Casimir effect in a two dimensional signature changing spacetime

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Abstract

We study the Casimir effect for free massless scalar fields propagating on a two-dimensional cylinder with a metric that admits a change of signature from Lorentzian to Euclidean. We obtain a nonzero pressure, on the hypersurfaces of signature change, which destabilizes the signature changing region and so alters the energy spectrum of scalar fields. The modified region and spectrum, themselves, back react on the pressure. Moreover, the central term of diffeomorphism algebra of corresponding infinite conserved charges changes correspondingly.

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1 Introduction

The Casimir effect is usually regarded as the most well-known manifestation of vacuum fluctuations in quantum field theory. In this effect, the presence of reflecting boundaries in the quantum vacuum alters the zero-point modes of the quantized fields, and results in the shifts of the vacuum expectation values of quantities such as energy densities and stresses. These shifts lead to vacuum forces which act on the reflecting boundaries. The particular features of these forces depend on the nature of the quantum field, the type of spacetime manifold and its dimensionality, the boundary geometries and the specific boundary conditions imposed on the field. Since the original work by Casimir in 1948 [1] many theoretical and experimental works have been done on this problem [3, 4, 5, 6, 7, 8, 9, 10, 11]. In general, there are several approaches to calculate the Casimir energy: mode summation [2], Green’s function method [4], heat kernel method [8], along with appropriate regularization schemes such as point separation [12, 13], dimensional regularization [14], and zeta function regularization [15, 16, 17, 18, 19]. Recently, general new methods have been obtained to compute the renormalized one-loop quantum energies and energy densities [20, 21].

On the other hand, signature changing spacetimes have recently been of particular importance as the specific geometries with interesting physical effects. The original idea of signature change was due to Hartle, Hawking and Sakharov [22]. This interesting idea would make it possible to have both Euclidean and Lorentzian metrics in path integral approach to quantum gravity. Later, it was shown that the signature change may happen in classical general relativity, as well [23]. There are two different approaches, continuous and discontinuous, to study the signature change in classical general relativity [23, 24]. In the continuous approach, the signature of metric changes continuously in passing from Euclidean to Lorentzian region. Hence, the metric becomes degenerate at the border of these regions. In the discontinuous approach, however, the metric becomes nondegenerate everywhere and is discontinuous at the border of Euclidean and Lorentzian regions.

The issue of propagation of quantum fields on signature-changing spacetimes has also been of some interest [24]. For example, Dray et al have shown that the phenomenon of particle production may happen for scalar particles propagating in a spacetime with heterotic signature. They have also obtained a rule for propagation of massless scalar fields on a two dimensional spacetime with signature change. Dynamical determination of the metric signature in spacetime of nontrivial topology is another interesting issue which has been studied in [25].

To the authors knowledge, no attempt has been done to study the Casimir effect within the geometries with signature change. A relevant work to the present paper is [26]. In this work, a model of free massless scalar fields on a two dimensional cylinder with a signature-changing strip has been studied and shown that the energy spectrum depends on the strip’s width and differs from the ordinary bosonic spectrum, for low energies. Moreover, It was shown that the diffeomorphism algebra of the corresponding infinite conserved charges is different from “Virasoro” algebra and approaches it at higher energies.

In this paper, we study the Casimir effect for the free massless scalar field propagating on the above two-dimensional signature-changing cylinder. We will obtain a nonzero pressure on the hypersurfaces of signature change which leads to instability in the signature-changing region. Therefore, depending on situation, Euclidean or Lorentzian region will grow or shrink,
and this will alter the energy spectrum and the diffeomorphism algebra discussed above.

Meanwhile, a lot of topics related to the Casimir effect have been explored in the context of string theory [10] [27]. The above two-dimensional signature-changing cylinder is topologically similar to a closed string, with two Euclidean and Lorentzian parts, propagating in a distributional way in the 2-dimensional target space, and the discontinuous nature of the model in classifying Euclidean and Lorentzian solutions with discrete symmetry motivates one to study it in the context of orbifolds [26] [28]. On the other hand, a closed string with two Euclidean and Lorentzian parts may be of some importance in the context of D-branes and related conformal field theories. Therefore, taking into account this similarity, the study of Casimir effect in our model may have nontrivial impacts on closed strings or D-branes.

In general, we believe the idea of Casimir effect in signature-changing spacetimes is novel and interesting. In the present paper this effect is inevitably limited to two-dimensional spacetime, which may be relevant to the study of closed bosonic strings with Euclidean and Lorentzian parts. But, further study of Casimir effect in 3+1 dimensional signature changing spacetimes may have more important physical implications, especially at early universe [29].

2 Casimir stress tensor in signature changing spacetime

We consider a free massless scalar field \( \phi \) which propagates on a two-dimensional manifold \( M = \mathbb{R} \times S^1 \) (the circle \( S^1 \) represents space and the real line \( \mathbb{R} \) represents time) with the following metric:

\[
ds^2 = -d\tau^2 + g(\sigma)d\sigma^2, \tag{1}
\]

where \( \tau \) is timelike coordinate and \( \sigma \) is a periodic spacelike coordinate with the period \( L \), and

\[
g(\sigma) = \begin{cases} 
-1 & 0 < \sigma < \sigma_0 + \text{mod } L, \\
+1 & \sigma_0 < \sigma < L + \text{mod } L,
\end{cases} \tag{2}
\]

where \( \sigma = 0, L \) and \( \sigma = \sigma_0 \) are the hypersurfaces of signature change. We assume the scalar field to satisfy specific junction conditions at these hypersurfaces. In the literature of signature change there are two kinds of junction conditions:

i) \( \phi \) and its derivatives are continuous across \( \sigma = \sigma_0 \), (Dray et al) [24],

ii) \( \phi \) is continuous but its derivatives vanish across \( \sigma = \sigma_0 \), (Hayward) [23].

