Arithmetic over the Gaussian Number Field on a Certain Family of Elliptic Curves with Complex Multiplication

By

Yoshihiro ÔNISHI* and Fumio SAIRAIJI**

Abstract

This work is a sequel of a previous work of one of the authors (Y.Ô), which treated certain congruence relation between an elliptic Gauss sum and a coefficient of power series expansion at the origin of the lemniscate sine function. We extend the previous result (in [O]) which concerned only for non-vanishing elliptic Gauss sums. We give new congruence relations between power series coefficients of the lemniscate cosine function, which hold if and only if the corresponding elliptic Gauss sum vanishes.

Introduction

In the paper [H1], Hurwitz gave the following result:

**Theorem 0.1.** Let \( p > 3 \) be a rational prime, and let \( h(-p) \) be the class number of the imaginary quadratic field \( \mathbb{Q}(\sqrt{-p}) \). Then we have

\[
h(-p) \equiv \begin{cases} -2B_{2p} \mod p & \text{if } p \equiv 3 \mod 4, \\ 2^{-1}E_{2p} \mod p & \text{if } p \equiv 1 \mod 4. \end{cases}
\]

Here \( B_{n} \) is the \( n \)-th Bernoulli number, and \( E_{n} \) is the \( n \)-th Euler number. Moreover, the absolutely smallest residue of the right hand side exactly equals to \( h(-p) \).

Each of these congruences is proved by expressing the value at \( s = 1 \) of the Dirichlet L-series \( L(s, \left( \frac{-p}{p} \right)) \) as a trigonometric Gauss sums, which is defined by a sort of Gauss sum.

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sum using suitable trigonometric function instead of the exponential function in the classical Gauss sum. Under the Birch Swinnerton-Dyer (BSD) conjecture, one of the authors gave in [O] an analogue of Theorem 0.1 by replacing Dirichlet $L$-series and the trigonometric Gauss sum by Hecke’s $L$-series and an elliptic Gauss sum, respectively, in which the class number is replaced by a square root of the conjectural order of the Tate-Shafarevich group, and the Bernoulli number or Euler number is done by certain coefficient of the power series expansion at the origin of the lemniscate sine function.

Elliptic Gauss sums were used, in order to compute numerically the $L$-series attached to some elliptic curves over $\mathbb{Q}$, in the famous original paper [BSD] by Birch and Swinnerton-Dyer. We wish to use them for investigation of $L$-series attached to some elliptic curves defined over $\mathbb{Q}(i)$.

The paper [O] is written about such investigation only for the case where the associated prime $\ell$ is congruent to 5 or 13 modulo 16, since the treated elliptic Gauss sum for that case does not vanish and it is directly relates the order of Tate-Shafarevich group. In this paper, we extend the result [O] to all the cases on modulo 16 of the primes $\ell$ congruent 1 modulo 4. The remarkable point is that, in the cases which do not treated in [O], the corresponding elliptic Gauss sums indeed vanish often, which means the associated Hecke $L$-series vanish as well. We verify such vanishing phenomenon by the tables in [A]. So that, the corresponding elliptic curve, which is defined over the Gaussian number field, is expected to be of positive Mordell-Weil rank.

We present certain Kummer type congruences (Theorem 7.1) on power series coefficients of the lemniscate cosine function which are valid if and only if the corresponding elliptic Gauss sum (hence the value at 1 of the corresponding Hecke $L$ series) vanishes.

The corresponding elliptic curve (see the defining equations (3.4), (4.4), and (5.4)) is additive reduction modulo $\lambda$ and our Kummer-type congruence is quite resemble to the Kummer congruence which guarantees the existence of the Kubota-Leopoldt $p$-adic $L$-function. So the authors hope that our result gives a hint for a construction of $p$-adic $L$-functions for an elliptic curve which is additive reduction modulo $p$.

This paper is organized as follows. From §1 to §2, we setup fundamental background. From §3 to §4, we review the results in [A] and [O]. In §5, we review the result for the rest cases which is omitted in [O]. In §6, we discuss some structure of the Mordell-Weil group of the curve and how our theory relates BSD conjecture. From §7 to §11, we give the main result (Theorem 7.1) and its proof. Especially, in §11, we show a two term congruence relation (see Theorem 7.2) which might be a hint to construct a $p$-adic ($\lambda$-adic) $L$-function for an elliptic curve that is additive reduction modulo $\lambda$.

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1 The lemniscate sine and cosine function

The inverse function

\[ t \mapsto u = \int_0^t \frac{dt}{\sqrt{1 - t^4}} \]

is called the lemniscate sine function, which is denoted by \( t = \text{sl}(u) \) and is expanded as

\[ \text{sl}(u) = u - \frac{1}{10} u^5 + \frac{1}{120} u^9 - \frac{11}{15600} u^{13} + \cdots = \sum_{n=0}^{\infty} C_n u^n \]

with \( C_n \) in \( \mathbb{Q} \). Then we have \( C_n = 0 \) if \( n \not\equiv 1 \mod 4 \) and \( n! C_n \) belongs to \( \mathbb{Z} \). It is an elliptic function whose period lattice is \( \Omega = (1 - i) \varpi \mathbb{Z}[i] \), where

\[ \varpi = 2 \int_0^1 \frac{dt}{\sqrt{1 - t^4}} = \int_1^\infty \frac{dx}{2\sqrt{x^3 - x}} = 2.62205 \cdots . \]

The divisor of \( \text{sl}(u) \) modulo \( \Omega \) is given by

\[ \text{div}(\text{sl}) = (0) + (\varpi) - \left( \frac{\varpi}{1 + i} \right) - \left( \frac{i\varpi}{1 + i} \right) . \]

Throughout this paper, we denote

\[ \varphi(u) = \text{sl} \left( (1 - i) \varpi u \right) . \]

The lemniscate cosine \( \text{cl}(u) \) is defined by

\[ \text{cl}(u) = \text{sl} \left( u + \frac{\varpi}{2} \right) . \]

Moreover, we use the notation

\[ \psi(u) = \text{cl} \left( (1 - i) \varpi u \right) . \]

Then both of functions \( \varphi(u) \) and \( \psi(u) \) have period lattice \( \mathbb{Z}[i] \) of \( \mathbb{C} \). We define the
\[ D_n \]’s by the expansion of \( \text{cl}(u) \) as
\[
\text{cl}(u) = \sum_{n=0}^{\infty} D_n u^n = 1 - u^2 + \frac{1}{2} u^4 - \frac{3}{10} u^6 + \frac{7}{40} u^8 - \ldots.
\]
Then, \( D_n = 0 \) for odd \( n \) and \( n! D_n \) is in \( \mathbb{Z} \).

2 The ray class field

We take a rational prime \( \ell \equiv 1 \mod 4 \), and we fix its decomposition \( \ell = \lambda \overline{\lambda} \) in \( \mathbb{Z}[i] \) with \( \lambda \equiv 1 \mod (1+i)^3 \). We fix a subset \( S \) of \( \mathbb{Z}[i] \) (sometimes called a \( 1/4 \)-set) such that \( (\mathbb{Z}[i]/(\lambda))^x \simeq S \cup -S \cup iS \cup -iS, |S| = (\ell - 1)/4 \). Moreover we define
\[
A = \varphi(\frac{1}{\lambda}), \quad \mathcal{O}_\lambda = \text{the ring of integers in } \mathbb{Q}(i, A),
\]
\[
\tilde{\lambda} = \gamma(S)^{-1} \prod_{r \in S} \varphi(\frac{r}{\lambda}),
\]
\[
\begin{cases}
\{ \pm 1, \pm i \} \ni \gamma(S) \equiv \prod_{r \in S} r \mod \lambda \text{ if } \ell \equiv 5 \mod 8, \\
\{ \pm i \} \ni \gamma(S)^2 \equiv \prod_{r \in S} r^2 \mod \lambda \text{ if } \ell \equiv 1 \mod 8.
\end{cases}
\]

Here, we have sign ambiguity of \( \gamma(S) \) in the case of \( \ell \equiv 1 \mod 8 \). In any case, we know (see for example p.106 of [A] or Lemma 1.11 [A]) that
\[
(2.2) \quad A \in \mathcal{O}_\lambda, \quad (A)^{\ell-1} = (\lambda), \quad \tilde{\lambda}^4 = -\lambda.
\]
Note that \( \mathbb{Q}(i, A) \) is the ray class field over \( \mathbb{Q}(i) \) of conductor \( ((1+i)^3\lambda) \) (see Takagi [T], §32, for instance).

Throughout this paper, we fix the identification
\[
(2.3) \quad \mathbb{Z}[i]_\lambda \simeq \mathbb{Z}_\ell,
\]
where the left hand side is the \( \lambda \)-adic completion of \( \mathbb{Z}[i] \). Moreover, we consider they are subring of the algebraic closure \( \overline{\mathbb{Q}}_\ell \) of \( \mathbb{Q}_\ell \). We use the following notation. For any element \( \alpha \) in the integer ring \( \mathbb{Z}_\ell \) of \( \overline{\mathbb{Q}}_\ell \), we denote the \( \ell \)-adic order by \( \text{ord}(\alpha) \). For example, for \( \alpha \) in \( \mathbb{Z}[i]_\lambda \), \( \text{ord}(\alpha) = n \) if and only if \( \lambda^n \) divides \( \alpha \) but \( \lambda^{n+1} \) does not.

3 Asai’s theory

In this section we recall the results from [A] in order to go to the rest cases smoothly. We assume here \( \ell \equiv 13 \mod 16 \) for simplicity. For the other cases, see [A]. We put \( \chi_{\lambda}(r) = \left( \frac{r}{\lambda} \right) \). In this case, the elliptic Gauss sum is defined by
\[
\text{egs}(\lambda) = \sum_{r \in S} \chi_{\lambda}(r) \varphi\left( \frac{r}{\lambda} \right).
\]
Since the terms of this summation are algebraic integers, so is \( \text{egs}(\lambda) \).
Theorem 3.1. ([A]) There exists an odd $A_\lambda$ in $\mathbb{Z}$ such that 
$$\text{egs}(\lambda) = A_\lambda \tilde{\lambda}^3,$$
where $\tilde{\lambda}$ is defined by (2.1). In particular, $\text{egs}(\lambda) \neq 0$.

