Efficient hybrid method for forward kinematics analysis of parallel robots based on signal decomposition and reconstruction

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Abstract
This article combines a new method based on signal decomposition and reconstruction with a fifth-order numerical algorithm and proposes an efficient hybrid method for solving forward kinematics problem of parallel manipulators. In this hybrid method, new method can first generate an approximate solution of the forward kinematics problem, which will be taken as initial guess for the fifth-order numerical technique. The answer with desired level of accuracy is then obtained. The superposition principle is proposed stating that each rod’s displacement required to drive the mobile platform with 6-degree-of-freedom coupled motion is approximately a sum of the rod’s displacement required to drive it with each of six basic motions including roll $q_1$, pitch $q_2$, yaw $q_3$, surge $q_4$, sway $q_5$, and heave $q_6$. By solving the equations established in accordance with the superposition principle, the approximate solution is also gained. Superposition principle is more applicable to the robots with small rotational workspace. The solution is directly solved under superposition principle. Even though the rotational workspace is expanded, the solution still meets the required accuracy of initial value of high-order iterative technique for obtaining its exact solution. The proposed method is then applied to a test bench for bogie parameters and a Stewart platform. Simulation results demonstrate that maximum absolute error of displacement along X-, Y-, and Z-axis is 0.174 mm and its relative error is about $3.4\%$. Using this method will reduce about 34.1% of the required number of iterations to solve the forward kinematics problem compared with improved hybrid method and up to 49.8% compared with the Newton–Raphson method.

Keywords
Parallel robots, rail vehicle, forward kinematics, high-order iterative, superposition principle

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Introduction
Parallel robots are closed-loop kinematic structures mechanically composed of a mobile platform with an end-effector, a base platform, and three or more articulated links connecting the mobile platform to base. Each link houses a linear actuator near the link base, enabling link length changes to control the final position and orientation of the mobile end-effector. Because of their unique properties like high precision and large load-to-weight ratios, parallel robots have been widely applied in different fields such as motion

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As one of the key research issues in parallel robots, the kinematic problem is divided into two lines of research, that is, the inverse kinematics problem (IKP) and the forward kinematics problem (FKP). Compared with the IKP mapping of the task workspace to joint space, the FKP mapping of the joint workspace to task space is relatively difficult. Moreover, obtaining a fast and precise forward kinematics method is an essential step in modeling and in the control of the parallel robots, especially for real-time applications. Therefore, finding an efficient way to solve the FKP and to reduce its computation time has always been one of researchers' priorities.

The coordinates of hinges of the mobile platform relative to the base platform are deduced from the coordinate transformation approach. Based on the relationship between these coordinates and the length of rods, there are some highly nonlinear forward kinematics equations of the workplace, which are not easy to be solved analytically. Parikh and Lam extended a neural-network-based hybrid strategy to the forward kinematics. This performing concept was then combined with a standard Newton–Raphson (NR) numerical technique to yield a hybrid solution strategy. Because the NR method is a second-order numerical algorithm, the desired accuracy can be achieved in less number of iterations and shorter time by higher-order methods. Therefore, Kardan and Akbarzade combined a new structure of artificial neural network same class one network with a third-order numerical algorithm and proposed an improved hybrid method (IHM) for solving the FKP of parallel manipulators.

In this article, by combining a new method based on signal decomposition and reconstruction with high-order iterative method together with the use of a new decomposition technique, a new hybrid strategy is proposed for solving the FKP of parallel robots. An effective way to increase the convergence speed of numerical algorithms is to provide them with more accurate initial guesses. In the proposed method, the new method based on signal decomposition and reconstruction is first used to identify the kinematic parameter like the end-effector’s position and orientation as the next step initial guess. Then, fifth-order numerical technique uses the approximate solution as an initial guess to find the exact answer in real time with the desired level of accuracy. The performance of the proposed method is evaluated by applying it to Stewart platform (SP) and test bench for bogie parameters (TBBP). Compared with the NR method and IHM, the proposed method achieves the desired accuracy in less number of iterations and a shorter time.

Therefore, the main contributions of this article can be summarized as follows:

1. This article combines a new method based on signal decomposition and reconstruction with a fifth-order numerical algorithm and proposes an efficient hybrid method (EHM) for solving FKP of parallel manipulators.

2. First, superposition principle with respect to the rod's displacement that each rod's displacement required to drive the mobile platform with 6-degree-of-freedom (DOF) coupled motion is a sum of the rod's displacements required to drive it with each of 6 single basic motions including roll $q_1$, pitch $q_2$, yaw $q_3$, surge $q_4$, sway $q_5$, and heave $q_6$ is proposed. Then, in accordance with the principle by which the order of magnitude of rod displacement required to drive each of 6 single basic motions should be as close as possible, operator matrix $\delta q_1, \delta q_2, \delta q_3, dq_4, dq_5$, and $dq_6$ are selected. In addition, specific ratio matrix derived from the mapping relationship between the position of the moving platform and the rod displacement is obtained. By solving the equations established in accordance with the superposition principle, the approximate solution is also obtained.

3. To improve the speed and the accuracy of the solution, high-order iterative method is used, which has better efficiency index to solve the system of nonlinear equations using the new decomposition technique.

