CONTROLLING COMPOSITION FACTORS OF A FINITE GROUP BY ITS CHARACTER DEGREE RATIO

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Abstract. For a finite nonabelian group $G$ let $\text{rat}(G)$ be the largest ratio of degrees of two nonlinear irreducible characters of $G$. We show that nonabelian composition factors of $G$ are controlled by $\text{rat}(G)$ in some sense. Specifically, if $S$ different from the simple linear groups $\text{PSL}_2(q)$ is a nonabelian composition factor of $G$, then the order of $S$ and the number of composition factors of $G$ isomorphic to $S$ are both bounded in terms of $\text{rat}(G)$. Furthermore, when the groups $\text{PSL}_2(q)$ are not composition factors of $G$, we prove that $|G : \mathcal{O}_\infty(G)| \leq \text{rat}(G)^2$ where $\mathcal{O}_\infty(G)$ denotes the solvable radical of $G$.

1. Introduction

For a finite group $G$, let $b(G)$ denote the maximum degree of an (ordinary) irreducible character of $G$. If $G$ is nonabelian, we denote by $c(G)$ the minimum degree of a nonlinear irreducible character of $G$, and write

$$\text{rat}(G) := \frac{b(G)}{c(G)}.$$ 

This ratio is referred to as the character degree ratio of $G$. Indeed, $\text{rat}(G)$ is the largest ratio of degrees of two arbitrary nonlinear irreducible characters of $G$. When $G$ is abelian, we adopt a convention that $\text{rat}(G) = 1$.

The character degree ratio was first introduced and studied by I.M. Isaacs in [10] in connection with character kernels of finite groups. It is clear that $\text{rat}(G) = 1$ if and only if $G$ has at most two character degrees and the structure of such a group is fairly simple. It was proved in [9, Theorem 12.5] that if $G$ has exactly two character degrees, say 1 and $m > 1$, then $G$ either has an abelian normal subgroup of index $m$ or is the direct product of a $p$-group and an abelian group. Consequently, a group with exactly two character degrees must be metabelian but not abelian. From this initial result, it is natural to ask whether one can get some structural information about a finite group from its character degree ratio.

For instance, Isaacs proved in [10, Theorem 6.1 and Corollary 6.5] that the derived length and the Fitting height of a solvable group $G$ are bounded above respectively.
by $3 + 4 \log_2(\text{rat}(G))$ and $3 + 2 \log_2(\text{rat}(G))$. This result indicates that the derived length and the Fitting height of a solvable group are controlled by its character degree ratio.

The main goal of this paper is to show that the nonabelian composition factors of an arbitrary finite group $G$ are somehow also controlled by $\text{rat}(G)$. Our first result is the following.

**Theorem A.** Let $G$ be a finite group and $S$ a nonabelian composition factor of $G$ different from the simple groups $\text{PSL}_2(q)$ with $q$ a prime power. Then the order of $S$ and the number of times that $S$ occurs as a composition factor of $G$ are both bounded in terms of $\text{rat}(G)$.

Let $S$ be a nonabelian composition factor of $G$. By $|S|$ is bounded in terms of $\text{rat}(G)$ we mean that there is an (increasing) function $f : \mathbb{Q}^+ \to \mathbb{R}$ such that $|S| < f(\text{rat}(G))$. The family of simple groups $\text{PSL}_2(q)$ is special in this context. It is known (see [17] for instance) that the set of character degrees of $\text{PGL}_2(q)$ is $\{1, q - 1, q, q + 1\}$ and thus $\text{rat}(\text{PGL}_2(q)) = (q + 1)/(q - 1) \to 1$ as $q \to \infty$. Therefore, the exclusion of $\text{PSL}_2(q)$ in Theorem A is necessary.

When $G$ is a solvable group, we have seen that the derived length of $G$ is $\text{rat}(G)$-bounded. However, the order $|G|$ or even the index $|G : F(G)|$ of the Fitting subgroup $F(G)$ of $G$ is not $\text{rat}(G)$-bounded, see Section 4 for some examples. We prove that if the simple groups $\text{PSL}_2(q)$ are not composition factors of $G$, then the order of the ‘non-solvable part’ of $G$ is polynomially bounded in terms of $\text{rat}(G)$.

**Theorem B.** Let $G$ be a finite group with no composition factor isomorphic to $\text{PSL}_2(q)$ with $q$ a prime power. Let $\mathcal{O}_\infty(G)$ denote the solvable radical of $G$. Then $|G : \mathcal{O}_\infty(G)| \leq \text{rat}(G)^{21}$.

Theorem A suggests that when $\text{rat}(G)$ is small enough, then $G$ does not have any nonabelian composition factor different from $\text{PSL}_2(q)$. We give an upper bound for $\text{rat}(G)$ so that every nonabelian composition factor $S \neq \text{PSL}_2(q)$ of $G$ disappears.

**Theorem C.** Let $G$ be a finite group with $\text{rat}(G) < 16/5$. Then the only possible non-abelian composition factors of $G$ are $\text{PSL}_2(q)$ where $q \geq 5$ is a prime power. Consequently, if the simple groups $\text{PSL}_2(q)$ are not composition factors of $G$, then $G$ is solvable.

Remark that the bound in Theorem C can not be improved as $\text{rat}(\text{PSL}_3(4)) = 16/5$.

