ON THE BOXICITY OF KNESER GRAPHS AND COMPLEMENTS OF LINE GRAPHS

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Abstract

An axis-parallel \(d\)-dimensional box is a cartesian product \(I_1 \times I_2 \times \cdots \times I_b\) where \(I_i\) is a closed sub-interval of the real line. For a graph \(G = (V, E)\), the boxicity of \(G\), denoted by \(\text{box}(G)\), is the minimum dimension \(d\) such that \(G\) is the intersection graph of a family \((B_v)_{v \in V}\) of \(d\)-dimensional boxes in \(\mathbb{R}^d\).

Let \(k\) and \(n\) be two positive integers such that \(n \geq 2k + 1\). The Kneser graph \(Kn(k, n)\) is the graph with vertex set given by all subsets of \(\{1, 2, \ldots, n\}\) of size \(k\) where two vertices are adjacent if their corresponding \(k\)-sets are disjoint. In this note we derive a general upper bound for boxicity of the Kneser graph of any graph \(G\), and as a corollary we derive that boxicity is at most \(\Theta(\log n\log k)\).

Keywords: Boxicity, Kneser Graphs, Line Graphs, Graph Theory

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1 Introduction

An axis-parallel \(d\)-dimensional box is a cartesian product \(I_1 \times I_2 \times \cdots \times I_b\) where \(I_i\) is a closed sub-interval of the real line. For a graph \(G = (V, E)\), the boxicity of \(G\), denoted by \(\text{box}(G)\), is the minimum dimension \(d\) such that \(G\) is the intersection graph of a family \((B_v)_{v \in V}\) of \(d\)-dimensional boxes in \(\mathbb{R}^d\). Boxicity has been introduced by Roberts \(19\) in 1969 and has been extensively studied since then, see for example \(1, 4, 5, 9, 10, 20\).

Let \(k\) and \(n\) be two positive integers such that \(n \geq 2k + 1\). The Kneser graph \(Kn(k, n)\) is the graph with vertex set given by all subsets of \(\{1, 2, \ldots, n\}\) of size \(k\) where two vertices are adjacent if their corresponding \(k\)-sets are disjoint. The notion of Kneser graph was born in 1955 in a paper of Kneser \(17\), where he conjectured that the chromatic number \(\chi(\text{Kn}(k, n))\) is equal to \(n - 2k + 2\). In 1978 Lovasz \(18\) settled this conjecture with a brilliant topological proof. Since then, several papers have focused on the properties of this family, see for example \(6, 11, 15\).

In this note we are interested in deriving bounds on the boxicity of Kneser graphs. In particular, we prove the following theorems.

**Theorem 1.1.** Fix two positive integers \(k, n\) with \(n \geq 2k + 1\). The boxicity of the Kneser graph \(Kn(k, n)\) is at most \(n - 2\). Moreover, if \(n \geq 2k^3 - 2k^2 + 1\), then \(\text{box}(Kn(k, n)) \geq n - \frac{13k^2 - 11k + 16}{2}\).

In general, less precise lower bounds can be obtained without the assumption \(n \geq 2k^3 - 2k^2 + 1\) by exploiting the relationship between boxicity and poset dimension proved by \(1\). We will have a quick glance at the main technique at the end of Section \(11\) and we invite the reader to consult the papers \(1, 12, 16\) for further details.

The second main part of this paper deals with the boxicity of complements of line graphs and in particular the boxicity of the Kneser graph \(Kn(2, n)\). The line graph \(L(G)\) of a graph \(G\) has vertex set \(E(G)\) and edge set \(\{(uv, vw) : uv, vw \in E(G)\}\). Denote by \(G^c\) the complement graph of a graph \(G\), by \(\delta(G)\) the minimum degree of \(G\), and by \(\Delta(G)\) the maximum degree of \(G\).

The aim of the next theorem is twofold: first, it gives a sharper lower bound on boxicity of \(Kn(2, n)\) than Theorem \(11\) and second, it generalises this lower bound to complements of line graphs, which are realised as induced subgraphs of \(Kn(2, n)\).
**Theorem 1.2.** Let $G$ be any graph on $n$ vertices of maximum degree $\Delta = \Delta(G) \geq 3$ and let $H = L(G)$ denote its line graph. Then, the boxicity of the complement of $H$ is at most $n - 2$. Moreover,

- $\text{box}(H^c) \geq \frac{|E(H)|}{12}$, if $\Delta = 3$;
- $\text{box}(H^c) \geq \frac{|E(H)|}{16}$, if $\Delta = 4$;
- $\text{box}(H^c) \geq \frac{2|E(H)|}{\Delta^2 + 3\Delta}$, if $\Delta \geq 5$.

**Remark 1.3.** If $\Delta(G) = 2$, then both $G$ and $H$ are unions of disjoint paths and cycles. For this particular kind of graphs, Corollary 3.3 in [7] together with Lemma 3 in [21] imply that $\text{box}(H^c) = \sum_{i=1}^{k} \left\lceil \frac{|E(H_i)|}{3} \right\rceil$, where $(H_i)_{i \in [k]}$ are the connected components of $H$.

**Corollary 1.4.** For every $n \geq 5$, the boxicity of the Kneser graph $Kn(2,n)$ is either $n-3$ or $n-2$.

**Proof of Corollary 1.4 assuming Theorem 1.2.** Note that $Kn(2,n)$ is the complement of the line graph of the complete graph $Kn$. Then, Theorem 1.2 applied for $G = Kn$ with $\Delta = n - 1$ shows the upper bound, and since $n \geq 5$, we also have

$$\frac{n(n-1)(n-2)}{32} \geq \frac{60}{32} - \frac{n(n^2 - 3n)}{(n-1)(n+2)} \geq (n-4).$$

Therefore, $\text{box}(G) \geq n - 3$, which proves the corollary.

### 1.1 Plan of the paper

In Section 2, we introduce several preliminary results. In Section 3, we prove the upper bound in Theorem 1.1. In Section 4, we prove the lower bound in Theorem 1.1 and we discuss how to obtain general lower bounds through some already known result about poset dimension. In Section 5, we prove Theorem 1.2. We conclude the paper with a related discussion in Section 6.

