Abstract

Ridge estimators regularize the squared Euclidean lengths of parameters. Such estimators are mathematically and computationally attractive but involve tuning parameters that can be difficult to calibrate. In this paper, we show that ridge estimators can be modified such that tuning parameters can be avoided altogether. We also show that these modified versions can improve on the empirical prediction accuracies of standard ridge estimators combined with cross-validation, and we provide first theoretical guarantees.

Keywords: Generalized linear models; high-dimensional estimation; ridge estimator
1 Introduction

High-dimensional estimators typically minimize an objective function that contains again two functions: a data-fitting function to ensure a good fit to the data and a penalty function to leverage additional information. Popular data-fitting functions are least-squares and negative log-likelihood; popular penalty functions are $\ell_1$ (lasso) (Tibshirani 1996) and $\ell_2^2$ (ridge) (Hoerl and Kennard 1970). The weighting between data-fitting and penalty function is finally determined by a tuning parameter, which needs to be calibrated to fit the specific estimator, data, and application at hand.

Known calibration schemes such as cross-validation (Stone 1974; Golub et al. 1979), stability selection (Meinshausen and Bühlmann 2010; Shah and Samworth 2013), and adaptive validation (Chichignoud et al. 2016; Li and Lederer 2019; Taheri et al. 2019) require two steps: compute the estimators or surrogates of them for a range of tuning parameters and then apply a rule to select among those candidate estimators. We now focus on the ridge estimators and pose the question of whether the calibration of their tuning parameters can instead be integrated into the estimation process directly.

In this paper, we modify standard ridge estimators such that the calibration of the tuning parameter is indeed part of the estimation process directly. We make use of two earlier lines of research: First, the edr (Huang et al. 2019), which shows that replacing $\ell_2^2$-regularization by $\ell_2$-regularization can make estimators amenable to recent techniques in high-dimensional theory. Second, the trex (Lederer and Müller 2015; Bien et al. 2018a,b; Lederer and Müller 2014), which proposes a way to integrate tuning parameter calibration into lasso-type estimators. However, while both of these lines of research focus on regularized least-squares in linear regression, we demonstrate that an inherent calibration of the ridge parameter is possible for a wide range of data-fitting functions and models.
We make three main contributions:

- We motivate alternative ridge estimators that dispense with tuning parameters (Section 2).
- We establish theoretical insights for these new estimators (Theorems 2.2, 2.3, and 3.2) and also for the underlying edr estimators (Theorems 2.1 and 3.1).
- We show that the tuning-free estimators can be readily computed (Section 4.1) and rival our outmatch standard pipelines empirically (Section 4.2).

2 Methodology

Standard methods to estimate a target parameter $\theta^* \in \mathbb{R}^p$ from data $Z$ are ridge-type estimators of the form

$$\hat{\theta}_{\text{ridge}}[\tau] \in \arg \min_{\theta \in \mathbb{R}^p} \left\{ D(\theta|Z) + \tau \|\theta\|_2^2 \right\},$$

(2.1)

where $D(\theta|Z) : \mathbb{R}^p \to \mathbb{R}$ is a data-fitting function and $\tau \in [0, \infty)$ is a tuning parameter. Ridge regularization, also known as Tikhonov regularization, can be traced back to (Tikhonov 1943). A common data-fitting function is the least-squares function $D(\theta|Z) := \|y - X\theta\|_2^2$ for regression data $Z = (y, X) \in \mathbb{R}^n \times \mathbb{R}^{n \times p}$, which leads to the usual ridge estimator (Hoerl and Kennard 1970). Well-known extensions of this estimator define $D(\theta|Z)$ as negative log-likelihood functions (Nelder and Wedderburn 1972).

A main challenge in the application of these estimators is the calibration of $\tau$. Our objective is, therefore, to rewrite the estimators such that we can avoid this tuning parameter.
Our first step is to change the $\ell_2^2$-prior function in (2.1) to $\ell_2$:

$$
\hat{\theta}_{edr}[\lambda] \in \arg\min_{\theta \in \mathbb{R}^p} \left\{ D(\theta|Z) + \lambda\|\theta\|_2 \right\}.
$$

(2.2)

These estimators generalize the edr estimator for linear regression (Huang et al. 2019). We will see in the following that the change from (2.1) to (2.2) allows us to apply standard techniques from modern high-dimensional theory while preserving the original estimators’ key features such as their computational simplicity.

Indeed, edr and ridge estimators are computational siblings. Assuming—for simplicity—here and in the following that the data-fitting function $D$ is convex and differentiable, we can define the “score” function as

$$
s(\theta) := -\frac{\partial D(\theta|Z)}{\partial \theta}
$$

(2.3)

and find the following (all proofs are deferred to Appendix A):

**Theorem 2.1** (Equivalence of edr and ridge). Edr estimator $\hat{\theta}_{edr}[\lambda]$ and ridge estimator $\hat{\theta}_{ridge}[\tau]$ are equivalent if the following two statements hold:

1. For each ridge estimator $\hat{\theta}_{ridge}[\tau]$ with $\tau \geq 0$, there exists a $\lambda = 2\tau \|\hat{\theta}_{ridge}[\tau]\|_2 \geq 0$ such that $\hat{\theta}_{edr}[\lambda] = \hat{\theta}_{ridge}[\tau]$;

2. For each edr estimator $\hat{\theta}_{edr}[\lambda]$ with $\lambda \geq 0$, there exists a $\tau = \lambda/(2\|\hat{\theta}_{edr}[\lambda]\|_2) \geq 0$ such that $\hat{\theta}_{ridge}[\tau] = \hat{\theta}_{edr}[\lambda]$. In particular, if $\hat{\theta}_{edr}[0] = 0_p$, then there exists $\tau = 0$ such that $\hat{\theta}_{ridge}[0] = \hat{\theta}_{edr}[0] = 0_p$.

Moreover, if $\hat{\theta}_{edr}[\lambda] = \hat{\theta}_{ridge}[\tau]$, then

$$
\lambda = \|s(\hat{\theta}_{edr}[\lambda])\|_2 = \|s(\hat{\theta}_{ridge}[\tau])\|_2.
$$

4
This result generalizes Theorem 2 of Huang et al. (2019) for the edr estimator in linear regression (that special case also follows from (Ahsen and Vidyasagar 2017, Theorem 5) for the clot estimator, which combines $\ell_1$- and $\ell_2$-regularization.). It shows first that the ridge and edr paths are remappings of each other and then gives a relationship between the edr tuning parameter and the score function. These two observations are crucial for the following.

Our second step is to modify the data-fitting function $D(\theta|Z)$ in (2.2) in a way that makes tuning parameters unnecessary. Our motivation comes from the trex in $\ell_1$-regularized linear regression (Lederer and Müller 2015):

$$
\hat{\theta}_{\text{trex}} \in \arg\min_{\theta \in \mathbb{R}^p} \left\{ \frac{\|y - X\theta\|_2^2}{\|X^\top (y - X\theta)\|_\infty/2} + \|\theta\|_1 \right\}.
$$

The idea of the trex is to amend the lasso estimator (Tibshirani 1996) with the additional factor $\|X^\top (y - X\theta)\|_\infty/2$ for an “inherent” calibration of the tuning parameter. This modification is unsuitable for us because our estimators include general data-fitting functions and a different regularizer, but we can still use that overall idea of complementing the data-fitting function with a factor. The factor is motivated by Theorem 2.1: we divide the objective function in (2.2) by $\lambda$ and then replace $\lambda$ by its “functional value” $\lambda = \|s(\hat{\theta}_{\text{edr}}[\lambda])\|_2$. We call the resulting estimator

$$
\hat{\theta}_{\text{t-ridge}} \in \arg\min_{\theta \in \mathbb{R}^p} \left\{ \frac{D(\theta|Z)}{\|s(\theta)\|_2} + \|\theta\|_2 \right\}
$$

the $t$-ridge. Similarly as the trex, the t-ridge does away with tuning parameters.

