Quasitriangular (\(G\)-cograded) multiplier Hopf algebras

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Abstract

We put the known results on the antipode of a usual quasitriangular Hopf algebra into the framework of multiplier Hopf algebras. We illustrate with examples which can not be obtained by using classical Hopf algebras. The focus of the present paper lies on the class of the so-called \(G\)-cograded multiplier Hopf algebras. By doing so, we bring the results of quasitriangular Hopf group-coalgebras (as introduced by Turaev) to the more general framework of multiplier Hopf algebras.

Introduction

The motivating example for quasitriangular Hopf algebras is given by the Hopf algebra \(H = U_q(g)\), for \(g\) a finite-dimensional semisimple Lie algebra over \(k = \mathbb{C}\), see [Dr1]. However \(H\) is not quasitriangular in the strict sense of the definition. The \(R\)-matrix lies in a completion of \(H \otimes H\) rather than in \(H \otimes H\) itself. The Hopf algebra \(H = U_q(g)\) is called ”topologically” quasitriangular. We believe that an
approach with multiplier Hopf algebras can evade this problem in purely algebraic terms. The notion of a quasitriangular multiplier Hopf algebra is introduced in [Z].

**Definition** [Z]. A regular multiplier Hopf algebra \((A, \Delta)\) is called quasitriangular if there exists an invertible multiplier \(R\) in \(M(A \otimes A)\) which is subject to

1. \((\Delta \otimes \iota)(R) = R^{13}R^{23}\) \((\iota \otimes \Delta)(R) = R^{13}R^{12}\)
2. \(R\Delta(a) = \Delta^{\text{cop}}(a)R\) for all \(a \in A\)
3. \((\varepsilon \otimes \iota)(R) = (i \otimes \varepsilon)(R) = 1 \in M(A)\).

Observe that in the equations (1) and (3) we extended a non-degenerate algebra homomorphism to the multiplier algebra. This technique is natural in the framework of multiplier Hopf algebras. For more details on these extensions, we refer to [VD1-Appendix A4].

The present paper is organized as follows.

In the Preliminaries, we review some recent results on multiplier Hopf algebras which are used throughout this paper.

In Section 2 we prove that the properties on the antipode of a quasitriangular Hopf algebras (as given in [Dri 2]) also hold for the antipode of a quasitriangular multiplier Hopf algebra. The general result is stated in Proposition 2.6. For a quasitriangular discrete multiplier Hopf algebra \(A\) we can express the inner automorphism \(S^4\) by using the modular multiplier in \(A\) and in the reduced dual \(\hat{A}\), see Proposition 2.9. To finish Section 2, we give infinite-dimensional examples which can not be obtained in the framework of usual Hopf algebras.

In Section 3 we consider a \(G\)-cograded multiplier Hopf algebra \(A\) elaborated with a crossing of \(G\) on \(A\), see Preliminaries. In Definition 3.1, we define when a \(G\)-cograded multiplier Hopf algebra with a crossing \(\pi\) is "\(\pi\)-quasitriangular". Throughout Section 3 we investigate the antipode of a \(G\)-cograded multiplier Hopf algebra which is \(\pi\)-quasitriangular. By using the crossing of \(G\) on \(A\) in an appropriate way, we prove that \(S^4\) is again an inner automorphism on \(A\), induced by a grouplike multiplier in \(M(A)\). The proofs make use of natural tools and techniques of multiplier Hopf algebra theory. An important class of \(\pi\)-quasitriangular multiplier Hopf algebras is given by the Drinfel’d double constructions for \(G\)-cograded multiplier Hopf algebras, see Theorem 3.11.1. We notice that the results of Section 3 apply to the so-called Hopf group-coalgebras as introduced by Turaev in [T]. Indeed, in [A-De-VD, Theorem 1.5], we have explained how to consider a Hopf group-coalgebra in the framework of group-cograded multiplier Hopf algebras. Therefore, the result of Zunino in [Zun, Theorem 5.6] is a special case of Theorem 3.11.1.

1. Preliminaries

Several important constructions and conceptions for classical Hopf algebras have
their best examples in the finite-dimensional cases, e.g. dualizing, Drinfel’d double, quasitriangularity, · · · . In [VD1], A. Van Daele introduced the much larger class of multiplier Hopf algebras. In this framework, one considers an algebra $A$ over the field $\mathbb{C}$, with or without an identity, but with a non-degenerate multiplication map $m$ (viewed as bilinear form) on $A \times A$. There is a homomorphism $\Delta$ from $A$ to the multiplier algebra $M(A \otimes A)$ of $A \otimes A$. Certain conditions on $\Delta$ (such as coassociativity) are imposed. The motivating example in [VD1] is the case where $A = K(G)$, the algebra of complex valued, finitely supported functions on an arbitrary group $G$. In $A = K(G)$, one uses the pointwise multiplication while the coproduct is given by the formula ($\Delta f(s, t) = f(st)$ with $s, t \in G$ and $f \in A$.

If $A$ has an identity, one has a usual Hopf algebra. Also for these multiplier Hopf algebras, there is a natural notion of left and right invariance for linear functionals (called integrals in Hopf algebra theory). We only consider regular multiplier Hopf algebras (i.e. with a bijective antipode). If $(A, \Delta)$ has invariant functionals, one can construct, in a canonical way, the restricted dual $(\hat{A}, \hat{\Delta})$. We notice that $(\hat{A}, \hat{\Delta})$ is in the same category and $\hat{A}$ is isomorphic to $A$. The duality theorems are investigated in [VD2]. It is shown that these theorems coincide with the usual duality for finite-dimensional Hopf algebras. In fact, the evaluation $\langle \hat{A}, A \rangle$ is a special case of a pairing of two multiplier Hopf algebras.

The Drinfel’d double construction, which is in the Hopf algebra framework only considered for finite-dimensional cases, can be done for any pairing $\langle A, B \rangle$ of two multiplier Hopf algebras. Essentially, two multiplier Hopf algebras $A$ and $B$ are paired if the product of $A$ (resp. coproduct of $A$) is dual to the coproduct of $B$ (resp. product of $B$). Regularity conditions are needed to write down this idea in a correct way. For details, we refer to [Dra-VD], [De-VD1]. The Drinfel’d double $D = A \bowtie B^{\text{cop}}$ of a pair $\langle A, B \rangle$ is a twisted tensor product algebra, installed on the tensor product $A \otimes B$. It is proven in [Dra-VD] that the following twist map $T$ is well-defined. The map $T : B \otimes A \to A \otimes B$ is given by the formula $T(b \otimes a) = \sum (a(1), S^{-1}(b(3))) a(3) b(1) a(2) \otimes b(2)$ for all $a \in A$ and $b \in B$. The comultiplication in $D = A \bowtie B^{\text{cop}}$ is given as follows. Take $a \in A$ and $b \in B$. $\Delta(a \bowtie b) = \Delta(a) \Delta^{\text{cop}}(b)$ considered as multiplier in $M(D \otimes D)$. Of course we have $\Delta^{\text{cop}}(a \bowtie b) = \Delta^{\text{cop}}(a) \Delta(b)$.

Recently, we considered the so-called $G$-cograded multiplier Hopf algebras, see [A-De-VD]. This class of multiplier Hopf algebras is a generalization of the Hopf group-coalgebras as introduced by Turaev in [T]. Let $G$ be any group.

**Definition** [A-De-VD]. A multiplier Hopf algebra $B$ is called $G$-cograded if we have

1. $B = \sum_{p \in G} B_p$ with $(B_p)_{p \in G}$ a family of subalgebras of $A$ such that $B_p B_q = 0$ if $p \neq q$,
(2) \( \Delta(B_{pq})(1 \otimes B_q) = B_p \otimes B_q \) and \( (B_p \otimes 1)\Delta(B_{pq}) = B_p \otimes B_q \) for all \( p,q \in G \).

Let \( \text{Aut}(B) \) denote the group of algebra automorphism on \( B \).

**Definition** [De-VD3]. By an action of the group \( G \) on \( B \), we mean a group homomorphism \( \pi : G \rightarrow \text{Aut}(B) \). We require that for all \( p \in G \)

\[
\pi_p \quad \text{respects the comultiplication on} \quad B \quad \text{in the sense that} \quad \Delta(\pi_p(b)) = (\pi_p \otimes \pi_p)(\Delta(b)).
\]

We call this action *admissible* if there is an action \( \rho \) of \( G \) on itself so that

\[
(\pi_p(B_q)) = B_{\rho_p(q)}
\]

\[
(\pi_{\rho_p(q)}) = \pi_{pqp^{-1}}
\]

for all \( p,q \in G \).

An admissible action is called a *crossing* if furthermore \( \rho_p(q) = pqp^{-1} \) for all \( p,q \in G \).

**Theorem** [De-VD3]. Let \( B \) be a \( G \)-cograded multiplier Hopf algebra and let \( \pi \) be an admissible action of \( G \) on \( B \). Then we have a deformation \( \tilde{B} \) of \( B \) in the following way. As algebra, \( \tilde{B} = B \). The comultiplication \( \tilde{\Delta} \) on \( \tilde{B} \) is defined as follows

\[
\tilde{\Delta}(b)(1 \otimes b') = (\pi_{q^{-1}} \otimes \iota)(\Delta(b)(1 \otimes b'))
\]

where \( b \in B \) and \( b' \in B_q \). The counit \( \tilde{\varepsilon} \) is the original counit \( \varepsilon \). The antipode \( \tilde{S} \) is given by the formula \( \tilde{S}(b) = \pi_{p^{-1}}(S(b)) \) for \( b \in B_p \).

There is a Drinfel’d double construction in the framework of \( G \)-cograded multiplier Hopf algebras in the following sense.

**Proposition** [De-VD3]. Let \( \langle A, B \rangle \) be a pairing of two regular multiplier Hopf algebras. Suppose that \( B \) is \( G \)-cograded. Then there exists subspaces \( \{A_p\}_{p \in G} \) of \( A \) such that

(1) \( A = \bigoplus_{p \in G} A_p \) and \( A_p A_q \subseteq A_{pq} \),
(2) \( \langle A_p, B_q \rangle = 0 \) whenever \( p \neq q \),
(3) \( \langle \Delta(A_p), B_q \otimes B_r \rangle = 0 \) if \( q \neq p \) or \( r \neq p \) where \( p,q,r \in G \).

