Breaking the current density threshold in spin-orbit-torque magnetic random access memory

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Spin-orbit-torque magnetic random access memory (SOT-MRAM) is a promising technology for the next generation of data storage devices. The main bottleneck of this technology is the high reversal current density threshold. This outstanding problem of SOT-MRAM is now solved by using a current density of constant magnitude and varying flow direction that reduces the reversal current density threshold by a factor of more than the Gilbert damping coefficient. The theoretical limit of minimal reversal current density and current density for a GHz switching rate of the new reversal strategy for CoFeB/Ta SOT-MRAMs are respectively of the order of $10^5$ A/cm$^2$ and $10^6$ A/cm$^2$ far below $10^7$ A/cm$^2$ and $10^8$ A/cm$^2$ in the conventional strategy. Furthermore, no external magnetic field is needed for a deterministic reversal in the new strategy.

Subject Areas: Magnetism, Nanophysics, Spintronics

I. INTRODUCTION

Fast and efficient magnetization reversal is of not only fundamentally interesting, but also technologically important for high density data storage and massive information processing. Magnetization reversal can be induced by magnetic field [1–3], electric current through direct [4–9] and/or indirect [10–22] spin angular momentum transfer from polarized itinerant electrons to magnetization, microwaves [23], laser light [24], and even electric fields [25]. While the magnetic field induced magnetization reversal is a matured technology, it suffers from scalability and field localization problems [8] for nanoscale devices. Spin transfer torque magnetic random-access memory is an attractive technology in spintronics [26], although Joule heating, device durability and reliability are challenging issues [11, 26]. In an spin-orbit-torque magnetic random access memory (SOT-MRAM) whose central component is a heavy-metal/ferromagnet bilayer, an electric current in the heavy-metal layer generates a pure spin current [14, 15] that flows perpendicularly into the magnetic layer. The spin current, in turn, produces spin-orbit torques (SOT) through spin angular momentum transfer [4, 5] and/or Rashba effect [10–22]. SOT-MRAM is a promising technology because writing charge current does not pass through the memory cells so that the cells do not suffer from the Joule heating and associated device damaging. In principle, such devices are infinitely durable due to negligible heating from spin current [11]. However, the reversal current density threshold (above $10^7$ A/cm$^2$ [14, 15] for realistic materials) in the present SOT-MRAM architecture is too high. To have a reasonable switching rate (order of GHz), the current density should be much larger than $10^8$ A/cm$^2$ [14, 15] that is too high for devices. In order to lower the minimal reversal current density as well as to switch magnetization states at GHz rate at a tolerable current density in SOT-MRAM, it is interesting to find new reversal schemes (strategies) that can achieve above goals. In this paper, we show that a proper current density pulse of time-dependent flow direction and constant magnitude, much lower than the conventional threshold, can switch a SOT-MRAM at GHz rate. Such a time-dependent current pulse can be realized by using two perpendicular currents passing through the heavy-metal layer. The theoretical limit of minimal reversal current density of the new reversal strategy for realistic materials can be of the order of $10^5$ A/cm$^2$, far below $10^7$ A/cm$^2$ in the conventional strategy that uses a direct current (DC), both based on macrospin approximation. The validity of the macrospin model is also verified by micromagnetic simulations.

II. MACROSPIN MODEL AND RESULTS

A. Model

Our new reversal strategy for an SOT-MRAM, whose central component is a ferromagnetic/heavy-metal bilayer lying in the $xy$-plane with initial spin along the $+z$-direction as shown in Fig. 1 uses a current density $J = J \cos \Phi x + J \sin \Phi y$ generated from two time-
dependent electric currents flowing along the \( x \)- and the \( y \)-directions, where \( \Phi \) is a time-dependent angle between \( \mathbf{J} \) and the \( x \)-axis and \( \mathbf{J} \) is a constant total current density. The magnetic energy density is \( \varepsilon = -K \cos^2 \theta \) with \( K \) being the anisotropy coefficient and \( \theta \) being the polar angle of the magnetization. In the absence of an electric current, the system has two stable states \( \mathbf{m} = +\hat{z} \) and \( \mathbf{m} = -\hat{z} \) where \( \mathbf{m} \) is the unit direction of magnetization \( \mathbf{M} = M\mathbf{m} \) of magnitude \( M \). The electric current generates a transverse spin current perpendicularly flowing into the ferromagnetic layer via the spin-Hall effect [10], and then produces an effective SOT on the magnetization.

