Abstract

We construct a moduli space for Riemann surfaces that is universal in the sense that it represents compact Riemann surfaces of any finite genus. This moduli space is stratified according to genus, and it carries a metric and a measure that induce a Riemannian metric and a finite volume measure on each stratum. Applications to the Plateau-Douglas problem for minimal surfaces of varying genus and to the partition function of Bosonic string theory are outlined. The construction starts with a universal moduli space of Abelian varieties. This space carries a structure of an infinite dimensional locally symmetric space which is of interest in its own right. The key to our construction of the universal moduli space then is the Torelli map that assigns to every Riemann surface its Jacobian and its extension to the Satake-Baily-Borel compactifications.

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Contents

1 Introduction 2
2 Satake-Baily-Borel compactification of the moduli space $M_g$ 5
3 Universal moduli spaces of abelian varieties 6
4 Universal moduli spaces of Riemann surfaces 10
5 Riemannian metrics 12
6 Integration on the universal moduli space of Riemann surfaces 16
1 Introduction

For every $g \geq 0$, let $\mathcal{M}_g$ be the moduli space of compact Riemann surfaces of genus $g$. When $g = 0$, $\mathcal{M}_g$ consists of only one point, which corresponds to the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$. When $g \geq 1$, it is known that $\mathcal{M}_g$ is noncompact since Riemann surfaces of positive genus can degenerate.

A basic result about $\mathcal{M}_g$ is that $\mathcal{M}_g$ is a complex orbifold and a quasi-projective variety. See [HM] and references there.

For many applications, we need to compactify $\mathcal{M}_g$, in particular, to obtain compactifications which are projective varieties over $\mathbb{C}$, or even defined over some specific number fields.

Besides the well-known Deligne-Mumford compactification $\overline{\mathcal{M}}_g^{DM}$ by adding stable Riemann surfaces of Euler characteristic $2 - 2g$, there is also the Satake-Baily-Borel compactification $\overline{\mathcal{M}}_g^{SBB}$ whose boundary points correspond to unions of compact Riemann surfaces of Euler characteristic strictly greater than $2 - 2g$.

$\overline{\mathcal{M}}_g^{SBB}$ is constructed by Satake-Baily-Borel compactification of the Siegel modular variety $\mathcal{A}_g$, which is also the moduli space of principally polarized abelian varieties of dimension $g$. The compactification $\overline{\mathcal{M}}_g^{SBB}$ will be described in more detail in the next section.

The two compactifications $\overline{\mathcal{M}}_g^{DM}$ and $\overline{\mathcal{M}}_g^{SBB}$ are different in several aspects.

1. $\overline{\mathcal{M}}_g^{DM}$ is a complex orbifold, but $\overline{\mathcal{M}}_g^{SBB}$ is highly singular when $g \geq 2$,

2. $\overline{\mathcal{M}}_g^{DM}$ is the moduli space of stable Riemann surfaces of Euler characteristic $2 - 2g$ for $g \geq 2$, but $\overline{\mathcal{M}}_g^{SBB}$ is not a moduli space for a modular functor (see [HM, p. 45]).

3. The boundary points of $\overline{\mathcal{M}}_g^{DM}$ correspond to Riemann surfaces with punctures by pinching along loops of Riemann surfaces $\Sigma_g$, and the boundary points of $\overline{\mathcal{M}}_g^{SBB}$ correspond to unions of compact Riemann surfaces which are obtained from the punctured Riemann surfaces in the boundary of $\overline{\mathcal{M}}_g^{DM}$ by forgetting (or filling in) the punctures. Therefore, there is a surjective map $\overline{\mathcal{M}}_g^{DM} \rightarrow \overline{\mathcal{M}}_g^{SBB}$, which is not injective when $g \geq 2$.

For many applications, especially those in algebraic geometry and arithmetic geometry, the above properties, especially the modular property, make $\overline{\mathcal{M}}_g^{DM}$ more desirable. On the other hand, for some applications to string theory and minimal surfaces, $\overline{\mathcal{M}}_g^{SBB}$ is more suitable for the reason that only compact Riemann surfaces appear on the boundary of $\overline{\mathcal{M}}_g^{SBB}$ and only compact Riemann surfaces matter, for the basic reason that the Riemann extension theorem can remove the punctures.

Let us explain this in more detail. Let us start with geometric analysis, with the Plateau-Douglas problem in Euclidean space (or some Riemannian manifold). Here, one wants to find conditions for a configuration of $k$ disjoint oriented Jordan curves in Euclidean space to bound a minimal surface of a given genus $g$ and $k$ boundary curves. A minimal surface in this context is
a harmonic and conformal map from some Riemann surface \( \Sigma \) satisfying the above Plateau type boundary condition \([DHS]\). In order to break the diffeomorphism invariance of the area integral, one works with the Dirichlet integral, that is, for a map \( h : C \to \mathbb{R}^N \), we consider

\[
S(h, C) := \int_C |dh|^2; \tag{1.1}
\]

by conformal invariance, we do not need to specify a conformal metric on \( \Sigma \). That is, when \( \rho(z)dzd\bar{z} \) is a Riemannian metric on \( \Sigma \) compatible with its conformal structure, then \( \int_C \rho(z)^{-1} \frac{\partial h}{\partial z} \frac{\partial h}{\partial \bar{z}} \rho(z)dzd\bar{z} = S(h, C) \) independently of the particular choice of \( \rho \).

When one takes a minimizing sequence for the Dirichlet integral (1.1), again such a sequence could degenerate and end up with a limit of smaller topological type, that is, an element of the boundary of \( \mathcal{M}_g \).

Since here we discuss surfaces with boundary that have to map to the given disjoint Jordan curves, we need moduli spaces of Riemann surfaces with boundary. These can be obtained from those without boundary as follows (see e.g. \([J1]\) for more details). Let \( \Sigma \) be a compact Riemann surface of genus \( g \) with \( k \) boundary curves. We then form its Schottky double \( \Sigma' \), a compact Riemann surface without boundary of genus \( 2g + k - 1 \), by reflecting \( \Sigma \) across its boundary, that is, by taking a copy \( \Sigma'' \) of \( \Sigma \) with the opposite orientation and identifying \( \Sigma \) and \( \Sigma'' \) along their corresponding boundaries. \( \Sigma' \) then possesses an anticonformal involution \( i \) that interchanges \( \Sigma \) and \( \Sigma'' \), leaving their common boundaries fixed. (When we equip \( \Sigma \) with a constant curvature metric for which all boundary curves are geodesic, we can also perform these constructions within the Riemannian setting. \( i \) then is an orientation reversing isometry whose fixed point is a collection of closed geodesics that constitute the common boundary of \( \Sigma \) and \( \Sigma'' \). Conversely, if we have a compact Riemann surface \( \Sigma' \) of genus \( 2g + k - 1 \) with such an involution that leaves \( k \) geodesics fixed, then \( \Sigma' \) can be seen as the union of two isometric surfaces of genus \( g \) with \( k \) boundary curves. From this construction, we see that the moduli space of genus \( g \) surfaces with \( k \) boundary curves is the moduli space of genus \( 2g + k - 1 \) surfaces without boundary with an involution that leaves \( k \) disjoint simple loops (closed geodesics with respect to a constant curvature metric) fixed. The moduli space of surfaces with such an involution \( \mathcal{M}_{g,k} \) is a totally real subspace of the moduli space \( \mathcal{M}_{2g+k-1} \), and all properties of the latter apply to the former with obvious modifications, except that \( \mathcal{M}_{g,k} \), as a totally real subspace, does not possess a complex structure. Therefore, in the sequel, we only discuss the spaces \( \mathcal{M}_g \).

