1. Introduction

The paper has two goals. First goal is to give a complete proof (see Theorem 8) of an old and unpublished result of María Ronco [7], which describes free objects in the category of Leibniz algebras satisfying the identity:

\[ [[x, x], y] = 0. \]

We call such algebras as Ronco algebras. Our interest with theory of Ronco algebras comes from the fact that it is a minimal category of Leibniz algebras (see [2, Theorem 11]) to Ronco algebras and so called Lie \( \mu \)-algebras, see Theorem 12.

2. Preliminaries on second Leibniz homology

In what follows \( K \) is a field of characteristic not equal 2. All vector spaces are taken over \( K \). Moreover, instead \( \otimes_K \) and \( \text{Hom}_K \) we will simply write \( \otimes \) and \( \text{Hom} \).

Recall that [5] a Leibniz algebra is a vector spaces \( L \), equipped with an operation \([-,-]: L \otimes L \to L \) such that

\[ [x, [y, z]] = [[x, y], z] - [x, [z, y]]. \]

We refer the reader to [5, 6, 8, 4, 9, 10] for more on Leibniz algebras, Leibniz homology, Leibniz representations and some conjectures about Leibniz algebras and Leibniz homology. The following identities are consequences of (1):

\[ [x, [y, y]] = 0, \quad [x, [y, z]] + [x, [z, y]] = 0. \]

The category of Leibniz algebras is denoted by \( \mathcal{LB} \). It is clear that Lie algebras are those Leibniz algebras \( L \) for which the identity \( [x, x] = 0 \) holds. Denote by \( \mathcal{L}^{\text{Lie}} \) the subspace of \( L \) generated by elements \( [x, x], x \in L \). Then \( \mathcal{L}^{\text{Lie}} \) is a two-sided ideal of \( L \) and the quotient \( L_{\text{Lie}} \) is a Lie algebra. Moreover the assignment \( L \mapsto L_{\text{Lie}} \) of the left adjoint functor to the inclusion of the category \( \mathcal{LB} \) of Lie algebras in \( \mathcal{LB} \) [5]. It follows that for any Leibniz algebra \( L \) one has an exact sequence

\[ \text{Sym}^2(L) \xrightarrow{\mu} L \to L_{\text{Lie}} \to 0 \]

where \( \text{Sym}^2 \) denotes the second symmetric power and \( \mu(x \otimes y) = [x, y] + [y, x], x, y \in g \). Here \( x \otimes y \) denotes the image of \( x \otimes y \) in \( \text{Sym}^2(L) \) under the canonical map \( L \otimes L \to \text{Sym}^2(L) \).

We refer to [5] for definition of the homology and cohomology of Leibniz algebras. Here we will need mostly the first and the second homologies, which can be defined by

\[ \text{HL}_1(L) = L_{\text{ab}} = L/[L, L], \]

\[ \text{HL}_2(L) = \frac{\text{Ker}([-,-]: L \otimes L \to L)}{\text{Im}(d: L \otimes L \otimes L \to L \otimes L \otimes L)} \]

where

\[ d(x \otimes y \otimes z) = [x, y] \otimes z - [x, z] \otimes y - x \otimes [y, z]. \]

Recall that [5] for any vector space \( a \) considered as an abelian Leibniz algebra, the vector space \( \text{Hom}(\text{HL}_2(L), a) \) classifies all central extensions of \( L \) by \( a \). From this fact and the classical Yoneda Lemma we obtain the following well-known result.

**Lemma 1.** For any Leibniz algebra \( L \) there exists a central extension

\[ 0 \to \text{HL}_2(L) \to \hat{L} \xrightarrow{p} L \to 0 \]

with properties:

(i) The homomorphism \( p \) induces an isomorphism \( \hat{L}_{ab} \cong L_{ab} \), where usual \( L_{ab} = L/[L, L] \).

