Abstract

We define a category of divided Dieudonné crystals which classifies \( p \)-divisible groups over schemes in characteristic \( p \) with certain finiteness conditions, including all \( F \)-finite noetherian schemes. For formally smooth schemes or locally complete intersections this generalizes and extends known results on the classical crystalline Dieudonné functor.

Contents

1 Introduction 2
2 Finiteness conditions on the Frobenius map 5
3 Dieudonné crystals 7
4 Torsion in PD envelopes 10
5 Frames and windows 12
6 Dieudonné theory over semiperfect rings 15
7 Dieudonné theory by \( p \)-root descent 22
8 Divided Dieudonné crystals 24
9 The divided Dieudonné functor 29
10 Explicit divided Dieudonné crystals 31
11 Divided Dieudonné crystals and displays 40
1 Introduction

For a scheme $X$ in characteristic $p$ we consider the crystalline Dieudonné functor from $p$-divisible groups to Dieudonné crystals,

$$D_X : BT(X) \to D(X),$$

which is outlined in [Gr1, Gr2] and constructed in [MM, BBM]. The Hodge filtration of a $p$-divisible group gives a refinement of $D_X$ to a functor

$$DF_X : BT(X) \to DF(X)$$

from $p$-divisible groups to Dieudonné crystals with an admissible filtration as defined in [Gr2], called filtered Dieudonné crystals in the following.

The functors $D_X$ and $DF_X$ can be expected to have good properties only if $X$ is PD torsion free in the sense that the universal PD envelopes of affine open subschemes of $X$ are torsion free. We will define a category of divided Dieudonné crystals which compensates the effect of PD torsion. Here ‘divided’ refers to the fact that the Frobenius operator of the crystal is divided by $p$ on the filtration module. It is clear that the image of the filtration under the Frobenius is divisible by $p$, but if torsion occurs, the division is not unique and thus carries additional information.

In more detail, following [FM] we consider the presheaf of rings $O_{X}^{\text{cris}}$ on the category of $X$-schemes defined as the direct image of the crystalline structure sheaf. It carries a natural PD ideal $I_{X}^{\text{cris}} \subseteq O_{X}^{\text{cris}}$ with quotient $O_X$ and an endomorphism $\sigma$ induced by the Frobenius of $X$. The definition of divided Dieudonné crystals is based on the following fact.

Lemma 1.1. (Lemma 8.10) There is a natural $\sigma$-linear map

$$\sigma_1 : T_{X}^{\text{cris}} \to O_{X}^{\text{cris}}$$

with $p\sigma_1 = \sigma$.

This means that the sheaf $O_{X}^{\text{cris}}$ carries a frame structure in the sense of [La1], and we define divided Dieudonné crystals as windows over this frame in the following sense.

Definition 1.2. A divided Dieudonné crystal over $X$ is a collection

$$\mathcal{M} = (\mathcal{M}, \mathcal{M}_1, \Phi, \Phi_1)$$

where $\mathcal{M}$ is a finite locally free $O_{X}^{\text{cris}}$-module with respect to the pr-topology, $\mathcal{M}_1 \subseteq \mathcal{M}$ is a submodule with $T_{X}^{\text{cris}}\mathcal{M} \subseteq \mathcal{M}_1$ such that $\mathcal{M}/\mathcal{M}_1$ is a finite locally free $O_X$-module, $\Phi : \mathcal{M} \to \mathcal{M}$ and $\Phi_1 : \mathcal{M}_1 \to \mathcal{M}$ are $\sigma$-linear maps with $\Phi_1(ax) = \sigma_1(a)\Phi(x)$ for local sections $a \in T_{X}^{\text{cris}}$ and $x \in \mathcal{M}$, such that $\Phi(\mathcal{M}) + \Phi_1(\mathcal{M}_1)$ generates $\mathcal{M}$.
The category of divided Dieudonné crystals will be denoted by \( \text{DD}(X) \). There is a forgetful functor \( \text{DD}(X) \to \text{DF}(X) \) from divided Dieudonné crystals to filtered Dieudonné crystals, which is an equivalence if \( X \) is PD torsion free; see Proposition 8.15.

The scheme \( X \) is called \( F \)-finite if the Frobenius morphism \( \phi : X \to X \) is finite, and \( X \) is called \( F \)-nilpotent if the kernel of \( \phi : \mathcal{O}_X \to \mathcal{O}_X \) is locally a nilpotent ideal. Every locally noetherian \( \mathbb{F}_p \)-scheme is \( F \)-nilpotent.

**Theorem 1.3.** For each scheme \( X \) over \( \mathbb{F}_p \) there is a functor

\[
\text{DD}_X : \text{BT}(X) \to \text{DD}(X)
\]

from \( p \)-divisible groups to divided Dieudonné crystals. The functor \( \text{DD}_X \) is an equivalence of categories if \( X \) is \( F \)-finite and \( F \)-nilpotent.

See Proposition 9.1 and Theorem 9.2.

The usual description of crystals by modules with a connection extends to a description of divided Dieudonné crystals by windows with a connection; we refer to Section 10 for details.

Definition 1.2 can be extended to schemes \( X \) where \( p \) is locally nilpotent, and again there is a functor \( \text{DD}_X \) as above. If \( p \geq 3 \) and the reduction \( X_{\overline{\mathbb{F}_p}} \) is \( F \)-finite and \( F \)-nilpotent, then the functor \( \text{DD}_X \) is an equivalence by an obvious extension of Theorem 1.3.

**Outline of the proof**

We will use a topology \( \text{pr} \) on the category of \( X \)-schemes which is generated by the Zariski topology and infinite successions of extractions of \( p \)-th roots (Definition 7.2). By \( \text{pr} \)-descent, Lemma 1.1 and Theorem 1.3 are reduced to the case where \( X = \text{Spec} R \) is affine and semiperfect in the sense that the Frobenius endomorphism of \( R \) is surjective. Then \( \mathcal{O}^{\text{cris}}_X(X) = A^{\text{cris}}(R) \) is the universal \( p \)-adic PD thickening of \( R \), and Lemma 1.1 follows from the corresponding statement for \( A^{\text{cris}} \), which is proved in [SW]. Moreover, divided Dieudonné crystals over \( X \) are equivalent to windows over the frame \( A^{\text{cris}}(R) \). Hence the results of [La5] give the functor \( \text{DD}_X \) and prove the equivalence if \( R \) is iso-balanced; this is a technical condition that implies \( F \)-nilpotence. The proof of the equivalence when \( R \) is \( F \)-nilpotent uses similar methods, based on a deformation to the perfect case (Section 6).

**Implications for the classical functors**

The following consequences of Theorem 1.3 are straightforward. We assume again that \( X \) is a scheme in characteristic \( p \) and refer to the main text for variants when \( p \) is nilpotent on \( X \).
Corollary 1.4. (Corollary 9.7) If $X$ is $F$-finite and $F$-nilpotent, then the functor $D_X$ is fully faithful up to isogeny, more precisely for $p$-divisible groups $G, H$ over $X$ the homomorphism

$$\text{Hom}(G, H) \to \text{Hom}(D_X(G), D_X(H))$$

is injective with cokernel annihilated by $p$.

Corollary 1.5. (Theorem 7.9) If $X$ is $F$-finite, $F$-nilpotent, and PD torsion free, then the functor $DF_X$ is an equivalence.

This result can also be proved directly by pr-descent from the semiperfect case, without explicit reference to divided Dieudonné crystals; this route will be followed in the main text. But even then, divided Dieudonné crystals appear implicitly in the form of windows over $A_{\text{cris}}$ since in the deformation argument used for the semiperfect case, rings with PD torsion may occur.

An l.c.i. scheme is a locally noetherian scheme whose complete local rings are complete intersection rings. An $F$-finite locally noetherian scheme is excellent, and an excellent l.c.i. scheme is PD torsion free. Hence Corollary 1.5 implies the following.

Corollary 1.6. (Corollary 7.10) If $X$ is an $F$-finite l.c.i. scheme, then the functor $DF_X$ is an equivalence.

This applies in particular if $X$ is $F$-finite and regular, or equivalently if $X$ is locally noetherian and has locally a finite $p$-basis. In that case, the finiteness condition can be omitted as follows.

Theorem 1.7. (Theorem 7.6) If $X$ has locally a $p$-basis, then the functor $D_X$ is an equivalence.

This applies for example when $X$ is smooth over a field. In Theorem 1.7 one can replace $D_X$ by $DF_X$ because the forgetful functor $DF(X) \to D(X)$ is an equivalence if $X$ has locally a $p$-basis. Moreover, this functor is fully faithful if $X$ is an l.c.i. scheme. Hence the conclusions of Theorem 1.7 and Corollary 1.6 mean that the functor $D_X$ is fully faithful with essential image $DF(X)$, which confirms \cite{Gr2} p. 108, \textsection 1 in these cases.

The proof of Theorem 1.7 is again a reduction to the semiperfect case by pr-descent, using that the relevant semiperfect rings have a simple explicit description that allows to apply a variant of the methods of \cite{La5}.

A weaker version of Corollary 1.6 was announced in \cite{La4}, based on a relative version of the theory of Dieudonné displays of \cite{Zi2, La2}.

Previously known properties of the Dieudonné functors

Corollaries 1.4, 1.5, 1.6 and Theorem 1.7 extend known results on the functors $D_X$ and $DF_X$, which we try to collect here.
1. If \( X \) is perfect (the case of an empty \( p \)-basis), the functor \( \mathbf{D}_X \) and its analogoue for finite group schemes are an equivalence by a result of Gabber, proved in [Be2] for perfect valuation rings; see also [La3].

2. If \( X \) is normal and locally irreducible and has locally a \( p \)-basis, then \( \mathbf{D}_X \) and its analogue for finite group schemes are fully faithful by [BM].

3. If \( X \) is a formally smooth formal scheme in characteristic \( p \) such that \( X_{\text{red}} \) is of finite type over a field with a finite \( p \)-basis, the functor \( \mathbf{D}_X \) is an equivalence by [dJ]. This can be deduced from the case of \( F \)-finite regular schemes, which is part of Theorem 1.7; see Remark 7.8.

4. If \( X \) is of finite type over a field with a finite \( p \)-basis, the functor \( \mathbf{D}_X \) is fully faithful up to isogeny by [dJ].

5. If \( X \) is a locally noetherian excellent l.c.i. scheme, then \( \mathbf{D}_X \) is fully faithful by [dJM]. Since \( F \)-finite and noetherian implies excellent, this covers the full faithfulness part of Corollary 1.6.

6. If \( X \) is the spectrum of a perfect ring modulo a finite regular sequence, the functor \( \mathbf{D}_X \) is fully faithful by [SW]. This is also a consequence of Corollary 1.5.

7. If \( \mathcal{O}_K \) is a complete discrete valuation ring of characteristic zero with perfect residue field of characteristic \( p \geq 3 \), Corollary 1.6 for the scheme \( X = \text{Spec} \mathcal{O}_K/p \) is equivalent to Breuil’s classification of \( p \)-divisible groups over \( \mathcal{O}_K \) by filtered modules in [Br]. Breuil’s result was extended to a relative setting (formally smooth over \( \mathcal{O}_K \)) in [Ki] and to complete regular local rings of higher dimension in [CL]; this corresponds to additional cases of Corollaries 1.5 and 1.6.

This article is organized as follows. In sections 2–5 we collect some generalities on \( F \)-finite schemes, the crystalline Dieudonné functor, schemes with torsion free PD envelopes, and the notion of frames, including a sheaf version. After the results of [La5] are extended to more general classes of semiperfect rings in section 6, the pr-topology is used to deduce Corollary 1.5 and Theorem 1.7 in section 7. The notion of divided Dieudonné crystals is introduced in section 8 and section 9 contains a proof of Theorem 1.3. An explicit description of divided Dieudonné crystals in terms of windows over PD envelopes with a connection is given in section 10. Finally, the relation between divided Dieudonné crystals and the display associated to a \( p \)-divisible group as in [Zi1, La3] is briefly discussed in section 11.

2 Finiteness conditions on the Frobenius map

An \( \mathbb{F}_p \)-algebra \( R \) is called \( F \)-finite if the Frobenius endomorphism \( \phi : R \to R \) is finite. An \( F \)-finite noetherian \( \mathbb{F}_p \)-algebra is excellent by [Ku1, Th. 2.5].
For reference we recall:

**Lemma 2.1.** For a noetherian $\mathbb{F}_p$-finite $\mathbb{F}_p$-algebra $R$ the following are equivalent:

1. $R$ has locally a finite $p$-basis,
2. $R$ is formally smooth over $\mathbb{F}_p$,
3. $R$ is formally smooth over $\mathbb{F}_p$ with respect to the $I$-adic topology for an ideal $I \subseteq \text{Rad}(A)$, where $\text{Rad}(R)$ is the Jacobson radical,
4. $R$ is regular.

**Proof.** (1) $\Rightarrow$ (2) follows from [EGA 0IV (21.2.7)] or [dJ, Lemma 1.1.2], and (2) $\Rightarrow$ (3) is clear. (3) $\Rightarrow$ (4) follows from [EGA 0IV (22.5.8)] or [Ma, Thm. 28.7] applied to the localizations of $A$ at all maximal ideals. To prove (4) $\Rightarrow$ (1) one can assume that $R$ is local. Let $a_1, \ldots, a_n \in R$ map to a $p$-basis of the residue field $R/m_R$ and let $a_{r+1}, \ldots, a_n$ be a minimal set of generators of the maximal ideal $m_R$. Since the completion $\hat{R}$ is isomorphic to a power series ring, $a_1, \ldots, a_n$ form a $p$-basis of $\hat{R}$, and thus a $p$-basis of $R$ by faithfully flat descent; note that $\hat{R} \otimes_{R, \phi} R \cong R$ via $\phi \otimes 1$.

**Remark 2.2.** A noetherian $\mathbb{F}_p$-algebra $R$ is regular iff $\phi : R \to R$ is flat by [Ku2, Thm. 2.1]. This will not be used here.

**Remark 2.3.** The basic rings in [dJ, 1.3.1] are noetherian $\mathbb{F}_p$-algebras $A$ which are $I$-adically complete and formally smooth for the $I$-adic topology such that $A/I$ is of finite type over a field $k$ with a finite $p$-basis. Such rings are $F$-finite and thus satisfy the equivalent conditions of Lemma 2.1.

**Definition 2.4.** An $\mathbb{F}_p$-algebra $R$ will be called $F$-nilpotent if the kernel of the Frobenius endomorphism $\phi : R \to R$ is a nilpotent ideal. An $\mathbb{F}_p$-scheme $X$ is called $F$-nilpotent if for each affine open subscheme $U = \text{Spec} R$ of $X$ the ring $R$ is $F$-nilpotent.

**Remark 2.5.** If the kernel of $\phi$ is finitely generated, then $R$ is $F$-nilpotent, in particular every noetherian $\mathbb{F}_p$-algebra is $F$-nilpotent.

Let $R$ be an $\mathbb{F}_p$-algebra. For a given family $(a_i)_{i \in I}$ of elements of $R$ we consider the rings

$$R_n = R[(a_i^{1/p^n})_{i \in I}] = R[(T_i)_{i \in I}]/((T_i^{p^n} - a_i)_{i \in I})$$

and

$$R_\infty = R[(a_i^{1/p^n})_{i \in I}] = \lim_{n \to \infty} R_n$$

with respect to the inclusions $R_n \to R_{n+1}$ defined by $T_i \mapsto T_i^{p^n}$.
Lemma 2.6. If $R$ is an $F$-nilpotent $\mathbb{F}_p$-algebra and $a_1, \ldots, a_r \in R$, then $R_\infty = R[[a_i^{1/p^\infty}]_{1 \leq i \leq r}]$ is $F$-nilpotent.

Proof. Let $R_n = R[[a_i^{1/p^n}]_{1 \leq i \leq r}]$ as a subring of $R_\infty$. The Frobenius $\phi$ of $R_n$ induces a homomorphism $\bar{\phi} : R_n \to R_{n-1}$, and the commutative diagram

$$
\begin{array}{ccc}
R_n & \longrightarrow & R_{n+1} \\
\downarrow \phi & & \downarrow \delta \\
R_{n-1} & \longrightarrow & R_n
\end{array}
$$

is cocartesian, thus $\phi : R_\infty \to R_\infty$ is the base change of $\bar{\phi} : R_1 \to R$. It suffices to show that the kernel of $\bar{\phi}$ is nilpotent. But this $\bar{\phi}$ factors as $R_1 = R[[a_i^{1/p^\infty}]] \to R[[a_i^{1/p}]] \to R$

where $\psi$ is the base change of $\phi : R \to R$ and $\pi$ maps $(a_i^{1/p})^{1/p}$ to $a_i$. The kernel of $\psi$ is nilpotent since this holds for the kernel of $\phi$, and the kernel of $\pi$ is generated by $(a_i^{1/p})^{1/p} - a_i$ for $i = 1, \ldots, r$, thus nilpotent as well. \hfill \Box

3 Dieudonné crystals

In this section we recall Dieudonné crystals, admissible filtrations, and the crystalline Dieudonné functor. As a base we take the PD scheme $(\Sigma, p\mathcal{O}_\Sigma, \gamma)$ with $\Sigma = \text{Spec } \mathbb{Z}_p$ where $\gamma$ are the canonical divided powers. Sometimes we will also consider $\Sigma_n = \text{Spec } \mathbb{Z}/p^n$, viewed as a PD subscheme of $\Sigma$.

Let $X$ be a scheme on which $p$ is locally nilpotent and $X_0 = X \times \text{Spec } \mathbb{F}_p$. Let $\text{CRIS}(X/\Sigma)$ be the big fppf crystalline site as in [BBM]. Its objects are PD thickenings $(U, T, \delta)$ compatible with $\Sigma$ where $U$ is an $X$-scheme and $p$ is locally nilpotent on $T$, and coverings of $(U, T, \delta)$ correspond to fppf coverings of $T$. The crystalline structure sheaf $\mathcal{O}_{X/\Sigma}$ on $\text{CRIS}(X/\Sigma)$ is defined by $\mathcal{O}_{X/\Sigma}(U, T, \delta) = \mathcal{O}_T(T)$. An $\mathcal{O}_{X/\Sigma}$-module $\mathcal{M}$ gives an $\mathcal{O}_T$-module $\mathcal{M}_{U,T,\delta}$, in particular an $\mathcal{O}_U$-module $\mathcal{M}_U = \mathcal{M}_{(U,U,0)}$, and the restriction of $\mathcal{M}$ to $\text{CRIS}(X_0/\Sigma)$ will be denoted by $\mathcal{M}_0$.

A Dieudonné crystal over $X$ is a triple $(\mathcal{M}, F, V)$ where $\mathcal{M}$ is a finite locally free $\mathcal{O}_{X/\Sigma}$-module and $F : \phi^*(\mathcal{M}_0) \to \mathcal{M}_0$ and $V : \mathcal{M}_0 \to \phi^*(\mathcal{M}_0)$ are linear maps with $VF = p$ and $VF = p$; here $\phi$ is the Frobenius map of $X_0$. Let $D(X)$ denote the category of Dieudonné crystals over $X$.

