Gluon Condensate from Superconvergent QCD Sum Rule*

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Sum rules for the nonperturbative piece of correlators (specifically, the vector current correlator) are discussed. The sum rule subtracting the perturbative part is of the superconvergent type. Thus it is dominated by the bound states and the low energy production cross section. It leads to a determination of the gluon condensate \( \langle \alpha_s G^2 \rangle \). We find

\[ \langle \alpha_s G^2 \rangle \simeq 0.048 \pm 0.030 \text{ GeV}^4. \]

1. SUM RULE

The potential, or more generally the spectrum of a system of heavy quarks cannot be directly discussed in terms of the OPE (operator product expansion). However, one can use dispersion relations to deduce a number of sum rules relating bound state properties to quantities obtainable via the OPE (“TPE-type” sum rules). One can then use the estimates of nonperturbative contributions to bound states energies and wave functions to actually go beyond the traditional analysis. Although the sum rules, being *global* relations, cannot discriminate details one can check consistency and even obtain reasonable estimates on nonperturbative quantities, specifically on the gluon condensate. This last is the aim of the present note, where we will use a method generalizing that proposed by Novikov\(^{[1]}\).

To do so we consider the correlator for the vector current of heavy quarks,

\[
\Pi_{\mu\nu} = (p^2 g_{\mu\nu} - p_{\mu} p_{\nu}) \Pi(p^2) = i \int d^4 x e^{ip \cdot x} \langle T J_\mu(x) J_\nu(0) \rangle ,
\]

where \( J_\mu = \bar{\psi} \gamma_\mu \psi \) and sum over omitted colour indices is understood. This will give information on triplet, \( t = 0 \) states; information on states with other quantum numbers would be obtained with other correlators. The function \( \Pi(t) \) satisfies a dispersion relation,

\[
\Pi(t) = \frac{1}{\pi} \int ds \frac{\rho(s)}{s - t} ,
\]

where \( \rho(s) \equiv \text{Im} \Pi(s) \). Actually, this equation should have been written with one subtraction.

We will not bother to do so as its contribution drops out for the quantities of interest for us here.

Let us denote by \( \Pi_{\text{p.t.}} , \rho_{\text{p.t.}} \) to the corresponding quantities calculated in perturbation theory, albeit to all orders, but nonperturbative effects are neglected in \( \Pi_{\text{p.t.}} , \rho_{\text{p.t.}} \). (In actual calculations we cannot of course include all orders. We will sum the one-gluon exchange to all orders which can be done explicitly in the nonrelativistic regime, and add one loop radiative corrections to this.) In particular, for example, the gluon condensate contribution is not included in the “p.t.” pieces.

At large \( t \), both spacelike and timelike, the OPE is applicable to \( \Pi(t) \), and we have the well-known results\(^{[2]}\),

\[
\Pi(t) \simeq \Pi_{\text{p.t.}}(t) + \frac{\langle \alpha_s G^2 \rangle}{12 \pi t^2} \tag{3}
\]

and

\[
\rho(s) \simeq \rho_{\text{p.t.}}(s) - \frac{N_c C_F}{128} \frac{\langle \alpha_s G^2 \rangle}{s^2} \left( 1 + \frac{v^2}{s} \right) \left( 1 - \frac{v^2}{s} \right) \frac{1}{v^2} \tag{4}
\]

with \( v = (1 - 4m^2/s)^{1/2} \) the velocity of the quarks. Moreover,

\[
\Pi_{\text{p.t.}}(t) \simeq \frac{N_c}{12 \pi^2} \left\{ \log \frac{-t}{\nu^2} + \frac{3C_F}{\beta_0} \log \log \frac{-t}{\nu^2} + \cdots \right\},
\]

\[
\text{Im} \Pi_{\text{p.t.}}(s) \simeq \frac{N_c}{12 \pi} \left\{ 1 + \frac{3C_F \alpha_s}{4\pi} + \cdots \right\} ;
\]

\( N_c = 3, C_F = 4/3 \).