In this paper, we assume the first junction conditions as the appropriate boundary conditions at each region. We assume the continuity of \( \phi \) as well as its derivatives at all times \( \tau \) as

\[
\begin{align*}
\phi^E \big|_{\Sigma, \Sigma'} & = \phi^L \big|_{\Sigma, \Sigma'}, \\
\partial_\sigma \phi^E \big|_{\Sigma, \Sigma'} & = -\epsilon \partial_\sigma \phi^L \big|_{\Sigma, \Sigma'},
\end{align*} \tag{3}
\]

where \( \Sigma, \Sigma' \) are the hypersurfaces of signature change, and \( \epsilon = \epsilon^\pm \) takes the values \( \pm 1 \) according to the orientation of the coordinates \( \tau \) and \( \sigma \) in both regions of different signatures. Assuming \( \epsilon^+ = +1 \) and \( \epsilon^- = +1 \) for Euclidean and Lorentzian regions respectively, the junction

\footnote{We assume \( L \) is the circumference of the circle with radius \( r \).}

\footnote{Notice that in this region the metric will be \( g_{\alpha\beta} = \text{diag}(-1, -1) \).}
conditions (3) are written as [26]

\[
\begin{align*}
\phi^E \mid_0 &= \phi^L \mid_L \\
\partial_\sigma \phi^E \mid_0 &= -\partial_\sigma \phi^L \mid_L.
\end{align*}
\]

By solving the wave equations

\[
\begin{align*}
(\partial_\tau^2 + \partial_\sigma^2) \phi^E(\sigma, \tau) &= 0, \\
(\partial_\tau^2 - \partial_\sigma^2) \phi^L(\sigma, \tau) &= 0,
\end{align*}
\]

in both Euclidean and Lorentzian regions and imposing the junction conditions (4), we obtain nontrivial solutions for \(\phi_\omega\), provided the continuous spectrum \(\omega\) satisfies the following “quantization condition” [26]

\[
cosh \omega \sigma_0 \cos(\omega(\sigma_0 - L)) = 1.
\]

It is shown in [26], that due to the same time evolution of the functions \(\Phi^E_\omega\) and \(\Phi^L_\omega\) one can construct a set of real distributional orthogonal and complete solutions on the arbitrary \(\tau = \text{const}\) hypersurface as

\[
\Phi_\omega(\sigma, \tau) = \Theta^+ \Phi^E_\omega(\sigma, \tau) + \Theta^- \Phi^L_\omega(\sigma, \tau),
\]

where \(\Phi^E_\omega = (\phi^E_\omega + \phi^{-E}_\omega)\), \(\Phi^L_\omega = (\phi^L_\omega + \phi^{-L}_\omega)\), and \(\Theta^+, \Theta^-\) are Heaviside distributions with support in Euclidean and Lorentzian regions, respectively. The solutions \(\Phi_\omega\) are then expanded as normal mode expansions [26].

One can also obtain the following expressions for the components of energy-momentum tensors associated with the scalar field \(\Phi(\sigma, \tau)\) in both Euclidean and Lorentzian regions [26]

\[
\begin{align*}
T^E_{00} &= [(\partial_\tau \Phi^E)^2 - (\partial_\sigma \Phi^E)^2], & T^E_{01} &= 2 \partial_\tau \Phi^E \partial_\sigma \Phi^E, \\
T^L_{00} &= [(\partial_\tau \Phi^L)^2 + (\partial_\sigma \Phi^L)^2], & T^L_{01} &= 2 \partial_\tau \Phi^L \partial_\sigma \Phi^L.
\end{align*}
\]

By introducing new coordinates \(\sigma^E_+, \sigma^E_-\) in the Euclidean region, and \(\sigma^L_+, \sigma^L_-\) in the Lorentzian one as

\[
\begin{align*}
\sigma^E_+ &= \tau + i\sigma, & \sigma^E_- &= \tau - i\sigma, \\
\sigma^L_+ &= \tau + \sigma, & \sigma^L_- &= \tau - \sigma,
\end{align*}
\]

The spectrum \(\omega\) in this model is obtained by solving the quantization condition which leads to real and \(\sigma_0\)-dependent values. It differs from ordinary spectrum (with pure Lorentzian signature) at low energies and coincides with the integer roots of \(\cos(\omega(L - \sigma_0))\), at high energies. Therefore, “sum over energies” approaches “sum over integers” at higher energies [26].

