Dynamical evolution of the electromagnetic perturbation coupling to Einstein tensor

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Abstract

We have investigated the wave dynamics of an electromagnetic perturbation coupling to Einstein tensor in the four-dimensional Reissner-Nordström black hole spacetime. Our results show that besides the dependence on the coupling between electromagnetic field and Einstein tensor, the wave dynamic equation of the perturbation strongly depends on the parity of the electromagnetic field itself, which is quite different from that of the usual electromagnetic perturbation without the coupling in the four-dimensional spacetime. Moreover, we also find that the electromagnetic perturbation with odd parity grows with exponential rate and the instability happens if the coupling strength is stronger than certain a critical value. However, the electromagnetic perturbation with even parity always decays in the Reissner-Nordström black hole spacetime.

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I. INTRODUCTION

One of the most important interactions is the coupling between the electromagnetic and gravitational fields. In the standard Einstein-Maxwell theory, the Lagrangian contains only the Einstein-Hilbert term and the quadratic term of Maxwell tensor. The latter is related directly to electromagnetic field, which can also be regarded as the coupling between Maxwell tensor and the spacetime metric tensor. But the interactions between electromagnetic field and curvature tensor are excluded in this standard electromagnetic theory. In general, such a kind of couplings related to spacetime curvature in Lagrangian could give rise to higher order terms both in the Maxwell and Einstein equations, which results in the more complicated and strange behaviors of the electromagnetic field in the background spacetime. Recently, a special kind of generalized Einstein-Maxwell theories of electromagnetic field coupling to curvature tensor have been investigated in [1–4]. It is shown that in these models the equations of motion both for the electromagnetic and gravitational fields are still second-order differential equations, but the couplings between Maxwell tensor and curvature tensor modify the coefficients of the second-order derivatives. This means that the modifications originating from the couplings change the properties of gravitational and electromagnetic waves propagated in the spacetime, which could yield time delays in the arrival of these waves [1]. The electromagnetic quantum fluctuations causing by the couplings with curvature could yield the inflation in the evolution of the early Universe [5–9]. Moreover, these coupled terms could be used as a mechanism to interpret the large scale magnetic fields observed in clusters of galaxies [10–12].

Weyl tensor is an important tensor in general relativity, which describes a type of gravitational distortion in the spacetime. The couplings between Maxwell field and Weyl tensor have been investigated extensively in the literature [13–21]. Considering that Weyl tensor is related to the curvature tensors $R_{\mu \nu \rho \sigma}$, $R_{\mu \nu}$ and the Ricci scalar $R$, the electrodynamics with the couplings involving Weyl tensor, in a sense, can be regarded as a special kind of generalized Einstein-Maxwell theory with the couplings among electromagnetic field and curvature tensors. The coupling terms involving Weyl tensor could be emerged naturally in quantum electrodynamics with the photon effective action arising from one-loop vacuum polarization on a curved background spacetime [14]. Moreover, the recent investigations also imply that these couplings should be appeared in the strong gravitational region of classical compact astrophysical objects with high mass density at the center of galaxies [15, 16]. The effects of the coupling between Maxwell field and Weyl tensor on holographic conductivity and charge diffusion in the anti-de Sitter spacetime are studied in [13], which is
shown that the presence of the coupling changes the universal relation with the $U(1)$ central charge observed at leading order. Moreover, it is found that the couplings with Weyl tensor modify the critical temperature at which holographic superconductors happen and change the order of the phase transition of the holographic superconductor\cite{17, 20}. The effects of such a kind of couplings on the transition between the holographic insulator phase and superconductor phase have been also investigated in the AdS soliton spacetime\cite{21}. Recently, we studied the dynamical evolution and Hawking radiation of the electromagnetic field coupling to Weyl tensor in the Schwarzschild black hole spacetime\cite{22, 23}. Our results show that both the dynamical evolution and Hawking radiation of the electromagnetic field depend not only on the coupling parameter, but also on the parity of the electromagnetic field.