### 2 Preliminaries

#### 2.1 Preliminaries on interval graphs

Let $V$ be a ground set and $\mathcal{F}$ a family of subsets of $V$. The intersection graph of $\mathcal{F}$ is the graph with vertex set $\mathcal{F}$ and edge set \{ $S_1S_2 : S_1, S_2 \in \mathcal{F}, S_1 \cap S_2 \neq \emptyset$ \} A graph $G$ is an interval graph if it can be represented as the intersection graph of a family of closed subintervals of the real line (see Golumbic [13] or Gyárfás [14] for a survey).

In view of the proof of the upper bound in Theorem 1.1, it will be useful to restate the geometric definition of boxicity in terms of interval graphs.

**Observation 2.1** ([4, Theorem 3]). A graph $G = (V,E)$ has boxicity at most $k$ if and only if there are $k$ interval graphs $I_i = (V,E_i)$ for $i \in [k]$ such that $E(G) = \bigcap_{i \in [k]} E(I_i)$, or equivalently $E(G^c) = \bigcup_{i \in [k]} E(I_i^c)$.

#### 2.2 Graph theoretic preliminaries

For a graph $G = (V,E)$ and a set $S \subseteq V$, denote

$$N_G(S) = \{ u \in V \setminus S \mid \forall v \in S, uv \in E \}.$$

Moreover, let

$$c(k,G) = \max_{S \subseteq V \mid |S| = k} |N_G(S)|.$$

Note that for any $i,j \in \mathbb{N}$, $c(i,G) = j$ if and only if $G$ contains a copy of the complete bipartite graph $K_{i,j}$ as a subgraph, and contains no copy of $K_{i,j+1}$.

The following lemma appears as Theorem 2 in [2].

**Lemma 2.2** ([2, Theorem 2]). Let $G$ be a non-complete graph on $n$ vertices. Then,

$$\text{box}(G) \geq \frac{|E(G^c)|}{\sum_{i=1}^{n-1} c(i,G^c)}.$$
2.3 Other preliminaries

We finish the preliminary section with two classical inequalities.

**Lemma 2.3** (Bernoulli’s inequality, see e.g. [3]). For every real number \( a > -1 \) and for every positive integer \( n \) we have that \((1 + a)^n \geq 1 + an\).

We finally state the Erdős-Ko-Rado Theorem [8] - one of the most fundamental results in set theory.

**Theorem 2.4** ([8]). Fix two positive integers \( k, n \) with \( 2k \leq n \). Let \( \mathcal{A} \) be a family of \( k \)-subsets of \([n]\) such that any pair of sets have a non-empty intersection. Then, \( |\mathcal{A}| \leq \binom{n-1}{k-1} \), and if \( 2k + 1 \leq n \), equality holds only for families of \( k \)-subsets of \([n]\), all containing a fixed element \( i \leq n \).

3 Proof of the upper bound in Theorem 1.1

Fix two positive integers \( k, n \) with \( 4 \leq 2k \leq n - 1 \). We will construct a covering of the complement of the graph \( Kn(k, n) \) with complements of \( n - 2 \) interval graphs, and then conclude by Theorem 2.4. Below, we adopt the convention that \( \binom{n}{b} = 0 \) if \( b < 0 \). For every \( i \in [n - 2] \), define the interval graph \( I_i \) as follows.

- To every set \( S \subseteq [n], |S| = k \), if neither of \( i, n - 1, n \) is in \( S \), assign the interval \( \mathbb{R} \) to \( S \).
- Assign each of the intervals \([2i, 2i + 1])_{0 \leq i \leq (n-3) - 1}\) to a different set among the \( \binom{n-3}{k-3} \)-subsets of \([n]\) containing \( \{i, n - 1, n\} \).
- Assign each of the intervals \( 2i + 2\binom{n-3}{k-3}, 2i + 2\binom{n-3}{k-3} + 1 \)\) to a different set among the \( \binom{n-3}{k-2} \)-subsets of \([n]\), containing \( i \) and \( n \), but not \( n - 1 \).
- Assign each of the intervals \( 2i + 2\binom{n-3}{k-3} + 2\binom{n-3}{k-2}, 2i + 2\binom{n-3}{k-3} + 2\binom{n-3}{k-2} + 1 \)\) to a different set among the \( \binom{n-3}{k-1} \)-subsets of \([n]\), containing \( i \), but neither \( n \) nor \( n - 1 \).
- Assign each of the intervals \( 2i + 2\binom{n-3}{k-3} + 2\binom{n-3}{k-2} + 2\binom{n-3}{k-3} + 2\binom{n-3}{k-2} + 2\binom{n-3}{k-1} + 1 \)\) to a different set among the \( \binom{n-3}{k-2} \)-subsets of \([n]\), containing \( i \) and \( n - 1 \), but not \( n \).
- Assign the interval \( 2\binom{n-3}{k-3} + 2\binom{n-3}{k-2}, 2\binom{n-3}{k-3} + 2\binom{n-3}{k-2} + 2\binom{n-3}{k-1} - 1 \)\) to all of the \( \binom{n-3}{k-2} \)-subsets of \([n]\), containing \( n - 1 \) and \( n \), but not \( i \).
- Assign the interval \( 2\binom{n-3}{k-3} + 2\binom{n-3}{k-3} + 2\binom{n-3}{k-2} + 2\binom{n-3}{k-1} - 1 \)\) to all of the \( \binom{n-3}{k-1} \)-subsets of \([n]\), containing \( n - 1 \), but neither \( n \) nor \( i \).
Figure 1: For $i \in [n - 2]$, the figure represents the positions of all finite intervals in $I_i$. Dotted lines correspond to a number of consecutive disjoint intervals, solid lines correspond to single intervals. On every line (solid or dotted) is denoted the exact subset of $\{i, n - 1, n\}$, which is included in the sets, corresponding to the particular interval or group of disjoint intervals.

