different in cases \( n = 2 \) and \( n = 3 \).

Paper [25] is devoted to LSC of the class of reaction-convecton equations

\[
u_t = \nabla \cdot (D(u)\nabla u) + K(u) \cdot \nabla u
\]

with the vector \( K(u) = (K^1, 0) \) and \( n = 2 \). All possible Lie symmetries are listed in Table 1 [25]. It turns out that there are only five nontrivial extensions of the principal (basic) Lie algebra of invariance

\[
A^{pr} = \langle X_0 = \frac{\partial}{\partial t}, \quad X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial x_2} \rangle
\]

We remind the reader that the Lie algebra \( \mathfrak{g} \) is called principal because one is the largest common algebra of invariance for all equations of the form (4). It should be also mentioned that the results derived in [25] were generalized in paper [26] on the class of equations (1) with \( n = 3 \), moreover the so called anisotropic case was also under study.
To the best of our knowledge, the classical problem, which consists in deriving a complete LSC of the class of RDC equations (1) with \( n > 1 \), is not solved at the present time. This problem cannot be solved as a consequence or generalization of the results obtained in the papers cited above. Moreover, it is well-known that multidimensional (in space) case is not reducible to 1D case in Lie symmetry analysis. For example, the so called conformal exponent \( k = -4/(n+2) \) in the diffusivity \( D(u) = u^k \) leads to absolutely different Lie symmetries of equation (2) for \( n = 1 \) and \( n = 2 \), hence the results obtained in [18, 19] cannot be generalized on the 2D case. Finally, it follows from the papers [23] and [24] that the case \( n = 2 \) is a special one comparing with \( n > 2 \). Thus, we examine here equations of the form (1) in the case of two space variables.

In Section 2, the determining equations (DEs) for finding Lie symmetries are constructed and the group of equivalence transformations (ETs) is identified. It is proved that continuous ETs of the class of RDC equations (1) \((n = 2)\) form the 10-parameter Lie group. In Section 3, necessary conditions for existing a nontrivial Lie symmetry of a given RDC equation of the form (1) are derived by using DEs and the group of ETs. In Section 4, sufficient conditions are obtained and, as a result, the LSC problem is solved up to the equivalence transformations. In Section 5, form-preserving transformations (FPTs) are constructed. Applying these transformations to the RDC equations derived in Section 4, we have shown that there are exactly 9 correctly-specified FPTs (they do not belong to ETs!), which allow us to map a RDC equation with nontrivial symmetry to another equation with the same symmetry. Finally, the main result of the paper is derived, namely: the canonical list of all inequivalent RDC equations admitting nontrivial Lie symmetry (up to any point transformations) is obtained. In Section 6, an example devoted to the construction of exact solutions for the porous-Fisher type equation with the Burgers term is presented. Finally, some discussion and conclusions are given in Section 7.

## 2 Determining equations for finding Lie symmetries and the group of equivalence transformations

In general, the this section and Section 3, 4 and 5 are devoted to realization of the following algorithm for solving LSC problem for the class of RDC equations

\[
 u_t = (D(u)u_x)_x + (D(u)u_y)_y + K^1(u)u_x + K^2(u)u_y + R(u). 
\]  

(6)

The algorithm consists of the following steps

1. Application of the classical Lie method for deriving the system of DEs.
2. Finding the principal algebra \( A^{pr} \).
3. Construction of the group of equivalence transformations (ETs) for the class of RDC equations in question.
4. Deriving necessary conditions for possible extensions of \( A^{pr} \) i.e. existence of nontrivial Lie symmetry.
5. Finding sufficient conditions for extensions of \( A^{pr} \) and deriving LSC for the class of equations (1) using the group of ETs.
6. Construction of form-preserving transformations (FPTs).
7. Deriving a complete LSC using FPTs.

Of course, this scheme can be modified in some cases, however, one may claim that it is a typical way to derive the so called canonical list of all inequivalent equations from the class in question admitting nontrivial Lie symmetry (see more details in Chapter 2 of [20]).

In this section, we implement the first, second and third steps, hence the following statement can be formulated.

**Theorem 1** The principal algebra \( A^{pr} \) of the class of RDC equations (7) is the three-dimensional Abelian algebra with the basic operators

\[
A^{pr} = < \frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y} >.
\]

**Proof.** To prove this statement we should construct the system of DEs for the class of equations in question. According to the classical Lie scheme (see e.g., [20, 27, 28]), we write the most general structure of Lie symmetry operator (another terminology is infinitesimal operator) in the form

\[
X = \xi^0(t, x, y, u)\frac{\partial}{\partial t} + \xi^1(t, x, y, u)\frac{\partial}{\partial x} + \xi^2(t, x, y, u)\frac{\partial}{\partial y} + \eta(t, x, y, u)\frac{\partial}{\partial u},
\]

where \( \xi^0, \xi^1, \xi^2, \eta \) are to-be-determined functions. On the other hand, the class of equations (6) is considered as the manifold

\[
S \equiv \left\{ (D(u)u_x)_x + (D(u)u_y)_y + K^1(u)u_x + K^2(u)u_y + R(u) - u_t = 0 \right\},
\]

in the prolonged space of the variables

\[
t, x, y, u, u_1, u_2
\]

where \( u \equiv (u_t, u_x, u_y) \) and \( u \equiv (u_{tt}, u_{tx}, u_{ty}, u_{xx}, u_{xy}, u_{yy}) \).

An equation of the form (3) is invariant under the transformations generated by the infinitesimal operator (8) when the following invariant criteria is satisfied:

\[
X_2 \left( (D(u)u_x)_x + (D(u)u_y)_y + K^1(u)u_x + K^2(u)u_y + R(u) - u_t = 0 \right) \bigg|_S = 0,
\]

where \( X_2 \) is the second prolongation of \( X \), which is calculated by the well-known formulae (see, e.g., [20, 27, 28]). Making straightforward calculations (nowadays it can be done using computer algebra packages, e.g., Maple), one arrives at a system of DEs for finding the coefficients of the infinitesimal operator (3). The system of DEs consists of the following differential equations:

\[
\xi^0_x = \xi^0_y = \xi^1_u = \xi^2_u = \eta_{uu} = 0,
\]

\[
\xi^1_x = \xi^2_y, \quad \xi^2_x + \xi^1_y = 0,
\]

\[
\eta{\dot{D}} = (2\xi^1_x - \xi^0_t)D,
\]

\[
\eta{\dot{K}}^1 = (\xi^1_x - \xi^0_t)K^1 - \xi^2_x K^2 - 2\eta_{ux}D - 2\eta_xD - \xi^1_t,
\]

\[
\eta{\dot{K}}^2 = \xi^2_x K^1 + (\xi^1_x - \xi^0_t)K^2 - 2\eta_{yu}D - 2\eta_yD - \xi^2_t,
\]
\[ \eta \dot{R} = (a - \xi^0_t) R - \triangle \eta D - \eta_x K^1 - \eta_y K^2 + \eta, \]  

(16)

where the Laplacian \( \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \) and the upper dot means differentiation w.r.t. \( u \). The linear equations (11)-(12) can be easily integrated. However, the general solution of (13)-(16) depends essentially on the form of the functions \( D(u), K^1(u), K^2(u) \) and \( R(u) \). In the case of finding the principal algebra, the problem simplifies because the functions \( \xi^0, \xi^1, \xi^2 \) and \( \eta \) should satisfy the system of DEs for arbitrary smooth functions \( D(u), K^1(u), K^2(u) \) and \( R(u) \). So, we immediately obtain \( \eta = 0 \) from Eq. (13), while the system of linear equations

\[ \xi^0 = \xi^1 = \xi^2 = \xi_0 = \xi_1 = \xi_2 = 0 \]  

(17)

is derived for the functions \( \xi^0, \xi^1 \) and \( \xi^2 \). As a result, the general solution of the system of DEs takes the form

\[ \xi^0 = c_0, \quad \xi^1 = c_1, \quad \xi^2 = c_2, \quad \eta = 0, \]  

(18)

where \( c_0, c_1 \) and \( c_2 \) are arbitrary constants. The Lie symmetry operator (8) with the coefficients (18) produces the the three-dimensional Abelian algebra (7).

The proof is now completed. \( \square \)

Now we turn to the group of ETs. In order to find this group for the class of RDC equations (6), the standard technique, which was formalized in [29] (see also the relevant chapters in [20] and [30]), can be applied.

**Theorem 2** The group of the continuous ETs of the class of RDC equations (6) is the 10-parameter Lie group

\[
\begin{align*}
t' & = e^{\theta_0} t + m_0, \\
x' & = e^{\theta_1} (x \cos \theta_2 - y \sin \theta_2) + g_1 t + m_1, \\
y' & = e^{\theta_1} (y \cos \theta_2 + x \sin \theta_2) + g_2 t + m_2, \\
u' & = e^{\theta_1} u + m, \\
D' & = e^{2\theta_1 - \theta_0} D, \\
K'^1 & = e^{-\theta_0} [e^{\theta_1} (K^1 \cos \theta_2 - K^2 \sin \theta_2) - q_1], \\
K'^2 & = e^{-\theta_0} [e^{\theta_1} (K^2 \cos \theta_2 + K^1 \sin \theta_2) - q_2], \\
R' & = e^{\theta_1} R,
\end{align*}
\]  

(19)

where \( g_1, g_2, \theta_0, \theta_1, \theta_2, \theta, m_0, m_1, m_2, \) and \( m \) are arbitrary group parameters.