In contrast to trex that requires elaborate algorithms (Bien et al. 2018a), we first show that the t-ridge is simply one element of the path of the ridge estimator.
Theorem 2.2 (T-ridge is on the ridge path). Let \( \hat{\theta}_{\text{t-ridge}} \) be a t-ridge estimator defined by (2.4). Define \( \lambda := \|s(\hat{\theta}_{\text{t-ridge}})\|_2 \). Then, there always exists a tuning parameter \( \tau := \lambda/(2\|\hat{\theta}_{\text{edr}}[\lambda]\|_2) \) such that \( \hat{\theta}_{\text{ridge}}[\tau] \) minimizes the objective function in (2.4) and \( \hat{\theta}_{\text{t-ridge}} = \hat{\theta}_{\text{edr}}[\lambda] = \hat{\theta}_{\text{ridge}}[\tau] \).

Theorem 2.2 also implies that t-ridge is on the path of edr. Moreover, under a mild technical assumption, the t-ridge estimator is unique.

Theorem 2.3 (Uniqueness of the t-ridge estimator). If \( D(\theta|Z) > 0 \) for any \( \theta \in \mathbb{R}^p \), the minimum of the t-ridge objective function in (2.4) is unique.

Such a result has not been established for the trex estimator. It ensures that the t-ridge estimator retains the uniqueness of the ridge estimator. Hence, calculating the t-ridge essentially amounts to a grid search on the ridge path—see Section 4.1 for details.

3 Applications in generalized linear models

We now apply the t-ridge estimator to generalized linear models and derive the first theoretical results.

3.1 T-ridge estimator for generalized linear models

In this section, we exemplify the t-ridge estimator for maximum regularized likelihood estimation in generalized linear models. We consider data \( Z = (y, X) \) that follow a conditional distribution

\[
y_i|\mathbf{x}_i, \beta^* \sim F \quad \text{with} \quad g(\mathbb{E}(y_i|\mathbf{x}_i, \beta^*)) = \mathbf{x}_i^\top \beta^*.
\]

(3.1)
Here, \( \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n \) is a vector of outcomes and \( \mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)^\top \in \mathbb{R}^{n \times p} \) a design matrix. The distribution \( F \) is assumed in the exponential family, \( g : \mathbb{R} \mapsto \mathbb{R} \) is a link function, and \( \mathbf{\beta}^* \in \mathbb{R}^p \) is the unknown regression vector. We allow for high-dimensional settings, that is, the number of parameters \( p \) may rival or even exceed the number of observations \( n \).

For every vector \( \mathbf{\beta} \in \mathbb{R}^p \), the density of \( y_i | \mathbf{x}_i, \mathbf{\beta} \) can be written as (Nelder and Wedderburn 1972)
\[
f(y_i | \mathbf{x}_i, \mathbf{\beta}) = \exp \left\{ \frac{y_i \mathbf{x}_i^\top \mathbf{\beta} - b(\mathbf{x}_i^\top \mathbf{\beta})}{d(\phi)} + c(y_i, \phi) \right\}.
\]
The inner product \( \mathbf{x}_i^\top \mathbf{\beta} \) is related to the mean of the distribution and the dispersion parameter \( \phi \in \mathbb{R} \) to the variance of the distribution; indeed, the mean of \( y_i | \mathbf{x}_i, \mathbf{\beta} \) is \( g^{-1}(\mathbf{x}_i^\top \mathbf{\beta}) \), and the variance of \( y_i | \mathbf{x}_i, \mathbf{\beta} \) is \( d(\phi) \) times the second derivative of the function \( b \) with respect to \( \mathbf{x}_i^\top \mathbf{\beta} \). Without loss of generality, each outcome \( y_i, i \in \{1, \ldots, n\} \) can be written as its’ true mean \( \mathbb{E}(y_i | \mathbf{x}_i, \mathbf{\beta}^*) = g^{-1}(\mathbf{x}_i^\top \mathbf{\beta}^*) \) plus a random noise. That is,
\[
y_i = g^{-1}(\mathbf{x}_i^\top \mathbf{\beta}^*) + \varepsilon_i, \quad \text{for } i \in \{1, \ldots, n\},
\]
where \( \varepsilon_i \in \mathbb{R} \) are the random noises. The real-valued functions \( b, c, d, \) and \( g^{-1} \) are specified by the concrete choice of the distribution \( F \); their forms for the most common distributions are given in Table 1. To be clear, we consider the canonical link function, which satisfies \( g^{-1}(z) = b'(z) \) for any \( z \in \mathbb{R} \).

Assuming that the \( y_i \)'s are independent, the log-likelihood function of \( F \) is
\[
\ln(\mathbf{\beta}|\mathbf{y}, \mathbf{X}) = \log \left( \prod_{i=1}^{n} f(y_i | \mathbf{x}_i^\top \mathbf{\beta}, \phi) \right) = \sum_{i=1}^{n} \left( \frac{y_i \mathbf{x}_i^\top \mathbf{\beta} - b(\mathbf{x}_i^\top \mathbf{\beta})}{d(\phi)} + c(y_i, \phi) \right).
\]
Table 1: common distributions with their $b$, $c$, $d$, and $g^{-1}$ functions

| distribution of $F$ | $b(x_i^\top \beta)$ | $g^{-1}(x_i^\top \beta)$ | $c(y_i, \phi)$ | $d(\phi)$ |
|---------------------|---------------------|---------------------|----------------|-------------|
| Gaussian            | $\frac{(x_i^\top \beta)^2}{2}$ | $x_i^\top \beta$ | $\frac{1}{2d(\phi)}y_i^2 - \frac{1}{2} \log (2\pi d(\phi))$ | $\sigma^2$ |
| Poisson             | $\exp(x_i^\top \beta)$ | $\exp(x_i^\top \beta)$ | $-\log(y_i!)$ | $1$ |
| Bernoulli           | $\log(1 + \exp(x_i^\top \beta))$ | $\frac{\exp(x_i^\top \beta)}{1 + \exp(x_i^\top \beta)}$ | $0$ | $1$ |

Omitting factors and summands that do not depend on $\beta$, we find the negative log-likelihood data-fitting term can be simplified as

$$D(\beta|(y, X)) = -\sum_{i=1}^{n} (y_i x_i^\top \beta - b(x_i^\top \beta)).$$

(3.2)

Observing that the derivative of function $b$ is $g^{-1}$, the corresponding score function (2.3) is

$$s(\beta) = -\sum_{i=1}^{n} x_i^\top (y_i - g^{-1}(x_i^\top \beta)).$$

(3.3)

In view of (2.4), this means that the t-ridge estimator is

$$\hat{\beta}_{\text{t-ridge}} \in \arg \min_{\beta \in \mathbb{R}^p} \left\{ -\sum_{i=1}^{n} (y_i x_i^\top \beta - b(x_i^\top \beta)) \middle\| \sum_{i=1}^{n} x_i^\top (y_i - g^{-1}(x_i^\top \beta)) \right\|_2 + \| \beta \|_2 \right\}.$$

(3.4)

As compared to the general form of the t-ridge in (2.4), the general parameter $\theta$ is specified to the regression vector $\beta$, the general data $Z$ is specified to the regression data $(y, X)$, and the data-fitting function $D$ and the score function $s$ are specified to (3.2) and (3.3), respectively.
3.2 Further theoretical insights

We now establish theoretical insights into the t-ridge estimator. Our two results are Theorem 3.1, which is a novel prediction guarantee for (a generalized version of) the related edr estimator, and Theorem 3.2, which is a prediction guarantee for the t-ridge estimator.