Returning to the discussion of (1.1) when the underlying Riemann surface \( \Sigma \) degenerates in \( \mathcal{M}_g \), the question arises which boundary of \( \mathcal{M}_g \) should we take here when we want to consider limits of sequences of degenerating Riemann surfaces and assign a value to the Dirichlet integral on such a limit. The key observation is that if we take a limit of harmonic maps, we should expect the limit also to be harmonic. That is, we get a harmonic map from a degenerated surface. When we consider that object as an element of the Deligne-Mumford compactification \( \overline{\mathcal{M}}_{g \mathscr{D}M} \), it would have two punctures. But a (bounded) harmonic map extends across such a puncture, and therefore, it does not feel the effect of that puncture. Thus, the punctures are irrelevant, and the natural domain for our harmonic map is a lower topological type Riemann surface without any punctures, that is, an element of the Satake-Baily-Borel \( \overline{\mathcal{M}}_{g \mathscr{BB}} \) rather than the Deligne-Mumford compactification \([JS]\). Moreover, if one wants to look at minimal surfaces of arbitrary genus, one should have a universal moduli space that contains Riemann surfaces of all possible genera. In some sense, we want to let \( g \to \infty \). But from the Deligne-Mumford compactification we would then
encounter boundary strata with ever more punctures, and in the limit infinitely many. This seems undesirable. Thus, this provides motivation for our construction of a universal moduli space.

String theory [Pol] can be considered as a quantization of the Plateau-Douglas problem just described [J2]. And when one wants to compute corresponding partition and correlation functions, one should take a sum over all possible genera, possibly with suitable weights for the different values of $g$, of suitable integrals over the individual $M_g$ [Hil], as we shall explain in more detail in Section 0. But then again, a boundary stratum of $M_g$ should be $M_{g-1}$ and or a product $M_{g_1} \times \cdots \times M_{g_k}$ with $g_1 + \cdots + g_k = g$, and not a space blown up from the latter by introducing additional puncture positions as in the Deligne-Mumford compactification. In any case, the situation for genus $g - 1$ should be structurally the same as for genus $g$, and not more complicated. Otherwise, we cannot meaningfully let $g \to \infty$.

For these applications, it is therefore important to consider surfaces of different genus together. For instance, for a general treatment of the Plateau-Douglas problem for minimal surfaces, we wish to have a Conley type index formula that involves minimal surfaces of all finite genera simultaneously. This is because given some configuration of Jordan curves in Euclidean space or a Riemannian manifold, we may not know a priori what the largest genus of a minimal surface bounded by this configuration is. In particular, we want to have a global Euler characteristic involving all critical points of the Dirichlet integral of any genus. Also, when we vary the configuration of Jordan curves, while the global Euler characteristic will stay the same, the solutions can change their genera or bifurcate in other ways. Likewise, when one wants to do string theory in a non-perturbative manner, one wants to have a formula that includes all finite genera simultaneously, instead of an expansion in terms of the genus.

The above consideration shows that it is desirable to construct a \textit{universal compactified moduli space} $\mathcal{M}_{\infty}$ satisfying the following properties:

1. $\mathcal{M}_{\infty}$ is a connected stratified complex analytic space such that the closure of each stratum is a projective variety.

2. For every $g$, $\mathcal{M}_{g}^{SBB}$ is embedded into $\mathcal{M}_{\infty}$, and $\mathcal{M}_{\infty}$ is the union of these subsets $\mathcal{M}_{g}^{SBB}$, $g \geq 0$.

3. There is a natural measure on $\mathcal{M}_{\infty}$ which induces compatible measures on different strata, and with respect to the induced measure, every stratum has finite volume.

We note that the connectedness condition in (1) excludes the trivial construction by taking the disjoint union of $\mathcal{M}_{g}^{SBB}$, $g \geq 0$.

The idea is to construct a natural embedding $\mathcal{M}_{g}^{SBB} \subset \mathcal{M}_{g+1}^{SBB}$ for every $g$, which are compatible for all $g$, then take their union under such inclusion obtain a desired space. This can be obtained by using another family of locally symmetric spaces $\mathcal{A}_g$ and their compactifications $\mathcal{A}_g^{SBB}$, and the Jacobian map $M_g \to \mathcal{A}_g$ allows us to pass such a construction to $\mathcal{M}_g^{SBB}$. This infinite dimensional locally symmetric space structure will allow us to define a (positive semi-definite) Riemannian metric and a natural measure on the universal moduli space $\mathcal{M}_{\infty}$. A slight modification, consisting in pulling back the metric from its Jacobian back to each individual Riemann surface instead of pulling the metric from the moduli space of principally polarized Abelian varieties back to the moduli space of Riemann surfaces, will even lead to a positive definite Riemannian metric.
2 Satake-Baily-Borel compactification of the moduli space $\mathcal{M}_g$

As mentioned before, the Deligne-Mumford compactification $\overline{\mathcal{M}}^{DM}_g$ is well-known, while the Satake-Baily-Borel compactification $\overline{\mathcal{M}}^{SBB}_g$ is less known. Hence, we give the definition of the latter in some detail.

Let

$$\mathfrak{h}_g = \{ X + iY \mid X, Y \text{ symmetric } n \times n \text{ matrices}, Y > 0 \}$$

be the Siegel upper half space of degree $g$. It is a Hermitian symmetric space of noncompact type. The symplectic group $Sp(2g, \mathbb{R})$ acts holomorphically and transitively on it with the stabilizer of $iI_g$ isomorphic to $U(g)$. Therefore,

$$\mathfrak{h}_g = Sp(2g, \mathbb{R})/U(g).$$

The Siegel modular group $Sp(2g, \mathbb{Z})$ acts properly and holomorphically on $\mathfrak{h}_g$, and the quotient $Sp(2g, \mathbb{Z})\backslash \mathfrak{h}_g$ is a complex orbifold and equal to $\mathcal{A}_g$. See the book [Nam] for more detail.

Since Abelian varieties can degenerate, $\mathcal{A}_g$ is noncompact. It is a quasi-projective variety and can be compactified to a normal projective variety $\mathcal{A}^{SBB}_g$. This compactification of $\mathcal{A}_g$ was first constructed as a topological space by Satake in [Sa] and as a normal projective space by Baily in [Ba]. Since this is a special case of the Baily-Borel compactification for general arithmetic locally Hermitian symmetric spaces in [BB], we call it Satake-Baily-Borel compactification and denote it by $\overline{\mathcal{A}}^{SBB}_g$.

Since $\mathfrak{h}_g$ is a Hermitian symmetric space of noncompact type, it can be embedded into its compact dual, under the Borel embedding. (Note that when $g = 1$, $\mathfrak{h}_1$ is the Poincaré upper halfplane $\mathbb{H}^2 = \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \}$, and its compact dual is the Riemann sphere $\mathbb{C} \cup \{ \infty \}$). Denote the closure of $\mathfrak{h}_g$ under this embedding by $\overline{\mathfrak{h}}_g$. Then the symplectic group $Sp(2g, \mathbb{R})$ acts on the compactification $\overline{\mathfrak{h}}_g$.

For every $g' < g$, we can embed $\mathfrak{h}_{g'}$ into the boundary of $\overline{\mathfrak{h}}_g$ in infinitely many different ways. The most obvious one, usually called the standard embedding [Nam], is as follows:

$$\mathfrak{h}_{g'} \hookrightarrow \overline{\mathfrak{h}}_g, \quad X' + iY' \mapsto \begin{pmatrix} X' & 0 \\ 0 & 0 \end{pmatrix} + i \begin{pmatrix} Y' & 0 \\ 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (2.1)

Under the action of $Sp(2g, \mathbb{R})$, we get other embeddings. To compactify $\mathcal{A}_g$, we only need the translates of $\mathfrak{h}_{g'}$, $g' < g$, under $Sp(2g, \mathbb{Z})$. These boundary components are called rational boundary components of $\mathfrak{h}_g$.

Denote the union of $\mathfrak{h}_g$ with these rational boundary components by $\overline{\mathfrak{h}}_{g, \mathbb{Q}}$. Then there is a Satake topology on $\overline{\mathfrak{h}}_{g, \mathbb{Q}}$ such that $Sp(2g, \mathbb{Z})$ acts continuously with a compact quotient, which is the compactification $\overline{\mathcal{A}}^{SBB}_g$.

