We study generating functions for the number of involutions in $S_n$ avoiding (or containing once) 132, and avoiding (or containing once) an arbitrary permutation $\tau$ on $k$ letters. In several interesting cases the generating function depends only on $k$ and is expressed via Chebyshev polynomials of the second kind. In particular, we establish that involutions avoiding both 132 and 12 \ldots k have the same enumerative formula according to the length than involutions avoiding both 132 and any double-wedge pattern possibly followed by fixed points of total length $k$. Many results are also shown with a combinatorial point of view.

1. Introduction

A permutation is a bijection from $[n] = \{1, 2, \ldots, n\}$ to $[n]$. Let $S_n$ be the set of permutations of length $n$.

Let $\alpha \in S_n$ and $\tau \in S_k$ be two permutations. We say that $\alpha$ contains $\tau$ if there exists a subsequence $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $(\alpha_{i_1}, \ldots, \alpha_{i_k})$ is order-isomorphic to $\tau$; in such a context $\tau$ is usually called a pattern. We say that $\alpha$ avoids $\tau$, or is $\tau$-avoiding, if such a subsequence does not exist. The set of all $\tau$-avoiding permutations in $S_n$ is denoted $S_n(\tau)$. For an arbitrary finite collection of patterns $T$, we say that $\alpha$ avoids $T$ if $\alpha$ avoids any $\tau \in T$; the corresponding subset of $S_n$ is denoted $S_n(T)$.

While the case of permutations avoiding a single pattern has attracted much attention, the case of multiple pattern avoidance remains less investigated. In particular, it is natural, as the next step, to consider permutations avoiding pairs of patterns $\tau_1, \tau_2$. This problem was solved completely for $\tau_1, \tau_2 \in S_3$ (see [SS]), for $\tau_1 \in S_3$ and $\tau_2 \in S_4$ (see [W]), and for $\tau_1, \tau_2 \in S_4$ (see [Bl, Km] and references therein). Several recent papers [CW, MV1, Kt, MV2] deal with the case $\tau_1 \in S_3, \tau_2 \in S_k$ for various pairs $\tau_1, \tau_2$. Another natural question is to study permutations avoiding $\tau_1$ and containing $\tau_2$ exactly $t$ times. Such a problem for certain $\tau_1, \tau_2 \in S_3$ and $t = 1$ was investigated in [Ro], and for certain $\tau_1 \in S_3, \tau_2 \in S_k$ in [RWZ, MV1, Kt]. The tools involved in these papers include continued fractions, Chebyshev polynomials, and Dyck words.

Chebyshev polynomials of the second kind (in what follows just Chebyshev polynomials) are defined by

$$U_r(\cos \theta) = \frac{\sin(r + 1)\theta}{\sin \theta}$$
for \( r \geq 0 \). Evidently, \( U_r(x) \) is a polynomial of degree \( r \) in \( x \) with integer coefficients. Chebyshev polynomials were invented for the needs of approximation theory, but are also widely used in various other branches of mathematics, including algebra, combinatorics, and number theory (see [\text{R}]).

*Dikey words* are words \( w \in \{x, \overline{x}\}^* \) verifying that \(|w|_x = |w|_{\overline{x}}\) and that for all \( w = w'w'' \), \(|w'|_x \geq |w''|_{\overline{x}}\).

Dikey words of length \( 2n \) are enumerated by the \( n \)th Catalan number \( C_n = \frac{1}{n+1} \binom{2n}{n} \) whose generating function is \( C(x) = \frac{1 - \sqrt{1 - 4x}}{2x} \).

We also consider words of \( \{a, b^2\}^* \) of length \( n \) enumerated by the \( n \)th Fibonacci number \( F_n \) with \( F_0 = F_1 = 1 \) and \( F_n = F_{n-1} + F_{n-2} \) whose generating function is \( F(x) = \frac{1}{1 - x - x^2} \).

Apparently, for the first time the relation between restricted permutations and Chebyshev polynomials was discovered by Chow and West in [\text{CW}], and later by Mansour and Vainshtein [\text{MV1}, \text{MV2}, \text{MV3}, \text{MV4}], Krattenthaler [\text{Kt}]. These results related to a rational function

\[
R_k(x) = \frac{2U_{k-1}(t)}{U_k(t)}, \quad t = \frac{1}{2\sqrt{x}} \]

for all \( k \geq 1 \).

An involution is a permutation such that its cycles are of length 1 or 2 that is \( \alpha \in S_n \) is an involution if and only if \( \alpha(\alpha_i) = i \) for all \( i \in [n] \).

Some authors considered involutions with forbidden patterns. Regev in [\text{Rd}] provided asymptotic formula for \( 12 \cdots k \)-avoiding involutions of length \( n \) and he also established that \( 1234 \)-avoiding involutions of length \( n \) are enumerated by Motzkin numbers. Gessel [\text{Ge}] exhibited the enumeration of such \( 12 \cdots k \)-avoiding involutions of length \( n \). Moreover, Gouyou-Beauchamps [\text{GB}] obtained by an entirely bijective proof very nice exact formulas for the number of \( 12345 \)-avoiding and \( 123456 \)-avoiding involutions of length \( n \).

Gire [\text{Gi}] studied some permutations with forbidden subsequences and established a one-to-one correspondence between 1-2 trees having \( n \) edges and permutations of length \( n \) avoiding patterns 321 and 231, the latter being allowed in the case where it is itself a subsequence of the pattern 3142. Guibert and Pergola and Pinzani [\text{G-PH}] also established bijections between 1-2 trees having \( n \) edges and vexillary involutions of length \( n \). So all these sets are enumerated by the \( n \)th Motzkin number \( \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{2n}{2i} C_i \). It remains a connected open problem: in [\text{Gu}] conjectures that \( 1432 \)-avoiding involutions of length \( n \) are also enumerated by the \( n \)th Motzkin number.

In this paper we present a general approach to the study of involutions in \( S_n \) avoiding 132 (or containing 132 exactly once), and avoiding (or containing exactly once) an arbitrary pattern \( \tau \in S_k \). As a consequence, we derive all the previously known results for this kind of problems, as well as many new results. Some results are also established by bijections as for an example a bijection between 132-avoiding involutions and primitive Dyck words.

The paper is organized as follows. The case of involutions avoiding both 132 and \( \tau \) is treated in Section 2. We present an explicit expression in terms of Chebyshev polynomials for several interesting cases. The case of involutions avoiding 132 and containing \( \tau \) exactly once is treated in section 3.
Here again we present an explicit expression in terms of Chebyshev polynomials for several interesting cases. Finally, the cases of involutions containing 132 exactly once and either avoiding or containing exactly once an arbitrary pattern \( \tau \) is treated in sections 4 and 5; respectively.

2. AVOIDING 132 AND ANOTHER PATTERN

Let \( I_T(n) \) denote the number of involutions in \( S_n(132) \) avoiding \( T \), and let \( I_T(x) = \sum_{n \geq 0} I_T(n)x^n \) be the corresponding generating function. The following proposition is the base of all the other results in this section, which holds immediately from definitions.

**Proposition 2.1.** For any involution \( \pi \in S_n(132) \) such that \( \pi_j = n \) holds either,

1. for \( 1 \leq j \leq \lceil n/2 \rceil \), \( \pi = (\beta, n, \gamma, \delta, j) \), where
   I. \( \beta \) is a 132-avoiding permutation of the numbers \( n-j+1, \ldots, n-2, n-1 \),
   II. \( \delta \) is a 132-avoiding permutation of the numbers \( 1, \ldots, j-2, j-1 \) such that \( \delta \cdot \beta \) is the identity permutation of \( S_{j-1} \),
   III. \( \gamma \) is a 132-avoiding involution of the numbers \( j+1, j+2, \ldots, n-j-1, n-j \);
2. for \( j = n \), \( \pi = (\beta, n) \) where \( \beta \) is an involution in \( S_{n-1}(132) \).

As a corollary of Proposition 2.1 we get the generating function for the number of involutions in \( S_n(132) \) as follows.

