REPRESENTATIONS OF CLASSICAL GROUPS ON THE LATTICE AND ITS APPLICATION TO THE FIELD THEORY ON DISCRETE SPACE-TIME

Miguel Lorente
Departamento de Física
Universidad de Oviedo
33007 Oviedo, Spain

We explore the mathematical consequences of the assumption of a discrete space-time. The fundamental laws of physics have to be translated into the language of discrete mathematics. We find integral transformations that leave the lattice of any dimension invariant and apply these transformations to field equations.

1. INTRODUCTION

The idea of a discrete space-time has been introduced by physicists in the past in several different ways. Heisenberg advocated a fundamental length, Snyder has proposed a position operator with discrete spectrum, Ahmavaara has carried out a physical model on a cubic lattice, which is embedded in a finite linear space over a Galois field.

Recently the use of discrete space-time variables in Lattice Gauge Theories have become very popular to construct models based on experimental date. An other succesful line of research deals with integrable models on discrete space-time.

Many other hypothesis can be mentioned but they are reduced to two types: i) those using the space-time lattice as an artificial model, and ii) those proposing the discreteness of space-time as some fundamental structure of reality (Lorente 1986c).

The first assumption is very old and of technical character and do not present conceptual problems. The second assumption is more recent, although is based on some old pressupositions, namely, the relational character of the structure of space and time in contradistinction to the absolute idea of space and time (Earman 1989). Both assumptions were defended in XVII century by Leibniz and Newton, respectively, and although the last one become more succesful the former remained as a consistent model (Jammer 1969). Acording to Leibniz “Time is the order of monads not existing simultaneously. Space is
the order or monads that coexist or exist simultaneously. A point changes its position when it changes its relations from some points to different ones” †. This assumption leads to some discrete structure, although Leibniz did not carried out all its consequences. Also Riemann is his 1854 Inaugural Dissertation discussed the possibility of a discrete manifold based on some intrinsic metric and Weyl took over Riemann’s ideas (Grüenbaum 1977). The relational character of space and time have been advocated recently by R. Penrose (1971), D. Finkelstein and C.F. von Weiszäcker (Castell 1986). We have also proposed some physical model based on the relational hypothesis of the space and time (Lorente 1974, 1976, 1986a,b, 1987).

In this paper we explore some mathematical consequences of the assumption of a discrete space-time in his most naive structure, namely, a hypercubic lattice. Obviously the mathematical tool must be the functions of discrete variables and difference equations. Therefore, the fundamental laws of physics have to be translated into the language of discrete mathematics. The objection which is usually raised against such discrete schemes is that they are not invariant under the Lorentz group. We try to overcome this difficulty, as Schild did (Schild 1949), finding all integral transformations that leave the lattice of any dimension invariant, and applying these transformations to the fundamental field equations.

In section 2 and 3, we introduce the Cayley parametrization of the classical groups with some easy examples that can be enlarged to all semisimple Lie groups. In section 4 we review the isomorphisms between real forms and its explicit calculation. In section 5 we describe the method to calculate all integral transformations that leave a hypercubic lattice invariant, using the results of sections 2 to 4. In section 6 and 7 we construct a lagrangian formalism for the Klein Gordon and Dirac field on the lattice. In section 8 we introduce the generators of the space and time displacement and Lorentz boost on the lattice to check the Lorentz invariance of the model and we “integrate” these generators via some “Taylor expansion”. From the physical point of view these schemes have a twofold interpretation. Either we take the continuous limit in order to recover the continuous character of physical laws or we keep the formulas as they appear as a consequence of the hypothesis of a discrete space-time.

2. CAYLEY’S RATIONAL PARAMETRIZATION OF SEMISIMPLE GROUPS

Let \( S \) be a semisimple Lie group of complex matrices \( S \), which leaves invariant some non-degenerate bilinear form. We call a matrix \( S \) of the semisimple group \( S \) non-exceptional if \( \det (E + S) \neq 0 \), where \( E \) is the unit matrix. Cayley (1846) has proved that every non-exceptional matrix \( S \) can be expressed as follows

\[
S = (E + X)^{-1}(E - X) = (E - X)(E + X)^{-1}
\]  

(2.1)

where \( X \) is also a non-exceptional matrix.

† According to Jammer, Leibniz’s Monadology was inspired by the philosopher Maimonides who in his *Guide for the Perplexed* describe some discrete structure of space and time.
If \( G \) is the coefficient matrix of the non-degenerate bilinear form, which is left invariant under the group \( S \), the non-exceptional matrices \( S \) satisfy the relation

\[
S^\dagger GS = G
\]

and because of (2.1) the corresponding matrices \( X \) will also satisfy

\[
X^\dagger G + GX = 0
\]

In order to obtain the independent parameters of the semisimple group \( S \) it is more convenient to work with expression (2.3), which is linear, rather than with expression (2.2), which is quadratic. If we diagonalize or reduce to the canonical form the coefficient matrix \( G \) we have a further simplification of (2.3). Taking the independent elements of the matrix \( X \) given by (2.3) to be the independent parameters, we obtain Cayley’s rational parametrization of the semisimple group \( S \). (Note that when the independent elements of \( X \) are complex their real and imaginary part should be taken as independent parameters.)

In Table 1 we give the explicit conditions on the non-exceptional matrices \( S \) and \( X \) for all classical groups, as derived from expression (2.2) and (2.3), respectively. The notation \( S^T \) means the transpose matrix and \( S^\dagger \) the adjoint. Also

\[
|E + S| \equiv \det(E + S)
\]

and

\[
J \equiv \begin{pmatrix} 0 & E_n \\ E_n & 0 \end{pmatrix} \quad \text{and} \quad I \equiv \begin{pmatrix} E_p & 0 \\ 0 & -E_q \end{pmatrix}
\]

where \( E_n, E_p, E_q \) are the unit matrix of order \( n, p, q \) respectively. The condition on the matrix \( X \) gives automatically the unimodularity condition

\[
\det(E + S) = \det(E - S)
\]

except in the groups \( SU(n+1) \) and \( SU(p, q) \) and therefore (2.4) imposes an extra condition on the parameters corresponding to these groups.