**Theorem** [De-VD3]. Let \( \langle A, B \rangle \) be a pair of multiplier Hopf algebras and assume that \( B \) is a \( G \)-cograded multiplier Hopf algebra. Let \( \pi \) be a admissible action of \( G \) on \( B \). The space \( D^\pi = A^{\text{cop}} \bowtie \tilde{B} \) is a multiplier Hopf algebra, called the Drinfel’d double, with the multiplication, the comultiplication, the counit and the antipode, depending on the pairing as well as on the action \( \pi \) in the following way.
• \((a \bowtie b)(a' \bowtie b') = (m_A \otimes m_B)(\iota \otimes T \otimes \iota)(a \otimes b \otimes a' \otimes b')\) where \(T(b \otimes a') = \sum (a'_3, S^{-1}(b_3)) (a'_1, \pi_{p-1}(b)) a'_2 \otimes b_2\) for all \(a' \in A\) and \(b \in B_p\).

• \(\Delta(a \bowtie b) = \Delta^{cop}(a) \tilde{\Delta}(b)\) where \(\Delta^{cop}(a)\) and \(\tilde{\Delta}(b)\) are considered as multipliers in \(M(D^\pi \otimes D^\pi)\).

• \(\varepsilon(a \bowtie b) = \varepsilon(a) \varepsilon(b)\)

• \(S(a \bowtie b) = T(\pi_{p-1}(S(b)) \otimes S^{-1}(a))\).

2. Quasitriangular multiplier Hopf algebras

Let \(A\) be any regular multiplier Hopf algebra. Let \(R\) denote a multiplier in \(M(A \otimes A)\) so that for all \(a \in A\) we have \((a \otimes 1)R\) and \(R(a \otimes 1)\) are elements in \(A \otimes M(A)\). We use the following notation

\[(a \otimes 1)R = \sum aR^{(1)} \otimes R^{(2)}\quad \text{and} \quad R(a \otimes 1) = \sum R^{(1)} a \otimes R^{(2)}\]
in \(A \otimes M(A)\).

To a multiplier \(R\) in \(M(A \otimes A)\) which satisfies the above conditions, we associate a left multiplier in the sense of the following definition.

2.1 Definition. Take \(R\) as above. We define the left multiplier \(u\) as follows. Take \(a \in A\). We set

\[ua = \sum S(R^{(2)}) R^{(1)} a \quad \text{in} \quad A.\]

It is easy to see that \(u\) is a left multiplier of \(A\).

Notice that we use the extension to \(M(A)\) of the anti-isomorphism \(S\) on \(A\).

In the following lemma we give a condition on \(R\) in order that \(u\) is a two-sided multiplier in \(M(A)\).

2.2 Lemma. Take \(R \in M(A \otimes A)\) as above. Suppose that for all \(a \in A\), \(R\Delta(a) = \Delta^{cop}(a)R\) in \(M(A \otimes A)\). Then we have

(1) \(u\) is a multiplier in \(M(A)\) so that \(ua = S^2(a)u\) for all \(a \in A\).

(2) If \(R\) is invertible in \(M(A \otimes A)\), then \(u\) is invertible in \(M(A)\). Now \(S^2\) is an inner automorphism. Take \(a \in A\), we have \(S^2(a) = uu^{-1} = S(u)^{-1}aS(u)\).

Proof. (1) The original proof of [Dri2] can be easily modified into the framework of multiplier Hopf algebras. Take any \(a, b, c\) in \(A\). We have in \(A \otimes A \otimes A\)
\[ \sum a(1)b \otimes R(2)a(2) \otimes ca(3) = \sum a(2)R(1)b \otimes a(1)R(2) \otimes ca(3). \]

Therefore,
\[ \sum S^2(ca(3))S(R(2)a(2))R(1)b = \sum S^2(ca(3))S(a(1)R(2))a(2)R(1)b. \]

So,
\[ \sum S^2(c)S(R(2))R(1)ab = \sum S^2(c)\sum S^2(a)S(R(2))R(1)b. \]

This means that \( u(ab) = S^2(a)(ub) \) for all \( a, b \in A \). So \( u \) is a multiplier in \( M(A) \) and the right multiplication with \( u \) is given by the formula \( S^2(a)u = ua \) for all \( a \in A \).

(2) Suppose that \( R \) is invertible in \( M(A \otimes A) \). Let \( V \in M(A \otimes A) \) so that \( RV = VR = 1 \otimes 1 \) we suppose again that for all \( a \in A \), \( V(a \otimes 1), (a \otimes 1)V \) are in \( A \otimes M(A) \). Define the left multiplier \( t \) of \( A \) as follows

\[ \text{Take } a \in A, \text{ we set } ta = \sum S^{-1}(V(2))V(1)a. \]

We can prove in a similar way as above that \( t \) is a multiplier in \( M(A) \). More precisely, we have for all \( a \in A \) that \( ta = S^{-2}(a)t \).

We now calculate that \( ut = 1 \) in \( M(A) \). It is sufficient to prove that these multipliers equal as left multipliers on \( A \). Take any \( a \in A \), we have

\[
(ut)a = u(\sum S^{-1}(V(2))V(1)a) = \sum S(V(2))uV(1)a = \sum S(V(2))S(R(2))R(1)V(1)a = a.
\]

Furthermore, \( ut = S^2(t)u = 1 \) and we conclude that \( u^{-1} = t = S^2(t) \).

From (1), we easily deduce that \( S^2 \) is inner and for all \( a \in A \), \( S^2(a) = uau^{-1} \). By applying the antipode on both sides of this equation and using that \( S(A) = A \), we easily become the second expression for \( S^2 \).

We now suppose that \( A \) is quasitriangular in the sense of [Z]. We use [Z, Definition 1] in a slightly different formulation.

**2.3 Definition.** A regular multiplier Hopf algebra is called quasitriangular if there is an invertible multiplier \( R \in M(A \otimes A) \) so that
\[
\begin{align*}
(1) \quad & R\Delta(a) = \Delta^{cap}(a)R \text{ for all } a \in A, \\
(2) \quad & (\Delta \otimes \iota)(R) = R^{13}R^{23} \\
(3) \quad & (\iota \otimes \Delta)(R) = R^{13}R^{12}.
\end{align*}
\]
We assume furthermore that for all \( a \), \( R(a \otimes 1) \) and \( (a \otimes 1)R \) are in \( A \otimes M(A) \).

The following results are known for usual quasitriangular Hopf algebras.

**2.4 Lemma.** Let \( R \) be a generalized \( R \)-matrix for \( A \) as in Definition 2.3. Then we have for all \( a \in A \)

1. \( (\varepsilon \otimes \iota)(R) = 1 = (\iota \otimes \varepsilon)(R) \) in \( M(A) \)
2. \( R^{-1}(a \otimes 1) = (\iota \otimes S^{-1})(R(a \otimes 1)) \)
   \[ (a \otimes 1)R^{-1} = (\iota \otimes S^{-1})(a \otimes 1)R \]
   \[ R^{-1}(a \otimes 1) = (S \otimes \iota)((S^{-1}(a) \otimes 1)R) \]
3. \( (S \otimes S)(R) = R \) in \( M(A \otimes A) \).

**Proof.** (1) This proof is the same as for usual Hopf algebras.

(2) We prove the expressions for \( R^{-1} \) as left multiplier. The other proofs are similar. Take \( a, b \in A \). Then we have

\[
R((\iota \otimes S^{-1})(R(a \otimes 1))) = (\iota \otimes m)(\iota \otimes \iota \otimes S^{-1})(((\iota \otimes \Delta^{cop})(R))(a \otimes 1 \otimes 1)) = a \otimes 1.
\]

Therefore, \( R^{-1}(a \otimes 1) = (\iota \otimes S^{-1})(R(a \otimes 1)) \) because \( R \) is supposed to be invertible.

Similarly, we calculate

\[
(b \otimes 1)R((S \otimes \iota)((S^{-1}(a) \otimes 1)R)) = (m \otimes \iota)((b \otimes S^{-1}(a) \otimes 1)((\Delta \otimes \iota)(R))) = ba \otimes 1.
\]

Therefore, \( R^{-1}(a \otimes 1) = (S \otimes \iota)((S^{-1}(a) \otimes 1)R) \).

(3) \[ ((S \otimes S)(R))(S(a) \otimes 1) = (S \otimes S)((a \otimes 1)R) \]
   \[ = (\iota \otimes S)((S \otimes \iota)((a \otimes 1)R)) = (\iota \otimes S)(R^{-1}(S(a) \otimes 1)) \]
   \[ = (\iota \otimes S)((\iota \otimes S^{-1})(R(S(a) \otimes 1))) = R(S(a) \otimes 1). \]

In the case of a usual Hopf algebra, we get the known results when we put \( a = 1 \). ■

**2.5 Corollary.** Let \( R \) be a generalized \( R \)-matrix for \( A \), in the sense of Definition 2.3. Let \( u \) be in \( M(A) \), defined by the formula \( ua = \sum S(R^{(2)})R^{(1)}a \) for all \( a \in A \).
Then $u$ is invertible in $M(A)$ and $u^{-1}$ is given as

\[ u^{-1}a = \sum S^{-2}(R^2)R^{(1)}a = \sum S^{-1}(R^2)S(R^{(1)})a = \sum R^{(2)}S^2(R^{(1)})a. \]

**Proof.** Combine the proof of Lemma 2.2(2) and Lemma 2.4 (2). ■

Let $A$ be a quasitriangular multiplier Hopf algebra. For all $a \in A$, we have $S^2(a) = auu^{-1} = S(u)^{-1}aS(u)$, see Lemma 2.2. Therefore, we also have

\[ S^4(a) = uS(u)^{-1}aS(u)u^{-1}, \quad \text{for all} \quad a \in A. \]

We now prove that $uS(u)^{-1}$ is a grouplike multiplier in $M(A)$. Our proof is slightly different from the original proof in [Dri2]. We only make use of the multiplication by multipliers (instead of using actions), combined by the Yang-Baxter equation

\[ R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12} \quad \text{in} \quad M(A \otimes A \otimes A). \]

For a proof of the Yang-Baxter equation for quasitriangular multiplier Hopf algebras, we refer to [Z, Proposition 3].

**2.6 Proposition.** Let $(A, R)$ be a quasitriangular multiplier Hopf algebra. Then for all $a \in A$

\[ S^4(a) = gag^{-1}, \]

where $g = uS(u)^{-1}$ is a grouplike multiplier in $M(A)$.