**Theorem 2.2.** (see [SS, Prop. 5]) Let \( C(t) \) be the generating function for the Catalan numbers; then

\[
I_\emptyset(x) = \frac{1}{1 - x - x^2C(x^2)}.
\]

**Proof.** Proposition 2.1 yields for all \( n \geq 1 \),

\[
I_\emptyset(n) = \sum_{j=1}^{\lceil n/2 \rceil} C_{j-1}I_\emptyset(n-2j) + I_\emptyset(n-1),
\]

where \( C_{j-1} \) is the \((j-1)\)th Catalan number. Besides \( I_\emptyset(0) = 1 \), therefore in terms of generating function we get that

\[
I_\emptyset(x) = 1 + x^2C(x^2)I_\emptyset(x) + xI_\emptyset(x).
\]

We can also prove this result by a bijective point of view.

Let \( P_{x,\overline{\tau}} = \{ w \in \{ x, \overline{x} \}^* : \text{ for all } w = w'w'', |w'|_x \geq |w''|_\overline{x} \} \) be the language of primitive Dyck words. The number of such words of \( P_{x,\overline{\tau}} \) of length \( n \) is the central binomial coefficient \( \binom{n}{\frac{n}{2}} \) with \( n \geq 0 \). Indeed, any primitive Dyck word \( w \) of \( P_{x,\overline{\tau}} \) can be uniquely written as \( w_0xw_1x \cdots xw_p \) where \( w_i \) is a Dyck word (that is \( w_i \in P_{x,\overline{\tau}} \) and \( |w_i|_x = |w_i|_\overline{x} \)) for all \( 0 \leq i \leq p \), but \( w \) can also be uniquely written as \( w_0xw_1x \cdots x |w_{\lceil n/2 \rceil}w_{\lceil n/2 \rceil+1}xw_{\lceil n/2 \rceil+2} \cdots xw_p \). So primitive Dyck words \( w \) of \( P_{x,\overline{\tau}} \) of length \( n \) are in bijection with bilateral words of \( \{ w \in \{ x, \overline{x} \}^* : |w|_x = |w|_{\overline{x}} \text{ or } |w|_x = |w|_{\overline{x}} - 1 \} \) of length \( n \) trivially enumerated by \( \binom{n}{\frac{n}{2}} \).

**Theorem 2.3.** There is a bijection \( \Phi \) between involutions in \( S_n(132) \) and primitive Dyck words of \( P_{x,\overline{\tau}} \) of length \( n \). Moreover, the number of fixed points of the involution corresponds to the difference between the number of letters \( x \) and \( \overline{x} \) into the primitive Dyck word.
Proof. Let $\pi$ be a 132-avoiding involution on $[n]$ having $p$ fixed points. According to Proposition 2.1 we have $\pi = \pi'\pi''x\pi'''$ with $|\pi'| = \frac{n-p}{2}$ ($\pi'$ has no fixed points and constitutes cycles with $\pi''$ or $\pi'''$), $\pi''$ does not contain fixed point and $\pi(x) = x$ ($x$ is the first fixed point). We obtain two 132-avoiding involutions on $[n + 1]$ from $\pi$: the first one is given by inserting a fixed point between $\pi'$ and $\pi''$, and the second one (iff $\pi$ has at least one fixed point) is given by modifying the first fixed point $x$ by a cycle starting between $\pi'$ and $\pi'''$. All 132-avoiding involutions can be obtained (and only once) by applying this rule, starting from the empty involution of length 0.

Let $w$ be a primitive Dyck word of $P_{x,\tau}$ of length $n$ such that $|w|_x - |w|_\tau = p$. So we have $w = w_0xw_1x\ldots xw_p$ where $w_i$ are Dyck words for all $0 \leq i \leq p$. We obtain two primitive Dyck words of length $n + 1$ from $w$: $xw$ and $xw_0xw_1x\ldots xw_p$ (iff $p > 0$). All primitive Dyck words can be obtained (and only once) by applying this rule, starting from the empty word of length 0.

Clearly, these two generating trees for the 132-avoiding involutions and the primitive Dyck words can be characterized by the following succession system:

\[
\begin{align*}
(0) & \sim (1) \\
(0) & \sim (1) \\
(p) & \sim (p + 1), (p - 1) \quad \text{if } p \geq 1
\end{align*}
\]

Figure 1 shows the bijection $\Phi$ between 132-avoiding involutions and primitive Dyck words (and the labels of the succession system which characterizes them) for the first values.

**Corollary 2.4.** The number of 132-avoiding involutions of length $n$ is $\binom{n}{\frac{n}{2}}$. Moreover, the number of 132-avoiding involutions of length $n$ having $p$ fixed points with $0 \leq p \leq n$ (and $p$ is odd iff $n$ is odd) is the ballot number $\binom{n}{\frac{n}{2}} - \binom{n+p}{\frac{n+p+1}{2}}$.

Proof. Indeed, the number of primitive Dyck words $w$ of $P_{x,\tau}$ according to the length and $|w|_x - |w|_\tau$ is given by the ballot number (or Delannoy number [3] or distribution $\alpha$ of the Catalan number [Kw]).

In particular, the number of 132-avoiding involutions of length $2n$ without fixed points is $C_n$ the $n$th Catalan number.

The following theorem is the base of all the other results in this section.

**Theorem 2.5.** Let $T$ set of patterns, $T' = \{(\tau, |\tau| + 1) : \tau \in T\}$, and let $S_T(x)$ be the generating function for the number of $T$-avoiding permutations in $S_n(132)$. Then

\[ I_{T'}(x) = \frac{1}{1 - x^2S_T(x^2)} + \frac{x}{1 - x^2S_T(x^2)} I_T(x). \]

Proof. Proposition 2.1 with definitions of $T'$ yields for $n \geq 1$,

\[ I_{T'}(n) = I_T(n - 1) + \sum_{j=1}^{\lfloor n/2 \rfloor} s_T(j - 1) I_{T'}(n - 2j), \]

where $s_T(j - 1)$ is the number of permutations in $S_{j - 1}(132, T)$. Hence, in terms of generating functions we have

\[ I_{T'}(x) - 1 = x \cdot I_T(x) + x^2S_T(x^2) \cdot \frac{I_{T'}(x) + I_{T'}(-x)}{2} + x^2S_T(x^2) \cdot \frac{I_{T'}(x) - I_{T'}(-x)}{2}, \]

so the theorem holds.
2.1. Avoiding 132 and 12...k.

Example 2.6. (see [SS]) Let us find $I_{123}(x)$; let $T' = \{123\}$ and $T = \{12\}$, so Theorem 2.5 gives

$$I_{123}(x) = \frac{1}{1 - x^2 S_{12}(x^2)} + \frac{x}{1 - x^2 S_{12}(x^2)} I_{12}(x),$$

where by definitions $S_{12}(x) = I_{12}(x) = \frac{1}{1-x}$, hence

$$I_{123}(x) = \frac{1 + x}{1 - 2x^2},$$

which means the number of involutions $I_{123}(n)$ is given by $2^{\lfloor n/2 \rfloor}$ for all $n \geq 0$.

Similarly, $I_{1234}(x) = \frac{1}{1 - x - x^2}$, so the number of involutions $I_{1234}(n)$ is given by $F_n$, the $n$th Fibonacci number.
The case of varying \( k \) is more interesting. As an extension of Example 2.3 let us consider the case \( T = \{12 \ldots k\} \).