Remark 3.2. (1) This theorem is proved by using the functional equation for the Hecke $L$-series corresponding to the suitable Hecke character associated to $\chi_\lambda$ and the formula of Cassels-Matthews (see [M]) for the classical quartic Gauss sum which appears as the root number of the functional equation. It is expected to have another prove the formula of Cassels-Matthews if we get a part of BSD conjecture including the parity of the order of the corresponding Tate-Shafarevich group.

(2) We call $A_\lambda$ the coefficient of $\text{egs}(\lambda)$ according to [A].

We recall the corresponding Hecke $L$-series. Still we are assuming $\ell \equiv 13 \mod 16$.

Taking $\{1, i\}$ as a set of complete representatives of $(\mathbb{Z}[i]/(1 + i)^2)\times$, we define 
$$\chi_\alpha(\alpha) = 2^2 \quad \text{for} \quad \alpha \equiv \varepsilon \mod (1 + i)^2, \quad \varepsilon \in \{1, i\}, \quad \tilde{\chi}((\alpha)) = \chi_\lambda(\alpha) \chi_0(\alpha) \varepsilon.$$
Then $\tilde{\chi}$ is a Hecke character of conductor $(1 + i)^2 \lambda$.

Theorem 3.3. We have 
$$L(1, \tilde{\chi}) = -2(1 - i)(2\lambda^{-1}) \text{egs}(\lambda).$$

Searching an elliptic curve whose conductor is the square of that of $\tilde{\chi}$ (see [ST], Theorem 12), we see that the elliptic curve corresponding to $L(s, \tilde{\chi})$ is 
$$(3.4) \quad E_{-\lambda} : y^2 = x^3 + \lambda x, \quad (\lambda \overline{\lambda} = \ell \equiv 13 \mod 16).$$
Deuring [D] showed that 
$$(3.5) \quad L_{E_{-\lambda}/\mathbb{Q}(i)}(s) = L(s, \tilde{\chi}) L(s, \overline{\chi}).$$
Especially, if $\ell \equiv 13 \mod 16$, then $\text{rank } E_{-\lambda}/\mathbb{Q}(i) = 0$, which is shown by Theorems 5.1, 5.3 and Coates-Wiles theorem in [CW]. Moreover, we recall the following result from [O].

Proposition 3.6. If the full statement of BSD conjecture for the curve $E_{-\lambda}$ is true, then 
$$\# \text{II}(E_{-\lambda}/\mathbb{Q}(i)) = A_\lambda \overline{\lambda}^2.$$

4 Some congruence on the coefficients of elliptic Gauss sums

The former part of the following theorem is proved in [O] and reproved a sophisticated method as Lemma 8.3 later. Let $C_n$ be the coefficient of $u^n$ defined by (1.1). Since $\frac{3}{\pi}(\ell - 1)! C_{\frac{3}{2}(\ell - 1)}$ is in $\mathbb{Z}$, $-\frac{1}{\pi} C_{\frac{3}{2}(\ell - 1)}$ is in $\mathbb{Z}_\ell$.

Theorem 4.1. ([O]) Assuming $\ell \equiv 13 \mod 16$, we have $A_\lambda \equiv -\frac{1}{\pi} C_{\frac{3}{2}(\ell - 1)}$ mod $\ell$. The absolutely minimal residue of the right hand side is exactly equal to $A_\lambda$.

The latter part of Theorem 4.1 follows from the former part and the following lemma which is proved in [O].
Lemma 4.2. For any \( \ell = \lambda \overline{\lambda} \equiv 1 \mod 4 \), we have \( |A_\lambda| < \frac{1}{2} \ell \).

Remark 4.3. Observing Kanou’s manmouth table, the behavior of \( |\text{egs}(\lambda)| \) with respect to \( \ell \to \infty \) is quite small. Indeed, the estimation \( |A_\lambda| < \ell^{1/4} \) is hopeful.

Joining Proposition 5.0 and Therem 4.1 together, we have a natural generalization of Hurwitz’ congruence in Theorem 0.1.

For the case of \( \ell \equiv 5 \mod 16 \), we have a similar story which is described in [A] and [O]. The corresponding elliptic curve for this case is

\[
\delta^\epsilon_\lambda : y^2 = x^3 - \frac{3}{4} \lambda x, \quad (\lambda \overline{\lambda} = \ell \equiv 5 \mod 16),
\]

for which we have rank \( \delta^\epsilon_\lambda (\mathbb{Q}(i)) = 0 \), and the corresponding congruence as (4.1).

So, from the next section, we proceed to the remaining case of \( \ell \equiv 1 \mod 8 \). About 18% of the 172 examples of this case in [A], \( \text{egs}(\lambda) = 0 \) holds, where the elliptic Gauss sum for this case is defined in the next section.

5 The foregoing researches in the case of \( \ell \equiv 1 \mod 8 \)

From now on, we always assume the prime \( \ell \) satisfies \( \ell \equiv 1 \mod 8 \), \( \ell = \lambda \overline{\lambda} \), \( \lambda \equiv 1 \mod (1 + i)^3 \), \( \chi_\lambda(\nu) = \left( \frac{\nu}{\lambda} \right) \). Then we see \( \chi_\lambda(i) = \overline{\chi}(1) = (-1)^\frac{\lambda i}{\overline{\lambda}} \). Using \( \psi(u) = \psi((1 - i) \overline{\nu}) \), the elliptic Gauss sum for this case is defined by

\[
\text{egs}(\lambda) = \sum_{\nu \in \mathbb{S} \setminus S} \chi_\lambda(\nu) \psi\left( \frac{\nu}{\lambda} \right).
\]

In this paper \( \varepsilon \) always denotes any element in \( \mu_4 \), where \( \mu_4 = \{1, -1, i, -i\} \). Recalling the canonical isomorphism \( \mu_4 \isom \left( \mathbb{Z}[i]/(1 + i)^3 \right)^\times \), we define the character \( \chi_\alpha \) by

\[
\chi_\alpha(\alpha) = \varepsilon \text{ if } \alpha \equiv \varepsilon \mod (1 + i)^3 \quad (\alpha \in \mathbb{Z}[i], (1 + i)\overline{f}(\alpha)).
\]

(Case 1) If \( \ell \equiv 1 \mod 16 \), \( \chi_\lambda(i) = 1 \). We define \( \chi_\lambda = \chi_\lambda \chi_\alpha \) and \( \overline{\chi}(\alpha) = \chi_\lambda(\alpha) \overline{\chi} \).

(Case 2) If \( \ell \equiv 0 \mod 16 \), \( \chi_\lambda(i) = -1 \). So defining \( \chi_\lambda = \chi_\lambda \chi_\alpha \) and \( \overline{\chi}(\alpha) = \chi_\lambda(\alpha) \overline{\chi} \).

In any case, we see \( \overline{\chi} \) is a Hecke character of conductor \( (1 + i)^3 \lambda \). Then, as in [A], we have the following expression:

\[
L(1, \overline{\chi}) = (-1)^\frac{\lambda i}{(1 + i)^3} \chi_\lambda(1 + i) 2^{-1} \lambda^{-1} \text{egs}(\lambda).
\]

Theorem 5.2. ([A]) Let \( \zeta_8 = \exp(2\pi i/8) \). There exists \( A_\lambda \) in \( \mathbb{Z}[\zeta_8] \) such that

\[
\text{egs}(\lambda) = A_\lambda \chi_\lambda^3.
\]

Here, \( A_\lambda \) is given by the table (5.5) below with some \( a_\lambda \) in \( \mathbb{Z} \).

This theorem is also proved by using the formula of Cassels-Matthew and the functional equation of \( L(s, \chi) \). In [A], it is observed by Asai that \( a_\lambda \) is in \( 2\mathbb{Z} \), but any proof of this is not known yet.

Searching the elliptic curve whose conductor is \( (1 + i)^3 \lambda \), which is square of that of \( \chi \) (ST, Theorem 12), we see the Hecke \( L \)-series associated to \( \text{egs}(\lambda) \) is a factor
of the $L$-series of the elliptic curve

$$E_{\lambda} : y^2 = x^3 - \lambda x, \quad (\lambda \equiv \ell \equiv 1 \text{ mod } 8).$$

We have the same equation as (5.5) for this case as well. The reduction type at $(1+i)$ is of type III, and one at $\lambda$ is of type $I_{2}^{*}$. Each Tamagawa number $\tau_{p}$ and the coefficients $A_{\lambda}$ of $\text{egs}(\lambda)$ is given as follows:

$$
\begin{array}{cccc}
\chi_{\lambda}(1+i) & A_{\lambda} & \tau_{(\lambda)} & (\ell \equiv 1 \text{ mod } 16) \\
& & & \\
\ell \equiv 1 \text{ mod } 16 & A_{\lambda} & \tau_{(\lambda)} & (1+i) \\
2 & 2 & 2 & 2 \\
4 & 4 & 2 & 2 \\
\ell \equiv 9 \text{ mod } 16 & A_{\lambda} & \tau_{(\lambda)} & (1+i) \\
2 & 2 & 2 & 2 \\
4 & 4 & 2 & 2 \\
\end{array}
$$

**Remark 5.6.** Assuming the full statement of BSD conjecture true, we have $(\frac{1}{2}a_{\lambda})^{2} = \# \text{III}(E_{\delta})$ if $a_{\lambda} \neq 0$.

Recall the numbers $D_{n}$ defined in (1.2). Since $(\frac{3}{4}(\ell - 1))! D_{\frac{3}{4}(\ell - 1)}$ is in $Z_{\ell}$, $-\frac{1}{2}D_{\frac{3}{4}(\ell - 1)}$ is in $Z_{\ell}$. We keep in mind that $Z[\zeta_{8}]$ is an Euclidean ring. Using the method of [O] and Lemma 4.2, the following is shown.

**Theorem 5.7.** Let $\tilde{\lambda}_{0}$ be a prime lying above $\lambda$ in $Q(\zeta_{8})$. We have

$$A_{\lambda} \equiv -\frac{1}{2}D_{\frac{3}{4}(\ell - 1)} \text{ mod } \tilde{\lambda}_{0},$$

where $A_{\lambda}$ is given by the table (5.5). Furthermore, $A_{\lambda}$ is the minimal residue in $\zeta_{8}Z[i]$ of the right hand side with respect to the absolute norm.