Einstein tensor is another important tensor in general relativity. The investigation indicates\cite{24} that in cosmology the coupling between Einstein tensor and scalar field $G^{\mu\nu} \partial_\mu \psi \partial_\nu \psi$ can solve naturally the problem of a graceful exit from inflation without any fine-tuned potential. Moreover, the cosmic evolution of a scalar field with the kinetic term coupling to more than one Einstein tensor discloses that the scalar field behaves nearly like a dynamic cosmological constant during the late time evolution of the Universe\cite{25}. The effects of the kinetic couplings with Einstein tensor in the cosmology have also been investigated in Refs.\cite{26, 29}. We\cite{30} studied the Hawking radiation and dynamical evolution for a scalar field coupling to Einsteins tensor in the background of the Reissner-Nordström black hole spacetime. It is shown that the presence of the coupling not only enhances the absorption probability and Hawking radiation of the black hole, and but also modifies the standard results of dynamical evolution of the scalar field in the background spacetime. The scalar hair from a derivative coupling of a scalar field to the Einstein tensor have been investigated in Braneworlds\cite{31} and the scalar-tensor theory\cite{32}. With this coupling, the study also shows\cite{33} that a phase transition of a Reissner-Nordström black hole to a hairy black hole occurs in asymptotically flat spacetime since an Abelian $U(1)$ gauge symmetry was broken in the vicinity of the horizon of the black hole as the coupling constant is large enough. These new features of the coupled matter fields may trigger more attention to focus on the study of the couplings among matter fields and spacetime curvature tensors in the more general cases.

In order to probe the full features of the electromagnetic field coupling to spacetime curvature tensors, we here will study the dynamical evolution of an electromagnetic perturbation coupling to Einstein tensor in the background spacetime and explore the effect of the coupling on the quasinormal modes of the perturbation and on the stability of the black hole. The plan of our paper is organized as follows: in Sec.II, we will construct a simple form of the coupling between Maxwell tensor and Einstein tensor and then derive the master equation
of the coupled electromagnetic perturbation in the four-dimensional static and spherical symmetric black hole spacetime. In Sec.III, we will study numerically the effects of the coupling on the quasinormal modes of the external electromagnetic perturbation in the Reissner-Nordström black hole spacetime and then examine if the black hole is stable or not in this case. We also make a comparison among the dynamical properties of perturbational fields coupling to gravitational tensors in some black hole spacetimes. To conclude, we present a summary in the last section.

II. WAVE EQUATION FOR THE ELECTROMAGNETIC PERTURBATIONS COUPLING TO EINSTEIN TENSOR

Theoretically, there exists many forms of the couplings between the Maxwell field and Einstein tensor. We here consider only one of the simplest forms of the coupling in the curved spacetime, which yields that the action describing the Maxwell field coupling to Einstein tensor has a form

$$S = \int d^4x \sqrt{-g} \left[ \frac{R}{16\pi G} - \frac{1}{4} \left( F_{\mu\nu} F^{\mu\nu} - 4\alpha G^{\mu\rho} F_{\mu\nu} F_{\nu\rho} \right) \right],$$

where $G_{\mu\nu}$ is the Einstein tensor and $\alpha$ is a coupling constant with dimensions of length-squared. $F_{\mu\nu}$ is the electromagnetic tensor, which is related to the electromagnetic vector potential $A_{\mu}$ by $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$. As in the case with Weyl corrections [13–21], it is expected that the Einstein-Maxwell equation must be corrected in this case since the coupling term with Einstein tensor is also a tensorial structure modifying the Maxwell term at leading order in derivatives. One of the advantages of the coupling form we chosen here is that the modified Einstein-Maxwell equation is not so complicated that the master equation for the electromagnetic perturbation can be decoupled to a second order differential equation. It is very crucial that we can study further the dynamical evolution of electromagnetic perturbation in the black hole spacetime.

Varying the action (1) with respect to $A_{\mu}$, it is easy to get the Maxwell’s equation describing the electromagnetic field coupling to Einstein tensor

$$\nabla_\mu \left( F^{\mu\nu} - 4\alpha G^{\mu\rho} F_{\nu\rho} \right) = 0.$$  

(2)

Obviously, the presence of the coupling term $G^{\mu\rho} F_{\nu\rho}$ in the equation of motion implies that the coupling between Maxwell field and Einstein tensor will modify the dynamical behaviors of the electromagnetic perturbation in the background spacetime.