**Theorem 2.7.** For all \( k \geq 1 \),

\[
I_{12 \ldots k}(x) = \frac{1}{x \cdot U_k \left( \frac{1}{2x} \right)} \sum_{j=0}^{k-1} U_j \left( \frac{1}{2x} \right).
\]

**Proof.** Immediately, the theorem holds for \( k = 1 \). Let \( k \geq 2 \); Theorem 2.4 gives

\[
I_{12 \ldots k}(x) = \frac{x}{1 - x^2 S_{12 \ldots (k-1)}(x^2)} + \frac{1}{1 - x^2 S_{12 \ldots (k-1)}(x^2)} I_{12 \ldots (k-1)}(x).
\]

On the other hand, the generating function for the sequence \( S_n(12, 12 \ldots (k-1)) \) is given by \( R_k(x) \) (see [CW] Th. 1) with \( R_k(x) = \frac{1}{1 - x R_{k-1}(x)} \) (see [MV]) we get that

\[
I_{12 \ldots k}(x) = R_k(x^2) + x R_k(x^2) I_{12 \ldots (k-1)}(x).
\]

Besides \( I_1(x) = R_0(x) = 1 \), hence by use induction on \( k \) and definitions of \( R_k(x) \) the theorem holds. \( \square \)

We consider now a combinatorial point of view for this result.

Let \( \pi \) be a 132-avoiding involution. Clearly, if \( \pi \) avoids 12...\( k \) then \( \pi \) has less than \( k \) fixed points. Moreover, if \( \pi \) of length \( n \) having less than \( k \) fixed points is obtained from an 132-avoiding involution \( \sigma \) of length less than \( n \) having \( k \) fixed points (and take \( \sigma \) as big as possible) by applying the rules described for bijection \( \Phi \) given by Theorem 2.3, then \( \pi \) contains a subsequence 12...\( k \) because the first fixed points of \( \sigma \) become cycles into \( \pi \) such that the beginning of these cycles and the last remaining fixed points of \( \sigma \) into \( \pi \) constitute a subsequence of type 12...\( k \). So the succession system \((\ast)\)

\[
\begin{align*}
(0) & \sim (1) \\
(p) & \sim (p+1), (p-1), 1 \leq p \leq k-2 \\
(k-1) & \sim (k-2)
\end{align*}
\]

characterizes the generating tree of the involutions avoiding both 132 and 12...\( k \).

It is easy to see that for \( k \) odd, the number of involutions of length \( 2m \) avoiding both 132 and 12...\( k \) is the twice of the number of involutions of length \( 2m-1 \) avoiding both 132 and 12...\( k \).

Moreover, the reader can note that the set of labels of this succession system is finite and so the corresponding generating function is rational. More precisely, we immediately deduce from the previous succession system that the number of involutions of length \( n \) avoiding both 132 and 12...\( k \) and having \( p \) fixed points is given by the \((p+1)\)th component of the vector given by \( V_k M_k^n \) where

\[
V_k = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \end{pmatrix}
\]

\( k \times k \) matrix.

In another way, we can see that as an automaton where the states are 0, 1, \ldots, \( k-1 \) and the transitions are arrows from \( i \) to \( i+1 \) for \( 0 \leq i < k-1 \) and from \( i \) to \( i-1 \) for \( 0 < i \leq k-1 \).
The bijection \( \Phi \) establishes a one-to-one correspondence between involutions of length \( n \) avoiding both 132 and 12 \ldots k and having \( p \) fixed points, and primitive Dyck words \( w = w_0xw_1x \ldots xw_p \) of \( P_{2k} \) of length \( n \) such that \( w_i \) is a Dyck word of height less than \( k - p + i \) (that is \( w_i \in P_{2k} \) and \( |w_i|_x = |w_i|_\pi \)) and for all \( w_i = w'^i \), \( |w'| = |w'|_\pi < k - p + i \) for all \( 0 \leq i \leq p \).

In particular involutions of length 2\( n \) avoiding both 132 and 12 \ldots k without fixed points are in bijection by \( \Phi \) with Dyck words of length 2\( n \) of height less than \( k \).

These Dyck words of bounded height was considered by Kreweras [Kw] and Viennot [V]. In particular, Dyck words of length \( 2n \) of height less than 1, 2, 3, 4, 5 are respectively enumerated by 0, 1, 2\(^{n-1} \), \( F_{n-2} \), \( \frac{2n+1}{3} \) for all \( n \geq 1 \).

We provide some simple bijections for special cases \( k = 3, 4, 5 \) (related to Example 2.6) by generating some well known words in the same way as involutions avoiding both 132 and 12 \ldots k.

Fist of all, we consider the case \( k = 3 \) and the words of \( \{a,b\}^* \) or \( a\{a,b\}^* \) enumerated by the powers of 2 we can generate from the empty word labeled (0) by the rules:

\[
\begin{align*}
\text{w}(0) & \leadsto aw(1) \\
aw(1) & \leadsto aw(2), bw(0) \\
w(2) & \leadsto bw(0)
\end{align*}
\]

such that the words labeled (0) start by \( a \) whereas the words labeled (1) or (2) start by \( b \).

So, words of \( \{a,b\}^n \) (respectively \( \{a,b\}^n \)) are in bijection with involutions avoiding both 132 and 123 of length 2\( n \) (respectively 2\( n + 1 \)) enumerated by 2\( n \) (respectively 2\( n \)).

Next we consider the case \( k = 4 \) and the words of \( \{a,b^2\}^* \) enumerated by the Fibonacci numbers we can generate from the empty word labeled (0) by the rules:

\[
\begin{align*}
\text{w}(0) & \leadsto aw(1) \\
aw(1) & \leadsto aw(2), b^2w(0) \\
w(2) & \leadsto b^2w(3), aw(1) \\
w(3) & \leadsto bw(2)
\end{align*}
\]

such that the words labeled (0) or (3) start by \( b^2 \) whereas the words labeled (1) or (2) start by \( a \).

So, words of \( \{a,b^2\}^* \) of length \( n \) are in bijection with involutions in \( S_n(132, 1234) \) enumerated by \( F_n \) the nth Fibonacci number.

Now we consider the case \( k = 5 \) and the words of \( \{a,b,c\}^*a \) or \( \{a,b,c\}^*a \cup b\{a,b,c\}^*a \) enumerated by the powers of 3 we can generate from the empty word labeled (0) by the rules:

\[
\begin{align*}
\text{w}(0) & \leadsto aw(1) \\
w(1) & \leadsto bw(2), w(0) \\
w = bw' = bb^*cw(2) & \leadsto w(3), aw(1) \\
w = bw' = bb^*aw(2) & \leadsto cw(3), w(1) \\
w(3) & \leadsto w(4), bw(2) \\
w(4) & \leadsto cw(3)
\end{align*}
\]

such that the words labeled (0) or (1) start by \( b^*a \), the words labeled (3) or (4) start by \( b^*c \), and the words labeled (2) start by \( b \) (and have one letter more than words labeled (0) or (4) at the same level).

So, words of \( \{a,b,c\}^n \) (respectively \( \{a,b,c\}^n \cup b\{a,b,c\}^n \)) are in bijection with involutions avoiding both 132 and 12345 of length 2\( n + 1 \) (respectively 2\( n + 2 \)) enumerated by 3\( n \) (respectively 2.3\( n \)).

Figure 2 (an output of the software forbid [Gu]) shows the first values for the number of involutions avoiding both 132 and 12 \ldots k for 3 \leq k \leq 5 according to the number of fixed points.

