AN ESTIMATOR FOR THE TAIL-INDEX OF GRAPHEX PROCESSES

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Abstract. Sparse exchangeable graphs resolve some pathologies in traditional random graph models, notably, providing models that are both projective and allow sparsity. In a recent paper, Caron and Rousseau (2017) show that for a large class of sparse exchangeable models, the sparsity behaviour is governed by a single parameter: the tail-index of the function (the graphon) that parameterizes the model. We propose an estimator for this parameter and quantify its risk. Our estimator is a simple, explicit function of the degrees of the observed graph. In many situations of practical interest, the risk decays polynomially in the size of the observed graph. Accordingly, the estimator is practically useful for estimation of sparse exchangeable models. We also derive the analogous results for the bipartite sparse exchangeable case.

1. Introduction

Most statistical models for relational data are based on vertex-exchangeable random graphs (also called graphon models), see Orbanz and Roy (2015) for a review. These models are interpretable and easy to work with, but have the considerable limitation that, almost surely, they generate dense graphs; that is, the number of edges scales quadratically with the number of vertices.

Sparse exchangeable random graphs (or grapheX processes) Caron and Fox (2017); Veitch and Roy (2015); Borgs et al. (2016) have been proposed as a generalization of the graphon models that preserve the key properties, but allow for sparsity. We recall these models for unipartite graphs and bipartite graphs in sections 2.1 and 3.1.

Efficient estimation remains challenging for these new models. A situation of particular interest is to define grapheX process models that are close analogues of dense models that have been already proven to be useful in practice; this is the approach of Todeschini and Caron (2016); Herlau et al. (2016), and a number of models of this kind are described in Borgs et al. (2016). Roughly speaking, we would like to combine efficient estimation procedures already developed in the dense case with some new procedures for estimating the additional parameters governing the sparse behaviour.

Recently, Caron and Rousseau (2017) have studied asymptotics for an important class of sparse exchangeable models (encompassing most known
examples). They derive the remarkable result that, for this model class, the sparsity (i.e. the scaling of number of edges with respect to number of vertices in the large graph limit) and the degree distribution of the graph are governed by a single parameter of the model, which we call the tail-index $\sigma$.

The main contribution of the present paper is to propose a simple estimator $\hat{\sigma}$ for $\sigma$ and to quantify its asymptotic risk. Our estimator is a simple, explicit function of the degree distribution of the graph. Moreover, we show that in many situations, the risk decays polynomially in the size of the graph—by contrast, the more direct estimator of Caron and Rousseau (2017) has risk that decays only logarithmically. The fast rate of convergence allows our estimator to be used as a plugin estimator for $\sigma$. This enables, for example, empirical Bayes procedures for estimating sparse exchangeable models; see the companion paper Veitch et al. (2017) for an example applying this idea to a complicated practical model.

We recall Caron and Rousseau’s findings in section 2.2 for simple graphs, and briefly discuss the counterparts results for bipartite graphs in section 3.2. The estimator and the main result for simple graphs are given in section 2.3, and the corresponding results for bipartite graphs are given in section 3.3.

Throughout the document, we use the notations $X_t \sim Y_t$ for $\lim_{t} X_t/Y_t = 1$. If $X_t$ and $Y_t$ are random variables, the limit is understood almost-surely. We also write $X_t \asymp Y_t$ if there is a constant $c \in \mathbb{R}$ such that $\lim_{t} X_t/Y_t = c$. Inequalities up to generic constants are denoted by the symbols $\lesssim$ and $\gtrsim$. Moreover, we recall that a function $\ell: \mathbb{R}_+ \to \mathbb{R}_+$ is called slowly varying if $\lim_{x \to \infty} \ell(cx)/\ell(x) = 1$ for all $c > 0$. Also, it is assumed that all random variables are defined on a common probability space $(\Omega, \mathcal{F}, P)$. The symbol $E$ will denote the expectation under $P$, and $E[\cdot | \cdot]$ and $P(\cdot | \cdot)$ stand respectively for conditional expectation and conditional distribution. For any set $S$ with finite cardinality, we write $|S|$ the number of elements in $S$.

2. Unipartite undirected graphs

An undirected graph $g$ is an ordered pair $g = (v, e)$ where $v$ is a set of vertices and $e$ is a set of unordered pairs of vertices called edges. We also consider graphs with no isolated vertices, that is for all $u \in v$ there exists $u' \in v$ such that $\{u, u'\} \in e$.

2.1. The model. The model for random undirected graphs we consider in that paper is the sparse exchangeable framework introduced by Caron and Fox (2017) and further studied in Veitch and Roy (2015, 2016); Todeschini and Caron (2016); Borgs et al. (2016); Borgs et al. (2017); Janson (2017); Caron and Rousseau (2017). Let $(X, \rho)$ be a measured space. By the mean of random graph $G = (V, E)$, it is understood $V \subseteq X$ and $E$ is a symmetric point process (Daley and Vere-Jones, 2003a,b) over $V \times V$, seen as a random edge set for $G$. Because we assume $G$ has no isolated vertices, $G$ is entirely determined by its edge set.
The model can be summarized as follows. First, denoting by $\lambda$ the Lebesgue measure on $\mathbb{R}_+$ and by $\text{PP}(m)$ the distribution of a Poisson process (Daley and Vere-Jones, 2003a,b) with mean intensity measure $m$, draw,

$$\Pi = \{(X_1, U_1), (X_2, U_2), \ldots\} \sim \text{PP}(\rho \times \lambda).$$

The random variables $\{U_1, U_2, \ldots\}$ are potential labels for vertices. By properties of Poisson processes, each of these labels is almost-surely unique. The random variables $\{X_1, X_2, \ldots\}$ are latent features that govern how vertices will connect. Let $B = (B_{i,j})_{(i,j) \in \mathbb{N}^2}$ be an array of binary variables that indicate the vertices $U_i$ and $U_j$ are connected if $B_{i,j} = 1$ and not connected if $B_{i,j} = 0$. Because we are considering undirected graphs, we always let $B_{i,j} = B_{j,i}$. Then, conditional on $\Pi$, for a jointly measurable, symmetric, function $W:\mathcal{X} \times \mathcal{X} \to [0,1]$ we let independently for all $i \in \mathbb{N}$ and all $j \leq i$,

$$B_{i,j} \mid \Pi \sim \text{Bernoulli}(W(X_i, X_j)).$$

Using the terminology of Veitch and Roy (2015); Borgs et al. (2016) we refer to the function $W$ as the graphon. For technical reasons, we always assume in that paper $W$ and $\rho$ are chosen so that

$$\int_{\mathcal{X}} \int_{\mathcal{X}} W(x, x') \rho(dx)\rho(dx') < +\infty.$$

The random edge set for $G$ is then the point process $E$ given in the random measure form by

$$E := \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} B_{i,j} \delta_{U_i, U_j}.$$ 

Equivalently, we shall write $\{U_i, U_j\} \in E$ if and only if $B_{i,j} = 1$ (and thus $B_{j,i} = 1$). The vertex set $V$ of $G$ is obtained from $E$ according to the rule $U_i \in V$ if and only if there exists $j \in \mathbb{N}$ such that $\{U_i, U_j\} \in E$.

The generative process for $G$ is called a graphex process. A sample from a graphex process has an almost-surely infinite (but countable) number of vertices. We view the model as defining a growing sequence of graphs indexed by the (continuous) sample size $\alpha \in \mathbb{R}_+$. The (almost-surely) finite graph $G_\alpha = (V_\alpha, E_\alpha)$ is obtained from $G$ by taking

$$E_\alpha := \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} B_{i,j} 1_{U_i \leq \alpha} 1_{U_j \leq \alpha} \delta_{U_i, U_j},$$

and $U_i \in V_\alpha$ if and only if $U_i \leq \alpha$ and there exists $j \in \mathbb{N}$ such that $\{U_i, U_j\} \in E_\alpha$. The parameter $\alpha > 0$ is referred to as the size of the subgraph $G_\alpha$ since it controls the law of $|V_\alpha|$ and $|E_\alpha|$.

2.2. Graphex marginal and tail-index. It is well-known (Caron and Fox, 2017; Veitch and Roy, 2015; Caron and Rousseau, 2017) that when $\rho(\mathcal{X}) < +\infty$ a graph $G$ drawn from a graphex process is almost-surely dense, in the sense that $|E_\alpha| \asymp |V_\alpha|^2$ almost-surely as $\alpha \to \infty$.

The most interesting situation happens when $\rho(\mathcal{X}) = \infty$ eventually. In that case, as shown by Caron and Rousseau (2017), the asymptotic behavior
of $G_\alpha$ as $\alpha \to \infty$ is determined by the tail behavior of the graphex marginal $\mu : \mathcal{X} \to \mathbb{R}_+$ defined by

$$
\mu(x) := \int_{\mathcal{X}} W(x, x') \rho(dx').
$$

More specifically, they show that if there is $\sigma \in [0, 1]$ and a slowly varying function $\ell$ such that as $z \to 0$,

$$
F(z) := \rho\{x \in \mathcal{X} : \mu(x) \geq z\} \sim z^{-\sigma \ell(z^{-1})},
$$

then, under mild supplementary assumptions on $W$, there are four regimes for the asymptotic behavior of $G_\alpha$:

- **Dense**: $\sigma = 0$ and $\lim_{t \to \infty} \ell(t) < \infty$. In that case $F$ has compact support and $|E_\alpha|/|V_\alpha|^2$ almost-surely.
- **Almost-dense**: $\sigma = 0$ and $\lim_{t \to \infty} \ell(t) = \infty$. In that case $|E_\alpha|/|V_\alpha|^2 \to 0$ almost-surely, but $|E_\alpha|/|V_\alpha|^{2-\epsilon} \to \infty$ almost-surely for every $\epsilon > 0$.
- **Sparse with power-law**: $\sigma \in (0, 1)$. In that case, $|E_\alpha| \asymp |\ell(\alpha)|^{2/\sigma}$ almost-surely. Moreover, the degree distribution $^1$ of $G$ has a power-law (Caron and Rousseau, 2017).
- **Very sparse**: $\sigma = 1$. In that case, $\ell$ has to go to zero sufficiently fast and $|E_\alpha|/|V_\alpha| \to \infty$ almost-surely while $|E_\alpha|/|V_\alpha|^{1+\epsilon} \to 0$ almost-surely for all $\epsilon > 0$.

We call $\sigma$ in equation (2) the tail-index of $\mu$. Clearly $\sigma$ is a crucial parameter of graphex processes since it governs the asymptotic behavior of $|E_\alpha|$, $|V_\alpha|$ as well as the degree distribution of $G_\alpha$.

2.3. **Estimation of the tail-index.** A naive estimator for $\sigma$ based on the asymptotic growth of $|E_\alpha|/|V_\alpha|$ has been proposed in Caron and Rousseau (2017). Limitations of the naive estimator are discussed later in section 3.4. We now describe our estimator. For some real numbers $p \in [0, 1]$, we first introduce the new random variables

$$
N_{p, \alpha} := \begin{cases} 
  p \sum_{U \in V_\alpha} \left( 1 - (1 - p) \sum_{U' \in V_\alpha \setminus \{U\}} 1_{E_\alpha \{U, U'\}} \right) & \text{if } 0 \leq p < 1, \\
  \sum_{U \in V_\alpha} \left\{ \sum_{U' \in V_\alpha \setminus \{U\}} 1_{E_\alpha \{U, U'\}} > 0 \right\} & \text{if } p = 1.
\end{cases}
$$

In general $N_{1, \alpha} \neq |V_\alpha|$ because $N_{1, \alpha}$ discards all vertices that have only self-edges, i.e. $N_{1, \alpha}$ counts vertices that are connected to at least one vertex distinct from themselves. Then, for a chosen value of $p \in (0, 1)$, we propose to use as an estimator for $\sigma$,

$$
\hat{\sigma}_{p, \alpha} := \begin{cases} 
  \frac{\log N_{1, \alpha} - \log N_{p, \alpha}}{-\log p} - 1 & \text{if } N_{p, \alpha} \geq 1, \\
  0 & \text{otherwise}.
\end{cases}
$$

$^1$Here the degree distribution is understood as $p_\alpha := (p_{\alpha, k})_{k \geq 1}$ where $p_{\alpha, k} := \left| V_\alpha \right|^{-1} \sum_{U \in V_\alpha} 1_{\left\{ \sum_{U' \in V_\alpha} 1_{E_\alpha \{U, U'\}} = k \right\}}$. It is said that $p_\alpha$ has a power law if there is a constant $c > 0$ and a slowly-varying function at infinity $L$ such that $p_{\alpha, k} \sim k^{-c} L(k)$ for $k$ large enough.
Notice that $\hat{\sigma}_{p,\alpha}$ is well-defined since $N_{1,\alpha} \geq N_{p,\alpha}$ almost-surely for all $0 \leq p \leq 1$. Also note that $\hat{\sigma}_{p,\alpha}$ does not depend explicitly on the sample size $\alpha$; this is important because $\alpha$ is generally considered to be unknown.

In practice, we observe better performance by computing the estimator for values of $p \in (0, 1)$ close to 1, see section 4.3 for illustration.

In order to be able to compute the risk of $\hat{\sigma}_{p,\alpha}$, we require further assumptions on the model. Our first assumption is analogous to the assumptions encountered in Veitch and Roy (2015); Caron and Rousseau (2017). We first introduce the 2-points correlation function $\nu: \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$,

$$\nu(x, x') := \int_X W(x, y)W(y, x')\rho(dy).$$

**Assumption 1.** If $0 \leq \sigma < 1$, we assume that there is a constant $\eta > \max(1/2, \sigma)$ and a constant $C \geq 0$ such that $\nu(x, x') \leq C\mu(x)\mu(x')^\eta$ for all $x, x' \in \mathcal{X}$. If $\sigma = 1$, we assume $\nu(x, x') \leq C\mu(x)\mu(x')$, and we set $\eta = 1$.