The organization of the paper is as follows. In the next section, we state a result on character degree ratio of simple groups, which will be crucial in the proofs of the main theorems. The proof of this result is presented in Section 4 for the alternating groups and in Section 5 for the simple groups of Lie type and simple sporadic groups. Theorems A, B, and C are proved respectively in Sections 2, 3, and 6. In the final section, we present some examples of solvable groups showing that $|G : F(G)|$ is not bounded in terms of $\text{rat}(G)$.
2. Composition factors and character degree ratio - Theorem A

We will prove Theorem A in this section. To do that, we assume the following important result on the character degree ratios of nonabelian simple groups and we postpone its proof until Sections 4 and 5.

**Theorem 1.** Let $S$ be a nonabelian simple group different from $\text{PSL}_2(q)$. Then $S$ has two non-principal irreducible characters $\alpha$ and $\beta$ extendible to $\text{Aut}(S)$ such that

$$\frac{\alpha(1)}{\beta(1)} > |S|^{1/14}.$$ 

This theorem will be proved in Section 4 for the alternating groups and in Section 5 for simple groups of Lie type and simple sporadic groups.

Now we observe an obvious but useful lemma.

**Lemma 2.** If $M$ is a normal subgroup of $G$, then $\text{rat}(G/M) \leq \text{rat}(G)$.

**Proof.** This is obvious since every irreducible character of $G/M$ can be viewed as an irreducible character of $G$. $\square$

The following is an important step towards the proofs of Theorems A and B.

**Proposition 3.** Let $N$ be a nonabelian minimal normal subgroup of $G$ such that the simple groups $\text{PSL}_2(q)$ are not direct factors of $N$. Then

$$\text{rat}(G)^{14} \geq \text{rat}(G/N)^{14} \cdot |N|.$$ 

**Proof.** Since $N$ is a nonabelian minimal normal subgroup of $G$, we see that $N = S \times \cdots \times S$, a direct product of $k$ copies of a nonabelian simple group $S$. By Theorem 1, there exist non-principal characters $\alpha, \beta \in \text{Irr}(S)$ such that $\alpha$ and $\beta$ extend to $\text{Aut}(S)$ and $\alpha(1)/\beta(1) > |S|^{1/14}$. By [1, Lemma 5], both the product characters $\beta \times \cdots \times \beta$ and $\alpha \times \cdots \times \alpha$ extend to $G$. Let $\psi$ and $\chi$ be their extensions, respectively. First we have

$$\psi(1) = \beta(1)^k$$ 

is a nontrivial character degree of $G$.

Next, since $\chi|_N = \alpha \times \cdots \times \alpha$, Gallagher’s Theorem [9, Corollary 6.17] guarantees that there is a bijection $\lambda \mapsto \lambda \chi$ between $\text{Irr}(G/N)$ and the set of irreducible characters of $G$ lying above $\alpha \times \cdots \times \alpha$. In particular, by taking $\lambda$ to be an irreducible character of $G/N$ of the largest degree, we deduce that

$$b(G/N)\chi(1) = b(G/N)\alpha(1)^k$$ 

is a character degree of $G$.

It follows that

$$\text{rat}(G) \geq \frac{b(G/N)\alpha(1)^k}{\beta(1)^k} \geq \text{rat}(G/N) \left(\frac{\alpha(1)}{\beta(1)} \right)^k.$$ 

Since $\alpha(1)/\beta(1) > |S|^{1/14}$, we obtain

$$\text{rat}(G) \geq \text{rat}(G/N)|S|^{k/14} = \text{rat}(G/N)|N|^{1/14},$$
as the proposition stated.

We are now ready to prove the first main result.

Proof of Theorem A. Let $S$ be a simple group and $N = S \times \cdots \times S$ a minimal normal subgroup of $G$. Lemma 2 and Proposition 3 imply that $\text{rat}(G) \geq \text{rat}(G/N)$ if $S$ is abelian or isomorphic to $\text{PSL}_2(q)$ (where $q$ is a prime power) and $\text{rat}(G) \geq \text{rat}(G/N) |N|^{1/14}$ otherwise. Applying this repeatedly to $N_{i+1}/N_i \triangleleft G/N_i$ for $i = 0, 1, ..., n - 1$ where $1 = N_0 < N_1 < \cdots < N_n = G$
is a chief series of $G$, we obtain that the product of the orders of all nonabelian chief factors which are not direct products of $\text{PSL}_2(q)$ is at most $\text{rat}(G)^{14}$. We deduce that, if $S \neq \text{PSL}_2(q)$ is a non-abelian composition factor of $G$, then the order of $S$ and the number of times that $S$ occurs as a composition factor of $G$ are both bounded in terms of $\text{rat}(G)$. □

3. Bounding the index of the solvable radical - Theorem B

We record a result of Maróti on bounding the order of a particular permutation group.

Lemma 4 (Maróti [14]). Let $G$ be a permutation group of degree $n$, and let $d$ be an integer not less than 4. If $G$ has no composition factors isomorphic to an alternating group of degree greater than $d$, then $|G| \leq d!(n-1)/(d-1)$.

This lemma can be used to prove a significant improvement of a result of Gluck in [7, Proposition 2.5].

Proposition 5. Let $N$ be a normal subgroup of $G$ such that $G/N$ is solvable. Then $|G : O_\infty(G)| \leq |N|^{1.43}$.

Proof. Step 1: We may assume that $O_\infty(G) = 1$.
If $O_\infty(G) \neq 1$, we replace $G$ by $G/O_\infty(G)$ and $N$ by $O_\infty(G)N/O_\infty(G)$. Then we use induction on $|G|$ and notice that $O_\infty(G/O_\infty(G)) = 1$.