- Assign each of the intervals
  \[
  \left[ 2 \binom{n - 3}{k - 3} + 2 \binom{n - 3}{k - 2}, 2 \binom{n - 3}{k - 3} + 4 \binom{n - 3}{k - 2} + 2 \binom{n - 3}{k - 1} - 1 \right]
  \]
  to a different set among the $\binom{n - 3}{k - 1}$ $k$-subsets of $[n]$, containing $n$, but neither $n - 1$ nor $i$.

Figure 1 shows a representation of the described intervals.

For every $i \in [n]$, denote by $K_i$ the complete graph on all vertices in $Kn(k, n)$, corresponding to sets, containing $i$. One may readily check that:

- $\bigcup_{j \in [n]} E(K_j) = E(Kn(k, n)^c)$,
- for every $i \in [n - 2]$, $E(K_i^c) \subset E(I_i^c) \subset E(Kn(k, n)^c)$, and
- the cliques $K_{n-1}$ and $K_n$ are both contained in $\bigcup_{i \in [n-2]} I_i^c$.

This shows that $\bigcup_{i \in [n-2]} E(I_i^c) = E(Kn(k, n)^c)$, concluding the proof.

### 4 Proof of the lower bound in Theorem 1.1

For any $k \geq 2$ and $n \geq 2k^3 - 2k^2 + 1$, fix $G = Kn(k, n)$ and $c(\cdot) = c(\cdot, G^c)$, where $c(\cdot, \cdot)$ was defined just before Lemma 2.2.

**Observation 4.1.** For any $i, j \in \mathbb{N}$, if $c(i) = j$, then $c(j + 1) < i \leq c(j)$.

**Proof.** The fact that $c(i) = j$ means that there is a copy of the complete bipartite graph $K_{i,j}$, included in $G^c$, but no copy of $K_{i,j+1}$ could be realised as a subgraph of $G^c$, which is equivalent to our claim.

**Corollary 4.2.** For any $i, j \in \mathbb{N}$, if $c(i) = j$ and $c(i + 1) < j$, then $c(j) = i$.

**Proof.** Fix $s = c(i+1)$. By Observation 4.1 for $(i,j)$ we have that $i \leq c(j)$, and since $s < j$, by the same result for $(i+1,j)$ we deduce that $c(j) \leq c(s+1) < i+1 \leq c(s)$, which proves the corollary.

A more visual interpretation of the last corollary is the following. Consider a Young diagram with columns of altitude $(c(i))_{1 \leq i \leq \left\lfloor \frac{n}{k} \right\rfloor - 1}$, that is, for every $i \in \left[ \left\lfloor \frac{n}{k} \right\rfloor - 1 \right]$, the column over $[i-1,i]$ has height $c(i)$ (see Figure 2).

Then, Observation 4.1 and Corollary 4.2 imply that this Young diagram is symmetric with respect to the line $y = x$.

It follows that, when computing $\sum_{i=1}^{\left\lfloor \frac{n}{k} \right\rfloor - 1} c(i)$, it is sufficient to compute the area of the diagram above the line $y = x$ and to multiply by two. Let $t = \max\{i : c(i) \geq i\}$. The expression of this area is given by

$$
\sum_{i=1}^{t} \left( c(i) - \frac{(i - 1/2)}{2} \right) = \sum_{i=1}^{t} c(i) - \frac{t^2}{2}.
$$

(1)

Call a bipartite graph with given parts $(V_1, V_2)$ balanced if $|V_1| = |V_2|$.
Lemma 4.3. The largest balanced complete bipartite graph, contained in $G^c$ as a subgraph, contains $2 \left\lfloor \frac{1}{2} \left( \frac{n-1}{k-1} \right) \right\rfloor$ vertices, or equivalently $t = \left\lfloor \frac{1}{2} \left( \frac{n-1}{k-1} \right) \right\rfloor$.

Proof. Denote by $s$ the number of vertices in every part of a largest balanced complete bipartite subgraph of $G^c$. First of all, there is a clique in $G^c$ that contains \( n - 1 k - 2 \), so $s \leq \left\lfloor \frac{1}{2} \left( \frac{n-1}{k-1} \right) \right\rfloor$.

To prove the upper bound, we argue by contradiction. Suppose that $2s \geq \left( \frac{n-1}{k-2} \right)$. Then, by Theorem 2.4 there are two vertices, corresponding to disjoint $k$-subsets $A$ and $B$ of $[n]$, and clearly these vertices must be contained in the same part. Then, each of the sets corresponding to a vertex in the other part must contain one element from both $A$ and $B$. Thus, there are at most $k^2 \left( \frac{n-2}{k-2} \right)$ vertices in every part, and since $n \geq 2k^3 \frac{k^2}{2} + 1$, one may deduce that

$$2s \leq 2k^2 \left( \frac{n-2}{k-2} \right) \leq \left( \frac{n}{k-1} \right).$$

This contradiction concludes the proof of the lemma.

Observation 4.4. For every triplet of positive integers $(a, b, c)$ such that $a > c$,\[ \binom{a}{b} - \binom{a-c}{b} \leq c \frac{a-1}{b-1}. \]

Proof. Using Pascal’s identity $c$ times we deduce that

$$\binom{a}{b} - \binom{a-c}{b} = \sum_{i=1}^{c} \binom{a-i+1}{b} - \binom{a-i}{b} = \sum_{i=1}^{c} \binom{a-i}{b-1}.$$  

Then, the result follows by the trivial inequality $\frac{a-i}{b-1} \leq \frac{a-1}{b-1}$ for every $i \geq 1$.

Corollary 4.5. For every $k \geq 2$ and $n \geq 2k^3 \frac{k^2}{2} + 1$,

$$\sum_{i=1}^{n-2} c(i) \leq k^2 \left( \frac{n-3}{k-3} \right) \left( \frac{n-1}{k-1} \right).$$

Proof. For every $i$ we have that $c(i) \leq c(1) = \binom{n}{k} - \binom{n-k}{k} - 1$, so

$$\sum_{i=1}^{n-2} c(i) \leq \left( \binom{n-2}{k-2} - \binom{n-2-k}{k-2} \right) \left( \binom{n}{k} - \binom{n-k}{k} - 1 \right) \leq k^2 \left( \frac{n-3}{k-3} \right) \left( \frac{n-1}{k-1} \right),$$

where the last inequality is achieved by two consecutive applications of Observation 4.4.
Fix a family $\mathcal{F}_i$ of $i$-k-subsets of $[n]$.