**FKP of parallel robots’ description**

Parallel robots are closed-loop kinematic structures. In particular, the 6-DOF parallel manipulator comprised a mobile platform with an end-effector, a fixed platform, and three or more articulated links connecting the mobile platform to a fixed platform and supports the mobile platform. The body frame $O_x - X_y Y_z Z_a$ with its origin $O_x$ at the geometric center of the platform is fixed to the moving platform. The inertial frame $O_y - X_b Y_b Z_b$ is attached to the base and coincides with the body frame when the manipulator is in its initial

![Figure 1. Definition of coordinate frame for the parallel manipulator.](image-url)
pose (also called central configuration). Figure 1 illustrates the coordinate frame for the parallel manipulator. The three rotational motions are denoted as roll $q_1$, pitch $q_2$, and yaw $q_3$ and the three translational motions of the body frame with respect to the inertial frame are denoted as surge $q_4$, sway $q_5$, and heave $q_6$. Therefore, the generalized coordinate matrices $A$ and $B$, whose elements are chosen to describe the position and orientation of the movable platform, can be written as

$$A = \begin{bmatrix} l_1 & l_2 & \ldots & l_n \\ W_1 & W_2 & \ldots & W_n \\ h_1 & h_2 & \ldots & h_n \\ 1 & 1 & \ldots & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} L_1 & L_2 & \ldots & L_n \\ W_1 & W_2 & \ldots & W_n \\ h_1 & h_2 & \ldots & h_n \\ 1 & 1 & \ldots & 1 \end{bmatrix}$$

where matrix $A$ represents the coordinate values of the upper hinge joint in the global coordinate system and matrix $B$ represents the coordinate values of the lower hinge joint in the global coordinate system. Therefore, $l, L$ represents the X-axis coordinate value of the upper hinge point and lower hinge point, respectively; $w, W$ represents the Y-axis coordinate value of the upper hinge point; and $h, H$ represents the Z-axis coordinate value of the hinge point. The forward kinematics function of 6-DOF parallel robots can be written as

$$F_i(q_1, q_2, q_3, q_4, q_5, q_6, G_i) = 0 \quad (i = 1, 2, \ldots, n)$$

where $F_i$ is a nonlinear vector function and $G_i$ is a vector of the robot chains' length which is known as the input to equation (2). Finally, $q_1 \ldots q_6$ is a 6-by-1 vector named end-effector’s orientation and position.

**Efficient method for solving the FKP of parallel robots**

**Kinematics model**

Define $R(q_1, q_2, q_3)_{3 \times 3}$ rotation transformation, where $q_1, q_2, \text{and } q_3$ represent the rotation Euler angle by

$$g_1(q_1) = \sqrt{(W_i - w_i \cdot \cos(q_1) + h_i \cdot \sin(q_1))^2 + (h_i \cdot \cos(q_1) - H_i + w_i \cdot \sin(q_1))^2 + (L_i - l_i)^2 - Gr_i}$$

The term $g_1(i = 1, 2, \ldots, n)$ represents the $i$th actuator rod’s displacement trajectory when the moving platform rotated a certain number of $q_1$ rad around X-axis

$$g_2(q_2) = \sqrt{(H_i - h_i \cdot \cos(q_2) + l_i \cdot \sin(q_2))^2 + (l_i \cdot \cos(q_2) - L_i + h_i \cdot \sin(q_2))^2 + (W_i - w_i)^2 - Gr_i}$$

The term $g_2(i = 1, 2, \ldots, n)$ represents the $i$th actuator rod’s displacement trajectory when moving platform only rotated a certain number of $q_2$ rad around Y-axis

which the upper platform rotates around X-, Y-, and Z-axis under Z-Y-X order; and $P(q_4, q_5, q_6)_{1 \times 3}$ is the position transformation. So, the homogeneous coordinate transformation matrix $T$ is constructed as

$$T = \begin{bmatrix} R(q_1, q_2, q_3) & P(q_4, q_5, q_6) \\ 0 & 1 \end{bmatrix}$$

$$T(q_1, q_2, q_3, q_4, q_5, q_6) = \begin{bmatrix} cq_2cq_3 & -cq_1sq_3 + sq_1sq_2cq_3 & sq_1sq_3 + cq_1sq_2cq_3 \\ cq_2sq_3 & cq_1cq_3 + sq_1sq_3 & -sq_1cq_3 + cq_1sq_2cq_3 \\ -sq_2 & sq_1cq_2 & cq_1cq_2 \end{bmatrix}$$

where $c$ represents cos, and $s$ represents sin.

**Superposition principle of cylinder displacement**

By inverse kinematic model, the rods’ displacement of parallel manipulator used to control the final position and orientation of the mobile end-effector are confirmed

$$G_i = \|RA_i + P - B_i\| = \|T(q_1, q_2, q_3, q_4, q_5, q_6) \cdot A_i - B_i\| \quad (i = 1, 2, \ldots, n)$$

where $G_i$ is the $i$th actuator rod’s displacement trajectory under 6-DOF coupled motion

$$g_{ij} = \|T(q_j) \cdot A_i - B_i\| \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, 6)$$

where $g_{ij}$ represents the $i$th actuator rod’s displacement trajectory under the $j$th basic single motion including roll $q_1$, pitch $q_2$, yaw $q_3$, surge $q_4$, sway $q_5$, and heave $q_6$.
The term \( g_{3i}(i = 1, 2, \ldots, n) \) represents the \( i \)th rod’s displacement trajectory when moving platform rotated a certain number of \( q_3 \) rad around Z-axis

\[
g_{3i}(q_3) = \sqrt{(L_i - l_i \cos(q_3) + w_i \sin(q_3))^2 + (w_i \cos(q_3) - W_i + l_i \sin(q_3))^2 + (H_i - h_i)^2 - Gr_i}
\]

The term \( g_{4i}(i = 1, 2, \ldots, n) \) represents the \( i \)th rod’s displacement trajectory when moving platform translated a certain amount of \( q_4 \) mm along X-axis

\[
g_{4i}(q_4) = \sqrt{(l_i - L_i + q_4)^2 + (H_i - h_i)^2 + (W_i - w_i)^2 - Gr_i}
\]

The term \( g_{5i}(i = 1, 2, \ldots, n) \) represents the \( i \)th rod’s displacement trajectory when moving platform translated a certain amount of \( q_5 \) mm along Y-axis

\[
g_{5i}(q_5) = \sqrt{(w_i - W_i + q_5)^2 + (H_i - h_i)^2 + (L_i - l_i)^2 - Gr_i}
\]

\[
G_1 = \sqrt{(q_6 - H - lq_2 + h + wq_1)^2 + [q_4 - L + hq_2 - wq_3 + l]^2 + [q_5 - W - hq_1 + w + lq_3]^2 - Gr}
\]