2.2. Avoiding 132 and 213 \ldots k.
Involutions $\pi \in S_n(132, 123)$ according to $|\{\pi(x) = x\}|$ for $1 \leq n \leq 15$

$=1$ $=2$ $=3$ $=4$ $=5$ $=6$ $=7$ $=8$ $=9$ $=10$ $=11$ $=12$ $=13$ $=14$ $=15$: [n]

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|-----|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|
| 1   |   |   |   |   |   |   |   |   |   | 0  | 0  | 0  | 0  | 0   |
| 2   |   | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0   | 0   | 0   | 0   | 0   |
| 3   |   | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0   | 0   | 0   | 0   | 0   |
| 4   |   | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0   | 0   | 0   | 0   | 0   |

Involutions $\pi \in S_n(132, 1234)$ according to $|\{\pi(x) = x\}|$ for $1 \leq n \leq 15$

$=1$ $=2$ $=3$ $=4$ $=5$ $=6$ $=7$ $=8$ $=9$ $=10$ $=11$ $=12$ $=13$ $=14$ $=15$: [n]

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|-----|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|
| 1   |   |   |   |   |   |   |   |   |   | 0  | 0  | 0  | 0  | 0   |
| 2   |   | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0   | 0   | 0   | 0   | 0   |
| 3   |   | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0   | 0   | 0   | 0   | 0   |
| 4   |   | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0   | 0   | 0   | 0   | 0   |

Involutions $\pi \in S_n(132, 12345)$ according to $|\{\pi(x) = x\}|$ for $1 \leq n \leq 15$

$=1$ $=2$ $=3$ $=4$ $=5$ $=6$ $=7$ $=8$ $=9$ $=10$ $=11$ $=12$ $=13$ $=14$ $=15$: [n]

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|-----|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|
| 1   |   |   |   |   |   |   |   |   |   | 0  | 0  | 0  | 0  | 0   |
| 2   |   | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0   | 0   | 0   | 0   | 0   |
| 3   |   | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0   | 0   | 0   | 0   | 0   |
| 4   |   | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0   | 0   | 0   | 0   | 0   |

\textbf{Figure 2.} The number of involutions avoiding both 132 and 12...k for $3 \leq k \leq 5$ according to the number of fixed points.

\textbf{Example 2.8.} Let us find $I_{213}(x)$; let $T' = \{213\}$ and $T = \{21\}$, so Theorem 2.7 gives

$$I_{213}(x) = \frac{1}{1 - x^2 S_{21}(x^2)} + \frac{x}{1 - x^2 S_{21}(x^2)} I_{21}(x),$$

where by definitions $S_{21}(x) = I_{21}(x) = \frac{1 - x}{1 - 2x}$, hence

$$I_{213}(x) = \frac{1 + x}{1 - 2x^2},$$

which means the number of involutions $I_{213}(n)$ is given by $2^{[n/2]}$ for all $n \geq 0$.

Similarly, $I_{2134}(x) = \frac{1}{1 - x^2 S_{21}(x^2)}$, so the number of involutions $I_{2134}(n)$ is given by $F_n$, the $n$th Fibonacci number.

We can easily prove by a combinatorial way that $I_{213}(n)$ is given by $2^{[n/2]}$.

An involution $\pi$ avoiding both 132 and 213 of length $n$ can be either $(n + 1 - i)(n + 2 - i) \ldots n\pi'12 \ldots i$ with $i \geq 1$ or 12...n such that $\pi'$ is also an involution avoiding both 132 and 213 (of length $n - 2i$, and if we subtract $i$ to each element). We code this recursive decomposition of an involution $\pi$ avoiding both 132 and 213 by a word of nonnegative integers formed by the successive positive numbers $i$ and whose last nonnegative integer is (the smallest integer of) the number of the fixed points in $\pi$ divided by 2. This coding is clearly bijective.

For example, the involutions $\epsilon, 1, 12, 21, 123, 321, 1234, 4231, 3412$ and 4321 are respectively coded by $0, 0, 1, 10, 1, 10, 2, 11, 20$ and 110. Moreover, the involution 21 19 20 16 17 18 15 14 9 10 11 12
Theorem 2.9. For all $k \geq 1$, 
\[ I_{213\ldots k}(x) = \frac{1}{x \cdot U_k \left( \frac{x}{2} \right)} \sum_{j=0}^{k-1} U_j \left( \frac{1}{2x} \right). \]

Therefore, Theorem 2.7 and Theorem 2.9 yields $I_{123\ldots k}(n) = I_{213\ldots k}(n)$. We establish a bijection for this result.

Theorem 2.10. There is a bijection between involutions avoiding both 132 and 12\ldots k of length $n$ and involutions avoiding both 132 and 2134\ldots k of length $n$, for any $k \geq 3$.

Moreover, two involutions in bijection have the same number of fixed points $p$ for all $0 \leq p \leq k - 3$ whereas the involutions avoiding both 132 and 12\ldots k having $k - 2$ or $k - 1$ fixed points correspond to the involutions avoiding both 132 and 2134\ldots k having $k - 2$ or more fixed points.

Proof. In order to establish this result we consider a generating tree for the involutions avoiding both 132 and 2134\ldots k which is characterized by the same succession system $(*)$ given in Subsection 2.1 characterizing a generating tree for the involutions avoiding both 132 and 12\ldots k.

So let $\pi$ be an involution avoiding both 132 and 2134\ldots k of length $n$ and let $q = |\{\pi(x) = x\}|$ be the number of fixed points of $\pi$. The label $(p)$ of $\pi$ is defined by $p = q$ if $q \leq k - 3$ or by $p = k - 2$ if $q \geq k - 2$ and $(n + k) \mod 2 = 0$ or by $p = k - 1$ if $q \geq k - 2$ and $(n + k) \mod 2 = 1$. Of course, the empty involution of length 0 has label (0). We obtain $\sigma$ an involution avoiding both 132 and 2134\ldots k of length $n + 1$ by applying the following rules:

- If $p \in [0, k - 3]$, we have $\pi = \pi'\pi''$ with $|\pi'| = \frac{n-k}{2}$, and then $\sigma$ obtained by inserting a fixed point between $\pi'$ and $\pi''$ has label $(p + 1)$.
- If $p = k - 2$, we have $\pi = \pi'x\pi''$ with $\pi(x) = x = \frac{n+4-k}{2}$, and then $\sigma$ obtained by inserting a fixed point between $\pi'$ and $x$ has label $(k - 1)$.
- If $p \in [1, k - 3]$, we have $\pi = \pi'\pi''x\pi'''$ with $|\pi'| = \frac{n-k}{2}$, $\pi(x) = x$ and $\pi(y) \neq y$ for all $1 \leq y < x$. Then $\sigma$ obtained by modifying the first fixed point $x$ by a cycle starting between $\pi'$ and $\pi'''$ (and ending in $x$) has label $(p - 1)$.
- If $p = k - 1$, we have $\pi = \pi'x(x+1)\pi''$ with $\pi(x) = x = \frac{n+3-k}{2}$, and then $\sigma$ obtained by inserting a fixed point between $\pi'$ and $x$ has label $(k - 2)$.
- If $p = k - 2$, we have $\pi = \pi'(x - j)(x - j + 1)\ldots(x + j)\pi''$ with $j \geq 0$, $\pi(x) = x = \frac{n+4-k}{2}$, $|\pi'| = \frac{n-k}{2} - j + e$ for all $e \in \pi'$. Then $\sigma$ obtained by modifying the $2j + 1$ fixed points between $\pi'$ and $\pi''$ by $j + 1$ consecutive cycles each of difference (between the index and the value) $j + 1$ that is $(\pi'_1 + 1)(\pi'_2 + 1)\ldots(\pi'_{e_{\pi'_{j+1}+1}} + 1)(\frac{n-k}{2} + 3)(\frac{n-k}{2} + 4)\ldots(\frac{n-k}{2} + 3 + j)(\frac{n-k}{2} - j + 2)(\frac{n-k}{2} - j + 3)\ldots(\frac{n-k}{2} + 2)\pi''$ has label $(k - 3)$.

\[ \square \]
Corollary 2.11. There is a bijection between permutations avoiding both 132 and 12...k of length n and permutations avoiding both 132 and 2134...k of length n, for any k ≥ 3.

Proof. By Proposition 2.1, we deduce that permutations π avoiding both 132 and 12...k (respectively 2134...k) of length n are in bijection with involutions without fixed points (π⁻¹ + n)π avoiding both 132 and 12...k (respectively 2134...k) of length 2n. Moreover, a particular case of Theorem 2.10 establishes a one-to-one correspondence between involutions avoiding both 132 and 2134...k without fixed points and involutions avoiding both 132 and 2134...k without fixed points. □

2.3. Avoiding 132 and (d + 1(d + 2)...k12...d).

Example 2.12. By Proposition 2.7 it is easy to obtain for n ≥ 1,

\[ I_{231} = n; \quad I_{321} = \lfloor n/2 \rfloor + 1. \]

We consider a combinatorial approach to show Example 2.12. Clearly, we have that involutions avoiding both 132 and 231 of length n are i(i − 1)...(i + 1)(i + 2)...n for all 1 ≤ i ≤ n and that involutions avoiding both 132 and 321 of length n are (i + 1)(i + 2)...(2i)12...i(2i + 1)(2i + 2)...n for all 0 ≤ i ≤ \( \lfloor n/2 \rfloor \).