Note that the action of $Sp(2g, \mathbb{Z})$ on $\overline{\mathfrak{h}}_{g, \mathbb{Q}}$ is not proper, but the quotient $Sp(2g, \mathbb{Z})\backslash \overline{\mathfrak{h}}_{g, \mathbb{Q}}$ is a Hausdorff topology. A good example to keep this in mind is to consider the case $g = 1$, or equivalently the case of $SL(2, \mathbb{Z})$ acting on the upper halfplane $\mathbb{H}^2$. The boundary of the closure $\overline{\mathbb{H}}^2$ of $\mathbb{H}^2$ in the extended complex plane $\mathbb{C} \cup \{ +\infty \}$ is equal to $\partial \mathbb{H}^2 = \mathbb{R} \cup \{ \infty \}$. In the boundary $\partial \mathbb{H}^2$, rational boundary components correspond to the rational points $\mathbb{Q} \cup \{ \infty \}$. On the other hand, the union $\overline{\mathbb{H}}^2 \cup \mathbb{Q} \cup \{ \infty \}$ with the induced subspace topology from $\overline{\mathbb{H}}^2$ is not the Satake topology. In fact, with respect to the induced subspace topology, its quotient by $SL(2, \mathbb{Z})$ is non-Hausdorff.
In the Satake topology of \( \mathbb{H}^2 \cup \mathbb{Q} \cup \{ \infty \} \), a basis of neighborhoods of every rational boundary point \( \xi \in \mathbb{Q} \cup \{ \infty \} \) is given by horoballs based at \( \xi \).

It is known that for every compact Riemann surface \( \Sigma_g \) of genus \( g \geq 1 \), the complex torus \( H^0(\Sigma, \Omega)^* / H_1(\Sigma, \mathbb{Z}) \) has a canonical principal polarization induced from the cup product of \( H_1(\Sigma, \mathbb{Z}) \times H^1(\Sigma, \mathbb{Z}) \to H^2(\Sigma, \mathbb{Z}) \cong \mathbb{Z} \). It is called the Jacobian variety of \( \Sigma_g \) and denoted by \( J(\Sigma_g) \).

Using this, we can define the Jacobian (or period, Torelli) map,

\[
J : \mathcal{M}_g \to \mathcal{A}_g, \quad \Sigma_g \mapsto J(\Sigma_g).
\]

By Torelli theorem, \( J \) is injective (see [GH] for example). By [Ba2], the image \( J(\mathcal{M}_g) \) is an algebraic subvariety. Therefore, the closure of \( J(\mathcal{M}_g) \) in \( \overline{\mathcal{A}_g}^{SBB} \) is a projective variety, and it gives a compactification of \( \mathcal{M}_g \), which is the Satake-Baily-Borel compactification \( \overline{\mathcal{M}_g}^{SBB} \) mentioned above.

### 3 Universal moduli spaces of abelian varieties

In this section, we construct an infinite dimensional locally symmetric space \( \mathcal{A}_\infty \) and its completion \( \overline{\mathcal{A}_\infty}^{SBB} \) such that:

1. For every \( g \), there is a canonical inclusion \( \mathcal{A}_g \hookrightarrow \mathcal{A}_\infty \).
2. For every \( g \), there is an canonical embedding \( \mathcal{A}_g \hookrightarrow \mathcal{A}_{g+1} \), and these inclusions are compatible for all \( g \).
3. \( \mathcal{A}_\infty \) is the union of these images \( \mathcal{A}_g \).
4. \( \overline{\mathcal{A}_\infty}^{SBB} \) is a stratified complex analytic space such that the closure of each stratum is a complex projective space.
5. \( \mathcal{A}_\infty \) is open and dense in \( \overline{\mathcal{A}_\infty}^{SBB} \). For every \( g \), the closure of \( \mathcal{A}_g \) in \( \overline{\mathcal{A}_\infty}^{SBB} \) is \( \overline{\mathcal{A}_g}^{SBB} \).
6. \( \overline{\mathcal{A}_\infty}^{SBB} \) is the union of the subspaces \( \overline{\mathcal{A}_g}^{SBB} \), \( g \geq 0 \).

For this purpose, we first construct an infinite dimensional symmetric space \( \mathfrak{h}_\infty \) from the family of Siegel upper halfspaces \( \mathfrak{h}_g \), \( g \geq 1 \).

For every \( g \geq 1 \), we embed \( \mathfrak{h}_g \) into \( \mathfrak{h}_{g+1} \) as follows:

\[
X + iY \in \mathfrak{h}_g \mapsto \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} + i \begin{pmatrix} Y & 0 \\ 0 & 1 \end{pmatrix} \in \mathfrak{h}_{g+1}.
\]  

(3.1)

Clearly, the image of \( \mathfrak{h}_g \) in \( \mathfrak{h}_{g+1} \) is a totally geodesic subspace. Then we obtain a direct sequence of increasing Hermitian symmetric spaces of noncompact type:

\[
\mathfrak{h}_1 \hookrightarrow \mathfrak{h}_2 \hookrightarrow \cdots.
\]

Then the direct limit with the natural topology \( \mathfrak{h}_\infty = \lim_{g \to \infty} \mathfrak{h}_g \) is an infinite dimensional smooth manifold locally based on \( \mathbb{C}^\infty \), the complex vector space of finite sequences with the finite topology. See [Gl] [Ha].
In our case, we can be more specific with the space $h_\infty$ about its nature as an infinite dimensional manifold, which will be important for applications we have in mind.

$$h_\infty = \{ \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} + i \begin{pmatrix} Y & 0 \\ 0 & I_\infty \end{pmatrix} | X, Y \text{ are } g \times g \text{ symmetric matrices for some } g, Y > 0 \}. \quad (3.2)$$

Abstractly, we have the following result.

**Proposition 3.1.** The limit space $h_\infty$ is an infinite dimensional Hermitian symmetric space.

**Proof.** Since each $h_g$ is a totally geodesic Hermitian subspace of a larger Hermitian symmetric space $h_{g+k}$, $k \geq 0$, it can be seen that every point in $h_\infty$ is an isolated fixed point of an involutive holomorphic isometry of $h_\infty$, and hence $h_\infty$ is an infinite dimensional symmetric space.

On the other hand, the description of $h_\infty$ in Equation 3.2 shows that it is not a usual infinite dimensional Hermitian symmetric space modelled on Hilbert manifolds as in [Kau] [Tu] [Up], or other completions and extensions in other papers.

For example, one natural extension of $h_\infty$ is to consider the symmetric space defined by

$$h'_\infty = \{ X + iY | X, Y \text{ are } \infty \times \infty \text{-symmetric matrixes whose entries satisfy some convergence properties, and finite major minors of } Y \text{ are positive definite.} \}$$

This is more common in the theory of infinite dimensional symmetric spaces. A more restricted extension of $h_\infty$ is to consider the symmetric space

$$h''_\infty = \{ X + iY | X, Y \in h'_\infty, X, Y - I_\infty \text{ have only finitely many nonzero entries.} \}$$

From Equation 3.2 it is clear that we have strict inclusion:

$$h_\infty \subset h''_\infty \subset h'_\infty.$$

As mentioned before, for each $g$, $Sp(2g, \mathbb{Z})$ acts properly, holomorphically and isometrically on $h_g$, and the quotient $Sp(2g, \mathbb{Z}) \backslash h_g$ is a Hermitian locally symmetric space and is equal to $A_g$.

We need to construct a discrete group $Sp(\infty, \mathbb{Z})$ which also acts properly, holomorphically and isometrically on $h_\infty$ such that the quotient $Sp(\infty, \mathbb{Z}) \backslash h_\infty$ gives the desired space $A_\infty$, an infinite dimensional Hermitian locally symmetric space.

