Whittaker functions and related stochastic processes

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Abstract. We review some recent results on connections between Brownian motion, Whittaker functions, random matrices and representation theory.

1. Harish-Chandra formula, Duistermaat-Heckman measure and Gelfand-Tsetlin patterns

Define $J_{\lambda}(x) = h(\lambda)^{-1} \det(e^{\lambda x})$, where $h(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j)$. For each $x$, $J_{\lambda}(x)$ is an analytic function of $\lambda$; in particular, $J_0(x) = \left(\prod_{j=1}^{\infty} j!\right) h(x)$. The functions $J_{\lambda}(x)$ play a central role in random matrix theory. For example, if $\Lambda$ and $X$ are Hermitian matrices with eigenvalues given by $\lambda$ and $x$, respectively, then

$$\int_{U(n)} e^{\text{tr}\Lambda U X U^*} \, dU = \frac{J_{\lambda}(x)}{J_0(x)},$$

where the integral is with respect to normalised Haar measure on the unitary group. This is known as the Harish-Chandra, or Itzykson-Zuber, formula.

Let $\beta = (\beta_i, \ t \geq 0)$ be a standard Brownian motion in $\mathbb{R}^n$ with drift $\lambda$. Denote by $\mathbb{P}_x$ the law of $\beta$ started at $x$ and by $\mathbb{E}_x$ the corresponding expectation. Set

$$\Omega = \{x \in \mathbb{R}^n : x_1 > x_2 > \cdots > x_n\}, \quad T = \inf\{t > 0 : \beta_t \notin \Omega\}.$$

For $\lambda, x \in \mathbb{R}^n$, write $\lambda(x) = \sum_i \lambda_i x_i$.

Proposition 1. For $x, \lambda \in \Omega$, $J_{\lambda}(x) = h(\lambda)^{-1} e^{\lambda(x)} \mathbb{P}_x(T = \infty)$.

Proof. This is well known, see for example [6]. The function $u(x) = \mathbb{P}_x(T = \infty)$, $x \in \Omega$, satisfies $\frac{1}{2} \Delta u + \lambda \cdot \nabla u = 0$, vanishes on the boundary of $\Omega$ and $\lim_{x \to \infty} u(x) = 1$. Here we write $x \to \infty$ to mean $x_i - x_{i+1} \to \infty$ for $i = 1, \ldots, n-1$. Hence $v(x) = e^{\lambda(x)} u(x)$ satisfies $\Delta v = \sum_i \lambda_i^2 v$, vanishes on the boundary of $\Omega$ and $\lim_{x \to \infty} e^{-\lambda(x)} v(x) = 1$. The function $\det(e^{\lambda x})$ also has these properties, so by uniqueness, $v(x) = \det(e^{\lambda x})$, as required.

The Harish-Chandra formula has the following interpretation. Pick $U$ at random according to the normalised Haar measure on $U(n)$ and let $\mu^x(dy)$ denote the law of the diagonal of the random matrix $UXU^*$. Then the integral becomes

$$\int_{U(n)} e^{\text{tr}\Lambda U X U^*} \, dU = \int_{\mathbb{R}^n} e^{\lambda(y)} \mu^x(dy).$$

Setting $m^x(dy) = J_0(x) \mu^x(dy)$, we obtain

$$\int_{\mathbb{R}^n} e^{\lambda(y)} m^x(dy) = J_{\lambda}(x).$$

The measure $m^x$ is known as the Duistermaat-Heckman measure associated with the point $x \in \Omega$. It has the following properties, which are well-known. The symmetric group $S_n$ acts naturally on $\mathbb{R}^n$ by permuting coordinates. The support of the measure $m^x$ is the convex hull of the set of images of $x$ under the action of $S_n$. It has a piecewise polynomial density. This comes from the fact, which we will now explain, that the

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Duistermaat-Heckman measure is the push-forward via an affine map of the Lebesgue measure on a higher dimensional polytope known as the Gelfand-Tsetlin polytope.

Let \( x \in \Omega \) and denote by \( GT(x) \) the polytope of Gelfand-Tsetlin patterns with bottom row equal to \( x \):

\[
GT(x) = \{ P_{k,j}, 1 \leq j \leq k \leq n : P_{k,j+1} \leq P_{k-1,j} \leq P_{k,j}, 1 \leq j < k \leq n, P_{n,n} = x \}.
\]

Define the type of a pattern \( P \) to be the vector

\[
\text{type } P = \left( P_{1,1}, P_{2,1} + P_{2,2} - P_{1,1}, \ldots, \sum_{j=1}^{n} P_{n,j} - \sum_{j=1}^{n-1} P_{n,j} \right).
\]

Consider the map from \( U(n) \) to \( GT(x) \) defined by \( U \mapsto P \) where: for each \( 1 \leq k \leq n \), \( P_k \) is the vector of eigenvalues of the \( k^{th} \) principal minor of \( UXU^* \). It is well-known (see, for example, \cite{2} or \cite{11} Section 5.6) for a more general statement) that the push-forward of Haar measure under this map is the standard Euclidean formula, due to Givental \cite{23, 28, 18}.

This follows from the identity (4) gives the following integral equation:

\[
J_\lambda(x) = \int_{GT(x)} e^{\lambda \text{type } P} dP.
\]

**2. Whittaker functions**

Set \( H = \Delta - 2 \sum_{i=1}^{n-1} e^{-\alpha_i(x)} \), where \( \alpha_i = \epsilon_i - \epsilon_{i+1} \), \( i = 1, \ldots, n-1 \). Write \( H = H^{(n)} \) for the moment; we will drop the superscript again later, whenever it is unnecessary. For convenience we define \( H^{(1)} = d^2/dx^2 \) and \( \psi^{(1)}_\lambda(x) = e^{\lambda x} \). Following \cite{18}, for \( n \geq 2 \) and \( \theta \in \mathbb{C} \), define a kernel on \( \mathbb{R}^n \times \mathbb{R}^{n-1} \) by

\[
Q^{(n)}_\theta(x, y) = \exp \left( \theta \left( \sum_{i=1}^{n} x_i - \sum_{i=1}^{n-1} y_i \right) - \sum_{i=1}^{n-1} (e^{\theta x_i - y_i} + e^{x_i-y_i}) \right).
\]

Denote the corresponding integral operator by \( Q^{(n)}_\theta \), defined on a suitable class of functions by

\[
Q^{(n)}_\theta f(x) = \int_{\mathbb{R}^{n-1}} Q^{(n)}_\theta(x, y) f(y) dy.
\]

The Whittaker functions \( \psi^{(n)}_\lambda, \lambda \in \mathbb{C}^n \) are defined recursively by

\[
\psi^{(n)}_{\lambda_1, \ldots, \lambda_n} = Q^{(n-1)}_{\lambda_n} \psi^{(n-1)}_{\lambda_1, \ldots, \lambda_{n-1}}.
\]

As observed in \cite{18}, the following intertwining relation holds:

\[
(H^{(n)} - \theta^2) \circ Q^{(n)}_\theta = Q^{(n)}_\theta \circ H^{(n-1)}.
\]

This follows from the identity \( (H^{(n)} - \theta^2) Q^{(n)}_\theta(x, y) = H^{(n-1)} Q^{(n)}_\theta(x, y) \), which is readily verified. Combining \cite{4} with the intertwining relation \cite{5} yields the eigenvalue equation:

\[
H^{(n)} \psi^{(n)}_\lambda = \left( \sum_{i=1}^{n} \lambda_i^2 \right) \psi^{(n)}_\lambda.
\]

Let us now drop the superscripts and write \( H = H^{(n)}, \psi_\lambda = \psi^{(n)}_\lambda \). Iterating \cite{4} gives the following integral formula, due to Givental \cite{23, 28, 18}.

\[
\psi_\lambda(x) = \int_{\Gamma(x)} e^{\mathcal{F}_\lambda(T)} \prod_{k=1}^{n} \prod_{i=1}^{k} dT_{k,i},
\]

where \( \Gamma(x) \) denotes the set of real triangular arrays \( (T_{k,i}, 1 \leq i \leq k \leq n) \) with \( T_{n,i} = x_i, 1 \leq i \leq n \), and

\[
\mathcal{F}_\lambda(T) = \sum_{k=1}^{n} \lambda_k \left( \sum_{i=1}^{k} T_{k,i} - \sum_{i=1}^{k-1} T_{k-1,i} \right) - \sum_{k=1}^{n-1} \sum_{i=1}^{k} \left( e^{T_{k,i}-T_{k+1,i}} + e^{T_{k+1,i+1}-T_{k,i}} \right).
\]
Now, it is shown in [3] that, for each \( \lambda \in \Omega \), the equation \( Hf = \sum_i \lambda_i^2 f \) has a unique solution \( f = f_\lambda \) such that \( e^{-\lambda(x)} f_\lambda(x) \) is bounded and \( \lim_{x \to +\infty} e^{-\lambda(x)} f_\lambda(x) = 1 \), where we write \( x \to +\infty \) to mean \( \alpha_i(x) = x_i - x_{i+1} \to +\infty \) for each \( i \). Moreover, by Feynman-Kac,

\[
f_\lambda(x) = e^{\lambda(x)E_x} \exp \left( -\sum_{i=1}^{n-1} \int_0^\infty e^{-\alpha_i(\beta_s)} ds \right),
\]

where \( \beta_s \) is a Brownian motion in \( \mathbb{R}^n \) with drift \( \lambda \) as in the previous section. The relation between the functions \( f_\lambda \) and the Whittaker functions \( \psi_\lambda \) is thus determined by the following proposition.