For $(U, T, \delta) \in \text{CRIS}(X_0/\text{Spec } \mathbb{F}_p)$, which implies that $T$ is an $\mathbb{F}_p$-scheme, the Frobenius morphism $\phi_T : T \to T$ has image in $U$ and thus induces a morphism $\phi_{U/T} : T \to U$; this holds since a PD ideal in characteristic $p$ is annihilated by the Frobenius. An admissible filtration for a Dieudonné
crystal \((\mathcal{M}, F, V)\) over \(X\) is a locally direct summand \(\text{Fil}^1 \mathcal{M}_X \subseteq \mathcal{M}_X\) such that for each \((U, T, \delta) \in \text{CRIS}(X_0/F_p)\) we have
\[
\phi^*_U((\text{Fil}^1 \mathcal{M}_X)_U) = \text{Ker}(F: \phi^*(\mathcal{M})_{(U,T,\delta)} \to \mathcal{M}_{(U,T,\delta)}) \tag{3.1}
\]
inside \(\phi^*_U(\mathcal{M}_U) = \phi^*(\mathcal{M})_{(U,T,\delta)}\). A variant of this definition appears in [Gr2, Chap. V]. The category of Dieudonné crystals over \(X\) with an admissible filtration, also called filtered Dieudonné crystals in the following, will be denoted by \(\text{DF}(X)\).

The crystalline Dieudonné functor

Let
\[D = D_X : \text{BT}(X) \to D(X) \tag{3.2}\]
be the contravariant crystalline Dieudonné functor from \(p\)-divisible groups to Dieudonné crystals as defined in [BBM] and [MM]. For \(G \in \text{BT}(X)\) there is a natural exact sequence of locally free \(\mathcal{O}_X\)-modules
\[0 \to \text{Lie}(G) \to D(G)_X \to \text{Lie}(G^\vee) \to 0,
\]
called the Hodge filtration of \(G\), which gives an extension of \(D_X\) to a functor
\[\text{DF}_X : \text{BT}(X) \to \text{DF}(X) \tag{3.3}\]
from \(p\)-divisible groups to filtered Dieudonné crystals defined by \(\text{Fil}^1 D(G)_X = \text{Lie}(G)^\vee\). The Hodge filtration is admissible by [BBM, Prop. 4.3.10].

Remark 3.1 (Reduction modulo \(p\)). The restriction functor of Dieudonné crystals \(D(X) \to D(X_0)\) is an equivalence, and lifts of a filtered Dieudonné crystal \(\mathcal{M} \in \text{DF}(X_0)\) to \(D(X)\) correspond to lifts of \(\text{Fil}^1 \mathcal{M}_{X_0} \subseteq \mathcal{M}_{X_0}\) to a locally direct summand of \(\mathcal{M}_{(X_0,X,\gamma)}\). If \(p \geq 3\), then lifts under the functor \(\text{BT}(X) \to \text{BT}(X_0)\) correspond to lifts of the Hodge filtration by the Grothendieck-Messing Theorem [Me], and thus the diagram
\[
\begin{array}{ccc}
\text{BT}(X) & \xrightarrow{DF_X} & \text{DF}(X) \\
\downarrow & & \downarrow \\
\text{BT}(X_0) & \xrightarrow{DF_{X_0}} & \text{DF}(X_0)
\end{array}
\]
is 2-cartesian. In particular, if \(p \geq 3\) and the functor \(\text{DF}_{X_0}\) is an equivalence, then \(\text{DF}_X\) is an equivalence as well.
Forgetting the filtration

Lemma 3.2. If $X$ is an $\mathbb{F}_p$-scheme which is reduced or a locally noetherian l.c.i. scheme, the forgetful functor $DF(X) \to D(X)$ is fully faithful.

Proof. We have to show that a Dieudonné crystal $(M, F, V)$ carries at most one admissible filtration. If $X$ is reduced this holds since the Frobenius endomorphism $\phi_R$ of a reduced $\mathbb{F}_p$-algebra $R$ is injective, so a direct summand of a projective $R$-module is determined by its scalar extension under $\phi_R$. In the l.c.i. case we can assume that $X = \text{Spec } R$ where $R$ is a complete local ring. Then $R = A/J$ where $A$ is a power series ring in finitely many variables over a field $k$ and the ideal $J$ is generated by a regular sequence $t_1, \ldots, t_r$. Let $J' = (t_1', \ldots, t_r')$ and $R' = A/J'$. There are divided powers $\delta$ on the ideal $J/J'$ with $\delta_p(t_i) = 0$. Indeed, this is clear when $A$ equals $\mathbb{F}_p[[T_1, \ldots, T_r]]$ with $t_i = T_i$, and the general case follows since the homomorphism $A \to A$ defined by $T_i \mapsto t_i$ is flat. The homomorphism $\phi_{R/R'} : R \to R'$ induced by $\phi_R$ is injective because it is a base change of $\phi_A : A \to A$, which is faithfully flat. Hence a direct summand of a projective $R$-module is determined by its scalar extension under $\phi_{R/R'}$, and the lemma follows.

Lemma 3.3. If $X$ is an $\mathbb{F}_p$-scheme which has locally a $p$-basis, the forgetful functor $DF(X) \to D(X)$ is an equivalence.

Proof. See [LJ] Prop. 2.5.2. We may assume that $X = \text{Spec } R$ where $R$ has a $p$-basis $(x_i)_{i \in I}$, and we have to show that for a Dieudonné crystal $(M, F, V)$ over $\text{Spec } R$ an admissible filtration $Q \subseteq M_R$ exists, which is then unique since $X$ is reduced. Let

$$Q' = \ker(F_R : \phi_R^*(M_R) \to M_R).$$

Since $R$ has a $p$-basis, every affine $(\text{Spec } A, \text{Spec } B, \delta)$ in $\text{CRIS}(X/\text{Spec } \mathbb{F}_p)$ maps non-uniquely to the trivial PD thickening $(X, X, 0)$. Indeed, the restriction of the given map $R \to A$ to the subring $\phi(R) \subseteq R$ can be lifted to a homomorphism $\phi(R) \to B$, and an extension of this map to $R$ is given by defining it on a $p$-basis of $R$. Hence a direct summand $Q \subseteq M_R$ is admissible iff $\phi_R^*(Q) = Q'$. Let $R' = R$ as an $R$-algebra via $\phi_R$, let $R'' = R' \otimes_R R'$, and let $M'$ and $M''$ denote the base change of $M$ to Dieudonné crystals over $\text{Spec } R'$ and $\text{Spec } R''$. Then $Q'$ is an admissible filtration for $M'$, and we have to show that $Q'$ descends to $R$, or equivalently that the scalar extensions of $Q'$ under the two homomorphisms $R' \to R''$ are equal. This holds because a Dieudonné crystal over $\text{Spec } R''$ carries at most one admissible filtration, i.e. the functor $DF(\text{Spec } R'') \to D(\text{Spec } R'')$ is fully faithful, which is a variant of Lemma 3.2. More precisely, $R''$ is isomorphic to $R'[[Y_i]_{i \in I}]/(\{Y_i^{p^r}\}_{i \in I})$ where $Y_i$ is the image of $x_i \otimes 1 - 1 \otimes x_i$; cf. Lemma 17.2 below. Let $B = R'[[Y_i]_{i \in I}]/(\{Y_i^{p^r}\}_{i \in I})$. Then the kernel of $B \to R''$ carries divided powers $\delta$ with $\delta(Y_i^{p^r}) = 0$, and the homomorphism
\( \phi_{B/R''} : R'' \to B \) is injective. Hence \( \text{DF}(\text{Spec } R'') \to \text{D}(\text{Spec } R'') \) is fully faithful as required. \( \square \)

**Remark 3.4.** In general, the functor \( \text{DF}_X \) is not fully faithful, for example when \( X = \text{Spec } R \) with \( R = k[X,Y]/I^2 \) where \( k \) is a field and \( I = \langle X,Y \rangle \); see [BO]. In this example the functor \( \text{DF}_X \) is not fully faithfully either because \( \text{DF}(X) \to \text{D}(X) \) is fully faithful. Proof: Let \( R' = k[X,Y]/I^{2p} \). Then \( R' \to R \) is a PD thickening with trivial divided powers, and the homomorphism \( \phi_{R'/R} : R \to R' \) is injective.

## 4 Torsion in PD envelopes

Let \( R \) be a ring in which \( p \) is nilpotent. In this section we collect some permanence properties of torsion in PD envelopes.

In the following, a presentation of \( R \) will be a surjective ring homomorphism \( A \to R \) where \( A \) is \( p \)-adic\(^1 \) and \( \mathbb{Z}_p \)-flat and \( A_0 = A/p \) has a \( p \)-basis. For a presentation \( A \to R = A/I \) we consider the \( p \)-adic completion of the PD envelope relative to the PD envelope relative to the PD ring \( (\mathbb{Z}_p, p\mathbb{Z}_p) \):

\[
D = D_\gamma(A \to R)^\wedge = D_{A,\gamma}(I)^\wedge. \tag{4.1}
\]

One particular choice is \( A = \mathbb{Z}_p[R]^\wedge \), the \( p \)-adic polynomial ring with set of variables \( R \). In that case we write \( D = D(R) \) as in [dJM] and call \( D(R) \) the universal \( p \)-adic PD envelope of \( R \).

It will be sufficient to consider the case where \( R \) is an \( \mathbb{F}_p \)-algebra because in general, with \( R_0 = R/p \) we have \( D_\gamma(A \to R) = D_\gamma(A \to R_0) \), and every presentation \( A \to R_0 \) can be lifted to a presentation \( A \to R \).

**Lemma 4.1.** If \( R \) is a semiperfect \( \mathbb{F}_p \)-algebra, i.e. the Frobenius \( \phi : R \to R \) is surjective, and if \( A_0 \) is perfect, then \( D \) coincides with \( A_{\text{cris}}(R) \), the universal \( p \)-adic PD thickening of \( R \).

**Proof.** One verifies that \( D \) has the universal property of \( A_{\text{cris}}(R) \). \( \square \)

**Proposition 4.2.** If \( R \) is fixed and the ring \( D \) of (4.1) is torsion free for one presentation \( A \to R \), then \( D \) is torsion free for every presentation \( A \to R \).

**Proof.** We can assume that \( pR = 0 \) and proceed in several steps.

Step 1. Let \( \pi : A \to R \) be a presentation and \( A' = A[\langle T_i \rangle_{i \in I}]^\wedge \) for a set \( I \), with an extension \( \pi' : A' \to R \) of \( \pi \). Let \( D \) and \( D' \) be the rings (4.1) associated to \( \pi \) and \( \pi' \). We claim that \( D \) is torsion free iff \( D' \) is torsion free. Let \( \tilde{\pi} : A' \to A \) be a homomorphism of \( A \)-algebras that lifts \( \pi' \). By a change of variables we may assume that \( \tilde{\pi}(T_i) = 0 \). Then \( E = D_\gamma(A' \to A)^\wedge \) is the \( p \)-adic PD polynomial algebra \( A[\langle T_i \rangle_{i \in I}]^\wedge \). Using [Be1] I Cor. 1.7.2 one shows that \( D' \cong D \otimes_A E = D[\langle T_i \rangle_{i \in I}]^\wedge \), and the claim follows.

\(^1\) \( p \)-adic always means \( p \)-adically complete and separated.
Step 2. Let again $A \to R$ and $D$ be given. Let $A_0^\infty = A_0^{\text{perf}}$ be the perfect hull of $A_0$, let $A^\infty = W(A_0^\infty)$ be the unique lift of $A_0^\infty$, and choose a homomorphism $A \to A^\infty$ that lifts $A_0 \to A_0^\infty$. Then $A/p^r \to A^\infty/p^r$ is faithfully flat. Let $R^\infty = R \otimes_A A^\infty$. If $(\tilde{x}_i)$ is a $p$-basis of $A_0$ with image $(\tilde{x}_i)$ in $R$ then $R^\infty = R[(\tilde{x}_i^{1/p^n})]$ as in (4.2), in particular $R^\infty$ depends only on the reductions $\tilde{x}_i$. The completed PD envelopes $D = D_1(A \to R)^\wedge$ and $D^\infty = D_2(A^\infty \to R^\infty)^\wedge$ are related by $D^\infty = D \hat{\otimes}_A A^\infty$, moreover $D^\infty \cong A_{\text{cris}}(R^\infty)$ by Lemma 4.1. Hence in this situation $D$ is torsion free iff $A_{\text{cris}}(R^\infty)$ is torsion free.

Step 3. One concludes as follows. The property that $D$ is torsion free is not changed if we adjoin polynomial variables to $A$ by step 1, so we can assume that a $p$-basis of $A_0$ generates $R$ as a ring. Then there is a surjective map $A' \to R$ where $A'$ is a completed polynomial ring over $\mathbb{Z}$, such that a $p$-basis of $A_0$ and a $p$-basis of $A_0$ have the same image in $R$. The rings $R^\infty$ of step 2 associated to $A$ and $A'$ coincide, hence $D$ is torsion free iff $A_{\text{cris}}(R^\infty)$ is torsion free iff $D'$ is torsion free by step 2. The property that $D'$ is torsion free depends only on $R$ by step 1.

It will be convenient to use the following terminology.

Definition 4.3. A ring $R$ in which $p$ is nilpotent is PD torsion free if for some (equivalently: any) presentation $A \to R$ the ring $D$ of (4.1) is torsion free. A scheme $X$ on which $p$ is locally nilpotent is PD torsion free if for each affine open subscheme $\text{Spec } R \subseteq X$ the ring $R$ is PD torsion free.

Note that $X$ is PD torsion free iff $X_0 = X \times \text{Spec } \mathbb{F}_p$ has this property.

Lemma 4.4 (dJM). An excellent l.c.i. scheme $X$ over $\mathbb{F}_p$ is PD torsion free.

Proof. One can assume that $X = \text{Spec } R$ where $R$ is local. Then the ring $D(R)$ is torsion free by [dJM] Lemma 4.7.

The following is implicit in the proof of [dJM] Lemma 4.7.

Proposition 4.4 (dJM). An excellent l.c.i. scheme $X$ over $\mathbb{F}_p$ is PD torsion free.

Proof. One can assume that $X = \text{Spec } R$ where $R$ is local. Then the ring $D(R)$ is torsion free by [dJM] Lemma 4.7.

The following is implicit in the proof of [dJM] Lemma 4.7.

Lemma 4.5. Let $A \to R$ and $A' \to R'$ be presentations of rings in which $p$ is nilpotent and let $D$ and $D'$ be the associated rings (4.1). Assume that $A' = A[T_1, \ldots, T_n]^{\wedge}$ such that the inclusion $A \to A'$ induces a homomorphism $u : R \to R'$. If $u$ is an l.c.i. homomorphism, then $D/p^r \to D'/p^r$ is flat.

Proof. We can assume that $pR = 0$. Assume first that $u$ is a complete intersection with respect to the $T_i$ in the sense that

$$R' = R[T_1, \ldots, T_n]/(f_1, \ldots, f_m)$$

where $f_1, \ldots, f_m$ is a regular sequence in every fibre. We consider the rings $\tilde{A} = D[T_1, \ldots, T_n]$ and $\tilde{R} = \tilde{A}/(g_1, \ldots, g_r)$ where $g_i$ is a lift of $f_i$, and the $p$-adic PD envelope relative to the given PD thickening $D \to R = D/I$

$$\tilde{D} = D(D, I, \delta)(\tilde{A} \to \tilde{R})^\wedge.$$
There is a natural homomorphism $\tilde{R} \to R'$. We claim that the kernel of the composition $\tilde{D} \to \tilde{R} \to R'$ carries divided powers which identify $\tilde{D}$ with $D'$. Let $A_r = A/p^r$ etc. Then Spec $\tilde{R}_r \to$ Spec $A_r$ is a regular embedding of flat $D_r$-schemes of finite type, which implies that $\tilde{D}_r$ is flat over $D_r$ by [BBM] Lemma 2.3.3 and its proof. Let $J_r = \text{Ker}(\tilde{D}_r \to \tilde{R}_r)$ and $I_r = \text{Ker}(D_r \to R)$. The divided powers on $I_r$ extend to $I_r\tilde{D}_r = I_r \otimes_{D_r} \tilde{D}_r$, and we have $J_r \cap I_r \tilde{D}_r = I_r \otimes_{D_r} J_r$. It follows that the divided powers on $I_r\tilde{D}_r$ and on $J_r$ are compatible and extend to $I_r\tilde{D}_r + J_r$, which is the kernel of $\tilde{D}_r \to R'$. One verifies that the resulting PD thickening $\tilde{D}_r \to R'$ has the universal property of $D'_r \to R'$, which proves the claim. Since $\tilde{D}_r$ is flat over $D_r$, the lemma is proved when $u$ is a complete intersection.

In general one can cover Spec $R'$ by open sets Spec $R'_i$ with $R'_i = R'_{g_i}$, equipped with the presentation $A'_i = A'[T_{n+1}] \to R'_i$ defined by $T_{n+1} \mapsto g_i$, such that $R \to R'_i$ is a complete intersection with respect to $T_1, \ldots, T_{n+1}$. Let $D'_i$ be the ring (111) associated to $A'_i \to R'_i$. Then by the first part of the proof, $D \to D'_i$ is flat modulo $p^r$ and $D' \to \prod D'_i$ is faithfully flat modulo $p^r$. Hence $D \to D'$ is flat modulo $p^r$.

**Corollary 4.6.** Let $R \to R'$ be a faithfully flat l.c.i. morphism of rings in which $p$ is nilpotent. Then $D(R)/p^r \to D(R')/p^r$ is faithfully flat. In particular, $R$ is PD torsion free if this holds for $R'$.

**Proof.** Note that $R \subseteq R'$ is a subring. Let $A = \mathbb{Z}_p[R]^\wedge$ and $D = D(R)$. For a finite subset $I \subseteq R' \setminus R$ which generates $R'$ as an $R$-algebra let $A_I = \mathbb{Z}_p[R \cup I]^\wedge$ and let $D_I$ be the ring (111) for the natural presentation $A_I \to R'$. Then $D/p^r \to D_I/p^r$ is flat by Lemma 4.5 and $D(R')/p^r$ is the filtered colimit over $I$ of $D_I/p^r$, thus flat over $D/p^r$. □

## 5 Frames and windows

We use the terminology of frames and windows as in [Zi3] and [La1] with some modifications adapted to the present situation.

**Definition 5.1.** A frame $\underline{A} = (A, I, R, \sigma, \sigma_1)$ consists of $p$-adic rings $A$ and $R = A/I$ for an ideal $I \subseteq A$, a ring homomorphism $\sigma : A \to A$ which induces the Frobenius on $A/p$, and a $\sigma$-linear map $\sigma_1 : I \to A$.