**Proof.** Let $\sigma$ denote the flip map on $A \otimes A$ and extend it to $M(A \otimes A)$. We calculate the multiplier $\Delta(u)\sigma(R)R$. Take any $a, b \in A$. Take $a_i, b_i \in A$ so that $a \otimes b = \sum \Delta(a_i)(1 \otimes b_i)$. Then we have

\[
\sum \Delta(u)\sigma(R)(1 \otimes b_i) = \sum \Delta(u)\sigma(R)R(1 \otimes b_i) = \sum \Delta(S(R^{(2)})R^{(1)}a_i)\sigma(R)(1 \otimes b_i) = \sum \Delta(S(R^{(2)}))\Delta(R^{(1)}a_i)\sigma(R)R(1 \otimes b_i)
\]

\[
\sum ((\sigma \circ (S \otimes S))\Delta(R^{(2)}))\sigma(R)R\Delta(R^{(1)}a_i)(1 \otimes b_i).
\]
Throughout the proof, the letters $R, T, U, V, W, Z$ are used to denote the $R$-matrix in $M(A \otimes A)$.

\[
\begin{align*}
&= \sum ((\sigma \circ (S \otimes S))(R^{(2)} \otimes U^{(2)})\sigma(R)R\Delta(U^{(1)}R^{(1)}a)_{i})(1 \otimes b_{i})
\ &= \sum ((\sigma \circ (S \otimes S))(R^{(2)}W^{(2)} \otimes U^{(2)}V^{(2)})\sigma(R)R(U^{(1)}R^{(1)}a \otimes V^{(1)}W^{(1)}b)
\ &= \sum (S(U^{(2)}V^{(2)})T^{(2)}Z^{(1)}U^{(1)}R^{(1)}a) \otimes (S(R^{(2)}W^{(2)})T^{(1)}Z^{(2)}V^{(1)}W^{(1)}b).
\end{align*}
\]

If we apply the Yang-Baxter equation to the underlined expression we get

\[\sum (S(V^{(2)}U^{(2)})T^{(2)}U^{(1)}Z^{(1)}R^{(1)}a) \otimes (S(R^{(2)}W^{(2)})T^{(1)}V^{(1)}Z^{(2)}W^{(1)}b)\]

We notice that for all $a \in A$, we have

\[
\sum U^{(1)}R^{(1)}a \otimes S(R^{(2)})U^{(2)} = U^{(1)}R^{(1)}a \otimes S^{-1}(U^{(2)})R^{(2)}
\]

\[
= (\iota \otimes S)(\sum U^{(1)}R^{(1)}a \otimes S^{-1}(U^{(2)})R^{(2)}) = a \otimes 1.
\]

Now the above expression can be simplified as follows.

\[
\sum S(U^{(2)})U^{(1)}a \otimes S(W^{(2)})W^{(1)}b = (u \otimes u)(a \otimes b).
\]

We obtain $\sigma(R)R\Delta(u) = \Delta(u)\sigma(R)R = u \otimes u$ in $M(A \otimes A)$.

Therefore,

\[
\Delta(S(u)) = (\sigma \circ (S \otimes S))\Delta(u)
\]

\[
= (\sigma \circ (S \otimes S))(\sigma(R)R)^{-1}(u \otimes u))
\]

\[
= (S(u) \otimes S(u))\sigma(((S \otimes S)(R)(S \otimes S)(\sigma(R)))^{-1})
\]

\[
= (S(u) \otimes S(u))\sigma((R\sigma(R))^{-1})
\]

\[
= (S(u) \otimes S(u))(\sigma(R)R)^{-1}.
\]

Now, it is easy to see that $uS(u)^{-1}$ is grouplike. ■

To finish this section, we suppose that $A$ is a discrete multiplier Hopf algebra, as introduced in [VD-Z]. A multiplier Hopf algebra $A$ is discrete if $A$ contains coin-tegrals. A left cointegral in $A$ is a non-zero element $h \in A$ such that $ah = \varepsilon(a)h$ for all $a \in A$. Right cointegrals are defined in a similar way.

In [VD-Z, Theorem 2.10] is proven that a discrete multiplier Hopf algebra $A$ always contains non-zero left (and right) invariant functionals. Recall that a non-zero linear map $\varphi$ on $A$ is called left invariant if $(\iota \otimes \varphi)\Delta(a) = \varphi(a)1$ for all $a \in A$. 


Similarly, non-zero right invariant functionals are considered. The reduced dual $\hat{A}$ is given as the following subspace of the full linear dual.

$$\hat{A} = \{ \varphi(a) \mid a \in A \} = \{ \varphi(a) \mid a \in A \}.$$

For more details on this duality, we refer to [VD2]. In the present paper, we make use of the so-called modular multiplier $\delta_A$ which is assigned to a multiplier Hopf algebra $A$ with non-zero left (right) invariant functionals. We have from [VD2, Propositions 3.8, 3.9] that the modular multiplier $\delta_A$ in $M(A)$ is uniquely determined such that for all $a \in A$

$$(\varphi \otimes \iota)\Delta(a) = \varphi(a)\delta_A \quad \delta_A \text{ is invertible and}$$

$$\Delta(\delta_A) = \delta_A \otimes \delta_A \quad \varepsilon(\delta_A) = 1 \quad S(\delta_A) = \delta_A^{-1}.$$

The evaluation map $\langle \hat{A}, A \rangle$ is an important case of a pairing between multiplier Hopf algebras. According to this pairing, we consider the right action of $\hat{A}$ on $A$, denoted as $A \triangleright \hat{A}$ which is defined in the following way. Take $a \in A$ and $g \in \hat{A}$, then

$$a \triangleright g = \sum \langle g, a_{(1)} \rangle a_{(2)}.$$ Observe that $a_{(1)}$ is covered by $g$ through the pairing. The action $A \triangleright \hat{A}$ is unital and extends to the multiplier algebra in an obvious way.

The Lemmas 2.7 and 2.8 give general results on discrete multiplier Hopf algebras which are used in Proposition 2.9.

2.7 Lemma. Let $h$ be a left cointegral in $A$. Then the evaluation $\hat{\varphi}_h = \langle \cdot, h \rangle$ is a left integral on $\hat{A}$. Furthermore, $ha = \langle \delta_A, a \rangle h$ for all $a \in A$.

Proof. Clearly $\hat{\varphi}_h$ is a linear functional on $\hat{A}$. For all $f \in \hat{A}$, we calculate the multiplier $(\iota \otimes \hat{\varphi}_h)\Delta(f)$ in $M(\hat{A})$. Take $g \in \hat{A}$, then by definition,

$$((\iota \otimes \hat{\varphi}_h)\Delta(f))g = (\iota \otimes \hat{\varphi}_h)(\Delta(f)(g \otimes 1)) = \sum f_{(2)}(h)f_{(1)}g.$$ For $a \in A$, we have

$$\langle \sum f_{(2)}(h)f_{(1)}g \rangle(a) = \sum f_{(2)}(h)f_{(1)}(a_{(1)})g(a_{(2)}) = \sum f(a_{(1)}h)g(a_{(2)}) = f(h)g(a).$$

Therefore, $((\iota \otimes \hat{\varphi}_h)\Delta(f))g = \hat{\varphi}_h(f)g$ and this means that $\hat{\varphi}_h$ is a left integral on $\hat{A}$.

As $\hat{\varphi}_h$ is a left integral on $\hat{A}$, we have that for all $f \in \hat{A}$, $(\hat{\varphi}_h \otimes \iota)\Delta(f) = \hat{\varphi}_h(f)\delta_A$ where $\delta_A$ is the modular multiplier in $\hat{A}$ as defined in [VD2]. Take any $g \in \hat{A}$, then
we have \( g = \varphi(x \cdot) \) where \( \varphi \) is a left integral on \( A \) and \( x \in A \). The equation
\[
\sum \langle f(1), h \rangle f(2)g = \langle f, h \rangle \delta_{\hat{A}}g
\]
yields for all \( f, g \in \hat{A} \). By using \((1 \otimes A)\Delta(A) = A \otimes A\), we easily become that \( ha = \langle \delta_{\hat{A}}, a \rangle h \) for all \( a \in A \). \( \blacksquare \)

2.8 Lemma. Let \( A \) be any discrete multiplier Hopf algebra. Let \( h \) denote a left cointegral in \( A \) and \( \delta_{\hat{A}} \) (resp. \( \delta_{\hat{A}} \)) is the modular multiplier in \( M(A) \) (resp. \( M(\hat{A}) \)). For all \( a \in A \), we have
\[
(1) \ (1 \otimes a)\Delta(h) = (S(a) \otimes 1)\Delta(h)
\]
\[
(2) \ \Delta(h)(a \otimes 1) = \Delta(h)(1 \otimes S(a \triangleright \delta_{\hat{A}}))
\]
\[
(3) \ \Delta(h)(a \otimes 1) = \langle \delta_{\hat{A}}, a \rangle^{-1} \sum S^2(h(2))a \otimes h(1)\delta_{\hat{A}}.
\]
Notice that the modular \( \langle \delta_{\hat{A}}, a \rangle \) has a meaning because \( \delta_{\hat{A}} \) is a grouplike multiplier.

Proof. (1) See [VD-Z, Proposition 2.5].
(2) Take any \( a, b \) in \( A \). Then we have
\[
\Delta(h)(a \otimes b) = \sum h(1)a(1) \otimes h(2)\varepsilon(a(2))b
\]
\[
= \sum h(1)a(1) \otimes h(2)a(2)S(a(3))b = \sum \Delta(ha(1))(1 \otimes S(a(2))b)
\]
\[
= \sum \langle \delta_{\hat{A}}, a(1) \rangle \Delta(h)(1 \otimes S(a(2))b) = \Delta(h)(1 \otimes S(a \triangleright \delta_{\hat{A}})b).
\]

Now the statement follows.
(3) Let \( \psi \) be a right integral on \( A \) so that \( \psi(h) = 1 \). Apply the operator \( \psi \otimes \iota \) on the both sides of the equation in (1). Then we get for all \( a \in A \)
\[
a = \sum \psi(S(a)h(1))h(2)
\]
and equivalently,
\[
S(a) = \sum \psi(ah(1))S^2(h(2)).
\]
We put \( \psi = \varphi \circ S \) for some left integral on \( A \). Then we have \( \varphi(h) = \langle \delta_{\hat{A}}, a \rangle^{-1} \) because \( \psi(a) = \varphi(a\delta_{\hat{A}}) \) for all \( a \in A \). Apply the operator \( \iota \otimes \varphi \) to both sides of the equation in (1). Then we get for all \( a \in A \)
\[
\sum \varphi(ah(2))h(1) = \langle \delta_{\hat{A}}, a \rangle^{-1}S(a).
\]
If we combine these equations, we obtain that
\[
\langle \delta_{\hat{A}}, a \rangle \sum \varphi(ah(2))h(1) = \sum \varphi(ah(1)\delta_{\hat{A}})S^2(h(2)).
\]
As $\varphi$ is faithful in the sense that $\varphi(ab) = 0$ for all $a \in A$ implies that $b = 0$, we have obtained that

$$\langle \hat{\delta} A, \delta A \rangle \sum h_{(2)} \otimes h_{(1)} b = \sum h_{(1)} \delta A \otimes S^2(h_{(2)}) b$$

for all $b \in A$. Now the statement follows. ■

We are ready to calculate the grouplike multiplier $g = uS(u)^{-1}$ for a discrete multiplier Hopf algebra which is quasitriangular.