**Proposition 2.** For \( \lambda \in \Omega \),

\[
\lim_{x \to +\infty} e^{-\lambda(x)} \psi_\lambda(x) = \prod_{i<j} \Gamma(\lambda_i - \lambda_j).
\]

**Proof.** We prove this by induction on \( n \) using the recursion [4]. Write \( \psi_\lambda = \psi_\lambda^{(n)} \) as before, setting \( \psi_\lambda^{(1)}(x) = e^{\lambda x} \). Then \( e^{-\lambda(x)} \psi_\lambda^{(1)}(x) = 1 \) and, for \( n \geq 2 \),

\[
e^{-\lambda(x)} \psi_\lambda^{(n)}(x) = \int_{\mathbb{R}^{n-1}} \exp \left( -\sum_{i=1}^{n} \lambda_i x_i + \lambda_n \left( \sum_{i=1}^{n} x_i - \sum_{i=1}^{n-1} y_i \right) - \sum_{i=1}^{n-1} \left( e^{y_i-x_i} + e^{x_{i+1}-y_i} \right) \right) \times \psi_{\lambda_1,\ldots,\lambda_{n-1}}^{(n-1)}(y_1, \ldots, y_{n-1}) dy_1 \ldots dy_{n-1} = \int_{\mathbb{R}^{n-1}} e^{-\sum_{i=1}^{n-1} (\lambda_i - \lambda_n) y_i} \exp \left( -\sum_{i=1}^{n-1} e^{y_i} - \sum_{i=1}^{n-1} e^{x_{i+1}-x_i-y_i} \right) \times e^{-\sum_{i=1}^{n-1} \lambda_i (x_i+y_i)} \psi_{\lambda_1,\ldots,\lambda_{n-1}}^{(n-1)}(x_1+y_1, \ldots, x_{n-1}+y_{n-1}) dy_1 \ldots dy_{n-1}.
\]

By induction, we immediately conclude that, for each \( n \), if \( x, \lambda \in \Omega \) then \( e^{-\lambda(x)} \psi_\lambda^{(n)}(x) \leq \prod_{i<j} \Gamma(\lambda_i - \lambda_j) \).

Here we are using

\[
\int_{\mathbb{R}^{n-1}} e^{-\sum_{i=1}^{n-1} (\lambda_i - \lambda_n) y_i} \exp \left( -\sum_{i=1}^{n-1} e^{y_i} - \sum_{i=1}^{n-1} e^{x_{i+1}-x_i-y_i} \right) dy_1 \ldots dy_{n-1}
\]

\[
\leq \int_{\mathbb{R}^{n-1}} e^{-\sum_{i=1}^{n-1} (\lambda_i - \lambda_n) y_i} \exp \left( -\sum_{i=1}^{n-1} e^{y_i} \right) dy_1 \ldots dy_{n-1} = \prod_{i=1}^{n-1} \Gamma(\lambda_i - \lambda_n).
\]

It follows, again by induction and using the dominated convergence theorem, that (10) holds for \( \lambda \in \Omega \).

**Corollary 1.** For \( \lambda \in \Omega \),

\[
\psi_\lambda(x) = \prod_{i<j} \Gamma(\lambda_i - \lambda_j) e^{\lambda(x)E_x} \exp \left( -\sum_{i=1}^{n-1} \int_0^\infty e^{-\alpha_i(\beta_s)} ds \right).
\]

**Corollary 2.** For \( x, \lambda \in \Omega \),

\[
J_\lambda(x) = \lim_{\beta \to \infty} \beta^{-n(n-1)/2} \psi_\lambda(\beta x).
\]

**Proof.** By Proposition 1 the statement is equivalent to

\[
\lim_{\beta \to \infty} \beta^{-n(n-1)/2} \psi_\lambda(\beta x) = h(\lambda)^{-1} e^{\lambda(x)P(T = \infty)}.
\]

This follows directly from (10) by Brownian re-scaling.

As shown in [3], the function \( \psi_\lambda \), which can be defined by (10), is a class-one Whittaker function, as defined by Jacquet [27] and Hashizume [25]. In the notation of the paper [3] we are taking \( \Pi = \{ \alpha_i/2, i = 1, \ldots, n-1 \} \), \( m(2\alpha_i) = 0 \), \( |\eta_\alpha|^2 = 1 \) and \( \psi_\mu(x) = 2^q k_\mu(x) \) where \( q = n(n-1)/2 \). In the paper [24], the relationship between Givental integral formula and a recursive integral formula due to Stade [52] based on Jacquet’s definition (see also [26]) is described.
Givental’s integral formula \((7)\) has a very similar structure to the formula \((3)\). Indeed, if we define the type of an array \((T_{k,i}, 1 \leq i \leq k \leq n)\) to be the vector

\[
type T = \left( T_{1,1}, T_{2,1} + T_{2,2} - T_{1,1}, \ldots, \sum_{j=1}^{n} T_{n,j} - \sum_{j=1}^{n-1} T_{n-1,j} \right),
\]

and a measure

\[
g(dT) = \prod_{k=1}^{n-1} \prod_{i=1}^{k} e^{-e^{T_{k,i} - T_{k+1,i}}} e^{-e^{T_{k,i+1} - T_{k+1,i}}} dT_{k,i} = e^{F_0(T)} \prod_{k=1}^{n-1} \prod_{i=1}^{k} dT_{k,i},
\]

then

\[
\psi_\lambda(x) = \int_{\Gamma(x)} e^{\lambda \cdot \text{type } T} g(dT).
\]

On the other hand, if we replace the functions \(e^{-e^{x-y}}\) in the reference measure \(g\) by the indicator functions \(1_{x<y}\) to get a new reference measure

\[
g_0(dT) = \prod_{k=1}^{n-1} \prod_{i=1}^{k} 1_{T_{k,i} < T_{k+1,i}}, 1_{T_{k,i+1} < T_{k,i}},
\]

then \((3)\) can be written as

\[
J_\lambda(x) = \int_{\Gamma(x)} e^{\lambda \cdot \text{type } T} g_0(dT).
\]

We note the following. If \(\lambda \in i\mathbb{R}^n\) then \(\psi_\lambda(x) = \psi_{-\lambda}(x)\); if \(\lambda \in i\mathbb{R}^n\) and \(\nu \in \mathbb{R}^n\), then \(|\psi_{\lambda+\nu}(x)| \leq \psi_\lambda(x)\). For each \(x \in \mathbb{R}^n\), \(\psi_\lambda(x)\) is an entire, symmetric function of \(\lambda \in \mathbb{C}^n\) \([21, 25, 31]\). There is a Plancherel theorem \([55, 49, 21, 31]\) which states that the integral transform

\[
(11) \quad \hat{f}(\lambda) = \int_{\mathbb{R}^n} f(x) \psi_\lambda(x) dx
\]

is an isometry from \(L_2(\mathbb{R}^n, dx)\) onto \(L^{sym}_2(i\mathbb{R}^n, s_n(\lambda) d\lambda)\), where \(L^{sym}_2\) is the space of \(L_2\) functions which are symmetric in their variables, \(\iota = \sqrt{-1}\) and \(s_n(\lambda) d\lambda\) is the *Sklyanin measure* defined by

\[
(12) \quad s_n(\lambda) = \frac{1}{(2\pi i)^{n/2}} \prod_{j \neq k} \Gamma(\lambda_j - \lambda_k)^{-1}.
\]

