THE REVERSE MATHEMATICS OF WQOS AND BQOS

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ABSTRACT. In this paper we survey wqo and bqo theory from the reverse mathematics perspective. We consider both elementary results (such as the equivalence of different definitions of the concepts, and basic closure properties) and more advanced theorems. The classification from the reverse mathematics viewpoint of both kinds of results provides interesting challenges, and we cover also recent advances on some long standing open problems.

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This paper is an update of [Mar05], which was written in 2000 and documented the state of the research about the reverse mathematics of statements dealing with wqos and bqos at the turn of the century. Since then, new work on the subject has been carried out and we describe it here. We however include also the results already covered by [Mar05], attempting to cover exhaustively the topic. We also highlight some open problems in the area.

In Section 1 we give a brief introduction to reverse mathematics for the reader whose interest in wqos and bqos originates elsewhere. The readers familiar with this research program can safely skip this section. In Section 2 we compare different characterizations of wqos and study their closure under basic operations, such as subset, product and intersection. Here even seemingly trivial properties provide interesting challenges for the reverse mathematician. The study of characterizations and closure under simple operations is repeated in Section 3 for bqos; the strength of some statements go all the way up to $\text{ATR}_0$ and apparently simple statements such as “3 is bqo” have escaped classification so far. In Section 4 we consider the minimality arguments which are one of the main proof techniques of the subject. Section 5 looks at structural results, such as the theorem by de Jongh and Parikh asserting the existence of a maximal linear extension of a wqo. Section 6 deals with what we might call the major results of wqo and bqo theory, such as Higman’s, Kruskal’s and Nash-Williams’ theorems, the minor graph theorem and Fraïssé’s...
conjecture. We end the paper with a section dealing with results about a topological version of wqos.

1. Reverse mathematics

Reverse mathematics is a wide ranging research program in the foundations of mathematics. The main goal of the program is to give mathematical support to statements such as “Theorem A is stronger than Theorem B” or “Theorems C and D are equivalent”. If taken literally the first statement does not make sense: since A and B are both true, they are logically equivalent. By the same token, the second statement is trivially true, and thus carries no useful information. However a clarification of these statements is possible by finding out precisely the minimal axioms needed to prove B and showing that they do not suffice to prove A, and by showing that these minimal axioms coincide for C and D. We are thus interested in proving equivalences between theorems and axioms, yielding equivalences and nonequivalences between different theorems, over a weak base theory.

Although we can label “reverse mathematics” any study of this kind (including the study of different forms of the axiom of choice over the base theory ZF), the term is usually restricted to the setting of subsystems of second order arithmetic. The language $\mathcal{L}_2$ of second order arithmetic has variables for natural numbers and variables for sets of natural numbers, constant symbols 0 and 1, binary function symbols for addition and product of natural numbers, symbols for equality and the order relation on the natural numbers and for membership between a natural number and a set. A model for $\mathcal{L}_2$ consists of a first order part (an interpretation for the natural numbers $\mathbb{N}$ equipped with $+\cdot$ and $\leq$) and a second order part consisting of a collection of subsets of $\mathbb{N}$. When the first order part is standard we speak of an $\omega$-model and we can identify the model with the subset of $\mathcal{P}(\omega)$ that constitutes its second order part.

Second order arithmetic is the $\mathcal{L}_2$-theory with classical logic consisting of the axioms stating that the natural numbers are a commutative ordered semiring with identity, the induction scheme for arbitrary formulas, and the comprehension scheme for sets of natural numbers defined by arbitrary formulas.

Hermann Weyl [Wey18] and Hilbert and Bernays [HB68, HB70] already noticed in their work on the foundations of mathematics that $\mathcal{L}_2$ is rich enough to express, using appropriate codings, significant parts of mathematical practice, and that many mathematical theorems are provable in (fragments of) second order arithmetic. Actually Weyl used a theory similar to what we now denote by $\text{ACA}_0^+$ (a slight strengthening of $\text{ACA}_0$, to be described below). Recently Dean and Walsh [DW17] traced the history of subsystems of second order arithmetic leading to [Fri75], where Harvey Friedman started the systematic search for the axioms that are sufficient and necessary to prove theorems of ordinary, not set-theoretic, mathematics. One of Friedman’s main early discoveries was that (in his words) “When the theorem is proved from the right axioms, the axioms can be proved from the theorem”. Friedman also highlighted the role of set-existence axioms, and this soon led to restricting the induction principles allowed in the various systems. The base system $\text{RCA}_0$ and the now well-known $\text{WKL}_0$, $\text{ACA}_0$, $\text{ATR}_0$, and $\Pi^1_1$-$\text{CA}_0$, were introduced in [Fri76]. Today, most of reverse mathematics research compares the strength of mathematical theorems by establishing equivalences, implications and nonimplications over $\text{RCA}_0$.

To describe $\text{RCA}_0$ and the other systems used in reverse mathematics let us also recall that formulas of $\mathcal{L}_2$ are classified in the usual hierarchies: those with no set quantifiers and only bounded number quantifiers are $\Delta^0_0$, while counting the number of alternating unbounded number quantifiers we obtain the classification of
all arithmetical (= without set quantifiers) formulas as $\Sigma^0_n$ and $\Pi^0_n$ formulas (one uses $\Sigma$ or $\Pi$ depending on the type of the first quantifier in the formula, existential in the former, universal in the latter). Formulas with set quantifiers in front of an arithmetical formula are classified by counting their alternations as $\Sigma^i_n$ and $\Pi^i_n$.

A formula is $\Delta^i_n$ in a given theory if it is equivalent in that theory both to a $\Sigma^i_n$ formula and to a $\Pi^i_n$ formula.

In RCA$_0$ the induction scheme and the comprehension scheme of second order arithmetic are restricted respectively to $\Sigma^0_1$ and $\Delta^0_1$ formulas. RCA$_0$ is strong enough to prove some basic results about many mathematical structures, but too weak for many others. The $\omega$-models of RCA$_0$ are the Turing ideals: subsets of $\mathcal{P}(\omega)$ closed under join and Turing reducibility. The minimal $\omega$-model of RCA$_0$ consists of the computable sets and is usually denoted by REC.

If a theorem $T$ is expressible in $L_2$ but unprovable in RCA$_0$, the reverse mathematician asks the question: what is the weakest axiom we can add to RCA$_0$ to obtain a theory that proves $T$? In principle, we could expect that this question has a different answer for each $T$, but already Friedman noticed that this is not the case. In fact, most theorems of ordinary mathematics expressible in $L_2$ are either provable in RCA$_0$ or equivalent over RCA$_0$ to one of the following four subsystems of second order arithmetic, listed in order of increasing strength: WKL$_0$, ACA$_0$, ATR$_0$, and $\Pi^1_1$-CA$_0$. This is witnessed in Steve Simpson’s monograph [Sim09] and summarized by the Big Five terminology. We thus obtain a neat picture where theories belonging to quite different areas of mathematics are classified in five levels, roughly corresponding to the mathematical principles used in their proofs. RCA$_0$ corresponds to “computable mathematics”, WKL$_0$ embodies a compactness principle, ACA$_0$ is linked to sequential compactness, ATR$_0$ allows for transfinite arguments, $\Pi^1_1$-CA$_0$ includes impredicative principles.

To obtain WKL$_0$ we add to RCA$_0$ the statement of Weak König’s Lemma, i.e., every infinite binary tree has a path, which is essentially the compactness of Cantor space. An equivalent statement, intuitively showing that WKL$_0$ is stronger than RCA$_0$ (a rigorous proof needs simple arguments from model theory and computability theory), is $\Sigma^0_1$-separation: if $\varphi(n)$ and $\psi(n)$ are $\Sigma^0_1$-formulas such that $\forall n \neg(\varphi(n) \land \psi(n))$ then there exists a set $X$ such that $\varphi(n) \iff n \in X$ and $\psi(n) \iff n \notin X$ for all $n$. WKL$_0$ and RCA$_0$ have the same consistency strength of Primitive Recursive Arithmetic, and are thus proof-theoretically fairly weak. Nevertheless, WKL$_0$ proves (and often turns out to be equivalent to) a substantial amount of classical mathematical theorems, including many results about real-valued functions and countable rings and fields, basic Banach space facts, etc. The $\omega$-models of WKL$_0$ are the Scott ideals, and their intersection consists of the computable sets.

ACA$_0$ is obtained from RCA$_0$ by extending the comprehension scheme to all arithmetical formulas. The statements without set variables provable in ACA$_0$ coincide exactly with the theorems of Peano Arithmetic, so that in particular the consistency strength of the two theories is the same. Within ACA$_0$ one can develop a fairly extensive theory of continuous functions, using the completeness of the real line as an important tool. ACA$_0$ proves (and often turns out to be equivalent to) also many basic theorems about countable fields, rings, and vector spaces. For example, ACA$_0$ is equivalent, over RCA$_0$, to the Bolzano-Weierstrass theorem on the real line. The $\omega$-models of ACA$_0$ are the Turing ideals closed under jumps, so that the minimal $\omega$-model of ACA$_0$ consists of all arithmetical sets.

ATR$_0$ is the strengthening of RCA$_0$ (and ACA$_0$) obtained by allowing to iterate arithmetical comprehension along any well-order. It can be shown [Sim09, Theorem V.5.1] that, over RCA$_0$, ATR$_0$ is equivalent to $\Sigma^1_1$-separation, which is exactly as $\Sigma^0_1$-separation but with $\Sigma^1_1$ formulas allowed. This is a theory at the outer limits...
of predicativism and proves (and often turns out to be equivalent to) many basic statements of descriptive set theory but also some results from advanced algebra, such as Ulm’s theorem.

$\Pi^1_1 \text{-CA}_0$ is the strongest of the big five systems, and is obtained from RCA$_0$ by extending the comprehension scheme to $\Pi^1_1$ formulas. Also this axiom scheme is equivalent to many results, including some from descriptive set theory, Banach space theory and advanced algebra, such as the structure theorem for countable Abelian groups.

In recent years there has been a change in the reverse mathematics main focus: following Seetapun’s breakthrough result that Ramsey theorem for pairs is not equivalent to any of the Big Five systems [SS95], a plethora of statements, mostly in countable combinatorics, have been shown to form a rich and complex web of implications and nonimplications. The first paper featuring complex and non-linear diagrams representing the relationships between statements of second order arithmetics appears to be [HS07]. Nowadays diagrams of this kind are a common feature of reverse mathematics papers. This leads to the zoo of reverse mathematics, a terminology coined by Damir Dzhafarov when he designed “a program to help organize relations among various mathematical principles, particularly those that fail to be equivalent to any of the big five subsystems of second-order arithmetics” (Hirschfeldt’s monograph [Hir15] highlights this new focus of the reverse mathematics program).