2.9 Proposition. Let $A$ be a discrete quasitriangular multiplier Hopf algebra. Let $\delta_A$ (resp. $\hat{\delta} A$) denote the modular multiplier in $M(A)$ (resp. $\hat{M}(A)$). Then we have

$$uS(u)^{-1} = \delta_A^{-1}((\iota \otimes \langle \hat{\delta} A^{-1}, \cdot \rangle)(R)) \text{ in } M(A).$$

Proof. In the right hand side of the statement the extension of the algebra homomorphism $(\iota \otimes \langle \hat{\delta} A^{-1}, \cdot \rangle)$ to the multiplier algebra $M(A \otimes A)$ is considered. Throughout the proof, $h$ denotes a left cointegral in $A$ and $u$ is the multiplier in $M(A)$, associated to $R$-matrix $R$. As $A$ is almost co-commutative, we have

$$R\Delta(h) = \Delta^{\text{cop}}(h)R \text{ in } M(A \otimes A)$$

This means that for all $a \in A$ we have

$$\sum R^{(1)} h_{(1)} a \otimes R^{(2)} h_{(2)} = \sum h_{(2)} R^{(1)} a \otimes h_{(1)} R^{(2)}.$$ 

By using Lemma 2.8, the left hand side of the equation (*) equals

$$\sum R^{(1)} S(R^{(2)}) h_{(1)} a \otimes h_{(2)} = \langle \hat{\delta} A, \delta A \rangle^{-1} S(u) \sum S^2(h_{(2)}) a \otimes h_{(1)} \delta A$$

$$= S(u) \sum S^2(h_{(2)}) \delta_A^{-1} a \otimes h_{(1)}.$$ 

By using Lemma 2.8(2), the right hand side of the equation (*) is given as

$$\sum h_{(2)} S(R^{(2)} \preceq \delta A) R^{(1)} a \otimes h_{(1)}.$$ 

As $(\iota \otimes \Delta)(R) = R^{13} R^{12}$, this last expression can be written as

$$\sum \langle \hat{\delta} A, R^{(2)} \rangle h_{(2)} u R^{(1)} a \otimes h_{(1)}.$$ 

The equation (*) can be written as

$$S(u) \sum S^2(h_{(2)}) \delta_A^{-1} a \otimes h_{(1)} = \sum \langle \hat{\delta} A, R^{(2)} \rangle h_{(2)} u R^{(1)} a \otimes h_{(1)}.$$
Let $\psi$ be a right integral on $A$ so that $\psi(h) = 1$. Apply the operator $(\iota \otimes \psi)$ on both sides of the above equation. We become that

$$S(u)\delta^{-1}_A = \sum \langle \delta_{\hat{A}}, R^{(2)} \rangle u R^{(1)} a.$$  

Therefore, we have in $M(A)$

$$S(u)\delta^{-1}_A = u(\iota \otimes \langle \delta_{\hat{A}}, \cdot \rangle)(R)$$

and so,

$$u^{-1}S(u) = ((\iota \otimes \langle \delta_{\hat{A}}, \cdot \rangle)(R))\delta_A.$$  

We easily obtain that $uS(u)^{-1} = \delta_A^{-1}(\iota \otimes \langle \delta_{\hat{A}}^{-1}, \cdot \rangle)(R)$. ■

2.10 Remark. Take $A$ a finite-dimensional Hopf algebra. In [Dri2] and [Ra] is given a similar result as in Proposition 2.9.

We now give examples which can not be obtained in the framework of usual Hopf algebras.

The dual multiplier Hopf algebra of a Ore-extension

General Ore-extensions of cyclic groups are introduced and elaborated in [B-D-G-M]. We consider the special case $A = (kC)_{2,-1,1}$.

As algebra, $A$ is generated by an invertible element $a$ and an element $b$, so that $b^2 = 0$. The multiplication in $A$ is induced by the commutation rule $ab = -ba$. The coalgebra structure is given as follows

$$\Delta(a) = a \otimes a, \quad \varepsilon(a) = 1, \quad S(a) = a^{-1}$$

$$\Delta(b) = (a \otimes b) + (b \otimes 1), \quad \varepsilon(b) = 0, \quad S(b) = -a^{-1}b$$

In [B-D-G-M] is proven that $A$ is co-Frobenius. A left integral is given by the functional $\varphi$ when $\varphi(a^{-1}b) = 1$ and $\varphi$ is zero elsewhere on the basis. The modular element in $A$ is given as $\delta_{\hat{A}} = a^{-1}$. This means that $(\varphi \otimes \iota)\Delta(a^m b^n) = \varphi(a^m b^n) a^{-1}$ for all $m \in \mathbb{Z}$ and $n = 0, 1$.

The dual multiplier Hopf algebra $\hat{A}$, defined as $\hat{A} = \{ \varphi(x) \mid x \in A \}$, is a discrete multiplier Hopf algebra, see [VD2]. This means that there are cointegrals in $\hat{A}$. We briefly describe the multiplier Hopf algebra $\hat{A}$. For more details, we refer to [De].

A linear basis of $\hat{A}$ is given by the functionals $\omega_{m,n}, m \in \mathbb{Z}$ and $n = 0, 1$ where $\omega_{m,n}(a^m b^n) = 1$ and $\omega_{m,n}$ is zero elsewhere on the basis of $A$. The product in $\hat{A}$ is induced by the following commutation rules.
\[
\begin{align*}
\omega_{p,0}\omega_{q,0} &= \delta_{p,q}\omega_{p,0} & \omega_{p,0}\omega_{q,1} &= \delta_{p,q,1}\omega_{q,1} \\
\omega_{q,1}\omega_{p,0} &= \delta_{q-p,0}\omega_{p,1} & \omega_{p,1}\omega_{q,1} &= 0 \\
\omega_{q,1}\omega_{p,0} &= \delta_{q-p,0}\omega_{p,1} & \omega_{p,1}\omega_{q,1} &= 0
\end{align*}
\]

where \( p, q \in \mathbb{Z} \).

The coalgebra structure in \( \hat{A} \) is given as follows.

\[
\begin{align*}
\Delta(\omega_{p,0}) &= \sum_{q \in \mathbb{Z}} \omega_{q,0} \otimes \omega_{p-q,0} \\
\Delta(\omega_{p,1}) &= \sum_{r, s = 0, 1} (-1)^{s(p-s)} \omega_{r,s} \otimes \omega_{p-r,1-s} \\
S(\omega_{p,0}) &= \omega_{-p,0} & S(\omega_{p,1}) &= (-1)^p \omega_{-p-1,1} \\
\varepsilon(\omega_{p,0}) &= \delta_{p,1} & \varepsilon(\omega_{p,1}) &= 0.
\end{align*}
\]

A left cointegral in \( \hat{A} \) is given by the functional \( \omega_{-1,1} \), see also Lemma 2.7. A left integral on \( \hat{A} \) is given by \( \hat{\varphi} \) which is defined on \( \hat{A} \) as follows:

\[
\hat{\varphi}(\omega_{p,0}) = 0 \text{ for all } p \in \mathbb{Z}, \quad \hat{\varphi}(\omega_{p,1}) = 1 \text{ for all } p \in \mathbb{Z}.
\]

The modular multiplier in \( M(\hat{A}) \) is given by

\[
\delta_{\hat{A}} = \sum_{p \in \mathbb{Z}} (-1)^p \omega_{p,0}.
\]

Furthermore, \( \hat{A} \) is quasitriangular. A generalized \( R \)-matrix is given by the multiplier \( R \) in \( M(\hat{A} \otimes \hat{A}) \)

\[
R = \sum_{p \in \mathbb{Z}} \delta^p_{\hat{A}} \otimes \omega_{p,0}.
\]

One can easily check that

1. \((\Delta \otimes \iota)(R) = R^{13}R^{23}\) in \( M(\hat{A} \otimes \hat{A} \otimes \hat{A}) \)
2. \((\iota \otimes \Delta)(R) = R^{13}R^{12}\) in \( M(\hat{A} \otimes \hat{A} \otimes \hat{A}) \)
3. \(R\Delta(\omega_{p,0}) = \Delta^{\text{cop}}(\omega_{p,0})R, \quad R\Delta(\omega_{p,1}) = \Delta^{\text{cop}}(\omega_{p,1})R.\)

It is easy to see that the multiplier \( R \) is covered by the elements \( \omega_{k,l} \) (\( k \in \mathbb{Z} \) and \( l = 0, 1 \)) in the sense that \( R(\omega_{k,l} \otimes 1), (\omega_{k,l} \otimes 1)R \) are in \( \hat{A} \otimes M(\hat{A}) \).

We have \( R^{-1} = R \). The multiplier \( u \) in \( M(\hat{A}) \) defining the square of the antipode as an inner automorphism (see Lemma 2.2) is given by the modular multiplier i.e., \( u = \delta_{\hat{A}} \). Therefore, \( uS(u)^{-1} = 1 \) in \( M(\hat{A}) \). This result indicates that \( S^4 \) is the identity on \( \hat{A} \), see Proposition 2.6. To finish, we notice that \( \hat{A} \) is a discrete quasitriangular multiplier Hopf algebra.