The term \( g_{6i}(i = 1, 2, \ldots, n) \) represents the \( i \)th rod’s displacement trajectory when moving platform translated a certain amount of \( q_6 \) mm along Z-axis

\[
g_{6i}(q_6) = \sqrt{(w_i - W_i + q_6)^2 + (H_i - h_i)^2 + (L_i - l_i)^2 - Gr_i}
\]

\[
g_{6i} + g_{2i} + g_{3i} + g_{4i} + g_{5i} + g_{6i} = \sum_{j=1}^{6} g_{ji} = G_i \quad (i = 1, 2, \ldots, n)
\]

where \( Gr_i \) is the initial length of the rod. Equation (13) is called the superposition principle.\(^{10,11}\) The final motion of parallel robots is the superposition of six basic single motions, so the final trajectory of each actuator under 6-DOF coupled motion is also the superposition of six trajectories of actuators under six basic single motions.

We prove the superposition principle as follows

Set \( \Delta_i = G_i - \sum_{j=1}^{6} g_{ji} \) \hspace{1cm} (14)

Different actuator rod’s trajectories have the same mathematical model as shown in equations (6)-(12). Take \( i = 1 \) as an example. According to the actual operational conditions and workspace analysis of TBBP proved in section “Test bench for bogie parameters,” \( q_1, q_2, \) and \( q_3 \) are all less than 0.1745 rad.

Therefore, \( \sin(q_i) = q_i(i = 1, 2, 3), \sin(q_i) \cdot \sin(q_k) = 0 \) \hspace{1cm} (i = 1, 2, 3; k = 1, 2, 3; i \neq k), \hspace{1cm} and \hspace{1cm} \cos(q_i) = 1 \hspace{1cm} (i = 1, 2, 3) \hspace{1cm} are deemed as applicable. Under these conditions, the rotational order does not matter. For instance, both transformation matrix Tzxy under Z-Y-X order and Tzxy under Z-X-Y order are equal to equation (15)

\[
T = \begin{bmatrix}
1 & -q_3 & q_2 & q_4 \\
q_3 & 1 & -q_1 & q_5 \\
-q_2 & q_1 & 1 & q_6 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Therefore, equation (5) is thus simplified as

\[
g_{11} = \sqrt{(W - w + hq_1)^2 + (h - H + wq_1)^2 + (L - l)^2 - Gr} = \frac{(W - w + hq_1)^2 + (h - H + wq_1)^2 + (L - l)^2 - Gr}{\sqrt{2q_1[h(W - w) - w(H - h)] + q_1^2(h^2 + w^2) + Gr + Gr^2}}
\]

We can define \( d_1 = 2q_1[h(W - w) - w(H - h)] + q_1^2(h^2 + w^2). \) Similarly, \( g_{21}, g_{31}, g_{41}, g_{51}, \) and \( g_{61} \) are all derived. Then, \( g_{j1} \) can be written as

\[
g_{j1} = \frac{d_j}{\sqrt{Gr^2 + d_j + Gr}}
\]

where

\[
\begin{align*}
d_1 &= 2q_1[h(W - w) - w(H - h)] + q_1^2(h^2 + w^2) \\
d_2 &= 2q_2[l(H - h) - h(L - l)] + q_2^2(l^2 + h^2) \\
d_3 &= 2q_3[w(L - l) - l(W - w)] + q_3^2(l^2 + w^2) \\
d_4 &= 2q_4(l - L) + q_4^2 \\
d_5 &= 2q_5(w - W) + q_5^2 \\
d_6 &= 2q_6(h - H) + q_6^2
\end{align*}
\]

Since \( Gr^2 \gg d_j, \Delta_1 \) can be expressed as

\[
\Delta_1 = G_1 - \frac{\sum_{j=1}^{6} d_j}{2Gr}
\]
By substituting equations (16)–(18) into equation (19), we get

$$\Delta_1 = G_1 - \sum_{j=1}^{6} g_{1j} = \frac{[q_6(q_{w1} - l_{q2}) + q_3(l_{q3} - h_{q1}) + q_4(h_{q2} - w_{q3})] - [(w_{q1} + l_{q2})^2 + (l_{q3} + h_{q1})^2 + (h_{q2} - w_{q3})^2]}{2Gr}$$  \hspace{1cm} (20)

Since $[(w_{q1} + l_{q2})^2 + (h_{q1} + l_{q3})^2 + (l_{q3} + h_{q1})^2] > 0$, then

$$\Delta_1 = G_1 - \sum_{j=1}^{6} g_{1j} < \frac{q_6(q_{w1} - l_{q2}) + q_3(l_{q3} - h_{q1}) + q_4(h_{q2} - w_{q3})}{2Gr}$$  \hspace{1cm} (21)

According to equation (19) and property of inequality

$$\frac{\Delta_1}{F_1} < \frac{[q_6(q_{w1} - l_{q2}) + q_3(l_{q3} - h_{q1}) + q_4(h_{q2} - w_{q3})]}{\sqrt{(q_{w1} - l_{q2})^2 + (l_{q3} - h_{q1})^2 + (h_{q2} - w_{q3})^2} + \sqrt{q_6^2 + (q_3^2 + q_4^2)}}$$

$$\frac{\Delta_1}{F_1} < \frac{Gr + \tilde{G} \cdot p_2 + p_4 + p_6}{2Gr} < \frac{\tilde{G} \cdot p_2 + p_4 + p_6}{Gr \cdot p_1 + p_3 + p_5}$$  \hspace{1cm} (23)

where

$$Z = p_1 + p_2 + p_3$$

$$p_1(q_{q1}, q_{q2}, q_{q3}) = q_6^2 + (q_{w1} - l_{q2})^2 + 2q_6(q_{w1} - l_{q2}) - 2(H - h)q_6 - 2(H - h)(q_{w1} - l_{q2})$$