| TABLE 1. Cayley’s decomposition of semisimple groups |
|-----------------------------------------------------|
| Group \( SO(2n) \) | Conditions on \( S \) | Conditions on \( X \) | Unimodularity | Parameters |
| \( S^T S = E \) | \( X^T + X = 0 \) | \( n(2n-1) \) |
| \( SO(2n+1) \) | \( S^T S = E \) | \( X^T + X = 0 \) | \( n(2n+1) \) |
| \( SU(n+1) \) | \( S^\dagger S = E \) | \( X^\dagger + X = 0 \) | \( |E + S| = |E - S| \) | \( n(n+2) \) |
| \( Sp(2n) \) | \( S^T J S = J \) | \( X^T J + JX = 0 \) | \( n(2n+1) \) |
| \( SO(p, q) \) | \( S^T IS = I \) | \( X^T I + IX = 0 \) | \( 1/2(p + q)(p + q - 1) \) |
| \( SU(p, q) \) | \( S^T IS = I \) | \( X^T I + IX = 0 \) | \( |E + S| = |E - S| \) | \( (p + q)^2 - 1 \) |
3. SOME EXAMPLES

3.1. The Rotation Group, SO(3)

From Table 1 the matrix $X$ is antisymmetric and it can be expressed in the following way

$$X = \frac{1}{m} \begin{pmatrix} 0 & n & -p \\ n & 0 & q \\ p & -q & 0 \end{pmatrix} \quad (3.1.1)$$

where $n, p, q$ are independent parameters and $m$ has been introduced for convenience. Using (3.1.1.) one obtains the Cayley parametrization of the non-exceptional matrix of the rotation group

$$S = \frac{1}{m^2 + n^2 + p^2 + q^2} \times \begin{pmatrix} m^2 - n^2 - p^2 + q^2 & -2mn + 2pq & 2mp + 2nq \\ 2mn + 2pq & m^2 - n^2 + p^2 - q^2 & -2mq + 2np \\ -2mp + 2nq & 2mq + 2np & m^2 + n^2 - p^2 - q^2 \end{pmatrix} \quad (3.1.2)$$

If we define $\alpha = m + in, \quad \beta = p - iq$ and then impose $m^2 + n^2 + p^2 + q^2 = 1$, the parametrization of the matrix $S$ given (3.1.2) is identical with the parametrization used by Wigner (1959) for the 3-dimensional rotation group. The parameters $\alpha$ and $\beta$ used by him are related to the parametrization of $SU(2)$, the covering group of $SO(3)$, in this way

$$S = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1 \quad (3.1.3)$$

In terms of the components of the axis of rotation $(a_1, a_2, a_3)$ and of the angle of rotation $\phi$, the Cayley parameters have the following geometrical interpretation

$$\frac{q}{a_1} = \frac{p}{a_2} = \frac{q}{a_3}, \quad \cos \phi = \frac{m^2 - n^2 - p^2 - q^2}{m^2 + n^2 + p^2 + q^2} \quad (3.1.4)$$

Similar parametrization and corresponding properties can be obtained for the $N$-dimensional rotation groups.

3.2. The Unitary Group SU(2)

From Table 1 the matrix $X$ is antihermitian and it can be expressed as

$$X = \frac{1}{l} \begin{pmatrix} ia & \rho \\ -\rho^* & ib \end{pmatrix} \quad (3.2.1)$$

where $a$ and $b$ are real parameters, $\rho = r + is$ and $l$ has been added for convenience.

From (2.5) and (3.2.1) one obtains

$$S = \frac{1}{\Delta} \begin{pmatrix} l^2 + ab - |\rho|^2 + i2lb & -2l\rho \\ 2l\rho^* & l^2 + ab - |\rho|^2 + i2la \end{pmatrix} \quad (3.2.2)$$
with $\Delta = l^2 - ab + |\rho|^2 + il^2(a + b)$.

The antihermiticity of $X$ gives

$$\det (E + X)^* = \det (E - X)$$

but it does not imply the unimodularity condition. (In the rotation group the antisymmetry of $X$ does imply the unimodularity of $S$.) If we impose $\det S = 1$, from (2.1) follows that

$$\det (E + X) = \det (E - X)$$

Both conditions, unitarity and unimodularity of $S$, give $\det(E + X) = \text{real}$, or

$$a + b = \text{Tr} \; X = 0 \quad (3.2.3)$$

Substituting (3.2.3) in (3.2.2) we obtain the general expression for the unitary unimodular matrices in two dimensions

$$S = \frac{1}{\Delta} \begin{pmatrix} l^2 - a^2 - r^2 - s^2 - i2la & -2lr - i2ls \\ 2lr - i2ls & l^2 - a^2 - r^2 - s^2 + i2la \end{pmatrix} \quad (3.2.4)$$

with $\Delta = l^2 + a^2 + r^2 + s^2$. Obviously the matrix (3.2.4) is equivalent to (3.1.3), but uses different parametrization.

### 3.3. The Proper Lorentz Group $\text{SO}(3.1)$

From Table 1 one obtains the traceless matrix

$$X = \frac{1}{m} \begin{pmatrix} 0 & n & -p & r \\ -n & 0 & q & s \\ p & -q & 0 & t \\ r & s & t & 0 \end{pmatrix} \quad (3.3.1)$$

where $n, p, q, r, s, t$ are real independent parameters and $m$ has been introduced as before. The unimodularity of $S$ does not impose further conditions on these parameters. From (2.1) one gets

$$S = \frac{1}{\Delta} \begin{pmatrix} m^2 - n^2 - p^2 + q^2 + r^2 - s^2 - t^2 + \lambda^2 & 2mn + 2pq + 2rs - 2\lambda t \\ -2mp + 2nq + 2rt + 2\lambda s & -2mr - 2ns + 2pt - 2\lambda q \\ -2mn + 2pq + 2rs + 2\lambda t & -2mq + 2np + 2st + 2\lambda r \\ m^2 - n^2 + p^2 - q^2 - r^2 + s^2 - t^2 + \lambda^2 & m^2 + n^2 - p^2 - r^2 - s^2 + t^2 + \lambda^2 \\ 2mq + 2np + 2st - 2\lambda r & -2mt - 2pr + 2qs - 2\lambda n \\ -2ms + 2nr - 2qt - 2\lambda p & \end{pmatrix} \quad (3.3.2)$$
where
\[ m\lambda = nt + ps + qr \]
and
\[ \Delta = m^2 + n^2 + p^2 + q^2 - r^2 - s^2 - t^2 - \lambda^2 \]

If \( \Delta > 0 \), since \( \det S = 1 \), one obtains the general expression for the non-exceptional matrices of the proper Lorentz group \( (S_{14} > 0) \).

If \( r = s = t = 0 \), one recovers expression (3.1.2) for the proper orthogonal group in 3-dimensions.

If \( n = p = q = 0 \) one is left with the non-exceptional matrices of the pure Lorentz transformations. In this case, comparison of (3.3.2) with a pure Lorentz transformation with velocity \( v \) along \( v \) gives (Møller, 1952)

\[
\frac{v_x}{r} = \frac{v_y}{s} = \frac{v_z}{t} = \frac{2mc}{m^2 + r^2 + s^2 + t^2}, \quad \left(1 - \frac{v^2}{c^2}\right)^{1/2} = \frac{m^2 - r^2 - s^2 - t^2}{m^2 + r^2 + s^2 + t^2} \tag{3.3.3}
\]

where \( c \) is the velocity of light in vacuum.

