Chern-Ricci invariance along $G$-geodesics

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Abstract

Over a compact oriented manifold, the space of Riemannian metrics and normalized positive volume forms admits a natural pseudo-Riemannian metric $G$, which is useful for the study of Perelman’s $W$ functional. We show that if the initial speed of a $G$-geodesic is $G$-orthogonal to the tangent space to the orbit of the initial point, under the action of the diffeomorphism group, then this property is preserved along all points of the $G$-geodesic. We show also that this property implies preservation of the Chern-Ricci form along such $G$-geodesics, under the extra assumption of complex anti-invariant initial metric variation and vanishing of the Nijenhuis tensor along the $G$-geodesic. More in general we show that the main obstruction to the invariance of the Chern-Ricci form is the vanishing of the Nijenhuis tensor. This result is useful for a slice type theorem needed for the proof of the dynamical stability of the Soliton-Kähler-Ricci flow.

1 Statement of the invariance result

We consider the space $\mathcal{M}$ of smooth Riemannian metrics over a compact oriented manifold $X$ of dimension $m$. We denote by $\mathcal{V}_1$ the space of positive smooth volume forms with integral one. Notice that the tangent space of $\mathcal{M} \times \mathcal{V}_1$ is

$$
T_{\mathcal{M} \times \mathcal{V}_1} = C^\infty(X, S^2T^*_X) \oplus C^\infty(X, \Lambda^m T^*_X)_{\Omega},
$$

where $C^\infty(X, \Lambda^m T^*_X)_{\Omega} := \{ V \in C^\infty(X, \Lambda^m T^*_X) \mid \int_X V = 0 \}$. We denote by $\text{End}_g(T_X)$ the bundle of $g$-symmetric endomorphisms of $T_X$ and by $C^\infty_0(X, \mathbb{R})$ the space of smooth functions with zero integral with respect to $\Omega$. We will use the fact that for any $(g, \Omega) \in \mathcal{M} \times \mathcal{V}_1$ the tangent space $T_{\mathcal{M} \times \mathcal{V}_1, (g, \Omega)}$ identifies with $C^\infty(X, \text{End}_g(T_X)) \oplus C^\infty_0(X, \mathbb{R})_{\Omega}$ via the isomorphism

$$(v, V) \mapsto (v_g^*, V_{\Omega}) := (g^{-1}v, V_{\Omega}).$$

In [Pal6], we consider the pseudo-Riemannian metric $G$ over $\mathcal{M} \times \mathcal{V}_1$, defined over any point $(g, \Omega) \in \mathcal{M} \times \mathcal{V}_1$ by the formula

$$
G_{g, \Omega}(u, U; v, V) = \int_X \left[ \langle u, v \rangle_g - 2U^\flat V_{\Omega} \right] \Omega,
$$

Key words: Almost complex manifolds, Chern-Ricci form, Bakry-Emery-Ricci tensor.

AMS Classification: 32Q60, 32Q15.
for all \((u, U), (v, V) \in T_{M \times V}\). The gradient flow of Perelman’s \(W\)-functional \[\text{Per}\] with respect to the structure \(G\) is a modification of the Ricci flow with relevant properties (see \[\text{Pal6, Pal7}\]). The \(G\)-geodesics exists only for short time intervals \((-\varepsilon, \varepsilon)\). This is because the \(G\)-geodesics are uniquely determined by the evolution of the volume forms and the latter degenerate in finite time (see section \[2\]). In \[\text{Pal6}\], we show that the space \(G\)-orthogonal to the tangent of the orbit of a point \((g, \Omega) \in M \times V\), under the action of the identity component of the diffeomorphism group is

\[
\mathbb{F}_{g, \Omega} := \{(v, V) \in T_{M \times V} | \nabla^* g_{\omega} v + \nabla g V = 0\},
\]

where \(\nabla^* g_{\omega}\) denotes the adjoint of the Levi-Civita connection with respect to the volume form \(\Omega\). In this paper we show the following conservative property.

**Proposition 1** Let \((g_t, \Omega_t)_{t \in (-\varepsilon, \varepsilon)} \subset M \times V\) be a \(G\)-geodesic such that \((g_0, \Omega_0) \in \mathbb{F}_{g_0, \Omega_0}\). Then \((g_t, \Omega_t) \in \mathbb{F}_{g_t, \Omega_t}\) for all \(t \in (-\varepsilon, \varepsilon)\).

We consider now a compact symplectic manifold \((X, \omega)\) and we denote by \(\mathcal{J}_\omega\) the space of smooth almost complex structures compatible with the symplectic form \(\omega\). We notice that the variations inside the space of metrics \(\mathcal{M}_\omega := -\omega \cdot \mathcal{J}_\omega \subset M\), at a point \(g = -\omega J\), are \(J\)-anti-invariant. Thus, in this set-up, it is natural to consider the sub-space

\[
\mathbb{F}^J_{g, \Omega} := \{(v, V) \in \mathbb{F}_{g, \Omega} | v = -J^* v J\}.
\]

With these notations we state the following result.

**Theorem 1** (Main result. The invariance of the Chern-Ricci form). Let \((X, J_0, g_0)\) be a compact almost-Kähler manifold with symplectic form \(\omega := g_0 J_0\). Then for any \(G\)-geodesic \((g_t, \Omega_t)_{t \in (-\varepsilon, \varepsilon)} \subset M \times V\), with initial speed \((g_0, \Omega_0) \in \mathbb{F}^{J_0}_{g_0, \Omega_0}\) holds the properties \(J_t := -\omega^{-1} g_t \in \mathcal{J}_\omega\), \((g_t, \Omega_t) \in \mathbb{F}^{J_t}_{g_t, \Omega_t}\) and the variation formulas

\[
\frac{d}{dt} \text{Ric}_{J_t}(\Omega_t) = -d \text{Tr}_{g_t} [\omega(\bullet \cdot N_{g_t}) g_t^*],
\]

\[
2 \frac{d}{dt} \text{Ric}_{J_t}(\Omega_t) = d \text{Tr}_{g_t} [\omega(\bullet \cdot \overline{\partial f_{X,J_t}} g_t^*)].
\]

In particular if the Nijenhuis tensor vanishes identically along the \(G\)-geodesic, then \(\text{Ric}_{J_t}(\Omega_t) = \text{Ric}_{J_0}(\Omega_0)\), for all \(t \in (-\varepsilon, \varepsilon)\).

Our unique interest in this result concerns the Fano case \(\omega = \text{Ric}_{J_{\omega}}(\Omega_0)\). In this case the space of \(\omega\)-compatible complex (integrable) structures \(\mathcal{J}_\omega\) embeds naturally inside \(M \times V\) via the Chern-Ricci form. (This is possible thanks to the \(\overline{\partial \partial}\)-lemma). The image of this embedding is

\[
S_{\omega} := \{(g, \Omega) \in \mathcal{M}_\omega \times V | \omega = \text{Ric}_J(\Omega), J = -\omega^{-1} g\},
\]
with $\mathcal{M}_\omega := -\omega J_\omega \subset \mathcal{M}$. It is well-known that the $J$-anti-linear endomorphism sections associated to the metric-variations in $\mathcal{M}_\omega$ at a point $g = -\omega J$, are $\overline{\partial} T_{X,J}$-closed. Thus, in the integrable set-up, it is natural to consider the sub-space

$$E^J_{g,\Omega}[0] := \left\{ (v, V) \in E^J_{g,\Omega} \mid \overline{\partial} T_{X,J} v = 0 \right\}.$$ 

It has been showed in [Pal6] that this is the space $G$-orthogonal to the tangent to the orbit of the point $(g, \Omega) \in S_\omega$, under the action of the identity component of the $\omega$-symplectomorphisms group. (See the identity 1.14 in [Pal6]). Furthermore the product $G_{g,\Omega}$ is positive over $E^J_{g,\Omega[0]}$, thanks to a result in [Pal6]. We conjecture the following slice type result.