**Lemma 4.6.** Suppose that the intersection of all sets in the family $\mathcal{F}_i$ contains exactly one element of $[n]$. Then, there are at most $\binom{n-1}{k-1} - i + (k-1)^2 \binom{n-3}{k-2}$ k-subsets of $[n]$ that are not contained in $\mathcal{F}_i$ and intersect each of the sets in $\mathcal{F}_i$.

**Proof.** Without loss of generality let the common element of all sets in $\mathcal{F}_i$ be 1. Let $A = \{a_1 = 1, a_2, \ldots, a_k\}$ be a member of $\mathcal{F}_i$, and for every $j \in [k] \setminus 1$, let $B_j$ be a set in $\mathcal{F}_i$, not containing $a_j$. Then, any k-subset of $[n]$ which intersects all members of $\mathcal{F}_i$ either contains 1 or it contains an element $a_j$ among $\{a_2, \ldots, a_k\}$, and at least one of the $k-1$ elements of $B_j$ different from 1. Thus, there are

\[
\binom{n-1}{k-1} - i \ksets \text{ outside } \mathcal{F}_i, \text{ containing 1, and at most}
\]

\[
(k-1)^2 \binom{n-3}{k-2}
\ksets \text{ of } [n], \text{ which intersect every element of } \mathcal{F}_i, \text{ but do not contain } 1. \text{ This proves the lemma.} \square
\]

**Lemma 4.7.** Suppose that the intersection of all sets in the family $\mathcal{F}_i$ is empty. Then, there are at most $k^2 \binom{n-2}{k-2}$ k-subsets of $[n]$ that are not contained in $\mathcal{F}_i$ and intersect each of the sets in $\mathcal{F}_i$.

**Proof.** Let $A = \{a_1, a_2, \ldots, a_k\}$ be an arbitrary set in $\mathcal{F}_i$, and for every $j \in [k]$, let $B_j$ be an arbitrary set in $\mathcal{F}_i$, not containing $a_j$. Counting all k-subsets, intersecting all members of $\mathcal{F}_i$, according to the first element in $A$ which they contain, say $a_i$, and then the first element of $B_j$ which they contain, we get at most $k^2 \binom{n-2}{k-2}$ k-subsets of $[n]$ outside $\mathcal{F}_i$, which intersect each of the k-subsets in $\mathcal{F}_i$.

**Lemma 4.8.** For every $k \geq 2$, $n \geq 2k^3 - 2k^2 + 1$ and $i \in \left[\binom{n-2}{k-2} - (\binom{n-2-k}{k-2} + 1, \binom{n-2}{k-2}\right]$, $c(i) = 2\binom{n-1}{k-1} - \binom{n-2}{k-2} - i$.

**Proof.** We prove that, whatever the choice of $\mathcal{F}_i$, the vertices in $G^r$, corresponding to the sets in $\mathcal{F}_i$, have at most $2\left(\binom{n-1}{k-1} - \binom{n-2}{k-2} - i\right)$ common neighbours. Since $i > \left(\binom{n-3}{k-3}\right)$, the sets in $\mathcal{F}_i$ could have at most two common elements. If the given $i$ sets contain two common elements, then let these elements be 1 and 2. We show that there is no k-subset $\{a_1, a_2, \ldots, a_k\} \subseteq [n] \setminus \{1, 2\}$ intersecting all the members of $\mathcal{F}_i$. Indeed, the number of all k-subsets of $[n]$, containing 1, 2 and at least one element among $\{a_1, a_2, \ldots, a_k\}$, is $\binom{n-2}{k-2} - \binom{n-2-k}{k-2} < i = |\mathcal{F}_i|$. Thus, any k-set, intersecting all members of $\mathcal{F}_i$, contains either 1 or 2, or both, and there are exactly $2\left(\binom{n-1}{k-1} - \binom{n-2}{k-2} - i\right)$ such subsets of $[n]$ outside $\mathcal{F}_i$.

By Lemma 4.6 and Lemma 4.7, it remains to verify that for every $i \in \left[\binom{n-2}{k-2} - (\binom{n-2-k}{k-2} + 1, \binom{n-2}{k-2}\right]$

\[
\max \left(\binom{k^2}{k-2} \binom{n-2}{k-2}, \binom{n-1}{k-1} - i + (k-1)^2 \binom{n-3}{k-2} \right) \leq \binom{n-1}{k-1} - \binom{n-2}{k-2} - i,
\]

which is equivalent to

\[
k^2 \binom{n-2}{k-2} \leq 2\binom{n-1}{k-1} - 2\binom{n-2}{k-2} = 2\binom{n-2}{k-1} \text{ and } (k-1)^2 \binom{n-3}{k-2} \leq \binom{n-1}{k-1} - \binom{n-2}{k-2} = \binom{n-2}{k-1}.
\]

Since $n \geq k^3$, we have

\[
2\binom{n-2}{k-1} = 2\frac{n-k}{k-1} \binom{n-2}{k-2} \geq k^2 \binom{n-2}{k-2},
\]

and the second inequality holds since

\[
\binom{n-2}{k-1} = \frac{n-2}{k-1} \binom{n-3}{k-2} \geq (k-1)^2 \binom{n-3}{k-2}.
\]

The lemma is proved. \square

**Lemma 4.9.** For every $k \geq 2$, $n \geq 2k^3 - 2k^2 + 1$ and $i \in \left[\binom{n-2}{k-2} + 1, \binom{n-1}{k-1} - \binom{n-1-k}{k-1}\right]$, $c(i) = \binom{n-1}{k-1} - i + (k-1)^2 \binom{n-3}{k-2}$. 

