Profinite genus of fundamental groups of torus bundles

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Abstract

In this paper we establish lower and upper bounds for the cardinality of the profinite genus of the fundamental group \( \pi_1(M_A) \cong (\mathbb{Z} \times \mathbb{Z}) \rtimes \mathbb{Z} \) of a torus bundle \( M_A \) in terms of the number of ideal classes of the order \( \mathbb{Z}[\lambda] \), where \( \lambda \) is an eigenvalue of the matrix \( A \) in \( \text{GL}_2(\mathbb{Z}) \).

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1 Introduction

Recently, the following question has attracted much attention in geometric group theory.

**Question 1:** Let \( M \) be a compact, orientable 3-manifold. To what extent is \( \pi_1(M) \) determined by its profinite completion?

In particular, [1, p. 138] highlights the following conjecture.

**Conjecture:** The fundamental group of a compact, orientable 3-manifold is determined by its profinite completion up to finitely many isomorphism classes.

W. P. Thurston, in 1982, has conjectured that there are only eight 3-dimensional model geometries, which are: \( \mathbb{E}^3, \mathbb{H}^3, \mathbb{S}^3, S^2 \times \mathbb{R}, H^2 \times \mathbb{R}, \text{SL}_2(\mathbb{R}), \text{Nil} \) and \( \text{Sol} \); this was proved by G. Y. Perelman in 2003. H. Wilton and P. Zalesskii [26] showed that the geometry of a 3-manifold is determined by the profinite completion of its fundamental group. In [27], they also proved that the profinite completion of the fundamental group of a 3-manifold \( M \) determines the Jaco–Shalen–Johannson decomposition of \( M \). H. Wilton and P. Zalesskii highlight that considering [26] and [27], the next step in addressing Question 1 is to consider the parts of the Jaco–Shalen–Johannson decomposition (i.e., the manifold modeled in one of the eight geometries) and point out that a definitive treatment of the case of 3-dimensional solvmanifolds would be a valuable addition to the literature.

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G. Wilkes [24] proved that all Seifert fiber spaces are distinguished from each other by their profinite completions apart from some previously-known examples due to Hempel of 3-manifolds with the geometry \( \mathbb{H}^3 \times \mathbb{R} \). Furthermore, in [25], he presented a complete answer to the Question 1 in relation to graph manifolds, not including the 3-dimensional solvmanifolds. In other words, G. Wilkes proved that a closed orientable 3-manifold with geometric structure modeled in one of the following geometries: \( S^3 \), \( \mathbb{E}^3 \), \( S^2 \times \mathbb{R} \), \( \text{SL}_2(\mathbb{R}) \) or \( \text{Nil} \), is determined by the profinite completion of its fundamental group. By contrast, L. Funar [9] constructed a family of torus bundles, with the geometry \( \text{Sol} \), which are not distinguished by the profinite completions of their fundamental groups.

Here we give an answer to Question 1 for 3-dimensional solvmanifolds addressing the remark of H. Wilton and P. Zalesskii [27]. We prove that there is a family of torus bundles, with the geometry \( \text{Sol} \), that are determined by the profinite completions of their fundamental groups.

A solvmanifold can be represented by a Mostow fibration (see for instance [15, Thm. 2.4]). For a 3-dimensional solvmanifold \( M \) that is not a nilmanifold, the corresponding Mostow fibration must have one of the following two forms,

\[
\mathbb{T}^2 \hookrightarrow M \to S^1 \quad \text{or} \quad S^1 \hookrightarrow M \to \mathbb{T}^2,
\]

where \( S^1 \) is the unit circle and \( \mathbb{T}^2 \) is a 2-dimensional torus. The 3-dimensional solvmanifolds \( M \) of type \( S^1 \hookrightarrow M \to \mathbb{T}^2 \) are determined by the profinite completions of their fundamental groups (see [24, Thm. 1.2]). Thus we concentrate on the case \( \mathbb{T}^2 \hookrightarrow M \to S^1 \) that leads to corresponding exact sequence of the fundamental groups,

\[
0 \to \mathbb{Z}^2 \to \pi_1(M) \to \mathbb{Z} \to 0.
\]

A 3-dimensional solvmanifold \( M \) of type \( \mathbb{T}^2 \hookrightarrow M \to S^1 \) is called a torus bundle. Its fundamental group is the semi-direct product \( \mathbb{Z}^2 \rtimes_A \mathbb{Z} \), where \( A \) is a matrix in \( \text{GL}_2(\mathbb{Z}) \). It is well known that a compact solvmanifold is uniquely determined by its fundamental group, up to diffeomorphism (see [17, Thm. A]). Thus, it makes sense to write \( M_A \) to denote the torus bundle \( M \) of fundamental group \( \mathbb{Z}^2 \rtimes_A \mathbb{Z} \), with \( A \) in \( \text{GL}_2(\mathbb{Z}) \).

We will denote by \( \text{tr}(A) \) and \( \text{det}(A) \) the trace and determinant of a matrix \( A \). If \( |\text{tr}(A)| > 2 \), then \( A \) is hyperbolic (i.e., neither of its eigenvalues has absolute value 1) and the torus bundle \( M_A \) admits the \( \text{Sol} \) geometry; if \( |\text{tr}(A)| \leq 2 \), the torus bundle \( M_A \) admits the \( \text{Nil} \) geometry (respectively the \( \text{Sol} \) geometry) if \( A \) has infinite order and is not hyperbolic (respectively \( A \) is a hyperbolic matrix); and if \( A \) has finite order, the torus bundle \( M_A \) admits the \( \mathbb{E}^3 \) geometry (see [22, Thm. 5.5]).

We denote by \( \hat{G} \) the profinite completion of a group \( G \). Let \( \mathcal{C} \) be a class of groups and \( G \in \mathcal{C} \). Following [12], we define the \( \mathcal{C} \)-genus of \( G \), denoted by \( \#(\mathcal{C}, G) \), as the set of isomorphism classes of groups belonging to \( \mathcal{C} \) that has the same profinite completion as the fixed group \( G \). Since the fundamental groups of the torus bundles are polycyclic, in this paper we restrict attention to the class \( \mathcal{P} \mathcal{F} \) of polycyclic-by-finite groups. For this reason, we calculate the profinite genus \( \#(\mathcal{C}, G_A) = \mathbb{Z}^2 \rtimes_A \mathbb{Z} \), where \( A \in \text{GL}_2(\mathbb{Z}) \). In what follows, \( \#(\mathcal{C}, G) \) denotes the cardinality of the set \( \#(\mathcal{C}, G) \).

In our study of the Question 1, we start with the case where \( \mathbb{Z}^2 \rtimes_A \mathbb{Z} \) is nilpotent.

**Theorem 1.1.** Let \( A \) be a matrix in \( \text{GL}_2(\mathbb{Z}) \) and consider the semi-direct product \( G_A = \mathbb{Z}^2 \rtimes_A \mathbb{Z} \). If all eigenvalues of \( A \) are equal to 1, then \( \#(\mathcal{P} \mathcal{F}, G_A) = 1. \)
In the case that $Z_2 \rtimes_A Z$ is not nilpotent, we establish lower and upper bounds for the cardinality of $g(PF, \pi_1(M_A))$ in terms of the number of ideal classes of the order $Z[\lambda]$, where $\lambda$ is an eigenvalue of the matrix $A$ in $GL_2(Z)$. To this end, we use a famous theorem of Latimer and MacDuffee [20, Thm. III. 13], which says that there is a one-to-one correspondence between the conjugacy classes of $n \times n$ matrices over $Z$ with same characteristic polynomial $f$ and the ideal classes of the order $Z[\lambda]$, where $\lambda$ is a root of $f$. More precisely, we get the following result.

**Theorem 1.2.** Let $A$ be a matrix in $GL_2(Z)$ and consider the group $G_A = Z_2 \rtimes_A Z$. Let $\lambda$ be an eigenvalue of $A$. Then,

(i) if $A$ has distinct eigenvalues, $\text{tr}(A) \neq 0$ and the conjugacy class of $A$ corresponds to a class of an invertible ideal of the order $Z[\lambda]$, then

\[
\begin{align*}
    h(Z[\lambda]) &\leq g(PF, G_A) \leq \tilde{h}(\lambda), & \text{if } \det(A) = -1 \\
    h(Z[\lambda])/2 &\leq g(PF, G_A) \leq \tilde{h}(\lambda), & \text{if } \det(A) = 1
\end{align*}
\]

where $h(Z[\lambda])$ is the order of the ideal class group and $\tilde{h}(\lambda)$ is the number of ideal classes of $Z[\lambda]$.

(ii) if $\text{tr}(A) = 0$ or $A$ has all its eigenvalues equal to $-1$, then

\[g(PF, G_A) = h(Z[\lambda]) = 1.\]

Let us mention two important consequences of the Theorems 1.1 and 1.2.

**Corollary 1.3.** Let $A \in GL_2(Z)$ with $\det(A) = -1$ and consider the semi-direct product $G_A = Z_2 \rtimes_A Z$. If $\text{tr}(A)^2 - 4\det(A)$ is square-free, then $g(PF, G_A) = h(Z[\lambda])$ where $\lambda$ is an eigenvalue of $A$.

**Corollary 1.4.** Let $A$ be a matrix in $GL_2(Z)$, as in Theorem 1.2 or Theorem 1.1, with eigenvalue $\lambda$. If $\tilde{h}(\lambda) = 1$, then the torus bundle $M_A$ is determined among 3-manifolds by the profinite completion of its fundamental group.

It is worth pointing out that, in 1801, Gauss [10] conjectured –and this remains open to this day– that there are infinitely many real quadratic fields with class number one.