Many elements of the zoo are connected to Ramsey theorem. By RT$_k^\ell$ we denote Ramsey theorem for sets of size $k$ and $\ell$ colors: for every coloring $c : [N]^k \to \ell$ (here $[X]^k$ is the set of all subsets of $X$ with exact $k$ elements, and $\ell$ is the set $\{0, \ldots, \ell - 1\}$) there exists an infinite homogenous set $H$, i.e., such that for some $i < \ell$ we have $c(s) = i$ for every $s \in [H]^k$. RT$_{\infty}^\ell$ is $\forall \ell$ RT$_k^\ell$. A classic result is that RT$_{\infty}^\ell$ is equivalent to ACA$_0$ over RCA$_0$ when $k \geq 3$ and $\ell \geq 2$ (see [Sim09, §III.7]). On the other hand, building on Seetapun’s result with the essential new step provided by Liu [Liu12], we now know that RT$_2^3$ and RT$_{\infty}^\omega$ are both incomparable with WKL$_0$ (see [Hir15, §6.2 and Appendix]). For any fixed $\ell$ the infinite pigeonhole principle for $\ell$ colors RT$_1^\ell$ is provable in RCA$_0$. On the other hand the full infinite pigeonhole principle RT$_{\infty}^1$ is not provable in RCA$_0$ and not even in WKL$_0$; in fact it is equivalent over RCA$_0$ to the principle known as $\Sigma^0_2$-bounding, which is intermediate in strength between $\Sigma^0_3$-induction and $\Sigma^0_2$-induction.

Two of the earliest examples of the zoo phenomenon play a significant role with respect to statements dealing with wqos. Both statements are fairly simple consequences of RT$_2^3$. CAC is the statement that any infinite partial order contains either an infinite antichain or an infinite chain, while ADS asserts that every infinite linear order has either an infinite ascending chain or an infinite descending chain. Hirschfeldt and Shore [HS07] showed that RT$_2^3$ is properly stronger than CAC, which in turn implies ADS. They also showed that none of these principles imply WKL$_0$ over RCA$_0$. The fact that CAC is properly stronger than ADS was first proved by Lerman, Solomon, and Towsnear [LST13], and then a simpler proof by Patey [Pat16]. These results support the idea that RT$_2^3$, in contrast to the big five, is not robust (Montalbán [Mon11] informally defined a theory to be robust “if it is equivalent to small perturbations of itself”).

Wqo and bqo theory represents an area of combinatorics which has always interested logicians. From the viewpoint of reverse mathematics, one of the reasons for this interest stems from the fact that some important results about wqos and bqos appear to use axioms that are within the realm of second order arithmetic, yet are much stronger than those necessary to develop other areas of ordinary mathematics (as defined in the introduction of [Sim09]). We will see that results about wqo and
bqo belong to both facets of reverse mathematics: some statements fit neatly in the big five picture, while some others provide examples of the zoo.

When dealing with wqo and bqo theory, at first sight the limitations of the expressive power of second-order arithmetic compel us to consider only quasi-orders defined on countable sets. This is actually not a big restriction because a quasi-order is wqo (resp. bqo) if and only if each of its restrictions to a countable subset of its domain is wqo (resp. bqo). The limitation mentioned above must be adhered to when we quantify over the collection of all wqos (or bqos), typically in statements of the form “for every wqo ...” However we can also consider specific quasi-orders defined on uncountable sets (such as the powerset of a countable set, the collection of infinite sequences of elements of a countable set, or the set of all countable linear orders); statements about these (with a fixed quasi-order) being wqo or bqo can be expressed in a natural way in second-order arithmetic (see Definition 6.3 below).

We often use $\leq_N$ for the order relation given by the symbol $\leq$ in the language of second order arithmetic. This notation helps to emphasize when we are comparing elements of a quasi-order via the quasi-order relation and when we are comparing them via the underlying structure of arithmetic. We use this notation when the distinction between these orders is not immediately clear from the context.

As usual in the reverse mathematics literature, whenever we begin a definition or statement with the name of a subsystem of second order arithmetic in parenthesis we mean that the definition is given, or the statement proved, within that subsystem.

2. Characterizations and basic properties of wqos

**Definition 2.1** ($\text{RCA}_0$). A quasi-order is a pair $(Q, \preceq)$ such that $Q$ is a set and $\preceq$ is a transitive reflexive relation on $Q$.

When there is no danger of confusion we assume that $Q$ is always equipped with the quasi-order $\preceq$ and that $\preceq$ is always a quasi-order on the set $Q$. Thus in our statements we often mention only $\preceq$ or only $Q$.

Partial orders are natural examples of quasi-orders: a partial order is a quasi-order which also satisfies antisymmetry. We can transform a quasi-order $Q$ into a partial order using the equivalence relation defined by $x \sim y$ if and only if $x \preceq y$ and $y \preceq x$. The quotient structure $Q/\sim$ is naturally equipped with a partial order which can be formed using $\Delta^0_1$ comprehension in $\text{RCA}_0$ (it suffices to identify an equivalence class with its least member with respect to $\leq_N$).

Much of the standard terminology and notation for partial orders is used also when dealing with quasi-orders. For example, we write $x \perp y$ to indicate that $x$ and $y$ are incomparable under $\preceq$ and we write $x < y$ if $x \preceq y$ and $y \not\preceq x$.

**Definition 2.2** ($\text{RCA}_0$). A set $A \subseteq Q$ is an antichain if $x \perp y$ for all $x \neq y \in A$. A set $C \subseteq Q$ is a chain if $x \preceq y$ or $y \preceq x$ for all $x,y \in C$.

A set $I \subseteq Q$ is an initial interval if $y \preceq x$ for some $x \in I$. The definition of final interval is symmetric, with $x \preceq y$ for some $x \in I$.

**Definition 2.3** ($\text{RCA}_0$). A quasi-order $(Q, \preceq)$ is linear if $Q$ is a chain.

If $\preceq$ is a quasi-order on $Q$ and $\preceq_L$ is a linear quasi-order on $Q$, then we say $\preceq_L$ is a linear extension of $\preceq$ if for all $x,y \in Q$, $x \preceq y$ implies $x \preceq_L y$ and $x \sim_L y$ implies $x \sim y$.

Notice that (provably in $\text{RCA}_0$) if $Q$ is a linear quasi-order then $Q/\sim$ is a linear order. Moreover, if $\preceq_L$ is a linear extension of $\preceq$ then $x \sim y$ if and only if $x \sim_L y$ and therefore the linear extensions of a quasi-order $Q$ correspond exactly to the linear extensions of the partial order $Q/\sim$.

We can now give the official definition of wqo within $\text{RCA}_0$. 

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**Definition 2.4** ($\text{RCA}_0$). A quasi-order $(Q, \preceq)$ is a wqo if $\preceq_L$ is a linear order on $Q$.

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**Definition 2.5** ($\text{RCA}_0$). A quasi-order $(Q, \preceq)$ is a bqo if $\preceq_L$ is a linear order on $Q$ for all $\preceq_L$ is a linear extension of $\preceq$.

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**Theorem 2.6** ($\text{RCA}_0$). The collection of all wqos (bqos) is a set and forms a linear order for all $\preceq_L$ is a linear extension of $\preceq$.

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**Corollary 2.7** ($\text{RCA}_0$). The theory of wqos (bqos) is $\Pi^1_1$-complete.

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**Example 2.8**. The reverse mathematics big five picture, while some others provide examples of the zoo.
Definition 2.4 (RCA₀). Let ≤ be a quasi-order on \(Q\). \((Q, \leq)\) is \(wqo\) if for every map \(f : \mathbb{N} \to Q\) there exist \(m <_\mathbb{N} n\) such that \(f(m) \leq f(n)\).

Definition 2.5 (RCA₀). An infinite sequence of elements of \(Q\) is a function \(f : A \to Q\) where \(A \subseteq \mathbb{N}\) is infinite.

- \(f\) is ascending if \(f(n) < f(m)\) for all \(n, m \in A\) with \(n <_\mathbb{N} m\). Similarly, \(f\) is descending if \(f(m) < f(n)\) whenever \(n, m \in A\) are such that \(n <_\mathbb{N} m\).

- A well-order is a linear quasi-order with no infinite descending sequences.

We say that \(f\) is a good sequence (with respect to \(\leq\)) if there exist \(m, n \in A\) such that \(m <_\mathbb{N} n\) and \(f(m) \leq f(n)\); if this does not happen we say that \(f\) is bad.

The following characterization of \(wqo\) is immediate, and easy to prove within RCA₀ using the existence of the enumeration of the elements of an infinite subset of \(\mathbb{N}\) in increasing order:

Fact 2.6 (RCA₀). Let \((Q, \leq)\) be a quasi-order. The following are equivalent:

1. \(Q\) is \(wqo\);
2. every sequence of elements of \(Q\) is good with respect to \(\leq\).

Wqos can be characterized by several other statements about quasi-orders. The systematic investigation of the axioms needed to prove the equivalences between these characterizations was started by Cholak, Marcone, and Solomon in [CMS04].

Let us begin with the characterizations which are provable in RCA₀.

Lemma 2.7 (RCA₀). Let \((Q, \leq)\) be a quasi-order. The following are equivalent:

1. \(Q\) is \(wqo\);
2. \(Q\) has the finite basis property, i.e., for every \(X \subseteq Q\) there exists a finite \(F \subseteq X\) such that \(\forall x \in X \exists y \in F. y \leq x\);
3. there is no infinite sequence of initial segments of \(Q\) which is strictly decreasing with respect to inclusion;
4. there is no infinite sequence of final segments of \(Q\) which is strictly increasing with respect to inclusion.

The equivalence between \((i)\) and \((ii)\) was already noticed by Simpson (see [Sim88, Lemma 3.2], where the finite basis property is stated in terms of partial orders rather than quasi-orders: full details with the current definition are provided in [Mar05, Lemma 4.8]).

The equivalence between \((iii)\) and \((iv)\) is immediate by taking complements with respect to \(Q\). To show that \((i)\) implies \((iii)\) start from an infinite sequences \(\{I_n : n \in \mathbb{N}\}\) of initial segments of \(Q\) which is strictly decreasing with respect to inclusion and for every \(n\) let \(f(n)\) be the \(\leq_{\mathbb{N}}\) minimum element of \(I_n \setminus I_{n+1}\): \(f\) is a bad sequence. To prove that \((iii)\) implies \((i)\) let \(F\) be a bad sequence with domain \(\mathbb{N}\) and set \(I_n = \{x \in Q : \forall i \leq n f(i) \not\leq x\}\) and \(I_n = \{x \in \mathbb{N}\}\) is an infinite strictly decreasing sequence of initial segments of \(Q\).

We now consider the characterizations of the notion of \(wqo\) which turn out to be more interesting from the reverse mathematics viewpoint.

Definition 2.8 (RCA₀). Let \((Q, \leq)\) be a quasi-order:

- \(Q\) is \(wqo(set)\) if for every \(f : \mathbb{N} \to Q\) there is an infinite set \(A\) such that for all \(n, m \in A\), \(n <_\mathbb{N} m \rightarrow f(n) \leq f(m)\);
- \(Q\) is \(wqo(anti)\) if it has no infinite descending sequences and no infinite antichains;
- \(Q\) is \(wqo(anti)\) if every linear extension of \(\leq\) is a well-order.