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Proof. Once again, we work with \( \mathcal{F}_i \). Since \( i > \binom{n-2}{k-2} \), the sets in \( \mathcal{F}_i \) could have at most one common element. Thus, by Lemma 4.6 and Lemma 4.7 it remains to prove that for every \( i \leq \binom{n-1}{k-1} - \binom{n-1-k}{k-1} \) we have

\[
\binom{n-1}{k-1} - i + (k-1)^2 \binom{n-3}{k-2} \geq k^2 \binom{n-2}{k-2}.
\]

Note that, on the one hand,

\[
\binom{n-1-k}{k-1} = \left( \prod_{j=0}^{k-2} \frac{n-1-j-k}{n-1-j} \right) \binom{n-1}{k-1} \geq \left( 1 - \frac{k}{n-k+1} \right)^{k-1} \binom{n-1}{k-1} \binom{n-2}{k-2},
\]

and on the other hand, by Bernoulli’s inequality (Lemma 2.3) and the assumption \( n \geq 2k^3 - 2k^2 + 1 \) we have

\[
\left( 1 - \frac{k}{n-k+1} \right)^{k-1} \binom{n-1}{k-1} \geq \left( 1 - \frac{k(k-1)}{2(k-1)^2(k+1)} \right) 2k^2 \geq 2k^2 \geq k^2.
\]

Lemma 4.10. For every \( k \geq 2, n \geq 2k^3 - 2k^2 + 1 \) and \( i \in \left[ \binom{n-1}{k-1} - \binom{n-1-k}{k-1} + 1, \frac{1}{2} \binom{n-1}{k-1} \right] \), \( c(i) = \binom{n-1}{k-1} - i \).

Proof. Once again, we work with \( \mathcal{F}_i \). Since \( i > \binom{n-2}{k-2} \), the sets in \( \mathcal{F}_i \) could have at most one common element. We consider two cases.

If the sets in \( \mathcal{F}_i \) all contain one common element, let this element be 1 without loss of generality. Then, since \( i \geq \binom{n-1}{k-1} - \binom{n-1-k}{k-1} + 1 \), any \( k \)-subset of \( [n] \), intersecting all members in the family, contains 1. Indeed, the number of all \( k \)-subsets of \( [n] \), containing 1 and containing some element among \( \{a_1, a_2, \ldots, a_k\} \) for any \( \{a_1, a_2, \ldots, a_k\} \subseteq [n] \setminus 1 \), is \( \binom{n-1}{k-1} - \binom{n-1-k}{k-1} < i \). Since the number of \( k \)-sets, containing 1 and not in \( \mathcal{F}_i \), is \( \binom{n-1}{k-1} - i \), we conclude that \( c(i) \geq \binom{n-1}{k-1} - i \).

If the sets in \( \mathcal{F}_i \) do not have a common element, by Lemma 4.7 there are at most \( k^2 \binom{n-2}{k-2} \) elements outside \( \mathcal{F}_i \), which intersect each of the \( k \)-sets in \( \mathcal{F}_i \). It remains to observe that, for \( n \geq 2k^3 - 2k^2 + 1 \), \( k^2 \binom{n-2}{k-2} \leq \frac{1}{2} \binom{n-1}{k-1} \) and so \( k^2 \binom{n-2}{k-2} \leq \binom{n-1}{k-1} - \frac{1}{2} \binom{n-1}{k-1} \leq \binom{n-1}{k-1} - i \), which proves the lemma.

Using 11 we deduce that

\[
\frac{1}{2} \sum_{i=1}^{\binom{n-1}{k-1}} c(i) \leq \frac{\binom{n-1}{k-1}}{2} \left( \frac{n-1}{k-1} \right)^2.
\]

We separate the above sum into four sums over the intervals \( \left[ 1, \frac{1}{2} \binom{n-1}{k-1} \right], \left[ \frac{1}{2} \binom{n-1}{k-1} - \frac{n-1-k}{k-1}, \frac{1}{2} \binom{n-1}{k-1} \right], \left[ \frac{n-2}{k-2} + 1, \frac{1}{2} \binom{n-1}{k-1} \right] \) and \( \left[ \frac{n-1}{k-1} - \frac{n-1-k}{k-1} + 1, \frac{1}{2} \binom{n-1}{k-1} \right] \). Then, by Lemma 4.5 we get that

\[
\sum_{i=1}^{\binom{n-1}{k-1} - \binom{n-2}{k-2}} c(i) \leq k^2 \binom{n-3}{k-3} \binom{n-1}{k-1} = \frac{k^2(k-1)(k-2)}{(n-1)(n-2)} \binom{n-1}{k-1} \leq \frac{k^2(k-1)^2}{(n-1)^2} \binom{n-1}{k-1}^2,
\]

where the last inequality holds since \( \frac{k-2}{n-2} \leq \frac{k-1}{n-1} \).
By Lemma 4.8 we get that

\[
\sum_{i=\binom{n-2}{k-2}+1}^{\binom{n-2}{k-2}-(n-2)^k+1} c(i) \leq \sum_{i=\binom{n-2}{k-2}+1}^{\binom{n-2}{k-2}-(n-2)^k+1} \binom{2(n-1)}{k-1} - \binom{n-2}{k-2} - i
\]

\[
\leq \binom{n-2}{k-2} \left( 2 \binom{n-1}{k-1} - \binom{n-2}{k-2} \right) - \sum_{i=\binom{n-2}{k-2}+1}^{\binom{n-2}{k-2}-(n-2)^k+1} i
\]

\[
\leq \frac{(n-2)(n-2)}{(k-2)} \left( 2 - \frac{k-1}{n-1} \right) \binom{n-1}{k-1} - \sum_{i=\binom{n-2}{k-2}+1}^{\binom{n-2}{k-2}-(n-2)^k+1} i
\]

\[
\leq \frac{(k-1)(n-2)}{(n-1)^2} \binom{n-1}{k-1} ^2 - \sum_{i=\binom{n-2}{k-2}+1}^{\binom{n-2}{k-2}-(n-2)^k+1} i.
\]

