Non-analyticity in the large $N$ renormalization group

Vipul Periwal†

The Institute for Advanced Study
Princeton, New Jersey 08540-4920

The flow of the action induced by changing $N$ is computed in large $N$ matrix models. It is shown that the change in the action is non-analytic. This non-analyticity appears at the origin of the space of matrices if the action is even.

† vipul@guinness.ias.edu
Given analogies between the double-scaling solutions[1] of matrix models and critical phenomena, it is natural to consider $N$ as a kind of cutoff in the string theory, associated with the matrix model by means of the Feynman diagram expansion of the matrix model[2]. One can motivate this interpretation of $N$ quite precisely. Double-scaling gives the following form for the universal part of the free energy of the matrix model, equivalent to the partition function to all orders in the genus expansion of the associated string theory:

$$F \sim \sum_{\chi} b_{\chi}(g_{c} - g)^{\chi \alpha} N^{\chi}$$

$$= \sum_{\chi} b_{\chi} N^{\chi} g_{c}^{\chi \alpha} \sum_{k} \left( \frac{g}{g_{c}} \right)^{k} \frac{\Gamma(\chi \alpha + 1)}{\Gamma(k + 1)\Gamma(\chi \alpha - k + 1)}$$

$$= \sum_{k} \left( \frac{g}{g_{c}} \right)^{k} \sum_{\chi} b_{\chi} (N g_{c}^{\alpha})^{\chi} \frac{\Gamma(\chi \alpha + 1)}{\Gamma(k + 1)\Gamma(\chi \alpha - k + 1)}.$$

Now, if we look at the series for large values of $k$, we find

$$\frac{\Gamma(\chi \alpha + 1)}{\Gamma(k + 1)\Gamma(\chi \alpha - k + 1)} \sim (-)^{k} \frac{k^{-\chi \alpha}}{\Gamma(-\chi \alpha)},$$

implying that the coefficients of $(-g/g_{c})^{k}$ for large $k$ are functions of $A \equiv k/N^{1/\alpha}g_{c}$. Since $k$ is the number of plaquettes on the triangulated surface, and $\alpha > 0$, it is natural to interpret $A$ as the renormalized area, with $N, k \to \infty$ as the continuum limit. In other words, one interprets $N^{-1/\alpha}$ as the area of a plaquette. It therefore is of interest to compute the renormalization group flow induced by varying $N$.

Having given this motivation, it is necessary to point out that the above manipulation is only of heuristic value. The asymptotic behaviour of the coefficients is only valid for $k$ much larger than $-\chi \alpha$, so one cannot deduce properties of the fixed renormalized area partition functions by looking at the large order behaviour of the free energy. It would be too much of a digression to say more about such an approach in the present paper.

Brézin and Zinn-Justin[3] recently considered a large $N$ renormalization group approach to matrix models, with different motivations in mind. They obtained results that suggested that qualitative properties of the double-scaling solutions could be reproduced by such considerations.

The precise problem is the following: Let $\Xi_{N+1}$ be an $(N + 1) \times (N + 1)$ Hermitian matrix, and let $\Xi_{N}$ be related to $\Xi_{N+1}$ as follows:

$$\Xi_{N+1} \equiv \left( \begin{array}{cc} \Xi_{N} & \psi \\ \psi^{T} & \xi \end{array} \right),$$
where $\psi$ is a vector with $N$ complex entries and $\xi$ is a real number. Let $V$ be a polynomial* of the form $\sum_{i=1}^{M} a_i x^{2i} / 2i$. Then, with $\Omega_N \equiv \text{vol} U_N$ (in the adjoint representation), we wish to find $\delta V$ such that

$$Z_{N+1} \equiv \int \frac{d\Xi_{N+1}}{\Omega_{N+1}} \exp \left[ -(N + 1) \text{tr} V(\Xi_{N+1}) \right] = \int \frac{d\Xi_{N}}{\Omega_{N}} \exp \left[ -N \text{tr} (V + \delta V)(\Xi_{N}) \right].$$

I shall show that the evolution of $V$ under changes of $N$ induces non-analytic changes in $V$ around the origin of the matrix integration domain. The resulting matrix model may not possess a simple surface interpretation, and more importantly, that the large $N$ renormalization group is not defined on the space of polynomials in the basic operators $\hat{\Theta}_n \equiv \text{tr} \Xi_n / N$.