* Contribution to Prof. L. Okun’s seventieth birthday
If we then define \( \Pi_{NP}, \rho_{NP} \) as the results of subtracting the perturbative parts,

\[
\Pi_{NP} \equiv \Pi - \Pi_{p.t.}; \rho_{NP} \equiv \rho - \rho_{p.t.},
\]

it follows from the OPE, Eq. (3), that \( \Pi_{NP} \) decreases at infinity like \( t^{-2} \) and hence satisfies a superconvergent dispersion relation. We thus have a first sum rule:

\[
\int ds \rho_{NP}(s) = 0. \tag{5}
\]

In fact it would appear that one still has another sum rule because of the following argument. At large \( t \), \( \Pi_{NP}(t) \) behaves like (cf. Eq. (3))

\[
\Pi_{NP}(t) \sim \frac{(\alpha_s G^2)}{12\pi t^2},
\]

while the contribution from the bound states to the dispersion relation (see below),

\[
\Pi_{NP: bound states}(t) \sim \frac{(\alpha_s G^2)}{t^2 \alpha_s^3} \tag{6}
\]

dominates over this. Therefore we have the extra relation,

\[
\int ds \, s \rho_{NP}(s) = 0. \tag{7}
\]

It turns out that (7) is actually equivalent to (5), up to radiative corrections. This is because the region where any of the integrals in (5), (7) are appreciably different from zero is for \( s \approx 4m^2(1 + O(\alpha_s^2)) \), so (7) differs from (5) by terms of order \( \alpha_s^2 \), smaller than the radiative corrections which neither (5) nor (7) take into account.

Let us return to the sum rule (5). The function \( \rho(s) \) consists of a continuum part, for \( s \) above threshold for open bottom production, and a sum of bound states. Both can be calculated theoretically provided that \( s \) is larger than a certain critical \( s(v_0) \), and \( n \) or equal than a critical \( n_0 \). \( s(v_0) \) and \( n_0 \) are defined as the points where the perturbation theoretic contribution to \( \rho \) and the nonperturbative one are of equal magnitude, and form the limits of the regions where a full theoretical evaluation is possible.

To be precise, for the continuum we use (4) so that above the critical \( s(v_0) \),

\[
\rho_{NP}^{\text{cont}}(s) = \frac{N_c C_F}{128} \frac{(\alpha_s G^2)}{s^2} \frac{(1 + \nu^2)(1 - \nu^2)^2}{\nu^3}, \quad s > s(v_0), \tag{8a}
\]

and \( v_0 \) is such that \( \rho_{NP}^{\text{cont}}(s(v_0)) \approx \rho_{p.t.}^{\text{cont}}(s(v_0)) \); numerically, and for \( b \bar{b}, v_0 \simeq 0.2 \). For the bound states \( \rho \) is proportional to the square of the wave function at the origin,

\[
\rho(s) = \frac{N_c}{M_n} |R_n(0)|^2 \delta(s - M_n).
\]

We may get \( \rho_{NP}^{b.s.}(s) \) and \( \rho_{NP}^{b.s.}(s) \) by splitting the residue \( |R_n(0)|^2 \) into a Coulombic piece,

\[
|R_{n}^{\text{Coul}}(0)|^2 = \frac{m^3 C_F^3 \alpha_s^3}{2 n^3} (1 - \delta_{n,s}^b),
\]

where the one loop corrections \( \delta_{n,s}^b \) may be found in ref. 3, and the (leading) nonperturbative correction are given by the Leutwyler–Voloshin analysis (refs. 4, 3). So we have

\[
|R_n(0)|^2 \approx |R_n^{\text{Coul}}(0)|^2 + |R_n^{\text{Coul}}(0)|^2 \delta_{n,s}^n;
\]

the numbers \( \delta_{n}^{NP} \) have been calculated by Leutwyler and Voloshin. For \( n = 1 \),

\[
\delta_{1}^{NP} = \frac{38.3 (\alpha_s G_s^2)}{m^3 C_F^3 \alpha_s^2},
\]

This is all we really need since, for bottomium, \( n_0 = 1 \). Thus we have

\[
\rho_{NP}^{b.s.}(s) = 3N_c C_F \pi m^3 \frac{(\alpha_s G_s^2)}{8 \alpha_s^3 m^2} \sum_{n=1}^{n_0} \frac{n_0}{M_n} \delta(s - M_n^2),
\]

\( n \leq n_0 \),

\[
\tag{8b}
\]

the \( \eta_n \) known in terms of the \( \delta_{n}^{NP} \).

