GENERALIZED ANDREWS-GORDON IDENTITIES

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ABSTRACT. In a recent paper, Griffin, Ono and Warnaar present a framework for Rogers-Ramanujan type identities using Hall-Littlewood polynomials to arrive at expressions of the form
\[ \sum_{\lambda: \lambda_1 \leq m} q^{a|\lambda|} P_{2\lambda}(1, q, q^2, \ldots ; q^n) = \text{"Infinite product modular function"} \]
for \( a = 1, 2 \) and any positive integers \( m \) and \( n \). A recent paper of Rains and Warnaar presents further Rogers-Ramanujan type identities involving sums of terms \( q^{a|\lambda|/2} P_{\lambda}(1, q, q^2, \ldots ; q^n) \). It is natural to attempt to reformulate these various identities to match the well-known Andrews-Gordon identities they generalize. Here, we find combinatorial formulas to replace the Hall-Littlewood polynomials and arrive at such expressions.

1. INTRODUCTION

In [1], the authors construct a general framework describing four doubly-infinite families of Rogers-Ramanujan type identities. In this context, the famous Rogers-Ramanujan identities [2]

(1.1) \[ \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q) \cdots (1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})} \]

and

(1.2) \[ \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1-q) \cdots (1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})} \]

are presented as a special case of their Theorem 1.1 through setting the parameters \((m, n) = (1, 1)\). Fixing only \( n = 1 \) gives rise to the \( i = 1 \) and \( i = m + 1 \) instances of the well-known Andrews-Gordon identities, [3]

(1.3) \[ \sum_{r_1 \geq \cdots \geq r_m \geq 0} \frac{q^{r_1^2 + \cdots + r_m^2 + r_1 + \cdots + r_m}}{(q)_{r_1-r_2} \cdots (q)_{r_{m-1}-r_m} (q)_{r_m}} = \frac{(q^{2m+3}; q^{2m+3})_{\infty}}{(q)_{\infty}} \cdot \theta(q^i; q^{2m+3}), \]
where we use the standard notation
\[(a)_k = (a; q)_k := \begin{cases} (1-a)(1-aq)\cdots(1-aq^{k-1}) & \text{if } k \geq 0 \\ \prod_{j=0}^{\infty}(1-aq^j) & \text{if } k = \infty \end{cases} \]
and
\[\theta(a; q) := (a; q)_{\infty}(q/a; q)_{\infty}.\]
For convenience, we also set
\[\theta(a_1, \ldots, a_n; q) := \theta(a_1; q) \cdots \theta(a_n; q).\]

The general identities in [1] are presented as sums over partitions \(\lambda = (\lambda_1, \lambda_2, \ldots)\) of associated Hall-Littlewood polynomials in the form
\[\sum_{\lambda: \lambda_1 \leq m} q^{\lvert \lambda \rvert/2} P_{2\lambda}(1, q, q^2, \ldots; q^n) = \text{“Infinite product modular function”},\]
with \(a = 1, 2\). Meanwhile, the identities in [4] take the form
\[\sum_{\lambda: \lambda_1 \leq m} C q^{\lvert \lambda \rvert/2} P_\lambda(1, q, q^2, \ldots; q^n) = \text{“Infinite product modular function”},\]
where \(C\) is a particular product of Pochhammer symbols. Given this framework, it is natural to ask to what extent we can reformulate these identities to look like the Andrews-Gordon identities stated above. Here, we recast the left-hand side of these identities, without reference to partitions or Hall-Littlewood polynomials, to arrive at such explicit expressions.

