LIMIT SETS OF WEIL-PETERSSON GEODESICS

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Abstract. In this paper we prove that the limit set of any Weil-Petersson geodesic ray with uniquely ergodic ending lamination is a single point in the Thurston compactification of Teichmüller space. On the other hand, we construct examples of Weil-Petersson geodesics with minimal non-uniquely ergodic ending laminations and limit set a circle in the Thurston compactification.

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1. Introduction

Given a surface $S$, let $\text{Teich}(S)$ denote the Teichmüller space of hyperbolic metrics on $S$, and $\text{Mod}(S)$ the mapping class group of $S$. Thurston compactified $\text{Teich}(S)$ by adjoining the space of projective measured laminations $\mathcal{PML}(S)$, and used this in his classification of elements of $\text{Mod}(S)$; [Thu88, FLP79]. On the other hand, $\text{Teich}(S)$ has two important, $\text{Mod}(S)$–invariant, unique-geodesic metrics, and hence has natural visual compactifications. These metrics have their own drawbacks—the Teichmüller metric is not negatively curved [Mas75] and Weil-Petersson metric is incomplete [Wol75]—and hence the standard results about visual compactification do not readily apply to any of these metrics. For example, the action of $\text{Mod}(S)$

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extends continuously to neither the Teichmüller visual boundary \cite{Ker80} nor the Weil-Petersson visual boundary \cite{Bro05}.

In \cite{Mas82}, Masur showed that the Thurston boundary and the Teichmüller visual boundary are not so different, proving that almost every Teichmüller ray converges to a point on the Thurston boundary (though positive dimensional families of rays based at a single point can converge to the same point). Lenzhen \cite{Len08} constructed the first examples of Teichmüller geodesic rays that do not converge to a unique point in the Thurston boundary, and recent constructions have illustrated increasingly exotic behavior \cite{LLR13, BLMR16, CMW14, LMR16}.

In this paper we begin an investigation into the behavior of how the Weil-Petersson visual compactification relates to Thurston’s compactification. Specifically, we study the behavior of Weil-Petersson geodesic rays in the Thurston compactification. Our results are stated in terms of the ending laminations of Weil-Petersson geodesic rays introduced by Brock, Masur and Minsky in \cite{BMM10}; see \S2.7. The first theorem is a version of Masur’s convergence for Teichmüller geodesics. We say that a lamination is uniquely ergodic if it admits a unique transverse measure, up to scaling. Moreover, we say that the lamination is minimal if every leaf of the lamination is dense in the lamination, and filling if the lamination intersects every simple closed geodesic on the surface nontrivially.

**Theorem 1.1.** Suppose that the ending lamination of a Weil-Petersson geodesic ray is minimal, filling, and uniquely ergodic. Then the ray converges in the Thurston compactification to the unique projective class of transverse measures on the ending lamination.

On the other hand, we prove that there are geodesic rays for which the ending lamination is minimal but non-uniquely ergodic, and whose limit sets are positive dimensional and in fact are non-simply connected.

**Theorem 1.2.** There exist Weil-Petersson geodesic rays with minimal, filling, non-uniquely ergodic ending laminations whose limit sets in the Thurston compactification are topological circles.

See Theorem 5.7 for a more precise statement. Without the minimality assumption, the construction of Weil-Petersson geodesic rays that do not limit to a single point requires some different ideas. This construction is given in \cite{BLMR17}.

1.1. **Outline of the paper.** Section 2 is devoted to background about Teichmüller theory, curve complexes and laminations. In Section 2.5 we state our technical results about sequences of curves on surfaces that limit to non-uniquely ergodic laminations. These results are minor variations of those in \cite{BLMR16}, and their proofs are sketched in the appendix of the paper. In Section 3 we construct explicit examples of non-uniquely ergodic laminations on punctured spheres, appealing to the results from
Section 2.5 In Section 4 we study the limiting picture of axes for pseudo-Anosov mapping classes of a punctured sphere arising in the construction of non-uniquely ergodic laminations in Section 3. In Section 5, we use this analysis to determine limit sets of our WP geodesic rays with non-uniquely ergodic ending laminations, and thus prove Theorem 1.2. In Section 6 we prove Theorem 1.1 about limit sets of geodesic rays with uniquely ergodic ending laminations.

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2. Background

Notation 2.1. Our notation for comparing quantities in this paper is as follows: Let $K \geq 1$ and $C \geq 0$. Given two functions $f, g : X \to \mathbb{R} \geq 0$, we write $f \asymp_{K,C} g$ if $1/K g(x) - C \leq f(x) \leq Kg(x) + C$ for all $x \in X$, $f \asymp_{K,C}^{+} g$ if $g(x) - C \leq f(x) \leq g(x) + C$ and $f \asymp_{K,C}^{-} g$ if $1/K g(x) \leq f(x) \leq Kg(x)$. The notation $f \asymp_{K,C}^{+} g$ means that $f(x) \leq g(x) + C$ for all $x \in X$ and $f \asymp_{K,C}^{-} g$ means that $f(x) \leq Kg(x)$ for all $x \in X$.

When the numbers $K, C$ are understood from the context we drop them from the notation.

2.1. Surfaces and subsurfaces: In this paper surfaces are connected, orientable and of finite type with boundaries or punctures. We denote a surface with genus $g$ and $b$ boundary curves or punctures by $S_{g,b}$ and define the complexity of the surface by $\xi(S_{g,b}) = 3g + b - 3$. The main surfaces we consider always have only punctures, however we consider subsurfaces of the main surfaces with both punctures and boundary curves.

2.2. Curves and laminations.

Notation 2.2. Throughout this paper, by a curve we mean the isotopy class of an essential, simple, closed curve. When convenient, we do not distinguish between a curve and a representative of the isotopy class. A multicurve is a collection of pairwise disjoint curves (that is, curves with pairwise disjoint representatives).

By a subsurface of $S$, we mean the isotopy class of an embedded essential subsurface: one whose boundary consists of essential curves and whose punctures agree with those of $S$.

We say that two curves $\alpha$ and $\beta$ overlap and denote it by $\alpha \triangleleft \beta$ if the curves $\alpha$ and $\beta$ cannot be represented disjointly on the surface $S$. Two multicurves $\sigma$ and $\tau$ overlap if there are curves $\alpha \in \sigma$ and $\beta \in \tau$ which overlap. A curve $\alpha$ and a subsurface $Y \subseteq S$ overlap, denoted by $\alpha \triangleleft Y$, if $\alpha$ cannot be realized disjointly from $Y$ (up to homotopy). A multicurve and a subsurface overlap if a component of the multicurve overlaps with the
subsurface. We say that two subsurfaces $Y$ and $Z$ overlap, and denote it by $Y \pitchfork Z$, if $\partial Y \pitchfork Z$ and $\partial Z \pitchfork Y$.

We refer the reader to [MM99, MM00] for background about the curve complex and subsurface projection maps. Denote the curve complex of a surface $S$ by $\mathcal{C}(S)$ and the set of vertices of the complex by $\mathcal{C}_0(S)$. The set $\mathcal{C}_0(S)$ is in fact the set of essential simple closed curves on $S$.

A pants decomposition on the surface $S$ is a multicurve with a maximal number of components. A (partial) marking $\mu$ on the surface consists of pants decomposition, called the base of $\mu$, and a choice transversal curves for (some) all curves in the base. For background about pants and marking graphs and hierarchical structures and (hierarchy) resolution paths in pants and marking graphs we refer the reader to [MM00]. We denote the pants graph of the surface $S$ by $P(S)$. Here we only recall that hierarchy paths are certain quasi-geodesics in $P(S)$ with quantifiers that only depend on the topological type of $S$.

Let $Y \subseteq S$ be an essential subsurface. The $Y$–subsurface projection coefficient of two multicurves, markings or laminations $\mu, \mu'$ is defined by

\[ d_Y(\mu, \mu') := \text{diam}_{\mathcal{C}(Y)}\left(\pi_Y(\mu) \cup \pi_Y(\mu')\right). \]

Here $\pi_Y$ is the subsurface projection (coarse) map and $\text{diam}_{\mathcal{C}(Y)}(\cdot)$ denotes the diameter of the given subset of $\mathcal{C}(Y)$. When $Y$ is an annular subsurface with core curve $\gamma$ we also denote $d_Y(\mu, \mu')$ by $d_\gamma(\mu, \mu')$.

Our results in this paper are formulated in terms of subsurface coefficients which can be thought of as an analogue of continued fraction expansions which provide a kind of symbolic coding for us.

We assume that the reader is familiar with basic facts about laminations and measured laminations on hyperbolic surfaces (see e.g. [PH92] for an introduction). We denote the space of measured laminations equipped with the weak$^*$ topology by $\mathcal{ML}(S)$ and the space of projective classes of measured laminations equipped with the quotient topology by $\mathcal{PML}(S)$.

Recall that a measurable lamination is uniquely ergodic if it supports exactly one transverse measure up to scale. Otherwise, the lamination is non-uniquely ergodic.

An important property of curve complexes is that they are Gromov hyperbolic [MM99]. By the result of Klarreich [Klar99] the Gromov boundary of the curve complex is homeomorphic to the quotient space of the space of (projective) measured laminations with minimal, filling supports by the measure forgetting map equipped with the quotient topology, denoted by $\mathcal{EL}(S)$.

The Masur-Minsky distance formula [MM00] provides a coarse estimate for the distance between two pants decompositions in the pants graph $P(S)$. More precisely, there exists a constant $M > 0$ depending on the topological type of $S$ with the property that for any threshold $A \geq M$ there are constants
$K \geq 1$ and $C \geq 0$ so that for any $P, Q \in P(S)$ we have

$$d(P, Q) \preceq_{K, C} \sum_{Y \subseteq S \text{ non-annular}} \{d_Y(P, Q)\}_A$$

where the cut-off function is defined by $\{x\}_A = \begin{cases} x & \text{if } x \geq A \\ 0 & \text{if } x \leq A \end{cases}$.

The following theorem is a straightforward consequence of [MM00, Theorem 3.1].

**Theorem 2.3.** (Bounded geodesic image) Given $K \geq 1$ and $C \geq 0$. Suppose that $\{\gamma_i\}_i$ is a sequence of curves which forms a $1$–Lipschitz $(K, C)$–quasi-geodesic in $C(S)$. Further, suppose that for a subsurface $Y \subseteq S, \gamma_i \pitchfork Y$ for all $i$. Then there is a constant $G > 0$ depending on $K, C$ so that

$$\text{diam}_{C(Y)} \left( \{\pi_Y(\gamma_i)\}_i \right) \leq G.$$ 

For the following inequality, see [Beh06] [Man13].

**Theorem 2.4.** (Behrstock inequality) There exists a constant $B_0 > 0$ such that for any two subsurfaces $Y, Z \subseteq S$ with $Y \pitchfork Z$ and a fixed marking or lamination $\mu$ we have that

$$\min \left\{d_Y(\partial Z, \mu), d_Z(\partial Y, \mu) \right\} \leq B_0.$$ 

We also need the following no back-tracking property of hierarchy paths, which follows from inequality (6.3) in [MM00 §6.3].

**Theorem 2.5.** There is a constant $C > 0$ depending only on the topological type of $S$ so that given a hierarchy path $\rho : [m, n] \to P(S)$ ($[m, n] \subset \mathbb{Z} \cup \{-\infty, +\infty\}$), for parameters $i_1 \leq i_2 \leq i_3 \leq i_4$ in $[m, n]$, and a non-annular subsurface $Y \subseteq S$ we have

$$d_Y(\rho(i_1), \rho(i_4)) \geq d_Y(\rho(i_2), \rho(i_3)) - C.$$ 

2.3. **Twist parameter.** We define the twist parameter of a curve $\delta$ about $\gamma$ at a point $X$ in Teichmüller space by

$$\text{tw}_\gamma(\delta, X) := d_\gamma(\mu, \delta)$$

where $\mu$ is a Bers marking at $X$ (for definition of Bers marking see §2.6).

Note that for a filling set of bounded length curves $\Gamma$ at $X$ we have $\text{tw}_\gamma(\delta, X) \gtrsim \text{diam}_{C(\gamma)}(\Gamma \cup \delta)$.

2.4. **The Thurston compactification.** Recall that a point in the Teichmüller space $\text{Teich}(S)$ is a marked complete hyperbolic surface $[f : S \to X]$. The mapping class group of $S$, denoted by $\text{Mod}(S)$, acts by remarking on $\text{Teich}(S)$ and the quotient is the moduli space of hyperbolic surfaces $\mathcal{M}(S)$.

Given a curve $\alpha \in C_0(S)$, the hyperbolic length of $\alpha$ at $[f : S \to X]$ is defined to be the hyperbolic length of the geodesic homotopic to $f(\alpha)$ in
Abusing notation and denoting the point in Teich(S) by X, we write the hyperbolic length simply as \( \ell_\alpha(X) \). For an \( \epsilon > 0 \), the \( \epsilon \)-thick part of Teichmüller space consists of points \( X \in \text{Teich}(S) \) with \( \ell_\alpha(X) \geq 2\epsilon \) for all curves \( \alpha \). The projection of this set to the moduli space is the \( \epsilon \)-thick part of moduli space.

The hyperbolic length function extends to a continuous function \( \ell(\cdot) : \text{Teich}(S) \times \mathcal{ML}(S) \rightarrow \mathbb{R} \).

Let \( \nu \) be a measurable lamination and \( \bar{\nu} \) a measured lamination with support \( \nu \). Moreover, denote the projective class of \( \bar{\nu} \) by \( [\bar{\nu}] \). The Thurston compactification, \( \hat{\text{Teich}}(S) = \text{Teich}(S) \cup \mathcal{PML}(S) \) is constructed so that a sequence \( \{X_n\}_{n=0}^\infty \subseteq \text{Teich}(S) \) converges to \( [\bar{\nu}] \in \mathcal{PML}(S) \) if and only if

\[
\lim_{n \to \infty} \frac{\ell_\alpha(X_n)}{\ell_\beta(X_n)} = \frac{i(\alpha, \bar{\nu})}{i(\beta, \bar{\nu})}
\]

for all simple closed curves \( \alpha, \beta \) with \( i(\bar{\nu}, \beta) \neq 0 \). Here and throughout this paper the bi-homogenous function \( i(\cdot, \cdot) \) denotes the geometric intersection number of two curves and its extension to the space of measured laminations \( \mathcal{ML}(S) \). See [Bon01, FLP79] for more details on the intersection function and Thurston compactification.

### 2.5. Sequences of curves.

In [LLR13] and [BLMR16] the authors studied infinite sequences of curves on a surface that limit to non-uniquely ergodic laminations. The novelty in this work is that local estimates on subsurface projections and intersection numbers are promoted to global estimates on these quantities. We require minor modifications of some of the key results from [BLMR16] so as to be applicable to the sequences of curves on punctured spheres described in §3. We state the results here and sketch their proofs in the appendix for completeness.

Given a curve \( \gamma \) let \( D_{\gamma} \) be the positive (left) Dehn twist about \( \gamma \).

**Definition 2.6.** Fix positive integers \( m \leq \xi(S) \) and \( b' \geq b > 0 \), and a sequence \( \mathcal{E} = \{e_k\}_{k=0}^\infty \subseteq \mathbb{N} \). We say that a sequence of curves \( \{\gamma_k\}_{k=0}^\infty \) satisfies \( \mathcal{P} = \mathcal{P}(\mathcal{E}) \) if the following hold:

1. any \( m \) consecutive curves are pairwise disjoint,
2. any consecutive \( 2m \) curves fill \( S \), and
3. for all \( k \geq m \), \( \gamma_{k+m} = D_{\gamma_k}^{e_k}(\gamma'_k) \), where \( \gamma'_k \) is a curve such that

\[
\begin{align*}
    i(\gamma'_{k+m}, \gamma_j) &\begin{cases}
        \leq b' & \text{for } j = k - m, \ldots, k + m - 1 \\
        = b & \text{for } j = k, k - 1 \\
        = 0 & \text{for } j = k + 1, \ldots, k + m - 1
    \end{cases}
\end{align*}
\]

**Remark 2.7.** The only real difference between this definition and the one given in our previous paper [BLMR16, §3] is that this one requires fewer of the intersection numbers to be nonzero.
For the remainder of this subsection, we will assume that Γ = {γ_k}_k satisfies P for some m, b, b' and E = {e_k}_k (see §3 for explicit examples). Furthermore, we assume there exists an a ≥ 1 such that e_{k+1} ≥ ae_k for all k ≥ 0.

The first result describes the behavior of {γ_k}_k in the curve complex of S and its subsurfaces. Let ℳ be the monoid generated by m and m + 1, that is

$$\mathcal{M} = \{im + j(m + 1) \mid i, j \in \mathbb{Z}^{≥0}\}.$$  

**Theorem 2.8.** There exist constants E, K, C > 0 such that if e_0 ≥ E, then {γ_k}_k is a 1–Lipschitz (K,C)–quasi-geodesic. In particular, there exists a ν ∈ ΕΛ(S) such that any accumulation point of {γ_k}_k in PML(S) is supported on ν.

Furthermore, there exists a constant R > 0, and for a marking µ another constant R(µ) > 0 depending on µ, so that for i < k < j, with k−i, j−k ∈ ℳ, we have γ_i \pitchfork γ_k, γ_j \pitchfork γ_k, and

$$d_{γ_k}(γ_i, γ_j), d_{γ_k}(γ_i, ν) \leq_R e_i \quad \text{and} \quad d_{γ_k}(µ, γ_j), d_{γ_k}(µ, ν) \leq_R R(µ)e_i.$$  

Also, for any i < j and a subsurface W which is not an annulus with core curve γ_k for some k we have

$$d_W(γ_i, γ_j), d_W(γ_i, ν) \leq R \quad \text{and} \quad d_W(µ, γ_j), d_W(µ, ν) \leq R(µ).$$

The next result provides estimates on intersection numbers for curves in our sequence. To describe the estimates, for all i<k, define the integers

$$A(i, k) := \prod_{\substack{i+m \leq j < k,\ j \equiv k \mod m}} b_{ej}$$

where the product is taken to be 1 whenever the index set is empty.

**Theorem 2.9.** If a > 1 is sufficiently large and e_{k+1} ≥ ae_k, then there exists κ_0 ≥ 1 such that if(γ_i, γ_k) ≤ κ_0 A(i, k) for all i < k, and

$$i(γ_i, γ_k) \asymp_{κ_0} A(i, k).$$

if k − i ≥ 2m and i ≡ k mod m, or if i ≤ 2m − 1 and k − i ≥ m^2 + m − 1.

For any curve δ, there exists κ(δ) ≥ 1 such that for all k sufficiently large

$$i(δ, γ_k) \asymp_{κ(δ)} A(0, k).$$

For reference, we also record the following simple fact (see [BLMR16, Lemma 5.6]).

**Lemma 2.10.** Suppose that E = {e_k}_k satisfies e_k ≥ ae_{k-1} for all k and some a > 1. Then whenever k < l, we have

$$\frac{A(i, k)}{A(i, l)} \leq a^{1-\lfloor \frac{|l-i|}{m} \rfloor}.$$
The final result tells us that \( \{\gamma_k\}_k \) splits into \( m \) subsequences, each projectively converging to a distinct ergodic measure on \( \nu \).

**Theorem 2.11.** If \( \{\gamma_k\}_k \) is as in Theorems 2.8 and 2.9, then the sequence determines a \( \nu \in \mathcal{EL}(\mathcal{S}) \) which is non-uniquely ergodic and supports \( m \) ergodic measures, \( \nu^0, \ldots, \nu^{m-1} \), given by

\[
\lim_{i \to \infty} \frac{\gamma_{h+im}}{A(0, h + im)} = \nu^h,
\]

for each \( h = 0, \ldots, m - 1 \).