If we define
\[
\begin{align*}
\alpha &= m - t + i(n - \lambda), \quad \beta = -p - r + i( q - s) \\
\gamma &= p - r + i( q + s), \quad \delta = m + t - i(n + \lambda)
\end{align*}
\]

and introduce these variables in the general expression of the proper Lorentz group in terms of the parameters of the \( SL(2, C) \) group (Naimark, 1964a)

\[
\begin{pmatrix}
\alpha & b \\
\gamma & \delta
\end{pmatrix}
\]

\( \alpha\delta - \beta\gamma = 1 \)

we obtain the expression (3.3.2) plus the condition

\[
\begin{align*}
m\lambda &= nt + ps + qr \\
\Delta &= m^2 + n^2 + p^2 + q^2 - r^2 - s^2 - t^2 - \lambda^2 = 1
\end{align*}
\]

4. ISOMORPHISM BETWEEN REAL FORMS

According to Cartan theory, there are some real forms of simple Lie groups of low dimensionality which are locally isomorphic (Helgason, 1978). We describe them by the bijection of \( R^n \) onto a set of matrices \( S \).

i) \( SL(2, R) \approx SO(2, 1) \). Define a set of \( 2\times2 \) real matrices \( A \), by the conditions \( A^T = A \), where \( A^T \) means transposed. The bijection of an element \( (x_0, x_1, x_2) \) of \( R^3 \) onto a matrix \( A \) is the following:

\[
A = \begin{pmatrix}
x_0 + x_2 \\
x_1 \\
x_0 - x_2
\end{pmatrix}
\]

(4.1)

The transformations \( A' = SAS^T \) with \( S \in SL(2, R) \), map \( A \) into itself. Since

\[
\det A = x_0^2 - x_1^2 - x_2^2 = \det A'
\]

(4.2)
this transformation induces the desired isomorphism.

(ii) $SL(2,C) \approx SO(3,1)$. Define $A$, a $2 \times 2$ complex matrix, by the condition $A^\dagger = A$, where $A^\dagger$ means the Hermitian conjugate matrix. The bijection of $(x_0, x_1, x_2, x_3)$ in $R^4$ onto $A$ is given by

$$A = \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix} \quad (4.3)$$

The transformation $A^\prime = SAS^\dagger$ with $S \in SL(2,C)$ maps $A$ into itself, as it is well known (Gel’fand et al., 1963). Since

$$\text{det} A = x_0^2 - x_1^2 - x_2^2 - x_3^2 = \text{det} A^\prime \quad (4.4)$$

this transformation induces the mentioned isomorphism.

(iii) $Sp(4,R) \approx SO(3,2)$. The matrix $A$ is a four-dimensional real matrix, satisfying $A^T J = JA$ and $Tr A = 0$, where

$$J \equiv \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix},$$

$E$ is the unit matrix of dimension 2.

The bijection of an element $(x_1, x_2, x_3, x_4, x_5)$, of $R^5$ onto $A$ is given by

$$A = \begin{pmatrix} x_1 & x_2 + x_3 & 0 & x_4 + x_5 \\ x_2 - x_3 & -x_1 & -x_4 - x_5 & 0 \\ 0 & x_4 - x_5 & x_1 & x_2 - x_3 \\ -x_4 + x_5 & x_2 + x_3 & -x_1 \end{pmatrix} \quad (4.5)$$

The transformation $A^\prime = SAS^{-1}$ with $S \in Sp(4,R)$ maps $A$ into itself, namely, $A^T J = JA^\prime, Tr A^\prime = 0$. Since

$$\text{det } A = (x_1^2 + x_2^2 - x_3^2 - x_4^2 + x_5^2)^2 = \text{det} A^\prime, \quad (4.6)$$

this transformation induces the desired isomorphism.

(iv) $Sp(1,1) \approx SO(4,1) A$ is defined by the four-dimensional complex matrix satisfying $A^T J = JA$, $A^\dagger K = K A$, $Tr A = 0$, with $K \equiv \text{diag}(1, -1, 1, -1)$.

The bijection of an element $(x_1, x_2, x_3, x_4, x_5)$ of $R^5$ onto $A$ is

$$A = \begin{pmatrix} x_1 & x_2 + ix_3 & 0 & x_4 + ix_5 \\ -x_2 + ix_3 & -x_1 & -x_4 - ix_5 & 0 \\ 0 & x_4 - ix_5 & x_1 & -x_2 + ix_3 \\ -x_4 + ix_5 & x_2 + ix_3 & -x_1 \end{pmatrix} \quad (4.7)$$

Given an element $S$ of the group $Sp(1,1)$, that is to say, $S^TJS = J, S^\dagger KS = K$, the transformation $A^\prime = SAS^{-1}$ maps $A$ into itself. Since

$$\text{det } A = (x_1^2 - x_2^2 - x_3^2 - x_4^2 + x_5^2)^2 = \text{det } A^\prime \quad (4.8)$$

this transformation induces the desired isomorphism.
(v) $SU(2,2) \approx SO(4,2)$. $A$ is defined by the four-dimensional complex matrix, satisfying $A^T = -A$, $A^* I = I \bar{A}$, with $\bar{A}$, the complex conjugate matrix of $A$, $A^*$ the dual matrix of $A$, namely, $(A^*)_{ab} = \frac{1}{2} \varepsilon_{abcd} A^{cd}$, and

$$I \equiv \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}$$

The bijection of an element $(x_1, x_2, x_3, x_4, x_5, x_6)$ of $R^6$ onto $A$ is (Beckers et al., 1978)

$$A = \begin{pmatrix} 0 & x_1 + ix_2 & x_3 + ix_4 & x_5 + ix_6 \\ -x_1 - ix_2 & 0 & x_5 - ix_6 & -x_3 + ix_4 \\ -x_3 - ix_4 & -x_5 + ix_6 & 0 & -x_1 + ix_2 \\ -x_4 - ix_6 & x_3 - ix_4 & x_1 - ix_2 & 0 \end{pmatrix} \quad (4.9)$$

The transformation $A' = SAS^T$, with $S$ satisfying $S^T IS = I$, maps $A$ into itself. Since

$$\det A = (-x_1^2 - x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2) = \det A' \quad (4.10)$$

this transformation belongs also to $SO(4,2)$.