As an extension of Example 2.12 let us consider the case \( T = \{[k,d]\} \), where \([k,d] = (d + 1, d + 2, \ldots, k, 1, 2, \ldots, d)\).

Theorem 2.13. For any k ≥ 2, k/2 ≥ d ≥ 1,

\[ I_{[k,d]} = \frac{1}{x(U_d(t) - U_{d-1}(t))} \left[ U_{d-1}(t) + \frac{U_{k-2d-1}(t)}{U_{k-d}(t)U_{k-d-1}(t)} \sum_{j=0}^{k-d-1} U_j(t) \right], \quad t = \frac{1}{2x}. \]

Proof. Proposition 2.1 yields, in the second case the generating function for the number of involutions \([k,d]-\)avoiding permutations is \( xI_{[k,d]}(x) \). In the first case, we assume that \( \gamma \) either (1) avoiding 12...k, or (2) containing 12...k. In (1), \( \beta \) and \( \delta \) avoiding 12...k, so the generating function for these number of involutions is \( x^2 R_{k-d-1}(x^2)I_{12...k}(x) \) (similarly Theorem 2.7). In (2), \( \beta \) and \( \delta \) avoiding 12...k, so the generating function for these number of involutions is \( x^2 R_{d-1}(x^2)(I_{[k,d]}(x) - I_{12...k}(x)) \) (the generating function for the number of involutions in \( S_n(132, [k,d]) \) such containing 12...k is given \( I_{[k,d]}(x) - I_{12...k}(x) \)). Therefore

\[ I_{[k,d]}(x) = 1 + xI_{[k,d]}(x) + x^2 R_{k-d-1}(x^2)I_{12...k}(x) + x^2 R_{d-1}(x^2)(I_{[k,d]}(x) - I_{12...k}(x)), \]

which means that

\[ I_{[k,d]}(x) = \frac{1}{1 - x - x^2 R_{k-d-1}(x^2)} \cdot (1 + x^2 I_{12...k}(x^2)R_{k-d-1}(x^2) - R_{d-1}(x^2)). \]

Hence, by use the identities \( R_k(x) = \frac{1}{1 - x R_{k-1}(x)} \) and \( R_a(x) - R_b(x) = \frac{U_{r+1}(t)}{\sqrt{2U_a(t)U_b(t)}} \), the theorem holds. □

Example 2.14. Theorem 2.13 yields for k = 4 and d = 2, the number of involutions \( I_{3412}(n) \) is given by \( F_n \) the nth Fibonacci number.
We consider a combinatorial approach to show Example 2.14. An involution \( \pi \) avoiding both 132 and 3412 of length \( n \) can be written as \( \pi = \pi_1(i + 1)(i + 2) \cdots n \) with \( 1 \leq i \leq n \) such that \( \pi' \) is also an involution avoiding both 132 and 3412 (of length \( n - 2 \), and if we subtract 1 to each element). We code \( \pi \) by a word of \( \{a, b\}^n \) of length \( n \) in that way: \( a \) if \( \pi_i = i \), \( b \) if \( \pi_i < i \) and nothing if \( \pi_i > i \) for all \( 1 \leq i \leq n \). This coding is clearly bijective.

Following [MV2] we say that \( \tau \in S_k \) is a wedge pattern if it can be represented as \( \tau = (\tau^1, \rho^1, \ldots, \tau^r, \rho^r) \) so that each of \( \tau^i \) is nonempty, \( (\rho^1, \rho^2, \ldots, \rho^r) \) is a layered permutation of \( 1, \ldots, s \) for some \( s \), and \( (\tau^1, \tau^2, \ldots, \tau^r) = (s + 1, s + 2, \ldots, k) \). For example, 645783912 and 456378129 are wedge patterns.

For a further generalization of Theorem 2.7, Theorem 2.9 and [MV2, Th. 2.6], consider the following definition. We say that \( \tau \in S_{2l} \) is a double-wedge pattern if there exist a wedge pattern \( \sigma \in S_{l-1} \) such that

\[
\tau = (\sigma^{-1} + l, 2l, \sigma, l) \text{ or } \tau = (\sigma + l, 2l, \sigma^{-1}, l).
\]

For example, the double-wedge patterns of length 10 are 6789(10)12345, 7689(10)21345, 7869(10)31245, 7896(10)12345, 8679(10)31245, 8796(10)23145, 8967(10)34125, 9678(10)23415 and 9768(10)32415.

**Theorem 2.15.** For any double-wedge pattern \( \tau \in S_{2l}(132) \)

\[
I_\tau(x) = I_{12\ldots(2l)}(x) = \frac{R_l(x^2)}{1 - xR_l(x^2)}.
\]

**Proof.** First of all, let us find the generating function \( I_\rho(x) \) where \( \rho = (\sigma^{-1} + l, 2l, \sigma, l) \). By use Proposition 2.1 we obtain in the first case \( xI_\rho(x) \), and in the second case \( x^2S_\sigma(x^2)I_\rho(x) \) where \( S_\sigma(x^2) \) is the generating function for the number of permutations in \( S_n(132, \sigma) \), therefore (1 for the empty permutation)

\[
I_\rho(x) = 1 + xI_\rho(x) + x^2S_\sigma(x^2)I_\rho(x).
\]

On the other hand, Mansour and Vainshtein proved \( S_\sigma(x) = R_{l-1}(x) \) for any wedge pattern \( \sigma \), so

\[
I_\rho(x) = \frac{1}{1 - x - x^2R_{l-1}(x^2)}.
\]

By use the identity \( R_l(x) = \frac{1}{1-xR_{l-1}(x^2)} \) we have

\[
I_\rho(x) = \frac{R_l(x)}{1 - xR_l(x^2)}.
\]

Now, let us find \( I_{12\ldots2l}(x) \) in terms of \( R_l(x) \). So, by use the identity

\[
\sum_{j=0}^{2l} U_j(t) = \frac{U_{2l}(t)U_{l-1}(t)}{U_l(t) - U_{l-1}(t)}
\]

and use the symmetric inverse operation, the first part of the theorem holds.

**Theorem 2.16.** For any wedge pattern \( \sigma \in S_{l-1} \) the generating function for the number of permutations in \( S_n(132, (\sigma^{-1} + l, 2l, \sigma, l, 2l + 1, \ldots, k)) \) (or \( S_n(132, (\sigma + l, 2l, \sigma^{-1}, l, 2l + 1, \ldots, k)) \), or \( S_n(132, (\sigma + l, 2l, \sigma, l, 2l + 1, \ldots, k)) \)) is given by \( R_k(x) \), for all \( k \geq 2l \).
Proof. Let $\tau = (\sigma^{-l} + 1, 2l, \sigma, l)$ and let $S_{\tau}(x)$ be the generating function for the number of permutations in $S_\alpha(123, \tau)$. By [MV2, Th. 1] we have
\[ S_{\tau}(x) = 1 + x(S_{\tau}(x) - S_{\sigma}^{-1}(x))S_{\sigma}(x) + xS_{\sigma}^{-1}(x)S_{\tau}(x). \]
On the other hand, by [MV2, Th. 2.6] and $\sigma$ a wedge pattern in $S_{\tau-1}(123)$ we have $S_{\sigma-1}(x) = S_{\sigma}(x) = R_{l-1}(x)$. Therefore, by use the identity $R_l(x) = \frac{1}{xR_{l-1}(x)}$ we get
\[ S_{\tau}(x) = \frac{R_l(x)(1 - xR_{l-1}(x))R_l(x)}{1 - xR_l^2(x)}. \]