### 2.6. Weil-Petersson metric.

The Weil-Petersson (WP) metric is an incomplete, mapping class group invariant, Riemannian metric with negative curvature on the Teichmüller space. The WP completion of Teichmüller space \( \overline{\text{Teich}}(\mathcal{S}) \) is a stratified CAT(0) space. Each stratum is canonically isometric to the product of Teichmüller spaces of lower complexity, each equipped with the WP metric. More precisely, for any possibly empty multicurve \( \sigma \) on \( \mathcal{S} \) the stratum \( \mathcal{S}(\sigma) \) consists of finite type Riemann surfaces appropriately marked by \( \mathcal{S}\setminus \sigma \), and this is isometric to the product of Teichmüller spaces of the connected components of \( \mathcal{S}\setminus \sigma \). An important property of completion strata is the following non-refraction property.

**Theorem 2.12.** (Non-refraction; [Wol10] [DW03]) The interior of the geodesic segment connecting a point \( X \in \mathcal{S}(\sigma) \) to a point \( Y \in \mathcal{S}(\sigma') \) lies in the stratum \( \mathcal{S}(\sigma \cap \sigma') \).

Let \( L_S > 0 \) be the Bers constant of \( \mathcal{S} \) (see [Bus10, §5]). Then each point \( X \in \text{Teich}(\mathcal{S}) \) has a pants decomposition \( P \) (Bers pants decomposition) with the property that the length of every curve in \( P \) with respect to \( X \) is at most \( L_S \). Any curve in a Bers pants decomposition is called a Bers curve. Moreover, a marking whose base is a Bers pants decomposition and has transversal curves with shortest possible length is called a Bers marking.

Brock [Bro03] showed that the coarse map

\[
Q : \text{Teich}(\mathcal{S}) \to P(\mathcal{S})
\]

which assigns to a point in the Teichmüller space a Bers pants decomposition at that point is a quasi-isometry.

Using the non-refraction property of completion strata Wolpert [Wol03] gives a picture for the limits of sequences of bounded length WP geodesic segments in Teichmüller space after remarkings. The following strengthening of the picture was proved in [Mod15, §4]. For reference, given a multicurve \( \sigma \) let \( \text{tw}(\sigma) < \text{Mod}(\mathcal{S}) \) denote the subgroup generated by positive Dehn twists about the curves in \( \sigma \).

**Theorem 2.13.** (Geodesic Limit) Given \( T > 0 \), let \( \zeta_n : [0, T] \to \text{Teich}(\mathcal{S}) \) be a sequence of geodesic segments parametrized by arc length. After possibly passing to a subsequence, there exist a partition \( 0 = t_0 < \ldots < t_{k+1} = T \)
of the interval \([0, T]\), multicurves \(\sigma_l, l = 0, \ldots, k + 1\), a multicurve \(\hat{\tau} = \sigma_l \cap \sigma_{l+1}\) for all \(l = 0, \ldots, k\) and a piecewise geodesic segment

\[
\hat{\zeta} : [0, T] \to \Teich(S)
\]

such that

1. \(\hat{\zeta}(t) \in S(\sigma_l)\) for each \(l = 0, \ldots, k + 1\),
2. \(\hat{\zeta}((t_l, t_{l+1})) \subset S(\hat{\tau})\) for each \(l = 0, \ldots, k\),
3. there exist elements \(\psi_n \in \Mod(S)\) which are either trivial or unbounded as \(n \to \infty\) and elements \(T_{l,n} \in \tw(\sigma_l - \hat{\tau})\) such that for any \(\gamma \in \sigma_l - \hat{\tau}\) the power of the positive Dehn twist \(D_\gamma\) about \(\gamma\) is unbounded as \(n \to \infty\), and we have:

\[
\lim_{n \to \infty} \psi_n(t) = \hat{\zeta}(t)
\]

for any \(t \in [0, t_1]\). Moreover, for each \(l = 1, \ldots, k\) let

\[
(2.10) \quad \varphi_{l,n} = T_{l,n} \circ \cdots \circ T_{1,n} \circ \psi_n,
\]

then

\[
\lim_{n \to \infty} \varphi_{l,n}(\zeta_n(t)) = \hat{\zeta}(t)
\]

for any \(t \in [t_l, t_{l+1}]\).

In [Mod15] controls on length-functions along WP geodesics in terms of subsurface coefficients are developed. The following are corollaries 4.10 and 4.11 in [Mod15]. Here we denote a Bers marking at a point \(X \in \Teich(S)\) by \(\mu(X)\).

**Theorem 2.14.** Given \(T, \epsilon_0\) and \(\epsilon < \epsilon_0\) positive, there is an \(N \in \mathbb{N}\) with the following property. Let \(\zeta : [0, T'] \to \Teich(S)\) be a WP geodesic segment parametrized by arc length, of length \(T' \leq T\), such that

\[
\sup_{t \in [0, T']} \ell_\gamma(\zeta(t)) \geq \epsilon_0.
\]

Then if \(d_\gamma(\mu(\zeta(0)), \mu(\zeta(T'))) > N\) we have

\[
\inf_{t \in [0, T']} \ell_\gamma(\zeta(t)) \leq \epsilon.
\]

**Theorem 2.15.** Given \(T, \epsilon_0, s\) positive with \(T > 2s\) and \(N \in \mathbb{N}\), there is an \(\epsilon \in (0, \epsilon_0)\) with the following property. Let \(\zeta : [0, T'] \to \Teich(S)\) be a WP geodesic segment parametrized by arc length of length \(T' \leq T\), such that

\[
J \subseteq [s, T' - s]
\]

be a subinterval. Suppose that for some \(\gamma \in C_0(S)\) we have

\[
\sup_{t \in [0, T']} \ell_\gamma(\zeta(t)) \geq \epsilon_0.
\]

Then, if \(\inf_{t \in J} \ell_\gamma(\zeta(t)) \leq \epsilon\), we have

\[
d_\gamma(\mu(\zeta(0)), \mu(\zeta(T'))) > N.
\]

In this paper we will frequently use the following result of Wolpert for estimating distance of a point and a completion stratum. It is part of [Wol08 Corollary 4.10].
Corollary 2.16. Let $X \in \text{Teich}(S)$ and let $\sigma$ be a multicurve, then

$$d_{\text{WP}}(X, S(\sigma)) \leq \sqrt{2\pi \sum_{\alpha \in \sigma} \ell_{\alpha}(X)}$$

and

$$d_{\text{WP}}(X, S(\sigma)) = \sqrt{2\pi \sum_{\alpha \in \sigma} \ell_{\alpha}(X) + O\left(\sum_{\alpha \in \sigma} \ell_{\alpha}(X)\right)^{5/2}}$$

where the constant of the $O$ notation depends only on an upper bound for the length of curves in $\sigma$ at $X$.

2.7. End invariants. Brock, Masur, and Minsky [BMM10] introduced the notion of ending lamination for Weil-Petersson geodesic rays as follows: Note that here and throughout the paper all geodesics would be parametrized by the arc length. Let $r: [a, b) \to \text{Teich}(S)$ be a complete WP geodesic ray (a ray whose domain cannot be extended to the left end point $b$). First, the weak$^*$ limit of an infinite sequence of weighted distinct Bers curves at times $t_i \to b$ is an ending measure of the ray $r$, and any curve $\alpha$ with $\lim_{t \to b} \ell_{\alpha}(r(t_i)) = 0$ is a pinching curve of $r$. Now the union of supports of ending measures and pinching curves of $r$ is the ending lamination of $r$ which we denote by $\nu(r)$.

Let $g: I \to \text{Teich}(S)$ be a WP geodesic, where $I \subseteq \mathbb{R}$ is an interval. Denote the left and right end points of $I$ by $a, b$, respectively, and let $c$ be a point in the interior of $I$. If $g$ is extendable to $b$ in $\text{Teich}(S)$, including the situation that $b \in I$, then the forward end invariant of $g$, denoted by $\nu^+$, is a (partial) Bers marking at $g(b)$. If not, the forward end invariant of $g$ (also called the forward ending lamination) is the ending lamination of the geodesic ray $g(t)|_{[c,b]}$ defined above. The backward end invariant (ending lamination) $\nu^-$ of $g$ is defined similarly considering the ray $g(-t)|_{(a,c]}$. Finally, the pair $(\nu^-, \nu^+)$ is called the end invariant of $g$.

For example, the end invariant of a geodesic segment $g: [a, b] \to \text{Teich}(S)$ is the pair of markings $(\mu(g(a)), \mu(g(b)))$.

For more detail about end invariants of WP geodesics and their application to study the geometry and dynamics of Weil-Petersson metric see [BMM10, BMM11, Mod15, Mod16, BM15, Ham15].

2.8. Bounded combinatorics. Given $R > 0$, a pair of (partial) markings or laminations $(\mu, \nu)$ has $R$–bounded combinatorics if for any proper subsurface $Y \subseteq S$ the bound

$$d_Y(\mu, \nu) \leq R$$

holds. If the bound holds only for non-annular subsurfaces of $S$ we say that the pair has non-annular $R$–bounded combinatorics.

The following theorem relates the non-annular bounded combinatorics of end invariants to the behavior of WP geodesics.
Theorem 2.17. For any \( R > 0 \) there is an \( \epsilon > 0 \) so that any WP geodesic ray \( r : [0, \infty) \to \text{Teich}(S) \) whose end invariant has non-annular \( R \)-bounded combinatorics visits the \( \epsilon \)-thick part of \( \text{Teich}(S) \) infinitely often.

Proof. The fact that an individual ray \( r \) visits an \( \epsilon \)-thick part of \( \text{Teich}(S) \) infinitely often is [BM15] Theorem 4.1. To show that \( \epsilon \) can be chosen uniformly for all geodesic rays \( r \) whose end invariants have non-annular \( R \)-bounded combinatorics consider a decreasing sequence \( \epsilon_n \to 0 \) and a sequence of WP geodesic rays \( r_n : [0, \infty) \to \text{Teich}(S) \) with non-annular \( R \)-bounded combinatorics end invariants \((\mu(r_n(0)), \nu_n^+)\) and assume that \( \epsilon_n \) is the largest number that \( r_n \) visits the \( \epsilon_n \)-thick part of Teichmüller space infinitely often. In particular, for each \( n \geq 1 \) there is a time \( t_n \) so that \( r_n([t_n, \infty)) \) does not intersect the \( 2\epsilon_n \)-thick part of \( \text{Teich}(S) \).

Since the end invariant of \( r_n \), has non-annular \( R \)-bounded combinatorics, a hierarchy path \( g_n \) between the end invariant is stable in the pants graph \( P(S) \) [BMM11] Theorem 4.3 [Mod15] Theorem 5.13, in particular \( g_n \) and \( Q(r_n) \), the image of \( r_n \) under Brock’s quasi-isometry (2.9), \( D \)-fellow travel in \( P(S) \) where the constant \( D \) depends only on \( R \). Theorem 2.5 then guarantees that any two points along \( g_n \) also satisfy the non-annular bounded combinatorics condition (2.11) with a larger constant.

Now for any two times \( t_1, t_2 \in [0, \infty) \), let \( i_1, i_2 \) be so that \( d(g_n(i_1), Q(r_n(t_1))) \) and \( d(g_n(i_2), Q(r_n(t_2))) \) are at most \( D \). Then from the distance formula (2.2) we see that all subsurface coefficients of the pair \((g_n(i_1), Q(r_n(t_1)))\) and the pair \((g_n(i_2), Q(r_n(t_2)))\) are bounded by \( \max\{A, KD + KC\} \) for a choice of threshold the \( A \) in the formula. This together with the fact that the pair \((g_n(i_1), g_n(i_2))\) has non-annular bounded combinatorics imply that the pair \((Q(r_n(t_1)), Q(r_n(t_2)))\) also satisfies the non-annular bounded combinatorics condition for some \( R' > R \) which depends only on \( R \).

For \( R' > 0 \) as above let the constants \( T_0 > 0 \) and \( \epsilon_0 > 0 \) be as in Lemma 4.2 of [BM15]. Let \( I_n \subset [t_n, \infty) \) be an interval of length \( T_0 \) which contains a time \( s_n \) so that \( r_n(s_n) \) is in the \( \epsilon_n \)-thick part of Teichmüller space and \( s_n \to \infty \) as \( n \to \infty \). By the previous paragraph the end points of \( r_n|_{I_n} \) have non-annular \( R' \)-bounded combinatorics, so by Lemma 4.2 of [BM15] after possibly passing to a subsequence the geodesic segments \( r_n|_{I_n} \) intersect the \( \epsilon_0 \)-thick part of Teichmüller space. But for \( n \) sufficiently large \( 2\epsilon_n < \epsilon_0 \) and by the choice of \( I_n \) the segment \( r_n|_{I_n} \) does not intersect the \( 2\epsilon_n \)-thick part, which contradicts that \( r_n|_{I_n} \) intersects the \( \epsilon_0 \)-thick part. The existence of a uniform \( \epsilon \) for all \( r \) with non-annular \( R \)-bounded combinatorics follows from this contradiction. \( \square \)

2.9. Isolated annular subsurfaces. In this section we recall the relevant aspects of the notion of an isolated annular subsurface along a hierarchy path from [Mod15] §6 and its consequences for our purposes in this paper.

Let \((\mu, \nu)\) be a pair of (partial) markings or laminations with non-annular \( R \)-bounded combinatorics. A hierarchy path \( g : [m, n] \to P(S), [m, n] \subseteq \mathbb{Z} \cup \{\pm \infty\} \), with end points \((\mu, \nu)\) is stable in the pants graph of \( S \) [BMM11].
In particular, \( Q(g) \) the image of a WP geodesic \( g \) with end invariant \((\mu, \nu)\) under Brock’s quasi-isometry \([2.9]\) has \( D\)-fellow travels \( g \) for a \( D > 0 \) depending only on \( R \). For a parameter \( i \in [m, n] \), we say that the time \( t \) corresponds to \( i \), if \( Q(g(t)) \) is within distance \( D \) of \( g(i) \), and vice versa.

Let \( i \in [m, n] \), and let \( Q \) be a pants decomposition within distance \( D \) of the point \( g(i) \), moreover let \( \gamma \) be a curve in \( Q \). By [Mod15, Definition 6.3] the annular subsurface with core curve \( \gamma \) is isolated at \( i \) along \( g \) and hence by [Mod15, Lemma 6.4] we have:

**Lemma 2.18.** (Annular coefficient comparison) There are positive constants \( \bar{w}, b \) and \( B \) depending on \( R \) and a constant \( L \) depending only on the topological type of \( S \), so that for the curve \( \gamma \), a time \( t \) corresponding to \( i \), any \( s \geq \bar{w} \) and \( s' \approx b \), we have:

\[
(2.12) \quad d_\gamma(\mu(g(t-s')), \mu(g(t+s'))) \approx B d_\gamma(g(i-s), g(i+s)),
\]

where \( \mu(\cdot) \) is a choice of Bers marking at the point. Moreover,

\[
(2.13) \quad \min\{\ell_\gamma(g(t-s')), \ell_\gamma(g(t+s'))\} \geq L.
\]

### 3. Sequences of curves on punctured spheres

In this section we construct a sequence of curves that satisfies the condition \( \mathcal{P} \) of Definition 2.6. This construction is a generalization of the one in [LLR13] to spheres with more punctures. Fix a sequence of positive integers \( \mathcal{E} = \{e_k\}_{k=0}^\infty \).

Let \( p \geq 5 \) be an odd integer and \( S = S_{0,p} \) be a sphere with \( p \) punctures. We visualize \( S \) as the double of a regular \( p \)-gon (with vertices removed), admitting an order \( p \) rotational symmetry, as in Figure 1. Let \( \rho: S \to S \) be the counterclockwise rotation by angle \( 4\pi/p \). Set \( m = \frac{p-1}{2} \).

Next, let \( \gamma_0 \) be a curve obtained by doubling an arc connecting two sides of the polygon adjacent to a common side. Then \( \{\rho^j(\gamma_0)\}_{j=0}^{p-1} \) is a set of \( p \) curves that pairwise intersection 0 or 2 times; see Figure 1. We let \( \alpha = \rho^m(\gamma_0) \), and recall that \( D_\alpha \) denotes the positive Dehn twist about the curve \( \alpha \). For \( k \geq 1 \), set

\[
(3.1) \quad \phi_k = D_\alpha^{e_k+m-1} \rho, \quad \Phi_k = \phi_1 \phi_2 \cdots \phi_k,
\]

(in particular, \( \Phi_0 = \text{id} \)).

Define a sequence of curves \( \Gamma = \Gamma(\mathcal{E}) = \{\gamma_k\}_{k=0}^\infty \), starting with \( \gamma_0 \), by the formula

\[
(3.2) \quad \gamma_k = \Phi_k(\gamma_0).
\]

Since a twist about \( \alpha \) has no effect on a curve disjoint from it, for \( 0 \leq j \leq 2m-1 \),

\[
(3.3) \quad \gamma_k = \Phi_{k-j}(\gamma_j) = \Phi_{k-m}(\alpha),
\]
for all $k \geq m$. See Figure 2 for a picture illustrating $2m + 1$ consecutive curves.

**Proposition 3.1.** The sequence $\Gamma(\mathcal{E}) = \{\gamma_k\}_{k=0}^{\infty}$ satisfies condition $\mathcal{P}(\mathcal{E})$ in Definition 2.6 for $m = \frac{p-1}{2}$ and $b' = b = 2$.

**Proof.** The proof boils down to showing that, after applying an appropriate homeomorphism, any $2m + 1 = p$ consecutive curves differ from $\gamma_0, \ldots, \gamma_{2m}$ only in the amount of relative twisting of $\gamma_0$ and $\gamma_{2m}$ around $\gamma_m$; see Figure 2 and compare with the construction from [LLR13]. We now explain this in more detail.

First, observe that for $j = 1, \ldots, p - 2 = 2m - 1$, $i(\alpha, \rho^j(\gamma_0)) = 0$. Thus

$$\gamma_j = \rho^j(\gamma_0),$$
for $j = 0, \ldots, 2m - 1$. By construction, any two of these curves intersect 0 or 2 times, while the first $m$ are pairwise disjoint. Furthermore, the entire set of $2m$ curves fills $S$; see Figure [1].

Next, for any $k \geq m$, applying $\Phi_{k-m}^{-1}$ to the $2m - 1$ consecutive curves $\{\gamma_{k-m}, \ldots, \gamma_{k+m-1}\}$, (3.3) implies

$$\Phi_{k-m}^{-1}(\gamma_{k+m+j}) = \gamma_j$$

for each $j = 0, \ldots, 2m - 1$. Since $k$ was arbitrary, it follows that any $m$ consecutive curves are pairwise disjoint, and any $2m$ consecutive fill $S$. Thus, conditions (i) and (ii) of $\mathcal{P}$ are satisfied.

For part (iii), let $\gamma_{k+m} = \Phi_{k-m}(\rho(2m-1)) = \Phi_{k-m}(\rho^{2m}(\gamma_0))$. Then, for $j = 0, \ldots, 2m - 1$, we may apply $\Phi_{k-m}^{-1}$, and we have

$$i(\gamma_{k+m}, \gamma_{j+m}) = i(\rho^{2m}(\gamma_0), \gamma_j) = i(\rho^{2m}(\gamma_0), \rho^j(\gamma_0)) = \begin{cases} 2 & \text{for } j = m, \\
0 & \text{otherwise,} \end{cases}$$

which implies the intersection number requirement for (iii), with $b' = b = 2$.