For each $g$, every element of $Sp(2g, \mathbb{R})$ can be written as a $2 \times 2$ block matrix consisting of $n \times n$ matrices,

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

such that

$$\begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} M = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix},$$

which is equivalent to the equations:

$$tAC = tCA, tBD = tDB, tAD - tCB = I_g.$$
Then we get an embedding

$$Sp(2g, \mathbb{R}) \rightarrow Sp(2(g+1), \mathbb{R}), \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \rightarrow \begin{pmatrix} (A & 0) \\ (0 & 1) \\ (C & 0) \\ (0 & 0) \end{pmatrix} \begin{pmatrix} (B & 0) \\ (0 & 0) \end{pmatrix} .$$

This induces an embedding

$$Sp(2g, \mathbb{Z}) \rightarrow Sp(2(g+1), \mathbb{Z}).$$

Under these embeddings of $h_g \hookrightarrow h_{g+1} + 1$ and $Sp(2g, \mathbb{Z}) \rightarrow Sp(2(g+1), \mathbb{Z})$, it is clear that the action of $Sp(2g, \mathbb{Z})$ on $h_g$ leaves the subspace $h_g$ stable, and we obtain a canonical embedding

$$A_g = Sp(2g, \mathbb{Z}) \setminus h_g \rightarrow A_{g+1} = Sp(2(g+1), \mathbb{Z}) \setminus h_{g+1}. \quad (3.3)$$

From these increasing sequences of groups $Sp(2g, \mathbb{R})$ and $Sp(2g, \mathbb{Z})$, we obtain two limit groups

$$Sp(\infty, \mathbb{R}) = \lim_{g \rightarrow \infty} Sp(2g, \mathbb{R}), \quad Sp(\infty, \mathbb{Z}) = \lim_{g \rightarrow \infty} Sp(2g, \mathbb{Z}).$$

These groups can be described explicitly as follows:

$$Sp(\infty, \mathbb{R}) = \left\{ \begin{pmatrix} A & B \\ 0 & I_\infty \end{pmatrix} \begin{pmatrix} 0 & 0 \\ C & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & I_\infty \end{pmatrix} \right\},$$

where for some $g \geq 1$, $A, B, C$ and $D$ are $g \times g$ block matrices, and they satisfy

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbb{R}).$$

Similarly,

$$Sp(\infty, \mathbb{Z}) = \left\{ \begin{pmatrix} A & B \\ 0 & I_\infty \end{pmatrix} \begin{pmatrix} 0 & 0 \\ C & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & I_\infty \end{pmatrix} \right\},$$

where $A, B, C, D$ are $g \times g$-block integral matrices and satisfy

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbb{Z}).$$

It is clear that $Sp(\infty, \mathbb{R})$ acts transitively on $h_\infty$, and hence $h_\infty$ can be considered as a symmetric space associated with the infinite dimensional Lie group $Sp(\infty, \mathbb{R})$.

Though $Sp(\infty, \mathbb{Z})$ is not a finitely generated group, it is countable and is a discrete subgroup of the Lie group $Sp(\infty, \mathbb{R})$. It acts properly on $h_\infty$, and the quotient space $Sp(\infty, \mathbb{Z}) \setminus h_\infty$ is Hausdorff, and it can be seen that

$$Sp(\infty, \mathbb{Z}) \setminus h_\infty = \lim_{g \rightarrow \infty} A_g, \quad (3.4)$$
where the right hand side is defined by the inclusion in Equation 3.3. Denote this limiting space by $A_\infty$. We call it the universal moduli space of principally polarized abelian varieties.

By the explicit description of $h_\infty$ in Equation 3.2, we can see that $h_\infty$ is a complex manifold and a Hermitian symmetric space, and the action of $Sp(\infty, \mathbb{R})$ on $h_\infty$ is holomorphic.

**Remark 3.2.** If we take the limit space $\lim_{g \to \infty} A_g$ directly, we only get a topological space, or some kind of infinite dimensional orbitfolds, since each $A_g$ is not smooth. The realization as a quotient of $h_\infty$ gives it more structures, which will be needed for constructing natural measures on $A_\infty$ and its completion $\overline{A}_\infty$. On the other hand, it might be possible to generalize the constructions [Gl] [Ha] to the setup of orbifolds. In any case, the triple of an infinite dimensional Lie group $Sp(\infty, \mathbb{R})$, an infinite dimensional symmetric space $h_\infty$, and an arithmetic group $Sp(\infty, \mathbb{Z})$ is appealing.

**Remark 3.3.** The group $Sp(\infty, \mathbb{Z})$ can be thought of as an arithmetic subgroup of the infinite dimensional Lie group $Sp(\infty, \mathbb{R})$. Though infinite dimensional symmetric spaces have been studied by many people, see for example, [Kau] [Tu] [Up], it seems that their quotients by analogues of arithmetic subgroups of linear algebraic groups have not been studied. On the other hand, it is known that arithmetic locally symmetric spaces have much richer structures than symmetric spaces of noncompact types, and they occur naturally as important spaces ranging from number theory, algebraic geometry, differential geometry, to topology etc.

**Proposition 3.4.** The space $A_\infty$ is a complex space and has a canonical stratification induced from the canonical stratification of the subspaces $A_{g+1} - A_g$, $g \geq 1$.

Next we construct the completion $\overline{A}_\infty$ of $A_\infty$. For this purpose, we follow the standard procedure of compactifications of arithmetic locally symmetric spaces in [BB] (see also [Bl] for other references).

As mentioned before, the closure of $h_g$ in its compact dual gives a compactification $\overline{h}_g$, and the standard embedding of $h_{g'}$, $g' < g$, into the boundary of $\overline{h}_g$ in Equation 2.1 and the translates by $Sp(2g, \mathbb{Z})$ of these standard boundary components give all the rational boundary components of $h_g$. The union of $h_g$ with the rational boundary components gives a partial compactification $\overline{h}_g, \mathbb{Q}$ with the Satake topology such that the quotient $Sp(2g, \mathbb{Z}) \setminus \overline{h}_g, \mathbb{Q}$ is $\overline{A}_g$.

For our purpose, we need to show that these constructions for $A_g$ are compatible with respect to natural embedding between them when $g$ increases.

**Proposition 3.5.** For every $g$, under the inclusion $h_g \hookrightarrow h_{g+1}$ in Equation 3.7, the closure of $h_g$ in $\overline{h}_{g+1}, \mathbb{Q}$ is equal to $\overline{h}_g, \mathbb{Q}$. Consequently, the closure of $A_g$ in $\overline{A}_{g+1}$ is equal to $\overline{A}_g$.

**Proof.** This can be seen from how the standard boundary components $h_{g'}$ of $h_g$, $g' < g$, fit together and degenerate inductively.

**Remark 3.6.** The compactification $\overline{A}_{g+1}$ can be written as a disjoint union

$$\overline{A}_{g+1} = A_{g+1} \sqcup A_g \sqcup \cdots \sqcup A_1 \sqcup \{0\}.$$

Note that $Sp(2, \mathbb{Z}) = SL(2, \mathbb{Z})$ and $h_1$ is the Poincaré upper half plane $\mathbb{H}^2$, and hence $A_1 \cong SL(2, \mathbb{Z}) \setminus \mathbb{H}^2$ is noncompact and can be compactified by adding a cusp point $\{\infty\}$, which is really $A_0$. This shows that $A_g$ is embedded into $\overline{A}_{g+1}$ in two different ways: as an interior space through
the embedding in Equation 3.1 and in the boundary through the above disjoint decomposition (or Equation 2.1).

Once we have the compatibility in Proposition 3.5 we can construct a completion \( \overline{\mathcal{A}}_{\infty}^{SBB} \) as follows.