The sums appearing on the left-hand sides of identities corresponding to those in [4] will range over various sets of decreasing integers \(s_i \geq \cdots \geq s_j\) related by \(s_i \geq s_{i+1}\) for \(0 \leq j \leq n - 1\). We will write \(s^{(j)} := s^{(j)}_1 + \cdots + s^{(j)}_m\) and use the convention that \(s^{(n)}_i = 0\) for all \(i\) and \(s^{(j)}_{m+1} = 0\) for all \(j\). For such a collection of integers, we define
\[A_{m,n}(s_*):= A_{m,n}(s^{(0)}_*, s^{(1)}_*, \ldots, s^{(n)}_*) \]
\[= -\frac{n}{2}s^{(0)} + s^{(1)} + \ldots + s^{(n-1)} + \frac{n}{2} \sum_{i=1}^{m} \sum_{a=1}^{n} (s^{(a)}_i - s^{(a)}_{i+1})^2 \]
and
\[B_{m,n}(s_*):= B_{m,n}(s^{(0)}_*, s^{(1)}_*, \ldots, s^{(n)}_*; q) \]
\[= \prod_{i=1}^{m} \prod_{a=1}^{n} \frac{(q)_{s^{(a)}_i - s^{(a)}_{i+1}}}{(q)_{s^{(a)}_i - s^{(a)}_{i+1}}(q)_{s^{(a)}_i - s^{(a)}_{i+1}}}.\]
When the identities include only Hall-Littlewood polynomials of even partitions as in (1.4), it will be more convenient to write our sums over integers \( r_1 \geq \cdots \geq r_m \geq 0 \) and collections \( s^{(j)}_1 \geq \cdots \geq s^{(j)}_{2m} \) for \( 1 \leq j \leq n - 1 \) satisfying \( s^{(j)}_i \geq s^{(j+1)}_i \) and \( r_{[i/2]} \geq s^{(1)}_i \). In this case, we define
\[
C_{m,n}(r_s, s_s) := C_{m,n}(r_s, s^{(1)}_s, \ldots, s^{(n)}_s) = A_{2m,n}(s^{(0)}_s = r_{[i/2]}, s^{(1)}_s, \ldots, s^{(n)}_s)
\]
and
\[
D_{m,n}(r_s, s_s; q) := D_{m,n}(r_s, s^{(1)}_s, \ldots, s^{(n)}_s; q) := B_{2m,n}(s^{(0)}_s = r_{[i/2]}, s^{(1)}_s, \ldots, s^{(n)}_s).
\]

The following is a reformulation of Theorem 1.1 of [1] which more closely resembles the Andrews-Gordon identities as stated in (1.3).

**Theorem 1.1.** For positive integers \( m \) and \( n \), let \( \kappa := 2m + 2n + 1 \) and \( n' := 2n - 1 \). Then we have
\[
\sum_{r_s, s_s} D_{m,n'}(r_s, s_s; q^{n'}) q^{C_{m,n'}(r_s, s_s) + r} = \frac{(q^{\kappa}; q^{\kappa})^n}{(q)^n} \prod_{i=1}^{n} \theta(q^{i+m}; q^{\kappa}) \prod_{1 \leq i < j \leq n} \theta(q^{j-i}; q^{i+j-1}; q^{\kappa})
\]

and
\[
\sum_{r_s, s_s} D_{m,n'}(r_s, s_s; q^{n'}) q^{C_{m,n'}(r_s, s_s) + 2r} = \frac{(q^{\kappa}; q^{\kappa})^m}{(q)^m} \prod_{i=1}^{m} \theta(q^{i}; q^{\kappa}) \prod_{1 \leq i < j \leq m} \theta(q^{j-i}; q^{i+j}; q^{\kappa})
\]

where the sums range over sets of indices \( r_1 \geq \cdots \geq r_m \geq 0 \) and \( s^{(j)}_1 \geq \cdots \geq s^{(j)}_{2m} \geq 0 \) for \( 1 \leq j \leq n' - 1 \) satisfying \( s^{(j)}_i \geq s^{(j+1)}_i \) and \( r_{[i/2]} \geq s^{(1)}_i \).

**Remark.** As promised, the Andrews-Gordon identities are easily recognized through setting \( n = 1 \). Since \( n' = 1 \), there are no \( s \) indices in the sum, giving
\[
C_{m,1}(r_s) = -r + \frac{1}{2} \sum_{i=1}^{2m} r_{[i/2]}^2 = -(r_1 + \ldots + r_m) + r_1^2 + \ldots + r_m^2
\]
and
\[
D_{m,1}(r_s; q) = \prod_{j=1}^{m} \frac{1}{(q)_{r_i - r_{i+1}}} = \frac{1}{(q)_{r_1 - r_2} \cdots (q)_{r_{m-1} - r_m} (q)_{r_m}}.
\]

Plugging this into the two identities in the theorem above results directly in (1.3) with \( i = 1 \) and \( i = m + 1 \) respectively.
One can obtain similar reformulations of Theorems 1.2 and 1.3 of \cite{1}, which we label respectively here.