An $\underline{A}$-window $\underline{M} = (M, M_1, \Phi, \Phi_1)$ consists of a finite projective $A$-module $M$, a submodule $M_1 \subseteq M$ with $IM \subseteq M_1$ such that $M/M_1$ is projective over $R$, and $\sigma$-linear maps $\Phi : M \to M$ and $\Phi_1 : M_1 \to M$ with $p\Phi_1 = \Phi$ and $\Phi_1(ax) = \sigma_1(a)\Phi(x)$ for $a \in A$ and $x \in M$, such that $M$ is generated by $\Phi(M) + \Phi_1(M_1)$.

Homomorphisms of $\underline{A}$-windows are module homomorphisms that preserve all data. The category of $\underline{A}$-windows is denoted by $\text{Win}(\underline{A})$. 
Remark 5.2. For an \( A \)-window \( M \) as in Definition 5.1 there is a decomposition \( M = L \oplus T \) with \( M_1 = L \oplus IT \), called a normal decomposition, and the pair \( (\Phi, \Phi_1) \) corresponds to a \( \sigma \)-linear isomorphism \( \Psi : L \oplus T \to M \) defined by \( \Psi = \Phi_1 \) on \( L \) and \( \Psi = \Phi \) on \( T \). Proof: The existence of \( \sigma_1 \) implies that \( x^p = 0 \) for each \( x \in \text{Ker}(A/p \to R/p) \), hence finite projective \( R \)-modules can be lifted to \( A \), and the decomposition of \( M \) follows. See [La1] Lemma 2.6] for more details. It also follows that \( I + pA \subseteq \text{Rad}(A) \), which means that \( A \) is a frame in the sense of [La1] Def. 2.1.

Remark 5.3 (Functoriality). Let \( A' = (A', I', R', \sigma, \sigma_1) \) be another frame. A frame homomorphism \( \alpha : A \to A' \) is a ring homomorphism \( A \to A' \) that induces a homomorphism \( R \to R' \) and commutes with \( \sigma \) and \( \sigma_1 \). It induces a base change functor \( \alpha^* : \text{Win}(A) \to \text{Win}(A') \); see [La1] Lemma 2.10. Universal property: If windows \( M \) over \( A \) and \( M' \) over \( A' \) are given, define an \( \alpha \)-homomorphism \( M \to M' \) to be a homomorphism of \( A \)-modules that preserves the filtration and commutes with \( \Phi \). Then \( \alpha \)-homomorphisms \( M \to M' \) correspond to homomorphisms of \( A' \)-windows \( \alpha^* M \to M' \).

Example 5.4. For a \( p \)-adic ring \( R \) the ring of \( p \)-typical Witt vectors \( W(R) \) carries a frame structure \( W(R) = (W(R), I(R), R, \sigma, \sigma_1) \) where \( \sigma \) is the Witt vector Frobenius and \( \sigma_1 \) is the inverse of the bijective Verschiebung homomorphism \( v : W(R) \to I(R) \). A window over \( W(R) \) is a 3\( n \)-display over \( R \) in the sense of [Zi1], also called display in the literature.

Example 5.5. Let \( R \) be a \( p \)-adic ring. A frame for \( R \) in the sense of [Zi3] is a \( p \)-adic PD thickening \( A \to R = A/I \) where \( A \) is torsion free, with a Frobenius lift \( \sigma \) on \( A \). Then \( \sigma(I) \subseteq pA \) and \( (A, I, R, \sigma, \sigma_1) \) with \( \sigma_1 = p^{-1} \sigma \) is a frame in the sense of Definition 5.1.

Let us recall a version of the deformation lemma for windows. An endomorphism \( f \) of an abelian group is called pointwise nilpotent if each element is annihilated by some power of \( f \).

Proposition 5.6. Let \( \alpha : A' \to A \) be a homomorphism of frames such that \( A' \to A \) is surjective with kernel \( N \) and \( R' \to R \) is bijective. If \( \sigma_1 : N \to N \) induces a pointwise nilpotent endomorphism of \( N/p \), then the base change functor \( \alpha^* : \text{Win}(A') \to \text{Win}(A) \) is an equivalence.

Proof. Since \( A' \) and \( A \) are \( p \)-adic, \( N \) is closed and hence complete for the \( p \)-adic topology of \( A' \). It follows that \( N \) is \( p \)-adic; see the proof of [SP Tag 0901]. For \( n \geq 0 \) we define a frame \( B_n = (A'/p^n N, I'/p^n N, R', \sigma, \sigma_1) \) as a quotient of \( A' \). Since \( \sigma \) is pointwise nilpotent on \( N/p^n N \) the projection homomorphism \( B_n \to B_0 = A \) induces an equivalence of windows by [La1] Thm. 3.2]. Since \( A' = \lim_{\leftarrow n} B_n \) taken componentwise, the result follows by [La1] Ie. 2.12].

We will also use a sheaf version of frames and windows.
Definition 5.7. Let $T$ be a topos.

A ring $B$ in $T$ is called $p$-adic if $B \hookrightarrow \varprojlim B/p^n$.

A frame in $T$ is a collection $\mathfrak{A} = (\mathcal{A}, \mathcal{T}, \mathcal{R}, \sigma, \sigma_1)$ where $\mathcal{A}$ and $\mathcal{R}$ are $p$-adic rings in $T$ with $\mathcal{R} = \mathcal{A}/\mathcal{I}$ for an ideal $\mathcal{I} \subseteq \mathcal{A}$, $\sigma : \mathcal{A} \to \mathcal{A}$ is a ring homomorphism which induces the Frobenius homomorphism on $\mathcal{A}/p$, and $\sigma_1 : \mathcal{T} \to \mathcal{A}$ is a $\sigma$-linear map with $p\sigma_1 = \sigma$.

If $\mathfrak{A}$ is a frame in $T$, an $\mathfrak{A}$-window is a collection $\mathfrak{M} = (\mathcal{M}, \mathcal{M}_1, \Phi, \Phi_1)$ where $\mathcal{M}$ is a finite locally free $\mathfrak{A}$-module, $\mathcal{M}_1 \subseteq \mathcal{M}$ is a submodule with $\mathcal{I}\mathcal{M} \subseteq \mathcal{M}_1$ such that $\mathcal{M}/\mathcal{M}_1$ is locally free over $\mathcal{R}$, and $\Phi : \mathcal{M} \to \mathcal{M}$ and $\Phi_1 : \mathcal{M}_1 \to \mathcal{M}$ are $\sigma$-linear maps with $p\Phi_1 = \Phi$ on $\mathcal{M}_1$ and $\Phi_1(ax) = \sigma_1(a)\Phi(x)$ for local sections $a \in \mathcal{T}$ and $x \in \mathcal{M}$, such that $\mathcal{M}$ is generated by $\Phi(\mathcal{M}) + \Phi_1(\mathcal{M}_1)$.

Homomorphisms of $\mathfrak{A}$-windows are module homomorphisms that preserve all data. The category of $\mathfrak{A}$-windows is denoted by $\text{Win}(\mathfrak{A}/T)$.

Remark 5.2 carries over to the sheaf version as follows.

Lemma 5.8. For $\mathfrak{A}$-modules $\mathcal{M}_1 \subseteq \mathcal{M}$ as in Definition 5.7, locally there is a decomposition of $\mathfrak{A}$-modules $\mathcal{M} = \mathcal{L} \oplus \mathcal{T}$ with $\mathcal{M}_1 = \mathcal{L} \oplus \mathcal{I}\mathcal{T}$. If such a decomposition is given, pairs $(\Phi, \Phi_1)$ such that $(\mathcal{M}, \mathcal{M}_1, \Phi, \Phi_1)$ is an $\mathfrak{A}$-window correspond to $\sigma$-linear isomorphisms $\Psi : \mathcal{L} \oplus \mathcal{T} \to \mathcal{M}$ defined by $\Psi = \Phi_1$ on $\mathcal{L}$ and $\Psi = \Phi$ on $\mathcal{T}$.

Proof. The existence of $\sigma_1$ implies that each $x$ in the kernel of $\mathcal{A}/p \to \mathcal{R}/p$ satisfies $x^p = 0$. It follows that the map of idempotents $\text{Idem}(M_n(\mathcal{A})) \to \text{Idem}(M_n(\mathcal{R}))$ is surjective as sheaves. Indeed, let $e \in M_n(\mathcal{R})$ be idempotent. Locally $e$ can be lifted to an element $y \in M_n(\mathcal{A})$, with image $y_\mathcal{R} \in M_n(\mathcal{R}/p^r)$. Since the kernel of $\mathcal{A}/p^r \to \mathcal{R}/p^r$ is a nil-ideal, there is a unique idempotent $\tilde{e}_r \in M_n(\mathcal{A}/p^r)$ which is a polynomial in $y_\mathcal{R}$ with integral coefficients such that $\tilde{e}_r$ and $e$ have equal image in $M_n(\mathcal{R}/p^r)$. The system $(\tilde{e}_r)_r$ gives an idempotent $\tilde{e} \in M_n(\mathcal{A})$ which maps to $e$. Now the surjective map of finite locally free $\mathcal{R}$-modules $\mathcal{M}/\mathcal{I}\mathcal{M} \to \mathcal{M}/\mathcal{M}_1$ splits locally, and the resulting decomposition of $\mathcal{M}/\mathcal{I}\mathcal{M}$ lifts to a decomposition of $\mathcal{M}$ locally. For the correspondence between $(\Phi, \Phi_1)$ and $\Psi$ see [La1] Lemma 2.6. \hfill \Box

Definition 5.1 and Definition 5.7 are related as follows. Let $\text{LF}(\mathfrak{A})$ denote the category of finite locally free $\mathfrak{A}$-modules.

Lemma 5.9. Let $\mathfrak{A}$ be a frame in $T$ as in Definition 5.7 and consider the global sections $\mathfrak{A} = \Gamma(T, \mathfrak{A})$ defined by $A = \Gamma(T, \mathcal{A})$ and $\mathcal{R} = \Gamma(T, \mathcal{R})$ etc. If $A \to R$ is surjective, then $\mathfrak{A}$ is a frame as in Definition 5.7. If moreover the functor of global sections induces equivalences $\text{LF}(\mathfrak{A}) \xrightarrow{\sim} (\text{finite projective } \mathfrak{A} \text{-modules})$ and $\text{LF}(\mathcal{R}) \xrightarrow{\sim} (\text{finite projective } \mathcal{R} \text{-modules})$, then it induces an equivalence $\text{Win}(\mathfrak{A}/T) \cong \text{Win}(\mathfrak{A})$. 

14
Proof. Let us show that the ring $A$ is $p$-adic. Let $A_n = \Gamma(T, A/p^n)$ and let $A'_n$ be the image of $A \to A_n$. Since $A$ is $p$-adic we have $A \cong \varprojlim A_n$, hence $A \cong \varprojlim A'_n$. Since $p^nA$ lies in the kernel of $A \to A'_n$ it follows that $A$ is $p$-adic; see the proof of [SP, Tag 090T]. Similarly $R$ is $p$-adic, and $A$ is a frame if $A \to R$ is surjective. Assume that the hypothesis on finite locally free modules holds. For given $M \in LF(A)$ let $M = M \otimes_A R \in LF(R)$, and consider the associated finite projective modules $M = \Gamma(T, M)$ over $A$ and $M = \Gamma(T, M)$ over $R$. Then $M = M \otimes_A A$ and $M = M \otimes_R R$, moreover $\bar{M} = M \otimes_A R$ since this holds after $\otimes_R R$. It follows that submodules $\mathcal{M}_1 \subseteq \mathcal{M}$ such that $\mathcal{M}/\mathcal{M}_1$ is a finite locally free $R$-module correspond to submodules $M_1 \subseteq M$ such that $M/M_1$ is finite projective over $R$, via $M_1 = \Gamma(T, \mathcal{M}_1)$. If $M_1 \subseteq M$ is given, choose a decomposition $M = L \oplus T$ with $M_1 = L \oplus IT$, which gives $\mathcal{M} = \mathcal{L} \oplus \mathcal{T}$ with $\mathcal{M}_1 = \mathcal{L} \oplus \mathcal{IT}$ for $\mathcal{L} = L \otimes_A A$ and $\mathcal{T} = T \otimes_A A$. Then window structures on $(\mathcal{M}, \mathcal{M}_1)$ and on $(\mathcal{M}_1, \mathcal{M})$ correspond to $\sigma$-linear isomorphisms $\Psi : \mathcal{L} \oplus \mathcal{T} \to \mathcal{M}$ and $\Psi' : \mathcal{L} \oplus \mathcal{T} \to \mathcal{M}$ by Remark [5.2] and Lemma [5.8] and the relation $\Psi' = \Gamma(T, \Psi)$ gives a bijective correspondence between the window structures.

Finally let us record an elementary fact.

Lemma 5.10. Let $\mathcal{A}$ be a frame in a topos as in Definition [5.7] such that $p \in \mathcal{T}$ and $\sigma_1(p) = 1$. For $\mathcal{A}$-windows $\mathcal{M}$, $\mathcal{N}$ the natural homomorphism $\rho : \text{Hom}(\mathcal{A}, \mathcal{M}, \mathcal{N}) \to \text{Hom}_{\mathcal{A}, \Phi}(\mathcal{M}, \mathcal{N})$ from window homomorphisms to $\Phi$-module homomorphisms is injective with cokernel annihilated by $p$.

Proof. Clearly $\rho$ is injective. Assume that $g : \mathcal{M} \to \mathcal{N}$ commutes with $\Phi$ and let $h = pg$. Then $h(M_1) \subseteq pM_1 \subseteq N_1$, and $h$ commutes with $\Phi_1$ since for $x \in M_1$ we have $\Phi_1(h(x)) = \Phi_1(pg(x)) = \sigma_1(p)\Phi(g(x)) = \Phi(g(x))$ and $h(\Phi_1(x)) = g(\Phi(x))$ using that $p\Phi_1 = \Phi$.

6 Dieudonné theory over semiperfect rings

Let $R$ be a semiperfect $\mathbb{F}_p$-algebra, i.e. the Frobenius $\phi_R : R \to R$ is surjective. The ring $R^\phi = \varprojlim(R, \phi_R)$ is perfect, and the projection $R^\phi \to R$ is surjective, thus

$$R = R^\phi / J$$

for an ideal $J$. The ring $A_{\text{fin}}(R) = W(R^\phi)$ is the universal pro-infinitesimal thickening of $R$, and $A_{\text{cris}}(R) = D_\gamma(A_{\text{fin}}(R) \to R)\wedge$ as in [4.7] is the universal $p$-adic PD thickening of $R$. Let $I_{\text{cris}}(R)$ be the kernel of $A_{\text{cris}}(R) \to R$. The Frobenius $\phi_R$ induces an endomorphism $\sigma$ of $A_{\text{cris}}(R)$.

By [SW, Lemma 4.1.8] there is a unique functorial $\sigma$-linear map $\sigma_1 : I_{\text{cris}}(R) \to A_{\text{cris}}(R)$ with $\sigma_1(p) = 1$, which gives a functorial frame

$$A_{\text{cris}}(R) = (A_{\text{cris}}(R), I_{\text{cris}}(R), R, \sigma, \sigma_1).$$

(6.1)
By [La5, Th. 6.3] there is a contravariant functor from $p$-divisible groups over $R$ to windows over this frame:
\[
\Phi^{\text{cris}}_R : \text{BT}(\text{Spec } R) \to \text{Win}(A_{\text{cris}}(R))
\] (6.2)
such that for $M = \Phi^{\text{cris}}_R(G)$ the module $M$ is the value of the Dieudonné crystal $D(G)$ at $A_{\text{cris}}(R)$. We recall the following facts:

**Remark 6.1.** By [La5, Th. 7.10] the functor $\Phi^{\text{cris}}_R$ is an equivalence if $R$ is iso-balanced. The ring $R$ is called balanced if $\phi(J) = J^p$, or equivalently if the ideal $\bar{J} = \text{Ker}(\phi_R)$ satisfies $\bar{J}^p = 0$, and $R$ is called iso-balanced if there is an ideal $I \subseteq R$ annihilated by a power of $\phi$ such that $R/I$ is balanced; then $I$ is nilpotent. If $J$ is finitely generated then $R$ is iso-balanced.

**Remark 6.2.** If the ring $A_{\text{cris}}(R)$ is torsion free, the category $\text{Win}(A_{\text{cris}}(R))$ is equivalent to the category of filtered Dieudonné crystals $DF(\text{Spec } R)$, and the functor $\Phi^{\text{cris}}_R$ corresponds to $DF_{\text{Spec } R}$; see [CL, Prop. 2.6.4]. This holds when $R$ is a complete intersection in the sense that $J$ is generated by a regular sequence. In that case $R$ is iso-balanced, so it follows that the functor $DF_{\text{Spec } R}$ is an equivalence.

We will generalize these results in two directions, using a variation of the methods of [La5].

### 6.1 $F$-nilpotent semiperfect rings

Let $R = R^\flat/J$ be a semiperfect $F_p$-algebra as above. The ring $R$ is $F$-nilpotent in the sense of Definition 2.4 iff $J^r \subseteq \phi(J)$ for some $r \geq 0$. One verifies that every iso-balanced semiperfect ring is $F$-nilpotent.

Assume that $J = K_0 \supseteq K_1 \supseteq \ldots$ is a decreasing sequence of ideals of $R^\flat$ such that $K_i^p \subseteq K_{i+1}$, this is called an admissible sequence of ideals in [La5, Def. 7.5]. Then the set
\[
W(K_s) = \{(a_0, a_1, \ldots) \in W(R^\flat) \mid a_i \in K_i\}
\]
is an ideal of $W(R^\flat)$, and the ring $A(K_s) = W(R^\flat)/W(K_s)$ is a straight weak lift of $R$ in the sense of [La5, Def. 7.3] by [La5, Lemma 7.6], which implies that it carries a natural frame structure
\[
\underline{A}(K_s) = (A(K_s), pA(K_s), R, \sigma, \sigma_1)
\]
as a quotient of the frame $\underline{W}(R^\flat)$ of Example 5.4. By [La5, Lemma 7.4] there is a natural frame homomorphism
\[
\varkappa : A_{\text{cris}}(R) \to \underline{A}(K_s).
\]
The composition of $\Phi^{\text{cris}}_R$ with the base change functor $\varkappa^*$ is a functor
\[
\Phi_{K_s} : \text{BT}(R) \to \text{Win}(\underline{A}(K_s)).
\]
The minimal admissible sequence $J_s$ is defined by $J_s = J^p$.  

Proposition 6.3. If \( R \) is \( F \)-nilpotent and \( K_\ast = J_\ast \) is the minimal admissible sequence, \( \varkappa \) induces an equivalence of the window categories.