For \(x, \mu \in \mathbb{R}^n\), denote by \(\sigma^\mu_\nu\) the probability measure on the set of real triangular arrays \((T_{k,i})_{1 \leq i \leq k \leq n}\) defined by

\[
\int f d\sigma^\mu_\nu = \psi_\mu(x)^{-1} \int_{\Gamma(x)} f(T) e^{F_\nu(T)} \prod_{k=1}^{n-1} \prod_{i=1}^{k} dT_{k,i}.
\]

Define a probability measure \(\gamma^\mu_\nu\) by

\[
\int_{\mathbb{R}^n} e^{\lambda \cdot \psi^\mu_\nu(x)} dy = \frac{\psi_{\mu+\lambda}(x)}{\psi_\mu(x)}, \quad \lambda \in \mathbb{C}^n.
\]

The probability measure \(\gamma^\mu = \gamma^\mu_0\) is the analogue of the (normalised) Duistermaat-Heckman measure in this setting. The integral operator \(K\) with kernel \(K(x, dy) = \psi_0(x) \gamma^\mu(dy)\) satisfies the intertwining relation \(HK = K\Delta\). We can write \(K(x, dy) = k(x, y) \rho_x(dy)\), where \(k\) is a smooth kernel from \(\mathbb{R}^n\) to \(\mathbb{R}^n_2 = \{y \in \mathbb{R}^n : \sum_i y_i = \sum_i x_i\}\) and \(\rho_x\) denotes the Euclidean measure on \(\mathbb{R}^n_2\). For \(n = 2\),

\[
k(x, y) = \exp(-e^{x_2-y_1} - e^{y_1-x_1})
\]

and, for \(n = 3\),

\[
k(x, y) = \psi_0^{(2)}(a, b) = 2K_0(2e^{(b-a)/2})
\]

where

\[
e^{-a} = e^{x_3-y_1-y_2} + e^{-x_1}, \quad e^b = e^{y_1} + e^{y_2} + e^{y_1+y_2-x_2} + e^{x_2},
\]

and \(K_0\) denotes the Macdonald function with index 0.
3. Interpretation of $\gamma^x$ in terms of Brownian motion

A reduced decomposition of an element $w \in S_n$ is a minimal expression of $w$ as a product of adjacent transpositions, that is, $w = s_{i_1} \ldots s_{i_r}$, where $s_i$ denotes the transposition $(i, i+1)$. We will also refer to the word $i = i_1 i_2 \ldots i_r$ as a reduced decomposition. By definition, any reduced decomposition has the same length $l(w)$, defined to be the length of $w$. There is a unique longest element in $S_n$, namely the permutation $w_0 = \left(\begin{array}{cccc}1 & 2 & \cdots & N \\ N & N-1 & \cdots & 1\end{array}\right)$.

Its length is $n(n-1)/2$, as can be seen by taking the reduced decomposition $i = 1 \ 21 \ 321 \ \ldots \ n \ n-1 \ \ldots \ 21$.

The symmetric group acts on $\mathbb{R}^n$ by permutation of coordinates, and as such is an example of a finite reflection group. It is generated by the hyperplane reflections $s_i$. Let $w$ depend only on $\alpha$ and $\eta$ in the coordinates $\mathbb{R}$.

The evolution of the triangular array $T_{k,j}$, $1 \leq j \leq k \leq n$ is given recursively as follows: $dT_{1,1} = d\eta^1$ and, for $k \geq 2$,

$$dT_{k,1} = dT_{k-1,1} + e^{T_{k-1,1}} dt$$
$$dT_{k,2} = dT_{k-1,2} + (e^{T_{k-1,2}} - e^{T_{k-1,1}}) dt$$
$$\vdots$$
$$dT_{k,k-1} = dT_{k-1,k-1} + (e^{T_{k-1,k-1}} - e^{T_{k-1,k-2}}) dt$$
$$dT_{k,k} = d\eta^k - e^{T_{k-1,k-1}} dt.$$}

The process, which is clearly Markov, contains a number of projections which are also Markov. For example, setting $\xi_k = T_{k,k}$, we have, for $k \leq n$,

$$d\xi_k = d\eta^k - e^{\xi_{k-1}} dt.$$}

This defines a simple interacting particle system on the real line, which has very nice properties. For example, in the coordinates $\sum_i \xi_i$ and $\xi_{i+1} - \xi_i$, $1 \leq i \leq n-1$, it has a product form invariant measure, that is, a product measure which is invariant.

A remarkable fact is that each row in the pattern $T_{k,j}$ is a Markov process with respect to its own filtration. This gives an interpretation of the measures $\gamma_\mu^x$ and $\sigma_\mu^x$ defined in the previous section.

**Theorem 1.** 411 $T_{w_0} \eta(t)$, $t > 0$ is a diffusion process in $\mathbb{R}^n$ with infinitesimal generator

$$\mathcal{L}_\mu = \frac{1}{2} \psi_\mu^{-1} \left(H - \sum_{i=1}^n \mu_i^2 \right) \psi_\mu = \frac{1}{2} \Delta + \nabla \log \psi_\mu \cdot \nabla.$$
For each $t > 0$, the conditional law of $\{T_{k,j}(t), \ 1 \leq j \leq k \leq n\}$, given $\{T_{w_0}\eta(t), \ s \leq t; \ T_{w_0}\eta(t) = x\}$, is $\sigma^x_t$ and the conditional law of $\eta(t)$, given $\{T_{w_0}\eta(t), \ s \leq t; \ T_{w_0}\eta(t) = x\}$, is $\gamma^x_t$. The law of $T_{w_0}\eta(t)$ is given by

$$\nu_t^\eta(dx) = e^{-\sum_{i} \lambda_i^2 t / 2} \psi_i(x) \theta_i(x) dx,$$

where

$$\theta_i(x) = \int_{\mathbb{R}^n} \psi_{\lambda}(x) e^{\sum_{i} \lambda_i^2 t / 2} s_i(\lambda) d\lambda.$$  \hfill (14)

In the case $n = 2$, this is equivalent to a theorem of Matsumoto and Yor \[37\]. Write $L = L_0$ and $\nu = \nu_0^I$. The diffusion with generator $L$ is the analogue of Dyson’s Brownian motion in this setting and the measures $\nu_t$ and $\theta_t$ (the latter requires normalization) are the analogues of the Gaussian unitary and Gaussian orthogonal ensembles, respectively. The diffusion with generator $L_\mu$ was introduced in \[3\]. When $\mu \in \mathbb{P}$, it can be interpreted as a Brownian motion in $\mathbb{R}^n$ killed according to the potential $\sum_i e^{x_i^2 - x_i}$ and then conditioned to survive forever \[29, 30\]. The path-transformation $T_{w_0}$ is closely related to the geometric (lifting of) the RSK correspondence introduced by A.N. Kirillov \[34\] and studied further by Noumi and Yamada \[40\]. A discrete-time version of the above theorem, which works directly in the setting of the geometric RSK correspondence, is given in \[14\]. In the discrete-time setting the Whittaker functions continue to play a central role. See also \[8, 10, 13, 24, 43, 46\] for further related developments.

### 4. Application to random polymers

The following model was introduced in \[44\]. The environment is given by a sequence $B_1, B_2, \ldots$ independent standard 1-dim Brownian motions. For up/right paths $\phi \equiv \{0 < t_1 < \ldots < t_{n-1} < t\}$ (as shown in Figure 1), define

$$E(\phi) = B_1(t_1) + B_2(t_2) - B_2(t_1) + \cdots + B_N(t) - B_N(t_{n-1}),$$

$$P(d\phi) = Z_n^\eta(\beta)^{-1} e^{\beta E(\phi)} d\phi, \quad Z_n^\eta(\beta) = \int e^{\beta E(\phi)} d\phi.$$

Set $X^\eta_t(t) = \log Z_n^\eta$ and, for $k = 2, \ldots, n$,

$$X^\eta_t(t) + \cdots + X^\eta_t(t) = \log \int e^{E(\phi_1) + \cdots + E(\phi_k)} d\phi_1 \cdots d\phi_k,$$

where the integral is over non-intersecting paths $\phi_1, \ldots, \phi_k$ from $(0, 1), \ldots, (0, k)$ to $(t, n-k+1), \ldots, (t, n)$.