(vi) $SL(2,Q) \approx SO(5,1)$. Let $A$ be a two-dimensional quaternion matrix defined by $A^\dagger = A$. The bijection of an element $(x_0, x_1, x_2, x_3, x_4, x_5)$ of $R^6$ onto $A$ is the following:

$$A = \begin{pmatrix} x_0 + x_1 & x_2 + x_3 i + x_4 j + x_5 k \\ x_2 - x_3 i - x_4 j - x_5 k & x_0 - x_1 \end{pmatrix} \quad (4.11)$$

with $(i, j, k)$ a basis for the quaternions. The transformation $A' = SAS^\dagger$, with $S \in SL(2,Q)$ maps $A$ into itself. Since

$$\det A = (x_0 - x_1^2 - x_2^2 - x_3^2 - x_4^2 - x_5^2)^2 = \det A' \quad (4.12)$$

this transformation induces the desired isomorphism (Barut et al., 1965)

(vii) $SL(4,R) \approx SO(3,3)$. $A$ is defined by the four-dimensional real matrix, satisfying $A^T = -A$. The bijection of an element $(x_1, x_2, x_3, x_4, x_5, x_6)$ of $R^6$ onto $A$ is the following:

$$A = \begin{pmatrix} 0 & -x_1 + x_4 & x_2 + x_5 & x_3 + x_6 \\ -x_1 - x_4 & 0 & x_3 - x_6 & -x_2 + x_5 \\ -x_2 - x_5 & -x_3 + x_6 & 0 & x_1 - x_4 \\ -x_3 - x_6 & x_2 - x_5 & -x_1 + x_4 & 0 \end{pmatrix} \quad (4.13)$$

The transformation $S' = SAS^T$, with $S \in SL(4,R)$, maps $A$ into itself. Since

$$\det A = (x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_5^2 - x_6^2)^2 = \det A' \quad (4.14)$$

this transformation induces the desired isomorphism.
5. INTEGRAL TRANSFORMATIONS

A transformation that leaves invariant an hypercubic lattice is called an integral transformation. It means that if we take an integral vector (the components of which are real integers or Gaussian numbers) the transformed vector is also integral.

It is easy to prove that a transformation of a classical group is integral if and only if all its components are integers.

Given the Cayley realization of a classical groups (2.1) the corresponding transformation is integral if and only if all the Cayley parameters are integers and det $(1 + X) = 1$. But this procedure do not exhaust all the integral transformations. Let us give some examples.

5.1 $SO(3)$

The Cayley parametrization is given by (3.1.2.). Therefore we require $m, n, p, q$ to be integer numbers and

$$m^2 + n^2 + p^2 + q^2 = 1 \quad (5.1.1)$$

The non-exceptional matrix satisfying both conditions is the unit matrix corresponding to $m = \pm 1, \; n = p = q = 0$

The other set of non trivial integral transformation are obtained in this way: from the Cayley transform

$$S = \frac{1 - X}{1 + X} = \frac{2}{1 + X} - 1 \quad (5.1.2)$$

it follows that $S$ is integral if $2(1 + X)^{-1}$ is integral.

From (3.1.2) we get:

$$\frac{2}{1 + X} = \frac{2}{m^2 + n^2 + p^2 + q^2} \begin{pmatrix} m^2 + q^2 & -mn + pq & mp + nq \\ mn + pq & m^2 + p^2 & -mq + np \\ -mp + nq & mq + np & m^2 + n^2 \end{pmatrix} \quad (5.1.3)$$

Obviously $2(1 + X)^{-1}$ is integral if $m, n, p, q$ are integers and $m^2 + n^2 + p^2 + q^2 = 2$. The solutions of this diophantine equation are:

$$\begin{cases} m = \pm 1, \; n = \pm 1, \; p = q = 0 \\ m = \pm 1, \; p = \pm 1, \; n = q = 0 \\ m = \pm 1, \; q = \pm 1, \; n = p = 0 \end{cases} \quad (5.1.4)$$

By similar considerations we obtain also the integral transformations with

$$m = \pm 1, \; n = \pm 1, \; p = \pm 1, \; q = \pm 1 \quad (5.1.5)$$

5.2 $SO(3,1)$

As before we require in (3.3.2) all the Cayley parameters $m, n, p, q, r, s, t$ to be integer numbers; after substituting $\lambda = (nt + ps + qr)/m$ the condition det $(1 + X) = 1$ becomes

$$m^2(m^2 + n^2 + p^2 + q^2 - r^2 - s^2 - t^2) - (nt + ps + qn)^2 = m^2 \quad (5.2.1)$$
This formidable diophantine equation can be simplified if we expand the inverse matrix appearing in the Caley transform (5.1.2), namely,

\[
\frac{1}{1 + X} = 1 - X + X^2 - X^3 + \ldots
\]

This series must be finite if matrix \( S \) is supposed to be an integral transformation, therefore \( X^r = 0 \) for some \( r \). Since \( X \) is a nilpotent matrix, it has zero as the only eigenvalue. From the secular equation it follows that the sum of all principal minors of the same order of the matrix \( X \) are equal to zero. This property applied to (3.3.1) gives the conditions

\[
\begin{align*}
&n^2 + p^2 + q^2 - r^2 - s^2 - t^2 = 0 \\
&\ (nt + ps + qr)^2 = 0
\end{align*}
\]  

(5.2.2)

substituting these values in (5.2.1) we get \( m = 1 \).

An other set of solutions are obtained from the condition \( 2(1 + X)^{-1} \) to be integral.

This condition together with (5.2.2) gives

\[
\frac{2}{1 + X} = \frac{2}{m^2} \times
\left( \begin{array}{cccc}
  m^2 + q^2 - s^2 - t^2 & -mn + pq + rs & mp + nq + rt & -mr = ns - pt \\
  mn + pq + rs & m^2 + p^2 - r^2 - t^2 & -mq + np + st & -ms - nr + qt \\
  -mp + nq + rt & mq + np + st & m^2 + n^2 - r^2 - s^2 & -mt + pr - qs \\
  -mr - ns + pt & -ms + nr - qt & -mt - pr + qs & m^2 + n^2 + p^2 + q^2
\end{array} \right)
\]

(5.2.3)

The condition of integral transformation requires that each matrix element multiplied by 2 must be divisible by \( m^2 \). Therefore the only solutions are:

\[
\begin{align*}
&m = 2 \ , \ n, t, p, s \ \text{odd integers}; \ q, r \ \text{even integers} \\
\text{or} \quad &m = 2 \ , \ n, t, q, r \ \text{odd integers}; \ p, s \ \text{even integers}
\end{align*}
\]  

(5.2.4)

The same method can be applied to other integral transformations corresponding to classical groups.