We first introduce a standard “margin condition” for the function $b$.

**Condition 3.1** (Margin condition for the function $b$). For each $x_i, i \in \{1, \ldots, n\}$, if a given vector $\beta \in \mathbb{R}^p$ such that $\|\beta - \beta^*\|_2 \leq \delta$ for some constants $\delta > 0$, then there exist a constant $C_i > 0$ with

$$b(x_i^\top \beta) - b(x_i^\top \beta^*) \geq b'(x_i^\top \beta^*)(x_i^\top \beta - x_i^\top \beta^*) + \frac{1}{C_i^2}(x_i^\top \beta - x_i^\top \beta^*)^2.$$ 

Notice that the function $b$ for generalized linear model is differentiable, so $b'(\cdot)$ always exists. Such a constant $C_i$ always exists in generalized linear models (van de Geer 2016, Section 11.6) once $x_i, \beta \in \mathbb{R}^p$ are given; for example, $C_i = 2, i \in \{1, \ldots, n\}$ in the case that $F$ satisfies the Gaussian distribution. The existence of the constants implies in particular that the function $b$ is strictly convex.

The following theorem shows that $\lambda^* := \|s(\beta^*)\|_2 = \|X^\top \varepsilon\|_2$, where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)^\top$ is the noise vector, is indeed an optimal tuning parameter for the edr estimator. This result further strengthens our motivation for the t-ridge estimator as the edr estimator calibrated inherently to that $\lambda$.

**Theorem 3.1** (Prediction error bound for the generalized edr). Consider data $(y, X)$ that follow (3.1) and the corresponding edr estimator

$$\hat{\beta}_{edr}[\lambda] \in \arg\min_{\beta \in \mathbb{R}^p} \left\{-\sum_{i=1}^n (y_i x_i^\top \beta - b(x_i^\top \beta)) + \lambda \|\beta\|_2\right\}$$ (3.5)
according to (2.2). With \( C = \max_{i \in \{1, \ldots, n\}} \{ C_i \} \) as in the margin condition 3.1 and define \( \lambda^* := \| s(\beta^*) \|_2 \), it holds for \( \lambda \geq \lambda^* \) that

\[
\| X (\hat{\beta}_{\text{edr}}[\lambda] - \beta^*) \|_2^2 \leq 2C^2 \lambda \| \beta^* \|_2.
\]

This bound complements the bound on the individual prediction errors \( |x_i^\top (\hat{\beta}_{\text{edr}}[\lambda] - \beta^*)| \) that has been derived previously (Huang et al. 2019, Lemma 1). But more interesting here is that it provides further support for the t-ridge estimator: The bound suggests that for accurate prediction with the edr estimator, the tuning parameter \( \lambda \) needs to be sufficiently small (since the bound is proportional to \( \lambda \)), but not too small (to satisfy the condition \( \lambda \geq \| X^\top \varepsilon \|_2 \)). In particular, the bound is optimized at \( \lambda = \| s(\beta^*) \|_2 = \| X^\top \varepsilon \|_2 \)—in line with our motivation for the t-ridge estimator.

The following theorem finally gives a bound on the prediction loss of the t-ridge estimator.

**Theorem 3.2** (Prediction error bound for the t-ridge). Consider data \( (y, X) \) that follow (3.1) and t-ridge estimator \( \hat{\beta}_{\text{t-ridge}} \) defined in (3.4). Let \( \hat{\lambda} := \| s(\hat{\beta}_{\text{t-ridge}}) \|_2 \) and \( \lambda^* := \| s(\beta^*) \|_2 \). With \( C = \max_{i \in \{1, \ldots, n\}} \{ C_i \} \) as in the margin condition 3.1, it holds that

\[
\| X (\hat{\beta}_{\text{t-ridge}} - \beta^*) \|_2^2 \leq 2C^2 \max\{ \lambda^*, \hat{\lambda} \} \| \beta^* \|_2.
\]

This bound parallels the one for the trex for \( \ell_1 \)-regularized linear regression (Bien et al. 2018b, Theorem 2). Moreover, it relates to Theorem 3.1; in particular, if \( \hat{\lambda} \leq \lambda^* \), then the t-ridge bound equals the edr bound at the optimal tuning parameter \( \lambda^* \)—without the t-ridge knowing that tuning parameter. Interestingly, one can check if this case applies in an extremely simple way:

**Lemma 3.1** (Relationship between \( \lambda^* \) and \( \hat{\lambda} \)). If \( D(\hat{\beta}_{\text{t-ridge}}|y, X) > 0 \), then \( \hat{\lambda} \geq \lambda^* \); if \( D(\hat{\beta}_{\text{t-ridge}}|y, X) < 0 \), then \( \hat{\lambda} \leq \lambda^* \).
Together with Theorem 3.2, this gives a concrete guarantee for the prediction accuracy of the t-ridge.

4 Algorithm and Numerical Analysis

In this section, we introduce a specific algorithm for the t-ridge estimator for maximum regularized likelihood estimation in generalized linear models. We then show that the t-ridge matches or even outperforms ridge combined with cross-validation, the standard pipeline in this context, in the three most common cases for the distribution $F$: Gaussian, Poisson, and Bernoulli.

4.1 Algorithm

The t-ridge’s objective function for maximum regularized likelihood estimation in generalized linear regression

$$f_{t\text{-ridge}} : \beta \rightarrow -\sum_{i=1}^{n} (y_i x_i^\top \beta - b(x_i^\top \beta)) + \|\beta\|_2$$

seems very hard to optimize, in particular, because it is non-convex. But Theorem 2.2 entails a very simple and effective optimization strategy: solve the ridge path with a standard algorithm and then select the solution that minimizes the t-ridge objective function (4.1).

It turns out that this strategy can be improved even further: one can use the differentiability of the objective function (4.1) to speed up the grid search over the ridge solution path. We proceed in three steps:

Step 1: Compute a stationary point $\hat{\beta}_{sp}$ of the t-ridge object function (4.1) with a standard algorithm such as the Fletcher-Reeves algorithm (Fletcher and Reeves 1964).
Step 2: Compute the ridge tuning parameter $\tau := \|s(\hat{\beta}_{sp})\|^2/(2\|\hat{\beta}_{sp}\|^2)$ and set $\tau_{\min} := \max\{0.05, (\tau - c)\}$ and $\tau_{\max} := \tau + c$ for a given range $c \in (0, \infty)$.

Step 3: Compute the ridge estimator with a standard algorithm for $m$ equally spaced tuning parameters in $[\tau_{\min}, \tau_{\max}]$, and then select the corresponding estimator that minimizes the t-ridge object function (4.1).

The underpinning idea is that the stationary points of (4.1) give a hint of what ridge estimators are relevant so that the search over the ridge path can be narrowed down to a small interval. And we indeed find empirically that the gain that is due to restricting the ridge path (Step 3) outweighs the computations of the stationary point (Step 1) and the two ridge estimators (Step 2).

Another statement of our approach is Algorithm 1.

Throughout, we set $c := 0.1$ and $m := 1000$, which leads to excellent results over a wide range of settings. As a technical detail, we set $\tau_{\min} := 10^{10}$ and $\tau_{\max} := 10^{11}$ if $\|\hat{\beta}_{sp}\|^2 = 0$ to avoid vanishing denominators, and for similar reasons, we set $\tau_{\min} := \max\{0.05, \tau - c\}$.

4.2 Numerical Analysis

We now show that our pipeline rivals $K$-fold cross-validation, the standard pipeline in this context. We compute the latter with the glmnet package in R with default settings (Friedman et al. 2010).

