We analyze the vacuum structure of SU(2) lattice gauge theories in $D = 2, 3, 4$, concentrating on the stability of ’t Hooft loops. High precision calculations have been performed in $D=3$; similar results hold also for $D=4$ and $D=2$. We discuss the impact of our findings on the continuum limit of Yang-Mills theories.
1. Introduction

Most of the ideas aimed at solving confinement in $SU(N)$ Yang-Mills theories involve topological degrees of freedom of some sort. Among these, $Z_N$ (i.e. center) vortices have received much attention in the literature in general and in lattice investigations in particular [2, 3].

The gauge group of pure Yang-Mills $SU(N)\!/Z_N$ possesses a non trivial first homotopy class, corresponding to its center, \( \pi_1(SU(N)\!/Z_N) = Z_N \). A super-selection rule will thus arise for the physical Hilbert space of gauge invariant states [4], with sectors labeled by a center vortex topological index \( n \in Z_N \). According to ’t Hooft’s original idea [1], the low temperature confinement phase should then correspond to a superposition of all topological sectors, while above the deconfinement transition vortex symmetry gets broken to the trivial sector \( n = 0 \); the ’t Hooft loop \( H \), dual to the Wilson loop \( W \), is the natural observable to describe the transition, “counting” the number of topological vortices piercing it. From \( H \) one can reconstruct the free energy for vortex creation, \( F = \Delta U - T \Delta S \), which should jump at \( T_c \); the monitoring of such behaviour across the deconfinement transition has received broad attention in the literature [5, 6, 7, 8, 9].

The natural choice to investigate \( F \) upon lattice discretization of Yang-Mills theories would be to define the partition function through the adjoint Wilson action \( Z \sim e^{\beta A \text{Tr}_A(U)} \), transforming under $SU(N)\!/Z_N$; in this case all topological sectors are dynamically included [10]. Universality should of course allow the equivalent use of the standard Wilson plaquette action \( S \sim \text{Tr}_F(U) \). In this case topology must however be introduced “by hand” summing over all twisted boundary conditions\(^1\). The “proper” partition function \( \tilde{Z} \) can then be defined through the weighted sum of all partition functions with fixed twisted b.c.\(^2\). Since each of them must be determined by independent simulations, their relative weights can only be calculated through indirect means [6, 7, 11].

There is however a loophole in the argument given above. The two partition functions \( Z \) and \( \tilde{Z} \) can only be shown to be equivalent when $Z_N$ magnetic monopoles are absent [10, 12]. Taking the explicit case of $SU(2)\!/Z_2 = SO(3)$, this translates into the constraint:

\[
\sigma_c = \prod_{P \in \partial c} \text{sign} (\text{Tr}_F U_P) = 1 \tag{1.1}
\]

being satisfied for every elementary 3-cube \( c \), where \( U_P \) denotes the plaquettes belonging to the cube surface \( \partial c \). This ensures that endpoints of open center vortices, $Z_2$ magnetic monopoles, are suppressed and only closed, i.e. topological $Z_2$ vortices winding around the boundaries can form.

The above condition is usually quoted when claiming that the bulk transition separating the strong and weak coupling regime along $\beta_A$ [13, 14, 15, 16, 17] constitutes an obstacle in defining the continuum limit for the adjoint Wilson action. This however assumes that topological sectors along the fundamental coupling $\beta$ are always well defined. We will show this not to be the case. Together with a set of established results demonstrating that above the adjoint bulk transition topological sectors are well defined and a physical continuum limit of the theory exists [8, 9, 18, 19, 20, 21, 22, 23, 24, 25], we can turn the argument around, casting doubts that investigations of vortex topology for the fundamental Wilson are well defined. Preliminary results had been presented in Ref. [26].

\(^1\)Such topological boundary conditions also play a rôle in lattice investigations of the string spectrum or large $N$ reduction.