**Theorem 1.2.** For positive integers \(m\) and \(n\), let \(\kappa := 2m + 2n + 2\) and \(n' := 2n\). Then we have

\[
\sum_{r_*, s_*} \mathcal{D}_{m, n'}(r_*, s_*; q^{n'}) q^{C_{m, n'}(r_*, s_*)+r} = \frac{(q^2; q^2)_\infty (q^\kappa/2; q^{\kappa/2})_\infty (q^\kappa; q^{\kappa})_\infty}{(q; q^2)_\infty (q^\kappa; q^{\kappa})_\infty} \prod_{i=1}^n \theta(q^i; q^{\kappa/2}) \prod_{1 \leq i < j \leq n} \theta(q^{j-i}, q^{j+i}; q^\kappa)
\]

\[
= \frac{(q^\kappa; q^\kappa)_\infty}{(q)_\infty^m} \prod_{i=1}^m \theta(q^i, q^\kappa) \prod_{1 \leq i < j \leq m} \theta(q^{j-i}, q^{j+i}; q^\kappa).
\]

**Theorem 1.3.** For positive integers \(m\) and \(n\) with \(n \geq 2\), let \(\kappa := 2m + 2n\) and \(n' := 2n\). Then we have

\[
\sum_{r_*, s_*} \mathcal{D}_{m, n'}(r_*, s_*; q^{n'}) q^{C_{m, n'}(r_*, s_*)+2r} = \frac{(q^\kappa; q^\kappa)_\infty}{(q^2; q^2)_\infty (q^\kappa/2; q^{\kappa/2})_\infty} \prod_{i=1}^{n-1} \theta(q^{j-i}, q^{j+i-1}; q^\kappa)
\]

\[
= \frac{(q^\kappa; q^\kappa)_\infty}{(q^2; q^2)_\infty} \prod_{i=1}^m \theta(q^i, q^\kappa) \prod_{1 \leq i < j \leq m} \theta(q^{j-i}, q^{j+i}; q^\kappa).
\]

We present an example of the identities that are obtained from these theorems.

**Example.** Setting \(n = 1\) in Theorem 1.2 and writing \(s_i\) for \(s_i^{(1)}\), we have that

\[
\sum_{r_1 \geq \cdots \geq r_m \geq 0 \atop s_1 \geq \cdots \geq s_m \geq 0 \atop r_{[1/2]} \geq s_i} q^{(r_1-s_1)^2+(r_1-s_2)^2+\cdots+(r_m-s_{2m-1})^2+(r_m-s_{2m})^2+s_1^2+\cdots+s_{2m}^2+s-r} \frac{(q^2; q^2)_{s_1-s_2} \cdots (q^2; q^2)_{s_{2m-1}-s_{2m}} (q^2; q^2)_{s_2} (q^2; q^2)_{r_1-r_2} \cdots (q^2; q^2)_{r_m-1-r_m}}{\theta(q; q^{m+2}) (q^2; q^2)_{r_1-s_3} (q^2; q^2)_{r_2-s_5} \cdots (q^2; q^2)_{r_m-1-s_{2m-1}}} \times \frac{(q^2; q^2)_{r_1-s_3} (q^2; q^2)_{r_2-s_5} \cdots (q^2; q^2)_{r_m-1-s_{2m-1}}}{(q^2; q^2)_{r_1-s_3} (q^2; q^2)_{r_2-s_5} \cdots (q^2; q^2)_{r_m-1-s_{2m-1}}} 
\]

\[
= \frac{(q^2; q^2)_\infty (q^{m+2}; q^{m+2})_\infty}{(q^2)_\infty^2} \theta(q; q^{m+2})
\]

\[
= \frac{(q^{2m+4}; q^{2m+4})_\infty^m}{(q)_\infty^m} \prod_{i=1}^m \theta(q^{i+1}, q^{2m+4}) \prod_{1 \leq i < j \leq m} \theta(q^{j-i}, q^{i+j+1}, q^{2m+4}).
\]
for any positive integer \(m\). Specializing to \((m, n) = (2, 1)\), one finds