The Theorems 1.1 and 1.2 are proved in Section 5. Sections 2, 3 and 4 contain preliminary results and definitions necessary for the proof of the main theorems. We conclude the paper in Section 6 with examples of applications of the Theorems 1.1 and 1.2.

## 2 Preliminary Results

Let $A$ be a matrix in $GL_2(Z)$. Consider the semi-direct product $G_A = N \rtimes_A C$, where $N \cong Z \times Z$ and $C \cong Z$ so that the generator $1 \in C$ acts on $N$ via $A$.

**Lemma 2.1.** Let $A$ be a matrix in $GL_2(Z)$ such that none of its eigenvalues is $1$. Let $G_A = N \rtimes_A C$ be the corresponding semi-direct product. Then,
(i) \((\text{id} - A)(N)\) is a subgroup of finite index in \(N\).

(ii) The derived subgroup of \(G_A\) is equal to \((\text{id} - A)(N)\).

**Proof.** We write the elements of \(G_A\) as pairs \((v, t) \in N \times C\) with multiplication

\[(v, t)(u, s) = (v + A' u, t + s)\]

(i) Consider the mapping \(f : N \to N\) given by \(f(v) = v - Av\). It is easy to see that \(f\) is a homomorphism. Therefore, \(f(N) = (\text{id} - A)(N)\) is a subgroup of \(N\). Since none of the eigenvalues of the matrix \(A\) is equal to \(1\), the equation \(Av = v\) has only the trivial solution \(v = 0\). This implies that \(f\) is injective. Hence, \((\text{id} - A)(N)\) has finite index in \(N\).

(ii) Note that the quotient \(\frac{G_A}{(\text{id} - A)(N)} = \frac{N}{(\text{id} - A)(N)} \ltimes_A C\) is an abelian group, since \(A\) acts trivially on \(N/(\text{id} - A)(N)\). Thus,

\[\left[ G_A, G_A \right] \leq (\text{id} - A)(N).\]

On the other hand, we have

\[(v - Av, 0) = (v, t)(Av, t)^{-1} = (v, t)((0, 1)(v, t)(0, 1)^{-1})^{-1} = (v, t)(0, 1)(v, t)^{-1}(0, 1)^{-1},\]

for each \(v \in N\) and \(t \in C\). Then, \((\text{id} - A)(N) \subseteq [G_A, G_A]\) and therefore \([G_A, G_A] = (\text{id} - A)(N)\).

\[\square\]

**Lemma 2.2.** Let \(A \neq I\) be a matrix in \(\text{GL}_2(\mathbb{Z})\). Then, \(G_A\) is a nilpotent group if and only if \(A\) has all the eigenvalues equal to 1. Furthermore, \(G_A\) has nilpotency class 2.

**Proof.** Note that, if the matrix \(A\) has at least one eigenvalue different from 1, then the center of the group \(G_A\) is trivial and, in consequence, \(G_A\) is not nilpotent.

Conversely, let \(A \in \text{GL}_2(\mathbb{Z})\) with all its eigenvalues equal to 1. By [6, Lem. 13.27],

\[\{1\} \leq \mathbb{Z} \leq \mathbb{Z}^2\]

is a series of \(\mathbb{Z}^2\) such that \(A\) acts on \(\mathbb{Z}^2/\mathbb{Z}\) and \(\mathbb{Z}\) as the identity. Thus, \(\mathbb{Z}\) is a central subgroup of \(G_A\), i.e., \(\mathbb{Z}\) is contained in the center \(Z_1(G_A)\) of \(G_A\). Note that, \(G_A\) is nilpotent of class 2 if and only if we have the following upper central series

\[\{1\} = Z_0(G_A) \leq Z_1(G_A) \leq Z_2(G_A) = G_A,\]

if and only if the group \(G_A/Z_1(G_A)\) is abelian. Which is the case, since the group

\[\frac{\mathbb{Z}^2}{\mathbb{Z}} \ltimes_A \mathbb{Z} \cong \frac{G_A}{\mathbb{Z}}\]
is abelian (because $A$ acts trivially on $\mathbb{Z}^2/\mathbb{Z}$) and

$$\frac{G_A}{Z_1(G_A)} \cong \frac{G_A/\mathbb{Z}}{Z_1(G_A)/\mathbb{Z}}.$$  

Lemma 2.3. Let $A$ be a matrix in $GL_2(\mathbb{Z})$. Then $G_A \cong G_{A^{-1}}$.

Proof. Consider the mapping $f : G_A \rightarrow G_{A^{-1}}$ defined by $f(v, t) = (v, -t)$. It is easy to see that $f$ is an isomorphism from $G_A$ to $G_{A^{-1}}$. \hfill \Box

Let $A$ and $B$ be matrices in $GL_2(\mathbb{Z})$. Consider the semi-direct products $G_A = N_1 \rtimes_A C$ and $G_B = N_2 \rtimes_B C$, where $N_1 \cong \mathbb{Z} \times \mathbb{Z} \cong N_2$ and $C \cong \mathbb{Z}$.

Lemma 2.4.

(i) Let $A$ and $B$ be matrices in $GL_2(\mathbb{Z})$ such that none of its eigenvalues is 1. If $f : G_A \rightarrow G_B$ is an isomorphism then $f(N_1) = N_2$.

(ii) Let $A$ and $B$ be matrices in $GL_2(\mathbb{Z})$ such that none of its eigenvalues is 1. If $f : \hat{G}_A \rightarrow \hat{G}_B$ is an isomorphism then $f(\hat{N}_1) = \hat{N}_2$.

Proof. (i) We have by Lemma 2.1 that the derived subgroup $[G_B, G_B]$ has finite index in $N_2$. Consider

$$L = \{ \alpha \in G_B : \exists n \in \mathbb{Z}, n \neq 0, \text{ such that } \alpha^n \in [G_B, G_B] \}.$$  

We claim that $L = N_2$. Indeed, since the index of $[G_B, G_B]$ in $N_2$ is finite we have $N_2 \subseteq L$. On the other hand, if $\alpha = (v, t) \in L$ ($v \in N_2, t \in C$), then there is $n \in \mathbb{Z}, n \neq 0$, such that $\alpha^n \in [G_B, G_B] \leq N_2$. Note that,

$$\alpha^n = (v, t)^n = (v + A'tx + \cdots + A^{(n-1)}tv, nt).$$  

Since $\alpha^n \in N_2$ and $n \neq 0$ we have $t = 0$. Therefore $L \subseteq N_2$ and hence $L = N_2$.

Let $u \in N_1$, then there is $n \in \mathbb{Z}, n \neq 0$, such that $u^n \in [G_A, G_A]$. Hence, $f(u)u^n \in [G_B, G_B]$ and so $f(u) \in N_2$. Therefore, $f(N_1) \subseteq N_2$. By a similar argument, $N_2 \subseteq f(N_1)$.

(ii) This follows from (i). \hfill \Box

Lemma 2.5 ([12], Prop 2.5). Let $H, N$ be groups and $\varphi_1, \varphi_2 : H \rightarrow \text{Aut}(N)$ be homomorphisms. Let $G_1 = N \times_{\varphi_1} H$ and $G_2 = N \times_{\varphi_2} H$ be the corresponding semi-direct products. The following statements hold.

1. Suppose that there exist $\Theta \in \text{Aut}(H)$ and $\bar{f} \in \text{Aut}(N)$ such that

$$\varphi_1(h)\bar{f}^{-1} = \varphi_2(\Theta(h))$$  

for all $h \in H$. Then the map $f : G_1 \rightarrow G_2$ defined by $f(n, h) = (\bar{f}(n), \Theta(h))$ is an isomorphism which satisfies $f(N) = N$ and $f(H) = H$. 

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2. Suppose now that \( N \) is abelian and that \( f : G_1 \to G_2 \) is an isomorphism such that \( f(N) = N \). Define \( \bar{f} = f|_N \) and \( \Theta : H \to H \) by inducing \( f \) to \( G_1/N = H \). Then the pair \((\bar{f}, \Theta)\) satisfies (1) and hence also defines an isomorphism from \( G_1 \) to \( G_2 \).

**Lemma 2.6** ([12], Cor 2.2). Let \( H, N \) be groups and \( \varphi_1, \varphi_2 : H \to \text{Aut}(N) \) homomorphisms. Let \( G_1 = N \rtimes_{\varphi_1} H \) and \( G_2 = N \rtimes_{\varphi_2} H \) be the corresponding semidirect products and let \( f : G_1 \to G_2 \) be an isomorphism such that \( f(N) = N \). Then \( \tilde{\varphi}_1(h)\bar{f}^{-1} = \tilde{\varphi}_2(f(h)) \) for all \( h \in H \), where \( \bar{f} \) is the image of the restriction \( f|_N \) in \( \text{Out}(N) \) and \( \tilde{\varphi}_i \) is the composition of \( \varphi_i \) with the natural homomorphism \( \text{Aut}(N) \to \text{Out}(N) \) for \( i = 1, 2 \). In particular, \( \tilde{\varphi}_1(H) \) and \( \tilde{\varphi}_2(H) \) are conjugate in \( \text{Out}(N) \).

**Remark 2.7.** Note that similar versions of Lemmas 2.5 and 2.6 hold for profinite groups and continuous homomorphisms. The same proofs presented in [12] applies to these reformulations.

We will denote by \( \mathfrak{F}(G) \) the set of finite quotients of a group \( G \).

**Proposition 2.8** ([8], p. 227). Let \( G \) and \( H \) be finitely generated groups. Then \( \hat{G} \) and \( \hat{H} \) are isomorphic as topological groups if and only if \( \mathfrak{F}(G) = \mathfrak{F}(H) \).

This allows to state version of Theorem of F. Grunewald and R. Scharlau.