$$p_2(q_{q1}, q_{q2}, q_{q3}) = q_6(q_{w1} - l_{q2})$$

$$p_3(q_{q1}, q_{q2}, q_{q3}) = q_3^2 + (l_{q3} - h_{q1})^2 + 2q_3(l_{q3} - h_{q1}) - 2(W - w)q_3 - 2(W - w)(l_{q3} - h_{q1})$$

$$p_4(q_{q1}, q_{q2}, q_{q3}) = q_3(l_{q3} - h_{q1})$$

$$p_5(q_{q2}, q_{q3}, q_{q4}) = q_4^2 + (h_{q2} - w_{q3})^2 + 2q_4(h_{q2} - w_{q3}) - 2(L - l)q_4 - 2(L - l)(h_{q2} - w_{q3})$$

$$p_6(q_{q2}, q_{q3}, q_{q4}) = q_4(h_{q2} - w_{q3})$$

$$\tilde{G} = \sqrt{(q_{w1} - l_{q2} - h + w_{q1})^2 + (l_{q3} - h_{q1})^2 + (h_{q2} - w_{q3})^2} + \sqrt{q_6^2 + (q_3^2 + q_4^2)}$$

Based on the actual situation, when $q_{w1} - l_{q2} \geq q_6$, $p_1$ is a higher-order infinite than $p_2$. When $q_{w1} - l_{q2} < q_6$, $p_1$ is also a higher-order infinite than $p_2$. Therefore, $p_1$ is always a higher-order infinite than $p_2$. Similarly, $p_3$ is always a higher-order infinite than $p_4$ and $p_5$ is always a higher-order infinite than $p_6$. $\tilde{G}$ is the final length of rod, so $|\tilde{G}/G_r|$ is less than 2. As independent variable $q_i (i = 1, 2, \ldots, 6)$ changes, $(\Delta_1/F_1) = 0$. So, similarly, $(\Delta_i/F_i) = 0$ ($i = 1, 2, \ldots, n$).

This conclusion shows that the final trajectory of rod displacement is a linear superposition of the trajectories under all the basic motions with a certain difference calculated by subtraction of the sum derived from superposition principle and coupling value. However, this difference brought by parallel mechanism’s nonlinearity negatively affects the accuracy of solution. According to the actual operational conditions and workspace analysis of parallel mechanism, $q_i (i = 1, 2, 3)$ are all less than 0.1745 rad. The superposition principle is fully satisfied.

Superposition principle is more applicable to the robots with small rotational workspace. The solution is directly solved under superposition principle. Even though the rotational workspace is expanded, the solution still meets the required accuracy of initial value of high-order iterative technique for obtaining its exact solution. The approach of linear decoupling based on signal decomposition and reconstruction can reduce the blindness on selecting initial value and then improve the high-order iteration method’s convergence speed to obtain the accurate real-time solution.

### Ratio matrix model

First, the principle of signal decomposition and reconstruction is applied to solve FKP of 6-DOF parallel manipulator. Superposition principle is proposed stating that each rod’s displacement required to drive the mobile platform with 6-DOF coupled motion is a sum of the rod’s displacements required to drive it with each of 6 single basic motions including roll $q_1$, pitch $q_2$, yaw $q_3$, surge $q_4$, sway $q_5$, and heave $q_6$, and it is proved in section “Superposition principle of cylinder displacement.”

Given differential operator impulse $q_{i}\dot{q}_{i}$ under the $j$th basic motion, the $i$th actuator rod’s displacement is $g_{ij}(q_{i})$. When differential operator impulse $k_{ij} \cdot q_{ij}$ is given under the $j$th basic motion, the displacement of
ith actuator rod’s displacement is $k_j(k_j, q_j)g_j(q_j)$. The ratio is a function of $k_j$ as the independent variable. Ratio matrix can be thus deduced by the following equation

$$M_j(k_j) = \frac{\partial g_j(k_j \cdot q_j)}{\partial g_j(q_j)} = k_j \cdot f(k_j, q_j) \quad (j = 1, 2, \ldots, 6; \quad i = 1, 2, \ldots, n) \quad (24)$$

The specific ratio matrix derived from the mapping relationship between the position of the moving platform and the displacement of the rod is obtained. According to the fact that different parallel manipulators have different geometric shapes, special ratio matrix is thus analytically obtained. Then, in accordance with the principle by which the order of magnitude of rod displacement required to drive the moving platform with each of 6 single basic motions should be as close as possible, differential operator matrix $J_j(\delta q_1, \delta q_2, \delta q_3, dq_4, dq_5, dq_6)$ is thus selected (see equations (25) and (26))

$$J_j(\delta q_1, \delta q_2, \delta q_3, dq_4, dq_5, dq_6) = \left\| T(q_j) \bullet A_i - B_i \right\| - G_i \quad (j = 1, 2, \ldots, 6; \quad i = 1, 2, \ldots, n) \quad (25)$$

$$J(\delta q_1, \delta q_2, \delta q_3, dq_4, dq_5, dq_6) = \begin{bmatrix} g_{11}(\delta q_1) & g_{12}(\delta q_1) & \cdots & g_{1n}(\delta q_1) \\ g_{21}(\delta q_2) & g_{22}(\delta q_2) & \cdots & g_{2n}(\delta q_2) \\ \vdots & \vdots & \ddots & \vdots \\ g_{61}(dq_6) & g_{62}(dq_6) & \cdots & g_{6n}(dq_6) \end{bmatrix} \quad (26)$$

In this article, a type of matrix arithmetic is defined

$$A = [a_{ij}]_{m \times n}, B = [b_{ij}]_{m \times n}, C = A \otimes B = [a_{ij} \cdot b_{ij}]_{m \times n}$$