The dimensions of the design matrix are $(n, p) \in \{(100, 300), (200, 500), (50, 1000)\}$. Each row $x_i \in \mathbb{R}^p$ of the design matrix $X \in \mathbb{R}^{n \times p}$ is sampled from a $p$-dimensional normal distribution with mean $0_p$ and covariance matrix $\Sigma$, where $\Sigma_{uv} = k^{|u-v|}$, $u, v \in \{1, \ldots, p\}$, and $k \in \{0, 0.2, 0.4\}$ is the magnitude of the mutual correlations ($0^0 := 1$). The columns of
Input: data \((y, X)\), range \(c\), and number of candidate ridge tuning parameters \(m\)

Output: \(\hat{\beta}_{t\text{-ridge}}\) from (3.4)

Compute a stationary point \(\hat{\beta}_{sp}\) of (4.1) using the Fletcher-Reeves algorithm

Set \(\tau := \|s(\hat{\beta}_{sp})\|_2/(2\|\hat{\beta}_{sp}\|_2)\)

if \(\hat{\beta}_{sp} = 0_p\) then
\[
\tau_{\text{min}} := 10^{10}; \tau_{\text{max}} := 10^{11}
\]
else
\[
\tau_{\text{min}} := \max\{0.05, \tau - c\}
\]
\[
\tau_{\text{max}} := \tau + c
\]
end if

for \(i = 1\) to \(m\) do
\[
\tau_i := \tau_{\text{min}} + (\tau_{\text{max}} - \tau_{\text{min}}) \cdot i/m
\]
Compute \(\hat{\beta}_{\text{ridge}}[\tau_i]\) and \(f_{t\text{-ridge}}(\hat{\beta}_{\text{ridge}}[\tau_i])\)
end for

Set \(\hat{\tau} \in \arg\min_{i \in \{1, \ldots, m\}} \{f_{t\text{-ridge}}(\hat{\beta}_{\text{ridge}}[\tau_i])\}\)

Return \(\hat{\beta}_{t\text{-ridge}} = \hat{\beta}_{\text{ridge}}[\hat{\tau}]\)

**Algorithm 1:** \(t\)-ridge in generalized linear models

the design matrix are then normalized to have Euclidean norm equal to one. The entries of the regression vector \(\beta^*\) are sampled i.i.d. from \(\mathcal{N}(0, 1)\) and then projected onto the row space of \(X\) to ensure identifiability (Shao and Deng 2012; Bühlmann 2013).

We run 100 experiments for each set of parameters and report the means of the relative prediction errors defined by \(\|X(\hat{\beta} - \beta^*)\|_2/\|X\beta^*\|_2\).
4.2.1 Gaussian case

We first generate Gaussian data, where the outcome vector

\[ y = X\beta^* + \epsilon \]

is the true signal \( X\beta^* \) plus the noise vector \( \epsilon \). The entries of the noise vector \( \epsilon \) are sampled i.i.d. from \( \mathcal{N}(0, \sigma^2) \), where \( \sigma^2 \) is set such that the signal-to-noise ratio

\[ \frac{\left( \sum_{i=1}^{n} (x_i^\top \beta^*)^2 - \left( \frac{\sum_{i=1}^{n} x_i^\top \beta^*}{n} \right)^2 \right)}{\sigma^2(n - 1)} \]

equals 10. According to Table 1, the t-ridge estimator (3.4) is

\[ \hat{\beta}_{t\text{-ridge}} \in \arg\min_{\beta \in \mathbb{R}^p} \left\{ \frac{\|y - X\beta\|^2}{2\|X^\top(y - X\beta)\|^2} + \|eta\|^2 \right\}. \]

Table 2: t-ridge outperforms \( K \)-fold cross-validated ridge (\( K \)-fold CV ridge) for \( K \in \{5, 10\} \) in prediction on Gaussian data with \( k = 0 \).

| Relative prediction error | n   | p   | Mean of relative errors (sd) |
|----------------------------|-----|-----|-----------------------------|
|                            |     |     | t-ridge | 5-fold CV ridge | 10-fold CV ridge |
| \( \frac{\|X(\hat{\beta} - \beta^*)\|_2}{\|X\beta^*\|_2} \) | 100 | 300 | 0.34 (0.03) | 0.53 (0.05) | 0.53 (0.04) |
|                            | 200 | 500 | 0.36 (0.02) | 0.51 (0.02) | 0.51 (0.02) |
|                            | 50  | 1000| 0.31 (0.04) | 0.57 (0.28) | 0.54 (0.28) |

Table 2 demonstrates that the t-ridge outperforms 5- and 10-fold cross-validated ridge. (The results for \( k \in \{0.2, 0.4\} \) are deferred to Table 5 in Appendix B).
Figure 1: Relative errors of the t-ridge estimator as compared to the maximum likelihood estimator $\|\hat{\beta}_{t\text{-ridge}} - \hat{\beta}_{\text{mle}}\|_2 / \|\hat{\beta}_{\text{mle}}\|_2$ for Gaussian data with $p = 20$ and $k = 0$. The t-ridge estimator quickly approximates the maximum likelihood estimator when the sample size $n$ increases.

Figure 1 confirms that the t-ridge converges rapidly to the unregularized maximum likelihood estimator $\hat{\beta}_{\text{mle}}$, which minimizes $D(\beta|y, X)$ defined in (3.2), in the relative error $\|\hat{\beta}_{t\text{-ridge}} - \hat{\beta}_{\text{mle}}\|_2 / \|\hat{\beta}_{\text{mle}}\|_2$ as $p = 20$ is fixed and the number of observations $n$ increases. Similar observations can be made in the Poisson and Bernoulli cases. These results suggest that t-ridge estimators can be applied without regard of the dimensionality of the problem.

### 4.2.2 Poisson case

We then generate Poisson data. According to Table 1, the t-ridge estimator (3.4) is

$$
\hat{\beta}_{t\text{-ridge}} \in \arg \min_{\beta \in \mathbb{R}^p} \left\{ \frac{- \sum_{i=1}^{n} (y_i x_i^\top \beta - \exp(x_i^\top \beta))}{\sum_{i=1}^{n} (y_i - \exp(x_i^\top \beta))_2^2} + \|eta\|_2 \right\}.
$$
Table 3: t-ridge outperforms K-fold cross-validated ridge (K-fold CV ridge) for \( K \in \{5, 10\} \) in prediction on Poisson data with \( k = 0 \).

| Relative prediction error | n  | p  | Mean of relative errors (sd) |
|----------------------------|----|----|-------------------------------|
|                            |    |    | t-ridge | 5-fold CV ridge | 10-fold CV ridge |
| \( \frac{\|X(\hat{\beta} - \beta^*)\|_2}{\|X\beta^*\|_2} \) | 100 | 300 | 0.60 (0.07) | 0.83 (0.06) | 0.82 (0.06) |
|                            | 200 | 500 | 0.65 (0.06) | 0.79 (0.05) | 0.79 (0.05) |
|                            | 50  | 1000| 0.84 (0.06) | 0.97 (0.03) | 0.97 (0.03) |

Table 3 shows that the t-ridge estimator outperforms 5- and 10-fold cross-validated ridge across all settings.

4.2.3 Bernoulli case

We finally generate Bernoulli data. According to Table 1, the t-ridge estimator (3.4) is

\[
\hat{\beta}_{t-ridge} \in \arg \min_{\beta \in \mathbb{R}^p} \left\{ -\sum_{i=1}^n \left( y_i x_i^T \beta - \left(1 + \exp(x_i^T \beta)\right)\right) \frac{\|\sum_{i=1}^n x_i^T \left(y_i - \frac{\exp(x_i \beta)}{1 + \exp(x_i \beta)}\right)\|_2}{\|X\beta^*\|_2} + \|\beta\|_2 \right\}
\]

Table 4 shows that the t-ridge estimators rival 5- and 10-fold cross-validated ridge across all settings.