### 6 An analogue of the congruent numbers

The following is well-known (see, for example, Koblitz’ book [K]).

**Proposition 6.1.** Let $n$ be a rational integer. For the elliptic curve $E_{n^{2}} : y^2 = x^3 - n^2x$, the following three are equivalent each other:

1. There exist $u, v$ in $Q$ such that $n^2 = u^2 - v^2$;
2. $n$ is a congruent number;
3. $\text{rank} \, E_{n^{2}}(Q) > 0$.

The claim (1) is a sort of paraphrase of the definition of congruent number for $n$. The equivalence of (2) and (3) is described as Proposition 18 in [K].

**Lemma 6.2.** Let $A$ be a square-free integer in $Z[i]$ not dividing 5. Then there are only two torsion points $(0, 0)$ and $\infty$ in the group of $Q(i)$-rational points on the elliptic curve $E_{A} : y^2 = x^3 - Ax$. 

$$E_{\lambda} : y^2 = x^3 - \lambda x.$$
Proof. This proof is a slight modification of the argument in [N]. Since \( A \) is square-free, the equation \( x^3 - Ax = 0 \) has only root \( x = 0 \) in \( \mathbb{Q}(i) \). Thus the \( \mathbb{Q}(i) \)-rational point of \( \mathcal{E}_A \) of order two is \((0,0)\). Let \((a,b)\) be a \( \mathbb{Q}(i) \)-rational point of \( \mathcal{E}_A \). The \( x \)-coordinate of \([1+i](a,b)\) is \( x_{1+i} = (\frac{b}{1+i})^2 \) and the \( x \)-coordinate of \([2](a,b)\) is \( x_2 = (\frac{a^2-A}{2})^2 \). Therefore, \( x \)-coordinate of any point in either \([1+i] \mathcal{E}_A(\mathbb{Q}(i)) \) or \([2] \mathcal{E}_A(\mathbb{Q}(i)) \) is square. Assume \((a,b)\) is of finite order. Then \( a \) and \( b \) belong to \( \mathbb{Z}[i] \) (see [N], p.14, Theorem 2 or [C], §11 and §12). If \((a,b)\) satisfies \([2](a,b) = (0,0)\), we have \( a^2 = -A \). It does not occur because \( A \) is square-free. Hence there does not exist any \( \mathbb{Q}(i) \)-rational point of order divided by 4. Assume that \((a,b)\) is \( \mathbb{Q}(i) \)-rational points of odd order. Since \( \mathcal{E}_A(\mathbb{Q}(i))/[1+i] \mathcal{E}_A(\mathbb{Q}(i)) \) is an abelian group of exponent two, \((a,b)\) is in \([1+i] \mathcal{E}_A(\mathbb{Q}(i)) \). Thus \( a \) is square in \( \mathbb{Z}[i] \). Since \([1+i](a,b)\) is of odd order and \( x_{1+i} \) is in \( \mathbb{Z}[i] \), we have \( a \mid b \) and \( 1+i \mid b \). As \( a \) is square and \( b^2 = a(a^2-A) \), we have \( a = f^2, b = f^2 g, a^2-A = f^2 g^2 \) for some \( f, g \) in \( \mathbb{Z}[i] \). Since \(-A = f^2(g^2-f^2)\) and \( A \) is square-free, \( f^2 \) is unit. Thus we have \( f^2 = \pm 1 \). Furthermore, \([2](a,b)\) is of odd order and \( x_2 \) is in \( \mathbb{Z}[i] \), we have \( 2b \mid a^2 + A \). Since \( f^2 \) is unit, we have \( 2g \mid 2f^2 - g^2 \). Thus we have \( 1+i \mid g \) and \( \frac{a^2}{1+i} \mid f^2 \). Since \( f^2 \) is an unit, \( g \) is equal to \( 1+i \) up to unit. Therefore we have \( A = \pm(-1 \pm 2i) \). They does not occur because \( A \) does not divide 5. Hence there does not exitsit any \( \mathbb{Q}(i) \)-rational point of odd order. This completes the proof. \( \square \)

Remark 6.3. In the last four cases of \( A \) in the proof above, we see the groups of \( \mathbb{Q}(i) \)-rational points of the curves are of rank 0 because the \( L \)-functions do not vanish at 1 (see the proof of Lemma 2.11 (b) p.105, [A]). So that they are finite groups due to [CW]. MAGMA says that the groups are of order 10 generated by \( \pm(1 \mp 2i, -1 \pm 3i) \).

We prove the following analogue of Proposition 6.1.

Proposition 6.4. Let \( \lambda \) be any Gaussian prime of degree 1 satisfying \( \lambda \equiv 1 \ \bmod \ (1+i)^3 \) and assume \( \lambda \nmid 5 \). The following three statements are equivalent:

1. There are infinitely many \( \mathbb{Q}(i) \)-rational points on \( \mathcal{E}_\lambda \), namely,
   \[
   \text{rank } \mathcal{E}_\lambda(\mathbb{Q}(i)) > 0;
   \]

2. The prime \( \lambda \) is of the form \(-\alpha^4 + \beta^2 i\) with \( \alpha, \beta \in \mathbb{Q}(i)\);

3. The prime \( \lambda \) is of the form \( u^4 - v^2 \) with \( u, v \in \mathbb{Q}(i)\).

Proof. (2)⇒(1). For the given expression \( \lambda = -\alpha^4 + \beta^2 i \), we see \( \alpha^2, \beta \) is a \( \mathbb{Q}(i) \)-rational point of infinite order on the curve \( \mathcal{E}_\lambda \) because of Lemma 6.2 and
   \[
   (-\alpha^2 i)^3 - \lambda(\alpha i)^2 = (-\alpha^2 i)^3 - (-\alpha^4 + \beta^2 i)(\alpha i)^2 = (\alpha \beta)^2.
   \]

(3)⇒(1). This is proved similarly. Indeed, if \( \lambda = u^4 - v^2 \), then \((x, y) = (u^2, uv)\) is a point of infinite order on \( \mathcal{E}_\lambda(\mathbb{Q}(i)) \) because of \( x^2 = \lambda x = u^6 - (u^4 - v^2) u^2 = (uv)^2 = y^2 \).

(1)⇒(3). Lemma 6.2 implies that the set of torsion points of \( \mathcal{E}_\lambda(\mathbb{Q}(i)) \) is \( \{(0,0), \infty\} \). So we assume there exists a non-torsion point \((a, b)\), namely \( b^2 = a^3 - \lambda a \), with \( a, b \)
in $Q(i)$. The duplication $[2](a, b)$ is given by
\[
\left(\frac{(a^2 + \lambda)^2}{4b^2}, \frac{a^6 - 5\lambda a^4 - 5\lambda^2 a^2 + \lambda^3}{8b^3}\right).
\]
We define
\[
u = \frac{a^2 + \lambda}{2b} (\neq 0), \quad v = \frac{a^4 - 6\lambda a^2 + \lambda^2}{4b^2}.
\]
Then the point $(u^2, uv)$ is on the curve, and $\lambda = u^4 - v^2$.
(1)$\Rightarrow$(2). This proof is given by 2-descent, which is a modification of the proof of Proposition 1.4 in Chapter X, [Si]. We put
\[
T_\lambda = \{ b \in Q(i)^\times/(Q(i)^\times)^2 \mid \text{ord}_p(b) \equiv 0 \mod 2 \text{ for all prime } p \not| \lambda \}.
\]
This is a subgroup of $Q(i)^\times/(Q(i)^\times)^2$ of order four generated by $i$ and $\lambda$. There is a homomorphism
\[
(6.5) \quad \delta_\lambda(Q(i)) \to T_\lambda \text{ defined by } (x, y) \mapsto \begin{cases} x & \text{if } x \neq 0, \\ \lambda & \text{if } x = 0, \\ 1 & \text{if } x = \infty. \end{cases}
\]
Indeed, if we put $(x_3, y_3) = (x_1, y_1) + (x_2, y_2)$ which is an addition on $\delta_\lambda$, and
\[
m = \begin{cases} \frac{(y_1 - y_2)/(x_1 - x_2)} & \text{if } x_1 \neq x_2, \\ \frac{(3x_1^2 + \lambda)/(2y_1)} & \text{if } x_1 = x_2, \end{cases}
\]
we have $x_1x_2x_3 = (-mx_1 + y_1)^2$. Hence, $x_3 \in x_1x_2 (Q(i)^\times)^2$ if $x_1x_2 \neq 0$. If $x_2 \neq 0$ and $x_1 = 0$, we have
\[
x_3 = \left(\frac{y_2}{x_2}\right)^2 - x_2 = \frac{-\lambda x_2}{x_2^2} \in \lambda x_2 (Q(i)^\times)^2
\]
because of $m^2 = x_1 + x_2 + x_3$. We show that the kernel of (6.5) is $[1 + i]\delta_\lambda(Q(i))$. Let $(x_1, y_1)$ be a point in $\delta_\lambda(Q(i))$, $(x_3, y_3) = [1 + i](x_1, y_1)$ and $(x_2, y_2) = [i](x_1, y_1) = (-x_1, iy_1)$. Then we see
\[
x_3 = m^2 - x_1 - x_2 = \left(\frac{y_1}{(1 + i)x_1}\right)^2 \in (Q(i)^\times)^2.
\]
Therefore $[1 + i]\delta_\lambda(Q(i))$ is contained in the kernel. Conversely, suppose $(x_3, y_3)$ is in $\delta_\lambda(Q(i))$ and $x_3$ is a square. Then, firstly, $m = y_1/((1 + i)x_1)$ is in $Q(i)$ because of the assumption on $x_3$. Secondly, as
\[
y_3 = -(m(x_3 - x_1) - y_1) = -m^3 - \frac{1 + i}{2} y_1
\]
is in $Q(i)$, $y_1$ is in $Q(i)$. Thirdly, as $m$ is in $Q(i)$, $x_1$ is also in $Q(i)$. Accordingly, the kernel is contained in $[1 + i]\delta_\lambda(Q(i))$. Therefore the induced homomorphism
\[
\delta_\lambda(Q(i))/[1 + i]\delta_\lambda(Q(i)) \to T_\lambda
\]
is bijective. Summing up, for a $Q(i)$-rational point $(x, y)$, either $x$ or the first coordinate of $(x, y) + (0, 0)$ is of the form $\alpha^2 i$. We write the obtained point as $(\alpha^2 i, \alpha\beta)$.
Then $\alpha^2 \beta^2 = -\alpha^4 i - \lambda \alpha^2 i$. This means $\lambda = -\alpha^4 + \beta^2 i$ and the proof has been completed.