\[ \frac{\partial \mathbf{m}}{\partial t} = -\mathbf{m} \times \mathbf{h}_{\text{eff}} + \alpha \mathbf{m} \times \frac{\partial \mathbf{m}}{\partial t} + \vec{\tau}, \] (2)

where \( \alpha \) is the Gilbert damping constant that is typically much smaller than unity. The effective field is \( \mathbf{h}_{\text{eff}} = -\nabla \varepsilon \) from energy density \( \varepsilon \). Time, magnetic field and energy density are respectively in units of \((\gamma M)^{-1}\), \( M \) and \( \mu_0 M^2 \), where \( \gamma \) and \( \mu_0 \) are respectively the gyromagnetic ratio and vacuum magnetic permeability. In this unit system, \( \alpha = \frac{\hbar}{2e \alpha d \mu_0 M^2} \theta_{\text{SH}} J \) becomes dimensionless.

The magnetization \( \mathbf{m} \) can be conveniently described by a polar angle \( \theta \) and an azimuthal angle \( \phi \) in the \( xyz \)-coordinate. In terms of \( \theta \) and \( \phi \), the generalized LLG equation becomes

\begin{align}
(1 + \alpha^2) \phi & = -\alpha K \sin 2\theta + a(1 - \alpha \beta) \cos \theta \sin(\Phi - \phi) + a(\alpha + \beta) \cos(\Phi - \phi) \equiv F_1, \quad (3a) \\
(1 + \alpha^2) \theta & = K \sin 2\theta - a(1 - \alpha \beta) \cos(\Phi - \phi) + a(\alpha + \beta) \cos \theta \sin(\Phi - \phi) \equiv F_2. \quad (3b)
\end{align}

B. Derivation of the Euler-Lagrange equation

The goal is to reverse the initial state \( \theta = 0 \) to the target state \( \theta = \pi \) by SOT. There are an infinite number of paths that connect the initial state \( \theta = 0 \) with the target state \( \theta = \pi \), and each of these paths can be used as a magnetization reversal route. For a given reversal route, there are an infinite number of current pulses that can reverse the magnetization. The theoretical limit of minimal current density \( J_c \) is defined as the smallest values of minimal reversal current densities of all possible reversal routes. Then it comes two interesting and important questions: 1) What is \( J_c \) above which there is at least one reversal route that the current density can reverse the magnetization along it? 2) For a given \( J > J_c \), what are the optimal reversal route and the optimal current pulse \( \Phi(t) \) that can reverse the magnetization at the highest speed?

Dividing Eq. (3b) by Eq. (3a), one can obtain the following constraint,

\[ G \equiv \frac{\partial \phi}{\partial \theta} \sin \theta F_1 - F_2 = 0. \quad (4) \]

The magnetization reversal time \( T \) is

\[ T = \int_0^\pi \frac{d\theta}{\theta} = \int_0^\pi \frac{1 + \alpha^2}{F_1} d\theta. \quad (5) \]