**Conjecture 1** Let $(X, J)$ be a Fano manifold and let $\omega \in 2\pi c_1 (X)$ be a Kähler form. Then the distribution $(g, \Omega) \in S_\omega \mapsto E^J_{g,\Omega[0]}$, with $J := -\omega^{-1} g$ is integrable over the space $S_\omega$, with leave at the point $(g, \Omega)$ given locally, in a neighborhood of this point, by $\Sigma^\omega_{g,\Omega} := \{ (\gamma, \mu) \in \text{Exp}_G (E^J_{g,\Omega}) \mid \nabla_{\gamma} \omega = 0 \}$. We invite the readers to compare with [F-S] for other approaches concerning slice type problems in the space of compatible complex structures. In view of the results in section 9 of [Pal6], the solution of this conjecture is crucial for the proof of the dynamical stability of the Soliton-Kähler-Ricci flow [Pal9]. An important ingredient for the proof of the main theorem is the general variation formula [11]. Particular cases of this variation formula have been intensively studied. See [Fu, Do, Mo, Ga, Pal5]. These formulas allow to establish an important moment map picture in Kähler-geometry. See [Fu] for the integrable case and [Do] for the almost complex case. In the last section we provide a formula relating the Bakry-Emery-Ricci tensor with the Chern-Ricci form.

## 2 Pure evolving volume nature of the $G$-geodesic equation

We remind that the equation of a $G$-geodesic $(g_t, \Omega_t)_{t \in (-\varepsilon, \varepsilon)}$, (see [Pal6]), rewrites under the form

$$\left\{ \begin{array}{l}
\frac{d}{dt} \hat{g}_t^* + \hat{\Omega}_t^* \hat{g}_t^* = 0, \\
\hat{\Omega}_t + \frac{1}{4} \left( |\hat{g}_t|_{g_t}^2 - 2(\hat{\Omega}_t^*)^2 - \int_X \left[ |\hat{g}_t|_{g_t}^2 - 2(\hat{\Omega}_t^*)^2 \right] \Omega_t \right) \Omega_t = 0. 
\end{array} \right.$$ 

The invariance of the scalar product of the speed of geodesics implies

$$G_t := G_{g_t, \Omega_t} (\dot{g}_t, \dot{\Omega}_t; g_t, \Omega_t) \equiv G_{g_0, \Omega_0} (\dot{g}_0, \dot{\Omega}_0; g_0, \Omega_0).$$ 

Therefore a solution of the system (S) satisfies also

$$\left\{ \begin{array}{l}
\frac{d}{dt} \hat{g}_t^* + \hat{\Omega}_t^* \hat{g}_t^* = 0, \\
\hat{\Omega}_t + \frac{1}{4} \left[ |\hat{g}_t|_{g_t}^2 - 2(\hat{\Omega}_t^*)^2 - G_0 \right] \Omega_t = 0. 
\end{array} \right.$$
The first equation in the system \((S_1)\) rewrites as
\[
\dot{g}^* = \frac{\Omega_0}{\Omega_t} \dot{g}_0^*,
\]
which provides the expression
\[
g_t = g_0 \exp \left( \dot{g}_0^* \int_0^t \frac{\Omega_0}{\Omega_s} ds \right).
\]
We set \(u_t := \Omega_t / \Omega_0\), and we observe the trivial identities
\[
|\dot{g}_t|^2 = \text{Tr}_R (\dot{g}_t^*)^2 = u_t^{-2} |\dot{g}_0|^2,
\]
\[
\dot{\Omega}_t^* = \dot{u}_t / u_t.
\]
We deduce that the system \((S_1)\) is equivalent to the system
\[
\begin{cases}
  g_t = g_0 \exp \left( \dot{g}_0^* \int_0^t u_s^{-1} ds \right), \\
  \Omega_t = u_t \Omega_0, \\
  4\ddot{u}_t + \frac{|\dot{g}_0|^2}{u_s} - 2a_t^2 - u_t G_0 = 0, \\
  u_0 = 1, \\
  \int_X \dot{u}_0 \Omega_0 = 0.
\end{cases}
\]
The solution \(u\) is given by the explicit formula
\[
u_t = 1 + \hat{u}_0 \sum_{k \geq 0} \frac{(G_0/2)^k}{(2k + 1)!} t^{2k + 1} - \frac{1}{4} N_0 \sum_{k \geq 1} \frac{(G_0/2)^{k-1}}{(2k)!} t^{2k},
\]
\[
N_0 := N_0 - G_0,
\]
\[
N_0 := |\dot{g}_0|^2 \gamma_0 - 2(\dot{\Omega}_0^*)^2.
\]
Thus the solution \((g_t, \Omega_t)_{t \in (-\infty, \epsilon)}\) of the system \((S_1)\) satisfies \(\int_X \Omega_t \equiv 1\). This implies \(G_t \equiv G_0\). We infer that the system \((S_1)\) is equivalent to the system \((S)\).

In the case \(G_0 > 0\), the previous formula for \(u_t\) reduces to the expression
\[
u_t = 1 + \Omega_0^* \gamma_0^{-1} \sinh (\gamma_0 t) - N_0 (2\gamma_0)^{-2} [\cosh (\gamma_0 t) - 1],
\]
with \(\gamma_0 := (G_0/2)^{1/2}\).
3 Conservative properties along $G$-geodesics

In this section we show proposition 1.

**Proof** We remind first the fundamental variation formula

$$2\left[(D_{g,\Omega} \nabla^* \cdot) (v, V)\right] = \frac{1}{2} \nabla_g |v|^2_g - 2 v_g^* \cdot (\nabla^*_g v_g^* + \nabla_g V^*_g),$$

(3.1)

obtained in [Pal8], (see the formula 19 in [Pal8]). Using (3.1) we develop the derivative

$$2 \frac{d}{dt} \left( \nabla^*_g \dot{g}_t^* + \nabla_g \dot{\Omega}_t^* \right) = - 2 \dot{g}_t^* \cdot \left( \nabla^*_g \dot{g}_t^* + \nabla_g \dot{\Omega}_t^* \right) + \frac{1}{2} \nabla_g |\dot{g}_t|^2_g,$$

$$+ 2 \nabla^*_g \frac{d}{dt} \dot{g}_t^* + 2 \nabla_g \frac{d}{dt} \dot{\Omega}_t^* - 2 \dot{g}_t^* \cdot \nabla_g \dot{\Omega}_t^*.$$

Writing the equations defining the $G$-geodesic $(g_t, \Omega_t)_{t \in (-\epsilon, \epsilon)}$, under the form

$$\begin{aligned}
\left\{ \begin{array}{l}
\frac{d}{dt} \dot{g}_t^* + \Omega_t^* \dot{g}_t^* = 0, \\
2 \frac{d}{dt} \dot{\Omega}_t^* + (\dot{\Omega}_t^*)^2 + \frac{1}{2} |\dot{g}_t|^2_g - \frac{1}{2} G_{g,\Omega} (\dot{g}_t, \dot{\Omega}_t; \dot{g}_t, \dot{\Omega}_t) = 0,
\end{array} \right.
\end{aligned}$$

we infer

$$2 \frac{d}{dt} \left( \nabla^*_g \dot{g}_t^* + \nabla_g \dot{\Omega}_t^* \right) = - 2 \dot{g}_t^* \cdot \left( \nabla^*_g \dot{g}_t^* + \nabla_g \dot{\Omega}_t^* \right) - 2 \nabla^*_g \left( \dot{\Omega}_t^* \dot{g}_t^* \right) - \nabla_g \left( \dot{\Omega}_t^* \right)^2 - 2 \dot{g}_t^* \cdot \nabla_g \dot{\Omega}_t^*,$$

and thus

$$2 \frac{d}{dt} \left( \nabla^*_g \dot{g}_t^* + \nabla_g \dot{\Omega}_t^* \right) = - 2 \left( \dot{g}_t^* + \dot{\Omega}_t^* \right) \cdot \left( \nabla^*_g \dot{g}_t^* + \nabla_g \dot{\Omega}_t^* \right).$$

Then the conclusion follows by Cauchy’s uniqueness. \qed

Let $J \subset C^\infty(X, \text{End}_\mathbb{R}(T_X))$ be the set of smooth almost complex structures over $X$. For any non degenerate closed 2-form $\omega$ over a symplectic manifold, we define the space $J^\omega_{ac}$ of $\omega$-compatible almost complex structures as

$$J^\omega_{ac} := \{ J \in J \mid - \omega J \in \mathcal{M} \}.$$

With these notations holds the following result.