RCA₀ proves quite easily some implications: every \(wqo(set)\) is \(wqo\), and every \(wqo\) is both \(wqo(anti)\) and \(wqo(anti)\). Cholak, Marcone, and Solomon showed that all other implications between these notions are not true in the \(\omega\)-model REC, and hence are not provable within RCA₀.
Theorem 2.9. The implications between the notions of wqo, wqo(set), wqo(anti) and wqo(ext) which are provable in RCA₀ are exactly the ones in the transitive closure of the diagram:

\[ \text{wqo(anti)} \] \[ \text{wqo(set)} \rightarrow \text{wqo} \] \[ \rightarrow \] \[ \text{wqo(ext)} \]

In fact the above diagram depicts the implications which hold in REC, and thus adding induction axioms to RCA₀ yields no other implications.

To show that every wqo implies wqo(set) fails in REC, it suffices to recall a classical construction (due to Denisov and Tennenbaum independently: see [Dow98]) of a computable linear order of order type \( \omega + \omega^* \) which does not have any infinite computable ascending or descending sequences.

Similarly, showing that REC does not satisfy that every wqo(ext) is wqo means building a computable partial order \((Q, \preceq)\) such that all its computable linear extensions are computably well-ordered (i.e., do not have infinite computable descending sequences) but there is a computable \( f : \mathbb{N} \rightarrow Q \) such that \( f(m) \npreceq f(n) \) for all \( m <_\mathbb{N} n \). In fact the partial order constructed in [CMS04, Theorem 3.21] using a finite injury construction is such that \( f(m) \perp f(n) \) for all \( m \neq n \), thus obtaining the stronger result that REC does not satisfy that every wqo(ext) is wqo(anti).

To show that wqo(anti) implies wqo does not hold in REC one needs to find a computable partial order \((Q, \preceq)\) with no computable infinite antichains and no computable infinite descending sequences but such that there exists a computable \( f : \mathbb{N} \rightarrow Q \) such that \( f(m) \npreceq f(n) \) for all \( m <_\mathbb{N} n \). The partial order built in [CMS04, Theorem 3.9] has the additional property of having a computable linear extension with a computable infinite descending sequence (see [CMS04, Corollary 3.10]). Hence REC does not satisfy that every wqo(anti) is wqo(ext).

One can improve the latter construction obtaining even more information. In fact, [CMS04, Theorem 3.11] shows that if \((X_i)_{i \in \mathbb{N}}\) is a sequence of uniformly \( \Delta^0_2 \), uniformly low sets there exists a computable partial order \((Q, \preceq)\), such that for all \( i \) no \( X_i \)-computable function lists an infinite antichain or an infinite descending sequence in \( Q \), but there exists a computable \( f : \mathbb{N} \rightarrow Q \) such that \( f(m) \npreceq f(n) \) for all \( m <_\mathbb{N} n \). Since for an appropriate choice of \((X_i)_{i \in \mathbb{N}}\) we have that the \( \omega \)-model \( \{ Y : \exists i (Y \preceq_T X_i) \} \) satisfies \( \text{WKL}_0 \), we obtain that \( \text{WKL}_0 \) does not prove that every wqo(anti) is wqo.

Further exploring the provability of the other implications in \( \text{WKL}_0 \), we notice that it is fairly easy to prove in \( \text{RCA}_0 \) that the statement that every wqo is wqo(set) implies \( \text{RT}^1_{<\infty} \) ([CMS04, Lemma 3.20]), and hence is not provable in \( \text{WKL}_0 \).

On the other hand, [CMS04, Theorem 3.17] shows that \( \text{WKL}_0 \) proves (using the fact, equivalent to \( \text{WKL}_0 \), that every acyclic relation is contained in a partial order) that every wqo(ext) is wqo. Putting the information mentioned above together we obtain the following picture regarding provability in \( \text{WKL}_0 \).

Theorem 2.10. The implications between the notions of wqo, wqo(set), wqo(anti) and wqo(ext) which are provable in \( \text{WKL}_0 \) are exactly the ones in the transitive
This leads to the following natural question, which has resisted any attempt so far.

**Question 2.11.** Consider the statements “every wqo(ext) is wqo” and “every wqo(set) is wqo(anti)”. Are they equivalent to WKL\(_0\) over RCA\(_0\)?

On the other hand, the statement “every wqo(anti) is wqo(set)” turns out to be equivalent to CAC over RCA\(_0\) ([CMS04, Lemma 3.3]). It follows that the statements “every wqo(anti) is wqo” and “every wqo is wqo(set)” are both provable from CAC.

**Theorem 2.12.** RCA\(_0\) + CAC proves the implications between the notions of wqo, wqo(set), wqo(anti) and wqo(ext) which are in the transitive closure of the diagram:

![Diagram showing the transitive closure of the implications between wqo, wqo(set), wqo(anti), and wqo(ext).](attachment:image)

The diagram of Theorem 2.12 is different from the ones of Theorems 2.9 and 2.10 in that it is unknown whether the missing implications can be proved in RCA\(_0\) + CAC.

**Question 2.13.** Does RCA\(_0\) + CAC proves “every wqo(ext) is wqo”?

Notice that a positive answer to Question 2.11 implies, since RCA\(_0\) + CAC does not prove WKL\(_0\), a negative answer to Question 2.13.

RCA\(_0\) easily proves that all well-orders and all finite quasi-orders are wqo (indeed for the latter fact the finite pigeonhole principle suffices). By Theorem 2.9 the same happens for wqo(anti) and wqo(ext). Regarding wqo(set) we have that, using the appropriate RT\(_1^3\), for any specific finite quasi-order RCA\(_0\) proves that the quasi-order is wqo(set). On the other hand, it is not difficult to see that, over RCA\(_0\), “every finite quasi-order is wqo(set)” is equivalent to RT\(_1^2\);\(\leq\infty\), while “every well-order is wqo(set)” is equivalent to ADS.

Wqos enjoy several basic closure properties. The study of these from the viewpoint of reverse mathematics was started in [Mar05] and [CMS04].

We first consider the basic property of closure under taking subsets. The proof of the following lemma is immediate.

**Lemma 2.14 (RCA\(_0\)).** Let \(\mathcal{P}\) be any of the properties wqo, wqo(anti) or wqo(set). If \((Q, \preceq)\) satisfies \(\mathcal{P}\) and \(R \subseteq Q\) then the restriction of \(\preceq\) to \(R\) satisfies \(\mathcal{P}\) as well.

If \(\mathcal{P}\) is wqo(ext) then the statement of Lemma 2.14 is slightly more difficult to prove, since the obvious proof of the reversal is based on the following fact: if \((Q, \preceq)\) is a partial order, \(R \subseteq Q, \preceq_L\) is a linear extension of the restriction of \(\preceq\) to \(R\), then there exists a linear extension of the whole \(\preceq\) which extends also \(\preceq_L\). WKL\(_0\) suffices to prove this statement, because we can consider \(\preceq \cup \preceq_L\), which is an acyclic relation, extend it to a partial order (here is the step using WKL\(_0\), see [CMS04, Lemma 3.16]), and then to a linear order (RCA\(_0\) suffices for this last step).
Question 2.15. Does RCA₀ suffice to prove that if \((Q, \preceq)\) is wqo(\text{ext}) and \(R \subseteq Q\) then the restriction of \(\preceq\) to \(R\) is also wqo(\text{ext})? Is this implication equivalent to WKL₀?

Let us now consider basic closure operations that involve two quasi-orders.

Definition 2.16 (RCA₀). If \(\preceq₁\) and \(\preceq₂\) are quasi-orders on \(Q₁\) and \(Q₂\) we may assume that \(Q₁ \cap Q₂ = \emptyset\) (or replace each \(Qᵢ\) by its isomorphic copy on \(Qᵢ \times \{i\}\)). We can define the sum quasi-order and the disjoint union quasi-order on \(Q₁ \cup Q₂\) (denoted by \(Q₁ + Q₂\) and by \(Q₁ \cup Q₂\) respectively) by
\[
\begin{align*}
x \preceq₁⁺ y & \iff (x \in Q₁ \land y \in Q₂) \lor (x, y \in Q₁ \land x \preceq₁ y) \lor (x, y \in Q₂ \land x \preceq₂ y); 
x \preceq₁⁻ y & \iff (x, y \in Q₁ \land x \preceq₁ y) \lor (x, y \in Q₂ \land x \preceq₂ y).
\end{align*}
\]
The product quasi-order on \(Q₁ \times Q₂\) is defined by
\[\big(x₁, x₂\big) \preceq \big(y₁, y₂\big) \iff x₁ \preceq₁ y₁ \land x₂ \preceq₂ y₂.\]
Moreover if \(\preceq₁\) and \(\preceq₂\) are quasi-orders on the same set \(Q\) then the intersection quasi-order on \(Q\) is defined by
\[x \preceq₁∩₁ y \iff x \preceq₁ y \land x \preceq₂ y.\]

The following lemma follows easily from the provability in RCA₀ of RT₁².

Lemma 2.17 (RCA₀). Let \(P\) be any of the properties wqo, wqo(\text{ext}), wqo(\text{anti}) or wqo(\text{set}). If \(Q₁\) and \(Q₂\) satisfy \(P\) then \(Q₁ + Q₂\) and \(Q₁ \cup Q₂\) satisfy \(P\) with respect to the sum and disjoint union quasi-orders.

The next lemma was first noticed in [Mar05] for wqos, and then extended to the other notions in [CMS04].

Lemma 2.18 (RCA₀). Let \(P\) be any of the properties wqo, wqo(\text{anti}) or wqo(\text{set}). The following are equivalent:
\[(i)\] if \(Q\) satisfies \(P\) with respect to the quasi-orders \(\preceq₁\) and \(\preceq₂\) then \(Q\) satisfies \(P\) with respect to the intersection quasi-order;
\[(ii)\] if \(Q₁\) and \(Q₂\) satisfy \(P\) then \(Q₁ \times Q₂\) satisfies \(P\) with respect to the product quasi-order.

The proof of \((i)\) implies \((ii)\) is based on the fact that products can be realized as intersections and works for wqo(\text{ext}) as well. The proof of \((ii)\) implies \((i)\) uses the fact that intersections can be viewed as subsets of products, and thus employs Lemma 2.14. In [CMS04] it is claimed that Lemma 2.18 holds also when \(P\) is wqo(\text{ext}), but it seems that this might depend on the answer to Question 2.15.

Question 2.19. Let \(P\) be wqo(\text{ext}). Does RCA₀ suffice to prove that \((ii)\) of Lemma 2.18 implies \((i)\)? Is this implication equivalent to WKL₀?

The following results are from [CMS04].

Lemma 2.20 (RCA₀). Let \(P\) be any of the properties wqo, wqo(\text{ext}), wqo(\text{anti}) or wqo(\text{set}).
\[
\bullet \text{If } Q \text{ is wqo(\text{set}) with respect to the quasi-orders } \preceq₁ \text{ and } \preceq₂ \text{ then } Q \text{ satisfies } P \text{ with respect to the intersection quasi-order;}
\bullet \text{if } Q₁ \text{ and } Q₂ \text{ are wqo(\text{set}) then } Q₁ \times Q₂ \text{ satisfies } P \text{ with respect to the product quasi-order.}
\]

Theorem 2.21. Let \(P₁\) be any of the properties wqo, wqo(\text{ext}) and wqo(\text{anti}). Let \(P₂\) be any of the properties wqo, wqo(\text{set}), wqo(\text{ext}) and wqo(\text{anti}).
\[
\bullet \text{WKL₀ does not prove that if } Q \text{ satisfies } P₁ \text{ with respect to the quasi-orders } \preceq₁ \text{ and } \preceq₂ \text{ then } Q \text{ satisfies } P₂ \text{ with respect to the intersection quasi-order;}
\]
• WKL₀ does not prove that if \( Q_1 \) and \( Q_2 \) satisfy \( \mathcal{P}_1 \) then \( Q_1 \times Q_2 \) satisfies \( \mathcal{P}_2 \) with respect to the product quasi-order.