Let $\eta = (B_n, \ldots, B_1)$. Then $X = T_{w_0}\eta$ and the following holds.

**Theorem 2.** \[41\] The process $X(t), t > 0$ is a diffusion in $\mathbb{R}^n$ with infinitesimal generator $L$. The distribution of $X(t)$ is given by $\nu_t$. For $s > 0$,

$$E e^{-sZ^\eta_t} = \int s^{-\sum \lambda_i} \prod_i \Gamma(\lambda_i) e^{\frac{s}{\sum_i \lambda_i^2 t_s} s_i(\lambda)} d\lambda,$$

where the integral is along (upwards) vertical lines with $\Re \lambda_i > 0$ for all $i$.  

\[\text{Figure 1. An up/right path } \phi \equiv \{0 < t_1 < \ldots < t_{n-1} < t\}.\]
The free energy for this model is given by \cite{44, 39}

$$\lim_{n \to \infty} \frac{1}{n} \log Z_n^w = \inf_{t > 0} [t - \Psi(t)],$$

almost surely, where $\Psi(z) = \Gamma'(z)/\Gamma(z)$. The conjectured KPZ scaling behaviour for the fluctuations of $\log Z_n^w$ was (essentially) established by Seppäläinen and Valkó \cite{50}; more recently, Borodin, Corwin and Ferrari \cite{8, 9} have proved the full KPZ universality conjecture for this model, namely that $\log Z_n^w$, suitably centered and rescaled, converges in law to the Tracy-Widom $F_2$ distribution of random matrix theory. See also \cite{51}.

5. Reduced double Bruhat cells and their parameterisations

The Weyl group associated with $GL(n)$ is the symmetric group $S_n$. Each element $w \in S_n$ has a representative $\bar{w} \in GL(n)$ defined as follows. Denote the standard generators for $\mathfrak{g}_n$ by $h_i, e_i$ and $f_i$. For example, for $n = 3$,

$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad h_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad f_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

For adjacent transpositions $s_i = (i, i + 1)$, define

$$\bar{s}_i = \exp(-e_i) \exp(f_i) \exp(-e_i) = (I - e_i)(I + f_i)(I - e_i).$$

In other words, $\bar{s}_i = \varphi_i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ where $\varphi_i$ is the natural embedding of $SL(2)$ into $GL(n)$ given by $h_i, e_i$ and $f_i$. For example, when $n = 3$,

$$\bar{s}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \bar{s}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$ 

Now let $w = s_{i_1} \ldots s_{i_r}$ be a reduced decomposition and define $\bar{w} = \bar{s}_{i_1} \ldots \bar{s}_{i_r}$. Note that $\bar{uv} = \bar{u} \bar{v}$ whenever $l(uv) = l(u) + l(v)$. For $n = 2$, $w_0 = s_1$ and

$$\bar{w}_0 = \bar{s}_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

For $n = 3$, $w_0 = s_1 s_2 s_1 = s_2 s_1 s_2$ is represented by

$$\bar{w}_0 = \bar{s}_1 \bar{s}_2 \bar{s}_1 = \bar{s}_2 \bar{s}_1 \bar{s}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Denote the upper (respectively lower) triangular matrices in $GL(n)$ by $B$ and $B_-$, and the upper (respectively lower) uni-triangular matrices in $GL(n)$ by $N$ and $N_-$. The group $GL(n)$ has two Bruhat decompositions

$$GL(n) = \bigcup_{u \in S_n} B \bar{u}B = \bigcup_{v \in S_n} B_- \bar{v}B_-.$$

The double Bruhat cells $G_{u,v}$ are defined, for $u, v \in S_n$, by

$$G_{u,v} = B \bar{u}B \cap B_- \bar{v}B_-.$$

The reduced double Bruhat cells $L_{u,v}$ are defined by

$$L_{u,v} = N \bar{u}N \cap B_- \bar{v}B_-.$$

We also define the opposite reduced double Bruhat cells $M_{u,v}$ by

$$M_{u,v} = B \bar{u}B \cap N_- \bar{v}N_-.$$
The reduced double Bruhat cell $L^{w,e}$ (where $e$ denotes the identity in $S_n$) admits the following parameterisations, one for each reduced decomposition of $w$. See [36, 3, 4, 16]. Set

$$Y_i(u) = \varphi_i \begin{pmatrix} u & 0 \\ 1 & u^{-1} \end{pmatrix}, \quad i = 1, \ldots, n - 1.$$  

Then, for any reduced decomposition $i = i_1 \ldots i_r$ of $w$, the map

$$(u_1, \ldots, u_r) \mapsto Y_{i_1}(u_1) \cdots Y_{i_r}(u_r)$$

defines a bijection between $\mathbb{C}_r^{r \neq 0}$ and $L^{w,e}$. This bijection has the property that the totally positive part $L^{w,v}$ of $L^{u,v}$ corresponds precisely to the subset $\mathbb{R}^r_{> 0}$ of $\mathbb{C}_r^{r \neq 0}$. There are explicit transition maps which relate the parameters $(u_1, \ldots, u_r)$ corresponding to different reduced decompositions of $w$.

In the case $n = 3$, the two representations of an element in $L^{w,v}$ corresponding to the words 121 and 212, denoting the corresponding parameters by $(u_1, u_2, u_3)$ and $(u'_1, u'_2, u'_3)$, respectively, are given by

$$
\begin{pmatrix}
u_1 u_3 \\
u_3 + u_2/u_1 & 0 & 0 \\
1 \\
\end{pmatrix}
\begin{pmatrix}
u_1' u_3' \\
u_3' + u_2'/u_1' & 0 & 0 \\
1 \\
\end{pmatrix} =
\begin{pmatrix}
u_1 u_3 & 0 & 0 \\
1 \\
\nu_3 + u_2/u_1 & 0 & 0 \\
1 \\
\end{pmatrix}
\begin{pmatrix}
u_1' u_3' & 0 & 0 \\
1 \\
\nu_3' + u_2'/u_1' & 0 & 0 \\
1 \\
\end{pmatrix}.
$$

The transition maps are given by

$$(v_1, \ldots, v_r) \mapsto X_{i_1}(v_1) \cdots X_{i_r}(v_r)$$

defines a bijection between $\mathbb{C}_r^{r \neq 0}$ and $M^{w,e}$. This bijection also has the property that the totally positive part $M^{w,v}$ of $M^{w,v}$ corresponds precisely to the subset $\mathbb{R}^r_{> 0}$ of $\mathbb{C}_r^{r \neq 0}$.

In the case $n = 3$, the two representations of an element in $M^{w,v}$ corresponding to the words 121 and 212, denoting the corresponding parameters by $(v_1, v_2, v_3)$ and $(v'_1, v'_2, v'_3)$, respectively, are given by

$$
\begin{pmatrix}
u_1 + v_3 \\
1 \\
\nu_2 + v_3 \\
\end{pmatrix}
\begin{pmatrix}
u_1' + v_3' \\
1 \\
\nu_2' + v_3' \\
\end{pmatrix} =
\begin{pmatrix}
u_1 + v_3 & 0 & 0 \\
1 \\
\nu_2 + v_3 & 0 & 0 \\
1 \\
\end{pmatrix}
\begin{pmatrix}
u_1' + v_3' & 0 & 0 \\
1 \\
\nu_2' + v_3' & 0 & 0 \\
1 \\
\end{pmatrix},
$$

with transition maps

$$(v_1, v_2, v_3) \mapsto \frac{v_2 v_3}{v_1 + v_3}, \quad v'_1 = v_1 + v_3, \quad v'_2 = v_1 v_3, \quad v'_3 = \frac{v_1 v_2}{v_1 + v_3}.$$  

We conclude this section with a simple lemma. Let $b \in G^{e,w}$ and write $b = an$ where $a = \text{diag} (a_1, \ldots, a_n)$, say, and $n \in N$. Then [16], for any $w \in S_n$, $b\bar{w}$ has a Gauss (or LDU) decomposition $b\bar{w} = [b\bar{w}]_- [b\bar{w}]_0 [b\bar{w}]_+$ and $n\bar{w}$ has a Gauss decomposition $n\bar{w} = [n\bar{w}]_- [n\bar{w}]_0 [n\bar{w}]_+$. Moreover, $[n\bar{w}]_- = [n\bar{w}]_- [n\bar{w}]_0 \in L^{w,v}$ and $[b\bar{w}]_- \in M^{w,e}$. Let $i = i_1 \ldots i_r$ be a reduced decomposition for $w$. Then we can write

$$[n\bar{w}]_- = Y_{i_1}(u_1) \cdots Y_{i_r}(u_r), \quad [b\bar{w}]_- = X_{i_1}(v_1) \cdots X_{i_r}(v_r).$$

Define $Z_i(u) = \varphi_i \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$. Set $a^0 = a$ and, for $1 \leq k \leq r$, $a^k = a^{k-1} Z_{i_k}(u_k)$. Write $a^k = \text{diag} (a_1^k, \ldots, a_n^k)$.