(ii) For any central extension

\[ 0 \to a \to \h \to L \to 0 \]

there exist a commutative diagram of Leibniz algebras and Leibniz algebra homomorphisms

\[
\begin{array}{ccc}
\hat{L} & \xrightarrow{p} & L \\
\downarrow f & & \downarrow id_L \\
0 & \xrightarrow{1} & 0
\end{array}
\]

\[
\begin{array}{ccc}
a & \xrightarrow{i} & \h \\
\downarrow & & \downarrow adm \\
0 & \xrightarrow{1} & L \\
\end{array}
\]
(iii) If $f' : \mathcal{L} \to \mathfrak{h}$ also fits in the commutative diagram, then there exists a unique linear map $\gamma : \mathcal{L}_{ab} \to a$ such that $f_1 = f + \gamma \circ p$.

In particular there exists a unique $\alpha$ in the commutative diagram

$$
\begin{array}{cccc}
0 & \longrightarrow & \text{HL}_2(\mathcal{L}) & \longrightarrow & \mathcal{L} & \longrightarrow & 0 \\
0 & \longrightarrow & a & \longrightarrow & \mathfrak{h} & \longrightarrow & L & \longrightarrow & 0 \\
\end{array}
$$

In this paper, we are mostly interested in the case when $\mathfrak{g}$ is a Lie algebra. Leibniz homology of Lie algebras was investigated in [8]. We will need some results from that paper. In order to state them we set

$$
\text{HR}_0(\mathfrak{g}) = H_0(\mathfrak{g}, \text{Sym}^2(\mathfrak{g})).
$$

In other words $\text{HR}_0(\mathfrak{g})$ is the quotient of $\text{Sym}^2(\mathfrak{g})$ by the relation $x \odot [y, z] = [x, y] \odot z$.

Observe that for abelian Lie algebra $\mathfrak{g}$ one has $\text{HR}_0(\mathfrak{g}) = \text{Sym}^2(\mathfrak{g})$. Hence for any Lie algebra $\mathfrak{g}$ the abelization map $\mathfrak{g} \to \mathfrak{g}_{ab}$ induces the homomorphism

$$
\text{HR}_0(\mathfrak{g}) \to \text{Sym}^2(\mathfrak{g}_{ab}).
$$

**Lemma 2.** Let $\mathfrak{g}$ be a free nil 2-class Lie algebra generated by a vector space $V$. That is

$$
\mathfrak{g} = V \bigoplus \Lambda^2(V), \quad [u, v] = u \wedge v, \quad \text{and} \quad [v, w] = 0, [w, w'] = 0, u, v, w, \in V, w, w' \in \Lambda^2(V),
$$

Then one has an exact sequence

$$
0 \to \Lambda^3(V) \to \text{HR}_0(\mathfrak{g}) \to \text{Sym}^2(V) \to 0.
$$

**Proof.** Clearly

$$
\text{Sym}^2(\mathfrak{g}) = \text{Sym}^2(V) \bigoplus V \otimes \Lambda^2(V) \bigoplus \text{Sym}^2(\Lambda^2(V))
$$

Now, take $w, w' \in \Lambda^2(V)$, assume $w' = u \wedge v$. Then

$$
w \odot w' = w \odot [u, v] = [w, u] \odot v = 0
$$

Thus the image of $\text{Sym}^2(\Lambda^2(V))$ in $\text{HR}_0(\mathcal{L})$ is zero. We also have

$$
u \odot v \wedge v' = u \odot [v, v'] = v' \odot u \wedge v
$$

where $u, v, v' \in V$. It follows that the kernel of the canonical map $\text{HR}_0(\mathfrak{g}) \to \text{Sym}^2(V)$ is the same as

$$
\frac{V \otimes \Lambda^2(V)}{u \odot v \wedge v' = v' \odot u \wedge v}
$$

The last question is isomorphic to $\Lambda^3(V)$ via the map $u \odot v \wedge v' \mapsto u \wedge v \wedge v'$. To see the last statement it suffices to check this fact in dimensions 1, 2, 3 because the functors in the questions are cubical. □

**Proposition 3.** Let $\mathfrak{g}$ be a Lie algebra. Then the following holds:

i) There is a natural isomorphism

$$
\text{HL}_2(\mathcal{L}) \cong H_1(\mathcal{L}, \mathcal{L}^{\text{ad}}),
$$

where the right hand side is the classical Lie algebra homology with coefficients in the adjoint representation.

ii) There is a natural homomorphism

$$
\text{HR}_0(\mathcal{L}) \to H_1(\mathcal{L}, \mathcal{L}^{\text{ad}})
$$

which is an isomorphism if $H_2(\mathcal{L}) = 0 = H_3(\mathcal{L})$.