From the increasing sequence of bordifications
\[
\overline{h}_{1,Q} \hookrightarrow \overline{h}_{2,Q} \hookrightarrow \overline{h}_{3,Q} \hookrightarrow \cdots ,
\]
we can form
\[
\overline{h}_{\infty,Q} = \lim_{g \to \infty} \overline{h}_{g,Q}.
\]
The space \( \overline{h}_{\infty,Q} \) also has a concrete realization similar to that of \( h_{\infty} \) in Equation 3.2:
\[
\overline{h}_{\infty,Q} = \{ \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} + i \begin{pmatrix} Y & 0 \\ 0 & I_{\infty} \end{pmatrix} | X + iY \in \overline{h}_{g,Q} \text{ for some } g \}.
\]

Taking the quotient by \( Sp(\infty,\mathbb{Z}) \), we obtain the desired completion of \( \mathcal{A}_{\infty} \),
\[
\overline{\mathcal{A}}_{\infty}^{SBB} = Sp(\infty,\mathbb{Z}) \backslash \overline{h}_{\infty,Q}.
\]

We note that
\[
\overline{\mathcal{A}}_{\infty}^{SBB} = \lim_{g \to \infty} \overline{\mathcal{A}}_{g}^{SBB} = \bigcup_{g \geq 0} \overline{\mathcal{A}}_{g}^{SBB}
\]
derived under the inclusion \( \overline{\mathcal{A}}_{g}^{SBB} \hookrightarrow \overline{\mathcal{A}}_{g+1}^{SBB} \).

Motivated by the decomposition of \( \overline{\mathcal{A}}_{g+1}^{SBB} \) in Remark 3.6 we can obtain a decomposition of \( \overline{\mathcal{A}}_{\infty}^{SBB} \) into an infinite dimensional interior and finite dimensional boundary pieces:

**Proposition 3.7.** The completion \( \overline{\mathcal{A}}_{\infty}^{SBB} \) admits the following decomposition,
\[
\overline{\mathcal{A}}_{\infty}^{SBB} = \mathcal{A}_{\infty} \bigsqcup (\mathcal{A}_0 \sqcup \mathcal{A}_1 \sqcup \mathcal{A}_2 \sqcup \cdots ),
\]
where the disjoint union
\[
\sqcup_{g \geq 0} \mathcal{A}_g
\]
is the boundary, and \( \mathcal{A}_{\infty} \) is the interior in some sense, which can also be decomposed into a non-disjoint union of \( \mathcal{A}_g, g \geq 0 \) (Equation 3.4). Note also that every \( \mathcal{A}_g \) can appear in \( \overline{\mathcal{A}}_{\infty}^{SBB} \) in two ways: either in the interior \( \mathcal{A}_{\infty} \), or in the boundary \( \sqcup_{g \geq 0} \mathcal{A}_g \).

### 4 Universal moduli spaces of Riemann surfaces

In this section, we construct a universal moduli space \( \mathcal{M}_{\infty} \) and its completion \( \overline{\mathcal{M}}_{\infty}^{SBB} \) by using the spaces \( \mathcal{A}_{\infty} \) and \( \overline{\mathcal{A}}_{\infty}^{SBB} \) constructed in the previous section.

Recall that for every \( g \), there is the embedding by the Jacobian map
\[
J : \mathcal{M}_g \to \mathcal{A}_g,
\]
which induces an embedding
\[
J : \overline{\mathcal{M}}_g^{SBB} \to \overline{\mathcal{A}}_g^{SBB}.
\]

We note the following description of the boundary of \( \overline{\mathcal{M}}_g^{SBB} \).
Proposition 4.1. The boundary of $\overline{M}_g^{SBB}$ is the union of $M_{g_1} \times \cdots \times M_{g_k}$, where $g_1 + \cdots + g_k \leq g$, $k \geq 1$. The equality occurs only when we pinch homologically trivial loops of compact Riemann surfaces $\Sigma_g$ of genus $g$.

To see this, we note that if we pinch a homologically nontrivial loop on $\Sigma_g$, then we get a surface $\Sigma_{g-1}$ on the boundary of $\overline{M}_g^{SBB}$. If we pinch one homologically trivial loop, then we get a union of two compact Riemann surfaces $\Sigma_{g_1}, \Sigma_{g_2}$, where $g_1 + g_2 = g$. By iteration, we get the above picture. (Note that if we use the Deligne-Mumford compactification of $M_g$, then we get punctured Riemann surfaces, and the locations of the punctures make the dimension of the boundary components bigger. Here we forget these punctures).

Proposition 4.2. For every pair of natural numbers $g' < g$, if the moduli space $M_{g'}$ appears in the boundary of $\overline{M}_g^{SBB}$ as in the above proposition, then the closure of $M_{g'}$ is equal to $\overline{M}_g^{SBB}$.

This is one nice inductive property of the compactification $\overline{M}_g^{SBB}$. No new types of Riemann surfaces and their moduli spaces appear.

For the purpose of constructing a universal moduli space of Riemann surfaces, a seemingly unfortunate fact is that there is no obvious inclusion of $M_g$ into $M_{g+1}$ as in the case of $A_g \subset A_{g+1}$. On the other hand, the Jacobian maps $J$ in Equations 4.1, 4.2 overcome this difficulty. Later, this turns out to be a nicer property in terms of constructing stratifications.

Consider the subspaces $J(M_g)$ of $A_g \subset A_\infty$ and $J(\overline{M}_g^{SBB})$ of $\overline{A}_g^{SBB} \subset \overline{A}_\infty^{SBB}$, $g \geq 1$. Define

$$M_\infty = \cup_{g \geq 1} J(M_g) \subset A_\infty,$$

and

$$\overline{M}_\infty^{SBB} = \cup_{g \geq 1} J(\overline{M}_g^{SBB}) \subset \overline{A}_\infty^{SBB}.$$

After defining these spaces, it is crucial to understand their properties. The next result is probably the most basic or the minimal requirement, otherwise we could take the trivial construction of a disjoint union of $\overline{M}_g^{SBB}$, which is definitely not what we want.

Proposition 4.3. The subspace $\overline{M}_\infty^{SBB} \subset \overline{A}_\infty^{SBB}$ is connected.

Proof. We want to show that $J(M_g) \subset A_g$ is contained in the closure of $J(M_{g+1})$ in $A_{g+1}$. Recall that $A_g$ is embedded into $A_{g+1}$ through the embedding of $h_g \hookrightarrow h_{g+1}$ in Equation 3.1. Suppose $\Sigma_1$ is the compact Riemann surface of genus 1 whose period in $h_1$ is equal to $i$ with respect to a suitable choice of basis of $H_1(\Sigma_1, \mathbb{Z})$. For any compact Riemann surface $\Sigma_g$ of genus $g$, $J(\Sigma_g)$ gives a point $p$ in $J(M_g) \subset A_g$. Let $p$ also denote the image of $p$ in $A_{g+1}$ under the above embedding $A_g \to A_{g+1}$. Then the disjoint union $\Sigma_g \cup \Sigma_1$ is mapped to the point $p \in A_{g+1}$.

Now if we pick two points on $\Sigma_g$ and $\Sigma_1$ and remove small disks around them depending on a small parameter $\varepsilon$ and glue them, we get a compact Riemann surface $\Sigma_{g+1,\varepsilon}$ of genus $g + 1$ with a short separating neck. Note that $J(\Sigma_{g+1,\varepsilon})$ is contained in $J(M_{g+1})$ in $A_{g+1}$. When $\varepsilon \to 0$, $J(\Sigma_{g+1,\varepsilon})$ converges to $J(\Sigma_g \cup \Sigma_1)$, which is the point $p$ in $A_{g+1}$ above. It follows that $p$ is in the closure of $J(M_{g+1})$, and hence every point of $J(M_g)$ is a limit of points of $J(M_{g+1})$.

