Abstract
In general relativity, an IDEAL (Intrinsic, Deductive, Explicit, ALgorithmic) characterization of a reference spacetime metric $g_0$ consists of a set of tensorial equations $T[g] = 0$, constructed covariantly out of the metric $g$, its Riemann curvature and their derivatives, that are satisfied if and only if $g$ is locally isometric to the reference spacetime metric $g_0$. The same notion can be extended to also include scalar or tensor fields, where the equations $T[g, \phi] = 0$ are allowed to also depend on the extra fields $\phi$. We give the first IDEAL characterization of cosmological FLRW spacetimes, with and without a dynamical scalar (inflaton) field. We restrict our attention to what we call regular geometries, which uniformly satisfy certain identities or inequalities. They roughly split into the following natural special cases: constant curvature spacetime, Einstein static universe, and flat or curved spatial slices. We also briefly comment on how the solution of this problem has implications, in general relativity and inflation theory, for the construction of local gauge invariant observables for linear cosmological perturbations and for stability analysis.

Keywords: differential invariants, cosmology, equivalence up to isometry
1. Introduction

In this work, we are interested in an intrinsic characterization of homogeneous and isotropic cosmological spacetimes (also known as Friedmann–Lemaître–Robertson–Walker or FLRW spacetimes), either with or without the presence of a scalar field (aka inflationary spacetimes). By a spacetime \((M, g)\), we mean a smooth manifold \(M\) with a Lorentzian metric \(g\). While ‘intrinsic’ generally does preclude direct reference to the form of the spacetime metric in a special coordinate system, it is a vague enough term to have multiple interpretations. To be specific, we refer to an IDEAL\(^5\) or Rainich-type characterization that has been used, for instance, in the works \([6, 12, 14, 15, 20, 27, 33]\). It consists of a list of tensorial equations \(T_a[g] = 0, a = 1, 2, \ldots, N\), constructed covariantly out of the metric \((g)\) and its derivatives (concomitants of the Riemann tensor) that are satisfied if and only if the given spacetime locally belongs to the desired class, possibly narrow enough to be the isometry class of a reference spacetime geometry. This notion has a natural generalization \(T_a[g, \phi] = 0\) to spacetimes equipped with scalar or tensor fields \((\phi)\), with equivalence still given by isometric diffeomorphisms that also transform the additional scalars or tensors into each other. A nice historical survey of this and other local characterization results can be found in \([22]\).

An IDEAL characterization neither requires the existence of any extra geometric structures, nor the translation of the metric and of the curvature into a frame formalism. Thus, it is an alternative to the Cartan–Karlhede characterization \([31, \text{chapter 9}]\), which is based on Cartan’s moving frame formalism. Intrinsic characterizations, of various types, have been of long standing and independent interest in geometry and general relativity. But, in addition, they can be helpful in deciding when a metric, given for instance by some complicated coordinate formulas, corresponds to one that is already known. In this regard, an IDEAL characterization is especially helpful if one would like to find an algorithmic solution to this recognition problem. In numerical relativity, the near-satisfaction of the tensor equations \(T_a[g] \approx 0\) may signal the local proximity of a numerical spacetime to a desired reference geometry. In addition, the approach to zero of \(T_a[g] \to 0\) could be used to study either linear or nonlinear stability of reference geometries, in an unambiguous and gauge independent way.

The following particular application should be noted. By the Stewart–Walker lemma \([32, \text{lemma 2.2}]\), the vanishing of a tensor concomitant \(T_a[g] = 0\) for a metric \(g\) implies that its linearization \(\mathcal{T}_a[h]\) \(\left(T_a[g + \epsilon h] = T_a[g] + \epsilon \mathcal{T}_a[g] + O(\epsilon^2)\right)\) is invariant under linearized diffeomorphisms. Thus, any quantity of the form \(\mathcal{T}_a[h]\) defines a gauge invariant observables in linearized gravity, when Einstein or Einstein-matter equations are linearly perturbed about a background solution \(g\). A straightforward argument shows that an IDEAL characterization provides a list \(\mathcal{T}_a[h], a = 1, \ldots, N\), of gauge invariant observables that is also complete (it suffices to check that \(T_a[g + h] = 0\) do not approach zero at \(O(h^2)\) or higher order). That is, the joint kernel of \(\mathcal{T}_a[h] = 0\) locally consists only of pure gauge modes \((h = \mathcal{L}_v g\) for some vector field \(v)\). The use of such local observables (given by differential operators) can be advantageous both in theoretical and practical investigations of classical and quantum field theoretical models because they cleanly separate the local (or ultraviolet) and global (or infrared) aspects of the theory. This is of particular and current relevance to some controversies in inflationary models of early universe cosmology \([23, 34]\). Despite their importance, complete lists of (linearized) local gauge invariant observables have been explicitly produced only in very few cases, by ad hoc methods. For instance, in the case of Einstein equations coupled to a single inflaton field, a complete list has been produced only recently \([17]\). On the other hand,

\(^5\)The acronym, explained in \([14]\) (footnote, p.2), stands for intrinsic, deductive, explicit and algorithmic.
linearising the equations of an IDEAL characterization provides a systematic method of construction. The results of this method can be compared to those of [17] and are equivalent [18]. Since these two sets of results naturally appear in rather different forms, a detailed comparison is beyond the scope of this work and will be presented elsewhere.

A similar geometric approach to the construction of gauge invariant linearized observables was taken in [11], using what we would call a partial IDEAL characterization of cosmological spacetimes. No proof of their completeness was ever given. In a sense, we complete the earlier literature in this regard.

In this work, we add the cases of FLRW and inflationary spacetimes to the (unfortunately still small) literature concerning IDEAL characterizations of isometry classes of individual reference geometries. Other IDEAL characterizations for geometries of interest in general relativity include Schwarzschild [12], Reissner–Nordström [13], Kerr [15], Lemaître–Tolman–Bondi [14], Stephani universes [16] (see references for complete lists and details) and of course the classic cases of constant curvature spaces, which are known to be fully characterized by the structure of the Riemann tensor (by theorems of Riemann and Killing–Hopf).

The synopsis of the paper is the following: in section 1.1 we fix our notation and we outline our main results on the IDEAL characterization of FLRW spacetimes (theorem 1.4) and inflationary geometries and prove our main theorems. Therefore we will not dwell on the mathematical proofs, but we will focus on the basic technique. In this subsection, our goal is to introduce our conventions and to outline our main results.

In section 1.1, we provide flowcharts for classifying spacetimes into FLRW and inflationary isometry classes, visually summarizing the contents of theorems 1.4 and 1.5. In section 2 we collect relevant information on the geometry of FLRW and inflationary spacetimes. No proof of their completeness was ever given. In a sense, we complete the earlier literature in this regard.

1.1. Main results

In this subsection, our goal is to introduce our conventions and to outline our main results. Therefore we will not dwell on the mathematical proofs, but we will focus on the basic technical tools, necessary to formulate and to understand the physical significance of our findings.

In this work, a spacetime or Lorentzian manifold \((M, g)\) will be a smooth finite dimensional manifold \(M\) (also Hausdorff, second countable, connected and orientable) of \(\dim M = n + 1 \geq 2\), with a Lorentzian metric \(g\) (with signature \(-+\cdots\)). A spacetime with scalar will consist of a triple \((M, g, \phi)\), where \((M, g)\) is a Lorentzian manifold and \(\phi: M \to \mathbb{R}\) is a smooth scalar field. Obviously, we could always consider the spacetime \((M, g)\) as the special spacetime with zero scalar, \((M, g, 0)\). In addition, with inflationary spacetimes, we will be assuming that the metric and the scalar field satisfy the coupled Einstein–Klein–Gordon equations, possibly with a nonlinear potential.

These observations should be kept in mind while reading the following

**Definition 1.1 (Locally isometric).** A spacetime with scalar \((M_1, g_1, \phi_1)\) is locally isometric at \(x_1 \in M_1\) to a spacetime with scalar \((M_2, g_2, \phi_2)\) at \(x_2 \in M_2\) if there exist open neighbourhoods \(U_1 \ni x_1, U_2 \ni x_2\) and a diffeomorphism \(\chi: U_1 \to U_2\) such that \(\chi(x_1) = x_2, \chi^* g_2 = g_1\) and \(\chi^* \phi_2 = \phi_1\). If we can choose \(U_1 = M_1\) and \(U_2 = M_2\) then they are (globally) isometric. If for every \(x_1 \in M_1\) there is \(x_2 \in M_2\) such that \((M_1, g_1, \phi_1)\) at \(x_1\) is locally isometric to \((M_2, g_2, \phi_2)\) at \(x_2\), we simply say that \((M_1, g_1, \phi_1)\) is locally isometric to \((M_2, g_2, \phi_2)\) (note the asymmetry in the definition). If \((M_1, g_1, \phi_1)\) is locally isometric to \((M_2, g_2, \phi_2)\), as well as vice versa, we say that they are locally isometric to each other (which constitutes an equiva-
lence relation). All spacetimes with scalar that are locally isometric to a reference \((M, g, \phi)\) constitute its local isometry class.

Our main results give an IDEAL characterization of local isometry classes of regular FLRW and inflationary spacetimes. In the following we give their precise definition, which is motivated in more detail in sections 2.3 and 2.4. Starting from the first case:

**Definition 1.2 (Regular FLRW spacetime).** Let us fix a constant \(\kappa \neq 0\). Denote by the triple \((m, \alpha, f)\), of a dimension \(m \geq 1\), a constant \(\alpha \in \mathbb{R}\) and a smooth positive function \(f : I \to \mathbb{R}\) defined on an interval \(I \subseteq \mathbb{R}\), the corresponding FLRW spacetime \((M, g, f) = (I \times F, -dt^2 + f^2 g^0)\) (definition 2.2), with \(\alpha\) the sectional curvature of \((F, g^0)\) and \(F \cong \mathbb{S}^m\) (when \(\alpha > 0\)) or \(F \cong \mathbb{R}^m\) (when \(\alpha \leq 0\)).

We call \((M, g)\) a regular FLRW spacetime if it belongs to one of the parametrized families identified below.

(a) Constant curvature spacetime, with spacetime sectional curvature \(K\):

\[
\text{CC}^m_K = \{ (m, K, \cosh(\sqrt{K}t)) \mid K > 0, I = \mathbb{R} \},
\]

\[
\{ (m, 0, 1) \mid K = 0, I = \mathbb{R} \},
\]

\[
\{ (m, K, \cosh(\sqrt{-K}t)) \mid K < 0, I = \mathbb{R} \}. \tag{1}
\]

(b) Einstein static universe, with spatial sectional curvature \(K \neq 0\):

\[
\text{ESU}^m_K = \{ (m, K, 1) \mid m > 1, I = \mathbb{R} \}. \tag{2}
\]

(c) Spatially flat constant scalar curvature spacetime, with spacetime scalar curvature \(m(m + 1)K\) and such that \(f^2(J) = J\):

\[
\text{CSC}^m_{K,J} = \{ (m, \alpha, f) \mid m > 1, f' \neq 0, \frac{f'' - f' \alpha}{f^2} + \frac{m + 1}{2} \left( \frac{f'^2}{f^2} - K \right) = 0 \}. \tag{3}
\]

(d) Generic constant scalar curvature spacetime, with spacetime scalar curvature \(m(m + 1)K\), normalized radiation density constant \(\Omega\) and such that \(\frac{f^2}{f(J)} = J\):

\[
\text{CSC}^m_{K,\Omega,J} = \{ (m, \alpha, f) \mid m > 1, \alpha \neq 0, f' \neq 0, \frac{f'^2}{f^2} + \frac{\alpha}{f^2} = K + \Omega \frac{|\alpha|^{(m+1)/2}}{f^{m+1}} \}. \tag{4}
\]

(e) Spatially flat FLRW spacetime with normalized pressure function \(P\) defined on an open interval \(J\), with \(0 < \frac{f'}{f(J)} = J\) and

\[
P(u) \left[ \frac{1}{2} \frac{\partial_u P(u)}{\partial u} - \frac{1}{2\kappa} \right] \neq 0 \tag{5}
\]

everywhere on \(J\):

\[
\text{FLRW}^m_{p,J} = \{ (m, 0, f) \mid \left( \frac{f''}{f} - \frac{f'^2}{f^2} \right) + \frac{m f'^2}{2 f^2} = -\kappa P \left( \frac{f'}{f} \right)^2 \}. \tag{6}
\]

(f) Generic FLRW spacetime with normalized energy function \(E\) defined on an open interval \(J\), with \(0 \notin \frac{f'}{f(J)} = J\) and
\[
\frac{\partial u}{\partial v} \left[ u \frac{\partial}{\partial v} E(u) - \frac{(m+1)}{2} \right] \neq 0
\]  

everywhere on \( J \):

\[
\text{FLRW}^m_{E,J} = \left\{ (m, \alpha, f) \mid m > 1, \alpha \neq 0, \frac{f'^2}{f^2} + \frac{\alpha}{f^2} = \kappa E(\alpha/f^2) \right\}.
\]

Next, we focus our attention to the inflationary spacetimes, following the more detailed motivation from sections 2.5 and 2.6:

**Definition 1.3 (Regular inflationary spacetime).** Let us fix a constant \( \kappa \neq 0 \). Denote by the quadruple \((m, \alpha, f, \phi)\), of dimension \( m > 1 \), constant \( \alpha \in \mathbb{R} \), and smooth functions \( f, \phi : I \to \mathbb{R} \) defined on an interval \( I \subseteq \mathbb{R} \), with \( f \) positive, the corresponding inflationary spacetime \((M, g, \phi) = (I \times F, -dt^2 + f^2 g^F, \phi)\) (definition 2.10), with \( \phi \) being the composition of standard projection \( I \times F \to I \) with \( \phi \), with \( \alpha \) the sectional curvature of \((F, g^F)\) and \( F \cong S^n \) (when \( \alpha > 0 \)) or \( F \cong \mathbb{R}^n \) (when \( \alpha \leq 0 \)).

We call \((M, g, \phi)\) a regular inflationary spacetime if it belongs to one of the parametrized families identified below.