**Proof** In order to find operator \( E \), we use the standard procedure (see, e.g., section 2.3.1 [20]), which is based on a modification of the classical Lie method. In the case of class (6), we should start from the infinitesimal operator

\[ E = \xi^0 \partial_t + \xi^1 \partial_x + \xi^2 \partial_y + \eta \partial_u + \zeta^0 \partial_D + \zeta^1 \partial_{K^1} + \zeta^2 \partial_{K^2} + \zeta^3 \partial_R, \]  

(20)

where the coefficients

\[ \xi^0 = \xi^1(t, x, y, u), \quad \xi^1 = \xi^2(t, x, y, u), \quad \eta = \eta(t, x, y, u), \]

\[ \zeta^0 = \xi^0(t, x, y, u, D, K^1, K^2, R), \quad \zeta^1 = \xi^1(t, x, y, u, D, K^1, K^2, R), \]

\[ \zeta^2 = \xi^2(t, x, y, u, D, K^1, K^2, R), \quad \zeta^3 = \xi^3(t, x, y, u, D, K^1, K^2, R) \]  

(21)

are to-be-determined functions. Note that the coefficients \( \zeta^j, j = 0, ..., 3 \) may depend on \( D, K^1, K^2 \) and/or \( R \) (in contrast to other coefficients of \( E \)).
Lie’s invariance criterion should be applied to system of equations consisting of (6) and a set of differential consequences of the functions \( D, K^1, K^2 \) and \( R \) with respect to the variables \( t, x, y, u_t, u_x \) and \( u_y \). Making relevant calculations, we arrive at the following system

\[
\begin{align*}
\xi^0_u &= \xi^0_t = \xi^2_x = \xi^0_y = \xi^0_t = \eta u_u = 0, \\
\zeta^0_t &= \zeta^0_y = \zeta^1_t = \zeta^1_x = \zeta^2_x = \zeta^2 = \zeta^3_t = \zeta^3_x = \zeta^3_y = \zeta^3 = \\
&= \xi^1_u = \zeta^1_y = \xi^1 = \xi^1_x = \zeta^1 = \zeta^1_t = \zeta^1_x = \zeta^2_u = \zeta^2_t = \zeta^2_u = \zeta^2_x = \zeta^2_y = \zeta^2 = \zeta^2 = \zeta^2 = \zeta^2 = \zeta^2 = \zeta^2
\end{align*}
\]

\[
\begin{align*}
\xi^1 &= \xi^2_y, \\
\xi^2_x &= \xi^1_y = 0, \\
\zeta^0 &= (\xi^0_t - 2\xi^1_t)D, \\
\zeta^1 &= \xi^1_t D - \xi^1_x K^1 + \xi^1_y K^2, \\
\zeta^2 &= \xi^2_t D + \xi^2_x K^1 + (\xi^2_y - 2\xi^1_t)K^2, \\
\zeta^3 &= (\eta u - 2\xi^1)R
\end{align*}
\]

to find the coefficients \( \xi^0, \xi^a, \eta, \zeta^0, \zeta^1, \zeta^2, \zeta^3 \) of operator \( (20) \).

Because the system of PDEs \( \text{(22)\text{--28)} \) is linear, its general solutions can be easily constructed, hence we obtain

\[
\begin{align*}
\xi^0 &= \kappa_0 t + d_0, \\
\xi^1 &= \kappa_1 x + g_1 t - c y + d_1, \\
\xi^2 &= \kappa_1 y + g_2 t + c x + d_2, \\
\eta &= \kappa u + d, \\
\zeta^0 &= (\kappa_0 - 2\kappa_1)D, \\
\zeta^1 &= g_1 D + (c - \kappa_1)K^1 - cK^2, \\
\zeta^2 &= g_2 D + cK^1 + (c - \kappa_1)K^2, \\
\zeta^3 &= (\kappa - 2\kappa_1)R
\end{align*}
\]

where \( c, \kappa, \kappa_0, \kappa_1, g_1, g_2, d_0, d_1, d_2 \) and \( d \) are arbitrary parameters.

Thus, the infinitesimal operator \( (20) \) with the coefficients from \( (29) \) generates the 10-dimensional Lie algebra. Finally, making standard calculations one can easily show that this algebra produces the 10-parameter Lie group of equivalence transformations \( \text{(13)} \).

The proof is now completed. \( \square \)

### 3 Necessary conditions for existence of a nontrivial Lie symmetry

Here we are searching for necessary conditions, which are needed for extension of the principal Lie algebra \( \text{(7)} \). In other words, we need to establish all possible forms of the functions \( D, K^1, K^2, \) and \( R \) leading to extension of Lie symmetry of the relevant RDC equations from class \( \text{(6)} \). The main result of this section can be formulated as follows.

**Theorem 3** If an arbitrary equation belonging to the class of RDC equations \( \text{(6)} \) admits a maximal algebra of invariance (MAI) of higher dimensionality than algebra \( \text{(7)} \), then the functions \( D, K^1, K^2, \) and \( R \) must possess the structures listed in Table \( \text{(1)} \) where \( \lambda_3, \lambda_4, \lambda_5, k, m, p \) and \( s \) are arbitrary constants. Any other RDC equation possessing a nontrivial Lie symmetry is reducible to one of those from Table \( \text{(1)} \) by an appropriate ET of the form \( \text{(15)} \).
Table 1: Necessary conditions for nontrivial Lie symmetries

| $D$ | $K^1$ | $K^2$ | $R$ | Restrictions |
|-----|-------|-------|-----|-------------|
| 1 $e^u$ | $e^{mu} \cos(pu)$ | $e^{mu} \sin(pu)$ | $\lambda_3 e^{(2m-s)u}$ | $(m, p) \neq (0, 0), m \neq s$ |
| 2 $u$ | $0$ | $0$ | $\lambda_3 e^{-u}$ | $m \neq k, (m, p) \neq (0, 0)$ |
| 3 $e^u$ | $u$ | $0$ | $\lambda_3 e^{-u}$ | $k \neq 0$ |
| 4 $u^k$ | $u^m \cos(pu)$ | $u^m \sin(pu)$ | $\lambda_3 u^{2m-k+1}$ | $k \neq 0$ |
| 5 $u^k$ | $\ln u$ | $0$ | $\lambda_3 u^{2m-k+1}$ | $k \neq 0$ |
| 6 $u^k$ | $\ln u$ | $0$ | $\lambda_3 u^{2m-k+1}$ | $k \neq 0$ |

Proof To prove this theorem, one needs to analyze the system of DEs (11)-(12). Obviously, Eqs. (11)-(12) are rather simple and can be easily integrated. In particular, Eqs. (11) allow us to obtain

$$\xi^0 = \xi^0(t), \quad \xi^i = \xi^i(t, x, y), \quad \eta = a(t, x, y) u + b(t, x, y),$$

(30)

where $a(t, x, y)$ and $b(t, x, y)$ are arbitrary functions, $i = 1, 2$.

The system of equations (13)-(16) from the formal point of view it is more complicated object than the general RDC equation (3). However, unknown functions in Eqs. (13)-(16) depend on different variables (e.g., $\xi^0$ depends on $t$ while $D$ depends on $u$ only) and it is a common peculiarity of such type systems, which allows us to work out an algorithm for their solving. Unfortunately, this algorithm usually is quite cumbersome and consists of examination of several inequivalent cases. Happily we can partly use the algorithm presented in Chapter 2 [20] for the solving similar system obtained for (1+1)-dimensional general RDC equation. In particular, to simplify the relevant calculations we introduce the so called structural constants as follows

$$a = k_1 \varphi, \quad b = k_2 \varphi, \quad 2\xi^1 - \xi^0 = k \varphi, \quad \xi^1 - \xi^0 = m \varphi, \quad \xi^2 = p \varphi, \quad a_x = \alpha_1 \varphi, \quad a_y = \alpha_2 \varphi, \quad b_x = \beta_1 \varphi, \quad b_y = \beta_2 \varphi, \quad \xi^1_1 = \gamma_1 \varphi, \quad \xi^2_1 = \gamma_2 \varphi, \quad a - \xi^0 = (2m + k_1 - k) \varphi, \quad \triangle a = h_1 \varphi,$$

(31)

where $k, m, p, k_1, n_1, h_1, \alpha_i, \beta_i$ and $\gamma_i (i = 1, 2)$ are some structural constants relating all unknown functions with the function $\varphi(t, x, y)$, which is arbitrary at the moment. Using notations (31), Eqs. (13)-(16) can be rewritten in the form

$$(k_1 u + k_2) \dot{D} = k D,$$

$$(k_1 u + k_2) K^1 = m K^1 - p K^2 - 2\alpha_1 D - 2(\alpha_1 u + \beta_1) \dot{D} - \gamma_1,$$

$$(k_1 u + k_2) K^2 = p K^1 + m K^2 - 2\alpha_2 D - 2(\alpha_2 u + \beta_2) \dot{D} - \gamma_2,$$

(32)

System (32) possesses a simpler structure comparing with Eqs. (13)-(16) because one does not involve the functions on the variables $t, x$ and $y$. On the other hand, this system produces all possible forms the functions $D, K^1, K^2$ and $R$ leading to extensions of the principal Lie algebra (7). The rest of the proof is devoted to solving system (32), which consists of four linear ODEs.
First of all, it can be noted that system (32) can be slightly simplified using the correctly-specified ETs of the form

\[
\begin{align*}
t' &= t, \quad x' = x + \theta_1 t, \quad y' = y + \theta_2 t, \quad u' = u, \quad D' = D, \quad K' = K^1 - \theta_1, \\
K' &= K^2 - \theta_2, \quad R' = R,
\end{align*}
\]

which are a particular case of (19). In fact, the parameters \(\theta_1\) and \(\theta_2\) can be chosen in a such way that system (32) can be transformed to the same form with \(\gamma_1 = \gamma_2 = 0\) (hereafter primes are skipped)

\[
\begin{align*}
(k_1u + k_2) \dot{D} &= kD, \\
(k_1u + k_2) \dot{K}^1 &= mK^1 - pK^2 - 2\alpha_1 D - 2(\alpha_1 u + \beta_1) \dot{D}, \\
(k_1u + k_2) \dot{K}^2 &= pK^1 + mK^2 - 2\alpha_2 D - 2(\alpha_2 u + \beta_2) \dot{D}, \\
(k_1u + k_2) \dot{R} &= (2m - k + k_1) \dot{R} - (h_1 u + n_1) D - (\alpha_1 u + \beta_1) R^1 - \\
&- (\alpha_2 u + \beta_2) K^2 + h_2 u + n_2,
\end{align*}
\]

provided

\[m^2 + p^2 \neq 0,\]

The possibility \(m = p = 0\) will be treated below when one comes up.

It can be noted that the first equation in (34) has the same structure as one in the system of DEs for the class of RDC equations (1) with \(n = 1\) (see Eq.(2.80) in [20]). Thus, the following five different case should be examined (see P.41 in [20]):

1) \(k_1 = 0, \quad k_2 = 0, \quad k = 0\);
2) \(k_1 = 0, \quad k_2 \neq 0, \quad k \neq 0\);
3) \(k_1 \neq 0, \quad k_2 = 0, \quad k \neq 0\);
4) \(k_1 = 0, \quad k_2 \neq 0, \quad k = 0\);
5) \(k_1 \neq 0, \quad k_2 = 0, \quad k = 0\).

Since Case 1) is rather trivial (the first equation in (34) simply vanishes) and leads to the first case of Table 1. Here the relevant analysis is omitted.

Consider Case 2), i.e. \(k_1 = 0, \quad k_2 \neq 0, \quad k \neq 0\) (without loosing a generality we can set \(k_2 = k = 1\)). The first two equations of system (34) immediately give \(a = 0, \quad b = \varphi\), hence

\[
\alpha_1 = \alpha_2 = h_1 = h_2 = 0
\]

and

\[
b = 2\xi_1^1 - \xi_1^0, \quad mb = \xi_1^1 - \xi_1^0, \quad pb = \xi_1^2, \quad \gamma_1 b = \xi_1^1, \quad \gamma_2 b = \xi_1^2.
\]

It follows from Eqs. (37) that

\[
(2m - 1) \xi_1^1 = (m - 1) \xi_1^0.
\]

Now we take differential consequences of (38) w.r.t.\(x\) and \(y\) and use Eq. (12). As a result, the equations

\[
m(2m - 1)b_x = 0, \quad m(2m - 1)b_y = 0.
\]

are derived. Finally, differentiating the last equation in (38) w.r.t.\(x\) and \(y\), and using Eqs. (39), the compatibility constrain

\[
m(m - 1)(2m - 1)b_t = 0
\]

is obtained.
The compatibility constraint (39) leads to four different subcases, which can be considered step by step. Namely, the following different subcases should be examined: (2i) \( m \neq 0 \), (2ii) \( m = \frac{1}{2} \), (2iii) \( m = k \), and (2iv) \( m = 1 \).