\[
\sum_{r_1 \geq r_2 \geq 0 \atop s_1 \geq \cdots \geq s_4 \geq 0 \atop r_1/r_2 \geq s_i} \frac{(q^2; q^2)_{r_1-s_1} q_{r_1-s_1} (r_1-s_1)^2 + (r_1-s_2)^2 + (r_2-s_3)^2 + (r_2-s_4)^2 + s_1^2 + s_2^2 + s_3^2 + s_4 + s_1 - r_1 - r_2}{(q^2; q^2)_{r_1-s_1} (q^2; q^2)_{r_2-s_3} (q^2; q^2)_{s_1-s_2} (q^2; q^2)_{s_2-s_3} (q^2; q^2)_{s_3-s_4} (q^2; q^2)_{s_4} (q^2; q^2)_{r_1-r_2}}
\]

\[
= \frac{(q^2; q^2)_\infty (q^4; q^4)_\infty}{(q^2)_\infty} \theta(q; q^4) = \frac{(q^8; q^8)_\infty^2}{(q^2)_\infty^2} \theta(q^2; q^8) \theta(q^4; q^8) \theta(q, q^4; q^8).
\]

We now turn to the identities of the form \((1.5)\). The following theorems are reformulations of Theorems 5.10–5.12 of [4].

**Theorem 1.4.** For positive integers \(m\) and \(n\) let \(\kappa := m + 2n + 1\) and \(n' := 2n\). Then we have

\[
\sum_{s_*} B_{m, n'}(s_*; q^{n'}) q^{A_{m, n'}(s_*) + \frac{1}{2}s(0)}
\]

\[
= \frac{(q^\kappa; q^\kappa)_\infty^{n-1} (q^\kappa/2; q^\kappa/2)_\infty}{(q; q)_\infty^{n-1} (q^{1/2}; q^{1/2})_\infty} \prod_{i=1}^{n} \theta(q^i; q^{\kappa/2}) \prod_{1 \leq i < j \leq n} \theta(q^{j-i}, q^{i+j}; q^\kappa),
\]

where the sum ranges over sets of decreasing integers \(s_i^{(j)} \geq \cdots \geq s_{m}^{(j)}\) for \(0 \leq j \leq n' - 1\) satisfying \(s_i^{(j)} \geq s_i^{(j+1)}\).

**Theorem 1.5.** For positive integers \(m\) and \(n\) let \(\kappa := m + 2n\) and \(n' := 2n - 1\). Then we have

\[
\sum_{s_*} B_{m, n'}(s_*; q^{n'}) q^{A_{m, n'}(s_*) + \frac{1}{2}s(0)}
\]

\[
= \frac{(q^\kappa; q^\kappa)_\infty^{n} (q^{m-1}/2; q^{m-1}/2)_\infty}{(q; q)_\infty^{n-1} (q^{1/2}; q^{1/2})_\infty} \prod_{i=1}^{n} \theta(q^{i+(m-1)/2}; q^\kappa) \prod_{1 \leq i < j \leq n} \theta(q^{j-i}, q^{i+j-1}; q^\kappa)
\]

and

\[
\sum_{s_*} B_{m, 2n}(s_*; q^{2n}) \left( \prod_{i=1}^{m-1} (-q^n; q^n)_{s_i^{(0)} - s_{i+1}^{(0)}} \right) q^{A_{m, 2n}(s_*) + \frac{1}{2}s(0)}
\]

\[
= \frac{(q^\kappa; q^\kappa)_\infty^{n-1} (q^{\kappa/2}; q^{\kappa/2})_\infty}{(q; q)_\infty^{n-1} (q^{1/2}; q^{1/2})_\infty} \prod_{i=1}^{n} \theta(q^{-1/2}; q^{\kappa/2}) \prod_{1 \leq i < j \leq n} \theta(q^{j-i}, q^{i+j-1}; q^\kappa).
\]
Theorem 1.6. For positive integers \(m\) and \(n\) let \(\kappa := m + 2n - 1\) and \(n' := 2n - 1\). Then we have