Proof. This is an application of Proposition 5.3: note that \( \varkappa \) is surjective. Let \( N \subseteq A_{\text{cris}}(R) \) be the kernel of \( \varkappa \). We have to show that \( \sigma_1 \) induces a pointwise nilpotent endomorphism of \( N/pN \). Let \( N_0 \subseteq A_{\text{cris}}(R) \) be the ideal generated by the elements \([a^{[n]}]\) for \( a \in J \) and \( n \geq 1 \) (not completed), where the exponent \([n]\) denotes the \( n \)-th divided power. Then \( N_0 \subseteq N \). Each element \( x \in A_{\text{cris}}(R) \) can be written as \( x = b + \sum_{i \geq 0} p^i y_i \) with \( b \in W(R^\flat) \) and \( y_i \in N_0 \), and we have \( x \in N \) if \( b \in N \) iff \( b \in W(J_\ast) \). It follows that \( N/pN \) is generated by \( N_0 \) and \( W(J_\ast) \). For \( a \in J \) we have

\[
\sigma_1([a^{[n]}]) = c_n[a^{[np]}] \quad \text{with} \quad c_n = (np)!/(n!p)
\]  

by [SW] Lemma 4.1.8, and \( c_n \) is divisible by \( p \) if \( n \geq p \), so \( \sigma_1^2 \) is zero on \( N_0/pN_0 \). Since \( R \) is \( F \)-nilpotent we have \( J^{p^r} \subseteq \phi(J) \) for some \( r \), and thus \( J^{p^r s} \subseteq \phi(J^{p^s}) \) for each \( s \). Hence for \( a \in W(J_\ast) \cap p^r W(R^\flat) \) we have \( \sigma_1(a) = \sigma(b) = pr_1(b) \) with \( b \in W(J_\ast) \) and thus \( \sigma_1(a) = 0 \) in \( N/pN \). Finally, for \( a \in J_\ast \) with \( 0 \leq i < r \) the element \( v^i([a]) \) of \( W(J_\ast) \) is mapped to \( pN_0 \) by \( \sigma_1^{i+2} \) by (6.3) again. It follows that \( \sigma_1^{r+1} \) is zero on \( N/pN \).

Let \( R_1 = R^\flat/p^r \) and \( A = A(J_\ast) \) and \( A_1 = A(J_{\ast+1}) \). Then \( R_1 \) is semiperfect with \( R_1^0 = R^\flat \), and \( A_1 \) gives a frame \( A_1 \) for \( R_1 \).

Proposition 6.4. There is a 2-cartesian diagram of categories:

\[
\begin{array}{ccc}
BT(R_1) & \longrightarrow & BT(R) \\
\Phi_{J_{\ast+1}} & & \Phi_J \\
\downarrow \downarrow & & \downarrow \downarrow \\
\text{Win}(A_1) & \longrightarrow & \text{Win}(A)
\end{array}
\]

Proof. There are natural surjective homomorphisms \( R_1 \rightarrow R \) and \( A_1 \rightarrow A \) which induce the horizontal functors. Let \( I \) be the kernel of \( A_1 \rightarrow A \rightarrow R \). We will define divided powers on \( I \) and a homomorphism \( \sigma_1 : I \rightarrow A_1 \) which gives a frame

\[
A' = (A_1, I, R, \sigma, \sigma_1).
\]

Let \( N \) be the kernel of \( A_1 \rightarrow A \). Then \( I = N + pA_1 \), and

\[
N = W(J_\ast)/W(J_{\ast+1}), \quad N \cap pA_1 = v(W(J_{\ast+1}))/v(W(J_{\ast+2})).
\]

It follows that \( \sigma_1 : pA_1 \rightarrow A_1 \) is zero on \( N \cap pA_1 \) and thus extends uniquely to a homomorphism \( \sigma_1 : I \rightarrow A_1 \) with \( \sigma_1(N) = 0 \). For \( a_1, \ldots, a_p \in W(J_\ast) \) we have \( a_1 \cdots a_p \in W(J_{\ast+1}) \) by the homogeneity properties of the Witt vector multiplication. Hence \( N^p = 0 \), and \( N \) carries the trivial divided powers \( \delta \) defined by \( \delta_p(x) = 0 \) for all \( x \in N \). For \( a \in W(J_{\ast+1}) \), the natural divided
powers $\gamma$ on $pW(R^p) = I(R^p)$ give $\gamma_p(v(a)) = (p^{p-1}/p!)v(a^p) \in W(J_{r+1})$ since $a^p \in W(J_{r+2})$. It follows that the given divided powers $\gamma$ on $pA_1$ and $\delta$ on $N$ coincide on $N \cap pA_1$ and thus extend to divided powers on $I$.

There are natural frame homomorphisms $\underline{A}_1 \rightarrow \underline{A}' \rightarrow \underline{A}$ over the ring homomorphisms $R_1 \rightarrow R = R$. Since $\sigma_1$ is zero on $N$, the homomorphism $\underline{A}' \rightarrow \underline{A}$ induces an equivalence of windows by Proposition 5.6. Next we want to define a frame homomorphism $\underline{A}_{\text{cris}}(R) \rightarrow \underline{A}'$.

The projection $W(R^p) \rightarrow A_1$ extends to a homomorphism $\tilde{x} : A_{\text{cris}}(R) \rightarrow A_1$ of $p$-adic PD thickenings of $R$ due to the divided powers on $I$. The composition of $\tilde{x}$ with either $A_1 \rightarrow A$ or $A_{\text{cris}}(R_1) \rightarrow A_{\text{cris}}(R)$ is the homomorphism $\tilde{x}$ of $A$ or $A_1$. The homomorphism $\tilde{x}$ commutes with $\sigma$ since $\sigma$ preserves the divided powers on both sides. We claim that $\tilde{x}$ is a frame homomorphism $A_{\text{cris}}(R) \rightarrow \underline{A}'$, which means that $\tilde{x}$ commutes with $\sigma_1$. This is a direct calculation: The ideal $I_{\text{cris}}(R)$ is generated by $p$ and the elements $[a]^{[n]}$ for $a \in J$ and $n \geq 1$, where $[n]$ means $n$-th divided power. Using (6.3) we get

$$\tilde{x}(\sigma_1([a]^{[n]})) = c_n \tilde{x}([a]^{[np]}) = c_n \tilde{x}([a])^{[np]} = 0$$

since $\tilde{x}([a]) \in N$ and $N^{[p]} = 0$, moreover

$$\sigma_1(\tilde{x}([a]^{[n]})) = \sigma_1(\tilde{x}([a])^{[n]}) = 0$$

since $\tilde{x}([a])^{[n]} \in N$ and $\sigma_1(N) = 0$.

The composition of $\Phi^R_{\text{cris}}$ with the base change functor $\tilde{x}^*$ is a functor $\Phi'$ such that the following diagram commutes.

$$\begin{array}{ccc}
BT(R_1) & \rightarrow & BT(R) \\
\downarrow^{\Phi_{J_{r+1}}} & \downarrow^{\Phi'} & \downarrow^{\Phi_{J_1}} \\
Win(A_1) & \rightarrow & Win(\underline{A}') \xrightarrow{\sim} Win(A)
\end{array}$$

For a $p$-divisible group $G$ over $R$ and $\underline{M}' = \Phi'(G)$, the module $M' \otimes_{A_1} R_1$ coincides with the value of the Dieudonné crystal $D(G)_{R_1 \rightarrow R}$, where $R_1 \rightarrow R$ is a PD thickening induced by the divided powers on $I \subseteq A_1$. The divided powers on $\text{Ker}(R_1 \rightarrow R)$ are trivial since $N$ maps surjectively to this ideal. By the Grothendieck-Messing Theorem it follows that lifts of $G$ to $R_1$ and lifts of $\underline{M}'$ to $\underline{A}$, correspond to lifts of the Hodge filtration in the same way, and the proposition follows.

**Corollary 6.5.** If $R$ is $F$-nilpotent, there is a 2-cartesian diagram of categories

$$\begin{array}{ccc}
BT(R^p) & \rightarrow & BT(R) \\
\downarrow^{\Phi_{\text{cris}}^R} & \downarrow^{\Phi_{J_1}} \\
Win(W(R^p)) & \rightarrow & Win(\underline{A})
\end{array}$$
Proof. Proposition 6.4 applied to \( R_n = R^p/J^p \) for \( n \geq 0 \) gives 2-cartesian squares

\[
\begin{array}{ccc}
BT(R_n) & \rightarrow & BT(R) \\
\downarrow_{\Phi_{J_{n+1}}} & & \downarrow_{\Phi_J} \\
Win(A_n) & \rightarrow & Win(A)
\end{array}
\]

with \( A_n = W(R^p)/W(J_{n+1}) \). Since \( R \) is \( F \)-nilpotent we have \( \phi(J) \subseteq J^r \) for some \( r \), which implies that \( R^p = \lim_{\rightarrow} R^p/J^p \) and thus \( W(R^p) = \lim_{\rightarrow} A_n \).

The proposition follows since the categories of \( p \)-divisible groups and windows preserve these limits by [La1, Lemma 2.12] and the obvious analogue of [Me, II Lemma 4.16]; see also [dJ, Lemma 2.4.4].

**Theorem 6.6.** For each \( F \)-nilpotent semiperfect \( F \)-algebra \( R \) the functor

\[
\Phi^{\text{cris}}_R : BT(R) \rightarrow Win(A^{\text{cris}}(R))
\]

is an equivalence.

**Proof.** The functor \( \Phi^{\text{cris}}_R : BT(R^p) \rightarrow Win(W(R^p)) \) is an equivalence by a result of Gabber; see [La3, Thm. 6.4]. The reduction functor \( Win(W(R^p)) \rightarrow Win(A) \) is essentially surjective by the proof of [La5, Thm. 5.7]. The 2-cartesian square of Corollary 6.5 implies that \( \Phi_J : BT(R) \rightarrow Win(A) \) is an equivalence, using [La5] Lemma 5.9. Proposition 6.3 gives the result.

### 6.2 Infinite complete intersections

For a perfect \( F \)-algebra \( S_0 \) and a set \( I \) we consider the semiperfect ring

\[
R = S_0[\{Y_i\}_{i \in I}]^{\text{per}}/(\{Y_i\}_{i \in I})
\]

where per means perfect hull. If the set \( I \) is finite, then \( R \) is a quotient of a perfect ring by a regular sequence, and the functors \( \Phi^{\text{cris}}_R \) and \( DF^{\text{Spec}}_R \) are equivalences by [La5] Cor. 5.11 & 5.13. In the following we verify that this also holds when \( I \) is infinite.

First we construct a lift of \( R \), i.e. a \( p \)-adic and \( \mathbb{Z}_p \)-flat ring \( A \) such that \( A/p = R \) with a Frobenius lift \( \sigma : A \rightarrow A \). Let \( S = R^p = \lim_{\rightarrow} (R, \phi) \) and \( R = S/J \) as earlier. By a slight abuse of notation we write

\[
Y_i = (Y_i^{p-r})_r \in S.
\]

Let \( K_0 \subseteq W(S) \) be the ideal generated by the elements \( \{Y_i\} \) for \( i \in I \), let \( K \subseteq W(S) \) be the closure of \( K_0 \) with respect to the limit topology in

\[
W(S) = \lim_{\rightarrow} W_n(S/\phi^n(J)),
\]

and let \( A = W(S)/K \). The Frobenius of \( W(S) \) induces \( \sigma : A \rightarrow A \).
Lemma 6.7. The ring $A$ is $p$-adic and $\mathbb{Z}_p$-flat and $A/p = R$.

Proof. We will write $A_n = A/p^n$. If the set $I$ is finite, then $K_0 = K$ and the lemma is easily verified using that the elements $Y_i$ of $S$ form a regular sequence. In the general case, by definition we have

$$A = \lim_{\longrightarrow} (W_n(S/\phi^n(J))/\bar{K}_0)$$

where $\bar{K}_0$ is the image of $K_0$. For fixed $n$, the ring in the limit stabilizes for $m \geq n - 1$ because the ideal $\phi^m(J)/\phi^{m+1}(J)$ is generated by all $Y_i^{p^m}$ for $i \in I$, and for $r \leq n - 1$ the shift $v^r([Y_i^{p^m}]) = p^r[Y_i^{p^{m-r}}]$ lies in $K_0$. The stable value is given by

$$W_n(S/\phi^{n-1}(J))/\bar{K}_0 \cong A_n.$$ 

Clearly $A_1 = R$. We have to show that $A_n$ is flat over $\mathbb{Z}/p^n$. For each finite subset $M \subseteq I$ let $R_M \subseteq R$ be the analogue of $R$ with $M$ in place of $I$, let $S_M = R_M^\circ$, and let $J_M$ be the kernel of $S_M \to R_M$, which is generated by $Y_i$ for $i \in M$, and let $K_M \subseteq W(S_M)$ be the ideal generated by $[Y_i]$ for $i \in M$. Then the ring

$$A_{M,n} = W_n(S_M/\phi^{n-1}(J_M))/\bar{K}_M = W_n(S_M)/\bar{K}_M$$

is flat over $\mathbb{Z}/p^n$. Now $R$ is the colimit over $M$ of $R_M$, and $\phi^{-n}$ induces an isomorphism $S/\phi^n(J) \cong R$. It follows that $A_n$ is the colimit over $M$ of $A_{M,n}$, so $A_n$ is flat over $\mathbb{Z}/p^n$. \hfill \Box

Lemma 6.8. The ring $A_{\text{cris}}(R)$ is $p$-torsion free.

Proof. If the set $I$ is finite, the lemma holds because the elements $Y_i$ of $S$ form a regular sequence. In general we note that $A_{\text{cris}}(R)/p^n$ is the PD envelope relative to $\Sigma_n$ of the kernel of $W_n(S) \to R$, or equivalently of $W_n(S/\phi^n(J)) \to R$; note that $W_n(\phi^n(J))$ maps to zero in every PD thickening of $R$ annihilated by $p^n$ since $v^r([a^{p^n}]) = p^r[a]^{p^{n-r}}$ becomes divisible by $p^n$ for $a \in J$. Using the notation of the proof of Lemma 6.7 the arrow

$$W_n(S/\phi^n(J)) \to R$$

is the colimit over all finite subsets $M \subseteq I$ of the arrows $W_n(S_M/\phi^n(J_M)) \to R_M$. The PD envelope relative to $\Sigma_n$ of the latter is flat over $\mathbb{Z}/p^n$, and the lemma follows. \hfill \Box

Lemma 6.7 implies that there is a frame $\mathfrak{A} = (A, pA, R, \sigma, \sigma_1)$ with $\sigma_1 = p^{-1}\sigma$. The universal property of $A_{\text{cris}}(R)$ gives a ring homomorphism $A_{\text{cris}}(R) \to A$, which is a frame homomorphism

$$\varphi : A_{\text{cris}}(R) \to A$$

since $A$ is torsion free. We have functors

$$\text{BT}(\text{Spec } R) \xrightarrow{\Phi_{\text{cris}}} \text{Win}(A_{\text{cris}}(R)) \xrightarrow{\varphi^*} \text{Win}(A)$$

20
where $\Phi^\text{cris}_R$ can be defined directly by evaluation of the Dieudonné crystal at $A^\text{cris}(R)$ since this ring is torsion free. The composition $\Phi_A = \pi^* \circ \Phi^\text{cris}_R$ is defined by evaluation of the Dieudonné crystal at $A$.

**Lemma 6.9.** The homomorphism $\pi$ induces an equivalence of windows.

**Proof.** Let $N \subseteq A^\text{cris}(R)$ be the kernel of $\pi$. By Proposition 5.6 it suffices to show that $\sigma$ induces a pointwise nilpotent endomorphism of $N/p$. Since $A$ is torsion free, $N/p$ is the kernel of $A^\text{cris}(R)/p \rightarrow A/p = R$, which is generated by the divided powers $[Y_i]^n$ for $i \in I$ and $n \geq 1$. Using (6.3) it follows that $\sigma^2 = 0$ on $N/p$.

Let $\tilde{I}$ be the kernel of $A \rightarrow R \xrightarrow{\phi} R$. One verifies that $\tilde{I}$ is a PD ideal, using that $\tilde{I}$ is generated by $p$ and $[Y_i^1/p]$ for $i \in I$, and $[Y_i^1/p]^p = 0$ in $A$.

**Lemma 6.10.** The divided powers on $\tilde{I}$ induce pointwise nilpotent divided powers $\delta$ on the ideal $\bar{J} = \tilde{I}/pA = \ker(\phi : R \rightarrow R)$.

**Proof.** The ideal $\bar{J}$ is generated by $Y_i^1/p$ for $i \in I$. Since $[Y_i^1/p] \in A$ is an inverse image of $Y_i$ with $[Y_i^1/p]^p = 0$, it follows that $\delta_p(Y_i^1/p) = 0$.

**Theorem 6.11.** For $R$ as in (6.4) the functor $\Phi^\text{cris}_R$ is an equivalence.

**Proof.** By Lemma 6.10 the hypotheses of [La5, Th. 5.7] are satisfied, which implies that the functor $\Phi_A = \pi^* \circ \Phi^\text{cris}_R$ is an equivalence. The result follows since $\pi^*$ is an equivalence by Lemma 6.9.

**Corollary 6.12.** For $R$ as in (6.4) the functor $DF_{\text{Spec } R}$ is an equivalence.

**Proof.** Since $A^\text{cris}(R)$ is torsion free by Lemma 6.8 the category $\text{Win}(A^\text{cris}(R))$ is equivalent to $DF(\text{Spec } R)$ such that $\Phi^\text{cris}_R$ corresponds to $DF_{\text{Spec } R}$.

### 6.3 Extension to rings where $p$ is nilpotent

Let $R$ be a ring in which $p$ is nilpotent such that $R_0 = R/p$ is semiperfect. There is a universal $p$-adic PD thickening $\pi : A^\text{cris}(R) \rightarrow R$, which gives a frame $A^\text{cris}(R)$ as in (6.1). Namely $A^\text{cris}(R) = A^\text{cris}(R_0)$ as a ring, $\pi$ is the unique lift of $A^\text{cris}(R_0) \rightarrow R_0$, and $\sigma_1 : I^\text{cris}(R) \rightarrow A^\text{cris}(R)$ is the restriction of $\sigma_1 : I^\text{cris}(R_0) \rightarrow A^\text{cris}(R_0)$. Windows over $A^\text{cris}(R)$ correspond to windows $M$ over $A^\text{cris}(R_0)$ together with a lift of $M_1 \subseteq M \otimes_{A^\text{cris}(R_0)} R_0$ to a direct summand of $M \otimes_{A^\text{cris}(R_0)} R$. Hence the Hodge filtration of a $p$-divisible group gives an extension of $\Phi^\text{cris}_{R_0}$ to a contravariant functor

$$\Phi^\text{cris}_R : BT(R) \rightarrow \text{Win}(A^\text{cris}(R)).$$

If $p \geq 3$ and $\Phi^\text{cris}_{R_0}$ is an equivalence, then $\Phi^\text{cris}_R$ is an equivalence.
7 Dieudonné theory by $p$-root descent

In this section we derive properties of the crystalline Dieudonné functor by descent from the semiperfect case, using a topology that allows infinite extractions of $p$-th roots.

