Transits of the QCD Critical Point

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Abstract

We analyze the evolution of hydrodynamic fluctuations in a heavy ion collision as the system passes close to the QCD critical point. We introduce two small dimensionless parameters $\lambda$ and $\Delta_s$ to characterize the evolution. $\lambda$ compares the microscopic relaxation time (away from the critical point) to the expansion rate $\lambda \equiv \tau_0/\tau_Q$, and $\Delta_s$ compares the baryon to entropy ratio, $n/s$, to its critical value, $\Delta_s \equiv (n/s - n_c/s_c)/(n_c/s_c)$. We determine how the evolution of critical hydrodynamic fluctuations depends parametrically on $\lambda$ and $\Delta_s$. Finally, we use this parametric reasoning to estimate the critical fluctuations and correlation length for a heavy ion collision, and to give guidance to the experimental search for the QCD critical point.

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I. INTRODUCTION

A. Overview and goals

The conjectured QCD critical point is a landmark point in the QCD phase diagram. This is the end point of a line of first-order phase transitions, which separates the Quark-Gluon Plasma (QGP) phase from hadronic matter. Due to the sign problem at finite baryon chemical potential, lattice QCD simulations have yet to confirm the existence of a critical point [1]. Nevertheless, the conjectured point in the phase diagram is theoretically well motivated, and has been found in various effective field theory models, see Refs. [2–4] for reviews. An intense experimental effort is underway to locate and to characterize the critical point through a beam energy scan (BES) of heavy ion collisions at the Relativistic Heavy Ion Collider (RHIC) [5, 6].

The experimental search for the QCD critical point will focus on fluctuations. The existence of a critical point in a heavy ion collision should lead to large correlations and enhanced fluctuations of conserved densities [7, 8]. These enhanced fluctuations should manifest themselves through the multiplicity fluctuations of the produced hadrons. However, the systems created in these nuclear collisions are rapidly expanding, and consequently thermodynamic fluctuations will not be fully equilibrated. In particular, it has been demonstrated previously that due to the expansion of the fireball and the physics of critical slowing down, the critical fluctuations can differ significantly from their equilibrium expectation [9, 10]. Further, in any real experiment the system will not pass directly through the critical point, and this again will limit the size of the critical fluctuations.

To quantify how the expansion of the system and missing the critical point will tame the critical fluctuations we will introduce two small parameters, $\lambda$ and $\Delta$, which characterize the evolution of the fireball:

$$\lambda \equiv \frac{\tau_0}{\tau_Q},$$  \hspace{1cm} (1)

$$\Delta = \frac{n_c}{s_c} \left( 1 - \frac{s}{n_c} \right).$$  \hspace{1cm} (2)

The first parameter $\lambda$ is the product of the microscopic relaxation time away from the critical point $\tau_0$ and the expansion rate $1/\tau_Q$ (more precise definitions of $\tau_0$ and $\tau_Q$ are given below). The second parameter $\Delta$ quantifies the deviation of the baryon number to entropy ratio $n/s$ from its critical value $n_c/s_c$ during the adiabatic expansion of the system. A primary goal of the current study is to determine how the magnitude of the critical fluctuations depends parametrically on these two small parameters.

In perfect equilibrium, the hydrodynamic fluctuations in the energy density (for example) are given by the textbook thermodynamic formula

$$\langle \delta e(t, x) \delta e(t, y) \rangle_{\text{equilibrium}} = T^2 C_v \delta^{(3)}(x - y),$$  \hspace{1cm} (3)

where $C_v$ is the specific heat at constant volume. In Fourier space this says that all wavenumbers have equal amplitude

$$\langle \delta e(t, k) \delta e(t, -k') \rangle_{\text{equilibrium}} = T^2 C_v (2\pi)^3 \delta^{(3)}(k - k').$$  \hspace{1cm} (4)

However, for an expanding system, even away from the critical point, the distribution of fluctuations will not follow this equilibrium form, since long wavelengths of conserved quantities take a long time to relax to equilibrium. The second goal of this paper is to determine
the wavelength which characterizes the enhanced specific heats near the critical point, and to specify how this wavelength depends on \( \lambda \) and \( \Delta_s \).

Away from the critical point, there is a length scale \( \ell_{\text{max}} \) where modes with wavelength longer than \( \ell_{\text{max}} \) fall out of equilibrium and reflect the expansion history rather than the equilibrium specific heat [11]. Indeed, the equilibration of hydrodynamic fluctuations is a diffusive process. The diffusion coefficient away from the critical point is of order \( D_0 \sim \ell_0^2/\tau_0 \) where \( \tau_0 \) is the relaxation time introduced above, and \( \ell_0 \) is a microscopic length. The maximum wavelength that can be equilibrated by diffusion over the total time \( \tau_Q \) is

\[
\ell_{\text{max}}^2 \sim \ell_0^2 \left( \frac{\tau_Q}{\tau_0} \right),
\]

or

\[
\ell_{\text{max}} \sim \frac{\ell_0}{\sqrt{\lambda}}.
\]

There is insufficient time to equilibrate modes longer than \( \ell_{\text{max}} \), and thus \( \ell_{\text{max}} \) provides a robust upper cutoff on the size of critically correlated domains in the expanding fireball.

Near a critical point the diffusion coefficient is not a constant value \( D_0 \), but rapidly approaches zero. Thus the length scale characterizing critical domains is necessarily smaller than \( \ell_{\text{max}} \). Modes with wavelength \( \ell \ll \ell_{\text{max}} \) (but still longer than \( \ell_0 \)) are equilibrated away from the critical point, but fall out of equilibrium as the system approaches the critical point. The emergent length scale, which arises from the competition between the expansion of the fireball and the diffusive equilibration of fluctuations, is known as the Kibble-Zurek length \( \ell_{\text{kz}} \). The Kibble-Zurek length is the correlation length at the time when the system falls out of equilibrium, and characterizes both the magnitude and distribution of fluctuations in an evolving critical system [12–15]. The importance of Kibble-Zurek length (and time) for the QCD critical point search has been identified in Ref. [16]. As we will see, the Kibble-Zurek length is of order

\[
\ell_{\text{kz}} \sim \frac{\ell_0}{\lambda^{0.18}},
\]

leading to an interesting hierarchy of scales \( \ell_0 \ll \ell_{\text{kz}} \ll \ell_{\text{max}} \). Both \( \ell_{\text{max}} \) and \( \ell_{\text{kz}} \) are estimated in the conclusions.

Beyond parametric estimates, we will determine the time evolution of hydrodynamic correlators (such as Eq. (3)) by evolving stochastic hydrodynamics for an expanding fluid in the vicinity of the QCD critical point. Specifically, following Ref. [11] (see also Ref. [17, 18]), we will write down and solve a set hydro-kinetic equations governing the evolution of hydrodynamic two point functions. The hydro-kinetic approach reformulates stochastic hydrodynamics as non-fluctuating hydrodynamics (describing a long wavelength background) coupled to a set of deterministic kinetic equations describing the phase space distribution of short wavelength thermodynamic fluctuations, see also Refs. [18, 19] for related developments. The hydro-kinetic formulation successfully describes non-trivial effects such as the hydrodynamic tails [11] and the renormalization of bulk viscosity [20], both of which are a consequence of the non-equilibrium evolution of thermodynamic fluctuations. We first extend this approach to a system with non-zero net baryon density, and then implement critical fluctuations as implied by the critical universality. We show how characteristic length scale \( \ell_{\text{kz}} \) emerges from the hydro-kinetic equations for an expanding fireball. See also Refs. [21–24] for previous studies of critical fluctuations based on stochastic hydrodynamics.

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1 In Ref. [11] the length scale \( \ell_{\text{max}} \) is parametrized by the wavenumber \( k_* \sim 1/\ell_{\text{max}} \).
B. Setup and outline

1. Setup

Consider the hydrodynamic evolution of a single fluid cell of QCD matter passing close to the critical point. In the rest frame of the material, the entropy and baryon number densities follow the equations of ideal hydrodynamics

\[
\begin{align*}
\partial_\tau s &= - s \nabla \cdot u, \\
\partial_\tau n &= - n \nabla \cdot u,
\end{align*}
\]

where \(\tau\) is the proper time of the fluid cell and \(\nabla \cdot u = \partial_\mu u^\mu\) is the expansion scalar. Since the system is close to the critical point only for a short period of time we may treat the expansion scalar as a constant, \(\partial_\mu u^\mu \equiv 1/\tau_Q\). Indeed, \(\tau_Q\) is of order the system’s lifetime, while the time scales for the critical dynamics \(t_{K}\) and \(t_{cr}\) will be parametrically smaller than \(\tau_Q\) justifying this approximation a-posteriori.

The entropy per baryon \(s/n\) in Eq. (8) is constant in time. We will refer to relative deviation of \(s/n\) from \(s_c/n_c\) as the “detuning” parameter, \(\Delta_s\). Close to the critical point

\[
\Delta_s \equiv \frac{n_c}{s_c} \left( \frac{s}{n} - \frac{s_c}{n_c} \right) \simeq \frac{\Delta s}{s_c} - \frac{\Delta n}{n_c},
\]

where \(\Delta n\) notates the deviation from the critical value

\[
\Delta n \equiv n - n_c,
\]

with an analogous notation for other quantities (e.g. \(\Delta \mu \equiv \mu - \mu_c\)). \(\Delta_s\) is a dimensionless number and is small for a system passing close to the critical point.

There is a time \(\tau_1\) where the baryon number reaches its critical value, \(n_c\). The entropy at this time differs from its critical value by \(\Delta \bar{s}/s_c \simeq \Delta_s\). For times close to \(\tau_1\), we can integrate the equations of motion Eq. (8) yielding

\[
\begin{align*}
\frac{\Delta n(t)}{n_c} &= - \frac{t}{\tau_Q}, \\
\frac{\Delta s(t)}{s_c} &= \Delta_s - \frac{t}{\tau_Q},
\end{align*}
\]

where we have defined \(t \equiv \tau - \tau_1\). Thermodynamics relates the deviation in the (average) energy density from its critical value to these two quantities

\[
\Delta e = T_c \Delta s + \mu_c \Delta n .
\]

In Fig. 1(a) we show a schematic picture of typical trajectory in the full QCD phase diagram, portrayed in the \((n, s)\)-plane\(^2\). In Fig. 1(b) we have rescaled the axes of (a) by \(n_c\) and \(s_c\) and expanded the region near the critical point. The detuning parameter \(\Delta_s\) is the intercept of \(45^\circ\) lines which label the trajectories of the system. Finally, in Fig. 1(c) (which is discussed more completely in Sect. II C) we have rescaled the \(\Delta n/n_c\) and \(\Delta s/s_c\) axes of Fig. 1(b) by \(\Delta_s\) and \(\Delta_s^{(1-\alpha)/\beta}\) respectively. Only in (c) does the fact that the system misses the critical point by an amount \(\Delta_s\) become important.

\(^2\) In this figure the coexistence line is shown as a flat line, which is a commonly used idealization [25]. This idealization is not essential to the parametric reasoning discussed in the text and illustrated in Fig. 1
FIG. 1. (a) A schematic trajectory of a heavy ion collision passing close to the critical point. The duration of panel (a) is of order $\Delta t \sim \tau_Q$. (b) A magnification of the critical region in figure (a) by $\Delta_s$. The duration of panel (b) is of order $\Delta t \sim \tau_Q \Delta_s$. In this regime the Ising magnetic field $h$ is negligibly small, and the susceptibilities scale as a power $\Delta n/n_c$. (c) In this panel we have rescaled the $\Delta n/n_c$ and $\Delta s/s_c$ axes of (b) by $\Delta^b$ and $\Delta_s$ respectively, with $b \equiv (1 - \alpha)/\beta \approx 2.7$. The duration of panel (c) is of order $\Delta t \sim t_{cr} \sim \tau_Q \Delta^b_s$. At the time $t_{cr}$ the system leaves the coexistence region and the equilibrium correlation length reaches its maximal value (see Eq. (84)). Only in panel (c) is the equation of state a nontrivial function (i.e. beyond simple powers) of the scaling variable $z \propto r/h^{1/\beta}$. 
2. Computational outline

The goal of the current paper is to determine how the distribution of hydrodynamic fluctuations evolves in time as the mean entropy and baryon number densities evolve according to Eq. (11), and the system passes close to the critical point with parameter $\Delta_s$. For several important (and related) reasons the primary object of study is the distribution of fluctuations in the entropy per baryon $\delta s \equiv n \delta(s/n)$.

$$N^{\delta s}(t, k) \equiv \int d^3x e^{i k \cdot (x-y)} \langle \delta s(t, x) \delta s(t, y) \rangle .$$  \hspace{1cm} (13)

First, this correlation function diverges near the critical point as the Ising magnetic susceptibility $\chi_{is}$, which has the largest critical exponent $\gamma \simeq 1.23$ (see [26] and Sect. II B 3). Second, $N^{\delta s}$ determines the specific heat at constant pressure $C_p$ in the limit $k \rightarrow 0$ (see [27] and Sect. II B 2). Finally, the $\delta s$ fluctuation is a diffusive eigen-mode of the linearized hydrodynamic equations, and therefore evolves independently of other hydrodynamic fluctuations. The associated heat diffusion coefficient $D_\delta$, which controls the relaxation of $\delta s$, is similar in magnitude to the baryon number diffusion coefficient $D_B$ (see [28, 29] and Sect. III A 1).

We will determine how the amplitude and the shape of the $N^{\delta s}$ distribution depend on the parameters $\lambda$ and $\Delta_s$.

We first need to describe how this correlation function would evolve in perfect equilibrium; this involves several ingredients as described in Sect. II. The time evolution of the overall amplitude of $N^{\delta s}$ in equilibrium is given by $C_p(t)$ which is related through universality to the Ising magnetic susceptibility $\chi_{is}$. In Sect. II A we describe how to map the QCD quantities $\Delta s$ and $\Delta n$ onto the phase diagram of the Ising model. Since the time dependence of $\Delta s$ and $\Delta n$ has already been prescribed in Eq. (11), once the QCD-to-Ising map is given, the time evolution of $\chi(t) \propto C_p(t)$ is fixed. The shape of the $N^{\delta s}$ distribution is controlled by the correlation length $\xi(t)$ which is also specified through universality. In equilibrium, the relaxation time parameter $\lambda$ plays no role, and the evolution of $N_0^{\delta s}$ is determined only by $\tau_Q$ and $\Delta_s$. As we show in Sect. II C, the relevant timescale for the non-trivial evolution of $C_p(t)$ and $\xi(t)$ is set by a crossing timescale:

$$t_{cr} \sim \tau_Q \Delta_s^b, \quad b \equiv \frac{1 - \alpha}{\beta} \simeq 2.7 .$$  \hspace{1cm} (14)

The equilibrium evolution of the $N^{\delta s}$ is summarized in Sect. II D, where the time dependence of the amplitude $C_p(t) \propto \chi_{is}(t)$ and correlation length $\xi(t)$ are shown in Fig. 2(a) and (b) respectively.