All instances of Theorem 2.21 follow easily (using Lemma 2.18 and Theorem 2.10 from Theorem 4.3 of [CMS04]). To state this theorem fix an \( \omega \)-model \( \mathcal{M} \) of WKL₀ which consists of the sets Turing reducible to a member of a sequence of uniformly \( \Delta^0_2 \), uniformly low sets. The theorem asserts the existence of computable partial orders \( \preceq_0 \) and \( \preceq_1 \) which are wqo in \( \mathcal{M} \) (i.e., \( \mathcal{M} \) contains no bad sequence with respect to either \( \preceq_0 \) or \( \preceq_1 \)) and such that \( \preceq_0 \cap \preceq_1 \) is an infinite antichain (so that the intersection is not wqo(anti)). The construction of \( \preceq_0 \) and \( \preceq_1 \) is by a finite injury argument.

Theorem 2.22 and Lemma 2.21 imply that RCA₀ + CAC proves the closure of wqos under product. On the other hand Frittaion, Marcone, and Shafer pointed out that this statement implies ADS and asked for a classification. Recently, Henry Towsner [How16] gave a typical zoo answer to this question by proving the following theorem.

**Theorem 2.22.** WKL₀ does not prove that the closure of wqos under product implies CAC, nor that ADS implies the closure of wqos under product.

Towsner starts by translating the statement in Ramsey-theoretic terms. Given the coloring \( c : [\mathbb{N}]^2 \to \ell \) we say that color \( i \) is transitive if \( c(k_0, k_2) = i \) whenever \( c(k_0, k_1) = c(k_1, k_2) = i \) for some \( k_1 \) satisfying \( k_0 < k_1 < k_2 \). Hirschfeldt and Shore [HS07] noticed that ADS is equivalent to the restriction of \( RT^2_2 \) to colorings such that both colors are transitive, while CAC is equivalent to the restriction of \( RT^2_2 \) to colorings with one transitive color. Towsner notices that the closure of wqos under product is equivalent to the following intermediate statement: if \( c : [\mathbb{N}]^2 \to 3 \) is such that colors 0 and 1 are transitive then there exists an infinite set \( H \) such that for some \( i < 2 \) we have \( c(s) \neq i \) for every \( s \in [H]^k \) (i.e., \( H \) avoids one of the transitive colors). Then he proceeds to construct Scott ideals with the appropriate properties: the first satisfies the above transitive color avoiding statement but not the restriction of \( RT^2_2 \) to colorings with one transitive color; the second satisfies for all \( \ell \) the restriction of \( RT^2_2 \) to colorings such that all color are transitive, but fails to satisfy the statement equivalent to the closure of wqos under product.

Special instances of the closure of wqos under product have been studied by Simpson [Sim88].

**Theorem 2.23** (RCA₀). Let \( \omega \) denote the order \( (\mathbb{N}, \leq_N) \). Then
1. the product of two copies of \( \omega \) is wqo with respect to the product quasi-order.
2. the following are equivalent:
   (i) \( \omega^\omega \) is well-ordered;
   (ii) for every \( k \in \mathbb{N} \) the product of \( k \) copies of \( \omega \) is wqo with respect to the (obvious generalization of the) product quasi-order.

Since \( \omega^\omega \) is the proof theoretic ordinal of RCA₀, it follows that RCA₀ does not prove the statement (ii) above.

Recently Hatzikiriakou and Simpson [HS17] proved that another statement dealing with wqos is equivalent to the fact that \( \omega^\omega \) is well-ordered. A Young diagram is a sequence of natural numbers \( (m_0, \ldots, m_k) \) such that \( m_i \geq m_{i+1} \) and \( m_k > 0 \). We denote by \( \mathcal{D} \) the set of all Young diagrams, and set \( \langle m_0, \ldots, m_k \rangle \preceq \langle n_0, \ldots, n_k \rangle \) if and only if \( k \leq h \) and \( m_i \leq n_i \) for all \( i \leq k \).

**Theorem 2.24** (RCA₀). The following are equivalent:
1. \( \omega^\omega \) is well-ordered;
2. \( (\mathcal{D}, \preceq_D) \) is wqo.
Theorems 2.23 and 2.24 are both motivated by the study of results about the non-existence of infinite ascending sequences of ideals in rings.

3. Characterizations and basic properties of bqos

To give the definition of bqo we need some terminology and notation for sequences and sets (here we follow Mar05). All the definitions are given in RCA0. Let $\mathbb{N}^{<\mathbb{N}}$ be the set of finite sequences of natural numbers. If $s \in \mathbb{N}^{<\mathbb{N}}$ we denote by $lh\ s$ its length and, for every $i < lh\ s$, by $s(i)$ its $(i+1)$-th element. Then we write this sequence as $s = (s(0), \ldots, s(lh\ s - 1))$. If $s, t \in \mathbb{N}^{<\mathbb{N}}$ we write $s \sqsubseteq t$ if $s$ is an initial segment of $t$, i.e., if $lh\ s \leq lh\ t$ and $\forall i < lh\ s\ s(i) = t(i)$. We write $s \sqsubset t$ if the range of $s$ is a subset of the range of $t$, i.e., if $\forall i < lh\ s\ \exists j < lh\ t\ s(i) = t(j)$. $s \sqsubseteq t$ and $s \sqsubset t$ have the obvious meanings. We write $s^{-}t$ for the concatenation of $s$ and $t$, i.e., the sequence $u$ such that $lh\ u = lh\ s + lh\ t$, $u(i) = s(i)$ for every $i < lh\ s$, and $u(lh\ s + i) = t(i)$ for every $i < lh\ t$. These notations are extended to infinite sequences (i.e., functions with domain $\mathbb{N}$) as well.

If $X \subseteq \mathbb{N}$ is infinite we denote by $[X]^{<\mathbb{N}}$ the set of all finite subsets of $X$. We identify an element of $[\mathbb{N}]^{<\mathbb{N}}$ with the unique element of $\mathbb{N}^{<\mathbb{N}}$ which enumerates it in increasing order, so that we can use the notation introduced above. If $k \in \mathbb{N}$, $[X]^k$ is the subset of $[X]^{<\mathbb{N}}$ consisting of the sets with exactly $k$ elements. Similarly $[X]^N$ stands for the collection of all infinite subsets of $X$. Note that $[X]^N$ does not formally exist in second order arithmetic, and is only used in expressions of the form $Y \in [X]^N$; here again we identify $Y$ with the unique sequence enumerating it in increasing order (notice that in RCA0 an element of $[X]^N$ exists as a set if and only if it exists as an increasing sequence, so that this identification is harmless). For $X \in [\mathbb{N}]^N$ let $X^{-} = X \setminus \{\min X\}$, i.e., $X$ with its least element removed. Similarly if $s \in [\mathbb{N}]^{<\mathbb{N}}$ is nonempty we set $s^{-} = s \setminus \{\min s\}$.

If $B \subseteq [\mathbb{N}]^{<\mathbb{N}}$ then base$(B)$ is the set
\[
\{ n : \exists s \in B \exists i < lh\ s\ s(i) = n \}.
\]
RCA0 does not prove the existence of base$(B)$ for arbitrary $B \subseteq [\mathbb{N}]^{<\mathbb{N}}$; indeed in Mar05 Lemma 1.4 it is shown that, over RCA0, ACA0 is equivalent to the assertion that base$(B)$ exists as a set for every $B \subseteq [\mathbb{N}]^{<\mathbb{N}}$. However this does not affect the possibility of defining blocks and barriers within RCA0: e.g., “base$(B)$ is infinite” (which is condition (1) in the definition of block below) can be expressed by $\forall m \exists n > m \exists s \in B$ $n \in s$. Similarly, when we say $X$ is a subset of base$(B)$ (for example in condition (2) of the definition of block), we mean $\forall x \in X \exists s \in B$ $x \in s$. After giving the definitions, Lemma 3.2 below will show that in fact RCA0 proves that base$(B)$ exists whenever $B$ is a block (and, a fortiori, a barrier).

Definition 3.1 (RCA0). A set $B \subseteq [\mathbb{N}]^{<\mathbb{N}}$ is a block if:

1. base$(B)$ is infinite;
2. $\forall X \in [\text{base}(B)]^{\mathbb{N}} \exists s \in B\ s \sqsubseteq X$;
3. $\forall s, t \in B\ s \sqsubset t$.

$B$ is a barrier if it satisfies (1), (2) and

3’. $\forall s, t \in B\ s \sqsubset t$.

Within RCA0 it is immediate that every barrier is a block and we can check that $[\mathbb{N}]^k$ (for $k > 0$), $\{ s \in [\mathbb{N}]^{<\mathbb{N}} : lh\ s = s(0) + 1 \}$ and $\{ s \in [\mathbb{N}]^{<\mathbb{N}} : lh\ s = s(s(0)) + 1 \}$ are barriers.

Notice that if $B$ is a block and $Y \in [\text{base}(B)]^{\mathbb{N}}$ then RCA0 proves that there exists a unique block $B' \subseteq B$ such that base$(B') = Y$: in fact $B' = \{ s \in B : s \sqsubseteq Y \}$. 
Moreover if \( B \) is a barrier then \( B' \) is also a barrier and we say that \( B' \) is a subbarrier of \( B \).

The following result is Lemma 5.5 of [CMS04].

**Lemma 3.2 (RCA\(_0\)).** If \( B \) is a block then \( \text{base}(B) \) exists as a set and \( B \) is isomorphic to a block \( B' \) with \( \text{base}(B') = \mathbb{N} \).

**Definition 3.3 (RCA\(_0\)).** Let \( s, t \in [\mathbb{N}]^{<\mathbb{N}} \): we write \( s \triangleleft t \) if there exists \( u \in [\mathbb{N}]^{<\mathbb{N}} \) such that \( s \subseteq u \) and \( t \subseteq u^\rightarrow \).

Notice that \( \langle 2, 4, 9 \rangle \triangleleft \langle 4, 9, 10, 14 \rangle \triangleleft \langle 9, 10, 14, 21 \rangle \) and \( \langle 2, 4, 9 \rangle \not\triangleleft \langle 9, 10, 14, 21 \rangle \), so that \( \triangleleft \) is not transitive.

**Definition 3.4 (RCA\(_0\)).** Let \( (Q, \preceq) \) be a quasi-order, \( B \) be a block and \( f : B \to Q \). We say that \( f \) is good (with respect to \( \preceq \)) if there exist \( s, t \in B \) such that \( s \triangleleft t \) and \( f(s) \preceq f(t) \). If \( f \) is not good then we say that it is bad. \( f \) is perfect if for every \( s, t \in B \) such that \( s \triangleleft t \) we have \( f(s) \preceq f(t) \).