7.1 A $p$-root topology

Let $R$ be a ring in which $p$ is nilpotent. For a family $(a_i)_{i \in I}$ of elements of $R$ we define $R[(a_i^{1/p^n})_{i \in I}]$ and $R[(a_i^{1/p^\infty})_{i \in I}]$ as in (2.1) and (2.2).

Definition 7.1. Let $R$ be a ring in which $p$ is nilpotent. A morphism $f : \text{Spec } R' \to \text{Spec } R$ is called

- an extraction of a $p$-th root if $R' \cong R[(a_i^{1/p})_{i \in I}]$ with $a_i \in R$,
- a simultaneous extraction of $p$-th roots if $R' \cong R[(a_i^{1/p})_{i \in I}]$ for some family $(a_i)_{i \in I}$ of elements of $R$,
- a $p$-root extension if $R' = \lim_{\to} B_i$ for a sequence of rings $R = B_0 \to B_1 \to B_2 \to \ldots$ where each $\text{Spec } B_{i+1} \to \text{Spec } B_i$ is a simultaneous extraction of $p$-th roots.

The main example of a $p$-root extension is $R \to R_\infty = R[(a_i^{1/p^\infty})_{i \in I}]$ for a family $(a_i)$ as above. If $R/p$ is generated by the $a_i$ as an algebra over $\phi(R/p)$, then $R_\infty/p$ is semiperfect. The class of $p$-root extensions is closed under composition and under base change, moreover $p$-root extensions are faithfully flat universal homeomorphisms. We consider the topology which is generated by the Zariski topology and $p$-root extensions:

Definition 7.2. Let $X$ be a scheme on which $p$ is locally nilpotent. A morphism $f : X' \to X$ is called a $p$-root morphism (pr-morphism) if $f$ is a $p$-root extension Zariski locally, i.e. for each $y \in X'$ there are affine open sets $U' \subseteq X'$ and $U \subseteq X$ with $y \in U'$ and $f(U') \subseteq U$ such that $f : U' \to U$ is a $p$-root extension. A pr-covering is a surjective family of pr-morphisms.

We denote by $X_{pr}$ the category of pr-sheaves on the category of $X$-schemes.

Remark 7.3. In [FJ] a morphism of $\mathbb{F}_p$-schemes is called a $p$-root-morphism or $p$-morphism if Zariski locally it is a finite succession of extractions of $p$-th roots. Thus the pr-morphisms used here a pro-version of the $p$-morphisms of [FJ]. For sheaves that commute with filtered colimits of rings, passing to the pro-version makes no difference.

For a scheme $X$ on which $p$ is locally nilpotent let $(X/\Sigma)_{\text{CRIS,pr}}$ be the topos of pr-sheaves on CRIS$(X/\Sigma)$, where a pr-covering of $(U,T,\delta) \in \text{CRIS}(X/\Sigma)$ corresponds to a pr-covering of $T$ as usual. A quasi-coherent (resp. finite locally free) crystal on $X$ is a quasi-coherent (resp. finite locally free) $\mathcal{O}_{X/\Sigma}$-module on $(X/\Sigma)_{\text{CRIS,Zar}}$, or equivalently on $(X/\Sigma)_{\text{CRIS,pr}}$; the equivalence holds by faithfully flat descent of modules.
one sees that the elements form an over the category of schemes on which p-
cally free crystals, of Dieudonné crystals, and of filtered Dieudonné crystals
This is straightforward, using that for a
Proof. g: Spec
PD thickening since B
follows.
Proof. We may assume that p
from

Remark 7.4 (Functoriality). For a morphism f: X' → X there is a morph-

Lemma 7.5. The fibered categories of quasi-coherent crystals, of finite loc-

crystals, of Dieudonné crystals, and of filtered Dieudonné crystals
over the category of schemes on which p is locally nilpotent are pr-stacks.
Proof. This is straightforward, using that for a p-root extension f: Spec R' →
Spec R and a PD thickening Spec R → Spec B there is a p-root extension g:
Spec B' → Spec B which lifts f, and Spec R' → Spec B' is naturally a
PD thickening since B → B' is flat.

7.2 Schemes with a local p-basis

Theorem 7.6. Let X be an \( \mathbb{F}_p \)-scheme which has locally a p-basis. Then
the functor

\[
D_X : BT(X) \to D(X)
\]

from p-divisible groups to Dieudonné crystals is an equivalence.

Proof. We may assume that X = Spec R where R has a p-basis \((x_i)_{i \in I}\). Let
R' = R\text{per} be the perfect hull of R. Then R' = R[(x_i^{1/p^\infty})_{i \in I}], in particular
Spec R' → Spec R is a p-root extension. Let R'' = R' \otimes_R R' and R''' =
R' \otimes_R R' \otimes_R R'. Let y_i = x_i \otimes 1 - 1 \otimes x_i \in R' \otimes_{\mathbb{F}_p} R'.

Lemma 7.7. If R''' is viewed as an R'-algebra by one of the factors, there
is an isomorphism of R'-algebras

\[
R''' \cong R'[(Y_i)_{i \in I}]^{\per}/((Y_i)_{i \in I})
\]

which sends \( y_i^{p^{-r}} \) to the image of \( y_i^{p^{-r}} \) under \( R' \otimes_{\mathbb{F}_p} R' \to R''' \).

Proof. The monomials \( x^a = \prod x_i^{a_i} \) with exponents \( a \in (0, 1) \cap \mathbb{Z}[1/p]^{(f)} \)
form a basis of \( R' \) as an \( R \)-module. Hence the elements \( 1 \otimes x^a \) form a basis
of \( R''' \) as an \( R' \)-module by the first factor. By writing out \( y^a = \prod y_i^{a_i} \)
as an \( R' \)-linear combination of the elements \( 1 \otimes x^a \) one sees that the elements \( y^a \)
form an \( R' \)-basis of \( R''' \) as well. Since \( y_i \) maps to zero in \( R''' \), the lemma
follows.

We continue the proof of Theorem 7.6. By Lemma 7.7 the ring \( R''' \) takes
the form \( \mathbb{F}_p \), with \( S_0 = R' \), and the same holds for \( R'' = R'' \otimes_R R''' \). Hence
the functor \( DF_{\text{Spec } T} \) is an equivalence for \( T \in \{ R', R'', R''' \} \) by Corollary 6.12. By faithfully flat descent of p-divisible groups and pr-descent of filtered
Dieudonné crystals under Spec \( R' \to Spec R \) (Lemma 7.5) it follows that
the functor \( DF_{\text{Spec } R} \) is an equivalence. Since the category \( DF(\text{Spec } R) \) is
equivalent to \( D(\text{Spec } R) \) by Lemma 3.3, this proves Theorem 7.6.
Remark 7.8. Theorem 7.6 extends [dJ, Main Theorem 1], which states that the functor $D_X$ is an equivalence if $X$ is a formal scheme which locally takes the form $X = \text{Spf} A$ for $A$ as in Remark 2.3. Indeed, Theorem 7.6 applies to $X = \text{Spec} A$, which has locally a finite $p$-basis by Lemma 2.1; moreover $BT(X) \cong BT(\mathfrak{X})$ and $D(X) \cong D(\mathfrak{X})$ by [dJ, Lemma 2.4.4 & Prop. 2.4.8].

7.3 Locally complete intersections

Theorem 7.9. If $X$ is an $F$-finite and $F$-nilpotent scheme over $\mathbb{F}_p$ which is PD torsion free then the functor $DF_X : BT(X) \to DF(X)$ from $p$-divisible groups to filtered Dieudonné crystals is an equivalence.

See Definitions 2.3 and 4.3 for $F$-nilpotent and PD torsion free schemes.

Corollary 7.10. If $X$ is an $F$-finite l.c.i. scheme over $\mathbb{F}_p$ then $DF_X$ is an equivalence.

Proof. Noetherian and $F$-finite implies excellent by [Ku1, Th. 2.5], hence $X$ is PD torsion free by Proposition 4.4; moreover noetherian implies $F$-nilpotent, so Corollary 7.10 follows from Theorem 7.9.

Proof of Theorem 7.9. We can assume that $X = \text{Spec} R$. Let $a_1, \ldots, a_r$ generate $R$ as an algebra over $\phi(R)$, using that $R$ is $F$-finite, and let

$$R' = R[\left( a_i^{1/p^{\infty}} \right)_{1 \leq i \leq r}]$$

and $R'' = R' \otimes_R R'$, $R''' = R' \otimes_R R' \otimes_R R'$. Each ring $T \in \{ R', R'', R''' \}$ is semiperfect and $F$-nilpotent by Lemma 2.6, hence the functor $\Phi_{\cris}^T$ is an equivalence by Theorem 6.6. Since $R$ is PD torsion free, the same holds for $T$ by Corollary 4.4 using a colimit, and $A_{\cris}(T)$ is torsion free by Lemma 4.1. Hence $\text{Win}(A_{\cris}(T))$ is equivalent to $DF(\text{Spec} T)$ by Remark 6.2, and the functor $DF_{\text{Spec} T}$ is an equivalence as well. Theorem 7.9 follows by faithfully flat descent of $p$-divisible groups and pr-descent of filtered Dieudonné crystals under $\text{Spec} R' \to \text{Spec} R$; see Lemma 7.5.

8 Divided Dieudonné crystals

Throughout this section let $X$ be a scheme on which $p$ is locally nilpotent. Let $X_{\text{pr}}$ and $(X/\Sigma)_{\text{CRIS, pr}}$ be defined as in Section 7.4.

Lemma 8.1. There are morphisms of topoi

$$X_{\text{pr}} \xrightarrow{i} (X/\Sigma)_{\text{CRIS, pr}} \xrightarrow{u} X_{\text{pr}}$$

with $u \circ i = \text{id}$ where $i_* = u^{-1}$ is defined by $i_*(F)(U, T, \delta) = F(U)$. 24
Proof. Let \( i^{-1}(G)(U) = G(U, U, 0) \) and \( u_*(G)(U) = \Gamma(\Spec \mathbb{Z}/p^n, G) \). Then \( i_* \) and \( i^{-1} \) preserve sheaves and \( i^{-1} \) is an exact left adjoint of \( i_* \), and \( u_* \) is a right adjoint of \( u^{-1} \) on the level of presheaves, moreover \( i^{-1} \circ u^{-1} = \text{id} \). Using that \( p \)-root extensions can be lifted to PD thickenings one verifies that \( u_* \) preserves sheaves and that \( u^{-1} \) is exact.

\[ \text{Remark 8.2. For } n \geq 1 \text{ let } X_n = X \times \Spec \mathbb{Z}/p^n. \] There are analogous morphisms of topoi

\[ (X_n)_{\text{pr}} \xrightarrow{i_n} (X_n/\Sigma_n)_{\text{CRIS, pr}} \xrightarrow{u_n} (X_n)_{\text{pr}} \]

with \((u_n)_*(G)(U) = \Gamma((U/\Sigma_n)_{\text{CRIS}}, G)\).

\[ \text{Remark 8.3. For a morphism of schemes } f : X' \to X \text{ the morphisms } i \text{ and } u \text{ of Lemma 8.1 are compatible with the morphisms } f : X'_{\text{pr}} \to X_{\text{pr}} \text{ and } f : (X/\Sigma)_{\text{CRIS, pr}} \to (X/\Sigma)_{\text{CRIS, pr}} \text{ of Remark 7.4. The same holds for } i_n \text{ and } u_n. \]

\[ \text{Remark 8.4. If } U = \Spec R \text{ is an affine } X\text{-scheme where } R/p \text{ is semiperfect, the category CRIS}(U/\Sigma) \text{ has the initial pro-object } \text{Spf } \mathcal{O}_{\text{cris}}(R) \text{, and } u_* (G)(U) = \varprojlim G(U, \Spec \mathcal{O}_{\text{cris}}(R)/p^n, \delta). \] This formula determines the sheaf \( u_*(G) \) since schemes \( U \) of this type form a basis of the pr-topology of \( X \).

The sheaf \( \mathcal{O}_{X}^{\text{cris}} \) and its locally free modules

Following Fontaine-Messing \[ \text{[FM]} \] we define \( \mathcal{O}_{X}^{\text{cris}} = u_* \mathcal{O}_{X/\Sigma} \), which is a ring in \( X_{\text{pr}} \). Then \( i \) and \( u \) are morphisms of ringed topoi

\[ (X_{\text{pr}}, \mathcal{O}_X) \xrightarrow{i} ((X/\Sigma)_{\text{CRIS, pr}}, \mathcal{O}_{X/\Sigma}) \xrightarrow{u} (X_{\text{pr}}, \mathcal{O}_{X}^{\text{cris}}) \]

in a natural way. We also consider the torsion version \( \mathcal{O}_{X,n}^{\text{cris}} = (u_n)_* \mathcal{O}_{X_n/\Sigma_n} \) in \( (X_n)_{\text{pr}} \). For a morphism \( f : X' \to X \) we have \( f^{-1} \mathcal{O}_{X}^{\text{cris}} = \mathcal{O}_{X'}^{\text{cris}} \) and thus an evident morphism of ringed topoi

\[ f : (X'_{\text{pr}}, \mathcal{O}_{X'}^{\text{cris}}) \to (X_{\text{pr}}, \mathcal{O}_{X}^{\text{cris}}). \]

The canonical PD ideal \( \mathcal{I}_{X/\Sigma} \subseteq \mathcal{O}_{X/\Sigma} \) is the kernel of \( \mathcal{O}_{X/\Sigma} \to i_* \mathcal{O}_X \). The functor \( u_* \) applied to this map gives a homomorphism \( \mathcal{O}_{X}^{\text{cris}} \to \mathcal{O}_X \) with kernel \( \mathcal{I}_{X}^{\text{cris}} = u_* \mathcal{I}_{X/\Sigma} \).

\[ \text{Lemma 8.5. The homomorphism } \mathcal{O}_{X}^{\text{cris}} \to \mathcal{O}_X \text{ is surjective, the sheaf } \mathcal{O}_{X}^{\text{cris}} \text{ is } p\text{-adic, i.e. } \mathcal{O}_{X}^{\text{cris}} \cong \varprojlim_n \mathcal{O}_{X}^{\text{cris}} / p^n, \text{ and we have } \mathcal{O}_{X}^{\text{cris}} / p^n \cong j_n* \mathcal{O}_{X,n}^{\text{cris}} \text{ where } j_n : (X_n)_{\text{pr}} \to X_{\text{pr}} \text{ is the natural morphism.} \]
Proof. One verifies directly that $O^\text{cris}_X = \lim_{\leftarrow n} j_{n*}O^\text{cris}_{X,n}$. If $U = \text{Spec } R$ is an affine $X$-scheme such that $R/p$ is semiperfect, then $O^\text{cris}_X(U) = A_{\text{cris}}(R)$ and $j_{n*}O^\text{cris}_{X,n}(U) = A_{\text{cris}}(R)/p^n$. Hence $j_{n*}O^\text{cris}_{X,n}$ coincides with the presheaf quotient $O^\text{cris}_X/p^n$ on a basis of the topology, thus $j_{n*}O_{X,n} \cong O^\text{cris}_X/p^n$ as sheaves. Since $A_{\text{cris}}(R) \to R$ is surjective, the lemma follows. □

Lemma 8.6. The functors $u_*$ and $u^*$ between $O_{X/\Sigma}$-modules and $O^\text{cris}_X$-modules induce inverse equivalences between the categories of finite locally free modules, $\text{LF}(O_{X/\Sigma}) \cong \text{LF}(O^\text{cris}_X)$.

Proof. If $M$ is a finite locally free $O^\text{cris}_X$-module, then $u^*M$ is a finite locally free $O_{X/\Sigma}$-module, moreover $M \to u_*u^*M$ is an isomorphism because this can be verified locally, and it holds for $M = O^\text{cris}_X$ by the definition of $O^\text{cris}_X$. Let $M$ be a locally free $O_{X/\Sigma}$-module. We claim that the $O^\text{cris}_X$-module $u_*M$ is finite locally free and that $u^*u_*M \to M$ is an isomorphism. This can be verified pr-locally, so let $X = \text{Spec } R$ such that $R/p$ is semiperfect. By passing to a pr-covering of $X$ we can assume that $\tilde{M} = M(X,X,0)$ is a finite free $R$-module. For $n \geq 1$ with $p^nR = 0$ let $A_n = A_{\text{cris}}(R)/p^n$. Then $M_n = M(X,\text{Spec } A_n, \delta)$ is a finite projective $A_n$-module with basis $(e_i)_n$, and $u_*(M)(X) = \lim_{\leftarrow n} M_n$ is a finite free module over $A_{\text{cris}}(R) = O^\text{cris}_X(X)$. If $X' \to X$ is a morphism and $X' = \text{Spec } R'$ where $R'/p$ is semiperfect, then an $R$-basis of $M(X,X,0)$ maps to an $R'$-basis of $M(X',X',0)$, which implies that an $A_{\text{cris}}(R)$-basis $(e_i)_n$ of $u_*(M)(X)$ maps to an $A_{\text{cris}}(R')$-basis of $u_*(M)(X')$. Hence $u_*M$ is free over $O^\text{cris}_X$ with basis $(e_i)_n$, and the assertion follows. □

Corollary 8.7. Let $X = \text{Spec } R$ such that $R/p$ is semiperfect. Then finite locally free $O^\text{cris}_X$-modules $M$ are equivalent to finite projective modules $M$ over $A = A_{\text{cris}}(R)$ by the functors $M = \Gamma(X,M)$ and $M = M \otimes_A O^\text{cris}_X$.

Proof. Finite locally free $O_{X/\Sigma}$-modules $M$ are equivalent to finite projective $A$-modules $M$ by the functors $M = \lim_{\leftarrow n} M(X,\text{Spec } A/p^n, \delta)$ and $M = M \otimes_A O_{X/\Sigma}$. Hence Lemma 8.6 gives the result. □

The frame structure on $O^\text{cris}_X$
The sheaf $O^\text{cris}_X$ carries a natural Frobenius lift

$$\sigma : O^\text{cris}_X \to O^\text{cris}_X$$

which can be defined as follows. If $U = \text{Spec } R$ is an affine $X$-scheme where $R/p$ is semiperfect, then the Frobenius $\phi : R/p \to R/p$ induces an endomorphism $\sigma$ of $A_{\text{cris}}(R/p) = A_{\text{cris}}(R) = O^\text{cris}_X(U)$. Since such $U$ form a basis of the pr-topology, this defines $\sigma$ as an endomorphism of $O^\text{cris}_X$. 26
Remark 8.8. Without reference to semiperfect rings the definition of $\sigma$ goes as follows. Assume first that $X$ is an $\mathbb{F}_p$-scheme. Then for each $X$-scheme $U$ the Frobenius morphism $\phi_U : U \to U$ induces an endomorphism $\sigma = \phi_U^*$ of $\mathcal{O}^{\text{cris}}_X(U) = \Gamma(U/\Sigma, \mathcal{O}_{U/\Sigma})$, thus an endomorphism $\sigma$ of $\mathcal{O}^{\text{cris}}_X$. In general this gives an endomorphism $\sigma$ of $\mathcal{O}^{\text{cris}}_{X_0}$, which is extended to $\mathcal{O}^{\text{cris}}_X$ using that $\mathcal{O}^{\text{cris}}_X = i_* \mathcal{O}^{\text{cris}}_{X_0}$ for the natural morphism $i : (X_0)_{\text{pr}} \to X_{\text{pr}}$.