**Remark 6.6.** We shall give some remarks on Proposition 6.4.

(1) A prime $\lambda$ of the form in (2) or (3) of Proposition 6.4 should be called a Gaussian congruent number.

(2) In the examples in [A] each of statements (1), (2), and (3) of Proposition 6.4 is satisfied if and only if $\text{egs}(\lambda) = 0$.

(3) In the examples of [A] such that $\text{egs}(\lambda) = 0$, except $\lambdaX = 4817 \equiv 1 \mod 16$, we can take $\alpha, \beta$ in $\mathbb{Z}[i]$. See Example 6.7 below.

(4) We summarize the situation as follows:

$$
\lambda \text{ is of the form } -\alpha^4 i + \beta^2 \iff \text{Prop. 6.4} \iff \text{BSD} \iff \text{Coates-Wiles} \iff L(1, \chi) = 0 \iff \text{Anal} \iff \text{egs}(\lambda) = 0.
$$

(5) In the proof of (1) $\Rightarrow$ (2), we show that $\delta_\lambda(Q(i))/[1 + i]\delta_\lambda(Q(i))$ is generated by $(0, 0)$ and at most one non-torsion point. Thus the $\mathbb{Z}[i]$-rank of $\delta_\lambda(Q(i))$ is at most one. If $\text{egs}(\lambda) = 0$, then $\delta_\lambda(Q(i))$ has $\mathbb{Z}[i]$-rank one, that is, MW-rank two.

**Example 6.7.** Take $\lambda = 41 + 56 i$, $\ell = \lambdaX = 4817 \equiv 1 \mod 16$. Then $\lambda = \alpha^4 - \beta^2$, where

$$
\alpha = \frac{-7i(2 + i)(4 + i)}{3(1 + 2i)(2 + 3i)}, \quad \beta = \frac{-(1 + i)^5(3 + 2i)(7 + 8i)(6 + 11i)}{3^2(1 + 2i)^2(2 + 3i)^2}
$$

and $P = (\alpha^2, \alpha \beta)$ is a point of infinite order. This is given by a calculation by MAGMA. It also says that the Mordell-Weil rank of $\delta_\lambda$ is 2. We know another rational point by MAGMA as follows. Let

$$
\alpha' = \frac{i(1 + 2i)(2 + 3i)}{3}, \quad \beta' = \frac{i 7(1 + i)(2 + i)(4 + i)}{3^2}.
$$

Then $\lambda = -\alpha'^4 + \beta'^2 i$ and $Q = (\alpha'^2, \alpha' \beta')$ is in $\delta_\lambda(Q(i))$ and $P = [1 + i] Q$. We do not know the point $Q$ generates how much part of the MW-group.
7 Vanishing EGS and Kummer-type congruence

We rewrite the expansion (1.2) of $\text{cl}(u)$. Namely, we define $G_n$ in $\mathbb{Z}$ by

$$
\text{cl}(u) = \sum_{n=0}^{\infty} G_n \frac{u^n}{n!} = 1 - 2u^2 + 12 \frac{u^4}{4!} - 216 \frac{u^6}{6!} + 7056 \frac{u^8}{8!} - 368928 \frac{u^{10}}{10!} + \cdots.
$$

Of course $G_n = n! D_n$. We denote by $H_\ell$ the Hasse invariant of $y^2 = x^3 - x$ at $\ell \equiv 1 \mod 4$, namely,

$$
H_\ell = \lambda + \lambda \equiv (\ell - 1) \left( \frac{\ell + 1}{\ell - 1} \right) \mod \ell.
$$

Our main result is the following theorem.

**Theorem 7.1.** The followings are equivalent:
1. $\text{egs}(\lambda) = 0$;
2. $\ell \mid G_{\frac{\ell}{4} - 1}$; (This is a special case of (3).)
3. Let $e$ be any positive integer satisfying $e \equiv \frac{3}{4}(\ell - 1) \mod (\ell - 1)$. Then for any $a \geq 0$,
   $$
   \sum_{r=0}^{a} \binom{a}{r} (-H_\ell)^{a-r} \frac{G_{e+r(\ell - 1)}}{e + r(\ell - 1)} \equiv 0 \mod \ell^{a - \left\lfloor \frac{a}{4} \right\rfloor + 1}.
   $$

In addition to this theorem, we have the following result.

**Theorem 7.2.** Assume that $\text{egs}(\lambda) = 0$. Let $b \geq 0$ be a given integer and $e$ be any positive integer satisfying $e \equiv \frac{3}{4}(\ell - 1) \mod (\ell - 1)$. Then the congruence

$$
G_{e+k(\ell - 1)} \equiv H_\ell^k \cdot \frac{G_e}{e} \mod \ell^{b+2}
$$

holds for any $k$ such that $\text{ord}(k) \geq b$.

8 $\ell$-adic explicit formula of an elliptic Gauss sum

Recall our identification of $\mathbb{Z}[i]_\lambda$ and $\mathbb{Z}_\ell$. As we treat a plenty of power series in $\mathbb{Z}_\ell[[x]]$ in this paper, we summarize convention on notation here. Let $f(x)$ and $g(x)$ be power series in $\mathbb{Z}_\ell[[x]]$. For a rational number $a$ in $\mathbb{Q}$, we write

$$
\text{if } f(x) \equiv g(x) \mod \lambda^a
$$

if all the coefficients of the terms in $f(x) - g(x)$ have $\ell$-adic order at least $a$. For a positive integer $m$, we write

$$
\text{if } f(x) \equiv g(x) \mod \text{deg } m
$$

if $f(x) - g(x)$ belongs to $x^m \mathbb{Z}_\ell[[x]]$. Moreover, we write

$$
\text{if } f(x) \equiv g(x) \mod \text{deg } m, \mod \lambda^a
$$

if all the coefficients of the terms of degree less than $m$ in $f(x) - g(x)$ have $\ell$-adic
order at least $a$. From now on, the number
\[d = \frac{3}{4}(\ell - 1)\]
appears frequently. Taking a primitive $(\ell - 1)$-th root $\zeta$ of 1 in $\mathbb{Z}_\ell$, we define
\[(8.1) \quad \text{Cl}(u) = \frac{1}{2} \sum_{j=0}^{\ell - 1} \zeta^{-\phi j} \text{Cl}(\zeta^j u).
\]
Then we have
\[\text{Cl}(u) = \frac{\ell - 1}{2} \sum_{a=0}^{\infty} G_{d+a(\ell-1)} \frac{u^{d+a(\ell-1)}}{(d + a(\ell - 1))!}.
\]
Thus $G_{d+a(\ell-1)}/(d + a(\ell - 1))$ is the coefficient of the power series expansion of $(\ell - 1)^{-1} \text{Cl}(u)/u$ of $u^n/n!$ with $n = d + a(\ell - 1) - 1$.

For a proof of Theorem 7.1, we give an $\ell$-adic explicit formula of $\text{egs}(\lambda)$ by using the Lubin-Tate formal group.

Let $\mathbf{LT}(x, y)$ be the Lubin-Tate formal group over $\mathbb{Z}_\ell$ corresponding to $\lambda$-plication $[\lambda]_{\mathbf{LT}}(x) = \lambda x + x^{\ell}$. Then non-zero points of the group $\mathbf{LT}[\lambda]$ of the $\lambda$-division points are roots of $\lambda + x^{\ell - 1} = 0$. Let $f_0(x)$ be the formal logarithm of $\mathbf{LT}(x, y)$. It follows from $[\lambda]_{\mathbf{LT}}(x) = f_0^{-1}(\lambda f_0(x)) \equiv x^{\ell} \mod \lambda$ that $\lambda f_0(x) \equiv f_0(x^{\ell}) \mod \lambda$ by Proposition 4.2 of Honda [Ho]. Namely, $\mathbf{LT}(x, y)$ is of type $\lambda - T$. Let $\mathbf{sl}(x, y)$ be the formal group defined by
\[\mathbf{sl}(x, y) = \text{sl}(\text{sl}^{-1}(x) + \text{sl}^{-1}(y))
\]
By the definition of $\mathbf{sl}(x, y)$, $\lambda$-plication $[\lambda]_{\mathbf{sl}}(x)$ satisfies
\[\text{Cl}(x) \circ \text{sl}(x) = \text{sl}(\lambda x).
\]
Thus $\lambda$ is a points of the group $\mathbf{sl}[\lambda]$ of $\lambda$-division points. As is well-known (see for instance, Proposition 8.2 of Lemmermeyer [Le] or Theorem 1.28 in [O2], which gives another proof by using the relation $\varphi(u) = \text{sl}(u^{-2})$, it holds that
\[\text{Cl}(x) \equiv x^{\ell} \mod \lambda.
\]
Hence, the formal group $\mathbf{sl}(x, y)$ is of type $\lambda - T$ as well. Since the formal group $\mathbf{LT}(x, y)$ is of the same type $\lambda - T$, there exists the unique strong isomorphism $\iota$ over $\mathbb{Z}_\ell$ from $\mathbf{LT}(x, y)$ to $\mathbf{sl}(x, y)$. Namely, there uniquely exists $\iota(x)$ in $\mathbb{Z}_\ell[[x]]$ such that
\[\iota(\mathbf{LT}(x, y)) = \mathbf{sl}(\iota(x), \iota(y)), \quad \iota(x) \equiv x \mod 2.
\]
Then there exists $\eta$ of the group $\mathbf{LT}[\lambda]$ of $\lambda$-division points of $\mathbf{LT}(x, y)$ such that
\[A = \varphi(1/\lambda) = \iota(\eta).
\]
We recall that $\eta^{\ell - 1} = -\lambda$ (see (2.2)). Since
\[\text{Cl}(u) = \varphi \circ \text{sl}(u), \quad \text{where } \varphi(x) = \sqrt{1 - x^2}/1 + x^2,
\]
we have
\[ \psi(1/\lambda) = \phi \circ \iota(\eta). \]

We note that \( \phi(x) \) is in \( \mathbb{Z}_m[[x]] \).