Finally, applying $\Phi_{k-m}$ to $\gamma_{k+m}$ we get

$$\Phi_{k-m}^{-1}(\gamma_{k+m}) = \Phi_{k-m}^{-1}(\Phi_{k-m}(\gamma_{k+m})) = \Phi_{k-m}^{-1}(\Phi_{k-m}(\rho^{2m}(\gamma_0))) = \Phi_{k-m}^{-1}(\rho^{2m}(\gamma_0)) = \rho^{2m}(\gamma_0) = \gamma_{k+m}.$$
the sequence of curves \( \{ f_k^e(\alpha) \}_{k=0}^{\infty} \) determines a projective measured lamination \([\bar{\nu}^+_e]\) and \( \{ f_k^e(\alpha) \}_{k=0}^{-\infty} \) determines a projective measured lamination \([\bar{\nu}^-_e]\).

A key ingredient in our construction of a Weil-Petersson geodesic ray in §5 will be a very precise understanding of the limiting picture of the axes \( g_e \) of the pseudo-Anosov mapping classes \( f_e \), as \( e \) tends to infinity. The main results of this section are Proposition 4.5 in which we describe a biinfinite piecewise geodesic in \( \text{Teich}(S) \) which approximate the geodesics \( g_e \) in the Hausdorff topology and gives us the necessary limiting picture for \( g_e \) as \( e \to \infty \).

Our analysis of the axes \( g_e \) of \( f_e \) begins with an analysis of the action of \( \rho \) on \( \text{Teich}(S) \) and certain strata in the Weil-Petersson completion. Observe that the quotient of \( S \) by \( \langle \rho \rangle \) is a sphere with one puncture and two cone points. A fixed point of \( \rho \) in \( \text{Teich}(S) \) is a \( \rho \)-invariant conformal structure on \( S \), or equivalently, a conformal structure obtained by pulling back a conformal structure on the quotient \( S/\langle \rho \rangle \). Since the sphere with three marked points is rigid, there is a unique such conformal structure, and hence exactly one fixed point \( Z \in \text{Teich}(S) \) for the action of \( \rho \).

**Proposition 4.1.** For the stratum \( S(v) \) defined by a multicurve \( v \), there exists a point \( X_0 \in S(v) \) so that

\[
d_{WP}(X_0, \rho(X_0)) = \inf_{Y \in S(v)} d_{WP}(Y, \rho(Y)).
\]

**Remark 4.2.** Note that unless \( S(v) \) is a point (i.e. \( v \) is a pants decomposition), \( \overline{S(v)} \) is not compact.

Now we define the function

\[
F : \overline{\text{Teich}(S)} \to \mathbb{R}^\geq,
\]

by \( F(X) = d_{WP}(X, \rho(X)) \). The proposition is then equivalent to showing that the restriction of \( F \) to the closure \( \overline{S(v)} \) attains a minimum value. We begin with a lemma.

**Lemma 4.3.** The function \( F : \overline{\text{Teich}(S)} \to \mathbb{R}^\geq \) is convex, 2-Lipschitz, and for any \( R > 0 \), \( F^{-1}([0, R]) \) is a bounded set.

**Proof.** Since the completion of the Weil-Petersson metric is CAT(0), the distance function on \( \overline{\text{Teich}(S)} \) is convex, and hence so is \( F \). The triangle inequality proves that \( F \) is 2-Lipschitz, since

\[
|F(Y) - F(X)| = |d_{WP}(Y, \rho(Y)) - d_{WP}(X, \rho(X))| \leq |d_{WP}(X,Y) + d_{WP}(X, \rho(Y)) - d_{WP}(X, \rho(X))| \\
\leq d_{WP}(X,Y) + d_{WP}(\rho(X), \rho(Y)) = 2d_{WP}(X,Y).
\]

Let \( Z \in \text{Teich}(S) \) be the fixed point of the action of \( \rho \) on \( \text{Teich}(S) \) and suppose that \( R_0 > 0 \) is sufficiently small so that \( B_{R_0}(Z) \), the closed ball
of radius $R_0$ in $\overline{\text{Teich}(S)}$ about $Z$, is contained in $\text{Teich}(S)$, and thus is compact. Let $R_1 > 0$ be the minimum value of $F$ on $\partial B_{R_0}(Z)$.

For any $Y \in \overline{\text{Teich}(S)} \setminus B_{R_0}(Z)$, let $Y_0$ be the unique point of intersection of the geodesic from $Z$ to $Y$ with the sphere $\partial B_{R_0}(Z)$. Then it follows that $d_{\text{WP}}(Y, Z) = R_0 + d_{\text{WP}}(Y, Y_0)$, and so convexity of $F$ implies

$$F(Y_0) \leq \frac{R_0}{d_{\text{WP}}(Y, Z)} F(Y) + \frac{d_{\text{WP}}(Y, Y_0)}{d_{\text{WP}}(Y, Z)} F(Z)$$

But then since $F(Z) = 0$ and $F(Y_0) \geq R_1$ we have

$$F(Y) \geq \frac{d_{\text{WP}}(Z, Y)}{R_0} F(Y_0) \geq \frac{d_{\text{WP}}(Z, Y)}{R_0} R_1.$$

Rearranging the above inequality, we have

$$d_{\text{WP}}(Z, Y) \leq \frac{R_0}{R_1} F(Y)$$

and hence if $R > R_0$ and $F(Y) \leq R$, then we have $d_{\text{WP}}(Z, Y) \leq \frac{R_0 R}{R_1}$. That is, $F^{-1}(\{0, R\}) \subset B_{R_0 R / R_1}(Z)$, as required. \hfill \□

**Proof of Proposition 4.1.** Any stratum in $\overline{S(v)}$ has the form $S(v')$ for a multicurve $v'$ containing $v$. Observe that the infimum of the function $F$ on any stratum $S(v')$ in $\overline{S(v)}$ is no less than the infimum of $F$ on $S(v)$. Let $S(v')$ in $\overline{S(v)}$ be a stratum in the closure having minimal dimension, so that the infimum of $F$ on $S(v')$ is equal to the infimum on $S(v)$. It suffices to show that the infimum of $F$ on $S(v')$ is realized on $S(v')$.

Let $\{X_n\}_{n=1}^{\infty} \subset S(v')$ be an infimizing sequence for $F$ on $S(v')$; that is

$$\lim_{n \to \infty} F(X_n) = \inf_{X \in S(v')} F(X).$$

Let $R < \infty$ be such that $F(X_n) \leq R$ for all $n \geq 1$. Lemma 4.3 then implies that there exists $D > 0$ such that $d_{\text{WP}}(Z, X_n) \leq D$ for all $n \geq 1$.

By the triangle inequality, the lengths of the geodesic segments $[X_1, X_n]$ are bounded by $2D$. Let $S_1, \ldots, S_m$ be the connected component of $S \setminus v'$. Then $S(v')$ is isometric to $\prod_{j=1}^{m} \text{Teich}(S_j)$ with the product of WP metrics on each factor. Let $\zeta_n : [0, T_n] \to \prod_{j=1}^{m} \text{Teich}(S_j)$ be the parametrization of $[X_1, X_n]$ by arc length. Let

$$\text{pr}_j : \prod_{j=1}^{m} \text{Teich}(S_j) \to \text{Teich}(S_j)$$

be the projection to the $j$-th factor and let $\zeta^j_n : [0, T^j_n] \to \text{Teich}(S_j)$ be parametrization by arc length of $\text{pr}_j \circ \zeta_n$. Note that $T^j_n \leq 2D$ for $j = 1, \ldots, m$. So for a fixed $j$, trimming the intervals and reparametrization we get a sequence of geodesic segments $\zeta^j_n : [0, T^j] \to \text{Teich}(S_j)$ of equal length. We may then apply Theorem 2.13 (Geodesic Limit Theorem) to
the sequence of geodesic segments \( \zeta_i^n : [0, T_i^j] \to \text{Teich}(S_j) \). Let the multicurves \( \sigma_i^j, i = 0, \ldots, k_j + 1, \) the multicurve \( \tau_i^j \), the partition \( t_0^j < \ldots < t_{k_j+1}^j \) and the piecewise geodesic \( \hat{\zeta}^j \) be as in the theorem. Also let the elements of mapping class group \( \psi_i^j \) and \( \varphi_{l,n}^j, l = 1, \ldots, k_j \) be as in the theorem. Note that by the theorem when \( k_j \geq 1 \) we have that \( \hat{\zeta}^j(t_1^j) \in S(\sigma_1^j) \) and \( \lim_{n \to \infty} \varphi_{1,n}(\zeta_i^j(t_1^j)) = \hat{\zeta}^j(t_1^j) \).

Since the geodesics \( \zeta_i^j \) have a common starting point \( pr_j(X_1) \), it follows that \( \psi_i^j \) is the identity map for all \( n \). Hence, if \( k_j = 0 \), then after possibly passing to a subsequence the points \( pr_j(X_n) \) converge.

First suppose that \( k_j = 0 \) for all \( j = 1, \ldots, m \), then after possibly passing to a subsequence all sequences \( pr_j(X_n) \) converge as \( n \to \infty \). As a result the points \( X_n \) converge and we are done.

Now we suppose that \( k_j \geq 1 \) for some \( j \), let \( \beta \in \sigma_1^j \), and we derive a contradiction. Note that \( pr_j(X_1) \) and \( pr_j(X_n) \) are in \( \text{Teich}(S_j) \). Claim 4.9 in the proof of Theorem 4.6 in [Mod15] tells us that for Bers markings \( \mu(pr_j(X_1)) \) and \( \mu(pr_j(X_n)) \) and curves \( \beta_n = (\varphi_{1,n}^j)^{-1}(\beta) \),

\[
d_{\beta_n}(\mu(pr_j(X_1)), \mu(pr_j(X_n))) \to \infty.
\]
as \( n \to \infty \).

Now recall that \( \varphi_{1,n}^j = \mathcal{T}_{1,n}^j \circ \psi_i^j \), also that \( \mathcal{T}_{1,n}^j \) is the composition of a power of the Dehn twist about the curve \( \beta_n \) and powers of Dehn twists about curves disjoint from \( \beta_n \). Moreover, as we saw above \( \psi_n \) is identity. Thus \( \beta_n \equiv \beta \) for all \( n \). Therefore the above limit becomes

\[
d_{\beta}(\mu(pr_j(X_1)), \mu(pr_j(X_n))) \to \infty
\]
as \( n \to \infty \). We may then choose a sequence \( \{n_k\}_{k=1}^\infty \) so that

\[
(4.2) \quad d_{\beta}(\mu(pr_j(X_{n_k})), \mu(pr_j(X_{n_k+1}))) \to \infty
\]
as \( k \to \infty \).

**Claim 4.4.** There exists a sequence of points \( \{Y_{n_k}\}_k \) on the geodesic segments \( [X_{n_k}, X_{n_k+1}] \) with the property that the distance between \( Y_{n_k} \) and \( S(v' \cup \beta) \) goes to 0.

**Proof.** It suffices to show that there is a sequence of points \( Y_{n_k} \) on \( [X_{n_k}, X_{n_k+1}] \) so that the distance between \( pr_j(Y_{n_k}) \) and \( S(\beta) \subset \text{Teich}(S_j) \) goes to 0. If the distance between \( pr_j(X_{n_k}) \) and \( S(\beta) \) goes to 0, the sequence \( Y_{n_k} = X_{n_k} \) is the desired sequence. Otherwise, there is a lower bound for the distance between \( pr_j(X_{n_k}) \) and \( S(\beta) \). Moreover, by Corollary 2.16 we have

\[
d_{WP}(pr_j(X_{n_k}), S(\beta)) \leq \sqrt{2\pi \ell_{\beta}(pr_j(X_{n_k}))}.
\]

Thus we obtain a lower bound for \( \ell_{\beta}(pr_j(X_{n_k})) \). Appealing to Theorem 2.14 the annular coefficient limit (4.2) provides a point \( Y_{n_k}^j \) on \( [pr_j(X_{n_k}), pr_j(X_{n_k+1})] \)
so that } \ell_\beta(Y_{n_k}^j) \to 0 \text{, and hence again by Corollary } 2.16 \text{ the distance between } Y_{n_k}^j \text{ and } S(\beta) \text{ goes to } 0. \text{ Now the points } Y_{n_k} \text{ on } [X_{n_k}, X_{n_k+1}] \text{ with } \text{pr}_j(Y_{n_k}) = Y_{n_k}^j \text{ are the desired points.} \quad \square

It follows from the above claim and the convexity of the function } F \text{ that }
(4.3) \quad F(Y_{n_k}) \leq \max \{ F(X_{n_k}), F(X_{n_k+1}) \}.

Therefore, \{Y_{n_k}\} is also an infimizing sequence for the function } F \text{ on } S(\nu). \text{ Let } Y_k' \text{ be the closest point to } Y_{n_k} \text{ in } S(\nu' \cup \beta). \text{ Since the distance of the points } Y_{n_k} \text{ and } S(\nu' \cup \beta) \text{ goes to } 0 \text{ we have that }
\quad d_{WP}(Y_{n_k}, Y_k') \to 0

and therefore } F(Y_{n_k}) \text{ and } F(Y_k') \text{ have the same limit since } F \text{ is } 2\text{-Lipschitz. Therefore } \{Y_k'\}_{k} \text{ is an infimizing sequence for the function } F \text{ in the stratum } S(\nu' \cup \beta), \text{ but this stratum has dimension less than that of } S(\nu'). \text{ This contradiction finishes the proof of the proposition.} \quad \square

Now we can describe the biinfinite piecewise geodesics } g_e^\omega \subset \overline{\text{Teich}(S)} \text{ which approximate the geodesics } g_e, \text{ the axes of the pseudo-Anosov mapping classes } f_e \text{ as follows. First, appealing to Proposition } 4.1 \text{, let } X_0 \in S(\rho^{-1}(\alpha)) \text{ be a point where the function } F(X) = d_{WP}(X, \rho(X)) \text{ is minimized in the closure of the stratum } S(\rho^{-1}(\alpha)). \text{ As already observed, on } S(\rho^{-1}(\alpha)), \text{ we have } f_e = D_\alpha^e \rho = \rho \text{ since } D_\alpha^e \text{ acts trivially on } \rho(S(\rho^{-1}(\alpha))) = S(\alpha). \text{ Consequently, } f_e(X_0) = \rho(X_0), \text{ and we may concatenate the geodesic segment } \omega = [X_0, \rho(X_0)] \text{ with its } f_e\text{-translates to produce an } f_e\text{-invariant, biinfinite piecewise geodesic in } \overline{\text{Teich}(S)}:

(4.4) \quad g_e^\omega = \cdots \cup f_e^{-2}(\omega) \cup f_e^{-1}(\omega) \cup \omega \cup f_e(\omega) \cup f_e^2(\omega) \cup \cdots

Proposition 4.5. \text{ The path } g_e^\omega \text{ is a biinfinite piecewise geodesic that fellow travels } g_e, \text{ and the Hausdorff distance between } g_e^\omega \text{ and } g_e \text{ tends to } 0 \text{ as } e \to \infty.

For the proof of the proposition we need the following theorem which is a characterization of the short curves along the geodesic } g_e. \text{ In the following let } E > 0 \text{ be the constant from Theorem } 2.8.

Theorem 4.6. \text{ There exists } \epsilon > 0 \text{ so that for all } e > E \text{ and every point of } g_e, \text{ at most one curve on } S \text{ has length less than } \epsilon, \text{ and such a curve is in the set } \{ f_e^k(\alpha) \}_{k \in \mathbb{Z}} \text{ (with } \alpha \text{ as in } 13). \text{ Moreover, let } t_e \text{ be the translation length of } f_e, \text{ then after reparametrization of } g_e \text{ we have that the minimal length of the curve } f_e^k(\alpha), \ k \in \mathbb{Z} \text{ along } g_e \text{ is realized at } kt_e \text{ and tends to zero as } e \to \infty.

Proof. \text{ Let } \nu_e^\pm, \text{ as before, be the laminations determined by the sequences of curves } \{ f_e^k(\alpha) \}_{k, \pm} \in \mathbb{Z}. \text{ There is a uniform bound for all subsurface coefficients of the pairs } (\nu_e^-, \nu_e^+) \text{ except those of } \{ f_e^k(\alpha) \}_{k \in \mathbb{Z}}. \text{ This follows from the fact}
that in Theorem 2.8, the upper bound $R$ depends only on the parameters from Definition 2.6 and the initial marking $\mu$ which is the same for all $f_e$.

Similarly we have

\[(4.5) \quad d_{\mathfrak{f}_k^k(\alpha)}(\nu_e^-,\nu_e^+)^\perp \leq e\]

for all $k \in \mathbb{Z}$, where the additive error is independent of $e$.

Let $g_e : [-\infty, +\infty) \to P(S)$ be a hierarchy path between the pair $(\nu_e^-,\nu_e^+)$ (see [MM00]). Since the pair has non-annular $R$–bounded combinatorics $g_e$ is stable in $P(S)$ [BMM11], Theorem 4.3 [Mod15], Theorem 5.13. Therefore, $g_e$ and $Q(g_e)$ the image of $g_e$ under Brock’s quasi isometry (2.9) $D$–fellow travel, where the constant $D \geq 0$ depends only on $R$.

**Lemma 4.7.** There is an $\epsilon_2 > 0$, so that for all $e > E$, $g_e$ visits the $\epsilon_2$–thick part of $\text{Teich}(S)$ infinitely often in both forward and backward times.

**Proof.** Let $\mu(g_e(0))$ be a Bers marking at $g_e(0)$, and let $i$ be so that $g_e(i)$ is within distance $D$ of $Q(g_e(0))$. Then all non-annular subsurface coefficients of the pair $(g_e(i),Q(g_e(0)))$ are bounded by $\max\{KD + KC,A\}$ by the distance formula (2.2) for a choice of threshold $A$. Moreover, by Theorem 2.5 all non-annular subsurface coefficients of the pair $(g_e(i),\nu_e^+)$ are bounded by an enlargement of $R$. Combining the bounds with the triangle inequality in the curve complex of each subsurface then implies that $(\mu(g_e(0)),\nu_e^+)$, the end invariant of $g_e|_{[0,\infty)}$, has non-annular bounded combinatorics, independent of $e$. Theorem 2.17 then guarantees that for an $\epsilon_2 > 0$, independent of $e$, the geodesic ray $g_e|_{[0,\infty)}$ visits the $\epsilon_2$–thick part of $\text{Teich}(S)$ infinitely often. The proof of that the geodesic ray $g_e|_{[0,-\infty)}$ visits the $\epsilon_2$–thick part of $\text{Teich}(S)$ infinitely often is similar. \hfill \Box

Now we prove the following:

**Lemma 4.8.** There exists $\epsilon_1 > 0$, depending only on $R$, so that for all $e > E$ the length of each curve $\gamma \notin \{f^k(\alpha)\}_k$ is bounded below by $\epsilon_1$ along $g_e$.