(a) Constant scalar, with scalar value \( \Phi \), on a constant curvature spacetime with scalar curvature \( K \):

\[
\text{CC}_K^{m} \text{CES} \Phi = \left\{ (m, \alpha, f, \Phi) \mid (m, \alpha, f) \in \text{CC}_K^{m} \right\}.
\]

(b) Constant energy scalar, with energy density \( \rho > 0 \) and \( J = \phi(I) \), on an Einstein static universe with spatial sectional curvature \( K = \frac{2}{m(m-1)}\kappa \rho \), or equivalently with cosmological constant \( \Lambda = \frac{(m-1)}{m}\kappa \rho \):

\[
\text{ESU}^{m}_K \text{CES} \rho, J = \left\{ (m, K, 1, \sqrt{2\rho/m}) \mid I = J/\sqrt{2\rho/m} \right\}.
\]

(c) Spatially flat massless minimally-coupled scalar spacetime, with cosmological constant \( \Lambda, J = \phi(I) \) and \( J' = \frac{\rho'}{\rho} = \frac{2\Lambda}{m(m-1)} < \frac{1}{\kappa} (J')^2 \):

\[
\text{MMS}^{m,0}_{\Lambda, J, J'} = \left\{ (m, 0, f, \phi) \mid \phi' < 0, \frac{f'}{f} \neq 0, \frac{\rho'}{\rho} = \frac{\alpha}{m(m-1)}, \frac{J'}{J} - \frac{\rho'}{\rho} + mJ' = \frac{2\Lambda}{m(m-1)} \right\}.
\]

(d) Generic massless minimally-coupled scalar spacetime, with cosmological constant \( \Lambda \), normalized scalar energy constant \( \Omega > 0 \), \( J = \phi(I) \) and \( J' = \frac{\rho'}{\rho} = \frac{2\Lambda}{m(m-1)} \neq 0 \):

\[
\text{MMS}^{m,\Omega}_{\Lambda, J, J'} = \left\{ (m, \alpha, f, \phi) \mid \alpha \neq 0, \frac{f'}{f} \neq 0, \phi' = -\sqrt{\Omega} \frac{\Omega n^2}{m}, \frac{\rho'}{\rho} = \frac{2\Lambda + \kappa \Omega n^2}{m(m-1)} \right\}.
\]

(e) Spatially flat nonlinear Klein–Gordon spacetime, with non-constant scalar self-coupling potential \( V : J \to \mathbb{R} \), with \( J = \phi(I) \), and expansion profile \( \Xi : J \to \mathbb{R} \), satisfying \( \Xi(u) \neq 0, \kappa_d \Xi \Xi(u) > 0 \) and \( \delta_V \Xi = 0 \) in the notation of (18):

\[
\text{NKG}^{m,0}_{V, \Xi, J} = \left\{ (m, 0, f, \phi) \mid \frac{f'}{f} = \Xi(\phi), \phi' = -\frac{(m-1)}{\kappa} \partial_\phi \Xi(\phi) \right\}.
\]
(f) Generic nonlinear Klein–Gordon spacetime, with non-constant scalar potential $V: J \to \mathbb{R}$, with $J = \phi(I)$, and expansion profile $(\Pi, \Xi): J \to \mathbb{R}^2$, satisfying $\Pi < 0$, $\Xi \neq 0$, $\kappa \frac{\Pi' + V}{m(m-1)} \neq \Xi^2$, and $\mathcal{E}_V(\Pi, \Xi) = 0$ in the notation of (20):

$$NKG_{\Pi,\Xi,I} = \left\{ (m, \alpha, f, \phi) \mid \alpha \neq 0, \frac{f'}{f} \neq 0, \phi' = \Pi(\phi), \frac{f'^2}{f^2} = \Xi(\phi), \frac{f'^2}{f^2} + \frac{2}{f} = \kappa \frac{\Pi' + V(\phi)}{m(m-1)} \right\}. \quad (14)$$

Below we directly give the list of tensor equations, covariantly constructed from the metric, the Riemann curvature, and its derivatives, that characterize the corresponding local isometry classes. Observe that an IDEAL characterization is not unique. Given one, many others can be produced by covariant and invertible transformations. Our choices are based on various conventions used in relativity and cosmology.

To be specific, our conventions for the relations between the metric $g_{ij}$, covariant derivative $\nabla_j$, Riemann curvature, as well as Ricci tensor and scalar are the following:

$$\nabla_j \nabla_i - \nabla_i \nabla_j \nabla_k v_k = R_{ijk} v_i, \quad R_{ijk} = R_{ijk} g_{lh},$$

$$R_{k} = R_{ijkl}^k, \quad R = R_{ijkl} g^{ij}, \quad B = R_{ijkl} g^{kj} g^{il}.$$  

It is convenient to define the following product (sometimes also known as the Kulkarni–Nomizu product) that builds an object with the symmetries of the Riemann tensor out of symmetric 2-tensors $A_{ij}, B_{ij}$:

$$(A \odot B)_{ijk} = A_{ik} B_{j} - A_{jk} B_{i} + A_{kj} B_{i} - A_{ji} B_{k}.$$  

Note that $A \odot B = B \odot A$ and $UU \odot UU = 0$, with $(UU)_{ij} = U_i U_j$ and $U_i$ any vector field. For dim $M = m + 1 > 2$, our formula for the Weyl tensor is

$$W_{ijk} = R_{jk} - \frac{1}{(m-1)} (g \odot R)_{ijk} + \frac{1}{2m(m-1)} \mathcal{R}(g \odot g)_{ijk}. \quad (16)$$

Note that $W_{ijk}$ vanishes precisely when $R_{jk} = (g \odot A)_{ijk}$ for some symmetric $A_{ij}$. As usual, we denote idempotent symmetrization and anti-symmetrization by $A_{(ij)} = \frac{1}{2}(A_{ij} + A_{ji})$, $A_{[ij]} = \frac{1}{2}(A_{ij} - A_{ji})$.

The first theorem classifies just the Lorentzian spacetime, without reference to a scalar field. The definitions for the various scalar and tensor fields introduced below may seem ad hoc, but they have straightforward geometrical meanings. The vector field $U^\nu$ plays the role of a future-pointing unit timelike vector field, orthogonal to the cosmological spatial slices. It is defined as a normalized gradient of a curvature scalar, with the choice of curvature scalar depending on the precise case being considered. The tensors $\Psi_{ij}$ and $\mathcal{D}_j$ encode in them the shear, twist and geodesic character of $U^\nu$ and are non-zero when the spacetime deviates from a generalized Robertson–Walker (GRW) spacetime (a possibly non-homogeneous geometry undergoing cosmological expansion or contraction). The expansion $\xi$ of the vector field $U^\nu$ also plays the role of the Hubble rate, while $\eta$ that of the Hubble acceleration. The tensor $\mathcal{E}_{ijkh}$ measures the deviation of the spatial slices from homogeneity and isotropy, while the scalar $\zeta$, together with $\mathcal{E}_{ijkh}$, measures the deviation of spatial slices from flatness.

**Theorem 1.4.** Consider a Lorentzian manifold $(M, g)$ of dim $M = m + 1 > 2$, $\kappa \neq 0$ a fixed constant. With $U$ a unit timelike vector field, consider the following notations, which are defined when possible:
\[ \xi := \frac{\nabla U_i}{m}, \quad \eta := U^i \nabla_i \xi, \quad \zeta := \frac{\mathcal{R} - 2m\eta + \frac{1}{2}m(m + 1)\xi^2}{m(m - 1)}, \]

\[ \mathcal{P}_{ij} := U^i \nabla_j \xi, \quad \mathcal{D}_{ij} := \nabla_i U_j - \frac{\nabla U_i}{m} (g_{ij} + U_iU_j), \]

\[ \mathcal{E}_{ijkh} := \mathcal{R}_{ijkh} - \left( g \odot \left[ \frac{\xi^2}{2} \nabla - \eta \mathcal{U} \right] \right)_{ijkh}, \]

\[ \mathcal{C}_{ijkh} := \mathcal{R}_{ijkh} - \left( g \odot \left[ \frac{(\xi^2 + \zeta)}{2} \nabla - (\eta - \zeta) \mathcal{U} \right] \right)_{ijkh}, \]

\[ U_R := -\frac{\nabla \mathcal{R}}{\sqrt{-g}}, \quad U_B := -\frac{\nabla \mathcal{B}}{\sqrt{-g}}, \]

(17)

Given \( x \in M \), table 1 gives the list of inequalities and equations (right column, written using the above notation, with a specific choice of \( U \)) that are satisfied on a neighborhood of \( x \) if and only if the Lorentzian manifold belongs to the corresponding local isometry class at \( x \) (left column) of a regular FLRW spacetime (definition 1.2). Each local isometry class belongs to a family, parametrized by real constants, intervals or functions (middle column). By continuity, each inequality need only be checked at \( x \).

In addition, since both theorem 1.4 and table 1 are densely packed with information, we include a graphical flowchart summaries of the same information in figure 1. The notation is the same as in the original theorems.

Finally, we state the theorem classifying inflationary spacetimes, those endowed with scalar and satisfying the coupled Einstein–Klein–Gordon equations, where the equation for the scalar \( \phi \) may be nonlinear due to a potential \( V(\phi) \). The reader is referred to the paragraph preceding theorem 1.4 for an explanation of the notation. The new scalar \( H \) roughly corresponds to the Hamilton–Jacobi equation of spatially flat single field inflation, while \( G \) is its generalization to the non-flat case. See the end of section 2.6 for a more detailed motivation.

**Theorem 1.5.** Consider an inflationary spacetime \((M, g, \phi)\) of \( \dim M = m + 1 > 2 \), \( \kappa \neq 0 \) a fixed constant. With \( U \) a unit timelike vector field, recall the notation of theorem 1.4, supplemented with

\[ (\cdot) := U^i \nabla_i (\cdot), \quad U_{\phi} := \frac{\nabla \phi}{\sqrt{-g}}, \]

(18)

\[ \mathcal{S}_V(\Xi) := (\partial_u \Xi)^2 - \kappa \frac{m \Xi^2}{(m - 1)} + \kappa^2 \frac{V}{(m - 1)^2}, \]

(19)

\[ \mathcal{G}_V(\Pi, \Xi) := \left( \frac{\partial_u \Xi + \kappa \Pi}{m(m - 1)} \right) \left( \kappa \frac{H^2 + V}{m(m - 1)} - \Xi^2 \right) + 2 \Xi \left( \kappa \frac{H^2 + V}{m(m - 1)} - \Xi^2 \right), \]

(20)

where \( \Xi = \Xi(u) \) and \( \Pi = \Pi(u) \). Let \( g \) and \( \phi \) satisfy the coupled Einstein–Klein–Gordon equations with scalar potential \( V(\phi) \),

\[ \nabla_i \nabla_i \phi - \frac{1}{2} \partial_u V(\phi) = 0, \]

(21)

\[ \text{Class. Quantum Grav. 35 (2018) 035013} \]

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R_{ij} - \frac{1}{2} R g_{ij} = \kappa \left( (\nabla_i \phi)(\nabla_j \phi) - \frac{1}{2} g_{ij} [ (\nabla \phi)^2 + V(\phi) ] \right). \quad (22)

Given \( x \in M \), Table 2 gives the list of inequalities and equations (right column) that are satisfied on some neighborhood of \( x \) if and only if the inflationary spacetime belongs to the corresponding local isometry class at \( x \) (left column) of a regular inflationary spacetime. Each
local isometry class belongs to a family, parametrized by real constants, intervals or functions (middle column). By continuity, each inequality needs only to be checked at \( x \).

In addition, since both theorem 1.5 and table 2 are densely packed with information, we include a graphical flowchart summaries of the same information in figure 2. The notation is the same as in the original theorems.

Note that when we make the choice \( U = U_0 \), it automatically follows that \( \phi' = -\sqrt{-(\nabla \phi)^2} < 0 \). This convention is common in the study of inflation, where \( \phi(t) \) starts off at a high value and then ‘rolls down hill’ as \( t \) increases. This is reflected in the inequalities in table 2.
Table 2. IDEAL characterization of local isometry classes of regular inflationary spacetimes (theorem 1.5).

| Class Parameters | Inequalities/equalities |
|-------------------|-------------------------|
| (a) Constant scalar | $V(u) = \frac{2}{n} \Lambda$, $K = \frac{2}{m(m-1)} \Lambda$, $R_{ij} - \frac{2}{m(m-1)} (g \otimes g)_{ij} = 0$, $\phi = \Phi$ |
| (b) Constant energy scalar | $V(u) = \frac{2(\mu-1)}{\mu} \rho$, $\rho > 0$, $K = \frac{2\rho}{m(m-1)}$, $(U = U_\phi)$, $(\nabla \phi)^2 < 0$, $\zeta > 0$, $\phi(x) \in J$, $\nabla_i U_j = 0$, $\xi_{ij} = 0$, $\nabla_\phi = 0$, $\zeta = \frac{2\rho}{m(m-1)}$ |
| (c) Spatially flat massless minimally-coupled scalar | $V(u) = \frac{2}{n} \Lambda$, $\frac{2\Lambda_{\mu}}{m(m-1)} < \frac{1}{n}(J')^2$, $(U = U_\phi)$, $(\nabla \phi)^2 < 0$, $\frac{1}{n}(\xi^2 - \frac{2\Lambda}{m(m-1)}) > 0$, $\phi(x) \in J$, $\xi(x) \in J'$, $\nabla_i U_j - \nabla \phi \xi_{ij} = 0$, $\xi_{ij} = 0$, $\eta + m\xi^2 = \frac{2\Lambda}{m(m-1)}$, $\xi^2 = \frac{2\Lambda + \xi}{m(m-1)}$ |
| (d) Generic massless minimally-coupled scalar | $V(u) = \frac{2}{n} \Lambda$, $0 \notin J'$, $\Omega > 0$, $(U = U_\phi)$, $(\nabla \phi)^2 < 0$, $\phi(x) \in J$, $\xi(x) \in J'$, $\nabla_i U_j - \nabla \phi \xi_{ij} = 0$, $\xi_{ij} = 0$, $\phi' = -\sqrt{\Omega} |\xi|^2$, $\xi^2 + \zeta = \frac{2\Lambda + \xi}{m(m-1)}$ |
| (e) Spatially flat nonlinear Klein–Gordon | $V, \Xi: J \to \mathbb{R}$, $\Xi(u) \neq 0$, $\frac{1}{n} \Xi'(u) > 0$, $V'(u) \neq 0$, $\nabla \phi (\Xi) = 0$, $(U = U_\phi)$, $(\nabla \phi)^2 < 0$, $\xi \neq 0$, $\frac{1}{n} \eta < 0$ |
| (f) Generic nonlinear Klein–Gordon | $V, \Xi, \Pi: J \to \mathbb{R}$, $\Pi < 0$, $\Xi \neq 0$, $\nabla^V_{\mu} \Xi \neq \Xi^2$, $V'(u) \neq 0$, $\nabla \phi (\Pi, \Xi) = 0$, $(U = U_\phi)$, $(\nabla \phi)^2 < 0$, $\xi \neq 0$, $\frac{1}{n} \eta < 0$ |
Our characterization theorems cover what we have called regular FLRW or inflationary spacetimes (definitions 1.2 and 1.3), which are required to satisfy the inequalities listed in tables 1 and 2 everywhere.

