AN EXTENSION OF THE POLYAK CONVEXITY PRINCIPLE WITH APPLICATION TO NONCONVEX OPTIMIZATION

AMOS UDERZO

Abstract. The main problem considered in the present paper is to single out classes of convex sets, whose convexity property is preserved under nonlinear smooth transformations. Extending an approach due to B.T. Polyak, the present study focuses on the class of uniformly convex subsets of Banach spaces. As a main result, a quantitative condition linking the modulus of convexity of such kind of set, the regularity behaviour around a point of a nonlinear mapping and the Lipschitz continuity of its derivative is established, which ensures the images of uniformly convex sets to remain uniformly convex. Applications of the resulting convexity principle to the existence of solutions, their characterization and to the Lagrangian duality theory in constrained nonconvex optimization are then discussed.

1. Introduction

In many fields of mathematics, persistence phenomena of specific geometrical properties under various kind of transformations have been often a subject of interest and study. Transformations, when possible formalized by mappings acting among spaces, sometimes have been classified on the base of aspects in a structure that they can preserve (whence the very term “morphism”). Convexity is a geometrical property which emerged in ancient times, at the very beginning of geometry, and since then remained unchanged for almost two millennia and half. This happened by virtue of the great variety of successful applications that it found in many different areas. In particular, the relevant role played by convexity in optimization and control theory is widely recognized. This led to develop a branch of mathematics, called convex analysis, that elected convexity as its main topic of study. In spite of such an interest and motivations, not much seems to be known up to now about phenomena of persistence of convexity under nonlinear transformations. Yet, advances in this direction would have a certain impact on the analysis of optimization problems. Historically, the first results somehow connected with the issue at the study relate to the numerical
range of quadratic mappings (namely, mappings whose components are quadratic forms) and can be found in [6]. A notable step ahead was made when the preservation of convexity of small balls under smooth regular transformations between Hilbert spaces was established by B.T. Polyak (see [15]). After that, some other contributions to understanding the phenomenon in a similar context were given by [3, 19]. Various applications of it to linear algebra, optimization and control theory are presented in [15, 16, 17, 19].

In the present paper, by following the approach introduced by B.T. Polyak, the study of classes of sets with persistent convexity properties is carried on. More precisely, the analysis here proposed focusses on the class of uniformly convex subsets of certain Banach spaces. An interest in similar classes of sets, in connection with the problem under study, appears already in [15], where strongly convex sets are actually mentioned. This seems to be rather natural, inasmuch as elements of such classes share the essential geometrical features of balls in a Hilbert space: nonempty interior, boundedness and, what plays a crucial role, a uniform rotundity, which implies a boundary consisting of only extreme points. The feature last mentioned is captured and quantitatively expressed by the notion of modulus of convexity of a set. According to the spirit of the Polyak’s approach, the main idea behind the investigations exposed in the paper is that, if the modulus of convexity of a given set matches the smoothness and the regularity property of a given nonlinear mapping, then the persistence of convexity under that mapping can be guaranteed.

The contents of the paper are arranged in the next sections as follows. In Section 2, the notion of modulus of convexity of a set and of uniformly convexity are recalled, along with several examples and related facts, useful for the subsequent analysis. Besides, the regularity behaviour of a nonlinear smooth mapping, namely openness at a linear rate, is entered as a crucial tool, along with the related exact bound. In Section 3, the main result of the paper, which is an extension of the aforementioned convexity principle due to B.T. Polyak, is established and some of its features are discussed. In Section 4, some applications of the main result to nonconvex constrained optimization problems are provided.

2. Notations and preliminaries

The basic notations in use throughout the paper are as follows. \( \mathbb{R} \) denotes the real number set. Given a metric space \( (X, d) \), an element \( x_0 \in X \) and \( r \geq 0 \), \( B(x_0, r) = \{x \in X : d(x, x_0) \leq r\} \) denotes the (closed) ball with center \( x_0 \) and radius \( r \). In particular, in a Banach space, the unit ball centered at the null vector will be indicated by \( \mathbb{B} \), whereas the unit sphere by \( \mathbb{S} \). The distance of \( x_0 \in X \) from a set \( S \subseteq X \) is denoted by \( \text{dist}(x_0, S) \). If \( S \subseteq X \), \( B(S, r) = \{x \in X : \text{dist}(x, S) \leq r\} \) denotes the (closed) \( r \)-enlargement of \( S \). The diameter of a set \( S \subseteq X \) is defined as \( \text{diam} S = \sup\{d(x_1, x_2) : x_1, x_2 \in S\} \). By \( \text{int} S, \ cl S \) and \( \text{bd} S \) the
topological interior, the closure and the boundary of a set \( S \) are marked, respectively. If \( S \) is a subset of a Banach space \((X, \| \cdot \|)\), \( \text{ext} \, S \) denotes the set of all extreme points of \( S \), in the sense of convex analysis, \( 0 \) stands for the null element of \( X \) and \([x_1, x_2]\) denotes the closed line segment with endpoints \( x_1, x_2 \in X \). Given a function \( h : X \to Y \) between metric spaces and a set \( U \subseteq X \), \( h \) is said to be Lipschitz continuous on \( U \) if there exists a constant \( \ell > 0 \) such that \( d(h(x_1), h(x_2)) \leq \ell d(x_1, x_2) \), for every \( x_1, x_2 \in U \). The infimum over all values \( \ell \) making the last inequality satisfied on \( U \) is called exact bound of Lipschitz continuity of \( h \) on \( U \) and is denoted by \( \text{lip}(h, U) \). The Banach space of all bounded linear operators between the Banach spaces \( X \) and \( Y \), equipped with the operator norm, is denoted by \((\mathcal{L}(X, Y), \| \cdot \|_\mathcal{L})\). If, in particular, it is \( Y = \mathbb{R} \), the simpler notation \((X^*, \| \cdot \|_*)\) is used. The null vector in a dual space is marked by \( 0^* \), whereas the unit sphere by \( S^* \), with \((\cdot, \cdot)\) marking the duality pairing a space and its dual. If \((x_n)_n\) is a sequence in \( X \), then \( x_n \xrightarrow{w} x \) indicates the convergence, as \( n \to \infty \), with respect to the weak topology \( \sigma(X, X^*) \) on \( X \). Given a mapping \( f : \Omega \to Y \), with \( \Omega \) open subset of \( X \), and \( x_0 \in \Omega \), the Gâteaux derivative of \( f \) at \( x_0 \) is denoted by \( Df(x_0) \). If \( f \) is Gâteaux differentiable at each point of \( \Omega \) and the mapping \( Df : \Omega \to \mathcal{L}(X, Y) \) is Lipschitz continuous on \( \Omega \), \( f \) is said to be of class \( C^{1,1}(\Omega) \).