Heaviside distributions have the property \(d\Theta^\pm = \pm \delta\), where \(\delta\) is the hypersurface Dirac distribution with support on the hypersurfaces of signature change.
we obtain

\[ T_{++}^E = (T_{00}^E - iT_{01}^E)/2, \quad T_{++}^L = (T_{00}^L + T_{01}^L)/2, \]

\[ T_{--}^E = (T_{00}^E + iT_{01}^E)/2, \quad T_{--}^L = (T_{00}^L - T_{01}^L)/2, \]

\[ T_{+-}^E = T_{-+}^E = 0, \quad T_{+-}^L = T_{-+}^L = 0. \]

Then by substituting the normal mode expansions of the solutions \( \Phi_\omega \) we obtain [26]

\[ T_{++}^E = \frac{1}{2}[(\partial_\omega \Phi - i \partial_\sigma \Phi)]^2 \]

\[ = 2 \sum_{\omega \omega'} f_{\omega}(\omega') \bar{f}_{\omega'}(\omega') \tilde{\alpha}_\omega \tilde{\alpha}_\omega', \]

\[ T_{--}^E = \frac{1}{2}[(\partial_\omega \Phi + i \partial_\sigma \Phi)]^2 \]

\[ = 2 \sum_{\omega \omega'} \bar{f}_{\omega}(\omega') \bar{f}_{\omega'}(-\omega) \tilde{\alpha}_\omega \tilde{\alpha}_\omega', \]

\[ T_{++}^L = \frac{1}{2}[(\partial_\omega \Phi + \partial_\sigma \Phi)]^2 \]

\[ = 2 \sum_{\omega \omega'} f_{\omega}(\omega') f_{\omega'}(-\omega) \tilde{\alpha}_\omega \tilde{\alpha}_\omega', \]

\[ T_{--}^L = \frac{1}{2}[(\partial_\omega \Phi - \partial_\sigma \Phi)]^2 \]

\[ = 2 \sum_{\omega \omega'} \bar{f}_{\omega}(\omega') \bar{f}_{\omega'}(-\omega) \tilde{\alpha}_\omega \tilde{\alpha}_\omega', \]

where \( \tilde{\alpha}^\dagger \) and \( \tilde{\alpha} \) are the creation and annihilation operators, respectively, and \(^5\)

\[ \tilde{f}_{\omega}(\pm) = [(a/b)_\omega + 1] \exp(-i \omega \sigma_\pm) / \sqrt{4\pi} < \Phi_\omega, \Phi_\omega >, \]

\[ \tilde{f}_{\omega}(\pm) = [(c/b)_\omega + (1/b)_{-\omega}] \exp(-i \omega \sigma_\pm) / \sqrt{4\pi} < \Phi_\omega, \Phi_\omega >, \]

with \( (a/b)_\omega, (c/b)_\omega, (1/b)_\omega \) given by [26]

\[ (a/b)_\omega = \frac{\sin(\sigma_0 - L)}{\cosh \sigma_0 + \sinh \sigma_0 - \cos(\sigma_0 - L)}, \]

\[ (c/b)_\omega = \frac{(1+i)\exp(2\pi i \sigma_0)\sinh \sigma_0 + \cosh \sigma_0 - \exp(\sigma_0 - L)) \cosh \sigma_0 + \sinh \sigma_0 - \cos(\sigma_0 - L))}{2(\cosh \sigma_0 + \sinh \sigma_0 - \cos(\sigma_0 - L))}, \]

\[ (1/b)_\omega = \frac{(1-i)\exp(-2\pi i \sigma_0)\sinh \sigma_0 + \cosh \sigma_0 - \exp(-\sigma_0 - L)) \cosh \sigma_0 + \sinh \sigma_0 - \cos(\sigma_0 - L))}{2(\cosh \sigma_0 + \sinh \sigma_0 - \cos(\sigma_0 - L))}. \]

Notice that in obtaining these results we have imposed the quantization condition (6). Now, substituting the normal mode expansions (11) into Eqs.(10) leads to

\[ T_{00}^E = T_{++}^E + T_{--}^E = 2 \sum_{\omega \omega'}(\tilde{f}_{\omega}(\omega') \tilde{f}_{\omega'}(\omega) + \tilde{f}_{\omega}(\omega') \tilde{f}_{\omega'}(\omega)) \tilde{\alpha}_\omega \tilde{\alpha}_\omega', \]

\[ T_{01}^E = i(T_{+-}^E - T_{-+}^E) = 2i \sum_{\omega \omega'}(\tilde{f}_{\omega}(\omega') \tilde{f}_{\omega'}(\omega) - \tilde{f}_{\omega}(\omega') \tilde{f}_{\omega'}(\omega)) \tilde{\alpha}_\omega \tilde{\alpha}_\omega', \]

\[ T_{00}^L = T_{++}^L + T_{--}^L = 2 \sum_{\omega \omega'}(\tilde{f}_{\omega}(\omega') \tilde{f}_{\omega'}(\omega) + \tilde{f}_{\omega}(\omega') \tilde{f}_{\omega'}(\omega)) \tilde{\alpha}_\omega \tilde{\alpha}_\omega', \]

\[ T_{01}^L = T_{+-}^L - T_{-+}^L = 2 \sum_{\omega \omega'}(\tilde{f}_{\omega}(\omega') \tilde{f}_{\omega'}(\omega) - \tilde{f}_{\omega}(\omega') \tilde{f}_{\omega'}(\omega)) \tilde{\alpha}_\omega \tilde{\alpha}_\omega'. \]