The optimization problem here is to find the optimal reversal route \( \phi(\theta) \) and the optimal current pulse \( \Phi(t) \)
such that $T$ is minimum under constraint (1). Using the Lagrange multiplier method, the optimal reversal route and the optimal current pulse satisfy the Euler-Lagrange equations

$$\frac{\partial F}{\partial \phi} = \frac{d}{d\theta} \left( \frac{\partial F}{\partial (\partial \phi/\partial \theta)} \right), \quad \frac{\partial F}{\partial \Phi} = \frac{d}{d\theta} \left( \frac{\partial F}{\partial (\partial \Phi/\partial \theta)} \right),$$

where $F = (1 + \alpha^2)/F_1 + \lambda G$ and $\lambda$ is the Lagrange multipliers which can be determined self-consistently by Eq. (6) and constrain (1). Given a current density of constant magnitude $J$, Eq. (1) may or may not have a solution of $\phi(\theta)$ that continuously passing through $\theta = 0$ and $\theta = \pi$. If such a solution exists, then $\phi(\theta)$ is the optimal path for the fastest magnetization reversal and the corresponding solution of $\Phi(t)$ is the optimal current pulse. The theoretical limit of minimal reversal current density is then the smallest current density $J_c$ below which the optimal reversal path does not exist.

C. The optimal current pulse and theoretical limit of minimal reversal current density

From Eqs. (3a), (3b) and (4) as well as $F = (1 + \alpha^2)/F_1 + \lambda G$, the Euler-Lagrange equation of (6) becomes

$$\lambda \frac{d}{d\theta} (F_1 \sin \theta) = 0,$$

$$\frac{1 + \alpha^2}{F_1} \frac{\partial F_1}{\partial \phi} - \lambda \frac{\partial G}{\partial \phi} = - \frac{1 + \alpha^2}{F_1} \frac{\partial F_1}{\partial \Phi} + \lambda \frac{\partial G}{\partial \Phi} = 0.$$  

From Eq. (7a), one has $\lambda \neq 0$ or $\lambda = 0$. If $\lambda \neq 0$, $F_1$ must be $F_1 = C/\sin \theta$ ($C \neq 0$) so that $(1 + \alpha^2) \theta = C/\sin \theta \rightarrow \infty$ as $\theta \rightarrow 0$ or $\pi$. This solution is not physical, and should be discarded. Therefore, the only allowed solution must be $\lambda = 0$, and one has $\partial F_1/\partial \Phi = 0$ according to Eq. (7a). Interestingly, this is exactly the condition of maximal $\theta = F_1/(1 + \alpha^2)$ as $\Phi$ varies. $\Phi$ satisfies $\tan(\Phi - \phi) = \frac{1 - \alpha \beta}{\alpha + \beta} \cos \theta$, or

$$\Phi = \tan^{-1} \left( \frac{1 - \alpha \beta}{\alpha + \beta} \cos \theta \right) + \phi + \pi \quad (\beta < -\alpha)$$

and

$$\Phi = \tan^{-1} \left( \frac{1 - \alpha \beta}{\alpha + \beta} \cos \theta \right) + \phi \quad (\beta > -\alpha).$$

Substituting Eq. (3) into the LLG equation (3), $\theta(t)$ and $\phi(t)$ are determined by the following equations,

$$\dot{\theta} = \frac{1}{1 + \alpha^2} \left[ a P(\theta) - \alpha K \sin 2\theta \right],$$

$$\dot{\phi} = \frac{1}{1 + \alpha^2} \left[ 2 K \cos \theta - a (\alpha + \beta)(1 - \alpha \beta) \sin \theta \right],$$

where $P(\theta) = \sqrt{(\alpha + \beta)^2 + (1 - \alpha \beta)^2 \cos^2 \theta}$. To reverse magnetization from $\theta = 0$ to $\theta = \pi$, $a$ must satisfy $a > \alpha K \sin(2\theta)/P(\theta)$ according to Eq. (6a) so that $\dot{\theta}$ is non-negative for all $\theta$. Obviously, $\dot{\theta} = 0$ at $\theta = \pi/2$ when

$$\beta = -\alpha. \quad \text{The magnetization reversal is not possible in this case, and } \beta = -\alpha \text{ is a singular point. The theoretical}

