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STATIC UNIVERSE MODEL EXISTING
DUE TO THE MATTER-DARK ENERGY COUPLING

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Abstract

The work investigates a static, isotropic and almost homogeneous Universe containing a real scalar field modeling the Dark-Energy (quintaessence) interacting with pressureless matter. It is argued that the interaction between matter and the Dark Energy, is essential for the very existence of the considered solution. Assuming the possibility that Dark-Energy can be furnished by the Dilaton (a scalar field reflecting the condensation of string states with zero angular momentum) we fix the value of scalar field at the origin to the Planck scale. It became possible to fix the ratio of the amount of Dark Energy to matter energy, in the currently estimated value \( \frac{0.7}{0.3} \) and also the observed magnitude of the Hubble constant. The small value of the mass for the scalar field chosen for fixing the above ratio and Hubble effect strength, results to be of the order of \( 10^{-29} \text{cm}^{-1} \), a small value which seems to be compatible with the zero mass of the Dilaton in the lowest approximations.

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I. INTRODUCTION

The assumption about the isotropic and homogeneous nature of our Universe, that is the Cosmological Principle, is central to modern Cosmology [1]. However, recent experimental observations suggest the possibility for the break down of the validity of the principle at large scales [2]. In the present work a static solution of the KG equation in interaction with matter is investigated which shows a large region of homogeneity close to a central symmetry point, but not at large distances. The existence of this static solution essentially rests on the presence of an interaction of the scalar field (modeling the quintessence) with the pressureless matter. The solution discussed here is a generalization of one formerly investigated in Refs. [3, 4] in the absence of matter. The special characteristics of the scalar field led to the proposal made in Ref. [3] of considering it as representing the Dilaton of the string theory [5–8]. This idea came from the fact that when you fix the value of the scalar field (which has dimension of mass) at the central symmetry point at the Planck scale, by also requiring an amount of Hubble effect similar to the experimental one, the radius of existence of the solution gets a value near to \( R = 10^{28} \text{cm} \). However, even more curious is that the values of KG mass of the fields obtained by fixing the above parameters, results to be of the order of \( 1/R \). That is, a very small value which seems compatible with a very tiny mass acquired by the Dilaton due to boundary conditions or non perturbative effects, which could deviate it from its known massless character in the first approximation.

Various static models of the Universe have been considered. Among them are the ones of Einstein, Le Maitre and de’Sitter, respectively. Originally, Einstein [1] examined a Universe filled of uniformly distributed matter but obtained a non-static metric. This result motivated him to introduce in his equations the Cosmological Constant term \( \lambda \), with the objective of allowing the obtaining of a static solution. However, this model contradicted the observations at that time and was disregarded.

The de’Sitter model considers the Universe as empty and is defined by a Cosmological Constant \( \lambda \equiv \frac{1}{r_s^2} \). The metric has the form:

\[
ds^2 = \left(1 - \frac{r^2}{r_s^2}\right)cdt^2 - \frac{1}{\left(1 - \frac{r^2}{r_s^2}\right)}dr^2 - r^2 (\sin^2 \theta d\varphi^2 + d\theta^2),
\]

which describes an homogeneous space of volume

\[
V = 2\pi^2 r_s^3.
\]

The high symmetry of this space-time is evidenced by the property that any point of it can be transformed in the origin of coordinates by a bijective transformation [9]. However, the fact that in this model the Universe is empty, is unreal. It can be underlined that experimentally and theoretically the DE has been associated to the presence of a cosmological constant [10]. However, in this modern view the Cosmological Constant coexists with matter.
In connection with the above fact, it has been observed that a centrally symmetric static scalar field satisfying the Einstein-Klein-Gordon equations (EKG) curves the space time in a form resembling the one in the de'Sitter space in a large neighborhood of the origin of coordinates [4]. The fact that the scalar field is more weakly varying along the radial domain when its value at the origin is lower is an interesting property to notice. The associated densities of energy and pressure are positive and negative respectively and weakly varying, approximating the presence of a positive Cosmological Constant. These properties suggested the idea advanced in [3] of considering the Dark Energy (DE) as described by a scalar field in this approximately homogeneous field configuration studied in [4]. This assumption will determine the abandoning of the Cosmological Principle in favour of what could be imagined as a kind of "Matryoshka" model of the Universe. In this conception, proposed in [3, 4], we could be living inside of a particular configuration in which the scalar field has a definite value resulting from the collapse of string matter in fermionic states. The presence of the DE is then assumed to be associated to the scalar field which could be radiated by the string matter in fermionic states under the extreme conditions of the collapse. The effective realization of this picture in Nature, could lead to the possibility that the astrophysical black-holes (for example, the ones which are expected to exist near the centres of the Galaxies) could simply be small Universes in which at their interior, the Dilaton field takes values differing from its external one, which is associated to the Dark Energy. This change could be produced by the collapse of fermion matter when falling into the hole by also radiating zero angular momentum modes, that is of the Dilaton. We encounter this picture as interesting and think its exploration is worth considering.