**Proof.** All results were proved in [8]. □

**Corollary 4.** For a free Lie algebra $\mathcal{L} = \bigoplus_{n \geq 1} \text{Lie}_n(V)$ one has

$$
\text{HL}_2(\mathcal{L}) \cong H_1(\mathcal{L}, \mathcal{L}^{\text{ad}}) \cong \text{HR}_0(\mathcal{L}) \cong \bigoplus_{n \geq 2} \text{Ker}(\text{Lie}_{n-1}(V) \otimes V \to \text{Lie}_n(V))
$$

**Proof.** Since Lie algebra homology vanishes on free Lie algebras in dimensions $\geq 2$ all statements except the last isomorphism follows from Proposition 3. To prove the last isomorphism, observe that if $\mathcal{L} = \bigoplus_{n \geq 1} \text{Lie}_n(V)$ is a free Lie algebra and $M$ is a $\mathcal{L}$-module, then any derivation $\mathcal{L} \to M$ uniquely defined by a linear map $V \to M$. Hence one has an exacts sequence:

$$
M \xrightarrow{\partial} \text{Hom}(V, M) \to H^1(\mathcal{L}, M) \to 0
$$

where $\partial(m)(v) = [v, m]$. By duality we also have an exact sequence

$$
0 \to H_1(\mathcal{L}, M) \to V \otimes M \xrightarrow{\delta} M
$$

where $\delta(v \otimes m) = [v, m]$. This implies the isomorphism $H_1(\mathcal{L}, \mathcal{L}^{\text{ad}}) \cong \bigoplus_{n \geq 2} R_n(V)$. □

For higher Leibniz homology of a free Lie algebra we refer to [8].
3. Ronco algebras

A Ronco algebra is a Leibniz algebra $X$ if it is a central extension of a Lie algebra. Thus if there exists a central extension

$$0 \to A \to \mathcal{L} \to \mathfrak{h} \to 0,$$

of Leibniz algebras, for which $\mathfrak{h}$ is a Lie algebra.

**Lemma 5.** A Leibniz algebra $\mathcal{L}$ is a Ronco algebra, iff

$$[[x, x], y] = 0.$$  \hfill (5)

holds for all $x, y \in \mathcal{L}$. If this the case, then the following identity holds:

$$[[x, y], z] + [[y, x], z] = 0.$$  \hfill (6)

**Proof.** We have a short exact sequence

$$0 \to \mathcal{L}^{\text{ann}} \to \mathcal{L} \to \mathcal{L}_{\text{Lie}} \to 0.$$  

By the relations (2) we have $[\mathcal{L}, \mathcal{L}^{\text{ann}}] = 0$. Thus if $[[x, x], y] = 0$ holds, then $\mathcal{L}^{\text{ann}}$ is a central subalgebra of $\mathcal{L}$ and hence $\mathcal{L}$ is a Ronco algebra. Conversely, assume $\mathcal{L}$ is a Ronco algebra and (1) is a central extension with $\mathfrak{h} \in \mathcal{L}/\mathcal{L}$. Then for any $x \in \mathcal{L}$ we have $[x, x] \in A$ and hence $[[x, x], y] \in [A, \mathcal{L}] = 0$.  \hfill $\square$

Thus any symmetric Leibniz algebra $\mathfrak{a}$ is a Ronco algebra. The collection of all Ronco algebras is denoted by $\mathfrak{R}$. Thus $\mathfrak{R}$ is a variety of Leibniz algebras. Hence it posses free algebras.