\text{limit of minimal reversal current density } J_c \text{ for } \beta \neq -\alpha \text{ is}

$$J_c = \frac{2 \alpha \delta K d}{\theta_{SH} h} - Q,$$

where $Q \equiv \max \{\sin 2\theta/P(\theta)\}$ for $\theta \in [0, \pi]$. In comparison with the current density threshold (14) (18) ($J_c^{\text{dc}}$) in the conventional strategy for $\beta = 0$,

$$J_c^{\text{dc}} = \frac{2 \alpha \delta K d}{\theta_{SH} h} (1 - \frac{H}{\sqrt{2K}}),$$

the minimal reversal current density is reduced by more than a factor of $\alpha$. Here $H \approx \frac{22}{\alpha}$ Oe in experiments) is a small external magnetic field needed for a deterministic reversal in conventional strategy. Using CoFeB/Ta parameters of $M = 3.7 \times 10^5$ A/m, $K = 5.0 \times 10^3$ J/m$^3$, $\theta_{SH} = 0.084$ and $d = 0.6$ nm. As a comparison, $J_c^{\text{dc}}$ is also plotted as the dashed lines.

$$\beta = -0.3.$$ 

FIG. 2. The log $\alpha$-dependence of $J_c$ for various $\beta$ are plotted as the solid curves for model parameters of $M = 3.7 \times 10^5$ A/m, $K = 5.0 \times 10^3$ J/m$^3$, $\theta_{SH} = 0.084$ and $d = 0.6$ nm. As a comparison, $J_c^{\text{dc}}$ is also plotted as the dashed lines.

$$\beta = 0.3,$$

$$\beta = -0.3.$$
Using the same parameters as those for Fig. 2 with $\alpha = 0.008$ and various $\beta$, Fig. 4 shows the optimal current pulses ((a)-(c)) and the corresponding fastest magnetization reversal routes ((d)-(f)) for $\beta = 0.3$ and $J = 1.92 \times 10^6$ A/cm$^2$, for $\beta = 0.1$ and $J = 9.0 \times 10^6$ A/cm$^2$, and for $\beta = 0.3$ and $J = 9.0 \times 10^6$ A/cm$^2$. It is known that Ta has less effect on the reversal process and optimal reversal path winds around the two stable states many times. Correspondingly, the driving current makes also many turns as shown by the multiple oscillations of $m_x$ and $m_y$. The number of spinning turns depends on how far $J$ is from $J_c$. The closer $J$ to $J_c$, the number of turns is larger. The number of turns is about 5 in Figs. 3(a) and 3(d) for $J \approx 15 J_c$, and one turn for $J > 50 J_c$ as shown in Figs. 3(b), 3(c), 3(e) and 3(f), so that the reversal is almost ballistic. The reversal time for $\beta = 0.1$ and $J = 9.0 \times 10^6$ A/cm$^2$ is about 3.3 nanoseconds, and for $\beta = 0.3$ and $J = 9.0 \times 10^6$ A/cm$^2$ is about 2.1 nanoseconds. Figure 4 is the reversal time as a function of current density $J$ under the optimal current pulse for the same parameters as those for Fig. 2. The reversal time quickly decreases to nanoseconds as current density increases. In a real experiment, there are many uncertainties so that the current pulse may be different from the optimal one. To check whether our strategy is robust against small fluctuations, we let the current pulse in Fig. 3(c) deviate from its exact value. Numerical simulations show that the magnetization reversal is not significantly influenced at least when the deviation between the real current and optimal current is less than five percents.
III. VERIFICATION OF MACROSPIN MODEL 