\(^5\) The scalar product \( < \Phi_\omega, \Phi_\omega > \) has been defined in [26]
3 VEV of Casimir stress tensor

The poisson bracket structure for $\Phi$ and its conjugate momentum $\Pi$ is given by equal time relations
\[
\{\Phi(\sigma), \Pi(\sigma')\} = \delta(\sigma - \sigma'),
\]
\[
\{\Phi(\sigma), \Phi(\sigma')\} = \{\Pi(\sigma), \Pi(\sigma')\} = 0.
\tag{15}
\]
By substituting the normal mode expansions of $\Phi$ and $\Pi$ together with the expansion of $\delta(\sigma - \sigma')$ we obtain
\[
[\tilde{\alpha}_\omega, \tilde{\alpha}_\omega^\dagger] = \omega \delta_{\omega + \omega', 0}.
\tag{16}
\]
If we assume $|0_L>$ to be the vacuum state for the signature changing cylinder $R \times S^1$ with circumference $L$, then
\[
\tilde{\alpha}_\omega|0_L> = 0,
\]
\[
<0_L|\tilde{\alpha}_\omega = 0,
\]
\[
<0_L|\tilde{\alpha}_\omega^\dagger|0_L> = \omega \delta_{\omega + \omega', 0}
\tag{17}
\]
and we obtain the following vacuum expectation values for the components of energy momentum tensors
\[
<0_L|T_{00}^E|0_L> = 2 \sum_{\omega' \omega}(\tilde{f}_\omega^E(+) \tilde{f}_{\omega'}^E(+) + \tilde{f}_{-\omega}^E(-) \tilde{f}_{-\omega'}^E(-))\omega \delta_{\omega + \omega', 0},
\]
\[
<0_L|T_{01}^E|0_L> = 2i \sum_{\omega' \omega}(\tilde{f}_\omega^E(+) \tilde{f}_{\omega'}^E(+) - \tilde{f}_{-\omega}^E(-) \tilde{f}_{-\omega'}^E(-))\omega \delta_{\omega + \omega', 0},
\]
\[
<0_L|T_{10}^L|0_L> = 2 \sum_{\omega' \omega}(\tilde{f}_\omega^L(+) \tilde{f}_{\omega'}^L(+) + \tilde{f}_{-\omega}^L(-) \tilde{f}_{-\omega'}^L(-))\omega \delta_{\omega + \omega', 0},
\]
\[
<0_L|T_{11}^L|0_L> = 2 \sum_{\omega' \omega}(\tilde{f}_\omega^L(+) \tilde{f}_{\omega'}^L(+) - \tilde{f}_{-\omega}^L(-) \tilde{f}_{-\omega'}^L(-))\omega \delta_{\omega + \omega', 0},
\]
\[
<0_L|T_{11}^E|0_L> = -<0_L|T_{00}^E|0_L>,
\]
\[
<0_L|T_{11}^L|0_L> = <0_L|T_{00}^L|0_L>.
\tag{18}
\]
Now, by the following normalization
\[
4\pi <\Phi_\omega, \Phi_\omega> = \frac{1}{2}(L - \sigma_0),
\tag{19}
\]
and inserting it into equations (12) and then applying the delta function \( \delta_{\omega+\omega',0} \) we obtain

\[
< 0_L|T_{00}^E|0_L >= \frac{2}{L-\sigma_0} \sum_{\omega=0}^{\infty} [(\frac{\omega}{\sigma}) \omega + 1] [(\frac{\omega}{\sigma}) - \omega + 1] \omega,
\]

\[
< 0_L|T_{01}^E|0_L >= < 0_L|T_{10}^E|0_L >= 0,
\]

\[
< 0_L|T_{11}^E|0_L >= -< 0_L|T_{00}^E|0_L >,\quad (20)
\]

\[
< 0_L|T_{00}^L|0_L >= \frac{2}{L-\sigma_0} \sum_{\omega=0}^{\infty} [(\frac{\omega}{\sigma}) \omega + (\frac{1}{2}) - \omega] [(\frac{\omega}{\sigma}) \omega + (\frac{1}{2}) - \omega] \omega,
\]

\[
< 0_L|T_{01}^L|0_L >= < 0_L|T_{10}^L|0_L >= 0,
\]

\[
< 0_L|T_{11}^L|0_L >= < 0_L|T_{00}^L|0_L >.
\]

By using the Heaviside distribution \( \Theta^+, \Theta^- \) we may write

\[
< 0_L|T_{00}|0_L >= \Theta^+ < 0_L|T_{00}^E|0_L > + \Theta^- < 0_L|T_{00}^L|0_L >,
\]

\[
< 0_L|T_{11}|0_L >= \Theta^+ < 0_L|T_{11}^E|0_L > + \Theta^- < 0_L|T_{11}^L|0_L >.
\]

(21)

The normal ordered expressions are then as follows

\[
< 0_L|:T_{00}:|0_L >= \Theta^+ < 0_L|:T_{00}^E:|0_L > + \Theta^- < 0_L|:T_{00}^L:|0_L >,
\]

\[
< 0_L|:T_{11}:|0_L >= \Theta^+ < 0_L|:T_{11}^E:|0_L > + \Theta^- < 0_L|:T_{11}^L:|0_L >,
\]