**Remark 2.1.** (i) In view of a subsequent employment, let us recall that, whenever \( f : \Omega \to Y \) is a mapping of class \( C^{1,1}(\Omega) \) between Banach spaces, with \( \Omega \) open subset of \( X \) and \( x_1, x_2 \in \Omega \) are such that \([x_1, x_2] \subseteq \Omega \), the following estimate holds true (see, for instance, [20] Lemma 2.7)

\[
\left\| \frac{f(x_1) + f(x_2)}{2} - f \left( \frac{x_1 + x_2}{2} \right) \right\| \leq \frac{\text{lip}(Df, \Omega)}{8} \|x_1 - x_2\|^2.
\]

(ii) It is not difficult to see that, if \( S \subseteq \Omega \) is a bounded set, i.e. \( \text{diam} \, S < +\infty \), and \( f \in C^{1,1}(\Omega) \), then it must be

\[
\sup_{x \in S} \|Df(x)\|_\mathcal{L} < +\infty.
\]

Therefore, letting \( \beta_S = \sup_{x \in S} \|Df(x)\|_\mathcal{L} \), as an immediate consequence of the mean-value theorem, one obtains

\[
\text{diam} \, f(S) \leq \beta_S \text{diam} \, S,
\]

that is \( f(S) \) is bounded too.

### 2.1. Uniformly convex sets.

**Definition 2.2.** (i) Let \( S \subseteq X \) be a nonempty, closed and convex subset of a real Banach space. The function \( \delta_S : [0, \text{diam} \, S) \to [0, +\infty) \) defined by

\[
\delta_S(\epsilon) = \sup \left\{ \delta \geq 0 : B \left( \frac{x_1 + x_2}{2}, \delta \right) \subseteq S, \forall x_1, x_2 \in S : \|x_1 - x_2\| = \epsilon \right\}
\]
is called *modulus of convexity* of the set $S$. Whenever the value of $\text{diam } S$ is attained at some pair $x_1, x_2 \in S$, the function $\delta_S$ will be meant to be naturally extended to $[0, \text{diam } S]$. 

(ii) After [13], a nonempty, closed and convex set $S \subseteq X$, with $S \neq X$, is said to be *uniformly convex* provided that

$$\delta_S(\epsilon) > 0, \quad \forall \epsilon \in \left\{ (0, \text{diam } S], \ 0, \text{diam } S) \right\},$$

if $\text{diam } S$ is attained on $S$, otherwise.

Since $\text{diam } S$ vanishes if $S$ is a singleton, Definition 2.2 (ii) does not exclude such kind of convex sets. Nevertheless, as singletons are of minor interest in connection with the problem at the issue, henceforth a uniformly convex set will be always assumed to contain at least two distinct points.

**Example 2.3.** (i) Balls in a uniformly convex Banach space may be viewed as a paradigm for the notion of uniform convexity for sets. Recall that, after [4], a Banach space $(X, \| \cdot \|)$ is said to be uniformly convex (or to have a uniformly convex norm) if

$$\delta_X(\epsilon) = \inf \left\{ 1 - \frac{x_1 + x_2}{2} : x_1, x_2 \in B, \ |x_1 - x_2| = \epsilon \right\} > 0, \forall \epsilon \in (0, 2].$$

The function $\delta_X$ is called modulus of convexity of the space $(X, \| \cdot \|)$. In fact, it is possible to prove that

$$\delta_B(\epsilon) = \delta_X(\epsilon), \quad \forall \epsilon \in (0, 2].$$

Such classes of Banach spaces as $l^p$ and $L^p$, with $1 < p < \infty$, are known to consist of uniformly convex spaces. In particular, every Hilbert space is uniformly convex. Since every uniformly convex Banach space must be reflexive (according to the Milman-Pettis Theorem), the spaces $l^1, L^1, L^\infty, C([0, 1])$ and $c_0$ fail to be. For $p \geq 2$, the exact expression of the modulus of convexity of the spaces $l^p$ and $L^p$ is given by

$$\delta_{l^p}(\epsilon) = \delta_{L^p}(\epsilon) = 1 - \left[ 1 - \left( \frac{\epsilon}{2} \right)^p \right]^{1/p}, \quad \forall \epsilon \in (0, 2].$$

For more details on uniformly convex Banach spaces and properties of their moduli the reader may refer to [5, 10, 12]. A useful remark enlightening the connection between the notions of uniform convexity for sets and uniform convexity of Banach spaces can be found in [1, Theorem 2.3]: a Banach space can contain a closed uniformly convex set iff it admits an equivalent uniformly convex norm. Such class of Banach spaces have been characterized in terms of superreflexivity in [9]. Throughout the present paper, the Banach space $(X, \| \cdot \|)$ will be supposed to be equipped with a uniformly convex norm.

(ii) After [18], any subset of a Banach space $S \subseteq X$ of the form

$$S = \bigcap_{x \in M} B(x, r) \neq \emptyset,$$

where $r$ is a fixed positive real and $M \subseteq X$ is an arbitrary subset, with $M \neq X$, is called $r$-convex or strongly convex of radius $r$. It is readily seen
that, if a Banach space \((X, \| \cdot \|)\) is uniformly convex with modulus \(\delta_X\), then any strongly convex set \(S \subseteq X\) with radius \(r\) is uniformly convex and its modulus of convexity satisfies the relation

\[
\delta_S(\varepsilon) \geq r \delta_X \left( \frac{\varepsilon}{r} \right), \quad \forall \varepsilon \in (0, \text{diam } S).
\]

(2.2)

(iii) Let \(\theta : [0, +\infty) \to [0, +\infty)\) be an increasing function vanishing only at 0. Recall that, according to [21], a function \(\varphi : X \to \mathbb{R}\) is said to be uniformly convex with modulus \(\theta\) if it holds

\[
\varphi(tx_1 + (1 - t)x_2) \leq t\varphi(x_1) + (1 - t)\varphi(x_2) - t(1 - t)\theta(\|x_1 - x_2\|),
\]

\forall x_1, x_2 \in X, \forall t \in [0, 1].

If, in particular, \(\theta(s) = \kappa s^2\), a uniformly convex function with such a modulus is called strongly convex. Sublevel sets of Lipschitz continuous uniformly convex functions are uniformly convex sets. More precisely, given \(\alpha > 0\), if \(\varphi\) is Lipschitz continuous on \(X\), with exact bound \(\text{lip}(\varphi, X) > 0\), then the set \([\varphi \leq \alpha] = \{x \in X : \varphi(x) \leq \alpha\}\) turns out to be uniformly convex with modulus

\[
\delta_{[\varphi \leq \alpha]}(\varepsilon) \geq \frac{\theta(\varepsilon)}{4\text{lip}(\varphi, X) + \varepsilon}, \quad \forall \varepsilon \in (0, \text{diam } [\varphi \leq \alpha]).
\]

(2.3)

Indeed, fixed \(\varepsilon \in (0, \text{diam } [\varphi \leq \alpha])\), take \(x_1, x_2 \in [\varphi \leq \alpha]\), with \(x_1 \neq x_2\) and \(\|x_1 - x_2\| = \varepsilon\), and set \(\bar{x} = \frac{1}{2}(x_1 + x_2)\). By the uniform convexity of \(\varphi\) with modulus \(\theta\) one has

\[
\varphi(\bar{x}) \leq \frac{\varphi(x_1) + \varphi(x_2)}{2} - \frac{\theta(\|x_1 - x_2\|)}{4}.
\]