For the electromagnetic perturbation in the background of a four-dimensional static and spherical symmetric
black hole spacetime with the metric

\[ ds^2 = f dt^2 - \frac{1}{f} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2, \]  

(3)

we can expand the electromagnetic potential \( A_\mu \) in vector spherical harmonics [34–36]

\[ A_\mu = \sum_{l,m} \left[ \begin{array}{c} 0 \\ 0 \\ a^{lm}(t,r) \partial_\phi Y_{lm} \\ -a^{lm}(t,r) \sin \theta \partial_\theta Y_{lm} \end{array} \right] + \left[ \begin{array}{c} j^{lm}(t,r)Y_{lm} \\ h^{lm}(t,r)Y_{lm} \\ k^{lm}(t,r) \partial_\phi Y_{lm} \\ k^{lm}(t,r) \partial_\theta Y_{lm} \end{array} \right], \]  

(4)

where the metric coefficient \( f \) is a function of polar coordinate \( r \). The first term in the electromagnetic vector potential (4) has parity \((-1)^{l+1}\) and the second term has parity \((-1)^l\), where \( l \) is the angular quantum number and \( m \) is the azimuthal number.

Since all of metric coefficients in the spacetime (3) are independent of time coordinate \( t \), one can separate the variables \( a^{lm}(t,r) \), \( h^{lm}(t,r) \), \( j^{lm}(t,r) \) and \( k^{lm}(t,r) \) as

\[ a^{lm}(t,r) = a^{lm}(r)e^{-i\omega t}, \quad h^{lm}(t,r) = h^{lm}(r)e^{-i\omega t}, \]

\[ j^{lm}(t,r) = j^{lm}(r)e^{-i\omega t}, \quad k^{lm}(t,r) = k^{lm}(r)e^{-i\omega t}. \]  

(5)

Substituting the electromagnetic potential (4) with the separating variables (5) into the modified Maxwell equation (2), one can obtain three independent coupled differential equations after some tedious calculations. Eliminating \( k^{lm}(r) \), we find that both of the wave equations for the electromagnetic perturbation with different parities can be simplified as the form of a second order differential equation

\[ \frac{d^2 \Psi(r)}{dr^2} + [\omega^2 - V(r)]\Psi(r) = 0, \]  

(6)

where \( r_* \) is the tortoise coordinate defined by \( dr_* = \frac{dr}{f} \). \( \Psi(r) \) is a wavefunction, which is a linear combination of the three functions appeared in the electromagnetic potential \( A_\mu \), i.e., \( j^{lm}(r), h^{lm}(r) \), and \( a^{lm}(r) \). The concrete form of \( \Psi(r) \) depends not only on the coupling between the field and Einstein tensor, but also on the parity of the electromagnetic perturbation itself. For the odd parity \((-1)^{l+1}\), the wavefunction \( \Psi(r) \) can be expressed as

\[ \Psi(r) = a^{lm} \sqrt{1 + \frac{4\alpha}{r^2}(rf' + f - 1)}, \]  

(7)

but for the even parity \((-1)^l\), it is given by

\[ \Psi(r) = \frac{r^2}{l(l+1)} \left( -i \omega h^{lm} - \frac{d^{lm}}{dr} \right) \sqrt{1 + \frac{4\alpha}{r^2}(rf' + f - 1)}. \]  

(8)
Similarly, the effective potential $V(r)$ in Eq. (6) depends also on the parity of the perturbation. The form of the potential $V(r)$ can be expressed as

$$V(r) = f \left\{ \frac{l(l+1)}{r^2} \alpha \left( h_0 + h_1 \alpha \right) \right\},$$

for the odd parity, and

$$V(r) = f \left\{ \frac{l(l+1)}{r^2} - \frac{\alpha \left( h_0 + h_2 \alpha \right)}{r^2 [r^2 + 4\alpha (r f'' + f - 1)]^2} \right\},$$

for the even parity, respectively. Here the quantities $h_0$, $h_1$ and $h_2$ are

$$h_0 = 2r^2 f [6f^2 + rf' (r^2 f'' + 2) + f (r^3 f^{(3)} - r^2 f'' - 4rf' - 6)],$$

$$h_1 = 4f [8f^3 + 2rf' (r^2 f'' + 2)(rf' - 1) + 2f^2 (3f^2 f^{(3)} + 2f'' + 2rf' - 8)$$

$$+ rf' (3r^2 f f'' - r^2 f'' + 2r f'' - 8r^2 f'^2 + 8)],$$

$$h_2 = 4rf \{ 2rf^2 (r^2 f''' - 4f + 2) + rf [2(f - 1) (rf^{(3)} + 5f''')] - 3r^2 f''' \}$$

$$+ 2f' (r^3 f f^{(3)} - r^2 f'' + 2f^2 - 2) \}.$$

It is clear that due to the coupling between Maxwell field and Einstein tensor, the effective potential is different for the electromagnetic perturbations with different parities in the four dimensional black hole spacetime. It is entirely different from one obtained in the usual case without the coupling between Maxwell field and Einstein tensor in which the effective potential of the electromagnetic perturbation is independent of the parity of field.