Step 2: We may assume that $N$ is a unique minimal normal subgroup of $G$.
Since $O_\infty(G) = 1$ and $G/N$ is solvable, it suffices to show that $N$ is a minimal normal subgroup of $G$. Assume not. Then there exists a minimal normal subgroup $M$ properly contained in $N$. Let $K \subseteq G$ such that $K/M := O_\infty(G/M)$. Then by induction on the order of $N$, we have

$$|G : K| = \left| \frac{G}{M} : O_\infty \left( \frac{G}{M} \right) \right| \leq |N/M|^{1.43}$$
and \[ |K : O_\infty(K)| \leq |M|^{1.43}. \]

Since \( O_\infty(K) \subseteq O_\infty(G) \), it follows that
\[ |G : O_\infty(G)| \leq |G : O_\infty(K)| = |G : K||K : O_\infty(K)| \leq |N/M|^{1.43}|M|^{1.43} = |N|^{1.43}, \]
which proves the proposition.

Let \( N = S_1 \times \cdots \times S_k \), a direct product of copies of a nonabelian simple group \( S \). Then \( G \) permutes the subgroups \( S_1, S_2, \ldots, S_k \) by conjugation. Let \( H \) denote the kernel of this action. Then \( H \) has an embedding into \( \text{Aut}(S_1) \times \cdots \times \text{Aut}(S_k) \) and \( G/H \) has an embedding into the symmetric group \( S_k \).

Step 3: \( |G| \leq (\sqrt[3]{24}|\text{Aut}(S)|)^k \).

We note that \( N \subseteq H \) and recall that \( G/N \) is solvable. Thus \( G/H \) is solvable as well. So \( G/H \) is a solvable permutation group of degree \( k \) and therefore
\[ |G/H| \leq 24^{(k-1)/3} \leq \sqrt[3]{24}^k \]
by Lemma 4. On the other hand, as \( H \) is embedded into \( \text{Aut}(S_1) \times \cdots \times \text{Aut}(S_k) \), we have
\[ |H| \leq |\text{Aut}(S)|^k. \]

Now we deduce that
\[ |G| = |G/H||H| \leq (\sqrt[3]{24}|\text{Aut}(S)|)^k. \]

Step 4: \( |G| \leq |N|^{1.43}. \)

It suffices to show that \( |\text{Out}(S)| \leq |S|^{0.43}/\sqrt[3]{24} \) for every nonabelian simple group \( S \). If \( S = A_5 \), \( \text{PSL}_2(7) \), or \( A_6 \) then we can check the inequality directly. So we may assume that \( |S| \geq 504 \). Then \( \sqrt[3]{24} \leq |S|^{0.171} \) and hence it suffices to show that \( |\text{Out}(S)| \leq |S|^{0.259} \) for every nonabelian simple group \( S \). But this is done by a routine check on the list of simple groups together with their outer automorphism groups, see [8] for instance. \( \square \)

Now we can prove Theorem B.

*Proof of Theorem B.* We argue by induction on the order of \( G \). Since \( \text{rat}(G/O_\infty(G)) \leq \text{rat}(G) \) by Lemma 2 we may assume that \( O_\infty(G) = 1 \). We want to show that \( |G| \leq \text{rat}(G)^{21} \).

If \( G \) is trivial then we are done, so assume that \( G \) is nontrivial. Then \( G \) has a nonabelian minimal normal subgroup \( N \). We note that the groups \( \text{PSL}_2(q) \) are not the direct factors of \( N \) since they are not the composition factors of \( G \). Let \( M \trianglelefteq G \) such that
\[ \frac{M}{N} = O_\infty(G/N). \]
As $O_\infty(M)$ is characteristic in $M$ and $M \leq G$, we see that $O_\infty(M) \leq G$. Thus $O_\infty(M) = 1$ since $G$ has trivial solvable radical. We note that $M/N$ is solvable. Therefore, Proposition 5 implies that

$$|M| = |M : O_\infty(M)| \leq |N|^{1.43}.$$  

Using induction hypothesis for $G/N$, we have

$$|G : M| = |G/N : O_\infty(G/N)| \leq \text{rat}(G/N)^{21}.$$  

It follows that

$$|G| \leq |M| \text{rat}(G/N)^{21} \leq |N|^{1.43} \cdot \text{rat}(G/N)^{21}.$$  

Together with Proposition 3, we deduce that

$$|G| \leq \text{rat}(G)^{21},$$  

which completes the proof. □

4. Character degree ratio of the alternating groups

In this section, we prove Theorem 1 for the alternating groups with $n \geq 7$ (note that $A_5 \cong \text{PSL}_2(5)$ and $A_6 \cong \text{PSL}_2(9)$ are excluded from the theorem). First, however, we review some of the basic results on characters of the symmetric and alternating groups, proofs of which can be found in [11] for instance.