**Lemma 1.** The following relations holds: $[b\bar{w}]_0 = a^r$ and, for $k = 1, \ldots, r$, $v_k = u_k^{-1} a_{i_k+1}^{k-1}/a_{i_k}^{k-1}$.

**Proof.** Note that $an\bar{w} = [b\bar{w}]_- = [b\bar{w}]_- [b\bar{w}]_0$, hence

$$a Y_{i_1}(u_1) \cdots Y_{i_r}(u_r) = X_{i_1}(v_1) \cdots X_{i_r}(u_r) [b\bar{w}]_0.$$  

The result follows by repeated application of the identity $a Y_i(u) = X_i(u) a'$, where $a' = a Z_i(u)$ and $v = u^{-1} a_{i+1}/a_i$.  

$\square$
6. An evolution on upper triangular matrices

As shown in [6], the path-transformations $T_w \eta$ can also be represented in terms of an evolution on the upper triangular matrices in $GL(n, \mathbb{R})$. Let $w = s_{i_1} \cdots s_{i_r}$ be a reduced decomposition and $\eta : (0, \infty) \to \mathbb{R}^n$ a continuous path. Set $\eta_0 = \eta$ and, for $k \leq r$,

$$\eta_k = T_{i_k} \cdots T_{i_1} \eta \quad x_k(t) = \log \int_0^t e^{-\alpha_k(\eta_{k-1}(s))} ds.$$  

Then $\eta_r = T_w \eta$ and, for each $k \leq r$, $\eta_k = \eta + \sum_{j=1}^k x_j \alpha_j$.

Write $\eta(t) = (\eta^n_1, \ldots, \eta^n_n)$. Define a path $b(t)$ taking values in $B$ by

$$b_{ij}(t) = e^{\eta^n_i(t)} \int_{0<s_{j-1}<s_j<...<s_i<t} \exp \left( -\sum_{k=i}^{j-1} \alpha_k(\eta(s_k)) \right) ds_i \cdots ds_{j-1}.$$  

If $\eta$ is smooth, the $b$ satisfies the ordinary differential equation

$$db = \left( \sum_{i=1}^n h_i dt^i + \sum_{i=1}^{n-1} e_i dt \right) b, \quad b(0) = I.$$  

If $\eta$ is a Brownian path (as in the next section) then $b$ satisfies the equation interpreted as a Stratonovich SDE.

When $n = 2$,

$$db = \left( \begin{array}{c} d\eta^1 \\ dt \\ \frac{dt^2}{d\eta^2} \end{array} \right) b, \quad b(t) = \left( \begin{array}{c} e^{\eta^1} \\ 0 \\ e^{\eta^2} \end{array} \right), \quad \int_0^t e^{\eta^1_s - \eta^1_t + \eta^2_t} ds.$$  

When $n = 3$,

$$db = \left( \begin{array}{c} d\eta^1 \\ dt \\ dt^2 \\ d\eta^3 \end{array} \right) b, \quad b(t) = \left( \begin{array}{c} e^{\eta^1} \\ \int_0^t e^{\eta^2_s - \eta^1_s + \eta^3_s} ds \\ \int_0^t e^{\eta^2_s + \eta^2_s - \eta^3_s} ds \\ 0 \\ e^{\eta^3} \end{array} \right).$$  

and the solution is given by

$$b(t) = \left( \begin{array}{cc} e^{\eta^1} & \int_0^t e^{\eta^2_u - \eta^1_u + \eta^3_u} du \\ 0 & e^{\eta^3} \end{array} \right) \left( \int_0^t e^{\eta^2_s - \eta^1_s + \eta^3_s} ds \right).$$

Write $b = an$, where $a = \text{diag}(e^{\eta^1}, \ldots, e^{\eta^n})$ and $n \in N$. Set $u_k = e^{x_k}$ and $v_k = e^{-x_k - \alpha_k(\eta_{k-1})}$.

**Theorem 3.** [6, 7] For each $t > 0$, $b(t)w$ has a Gauss decomposition $bw = [bw] - [bw]_0 [bw]_+ + \text{exp}(T_w \eta(t))$. Moreover, $[bw]_0 = Y_{i_1}(u_{i_1}) \cdots Y_{i_r}(u_{i_r}) \in L_{>0}^{w,e}$.

By Lemma[7] we also have $[bw]_+ = X_i(u_{i_1}) \cdots X_i(u_{i_r}) \in M_{>0}^{w,e}$.

**6.1. The case $n = 2$.** From the definitions: $\alpha_1 = e_1 - e_2$, $w_0 = s_{i_1} = s_{i_1} - e_2$,

$$u := u_1 = e^{x_1} = \int_0^t e^{-\eta^1_s + \eta^2_s} ds \quad v := v_1 = e^{-\eta^1_t} = e^{-\eta^1_t + \eta^2_t} u^{-1}$$

$$e^{T_{w_0} \eta} = (e^{\eta^1} u, e^{\eta^2} u^{-1}) = \left( \int_0^t e^{\eta^2_s + \eta^1_s - \eta^1_s} ds, \int_0^t e^{-(\eta^1_s + \eta^2_s - \eta^2_s)} ds \right).$$

$$b = \left( \begin{array}{c} e^{\eta^1} \\ 0 \\ e^{\eta^2} \end{array} \right), \quad \left( \begin{array}{c} e^{\eta^1} \\ 0 \\ e^{\eta^2} \end{array} \right) = \left( \begin{array}{c} e^{\eta^1} \\ 0 \\ e^{\eta^2} \end{array} \right), \quad \left( \begin{array}{c} 1 \\ u \\ 1 \end{array} \right) = an$$

Taking $\bar{w}_0 = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right)$, we see that

$$b\bar{w}_0 = \left( \begin{array}{c} e^{\eta^1} u - e^{\eta^1} \\ e^{\eta^2} \\ 0 \end{array} \right) = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \left( \begin{array}{c} e^{\eta^1} u \\ 0 \end{array} \right) \left( \begin{array}{c} 1 \\ -u^{-1} \end{array} \right)$$

and

$$nw_0 = \left( \begin{array}{cc} 1 & u \\ 0 & 1 \end{array} \right) \left( \begin{array}{c} 0 \\ -1 \end{array} \right) = \left( \begin{array}{c} u \\ 1 \end{array} \right) = \left( \begin{array}{c} u \\ 0 \end{array} \right) \left( \begin{array}{c} 1 \\ -u^{-1} \end{array} \right) \left( \begin{array}{c} 1 \\ 0 \end{array} \right)$$
Hence
\[ [b\tilde{w}]_0 = e^{T_{w_0}} \eta, \quad [b\tilde{w}]_\infty = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} = X_1(v), \quad [n\tilde{w}]_0 = \begin{pmatrix} u & 0 \\ 1 & u^{-1} \end{pmatrix} = Y_1(u) \]
as claimed.