After specifying how the equilibrium expectation evolves we will write down a dynamical evolution equation for $N^{\delta s}$ by analyzing stochastic hydrodynamics in the expanding critical background – see Sect. III. The diffusion coefficient entering in this evolution equation determines a relaxation rate $\Gamma_{\delta}$ for the $\delta s$ mode, which approaches zero near the critical point, $\Gamma_{\delta} \propto \xi^{-z}$ with $z = 4 - \eta$. Comparing the relaxation rate to the rate of change of the equilibrium expectation yields an emergent Kibble-Zurek timescale

$$t_{kz} \sim \tau_Q \lambda^{-\alpha z/(1+\alpha z)}, \quad a \equiv \frac{1}{1 - \alpha} \simeq a \simeq 1.12,$$  \hspace{1cm} (15)

which sets the timescale for the non-equilibrium evolution of the fluctuations. The Kibble-Zurek time is described more completely in Sect. III C.
Our final numerical result for the time evolution of $N^{s\bar{s}}$ when the system passes directly through the critical point ($t_{cr} = 0$) is shown in Fig. 3 of Sect. III D. When the system misses the critical $N^{s\bar{s}}(t,k)$ generally depends on the ratio of $t_{cr}$ and $t_{kz}$ leading to Fig. 4. Numerical estimates for the magnitude of $N^{s\bar{s}}$ and the correlation length are discussed in the conclusions.

II. TRANSITS OF THE CRITICAL POINT: EQUILIBRIUM

In this section we will analyze the equilibrium fluctuations of $\hat{s}$ close to critical point during a transit of the QCD critical point. Subsequently in Sect. III we will analyze the dynamics of the system to determine the corresponding non-equilibrium distribution $N^{s\bar{s}}$.

A. Mapping the QCD equation of state onto the Ising model

To map the QCD equation of state onto the Ising model, we need to relate the temperature and chemical potential in QCD to the temperature and magnetic field of Ising system. Alternatively we may work with extensive variables and map the energy and number densities of QCD to the energy density and magnetization of the Ising model. Since the time dependence of the QCD extensive variables have already been specified in Eqs. (11) and (12), the system’s trajectory in the Ising phase diagram is completely determined once this map is given.

The extensive thermodynamic variables in QCD phase diagram are denoted generically with $x^a$

$$x^a \equiv (e \ n),$$

while the corresponding thermodynamically conjugate variables are denoted with capital letters $X_a = -\partial s/\partial x^a$

$$X_a = (-\beta \ \hat{\mu}).$$

Here $\hat{\mu} = \mu/T$ and $\beta = 1/T$. Near the critical point the entropy can be written as a regular piece plus a singular piece\(^3\), $s = s_{reg} + s_{sing}$, where the regular piece is

$$s_{reg} = s_c + \beta_c \Delta e - \hat{\mu}_c \Delta n.$$  

Then from Eq. (17) the singular part of the entropy density satisfies

$$\Delta X_a = -\frac{\partial s_{sing}}{\partial x^a} = (-\Delta \beta \ \Delta \hat{\mu}),$$

with $\Delta \beta \equiv \beta - \beta_c$ etc, so that

$$ds_{sing}(x) = -\Delta X_a(x) \, dx^a.$$  

Equilibrium fluctuations in QCD are treated as in Ref. [27]. In each subsystem of volume $V$ which is large compared to the cube of the correlation length, the probability of a fluctuation $x^a \rightarrow x^a + \delta x^a$ is Gaussian and given by

$$P \propto e^{\Delta S(2)} \quad \Delta S(2) = -\frac{1}{2} V \, S_{ab}(x) \, \delta x^a \delta x^b,$$

\(^3\) Strictly speaking it is the free energy and not the entropy which may be clearly divided into regular and singular pieces. $s_{reg}$ and $s_{sing}$ are determined from the corresponding free energies with the relation $s = \beta p + \beta e - \hat{\mu} n$, where $e$ and $n$ are derivatives of the free energy with respect to $-\beta$ and $\hat{\mu}$.\n
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Here the matrix $S_{ab}$ is given by equilibrium thermodynamics

$$S_{ab}(x) = \frac{\partial X_a(x)}{\partial x^b} = -\frac{\partial^2 s(x)}{\partial x^a \partial x^b}.$$  

(22)

Finally if the $\delta x(r)$ is a function of space, the probability becomes a functional and takes the form

$$P[\delta x] \propto e^{\Delta S(2)}, \quad \Delta S(2) = -\frac{1}{2} \int d^3 r S_{ab}(x) \delta x^a(r) \delta x^b(r).$$  

(23)

The extensive variables in the Ising model (the energy density and the magnetization) are denoted generically with $x^A$, distinguished from QCD case by the uppercase index:

$$x^A \equiv (\epsilon \ \psi).$$  

(24)

Here $\epsilon \equiv (E - E_c)/T_c$ is the deviation of Ising energy density from the critical one relative to the Ising critical temperature $T_c$, while $\psi$ is the spin density (the order parameter). The thermodynamically conjugate variables are $X_A = -\partial s_{\text{sing}}/\partial x^A$

$$X_A = (r \ h),$$  

(25)

where $r = (T - T_c^\text{Ising})/T_c^\text{Ising}$ denotes the reduced temperature, and $h = H/T_c^\text{Ising}$ is the reduced magnetic field (see Appendix A). The singular part of the Ising entropy is

$$ds_{\text{sing,Is}}(x) = -X_A(x)dx^A,$$  

(26)

and the equilibrium quadratic functional reads

$$\Delta S(2) = -\frac{1}{2} \int d^3 r S_{AB}(x) \delta x^A(r) \delta x^B(r).$$  

(27)

The mapping between $x^a$ and $x^A$ or $X_a$ and $X_A$ is not universal but is analytic [26]. Therefore, in the vicinity of the critical point, $\Delta X_a$ and $X_A$ are related through a linear transformation specified by 2-by-2 matrix $M$:

$$X_A = \Delta X_b \tilde{M}_A^b,$$  

$$\tilde{M}_b^A = \frac{\partial X_A}{\partial X_b}. \quad (28)$$

Similarly, the extensive variable are related with a 2-by-2 matrix $M$

$$x^A = M^A_b \Delta x^b,$$  

$$M_b^A = \frac{\partial x^A}{\partial x^b}. \quad (29)$$

The matrices $M$ and $\tilde{M}$ are inverses of each other. Indeed, the probability of a fluctuation in the extensive QCD parameters $\delta \epsilon, \delta n$ must be the same as a corresponding fluctuation in $\delta \epsilon, \delta \psi$ in the Ising system in order to have universal behavior. The decrease in entropy per volume $\Delta S(2)$ due to a fluctuation must be the same in both systems:

$$\delta h \delta \psi + \delta r \delta \epsilon = \delta \bar{\mu} \delta n - \delta \beta \delta \epsilon$$  

(30)

i.e. $\delta X_A \delta x^A = \delta X_a \delta x^a$. From eq. (30) we see that $M$ and $\tilde{M}$ are inverse matrices of each other

$$M_b^A \tilde{M}_a^C = \delta^B_C.$$  

(31)
With this relation we also see that singular parts of the entropy differential \( ds_{\text{sing}} \) of the QCD and Ising systems agree.

Of the four parameters in the two-by-two matrix \( \hat{M} \) (or \( M \)), two of the parameters are just scale factors, while the remaining two parameters determine the directions of changing \( \tau \) and \( h \) in the QCD \( T, \mu \) plane. The line \( h = 0 \) is the coexistence line in the Ising system, and must correspond to the coexistence curve, \( T_{\text{cx}}(\mu) \), in the QCD phase diagram. Thus, knowledge of \( T_{\text{cx}}(\mu) \) places a constraint on the remaining two directional parameters of \( M \), which is found by setting \( dh = 0 \) (i.e., constant \( h \)) in Eq. (28)

\[
\frac{T'_{\text{cx}}}{1 - (\mu_c/T_c)T''_{\text{cx}}(\mu)} \left( \frac{1}{T_c} \hat{M}_h^\mu \right) = M^e_e ,
\]

or equivalently

\[
\frac{T'_{\text{cx}}}{1 - (\mu_c/T_c)T''_{\text{cx}}(\mu)} \left( \frac{1}{T_c} \hat{M}_n^\mu \right) = -\hat{M}_n^\mu .
\]

Following previous works [25], we ignore the \( \mu \) dependence of \( T_{\text{cx}}(\mu) \) and set \( T'_{\text{cx}}(\mu) = 0 \), and thus \( M^e_e = 0 \) and \( \hat{M}_n^\mu = 0 \). For maximum simplicity we will also take the direction of increasing \( h \) in the Ising model to correspond with the \( T \) direction of QCD by setting \( M^\mu_e = -M^\mu_n \mu_c \). With these choices, the map is determined by two positive dimensionless scale factors, \( (T_cM^\mu_e) \) and \( -M^\mu_n \), leading to the definition

\[
A_s \equiv (T_cM^\mu_e) , \quad A_n \equiv -M^\mu_n .
\]

The intensive parameters of the Ising model and QCD are related after elementary algebra

\[
r = -\frac{1}{A_n T_c} \Delta \mu ,
\]

\[
h = \frac{1}{A_s T_c} \Delta T .
\]

In terms of the extensive parameters, this means that the QCD entropy is proportional to the order parameter

\[
\epsilon = -A_n \Delta n ,
\]

\[
\psi = A_s \Delta s ,
\]

where we have used \( T_c \Delta s = \Delta \epsilon - \mu_c \Delta n \). Finally, as discussed more completely below, the Ising energy density and magnetization, \( (\epsilon, \psi) \), are determined up to two normalization constants, \( (\mathcal{M}_0 h_0, \mathcal{M}_0) \). These constants can always be adjusted by redefining the mapping parameters, and we will conventionally choose

\[
\mathcal{M}_0 h_0 \equiv n_c ,
\]

\[
\mathcal{M}_0 \equiv s_c ,
\]

so that the scale factors \( (A_n, A_s) \) are of order unity. Thus, our final specification for how \( (\epsilon, \psi) \) are related to \( (\Delta n, \Delta s) \) reads

\[
\frac{\epsilon}{\mathcal{M}_0 h_0} = -A_n \frac{\Delta n}{n_c} ,
\]

\[
\frac{\psi}{\mathcal{M}_0} = A_s \frac{\Delta s}{s_c} .
\]
Our conclusions will be largely independent of the precise form of the mapping between QCD and Ising model. What is important in what follows is that $A_n$ and $A_s$ are positive, dimensionless, and of order unity constants. Further Eq. (42) together with the time dependence of $\Delta n$ and $\Delta s$ given in Eq. (11) fully specify how the QCD system evolves in the Ising model plane as a function of time.

B. The QCD specific heat $C_p$ and the speed of sound near the critical point

Given the Ising equation of state and the corresponding states in the QCD medium, we may compute how the QCD specific heats and the speed of sound are related to the Ising susceptibilities near the critical point. As we will review, the critical behavior of the speed of sound and the specific heat at constant pressure, $C_p$, are independent of the details of the mapping matrix $M^{A_b}$. $C_p$ determines the fluctuations in the entropy per baryon $\hat{s}$, and is the most rapidly divergent equilibrium susceptibility near the QCD critical point.

1. The Ising model susceptibilities

The Ising model susceptibilities determine the fluctuations in the extensive quantities $x^A$, and are given by the matrix

$$G_{is}^{AB} = \frac{1}{V} \frac{\partial^2 \log Z_{\text{sing}}}{\partial X_A \partial X_B} \bigg|_{X_A=0} = V \left\langle \delta x^A \delta x^B \right\rangle. \quad (43)$$

The conventional names for the entries of this matrix are

$$G_{is}^{11} \equiv C_H,$$

$$G_{is}^{22} \equiv \chi_{is}, \quad (44)$$

$$\det G_{is}^{AB} \equiv \chi_{is} C_M,$$

$$C_M = G_{is}^{11} - \frac{(G_{is}^{12})^2}{G_{is}^{22}}, \quad (46)$$

where $C_H$ is the specific heat at constant magnetic field, and $C_M$ is the specific heat at constant magnetization. Straightforward algebra (see Appendix A for details) yields explicit expressions for these quantities in terms of the commonly used $R, \theta$ parametrization – see Eq. (204) in Appendix A. As seen from the appended expressions, the Ising susceptibility $\chi_{is}$ and specific heat $C_M$ diverge as

$$\chi_{is} \propto R^{-\gamma}, \quad \text{with} \quad \gamma = 1.24, \quad (47)$$

$$C_M \propto R^{-\alpha}, \quad \text{with} \quad \alpha = 0.11. \quad (48)$$

where $R \to 0$ near the critical point. From a perspective of heavy ion collisions, the critical exponent $\alpha$ is so small that it will probably never be observed, and we will focus on susceptibility $\chi_{is}$.

The inverse matrix determines the corresponding fluctuations of the intensive parameters

$$S_{is}^{AB} \equiv \left( G_{is}^{-1} \right)_{AB} = \frac{1}{\chi_{is} C_M} \begin{pmatrix} \chi_{is} & -G_{is}^{12} \\ -G_{is}^{12} & C_H \end{pmatrix} = V \langle \delta X_A \delta X_B \rangle, \quad (49)$$
which follows from the definition, \( X_A = -\partial S/\partial x^A \). We note the correlations between the extensive and intensive variables are simple

\[
V \langle \delta x^A \delta X_B \rangle = \delta^A_B, \tag{50}
\]

reflecting the relation, \( S^{is} = G^{is-1} \).

Finally, let us discuss the wavenumber dependence of the Ising correlation functions. Near the critical point the correlation function of magnetization,

\[
\langle \psi(k)\psi(k') \rangle \equiv \chi_{is}(k) (2\pi)^3 \delta^{(3)}(k - k'), \tag{51}
\]

takes the form

\[
\chi_{is}(k) = \chi_{is} K_{\chi}(k\xi), \tag{52}
\]

where \( K_{\chi}(k\xi) \) is a static universal function with unit normalization\(^4\), \( K_{\chi}(0) = 1 \). \( K_{\chi} \) has been studied extensively [31], and for \( k \gg \xi^{-1} \) takes the asymptotic form

\[
\chi_{is} K_{\chi}(k\xi) = \frac{C_\infty}{k^{2-\eta}}, \tag{53}
\]

where \( \eta \simeq 0.036 \) is the critical exponent, and the constant \( C_\infty \) is independent of \( \xi \). We will use the simple Ornstein-Zernicke form [26]

\[
K_{\chi}(k\xi) = \frac{1}{1 + (k\xi)^{2-\eta}}, \tag{54}
\]

which has the correct limits for \( k \ll \xi^{-1} \), and \( k \gg \xi^{-1} \).

2. The QCD susceptibilities

The corresponding QCD susceptibility matrices are

\[
G^{ab} = \frac{1}{V} \frac{\partial^2 \log Z_{sing}}{\partial X_a \partial X_b}, \quad S_{ab} \equiv (G^{ab})^{-1}, \tag{55}
\]

which determine the QCD fluctuations \( \langle \delta x^a \delta x^b \rangle \), and \( \langle \delta X_a \delta X_b \rangle \) respectively. The matrix \( G^{ab} \) determines the speed of sound \( c_s^2 \) and the fluctuations in the entropy per baryon as we review below.