For a positive integer $n$, a partition $\lambda$ of $n$ is defined to be a nonincreasing sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ that sum to $n$. We adopt the “exponential” notation for repeated terms in a partition, so that for instance the partition $(5, 4, 4, 4, 3, 3, 1)$ would be written $(5, 4^3, 3^2, 1)$. Recall that the ordinary irreducible representations of the symmetric group $S_n$ are indexed by the partitions of $n$, and if $\lambda$ is a partition of $n$, we let $\chi_\lambda$ denote the irreducible character of $S_n$ corresponding to $\lambda$. For the node $(i, j)$ in row $i$ and column $j$ of the Young diagram of $\lambda$, we let $h_\lambda(i, j)$ denote the hook length of the node $(i, j)$, which is computed by counting the number of nodes of $\lambda$ directly to the right and directly below $(i, j)$, including $(i, j)$ itself. We denote by $H_\lambda$ the product of all the hook lengths of $\lambda$, and we note that the hook length formula shows that

$$\chi_\lambda(1) = \frac{n!}{H_\lambda}.$$  

Recall that the conjugate $\lambda'$ of the partition $\lambda$ is the partition obtained by interchanging the rows and the columns of $\lambda$. We say $\lambda$ is self-conjugate if $\lambda' = \lambda$. If $\lambda$ is a partition of $n$, then $\chi_\lambda$ restricts irreducibly to $A_n$ if and only if $\lambda$ is not self-conjugate.

For $n \geq 4$, the partition $(n - 1, 1)$ is certainly not self-conjugate, and $\chi_{(n-1,1)}(1) = n - 1$. Thus to prove Theorem 1 for the alternating groups, it suffices to find a
character $\chi$ of $A_n$ such that $\chi(1) > (n!)^{1/14}(n - 1)$. Note that for a non self-conjugate partition $\lambda$ of $n$, $\chi_{\lambda}(1) > (n!)^{1/14}(n - 1)$ if and only if
$$\frac{(n!)^{13/14}}{n - 1} > H_{\lambda}.$$ To prove Theorem 1 for the alternating groups, then, it is enough to prove the following:

**Proposition 6.** Let $n \geq 7$ be an integer. Then there exists a partition $\lambda$ of $n$ that is not self-conjugate such that
$$\frac{(n!)^{13/14}}{n - 1} > H_{\lambda}.$$ In particular, Theorem 1 is true for the alternating groups.

We begin by proving a lower bound on $(n!)^{13/14}/(n - 1)$.

**Lemma 7.** For any positive integer $n \geq 15$, we have
$$\frac{(n!)^{13/14}}{n - 1} > 1.35 \left(\frac{n}{e}\right)^{25n/28}.$$ **Proof.** By Stirling’s formula, we have
$$(n!)^{13/14} > \left(\frac{n}{e}\right)^{13n/14} (2\pi n)^{13/28}.$$ Thus
$$\frac{(n!)^{13/14}}{n - 1} > \frac{\left(\frac{n}{e}\right)^{13n/14}}{n} (2\pi)^{13/14} \left(\frac{2\pi}{e^{13}}\right)^{1/28}.$$ Note that $\left(\frac{2\pi}{e^{13}}\right)^{1/28} > 1.35$. Moreover, since $n \geq 15$, we have $\frac{13n}{14} - \frac{15}{28} > \frac{25n}{28}$. Thus
$$\frac{(n!)^{13/14}}{n - 1} > 1.35 \left(\frac{n}{e}\right)^{25n/28},$$ as claimed. □

We will now work to develop non self-conjugate partitions with large degree, by finding non self-conjugate partitions with “small” $H_{\lambda}$. Recall that if $\lambda$ and $\mu$ are partitions, we say that $\lambda \leq \mu$ if, for each $i$, $\lambda_i \leq \mu_i$.

**Definition 1.** Let $m$ be a positive integer. Define $\Gamma_m$ to be the set of partitions $\lambda$ such that $(m^m) \leq \lambda \leq ((m + 2)^m)$.

Note that if $\lambda \in \Gamma_m$ and $\lambda$ is a partition of $n$, then $m^2 \leq n \leq m^2 + 2m = (m+1)^2 - 1$. Thus for each $n$, there is a unique value of $m$ such that there is a partition of $n$ contained in $\Gamma_m$. Moreover, with the exception the partition $(m^m)$ of $n = m^2$, every partition in $\Gamma_m$ is not self-conjugate.
Lemma 8. Let $m$ be a positive integer and suppose $\lambda \in \Gamma_m$. Then $H_\lambda < (m+1)^{(m+1)^2}$.

Proof. Note that if $\lambda \in \Gamma_m$, then $\lambda$ is properly contained in the partition $\mu_m = ((m+2)^m)$. Thus $H_\lambda < H_{\mu_m}$.

Observe that the hook lengths in the first column of $\mu_m$ are $(2m+1, 2m, \ldots, m+2)$, and thus by the geometric mean inequality we have the product of the hook lengths in the first column of $\mu_m$ is

$$\prod_{i=2}^{m+1} (m + i) \leq \left(\frac{(2m+1) + (2m) + \cdots + (m+2)}{m}\right)^m = \left(\frac{3m + 3}{2}\right)^m.$$

Similarly the product of hook lengths in column $j$ of $\mu_m$, for $1 \leq j \leq m+2$, is bounded above by

$$\left(\frac{3m + 5 - 2j}{2}\right)^m.$$

Thus

$$H_\lambda < \left[\left(\frac{3m + 3}{2}\right) \cdots \left(\frac{m + 1}{2}\right)\right]^m \leq (m+1)^{m^2+2m} < (m+1)^{(m+1)^2},$$

where we are again using the geometric mean inequality, and we are done. \hfill \square

Lemma 9. Suppose $n \geq 64$. If $\lambda \in \Gamma_m$ is a partition of $n$ (thus $m \geq 8$) then

$$\frac{(n!)^{13/14}}{n-1} \geq H_\lambda.$$

Proof. By Lemma 7 we have

$$\frac{(n!)^{13/14}}{n-1} > 1.35 \left(\frac{n}{e}\right)^{25n/28} > \left(\frac{n}{e}\right)^{25n/28}.$$