\[
\sum_{s_*} B_{m,n'}(s_*; q^{n'}) \left( \prod_{i=1}^{m-1} (-q^{n-1/2}; q^{n-1/2})_{s_i}^{(0)} \right) q^{A_{m,n'}(s_*) + \frac{1}{2}s^{(0)}}
\]

\[
= \frac{(q^{\kappa}; q^{\kappa})_{\infty}^{n}}{(q; q)^{m-1}(q^{1/2}; q^{1/2})_{\infty}} \prod_{i=1}^{n} \theta(q^{j+m/2-1/2}; q^{\kappa}) \prod_{1 \leq i < j \leq n} \theta(q^{j-i}, q^{i+j-2}; q^{\kappa}).
\]

Remark. As the authors mention in [4], setting \(n = 1\) in Theorem 1.6 gives rise to Bressoud’s even modulus identities in [5]. Indeed, this is easily recognizable using the parameters defined in (1.6) and (1.7). Writing \(s_i\) for \(s_i^{(0)}\), we have

\[
A_{m,1}(s_*) = -\frac{1}{2}(s_1 + \ldots + s_m) + \frac{1}{2}(s_1^2 + \ldots + s_m^2)
\]

and

\[
B_{m,1}(s_*; q) = \prod_{j=1}^{m} \frac{1}{(q)_s q_{s_i + s_{i+1}}}
\]

Thus, the left hand side above is

\[
\sum_{s_1 \geq \ldots \geq s_m \geq 0} \frac{1}{(q; q)_{s_m}} \prod_{i=1}^{m-1} \frac{(-q^{1/2}; q^{1/2})_{s_i} (-q^{1/2}; q^{1/2})_{s_i + 1}}{(q; q)_{s_i} (q; q)_{s_i + 1}} q^{\frac{1}{2}(s_1^2 + \ldots + s_m^2)}.
\]

Putting \(q^2\) for \(q\) and using the fact that \((q^2; q^2)_k = \frac{1}{(q; q)_k} (q; q^2)_k\), we obtain

\[
\sum_{s_1 \geq \ldots \geq s_m \geq 0} \frac{q^{s_1^2 + \ldots + s_m^2}}{(q)_{s_1 - s_2} \ldots (q)_{s_m - s_{m-1}} (q^2; q^2)_{s_m}} = \frac{q^{2m+2} (q^{2m+2}; q^{2m+2})_{\infty}}{(q)_{\infty}} \theta(q^{m+1}, q^{2m+2}).
\]

These are the corresponding even moduli identities to the odd-modulus Andrews-Gordon identities.

For integers \(s_1, \ldots, s_m\) we write \(\text{alt}(s_*) := s_1 - s_2 + \ldots \pm s_m\). The following is a reformulation of Theorem 5.14 of [4].

Theorem 1.7. For positive integers \(m\) and \(n\), let \(\kappa := 2m + 2n\). Then we have

\[
\sum_{s_*} B_{2m,2n}(s_* \bigg| \prod_{i=1}^{2m-1} (q^{2n}; q^{2n})_{\left\lfloor s_i^{(0)} - s_{i+1}^{(0)} \right\rfloor} \bigg| q^{A_{2m,2n}(s_*) + \frac{1}{2}s^{(0)} + \text{alt}(s_0^{(0)})})
\]

\[
= \frac{(q^{\kappa}; q^{\kappa})_{\infty}^{n} (-q^{\kappa/2}; q^{\kappa})_{\infty}^{n}}{2(q; q)^{n}_{\infty}} \prod_{i=1}^{n} \theta(-q^{i-1}, q^{i+\kappa/2-1}; q^{\kappa}) \prod_{1 \leq i < j \leq n} \theta(q^{j-i}, q^{i+j-2}; q^{\kappa}).
\]
where the prime on the sum denotes the restriction “$s_{i}^{(0)} - s_{i+1}^{(0)}$ is even for $i = 1, 3, \ldots, 2m - 1.”$

This paper is organized as follows. In the next section we define the Hall-Littlewood polynomials and recall a key formula from [6,7]. This allows us to prove two lemmas re-expressing the Hall-Littlewood polynomials that appear in (1.4) and (1.5). We then apply these to prove Theorems 1.1–1.7 in the following section.