\(^2\)See e.g. Ref. [11], Chapt. 3.
2. Setup

We will investigate as a test case the $SU(2)$ fundamental Wilson action with periodic b.c.:

$$ S = \beta \sum_{x, \mu > \nu} [1 - \text{Tr}_F(U_{\mu\nu}(x))] $$

(2.1)

in $D = 2, 3, 4$ dimensions. Different groups or b.c. can be considered as well and won’t change the main results given below.

The twist operator, measuring the number of topological vortices piercing the ’t Hooft loop in the $\mu, \nu$ planes, can be constructed via [5]:

$$ z_{\mu\nu} = \frac{1}{L^{D-2}} \sum_{y_{\perp \mu \nu \text{plane}}} \prod_{x \in \mu \nu \text{plane}} \text{sign}(\text{Tr}_F(U_{\mu\nu}(x, y))). $$

(2.2)

When topological sectors are well defined $z_{\mu\nu}$ takes values $\pm 1$ for all fixed $\mu$ and $\nu$; e.g. in the case at hand, i.e. periodic b.c., the topological sector must be trivial and one should always have $z_{\mu\nu} = 1 \ \forall \mu, \nu$. It is now easy to define an order parameter $z$ such that $z = 1$ if, whatever the b.c., the vortex topology takes the correct value expected in the continuum theory, while $z = 0$ when $\mathbb{Z}_2$ monopoles are still present and open $\mathbb{Z}_2$ center vortices dominate the vacuum, making the identification of topological sectors ill defined at best:

$$ z = 1 - |z_{12} - \langle z_{12} \rangle| \quad D = 2 $$

(2.3)

$$ z = \frac{2}{D(D-1)} \sum_{\mu > \nu = 1}^D \langle |z_{\mu\nu}| \rangle \quad D \geq 3. $$

(2.4)

In the following we will monitor $z$ as a function of $\beta$ and the volume $L^D$. The continuum limit of our lattice discretization is of course defined by taking the thermodynamic limit $L \to \infty$ first and then the weak coupling limit $\beta \to \infty$.

3. Results

The $D = 2$ case offers an interesting cross-check of the numerical results in higher dimensions, since here everything can be calculated analytically. For the order parameter $z$ and its susceptibility $\chi$ we have, for fixed volume $L^2$:

$$ \langle z \rangle_L = e^{-4L^2 p(\beta)}; \quad \chi_L = L^2 \left[ e^{-4L^2 p(\beta)} - e^{-8L^2 p(\beta)} \right] $$

(3.1)

$$ p(\beta) = \frac{1}{2} \left[ 1 - \frac{L(\beta)}{\hat{L}(\beta)} \right] = \sqrt{\frac{2\beta}{\pi}} e^{-\beta} (1 + \mathcal{O}(1/\beta)),$$

(3.2)

where $L$ and $\hat{L}$ denote the modified Struve and Bessel functions, respectively. Plotting the above functions (see Fig. (1)) we can clearly distinguish a “strong” coupling regime, where the topology is ill defined, and a “weak” coupling one, where $z$ takes the correct value it should have in the continuum theory. The finite size scaling analysis can be performed exactly, giving for the susceptibility peaks and the corresponding pseudo-critical coupling:

$$ \beta_c(L) = \ln L^2 + \frac{1}{2} \ln \ln L^2 + \mathcal{O}(1); \quad \chi_L(\beta_c(L)) = \frac{L^2}{4}. $$

(3.3)
From the above equation it is easy to extract the critical behaviour of the correlation length around the critical coupling $\beta_c = \infty$, including logarithmic corrections:

$$\xi \sim \sqrt{2} \frac{\pi \ln 2}{2\beta} e^{\frac{1}{2}\beta}$$

i.e. an essential scaling\(^3\) with critical exponents $\nu = 1$, $\eta = 0$ and $r = 0$.

Summarizing, although for any fixed volume $L^2$ one can always find a coupling above which the topology corresponds to that dictated by the boundary conditions, taking the thermodynamic limit first, as one should, the “strong” coupling regime extends to $\beta = \infty$ and the system is always in the disordered phase. A vortex topology cannot be defined.