Therefore, for an arbitrary \(\eta > 0\), by the Lipschitz continuity of \(\varphi\) on \(X\), one finds

\[
\varphi(x) = \varphi(x) - \varphi(\bar{x}) + \varphi(\bar{x}) \\
\leq (\text{lip}(\varphi, X) + \eta) \frac{\theta(\varepsilon)}{4(\text{lip}(\varphi, X) + \eta)} + \alpha - \frac{\theta(\varepsilon)}{4} \leq \alpha,
\]

for every \(x \in B\left(\bar{x}, \frac{\theta(\varepsilon)}{4(\text{lip}(\varphi, X) + \eta)}\right)\). Thus, it results in

\[
B\left(\bar{x}, \frac{\theta(\varepsilon)}{4(\text{lip}(\varphi, X) + \eta)}\right) \subseteq [\varphi \leq \alpha],
\]

so

\[
\delta_{[\varphi \leq \alpha]}(\varepsilon) \geq \frac{\theta(\varepsilon)}{4(\text{lip}(\varphi, X) + \eta)}.
\]

The estimate in (2.3) follows by arbitrariness of \(\eta\).

It is not difficult to see that, given two subsets \(S_1\) and \(S_2\) of \(X\), it is \(\delta_{S_1 \cap S_2} \geq \min\{\delta_{S_1}, \delta_{S_2}\}\). Therefore, the class of uniformly convex sets is closed under finite intersection. In the next remark, some known facts about uniformly convex sets are collected, which will be relevant to the subsequent analysis.
Remark 2.4. (i) Every uniformly convex set, which does not coincide with the entire space, is bounded (see [1]).

(ii) Directly from Definition 2.2, it follows that every uniformly convex set has nonempty interior. This fact entails that, while uniformly convex subsets are compact if living in finite-dimensional spaces, they can not be so in infinite-dimensional Banach spaces.

(iii) As a consequence of Definition 2.2, if any uniformly convex set \( S \) admits a modulus of convexity of power type 2, i.e. such that
\[
\delta_S(\varepsilon) \geq c\varepsilon^2, \quad \forall \varepsilon \in (0, \text{diam } S),
\]

for some \( c > 0 \), then it fulfills the following property: for every \( \tilde{c} \in (0, c) \) it holds
\[
B \left( \frac{x_1 + x_2}{2}, \tilde{c}\|x_1 - x_2\| \right) \subseteq S, \quad \forall x_1, x_2 \in S.
\]

It is worth noting that this happens for the balls in any Hilbert space or in the Banach spaces \( l^p \) and \( L^p \), with \( 1 < p < 2 \), where the following estimate is known to hold
\[
\delta_{l^p}(\varepsilon) = \delta_{L^p}(\varepsilon) > \frac{p-1}{8}\varepsilon^2, \quad \forall \varepsilon \in (0, 2]
\]
(see, for instance, [12]).

(iv) For every uniformly convex set \( S \), a constant \( \beta > 0 \) can be proved to exist such that
\[
\delta_S(\varepsilon) \leq \beta\varepsilon^2, \quad \forall \varepsilon \in (0, \text{diam } S)
\]
(see [1]). Thus, a modulus of convexity of the power 2 is a maximal one.

The next proposition provides a complete characterization of uniform convexity for subsets of a finite-dimensional Euclidean space in terms of extremality of their boundary points. Below, a variational proof of this fact is provided.

Proposition 2.5. A convex compact subset \( S \subseteq \mathbb{R}^n \), with nonempty interior, is uniformly convex iff \( \text{ext } S = \text{bd } S \).

Proof. Observe that by compactness of \( S \), it is \( \text{bd } S \neq \emptyset \). Actually, the Krein-Milman theorem ensures that \( \text{ext } S \neq \emptyset \) also. Clearly, it is \( \text{ext } S \subseteq \text{bd } S \). To begin with, assume that \( S \) is uniformly convex. Take any \( \bar{x} \in \text{bd } S \). If it were \( \bar{x} \notin \text{ext } S \), then there would exist \( x_1, x_2 \in S \setminus \{\bar{x}\} \), with \( x_1 \neq x_2 \), such that \( \bar{x} = \frac{x_1 + x_2}{2} \). Observe that, as \( \bar{x} \in \text{bd } S \), the inclusion \( B(\bar{x}, \delta) \subseteq S \) can be true only for \( \delta = 0 \). Thus \( \delta_S(\|x_1 - x_2\|) = 0 \), contradicting the fact that \( S \) is uniformly convex.

Conversely, assume that the equality \( \text{ext } S = \text{bd } S \) holds true. Fix an arbitrary \( \varepsilon \in (0, \text{diam } S) \) (under the current hypotheses the value \( \text{diam } S \) is attained on \( S \)). Notice that, since \( S \) is compact, the set
\[
S_\varepsilon^2 = \{(x_1, x_2) \in S \times S : \|x_1 - x_2\| = \varepsilon\}
\]
is still compact. Define the function \( \vartheta : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty) \) by setting
\[
\vartheta(x_1, x_2) = \text{dist} \left( \frac{x_1 + x_2}{2}, \mathbb{R}^n \setminus \text{int} S \right).
\]
Since such a function is continuous on \( \mathbb{R}^n \times \mathbb{R}^n \), it attains its global minimum over \( S_\epsilon^2 \) at some point \((\hat{x}_1, \hat{x}_2) \in S_\epsilon^2\), with \( \hat{x}_1 \neq \hat{x}_2 \) as \( \|\hat{x}_1 - \hat{x}_2\| = \epsilon \). If it were \( \vartheta(\hat{x}_1, \hat{x}_2) = 0 \), then it would happen that
\[
\frac{\hat{x}_1 + \hat{x}_2}{2} \in \text{bd} S.
\]
The last inclusion contradicts the fact that \( \frac{\hat{x}_1 + \hat{x}_2}{2} \) is an extreme point of \( S \). Therefore, one deduces that \( \vartheta(\hat{x}_1, \hat{x}_2) > 0 \). As it is true that
\[
\delta_S(\epsilon) = \min_{(x_1, x_2) \in S_\epsilon^2} \vartheta(x_1, x_2) > 0,
\]
the requirement in Definition 2.2 (ii) turns out to be satisfied. The arbitrariness of \( \epsilon \in [0, \text{diam } S] \) completes the proof. \( \square \)

Proposition 2.5 can not be extended to infinite-dimensional spaces, where balls with \( \text{ext } \mathcal{B} = \text{bd } \mathcal{B} \) can exist, yet failing to be uniformly convex (see [5]).