I use a measure that agrees explicitly with standard matrix model results. My results are physically in accord with what one expects in random matrix physics[4]. In an eigenvalue picture, integrating out one eigenvalue redefines the potential felt by the other eigenvalues. In particular, suppose we look at the effective potential felt by the remaining eigenvalues near the origin matrices. The non-analyticity arises from the following physics: to leading order in $N$, the evaluation of the contribution to $\delta V$ due to the eliminated eigenvalue is given by this eigenvalue attempting to be as far from the origin as possible, consistent with the fact that it is confined by the potential, $V$. There are two such positions available to this eigenvalue. However, only one of these saddlepoints contributes to leading order in $N$. It is precisely the discrete change in the dominant saddlepoint value for an infinitesimal change in the matrix $\Xi_N$ that leads to the non-analyticity in the effective potential. Of course, as with all non-analytic behaviour associated with an ‘infinite volume’ (here, the limit of large $N$), the non-analyticity is ameliorated when ‘finite-size’ effects (here, $1/N$ corrections) are accounted for.

While the models considered are simple enough to allow analytic solution, the result obtained here is concrete evidence for the subtlety of the large $N$ limit in general. In particular, the extraction of physics at finite $N$, including $1/N$ corrections, for theories with fields transforming in the adjoint representation, may be more subtle than one might expect naively.

I now turn to the solution of the problem stated above. The first observation is that the $U_{N+1}$ symmetry can be broken to $U_N$, with the eliminated generators used to set $\psi = 0$. This produces a Jacobian factor in the measure so

$$Z_{N+1} = \int \frac{d\Xi_{N}}{\Omega_{N}} \exp \left[ -(N + 1) \text{tr} V(\Xi_{N}) \right] \int d\xi \exp \left[ -(N + 1) V(\xi) \right] \det (\xi - \Xi_{N})^2.$$

* $V$ will be an even polynomial throughout this paper for simplicity.
Thus
\[ N \text{tr} \delta V = \text{tr} V - \ln \int d\xi \exp \left[ -(N + 1)V(\xi) \right] \det(\xi - \Xi_N)^2. \]

It remains therefore to evaluate the integral over \( \xi \). This integral can be evaluated by saddlepoints since the determinant is that of an \( N \)-dimensional matrix, and \( N+1 \) multiplies \( V \) in the measure. This evaluation is non-perturbative as far as \( V \) is concerned for we need not make any assumptions about the coefficients in \( V \).

We want to figure out the changes in the polynomial \( V \) when we integrate out some of the degrees of freedom. We may assume that \( \hat{\Theta}_n \) are small in the following, since the coefficients of a polynomial can be obtained from its behaviour at the origin. The saddlepoint value of \( \xi_s \) satisfies
\[ \xi_s V'(\xi_s) = 2 \frac{1}{N+1} \text{tr} \frac{1}{\xi_s - \Xi_N} = 2 \frac{N}{N+1} \left[ 1 + \frac{\hat{\Theta}_1}{\xi_s} + \frac{\hat{\Theta}_2}{\xi_s^2} + \ldots \right]. \] (1)

A very similar equation appears in double-scaling solutions of vector models [5] as well—this should not come as a surprise since we have essentially integrated out a vector here. We can solve this equation for \( \xi_s = \xi_s(a_i, \hat{\Theta}_n) \). If we are interested in perturbation theory, in other words in the surfaces provided by the Feynman diagram expansion, it is of interest to solve eq. 1 when \( a_i, i > 1 \) are assumed small. To leading order in \( N \) we can set \( N/(N+1) \approx 1 \) in eq. 1, since the error is of the same order as corrections to the saddlepoint value of the integral.

Eq. 1 has two solutions with the same assumptions as above. Since we consider \( V \) even, these are related by \( \xi_s \to -\xi_s \). Then the change in \( V \) is
\[ \text{tr} \delta V(\Xi_N) = \frac{1}{N} \left[ \text{tr} V(\Xi_N) - \ln \left\{ \exp \left[ -(N + 1)V(\xi_s) \right] \left( \det(\xi_s - \Xi_N)^2 + \det(\xi_s + \Xi_N)^2 \right) \right\} \right] \]
\[ = \frac{1}{N} \text{tr} V(\Xi_N) + \frac{N+1}{N} V(\xi_s) \]
\[ - \ln \left\{ \exp \text{tr} \ln (\xi_s - \Xi_N)^2 + \exp \text{tr} \ln (\xi_s + \Xi_N)^2 \right\}^{1/N} \]
\[ = V(\xi_s) + \frac{1}{N} \left[ \text{tr} V(\Xi_N) + V(\xi_s) - \max \left( \text{tr} \ln(\xi_s - \Xi_N)^2, \text{tr} \ln(\xi_s + \Xi_N)^2 \right) \right], \]
up to terms higher order in \( 1/N \). The maximum function arises as the limit \( \lim_{n \to \infty} (a^n + b^n)^{1/n} = \max(a,b) \). The monotonicity of the exponential function implies \( \exp \max(a,b) = \max(e^a, e^b) \). We have thus derived non-analytic behaviour in \( \delta V \). Terms involving \( \hat{\Theta}_n \) were neglected in eq. 1, consistent with the fact that I am investigating the change in the potential at \( \Xi_N \approx 0 \).
Very close to the origin, up to constants,

\[ \text{tr} \delta V \approx -2 \left| \frac{\hat{\Theta}_1}{\xi_s} \right|, \]  