5.3. The method of isomorphism between real forms

Section 4 can be used to find integral transformations. Take the isomorphism between \( SL(2, C) \) and \( SO(3, 1) \). A spin transformation of the group \( SL(2, C) \) corresponds to a Lorentz transformation with Cayley parametrization after the identification (3.3.4).

An spin transformation is integral if the corresponding Lorentz transformation is integral. Schild has solved completely the problem of classifying all integral Lorentz transformation by the following theorem (Schild 1949):

A spin transformation \( \lambda_{ij} \ (i, j = 1, 2) \) is integral if and only if one of the following four conditions is satisfied:

I. \( \lambda_{ij} \) are Gaussian integers such that \( \lambda_{11}\lambda_{22} - \lambda_{21}\lambda_{12} = 1 \) and such that \( \lambda_{11} + \lambda_{21} + \lambda_{12} + \lambda_{22} \) is even.
II. \( \lambda_{ij} = \mu_{ij}/(1 + i) \), where \( \mu_{ij} \) are odd integers such that \( \mu_{11}\mu_{22} - \mu_{21}\mu_{12} = 2i \)

III. \( \lambda_{ij} \) are integers such that \( \lambda_{11}\lambda_{22} - \lambda_{21}\lambda_{12} = i \) and such that \( \lambda_{11} + \lambda_{12} + \lambda_{21} + \lambda_{22} \) is even.

IV. \( \lambda_{ij} = \mu_{ij}/(1 + i) \) where \( \mu_{ij} \) are odd integers such that \( \mu_{11}\mu_{22} - \mu_{21}\mu_{12} = -2 \)

(For the definition and properties of Gaussian numbers see Schil 1949).

We can compare this classification with our results of section 5.2. It is easy to check with the help of (3.3.4) that Case I corresponds to \( m = 1 \) and (5.2.2) and Case II corresponds to \( m = 2 \) and (5.2.4).

Cases III and IV are obtained from I and II multiplying \( \alpha, \beta \) or \( \gamma, \delta \) by \( i \) in (3.3.4). It corresponds to interchange first and second row in (3.3.2), multiplying the first one by -1. If we choose to multiply \( \alpha, \gamma \) or \( \beta, \delta \) by \( i \) in (3.3.4) we obtain the corresponding transformation (3.3.2) with the first and second column (multiplied by \(-1\)) interchanged.

In order to complete the algorithm to calculate the \( SL(2, C) \) transformations with Gaussian numbers, we have to solve the diophantine equation \( \lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21} = 1 \). As before we write the general element of \( SL(2, C) \) as a finite series, say

\[
S = \frac{1}{1 + X} = 1 - X \quad \text{with} \quad X^2 = 0 \quad (5.3.1)
\]

the condition for \( X \) to be a nilpotent matrix requires

\[
\det X = 0 \quad \text{tr} X = 0 \quad , \quad (5.3.2)
\]

the most general solution of which is

\[
X = \begin{pmatrix} zw & -zw^2 \\ z & -zw \end{pmatrix} \quad (5.3.3)
\]

with \( z, w \) Gaussian integers. Therefore

\[
S = \begin{pmatrix} 1 - zw & -zw^2 \\ z & 1 + zw \end{pmatrix} \equiv \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} \quad (5.3.4)
\]

that can be enlarged to the general solution

\[
S = \begin{pmatrix} \lambda_{11} + p\lambda_{21} & \lambda_{12} + p\lambda_{22} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} \quad (5.3.5)
\]

with \( p \), an arbitrary Gaussian integer.

6. A LAGRANGIAN MODEL FOR THE KLEIN-GORDON FIELD ON THE LATTICE

Let us define the real scalar field \( \phi(x, t) \) on the grid points of a \((1 + 1)\)-dimensional lattice as \( \phi(\varepsilon j, \tau n) \equiv \phi_j^n \) where \( \varepsilon, \tau \) are the fundamental space and time interval and \( j, n \) are integer numbers. We want to associate to this field a Lagrangian, such that the equations of motion are recovered from the Euler-Lagrange difference equations.
A suitable Lagrangian for the Klein-Gordon field is:

\[
L_n = -\frac{1}{2} \sum_{j=0}^{N-1} \left\{ \frac{1}{\varepsilon^2} \left( \tilde{\nabla}_j \Delta_j \tilde{\nabla}_n \phi_j^n \right)^2 - \frac{1}{\tau^2} \left( \tilde{\nabla}_j \tilde{\nabla}_j \nabla_n \phi_j^n \right)^2 + M^2 \left( \tilde{\nabla}_j \tilde{\nabla}_n \phi_j^n \right)^2 \right\}
\]

\[
\equiv \varepsilon \sum_{j=0}^{N-1} L_n \quad (6.1)
\]

where periodic boundary conditions for the field are supposed, \( \Delta_j(\nabla_j) \) are the forward (backward) difference operator and \( \tilde{\Delta}_j(\tilde{\nabla}_j) \) are the forward (backward) average operator with respect to the space indices:

\[
\Delta_j f_j = f_{j+1} - f_j \quad \nabla_j f_j = f_j - f_{j-1} \quad (6.2)
\]

\[
\tilde{\Delta}_j f_j = \frac{1}{2} (f_{j+1} + f_j) \quad \tilde{\nabla}_j f_j = \frac{1}{2} (f_j + f_{j-1}) \quad (6.3)
\]

and similar for the time indices. In the limit \( j \to \infty, \varepsilon \to 0, j\varepsilon \to x, n \to \infty, \tau \to 0, n\tau \to t \) we have

\[
\tilde{\nabla}_j \tilde{\Delta}_j \tilde{\nabla}_n \phi_j^n \to \phi(x,t) \quad (6.4)
\]

\[
\frac{1}{\varepsilon} \tilde{\nabla}_j \Delta_j \tilde{\nabla}_n \phi_j^n \to \frac{\partial \phi}{\partial x}(x,t) \quad (6.5)
\]

\[
\frac{1}{\tau} \tilde{\nabla}_j \tilde{\Delta}_j \nabla_n \phi_j^n \to \frac{\partial \phi}{\partial t}(x,t) \quad (6.6)
\]

Taking the variations of the Lagrangian density \( L_n \) with respect to the time difference of the field, we get

\[
\frac{\partial L_n}{\partial \left( \frac{1}{\tau} \nabla_n \tilde{\nabla}_j \tilde{\nabla}_n \phi_j^n \right)} = \frac{1}{\tau} \nabla_n \tilde{\Delta}_j \tilde{\nabla}_j \phi_j^n \equiv \tilde{\nabla}_n \tilde{\Delta}_j \tilde{\nabla}_n \pi_j^n \quad (6.7)
\]

with \( \pi_j^n \) as the conjugate momentum.