Remark 4.4. Adding a compact Riemann surface $\Sigma_1$ of genus 1 to $\Sigma_g$ to obtain a compact Riemann surface $\Sigma_{g+1,\varepsilon}$ is one natural way to relate $M_g$ to $M_{g+1}$. This was used in the formulation of stability results in [Har] and [MW] on homology and cohomology of $M_g$, or mapping class groups.
Proposition 4.5. The subspace $\overline{M}_{sBB}^\infty \subset \overline{A}_{sBB}^\infty$ has a canonical stratification such that the closure of each stratum is a projective variety over $\mathbb{C}$, and is a union of $\overline{M}_{g}^{\simeq}$, $g \geq 0$, though $\overline{M}_{g}^{\simeq}$ can appear in many different ways in $\overline{M}_{sBB}^\infty$.

Proof. We note that each $\overline{M}_{g}^{\simeq}$ has a canonical stratification by Proposition 4.1. Since $\overline{M}_{sBB}^\infty$ is the union of $\overline{M}_{g}^{\simeq} = J(\overline{M}_{g}^{\simeq})$ for $g \geq 0$, the above Proposition shows that $\overline{M}_{g}^{\simeq}$ is contained in $\overline{M}_{g+1}^{\simeq}$. Then by considering $\overline{M}_{g+1}^{\simeq} - \overline{M}_{g}^{\simeq}$, we obtain a desired stratification.

In comparison to Proposition 3.7 about the decomposition of $A_{sBB}^\infty$, we have the following result.

Theorem 4.6. For every $g$, there is only one way to embed $M_{g}$ into $M_{g+1}^{\simeq}$, which is also equal to the closure of $M_{g+1}$ inside $\overline{M}_{sBB}^\infty$. Under this inclusion, we get an increasing sequence of spaces:

$$M_{1}^{\simeq} \hookrightarrow M_{2}^{\simeq} \hookrightarrow M_{3}^{\simeq} \hookrightarrow \cdots,$$

and

$$\overline{M}_{sBB}^\infty = \lim_{g \to \infty} \overline{M}_{g}^{\simeq} = \bigcup_{g \geq 1} \overline{M}_{g}^{\simeq}.$$

Remark 4.7. Note that the unique embedding of $M_{g}$ into the compactification $\overline{M}_{g+1}^{\simeq}$ is in some sense a nicer property for this family of $M_{g}$ than the family of $A_{g}$ since there are two different embeddings of $A_{g}$ into $\overline{A}_{g+1}^{\simeq}$, as pointed out in Proposition 3.7.

Remark 4.8. The relation between the inductive limit $\lim_{g \to \infty} \overline{M}_{g}^{\simeq}$ from the general construction and the space constructed in this paper through $A_{sBB}^\infty$ can also be seen as follows. A Riemann surface of genus $g$ on one hand is a degenerated Riemann surface $\Sigma$ of genus $g+1$ where a homologically nontrivial loop, i.e., a non-separating loop, has been pinched to a point. Therefore, its Jacobian $J(\Sigma)$, an Abelian variety of dimension $g$, is also a degenerated Abelian variety of dimension $g+1$. Alternatively, we can identify $J(\Sigma)$ with an Abelian variety of dimension $g+1$ by multiplying it with a normalized Abelian variety of dimension 1. This would correspond to viewing $\Sigma$ as a Riemann surface of genus $g+1$ by taking its disjoint union with a standard Riemann surface of genus 1. Of course, there is the issue of the choice of normalization here for that Riemann surface of genus 1. But the advantage of the construction is that we no longer need to go to the boundary of the moduli space for Riemann surfaces of genus $g+1$ or of principally polarized Abelian varieties of dimension $g+1$ to get the objects of genus/dimension $g$, but can stay within the interior. And we can interpolate between the two construction by degenerating the Riemann surfaces of genus 1 that had been added as a factor/component. (We don’t need to address the other way a Riemann surface of genus $g+1$ can be degenerate, by pinching a homologically trivial loop, because in that case, the genus and the dimension of the Jacobian do not drop, and therefore, the corresponding degeneration stays in the interior of the moduli space $A_{g}$ anyway.)

5 Riemannian metrics

First, we note that each irreducible symmetric space has a unique invariant Riemannian metric up to scaling. On each $h_{g}$, we can choose the invariant Riemannian metric such that under the
canonical embedding $\mathbb{H}^2 = \mathfrak{h}_1 \hookrightarrow \mathfrak{h}_g$ as in (or induced inductively from) Equation 3.1, the induced metric on $\mathbb{H}^2$ is the Poincaré hyperbolic metric.

**Proposition 5.1.** *With the above normalization of invariant metrics on $\mathfrak{h}_g$, for every $g \geq 1$, the embedding $\mathfrak{h}_g \hookrightarrow \mathfrak{h}_{g+1}$ in Equation 3.1 is an isometric embedding, and the infinite dimensional Siegel space $\mathfrak{h}_\infty$ has an invariant Riemannian metric, which induces the normalized invariant Riemannian metric on each embedded interior subspace $\mathfrak{h}_g$ in $\mathfrak{h}_\infty$. On the completion $\overline{\mathfrak{h}_\infty}$, we can put on a stratified Riemannian metric so that on each standard boundary component $\mathfrak{h}_g$ and hence every rational boundary component, the induced metric is the above normalized invariant metric on $\mathfrak{h}_g$. Though the boundary strata $\mathfrak{h}_g$ are at infinite distance from interior points of $\overline{\mathfrak{h}_\infty}$, i.e., points in $\mathfrak{h}_\infty$ (for example, from the interior points contained in any interior subspace of $\mathfrak{h}_{g'}$ of $\mathfrak{h}_\infty$), it is no problem since these metrics on the boundary strata are compatible in tangential directions in the following sense.

Suppose $\mathfrak{h}_g$ is a rational boundary component, i.e., contained in the boundary of $\overline{\mathfrak{h}_\infty}$. Then we have families of “parallel” subspaces $\mathfrak{h}_g$ inside $\mathfrak{h}_\infty$ which converge to the boundary component $\mathfrak{h}_g \subset \overline{\mathfrak{h}_\infty}$. For example, for the standard boundary component $\mathfrak{h}_g$ of $\overline{\mathfrak{h}_\infty}$, we can push the canonically embedded interior subspaces $\mathfrak{h}_g$ in $\mathfrak{h}_\infty$ towards the boundary component.

If $v$ is a tangent vector to such an interior subspace $\mathfrak{h}_g$, then it is also a tangent vector to the boundary $\mathfrak{h}_g$. An important point is that the norms of $v$ are the same. This means that we have a compatible stratified Riemannian metric on different stratification components of the completion $\overline{\mathfrak{h}_\infty}$, and hence also on the completion $\overline{\mathcal{A}}^{SBB}$.

Now when we decompose $\overline{\mathcal{A}}^{SBB}_\infty$ into the disjoint union

$$\overline{\mathcal{A}}^{SBB}_\infty = \mathcal{A}_\infty \coprod \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2 \cup \cdots \cup \mathcal{A}_g \cup \cdots,$$

we can use the Riemannian metric to define a measure on each of the boundary piece $\mathcal{A}_g$. (Note that they are disjoint).

For the interior $\mathcal{A}_\infty$, which is a non-disjoint union of $\mathcal{A}_0, \cdots, \mathcal{A}_g, \cdots$, and for any finite dimensional analytic subspace $K$ in $\mathcal{A}_\infty$, we can use the Riemannian metric of $\mathcal{A}_\infty$ or $\mathfrak{h}_\infty$ to define a measure on $K$, or suppose $K$ is contained in some $\mathcal{A}_g$, then we can use Riemannian measure of $\mathcal{A}_g$ and restrict it to $K$.

The double appearance of $\mathcal{A}_g$ in $\overline{\mathcal{A}}^{SBB}_\infty$ above in might not be so nice. On the other hand, this does not occur for $\overline{\mathcal{M}}^{SBB}_\infty$.