**Proof.** Suppose that for a $t \in \mathbb{R}$ the length of $\gamma$ at $g_e(t)$ is less than the Bers constant. Then, $\gamma \notin Q(g_e(t))$ and thus $\gamma$ is isolated at some $i$ along $g_e$; for the discussion about isolated annular subsurfaces see (2.9) By Lemma 2.18 there are constants $\tilde{w}, b$ depending only on $R$ and a constant $L$ such that for any $s > \tilde{w}$ and $s' \geq b, s$,

$$\min\{\ell_\gamma(g_e(t-s')),\ell_\gamma(g_e(t+s'))\} \geq L.$$

Fix $s, s'$ as above and fix $u < s'$. Let $J = [t - s' + u, t + s' - u]$. Then, Theorem 2.17 applies to the geodesic segment $g_e|_{[t+s'-u,t+s']}$, and implies that for any integer $N \geq 1$, there is an $\epsilon \in (0, L/2)$ so that

\[(4.6) \quad \inf_{r \in J} \ell_\gamma(g_e(r)) < \epsilon, \text{ then } d_\gamma(\mu(g_e(t-s')),\mu(g_e(t+s'))) > N,
\]

where $\mu(\cdot)$ denotes a Bers marking at the given point.
According to Lemma \textbf{2.18} there is a constant $B > 0$ depending only on $R$ such that
\begin{equation}
(4.7) \quad d_{\gamma}(\mu(g_e(t-s')), \mu(g_e(t+s'))) \leq B d_{\gamma}(g_e(i-s), g_e(i+s)).
\end{equation}

Further, suppose that $\gamma$ is not in the set $\{f_k^e(\alpha)\}_{k \in \mathbb{Z}}$. Then the upper bound for $d_{\gamma}(\nu^e_-, \nu^e_+)$ and Theorem \textbf{2.5} for the parameters $-\infty, i-s, i+s, \infty$ of $g_e$ give us an upper bound for the subsurface coefficient
\begin{equation}
\frac{1}{d_{\gamma}(g_e(i-s), g_e(i+s))}
\end{equation}
depending only on $R$. So by (4.7) we get an upper bound for
\begin{equation}
\frac{1}{d_{\gamma}(\mu(g_e(t-s')), \mu(g_e(t+s')))}
\end{equation}
depending only on $R$. On the other hand, since $t \in J$ by (4.6) if $\ell_+(g(t))$ gets arbitrary small, then $d_{\gamma}(\mu(g_e(t-s')), \mu(g_e(t+s')))$ would become arbitrary large, which contradicts the upper bound we just obtained. Therefore, there is a lower bound $\epsilon_1 > 0$ for the length of $\gamma$ at time $t$ which depends only on $R$. Since $t$ was arbitrary the proof of the lemma is complete. \hfill $\Box$

The length of each one of the curves in the set $\{f_{k}^e(\alpha)\}_{k \in \mathbb{Z}}$ is strictly convex along $g_e$ ([Wol08]), and so has a unique minimum. The unique minimum for $f_{k}^e(\alpha)$ occurs at the $f_{k}^e$–image of the point where $\alpha$ is minimized. Thus, we can parameterize $g_e$ by arc length so that for $t_e$ the WP translation length of $f_{k}^e$, the length of the curve $f_{k}^e(\alpha)$ is minimized at $g_e(kt_e)$.

By Lemma \textbf{4.7} there is an $\epsilon_2 > 0$ so that for all $e>E$, $g_e$ visits the $\epsilon_2$–thick part infinitely often in both forward and backward times. Let $t'_e \in (0,t_e)$ be a time for which $g_e(t'_e)$ is in the $\epsilon_2$–thick part. But then $g_e(kt_e + t'_e)$ is in the thick part for all $k$. In particular, by convexity of the length of $\alpha$, it follows that outside the interval $(-t_e + t'_e, 2e_2)$, $2e_2$ is a uniform lower bound for the length of $\alpha$. Likewise, the length of $f_{k}^e(\alpha)$ is uniformly bounded below by $2\epsilon_2$ outside the interval $((k-1)t_e + t'_e, kt_e + t'_e)$. Consequently, for $k \neq k'$, the curves $f_{k}^e(\alpha)$ and $f_{k'}^e(\alpha)$ cannot simultaneously have length less than $2\epsilon_2$.

As we saw in Lemma \textbf{4.8} the only curves which can get shorter than $\epsilon_1$ along $g_e$ are $\{f_{k}^e(\alpha)\}_{k \in \mathbb{Z}}$. Moreover, since we saw above that two of these curves cannot get shorter than $\epsilon_2$ at the same time, the first statement of the theorem holds for $\epsilon = \min\{\epsilon_1, 2\epsilon_2\}$.

Let the laminations $\nu^e_\pm$ be as before, and let $g_e$ be a hierarchy path between $\nu^e_-$ and $\nu^e_+$. Recall that $g_e$ is stable and that $g_e$ and $Q(g)$, $D$–fellow travel for a $D$ that depends only on $R$.

Note that by (4.5) each curve $f_{k}^e(\alpha)$ $(k \in \mathbb{Z})$ for $e$ sufficiently large has a big enough subsurface coefficient that $f_{k}^e(\alpha)$ is in $g_e(i)$ for an $i$ in the domain of $g_e$ by Lemma \textbf{6.2 (Large link)}. Thus $f_{k}^e(\alpha)$ is isolated at $i$ along $g_e$ (see \textbf{2.9}). Let $t$ be a time so that $Q(g_e(t))$ is within distance $D$ of $g_e(i)$. Then for constants $\tilde{w}, b, B$ from Lemma \textbf{2.18} and any $s \geq \tilde{w}$ and $s' \approx_b s$ we
have that
\[ d_\gamma(\mu(g_e(t - s')), \mu(g_e(t + s'))) \geq_{B} d_\gamma(g_e(i - s), g_e(i + s)). \]
Thus the bound \( d_{f^k_e(\alpha)}(\nu_e, \nu_e^+) \geq e \) and Theorem 2.5 for the parameters \(-\infty, i - s, i + s, \infty\) of \( g_e \) imply that
\[ d_\gamma(\mu(g_e(t - s')), \mu(g_e(t + s'))) \geq e. \]
Moreover, for the constant \( L > 0 \) form Lemma 2.18 we have that
\[ \min\{\ell_\gamma(g_e(t - s')), \ell_\gamma(g_e(t + s'))\} \geq L. \]

Now fix \( s, s' \), then Theorem 2.14 applies to geodesic segment \( g_e|[t - s', t + s']\) and implies that
\[ \inf_{r \in [t - s', t + s']} \ell_{f^k_e(\alpha)}(g_e(t)) \rightarrow 0 \]
as \( e \rightarrow \infty \). But the minimal length of \( f^k_e(\alpha) \) is realized at \( k\ell_e \) so
\[ \lim_{e \rightarrow \infty} \ell_{f^k_e(\alpha)}(g_e(k\ell_e)) = 0. \]
This completes the proof of the second statement of the theorem.

We continue to use \( t_e > 0 \) to denote the WP translation length of \( f_e \) and assume the geodesic \( g_e \) is parameterized as in the proof of the theorem above. Then, in particular the minimal length of \( f^k_e(\alpha) \) along \( g_e \) is realized at time \( k\ell_e \) and \( \ell_{f^k_e(\alpha)}(g_e(k\ell_e)) \rightarrow 0 \) as \( e \rightarrow \infty \). Likewise, \( \{(k - 1)t_e + t'_e\}_{k \in \mathbb{Z}} \) denotes times when \( g_e \) intersects the fixed thick part of \( \text{Teich}(S) \). Also, note that the minimum of the length of \( f^k_e(\alpha) \) is realized at \( k\ell_e \) \((k \in \mathbb{Z})\) and \( \lim_{e \rightarrow \infty} \ell_{f^k_e(\alpha)}(k\ell_e) = 0. \)

To prove Proposition 4.3, we also need the following lemma about the limit of translation length of \( f_e \).

**Lemma 4.9.** The translation distance \( t_e \) of \( f_e \) limits to \(|\omega|\) the length of \( \omega \); that is,
\[ \lim_{e \rightarrow \infty} t_e = d_{\text{WP}}(X_0, \rho(X_0)) = |\omega|, \]
where \( X_0 \in \overline{S}(\rho^{-1}(\alpha)) \), as before, is the point where \( d_{\text{WP}}(X_0, \rho(X_0)) = \inf_{X \in S(\rho^{-1}(\alpha))} d_{\text{WP}}(X, \rho(X)) \).

**Proof.** Let \( Y_e \in S(\rho^{-1}(\alpha)) \) be the closest point to \( g_e(-t_e) \) (and hence closest to the entire geodesic \( g_e \)). Then \( f_e(Y_e) = D^\alpha\rho(Y_e) = \rho(Y_e) \), and hence
\[ d_{\text{WP}}(f_e(Y_e), Y_e) = d_{\text{WP}}(\rho(Y_e), Y_e) \geq d_{\text{WP}}(X_0, \rho(X_0)). \]
Moreover, \( \rho^{-1}(\alpha) = f^{-1}_e(\alpha) \) so the minimal length of \( \rho^{-1}(\alpha) \) along \( g_e \) is realized at time \(-t_e\) and \( \ell_{\rho^{-1}(\alpha)}(g_e(-t_e)) \rightarrow 0 \) as \( e \rightarrow \infty \). By Corollary 2.16, the distance between \( g_e(-t_e) \) and \( \mathcal{S}(\rho^{-1}(\alpha)) \) is bounded above by \( \sqrt{2\pi\ell_{\rho^{-1}(\alpha)}(g_e(-t_e))} \), so we obtain
\[ d_{\text{WP}}(Y_e, g_e(-t_e)) = d_{\text{WP}}(f_e(Y_e), g_e(0)) \rightarrow 0. \]
as $e \to \infty$. It follows then from the triangle inequality that
\[
\liminf_{e \to \infty} t_e = \liminf_{e \to \infty} \text{d}_{\text{WP}}(g_e(-t_e), g_e(0)) \\
\geq \liminf_{e \to \infty} \left( \text{d}_{\text{WP}}(Y_e, f_e(Y_e)) - \text{d}_{\text{WP}}(g_e(-t_e), Y_e) \\
- \text{d}_{\text{WP}}(f_e(Y_e), g_e(0)) \right) \\
\geq \text{d}_{\text{WP}}(X_0, \rho(X_0)).
\]

On the other hand, since $g_e$ is the geodesic axis of $f_e$, $t_e$ is less than the distance that $f_e$ translates along $g^e_\omega$, which is precisely $|\omega| = \text{d}_{\text{WP}}(X_0, \rho(X_0))$. That is, $t_e \leq \text{d}_{\text{WP}}(X_0, \rho(X_0))$, and hence
\[
\limsup_{e \to \infty} t_e \leq \text{d}_{\text{WP}}(X_0, \rho(X_0)).
\]

Combining this with the above, we have $\lim_{e \to \infty} t_e = \text{d}_{\text{WP}}(X_0, \rho(X_0))$, completing the proof of the lemma.

We are now ready for the proof of Proposition 4.5.

**Proof of Proposition 4.5.** We recall that $g_e$ intersects a fixed thick part of Teichmüller space, independent of $e$, at the times $(k-1)t_e + t'_e$, for all $k \in \mathbb{Z}$. Denote the closest point on $g^e_\omega$ to the point $g_e((k-1)t_e + t'_e)$ by $X_{e,k}$. The distance between $X_{e,k}$ and $g_e((k-1)t_e + t'_e)$ must tend to zero as $e \to \infty$. Otherwise, the strict negative curvature in the thick part of $\text{Teich}(S)$ would imply a definite contraction factor $\delta < 1$ for the closest point projection to $g_e$ restricted to $g^e_\omega$ for all $e$ sufficiently large. Since $X_{e,k+1} = f_e(X_{e,k})$, $\text{d}_{\text{WP}}(X_{e,k}, X_{e,k+1}) = |\omega|$. Now by the contraction of the projection on $g_e$ and Lemma 4.9 we would have that
\[
|\omega| = \lim_{e \to \infty} t_e \leq \delta|\omega|
\]
an obvious contradiction. The sequence of points $\{X_{e,k}\}_{k \in \mathbb{Z}}$ is $f_e$-invariant and its distance to $g_e$ tends to 0 as $e \to \infty$. Appealing to the CAT(0) property of $\text{Teich}(S)$, the farthest point of $g^e_\omega$ to $g_e$ must also have distance tending to 0, and hence the Hausdorff distance between $g_e$ and $g^e_\omega$ tends to 0, as desired.

**Corollary 4.10.** The point $X_0 \in \overline{S(\rho^{-1}(\alpha))}$ where the minimum of the function $F(X) = \text{d}_{\text{WP}}(X, \rho(X))$ (restricted to $\overline{S(\rho^{-1}(\alpha))}$) is realized lies in $S(\rho^{-1}(\alpha))$. Moreover,
\[
\lim_{e \to \infty} g_e([-t_e, 0]) = \omega.
\]

**Proof.** First recall that $\ell_{f_e^{-1}(\alpha)}(g_e(-t_e)) = \ell_{\rho^{-1}(\alpha)}(g_e(-t_e))$ and that $\ell_{\alpha}(g_e(0))$ goes to 0 as $e \to \infty$. The distance between the point $g_e(-t_e)$ and the stratum $S(\rho^{-1}(\alpha))$ by Corollary 2.16 is bounded above by $\sqrt{2\pi \ell_{\rho^{-1}(\alpha)}(g_e(-t_e))}$, and hence tends to zero. Thus the point $g_e(-t_e)$ converges to the closure of $S(\rho^{-1}(\alpha))$. From Theorem 4.6 the only curve which is very short (has
length less than \( \epsilon \) at \( g_e(-t_e) \) is \( \rho^{-1}(\alpha) \), so the point \( g_e(-t_e) \) converges to \( S(\rho^{-1}(\alpha)) \). Similarly we can see that \( g_e(0) \) converges to \( S(\alpha) \).

Moreover, since \( g_e \) is a geodesic and \( g_e(0) = \rho(g_e(-t_e)) \) the point \( g_e(-t_e) \) converges to \( X_0 \) at which the minimum of the function \( F \) is realized. Also, \( g_e(0) \) converges to \( \rho(X_0) \).

By the non-refraction of property of WP geodesics (Theorem 2.12) then the interior of \( \omega \) lies in \( \text{Teich}(S) \). The limiting behavior of the geodesic follows from the CAT(0) property of the metric on \( \overline{\text{Teich}(S)} \).

The geodesic axis \( g_e \) descends to a closed geodesic \( \hat{g}_e \) in \( \mathcal{M}(S_0,p) \) and \( \omega \) descends to a geodesic segment \( \hat{\omega} \) in \( \overline{\mathcal{M}(S_0,p)} \). The previous corollary immediately implies the following.

**Corollary 4.11.** As \( e \to \infty \), we have convergence \( \hat{g}_e \to \hat{\omega} \subset \overline{\mathcal{M}(S_0,p)} \).

For any \( e > 0 \), we let \( \delta_e \) denote the geodesic segment from the midpoint of \( \omega \) to the midpoint of \( f_e(\omega) \). We also let \( \omega^- \) and \( \omega^+ \) denote the first and second half-segments of \( \omega \), respectively (so \( \omega = \omega^- \cup \omega^+ \) and \( \omega^- \cap \omega^+ \) is the midpoint of \( \omega \)). Our construction in the next section will use the following.

**Lemma 4.12.** Given \( \epsilon > 0 \), there exists \( N > 0 \) so that for all \( e \geq N \), the triangle with sides \( \delta_e, \omega^+ \), and \( f_e(\omega^-) \) has angles less than \( \epsilon \) at the endpoints of \( \delta_e \), and the Hausdorff distance between \( \delta_e \) and \( \omega^+ \cup f_e(\omega^-) \) is at most \( \epsilon \).

**Proof.** By Proposition 4.5 and Corollary 4.10, the segment \( g_e([\frac{-t_e}{2}, \frac{t_e}{2}]) \) can be made as close as we like to \( \omega^+ \cup f_e(\omega^-) \). Since \( g_e([\frac{-t_e}{2}, \frac{t_e}{2}]) \) and \( \delta_e \) are both geodesics in a CAT(0) space, and since their endpoints become closer and closer as \( e \) tends to infinity, it follows that the distance between \( g_e([\frac{-t_e}{2}, \frac{t_e}{2}]) \) and \( \delta_e \) tends to zero as \( e \to \infty \). Therefore, the distance between \( \delta_e \) and \( \omega^+ \cup f_e(\omega^-) \) tends to zero as \( e \to \infty \). This proves the second statement of the lemma.

Short initial segments of \( \delta_e \) and \( \omega^+ \) are both geodesics in a Riemannian manifold, they have a common initial point, and the initial segment of \( \delta_e \) converges to that of \( \omega^+ \) as \( e \to \infty \). It follows that the angle between \( \delta_e \) and \( \omega^+ \) tends to zero as \( e \to \infty \). A similar argument (composing with \( f_e^{-1} \)) shows that the angle at endpoint of \( \delta_e \) and \( f_e(\omega^-) \) tends to zero as \( e \to \infty \). This proves the first statement of the lemma. \( \square \)

5. **The non-uniquely ergodic case**

Given a sequence of integers \( \mathcal{E} \subset \mathbb{N} \) in this section first we construct a WP geodesic ray \( r \) that is strongly asymptotic to the piecewise geodesic \( g_{\mathcal{E}}^r \) in \( \overline{\text{Teich}(S)} \) similar to the construction in [4] but now for a non-constant sequence \( \mathcal{E} \); see (5.3). The proof of strong asymptoticity involves producing regions with definite total negative curvature on ruled surfaces and an application of the Gauss-Bonnet Theorem (c.f. [BMM11, Mod16]). The asymptoticity to \( g_{\mathcal{E}}^r \) helps us to develop good control on lengths of curves
along \( r \) in §5.2 and determine the limit set of \( r \) in the Thurston compactification of Teichmüller space in §5.3. In §5.4 we prove a technical result required for determining the limit sets of rays in §5.3.

5.1. **Infinite geodesic ray.** Consider a sequence \( \mathcal{E} = \{e_k\}_{k \in \mathbb{N}} \) with \( e_0 > E \) and \( e_{k+1} \geq a e_k \) for some \( a > 1 \) and all \( k \), to which we will impose further constraints later. We write \( \phi_k, \Phi_k = \phi_1 \cdots \phi_{k-1}\phi_k \), and \( \Gamma(\mathcal{E}) = \{\gamma_k\}_{k=0}^{\infty} \) as in (3.1) and (3.2) in §3. Recall from (3.3) that \( \gamma_k = \Phi_{k-m}(\alpha) \) for all \( k \geq m \). Moreover, recall that the sequence \( \{\gamma_k\}_{k} \) converges to a minimal non-uniquely ergodic lamination \( \nu \) in \( \mathcal{EL}(S) \) by Corollary 3.2.

Let \( \omega \) denote the Weil-Petersson geodesic segment connecting the point \( X_0 \in \mathcal{S}(\rho^{-1}(\alpha)) \) to \( \rho(X_0) \in \mathcal{S}(\alpha) \) as in §4. Note that \( X_0 \in \mathcal{S}(\rho^{-1}(\alpha)) \) and so

\[
\Phi_k(X_0) \in \mathcal{S}(\Phi_k(\rho^{-1}(\alpha))) = \mathcal{S}(\Phi_k(\rho^{-1}))(\alpha)) = \mathcal{S}(\gamma_{k+m-1}).
\]

Write \( \delta_k \) for the geodesic segment connecting midpoints of \( \omega \) and \( \phi_k(\omega) \) (compare with §4 and Lemma 4.12). The endpoint of \( \delta_k \) on \( \omega \) will be called its initial endpoint, and the one on \( \phi_k(\omega) \) its terminal endpoint. The image of \( \delta_k \) under any mapping class will have its endpoints labelled as initial and terminal according to those of \( \delta_k \).

With this notation, we claim that the terminal endpoint of \( \Phi_k(\delta_{k+1}) \) is the same as the initial endpoint of \( \Phi_{k+1}(\delta_{k+2}) \). Indeed, applying \( \Phi_k^{-1} \) to this pair of arcs, we have \( \delta_{k+1} \) and \( \phi_k(\delta_{k+2}) \). The terminal endpoint of \( \delta_{k+1} \) is the midpoint of \( \phi_{k+1}(\omega) \). This is the \( \phi_{k+1} \)-image of the midpoint of \( \omega \), which is also the \( \phi_{k+1} \)-image of the initial endpoint of \( \delta_{k+1} \), as claimed.