By use the definitions of Chebyshev polynomials of the second kind it is easy to see
\[ \frac{R_l(x)(1 - xR_{l-1}(x))R_l(x)}{1 - xR_l^2(x)} = R_{2l}(x), \]
hence by use again Theorem [MV2, Th. 1] we have $S_{(\tau, 2l+1, \ldots, k)}(x) = R_k(x)$. Similarly we obtain the other cases. \hfill \Box

As a corollary of Theorem 2.15 we have

**Theorem 2.17.** For any double wedge pattern $\tau \in I_{2l}(132)$
\[ I_{(\tau, 2l+1, 2l+2, \ldots, k)}(x) = I_{12..k}(x). \]

**Proof.** Since, if $S_{\beta}(x) = S_{\gamma}(x)$ and $I_{\beta}(x) = I_{\gamma}(x)$, then Theorem 2.15 yields $I_{\tau}(x) = I_{\beta}(x)$, and by use [MV2, Th. 1] we have $S_{\tau}(x) = S_{\beta}(x)$, where $\tau' = (\tau_1, \ldots, \tau_p, p + 1)$ and $\beta' = (\tau_1, \ldots, \tau_p, p + 1)$ two patterns in $S_{p+1}$. Hence, the theorem holds by use Theorem 2.15, Theorem 2.16, and induction on $p$. \hfill \Box

In view of Theorem 2.15 and Theorem 2.17 it is a challenge to find a bijective proof.

### 2.4. Avoiding 132 and two other patterns.

Now, let us restrict more than two patterns (132 and two other patterns).

**Example 2.18.** Let us find $I_{123,213}(x)$; let $T' = \{123, 213\}$ and $T = \{12, 21\}$, so Theorem 2.13 gives
\[ I_{123,213}(x) = \frac{1}{1 - x^2S_{12,21}(x^2)} + \frac{x}{1 - x^2S_{12,21}(x^2)}I_{12,21}(x), \]
where by definitions $S_{12,21}(x) = I_{12,21}(x) = 1 + x$, hence
\[ I_{123,213}(x) = \frac{1 + x + x^2}{1 - x^2 - x^4}, \]
which means the number of involutions $I_{123,213}(2n)$ is given by $F_{n+1}$, and $I_{123,213}(2n+1)$ is given by $F_n$ for all $n \geq 0$, where $F_m$ is the $m$th Fibonacci number.

We consider a combinatorial approach to show Example 2.18 by distinguishing the cases of odd and even length.

An involution $\pi$ avoiding 132, 123 and 213 of length $2n + 1$ can be written either $(2n + 1)\pi'1$ or $(2n)(2n + 1)\pi''21$ or 1 (if $n = 0$) such that $\pi'$ and $\pi''$ are also involutions avoiding 132, 123 and 213 (of length $2n - 1$ for $\pi'$ if we subtract 1 to each element, of length $2n - 3$ for $\pi''$ if we subtract 2 to each element). We code $\pi$ by a word of $\{a, b^2\}^*$ of length $n$ in that way: $a$ if $\pi_i = 2n + 2 - i$, $b^2$ if $\pi_i = 2n + 1 - i$ and nothing if $\pi_i = 2n + 3 - i$ for all $1 \leq i \leq n$. This coding is clearly bijective.
An involution \( \pi \) avoiding 132, 123 and 213 of length 2\( n \) can be written either \((2n)\pi'1\) (that includes 21 if \( n = 1 \)) or \((2n-1)(2n)\pi''121\) or 12 (if \( n = 1 \)) or the empty involution (if \( n = 0 \)) such that \( \pi' \) and \( \pi'' \) are also involutions avoiding 132, 123 and 213 (of length \( 2n - 2 \) for \( \pi' \) if we subtract 1 to each element, of length \( 2n - 4 \) for \( \pi'' \) if we subtract 2 to each element). We code \( \pi \) by a word of \( \{a, b^2\}^* \) of length \( n + 1 \) in that way: \( a \) if \( \pi_i = 2n + 1 - i \) for all \( 1 \leq i \leq n - 1 \), \( b^2 \) if \( \pi_n = n + 1 \), \( b^2 \) if \( \pi_i = 2n - i \) for all \( 1 \leq i \leq n - 2 \), \( b^2a \) if \( \pi_{n-1} = n + 1 \) and \( aa \) if \( \pi_n = n \). Moreover, the empty involution is coded by \( a \). This coding is clearly bijective.

Using definitions and Theorem 2.3 it is easy to see the following.

**Corollary 2.19.** For all \( k \geq 1 \),

\[
I_{123...k,213}(x) = I_{(k-1)...21k,123}(x) = \frac{1 + x + x^2 + \cdots + x^{k-1}}{1 - x^2 - x^4 - \cdots - x^{2(k-1)}}.
\]

**Example 2.20.** Using Proposition 2.4 it is easy to see for \( n \geq 1 \),

\[
I_{213,321}(n) = \frac{1}{2}((-1)^n + 3), \quad I_{213,4321}(n) = [n/2] + 1.
\]

We consider a combinatorial approach to show Example 2.20. Clearly, we have that involutions avoiding 132, 213 and 321 of length \( n \) are \( 12\ldots n \) and also \((m+1)(m+2)\ldots n12\ldots m \) if \( n = 2m \) with \( m \geq 1 \). We also have that involutions avoiding 132, 213 and 4321 of length \( n \) are \((n+1-i)(n+2-i)\ldots(j+1)i(j+2)\ldots(n-i)12\ldots i \) for all \( 0 \leq i \leq \lfloor n/2 \rfloor \).

### 3. Avoiding 132 and containing another pattern

Let \( I_\tau(n) \) denote the number of involutions in \( S_n(132) \) containing \( \tau \) exactly \( r \) times, and let \( I_\tau^r(x) = \sum_{n \geq 0} I_\tau(n)x^n \) be the corresponding generating function. Let us start by the following example.

**Example 3.1.** By Proposition 2.4 it is easy to see

\[
I_{12}^1(x) = x I_1^1(x) + x^2 I_{12}^1(x),
\]

which means \( I_{12}^1(x) = \frac{x^2}{1-x^2} \).

As extension of Example 3.1 let us consider the case \( \tau = 12\ldots k \).

**Theorem 3.2.** For all \( k \geq 1 \);

\[
I_{12\ldots k}^1 = \frac{1}{U_k(\frac{1}{2x})}.
\]

**Proof.** By Proposition 2.4 we have for \( n \geq k \),

\[
I_{12\ldots k}^1(n) = I_{12\ldots(k-1)}^1(n-1) + \sum_{j=1}^{[n/2]} s_{12\ldots(k-1)}(j - 1) I_{12\ldots k}^1(n-2j),
\]

where \( s_{12\ldots k}(j-1) \) is the number of \( 12\ldots k \)-avoiding permutations in \( S_{j-1}(132) \). Besides \( I_{12\ldots k}^1(n) = 0 \) for all \( n \leq k - 1 \), and \( I_{12\ldots k}^1(k) = 1 \). Similarly as proof of Theorem 2.7 we have

\[
I_{12\ldots k}^1(x) = x R_k(x^2) I_{12\ldots(k-1)}^1(x).
\]

Hence, by induction on \( k \) with initial condition \( I_1^1 = x \), the theorem holds. \( \square \)
Similarly as Theorem 3.2 we have an explicit formula when \( \tau = 213 \ldots k \) or \( \tau = 23 \ldots k1 \).

**Theorem 3.3.** For all \( k \geq 2 \),
\[
I^1_{121 \ldots k} = \frac{1-x^2}{U_k \left( \frac{x}{12} \right)}, \quad I^1_{23 \ldots k1}(x) = \frac{x^3}{(1-x)U_{k-2} \left( \frac{x}{12} \right)}.
\]

More generally, by Proposition 2.1 and the argument proof of Theorem 2.7 we get

**Theorem 3.4.** For any \( k, r \geq 1 \)
\[
I^r_{12 \ldots k}(x) = xI^r_{12 \ldots (k-1)}(x) + x^2 \sum_{2a+b=r} S^n_{12 \ldots (k-1)}(x^2)I^b_{12 \ldots k}(x),
\]
where \( S^n_{12 \ldots (k-1)}(x) \) is the generating function for the number of permutations in \( S_n \) containing \( 12 \ldots (k-1) \) exactly \( r \) times.