By Lemma 4.9 we get that

\[
\sum_{i=\binom{n-1}{k-1}+1}^{\binom{n-1}{k-1}-(n-1)^k+1} c(i) \leq \sum_{i=\binom{n-1}{k-1}+1}^{\binom{n-1}{k-1}-(n-1)^k+1} \left( \binom{n-1}{k-1} - i + (k-1)^2 \binom{n-3}{k-2} \right)
\]

\[
\leq \left( \binom{n-1}{k-1} - \binom{n-1-k}{k-1} - \binom{n-2}{k-2} \right) \left( \binom{n-1}{k-1} + (k-1)^2 \binom{n-3}{k-2} \right) - \sum_{i=\binom{n-1}{k-1}+1}^{\binom{n-1}{k-1}-(n-1)^k+1} i
\]

\[
\leq \frac{(k-1)(n-2)}{(k-2)} \left( 1 + \frac{(k-1)^3}{n-1} \right) \binom{n-1}{k-1} ^2 - \sum_{i=\binom{n-1}{k-1}+1}^{\binom{n-1}{k-1}-(n-1)^k+1} i
\]

Also, by Lemma 4.10 we have

\[
\sum_{i=\binom{n-1}{k-1}-(n-1)^k+1}^{\binom{n-1}{k-1}+(n-1)^k+1} c(i) \leq \sum_{i=\binom{n-1}{k-1}-(n-1)^k+1}^{\binom{n-1}{k-1}+(n-1)^k+1} \left( \binom{n-1}{k-1} - i \right)
\]

\[
\leq \left( \binom{n-1-k}{k-1} - \frac{1}{2} \binom{n-1}{k-1} \right) \binom{n-1}{k-1} - \sum_{i=\binom{n-1}{k-1}+(n-1)^k+1}^{\binom{n-1}{k-1}+(n-1)^k+1} i
\]

\[
\leq \frac{1}{2} \binom{n-1}{k-1} ^2 - \sum_{i=\binom{n-1}{k-1}-(n-1)^k+1}^{\binom{n-1}{k-1}+(n-1)^k+1} i.
\]
Focusing on the sums which are not yet developed, we have that

\[
\left( \sum_{i=1}^{\binom{n-1}{k}} i \right) + \frac{1}{2} \left\lfloor \frac{1}{2} \left( \frac{n-1}{k-1} \right) \right\rfloor^2
\]

is bounded from below by

\[
\left( \sum_{i=1}^{\binom{n-1}{k}} i \right) - \left( \sum_{i=1}^{\binom{n-1}{k-2}} i \right) + \frac{1}{2} \left\lfloor \frac{1}{2} \left( \frac{n-1}{k-1} \right) \right\rfloor^2
\]

\[
\geq \left( \sum_{i=1}^{\binom{n-1}{k-1}} i \right) + \frac{1}{2} \left\lfloor \frac{1}{2} \left( \frac{n-1}{k-1} \right) \right\rfloor^2
\]

\[
\geq \frac{1}{2} \left[ \left( \frac{n-1}{k-1} \right)^2 \right] + \left( \frac{n-1}{k-1} \right) - \frac{1}{2} \left( \frac{n-2}{k-2} - \left( \frac{n-2-k}{k-2} \right) + 1 \right)^2 + \frac{1}{2} \left( \frac{n-1}{k-1} \right)^2
\]

\[
\geq \left( \frac{1}{2} \left( \frac{n-1}{k-1} \right)^2 - \frac{1}{2} \right) + \left( \frac{1}{2} \left( \frac{n-1}{k-1} \right) - \frac{1}{2} - \frac{1}{2} \left( \frac{n-2}{k-2} - \left( \frac{n-2-k}{k-2} \right) + 1 \right)^2
\]

\[
\geq \left( \frac{1}{4} \left( \frac{n-1}{k-1} \right)^2 - \frac{1}{4} \left( \frac{n-1}{k-1} \right) - \frac{1}{2} \left( \frac{n-2}{k-2} - \left( \frac{n-2-k}{k-2} \right) + 1 \right)^2
\]

\[
\geq \left( \frac{1}{4} - \frac{k-1}{4(n-1)} - \frac{(k-1)^2}{2(n-1)^2} \right) \left( \frac{n-1}{k-1} \right)^2.
\]

We conclude that \( \sum_{i=1}^{\binom{n}{k}} c(i) \) is bounded from below by

\[
2 \left( \frac{k^2(k-1)^2}{(n-1)^2} + \frac{2(k-1)}{n-1} + \frac{(k-1)^2}{(n-1)^2} \left( 1 + \frac{(k-1)^3}{n-1} \right) + \frac{1}{2} \left( \frac{n-1}{k-1} \right)^2 \right)
\]

\[
= 2 \left( \frac{1}{4} + \frac{(k-1)^2 + 9(k-1)/4}{n-1} + \frac{(k-1)^2}{(n-1)^2} \left( 1 + \frac{(k-1)^3}{n-1} \right) + \frac{1}{2} \left( \frac{n-1}{k-1} \right)^2 \right)
\]

\[
\leq 2 \left( \frac{1}{4} + \frac{(k-1)^2 + 9(k-1)/4}{n-1} + \frac{(k-1)^2}{(n-1)^2} \left( 1 + \frac{(k-1)^3}{n-1} \right) + \frac{1}{2} \left( \frac{n-1}{k-1} \right)^2 \right)
\]

\[
\leq \left( \frac{1}{2} + \frac{2(k-1)^2 + 9(k-1)/2}{n-1} + \frac{(k-1)^2}{(n-1)^2} \left( 1 + \frac{(k-1)^3}{n-1} \right) \right) \left( \frac{n-1}{k-1} \right)^2.
\]

We conclude by (2) and Lemma 2.2 that box(G) is at least

\[
\frac{\left| E(G') \right|}{\left( \frac{1}{2} + \frac{3k^2 - 5k/2 + 5/2}{n-1} \right) \left( \frac{n-1}{k-1} \right)^2}
\]

\[
\geq \frac{\sum_{i=1}^{\binom{n}{k}} \left( \frac{n}{k} - \binom{n-k}{k} \right)}{\left( \frac{1}{2} + \frac{3k^2 - 5k/2 + 5/2}{n-1} \right) \left( \frac{n-1}{k-1} \right)^2} = \frac{n}{1 + 6k^2 - 5k + 5} \left( \frac{n}{k-1} \right)^{\binom{n}{k} - \binom{n-k}{k}}.
\]