We start from the most general one (2i). Obviously Eqs. (39) and (40) immediately give \( b = \text{const} \), hence

\[
\beta_1 = \beta_2 = n_1 = n_2 = 0. \tag{41}
\]

Thus, system (32) takes the form

\[
\dot{D} = D, \quad \dot{K}_1 = mK_1 - pK^2, \quad \dot{K}_2 = pK_1 + mK^2, \quad \dot{R} = (2m - 1)R. \tag{42}
\]

The general solution of the latter is

\[
D = \lambda_0 e^u, \quad K_1 = \lambda_1 e^{mu} \cos(pu + \lambda_2), \quad K_2 = \lambda_1 e^{mu} \sin(pu + \lambda_2), \quad R = \lambda_3 e^{(2m-1)u}. \tag{43}
\]

Hereafter \( \lambda_0 \neq 0 \), \( \lambda_1 \neq 0 \), \( \lambda_2 \), \( \lambda_3 \) are arbitrary constants. However, three of these constants can be reduced to \( \lambda_0 = \lambda_1 = 1 \), \( \lambda_2 = 0 \) using the equivalence transformation

\[
t \to \theta_0 t, \quad x \to \theta_1 x, \quad y \to \theta_1 y, \quad u \to u + \theta_2. \tag{44}
\]

where

\[
\theta_0 = \frac{\lambda_0 e^{(2m-1)\frac{\lambda_2}{p}}}{\lambda_1^2}, \quad \theta_0 = \frac{\lambda_0 e^{(m-1)\frac{\lambda_2}{p}}}{\lambda_1}, \quad \theta_0 = -\frac{\lambda_2}{p}.
\]

Thus, the second case with \( s = 1 \) of Table 1 is identified.

Consider subcase (2ii) \( m = 0 \). In a quite similar way as it was done in case (2i), using system (31), one may extract the restrictions

\[
\beta_1 = \beta_2 = n_1 = n_2 = 0. \tag{45}
\]

Thus, system (32) takes the form

\[
\dot{D} = D, \quad \dot{K}_1 = -pK^2 - \gamma_1, \quad \dot{K}_2 = pK_1 - \gamma_2, \quad \dot{R} = -R. \tag{46}
\]

The general solution of (46) depends on the constant \( p \). Assuming \( p \neq 0 \), we arrive at the following general solution of (46)

\[
D = \lambda_0 e^u, \quad K_1 = \lambda_1 \cos(pu + \lambda_2), \quad K_2 = \lambda_1 \sin(pu + \lambda_2), \quad R = \lambda_3 e^{-u},
\]

So, the formulae (43) are valid also for \( m = 0 \). Moreover, using ET (44) the coefficients in the above formulae are reducible to \( \lambda_0 = \lambda_1 = 1 \), \( \lambda_2 = 0 \).

Assuming \( p = 0 \), we note that restriction (45) is broken, hence the constants \( \gamma_1 \) and \( \gamma_2 \) can be non-zero in (46). So, the general solution of (46) is

\[
D = \lambda_0 e^u, \quad K_1 = \lambda_1 u, \quad K_2 = \lambda_2 u, \quad R = \lambda_3 e^{-u}.
\]

Now we again use the following ET

\[
t \to \frac{1}{\lambda_0} t, \quad x \to \frac{\lambda_0}{\lambda_1}(\lambda_1 x + \lambda_2 y), \quad y \to \frac{\lambda_0}{\lambda_2}(\lambda_1 y - \lambda_2 x), \quad u \to u. \tag{47}
\]

in order to simplify the coefficients as follows \( \lambda_0 = \lambda_1 = 1 \), \( \lambda_2 = 0 \).
Thus, the third case of Table 1 is identified.

Consider subcase (2iii) \( m = \frac{1}{2} \). In a quite similar way the corresponding system and its general solution

\[
D = e^u, \quad K^1 = e^{\frac{1}{2}u} \cos pu, \quad K^2 = e^{\frac{1}{2}u} \sin pu, \quad R = \lambda_3
\]

were constructed. So, formulae (43) are valid also for \( m = \frac{1}{2} \).

Finally, subcase (2iv) \( m = 1 \) was examined. It was shown that system (32) takes the form

\[
\dot{D} = D, \quad \dot{K}^1 = K^1 - pK^2, \quad \dot{K}^2 = K^2 + pK^1, \quad \dot{R} = R + q_2.
\]

The latter possesses the general solution

\[
D = \lambda_0 e^u, \quad K^1 = \lambda_1 e^u \cos (pu + \lambda_2), \quad K^2 = \lambda_4 e^u \sin (pu + \lambda_2), \quad R = \lambda_3 e^u + \lambda_4,
\]

where \( \lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \) are arbitrary constants. Here three coefficients are again reducible to \( \lambda_0 = \lambda_1 = 1, \lambda_2 = 0 \) via application of ET (44).

As a result, the fourth case of Table 1 is identified.

Thus, the cases 1, 2 (with \( s \neq 0 \)), 3 and 4 of Table 1 are identified. All other cases of Table 1 were derived in a very similar way by the examination of Cases 3)–5).

The proof is now completed. \( \square \)
4 Lie symmetry classification using the equivalence transformations

As one may note, the steps 1–4 of the algorithm presented in Section 2 are already realized. Here we are going to identify sufficient conditions needed for extension of the principal Lie algebra (7) (see step 5 of the algorithm). Having this done and taking into account the group of ETs (19), we can complete LSC for the class of RDC equations (6).

**Theorem 4** All possible nontrivial MAI (i.e., Lie algebras of dimensionality four and higher) of RDC equations of the form (6) depending on the functions $D, K^1, K^1$ and $R$ are presented in Table 2. Any other equation of the form (6) with a nontrivial Lie symmetry is reduced by an ET from $E$ (17) to one of 32 equations listed in Table 2.

| Equation | MAI | Restrictions |
|----------|-----|--------------|
| 1 $u_t = (D(u)u_x)_x + (D(u)u_y)_y + R(u)$ | $< \partial_t, \partial_x, \partial_y, J_{12} >$ | $D - \forall, R - \forall$ |
| 2 $u_t = (D(u)u_x)_x + (D(u)u_y)_y$ | $< \partial_t, \partial_x, \partial_y, J_{12}, D_0 >$ | $D - \forall$ |
| 3 $u_t = \Delta u + \gamma_1$ | $< \partial_t, \partial_x, \partial_y, J_{12}, G_x, G_y, I, D_0, \Pi, Q_{12} >$ | $\gamma_1 > 0$ |
| 4 $u_t = \Delta u + \gamma_1 u$ | $< \partial_t, \partial_x, \partial_y, J_{12}, G_x, G_y, I, D_0 + 2\gamma_1 t \partial_y, \Pi + \gamma_1 t (2t + \frac{x^2 + y^2}{4}) \partial_y, Q_{12} >$ | $\gamma_1 > 0$ |
| 5 $u_t = \Delta u + \gamma_1 u \ln u$ | $< \partial_t, \partial_x, \partial_y, J_{12}, e^{\gamma_1 t} I, G_x, G_y >$ | $\gamma_1 > 0$ |
| 6 $u_t = (e^{nu} u_x)_x + (e^{nu} u_y)_y$ | $< \partial_t, \partial_x, \partial_y, J_{12}, D_0, D_2 >$ | $\delta = 1$ |
| 7 $u_t = (e^{nu} u_x)_x + (e^{nu} u_y)_y + \gamma_1 e^{nu}$ | $< \partial_t, \partial_x, \partial_y, J_{12}, (\delta - m) D_0 - 2D_4 >$ | $m \neq 0$ |
| 8 $u_t = (e^{nu} u_x)_x + (e^{nu} u_y)_y + \gamma_1 e^{nu}$ | $< \partial_t, \partial_x, \partial_y, J_{12}, (\delta - m) D_0 - 2D_4 >$ | $m \neq 0$ |
| 9 $u_t = (e^{nu} u_x)_x + (e^{nu} u_y)_y + \gamma_1$ | $< \partial_t, \partial_x, \partial_y, J_{12}, D_0, D_2, T_2 >$ | $\delta = 1$ |
| 10 $u_t = (e^{nu} u_x)_x + (e^{nu} u_y)_y + \gamma_1 e^{nu} + \gamma_2$ | $< \partial_t, \partial_x, \partial_y, J_{12}, D_0, D_1 >$ | $m \neq 1$ |
| 11 $u_t = (u^k u_x)_x + (u^k u_y)_y$ | $< \partial_t, \partial_x, \partial_y, J_{12}, D_0, D_1 >$ | $k \neq 1; 0$ |
| 12 $u_t = (u^k u_x)_x + (u^k u_y)_y + \gamma_1 u^m$ | $< \partial_t, \partial_x, \partial_y, J_{12}, (m - 1) D_0 - D_1 >$ | $m \neq 1$ |
| 13 $u_t = (u^k u_x)_x + (u^k u_y)_y + \gamma_1 u^m$ | $< \partial_t, \partial_x, \partial_y, J_{12}, D_0, D_1, T_1 >$ | $k \neq 1; 0$ |
| 14 $u_t = (u^k u_x)_x + (u^k u_y)_y + \gamma_1 u^m + \gamma_2 u$ | $< \partial_t, \partial_x, \partial_y, J_{12}, T_1 >$ | $k \neq 0$ |
| 15 $u_t = (u^{-1} u_x)_x + (u^{-1} u_y)_y$ | $< \partial_t, \partial_x, \partial_y, J_{12}, J_{12}, X >$ | $k = -1$ |
| 16 $u_t = (u^{-1} u_x)_x + (u^{-1} u_y)_y + \gamma_1 u$ | $< \partial_t, \partial_x, \partial_y, J_{12}, T_1, X >$ | $k = -1$ |
| 17 $u_t = (e^{nu} u_x)_x + (e^{nu} u_y)_y + e^{nu} u_x \cos (pu) + e^{nu} u_y \sin (pu) + \sigma e^{(2m-\delta)u}$ | $< \partial_t, \partial_x, \partial_y, J_{12}, D_0 + D_4 + p J_{12} >$ | $(m, p) \neq (0, 0)$ |
| Equation | Description | Conditions |
|----------|-------------|------------|
| $u_t = (e^{u}u_x)_x + (e^{u}u_y)_y + u u_x + \sigma e^{-u}$ | $< \partial_t, \partial_x, \partial_y$, $D_0 - D_4 - t \partial_x >$ | $\delta = 1$ |
| $u_t = (e^{u}u_x)_x + (e^{u}u_y)_y + u^2[u_x \cos(p \ln u) + u_y \sin(p \ln u)] + \sigma e^{u}$ | $< \partial_t, \partial_x, \partial_y$, $D_4 + p J_{12}$ | $\delta = 1$ |
| $u_t = (u^k u_x)_x + (u^k u_y)_y + u^k u_x + \sigma u^{-k+1}$ | $< \partial_t, \partial_x, \partial_y$, $T_2$ | $\delta = 1$ |
| $u_t = (u^k u_x)_x + (u^k u_y)_y + u^k u_x + \lambda_1 u^{-k+1}$ | $< \partial_t, \partial_x, \partial_y$, $D_3 + p J_{12}$ | $m \neq k, (m, p) \neq (0, 0)$ |
| $u_t = (u^k u_x)_x + (u^k u_y)_y + u^k u_x + \lambda_2 u^{-k+1} + \gamma_1 u$ | $< \partial_t, \partial_x, \partial_y$, $D_3$ | $k \neq 0, k^2 \lambda_3 \neq 4(k + 1)$ |
| $u_t = (u^k u_x)_x + (u^k u_y)_y + u^k u_x + 4 \frac{k+1}{k} u^{-k+1}$ | $< \partial_t, \partial_x, \partial_y$, $D_3, R_1, R_2$ | $\delta = 1$ |
| $u_t = (u^k u_x)_x + (u^k u_y)_y + 4 \frac{k+1}{k} u^{-k+1} + \gamma_1 u$ | $< \partial_t, \partial_x, \partial_y$, $T_1$ | $\delta = 1$ |
| $u_t = (u^k u_x)_x + uu_x + \gamma_1 u$ | $< \partial_t, \partial_x, \partial_y$, $\gamma_1 >$ | $\delta = 1$ |
| $u_t = \Delta u + uu_x + \gamma_1 u$ | $< \partial_t, \partial_x, \partial_y$, $\gamma_1 >$ | $\delta = 1$ |
| $u_t = \Delta u + uu_x + \gamma_1 u$ | $< \partial_t, \partial_x, \partial_y$, $\gamma_1 >$ | $\delta = 1$ |
| $u_t = \Delta u + uu_x + \gamma_1 u$ | $< \partial_t, \partial_x, \partial_y$, $\gamma_1 >$ | $\delta = 1$ |
| $u_t = \Delta u + uu_x + \gamma_1 u$ | $< \partial_t, \partial_x, \partial_y$, $\gamma_1 >$ | $\delta = 1$ |