The similar behaviors of the effective potential is also found in the case of the electromagnetic perturbation with Weyl corrections in the four dimensional spacetime [22]. It could be explained by a fact that the coupling terms $C_{\mu \nu \rho \sigma} F^{\mu \nu} F^{\rho \sigma}$ and $G^{\mu \nu} F_{\mu \nu} F^{\rho \nu}$ can be treated as a kind of general classical couplings between the gravitational tensor and electromagnetic field since both of the Weyl tensor $C_{\mu \nu \rho \sigma}$ and the Einstein tensor $G_{\mu \nu}$ are functions of the curvature tensors of background spacetime. Moreover, we also find that the effective potentials (9) and (10) recover to the usual form for the electromagnetic perturbation without coupling to Einstein tensor as the coupling constant $\alpha = 0$, in which the effective potentials for different parities share the same form.

III. WAVE DYNAMICS OF THE ELECTROMAGNETIC PERTURBATION COUPLING TO EINSTEIN TENSOR IN THE REISSNER-NORDSTROM BLACK HOLE SPACETIME

We are now in position to study the wave dynamics of the electromagnetic perturbation coupling to Einstein tensor in a four dimensional Reissner-Nordström black hole spacetime. Here, we choose the Reissner-Nordström black hole spacetime as a background metric mainly because it is the simplest black hole with the
nonzero components of the Einstein tensor \( G_{\mu\nu} \) in general relativity theory. Moreover, we assume that the coupling exists only for the external electromagnetic perturbation and it doesn’t exist for the static electric field carried by Reissner-Nordström black hole itself. In a sense, the external electromagnetic perturbation coupling to Einstein tensor can be treated as a vector perturbational field satisfied the corrected Maxwell equation (2). For convenience, we still call it as the electromagnetic perturbation in the subsequent part of the paper.

For a four dimensional Reissner-Nordström black hole spacetime, the metric function is
\[
f(r) = 1 - \frac{2M}{r} + \frac{q^2}{r^2}.
\]
Substituting it into Eqs.(9) and (10), one can obtain the effective potentials for the electromagnetic perturbations
\[
V(r)_{\text{odd}} = (1 - \frac{2M}{r} + \frac{q^2}{r^2}) \left\{ \frac{l(l+1)}{r^2} \left( \frac{r^4 + 4aq^2}{r^4 - 4aq^2} \right) - \frac{8aq^2[(5r^2 - 12Mr + 7q^2)r^4 - 4aq^2(3r^2 - 8Mr + 5q^2)]}{r^4(r^4 - 4aq^2)^2} \right\},
\]
(14)
for the odd parity and
\[
V(r)_{\text{even}} = (1 - \frac{2M}{r} + \frac{q^2}{r^2}) \left\{ \frac{l(l+1)}{r^2} + \frac{8aq^2[(5r^2 - 12Mr + 7q^2)r^4 + 4aq^2(r^2 - q^2)]}{r^4(r^4 - 4aq^2)^2} \right\},
\]
(15)
for the even parity, respectively. From above formulas, it is easy to find that the effective potentials for electromagnetic perturbations (14) and (15) are discontinuous and divergent at the same point \( r^4 - 4aq^2 = 0 \).