It remains to note that

\[
\frac{1}{1 + 6k^2 - 5k + 5} \geq 1 - \frac{6k^2 - 5k + 5}{n-1}
\]

and by Pascal’s identity applied \( k \) times

\[
\binom{n}{k} - \binom{n-k}{k} = \sum_{i=1}^{k} \binom{n-i}{k-1} = \sum_{i=1}^{k} \binom{n-1}{k-1} \frac{\prod_{j=1}^{i-1} \left( n-k+1-j \right)}{\prod_{j=1}^{i-1} \left( n-j \right)} \geq \sum_{i=1}^{k} \binom{n-1}{k-1} \left( 1 - \frac{k-1}{n-k+1} \right)^{i-1}.
\]

Since \( n \geq 2k - 1 \), we have that \( \frac{k-1}{n-k+1} < 1 \), so by Lemma 2.3 \( \left( 1 - \frac{k-1}{n-k+1} \right)^{i-1} \geq 1 - \frac{(k-1)(i-1)}{n-k+1} \), so we conclude that (3) is bounded from below by

\[
\sum_{i=1}^{k} \binom{n-1}{k-1} \left( 1 - \frac{(k-1)(i-1)}{n-k+1} \right) \geq \binom{n-1}{k-1} \left( k - \frac{(k-1)^2}{2(n-k+1)} \right).
\]
Thus, we get that $\text{box}(G)$ is at least

$$\frac{n}{1 + \frac{6k^2 - 5k + 5}{2k(n-k+1)}} \left(\begin{pmatrix} n \\ k \end{pmatrix} - \begin{pmatrix} n-k \\ k \end{pmatrix} \right) \geq n \left(1 - \frac{6k^2 - 5k + 5}{2k(n-k+1)} \right) \left(1 - \frac{(k-1)^2}{2(n-k+1)} \right) \geq n \left(1 - \frac{6k^2 - 5k + 5}{2k(n-k+1)} \right) \geq n \left(1 - \frac{6k^2 - 5k + 8}{2n} \right) \geq n \left(1 - \frac{6k^2 + 5k - 8}{2(n-k+1)} \right) = n - \frac{13k^2 - 11k + 16}{2},$$

where the third line comes from the inequalities

$$\frac{6k^2 - 5k + 5}{2n-1} \leq 3 \quad \text{and} \quad \frac{(k-1)^2}{(n-k+1)} \leq 1,$$

which are ensured by the assumption that $n \geq 2k^3 - 2k^2 + 1$. The proof of Theorem 1.1 is finished. \(\square\)

**Remark 4.11.** In general, for any positive integers $k, n$ with $n \geq 2k + 1$, lower bounds on the boxicity of the Kneser graph $Kn(k,n)$ can be easily derived thanks to the remarkable connection between graph boxicity and poset dimension, shown in [11]. To explain the approach, we define the extended double cover of a graph $G$ to be the graph $G_e$ with vertex set $V_1 \cup V_2$, where $V_1 = \{u_1 : u \in V(G)\}$ and $V_2 = \{u_2 : u \in V(G)\}$ are two disjoint copies of $V(G)$, and edge set $\{u_1v_2 : u_1 \in V_1, v_2 \in V_2, u \equiv v\}$ or $\equiv w \in E(G)\}$. Fix $G = Kn(k,n)$, and let $S_u$ be the $k$-set which corresponds to $u$ in the construction of $Kn(k,n)$. For every $u \in V(G)$, associate $S_u$ to $u_1 \in V_1$ and $[n] \setminus S_u$ to $u_2 \in V_2$. Then, let $G_e$ be the subgraph of $G_e$, induced by the vertices in $V_1 \cup V_2$, whose corresponding sets contain the element 1. Clearly, $\text{box}(G_e) \leq \text{box}(G_e)$, and by Lemma 2 in [11] we have $\text{box}(G_e) - 2 \leq \text{box}(G)$.

Since $G_e$ is the comparability graph of the poset $(k-1, n-k-1; n-1)$ (with elements the subsets of $[n-1]$, of size $k-1$ or $n-k-1$, partially ordered by the inclusion relation), combining Theorem 1 in [2], and Propositions 2.1 and 2.2 in [17], one may deduce that

$$\text{box}(G) \geq n - \frac{2k-1}{2}, \text{ if } 2k+1 \leq n \leq 3k+1,$$

and

$$\text{box}(G) \geq n - \frac{k-4}{2}, \text{ if } n \geq 3k+2.$$

For values of $n$ “close” to $2k+1$, better lower bounds may be deduced by Theorem 4.5 in [10]. In particular, using crucially the fact that $(k-1, n-k-1; n-1)$ contains an isomorphic copy of $(1, n-2k+1; n-k+1)$ as an induced subposet, there is a positive constant $c > 0$ such that

$$c2^{n-2k+1} \log \log n - 2 \leq \text{box}(G) \text{ if } n - 2k + 1 \leq \log \log n - \log \log \log n.$$

## 5 Proof of Theorem 1.2

Recall that in Theorem 1.2 $G$ is a graph of maximum degree $\Delta \geq 3$, and line graph $H = L(G)$. Denote for brevity $c(i) = c(i, H)$. 

**Proof of Theorem 1.2**. The upper bound is a consequence of Theorem 1.1 and the observation that the complement of the line graph of $G$ is an induced subgraph of $Kn(2, n)$.

In the remainder of the proof, we show that

$$\sum_{i=1}^{\lfloor |V(H)| \rfloor - 1} c(i) \leq 12 \cdot \mathbb{1}_{\Delta=3} + 16 \cdot \mathbb{1}_{\Delta=4} + \frac{\Delta(\Delta+3)}{2} \mathbb{1}_{\Delta \geq 5}. \quad (4)$$

The lower bound then directly follows by Lemma 2.2 applied for $H^c$.  