2. Geometry of FLRW and inflationary spacetimes

Definition 2.1. Let \((F, g^F)\) be a \(m\)-dimensional Riemannian manifold, \(m \geq 1\), \(I \subseteq \mathbb{R}\) an open interval with standard coordinate \(t\) and endowed with the usual reversed metric \(-dt^2\) and \(f \in C^\infty(I)\), with \(f > 0\). A generalized Robertson Walker (GRW spacetime) is a product manifold \(M = I \times F\) endowed with the metric \(g\) defined as
\[
g = -\pi_t^2 dt^2 + (f \circ \pi_t)^2 \pi_F^* g^F
\]  
(23)
where \(\pi_t\) and \(\pi_F\) are respectively the projections on \(I\) and \(F\). Furthermore \(I\) is called the base, \(F\) the fiber and \(f\) the warping function (also scale factor, in the literature on cosmology).

To simplify notation in the sequel, let us introduce the notation \(\tilde{T} = \pi_t^* T\) for any completely covariant tensor \(T\) defined on \(F\).

The definition implies that around every point of \(M = I \times F\), there exists a coordinate system \((x^0, x')\) adapted to the product structure, such that, denoting \(t = x^0\),
\[
g_{ij} = -(dt)(dt) + f^2(t)g_{ij}^F,
\]  
(24)
where \(g_{ij}^F\) depends only on the \(x'\) coordinates with \(i > 0\) and \(g_{ij}^F(\lambda)^j = 0\). The only obstacle to making the last statement global on \(M\) is that the \(F\) factor may not admit a global coordinate system.

Definition 2.2. A Friedmann–Lemaître–Robertson–Walker (FLRW) spacetime is a Lorentzian manifold \((M, g)\) that is a GRW spacetime (definition 2.1) where the fiber \((F, g^F)\) is simply connected, complete and has constant curvature \(\alpha\) (some constant), that is, the Riemann curvature tensor \(R^F_{ijkh}\) of \((F, g^F)\) has the form
\[
R^F_{ijkh} = \alpha(g^F_{ik}g^F_{jh} - g^F_{ih}g^F_{jk}).
\]  
(25)
When \(\text{dim} M = 2\), only \(\alpha = 0\) is possible, since any 2-dimensional \((F, g^F)\) is flat.

It is well known that any simply connected, complete Riemannian manifold of constant curvature, meaning that its Riemann curvature tensor is of the form (25), is isometric to either a round sphere (\(\alpha > 0\)), flat Euclidean space (\(\alpha = 0\)), or a hyperbolic space (\(\alpha < 0\)) [36, section 2.4]. If the complete and simply connected hypotheses are dropped, then a constant curvature Riemannian manifold is still locally isometric to one of these model spaces.

Similarly, in the sequel, we will be interested in Lorentzian spacetimes that are locally isometric (definition 1.1) to GRW or FLRW models.

2.1. Riemann curvature in GRW spacetimes

Below, we describe the Riemann curvature \(R_{ijkh}\) in a GRW spacetime, in terms of the curvature of \((F, g^F)\), the warping function \(f\) and the vector field \(U_i = -(dt)_i\). For reference, let us denote the Riemann tensor on the \((F, g^F)\) factor by \(R^F_{ijkh}\), with \(R^F_{ij} = (g^F)^{ij}R^F_{ijkh}\) and \(\nabla^F = (g^F)^{ij}\partial_i R^F_{jk}\) denoting respectively the corresponding Ricci tensor and scalar. Recall also the notation \(\nabla^F = \pi^*_F \nabla^F\), \(R^F = \pi^*_F R^F\) and \(\bar{R}^F = \pi^*_F \bar{R}^F\).

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Figure 2. IDEAL characterization of local isometry classes of regular inflationary spacetimes (theorem 1.5, table 2).
Adapting the more general results on the covariant derivative on warped products [26, proposition 7.35], the action of the spacetime covariant derivative is determined by

\[ \nabla_i(fU_j) = f'g_{ij} \]

\[ \nabla_i \tilde{X}_j = \tilde{\nabla}_iX_j - 2f'fU_i(\tilde{X}_j), \]

for any \(X_j\) defined on \(F\). Recalling the notation already used in the introduction, for any \(U_i\) we can define the temporal derivative \((-') := U^\mu \nabla_\mu (-)\) and also

\[ \xi := \frac{\nabla^i U_i}{m}, \quad \eta := \xi' = U^i \nabla_i \xi. \]

(27)

With our choice of \(U\) on a GRW spacetime, we will be making repeated use of the identities

\[ \xi = f'f, \quad \eta = f''f - f'f^2. \]

(28)

Geometrically \(\xi\) is called the expansion of the vector field \(U\), while its GRW value \(f'/f\) is known as the Hubble rate in the literature on cosmology.

Next, adapting the more general result [26, proposition 7.42] of how to write the Riemann tensor of a warped product manifold in terms of the curvatures of its factors and the warping function, it is possible to give the following general expression for the Riemann tensor of GRW spacetimes:

\[ R_{ijkh} = f^2\tilde{R}_{ijkh}^F + \left( g \odot \left[ \frac{1}{2}f'^2g - \left( \frac{f''}{f} - \frac{f'^2}{f^2} \right) dt^2 \right] \right)_{ijkh} \]

\[ = f^2\tilde{R}_{ijkh}^F + \left( g \odot \left[ \frac{\xi^2}{2}g - \eta gU \right] \right)_{ijkh}, \]

(29)

where \((dt^2)_{ij} = (dt)_i(dt)_j\) and where we have used the product notation (15). When \(m = 1\), the tensors \(g \odot UU\) and \(g \odot g\) are no longer linearly independent, in fact \(g \odot UU = -\frac{1}{2}g \odot g\).

Moreover, the Riemann curvature for a 1-dimensional \((F, g_F)\) is always zero. Hence, in the special \(m = 1\) case we have the simplification

\[ R_{ijkh} = \frac{(\eta + \xi^2)}{2} + (g \odot g)_{ijkh} = f''f \frac{1}{2}(g \odot g)_{ijkh}. \]

(30)

As a consequence, using the identities

\[ g^{k \bar{h}}\tilde{R}^F_{k \bar{h}j} = \frac{1}{f^2} \pi^F_k((g^F)^{k \bar{h}} R^F_{\bar{h}j}) = \frac{1}{f^2} \tilde{R}^F_{kj}, \]

\[ g^{j \bar{j}}\tilde{R}^F_j = \frac{1}{f^2} \pi^F_k((g^F)^j \tilde{R}^F_{\bar{j}j}) = \frac{1}{f^2} \tilde{R}^F_j, \]

(31)

(32)

we get the following formulas for the Ricci tensor \(R_{ij} = g^{k \bar{h}}R_{k \bar{h}j}\) and scalar \(R = g^{ij}R_{ij}\):

\[ R_{ij} = \tilde{R}^F_{ij} - (m - 1) \left( \frac{f''}{f} - \frac{f'^2}{f^2} \right) U_iU_j + \left( \frac{f''}{f} + (m - 1)\frac{f'^2}{f^2} \right) g_{ij} \]

\[ = \tilde{R}^F_{ij} - (m - 1)\eta U_iU_j + (\eta + m\xi^2)g_{ij}, \]

(33)
\[ R = \frac{1}{f^2} \tilde{R}^F + 2m \frac{f''}{f} + m(m-1) \frac{f'^2}{f^2} = \frac{1}{f^2} \tilde{R}^F + 2m \eta + m(m+1) \xi^2. \] (34)

For completeness, we also compute the value of the scalar square of the Ricci tensor:

\[ B = \frac{\tilde{B}^F}{f^4} + 2 \left( \frac{f''}{f} + (m-1) \frac{f'^2}{f^2} \right) \tilde{R}^F + m \left( \frac{f''}{f} + (m-1) \frac{f'^2}{f^2} \right)^2 + m^2 \frac{f'^2}{f^2} = \frac{\tilde{B}^F}{f^4} + 2(\eta + m \xi^2) \tilde{R}^F + m(\eta + m \xi^2)^2 + m^2(\eta + \xi^2)^2. \] (35)

where \( B^F = (g^F)^{ik}(g^F)^{jh}R^F_{ijhk}. \)

The above formulas motivate the following definitions, which can be used to isolate the spatial curvature \( R^F_{ijhk} \) from the knowledge of the spacetime curvature \( R_{ijhk} \) and of \( U_i \).

**Definition 2.3.** Consider a Lorentzian manifold \((M, g)\) with a unit timelike vector field \( U \).

Recall also the scalars \( \xi \) and \( \eta \) scalars from (27).

(a) We define the zero (spatial) curvature deviation (ZCD) tensor as

\[ \mathcal{Z}_{ijkh} := R_{ijkh} - \left( g \odot \left[ \frac{\xi^2}{2} g - \eta U U \right] \right)_{ijkh}. \] (36)

(b) Provided \( m > 1 \), we define the spatial curvature scalar as

\[ \zeta := \frac{\mathcal{Z}_{ij} \mathcal{Z}^{ij}}{m(m-1)} = \frac{R - 2m \eta - m(m+1) \xi^2}{m(m-1)} \] (37)

and if \( m = 1 \), we set \( \zeta = 0 \).

(c) We define the constant (spatial) curvature deviation (CCD) tensor as

\[ \mathcal{C}_{ijkh} := R_{ijkh} - \left( g \odot \left[ \frac{\xi^2}{2} g - (\eta - \zeta) U U \right] \right)_{ijkh}. \] (38)

On GRW spacetimes, the usefulness of these definitions lies in the identities

\[ \mathcal{Z}_{ijkh} = f^2 \tilde{R}^F_{ijkh}, \quad \zeta = \frac{1}{m(m-1)} \frac{\tilde{R}^F}{f^2}, \] (39)

\[ \mathcal{C}_{ijkh} = f^2 \left( \tilde{R}^F_{ijkh} - \frac{1}{m(m-1)} \frac{\tilde{R}^F}{f^2} \left( \tilde{g}^F \odot \tilde{g}^F \right)_{ijkh} \right). \] (40)
2.2. Riemann curvature in FLRW spacetimes

Next, we specialize the main formulas obtained in the preceding section from GRW to FLRW spacetimes (definition 2.2), by making use of their spatial curvature structure

\[R^F_{\ell m} = \frac{\alpha}{2} (g^F \odot g^F)_{\ell m}, \quad R^F = (m - 1)\alpha g^F, \quad R^F = m(m - 1)\alpha, \quad (41)\]

and of the identity

\[
f^2 \frac{1}{2} (\tilde{g}^F \odot \tilde{g}^F)_{\ell m} = \frac{1}{f^2} \frac{1}{2} ((g + UU) \odot (g + UU))_{\ell m} = \frac{1}{f^2} \left( (g \odot UU)_{\ell m} + \frac{1}{2} (g \odot g)_{\ell m} \right) = \frac{1}{f^2} \left( g \odot \left[ \frac{1}{2} g + UU \right] \right)_{\ell m}, \quad (42)\]

where we have recalled that \(f^2 \tilde{g}^F = g + UU\). Recall also the definitions of \(U_i = (dr)_i\), the scalars \(\xi\) and \(\eta\) from (27), and note the identity

\[
\zeta = \frac{\alpha}{f^2} \quad (43)\]

for the spatial curvature scalar (definition 2.3) when \(m > 1\). When \(m = 1\), we always have \(R^F_{\ell m} = 0\), so it is consistent to take \(\zeta = 0\), as we do.

Thus, for FLRW spacetimes of spatial sectional curvature \(\alpha\), we have

\[
R_{\ell m} = \left( g \odot \left[ \frac{(\xi^2 + \zeta)}{2} g - (\eta - \zeta)UU \right] \right)_{\ell m}, \quad (44)\]

\[
R_{ij} = -(m - 1)(\eta - \zeta)U_iU_j + [(\eta - \zeta) + m(\xi^2 + \zeta)]g_{ij}, \quad (45)\]

\[
R = m \left[ 2(\eta - \zeta) + (m + 1)(\xi^2 + \zeta) \right], \quad (46)\]

\[
B = m[(\eta - \zeta) + m(\xi^2 + \zeta)]^2 + m^2(\eta + \xi^2)^2, \quad (47)\]

where we have also used \(B^F = m(m - 1)^2\alpha^2\). In the special \(m = 1\) case, the above formulas simplify to

\[
R_{\ell m} = \left( \eta + \xi^2 \right) (g \odot g)_{\ell m}, \quad (48)\]

\[
R_{ij} = (\eta + \xi^2)g_{ij}, \quad (49)\]

\[
R = 2(\eta + \xi^2), \quad (50)\]

\[
B = 2(\eta + \xi^2)^2. \quad (51)\]

Because of the frequent appearance of the combinations \(\eta - \zeta = \frac{f''}{f} - \frac{f'}{f^2} - \frac{\alpha}{f^2}\) and \(\xi^2 + \zeta = \frac{f'^2}{f^2} + \frac{\alpha}{f^2}\), in the sequel we will need the identity

\[(\xi^2 + \zeta) = 2\xi(\eta - \zeta) \quad \text{or} \quad \left( \frac{f'^2}{f^2} + \frac{\alpha}{f^2} \right)' = 2\frac{f''}{f} \left( \frac{f'^2}{f^2} - \frac{\alpha}{f^2} \right). \quad (52)\]
2.3. Perfect fluid interpretation

An arbitrary FLRW spacetime will in general not satisfy the vacuum Einstein equations. But it could be interpreted, when \( m > 1 \), as a solution of Einstein equations with a perfect fluid stress energy tensor

\[
R_{ij} - \frac{1}{2} R g_{ij} + \Lambda g_{ij} = \kappa T_{ij} = \kappa (\rho + p) U_i U_j + \kappa p g_{ij},
\]

where \( \Lambda \) is the cosmological constant, \( \rho \) is the energy density and \( p \) is the pressure. When \( m = 3 \), the coupling constant usually has the value \( \kappa = \frac{8 \pi G}{c^4} \), where \( G \) is Newton’s constant and \( c \) the speed of light. In other dimensions, there are at least two conventions: either keeping the value of \( \kappa \) the same, or setting it to \( \kappa \frac{m}{2 \sigma_m G / c^4} \), where \( \sigma_m = \frac{2 \pi^{\frac{m}{2}}}{\Gamma(m - \frac{1}{2})} \) is the area of the unit \((m - 1)\)-sphere. We will simply keep it as an unspecified but fixed constant \( \kappa \neq 0 \). The cosmological constant could of course be shifted to \( \Lambda \mapsto \Lambda \) by the redefinitions \( p \mapsto p - \Lambda / \kappa, \rho \mapsto \rho + \Lambda / \kappa \). When \( m = 1 \), the fluid interpretation is no longer possible, simply because the Einstein tensor \( R_{ij} - \frac{1}{2} R g_{ij} \) is identically zero in two spacetime dimensions.