(22)

such that

\[
< 0_L|:T_{00}^E:|0_L >= < 0_L|T_{00}^E|0_L > - \lim_{L' \to \infty} < 0_L|T_{00}^E|0_{L'} >,
\]

\[
< 0_L|:T_{00}^L:|0_L >= < 0_L|T_{00}^L|0_L > - \lim_{L' \to \infty} < 0_L|T_{00}^L|0_{L'} >,
\]

\[
< 0_L|:T_{11}^E:|0_L >= < 0_L|T_{11}^E|0_L > - \lim_{L' \to \infty} < 0_L|T_{11}^E|0_{L'} >,
\]

\[
< 0_L|:T_{11}^L:|0_L >= < 0_L|T_{11}^L|0_L > - \lim_{L' \to \infty} < 0_L|T_{11}^L|0_{L'} >,
\]

(23)

where the second terms are introduced to remove the ultraviolet divergences in the first terms [2]. Since we have Euclidean and Lorentzian quantities in the second terms, then the state \( |0_{L'} > \) should have the property: \( |0_{L'} > \to |0 > \) as \( L' \to \infty \), such that \( |0 > \) is the vacuum state of an \( R \times R \) signature changing spacetime. The Casimir energy is measured with respect to this state.

4 Regularization of VEV of Casimir stress tensor

Because both terms on the r.h.s of above equations are individually divergent they have to be subtracted by careful analysis. By introducing the cut-off to the sums in 00 components we
obtain
\[ <0_L|T_{00}^E|0_L>_{\text{cut-off}} = \frac{2}{L-\sigma_0} \sum_{\omega=0}^{\infty} [(\frac{\alpha}{\lambda})_\omega + 1] [([\frac{\alpha}{\lambda}]_\omega + 1) \omega e^{-\alpha \omega}, \]
\[ <0_L|T_{00}^E|0_L>_{\text{cut-off}} = \frac{2}{L-\sigma_0} \sum_{\omega=0}^{\infty} [(\frac{\alpha}{\lambda})_\omega + 1] [([\frac{\alpha}{\lambda}]_\omega + 1) \omega e^{-\alpha \omega}. \]

The 11 components will be easily obtained in terms of 00 components through Eqs.(20). We now break down each sum into two separate sums
\[ <0_L|T_{00}^E|0_L>_{\text{cut-off}} = \frac{2}{L-\sigma_0} \sum_{\omega<\omega} [(\frac{\alpha}{\lambda})_\omega + 1] [([\frac{\alpha}{\lambda}]_\omega + 1) \omega e^{-\alpha \omega} \]
\[ + \frac{2}{L-\sigma_0} \sum_{\omega=N}^{\infty} [(\frac{\alpha}{\lambda})_\omega + 1] [([\frac{\alpha}{\lambda}]_\omega + 1) \omega e^{-\alpha \omega}, \]
\[ <0_L|T_{00}^E|0_L>_{\text{cut-off}} = \frac{2}{L-\sigma_0} \sum_{\omega<\omega} [(\frac{\alpha}{\lambda})_\omega + 1] [([\frac{\alpha}{\lambda}]_\omega + 1) \omega e^{-\alpha \omega} \]
\[ + \frac{2}{L-\sigma_0} \sum_{\omega=N}^{\infty} [(\frac{\alpha}{\lambda})_\omega + 1] [([\frac{\alpha}{\lambda}]_\omega + 1) \omega e^{-\alpha \omega}, \]

where \( N \) is the root of quantization condition so that for any \( \omega \geq N \) the spectrum almost coincides with integer values. As is shown in Fig.2 in [26], it is easily seen from quantization condition that whatever can \( \sigma_0 \) be, the real spectrum of \( \omega \) approaches the integer one, generally at higher values of \( N \). Hence, we assume \( N \) to be sufficiently large so that the first sums over \( \omega < N \) be (finite) sums over real values and the second sums over \( \omega \geq N \) be almost (infinite) sums over integer values. Thus, we may discard the cut-off \( e^{-\alpha \omega} \) from the finite sums and keep them just for infinite sums.

Now, we consider the second sum in \( <0_L|T_{00}^E|0_L>_{\text{cut-off}} \). Since each of the terms \([([\frac{\alpha}{\lambda}]_\omega + 1)] \) and \([([\frac{\alpha}{\lambda}]_\omega + 1)] \) approaches 1 for large \( \omega \) (because \( ([\frac{\alpha}{\lambda}]_\omega + 1) \) almost vanish for large \( \omega \) ) then this sum goes like
\[ \frac{2}{L-\sigma_0} \sum_{\omega=N}^{\infty} \omega e^{-\alpha \omega}. \]