To obtain the Euler-Lagrangian equation we take the time difference of the last expression to be equal to the variation of the Lagrangian density with respect to the average field:

\[
\frac{1}{\tau^2} \Delta_n \nabla_n \tilde{\Delta}_j \tilde{\nabla}_j \phi_j^n = \tilde{\Delta}_n \frac{\partial L_n}{\partial \left( \tilde{\nabla}_j \tilde{\nabla}_n \phi_j^n \right)} = \tilde{\Delta}_n \left[ \frac{1}{\varepsilon^2} \nabla_j \Delta_j \nabla_n \phi_j^n - M^2 \tilde{\nabla}_j \tilde{\nabla}_n \phi_j^n \right] \quad (6.8)
\]

where \( \tilde{\Delta}_n \) has been introduce for homogeneity in the last equality and integration by parts have been used.

The last expression is the wave equation for the Klein-Gordon field on the lattice.

The “plane wave” solutions, the Fourier decomposition of the fields in terms of a complete set of orthogonal functions on the lattice have been given elsewhere (Lorente, 1992).
With the help of the conjugate field defined in (6.7) we can construct an Hamiltonian density on the lattice in the usual way.

\[
\mathcal{H}_n = \left( \nabla_j \Delta_j \nabla_n \pi_j^n \right) \frac{1}{\tau} \nabla_n \Delta_j \nabla_j \phi_j^n - \mathcal{L}_n
\]  

(6.9)

from which the Hamiltonian follows:

\[
H_n = \varepsilon \sum_{j=0}^{N-1} \frac{1}{2} \left\{ \left( \nabla_j \Delta_j \nabla_n \pi_j^n \right)^2 + \frac{1}{\varepsilon^2} \left( \nabla_j \Delta_j \nabla_n \phi_j^n \right)^2 + M^2 \left( \nabla_j \Delta_j \nabla_n \phi_j^n \right)^2 \right\}
\]

\[
= \varepsilon \sum_{j=0}^{N-1} \mathcal{H}_n
\]

(6.10)

Taking the variation of the Hamiltonian density with respect to the averaged fields and its conjugate momentum, the Hamilton equations of motion are obtained:

\[
\frac{1}{\tau} \Delta_n \left( \nabla_j \Delta_j \nabla_n \phi_j^n \right) = \frac{\partial \mathcal{H}_n}{\partial \left( \nabla_j \Delta_j \nabla_n \phi_j^n \right)} = \Delta_n \left( \nabla_j \Delta_j \nabla_n \pi_j^n \right)
\]

(6.11)

\[
\frac{1}{\tau} \Delta_n \left( \nabla_j \Delta_j \nabla_n \pi_j^n \right) = -\frac{\partial \mathcal{H}_n}{\partial \left( \nabla_j \Delta_j \nabla_n \phi_j^n \right)} = \Delta_n \left\{ \frac{1}{\varepsilon^2} \nabla_j \Delta_j \nabla_n \phi_j^n - M^2 \nabla_j \Delta_j \nabla_n \phi_j^n \right\}
\]

(6.12)

where integration by parts has been used. Again, the Hamiltonian equations of motion lead to the wave equation (6.8). Notice that (6.11) and (6.12) can be simplified by the time average operator \( \nabla_n \), namely:

\[
\frac{1}{\tau} \Delta_n \left( \nabla_j \Delta_j \nabla_n \phi_j^n \right) = \Delta_n \left( \nabla_j \Delta_j \pi_j^n \right)
\]

(6.13)

\[
\frac{1}{\tau} \Delta_n \left( \nabla_j \Delta_j \pi_j^n \right) = \Delta_n \left( \frac{1}{\varepsilon^2} \nabla_j \Delta_j \phi_j^n - M^2 \nabla_j \Delta_j \phi_j^n \right)
\]

(6.14)

which can be obtained from the following Hamiltonian

\[
H_n = \varepsilon \sum_{j=0}^{N-1} \frac{1}{2} \left\{ \left( \nabla_j \Delta_j \pi_j^n \right)^2 + \frac{1}{\varepsilon^2} \left( \nabla_j \Delta_j \phi_j^n \right)^2 + M^2 \left( \nabla_j \Delta_j \phi_j^n \right)^2 \right\}
\]

(6.15)

For the quantization of the Klein-Gordon field we introduce the equal time commutation relations:

\[
[\nabla_j \Delta_j \phi_j^n, \nabla_j' \Delta_j' \pi_j'^n] = \frac{1}{\varepsilon} \delta_{jj'}
\]

(6.16)

\[
[\nabla_j \Delta_j \phi_j^n, \nabla_j' \Delta_j' \phi_j'^n] = 0 = [\nabla_j \Delta_j \pi_j^n, \nabla_j' \Delta_j' \pi_j'^n]
\]

(6.17)
from which the Heisenberg equations of motion, the Fourier decomposition in terms of the creation and annihilation operators can be deduced in the usual way (Lorente 1992).

7. A LAGRANGIAN FOR THE DIRAC FIELD ON THE LATTICE

We can repeat the same steps as in the previous case for the Dirac field on the \((1 + 1)\)-dimensional lattice \(\psi_{\alpha}(j\varepsilon, n\tau) \equiv \psi^\dagger_{\alpha j}\).

A suitable Lagrangian density is

\[
\mathcal{L}_n = -\tilde{\Delta}_j \tilde{\Delta}_n \psi^\dagger_j \left\{ \gamma_4 \gamma_1 \frac{1}{\varepsilon} \tilde{\Delta}_j \Delta_n \psi^n_j - i \frac{1}{\tau} \gamma_4 \tilde{\Delta}_j \Delta_n \psi^n_j + M \gamma_4 \tilde{\Delta}_j \tilde{\Delta}_n \psi^n_j \right\}
\]

(7.1)

with

\[
\gamma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad i \gamma_1 \gamma_4 = \gamma_5 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

(7.2)

leading to the Euler-Lagrange equation

\[
\frac{\partial \mathcal{L}_n}{\partial \left( \frac{1}{\tau} \Delta_n \tilde{\Delta}_j \psi^n_j \right)} = i \tilde{\Delta}_j \tilde{\Delta}_n \psi^\dagger_j \equiv \tilde{\Delta}_j \tilde{\Delta}_n \pi^n_j
\]

(7.3)

\[
\frac{1}{\tau} \tilde{\nabla}_n \left( \tilde{\Delta}_j \tilde{\Delta}_n \pi^n_j \right) = \tilde{\nabla}_n \frac{\partial \mathcal{L}_n}{\partial \left( \tilde{\Delta}_n \tilde{\Delta}_j \psi^n_j \right)} = \tilde{\nabla}_n \left\{ \frac{1}{\varepsilon} \Delta_j \tilde{\Delta}_n \psi^\dagger_j \gamma_4 \gamma_1 - M \tilde{\Delta}_j \Delta_n \psi^\dagger_j \gamma_4 \right\}
\]

(7.4)

Substituting (7.3) in (7.4), taking the adjoint operation of both sides and multiplying by \(\gamma_4\) from the right we obtain the Dirac equation on the lattice

\[
\tilde{\nabla}_n \left\{ \gamma_1 \frac{1}{\varepsilon} \tilde{\Delta}_j \tilde{\Delta}_n - i \gamma_4 \frac{1}{\tau} \Delta_n \tilde{\Delta}_j + M \tilde{\Delta}_j \tilde{\Delta}_n \right\} \psi^n_j = 0
\]

(7.5)

The plane wave solutions and the Fourier decomposition in terms of a complete set of orthogonal functions on the lattice have been given elsewhere (Lorente 1991).