Defining \( T = g^{ij} T_{ij} \), an equivalent form of Einstein’s equations is

\[
R_{ij} = \kappa T_{ij} - \frac{\kappa}{m - 1} T g_{ij} = \kappa (\rho + p) U_i U_j + \frac{\kappa}{m - 1} p g_{ij}.
\]

Hence, for FLRW spacetimes, these equations translate to

\[
\frac{f''}{f} - \frac{f'^2}{f^2} + \frac{\alpha}{f^2} = - \frac{1}{m - 1} \kappa (\rho + p), \quad \kappa \rho = \frac{m(m - 1)}{2} \left( \frac{f'^2}{f^2} + \frac{\alpha}{f^2} \right), \quad (55)
\]

On the top-left we have the Friedmann equation, while on the bottom-left we have the acceleration equation. These equations agree with the formulas previously obtained in [4], which was one of the first to consider perfect fluid cosmologies in higher spacetime dimensions. The Bianchi identity \( \nabla^i (R_{ij} - \frac{1}{2} R g_{ij}) = 0 \) implies the stress–energy conservation \( \nabla^i T_{ij} = 0 \) condition, which translates to the energy conservation or continuity equation

\[
\rho' + m f^i f^j (\rho + p) = 0.
\]

2.4. Special FLRW classes

Below, we list the forms of FLRW spacetimes (definition 2.2) satisfying some special geometric conditions. Throughout this section, consider an FLRW spacetime \((M, g)\), \( \dim M = m + 1 \geq 2 \), with warping function \( f : I \to \mathbb{R} \) and spatial sectional curvature \( \alpha \). Whenever parameters are present, they must be chosen to respect \( f(t) > 0 \) for all \( t \in I \), even if not explicitly indicated, as well as \( \alpha = 0 \) when \( m = 1 \).
Lemma 2.4. The complete list of possible triples \((m, \alpha, f(t))\) satisfying the flat (or Minkowski space) condition, \(R_{ijkl} = 0\), consists of

\[
\begin{align*}
\frac{\dot{f}'}{f'} &= 0: & (m, 0, A) & (A > 0); \\
\frac{\ddot{f}'}{f'} &\neq 0: & m = 1: & \frac{\dot{f}'}{f'} = 0 \quad (1, 0, A(t - t_0)) & (A \neq 0);
& m > 1: & \frac{\dot{f}'}{f'} + \frac{\alpha}{f} = 0 \quad (m, \alpha, \pm \sqrt{-\alpha}(t - t_0)) & (\alpha < 0).
\end{align*}
\]

Proof. From equation (44), the necessary and sufficient conditions are \(\frac{\ddot{f}'}{f'} = 0\) and \(\left(\frac{\dot{f}'}{f'} + \frac{\alpha}{f}\right) = 0\), when \(m > 1\), or only \(f''/f = 0\), when \(m = 1\). It is easy to see that the desired conclusion exhausts the solutions of these equations under the constraint that \(f(t) \neq 0\) everywhere.

Lemma 2.5. The complete list of possible triples \((m, \alpha, f(t))\) satisfying the constant curvature (or (anti-)de Sitter space) condition, \(R_{ijkl} = \frac{K}{2}(g \circ g)_{ijkl}\), with sectional curvature \(K \neq 0\), consists of \((A, K)\) constant

\[
\begin{align*}
\frac{\dot{f}'}{f'} &= 0: & \frac{\ddot{f}'}{f'} + \frac{\alpha}{f} = K, & (m, 0, K) & (m > 1, K \neq 0, A > 0).
\end{align*}
\]

Proof. Again, referring to equation (44), the necessary and sufficient conditions are \(\frac{\ddot{f}'}{f'} = \frac{\dot{f}'}{f'} - \frac{\alpha}{f} = K\), when \(m > 1\), or only \(f''/f = K\), when \(m = 1\). If \(m > 1\) and \(f'/f = 0\), we must have \(K = \alpha/f^2 = 0\), which contradicts the \(K \neq 0\) hypothesis. Otherwise, it is easy to see that the desired conclusion exhausts the solutions of these equations under the constraint that \(f(t) \neq 0\) everywhere.

Lemma 2.6. The complete list of possible triples \((m, \alpha, f(t))\) satisfying both conditions \(R' = 0\) and \(B' = 0\), but not of constant curvature, consists of \((A, K)\) constant

\[
(m, KA^2, A) \quad (m > 1, K \neq 0, A > 0).
\]

This is the Einstein static universe [35, section 16.2] with spatial sectional curvature \(K\), which solves the Einstein equation, \(R_g - \frac{1}{2}Rg + \Lambda g = 0\), with the cosmological constant \(\Lambda = \frac{(m-1)(m-2)}{2}K\).
Proof. From equations (46) and (47), both $R' = 0$ and $B' = 0$ are third order equations in $f$. Eliminating $f'''$ from both of them, we obtain the integrability condition

$$m^2(m - 1)^2f' \left( \frac{f''}{f} - \frac{f'^2}{f^2} - \frac{\alpha}{f} \right) = 0. \quad (58)$$

Obviously, it is trivial when $m = 1$. This is not surprising, because then $R$ is the only independent curvature component and $R' = 0$ already implies that the spacetime is of constant curvature, which is excluded by the hypotheses.

Further, this integrability condition splits into the cases $f'/f = 0$ and $f'/f \neq 0$. In the latter, it implies $\frac{f''}{f} - \frac{f'^2}{f^2} - \frac{\alpha}{f} = 0$ and $(\frac{f'^2}{f^2} + \frac{\alpha}{f})' = 0$ (see equation (52)). But these are precisely the necessary and sufficient conditions for the spacetime to be of constant curvature (lemma 2.5), which is excluded by our hypotheses. Thus, we are left with the only possibility $f'/f = 0$ and the desired conclusion clearly exhausts the solutions of this equation. \hfill $\Box$

Lemma 2.7. The complete list of possible triples $(m, \alpha, f(t))$ satisfying the constant scalar curvature condition $R' = 0$, but with $R' \neq 0$, consists of

$$\begin{cases} 
\alpha = 0 : & (m, 0, f) \quad \left( m > 1, \frac{f''}{f} - \frac{f'^2}{f^2} + \frac{(m+1)}{2} \left( \frac{f'^2}{f^2} - K \right) = 0, \ K = K \neq 0 \right) ; \\
\alpha \neq 0 : & (m, \alpha, f) \quad \left( m > 1, \frac{f'^2}{f^2} + \frac{\alpha}{f} = K + \kappa \Omega \frac{m+1}{m+2}, \ \Omega \neq 0 \right). 
\end{cases}$$

These are FLRW spacetimes with cosmological constant $\Lambda = \frac{m(m-1)}{2} K$ and radiation perfect fluid of energy density $\Omega_r = \frac{1}{K} \left( \frac{f'^2}{f^2} + \frac{\alpha}{f} - K \right)$, where $\Omega_r = C/f^{m+1}$ for some constant $C$. We refer to $\Omega_r$ as the radiation energy density because the term with the power law $1/f^{m+1}$ in the Friedmann equation

$$\frac{f'^2}{f^2} + \frac{\alpha}{f} = K + \kappa \frac{C}{f^{m+1}}, \quad (59)$$

when considered by itself gives rise to the constitutive relation $p_r(\rho) = \rho/m$, which is characteristic of radiation in thermal equilibrium [25]. If $\Omega_\alpha = \alpha/f^2$ is the energy density due to spatial curvature, when it is nonzero, the ratio $\Omega = \Omega_r/\Omega_\alpha$ defines our normalized radiation density constant $\Omega$.

Proof. If $f'/f = 0$, then $R = m(m - 1)\alpha/f^2$ and $B = m(m - 1)^2\alpha^2/f^4$. Hence $R' = 0$ implies $B' = 0$. The same implication holds if $m = 1$ (see proof of lemma 2.5). Therefore, by the $B' \neq 0$ hypothesis, we can assume that $m > 1$ and $f'/f \neq 0$.

From equation (46), the constant scalar curvature condition $R = m(m + 1)K$ (with some constant $K$)

$$2 \left( \frac{f''}{f} - \frac{f'^2}{f^2} - \frac{\alpha}{f} \right) + (m + 1) \left( \frac{f'^2}{f^2} + \frac{\alpha}{f} \right) = (m + 1)K, \quad (60)$$

after multiplying both sides by the integrating factor $(f'/f)f^{m+1}$ and using identity (52), it is equivalent to
\[ f^{m+1} \left( \frac{f'}{f^2} + \frac{\alpha}{f^2} - K \right) = \kappa C, \]  
(61)

for some constant \( C \). If \( C = 0 \), we are back to the case of constant curvature (lemma 2.5), which is excluded by the \( B' \neq 0 \) hypothesis. When \( \alpha \neq 0 \), we can normalize this constant as \( C = \Omega |\alpha|^\frac{m+1}{2} \), with some \( \Omega \neq 0 \). Thus, the desired conclusion clearly consists of the necessary and sufficient conditions for \( R' = 0 \) and \( B' \neq 0 \) to hold.

**Lemma 2.8.** For any triple \((m, \alpha, f(t))\) for which \( \alpha = 0 \), \((f'f^{-1})' \neq 0\) and \((\nabla R)^2 < 0\), there is a unique smooth function \( P : J \to \mathbb{R} \), where \( J = \frac{e^C}{f'}(I), I \subseteq \mathbb{R} \) and

\[ \frac{f''}{f} - \frac{f'^2}{f^2} + \frac{mf^2}{f} = -\kappa P \left( (f'/f)^2 \right). \]  
(62)

The function \( P(u) \) will also satisfy the following condition for each \( u \in J \):

\[ P(u) \left[ \kappa \partial_u P(u) - \frac{1}{2} \right] \neq 0. \]  
(63)

We will call \( P \) the normalized pressure function because, when \( m > 1 \), the spacetime admits a perfect fluid interpretation (section 2.3) with energy density \( \kappa \rho(t) = \frac{m(m-1)}{2} (f'/f)^2 \), pressure \( p(t) = (m-1)P \left( (f'/f)^2 \right) \), which admits the constitutive relation \( p = p(\rho) \), where \( p(\rho) = (m-1)P \left( \frac{2}{m(m-1)} \kappa \rho \right) \).

When \( m = 1 \), the triviality of the Einstein equations does not allow such an interpretation, so without loss of generality the function \( P \) simply determines the differential equation satisfied by \( f \).

**Proof.** Under our hypotheses, the existence of a unique function \( P(u) \) is an elementary consequence of the implicit function theorem. If \( P(u) = 0 \), then we are back to the case of flat or constant curvature spacetime (lemmas 2.4 and 2.5), while \( P(u) = \frac{1}{2C} [u - (m+1)\kappa] \) brings us back to the \( R' = 0 \) case (lemma 2.7), both of which contradict the \((\nabla R)^2 < 0\) hypothesis. For any other value of \( P(u) \), we have \( \nabla R \neq 0 \), which can then only be timelike.

**Lemma 2.9.** For any triple \((m, \alpha, f(t))\) for which we have \( \alpha \neq 0 \), \( f'f^{-1} \neq 0 \) and \((\nabla R)^2 < 0\), there is a unique smooth function \( E : J \to \mathbb{R} \), where \( J = \frac{e^C}{f'}(I), I \subseteq \mathbb{R} \) and

\[ \frac{f'^2}{f^2} + \frac{\alpha}{f^2} = \kappa E(\alpha/f^2). \]  
(65)

The function \( E(u) \) will also satisfy the following conditions for each \( u \in J \):

\[ \kappa E(u) - u > 0, \quad \partial_u \left[ u \partial_u E(u) - \frac{(m+1)}{2} E(u) \right] \neq 0. \]  
(66)

We will call \( E \) the normalized energy function because, when \( m > 1 \), the spacetime admits a perfect fluid interpretation (section 2.3) with energy density \( \rho(t) = \frac{m(m-1)}{2} E(\alpha/f^2) \) and
pressure \( p(t) = -(f \phi')/(mf') - \rho \) given by the continuity equation (57). When \( m = 1 \), the triviality of the Einstein equations does not allow such an interpretation, so without loss of generality the function \( E \) simply determines the differential equation satisfied by \( f \).

**Proof.** Under our hypotheses, the existence of a unique function \( E(u) \) is an elementary consequence of the implicit function theorem. Since \( f^2/f' > 0 \), we must also have \( u \partial_uu - \frac{m+1}{2}E(u) = 0 \). Thus, the second inequality in (66) is sufficient to ensure that \( \nabla R \neq 0 \), which can then only be timelike.

\[ \square \]

2.5. Scalar field

In this section, we will be interested in the geometry of Lorentzian spacetimes that are endowed with a scalar field and satisfying the coupled Einstein equations. To make non-trivial use of Einstein equations, throughout this section, we will assume that the spacetime dimension is \( m + 1 > 2 \). This information will later be used in section 3.4 to classify the local isometry classes (definition 1.1) of such spacetimes.

**Definition 2.10.** We call a spacetime with scalar \((M, g, \phi)\), with \( \dim M = m + 1 > 2 \), an inflationary spacetime when \((M, g)\) can be put in FLRW form (23), \((M, g) \cong (I \times F, -dt^2 + f^2g^F)\) such that \( \phi = \phi(t) \) is only a function of the \( t \)-coordinate, and for some constant \( \Lambda \) and smooth function \( V(\phi) \) the coupled Einstein–Klein–Gordon equations are satisfied

\[ \nabla^i\nabla_i\phi - \frac{1}{2}g_{ij}V(\phi) = 0, \]  

\[ R_{ij} - \frac{1}{2}g_{ij}R + \Lambda g_{ij} = \kappa T_{ij}, \]  

where \( T_{ij} = (\nabla_i\phi)(\nabla_j\phi) - \frac{1}{2}g_{ij}(\nabla\phi)^2 + V(\phi). \)

Equation (67) is in general the nonlinear Klein–Gordon equation with \( V(\phi) \) the self-coupling potential, though in the special case that the potential is a quadratic polynomial it becomes linear. It is easy to see that we can set \( \Lambda \mapsto 0 \) by the redefinition \( V(\phi) \mapsto V(\phi) + \frac{2}{\kappa}\Lambda \). We will adopt this convention from now on.