**Proposition 2.9** ([11], p. 163). Let \( G_1 \) and \( G_2 \) be finitely generated torsion-free nilpotent groups of nilpotency class 2, such that the Hirsch number \(^1\) of \( G_1 \) is smaller than or equal to 5 and \( \hat{G}_1 \cong \hat{G}_2 \). Then \( G_1 \cong G_2 \).

3 The Ideal Class Group

In this section we recall definitions and results of Number Theory that will be used in the paper.

**Definition 3.1.** Let \( K \) be an algebraic number field of degree \( n \). An order of \( K \) is a subring \( \mathcal{O} \) of the ring of integers of \( K \) which contains an integral basis of length \( n \). The ring of integers of \( K \), which we denote by \( \mathcal{O}_K \), is called the maximal order of \( K \).

Set
\[
  r := \begin{cases} 
    1, & \text{if } m \not\equiv 1 \pmod{4} \\
    2, & \text{if } m \equiv 1 \pmod{4}
  \end{cases}
\]
and define \( \omega_0 := (r - 1 + \sqrt{m})/r \).

**Proposition 3.2** ([14], Thm. 4.17). If \( \mathcal{O} \) is any order of a quadratic field \( K \), then \( \mathcal{O} \) is a \( \mathbb{Z} \)-module with basis \( \{1, \omega\} \), where \( \omega := f\omega_0 \) for some \( f \in \mathbb{Z} \).

Then \( \mathcal{O} \) has finite index in \( \mathcal{O}_K \). The index \( f = |\mathcal{O}_K : \mathcal{O}| \) is called the conductor of the order \( \mathcal{O} \).

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\(^1\)This is the number of infinite cyclic factors in a series with cyclic or finite factors.
Lemma 3.3. Let $A$ be a matrix in $\text{GL}_2(\mathbb{Z})$. If $\text{tr}(A) \neq 0$ and $A$ has distinct eigenvalues $\lambda_1$ and $\lambda_2$, then $\mathbb{Q}(\lambda_1, \lambda_2)$ is a quadratic field.

Proof. Let $A \in \text{GL}_2(\mathbb{Z})$. Then the characteristic polynomial of $A$ is $p(x) = x^2 - \text{tr}(A)x + \det(A)$ and

$$
\lambda_{\pm} = \frac{\text{tr}(A) \pm \sqrt{\text{tr}(A)^2 - 4\det(A)}}{2}
$$

are its eigenvalues. Note that $A$ has equal eigenvalues if and only if $\text{tr}(A) = \pm 2$ and $\det(A) = 1$. Since $A$ has distinct eigenvalues and $\text{tr}(A) \neq 0$, we conclude that the possible values for the trace and the determinant of the matrix $A$ are:

(i) $\text{tr}(A) = \pm 1$ and $\det(A) = \pm 1$ or

(ii) $\text{tr}(A) = \pm 2$ and $\det(A) = -1$ or

(iii) $|\text{tr}(A)| > 2$ and $\det(A) = \pm 1$.

The cases (i) and (ii) are trivial. If $A$ satisfies (iii), then the eigenvalues of $A$ are irrational. Indeed, suppose that there is an integer $k$ such that $\text{tr}(A)^2 - 4\det(A) = k^2$. This gives $\text{tr}(A)^2 - k^2 = \pm 4$, which implies $k = \text{tr}(A) \pm 1$ and so $2(\pm \text{tr}(A)) - 1 = \pm 4$, which is a contradiction, because $|\text{tr}(A)| > 2$. Therefore, we conclude that $K = \mathbb{Q}(\lambda_+, \lambda_-)$ is a quadratic field, i.e., there is a square-free integer $d$ such that $K = \mathbb{Q}(\sqrt{d})$. \hfill \qed

Lemma 3.4. Let $A \in \text{GL}_2(\mathbb{Z})$ with distinct eigenvalues. If $\text{tr}(A)^2 - 4\det(A)$ is square-free, then $\mathbb{Z}[\lambda] = \mathcal{O}_K$, where $K = \mathbb{Q}(\lambda)$ and $\lambda$ is an eigenvalue of $A$.

Proof. Note that $\{1, \lambda\}$ is a $\mathbb{Z}$-basis for $\mathbb{Z}[\lambda]$ and that the discriminant of $\{1, \lambda\}$ is $D = \text{tr}(A)^2 - 4\det(A)$. Since $D$ is square-free, it follows that $\{1, \lambda\}$ is a $\mathbb{Z}$-basis for $\mathcal{O}_K$ by [2, Thm. 7.1.8]. Therefore, $\mathbb{Z}[\lambda] = \mathcal{O}_K$. \hfill \qed

Proposition 3.5 ([7], §16.3, Prop. 14 and Cor. 19). The ring of integers $\mathcal{O}_K$ of an algebraic number field $K$ is a Dedekind Domain. In particular, if $I$ is a nonzero ideal of $\mathcal{O}_K$, then every ideal in the quotient $\mathcal{O}_K/I$ is principal.

Definition 3.6. The norm $N(I)$ of an $\mathcal{O}$-ideal $I$ is the index $|\mathcal{O}/I|$.

Proposition 3.7 ([14], Prop. 4.23 and Thm. 4.24). Let $\mathcal{O}$ be an order in a quadratic field $K$. Then,

1. $I$ is an ideal of $\mathcal{O}$ if and only if $I$ is a $\mathbb{Z}$-module with basis $\{a, b + c\omega\}$, where $\omega$ is as in Proposition 3.2, $a, b, c \in \mathbb{Z}$, $a > 0$, $c > 0$, $0 \leq b < a$, $c \mid a$, $c \mid b$, and $ac$ divides the norm of $b + c\omega$.

2. $N(I) = ac$.

Definition 3.8. A fractional ideal of an order $\mathcal{O}$ is a finitely generated $\mathcal{O}$-submodule $I \neq 0$ of $K$ such that $dI \subseteq \mathcal{O}$ for some $d \in \mathcal{O} \setminus \{0\}$.

The ideals of $\mathcal{O}$ are also fractional ideals of $\mathcal{O}$ (take $d = 1$). For clarity these are occasionally called the integral ideals of $\mathcal{O}$.

For any $a \in K \setminus \{0\}$ the cyclic $\mathcal{O}$-module $(a) := a\mathcal{O} = \{ax : x \in \mathcal{O}\}$ is called the principal fractional ideal generated by $a$. 7
Definition 3.9. Two fractional ideals $I$ and $J$ are equivalent $(I \sim J)$ if there is $\alpha \in K \setminus \{0\}$ such that $J = \alpha I$.

The equivalence established in Definition 3.9 is an equivalence relation in the set of all the fractional ideals of $\mathcal{O}$ and therefore partitions those ideals into distinct ideal classes.

Proposition 3.10 ([5], Thm. 20.6). The number of classes of fractional $\mathcal{O}$-ideals is finite.

Definition 3.11. A fractional $\mathcal{O}$-ideal $I$ is invertible if there is another fractional $\mathcal{O}$-ideal $J$ such that $IJ = (\alpha)$, where $\alpha \in \mathcal{O}$.

Proposition 3.12 ([19], §12). Let $\mathcal{O}$ be an order. The set $I(\mathcal{O})$ of invertible fractional $\mathcal{O}$-ideals form an abelian group. Moreover, the fractional principal $\mathcal{O}$-ideals give a subgroup $P(\mathcal{O}) \subseteq I(\mathcal{O})$.

Definition 3.13. The quotient group $H(\mathcal{O}) = I(\mathcal{O})/P(\mathcal{O})$ is called the ideal class group (or Picard group) of the order $\mathcal{O}$. We denote by $[I]$ the class of an ideal $I \in I(\mathcal{O})$.

Since the fractional $\mathcal{O}$-ideals $I$ and $J$ are in the same class of $H(\mathcal{O})$ if and only if $I \sim J$, we have by Proposition 3.10 the following corollary.

Corollary 3.14. The ideal class group $H(\mathcal{O})$ is finite. Its order will be denoted by $h(\mathcal{O})$.

Given an order $\mathcal{O}$ of a quadratic field $K$ with conductor $f$, we say that a nonzero $\mathcal{O}$-ideal $I$ is prime to $f$ provided that $I + f\mathcal{O} = \mathcal{O}$.

Proposition 3.15 ([4], Lem. 7.18, Prop. 7.4 and [14], Thm. 4.37). Let $\mathcal{O}$ be an order of conductor $f$.

1. An $\mathcal{O}$-ideal $I$ is prime to $f$ if and only if its norm $N(I)$ is relatively prime to $f$, i.e., $(N(I), f) = 1$.

2. Every $\mathcal{O}$-ideal prime to $f$ is invertible.

3. If $I$ is an invertible $\mathcal{O}$-ideal, then there is always some $\mathcal{O}$-ideal $J$ such that $I \sim J$ and $(N(J), f) = 1$.

Lemma 3.16. Let $\mathcal{O}$ be an order of a quadratic field $K$ with conductor $f$. Then each class of ideals in $H(\mathcal{O})$ contains an ideal that is prime to $f$.

**Proof.** Let $[I] \in H(\mathcal{O})$. We can assume that $I$ is an integral ideal of $\mathcal{O}$. Since $I$ is invertible, it follows from Proposition 3.15 (3) that there is an $\mathcal{O}$-ideal $J$ such that $J \sim I$ and $(N(J), f) = 1$. By Proposition 3.15, we have $J$ is an invertible ideal that is prime to $f$ that belongs to the class of $I$. 