With the help of the conjugate field \(\pi^n_j = i \psi^\dagger_j\) we can construct the Hamiltonian density

\[
\mathcal{H}_n = \left( \tilde{\Delta}_n \tilde{\Delta}_j \pi^n_j \right) \frac{1}{\tau} \Delta_n \tilde{\Delta}_j \psi^n_j - \mathcal{L}_n
\]

\[
= \left( \tilde{\Delta}_n \tilde{\Delta}_j \psi^\dagger_j \right) \gamma_4 \gamma_1 \tilde{\Delta}_n \Delta_j \psi^n_j + M \left( \tilde{\Delta}_n \tilde{\Delta}_j \psi^\dagger_j \right) \tilde{\Delta}_n \tilde{\Delta}_j \psi^n_j
\]

(7.6)

from which the Hamiltonian equations of motion can be derived leading again to the Dirac equation.

As in the Klein-Gordon case, one can simplify the Hamilton equation by \(\tilde{\Delta}_n\) which can be deduced from the new Hamiltonian

\[
H_n = \varepsilon \sum_{j=0}^{N-1} \tilde{\Delta}_j \psi^\dagger_j \left\{ \frac{1}{\varepsilon} \gamma_4 \gamma_1 \Delta_j \psi^n_j + M \gamma_4 \tilde{\Delta}_j \psi^n_j \right\}
\]

(7.7)
For the quantization of the Dirac field we require the equal time anticommutation relations
\[
[\Delta_j \psi_{\alpha j}^n, \Delta_j' \psi_{\beta j'}^{n^*}]_+ = \frac{1}{\varepsilon} \delta_{\alpha \beta} \delta_{jj'}
\] (7.8)
with other anticommutations vanishing. If we plug \(H_n\) in the Heisenberg equations of motion, we obtain the Dirac equation, from which the plane wave solutions and the Fourier decomposition in terms of the creation and annihilation operators can be obtained (Lorente 1991).

8. CONSERVATION LAWS AND LORENTZ INVARIANCE

As in the continuous case we can make the connection between symmetries and conservation laws in the language of generators. The condition for symmetry of the Lagrangian under space and time displacement and pure Lorentz transformation is that the generators are constant of the motion (Yamamoto 1991). In the case of the Klein-Gordon fields the generators of the (one step) space and time translations and Lorentz boost can be taken as:

\[
P = - \sum_{j=0}^{N-1} \frac{1}{2} \left\{ \left( \tilde{\nabla}_j \tilde{\Delta}_j \pi_j^n \right) \left( \tilde{\nabla}_j \Delta_j \phi_j^n \right) + \left( \tilde{\nabla}_j \Delta_j \phi_j^n \right) \left( \tilde{\nabla}_j \tilde{\Delta}_j \pi_j^n \right) \right\} \] (8.1)
\[
H = \varepsilon \sum_{j=0}^{N-1} \frac{1}{2} \left\{ \left( \tilde{\nabla}_j \tilde{\Delta}_j \pi_j^n \right)^2 + \frac{1}{\varepsilon^2} \left( \tilde{\nabla}_j \Delta_j \phi_j^n \right)^2 + M^2 \left( \tilde{\nabla}_j \Delta_j \phi_j^n \right)^2 \right\} \] (8.2)
\[
K = \varepsilon \sum_{j=0}^{N-1} \frac{1}{2} \varepsilon j \left\{ \left( \tilde{\Delta}_j \tilde{\nabla}_j \tilde{\Delta}_j \pi_j^n \right)^2 + \frac{1}{\varepsilon^2} \left( \tilde{\Delta}_j \tilde{\nabla}_j \Delta_j \phi_j^n \right)^2 + M^2 \left( \tilde{\Delta}_j \tilde{\nabla}_j \Delta_j \phi_j^n \right)^2 \right\} \]
\[- \varepsilon \sum_{j=0}^{N-1} n \tau \frac{1}{2} \left\{ \left( \tilde{\Delta}_j \tilde{\nabla}_j \phi_j^n \right) \left( \tilde{\Delta}_j \nabla_j \phi_j^n \right) + \left( \tilde{\Delta}_j \nabla_j \phi_j^n \right) \left( \tilde{\Delta}_j \tilde{\nabla}_j \phi_j^n \right) \right\} \] (8.3)

Using (6.13) and (6.14) it can be proved that these operators are constant of time
\[
\frac{1}{\tau} \Delta_n P = \frac{1}{\tau} \Delta_n H = \frac{1}{\tau} \Delta_n K = 0 \] (8.4)

In order to check the Lorentz invariance of quantized field scheme on the lattice, one can prove with the help of (6.16) and (6.17) that these operators satisfy the standard commutation relations:
\[
[H, P] = 0 \quad , \quad [K, H] = iP \quad , \quad [K, P] = iH \] (8.5)

For the Dirac quantum fields, the generators of the (one step) space and time translations and Lorentz boost can be taken as
\[
P = -i \sum_{j=0}^{N-1} \left( \tilde{\Delta}_j \psi_{\alpha j}^{n^*} \right) \left( \Delta_j \psi_{\alpha j}^n \right) \] (8.6)
\[ H = \varepsilon \sum_{j=0}^{N-1} \Delta_j \psi_{\alpha j}^+ \left\{ (\gamma_4 \gamma_1)_{\alpha \beta} \frac{1}{\varepsilon} \Delta_j \psi_{\beta j}^n + M (\gamma_4)_{\alpha \beta} \Delta_j \psi_{\beta j}^n \right\} \quad (8.7) \]

\[ M_{14} = i \varepsilon \sum_{j=0}^{N-1} \Delta_j \nabla_j \psi_{\alpha j}^+ \left\{ (\gamma_4 \gamma_1)_{\alpha \beta} \frac{1}{\varepsilon} \Delta_j \nabla_j \psi_{\beta j}^n + M (\gamma_4)_{\alpha \beta} \Delta_j \nabla_j \psi_{\beta j}^n \right\} \]

\[ - i \varepsilon \sum_{j=0}^{N-1} \tau n \left\{ \Delta_j \psi_{\alpha j}^+ \nabla_j \psi_{\alpha j} + \varepsilon \sum_{j=0}^{N-1} \Delta \psi_{\alpha j}^+ \frac{1}{2i} (\gamma_1 \gamma_4)_{\alpha \beta} \Delta_j \psi_{\beta j}^n \right\} \quad (8.8) \]