2.2. Openness at a linear rate. In the next definition, some notions and related results are recalled, which describe quantitatively a certain surjective behaviour of a mapping. Such a local property, in a synergical interplay with other features (\( C^{1,1} \)-smoothness and uniform convexity) of the involved objects, allows one to achieve the main result in the paper.

Definition 2.6. Let \( f : X \to Y \) be a mapping between two metric spaces and \( x_0 \in X \). The mapping \( f \) is said to be open at a linear rate around \( x_0 \) if there exist positive reals \( \delta, \zeta \) and \( \sigma \) such that
\[
\text{(2.5)} \quad f(\mathcal{B}(x, r)) \supseteq \mathcal{B}(f(x), \sigma r) \cap \mathcal{B}(f(x_0), \zeta), \quad \forall x \in \mathcal{B}(x_0, \delta), \forall r \in [0, \delta].
\]

It is well known (see, for instance, [8, 13]) that the property of openness at a linear rate for a mapping \( f \) around \( x_0 \) can be equivalently reformulated as follows: there exist positive reals \( \delta \) and \( \kappa \) such that
\[
\text{(2.6)} \quad \text{dist} \left( x, f^{-1}(y) \right) \leq \kappa d(y, f(x)), \quad \forall x \in \mathcal{B}(x_0, \delta), \forall y \in \mathcal{B}(f(x_0), \delta).
\]
Whenever the inequality (2.6) holds, \( f \) is said to be metrically regular around \( x_0 \). The infimum over all values \( \kappa \) for which there exists \( \delta > 0 \) such that (2.6) holds true is called exact regularity bound of \( f \) around \( x_0 \) and it will be denoted by \( \text{reg}(f, x_0) \), with the convention that \( \text{reg}(f, x_0) = +\infty \) means that \( f \) fails to be metrically regular around \( x_0 \).

Remark 2.7. (i) It is convenient to note that, whenever \( f \) is continuous at \( x_0 \), the inclusion defining the openness of \( f \) at a linear rate around \( x_0 \) takes the simpler form: there exists positive \( \delta \) and \( \sigma \) such that
\[
\text{(2.7)} \quad f(\mathcal{B}(x, r)) \supseteq \mathcal{B}(f(x), \sigma r), \quad \forall x \in \mathcal{B}(x_0, \delta), \forall r \in [0, \delta].
\]
(ii) From the inclusion (2.7) it is clear that, whenever a mapping \( f \) is open at a linear rate around \( x_0 \) and continuous at the same point, it holds
\[
(2.8) \quad f(\text{int } S) \subseteq \text{int } f(S),
\]
provided that \( S \subseteq B(x, \delta) \), where \( \delta \) is as above. Indeed, if it is \( x \in \text{int } S \), then for some \( r \in (0, \delta) \) it must be \( B(x, r) \subseteq S \). Therefore, one gets
\[
B(f(x), \sigma r) \subseteq f(B(x, r)) \subseteq f(S).
\]
In turn, from the inclusion (2.8), one deduces
\[
f^{-1}(y) \cap S \subseteq \text{bd } S, \quad \forall y \in \text{bd } f(S).
\]

As the behaviour formalized by openness at a linear rate/metric regularity plays a crucial role in a variety of topics in variational analysis, it has been widely investigated in the past decades and several criteria for detecting the occurrence of it are now at disposal. In the case of smooth mappings between Banach spaces, the main criterion for openness at a linear rate/metric regularity, known under the name of Lyusternik-Graves theorem, can be stated as follows (see [8, 13]).

**Theorem 2.8 (Lyusternik-Graves).** Let \( f : X \rightarrow Y \) be a mapping between Banach spaces. Suppose that \( f \) is strictly differentiable at \( x_0 \in X \). Then, \( f \) is open at a linear rate around \( x_0 \) iff \( Df(x_0) \) is onto, i.e. \( Df(x_0)(X) = Y \).

The above criterion is usually complemented with the following (primal and dual) estimates of the exact regularity bound, which are relevant for the present analysis:
\[
\text{reg}(f, x_0) = \sup_{\|y\| \leq 1} \inf \{\|x\| : x \in Df(x_0)^{-1}(y)\}
\]
and
\[
\text{reg}(f, x_0) = \left( \inf_{\|y^*\| = 1} \|Df(x_0)^*y^*\|_* \right)^{-1} = (\text{dist}(0^*, Df(x_0)^*(S^*)))^{-1},
\]
where \( \Lambda^* \in \mathcal{L}(Y^*, X^*) \) denotes the adjoint operator to \( \Lambda \in \mathcal{L}(X, Y) \) and the conventions
\[
\inf \emptyset = +\infty \quad \text{and} \quad 1/0 = +\infty
\]
are adopted. Remember that \( \Lambda \in \mathcal{L}(X, Y) \) is onto iff \( \Lambda^* \) has bounded inverse.

3. **An extension of the Polyak convexity principle**

Given \( c > 0 \), let us introduce the following subclasses of uniformly convex subsets of \( X \), with modulus of convexity of power type 2:
\[
\mathcal{UC}_c^2(X) = \{ S \subseteq X : \delta_S(\epsilon) \geq c\epsilon^2, \forall \epsilon \in (0, \text{diam } S) \}
\]
and
\[ UC^2(X) = \bigcup_{c > 0} UC^2_c(X). \]

**Remark 3.1.** In the proof of the next theorem the following fact, which can be easily proved by an iterative bisection procedure, will be used: any closed subset \( V \) of a Banach space is convex iff \( \frac{y_1 + y_2}{2} \in V \), whenever \( y_1, y_2 \in V \). It is easy to check that if \( V \) is not closed, this mid-point property does not imply the convexity of \( V \).

Below, the main result of the paper is established.

**Theorem 3.2.** Let \( f : \Omega \rightarrow \mathbb{Y} \) be a mapping between Banach spaces, with \( \Omega \) open nonempty subset of \( X \). Let \( x_0 \in \Omega \) and \( c > 0 \) such that:

(i) \( f \in C^{1,1}(\text{int } B(x_0, r_0)) \), for some \( r_0 > 0 \);

(ii) \( Df(x_0) \) is onto;

(iii) it holds
\[ \frac{\text{reg}(f, x_0) \cdot \text{lip}(Df, \text{int } B(x_0, r_0))}{8} < c. \]

Then, there exists \( \rho \in (0, r_0) \) such that, for every \( S \in UC^2_c(X) \), with \( S \subseteq \text{int } B(x_0, \rho) \) and \( f(S) \) closed, it is \( f(S) \in UC^2(\mathbb{Y}) \).