The sum rule (5) can then be written schematically as

\[
\int_{s(v_0)}^{\infty} \rho_{NP} + \sum_{n=1}^{n_0} \text{Residue of } \rho_{NP} = \left\{ \int_{s(v_0)}^{\infty} \rho_{NP} + \sum_{n=n_0+1}^{\infty} \text{Residue of } \rho_{NP} \right\}.
\]

The left hand side is given in terms of \( \langle \alpha_s G_s^2 \rangle \) by Eqs. (8); the right hand side can be connected with experiment with the following argument. The sum over higher bound states,

\[
\sum_{n=n_0+1}^{\infty} \text{Residue of } \rho_{NP}.
\]

may be identified as the difference between the sum over the experimental residues of the poles of the bound states, and what we would get by a Coulombic formula, for all \( n \geq n_0 + 1 \). Certainly, this Coulombic formula will not be valid for large \( n \).
because here the radiative corrections will become large; but, because the residues decrease like $1/n^3$ the contribution of these states will be negligible. We write this decomposition as

\[
\begin{align*}
(\text{bound states with } n > n_0) & = \rho_{\text{b.s., } n>n_0}^{\text{cont}}(s) \\
& - \rho_{\text{Coulombic, } n>n_0}^{\text{b.s.}}(s).
\end{align*}
\]

As for the continuum piece below $s(v_0)$ we may likewise interpret it as the difference between experiment and a perturbative evaluation, which we write as

\[
\rho_{\text{NP}}^{\text{cont}}(s) = \rho_{\exp}^{\text{cont}}(s) - \rho_{\text{p.t.}}^{\text{cont}}(s), \quad s < s(v_0),
\]

and, because we are close to threshold, we have

\[
\rho_{\text{p.t.}}^{\text{cont}}(s) = \frac{N_c C_F}{8} \frac{1}{1 - e^{-\pi c_F a_s/v}} (1 + \delta_{\text{cont}} a_s)
\]

\[
\sim \frac{N_c C_F a_s}{8} (1 + \delta_{\text{cont}} a_s)
\]

and the value of the one loop radiative correction $\delta_{\text{cont}} a_s$ may be found in ref. 5.

Taking everything into account the sum rule (5) becomes,

\[
\sum_{n=n_0+1} \frac{1}{m^2 M_n} |R_n^{\text{exp}}(0)|^2 + f_{\text{back}}(v_0) = 2C_F^3 \left\{ \alpha_s^3 \sum_{n=n_0+1} \frac{1}{n^3} - \frac{\pi (\alpha_s G^2)}{m^4 a_s^3} \sum_{1}^{n_0} \lambda_n n^5 \right\} + \frac{2}{3} \left\{ 8 \epsilon_2 a_s^3 + \frac{\langle \alpha_s G^2 \rangle}{48c_1 a_s^3 m^2} \right\}.
\]

We have defined $v_0 \equiv \epsilon a_s$ and the expression is valid up to corrections of relative order $\alpha_s$. The function $f_{\text{back}}(v_0)$ is the contribution of the background which, when added to the resonances above threshold (included in the sum in the l.h.s. of (9)), give the experimental value of $\int_{\text{threshold}}^{s(v_0)} \rho_{\text{NP}}$. The function $f_{\text{back}}$ would be obtained by integrating the cross sections for production of $Y + G$ and $B B$, where by $G$ we mean a “glueball” decaying into $2\pi$, and $B$ is any of the states $B^0, B^\pm, B^*$. Because we may assume that the structure is provided by the resonances, we can take $f_{\text{back}}$ given by phase space only. So we have

\[
f_{\text{back}}(v_0) = f_1 v_0^{5/2} + f_2 v_0^3
\]

where the first term refers to the channel $Y + G$, and the second to $BB$. We have in this expression neglected $m_G$.

2. NUMEROLOGY

In principle the procedure would appear straightforward. One would fit the resonance and bound state residues and $J$ to the data, and then, after substituting into (9), obtain a determination of $\langle \alpha_s G^2 \rangle$. In practice, however, things do not work out so nicely. The quality of the experimental data does not allow any precise determination of the constants $f_{1,2}$; any values in the range $f_{1,2} \sim 0.03, 0.1$ would do the job. Secondly, the effective dependence of $\langle \alpha_s G^2 \rangle$ in Eq. (9) on experiment is proportional to $\alpha_s^3$: so the result will depend very strongly on the value of $\alpha_s$ we choose. This is particularly true because radiative corrections to the nonperturbative contribution to the bound states have not been calculated, so there is not even a “natural” renormalization point.