**Lemma 1** Let $J_0 \in J^\omega_{ac}$ and let $(g_t, \Omega_t)_{t \in (-\epsilon, \epsilon)} \subset \mathcal{M} \times \mathcal{V}_1$ be a $G$-geodesic such that $g_0 = -\omega J_0$ and $g_t^* J_0 = -J_0 g_t^*$. Then $J_t := -\omega^{-1} g_t \in J^\omega_{ac}$, for all $t \in (-\epsilon, \epsilon)$. 


Proof

Using the identity $\dot{J}_t = J_t \dot{g}_t^*$ and the $G$-geodesic equation

\[ \frac{d}{dt} \dot{g}_t^* + \dot{\Omega}_t^* \dot{g}_t^* = 0, \]

we obtain the variation formula

\[ \frac{d}{dt} (J_t \dot{g}_t^* + \dot{g}_t^* J_t) = \left( J_t \dot{g}_t^* + \dot{g}_t^* J_t \right) \cdot \left( \dot{g}_t^* - \dot{\Omega}_t^* \right). \]

This implies $\dot{g}_t^* J_t = -J_t \dot{g}_t^*$, for all $t \in (-\varepsilon, \varepsilon)$, by Cauchy’s uniqueness. We deduce in particular the evolution identity $2 \dot{J}_t = [J_t, \dot{g}_t^*]$. Then $J_t^2 = -\mathbb{I}_{T_X}$, thanks to lemma 4 in [Pal3]. We infer the required conclusion. \( \square \)

4 The first variation of the $\Omega$-Chern-Ricci form

Let $(X, J)$ be an almost complex manifold. Any volume form $\Omega > 0$ induces a hermitian metric $h_\Omega$ over the canonical bundle $K_{X,J} := \Lambda^{n,0}_J T^*_X$, which is given by the formula

\[ h_\Omega(\alpha, \beta) := \frac{n! \Omega^{n^2} \alpha \wedge \overline{\beta}}{\Omega}. \]

We define the $\Omega$-Chern-Ricci form

\[ \text{Ric}_J(\Omega) := -i C_{h_\Omega}(K_{X,J}), \]

where $C_h(F)$ denotes the Chern curvature of a hermitian vector bundle $(F, \overline{\partial}^F, h)$, equipped with a $(0,1)$-type connection. Consider also a $J$-invariant hermitian metric $\omega$ over $X$. We remind that the $\omega$-Chern-Ricci form is defined by the formula

\[ \text{Ric}_J(\omega) := \text{Tr}_\omega [JC_{\omega}(T_{X,J})]. \]

The fact that the metric $h_{\omega^n}$ over $K_{X,J}$ is induced by the metric $\omega$ over $T_{X,J}$ implies, by natural functorial properties, the identity $\text{Ric}_J(\omega) = \text{Ric}_J(\omega^n)$. Let now

\[ KS := \left\{(J, g) \in J \times \mathcal{M} \mid g = J^* g J, \nabla g J = 0\right\}, \]

be the space of Kähler structures over a compact manifold $X$. We remind that if $A \in \text{End}_\mathbb{R}(T_X)$, then its transposed $A^T_g$ with respect to $g$ is given by $A^T_g = g^{-1} A^* g$. We observe that the compatibility condition $g = J^* g J$, is equivalent to the condition $J^T_g = -J$. We define also the space of almost Kähler structures as

\[ AKS := \left\{(J, g) \in J \times \mathcal{M} \mid g = J^* g J, d (g J) = 0\right\}. \]

With these notations hold the following first variation formula for the $\Omega$-Chern-Ricci form. (Compare with [Fu, Do, Mol, Ga, Pal5].)
Proposition 2 Let \((J_t, g_t)_t \subset AKS\) and \((\Omega_t)_t \subset V\) be two smooth paths such that \(J_t = (J_t)_g\). Then hold the first variation formula
\[
2 \frac{d}{dt} \text{Ric}_{J_t}(\Omega_t) = L_{\partial_t} \Omega_t + \partial_t \Omega_t + \Omega_t \omega_t, \tag{4.1}
\]
with \(\omega_t = g_t J_t\).

Proof STEP I. Local expressions. We consider first the case of constant volume form \(\Omega\). We remind a general basic identity. Let \((L, \Omega, g)\) be a (0, 1)-type connection, equipped with a \((0, 1)\)-type connection, over an almost complex manifold \((X, J)\) and let \(D_{L,h} = \partial_{L,h} + \nabla_L\) be the induced Chern connection. In explicit terms \(\partial_{L,h} := \partial_{L,h} \circ \nabla_L\). We observe that for any local non-vanishing \(\sigma \in C^\infty(U, L \times 0)\) over an open set \(U \subset X\), holds the identity
\[
\sigma^{-1} \partial_{L,h} \sigma(\eta) = |\sigma|_h^{-2} h(\partial_{L,h} \sigma(\eta), \sigma) = |\sigma|_h^{-2} \left[ \eta_1, 0 \right] \sigma|_h^2 - h(\sigma, \nabla_L \sigma(\eta)) \right] = \eta_1, 0 \log |\sigma|_h^2 - \sigma^{-1} \nabla_L \sigma(\eta),
\]
for all \(\eta \in T_X \otimes \mathbb{C}\). We infer the formula
\[
i \sigma^{-1} \partial_{L,h} \sigma = i \partial_j \log |\sigma|_h^2 + 2 \Re(i \sigma^{-1} \nabla_L \sigma).
\]
In the case \(L = K_{X,J} := \Lambda^{n,0}_{J,t} T_X^*\) and \(h \equiv h_{11}\) we get for all
\[
\beta_t = \beta_1, 0 \wedge \ldots \wedge \beta_n, 0 \in C^\infty(U, K_{X,J} \times 0),
\]
with \(\beta^{1,0}_{r,t} := \beta_r, 0 \wedge \ldots \wedge \beta_n, 0, r = 1, \ldots, n,\) the formula for the 1-form \(\alpha_t\),
\[
\alpha_t := i \beta^{-1}_t D_{K_{X,J}, h_{11}} \beta_t = i \partial_j \log \frac{i^n \beta_t \wedge \beta_t}{\Omega} + 2 \Re(i \beta^{-1}_t \nabla_{K_{X,J}, \beta_t}).
\]
We also notice the local expression \(\text{Ric}_{J_t}(\Omega) = -i C_{h_{11}} (K_{X,J}) = -d \alpha_t\). In order to expand the time derivative of the expression
\[
\alpha_t(\eta) = i \eta_1, 0 \log \frac{i^n \beta_t \wedge \beta_t}{\Omega} + 2 \Re \left[ i \beta^{-1}_t \sum_{r=1}^n \beta^{1,0}_{r,t} \wedge \ldots \wedge \left( \eta_1, 0 - \nabla_{J_t} \beta^{1,0}_{r,t} \right) \wedge \ldots \wedge \beta^{1,0}_n \right],
\]
we observe first the formula
\[
2 \frac{d}{dt} \left( \partial_{J_t} \beta^{1,0}_{J_t} \right) = J_t \partial_t \left[ (d - 2 \nabla_{J_t}) \beta^{1,0}_{J_t} \right] - i \left[ \beta \left( \beta \cdot J_t \right) \right]^{1,1}.
\tag{4.2}
\]
We notice indeed that for bi-degree reasons holds the identity

\[ 2\partial_{J^{1,0}_{j,t}} = 2\left(d\beta_{1,0}^{1,0}\right)_{J_{t}} = d\beta_{1,0}^{1,0} + J_{t}^{*}d\beta_{1,0}^{1,0}J_{t}. \]

Then time deriving the latter we infer the required formula (4.2).