Taking a primitive \((\ell - 1)\)-th root \( \zeta \) of 1 in \( \mathbb{Z}_{\ell} \), we define
\[ \text{Sl}(u) = \frac{1}{4} \sum_{j=0}^{\ell-1} \zeta^{-dj} \text{sl}(\zeta^j u). \]

Then we have
\[ \text{Sl}(u) = \frac{\ell - 1}{4} \sum_{a=0}^{\infty} C_{d+a(\ell-1)} u^{d+a(\ell-1)} \]
as well as
\[ \text{Cl}(u) = \frac{\ell - 1}{2} \sum_{a=0}^{\infty} D_{d+a(\ell-1)} u^{d+a(\ell-1)}. \]

**Lemma 8.2.**
1. If \( \ell \equiv 3 \mod 8 \), the equation \( \text{egs}(\lambda) = (\text{Sl} \circ f_0)(\eta) \) holds.
2. If \( \ell \equiv 1 \mod 8 \), the equation \( \text{egs}(\lambda) = (\text{Cl} \circ f_0)(\eta) \) holds.

Proof. It follows from \( [\zeta]_{\mathbb{Z}_m}(x) = \text{sl}(\zeta x^{-1}) \in \mathbb{Z}_m[[x]] \) that \( \text{sl}^{-1}(x) \) is in \( \mathbb{Z}_{\ell}[\zeta]_{\mathbb{Z}_m}[[x]] \), and from \( \text{Sl}^{-1} \circ \iota(x) = f_0(x) \) that \( \text{Sl} \circ f_0(x) \) is also in \( \mathbb{Z}_{\ell}[x] \). Since \( \text{Cl}(x) = \phi \circ \text{Sl}(x) \) and \( \phi(x) \) is in \( \mathbb{Z}_{\ell}[x] \), we see \( \text{Cl}(\zeta x^{-1}) = \phi([\zeta]_{\mathbb{Z}_m}(x)) \). Hence, \( \text{Cl} \circ f_0(x) \) is in \( \mathbb{Z}_{\ell}[x] \). For \( \alpha \) in \( \mathbb{Z} \) coprime to \( \lambda \), we convert, if necessary, \( \zeta \) another \((p-1)\)-th root of 1 as satisfying \( \alpha \equiv \zeta \mod \lambda \). Then \( \varphi(\alpha/\lambda) = [\alpha]_{\mathbb{Z}_m}(A) = [\alpha]_{\mathbb{Z}_m}(A). \) Since \( A = \iota(\eta) \) and \( \text{Sl}^{-1} \circ \iota(x) = f_0(x) \), we have \( \varphi(\alpha/\lambda) = (\text{Sl} \circ \zeta x^{-1})(A) = (\text{Sl} \circ \zeta f_0)(\eta) \). We also have \( \psi(\alpha/\lambda) = (\text{Cl} \circ \zeta f_0)(\eta) \). Since \( \chi(\lambda) = \chi(\zeta) = \zeta^{-d} \), we have
\[ \text{egs}(\lambda) = \frac{1}{4} \sum_{a=1}^{\ell-1} \chi(\alpha) \varphi(\alpha/\lambda) = (\text{Sl} \circ f_0)(\eta) \]
in the case of \( \ell \equiv 5 \mod 8 \), and
\[ \text{egs}(\lambda) = \frac{1}{2} \sum_{a=1}^{\ell-1} \chi(\alpha) \psi(\alpha/\lambda) = (\text{Cl} \circ f_0)(\eta) \]
in the case of \( \ell \equiv 1 \mod 8 \). This completes the proof of Lemma 8.2. \( \square \)

**Lemma 8.3.**
1. If \( \ell \equiv 5 \mod 8 \), it holds that
\[ \text{egs}(\lambda) \equiv \frac{\ell - 1}{4} C_d \eta^d \mod \eta^\ell. \]
2. If \( \ell \equiv 1 \mod 8 \), it holds that
\[ \text{egs}(\lambda) \equiv \frac{\ell - 1}{2} D_d \eta^d \mod \eta^\ell. \]

Proof. Since \( \lambda f_0(x) = f_0 \circ \lambda \iota \iota = f(\lambda x + x^d) \), we have \( \lambda f_0(x) \equiv f_0(\lambda x) \mod \ell \). Thus we have \( f_0(x) \equiv x \mod \deg \ell \) and
\[ \text{Sl} \circ f_0(x) \equiv \text{Sl}(x) \equiv \frac{\ell - 1}{2} C_d x^d \mod \deg \ell. \]
Similarly we have
\[ Cl \circ f_0(x) \equiv Cl(x) \equiv \frac{\ell - 1}{4} D_d x^d \mod \deg \ell. \]
Since \( Sl \circ f_0(x) \) and \( Cl \circ f_0(x) \) belong to \( \mathbb{Z}_\ell[[x]] \), the assertion follows. \( \square \)

**Proof of Theorem 5.7.** Because of \( \lambda = \gamma(S)^{-1} \prod_{r \in S} \varphi(r/\lambda) \equiv \eta^{\frac{\ell - 1}{2}} \mod \eta^{\frac{\ell + 1}{2}} \) and \( \text{deg}(\lambda) = A_\lambda \lambda^3 \), we have
\[ A_\lambda \equiv -\frac{1}{4} C_d \mod \eta^{\frac{\ell - 1}{2}} \]
by using Lemma 9.2. Since both sides are rational number, we have the assertion of Theorem 5.7. \( \square \)

**9 Application of the Hochschild formula**

In this section, we use the following formula known as the Hochschild formula. For a proof of this formula, see Matsumura [Ma], p.197, Theorem 25.5.

**Lemma 9.1.** Let \( R \) be a commutative ring of characteristic \( \ell \). Let \( \delta \) be a derivation over \( R \). Then, for any element \( b \) in \( R \), we have
\[ (b\delta)^{\ell} = b^{\ell} \delta^\ell + ((b\delta)^{\ell-1}(b)) \cdot \delta. \]

We put \( u = f_0(x) \). By the definition of \( H_\ell \), we have \( (\lambda - T)(\lambda - T) = \ell - H_\ell T + T^2 \). Since \( \lambda - T \) is a unit in \( \mathbb{Z}_\ell[[T]] \), any formal group over \( \mathbb{Z}_\ell \) of type \( \lambda - T \) is also of type \( \ell - H_\ell T + T^2 \).

**Lemma 9.2.** Let \( \phi(x) \) be a power series in \( \mathbb{Z}_\ell[[x]] \). Then
\[ \left( \frac{d}{dx} \right)^\ell \left( H_\ell \frac{d}{dx} \right) \phi(x) \in \ell \mathbb{Z}_\ell[[x]]. \]  

**Proof.** Since \( du/dx = f_0'(x) \) is in \( \mathbb{Z}_\ell[[x]]^x \), \( \frac{d}{dx} \frac{d}{du} \) is a derivation on \( \mathbb{Z}_\ell[[x]] \). Since \( \mathbf{LT}(x, y) \) is of type \( \ell - H_\ell T + T^2 \), there exists \( h(x) \) in \( \mathbb{Z}_\ell[[x]] \) such that
\[ \ell f_0(x) - H_\ell f_0(x^\ell) + f_0(x^{\ell^2}) = th(x). \]
This yields that
\[ f_0'(x) - H_\ell f_0'(x^\ell) x^{\ell-1} \equiv h'(x) \mod \ell. \]
Differentiating this \( \ell - 1 \) times by \( x \), we have
\[ f_0^{(\ell)}(x) - H_\ell f_0^{(\ell)}(x^\ell)(\ell - 1)! \equiv h^{(\ell)}(x) \mod \ell. \]
By \( (\ell - 1)! \equiv -1 \mod \ell \), \( f_0^{(\ell)}(x) \) in \( \mathbb{Z}_\ell[[x]] \), and \( h^{(\ell)}(x) \equiv 0 \mod \ell \), we have
\[ f_0^{(\ell)}(x) + H_\ell (f_0'(x))^\ell \equiv 0 \mod \ell. \]
Let \( \phi(x) \) be a power series in \( \mathbb{Z}_\ell[[x]] \).
\[ 0 \equiv \left( \frac{d}{dx} \right)^\ell \phi(x) \equiv \left( \frac{d}{dx} \frac{du}{dx} \right)^{\ell} \phi(x) \mod \ell. \]
By using the Hochschild formula (Lemma 9.1), we have
\[ 0 \equiv \left( \frac{du}{dx} \right)^\ell \frac{d^\ell \phi}{du^\ell} + \left( \frac{du}{dx} \right) \frac{d^{\ell-1} \phi}{du^{\ell-1}} \frac{d}{dx} \frac{d^\ell \phi}{du^\ell} \equiv \left( \frac{du}{dx} \right)^\ell \frac{d^\ell \phi}{du^\ell} + \frac{d^\ell u}{dx^\ell} \cdot \frac{d\phi}{du} \mod \ell. \]

Thus we have
\[ \frac{d^\ell \phi}{du^\ell} \equiv 0 \mod \ell \]
and by (9.4) we have
\[ \frac{d^\ell \phi}{du^\ell} - H_\ell \frac{d\phi}{du} \equiv 0 \mod \ell \mathbf{Z}_\ell[[x]]. \]
This completes the proof of Lemma 9.2.

Let \( c \) be an integer. Let \( \phi(x) \) be any element in \( \ell^c \mathbf{Z}_\ell[[x]] \). We define the expansion of \( \phi \circ \text{sl}(u) \) by
\[ \phi \circ \text{sl}(u) = \sum_{k \geq 0} b_k x^k \quad (b_k \in \mathbb{Q}_\ell). \]
We denote
\[ \Omega_c = \left( \frac{d}{du} \right)^\ell - H_\ell \frac{d}{du}. \]
For any non-negative integer \( a \), we see \( \Omega_c^a \phi(x) \) in \( \ell^{a+c} \mathbf{Z}_\ell[[x]] \) by (9.5). Since
\[ \Omega_c^a \left( \sum_{k \geq 0} b_k x^k \right) = \sum_{k \geq 0} \left( \sum_{r=0}^a \binom{a}{r} (-H_\ell)^{a-r} b_{k+a+r(p-1)} \right) \frac{x^k}{k!} \in \ell^{a+c} \mathbf{Z}_\ell[[x]] \subset \ell^{a+c} \mathbf{Z}_\ell[[u]], \]
we have
\[ \sum_{r=0}^a \binom{a}{r} (-H_\ell)^{a-r} b_{k+a+r(p-1)} \equiv 0 \mod \ell^{a+c}. \]

10 Proof of the main theorem
We prove the implications (1) \( \Rightarrow \) (3) \( \Rightarrow \) (2) in Theorem 7.2.

Lemma 10.1. If \( \text{egs}(\lambda) = 0 \), then \( (\text{Cl} \circ f_0)(x)/(\lambda x + x^\ell) \) is in \( \mathbf{Z}_\ell[[x]] \).