To write down the formulas relating the speed of sound to \( G^{ab} \), we define derivatives of the pressure

\[
p^a \equiv \frac{\partial p}{\partial X_a}, \quad (p^e, p^n) = \left(-\frac{\partial p}{\partial \beta}, \frac{\partial p}{\partial \hat{\mu}}\right) = \left(\frac{w}{\beta}, \frac{n}{\beta}\right), \tag{56}
\]

and then the speed of sound, \( c_s^2 = (\partial p/\partial e)_{n/s} \), is given by

\[
c_s^2 = \left(\frac{\partial p}{\partial e}\right)_n + \frac{n}{w} \left(\frac{\partial p}{\partial n}\right)_e, \tag{57a}
\]

\[
= \frac{\beta}{w} p^a S_{ab} p^b. \tag{57b}
\]

\(^4\) In principle, \( K_{\chi} \) will be different inside and outside coexistence regime [30]. While including such dependence is straightforward, we will neglect this refinement in the current study.
As usual, $w \equiv e + p$ is the enthalpy density. From this expression we see that fluctuations in the pressure $\delta p = p^a \delta X_a$ determine the speed of sound

$$V \langle \delta p^2 \rangle = p^a S_{ab} p^b = \frac{wc_s^2}{\beta}. \quad (58)$$

The fluctuations in the entropy per baryon will play a central role in what follows, and thus we define

$$\delta \hat{s} \equiv n \delta \left( \frac{s}{n} \right) = \delta s - \frac{s}{n} \delta n. \quad (59)$$

The fluctuations in $\hat{s}$ can be written in terms of $\delta e$ and $\delta n$

$$T \delta \hat{s} = \delta e - \frac{w}{n} \delta n, \quad (60)$$

and are uncorrelated with the fluctuations in the pressure

$$\langle \delta p \delta \hat{s} \rangle = 0, \quad (61)$$

which can be derived from Eq. (50) and Eq. (56). A more complete discussion of this and the thermodynamic relations in the rest of this section is given in Refs. [26, 27]. The fluctuations in $\hat{s}$ are determined by the specific heat at constant pressure, $C_p \equiv nT \left( \frac{\partial (s/n)}{\partial T} \right)_p$, via

$$N^{\hat{s}\hat{s}} \equiv V \langle (\delta \hat{s})^2 \rangle = V \left\langle \left( \delta \hat{s} - \frac{s}{n} \delta n \right)^2 \right\rangle = C_p. \quad (62)$$

Straightforward analysis shows that $C_p$ is related to determinant of the susceptibility matrix

$$\left( \frac{nT}{w} \right)^2 C_p = \frac{\beta c_s^2}{w} \det G^{ab}. \quad (63)$$

The specific heat at constant pressure is also related to the specific heat at constant volume $C_V \equiv T(\partial s/\partial T)_n$ through the familiar relation

$$\left( \frac{nT}{w} \right)^2 C_p = T \frac{\partial n}{\partial \mu} \left( \frac{TC_V c_s^2}{w} \right). \quad (64)$$

In the low density limit, $n \rightarrow 0$, the final factor on the r.h.s. approaches unity, $(TC_V c_s^2/w) \rightarrow 1$. Eq. (64) leads to an important relation, Eq. (121) below, between the baryon number diffusion coefficient and the diffusion coefficient of $\hat{s}$.

In practice, both theoretically and experimentally, it is easier to work with the correlation function of $\hat{s}$ rather than fluctuations of $\hat{s}$ in a finite volume $V$

$$N^{\hat{s}\hat{s}}(t, k) \equiv \int d^3 x e^{ik \cdot (x - y)} \langle \delta \hat{s}(t, x) \delta \hat{s}(t, y) \rangle. \quad (65)$$

In equilibrium, Eq. (62) predicts that $N^{\hat{s}\hat{s}}(t, k)$ approaches $C_p$ as $k \rightarrow 0$. 

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We have now specified how the speed of sound and specific heats are related to the QCD susceptibility matrix $G^{ab}$. The QCD susceptibilities are related to the corresponding Ising quantities with the mapping matrices of Sect. II A.

\begin{equation}
G^{ab} = \bar{M}^a_A \bar{M}^b_B G^{AB}_{is}.
\end{equation}

As we will now review, near the critical point the speed of sound, $c_s^2$, approaches zero as $C^{-1}_M \propto R^\alpha$, while the specific heat, $C_p$, diverges as $\chi_{is} \propto R^{-\gamma}$ [26]. This is independent of the details of the mapping matrix $M^a_A$. From a practical perspective this means that the softening of the equation of state near the critical point will probably be too small to observe (since $\alpha$ is small), and the experimental heavy ion program should focus on the fluctuations in $s/n$ which reflects the diverging value of the specific heat $C_p \propto \chi_{is}$.

To review how the speed of sound behaves near the Ising critical point, we first note that by inserting unity of the form $\bar{M}^a_A \bar{M}^b_B \epsilon^n_n = \delta^a_c$ into Eq. (58), we can express the speed of sound near the critical point as

\begin{equation}
c^2_s = \frac{\beta}{w} p^n A S_{AB} p^B \simeq \frac{\beta}{w} \left( \frac{\partial p}{\partial r} \right)^2 \frac{1}{C_M},
\end{equation}

where we define $p^n = (\partial p/\partial X^A)$ and thus $(\partial p/\partial r)_h = M^n_r p^n + M^r_r p^r$, is derivative of the QCD pressure in the direction of reduced Ising temperature. We note that $(\partial p/\partial r)$ remains finite near the critical point. In approximating Eq. (67), we recognized that near the critical point $\chi_{is}$ is strongly divergent, and thus the $rr$ component in $p^n A S_{AB} p^B$ dominates the sum. This shows (as claimed) that the speed of sound approaches zero like the Ising specific heat $C^\alpha_M$, i.e. as $R^\alpha$. In the case of the simple mapping described in Sect. II A we have

\begin{equation}
c^2_s = \frac{\beta (T_c n_c A^n)}{w C_M^2}.
\end{equation}

In the rest of this paper we will focus on the specific heat $C_p$ which exhibits a much more dramatic behavior, diverging as $R^{-\gamma}$ near the critical point.

The behavior of $C_p$ near the critical point is determined by the determinant in Eq. (63) and the relation between the determinants of the QCD and Ising systems

\begin{equation}
det G^{ab} = (\det \bar{M})^2 \det G^{AB}_{is}
\end{equation}

Thus since $\det G^{AB}_{is} = \chi_{is} C_M$, we find with Eqs. (63) and (67) that

\begin{equation}
C_p = \frac{\left( \frac{1}{T_c n_c \partial p} \right)^2}{(T_c \det \bar{M})^2} \chi_{is}.
\end{equation}

The factors $(T_c \det \bar{M})$ and $(\partial p/\partial r)/n_c T_c$ are both dimensionless and of order unity. Thus, independently of the details between the QCD and Ising variables, the specific heat $C_p$ is proportional to the Ising susceptibility $\chi_{is}$ and diverges as $R^{-\gamma}$. For the simple mapping of Sect. II A the specific heat takes the particularly simple form

\begin{equation}
C_p = \frac{\chi_{is}}{A_s^2},
\end{equation}

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which we will assume in what follows.

Finally, later we will study the correlation function $N^{\hat{s}\hat{s}}(t,k)$ as a function of $k$. In equilibrium, this will take the form

$$N^{\hat{s}\hat{s}}_0(t,k) = \frac{\chi_{\hat{s}\hat{s}}}{A^2_0 \left[ 1 + (k\xi)^2 - \eta \right]},$$

where we have adopted for simplicity Ornstein-Zernicke form, which has the properties discussed in Sect. II B 1.

At this point we need to determine how the parameters $\chi_{\hat{s}\hat{s}}(t)$ and $\xi(t)$ depend on time when $\Delta n$ and $\Delta s$ follow the adiabatic trajectory parametrized by Eq. (11). We will turn to this task in the next section.

C. The timescale for the scaling regime during a transit of the QCD critical point

We have now specified how the extensive Ising variables $(\epsilon, \psi)$ are determined by the QCD quantities $(\Delta n, \Delta s)$ with Eq. (42). We also have specified how the extensive QCD quantities depend on time in Eq. (11). Finally, the Ising equation of state determines the time dependence of the corresponding susceptibilities and correlation lengths, from the time dependent extensive Ising variables. In this section we will show how the scaling form of the Ising equation of state leads to a characteristic scaling form in time for these quantities.

Outside of the coexistence region, the scaling of the Ising equation of state implies the following scaling forms for the extensive variables $(\epsilon, \psi)$ as a function of $(r, h)$

$$\epsilon = M_0 h_0 |r|^{1-\alpha} f_\epsilon(z),$$

$$\psi = M_0 |r|^\beta f_\psi(z),$$

Here $(M_0 h_0, M_0) \equiv (n_c, s_c)$ are two (conventional) constants described above, and below $f_X(z)$ denotes a generic universal scaling function of the variable $z \propto r/|h|^{1/\delta}$ (see Eq. (193) in Appendix A for a complete definition of $z$.) All susceptibilities and correlation lengths take this generic form, and no additional constants need to be introduced. In practice, given $(\epsilon, \psi)$ we numerically determine $(R, \theta)$ from the Ising parametrization described in Appendix A, and then evaluate all other thermodynamic functions.

As $z \to z_0 \equiv -\infty$, the system approaches the coexistence region, and $f_\epsilon(z)$ and $f_\psi(z)$ approach $-1$ and $1$ by convention. Inside the coexistence region the energy density is related to the temperature by

$$\epsilon = -M_0 h_0 |r|^{1-\alpha},$$

and the magnetization lies in the range $(-\psi_0, \psi_0)$ where

$$\psi_0 = M_0 |r|^{\beta}.$$

These expressions for the extensive quantities in terms of the intensive ones may be inverted. We define a new scaling variable based on extensive variables

$$u \equiv \frac{\epsilon}{M_0 h_0 \left( \frac{M_0}{|\psi|} \right)^b},$$

A handy Mathematica notebook which evaluates all universal Ising thermodynamic variables and correlation lengths is made available as part of this work.
where $b = (1 - \alpha)/\beta \simeq 2.7$, and then outside the coexistence region

\begin{equation}
  r = \left( \frac{|\epsilon|}{M_0 h_0} \right)^a f_r(u),
\end{equation}

\begin{equation}
  z = f_z(u),
\end{equation}

with $a = 1/(1 - \alpha) \simeq 1.12$. The system is in the coexistence region for $u < -1$.

The advantage of a scaling variable based on extensive quantities is that the extensive quantities depend on time in a simple way. Indeed, the scaling variable $u$ is approximately linear in time

\begin{equation}
  u = \frac{A_n t/\tau_Q}{(A_s (\Delta_s - t/\tau_Q))^b} \simeq \frac{A_n}{A_s^b} \frac{t}{\tau_Q \Delta_s^b}.
\end{equation}

In the last step, we recognized that in order to see the detailed scaling structure in the equation of state (which is parametrized by $f_r(u)$ in (77)), we must have $|u| \sim |z| \sim |\theta| \sim 1$. For $|u| \sim 1$, $|t/\tau_Q| \sim \Delta_s$ and is small compared $\Delta_s$ in this regime. From the last equality of Eq. (79), the system crosses the detailed scaling regime over a time period of order

\begin{equation}
  t_{cr} \sim \tau_Q \Delta_s^b.
\end{equation}

Parametrically outside of this time window the scaling functions such as $f_r(u)$ may be treated as constants. Inside of this time window the QCD parameters are of order

\begin{equation}
  \frac{\Delta_s}{s_c} \sim \Delta_s, \quad \frac{\Delta_n}{n_c} \sim \Delta_s^b.
\end{equation}

Accordingly, in Fig. 1(c) we have rescaled the $x$ and $y$ axis by $\Delta_s^b$ and $\Delta_s$, which flattens the 45o trajectory lines in Fig. 1(b). It is only in this regime that the detailed scaling structure of the Ising equation of state (as recorded by the $(R, \theta)$ parametrization) is really necessary.

To simplify notation we absorb the mapping constants into the definition of the parameters defining

\begin{equation}
  \tau_Q \equiv \frac{\tau_Q}{A_n},
\end{equation}

\begin{equation}
  \Delta_s \equiv A_s \Delta_s.
\end{equation}

The crossing time is defined as the time when the system leaves the coexistence region (see Fig. 1(c))

\begin{equation}
  t_{cr} \equiv -\tau_Q \Delta_s^b, \quad \text{with} \quad u = \frac{t}{|t_{cr}|},
\end{equation}

so $u = -1$ corresponds to $t = t_{cr}$. The Ising energy and order parameter have a simple time dependence

\begin{equation}
  \epsilon = M_0 h_0 \frac{t}{\tau_Q}, \quad \psi = M_0 \Delta_s.
\end{equation}

The scaling of the Ising susceptibility and other thermodynamic quantities with with $\epsilon$ and $u$ imply a specific scaling in time. For instance, using the Ising parametrization in Appendix A, the susceptibility behaves as

\begin{equation}
  \chi_{is} = \chi_0 \left( \frac{|\epsilon|}{M_0 h_0} \right)^{-a\gamma} f_\chi(u),
\end{equation}

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where
\[ \chi_0 \equiv 0.365 \frac{s_c^2}{n_c}. \] (87)
is the typical size of \( C_p \) away from the critical point, and we recall that \( s_c^2/n_c = M_0/h_0 \). The scaling function is continuous and takes the form
\[ f_\chi (u) = \begin{cases} 1 & u < -1 \\ f_\chi (u) & u > -1 \end{cases}, \] (88)
with limiting values
\[ f_\chi (-1) = 1, \quad f_\chi (u) \xrightarrow{u \to \infty} f_\chi^+ \equiv 1.954. \] (89)
The combination \(|u|^{-a\gamma} f_\chi (u)\) is regular and decreasing for \( u > -1 \). Thus, the equilibrium susceptibility as a function of time takes the following form
\[ \chi_{is} = \chi_0 \left( \frac{|t|}{\tau_Q} \right)^{-a\gamma} f_\chi \left( \frac{t}{|t_{cr}|} \right), \] (90a)
which can be written as function \( t/|t_{cr}| \) using Eq. (84)
\[ \chi_{is} = \chi_0 \left( \frac{|\epsilon|}{M_0 h_0} \right)^{-a\nu} f_\xi \left( \frac{t}{|t_{cr}|} \right). \] (90b)
Eq. 90b is plotted in Fig. 2(a). To evaluate \(|u|^{-a\nu} f_\xi (u)\) in practice, we determine the \((R, \theta)\) associated with \((\epsilon, u)\) numerically – see Appendix A.

The correlation length follows a similar pattern. The equilibrium correlation length in the Ising model takes the scaling form (see Appendix A)
\[ \xi(t) = \ell_o \left( \frac{|\epsilon|}{M_0 h_0} \right)^{-a\nu} f_\xi (u), \] (91)
where
\[ \ell_o \equiv 0.365 n_c^{-1/3}, \] (92)
is of order the inter-particle spacing, and we recall that \( M_0 h_0 = n_c \). The limiting values of the analogous scaling function \( f_\xi (u) \) are
\[ f_\xi (-1) = 1, \quad f_\xi (u) \xrightarrow{u \to \infty} f_\xi^+ \equiv 1.222, \] (93a)
and \(|u|^{-a \nu} f_\xi (u)\) is regular and decreasing for \( u > -1 \). The equilibrium correlation length as a function of time takes form
\[ \xi(t) = \ell_0 \left( \frac{|t|}{\tau_Q} \right)^{-a\nu} f_\xi \left( \frac{t}{|t_{cr}|} \right) \] (94a)
or after using the definition of \( t_{cr} \) (Eq. (84))
\[ \xi(t) = \ell_0 \left( \frac{|t|}{\tau_Q} \right)^{-\nu/\beta} f_\xi \left( \frac{t}{|t_{cr}|} \right). \] (94b)
Eq. 94b is plotted in Fig. 2(b). To evaluate \(|u|^{-a\nu} f_\xi (u)\) in practice we use the numerical data on the Ising model from Engels, Fromme and Seniuch [32] – see Appendix A.
FIG. 2. The Ising susceptibility and correlation length as a function of time during a transit of the QCD critical point along an adiabatic trajectory characterized by $\bar{\Delta}_s$. The time axis has been rescaled by $t_{cr} \sim \tau_Q \Delta_s^b$, see Eq. (84). The $y$ axes have been rescaled by an appropriate power of $\Delta_s$ so that the curve is independent of $\Delta_s$.