Therefore, in this work we address the finding of a static solution related to the one discussed in [3, 4], but also including pressureless matter, in order to approach the discussion of the physical properties of these configurations in a situation more closely related to the physical conditions in the Universe.

An important aspect which emerged in the first examination of the problem, is that the co-existence of the scalar field as described by the EKG equations including also the dust energy momentum tensor does not allow the existence of static solutions, at least in centrally symmetric configurations [11]. However, in this work we will argue that such a solution exists if the interaction between the matter and the Dark Energy (or Dilaton condensate) is included. The introduction of the coupling does not damage the almost homogenous character of the solution in a relative large region around the origin of the central symmetry, being far away from the limits of the Universe. An interesting outcome is that the distributions of matter and of Dark Energy both show a very close behavior. That is, the scalar field (Dark Energy) is able to sustain an amount of matter being almost proportional between them.

The hypothesis of a pressureless matter (Dust Filled Universe) [11] is employed here. It constitutes a reasonable assumption in the framework of modern Cosmology [1], reflecting the idea that the astrophysical systems are formed by large collections of galaxies and these again
are grouped into clusters of such collections, which can reasonably be considered as being weakly bounded among them. This structure suggests that the pressureless condition of the gas of galaxies is a reasonable assumption.

II. FIELD EQUATIONS

Given the isotropic and stationary character of the solution which is searched, let us propose the structure of the metric in the standard form

\[ ds^2 = v(\rho)dx^2 - u(\rho)^{-1}d\rho^2 - \rho^2(sin^2\theta d\varphi^2 + d\theta^2), \]

\[ x^0 = ct, \quad x^1 = \rho, \]

\[ x^2 = \varphi, \quad x^3 = \theta, \]

from which the components of the Einstein tensor \( G_{\mu\nu} \) can be computed. Since the metric tensor is diagonal and only depending on \( \rho \), the only non vanishing components of \( G_{\mu\nu} \) result in

\[ G^0_0 = \frac{u'}{\rho} - \frac{1 - u}{\rho^2}, \]

\[ G^1_1 = \frac{u v'}{\rho} - \frac{1 - u}{\rho^2}, \]

\[ G^2_2 = G^3_3 = \frac{u'}{2v} + \frac{u'v'}{v} + \frac{u''}{v} - \frac{u'}{u} + \frac{u'}{v}. \]

The components \( G^0_0 \) and \( G^3_3 \) generate second order equations in the temporal component of the metric, which explicitly do not play an important role thanks to the Bianchi identities [1]:

\[ G^\nu_{\mu\nu} = 0. \]

which will be employed below. Assumed the satisfaction of the Einstein equations the \( G^\nu_{\mu} \) tensor can be substituted by the energy momentum tensor \( T^\nu_{\mu} \). Equation (3) is interpreted as a set of dynamical equations for the parameters, that is \( e, p \) and \( \phi \).

A. Matter and Dilaton Dark Energy

In this section let us sketch the way followed for obtaining two of the necessary equations needed to show the existence of the mentioned static model for the Universe: the Bianchi relations (3) and the static equation for the scalar field coupled to matter.

Let us write the action for the scalar field matter in the given space time in the form

\[ S_{\text{mat-}\phi} = \int L\sqrt{-g}d^4x, \]
where \( g \) is the determinant of the metric tensor, and consider that the Lagrangian density takes the form:

\[
L = \frac{1}{2} (g^{\alpha\beta} \phi,_{\alpha} \phi,_{\beta} + m^2 \phi^2) + j \phi + L_{e,p}.
\]  
(5)

The first and third terms of the right member of (5) are the Lagrangian densities of the KG scalar field and the dust-like matter respectively; while the second term is an interaction term between both which is added. The strength of the interaction will be represented by the constant source \( j \).

Based on the argument given in the last paragraph of section 1, we will consider for the matter the perfect fluid expression \([1]\):

\[
(T_{e,p})^\mu_\nu = p \delta^\nu_\mu + u^\nu u_\mu (p + e),
\]  
(6)

where \( p \) is the pressure of the matter. Note that we will assume pressure-less matter \( p = 0 \). However, for reference purposes, we will get the expression for a general pressure \( p \) up to the end when the limit \( p = 0 \) will be fixed.