**Proof.** The proof is divided into two parts.

**First part:** Let us show that \( f(S) \) is convex. According to the hypothesis (iii), it is possible to fix positive reals \( \kappa \) and \( \ell \) in such a way that \( \kappa > \text{reg}(f, x_0) \), \( \ell > \text{lip}(Df, \text{int } B(x_0, r_0)) \), and the following inequality is fulfilled
\[ \frac{\kappa \ell}{8} < c. \]

By virtue of hypotheses (i) and (ii), as \( f \) is in particular strictly differentiable at \( x_0 \), it is possible to invoke the Lyusternik-Graves theorem, ensuring that \( f \) is metrically regular around \( x_0 \). This means that there exist positive reals \( \tilde{\kappa} \) and \( \tilde{r} \) such that
\[ \text{reg}(f, x_0) < \tilde{\kappa} < \kappa, \quad \tilde{r} \in (0, r_0), \]
and
\[ \text{dist} (x, f^{-1}(y)) \leq \tilde{\kappa} \|y - f(x)\|, \quad \forall x \in B(x_0, \tilde{r}), \forall y \in B(f(x_0), \tilde{r}). \]

Besides, by the continuity of \( f \) at \( x_0 \), corresponding to \( \tilde{r} \) there exists \( r_* \in (0, r_0) \) such that
\[ f(x) \in B(f(x_0), \tilde{r}), \quad \forall x \in B(x_0, r_*). \]

Then, take \( \rho \in (0, \min\{\tilde{r}, r_*\}) \). Notice that, in the light of Remark 2.7, up to a further reduction in the value of \( \rho \), one can assume that for some \( \sigma > 0 \) it holds
\[ f(B(x, r)) \supseteq B(f(x), \sigma r), \quad \forall x \in B(x_0, \rho), \forall r \in [0, \rho]. \]
Now, take an arbitrary element $S \in UC_c^2(X)$, with $S \subseteq B(x_0, \rho)$ and such that $f(S)$ is closed. According to Remark 2.1, the convexity of $f(S)$ can be proved by showing that for every $y_1, y_2 \in f(S)$, with $y_1 \neq y_2$, it holds $\frac{y_1 + y_2}{2} \in f(S)$. To this aim, let $x_1, x_2 \in S$ be such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. For convenience, set

$$\bar{x} = \frac{x_1 + x_2}{2} \quad \text{and} \quad \bar{y} = \frac{y_1 + y_2}{2}.$$ 

Notice that, as it is $y_1 \neq y_2$, it must be also $x_1 \neq x_2$. Moreover, as $S \subseteq B(x_0, \rho) \subseteq B(x_0, r_\ast)$, one has $y_1, y_2 \in B(f(x_0), \bar{r})$ and therefore, by the convexity of a ball, one has also $\bar{y} \in B(f(x_0), \bar{r})$. Thus, since $\bar{x} \in B(x_0, \bar{r})$ and $y \in B(f(x_0), \bar{r})$, the inequality (3.2) applies. According to it, one finds

$$(3.4) \quad \text{dist} (\bar{x}, f^{-1}(\bar{y})) \leq \kappa \|\bar{y} - f(\bar{x})\|.$$ 

If $\bar{y} = f(\bar{x})$ the proof of the convexity of $f(S)$ is complete, because $\bar{x} \in S$. Otherwise, it happens that $\|\bar{y} - f(\bar{x})\| > 0$, so the inequality (3.4) entails the existence of $\hat{x} \in f^{-1}(\bar{y})$ such that

$$\|\hat{x} - \bar{x}\| < \kappa \|\bar{y} - f(\bar{x})\|.$$ 

By taking account of the estimate (2.1) in Remark 2.1 (i), as it is $[x_1, x_2] \in B(x_0, \rho) \subseteq \text{int} B(x_0, r_0)$, one consequently obtains

$$\|\hat{x} - \bar{x}\| < \kappa \frac{\ell}{8} \|x_1 - x_2\|^2,$$

that is $\hat{x} \in B(\bar{x}, \frac{\kappa \ell}{8} \|x_1 - x_2\|^2)$. Since $S \in UC_c^2(X)$ and the inequality (3.1) is in force, in the light of what observed in Remark 2.1 (iii) it follows

$$B\left(\bar{x}, \frac{\kappa \ell}{8} \|x_1 - x_2\|^2\right) \subseteq S,$$

with the consequence that $\hat{x} \in S$ and hence $\bar{y} = f(\hat{x})$ turns out to belong to $f(S)$.

Second part: Let us prove now the assertion in the thesis. According to what noted in Remark 2.1 (ii), under the above hypotheses $f(S)$ is bounded. Fix $\epsilon \in (0, \text{diam} f(S))$ and take arbitrary $y_1, y_2 \in f(S)$, with $\|y_1 - y_2\| = \epsilon$. Let $\bar{y}, x_1, x_2, \bar{x}$ and $\hat{x}$ be as in the first part of the proof (it may happen that $\hat{x} = \bar{x}$). In order to prove that $f(S) \subseteq UC^2(Y)$, it is to be shown that, independently of $y_1, y_2 \in f(S)$ and $\epsilon$, there exists $\gamma > 0$ such that $B(\gamma, \gamma \epsilon^2) \subseteq f(S)$. Again recalling Remark 2.1 (ii), it is possible to define the positive real value

$$\beta = \sup_{x \in S} \|Df(x)\|_\infty + 1 < +\infty.$$ 

By virtue of inequality (3.1), it is possible to pick $\eta \in (0, c - \frac{\kappa \ell}{8})$ in such a way that

$$\hat{x} \in B\left(\bar{x}, \frac{\kappa \ell}{8} \|x_1 - x_2\|^2\right) \subseteq B\left(\bar{x}, \left(\frac{\kappa \ell}{8} + \eta\right) \|x_1 - x_2\|^2\right) \subseteq S.$$
From the last chain of inclusions, it readily follows that

\[ B(\hat{x}, \eta \|x_1 - x_2\|^2) \subseteq S. \]

Since, by the mean-value theorem, it is

\[ \|y_1 - y_2\| \leq \beta \|x_1 - x_2\|, \]

one obtains

\[ \epsilon^2 = \|y_1 - y_2\|^2 \leq \beta^2 \|x_1 - x_2\|^2, \]

and hence \( B(\hat{x}, \eta \epsilon^2 / \beta^2) \subseteq S \). Now, recall that \( f \) is open at a linear rate around \( x_0 \). Accordingly, as \( S \subseteq B(x_0, \rho) \), up to a further reduction in the value of \( \eta > 0 \) in such a way that \( \eta \text{diam}^2 f(S) / \beta^2 < \rho \), one finds

\[ B(\bar{y}, \sigma \eta \epsilon^2 / \beta^2) \subseteq f(B(\hat{x}, \eta \epsilon^2 / \beta^2)) \subseteq f(S) \]

(remember the inclusion \((\ref{3.3})\)). Thus, since by construction \( \sigma, \eta \) and \( \beta \) are independent of \( y_1, y_2 \) and \( \epsilon \), one can conclude that

\[ \delta_{f(S)}(\epsilon) \geq \frac{\sigma \eta \epsilon^2}{\beta^2}. \]

By arbitrariness of \( \epsilon \in (0, \text{diam} f(S)) \), this completes the proof. \( \square \)

A first comment to Theorem 3.2 concerns its hypothesis (iii), which seems to find no counterpart in the convexity principle due to B.T. Polyak (see \([\ref{15}, \text{Theorem 2.1}]\)). Such hypothesis postulates a uniform convexity property of \( S \), which must be quantitatively adequate to the metric regularity of \( f \) and to the Lipschitz continuity of \( Df \) around \( x_0 \). Matching this condition is guaranteed for strongly convex sets (in particular, for balls) with a sufficiently small radius, provided that the underlying Banach space fulfills a certain uniform convexity assumption. This fact is clarified by the following

**Corollary 3.3.** Let \( f : \Omega \rightarrow \mathbb{Y} \) be a mapping between Banach spaces, with \( \Omega \) open nonempty subset of \( X \). Let \( x_0 \in \Omega \) be such that:

(i) \( (X, \| \cdot \|) \) admits a modulus of convexity of power type 2;

(ii) \( f \in C^1_{\text{int} B(x_0, r_0)} \), for some \( r_0 > 0 \);

(iii) \( Df(x_0) \) is onto.