**STEP II. Local choices.** We fix an arbitrary time \( \tau \). We want to compute the time derivative \( \dot{\alpha}_{\tau}(\eta) \). We take the open set \( U \subset X \) relatively compact. Then for a sufficiently small \( \varepsilon > 0 \), the bundle map

\[ \varphi_{t} := \det_{c}^{n^{1,0}}_{j_{t}} : \Lambda_{j_{t}}^{n,0}T_{U}^{*} \rightarrow \Lambda_{j_{t}}^{n,0}T_{U}^{*} \]

\[ \beta_{1} \wedge \ldots \wedge \beta_{n} \rightarrow \beta_{t} := \beta_{1,0}^{1,0} \wedge \ldots \wedge \beta_{n,0}^{1,0}, \]

is an isomorphism for all \( t \in (\tau - \varepsilon, \tau + \varepsilon) \). We set for notations simplicity \( D_{t} := D_{KX,J^{1,0}H_{0}} \). We consider also the connection \( D_{\varphi_{t}} := \varphi_{t}^{*}D_{t} \) over the bundle \( \Lambda_{j_{t}}^{n,0}T_{U}^{*} \). Explicitly \( D_{\varphi_{t}}\beta = \varphi_{t}^{-1}D_{t}\beta \). Then the expression \( D_{t}\beta_{t} = \alpha_{t} \otimes \beta_{t} \) implies \( D_{\varphi_{t}}\beta = \alpha_{t} \otimes \beta \). We deduce that

\[ \dot{\alpha}_{t} = \frac{d}{dt}D_{\varphi_{t}}, \]

is independent of the choice of \( \beta \). We want to compute \( \dot{\alpha}_{\tau} \) at an arbitrary point \( p \in U \).

**STEP IIa. The Kähler case.** (We consider first this case since is drastically simpler). Let \( \nabla_{g_{\tau}} \) be the Levi-Civita connection of \( g_{\tau} \). Using parallel transport and the Kähler assumption \( \nabla_{g_{\tau}}J_{\tau} = 0 \), we can construct (up to shrinking \( U \) around \( p \)), a frame \( (\beta_{r})_{r=1}^{n} \subset C^{\infty}(U, \Lambda_{j_{t}}^{1,0}T_{U}^{*}) \), satisfying \( \nabla_{g_{\tau}}\beta_{r}(p) = 0 \), for all \( r = 1,\ldots,n \), and the identity

\[ \omega_{\tau} = \frac{i}{2} \sum_{r=1}^{n} \beta_{r} \wedge \bar{\beta}_{r}, \]

over \( U \). Then \( dV_{g_{\tau}} = 2^{-n}i^{n^{2}}\beta_{r} \wedge \bar{\beta}_{r} \). We set now \( f_{\tau} := \log dV_{g_{\tau}} \). The identity \( d\beta_{r} = \text{Alt} \nabla_{g_{\tau}}\beta_{r} \wedge \bar{\beta}_{r} \) implies \( d\beta_{r}(p) = 0 \). Then formula (4.2) implies the identity at the point \( p \),

\[ \frac{d}{dt}_{|t=\tau} \left( J_{\tau}^{1,0}_{r,t} \right) = -i\beta_{r} \left( \nabla_{g_{\tau}}\beta_{r} \right)_{j_{t}}^{1,1} \]

\[ = -i\beta_{r} \text{Alt} \left( \nabla_{g_{\tau}}^{1,0}_{j_{t}}\beta_{r} \right). \]

(The last equality follows from the Kähler assumption). We deduce

\[ \eta_{j_{t}}^{1,0} - 2\frac{d}{dt}_{|t=\tau} \left( J_{\tau}^{1,0}_{r,t} \right) = i\beta_{r} \nabla_{g_{\tau}}^{1,0}_{j_{t}}\beta_{r} \cdot \eta_{j_{t}}^{0,1} = i\beta_{r} \nabla_{g_{\tau}}^{1,0}_{j_{t}}\beta_{r} \cdot \eta, \]

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at the point \( p \). (The last equality follows also from the Kähler assumption).

Using this last identity we obtain the expression at the point \( p \),

\[
\dot{\alpha}_\tau (\eta) = \frac{1}{2} J_{\tau} \cdot \eta \cdot f_{\tau} + \eta^{1,0}_{\tau} \Re \left( \beta_{1}^{-1} \sum_{r=1}^{n} \beta_1 \wedge \ldots \wedge \left( \beta_r J_{\tau} \right)^{1,0} \wedge \ldots \wedge \beta_n \right) \\
- \Re \left( \beta_{1}^{-1} \sum_{r=1}^{n} \beta_1 \wedge \ldots \wedge \beta_r \nabla_{g_{\tau}} J_{\tau} \cdot \eta \wedge \ldots \wedge \beta_n \right) \\
= - \frac{1}{2} \left( \text{Tr}_{\mathbb{R}} \nabla_{g_{\tau}} \dot{J}_{\tau} - df_{\tau} \cdot \dot{J}_{\tau} \right) (\eta),
\]

thanks to the elementary identities \( (\beta_r J_{\tau})^{1,0} = 0 \), \( \text{Tr}_{\mathbb{C}} A = \text{Tr}_{\mathbb{C}} A^* \), and \( \text{Tr}_{\mathbb{R}} B = 2 \Re (\text{Tr}_{\mathbb{C}} B^{1,0}) \), for all \( B \in \text{End}_{\mathbb{R}}(T_X) \). Using now the symmetry identities \( \dot{J}_{\tau} = (\dot{J}_{\tau})^T_g \) and \( \nabla_{g_{\tau}, \xi} \dot{J}_{\tau} = (\nabla_{g_{\tau}, \xi} J_{\tau})^T_{g_{\tau}} \), we obtain

\[
2 \dot{\alpha}_\tau = \nabla_{g_{\tau}} \dot{J}_{\tau} - g_{\tau},
\]

over \( U \). We conclude, thanks to the Kähler condition and Cartan’s identity, the required formula for arbitrary time \( t \), in the case of constant volume form.