Considering that the effective potential should be continuous in the physical region outside the black hole, we here must impose a constraint on the coupling constant \( \alpha \) for the electromagnetic perturbations, i.e., \( r^4 - 4aq^2 > 0 \), which ensures that the effective potentials (14) and (15) are continuous outside the outer event horizon of Reissner-Nordström black hole. Comparing it with one obtained in the case of electromagnetic perturbation with Weyl corrections [22], we find that here the constraint on the coupling constant \( \alpha \) is independent of the parities of electromagnetic fields, whereas in the case with Weyl corrections such constraint depends on the parity of the electromagnetic field. The dependence of the effective potentials \( V(r)_{\text{odd}} \) and \( V(r)_{\text{even}} \) on the coupling constant \( \alpha \) for fixed \( l \) is shown in Figs.(1) and (2). For the effective potentials \( V(r)_{\text{odd}} \), one can find that the peak height of the potential barrier increases with the coupling constant \( \alpha \) fixed \( l \). For the effective potentials \( V(r)_{\text{even}} \), we find that with the increase of the coupling constant \( \alpha \) it increases for the positive \( \alpha \) and decrease for the negative \( \alpha \). Moreover, we also find that near the outer event horizon there exits the negative gap in the effective potential \( V(r)_{\text{odd}} \) only for the certain negative value of \( \alpha \) and in the potential \( V(r)_{\text{even}} \) only for the certain positive value of \( \alpha \). These differences between the effective potentials \( V(r)_{\text{odd}} \) and \( V(r)_{\text{even}} \) imply that the features of the wave dynamics of the electromagnetic
perturbation coupling to Einstein tensor would depend heavily on the parities of the perturbation fields. In the following section, we will examine those values of the coupling constant $\alpha$ for which the negative gap appears in the effective potentials and investigate the stability of the Reissner-Nordström black hole under the electromagnetic perturbation coupling to Einstein tensor.

In order to investigate the dependence of electromagnetic quasinormal modes on the coupling constant $\alpha$, we calculate the fundamental quasinormal modes ($n = 0$) of two electromagnetic perturbations with different parities by using the WKB approximation method [37, 38]. The change of the fundamental quasinormal frequencies of electromagnetic perturbation with the coupling constant $\alpha$ and the charge $q$ is plotted in Figs. (3)-(6). With the increase of the coupling constant $\alpha$, we find that the real parts of fundamental quasinormal frequencies increase for the electromagnetic perturbation with odd parity. For the perturbation with even parity, the real parts also increase monotonously with respect to $\alpha$ in the case with the smaller value of $q$. 

FIG. 1: Variation of the effective potential $V(r)_{\text{odd}}$ with the polar coordinate $r$ for fixed $l = 1$ (left), $l = 2$ (middle) and $l = 3$ (right). The long-dash-dotted, dashed, short-dash-dotted, solid and dotted lines are corresponding to the cases with $\alpha = -15, -10, -5, 0, 5$, respectively. We set $2M = 1$ and $q = 0.2$.

FIG. 2: Variation of the effective potential $V(r)_{\text{even}}$ with the polar coordinate $r$ for fixed $l = 1$ (left), $l = 2$ (middle) and $l = 3$ (right). The long-dash-dotted, dashed, short-dash-dotted, solid and dotted lines are corresponding to the cases with $\alpha = -15, -10, -5, 0, 5$, respectively. We set $2M = 1$ and $q = 0.2$. 

[37, 38]
FIG. 3: Effects of the coupling parameter $\alpha$ on the real parts of the fundamental quasinormal modes of electromagnetic perturbation with the odd parity (the top row) or the even parity (the bottom row) in the Reissner-Nordström black hole spacetime for fixed $q$. We set $2M = 1$.

but for the case with the larger $q$, they decrease first and then increase. The changes of the imaginary parts with $\alpha$ are shown in Fig. (4). For the smaller $q$, we find that the imaginary parts increase with $\alpha$ for the perturbation with odd parity and decrease for the perturbation with even one. For the larger $q$, the imaginary parts first increase and then decrease for the odd parity electromagnetic perturbation. However, for the even parity electromagnetic perturbation, they first increase and then decrease and finally increase with $\alpha$ again. Moreover, the dependence of the fundamental quasinormal frequencies on the charge $q$ are plotted in Figs. (5) and (6) for different $\alpha$. When $\alpha > 0$, one can obtain easily that the real parts of two perturbations with different parities increase with $q$. When $\alpha < 0$, we find that for $l = 1$, the real part for the even parity perturbation increase with $q$ first decrease and then increase. However, the real part for the odd parity perturbation increase with $q$ in the case with the weaker coupling, and it first decrease and then increase in the case with the stronger one in this case. For other values of $l$, the real part increases with $q$ for the even parity perturbation and decreases for odd one in the case with negative $\alpha$. From Fig.(6), one can find that for the positive $\alpha$, the imaginary part of the odd parity perturbation increases first and then decrease with $q$. The changes of imaginary part of the even parity perturbation is just the opposite to that
FIG. 4: Effects of the coupling parameter $\alpha$ on the imaginary parts of the fundamental quasinormal modes of electromagnetic perturbation with the odd parity (the top row) or the even parity (the bottom row) in the Reissner-Nordström black hole spacetime for fixed $q$. We set $2M = 1$.