We can now give the definition of bqo:

**Definition 3.5 (RCA\(_0\)).** Let \( (Q, \preceq) \) be a quasi-order.

- \( Q \) is bqo if for every barrier \( B \) every \( f : B \to Q \) is good with respect to \( \preceq \);
- \( Q \) is bqo(block) if for every block \( B \) every \( f : B \to Q \) is good with respect to \( \preceq \).

An alternative definition of bqo was given by Simpson in [Sim85a]. A block \( B \) represents an infinite partition of \( [\text{base}(B)]^\mathbb{N} \) into clopen sets with respect to the topology that \( [\text{base}(B)]^\mathbb{N} \) inherits from \( \mathbb{N}^\mathbb{N} \). Thus any \( f : B \to Q \) represents a continuous function \( F : [\text{base}(B)]^\mathbb{N} \to Q \) where \( Q \) has the discrete topology; \( f \) is good if for some \( X \in [\text{base}(B)]^\mathbb{N} \) we have \( F(X) \preceq F(X^-) \). Therefore \( (Q, \preceq) \) is bqo if and only if for every continuous function \( F : [\text{base}(B)]^\mathbb{N} \to Q \) there exists \( X \in [\text{base}(B)]^\mathbb{N} \) such that \( F(X) \preceq F(X^-) \). Moreover if we replace continuous with Borel we are still defining the same notion (this follows from the fact, originally proved by Mathias, that for every Borel function \( F : [\text{base}(B)]^\mathbb{N} \to Q \) there exists \( X \in [\text{base}(B)]^\mathbb{N} \) such that the restriction of \( F \) to \( [X]^\mathbb{N} \) is continuous). We are not discussing these alternative characterizations of bqo here, but they have been exploited by Montalbán in his proof of Theorem 6.28 below.

It is easy to see (using the barrier \( [\mathbb{N}]^1 \) and the fact that \( \langle m \rangle \not\triangleleft \langle n \rangle \) if and only if \( m < n \) that RCA\(_0\) proves that every bqo is wqo.

Lemma 3.2 shows that within RCA\(_0\) we can restrict the definition of bqo and bqo(block) to functions with domain barriers or blocks with base \( \mathbb{N} \). It is also immediate that every bqo(block) is also a bqo. For the opposite implication, we have the following result [CMS04, Theorem 5.12].

**Lemma 3.6 (WKL\(_0\)).** Every bqo is bqo(block).

The natural proof that every bqo is bqo(block) uses the clopen Ramsey theorem, which is equivalent to ATR\(_0\), to show that every block contains a barrier. The proof of Lemma 3.6 instead exploits a construction originally appeared in [Mar94] and builds a barrier which is connected to, but in general not included in, the original block.

Lemma 3.6 leads to the following question:

**Question 3.7.** Is “every bqo is bqo(block)” equivalent to WKL\(_0\) over RCA\(_0\)?

Another characterization of bqos corresponds to the wqo(set) characterization of wqos.
Definition 3.8. A quasi-order \((Q, \preceq)\) is bqo(set) if for every barrier \(B\) and every \(f : B \to Q\) there exists a subbarrier \(B' \subseteq B\) such that \(f\) restricted to \(B'\) is perfect with respect to \(\preceq\).

RCA\(_0\) trivially proves that every bqo(set) is bqo, while the reverse implication is known to be much stronger (see [Mar05, Theorem 4.9], which revisits [Sim09, Lemma V.9.5]).

Theorem 3.9 (RCA\(_0\)). The following are equivalent:

(i) \(\text{ATR}_0\);

(ii) every bqo is bqo(set).

It is easy to realize that RCA\(_0\) suffices to prove that every well-order is bqo, and even bqo(block) (see [Mar05, Lemma 3.1]). Dealing with finite quasi-orders is however more problematic. Let \(n\) denote the partial order consisting of \(n\) mutually incomparable elements, and notice that if \(n\) is bqo, or bqo(block), or bqo(set), then every quasi-order with the same number of elements has the same property. The following results are from [Mar05, Lemma 3.2, Theorem 5.11 and Theorem 4.9].

Theorem 3.10. \((1)\) RCA\(_0\) proves that \(2\) is bqo and bqo(block);

(2) \(\text{ATR}_0\) proves that \(3\) is bqo;

(3) for any fixed \(n \geq 3\), RCA\(_0\) proves that \(3\) is bqo is equivalent to \(n\) is bqo;

(4) for any fixed \(n \geq 2\), RCA\(_0\) proves that \(n\) is bqo(set) is equivalent to \(\text{ATR}_0\).

Item (3) above leads to the following question, which was already stated as Problem 3.3 in [Mar05].

Question 3.11. What is the strength of the statement “\(3\) is bqo”?

Over the years, the author has involved several colleagues in trying to attack this problem, but no progress has been made. We devote some time to explain the situation. The \(\preceq\) relation can be viewed as defining a graph with the elements of \([\mathbb{N}]^\mathbb{N}\) as vertices. The assertion that \(n\) is bqo amounts to state that the subgraph whose set of vertices is a barrier is not \(n\)-colorable. Indeed, the proof of item (1) of Theorem 3.10 amounts to the definition within RCA\(_0\) of a cycle of odd length inside any barrier or block. It is much more difficult to show that a graph is not 3-colorable, and this accounts for the increased difficulty in showing that \(3\) is bqo. A first step in beginning to answer Question 3.11 would be showing that the \(\omega\)-model REC does not satisfy that every barrier is 3-colorable. To this end one cannot use a computable barrier \(B\): in fact being 3-colorable is an arithmetic property, and hence surely false for \(B\) in REC. What is needed is some \(B \subseteq [\mathbb{N}]^\mathbb{N}\) which looks like a barrier in REC (i.e., which satisfies (1) and (3’) of Definition 3.1 and is such that for every computable \(X \in [\text{base}(B)]^\mathbb{N}\) there exists \(s \in B\) with \(s \sqsubseteq X\), but is 3-colorable.

Moving now to the basic closure properties of bqos, we start by noticing the following obvious fact, which mirrors the results of Lemma 2.14 about wqos.

Lemma 3.12 (RCA\(_0\)). Let \(\mathcal{P}\) be any of the properties bqo, bqo(block) or bqo(set). If \((Q, \preceq)\) satisfies \(\mathcal{P}\) and \(R \subseteq Q\) then the restriction of \(\preceq\) to \(R\) satisfies \(\mathcal{P}\) as well.

Only part of Lemma 2.17 has an analogous for bqos.

Lemma 3.13 (RCA\(_0\)). Let \(\mathcal{P}\) be any of the properties bqo, bqo(block) or bqo(set). If \(Q_1\) and \(Q_2\) satisfy \(\mathcal{P}\) then \(Q_1 + Q_2\) satisfies \(\mathcal{P}\) with respect to the sum quasi-order.

When \(\mathcal{P}\) is bqo this is [Mar05, Lemma 5.14]. The proof shows that for any \(f : B \to Q_1 + Q_2\) there is a subbarrier \(B'\) such that the restriction of \(f\) to \(B'\) has
range in \( Q \), for some \( i \): this yields the result also when \( P \) is bqo(set). Moreover the proof works also for blocks, thus taking care of the case when \( P \) is bqo(block).

The closure under disjoint unions of bqos is much stronger than the corresponding property for wqos. In fact we have

**Lemma 3.14 (RCA\(_0\)).** Let \( P \) be any of the properties bqo, bqo(block) or bqo(set). The following are equivalent:

(i) if \( Q_1 \) and \( Q_2 \) satisfy \( P \) then \( Q_1 \cup Q_2 \) satisfies \( P \) with respect to the disjoint union quasi-order;

(ii) if \( Q \) satisfies \( P \) with respect to the quasi-orders \( \preceq_1 \) and \( \preceq_2 \) then \( Q \) satisfies \( P \) with respect to the intersection quasi-order;

(iii) if \( Q_1 \) and \( Q_2 \) satisfy \( P \) then \( Q_1 \times Q_2 \) satisfies \( P \) with respect to the product quasi-order.

All these statements are provable in \( \text{ATR}_0 \). When \( P \) is bqo or bqo(block) they imply \( \text{ACA}_0 \), when \( P \) is bqo(set) they are equivalent to \( \text{ATR}_0 \).

The equivalence between the three statements for bqo is Lemma 5.16 of [Mar05]; the implication from (i) to (iii) uses Theorem 6.10 below. The same proof works also for bqo(block) and bqo(set). Provability in \( \text{ATR}_0 \) follows easily from the clopen Ramsey theorem. The implication towards \( \text{ACA}_0 \) is Lemma 5.17 of [Mar05] (which uses the proof of Theorem 5.10 below) when we are dealing with bqos, and works also for bqo(block). The implication towards \( \text{ATR}_0 \) is immediate from item (4) of Theorem 6.10 because (i) for bqo(set) implies that 2 is bqo(set).

**Question 3.15.** What is the strength of statements (i)–(iii) of Lemma 3.14, when \( P \) is bqo or bqo(block)?

Since the statements imply \( \text{ACA}_0 \), by Lemma 5.6 there is a single answer for bqo and bqo(block). Since (i) for bqo implies that 3 is bqo, Questions 3.13 and 3.14 are connected.

4. Minimality arguments

One of the main tools of wqo theory is the minimal bad sequence lemma (apparently isolated for the first time in [NW63]). The idea is to prove that a quasi-order is wqo by showing that if there exists a bad sequence then there is one with a minimality property, and eventually reaching a contradiction from the latter assumption. To state the lemma in its general form we need the following definitions.

**Definition 4.1 (RCA\(_0\)).** Let \((Q, \preceq)\) be a quasi-order. A transitive binary relation \( \preceq' \) on \( Q \) is compatible with \( \preceq \) if for every \( x, y \in Q \) we have that \( x \preceq' y \) implies \( x \preceq y \). We write \( x \preceq' y \) for \( x \preceq' y \lor x = y \). In this situation, if \( A, A' \in [N]^\mathbb{N}, f : A \to Q \), and \( f' : A' \to Q \) we write \( f \preceq f' \) if \( A \subseteq A' \) and \( \forall n \in A \ f(n) \preceq f'(n) \); we write \( f < f' \) if \( f \preceq f' \) and \( \exists n \in A f(n) < f'(n) \). \( f \) is minimal bad with respect to \( \preceq' \) if it is bad with respect to \( \preceq \) and there is no \( f' < f \) which is bad with respect to \( \preceq \).

**Statement 4.2 (minimal bad sequence lemma).** Let \((Q, \preceq)\) be a quasi-order and \( \preceq' \) a well-founded relation which is compatible with \( \preceq \): if \( A' \in [N]^\mathbb{N} \) and \( f' : A' \to Q \) is bad with respect to \( \preceq \) then there exists \( f : A \to Q \) such that \( f \preceq f' \) and \( f \) is minimal bad with respect to \( \preceq' \).