On an FLRW background, when \( \phi = \phi(t) \), the stress energy tensor and the wave operator are given by

\[ T_{ij} = \phi^2U_iU_j + \frac{1}{2} [\phi'^2 - V(\phi)]g_{ij}, \]  

\[ \nabla^i\nabla_i\phi = -\phi'' - m^2f f' \phi'. \]

Hence, the coupled Einstein–Klein–Gordon equations reduce to the system of ODEs

\[ \frac{f'^2}{f^2} + \frac{\alpha}{f^2} = \kappa \frac{\phi'^2 + V(\phi)}{m(m-1)}, \]
\[
\frac{f''}{f} - \frac{f'^2}{f^2} - \frac{\alpha}{f^2} = -\kappa \frac{\phi'^2}{(m-1)},
\]
(72)

\[
\phi'' + \frac{1}{2} \partial_\phi V(\phi) = -mf''/f',
\]
(73)

which we will refer to as the Friedmann equation (71), the (Einstein) acceleration equation (72), and the nonlinear Klein–Gordon equation. When \( \phi' \neq 0 \), the nonlinear Klein–Gordon equation is not independent from the other two and follows from the continuity equation (57) applied to this situation. Note that the potential \( V(\phi) \) can be isolated from the following combination of the Friedmann and acceleration equations:

\[
\kappa \frac{V(\phi)}{(m-1)} = \left( \frac{f''}{f} - \frac{f'^2}{f^2} - \frac{\alpha}{f^2} \right) + m \left( \frac{f'^2}{f^2} + \frac{\alpha}{f^2} \right)
\]

\[
= \frac{f''}{f} + (m-1) \left( \frac{f'^2}{f^2} + \frac{\alpha}{f^2} \right).
\]
(74)

While we will eventually give a characterization of local isometry classes of inflationary spacetimes with a specific scalar potential \( V(\phi) \), it is an interesting question how to recognize when an FLRW spacetime can be interpreted as part of a solution to an Einstein–Klein–Gordon system with some potential \( V(\phi) \). This is a coarser version of the question that asks for a Rainich-type characterization with a specific potential \( V(\phi) \). The latter finer question was answered in theorem 4 of [20], on which we base the following considerations.

Our starting point are the equations

\[
-\kappa \frac{\phi'^2}{(m-1)} = \frac{f''}{f} - \frac{f'^2}{f^2} - \frac{\alpha}{f^2},
\]
(75)

\[
\kappa \frac{V(\phi)}{(m-1)} = \left( \frac{f''}{f} - \frac{f'^2}{f^2} - \frac{\alpha}{f^2} \right) + m \left( \frac{f'^2}{f^2} + \frac{\alpha}{f^2} \right).
\]
(76)

To answer our question, we will be happy with some reasonable conditions on a given \((\alpha,f)\) for the existence of \(\phi(t)\) and \(V(\phi)\) such that the above equations are satisfied. Supposing that the potential \( V(\phi) \) has a smooth inverse, \(V(\phi) = u \iff \phi = W(u)\), we have the relation \((V(\phi))'/\phi' = 1/W'(V(\phi))\), which is of course consistent only if both expressions remain both finite and non-zero. On the other hand, knowing \(W'(u)\), we can recover \(W\) up to the ambiguity \(W(u) \rightarrow W(u) + \phi_0\), which determines \(V\) up to the ambiguity \(V(\phi) \rightarrow V(\phi - \phi_0)\). Thus, under the hypotheses

\[
\frac{1}{\kappa} \left( \frac{f''}{f} - \frac{f'^2}{f^2} - \frac{\alpha}{f^2} \right) > 0,
\]
(77)

\[
\left[ \left( \frac{f''}{f} - \frac{f'^2}{f^2} - \frac{\alpha}{f^2} \right) + m \left( \frac{f'^2}{f^2} + \frac{\alpha}{f^2} \right) \right]' \neq 0,
\]
(78)

using the last left-hand-side as the independent variable in an application of the implicit function theorem, we define functions \(W\) by the formula
\[
\frac{(m-1)}{\kappa} \left[ \frac{f''}{f} - \frac{f'^2}{f^2} - \frac{\alpha}{f^2} \right] + m \left( \frac{f'^2}{f^2} + \frac{\alpha}{f^2} \right) \right]^\prime \\
\pm \sqrt{-\frac{(m-1)}{\kappa} \left[ \frac{f''}{f} - \frac{f'^2}{f^2} - \frac{\alpha}{f^2} \right]}
\]
\[= \frac{1}{W'} \left( \frac{(m-1)}{\kappa} \left[ \frac{f''}{f} - \frac{f'^2}{f^2} - \frac{\alpha}{f^2} \right] + m \left( \frac{f'^2}{f^2} + \frac{\alpha}{f^2} \right) \right), \quad (79)
\]

which fixes \( W \) uniquely up to the ambiguity, \( W(u) \mapsto \pm W(u) + \phi_0 \). Hence, we can let \( V(\phi) = W^{-1}(\phi) \) and
\[
\phi(t) = W \left( \frac{(m-1)}{\kappa} \left[ \frac{f''}{f} - \frac{f'^2}{f^2} - \frac{\alpha}{f^2} \right] + m \left( \frac{f'^2}{f^2} + \frac{\alpha}{f^2} \right) \right), \quad (80)
\]
which are unique up to the ambiguity \( V(\phi) \mapsto V(\pm [\phi - \phi_0]) \) and \( \phi(t) \mapsto \pm [\phi(t) - \phi_0] \). With these definitions for \( \phi(t) \) and \( V(t) \), \((\alpha, f)\) will satisfy the desired coupled Einstein–Klein–Gordon equations. Thus, any FLRW spacetime satisfying the inequalities (77) and (78) can be thought of as part of a solution of the Einstein–Klein–Gordon equations with some non-constant potential. On the other hand, the conditions on \( \alpha \) and \( f \) be part of a solution of Einstein–Klein–Gordon equations with a constant potential are considered in lemma 2.13.

2.6. Special inflationary classes

Below, we list the forms of inflationary spacetimes (definition 2.10) satisfying some special geometric conditions. Throughout this section, consider an inflationary spacetime \((M, g, \phi)\), \(\dim M = m + 1 > 2\), with scalar field \(\phi : I \to \mathbb{R}\), warping function \(f : I \to \mathbb{R}\) and spatial sectional curvature \(\alpha\). Whenever parameters are present, they must be chosen to respect \(\alpha > 0\) for all \(t \in I\), even if not explicitly indicated.

**Lemma 2.11.** *The complete list of possible quadruples \((m, \alpha, f(t), \phi(t))\) satisfying the constant scalar condition, \(\phi(t) = \Phi\), as well as \(f'/f \neq 0\), consists of \((m, \alpha, f, \Phi)\) with \((m, \alpha, f)\) satisfying the constant curvature condition, \(R_{\ell k h} = \frac{K}{2}(g \otimes g)_{\ell k h}\), with some spacetime sectional curvature constant \(K\). The Einstein–Klein–Gordon equations are satisfied with the choice \(V(\phi) = \frac{\kappa}{2} \Lambda\), where the cosmological constant \(\Lambda = -\frac{m(m-1)}{2} K\).*

**Proof.** Since \(\phi' = 0\), the Einstein–Klein–Gordon equations reduce to \(R_{ij} - \frac{1}{2} R g_{ij} = -\Lambda g_{ij}\), or \(R_{ij} = mK g_{ij}\), with \(K = \frac{2}{m(m-1)} \Lambda\), which together with the FLRW property is precisely the necessary and sufficient to be of constant curvature. \(\square\)

Further on, in several cases, we will require the condition \(f'/f \neq 0\). So first, we explore the special case \(f'/f = 0\), of static backgrounds. We know from lemma 2.6 that the only static FLRW backgrounds are flat or Einstein static universes, with the flat case already covered by lemma 2.11. What is special about this case is that the energy \(\frac{1}{2} (\phi'^2 + V(\phi))\) of the scalar field is conserved. It turns out that the converse is also true and it is only consistent with \(V(\phi)\) being constant.

**Lemma 2.12.** *The complete list of possible quadruples \((m, \alpha, f(t), \phi(t))\) satisfying the constant energy condition \(\frac{1}{2} (\phi'^2 + V(\phi)) = \rho\), with some constant \(\rho\), but with \((m, \alpha, f(t))\) not of constant curvature, consists of
\[
(m, KA^2, A, \pm \sqrt{2\rho/m(t - t_0)}) \quad (A > 0, \rho > 0).
\]

(81)
Proof. We can presume that \( \phi' \neq 0 \), since otherwise the spacetime is of constant curvature (lemma 2.11). The Friedmann equation (71) reduces to \( f'^2/f^2 + \alpha/f^2 = \kappa \rho \). Using the identity (52) and the acceleration equation (72), we conclude that \( f'/f = 0 \). Plugging this conclusion back into the Friedmann and acceleration equation, we find that each of \( K = \alpha/f^2 \), \( \frac{2}{\kappa} \Lambda = V(\phi) \) and \( \phi'^2 \) must be individually constant, with \( K \) interpreted as the spatial sectional curvature and \( \Lambda \) the cosmological constant. If we take \( \rho \) as an independent constant, the rest are given by \( K = \frac{2}{m(m-1)} \kappa \rho, \phi'^2 = \frac{(m-1)}{\kappa} K = \frac{2}{m} \rho \) and \( \Lambda = \frac{(m-1)}{m} \kappa \rho \). \( \square \)

Whenever the scalar potential \( V(\phi) \) is a constant, the Klein–Gordon equation is just the wave equation \( \nabla \nabla \phi = 0 \), which we also call the \textit{massless minimally-coupled Klein–Gordon equation}.

Lemma 2.13. The complete list of possible quadruples \((m, \alpha, f(i), \phi(t))\) with \( V(\phi) = \frac{2}{\kappa} \Lambda \) a constant, where the scalar field is not constant nor of constant energy, consists of

\[
\begin{align*}
\alpha = 0: & \quad (m, 0, f, \phi) \left( \frac{f''}{f} - \frac{f'^2}{f^2} - \frac{\alpha}{f^2} \right) + m \left( \frac{f'^2}{f^2} + \frac{\alpha}{f^2} \right) = 2 \frac{\Lambda}{(m-1)} , \\
\alpha \neq 0: & \quad (m, \alpha, f, \phi) \left( \frac{f''}{f} + \frac{\alpha}{f^2} \right) + \frac{\kappa \phi'^2 + 2 \Lambda}{m(m-1)} \Omega = \frac{\alpha}{f^2} , \\
& \quad \phi' = \pm \sqrt{\Omega} \frac{\alpha}{f^2} , \quad \Omega > 0
\end{align*}
\]

Proof. Recall from (76) that a constant potential \( V(\phi) = \frac{2}{\kappa} \Lambda \) implies the equation

\[
\left( \frac{f''}{f} - \frac{f'^2}{f^2} - \frac{\alpha}{f^2} \right) + m \left( \frac{f'^2}{f^2} + \frac{\alpha}{f^2} \right) = 2 \frac{\Lambda}{(m-1)} ,
\]

which is also supplemented by the Friedmann equation (71)

\[
\frac{f'^2}{f^2} + \frac{\alpha}{f^2} = \frac{\kappa \phi'^2 + 2 \Lambda}{m(m-1)}
\]

is clearly equivalent to the Einstein equations with a massless minimally-coupled scalar field stress energy tensor and, because of our hypothesis that \( \phi' \neq 0 \) and the comments below equation (73), which are equivalent to the full coupled Einstein–Klein–Gordon system. Setting \( \alpha = 0 \) completes the proof of the first part of the lemma.

The hypothesis of non-constant energy and lemma 2.12 imply that \( f'/f \neq 0 \). Thus, we obtain the following equivalent form of (82) after multiplying it by the integrating factor \( 2(f'/f)^{2m} \):

\[
f^{2m} \left( \frac{f'^2}{f^2} + \frac{\alpha}{f^2} - \frac{2}{m(m-1)} \Lambda \right) = \frac{\kappa}{m(m-1)} C ,
\]

for some constant \( C \). When \( \alpha \neq 0 \), we can normalize \( C \) by a power of \(|\alpha|\) to get

\[
\frac{f'^2}{f^2} + \frac{\alpha}{f^2} = \frac{2 \Lambda + \kappa \Omega^{1/2}}{m(m-1)} ,
\]

for some constant \( C \).
with another constant \( \Omega \). Provided that \( \Omega > 0 \), we can determine \( \phi(t) \) by the equation
\[
\frac{1}{f^m} (f^m \phi')' = \phi'' + m^2 f \phi' = 0.
\] (86)

With the above expression for \( \phi' \), plugging it into the Friedmann equation gives exactly equation (85). This observation completes the proof of the second part of the lemma.

Next, we will transform the Einstein–Klein–Gordon equations (71)–(73) under the hypothesis that \( \phi' \neq 0 \) everywhere. If we use the Friedmann equation to eliminate \( \phi' \) from the acceleration equation, while also multiplying the Klein–Gordon equation by \( \phi' \) and adding to it a multiple of the acceleration equation, they can be equivalently expressed as
\[
\frac{\kappa \phi'^2 + V(\phi)}{m(m-1)} - \frac{f'^2}{f^2} = \frac{\alpha}{f^2},
\] (87)
\[
\left(\frac{f'}{f}\right)' + \kappa \frac{\phi'^2}{m(m-1)} = \left(\frac{\kappa \phi'^2 + V(\phi)}{m(m-1)} - \frac{f'^2}{f^2}\right),
\] (88)
\[
\left(\kappa \frac{\phi'^2 + V(\phi)}{m(m-1)} - \frac{f'^2}{f^2}\right)' = -2f' \left(\kappa \frac{\phi'^2 + V(\phi)}{m(m-1)} - \frac{f'^2}{f^2}\right).
\] (89)

The equations (88) and (89) are second order, while (87) is first order. To see that there are no integrability conditions, note that differentiating the first order equation gives the identity
\[
\left[f^2 \left(\kappa \frac{\phi'^2 + V(\phi)}{m(m-1)} - \frac{f'^2}{f^2}\right)\right]' = f^2 \left[\left(\kappa \frac{\phi'^2 + V(\phi)}{m(m-1)} - \frac{f'^2}{f^2}\right)' + 2f' \left(\kappa \frac{\phi'^2 + V(\phi)}{m(m-1)} - \frac{f'^2}{f^2}\right)\right],
\] (90)
where the right-hand-side is clearly proportional to (89).