Remark 8.9. If $X$ is an $\mathbb{F}_p$-scheme, for an $\mathcal{O}^{\text{cris}}_X$-module $\mathcal{M}$ there is a natural homomorphism $w : \sigma^*(\mathcal{M}) \to \phi^*(\mathcal{M})$ defined as follows (here $\sigma^*$ means scalar extension under $\sigma$ and $\phi^*$ means inverse image under $\phi$). For each scheme $U \to X$ the Frobenius $\phi_U : U \to U$ induces a homomorphism

$$\phi_U^* : \mathcal{M}(U \to X) \to \mathcal{M}(U \to X \xrightarrow{\phi} X),$$

which gives a $\sigma$-linear map $\mathcal{M} \to \phi^*(\mathcal{M})$, whose linearization is $w$. The homomorphism $w$ is an isomorphism if $\mathcal{M}$ is quasi-coherent, in particular if $\mathcal{M}$ is finite locally free. Indeed, $w$ is an isomorphism for $\mathcal{M} = \mathcal{O}^{\text{cris}}_X$, and thus for quasi-coherent $\mathcal{M}$ since the construction is local and the functors $u^*$ and $\phi^*$ are right exact.

Lemma 8.10. There is a unique $\sigma$-linear homomorphism

$$\sigma_1 : \mathcal{T}^{\text{cris}}_X \to \mathcal{O}^{\text{cris}}_X$$

with $p\sigma_1 = \sigma$ which is functorial in $X$. If $p\mathcal{O}_X = 0$ then $\sigma_1(p) = 1$.

Proof. For $U = \text{Spec} \mathcal{R}$ such that $\mathcal{R}/p$ is semiperfect, by [SW] Lemma 4.1.8 there is a unique functorial $\sigma$-linear map $\sigma_1 : I^{\text{cris}}(\mathcal{R}/p) \to A^{\text{cris}}(\mathcal{R}/p)$ with $\sigma_1(p) = 1$ and thus $p\sigma_1 = \sigma$, moreover a functorial $\sigma_1$ with $p\sigma_1 = \sigma$ satisfies $\sigma_1(p) = 1$ since $A^{\text{cris}}(\mathcal{F}_p) = \mathbb{Z}_p$ is torsion free. Since $A^{\text{cris}}(\mathcal{R}) = A^{\text{cris}}(\mathcal{R}/p)$ with $I^{\text{cris}}(\mathcal{R}) \subseteq I^{\text{cris}}(\mathcal{R}/p)$, by restriction we get $\sigma_1 : I^{\text{cris}}(\mathcal{R}) \to A^{\text{cris}}(\mathcal{R})$. If $U$ is an $X$-scheme this is a homomorphism $\sigma_1 : \mathcal{T}^{\text{cris}}_X(U) \to \mathcal{O}^{\text{cris}}_X(U)$, which extends to a homomorphism $\sigma_1 : \mathcal{T}^{\text{cris}}_X \to \mathcal{O}^{\text{cris}}_X$ since affine $X$-schemes with semiperfect reduction form a base of the pr-topology of $X$.

Hence we have a frame in $X_{\text{pr}}$ in the sense of Definition 5.7

$$\mathcal{O}^{\text{cris}}_X = (\mathcal{O}^{\text{cris}}_X, \mathcal{T}^{\text{cris}}_X, \mathcal{O}_X, \sigma, \sigma_1).$$

Definition 8.11. A divided Dieudonné crystal over $X$ is a window over $\mathcal{O}^{\text{cris}}_X$ as in Definition 5.7. The category of divided Dieudonné crystals over $X$ will be denoted by $\text{DD}(X) = \text{Win}(\mathcal{O}^{\text{cris}}_X / X_{\text{pr}})$.

Lemma 8.12. Divided Dieudonné crystals over schemes $X$ in which $p$ is locally nilpotent form a pr-stack.

Proof. This is immediate from the definition.
The following is a special case of a more general result for arbitrary rings $R$ in Corollary 10.14 below, where also a connection appears.

**Proposition 8.13.** If $X = \text{Spec } R$ where $R/p$ is semiperfect, taking global sections gives an equivalence

$$\text{DD}(\text{Spec } R) \sim - \to \text{Win}(A_{\text{cris}}(R))$$

between divided Dieudonné crystals over $X$ and windows over $A_{\text{cris}}(R)$.

**Proof.** We have $A_{\text{cris}}(R) = \Gamma(X, \mathcal{O}^{\text{cris}}_X)$ and $R = \Gamma(X, \mathcal{O}_X)$. Hence the result follows from Lemma 5.9, using Corollary 8.7 and the equivalence between finite locally free $\mathcal{O}_X$-modules and finite projective $R$-modules. \qed

**Relation with filtered Dieudonné crystals**

**Proposition 8.14.** The scheme $X$ is PD torsion free (Definition 4.3) iff the ring $\mathcal{O}^{\text{cris}}_X(U)$ is torsion free for every pr-morphism $U \to X$.

**Proof.** One reduces to the case that $X = \text{Spec } R$ over $\mathbb{F}_p$. Assume that $\mathcal{O}^{\text{cris}}_X(U)$ is torsion free for every pr-morphism $U \to X$. Let $A \to R$ be a presentation with $p$-adic PD envelope $D$ as in (4.1), and define $A^\infty$ and $R^\infty$ as in the proof of Proposition 4.2. We have an injective ring homomorphism $D \to D \hat{\otimes}_A A^\infty = A_{\text{cris}}(R^\infty) = \mathcal{O}^{\text{cris}}_X(\text{Spec } R^\infty)$. Since $\text{Spec } R^\infty \to \text{Spec } R$ is a pr-morphism, $\mathcal{O}^{\text{cris}}_X(\text{Spec } R^\infty)$ is torsion free, thus $D$ is torsion free.

Now assume that $R$ is PD torsion free and let $U \to X$ be a pr-morphism. We have to show that $\mathcal{O}^{\text{cris}}_X(U)$ is torsion free. By passing to a pr-covering of $U$ we can assume that $U = \text{Spec } R'$ with semiperfect $R'$ and that $U \to X$ factors as $U \to Z \to X$ where $Z \to X$ is an open immersion and $U \to Z$ is a $p$-root covering. Then $R'$ is PD torsion free by Corollary 4.6 using a colimit, thus $\mathcal{O}^{\text{cris}}_X(U) = A_{\text{cris}}(R')$ is torsion free. \qed

**Proposition 8.15.** There is a natural functor

$$\rho_X : \text{DD}(X) \to \text{DF}(X)$$

from divided Dieudonné crystals to filtered Dieudonné crystals, which is an equivalence if $X$ is PD torsion free.

**Proof.** By passing to a pr-covering of $X$ we can assume that $X = \text{Spec } R$ with semiperfect $R/p$, because divided Dieudonné crystals and filtered Dieudonné crystals form pr-stacks by Lemmas 7.5 and 8.12, the condition that $X_0$ is PD torsion free passes to a pr-covering and means that $A_{\text{cris}}(R)$ is torsion free; see Proposition 8.14.

By Proposition 8.13, a divided Dieudonné crystal over $X$ corresponds to a window $\underline{M} = (M, M_1, \Phi, \Phi_1)$ over $\underline{A} = A_{\text{cris}}(R)$. Let $F : M^\sigma \to M$ be the
linearization of $\Phi$. There is a unique homomorphism $V : M \to M^\sigma$ with $V(\Phi(x)) = p \otimes x$ and $V(\Phi_1(x)) = 1 \otimes x$, which implies $VF = p$ and $FV = p$; see for example [CL, Remark 2.1.4]. The triple $(M, F, V)$ corresponds to a Dieudonné crystal over $X$. Let $M_R = M \otimes_{A_{cris}(R)} R$ and let $\text{Fil}^1 M_R \subseteq M_R$ be the image of $M_1$. We want to define $\rho_X(M) = (M, F, V, \text{Fil}^1 M_R)$ and have to verify that the filtration is admissible in the sense of (3.1). The Frobenius of $A/p$ induces $\bar{\phi} : R/p \to A/p$, and (3.1) holds iff $\bar{\phi}^*(\text{Fil}^1 M_R/p)$ is equal to $\text{Ker}(F : M_\sigma^{\sigma} A/p \to M_\sigma A/p)$. These are two direct summands of $M_\sigma^{\sigma}/p$ of the same rank, and the inclusion $\subseteq$ holds since $\Phi = p\Phi_1$ on $M_1$. Equality follows, and $\rho_X$ is defined.

If $A_{cris}(R)$ is torsion free, $\rho_X$ is an equivalence by [CL, Prop. 2.6.4].

**Corollary 8.16.** The functor $\rho_X$ is an equivalence if $X_0 = X \times \text{Spec} F_p$ is an excellent l.c.i. scheme.

**Proof.** Use Propositions 4.4 and 8.15.

**Lemma 8.17** (Reduction modulo $p$). There is a 2-cartesian diagram of categories:

$$
\begin{array}{ccc}
\text{DD}(X) & \xrightarrow{\rho_X} & \text{DF}(X) \\
\downarrow & & \downarrow \\
\text{DD}(X_0) & \xrightarrow{\rho_{X_0}} & \text{DF}(X_0)
\end{array}
$$

**Proof.** This holds because lifts under both vertical functors correspond to lifts of the Hodge filtration. In more detail, by pr-descent we may assume that $X = \text{Spec} R$ where $R/p$ is semiperfect. Let $A = A_{cris}(R) = A_{cris}(R_0)$. Then lifts of an $A_{cris}(R_0)$-window $\mathcal{M}$ to an $A_{cris}(R)$-window correspond to lifts of $\bar{M}_1 \subseteq M \otimes_A R_0$ to a direct summand of $M \otimes_A R$, which also correspond to lifts of $\rho_{X_0}(\mathcal{M})$ to $\text{DF}(X)$. □

9 The divided Dieudonné functor

**Proposition 9.1.** For each scheme $X$ on which $p$ is locally nilpotent there is a functor

$$
\text{DD}_X : \text{BT}(X) \to \text{DD}(X)
$$

from $p$-divisible groups to divided Dieudonné crystals, which is compatible with base change in $X$, with an isomorphism $\text{DF}_X \cong \rho_X \circ \text{DD}_X$.

Here $\text{DF}_X$ is the filtered Dieudonné functor of (3.3), and $\rho_X$ is the functor of Proposition 8.15. We call $\text{DD}_X$ the divided Dieudonné functor.

**Proof.** One can assume that $X = \text{Spec} R$ where $R/p$ is semiperfect because $p$-divisible groups, divided Dieudonné crystals, and filtered Dieudonné crystals form pr-stacks by faithfully flat descent, Lemma 8.12, and Lemma 7.5.
respectively. Then divided Dieudonné crystals over $X$ are equivalent to $\mathcal{A}_{\text{cris}}(R)$-windows by Proposition 8.13 and the required functor $\mathbf{DD}_X$ is the functor $\Phi_{\text{cris}}^R$ of (6.2) and (6.5). The isomorphism $\mathbf{DF}_X \cong \rho_X \circ \mathbf{DD}_X$ is part of the construction of $\Phi_{\text{cris}}^R$ in [La5, Thm. 6.3] when $R$ is an $\mathbb{F}_p$-algebra, and it extends to the general case because for $G \in \text{BT}(X)$ with reduction $G_0 \in \text{BT}(X_0)$ the lifts $\mathbf{DF}_X(G)$ of $\mathbf{DF}_{X_0}(G_0)$ and $\Phi_{\text{cris}}^R(G_0)$ are both given by the Hodge filtration of $G$.

**Theorem 9.2.** For an $F$-finite and $F$-nilpotent $\mathbb{F}_p$-scheme $X$ the divided Dieudonné functor $\mathbf{DD}_X$ is an equivalence.

**Proof.** We can assume that $X = \text{Spec } R$ and define $R'$, $R''$, $R'''$ as in the proof of Theorem 7.9, using that $R$ is $F$-finite. Each $T \in \{ R', R'', R''' \}$ is semiperfect and $F$-nilpotent by Lemma 2.6, hence the functor $\Phi_{\text{cris}}^T$ is an equivalence by Theorem 6.6, and the functor $\mathbf{DD}_{\text{Spec } T}$ is an equivalence by Proposition 8.13. By faithfully flat descent for $p$-divisible groups and pr-descent for divided Dieudonné crystals (Lemma 8.12) it follows that $\mathbf{DD}_{\text{Spec } R}$ is an equivalence. □

**Remark 9.3.** Theorem 7.9 follows from Theorem 9.2 and Proposition 8.15.

**Corollary 9.4.** Assume that $p \geq 3$. Then for every scheme $X$ on which $p$ is locally nilpotent such that $X_0 = X \times \text{Spec } \mathbb{F}_p$ is $F$-finite and $F$-nilpotent, the divided Dieudonné functor $\mathbf{DD}_X$ is an equivalence.

**Proof.** By Remark 3.1 and Lemma 8.17 for $p \geq 3$ the diagram

$$
\begin{array}{ccc}
\text{BT}(X) & \mathbf{DD}_X & \text{DD}(X) \\
\downarrow & & \downarrow \\
\text{BT}(X_0) & \mathbf{DD}_{X_0} & \text{DD}(X_0)
\end{array}
$$

is 2-cartesian, and $\mathbf{DD}_{X_0}$ is an equivalence by Theorem 9.2. □

**Corollary 9.5.** Let $X$ be a scheme with $p' \mathcal{O}_X = 0$ such that $X_0 = X \times \text{Spec } \mathbb{F}_p$ is $F$-finite and $F$-nilpotent. For $p$-divisible groups $G$ and $H$ over $X$ with Dieudonné crystals $\mathbf{D}_X(G) = (\mathcal{M}, F, V)$ and $\mathbf{D}_X(H) = (\mathcal{N}, F, V)$, the homomorphism

$$
\text{Hom}(G, H) \rightarrow \text{Hom}_F(\mathcal{M}, \mathcal{N}) \quad (9.1)
$$

is injective with cokernel annihilated by $p^{r-1}$. In particular, the functor $\mathbf{D}_X$ is fully faithful up to isogeny.

**Proof.** We can assume that $X = X_0$ since this does not change the target of (9.1), and $\text{Hom}(G, H) \rightarrow \text{Hom}(G_0, H_0)$ is injective with cokernel annihilated by $p^{r-1}$; see for example [Ka, Lemma 1.1.3] and [La2, Lemma 3.2].
Let $M' = \mathbf{D} \mathcal{D}_X(G)$ and $N' = \mathbf{D} \mathcal{D}_X(H)$ be the divided Dieudonné crystals associated to $G$ and $H$. Under the equivalence of Lemma 8.6, the homomorphism of $O_X/\Sigma$-modules $F : \phi^* M \to M$ corresponds to a homomorphism of $O_{X,\text{cris}}$-modules $F' : \phi^* M' \to M'$, and the composition of $F'$ with the isomorphism $\psi : \phi^* M' \cong \phi^* M'$ of Remark 8.9 is the homomorphism $\Phi : \sigma^* M' \to M'$ which is part of $M'$. Thus (9.1) factors as

$$\text{Hom}(G, H) \to \text{Hom}(M', N') \to \text{Hom}_\Phi(M', N').$$

Here the first arrow is an isomorphism by Theorem 9.2, and the second arrow is injective with cokernel annihilated by $p$ by Lemma 5.10.  

10 Explicit divided Dieudonné crystals

Let $R$ be a ring in which $p$ is nilpotent. In this section we describe divided Dieudonné crystals over $\text{Spec} \ R$ by windows with a connection. The procedure is straightforward. First we construct the relevant frame, then define windows with a connection, translate the connection to an HPD stratification, and finally relate this with divided Dieudonné crystals.

A frame structure on PD envelopes

Assume that $R = A/I$ where $A$ is a $p$-adic and $\mathbb{Z}_p$-flat ring, and $\sigma : A \to A$ is a Frobenius lift. Later we will assume that $A/p$ has a $p$-basis, which means that $A \to R$ is a presentation as in section 4. Let $D = D_{\gamma}(A \to R)^!$ be the $p$-adic PD envelope as in (1), let $\overline{I}$ be the kernel of $D \to R$, for $a \in \overline{I}$ let $a^{[n]} \in D$ denote the $n$-th divided power, and let $\sigma : D \to D$ be the unique extension of $\sigma$. This is a Frobenius lift on $D$ because the Frobenius of $D/p$ preserves the divided powers and thus equals the reduction of $\sigma$.

Lemma 10.1. There is a unique $\sigma$-linear map $\sigma_1 : \overline{I} \to D$ with $p \sigma_1 = \sigma$ which is functorial in $(A \to R, \sigma)$. We obtain a frame

$$D = (D, \overline{I}, R, \sigma, \sigma_1)$$

functorially associated to $(A \to R, \sigma)$.

If $R$ is semiperfect and $A_0 = A/p$ is perfect, then $D = A_{\text{cris}}(R)$, and the restriction of Lemma 10.1 to this case is [SW, Lemma 4.1.8]. The proof of Lemma 10.1 in general will be similar, using a reduction to a universal case.

Proof. We may assume that $R$ is an $\mathbb{F}_p$-algebra because passing to $R/p$ does not change $D$ and enlarges $\overline{I}$. Since $A$ is torsion free there is a well-defined map $\tau : A \to A$ such that $\sigma(a) = a^p + p\tau(a)$ for $a \in A$.

If $D$ is torsion free, then $\sigma_1$ exists and is given by

$$\sigma_1(a) = (p - 1)! a^{[p]} + \tau(a), \quad \sigma_1(a^{[n]}) = (p^{n-1}/n!)\sigma_1(a)^n \quad (10.1)$$

31
for \( a \in I \) and \( n \geq 1 \).

By functoriality it follows that \( \sigma_1 \) is unique in general. Namely, let \( A' = \mathbb{Z}_p[T_0,T_1,\ldots] \) with \( \sigma(T_i) = T_i^p + pT_{i+1} \), thus \( \tau(T_i) = T_i^{\gamma} \), and let \( R' = A'/I' \) for \( I' = (p,T_0) \). Then \( D' = D_\gamma(A' \to R') \) is torsion free. For each \( a \in I \) there is a unique homomorphism \( A' \to A \) which commutes with \( \sigma \) such that \( T_0 \mapsto a \) and thus \( T_i \mapsto \tau^i(a) \). It induces a homomorphism \( D' \to D \) with \( T_0^{[n]} \mapsto a^{[n]} \), and it follows that \([10,1]\) holds if a functorial \( \sigma_1 \) exists.

Let us prove that \( \sigma_1 \) exists. We re-define \( D = D_\gamma(A \to R) \) without \( p \)-adic completion, and \( \bar{I} = \text{Ker}(D \to R) \) accordingly. A functorial \( \sigma_1 : \bar{I} \to D \) will extend to the \( p \)-adic completion.