**Lemma 6.** Take a free Lie algebra $\mathcal{L}$ on a vector space $V$, that is $\mathcal{L} = \bigoplus_{n \geq 1} \text{Lie}_n(V)$, $\text{Lie}_1(V) = V$ and consider the central extension $0 \to \text{HL}_2(\mathcal{L}) \to \tilde{\mathcal{L}} \to \mathcal{L} \to 0$ as in Lemma[1]. Choose a linear section $s : \mathcal{L} \to \tilde{\mathcal{L}}$ of $p$. If we identify $s(V)$ and $V$, then $\tilde{\mathcal{L}}$ is free object in $\mathfrak{R}$ generated by $V$.

**Proof.** Take $\mathfrak{h} \in \mathfrak{R}$ and a linear map $f : V \to \mathfrak{h}$. We have to show that $f$ has a unique extension (still denoted by $f$) $\tilde{\mathcal{L}} \to \mathfrak{h}$ which is Leibniz algebra homomorphism. First we show the uniqueness. Assume $f$ has two extensions $f_1, f_2 : \tilde{\mathcal{L}} \to \mathfrak{h}$. Since $(-)_{\text{Lie}}$ is a functor we obtain corresponding homomorphisms of Lie algebras $(f_1)_{\text{Lie}}, (f_2)_{\text{Lie}} : \mathcal{L} \to \mathfrak{h}_{\text{Lie}}$. Since both of them agree on $V$ and $\mathcal{L}$ is free on $V$, we see that $(f_1)_{\text{Lie}} = (f_2)_{\text{Lie}}$. Denote this common value by $f_{\text{Lie}}$. Then we have $f_2 = f_1 + \gamma \circ p$, thanks to Lemma[1] for a linear map $\gamma : \mathcal{L}_{\text{ab}} \to \mathfrak{a}$. Since $f_1 = f_2$ on $V$ and the composite $V \to \mathcal{L} \to \mathcal{L}_{\text{ab}}$ is an isomorphism, we see that $\gamma = 0$. Thus $f = f_2$ on $\tilde{\mathcal{L}}$ and the uniqueness follows.

For existence, observe that $f$ has a unique extension to a homomorphism of Lie algebras $f' : \mathcal{L} \to \mathfrak{h}_{\text{Lie}}$, because $\mathcal{L}$ is free Lie algebra. Since $\mathfrak{h} \in \mathfrak{R}$ the short exact sequence

$$0 \to \mathfrak{h}^{\text{ann}} \to \mathfrak{h} \to \mathfrak{h}_{\text{Lie}} \to 0$$

is a central extension of the Lie algebra $\mathfrak{h}_{\text{Lie}}$. Thus by Lemma[1] we can form the following commutative diagram of central extensions

\[
\begin{array}{ccc}
0 & \rightarrow & \text{HL}_2(\mathcal{L}) \\
\downarrow & & \downarrow p \\
0 & \rightarrow & \tilde{\mathcal{L}} \\
\downarrow g' & & \downarrow id \\
0 & \rightarrow & \mathcal{L} \\
\downarrow id & & \downarrow \gamma \\
0 & \rightarrow & \mathfrak{h} \\
\end{array}
\]

where $\mathfrak{h}'$ is the pull-back in the corresponding diagram. It follows that for any $v \in V$, we have $f(v) - gg'(v) \in \mathfrak{a}$. Since $V \cong \mathcal{L}_{\text{ab}}$ is an isomorphism, we have a unique linear map $\gamma : \mathcal{L}_{\text{ab}} \to \mathfrak{a}$ for which $\gamma(v) = f(v) - gg'(v)$, $v \in V$. It follows that $gg' + \gamma \circ p$ is a Leibniz homomorphism, which extends the map $f : V \to \mathfrak{h}$.  \hfill $\square$

**Corollary 7.** The theory Ronco of Ronco algebras fits in the linear extension of algebraic theories [1]:

$$0 \rightarrow \text{HR}_0 \rightarrow \text{Ronco} \rightarrow \text{Lie} \rightarrow 0$$

Recall that [5] the free Leibniz algebra generated by a vector space $V$ is

$$\text{Leib}(V) = V \bigoplus V^\otimes 2 \bigoplus V^\otimes 3 \bigoplus \cdots = \bigoplus_{n \geq 1} V^\otimes n,$$

where the bracket is uniquely defined by the property

$$[\omega, v] = \omega \otimes v.$$  

Here $\omega \in V^\otimes n$ and $v \in V$.