In the same way for \( <0_L|T_{00}^E|0_L>_{\text{cut-off}} \), it is easily shown that each of the terms \([([\frac{\alpha}{\lambda}]_\omega + ([\frac{\alpha}{\lambda}]_\omega + 1)] \) and \([([\frac{\alpha}{\lambda}]_\omega + ([\frac{\alpha}{\lambda}]_\omega + 1)] \) approaches 1 for large integer-like values \( \omega \geq N \), and the second sum goes like
\[ \frac{2}{L-\sigma_0} \sum_{\omega=N}^{\infty} \omega e^{-\alpha \omega}, \]

as well. Therefore, we calculate this sum for both regions. We know that \( \omega \geq N \) denotes for integers, hence we redefine \( \omega = N \) to (integer) \( \Omega = 0 \). To this end, we note that \( \omega \geq N \) indicates, by definition, the integer roots of \( \cos(\Omega(L-\sigma_0)) = 0 \) in the quantization condition, from which we obtain \( \omega \geq n \frac{1}{L-\sigma_0} \pi \) and \( N = n \frac{1}{L-\sigma_0} \pi \). This is equal to \( n = \frac{N(L-\sigma_0)}{\pi} - \frac{1}{2} \) with integer \( n \). Therefore, we may define \( \Omega = n - \frac{N(L-\sigma_0)}{\pi} + \frac{1}{2} \), with \( \Omega = 0, 1, 2, \ldots \). We also obtain \( \omega \) in terms of \( \Omega \) as \( \omega = N + \frac{\pi \Omega}{L-\sigma_0}. \)

Therefore, the sum (26) or (27) in Euclidean and Lorentzian regions is written as
\[ \frac{2}{L-\sigma_0} \sum_{\Omega=0}^{\infty} (N + \frac{\pi \Omega}{L-\sigma_0}) e^{-\alpha(N+\frac{\pi \Omega}{L-\sigma_0})}, \]
which, after some calculations, leads to\(^6\)

\[
\frac{2}{L - \sigma_0} \sum_{\Omega=0}^{\infty} (N + \frac{\pi \Omega}{L - \sigma_0}) e^{-\alpha (N + \frac{\pi \Omega}{L - \sigma_0})} = \frac{2}{L - \sigma_0} e^{-\alpha N} (N + \frac{\pi \Omega}{L - \sigma_0}) e^{-\alpha (N + \frac{\pi \Omega}{L - \sigma_0})} - N. \quad (28)
\]

First, we focus on the Euclidean calculations. By using Eqs.\((22), (23)\) for the Euclidean region we find

\[
\Theta^+ < 0_L : T_{00}^{E} : |0_L \rangle = \Theta^+ \left\{ \frac{2}{L - \sigma_0} \sum_{\omega=0}^{N-1} \left[ \sum_{\omega=0}^{N-1} \omega + \lim_{L \rightarrow \infty} e^{-\alpha N} \frac{(N + \frac{\pi \Omega}{L - \sigma_0}) e^{-\alpha (N + \frac{\pi \Omega}{L - \sigma_0})} - N}{(e^{-\alpha (N + \frac{\pi \Omega}{L - \sigma_0})} - 1)^2} \right] \right\}, \quad (29)
\]

or

\[
\Theta^+ \left\{ \frac{2}{L - \sigma_0} \sum_{\omega=0}^{N-1} \omega + \lim_{L \rightarrow \infty} e^{-\alpha N} \frac{(N + \frac{\pi \Omega}{L - \sigma_0}) e^{-\alpha (N + \frac{\pi \Omega}{L - \sigma_0})} - N}{(e^{-\alpha (N + \frac{\pi \Omega}{L - \sigma_0})} - 1)^2} \right\}, \quad (30)
\]

where the finite sum has appeared without cut-off. In obtaining \(\sum_{\omega=0}^{N-1} \omega\) in the second line of (29), we have used \(\lim_{L \rightarrow \infty} (\frac{a}{\Omega} \pm 1) \equiv \lim_{\Omega \rightarrow \infty} (\frac{a}{\Omega} \pm 1) = 0\) in the finite sum. We now expand

\[
e^{-\alpha N} (N + \frac{\pi \Omega}{L - \sigma_0}) e^{-\alpha (N + \frac{\pi \Omega}{L - \sigma_0})} - N = \lim_{L \rightarrow \infty} e^{-\alpha N} (N + \frac{\pi \Omega}{L - \sigma_0}) e^{-\alpha (N + \frac{\pi \Omega}{L - \sigma_0})} - N \]

\[
e^{-\alpha (N + \frac{\pi \Omega}{L - \sigma_0})} - 1)^2 \]

terms in the second bracket of (30) about \(\alpha = 0\). After some calculation we obtain

\[
\frac{2}{L - \sigma_0} e^{-\alpha N} \frac{(N + \frac{\pi \Omega}{L - \sigma_0}) e^{-\alpha (N + \frac{\pi \Omega}{L - \sigma_0})} - N}{(e^{-\alpha (N + \frac{\pi \Omega}{L - \sigma_0})} - 1)^2} \]

\[
= \frac{2}{\alpha^2 \pi} - (\frac{\alpha^2 N^2}{2} - \alpha N + 1) \frac{\pi}{6(L - \sigma_0)^2} + \frac{13}{6(L - \sigma_0)} N - \frac{N^2}{\pi} (1 - \alpha N).
\]

Substituting this result into the above bracket and taking \(\alpha \rightarrow 0\) leads to

\[
\frac{2}{L - \sigma_0} e^{-\alpha N} (N + \frac{\pi \Omega}{L - \sigma_0}) e^{-\alpha (N + \frac{\pi \Omega}{L - \sigma_0})} - N \]

\[
\lim_{L \rightarrow \infty} e^{-\alpha N} (N + \frac{\pi \Omega}{L - \sigma_0}) e^{-\alpha (N + \frac{\pi \Omega}{L - \sigma_0})} - N \]

\[
\frac{\pi}{6(L - \sigma_0)^2} \frac{13}{6(L - \sigma_0)} N.
\]