D. Summary of the equilibrium expectation

To conclude this section let us collect and review the equilibrium formulas. $N_{\hat{s}\hat{s}}(t, k)$ in equilibrium takes the approximate form, from Eqs. (71) and (72),

$$N_{\hat{s}\hat{s}}(t, k) = \frac{1}{A_s^2} \frac{\chi_{\text{Is}}(t)}{1 + (k\xi(t))^2-\eta}, \quad \text{(95)}$$

where $A_s$ is a constant determined by the mapping between QCD and the Ising model. The specific heat and equilibrium correlation length are universal functions of time as shown in Fig. 2, and the timescale for their evolution is set by $t_{cr} \sim \tau_Q \Delta_s^b$. In the next section we will describe how the system evolves according to stochastic hydrodynamics, and tries to approach this time dependent equilibrium expectation.

III. TRANSITS OF THE CRITICAL POINT: DYNAMICS

The primary purpose of this work is to discuss the fluctuations of thermodynamic variables (e.g. $e, n$) for a system transiting close to the QCD critical point. Specifically, we will focus on the time evolution of the correlation functions of the thermodynamic variables, which quantify the fluctuations with a specific wave number $k$. In the previous section, we have analyzed the equilibrium behavior of these correlations, and now we will study their dynamical evolution.

We first determine this evolution in the hydrodynamic regime, $k \ll \xi^{-1}$. To this end, we start from fluctuating hydrodynamics, and derive a set of relaxation equations for the correlations, which we refer to as the hydro-kinetic equations [11]. In the previous section, we
showed that critical fluctuations are more enhanced in the $\hat{s}$ mode than in any other combination of thermodynamic variables. When we apply the hydro-kinetic equations (Eq. (122) below) to a system near a critical point, we find that the equilibration of the $\hat{s}$ correlator $N^{\hat{s}\hat{s}}$ is independent of the other hydrodynamic modes, allowing us to focus on $N^{\hat{s}\hat{s}}$.

The description of $N^{\hat{s}\hat{s}}$ near a critical point, even in equilibrium, involves an additional length scale. As we have seen in Eq. (95), the behavior of $N^{\hat{s}\hat{s}}$ in equilibrium exhibits a non-trivial dependence on the wavenumber $k$, and such dependence is characterized by the correlation length $\xi$. To model the off-equilibrium evolution of $N^{\hat{s}\hat{s}}$ in the scaling region, we need to extend the hydro-kinetic equations to larger $k$, $\xi^{-1} \ll k \ll \ell_0^{-1}$. This is done schematically in Sect. III B – see Eq. (129). It should be made clear that Eq. (129) is simply a rough model we will use to describe the dynamics of $N^{\hat{s}\hat{s}}$ in the scaling regime, and we defer a systematic treatment to future work. In Sect. III C, we estimate the characteristic time and length scales of $N^{\hat{s}\hat{s}}$. Finally, we evaluate $N^{\hat{s}\hat{s}}$ numerically by solving Eq. (129) numerically to determine the time evolution fluctuations during a transit of the critical point.

A. The evolution of fluctuations for a fluid with finite baryon density

1. The derivation of hydro-kinetic equations

We begin by considering the fluctuations around a uniform static fluid background of the extensive thermodynamic variables $e(t, x) = e + \delta e(t, x)$, $n(t, x) = n + \delta n(t, x)$, and momentum $\vec{g}(t, x) \equiv w\vec{u}(t, x)$, where $\vec{u}(t, x)$ denotes the fluid velocity. In $k$-space, the fluctuations of longitudinal momentum $g \equiv \vec{g} \cdot \hat{k}$ will mix with $\delta e, \delta n$ at finite density, and we will denote them collectively as\(^6\):

$$\delta x^a \equiv (\delta e, \delta n, g) .$$

Transverse components of the momentum, $\vec{g}_T \cdot \hat{k} = 0$, decouple from $\delta x^a$ modes in the linear regime (see Eq. (100) below).

We are interested in the equal-time correlation function $N^{\hat{a}\hat{b}}(t, \vec{k})$ in $k$-space:

$$\langle \delta x^\hat{a}(t, \vec{k}) \delta x^\hat{b}(t, -\vec{k}') \rangle \equiv (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}') N^{\hat{a}\hat{b}}(t, \vec{k}) ,$$

The equilibrium values of $N^{\hat{a}\hat{b}}$, namely $N^{\hat{a}\hat{b}}_0$, are given by the susceptibility matrix:

$$N^{\hat{a}\hat{b}}_0 = (S^{\hat{a}\hat{b}})^{-1} ,$$

where

$$S^{\hat{a}\hat{b}} = \begin{pmatrix} S_{ee} & S_{en} & 0 \\ S_{ne} & S_{nn} & 0 \\ 0 & 0 & \frac{\beta}{w} \end{pmatrix} ,$$

and where $S_{ee}, S_{en}, S_{nn}$ are defined in Eq. (22).

\(^6\) The bar in $x^a$ and $X^a$ indicate that the longitudinal momentum and velocity are appended to the set $x^a$ and $X^a$ defined in Sect. II A.
In order to derive a relaxation equation for $N^{ab}(t, k)$, we consider the linearized stochastic hydrodynamic equations in the $k$-space:

\[
\begin{align*}
\frac{\partial}{\partial t} \delta e(t, k) &= -i \vec{k} \cdot \vec{g}, \\
\frac{\partial}{\partial t} \delta n(t, k) &= -\frac{n}{w} i \vec{k} \cdot \vec{g} - \frac{\eta k^2}{w} \delta m - \xi_n, \\
\frac{\partial}{\partial t} \bar{g}(t, k) &= -i \vec{k} \delta p - \frac{\eta k^2}{w} g - \frac{\zeta + \frac{1}{3} \eta}{w} \bar{k} \cdot \vec{g} - \bar{\xi}.
\end{align*}
\]

The noise terms are introduced above to describe dynamics of hydrodynamic fluctuations\(^7\), and the noise correlations are constrained by the fluctuation-dissipation theorem (see for example Ref. [27]):

\[
\begin{align*}
\langle \xi_i(t, k) \xi^i(t', -k') \rangle &= 2T \left[ \eta k^2 \delta^{ij} + \left( \zeta + \frac{1}{3} \eta \right) k^i k^j \right] (2\pi)^3 \delta^{(3)}(k - k') \delta(t - t'), \\
\langle \xi_n(t, k) \xi_n(t', -k') \rangle &= 2T \lambda_B k^2 (2\pi)^3 \delta^{(3)}(k - k') \delta(t - t'), \\
\langle \xi_i(t, k) \xi_n(t', -k') \rangle &= 0.
\end{align*}
\]

As usual, shear viscosity, bulk viscosity and baryon conductivity are denoted by $\eta, \zeta, \lambda_B$ respectively.

From the hydrodynamic equation (100), we write the equation for $x^{\bar{a}}$ in a compact fashion:

\[
\begin{align*}
\frac{\partial}{\partial t} \delta x^{\bar{a}}(t, k) &= -i k L^{\bar{a}b} \delta X_b + k^2 \Lambda^{\bar{a}b} \delta X_b + \xi^{\bar{a}}, \\
&= -i k L^{\bar{a}b} \delta x^b + k^2 D^{\bar{a}b} \delta x^b + \xi^{\bar{a}},
\end{align*}
\]

with noise correlator

\[
\langle \xi^{\bar{a}}(t, k) \xi^{\bar{b}}(t', -k') \rangle = 2k^2 \Lambda^{\bar{a}b} (2\pi)^3 \delta^{(3)}(k - k') \delta(t - t').
\]

Here the matrices are

\[
L^{\bar{a}b} = \begin{pmatrix} 0 & 0 & p^e \\ 0 & 0 & p^n \\ p^e & p^n & 0 \end{pmatrix}, \quad \Lambda^{\bar{a}b} = T \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda_B & 0 \\ 0 & 0 & \zeta + \frac{1}{3} \eta \end{pmatrix}.
\]

with $(p^e, p^n) = (w/\beta, n/\beta)$ defined in Eq. (56). Generalizing the discussion in Sect. II A, we have introduced conjugate variables through the relation $\delta X_a = S_{ab} \delta x^b$

\[
X_{\bar{a}} \equiv \begin{pmatrix} -\beta, \hat{m}, \frac{\beta g}{w} \end{pmatrix}.
\]

In the second line of Eq. (102), we have further defined:

\[
\begin{align*}
L^{\bar{a}} \equiv L^{\bar{a}b} S_{b\bar{e}}, \\
D^{\bar{a}} \equiv \Lambda^{\bar{a}b} S_{b\bar{e}}.
\end{align*}
\]

---

\(^7\) We use the Landau fluid frame throughout, and therefore the noise is absent in the first equation of Eq. (100).
By carefully averaging out the noise, we obtain the following equation for $N^{\alpha\beta}$ from Eq. (102)

$$\frac{\partial}{\partial t} N(t, \kappa) = -ik(L \cdot N - N \cdot L^T) - k^2(D \cdot N + N \cdot D^T) + 2k^2 \Lambda$$

$$= -ik(L \cdot N - N \cdot L^T) - k^2(D \cdot N + N \cdot D^T) + k^2(D \cdot N_0 + N_0 \cdot D^T),$$

where in the second line of Eq. (107), we have used the relation (106) and $N_0 = S^{-1}$. The last term on the R.H.S. of Eq. (107) arises from the noise $\xi^\alpha$ and acts as a source. The correlations will propagate and dissipate, as described by the first and second terms on the R.H.S. of Eq. (107) respectively. When $N = N_0$, the propagation term vanishes, i.e. $L \cdot N_0 - N_0 \cdot L = 0$, and the last two terms on the R.H.S of Eq. (107) balance with each other. Therefore, $N_0$ is a static solution to Eq. (107) as it should be.

Following Ref. [11] and for later convenience, we will consider the fluctuations in $\delta x^{(\alpha)}$, which is given by a specific linear combination of $\delta x^\alpha$, namely $\delta x^{(\alpha)} \equiv \delta x^\alpha e^{(\alpha)}_\bar{a}$. Here $e^{(\alpha)}_\bar{a}$ is defined as the left eigenvectors for the non-hermitian matrix $L$:

$$\sum_{\bar{a}} e^{(\alpha)}_\bar{a} L^\bar{a}_b = \lambda^\alpha e^{(\alpha)}_b, \quad \sum_{\bar{b}} L^\bar{a}_b e^{(\alpha)}_\bar{b} = \lambda^\alpha e^{(\alpha)}_\bar{a},$$

(108)

where $\lambda^\alpha$ are corresponding eigenvalues, and where we have also introduced right eigenvectors $e^{(\alpha)}_\alpha$. Here $e^{(\alpha)}_\alpha$ and $e^{(\alpha)}_\bar{a}$ satisfy the orthogonality relations:

$$\sum_{\bar{a}} e^{(\alpha)}_\bar{a} e^{(\beta)}_{\bar{a}} = \delta^{\alpha}_{\beta}, \quad \sum_{\alpha} e^{(\alpha)}_\alpha e^{(\alpha)}_b = \delta^b_\bar{a}. 

(109)$$

Consequently, $L^\alpha_{\beta}$ is diagonalized as

$$L^\alpha_{\beta} \equiv e^{(\alpha)}_\bar{a} L^\bar{a}_b e^{(\beta)}_\bar{b} = \lambda^\alpha \delta^{\alpha}_{\beta}. 

(110)$$

We denote the three eigen-modes by $\alpha = +, -, \hat{s}$ for reasons which will become obvious shortly. In what follows, we will consider the correlation functions of those modes:

$$\langle \delta x^{\alpha}(t, \kappa)\delta x^{\beta}(t, -\kappa') \rangle \equiv (2\pi)^3 \delta^{(3)}(\kappa - \kappa')N^{\alpha\beta}(t, \kappa).$$

(111)

To better understand the physical meaning of $\delta x^{(\alpha)}$, we write down the eigenvalues

$$\lambda^\pm = \pm c_s, \quad \lambda^\hat{s} = 0,$$

(112)

and specific form of the eigenvectors:

$$e^{(\pm)} = \frac{1}{\sqrt{2}} \left( \frac{1}{w} \pm c_s \right), \quad e^{(s)} = \frac{nT}{c_s^2 w} \left( \frac{\partial p}{\partial n}, -\frac{\partial p}{\partial e}, 0 \right),$$

(113a)

$$e^{(\pm)} = \frac{1}{\sqrt{2}c_s^2} \left( \frac{\partial p}{\partial e}, \frac{\partial p}{\partial n}, \pm c_s \right), \quad e^{(s)} = \left( \frac{1}{T}, -\frac{w}{nT}, 0 \right).$$

(113b)

Consequently,

$$\delta x^{(\pm)} = \frac{1}{\sqrt{2}c_s^2} (\delta p \pm c_s g), \quad \delta x^{(s)} = \delta \hat{s}. \quad (114)$$

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It should be clear now that those two modes with eigenvalues $\pm cs$ correspond to two propagating sound modes, and the mode with zero eigenvalue is identical to the $\hat{s}$ mode. To find the equilibrium variances of these fluctuations we evaluate $N^\alpha_0 = \bar{e}_a^{(\alpha)} N_0^{\hat{s} \hat{s}} \bar{e}_b^{(\beta)}$ and find the non-zero components

$$N^+_0 = N^-_0 = \frac{w}{\beta c_s^2}, \quad N^\hat{s} \hat{s} = C_p,$$

which should be compared with Eqs. (58), (61), and (62) of the previous section. Note that the fluctuations of $\hat{s}$ are uncorrelated with the pressure fluctuations $\delta x^{(\pm)}$.