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2. Hall-Littlewood $q$-series

Our proofs of Theorems 1.1–1.7 rely on explicit combinatorial formulas for the Hall-Litltlewood polynomials appearing on the left-hand side of the identities in [1] and [4]. After defining these objects, we state and prove these two formulas as Lemmas 2.1 and 2.2.

A partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a decreasing sequence of non-negative integers $\lambda_1 \geq \lambda_2 \geq \cdots$ with a finite number $l(\lambda)$ of nonzero terms. By $2\lambda$ we mean the partition $(2\lambda_1, 2\lambda_2, \ldots)$. To each partition, one can associate a Ferrers-Young diagram whose $i$th row consists of $\lambda_i$ boxes. The conjugate partition $\lambda'$ is defined to be the partition associated to the transpose of the Ferrers-Young diagram of $\lambda$. The multiplicity $m_i = m_i(\lambda)$ of an integer $i$ is the number of times it appears in the partition and is equal to $\lambda_{i+1}' - \lambda_i'$. Given two partitions $\lambda, \mu$ we write $\lambda \subseteq \mu$ if the Ferrers-Young diagram for $\lambda$ is contained in that of $\mu$, in other words if $\lambda_i \leq \mu_i$ for all $i$. Given a partition $\lambda$, with $\lambda_1 \leq n$, the associated Hall-Littlewood polynomial is defined as

$$P_{\lambda}(x_1, \ldots, x_n; q) = \prod_{i=0}^{n} \frac{(1-q)^{m_i}}{(q)_{m_i}} \sum_{w \in \mathfrak{S}_n} w \left( x_1^{\lambda_1} \cdots x_n^{\lambda_n} \prod_{i<j} x_i - qx_j \right),$$

where $m_0 := n - l(\lambda)$ and the symmetric group $\mathfrak{S}_n$ acts by permuting the $x_i$. One can extend this definition to symmetric functions in countably many variables as follows. If $p_r = x_1^r + x_2^r + \ldots$ is the $r$-th power sum and $p_{\lambda} = \prod_{i \geq 1} p_{\lambda_i}$, then the set $\{p_{\lambda}(x_1, \ldots, x_n)\}_{l(\lambda) \leq n}$ is a $\mathbb{Q}$-basis for the ring of symmetric functions in the $x_i$. Let $\phi_{q}$ be the ring homomorphism determined by $\phi_{q}(p_r) = p_r/(1 - q^r)$. Then we define
the modified Hall-Littlewood polynomials by $P'_\lambda := \phi_q(P_\lambda)$ and

(2.1) \[ Q'_\lambda(x_1, \ldots, x_n; q) := P'_\lambda(x_1, \ldots, x_n; q) \prod_{i \geq 1}(q)_{\lambda'_i - \lambda_{i+1}}. \]

From the fact that

\[ \phi_q(p_r(1, q, \ldots, q^{n-1})) = \frac{1 - q^{nr}}{1 - q^r} \cdot \frac{1}{1 - q^{nr}} = p_r(1, q, q^2, \ldots), \]

we see that

(2.2) \[ P_\lambda(1, q, q^2, \ldots; q^n) = P'_\lambda(1, q, \ldots, q^{n-1}; q^n). \]

We recall the following combinatorial formula for the modified Hall-Littlewood polynomials [6, 7],

(2.3) \[ Q'_\lambda(x_1, \ldots, x_n; q) = \sum_{\lambda_1} \prod_{i=1}^{\lambda_1} \prod_{a=1}^{n} x_{a}^{\mu_i(a) - \mu_i(a)} \cdot \frac{(q)_{\mu_i(a) - \mu_i(a)}}{(q)_{\mu_i(a) - \mu_i(a)}}, \]

where the sum is over partitions $0 = \mu^{(n)} \subseteq \cdots \subseteq \mu^{(1)} \subseteq \mu^{(0)} = \lambda'$. We will combine (2.1)-(2.3) to arrive at the following expression for the Hall-Littlewood polynomials appearing in the sum sides of the identities we wish to rewrite.