To illustrate the formula in Proposition 2.9, we calculate in \( M(\hat{A}) \)
The Drinfel’d double of a Ore-extension

We start with the Ore-extension $A = (kC)_{2,-1,1}$. In previous example, we considered the dual multiplier Hopf algebra $\hat{A}$. By the construction of $\hat{A}$, we have that $\langle \hat{A}, A \rangle$ is a pairing of multiplier Hopf algebras in the sense of [Dra-VD]. We consider the Drinfel’d double of the pair $\langle \hat{A}, A \rangle$. Let $D$ denote $D = \hat{A} \bowtie A^{\text{cop}}$. For the general construction of the Drinfel’d double of a pair of multiplier Hopf algebras, we refer to [Dra-VD] and [De-VD1]. One can show that the maps

$$\hat{A} \rightarrow M(\hat{A} \bowtie A) : \omega_{k,l} \mapsto \omega_{k,l} \bowtie 1, \quad k \in \mathbb{Z}, \ l = 0, 1,$$

$$A \rightarrow M(\hat{A} \bowtie A) : a^{p}b^{q} \mapsto 1 \bowtie a^{p}b^{q}, \quad p \in \mathbb{Z}, \ q = 0, 1$$

are algebra embeddings, so that $\omega_{k,l} \bowtie a^{p}b^{q} = (\omega_{k,l} \bowtie 1)(1 \bowtie a^{p}b^{q})$ and $(1 \bowtie a^{p}b^{q})(\omega_{k,l} \bowtie 1) = T(a^{p}b^{q} \bowtie \omega_{k,l})$ where $T$ is the twist map $T : A \otimes \hat{A} \rightarrow \hat{A} \otimes A$ defined by the formula

$$T(x \otimes y) = \sum \langle y_{(1)}, S^{-1}(x_{(3)}) \rangle \langle y_{(2)}, x_{(3)} \rangle y_{(2)} \otimes x_{(2)}$$

for all $x \in A$ and $y \in \hat{A}$.

If we identify $\hat{A}$ and $A$ with their images in $\hat{A} \otimes A$, we find that $D = \hat{A} \bowtie A^{\text{cop}}$ is the algebra generated by $\hat{A}$ and $A$ with the following commutating relations

$$a\omega_{p,0} = \omega_{p,0}a \quad b\omega_{p,0} = \omega_{p-1,0}b \quad a\omega_{p,1} = -\omega_{p,1}a \quad b\omega_{p,1} = \omega_{p,0} - (-1)^{p}\omega_{p,0}a - \omega_{p-1,1}b$$

where $p \in \mathbb{Z}$.

The comultiplication in $D = \hat{A} \bowtie A^{\text{cop}}$ is given by the formula

$$\Delta(\omega_{p,k}a^{r}b^{l}) = \Delta(\omega_{p,k})\Delta^{\text{cop}}(a^{r}b^{l})$$

for all $p, r \in \mathbb{Z}$ and $k, l = 0, 1$. One can check that $R = \sum_{p \in \mathbb{Z}, k=0,1} \omega_{p,k} \otimes a^{p}b^{k}$ in $M(D \otimes D)$ is a generalized $R$-matrix for $D$. This result also follows from [De-VD2,
Theorem 4.7]. Remark that for all \( p \in \mathbb{Z} \) and \( k = 0,1 \) we have \( R(\omega_{p,k} \otimes 1) \) and \( (\omega_{p,k} \otimes 1)R \) in \( D \otimes D \).

We calculate the associated multiplier \( u \) in \( M(D) \) in the sense of Lemma 2.2.

\[
\begin{align*}
 u &= \sum_{p \in \mathbb{Z}, k = 0,1} S^{-1}(a^p b^k) \omega_{p,k} \\
\end{align*}
\]

that is,

\[
\begin{align*}
 u &= \sum_{p \in \mathbb{Z}} a^{-p} \omega_{p,0} + \sum_{p \in \mathbb{Z}} (-1)^p a^{-1-p} b \omega_{p,1}. \\
\end{align*}
\]

By using the commutation rules in \( D \), we obtain

\[
\begin{align*}
 u &= \sum_{p \in \mathbb{Z}} (-1)^p \omega_{p,0} a^{-p+1} + \sum_{p \in \mathbb{Z}} \omega_{p,1} a^{-p-2} b. \\
\end{align*}
\]

Following Corollary 2.5, the inverse multiplier \( u^{-1} \) in \( M(D) \) is given as

\[
\begin{align*}
 u^{-1} &= \sum_{p \in \mathbb{Z}, k = 0,1} a^p b^k S^2(\omega_{p,k}) \\
\end{align*}
\]

that is,

\[
\begin{align*}
 u^{-1} &= \sum_{p \in \mathbb{Z}} a^p \omega_{p,0} - \sum_{p \in \mathbb{Z}} a^p b \omega_{p,1}. \\
\end{align*}
\]

Again, by using the commutation rules in \( D \), we get

\[
\begin{align*}
 u^{-1} &= \sum_{p \in \mathbb{Z}} (-1)^p \omega_{p,0} a^{p+1} + \sum_{p \in \mathbb{Z}} (-1)^p \omega_{p+1,1} a^p b. \\
\end{align*}
\]

Therefore, we have

\[
\begin{align*}
 S(u)^{-1} &= \sum_{p \in \mathbb{Z}} (-1)^p a^{-p-1} \omega_{-p,0} - \sum_{p \in \mathbb{Z}} (-1)^p a^{-1-p} b \omega_{-p,1}. \\
\end{align*}
\]

By the commutation rules in \( D \), we obtain

\[
\begin{align*}
 S(u)^{-1} &= \sum_{p \in \mathbb{Z}} \omega_{p,0} a^p - \sum_{p \in \mathbb{Z}} \omega_{p,1} a^p b. \\
\end{align*}
\]

The grouplike multiplier \( g = uS(u)^{-1} \) which installs \( S^4 \) as an inner automorphism, in the sense of Theorem 2.6, is given by the expression

\[
\begin{align*}
 g &= uS(u)^{-1} = \sum_{p \in \mathbb{Z}} (-1)^p \omega_{p,0} a^{-1} - \sum_{p,q \in \mathbb{Z}} (-1)^p \omega_{p,0} a^{-p-1} \omega_{q,1} a^q b \\
&\quad + \sum_{p,q \in \mathbb{Z}} \omega_{p,1} a^{-p-2} b \omega_{q,0} a^q - \sum_{p,q \in \mathbb{Z}} \omega_{p,1} a^{-p-2} b \omega_{q,1} a^q b. \\
\end{align*}
\]
By using the commutation rules in $D$, we obtain

$$g = uS(u)^{-1} = \sum_{p \in \mathbb{Z}} (-1)^p \omega_{p,0} a^{-1} = \delta_{\tilde{A}} \bowtie \delta_A.$$ 

Observe that $g$ is grouplike, but not trivial.

### 3. Quasitriangular $G$-cograded multiplier Hopf algebras

Let $G$ be a group and let $A$ be a $G$-cograded multiplier Hopf algebra elaborated with an admissible action $\pi$. We use the deformed multiplier Hopf algebra $(\tilde{A}, \tilde{\Delta})$ as reviewed in the preliminaries.

#### 3.1 Definition.

Let $A$ be a $G$-cograded multiplier Hopf algebra with an admissible action $\pi$. We call $A$ $\pi$-quasitriangular if there is an invertible multiplier $R \in M(A \otimes A)$ so that

1. $(\pi_p \otimes \pi_p)(R) = R$ for all $p \in G$
2. For all $a \in A$, $R\Delta(a) = \Delta^{\text{cop}}(a)R$ in $M(A \otimes A)$
3. $(\tilde{\Delta} \otimes i)(R) = R^{13}R^{23}$
4. $(i \otimes \Delta)(R) = R^{13}R^{12}$

We furthermore suppose that $R(a \otimes 1), (a \otimes 1)R \in A \otimes M(A)$ for all $a \in A$. We call $R$ the generalized $\pi$-matrix for $A$. We denote $R(a \otimes 1) = \sum R^{(1)}a \otimes R^{(2)}$ in $A \otimes M(A)$ and $(a \otimes 1)R = \sum aR^{(1)} \otimes R^{(2)}$.

If $\pi$ is the trivial action, we recover Definition 2.3. The Drinfel’d double construction in the framework of $G$-cograded multiplier Hopf algebras is again the prototype of $\pi$-quasitriangularity. This statement will be proven in Theorem 3.11.1.

#### 3.2 Lemma.

Let $(A, R)$ be a $\pi$-quasitriangular $G$-cograded multiplier Hopf algebra. Then we have

1. $(\varepsilon \otimes i)(R) = (i \otimes \varepsilon)(R) = 1$ in $M(A)$
2. $R^{-1}(a \otimes 1) = (i \otimes S^{-1})(R(a \otimes 1))$,
   $$R^{-1}(a \otimes 1) = (\tilde{S} \otimes i)((\tilde{S}^{-1}(a) \otimes 1)R)$$
3. $(\tilde{S} \otimes S)(R) = R$.

**Proof.** The proofs of this lemma are very similar to the proofs of Lemma 2.4.
The generalized $\pi$-matrix for a $G$-cograded multiplier Hopf algebra satisfies the Yang-Baxter equation in the sense of the following proposition.

### 3.3 Proposition

Let $A$ be a $\pi$-quasitriangular multiplier Hopf algebra. The $\pi$-Yang-Baxter Equation for the $\pi$-matrix $R$ is given as follows. Take $a \in A$ and $b \in A_p$, for any $p \in G$. Then we have

$$ (R^{23}R^{13}R^{12})(a \otimes b \otimes 1) = (R^{12}(i \otimes i \otimes \pi_{p^{-1}})(R^{13})R^{23})(a \otimes b \otimes 1) = (R^{12}(\pi_p \otimes i \otimes i)(R^{13})R^{23})(a \otimes b \otimes 1). $$

**Proof.** Clearly we have

$$ R^{23}R^{13}R^{12} = R^{23}((i \otimes \Delta)(R)) \overset{(*)}{=} ((i \otimes \tilde{\Delta}^{\text{cop}})(R))(1 \otimes R) $$

We here explain $(*)$. For all $a, b \in A$

$$ ((i \otimes \tilde{\Delta}^{\text{cop}})(a \otimes b))(1 \otimes R) = (a \otimes 1 \otimes 1)(1 \otimes \tilde{\Delta}^{\text{cop}}(b)R) = (a \otimes 1 \otimes 1)(1 \otimes R \Delta(b)) = (1 \otimes R)((i \otimes \Delta)(a \otimes b)). $$

Let $U = R = T$ denote the generalized $\pi$-matrix for $A$. The Equation $(*)$ means that for all $a \in A$ and $b \in A_p$, $p \in G$, we have

$$ (R^{23}R^{13}R^{12})(a \otimes b \otimes 1) = \sum ((i \otimes \tilde{\Delta}^{\text{cop}})(R))(a \otimes R^{(1)}b \otimes R^{(2)}). $$

So we have proven the first equation of the statement. The second equation easily follows by using condition (1) in Definition 3.1. $\blacksquare$

We now consider a $G$-cograded multiplier Hopf algebra $A$. Let $\pi$ be a crossing of $G$ on $A$. This means that for all $p, q \in G$, we have $\pi_p(A_q) = A_{pq^{-1}}$. Suppose that $R$ is a generalized $\pi$-matrix for $A$. We bring the results on the antipode, as proven in Section 2, to this framework.