We can now determine the dynamical equation for $N^{\alpha\beta}$ by expressing Eq. (107) in the eigen-basis of $L$, after defining the matrix elements

$$D^\alpha_\beta \equiv \bar{e}_a^{(\alpha)} D_{\bar{b}}^{\hat{s}} \bar{e}_b^{(\beta)},$$

In the eigen-basis of $L$ the diagonal components $(L \cdot N - N \cdot L)^{\alpha\alpha}$ vanish, and $N^{\alpha\alpha}$ will dissipate but will not oscillate as a function of time. By contrast, the off-diagonal components of $(L \cdot N - N \cdot L)^{\alpha\beta}$ are found to be proportional to $c_s$, and rotate rapidly. This observation allows us to neglect off-diagonal components of $N^{\alpha\beta}$ and to focus on the evolution of $N^{\alpha\alpha}$. This kinetic (or WKB) approximation to the linearized hydrodynamic wave equations is described in greater detail in Refs. [11, 33]. Taking the diagonal components of Eq. (107), we find:

$$\partial_t N^{\alpha\alpha}(t, k) = -2D_{\alpha} k^2 \left( N^{\alpha\alpha}(t, k) - N^{\alpha\alpha}_0 \right),$$

where we have used the fact that $N^{\alpha\beta}_0$ is a diagonal matrix. The diffusion coefficients $D_{\alpha} \equiv D^\alpha_\alpha$ can be found by explicit calculation:

$$D_{\pm} = \frac{1}{2} \left[ \frac{\lambda_B}{wc_s^2} \left( \frac{\partial p}{\partial n} \right)_e + \frac{1}{w} \left( \z + \frac{4}{3} \eta \right) \right],$$

$$D_{\hat{s}} = \frac{T \lambda_B}{(nT/w)^2 C_p}.$$

It is useful to define the thermal conductivity $\lambda_T$ with a Franz-Wiedemann type relation

$$\lambda_T \equiv \frac{T \lambda_B}{(nT/w)^2},$$

so that

$$D_{\hat{s}} = \frac{\lambda_T}{C_p}.$$

Eq. (117) extends the hydro-kinetic equations of a charge-neutral fluid [11] to finite baryon density (see also Refs. [18, 28]). The equilibration rate of $N^{\hat{s} \hat{s}}$ is controlled by diffusion coefficient $D_{\hat{s}}$ in Eq. (120). Using the thermodynamic relation, Eq. (64), and the definition of the baryon number diffusion coefficient, $D_B = \lambda_B / (\partial n / \partial \mu)_T$, we can relate $D_{\hat{s}}$ to $D_B$

$$D_{\hat{s}} = D_B \left( \frac{w}{TC_v c_s^2} \right).$$
The coefficient in parenthesis approaches unity as \( n \to 0 \), and is never far from unity for the baryon densities explored at RHIC. Thus, \( D_\delta \) can be estimated from the baryon diffusion coefficient, \( D_B \).

So far, we have derived a kinetic equation (117) which describes the evolution of fluctuations around a uniform static background. We now sketch the steps needed to extend our analysis to an expanding hydrodynamic background, referring to the literature for a more complete treatment [11]. First, we need to take into account that \( N_0^{\alpha\alpha}(t) \) as well as \( D_\alpha(t) \) will in general depend on \( t \). Second, we have to introduce gradient terms which account for the expansion of the system. The explicit expression of such gradient terms is not important for the subsequent discussion. What is important, though, is that these terms are proportional to \( 1/\tau_Q \), where \( \tau_Q \) is the expansion rate we introduced earlier. Therefore in an expanding fluid background, the hydro-kinetic equation takes the form (schematically)

\[
\partial_t N^{\alpha\alpha}(t, k) = -2D_\alpha(t) k^2 [N^{\alpha\alpha}(t, k) - N_0^{\alpha\alpha}(t)] + [\text{terms } \propto 1/\tau_Q].
\] (122)

The dynamics of \( N^{\alpha\alpha} \) as described by Eq. (122) is driven by the competition between the expansion of the system and the equilibration of thermal fluctuations. Since the equilibration of \( N^{\alpha\alpha} \) is achieved by diffusion with rate \( \propto D k^2 \), \( N^{\alpha\alpha} \) will depend non-trivially on wavelength, although the equilibrium expectation \( N_0^{\alpha\alpha} \) is \( k \)-independent. Away from the critical point, we can estimate a non-equilibrium length scale, \( \ell_{\text{neq}} \sim \ell_{\text{max}} \), which divides the non-equilibrium and equilibrium fluctuations of the system, characterizing the transition between the two regimes. Wavelengths longer than \( \ell_{\text{max}} \) are too long to equilibrate by diffusion over a time \( \tau_Q \). Recalling the introduction, we parametrize the diffusion constant away from the critical point as \( D_0 \sim \ell_0^2 / \tau_0 \),

\[
D_0 \sim \frac{\ell_0^2}{\tau_0},
\] (123)

where \( \tau_0 \) is the microscopic relaxation time, and \( \ell_0 \) is a microscopic length. Equating the diffusion rate of a mode of wavenumber \( k \sim 1/\ell_{\text{max}} \) with the expansion rate \( \sim 1/\tau_Q \)

\[
D_0 k^2 \sim 1/\tau_Q,
\] (124)

we obtain Eq. (6) as advertised in the introduction.

As we discuss below, when the system approaches the critical point, the length scale \( \ell_{\text{neq}} \) (which separates the non-equilibrium and equilibrium fluctuations of the system) will decrease, and a shorter length \( \ell_{\text{eq}} \) will replace \( \ell_{\text{max}} \).

2. Evolution of fluctuations in the hydrodynamic regime near a critical point

Let us now apply the general kinetic equation obtained in the previous section, Eq. (122), to a system passing close to the QCD critical point. Because of criticality two new features emerge which simplify Eq. (122). First, since \( N_0^{\alpha\alpha} \) will become singular near a critical point, the percent change per time of \( N_0^{\alpha\alpha} \) will become much larger than \( 1/\tau_Q \) (see below), and the gradient terms proportional to \( 1/\tau_Q \) in Eq. (122) can be safely neglected. Second, a hierarchy of relaxation rates emerges near a critical point with \( D_\delta \ll D_\pm \) [18]. This is because \( D_\delta \) is inversely proportional to \( C_p \), which is the most divergent susceptibility near the critical point. Thus, the \( \delta \) mode will be the first to fall out of equilibrium during a transit of the
critical point. For these reasons, we will concentrate on the evolution of the $N^{\hat{s}\hat{s}}$ from now on, and write the equation for $N^{\hat{s}\hat{s}}$ from now on, and write the equation for $N^{\hat{s}\hat{s}}$ as

$$\partial_t N^{\hat{s}\hat{s}}(t, k) = -2D_s(t)k^2 \left[ N^{\hat{s}\hat{s}}(t, k) - C_p(t) \right].$$

Eq. (125) is valid in the hydrodynamic region $k \ll 1/\xi$. We will extend Eq. (125) to the scaling regime in the next section.

**B. Evolution of fluctuations in the scaling regime near a critical point**

Before continuing, let us review the equilibrium result for $N^{\hat{s}\hat{s}}$ which is notated with $N_0^{\hat{s}\hat{s}}$. As derived in Sect. II, the equilibrium correlator takes the form

$$N_0^{\hat{s}\hat{s}}(t, k) = \frac{C_p(t)}{1 + (k\xi(t))^{2-\eta}},$$

where $C_p(t) = \chi_{is}(t)/A_s^2$ and $\xi(t)$ are the time dependent susceptibility and correlation length respectively. The interpolating form for the $k$-dependence captures two limits: the low-$k$ hydrodynamic limit $k \xi \ll 1$, and the high-$k$ scaling limit $k \xi \gg 1$. In the high-$k$ scaling limit, the equilibrium correlation functions are power laws $N_0^{\hat{s}\hat{s}} \propto k^{-(2-\eta)}$ and are independent of $\xi(t)$.

We will introduce a dynamical model to describe the non-equilibrium evolution of $N^{\hat{s}\hat{s}}(t, k)$ for the full range of momenta, including $k \xi \sim 1$. Using fluctuating hydrodynamics we derived a hydro-kinetic equation for $N^{\hat{s}\hat{s}}$ which applies in the hydrodynamic regime where $k \ll 1/\xi$. To generalize this relaxation equation to modes in the scaling regime $\xi^{-1} \ll k \ll \ell_0^{-1}$, let us first write the small $k$ hydrodynamic equation (125) more explicitly

$$\partial_t N^{\hat{s}\hat{s}}(t, k) = -2\left( \frac{\lambda_T}{C_p} \right) k^2 \left[ N^{\hat{s}\hat{s}}(t, k) - C_p(t) \right], \quad (k \xi \ll 1).$$

Here $\lambda_T$ is the thermal conductivity described in Sect. III A 1. Observe that Eq. (127) follows the general pattern that the relaxation rate is proportional to the transport coefficient (i.e. $\lambda_T$) divided by the corresponding susceptibility (i.e. $C_p$). We expect this pattern will still hold for finite $k$. Thus, as a model for $k \xi \sim 1$ we will replace the specific heat in (127) with its $k$-dependent form

$$C_p \rightarrow \frac{C_p}{1 + (k\xi)^{2-\eta}}.$$

and treat the conductivity $\lambda_T$ as a constant. The model takes the form of a $k$-dependent relaxation time equation

$$\partial_t N^{\hat{s}\hat{s}}(t, k) = -2\Gamma_{\hat{s}}(t, k) \left[ N^{\hat{s}\hat{s}}(t, k) - N_0^{\hat{s}\hat{s}}(t, k) \right],$$

where

$$\Gamma_{\hat{s}}(t, k) \equiv \left( \frac{\lambda_T}{C_p\xi^2} \right) (k\xi)^2(1 + (k\xi)^{2-\eta}).$$

The model reduces to the hydrodynamic limit in (127) for $k \xi \ll 1$, and will approach the universal scaling form for $k \xi \gg 1$. 

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In the next paragraph we will discuss the limitations of the model, after describing the behavior of relaxation rate $\Gamma_s$ during a transit of the critical point. We have already mentioned that for $k \ll \xi^{-1}$, the relaxation rate is small and approaches zero as $k^2$ as is typical of conserved quantities. We now turn to the relaxation rate at $k \sim \xi^{-1}$ and $k \gg \xi^{-1}$. Consider for simplicity the behavior of $\Gamma_s$ when the system passes directly through the critical point, $t_{cr} \to 0$ with $t < 0$. In this case (from eqs. (86) and (91)) the specific heat follows the power law

$$ C_p = \frac{\chi_0}{A_s} \left( \frac{\xi}{\ell_0} \right)^{2-\eta}, \quad (131) $$

and the relaxation rate for $k\xi = 1$ depends on the correlation length $\xi$ as

$$ \Gamma_s(t, \xi^{-1}) = \frac{2}{\tau_0} \left( \frac{\xi}{\ell_0} \right)^{-4+\eta}, \quad (132) $$

where we have defined a typical microscopic timescale $\tau_0$ using the previously defined constants

$$ \frac{1}{\tau_0} \equiv A_s^2 \left( \frac{\lambda_T}{\lambda_0 \ell_0^2} \right). \quad (133) $$

$\tau_0$ and $\ell_0$ set the diffusion coefficient away from the critical point, $D_0 \equiv \lambda_T/C_{p,0} = \ell_0^2/\tau_0$. For $k \gg \xi^{-1}$, the relaxation rate is large, scales with a power of $k$, and is independent of the correlation length

$$ \Gamma_s(t, k) \bigg|_{k\xi \gg 1} = \frac{1}{\tau_0} (\ell_0 k)^{4-\eta}. \quad (134) $$

Now let us discuss the limitations of the model. In general the conductivity $\lambda_T$ will scale with the correlation length $\xi$ as

$$ \lambda_T = \lambda_{T0} \left( \frac{\xi}{\ell_0} \right)^{x_\lambda}, \quad (135) $$

where $\lambda_{T0}$ is the typical thermal conductivity away from the critical point, and the exponent $x_\lambda$ results from the renormalization of the conductivity by the thermal fluctuations of the system. Such a renormalization (which ultimately is a resummation of the non-linear interactions of the stochastic system) is neglected in the current model based on linearized hydrodynamics, which evolves the two point functions and neglects the evolution of higher point functions. Thus the model corresponds to “model B” according to the classification of Hohenberg and Halperin [34], while the dynamical universality class of the QCD critical point is “model H” [35] where the conductivity is renormalized with critical exponent $x_\lambda = 0.946$. In addition, the renormalized conductivity will in general depend on $k$ as $\lambda_T K_\lambda(k\xi)$, where $K_\lambda(k\xi)$ is another dynamical scaling function with fixed normalization, $K_\lambda(0) = 1$. The scaling function $K_\lambda(k\xi)$ has been studied extensively [26, 36, 37], and its asymptotic behavior is also related to critical exponent $x_\lambda$

$$ K_\lambda(k\xi) \sim (k\xi)^{-x_\lambda}, \quad k\xi \gg 1. \quad (136) $$

Thus, the relaxation rate at large $k$ is generally expected to scale with the dynamical critical exponent $z \equiv 4 - \eta - x_\lambda$

$$ \Gamma_s(t, k) \bigg|_{k\xi \gg 1} \sim \frac{1}{\tau_0} (\ell_0 k)^z, \quad z \equiv 4 - \eta - x_\lambda. \quad (137) $$
In comparison with Eq. (137), the current model (134) has the dynamical critical exponent
\[ z = 4 - \eta, \]  
which we will use in the numerical work below. While it is straightforward to refine the model and to input \( x_\lambda \) and \( K_\lambda \) from “model H”, we will continue to use the “model B” results, which are sufficient for the our illustrative purpose. It would be interesting to simulate a stochastic non-linear Landau-Ginzburg functional which would naturally reproduce the correct dynamical critical exponents of model \( H \), and correctly describe the non-linear and non-equilibrium evolution of the system during the expansion.

C. Kibble-Zurek scaling and missing the critical point

Before solving Eq. (130) numerically, let us analyze the timescales associated with this evolution. As noted in the previous subsection, low momentum modes with \( k \ll \xi^{-1} \) have a small relaxation rate and are out-of-equilibrium even away from the critical point. On the other hand, high momentum modes with \( k \gg \xi^{-1} \) have a large relaxation rate and are always in equilibrium. We will focus on modes with \( k \sim \xi^{-1} \) where the relaxation rate as a function of time follows the pattern described in Sect. II C for \( \chi(t) \) and \( \xi(t) \) (see Eq. (130)). Specifically, from eqs. (86), (91), and (130), \( \Gamma_\hat{s} \) takes the form
\[ \Gamma_\hat{s}(t,\xi^{-1}) = \frac{1}{\tau_0} \left( \frac{|t|}{\bar{\tau}Q} \right)^{a\nu z} f_\Gamma \left( \frac{t}{|t_{cr}|} \right), \]  
where
\[ f_\Gamma \equiv \frac{1}{f_\chi f_\xi^2}. \]  
is a universal function. Following the pattern described in Sect. II C, \( f_\Gamma \) has the following limits
\[ f_\Gamma(u) = \begin{cases} 1 & u < -1, \\ f_\Gamma^+ \equiv 0.3427 & u \to +\infty, \end{cases} \]  
and \( |u|^{a\nu z} f_\Gamma(u) \) is regular for \( u > -1 \).