**STEP IIb. The almost Kähler case.** We remind first that in this case holds the classical identity

\[
g\left( \nabla_{g_{\tau}} J \cdot \eta, \mu \right) = -2 g \left( J \xi, N_j (\eta, \mu) \right), \tag{4.3}
\]

where \( N_j \) is the Nijenhuis tensor, defined by the formula

\[
4 N_j (\xi, \eta) := [\xi, \eta] + J[\xi, J\eta] + J[J\xi, \eta] - [J\xi, J\eta].
\]

The identity (4.3) combined with the identity \( N_j (J\eta, \mu) = -J N_j (\eta, \mu) \), implies

\[
\nabla_{g_{\tau}, \xi} J = -J \nabla_{g_{\tau}} J. \tag{4.4}
\]

We consider also the Chern connection \( D^\omega_{\Lambda^1_{J,0} T_X} \) of the complex vector bundle \( \Lambda^1_{J,0} T_X \) with respect to the hermitian product

\[
\langle \alpha, \beta \rangle_{\omega} := \frac{1}{2} \text{Tr}_{\omega} (i \alpha \wedge \bar{\beta}).
\]

This connection is obviously the dual of the Chern connection \( D^\omega_{T^1_{X,j}} T_X \) of the hermitian vector bundle \( (T^{1,0}_{X,j}, \omega) \). We denote by \( D^\omega_{T X, j} \) the Chern connection of \( (T_{X,j}, \omega) \). By abuse of notations, we denote with the same symbol its complex linear extension over \( T_X \otimes_{\mathbb{R}} \mathbb{C} \). The latter satisfies the formula

\[
D^\omega_{T X, j} \xi = D^\omega_{T^1_{X,j}} \xi^j + \overline{D^\omega_{T^1_{X,j}} \xi^j}, \quad \forall \xi \in C^\infty (X, T_X \otimes_{\mathbb{R}} \mathbb{C}).
\]
In the almost Kähler case, $D_{T_X, J}^\omega$ is related to the Levi-Civita connection $\nabla_g$ (see [Pal2] and use identity (4.3)), via the formula

$$D_{T_X, J}^\omega \xi \eta = \nabla_g \xi \eta - \frac{1}{2} J \nabla_g J \cdot \xi \eta,$$

for all $\xi, \eta \in C^\infty (X, T_X \otimes \mathbb{C})$. Thus

$$D_{T_X, J}^{\omega_0} \xi^1_0 = \nabla_g \xi^1_0 - \frac{1}{2} J \nabla_g J \cdot \xi^1_0,$$

and

$$D_{A^1_T X, J}^{\omega_1, 0} \xi^1_0 = \nabla_g \beta^1_0 \cdot \xi^1_0 + \frac{1}{2} i \beta^1_0 \cdot \nabla_g J \cdot \xi^1_0$$

$$= \nabla_g \beta^1_0 \cdot \xi^1_0,$$

since $\nabla_g J \cdot J = -J \nabla_g J$. We apply now these considerations to the almost Kähler structure $(J_T, g_T)$. Using parallel transport, we can construct a complex frame $(\beta_r)_{r=1}^n \subset C^\infty (U, \Lambda^1_{\beta}, T^* U)$, satisfying $D_{A^1_T X, J}^{\omega_1, 0} \beta_r (p) = 0$, for all $r = 1, \ldots, n$, and the identity

$$\omega_r = \frac{i}{2} \sum_{r=1}^n \beta_r \wedge \bar{\beta}_r,$$

over $U$. Then as before, holds the identity $dV_{g_r} = 2^{-n} i^n \beta_r \wedge \bar{\beta}_r$. We infer

$$\overline{\partial}_J \beta_r (p) = 0,$$

$$\left( \nabla_{g_r} \beta_r \cdot \xi^1_{1r} \right) (p) = 0,$$

$$\partial_J \beta_r (p) = 0,$$

for all $r = 1, \ldots, n$. The last one follows indeed from the elementary identities

$$\partial_J \beta_r (\xi, \eta) = d \beta_r (\xi, \eta) = \nabla_{g_r} \xi \beta_r \cdot \eta - \nabla_{g_r} \eta \beta_r \cdot \xi,$$

for all $\xi, \eta \in C^\infty (X, T^1_{X, J_1})$. We observe now that formula (4.2) writes as

$$2 \frac{d}{dt} \left( \overline{\partial}_J \beta_r^1 \right) = J_r \cdot J_r \left[ (\partial_{J_r} - \overline{\partial}_{J_r}) \beta_r^1 - \beta_r^1 \cdot N_{J_r} \right] - i \left[ d \left( \beta_r \cdot J_r \right) \right]_{J_r}.  \tag{4.2}$$

Thus at the point $p$ holds the identity

$$2 \frac{d}{dt} \left|_{t=\tau} \right. \left( \overline{\partial}_J \beta_r^1 \right) \left( \eta_{J_r}, \mu \right) = - \beta_r \cdot N_{J_r} \left( \eta_{J_r}, \mu \right)$$

$$- i \left[ d \left( \beta_r \cdot J_r \right) \right]_{J_r} \left( \eta_{J_r}, \mu \right)$$

$$= i \beta_r \cdot N_{J_r} \left( \eta_{J_r}, \mu \right)$$

$$- i \left[ d \left( \beta_r \cdot J_r \right) \right] \left( \eta_{J_r}, \mu \right).$$
We set for notations simplicity $\eta_r^{0,1} := \eta_r^{0,1}, \mu_r^{1,0} := \mu_r^{1,0}$ and we observe the expansion

\[
d(\beta_r \cdot \dot{J}_r) (\eta_r^{0,1}, \mu_r^{1,0}) = \nabla_{g_r, \eta_r^{0,1}} \beta_r \cdot \dot{J}_r \mu_r^{1,0} + \beta_r \cdot \nabla_{g_r, \eta_r^{0,1}} \dot{J}_r \cdot \mu_r^{1,0} - \nabla_{g_r, \mu_r^{1,0}} \beta_r \cdot \dot{J}_r \eta_r^{0,1} - \beta_r \cdot \nabla_{g_r, \mu_r^{1,0}} \dot{J}_r \cdot \eta_r^{0,1}.
\]

We notice also the trivial identity $\beta_r \cdot \dot{J}_r \mu_r^{1,0} = \beta_r \cdot (\dot{J}_r \mu_r)^{0,1} = 0$, over $U$. Taking a covariant derivative of this we infer

\[0 = \nabla_{g_r, \eta_r^{0,1}} \beta_r \cdot \dot{J}_r \mu_r^{1,0} + \beta_r \cdot \nabla_{g_r, \eta_r^{0,1}} \dot{J}_r \cdot \mu_r^{1,0} + \beta_r \cdot \dot{J}_r \nabla_{g_r, \eta_r^{0,1}} \mu_r^{1,0}.
\]

The identity (4.4) implies

\[
\nabla_{g_r, \eta_r^{0,1}} \mu_r^{1,0} = \left( \nabla_{g_r, \eta_r^{0,1}} \mu - \frac{i}{2} \nabla_{g_r, \eta} \cdot \mu \right)^{1,0}_{\dot{\eta}_r}.
\]

Thus

\[
\beta_r \cdot \dot{J}_r \nabla_{g_r, \eta_r^{0,1}} \mu_r^{1,0} = \beta_r \cdot \left[ \dot{J}_r \left( \nabla_{g_r, \eta_r^{0,1}} \mu - \frac{i}{2} \nabla_{g_r, \eta} \cdot \mu \right) \right]^{0,1}_{\dot{\eta}_r} = 0,
\]

and

\[
d(\beta_r \cdot \dot{J}_r) (\eta_r^{0,1}, \mu_r^{1,0}) = -\beta_r \cdot \nabla_{g_r, \mu_r^{1,0}} \dot{J}_r \cdot \eta_r^{0,1}, \quad (4.9)
\]

at the point $p$, since

\[
\nabla_{g_r, \mu_r^{1,0}} \beta_r \cdot \dot{J}_r \eta_r^{0,1} = \nabla_{g_r, \mu_r^{1,0}} \beta_r \cdot \left( \dot{J}_r \eta \right)^{1,0}_{\dot{\eta}_r} = 0,
\]

at $p$ thanks to (4.7). Taking a covariant derivative of the identity

\[
\dot{J}_r J_r + J_r \dot{J}_r = 0,
\]

we obtain

\[
\nabla_{g_r} \dot{J}_r J_r + \dot{J}_r \nabla_{g_r} J_r + \nabla_{g_r} J_r \dot{J}_r + J_r \nabla_{g_r} \dot{J}_r = 0,
\]

and thus

\[
2 \beta_r \cdot \nabla_{g_r, \mu_r^{1,0}} \dot{J}_r \cdot \eta_r^{0,1} = 2 \beta_r \cdot \left( \nabla_{g_r, \mu_r^{1,0}} \dot{J}_r \cdot \eta \right)^{1,0}_{\dot{\eta}_r}
\]

\[ - i \beta_r \cdot \left( \dot{J}_r \nabla_{g_r, \mu_r^{1,0}} J_r + \nabla_{g_r, \mu_r^{1,0}} J_r \dot{J}_r \right) \eta
\]

\[ = 2 \beta_r \cdot \nabla_{g_r, \mu_r^{1,0}} \dot{J}_r \cdot \eta - i \beta_r \cdot \dot{J}_r \nabla_{g_r, \mu} J_r \cdot \eta,
\]