Concatenating segments of this type defines a half-infinite path:

\[
R_\mathcal{E} = \delta_1 \cup \Phi_1(\delta_2) \cup \Phi_2(\delta_3) \cup \Phi_3(\delta_4) \cup \cdots.
\]

This path fellow-travels the concatenation of \( \omega \) and its translates:

\[
g^\omega_\mathcal{E} = \omega \cup \Phi_1(\omega) \cup \Phi_2(\omega) \cup \Phi_3(\omega) \cup \cdots.
\]

By (5.1), projecting \( R_\mathcal{E} \) and \( g^\omega_\mathcal{E} \) to the curve complex (via the systole map) gives paths fellow traveling \( \{\gamma_k\}_{k=0}^{\infty} \). By Proposition 3.1 and Theorem 2.8 it follows that these are quasi-geodesics in the curve complex. Since the projection to the curve complex is coarsely Lipschitz, so \( R_\mathcal{E} \) and \( g^\omega_\mathcal{E} \) are also quasi-geodesics.

We will also be interested in a truncation of \( R_\mathcal{E} \) after \( k \) steps:

\[
R_k^\mathcal{E} = \delta_1 \cup \Phi_1(\delta_2) \cup \cdots \cup \Phi_{k-1}(\delta_k),
\]

and let \( r_k \) denote the geodesic segment connecting the initial and terminal point of the broken geodesic segment \( R_k^\mathcal{E} \); see Figure 3.

The angle between consecutive segments \( \Phi_{k-1}(\delta_k) \) and \( \Phi_k(\delta_{k+1}) \) will be denoted \( \theta_k \). Applying \( \Phi_{k-1} \), this is the same as the angle between \( \delta_k \) and \( \phi_k(\delta_{k+1}) \). Observe that the angle \( \theta_k \) is at least \( \pi \) minus the sum of the angle between \( \delta_k \) and \( \phi_k(\omega) \) and \( \phi_k(\delta_{k+1}) \) and \( \phi_k(\omega) \) (with appropriate directions chosen). Since \( \phi_k = D^\lambda_{\alpha_{k+m-1}} \rho = f_{\alpha_{k+m-1}} \), by taking \( e_{k+m-1} \) and \( e_{k+m} \)
sufficiently large, appealing to Lemma 4.12 we can ensure that \( \theta_k \) is as close to \( \pi \) as we like. In particular, we additionally assume that our sequence \( \{e_k\}_k \) grows fast enough that

\[
\sum_{k=1}^{\infty} \pi - \theta_k < 1.
\]

We can also (clearly) assume that the integers \( e_k \) are all large enough so that \( \theta_k \geq \frac{\pi}{2} \) for all \( k \).

**Remark 5.1.** While we have imposed growth conditions here to control angles, it is worth mentioning that these are in addition to those conditions already imposed to prove non-unique ergodicity.

**Proposition 5.2.** The geodesic segments \( r_k \) limit to a geodesic ray \( r \) as \( k \to \infty \), and all three of \( r \), \( R_E \), and \( g_{\omega}^\infty \) are strongly asymptotic (the distance between any pair of them tends to zero).

**Proof.** According to the last part of Lemma 4.12, \( R_E \) and \( g_{\omega}^\infty \) are strongly asymptotic. Therefore, it suffices to prove that \( r_k \) has a limit \( r \), and that this is asymptotic to \( R_E \). Before proceeding, we note that \( \Phi_k(X_0) \) lies in the stratum \( S(\Phi_k(\rho^{-1}(\alpha))) = S(\Phi_{k-1}(\alpha)) = S(\gamma_{k+m}) \).

Let \( \{v_i\}_{i=0}^{\infty} \) denote the concatenation points of \( R_E \). Denote by \( P_k \) a ruled polygon bounded by \( r_k \) and \( R_k \). This polygon has vertices \( v_0, \ldots, v_k \). Let \( \theta_i \) denote the interior angles of \( P_k \) at \( v_i \), for \( i = 0, \ldots, k \), and observe that for \( 0 < i < k \), we have \( \theta_i \leq \theta^k \). In addition, there are constants \( c_0 < 0 \) and \( d_0 > 0 \) so that the \( d_0 \)-neighborhood of \( v_i \) in \( P_k \) has Gaussian curvature \( K \leq c_0 \). Consequently, for any \( d < d_0 \), if \( r_k \) is disjoint from the \( d \)-neighborhood \( N_d(v_i) \) of \( v_i \), then since \( \theta^k \geq \frac{\pi}{2} \), \( N_d(v_i) \cap P_k \) contains a quarter-sector of a disk of radius \( d \) centered at \( v_i \) in a surface of curvature at most \( c_0 \). Therefore, the integral of the curvature \( K \) over \( N_d(v_i) \cap P_k \) satisfies

\[
\int_{N_d(v_i) \cap P_k} K \, dA \leq \frac{c_0 \pi d^2}{4}.
\]
By the Gauss-Bonnet Theorem (see e.g. [Cha06, Theorem V.2.5]), we have
\[ \int_{P_k} K dA + \sum_{i=0}^{k} (\pi - \theta_i^k) = 2\pi \]
which implies
\[ \theta_0^k + \theta_k^k - \sum_{i=1}^{k-1} (\pi - \theta_i^k) = \int_{P_k} K dA. \]

For any \( d > 0 \), let \( i_1, i_2, \ldots, i_j \) denote those indices \( i \) for which \( R_{k}^E \) is more than \( d \) away from \( v_i \). Then by our assumption on the angles \( \theta_i \) in (5.4) we have
\[ \theta_0^k + \theta_k^k - \sum_{i=1}^{k-1} (\pi - \theta_i^k) \leq \int_{P_k} K dA \leq \sum_{\ell=1}^{j} \int_{N_{v_i} \cap P_k} K dA \leq \frac{j c_0 \pi d^2}{4}. \]

Since \( c_0 < 0 \), this implies
\[ j \leq \frac{4(\theta_0^k + \theta_k^k) - 1}{c_0 \pi d^2} \leq \frac{4|1 - (\theta_0^k + \theta_k^k)|}{|c_0| \pi d^2} \leq \frac{4(1 + 2\pi)}{|c_0| \pi d^2}. \]

This bounds the number of vertices along \( R_{k}^E \) that can be further than \( d \) away from \( r_k \) by some number \( J(d) \), which is independent of \( k \). Therefore, for any \( N > 0 \) and \( k, k' \geq N + 2J(d) + 1 \), there is a vertex \( v_i \) of \( R_E \) with \( N \leq i \leq \min\{k, k'\} \) so that \( r_k \) and \( r_{k'} \) contain points \( x_k \) and \( x_{k'} \), respectively, which are within distance \( d \) of \( v_i \). Therefore, \( x_k \) and \( x_{k'} \) are within distance 2\( d \) of each other. Since \( R_E \) is a quasi-geodesic, the distance from \( v_i \) to \( v_0 \) tends to infinity with \( i \). Consequently, as \( N \) tends to infinity, the distance from \( x_k \) and \( x_{k'} \) to \( v_0 \) also tends to infinity. By convexity of the distance function between two geodesic segments in a CAT(0) space, it follows that for any \( D > 0 \), the initial segments of \( \{r_k\}_k \) of length \( D \) form a Cauchy sequence in the topology of uniform convergence. By completeness of \( \text{Teich}(S) \), these segments of length \( D \) converge. Letting \( D \) tend to infinity, it follows that \( r_k \) converges (locally uniformly) to a geodesic ray \( r \).

For any \( d > 0 \), suppose that \( v_i \) is a vertex of \( R_E \) further than 2\( d \) away from any point of \( r \). For \( k \) sufficiently large, it follows that \( r_k \) is further than \( d \) from \( v_i \). Since there are at most \( J(d) \) of the latter indices \( i \), it follows that \( r \) must come closer than 2\( d \) from all but \( J(d) \) vertices. In particular, there exists \( N(d) \) so that for all \( i \geq N(d) \), \( r \) comes within 2\( d \) of \( v_i \). By convexity of the distance between geodesics in the WP metric, the distance of any point on \( R_E \) lying between consecutive vertices \( v_i \) and \( v_{i+1} \) (for \( i \geq N(d) \)) and \( r \) is no more than 2\( d \). Therefore, the tail of \( R_E \) starting at \( v_{N(d)} \) is within
Hausdorff distance $2d$ from some tail of $r$. Since $d$ was arbitrary, it follows that $R_\varepsilon$ and $r$ are strongly asymptotic, as required.

In the rest of this section let $r : [0, \infty) \to \Teich(S)$ be the geodesic ray from Proposition 5.2.

5.2. Curves along $r$. The following lemma is a straightforward consequence of the setup of curves $\{\gamma_k\}_k$ in §3 and the choice of the segment $\omega$ in the previous section which we record as a convenient reference.

**Lemma 5.3.** For any $k \geq m - 1$ the initial and terminal endpoints of $\Phi_{k-m+1}(\omega)$ are in the strata $S(\gamma_k)$ and $S(\gamma_{k+1})$, respectively. Furthermore, for any compact subsegment $I \subset \text{int}(\omega)$, the $2m$ consecutive curves

$$\{\gamma_{k-m+1}, \ldots, \gamma_{k+m}\}$$

have bounded length on $\Phi_{k-m+1}(I)$, with the bound depending on the choice of interval $I$, but independent of $k$.

**Proof.** Recall that $\alpha = \rho^m(\gamma_0) = \rho(\gamma_{m-1})$, and hence $X_0 \in S(\alpha) = S(\gamma_{m-1})$. Consequently $\Phi_{k-m+1}(X_0) \in S(\gamma_k)$, since $\Phi_{k-m+1}(\gamma_{m-1}) = \gamma_k$; see [3.4]. Thus the initial endpoint of $\Phi_{k-m+1}(\omega)$ is in $S(\gamma_k)$. Since the terminal endpoint of $\Phi_{k-m+1}(\omega)$ is the initial endpoint of $\Phi_{k-m+2}(\omega)$, this common endpoint lies in $S(\gamma_{k+1})$, proving the first statement.

The compact subsegment $I \subset \text{int}(\omega)$ is entirely contained in Teichmüller space, and hence the curves $\gamma_0, \ldots, \gamma_{2m-1}$ have bounded length in $I$. Since the $\Phi_{k-m+1}$-image of these curves are precisely those listed in the lemma, the second statement also follows.

**Theorem 5.4.** There exists a sequence $\{t_k\}_{k=1}^\infty$ which is eventually increasing, such that $\lim_{k \to \infty} \ell_{\gamma_k}(r(t_k)) = 0$. Furthermore, for any $\epsilon > 0$ sufficiently small, the set of curves with length less than $\epsilon$ along $r$ is contained in $\{\gamma_k\}_{k=0}^\infty$ and contains a tail of this sequence, $\{\gamma_k\}_{k \geq N}$, for some $N = N(\epsilon) \in \mathbb{Z}$.

**Proof.** Since $r$ is strongly asymptotic to $g_\varepsilon^\omega$ by Proposition 5.2 we may choose $t_k$ so that

$$d_{\text{WP}}(r(t_k), S(\gamma_k)) \to 0$$

as $k \to \infty$. Then by the formula

$$d_{\text{WP}}(r(t_k), S(\gamma_k)) = \sqrt{2\pi \ell_{\gamma_k}(r(t_k))} + O\left(\ell_{\gamma_k}(r(t_k))^{5/2}\right)$$

from Corollary 2.16, where the constant of the $O$ notation depends only on an upper bound for the length of $\gamma_k$ at the point $r(t_k)$, we see that

$$\lim_{k \to \infty} \ell_{\gamma_k}(r(t_k)) = 0.$$

Since $g_\varepsilon^\omega$ passes through the strata $\{S(\gamma_k)\}_k$ in order (i.e. $g_\varepsilon^\omega$ intersects $S(\gamma_k)$ before $S(\gamma_{k+1})$), the times when $r$ comes close to $\{S(\gamma_k)\}_k$ also occur in order. This proves the first statement.
For the second statement, we note that the first statement implies that for any \( \epsilon > 0 \), there exists \( N(\epsilon) > 0 \) so that for all \( k \geq N(\epsilon) \), \( \gamma_k \) has length less than \( \epsilon \) at some point along \( r \) (in fact, at the point \( r(t_k) \)). Moreover, by Lemma 5.3, \( \Phi_{k-m+1}(\omega) \) goes from \( S(\gamma_k) \) to \( S(\gamma_{k+1}) \), and no other curves become very short along \( \Phi_{k-m+1}(\omega) \). Again appealing to the fact that \( r \) is asymptotic to \( g_\epsilon^2 \), it follows that for \( k \) sufficiently large, the only curves of length less than \( \epsilon \) on \( r([t_k, t_{k+1}]) \) are \( \gamma_k \) and \( \gamma_{k+1} \). Therefore, for \( \epsilon \) sufficiently small, the only curves that can have length less than \( \epsilon \) along \( r \) are from \( \{\gamma_k\} \).

Because \( r \) is asymptotic to \( g_\epsilon^2 \), there is a version of Lemma 5.3 for \( r \). We will consider sequences \( \{s_k\}_k \subset [0, \infty) \) satisfying one of the following:

(C1) There exists \( \epsilon > 0 \) such that \( t_k + \epsilon < s_k < t_{k+1} - \epsilon \), or

(C2) \( \lim_{k \to \infty} |t_{k+1} - s_k| = 0 \).

**Corollary 5.5.** Suppose that \( \{s_k\}_k \subset [0, \infty) \) is a sequence.

- If \( \{s_k\}_k \) satisfies (C1), then the \( 2m \) consecutive curves \( \gamma_{k-m+1}, \ldots, \gamma_{k+m} \) have bounded length at \( r(s_k) \), independent of \( k \), but depending on \( \epsilon \).
- If \( \{s_k\}_k \) satisfies (C2), then \( \lim_{k \to \infty} \ell_{\gamma_{k+1}}(r(s_k)) = 0 \), and the \( 2m - 1 \) consecutive curves \( \gamma_{k-m+2}, \ldots, \gamma_{k+m} \) have bounded length at \( r(s_k) \), independent of \( k \).

**Proof.** Suppose that we are in case (C1). Then there exists a compact interval \( I \subset int(\omega) \) so that the Hausdorff distance between \( \Phi_{k-m+1}(I) \) and \( r([t_k + \epsilon, t_{k+1} - \epsilon]) \) tends to zero as \( k \to \infty \). By Lemma 5.3, \( \gamma_{k-m+1}, \ldots, \gamma_{k+m} \) have bounded length along \( \Phi_{k-m+1}(I) \). Since \( \Phi_{k-m+1}(I) \) remains bounded away from the completion strata of \( \text{Teich}(S) \), \( \gamma_{k-m+1}, \ldots, \gamma_{k+m} \) also have bounded length along \( r([t_k + \epsilon, t_{k+1} - \epsilon]) \), as required.

For case (C2), the assumptions imply that \( d_{WP}(r(s_k), S(\gamma_{k+1})) \to 0 \) as \( k \to \infty \), and hence \( \ell_{\gamma_{k+1}}(r(s_k)) \to 0 \) as \( k \to \infty \). The bound on the lengths of \( \gamma_{k-m+2}, \ldots, \gamma_{k+m} \) follows from case (C1) and convexity of the length-functions ([Wol08]). Indeed, from case (C1), we know that the curves \( \gamma_{k-m+2}, \ldots, \gamma_{k+m} \) have uniformly bounded lengths at \( r([t_k + \frac{t_{k+1}}{2}, t_{k+1} + \frac{t_{k+2}}{2}]) \) by convexity of length-functions.

As another application of Theorem 5.4, we can identify the ending lamination of \( r \).

**Corollary 5.6.** The lamination \( \nu \) is the ending lamination of the ray \( r \).

**Proof.** By Theorem 5.4, \( \lim_{k \to \infty} \ell_{\gamma_k}(r(t_k)) = 0 \). Since, by Theorem 2.11, the subsequence

\[ \{\gamma_k \mid k \equiv 0 \mod m\} \]

converges to \( \tilde{\nu}_0 \) in \( \mathcal{ML}(S) \) (after appropriately scaling), it follows that the ending lamination of \( r \) contains \( \nu \). Moreover, \( \nu \in \mathcal{EL}(S) \), and hence \( \nu \) is the ending lamination of \( r \).
5.3. The Limit set. By Corollary 5.6 the ending lamination of \( r \) is the minimal non-uniquely ergodic lamination \( \nu \). Let \( \bar{\nu}^h, h = 0, \ldots, m - 1 \), be the ergodic measures supported on \( \nu \) as in Theorem 2.11. Theorem 1.2 follows immediately from the following theorem.

Theorem 5.7. The limit set of \( r \) in \( PML(S) \) is the concatenation of the edges

\[
[\bar{\nu}^0, \bar{\nu}^1], \ldots, [\bar{\nu}^{m-1}, \bar{\nu}^0]
\]

in the 1–skeleton of the simplex of projective measures supported on \( \nu \).

We will reduce this to a more technical statement, and then in the next subsection, prove that technical statement. As we will be exclusively interested in lengths of curves along \( r \), for any curve \( \delta \) and \( s \in [0, \infty) \), we write

\[
\ell_\delta(s) = \ell_\delta(r(s)).
\]

Our main technical result is the following theorem.

Theorem 5.8. Suppose that \( \{s_k\}_k \subset [0, \infty) \) is a sequence.

- If \( \{s_k\}_k \) satisfies (C1), then there exists \( x_k > 0 \) such that for any simple closed curve \( \delta \), we have

\[
\lim_{k \to \infty} \frac{x_k i(\delta, \gamma_k)}{\ell_\delta(s_k)} = 1.
\]

- If \( \{s_k\}_k \) satisfies (C2), then there exist \( x_k, y_k \geq 0 \) with \( x_k + y_k > 0 \) such that for any simple closed curve \( \delta \), we have

\[
\lim_{k \to \infty} \frac{x_k i(\delta, \gamma_k) + y_k i(\delta, \gamma_{k+1})}{\ell_\delta(s_k)} = 1.
\]

Proof of Theorem 5.7 assuming Theorem 5.8. We will pass to subsequences in the following, and to avoid double subscripts, for a subsequence of a sequence \( \{c_k\}_k \), we simply write \( \{c_k\}_{k \in J} \), where \( J \) is the index set defining the subsequence. Likewise \( \lim_{k \in J} c_k \) will denote the limit of the subsequence as the indices from \( J \) tend to infinity.

Now suppose that \( \bar{\mu} \in PML(S) \) is a limit point of the ray \( r \). That is, for some sequence of times \( \{s_j\}_j \subset [0, \infty) \) and any two curves \( \delta, \delta' \), we have

\[
\lim_{j \to \infty} \frac{\ell_\delta(s_j)}{\ell_{\delta'}(s_j)} = \frac{i(\delta, \bar{\mu})}{i(\delta', \bar{\mu})}
\]

(see [2.4]). Since \( s_j \) must tend to infinity as \( j \to \infty \), by passing to a subsequence, we may assume that the sequence is increasing, and there exists an increasing sequence \( \{k_j\}_j \) such that either \( |s_j - t_{k_j}| \to 0 \) or else there exists \( \epsilon > 0 \) so that \( t_{k_j} + \epsilon < s_j < t_{k_j+1} - \epsilon \). Consequently, after reindexing, we assume (as we may) that our sequence is a subsequence \( \{s_k\}_{k \in J} \) of some sequence \( \{s_k\}_{k=1}^\infty \) satisfying either (C1) or (C2).