The following result first was obtained by María Ronco [7] in her unpublished notes written around 1995.
Theorem 8. Let $V$ be a vector space. Then there exists a unique bracket $-,-$ on
\[ \mathcal{R}(V) = V \bigoplus_{n \geq 1} \text{Lie}_n(V) \otimes V \]
such that the canonical map $\text{Leib}(V) \to \mathcal{R}(V)$, induced by the projection
\[ V^\otimes n \to \text{Lie}_n^{-1}(V) \otimes V, v_1 \otimes \cdots \otimes v_n \mapsto \{\{v_1,v_2\},\ldots,v_{n-1}\} \otimes v_n \]
preserves the bracket. Moreover $\mathcal{R}(V)$ is a free object Ronco algebra generated by $V$. Here $\{-,-\}$ is the bracket on $\text{Lie}(V)$.

Proof. We keep the notation from Lemma 6. Thus $L$ denotes the free Lie algebra generated by $V$ and $\hat{L}$ is the free algebra in $\mathcal{R}$ generated by $V$. Then we have a canonical epimorphism $f : \text{Leib}(V) \to \hat{L}$. Thanks to Lemma 9 below the map $f$ factors through the map
\[ g : \mathcal{R}(V) \to \hat{L}. \]
By construction of $\mathcal{R}(V)$ and by Lemma 6 this map fits in the following diagram
\[
\begin{array}{ccc}
0 & \longrightarrow & \bigoplus_{n \geq 1} R_n(V) \\
\downarrow & & \downarrow g \\
\mathcal{R}(V) & \longrightarrow & L \\
\downarrow & & \downarrow id \\
0 & \longrightarrow & \text{HL}_2(L) \\
\end{array}
\]
By Proposition 3 the first vertical map is an isomorphism. Hence $g$ is also an isomorphism and the result follows. \hfill \Box

Lemma 9. Let $f : L \to L'$ be a Leibniz algebra homomorphism. If $L' \in \mathcal{R}$ and $x \in \text{Ker}(L \to \text{Lie}_L)$, then $f([x,y]) = 0$ for all $y \in L$.

Proof. By the exact sequence (3), $x$ is a linear combination of elements of the form $[u,v] + [v,u]$. Hence the result follows from the identity (6). \hfill \Box

4. $\mu$-ALGEBRAS

Definition 10. A $\mu$-algebra is a vector space $m$ equipped with two binary operations $\mu : m \otimes m \to m$ and $\{-,-\} : m \otimes m \to m$
such that the following identities hold
\begin{enumerate}
  \item $xy = yx,$
  \item $(xyz) = 0 = (xy)z$ \\
  \item $\{x,x\} = 0$ and hence $\{x,y\} + \{y,x\} = 0$,
  \item $\{xy,z\} = 0$ \\
  \item $\{x,\{y,z\}\} + \{z,\{x,y\}\} + \{y,\{z,x\}\} = x\{y,z\}$
\end{enumerate}
Here $xy = \mu(x \otimes y)$.

A $\mu$-algebra is symmetric if $x\{y,z\} = 0$. So, in this case $\{-,-\}$ is a Lie algebra structure. These algebras were studied in [2].

Lemma 11. The function $(x,y,z) \mapsto x\{y,z\}$ is skew-symmetric. In particular, $x\{y,z\} = 0$ and hence $x\{y,z\} + y\{x,z\} = 0$.