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thanks to (4.4) and the fact that $\beta_r$ is of type $(1,0)$ with respect to $J_\tau$. We deduce

$$-id \left( \beta_r \cdot \dot{J}_\tau \right) \left( \eta_r^{0,1}, \mu_r^{1,0} \right) = i \beta_r \cdot \left( \nabla^{1,0}_{g_r,J_\tau,\mu} \dot{J}_\tau - \frac{1}{2} J_\tau \dot{J}_\tau \nabla g_r, J_\tau \right) \eta,$$

thanks to (4.9), and thus

$$\eta_r^{0,1} - 2 \frac{d}{dt} \left( \overline{\beta}_r^{1,0}_{J_\tau, t} \right) = i \beta_r \cdot N_{J_\tau} \left( \eta, \dot{J}_\tau \right)$$

$$+ i \beta_r \cdot \left( \nabla^{1,0}_{g_r,J_\tau,\mu} \dot{J}_\tau - \frac{1}{2} J_\tau \dot{J}_\tau \nabla g_r, J_\tau \right) \eta. \quad (4.10)$$

Using (4.6) and (4.10) we obtain

$$2 \dot{\alpha}_\tau (\eta) = \dot{J}_\tau \eta + f_\tau$$

$$+ 2 \text{Re} \left\{ i \beta_r^{-1} \sum_{l=1}^n \beta_1 \wedge \ldots \wedge \left[ \eta_r^{0,1} - 2 \frac{d}{dt} \left( \overline{\beta}_r^{1,0}_{J_\tau, t} \right) \right] \wedge \ldots \wedge \beta_n \right\}$$

$$= d f_\tau \cdot \dot{J}_\tau \eta$$

$$- 2 \text{Re} \text{Tr}_c \left[ N_{J_\tau} \left( \eta, \dot{J}_\tau \right) + \left( \nabla^{1,0}_{g_r,J_\tau,\mu} \dot{J}_\tau - \frac{1}{2} J_\tau \dot{J}_\tau \nabla g_r, J_\tau \right) \eta \right]$$

$$= d f_\tau \cdot \dot{J}_\tau \eta - \text{Tr}_n \left[ N_{J_\tau} \left( \eta, \dot{J}_\tau \right) + \left( \nabla g_r \dot{J}_\tau - \frac{1}{2} J_\tau \dot{J}_\tau \nabla g_r, J_\tau \right) \eta \right].$$

We show now the identity

$$2 \text{Tr}_n \left[ N_{J_\tau} \left( \eta, \dot{J}_\tau \right) \right] = \text{Tr}_n \left( \dot{J}_\tau \nabla g_r, J_\tau, \eta \right). \quad (4.11)$$

Indeed, let $(e_k)_{k=1}^{2n} \subset T_{X,p}$ be a $g_r$ $(p)$-orthonormal basis. Using (4.3), (4.4) and the symmetry assumption $\dot{J}_\tau = (\dot{J}_\tau)_g^T$, we obtain

$$2g \left( e_k, N_{J_\tau} \left( \eta, \dot{J}_\tau e_k \right) \right) = g \left( \nabla g_r, J_\tau e_k, J_\tau, \eta e_k \right)$$

$$= -g \left( \dot{J}_\tau J_\tau \nabla g_r, J_\tau, \eta, e_k \right)$$

$$= g \left( J_\tau J_\tau \nabla g_r, J_\tau, \eta, e_k \right),$$

and thus the required identity (4.11). We infer the formula

$$2 \dot{\alpha}_\tau (\eta) = - \text{Tr}_n \left( \nabla g_r, \dot{J}_\tau, \eta \right) + d f_\tau \cdot \dot{J}_\tau \eta,$$
over $U$. Using the symmetry identities $\dot{J}_\tau = (\dot{J}_\tau)^T_g$, and $\nabla_{g_\tau, \xi} \dot{J}_\tau = (\nabla_{g_\tau, \xi} \dot{J}_\tau)^T_g$, we infer
\[
2\dot{\alpha}_\tau = \nabla_{g_\tau}^* \dot{J}_\tau - g_\tau
\]
\[
= -J_\tau \nabla_{g_\tau}^* \dot{J}_\tau - \omega_\tau,
\]
over $U$. We conclude the required variation formula for arbitrary time $t$, in the case of constant volume form. In the case of variable volume forms, we fix an arbitrary time $\tau$ and we time derive at $t = \tau$ the decomposition
\[
\text{Ric}_{J_t}(\Omega_t) = \text{Ric}_{J_t}(\Omega_\tau) - dd^c_{J_t} \log \frac{\Omega_t}{\Omega_\tau}.
\]
We obtain
\[
\frac{d}{dt} \bigg|_{t=\tau} \text{Ric}_{J_t}(\Omega_t) = \frac{d}{dt} \bigg|_{t=\tau} \text{Ric}_{J_t}(\Omega_\tau) - dd^c_{J_\tau} \dot{\Omega}_\tau^*,
\]
and thus
\[
2 \frac{d}{dt} \text{Ric}_{J_t}(\Omega_t) = d \left[ \left( J_t \nabla_{g_t}^* \dot{J}_t - \nabla_{g_t}^* \dot{\Omega}_t^* \right) - \omega_t \right],
\]
thanks to the variation formula for the fixed volume form case. The conclusion follows from Cartan’s identity for the Lie derivative of differential forms.

We infer the following corollary.

**Corollary 1** Let $\omega$ be a symplectic form and let $\langle J_t, \Omega_t \rangle \subset J^{ac}_\omega \times \mathcal{V}$ be an arbitrary smooth family. Then holds the variation formulas
\[
2 \frac{d}{dt} \text{Ric}_{J_t}(\Omega_t) = -L_{\nabla_{g_t}^* \dot{g}_t^* + \nabla_{g_t}^* \dot{\Omega}_t^*} \omega - 2d \text{Tr}_{g_t} \left[ \omega(\cdot \nabla_{N_j} \dot{g}_t^*) \dot{g}_t^* \right], \quad (4.12)
\]
and
\[
2 \frac{d}{dt} \text{Ric}_{J_t}(\Omega_t) = -L_{\nabla_{g_t}^* \dot{g}_t^* + \nabla_{g_t}^* \dot{\Omega}_t^*} \omega + d \text{Tr}_{g_t} \left[ \omega(\cdot \nabla_{\mathcal{T}_{X, g_t}} \dot{g}_t^*) \dot{g}_t^* \right], \quad (4.13)
\]
with $g_t := -\omega J_t$.