**Stratified Riemannian metric on $\overline{\mathcal{M}}^{SBB}_\infty$**

To construct a measure on $\overline{\mathcal{M}}^{SBB}_\infty$, we use the embedding

$$\overline{\mathcal{M}}^{SBB}_\infty \subset \overline{\mathcal{A}}^{SBB}_\infty.$$

We can pull back the stratified Riemann metric on $\overline{\mathcal{A}}^{SBB}_\infty$ to $\overline{\mathcal{M}}^{SBB}_\infty$. Since, however, this is not an immersion, the pull back of the stratified Riemann metric of $\overline{\mathcal{A}}^{SBB}_\infty$ is not everywhere positive definite on $\overline{\mathcal{M}}^{SBB}_\infty$ [Rau1, Rau2]. We will address that issue in a moment and first investigate the properties of this pull-back metric.
To describe this metric, we use the following \textit{disjoint decomposition}:

\[ M_{\infty}^{SBB} = \bigsqcup_{g \geq 1} M_g \prod_{k \geq 2, g_1 + \ldots + g_k = g} M_{g_1} \times \ldots \times M_{g_k}. \] (5.2)

Note that in the above decomposition, we remove the distinguished boundary component \( M_{g-1} \) of \( M_g^{SBB} \) and its boundary components in Proposition 4.1 in order to avoid repetition, and group other components together with \( M_g \).

This distinguished boundary component \( M_{g-1} \) of \( M_g^{SBB} \) is also at infinite distance from the interior points of \( M_g \) and hence of \( M_g^{SBB} \). (There are other boundary components at infinite distance from the interior points which result from pinching homologically nontrivial loops. They appear in the boundary of \( M_{g-1} \)). On the other hand, the Riemannian metric on the boundary component \( M_{g-1} \) and the Riemannian metric on the interior of \( M_g^{SBB} \) are compatible in a similar sense as described above, i.e., when interior points of \( M_g^{SBB} \) converge to a point in \( M_{g-1} \), the norms of tangential vectors of \( M_{g-1} \) converge. Therefore, we can take the corresponding measures on all these strata, in particular, \( M_g \), to get a compatible stratified measure on \( M_g^{SBB} \).

We now address the issue of the non-positive definiteness, following \cite{HJ1}. The Jacobian map \( J \) that we have used for the embedding \( M_g \rightarrow A_g \) associates to each marked Riemann surface \( \Sigma \) its Jacobian \( J(\Sigma) \), a principally polarized Abelian variety. In particular, \( J(\Sigma) \) has canonical flat metric.

Instead of using the Jacobian map in order to map \( M_g \) to \( A_g \), we may use a related period map, which is a map from each individual Riemann surface \( \Sigma \) to \( J(\Sigma) \),

\[ p_\Sigma : \Sigma \rightarrow J(\Sigma), \]

in order to obtain a metric on \( \Sigma \) by pulling back the flat metric of \( J(\Sigma) \). This metric on \( \Sigma \) which is called the \textit{Bergman metric} can also be described as follows. Let \( \theta_1, \ldots, \theta_g \) be an \( L^2 \)-orthonormal basis of the space of holomorphic 1-forms on \( \Sigma \), i.e.

\[ \frac{-1}{2} \int_\Sigma \theta_i \wedge \bar{\theta}_j = \delta_{ij}. \] (5.3)

Note that since the sum of the squares of an orthonormal basis of holomorphic one-forms (orthonormality is conformally invariant for one-forms), (5.3) does not depend on the prior choice of a metric on \( \Sigma \).

The Bergman metric then is simply given by

\[ \rho_B(z)dzd\bar{z} := \sum_{i=1}^g \theta_i \bar{\theta}_i. \]

We can form the \( L^2 \)-product of holomorphic quadratic differentials

\[ (\omega_1 dz^2, \omega_2 dz^2)_B := \frac{-1}{2} \int_\Sigma \omega_1(z) \bar{\omega}_2(z) \frac{1}{\rho_B(z)} dz \wedge d\bar{z}. \] (5.4)

This induces a Riemannian metric on the Teichmüller space \( T_g \) and therefore also on its quotient \( M_g \) (or, more precisely, on a finite cover of \( M_g \) that does not possess quotient singularities).
(Note that $T_g$ is a complex manifold and is the universal covering of $M_g$ as an orbifold.) For simplicity, we shall also call this metric on $M_g$ the Bergman metric. As shown in [HJ1], this metric dominates the Siegel metric just described, i.e., the metric using the map $J : M_g \to A_g$. In order to understand this result, one should note that both metrics are induced by the Jacobian map that associates to each Riemann surface its Jacobian. For the metric $(\cdot, \cdot)_B$, we use the period map on each individual Riemann surface $\Sigma$, and pull back the flat metric on its Jacobian $J(\Sigma)$, and then form an $L^2$-product, analogous to the construction of the Weil-Petersson metric that works with the hyperbolic instead of the Bergman metric on $\Sigma$. For the Siegel metric, in contrast, we do not pull back the flat metrics on the individual Jacobians, but rather the symmetric metric on the moduli space $A_g$ of principally polarized Abelian varieties. In contrast to the Siegel metric, $(\cdot, \cdot)_B$ is positive definite everywhere. It has the same asymptotic behavior as the former, however. We summarize our considerations in the following

**Theorem 5.2.** There exists a metric on $\overline{M}_g^{SBB}$ that induces a Riemannian metric on each stratum $M_g$. This metric has the following properties

- Boundary points of $M_g$ corresponding to pinching a homologically nontrivial loop are at infinite distance from the interior.
- Boundary points of $M_g$ corresponding to pinching a homologically (but not homotopically) trivial loop have finite distance from the interior.
- The metric forgets the punctures, i.e. the boundary components have codimension 3, again with the exception of the one where one component of the limit Riemann surface has genus 1.

**Corollary 5.3.** There exists a measure on $\overline{M}_g^{SBB}$ that induces a finite volume measure on each stratum $M_g$.

**Proof.** Both the Siegel metric and the Bergman metric induce measures on the strata of $\overline{M}_g^{SBB}$. Even though the Siegel metric is not positive definite, it is degenerate only on the hyperelliptic locus because that is where the Jacobian map $J : M_g \to A_g$ is not an immersion as orbifolds, but this hyperelliptic locus is a quasiprojective subvariety of lower dimension. Thus, we find not only one, but two measures satisfying the claim. The measure induced by the Bergmann dominates that induced by the Siegel metric.

Thus, for the purposes of integration theory in Section 6, we could use either one. In order to get convergence of integrals on $\overline{M}_\infty^{SBB}$, we shall have to choose weights for these measures on the components $M_g$ depending on $g$.

**Remark 5.4.** When we shall combine measures on subspaces of a common ambient space to define a global measure in Section 6 below, one issue is the compatibility. The above explanation shows that the canonical Riemannian metric on $h_\infty$ and $h_\infty^Q$ serves as a gauge to coordinate different components $M_g$. When the dimension of the strata jumps, it does not affect distance functions much, but it has a big impact on measures by noticing that a subspace of smaller dimension usually has zero measure with respect to an absolutely continuous measure on an ambient space such as Riemannian measures. Therefore, measures on different subspaces need to be adjusted according to their dimensions. This is what we shall now turn to.
6 Integration on the universal moduli space of Riemann surfaces

The considerations in this section will apply to both the inductive limit of the $\overline{M}_g^{SBB}$ for $g \to \infty$ and the space $\overline{M}_\infty \subset \overline{M}_g^{SBB}$ constructed in this paper. They will not directly apply to $\overline{M}_\infty^{SBB}$, because that space is not a disjoint union of strata corresponding to the different values of $g$.

We want to construct a measure on our space with respect to which every stratum has finite volume. This measure will be inductively built from measures on the spaces $\overline{M}_g^{SBB}$. First of all, we note that $A_g$ is a finite volume quotient of the Siegel upper half space with its Riemann-Lebesgue measure induced by its natural Riemannian metric induced by the symmetric structure. This then induces a measure $\mu_g$ on $M_g$. That latter measure also has finite volume, essentially for the same reason that the Poincaré metric on the punctured disk has locally finite volume near the puncture.

The subsequent constructions will work for both the Siegel and the Bergman metric as explained in Section 5.