Finally, we have the following expression for the Euclidean region

\[
<T_{00}^{E} : |0_L \rangle = \frac{2}{L - \sigma_0} \sum_{\omega=0}^{N-1} \omega + \lim_{L \rightarrow \infty} e^{-\alpha N} \frac{(N + \frac{\pi \Omega}{L - \sigma_0}) e^{-\alpha (N + \frac{\pi \Omega}{L - \sigma_0})} - N}{(e^{-\alpha (N + \frac{\pi \Omega}{L - \sigma_0})} - 1)^2} \]

\[
= \frac{\pi}{6(L - \sigma_0)^2} \frac{13}{6(L - \sigma_0)} N. \quad (31)
\]

\(^6\)We have used \(\sum_{n=0}^{\infty} ne^{-2\pi an/L} = e^{2\pi a/L} (e^{2\pi a/L} - 1)^{-2} \) [2].
In the same way, the calculations for the Lorentzian region lead to

\[
<0_L|: T^L_{00} : |0_L> = \left\{ \frac{2}{L - \sigma_0} \sum_{\omega=0}^{N-1} \left[ \left( \frac{c}{b} \right)_\omega + \left( \frac{1}{b} \right)_\omega \right] \left[ \left( \frac{c}{b} \right)_\omega + \left( \frac{1}{b} \right)_\omega \right]^* - 1 \right\} \omega - \frac{\pi}{6(L - \sigma_0)^2} - \frac{13}{6(L - \sigma_0)} N. \tag{32}
\]

Note that, since \( N \) is not uniquely determined, then the expectation values (31), (32) are evaluated approximately. Therefore, for a given \( \sigma_0 \), replacing \( N \) by \( N + 1 \) or \( N - 1 \) leads to better or worse approximation, respectively. This is because, as we go to higher values of \( N \) the real spectrum coincides with integer one with better approximation.

5 Finite energy density and pressure in Euclidean and Lorentzian regions

By using Eqs.(20) for the 11 components we have

\[
<0_L|: T^E_{11} : |0_L> = -<0_L| : T^E_{00} : |0_L>,\]

\[
<0_L| : T^L_{11} : |0_L> = <0_L| : T^L_{00} : |0_L>. \tag{33}
\]

Therefore, the state \( |0_L> \) contains the finite energy density and pressure in the Euclidean and Lorentzian regions as follows:

\[
\begin{align*}
\{ \rho^E &= <0_L| : T^E_{00} : |0_L>, \\
\rho^L &= <0_L| : T^L_{00} : |0_L>, \\
\} \quad \{ p^E &= <0_L| : T^E_{11} : |0_L> = -\rho^E; \\
p^L &= <0_L| : T^L_{11} : |0_L> = \rho^L. \tag{34}
\end{align*}
\]

We then find that the total pressure acting on the signature changing hypersurfaces \( \sigma = 0, L \) and \( \sigma = \sigma_0 \) is given by

\[
p^T = p^L - p^E = \rho^L + \rho^E, \tag{36}
\]

which is generally nonzero according to Eqs.(31), (32) and (34). This nonzero pressure causes instability in the location of \( \sigma_0 \) relative to \( \sigma = 0 \). Depending on the initial location of \( \sigma_0 \), the corresponding value and sign of the pressure may lead one of the regions (\( L \) or \( E \)) to grow or shrink. It is very hard to judge about the exact behavior of the pressure from Eqs.(31), (32), because it depends on \( N \), the location of \( \sigma_0 \), and the complicate functions \((a/b)\omega,(c/b)\omega, (1/b)\omega\) in which the energy spectrum \( \omega \), itself, depends on \( \sigma_0 \) through the quantization condition (6).

Nevertheless, one may evaluate the situation in the two limits of \( \sigma_0 \). In the limit \( \sigma_0 \to 0 \), the term \( \frac{\pi}{6(L - \sigma_0)^2} \) may be neglected in comparison with two other terms. Therefore, there is a competition between the first sums and third terms in Eqs.(31), (32). And, upon this competition the pressure may cause the Euclidean region to grow or shrink. On the other hand, in the limit \( \sigma_0 \to L \), the term \( \frac{\pi}{6(L - \sigma_0)^2} \) may dominate the other two terms and the pressure \( p^T = \rho^L + \rho^E \) becomes negative, \( p^L < p^E \), which means the Euclidean region is growing (with increasing pressure) toward \( \sigma_0 = L \). Fortunately, in this case, there is no
divergency problem at \( \sigma_0 = L \). This is because, once the circle is completely covered by Euclidean metric, the quantization condition and all subsequent calculations break down.