**Remark 1** In Table 2, $k$, $m$, $p$ and $q$, are arbitrary constants, $\sigma \in \{-1, 0, 1\}$, $\delta \in \{0, 1\}$, $\gamma_1, \gamma_2 \in \{-1, 1\}$ and the following designations for Lie symmetry operators are introduced:

- $J_{12} = y \partial_x - x \partial_y$, $D_0 = 2t \partial_t + x \partial_x + y \partial_y$,
- $D_1 = k x \partial_x + ky \partial_y + 2u \partial_u$, $D_2 = x \partial_x + y \partial_y + 2u \partial_u$,
- $D_3 = k \partial_t - u \partial_u$, $D_4 = t \partial_t - u \partial_u$,
- $G_x = t \partial_x - \frac{1}{2} x I$, $G_y = t \partial_y - \frac{1}{2} y I$, $I = u \partial_u$,
- $\Pi = t^2 \partial_t + tx \partial_x + ty \partial_y - (t + \frac{k}{x} + \frac{p}{y}) u \partial_u$,
- $G_x = e^{-\gamma_1 k t} (\partial_x - \frac{1}{2} \gamma_1 u \partial_u)$, $G_y = e^{\gamma_1 k t} (\partial_y - \frac{1}{2} \gamma_1 u \partial_u)$,
- $T_1 = e^{-\gamma_1 t} (\partial_t + \gamma_1 u \partial_u)$, $T_2 = e^{-\gamma_1 t} (\partial_t + \gamma_1 u \partial_u)$,
- $G_1 = e^{\gamma_1 t} (\partial_x - \gamma_1 u \partial_u)$, $G_2 = e^{\gamma_1 t} (\partial_x - \gamma_1 u \partial_u)$,
- $G_0 = t \partial_x - u \partial_u$, $G_1 = t \partial_x - u \partial_u$,
Now we consider two possible possibilities. Also taking into account) has the form
\[ PDE \]

\[ \text{Subcases} (i) \]

Obviously, the general solution of (50)-(54) essentially depends on the parameter \( \lambda_t = \Delta \beta \pm \beta \).

\[ X_\infty = A(x,y)\partial_x + B(x,y)\partial_y - 2uA_x\partial_u, \]

where \( A(x,y) \) and \( B(x,y) \) are arbitrary functions satisfying the Cauchy-Riemann system \( A_x = B_y, \quad A_y = -B_x. \)

**Proof.** First of all, we note that Cases 1 and 2 present two subclasses of the general class of RDC equations \([3]\), when \( K^1 = K^2 = 0 \). LSC of the first subclass was firstly derived in \([24]\), while the second subclass was examined earlier in \([23]\). So, Cases 3–16 present the results derived earlier in \([23]\) and \([24]\). They can be formally identified by examination of Case 1 from Table \([1]\).

Examination of Cases 3, 6, 8 and 9 of Table \([1]\) is rather simple because \( K^2 = 0 \) and the function \( K^1 \) possesses a simple structure in each case. Notably these cases lead to the results derived in \([25]\) if one assumes additionally that the relevant lambda-s vanish. The detailed analysis involving all possible values of lambda-s lead to the results listed in Cases 18, 22, and 28–32 of Table \([2]\).

The equations arising in Cases 2, 4, 5 and 7 of Table \([1]\) are absolutely new and their examination is very nontrivial. Here we present the detailed examination of Case 7. In this case, the coefficients of Eq. \((6)\) are specified as follows

\[ D = u^k, \quad K^1 = u^k \cos(p \ln u), \quad K^2 = u^k \sin(p \ln u), \quad R = u(\lambda_3 u^k + \lambda_4), k \neq 0. \]  

Substituting the functions from \((49)\) into the subsystem of DEs \((13)-(16)\), one obtains

\[ b = 0, \quad ka = 2\xi_1^1 - \xi_0^0, \quad \xi_1^0 = 0, \quad \xi_2^0 = 0, \]  

\[ \xi_1^1 \cos(p \ln u) - (pa - \xi_2^2) \sin(p \ln u) = -2(k + 1)a_x, \]  

\[ (pa - \xi_2^2) \cos(p \ln u) + \xi_1^1 \sin(p \ln u) = -2(k + 1)a_y, \]  

\[ a_x \cos(p \ln u) + a_y \sin(p \ln u) + 2\lambda_3 \xi_1^1 = 0, \]  

\[ a_t = \lambda_4 \xi_1^0. \]  

Obviously, the general solution of \((50)-(54)\) essentially depends on the parameter \( p \), hence two subcases \((i)\) \( p \neq 0 \) and \((ii)\) \( p = 0 \) should be examined.

Assuming \( p \neq 0 \) and solving Eqs. \((50)-(54)\), we arrive at the linear system of the first-order PDE

\[ b = 0, \quad a_x = 0, \quad a_y = 0, \quad \xi_1^1 = 0, \quad \xi_1^0 = 0, \quad \xi_2^0 = pa, \quad ka + \xi_1^0 = 0, \]  

\[ \lambda_4 \xi_1^0 = 0. \]  

Now we consider two possible possibilities.

If \( \lambda_4 \neq 0 \) then Eq. \((50)\) gives \( \xi_1^0 = 0 \), hence Eqs. \((55)\) produce \( a = b = 0, \quad \xi_1^1 = \xi_1^2 = \xi_2^1 = \xi_2^2 = 0 \). As a result, we realize that this possibility leads only to the principal algebra \( A_{pr} \).

If \( \lambda_4 = 0 \) then the general solution of system \((55)-(56)\) (note that Eqs. \((11)-(12)\) should be also taking into account) has the form

\[ \xi_0^0 = k c_0 t + d_0, \quad \xi_1^1 = p c_0 y + d_a, \quad \xi_2^2 = -p c_0 x + d_a, \quad \eta = -c_0 u. \]  

(57)
The operator $X(8)$ with coefficients (67) produces the Lie algebra with the basic operators

$$\langle \partial_t, \partial_x, \partial_y, D_3 + pJ_{12} \rangle.$$  

So, Case 23 of Table 2 is identified (an arbitrary $\lambda_3$ can be reduced to the three values $\sigma$ by an ET from $E$).

Consider subcase (ii) $p = 0$. Now Eq. (3) with (19) takes the form

$$u_t = (u^k u_x)_x + (u^k u_y)_y + u^k u_x + u(\lambda_3 u^k + \lambda_4).$$  

(58)

In order to simplify further calculations, we apply the following ET of the form (19)

$$k^2 \frac{(k+1)^2}{16(k+1)^2} t \rightarrow t, \quad \frac{k}{4(k+1)} x \rightarrow x, \quad \frac{k}{4(k+1)} y \rightarrow y, \quad u \rightarrow u,$$

$$\lambda_3 \rightarrow \lambda_3, \quad \lambda_4 \rightarrow \lambda_4,$$

(59)

which transform Eq. (58) into

$$u_t = (u^k u_x)_x + (u^k u_y)_y + 4\frac{k+1}{k} u^k u_x + u(\lambda_3 u^k + \lambda_4), \quad k \neq -1; 0.$$

(60)

Thus, taking into account formulae (59), equations (50)-(54) with $p = 0$ reduce to the form

$$b = \xi_t^1 = \xi_t^2 = 0,$$

(61)

$$a = \frac{1}{2(k+1)^2} \xi_x^1 - \frac{k}{16(k+1)^2} \xi_t^0,$$

(62)

$$2(k+1)a_x = -\xi_x^1,$$

(63)

$$2(k+1)a_y = -\xi_y^1,$$

(64)

$$\lambda_3 - 4\frac{k+1}{k} \xi_x^1 = 0,$$

(65)

$$\xi_{tt}^0 + k\lambda_4 \xi_t^0 = 0.$$

(66)

The general solution of (61)-(66) essentially depends on the parameters $\lambda_3$ and $\lambda_4$. All the inequivalent subcases are

1) $\lambda_3 \neq 4\frac{k+1}{k^2}$, $\lambda_4 = 0$;

2) $\lambda_3 \neq 4\frac{k+1}{k^2}$, $\lambda_4 \neq 0$;

3) $\lambda_3 = 4\frac{k+1}{k^2}$, $\lambda_4 = 0$;

4) $\lambda_3 = 4\frac{k+1}{k^2}$, $\lambda_4 \neq 0$.