As usual \( u^\nu \) denote the contra-variant components of the 4-velocity of the fluid in the system of reference under consideration. In addition since we search for static configurations the 4-velocity takes the simple form \( u^\nu = \delta^\nu_0 \).

From the Lagrangian \( L \) in (5) and the above remarks the energy momentum tensor of the scalar field coupled with the matter gets the form

\[
T^\nu_\mu = -\frac{\delta^\nu_\mu}{2} (g^{\alpha\beta} \phi,_{\alpha} \phi,_{\beta} + m^2 \phi^2 + 2 j \phi) + g^{\alpha\nu} \phi,_{\alpha} \phi,_{\mu} + p \delta^\nu_\mu + \delta^\nu_0 \delta^0_\mu (p + e).
\]  
(7)

From equation (7), the Bianchi relation for \( \mu = 1 \) in (3) transforms in

\[-\phi j' + p' + \frac{j'}{2 \nu} (p + e) = 0.\]

In our case this is the only one of the four Bianchi relations which is different from zero.

The dynamical equation for the scalar field which determines the extremum of the action \( S_{mat,\phi} \), takes the form

\[
\delta S_{mat,\phi} = D \frac{\partial L}{\partial x^\nu} \frac{\partial L}{\partial \phi} - \frac{\partial L}{\partial \phi} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} (\sqrt{-g} g^{\mu\nu} \phi,_{\mu}) - m^2 \phi - j = 0,
\]  
(8)

which after introducing the components of the metric tensor simplifies to

\[u \phi'' + u \phi' \left( \frac{1}{2} \frac{v'}{v} + \frac{1}{2} \frac{u'}{u} + \frac{2}{\rho} \right) - m^2 \phi - j = 0.\]

Note that if \( u = v = 1 \), that is, in the Minkowski space, relation (9) reduces to the static KG equation for scalar field interacting with an external source \( j \). It might be helpful to notice that natural units

\([e] = [p] = cm^{-4}, [m] = cm^{-1}, [\phi] = cm^{-1}\)

are employed.
B. Einstein equations

The extremum of the action $S_{\text{mat-}\phi}$ with respect to the metric leads to the Einstein equations in the absence of a Cosmological Constant

$$G^\nu_\mu = GT^\nu_\mu,$$

where in natural units $G = 8\pi \times l_p^2$ and $l_p = 1.61 \times 10^{-33} \text{cm}$ is the Planck length.

From relation (7), the Einstein equations (9) take the form

$$\frac{u'}{\rho} - \frac{1 - u}{\rho^2} = -G[\frac{1}{2}(u\phi^2,\rho + m^2\phi^2 + 2j\phi) + e],$$

$$\frac{uv'}{v\rho} - \frac{1 - u}{\rho^2} = G[\frac{1}{2}(u\phi^2,\rho - m^2\phi^2 - 2j\phi) + p].$$

As mentioned above, the third Einstein equation is not needed for determining the solution, because it is implied by the other equations. This expression only imposes the continuity of the derivative of $v$ with respect to the radial variable since it is a second order differential equation.

We will assume that $j$, which gives the form of the interaction term between the dark energy and matter, is of the form:

$$j = ge^{\Phi},$$

where $g$ is a coupling constant for the interaction matter-scalar field. In the natural system of units $[g] = \text{cm}^{-1}$.

C. Working equations

With the aim of working with dimensionless forms of the equations (10) and (11), let us define the new variables and parameters

$$r \equiv m\rho, \quad \Phi \equiv \sqrt{8\pi l_p}\phi,$n

$$J \equiv \frac{\sqrt{8\pi l_p}}{m^2}j, \quad \epsilon \equiv \frac{8\pi l_p^2}{m^2}e, \quad \gamma \equiv \frac{g}{m}.$$

Let us now fix the mass of the Dilaton field to the value estimated in Ref. [3] in order to reproduce the observed strength of the Hubble effect in the regions near the origin. Interestingly, this value resulted in a very small quantity, $m = 4 \times 10^{-29} \text{cm}^{-1}$. This mass is compatible with the zero mass Dilaton in the lowest approximation. Also the mass is of the order of the inverse of the estimated radius of the Universe, like it was observed in Ref. [3].
Therefore, the set of working equations will be

\[
\begin{align*}
\frac{u_r}{r} - \frac{1-u}{r^2} &= -\frac{1}{2}(u\Phi^2 + \Phi^2) - J\Phi - \epsilon, \\
\frac{uv_r}{vr} - \frac{1-u}{r^2} &= -\frac{1}{2}(-u\Phi^2 + \Phi^2 + 2J\Phi), \\
\epsilon v_r - \Phi J_r &= 0, \\
u\Phi_{,rr} - \Phi - J &= -u\Phi_r (\frac{1}{2} v_r + \frac{1}{2} u_r + \frac{2}{r}).
\end{align*}
\]