The generalization of the minimal bad sequence lemma to bqos is known as the minimal bad array lemma (the maps of definition 5.3 are sometimes called arrays) or the forerunning technique (this method was explicitly isolated and clarified in [Lav78]). Again, we need some preliminary definitions.
Definition 4.3 (RCA₀). Let \((Q, \preceq)\) be a quasi-order and \(<'\) be compatible with \(\preceq\) in the sense of definition \(\text{[3]}\). If \(B\) and \(B'\) are barriers, \(f : B \to Q\), and \(f' : B' \to Q\) we write \(f \preceq f'\) if base\((B)\) \(\subseteq\) base\((B')\), and for every \(s \in B\) there exists \(s' \in B'\) such that \(s' \preceq s\) and \(f(s) \preceq f'(s')\). We write \(f <' f'\) if \(f \preceq f'\) and for some \(s \in B\), \(s' \in B'\) with \(s' \preceq s\) we have \(f(s) <' f'(s')\). \(f\) is minimal bad with respect to \(<'\) if it is bad with respect to \(\preceq\) and there is no \(f <' f\) which is bad with respect to \(\preceq\).

Statement 4.4 (minimal bad array lemma). Let \((Q, \preceq)\) be a quasi-order and \(<'\) a well-founded relation which is compatible with \(\preceq\). If \(B'\) is a barrier and \(f' : B' \to Q\) is bad with respect to \(\preceq\) then there exist a barrier \(B\) and \(f : B \to Q\) such that \(f \preceq f'\) and \(f\) is minimal bad with respect to \(<'\).

A milder generalization of the minimal bad sequence lemma is also useful: it was actually the first version of the minimal bad array lemma proved for a specific quasi-order by Nash-Williams in \([NW65]\) and was isolated in \([Mar94]\).

Definition 4.5 (RCA₀). Let \((Q, \preceq)\) be a quasi-order and \(<'\) be compatible with \(\preceq\) in the sense of definition \(\text{[3]}\). If \(B\) and \(B'\) are barriers, \(f : B \to Q\), and \(f' : B' \to Q\) we write \(f \preceq_i f'\) if \(B \subseteq B'\) and \(\forall s \in B\) \(f(s) \preceq f'(s)\). We write \(f <'_i f'\) if \(f \preceq_i f'\) and \(\exists s \in B\) \(f(s) <' f'(s)\). \(f\) is locally minimal bad with respect to \(<'\) if it is bad with respect to \(\preceq\) and there is no \(f <'_i f\) which is bad with respect to \(\preceq\).

Statement 4.6 (locally minimal bad array lemma). Let \((Q, \preceq)\) be a quasi-order and \(<'\) a well-founded relation which is compatible with \(\preceq\): if \(B'\) is a barrier and \(f' : B' \to Q\) is bad with respect to \(\preceq\) then there exist a barrier \(B\) and \(f : B \to Q\) such that \(f \preceq f'\) and \(f\) is locally minimal bad with respect to \(<'\).

The minimal bad sequence lemma and the locally minimal bad array lemma have been shown to be equivalent to the strongest of the big five by Simpson and Marcone in \([Mar90]\; Theorem 6.5].

Theorem 4.7 (RCA₀). The following are equivalent:

(i) \(\Pi^1_1\text{-CA}_0\);
(ii) the minimal bad sequence lemma;
(iii) the locally minimal bad array lemma.

On the other hand, the proofs of the minimal bad array lemma use very strong set-existence axioms: a crude analysis shows that they can be carried out within \(\Pi^1_2\text{-CA}_0\).

Question 4.8. What is the axiomatic strength of the minimal bad array lemma?

5. Structural results

In this section we consider theorems showing that wqos satisfy specific properties as partial orders.

The better known of these theorems is due to de Jongh and Parikh \([JP77]\) (an exposition of essentially the original proof appears in \([Har05]\; §8.4]; a proof based on the study of the partial order of the initial segments of the wqo is included in \([Fra00]\; §4.11]; proofs with a strong set-theoretic flavor appear as \([KT90]\; Theorem 4.7] and \([BG08]\; Proposition 52]).

Statement 5.1 (maximal linear extension theorem). If \((Q, \preceq)\) is wqo, then there exist a linear extension \(\preceq_L\) of \(Q\) which is maximal, meaning that any linear extension of \(Q\) embeds in an order-preserving way into \(\preceq_L\).

A less known result is due to Wolk \([Wol67]\; Theorem 9], actually Wolk’s statement is slightly stronger) and also appears as \([KT90]\; Theorem 4.9] and \([Har05]\; Theorem 8.1.7].
Statement 5.2 (maximal chain theorem). If \((Q, \preceq)\) is wqo, then there exist a chain \(C \subseteq Q\) which is maximal, meaning that every chain contained in \(Q\) embeds in an order-preserving way into \(C\).

Marcone and Shore [MS11] studied the strength of the maximal linear extension theorem and of the maximal chain theorem.

Theorem 5.3 (\(\text{RCA}_0\)). The following are equivalent:

(i) \(\text{ATR}_0\);
(ii) the maximal linear extension theorem;
(iii) the maximal chain theorem.

The proofs of the two theorems within \(\text{ATR}_0\) differ from the proofs found in the literature: to avoid using more induction than available in \(\text{ATR}_0\) one fixes a wqo \(Q\) and looks respectively at the tree of finite bad sequences in \(Q\)

\[
\text{Bad}(Q) = \{ s \in Q^{<\mathbb{N}} : \forall i, j < \text{lh} s(i < j \rightarrow s(i) \not\preceq s(j)) \}
\]

and at the tree of descending sequences in \(Q\)

\[
\text{Desc}(Q) = \{ s \in Q^{<\mathbb{N}} : \forall i, j < \text{lh} s(i < j \rightarrow s(j) \prec s(i)) \}
\]

(Here \(Q^{<\mathbb{N}}\) is the set of finite sequences of elements of \(Q\).) Since \(Q\) is wqo both these trees are well-founded and \(\text{ATR}_0\) can compute their rank functions. Focusing on the maximal linear extension theorem (the other proof follows the same strategy), by recursion on the rank of \(s \in \text{Bad}(Q)\) we assign to \(s\) a maximal linear extension of the restriction of \(\preceq\) to \(\{ x \in Q : s \langle x \rangle \in \text{Bad}(Q) \}\); when \(s\) is the empty sequence we have the maximal linear extension of \(Q\).

The two reversals contained in Theorem 5.3 have quite different proofs. The proof that the maximal chain theorem implies \(\text{ATR}_0\) is very simple (using the well-known equivalence between \(\text{ATR}_0\) and comparability of well-orders), while the proof that the maximal linear extension theorem implies \(\text{ATR}_0\) is more involved. In fact there is first a bootstrapping, showing that the maximal linear extension theorem implies \(\text{ACA}_0\). To this end it is useful a partial order \(Q\) such that the existence of any bad sequence in \(Q\) implies \(\text{ACA}_0\); thus if \(\text{ACA}_0\) fails then \(Q\) is wqo, we can apply the theorem and reach a contradiction from the existence of a maximal linear extension. We can now argue within \(\text{ACA}_0\) and, assuming the failure of \(\text{ATR}_0\) and using Theorem 6.23 below, build a wqo \(Q'\) which cannot have a maximal linear extension. The difference of the two proofs is no accident. In fact a theorem of Montalbán [Mon07] states that every computable wqo has a computable maximal linear extension (this implies that in showing that the maximal linear extension theorem implies anything unprovable in \(\text{RCA}_0\) the use of partial orders that are not really wqos is unavoidable), while Marcone, Montalbán and Shore [MMS12, Theorem 3.3] showed that for every hyperarithmetic set \(X\) there is a computable wqo \(Q\) with no \(X\)-computable maximal chain.

Another kind of structural theorems about quasi-orders concerns the decomposability of the quasi-order in finite pieces which are simple.

Definition 5.4 (\(\text{RCA}_0\)). Let \((Q, \preceq)\) be a quasi-order. \(I \subseteq Q\) is an ideal if

- \(\forall x, y \in I (x \in I \land y \preceq x \rightarrow y \in I)\);
- \(\forall x, y \in I \exists z \in I (x \preceq z \land y \preceq z)\).

Bonnet [Bon75, Lemma 2] (see also [Fra00, §4.7.2]) proved that a partial order has no infinite antichains if and only if every initial interval is a finite union of ideals (this result follows also from [ET43, Theorem 1]). In [FM14, Theorem 4.5] Frittaion and Marcone studied the left to right direction of Bonnet’s result and proved, among other things, the following equivalence.
Theorem 5.5 (RCA₀). The following are equivalent:

(i) ACA₀;
(ii) Higman’s theorem;
(iii) if Q is wqo then (P₁(Q), ≤vt) is wqo.

6. Major theorems about wqos and bqos

In this section we consider the major theorems about wqos and bqos, starting with Higman’s basic result, first proved in [Hig52] and then rediscovered many times.

Definition 6.1 (RCA₀). If (Q, ≤) is a quasi-order we define a quasi-order on Q≤N by setting s ≤vt t if and only if there exists an embedding of s into t, i.e., a strictly increasing f : lh s ↠ lh t such that s(i) ≤ f(i) for every i < lh s (here lh s is the length of the sequence s).

Statement 6.2 (Higman’s theorem). If Q is wqo then (Q≤N, ≤vt) is wqo.

Before analyzing Higman’s theorem from the reverse mathematics viewpoint, let us introduce other constructions of new quasi-orders starting from the one on X.

We denote by P(X) and P₁(X) the powerset of X and the set of all finite subsets of X. If X is infinite P(X) does not exist as a set in second order arithmetic, but we can define and study relations between elements of P(X). A quasi-order on P(X) is just a formula ϕ with two distinguished set variables such that ϕ(Y, Z) holds and ϕ(Z, W) imply ϕ(Y, W) whenever Y, Z, W ⊆ X. We use symbols like ≤ and infix notation to denote quasi-orders on P(X).

Definition 6.3 (RCA₀). If ≤ is a quasi-order on P(X), a sequence (Xₙ)ₙ∈ℕ of elements of P(X) is good (with respect to ≤) if there exist m <N n such that Xₘ ≤ Xₙ. Every such sequence is good we say that ≤ is wqo.

Analogously, a sequence (Xₙ)ₙ∈ℕ of elements of P(X) indexed by a barrier B is good (with respect to ≤) if there exist s, t ∈ B such that s <t t and Xₛ ≤ Xₜ. Every such sequence is good we say that ≤ is bqo.

The following two quasi-orders are called the Hoare quasi-order and the Smyth quasi-order in the computer science literature. (Here we follow the computer science notation: in [Mar05] ≤⁹ was written as ≤³VT and ≤⁴ as ≤⁷VT.)

Definition 6.4 (RCA₀). Let (Q, ≤) be a quasi-order. If X, Y ∈ P(Q) let

X ≤⁹ Y ⇐⇒ ∀x ∈ X ∃y ∈ Y x ≤ y and
X ≤⁴ Y ⇐⇒ ∀y ∈ Y ∃x ∈ X x ≤ y.

Theorem 6.5 (RCA₀). The following are equivalent:

(i) ACA₀;
(ii) Higman’s theorem;
(iii) if Q is wqo then (P₁(Q), ≤⁹) is wqo.