Also, it is well-known (Feller, 1971, Chapter XIII; Caron and Rousseau, 2017) that under equation (2) the asymptotic equivalence

$$\int_{\mathbb{R}_+} (1 - e^{-\alpha z}) F(dz) \sim \begin{cases} \alpha^\sigma \ell(\alpha) \Gamma(1 - \sigma) & \text{if } 0 \leq \sigma < 1, \\ \alpha \int_\alpha^\infty z^{-1}\ell(z)dz & \text{if } \sigma = 1 \end{cases}$$

holds as $\alpha \to \infty$. This relation is already enough to show consistency of $\hat{\sigma}_{p,\alpha}$. However, further assumptions on $F$ are required to quantify the risk of the estimator. In the present paper, we shall assume the following assumption.

**Assumption 2.** We assume that equation (2) holds. We furthermore assume that for all $p \in (0, 1)$ there is a sequence $(\Gamma_{p,\alpha})_{\alpha > 0}$ such that

$$\left| \frac{\int_{\mathbb{R}_+} (1 - e^{-p\alpha z}) F(dz)}{p^\sigma \int_{\mathbb{R}_+} (1 - e^{-\alpha z}) F(dz)} - 1 \right| \leq \Gamma_{p,\alpha}.$$ 

The purpose of assumption 2 is the following. The integral in equation (3) plays a determining role in computing the risk of $\hat{\sigma}_{p,\alpha}$. However, equation (3) says nothing about the rate at which the integral approaches $\alpha^\sigma \ell(\alpha)$. Moreover, we know that $\ell(p\alpha)/\ell(\alpha) \to 1$ as $\alpha \to \infty$ by the assumptions on $\ell$, but again, the rate of convergence is unknown. These two rates are encapsulated in assumption 2, allowing us to quantify the risk of $\hat{\sigma}_{p,\alpha}$. Bounds on $\Gamma_{p,\alpha}$ for various examples are given in section 2.5. These examples show that in many cases $\Gamma_{p,\alpha}$ has polynomial decay.

We are now in position to state the main theorem of this section, whose proof is deferred to appendix A.

**Theorem 1.** Under assumptions 1 and 2, there is a constant $C'' > 0$ depending only on $p$ and $W$ such that for all $\alpha \geq 1$ it holds

$$\mathbb{E}[(\hat{\sigma}_{p,\alpha} - \sigma)^2] \leq \frac{\Gamma_{p,\alpha}^2}{(-\log p)^2} + C'' \log(\alpha)^2 \max\left\{ \alpha^{-(1+\sigma)}, \alpha^{1-2\eta} \right\}.$$
The bound in the previous theorem is relatively sharp in the sense that it is easily seen by inspection of the proof of the theorem that if we also have \( \nu(x, y) \gtrsim \mu(x)^\eta \mu(y)^\eta \), then
\[
\mathbb{E}[(\hat{\sigma}_{p,\alpha} - \sigma)^2] \gtrsim \max \left\{ \alpha^{-(1+\sigma)}, \alpha^{1-2\eta} \right\}.
\]

2.4. Interpretation of the estimator. It is interesting to discuss the genesis of our estimator \( \hat{\sigma}_{p,\alpha} \) since it gives a nice interpretation. We assume here for simplicity that \( \sigma \in (0, 1) \). In that case, inspection of the proofs in Caron and Rousseau (2017) shows \(^2\) that almost-surely as \( \alpha \to \infty \),
\[
N_{1,\alpha} \asymp \alpha^{1+\sigma} \ell(\alpha).
\]
Then, if we could observe \( G_\alpha \) and \( G_{p\alpha} \) for some \( p \in (0, 1) \), we could estimate \( \sigma \) by noticing that almost-surely as \( \alpha \to \infty \),
\[
\frac{N_{1,p\alpha}}{N_{1,\alpha}} \sim p^{1+\sigma} \frac{\ell(p\alpha)}{\ell(\alpha)} \sim p^{1+\sigma},
\]
because \( \ell(p\alpha)/\ell(\alpha) \to 1 \) as \( \alpha \to \infty \) by the slowly-varying function assumption.

Unfortunately, we in general have only one sample \( G_\alpha \) at our disposal. However, it is known (Borgs et al., 2017; Veitch and Roy, 2016) that if \( G_\alpha \) is a size \( \alpha \) subgraph of a graphex process \( G \), then the random subgraph \( \tilde{G}_{p\alpha} = (\tilde{V}_{p\alpha}, \tilde{E}_{p\alpha}) \) obtained from \( G_\alpha \) by selecting vertices at random with probability \( p \) and keeping only non-isolated vertices, has the same distribution as \( G_{p\alpha} \). We should emphasize here that same distribution does not mean \( \tilde{G}_{p\alpha} = G_{p\alpha} \); nevertheless, the intuition is useful. The random variable \( N_{p,\alpha} \) is precisely equal to \( \mathbb{E}[|\tilde{V}_{p\alpha}| \mid G_\alpha] \).

Rephrased, \( \sigma \) controls the rate of growth of the number of vertices. Intuitively, the number of vertices in the graph at smaller sizes can be estimated as \( N_{p,\alpha} \); this allows us to estimate the growth rate, and thus also to estimate \( \sigma \).

2.5. Examples. Here we present some examples to show our assumptions are in general not too silly. We consider the examples which were originally given in Caron and Rousseau (2017), because they cover a large spectrum of behavior. From sections 2.5.1 to 2.5.4, \( \rho \) is taken to be the Lebesgue measure on \( \mathbb{R}_+ \) with Borel \( \sigma \)-algebra.

2.5.1. Dense graphs. Consider the function \( W(x, y) = (1-x)(1-y)1_{x \leq 1}1_{y \leq 1} \). Then \( \mu(x) = 0.5(1-x)1_{x \leq 1} \) and the graph is almost-surely dense. Assumption 1 is trivially satisfied with \( \eta = 1 \). Also, it follows from appendix D.1 that assumption 2 is satisfied with \( \Gamma_{p,\alpha} \propto \alpha^{-1} \): the risk is polynomially decreasing on this example.

\(^2\) Indeed, they state the proof for \( |V_\alpha| \), but it is easily seen that the result for \( N_{1,\alpha} \) follows from their proof by assuming \( W(x, x) = 0 \) for all \( x \in \mathcal{X} \).
2.5.2. Sparse, almost dense graphs without power law. Consider the function $W(x, y) = \exp(-x - y)$. Then $\mu(x) = \exp(-x)$ and assumption 1 is trivially satisfied with $\eta = 1$ and $\sigma = 0$. Moreover, it follows from appendix D.2 that assumption 2 is satisfied with $\Gamma_{p, \alpha} \propto 1 / \log(\alpha)$: the risk is logarithmically decreasing on this example.

2.5.3. Sparse graphs with power law, separable. For $0 < \sigma < 1$, consider the function $W(x, y) = (1 + x)^{-1/\sigma}(1 + y)^{-1/\sigma}$. Then obviously $\mu(x) = \sigma(1 + x)^{-1/\sigma}/(1 - \sigma)$ and assumption 1 holds trivially with $\eta = 1$. Moreover, from appendix D.3, assumption 2 holds with $\Gamma_{p, \alpha} \propto \alpha^{-\sigma}$: the risk is polynomially decreasing on this example.

2.5.4. Sparse graphs with power law, non separable. For $0 < \sigma < 1$, consider the function $W(x, y) = (1 + x + y)^{-1/\sigma}$. It is shown in Caron and Rousseau (2017) that assumption 1 holds with $\eta = (1 + \sigma)/2$. We have $\mu(x) = \sigma(1 + x)^{-1/\sigma}$, which is up to a constant the same marginal as the previous example, so that assumption 2 is verified with $\Gamma_{p, \alpha} \propto \alpha^{-\sigma}$ and the risk is polynomially decreasing on this example.

2.5.5. Generalized Gamma Process. For $0 < \sigma < 1$, let consider $\rho(dx) = x^{-1-\sigma}\exp(-x)/\Gamma(1 - \sigma)$ and $W(x, y) = 1 - \exp(-xy)$. This corresponds to the model considered in Caron and Fox (2017). Then,

$$\mu(x) = \frac{(1 + x)^{\sigma} - 1}{\sigma},$$

and, as $z \to 0$,

$$z^\sigma F(z) = \int_{\mathbb{R}_+} 1\{x \geq (1 + \sigma z)^{1/\sigma} - 1\} \frac{x^{-1-\sigma}e^{-x}dx}{\Gamma(1 - \sigma)} \to \frac{1}{\sigma \Gamma(1 - \sigma)}.$$

Not surprisingly, that means the graph is almost-surely sparse with a power-law degree distribution. Moreover,

$$\nu(x, y) = \int_{\mathbb{R}_+} (1 - e^{-xz})(1 - e^{-zy}) \frac{z^{-1-\sigma}e^{-z}dz}{\Gamma(1 - \sigma)} = \frac{-1 + (1 + x)^\sigma + (1 + y)^\sigma - (1 + x + y)^\sigma}{\sigma}.$$ 

Hence assumption 1 is satisfied with $\eta = 1$, because,

$$\frac{\nu(x, y)}{\mu(x)\mu(y)} = \frac{-1 + (1 + x)^\sigma + (1 + y)^\sigma - (1 + x + y)^\sigma}{((1 + x)^\sigma - 1)((1 + y)^\sigma - 1)} \leq 1 - \sigma.$$

Furthermore, the computations in appendix D.4 show that assumption 2 is satisfied with $\Gamma_{p, \alpha} \propto \alpha^{-\sigma}$. 
3. Bipartite undirected graphs

The model for undirected graphs described in the previous section can also be extended to bipartite graphs (Caron, 2012; Caron and Fox, 2017). Here we understand an undirected bipartite graph \( g = (v, w, e) \) as an ordered triple where \( u, v \) are sets of vertices, and \( e \subseteq v \times w \) is a set of edges. Informally, \( v \) and \( w \) are referred to as the parts of the graph, and only edges between vertices belonging to different parts are allowed.

3.1. The model. The model for bipartite graphs we consider is a natural extension of section 2.1 already considered in (Caron, 2012; Caron and Fox, 2017). Let \((X, \rho)\) and \((Y, \psi)\) be measured spaces. By the mean of a random bipartite graph \( G = (V, W, E) \), it is understood \( V \subseteq X \), \( W \subseteq Y \) and \( E \) is a point process over \( V \times W \), seen as a random edge set for \( G \).

The model can be summarized as follows. First, draw independently,

\[
\Pi_v = \{(X_1, U_1), (X_2, U_2), \ldots\} \sim \text{PP}(\rho \times \lambda),
\]

\[
\Pi_w = \{(Y_1, T_1), (Y_2, T_2), \ldots\} \sim \text{PP}(\psi \times \lambda).
\]

The random variables \( \{U_1, U_2, \ldots\} \) and \( \{T_1, T_2, \ldots\} \) are potential labels for vertices of the different parts. As for the unipartite case, let \( B = (B_{i,j})_{(i,j) \in \mathbb{N}^2} \) be an array of binary variables that indicate the vertices \( U_i \) and \( T_j \) are connected if \( B_{i,j} = 1 \) and not connected if \( B_{i,j} = 0 \). Then, conditional on \( (\Pi_v, \Pi_w) \), for a jointly measurable function \( W : X \times Y \to [0,1] \), we let independently for all \( (i,j) \in \mathbb{N}^2 \),

\[
B_{i,j} \mid \Pi_v, \Pi_w \sim \text{Bernoulli}(W(X_i, Y_j)).
\]

Note that \( W \) is no longer assumed to be symmetric, we in fact don’t necessarily have \( X = Y \). However, we still assume that

\[
\int_X \int_Y W(x,y) \rho(dx)\psi(dy) < +\infty.
\]

The edge set of \( G \) is then the point process \( E := \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} B_{i,j} \delta_{U_i, T_j} \). Equivalently, we write \((U_i, T_j) \in E\) if and only if \( B_{i,j} = 1 \). The vertex sets \((V, W)\) are obtained from \( E \) according to the rule \( U_i \in V \) if and only if there is \( j \in \mathbb{N} \) such that \((U_i, T_j) \in E\); similarly \( T_j \in W \) if and only if there is \( i \in \mathbb{N} \) such that \((U_i, T_j) \in E\).

The previous process generates a bipartite graph with an infinite number of vertices in both parts. For real numbers \( s, \alpha > 0 \), a subgraph \( G_{s,\alpha} = (V_{s,\alpha}, W_{s,\alpha}, E_{s,\alpha}) \) with an almost-sure finite number of vertices (see next section) is obtained from \( G \) by taking

\[
E_{s,\alpha} := \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} B_{i,j} 1_{U_i \leq s} 1_{T_j \leq \alpha} \delta_{U_i, T_j},
\]

and \( U_i \in V_{s,\alpha} \) if and only if \( U_i \leq s \) and there exists \( j \in \mathbb{N} \) such that \( T_j \leq \alpha \) and \((U_i, T_j) \in E_{s,\alpha}\); similarly \( T_j \in W_{s,\alpha} \) if and only if \( T_j \leq \alpha \) and there exists \( i \in \mathbb{N} \) such that \( U_i \leq s \) and \((U_i, T_j) \in E_{s,\alpha}\).
3.2. Asymptotic behavior. We state here the counterparts of the results of Caron and Rousseau (2017) for the bipartite situation. Proofs are given in appendix B.

The first result is about the asymptotic behavior of $|E_{s,\alpha}|$ in the large graph limit.