Consider subcase 1) $\lambda_3 \neq 4\frac{k+1}{k^2}$, $\lambda_4 = 0$. The general solution of system (61)-(66) and the remaining equations (11)-(12) from the system of DEs is formed by the functions

$$\xi^0 = k\eta t + d_0, \quad \xi^a = d_a, \quad \eta = -c_0 u.$$

(67)

The operator $X(8)$ with coefficients (67) produces the Lie algebra with the basic operators

$$\langle \partial_t, \partial_x, \partial_y, D_3 \rangle.$$  

So, Case 24 of Table 2 is identified.

Consider subcase 2) $\lambda_3 \neq 4\frac{k+1}{k^2}$, $\lambda_4 \neq 0$. Using ET

$$t \rightarrow \frac{\gamma_1 t}{\lambda_4}, \quad x \rightarrow x, \quad y \rightarrow y, \quad u \rightarrow \left(\frac{\lambda_4}{\gamma_1}\right)^{\frac{1}{\gamma_1}} u.$$
we can make \( \lambda_4 = \gamma_1 \). So, the general solution of system (61)-(66), (11), (12) has the form
\[
\xi^0 = c_0 e^{-\gamma_1 kt} + d_0, \quad \xi^a = d_a, \quad \eta = \gamma_1 c_0 e^{-\gamma_1 kt} u.
\] (68)

The operator \( X \) with coefficients (68) produces the Lie algebra with the basic operators
\[
\langle \partial_t, \partial_x, \partial_y, T_1 \rangle.
\]

So, Case 25 of Table 2 is identified.

Consider subcase 3) \( \lambda_3 \neq 4k^2 + 1, \lambda_4 = 0 \). The linear ODE
\[
\xi_1 \frac{\partial^2}{\partial x^2} + \xi_1 \frac{\partial}{\partial x} = 0.
\]
(69)
can be derived from Eqs. (62)-(65) in this subcase, which possesses the general solution
\[
\xi_1 = \varphi(y)e^{-x} + \psi(y).
\] (69)

Moreover, taking into account Eqs. (12) and (69) we obtain
\[
\xi_2 = \dot{\varphi}(y)e^{-x} - \dot{\psi}(y)x + \chi(y),
\] (70)
where \( \varphi = \varphi(y), \psi = \psi(y) \) and \( \chi = \chi(y) \) are arbitrary functions at the moment.

Now we substitute (69)-(70) into Eqs. (12) and arrive at the linear ODE system
\[
\varphi_{yy} + \varphi = 0, \quad \psi_{yy} = 0, \quad \chi_{yy} = 0,
\] (71)
which possesses the general solution
\[
\varphi = c_1 \cos y + c_2 \sin y, \quad \psi = c_3 y + c_4, \quad \chi = c_5.
\]

So, we arrive at
\[
\xi_1 = e^{-x}(c_1 \cos y + c_2 \sin y) + c_3 y + d_1,
\] (72)
\[
\xi_2 = e^{-x}(-c_1 \sin y + c_2 \cos y) - c_3 x + d_2,
\] (73)

The coefficient \( \xi^0 \) of the operator \( X \) can be easily derived from Eq. (66):
\[
\xi^0 = kc_0 t + d_0.
\] (74)

In the above formulae \( c_0, c_1, c_2, c_3, c_4, c_5, d_0, d_1 \) and \( d_2 \) are arbitrary constants.

In order to find the coefficient \( \xi^0 \) of the operator \( X \), we use (62), (74):
\[
a = -\frac{1}{2(k+1)} e^{-x}(c_1 \cos y + c_2 \sin y) - \frac{k}{16(k+1)^2} c_0.
\] (75)

So, taking into account (30), (61) and (75), we obtain
\[
\eta = \left[ \frac{1}{2(k+1)} e^{-x}(c_1 \cos y + c_2 \sin y) + \frac{k}{16(k+1)^2} c_0 \right] u.
\] (76)

Substituting (72) and (73) into (71), we arrive at the restriction \( c_3 = 0 \). So, the operator \( X \) with coefficients (71), (72), (73) (under the restriction \( c_3 = 0 \)) and (76) produces the Lie algebra with the basic operators
\[
\langle \partial_t, \partial_x, \partial_y, D_3, R_1, R_2 \rangle.
\]
Thus, Case 26 of Table 2 is identified.

Finally, we examine subcase 4) \( \lambda_3 \neq 4 \frac{k+1}{\lambda_4}, \lambda_4 \neq 0 \). Using ET

\[
t \mapsto \frac{t}{\lambda_4}, \quad x \mapsto x, \quad y \mapsto y, \quad u \mapsto |\lambda_4|^{\frac{1}{k}}u,
\]

we can set \( \lambda_4 = \gamma_1 \) without losing a generality. So, solving Eq. (66) we obtain

\[
\xi^0 = c_0 e^{-\gamma_1 kt} + d_0. \tag{77}
\]

Eqs. (30), (61) and (77) give

\[
\eta = -\left[ \frac{1}{2(k+1)} e^{-\gamma_1 x} (c_1 \cos y + c_2 \sin y) + \gamma_1 c_0 e^{-\gamma_1 kt} \right] u. \tag{78}
\]

The coefficients \( \xi^1 \) and \( \xi^2 \) again are given by (72)–(73) with \( c_3 = 0 \).

The operator \( X \) with coefficients (77), (72)–(73) (under the restriction \( c_3 = 0 \)) and (78) produces the Lie algebra with the basic operators

\[
\langle \partial_t, \partial_x, \partial_y, T_1, R_1, R_2 \rangle.
\]

So, Case 27 of Table is identified.

Thus, Cases 23-27 of Table 2 have been identified by examination of the RDC equation (6) with the coefficients listed in Case 7 of Table 1.

Cases 17-22 of Table 2 have been obtained by a similar analysis of the equations with the coefficients listed in Cases 2-6 of Table 1.

Finally, Cases 28-32 of Table 2 have been identified by the analysis of the RDC equations with the coefficients listed in Cases 8-9 of Table 1.

The proof is now completed. \( \square \)

Thus, we can state that the first five steps of the LSC algorithm presented in Section 2 have been realized. As a result, we have derived LSC of the class of RDC equations (10) based on the group of ETs (19). Such classification is often called LSC via the Lie-Ovsiannikov algorithm (see [20] for discussion on this matter). However, it is well-known that the Lie-Ovsiannikov algorithm does not lead to the so called canonical list of the PDEs admitting nontrivial Lie symmetry. In fact, the number of relevant equations often can be reduced by implementation of the last two steps of the algorithm from Section 2. In the next section, it will be proved that 10 equations among 32 those from Table 2 are reducible to other equations from the same table by appropriate FPTs.

5 Lie symmetry classification using the form-preserving transformations

Now we turn to notion of a form-preserving transformation (FPT). Roughly speaking, a FPT is a local substitution, which reduces some PDE from the given class to another PDE belonging to the same class. The rigorous definition can be as follows (see [20], P.32).

Definition. A non-degenerate point transformation given by the formulae

\[
t^* = f(t, x, u), \quad x^*_a = g_a(t, x, u), \quad u^* = h(t, x, u), \quad (a = 1, \ldots, n), \tag{79}
\]
which maps at least one equation of the form (1) into an equation belonging to the same class, is called the FPT of the PDE class (1).

Comparing this definition with the well-known definition of ETs, one immediately notes that each ET is automatically a FPT but not vice versa. In contrast to the ETs, a set of all possible FPTs for the given class of PDEs usually do not form a Lie group. However, a subset of FPTs may generate a group of ETs on a subclass of the given class (see, e.g. example in [20], Section 2.3.2). This is a reason why FPTs are also called additional equivalence transformations. To the best of our knowledge, the 1992 paper [31] was the first, in which FPTs were used to solve LSC problem for a class PDEs (the authors used the terminology ‘admissible transformations’).

Let us construct the set of FPTs for the class of RDC equations (6). We start from the most general form of point transformations

$$\tau = a(t, x, y, u), \quad x^* = b^1(t, x, y, u), \quad y^* = b^2(t, x, y, u), \quad v = c(t, x, y, u). \quad (80)$$

Now we assume that there exists a FPT of the form (80), which relates an equation from the class (6) with another one from the same class, say, of the form

$$v_\tau = (d(v)v_{x^*}x^* + (d(v)v_{y^*}y^*) + k^1(v)v_{x^*} + k^2(v)v_{y^*} + r(v), \quad (81)$$

Here $u = u(t, x, y)$ and $v = v(\tau, x^*, y^*)$ are unknown functions, while $a(t, x, y, u), \quad b^1(t, x, y, u), \quad b^2(t, x, y, u), c(t, x, y, u) \quad d = d(v), \quad k^1 = k^1(v), \quad k^2 = k^2(v), \quad r = r(v)$ are some given functions.

**Theorem 5** An arbitrary RDC equation of the form (6) can be reduced to another equation of the same form (7) by the local nondegenerate transformation (80) with the correctly-specified smooth functions $a, b^1, b^2$ and $c$ if and only if these functions are of the form

$$\tau = a(t), \quad x^* = b^1(t, x, y), \quad y^* = b^2(t, x, y), \quad v = M(t, x, y)u + N(t, x, y), \quad (82)$$

and and the following equalities take place

$$b^2_x = \pm b^1_y, \quad (83)$$

$$b^2_y = \mp b^1_x, \quad (84)$$

$$\left[ (b^1_x)^2 + (b^1_y)^2 \right] D(u) = \dot{a}d(v), \quad (85)$$

$$b^1_t + \frac{\partial}{\partial t} D(u) \left[ b^1_x(M_xu + N_x) + b^1_y(M_yu + N_y) \right] D(u) - b^1_xK^1(u) - b^1_yK^2(u) = -\dot{a}k^1(v), \quad (86)$$

$$b^2_t + \frac{\partial}{\partial t} D(u) \left[ b^2_x(M_xu + N_x) + b^2_y(M_yu + N_y) \right] D(u) - b^2_xK^1(u) + b^2_yK^2(u) = -\dot{a}k^2(v), \quad (87)$$

$$M_tu + N_t - (\triangle M(t, x, y)u + \triangle N(t, x, y)D(u) + \frac{\partial}{\partial x} \left[ \left( (M_xu + N_x)^2 + (M_yu + N_y)^2 \right) D(u) \right] - (M_xu + N_x)K^1(u) - (M_yu + N_y)K^2(u) + MR(u) = \dot{a}r(v). \quad (88)$$

provided

$$\dot{a}M \left[ (b^1_x)^2 + (b^1_y)^2 \right] \neq 0. \quad (89)$$
Proof. Firstly we note that any FPT [82] must be nondegenerate, i.e., its Jacobian is nonvanish:

\[
J = \begin{vmatrix}
    a_t & a_x & a_y & a_u \\
    b_t & b_x & b_y & b_u \\
    c_t & c_x & c_y & c_u \\
\end{vmatrix} \neq 0. \tag{90}
\]