Proof. Assume \( \text{egs}(\lambda) = 0 \) and put \( (\text{Cl} \circ f_0)(x) = \sum_{n=0}^\infty b_n x^n \) with \( b_n \) in \( \mathbf{Z}_\ell \).
Then, \( (\text{Cl} \circ f_0)(\eta) = \sum_{n=0}^\infty b_n \eta^n = 0 \) by Lemma 8.2. Therefore,
\[ (\text{Cl} \circ f_0)(x) = \sum_{n=0}^\infty b_n x^n = \sum_{n=0}^\infty b_n \eta^n = (x - \eta) \sum_{n=1}^\infty b_n \frac{x^n - \eta^n}{x - \eta} \in \mathbf{Z}_\ell[[\eta]][[x]] \]
because \( x - \eta \) divides \( x^n - \eta^n \). Similarly, any conjugate of \( x - \eta \) divides \( (\text{Cl} \circ f_0)(x) \) and \( x \) divides \( (\text{Cl} \circ f_0)(x) \). Hence, the assertion follows.

Lemma 10.2. Let \( \nu \) be a positive integer. Assume \( \text{egs}(\lambda) = 0 \). If \( n < \nu \ell (\ell - 1) \), then the coefficient in \( \chi^{\nu-1}(\text{Cl} \circ f_0)(x)/f_0(x) \) of \( x^n \) belongs to \( \ell \mathbf{Z}_\ell \).

Proof. Since \( f_0(x) \) is in \( \mathbf{Z}_\ell[[x]] \), it is seen that \( \xi^{-1} f_0(\xi x) \) is in \( \mathbf{Z}_\ell[[\xi]][[x]] \) for any \( \ell \)-adic algebraic integer \( \xi \) with \( \text{ord}(\xi) = 1/(\ell - 1) \) by calculating the \( \ell \)-adic order of
each coefficients of its expansion. We put
\[ g(x) = \frac{\lambda x + x^\ell}{f_0(x)} = \frac{\lambda x + x^\ell}{f_0(\lambda x + x^\ell)}. \]

Then
\[ g(\xi^j x) = \frac{\xi(\lambda \xi^{j-1} x + x^\ell)}{f_0(\xi(\lambda \xi^{j-1} x + x^\ell))} \in \mathbb{Z}[\xi][[[x]]], \]

because \( \text{ord}(\lambda \xi^{j-1}) = 1 - 1/\ell. \) Thus the \( \ell \)-adic order of the coefficient of \( x^n \) of \( g(x) \) is greater than or equal to \(- \left\lfloor \frac{n}{\ell} \right\rfloor \). Therefore, we see
\[ \lambda^\nu g(x) \equiv 0 \mod \nu\ell - 1, \mod \lambda. \]

Thus, each coefficient of the terms of degree less than \( \nu\ell - 1 \) in
\[ \lambda^{\nu - 1} \frac{\text{Cl}(u)}{u} = \lambda^{\nu - 1} \frac{\text{Cl} f_0(x)}{f_0(x)} = \frac{\text{Cl} f_0(x)}{\lambda x + x^\ell}, \lambda^\nu g(x) \]
is in \( \ell \mathbb{Z} \) by Lemma \[10.4\].

**Lemma 10.3.** Assume that \( \text{egs}(\lambda) = 0 \). Then we have for \( a < \nu \ell \) that
\[ \sum_{r=0}^{a} \binom{n}{r} (-H_\ell)^{a-r} \frac{G_{d+r(\ell-1)}}{d+r(\ell-1)} \equiv 0 \mod \ell^{a-\nu+2}. \]

**Proof.** We denote by \( \phi(x) \) the sum of the terms in \( \lambda^{\nu - 1}(\text{Cl} f_0(x))/f_0(x) \) of degree less than \( \nu\ell - 1 \). The assertion of Lemma \[10.2\] is amount to the same thing to say \( \phi(x) \) in \( \ell \mathbb{Z}[[x]] \). Now, the last argument in the previous section is applied by plugging \( c = 1 \) and \( b_k = \lambda^{\nu - 1}G_k/k \) for \( k < \nu\ell - 1 \). By using \[10.3\], we have
\[ \lambda^{\nu - 1} \sum_{r=0}^{a} \binom{n}{r} (-H_\ell)^{a-r} \frac{G_{d+r(\ell-1)}}{d+r(\ell-1)} \equiv 0 \mod \ell^{a+1} \]
for \( d + a(\ell-1) < \nu\ell - 1 \), that is, for \( a < \nu \ell \), hence the desired congruence.

We take \( \nu = \lfloor a/\ell \rfloor + 1 \) in Lemma \[10.3\]. Then \( \nu \) satisfies \( a < \nu \ell \) for any \( a \geq 0 \). Therefore, we conclude that
\[ \sum_{r=0}^{a} \binom{n}{r} (-H_\ell)^{a-r} \frac{G_{d+r(\ell-1)}}{d+r(\ell-1)} \equiv 0 \mod \ell^{a-\lfloor a/\ell \rfloor + 1} \]
for any \( a \geq 0 \). By expanding \( \{(x+1)-1\}^k(x+1)^a \), we have
\[ \sum_{j=0}^{\min(k-a,0)} \sum_{m=0}^{j+a} (-1)^{k-j} \binom{k}{j} \binom{j+a}{m} x^m = \sum_{r=0}^{a} \binom{a}{r} x^{r+k} \]
for any \( k \geq 0 \). By using the structure of \[10.5\], we simplify a linear combination of the sum of the left hand side of \[10.4\] for various \( a \)'s as follows:
\[ \sum_{j=0}^{k} (-1)^{k-j}(-H_\ell)^{k-j} \sum_{m=0}^{j+a} \binom{j+a-m}{m}(-H_\ell)^{j+a-m} \frac{G_{d+m(\ell-1)}}{d+m(\ell-1)} \]
Arithmetic on a Certain Family of Elliptic Curves

\[= \sum_{j=0}^{k} \sum_{m=0}^{j+a} (-1)^{k-j} \binom{k}{j} \binom{j+a}{m} (-H_{t})^{k+a-m} \frac{G_{d+m(\ell-1)}}{d+m(\ell-1)}\]

\[= \sum_{r=0}^{a} \binom{a}{r} (-H_{t})^{a-r} \frac{G_{d+(r+k)(\ell-1)}}{d+(r+k)(\ell-1)}.\]

Since the exponent \(a - \lfloor a/\ell \rfloor + 1\) of the modulus in (10.3) is a monotone increasing function on \(a\), we have

\[\sum_{r=0}^{a} \binom{a}{r} (-H_{t})^{a-r} \frac{G_{d+(r+k)(\ell-1)}}{d+(r+k)(\ell-1)} \equiv 0 \mod \ell^{a-\lfloor a/\ell \rfloor + 1}.\]

This is no other than the assertion (3) of Theorem 7.1. Thus (1) implies (3). Plugging \(a = k = 0\), we have

\[\frac{G_{d}}{d} \equiv 0 \mod \ell.\]

Thus (3) implies (2) in Theorem 7.1.

11 The two term congruence

In this section, we show Theorem 7.2.

Proof of Theorem 7.2 We show (7.3) by using the induction on \(b\).

(i) When \(b = 0\), the assertion follows from Theorem 7.1. Indeed, by taking \(a = 1\) in (3) of Theorem 7.1, we have

\[\frac{G_{e+\ell-1}}{e + \ell - 1} \equiv H_{e} \frac{G_{e}}{e} \mod \ell^{2}\]

for any \(e\) satisfying the assumption in Theorem 7.2. This is the case of \(b = 0\) of (7.3).

(ii) We assume that there exists some integer \(c > 0\) such that (7.3) holds for any \(b < c\). Then, by taking \(a = \ell c\) in (3) of Theorem 7.1, we have

\[(11.1) \sum_{r=0}^{\ell c} \binom{\ell c}{r} (-H_{t})^{e-r} \frac{G_{e+r(\ell-1)}}{e+r(\ell-1)} \equiv 0 \mod \ell^{e-e-c-1+1}.\]

For \(1 \leq r \leq \ell c - 1\), we have

\[\binom{\ell c}{r} (-H_{t})^{e-r} \frac{G_{e+r(\ell-1)}}{e+r(\ell-1)} + \binom{\ell c}{r} (-H_{t})^{e-r} \frac{G_{e+(\ell c-r)(\ell-1)}}{e+(\ell c-r)(\ell-1)}\]

\[= \binom{\ell c}{r} (-H_{t})^{e-r} \frac{G_{e+r(\ell-1)}}{e+r(\ell-1)} + (-H_{t})^{e-r} \frac{G_{e+(\ell c-r)(\ell-1)}}{e+(\ell c-r)(\ell-1)}\].