Lemma 3.17. Let $\mathcal{O}$ be an order of a quadratic field $K$ with conductor $f$. For each $\mathcal{O}$-ideal $J$ prime to $f$, the natural ring homomorphism $\mathcal{O}/J \to \mathcal{O}_K/J\mathcal{O}_K$ is an isomorphism.
Proof. Since $J$ is prime to $f$, we have by definition that $J + f\mathcal{O} = \mathcal{O}$. Thus,

$$J\mathcal{O}_K + f\mathcal{O}_K = (J + f\mathcal{O})\mathcal{O}_K = \mathcal{O}\mathcal{O}_K = \mathcal{O}_K.$$ 

We claim that $J\mathcal{O}_K \cap \mathcal{O} = J$. Indeed, note that

$$J \subseteq J\mathcal{O}_K \cap \mathcal{O} \subseteq (J\mathcal{O}_K \cap \mathcal{O})(J + f\mathcal{O}) \subseteq J + J \cdot f\mathcal{O}_K \subseteq J + f\mathcal{O}_K \subseteq \mathcal{O}.$$ 

This gives $J\mathcal{O}_K \cap \mathcal{O} = J$. Now consider natural homomorphism

$$\varphi: \mathcal{O} \to \frac{\mathcal{O}_K}{J\mathcal{O}_K}.$$ 

As $J\mathcal{O}_K + f\mathcal{O}_K = \mathcal{O}_K$, we conclude that $\varphi$ is surjective and from $J\mathcal{O}_K \cap \mathcal{O} = J$ we have $\ker(\varphi) = J$. Therefore, the statement follows from the First Isomorphism Theorem for rings. \qed

Lemma 3.18. Let $\mathcal{O}$ be an order of a quadratic field. If $I$ is an invertible $\mathcal{O}$-ideal, then $|I/mI| = |\mathcal{O}/m\mathcal{O}|$ for each positive integer $m$.

Proof. For each $m \in \mathbb{N}$, we have $mI \subseteq I \subseteq \mathcal{O}$. Then

$$\left| \frac{\mathcal{O}}{mI} \right| = \left| \frac{\mathcal{O}}{I} \right| \left| \frac{I}{mI} \right|.$$ 

Thus,

$$\left| \frac{I}{mI} \right| = \frac{|\mathcal{O}/mI|}{|\mathcal{O}/I|} \tag{2}$$ 

As $mI = m\mathcal{O}I$, we have by [14, Thm. 4.36] that

$$\left| \frac{\mathcal{O}}{mI} \right| = N(mI) = N(m\mathcal{O})N(I) = \left| \frac{\mathcal{O}}{m\mathcal{O}} \right| \left| \frac{\mathcal{O}}{I} \right|. \tag{3}$$ 

Substituting (3) into (2) yields

$$\left| \frac{I}{mI} \right| = \left| \frac{\mathcal{O}}{m\mathcal{O}} \right|. \quad \square$$ 

Let $I$ be an $\mathcal{O}$-ideal and $m \in \mathbb{N}$. Note that $I/mI$ is an $\mathcal{O}/m\mathcal{O}$-module with the action of $\mathcal{O}/m\mathcal{O}$ on $I/mI$ given by

$$(\theta + m\mathcal{O}) \cdot (\alpha + mI) := \theta \alpha + mI.$$ 

This action is well defined because $I$ is an $\mathcal{O}$-module.
Lemma 3.19. Let $\mathcal{O}$ be an order of a quadratic field $K$ with conductor $f$. If $I$ is an $\mathcal{O}$-ideal prime to $f$, then
\[
\frac{I}{p^i I} \cong \frac{\mathcal{O}}{p^i \mathcal{O}}
\]
as $\mathcal{O}/p^i \mathcal{O}$-modules for each prime number $p$ and $i \in \mathbb{N}$.

Proof. We divide the proof in two cases.

Case 1. If $p$ does not divide $f$.

By Proposition 3.7, \{a, b + c\} is a $\mathbb{Z}$-basis for $I$ and $N(I) = ac$, where $a, b, c \in \mathbb{Z}$. Hence, \{p^i a, p^i (b + c\omega)\} is a $\mathbb{Z}$-basis for $p^i I$ and $N(p^i I) = p^{2i}ac$. Since $I$ is prime to $f$, we have $(ac, 1) = 1$ by Proposition 3.15 (1). This gives $(p^{2i}ac, f) = 1$ since $p$ does not divide $f$. Thus, $p^i I$ are invertible ideals prime to $f$ by Proposition 3.15. Now, we have by Lemma 3.17 that
\[
\frac{\mathcal{O}}{p^i I} \cong \frac{\mathcal{O}_K}{p^i \mathcal{O}_K}
\]
as rings. From Proposition 3.5, it follows that $I/p^i I$ is a principal ideal of $\mathcal{O}/p^i I$. Since
\[
\left| \frac{I}{p^i I} \right| = \left| \frac{\mathcal{O}}{p^i \mathcal{O}} \right|
\]
by Lemma 3.18 and $I/p^i I$ and $\mathcal{O}/p^i \mathcal{O}$ are cyclic $\mathcal{O}/p^i \mathcal{O}$-modules, we have
\[
\frac{I}{p^i I} \cong \frac{\mathcal{O}}{p^i \mathcal{O}}
\]
as $\mathcal{O}/p^i \mathcal{O}$-modules.

Case 2. If $p$ divides $f$.

Consider the mapping $\varphi : I \to \mathcal{O}/p^i \mathcal{O}$ defined by $\varphi(\alpha) = \alpha a + p^i \mathcal{O}$. It is easy to see that $\varphi$ is a group homomorphism. Now let us prove that $\varphi$ is a bijection.

- $\varphi$ is surjective.

To prove the statement it is sufficient to prove that $aI + p^i \mathcal{O} = \mathcal{O}$. From $(a, p) = 1$ we conclude that $(a^2, p^i) = 1$. Note that \{p^i, p^i \omega\} and \{a^2, a(b + c\omega)\} are $\mathbb{Z}$-bases for $p^i \mathcal{O}$ and $aI$, respectively. As the Diophantine equation $xa^2 + yp^i = 1$ has an integer solution (because $(a^2, p^i) = 1$), we have $1 \in aI + p^i \mathcal{O}$. Thus, $\mathcal{O} \subseteq aI + p^i \mathcal{O}$ and consequently $aI + p^i \mathcal{O} = \mathcal{O}$.

- $\ker(\varphi) = p^i I$.

Remember that \{a, b + c\omega\} is a $\mathbb{Z}$-basis for $I$ and that \{1, \omega\} is a $\mathbb{Z}$-basis for $\mathcal{O}$. The result is obvious if $a = 1$. Suppose $a \neq 1$. Since $p \nmid a$, we have $p^i \nmid I$.

Let $\beta \in \ker(\varphi)$. Then $\beta a \in p^i \mathcal{O}$. So there is $\alpha \in \mathcal{O}$ such that $\beta a = p^i \alpha$. We claim that $\beta \in p^i I$. Indeed, we have
\[
\beta = ua + v(b + c\omega) \quad \text{and} \quad \alpha = r + s\omega
\]
where \( u, v, r, s \in \mathbb{Z} \). Then,
\[
    r + s\omega = \alpha = \frac{a}{p^i} = \frac{a}{p^i}(ua + vb) + \frac{a}{p^i}vc\omega.
\]

We thus get
\[
    r = \frac{a}{p^i}(ua + vb) \quad \text{and} \quad s = \frac{acv}{p^i}.
\]

As \( p \nmid ac \) and \( s \in \mathbb{Z} \), we have \( p^i \nmid v \). Furthermore, from \( p \nmid a^2 \) we conclude that \( p^i \nmid u \), since \( r \in \mathbb{Z} \). This gives \( u = p^i u' \) and \( v = p^i v' \), where \( u', v' \in \mathbb{Z} \). Therefore,
\[
    \beta = ua + v(b + c\omega) = p^i(u'a + v'(b + c\omega)) \in p^iI,
\]
which implies \( \ker(\varphi) \subseteq p^iI \). Clearly \( p^iI \subseteq \ker(\varphi) \) and so \( \ker(\varphi) = p^iI \).

Therefore, it follows from the previous points that the mapping \( \tilde{\varphi} : I/p^iI \to \mathcal{O}/p^i\mathcal{O} \) defined by \( \tilde{\varphi}(\alpha + p^iI) = \alpha a + p^i\mathcal{O} \) is an isomorphism of abelian groups. Since
\[
    \tilde{\varphi}((\theta + p^i\mathcal{O}) \cdot (\alpha + p^iI)) = \tilde{\varphi}(\theta \alpha + p^iI) = \theta \alpha a + p^i\mathcal{O} = (\theta + p^i\mathcal{O})(\alpha a + p^i\mathcal{O}) = (\theta + p^i\mathcal{O})\tilde{\varphi}(\alpha + p^iI),
\]
it follows that \( \tilde{\varphi} \) is an isomorphism of \( \mathcal{O}/p^i\mathcal{O} \)-modules. \( \square \)

## 4 Conjugation in \( \text{GL}_2(\mathbb{Z}) \) and \( \text{GL}_2(\hat{\mathbb{Z}}) \)

Let \( A \) and \( B \) be matrices in \( \text{GL}_2(\mathbb{Z}) \). Consider the semi-direct products \( G_A = N_1 \rtimes_A C \) and \( G_B = N_2 \rtimes_B C \), where \( N_1 \cong \mathbb{Z} \times \mathbb{Z} \cong N_2 \) and \( C \cong \mathbb{Z} \).

**Lemma 4.1.** Let \( A \) and \( B \) be matrices in \( \text{GL}_2(\mathbb{Z}) \) such that none of its eigenvalues is 1. Then \( G_A \) and \( G_B \) are isomorphic if and only if \( A \) is conjugate in \( \text{GL}_2(\mathbb{Z}) \) to \( B \) or \( B^{-1} \).