In [Kt] found an explicit formula for \( S^n_{12 \ldots k}(x) \), so Theorem 3.4 yields a recurrence for \( I^r_{12 \ldots k}(x) \). For example, following [Kt] ([MV1, Th. 3.1]) we have a recurrence for \( I^r_{12 \ldots k}(x) \) where \( r = 1, 2, \ldots, 2k \).

**Theorem 3.5.** Let \( k \geq 1 \); for \( r = 1, 2, \ldots, 2k \)
\[
I^r_{12 \ldots k}(x) = xI^r_{12 \ldots (k-1)}(x) + x^2 \sum_{2a+b=r, \ a>0} x^{a-1}I^b_{12 \ldots k}(x) \frac{U_{a-1} \left( \frac{x}{12} \right)}{U_{a-1} \left( \frac{x}{12} \right)}.
\]

The above Theorem yields for \( r = 2 \) an explicit formula for \( I^2_{12 \ldots k}(x) \).

**Corollary 3.6.** For all \( k \geq 1 \),
\[
I^2_{12 \ldots k}(x) = \frac{1}{U_k \left( \frac{x}{12} \right)} \sum_{i=1}^{k} \frac{U_i \left( \frac{1}{12} \right)}{U_{k+1-i} \left( \frac{x}{12} \right) U_{k-i} \left( \frac{x}{12} \right)}.
\]

**4. CONTAINING 132 ONCE AND AVOIDING ANOTHER PATTERN**

We first relate involutions containing 132 once to 132-avoiding involutions.

**Theorem 4.1.** There is a bijection \( \Psi \) between involutions containing 132 exactly once of length \( n \) having \( p \) fixed points with \( 1 \leq p \leq n \) and 132-avoiding involutions of length \( n - 2 \) having also \( p \) fixed points.

**Proof.** Let \( \pi = \pi' xz \pi'' y \pi''' \) with \( \pi(x) = x, \pi(y) = z \) and \( 1 + x = y < z \) be an involution containing 132 once (that is subsequence \( xzy \)) of length \( n \) having \( p \) fixed points. We replace the subsequence \( xzy \) by a fixed point between \( \pi' \) and \( \pi'' \) in order to obtain an 132-avoiding involution of length \( n - 2 \) having \( p \) fixed points. Note that the only possibility to have exactly once 132 subsequence is a cycle with a fixed point just to its left. Moreover, \( y = x + 1 \) in order to forbid another 132 subsequence and cycles are only allowed from \( \pi' \) to \( \pi'' \) and from \( \pi' \) to \( \pi''' \) (and not from \( \pi'', \pi', \pi'' \), \( \pi'' \), \( \pi''' \), \( \pi'' \), \( \pi''' \)) whereas fixed points can uniquely be into \( \pi''' \). Clearly the involution we obtain avoids 132 and in particular, the fixed point \( z = 2 \) cannot be a part of an 132-subsequence because it cannot be the 3 or 2 (all the elements on its left are greater than it) and it cannot be the 1 (there is no cycle starting on its right).

Let \( \sigma = \sigma' \sigma'' \sigma'''(\sigma'') \) with \( \sigma(t) = t \) and \( \sigma(i) \neq i \) for all \( 1 \leq i < t \) (that is \( t \) is the first fixed point), \( \sigma'(i) > t \) for all \( 1 \leq i < |\sigma'| \) (all the elements of \( \sigma' \) are cycles ending into \( \sigma''' \)), \( \sigma''(i) \in [|\sigma'\sigma''| + 1, t - 1] \).
for all $1 \leq i \leq |\sigma'|$ and $\sigma''(i) \in [|\sigma'| + 1, |\sigma'| + 1]$ for all $1 \leq i \leq |\sigma''|$ ($\sigma''\sigma'''$ is entirely constituted by cycles from $\sigma''$ to $\sigma'''$) be an 132-avoiding involution of length $n - 2$ having $p$ fixed points. We modify the fixed point $t$ by a cycle starting between $\sigma''$ and $\sigma'''$ (and ending between $\sigma'''$ and $\sigma''''$) and by adding a fixed point just to the right of $\sigma'''$ in order to obtain an involution containing 132 once of length $n$ having $p$ fixed points. Proposition 2.1 leads immediately to the decomposition of $\sigma$. The involution we obtain contains 132 exactly once that is the subsequence we modify and insert. There is no other 132-subsequence and in particular, the fixed point inserted and the start of the new cycle cannot be the 3 or 2 of another 132-subsequence (all the elements on their left are greater than them), the fixed point inserted and the start of the new cycle cannot be the 1 of another 132-subsequence (there is no cycle starting on their right), the end of the new cycle cannot be the 3 or another 132-subsequence (because in that case the 2 must be connected to $\sigma'$ and the 1 must be the fixed point inserted or an element of $\sigma'''$ that forms an 231-subsequence), and the end of the new cycle cannot be the 2 of another 132-subsequence (because in that case the 1 must be an element of $\sigma''\sigma'''$ or the start of the new cycle and the 3 must be the fixed point inserted or an element of $\sigma'''$ that forms an 312-subsequence excepted for the fixed point inserted and the new cycle).

So we have established a bijection between $\pi$ an involution containing 132 once and $\sigma$ an 132-avoiding involution where $t = z - 2$, $\pi'$ corresponds to $\sigma'\sigma''$, $\pi'' = \sigma'''$ and $\pi'''$ corresponds to $\sigma''''$.

For example, the involution $22\ 19\ 17\ 18\ 16\ 12\ 11\ 13\ 9\ 14\ 7\ 6\ 8\ 10\ 15\ 5\ 3\ 4\ 2\ 20\ 21\ 1\ 23$ containing 132 once (the subsequence $9\ 14\ 10$) corresponds to the 132-avoiding involution $20\ 17\ 15\ 16\ 14\ 10\ 9\ 11\ 7\ 6\ 8\ 12\ 13\ 5\ 3\ 4\ 2\ 18\ 19\ 1\ 21$.

**Corollary 4.2.** The number of involutions containing 132 exactly once of length $n$ having $p$ fixed points with $1 \leq p \leq n$ is the ballot number $\binom{n-2}{\frac{p}{2}-1} - \binom{n-2}{\frac{p}{2}}$. Moreover, the number of involutions containing 132 exactly once of length $n$ is $\binom{n-2}{\frac{p}{2}}$.

**Proof.** We immediately deduce this result from bijection $\Psi$ of Theorem 1.1 and Corollary 2.1. In fact, the number of involutions containing 132 once of length $n$ is either the number of 132-avoiding involutions of length $n - 2$ if $n$ is odd or the number of 132-avoiding involutions of length $n - 2$ having more than one fixed point if $n$ is even.

Of course, some of the following results can immediately be obtained from bijection $\Psi$ of Theorem 1.1 and results of Section 2.

Let $J_r(n)$ denote the number of involutions in $S_n(\tau)$ such containing 132 exactly once, and let $J_r(x) = \sum_{n \geq 0} J_r(n)x^n$ be the corresponding generating function. The following proposition is the base of all the other results in this section, which holds immediately from definitions.

**Proposition 4.3.** Let $\pi$ an involution in $S_n$ such that contains 132 exactly once, and let $\pi_j = n$. Then holds either

1. or $\pi_n = n$;
2. or $\pi = (\pi', n, \pi'', \pi'''', j)$ where $1 \leq j \leq n/2$, $\pi'''' = \pi'^{-1}$ and $\pi'$ avoids 132.
3. or $\pi = (\pi', m, 2m + 1, \pi'', m + 1)$ where $n = 2m + 1$, $\pi''' = \pi'^{-1}$ and $\pi' \in S_{m-1}(132)$.