(2)

which is an even function, as expected. Naively one could write

\[ \text{tr} \delta V = \left(1 + \frac{1}{N}\right) V(\xi_s) - 2\ln|\xi_s| + \frac{1}{N}\text{tr}V(\Xi_N) - 2\left\{ \frac{\hat{\Theta}_1}{\xi_s} + \frac{\hat{\Theta}_3}{3\xi_s^3} + \ldots \right\} - \frac{\hat{\Theta}_2}{2\xi_s^2} - \frac{\hat{\Theta}_4}{4\xi_s^4} - \ldots \} . \]

This expression neglects the presence of \( \hat{\Theta}_i \) in eq. 1, hence of terms that are nonlinear in \( \hat{\Theta}_i \) in \( \delta V \). The linear term displayed in eq. 2 giving the behaviour of \( \delta V \) near the origin is unchanged by such corrections.

Since the non-analyticity arises in the Gaussian theory as well, one might attempt to define a relative renormalization, by subtracting the non-analyticity found in a Gaussian theory. However, it is easy to see that this does not remove the non-analytic behaviour. Consider \( V = ax^2/2 + bx^4/4 \), then

\[ \xi_s = \pm \sqrt{\frac{2}{a}} \left( 1 - \frac{b}{a^2} + \ldots \right) . \]

If we ignore the non-analyticity, we derive

\[ \delta a = 2a \frac{b}{N} \left( 1 + \frac{b}{a^2} - \frac{2b^2}{a^4} + \ldots \right) , \]

\[ \delta b = b \frac{a^2}{2b} + 3 - \frac{2b}{a^2} + \ldots \].

If we rescale \( a \) to its original value, we find

\[ \tilde{b} = b + \frac{1}{N} \left( a^2 - b - \frac{6b^2}{a^2} + \ldots \right) + \ldots . \]

Thus, if we now subtract the renormalization of \( b \) that would occur even in a free theory, we find the result of Ref. 3 (eq. 43). It therefore becomes clear how to make contact with the results of Ref. 3. Note that the saddlepoint evaluation of the \( \xi \) integral is valid for \( b > -a^2/4 \), which implies that the critical value \( b_c = -a^2/6 \) is within the validity of the evaluation.

It should be stressed that in the theory of the renormalization group, approximate recursion relations are extremely important, so the renormalization found by Brézin and Zinn-Justin may well be the appropriate approach to the large \( N \) renormalization group.
However, it is important to understand the nature of the approximation. It is in this context that the calculation given in this paper is of interest. Furthermore, it was shown here that there are terms induced in the potential that do not allow a perturbative surface interpretation. However, if we started with a potential that was not even, we would obtain non-analytic behaviour at other points in the integration domain. The flow of even a Gaussian integral is rather subtle when the large $N$ symmetry group’s volume is taken into account. It would be fascinating if this was related to the properties of the topological ‘critical’ point found by Witten[6].

I am grateful to O. Lechtenfeld, R. Myers and C. Nappi for helpful conversations, and to J. Zinn-Justin for an e-mail communication. This work was supported by D.O.E. grant DE-FG02-90ER40542.

References

1. E. Brézin and V. Kazakov, Phys. Lett. 236B (1990) 144; M. Douglas and S. Shenker, Nucl. Phys. B335 (1990) 635; D. Gross and A. Migdal, Phys. Rev. Lett. 64 (1990) 127
2. G. ’t Hooft, Nucl. Phys. B72 (1974) 461
3. E. Brézin and J. Zinn-Justin, E.N.S.–Saclay preprint LPTENS 92/19/SPhT/92-064 (1992)
4. M.L. Mehta, Random matrices, Academic Press (New York, 1967)
5. A. Anderson, R.C. Myers and V. Periwal, Phys. Lett. 254B (1991) 89, Nucl. Phys. B360 (1991) 463; S. Nishigaki and T. Yoneya, Nucl. Phys. B348 (1991) 787; P. di Vecchia, M. Kato and N. Ohta, Nucl. Phys. B357 (1991) 495
6. E. Witten, Nucl. Phys. B340 (1990) 281