**Proof.** It follows from Lemma 2.4 (i) that if \( f : G_A \to G_B \) is an isomorphism, then the restriction of \( f \) to \( N_1 \) defines an isomorphism \( P : N_1 \to N_2 \). Then, \( P \in \text{GL}_2(\mathbb{Z}) \). Write \( f(0, 1) = (u, t) \), where \( u \in N_2 \) and \( t \in C \). Note that,
\[
    (0, 1)(v, 0)(0, -1) = (Av, 0).
\]

Computing \( f \) in equation (4), we have
\[
(PAv, 0) = f((0, 1)(v, 0)(0, -1)) = f(0, 1)f(v, 0)f(0, -1) = (u, t)(Pv, 0)(u, t)^{-1} = (u + B^tPv, t)(B^{-t}(-u), -t) = (B^tPv, 0).
\]

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Thus, \( PAP^{-1} = B^t \). Now, let us show that \( t = \pm 1 \). To this end, note that

\[
f(-P^{-1}u, 1) = f((P^{-1}(-u), 0)(0, 1)) \\
= (PP^{-1}(-u), 0)f(0, 1) \\
= (-u, 0)(u, t) \\
= (0, t).
\]

Since \((0, 1)^t = (0, t)\), implies that there is \( \beta \in G_A \) such that \( \beta^t = (-P^{-1}u, 1) \). Then, \( t \) divides 1 and hence \( t = \pm 1 \).

Conversely, if there is \( P \in \text{GL}_2(\mathbb{Z}) \) such that \( PAP^{-1} = B \) or \( PAP^{-1} = B^{-1} \), then \( G_A \cong G_B \) or \( G_A \cong G_B^{-1} \) by Lemma 2.5. As \( G_B \cong G_B^{-1} \) by Lemma 2.3, we have \( G_A \cong G_B \) in both cases.

The next lemma is a reformulation of Lemma 4.1 for profinite groups.

**Lemma 4.2.** Let \( A \) and \( B \) be matrices in \( \text{GL}_2(\mathbb{Z}) \) such that none of its eigenvalues is 1. Then \( \hat{G}_A \) and \( \hat{G}_B \) are isomorphic if and only if \( A \) is conjugate in \( \text{GL}_2(\hat{\mathbb{Z}}) \) to \( B \) or \( B^{-1} \).

**Proof.** It follows from Lemma 2.4 (ii) that if \( f : \hat{G}_A \to \hat{G}_B \) is an isomorphism, then the restriction of \( f \) to \( \hat{N}_1 \) defines an isomorphism \( P : \hat{N}_1 \to \hat{N}_2 \). Then, \( P \in \text{GL}_2(\hat{\mathbb{Z}}) \).

Suppose that \( f(0, 1) = (u, t) \), where \( u \in \hat{N}_2 \) and \( t \in \hat{C} \). Note that,

\[
(0, 1)(v, 0)(0, -1) = (Av, 0).
\]

By similar arguments as in the proof of Lemma 4.1, we concluded that \( PAP^{-1} = B^t \)

where \( t \) is a unit in \( \hat{C} \). Since \( \det(B) = \pm 1 \in \mathbb{Z} \) and

\[
\det(B^t) = (\det(B))^t \in \mathbb{Z},
\]

we have \( t = \pm 1 \).

Conversely, if there is \( P \in \text{GL}_2(\hat{\mathbb{Z}}) \) such that \( PAP^{-1} = B \) or \( PAP^{-1} = B^{-1} \), then \( \hat{G}_A \cong \hat{G}_B \) or \( \hat{G}_A \cong \hat{G}_B^{-1} \) by profinite version of the Lemma 2.5. We have by Lemma 2.3 that \( N_2 \times_{B^{-1}} C \cong N_2 \times B(C) \) and so \( \hat{N}_2 \times_{B^{-1}} \hat{C} \cong \hat{N}_2 \times B \hat{C} \). Therefore, \( \hat{G}_A \cong \hat{G}_B \). \( \square \)

**Lemma 4.3.** Let \( A \) and \( B \) be matrices in \( \text{GL}_2(\mathbb{Z}) \). If \( A \) and \( B \) are conjugate in \( \text{GL}_2(\hat{\mathbb{Z}}) \) then,

(i) \( \det(A) = \det(B) \)

(ii) \( \text{tr}(A) = \text{tr}(B) \)

(iii) \( A \) and \( B \) have the same characteristic polynomial.

**Proof.** If \( A \) and \( B \) are conjugate in \( \text{GL}_2(\hat{\mathbb{Z}}) \), then for each positive integer \( m \), \( \det(A) \) is congruent to \( \det(B) \) modulo \( m \) and \( \text{tr}(A) \) is congruent to \( \text{tr}(B) \) modulo \( m \). Thus, we conclude that \( \det(A) = \det(B) \) and \( \text{tr}(A) = \text{tr}(B) \).

Since the characteristic polynomials of \( A \) and \( B \) are \( p_A(x) = x^2 - \text{tr}(A)x + \det(A) \) and \( p_B(x) = x^2 - \text{tr}(B)x + \det(B) \), (iii) follows from (i) and (ii). \( \square \)
Proposition 4.4. Let $A$ and $B$ be matrices in $\text{GL}_2(\mathbb{Z})$ and consider the semi-direct products $G_A = N_1 \rtimes_A C$ and $G_B = N_2 \rtimes_B C$ with $N_1 \cong \mathbb{Z}^2 \cong N_2$ and $C \cong \mathbb{Z}$. Then, each isomorphism class in $\mathfrak{g}(G_A)$ contains a group $G_B$ so that $A$ and $B$ have the same eigenvalues.

Proof. Let $G_B$ be a representative of any isomorphism class in $\mathfrak{g}(G_A)$. Suppose first that $A$ has at least one eigenvalue equal to 1. Since $\det(A) = \pm 1$, we conclude that the possible eigenvalues of $A$ are or 1 and $-1$ or all equal to 1. If the eigenvalues of $A$ are 1 and $-1$, then $A$ has order 2 and hence $G_A$ is a 3-dimensional Bieberbach group with holonomy group of order 2 (see [23, Thm. 3.2]). By [18, Prop. 3.7 and Lem 3.3] we see that $G_B$ is also a 3-dimensional Bieberbach group with holonomy group of order 2 and $A$ is conjugate to $B$ in $\text{GL}_2(\hat{\mathbb{Z}})$. Therefore, it follows from Lemma 4.3 that $A$ and $B$ have the same eigenvalues. Now, if $A$ has all the eigenvalues equal to 1, then $G_A$ is a nilpotent group (Lemma 2.2). Hence, $\hat{G}_A$ and $\hat{G}_B$ are nilpotent groups, which implies that $B$ has all the eigenvalues equal to 1 (otherwise, the center of $\hat{G}_B$ is trivial, and consequently $\hat{G}_B$ is not nilpotent).

Suppose now that none of the eigenvalues of $A$ is 1. Then none of the eigenvalues of $B$ is 1, by what has already been proven in the previous paragraph. Hence, $A$ is conjugate to $B$ or $B^{-1}$ in $\text{GL}_2(\hat{\mathbb{Z}})$ (Lemma 4.2). By Lemma 2.3 we can assume that $A$ is conjugate to $B$ in $\text{GL}_2(\hat{\mathbb{Z}})$. Using again Lemma 4.3, we conclude that $A$ and $B$ have the same eigenvalues. \hfill \Box

Lemma 4.5. Let $A \in \text{GL}_2(\mathbb{Z})$. Then $\text{tr}(A^{-1}) = \text{tr}(A)/\text{det}(A)$.

Proof. Let $A \in \text{GL}_2(\mathbb{Z})$. It is easy to see that the characteristic polynomial of $A$ has the form $p(x) = x^2 - \text{tr}(A)x + \text{det}(A)$ and that

$$\lambda_{\pm} = \frac{\text{tr}(A) \pm \sqrt{\text{tr}(A)^2 - 4\text{det}(A)}}{2}$$

are the eigenvalues. Note that,

$$\frac{\text{det}(A)}{\lambda_+} = \lambda_-.$$

Since the trace of $A^{-1}$ is equal to the trace of the diagonal matrix $A^{-1}$, we have

$$\text{tr}(A^{-1}) = \frac{1}{\lambda_+} + \frac{1}{\lambda_-} = \frac{1}{\lambda_+} + \frac{\lambda_+}{\text{det}(A)} = \frac{2}{\text{tr}(A) + \sqrt{\text{tr}(A)^2 - 4\text{det}(A)}} + \frac{\text{tr}(A) + \sqrt{\text{tr}(A)^2 - 4\text{det}(A)}}{2\text{det}(A)} = \frac{\text{tr}(A)}{\text{det}(A)}.$$

\hfill \Box
Corollary 4.6. Let \( A \in \text{GL}_2(\mathbb{Z}) \) such that \( \det(A) = -1 \) and none of its eigenvalues is 1. Then the cardinality of the genus \( g(G_A) \) is exactly the number of conjugacy classes of matrices in \( \text{GL}_2(\mathbb{Z}) \) in the conjugacy class of \( A \) in \( \text{GL}_2(\hat{\mathbb{Z}}) \).

Proof. Let \( G_B \in g(G_A) \). By Proposition 4.4 none of the eigenvalues of \( B \) is 1. From Lemma 4.2, implies that \( A \) is conjugate in \( \text{GL}_2(\hat{\mathbb{Z}}) \) to \( B \) or \( B^{-1} \). By Lemma 2.3 we can assume that \( A \) is conjugate to \( B \) in \( \text{GL}_2(\hat{\mathbb{Z}}) \). Thus \( \det(A) = \det(B) \) and \( \text{tr}(A) = \text{tr}(B) \) by Lemma 4.3. Since \( \det(A) = -1 \), we have \( \text{tr}(B^{-1}) = -\text{tr}(B) \) (Lemma 4.5). Therefore, the statement follows by Lemmas 4.1 and 4.2. \( \square \)

Thus, if \( G_B \in g(G_A) \), we can assume that \( A \) and \( B \) have the same characteristic polynomial. Therefore, to study the extent to which matrices \( A \) and \( B \) are conjugate in \( \text{GL}_2(\mathbb{Z}) \), we need to study the set of conjugacy classes in \( \text{GL}_2(\mathbb{Z}) \) of matrices which have the same characteristic polynomial. For this purpose, the next theorem tells us that it is enough to study the ideal classes of orders of quadratic fields.