### 3.4 Definition

Take the assumptions as above. Define the left multiplier $\tilde{u}$ of $A$ as follows

$$ \tilde{u}a = \sum \tilde{S}(R^{(2)})R^{(1)}a \quad \text{for all } a \in A. $$
3.5 Remark. Take $a \in A_p$, then $	ilde{u}a = \sum \pi_p(S(R^{(2)}))R^{(1)}a$.

3.6 Lemma. Take the notations as above. Take $p \in G$ and $a \in A_p$. Then we have

(1) $	ilde{u}$ is a two-sided multiplier in $M(A)$, the right multiplication with $	ilde{u}$ is given as follows

$$a\tilde{u} = \tilde{u}\pi_p^{-1}(S^{-2}(a)).$$

Furthermore, the multiplier $	ilde{u}$ is invariant for each automorphism $\pi_p$.

(2) $	ilde{u}$ is invertible in $M(A)$. The square of the antipode is given as follows

$$S^2(a) = \tilde{u}(\pi_p^{-1}(a))\tilde{u}^{-1}$$

or equivalently,

$$S^2(a) = S(\tilde{u})^{-1}(\pi_p(a))S(\tilde{u}).$$

Proof. (1) Take $p \in G$. Let $a, b, c \in A_p$

$$\sum R^{(1)}a_{(1)}b \otimes R^{(2)}a_{(2)} \otimes ca_{(3)} = \sum a_{(2)}R^{(1)}b \otimes \pi_p^{-1}(a_{(1)})R^{(2)} \otimes ca_{(3)}.$$ 

Therefore,

$$\sum R^{(1)}a_{(1)}b \otimes \pi_p(R^{(2)}a_{(2)}) \otimes \pi_p(ca_{(3)})$$

$$= \sum a_{(2)}R^{(1)}b \otimes a_{(1)}\pi_p(R^{(2)}) \otimes \pi_p(ca_{(3)}).$$

This implies that

$$\sum \pi_p(S^2(ca_{(3)}))\pi_p(S(R^{(2)}a_{(2)}))R^{(1)}a_{(1)}b$$

$$= \sum \pi_p(S^2(ca_{(3)}))\pi_p(S(R^{(2)}))S(a_{(1)})a_{(2)}R^{(1)}b.$$ 

The left hand side of the above equation can be written as

$$\sum \pi_p(S(R^{(2)}a_{(2)}S(ca_{(3)})))R^{(1)}a_{(1)}b$$

$$= \sum \pi_p(S(R^{(2)}S(c)))R^{(1)}ab = \pi_p(S^2(c))(\tilde{u}ab).$$
The right hand side of the above equation can be written as
\[ \sum \pi_p(S^2(ca))\pi_p(S(R^{(2)}))R^{(1)}b = \pi_p(S^2(c))\pi_p(S^2(a)(\tilde{u}b)). \]

Therefore, we have for all \(a, b \in A_p\)
\[ \tilde{u}(ab) = \pi_p(S^2(a))(\tilde{u}b). \]

So, \(\tilde{u}\) is a two-sided multiplier in \(M(A)\) where the right multiplication with \(\tilde{u}\) is given as follows
\[ a\tilde{u} = \tilde{u}\pi_p^{-1}(S^{-2}(a)) \text{ for all } a \in A_p. \]

To finish we remark that \(\tilde{u}\) is invariant for each automorphism \(\pi_p\) because we have \((\pi_p \otimes \pi_p)(R) = R\).

(2) We now prove that \(\tilde{u}\) is invertible in \(M(A)\). We define the left multiplier \(\tilde{t}\) as follows
\[ \tilde{ta} = \sum R^{(2)}S^2(R^{(1)})a \text{ for all } a \in A. \]

First one can prove, in a similar way as in (1), that \(\tilde{t}\) is a multiplier in \(M(A)\). Notice that for \(a \in A_p\), we obtain \(a\tilde{t} = \tilde{t}\pi_p(S^2(a))\).

We now prove that \(\tilde{t}\) is the inverse of \(\tilde{u}\) in \(M(A)\). Take \(a \in A_p\). Let \(R\) and \(V\) denote the \(\pi\)-matrix in \(M(A \otimes A)\).

\[
\begin{align*}
\tilde{t}\tilde{u}a &= \sum R^{(2)}S^2(R^{(1)})\tilde{u}a = \sum R^{(2)}\tilde{u}\pi_p^{-1}(R^{(1)}\pi_p(a)) \\
&= \sum R^{(2)}\pi_p(S(V^{(2)}))V^{(1)}\pi_p^{-1}(R^{(1)}\pi_p(a)) \\
&= \sum S(S^{-1}(V^{(1)}\pi_p^{-1}(R^{(1)}\pi_p(a))))\pi_p(V^{(2)})S^{-1}(R^{(2)}) \\
&\overset{(*)}{=} S(S^{-1}(a)) = a.
\end{align*}
\]

We explain (*). From Proposition 3.2(2), we have
\[
\begin{align*}
\left( \sum V^{(1)} \otimes V^{(2)} \right)(R^{(1)}a \otimes S^{-1}(R^{(2)})) &= a \otimes 1 \\
\Rightarrow \sum V^{(1)}R^{(1)}a \otimes V^{(2)}S^{-1}(R^{(2)}) &= a \otimes 1 \\
\Rightarrow \sum S^{-1}(V^{(1)}R^{(1)}a)\pi_p(V^{(2)})S^{-1}(\pi_p(R^{(2)})) &= S^{-1}(a) \\
\Rightarrow \sum S^{-1}(V^{(1)}\pi_p^{-1}(R^{(1)}\pi_p(a)))\pi_p(V^{(2)})S^{-1}(R^{(2)}) &= S^{-1}(a).
\end{align*}
\]

As \(\tilde{t}\tilde{u}\) and 1 equal as left multipliers, they also equal as multipliers in \(M(A)\).
We also have for all \( a \in A_p \)

\[
a = \tilde{t}\tilde{u} = \pi_{p^{-1}}(S^{-2}(\tilde{u}a))\tilde{t} = \pi_{p^{-1}}(S^{-2}(\pi_p(S^2(a))\tilde{u}))\tilde{t} = aS^{-2}(\tilde{u})\tilde{t}.
\]

Therefore, \( S^{-2}(\tilde{u})\tilde{t} = 1 \) and also \( \tilde{u}S^2(\tilde{t}) = 1 \) as multipliers in \( M(A) \). So we obtained that \( \tilde{u} \) is invertible, more precisely, \( (\tilde{u})^{-1} = \tilde{t} = S^2(\tilde{t}) \).

To finish, we observe that the expressions for \( S^2 \) as given in the statement of this lemma easily follow from part (1). ■

By using Lemma 3.6, we easily see that \( S^4 \) is a usual inner automorphism. Indeed, take \( a \in A_{pq} \). Then we have

\[
S^4(a) = S^2(S^2(a)) = \tilde{u}(\pi_{p^{-1}}(S^2(a)))\tilde{u}^{-1} = \tilde{u}(\pi_{p^{-1}}(S(\tilde{u})^{-1}\pi_p(a)S(\tilde{u})))\tilde{u}^{-1} = \tilde{u}S(\tilde{u})^{-1}aS(\tilde{u})\tilde{u}^{-1}.
\]

In Proposition 3.10, we prove the much stronger result that \( \tilde{u}S(\tilde{u})^{-1} \) is a group-like multiplier in \( M(A) \). First we need to prove some technical lemmas.

### 3.7 Lemma

Take \( a \in A_{pq} \) and \( b \in A_q \), then we have

\[
((i \otimes \pi_{p^{-1}})(\sigma(R)))R\Delta(a)(1 \otimes b) = ((\pi_{pq}^{-1} \otimes \pi_{p^{-1}})\Delta(a))((i \otimes \pi_{p^{-1}})(\sigma(R)))R(1 \otimes b).
\]

**Proof.** Observe that this result is trivial done if \( \pi \) is the trivial action of \( G \) on \( A \). Take \( a \in A_{pq} \) and \( b \in A_q \). Then we have

\[
R\Delta(a) = \tilde{\Delta}^{\text{cop}}(a)R
\]

\[
\Rightarrow R\Delta(a)(1 \otimes b) = \tilde{\Delta}^{\text{cop}}(a)R(1 \otimes b)
\]

\[
\Rightarrow R\Delta(a)(1 \otimes b) = \sum(a_{(2)} \otimes \pi_{p^{-1}}(a_{(1)}))R(1 \otimes b) = ((i \otimes \pi_{p^{-1}})\Delta^{\text{cop}}(a))R(1 \otimes b).
\]

Therefore,

\[
((i \otimes \pi_{p^{-1}})(\sigma(R)))R\Delta(a)(1 \otimes b)\]

\[
= ((i \otimes \pi_{p^{-1}})(\sigma(R\Delta(a))))R(1 \otimes b)
\]
\[(i \otimes \pi_{p^{-1}})\sigma(\tilde{\Delta}^{cop}(a)R))R(1 \otimes b) = ((i \otimes \pi_{p^{-1}})(\tilde{\Delta}(a))((i \otimes \pi_{p^{-1}})(\sigma(R)))R(1 \otimes b)\]

we notice that \(\pi_{p^{-1}}(a_{(2)}) \in A_q \Rightarrow a(2) \in A_{pq^{-1}}\)

\[= ((\pi_{pq^{-1}p^{-1}} \otimes \pi_{p^{-1}})\Delta(a))((i \otimes \pi_{p^{-1}})(\sigma(R)))R(1 \otimes b).\]

\[\Box\]

3.8 Lemma. Take any \(p, q \in G\). Let \(a \in A_p\) and \(b \in A_q\). Then we have

\[\Delta(\tilde{u})((i \otimes \pi_{q^{-1}})(\sigma(R)))((\pi_{p^{-1}} \otimes \pi_{q^{-1}})(R))(a \otimes b) = (\tilde{u} \otimes \tilde{u})(a \otimes b).\]

Proof. We write \(a \otimes b = \sum \Delta(a_i)(1 \otimes b_i)\) where \(a_i \in A_{pq}\) and \(b_i \in A_q\). By using Lemma 3.7, we become that

\[\sum ((\pi_{pq^{-1}p^{-1}} \otimes \pi_{p^{-1}})\Delta(a_i)((i \otimes \pi_{p^{-1}})(\sigma(R)))R(1 \otimes b_i)\]