Based on the superposition principle, the equations used to solve the real-time position are established (see equation (27))

$$\begin{bmatrix} \cdots M_{11}(k_1) & \cdots M_{12}(k_1) & \cdots & \cdots M_{1n}(k_1) \\ \cdots M_{21}(k_2) & \cdots M_{22}(k_2) & \cdots & \cdots M_{2n}(k_2) \\ \vdots & \vdots & \ddots & \vdots \\ \cdots M_{61}(k_6) & \cdots M_{62}(k_6) & \cdots & \cdots M_{6n}(k_6) \end{bmatrix} \cdot \begin{bmatrix} \cdots g_{11}(\delta q_1) & \cdots g_{12}(\delta q_1) & \cdots & \cdots g_{1n}(\delta q_1) \\ \cdots g_{21}(\delta q_2) & \cdots g_{22}(\delta q_2) & \cdots & \cdots g_{2n}(\delta q_2) \\ \vdots & \vdots & \ddots & \vdots \\ \cdots g_{61}(dq_6) & \cdots g_{62}(dq_6) & \cdots & \cdots g_{6n}(dq_6) \end{bmatrix} = \begin{bmatrix} G_1 - G_{r_1} \\ G_2 - G_{r_2} \\ \vdots \\ G_n - G_{r_n} \end{bmatrix} \quad \text{T}$$

(27)

where $C$ is the identification matrix $C = [c_1, c_2, c_3, c_4, c_5, c_6]$, generally defined as $[1, 1, 1, 1, 1, 1]$.}

### High-order iterative method

To solve the nonlinear equations without divergence by Taylor expansion and Newton-type iteration, Yang et al. proposed a modified global Newton–Raphson (MGNR) algorithm. By Adomian decomposition, Darvishi and Barati extended a third-order algorithm that does not require second derivative of the equations and they proved that this algorithm benefits from a third-order convergence. The algorithm was then applied by Kardan and Akbarzadeh to solve FKP of parallel robots. Moreover, Waseem et al. constructed a sequence of high-order iterative methods for nonlinear equations system through a new decomposition technique mainly due to Daftardar-Gejji and Jafari and proved that the methods were higher-order convergent and had better efficiency index. This decomposition of nonlinear operator $N(X)$ is quite different from the Adomian decomposition, and it can be decomposed as

$$N \left( \sum_{i=0}^{\infty} X_i \right) = N(X_0) + \sum_{i=1}^{\infty} \left\{ N \left( \sum_{j=0}^{i} X_j \right) - N \left( \sum_{j=0}^{i-1} X_j \right) \right\} \quad (28)$$

In this work, this iterative algorithm is used to solve the nonlinear equation system of FKP of parallel robots. For a given $X(0)$, the approximate solution $X(k + 1)$ is computed using the following iterative schemes

$$Y^{(i)} = X^{(i)} - (F(X^{(i)}))^{-1} F(X^{(i)})$$

Algorithm 1

$$X^{(k+1)} = Z^{(k)} - (F(X^{(k)}))^{-1} F(Z^{(k)}), \quad k = 0, 1, 2, \ldots \quad (29)$$

$$Z^{(i)} = Y^{(i)} - (F(X^{(i)}))^{-1} F(Y^{(i)})$$

Algorithm 2

$$W^{(i)} = Z^{(i)} - (F(X^{(i)}))^{-1} F(Z^{(i)})$$

$$X^{(k+1)} = W^{(i)} - (F(X^{(i)}))^{-1} F(W^{(i)}), \quad k = 0, 1, 2, \ldots \quad (30)$$

where $F'$ is the Jacobian matrix of the system. The new method based on signal decomposition and reconstruction can first generate an approximate solution of the FKP, which will be taken as initial guess for the fifth-order numerical technique. The fifth-order numerical technique has better efficiency index. Using the numerical algorithm to iterate, the termination condition is as follows

$$\|X^{(k+1)} - X^{(k)}\|_\infty = \text{max}(x^1, x^2, \ldots, x^n) \leq E$$

(31)
Algorithm 1 has been proved to be the fourth-order convergence and Algorithm 2 to be the fifth-order convergence. In the following sections of this article, the better performance in less number of iterations and shorter time than the existing method will be proved.

Hydraulic actuator model

Researchers have already discussed the influence of the linearized servo-valve dynamics model and obtained a satisfactory result;\textsuperscript{14} so in this article, the servo-valve dynamics\textsuperscript{15} is neglected and hence input signal \( u \) is treated directly as the displacement of spool valve.

Assuming there is no external leakage,\textsuperscript{16} the cylinder dynamics can be simplified as

\[
\frac{1}{\beta} \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} -\frac{A_1}{V_1^2} & \frac{C_{d_1}}{V_1^2} \\ \frac{C_{d_2}}{V_2^2} & -\frac{A_2}{V_2^2} \end{bmatrix} \begin{bmatrix} \dot{x}_L \\ p_1 - p_2 \end{bmatrix} + \begin{bmatrix} \frac{k_{11} \sqrt{\Delta p_1}}{V_1} \\ \frac{k_{12} \sqrt{\Delta p_2}}{V_2} \end{bmatrix} u
\]

(32)

where \( A_1 \) is the cross-sectional area of the piston on which pressure \( p_1 \) acts; \( A_2 \) is the annular area on which pressure \( p_2 \) acts; \( \beta \) is the effective bulk modulus; \( x_L \) is the rod displacement; \( C_{d_k} \) is the coefficient of the internal leakage of the cylinder; \( k_{11} \) is the flow gain coefficient of the servo-valve port 1; and \( k_{12} \) is the flow gain coefficient of the servo valve\textsuperscript{17} port 2. \( \Delta p_i \) is defined as

\[
\Delta p_1 = \begin{cases} p_r - p_1 & \text{for } x_v > 0 \\ p_1 - p_r & \text{for } x_v < 0 \end{cases}
\]

(33)

\[
\Delta p_2 = \begin{cases} p_2 - p_r & \text{for } x_v > 0 \\ p_r - p_2 & \text{for } x_v < 0 \end{cases}
\]

(34)

where \( p_r \) is the supply pressure of the fluid and \( p_r \) is the tank or reference pressure. \( V_1 \) is the total control volume of the first chamber, and \( V_2 \) is the total control volume of the second chamber. \( V_1 \) and \( V_2 \) can be obtained by the following equation

\[
\begin{align*}
V_1 &= V_{1i} + A_1 x_L \\
V_2 &= V_{2i} + A_2 x_L
\end{align*}
\]

(35)

where \( V_{1i} \) is the initial volume of the first chamber and \( V_{2i} \) is the initial volume of the second chamber.