Another approach to find the generating function of involutions in $S_n$ containing 132 exactly once is by use Proposition 4.3.
Theorem 4.4. Let $C(t)$ be the generating function for the Catalan numbers; then

$$J_\emptyset(x) = \frac{x^3C(x^2)}{1-x-x^2C(x^2)}.$$  

Proof. According to Proposition 4.3 with terms of generating functions we get the following: the first part of the proposition yields $xJ_\emptyset(x)$, the second part of the proposition yields $x^2C(x^2)J_\emptyset(x)$, and the third part of the proposition gives $x^3C(x^2)$. Hence

$$J_\emptyset(x) = xJ_\emptyset(x) + x^2C(x^2)J_\emptyset(x) + x^3C(x^2).$$

Example 4.5. From Proposition 4.3 it is easy to see that, the number of the involutions in $S_n(123)$ and containing $132$ exactly once is $2^{(n-3)/2}$ for $n$ odd, otherwise is $0$. Also, $J_{1234}(n) = F_{n-3}$ the $(n-3)$th Fibonacci number, $J_{12345}(n) = 3^{(n-3)/2}$.

Again, the case of varying $k$ is more interesting. As an extension of Example 4.5 let us consider the case $\tau = 12\ldots k$.

Theorem 4.6. For all $k \geq 1$,

$$J_{12\ldots k}(x) = \frac{x}{U_k \left(\frac{1}{2x}\right)} \sum_{j=1}^{k-2} U_j \left(\frac{1}{2x}\right).$$

Proof. Proposition 4.3 with use the generating function of permutations in $S_n(132, 1\ldots k)$ given by $R_k(x)$, yields

$$J_{12\ldots k}(x) = xJ_{12\ldots (k-1)}(x) + x^2R_{k-1}(x^2)J_{12\ldots k}(x) + x^3R_{k-1}(x^2)R_k(x^2).$$

By use the relation $R_k(y) = 1/(1-yR_{k-1}(y))$ we get that

$$J_{12\ldots k}(x) = xR_k(x^2)J_{12\ldots (k-1)}(x) + x^3R_{k-1}(x^2)R_k(x^2),$$

so induction on $k$ with Example 4.5 gives the theorem.

Similarly, we obtain another case $\tau = 213\ldots k$.

Theorem 4.7. For all $k \geq 3$,

$$J_{213\ldots k}(x) = \frac{x}{U_k \left(\frac{1}{2x}\right)} \left[xU_2 \left(\frac{1}{2x}\right) + \sum_{j=2}^{k-2} U_j \left(\frac{1}{2x}\right)\right].$$

Proof. Similarly as proof of Theorem 4.6 with use the generating function for the number of $213\ldots k$-avoiding permutations in $S_n(132)$ is given by $R_k(x)$ (see [MV2]), we obtain that

$$J_{213\ldots k}(x) = xR_k(x^2)J_{213\ldots (k-1)}(x) + x^3R_{k-1}(x^2)R_k(x^2),$$

and by induction with $J_{213}(x) = x^4R_3(x^2)$ (it is easy to see) the theorem holds.

Example 4.8. Theorem 4.7 yields $J_{2134}(2n+3) = J_{2134}(2n+4) = F_{2n}$ the $(2n)$th Fibonacci number for all $n \geq 0$. 


Example 4.9. Proposition 4.3 yields, \( J_{231}(n) = 1 \) for all \( n \geq 1 \), and \( J_{2341}(n) = 2^{\left\lceil (n-1)/2 \right\rceil} - 1 \) for all \( n \geq 1 \).

Once again, the case of varying \( k \) is more interesting. As an extension of Example 4.9 let us consider the case \( \tau = 23\ldots k1 \).

Theorem 4.10. For all \( k \geq 3 \),

\[
J_{23\ldots k1}(x) = \frac{x^2 U_{k-3} \left(\frac{1}{2x}\right)}{(1-x)U_{k-2} \left(\frac{1}{2x}\right)} \left[ 1 + \frac{1}{U_{k-1} \left(\frac{1}{2x}\right)} \sum_{j=1}^{k-3} U_j \left(\frac{1}{2x}\right) \right].
\]

Proof. Similarly as proof Theorem 4.6 we have that

\[
J_{23\ldots k1}(x) = xJ_{23\ldots k1}(x) + x^2 R_{k-2}(x^2)J_{12\ldots (k-1)}(x) + x^3 R_{k-2}(x^2),
\]

so by using Theorem 4.6 the theorem holds.

More generally, we present an explicit expression when \( \tau = [k,d] \) as follows.

Theorem 4.11. For \( k \geq 4, 2 \leq d \leq k/2 \),

\[
J_{[k,d]}(x) = \frac{R_d(x^2)}{1-xR_d(x^2)} \left[ x^2 R_{k-d-1}(x^2) + \frac{x^2(R_{k-d-1}(x^2) - R_{d-1}(x^2))}{U_{k-d} \left(\frac{1}{2x}\right)} \sum_{j=1}^{k-d-2} U_j \left(\frac{1}{2x}\right) \right].
\]

Proof. According to Proposition 4.3 in terms of generating functions we get the following. In first case \( xJ_{[k,d]}(x) \). In the third case, if \( \pi' \) contains \( 12\ldots(k-d-1) \) then \( \pi \) contains \([k,d] \) which is a contradiction, we get that \( \pi' \) avoids \( 12\ldots(k-d-1) \), hence \( x^3 R_{k-d-1}(x^2) \). Finally, in the second case, let us observe two subcases \( \pi'' \) contains \( 12\ldots(k-d) \) or avoids \( 12\ldots(k-d) \); so by use the same argument of the third case we get

\[
x^2 R_{k-d-1}(x^2)J_{12\ldots (k-d)}(x) + x^2 R_{d-1}(x^2)(J_{[k,d]}(x) - J_{12\ldots (k-d)}(x)).
\]

Therefore, if we add all these cases we get \( J_{[k,d]}(x) \). Hence, by Theorem 4.6 this theorem holds.

5. Containing 132 once and containing another pattern

Let \( J^r_\tau(n) \) denote the number of involutions in \( S_n \) such containing 132 exactly once and containing \( \tau \) exactly \( r \) times. Let \( J^r_\tau(x) = \sum_{n\geq 0} J^r_\tau(n)x^n \) be the corresponding generating function. Let us start be the following result.

Theorem 5.1. For all \( k \geq 1 \),

\[
J^1_{12\ldots k}(x) = 0.
\]

Proof. By Proposition 4.3 it is easy to see

\[
J^1_{12\ldots k}(x) = xJ^1_{12\ldots (k-1)}(x) + x^2 R_{k-1}(x^2)J_{12\ldots k}(x).
\]

with \( J^1_{12}(x) = 0 \), hence induction on \( k \) gives the theorem.

Similarly as Theorem 5.1 we have another case where \( \tau = 23\ldots k1 \).
Theorem 5.2. For all $k \geq 1$, 
\[ J_{1\ldots k1}^1(x) = 0. \]

Example 5.3. Proposition 4.3 yields the following. The number of involutions $J_{21}^1(n) = 1$ for $n \geq 3$, and $J_{213}(n) = 2^{(n-8)/2}(1 + (-1)^n)$. Once again, the case of varying $k$ is more interesting. As an extension of Example 5.3 let us consider the case $\tau = 213\ldots k$.

Theorem 5.4. For all $k \geq 3$, 
\[ J_{1\ldots k213}(x) = xJ_{1\ldots k-1213}(x) + x^2R_{k-1}(x^2)J_{213\ldots k}(x). \]

Proof. By Proposition 4.3 it is easy to obtain
\[ J_{1\ldots k213}(x) = xJ_{1\ldots k-1213}(x) + x^2R_{k-1}(x^2)J_{213\ldots k}(x). \]

with $J_{1213}(x) = x^3/(1 - x)$ (which is yield directly from definitions), hence induction on $k$ gives the theorem.

\[ \square \]

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