Most proofs of Higman’s theorem are based on the minimal bad sequence lemma. Theorem 5.5 implies that such a proof cannot be carried out in ACA₀. In fact, the provability of Higman’s theorem in ACA₀ is based on the technique of reification of wqos by well-orders ([HP77, Sch79], see also [KT90]) and follows from the results in Section 4 of [Sim88] (see [Cle96] Theorem 3) for details). A reification of Q by the linear order (X, ≤ₓ) is a map ρ from Bad(Q) to X such that ρ(t) <ₓ ρ(s) whenever s ⊆ t. Thus, if X is a well-order then ρ is an approximation to the rank function on B(Q), and suffices to witness that Bad(T) is well-founded and hence Q is wqo.

ACA₀ is used twice in this proof: first to show that every wqo admits a reification by a well-order and then to show that ωω₀⁺⁺ is a well-order when α is a well-order.
Theorem 6.6 (RCA0). If \( Q \) is bqo then \( (\mathcal{P}(Q), \preceq^3) \) and \( (\mathcal{P}(Q), \preceq^1) \) (and hence, a fortiori, also \( (\mathcal{P}(Q), \preceq^2) \)) are bqo.

Theorem 6.7 (ACA0). If \( Q \) is bqo then \( (\mathcal{P}(Q), \preceq^3) \) is bqo.

Question 6.8. Is the statements “if \( Q \) is bqo then \( (\mathcal{P}(Q), \preceq^3) \) is bqo” equivalent to ACA0 over RCA0?

Trying to answer affirmatively the previous question, one is faced with the problem of applying the statement to a quasi-order \( Q \) which is proved to be bqo within RCA0. Such a \( Q \) must be infinite (otherwise \( \mathcal{P}(Q) = P(Q) \) and Theorem 6.6 applies) and, unless the answer to Question 5.11 is RCA0, with antichains of size at most 2.

More results about the Hoare and Smyth quasi-orders (obtained by weakening the conclusion) will be discussed in Section 7 below.

Another important result about wqos is Kruskal’s theorem [Kru60], establishing a conjecture of Vázsonyi from the 1930’s popularized by Erdős. This theorem deals with trees viewed as partial orders: for our purposes we can represent them in second-order arithmetic as subsets of \( \mathbb{N}^{<\omega} \) closed under initial segments.

Definition 6.9 (RCA0). Let \( \mathcal{T} \) be the set of all finite trees. If \( T_0, T_1 \in \mathcal{T} \) let \( T_0 \preceq T_1 \) if and only if there exists a homeomorphic embedding of \( T_0 \) in \( T_1 \), that is, an injective \( f : T_0 \to T_1 \) such that \( f(s \wedge t) = f(s) \wedge f(t) \) for every \( s, t \in T_0 \) (where \( s \wedge t \) is the greatest lower bound of \( s \) and \( t \), which is the longest common initial segment of the two sequences).

If \( Q \) is a set let \( \mathcal{T}^Q \) be the set of finite trees labelled with elements of \( Q \), that is, pairs \( (T, \ell) \) such that \( T \in \mathcal{T} \) and \( \ell \) is a function from \( T \) to \( Q \).

If \( (Q, \preceq) \) is a quasi-order and \( (T_0, \ell_0), (T_1, \ell_1) \in \mathcal{T}^Q \) let \( (T_0, \ell_0) \preceq_{\mathcal{T}^Q} (T_1, \ell_1) \) if and only if there exists a homeomorphic embedding \( f \) of \( T_0 \) in \( T_1 \) such that \( \ell_0(s) \preceq \ell_1(f(s)) \) for every \( s \in T_0 \).

RCA0 easily shows that \( \preceq_{\mathcal{T}^Q} \) and \( \preceq_{\mathcal{T}^Q} \) are quasi-orders.

Statement 6.10 (Kruskal’s theorem). If \( Q \) is wqo then \( (\mathcal{T}^Q, \preceq_{\mathcal{T}^Q}) \) is wqo.

The usual proof of Kruskal’s theorem uses the minimal bad sequence lemma and can be carried out in \( \Pi^1_1 \)-CA0 using Theorem 4.7. On the other hand, this statement is \( \Pi^1_2 \) and hence cannot imply \( \Pi^1_1 \)-CA0 (see [Mar06, Corollary 1.10]).

Harvey Friedman proved the following striking result (see [Sim85b]).

Theorem 6.11. ATR0 does not prove that \( (\mathcal{T}, \preceq_{\mathcal{T}^Q}) \) is wqo. A fortiori Kruskal’s theorem is not provable in ATR0.
To prove this theorem we build a map $\psi$ between $\mathcal{T}$ and a certain primitive recursive notation system for the ordinals less than $\Gamma_0$, and show that $\text{ACA}_0$ proves that $\psi(T_0) \leq_\varphi \psi(T_1)$ (where $\leq$ is the order on the ordinal notation system) whenever $T_0 \preceq_\mathcal{T} T_1$. Thus $\text{ACA}_0$ proves that if $(\mathcal{T}, \preceq_\mathcal{T})$ is wqo then the system of ordinal notations is a well-order. Since $\Gamma_0$ is the proof-theoretic ordinal of $\text{ATR}_0$, it follows that $\text{ATR}_0$ does not prove that $(\mathcal{T}, \preceq_\mathcal{T})$ is wqo.

A lower bound for Kruskal’s theorem is provided by the following theorem, that apparently has never been explicitly stated.

**Theorem 6.12 (RCA$_0$).** Kruskal’s theorem implies ATR$_0$.

**Sketch of proof.** We use the fact that ATR$_0$ is equivalent, over RCA$_0$, to the statement that if $X$ is a well-order then $\varphi(X, 0)$ is a well-order, where $\varphi$ is the formalization of the Veblen function on the ordinals. This theorem was originally proved by H. Friedman (unpublished) and then given a proof-theoretic proof in [RW93] and a computability-theoretic proof in [MM11]. We follow the notation of the latter paper.

To prove our theorem first notice that Kruskal’s theorem generalizes Higman’s theorem, so that we can argue in $\text{ACA}_0$. Given a well-order $X$ we can mimic the construction of the proof of Theorem 6.11 using $X$ as the set of labels for the finite trees. In this way we define a map $\psi$ between $\mathcal{T}^X$ and the ordinals less than the first fixed point for the Veblen function strictly larger than $X$. We then show that $\text{ACA}_0$ proves that $\psi(T_0) \leq_\varphi \psi(T_1)$ whenever $T_0 \preceq_\mathcal{T} T_1$. Since our hypothesis implies that $(\mathcal{T}^X, \preceq_\mathcal{T}^X)$ is wqo we obtain that $(\varphi(X, 0), \leq_\varphi)$ is a well-order, as needed. □

Thus Kruskal’s theorem is properly stronger than ATR$_0$ and provable in, but not equivalent to, $\Pi^1_2$-CA$_0$. In an attempt to classify statements of this kind, Henry Towsner [Tow13] introduced a sequence of intermediate systems based on weakening the leftmost path principle (which is equivalent to $\Pi^1_2$-CA$_0$). Towsner tested his approach by looking at various statements and, by analyzing Nash-Williams’ proof of Kruskal’s theorem, obtained the following result.

**Theorem 6.13.** Kruskal’s theorem is provable in Towsner’s system $\Sigma_2$-LPP$_0$.

Unfortunately no reversal to Towsner’s systems are known, so we do not know whether the upper bound for the strength of Kruskal’s theorem provided by the previous theorem is optimal.

Rathjen and Weiermann [RW93] carried out a detailed proof-theoretic analysis of the statement “$(\mathcal{T}, \preceq_\mathcal{T})$ is wqo” (beware that Rathjen and Weiermann call Kruskal’s theorem this statement) showing that it is equivalent over $\text{ACA}_0$ to the uniform $\Pi^1_2$ reflection principle of the theory obtained by adding transfinite induction for $\Pi^1_2$ formulas to $\text{ACA}_0$.

Harvey Friedman, inspired by ordinal notation systems, introduced a refinement of $\preceq_\mathcal{T}$ (obtained by requiring that the homeomorphic embedding satisfies a “gap condition”) and proved that it still yields a wqo on $\mathcal{T}$. Friedman himself [Sim85b] showed, generalizing the technique of theorem 6.11 to larger ordinals, that this wqo statement is not provable in $\Pi^1_2$-CA$_0$.

The most striking instance of this unprovability phenomenon is provided by the graph minor theorem, proved by Robertson and Seymour in a long series of papers (see [Tho95] Section 5) or [Die17] Chapter 12 for overviews.

**Definition 6.14 (RCA$_0$).** If $\mathcal{G}$ is the set of all finite directed graphs (allowing loops and multiple edges) define a quasi-order on $\mathcal{G}$ by setting $G_0 \preceq_m G_1$ if and only if $G_0$ is isomorphic to a minor of $G_1$ (recall that a minor is obtained by deleting edges and vertices and contracting edges).
Statement 6.15 (graph minor theorem). $\preceq_m$ is wqo on $\mathcal{G}$.

Friedman’s generalization of Kruskal’s theorem mentioned above plays a significant role in some steps of the proof of the graph minor theorem, which uses iterated applications of the minimal bad sequence lemma. This proof cannot be carried out in $\Pi^1_1$-$\text{CA}_0$ and the following theorem (proved by Friedman, Robertson and Seymour [FRS87] well before the completion of the proof of the graph minor theorem) shows that there is no simpler proof.

Theorem 6.16. The graph minor theorem (and even special cases where $\preceq_m$ is restricted to some subset of $\mathcal{G}$) is not provable in $\Pi^1_1$-$\text{CA}_0$.

This theorem is proved once more generalizing the technique of theorem 6.11 to larger ordinals. Notice also that the graph minor theorem is a $\Pi^1_1$ statement, and therefore does not imply any set-existence axiom (in fact it holds in every $\omega$-model). More recently Rathjen and Krombholz [Kro18] analyzed more in detail the proof by Robertson and Seymour in search of upper bounds for the proof-theoretic strength of this statement, showing that it can be carried out in the system obtained by adding transfinite induction for $\Pi^1_2$ formulas to $\Pi^1_1$-$\text{CA}_0$.

It is well-known that Higman’s theorem does not extend to infinite sequences, and the canonical counterexample is Rado’s partial order. The notion of bqo was developed by Nash-Williams as a way of ruling out Rado’s example and its generalizations. Indeed, one of the first theorems of the subject is a generalization of Higman’s theorem [NW68].

Definition 6.17 (RCA_0). If $(Q, \preceq)$ is a quasi-order we can extend the quasi-order $\preceq^*$ of Definition 6.1 from $Q^{<\omega}$ to $\tilde{Q}$, the set of all countable sequences of elements of $Q$ (i.e., the set of all functions from a countable well-order to $Q$).

Statement 6.18 (Nash-Williams’ theorem). If $Q$ is bqo then $(\tilde{Q}, \preceq^*)$ is bqo.

Notice that $\tilde{Q}$ is uncountable, and hence we express “$(\tilde{Q}, \preceq^*)$ is bqo” in a way similar to Definition 6.3.

The following theorem is [Mar96, Theorem 4.5].

Theorem 6.19. $\Pi^1_1$-$\text{CA}_0$ proves Nash-Williams’ theorem.

The most natural proof of Nash-Williams’ theorem uses the minimal bad array lemma, and therefore to prove Theorem 6.19 a new argument is needed. This is obtained by using the locally minimal bad array lemma (provable in $\Pi^1_1$-$\text{CA}_0$ by Theorem 4.7) to establish the following weak version of Nash-Williams’ theorem.