Taken together, the results in the Gaussian, Poisson, and Bernoulli case suggest that the t-ridge estimator is an alternative to standard pipelines for data of any dimension and type.
Table 4: t-ridge rivals $K$-fold cross-validated ridge ($K$-fold CV ridge) for $K \in \{5, 10\}$ in prediction on Bernoulli data with $k = 0$.

| Relative prediction error | n   | p   | Mean of relative errors (sd) |
|---------------------------|-----|-----|-----------------------------|
|                           |     |     | t-ridge | 5-fold CV ridge | 10-fold CV ridge |
| $\frac{||X(\hat{\beta}-\beta^*)||_2}{||X\beta^*||_2}$ |     |     | 100 300 | 0.87 (0.06) | 0.88 (0.07) | 0.88 (0.07) |
|                           |     |     | 200 500 | 0.86 (0.06) | 0.87 (0.07) | 0.86 (0.06) |
|                           |     |     | 50 1000 | 0.88 (0.08) | 0.91 (0.09) | 0.90 (0.10) |
5 Discussion

We have shown that the calibration of the tuning parameter can be incorporated directly into the formulation of ridge estimators. Since our approach in Section 2, called t-ridge, requires essentially only that the data-fitting function is differentiable, it can be applied to a wide variety of ridge estimators.

As an example, we have detailed the t-ridge estimator in Section 3 for generalized linear models, and we complemented the theoretical insights of Section 2 to corroborate the estimator’s motivation further. We expect that these mathematical insights will also be of use for tuning parameter calibration beyond the ridge estimator. We have also shown in Section 4 that the t-ridge estimator can be implemented efficiently and that it can outperform standard pipelines empirically across different types of data and dimensions.

We finally expect that tuning-free estimators such as trex and t-ridge can also be valuable for post-selection problems (Taylor et al. 2014; Taylor and Tibshirani 2017) since the inclusion of calibration schemes can be difficult in such problems.

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References

Ahsen, M. E. and Vidyasagar, M. (2017), ‘Two new approaches to compressed sensing exhibiting both robust sparse recovery and the grouping effect’, 2017 Indian Control Conference (ICC).
Bien, J., Gaynanova, I., Lederer, J. and Müller, C. L. (2018a), ‘Non-convex global minimization and false discovery rate control for the trex’, *Journal of Computational and Graphical Statistics* 27(1), 23–33.

Bien, J., Gaynanova, I., Lederer, J. and Müller, C. L. (2018b), ‘Prediction error bounds for linear regression with the trex’, *Test* 28(2), 451–474.

Bühlmann, P. (2013), ‘Statistical significance in high-dimensional linear models’, *Bernoulli* 19(4), 1212–1242.

Chichignoud, M., Lederer, J. and Wainwright, M. J. (2016), ‘A practical scheme and fast algorithm to tune the lasso With optimality guarantees’, *Journal of Machine Learning Research* 17(231), 1–20.

Fletcher, R. and Reeves, C. M. (1964), ‘Function minimization by conjugate gradients’, *The Computer Journal* 7(2), 149–154.

Friedman, J., Hastie, T. and Tibshirani, R. (2010), ‘Regularization paths for generalized linear models via coordinate descent’, *Journal of Statistical Software* pp. 1–22.

Golub, G. H., Heath, M. and Wahba, G. (1979), ‘Generalized cross-validation as a method for choosing a good ridge parameter’, *Technometrics* 21(2), 215–223.

Hoerl, A. E. and Kennard, R. W. (1970), ‘Ridge regression: biased estimation for nonorthogonal problems’, *Technometrics* 12(1), 55–67.

Huang, S.-T., Düren, Y., Hellton, K. H. and Lederer, J. (2019), ‘Tuning parameter calibration for prediction in personalized medicine’, *arXiv e-prints arXiv:1909.10635v3*. 

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Kuhn, H. W. and Tucker, A. W. (1951), Nonlinear programming, in ‘Proceedings of the second Berkeley Symposium on Mathematical Statistics and Probability’, University of California Press, Berkeley, Calif., pp. 481–492.

Lederer, J. and Müller, C. L. (2014), ‘Topology adaptive graph estimation in high dimensions’, arXiv e-prints arXiv:1410.7279.

Lederer, J. and Müller, C. L. (2015), ‘Don’t fall for tuning parameters: tuning-free variable selection in high dimensions with the trex’, Proceedings of the Twenty-Ninth AAAI conference on Artificial Intelligence.

Li, W. and Lederer, J. (2019), ‘Tuning parameter calibration for \(\ell_1\)-regularized logistic regression’, Journal of Statistical Planning and Inference 202, 80–98.

Meinshausen, N. and Bühlmann, P. (2010), ‘Stability selection’, Journal of the Royal Statistical Society: Series B (Statistical Methodology) 72(4), 417–473.

Nelder, J. A. and Wedderburn, R. W. M. (1972), ‘Generalized linear models’, Journal of the Royal Statistical Society. Series A (General) 135(3), 370–384.

Shah, R. D. and Samworth, R. J. (2013), ‘Variable selection with error control: another look at stability selection’, Journal of the Royal Statistical Society: Series B (Statistical Methodology) 75(1), 55–80.

Shao, J. and Deng, X. (2012), ‘Estimation in high-dimensional linear models with deterministic design matrices’, Annals of Statistics 40(2), 812–831.

Stone, M. (1974), ‘Cross-validatory choice and assessment of statistical predictions’, Journal of the Royal Statistical Society. Series B (Statistical Methodology) 36(2), 111–133.
Appendix A  Proofs

A.1 Proof of Theorem 2.1

Proof. We first show 1., that is, for each ridge estimator $\hat{\theta}_{\text{ridge}}[\tau]$ with $\tau \geq 0$, there always exists a $\lambda \geq 0$ satisfying $\lambda = 2\tau \|\hat{\theta}_{\text{ridge}}[\tau]\|_2$ such that $\hat{\theta}_{\text{edr}}[\lambda] = \hat{\theta}_{\text{ridge}}[\tau]$.

Using the Karush–Kuhn–Tucker (Kuhn and Tucker 1951) conditions for both (2.1) and (2.2), we have

$$-s(\hat{\theta}_{\text{ridge}}[\tau]) + 2\tau \hat{\theta}_{\text{ridge}}[\tau] = 0_p.$$
and

$$-s(\hat{\theta}_{edr}[\lambda]) + \lambda \frac{\partial \| \theta \|_2}{\partial \theta} \bigg|_{\theta = \hat{\theta}_{edr}[\lambda]} = 0_p, \quad \text{where the sub-differential of the } \ell_2 \text{ norm with respect to } \theta \text{ is defined as}$$

$$\frac{\partial \| \theta \|_2}{\partial \theta} := \begin{cases} \frac{\theta}{\| \theta \|_2} & \text{if } \theta \neq 0_p \\ \{ \kappa \in \mathbb{R}^p : \| \kappa \|_2 \leq 1 \} & \text{if } \theta = 0_p. \end{cases}$$

By rearrangement, we obtain

$$s(\hat{\theta}_{ridge}[\tau]) = 2\tau \hat{\theta}_{ridge}[\tau], \quad (A.1)$$

and

$$s(\hat{\theta}_{edr}[\lambda]) = \lambda \frac{\partial \| \theta \|_2}{\partial \theta} \bigg|_{\theta = \hat{\theta}_{edr}[\lambda]} \quad (A.2)$$

If \( \hat{\theta}_{ridge}[\tau] = 0_p \), by (A.1) we have

$$s(\hat{\theta}_{ridge}[\tau]) = s(0_p) = 0_p. \quad (A.3)$$

By setting \( \lambda = 0 \), (A.2) yield

$$s(\hat{\theta}_{edr}[0]) = 0_p.$$

The convexity of (2.1) and (2.2) imply that \( \hat{\theta}_{edr}[0] = 0_p = \hat{\theta}_{ridge}[\tau] \). In addition, taking \( \ell_2 \) norm on both sides of equations (A.1) and (A.2) yields

$$\| s(\hat{\theta}_{ridge}[\tau]) \|_2 = 0 = \| s(\hat{\theta}_{edr}[0]) \|_2.$$