The nonzero pressure obtained above and the consequent change in the signature changing region will certainly change the energy spectrum of the scalar fields through the quantization condition

\[
\cosh \omega \sigma_0 \cos \omega (\sigma_0 - L) = 1.
\]

The modified signature changing region \( \sigma_0 \) and energy spectrum \( \omega \) back react on the pressure through Eqs. (31), (32). The central term of the algebra corresponding to infinite conserved charges [26]

\[
[L_\omega, L_{\omega'}] = (\omega - \omega')L_{\omega + \omega'} + C(\omega, \omega')
\]

is correspondingly changed through

\[
C(\omega, \omega') = \delta_{\omega+\omega',0}f(\omega, \omega', \sigma_0) - 4 \sum_{\omega_1, \omega_2 > 0} \omega_1^2 \omega_2^2 C_{\omega_1, \omega_2}^{\omega' \omega} C_{\omega_1, \omega_2}^\omega + \sum_{\omega_1 > 0} \omega_1^2 (\omega - \omega_1)^2 C_{\omega_1, \omega - \omega_1}^{\omega' \omega}
\]

where

\[
f(\omega, \omega', \sigma_0) = 3 \sum_{l=-n}^0 [-2(l + k)a - \omega - \omega'][2(l + k)a + \omega]^2[|(2(l + k)a + \omega + \omega')(2(l + k)a)|]^{1/2}
\]

and

\[
N = (2k - 1)a, \quad \omega = (2n - 1)a, \quad a = \frac{\pi}{2(2\pi - \sigma_0)}
\]

with \( k \) and \( n \) as integers.

It is seen that in the special case \( N = 0 \) the first sums and the last terms vanish in (31), (32) and these lead to the standard result \(-\frac{\pi^2 L^2}{48\pi^2}\) for the pure Lorentzian metric \( \sigma_0 = 0 \) on the cylinder [2]. In fact, \( N = 0 \) corresponds to \( \omega = N = 0 \) which means \( \omega \) is an integer starting from zero; a case which occurs only in the pure Lorentzian region.

6 Conclusion

We have studied a two-dimensional model in which the spacetime is a cylinder (circle \( \times \) real number) with the circle representing space and the real line representing time. Moreover, we have assumed that this manifold admits a signature change of the type which had already been reported in [26].

We were interested in studying the Casimir effect for the real massless scalar fields propagating over this manifold. To this end, we have considered the expressions for the components of energy-momentum tensors associated with the real scalar field and calculated the corresponding vacuum expectation values. These expressions are found to be infinite, hence a regularization scheme is used to make them finite. By introducing a convenient cut-off and a regularization scheme, we obtain the finite expressions for the vacuum expectation values of the energy momentum tensors. These provide us with the finite energy densities and pressures in both Euclidean and Lorentzian regions so that the net pressure on the signature changing
hypersurfaces is obtained. This pressure causes instability in the signature changing region $\sigma_0$ and this instability alters the energy spectrum through the quantization condition. The modified $\sigma_0$ and spectrum $\omega$ themselves back react on the pressure through Eqs. (31), (32). Moreover, the central term of diffeomorphism algebra of real massless scalar fields obtained in [26] is altered due to modifications in $\sigma_0$ and spectrum $\omega$.

The action for free massless scalar field propagating on the signature changing background

$$S = \frac{1}{2} \int dt \int_{\text{Lorentzian}} d\sigma \sqrt{|g|} g^{\mu\nu} \partial_\mu \phi^L_\omega \partial_\nu \phi^L_\omega + \frac{1}{2} \int dt \int_{\text{Euclidean}} d\sigma \sqrt{|g|} g^{\mu\nu} \partial_\mu \phi^E_\omega \partial_\nu \phi^E_\omega$$

may be rewritten in the form of string action

$$S \sim \frac{1}{2} \int dt \int d\sigma \sqrt{|g|} g^{\mu\nu} \partial_\mu \phi^a_\omega \partial_\nu \phi^b_\omega \eta_{ab}$$

with the distribution $g_{\mu\nu} = \Theta^+ g^{\mu\nu}_E + \Theta^- g^{\mu\nu}_L$ and $\phi^a_\omega = (\phi^E_\omega, \phi^L_\omega)$ with $\eta_{ab} = \text{diag} (\Theta^+, \Theta^-)$. In this way it looks like we have a closed string with Euclidean and Lorentzian parts propagating in a distributional way in the two-dimensional target space $(\phi^E_\omega, \phi^L_\omega)$ [26]. The discontinuous nature of the model in classifying Euclidean and Lorentzian solutions $\Phi^E_\omega, \Phi^L_\omega$ with discrete symmetry under $\omega \leftrightarrow -\omega$ in each class motivates one to study it in the context of orbifolds. For example if we suppose the target space $M$ to be $\phi^a_\omega = (\phi^E_\omega, \phi^E_{-\omega}, \phi^L_\omega, \phi^L_{-\omega})$ and assume a permutation of it $\pi = (\phi^E_\omega, \phi^E_{-\omega})(\phi^L_\omega, \phi^L_{-\omega})$, then regarding the definition of an orbifold as the object one obtains by dividing a manifold by the action of a discrete group, it seems to be possible to define an orbifold $M/\pi$ which results in $\phi^a_\omega = (\phi^E_\omega, \phi^L_\omega)$. In this way, perhaps at a formal level, we may have a string on an orbifold [26].

Therefore, the study of Casimir effect in the present model may provide important results relevant to the study of closed bosonic strings. It is also appealing to proceed with the idea of Casimir effect in different 3+1 dimensional signature changing spacetimes to investigate what secondary effects may be produced by the Casimir effect [29].

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