Therefore,\n
\[(i \otimes \pi_{p^{-1}})\sigma(\tilde{\Delta}(a))((i \otimes \pi_{p^{-1}})(\sigma(R)))R(1 \otimes b)\]

\[= \sum ((\pi_{pq^{-1}p^{-1}} \otimes \pi_{p^{-1}})\Delta(\tilde{u}a_i)((i \otimes \pi_{p^{-1}})(\sigma(R)))R(1 \otimes b_i)\]

\[= \sum ((\pi_{pq^{-1}p^{-1}} \otimes \pi_{p^{-1}})\Delta(\pi_{pq}(S(R^{(2)}))R^{(1)}a_i)((i \otimes \pi_{p^{-1}})(\sigma(R)))R(1 \otimes b_i)\]

\[= \sum ((\pi_p \otimes \pi_q)\Delta(S(R^{(2)})))((\pi_{pq^{-1}p^{-1}} \otimes \pi_{p^{-1}})\Delta(R^{(1)}a_i))((i \otimes \pi_{p^{-1}})(\sigma(R)))R(1 \otimes b_i)\]

By using Lemma 3.7 to the underlined expression, the above formula becomes

\[\sum ((\pi_p \otimes \pi_q)\Delta(S(R^{(2)})))((i \otimes \pi_{p^{-1}})(\sigma(R)))R\Delta(0^{(1)}a_i)(1 \otimes b_i).\]

In the following, we use the letters \(R, U, T, V, W, Z\) to denote the generalized \(\tau\)-matrix for \(A\). The foregoing expression equals

\[\sum ((\pi_p \otimes \pi_q)\Delta(S(R^{(2)}V^{(2)})))((U^{(2)} \otimes \pi_{p^{-1}}(U^{(1)}))(T^{(1)} \otimes T^{(2)})(\pi_q(R^{(1)}) \otimes V^{(1)}))\]

\[= \sum (\pi_p(S(W^{(2)}Z^{(2)})) \otimes \pi_q(S(R^{(2)}V^{(2)})))((U^{(2)}T^{(1)} \otimes \pi_{p^{-1}}(U^{(1)}))T^{(2)})\]

\[= \sum (S(\pi_p(W^{(2)}Z^{(2)}))U^{(2)}\pi_q(W^{(1)}))\pi_q(R^{(1)})a \otimes (\pi_q(S(R^{(2)}V^{(2)})))\pi_{p^{-1}}(U^{(1)}))T^{(2)}Z^{(1)}V^{(1)}b)\]
We apply the $\pi$-Yang Baxter equation to the underlined expressions in the sense of Proposition 3.3.

$$= \sum (S(\pi_p(W^{(2)}Z^{(2)}))U^{(2)}Z^{(1)}T^{(1)}\pi_q(R^{(1)}))a \otimes (\pi_q(S(R^{(2)}V^{(2)})))\pi_{p-1}(U^{(1)})W^{(1)}T^{(2)}V^{(1)}b).$$

We notice that for all elements $x \in A$, we have

$$\sum \pi_{p-1}(U^{(1)})W^{(1)}x \otimes S(\pi_p(W^{(2)}))\pi_p(U^{(2)}) = (i \otimes \pi_p)(\sum U^{(1)}W^{(1)}x \otimes S(W^{(2)})U^{(2)}) = (i \otimes \pi_p)(i \otimes S)(x \otimes 1) = x \otimes 1.$$

Therefore, the above expression can be simplified as follows

$$\sum S(\pi_p(Z^{(2)}))Z^{(1)}T^{(1)}\pi_q(R^{(1)})a \otimes \pi_q(S(R^{(2)}V^{(2)}))T^{(2)}V^{(1)}b = \sum S(\pi_p(Z^{(2)}))Z^{(1)}a \otimes \pi_q(S(V^{(2)}))V^{(1)}b = (\tilde{u} \otimes \tilde{u})(a \otimes b).$$

So we have proven that for all $a \in A_p$ and $b \in A_q$, we have

$$((\pi_{pq}^{-1} \otimes \pi_{p-1}^{})(\sigma(R)))((i \otimes \pi_{p-1}^{})(\sigma(R)))(a \otimes b) = (\tilde{u} \otimes \tilde{u})(a \otimes b).$$

The above equation can be equivalently written as

$$\Delta(\tilde{u})((\pi_{pq}^{-1} \otimes i)(\sigma(R)))((\pi_{pq}^{-1} \otimes \pi_p^{})(R))(\pi_{pq}^{-1}(a) \otimes \pi_p^{}(b)) = (\tilde{u} \otimes \tilde{u})(\pi_{pq}^{-1}(a) \otimes \pi_p^{}(b)).$$

As $\pi$ is a crossing of $G$ on $A$, this latter equation can be written as follows. For all $a \in A_p$ and $b \in A_q$, we have

$$\Delta(\tilde{u})((i \otimes \pi_{q-1}^{})(\sigma(R)))((\pi_{p-1}^{} \otimes \pi_{q-1}^{})(R))(a \otimes b) = (\tilde{u} \otimes \tilde{u})(a \otimes b).$$

\[\blacksquare\]

3.9 Lemma. Take any $p, q \in G$. Let $a \in A_p$ and $b \in A_q$. Then we have

$$((\pi_p \otimes i)(\sigma(R)))R\Delta(\tilde{u})(a \otimes b) = (\tilde{u} \otimes \tilde{u})(a \otimes b).$$
Proof. Following Lemma 3.7, we have for all $x \in A_{pq}$ and $b \in A_q$
\[
\left( (\pi_{pq}^{-1} \otimes i)\sigma(R) \right) \left( (\pi_{pq}^{-1} \otimes \pi_p)(\Delta(x)) \right) (1 \otimes \pi_p(b))
= \Delta(x)\left( (\pi_{pq}^{-1} \otimes i)\sigma(R) \right) \left( (\pi_{pq}^{-1} \otimes \pi_p)(R) \right) (1 \otimes \pi_p(b)).
\]

The above equation can also be written as follows. For all $x \in A_{qp}$ and $b \in A_q$, we have
\[
\left( (\pi_q \otimes i)\sigma(R) \right) \left( (\pi_q \otimes \pi_p)(\Delta(x)) \right) (1 \otimes b)
= \Delta(x)\left( (\pi_q \otimes i)\sigma(R) \right) \left( (\pi_q \otimes \pi_p)(R) \right) (1 \otimes b).
\]

Therefore, we have for all $x \in M(A), b \in A_q$ and $a \in A_{qpq}^{-1}$
\[
(\pi_q \otimes \pi_p)((\pi_p \otimes i)(\sigma(R))R\Delta(x))(a \otimes b)
= \Delta(x)((i \otimes \pi_q^{-1})(\sigma(R))((\pi_{pq}^{-1} \otimes \pi_q^{-1})(R))(a \otimes b).
\]

We set $x = \tilde{u}$ and use Lemma 3.8. Then we have
\[
(\pi_q \otimes \pi_p)((\pi_p \otimes i)(\sigma(R))R\Delta(\tilde{u}))(a \otimes b) = (\tilde{u} \otimes \tilde{u})(a \otimes b).
\]

Now, the formula in the statement easily follows. ■

3.10 Proposition. The multiplier $\tilde{u} S(\tilde{u})^{-1}$ is grouplike in $M(A)$.

Proof. First, we calculate the multiplier $\Delta(S(\tilde{u}))$ in $M(A \otimes A)$ by making use of Lemma 3.9. Let $p, q$ be in $G$ and take $a \in A_p$ and $b \in A_q$,
\[
\Delta(S(\tilde{u}))(a \otimes b) = ((\sigma \circ (S \otimes S))(S(\tilde{u}))(a \otimes b)
= (((\sigma \circ (S \otimes S))(R^{-1}((\pi_{pq}^{-1} \otimes i)(\sigma(R^{-1})))(\tilde{u} \otimes \tilde{u}))(a \otimes b)
= (S(\tilde{u}) \otimes S(\tilde{u}))(i \otimes \pi_{pq}^{-1})((S \otimes (S)(R^{-1}))((S \otimes S)(S)(R^{-1}))(a \otimes b).
\]

Therefore, we have
\[
\Delta(S(\tilde{u})^{-1})(a \otimes b)
= ((S \otimes S)(R)((S \otimes S)(S)(R)))(S(\tilde{u})^{-1} \otimes S(\tilde{u})^{-1})(a \otimes b).
\]

As $(\tilde{S} \otimes S)(R) = R$, we have for all $a \in A_p$ and $b \in A_q$
\[
\Delta(S(\tilde{u})^{-1})(a \otimes b)
= ((i \otimes \pi_{pq}^{-1})(\sigma(R))((\pi_{pq}^{-1} \otimes \pi_{pq}^{-1})(R))(S(\tilde{u})^{-1} \otimes S(\tilde{u})^{-1})(a \otimes b).
\]
Take $p, q$ in $G$ and $a \in A_p, b \in A_q$, we combine the above result with Lemma 3.8 to obtain

$$
\Delta(\tilde{u}S(\tilde{u})^{-1})(a \otimes b) = (\tilde{u} \otimes \tilde{u})((\pi_{p^{-1}} \otimes \pi_q^{-1})(R^{-1}))((i \otimes \pi_q^{-1})(\sigma(R^{-1})))((i \otimes \pi_{q^{-1}})(\sigma(R)))
$$

$$
= (\pi_{p^{-1}} \otimes \pi_q^{-1})((\tilde{u})^{-1} \otimes S(\tilde{u})^{-1})(a \otimes b).
$$

So we have proven that $\tilde{u}S(\tilde{u})^{-1}$ is grouplike in $M(A)$.

3.11 The Drinfel’d double construction for $G$-cograded multiplier Hopf algebras

Let $\langle A, B \rangle$ be a pairing of multiplier Hopf algebras. Suppose that $B$ is $G$-cograded. We put $B = \bigoplus_{p \in G} B_p$. Let $\pi$ denote a crossing of $G$ on $B$. This means that for all $p, q \in G$ we have $\pi_p(B_q) = B_{pq^{-1}}$.

The Drinfel’d double construction, denoted by $D^\pi$, is reviewed in the preliminaries. We have that $D^\pi$ is again a $G$-cograded multiplier Hopf algebra with a nontrivial crossing of $G$ on $D^\pi$, see [De-Vd3, Prop. 3.13]. More precisely, we put $D^\pi_p = A \bowtie B_p$ for all $p \in G$. Define the automorphism $\pi'_p$ on $A$ by the formula $\langle \pi'_p(a), b \rangle = \langle a, \pi_{p^{-1}}(b) \rangle$. The maps $\{\pi'_p \otimes \pi_p, p \in G\}$ provide $D^\pi$ with a natural crossing of $G$ on $D^\pi$.