We will use the following standard representation of \( D \). Let \( P \) be the polynomial algebra over \( A \) with variables \( T_{a,m} \) for \( a \in I \) and \( m \geq 0 \). The homomorphism of \( A \)-algebras \( P \to D \) defined by \( T_{a,m} \mapsto a^{[m]} \) is surjective, thus \( D = P/\bar{\gamma} \), and the kernel \( N \) is generated by the relations that correspond to the PD axioms for the elements \( a^{[m]} \) as in [Ro, page 249, (1)–(4)] and [Be1 page 43, i)–ii)]; explicitly these relations are

\[
\begin{align*}
T_{a,m} - 1, & \quad T_{a,n} - c^nT_{a,n}, & \quad T_{a,n}T_{a,m} - (n+m)T_{a,n+m} \\
T_{a+b,n} - \sum_{0 \leq i \leq n} T_{a,i}T_{b,n-i}, & \quad T_{a,1} - a, & \quad T_{p,n} - p^n/n!
\end{align*}
\]

for \( a, b \in I \) and \( c \in A \). Let \( J \subseteq P \) be the kernel of \( P \to D \to R \), so \( J \) is the set of all polynomials with constant term in \( I \), and \( \bar{I} = J/\bar{\gamma} \).

Let \( \bar{\sigma} \) be the composition \( A \to D \xrightarrow{\sigma} D \) and let \( \bar{\sigma}_1 : \bar{I} \to D \) be the map of sets defined by \( \bar{\sigma}_1(a) = (p-1)!a^{[p]} + \tau(a) \). Then \( \bar{\sigma}_1 \) is \( \bar{\sigma} \)-linear. This can be verified either by a direct computation using the formulas for \( \tau(ab) \) and \( \tau(a+b) \) deduced from the fact that \( \sigma : A \to A \) is a ring homomorphism, or one can use a universality argument.

Let \( \tilde{\sigma} \) be the composition \( P \to D \xrightarrow{\sigma} D \) and let \( \tilde{\sigma}_1 : J \to D \) be the \( \tilde{\sigma} \)-linear map which extends \( \sigma_1 : I \to D \) by \( \tilde{\sigma}_1(T_{a,m}) = (p^{m-1}/m!){\sigma}_1(a)^m \) and \( \tilde{\sigma}_1(T_{a_1,m_1} \cdots T_{a_s,m_s}) = p^{s-1} \prod \tilde{\sigma}_1(T_{a_i,m_i}) \). One verifies that \( \tilde{\sigma}_1 \) is \( \tilde{\sigma} \)-linear. We have to show that \( \tilde{\sigma}_1 \) annihilates \( N \); then it induces the desired map \( \sigma_1 : \bar{I} \to D \). This holds if \( D \) is torsion free. The general case follows because \( \tilde{\sigma}_1 \) is functorial in the triple \( (A \to R, \sigma) \), and for each \( x \in N \) there is a triple \( (A' \to R', \sigma) \) with a homomorphism to \( (A, R, \sigma) \) such that the PD envelope \( D' = D_\gamma(A' \to R') \) is torsion free and such that \( x \) lies in the image of the resulting map \( N' \to N \).

Indeed, by the explicit description of the generators of \( N \) it suffices to verify that for finite sets of elements \( a_1,\ldots, a_r \in A \) and \( b_1,\ldots, b_s \in I \) there is a triple \( (A' \to R', \sigma) \) with torsion free \( D' \) and with a homomorphism to \( (A \to R, \sigma) \) such that all \( a_i \) lie in the image of \( A' \) and all \( b_i \) lie in the image of \( I' = \text{Ker}(A' \to R') \). This is solved by

\[
A' = \mathbb{Z}_p[(T_{ij})_{1 \leq i \leq r, j \geq 0}, (Y_{ij})_{1 \leq i \leq s, j \geq 0}]
\]

with \( \tau(T_{ij}) = T_{i(j+1)} \) and \( \tau(Y_{ij}) = Y_{i(j+1)} \) where \( I' \) is generated by \( p \) and all \( Y_{ij} \), and \( A' \to A \) is defined by \( T_{ij} \mapsto \tau^i(a_i) \) and \( Y_{ij} \mapsto \tau^j(b_i) \). \( \square \)
Windows with a connection

Let \((A \to R, \sigma)\) be as above and assume now that \(A/p\) has a \(p\)-basis. Let \((x_i)\) in \(A\) map to a \(p\)-basis of \(A/p\). Then the module of continuous differentials \(\hat{\Omega}_A\) is topologically free with basis \((dx_i)\). The derivation \(d : A \to \hat{\Omega}_A\) extends uniquely to a PD derivation

\[
\text{d} : D \to \hat{\Omega}_D := D \otimes_A \hat{\Omega}_A,
\]

which means that \(d(a^{[n]}) = a^{[n-1]} \otimes da\) for \(a \in \tilde{I}\) and \(n \geq 1\), and this is the universal PD derivation of \((D, \tilde{I})\) relative to \((\mathbb{Z}_p, \gamma)\) by [SpIR, Tag 07HW]. Since \(\sigma\) is a Frobenius lift, the endomorphism \(d\sigma\) of \(\hat{\Omega}_A\) is divisible by \(p\), i.e. there is a well-defined \(\sigma\)-linear map \((d\sigma)_1 : \hat{\Omega}_A \to \hat{\Omega}_A\) with \(d\sigma = p(d\sigma)_1\). It induces a \(\sigma_D\)-linear map \((d\sigma)_1 : \hat{\Omega}_D \to \hat{\Omega}_D\).

**Definition 10.2.** A connection on a \(D\)-window \(M = (M, M_1, \Phi, \Phi_1)\) is a connection \(\nabla : M \to M \otimes D \hat{\Omega}_D\) such that \(\Phi\) and \(\Phi_1\) are horizontal in the sense that the following diagrams commute.

\[
\begin{array}{ccc}
M \nabla & \to & M \otimes D \hat{\Omega}_D \\
\Phi \downarrow & & \Phi \otimes d\sigma \\
M \nabla & \to & M \otimes D \hat{\Omega}_D
\end{array}
\]

(10.2)

\[
\begin{array}{ccc}
M_1 \nabla & \to & M \otimes D \hat{\Omega}_D \\
\Phi_1 \downarrow & & \Phi \otimes (d\sigma)_1 \\
M \nabla & \to & M \otimes D \hat{\Omega}_D
\end{array}
\]

(10.3)

We denote by \(\text{Win}(D)_{\nabla}\) the category of windows with a connection.

**Remark 10.3.** If \(D\) is torsion free, (10.2) implies (10.3). If \(R\) is an \(\mathbb{F}_p\)-algebra, (10.3) implies (10.2) because for \(x \in M\) we have \(px \in M_1\) and \(\Phi(x) = \Phi_1(px)\).

**Lemma 10.4.** A connection \(\nabla\) on a \(D\)-window \(M\) is necessarily integrable and topologically quasi-nilpotent.

**Proof.** Let \(G = \nabla \circ \nabla : M \to M \otimes_D \Lambda^2 \hat{\Omega}_D\), where \(\Lambda^2\) is taken in the topological sense. Then \(G\) is \(D\)-linear and the following diagrams commute.

\[
\begin{array}{ccc}
M_1 \xrightarrow{G} & M \otimes_D \Lambda^2 \hat{\Omega}_D & \quad \quad M \xrightarrow{G} & M \otimes_D \Lambda^2 \hat{\Omega}_D \\
\Phi_1 \downarrow & \Phi \otimes p \Lambda^2(d\sigma)_1 & \quad \quad \Phi \downarrow & \Phi \otimes p \Lambda^2(d\sigma)_1 \\
M \xrightarrow{G} & M \otimes_D \Lambda^2 \hat{\Omega}_D & \quad \quad M \xrightarrow{G} & M \otimes_D \Lambda^2 \hat{\Omega}_D
\end{array}
\]

33
Since $M$ is generated by the images of $\Phi_1$ and $\Phi$ as a $D$-module we deduce: if $G(M)$ lies in $p^r M \otimes_D \Lambda^2 \hat{\Omega}_D$ for some $r \geq 0$ then the same holds for $r + 1$. Hence $G = 0$ and $\nabla$ is integrable. Let $\theta_i : \hat{\Omega}_D \to D$ be given by $dx_i \mapsto 1$ and $dx_j \mapsto 0$ for $i \neq j$. Let $N_i : M \to M$ be $\nabla$ composed with $id_M \otimes \theta_i$. Then $N_i$ commute, and for each $x \in M$ and $r \geq 1$, almost all $N_i(x)$ are zero mod $p^r$. We have to show that for each $x \in M$ and each $i$ some power $N_i^m(x)$ is zero in $M/pM$. It suffices to consider $x \in \Phi_1(M)$ or $x \in \Phi(M)$ since these generate $M$. If $x \in \Phi_1(M)$ then $\nabla(x)$ lies in the image of $D\Phi(M) \otimes \hat{\Omega}_D$ by (10.3), thus $N_i(x) \in D\Phi(M)$. Since the derivation $N_i : D \to D$ is nilpotent modulo $p$, it suffices to consider $x \in \Phi(M)$. Then $\nabla(x) = 0 \mod p$ by (10.2).

Windows with an HPD stratification

We keep the assumptions on $(A \to R, \sigma)$. The definition of an HPD stratification on a $D$-window is straightforward: For $m \geq 0$ let

$$A(m) = A \otimes_{\mathbb{Z}_p} \ldots \otimes_{\mathbb{Z}_p} A$$

with $m + 1$ factors and let

$$D(m) = D_\gamma(A(m) \to R)^\wedge.$$

We define $\sigma : A(m) \to A(m)$ by $\sigma$ on the factors. Lemma 10.1 gives frames

$$\mathcal{D}(m) = (D(m), I(m), \sigma, \sigma_1)$$

for $m \geq 0$, which form a cosimplicial frame $\mathcal{D}(*)$ with $\mathcal{D}(0) = \mathcal{D}$.

Let $p_0, p_1 : \mathcal{D} \to \mathcal{D}(1)$ be the homomorphisms that correspond to the first and second coordinate, and define $q_i : \mathcal{D} \to \mathcal{D}(2)$ for $0 \leq i \leq 2$ and $q_{ij} : \mathcal{D}(1) \to \mathcal{D}(2)$ for $0 \leq i < j \leq 2$ similarly.

**Definition 10.5.** An HPD stratification on a $D$-window $M$ is an isomorphism of $\mathcal{D}(1)$-windows $\varepsilon : p_0^* M \cong p_1^* M$ such that $q_{12}^* \varepsilon = q_{12}^* \circ q_{01}^* \varepsilon$ over $\mathcal{D}(2)$. Let $\text{Win}(\mathcal{D})^{\text{HPD}}$ be the category of $D$-windows with an HPD stratification.

**Proposition 10.6.** There is an equivalence $\text{Win}(\mathcal{D})^\nabla \cong \text{Win}(\mathcal{D})^{\text{HPD}}$.

The proof is standard, but some care is required because the base change of windows is not just the tensor product in all components. Let us first recall the explicit description of the rings $D(m)$. Let

$$E(m) = D_\gamma(A(m) \to A)^\wedge.$$

Since $E(m)$ is an augmented PD $A$-algebra, [Be1, I Cor. 1.7.2] implies that

$$E(m) \otimes_A E(n) \cong E(m + n)$$ (10.4)
with respect to \( A \to E(m) \) by the last factor and \( A \to E(n) \) by the first factor, and
\[
E(m) \hat{\otimes} A D \cong D(m)
\] (10.5)
with respect to \( A \to E(m) \) by any of the factors; this is also a consequence of [Be1, IV Cor. 1.3.5]. It follows that
\[
D(m) \hat{\otimes} D(n) \cong D(m + n).
\] (10.6)
Recall that \((x_i)\) in \( A \) map to a \( p \)-basis of \( A/p \). Let \( \xi_i = (x_i \otimes 1 - 1 \otimes x_i) \in A(1) \).

**Lemma 10.7.** As an \( A \)-algebra by one of the factors, \( E(1) \) is the \( p \)-adic completion of the PD polynomial algebra \( A \langle \xi_i \rangle \wedge \) in the variables \( \xi_i \).

See [BM, Cor. 1.3.2]. We give a direct proof for completeness.

**Proof.** Let \( B = A[\xi_i] \) as a polynomial ring and let \( E' = D_*(B \to A)\wedge \), which is the ring \( A \langle \xi_i \rangle \wedge \) of the lemma. There is a natural PD homomorphism \( E' \to E(1) \). Since \( x_i \) form a \( p \)-basis of \( A/p \) and \( x_i, \xi_i \) form a \( p \)-basis of \( A(1)/p \) the following diagram is cocartesian.

\[
\begin{array}{c}
B/p \longrightarrow A(1)/p \\
\phi \downarrow \quad \phi \downarrow \\
B/p \longrightarrow A(1)/p
\end{array}
\]

If \( S \to A/p \) is a PD thickening of \( \mathbb{F}_p \)-algebras and \( f : B/p \to S \) is a homomorphism of thickenings of \( A/p \), the composition \( f \circ \phi \) factors into \( B/p \to S \to A/p \to S \) where the last arrow is induced by the Frobenius of \( S \), and similarly for \( A(1) \) in place of \( B \). It follows that \( f \) extends to a unique homomorphism \( A(1)/p \to S \). Hence \( E'/p \to E(1)/p \) is bijective. Since \( E' \) is \( \mathbb{Z}_p \)-flat and both sides are \( p \)-adic it follows that \( E' \to E(1) \) is bijective. \( \square \)

For \( 1 \leq j \leq m \) let \( \xi_{ij} = q_{j-1,j}(\xi_i) \) in \( A(m) \). Lemma 10.7 and (10.6) give:

**Lemma 10.8.** \( D(m) \) viewed as a \( D \)-algebra by any of the factors is the \( p \)-adic completion of the free PD polynomial algebra \( D \langle \xi_{ij} \rangle \wedge \).

We consider the following truncations of the PD envelopes \( E(m) \) and \( D(m) \). Let \( J \) be the kernel of \( A(1) \to A \), let \( \bar{J} \) be the kernel of \( E(1) \to A \), and let \( \bar{J}^m \subseteq E(1) \) be the \( p \)-adic closure of the \( m \)-th PD power of \( J \). For \( r \geq 1 \) we set
\[
E(1)_r = E(1)/\bar{J}^r, \quad D(1)_r = D(1) \hat{\otimes} E(1) E(1)_r,
\]
and
\[
D(m)_r = D(1)_r \hat{\otimes} D \ldots \hat{\otimes} D(1)_r
\]
as a quotient of $D(m) = D(1) \hat{\otimes} D \ldots \hat{\otimes} D D(1)$. Then (10.5) implies that $D(1)_{r}$ is isomorphic to $D \hat{\otimes} E E(1)_{r}$ with respect to the homomorphisms $E \to E(1)_{r}$ by either of the two factors. In particular, $D(1)_{2} \cong D \oplus \tilde{\Omega}_{D}$ and thus $D(m)_{2} = D \oplus (\tilde{\Omega}_{D})^{\otimes m}$.

**Lemma 10.9.** For $m, r \geq 1$ there is a well-defined frame

$$D(m)_{r} = (D(m)_{r}, \tilde{I}(m)_{r}, R, \sigma, \sigma^{1})$$

as a quotient of $D(m)$.

**Proof.** Let $N$ be the kernel of $D(m) \to D(m)_{r}$. We have to show that $\sigma_{1}(N) \subseteq N$. For the ring $R' = A/p$ in place of $R$ we get an analogous frame $D'(m) = E(m)$ and an ideal $N' = \text{Ker}(E(m) \to E(m)_{r})$. There is a frame homomorphism $E(m) \to D(m)$, and $N$ is topologically generated by the image of $N'$. Hence it suffices to show that $\sigma_{1}$ preserves $N'$. But $E(m)_{r}$ is a topologically free $A$-module, thus $\mathbb{Z}_{q}$-flat, hence $N' \cap pE(m) = pN'$. Since $\sigma$ preserves $N'$ it follows that $\sigma_{1}$ preserves $N'$.

**Lemma 10.10.** For $m \geq 1$ the homomorphism $j = j_{0m} : D(1) \to D(m)$ induces an injective homomorphism $D(1)_{m+1} \to D(m)_{2}$.

**Proof.** We have $j(\xi) = \xi_{1} + \xi_{2} + \ldots + \xi_{m}$, so the image of $j(\xi^{r}) = j(\xi)^{r}$ in $D(m)_{2}$ is equal to $s_{r}(\xi_{1}, \ldots, \xi_{m})$, where $s_{r}$ is the $r$-th elementary symmetric polynomial when $r \leq m$ and $s_{r} = 0$ if $r > m$. For a finite product $a = \prod \xi_{i}^{r}$ of total degree $r = \sum r_{i}$ the image of $j(a)$ in $D(m)_{2}$ is $\prod s_{r_{i}}(\xi_{i1}, \ldots, \xi_{im})$, which is zero if $r > m$. One verifies that for $r \leq m$ the resulting elements $j(a)$ of $D(m)_{2} = D \oplus (\tilde{\Omega}_{D})^{\otimes m}$ are $D$-linearly independent.

**Proof of Proposition 10.6.** Let $M = (M, M_{1}, \Phi, \Phi^{1})$ be a $D$-window. An HPD stratification $\varepsilon$ on the $D$-module $M$ is equivalent to an integrable and topologically quasi-nilpotent connection $\nabla$ on $M$, and $\Phi$ commutes with $\varepsilon$ iff $\Phi$ is horizontal under $\nabla$ as in (10.2). We have to show that $\varepsilon$ is a window isomorphism, i.e. it commutes with $\Phi$ and $\Phi^{1}$, iff $\Phi$ and $\Phi^{1}$ are horizontal under $\nabla$ in the sense that (10.2) and (10.3) commute.