Using (7.5) and (7.8) it can be proved that these operators are constant of time

\[ \frac{1}{\tau} \Delta_n P = \frac{1}{\tau} \Delta_n H = \frac{1}{\tau} \Delta_n M_{14} = 0 \quad (8.9) \]

and that they satisfy the standard commutation relations

\[ [H, P] = 0 \quad , \quad [H, M_{14}] = P \quad , \quad [P, M_{14}] = H \quad (8.10) \]

We can convince ourselves that \( H \) and \( P \) are the generators of the time and space displacement by iteration of the Heisenberg equations of motion as in the continuous case. For the operator \( H \) we have

\[ \frac{1}{\tau} \left[ \Delta_n \left( \nabla_j \Delta_j \phi_j^0 \right), H \right] = \frac{1}{\tau} \left( \nabla_j \Delta_j \phi_j^0 \right) \quad (8.11) \]

where we have fixed the time index, say \( n = 0 \).

By iteration we have

\[ \sum_{k=1}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \left( \frac{\tau}{i} \right)^k \left( \Delta_n \right)^k \left[ \nabla_j \Delta_j \phi_j^0, H, \cdots, H \right] = \sum_{k=1}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \Delta_n^{(k)} \left( \nabla_j \Delta_j \phi_j^0 \right)_{n=0} \]

\[ = \nabla_j \Delta_j \phi_j^n - \nabla_j \Delta_j \phi_j^0 \quad (8.12) \]

which can be taken as the “Taylor expansion” on the lattice, namely,

\[ \nabla_j \Delta_j \phi_j^n = \nabla_j \Delta_j \phi_j^0 + \sum_{k=1}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \Delta_n^{(k)} \nabla_j \Delta_j \phi_j^n \bigg|_{n=0} \]

\[ = \nabla_j \Delta_j \phi_j^n \quad (8.13) \]

In the limit \( n \to \infty, \quad \tau \to 0, \quad n \tau \to t \) the expression (8.12) becomes

\[ \phi(x, 0) + \sum_{k=1}^{\infty} \frac{t^k}{k!} \left[ \phi(x, 0), H, \cdots, H \right] = \phi(x, 0) + \sum_{k=1}^{\infty} \frac{t^k}{k!} \frac{\partial^k}{\partial t^k} \phi(x, t) \bigg|_{t=0} = \phi(x, t) \quad (8.14) \]

But this expression is precisely the expansion of the continuous time translations generated by the operator \( H \)

\[ e^{iHt} \phi(x, 0) e^{-iHt} = \phi(x, t) \quad (8.15) \]
For the operator $P$ we have

$$\frac{1}{i} \left[ \tilde{\nabla}_j \tilde{\Delta}_j \phi^n_0, P \right] = \frac{1}{\varepsilon} \tilde{\nabla}_j \Delta_j \phi^n_0$$  \hspace{1cm} (8.16)

where we have fixed the space index, say $j = 0$. By iteration we have

$$\tilde{\nabla}_j \tilde{\Delta}_j \phi^n_0 + \sum_{k=1}^{j} \binom{j}{k} \varepsilon^k (\tilde{\Delta}_j)^{k-1} \left[ \tilde{\Delta}_j \phi^n_0, P \right]_{\text{times}} = \tilde{\nabla}_j \tilde{\Delta}_j \phi^n_0 + \sum_{k=1}^{j} \binom{j}{k} \Delta_j^{(k)} \left( \tilde{\nabla}_j \phi^n_j \right)_{j=0} = \tilde{\nabla}_j \tilde{\Delta}_j \phi^n_j$$  \hspace{1cm} (8.17)

which correspond to the $j$ step space translation on the lattice. In the limit (8.17) becomes the continuous space translation generated by the operator $P$.

APPENDIX. THE EINSTEIN DE BROGLIE RELATION ON THE LATTICE

In order to make connection of our scheme with the Einstein-de Broglie relations $E = \hbar \omega$, $p = \hbar k$ we take the discrete plane waves solutions of (6.8).

$$f^n_j (k, \omega) = \left( 1 + \frac{1}{2} i \varepsilon k \right)^j \left( 1 - \frac{1}{2} i \tau \omega \right)^n$$  \hspace{1cm} (A.1)

Obviously, we have for the period $T$ and wave length $\lambda$

$$T = N \tau \hspace{0.5cm}, \hspace{0.5cm} \lambda = N \varepsilon$$  \hspace{1cm} (A.2)

and for the phase velocity

$$v_p = \frac{\lambda}{T} = \frac{\varepsilon}{\tau}$$  \hspace{1cm} (A.3)

If we impose the boundary conditions

$$f^n_0 (k, \omega) = f^n_N (k, m)$$  \hspace{1cm} (A.4)

the wave number and the angular frequency can be defined as

$$k_m = \frac{2}{\varepsilon} \tan \frac{\pi m}{N}, \hspace{0.5cm} \omega_m = \frac{2}{\tau} \tan \frac{\pi m}{N}, \hspace{0.5cm} m = 0, 1, \ldots, N - 1$$  \hspace{1cm} (A.5)

Substituting the Einstein-de Broglie relations in the relativistic expression $E^2 - p^2 = M^2$ (we use natural units $\hbar = c = 1$), we obtain

$$\omega_m^2 - k_m^2 = \omega_m^2 \left( 1 - \frac{\tau^2}{\varepsilon^2} \right) = \omega_m^2 \left( 1 - \frac{1}{v_p^2} \right) = M^2$$  \hspace{1cm} (A.6)
Since the phase velocity and group velocity satisfy \( v_p v_g = 1 \), we have finally

\[
\omega_m^2 = \frac{M^2}{1 - v_g^2}
\]

(A.7)

giving a discrete mass spectrum due to the lattice.

ACKNOWLEDGMENT

We want to express our gratitude to Bruno Gruber for the invitation to participate in this Symposium in honour of L. Biedenharn and for the opportunity to present these ideas and discuss them during the symposium. This work has been partially supported by Vicerrectorado de Investigación de Universidad de Oviedo and by D.G.I.C.Y.T. (PS 89-0171).

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