If \( \hat{\theta}_{ridge}[\tau] \neq 0_p \), letting \( \lambda = 2\tau \| \hat{\theta}_{ridge}[\tau] \|_2 \) and by (A.1), we have

$$s(\hat{\theta}_{ridge}[\tau]) = 2\tau \| \hat{\theta}_{ridge}[\tau] \|_2 \frac{\hat{\theta}_{ridge}[\tau]}{\| \hat{\theta}_{ridge}[\tau] \|_2} = \lambda \frac{\hat{\theta}_{ridge}[\tau]}{\| \hat{\theta}_{ridge}[\tau] \|_2}.$$
Comparing the equation above with (A.2), we observe that \( \hat{\theta}_{\text{edr}}[\lambda] = \hat{\theta}_{\text{ridge}}[\tau] \) can be a solution of (A.2). Furthermore, by taking \( \ell_2 \) norm on both sides of equations (A.1) and (A.2), we have

\[
\|s(\hat{\theta}_{\text{ridge}}[\tau])\|_2 = \lambda = \|s(\hat{\theta}_{\text{edr}}[\lambda])\|_2
\]
as desired.

Secondly, we prove 2., that is, for each edr estimator \( \hat{\theta}_{\text{edr}}[\lambda] \) with \( \lambda \geq 0 \), there always exists a \( \tau \geq 0 \) satisfying \( \tau = \lambda/(2\|\hat{\theta}_{\text{edr}}[\lambda]\|_2) \) such that \( \hat{\theta}_{\text{ridge}}[\tau] = \hat{\theta}_{\text{edr}}[\lambda] \).

If \( \hat{\theta}_{\text{edr}}[\lambda] = 0_p \) and \( \lambda \neq 0 \), we can find \( \tau = \lambda/(2\|\hat{\theta}_{\text{edr}}[\lambda]\|_2) = \infty \) such that \( \hat{\theta}_{\text{ridge}}[\infty] = 0_p \), which is shown in the following. Let \( \tau = \infty \), then by the definition in (2.1), we have for any \( \theta \in \mathbb{R}^p \)

\[
D(\hat{\theta}_{\text{ridge}}[\infty]|Z) + \infty \cdot \|\hat{\theta}_{\text{ridge}}[\infty]\|_2^2 \leq D(\theta|Z) + \infty \cdot \|\theta\|_2^2.
\]
However, we observe that

\[
D(0_p|Z) + \infty \cdot \|0_p\|_2^2 = D(0_p|Z) + \lambda \|0_p\|_2^2 \\
\leq D(\theta|Z) + \lambda \|\theta\|_2^2 \\
\leq D(\theta|Z) + \infty \cdot \|\theta\|_2^2
\]
holds for any vector \( \theta \in \mathbb{R}^p \). By the convexity of the data-fitting function \( D \), we know that \( \hat{\theta}_{\text{ridge}}[\tau] \) is unique and hence, \( \hat{\theta}_{\text{ridge}}[\tau] = 0_p \).

If \( \hat{\theta}_{\text{edr}}[0] = 0_p \) and \( \lambda = 0 \), then we have

\[
D(0_p|Z) + 0 \cdot \|0_p\|_2^2 = D(0_p|Z) + 0 \cdot \|0_p\|_2 \\
\leq D(\theta|Z) + 0 \cdot \|\theta\|_2 \\
= D(\theta|Z) + 0 \cdot \|\theta\|_2^2
\]
Hence, \( \hat{\theta}_{\text{ridge}}[0] = 0_p \).
If $\hat{\theta}_{edr}[\lambda] \neq 0$, letting $\tau = \lambda/(2\|\hat{\theta}_{edr}[\lambda]\|_2)$ and by (A.2), we have
\[
s(\hat{\theta}_{edr}[\lambda]) = 2\frac{\lambda}{2\|\hat{\theta}_{edr}[\lambda]\|_2} \hat{\theta}_{edr}[\lambda]
= 2\tau \hat{\theta}_{edr}[\lambda].
\]
Comparing the equation above with (A.1), we observe that $\hat{\theta}_{ridge}[\tau] = \hat{\theta}_{edr}[\lambda]$ can be a solution of (A.1). Again, by taking $\ell_2$ norm on both sides of equations (A.1) and (A.2), we also have
\[
\|s(\hat{\theta}_{ridge}[\tau])\|_2 = \lambda = \|s(\hat{\theta}_{edr}[\lambda])\|_2
\]
as desired.

A.2 Proof of Theorem 2.2

Proof. We prove the theorem by two steps:

1. $\hat{\theta}_{t-ridge} = \hat{\theta}_{edr}[\lambda]$ where $\lambda = \|s(\hat{\theta}_{t-ridge})\|_2$;

2. $\hat{\theta}_{edr}[\lambda] = \hat{\theta}_{ridge}[\tau]$ where $\tau = \lambda/(2\|\hat{\theta}_{edr}[\lambda]\|_2)$.

Firstly, we prove 1.. By the definition of t-ridge estimator in (2.4), we have
\[
\frac{D(\hat{\theta}_{t-ridge}|Z)}{\|s(\hat{\theta}_{t-ridge})\|_2} + \|\hat{\theta}_{t-ridge}\|_2 \leq \frac{D(\hat{\theta}_{edr}[\lambda]|Z)}{\|s(\hat{\theta}_{edr}[\lambda])\|_2} + \|\hat{\theta}_{edr}[\lambda]\|_2.
\]
Notice that $\lambda = \|s(\hat{\theta}_{t-ridge})\|_2 \neq 0$. By Theorem 2.1, we have
\[
\|s(\hat{\theta}_{t-ridge})\|_2 = \lambda = \|s(\hat{\theta}_{edr}[\lambda])\|_2.
\]
Using this relation and multiplying $\lambda$ on the both sides of above inequality yields
\[
D(\hat{\theta}_{t-ridge}|Z) + \lambda\|\hat{\theta}_{t-ridge}\|_2 \leq D(\hat{\theta}_{edr}[\lambda]|Z) + \lambda\|\hat{\theta}_{edr}[\lambda]\|_2.
\]
On the other hand, by the definition of edr estimator, we have

$$ D(\hat{\theta}_{\text{edr}}[\lambda]|Z) + \lambda \|\hat{\theta}_{\text{edr}}[\lambda]\|_2 \leq D(\hat{\theta}_{\text{t-ridge}}|Z) + \lambda \|\hat{\theta}_{\text{t-ridge}}\|_2. $$

Combining the two inequalities above yields

$$ D(\hat{\theta}_{\text{t-ridge}}|Z) + \lambda \|\hat{\theta}_{\text{t-ridge}}\|_2 = D(\hat{\theta}_{\text{edr}}[\lambda]|Z) + \lambda \|\hat{\theta}_{\text{edr}}[\lambda]\|_2. $$

Since we assume the data-fitting function $D$ is convex, the objective function of edr method is also convex, which means edr has a unique global minimum. Hence, we have $\hat{\theta}_{\text{t-ridge}} = \hat{\theta}_{\text{edr}}[\lambda]$, which proved 1..