Sample robots

SP

The SP is composed of a base platform and a mobile platform supported by six actuators connecting together the base with the moving platform. As Figures 1 and 2 show, SP is a 6-DOF parallel robot with three rotations around X-, Y-, and Z-axis and three translations along X-, Y-, and Z-axis. The body frame \( O_a - X_aY_aZ_a \) with its origin \( O_a \) coincides with the geometric center of the movable platform. The inertial frame \( O_b - X_bY_bZ_b \) fixed to the base platform coincides with the body frame. Considering a symmetric SP, the upper, and the lower platform share identical shape and size. The projection of the upper platform on the basic plane coincides with the lower platform at its initial position. The parameters and the coordinate systems required for kinematic description of this robot are included in matrices A and B

\[
A = \begin{bmatrix}
298.5 & -123.27 & -175.23 & -175.23 & -123.27 & 298.5 \\
-30 & -273.51 & -243.51 & 243.51 & 273.51 & 30 \\
335 & 335 & 335 & 335 & 335 & 335 \\
1 & 1 & 1 & 1 & 1 & 1 \\
175.23 & 123.27 & -298.5 & -298.5 & 123.27 & 175.23 \\
-243.51 & -273.51 & -30 & 30 & 273.51 & 243.51 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}
\]

(36)

\[
B = \begin{bmatrix}
-243.51 & -273.51 & -30 & 30 & 273.51 & 243.51 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}
\]

According to SP’s geometrical characteristics, substitute its structure parameter matrices A and B into equation (24). SP’s ratio matrix \( M \) is thus obtained
in the global coordinate system, and matrix \( M(k_1, k_2, k_3, k_4, k_5, k_6) = \)
\[
\begin{bmatrix}
  k_1 & k_1 & k_1 & k_1 & k_1 & k_1 \\
  k_2 & k_2 & k_2 & k_2 & k_2 & k_2 \\
  k_3 & k_3 & k_3 & k_3 & k_3 & k_3 \\
  k_4 & k_4 & k_4 & k_4 & k_4 & k_4 \\
  k_5 & k_5 & k_5 & k_5 & k_5 & k_5 \\
  k_6 & k_6 & k_6 & k_6 & k_6 & k_6 
\end{bmatrix}
\]
(37)

\[
A = \begin{bmatrix}
  400 & 400 & -400 & -400 & 365 & 365 & 0 \\
  -1100 & 1100 & 1100 & -1100 & -1700 & 1700 & 2070 \\
  -990 & -990 & -990 & -990 & 0 & 0 & 0 \\
  1 & 1 & 1 & 1 & 1 & 1 & 1 \\
  -1100 & 1100 & 1100 & -1100 & -1700 & 1700 & 5000 \\
  -3920 & -3920 & -3920 & -3920 & 0 & 0 & 0 \\
  1 & 1 & 1 & 1 & 1 & 1 & 1 
\end{bmatrix}
\]
(38)

\[
B = \begin{bmatrix}
  400 & 400 & -400 & -400 & 3920 & 3920 & 3920 \\
  -1100 & 1100 & 1100 & -1100 & -3650 & 3650 & 3650 \\
  -990 & -990 & -990 & -990 & 0 & 0 & 0 \\
  1 & 1 & 1 & 1 & 1 & 1 & 1 \\
  -1100 & 1100 & 1100 & -1100 & -3095 & 3095 & 3095 \\
  -3920 & -3920 & -3920 & -3920 & 0 & 0 & 0 \\
  1 & 1 & 1 & 1 & 1 & 1 & 1 
\end{bmatrix}
\]

\[
M(k_1, k_2, k_3, k_4, k_5, k_6) = \begin{bmatrix}
  k_1 & k_1 & k_1 & k_1 & k_1 & k_1 \\
  k_2 & k_2 & k_2 & k_2 & k_2 & k_2 \\
  k_3 & k_3 & k_3 & k_3 & k_3 & k_3 \\
  k_4 & k_4 & k_4 & k_4 & k_4 & k_4 \\
  k_5 & k_5 & k_5 & k_5 & k_5 & k_5 \\
  k_6 & k_6 & k_6 & k_6 & k_6 & k_6 
\end{bmatrix}
\]
(39)

**TBBP**

Made as a special detecting device for testing the relevant parameters of a bogie, TBBP is composed of two independent motion platforms, that is, 1# 6-DOF Motion Platform and 2# 6-DOF Motion Platform (Figure 3). Seven identical hydraulics actuators, including four vertical ones, two longitudinal ones, and a single transverse one support both the two platforms. The 2# 6-DOF motion platform coordinate system attached to the mobile platform is established first. Figure 3 illustrates the serial numbers of the seven actuators mounted on the 6-DOF motion platform. Actuator 1, 5 and actuator 1, 6 are longitudinal and parallel, while actuator 1, 7 is transverse. These three actuators are coplanar. The centerlines of the three actuators intersect at two intersections. The midpoint of the line determined by these two intersection points is set as origin of global coordinate system, marked as \( O_1 \). As Figure 3 shows, the direction of forward motion of vehicle is X-axis, transverse direction is Y-axis, and vertical direction is Z-axis. Therefore, matrix \( A \) represents the coordinate values of the seven actuators’ upper hinge joint in the global coordinate system, and matrix \( B \) represents the coordinate values of the seven actuators’ lower hinge joint in the global coordinate system.