**Lemma 2.1.** Given a positive integer $m$ and a partition $\lambda$ with $\lambda_1 \leq m$, let $s_i^{(0)} = \lambda_i'$. Then for any positive integer $n$, we have

\[ P_\lambda(1, q, q^2, \ldots; q^n) = \sum_{s_i^{(j)} \geq 1} \mathcal{B}_{m,n}(s_*; q^n)q^A_{m,n}(s_*), \]

where the sum ranges over decreasing sets of integers $s_i^{(j)}$ with $1 \leq j \leq n - 1$ and $A_{m,n}(s_*)$ and $\mathcal{B}_{m,n}(s_*; q)$ are defined in (1.6) and (1.7).

**Proof.** For convenience, let $s_i^{(n)} = 0$ and $s_i^{(m+1)} = 0$ for all $i$ and $j$. The conditions on the indices $s_i^{(j)}$ are equivalent to the condition that the partitions $\mu^{(j)}$ defined by $\mu^{(j)} = (s_i^{(j)}, \ldots, s_m^{(j)})$ satisfy $0 = \mu^{(n)} \subseteq \cdots \subseteq \mu^{(1)} \subseteq \mu^{(0)} = \lambda'$. Thus, from (2.3) we have

\[ Q'_\lambda(1, q, \ldots, q^{n-1}; q^n) = \sum_{s_i^{(j)} \geq 1} \prod_{i=1}^{m} \prod_{a=1}^{n} q^{(a-1)(s_i^{(a-1)} - s_i^{(a)})} q^{(a-1)_{s_i^{(a-1)}} - s_i^{(a)}}, \]

or

\[ \prod_{i=1}^{m} \prod_{a=1}^{n} \frac{q^{(a-1)(s_i^{(a-1)} - s_i^{(a)})} q^{(a-1)_{s_i^{(a-1)}} - s_i^{(a)}}}{(q^n; q^n)_{s_i^{(a-1)} - s_i^{(a)}}(q^n; q^n)_{s_i^{(a-1)} - s_i^{(a)}}}. \]
Recall that we write \( s^{(j)} = s_1^{(j)} + \ldots + s_n^{(j)} \). The power of \( q \) appearing in a term of the sum corresponding to an index set \( s_i^{(j)} \) is given by

\[
\sum_{i=1}^m \sum_{a=1}^n \left( (a-1)(s_i^{(a-1)} - s_i^{(a)}) + n \left( s_i^{(a-1)} - s_i^{(a)} \right)^2 \right)
\]

\[
= \sum_{i=1}^m \sum_{a=1}^n \left( a - \frac{n}{2} \right) (s_i^{(a-1)} - s_i^{(a)}) + \frac{n}{2} \sum_{i=1}^m \sum_{a=1}^n (s_i^{(a-1)} - s_i^{(a)})^2
\]

\[
= \frac{n}{2} (s^{(0)} + s^{(1)} + \ldots + s^{(n-1)}) + \frac{n}{2} \sum_{i=1}^m \sum_{a=1}^n (s_i^{(a-1)} - s_i^{(a)})^2
\]

\[
= A_{m,n}(s_*)
\]

In addition, this power of \( q \) is multiplied the following product of Pochhammer symbols

\[
\prod_{i=1}^m \prod_{a=1}^n \frac{(q^n; q^n)_{s_i^{(a-1)} - s_i^{(0)}}}{(q^n; q^n)_{s_i^{(a)} - s_i^{(a+1)}}} = B(s_*; q^n) \prod_{j=1}^m (q^n; q^n)_{s_j^{(0)} - s_j^{(1)}}.
\]