Suppose first that \( \{s_k\}_{k=1}^\infty \) satisfies (C1), and pass to a further subsequence (with index set still denoted \( J \) for simplicity) so that all \( k \in J \) are...
and consider the arcs the simplex of projective classes of measures on \( \{ \} \) used in the proof of Theorem 5.7. Throughout what follows, we assume that 5.4. Proof of Theorem 5.8. For any subsequence so that all \( k \) subsequence so that all \( k \equiv h \mod m \) for all \( k \in J \), we have \( \lim r(s_k) = [\bar{\nu}^h] \) in the Thurston topology. Consequently, all the vertices of the simplex are in fact accumulation points.

Next, suppose that \( \{ s_k \}_{k=1}^\infty \) satisfies (C2), and again pass to yet another subsequence so that all \( k \in J \) are congruent to some \( h \in \{ 0, \ldots, m-1 \} \mod m \). In this case, we must pass to yet another subsequence so that \( [x_k \gamma_k + y_k \gamma_{k+1}] \) converges to some \( [\bar{\mu}'] \in \mathcal{PML}(S) \), for \( k \in J \). Note that, since by Theorem 2.11 \( [\gamma_k] \to [\bar{\nu}^h] \) and \( [\gamma_{k+1}] \to [\bar{\nu}^{h+1}] \) (where we replace \( h+1 \) by 0, if \( h+1 = m \)), we have \( [\bar{\mu}'] \in [[\bar{\nu}^h], [\bar{\nu}^{h+1}]] \)). Then, by similar reasoning we have

\[
\frac{i(\delta, \bar{\mu})}{i(\delta', \bar{\mu})} = \lim_{k \in J} \frac{i(\delta, s_k)}{i(\delta', s_k)} = \lim_{k \in J} \frac{x_k i(\delta, \gamma_k) + y_k i(\delta, \gamma_{k+1})}{x_k i(\delta', \gamma_k) + y_k i(\delta', \gamma_{k+1})} = \frac{i(\delta, \bar{\mu}')}{i(\delta', \bar{\mu}')},
\]

where the second to last equality follow from bilinearity of intersection number, while the last equality follows since \( [x_k \gamma_k + y_k \gamma_{k+1}] \to [\bar{\mu}'] \) in \( \mathcal{PML}(S) \), for \( k \in J \). Thus again we see that \( [\bar{\mu}] = [\bar{\mu}'] \).

So, the limit set of \( r \) is contained in the required loop in the 1–skeleton of the simplex of projective classes of measures on \( \nu \). If we fix \( h \in \{ 0, \ldots, m-1 \} \) and consider the arcs

\[
\{ r((t_k + t_{k+1})/2, t_{k+1} + t_{k+2}/2) : k \equiv h \mod m \}
\]

it follows that the initial endpoints converge to \( [\bar{\nu}^h] \) while the terminal endpoints converge to \( [\bar{\nu}^{h+1}] \) (again replacing \( h+1 \) with 0 if \( h+1 = m \)). Moreover, the accumulation set of this sequence of arcs is a connected subset of \( [[\bar{\nu}^h], [\bar{\nu}^{h+1}]] \). Any such set is necessarily the entire 1–simplex. Therefore, the ray \( r \) accumulates on the entire loop, as required.

5.4. Proof of Theorem 5.8. Here we prove the required technical theorem used in the proof of Theorem 5.7. Throughout what follows, we assume that \( \{ s_k \}_{k} \) satisfies (C1) or (C2). Many of the estimates can be carried out for both cases simultaneously.
From Corollary 5.5, the \(2m - 1\) curves \(\gamma_{k-m+2}, \ldots, \gamma_{k+m}\) have bounded lengths in \(r(s_k)\), and since
\[
\gamma_{k-m+2}, \ldots, \gamma_k, \gamma_{k+2}, \ldots, \gamma_{k+m}
\]
fill \(S - \gamma_{k+1}\), there is a pants decomposition \(P_k\) containing the \(m\)-component multicurve
\[
\sigma_k := \gamma_k \cup \ldots \cup \gamma_{k+m-1}
\]
such that \(\ell_\beta(s_k)\) is bounded for all \(\beta \in P_k\), independent of \(k\) (though the bounds depend on \(\epsilon\) in case (C1)). Write \(P_k^* = P_k \setminus \sigma_k\).

For an arbitrary curve \(\delta\) and a curve \(\beta \in P_k\), the contribution to the length of \(\delta\) from \(\beta\) in \(r(s_k)\) is defined by the equation:
\[
(5.5) \quad \ell_\delta(s_k, \beta) := i(\delta, \beta)\left( w_{s_k}(\beta) + tw_\beta(\delta, s_k)\ell_\beta(s_k) \right)
\]
where \(tw_\beta(\delta, s_k)\) is the twist of \(\delta\) about \(\beta\) at \(r(s_k)\) as is defined in (2.3), and \(w_{s_k}(\beta)\) is the width of the largest embedded tubular neighborhood of \(\beta\) in \(r(t_k)\) (i.e. the minimal distance between boundary components of the neighborhood). By [Bus10 §4.1], we have
\[
(5.6) \quad w_{s_k}(\beta) = 2 \log \left( \frac{1}{\ell_\beta(s_k)} \right).
\]

The following estimate for the hyperbolic length of a curve \(\delta\) from [CRS08, Lemmas 7.2, 7.3] will be our primary tool.

**Theorem 5.9.** Suppose that the sequence \(\{s_k\}_k\) satisfies (C1) or (C2). Then, for any curve \(\delta\) we have
\[
(5.7) \quad \left| \ell_\delta(s_k) - \sum_{\beta \in P_k} \ell_\delta(s_k, \beta) \right| = O\left( \sum_{\beta \in P_k} i(\delta, \beta) \right).
\]

Here the constant of the \(O\) notation depends only on the upper bound for the length of the curves in \(P_k\).

The proof of Theorem 5.8 now follows from estimating various terms in the sum in the above theorem, and finding that one (in case (C1)) or two (in case (C2)) dominate not only the other terms, but also the error term on the right.

Recall that for any simple closed curve \(\delta\), Theorem 2.9 implies that for all \(j\) sufficiently large we have
\[
(5.8) \quad i(\delta, \gamma_j) \asymp A(0, j)
\]
where the multiplicative error depends on \(\delta\), but not on \(j\). Combining (5.8) and Lemma 2.10 we see that for all \(0 \leq h \leq m - 1\), we have
\[
(5.9) \quad \lim_{k \to \infty} \frac{i(\delta, \gamma_{k+h})}{A(0, k + m)} = 0.
\]

Observe that the curves \(\gamma_{k+h}\) here are precisely the components of \(\sigma_k\). It turns out that the intersection numbers with the other curves in \(P_k\) (not
just those in $\sigma_k$), are also controlled by $A(0, k + m)$. This is essentially the Weil-Petersson analogue of [BLMR16, Theorem 9.15].

**Lemma 5.10.** For any $\beta_k \in P_k$ we have

$$\lim_{k \to \infty} \frac{i(\delta, \beta_k)}{A(0, k + m)} = 0.$$  

**Proof.** By (5.9) it suffices to prove the lemma when $\beta_k \in P^c_k$, for all $k$.

Let $\mu$ be any fixed marking on $\Sigma$ and let $Y_k$ be the component of the complement $\Sigma \setminus \sigma_k$ that contains $\beta_k$.

**Claim 5.11.** There exists $I > 0$, depending only on $\mu$ and $\delta$ so that $i(\pi_Y(\delta), \beta_k) \leq I$.

Since $\delta$ and $\mu$ are a fixed curve and marking, we can assume that their projections to all subsurfaces are uniformly close. Let $Z_k \subseteq Y_k$ be any subsurface with $\beta_k \pitchfork Z_k$, and observe that since $Z_k$ is disjoint from the $m$ consecutive curves in $\sigma_k$, Theorem 2.8 implies that it cannot be an annulus with core curve in the sequence $\{\gamma_i\}_i$. By Corollary 5.5, at the point $r(s_k)$ the $2m - 1$ curves $\gamma_{k-m+2}, \ldots, \gamma_{k+m}$ have length bounded independent of $k$, and hence $i(\beta_k, \gamma_l)$ is uniformly bounded for each $l = k - m + 2, \ldots, k + m$.

Since these curves fill $\Sigma \setminus \gamma_{k+1}$ and $\gamma_{k+1} \in \sigma_k$, $\pi_{Z_k}(\gamma_l) \neq \emptyset$ for some $k - m + 2 \leq l \leq k + m$, and hence $d_{Z_k}(\gamma_l, \beta_k)$ is uniformly bounded. Thus, by the triangle inequality and (2.5) we have

$$d_{Z_k}(\beta_k, \delta) \preceq d_{Z_k}(\gamma_l, \mu) \leq R(\mu).$$

Since this holds for all subsurfaces $Z_k \subseteq Y_k$, [CR07, Corollary D] tells us that $i(\pi_Y(\delta), \beta_k)$ is uniformly bounded, as required.

Every arc of $\pi_Y(\delta)$ comes from a pair of intersection points with curves in $\sigma_k$. Consequently, taking $\kappa(\delta)$ as the second paragraph of Theorem 2.9 and noting that $A(0, j)$ is increasing in $j$ we have

$$i(\delta, \beta_k) \leq I \sum_{d=k}^{k+m-1} i(\delta, \gamma_d) \preceq_{\kappa(\delta)} I \sum_{d=k}^{k+m-1} A(0, d) \leq m I A(0, k + m - 1).$$

Thus, setting $K = m I \kappa(\delta)$ the proof of lemma is complete. \qed

Next, we estimate the various terms of $\ell_\delta(s_k, \beta)$ for $\beta \in P_k$.

**Lemma 5.12.** Suppose that $\{s_k\}_k$ is a sequence satisfying either (C1) or (C2). Then for all $k$ sufficiently large and $\beta \in P_k$, we have

$$\text{tw}_\beta(\delta, s_k) \preceq \text{tw}_\beta(\gamma_0, s_k) \begin{cases} e_k & \text{if } \beta = \gamma_k \\ 0 & \beta \neq \gamma_k \text{ or } \gamma_{k+1}. \end{cases}$$

If $\{s_k\}_k$ satisfies (C1), then for all $k$ sufficiently large, $\text{tw}_{\gamma_{k+1}}(\delta, s_k) \preceq 0$. 
Proof. By Theorem 2.8 and Proposition 3.1 \( \{\gamma_k\}_k \) is a quasi-geodesic ray in \( \mathcal{C}(S) \) (the curve complex of \( S \)). Thus, for any fixed curve \( \delta \) and \( j \) sufficiently large, Theorem 2.3 implies that \( d_{\gamma_j}(\gamma_0, \delta) \doteqdot 0 \). To see this, note that for \( j \) sufficiently large the curve complex distance between \( \gamma_j \) and every curve on a geodesic connecting \( \gamma_0 \) to \( \delta \) is at least 3 and hence \( \gamma_j \) intersects all curves on the connecting geodesic. Thus Theorem 2.3 implies a uniform bound on \( d_{\gamma_j}(\gamma_0, \delta) \). Since each \( \beta \in P_k \) is within distance 1 of \( \gamma_k \), similarly we have \( d_{\beta}(\gamma_0, \delta) \doteqdot 0 \) for all \( \beta \in P_k \), once \( k \) is sufficiently large.

Suppose that \( \{s_k\}_k \) satisfies (C1). The filling set of curves \( \gamma_{k-m+1}, \ldots, \gamma_{k+m} \) have bounded length in \( r(s_k) \). So, for all \( k \) sufficiently large and \( \beta \in P_k \)

\[
\text{tw}_{\beta}(\delta, s_k) \doteqdot \text{tw}_{\beta}(\gamma_0, s_k) \doteqdot \text{diam}_{\mathcal{C}(\beta)}(\gamma_0 \cup \gamma_{k-m+1} \cup \cdots \cup \gamma_{k+m}).
\]

If \( \beta \in P^c_k \), then \( \beta \notin \{\gamma_j\}_j \), and so the term on the right is uniformly close to 0 by Theorem 2.8. If \( \beta = \gamma_j \neq \gamma_k \), then \( j > k \), and the only curves in the set \( \gamma_{k-m+1}, \ldots, \gamma_{k+m} \) which actually intersect \( \gamma_j \) nontrivially must have index less than \( j \). In this case, Theorem 2.8 implies that the term on the right is also uniformly close to 0. When \( \beta = \gamma_k \), again appealing to Theorem 2.8 the right-hand side is estimated (up to a bounded additive error) by

\[
d_{\gamma_k}(\gamma_0, \gamma_{k+m}) \doteqdot e_k.
\]

This proves the lemma when \( \{s_k\}_k \) satisfies (C1). The proof when \( \{s_k\}_k \) satisfies (C2) is nearly identical since the curves \( \gamma_{k-m+2}, \ldots, \gamma_{k+m} \) have bounded length and fill \( S \setminus \gamma_{k+1} \), so the only curve whose twisting we can no longer estimate is \( \gamma_{k+1} \). Since the conclusion of the lemma is silent regarding the twisting about this curve in case (C2), we are done. \( \Box \)

Proof of Theorem 5.8. In either case that \( \{s_k\}_k \) satisfies (C1) or (C2), define \( x_k = w_{s_k}(\gamma_k) + \text{tw}_{\gamma_k}(\gamma_0, s_k, \ell_{\gamma_k}(s_k)) \). Observe that \( \ell_{\gamma_k}(s_k) \doteqdot 1 \), so by (5.6) \( w_{s_k}(\gamma_k) \doteqdot 1 \) (these estimates depend on \( \epsilon \) in case (C1), but not \( k \)). Moreover, from Lemma 5.12 for any curve \( \delta \) and \( k \) sufficiently large we have

\[
\text{tw}_{\gamma_k}(\gamma_0, s_k) \doteqdot \text{tw}_{\gamma_k}(\delta, s_k) \doteqdot e_k \to \infty
\]
as \( k \to \infty \). Consequently, we have \( x_k \doteqdot e_k \) and

\[
\frac{\ell_{\delta}(s_k, \gamma_k)}{x_k i(\delta, \gamma_k)} = \frac{w_{s_k}(\gamma_k) + \text{tw}_{\gamma_k}(\delta, s_k, \ell_{\gamma_k}(s_k))}{w_{s_k}(\gamma_k) + \text{tw}_{\gamma_k}(\gamma_0, s_k, \ell_{\gamma_k}(s_k))} \to 1
\]
as \( k \to \infty \). Combining this with (5.5) and using the setup of integers \( A(0, k) \), for large \( k \), we have

\[
\ell_{\delta}(s_k, \gamma_k) \doteqdot x_k i(\delta, \gamma_k) \doteqdot e_k i(\delta, \gamma_k) \doteqdot A(0, k + m)
\]

Now suppose that we are in case (C1) and \( \beta_k \in P_k \), but \( \beta_k \neq \gamma_k \). As for \( \gamma_k \) above, we have \( \ell_{\beta_k}(s_k) \doteqdot 1 \doteqdot w_{s_k}(\beta_k) \) (with errors depending on \( \epsilon \), but
not \( k \)). Combining this with (5.8) and Lemma 5.12 we have

\[ \ell_\delta(s_k, \beta_k) \sim i(\delta, \beta_k). \]

Therefore, by (5.11) and Lemma 5.10, we have

\[ (5.12) \quad \ell_\delta(s_k, \beta_k) \xrightarrow{k \to \infty} i(\delta, \beta_k) \]

as \( k \to \infty \).

Combining Theorem 5.9 with Lemma 5.10, (5.10), and (5.12), for any curve \( \delta \) we have

\[ \lim_{k \to \infty} \ell_\delta(s_k) = \lim_{k \to \infty} \frac{\ell_\delta(s_k, \gamma_k)}{x_k i(\delta, \gamma_k)} + \frac{1}{x_k i(\delta, \gamma_k)} \left( \sum_{\beta_k \in P_k} \ell_\delta(s_k, \beta_k) + O \left( \sum_{\beta_k \in P_k} i(\delta, \beta_k) \right) \right) = 1, \]

as required.

When \( \{s_k\}_k \) satisfies (C2), \( x_k \) is defined as above, and we define

\[ y_k = w s_k(\gamma_{k+1}) + tw \gamma_{k+1}(\gamma_0, s_k) \ell_{\gamma_{k+1}}(s_k). \]

According to Corollary 5.5, \( \ell_{\gamma_{k+1}}(s_k) \to 0 \) as \( k \to \infty \), and so by (5.6) we have

\[ w s_k(\gamma_{k+1}) \to \infty \]

as \( k \to \infty \). Moreover, Lemma 5.12 ensures that for any curve \( \delta \) and \( k \) sufficiently large, we have

\[ tw \gamma_{k+1}(\gamma_0, s_k) \xrightarrow{\gamma \to \infty} tw \gamma_{k+1}(\delta, s_k). \]

Therefore,

\[ \lim_{k \to \infty} \frac{\ell_\delta(s_k, \gamma_{k+1})}{y_k i(\delta, \gamma_{k+1})} = 1, \]

and combining this with (5.10) we have

\[ \lim_{k \to \infty} \frac{\ell_\delta(s_k, \gamma_k) + \ell_\delta(s_k, \gamma_{k+1})}{x_k i(\delta, \gamma_k) + y_k i(\delta, \gamma_{k+1})} = 1. \]

Because the estimate (5.12) still holds for any curve \( \beta_k \in P_k \) where \( \beta_k \neq \gamma_k \) or \( \beta_k \neq \gamma_{k+1} \), we can again apply Theorem 5.9 and Lemma 5.10 to deduce that for any curve \( \delta \) we have

\[ \lim_{k \to \infty} \frac{\ell_\delta(s_k)}{x_k i(\delta, \gamma_k) + y_k i(\delta, \gamma_{k+1})} = \lim_{k \to \infty} \frac{\ell_\delta(s_k, \gamma_k) + \ell_\delta(s_k, \gamma_{k+1})}{x_k i(\delta, \gamma_k) + y_k i(\delta, \gamma_{k+1})} = 1, \]

completing the proof in case (C2), and hence in general. \( \square \)
6. The uniquely ergodic case

Let \( r : [0, \infty) \rightarrow \text{Teich}(S) \) be a WP geodesic ray, and denote the ending lamination of \( r \) by \( \nu \); see \([2.7]\). The following immediately implies Theorem 1.1 from the introduction.

**Theorem 6.1.** Suppose that \( \nu \) is uniquely ergodic, then the limit set of \( r \) in \( \mathcal{PML}(S) \) (Thurston boundary) is the point \([\bar{\nu}]\).

**Proof.** The proof of the theorem closely follows Masur’s proof of the analogous fact for Teichmüller geodesics \([\text{Mas82}]\) Theorem 1]. Assuming \([\bar{\xi}]\) is any accumulation point of \( r \), let \( \{t_i\}_i \) be a sequence of times so that \( r(t_i) \rightarrow [\bar{\xi}] \) as \( i \rightarrow \infty \) in the Thurston compactification of Teichmüller space \( \text{Teich}(S) \cup \mathcal{PML}(S) \); see \([2.4]\). According to the *Fundamental Lemma of* \([\text{FLP79}]\) exposé 8], there exists a sequence \( \{\bar{\mu}_i\}_i \subset \mathcal{ML}(S) \), such that
\[
i(\bar{\mu}_i, \delta) \leq \ell_\delta(r(t_i)),
\]
for all simple closed curves \( \delta \), as well as a sequence of positive real numbers \( \{b_i\}_i \), so that \( b_i\bar{\mu}_i \rightarrow \bar{\xi} \in \mathcal{ML}(S) \) and \( b_i \rightarrow 0 \), as \( i \rightarrow \infty \).