Since we are assuming that \( \phi' \neq 0 \), we can use \( \phi \) as the independent variable and convert all \( t \)-derivatives as \((-\cdot)' = \phi' \partial_{\phi} (\cdot)\). Denoting \( \pi = \phi' \) and \( \xi = f'/f \), we get the equations
\[
f^2 \left(\kappa \frac{\pi^2 + V(\phi)}{m(m-1)} - \xi^2\right) = \alpha,
\] (91)
\[
\pi \left[\partial_{\phi} \xi + \kappa \frac{\pi}{(m-1)}\right] = \left(\kappa \frac{\pi^2 + V(\phi)}{m(m-1)} - \xi^2\right),
\] (92)
\[
\partial_{\phi} \left(\kappa \frac{\pi^2 + V(\phi)}{m(m-1)} - \xi^2\right) = -2 \frac{\xi}{\pi} \left(\kappa \frac{\pi^2 + V(\phi)}{m(m-1)} - \xi^2\right),
\] (93)
where \( \xi, \pi \) and \( f \) are now all considered as functions of \( \phi \). With fixed \( V(\phi) \), the system (92) and (93) closes in the \( (\pi, \xi) \) variables, with the symmetry \( (\pi, \xi) \mapsto (-\pi, -\xi) \) corresponding to the coordinate transformation \( t \mapsto -t \), and can be solved for the highest derivatives \( \partial_{\phi} \xi \) and \( \partial_{\phi} \pi \) (always assuming that \( \pi \neq 0 \)). In the notation of (20), we can use the short-hand \( \Theta_V(\pi, \xi) = 0 \)
for this system. Hence the space of solutions $\xi = \Xi(\phi), \pi = \Pi(\phi)$, will be two-dimensional. We will always leave these parameters implicit in the choice of the solution $(\Pi(\phi), \Xi(\phi))$. With $(\Pi, \Xi)$ fixed, the equations $\phi' = \Pi(\phi), f'/f = \Xi(\phi)$ and $\alpha = f^2[\kappa \frac{(\Pi(\phi) + V(\phi))}{m(m-1)} - \Xi^2(\phi)]$ have a two-dimensional family of solutions, parametrized essentially by the transformations

$$ (\alpha, f(t), \phi(t)) \mapsto (A^2 \alpha, Af(t-t_0), \phi(t_0-t_0)), $$

which are the isometries preserving FLRW form (proposition 3.6). So the parameters determining $(\alpha, f, \phi)$ that are invariant under these transformations are essentially exhausted by the choice of $V(\phi)$ and the solution $(\Pi, \Xi)$. We summarize as follows.

**Lemma 2.14.** For any quadruple $(m, \alpha, f(t), \phi(t))$ for which $\alpha \neq 0$, $f'/f \neq 0$ and $(\nabla \phi)^2 < 0$, there is a unique smooth function $(\Pi, \Xi): J \rightarrow \mathbb{R}^2$, where $J = \phi(I)$ and

$$ \phi' = \Pi(\phi), \quad \frac{f'}{f} = \Xi(\phi), \quad \frac{\alpha}{f^2} = \frac{\Pi^2(\phi) + V(\phi)}{m(m-1)} - \Xi^2(\phi). $$

For each $u \in J$, these functions will also satisfy $\Pi(u) \neq 0, \Xi(u) \neq 0$ and $\frac{\Pi'(u) + V(u)}{m(m-1)} \neq \Xi^2(u)$, they will satisfy $\mathcal{S}_V(\Pi, \Xi) = 0$, in the notation of (20).

When $\alpha = 0$, the above discussion can be greatly simplified. The Einstein–Klein–Gordon system reduces to the following equivalent forms, using the same notation as above and always supposing that $\pi \neq 0$ everywhere:

$$ \begin{cases} \kappa \frac{\pi + V(\phi)}{m(m-1)} - \xi^2 = 0 \\ \pi \left[ \frac{\partial_\phi \xi + \kappa \pi}{m(m-1)} \right] = 0 \end{cases} \iff \left( \frac{(\partial_\phi \xi)^2}{\pi} = \frac{\kappa m(m-1) \xi^2 - \kappa V(\phi)}{m(m-1)} \right), \quad \pi = -\frac{(m-1)}{\kappa} \partial_\phi \xi $$

In a way, this simplification comes from eliminating $\pi = \phi'$ from the equations. In the notation of (18), we use the short-hand $\mathcal{S}_V(\xi) = 0$ for the equation satisfied by $\xi(\phi)$, which retains the symmetry $\xi \mapsto -\xi$. With $V(\phi)$ fixed, under the hypothesis $\partial_\phi \xi \neq 0$, this equation will have a one-dimensional family of solutions $\xi = \Xi(\phi)$. We will always leave the corresponding parameter implicit in the choice of the solution $\Xi(\phi)$. With $\Xi$ fixed, the equations $\phi' = -\frac{(m-1)}{\kappa} \partial_\phi \Xi(\phi), \quad f'/f = \Xi(\phi)$ have a two-dimensional family of solutions, again parametrized by the transformations (94). So the parameters determining $(f, \phi)$ that are invariant under these transformations are essentially exhausted by the choice of $V(\phi)$ and the solution $\Xi$. We summarize as follows.

**Lemma 2.15.** For any quadruple $(m, \alpha, f(t), \phi(t))$ for which $\alpha = 0$, $f'/f \neq 0$ and $(\nabla \phi)^2 < 0$, there is a unique smooth function $\Xi: J \rightarrow \mathbb{R}$, where $J = \phi(I)$ and

$$ \phi' = -\frac{(m-1)}{\kappa} \partial_\phi \Xi(\phi), \quad \frac{f'}{f} = \Xi(\phi). $$

For each $u \in J$, these functions will also satisfy $\Xi(u) \neq 0, \partial_\phi \Xi^2(u) \neq 0$ and it will satisfy $\mathcal{S}_V(\Xi) = 0$, in the notation of (18).

In the spatially flat ($\alpha = 0$) case, the equation $\mathcal{S}_V(\Xi) = 0$ is sometimes known as the *Hamilton–Jacobi equation* of single field inflation [24, 28]. The more general system $\mathcal{S}_V(\Pi, \Xi) = 0$ needed in the generic case ($\alpha \neq 0$) does not seem to have been considered before. In the cosmology literature, in the case of non-zero $\alpha$, an alternative system of equations has been used [30], though one less convenient for our purposes. There, a complex
scalar field $Z(\phi)$ is introduced, and plays the role of a ‘super-potential’ (in the sense of super-symmetry) for a ‘pseudo-Killing’ spinor. The isometry class of $(\alpha, f, \phi)$ determines the integrability conditions for $Z(\phi)$, an algebraic relation between $\phi$, $Z(\phi)$ and $Z'(\phi)$.

3. Geometric characterization

In this section, we leverage the information from section 2 to give necessary and sufficient conditions to belong to the local isometry class of a regular FLRW or inflationary spacetime, eventually proving our main theorems 1.4 and 1.5.

The resulting systems of conditions will be of the IDEAL type, as discussed in the Introduction, consisting of a list \{$T_a[g, \phi] = 0$, $a = 1, \ldots, N$, of tensor equations built covariantly out of a metric $g$, a scalar field $\phi$, and their derivatives. Each set of equations will consist of roughly three parts: for the GRW structure, for the FLRW structure, and for the specific isometry subclass.

3.1. Special cases

The two cases of FLRW spacetimes whose local isometry classes need to be characterized separately from the general pattern given in the sequel are the constant curvature spacetimes (lemmas 2.4 and 2.5) and Einstein static universes (lemma 2.6).

**Proposition 3.1.** Consider a Lorentzian manifold $(M, g)$, $\dim M = m + 1 \geq 2$.

(a) Given a fixed constant $K$, if $(M, g)$ everywhere satisfies

$$R_{\dot{k}\dot{h}} - K \frac{1}{2} (g \odot g)_{\dot{k}\dot{h}} = 0,$$

then it is locally isometric to any other spacetime satisfying the same condition. (b) Given a fixed constant $K$, if $m > 1$ and $(M, g)$ everywhere satisfies

$$W_{\dot{k}\dot{h}} = 0, \quad R^j [R_j^i - (m - 1)Kg_j^i] = 0,$$

$$\nabla_i R_{jk} = 0, \quad R - m(m - 1)K = 0,$$

while the 1-dimensional kernel of $R^j_i$ is timelike, it is locally isometric to an Einstein static universe with spatial sectional curvature $K$. The value $K = 0$ coincides with the flat case, $R_{\dot{i}\dot{j}h} = 0$.

**Proof.**

(a) This is standard; see for instance theorem 2.4.11 in [36].

(b) When $m = 1$, spatial slices are always flat, hence it is impossible to have $K \neq 0$ spatial sectional curvature. When $K = 0$, we are back in the flat case, characterized by $R_{\dot{k}\dot{h}} = 0$, a special case of part (a). This is why we take $m > 1$. Direct calculation (see 2.2) shows that the above equations hold when $(M, g)$ is an Einstein static universe with spatial sectional curvature $K \neq 0$.

Conversely, assume that we only know about $(M, g)$ that the above equations hold, with $K \neq 0$. The algebraic equations on the $R^j_i$ tangent space endomorphism guarantee that it is diagonalizable with precisely two distinct eigenvalues, 0 and $(m - 1)K$, with the kernel being
1-dimensional. Since $R_{ij}$ is symmetric, the kernel can only be either timelike or spacelike (not null)$^6$, with the hypotheses constraining it to be timelike. Since $R_{ij}$ is also covariantly constant, so is any unit vector $U^i$ in its kernel. That is, $R_{ij} \nabla X U^j = \nabla X (R_{ij} U^j) = 0$ for any $X$, which implies that $\nabla_X U^i = A_X U^i$ and $A_X = -U_j \nabla X U^j = -\frac{1}{2} \nabla_X (U^i U^j) = 0$. This gives us the desired $\nabla_X U^i = 0$ conclusion.

The existence of a covariantly constant unit vector $U^i$ implies that for any $x \in M$ and contractible open neighborhood $O \ni x$, the holonomy action of $(O, g|_{O})$ at $x$ leaves invariant the subspace spanned by $U^i$ at $x$ as well as its orthogonal complement (simply note that contraction with $U^i$ commutes with parallel transport). Under these conditions (proposition IV.5.2 in [19]), it is possible to locally factor $(O, g|_{O})$ into a direct product of a 1-dimensional and an $m$-dimensional pseudo-Riemannian manifold, $(I, -dr^2) \times (F, g^F)$, with $g^F$ of Riemannian signature.

Furthermore, the algebraic conditions on $W_{ijkh}$ and $R_{ij}$ imply that $W_{ijkh} = 0$ and $R_{ij} = (m-1)K g_{ij}$, which means that the spatial factor $(F, g^F)$ is locally of constant curvature with sectional curvature $K$. In other words, we can locally describe $(M, g)$ as an FLRW spacetime with $\alpha = K$ and $f(t) = 1$, which belongs precisely to the desired Einstein static universe class.

3.2. FLRW spacetimes

An FLRW spacetime (definition 2.2) is a GRW spacetime (definition 2.1) whose spatial slices have constant curvature (equation (25)). GRW spacetimes have been geometrically characterized in two different but related ways by the existence of a spatially conformal vector field $U$ by Sánchez [29] and of a concircular vector field $v$ by Chen [5]. Given Chen’s vector field $v$, the vector field $U = v/\sqrt{-v^2}$ satisfies the conditions of Sánchez. A recent survey of these and related geometric characterization results of GRW spacetimes can be found in [21].

Chen’s condition is somewhat simpler, but we will only be able to make use of it to characterize spatially curved, but not spatially flat FLRW spacetimes. In one case it will be possible to produce Chen’s vector field $v$ directly from the spacetime curvature, in the other not. Sánchez’s conditions work equally well also in the spatially flat case. So, motivated by providing the simplest set of equations when possible, we present both characterizations.

**Proposition 3.2 (Sánchez’s conditions).** Let $(M, g)$ be a Lorentzian manifold, $\dim M = m + 1 \geq 2$. It is locally GRW at $x \in M$ if and only if there exists, on a neighborhood of $x$, a unit timelike vector field $U$ that satisfies the conditions

\[ \Psi_{jk} := \frac{\nabla_j U_k \nabla_k U_j}{m} = 0, \quad (101) \]

\[ \mathcal{D}_{ij} := \nabla_i U_j - \frac{\nabla_k U^k}{m} (g_{ij} + U_i U_j) = 0. \quad (102) \]

**Proof.** In one direction, given an FLRW metric in the form (23), direct calculation shows that the above conditions are satisfied with $U^i = (\partial_t)^i$.

---

6. Suppose the 1-dimensional kernel $N$ of $R_i^j$ is null. From its invariant factors and the symmetry of $R_{ij}$, we have the following splittings of invariant subspaces, $N^+ = N \oplus S$ and $S^+ = N \oplus N'$, where $S$ is necessarily spacelike, meaning that $N'$ is 1-dimensional and has a non-zero eigenvalue. But, by the well-known Segre classification [31, section 5.1], on $S^+$, $R_i^j$ can either have only a single degenerate eigenvalue or no null eigenvectors.
In the other direction, Sánchez’ theorem 2.1 from [29] shows that locally \((M, g)\) can be put into the form (23), with \(U' = (\partial_t)'\). Sánchez’ original conditions look more complicated, but they follow from ours by easy algebraic manipulations. Sánchez’ hypotheses also include connectedness and simple connectedness. But, from the proof, these can all be dropped for the local result that we want.

We have based the above result on the characterization of GRW spacetimes that Sánchez obtained independently [29, theorem 2.1] in the process of a detailed investigation of the geometry of GRW spacetimes. However, this characterization (existence of a shear-free, \(\mathcal{D}_{(\phi)} = 0\), and twist-free, \(\mathcal{D}_{[\phi]} = 0\), vector field \(U\), with expansion \(\xi\) constant in directions orthogonal to \(U\), \(\mathcal{P}_{ij} = 0\)), at least when applied to FLRW spacetimes, has been known already as far back as [8, theorem 2.5.1, 9], and has been referenced for instance in [11, section III.B], [10, section 5.1]. Another independent source for these conditions seems to be the unpublished thesis [7], which has been referenced in at least [2, p.124].