**Proposition 1.** Suppose that $W$ satisfies equation (1). Then, $\mathbb{P}$-almost-surely as $\min(s,\alpha) \to \infty$,

$$|E_{s,\alpha}| \sim s\alpha \int_{\mathcal{X}} \int_{\mathcal{Y}} W(x, y) \psi(dy) \rho(dx).$$

We now state the results for the behavior of $|V_{s,\alpha}|$ and $|W_{s,\alpha}|$ in the large graph limit. As for the unipartite case, these behaviors are determined by the tails of the graphex marginals $\mu_v : \mathcal{X} \to \mathbb{R}_+$ and $\mu_w : \mathcal{Y} \to \mathbb{R}_+$ defined by

$$\mu_v(x) := \int_{\mathcal{Y}} W(x, y) \psi(dy), \text{ and } \mu_w(y) := \int_{\mathcal{X}} W(x, y) \rho(dx).$$

We will state the results only for the $v$-part of the graph, because the corresponding statements for the $w$-part are obtained by a trivial symmetry argument. We make use of the assumption that there is a constant $\sigma \in [0, 1]$ and a slowly-varying function $\ell_v$, such that as $z \to 0$,

$$F_v(z) := \rho\{x \in \mathcal{X} : \mu_v(x) \geq z\} \sim z^{-\sigma} \ell_v(z^{-1}).$$

Then we have the following proposition. There, when $\sigma = 1$ it is assumed that $\ell_v$ has enough decay at infinity so that the integral involved in the proposition converges. Moreover, we make here use of the 2-points correlation functions $\nu_v : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ defined by

$$\nu_v(x, x') := \int_{\mathcal{Y}} W(x, y)W(x', y) \psi(y).$$

**Proposition 2.** Suppose that $W$ satisfies equation (4) and $\nu_v$ satisfies equation (5) and the same conditions as in assumption 1. Then $\mathbb{P}$-almost-surely as $\alpha \to \infty$,

$$|V_{s,\alpha}| \sim \begin{cases} \alpha^\sigma \ell_v(\alpha) \Gamma(1 - \sigma) & \text{if } 0 \leq \sigma < 1, \\ \alpha \int_0^\infty z^{-1} \ell_v(z) \, dz & \text{if } \sigma = 1. \end{cases}$$

Finally, we give the behavior of the degree distributions in the large graph limit. We consider the degree distribution for the $v$-part of the graph; that is for each $k \in \mathbb{N}$ the number of vertices belonging to the $v$-part that are connected to $k$ vertices belonging to the $w$-part. More specifically, for every $k \in \mathbb{N}$, we let

$$D_k^{(w)} := \sum_{i \in \mathbb{N}} 1_{U_i \leq s} \{ \sum_{j \in \mathbb{N}} 1_{T_{ij} \leq \alpha} B_{i,j} = k \}.$$


Proposition 3. Suppose that $W$ satisfies equation (4) and $\nu_v$ satisfies equation (5) and the same conditions as in assumption 1. Then $\mathbb{P}$-almost-surely as $\alpha \to \infty$,

$$D_k^{(v)} \sim \begin{cases} s \times o(\ell_v(\alpha)) & \text{if } \sigma = 0, \\ s^\sigma \sigma \Gamma(j - \sigma)\ell_v(\alpha)/k! & \text{if } 0 < \sigma < 1, \\ so \left\{1_{k=1} \int_\alpha^\infty z^{-1}\ell_v(z) \, dz + \frac{\ell_v(\alpha)}{k(k-1)} 1_{k \geq 2}\right\} & \text{if } \sigma = 1. \end{cases}$$

We would like to emphasize that the results of propositions 2 and 3 only require $\alpha \to \infty$ to hold, while proposition 1 is valid only if $\min(s, \alpha) \to \infty$.

3.3. Tail-indexes estimation. As for the bipartite case, the parameters $\sigma$ plays a crucial role in the asymptotic behavior of the graphex process. Here we show how the estimator proposed in section 2.3 translates to the bipartite case. We assume for simplicity $p = 1/2$.

$$\hat{\sigma}_{s,\alpha} := \frac{\log |V_{s,\alpha}| - \log \left\{2^{-1} \sum_{U \in V_{s,\alpha}} \left(1 - 2^{-1} \sum_{T \in W_{s,\alpha}} 1_{E_{s,\alpha}(U,T)}\right)\right\}}{\log 2}$$

Then we have the main theorem of this section, whose proof is to be found in appendix C.

Theorem 2. Assume $\nu_v$ satisfies assumption 1 and that assumption 2 holds with $F$ replaced by $F_v$. Then there is a constant $C'' > 0$ depending only on $W$ such that for all $\alpha, s \geq 1$ it holds

$$\mathbb{E}[(\hat{\sigma}_{s,\alpha} - \sigma)^2] \leq \frac{\Gamma_{1/2,\alpha}^2}{(\log 2)^2} + C'' \log(\alpha)^2 \max\left\{\frac{1}{s \sigma}, \alpha^{1-2\eta}\right\}.\]$$

3.4. Discussion. Caron and Rousseau (2017) propose a naive estimator for $\sigma$ in the unipartite case. Their estimator can be tuned to handle the bipartite case as follows. Here we assume $\sigma \in (0,1)$ for simplicity. Let $W = \int_x \int_y W(x, y) \psi(dy) \rho(dx)$. Then Caron and Rousseau’s estimator for the parameters is based on the results of propositions 1 and 2, implying that $\mathbb{P}$-almost-surely as $\min(s, \alpha) \to \infty$,

$$\frac{\log |V_{s,\alpha}|}{\log |E_{s,\alpha}|} \sim \frac{\sigma}{\log(\sigma W)} + \frac{\log(s \ell_v(\alpha) \Gamma(1 - \sigma))}{\log(\sigma W)}.$$

If the behaviour of $\log s/\log \alpha$ is known as $\min(s, \alpha) \to \infty$, the previous equation can be used to construct a consistent estimator of $\sigma$. The main limitation of this approach is then clear: $\log s/\log \alpha$ is usually not known, and it is further unclear how to estimate this quantity from a single observation. That is, this naive estimator can not actually be computed in practice.

A second limitation of these estimators comes from the fact that they only use information about the graph size. Doing so, one cannot hope to achieve a rapidly decreasing risk of estimating $\sigma$ as $\min(s, \alpha) \to \infty$, even when $\log s/\log \alpha$ is known: the risk is necessarily bounded by a multiple constant.
of $1 / \log \min(\alpha, s)$. Note that this limitation is also true in the unipartite case. Our estimator overcomes this issue by incorporating information from the degree distribution of the graph.

Finally, observe that our estimator in the bipartite case only requires $\alpha \to \infty$, but not necessarily $\min(s, \alpha) \to \infty$.

4. Simulation results

We test our estimator for simple graphs in practice by simulating from the examples of section 2.5. For each example, we generate Monte Carlo (MC) samples according to the following procedure. For each sample:

(1) Generate the first $n$ jumps $\{X_1, \ldots, X_n\}$ of a fresh new $\alpha$-rate Poisson process on $\mathbb{R}_+$. This is done by drawing $n$ independents Gamma$(1, \alpha)$ random variables and letting $X_1 = \Gamma_1$, $X_2 = \Gamma_1 + \Gamma_2$, $\ldots$. For the Generalized Gamma Process (GGP) example, we transform $\{X_1, \ldots, X_n\}$ according to Ferguson and Klass (1972)’s inverse Lévy tail trick. In all cases, this generates the jumps in increasing order. We take advantage of the fact that $\mu$ is monotonically decreasing in all the examples to stop whenever $\mu(X_n) \leq 10^{-6}$.

(2) We generate the graph according to the procedure of section 2.1. It is unnecessary to store the whole graph in memory, because our statistics of interest involve only the number of edges, the number of vertices, and the degree distribution of the graph.

Notice that this is a rather inefficient way of getting MC samples, especially when $\mu$ is slowly decaying (i.e. in the sparse case). Labels $\{U_1, U_2, \ldots\}$ can be obtained by sampling iid uniformly from $[0, \alpha]$, although we don’t need them in general.

For different sets of parameters, we simulated MC samples to get estimates of $\mathbb{E}[E_\alpha]$, $\mathbb{E}[V_\alpha]$ and $\mathbb{E}[(\hat{\sigma}_{p,\alpha} - \sigma)^2]^{1/2}$. We also compare our estimator with the naive estimator from Caron and Rousseau (2017). We recall that their estimator is defined by

$$\hat{\sigma}_{CR} := \frac{2 \log |V_\alpha|}{\log |E_\alpha|} - 1.$$ 

In all simulations we picked $\sigma = 0.3$ for the sparse separable and sparse non-separable example, while the GGP example corresponds to a choice of $\sigma' = 0.5$. Results are summarized in the next sections.

4.1. Number of edges and number of vertices. We estimated $\mathbb{E}|E_\alpha|$ and $\mathbb{E}|V_\alpha|$ on the examples for various values of $\alpha$. The raw numbers are given in tables 1 and 2. Of course, these results are only of little interest since these expectations are available in closed form and computable either exactly, or at least with high precision using numerical integration. However, we believe having these numbers in mind is helpful to interpret the result of the next section in term of $|E_\alpha|$ and $|V_\alpha|$ instead of $\alpha$. 
Table 1. Average number of edges over 10 000 MC samples for the examples described in section 2.5 and various values of $\alpha$.

| $\alpha$         | Dense  | 25  | 50  | 100  | 200  | 400  | 800  |
|------------------|--------|-----|-----|------|------|------|------|
| $\alpha=25$      | 155.35 | 624.97 | 626.15 | 9976.05 | 39988.50 | 40016.50 | 160226 |
| $\alpha=50$      | 500.00 | 2495.29 | 2509.99 | 9999.26 | 159967 | 639409 | 82068.90 |
| $\alpha=100$     | 9976.05 | 457.36 | 9976.05 | 39988.50 | 40016.50 | 160226 |
| $\alpha=200$     | 2626.10 | 112.38 | 7328.27 | 39988.50 | 159967 | 639409 | 82068.90 |
| $\alpha=400$     | 1174.88 | 23.99 | 1174.88 | 2626.10 | 582214 | 582214 | 82068.90 |
| $\alpha=800$     | 4445.46 | 639409 | 582214 | 582214 | 582214 | 582214 | 82068.90 |

Table 2. Average number of vertices over 10 000 MC samples for the examples described in section 2.5 and various values of $\alpha$.

| $\alpha$         | Dense  | 25  | 50  | 100  | 200  | 400  | 800  |
|------------------|--------|-----|-----|------|------|------|------|
| $\alpha=25$      | 22.94  | 47.98 | 98.08 | 198.11 | 397.93 | 798.19 |
| $\alpha=50$      | 94.93  | 224.54 | 517.63 | 1174.88 | 2626.10 | 5804.23 |
| $\alpha=100$     | 41.06  | 112.38 | 299.82 | 782.73 | 2017.87 | 5137.58 |
| $\alpha=200$     | 34.20  | 95.54 | 257.71 | 677.67 | 1749.09 | 4445.46 |
| $\alpha=400$     | 200.92 | 598.13 | 1748.8 | 5031.92 | 14333 | 40408.5 |
| $\alpha=800$     | 200.92 | 598.13 | 1748.8 | 5031.92 | 14333 | 40408.5 |

Figure 1 displays $E|E_{\alpha}|$ and $E|V_{\alpha}|$ as a function of $\alpha$. We find here clearly that $E|E_{\alpha}| \propto \alpha^2$ and $E|V_{\alpha}| \propto \alpha^{1+\sigma}(\alpha)$.

4.2. Risk estimation for varying $\alpha$. We also estimated $E[(\hat{\sigma}_{0.5,\alpha} - \sigma)^2]^{1/2}$ and $E[(\hat{\sigma}_{CR} - \sigma)^2]^{1/2}$ on the examples for various values of $\alpha$. The raw numbers are given in tables 3 and 4.
Table 3. Estimation of $\mathbb{E}[(\hat{\sigma}_{0.5,\alpha} - \sigma)^2]^{1/2}$ using 10 000 MC samples for the examples described in section 2.5 and various values of $\alpha$.

| $\alpha$ | 25   | 50   | 100  | 200  | 400  | 800  |
|-----------|------|------|------|------|------|------|
| Dense     | 0.1620 | 0.0698 | 0.0327 | 0.0150 | 0.0079 | 0.0039 |
| Almost-dense | 0.2971 | 0.2440 | 0.2081 | 0.1814 | 0.1608 | 0.1443 |
| Sparse sep. | 0.2440 | 0.1658 | 0.1168 | 0.0875 | 0.0654 | 0.0488 |
| Sparse non-sep. | 0.2947 | 0.1945 | 0.1356 | 0.0966 | 0.0686 | 0.0462 |
| GGP       | 0.1219 | 0.0801 | 0.0502 | 0.0283 | 0.0110 | 0.0050 |

Table 4. Estimation of $\mathbb{E}[(\hat{\sigma}_{\text{CR}} - \sigma)^2]^{1/2}$ using 10 000 MC samples for the examples described in section 2.5 and various values of $\alpha$.

| $\alpha$ | 25   | 50   | 100  | 200  | 400  | 800  |
|-----------|------|------|------|------|------|------|
| Dense     | 0.2719 | 0.2125 | 0.1757 | 0.1500 | 0.1305 | 0.1154 |
| Almost-dense | 0.4240 | 0.3882 | 0.3590 | 0.3349 | 0.3144 | 0.2967 |
| Sparse sep. | 0.2963 | 0.2535 | 0.2230 | 0.1994 | 0.1802 | 0.1643 |
| Sparse non-sep. | 0.3483 | 0.2947 | 0.2575 | 0.2285 | 0.2051 | 0.1851 |
| GGP       | 0.1805 | 0.1577 | 0.1392 | 0.1239 | 0.1103 | 0.0982 |

We pictured in fig. 2 the risk of $\hat{\sigma}_{0.5,\alpha}$ and $\hat{\sigma}_{\text{CR}}$ as functions of $\alpha$ for all the simulated examples. As expected, the risk of $\hat{\sigma}_{0.5,\alpha}$ has polynomial decay every-time but the almost dense example, while the risk of $\hat{\sigma}_{\text{CR}}$ remains logarithmically decreasing on all examples. We note, however, that even in the situation when the risk has logarithm decay, $\hat{\sigma}_{0.5,\alpha}$ seems to perform slightly better than $\hat{\sigma}_{\text{CR}}$.