Let \( \bar{\nu} \) be a transverse measure on \( \nu \). Since \( \nu \) is uniquely ergodic, and hence minimal, there exists a sequence of Bers curves \( \{\gamma_i\}_i \) and positive real numbers \( \{c_i\}_i \), so that \( c_i\gamma_i \rightarrow \bar{\nu} \in \mathcal{ML}(S) \) and \( c_i \rightarrow 0 \), as \( i \rightarrow \infty \). Since the \( \gamma_i \) are Bers curves, by the inequality above there exists \( C > 0 \) so that
\[
i(\bar{\mu}_i, \gamma_i) \leq \ell_{\gamma_i}(r(t_i)) \leq C.
\]
Consequently, by continuity of the intersection form, we have
\[
i(\bar{\xi}, \bar{\nu}) = \lim_{i \rightarrow \infty} i(b_i\bar{\mu}_i, c_i\gamma_i) = \lim_{i \rightarrow \infty} b_ic_i i(\bar{\mu}_i, \gamma_i) \leq C \lim_{i \rightarrow \infty} b_ic_i = 0,
\]
and hence \( i(\bar{\xi}, \bar{\nu}) = 0 \). Because \( \nu \) is uniquely ergodic, \([\text{Mas82}]\) Lemma 2] implies \( \bar{\xi} \) is a multiple of \( \bar{\nu} \), and therefore \( [\bar{\xi}] = [\bar{\nu}] \). Since \( [\bar{\xi}] \) was an arbitrary accumulation point of \( r \) in \( \mathcal{PML}(S) \), the proof is complete. \( \square \)

7. Appendix

In this appendix we provide the proofs of the results of \([2.5]\) about sequences of curves. As we mentioned there, many of the proofs closely follow the ones in \([\text{BLMR16}]\), while others have been streamlined since the writing of that paper. Here we mainly outline the proofs that are similar, incorporating the required changes, and otherwise provide the streamlined proofs.

7.1. Subsurface coefficient estimates. In the next Lemma, \( B_0 \) is the constant from Theorem 2.4

**Lemma 7.1.** (Local to Global) Fix any \( B \geq B_0 + 1 \), and let \( \{\delta_k\}_{k=0}^\omega (\omega \in \mathbb{Z}^{\geq 0} \cup \{\infty\}) \) be a (finite or infinite) sequence of curves in \( \mathcal{C}(S) \), with the property that \( \delta_{k-1} \cap \delta_k, \delta_{k+1} \cap \delta_k \) and that \( d_{\delta_k}(\delta_{k-1}, \delta_{k+1}) \geq 3B \) for all \( k \geq 1 \). Then for all \( 0 \leq i < k < j \), we have that \( \delta_i \cap \delta_j, \delta_j \cap \delta_k \) and that
\[
|d_{\delta_k}(\delta_i, \delta_j) - d_{\delta_k}(\delta_{k-1}, \delta_{k+1})| \leq 2B_0.
\]
Proof. To simplify the notation, write \( d_k(i, j) = d_\delta(i_j, \delta_j) \). The proof is by induction on \( n = j - i \). The base case is \( n = 2 \), in which case \( i = k - 1 \), \( j = k + 1 \), and the conclusions of the lemma hold trivially.

We suppose that \( \gamma_i \nparallel \gamma_k \) and that (7.1) holds for all \( i, j \) with \( i < k < j \) and \( j - i \leq n \), and prove them for \( n + 1 \). To that end, suppose that \( 0 \leq i < k < j \) are such that \( j - i = n + 1 \). We claim that \( d_k(i, k - 1) \leq B_0 \). To see this, note that if \( i = k - 1 \), then the claim holds obviously. Otherwise, \( i < k - 1 < k \) and \( k - i \leq n \), so by hypothesis of the induction \( \gamma_i \nparallel \gamma_k \) and

\[
d_{k-1}(i, k) \geq e_k \geq 3B - 2B_0 > B_0,
\]

note that \( \delta_{k-1} \nparallel \delta_k \) by assumptions of the lemma. Then Theorem 2.4 implies that

\[
d_k(i, k - 1) \leq B_0.
\]

By a similar reasoning, we have that \( \gamma_0 \nparallel \gamma_j \) finishing the proof of lemma by induction.

Set \( B = \max\{3, B_0 + 1, G_0\} \), where \( G_0 \) is the constant from Theorem 2.3 for a geodesic in \( \mathcal{C}(S) \). Set \( E_0 = 3B + 4 \), and for the remainder of this subsection assume that the sequence \( \Gamma(\mathcal{E}) = \{\gamma_k\}_{k=0}^\infty \) satisfies \( \mathcal{P}(\mathcal{E}) \) from Definition 2.6 where \( \mathcal{E} = \{e_k\}_{k=0}^\infty \) satisfies \( e_k \geq a\varepsilon_{k-1} \) for some \( a \geq 1 \) and all \( k \), and \( e_0 \geq E_0 \) (and hence \( e_k \geq E_0 \) for all \( k \)). Also throughout this subsection let \( \mathbb{M} \) be the monoid generated by \( \{m, m+1\} \). A simple arithmetic computation shows that any integer which is greater than or equal to \( m^2 - 1 \) is in \( \mathbb{M} \).

**Lemma 7.2.** For all \( i < k < j \) such that \( k - i, j - k \in \mathbb{M} \), e.g. if \( k - i, j - k \geq m^2 - 1 \), we have that \( \gamma_i \nparallel \gamma_k, \gamma_j \nparallel \gamma_k \) and that

\[
|d_{\gamma_k}(\gamma_i, \gamma_j) - e_k| \leq 2B_0 + 4.
\]

**Proof.** As in the previous proof, we write \( d_k(i, j) = d_{\gamma_k}(\gamma_i, \gamma_j) \), and also write \( i(i, j) = (\gamma_i, \gamma_j) \) and \( \pi_k(j) = \pi_{\gamma_k}(\gamma_j) \).

We make a few observations from Definition 2.6. First, \( i(j, j+1) = 0 \) for all \( j \), and hence if \( \pi_k(j), \pi_k(j+1) \neq \emptyset \) for some \( k \), then \( d_k(j, j+1) = 1 \). Second, for all \( k, i(k, k+m), i(k, k + m + 1) \neq 0 \). Consequently, \( \pi_k(j) \neq \emptyset \) for all \( j, k \) with \( |j - k| \in \{m, m+1\} \). Finally, observe that \( |d_k(k - m, k + m) - e_k| \leq 2 \). Thus, if \( i < k < j \) and \( k - i, j - k \in \{m, m+1\} \), by the triangle inequality

\[
|d_k(i, j) - e_k| \leq 4.
\]

Now, for any sequence of integers \( \{k_j\}_j \) such that \( k_{j+1} - k_j \in \{m, m+1\} \) for all \( j \), the sequence \( \{\gamma_k\}_j \) has the property that \( \gamma_{k_{j+1}} \nparallel \gamma_{k_j} \) and that \( d_{\gamma_k}(k_{j-1}, k_{j+1}) \geq 3B \) by (7.2) and since \( e_k \geq E_0 \). Hence the sequence \( \{\gamma_k\}_j \) satisfies the assumptions of Lemma 7.1. Then by the lemma, for any \( \ell \) with \( i < \ell < j \), \( \gamma_\ell \nparallel \gamma_i, \gamma_\ell \nparallel \gamma_j \) and

\[
|d_{\gamma_k}(k_i, k_j) - e_k| \leq 2B_0 + 4.
\]
Therefore, if \( i < k < j \) and \( k - i, j - k \in \mathbb{M} \), then

\[ |d_k(i, j) - e_k| \leq 2B_0 + 4. \]

This completes the proof of the inequality in the statement of the lemma. \( \square \)

**Lemma 7.3.** The map \( k \mapsto \gamma_k \) is a 1-Lipschitz, \((K, C)\)-quasi-geodesic, where \( K = C = 2m^2 + 2m - 1 \).

**Proof.** First, suppose that \( i < j \) with \( j - i \geq 2m^2 + 2m - 1 \). Then by Lemma 7.2 for each \( m \) \( \in \{i + m^2, i + m^2 + 1, \ldots, j - m^2\} \), we have that \( \gamma_k \cap \gamma_i, \gamma_k \cap \gamma_j \) and that

\[ \frac{d_{\gamma_k} (\gamma_i, \gamma_j)}{e_k} \geq 2B_0 - 4 \geq B \geq 3. \]

Thus the curves \( \gamma_i, \gamma_j \) fill the annulus with core curve \( \gamma_k \). This implies that any curve that intersects \( \gamma_k \) must intersect one of \( \gamma_i \) or \( \gamma_j \). Moreover, the \( 2m \) curves \( \gamma_k \) for \( k = i + m^2, \ldots, j - m^2 \) fill \( S \), \( \gamma_i, \gamma_j \) also fill \( S \).

Next, suppose that \( j > i + 2m^2 + 2m - 1 \) and write \( j = i + q(2m^2 + 2m - 1) + r \), where \( q, r \) are nonnegative integers with \( 0 \leq r < 2m^2 + 2m - 1 \). Set the curve \( \delta_k = \gamma_{i+k(2m^2+2m-1)} \), for \( k = 1, \ldots, q - 1 \). Then the curves

\[ \gamma_i, \delta_1, \ldots, \delta_{q-1}, \gamma_j \]

form a sequence in \( C(S) \). As we saw above any two distinct curves in the sequence fill \( S \), and by (7.3) for all \( 0 < k < q \) we have

\[ d_{\delta_k} (\gamma_i, \gamma_j) \geq B > G_0. \]

So, by Theorem 2.3, a geodesic from \( \gamma_i \) to \( \gamma_j \) must have a vertex disjoint from \( \delta_k \), for all \( k = 0, \ldots, q \). Since any two curves \( \delta_k, \delta_{k'} \) in the sequence fill \( S \), no curve can be disjoint from more than one of them, and hence the geodesic must contain at least \( q + 1 \) vertices, so

\[ d(\gamma_i, \gamma_j) \geq q = \frac{j - i - r}{2m^2 + 2m - 1} \geq \frac{1}{K}(j - i - C) \]

where \( K = 2m^2 + 2m - 1 \) and \( C = 2m^2 + 2m - 1 \). Since this inequality trivially holds if \( j - i < C \) and \( i < j \), the required lower bound follows. Moreover, since \( \gamma_k, \gamma_{k+1} \) are disjoint, the map is 1-Lipschitz. Finally the upper bound for the \((K, C)\)-quasi-geodesic is immediate. \( \square \)

For each \( k \geq 0 \), let \( \mu_k := \{\gamma_k, \ldots, \gamma_{k+2m-1}\} \).

**Lemma 7.4.** There exists \( M > 0 \) such that for any subsurface \( W \subsetneq S \) which is neither \( S \) nor an annulus with core curve some \( \gamma_k \), we have

\[ d_W (\mu_i, \mu_j) \leq M \]

for all \( i, j \).

**Proof.** First, let \( \mu'_k = \{\gamma_k, \ldots, \gamma_{k+2m-2}, \gamma'_{k+2m-1}\} \) where \( \gamma'_{k+2m-1} \) is as in Definition 2.6. From the definition, any curve in \( \mu_k \) and curve in \( \mu'_k \) have
uniformly bounded intersection number (bounded by $m^2 b_2$). Consequently, there exists $M_0 > 0$ such that for any subsurface $W$ and any $k$, we have

$$d_W(\mu_k, \mu_{k+1}') \leq M_0.$$  

Next, observe that $\mu_{k+1}'$ and $\mu_{k+1}$ differ by Dehn twisting $\gamma_{k+2m}'$ about $\gamma_{k+m}$ (which has zero intersection number with all curves in $\mu_{k+1}'$ except $\gamma_{k+2m}$). Therefore, there exists another constant $M_1 > 0$ so that as long as $W$ is not the annulus with core $\gamma_{k+m}$, we have

$$d_W(\mu_{k+1}', \mu_{k+1}) \leq M_1.$$  

Indeed, for any such $W$, $\pi_W(\mu_{k+1}') \cap \pi_W(\mu_{k+1}) \neq \emptyset$, and so we may take $M_1$ to be at most the sum of the diameters of $\pi_W(\mu_{k+1}')$ and $\pi_W(\mu_{k+1})$ which is at most 4.

From these two inequalities, we see that for any subsurface $W$ which is not the annulus with core $\gamma_{k+m}$, the triangle inequality implies

$$d_W(\mu_k, \mu_{k+1}) \leq M_0 + M_1.$$  

From this it follows that for any $D > 0$ and $|j - i| \leq D$,

$$d_W(\gamma_i, \gamma_j) \leq D(M_0 + M_1)$$  

whenever $W$ is not an annulus with core curve $\gamma_k$, for some $k$.

Finally, suppose that $W$ is any subsurface which is not $S$ and not an annulus with core curve $\gamma_k$, for some $k$. By Lemma 7.3, $k \mapsto \gamma_k$ is a quasi-geodesic in $C(S)$. So there is a uniform bound for the number its vertices that are within distance 1 of $\partial W$. Consequently, there exists $D_0 > 0$ (independent of $W$) and $i_0$ so that if $k \notin [i_0, i_0 + D_0]$, then $\pi_W(\gamma_k) \neq \emptyset$. By Theorem 2.3, there exists $G = G(K, C)$ so that the projections of both sequences $\{\gamma_k\}_{k=0}^{i_0}$ and $\{\gamma_k\}_{k=i_0+D_0}$ to $W$ have diameter at most $G$. Combining this with (7.4) and setting $M = 2G + D_0(M_0 + M_1)$, we have

$$\text{diam}_{C(W)}(\{\gamma_k\}_{k=0}^{\infty}) \leq 2G + D_0(M_0 + M_1) = M.$$  

Since $M$ is independent of the subsurface $W$, this completes the proof. \qed

Proof of Theorem 2.8. The fact that $\{\gamma_k\}_k$ is a 1-Lipschitz $(K, C)$-quasi-geodesic is Lemma 7.3, Klarreich’s work [Kla99, Theorem 4.1] describing the Gromov boundary of the curve complex then implies that there exists a $\nu \in \mathcal{E}L(S)$, so that any accumulation point of $\{\gamma_k\}_k$ in $\mathcal{PML}(S)$ is supported on $\nu$.

Any accumulation point of $\{\gamma_k\}_k$ in the Hausdorff topology of closed subset of $S$ contains $\nu$, and hence for a subsurface $W \subseteq S$, $\pi_W(\nu) \subseteq \pi_W(\gamma_k)$ for all $k$ sufficiently large. Consequently, the equations on the left of (2.4) and (2.5) follow from Lemmas 7.2 and 7.4 respectively, setting $R = \max\{M, 2B_0 + 4\}$. For any marking $\mu$, the pairwise intersection between curves in $\mu$ and in $\mu_0$ are bounded by some finite number, and hence $d_W(\mu, \mu_0)$ is uniformly bounded by some constant $D > 0$, independent of $W$. Setting $R(\mu) = R + D$, the equations on the right-hand side of (2.4) and
(2.5) then follow from those on the left-hand side, together with the triangle inequality. □

7.2. Intersection number estimates. We now assume that \( \mathcal{E} = \{ e_k \}_k \) grows exponentially, with \( e_k \geq a e_{k-1} \) for some \( a > 1 \) and all \( k \geq 1 \). The aim is to estimate intersection numbers in terms of the numbers \( A(i, k) \) defined in (2.6). We begin with the upper bound.

Lemma 7.5. If \( \Gamma(\mathcal{E}) = \{ \gamma_k \}_k \) satisfies \( P(\mathcal{E}) \) with \( e_k \geq a e_{k-1} \) for some \( a > 1 \), then there exists \( \kappa > 0 \) such that

\[
i(\gamma_i, \gamma_k) \leq \kappa A(i, k).
\]

Moreover, we may take \( \kappa \) to be decreasing as a function of \( a \).

Sketch of proof. The proof is a rather complicated induction, but is essentially identical to the proof of Proposition 5.5 from [BLMR16]. We sketch the proof for completeness.

We first recall from [FLP79, exposé 4], that for any simple closed curves \( \beta, \delta, \delta' \) and integer \( e \), we have

\[
\left| i(D^e_\delta(\delta'), \delta) - |e|i(\delta', \beta)i(\delta, \beta) \right| \leq i(\delta, \delta').
\]

Since \( \gamma_{k+m} = D^{e_k}_{\gamma_k}(\gamma'_{k+m}) \) and \( i(\gamma'_{k+m}, \gamma_k) = b \), we can apply this to estimate \( i(\gamma_i, \gamma_{k+m}) \) to obtain

\[
\left| i(\gamma_i, \gamma_{k+m}) - be_k i(\gamma_i, \gamma_k) \right| \leq i(\gamma_i, \gamma'_{k+m}).
\]

The right-hand side can be bounded as follows. Since the curves \( \gamma_{k-m}, \ldots, \gamma_{k+m-1} \) fill \( S \), these cut any simple closed curve \( \delta \) into \( N \) arcs, where

\[
N = \sum_{j=k-m}^{k+m-1} i(\delta, \gamma_j).
\]

Now apply this cutting procedure to both \( \gamma'_{k+m} \) and \( \gamma_i \). Any pair of resulting arcs (one from \( \gamma'_{k+m} \) and one from \( \gamma_j \)) are either disjoint, intersect at most once if they lie in a complementary disk, or intersect at most twice if they lie in a once-punctured complementary disk. Thus

\[
i(\gamma_i, \gamma'_{k+m}) \leq 2 \sum_{j=k-m}^{k+m-1} i(\gamma_i, \gamma_j) \sum_{j=k-m}^{k+m-1} i(\gamma'_{k+m}, \gamma_j).
\]

Moreover, the assumption on the values of intersection numbers of \( \gamma'_{k+m} \) and the curves \( \gamma_j, j = k-m, \ldots, k+m-1 \) from Definition 2.6 implies that

\[
\sum_{j=k-m}^{k+m-1} i(\gamma_j, \gamma'_{k+m}) \leq (m+1)b'.
\]
Consequently, setting \( B = 2(m + 1)b' \), we have

\[
\left| i(\gamma_i, \gamma_{k+m}) - be_k i(\gamma_i, \gamma_k) \right| \leq B \sum_{j=k-m}^{k+m-1} i(\gamma_i, \gamma_j),
\]

and hence

\[
i(\gamma_i, \gamma_{k+m}) \leq be_k i(\gamma_i, \gamma_k) + B \sum_{j=k-m}^{k+m-1} i(\gamma_i, \gamma_j).
\]

The goal is to show that \( i(\gamma_i, \gamma_{k+m}) \leq \kappa A(i, k + m) \) for some \( \kappa > 0 \). The proof is by induction on \((k + m) - i\), and the constant \( \kappa \) is actually a limit of an increasing sequence of constants \( K(1) < K(2) < K(3) < \ldots \). To see what these constants should be, we assume by induction that \( i(\gamma_i, \gamma_j) \leq K(j - i)A(i, j) \) for all \( i < j \) with \( j - i < k + m \), then dividing both sides of the inequality above by \( A(i, k + m) \), we have

\[
\frac{i(\gamma_i, \gamma_{k+m})}{A(i, k + m)} \leq \frac{be_k i(\gamma_i, \gamma_k)}{A(i, k + m)} + B \sum_{j=k-m}^{k+m-1} \frac{i(\gamma_i, \gamma_j)}{A(i, k + m)}
\]

\[
\leq K(k - i) \frac{be_k A(i, k)}{A(i, k + m)} + B \sum_{j=k-m}^{k+m-1} \frac{i(\gamma_i, \gamma_j)}{A(i, k + m)}
\]

\[
\leq K(k + m - 1 - i) \left( 1 + B \sum_{j=k-m}^{k+m-1} a^{-\frac{k-i}{m}} \right)
\]

\[
= K((k + m - i) - 1)(1 + 2mB a^{-\frac{k-i}{m}})
\]

The right-hand side above suggests the recursive/inductive definition

\[
K(k + m - i) := K((k + m - i) - 1)(1 + 2mB a^{-\frac{k-i}{m}}).
\]

(Note that the right-hand side depends only on the difference \( k - i \) and not on \( i \) and \( k \) independently). For \( a > 1 \), one shows (using a comparison with a geometric series, after taking logarithms) that \( K(j) \) so defined is bounded, and since it is increasing, it converges to a constant \( \kappa > 0 \). Since \( K(j) < \kappa \) for all \( j \), the inequality above proves the lemma. See \cite{BLMR16} Proposition 5.5] for the details.