Proposition 4.7 (Latimer-MacDuffee, [20], Thm. III. 13). Let \( f(x) \in \mathbb{Z}[x] \) be a monic polynomial of degree \( n \) that is irreducible over \( \mathbb{Q} \) and let \( \lambda \) be a root of \( f(x) \). Then there is a one-to-one correspondence between the \( \mathbb{Z} \)-similarity classes of \( n \times n \) matrices over \( \mathbb{Z} \) with characteristic polynomial \( f \) and the ideal classes of the order \( \mathbb{Z}[\lambda] \).

Here we are interested in the case in which \( n = 2 \). Let

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z})
\]

and let \( \lambda \) be an eigenvalue of \( A \). Consider the \( \mathbb{Z} \)-module

\[
I_A = \langle b, \lambda - a \rangle = \{mb + n(\lambda - a): m, n \in \mathbb{Z} \}.
\]

(5)

Proposition 4.8 ([13], Thm. 1). Let \( A \) be a matrix in \( \text{GL}_2(\mathbb{Z}) \). Then, \( I_A \) is an ideal of the order \( \mathbb{Z}[\lambda] \).

We have the following version of Proposition 4.7 for \( \text{GL}_2(\mathbb{Z}) \).

Proposition 4.9 ([13], Thm. 2). Let \( A \) and \( B \) be matrices in \( \text{GL}_2(\mathbb{Z}) \) with the same characteristic polynomial \( f(x) \). Then, \( A \) and \( B \) are conjugate matrices in \( \text{GL}_2(\mathbb{Z}) \) if and only if the corresponding ideals \( I_A \) and \( I_B \) are in the same ideal class of \( \mathbb{Z}[\lambda] \), where \( \lambda \) is a root of \( f(x) \).

Let \( \lambda \) be an eigenvalue of the matrix \( A \in \text{GL}_2(\mathbb{Z}) \). For any fractional \( \mathbb{Z}[\lambda] \)-ideal \( I \), the multiplication by \( \lambda \) is a \( \mathbb{Z} \)-linear map, i.e., \( m_{I,\lambda} : I \to I \) defined by \( x \mapsto x\lambda \). Since \( I \) is a \( \mathbb{Z} \)-module, choosing a \( \mathbb{Z} \)-basis we can represent \( m_{I,\lambda} \) by a \( 2 \times 2 \) matrix \( [m_{I,\lambda}] \) over \( \mathbb{Z} \). Note that, if \( \lambda \) is a unit of \( \mathbb{Z}[\lambda] \), then \( m_{I,\lambda} \) is a bijection (because the mapping \( m_{I,\lambda^{-1}} : I \to I \) given by \( x \mapsto x\lambda^{-1} \) is an inverse to \( m_{I,\lambda} \)). Thus, if \( \lambda \) is a unit of \( \mathbb{Z}[\lambda] \), then \([m_{I,\lambda}] \in \text{GL}_2(\mathbb{Z})\).

Lemma 4.10. Let \( A \in \text{GL}_2(\mathbb{Z}) \) with distinct eigenvalues and \( \text{tr}(A) \neq 0 \). Let \( I \) and \( J \) be ideals in distinct classes of \( H(\mathbb{Z}[\lambda]) \), where \( \lambda \) is an eigenvalue of \( A \). Then, the matrices \([m_{I,\lambda}]\) and \([m_{J,\lambda}]\) are not conjugate in \( \text{GL}_2(\mathbb{Z}) \).
Proof. Initially, since $\lambda$ is a root of the polynomial $p(x) = x^2 - \text{tr}(A)x + \det(A)$, we have that $\lambda$ is an algebraic integer of the quadratic field $K = \mathbb{Q}(\lambda)$. As $\det(A) = \pm 1$, implies that the norm of $\lambda$ is equal to $\pm 1$ and hence $\lambda$ is a unit of the order $\mathbb{Z}[\lambda]$.

Let $B_1 = \{v_1, w_1\}$ and $B_2 = \{v_2, w_2\}$ be fixed $\mathbb{Z}$-bases for the $\mathbb{Z}$-modules $I$ and $J$, respectively. Consider the matrix representations $[m_{I,\lambda}]$ and $[m_{J,\lambda}]$ of the $\mathbb{Z}$-linear automorphisms $m_{I,\lambda} : I \to I$ and $m_{J,\lambda} : J \to J$ defined by the multiplications by $\lambda$ with respect to the bases $B_1$ and $B_2$, respectively. Let us denote by $\{e_1, e_2\}$ the canonical basis of $\mathbb{Z}^2$. Now, consider the $\mathbb{Z}$-linear isomorphisms $f : I \to \mathbb{Z}^2$ and $g : J \to \mathbb{Z}^2$ such that $f(v_1) = e_1 = g(v_2)$ and $f(w_1) = e_2 = g(w_2)$. It follows by construction that the diagrams

\[
\begin{array}{ccc}
I & \xrightarrow{f} & \mathbb{Z}^2 \\
\downarrow[m_{I,\lambda}] & & \downarrow[m_{I,\lambda}] \\
I & \xrightarrow{f} & \mathbb{Z}^2
\end{array}
\quad \quad \quad
\begin{array}{ccc}
J & \xrightarrow{g} & \mathbb{Z}^2 \\
\downarrow[m_{J,\lambda}] & & \downarrow[m_{J,\lambda}] \\
J & \xrightarrow{g} & \mathbb{Z}^2
\end{array}
\]

are commutative.

Suppose that the matrices $[m_{I,\lambda}]$ and $[m_{J,\lambda}]$ are conjugate in $\text{GL}_2(\mathbb{Z})$, i.e., there is $P \in \text{GL}_2(\mathbb{Z})$ such that $P^{-1}[m_{I,\lambda}]P = [m_{J,\lambda}]$. Thus, we obtain the following diagram of $\mathbb{Z}$-linear isomorphisms

\[
\begin{array}{ccc}
I & \xrightarrow{f} & \mathbb{Z}^2 \\
\downarrow[m_{I,\lambda}] & & \downarrow[m_{I,\lambda}] \\
I & \xrightarrow{f} & \mathbb{Z}^2
\end{array}
\quad \quad \quad
\begin{array}{ccc}
J & \xrightarrow{g} & \mathbb{Z}^2 \\
\downarrow[m_{J,\lambda}] & & \downarrow[m_{J,\lambda}] \\
J & \xrightarrow{g} & \mathbb{Z}^2
\end{array}
\quad \quad \quad
\begin{array}{ccc}
2 & \xrightarrow{P} & 2 \\
\downarrow & & \downarrow \\
2 & \xrightarrow{P} & 2
\end{array}
\]

As each square in the diagram is commutative, we have that the whole diagram is commutative. We claim that the composite map $g^{-1}Pf : I \to J$ is an $\mathbb{Z}[\lambda]$-module isomorphism. Indeed, note that for every $\alpha \in I$ we have

$$g^{-1}Pf(\alpha\lambda) = g^{-1}Pf(m_{I,\lambda}(\alpha)) = m_{J,\lambda}g^{-1}Pf(\alpha) = g^{-1}Pf(\alpha)\lambda.$$ 

Since the composite map $g^{-1}Pf : I \to J$ is clearly a $\mathbb{Z}$-linear isomorphism, it follows that $g^{-1}Pf : I \to J$ is an $\mathbb{Z}[\lambda]$-module isomorphism. Now consider an extension of $g^{-1}Pf : I \to J$ to a $K$-isomorphism $\psi : KI \to KJ$. Thus, we have

$$\psi(m\xi) = \psi(m)\xi, \quad m \in KI, \quad \xi \in K.$$ 

In particular, if $m = 1$ and restricting $\xi$ to $I$, we have

$$g^{-1}Pf(\xi) = \psi(\xi) = \psi(1)\xi, \quad \xi \in I.$$ 

Therefore, $g^{-1}Pf : I \to J$ is the multiplication by $\psi(1)$ and so $I = \psi(1)J$. Then $I$ and $J$ are in the same class of ideals, a contradiction. Therefore, the matrices $[m_{I,\lambda}]$ and $[m_{J,\lambda}]$ are not conjugate in $\text{GL}_2(\mathbb{Z})$.\qed

Let $A \in \text{GL}_2(\mathbb{Z})$ be a fixed matrix. We define

$$\approx_A - \text{class} := \{B \in \text{GL}_2(\mathbb{Z}) : B \text{ is conjugate to } A \text{ in } \text{GL}_2(\mathbb{Z})\}.$$ 

Lemma 4.11. Let $A$ be a matrix in $\text{GL}_2(\mathbb{Z})$ such that its eigenvalues are distinct and $\text{tr}(A) \neq 0$. Let $\lambda$ be an eigenvalue of $A$. If the conjugacy class of $A$ corresponds to a class $[I_A]$ of an invertible ideal $I_A$ of $\mathbb{Z}[\lambda]$, where $I_A$ is as in (5), then there are at least $h(\mathbb{Z}[\lambda])$ matrices in $\approx_A - \text{class}$ that are not conjugate to each other.
Proof. Let $I_1, \ldots, I_k$ be ideals in distinct classes of the group $H(\mathbb{Z}[\lambda])$. By Lemma 3.16, we can assume that each $I_i$ is prime to the conductor $f = \mathcal{O}_K : \mathbb{Z}[\lambda]$. Let $B_i$ be a fixed $\mathbb{Z}$-basis for each $I_i$ and consider $A_i \in \text{GL}_2(\mathbb{Z})$ the matrix of multiplication by $\lambda$ with respect to the basis $B_i$, $i = 1, \ldots, k$. By Lemma 4.10, $A_i$ is not conjugate to $A_j$ in $\text{GL}_2(\mathbb{Z})$, whenever $i \neq j$.