Using the Legendre’s formula which gives the exact \(\ell\)-adic order for the factorial of any positive integer, it is easily proved that (see also Dickson [13], p.270)

\[\text{ord}\left(\binom{\ell c}{r}\right) = c - \text{ord}(r)\]

provided that \(1 \leq r \leq \ell c - 1\). We note that \(\text{ord}(r) = \text{ord}(\ell c - r) < c\) for \(1 \leq r \leq \ell c - 1\).
Then we have

\[
(-H_t)^{e-r} \frac{G_{e+r}(t-1)}{e + r(\ell - 1)} + (-H_t)^d \frac{G_{e}(t-1)}{e + (\ell^c - r)(\ell - 1)} \equiv \{-1\}^{e-r} H_t^d e G_e e + \{1\}^{e} H_t^d e G_e e \mod \ell^{2+\ord(r)}
\]

\[
\equiv \{-1\}^{e-2r} + 1 \{1\}^{e} H_t^d e G_e e \equiv 0 \mod \ell^{2+\ord(r)}
\]

by using the assumption of the induction. As \((c - \ord(r)) + (2 + \ord(r)) = 2 + c\) we see

\[
\left(\frac{\ell^c}{t}\right) (-H_t)^{e-r} \frac{G_{e+r}(t-1)}{e + r(\ell - 1)} + \left(\frac{\ell^c}{t}\right) (-H_t)^d \frac{G_{e}(t-1)}{e + (\ell^c - r)(\ell - 1)} \equiv 0 \mod \ell^{2+c}.
\]

By taking the summation on \(r\) such that \(1 \leq r \leq (\ell^c - 1)/2\), we have

\[
\sum_{r=1}^{\ell^c - 1/2} \left(\frac{\ell^c}{t}\right) (-H_t)^{e-r} \frac{G_{e+r}(t-1)}{e + r(\ell - 1)} \equiv 0 \mod \ell^{2+c}.
\]

Since \(\ell^c - \ell^c - 1 \geq 2 + c\) holds for \(c \geq 0\), we have

\[
\frac{G_{e+r}(t-1)}{e + \ell^c(\ell - 1)} \equiv H_t^d e G_e e \mod \ell^{2+c}
\]

for any \(c \geq 1\) by (11.1) and (11.2). Thus the assertion (7.3) follows in the case of \(b = c\).

(iii) By the induction, (7.3) holds for any \(b \geq 0\).

\[\square\]

**Remark 11.3.** (1) On the classical Bernoulli numbers, if \(b \leq d\), then

\[
\frac{B_d}{d} \equiv \frac{B_{d+kp^{-1}(p-1)}}{d+kp^{-1}(p-1)} \mod p^b
\]

Here the condition \(b \leq d\) is essential. However, for any \(b, d,\) and \(k\), it is known that

\[
(1 - p^{d-1}) \frac{B_d}{d} \equiv (1 - p^{d+kp^{-1}(p-1)-1}) \frac{B_{d+kp^{-1}(p-1)}}{d+kp^{-1}(p-1)} \mod p^b.
\]

Of course, the extra factors are no other than Euler \(p\)-factors of the Riemann \(\zeta\)-function.

This consideration suggests that the reason why Theorem 7.2 holds without condition on \(b\) and \(d\) is that the Euler \(\lambda\)-factor of the Hecke \(L\)-function for \(\delta_{b,\lambda}\) is 1.

(2) On Kubota-Leopoldt \(p\)-adic \(L\)-function, it is fundamental that the special values of the corresponding complex \(L\)-function is given by (generalized) Bernoulli numbers and they satisfy (11.4) involving Euler \(p\)-factor of the complex \(L\)-series. However the congruence (7.2) is a relation on the numbers which are not exactly the special values but only their residues modulo some power of \(\ell\).

### 12 Central value of the Hecke \(L\)-function

In this section we refer to Koblitz [K]. We modify §5 and §6 of Chapter 2 in [K].

Put \(\mathcal{O} = \mathbb{Z}[i]\) and take \(\beta\) in \(\mathcal{O}\). Let \(\tilde{\chi}\) be a Hecke character of modulus \((\beta)\) of
weight one. Namely, \( \tilde{\chi}(\nu) = \chi_1(\nu)\mathcal{P} \), where \( \chi_1 \) is a character form \((\mathcal{O}/(\beta))^{\times}\) to \( \mathbb{C}^{\times} \) satisfying \( \chi_1(i) = i \). We define the Hecke \( L \)-function by

\[
L(s, \tilde{\chi}) = \sum_{\mathfrak{a}} \frac{\tilde{\chi}(\mathfrak{a})}{|\mathfrak{a}|^s} = \frac{1}{4} \sum_{\nu \in \mathcal{O}} \chi_1(\nu)\mathcal{P} \sum_{\gamma \bmod \beta} \chi_1(\gamma) \sum_{\alpha \in \mathcal{O}} \frac{\gamma + \alpha\beta}{\gamma + \alpha\beta} |\nu|^{2s},
\]

where \( \mathfrak{a} \) runs over the non-zero integral ideals of \( \mathcal{O} \) and \( Na = \# \mathcal{O}/\mathfrak{a} \) is the norm of \( \mathfrak{a} \).

We use a method to get the following classically known fact.

**Theorem 12.1.** The function defined by

\[
\Lambda(s, \tilde{\chi}) = \left( \frac{-2\pi}{\sqrt{4N(\beta)}} \right)^{-s} \Gamma(s) L(s, \tilde{\chi})
\]

satisfies

\[
\Lambda(s, \tilde{\chi}) = C(\tilde{\chi}) \Lambda(2-s, \tilde{\chi}),
\]

where \( C(\tilde{\chi}) = -i \beta^{-1} \sum_{\lambda \bmod \beta} \chi_1(\lambda) e^{2\pi i \text{Re}(\lambda/\beta)} \).

We do not need the result above itself but the following bi-product of its proof.

**Lemma 12.2.** We have the estimation

\[
L(1, \tilde{\chi}) < \frac{4}{\exp \left( \frac{\pi}{|\beta|} \right) - 1}.
\]

**Proof.** First of all, we note that \( \left( \frac{2\pi}{\sqrt{4N(\beta)}} \right)^{-s} = |\beta|^{-s} \pi^{-s} \) because of \( |\beta| = \sqrt{N(\beta)} \). We give an outline of proof which is divided into four steps.

(Step 1) We define

\[
F(t, \tilde{\chi}) = \frac{1}{4} \sum_{\nu \in \mathcal{O}} \chi_1(\nu)\mathcal{P} e^{-|\nu|^2}.
\]

Then, by using \( \int_0^\infty e^{-c t} t^s \frac{dt}{t} = e^{-s} \Gamma(s) \), we have

\begin{align}
\pi^{-s} \Gamma(s) L(s, \tilde{\chi}) &= \int_0^\infty F(t, \tilde{\chi}) t^s \frac{dt}{t}. \\
&= \int_0^\infty \int_0^\infty e^{-(s+t)} t^s \frac{dt}{t} \frac{d\nu}{|\mathfrak{a}|^s}.
\end{align}

(Step 2) For \( u \) in \( \mathbb{R}^2 \) and \( w \) in \( \mathbb{C}^2 \), we define \( \theta_u(t) = \sum_{m \in \mathbb{Z}} (m+u) \cdot w e^{-\pi t |m+u|^2} \) and \( \theta^w(t) = \sum_{m \in \mathbb{Z}} m \cdot w e^{2\pi i m} e^{-\pi t |m|^2} \), where \( \cdot \) stands for the inner product,

holds (cf. [K], p.85, (5.16)). Then,

\[
\theta_u(t) = -\frac{i}{t} \theta^w \left( \frac{1}{t} \right).
\]

Applying \( [\text{[12.3]}] \), \( F(t, \tilde{\chi}) \) is rewritten

\[
F(t, \tilde{\chi}) = \frac{1}{4} \sum_{\gamma \bmod \beta} \chi_1(\gamma) \sum_{\alpha \in \mathcal{O}} \frac{\beta + \alpha + \gamma}{\beta + \alpha + \gamma} e^{-\pi t |\beta + \alpha + \gamma|^2} = \frac{\beta}{4} \sum_{\gamma \bmod \beta} \chi_1(\gamma) \sum_{\alpha \in \mathcal{O}} \frac{\gamma + \beta}{\gamma + \beta} e^{-\pi t |\beta|^2 |\gamma + \beta|^2}
\]

By using a vector in \( \mathbb{R}^2 \), we write the inner sum. For a given \( \gamma \), we put \( \frac{\gamma}{\beta} = u_1 + u_2 i \)
with \( u = (u_1, u_2) \) in \( \mathbb{Q}^2 \), \( \alpha = m_1 + m_2i \) with \( m = (m_1, m_2) \) in \( \mathbb{Z}^2 \). Then we have

\[
\sum_{\alpha \in \mathcal{O}} \left( \frac{\alpha + \gamma}{\beta} \right) e^{-\pi t |\beta|^2} e^{\pi |\alpha+\beta|^2} = \sum_{m \in \mathbb{Z}^2} (m+u) \cdot (1,\bar{i}) e^{-\pi |\beta|^2} |m+u|^2 \\
= \sum_{m \in \mathbb{Z}^2} (m+u) \cdot (1,-i) e^{-\pi |\beta|^2} |m+u|^2.
\]

We put \( w = (1,-i) \). Then we have

\[
F(t, \bar{\chi}) = \frac{\beta}{4} \sum_{\gamma \mod \beta} \chi_1(\gamma) \theta_\gamma(|\beta|^2 t).
\]

We note \( v \) depends on \( \gamma \).