Then, there exists \( \rho \in (0, r_0) \) such that, for every \( r \)-convex set \( S \), with \( r \in (0, \rho) \) and \( f(S) \) closed, it holds \( f(S) \in UC^2(\mathbb{Y}) \).

**Proof.** By virtue of the hypothesis (i), according to Example \([\ref{2}, \text{Example 2.2}]\) (ii), any \( r \)-convex set \( S \) belongs to \( UC^2(X) \), for every \( r > 0 \). More precisely, on account of the inequality \((\ref{2.2})\), one has

\[ \delta_S(\epsilon) \geq r \delta_X \left( \frac{\epsilon}{r} \right) \geq \frac{\gamma \epsilon^2}{r}, \quad \forall \epsilon \in (0, 2r], \]

for some \( \gamma > 0 \). Therefore, in order for the hypothesis (iii) of Theorem 3.2 to be satisfied, it suffices to take

\[ r < \frac{8 \gamma}{\text{reg}(f, x_0) \cdot \text{lip}(Df, \text{int} B(x_0, r_0)) + 1}. \]
refinement \[ Theorem 3.2 \]

On the other hand, notice that Theorem 3.2 does not make any direct assumption on the Banach space \((X, \| \cdot \|)\) (nonetheless, take into account what remarked at the end of Example 2.3 (i)). Furthermore, since any ball \(B(x_0, r)\) is a \(r\)-convex sets, it should be clear that Corollary 3.3 allows one to embed in the current theory the Polyak convexity principle and its refinement [20, Theorem 3.2].

Another comment to Theorem 3.2 deals with the topological assumption on the image \(f(S)\). Of course, whenever \(X\) is a finite-dimensional Euclidean space, \(f(S)\) is automatically closed, because \(S\) is compact and \(f\) is continuous on \(S\). In an infinite-dimensional setting, the same issue becomes subtler. Below, a condition exploiting the Kadec-Klee property, which is valid in any uniformly convex space, is formulated. Recall that a Banach space \((X, \| \cdot \|)\) is said to have the the Kadec-Klee property if the weak and the norm topologies coincide on \(S\) (see [10]).

**Proposition 3.4.** Let \(f : \Omega \rightarrow \mathbb{Y}\) be a mapping between Banach spaces, with \(\Omega\) open nonempty subset of \(X\), and let \(x_0 \in \Omega\). Suppose that:

(i) \(f\) is continuous in \(B(x_0, r_0)\), for some \(r_0 > 0\);

(ii) for each sequence \((x_n)\) in \(B(x_0, r_0)\), with \(x_n \xrightarrow{w} \tilde{x}\) as \(n \rightarrow \infty\),

it is \(\limsup_{n \rightarrow \infty} \|x_n\| \leq \|\tilde{x}\|\);

(iii) \((X, \| \cdot \|)\) is uniformly convex;

(iv) \(f\) is metrically regular around \(x_0\).

Then, there exists \(\rho \in (0, r_0)\) such that, for every closed convex set \(S \subseteq B(x_0, \rho)\), \(f(S)\) is closed.

**Proof.** Since by the hypothesis (iv) \(f\) is metrically regular around \(x_0\), there exist positive real \(r \in (0, r_0)\) and \(\kappa\) such that

\[
(3.5) \quad \text{dist} \ (x, f^{-1}(y)) \leq \kappa\|f(x) - y\|, \quad \forall x \in B(x_0, r), \ \forall y \in B(f(x_0), r).
\]

By the continuity of \(f\) at \(x_0\), there exists \(\rho \in (0, r)\) such that

\[
f(x) \in B(f(x_0), r), \quad \forall x \in B(x_0, \rho).
\]

Thus, whenever \(S \subseteq B(x_0, \rho)\), one has \(f(S) \subseteq B(f(x_0), r)\).

Now, suppose that \(S \subseteq B(x_0, \rho)\) is a closed convex set. Let \((y_n)_n\) be a sequence in \(f(S)\), such that \(y_n \rightarrow y\) as \(n \rightarrow \infty\). As \(y_n \in f(S)\), there exists a sequence \((x_n)_n\) in \(S\) such that \(y_n = f(x_n)\), for each \(n \in \mathbb{N}\). Notice that, since \(x_n \in S \subseteq B(x_0, \rho) \subseteq B(x_0, r)\) and \(y \in \text{cl} f(S) \subseteq B(f(x_0), r)\), the inequality (3.5) applies, namely

\[
(3.6) \quad \text{dist} \ (x_n, f^{-1}(y)) \leq \kappa\|f(x_n) - y\| = \kappa\|y_n - y\|, \quad \forall n \in \mathbb{N}.
\]

This shows that \(\text{dist} \ (x_n, f^{-1}(y)) \rightarrow 0\) as \(n \rightarrow \infty\). As a closed convex set, \(S\) is also weakly closed. Moreover, as \(S\) is bounded in a reflexive Banach space, it is weakly compact. Since \(S\) is also sequentially weakly compact by the Eberlein-Šmulian theorem, one can assume that \((x_n)_n\) weakly converges to some \(\tilde{x} \in S\). If this is true, it is known that \(\|\tilde{x}\| \leq \liminf_{n \rightarrow \infty} \|x_n\|\)
By consequence, from the hypothesis (ii) one obtains that $\|x_n\| \to \|\tilde{x}\|$. Therefore, the Kadec-Klee property, which can be invoked because of the hypothesis (iii), enables one to assert that $x_n \to \tilde{x}$. So, by taking into account the inequality (3.6), one finds
\[
\text{dist}\left(\tilde{x}, f^{-1}(y)\right) \leq d(\tilde{x}, x_n) + \text{dist}\left(x_n, f^{-1}(y)\right) \to 0.
\]
Since $f$ is continuous, $f^{-1}(y)$ is closed and hence $\text{dist}\left(\tilde{x}, f^{-1}(y)\right) = 0$ entails $\tilde{x} \in f^{-1}(y)$. This leads to conclude that $y \in f(S)$, thereby completing the proof.