Let $p_{i} : D \to D(1)_{2}$ be the reduction of $p_{i} : D \to D(1)$ for $i = 0, 1$. The isomorphism $\varepsilon : p_{0}^{*}M \cong p_{1}^{*}M$ of $D(1)$-modules induces an isomorphism $\varepsilon : p_{0}^{*}M \cong p_{1}^{*}M$ of $D(1)_{2}$-modules. We claim that $\varepsilon$ is a window isomorphism iff this holds for $\varepsilon$. Assume the latter holds. The intersection of the kernels of $D(1) \to D(1)_{r+1}$ for varying $r$ is zero. Hence it suffices to show that the reduction $\varepsilon_{r+1}$ over $D(1)_{r+1}$ of $\varepsilon$ is a window isomorphism over $D(1)_{r+1}$. Let $q = q_{0} : D(1) \to D(r)$, which induces $\tilde{q} : D(1)_{r+1} \to D(r)_{2}$ by Lemma (10.10). Then $q^{*}\varepsilon$ is the composition of all $q^{*}_{ij} \varepsilon$, and the reduction $q_{r+1}^{*} \varepsilon_{r+1}$ is the composition of all $q_{r+1}^{*} \varepsilon$. Since $\varepsilon$ is a window isomorphism and since $\varepsilon$ is injective it follows that $\varepsilon_{r+1}$ is a window isomorphism as required.
The homomorphism $\bar{\varepsilon}$ corresponds to a $p_0$-linear map $\bar{\varepsilon}' : M \to \bar{p}_1^* M$, and $\bar{\varepsilon}$ is an isomorphism of $D(1)_2$-windows iff $\bar{\varepsilon}'$ is a homomorphism of windows relative to $p_0 : D \to D(1)_2$. Let us make this condition explicit. Let $K$ be the kernel of $D(1)_2 \to D$, thus $K \cong \hat{\Omega}_D$. Then $D(1)_2 = D \oplus K$ via $\bar{p}_1 : D \to D(1)$, and $\bar{p}_0 : D \to D \oplus K$ is given by $\bar{p}_0(a) = (a, da)$ with $da = a \otimes 1 - 1 \otimes a \in K$. The $D(1)_2$-window $\bar{p}_1^* M$ consists of the modules $\bar{p}_1^* M = M \oplus (M \otimes D, \bar{p}_1 K)$, $(\bar{p}_1^* M)_1 = M_1 \oplus (M \otimes D, \bar{p}_1 K)$, and the homomorphism $\Phi_1 : (\bar{p}_1^* M)_1 \to \bar{p}_1^* M$ is given by $\Phi_1 = \Phi_1 \oplus (\Phi \otimes \sigma_1)$, while $\Phi = \Phi \oplus (\Phi \otimes \sigma)$. Moreover $\bar{\varepsilon}'(x) = x + \nabla(x)$ for $x \in M$ under the identification $K \cong \Omega_D$. Under this identification, $(da)_1 : \hat{\Omega}_D \to \hat{\Omega}_D$ corresponds to $\sigma_1 : K \to K$. Indeed, this is clear when $D(1)_2$ is torion free, for example for $E(1)_2$ in place of $D(1)_2$, and the general case follows. As a consequence, $\bar{\varepsilon}$ is a window isomorphism iff $\bar{\varepsilon}'$ is a window homomorphism iff $\Phi$ and $\Phi_1$ are horizontal under $\nabla$.

HPD stratifications and divided Dieudonné crystals

We keep the assumptions on $(A \to R, \sigma)$. Let $X = \text{Spec } R$.

**Proposition 10.11.** There is an equivalence $\text{DD}(X) \cong \text{Win}(D)_{\text{HPD}}$ between divided Dieudonné crystals over $X$ and windows over $D$ with an HPD stratification.

**Remark 10.12.** This extends the usual equivalence between finite locally free $O_{X/S}$-modules and finite projective $D$-modules with an HPD stratification. More precisely, if $M \in \text{DD}(X)$ corresponds to $(M, \varepsilon) \in \text{Win}(D)_{\text{HPD}}$, the underlying $O_{X/S}$-module $M$ is equivalent to the $O_{X/S}$-module $u^* M$ by Lemma 8.6 and $M$ is the value of $u^* M$ at the $p$-adic PD thickening $D \to R$, equipped with the canonical HPD stratification. A slight technical complication will be that we have to use the pr-topology instead of the Zariski topology, and etale morphisms lift uniquely over PD thickenings, but pr-morphisms lift only non-uniquely.

**Lemma 10.13.** Let $(B, \sigma)$ be a $p$-adically complete and $\mathbb{Z}_p$-flat ring with a Frobenius lift and let $S$ be a perfect ring. Each homomorphism $B/p \to S$ lifts to a unique homomorphism $B \to W(S)$ that commutes with $\sigma$.

**Proof.** There is a unique homomorphism $B \to W(B)$ which is a section of $W(B) \to B$ and commutes with $\sigma$; see [Laz, VII Prop. 4.12]. The composition $B \to W(B) \to W(B/p) \to W(S)$ is the desired lift. It is unique because the arrow $B \to W(B/p)$ is functorial in $(B, \sigma)$.

37
Proof of Proposition [10.17]. One reduces to the case that $R$ is an $\mathbb{F}_p$-algebra because on both sides, lifts from $R_0$ to $R$ correspond to lifts of the Hodge filtration. First, we construct a functor

$$F : \text{Win}(D)^{\text{HPD}} \to \text{DD}(X).$$

Let $(M, \varepsilon) \in \text{Win}(D)^{\text{HPD}}$. Let $R \to R'$ be a ring homomorphism with semiperfect $R'$. Assume that $S \to R'$ is a surjective ring homomorphism where $S$ is perfect, and assume that $g_0 : A/p \to S$ lifts $A/p \to R \to R'$. Such pairs $(S, g_0)$ exist, for example $S = R' \flat = \lim_{\leftarrow} (R', \phi)$ allows a lift $g_0 : A/p \to S$ using that $A/p$ has a $p$-basis. If $(S, g_0)$ is given, $g_0$ lifts to a unique homomorphism $g_1 : A \to W(S)$ that commutes with $\sigma$ by Lemma [10.13] and $g_1$ induces a frame homomorphism

$$g : D \to A_{\text{cris}}(R')$$

by Lemmas [10.1] and [4.1]. We define an $A_{\text{cris}}(R')$-window $M_{R'}$ by $M_{R'} = g^* M$, and verify that this is independent of $(S, g_0)$ by the usual argument: Let $(S', g'_0)$ be another choice of $(S, g_0)$, and $g' : D \to A_{\text{cris}}(R')$ the corresponding frame homomorphism. Then

$$h_0 = g_0 \otimes g'_0 : A(1)/p \to S \otimes S_1$$

gives a homomorphism of frames

$$h : D(1) \to A_{\text{cris}}(R')$$

with $h \circ p_0 = g$ and $h \circ p_1 = g'$. The isomorphism $\varepsilon : p_0^* M \cong p_1^* M$ induces an isomorphism $g^* M \cong g'^* M$, which proves independence of $(S, g_0)$.

The construction of $M_{R'}$ is functorial in $R'$. By Proposition [8.13], $M_{R'}$ corresponds to a divided Dieudonné crystal $M_{X'}$ over $X' = \text{Spec} R'$. Since affine semiperfect $X$-schemes form a base of the pr-topology of $X$, the system $(M_{X'})_{X'}$ descends to a unique divided Dieudonné crystal $M$ over $X$, and the functor $F$ is defined by

$$F(M, \varepsilon) = M.$$ 

Next we will define a functor

$$G : \text{DD}(X) \to \text{Win}(D)^{\text{HPD}}.$$ 

We need a number of rings in order to use descent. Let $A^{(1)}$ be the $p$-adic completion of $\lim(A, \sigma)$. Then $A^{(1)}/p$ is the perfection of $A/p$, and $A^{(1)} = W(A^{(1)}/p)$. Let $R^{(1)} = R \otimes_A A^{(1)}$. Since $A$ has a $p$-basis, the morphisms $A/p^r \to A^{(1)}/p^r$ and $R \to R^{(1)}$ are faithfully flat. For $n \geq 0$ let

$$A^{(n)} = A^{(1)} \otimes_A \ldots \otimes_A A^{(1)}.$$
with \( n \) factors, \( R^{(n)} = R \otimes_A A^{(n)} \), and
\[
D^{(n)} = D_\gamma(A^{(n)} \to R^{(n)})^\wedge.
\]
Then \( D^{(n)} = D \hat{\otimes}_A A^{(n)} \) since \( A/p^r \to A^{(n)}/p^r \) is flat, hence
\[
D^{(n)} = D^{(1)} \hat{\otimes} \ldots \hat{\otimes} D^{(1)}
\]
with \( n \) factors, and similarly
\[
R^{(n)} = R^{(1)} \otimes \ldots \otimes R^{(1)}
\]
with \( n \) factors. As a variant, for \( n \geq 1 \) we also consider the rings
\[
\tilde{A}^{(n)} = A^{(1)} \hat{\otimes}_{\mathbb{Z}_p} \ldots \hat{\otimes}_{\mathbb{Z}_p} A^{(1)}
\]
with \( n \) factors. The ring \( \tilde{A}^{(n)}/p \) is perfect, hence
\[
A_{\text{cris}}(R^{(n)}) = D_\gamma(\tilde{A}^{(n)} \to R^{(n)})^\wedge
\]
by Lemma \[4.1\]. The rings \( \tilde{A}^{(n)} \) and \( A^{(n)} \) carry a natural Frobenius lift \( \sigma \) induced by \( \sigma : A \to A \). The projection \( \tilde{A}^{(n)} \to A^{(n)} \) commutes with \( \sigma \) and thus extends to a frame homomorphism
\[
h^{(n)} : A_{\text{cris}}(R^{(n)}) \to D^{(n)}
\]
by Lemma \[10.1\]. Thus we have the following commutative diagram of frames, where \( h^{(1)} \) is an isomorphism since \( \tilde{A}^{(1)} = A^{(1)} \).
\[
\begin{array}{c}
\tilde{A}_{\text{cris}}(R^{(1)}) \cong A_{\text{cris}}(R^{(2)}) = A_{\text{cris}}(R^{(3)}) \\
\downarrow h^{(1)} \downarrow h^{(2)} \downarrow h^{(3)} \\
D = D^{(0)} \longrightarrow D^{(1)} = D^{(2)} = D^{(3)}
\end{array}
\]
Each homomorphism \( \alpha : D^{(n)} \to D^{(n+1)} \) that appears in the lower row of this diagram induces an isomorphism
\[
D^{(n)} \otimes_{D^{(n)}} D^{(n+1)} \cong D^{(n+1)}
\]
where \( \otimes \) is taken component-wise, and consequently the base change of windows under \( \alpha \) is given by \( \alpha^*M = M \otimes_{D^{(n)}} D^{(n+1)} \).

Now let \( \mathcal{M} \in \text{DD}(X) \). For \( n \geq 1 \), the base change of \( \mathcal{M} \) to \( \text{Spec } R^{(n)} \) corresponds to an \( \tilde{A}_{\text{cris}}(R^{(n)}) \)-window \( M_{\tilde{R}^{(n)}} \) by Proposition \[8.13\] which gives a \( D^{(n)} \)-window \( \mathcal{M}^{(n)} = (h^{(n)})^*M_{\tilde{R}^{(n)}} \). These \( \mathcal{M}^{(n)} \) for \( 1 \leq n \leq 3 \) define a descent datum on \( \mathcal{M}^{(1)} \), which thus descends to a \( D \)-window \( M \) by faithfully flat descent applied to the components of \( \mathcal{M}^{(1)} \).
For the same $\mathcal{M}$, if we take $A(m) \to R$ with $m \geq 0$ instead of $A \to R$, the same construction gives a $D(m)$-window $\mathcal{M}(m)$. This will involve obvious frames $D(m)\to D(m')$ given by the cosimplicial structure, $\mathcal{M}(m')$ is the base change of $\mathcal{M}(m)$ in a compatible way. This gives a HPD stratification $\varepsilon$ on $\mathcal{M} = \mathcal{M}(0)$, and we can define the functor $G$ by

$$G(\mathcal{M}) = (\mathcal{M}, \varepsilon).$$

Assume that $F(\mathcal{M}, \varepsilon) = \mathcal{M}$ and $G(\mathcal{M}) = (\mathcal{N}, \varepsilon)$. Then the $\mathcal{M}(n)$ used in the construction of $G(\mathcal{M})$ are the base change of $\mathcal{M}$ under the frame homomorphisms $D \to A_{\text{cris}}(R(n))$ induced by the homomorphisms $A \to \tilde{A}(n)$ defined by the choice of one component; this is independent of the choice using the HPD stratification $\varepsilon$ as explained above. It follows that $\mathcal{N} \cong \mathcal{M}$, and one verifies that this isomorphism preserves the stratifications, thus $G \circ F \cong \text{id}$. It remains to show that $G$ is fully faithful. Assume that $G(\mathcal{M}) = (\mathcal{M}, \varepsilon)$ and $G(\mathcal{N}) = (\mathcal{N}, \varepsilon)$. We have a commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \text{Hom}(\mathcal{M}, \mathcal{N}) & \longrightarrow & \text{Hom}(\mathcal{M}_{R(1)}, \mathcal{N}_{R(1)}) & \longrightarrow & \text{Hom}(\mathcal{M}_{R(2)}, \mathcal{N}_{R(2)}) \\
& & G & \longrightarrow & \longrightarrow & \longrightarrow & \longrightarrow \\
0 & \longrightarrow & \text{Hom}(\mathcal{M}, \mathcal{N}) & \longrightarrow & \text{Hom}(\mathcal{M}(1), \mathcal{N}(1)) & \longrightarrow & \text{Hom}(\mathcal{M}(2), \mathcal{N}(2))
\end{array}
$$

Clearly the arrow labelled $G$ is injective. We have to show that an $f \in \text{Hom}(\mathcal{M}, \mathcal{N})$ which commutes with the HPD stratifications maps to zero in $\text{Hom}(\mathcal{M}_{R(2)}, \mathcal{N}_{R(2)})$. This condition depends only on the module homomorphism $f : M \to N$. Since the proposition holds for finite locally free modules in place of windows, the assertion follows.

**Corollary 10.14.** Let $R$ be a ring in which $p$ is nilpotent. For a presentation $A \to R$ with a Frobenius lift $\sigma : A \to A$ and the associated frame $D$ there is an equivalence $\text{DD}(\text{Spec } R) \cong \text{Win}(D)^\nabla$ between divided Dieudonné crystals over $X$ and windows over $D$ with a connection.

**Proof.** Propositions 10.6 and 10.11.

Using Proposition 9.1, Theorem 9.2 and Corollary 9.4 it follows:

**Corollary 10.15.** There is a functor $\text{BT}(\text{Spec } R) \to \text{Win}(D)^\nabla$, which is an equivalence if $R/p$ is $F$-finite and $F$-nilpotent and if $p \geq 3$ or $pR = 0$. 

## 11 Divided Dieudonné crystals and displays

Finally we mention a link between divided Dieudonné crystals and the displays of [Zi1] (called $3n$-displays in loc.cit.). Let $X$ be a scheme on which $p$
is locally nilpotent. As earlier we consider \( X_{pr} \), the category of pr-sheaves on the category of \( X \)-schemes. There is a frame

\[
\mathcal{W}(\mathcal{O}_X) = (W(\mathcal{O}_X), I(\mathcal{O}_X), \mathcal{O}_X, \sigma, \sigma_1)
\]

in \( X_{pr} \) whose value in \( \text{Spec } R \to X \) is the Witt frame \( \mathcal{W}(R) \) of Example 5.5. A window over \( \mathcal{W}(\mathcal{O}_X) \) will be called a display over \( X \), and \( \text{Disp}(X) \) denotes the category of displays over \( X \). A display over \( X = \text{Spec } R \) is a display over \( R \) in the usual sense, i.e. a window over \( \mathcal{W}(R) \), by faithfully flat descent of displays; see [Zi1, Theorem 37]. By [La3] there is a contravariant functor

\[
\Phi_X : \text{BT}(X) \to \text{Disp}(X),
\]

which restricts to an equivalence between formal (resp. unipotent) \( p \)-divisible groups and \( F \)-nilpotent (resp. \( V \)-nilpotent) displays. In fact, the functor considered in loc.cit. is covariant, but we get a contravariant functor by taking the dual on either side.

**Lemma 11.1.** There is a natural frame homomorphism \( \pi : \mathcal{O}^{\text{cris}}_X \to \mathcal{W}(\mathcal{O}_X) \) over the identity of \( \mathcal{O}_X \).

**Proof.** It suffices to define for each \( \text{Spec } R \to X \) with semiperfect \( R \) a frame homomorphism \( \pi : \mathcal{A}^{\text{cris}}(R) \to \mathcal{W}(R) \). The ring homomorphism \( \pi \) is the unique homomorphism \( \mathcal{A}^{\text{cris}}(R) \to \mathcal{W}(R) \) of \( p \)-adic PD thickenings of \( R \). One verifies that \( \pi \) is a frame homomorphism, using that for \( a \in J = \text{Ker}(R' \to R) \), the elements \( [a]^{[n]} \) lie in the kernel of \( \mathcal{A}^{\text{cris}}(R) \to \mathcal{W}(R) \), and that \( \sigma_1([a]^{[n]}) = \left( \frac{p^n!}{n!} \right) [a]^{[pn]} \).

We get a sequence of functors

\[
\text{BT}(R) \xrightarrow{DD_X} \text{DD}(X) \xrightarrow{\pi^*} \text{Disp}(X).
\]

The composition coincides with the functor \( \Phi_X \) by the uniqueness properties of \( \Phi_X \) as in [La3, Proposition 2.1].

A window \( M \) over a frame \( \mathcal{A} \) (or over a frame \( \mathcal{A} \) in a topos) is called \( F \)-nilpotent if the endomorphism \( \Phi : M \to M \) is nilpotent on \( M/pM \) (or locally nilpotent). We denote by \( \text{Win}(\mathcal{A})_{\text{nil}} \subseteq \text{Win}(\mathcal{A}) \) the full subcategory of all \( F \)-nilpotent windows.

**Proposition 11.2.** A frame homomorphism \( \alpha : \mathcal{A}' \to \mathcal{A} \) such that \( \mathcal{A}' \to A \) is surjective and \( R' \to R \) is bijective induces an equivalence of \( F \)-nilpotent windows.

**Proof.** We define \( B_n \) as in the proof of Proposition 5.6. The frame homomorphism \( B_n \to \mathcal{A} \) with kernel \( \overline{N} \) induces an equivalence of \( F \)-nilpotent windows by [La1, Theorem 10.3], applied to the sequence of ideals \( \overline{N} \supseteq p\overline{N} \supseteq \ldots \supseteq p^n\overline{N} = 0 \), and the assertion follows by [La1, Lemma 2.11].
The above functors restrict to functors

$$\text{BT}(R)_f \xrightarrow{\text{DD}_X} \text{DD}(X)_{\text{nil}} \xrightarrow{\pi^*} \text{Disp}(X)_{\text{nil}}$$

between the categories of formal $p$-divisible groups and $F$-nilpotent divided Dieudonné crystals or displays. Here the composite functor is an equivalence by [La3, Theorem 5.1].

**Proposition 11.3.** The functor $\text{DD}(X)_{\text{nil}} \xrightarrow{\pi^*} \text{Disp}(X)_{\text{nil}}$ is an equivalence.

**Proof.** By pr-descent it suffices to show that for each semiperfect ring $R$ the frame homomorphism $\pi : A_{\text{cris}}(R) \rightarrow W(R)$ induces an equivalence of $F$-nilpotent windows. This follows from Proposition 11.2.

As a consequence, $\text{DD}_X : \text{BT}(R)_f \rightarrow \text{DD}(X)_{\text{nil}}$ is an equivalence as well. This is more general than the restriction of Theorem 9.2 to formal $p$-divisible groups because there are no finiteness conditions on $X$.

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