FIG. 5: Effects of the charge $q$ on the real parts of the fundamental quasinormal modes of electromagnetic perturbation with the odd parity (the top row) or the even parity (the bottom row) in the Reissner-Nordström black hole spacetime for different $\alpha$. We set $2M = 1$. 
FIG. 6: Effects of the charge $q$ on the imaginary parts of the fundamental quasinormal modes of electromagnetic perturbation with the odd parity (the top row) or the even parity (the bottom row) in the Reissner-Nordström black hole spacetime for different $\alpha$. We set $2M = 1$.

with the odd parity. However, for the negative $\alpha$, the imaginary part decreases monotonously with $q$ for the odd parity perturbation, and it increases first and then decreases for the even one. These results differed from the standard results of the quasinormal modes indicate that the coupling modify the dynamical properties of the electromagnetic perturbations in the background of a Reissner-Nordström black hole.

In the subsequent section, we will study the dynamical evolution of the electromagnetic perturbation coupling to Einstein tensor in time domain \[39\] and examine the stability of the Reissner-Nordström black hole in this cases. With the help of the light-cone variables $u = t - r_*$ and $v = t + r_*$, one can find that the wave equation

$$-rac{\partial^2 \psi}{\partial t^2} + \frac{\partial^2 \psi}{\partial r_*^2} = V(r)\psi,$$

(16)

can be rewritten as

$$4\frac{\partial^2 \psi}{\partial u \partial v} + V(r)\psi = 0.$$ (17)

This two-dimensional wave equation \([17]\) can be integrated numerically by using the finite difference method
Making use of Taylor’s theorem, one can find that the wave equation (17) can be discretized as
\[
\psi_N = \psi_E + \psi_W - \psi_S - \delta u \delta v V \left( \frac{v_N + v_W - u_N - u_E}{4} \right) \psi_W + \psi_E + O(\epsilon^4) = 0.
\]
(18)

Here the points \(N, W, E\) and \(S\) are defined as: \(N: (u + \delta u, v + \delta v)\), \(W: (u + \delta u, v)\), \(E: (u, v + \delta v)\) and \(S: (u, v)\).

The parameter \(\epsilon\) is an overall grid scalar factor, so that \(\delta u \sim \delta v \sim \epsilon\). As in [39], we can set \(\psi(u, v = v_0) = 0\) and use a Gaussian pulse as an initial perturbation, centered on \(v_c\) and with width \(\sigma\) on \(u = u_0\) as
\[
\psi(u = u_0, v) = e^{-\frac{(v - v_c)^2}{2\sigma^2}}.
\]
(19)

In figs. (7) and (8), we show the dynamical evolution of the external electromagnetic perturbation coupling to

FIG. 7: The dynamical evolution of an electromagnetic perturbation with odd parity in the background of a Reissner-Nordström black hole spacetime. The figures from left to right are corresponding to \(l = 1, 2\) and \(3\). The dotted, solid, dash-dotted and dashed lines are corresponding to the cases with \(\alpha = 5, 0, -9, -15\), respectively. We set \(2M = 1\) and \(q = 0.2\). The constants in the Gauss pulse (19) \(v_c = 10\) and \(\sigma = 3\).

FIG. 8: The dynamical evolution of an electromagnetic perturbation with even parity in the Reissner-Nordström black hole spacetime. The figures from left to right are corresponding to \(l = 1, 2\) and \(3\). The dashed, dotted, dash-dotted and solid lines are corresponding to the cases with \(\alpha = 5, 0, -10, -30\), respectively. We set \(2M = 1\) and \(q = 0.2\). The constants in the Gauss pulse (19) \(v_c = 10\) and \(\sigma = 3\).