Turning now to the $D = 3$ case and using Monte-Carlo simulations to calculate $z$ we basically get the same picture. In Fig. (2) we show the susceptibility $\chi$ and its FSS with an essential scaling Ansatz, again with $\beta_c = \infty$:

$$\beta_c(L) \sim A \ln L^2 + B \ln \ln L^2; \quad \chi_L(\beta_c(L)) \sim C L^2 \ln^{-2r} L$$

The result is the same, i.e. in the continuum limit the theory is always in disordered phase and no vortex topology can be defined. The values obtained for $\beta_c$ are well within the scaling region and for fixed volumes $L^3$ they are always lower than the pseudo-critical coupling at which the “finite temperature” transition for time length $L$ would be measured.

The situation is inverted at $D = 4$. Here we still get the same result as above for our order parameter $z$, i.e. an essential scaling as in Eq. (3.5). The values of the pseudo-critical coupling $\beta_c$ are however higher than the values measured for the deconfinement transition at time length $L$, cfr. Fig. (3), making the scale at which topological vortices stabilize way above any physical scale involved in the process.

\(^3\)Compare with the critical behaviour of the XY model, $\xi \sim \exp(\beta t - \nu)$ and $\chi \sim \frac{2}{\pi} \ln^{-2\nu} \xi$, with $t = |T/T_c - 1|$ the reduced temperature and $\nu = 1/2$, $\eta = 1/4$ $r = -1/16$ the critical exponents [27].
Figure 2: Left: susceptibility $\chi$ of the order parameter $z$ in $D=3$. Right: same with FSS as in Eq. (3.5).

Figure 3: Left: susceptibility $\chi$ of the order parameter $z$ in $D=4$. Right: same with FSS as in Eq. (3.5).

4. Conclusions

We have shown that $\forall D \leq 4$ the vortex topology for the standard Wilson action is always ill defined in the continuum limit. This is driven by a too slow fall-off of discretization artefacts density, i.e. $\mathbb{Z}_2$ magnetic monopoles Eq. (1.1). This means that for any fixed $\beta$ there always exists a lattice size $L$ for which enough open center vortices can form, spoiling the identification of topological sectors and making a measurement of the conjectured super-selection rule in the thermodynamic limit ill-defined.

The details of such bulk effect will of course strongly depend on the discretization chosen. For example, the separation among the regimes in $D=3$ and $D=4$ are substantially different. While in $D=3$ the deconfinement transition for fixed length $L$ always lies in the spurious phase above the pseudo-critical coupling $\beta_c(L)$ Eq. (3.5), for $D=4$ the latter is always way above the physical scale, making the physical volumes quite small.
Other choices for the discretization would of course change the picture. For example, in $D = 4$ one could define the theory through the adjoint Wilson action, for which, above the the bulk transition [13, 14, 15, 16], topology is well defined Ref. [8, 9]. There it was however found that $F \neq 0$ in the confined phase, calling for more investigations of the standard center vortices symmetry breaking argument as a model for confinement. Simulations in this case are however technically quite demanding.

Another choice, which would in principle work in any dimensions, would be to define the discretized theory through a Positive Plaquette Model [28]. In this case the $\mathbb{Z}_2$ magnetic monopole constraint is always satisfied while the order parameter $z \equiv 1$ by construction. As was proven in Ref. [25], this is indeed equivalent to simulating the adjoint Wilson action in a fixed topological sector. This model should then be the discretization of choice if one is interested in investigating the role of center vortices.

We would like to stress that none of the above results contradicts universality. First, universality hold as long as no discretization artifacts obstacle the continuum limit. In our case, center monopoles and the associated open center vortices spoil the equivalence among different discretizations. Second, all physical properties measurable in “experiments”, like glueball masses and the critical exponents at the transition, should be reflected by physical observables which can be defined irrespectively of the discretization chosen; this however does not mean that everything that can be defined in a given discretization should acquire a physical meaning.

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