On the other hand, by Lemma 3.19

$$\frac{I_i}{p^l I_i} \cong \frac{\mathbb{Z}[\lambda]}{p^l \mathbb{Z}[\lambda]}$$

as $\mathbb{Z}[\lambda]/p^l \mathbb{Z}[\lambda]$-modules, for every prime $p$ and $l \in \mathbb{N}$. Now consider the canonical images $\overline{A}_i, \overline{A}_1$ in $\text{GL}_2(\mathbb{Z}/p^l \mathbb{Z})$ of the matrices $A_i, A_1$. From (6), we see that $\overline{A}_i$ is conjugate to $A_1$ in $\text{GL}_2(\mathbb{Z}/p^l \mathbb{Z})$ for every $l \in \mathbb{N}$ and $i = 1, \ldots, k$. Hence, the matrices $A_i$ and $A_1$ are conjugate in $\text{GL}_2(\mathbb{Z}_p)$ for every prime $p$ and $i = 1, \ldots, k$. Consequently, $A_i$ is conjugate to $A_1$ in

$$\text{GL}_2(\bar{\mathbb{Z}}) = \prod_p \text{GL}_2(\mathbb{Z}_p),$$

for each $i = 1, \ldots, k$.

Since the conjugacy class of the matrix $A$ corresponds to a class $[I_A]$ of an invertible ideal $I_A$ of $\mathbb{Z}[\lambda]$, it follows by Proposition 4.7 that there is $j \in \{1, \ldots, k\}$ such that $A$ is conjugate to $A_j$ in $\text{GL}_2(\mathbb{Z})$. Therefore, there are at least $h(\mathbb{Z}[\lambda])$ matrices in $\approx_A$-class that are not conjugate to each other. \qed

The next result will be useful to us, later on.

Lemma 4.12 ([3], Lem 3.2). Let $A$ and $B$ be matrices in $\text{GL}_2(\mathbb{Z})$. If $\det(A) = \det(B)$ and $\text{tr}(A) = \text{tr}(B) \neq 0$, then $A$ is conjugate to $B$ in $\text{GL}_2(\mathbb{Z})$ if and only if $A^2$ is conjugate to $B^2$ in $\text{GL}_2(\mathbb{Z})$.

5 Proof of main results

Proof of Theorem 1.1. If $A = I$, then the group $G_A \cong \mathbb{Z}^3$ and the result follows from [21, Prop. 3.1]. Now suppose $A \neq I$ and that its eigenvalues are all equal to 1. Let $B \in \text{GL}_2(\mathbb{Z})$ such that $\overline{G}_A \cong \overline{G}_B$. By Proposition 4.4 we can assume that $B$ also has all its eigenvalues equal to 1. By Lemma 2.2, $G_A$ and $G_B$ are nilpotent groups with nilpotency class 2. Also, note that $G_A$ and $G_B$ has Hirsch number 3. Since $G_A$ and $G_B$ are finitely generated and torsion-free, it follows by Proposition 2.9 that $G_A \cong G_B$, and $\#(\mathcal{P}, G_A) = 1$ as claimed. \qed

Proof of Theorem 1.2. Let $A$ and $B$ be matrices in $\text{GL}_2(\mathbb{Z})$ and let $\lambda$ be an eigenvalue of $A$. Consider the semi-direct products $G_A = N_1 \rtimes_A \mathbb{Z}$ and $G_B = N_2 \rtimes_B \mathbb{Z}$, where $N_1 \cong \mathbb{Z} \times \mathbb{Z} \cong N_2$ and suppose that

$$\overline{G}_B \cong \overline{G}_A.$$

(i) Let us assume that all eigenvalues of $A$ are distinct, $\text{tr}(A) \neq 0$ and that the conjugacy class of $A$ in $\text{GL}_2(\mathbb{Z})$ corresponds to a class $[I_A]$ of invertible ideals of $\mathbb{Z}[\lambda]$. Then, none of the eigenvalues of $A$ are equal to 1, and since $\overline{G}_A \cong \overline{G}_B$, we can assume that $A$ and $B$ have the same eigenvalues by Proposition 4.4. By
Lemma 4.11 there are at least \( h(Z[\lambda]) \) matrices in \( \approx_A \)-class that are not conjugate to each other in \( GL_2(Z) \) but are conjugate in \( GL_2(\hat{Z}) \). We now consider two cases to establish the lower bound.

**Case (a) If \( \det(A) = -1 \).**

Note that, if \( A, B \in \approx_A \)-class we have \( \det(A) = \det(B) = -1 \) and \( \text{tr}(A) = \text{tr}(B) \) by Lemma 4.3. Thus, \( \text{tr}(B^{-1}) = -\text{tr}(B) \) by Lemma 4.5 and hence \( B^{-1} \) is not conjugate to any matrix of \( \approx_A \)-class. Therefore,

\[
h(Z[\lambda]) \leq \#g(\mathcal{P}, G_A)
\]

by Lemmas 4.1 and 4.2.

**Case (b) If \( \det(A) = 1 \).**

If there are one or two conjugacy classes in \( \approx_A \)-class, then the result immediately follows from Lemma 4.1. Now suppose that \( \approx_A \)-class has more than two matrices that are not conjugate to each other and that \( A, B, C \in GL_2(Z) \) are three of these matrices. Note that, if \( A^{-1} \) is conjugate to \( B \) in \( GL_2(Z) \) and \( C^{-1} \) is also conjugate to \( B \) in \( GL_2(Z) \), then \( A^{-1} \) is conjugate to \( C^{-1} \) in \( GL_2(Z) \), which implies that \( A \) is conjugate to \( C \) in \( GL_2(Z) \), a contradiction. Therefore, \( A^{-1} \) and \( C^{-1} \) cannot be conjugate to more than one matrix of the subset of \( \approx_A \)-class formed by the matrices that are not conjugate to each other. Thus, by Lemmas 4.1 and 4.2 we have

\[
h(Z[\lambda])/2 \leq \#g(\mathcal{P}, G_A).
\]

On the other hand, by Proposition 4.4 we can assume that \( A \) and \( B \) have the same characteristic polynomial \( p_A(x) \). Now, Theorem 4.7 tells us that there is a one-to-one correspondence between the conjugacy classes of matrices in \( GL_2(Z) \) that have characteristic polynomial \( p_A(x) \) and the ideal classes of the order \( Z[\lambda] \). Hence, if \( \tilde{h}(\lambda) \) denotes the number of ideal classes of the order \( Z[\lambda] \), it follows by Lemma 4.1 that

\[
\#g(\mathcal{P}, G_A) \leq \tilde{h}(\lambda).
\]

Therefore,

\[
\begin{cases}
  h(Z[\lambda]) \leq \#g(\mathcal{P}, G_A) \leq \tilde{h}(\lambda), & \text{if } \det(A) = -1 \\
  h(Z[\lambda])/2 \leq \#g(\mathcal{P}, G_A) \leq \tilde{h}(\lambda), & \text{if } \det(A) = 1.
\end{cases}
\]

(ii) We divide the proof into two cases.

**Case 1. If \( A \) has all its eigenvalues equal to \(-1\).**

It is easy to see that the groups \( G_{B^2} = N_2 \rtimes_{B^2} Z \) and \( G_{A^2} = N_1 \rtimes_{A^2} Z \) are subgroups of \( G_B \) and \( G_A \) (corresponding to the replacement of the second factor \( Z \) by \( 2Z \)). Thus, the following sequences

\[
1 \rightarrow G_{B^2} \rightarrow G_B \rightarrow C_B \rightarrow 1 \quad \text{and} \quad 1 \rightarrow G_{A^2} \rightarrow G_A \rightarrow C_A \rightarrow 1
\]
are exact, where \( C_B \cong \mathbb{Z}/2\mathbb{Z} \cong C_A \). Note that the epimorphism \( p_i : G_i \to C_i \) induces an epimorphism of profinite groups \( \hat{p}_i : \hat{G}_i \to C_i \) where \( \ker(\hat{p}_i) = \hat{G}_{i2} \), for \( i \in \{A, B\} \). Hence the sequences

\[
1 \to \hat{G}_{B2} \to \hat{G}_B \xrightarrow{\hat{p}_B} C_B \to 1 \quad \text{and} \quad 1 \to \hat{G}_{A2} \to \hat{G}_A \xrightarrow{\hat{p}_A} C_A \to 1
\]

are exact.