Having transformation (80) one can express the derivatives of the function \( u \) by the well-known formulas (usually they are presented in the case of two independent variables but those formulae can be directly extended on three or more variables)

\[
u_t = -v_x a_t + v_x b_x^1 + v_y b_y^1 - c_x, \tag{91}
\]

\[
u_x = -v_x a_x + v_x b_x^1 + v_y b_y^1 - c_x, \tag{92}
\]

\[
u_y = -v_x a_y + v_x b_x^1 + v_y b_y^1 - c_y. \tag{93}
\]

\[
u_{xx} = -\frac{1}{2} \left[(a_x + a_u u_x)^2 v_{xx} + 2(a_x + a_u u_x)(b_x^1 + b_u^1 u_x) v_{xxx} + 2(a_x + a_u u_x)(b_x^2 + b_u^2 u_x) v_{xxy} + (a_x + 2a_u u_x) v_{xyy} + (b_x^1 + b_u^1 u_x)(b_x^2 + b_u^2 u_x) v_{xxy} + (b_x^2 + b_u^2 u_x) v_{xyy} + (b_x^1 + b_u^1 u_x) v_{xyy}ight], \tag{94}
\]

\[
u_{yy} = -\frac{1}{2} \left[(a_y + a_u u_y)^2 v_{xx} + 2(a_y + a_u u_y)(b_y^1 + b_u^1 u_y) v_{xxx} + 2(a_y + a_u u_y)(b_y^2 + b_u^2 u_y) v_{xxy} + (a_y + 2a_u u_y) v_{xyy} + (b_y^1 + b_u^1 u_y)(b_y^2 + b_u^2 u_y) v_{xxy} + (b_y^2 + b_u^2 u_y) v_{xyy} + (b_y^1 + b_u^1 u_y) v_{xyy}ight], \tag{95}
\]

where \( A = v_x a_t + v_x b_x^1 + v_y b_y^1 - c_x, u_x = \frac{\partial}{\partial x}, v_x = \frac{\partial}{\partial x}, y, v_x^* = \frac{\partial}{\partial x^*}, u_{xx} = \frac{\partial^2}{\partial x^2}, u_{yy} = \frac{\partial^2}{\partial y^2}, v_x v_x^* = \frac{\partial^2}{\partial x^2}, \) and \( v_y v_y^* = \frac{\partial^2}{\partial y^2}. \)

Substituting (91)–(95) into (9) one arrives at a very cumbersome expression. Let us assume that (80) is a FPT. So, the expression obtained must be reducible to an equation of the form (81). In the particular case, the coefficient next to the second-order derivative \( v_{xx}, v_{xxx}, v_{xyy}, v_{xxy} \) and \( v_{yy}, v_{xyy} \) should vanish and the coefficients next to the derivatives \( v_{xxy}, v_{xyy} \) must be equal to \( u_{xx} \) and \( v_{xx} \), respectively, hence one obtains the system of PDEs

\[
a_x + a_u u_x = 0, a_y + a_u u_y = 0, \tag{96}
\]

\[
(b_x^1 + b_u^1 u_x)(b_x^2 + b_u^2 u_x) = 0, \tag{97}
\]

\[
(b_x^1 + b_u^1 u_y)(b_x^2 + b_u^2 u_y)(b_y^2 + b_u^2 u_x) = 0, \tag{98}
\]

Equations (96) immediately give

\[
a_x = a_y = a_u = 0, \tag{99}
\]

while Eqs. (95) lead to

\[
b_x^1 b_x^2 + b_y^1 b_y^2 = 0. \tag{100}
\]
\[ b_1^1b_2^1 + b_1^1b_2^1 = 0, \quad b_1^2b_2^2 + b_1^2b_2^2 = 0, \quad (101) \]
\[ b_1^2b_2^2 = 0, \quad (102) \]
\[ (b_1^1)^2 + (b_1^1)^2 = (b_2^1)^2 + (b_2^1)^2, \quad (103) \]
\[ b_1^1b_1^1 = b_2^1b_2^1, \quad b_1^2b_1^1 = b_2^2b_2^1 \quad (104) \]
\[ b_1^1b_1^1 = (b_2^1)^2, \quad (105) \]

Obviously, Eqs. (102) and (105) are equivalent to
\[ b_1^1 = b_2^2 = 0. \quad (106) \]

So, using the derived restrictions (99) and (106) we can specify FPT in question as follows
\[ \tau = a(t), \quad x^* = b^1(x, y), \quad y^* = b^2(x, y), \quad v = c(x, y, u), \quad (107) \]
i.e. the first three formulae in (81) are derived.

Now one observes that formulae (91)-(95) can be simplified essentially if one takes into account
(106), namely:
\[ u_t = \frac{1}{c_u}(\Delta v + v_t^1v_x^1 + b_1^1v_y^1 - c_1), \quad (108) \]
\[ u_x = \frac{1}{c_u}(b_1^1v_x^1 + b_2^2v_y^2 - c_2), \quad (109) \]
\[ u_y = \frac{1}{c_u}(b_1^1v_x^1 + b_2^2v_y^2 - c_3), \quad (110) \]
\[ \Delta u = \frac{1}{c_u}\left[ \left( (b_1^1)^2 + (b_1^1)^2 \right) \Delta v - \frac{2c_b}{c_u} \left[ \left( (b_1^1)^2 + (b_1^1)^2 \right) v^1_xv^1_x + 2(b_1^1b_2^1 + b_1^1b_2^1) v^1_xv^1_y + \left( (b_2^2)^2 + (b_2^2)^2 \right) v^1_yv^1_y \right] + \left( \Delta b^1 - \frac{2c_u}{c_b}b_1^1 - \frac{2c_u}{c_b}b_1^1b_2^1 + \frac{2c_u}{c_b}(b_1^1c_x + b_1^1c_y) \right) v^1_x + \left( \Delta b^2 - \frac{2c_u}{c_b}b_2^2 - \frac{2c_u}{c_b}b_1^1b_2^1 + \frac{2c_u}{c_b}(b_2^2c_x + b_2^2c_y) \right) v^1_y + \frac{2c_u}{c_b}(e_xc_x + e_yc_y) - \Delta c \right] \right]. \quad (111) \]

Finally, substituting the right-hand-sides from (108)-(111) into (6), we note that the expression obtained is reducible to Eq. (81) only under the condition
\[ c_{uu} = 0, \quad (112) \]
i.e. the last formula in (81) is derived, and the equalities (88)–(88) should take place.

It can be also easily shown that Eqs. (101) and (103) are equivalent to Eqs. (83)–(84), while the restriction (89) immediately follows from (90) because of formulae (82).

The proof is now completed. \(\square\)

**Remark 2** Using Theorem 5 one can derive the discrete equivalence transformation
\[ t^* = t, \quad x^* = -x, \quad y^* = y, \quad u^* = u; \quad (113) \]
\[ t^* = t, \quad x^* = x, \quad y^* = y, \quad u^* = u; \quad (114) \]
\[ t^* = -t, \quad x^* = x, \quad y^* = y, \quad u^* = u; \quad (115) \]
\[ t^* = t, \quad x^* = x, \quad y^* = -y, \quad u^* = u; \quad (116) \]
\[ t^* = t, \quad x^* = y, \quad y^* = x, \quad u^* = u, \quad (117) \]

which take place for Eq. (2) with arbitrary smooth coefficients \(D, K_1, K_2, R\). However, if one takes into account the restriction \(D(u) > 0\) then the discrete transformation (113) is not valid, while (116) is valid only under the additional restriction \(D(-u) > 0\).
To complete the last step of the algorithm (see Section 2), we need to apply Theorem 5 to the equations listed in Table 2 in order to identify those pairs of them, which are reducible one to another by an appropriate FPT.

**Theorem 6** There are 9 equations in Table 2 which are reducible to other equations from the same table by an appropriate FPT of the form (82). All the equations and the corresponding transformations are presented in Cases 1–9 of Table 3.

### Table 3: Simplification of the RDC equations form (6) by means of FPTs

| Case | RDC equation | FPT | Canonical form of RDC equation |
|------|--------------|-----|-------------------------------|
| 1    | $u_t = \Delta u + \gamma_1$ | $\tau = t, x^* = x, y^* = y, v = u - \gamma_1t$ | $v_\tau = \Delta v$ |
| 2    | $u_t = \Delta u + \gamma_1u$ | $\tau = t, x^* = x, y^* = y, v = e^{\gamma_1t}u$ | $v_\tau = \Delta v$ |
| 3    | $u_t = (e^u u_x)_x + (e^u u_y)_y + +\sigma e^u + \gamma_1$ | $\tau = \gamma_1 e^{\gamma_1t}, x^* = x, y^* = y, v = u + \gamma_1t$ | $v_\tau = (e^{\gamma_1u_x})_x + (e^{\gamma_1u_y})_y + \sigma e^u$ |
| 4    | $u_t = (k u_x)_x + (k u_y)_y + +\sigma k^2 + \gamma_1 u$ | $\tau = \gamma_1 e^{\gamma_1t}, x^* = x, y^* = y, v = e^{-\gamma_1t}u$ | $v_\tau = (e^{k u_x})_x + (e^{k u_y})_y + \sigma k^{k+1}$ |
| 5    | $u_t = (e^u u_x)_x + (e^u u_y)_y + +e^u u_x + \sigma e^u + \gamma_1$ | $\tau = \gamma_1 e^{\gamma_1t}, x^* = x, y^* = y, v = u - \gamma_1t$ | $v_\tau = (e^{k u_x})_x + (e^{k u_y})_y + e^u v_x + \sigma e^u$ |
| 6    | $u_t = (k u_x)_x + (k u_y)_y + +4k u_x + \sigma k^2 + \gamma_1 u$ | $\tau = \gamma_1 e^{\gamma_1t}, x^* = x, y^* = y, v = e^{-\gamma_1t}u$ | $v_\tau = (e^{k u_x})_x + (e^{k u_y})_y + e^{k u_x} + \sigma k^{k+1}$ |
| 7    | $u_t = (k u_x)_x + (k u_y)_y + +4k u_x + 4k^2 + \gamma_1 u$ | $\tau = \gamma_1 e^{\gamma_1t}, x^* = x, y^* = y, v = e^{-\gamma_1t}u$ | $v_\tau = (e^{k u_x})_x + (e^{k u_y})_y + e^{k u_x} + \sigma k^{k+1}$ |
| 8    | $u_t = \Delta u + uu_x + \sigma$ | $\tau = t, x^* = x + \gamma_1 \frac{t^2}{2}, y^* = y, v = u - \gamma_1t$ | $v_\tau = \Delta v + vv_x$ |
| 9    | $u_t = \Delta u + \ln uu_x + \sigma u$ | $\tau = t, x^* = x + \gamma_1 \frac{t^2}{2}, y^* = y, v = e^{-\gamma_1t}u$ | $v_\tau = \Delta v + \ln vv_x$ |

**Sketch of the proof** of this theorem is similar to that of Theorem 2.11 [20]. Of course, Cases 1–4 of Table 3 involve the well-known substitutions for the linear and nonlinear RD, which were identified many years ago.