10
First, since $H$ has maximum degree at most $2(\Delta - 1)$ we have $c(1) \leq 2(\Delta - 1)$. Then, we prove that $c(2) \leq \max(\Delta - 1, 4)$. Indeed, let $\{i_1, i_2\}$ and $\{j_1, j_2\}$ be two distinct vertices of $H$. If the two sets have a common element, assume that $i_1 = j_1$. Then, the common neighbours of $\{i_1, i_2\}$ and $\{j_1, j_2\}$ belong to the set $\{(i_1, k) : v_i, v_k \in E(G)\} \cup \{i_2, j_2\}$. Otherwise, $\{i_1, i_2\}$ and $\{j_1, j_2\}$ have no common element, in which case they can have at most four neighbours: $\{i_1, j_1\}$, $\{i_1, j_2\}$, $\{i_2, j_1\}$ and $\{i_2, j_2\}$.

We divide the remainder of the proof in two cases according to the value of $\Delta$. Assume first that $\Delta \geq 5$.

**Claim.** For every $i$ between 3 and $\Delta - 1$, we have that $c(i) \leq \max(\Delta - i, 2)$.

**Proof.** Consider a set $S$ of $i \in [3, \Delta - 1]$ vertices of $H$. If all vertices in $S$ correspond to edges of $G$, containing a fixed vertex, then there are at most $\Delta - i$ vertices of $H$, connected to every vertex in $S$.

If $S$ contains two vertices of $H$, which correspond to disjoint edges of $G$, say $\{i_1, i_2\}$ and $\{j_1, j_2\}$, there are several cases to consider:

- If some of the remaining sets is disjoint from both $\{i_1, i_2\}$ and $\{j_1, j_2\}$, then the vertices in $S$ have no common neighbour in $H$;
- If some of the remaining sets intersects both $\{i_1, i_2\}$ and $\{j_1, j_2\}$, then let without loss of generality this set be $\{i_1, j_1\}$. In this case, the vertices in $S$ have at most two common neighbours, corresponding to $\{i_1, j_2\}$ and $\{i_2, j_1\}$;
- It remains the case when none of the remaining sets intersects either $\{i_1, i_2\}$ or $\{j_1, j_2\}$. In this case, assume without loss of generality that $\{i_1, k\}$ is in $S$ for some $k \notin \{i_2, j_1, j_2\}$. It means that the vertices in $S$ must have at most two common neighbours in $H$: $\{i_1, j_1\}$ and $\{i_1, j_2\}$.

It remains to consider the possibility that all vertices in $S$ correspond to sets, which intersect non-trivially, but do not have a common element. Then one must have $i = 3$ and three sets $\{i_1, i_2\}$, $\{i_2, i_3\}$ and $\{i_1, i_3\}$. In this case the vertices in $S$ have no common neighbour in $H$. This finishes the proof of the claim.

It remains to note that $c(\Delta) < 2$ since $c(2) \leq \Delta - 1$. Thus, choosing among the neighbours of any given vertex in $H$, one may observe that $c(\Delta) = \cdots = c(\min(\lvert V(H) \rvert - 1, 2(\Delta - 1)) \leq 1$ and $c(i) = 0$ for every $i > 2(\Delta - 1)$. Summing up, we get

\[
\sum_{i=1}^{\lvert V(H) \rvert - 1} c(i) \\
\leq 2(\Delta - 1) + \Delta - 1 + \sum_{i=3}^{\Delta-2} (\Delta - i) + 2 + \sum_{i=\Delta}^{2(\Delta-1)} 1 \\
= 3\Delta - 3 + \frac{(\Delta - 3)(\Delta - 2)}{2} - 1 + 2 + \Delta - 1 \\
= \frac{\Delta(\Delta + 3)}{2}.
\]

The first case is proved.

Taking advantage of the first part of the proof, we can mimic the same arguments to obtain upper bounds for the case $\Delta \in \{3, 4\}$. In particular,

\[
\sum_{i=1}^{\lvert V(H) \rvert - 1} c(i) \leq 4 + 4 + 2 + 2 = 12, \text{ if } \Delta = 3,
\]

and

\[
\sum_{i=1}^{\lvert V(H) \rvert - 1} c(i) \leq 6 + 4 + 2 + 1 + 1 = 16, \text{ if } \Delta = 4.
\]

Let us explain in more detail the last two cases. We already know that $c(1) \leq 2(\Delta - 1)$ and $c(2) \leq 4$, so by Observation \[\square\] $c(i_1) = 0$ for every $i_1 \geq 2\Delta - 1$ and $c(i_2) \leq 1$ for every $i_2 \geq 5$. Thus, it remains to show that $c(3) \leq 2$ in both cases, having $c(4) \leq c(3)$ then it proves our claim.

Indeed, on the one hand, if three edges of $G$ have a common endvertex, then at most one other edge may be adjacent to all three since $\Delta \leq 4$. On the other hand, if three edges of $G$ are disjoint or if they form a $K_3$, then
no edge is adjacent to all of them. It remains the case when two of the edges (say $uv, uw \in E(G)$) have a common endvertex (in this case $u$), not adjacent to the third edge. Then, if $\Delta = 3$, the only edges of $G$ that may possibly be adjacent to all three edges are $vw$ and the third edge of $G$, containing $u$. If $\Delta = 4$, there may possibly be at most three edges of $G$ of the form $uw_1, uw_1$ and $vw$, adjacent to both $uv$ and $uw$, but no edge of $G$ different from $uv$ and $uw$ may be adjacent to $uw_1, uw_1$ and $vw$ at the same time.

This concludes the proof of the theorem.

6 Conclusion and further questions

In this paper we studied the boxicity of Kneser graphs. Finding the right value of the boxicity of $Kn(k,n)$ for every choice of $k,n$ with $n \geq 2k+1$ seems an interesting, but also quite challenging problem. An easier, but nonetheless intriguing question is whether the lower bound on $Kn(k,n)$ in Theorem 1 could be improved to $n - \Theta(1)$ when $n \geq Ck^3$ for a large enough constant $C$. Moreover, in the case $k = 2, n \geq 5$, we proved that $\text{box}(Kn(2,n))$ is either $n - 3$ or $n - 2$, and with the help of a SageMath program we could show that the boxicity of the Petersen graph, corresponding to $Kn(2,5)$, is 3. This result suggests that the right value of $\text{box}(Kn(2,n))$ might be $n - 2$ for every $n \geq 5$.

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