Since none of the eigenvalues of the matrices \( A \) and \( B \) are equal to 1 (Proposition 4.4), we have by Lemma 2.4 that the restriction of any isomorphism \( f : \hat{G}_B \to \hat{G}_A \) to \( \hat{N}_2 \) is an isomorphism onto \( \hat{N}_1 \). Hence \( f \) induces an isomorphism \( \sigma : C_B \to C_A \). Thus, we get the following commutative diagram where the restriction of map \( f \) for \( \hat{G}_{B2} \) is an isomorphism from \( \hat{G}_{B2} \) to \( \hat{G}_{A2} \)

\[
\begin{array}{cccccc}
1 & \longrightarrow & \hat{G}_{B2} & \longrightarrow & \hat{G}_B & \xrightarrow{\hat{p}_B} & C_B & \longrightarrow & 1 \\
| & | & | & | & | & | & | & | & | \\
1 & \longrightarrow & \hat{G}_{A2} & \longrightarrow & \hat{G}_A & \xrightarrow{\hat{p}_A} & C_A & \longrightarrow & 1
\end{array}
\]

Note that \( A^2 \) has all its eigenvalues equal to 1, hence there is an isomorphism \( \phi : \hat{G}_{B2} \to \hat{G}_{A2} \) by Theorem 1.1. Since \( N_2 \times_B 2\mathbb{Z} = G_{B2} \) and \( N_1 \times_A 2\mathbb{Z} = G_{A2} \) and none of the eigenvalues of matrices \( B \) and \( A \) are equal to 1, it follows from Lemma 2.4 that \( \phi(N_2) = N_1 \). We conclude from Lemma 2.6 that the matrices \( B^2 \) and \( A^2 \) are conjugate in \( \text{GL}_2(\mathbb{Z}) \). Again, we can assume that \( A \) and \( B \) are conjugate in \( \text{GL}_2(\mathbb{Z}) \) (see Lemmas 2.3 and 4.2). We thus get \( \text{tr}(B) = \text{tr}(A) \) and \( \text{det}(B) = \text{det}(A) \) (Lemma 4.3). Lemma 4.12 now shows that \( B \) and \( A \) are conjugate in \( \text{GL}_2(\mathbb{Z}) \). Hence \( G_B \cong G_A \) (Lemma 4.1). Therefore, \( \#g(\mathcal{P} \mathcal{F}, G_A) = h(\mathbb{Z}) = 1 \), since \( \mathbb{Z} \) is a principal ideal domain.

**Case 2.** If \( \text{tr}(A) = 0 \).

First, we assume that \( \text{det}(A) = 1 \). In this case, the eigenvalues of \( A \) are distinct and different from 1. From the proof of Proposition 4.4, we see that none of the eigenvalues of \( B \) is 1. As \( \hat{G}_A \cong \hat{G}_B \), it follows by Lemmas 2.3 and 4.2 that we can assume that \( A \) and \( B \) are conjugate in \( \text{GL}_2(\mathbb{Z}) \). This gives \( \text{tr}(B) = \text{tr}(A) = 0 \) and \( \text{det}(B) = \text{det}(A) = 1 \) (Lemma 4.3). Now, by [13, Thm. 3] we see that every matrix with trace equal to zero and determinant equal to 1 is conjugate in \( \text{GL}_2(\mathbb{Z}) \) to

\[
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}.
\]

Hence, the matrices \( A \) and \( B \) are conjugate in \( \text{GL}_2(\mathbb{Z}) \) and so \( G_A \cong G_B \) by the Lemma 4.1. Therefore, \( \#g(\mathcal{P} \mathcal{F}, G_A) = 1 \). On the other hand, since the characteristic polynomial of \( A \) is \( p(x) = x^2 + 1 \), we have that \( \sqrt{-1} \) and \( -\sqrt{-1} \) are the eigenvalues of \( A \). From [2, Thm. 5.4.2] we see that \( \mathbb{Z}[\sqrt{-1}] \) is the ring of integers of the field \( \mathbb{Q}(\sqrt{-1}) \). Thus, \( \#g(\mathcal{P} \mathcal{F}, G_A) = h(\mathbb{Z}[\sqrt{-1}]) = 1 \) by [2, p. 325].
Finally, suppose that $\det(A) = -1$. By [13, Thm. 3] we have that $A$ is conjugate in $\text{GL}_2(\mathbb{Z})$ to

$$ P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. $$

Since $P^2 = I$, it follows that $G_A$ is a 3-dimensional Bieberbach group with holonomy group $\mathbb{Z}/2\mathbb{Z}$ (see [23, Thm. 3.2]). Therefore, by [18, Cor. 1.2] we have $\# g(\mathcal{P}F, G_A) = 1$. In this case, the characteristic polynomial of $A$ is $p(x) = x^2 - 1$, so 1 and $-1$ are the eigenvalues of $A$. Thus, as in Case 1, we see that $\# g(\mathcal{P}F, G_A) = h(\mathbb{Z}) = 1$.

**Proof of Corollary 1.3.** By Lemma 3.4, we have $h(\mathbb{Z}[\lambda]) = \tilde{h}(\lambda)$. Therefore, $\# g(\mathcal{P}F, G_A) = h(\mathbb{Z}[\lambda])$ by Theorem 1.2.

**Proof of Corollary 1.4.** This is an immediate consequence of the Theorems 1.1 and 1.2. □

### 6 Examples

In this section we give some examples of applications of the Theorems 1.1 and 1.2.

**Example 6.1.** Consider

$$ A = \begin{pmatrix} 2 & 1 \\ 5 & 2 \end{pmatrix} \in \text{GL}_2(\mathbb{Z}) $$

with characteristic polynomial $f_A(x) = x^2 - 4x - 1$. So the eigenvalues of $A$ are $\lambda_{\pm} = 2 \pm \sqrt{5}$. Since $5 \equiv 1 \pmod{4}$, it follows by [2, Thm. 5.4.2] that the order $\mathbb{Z}[\lambda_{\pm}] = \mathbb{Z}[\sqrt{5}]$ is not maximal, i.e., $\mathbb{Z}[\sqrt{5}]$ is not the ring of integers of the quadratic field $\mathbb{Q}(\sqrt{5})$. Consider $I = \langle 2, 1 + \sqrt{5} \rangle$. By Proposition 3.7 (1), we see that $I$ is an ideal of $\mathbb{Z}[\sqrt{5}]$. By [16, Ex. 1.32], $I$ is not invertible in $\mathbb{Z}[\sqrt{5}]$. This implies that the order $h(\mathbb{Z}[\sqrt{5}])$ of the ideal class group of $\mathbb{Z}[\sqrt{5}]$ is strictly less than the number of ideal classes $\tilde{h}(\lambda_{\pm})$ of $\mathbb{Z}[\sqrt{5}]$. As the corresponding ideal to the matrix $A$ is $I_A = \langle 1, \lambda_{\pm} - 2 \rangle = \mathbb{Z}[\sqrt{5}]$ (see (5)), which is invertible, and $\det(A) = -1$, it follows by Theorem 1.2 that holds the following inequality

$$ h(\mathbb{Z}[\sqrt{5}]) \leq \# g(\mathcal{P}F, \mathbb{Z}^2 \rtimes_A \mathbb{Z}) \leq \tilde{h}(\lambda_{\pm}) \quad \text{with} \quad h(\mathbb{Z}[\sqrt{5}]) < \tilde{h}(\lambda_{\pm}). $$

The next example shows that torus bundles modeled on the geometry $\text{Sol}$ are not determined by the profinite completions of their fundamental groups.

**Example 6.2.** Consider

$$ B = \begin{pmatrix} 4 & 7 \\ 3 & 5 \end{pmatrix} \in \text{GL}_2(\mathbb{Z}). $$

Note that the characteristic polynomial of $B$ is $f_B(x) = x^2 - 9x - 1$ and that

$$ \lambda_{\pm} = \frac{9 \pm \sqrt{85}}{2} $$
are the eigenvalues. Since $85$ is square-free and $\det(B) = -1$, it follows from Corollary 1.3 that
\[ \#g(\mathcal{P}F, \mathbb{Z}^2 \rtimes_B \mathbb{Z}) = h(\mathbb{Z}[\lambda_+]). \]
Now, by [2, Table 8] we see that $h(\mathbb{Z}[\lambda_+]) = 2$. As $|\text{tr}(B)| > 2$, it follows that the torus bundle $M_B$, such that $\pi_1(M_B) \cong \mathbb{Z}^2 \rtimes_B \mathbb{Z}$, is modeled on the geometry $\text{Sol}$ (Cf. [22, Thm. 5.5]).

In contrast, Corollary 1.4 tells us that there is a family of torus bundles, modeled on the geometry $\text{Sol}$, that are determined by the profinite completions of their fundamental groups.

**Example 6.3.** Consider the matrices
\[ C = \begin{pmatrix} 2 & 7 \\ 1 & 3 \end{pmatrix}, \quad D = \begin{pmatrix} 5 & 1 \\ 11 & 2 \end{pmatrix} \in \text{GL}_2(\mathbb{Z}). \]
Since the trace of these matrices are absolutely greater than 2, we have that the torus bundles $M_C$ and $M_D$, such that
\[ \pi_1(M_C) \cong \mathbb{Z}^2 \rtimes_C \mathbb{Z} \text{ and } \pi_1(M_D) \cong \mathbb{Z}^2 \rtimes_D \mathbb{Z}, \]
are all modeled on the geometry $\text{Sol}$ (Cf. [22, Thm. 5.5]). Now, note that the characteristic polynomials of $C$ and $D$ are
\[ f_C(x) = x^2 - 5x - 1, \quad f_D(x) = x^2 - 7x - 1, \]
respectively. Hence, the eigenvalues of $C$ and $D$ are
\[ \lambda_{\pm,C} = \frac{5 \pm \sqrt{29}}{2}, \quad \lambda_{\pm,D} = \frac{7 \pm \sqrt{53}}{2}, \]
respectively. Since 29 and 53 are square-free numbers and $\det(C) = \det(D) = -1$, it follows from Corollary 1.3 with [2, Table 8] that
\[ \#g(\mathcal{P}F, \mathbb{Z}^2 \rtimes_C \mathbb{Z}) = 1 = \#g(\mathcal{P}F, \mathbb{Z}^2 \rtimes_D \mathbb{Z}). \]
Therefore, the torus bundles $M_C$ and $M_D$ are determined among 3-manifolds by the profinite completion of their fundamental groups.

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