### 6.2. The case \( n = 3 \)

From the definitions:
\[ \alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - e_3, \quad w_0 = s_1 s_2 s_1 = s_2 s_1 s_2. \]

For the reduced decomposition \( w_0 = s_1 s_2 s_1 \), we have
\[ u_1 = e^{x_1} = \int_0^t e^{-\eta^1(s) + \eta^2(s)} ds, \quad v_1 = e^{y_1} = e^{\eta^1 u_1}, \]
\[ u_2 = e^{x_2} = \int_0^t e^{-\eta^1(s) + \eta^2(s)} ds, \quad v_2 = e^{y_2} = e^{\eta^2 u_2/u_1}, \]
\[ u_3 = e^{x_3} = \int_0^t e^{-\eta^2(s) + \eta^3(s)} ds, \quad v_3 = e^{y_3} = e^{-\eta^1 + \eta^2} u_2/u_1. \]

\[ b = \left( \begin{array}{cccc}
\eta^2 & \int_0^t e^{-\eta^1(s) + \eta^2(s)} ds & \int_0^t e^{-\eta^1(s) + \eta^2(s)} ds & \int_0^t e^{-\eta^1(s) + \eta^2(s) + \eta^3(s)} ds \\
0 & e^{\eta^2} & 0 & e^{\eta^3} \\
0 & 0 & e^{\eta^3} & 0 \\
0 & 0 & 0 & e^{\eta^3} \\
\end{array} \right) = \begin{pmatrix} u_1 & u_1 u_3 \\
0 & u_3 + u_2/u_1 \end{pmatrix} = an. \]

The identity
\[ \int_0^t e^{\eta^1 u_3 - \eta^2 u_2 + \eta^3 u_1} ds = u_3 + u_2/u_1 \]
follows from (15). Now,
\[ \tilde{w}_0 = \begin{pmatrix} 0 & 0 & 1 \\
0 & -1 & 0 \\
1 & 0 & 0 \end{pmatrix}, \]
so we have
\[ b\tilde{w}_0 = \begin{pmatrix} \eta^2 u_3 & -\eta^1 u_1 & e^{\eta^1} \\
e^{-\eta^1 u_3} u_1 + u_2/u_1 & -e^{\eta^2} & 0 \\
e^{\eta^3} & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\
v_1 + v_3 & 0 & 0 \\
v_2 v_3 & v_2 & 1 \end{pmatrix} \begin{pmatrix} u_1 u_3 & -u_1 & 1 \\
0 & u_3 + u_2/u_1 & -1 \\
0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1/u_3 & 1/u_1 u_3 \\
0 & 1 & -u_3 u_2 - 1/u_1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 \end{pmatrix} \]

and
\[ n\tilde{w}_0 = \begin{pmatrix} u_1 u_3 & -u_1 & 1 \\
0 & u_3 + u_2/u_1 & -1 \\
0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 u_3 & 0 & 0 \\
0 & u_2/u_1 u_3 & 0 \\
1 & 1/u_3 & 1/u_2 \end{pmatrix} \begin{pmatrix} 1 & -1/u_3 & 1/u_1 u_3 \\
0 & 1 & -u_3 u_2 - 1/u_1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 \end{pmatrix} \]
Thus \([b\tilde{w}_0]_0 = e^{Tw_0}\eta\),

\[
[b\tilde{w}_0]_- = \begin{pmatrix}
1 & 0 & 0 \\
v_1 + v_3 & 1 & 0 \\
v_2v_3 & v_2 & 1
\end{pmatrix} = X_1(v_1)X_2(v_2)X_3(v_3),
\]

\[
[u\tilde{w}_0]_- = \begin{pmatrix}
u_1u_3 \\
u_3 + u_2/u_1 \\
u_2/u_1u_3 \end{pmatrix} = Y_1(u_1)Y_2(u_2)Y_3(u_3)
\]

as claimed.

### 6.3. Evolution of the Lusztig parameters

As before, we introduce a probability measure \(\mathbb{P}\) under which \(\eta\) is a Brownian motion in \(\mathbb{R}^n\) with a drift \(\mu\) and \(\eta(0) = 0\). For each \(k \leq n\), set

\((T_{k,1}, \ldots, T_{k,k}) = T^{(k)}(\eta^1, \ldots, \eta^k)\).

Note that this is given in terms of the principal minors \(b^{(k)}\), \(k \leq n\), of \(b\) by

\(T^{(k)}(\eta^1, \ldots, \eta^k) = \log[b^{(k)} w_0^{(k)}]_0\),

where \(w_0^{(k)}\) denotes the longest element in \(S_k\). The evolution of the triangular array \(T_{k,j}\), \(1 \leq j \leq k \leq n\) is given by [13]. As remarked earlier, this process contains a number of projections which are also Markov. In particular, setting \(\xi_k = T_{k,k}\), we have, for \(k \leq n\),

\[d\xi_k = d\eta^k - e^{\xi_{k-1} - \xi_k} dt.\]

This defines a simple interacting particle system on the real line which, in the coordinates \(\sum_i \xi_i\) and \(\xi_{i+1} - \xi_i\), \(1 \leq i \leq n - 1\), has a product form invariant measure. There is an extension of this process, involving the Lusztig parameters, which is also Markov and, moreover, also has a product form invariant measure. Let \(v_1, \ldots, v_q\) be the Lusztig parameters corresponding to a reduced decomposition \(w_0 = s_{i_1} \ldots s_{i_q}\), that is,

\([b\tilde{w}_0]_- = X_{i_1}(v_1) \cdots X_{i_q}(v_q)\).

Set \(y_k = -\log v_k\). The evolution of \(y_k\), \(1 \leq k \leq q\), is given by

\[dy_k = d\alpha_{i_k}(\eta_{k-1}) + e^{-y_k} dt,
\]

where \(\eta_k = T_{ik} \cdots T_{i_1}\eta\). Setting \(x_k = y_k - \alpha_{i_k}(\eta_{k-1})\), note that \(dx_k = e^{-y_k} dt\) and \(\eta_k = \eta + \sum_{j=1}^{k} x_j \alpha_j\).

Hence,

\[dy_k = d\alpha_{i_k}(\eta) + \sum_{j=1}^{k-1} \alpha_{i_k}(\alpha_{i_j})e^{-y_j} dt + e^{-y_k} dt.
\]

Let \(\beta_1 = \alpha_{i_1}\) and, for \(2 \leq k \leq q\), \(\beta_k = s_{i_{k-1}} \ldots s_{i_1}\alpha_{i_k}\). Set \(\theta_k = -\beta_k(\mu)\). If \(\mu \in w_0\Omega = -\Omega\), then \(\theta_k > 0\) for all \(k\) and the diffusion has stationary distribution given by the product measure

\[\pi = \bigotimes_{k=1}^{q} \Gamma(\theta_k)^{-1} g_{\theta_k},\]

where \(g_{\theta}(dx) = \exp(-\theta x - e^{-x})dx\). This can be seen as a consequence of the following fact, which is the analogue in this setting of the output theorem for the \(M/M/1\) queue [44]. Let \(x_t\) be a standard one-dimensional Brownian motion with negative drift \(-\theta\), and consider the one-dimensional diffusion

\[dy = \sqrt{2} dx + e^{-y} dt.
\]

This has a unique invariant distribution \(\Gamma(\theta)^{-1} g_{\theta}\). If we start this diffusion in equilibrium and define \(\tilde{x}_t = x_t + 2(y_0 - y_t)\), then \(\tilde{x}\) has the same law as \(x\) and, moreover, \(\tilde{x}_s, s \leq t\) is independent of \(y_u, u \geq t\), for all \(t\). It follows that the measure \(\pi\) is invariant. For an analytic proof of this fact, see [45]. See also [7] Proposition 5.9, where the equivalent property is proved in the ‘zero-temperature’ setting.

If we choose the reduced decomposition \(i = 1 2 3 21 n - 1 n - 2 \ldots 21\), and define, for \(m \leq n - 1\) and \(1 \leq i \leq n - m\), \(q_{m,i} = T_{i+m,i+1} - T_{i+m-1,i}\), then

\[(y_1, y_2, \ldots, y_q) = (q_{1,1}, q_{1,2}, \ldots, q_{1,n}, q_{2,1}, \ldots, q_{2,n-1}, \ldots, q_{n-1,1}).\]
Note that $q_{1,i} = \xi_{i+1} - \xi_i$, for $1 \leq i \leq n - 1$. In these coordinates, the evolution is given by
\[
dq_{m,i} = d\alpha_i(\eta) + e^{-q_{m,i}}dt + \sum_{l=1}^{m-1} (2e^{-q_{l,i}} - e^{-q_{l,i+1}} - e^{-q_{l,i-1}})dt,
\]
with the conventions that the empty sum is zero and $q_{0,0} = +\infty$. Setting $\theta_{m,i} = \mu_{m+i} - \mu_m$, an invariant measure for this diffusion is given by the product measure $\otimes_{m,i} g_{\theta_{m,i}}$. The dynamics of this process can be viewed as a network, as follows. Consider the dynamics
\[
dQ = d(A - S) + e^{-Q}dt, \quad dD = dA - dQ, \quad dT = dS + dQ.
\]
We think of $A, S$ as the input and $D, T$ as the output, and represent this system graphically as:

```
A
S Q D
T
```