**Proof** We have $\dot{g}_t^* = -\dot{J}_t^* \dot{J}_t$ and thus the property $\dot{J}_t = (\dot{J}_t)^T_{g_t}$, which allows to apply (4.11). We notice now the equality
\[
\nabla_{g_t}^* \left( J_t \dot{J}_t \right) = -\nabla_{g_t, e_k} J_t \dot{J}_t e_k + J_t \nabla_{g_t}^* \dot{J}_t,
\]
with respect to a $g_t$-orthonormal local frame of $T_X$. We deduce
\[
2\dot{\alpha}_t = -\left( \nabla_{g_t}^* \left( J_t \dot{J}_t \right) + \nabla_{g_t, e_k} J_t \dot{J}_t e_k \right) - \omega.
\]
The identity (4.3) implies
\[ \omega(\nabla_{g_t,e_k} J_t \cdot \dot{J}_t e_k, \xi) = 2 \omega(e_k, N_{J_t} (\dot{J}_t e_k, J_t \xi)) \]
\[ = 2 \omega(e_k, N_{J_t} (J_t \dot{J}_t e_k, \xi)) \]
\[ = -2 d \text{Tr}_{g_t} \left[ \omega(\xi \sim N_{J_t} \dot{g}_t) \right]. \]

We infer the variation formula (4.12). In order to show (4.13) we notice first the identity
\[ \omega(e_k, N_{J_t} (\dot{g}_t^{-1} e_k, \xi)) \equiv 0, \text{ for arbitrary real local frame } (e_k) \text{ of } T_X. \]

Time deriving this we obtain
\[ \omega(e_k, N_{J_t} (\dot{g}_t^* e_k, \xi)) = \omega(e_k, \dot{N}_{J_t} (e_k, \xi)), \]
with respect to our \( g_t \)-orthonormal local frame. Then the general formula
\[ 2 \frac{d}{dt} N_{J_t} = \overline{\partial}_{T_X,J_t} (J_t \dot{J}_t) + J_t \dot{J}_t N_{J_t} - (J_t \dot{J}_t) \sim N_{J_t}, \]
(see the proof of lemma 7 in [Pal3]), implies
\[ \omega(e_k, N_{J_t} (\dot{g}_t^* e_k, \xi)) = \omega(e_k, \dot{N}_{J_t} (e_k, \xi)) \]
\[ - \omega(e_k, \partial_{T_X,J_t} \dot{g}_t^* (e_k)) \]
\[ = \omega(e_k, N_{J_t} (\dot{g}_t^* e_k, \xi)) - \omega(\dot{g}_t^* e_k, N_{J_t} (e_k, \xi)) \]
\[ - \omega(\partial_{T_X,J_t} \dot{g}_t^* (\xi, e_k), e_k). \]

Assuming for simplicity that the \( g_t \)-orthonormal local frame diagonalizes \( \dot{g}_t^* \) we deduce the identity
\[ 2 \text{Tr}_{g_t} \left[ \omega(\bullet \sim N_{J_t} \dot{g}_t) \right] = - \text{Tr}_{g_t} \left[ \omega(\bullet \sim \overline{\partial}_{T_X,J_t} \dot{g}_t) \right], \]
which implies the variation formula (4.13). \( \square \)

Combining lemma 1, proposition 1 and corollary 1 we deduce the main theorem 1.

5 The decomposition of the Bakry-Emery-Ricci tensor

We compare first the Riemannian Ricci tensor \( \text{Ric}(g) \) with the \( \omega \)-Chern-Ricci tensor.
Lemma 2 Let \((X, J, g)\) be an almost Kähler manifold with symplectic form \(\omega := gJ\). Then holds the identity

\[
\text{Ric}_J(\omega)(\xi, J\eta) = \text{Ric}(g)(\xi, \eta) + \omega(\nabla^*_g \overline{\nabla}_g J \xi, \eta) + \frac{1}{4} \text{Tr}_\mathbb{R}(\nabla_{g, \xi} J \cdot \nabla_{g, \eta} J),
\]

where \(\overline{\nabla}_g J (\xi, \eta) := \nabla_g J (\eta, \xi)\).

Proof Using formula (4.5), the standard Curvature identity

\[
(\nabla_{g, \xi} \nabla_{g, \eta} - \nabla_{g, \eta} \nabla_{g, \xi} - \nabla_{g, [\xi, \eta]} \mu = \mathcal{R}_g (\xi, \eta) \mu,
\]
a similar one for the Chern curvature \(\mathcal{C}_\omega (T_X, J)\) and the trivial equality

\[
\nabla^2_{g, \xi, \eta, J} - \nabla^2_{g, \eta, \xi, J} = [\mathcal{R}_g (\xi, \eta), J],
\]
we obtain the relation

\[
\mathcal{C}_\omega (T_X, J) (\xi, \eta) \mu = \mathcal{R}_g (\xi, \eta)_J^{1, 0} \mu - \frac{1}{4} (\nabla_{g, \xi} J \cdot \nabla_{g, \eta} J - \nabla_{g, \eta} J \cdot \nabla_{g, \xi} J) \mu. \tag{5.3}
\]

Let now \((e_k)_{k=1}^n \subset T_{X, p}\) be a \(\omega (p)\)-orthonormal and \(J (p)\)-complex basis. Then

\[
\text{Ric}_J(\omega)(\xi, J\eta) = \sum_{k=1}^n g \left( J\mathcal{C}_\omega (T_X, J)(\xi, J\eta) e_k, e_k \right)
\]

\[
= - \sum_{k=1}^n g \left( \mathcal{C}_\omega (T_X, J)(\xi, J\eta) e_k, Je_k \right),
\]
thanks to the identity \([\mathcal{C}_\omega (T_X, J)(\xi, \eta), J] = 0\). Using formula (5.3) we obtain

\[
\text{Ric}_J(\omega)(\xi, J\eta) = - \sum_{k=1}^n g \left( \mathcal{R}_g (\xi, J\eta)_J^{1, 0} e_k, Je_k \right)
\]

\[
+ \frac{1}{4} \sum_{k=1}^n g \left( (\nabla_{g, \xi} J \cdot \nabla_{g, J\eta} J - \nabla_{g, J\eta} J \cdot \nabla_{g, \xi} J) e_k, Je_k \right).
\]

We notice now the equalities

\[
-g \left( \mathcal{R}_g (\xi, J\eta)_J^{1, 0} e_k, Je_k \right) = -\frac{1}{2} g \left( \mathcal{R}_g (\xi, J\eta) e_k, Je_k \right)
\]

\[
+ \frac{1}{2} g \left( \mathcal{R}_g (\xi, J\eta) Je_k, e_k \right)
\]