Examining the relaxation time equation (129), the dynamical evolution of \( N_\hat{s}(t,k) \) is controlled by a competition between the relaxation rate \( \Gamma_\hat{s}(t,k) \) and the rate of change of the equilibrium expectation \( N_0\hat{s}(t,k) \). First we analyze the limit \( t_{cr} \to 0 \) and \( t < 0 \), where the relaxation rate takes the scaling form
\[ \Gamma_\hat{s}(t,\xi^{-1}) = \frac{1}{\tau_0} \left( \frac{|t|}{\bar{\tau}Q} \right)^{a\nu z}, \]  
reflecting the equilibrium scaling of \( \chi \) and \( \xi \) in this limit
\[ C'_p = \frac{\chi_0}{A^2_\hat{s}} \left( \frac{|t|}{\bar{\tau}Q} \right)^{-a\gamma}, \]  
\[ \xi = \ell_0 \left( \frac{|t|}{\bar{\tau}Q} \right)^{-a\nu}. \]
For \( t > 0 \) these forms are multiplied by the order one factors, \( f_\Gamma^+ \), \( f_\chi^+ \) and \( f_\xi^+ \), respectively. When \( t \to 0 \), the system approaches the critical point, and the relaxation rate decreases exhibiting critical slowing down. By contrast, the percent change per time of the equilibrium susceptibility \( C_p \) is of order
\[
\left| \frac{\partial_t C_p}{C_p} \right| \sim \left| \frac{1}{t} \right|,
\]
which diverges near the critical point. Consequently, the system will inescapably fall off equilibrium at some time \( t_{\text{kz}} \) (the Kibble-Zurek time), which can be determined by comparing these competing rates
\[
\frac{1}{t_{\text{kz}}} = \frac{1}{\tau_0} \left( \frac{t_{\text{kz}}}{\bar\tau_Q} \right)^{\alpha \nu / (\alpha \nu + 1)}.
\]
Solving for \( t_{\text{kz}} \) we find
\[
t_{\text{kz}} = \tau_0 \left( \frac{\tau_0}{\bar\tau_Q} \right)^{-\alpha \nu / (\alpha \nu + 1)},
\]
which is an intermediate scale \( \tau_0 \ll t_{\text{kz}} \ll \bar\tau_Q \). Indeed, since \( \bar\lambda \equiv \tau_0 / \bar\tau_Q \ll 1 \), the timescales \( \tau_0, \ t_{\text{kz}}, \) and \( \bar\tau_Q \) are widely separated:
\[
\bar\lambda \ll \bar\lambda^{1/(\alpha \nu + 1)} \ll 1.
\]

The Kibble-Zurek time \( t_{\text{kz}} \) characterizes the temporal evolution of \( N^{\hat{s} \hat{s}} \) during a transit of the critical point. Let us introduce an associated length scale \( \ell_{\text{kz}} \) (the Kibble-Zurek length), which is defined as the value of correlation length \( \xi \) at \( t = -t_{\text{kz}} \)
\[
\ell_{\text{kz}} \equiv \ell_0 \left( \frac{\tau_0}{\bar\tau_Q} \right)^{-\alpha \nu / (\alpha \nu + 1)} = \ell_0 \bar\lambda^{-\alpha \nu / (\alpha \nu + 1)}.
\]
Modes with \( k \lesssim \ell_{\text{kz}}^{-1} \) will fall out equilibrium for \( |t| \sim t_{\text{kz}} \), while modes with \( k \gg \ell_{\text{kz}}^{-1} \) will remain equilibrated. We therefore expect that \( \ell_{\text{kz}} \) will characterize the momentum dependence of \( N^{\hat{s} \hat{s}}(t,k) \). \( \ell_{\text{kz}} \) is also an intermediate scale, \( \ell_0 \ll \ell_{\text{kz}} \ll \ell_{\text{max}} \sim \ell_0 \lambda^{-1/2} \), where \( \ell_{\text{max}} \) is the maximum wavelength that can be equilibrated away from the critical point.

Finally, since the evolution is “frozen” for \( t \gtrsim -t_{\text{kz}} \), the magnitude of \( N^{\hat{s} \hat{s}}(t,k) \) can be estimated by the value of \( C_p \) at \( t = -t_{\text{kz}} \)
\[
N^{\hat{s} \hat{s}} \sim \frac{\chi_{\text{kz}}}{A_s} \equiv C_{p,\text{kz}} \quad \text{and} \quad \chi_{\text{kz}} \equiv \chi_0 \left( \frac{\ell_{\text{kz}}}{\ell_0} \right)^{2-\eta}.
\]

Thus, \( \ell_{\text{kz}} \) also determines the magnitude of fluctuations during a transit of the critical point through the definition of \( \chi_{\text{kz}} \propto \ell_{\text{kz}}^{2-\eta} \).

The qualitative discussion in the preceding paragraphs motivates us to introduce a rescaled two point function
\[
N^{\hat{s} \hat{s}} \equiv \frac{\chi_{\text{kz}}}{A_s^2} \tilde{N}^{\hat{s} \hat{s}}(\tilde{t},k\ell_{\text{kz}};t/|t_{\text{cr}}|),
\]
where we anticipate \( \tilde{N}^{\hat{s} \hat{s}} \) will be of order unity, and will depend on the rescaled time
\[
\tilde{t} \equiv \frac{t}{t_{\text{kz}}}.
\]
Substituting (151) into (129), we obtain an equation for $\bar{N}^{\bar{s}}$:

$$
\partial_t \bar{N}^{\bar{s}} = -2 |\bar{t}| \alpha v \frac{(k\xi)^2}{K_X(k\xi)} \left[ \bar{N}^{\bar{s}} - |\bar{t}|^{-\alpha} K_X(k\xi) \right], \quad \bar{t} \leq -t_{cr}/t_{kz},
$$

(153a)

$$
\partial_t \bar{N}^{\bar{s}} = -2 |\bar{t}| \alpha v \frac{(k\xi)^2}{K_X(k\xi)} \left[ \bar{N}^{\bar{s}} - |\bar{t}|^{-\alpha} f_X K_X(k\xi) \right], \quad \bar{t} \geq -t_{cr}/t_{kz},
$$

(153b)

where

$$
k\xi = \begin{cases} 
    k\ell_{kz} |\bar{t}|^{-\alpha} & \bar{t} \leq -t_{cr}/t_{kz}, \\
    k\ell_{kz} |\bar{t}|^{-\alpha} f_{\xi} & \bar{t} \geq -t_{cr}/t_{kz},
\end{cases}
$$

(153c)

and $K_X$ is given Eq. (54). The three scaling functions $f_{\Gamma}$, $f_X$ and $f_{\xi}$ take the form

$$
f_{\Gamma} \left( \frac{t_{kz}}{|t_{cr}|} \bar{t} \right), \quad f_X \left( \frac{t_{kz}}{|t_{cr}|} \bar{t} \right), \quad f_{\xi} \left( \frac{t_{kz}}{|t_{cr}|} \bar{t} \right).
$$

(153d)

Thus, $\bar{N}^{\bar{s}}$ only depends on scaling variables $\bar{t} = t/t_{kz}, k\ell_{kz} t_{kz}/t_{cr}$. When the system passes directly through the critical point $t_{cr} \to 0$, the quantities $f_{\Gamma}$, $f_X$ and $f_{\xi}$ approach universal constants ($f_{\Gamma}^0$, $f_X^0$, and $f_{\xi}^0$), and the correlation function $\bar{N}^{\bar{s}}$ is only a function of $t/t_{kz}$ and $k\ell_{kz}$. When the system misses the critical point by an amount $\Delta \approx t_{Q}\Delta_{Q}$, and the correlation function for $t > -t_{cr}$ additionally depends on the ratio $t_{cr}/t_{kz}$. We will present numerical results for $\bar{N}^{\bar{s}}$ in the next section by solving Eq. (153).

D. Transits of a critical point: numerical evaluation

Now we will determine $\bar{N}^{\bar{s}}$ by solving Eq. (153) numerically. First, we evaluate $\bar{N}^{\bar{s}}$ when the system passes directly through the critical point by setting $t_{cross}$ to zero. In Fig. 3(a) and (b) we plot $\bar{N}^{\bar{s}}$ as function of $\bar{k}$ for representative times before and after the critical point respectively. For comparison, we plot the corresponding equilibrium expectation with dashed curves.

As seen in the figure, the fluctuations recorded by $\bar{N}^{\bar{s}}$ are maximal at a given wavenumber $k_{eq}$ corresponding to a definite wavelength, $l_{eq} \equiv k_{eq}^{-1} \sim \ell_{kz}$. This is in contrast with the behavior of the equilibrium fluctuations $N^{\bar{s}}_0$ (the dashed curves) which increase monotonically as $k \to 0$. The maximum is the result of a competition between the hydrodynamic behavior at small $k$, and the critical scaling behavior at large $k$. Modes with $k \ll l_{eq}^{-1}$ equilibrate slowly (diffusively), reflecting the fact that the total charge is conserved and does not fluctuate. Consequently, the system does not respond to the increasing critical susceptibility at small $k$, and the magnitude of $\bar{N}^{\bar{s}}$ in the hydrodynamic region remains small compared to the equilibrium specific heat $C_p$. At large $k$, the relaxation rate grows as $k^{2}$ and becomes very large. Thus, the large $k$ tail of $\bar{N}^{\bar{s}}$ is always close to the equilibrium expectation, which vanishes as $1/k^{2}$. To summarize, $\bar{N}^{\bar{s}}$ will become small at both small

---

8 We need to specify the initial conditions of $\bar{N}^{\bar{s}}$ at an initial time $tI/t_{kz}$, where $tI < 0$ is the time when system enters the critical region. However, we are working in the parametric regime where $t_{Q}/t_{kz} \to \infty$, and the time $tI$ is of order $t_{Q}$. Therefore, $tI/t_{kz}$ should be taken to negative infinity; we take $tI/t_{kz} \sim -40$ in practice. Non-equilibrium effects will not be important at this early time, and consequently we initialize $\bar{N}^{\bar{s}}$ in equilibrium.
The time evolution of the correlation function of entropy per baryon fluctuations $\delta \hat{s} \equiv \delta s - (s/n) \delta n$, when the system passes directly through the critical point, $t_{cr}/t_{kz} = 0$. The wavenumber is measured in units of $\ell_{kz}^{-1} (k = k\ell_{kz})$, and $N^{ss}$ has been rescaled by the specific heat $C_p$ at the Kibble-Zurek time $C_{p,kz} (N^{ss} \equiv N^{ss}/C_{p,kz})$.

(a) The time evolution in the coexistence region $t < 0$ (before the critical point). (b) The time evolution after the system has left the coexistence region $t > 0$ (after the critical point).

$N^{ss}$ and large $k$, naturally exhibiting maximum at some intermediate wavenumber $\ell_{neq}^{-1}$. This scale characterizes $N^{ss}$ in the sense that wavenumbers significantly larger than $\ell_{neq}^{-1}$ are in equilibrium, while those smaller than $\ell_{neq}^{-1}$ are out of equilibrium.

From Fig. 3, the fluctuations grow with time for $t < 0$, and then return to their typical size after passing the critical point, $t > 0$. However, as we approach the critical point, the growth in $N^{ss}$ for $t > -t_{kz}$ is modest when compared to the rapid growth of $C_p$ (the dashed curves at $k = 0$). The system is exhibiting critical slowing down, and lags behind its equilibrium expectation.

The slow evolution of $N^{ss}$ implies that the system can remember the magnitude of the critical fluctuations even after passing through the critical point. Indeed for $t > 0$, $N^{ss}_{\text{max}}$ is even larger than its equilibrium expectation. Similar observations about the “memory effect” of critical fluctuations have been made in previous studies [9, 10]. The distinctive feature of $N^{ss}$, namely the maximum at a specific wavenumber $\ell_{neq}^{-1}$, is remembered for $t > 0$. It remains to be seen which experimental observables provide access to this interesting structure – see Sec. IV 4 for a preliminary proposal.

We now turn to finite detuning case shown in Fig. 4. In Fig. 4 (a,b), we show our results for $N^{ss}$ at $t_{cr}/t_{kz}=1$. The qualitative features are similar to the $t_{cr}/t_{kz}=0$ case, but the magnitude of the fluctuations is reduced. For still larger detuning $t_{cr}/t_{kz}=3$ shown in (c,d), the fluctuations are reduced even further. In the large detuning regime the equilibrium scaling of the specific heat at the crossing time $t_{cr}$ determines the magnitude of the fluctuations rather than the relaxation dynamics. Thus, the magnitude of the critical fluctuations are independent of $\lambda$ in this regime. Straightforward analysis based the previous sections (see
Sect. II C) shows that at $t_{cr}$ the fluctuations are of order

$$N^{\hat{s}\hat{s}} \sim C_{p,kz} \left( \frac{t_{cr}}{t_{kz}} \right)^{-a\gamma} \sim \frac{\lambda_0}{A_s} \Delta_s^{-\gamma/\beta}.$$  (154)

The wavenumber $\ell_{neq}^{-1}$ where the system transitions from the non-equilibrium behavior at small $k$ to equilibrium behavior at large $k$ is also reduced relative to $\ell_{kz}^{-1}$. Equating the relaxation rate at the crossing time to the rate of change in equilibrium, $\Gamma(t_{cr}, \ell_{neq}) \sim t_{cr}^{-1}$,
shows that
\[ \ell_{\text{neq}} \sim \ell_{\text{kz}} \left( \frac{t_{\text{cr}}}{t_{\text{kz}}} \right)^{(a\nu(z-2)+1)/2} \sim \ell_0 \lambda^{-1/2} \Delta_s^{(a\nu(z-2)+1)/2a\beta} \, . \tag{155} \]

Numerically these exponents evaluate to
\[ N^{\hat{s}\hat{s}} \sim \chi_0 \bar{\Delta}_s^{-3.8} \, , \tag{156} \]
\[ \ell_{\text{neq}} \sim \ell_0 \lambda^{-1/2} \bar{\Delta}_s^{3.26} \, . \tag{157} \]

When the detuning \( \Delta_s \) approaches unity, the non-equilibrium length \( \ell_{\text{neq}} \) approaches \( \ell_{\text{max}} = \ell_0 \lambda^{-1/2} \). Modes with wavelength longer than \( \ell_{\text{max}} \) remember the initial conditions at \( t = -\tau_Q \), and are unaffected by the transit of the critical point.

Summarizing this subsection, we have evaluated the fluctuations in the entropy to baryon number, \( N^{\hat{s}\hat{s}} \), for a system which passes directly through the critical point \( (t_{\text{cr}}/t_{\text{kz}} = 0) \), and which misses the critical point \( (t_{\text{cr}}/t_{\text{kz}} \neq 0) \). When \( t_{\text{cr}}/t_{\text{kz}} \) is not significantly larger than unity, the wavenumber dependence of \( N^{\hat{s}\hat{s}} \) is qualitatively different from its equilibrium expectation, and from earlier work. Previously, the non-equilibrium variance of the order parameter field has been evaluated for “model A” [15]. In this case the order parameter is not conserved, and its relaxation rate remains finite at \( k = 0 \). By contrast, the order parameter for QCD is conserved, and the relaxation rate vanishes as \( k \to 0 \). Because of this fundamental difference \( N^{\hat{s}\hat{s}} \) develops a maximum around \( k \sim \ell_{\text{kz}}^{-1} \), and the critical fluctuations will be most pronounced at the corresponding wavelength \( \sim \ell_{\text{kz}} \). This feature is absent in the study of Ref. [15].

IV. DISCUSSION

In this paper we have studied how the QCD medium created in a heavy ion collision will evolve during a transit of the conjectured critical point. We have defined two parameters, which are repeated here for convenience. The first is the “detuning parameter”
\[ \Delta_s \equiv \frac{n_c}{s_c} \left( \frac{s}{n} - \frac{s_c}{n_c} \right) \, , \tag{158} \]
and the second is the ratio of the microscopic time scale \( \tau_0 \) to the expansion rate \( \tau_Q^{-1} \)
\[ \lambda \equiv \tau_0 \partial_\mu u^\mu \equiv \frac{\tau_0}{\tau_Q} \, . \tag{159} \]

Then we asked how the critical hydrodynamic fluctuations in the system depend on these two parameters during the transit. These two parameters quantify how missing the critical point and finite relaxation rates will regulate the growth of critical fluctuations. This conclusion is organized around explaining Fig. 5 which summarizes our results.

1. Object of study and its behavior away from the critical point

First we explained that observable of primary interest is the fluctuations in the entropy per baryon (multiplied by \( n \))
\[ \delta s \equiv n \delta \left( \frac{s}{n} \right) = \delta s - \frac{s}{n} \delta n \, . \tag{160} \]
From an experimental point of view, it may be easier to work with the fluctuations in the baryon number per entropy (multiplied by $s$):

$$\delta \hat{n} \equiv s \delta \left( \frac{n}{s} \right) = \delta n - \frac{n}{s} \delta s. \quad (161)$$

which contains the same physical content.