Figure 3 shows the risks for both estimators as functions of $\alpha$, normalized so that the risk equal 1 when $\alpha = 25$. It is seen from the figure that the simulation results are consistent with the results of section 2.5 for the dense, almost dense, sparse separable and GGP examples. For the sparse non-separable example, however, because $\eta = (1 + \sigma)/2 < 1$, we expected to observe a risk decaying slower than the separable example, which is not the case. There’s at least two explanations for that. First of all, maybe the difference becomes visible only for larger values of $\alpha$. Second, it is plausible that bounding the variance of $\hat{\sigma}_{p,\alpha}$ using assumption 1 on the 2-points correlation function is too pessimistic, and the true variance is indeed much smaller than the upper bound derived in theorem 1.

Figure 4 is similar to fig. 3, but this time the normalized risk is plotted as a function of $|V_\alpha|$.

4.3. Risk estimation for various values of $p$. Finally, we ran the simulations for a fixed value of $\alpha = 50$, but estimated $\mathbb{E}[(\hat{\sigma}_{p,50} - \sigma)^2]^{1/2}$ for
Figure 2. Comparative plot of $\mathbb{E}[(\hat{\sigma}_{0.5,\alpha} - \sigma)^2]^{1/2}$ against $\mathbb{E}[(\hat{\sigma}_{CR} - \sigma)^2]^{1/2}$ using 10 000 MC samples for the examples described in section 2.5.
Figure 3. Comparative plot of $E[(\hat{\sigma}_{0.5,\alpha} - \sigma)^2]^{1/2}$ (left) against $E[(\hat{\sigma}_{CR} - \sigma)^2]^{1/2}$ (right) as functions of $\alpha$ using 10 000 MC samples for the examples described in section 2.5. Here the risks are normalized to equal 1 when $\alpha = 25$.

Figure 4. Comparative plot of $E[(\hat{\sigma}_{0.5,\alpha} - \sigma)^2]^{1/2}$ (left) against $E[(\hat{\sigma}_{CR} - \sigma)^2]^{1/2}$ (right) as functions of $|V_\alpha|$ using 10 000 MC samples for the examples described in section 2.5. Here the risks are normalized to equal 1 when $\alpha = 25$.

Various values of $p$ ranging from 0.1 to 0.99. Raw numbers are given in table 5 and illustration on fig. 5. It is not surprising to find here that the risk is slightly better for values of $p$ close to one, since this tends to reduce the ratio $\Gamma_{p,\alpha}/\log(1/p)$ on these examples. There is no guarantee, however, that in general $\Gamma_{p,\alpha}/\log(1/p)$ is minimized for $p \to 1$. Nevertheless, we would recommend to compute the estimator for large value of $p$ (say $p = 0.9$) to not deteriorate too much $\Gamma_{p,\alpha}$. 
Table 5. Estimation of $E[(\hat{\sigma}_{p,50} - \sigma)^2]^{1/2}$ using 10 000 MC samples for the examples described in section 2.5 and various values of $p$.

| $p$    | 0.1   | 0.3   | 0.5   | 0.7   | 0.9   | 0.99  |
|--------|-------|-------|-------|-------|-------|-------|
| Dense  | 0.1938| 0.0941| 0.0698| 0.0594| 0.0547| 0.0539|
| Almost-dense | 0.3148| 0.2617| 0.2440| 0.2345| 0.2276| 0.2257|
| Sparse sep. | 0.2552| 0.1880| 0.1658| 0.1531| 0.1478| 0.1438|
| Sparse non-sep. | 0.3026| 0.2229| 0.1945| 0.1800| 0.1695| 0.1671|
| GGP    | 0.1296| 0.0927| 0.0801| 0.0727| 0.0674| 0.0658|

Figure 5. Estimation of $E[(\hat{\sigma}_{p,50} - \sigma)^2]^{1/2}$ using 10 000 MC samples for the examples described in section 2.5 and various values of $p$.

Appendix A. Proof of Theorem 1

The starting point of the proof is to decompose the risk onto a deterministic bias term and some stochastic variance terms. Then, when $N_{p,\alpha} = 0$ we have

$$ (\hat{\sigma}_{p,\alpha} - \sigma)^2 \leq 1. \quad (6) $$

In the situation where $N_{p,\alpha} \geq 1$, we have

$$ -\log(p)\hat{\sigma}_{p,\alpha} = \log \frac{E_{N_{1,\alpha}}}{E_{N_{p,\alpha}}} + \log p $$

$$ + \log \left( 1 + \frac{N_{1,\alpha} - E_{N_{1,\alpha}}}{E_{N_{1,\alpha}}} \right) - \log \left( 1 + \frac{N_{p,\alpha} - E_{N_{p,\alpha}}}{E_{N_{p,\alpha}}} \right). $$

To ease notations, we define,

$$ b_{\sigma,\alpha} := \frac{\log(E_{N_{1,\alpha}}/E_{N_{p,\alpha}})}{-\log p} - 1 - \sigma, \quad Z_p := \frac{N_{p,\alpha} - E_{N_{p,\alpha}}}{E_{N_{p,\alpha}}}. $$
Moreover, because $N_{1,\alpha} \geq N_{p,\alpha} \geq 1$. Then when $N_{p,\alpha} \geq 1$, 

\[(\hat{\sigma}_{p,\alpha} - \sigma)^2 = \left( b_{\sigma,\alpha} + \frac{\log(1 + Z_1)}{-\log p} - \frac{\log(1 + Z_p)}{-\log p} \right)^2 \]

\[= b_{\sigma,\alpha}^2 + 2b_{\sigma,\alpha} \frac{\log(1 + Z_1)}{-\log p} - 2b_{\sigma,\alpha} \frac{\log(1 + Z_p)}{-\log p} \]

\[+ \frac{\log^2(1 + Z_1)}{(-\log p)^2} + \frac{\log^2(1 + Z_p)}{(-\log p)^2} - 2 \frac{\log(1 + Z_1) \log(1 + Z_p)}{(-\log p)^2}. \]

We now introduce functions $\varphi_1, \varphi_2 : (-1, \infty) \to \mathbb{R}_+$ such that,

\[\varphi_1(z) := \frac{\log(1 + z)}{z}, \quad \varphi_2(z) := -\frac{\log(1 + z) - z}{z^2}.\]

The functions $\varphi_1$ and $\varphi_2$ are non-negative and monotonically decreasing on $(-1, \infty)$. We can therefore write when $N_{p,\alpha} \geq 1$,

\[(7) \quad (\hat{\sigma}_{p,\alpha} - \sigma)^2 = b_{\sigma,\alpha}^2 + \frac{Z_1^2 \varphi_1(Z_1)^2}{(-\log p)^2} + \frac{Z_p^2 \varphi_1(Z_p)^2}{(-\log p)^2} - 2 \frac{Z_1 Z_p \varphi_1(Z_1) \varphi_1(Z_p)}{(-\log p)^2} \]

\[+ \frac{2b_{\sigma,\alpha} Z_1}{-\log p} - \frac{2b_{\sigma,\alpha} Z_1^2 \varphi_2(Z_1)}{-\log p} - \frac{2b_{\sigma,\alpha} Z_p}{-\log p} + \frac{2b_{\sigma,\alpha} Z_p^2 \varphi_2(Z_p)}{-\log p}. \]

But, by Young’s inequality we have $2|Z_1 Z_p \varphi_1(Z_1) \varphi_1(Z_p)| \leq Z_1^2 \varphi_1(Z_1)^2 + Z_p^2 \varphi_2(Z_p)^2$. Combining the Young inequality estimate with equations (6) and (11) gives

\[
E[(\hat{\sigma}_{p,\alpha} - \sigma)^2] = E[(\hat{\sigma}_{p,\alpha} - \sigma)^2 1_{N_{p,\alpha} = 0}] + E[(\hat{\sigma}_{p,\alpha} - \sigma)^2 1_{N_{p,\alpha} \geq 1}] \]

\[\leq \mathbb{P}(N_{p,\alpha} = 0) + b_{\sigma,\alpha}^2 + \frac{2E[Z_1^2 \varphi_1(Z_1)^2 1_{N_{p,\alpha} \geq 1}]}{(-\log p)^2} + \frac{2b_{\sigma,\alpha} E[Z_1 1_{N_{p,\alpha} \geq 1}]}{-\log p} \]

\[+ \frac{2E[Z_p^2 \varphi_2(Z_p)^2 1_{N_{p,\alpha} \geq 1}]}{(-\log p)^2} - \frac{2b_{\sigma,\alpha} E[Z_p 1_{N_{p,\alpha} \geq 1}]}{-\log p} + \frac{2b_{\sigma,\alpha} E[Z_1^2 \varphi_2(Z_1) 1_{N_{p,\alpha} \geq 1}]}{-\log p} - \frac{2b_{\sigma,\alpha} E[Z_1 \varphi_2(Z_1) 1_{N_{p,\alpha} \geq 1}]}{-\log p}. \]

Moreover, because $E[Z_1] = E[Z_p] = 0$ we get by Hölder’s inequality the following estimates.

\[E[Z_1 1_{N_{p,\alpha} \geq 1}] = E[Z_1 (1 - 1_{N_{p,\alpha} = 0})] \leq E[|Z_1| 1_{N_{p,\alpha} = 0}] \leq \sqrt{E[Z_1^2]} \mathbb{P}(N_{p,\alpha} = 0) \]

\[E[Z_p 1_{N_{p,\alpha} \geq 1}] = E[Z_p (1 - 1_{N_{p,\alpha} = 0})] = \mathbb{P}(N_{p,\alpha} = 0). \]

Also, because $E[N_{p,\alpha}] > 0$ (see proposition 4 below), by Chebychev inequality, for any $t < 1$,

\[\mathbb{P}(N_{p,\alpha} = 0) \leq \mathbb{P}(N_{p,\alpha} < \mathbb{E}N_{p,\alpha}(1 - t)) \leq \mathbb{P}(Z < -t) \leq \frac{E[Z_p^2]}{t^2}.\]
Since this is true for all $t < 1$, we certainly have $\mathbb{P}(N_{p,\alpha} = 0) \leq \mathbb{E}[Z_p^2]$. Furthermore, $N_{1,\alpha} \geq N_{p,\alpha} \geq 1$ implies that $Z_1 \geq -1 + (\mathbb{E}N_{1,\alpha})^{-1}$, then

$$
\varphi_1(Z_1)1_{N_{p,\alpha} \geq 1} \leq \frac{\log \mathbb{E}N_{1,\alpha}}{1 - (\mathbb{E}N_{1,\alpha})^{-1}}.
$$

Also,

$$
\varphi_2(Z_1) = \varphi_1(Z_1) - \frac{(1 + Z_1) \log (1 + Z_1) - Z_1}{Z_1^2} \leq \varphi_1(Z_1).
$$

Obviously, the same estimates hold for $Z_p$. Combining all the previous estimates yield the bound (where $(x)_+ = x$ if $x \geq 0$ and $(x)_+ = 0$ if $x < 0$).

$$
\mathbb{E}[(\hat{\sigma}_{p,\alpha} - \sigma)^2] \leq b^2_{\sigma,\alpha} + \left[1 + \frac{2(-b_{\sigma,\alpha})_+}{-\log p}\right] \mathbb{E}[Z_p^2] + \frac{2(b_{\sigma,\alpha})_+}{-\log p} \sqrt{\mathbb{E}[Z_p^2] \mathbb{E}[Z_p^2]}
$$

$$
+ \left[\frac{\log \mathbb{E}N_{1,\alpha}}{1 - (\mathbb{E}N_{1,\alpha})^{-1}}\right]^2 \frac{2\mathbb{E}[Z_p^2]}{(-\log p)^2} + \left[\frac{\log \mathbb{E}N_{1,\alpha}}{1 - (\mathbb{E}N_{1,\alpha})^{-1}}\right] \frac{2(-b_{\sigma,\alpha})_+ \mathbb{E}[Z_p^2]}{-\log p}
$$

$$
+ \left[\frac{\log \mathbb{E}N_{p,\alpha}}{1 - (\mathbb{E}N_{p,\alpha})^{-1}}\right]^2 \frac{2\mathbb{E}[Z_p^2]}{(-\log p)^2} + \left[\frac{\log \mathbb{E}N_{p,\alpha}}{1 - (\mathbb{E}N_{p,\alpha})^{-1}}\right] \frac{2(b_{\sigma,\alpha})_+ \mathbb{E}[Z_p^2]}{-\log p}.
$$

Then the conclusion of theorem 1 follows from the estimates in appendices A.1 and A.2.