Then the evolution of the $q_{m,i}$ can be represented as in Figure 2. To see directly from this picture the product-form invariant measure, note that, if $A$ and $S$ are independent standard one-dimensional Brownian motions with respective drifts $\lambda$ and $\sigma$, with $\lambda < \sigma$, then the diffusion $Q$ has invariant distribution $\Gamma(\theta)^{-1}g_{\theta}$, where $\theta = \sigma - \lambda$. Moreover, if we start this diffusion in equilibrium, then $D_t = A_t + Q_0 - Q_t$ and $T_t = S_t - Q_0 + Q_t$ are independent standard one-dimensional Brownian motions with respective drifts $\lambda$ and $\sigma$, and for each $t > 0$, $(D_s, T_s), s \leq t$ is independent of $Q_u, u \geq t$. The analogue of this fact in the setting of Poissonian queueing networks is the cornerstone of classical queueing theory. It is called the output, or Burke, theorem. Finally, we remark that the dynamics indicated by Figure 2 is the analogue, in this setting, of the dynamical interpretation given in [42] of the RSK correspondence as a kind of ‘queueing network’.

```
\begin{tikzpicture}
    \node (q11) at (0,0) {$q_{1,1}$};
    \node (q12) at (1,1) {$q_{1,2}$};
    \node (q13) at (2,1) {$q_{1,3}$};
    \node (q14) at (3,1) {$q_{1,4}$};
    \node (eta1) at (-1,0) {$\eta^1$};
    \node (eta2) at (-1,1) {$\eta^2$};
    \node (eta3) at (-1,2) {$\eta^3$};
    \node (eta4) at (-1,3) {$\eta^4$};
    \node (eta) at (-1,0) {$\eta^1$};
    \node (eta2) at (-1,1) {$\eta^2$};
    \node (eta3) at (-1,2) {$\eta^3$};
    \node (eta4) at (-1,3) {$\eta^4$};
    \node (T1) at (0,-1) {$T_{2,1}$};
    \node (T2) at (1,-1) {$T_{3,1}$};
    \node (T3) at (2,-1) {$T_{4,1}$};
    \node (T4) at (3,-1) {$T_{4,2}$};
    \node (T5) at (4,-1) {$T_{4,3}$};
    \node (T6) at (5,-1) {$T_{4,4}$};
    \node (T7) at (6,-1) {$T_{2,2}$};
    \node (T8) at (7,-1) {$T_{3,2}$};
    \node (T9) at (8,-1) {$T_{4,2}$};
    \node (T10) at (9,-1) {$T_{4,3}$};
    \node (T11) at (10,-1) {$T_{4,4}$};
    \draw[->] (q11) -- (eta1);\draw[->] (eta1) -- (eta2);\draw[->] (eta2) -- (eta3);\draw[->] (eta3) -- (eta4);
    \draw[->] (eta4) -- (q14);\draw[->] (eta4) -- (T1);\draw[->] (T1) -- (q21);\draw[->] (eta4) -- (T2);\draw[->] (T2) -- (q22);\draw[->] (eta4) -- (T3);\draw[->] (T3) -- (q31);\draw[->] (eta4) -- (T4);\draw[->] (T4) -- (q41);
    \draw[->] (eta2) -- (T5);\draw[->] (T5) -- (q21);\draw[->] (eta2) -- (T6);\draw[->] (T6) -- (q22);\draw[->] (eta2) -- (T7);\draw[->] (T7) -- (q31);\draw[->] (eta2) -- (T8);\draw[->] (T8) -- (q41);
    \draw[->] (eta3) -- (T9);\draw[->] (T9) -- (q21);\draw[->] (eta3) -- (T10);\draw[->] (T10) -- (q22);\draw[->] (eta3) -- (T11);\draw[->] (T11) -- (q31);
\end{tikzpicture}
```

**Figure 2.** Graphical representation of the evolution of Lusztig parameters

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7. From the Feynman-Kac formula to Givental’s integral formula

The fact that the evolution equation [13] for the Lusztig parameters has a product form invariant measure sheds some light on the relation between the Feynman-Kac formula [10] and the integral formula of Givental. It follows from this that, for any given reduced decomposition of $\nu_0$, the random variables

$$\int_0^\infty e^{-\alpha_i(\beta_s)} \, ds, \quad i = 1, \ldots, n - 1$$

can be expressed, via the transition maps, as rational functions of a collection of $q = n(n-1)/2$ independent Gamma-distributed random variables with respective parameters $\theta_i$, $k \leq q$, defined as above with $\beta = -\eta$. Note that $\beta$ is a Brownian motion with drift $\lambda = -\mu \in \Omega$. Since the sets $\{\theta_k, k \leq q\}$ and $\{\lambda_i - \lambda_j, i < j\}$ are the same, this allows [10] to be written as a $q$-dimensional integral

$$\psi_\lambda(x) = \prod_{i<j} \Gamma(\lambda_i - \lambda_j) e^{\lambda_i(x)} \mathbb{E}_x \exp \left( -\sum_{i=1}^{n-1} \int_0^\infty e^{-\alpha_i(\beta_s)} \, ds \right)$$

$$\times \prod_{i=1}^q \psi_i^{\theta_i} e^{-\theta_1 v_1 \cdots v_q} \, dv_1.$$  \hspace{1cm} (19)

For example, when $n = 3$ and $i = 121$, we have

$$\theta_1 = \lambda_1 - \lambda_2, \quad \theta_2 = \lambda_1 - \lambda_3, \quad \theta_3 = \lambda_2 - \lambda_3,$$

and, using [10],

$$r_1(v_1, v_2, v_3) = \frac{1}{v_1}, \quad r_2(v_1, v_2, v_3) = \frac{1}{v_1^2} = \frac{v_1 + v_3}{v_2 v_3}.$$  

In this case, the integral formula (19) becomes

$$\psi_\lambda(x) = e^{\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3} \int_{\mathbb{R}^3} v_1^{-\lambda_1 - 1} v_2^{-\lambda_2 - 1} v_3^{-\lambda_3 - 1} \times \exp \left( -v_1 - v_2 - v_3 - e^{-v_1 + x_2} \frac{1}{v_1} - e^{-x_2 + x_3} \frac{v_1 + v_3}{v_2 v_3} \right) \, dv_1 dv_2 dv_3.$$  

Under the change of variables

$$v_1 = e^{T_{12} - T_{21}}, \quad v_2 = e^{T_{32} - T_{23}}, \quad v_3 = e^{T_{22} - T_{11}},$$

where $T = (T_{ki}, 1 \leq i \leq k \leq 3)$ is an array with $(T_{31}, T_{32}, T_{33}) = (x_1, x_2, x_3)$, this integral becomes

$$\psi_\lambda(x) = \int_{\mathbb{R}^3} e^{\lambda_1 (T_{31} + T_{32} + T_{33} - T_{21} - T_{11}) + \lambda_2 (T_{21} + T_{22} - T_{11}) + \lambda_3 T_{11}} \times \exp \left( -e^{T_{32} - T_{21}} - e^{T_{33} - T_{22}} - e^{T_{22} - T_{11}} - e^{T_{21} - T_{31}} - e^{T_{11} - T_{22}} - e^{T_{22} - T_{11}} \right) \, dT_{11} dT_{21} dT_{22}.$$  

Since $\Psi_\lambda(x)$ is a symmetric function of $\lambda$ we see that this agrees with Givental’s integral formula [7]. We note that this is reminiscent of the derivation of Givental’s formula given in [21] (see also [18, 19]).