Proof. This is a direct consequence of $v)$. \hfill \Box

Theorem 12. The category of $\mu$-algebras is equivalent to the category of Ronco algebras, under this equivalence symmetric $\mu$-algebra corresponds to symmetric Leibniz algebras.

The theorem is a trivial consequence of Propositions 13 and 14.

Proposition 13. For a Ronco algebra $L$, we put:
\[ 2\{x,y\} = [x,y] - [y,x], \]
\[ 2xy = [x,y] + [y,x]. \]
(Thus $[x,y] = \{x,y\} + xy$.) Then $L$ together with operations $\{-,-\}$ and $\mu(x,y) = xy$ is a $\mu$-algebra.

Proof. The relations i) and iii) are obvious. By the relation (2) and (3) we have
\[ [xy,z] = 0 \quad \text{and} \quad [x,yz] = 0. \]
Hence
\[ 2(xy)z = [xy,z] + [z,xy] = 0 + 0 = 0. \]
Similarly,
\[ 2xz = [x,zy] + [yz,x] = 0 \]
and ii) follows. Quite similarly,
\[ 2\{x,y\} = [xy,z] - [z,xy] = 0 - 0 = 0 \]
and iv) follows.
Next, we have
\[ 4x\{y, z\} = [x, [y, z] - [z, y] + [[y, z] - [z, y], x] \]
By (2) and (6) we can rewrite:
\[ 4x\{y, z\} = 2[x, [y, z]] + 2[[y, z], x] = 2[x, [y, z]] - 2[[x, z], y] + 2[[y, z], x]. \]
Let us use (6) once more, to obtain
\[ 2x\{y, z\} = [[x, y], z] + [[z, x], y] + [[y, z], x] = \sum_{\text{cyclic}} [[x, y], z] \]
Next, we consider
\[ 4\{x, \{y, z\}\} = [x, [y, z] - [z, y]] - [[y, z] - [z, y], x] = 2[x, [y, z]] + 2[[z, y], x] \]
and hence
\[ 2\{x, \{y, z\}\} = [[x, y], z] - [[x, z], y] + [[z, x], y] + [[z, y], x]. \]
So, we proved
\[ 2\{x, \{y, z\}\} = [[x, y], z] + [[z, x], y] - [[y, z], x] \]
This implies
\[ 2 \sum_{\text{cyclic}} \{x, \{y, z\}\} = \sum_{\text{cyclic}} [[x, y], z] \]
This together (7) implies v).

\[ \square \]

**Proposition 14.** Let \( \mathfrak{m} \) be a \( \mu \)-algebra. We put
\[ [x, y] = \{x, y\} + xy. \]
Then \( [\cdot, \cdot] \) defines a Ronco algebra structure on \( \mathfrak{m} \).

**Proof.** We have
\[ [x, [y, z]] = x(yz + \{y, z\}) + \{x, yz + \{y, z\}\} = x\{y, z\} + \{x, [y, z]\} \]
Quite similarly
\[ [[x, y], z] = (xy + \{x, y\})z + \{xy + \{x, y\}, z\} = \{x, y\}z + \{\{x, y\}, z\} \]
and
\[ [[x, z], y] = \{x, z\}y + \{\{x, z\}, y\} \]
Hence
\[ [[x, y], z] - [[x, z], y] + [[z, x], y] = x\{y, z\} - \{x, y\}z + \{x, z\}y + \{x, [y, z]\} - \{\{x, y\}, z\} + \{\{x, z\}, y\} \]
By i) of Definition (10) and Lemma (11) the first line equals to \(-x\{y, z\}\). By iii) of Definition (10) the second line equals to
\[ \{x, \{y, z\}\} + \{z, \{x, y\}\} + \{y, \{z, x\}\} \]
which is equal to \(x\{y, z\}\) thanks to vi) of Definition (10). It follows that \([x, [y, z]] - [[x, y], z] + [[z, x], y] = 0\) and hence \( \mathfrak{m} \) is a Leibniz algebra. We also have \([x, x] = xx\), hence \([[x, x], y] = [xx, y] = 0 \) and the result follows.

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