**Proposition 3.3 (Chen’s conditions).** Consider a Lorentzian manifold \((M, g)\), \(\dim M = m + 1 \geq 2\). It is locally GRW at \(x \in M\) if and only if there exists, on a neighborhood of \(x\), a timelike vector field \(v\) and a scalar \(\mu\) that satisfy the condition

\[
\nabla_i v_j = \mu g_{ij}. \tag{103}
\]

A vector field satisfying (103) is called concircular.

**Proof.** In one direction, given GRW metric in the form (23), direct calculation shows that we can take \(v' = f(\partial_t)'\) and \(\mu = f'\).

Chen’s theorem 1 from [5] shows that locally \((M, g)\) can be put into the form (23), with \(v' = f(\partial_t)'\). Chen stated this result for \(m + 1 \geq 3\). However, the same proof also works when \(m + 1 = 2\). It is easiest to see by showing that the concircular condition (103) implies that \(U' = v'/\sqrt{-\sigma^2}\) satisfies Sánchez’ conditions, independently of the dimension. Let \(\phi = \sqrt{-\sigma^2}\), so that \(v' = \phi U'\). From the \(U'\nabla_i U_j = 0\) identity, the concircular condition decomposes into

\[
U_i \nabla_i \phi + \phi \nabla_i U_j = -\mu U_i U_j + \mu (g_{ij} + U_i U_j)
\]

\[
\iff \nabla_i \phi = -\mu U_i, \quad \nabla_i U_j = \frac{\mu}{\phi} (g_{ij} + U_i U_j). \tag{104}
\]

Then \(U_i U_j = 0\) implies \(U_i \nabla_j \phi = 0\), and \(\nabla_i \nabla_j \phi = 0\) implies \(U_i \nabla_j \mu = 0\). Finally, noting that \(\mu = U^i \nabla_i \phi = \frac{\phi}{\mu} \nabla^i U_i\) and eliminating both \(\phi\) and \(\mu\) gives us Sánchez’ conditions \(\mathcal{P}_{ij} = 0\) and \(\mathcal{D}_{[\phi]} = 0\).

The concircular condition can be rewritten slightly for our convenience.

**Lemma 3.4.** Let \(U\) be a vector field, \(v\) and \(\phi\) smooth functions, with \(\phi > 0\), and \(k\) a constant. Then the condition

\[
\nabla_i U_j + k \frac{\nabla_i \phi}{\phi} U_j = \nu g_{ij} \tag{105}
\]

implies that \(v = \phi^k U\) is a concircular vector field. In particular, \(U_i \nabla_j \phi = 0\).

**Proof.** The concircular condition with \(v = \phi^k U\) and \(\mu = \phi^k \nu\) is equivalent to \(\phi^{-k} \nabla_i (\phi^k U_j) = \phi^{-k} \mu g_{ij}\), which when expanded gives precisely equation (105). In GRW form (23), \(\phi^k U' = f(\partial_t)'\) and \(\nabla_i f = f' U_i\), from which follows the desired condition on \(\nabla_i \phi\).
Proposition 3.5. Consider a GRW spacetime \((M, g) \cong (I \times F, -dt^2 + f^2 \tilde{g}^F)\), \(\dim M = m + 1 \geq 2\). Set \(U^i = (\partial_i)\) and recall the notation of definition 2.3.

The \((F, gF)\) factor is locally of constant curvature if and only if the CCD tensor (see definition 2.3) vanishes and the spatial scalar curvature is constant,

\[ \mathcal{E}_{ijk} = 0 \quad \text{and} \quad U_i \nabla_j \zeta = 0. \]  

(106)

If in addition the spatial scalar curvature or equivalently the ZCD tensor (see definition 2.3) also vanishes, \(\zeta = 0\) or \(\mathfrak{Z}_{ijk} = 0\),

\[ \mathfrak{Z}_{ijk} = 0 \]  

(107)

then \((F, gF)\) is actually flat.

Proof. From equation (40), \(\mathcal{E}_{ijk} = 0\) is equivalent to

\[ R^F_{ijkh} = \frac{1}{m(m-1)} \frac{\mathcal{R}^F}{2} (f^F \otimes g^F)_{ijkh}. \]  

(108)

while \(U_i \nabla_j \zeta = 0\) and (39) imply that \(\mathcal{R}^F\) is a constant. Hence, \((F, gF)\) is of constant curvature. Furthermore, either of the conditions \(\zeta = 0\) or \(\mathfrak{Z}_{ijk} = 0\) implies that \(R^F_{ijkh} = 0\) and hence that \((F, gF)\) is flat.

\(\square\)

3.3. FLRW local isometry classes

Within the class of FLRW spacetimes, two metrics in the form (23) with different \((\alpha, f)\) parameters may or may not be isometric. Below, we give the results that allow us to classify FLRW metrics into isometry classes.

The obvious form-preserving transformations, time translation, reflection and rescaling, relate any FLRW metric to a 2-parameter family of (locally) isometric metrics. We state this result directly for FLRW spacetimes with scalar, which will come in useful later in section 3.4. As mentioned in the introduction, we can reduce to the case of no scalar field by setting the scalar field to zero.

Proposition 3.6. Consider two inflationary spacetimes \((M_i, g_i, \phi_i), i = 1, 2\), with corresponding spatial sectional curvature, warping function and scalar field triples \((\alpha_i, f_i, \phi_i), i = 1, 2\). If for every \(x \in M_1\) with \(t_1 = t(x)\) in the domain of \((f_1, \phi_1)\) there exists an open interval \((t_1 - \delta, t_1 + \delta)\) still in the domain of \((f_1, \phi_1)\), with \(\delta > 0\), and an interval \((t_2 - \delta, t_2 + \delta)\) in the domain of \((f_2, \phi_2)\) such that

\[
\begin{align*}
\alpha_1 &= A^2 \alpha_2, \\
\phi_1(t) &= \phi_2(st + t_0), \\
\end{align*}
\]

(109)

for some constants \(s \in \{+1, -1\}, A \neq 0\) and every \(t \in (t_1 - \delta, t_1 + \delta)\), then \((M_1, g_1, \phi_1)\) is locally isometric to \((M_2, g_2, \phi_2)\) at \(x \in M_1\).

Proof. The result follows from noting that an FLRW metric in standard form \(-dt^2 + f(t)^2 \tilde{g}^F\) is locally isometric to each of \(-dt^2 + f(-t)^2 \tilde{g}^F, -dt^2 + f(t + t_0)^2 \tilde{g}^F\) and to \(-dt^2 + (Af(t))^2 (\tilde{g}^F/A^2)\).
We will now show that, under certain conditions, two FLRW metrics with parameters \((\alpha_1, f_1)\) and \((\alpha_2, f_2)\) are locally isometric if and only if they belong to the same 2-parameter family as in proposition 3.6. To describe such a 2-parameter family of \((\alpha, f)\) intrinsically, we will look for a differential equation satisfied by every element of that family and only elements of that family. Heuristically, we should look for either a second order equation for \(f\) or a first order equation for \(f\) depending also on the parameter \(\alpha\), either of which will generically have a 2-parameter general solution.

The following helpful lemma follows easily from standard ODE existence and uniqueness theory [1].

**Lemma 3.7.** Consider a smooth real function \(G\) defined on an open interval \(J\), two nonzero real constants \(\alpha_1\) and \(\alpha_2\), and two nowhere vanishing smooth real functions \(f_1(t)\) and \(f_2(t)\) defined respectively on the open intervals \(I_1\) and \(I_2\).

(a) Suppose \(G > 0\) and that the pairs \((\alpha_1, f_1)\) and \((\alpha_2, f_2)\) both satisfy the differential equation

\[
(f'/f)^2 = G(\alpha/f^2)
\]

and that there exist \(t_1 \in I_1\) and \(t_2 \in I_2\) such that \(\frac{\alpha_1}{f_1(t_1)} = \frac{\alpha_2}{f_2(t_2)} \in J\). Then there exist constants \(s \in \{+1, -1\}, t_0, A \neq 0\) and \(\delta > 0\) such that \(t_2 = st_1 + t_0\) as well as

\[
\alpha_1 = A^2 \alpha_2 \quad \text{and} \quad f_1(t) = Af_2(st + t_0)
\]

for every \(t \in (t_1 - \delta, t_1 + \delta)\).

(b) Suppose that the functions \(f_1\) and \(f_2\) both satisfy the differential equation

\[
f''/f = G ((f'/f)^2)
\]

and that there exist \(t_1 \in I_1\) and \(t_2 \in I_2\) such that \(\frac{f_1(t_1)}{f_1(t_2)} = \frac{f_2(t_1)}{f_2(t_2)} \in J\). Then there exist constants \(s \in \{+1, -1\}, t_0, A \neq 0\) and \(\delta > 0\) such that \(t_2 = st_1 + t_0\) as well as

\[
f_1(t) = Af_2(st + t_0)
\]

for every \(t \in (t_1 - \delta, t_1 + \delta)\).

We are finally in a position to define and classify all regular FLRW spacetimes into families and to describe the parameters needed to identify an isometry class within each family.

**Lemma 3.8.** Two regular FLRW spacetimes (those belonging to one of the families identified in definition 1.2) are isometric to each other (definition 1.1) if and only if they belong to the same parametrized family and the corresponding parameters are identical.

**Proof.** Let us fix \(m\), noting that two isometric spacetimes must have the same dimension. To show that two spacetimes cannot be isometric, it is sufficient to point out an identity or inequality that is satisfied by curvature scalars or tensors on one spacetime but not on the other. With that in mind, recall (in the notation of theorem 1.4) that for FLRW spacetimes, \(\xi = f'/f\), \(\eta = f''/f - f'^2/f^2\) and \(\zeta = \alpha/f^2\), which are all curvature scalars as long as they are defined with respect to a vector field \(U\) that is also defined from pure, such as the choices \(U = U_R\) or \(U_B\). To show that all the representatives of a family with identical parameters are all isometric to each other, there will be two possibilities to consider. Either the representative is unique,
which is the trivial case. Or, all representatives are selected by satisfying a differential equation. By invoking lemma 3.7, we can be sure that two solutions to such an equation (with all parameters fixed), if they can be matched up at at least one point, are in fact locally isometric around that point. If the domains of these solutions can also be matched up, then it is clear that they are also globally isometric.

(a) For each $K$, there is a unique representative in $CC_{K}^{m}$. The scalar curvature $R = m(m + 1)K$ distinguishes the different values of $K$.

(b) Again, for each $K \neq 0$, there is a unique representative in $ES_{K}^{m}$. The scalar curvature $R = m(m - 1)K$ distinguishes the different values of $K$. Comparing the formulas from section 2.2 and proposition 3.1(b), the structure of the Ricci tensor $R_{ij}$ distinguish $ES_{K}^{m}$ from any spacetime of constant curvature.

(c) The representatives of the class $CSC_{K,\Omega,J}^{m}$ satisfy an equation like in lemma 3.7(b). The scalar curvature $R = m(m + 1)K$ distinguishes the different values of $K$, and setting $U = U_{B}$ the range $J = \xi^{2}(I)$ distinguishes the different intervals $\xi$. Also, from lemma 2.7, $(\nabla B)^{2} < 0$ distinguishes these spacetimes from those of parts (a) and (b), where $B' = 0$.

(d) The representatives of the class $CSC_{E,\Omega,J}^{m}$ satisfy an equation like in lemma 3.7(a). The scalar curvature $R = m(m + 1)K$ distinguishes the different values of $K$, and setting $U = U_{B}$, the constant $\kappa \Omega = (\xi^{2} + \zeta - K)/|\zeta|^{\frac{2}{3}}$ (lemma 2.7) and range $J = \zeta(I)$ distinguishes the different values of $\Omega$ and $J$. Again, $(\nabla B)^{2} < 0$ distinguishes these spacetimes from those of parts (a) and (b), while $\zeta \neq 0$ distinguishes them from those of part (c) where $\zeta = 0$.

(e) The representatives of the class $FLRW_{E,S}^{m,0}$ satisfy an equation like in lemma 3.7(b). Setting $U = U_{R}$, the identity $\eta + \frac{\omega}{n} \xi^{2} = -\kappa P(\xi^{2})$ and the range $J = \xi^{2}(I)$ distinguish different values of the $P$ and $J$ parameters. Also, combining the constraints on $P$ and lemma 2.8, $(\nabla R)^{2} < 0$ distinguishes these spacetimes from those of parts (a), (b), (c) and (d), where $R' = 0$.

(f) The representatives of the class $FLRW_{E,S}^{m}$ satisfy an equation like in lemma 3.7(b). Setting $U = U_{R}$, the identity $\xi^{2} + \zeta = \kappa E(\zeta)$ and the range $J = \zeta(I)$ distinguish different values of the $E$ and $J$ parameters. Again, combining the constraints on $E$ and lemma 2.9, $(\nabla R)^{2} < 0$ distinguishes these spacetimes from those of parts (a), (b), (c) and (d), while $\zeta \neq 0$ distinguishes them from those of part (e), where $\zeta = 0$.

We are now finally in a position to prove our main result about IDEAL characterizations of regular FLRW spacetimes.

Proof of theorem 1.4. The goal is to prove that, for each of the cases listed in table 1, a spacetime satisfies the listed equations (and inequalities) if and only if it is locally isometric (definition 1.1) to one of the regular FLRW spacetimes listed in definition 1.2. In one direction (a regular FLRW spacetime satisfies the corresponding conditions), this is essentially the content of lemma 3.8. It remains to show the converse.

(a) The constant curvature case is standard (proposition 3.1(a)).

(b) We have already proven the desired conclusion in the Einstein static universe case in proposition 3.1(b).

(c) and (e) With the appropriate definition of the unit timelike vector field $U$, according to proposition 3.2, the equations $\Psi_{\alpha} = 0$ and $\Omega_{\alpha} = 0$ are sufficient to locally put the spacetime in GRW form (23), while according to proposition 3.5 the equation $3_{\alpha \mu \beta} = 0$ implies that the spatial slices are flat and hence the spacetime is locally FLRW. The remaining
conditions place the spacetime in the unique corresponding local regular FLRW isometry class, as per lemmas 3.8(c) and (e).