![Figure 3. TBBP structure.](image)

According to TBBP’s geometrical characteristics, its structural parameter matrices \( A \) and \( B \) can be substituted into equation (24). TBBP’s ratio matrix \( M \) is thus obtained:

\[
M(k_1, k_2, k_3, k_4, k_5, k_6) = \begin{bmatrix}
  k_1 & k_1 & k_1 & k_1 & k_1 & k_1 \\
  k_2 & k_2 & k_2 & k_2 & k_2 & k_2 \\
  k_3 & k_3 & k_3 & k_3 & k_3 & k_3 \\
  k_4 & k_4 & k_4 & k_4 & k_4 & k_4 \\
  k_5 & k_5 & k_5 & k_5 & k_5 & k_5 \\
  k_6 & k_6 & k_6 & k_6 & k_6 & k_6 
\end{bmatrix}
\]

TBBP is a 6-DOF redundant structure. The workspace of such a bench is limited due to the constraint of actuator’s range, constraint of kinematic pair’s rotational range, and the geometrical interference (either intersection between two actuators or between actuators and bench) during the dynamic motion. On the basis of the range of actuator and the inverse kinematics model, the maximum rotational angles \( q_i (i = 1, 2, 3) \) can be calculated and are less than 0.1745 rad, which are called reachable workspace. Additionally, 30-T actuator and 50-T actuator are selected to correspond to different loads. The corresponding maximum rotational angle \( \delta_{\text{max}} \) was calculated to be 23.57° and 29.83°, respectively. To sum up, the workspace of TBBP is that all rotational angle values \( q_i (i = 1, 2, 3) \) are less than 0.1745 rad.

**Simulation of EHM and result**

In this section, EHM is used to analyze the FKP of TBBP and SP, and their performances are evaluated in comparison with the NR algorithm, as well as the IHM. To assess the EHM performance, its outputs calculated by single iteration of numerical algorithm are shown from Figures 4–6. Convert the desired robot’s trajectories into rod displacement by inverse kinematic. Transfer the rod displacement to electro-hydraulic servo actuators to make sure that robot will act in accordance with targeted trajectory. A more accurate solution is next taken as the initial value of the fifth-order numerical technique and calculates the solution with the desired accuracy with less number of iterations.
and in shorter time. Parallel mechanism can track a parameterized reference trajectory which can rotate around upper platform’s geometric center or around any point in workspace. The coordinate difference between that point in workspace and the geometric center of the upper platform is called the offset. In this article, a way to identify the offset based on Fourier series is provided as follows.

**Offset identification based on Fourier series**

When the hydraulic actuator has a periodic motion, the displacement function of the actuator can be described by a periodic function \( f(t) \) relevant to time \( t \). According to the sufficient condition for the Dirichlet theory, the displacement function \( f(t) \) can be expanded as the convergent Fourier series\(^{18}\).
\[ f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(\omega t) + b_n \sin(\omega t)) \quad (40) \]

where

\[ a_0 = \frac{1}{\pi} \int_{0}^{2\pi} f(t) dt \]
\[ a_n = \frac{1}{\pi} \int_{0}^{2\pi} f(t) \cos(\omega t) dt \]
\[ b_n = \frac{1}{\pi} \int_{0}^{2\pi} f(t) \sin(\omega t) dt \]

where \( t \) is the period of the signal and \( \omega = (2\pi/T) \) is the fundamental frequency, where fundamental frequency’s constant multiplies \( 2\omega, 3\omega, \ldots \) are harmonics.

When \( t = 0 \), initial solution of function can be obtained directly from equation (41)

\[ f(0) = a_0 + a_1 + \cdots + a_n \quad (41) \]

The input signal of hydraulic actuator is always a trigonometric function without harmonics and with initial solution, and the initial solution represents the offset of the actuator. So, the convergent Fourier series of \( f(t) \) is written as

\[ f(t) = a_0 + a_1 \cos(\omega t) + b_1 \sin(\omega t) \quad (42) \]

where \( a_0 \) is the offset of the input signal, and \( a_1 \) and \( b_1 \) are the amplitude of the input signal. Randomly sample at least four points of the actuator’s displacement function. By substituting the random points into equation (42), offset \( a_0 \), amplitude \( a_1 \) and \( b_1 \), and frequency \( \omega \) can be determined as follows

\[ f(t_i) = a_0 + a_1 \cos(\omega t_i) + b_1 \sin(\omega t_i) \quad (43) \]

where \( i = 1, 2, 3, 4 \) \( f(t_i) \) are the random sampling points and \( t_i \) is the corresponding time to \( f(t_i) \).

**Bench test and results analysis**

Given that the centroid of the moving platform moves on a circular trajectory in the \( y = 3 \) plane around the origin of the coordinate system, additionally, all its three orientational degrees of freedom \( q_1, q_2, \) and \( q_3 \) are actuated. In the case of the SP, the centroid of the moving platform moves on a circular path in the XY plane around any point in the workspace. Note that
although SP is assumed to follow a circular trajectory, all the three orientation degrees of freedom $q_1$, $q_2$, and $q_3$ are actuated (see Figures 4–6).

Four cycles of TBBP's desired trajectory can be discretized into 12,561 points and five cycles of Steward platform's desired trajectory can be discretized into 15,701 points. FKP was solved for each point; the simulated value (SV) solved by the proposed method was compared with the theoretical value (TV). The results are shown in Figures 5 and 6. Detailed solutions of IKP are presented in section “Efficient method for solving the FKP of parallel robots.” Inv represents the target trajectory, and xp represents the calculated data using the EHM.

Figures 6 and 7 compare the EHM outputs from the solution of FKP with the target ones. The maximum absolute error of the angle around X-, Y-, and Z-axis is of the order of $10^{-3}$ units and its relative error is about 1.9%. The maximum absolute error of displacement along X-, Y-, and Z-axis is 0.174 mm and its relative error is about 3.4%.