A.1. Study of the bias term $b_{\sigma,\alpha}$. The estimate for the bias term follows from the expectation of $N_{p,\alpha}$ for $p \in [0,1]$ given in the next proposition. Remark that from the definition of $N_{p,\alpha}$, we have

$$
N_{p,\alpha} = \begin{cases}
  p \sum_i 1_{U_i \leq \alpha} \left(1 - (1 - p)^{\sum_j \mathbb{1}_{j \neq i} B_{i,j} U_j \leq \alpha} \right) & \text{if } 0 < p < 1, \\
  \sum_i 1_{U_i \leq \alpha} \{\sum_j \mathbb{1}_{j \neq i} B_{i,j} U_j \leq \alpha > 0} & \text{if } p = 1.
\end{cases}
$$

Proposition 4. For any $p \in [0,1]$ and $\alpha > 0$ it holds,

$$
\mathbb{E}[N_{p,\alpha}] = p\alpha \int_{X} \left(1 - e^{-p\alpha \mu(x)}\right) \rho(dx).
$$

Proof. We assume here that $p \neq 1$. The case $p = 1$ follows from the steps, or by taking cautiously the limit $p \to 1$ in the proof. Then we have,

$$
\mathbb{E}[N_{p,\alpha}] = p\mathbb{E} \left[\sum_i 1_{U_i \leq \alpha} \left(1 - (1 - p)^{\sum_j \mathbb{1}_{j \neq i} B_{i,j} U_j \leq \alpha} \right) \right]
$$

$$
= p\mathbb{E} \left[\sum_i 1_{U_i \leq \alpha} \left(1 - \mathbb{E} \left[(1 - p)^{\sum_j \mathbb{1}_{j \neq i} B_{i,j} U_j \leq \alpha} \mid \Pi \right]\right) \right]
$$

$$
= p\mathbb{E} \left[\sum_i 1_{U_i \leq \alpha} \left(1 - \prod_{j \neq i} \mathbb{E} \left[(1 - p)^{B_{i,j} U_j \leq \alpha} \mid \Pi \right]\right) \right],
$$
where the last line follows from independence and dominated convergence theorem to take care of the infinite product. It is easily seen from the definition of \( B \) that for all \( i \neq j \),

\[
\mathbb{E}[(1 - p)B_{i,j}1_{U_j \leq \alpha} | \Pi] = 1 - pW(X_i, X_j)1_{U_j \leq \alpha}.
\]

It follows, invoking the Slyvniak-Mecke formula (Daley and Vere-Jones, 2003b, Chapter 13),

\[
\mathbb{E}[N_{p,\alpha}] = p\mathbb{E}\left[ \sum_i 1_{U_i \leq \alpha} \left( 1 - \prod_{j \neq i} (1 - pW(X_i, X_j)1_{U_j \leq \alpha}) \right) \right]
= p\alpha \int_{\mathcal{X}} \left( 1 - \mathbb{E} \left[ \prod_j (1 - pW(x, X_j)1_{U_j \leq \alpha}) \right] \right) \rho(dx).
\]

Then the conclusion of the proposition follows by Campbell’s formula (Kingman, 1993, Section 3.2) because,

\[
\mathbb{E} \left[ \prod_j (1 - pW(x, X_j)1_{U_j \leq \alpha}) \right] = \mathbb{E} \left[ \exp \left\{ \sum_j 1_{U_j \leq \alpha} \log(1 - pW(x, x_j)) \right\} \right]
= \exp \left\{ -p\alpha \int_{\mathcal{X}} W(x, x') \rho(dx') \right\}.
\]

It is clear from the result of the previous proposition that we have,

\[
\log \frac{\mathbb{E}[N_{1,\alpha}]}{\mathbb{E}[N_{p,\alpha}]} = \log \left[ \frac{\int_{\mathbb{R}_+} (1 - e^{-\alpha z}) F(dz)}{p\int_{\mathbb{R}_+} (1 - e^{-p\alpha z}) F(dz)} \right]
= -(1 + \sigma) \log p - \log \left[ \frac{\int_{\mathbb{R}_+} (1 - e^{-p\alpha z}) F(dz)}{p^\sigma \int_{\mathbb{R}_+} (1 - e^{-\alpha z}) F(dz)} \right].
\]

It then follows from assumption 2 that,

\[
|b_{\sigma,\alpha}| \leq \frac{\Gamma_{p,\alpha}}{-\log p}.
\]

A.2. Variance estimates for \( N_{p,\alpha} \). We now compute the estimate for \( \mathbb{E}[Z_1^2] \) and \( \mathbb{E}[Z_p^2] \) in the proof of the main theorem. We assume here that \( p \neq 1 \). The case \( p = 1 \) follows by taking cautiously the limit \( p \to 1 \) in the proof. To shorten coming equations, we write \( q := 1 - p \). We then have,

\[
N_{p,\alpha}^2 = p^2 \sum_i 1_{U_i \leq \alpha} \left( 1 - q^2 \sum_{j \neq i} B_{i,j}1_{U_j \leq \alpha} \right)^2
+ p^2 \sum_i \sum_{k \neq i} 1_{U_i \leq \alpha} 1_{U_k \leq \alpha} \left( 1 - q \sum_{j \neq i} B_{i,j}1_{U_j \leq \alpha} \right) \left( 1 - q \sum_{j \neq k} B_{k,j}1_{U_j \leq \alpha} \right).
\]
We call \( p^2 S_1 \) the first term of the rhs of the previous equation, and \( p^2 S_2 \) the second term, so that \( N_{p,\alpha}^2 = p^2 S_1 + p^2 S_2 \). That is,

\[
S_1 := \sum_i 1_{U_i \leq \alpha} \left( 1 - q \sum_{j \neq i} B_{i,j} 1_{U_j \leq \alpha} \right)^2
\]

\[
S_2 := \sum_i \sum_{k \neq i} 1_{U_i \leq \alpha} 1_{U_k \leq \alpha} \left( 1 - q \sum_{j \neq i} B_{i,j} 1_{U_j \leq \alpha} \right) \left( 1 - q \sum_{j \neq k} B_{k,j} 1_{U_j \leq \alpha} \right),
\]

A.2.1. Computation of \( \mathbb{E}[S_1] \). Expanding the square,

\[
S_1 = \sum_i 1_{U_i \leq \alpha} \left( 1 - 2q \sum_{j \neq i} B_{i,j} 1_{U_j \leq \alpha} + q^2 \sum_{j \neq i} B_{i,j} 1_{U_j \leq \alpha} \right)
\]

By independence, and using the dominated convergence theorem to take care of the infinite product,

\[
\mathbb{E}[S_1 | \Pi] = \sum_i 1_{U_i \leq \alpha} \left( 1 - 2 \prod_{j \neq i} \mathbb{E} \left[ q^{B_{i,j}} 1_{U_j \leq \alpha} | \Pi \right] \right)
\]

\[
+ \prod_{j \neq i} \mathbb{E} \left[ q^{2B_{i,j}} 1_{U_j \leq \alpha} | \Pi \right].
\]

From the definition of \( \mathcal{B} \) we get that,

\[
\mathbb{E}[q^{B_{i,j}} 1_{U_j \leq \alpha} | \Pi] = 1 - pW(X_i, X_j) 1_{U_j \leq \alpha},
\]

\[
\mathbb{E}[q^{2B_{i,j}} 1_{U_j \leq \alpha} | \Pi] = 1 - p(1 + q)W(X_i, X_j) 1_{U_j \leq \alpha}.
\]

Therefore,

\[
\mathbb{E}[S_1] = \mathbb{E} \left[ \sum_i 1_{U_i \leq \alpha} \left\{ 1 - 2 \prod_{j \neq i} (1 - pW(X_i, X_j) 1_{U_j \leq \alpha}) \right\} \right]
\]

\[
+ \prod_{j \neq i} (1 - p(1 + q)W(X_i, X_j) 1_{U_j \leq \alpha}) \right\} \right] \rho(dx).
\]

Invoking the Slyvniak-Mecke theorem (Daley and Vere-Jones, 2003b, Chapter 13),

\[
\mathbb{E}[S_1] = \alpha \int_{\mathcal{X}} \left\{ 1 - 2 \mathbb{E}[\prod_j (1 - pW(x, X_j) 1_{U_j \leq \alpha})] \right\} \rho(dx).
\]

Therefore, by Campbell’s formula (Kingman, 1993, Section 3.2) (see also the proof of proposition 4),

\[
\mathbb{E}[S_1] = \alpha \int_{\mathcal{X}} \left\{ 1 - 2e^{-p\alpha\mu(x)} + e^{-p(1+q)\alpha\mu(x)} \right\} \rho(dx)
\]
That is,
\[
E[S_1] = \alpha \int_\mathcal{X} \left(1 - e^{-p_1 \mu(x)}\right)^2 \rho(dx) + \alpha \int_\mathcal{X} e^{-p(1+q)\mu(x)} \left(1 - e^{-p^2 \mu(x)}\right) \rho(dx).
\]

A.2.2. Computation of \(E[S_2]\). We start with the expansion of the product in the expression of \(S_2\),
\[
S_2 = \sum_i \sum_{k \neq i} 1_{U_i \leq \alpha} 1_{U_k \leq \alpha} \left(1 - q \sum_{j \neq i} B_{i,j} 1_{U_j \leq \alpha} - q \sum_{j \neq k} B_{k,j} 1_{U_j \leq \alpha}ight)
\]
\[
= \sum_i \sum_{k \neq i} 1_{U_i \leq \alpha} 1_{U_k \leq \alpha} \left(1 - q \sum_{j \neq i} B_{i,j} 1_{U_j \leq \alpha} - q \sum_{j \neq k} B_{k,j} 1_{U_j \leq \alpha}ight)
\]
\[
+ q^2 B_{i,k} + \sum_{j \neq i, j \neq k} B_{i,j} 1_{U_j \leq \alpha} + \sum_{j \neq k, j \neq i} B_{k,j} 1_{U_j \leq \alpha}.
\]

The following is justified by independence and dominated convergence theorem,
\[
E[S_2 | \Pi] = \sum_i \sum_{k \neq i} 1_{U_i \leq \alpha} 1_{U_k \leq \alpha} \left(1 - \prod_{j \neq i} E[q^{B_{i,j}} 1_{U_j \leq \alpha} | \Pi] - \prod_{j \neq k} E[q^{B_{k,j}} 1_{U_j \leq \alpha} | \Pi] + \prod_{j \neq i, k} E[q^{(B_{i,j} + B_{k,j})} 1_{U_j \leq \alpha} | \Pi]\right).
\]

That is, because of equations (8) and (9),
\[
E[S_2 | \Pi] = \sum_i \sum_{k \neq i} 1_{U_i \leq \alpha} 1_{U_k \leq \alpha} \left(1 - \prod_{j \neq i} (1 - pW(X_i, X_j) 1_{U_j \leq \alpha})
\right.
\]
\[
- \prod_{j \neq k} (1 - pW(X_k, X_j) 1_{U_j \leq \alpha}) + (1 - p(1+q)W(X_i, X_k))
\]
\[
\times \prod_{j \neq i, k} (1 - pW(X_i, X_j) 1_{U_j \leq \alpha})(1 - pW(X_k, X_j) 1_{U_j \leq \alpha}) \bigg) \bigg).\]

We rewrite the previous in a slightly different form in order to be able to use the Slyvniak-Mecke theorem (Daley and Vere-Jones, 2003b, Chapter 13),
\[
E[S_2 | \Pi] = \sum_i \sum_{k \neq i} 1_{U_i \leq \alpha} 1_{U_k \leq \alpha} \left(1 - \prod_{j \neq i} (1 - pW(X_i, X_j) 1_{U_j \leq \alpha})
\right.
\]
\[
- \frac{1 - pW(X_k, X_i)}{1 - pW(X_k, X_k)} \prod_{j \neq i} (1 - pW(X_k, X_j) 1_{U_j \leq \alpha})
\]
\[
+ (1 - p(1+q)W(X_i, X_k))
\]
\[
\times \prod_{j \neq i} (1 - pW(X_i, X_j) 1_{U_j \leq \alpha})(1 - pW(X_k, X_j) 1_{U_j \leq \alpha}) \bigg) \bigg).\]
Then we can apply the Slyvniak-Mecke theorem to find that,

\[
\mathbb{E}[S_2] = \alpha \int_{\mathcal{X}} \mathbb{E} \left[ \sum_k 1_{U_k \leq \alpha} \left( 1 - \prod_j (1 - pW(x, X_j)1_{U_j \leq \alpha}) \right. \right.
\]
\[
\left. - \frac{1 - pW(X_k, x)}{1 - pW(X_k, X_k)} \prod_j (1 - pW(X_k, X_j)1_{U_j \leq \alpha}) \right.
\]
\[
\left. + (1 - p(1 + q)W(x, X_k)) \right. \times \left. \frac{\prod_j (1 - pW(x, X_j)1_{U_j \leq \alpha})(1 - pW(X_k, X_j)1_{U_j \leq \alpha})}{(1 - pW(X_k, X_k))} \right) \rho(dx).
\]

Again, we rewrite the previous in a more convenient form to apply the Slyvniak-Mecke theorem a second time,

\[
\mathbb{E}[S_2] = \alpha \int_{\mathcal{X}} \mathbb{E} \left[ \sum_k 1_{U_k \leq \alpha} \right.
\]
\[
\left. 1 - (1 - pW(x, X_k)1_{\prod_j (1 - pW(x, X_j)1_{U_j \leq \alpha})}) \right.
\]
\[
\left. - (1 - pW(X_k, x)) \prod_j (1 - pW(X_k, X_j)1_{U_j \leq \alpha}) + (1 - p(1 + q)W(x, X_k)) \times \right.
\]
\[
\left. \left. \prod_j (1 - pW(x, X_j)1_{U_j \leq \alpha})(1 - pW(X_k, X_j)1_{U_j \leq \alpha}) \right] \right) \rho(dx).
\]