Statement 6.20 (generalized Higman’s theorem). If $Q$ is bqo then $(Q^{<\omega}, \preceq^*)$ is bqo.

Assuming the generalized Higman’s theorem, we can prove Nash-Williams’ theorem in $\text{ATR}_0$. Thus the proof of Theorem 6.19 yields the following result.

Theorem 6.21 (ATR_0). The following are equivalent:

(i) Nash-Williams’ theorem;

(ii) the generalized Higman’s theorem.

Nash-Williams’ theorem cannot imply $\Pi^1_1$-$\text{CA}_0$, even over $\text{ATR}_0$ [Mar96, Theorem 5.7]. In fact, the proof of Theorem 6.19 actually establishes a $\Pi^1_2$ statement that, over $\text{ATR}_0$, implies Nash-Williams’ theorem. The argument mentioned before Theorem 6.11 then establishes the assertion. (Both Nash-Williams’ theorem and the generalized Higman’s theorem are $\Pi^1_3$ statements, so we cannot apply the argument directly.) Towsner [Tow13] looked also at the proof of the locally minimal bad array lemma.
Theorem 6.22. The generalized Higman’s theorem, and therefore also Nash-Williams’ theorem, is provable in Towsner’s system \( \text{TLPP}_0 \).

\( \text{TLPP}_0 \) is much stronger than the system \( \Sigma^2_2-\text{LPP}_0 \) appearing in Theorem 6.13. Unfortunately, as already mentioned, no reversal to Towsner’s systems are known, so Theorem 6.22 provides just an upper bound for the strength of Nash-Williams’ theorem. Regarding lower bounds, Shore [Sho93] proved the following important result.

Theorem 6.23 (\( \text{RCA}_0 \)). The following are equivalent:

(i) \( \text{ATR}_0 \)

(ii) every infinite sequence of countable well-orders contains two distinct elements which are comparable with respect to embeddability (as defined in Definition 6.25 below).

It is immediate that Nash-Williams’ theorem implies (ii), and hence \( \text{ATR}_0 \), within \( \text{RCA}_0 \).

Question 6.24. Is Nash-Williams’ theorem equivalent to \( \text{ATR}_0 \)?

A positive answer to this question was conjectured in [Mar96, Mar05].

Connected to Nash-Williams’ theorem is one of the most famous achievements of bqp theory, Laver’s proof [Lav71] of Fraïssé’s conjecture [Fra48]. Laver actually proved a stronger result (even stronger than the one we state below) and we keep the two statements distinct.

Definition 6.25 (\( \text{RCA}_0 \)). If \( L \) is the set of countable linear orderings define the quasi-order of embeddability on \( L \) by setting \( L_0 \preceq L_1 \) if and only if there exists an order-preserving embedding of \( L_0 \) in \( L_1 \), i.e., an injective \( f : L_0 \rightarrow L_1 \) such that \( x <_{L_0} y \) implies \( f(x) <_{L_1} f(y) \) for every \( x, y \in L_0 \).

Statement 6.26 (Fraïssé’s conjecture). \((L, \preceq_L)\) is wqo.

Statement 6.27 (Laver’s theorem). \((L, \preceq_L)\) is bqo.

Again, \( L \) is uncountable, and hence we express “\((L, \preceq_L)\) is wqo (bqo)” by imitating Definition 6.3.

The strength of Fraïssé’s conjecture is one of the most important open problems about the reverse mathematics of wqo and bqp theory. All known proofs of Fraïssé’s conjecture actually establish Laver’s theorem. Basically only one proof was known until 2016: this proof uses the minimal bad array lemma and can be carried out in \( \Pi^1_2-\text{CA}_0 \). Recently Antonio Montalbán [Mon17] made a major breakthrough by finding a new proof, which avoids any form of “minimal bad” arguments. This proof is based on Montalbán’s earlier analysis of Fraïssé’s conjecture [Mon06] and uses the Ramsey property for subsets of \([\mathbb{N}]^{\omega}\) and determinacy, yielding the following result.

Theorem 6.28. \( \Pi^1_2-\text{CA}_0 \) proves Fraïssé’s conjecture and Laver’s theorem.

Montalbán defines \( \Delta^0_3 \)-bqp by using \( \Delta^0_2 \) functions in Simpson’s definition of bqp given after Definition 3.5. Using the fact that \( \Sigma^0_2 \) sets are Ramsey (which is known to be equivalent to \( \Pi^1_1-\text{CA}_0 \)), he then shows that this notion is equivalent to bqp. Within \( \text{ATR}_0 \), using \( \Delta^0_2 \)-determinacy (which is equivalent to \( \text{ATR}_0 \)) Montalbán proves that if \( 3 \) is \( \Delta^0_3 \)-bqp then Laver’s theorem holds. Since \( 3 \) is bqp is provable in \( \text{ATR}_0 \) by item (2) of Theorem 3.10 the proof of Theorem 6.28 in \( \Pi^1_2-\text{CA}_0 \) is then complete.
Theorem 6.23 entails that Fraïssé’s conjecture (and a fortiori Laver’s theorem) implies ATR₀. Moreover Fraïssé’s conjecture is a Π₁^1 statement and the usual considerations yield that ATR₀ plus Fraïssé’s conjecture cannot imply Π₁^1-CA₀. Montalbán’s proof shows that to prove Fraïssé’s conjecture in any theory weaker than Π₁^1-CA₀ it suffices to prove that 3 is Δ₂^0-bqo. Thus an unexpected connection with Question 5.11 comes up. Indeed, Montalbán shows that by mimicking the proof of item (1) of Theorem 5.10 it is easy to see that ATR₀ proves that 2 is Δ₂^0-bqo.

**Question 6.29.** Is Fraïssé’s conjecture equivalent to ATR₀? Is “3 is Δ₂^0-bqo” provable in ATR₀?

A couple more results about Fraïssé’s conjecture are worth mentioning. First, Montalbán [Mont06] showed that Fraïssé’s conjecture is equivalent, over RCA₀ plus Σ₁^-induction, to a result about countable linear orders known as Jullien’s theorem. Therefore if the answer to Question 6.29 is negative then Fraïssé’s conjecture defines a system intermediate between ATR₀ and Π₁^1-CA₀ which is equivalent to other mathematical theorems.

On the other hand, Marcone and Montalbán [MM09] studied the restriction of Fraïssé’s conjecture to linear orders of finite Hausdorff rank. To state the result recall that ACA₀⁺ and ACA₀’ are obtained by adding to RCA₀ respectively “for every X, X^(ω) (the arithmetic jump of X) exists” and “for every X and k, X^(k) exists”. ACA₀⁺ is strictly weaker than ATR₀ but strictly stronger than ACA₀’, which in turn is strictly stronger than ACA₀. The ordinal 𝜙₂(0) is the first fixed point of the ϵ function: in RCA₀ we can define a linear order representing this ordinal, but showing that it is a well-order requires much stronger theories, since this is the proof-theoretic ordinal of ACA₀⁺.

**Theorem 6.30.** ACA₀⁺ plus “𝜙₂(0) is a well-order” proves the restriction of Fraïssé’s conjecture to linear orders of finite Hausdorff rank, which in turn implies, over RCA₀, ACA₀’ plus “𝜙₂(0) is a well-order”.

### 7. A Topological Version of WQOs

Recall that Q wqo does not imply that P(Q) with respect to ≤^5 or P₁(Q) with respect to ≤^6 is wqo. However we can still draw some conclusions about these partial orders if we weaken the conclusion, using a topological notion.

If (Q, ≤) is quasi-order we can use ≤ to define a number of different topologies on Q. These include the Alexandroff topology (whose closed sets are the initial intervals of Q) and the upper topology (whose basic closed sets are of the form \{ x ∈ Q : ∃ y ∈ F x ≤ y \} for F ⊆ Q finite). The topological notion that turns out to be relevant is the following: a topological space is Noetherian if it contains no infinite strictly descending sequences of closed sets. It turns out that Q wqo if and only if the Alexandroff topology on Q is Noetherian, and that Q wqo implies that the upper topology on Q is Noetherian. Goubault-Larrecq [GL07] proved that Q wqo does imply that the upper topologies of P(Q) with respect to both ≤^5 and ≤^6 are Noetherian. Frattaion, Hendtlass, Marcone, Shafer, and Van der Meeren [FHM⁺16] studied these results from the viewpoint of reverse mathematics, providing along the way proofs that have a completely different flavor from the category-theoretic arguments used by Goubault-Larrecq.

Before describing the results from [FHM⁺16] we need to explain the set-up, which in this case is not obvious because it is necessary to formalize statements about topological spaces which do not fit in the frameworks usually considered in subsystems of second-order arithmetic. (If Q is not an antichain then the Alexandroff and upper topologies are not T₁ and are thus very different from complete separable metric spaces.) First notice that if the quasi-order Q is countable the
Alexandroff and upper topology can be defined in RCA$_0$ within the framework of countable second-countable spaces introduced by Dorais [Dor11]. Expressing the fact that a countable second-countable space is Noetherian, as well as the connection mentioned above between $Q$ wqo and the fact that these topologies are Noetherian are also straightforward in RCA$_0$. However this still does not suffice to tackle all of Goubault-Larrecq’s results, because some of them deal with topologies defined on the uncountable space $P(Q)$. To express that the upper topology of $P(Q)$ with respect to either $\preceq^\flat$ or $\preceq^\sharp$ is Noetherian, the authors of [FHM+16] devise a way of representing these topological spaces. This representation shares some features with other well-established representations of topological spaces, including the familiar separable complete metric spaces and the countably based MF spaces introduced by Mummert [Mum06]. In this set-up the main results are the following [FHM+16, Theorem 4.7].

**Theorem 7.1 (RCA$_0$).** The following are equivalent:

(i) ACA$_0$;
(ii) if $Q$ is wqo, then the Alexandroff topology of $P_1(Q)$ with respect to $\preceq^\flat$ is Noetherian;
(iii) if $Q$ is wqo, then the upper topology of $P_1(Q)$ with respect to $\preceq^\flat$ is Noetherian;
(iv) if $Q$ is wqo, then the upper topology of $P_1(Q)$ with respect to $\preceq^\sharp$ is Noetherian;
(v) if $Q$ is wqo, then the upper topology of $P(Q)$ with respect to $\preceq^\flat$ is Noetherian;
(vi) if $Q$ is wqo, then the upper topology of $P(Q)$ with respect to $\preceq^\sharp$ is Noetherian.

In [GL13, Section 9.7] Goubault-Larrecq supports his claim that Noetherian spaces can be thought of as topological versions of wqos, by proving the following results. Starting from a topological space $X$ he introduces topologies on $X^{<\omega}$ and $T_X$ and proves the topological versions of Higman’s and Kruskal’s theorems, stating that if $X$ is Noetherian then $X^{<\omega}$ and $T_X$ are Noetherian. If $X$ is a countable second-countable space then so are $X^{<\omega}$ and $T_X$, which leads to the following so far unexplored question.

**Question 7.2.** What is the strength of the topological versions of Higman’s and Kruskal’s theorems restricted to countable second-countable spaces?

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