By Lemma 8 we have $H_\lambda < (m+1)^{(m+1)^2}$. Note that since $m \geq 8$, then we have $\frac{81}{64}n \geq \frac{81}{64}m^2 \geq (m+1)^2$. Therefore

$$H_\lambda \leq (m+1)^{(m+1)^2} \leq \left[\left(\frac{81}{64}n\right)^{1/2}\right]^{(m+1)^2} \leq \left[\left(\frac{81}{64}n\right)^{1/2}\right]^{81n/64} = \left(\frac{81}{64}n\right)^{81n/128}.$$

Thus it is enough to show that $\left(\frac{81}{64}n\right)^{81n/128} \leq \left(\frac{n}{e}\right)^{25n/28}$. After taking $n$th roots of both sides, this is equivalent to showing $\left(\frac{81}{64}n\right)^{81/128} \leq \left(\frac{n}{e}\right)^{25/28}$. However, basic algebra shows this inequality is true for all $n \geq 55$. \hfill \square

We are now ready to prove Proposition 6, which will prove Theorem 1 for $A_n$ with $n \geq 7$. 
Proof of Proposition 6. Note that the partition \((m^m) \in \Gamma_m\) is self-conjugate, so we may not use that partition if \(n = m^2\). However, by the branching rule (see \([11]\) for instance), \(\chi_{(m^m)}(1) = \chi_{(m^{m-1},m^{-1})}(1) \leq \chi_{(m+1,m^{m-2},m^{-1})}(1)\), and the partition \((m+1,m^{m-2},m^{-1})\) of \(m^2\) is not self-conjugate for \(m > 1\).

Thus if \(n \geq 64\) (i.e. \(m \geq 8\)), we are done by the previous lemma. To finish the proof, we need to exhibit non self-conjugate partitions for all \(n\) such that \(7 \leq n \leq 63\) that have large enough degree. However, this is checked as follows: for instance, for all \(n\) such that \(49 \leq n \leq 63\), there is a non self-conjugate partition \(\lambda\) of \(n\) which contains the partition \((7^7)\) with degree at least

\[
\chi_{(7^7)}(1) \approx 4.75 \times 10^{23} > 1.07 \times 10^8 \approx (63 - 1)(63!)^{1/14} \geq (n - 1)(n!)^{1/14}.
\]

Similar calculations can be easily done for partitions in \(\Gamma_6, \Gamma_5, \Gamma_4,\) and \(\Gamma_3\). The partitions \((4,2,2)\) and \((3,2,2)\) of \(8\) and \(7\), respectively, complete the proof. □

We mention here that the above method could be extended to prove better bounds in Theorem 1 for the alternating groups \(A_n\) in terms of powers of \(n!\). However, the best bound for the other simple groups is on the order of \(|S|^{1/14}\), so we do not feel it necessary to improve our bound for the alternating groups.

5. Character degree ratio of the simple groups of Lie type

We now prove Theorem 1 for the simple groups of Lie type and then finish off the proof of the theorem. We will choose the larger character \(\alpha\) to be the so-called Steinberg character and the smaller character \(\beta\) usually to be the smallest unipotent character. We mainly follow the notation in \([2]\) for finite groups of Lie type and their characters, especially unipotent characters.

We make essential use of the Lusztig’s classification of unipotent characters of finite groups of Lie type. We summarize some main points here very briefly. Let \(G\) be a simple algebraic group over \(\mathbb{F}_q\) and \(F\) a Frobenius endomorphism on \(G\). Set \(G := G^F\). Let \(G^*\) be the dual group of \(G\), \(F^*\) the dual Frobenius endomorphism, as defined in \([4] \text{§13.10}\) and denote \(G^* := G^{F^*}\). Lusztig’s classification (see Chapter 13 of \([4]\)) states that the set of irreducible complex characters of \(G\) is partitioned into Lusztig series \(\mathcal{E}(G, (s))\) associated to various conjugacy classes \((s)\) of semisimple elements of \(G^*\). In fact, \(\mathcal{E}(G, (s))\) consists of irreducible constituents of the Deligne-Lusztig character \(R_T^G(\theta)\), where \((\mathcal{T}, \theta)\) is of the geometric conjugacy class associated to \((s)\). The characters in \(\mathcal{E}(G, (1))\) are called unipotent characters of \(G\).

By the results of Lusztig, unipotent characters of isogenous groups have the same degrees. These degrees are polynomials in \(q\) and they are essentially products of cyclotomic polynomials evaluated at \(q\). The unipotent characters of the linear and unitary groups are labeled by partitions while those of the symplectic and orthogonal groups are labeled by the so-called symbols. For finite classical groups, the degrees of unipotent characters are given by some delicate combinatorial formulas. On the other
hand, the degrees of unipotent characters of exceptional groups have been computed explicitly, see \[2, \S13.8\] for more details.

We need the following result, due to Lusztig, that appears as Theorem 2.5 of [13].

**Lemma 10** (Lusztig [12]). Let $\chi$ be a unipotent character of a simple group of Lie type $S$. Then $\chi$ is $\text{Aut}(S)$-invariant, except in the following cases:

(i) In $S = P\Omega_{2n}^+(q)$ with even $n \geq 4$, the graph automorphism of order 2 interchanges the two unipotent characters in all pairs labeled by the same degenerate symbols of defect 0 and rank $n$.

(ii) In $S = P\Omega_8^+(q)$, the graph automorphism of order 3 has two orbits with characters labeled by the symbols $\{(2 2) \cdot (0 1 4), (1 2) \cdot (0 1 4)\}$.