Since the second equality $\hat{\theta}_{\text{edr}}[\lambda] = \hat{\theta}_{\text{ridge}}[\tau]$ can be obtained directly by Theorem 2.1, we finish the prove. \qed

### A.3 Proof of Theorem 2.3

**Proof.** We prove this theorem by contradiction. Suppose there are two t-ridge estimators $\hat{\theta}'_{\text{t-ridge}}$ and $\hat{\theta}''_{\text{t-ridge}}$ such that $\hat{\theta}'_{\text{t-ridge}} \neq \hat{\theta}''_{\text{t-ridge}}$. Let $\lambda' = \|s(\hat{\theta}'_{\text{t-ridge}})\|_2$ and $\lambda'' = \|s(\hat{\theta}''_{\text{t-ridge}})\|_2$. Note that the definition of t-ridge object function implies that the $\ell_2$ norm of the score function with respect to the t-ridge estimator is non-zero. Hence, $\lambda' = \lambda'' \neq 0$. By Theorem 2.2, we have

$$ \hat{\theta}'_{\text{t-ridge}} = \hat{\theta}_{\text{edr}}[\lambda'] = \hat{\theta}_{\text{edr}}[\lambda''] = \hat{\theta}''_{\text{t-ridge}}, $$

which produces a contradiction with $\hat{\theta}'_{\text{t-ridge}} \neq \hat{\theta}''_{\text{t-ridge}}$.

So, we aim to show in the following that if both $\hat{\theta}'_{\text{t-ridge}}$ and $\hat{\theta}''_{\text{t-ridge}}$ minimize the objective function of t-ridge, then $\lambda' = \lambda''$. By Theorem 2.2, we know that $\hat{\theta}'_{\text{t-ridge}} = \hat{\theta}_{\text{edr}}[\lambda']$ and
\tilde{\theta}_{\text{t-ridge}} = \tilde{\theta}_{\text{edr}}[\lambda']$. Since \(\tilde{\theta}_{\text{t-ridge}}\) and \(\tilde{\theta}_{\text{t-ridge}}''\) are minimums of t-ridge, we have

\[
\frac{D(\tilde{\theta}_{\text{edr}}[\lambda']|Z)}{\lambda'} + \|\tilde{\theta}_{\text{edr}}[\lambda']\|_2 = \frac{D(\tilde{\theta}_{\text{edr}}[\lambda''][Z])}{\lambda''} + \|\tilde{\theta}_{\text{edr}}[\lambda''][\lambda']\|_2
\]

\[
= \frac{1}{\lambda''}(D(\tilde{\theta}_{\text{edr}}[\lambda''][Z]) + \lambda''\|\tilde{\theta}_{\text{edr}}[\lambda''][\lambda']\|_2)
\]

\[
\leq \frac{1}{\lambda''}(D(\tilde{\theta}_{\text{edr}}[\lambda'][Z]) + \lambda'\|\tilde{\theta}_{\text{edr}}[\lambda'][\lambda']\|_2)
\]

\[
= \frac{D(\tilde{\theta}_{\text{edr}}[\lambda'][Z])}{\lambda''} + \|\tilde{\theta}_{\text{edr}}[\lambda'][\lambda']\|_2.
\]

This implies

\[
\frac{D(\tilde{\theta}_{\text{edr}}[\lambda'][Z])}{\lambda'} \leq \frac{D(\tilde{\theta}_{\text{edr}}[\lambda'][Z])}{\lambda''}.
\]

Similarly, we can obtain

\[
\frac{D(\tilde{\theta}_{\text{edr}}[\lambda''][Z])}{\lambda''} \leq \frac{D(\tilde{\theta}_{\text{edr}}[\lambda'][Z])}{\lambda'}.
\]

By the assumption that \(D(\theta|Z) > 0\) for all \(\theta \in \mathbb{R}^p\), we have \(D(\tilde{\theta}_{\text{edr}}[\lambda'][Z]) > 0\) and \(D(\tilde{\theta}_{\text{edr}}[\lambda''][Z]) > 0\). Hence, the two inequalities above yield \(\lambda' = \lambda''\) and we get the desired result.

\[\square\]

### A.4 Proof of Theorem 3.1

**Proof.** By the definition of (3.5), we have

\[-\sum_{i=1}^{n} (y_i x_i^\top \tilde{\beta}_{\text{edr}}[\lambda] - b(x_i^\top \tilde{\beta}_{\text{edr}}[\lambda])) + \lambda\|\tilde{\beta}_{\text{edr}}[\lambda]\|_2 \leq -\sum_{i=1}^{n} (y_i x_i^\top \beta^* - b(x_i^\top \beta^*)) + \lambda\|\beta^*\|_2.\]

By arranging, we have

\[\frac{1}{n}\sum_{i=1}^{n} (b(x_i^\top \tilde{\beta}_{\text{edr}}[\lambda]) - b(x_i^\top \beta^*)) \leq \frac{1}{n}\sum_{i=1}^{n} y_i x_i^\top (\tilde{\beta}_{\text{edr}}[\lambda] - \beta^*) + \lambda(\|\beta^*\|_2 - \|\tilde{\beta}_{\text{edr}}[\lambda]\|_2).\]
The margin condition on \( b \) yields
\[
b(x_i^\top \hat{\beta}_{\text{edr}}[\lambda]) - b(x_i^\top \beta^*) \geq b'(x_i^\top \beta^*)(x_i^\top \hat{\beta}_{\text{edr}}[\lambda] - x_i^\top \beta^*) + \frac{1}{C^2} (x_i^\top \hat{\beta}_{\text{edr}}[\lambda] - x_i^\top \beta^*)^2,
\]
where \( C = \max_{i \in \{1, \ldots, n\}} \{C_i\} \). Notice \( b'(x_i^\top \beta^*) = g^{-1}(x_i^\top \beta^*) \) and \( y_i = g^{-1}(x_i^\top \beta^*) + \varepsilon_i \), we can obtain
\[
\frac{1}{C^2} \sum_{i=1}^n (x_i^\top (\hat{\beta}_{\text{edr}}[\lambda] - \beta^*))^2 \leq \sum_{i=1}^n \varepsilon_i (x_i^\top (\hat{\beta}_{\text{edr}}[\lambda] - \beta^*)) + \lambda (\|\beta^*\|_2 - \|\hat{\beta}_{\text{edr}}[\lambda]\|_2).
\]
We write it as the matrix form
\[
\frac{1}{C^2} \|X(\hat{\beta}_{\text{edr}}[\lambda] - \beta^*)\|_2^2 \leq \langle X^\top \varepsilon, \hat{\beta}_{\text{edr}}[\lambda] - \beta^* \rangle + \lambda (\|\beta^*\|_2 - \|\hat{\beta}_{\text{edr}}[\lambda]\|_2).
\]
Using the Hölder’s inequality on the first term of right hand side, we get
\[
\frac{1}{C^2} \|X(\hat{\beta}_{\text{edr}}[\lambda] - \beta^*)\|_2^2 \leq \|X^\top \varepsilon\|_2 \|\hat{\beta}_{\text{edr}}[\lambda] - \beta^*\|_2 + \lambda (\|\beta^*\|_2 - \|\hat{\beta}_{\text{edr}}[\lambda]\|_2).
\]
By assumption \( \lambda \geq \lambda^* \) and triangle inequality, we obtain
\[
\|X(\hat{\beta}_{\text{edr}}[\lambda] - \beta^*)\|_2^2 \leq \lambda C^2 (\|\hat{\beta}_{\text{edr}}[\lambda] - \beta^*\|_2 + \|\beta^*\|_2 - \|\hat{\beta}_{\text{edr}}[\lambda]\|_2)
\]
\[
\leq 2\lambda C^2 \|\beta^*\|_2^2
\]
as desired. \( \square \)