\[
- g \left( \mathcal{R}_g (\xi, J\eta) e_k, Je_k \right).
\]
thanks to the anti-symmetry identity \((\mathcal{R}_g (\xi, J\eta))^T_g = -\mathcal{R}_g (\xi, J\eta)\). Using the first Bianchi identity and the identity (4.4), we infer
\[
\text{Ric}_j (\omega) (\xi, J\eta) = \sum_{k=1}^n g (\mathcal{R}_g (J\eta, e_k) \xi + \mathcal{R}_g (e_k, \xi) J\eta, J e_k) \\
+ \frac{1}{4} \sum_{k=1}^n g \left( (\nabla_{g, J\xi} J \cdot \nabla_{g, J\eta} J + \nabla_{g, J\eta} J \cdot \nabla_{g, J\xi} J) e_k, e_k \right) \\
= \sum_{k=1}^n [\mathcal{R}_g (J\eta, e_k, J e_k, \xi) + g (\mathcal{R}_g (e_k, \xi) J\eta, J e_k)] \\
+ \frac{1}{4} \text{Tr}_n (\nabla_{g, J\xi} J \cdot \nabla_{g, J\eta} J),
\]
where \(\mathcal{R}_g \in C^\infty (X, S^2_R(\Lambda^2_T X))\) is the Riemannian curvature form. Using its symmetry properties we have
\[
\mathcal{R}_g (J\eta, e_k, J e_k, \xi) = \mathcal{R}_g (J e_k, \xi, J\eta, e_k) \\
= -\mathcal{R}_g (J e_k, \xi, e_k, J\eta) \\
= -g (\mathcal{R}_g (J e_k, \xi) J\eta, e_k),
\]
We infer the equality
\[
\text{Ric}_j (\omega) (\xi, J\eta) = -\sum_{k=1}^n [g (J \mathcal{R}_g (J e_k, \xi) J\eta, J e_k) + g (J \mathcal{R}_g (e_k, \xi) J\eta, e_k)] \\
+ \frac{1}{4} \text{Tr}_n (\nabla_{g, J\xi} J \cdot \nabla_{g, J\eta} J),
\]
Using (5.2) we deduce
\[
\text{Ric}_j (\omega) (\xi, J\eta) = \sum_{k=1}^n [g (\mathcal{R}_g (J e_k, \xi) \eta, J e_k) + g (\mathcal{R}_g (e_k, \xi) \eta, e_k)] \\
+ \sum_{k=1}^n g \left( J \left[ \nabla^2_{g, J\xi, J e_k} J - \nabla^2_{g, J e_k, \xi} J \right] \eta, J e_k \right) \\
+ \sum_{k=1}^n g \left( J \left[ \nabla^2_{g, J e_k, \xi} J - \nabla^2_{g, e_k, \xi} J \right] \eta, e_k \right) \\
+ \frac{1}{4} \text{Tr}_n (\nabla_{g, J\xi} J \cdot \nabla_{g, J\eta} J),
\]
and thus

\[ \text{Ric}_J(\omega)(\xi, J\eta) = \text{Ric}(g)(\xi, \eta) \]

\[ + \sum_{k=1}^n g \left( [\nabla^2_{g,\xi,Je_k} J - \nabla^2_{g,Je_k,\xi}] \eta, e_k \right) \]

\[ - \sum_{k=1}^n g \left( [\nabla^2_{g,\xi,e_k} J - \nabla^2_{g,e_k,\xi}] \eta, J e_k \right) \]

\[ + \frac{1}{4} \text{Tr}_x \left( \nabla_{g,\xi} J \cdot \nabla_{g,\eta} J \right). \quad (5.4) \]

We notice now that the identity \( J^T_g = -J \), implies \( \left( \nabla^2_{g,\xi,\eta} J \right)^T_g = -\nabla^2_{g,\xi,\eta} J \).

Using this we obtain

\[ \mathfrak{X}_k := g \left( [\nabla^2_{g,\xi,Je_k} J - \nabla^2_{g,Je_k,\xi}] \eta, e_k \right) \]

\[ + g \left( [\nabla^2_{g,e_k,\xi} J - \nabla^2_{g,\xi,e_k} J] \eta, J e_k \right) \]

\[ = g \left( \eta, [\nabla^2_{g,Je_k,\xi} J - \nabla^2_{g,\xi,Je_k} J] e_k + [\nabla^2_{g,\xi,e_k} J - \nabla^2_{g,e_k,\xi} J] J e_k \right) \]

\[ = g \left( J \eta, J \left[ [\nabla^2_{g,Je_k,\xi} J - \nabla^2_{g,\xi,Je_k} J] e_k + J [\nabla^2_{g,\xi,e_k} J - \nabla^2_{g,e_k,\xi} J] J e_k \right] \right). \]

Taking a covariant derivative of the identity \( \nabla_g J \cdot J = -J \nabla_g J \) we infer

\[ J \nabla^2_{g,\xi,\eta} J = -\nabla^2_{g,\xi,\eta} J \cdot J - \nabla_g \xi \nabla_g \eta J - \nabla_g \eta \nabla_g \xi J. \]

Using this we deduce

\[ J \left[ [\nabla^2_{g,Je_k,\xi} J - \nabla^2_{g,\xi,Je_k} J] e_k + J [\nabla^2_{g,\xi,e_k} J - \nabla^2_{g,e_k,\xi} J] J e_k \right] \]

\[ = [\nabla^2_{g,\xi,Je_k} J - \nabla^2_{g,Je_k,\xi}] J e_k + [\nabla^2_{g,\xi,e_k} J - \nabla^2_{g,e_k,\xi}] e_k \]

\[ - \nabla_g J e_k \cdot \nabla_g \xi J e_k - \nabla_g \xi \nabla_g J e_k \]

\[ + \nabla_g J e_k \cdot \nabla_g \xi J e_k + \nabla_g J e_k \cdot \nabla_g \xi J e_k \]

\[ - \nabla_g \xi J \cdot \nabla_g e_k J e_k - \nabla_g e_k J \cdot \nabla_g \xi J e_k \]

\[ + \nabla_g e_k J \cdot \nabla_g \xi J \cdot J e_k + \nabla_g \xi J \cdot \nabla_g e_k J \cdot J e_k \]

\[ = [\nabla^2_{g,\xi,Je_k} J - \nabla^2_{g,Je_k,\xi}] J e_k + [\nabla^2_{g,\xi,e_k} J - \nabla^2_{g,e_k,\xi}] e_k. \]
by obvious diagonal cancellations. We notice now that the identity (4.4) implies
\( \nabla^* \xi = 0. \) A covariant derivative of this identity implies
\( \text{Tr}_g (\xi \nabla^* \xi) = 0. \) We deduce
\[
\sum_{k=1}^n \mathcal{X}_k = g \left( \nabla^* \nabla \xi, J \eta \right).
\]
This combined with (5.4) implies the required formula (5.1).

Let \( \Omega > 0 \) be a smooth volume form over an oriented Riemannian manifold \((X, g)\). We define the \( \Omega \)-Bakry-Emery-Ricci tensor of \( g \) as
\[
\text{Ric}_g(\Omega) := \text{Ric}(g) + \nabla_g d \log \frac{dV_g}{\Omega}.
\]
We set for notations simplicity
\[
[\text{Tr}_x (\nabla g \cdot J) \cdot \nabla g \cdot J] (\xi, \eta) := \text{Tr}_x (\nabla g \xi \cdot \nabla g \eta).
\]

**Lemma 3** Let \((X, J, g)\) be an almost Kähler manifold with symplectic form \( \omega := gJ \) and let \( \Omega > 0 \) be a smooth volume form. Then hold the decomposition formula
\[
\text{Ric}_g(\Omega) = -\text{Ric}_J(\Omega)J + \omega(\bullet, \nabla^* g J \bullet) - \frac{1}{4} \text{Tr}_x (\nabla g \bullet J \cdot \nabla g \bullet J)
+ g \left( \partial_{T_X,J} \nabla g \log \frac{dV_g}{\Omega} - \nabla g \log \frac{dV_g}{\Omega} \nabla J \right), \tag{5.5}
\]

**Proof.** This formula follows directly from the identities
\[
\text{Ric}_J(\Omega) = \text{Ric}_J(\omega^n) + dd^c f \log \frac{\omega^n}{\Omega},
\]
\[
\text{Ric}_J(\omega^n) = \text{Ric}_J(\omega), \text{ the identity (5.1) and the decomposition formula}
\]
\[
\nabla g df = -dd^c f \cdot J + g \left( \partial_{T_X,J} \nabla g f - \nabla g f \nabla J \right),
\]
for all twice differentiable function \( f \). The latter follows from a straightforward modification of the proof of lemma 29 in [Pal4]. □

**Acknowledgments.** I warmly thank the referee for pointing out a few inaccuracies in the original version of this manuscript.
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