(d) and (f) With the appropriate definition of the unit timelike vector field $U$, according to proposition 3.3 and lemma 3.4, the equation $\nabla_i U_j - \frac{\kappa}{m} U_j - \xi g_{ij} = 0$ is sufficient to locally put the spacetime in GRW form (23) and show that $\zeta$ is constant along the spatial slices, while according to proposition 3.5 the additional equation $\xi_{ijkh} = 0$ implies that the spatial slices are of constant curvature and hence the spacetime is locally FLRW. The remaining conditions place the spacetime in the unique corresponding local regular FLRW isometry class, as per lemma 3.8(c) and (e).

3.4. Inflationary local isometry classes

Within the class of inflationary spacetimes $(M, g, \phi)$, two spacetimes in the form (23) and with $\phi = \phi(t)$, with different $(\alpha, f, \phi)$ parameters may or may not be isometric. Below, we give the results that allow us to classify inflationary spacetimes into isometry classes (definition 1.1).

Recall that proposition 3.6 gives a sufficient condition for local isometry. We will now show that, under certain conditions, two inflationary spacetimes with parameters $(\alpha_i, f_i, \phi_i)$, $i = 1, 2$, are locally isometric if and only if they belong to the same 2-parameter family as in proposition 3.6. As in section 3.3, we will look for an ODE system, jointly satisfied by any $\alpha_i$, $f_i$, $\phi_i$, according to proposition 3.3 and lemma 3.4, the equation $\xi_{ijkh} = 0$ implies that the spatial slices are of constant curvature and hence the spacetime is locally FLRW. The remaining conditions place the spacetime in the unique corresponding local regular FLRW isometry class, as per lemma 3.8(c) and (e).

Lemma 3.9. Consider a smooth real function $V: J \to \mathbb{R}$ defined on an open interval, two non-zero real constants $\alpha_i$, $i = 1, 2$, and two pairs of smooth real functions $(f_i, \phi_i)$ defined on intervals $I_i$, $i = 1, 2$, with either $f_i$ nowhere vanishing.

(a) Suppose that $\Pi, \Xi: J \to \mathbb{R}$ are smooth real functions that satisfy the $\mathcal{G}_V(\Pi, \Xi) = 0$, in the notation of (20). Suppose also that the triples $(\alpha_i, f_i, \phi_i)$, $i = 1, 2$, both satisfy the system of differential equations

\[
\begin{align*}
\phi' &= \Pi(\phi), \\
\xi' &= \Xi(\phi), \\
\phi &= \kappa \frac{\Pi(\phi) + V(\phi)}{m(m-1)} - \Xi(\phi),
\end{align*}
\] (114)

and that there exist $t_i \in I_i$, $i = 1, 2$, such that $\phi(t_1) = \phi(t_2) \in J$ and $\frac{\alpha_1}{\alpha_2} = \frac{\phi(t_1)}{\phi(t_2)}$. Then there exist constants $t_0$, $A \neq 0$ and $\delta > 0$ such that

\[
\alpha_1 = A^2 \alpha_2, \quad f_1(t) = Af_2(t + t_0) \quad \text{and} \quad \phi_1(t) = \phi_2(t + t_0)
\] (115)

for every $t \in (t_1 - \delta, t_1 + \delta)$.

(b) Suppose that $\Xi: J \to \mathbb{R}$ is a smooth real function. Suppose also that the pairs $(f_i, \phi_i)$, $i = 1, 2$, both satisfy the system of differential equations

\[
\begin{align*}
\phi' &= -\frac{(m-1)}{\kappa} \partial_{\phi} \Xi(\phi), \\
\xi' &= \Xi(\phi),
\end{align*}
\] (116)

and that there exist $t_i \in I_i$, $i = 1, 2$, such that $\phi(t_1) = \phi(t_2) \in J$. Then there exist constants $t_0$, $A \neq 0$ and $\delta > 0$ such that

\[\text{32}\]
\[ f_1(t) = A f_2(t + t_0) \quad \text{and} \quad \phi_1(t) = \phi_2(t + t_0) \]  
(117)

for every \( t \in (t_1 - \delta, t_1 + \delta) \).

We are finally in a position to define and classify all regular inflationary spacetimes into families and to describe the parameters needed to identify an isometry class within each family.

**Lemma 3.10.** Two regular inflationary spacetimes (those belonging to one of the families identified in definition 1.3) are isometric to each other (definition 1.1) if and only if they belong to the same parametrized family and the corresponding parameters are identical.

The following proofs are very much analogous to the proofs of lemma 3.8 and theorem 1.4, but we will write them in a mostly self-contained way.

**Proof of lemma 3.10.** Let us fix \( m \), noting that two isometric spacetimes must have the same dimension. To show that two spacetimes with scalar cannot be isometric, it is sufficient to point out an identity or inequality that is satisfied by curvature scalars or tensors, possibly together also with scalars or tensors covariantly obtained from the scalar field, on one spacetime but not on the other. With that in mind, recall (in the notation of theorems 1.4 and 1.5), that for inflationary spacetimes \( \xi = f'/f, \eta = f''/f - f'/f^2 \) and \( \zeta = \alpha/f^2 \), which are all curvature scalars, as long as they are defined with respect to a vector field \( U \) that is also defined from either pure curvature or from the scalar field, such as the choices \( U = U_\xi, U_\eta \) or \( U_\phi \). To show that all the representatives of a family with identical parameters are all isometric to each other, there will be two possibilities to consider. Either the representative is unique, which is the trivial case. Or, all representatives are selected by satisfying a differential equation. By invoking lemmas 3.9 or 3.7, we can be sure that two solutions to such an equation (with all parameters fixed), if they can be matched up at at least one point, are in fact locally isometric around that point. If the domains of these solutions can also be matched up, then it is clear that they are also globally isometric.

(a) For each \( \Lambda \) (hence \( K = \frac{2}{m(m-1)} \Lambda \)) and \( \Phi \), there is a unique representative in \( \text{ESU}^\alpha \text{CES}_\Phi \).

The scalar curvature \( R = m(m + 1)K \) and the scalar field \( \phi = \Phi \) distinguish the different values of these parameters.

(b) For each \( \rho > 0 \) (hence \( K = \frac{2}{m(m-1)} \kappa \rho \)) and interval \( J \subseteq \mathbb{R} \), there is a unique representative in \( \text{ESU}^\alpha \text{CES}_\rho J \). The scalar curvature \( R = m(m - 1)K \) and the range \( J = \phi(I) \) distinguish different values \( \rho \) and \( J \). The condition \((\nabla \phi)^2 < 0\) distinguishes these spacetimes from those in part (a), where \( \phi' = 0 \).

(c) The representatives of \( \text{MMS}^\alpha_{\Lambda, J'} \) satisfy the equations \( f''/f + (m - 1)f'/f^2 = \frac{2\Lambda}{m(m-1)} \) which is like in lemma 3.7(a), and

\[ \phi' = -\sqrt{\frac{1}{\kappa} \left( \frac{f'^2}{f^2} - \frac{2\Lambda}{m(m-1)} \right)} \],

(118)

since by hypothesis \( \phi' < 0 \). Thus, the first equation shows that the underlying Lorentzian spacetimes are isometric for identical \( \Lambda \) and \( J' \). The second equation shows, by applying once again standard ODE existence and uniqueness theory, that the inflationary spacetimes are also isometric (as spacetimes with scalar) for identical \( J \). With the choice \( U = U_\rho \), the curvature scalars \( \eta + m\zeta^2 = \frac{2\Lambda}{m(m-1)} \) and the ranges of \( J = \phi(I) \), \( J' = \xi(I) \) distinguishes different \( \Lambda \), \( J \) and \( J' \). The implication that \( \xi = f'/f \neq 0 \) and \( \phi' \neq 0 \) distinguishes these spacetimes from those of parts (a) and (b).
(d) The representatives of the class $\text{MMS}_{3,\omega}$ satisfy an equation like in lemma 3.9(a), namely

$$\phi' = -\sqrt{\frac{\alpha}{m}} \frac{\phi^2}{f}, \quad f' = \pm \sqrt{\frac{2\Lambda + \kappa\alpha/\alpha^{2^m}}{m(m-1)}} - \frac{\alpha}{f^2},$$

(119)

where the $\pm$ sign is determined by whether $0 < J$ or $J < 0$. With the choice $U = U_\phi$, the curvature scalars $\eta + m\xi^2 = \frac{2\Lambda}{(m-1)^2} - \frac{\kappa\alpha/\alpha^{2^m}}{m(m-1)}$ and the range $J = \phi(I)$ distinguish different $\Lambda$, $\Omega$ and $J$. The implication that $\xi = f'/f \neq 0$ and $\phi' \neq 0$ distinguishes these spacetimes from those of parts (a) and (b), while $\zeta = \frac{\Pi(\phi)}{m(m-1)} - \Xi(\phi) \neq 0$ distinguishes them from those of part (c), where $\zeta = 0$.

(e) The representatives of the class $\text{NKG}_{3,\omega}$ satisfy an equation like in lemma 3.9(b). With the choice $U = U_\phi$, the identities $\phi' = \frac{\alpha}{m-1} \partial_\alpha \Xi(\phi), \xi = \Xi(\phi)$ and the range $J = \phi(I)$ distinguish different $\Xi$ and $J$. It is important to note that for any solution of $\mathcal{S}_V(\Xi)$, $-\Xi$ is also a solution that defines another spacetime isometric to a given one via $t \mapsto -(t - t_0)$ for some $t_0$. We have broken this degeneracy by the $\xi \partial_\alpha \Xi(u) > 0$ requirement (due to using $U_\phi$ and not $-U_\phi$), so distinct $\Xi$ imply non-isometric spacetimes. The identity $\eta + m\xi^2 = \frac{\kappa V(\phi)}{m(m-1)}$, with non-constant $V(\phi)$, distinguishes these spacetimes from those in parts (a), (b), (c) and (d), where the left-hand-side would have been constant.

(f) The representatives of the class $\text{NKG}_{0,\omega}$ satisfy an equation like in lemma 3.9(a). With the choice $U = U_\phi$, the identities $\phi' = \Pi(\phi), \xi = \Xi(\phi)$ and range $J = \phi(I)$ distinguish different $\Xi, \eta$ and $J$. It is important to note that for any solution of $\mathcal{S}_V(\Xi)$, $-\Xi$ is also a solution that defines another spacetime isometric to a given one via $t \mapsto -(t - t_0)$ for some $t_0$. We have broken this degeneracy by the $\Pi < 0$ requirement (due to using $U_\phi$ and not $-U_\phi$), so distinct $\Xi$ imply non-isometric spacetimes. The identity $\eta + m\xi^2 = \frac{\kappa V(\phi)}{m(m-1)}$, with non-constant $V(\phi)$, distinguishes these spacetimes from those in parts (a), (b), (c), and (d), where the left-hand-side would have been constant, while $\xi \neq 0$ distinguishes them from those of part (e), where $\zeta = 0$.

We are now finally in a position to prove our main result about IDEAL characterizations of regular inflationary spacetimes.

**Proof of theorem 1.5.** The goal is to prove that, for each of the cases listed in table 2, a spacetime satisfies the listed equations (and inequalities) if and only if it is locally isometric (definition 1.1) to one of the regular inflationary spacetimes listed in definition 1.3. In one direction (a regular inflationary spacetime satisfies the corresponding condition), this is essentially the content of lemma 3.10. It remains to show the converse.

(a) When $\phi = \Phi$ is a constant, so is $V(\phi) = \frac{1}{2} \Lambda$, which we have parametrized for our convenience with $\Lambda$. Then the Einstein–Klein–Gordon equations become the cosmological vacuum equations $R_0 = \frac{1}{2} R g_{ij} + \Lambda g_{ij} = 0$, which under the FLRW hypotheses have only the constant curvature solution.

(b) The existence of a timelike covariantly constant vector $U = U_\phi, \nabla_i U_j = 0$, implies that the spacetime decomposes into a direct sum, with one of the factors being of constant curvature, since the CCD tensor $\mathcal{C}_{ijk} = 0$ (see definition 2.3) vanishes and the spatial scalar curvature $\zeta = \frac{2\kappa}{m(m-1)} + \rho$ is constant (proposition 3.5); see the proof of proposition
3.1(b) for details. The conclusion, as desired, is that the spacetime is an Einstein static universe and the equation $\phi' = -\sqrt{2\rho/m}$ means that we can choose the time coordinate to put $\phi(t)$ precisely into the form in lemma 2.12.

(c) and (d) With the vector field $U = U_\phi$, according to proposition 3.3 and lemma 3.4, the equation $\nabla_i U_j - \sum \phi'_m U_j - \xi g_{ij} = 0$ is sufficient to locally put the spacetime into GRW form (23) and show that $\phi'$ is constant along spatial slices. In case (c), the equation $\phi' = -\sqrt{\Omega} |\zeta|^2$ shows that $\zeta$ is also constant on spatial slices, and together with the vanishing of the CCD tensor $C_{ijkh} = 0$ this implies that the spatial slices are of constant curvature. In both cases we have referred to proposition 3.5, and in both cases we have established that the spacetime is locally FLRW. Now, recalling the identities $\xi = f'/f$, $\eta = f''/f - f'^2/f^2$ and $\zeta = \alpha/f^2$, the remaining conditions in each case clearly show that the spacetime is locally isometric to the corresponding reference class in definitions 1.3(c) or (d).

(e) and (f) With the vector field $U = U_\phi$, according to proposition 3.2, the equations $\mathcal{P}_{ij} = 0$ and $\mathcal{D}_{ij} = 0$ are sufficient to locally put the spacetime into GRW form (23). In case (e), the vanishing of the ZCD tensor $Z_{ijkh} = 0$ implies that the spatial slices are flat. In case (f), the equations $\phi' = \Pi(\phi)$, $\xi = \Xi(\phi)$ show that $\zeta = \kappa \phi'^2 + V(\phi)/m(m-1)$ is then constant along spatial slices (slices of constant $\phi$), and together with the vanishing of the CCD tensor $C_{ijkh} = 0$ this implies that the spatial slices are of constant curvature. In both cases we have referred to proposition 3.5, and in both cases we have established that the spacetime is locally FLRW. Now, recalling the identities $\xi = f'/f$, $\eta = f''/f - f'^2/f^2$ and $\zeta = \alpha/f^2$, the remaining conditions in each case clearly show that the spacetime is locally isometric to the corresponding reference class in definitions 1.3(e) or (f).

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ORCID IDs

Igor Khavkine ✔️ https://orcid.org/0000-0003-4255-6579

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