The peculiarity of the RDC equations in Cases 5–9 is such that each equation contains the convective, which involves only the derivative w.r.t. $x$ (no terms involving $u_y$). It turns out that FPTs constructed for the (1+1)-dimensional analogs of these equations in [19] (see also Table 2.6 in [20]) are valid also for (1+2)-dimensional equations (the second space variable $y$ is unchangeable).

In particular, the most nontrivial FPT occurs in Case 9 of Table 3. On the other hand, one notes that it is nothing else but the substitution listed in Case 16 of Table 2.6 [20] with the formal additional transformation $y^* = y$. Substituting these formulae into Eqs. (83)-(88) and
taking into account that
\[ D(u) = d(v) = 1, \quad K^1 = \ln u, \quad k^1 = \ln v, \quad K^2 = k^2 = 0, \quad R(u) = \sigma u, \quad r(v) = 0, \]
one easily checks that the substitution
\[ \tau = t, \quad x^* = x + \frac{1}{2} \sigma t^2, \quad y^* = y, \quad v = e^{-\sigma t} u \]
is indeed FPT, which relates two RDC equations listed in Case 9 of Table 3.

The proof is now completed. \( \square \)

Now we formulate the main theorem presenting the canonical list of RDC equations from class (6) possessing nontrivial Lie symmetries.

**Theorem 7** All possible RDC equations of the form admitting nontrivial Lie symmetries are reduced to one of the 22 canonical equations listed in the second column of Table 4 by the relevant FPTs presented in Theorem 5. The relevant MAIs of the canonical RDC equations are listed in the third column of Table 4.

| Equation | MAI | Restrictions |
|----------|-----|--------------|
| 1 \( u_t = (D(u)u_x)_x + (D(u)u_y)_y + +R(u) \) | \( < \partial_t, \partial_x, \partial_y, J_{12} > \) | \( D - \forall, R - \forall \) |
| 2 \( u_t = (D(u)u_x)_x + (D(u)u_y)_y \) | \( < \partial_t, \partial_x, \partial_y, J_{12}, D_0 > \) | \( D - \forall \) |
| 3 \( u_t = \Delta u \) | \( < \partial_t, \partial_x, \partial_y, J_{12}, G_x, G_y, I, D_0, \Pi, G_k^1 > \) | |
| 4 \( u_t = (e^{\alpha u_x})_x + (e^{\alpha u_y})_y \) | \( < \partial_t, \partial_x, \partial_y, J_{12}, J_{12}, e^{\alpha t} I, G_x, G_y > \) | |
| 5 \( u_t = (e^{\alpha u_x})_x + (e^{\alpha u_y})_y \) | \( < \partial_t, \partial_x, \partial_y, J_{12}, J_{12}, D_0, D_2 > \) | \( \delta = 1 \) |
| 6 \( u_t = (e^{\alpha u_x})_x + (e^{\alpha u_y})_y + +e^{mu}[u_x \cos(pu) + u_y \sin(pu)] + \sigma e^{(2m-\delta)u} \) | \( < \partial_t, \partial_x, \partial_y, J_{12}, J_{12}, (m-1)D_0 - D_2 > \) | \( m \neq \delta \) |
| 7 \( u_t = (u^{k+1}u_x)_x + (u^{k+1}u_y)_y \) | \( < \partial_t, \partial_x, \partial_y, J_{12}, D_0, D_1 > \) | \( k \neq -1; 0 \) |
| 8 \( u_t = (u^{k+1}u_x)_x + (u^{k+1}u_y)_y + +u^{mu}[u_x \cos(pu) + u_y \sin(pu)] + \sigma e^{u} \) | \( < \partial_t, \partial_x, \partial_y, J_{12}, D_0, D_2 > \) | \( m \neq \delta \) |
| 9 \( u_t = (u^{k+1}u_x)_x + (u^{k+1}u_y)_y \) | \( < \partial_t, \partial_x, \partial_y, J_{12}, D_0, X > \) | \( k = -1 \) |
| 10 \( u_t = (e^{\alpha u_x})_x + (e^{\alpha u_y})_y + +e^{mu}[u_x \cos(pu) + u_y \sin(pu)] + \sigma e^{(2m-\delta)u} \) | \( < \partial_t, \partial_x, \partial_y, J_{12}, D_0, D_1 > \) | \( (m,p) \neq (0,0) \) |
| 11 \( u_t = (e^{\alpha u_x})_x + (e^{\alpha u_y})_y + +u^{mu}[u_x \cos(pu) + u_y \sin(pu)] + \sigma e^{u} \) | \( < \partial_t, \partial_x, \partial_y, J_{12}, D_0, D_1 > \) | \( \delta = 1 \) |
| 12 \( u_t = (e^{\alpha u_x})_x + (e^{\alpha u_y})_y + +u^{mu}[u_x \cos(pu) + u_y \sin(pu)] + \sigma e^{u} \) | \( < \partial_t, \partial_x, \partial_y, J_{12}, D_0, D_1 > \) | \( \delta = 1 \) |
| 13 \( u_t = (u^{k+1}u_x)_x + (u^{k+1}u_y)_y + +u^{mu}[u_x \cos(pu) + u_y \sin(pu)] + \sigma e^{(2m-\delta+1)u} \) | \( < \partial_t, \partial_x, \partial_y, (m-k)D_0 + +D_3 + pJ_{12} > \) | \( m \neq k, \quad (m,p) \neq (0,0) \) |
| 14 \( u_t = (u^{k+1}u_x)_x + (u^{k+1}u_y)_y + +\ln uu_x + \sigma u^{-k+1} \) | \( < \partial_t, \partial_x, \partial_y, kD_0 - D_3 - t\partial_x > \) | \( k \neq 0 \) |
Sketch of the proof.

If one compare the equations and MAIs listed in Table 2 with those from Table 4 then can be identified that the 20 cases are identical in the both tables. In fact, Cases 1–3, 6–8, 11, 12, 15, 17–19, 21–24, 26, 28, and 31–32 from Table 2 are exactly Cases 1–3, 4–6, 7, 8, 9, 10–12, 13–16, 17, 18, and 21–22 from Table 4, respectively.

The remaining 12 cases from Table 2 (Cases 4–5, 9–10, 13–14, 16, 20, 25, 27, and 29–30) are reducible to 11 cases in Table 4 using the relevant FPTs as listed in Table 5. As one may note making a simple analysis of Table 5, only two additional cases (see the last two lines) should be added to Table 4 (see 19 and 20 therein). As a result Table 4 contains exactly 22 equations.

The sketch of the proof is now completed. □

Remark 3 The (1+2)-dimensional Burgers equation

\[ u_t = \Delta u + uu_x + uu_y, \]

which is a natural two-dimensional generalization of the famous Burgers equation \( u_t = u_{xx} + uu_x \), is obtainable from the equation listed in Case 19 of Table 4 by the equivalence transformation

\[ t \to t, \quad x \to \frac{1}{2}(x + y), \quad y \to \frac{1}{2}(x - y), \quad u \to u. \]

Lie symmetries of the (1+2)-dimensional Burgers equation (in the form listed in Case 19 of Table 4) were found for the first time in [25] while symmetry reductions and exact solutions are presented in [39].
Table 5: Mapping the nonlinear RDC equations from Table 2 to their canonical forms using FPTs from Table 3. The numbers in three columns refer to the relevant cases in Tables 2, 3 and 4.

| RDC equation in Table 2 | FPTs in Table 3 | Canonical form in Table 4 |
|-------------------------|-----------------|--------------------------|
| 4                       | 1               | 3                        |
| 5                       | 2               | 3                        |
| 9                       | 3               | 5                        |
| 10                      | 3               | 6                        |
| 13                      | 4               | 7                        |
| 14                      | 4               | 8(with $m = k + 1$)       |
| 16                      | 4(with $k = -1$) | 9                        |
| 20                      | 5               | 10(with $p = 0$, $m = \delta = 1$) |
| 25                      | 6               | 16                       |
| 27                      | 7               | 17                       |
| 29                      | 8               | 19                       |
| 30                      | 9               | 20                       |

6 Examples of exact solutions of a generalization of the porous-Fisher equation

Here we examine the nonlinear equation

$$u_t = (uu_x)_x + (uu_y)_y + \lambda_1 uu_x + \lambda u(1 - u) \quad (118)$$

where $\lambda_1$ and $\lambda$ are arbitrary constants. Eq. (118) with $\lambda_1 = 0$ coincides with the so called porous-Fisher equation (see, e.g., [33, 34, 11, 35] and its generalization on reaction-diffusion systems [36]), which is a generalization of the famous 2D Fisher equation [32]

$$u_t = u_{xx} + u_{yy} + \lambda u(1 - u), \quad \lambda > 0. \quad (119)$$

Physically Eq. (118) with $\lambda_1 = 0$ describes the population dispersing to regions of lower density more rapidly as the population gets more crowded and has been extensively studied in the 1D approximation. (see, e.g., [11, 37] and references therein).

On the other hand, Eq. (118) can be thought as a generalization of the Murray equation

$$u_t = u_{xx} + u_{yy} + \lambda_1 uu_x + \lambda u(1 - u) \quad (120)$$

which was intensively studied in [11, 20, 33] in the 1D approximation.

It can be noted that Eq. (118) with $\lambda_1 = -\lambda = 8$ is nothing else but the RDC equation listed in Case 27 of Table 2 under the restriction $k = 1$. Moreover, this equation can be simplified via FPT listed in Case 7 of Table 3, namely

$$t \to -\frac{e^{-8t}}{8}, \quad x \to x, \quad y \to y, \quad u \to e^{-8t}u \quad (121)$$
to the form

\[ u_t = (uu_x)_x + (uu_y)_y + 8uu_x + 8u^2. \]  \tag{122}

Now we realize that Eq. (122) possesses the six-dimensional Lie algebra of invariance \( AL_6 \) (see Case 17 in Table 4), which is the largest for nonlinear RDC equations with non-zero convection terms. This Lie algebra is generated by the basic operators

\[ \partial_t, \partial_1, \partial_2, D_3 = t\partial_t - u\partial_u, \]

\[ R_1 = e^{-x}(\cos y\partial_x - \sin y\partial_y - \frac{2}{k} \cos yu\partial_u), \quad R_2 = e^{-x}(\sin y\partial_x + \cos y\partial_y - \frac{2}{k} \sin yu\partial_u). \]  \tag{123}

It should be noted that Eq. (122) admits two operators, \( R_1 \) and \( R_2 \), with very unusual structure and there are not \((1+1)\)-dimensional RDC equations admitting such kind of operators.