BY MICROMAGNETIC SIMULATION

In our analysis, the memory cell is treated as a macrospin. A nature question is how good the macrospin model is for a realistic memory device. To answer this question, we carried out micromagnetic simulations by using Newton-Raphson algorithm [30] for two memory cells of 150 nm×150 nm×0.6 nm (Figs. 5(a), (b), (d) and (e)) and 250 nm×250 nm×0.6 nm (Figs. 5(c) and (f)). To model the possible edge pinning effect due to magnetic dipole-dipole interaction, we consider square-shape devices instead of cylinder shape device whose edge pinning is negligible. To make a quantitative comparison, the material parameters are the same as those used in Fig. 3. In our simulations, the unit cell size is 2 nm×2 nm×0.6 nm. For a fair comparison, the optimal current pulses shown in Figs. 5(a) and (c) of respective current density \( J = 1.92 \times 10^6 \) A/cm\(^2\) and \( J = 9.0 \times 10^6 \) A/cm\(^2\) were applied to the memory cell of 150 nm×150 nm×0.6 nm. The symbols in Figs. 5(a) and (b) are the time evolution of averaged magnetization \( m_x, m_y \) and \( m_z \) while the solid lines are the theoretical predictions of macrospin model shown in Figs. 5(d) and (f). The perfect agreements prove the validity of the macrospin approximation for our device of such a size. To further verify that the memory device can be treated as a macrospin, Figs. 5(d) and (e) are the spin configurations in the middle of the reversal at \( t = 5.5 \) ns for Fig. 5(a) and at \( t = 1.2 \) ns for Fig. 5(b). The fact that all spins align almost in the same direction verifies the validity of the macro spin model. In real experiments, non-uniformity of current density is inevitable. To demonstrate the macrospin model is still valid, we let current density linearly varies from \( 9 \) to \( 9 \times 10^6 \) A/cm\(^2\) in the leftmost column of cells to \( 8.5 \times 10^6 \) A/cm\(^2\) on the rightmost column of cells. As expected, there is no noticeable difference with the data shown in Figs. 5(b) and (e).

For the large memory device of 250 nm×250 nm×0.6 nm, the optimal current pulse shown in Fig. 5(c) of current density \( J = 9.0 \times 10^6 \) A/cm\(^2\) was considered. The time evolution of averaged magnetization \( m_x, m_y \) and \( m_z \) are plotted in Fig. 5(c), with the symbols for simulations and solid lines for the macrospin model. They agree very well although there is a small deviation for device of such a large size. Figure 5(f) is the spin configurations in the middle of the reversal at \( t = 1.2 \) ns for Fig. 5(c). The macrospin model is not too bad although all spins are not perfectly aligned in this case.

In summary, for a normal SOT-MRAM device of size less than 300 nm [13-15], macrospin model describes magnetization reversal well. However, for a larger sample size and lower current density (\( J < 10^6 \) A/cm\(^2\) for the same material parameters as those used in Fig. 5), only the spins in sample center can be reversed while the spins near sample edges are pinned.

IV. DISCUSSION

Obviously, the strategy present here can easily be generalized to the existing spin-transfer torque MRAM. The mathematics involved are very similar, and one expects a substantial current density reduction is possible there if a proper optimal current pulse is used. Of course, how to generate such a current pulse should be much more challenge than that for SOT-MRAM where two perpendicular currents can be used. In the conventional strategy that uses a DC-current, a static magnetic field along the perpendicular currents can be used. In the conventional strategy which uses a DC-current, a static magnetic field along current flow is required for a deterministic magnetization reversal [13-14] so that the system falls into the target state by itself through the damping. Therefore, one would like to use materials with larger damping in the conventional strategy in order to speed up this falling process. In contrast, our strategy prefers low damping materials, and reversal is almost ballistic when current density is large enough (> \( 50J_c \) in the current case). To reverse the magnetization from \( \theta = \pi \) to \( \theta = 0 \), one only needs to reverse the current direction of the optimal current pulse. One should notice that the Euler-Lagrange equation allows us to easily obtain the optimal reversal current pulse and theoretical limit.
of the minimal reversal current density for an arbitrary magnetic cell such as in-plane magnetized layer and biaxial anisotropy.

V. CONCLUSION

In conclusion, we investigated the magnetization reversal of SOT-MRAMs, and propose a new reversal strategy whose minimal reversal current density is far below the existing current density threshold. For popular CoFeB/Ta system, it is possible to use a current density less than $10^6 \text{ A/cm}^2$ to reverse the magnetization at GHz rate, in comparison with order of $J \approx 10^8 \text{ A/cm}^2$ in the conventional strategy.

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