There are several reasons (discussed in Sect. II B 3 and Sect. III A 1) why $\delta \hat{n}$ is the relevant quantity. First, $\delta \hat{n}$ is an eigenmode of linearized hydrodynamics, and its fluctuations are proportional to the specific heat at constant pressure. Specifically, the two point functions are defined as

$$N^{\hat{n}\hat{n}}(t, \mathbf{k}) \equiv \int d^3 x \ e^{i \mathbf{k} \cdot \mathbf{x}} \langle \delta \hat{n}(t, \mathbf{x}) \delta \hat{n}(t, \mathbf{0}) \rangle, \quad (162)$$

and in equilibrium $N^{\hat{n}\hat{n}}$ determines $C_p$ from its small $k$ limit

$$N^{\hat{n}\hat{n}}_0(t, \mathbf{0}) \big|_{eq} = V \left\langle (\delta \hat{n})^2 \right\rangle_{eq} = \left( \frac{n}{s} \right)^2 C_p. \quad (163)$$

In the body of the text we have worked with $N^{\hat{n} \hat{k}}(t, \mathbf{k})$ which is proportional $N^{\hat{n}\hat{n}}(t, \mathbf{k})$

$$N^{\hat{n}\hat{n}}(t, \mathbf{k}) = \left( \frac{n}{s} \right)^2 N^{\hat{n} \hat{k}}(t, \mathbf{k}). \quad (164)$$
As the temperature approaches its critical value, \((n/s)^2 C_p\) will always diverge with the largest critical exponent of the Ising susceptibility matrix, \(\gamma \simeq 1.23\). By contrast, the squared speed of sound approaches zero with the critical exponent \(\alpha \simeq 0.11\), which is too slow to be of practical interest for the heavy ion program. As discussed in Sect. II B, these statements about \(C_p\) and \(c_s^2\) are independent of the detailed mapping matrix between the QCD and Ising variables.

Now let us describe the behavior of \(N^{\hat{n}}\) away from the critical point as illustrated in Fig. 5. Away from the critical point, the fluctuations in \(\delta \hat{n}\) scale as the fluctuations in \(\delta n\), which can be reasonably expected to be roughly Poissonian, \(V \langle (\delta n)^2 \rangle \sim n\). This leads to a Poisson estimate for these fluctuations

\[
\frac{N^{\hat{n}}(t, k)}{n} \sim 1, \quad \text{with} \quad \ell_0^{-1} \ll k \ll \ell_{\text{max}}^{-1}.
\]

Here \(\ell_0\) denotes a typical microscopic length scale, and \(\ell_{\text{max}}\) is discussed below. Searches for critical fluctuations will look for enhancements at fixed \(\ell_{\text{max}}\) to this baseline expectation that change non-monotonically with the (mean) \(n/s\).

Note that the Poissonian expectation in Eq. (165) is independent of \(k\) for all equilibrated modes with wavenumber smaller than the inverse correlation length \(\sim \ell_0^{-1}\). As discussed in the introduction, modes with wavelength longer than a “non-equilibrium” length, \(\ell_{\text{neq}} \sim \ell_{\text{max}} \sim \ell_0 \lambda^{-1/2}\), are always out of equilibrium even away from the critical point [11], and will not show critical behavior. We will see that when the system approaches the critical point, modes with wavelength shorter than \(\ell_{\text{max}}\) will begin to fall out of equilibrium, and the non-equilibrium length \(\ell_{\text{neq}}\) will decrease. This shown by the grey region of Fig. 5(b).

2. How missing the critical point regulates the critical fluctuations

In Sect. II C we determined how the equilibrium susceptibilities in QCD are regulated in time as the medium passes close the critical point during an adiabatic expansion with a detuning parameter \(\Delta_s\). This time evolution follows a specific pattern, which is a reflection of the scaling of the equilibrium equation of state. For example, the equilibrium specific heat \((n/s)^2 C_p\) (which diverges like the Ising susceptibility \(\chi_{\text{Is}} \propto r^{-\gamma}\)) has the following time dependence for an adiabatic trajectory near the critical point

\[
\frac{N^{\hat{n}}(t, 0)}{n} \bigg|_{\text{eq}} = c_0 n \left| \frac{t}{\tau_Q} \right|^{-a\gamma} f_X \left( \frac{t}{|t_{\text{cr}}|} \right).
\]

Here \(f_X(t/|t_{\text{cr}}|)\) is a known universal scaling function of order unity which can be determined by the \((R, \theta)\) parameterization of the Ising Model susceptibility. \(c_0\) is a dimensionless and order one non-universal constant, and the “crossing time” is

\[
t_{\text{cr}} \equiv -c_1 \tau_Q \Delta_s^{1/\alpha \beta} < 0,
\]

where \(c_1\) is another (dimensionless and order one) non-universal constant\(^{10}\). \(t/|t_{\text{cr}}|\) plays the role of the scaling variable, and the scaling function \(f_X(t/|t_{\text{cr}}|)\) approaches a (universal)

---

9 For example, we may estimate \(N^{\hat{n}}\) for a hadron gas. For a hadron gas at a temperature of \(T \simeq 155\) and \(s/n \simeq 25\) (corresponding to the chemical freezeout conditions at \(\sqrt{s_{NN}} = 12.5\) GeV) we find \(nC_p/s^2 \simeq 0.65\).

10 Explicit expressions for these constants are given in the text \((c_0 = 0.365 A_n^{-a\gamma} \text{ and } c_1 = A_s/A_n)\) in terms of the mapping matrix \(M^A_b\) between the QCD and Ising variables described in Sect. II A.
constant for $t/|t_{cr}| \to \pm \infty$. From Eq. (166) we see that the specific heat grows like a power until the scaling variable $t/|t_{cr}|$ approaches $-1$. For $t/|t_{cr}| \sim -1$ the system becomes aware that adiabatic trajectory will miss the critical point by $\Delta_s$, and this stops the growth of the specific heat. Setting $t$ to $t_{cr}$, we can estimate the maximum magnitude of equilibrium critical fluctuations relative to the Poissonian expectation

$$\frac{N_0 \hat{n}(t, k)}{n} \sim \Delta_s^{-\gamma/\beta}, \quad (168)$$

Here the wavelengths of interest $k^{-1}$ are of order the correlation length at the crossing time

$$k^{-1} \sim \xi(t_{cr}) \sim \ell_0 \Delta_s^{-\nu/\beta}.$$  \quad (169)

Sufficiently long wavelength modes are always out of equilibrium and will not show the enhancement in Eq. (168). Sect. III D estimates that for $\Delta_s$ small (but larger than a $\Delta_{kz}$ discussed below) the non-equilibrium length is of order $\ell_{neq} \sim \ell_0 \lambda^{-1/2} \Delta_s^{3.26}$. Fig. 5 shows how the correlation length $\xi(t_{cr})$ and the non-equilibrium length $\ell_{neq}$ come together as we begin to approach the critical point.

The estimate in Eq. (168) realizes one of the goals of this paper, i.e. to parametrically estimate how missing the critical point limits the critical fluctuations. However, the analysis in the next section shows (unfortunately) that non-equilibrium physics will set in well before the critical fluctuations are regulated by a finite missing parameter $\Delta_s$. Thus, the non-equilibrium dynamics will regulate the critical fluctuations well below the equilibrium estimate in Eq. (168). For this reason we will refrain from substituting numbers into Eq. (168).

3. How critical slowing down regulates the critical fluctuations

In Sect. III we estimated how the finite relaxation time limits the growth of critical fluctuations. For conserved (or approximately conserved) quantities such as $n/s$ the relaxation time depends on the wavelength of the mode of interest, with longer wavelengths modes taking longer to relax. For $k \sim \xi^{-1}$, the typical relaxation time increases near the critical point as

$$\tau_R(\xi) \equiv \tau_0 \left( \frac{\xi}{\ell_0} \right)^z,$$  \quad (170)

where $z \equiv 4-\eta \simeq 4$ in our setup\textsuperscript{11}, and $\tau_0$ is the microscopic time. We then find that modes with $k \sim \xi^{-1}$ fall out of equilibrium at the Kibble-Zurek time

$$t_{kz} \sim \tau_0 \left( \frac{\tau_0}{\tau_Q} \right)^{-a\nu z/(a\nu z+1)}, \quad \frac{a\nu z}{a\nu z + 1} \simeq 0.74,$$  \quad (171)

where $\nu \simeq 0.63$. The correlation length at this time is

$$\ell_{kz} \sim \ell_0 \left( \frac{\tau_0}{\tau_Q} \right)^{-a\nu/(a\nu z+1)}, \quad \frac{a\nu}{a\nu z + 1} \simeq 0.19.$$  \quad (172)

\textsuperscript{11} We have defined $\tau_R(\xi) \equiv 1/\Gamma_s(t, \xi^{-1})$ used in the body of the text, e.g. Eq. (132). The dynamical exponent $z = 4 - \eta$ is modified to $z = 3 - \eta$ in a more refined treatment where the conductivity $\lambda_B$ is renormalized by critical fluctuations.
Let us compare the \( t_{kz} \) and \( t_{cr} \) timescales. The Kibble-Zurek dynamics will begin to regulate the growth of critical fluctuations before the scaling behavior of the equation of state whenever \( t_{kz} \gg t_{cr} \). In this limit \( \Delta_s \to 0 \) and the scaling structure of the equation of state is irrelevant, since the system falls out of equilibrium before reaching the detailed scaling regime. Comparing Eq. (171) and Eq. (167) we see that \( t_{kz} \gg t_{cr} \) whenever \( \Delta_s \) is less than a certain threshold

\[
\Delta_s < \Delta_{kz} \equiv \lambda^{a\beta/(avz+1)}.
\]

(173)

As shown in Fig. 5, for \( \Delta_s < \Delta_{kz} \) the non-equilibrium length is set by \( \ell_{kz} \) and the magnitude of the fluctuations is of order the equilibrium susceptibility at \( t_{kz} \). Substituting numbers, with \( a \approx 1.12, z \approx 3.96, \) and \( \beta = 0.32 \), we find

\[
\Delta_{kz} = 0.86 \left( \frac{\lambda}{0.2} \right)^{0.096}.
\]

(174)

Clearly the strikingly small power, 0.096, makes the value \( \Delta_{kz} \) remarkably insensitive to the value of \( \lambda \). Thus, for realistic heavy-ion collisions with a finite \( \lambda \), the detailed equilibrium scaling of the equation of state has a limited range of validity, \( \Delta_{kz} \ll \Delta_s \ll 1 \). Essentially, if one is close enough to the critical point, then the dynamics will always be out of equilibrium. Thus, to simulate the evolution of trajectories with \( \Delta_s < \Delta_{kz} \), inputting an equation of state with the detailed scaling behavior (see Ref. [38]) into the hydrodynamic codes is not really necessary or sufficient. It is essential to simulate the non-equilibrium evolution of the system, along the lines of this work and Ref. [18].

Let us estimate the Kibble-Zurek timescale. We have defined a small parameter \( \lambda \), and the three time scales in our problem,

\[
\tau_0 \ll t_{kz} \ll \tau_Q ,
\]

(175)

are of relative size

\[
\tau_0 \ll \tau_0 \lambda^{-0.74} \ll \tau_0 \lambda^{-1} .
\]

(176)

Taking\(^{12} \) \( \tau_0 \approx 1.8 \text{ fm} \) and \( \lambda = 0.2 \), we find a relatively long time for \( t_{kz} \):

\[
1.8 \text{ fm} \ll 5.8 \text{ fm} \ll 8.9 \text{ fm} .
\]

(177)

Thus, if the system freezes out over a time of \( t_{kz} \approx 5.8 \text{ fm} \), then the critical enhancement of fluctuations estimated below may be visible.

Similarly, the system has the length scales

\[
\ell_0 \ll \ell_{kz} \ll \ell_{\max} ,
\]

(178)

which are of relative size

\[
\ell_0 \ll \ell_0 \lambda^{-0.18} \ll \ell_0 \lambda^{-1/2} .
\]

(179)

The microscopic length \( \ell_0 \) is of order the inter-particle spacing. For a hadronic gas with \( n/s = 25 \) and a chemical freezeout temperature \( T \approx 155 \text{ MeV} \) this length is approximately, \( \ell_0 \approx 1.2 \text{ fm} \). Taking \( \lambda = 0.2 \) we find that the three length scales are of order

\[
1.2 \text{ fm} \ll 1.6 \text{ fm} \ll 2.7 \text{ fm} .
\]

(180)

\(^{12}\) We have estimated the hadron density below using a thermal model. Then we multiplied the distance by the typical quasi particle velocity \( \sqrt{3c_s^2} \) to arrive at this estimate.
Comparing these numbers, we see that the correlation length at freezeout is at most twice the inter-particle spacing at these low densities.

Let us estimate the magnitude of the critical fluctuations when the Kibble-Zurek dynamics regulates the growth. The timescales and length-scales are set by the Kibble-Zurek time and length. Substituting $t_{kz}$ from Eq. (171) into Eq. (166) (with $c_0 \sim f_\chi \sim 1$), we find that the magnitude of the fluctuations relative to our Poisson expectation are enhanced by

$$\frac{N^{\hat{n}_\mu}(t_{kz}, \mathbf{k})}{n} \bigg|_{k \sim \ell_{kz}^{-1}} \sim \lambda^{-\gamma a/(\alpha v z + 1)}.$$  \hfill (181)

Numerically for $\lambda = 0.2$ we find a somewhat anemic 80% enhancement

$$\frac{N^{\hat{n}_\mu}(t_{kz}, \mathbf{k})}{n} \bigg|_{k \sim \ell_{kz}^{-1}} \sim 1.8 \left(\frac{\lambda}{0.2}\right)^{-0.37}.$$  \hfill (182)

This enhancement $\propto \lambda^{-0.37}$ is illustrated in Fig. 5, and is the largest one could reasonably expect in a heavy ion collision.

4. **How this analysis can inform the experimental search for the critical point**

We have analyzed the relevant length scales for the critical point search. In heavy ion collisions the longest wavelengths are long range in rapidity, and are described with hydrodynamics. These long wavelength modes, such as the elliptic and triangular flow, are not equilibrated and depend on the initial conditions. Only wavelengths smaller than a characteristic scale $\ell_{\text{max}}$ equilibrate during an expansion away from the critical point. Only modes with (wavelength) $\ll \ell_{\text{max}}$ can possibly exhibit critical properties. The typical wavelength for enhanced critical fluctuations is set by the Kibble-Zurek length $\ell_{kz}$, and this length is only somewhat larger than the inter-particle spacing in practice. Such short lengths are associated with non-flow correlations. Thus, if critical fluctuations are to be seen then one must carefully examine the non-flow correlations to look for modifications as the mean baryon number to entropy ratio is changed in the event.

The current measurements of kurtosis are essentially a measure of the probability of finding a baryon at mid-rapidity while keeping the particle number (entropy) fixed. It seems to us that the modifications of this quantity with beam energy are mostly a measurement of baryon transport in the initial state, and are perhaps unrelated to the critical fluctuations.