Let $C \subseteq Y$ be a closed convex cone with apex at $0$ and let $S \subseteq X$ be nonempty and convex. Recall that a mapping $f : S \to Y$ is said to be convex-like on $S$ with respect to $C$ if for every $x_1, x_2 \in S$ and $t \in [0, 1]$, there exists $x_t \in S$ such that
\[
(1 - t)f(x_1) + tf(x_2) \in f(x_t) + C.
\]
Convex-likeness is a generalization of the notion of $C$-convexity of mappings taking values on partially ordered vector spaces. It should be evident that, when $Y = \mathbb{R}$, $C = [0, +\infty)$ and $x_t = (1 - t)x_1 + tx_2$, the above inclusion reduces to the well-known inequality defining the convexity of a functional. The class of convex-like mappings has found a large employment in optimization and related topics. For instance, if $\mathbb{R}^m$ and $C = \mathbb{R}^m_+$ it is readily seen that this class includes all mappings $f = (f_1, \ldots, f_m)$, having each component $f_i : S \to \mathbb{R}$, $i = 1, \ldots, m$ convex on a convex set. The next corollary, which can be achieved as a direct consequence of Theorem 3.2, reveals that any $C^{1,1}$ smooth mapping behave as a convex-like mapping on uniformly convex sets near a regular point.

**Corollary 3.5.** Let $f : \Omega \to Y$ be a mapping between Banach spaces, $x_0 \in \Omega$ and $c > 0$. If $f$, $x_0$ and $c$ satisfy all hypotheses of Theorem 3.2, then there exists $\rho > 0$ such that, for every $S \in UC_2\rho(X)$, with $S \subseteq B(x_0, \rho)$ and $f(S)$ closed, and every cone $C \subseteq Y$, the mapping $f : S \to Y$ is convex-like on $S$ with respect to $C$.

**Proof.** The thesis follows at once by Theorem 3.2 from being
\[
(1 - t)f(x_1) + tf(x_2) \in f(S) \subseteq f(S) + C, \quad \forall x_1, x_2 \in S, \forall t \in [0, 1].
\]

## 4. Applications to Optimization

Throughout this section, applications of Theorem 3.2 will be considered to the study of constrained optimization problems, having the following format
\[
(P) \quad \min_{x \in S} \varphi(x) \quad \text{subject to} \quad g(x) \in C,
\]
where $\varphi : X \to \mathbb{R}$ and $g : X \to Y$ are given functions between Banach spaces, $S \subseteq X$ and $C \subseteq Y$ are given (nonempty) closed and convex sets.
Such a format is frequently employed in the literature for subsuming under a general treatment a broad spectrum of finite and infinite-dimensional extremum problems. The feasible region of problem \((P)\) will be henceforth denoted by \(R\), i.e. \(R = S \cap g^{-1}(C)\).

According to a long-standing approach in optimization, now recognized as ISA (acronym standing for Image Space Analysis), the analysis of several issues related to problem \((P)\) can be performed by associating with \((P)\) and with an element \(x_0 \in R\) the mapping \(f_{P,x_0} : X \to R \times Y\), which is defined by

\[
f_{P,x_0}(x) = (\varphi(x) - \varphi(x_0), g(x))
\]

(see, for instance, [11] and references therein). Such issues as the solution existence, optimality conditions, duality, and so on, can be investigated by studying relationships between the two subsets of the space \(R \times Y\) associated with \((P)\).

Remark 4.1. Directly from the above constructions, it is possible to prove the following well-known facts:

(i) \(x_0 \in R\) is a global solution to \((P)\) iff \(f_{P,x_0}(S) \cap Q = \emptyset\);

(ii) if the mapping \(f_{P,x_0}\) is open at a linear rate around \(x_0\), then \(x_0 \in R\) can not be a local solution to \((P)\).

The next theorem, which extends a similar result established in [20, Theorem 3.2], provides an answer to the question of solution existence and, at the same time, furnishes an optimality condition for detecting a solution. In order to formulate such a theorem, let us denote by \(N(C, \bar{y}) = \{y^* \in Y^* : \langle y^*, y - \bar{y} \rangle \leq 0, \quad \forall y \in C\}\) the normal cone to \(C\) at \(\bar{y}\) in the sense of convex analysis. Besides, let us denote by \(L : Y^* \times X \to \mathbb{R}\) the Lagrangian function associated with problem \((P)\), i.e.

\[
L(y^*, x) = \varphi(x) + \langle y^*, g(x) \rangle.
\]

The proof, whose main part is given for the sake of completeness, adapts an argument already exploited in [20]. It derives solution existence from the weak compactness of the problem image and the optimality condition by a linear separation technique.

**Theorem 4.2.** Given a problem \((P)\), let \(x_0 \in g^{-1}(C)\) and let \(c\) be a positive real. Suppose that:

(i) \((Y, \| \cdot \|)\) is a reflexive Banach space;

(ii) \(\varphi, g \in C^{1,1}(\text{int } B(x_0, r_0))\), for some \(r_0 > 0\) and \(Df_{P,x_0}(x_0)\) is onto;

(iii) it holds

\[
(4.1) \quad \frac{\text{reg}(f_{P,x_0}, x_0) \cdot \text{lip}(Df_{P,x_0}, \text{int } B(x_0, r_0))}{8} < c.
\]

Then, there exists \(\rho \in (0, r_0)\) such that, for every \(S \in UC_C^2(X)\), with \(x_0 \in S \subseteq B(x_0, \rho)\) and \(f_{P,x_0}(S)\) closed, one has

(t) there exists a global solution \(\bar{x}_S \in R\) to \((P)\);

(tt) \(\bar{x}_S \in \text{bd } S\) and hence \(\bar{x}_S \in \text{bd } R\);
(ttt) there exists \( y_S^\ast \in N(C, g(\bar{x}_S)) \) such that
\[
L(y_S^\ast, \bar{x}_S) = \min_{x \in S} L(y_S^\ast, x).
\]

**Proof.** (t) Under the hypotheses (ii) and (iii), one can apply Theorem 3.2. If \( \rho > 0 \) is as in the thesis Theorem 3.2, fix a set \( S \in UC_\rho^2(\mathbb{X}) \) satisfying all requirements in the above statement. Then its image \( f_{p,x_0}(S) \) turns out to be a convex, closed and bounded subset of \( \mathbb{R} \times \mathbb{Y} \), with nonempty interior. The existence of a global solution to (P) will be achieved by proving that an associated minimization problem in the space \( \mathbb{R} \times \mathbb{Y} \) does admit a global solution. To do so, define
\[
\tau = \inf \{ t : (t, y) \in f_{p,x_0}(S) \cap Q \}.
\]
Notice that \( x_0 \in R \). Since \( Df_{p,x_0}(x_0) \) is onto, by the Lyusternik-Graves theorem the mapping \( f_{p,x_0} \) too is open at a linear rate around \( x_0 \). Thus, as observed in Remark 4.1 (ii), \( x_0 \) fails to be a solution to (P). Consequently, according to Remark 4.1 (i), it must be
\[
f_{p,x_0}(S) \cap Q \neq \emptyset.
\]
This implies that \( \tau < +\infty \). Furthermore, if setting
\[
(4.2) \quad \bar{\tau} = \inf \{ t : (t, y) \in f_{p,x_0}(S) \cap \text{cl} Q \},
\]
then \( \bar{\tau} \) is actually attained at some \( (\bar{t}, \bar{y}) \in f_{p,x_0}(S) \cap \text{cl} Q \).