The good agreement between calculated and target paths, as well as the small errors, highlights the excellent performance of the proposed method, which has been able to achieve such good accuracies. In the case of TBBP, absolute errors of orientation around X-, Y-, and Z-axis are less than 0.002706 rad and its relative error is about 1.9%, while the position errors along the same axis are all less than 0.174 mm with relative error about 3.4%. For SP, absolute errors of orientation around X-, Y-, and Z-axis are all less than 0.0008 rad and relative error is about 2.3%, while the position errors along the same axis are all less than 0.15 mm and its relative error is only 0.7%. The effectiveness of the proposed method is proved by running two typical trajectories on TBBP and SP, respectively.

To obtain higher accuracies, the high-order numerical algorithm should be adopted. On the basis of the similar initial guess acquired by the new method but using different iterative algorithm, Tables 1 and 2 compare the average number of iterations required by the NR method, the IHM, and EHM proposed in this article to achieve different levels of accuracy.

Clearly, with the increase in the required level of accuracy, the average number of iterations increases for all methods. By selecting any level of accuracy for analysis, the high-order iteration method requires a lower average number of iterations compared with the
other methods. Especially in the case of SP, when accuracy level is 1, high-order iteration method reduces the average number of iterations up to 46.83% compared with IHM.

Regarding the obtained results, the convergence order of the numerical algorithm used in each method deemed as the decisive factor. Both the NR method and the IHM used the second-order and the third-order convergence NR algorithm, respectively, while EHM used a fifth-order convergence numerical algorithm. The difference between EHM, with better efficiency index, and other methods increases by increasing the level of accuracy. Therefore, fifth-order numerical algorithm is preferred in this article.

In the case of SP, each iteration of the NR algorithm takes about 0.8 ms, while each iteration of the fifth-order algorithm takes about 1.4 ms. These results are obtained by running the simulations on a PC with 3.6 GHz Quad-Core processor and 10 GB RAM. Note that with a different computer configuration, running time and calculation speed are different. From Table 2, with the increase in the accuracy level, EHM reduces the average number of iterations to solve the FKP up to 34.1% compared with IHM and up to 49.8% compared with the NR method. Based on the results of those simulations, it may be concluded that at higher accuracy levels, EHM achieves the target accuracy with less number of iterations and in shorter time.

To further verify the validity of the proposed method, an experimental measurement of the position of vehicle ends has been carried out. As Figure 7 shows, two adjacent vehicle ends are fixed onto the TBBP, meanwhile the relative position measurement device of vehicle ends is installed in the middle of the tight-lock vestibule diaphragms and includes two triangular frames. The two ends of the triangular frames are connected with the two vehicle ends, respectively. The 6-DOF parallel manipulator’s coupled motion was adopted to simulate the vehicle body moving along linear track, transition curve track, and circular tracks. Displacement sensors and the angular encoders collected the data related to relative position and orientation between the two vehicle ends during the test.

The relative position of the vehicle’s end calculated by the proposed method is compared with the measured actual ones by the measurement device. To verify the accuracy of the test, two cycles were carried out. Figure 8 shows that test results are in good agreement with the simulation results; CV is the theoretical value and MV is the measured value. The relative error representing the difference between the two displacements of 2# vehicle’s end relative to 1# vehicle’s end along Y1-axis is within 5.3%. When the train is moving at constant speed, the lateral relative motion of the adjacent vehicle’s ends greatly influences the performance of the train. In addition, the relative errors of the angles between X1-, Y1-, and Z1-axis and between X2-, Y2-, and Z2-axis are within 5%. The relative motion between the two adjacent vehicle’s ends along longitudinal direction and vertical direction little influences the performance of the train. By using a sensor with high precision and minimizing the error of test device installation, the error of measuring device at the vehicle’s ends will also reduce and the measurement results will be more accurate. The proposed method is further validated and the demand of precision and real time are satisfied.

### Conclusion

In this article, an EHM to reduce the duration of the forward kinematics analysis of parallel robots and to improve the accuracy of the solution is proposed. By the new method based on signal decomposition and reconstruction, an approximate solution of the FKP is generated. This solution is next taken as initial guess for the high-order numerical technique, which has been demonstrated having better efficiency index and solve the calculations with target level of accuracy.

To further verify the validity of the proposed method, two representative parallel robots, that is, SP and TBBP, are evaluated. The simulation results demonstrate that by comparing the EHM outputs with the target ones, the maximum absolute error of the angle around X-, Y-, and Z-axis is of the order of $10^{-3}$ units and its relative error is about 1.9%. The

| Accuracy level | Stop criteria | Average number of iterations | Improvement (%) |
|----------------|---------------|-----------------------------|-----------------|
|                | Emax          | NR  | IHM | Fourth-order | EHM | Over NR | Over IHM | Over fourth-order |
| 1              | $10^{-3}$     | 3.825 | 2.95 | 2.84      | 1.51 | 22.87   | 3.72    | 46.83           |
| 2              | $10^{-5}$     | 4.24  | 3    | 2.98      | 2.06 | 29.24   | 0.66    | 30.87           |
| 3              | $10^{-7}$     | 4.66  | 3.50 | 2.99      | 2.36 | 24.89   | 14.57   | 21.07           |
| 4              | $10^{-9}$     | 4.84  | 3.759| 2.99      | 2.50 | 22.33   | 20.45   | 16.38           |
| 5              | $10^{-11}$    | 5.05  | 3.89 | 3.05      | 2.56 | 22.97   | 21.59   | 16.06           |

IHM: improved hybrid method; EHM: efficient hybrid method.
maximum absolute error of displacement along X-, Y-, and Z-axis is only 0.174 mm with relative error about 3.4%. This method will reduce by 34.1% the required number of iterations to solve the FKP compared with IHM and up to 49.8% compared with the NR method. In both cases, the EHM method ensures a high level of accuracy of the solution. Using the TBBP, an experimental measurement of the position of vehicle’s ends has been carried out. The results show that the maximum relative error of the displacement of 2# vehicle’s end relative to 1# vehicle’s end along Y1-axis is about 5.3%. The proposed method is thus verified and shown to be capable of achieving the desired accuracy with less number of iterations and in shorter time. It should be noted that the proposed method is valid in general and may be applied to more complex robots.

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