### A.5 Proof of Theorem 3.2

**Proof.** By Theorem 2.2, we know that t-ridge is on the ridge path and hence, it is also on the edr path by Theorem 2.1 such that \( \hat{\beta}_{\text{t-ridge}} = \hat{\beta}_{\text{edr}}[\hat{\lambda}] \). According to the definition of \( \hat{\beta}_{\text{edr}} \), we have
\[
-\sum_{i=1}^n (y_i x_i^\top \hat{\beta}_{\text{edr}}[\hat{\lambda}] - b(x_i^\top \hat{\beta}_{\text{edr}}[\hat{\lambda}])) + \lambda^* \|\hat{\beta}_{\text{edr}}[\hat{\lambda}]\|_2 \leq -\sum_{i=1}^n (y_i x_i^\top \beta^* - b(x_i^\top \beta^*)) + \lambda^* \|\beta^*\|_2.
\]
Following the proof of Theorem 3.1 (the penultimate inequality), we have
\[
\frac{1}{C^2} \|X(\hat{\beta}_{\text{edr}}[\hat{\lambda}] - \beta^*)\|_2^2 \leq \lambda^* \|\hat{\beta}_{\text{edr}}[\hat{\lambda}] - \beta^*\|_2 + \hat{\lambda}(\|\beta^*\|_2 - \|\hat{\beta}_{\text{edr}}[\hat{\lambda}]\|_2)
\]
\[
\leq 2 \max\{\lambda^*, \hat{\lambda}\} \|\beta^*\|_2.
\]
Multiplying $C^2$ on both sides yields the desired result.

\[\square\]

### A.6 Proof of Lemma 3.1

**Proof.** According to the definition of t-ridge in (3.4), we have
\[
D(\hat{\beta}_{\text{t-ridge}}|y, X) \lambda + \|\hat{\beta}_{\text{t-ridge}}\|_2 \leq D(\hat{\beta}_{\text{edr}}[\lambda^*]|y, X) + \|\hat{\beta}_{\text{edr}}[\lambda^*]\|_2.
\]
If we multiply $\lambda^*$ on both sides of the above inequality, we obtain
\[
\frac{\lambda^*}{\lambda} D(\hat{\beta}_{\text{t-ridge}}|y, X) + \lambda^* \|\hat{\beta}_{\text{t-ridge}}\|_2 \leq D(\hat{\beta}_{\text{edr}}[\lambda^*]|y, X) + \lambda^* \|\hat{\beta}_{\text{edr}}[\lambda^*]\|_2.
\]
Hence,
\[
\left(\frac{\lambda^*}{\lambda} - 1\right) D(\hat{\beta}_{\text{t-ridge}}|y, X) + D(\hat{\beta}_{\text{t-ridge}}|y, X) + \lambda^* \|\hat{\beta}_{\text{t-ridge}}\|_2 \leq D(\hat{\beta}_{\text{edr}}[\lambda^*]|y, X) + \lambda^* \|\hat{\beta}_{\text{edr}}[\lambda^*]\|_2.
\]
By the definition of edr in (3.5), we know that
\[
D(\hat{\beta}_{\text{edr}}[\lambda^*]|y, X) + \lambda^* \|\hat{\beta}_{\text{edr}}[\lambda^*]\|_2 \leq D(\hat{\beta}_{\text{edr}}[\lambda^*]|y, X) + \lambda^* \|\hat{\beta}_{\text{edr}}[\lambda^*]\|_2.
\]
Combining this with the previous inequality yields
\[
\left(\frac{\lambda^*}{\lambda} - 1\right) D(\hat{\beta}_{\text{t-ridge}}|y, X) \leq 0.
\]
If $D(\hat{\beta}_{\text{t-ridge}}|y, X) > 0$, this implies that $\hat{\lambda} \geq \lambda^*$. If $D(\hat{\beta}_{\text{t-ridge}}|y, X) < 0$, this implies that $\hat{\lambda} \leq \lambda^*$. We finish the proof.

\[\square\]
Appendix B  Additional Simulations

Tables 5–7 give the results for the remaining settings described in Section 4. These results further corroborate our conclusion that the t-ridge estimator is a contender across dimensions and data types.

Table 5: t-ridge outperforms $K$-fold cross-validated ridge ($K$-fold CV ridge) for $K \in \{5, 10\}$ in prediction on Gaussian data with $k \in \{0.2, 0.4\}$.

| Relative prediction error | n   | p   | k    | Mean of relative errors (sd) |
|---------------------------|-----|-----|------|-----------------------------|
|                           |     |     |      | t-ridge | 5-fold CV ridge | 10-fold CV ridge |
| $\frac{|X(\tilde{\beta}-\beta^*)|_2}{|X\beta^*|_2}$ | 100  | 300 | 0.2  | 0.35 (0.03) | 0.54 (0.06) | 0.53 (0.05) |
|                           | 100  | 300 | 0.4  | 0.36 (0.03) | 0.53 (0.03) | 0.53 (0.03) |
|                           | 200  | 500 | 0.2  | 0.36 (0.02) | 0.50 (0.02) | 0.50 (0.02) |
|                           | 200  | 500 | 0.4  | 0.38 (0.02) | 0.50 (0.02) | 0.50 (0.02) |
|                           | 50   | 1000| 0.2  | 0.32 (0.03) | 0.52 (0.26) | 0.50 (0.25) |
|                           | 50   | 1000| 0.4  | 0.32 (0.04) | 0.55 (0.25) | 0.54 (0.26) |
Table 6: \( t \)-ridge outperforms \( K \)-fold cross-validated ridge (\( K \)-fold CV ridge) for \( K \in \{5, 10\} \) in prediction on Poisson data with \( k \in \{0.2, 0.4\} \).

| Relative prediction error | n   | p   | k   | Mean of relative errors (sd) |
|---------------------------|-----|-----|-----|-----------------------------|
|                           |     |     |     | t-ridge | 5-fold CV ridge | 10-fold CV ridge |
| \( \frac{\|X(\hat{\beta} - \beta^*)\|_2}{\|X\beta^*\|_2} \) |     |     |     | t-ridge | 5-fold CV ridge | 10-fold CV ridge |
| 100 300 0.2               | 0.63 (0.08) | 0.82 (0.05) | 0.82 (0.06) |
| 100 300 0.4               | 0.64 (0.08) | 0.82 (0.05) | 0.81 (0.05) |
| 200 500 0.2               | 0.66 (0.07) | 0.79 (0.05) | 0.78 (0.05) |
| 200 500 0.4               | 0.66 (0.07) | 0.78 (0.04) | 0.77 (0.04) |
| 50 1000 0.2               | 0.86 (0.05) | 0.96 (0.03) | 0.96 (0.03) |
| 50 1000 0.4               | 0.86 (0.04) | 0.96 (0.03) | 0.96 (0.03) |
Table 7: t-ridge rivals K-fold cross-validated ridge (K-fold CV ridge) for $K \in \{5, 10\}$ in prediction on Bernoulli data with $k \in \{0.2, 0.4\}$.

| Relative prediction error | n   | p   | k   | Mean of relative errors (sd) |
|----------------------------|-----|-----|-----|-----------------------------|
|                            |     |     |     | t-ridge | 5-fold CV ridge | 10-fold CV ridge |
| $\frac{\|X(\hat{\beta}-\beta^*)\|_2}{\|X\beta^*\|_2}$ |     |     |     | t-ridge | 5-fold CV ridge | 10-fold CV ridge |
| 100 300 0.2                 | 0.86 (0.06) | 0.88 (0.07) | 0.88 (0.08) |
| 100 300 0.4                 | 0.85 (0.06) | 0.86 (0.08) | 0.85 (0.08) |
| 200 500 0.2                 | 0.86 (0.05) | 0.87 (0.06) | 0.86 (0.06) |
| 200 500 0.4                 | 0.83 (0.05) | 0.84 (0.06) | 0.84 (0.06) |
| 50 1000 0.2                 | 0.87 (0.08) | 0.90 (0.09) | 0.89 (0.09) |
| 50 1000 0.4                 | 0.86 (0.08) | 0.88 (0.09) | 0.88 (0.09) |