In this section we investigate when $D^\pi$ is a $\pi$-quasitriangular $G$-cograded multiplier Hopf algebra in the sense of Definition 3.1.

We suppose that the pairing $\langle A, B \rangle$ has a canonical multiplier $W$ in $M(B \otimes A)$. This means that $W$ is invertible in $M(B \otimes A)$ and for all $a \in A$ and $b \in B$, we have $\langle W(a), b \otimes 1 \rangle = \langle a, b \rangle$. From the definition, $W$ is unique in $M(B \otimes A)$. Following [De-Vd2, Proposition 4.3], we have for all $a \in A$ and $b \in B$, $(\langle a, \cdot \rangle \otimes i_A)(W) = a$ and $(i_B \otimes \langle \cdot, b \rangle)(W) = b$. As $\langle A_p, B_q \rangle = 0$ whenever $p \neq q$, the following ”covering” conditions on $W$ are quite natural.

For all $b \in B_p$, we assume

$$
W(b \otimes 1) \in B_p \otimes A_p \quad \text{and} \quad (b \otimes 1)W \in B_p \otimes A_p.
$$

We will use the following notation

$$
W(b \otimes 1) = \sum W^{(1)}b \otimes W^{(2)} \quad \text{and} \quad (b \otimes 1)W = \sum bW^{(1)} \otimes W^{(2)}.
$$

Recall that $A$ and $B$ are embedded in $M(D^\pi)$ in the following way

$$
A \rightarrow M(D^\pi) : a \mapsto a \bowtie 1 \quad \text{and} \quad B \rightarrow M(D^\pi) : b \mapsto 1 \bowtie b.
$$
Therefore, we have the (non-degenerate) embedding

\[ B \otimes A \longrightarrow M(D^\pi \otimes D^\pi) : b \otimes a \mapsto (1 \bowtie b) \otimes (a \bowtie 1). \]

By extending this latter algebra embedding to \( M(B \otimes A) \), it makes sense to consider \( W \in M(B \otimes A) \) as a multiplier in \( M(D^\pi \otimes D^\pi) \). The following theorem generalizes [Zun, Theorem 5.6]

3.11.1. Theorem. Take the notations and assumptions as above. The Drinfel’d double \( D^\pi \) of the pair \( \langle A, B \rangle \) is \( \pi \)-quasitriangular for the natural crossing of \( G \) on \( D^\pi \). A \( \pi \)-matrix is given by the embedding of the canonical multiplier of the pair \( \langle A, B \rangle \).

Proof. (1) For all \( p \in G \), we have \( (\pi_p \otimes \pi'_p)(W) = W \) in \( M(B \otimes A) \). Therefore, we have

\[
((\pi'_p \otimes \pi_p) \otimes (\pi'_p \otimes \pi_p))(W) = W \quad \text{in} \quad M(D^\pi \otimes D^\pi).
\]

(2) We prove

\[
(\tilde{\Delta}_D \otimes i_D)(W) = W^{13}W^{23} \quad \text{in} \quad M(D^\pi \otimes D^\pi \otimes D^\pi).
\]

Recall that for all \( a, x \in A, b \in B \) and \( y \in B_q \), we have

\[
\tilde{\Delta}(a \bowtie b)((1 \bowtie 1) \otimes (x \bowtie y)) = \sum (\pi'_q(a_{(2)}) \bowtie b_{(1)}) \otimes (a_{(1)} \bowtie b_{(2)})(x \bowtie y)
\]

Therefore,

\[
(\tilde{\Delta}_D \otimes i_D)(W) = (\Delta_B \otimes i_A)(W) \quad \text{in} \quad M(D^\pi \otimes D^\pi \otimes D^\pi)
\]

By [De-VD2, Proposition 4.4], this expression equals \( W^{13}W^{23} \).

(3) We prove

\[
(i_D \otimes \Delta_D)(W) = W^{13}W^{12} \quad \text{in} \quad M(D^\pi \otimes D^\pi \otimes D^\pi).
\]

Indeed,

\[
(i_D \otimes \Delta_D)(W) = (i_B \otimes \Delta^\text{cop}_A)(W) \quad \text{in} \quad M(D^\pi \otimes D^\pi \otimes D^\pi)
\]

Again by [De-VD2, Proposition 4.4], this expression equals \( W^{13}W^{12} \).

(4) We prove that for all \( a \in A \) and \( b \in B \)
\[
W\Delta(a \triangleright b) = \tilde{\Delta}^{\text{cop}}(a \triangleright b)W \quad \text{in} \quad M(D^{\pi} \otimes D^{\pi}).
\]

Take \(x, a \in A\), \(b \in B\), \(y \in B_{p}\) and \(y' \in B_{q}\).

\[
W\Delta(a \triangleright b)((1 \triangleright y') \otimes (x \triangleright y)) = \sum((1 \triangleright W^{(1)})(a(2) \triangleright \pi_{p^{-1}}(b(1))y')) \otimes ((W^{(2)}a(1) \triangleright b(2))(x \triangleright y)).
\]

Again use [De-VD2, Proposition 4.4] and let the canonical multiplier be denoted by the letters \(W, T, V\). Then the expression above can be written as

\[
\sum((a(2), S^{-1}(T^{(1)}))(a(4), \pi_{q^{-1}}(W^{(1)}))(a(3) \triangleright V^{(1)}\pi_{p^{-1}}(b(1))y') \otimes ((W^{(2)}V^{(2)}T^{(2)}a(1) \triangleright b(2))(x \triangleright y))) = (a(1) \triangleright V^{(1)}\pi_{p^{-1}}(b(1))y') \otimes ((\pi'_{q}(a(2))V^{(2)} \triangleright b(2))(x \triangleright y)).
\]

Again, take \(a, x \in A\), \(b \in B_{pq}\), \(y \in B_{p}\) and \(y' \in B_{q}\). We calculate the expression

\[
\tilde{\Delta}^{\text{cop}}(a \triangleright b)W((1 \triangleright y') \otimes (x \triangleright y)) = \tilde{\Delta}^{\text{cop}}(a \triangleright b)((1 \triangleright W^{(1)}y') \otimes (W^{(2)}x \triangleright y)) = \sum(a(1) \triangleright b(2)W^{(1)}y') \otimes ((\pi'_{q}(a(2)) \triangleright b(1))(W^{(2)} \triangleright 1)(x \triangleright y)) = \sum(W^{(2)}, S^{-1}(b(3))\langle T^{(2)}, \pi_{p^{-1}}(b(1))\rangle(a(1) \triangleright b(4)W^{(1)}V^{(1)}T^{(1)}y') \otimes ((\pi'_{q}(a(2))V^{(2)} \triangleright b(2))(x \triangleright y)) = \sum(a(1) \triangleright V^{(1)}\pi_{p^{-1}}(b(1))y') \otimes ((\pi'_{q}(a(2))V^{(2)} \triangleright b(2))(x \triangleright y)).
\]

So, we have proven for all \(a \in A\) and \(b \in B\)

\[
W\Delta(a \triangleright b) = \tilde{\Delta}^{\text{cop}}(a \triangleright b)W \quad \text{in} \quad M(D^{\pi} \otimes D^{\pi}).
\]

We now conclude that the embedding of \(W\) in \(M(D^{\pi} \otimes D^{\pi})\) makes \(D^{\pi}\) into a \(\pi\)-quasitriangular multiplier Hopf algebra, in the sense of Definition 3.1. ■

3.11.2. "Finite type" Hopf group-coalgebras We give concrete data to illustrate Theorem 3.11.1. Let \(G\) be any group. Consider a "finite type" Hopf \(G\)-coalgebra as given in [T-Section 11]. In [A-De-VD, Theorem 1.5] we have shown how to consider any Hopf group-coalgebra into the framework of \(G\)-cograded multiplier
Hopf algebras. In this framework, a "finite type" Hopf $G$-coalgebra is dealt as a $G$-cograded multiplier Hopf algebra $A = \bigoplus_{p \in G} A_p$ where each $A_p$ is a finite-dimensional algebra with a unit. A crossing in the sense of [T-Section 11] defines a crossing on $A$. In [A-De-VD, Theorem 3.9] we have constructed an integral on $A$. Therefore, we can consider the reduced dual multiplier Hopf algebra which is given by the Hopf algebra $A^* = \bigoplus_{p \in G} (A_p)'$, see [A-De-VD, Corollary 3.4].

Consider the pair $\langle A^*, A \rangle$. Let $\{f_{pi}\} \subset (A_p)'$ and $\{e_{pi}\} \subset A_p$ be dual bases. Then the canonical multiplier of the pair $\langle A^*, A \rangle$ is given as

$$W = \sum_{p,i} e_{pi} \otimes f_{pi} \text{ in } M(A \otimes A^*).$$

Clearly, $W(a \otimes 1)$ and $(a \otimes 1)W$ are in $A \otimes A^*$ for all $a \in A$. By Theorem 3.11.1, we obtain that the embedding of $W$ in $M(D^\pi \otimes D^\pi)$ is a $\pi$-quasitriangular structure for $D^\pi$, considered for the natural crossing $\{\pi'_p \otimes \pi_p, p \in G\}$. The embedding of $W$ in $M(D^\pi \otimes D^\pi)$ is considered as $W = \sum_{p \in G} (1 \otimes e_{pi}) \otimes (f_{pi} \otimes 1)$. This result is also in [Zun, Theorem 5.6].

Take the "finite type" Hopf group-coalgebras where $\dim(A_p) = 1$ for all $p \in G$. The $G$-cograded multiplier Hopf algebra $A$ is given as $A = K(G)$, the multiplier Hopf algebra of all complex-valued functions with finite support in $G$, see Preliminaries. The reduced dual multiplier Hopf algebra $A^*$ is now given by the usual group Hopf algebra $A^* = k[G]$. The canonical multiplier of the pair $\langle k[G], K(G) \rangle$ is given as $W = \sum_{p \in G} \delta_p \otimes u_p$ in $M(K(G) \otimes k[G])$. The crossing of $G$ on $K(G)$ is given as $\pi_p(\delta_q) = \delta_{pq}^{-1}$ for all $p, q \in G$. We have $\widehat{K(G)} = K(G)^{cop}$. The Drinfel’d double is just a usual tensor product $D^\pi = k[G] \otimes K(G)^{cop}$. An easy calculation shows that $\tilde{u} = \sum_p u_p^{-1} \otimes \delta_p$ in $M(D^\pi)$. We have in this case $\tilde{u}S(\tilde{u})^{-1} = u_e \otimes \sum_{p \in G} \delta_p = 1$ in $M(D^\pi)$.

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