(12)  
(13)  
(14)  
(15)

D. The solutions near the center of symmetry

We will search for smooth solutions around the origin. Thus, the continuity of the derivatives \(v\) and \(\phi\), in all places including the origin, will be required. Thus considering the equations in a neighborhood of the origin the asymptotic field values can be obtained in the form

\[
\begin{align*}
\frac{u}{u_1} &= 1 + u_1 r^2, \\
\frac{v}{v_1} &= 1 + v_1 r^2, \\
\Phi &= \Phi_0 + \Phi_1 r^2, \\
\epsilon &= \epsilon_0 + \epsilon_1 r^2,
\end{align*}
\]

where \(u_1, v_1, \Phi_1, \epsilon_1\) are given by the relations

\[
\begin{align*}
u_1 &= -\frac{1}{3}(\Phi_0^2 + J_0 \Phi_0 + \epsilon_0), \\
\Phi_1 &= -\frac{1}{6}(\Phi_0 + J_0), \\
\epsilon_1 &= -\frac{\epsilon_0^2}{3\gamma \Phi_0} (\Phi_0^2 + J_0 \Phi_0 - \frac{\epsilon_0}{2}), \\
J_0 &= \gamma \epsilon_0^2.
\end{align*}
\]

(16)  
(17)  
(18)  
(19)

Note that the spacial dependence of the metric has an exact homogeneous structure near the center of symmetry. The quantities \(\Phi_0, \epsilon_0\) and the dimensionless coupling constant \(\gamma\) remains as free parameters. Future extensions of this work, are considered to optimize the parameters aiming to compare the predictions of the model with redshift vs. distance in the supernovae obervations. In what follows we will illustrate only the general behavior of the solutions for some physically motivated values of the parameters.

E. Solutions

Let us consider the numerical solutions of the equations (12)-(15), selecting the parameter values \(\gamma = -0.75, \Phi_0 = 2.2\) and \(\epsilon_0 = 1\). These specific values correspond to a coupling constant
FIG. 1: The radial contraviant component of the metric $g^{11} \equiv u(r)$ behaves basically as the as the as in the deSitter Universe having the size $R \equiv 0.25 \times 10^{29}$ cm.

FIG. 2: Temporal component of the metric $g_{00} \equiv v(r)$. Its decreasing behaviour show the redshift of the arriving to the central regions form the far regions. The radius of the singularity at the far away region is $R \equiv 0.25 \times 10^{29}$ cm.

$g = 2.9 \times 10^{-29} \text{cm}^{-1}$, a value of the scalar field at the origin $\phi_0 = 2.7 \times 10^{32} \text{cm}^{-1}$ (that is at the Planck scale) and at a matter energy density of $\rho = 2.3 \times 10^{7} \text{cm}^{-4}$. The numerical solutions of the equations (12)-(15) are illustrated in figures (1)-(4).

These parameters were a priori selected with the aim of fixing the estimated value of 0.7/0.3 for the ratio of the Dark Energy to the matter energy content in the Universe [12]and the approximate value of the Hubble effect.

From Fig.(1) the global similarity between the space-time being studied and the de Sitter static solution can be observed. Moreover, due to the chosen value of the Dilaton mass suggested in ([3]), the size of the Universe (defined as the radial distance at which the singularity of the structure appears) is of the order of the estimated value $10^{29} \text{cm}$. In Fig.(2) the dependence form of the temporal metric is shown, it evidences that the observer near the origin measure a redshift which was fixed to have a value being close to the one observed nowadays.
FIG. 3: The matter distribution $\epsilon(r)$ is slowly varying with the radial distance. The coupling between the scalar field and the matter $J\Phi$ is central in allowing the existence of the static solution, in which also the matter to Dark energy content ratio is also slowly varying. The radial singularity defining the end of the space time is at $R = 0.25 \times 10^{29}$ cm.

FIG. 4: The scalar field slowly varies with the radial component and behaves very closely with the matter density $\epsilon(r)$; The radial singularity defining the end of the space time is at $R = 0.25 \times 10^{29}$ cm. There is no static metric with Dilaton and matter in coexistence without interaction.

Figures (3) and (4) illustrate the distribution of energy and scalar field respectively. Note the similarity between both magnitudes. That is, the existence of the coupling not only allows the existence of the static solution, but in addition it also produces a configuration in which the proportion of matter and dark energy is more or less approximately constant over large regions of the space time.

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