Einstein tensor in the Reissner-Nordström black hole spacetime, which indicates that the dynamical evolution of the coupled electromagnetic perturbation depends heavily on the parity of the perturbation field. For the electromagnetic perturbation with odd parity, we find that for the positive \(\alpha\), the coupled electromagnetic
FIG. 9: The change of the threshold value $\alpha_c$ with the multipole number $l$ for fixed $q$. The points $l = 1 \sim 15$ are fitted by the function $\alpha_c = a \sqrt{\frac{l}{l+1}} + b$, where the coefficients $a$ and $b$ are numerical constants with dimensions of length-squared and their values depend on the charge $q$ of black hole.

perturbation decays as that without coupling to Einstein tensor. However, for the negative $\alpha$, we find that if the coupling constant $\alpha$ is below the critical value $\alpha_c$, the coupled electromagnetic field does not decay any longer, but it grows with exponential rate, which means that the instability happens in this case. Moreover, we also find that in the instability region the stronger coupling leads to that the instability growth appears more quickly and the growth rate becomes stronger. This could be explained by a fact that as $\alpha < 0$, the stronger coupling drops down the peak of the potential barrier and increases the negative gap near the black hole horizon for the odd parity electromagnetic perturbation, which could result in that the effective potential is non-positive definite. For the electromagnetic perturbation with even parity, it always decays in the allowed range of $\alpha$. The main reason is that in its effective potential the negative gap near the black hole horizon is very small, which is not enough to yield the instability to be excited. These dynamical properties of electromagnetic perturbation coupling to Einstein tensor are very similar to those of electromagnetic field with Weyl corrections [22].

For the odd parity electromagnetic perturbation coupling to Einstein tensor, the dependence of the critical value $\alpha_c$ on the multipole number $l$ is plotted in Fig.(9), which shows that the critical value $\alpha_c$ can be fitted best by the function

$$\alpha_c \simeq a \sqrt{\frac{l}{l+1}} + b,$$

where the coefficients $a$ and $b$ are numerical constants with dimensions of length-squared and their values depend on the charge $q$ of black hole. Obviously, one can find that the threshold value $\alpha_c$ is negative and the absolute value $|\alpha_c|$ decreases with the multipole number $l$ and $q$, which means that for the higher $l$ and $q$ the instability happens more easily. The changes of the numerical constants $a$ and $b$ with the charge

$q$.
FIG. 10: The changes of the numerical constants $a$, $b$ and $a+b$ with the charge $q$, which are fitted by the functions $a \simeq -5.0168 + \frac{1.5440}{q^2} - \frac{0.4573}{q}$, $b \simeq 5.8490 - \frac{1.7956}{q^2} + \frac{0.5015}{q}$ and $a+b \simeq -\frac{r_+^4}{4q^2}$, respectively.

$q$ are presented in Fig. (10), which tells us that the values of $a$, $b$ and $a+b$ are fitted by the functions $a \simeq -5.0168 + \frac{1.5440}{q^2} - \frac{0.4573}{q}$, $b \simeq 5.8490 - \frac{1.7956}{q^2} + \frac{0.5015}{q}$ and $a+b \simeq -\frac{r_+^4}{4q^2}$, respectively. Therefore, for a Schwarzschild black hole, one can find that the threshold value $\alpha_c$ tends to infinity for arbitrary $l$ since the charge $q$ disappears, which implies that for the electromagnetic field coupling to Einstein’s tensor, it always decays and the instability doesn’t happen in the Schwarzschild black hole spacetime. It is not surprising because for a Schwarzschild black hole all the components of the Einstein tensor vanish. Moreover, from Eq. (20) and Fig.(10), we can obtain that as $l \to \infty$ the threshold value $\alpha_c \to a + b \simeq \frac{r_+^4}{4q^2}$. The main reason is that in this case the potential in the equation of motion (14) becomes

$$V(r)|_{l \to \infty} = f \frac{l(l+1)}{r^2} \left( \frac{r^4 + 4\alpha q^2}{r^4 - 4\alpha q^2} \right),$$

and then the integration [40],

$$\int_{r_+}^{\infty} \frac{V(r)|_{l \to \infty}}{f} dr = \int_{r_+}^{\infty} \left( \frac{r^4 + 4\alpha q^2}{r^4 - 4\alpha q^2} \right) \frac{l(l+1)}{r^2} dr,$$

is positive definite as $\alpha < -\frac{r_+^4}{(4q^2)}$, which implies that the threshold value $\alpha_c = -\frac{r_+^4}{(4q^2)}$ as $l \to \infty$.