Now, as the set \( f_{p,x_0}(S) \) is closed, convex and bounded, so is its subset \( f_{p,x_0}(S) \cap \text{cl} Q \). The boundedness of the latter implies that \( \bar{\tau} > -\infty \). Moreover, by virtue of the hypothesis (i), \( f_{p,x_0}(S) \cap \text{cl} Q \) turns out to be weakly compact. Since the projection mapping \( \Pi_R : \mathbb{R} \times \mathbb{Y} \rightarrow \mathbb{R} \), given by \( \Pi_R(t, y) = t \) is continuous and convex, it is also weakly l.s.c., with the consequence that the infimum defined in (4.2) is actually attained at some \( (\bar{t}, \bar{y}) \in f_{p,x_0}(S) \cap \text{cl} Q \). This means that there exists \( \bar{x}_S \in S \) such that
\[
\tau = \bar{\tau} = \varphi(\bar{x}_S) \quad \text{and} \quad \bar{y} = g(\bar{x}_S) \in C.
\]
Let us show that \( \bar{x}_S \) is a global solution to (P). Assume to the contrary that there is \( \hat{x} \in R \) such that \( \varphi(\hat{x}) < \varphi(\bar{x}_S) \). Then, one finds
\[
\hat{t} = \varphi(\hat{x}) - \varphi(x_0) = \varphi(\hat{x}) - \varphi(\bar{x}_S) + \varphi(\bar{x}_S) - \varphi(x_0) < \varphi(\bar{x}_S) - \varphi(x_0) = \bar{t} = \bar{\tau} = \tau.
\]
Since it is \( \hat{x} \in R \), then \( \hat{x} \in S \) and \( \hat{y} = g(\hat{x}) \in C \), wherefrom one has \( (\hat{t}, \hat{y}) \in f_{p,x_0}(S) \cap Q \), which contradicts the definition of \( \tau \).
To prove that $\bar{x}_S$ belongs to $\text{bd } S$, notice that $(\bar{t}, \bar{y}) = f_{P,x_0}(\bar{x}_S) \in \text{bd } f_{P,x_0}(S)$. Then, by recalling what mentioned in Remark 2.7 (ii), this assertion follows from the openness at a linear rate of $f_{P,x_0}$ around $x_0$.

Again remembering Remark 4.1 (i), by the global optimality of $\bar{x}_S$, it results in

$$(4.3) \quad f_{P,\bar{x}_S}(S) \cap Q = \emptyset.$$ 

As one readily checks, it holds

$$f_{P,\bar{x}_S}(S) = f_{P,x_0}(S) + (\varphi(x_0) - \varphi(\bar{x}_S), 0),$$

that is to say $f_{P,\bar{x}_S}(S)$ is a translation of $f_{P,x_0}(S)$. Therefore, $f_{P,\bar{x}_S}(S)$ too is a closed, bounded, convex subset of $\mathbb{R} \times \mathbb{Y}$, with nonempty interior. Since (4.3) is true, the Eidelheit theorem makes it possible to linearly separate $f_{P,\bar{x}_S}(S)$ and $\text{cl } Q$. In other terms, this means the existence of a pair $(\gamma, y^*) \in (\mathbb{R} \times \mathbb{Y}) \setminus \{(0, 0^*)\}$ and $\alpha \in \mathbb{R}$ such that

$$\gamma(\varphi(x) - \varphi(\bar{x}_S)) + (y^*, g(x)) \geq \alpha, \quad \forall x \in S,$$

and

$$\gamma t + (y^*, y) \geq \alpha, \quad \forall (t, y) \in \text{cl } Q = (-\infty, 0] \times C.$$ 

The rest of the proof relies on a standard usage of the last inequalities and does not need to devise any specific adaptation.

Theorem 4.2 describes the local behaviour of a nonlinear optimization problem $(P)$ near a point $x_0 \in g^{-1}(C)$, around which the condition (4.1) linking the modulus of convexity of $S$, the regularity behaviour of $f_{P,x_0}$ and the Lipschitz continuity of its derivative happens to be satisfied: $(P)$ admits a global solution, which lies at the boundary of the feasible region and can be detected by minimizing the Lagrangian function. The reader should notice that globality of a solution and its characterization as a minimizer of a the Lagrangian function are phenomena typically occurring in convex optimization. Instead, they generally fail to occur in nonlinear optimization, where optimality conditions are usually only necessary and expressed in terms of Lagrangian stationary by means of first-order derivative.

Another typical phenomenon arising in convex optimization is the vanishing of the duality gap, i.e. the vanishing of the value

$$\text{gap } (P) = \inf_{x \in S} \sup_{y^* \in C^\circ} L(y^*, x) - \sup_{y^* \in C^\circ} \inf_{x \in S} L(y^*, x),$$

where $C^\circ = \{y^* \in \mathbb{Y}^* : \langle y^*, y \rangle \leq 0\}$ is the dual cone to $C$. Such a circumstance, which can be proved to take place in convex programming under proper qualification conditions, is known as strong (Lagrangian) duality. In the current setting, it can be readily achieved as a consequence of Theorem 4.2, without the need of extra assumptions, apart from the cone structure now imposed on the set $C$. 
Corollary 4.3. Given a problem \((P)\), suppose that \(C\) is a closed convex cone. Under the hypothesis of Theorem 4.2, it holds
\[
gap (P) = 0
\]
and there exists a pair \((y^*_S, \bar{x}_S)\) in \(C^\circ \times R\), which is a saddle point of \(L\), i.e.
\[
L(y^*, \bar{x}_S) \leq L(y^*_S, \bar{x}_S) \leq L(y^*_S, x), \quad \forall (y^*, x) \in C^\circ \times S.
\]

Proof. Let \(\bar{x}_S\) and \(y^*_S\) be as in the thesis of Theorem 4.2. Since \(C\) is a closed convex cone, \(2g(\bar{x}_S)\) and \(0\) belong to \(C\). By recalling that \(y^*_S \in N(C, g(\bar{x}_S))\), one has
\[
\langle y^*_S, y - g(\bar{x}_S) \rangle \leq 0, \quad \forall y \in C.
\]
By replacing \(y\) with \(2g(\bar{x}_S)\) and \(0\) in last inequality, one easily shows that \(\langle y^*_S, g(\bar{x}_S) \rangle = 0\) and hence \(y^*_S \in C^\circ\). The rest of the thesis then follows at once. \(\square\)

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(A. Uderzo) Dept. of Mathematics and Applications, University of Milano-Bicocca, Italy

E-mail address: amos.uderzo@unimib.it