(iii) In $S = \text{Sp}_4(2^{2f+1})$, the graph automorphism of order 2 interchanges the two unipotent characters labeled by $(1 2)$ and $(0 1 2)$.

(iv) In $S = G_2(3^{2f+1})$, the graph automorphism of order 2 interchanges the two unipotent characters denoted by $\phi_1, \phi_{1,3}$.

(v) In $S = F_4(2^{2f+1})$, the graph automorphism of order 2 has eight orbits of length 2 with characters denoted by

\[
\{\phi_{8,3}, \phi_{8,3}''\}, \{\phi_{8,9}, \phi_{8,9}''\}, \{\phi_{2,4}, \phi_{2,4}''\}, \{\phi_{2,16}, \phi_{2,16}''\}, \\
\{\phi_{9,6}, \phi_{9,6}''\}, \{\phi_{1,12}, \phi_{1,12}''\}, \{\phi_{4,7}, \phi_{4,7}''\}, \{B_2, \epsilon\}, \{B_2, \epsilon''\}.
\]

Malle has observed in [13, Theorem 2.4] a very important property of unipotent characters.

**Lemma 11** (Malle [13]). Let $S$ be a simple group of Lie type. Then every unipotent character of $S$ extends to its inertial group in $\text{Aut}(S)$.

Combining Lemmas [10] and [11] we see that almost all of the unipotent characters of a simple group of Lie type are extendible to the full automorphism group. The character table of the Tits group is available in [3] and we will treat it as a sporadic simple group rather than a simple group of Lie type.

**Proposition 12.** Theorem [1] is true for the simple groups of Lie type.

**Proof.** For each family of simple groups of Lie type $S$, we will create two unipotent characters that are extendible to $\text{Aut}(S)$ and their degree ratio is greater than $\frac{|S|}{14}$. These characters of course cannot be those singled out in Lemma [10]. Indeed, the character of larger degree will be chosen to be the Steinberg character of $S$, denoted
by \( \text{St}_S \). Assuming that \( p \) is the defining characteristic of \( S \), then it is well-known that \( \text{St}_S \) is extendible to \( \text{Aut}(S) \) and \( \text{St}_S(1) = |S|_p \), see [3].

1) Linear groups \( \text{PSL}_n(q) \) and unitary groups \( \text{PSU}_n(q) \) in dimension \( n \geq 3 \): The unipotent characters of \( S \) are labeled by partitions of \( n \), see [2, p. 465]. We consider the character \( \chi^\lambda \) labeled by the particular partition \( \lambda = (n - 1, 1) \). This character is extendible to \( \text{Aut}(S) \) by Lemmas 10 and 11 and has degree

\[
\chi^\lambda(1) = \frac{q^n - q}{q - 1} \quad \text{if } S = \text{PSL}_n(q)
\]

and

\[
\chi^\lambda(1) = \frac{q^n + q(-1)^n}{q + 1} \quad \text{if } S = \text{PSU}_n(q).
\]

We note that \( \text{St}_S(1) = q^{n(n-1)/2} \) and it is easy to check that \( \text{St}_S(1)/\chi^\lambda(1) > |S|^{1/14} \) when \( n = 3 \). So let us assume that \( n \geq 4 \). We have

\[
\frac{\text{St}_S(1)}{\chi^\lambda(1)} \geq \frac{q^{n(n-1)/2}}{(q^n - q)/(q - 1)} > \frac{q^{n(n-1)/2}}{q^n} = q^{(n-3)/2}.
\]

On the other hand, we see that \(|\text{PSL}_n(q)| < |\text{PSU}_n(q)| < q^{n^2-1} \). Since it is clear that \( q^{n(n-3)/2} > q^{(n^2-1)/14} \) for every \( n \geq 4 \), we conclude that

\[
\frac{\text{St}_S(1)}{\chi^\lambda(1)} > |S|^{1/14}.
\]

2) Symplectic groups \( \text{PSp}_{2n}(q) \) and orthogonal groups \( \Omega_{2n+1}(q) \) with \( n \geq 2 \): According to [2, p. 466], the unipotent characters of these groups are labeled by symbols of the form

\[
\left( \begin{array}{c} \lambda \\ \mu \end{array} \right) = \left( \begin{array}{c} \lambda_1 & \lambda_2 & \lambda_3 & \ldots & \lambda_a \\ \mu_1 & \mu_2 & \ldots & \mu_b \end{array} \right)
\]

where \( 0 \leq \lambda_1 < \lambda_2 < \ldots < \lambda_a \), \( 0 \leq \mu_1 < \mu_2 < \ldots < \mu_b \), \( a - b \) is odd and positive, \((\lambda_1, \mu_1) \neq (0, 0)\), and

\[
\sum_i \lambda_i + \sum_j \mu_j - \frac{(a + b - 1)}{2} = n.
\]

Now the character labeled by the symbol \( \left( \begin{array}{c} \lambda \\ \mu \end{array} \right) = \left( \begin{array}{c} 0 & 1 \\ n \end{array} \right) \) is extendible to \( \text{Aut}(S) \) and its degree is

\[
\chi^{\left( \begin{array}{c} \lambda \\ \mu \end{array} \right)}(1) = \frac{(q^n - 1)(q^n - q)}{2(q + 1)} < q^{2n}.
\]

We note that \( \text{St}_S(1) = q^{n^2} \) and it follows that

\[
\frac{\text{St}_S(1)}{\chi^{\left( \begin{array}{c} \lambda \\ \mu \end{array} \right)}(1)} > \frac{q^{n^2}}{q^{2n}} = q^{n^2 - 2n}.
\]
As $|\text{PSp}_{2n}(q)| = |\Omega_{2n+1}(q)| \leq q^{2n^2+n}$, the proposition follows in this case.