Applying the Slyvniak-Mecke theorem to the previous,

\[
\mathbb{E}[S_2] = \alpha^2 \int_{\mathcal{X}} \int_{\mathcal{X}} \left\{ 1 - (1 - pW(x, x')) \mathbb{E} \left[ \prod_j (1 - pW(x, X_j)1_{U_j \leq \alpha}) \right] \right.
\]
\[
\left. - (1 - pW(x, x')) \mathbb{E} \left[ \prod_j (1 - pW(x', X_k)1_{U_j \leq \alpha}) \right] + (1 - p(1 + q)W(x, x')) \right.
\]
\[
\left. \times \mathbb{E} \left[ \prod_j (1 - pW(x, X_j)1_{U_j \leq \alpha})(1 - pW(x', X_j)1_{U_j \leq \alpha}) \right] \right\} \rho(dx) \rho(dx').
\]

Using Campbell's formula (Kingman, 1993, Section 3.2) (see also the proof of proposition 4), and defining \( \nu(x, x') = \int_{\mathcal{X}} W(x, y)W(y, x') \rho(dy) \),

\[
\mathbb{E}[S_2] = \alpha^2 \int_{\mathcal{X}} \int_{\mathcal{X}} \left\{ 1 - (1 - pW(x, x')) e^{-p\alpha \mu(x)} - (1 - pW(x, x')) e^{-p\alpha \mu(x')} \right.
\]
\[
\left. + (1 - p(1 + q)W(x, x')) e^{-p\alpha \mu(x)} e^{p\alpha \mu(x')} + p^2 \alpha \nu(x, x') \right\} \rho(dx) \rho(dx').
\]

A.2.3. Bound on the variance. We now provide the bound for \( \mathbb{E}[Z_p^2] \) using the expression of \( \mathbb{E}[S_1], \mathbb{E}[S_2] \) and proposition 4. Proposition 4 gives

\[
\mathbb{E}[N_{p, \alpha}]^2 = (p\alpha)^2 \int_{\mathcal{X}} \int_{\mathcal{X}} \left\{ 1 - e^{-p\alpha \mu(x)} \right\} \left( 1 - e^{-p\alpha \mu(x')} \right) \rho(dx) \rho(dx')
\]
\[
= (p\alpha)^2 \int_{\mathcal{X}} \int_{\mathcal{X}} \left\{ 1 - e^{-p\alpha \mu(x)} - e^{-p\alpha \mu(x')} + e^{-p\alpha \mu(x)} e^{-p\alpha \mu(x')} \right\} \rho(dx) \rho(dx').
\]
Combining this with the expression for \( \mathbb{E}[S_1] \) and \( \mathbb{E}[S_2] \), we get

\[
\mathbb{E}[N_{p,\alpha}^2] - \mathbb{E}[N_{p,\alpha}]^2 = \alpha \int_X e^{-p(1+q)\alpha \mu(x)} \left( 1 - e^{-p^2\alpha \mu(x)} \right) \rho(dx) + \alpha \int_X \left\{ 1 - e^{-p\alpha \mu(x)} \right\}^2 \rho(dx) + 2p^3 \alpha^2 \int_X \mu(x) e^{-p\alpha \mu(x)} \rho(dx) + p^2 \alpha^2 \int_X \int_X e^{-p\alpha \mu(x) - p\alpha \mu(x')} + p^2 \alpha \nu(x,x') \left\{ 1 - e^{-p^2\alpha \nu(x,x')} \right\} \rho(dx) \rho(dx') - p^3(1+q)\alpha^2 \int_X \int_X W(x,x') e^{-p\alpha \mu(x) - p\alpha \mu(x')} + p^2 \alpha \nu(x,x') \rho(dx) \rho(dx').
\]

By Hölder’s and Young’s inequality, we have \( 2\nu(x,x') \leq 2\sqrt{\mu(x)\mu(x')} \leq \mu(x) + \mu(x') \). Moreover, \( p^2 \leq p \), and \( 1-e^{-x} \leq x \) imply,

\[
\int_X \int_X e^{-p\alpha \mu(x) - p\alpha \mu(x')} + p^2 \alpha \nu(x,x') \left\{ 1 - e^{-p^2\alpha \nu(x,x')} \right\} \rho(dx) \rho(dx') \leq p^2 \alpha \int_X \int_X e^{-p\alpha \mu(x)} e^{-p\alpha \mu(x')} \nu(x,x') \rho(dx) \rho(dx').
\]

If \( \sigma = 1 \) it follows from assumption 1 that,

\[
\int_X \int_X e^{-p\alpha \mu(x) - p\alpha \mu(x')} + p^2 \alpha \nu(x,x') \left\{ 1 - e^{-p^2\alpha \nu(x,x')} \right\} \rho(dx) \rho(dx') \leq Cp^2 \alpha \left\{ \int_X \mu(x) \rho(dx) \right\}^2,
\]

which is finite because of equation (1). Now if \( 0 \leq \sigma < 1 \), it follows from assumption 1 and Caron and Rousseau (2017, Lemma B.4) that,

\[
\int_X \int_X e^{-p\alpha \mu(x) - p\alpha \mu(x')} + p^2 \alpha \nu(x,x') \left\{ 1 - e^{-p^2\alpha \nu(x,x')} \right\} \rho(dx) \rho(dx') \leq Cp^2 \alpha^{-2\eta} \left\{ \int_X \alpha^\eta \mu(x)^\eta e^{-p\alpha \mu(x)} \rho(dx) \right\}^2 \lesssim \alpha^{1-2\eta+2\ell(\alpha)}.
\]

Moreover, using Caron and Rousseau (2017, Lemma B.4), it is easily seen that all other terms involved in \( \mathbb{E}[N_{p,\alpha}^2] - \mathbb{E}[N_{p,\alpha}]^2 \) are bounded by a multiple constant of \( \alpha^{1+\sigma} \) for any \( \sigma \in [0,1] \). Since we set \( \eta = 1 \) when \( \sigma = 1 \), it follows the estimate,

\[
\mathbb{E}[N_{p,\alpha}^2] - \mathbb{E}[N_{p,\alpha}]^2 \lesssim \alpha^{1+\sigma} + \alpha^{3-2\eta+2\ell(\alpha)}.
\]

The conclusion follows because \( \mathbb{E}[N_{p,\alpha}]^2 \geq \alpha^{2+2\ell(\alpha)} \) by proposition 4.

**Appendix B. Proofs of asymptotic results for bipartite graphs**

The results in section 3.2 are straightforward adaptations of the proofs in Caron and Rousseau (2017). We then just give the proof for proposition 3 since the argument for the bipartite version can be made more direct.
The argument runs as follows. We first provide the asymptotic behavior of \( \mathbb{E}[D_k^{(v)}] \) in appendix B.1 and then the asymptotic behavior of the variance in appendix B.2. The conclusion follows by Caron and Rousseau (2017, Lemma B.1), which provides the sufficient conditions under which \( D_k^{(v)} \sim \mathbb{E}[D_k^{(v)}] \mathbb{P} \)-almost-surely as \( \alpha \to \infty \).

In the whole section, we sill use the random variables \((N_i)_{i \in \mathbb{N}}\) defined as follows. For every \( i \in \mathbb{N} \),

\[
N_i := \sum_{j \in \mathbb{N}} 1_{T_j \leq \alpha B_{i,j}}.
\]

**B.1. Expectation of \( D_k^{(v)} \).** It is easily seen that,

\[
\mathbb{E}[D_k^{(v)}] = \mathbb{E}\left[ \sum_{i \in \mathbb{N}} 1_{U_i \leq s} \mathbb{P}(N_i = k \mid \Pi_v, \Pi_w) \right] = \mathbb{E}\left[ \sum_{i \in \mathbb{N}} 1_{U_i \leq s} \mathbb{P}(N_i = k \mid \Pi_v) \right] = \frac{\alpha^k}{k!} \mathbb{E}\left[ \sum_{i \in \mathbb{N}} 1_{U_i \leq s} \mu_v(X_i)^k \exp\{-\alpha \mu_v(X_i)\} \right],
\]

where the last line follows by lemma 1 below. By Campbell’s formula (Kingman, 1993, Section 3.2),

\[
\mathbb{E}[D_k^{(v)}] = \frac{s\alpha^k}{k!} \int_X \mu_v(x)^k e^{-\alpha \mu_v(x)} \rho(dx).
\]

Then the conclusion follows by straightforward adaption of Caron and Rousseau (2017, Lemma B.2).

**B.2. Variance of \( D_k^{(v)} \).** Proceeding as for the expectation, we find that,

\[
\mathbb{E}[(D_k^{(v)})^2] = \mathbb{E}\left[ \sum_{i \in \mathbb{N}} \sum_{i' \in \mathbb{N}} 1_{U_i \leq s} 1_{U_{i'} \leq s} \mathbb{P}(N_i = k, N_{i'} = k \mid \Pi_v) \right].
\]

Then using the Slyvniak-Mecke theorem, (Daley and Vere-Jones, 2003b, Chapter 13),

\[
\mathbb{E}[(D_k^{(v)})^2] = \mathbb{E}[D_k^{(v)}] + \mathbb{E}\left[ \sum_{i \neq i'} \sum_{i \in \mathbb{N}} 1_{U_i \leq s} 1_{U_{i'} \leq s} \mathbb{P}(N_i = k, N_{i'} = k \mid \Pi_v) \right]
\]

Then from the result of lemma 2 below,

\[
\mathbb{E}[(D_k^{(v)})^2] \leq \mathbb{E}[D_k^{(v)}] + s \int_X \mathbb{E}\left[ \sum_{i} 1_{U_i \leq s} e^{-\alpha \mu_v(x) - \alpha \mu_v(X_i) + \alpha \nu_v(x,X_i)} \right.
\]

\[
\times \sum_{m=0}^{k} \frac{\alpha^{k+m}}{(m!)^2 (k-m)!} \mu_v(X_{i'})^m \mu_v(x)^m \nu_v(x,X_{i'})^{k-m} \]

\[
\rho(dx).
\]
By Campbell’s formula (Kingman, 1993, Section 3.2),

\[
\mathbb{E}[(D_k^{(v)})^2] \leq \mathbb{E}[D_k^{(v)}] + s^2 \int \int e^{-\alpha \mu_v(x) - \alpha \mu_v(x') + \alpha \nu_v(x,x')} \\
\times \sum_{m=0}^{k} \frac{\alpha^{k+m}}{(m!)^2(k-m)!} \mu_v(x)^m \mu_v(x')^m \nu_v(x,x')^{k-m} \rho(dx)\rho(dx').
\]

Using the expression for the expectation obtained in appendix B.1, we deduce that

\[
\begin{align*}
\mathbb{E}[(D_k^{(v)})^2] - \mathbb{E}[D_k^{(v)}]^2 &\leq \mathbb{E}[D_k^{(v)}] \\
&+ \frac{s^2 \alpha^{2k}}{(k!)^2} \int \int e^{-\alpha \mu_v(x) - \alpha \mu_v(x') + \alpha \nu_v(x,x')} \\
&\times \mu_v(x)^k \mu_v(x')^k \left\{ 1 - e^{-\alpha \nu_v(x,x')} \right\} \rho(dx)\rho(dx') \\
&+ s^2 \sum_{m=0}^{k-1} \frac{\alpha^{k+m}}{(m!)^2(k-m)!} \int \int e^{-\alpha \mu_v(x) - \alpha \mu_v(x') + \alpha \nu_v(x,x')} \\
&\times \mu_v(x)^m \nu_v(x,x')^{k-m} \rho(dx)\rho(dx').
\end{align*}
\]

But \( \nu(x,x') \leq (\mu(x) + \mu(x'))/2 \) by Hölder’s inequality, \( 1 - e^{-x} \leq x \) for any \( x \geq 0 \), and \( \nu_v(x,x') \leq C \mu_v(x)^\eta \mu_v(x')^\eta \) under assumption 1. Then,

\[
\begin{align*}
\mathbb{E}[(D_k^{(v)})^2] - \mathbb{E}[D_k^{(v)}]^2 &\leq \mathbb{E}[D_k^{(v)}] \\
&+ C \frac{s^2 \alpha^{1-2\eta}}{(k!)^2} \left\{ \int e^{-\alpha \mu_v(x)} [\alpha \mu_v(x)]^{k+\eta} \rho(dx) \right\}^2 \\
&+ s^2 \sum_{m=0}^{k-1} \frac{C^{k-m} \alpha^{-(2\eta-1)(k-m)}}{(m!)^2(k-m)!} \left\{ \int e^{-\alpha \mu_v(x)} [\alpha \mu_v(x)]^{m(1-\eta)+\eta k} \rho(dx) \right\}^2.
\end{align*}
\]

By straightforward adaption of Caron and Rousseau (2017, Lemma B.4), we conclude that there exists a \( c > 0 \) such that \( \mathbb{E}[(D_k^{(v)})^2] - \mathbb{E}[D_k^{(v)}]^2 \leq \alpha^{-c} \mathbb{E}[D_k^{(v)}]^2 \).

B.3. Auxiliary results. In this section, we state a certain number of auxiliary results that are used in appendices B.1 and B.2.

Notice that conditional on \((\Pi_v, \Pi_w)\), the law of \( \{N_i\}_{i \in I} \) for every finite subset \( I \subset \mathbb{N} \) is characterized by the following Laplace transform, for every \( t \in \mathbb{C}^I \) for which the rhs is defined,

\[
\mathbb{E}[e^{-\sum_{i \in I} t_i N_i} | \Pi_v, \Pi_w] = \prod_{j \in N} \prod_{i \in I} \left[ 1 + (e^{-t_i} - 1)W(X_i, Y_j)1_{T_j \leq \alpha} \right] = \exp \left\{ \sum_{i \in I} \sum_{j \in N} 1_{T_j \leq \alpha} \log[1 + (e^{-t_i} - 1)W(X_i, Y_j)] \right\}.
\]
We use the previous result to characterize (by Fourier inversion) the distribution of \( N_i \) and the joint distribution of \((N_i, N'_i)\) for \( i \neq i' \), conditional on \( \Pi_v \).