In order to measure the expected critical point signal, one should divide the system at different beam energies into different event classes with a specified $n/s$ in a large mid-rapidity detector. (The proton to pion ratio can be used as a proxy for $n/s$.) If the system passes close to the critical point, the short range (connected) two point functions should change as the $n/s$ event class is scanned. These changes in the two point functions should be largely independent of centrality and beam energy, but should depend only on the mean $n/s$ of the event class. The presence of a critical point leads to short range spatial correlations of size of order $\ell_{kz}$. In momentum space this corresponds to a momentum difference of order $\Delta p \sim \hbar/\ell_{kz} \sim 50\text{MeV}$. Thus, the presence of a critical point there will enhance the short-range, almost HBT-like, correlations.

Any non-monotonic changes in the non-flow correlation strength in this fixed momentum range with the mean $n/s$ would certainly be remarkable. We plan to investigate such correlations in future work, and encourage our experimental colleagues to do the same.
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Appendices

A. THE ISING EQUATION OF STATE AND CORRELATION LENGTH

In this section we will parametrize the Ising equation of state with the familiar \((R, \theta)\) form.

1. Preliminaries

The free energy is the log of the partition function

\[
F(T, H) = - T \log \frac{Z(T, H)}{V}, \quad \text{with} \quad dF = - S dT - \psi dH, \tag{183}
\]

and thus

\[
\frac{d \log Z(T, H)}{V} = \frac{\mathcal{E}}{T^2} dT + \frac{\psi}{T} dH, \tag{184}
\]

where the energy density is \(\mathcal{E} = F - T \frac{\partial F}{\partial T}\). Near the critical point \(Z(T, H)\) is the product of a regular contribution and a singular contribution, \(Z_{\text{reg}} \times Z_{\text{sing}}\). The regular part is expanded in a Taylor series near the critical point, keeping only linear terms

\[
\frac{\Delta \log Z_{\text{reg}}}{V} = \frac{\mathcal{E}_c}{T_c^2} \Delta T = - \frac{\Delta F_{\text{reg}}}{T_c}, \quad \Delta F_{\text{reg}} = S_c \Delta T. \tag{185}
\]

Due to the \(Z_2\) symmetry of the Ising model, the regular part starts as \(H^2\) which can be neglected close to the critical point. Given Eq. (184) and Eq. (185) the singular contribution \(Z_{\text{sing}}\) satisfies

\[
d \log Z_{\text{sing}} = - \frac{d F_{\text{sing}}}{T_c} = \epsilon d r + \psi d h, \quad \frac{d F_{\text{sing}}}{T_c} = - s d r - \psi d h, \tag{186}
\]

where we have defined \(r = (T - T_c)/T_c\), \(\epsilon = (\mathcal{E} - \mathcal{E}_c)/T_c\), \(h = H/T_c\), and \(s = S(T, H) - S_c\). Thus, near the critical point we have \(\epsilon = s\) which follows from the definition of \(\mathcal{E}, \epsilon,\) and the decomposition of \(Z = Z_{\text{reg}} \times Z_{\text{sing}}\) into regular and singular parts.

The free energy \(F\) is the Legendre transform of the Gibbs free energy \(G(T, \psi) = F + \psi H\). The singular part satisfies

\[
\log Z_{\text{sing}}(r, h) = - \frac{G_{\text{sing}}(r, \psi)}{T_c} + \psi h, \tag{187}
\]

and the reduced magnetic field \(h\) is related to \(G_{\text{sing}}(r, \psi)/T_c\) by the thermodynamic relations, \(h = (\partial(G_{\text{sing}}/T_c)/\partial \psi)_r\).

---

\(13\) Relative to Ref. [25], but in accord with Ref. [26], we have reversed the roles of \(F\) (what we call the free energy) and \(G\) (what we call the Gibbs free energy).
2. The \((R, \theta)\) parameterization

Following previous authors [25, 26], we parametrize the Ising equation of state outside of the coexistence region with two auxiliary variables \((R, \theta)\) with \(\theta^2 \leq \theta_0^2\)

\[
\begin{align*}
 r &= (1 - \theta^2) R, \\
 \frac{h}{h_0} &= c_h \theta \left(1 - \theta^2/\theta_0^2\right) R^{3\delta}.
\end{align*}
\]

Then the equation of state takes the form [26]

\[
\frac{\psi}{M_0} = c_M \theta R^\beta,
\]

where \(\delta\) and \(\beta\) are critical exponents. \(\theta_0\) demarcates the boundary of the coexistence region and is approximately

\[
\theta_0 = \left(\frac{\delta - 3}{(\delta - 1)(1 - 2\beta)}\right)^{1/2} \simeq 1.166.
\]

As discussed in Sect. II A, the dimensionful constants \(M_0 h_0\) and \(M_0\) are chosen conventionally to be \((n_c, s_c)\) so that mapping matrix \(M^A_b\) is of order unity. The constants \(c_h\) and \(c_M\) will be chosen to maintain the convenient normalization conventions adopted in Sect. II C: namely that on coexistence line \(\psi/M_0 = |r|\beta\) and \(\epsilon/(M_0 h_0) = -|r|^{1-\alpha}\). Thus

\[
c_M = \frac{(\theta_0^2 - 1)^3}{\theta_0^3} \simeq 0.6145,
\]

and \(c_M c_h\) is given below in Eq. (203).

The dimensionless scaling variable \(\theta\) is directly related to the scaling variable used in Ref. [32]

\[
z = \left(\frac{r}{r_S}\right) \left(\frac{h_S}{h/(c_h h_0)}\right)^{1/3\delta} = 1.901 \frac{(1 - \theta_0^2)}{[\theta(1 - (\theta/\theta_0)^2)]^{1/3\delta}},
\]

where we defined

\[
r_S = \frac{\theta_0^2 - 1}{\theta_0^{1/3}} \simeq 0.225, \quad \text{and} \quad h_S = \frac{\theta_0^2 - 1}{\theta_0^2} = 0.265.
\]

Following Ref. [25], we can integrate the equation of state, Eq. (190), to determine the singular part of the grand sum, \(\log Z(r, h)\), which subsequently determines all thermodynamic quantities and susceptibilities through differentiation. Parametrizing \(G_{\text{sing}}/T_c\) as

\[
\frac{1}{h_0 M_0} \frac{G_{\text{sing}}(r, \psi)}{T_c} = c_h c_M R^{2-\alpha} g(\theta),
\]

\(\Delta\) The differences between our parameterization (taken from Ref. [26]) and the parametrization used in Ref. [25] are minor. We have neglected the fifth order term in the polynomial expansion of \(\tilde{h}(\theta) \simeq \theta(1 - \theta^2/\theta_0^2)\), and taken an analytic expression (valid to \(\epsilon^2\) in the \(\epsilon\) expansion) for the first zero \(\theta_0\) of \(\tilde{h}(\theta)\) [26]. With this simplified parametrization the specific heat \(C_M\) is only a function of \(R\) and the susceptibilities take a compact form. The numerical accuracy of this parametrization is more than sufficient for heavy ion physics.

\(\Delta\) Here our \((h/c_h h_0)\) and \(h_S\) are denoted by \(H\) and \(H_0\) respectively by Ref. [32].
A differential equation is easily obtained for \( g(\theta) \):

\[
(1 - \theta^2)g'(\theta) + 2(2 - \alpha)\theta g(\theta) = (2\beta \theta^2 + (1 - \theta^2))\theta(1 - (\theta/\theta_0)^2)^2.
\] (196)

Integrating the differential equation we find

\[
g(\theta) = \frac{(2\beta - 1) (\theta^2 - 1)^2}{2\alpha \theta_0^2} + \frac{(\theta^2 - 1) ((1 - 2\beta) \theta_0^2 + 4\beta - 1)}{2(\alpha - 1) \theta_0^2} - \frac{\beta (\theta_0^2 - 1)}{(\alpha - 2) \theta_0^2},
\] (197)

up to a homogeneous solution which does not contribute to the singular behavior [25].

From these expressions first derivatives can be obtained

\[
(-s \ h) = \frac{\partial (G_{\text{sing}}/T_c)}{\partial (r, \psi)} - \frac{\partial (R, \theta)}{\partial (r, \psi)} \left( \frac{\partial (R, \theta)}{\partial (r, \psi)} \right)^{-1},
\] (198)

where in practice this Jacobian matrix is evaluated through its inverse

\[
\left( \frac{\partial (R, \theta)}{\partial (r, \psi)} \right) = \left( \frac{\partial (r, \psi)}{\partial (R, \theta)} \right)^{-1}.
\] (199)

The singular entropy density and the singular energy density take the form

\[
\epsilon = \frac{s}{M_0 h_0} = c_{Mch} R^{1 - \alpha} f_{\epsilon}(\theta)
\] (200)

with

\[
f_{\epsilon}(\theta) = \frac{\beta(1 - \delta)(- (1 - \alpha)(2\beta - 1)\theta_0^2 + \alpha + 2\beta - 1)}{2(1 - \alpha)\alpha},
\] (201)

\[
\approx 1.496 - 1.951\theta^2.
\] (202)

From our requirement that on the coexistence line that \( \epsilon/(M_0 h_0) = -|r|^{1-\alpha} \) we find

\[
c_{Mch} = -\frac{(\theta_0^2 - 1)^{1-\alpha}}{f_{\epsilon}(\theta_0)} \approx 0.3486.
\] (203)

In a similar way the susceptibility matrix can be computed by taking second derivatives of the partition function, yielding

\[
\frac{C_M}{M_0 h_0} = c_{Mch} \frac{\gamma(\gamma - 1)}{2\alpha} R^{-\alpha},
\] (204a)

\[
\frac{\chi}{(M_0/ h_0)} = c_{Mch} \left[ 1 + (2\beta\delta - 3)(\delta - 1)\theta_0^2/\delta - 3 \right]^{-1} R^{-\gamma},
\] (204b)

\[
\frac{C_H}{M_0 h_0} = c_{Mch} \frac{\gamma}{2\alpha} \left[ \frac{(2\beta - 1)(\delta - 1)(\beta(\delta + 3) - 3)\theta_0^2 + (\delta - 3)(\gamma - 1)}{(\delta - 1)(2\beta\delta - 3)\theta_0^2 + (\delta - 3)} \right] R^{-\alpha}.
\] (204c)

It is particularly noteworthy that \( C_M \) is independent of the angle \( \theta \).
3. The correlation length

To evaluate the correlation length we used the numerical data from Engels, Fromme and Seniuch (EFS) [32] which is expressed in terms of the scaling variable \( z \) given in Eq. (193). The correlation length takes the scaling form

\[
\xi(h, z) = \left( \frac{h/(c_{h}h_{0})}{h_{S}} \right)^{-\nu/\beta \delta} g_{\xi}(z),
\]

(205)

where \( g_{\xi}(z) \) is a universal function (up to its normalization), which was determined numerically through precise simulations of the Ising model. Even its normalization is not independent of the non-universal parameters, \( \mathcal{M}_{0} \) and \( h_{0} \), introduced previously.

Since \( g_{\xi}(z) \propto z^{-\nu} \) for \( z \) large, the correlation length at zero field and \( T > T_{c} \) behaves as

\[
\xi \rightarrow \xi_{+}r^{-\nu},
\]

(206)

where we have used the definition of \( z \) given in Eq. (193). The length scale \( \xi_{+} \) is not independent of \( \mathcal{M}_{0} \) and \( h_{0} \), but is fixed from the scaling form the free energy

\[
- \frac{F_{\text{sing}}}{T_{c}} = \frac{\log Z_{\text{sing}}(r, h)}{V} = \xi^{-d} F_{\text{sing}}(z),
\]

(207)

where \( d = 3 \) notates the number of spatial dimensions, and \( F_{\text{sing}}(z) \) is a universal function. Comparison with Eq. (195) suggests that \( \mathcal{M}_{0}h_{0}(\xi_{+})^{d} \) should be a universal constant [26]. Indeed, EFS relate \( \xi_{+} \) to the parameters of the equation of state, \( \mathcal{M}_{0} \) and \( h_{0} \), introduced above. Translating their ratio into the current notation we have\(^{16}\)

\[
(c_{M}c_{h}) \mathcal{M}_{0}h_{0} \xi_{+}^{d} = 0.1231.
\]

(209)

In EFS, the numerical data for a normalized \( g_{\xi}(z) \) is presented by comparing it to the scaling function of the susceptibility. Specifically, the susceptibility \( \chi \) (see Eq. (204b)) is written

\[
\chi = \frac{1}{h_{S}} \left( \frac{h/(c_{h}h_{0})}{h_{S}} \right)^{1/\delta - 1} f_{\chi}(z),
\]

(210)

where \( h_{S} \) and its relation to the notation of EFS is given in Eq. (193) and the corresponding footnote. \( f_{\chi}(z) \) has the asymptotic form

\[
f_{\chi} \rightarrow \infty R_{\chi}z^{-\gamma},
\]

(211)

with \( R_{\chi} \simeq 1.723 \). \( g_{\xi}(z) \) is normalized and scaled by \( f_{\chi}(z) \)

\[
g_{\xi}(z) = g_{\xi}(0) (f_{\chi}(z))^{1/2} \left( \frac{g_{\xi}^{2}(z)}{f_{\chi}(z)} \right)^{1/2}.
\]

(212)

\(^{16}\) They define the parameters \( B \) and \( C_{+} \) which in the current notation read:

\[
B = \frac{\theta_{0}}{(\theta_{0}^{2} - 1)^{\beta}} (c_{M}\mathcal{M}_{0}) \simeq 1.6274 (c_{M}\mathcal{M}_{0}), \quad C_{+} = \frac{c_{M}\mathcal{M}_{0}}{c_{h}h_{0}}.
\]

(208)

They numerically determine the amplitude ratio \( Q_{c} = B^{2}(\xi_{+})^{d}/C_{+} = 0.326 \) which determines Eq. (209).
We fit the numerical data in Fig. 11 of EFS with

\[
\frac{\hat{g}_2^2(z)}{f_X(z)} = \frac{(u_+ + u_-) - (u_- - u_+) \tanh((z - x_0)/\sigma)}{2((z - x_0)^2 + 1)^{\nu/2}},
\]

which has the correct asymptotics

\[
\frac{\hat{g}_2^2(z)}{f_X(z)} \xrightarrow{z \to \pm \infty} u_\pm z^{-\nu/2}.
\]

The parameters are \( u_+ \), \( u_- \), and \( \sigma \) from the fit are

\[
 u_+ = 4.133, \quad u_- = 5.32, \quad \sigma = 3, \quad x_0 = 0.3431.
\]

The value \( x_0 = 0.3431 \) is constrained by the universality requirement that at \( z = 0 \) we have \( \hat{g}_2^2/f_X = \delta \). The slight deviation in our fitted values of \( u_+ \) and \( u_- \) from the asymptotic values quoted by EFS (\( u_+ = 4.001 \) and \( u_+/u_- \approx 0.75 \) respectively) stems from a desire to have a somewhat better fit over the full range in \( z \). Finally, with the functional form given in Eq. (213) and the normalization in Eq. (209), the value of \( g_\xi(0) \) of zero can be determined

\[
g_\xi(0) = 0.4838 \left( \frac{c_M c_h M_0 h_0}{d} \right)^{1/d},
\]

where we have unraveled the nested definitions to establish that \( \xi_+ = g_\xi(0)r_\nu S^\nu(u_+ R_X)^{1/2} \).

Summarizing, we use Eqs. (205), (212), (213), and (216) to evaluate the correlation length for any given value of \( h, r \).

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Note that $S$ in the non-relativistic literature typically denotes the entropy per particle $S/N$
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