Finally, we make a comparison among the dynamical properties of perturbational fields coupling to gravitational tensors in some black hole spacetimes. For the scalar field coupling to Einstein tensor in the Reissner-Nordström black hole spacetime [30], we find that the partial wave $l = 0$ always decays for arbitrary coupling constant and the nonzero-$l$ partial waves grow with exponential rate as the coupling constant $\eta$ is larger than the certain a critical $\eta_c$. For the electromagnetic field coupling to Einstein tensor or Weyl tensor [22], we find that the dynamical properties of perturbations depend sharply on the parities of perturbational fields themselves and the coupling strength between perturbational fields and gravitational tensors. For the electromagnetic perturbation with even parity, it always decays for arbitrary multipole number $l$ and coupling
constant $\alpha$. For the electromagnetic perturbation with odd parity grows with exponential rate and the instability occurs if the coupling constant $\alpha$ is smaller than certain a negative critical value $\alpha_c$. These dynamical properties of the odd parity electromagnetic field coupling to Einstein tensor or Weyl tensor are similar to those of the nonzero-$l$ partial waves of a scalar field coupling to Einstein tensor in the Reissner-Nordström black hole spacetime. Moreover, we find also that the critical coupling constant and the change of quasinormal frequencies with the coupling constant depend on the types of the coupled gravitational tensors, the black hole parameters, the multipole number $l$ and spin $s$ of perturbational fields. Furthermore, in the case with the coupling to Einstein tensor, the constraint on the coupling constant $\alpha$ is independent of the parities of electromagnetic fields, whereas in the case with Weyl corrections such constraint depends on the parity of the electromagnetic field.

**IV. SUMMARY**

We have studied the wave dynamics of an electromagnetic perturbation coupling to Einstein tensor in the four-dimensional Reissner-Nordström black hole spacetime. We find that the wave dynamic equation of the electromagnetic perturbation strongly depends on the parity of the perturbational field itself and the coupling strength between electromagnetic field and Einstein tensor. With the increase of the coupling constant $\alpha$, we find that the real parts of fundamental quasinormal frequencies increase for the electromagnetic perturbation with odd parity. For the perturbation with even parity, the real parts also increase monotonously with respect to $\alpha$ in the case with the smaller value of $q$, but for the case with the larger $q$, they decrease first and then increase. The changes of the imaginary parts with $\alpha$ are more complicated. The dependence of quasinormal frequencies on the black hole charge $q$ is also modified due to the coupling between electromagnetic perturbation and Einstein tensor. Moreover, we also find that the electromagnetic perturbation with even parity always decays for arbitrary multipole number $l$ and coupling constant $\alpha$. However, the electromagnetic perturbation with odd parity grows with exponential rate if the coupling constant $\alpha$ is smaller than certain a negative critical value $\alpha_c$, which can be fitted best by the function $\alpha_c \simeq a \sqrt{\frac{l}{l+1}} + b$ with two numerical constants (with dimensions of length-squared) $a, b$ depended on the charge $q$. The absolute value of $\alpha_c$ decreases with the multipole number $l$ and the charge $q$, which means that in the case with the higher $l$ and the larger $q$ the instability appears more easily and the growth rate becomes more stronger.

We also make a comparison among the dynamical properties of perturbational fields coupling to gravitational tensors in some black hole spacetimes. For the scalar field coupling to Einstein tensor in the Reissner-
Nordström black hole spacetime, we find that the partial wave \( l = 0 \) always decays for arbitrary coupling constant, which is similar to that of the even parity electromagnetic perturbation coupling to Einstein tensor or Weyl tensor. The nonzero-\( l \) partial waves of the coupled scalar field share the similar properties of the odd parity electromagnetic perturbation coupling to Einstein tensor or Weyl tensor in the black hole spacetime. These perturbations grow with exponential rate as the coupling strength is stronger than the certain a critical value. Moreover, we find also that the critical coupling constant and the change of quasinormal frequencies with the coupling constant depend on the types of the coupled gravitational tensors, the black hole parameters, the multipole number \( l \) and spin \( s \) of perturbational fields. Furthermore, the constraint on the coupling constant \( \alpha \) is independent of the parities of electromagnetic fields in the case with the coupling to Einstein tensor, but depends on the parity of the electromagnetic field in the case with Weyl corrections. It would be of interest to generalize our study to other black hole spacetimes, such as rotating black holes etc. Work in this direction will be reported in the future.

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