**Lemma 1.** Assume that \( k \in \mathbb{Z}_+ \). Then, for every \( i \in \mathbb{N} \),

\[
\mathbb{P}(N_i = k \mid \Pi_v) = e^{-\alpha \mu_v(X_i)} \frac{\alpha^k \mu_v(X_i)^k}{k!}.
\]

**Proof.** It is easily seen from equation (10) and Campbell’s formula (Kingman, 1993, Section 3.2) that \( N_i \) has Laplace transform (conditional on \( \Pi_v \)), for all \( t \in \mathbb{C} \) that makes sense,

\[
t \mapsto \exp \{ \alpha \mu_v(X_i)(e^{-t} - 1) \}.
\]

One recognizes the Laplace transform of a Poisson distribution. Hence the results follow. \( \square \)

**Lemma 2.** For any \( k \in \mathbb{N} \),

\[
\mathbb{P}(N_i = k_1, N_{i'} = k_2 \mid \Pi_v) \leq e^{-\alpha \mu_v(X_i) - \alpha \mu_v(X_{i'}) + \alpha \nu_v(X_i, X_{i'})} \times \sum_{m=0}^{k} \frac{\alpha^{k+m}}{(m!)^2(k-m)!} \mu_v(X_{i'})^m \mu_v(X_i)^m \nu_v(X_i, X_{i'})^{k-m}.
\]

**Proof.** From Campbell’s formula (Kingman, 1993, Section 3.2), we deduce from equation (10) that for any \((\xi_1, \xi_2) \in \mathbb{R}^2\),

\[
\mathbb{E}[e^{i\xi_1 N_i + i\xi_2 N_{i'} \mid \Pi_v}] = \exp \left\{ \alpha \mu_v(X_i)(e^{i\xi_1} - 1) + \alpha \mu_v(X_{i'})(e^{i\xi_2} - 1) + \alpha \nu_v(X_i, X_{i'})(e^{i\xi_1} - 1)(e^{i\xi_2} - 1) \right\}.
\]

Using orthogonality of trigonometric polynomials, it is easily seen that

\[
\mathbb{P}(N_i = k, N_{i'} = k \mid \Pi_v) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \mathbb{E}[e^{i\xi_1 N_i + i\xi_2 N_{i'} \mid \Pi_v}] e^{-i(\xi_1 k + \xi_2 k)} \, d\xi_1 d\xi_2.
\]

We first compute the inner integral in equation (11). To do so, write the intermediate integral

\[
I(\xi_2) := \frac{1}{2\pi} \int_0^{2\pi} \exp \left\{ \alpha \mu_v(X_i)e^{i\xi} + \alpha \nu_v(X_i, X_{i'})(e^{i\xi_2} - 1)e^{i\xi} - i\xi k \right\} \, d\xi.
\]

It is convenient to remark that \( I(\xi_2) \) is a contour integral along \( \Gamma := \{ z \in \mathbb{C} : |z| = 1 \} \). Then,

\[
I(\xi_2) = \frac{1}{2\pi} \int_{\Gamma} \exp \left\{ \alpha [\mu_v(X_i) + \nu_v(X_i, X_{i'})(e^{i\xi_2} - 1)]z \right\} \frac{1}{iz^{k+1}} \, dz.
\]

The complex exponential is analytic in the whole complex plane, thus the integrand in the previous is holomorphic everywhere except at zero where it
has a pole of order $k+1$. Thus, by the residue theorem we have

$$
I(\xi_2) = \frac{1}{k!} \lim_{z \to 0} \frac{d^k}{dz^k} \exp \left\{ \alpha [\mu_\nu(X_i) + \nu_\nu(X_i, X_{i'}) (e^{i\xi_2} - 1)] z \right\}
$$

$$
= \frac{\alpha^k}{k!} [\mu_\nu(X_i) + \nu_\nu(X_i, X_{i'}) (e^{i\xi_2} - 1)]^k.
$$

We introduce the new intermediate integral,

$$
J := \frac{1}{2\pi} \int_{0}^{2\pi} I(\xi) \exp \left\{ \alpha \mu_\nu(X_{i'}) e^{i\xi} - \alpha \nu_\nu(X_i, X_{i'}) e^{i\xi} - i\xi k \right\} d\xi
$$

$$
=: \frac{\alpha^k}{2\pi k!} \int_{i\xi k+1} K(z) \frac{d^k}{dz^k} K(z) dz,
$$

where,

$$
K(z) := [\mu_\nu(X_i) - \nu_\nu(X_i, X_{i'}) + z\nu_\nu(X_i, X_{i'})]^k
\times \exp \{ \alpha \mu_\nu(X_{i'}) z - \alpha \nu_\nu(X_i, X_{i'}) z \}.
$$

Again, the integrand $z \mapsto -iK(z)z^{-(k+1)}$ is holomorphic in the whole complex plane, except at zero where it has a pole of order $k+1$. This time, however, the computation of the residue at zero is more involved, but still doable. Near $z = 0$, we have the Taylor series expansion,

$$
\exp \{ \alpha \mu_\nu(X_{i'}) z - \alpha \nu_\nu(X_i, X_{i'}) z \}
= \sum_{m=0}^{k} \frac{\alpha^m z^m}{m!} [\mu_\nu(X_{i'}) - \nu_\nu(X_i, X_{i'})]^m + o(z^k),
$$

and by Newton’s binomial formula,

$$
[\mu_\nu(X_i) - \nu_\nu(X_i, X_{i'}) + \nu_\nu(X_i, X_{i'}) z]^k
$$

$$
= \sum_{p=0}^{k} \binom{k}{p} [\mu_\nu(X_i) - \nu_\nu(X_i, X_{i'})]^k p \nu_\nu(X_i, X_{i'})^p z^p.
$$

Therefore, as $z \to 0$,

$$
K(z) = o(z^k) + \sum_{m=0}^{k} \sum_{p=0}^{k} \binom{k}{p} \frac{\alpha^m}{m!} [\mu_\nu(X_{i'}) - \nu_\nu(X_i, X_{i'})]^m
\times [\mu_\nu(X_i) - \nu_\nu(X_i, X_{i'})]^k p \nu_\nu(X_i, X_{i'})^p z^m + p.
$$

Using the previous expression, it is easily seen that

$$
\lim_{z \to 0} \frac{d^k}{dz^k} K(z) = k! \sum_{m+p=k} \binom{k}{p} \frac{\alpha^m}{m!} [\mu_\nu(X_{i'}) - \nu_\nu(X_i, X_{i'})]^m
\times [\mu_\nu(X_i) - \nu_\nu(X_i, X_{i'})]^k p \nu_\nu(X_i, X_{i'})^p.
$$
Then,
\[
\lim_{z \to 0} \frac{q^k}{dz^k} K(z) \leq k! \sum_{m=0}^{k} \binom{k}{m} \frac{\alpha^m}{m!} \mu_v(X_i) m \mu_v(X_i) m \nu_v(X_i, X_i') k^{-m}.
\]
Therefore, by the residue theorem and after simple algebra,
\[
J \leq \sum_{m=0}^{k} \frac{\alpha^{k+m}}{(m!)^2 (k-m)!} \mu_v(X_i) m \mu_v(X_i) m \nu_v(X_i, X_i') k^{-m}.
\]
Using this last result and equation (11), we get the following expression for the joint distribution at \((k, k)\).

**Appendix C. Proof of the risk bound for the bipartite estimator**

The proof of theorem 2 follows the exact same line as for the theorem 1. Here we only state the expectation and variance estimate for the random variable

\[
M_{s, \alpha} := \frac{1}{2} \sum_{i \in \mathbb{N}} 1_{U_i \leq \alpha} \left( 1 - 2^{-\sum_{j \in \mathbb{N}} 1_{T_j \leq \alpha} B_{i,j}} \right).
\]

**C.1. Expectation of \(M_{s, \alpha}\).** By definition of the model and applying Campbell’s formula (Kingman, 1993, Section 3.2) a couple of times,

\[
\mathbb{E}[M_{s, \alpha}] = \frac{1}{2} \mathbb{E} \left[ \sum_{i \in \mathbb{N}} 1_{U_i \leq s} \left( 1 - \Pi_{j \in \mathbb{N}} \mathbb{E} \left[ 2^{-R_{i,j}} 1_{T_j \leq \alpha} | \Pi_v, \Pi_w \right] \right) \right]
\]

\[
= \frac{1}{2} \mathbb{E} \left[ \sum_{i \in \mathbb{N}} 1_{U_i \leq s} \left( 1 - \Pi_{j \in \mathbb{N}} \{ 1 - \frac{1}{2} W(X_i, Y_j) 1_{T_j \leq \alpha} \} \right) \right]
\]

\[
= \frac{1}{2} \mathbb{E} \left[ \sum_{i \in \mathbb{N}} 1_{U_i \leq s} \left( 1 - \prod_{j \in \mathbb{N}} \{ 1 - \frac{1}{2} W(X_i, Y_j) 1_{T_j \leq \alpha} \} \right) \right]
\]

\[
= \frac{1}{2} \mathbb{E} \left[ \sum_{i \in \mathbb{N}} 1_{U_i \leq s} \left( 1 - e^{-\alpha \mu_v(X_i)/2} \right) \right]
\]

Therefore,

\[
\mathbb{E}[M_{s, \alpha}] = \frac{s}{2} \int_{X} \left( 1 - e^{-\frac{\alpha}{2} \mu_v(x)} \right) \rho(dx).
\]

As \(\alpha \to \infty\), under our assumptions and by Caron and Rousseau (2017, Lemma B.2),

\[
\mathbb{E}[M_{s, \alpha}] \sim \begin{cases} 
  s \alpha^\sigma \ell_v(\alpha) \Gamma(1 - \sigma)/2 & \text{if } 0 \leq \sigma < 1, \\
  (s \alpha/2) \int_{\alpha}^{\infty} z^{-\sigma} \ell_v(z) \, dz & \text{if } \sigma = 1.
\end{cases}
\]
C.2. Variance of $M_{s,\alpha}$. We write that,

$$
E[M_{s,\alpha}^2] = \frac{1}{4}E \left[ \sum_{i\in\mathbb{N}} 1_{U_i \leq s} \left( 1 - 2^{-\sum_j 1_{T_j \leq \alpha B_{i,j}}} \right)^2 \right] + \frac{1}{4}E \left[ \sum_{i\in\mathbb{N}} \sum_{k\neq i} 1_{U_i \leq s} 1_{U_k \leq s} \left( 1 - 2^{-\sum_j 1_{T_j \leq \alpha B_{i,j}}} \right) \left( 1 - 2^{-\sum_m 1_{T_m \leq \alpha B_{k,m}}} \right) \right].
$$

The first term in the rhs of the previous equation is obviously bounded by $E[M_{s,\alpha}]/4$. We now take care of the second term. It is clear that,

$$
E \left[ \left( 1 - 2^{-\sum_j 1_{T_j \leq \alpha B_{i,j}}} \right) \left( 1 - 2^{-\sum_m 1_{T_m \leq \alpha B_{k,m}}} \right) \mid \Pi_v, \Pi_w \right]
= 1 - \prod_j E[2^{-B_{i,j}1_{T_j \leq \alpha}} \mid \Pi_v, \Pi_w] - \prod_m E[2^{-B_{k,m}1_{T_m \leq \alpha}} \mid \Pi_v, \Pi_w] + \prod_j \left( 1 - \frac{1}{2} W(X_i, Y_j) 1_{T_j \leq \alpha} \right) \prod_m \left( 1 - \frac{1}{2} W(X_k, Y_m) 1_{T_m \leq \alpha} \right).
$$

Note that the computation here are made slightly simpler as for the unipartite case since $B$ is not a symmetric array and $\Pi_v$ and $\Pi_w$ are independent. By Campbell’s formula (Kingman, 1993, Section 3.2),

$$
E \left[ \left( 1 - 2^{-\sum_j 1_{T_j \leq \alpha B_{i,j}}} \right) \left( 1 - 2^{-\sum_m 1_{T_m \leq \alpha B_{k,m}}} \right) \mid \Pi_v \right]
= 1 - e^{-\mu_v(X_i)/2} - e^{-\mu_v(X_k)/2} + e^{-\mu_v(X_i)/2 - \mu_v(X_k)/2} + e^{-\mu_v(X_i)/2 + \alpha_v(X_i, X_k)/4}.
$$

Then, by the Slyvniak-Mecke theorem (Daley and Vere-Jones, 2003b, Chapter 13),

$$
E[M_{s,\alpha}^2] \leq \frac{1}{4} E[M_{s,\alpha}] + \frac{s}{4} \int X \left[ \sum_{i\in\mathbb{N}} 1_{U_i \leq s} \left( 1 - e^{-\mu_v(X_i)/2} - e^{-\mu_v(x)/2} + e^{-\mu_v(X_i)/2 - \mu_v(x)/2} + e^{-\mu_v(X_i)/2 + \alpha_v(x, x)/4} \right) \right] \rho(dx).
$$

By Campbell’s formula again,

$$
E[M_{s,\alpha}^2] \leq \frac{1}{4} E[M_{s,\alpha}] + \frac{s^2}{4} \int X \int X \left( 1 - e^{-\mu_v(x)/2} - e^{-\mu_v(x)/2} + e^{-\mu_v(x)/2 - \mu_v(x)/2} + e^{-\mu_v(x)/2 + \alpha_v(x, x)/4} \right) \rho(dx) \rho(dx').
$$
Using equation (12), we find that,
\[
E[M^2_{s,\alpha}] - E[M_{s,\alpha}]^2 \leq \frac{1}{4}E[M_{s,\alpha}]
\]
\[
+ \frac{s^2}{4} \int X \int X e^{-\frac{\mu v(x')}{2} - \frac{\mu v(x)}{2} + \frac{\alpha v(x',x)}{4}} \left\{ 1 - e^{-\frac{\alpha v(x',x)}{4}} \right\} \rho(dx)\rho(dx').
\]
Moreover, \(\nu_v(x', x) \leq (\mu_v(x) + \mu_v(x'))/2\) by Hölder’s inequality, \(1 - e^{-x} \leq x\) for all \(x \geq 0\), and \(\nu_v(x', x) \leq C\mu_v(x)^\eta \mu_v(x')^\eta\) by assumption 2. Then,
\[
E[M^2_{s,\alpha}] - E[M_{s,\alpha}]^2 \leq \frac{1}{4}E[M_{s,\alpha}] + \frac{C\alpha s^2}{16} \left( \frac{8}{3\alpha} \right)^{2\eta} \left\{ \int X e^{-\frac{3\alpha v_v(x)}{8} \left[ \frac{3\alpha \mu_v(x)}{8} \right]} \rho(dx) \right\}^2.
\]
Henceforth, by a simple adaption of Caron and Rousseau (2017, Lemma B.2 and B.4), for \(\alpha\) large enough,
\[
E[M^2_{s,\alpha}] - E[M_{s,\alpha}]^2 \lesssim \begin{cases} s\alpha^\eta \ell_v(\alpha) + s^2 \alpha^{1-2\eta+2\sigma} \ell_v(\alpha)^2 & \text{if } 0 \leq \sigma < 1, \\ \sigma \int_{\alpha}^{\infty} z^{-1} \ell_v(z) \, dz + \alpha s^2 & \text{if } \sigma = 1. \end{cases}
\]
**Appendix D. Proof of examples rates bounds**

Here we prove the bound on \(\Gamma_{p,\alpha}\) for the various examples considered in section 2.5.