8. Fundamental Whittaker functions

The eigenvalue equation [6] also has series solutions known as fundamental Whittaker functions. Define a collection of analytic functions $a_{n,m}(\nu)$, $n \geq 2$, $m \in (\mathbb{Z}^+)^{n-1}$, $\nu \in \mathbb{C}^n$ recursively by

$$a_{2,m}(\nu) = \frac{1}{m! \Gamma(\nu_1 - \nu_2 + m + 1)},$$

and for $n > 2$,

$$a_{n,m}(\nu) = \sum_k a_{n-1,k}(\mu) \prod_{i=1}^{n-1} \frac{1}{(m_i - k_i)! \Gamma(\nu_i - \nu_n + m_i - k_i - 1)},$$

where
where \( \mu_i = \nu_i + \nu_n/(n-1) \), \( i \leq n-1 \), and the sum is over \( k \in (\mathbb{Z}_+)^{n-2} \) satisfying \( k_i \leq m_i, \ 1 \leq i \leq n-2 \), with the convention that \( k_0 = k_{n-1} = 0 \). Then [26] Theorem 15 for each \( n \), \( a_{n,m}(\nu) \) satisfies the recursion

\[
\sum_{i=1}^{n-1} m_i^2 - \sum_{i=1}^{n-2} m_im_{i+1} + \sum_{i=1}^{n-1} (\nu_i - \nu_{i+1})m_i \] \( a_{n,m}(\nu) = \sum_{i=1}^{n-1} a_{n,m-e_i}(\nu) \),

with the convention that \( a_{n,m} = 0 \) for \( m \notin (\mathbb{Z}_+)^{n-1} \), and \( a_{n,0}(\nu) = \prod_{i<j} \Gamma(\nu_i - \nu_j + 1)^{-1} \). Writing

\[
m'(n) = \sum_{i=1}^{n-1} m_i(e_i - e_{i+1}) \text{, the series}
\]

\[
m_{nu}(x) = \sum_{m} a_{n,m}(\nu)e^{-(m'+nu,x)}
\]

is a fundamental Whittaker function as defined by Hashizume [25], and satisfies the eigenvalue equation [8]. We adopt a slightly different normalisation than the ones used in the papers [25] or [26]. Note that, for each \( x \in \mathbb{R}^n \), \( m_{nu}(x) \) is an analytic function of \( \nu \). Moreover:

**Proposition 3.**

\[
\psi_{nu}(x) = \prod_{i<j} \frac{\pi}{\sin \pi(\mu_i - \mu_j)} \sum_{w \in S_n} (-1)^w m_{-nu}(x).
\]

**Proof.** This comes from [3]. In the notation of that paper we are taking \( \Pi = \{ \alpha_i/2, \ i = 1, \ldots, n-1 \} \), \( m(2\alpha) = 0, |\eta_\alpha|^2 = 1 \) and \( \psi_{nu}(x) = 2^q k_{nu}(x) \) where \( q = q(n(n-1)/2) \).

Now consider the function \( \theta_t(x) \) defined by (14). Note that we can write

\[
s_n(\lambda) = \frac{1}{(2\pi i)^n n!} h(\lambda) \prod_{i>j} \frac{\sin \pi(\lambda_i - \lambda_j)}{\pi}.
\]

**Corollary 3.**

\[
\theta_t(x) = \frac{1}{(2\pi i)^n} \int_{\mathbb{R}^n} m(x)h(\lambda)e^{\sum_i \lambda_i^2t/2}d\lambda.
\]

9. Relativistic Toda and \( q \)-deformed Whittaker functions

The algebraic structure underlying Theorem 1 is an intertwining relation between certain differential operators associated with the open quantum Toda chain with \( n \) particles. This structure should carry over to the setting of Ruijsenaars’ relativistic Toda difference operators and \( q \)-deformed Whittaker functions [47, 15, 22]. A recent (related, but different) development along these lines is given in [8]. We will describe here the \( q \)-analogue of Theorem 1 in the rank one case, which corresponds to \( n = 2 \).

In the case \( n = 2 \), the Whittaker function is given by

\[
\psi_\lambda(x) = 2\exp\left(\frac{1}{2}(\lambda_1 + \lambda_2)(x_1 + x_2)\right) K_{\lambda_1-\lambda_2}\left(2e^{(x_2-x_1)/2}\right),
\]

where \( K_{\nu}(z) \) is the Macdonald function. In this case, Theorem 1 is equivalent to the following theorem of Matsumoto and Yor [37].

**Theorem 4.**

1. Let \( \{B_t(\mu), t \geq 0\} \) be a Brownian motion with drift \( \mu \), and define

\[
Z^{(\mu)}_t = \int_0^t e^{2B_s(\mu) - B_s(\mu)} ds.
\]

Then \( \log Z^{(\mu)} \) is a diffusion process with infinitesimal generator

\[
\frac{1}{2} d^2}{dx^2} + \left( \frac{d}{dx} \log K_{\mu}(e^{-x}) \right) \frac{d}{dx}
\]

(2) The conditional law of \( B_t(\mu) \), given \( \{Z_s(\mu), s \leq t; Z_t(\mu) = z\} \), is given by the generalized inverse Gaussian distribution

\[
\frac{1}{2} K_{\mu}(1/z)^{-1} e^{\mu x} \exp (-\cosh(x)/z) dx.
\]
Let $0 \leq q < 1$. Denote the $q$-Pochhammer symbol by $(q)_n = (q; q)_n = (1 - q) \cdots (1 - q^n)$ with the conventions that $(q)_0 = 1$ and $(0)_n = 1$. In what follows we also adopt the convention that $0^0 = 1$.

For $\lambda \in \mathbb{C}$ and $z \geq 0$, define
\[
\psi_\lambda(z) = \sum_{y=0}^{z} \frac{q^{\lambda(2y-z)}}{(q)_y(q)_z-y}.
\]
This is a $q$-deformed Whittaker function associated with $\mathfrak{sl}_2$ \cite{22}. It satisfies the difference equation
\[
(1 - q^{z+1})\psi_\lambda(z + 1) + \psi_\lambda(z - 1) = (q^\lambda + q^{-\lambda})\psi_\lambda(z)
\]
where we set $\psi_\lambda(-1) = 0$, and is related to the $q$-Hermite polynomials by
\[
(q)_z \psi_\lambda(z) = H_z \left( \frac{q^\lambda + q^{-\lambda}}{2}, q \right).
\]

Fix $0 \leq q < 1$, $0 \leq p \leq 1$ and let $(Y_n, Z_n)_{n \geq 0}$ be a Markov chain with state space $\{(y, z) \in \mathbb{Z}^2 : z \geq y \geq 0\}$ and transition probabilities given by
\[
\Pi((y, z), (y + 1, z + 1)) = p, \quad \Pi((y, z), (y, z + 1)) = (1 - p)q^y, \quad \Pi((y, z), (y - 1, z - 1)) = (1 - p)(1 - q^y).
\]
Note that $Y$ is itself a Markov chain with transition probabilities
\[
P(y, y + 1) = p, \quad P(y, y) = (1 - p)q^y, \quad P(y, y - 1) = (1 - p)(1 - q^y),
\]
and $X = 2Y - Z$ is a simple random walk on the integers which increases by one with probability $p$ and decreases by one with probability $1 - p$. Choose $\nu \in \mathbb{R}$ such that $p = q^\nu/(q^\nu + q^{-\nu})$.

**Theorem 5.** Let $Y_0 = Z_0 = 0$. The process $(Z_n, n \geq 0)$ is a Markov chain with transition probabilities
\[
Q(z, z + 1) = \frac{1 - q^{z+1}}{q^\nu + q^{-\nu}} \frac{\psi_\nu(z + 1)}{\psi_\nu(z)}, \quad Q(z, z - 1) = \frac{1}{q^\nu + q^{-\nu}} \frac{\psi_\nu(z - 1)}{\psi_\nu(z)}.
\]
Moreover, for each $n \geq 0$, the conditional distribution of $Y_n$, given $\sigma\{Z_m, m \leq n\}$ and $Z_n = z$, is given by
\[
\pi_z(y) = \psi_\nu(z)^{-1} \frac{q^{\nu(2y-z)}}{(q)_y(q)_z-y}, \quad y = 0, 1, \ldots, z.
\]

The proof is straightforward using the theory of Markov functions, by which it suffices to check that the transition operators $\Pi$ and $Q$ satisfy the intertwining relation $QK = K\Pi$ where
\[
K(z, (y, z')) = \frac{\delta_{z, z'} q^{\nu(2y-z)}}{\psi_\nu(z)(q)_y(q)_z-y}.
\]
This intertwining relation is readily verified. When $q = 0$ and $\nu = 0$, $\psi_\nu(z) = z$ and the above theorem can be interpreted as the discrete version of Pitman’s ‘$2M - X$’ theorem, which states that if $X_n$ is a simple symmetric random walk and $M_n = \max_{m \leq n} X_m$, then $2M - X$ is a Markov chain with transition probabilities $Q(z, z + 1) = (z + 1)/2z$, $Q(z, z - 1) = (z - 1)/2z$. When $q \to 1$, it should rescale to Theorem 4.

The analogue of the output/Burke theorem in the setting of Theorem 5 is the following. If $p < 1/2$, then the Markov chain $Y$ has a stationary distribution. If $Y_0$ is chosen according to this distribution and $Z_0 = 0$, the process $(Z_n, n \geq 0)$ is a simple random walk on the integers which increases by one with probability $p$ and decreases by one with probability $1 - p$.

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