It is well-known that Lie symmetries allow to reduce the given PDE to that of lower dimensionality. In the case of Eq. (122), there are a wide range of possibilities to make such reductions because the equation in question admits the six-dimensional Lie algebra of invariance. Generally speaking, one should construct the so called optimal system of inequivalent (non-conjugate) subalgebras of \( AL_6 \) (see for details [40, 41, 42]). It is a nontrivial problem in the case of Lie algebras of high dimensionality and its solving lies beyond scopes of this paper.

On the other hand, there is a straightforward technique for deriving a set of reduced equations using the known Lie symmetry of the given PDE. In the case of Eq. (122), one should take the most general form of Lie’s operator belonging to \( AL_6 \):

\[ X = d_0\partial_t + d_1\partial_x + d_2\partial_y + c_0D_3 + c_1R_1 + c_2R_2 \]  \tag{124}

(here coefficients are arbitrary parameters) and solve the corresponding invariance surface condition

\[ X\Phi|_{\Phi=0} = 0, \]  \tag{125}

where \( \Phi = u(t, x, y) - u_0(t, x, y) \) and \( u_0(t, x, y) \) is an arbitrary solution of Eq. (122). As a result, we arrive at a linear first-order PDE, which is equivalent to the system of three ODEs

\[ \frac{dt}{c_0t + d_0} = \frac{dx}{e^{-x}z(y)+d_1} = \frac{du}{e^{-x}z(y)+d_2} = \frac{du}{-(\frac{2}{k}e^{-x}z(y)+c_0)u}, \]  \tag{126}

where the notation \( z(y) = c_1 \cos y + c_2 \sin y \) is introduced. Obviously that the form of solutions of system (126) depends essentially on the six parameters arising therein.

Here we examine the \( c_0 = d_0 = d_1 = d_2 = 0 \) corresponding the symmetry reduction via the operators \( R_1 \) and \( R_2 \). In this case, the direct integration of system (126) leads to the first integrals

\[ J_1 = t, \quad J_2 = \dot{z}(y)e^x, \quad J_3 = e^{2x}u. \]  \tag{127}

As a result, we arrive at the ansatz

\[ u = e^{-2x}\varphi(t, \omega), \quad \omega = \dot{z}(y)e^x, \]  \tag{128}

where \( \varphi = \varphi(t, \omega) \) is a new unknown function. Substituting ansatz (128) into Eq. (122), one obtains the reduced equation

\[ \varphi_t = (c^2_1 + c^2_2)(\varphi\varphi_\omega)\omega, \]  \tag{129}

which is reducible to the form

\[ \varphi_t = (\varphi\varphi_\omega)\omega. \]  \tag{130}
by the time-scaling $t \rightarrow \frac{t}{c_1^2 + c_2^2}$. We set $c_1^2 + c_2^2 = 1$ in what follows, hence $z(y) = \sin(y + y_0)$, $y_0 \in \mathbb{R}$. Thus, the symmetry reduction of Eq. (122) via the operators $R_1$ and $R_2$ gives nothing else but the porous diffusion equation (130), which is often called the Boussinesq equation.

Because the Boussinesq equation was extensively studied by many authors (see e.g. Section 4.2 in [20] and references therein) its exact solutions were constructed in several works and practically all of them are summarized in the handbook [43].

For example, the Boussinesq equation possesses the plane wave solution

$$\varphi = p(\omega + pt) + c_3$$

(131)

and

$$\varphi = c_4 t^{-1/3} - \frac{1}{6} t^{-1} \omega^2,$$

(132)

which are obtainable via the further Lie symmetry reduction of Eq. (130) to ODEs (this equation admits four-dimensional MAI [44]). Notably the exact solution (132) for the first time was derived in [45].

Thus, using ansatz (128) and solutions (131) and (132), we obtain exact solutions

$$u = e^{-2x - 8t} \left[ p \cos(y + y_0) e^x - \frac{5}{8} e^{-8t} + c_3 \right];$$

(133)

and

$$u = \frac{4}{3} \left[ \cos^2(y + y_0) + c_5 e^{-\frac{16}{3}t - 2x} \right]$$

(134)

of the nonlinear RDC equation (118) with $\lambda_1 = -\lambda = 8$ (here $c_5 = -\frac{3}{2} c_4 (c_1^2 + c_2^2)^{-\frac{1}{2}}$ can be thought as new arbitrary constant). It should be stressed that we have constructed two nontrivial exact solutions of (118) using the relatively simple solutions (131) and (132) of the the Boussinesq equation.

The both exact solutions have interesting asymptotical behaviour. In fact, solution (131) tends to zero provided $t \rightarrow \infty$. It means that the solution describes extinction of particles (population of spices, cells etc.). Solution (131) possesses another time asymptotic because $u \rightarrow \frac{4}{3} [\cos^2(y + y_0)]$ as $t \rightarrow \infty$. So, the solution describes such processes, which tend to the periodical steady-state w.r.t. the variable $y$. Notably both solutions are periodic w.r.t. the variable $y$ and have exponential growth/decay w.r.t. the variable $x$.

Finally, it should be noted that exact solutions of Eq. (118) in the 1D approximation were constructed and analyzed for the first time in [46] (see [20] for more details).

7 Concluding remarks

The main result of this work consists of solving the LSC problem for the class of RDC equations of the form (7). The class contains as particular cases several subclasses of RDC equations, which has been examined in earlier papers [21, 22, 23, 24, 25, 26]. Here this problem is solved for the first time for the most general class of such equations (see Theorem 7). All the specific equations and Lie symmetries identified in the above cited papers follow as particular cases from the results derived in this paper. Another important result from applicability point of view consists in deriving a new list of point transformations presented in Table 3. In fact, the form-preserving transformations listed therein allow us to identified hidden relations between RDC equations. For example, the standard Burgers equation is equivalent to that with a constant source (see, case 8 in Table 3).
The LSC problem for the class of RDC equations was solved using two methods (algorithms). The first one is based on the group of ETs and was firstly developed and applied by Ovsiannikov for with \( n = 1 \). In the case of class, this method lead to 32 different RDC equations possessing different MIAs presented in Table 2 (notable the first two equations in the table are subclasses of class, however, we treat them as equations containing arbitrary functions as parameters).

The second method of LSC is based on the set of form-preserving transformations, which do not form any group. Although FPTs were firstly used for solving LSC problem in 1990s, the method can be still thought as relatively new (see Section 2.3 in for more details and references). This method allowed us to make a further reduction of number of RDC equations possessing nontrivial Lie symmetries. As a result, Theorem 6 was proved, which says that there are 22 RDC equations admitting four- and higher-dimensional MIAs listed in Table 4. It means that there are exactly 22 RDC equations with nontrivial Lie symmetries, which are inequivalent up to any point transformation, and each other RDC equation possessing a nontrivial Lie symmetry is reducible to one of those from Table 4. Moreover this list of the RDC equations cannot be reduced by any point transformations. In this sense these 22 equations form a canonical list of the RDC equations with nontrivial Lie symmetries.

It should be noted that the results of LSC were verified using the computer algebra package Maple 18. It means that each RDC equation listed in Tables 2 and 4 was tested and the Lie symmetries obtained coincide with those from Tables 2 and 4.

Now we present a comparison of our results with those derived earlier. One notes that the RD equations and the relevant MIAs listed in cases 2–9 of Table 4 were identified earlier in papers. In particular, it was established for the first time in that the so called conformal exponent \(-4/(n + 2)\) in the case of two space variables, i.e. \( n = 2 \), leads to infinite-dimensional MIA (see case 9 of Table 4).

The diffusion-convection equations 1–5 listed in Table 1 follow as particular cases from the equations listed in cases 16, 15, 11, 10 and 19 of Table 4 respectively.

It should be also stressed that four RDC equations arising in Cases 10, 12–13 and 15 of Table 4 are absolutely new equations, which have no analogs among (1+1)-dimensional RDC equations (see or Table 2.7 in for comparison). The nonlinear equations arising in Cases 17–18, and 20–22 of Table 4 are also new (1+2)-dimensional RDC equations. On the other hand, they have analogs among (1+1)-dimensional RDC equations.

It is quite interesting that Eq. (122) (it is a particular case of the equation listed in Case 17 of Table 4) admits six-dimensional Lie algebra involving two operators, \( R_1 \) and \( R_2 \), with very unusual structure. However, calculating Lie brackets, one may show that the subalgebra with the basic operators \( \partial_x, \partial_y, R_1 \) and \( R_2 \) is nothing else but the extended Euclid algebra \( AE_1(2) \). In fact, the operators \( R_1 \) and \( R_2 \) should be treated as the space translation operators while \( \partial_y \) and \( \partial_x \) are the operators of rotations and scale transformations, respectively. So, \( AL_6 \) is the direct sum of \( AE_1(2) \) and the two-dimensional Abelian algebra \( \langle \partial_x, D_0 \rangle \). It is worth to note that the Lie symmetries \( R_1 \) and \( R_2 \) do not occur as those of (1+1)-dimensional RDC equations. Moreover, we foresee that such symmetries are not allowed by any (1+n)-dimensional RDC equation with \( n > 2 \).

The Lie symmetries obtained in this work can be widely used for finding exact solutions of the nonlinear RDC equations arising in applications. It can happen that an equation possessing a nontrivial Lie symmetry is not explicitly listed in Tables 2 and 4. However, such equation must be reducible to one of those in these tables by appropriate ET or/and FPT. In Section 6, an example is presented. In fact, we have identified how the porous-Fisher equation with the
Burgers term \((118)\) (with the correctly-specified coefficients) can be derived from Eq.\((122)\). Using the highly nontrivial Lie symmetries of Eq.\((122)\), exact solutions of the equation in question have been constructed and their properties identified. It was also shown that this 2D nonlinear PDE is reducible to the 1D Boussinesq equation \((130)\) by the Lie symmetry reduction. The authors are going to present more interesting (from the applicability point of view) examples in a forthcoming paper.

It should be also noted that there are RDC type equations involving the so called gradient-dependent diffusivity (see Eq.\((1)\) with \(D(|\nabla u|)\)), which arise in many mathematical models describing real-world processes (such as non-Newtonian flow, image processing etc.). The recent papers \([17]\) and \([18]\) present new results for solving the LSC problem for some classes of equations with gradient-dependent diffusivity in the case \(n > 1\). One may note that Lie symmetry properties of nonlinear equations with the standard diffusivity and gradient-dependent diffusivity are essentially different.

Finally, we point out that the LSC problem (the group classification) for the class of multi-dimensional \((n > 2)\) RDC equations \((1)\) is still open. We foresee that solving the LSC problem in the case of three and more space variables will lead to new results.

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