3) Orthogonal groups $\text{PO}^+_{2n}(q)$ with $2n \geq 8$: By [2, p. 471], the unipotent characters of these groups are labeled by symbols of the same form as the symplectic groups except that $a - b \equiv 0 \pmod{4}$ and

$$\sum_i \lambda_i + \sum_j \mu_j - \left[ \left( \frac{a + b - 1}{2} \right)^2 \right] = n.$$  

We consider the character labeled by the symbol $(\lambda) = (n^{-1})$. This character is Aut($S$)-invariant by Lemma 10 and therefore extendible to Aut($S$) by Lemma 11. Its degree is

$$\chi_{(\mu)}(1) = \frac{(q^n - 1)(q^n + q)}{q^2 - 1} < q^{2n}.$$  

Since $\text{St}_S(1) = q^{n(n-1)}$ and $|\text{PO}^+_{2n}(q)| < q^{2n^2-n}$, we are done in this case.

4) Orthogonal groups $\text{PO}^-_{2n}(q)$ with $2n \geq 8$: From [2, p. 475], we know that the unipotent characters of these groups are labeled by symbols of the same form as the orthogonal groups of plus type in even dimension except that $a > b$ and $a - b \equiv 2 \pmod{4}$. We consider the character labeled by the symbol $(\lambda) = (1 \ n^{-1})$. This character has degree

$$\chi_{(\mu)}(1) = \frac{(q^n+1)(q^n-q)}{q^2-1} < q^{2n}$$

and we are also done since $|\text{PO}^-_{2n}(q)| < q^{2n^2-n}$.

5) Exceptional groups: Since all unipotent characters of exceptional groups are explicitly computed, it is easy to find two unipotent characters that we need. Table 1 displays these characters as required. In the table, we denote by $\Phi_k := \Phi_k(q)$ the $k$th cyclotomic polynomial evaluated at $q$, see [2, §13.9]. We choose the bounding constant $1/14$ since for the Ree groups $^2G_2(q)$, the ratio $q^3/(q^2 - 1)\sqrt{q}/3$ is approximately equal to $\sqrt{3q}$ and $|^2G_2(q)|$ is approximately equal to $q^7$. □

Now we can complete the proof of Theorem 1.

Proof of Theorem 1. The simple groups of Lie type have been handled above. The alternating groups are treated in Section 4. Finally, the sporadic simple groups and the Tits group can be checked directly by using [3]. □

6. Character degree ratio and solvability - Theorem C

This section is devoted to the proof of Theorem C. We start with an analogue of Theorem 1.
Table 1. Unipotent characters of simple exceptional groups of Lie type.

| Group | Characters | Degrees |
|-------|------------|---------|
| $^2B_2(q)$ | $^2B_2[a]$ | $(q - 1)\sqrt{q/2}$ |
| St | | $q^2$ |
| $^3D_4(q)$ | $\psi_{1,3}$ | $q\Phi_{12}$ |
| St | | $q^{12}$ |
| $G_2(q)$ | $\phi_{2,1}$ | $q\Phi_2^2\Phi_3/6$ |
| St | | $q^6$ |
| $^2G_2(q)$ | cuspidal 1 | $(q^2 - 1)\sqrt{q/3}$ |
| St | | $q^3$ |
| $F_4(q)$ | $\phi_{4,1}$ | $q\Phi_2^2\Phi_6^2\Phi_8/2$ |
| St | | $q^{24}$ |
| $^2F_4(q)$ | $\epsilon'$ | $q\Phi_{6}\Phi_{12}$ |
| St | | $q^{12}$ |
| $E_6(q)$ | $\phi_{6,1}$ | $q\Phi_8\Phi_9$ |
| St | | $q^{36}$ |
| $^2E_6(q)$ | $\phi_{2,4}$ | $q\Phi_8\Phi_{18}$ |
| St | | $q^{36}$ |
| $E_7(q)$ | $\phi_{7,1}$ | $q\Phi_7\Phi_{12}\Phi_{14}$ |
| St | | $q^{63}$ |
| $E_8(q)$ | $\phi_{8,1}$ | $q\Phi_4^2\Phi_8\Phi_{12}\Phi_{20}\Phi_{24}$ |
| St | | $q^{120}$ |

Lemma 13. Let $S$ be a simple group of Lie type different from $\text{PSL}_2(q)$ for any prime power $q$. Then there exist non-principal irreducible characters $\alpha$ and $\beta$ of $S$ so that both $\alpha$ and $\beta$ extend to $\text{Aut}(S)$ and

$$\frac{\alpha(1)}{\beta(1)} \geq \frac{16}{5}.$$ 

Proof. We will essentially use the same characters as in the proof of Theorem 11. For the linear groups $\text{PSL}_n(q)$ with $n \geq 4$, we have

$$\text{rat}(S) \geq \frac{q^{n(n-1)/2}}{(q^n - q)(q - 1)} > q^{\frac{n(n-3)}{2}} \geq q^2 > \frac{16}{5}.$$ 