Since we are assuming fewer nonzero intersection numbers in the current work than in \cite{BLMR16}, the lower bounds we obtain are weaker, and the proof is slightly more complicated than the one in §5 of \cite{BLMR16}. Fortunately, the weaker estimates suffice for our purposes.

**Lemma 7.6.** If \( e_k \geq ae_{k-1} \) for \( a > 1 \) sufficiently large and all \( k \geq 0 \), then there exists \( \kappa' > 0 \) such that

\[
i(\gamma_i, \gamma_k) \geq \kappa' A(i, k)
\]
whenever

(1) \(0 \leq i \leq 2m - 1\) and \(k \geq i + m^2 + m - 1\), or
(2) \(k - i \geq 2m\) and \(i \equiv k \mod m\).

**Proof.** Fix some constant \(a_0 > 1\) and assume to begin that \(e_k \geq a_0 e_{k-1}\), and let \(\kappa > 0\) be the constant from Lemma 7.5. We will eventually take a larger \(a > a_0\), but we observe that upper bound on intersection numbers from Lemma 7.5 remains valid with this choice of \(\kappa\).

Now, recalling that \(A(i, k + m) = b e_k A(i, k)\), dividing (7.6) by \(A(i, k + m)\), we have

\[
\frac{i(\gamma_i, \gamma_{k+m})}{A(i, k + m)} \geq \frac{i(\gamma_i, \gamma_k)}{A(i, k)} - B \sum_{j=k-m}^{k+m-1} \frac{i(\gamma_i, \gamma_j)}{A(i, k + m)}.
\]

Combining this with Lemmas 2.10 and 7.5, we have

\[
\frac{i(\gamma_i, \gamma_{k+m})}{A(i, k + m)} \geq \frac{i(\gamma_i, \gamma_k)}{A(i, k)} - \kappa B \sum_{j=k-m}^{k+m-1} \frac{A(i, j)}{A(i, k + m)} \geq \frac{i(\gamma_i, \gamma_k)}{A(i, k)} - 2m\kappa B a^{-\left\lfloor \frac{k-i}{m} \right\rfloor}
\]

Recursively substituting, we see that for any \(n \geq 0\) such that \(i < k - nm\),

\[
(7.7) \quad \frac{i(\gamma_i, \gamma_{k+m})}{A(i, k + m)} \geq \frac{i(\gamma_i, \gamma_{k-nm})}{A(i, k - nm)} - 2m\kappa B \sum_{s=0}^{n} a^{-\left\lfloor \frac{k-i}{m} \right\rfloor + s}
\]

In this situation, taking \(a > a_0 > 1\) sufficiently large, we can make the term

\[
2m\kappa B \sum_{s=0}^{n} a^{-\left\lfloor \frac{k-i}{m} \right\rfloor + s}
\]

as small as we like, independent of \(n\) (since this is a partial sum of a geometric series with common ratio \(a\)). Specifically, we choose \(a > a_0 > 1\) large enough so that for all \(k - i \geq m\) and \(n \geq 0\), the sum is bounded above by \(\kappa'\), where

\[
\kappa' = \frac{1}{2} \min \left\{ \frac{1}{A(i, j)} \mid 0 \leq i \leq 2m - 1 \text{ and } i < j \leq m^2 + 3m - 2 \right\}.
\]

Now, suppose that \(0 \leq i \leq 2m - 1\) and that \(k \geq i + m^2 + m - 1\), and write \(k = j + m\) (so \(j \geq i + m^2 - 1\)). Let \(n \geq 0\) be such that

\[
i + m^2 - 1 \leq j - nm \leq m^2 + 3m - 2,
\]

which is possible since \(i + m^2 - 1 \leq m^2 + 2m - 2\). By Lemma 7.2 \(\gamma_i \cap \gamma_{j-nm}\), so \(i(\gamma_i, \gamma_{j-nm}) \geq 1\), and therefore

\[
\frac{i(\gamma_i, \gamma_{j+m})}{A(i, j + m)} \geq \frac{i(\gamma_i, \gamma_{j-nm})}{A(i, j - nm)} - 2m\kappa B \sum_{s=0}^{n} a^{-\left\lfloor \frac{j-i}{m} \right\rfloor + s} \geq 2\kappa' - \kappa' = \kappa'.
\]

That is, \(i(\gamma_i, \gamma_k) \geq \kappa' A(i, k)\), proving part (i).
Next, let \( k, i \) be any two positive integers such that \( k - i \geq 2m \) and \( i \equiv k \mod m \). Write \( k = j + m \) and let \( n \geq 0 \) be such that \( j - nm = i + m \). Then by (7.7) we have

\[
\frac{i(\gamma_i, \gamma_{i + m})}{A(i, j + m)} \geq \frac{i(\gamma_i, \gamma_{i - nm})}{A(i, j - nm)} - 2mkB \sum_{s=0}^{n} a^{-\lfloor \frac{i - s}{m} \rfloor + s} \leq \frac{\gamma_i, \gamma_{i + m}}{1}
\]

The last inequality follows from the fact that \( \kappa' \leq \frac{1}{2} \) and \( i(\gamma_i, \gamma_{i + m}) = b \geq 1 \). This proves (ii), and completes the proof of the lemma.

\[\square\]

**Proof of Theorem 2.9.** The first paragraph of the theorem follows immediately from Lemmas 7.5 and 7.6, setting \( \kappa_0 = \max\{\kappa, \frac{1}{\rho}\} \). The second paragraph is the next lemma. \[\square\]

**Lemma 7.7.** For any curve \( \delta \), there exists \( \kappa(\delta) > 0 \) and \( N(\delta) \in \mathbb{Z} \) so that for all \( k \geq N(\delta) \), we have

\[i(\delta, \gamma_k)^* \approx_{\kappa(\delta)} A(0, k)\]

We only sketch the proof since this is exactly the same statement and proof as in Lemma 5.11 from [BLMR16].

**Sketch of proof.** The idea is that from the estimates in Lemma 7.5 and 7.6 we have some \( \kappa_0, n_0 > 0 \) so that for all \( 0 \leq i \leq 2m - 1 \) and \( k \geq n_0 \),

\[i(\gamma_i, \gamma_k)^* \approx_{\kappa_0} A(0, k)\]

Since \( \mu_0 = \gamma_0 \cup \ldots \cup \gamma_{2m-1} \) fills \( S \), this means that the set of laminations \( \{\frac{\gamma_k}{A(0,k)}\}_{k \geq n_0} \subset \mathcal{ML}(S) \) form a compact subset of \( \mathcal{ML}(S) \). By Theorem 2.9 this sequence can only accumulate on points of \( \mathcal{ML}(S) \) supported on \( \nu \).

So, for any curve \( \delta \), there is a compact neighborhood of the accumulation points and a number \( \kappa(\delta) > 0 \) on which intersection number with \( \delta \) lies in the interval \( \left[ \frac{1}{\kappa(\delta)}, \kappa(\delta) \right] \). But then for \( k \) sufficiently large, \( \frac{\gamma_k}{A(0,k)} \) is in this neighborhood and hence

\[\frac{i(\delta, \gamma_k)}{A(0, k)} = \left( \delta, \frac{\gamma_k}{A(0, k)} \right)^* \approx_{\kappa(\delta)} 1\]

as required. \[\square\]

**7.3. Convergence in \( \mathcal{ML}(S) \).** Lemma 7.7 together with (7.5), are the key ingredients in the proof of the following, which is identical to Lemma 5.13 from [BLMR16].

**Proposition 7.8.** For each \( h = 0, \ldots, m - 1 \),

\[\lim_{i \to \infty} \frac{\gamma_{h+im}}{A(0, h + im)} = \tilde{\nu}^h\]

in \( \mathcal{ML}(S) \), where \( \tilde{\nu}^h \) is a measure supported on \( \nu \).
Sketch of proof. Applying (7.5) to estimate \(i(\gamma_k, \delta)\) using the fact that 
\[ \gamma_{k+m} = D_k \gamma_{k+m}, \]
and applying Lemma 7.7 we may argue as in the proof of Lemma 7.5 we have
\[
|i(\gamma_{k+m}, \delta) - b_k i(\gamma_k, \delta)| \leq i(\gamma_{k+m}, \delta) \leq B \sum_{l=k-m}^{k+m-1} i(\gamma_l, \delta)
\]
\[
\leq B\kappa(\delta) \sum_{l=k-m}^{k+m-1} A(0, l).
\]
Dividing both side by \(A(0, k+m)\) and applying Lemma 2.10 this implies
\[
\left| i\left( \frac{\gamma_{k+m}}{A(0, k+m)}, \delta \right) - i\left( \frac{\gamma_k}{A(0, k)}, \delta \right) \right| \leq 2mB\kappa(\delta) a^{-1} \frac{k}{m}.
\]
From this and a geometric series argument, we deduce that for all \(h = 0, \ldots, m-1\), the sequence \(\left\{ i\left( \frac{\gamma_{h+i m}}{A(0, h+i m)}, \delta \right) \right\}_{i=0}^{\infty}\) is a Cauchy sequence of real numbers, and hence converges. By Lemma 7.7 the limit is nonzero, and since this is true for every simple closed curve \(\delta\), the sequence \(\left\{ \frac{\gamma_{h+i m}}{A(0, h+i m)} \right\}_{i=0}^{\infty}\) converges to some \(\nu^h \in \mathcal{ML}(S)\), supported on \(\nu\) by Theorem 2.8.

The next lemma is the analog of Theorem 6.1 from [BLMR16]. The proof is essentially the same, but since the required intersection number estimates are weaker here, we sketch the proof nonetheless.

**Lemma 7.9.** For each \(h, h' \in \{0, \ldots, m-1\}\) with \(h \neq h'\), we have
\[
\lim_{i \to \infty} \frac{i(\gamma_{h+i m}, \nu^h)}{i(\gamma_{h+im}, \nu^{h'})} = \infty.
\]
Consequently, \(\nu^h\) is not absolutely continuous with respect to \(\nu^{h'}\).

**Sketch of Proof.** As in the proof of [BLMR16, Theorem 6.1], it clearly suffices to prove that for \(i\) sufficiently large,
\[
i(\gamma_{h+i m}, \nu^h) \asymp 1
\]
and
\[
\lim_{i \to \infty} i(\gamma_{h+i m}, \nu^{h'}) = 0.
\]
Appealing to Proposition 7.8 to estimate \(\nu^h\) by \(\frac{\gamma_{h+i m}}{A(0, h+i m)}\) for large \(k \gg i\), and Theorem 2.9 we have
\[
i(\gamma_{0+i m}, \nu^h) \asymp A(0, h+i m) \frac{A(h+i m, k m)}{A(0, h+i m)} = 1
\]
The last equality here follows from a simple calculation using the formula (2.6) for \(A(i, j)\) (see the proof of [BLMR16, Theorem 6.1] for details). The multiplicative error here can be made arbitrarily close to \(\kappa_0^2\) (taking \(k\) sufficiently large).
Similarly, we estimate $\bar{\nu}^{h'} = \frac{\gamma_{h'+km}}{A(0, h'+km)}$ and apply Theorem 2.9 to obtain

$$i(\gamma_0, \gamma_{h+(i+1)m}) i(\gamma_{h+im}, \bar{\nu}^{h'}) \prec A(0, h'+(i+1)m) \frac{A(h+im,h'+km)}{A(0, h'+km)} \prec a^{-i}$$

The first multiplicative error can be made arbitrarily close to $\kappa_0^2$ (again by taking $k$ sufficiently large). The second bound follows from a calculation and Lemma 2.10 with multiplicative error depending only on whether $h > h'$ or $h' > h$ (see [BLMR16] for details).

Proof of Theorem 2.11. All that remains is to prove that $\bar{\nu}^0, \ldots, \bar{\nu}^{m-1}$ are ergodic measures. At this point, the proof is identical to the proof of the analogous statement Theorem 6.7 from [BLMR16], appealing to the facts proved so far. This proof involves a detailed analysis of Teichmüller geodesics, drawing specifically on results of Lenzhen-Masur [LM10] and the fourth author [Raf05]. As this would take us too far afield of the current discussion, we refer the reader to that paper for the details. □

References

[Beh06] Jason A. Behrstock, Asymptotic geometry of the mapping class group and Teichmüller space, Geom. Topol. 10 (2006), 1523–1578.

[BLMR16] Jeffrey Brock, Christopher Leininger, Babak Modami, and Kasra Rafi, Limit sets of Teichmüller geodesics with minimal non-uniquely vertical laminations, II, J. Reine Angew. Math. to appear. arXiv:1601.03368 (2016).

[BLMR17] , Limit sets of Weil-Petersson geodesics with nonminimal ending laminations, arXiv:1711.01663 (2017).

[BM15] Jeffrey Brock and Babak Modami, Recurrent Weil-Petersson geodesic rays with non-uniquely ergodic ending laminations, Geom. Topol. 19 (2015), no. 6, 3565–3601.

[BMM10] Jeffrey Brock, Howard Masur, and Yair Minsky, Asymptotics of Weil-Petersson geodesics. I. Ending laminations, recurrence, and flows, Geom. Funct. Anal. 19 (2010), no. 5, 1229–1257.

[BMM11] , Asymptotics of Weil-Petersson geodesics II: bounded geometry and unbounded entropy, Geom. Funct. Anal. 21 (2011), no. 4, 820–850.

[Bon01] Francis Bonahon, Geodesic laminations on surfaces, Laminations and foliations in dynamics, geometry and topology (Stony Brook, NY, 1998), Contemp. Math., vol. 269, Amer. Math. Soc., Providence, RI, 2001, pp. 1–37.

[Bro03] Jeffrey F. Brock, The Weil-Petersson metric and volumes of 3-dimensional hyperbolic convex cores, J. Amer. Math. Soc. 16 (2003), no. 3, 495–535.

[Bro05] , The Weil-Petersson visual sphere, Geom. Dedicata 115 (2005), 1–18.

[Bus10] Peter Buser, Geometry and spectra of compact Riemann surfaces, Modern Birkhäuser Classics, Birkhäuser Boston Inc., Boston, MA, 2010, Reprint of the 1992 edition.

[Cha06] Isaac Chavel, Riemannian geometry, second ed., Cambridge Studies in Advanced Mathematics, vol. 98, Cambridge University Press, Cambridge, 2006, A modern introduction.

[CMW14] Jon Chaika, Howard Masur, and Michael Wolf, Limits in PMF of Teichmüller geodesics, J. Reine Angew. Math. to appear, arXiv:1406.0564 (2014).

[CR07] Young-Eun Choi and Kasra Rafi, Comparison between Teichmüller and Lipschitz metrics, J. Lond. Math. Soc. (2) 76 (2007), no. 3, 739–756.
[CRS08] Young-Eun Choi, Kasra Rafi, and Caroline Series, Lines of minima and Teichmüller geodesics, Geom. Funct. Anal. 18 (2008), no. 3, 698–754.

[DW03] Georgios Daskalopoulos and Richard Wentworth, Classification of Weil-Petersson isometries, Amer. J. Math. 125 (2003), no. 4, 941–975.

[FLP79] A. Fathi, F. Laudenbach, and V. Poenaru, Travaux de Thurston sur les surfaces, Astérisque No. 66-67 (1979), 1–286.

[Ham15] Ursula Hamenstädt, Weil-Petersson flow and Teichmüller flow, arXiv:1505.01113 (2015).

[Ker80] Steven P. Kerckhoff, The asymptotic geometry of Teichmüller space, Topology 19 (1980), no. 1, 23–41.

[Kla99] Erica Klarreich, The boundary at infinity of the curve complex, preprint (1999).

[Len08] Anna Lenzhen, Teichmüller geodesics that do not have a limit in PMF, Geom. Topol. 12 (2008), no. 1, 177–197.

[LLR13] Christopher Leininger, Anna Lenzhen, and Kasra Rafi, Limit sets of Teichmüller geodesics with minimal non-uniquely ergodic vertical foliation, J. Reine Angew. Math. to appear. arXiv: 1312.2305 (2013).

[LM10] Anna Lenzhen and Howard Masur, Criteria for the divergence of pairs of Teichmüller geodesics, Geom. Dedicata 144 (2010), 191–210.

[LMR16] Anna Lenzhen, Babak Modami, and Kasra Rafi, Two-dimensional limit sets of teichmueller geodesics, J. Mod Dyn. to appear. arXiv:1608.07945 (2016).

[Man13] Johanna Mangahas, A recipe for short-word pseudo-Anosov, Amer. J. Math. 135 (2013), no. 4, 1087–1116.

[Mas75] H.A. Masur, On a class of geodesics in Teichmüller space, Ann. of Math. (2) 102 (1975), no. 2, 205–221.

[Mas82] Howard Masur, Two boundaries of Teichmüller space, Duke Math. J. 49 (1982), no. 1, 183–190.

[MM99] Howard A. Masur and Yair N. Minsky, Geometry of the complex of curves. I. Hyperbolicity, Invent. Math. 138 (1999), no. 1, 103–149.

[MM00] H. A. Masur and Y. N. Minsky, Geometry of the complex of curves. II. Hierarchical structure, Geom. Funct. Anal. 10 (2000), no. 4, 902–974.

[Mod15] Babak Modami, Prescribing the behavior of Weil–Petersson geodesics in the moduli space of Riemann surfaces, J. Topol. Anal. 7 (2015), no. 4, 543–676.

[Mod16] _____, Asymptotics of a class of Weil–Petersson geodesics and divergence of Weil–Petersson geodesics, Algebr. Geom. Topol. 16 (2016), no. 1, 267–323.

[PH92] R. C. Penner and J. L. Harer, Combinatorics of train tracks, Annals of Mathematics Studies, vol. 125, Princeton University Press, Princeton, NJ, 1992.

[Raf05] Kasra Rafi, A characterization of short curves of a Teichmüller geodesic, Geom. Topol. 9 (2005), 179–202.

[Thu88] William P. Thurston, On the geometry and dynamics of diffeomorphisms of surfaces, Bull. Amer. Math. Soc. (N.S.) 19 (1988), no. 2, 417–431. MR 956596

[Wol75] Scott Wolpert, Noncompleteness of the Weil-Petersson metric for Teichmüller space, Pacific J. Math. 61 (1975), no. 2, 573–577. MR 0422692

[Wol03] Scott A. Wolpert, Geometry of the Weil-Petersson completion of Teichmüller space, Surveys in differential geometry, Vol. VIII (Boston, MA, 2002), Surv. Differ. Geom., VIII, Int. Press, Somerville, MA, 2003, pp. 357–393.

[Wol08] _____, Behavior of geodesic-length functions on Teichmüller space, J. Differential Geom. 79 (2008), no. 2, 277–334.

[Wol10] _____, Families of Riemann surfaces and Weil-Petersson geometry, CBMS Regional Conference Series in Mathematics, vol. 113, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2010.
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