Thus, we have

\[
Q'_\lambda(1, q, \ldots, q^{n-1}; q^n) = \sum_{\substack{s_1^{(j)} \geq \ldots \geq s_m^{(j)} \\ s_i^{(j)} \geq s_i^{(j+1)}}} \prod_{j=1}^m (q^n; q^n)_{s_j^{(0)} - s_j^{(1)}} B_{m,n}(s_*; q^n) q^{A_{m,n}(s_*)}
\]

\[
= \prod_{j=1}^m (q^n; q^n)_{s_j^{(0)} - s_j^{(1)}} \sum_{\substack{s_1^{(j)} \geq \ldots \geq s_m^{(j)} \\ s_i^{(j)} \geq s_i^{(j+1)}}} B_{m,n}(s_*; q^n) q^{A_{m,n}(s_*)},
\]

so using \( \text{(2.2)} \) and \( \text{(2.1)} \), we may write

\[
P_\lambda(1, q, q^2, \ldots; q^n) = P'_\lambda(1, q, \ldots, q^{n-1}; q^n)
\]

\[
= \frac{Q'_\lambda(1, q, \ldots, q^{n-1}; q^n)}{\prod_{j=1}^m (q^n; q^n)_{s_j^{(0)} - s_j^{(1)}}}
\]

\[
= \sum_{\substack{s_1^{(j)} \geq \ldots \geq s_m^{(j)} \\ s_i^{(j)} \geq s_i^{(j+1)}}} B_{m,n}(r_*; s_*; q^n) q^{A_{m,n}(r_*, s_*)}.
\]

\( \square \)
We also provide the following formula for Hall-Littlewood polynomials of even partitions.

**Lemma 2.2.** Given a positive integer $m$ and a partition $\lambda$ with $\lambda_1 \leq m$, let $r_i = \lambda'_i$. Then for any positive integer $n$, we have

$$P_{2\lambda}(1, q, q^2, \ldots ; q^n) = \sum_{\substack{r_i \geq \cdots \geq r_{2m} \geq 0 \\ s_i^{(j)} \geq \cdots \geq s_{2m}^{(j)}}} q^{\frac{r_i}{2}} \mathcal{D}_{m,n}(r*, s*; q^n) q^{C_{m,n}(r*, s*)},$$

where the sum ranges over sets of decreasing integers $s_i^{(j)}$ for $1 \leq j \leq n - 1$ and $C_{m,n}(r*, s*)$ and $\mathcal{D}_{m,n}(r*, s*; q)$ are defined in (1.8) and (1.9).

**Proof.** Applying the previous lemma to the partition $2\lambda$ and recalling the definitions of $C_{m,n}(r*, s*)$ and $\mathcal{D}_{m,n}(r*, s*; q)$ results directly in this expression. \qed

### 3. Proof of Theorems

The proofs of Theorems 1.1–1.3 follow immediately from Lemma 2.2 and the respectively labeled theorems of [1]. A sum over all partitions $\lambda$ with $\lambda_1 \leq m$ is the same as a sum over all partitions whose conjugates have length $l(\lambda') \leq m$. We may represent these partitions by their conjugates, which are specified by indices $r_1 \geq \cdots \geq r_m \geq 0$. This shows in each case that

$$\sum_{r_i, s*} \mathcal{D}_{m,n'}(r*, s*; q^{n'}) q^{C_{m,n'}(r*, s*) + ar} = \sum_{\lambda, \lambda_1 \leq m} q^{a|\lambda|} P_{2\lambda}(1, q, q^2, \ldots ; q^{n'})$$

for $a = 1, 2$.

The proofs of Theorems 1.4–1.7 follow from Theorems 5.10–5.12, 5.14 of [4] respectively by rewriting the sums that appear there in a similar way. In this case, we represent the sum over all partitions $\lambda$ with $\lambda_1 \leq m$ as a sum over their conjugates, which we specify by $s_1^{(0)} \geq \cdots \geq s_m^{(0)}$, and use Lemma 2.1 to rewrite the Hall-Littlewood polynomials. We note that with this notation we have $|\lambda| = s_1^{(0)} + \cdots + s_m^{(0)} = s^{(0)}$ and $m_i(\lambda) = s_i^{(0)} - s_{i+1}^{(0)}$. In addition, the number of odd partitions of $\lambda$, written as odd($\lambda$) in Theorem 5.14 of [4], is equal to

$$\sum_{i \text{ odd}} m_i(\lambda) = \sum_{i \text{ odd}} s_i^{(0)} - s_{i+1}^{(0)} = s_1 - s_2 + s_3 - \ldots \pm s_m = \text{alt}(s^{(0)}).$$

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