**D.1. Dense graphs example.** For any \(\alpha > 0\),
\[
\int_{\mathbb{R}_+} \left( 1 - e^{-\alpha u} \right) \, dx - \int_{\mathbb{R}_+} \left( 1 - e^{-p\alpha u(x)} \right) \, dx
\]
\[
= \int_0^1 e^{-p\alpha u/2} \left\{ 1 - e^{-(1-p)\alpha u/2} \right\} \, du \lesssim \alpha^{-1}.
\]
On the other hand, we have for \(\alpha\) large enough,
\[
\int_0^1 \left( 1 - e^{-\alpha u/2} \right) \, du \gtrsim 1.
\]
Hence, \(\Gamma_{p,\alpha} \propto \alpha^{-1}\).

**D.2. Sparse, almost dense graphs without power law.** For any \(\alpha > 0\),
\[
\int_{\mathbb{R}_+} \left( 1 - e^{-\alpha u} \right) \, dx - \int_{\mathbb{R}_+} \left( 1 - e^{-p\alpha u(x)} \right) \, dx
\]
\[
= \int_0^1 e^{-p\alpha u} \left\{ 1 - e^{-(1-p)\alpha u} \right\} \, \frac{du}{u}
\]
\[
\leq \alpha(1-p) \int_0^{1/\alpha} e^{-p\alpha u} \, du + \int_{1/\alpha}^1 e^{-p\alpha u} \, du \lesssim 1.
\]
On the other hand, when $\alpha > 0$ is large enough,
\[
\int_0^1 (1 - e^{-\alpha u}) \frac{du}{u} = (1 - e^{-\alpha}) \log(\alpha) - \int_0^\alpha \log(v) e^{-v} dv \\
\geq (1 - e^{-\alpha}) \log(p\alpha) - \int_1^\alpha \log(v) e^{-v} dv \\
\geq (1 - e^{-\alpha}) \log(\alpha) - \log(\alpha) \int_1^\alpha e^{-v} dv \geq \log(\alpha).
\]
Hence $\Gamma_{p,\alpha} \propto 1/\log \alpha$.

D.3. **Sparse graphs with power law.** Let $b := 1/(1 - \sigma)$, then for any $\alpha > 1$,
\[
p^\sigma \int_{\mathbb{R}_+} \left(1 - e^{-\alpha \mu(x)}\right) dx - \int_{\mathbb{R}_+} \left(1 - e^{-p\alpha \mu(x)}\right) dx \\
= \sigma \left(\frac{\sigma}{1 - \sigma}\right) \alpha^\sigma \left\{ p^\sigma \int_0^{\frac{\alpha}{1 - \sigma}} (1 - e^{-u}) u^{-1 - \sigma} du - \int_0^{\frac{\alpha}{1 - \sigma}} (1 - e^{-pu}) u^{-1 - \sigma} du \right\} \\
= \sigma \left(\frac{\sigma}{1 - \sigma}\right) \alpha^\sigma \int_{\frac{\alpha}{1 - \sigma}}^{\frac{\alpha}{1 - \sigma} b} (1 - e^{-u}) u^{-1 - \sigma} du \leq \frac{\sigma(1 - p)}{p}.
\]
On the other hand, when $\alpha$ is large enough,
\[
\int_{\mathbb{R}_+} \left(1 - e^{-\alpha \mu(x)}\right) dx = \alpha^\sigma \left(\frac{\sigma}{1 - \sigma}\right) \int_0^{\frac{\alpha}{1 - \sigma}} (1 - e^{-v}) v^{-1 - \sigma} dv \\
\geq \alpha^\sigma \left(\frac{\sigma}{1 - \sigma}\right) \int_0^{\frac{\alpha}{1 - \sigma}} e^{-v} v^{-\sigma} dv \geq \alpha^\sigma.
\]
Hence $\Gamma_{p,\alpha} \propto \alpha^{-\sigma}$.

D.4. **Generalized Gamma Process.** Note that in this example, we have for any $p \in [0, 1]$,
\[
\left(1 - e^{-p\alpha \mu(x)}\right) \rho(dx) = \int_{\mathbb{R}_+} \left(1 - e^{-u}\right) \left(1 + \frac{\sigma u}{p\alpha}\right)^{-1 + 1/\sigma} \\
\times \left(\left(1 + \frac{\sigma u}{p\alpha}\right)^{1/\sigma} - 1\right)^{-1 - \sigma} \exp \left\{ - \left(\left(1 + \frac{\sigma u}{p\alpha}\right)^{1/\sigma} - 1\right) \right\} \frac{du}{p\alpha}.
\]
We will find a bound on $\Gamma_{p,\alpha}$ by first lower bounding,
\[
A := \int_{\mathbb{R}_+} \left(1 - e^{-\alpha \mu(x)}\right) \rho(dx),
\]
and then upper bounding,
\[
B = \left| p^\sigma \int_{\mathbb{R}_+} \left(1 - e^{-\alpha \mu(x)}\right) \rho(dx) - \int_{\mathbb{R}_+} \left(1 - e^{-p\alpha \mu(x)}\right) \rho(dx) \right|.
\]
From the two bounds in appendices D.4.1 and D.4.2, we will conclude that
\( \Gamma_{p,\alpha} \propto \alpha^{-\sigma} \).

D.4.1. Lower bound on \( A \). Since the integrand is a positive function, we lower bound \( A \) by integrating on a smaller set. It is clear that when \( u \in [0, \alpha] \),
\[
\left( 1 + \frac{\sigma u}{\alpha} \right)^{-1+1/\sigma} \geq 1, \quad \text{and} \quad \left( 1 + \frac{\sigma u}{\alpha} \right)^{1/\sigma} - 1 \lesssim \frac{u}{\alpha}.
\]
Therefore, from equation (13) we deduce that,
\[
A \gtrsim \alpha^\sigma \int_{[0,\alpha]} (1 - e^{-u}) u^{-1-\sigma} \, du \gtrsim \alpha^\sigma.
\]

D.4.2. Upper bound on \( B \). We first write,
\[
\int_{\mathbb{R}^+} \left( 1 - e^{-p\alpha \mu(x)} \right) \rho(dx) = \int_{0}^{\alpha} \left( 1 - e^{-p\alpha \mu(x)} \right) \rho(dx) + \int_{\alpha}^{\infty} \left( 1 - e^{-p\alpha \mu(x)} \right) \rho(dx) =: C_{p,\alpha} + D_{p,\alpha}.
\]

It is easily seen from equation (13) that for some constant \( c > 0 \) (eventually depending on \( \sigma \) and \( p \), but not \( \alpha \)),
\[
D_{p,\alpha} \lesssim \int_{\alpha}^{\infty} (1 - e^{-u}) \left( \frac{u}{\alpha} \right)^{-1+1/\sigma} \left( \frac{u}{\alpha} \right)^{(1-\sigma)/\sigma} e^{-c(u/\alpha)^{1/\sigma}} \, du \\frac{\alpha}{\alpha}
= \alpha \int_{\alpha}^{\infty} (1 - e^{-u}) u^{-2} e^{-c(u/\alpha)^{1/\sigma}} \, du
\leq \int_{1}^{\infty} u^{-2} e^{-cu^{1/\sigma}} \, du \lesssim 1.
\]

It turns out that \( B \lesssim |p^\sigma C_{1,\alpha} - C_{p,\alpha}| + 1 \). We now consider the function,
\[
F(p, \sigma, \alpha, u) := \frac{1}{p\alpha} \left( 1 + \frac{\sigma u}{p\alpha} \right)^{-1+1/\sigma} \\
\times \left( \left( 1 + \frac{\sigma u}{p\alpha} \right)^{1/\sigma} - 1 \right)^{-1-\sigma} \exp \left\{ - \left( 1 + \frac{\sigma u}{p\alpha} \right)^{1/\sigma} - 1 \right\}.
\]

With a little bit of effort, it is seen that for any \( u \in [0, \alpha] \),
\[
F(p, \sigma, \alpha, u) = u^{-1-\sigma}(\alpha p)^{\sigma} + u^{-\sigma} O(\alpha^{-1+\sigma}).
\]

Hence, \( p^\sigma F(1, \sigma, \alpha, u) - F(p, \sigma, \alpha u) = u^{-\sigma} O(\alpha^{-1+\sigma}) \) whenever \( u \in [0, \alpha] \). It follows that,
\[
|p^\sigma C_{1,\alpha} - C_{p,\alpha}| \lesssim \alpha^{-1+\sigma} \int_{0}^{\alpha} (1 - e^{-u}) u^{-\sigma} \, du
= \int_{0}^{1} (1 - e^{-\alpha u}) u^{-\sigma} \, du \leq \frac{1}{1 - \sigma}.
\]
Henceforth, $B \lesssim 1$.

REFERENCES

C. Borgs, J. T. Chayes, H. Cohn, and N. Holden. Sparse exchangeable graphs and their limits via graphon processes. ArXiv e-prints, 1 2016.

C. Borgs, J. Chayes, H. Cohn, and V. Veitch. Sampling perspectives on sparse exchangeable graphs. Preprint, 2017.

Francois Caron. Bayesian nonparametric models for bipartite graphs. In F. Pereira, C. J. C. Burges, L. Bottou, and K. Q. Weinberger, editors, Advances in Neural Information Processing Systems 25, pages 2051–2059. Curran Associates, Inc., 2012. URL http://papers.nips.cc/paper/4837-bayesian-nonparametric-models-for-bipartite-graphs.pdf.

François Caron and Judith Rousseau. On sparsity and power-law properties of graphs based on exchangeable point processes. arXiv preprint arXiv:1708.03120, 2017.

François Caron and Emily B. Fox. Sparse graphs using exchangeable random measures. Journal of the Royal Statistical Society: Series B (Statistical Methodology), 79(5):1295–1366, 2017. ISSN 1467-9868. doi: 10.1111/rssb.12233. URL http://dx.doi.org/10.1111/rssb.12233.

Daryl J Daley and David Vere-Jones. An introduction to the theory of point processes: volume I: elementary theory and methods. Springer Science & Business Media, second edition, 2003a.

Daryl J Daley and David Vere-Jones. An introduction to the theory of point processes: volume II: general theory and structure. Springer Science & Business Media, second edition, 2003b.

William Feller. An introduction to probability theory and its applications, volume II, volume 2. Wiley, New York, 1971.

Thomas S Ferguson and Michael J Klass. A representation of independent increment processes without gaussian components. The Annals of Mathematical Statistics, 43(5):1634–1643, 1972.

Tue Herlau, Mikkel N Schmidt, and Morten Mørup. Completely random measures for modelling block-structured sparse networks. In D. D. Lee, M. Sugiyama, U. V. Luxburg, I. Guyon, and R. Garnett, editors, Advances in Neural Information Processing Systems 29, pages 4260–4268. Curran Associates, Inc., 2016. URL http://papers.nips.cc/paper/6521-completely-random-measures-for-modelling-block-structured-sparse-networks.pdf.

S. Janson. On convergence for graphexes. ArXiv e-prints, 2 2017.

J. F. C. Kingman. Poisson Processes. Oxford University Press, 1993.

P. Orbanz and D.M. Roy. Bayesian models of graphs, arrays and other exchangeable random structures. Pattern Analysis and Machine Intelligence, IEEE Transactions on, 37(2):437–461, 2 2015. ISSN 0162-8828. doi: 10.1109/TPAMI.2014.2334607.
A. Todeschini and F. Caron. Exchangeable Random Measures for Sparse and Modular Graphs with Overlapping Communities. *ArXiv e-prints*, 2 2016.

V. Veitch and D. M. Roy. The class of random graphs arising from exchangeable random measures. *ArXiv e-prints*, 12 2015.

V. Veitch and D. M. Roy. Sampling and Estimation for (Sparse) Exchangeable Graphs. *ArXiv e-prints*, 11 2016.

V. Veitch, E. Sharma, Z. Naulet, and D.M. Roy. Exchangeable modelling of relational data: Checking sparsity, data splitting, and sparse exchangeable poisson matrix factorization. *arXiv*, 11 2017.