(Step 3) By using the functional equation \((12.4)\), we prove that of \( F(t, \bar{\chi}) \).

\[
F\left( \frac{1}{|\beta|^2 t}, \bar{\chi} \right) = \frac{\beta}{4} \sum_{\gamma \mod \beta} \chi_1(\gamma) \theta_\gamma(1) = \frac{\beta}{4} \sum_{\gamma \mod \beta} \chi_1(\gamma)(-it^2) \theta_\gamma(t).
\]

By the definition of \( \theta_\nu(t) \), we calculate the right hand side.

\[
F\left( \frac{1}{|\beta|^2 t}, \bar{\chi} \right) = \frac{\beta}{4} \sum_{\gamma \mod \beta} \chi_1(\gamma)(-it^2) \sum_{m \in \mathbb{Z}^2} m \cdot (1, -i) e^{2\pi im \cdot u} e^{-\pi |\beta|^2 |m|^2}
\]

\[
= \frac{\beta}{4} (-it^2) \sum_{m \in \mathbb{Z}^2} m \cdot (1, -i) e^{-\pi |\beta|^2 |m|^2} \sum_{\gamma \mod \beta} \chi_1(\gamma) e^{2\pi im \cdot u}.
\]

It follows from \( m \cdot u = m_1 u_1 + m_2 u_2 = \Re((m_1 - m_2i)(u_1 + u_2i)) = \Re(\overline{\alpha} \overline{\nu}) \) that the sum inside is essentially Gauss sum and

\[
\sum_{\gamma \mod \beta} \chi_1(\gamma) e^{2\pi im \cdot u} = \chi_1(\overline{\alpha}) \sum_{\gamma \mod \beta} \chi_1(\overline{\nu}) e^{2\pi i \Re(\overline{\gamma}/\overline{\beta})}
\]

\[
= \chi_1(\overline{\alpha}) \sum_{\gamma \mod \beta} \chi_1(\gamma) e^{2\pi i \Re(\gamma/\beta)} = \chi_1(\overline{\alpha}) i|\beta| C(\overline{\chi}).
\]

Since \( \sum_{m \in \mathbb{Z}^2} m \cdot (1, -i) e^{-\pi |\beta|^2 |m|^2} = \sum_{\alpha \in \mathcal{O}} \alpha e^{-\pi |\alpha|^2} = \sum_{\alpha \in \mathcal{O}} \alpha e^{-\pi |\alpha|^2} \) and \( \bar{\chi}(\nu) = \chi_1(\nu) \nu \),

\[
F\left( \frac{1}{|\beta|^2 t}, \bar{\chi} \right) = \frac{\beta |\beta|^2}{4} t^2 C(\overline{\chi}) \sum_{\alpha \in \mathcal{O}} \chi_1(\overline{\alpha}) |\alpha e^{-\pi |\alpha|^2}| = |\beta|^2 t^2 C(\overline{\chi}) F(t, \bar{\chi}).
\]

(Step 4) From \((12.3)\), we have

\[
\pi^{-s} \Gamma(s) L(s, \bar{\chi}) = \int_0^\infty t^s F(t, \bar{\chi}) \frac{dt}{t} = \int_0^{\frac{\pi}{|\beta|^2}} t^s F(t, \bar{\chi}) \frac{dt}{t} + \int_{\frac{\pi}{|\beta|^2}}^\infty t^s F(t, \bar{\chi}) \frac{dt}{t},
\]

in which the former integration is rewritten as

\[
\int_0^{\frac{\pi}{|\beta|^2}} t^s F(t, \bar{\chi}) \frac{dt}{t} = |\beta|^{-2s} \int_{|\beta|^2}^\infty v^{-s} F\left( \frac{1}{|\beta|^2 v}, \bar{\chi} \right) \frac{dv}{v} = |\beta|^{-2s} C(\overline{\chi}) \int_{|\beta|^2}^\infty v^{2-s} F(v, \bar{\chi}) \frac{dv}{v}
\]
by replacing $t = \frac{\beta}{1 - e^t}$ and $\frac{dt}{\pi} = -\frac{dt}{e^t}$. In the case of $s = 1$, we have

$$
\pi^{-1}L(1, \chi) = C(\chi) \int_{m=1}^{\infty} F(t, \chi) dt + \int_{m=1}^{\infty} F(t, \chi) dt.
$$

We put

$$
L(s, \chi) = \sum_{m=1}^{\infty} \frac{b_m}{m^s} \quad (b_m \in \mathcal{O}).
$$

Then we have

$$
L(s, \chi) = \sum_{m=1}^{\infty} \frac{b_m}{m^s}, \quad F(t, \chi) = \sum_{m=1}^{\infty} b_m e^{-\pi mt}, \quad F(t, \chi) = \sum_{m=1}^{\infty} b_m e^{-\pi mt}.
$$

It follows from $|b_m| = |b_m|$ that

$$
|F(t, \chi)| \leq \sum_{m=1}^{\infty} |b_m| e^{-\pi mt}, \quad |F(t, \chi)| \leq \sum_{m=1}^{\infty} |b_m| e^{-\pi mt}.
$$

As $|C(\chi)| = 1$, we see

$$
\pi^{-1} |L(1, \chi)| \leq 2 \int_{m=1}^{\infty} \sum_{m=1}^{\infty} |b_m| e^{-\pi mt} dt = 2 \sum_{m=1}^{\infty} |b_m| \pi m e^{-\pi m/\beta}.
$$

Multiplying by $\pi$ on both sides and by using $|b_m| \leq \sigma_0(m) \sqrt{m} \leq 2m$, where $\sigma_0(m)$ denotes the number of positive divisors of $m$ (cf. [K], p. 96, Problem 4 of p. 97), we have

$$
|L(1, \chi)| \leq 4 \sum_{m=1}^{\infty} e^{-\pi m/\beta} = \frac{4e^{-\pi/\beta}}{1 - e^{-\pi/\beta}} = \frac{4}{e^{\pi/\beta} - 1}
$$

as desired. \qed

### 13 Estimate of the coefficients of elliptic Gauss sums

In this section we show (2) implies (1) in Theorem 7.1. At first we prove Lemma 4.2 whose proof has been reserved.

**Proof of Lemma 4.2** Since $1/(\exp(\pi/\beta) - 1) < |\beta|/\pi$, we have $L(1, \chi) < 4 \times |\beta|/\pi$. For $\ell \equiv 1 \mod 8$ and the conductor $(\beta) = ((1 + \bar{i})^3 \lambda)$, we see

$$
4 \frac{2\sqrt{2} \cdot |\lambda|}{\pi} > L(1, \chi) = \varpi \frac{1}{2} |A_\lambda||\lambda|^{-1} |\lambda|^{3} = \varpi \frac{1}{2} |A_\lambda||\lambda|^{-1} \lambda^{\frac{3}{2}}
$$

from Theorem 5.2, 5.1, and Lemma 12.2. So that we have $|A_\lambda| < (16\sqrt{2}/\pi \varpi) |\lambda|^{3/2}$. The right hand side is smaller than $\frac{1}{2} \ell$ for $\ell \geq 97$ because $97^{\frac{3}{2}} = 5.55 \cdots > \frac{32\sqrt{2}}{\pi \varpi} = 5.49 \cdots$. For $\ell < 97$, the inequality $|A_\lambda| < \ell/2$ actually holds by the tables in [A]. \qed

**Proof of (2) ⇒ (1) of Theorem 7.1** Assume that $\ell | G_0$. Then by Lemma 8.3 we have $\ell | \text{egs}(\lambda)$. By Theorem 5.2, we have $\lambda_0 | A_\lambda$, where $\lambda_0$ is the prime defined in Theorem 6.1, and we see $\ell | a_\lambda$. On the other hand, by Theorem 5.2 and Lemma 4.2, we have $|a_\lambda| \leq |A_\lambda| \leq \ell/2$. Thus we have $a_\lambda = A_\lambda = 0$ and $\text{egs}(\lambda) = 0$. \qed
References

[A] Asai, T., Elliptic Gauss sums and Hecke $L$-values at $s = 1$, RIMS Kôkyûroku Bessatsu, 4(2007) 79-121.

[BSD] Birch, B.J. and Swinnerton-Dyer, H.P.F., Notes on elliptic curves II, J. reine und angew. Math., 218(1965) 79-108.

[CW] Coates, J. and Wiles, A., On the conjecture of Birch and Swinnerton-Dyer, Invent. math., 39(1977) 223-251.

[C] Cassels, J.W.S., Lectures on elliptic curves, London Math. Soc. Student Texts 24, Cambridge Univ. Press, 1991.

[D] Deuring, M., Die Zetafunktionen einer algebraischen Kurve von Geschlechte Eins (III), Nachr. Acad. Wiss. Göttingen, (1956)37-76

[Di] Dickson, L.E., History of the theory of numbers I, Carnegie Institution of Washington 1919.

[Ha] Hazewinkel, T., Formal groups and applications, Academic Press, 1978, reprinted by A.M.S. Chelsea publishing, 2012.

[Ho] Honda, T., On the theory of commutative formal groups, J. Math. Soc. Japan, 22 (1970), 213-246.

[H1] Hurwitz, A., Über die Anzahl der Klassen binärer quadratischer Formen von negativer Determinante, Acta Math., 19(1985) 351-384.

[H2] Hurwitz, A., Über die Entwicklungskoeffizienten der lemniskatischen Funktionen, Nachr. Acad. Wiss. Göttingen, (1897)273-276, (Werke, Bd.II, pp.338–341).

[H3] Hurwitz, A., Über die Entwicklungskoeffizienten der lemniskatischen Funktionen, Math. Ann., 51 (1899) 196–226, (Werke, Bd.II, pp.342–373).

[K] Koblitz, N., Introduction to elliptic curves and modular forms (2nd ed.), G.T.M. 97, 1993.

[Le] Lemmermeyer, F., Reciprocity laws, Springer-Verlag Berlin Heiderberg 2010.

[L] Lutz, E., Sur l'équation $y^2 = x^3 − Ax − B$ dans les corps $p$-adiques, J. reine und angew. Math., 177(1937) 238-247.

[Ma] Matsumura, H., Commutative ring theory, Cambridge studies in advanced mathematics 8, Cambridge Univ. Press 1986.

[M] Matthews, C.R., Gauss sums and elliptic functions, II The quartic sum, Invent. math., 54(1979) 23-52.

[N] Nagell, T., Solution de quelques problèmes dans la théorie arithmétique des cubiques planes du premier genre, Skrifter utg. av det Norske Viden3k .-Akad i Oslo, Mat.-Naturv. KL.(1935), No.1, 1-25.

[O] Ônishi, Y., Congruence relations connecting Tate-Shafarevich groups with Hurwitz numbers, Interdisciplinary Information Sciences, 16(2010)71-86.

[O2] Ônishi, Y., Integrality of coefficients of division polynomials for elliptic Functions, http://www2.meijo-u.ac.jp/~yonishi/index.html#publications, (2011)

[ST] Serre, J.-P. and Tate, J., Good reduction of Abelian varieties, Ann. of Math., 2nd Ser., 88(1968) 492-517 (=J.-P.Serre : Oeuvres ,Tom 2, 1986, Springer-Verlag, pp.472-497) (= Collected Works of John Tate ,Part 1, AMS 2010, pp.377-402)

[Si] Silverman, J., The arithmetic of elliptic curves (2nd ed.), G.T.M.106, Springer-Verlag. 2009

[T] Takagi, T., Uber eine Theorie des relative Abel'schen Zahlkörpers, J. College of Science, Imperial Univ. of Tokyo 41(1920) 1-133, (Especially §32). (= Collected Papers, Iwanami Shoten 1973, pp.73-167).