These two difficulties may be partially overcome with the following tricks. First of all, since we are assuming that the $n = 1$ bound state is described with the bound state analysis as discussed in ref. 3, we may fix the value of $\alpha_s$ that produces such agreement. This means that we will take $0.35 \leq \alpha_s \leq 0.4$. Secondly, we may alter the treatment of the continuum in the following manner. We split not from $v_0$, but from $v_1$, arbitrary provided only that $v_1 > v_0$. Thus, for $s \leq s(v_1)$, we use $\rho_{\text{NP}}^{\text{cont}}(s) = \rho_{\exp}^{\text{cont}}(s) - \rho_{\text{p.t.}}^{\text{cont}}(s)$, and for $s \geq s(v_1)$ we take the theoretical expression

\[
\rho_{\text{NP}}^{\text{cont}}(s) = \frac{N_c C_F}{128} \frac{\langle \alpha_s G^2 \rangle}{s^2} \frac{(1 + v^2)(1 - v^2)^2}{v^5}.
\]

The sum rule is thus written as

\[
\sum_{n=2} \frac{1}{m^2 M_n} |R_n^{\text{exp}}(0)|^2 + f_{\text{back}}(v_1) = 2C_F^3 \left\{ [\zeta(3) - 1] a_s^3 - \frac{4.9 \langle \alpha_s G^2 \rangle}{m^4 a_s^3} \right\} + \frac{2}{3} \left\{ 8 \epsilon_2 a_s^3 + \frac{\langle \alpha_s G^2 \rangle}{48c_1 a_s^3 m^2} \right\}, \quad \epsilon_1 a_s = v_1.
\]

Then we may profit from the fact that the sum rule should be valid for all values of $v_1$ to fix $f_{1,2}$ requiring this independence, at least in the mean. That is to say, that when we increase $v_1$ past a particle threshold from $Y(2)$ to $Y(6)$ the variation of the corresponding determinations of
\langle \alpha_s G^2 \rangle \) around their average be minimum. The calculation may be further simplified replacing \( f_{\text{back}}(v_1) \rightarrow 2f_0 v_1^{2.75} \).

The results of the analysis are summarized in the following tables, where the column “Res” indicates at which resonance the cut in \( v_1 \) occurs. We have taken two rather extreme values of \( f_0 \).

| Res. | \( v_1 \) | \( \langle \alpha_s G^2 \rangle \) |
|------|---------|-------------------|
| \( \Upsilon(2) \) | 0.21 | 0.014 |
| \( \Upsilon(3) \) | 0.34 | 0.034 |
| \( \Upsilon(4) \) | 0.40 | 0.048 |
| \( \Upsilon(5) \) | 0.43 | 0.039 |
| \( \Upsilon(6) \) | 0.46 | 0.046 |

For \( \alpha_s = 0.35, f_0 = 0.04 \)

| Res. | \( v_1 \) | \( \langle \alpha_s G^2 \rangle \) |
|------|---------|-------------------|
| \( \Upsilon(2) \) | 0.21 | 0.037 |
| \( \Upsilon(3) \) | 0.34 | 0.057 |
| \( \Upsilon(4) \) | 0.40 | 0.067 |
| \( \Upsilon(5) \) | 0.43 | 0.048 |
| \( \Upsilon(6) \) | 0.46 | 0.052 |

For \( \alpha_s = 0.40, f_0 = 0.09 \)

This derivation shows very clearly the kind of errors one encounters. To the variations that may be called “statistical”, apparent in the different values found in the tables above

\[
0.014 \leq \langle \alpha_s G^2 \rangle \leq 0.067
\]

we have to add “systematic” ones, e.g., the influence of the not calculated radiative corrections, easily of some 30%: not to mention our including the Coulombic wave functions at the origin for large values of \( n \), or the lack of definition of the expression “perturbation theory to all orders” because of renormalon ambiguities. Given all these uncertainties, which do even make it dubious that one can really define with precision the condensate in terms of experimental observables, it is not surprising that one cannot pin down the gluon condensate with more accuracy than an estimate, taking into account above figures, of

\[
\langle \alpha_s G^2 \rangle \simeq 0.048 \pm 0.03 \text{GeV}^4.
\]

To get this average we have taken into account all determinations in the tables above, excluding the lowest (\( \Upsilon(2) \)) and highest, \( \Upsilon(6) \). This is slightly larger than old averages, and slightly lower than more recent ones\(^6\) which tended to give, respectively, \( \langle \alpha_s G^2 \rangle \simeq 0.042, \langle \alpha_s G^2 \rangle \simeq 0.065 \text{GeV}^4 \).

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