For $\text{PSL}_3(q)$, we still have the same inequality as long as $q \geq 4$. For $\text{PSL}_3(3)$ we use the characters of degrees 39 and 12 instead and note that $39/12 > 16/5$. We remark that $\text{PSL}_3(2) \cong \text{PSL}_2(7)$ is excluded from our consideration.
Similarly for the unitary groups \( \text{PSU}_n(q) \) with \( n \geq 4 \), we have
\[
\text{rat}(S) \geq \frac{q^{n(n-1)/2}}{(q^n + q(-1)^n)/(q + 1)} \geq q^2 > 16/5.
\]
On the other hand, for \( \text{PSU}_3(q) \) with \( q \neq 2 \), we have
\[
\text{rat}(S) \geq \frac{q^3}{q^2 - q} > \frac{16}{5}.
\]

For the symplectic groups \( \text{PSp}_{2n}(q) \) and orthogonal groups \( \Omega_{2n+1}(q) \) with \( n \geq 2 \), we have
\[
\text{rat}(S) \geq \frac{q^{n^2}}{(q^n - 1)(q^n - q)/2(q + 1)} > 2q^{n^2 - 2n + 1} \geq 4.
\]
For the orthogonal groups \( \text{PΩ}^\pm_{2n}(q) \) with \( n \geq 4 \), we have
\[
\text{rat}(S) > \frac{q^{n(n-1)}}{q^{2n}} \geq q^4 > 16/5.
\]

For the exceptional groups of Lie type, the characters in Table 1 show that \( \text{rat}(G) > 16/5 \). This is easily checked for the “small” values of \( q \). For the larger values, this can be seen by recalling that \( \Phi_n \) has degree \( \phi(n)! \) and, for \( n < 105 \), the coefficients of \( \Phi_n \) are \( \pm 1 \). Thus \( \Phi_n(q) \leq (\phi(n) + 1)q^{\phi(n)} \), which allows one to obtain the bounds \( \text{rat}(G) > 16/5 \).

The sporadic simple groups may be easily checked using [3]. Finally, for the alternating groups, it is enough to show that there exists a character of \( A_n \) of degree at least \( 16(n - 1)/5 \). However, we have already shown that there exists a character degree of \( A_n \) at least \( (n!)^{1/14}(n - 1) \), and for \( n \geq 11 \), we have \( (n!)^{1/14} > 16/5 \). The result can be checked for \( 7 \leq n \leq 10 \) using the character tables given in [3, 11], and we are done.

**Proof of Theorem C.** By applying Lemma 13, the proof goes along the same lines as that of Theorem A. We leave details to the reader.

\[ \square \]

7. Examples

In this section we give examples to show that the index of the Fitting subgroup need not be bounded in terms of \( \text{rat}(G) \), even if \( G \) is solvable. This is in contrast to the situation with character degrees, where bounds on \( |G : F(G)| \) are known in terms of the largest character degree \( b(G) \). For instance, it is shown by Gluck [7] that there is a constant \( L \) such that \( |G : F(G)| \leq b(G)^L \) for every finite group \( G \). When \( G \) is solvable, Moretó and Wolf [15] proved that \( |G : F(G)| \leq b(G)^3 \).

\(^{1}\text{Here } \phi(n) \text{ is the Euler’s totient function that counts the number of positive integers less than or equal to } n \text{ that are relatively prime to } n.\)
Recall that \( \text{rat}(G) \) is defined to be 1 if \( G \) has at most one nonlinear character degree. We first give an example of an infinite family of solvable groups, each with exactly one nonlinear character degree, such that \( |G : F(G)| \) can be arbitrarily large.

Let \( m \) be any positive integer with \( m > 1 \), and let \( p \) be a prime of the form \( md + 1 \) for some positive integer \( d \) (note there are infinitely many such primes by Dirichlet’s theorem). Let \( X \) be the Frobenius group formed by the action of \( \mathbb{F}_p^\ast \) on \( F := \mathbb{F}_p \). Let \( G \) be the subgroup of \( X \) containing \( F \) such that \( |G : F| = m \). Then the character degrees of \( G \) are precisely 1 and \( m \), although the character degree \( m \) could have large multiplicity. Thus \( \text{rat}(G) = 1 \) in this case. As \( m \) goes to infinity, the index \( |G : F(G)| \) is arbitrarily large, thus showing that \( |G : F(G)| \) is arbitrarily large.

One might want an example in which \( G \) has more than one nonlinear character degree, which we now provide. To do so, we examine a specific case of a more general example constructed in Section 3 of [10], and more details can be found in that paper. Let \( m = p^i \) for some prime \( p \) and some positive integer \( i \), and let \( n = p^i + 1 \). Let \( S \) be a Sylow \( p \)-subgroup of \( SU_3(p^2i) \). Let \( N \) be a maximal subgroup of \( \mathbb{Z}(S) \) and define \( P = S/N \). Note that \( P \) is extraspecial. Let \( \lambda \) be an element of order \( p^i + 1 \) of the multiplicative group of the field of order \( p^{2i} \), and define \( H \) to be the group generated by \( \lambda \). Then \( H \) acts Frobeniusly on \( P/\mathbb{Z}(P) \) and centralizes \( \mathbb{Z}(P) \). Let \( G \) be the semidirect product of \( H \) acting on \( P \). Note that \( G \) is solvable, and it can be shown that the character degrees of \( G \) are 1, \( p^i \), and \( p^i + 1 \) and that \( |G : F(G)| = p^i + 1 \). Thus as \( p \) or \( i \) goes to infinity, we see that \( \text{rat}(G) \) goes to 1 while \( |G : F(G)| \) goes to infinity.

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