We should recall that the different boundary components of $M_g$ behave differently with respect to our metric, be it the Siegel or the Bergman metric. The metric on $M_g$ induced from the invariant metric on $A_g$ under the Jacobian map $J$ is not of Poincaré type near a boundary component $M_i \times M_{g-i}$, since the induced metric here is not complete near such boundary points. For the boundary component $M_{g-1}$ lying on the boundary of $A_g$, it is complete. Nevertheless, our construction of measures will work in either situation, although the behavior of the measure will be different according to the type of boundary component.

We can then build the measure

$$\mu = \sum_g \lambda_g \mu_g$$

with positive real numbers $\lambda_g$ to be chosen. This means that

$$\mu(A) = \sum_g \lambda_g \mu_g(A \cap M_g)$$

for every measurable subset $A$ of our space, where again measurability requires measurability of the intersection with every stratum.

Again, we should point out that the space $\overline{M}_\infty^{SBB}$ is not simply a disjoint union of the different $M_g$. The stratification is a bit more complicated. For example, $M_i \times M_{g-i}$, $1 \leq i \leq g - 1$, and products of more factors appear also in the boundary of $\overline{M}_g^{SBB}$. Nevertheless, this does not affect our construction.

(The principle of the construction can easily be seen by considering the closed unit interval $[0,1]$ and equip it with the $\lambda_1$ times the Lebesgue measure on the open interval $(0,1)$ plus $\lambda_0$ times the Dirac measures at the boundary points 0 and 1.)

The question then arises how to determine the $\lambda_g$. For the convergence of series as they occur in string theory, the $\lambda_g$ should be sufficiently rapidly decaying functions of $g$. In string theory, this is achieved as follows. Let $\Sigma$ be a Riemann surface of genus $g$, and $h : \Sigma \to \mathbb{R}^N$ be a Sobolev function. Then its Dirichlet integral (1.1), also called the Polyakov action (see e.g. [Pol, J2]), in string theory, is

$$S(h, \Sigma) = \int_\Sigma |dh|^2$$

(6.3)
where by conformal invariance, we do not need to specify a conformal metric on Σ, as explained above. The string action then is defined as

$$S_{\text{string}}(h, \Sigma) := S(h, \Sigma) + \alpha(2g - 2)$$

(6.4)

for some positive constant \(\alpha\) that needs to be determined by considerations from physics which are not relevant for our current purposes (see [HP, Pol].

**Theorem 6.1.** For \(N = 26\), the string partition function can be written as

$$\sum_{g \geq 0} \int \exp(S_{\text{string}}(h, \Sigma))dh\Sigma$$

(6.5)

with a functional integration which for the component \(\Sigma\) is carried out over \(\mathcal{M}_g\).

**Proof.** For \(N = 26\), the conformal anomalies cancel (see e.g. [Pol, J2]), and consequently, for each genus \(g\), the partition function can be written as an integral over the moduli space of Riemann surfaces of genus \(g\) and an integral over the field \(h\), and for each \(g\), the value of this integral is finite. The term \(\alpha(2g - 2)\) in (6.3) then ensures the convergence of the resulting series.

As we have just explained, measures can be constructed on both \(\lim_{g \to \infty} \mathcal{M}_g^{SBB}\) via the general construction of direct limit and as a subspace of the infinite dimensional locally symmetric space \(\mathcal{A}_\infty\) and its completion \(\overline{\mathcal{A}}^{SBB}_\infty\).

We shall now explain that the additional structure coming from \(\overline{\mathcal{A}}^{SBB}_\infty\) and the embedding \(J: \overline{\mathcal{M}}^{SBB}_\infty \to \overline{\mathcal{A}}^{SBB}_\infty\) will shed some light on how natural the above construction is.

First, we point out that infinite dimensional symmetric spaces based on Hilbert spaces have natural Riemannian metrics, and their submanifolds also have induced Riemannian metrics. Unlike the case of finite dimensional cases, Riemannian metrics on infinite dimensional manifolds do not automatically induce measure and integration theory. Integration on infinite dimensional manifolds based on Hilbert and Banach spaces seems complicated. See [Ku, Wei], for example, for more information and references.

On the other hand, spaces in this paper such as \(\mathfrak{h}_\infty, \mathcal{A}_\infty^{SBB}\) and \(\overline{\mathcal{M}}^{SBB}_\infty\) have filtrations and stratifications by finite dimensional submanifolds, and we are only interested in finite dimensional subspaces at each step in some sense, and the general idea of analysis on stratified spaces will help. One important point is to use an invariant metric on the infinite dimensional symmetric space \(\mathfrak{h}_\infty\) and its completion \(\mathfrak{h}_\infty, \mathbb{Q}\) to coordinate metrics (see the previous section) and hence measures on these different strata. See the book [Pf] for some related information and references.

This is precisely what we have achieved here, as we have constructed a measure on \(\overline{\mathcal{A}}^{SBB}_\infty\) and \(\overline{\mathcal{M}}^{SBB}_\infty\) by using a stratified Riemann metric on \(\mathfrak{h}_\infty, \mathbb{Q}\).

7 Infinite dimensional locally symmetric spaces and stable cohomology of arithmetic groups

The Siegel upper half plane \(\mathfrak{h}_g\) is one important generalization of the Poincaré upper half plane \(\mathbb{H}^2 = SL(2, \mathbb{R})/SO(2)\).
Another important generalization is $X_n = SL(\infty, \mathbb{R})/SO(n), \ n \geq 2$. The arithmetic subgroup $SL(n, \mathbb{Z})$ acts properly on $X_n$, and the quotient $SL(n, \mathbb{Z}) \backslash X_n$ is the moduli space of positive definite $n \times n$-matrices (or quadratic forms in $n$ variables) of determinant 1, and which is also the moduli space of flat tori of volume 1 in dimension $n$.

Clearly there is an embedding

$$X_n \hookrightarrow X_{n+1}, \ A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix},$$

and an embedding

$$SL(n, \mathbb{R}) \hookrightarrow SL(n+1, \mathbb{R}), \ A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$ 

Then the procedure in §2 goes through, and we can construct the infinite dimensional symmetric space $X_\infty = \lim_{n \to \infty} X_n$, the infinite dimensional Lie group $SL(\infty, \mathbb{R}) = \lim_{n \to \infty} SL(n, \mathbb{R})$, and its arithmetic subgroup $SL(\infty, \mathbb{Z}) = \lim_{n \to \infty} SL(n, \mathbb{Z})$.

These spaces and groups can be realized concretely as follow s:

$$X_\infty = \{ \begin{pmatrix} A & 0 \\ 0 & I_\infty \end{pmatrix} \mid A \text{ is a positive definite } n \times n\text{-matrix for some } n \geq 1, \det A = 1 \},$$

$$SL(\infty, \mathbb{R}) = \begin{pmatrix} A & 0 \\ 0 & I_\infty \end{pmatrix} \mid A \in SL(n, \mathbb{R}) \text{ for some } n \geq 1 \},$$

and

$$SL(\infty, \mathbb{Z}) = \begin{pmatrix} A & 0 \\ 0 & I_\infty \end{pmatrix} \mid A \in SL(n, \mathbb{Z}) \text{ for some } n \geq 1 \}.$$ 

The quotient

$$SL(\infty, \mathbb{Z}) \backslash X_\infty$$

is an infinite dimensional locally symmetric space which contains every $SL(n, \mathbb{Z}) \backslash X_n$.

It is known that each $SL(n, \mathbb{Z}) \backslash X_n$ has a minimal Satake compactification

$$\overline{SL(n, \mathbb{Z}) \backslash X_n}^S = SL(n, \mathbb{Z}) \backslash X_n \sqcup SL(n-1, \mathbb{Z}) \backslash X_{n-1} \sqcup \cdots \sqcup \{ \infty \},$$

which can be obtained from the quotient by an $SL(n, \mathbb{Z})$-action on a completion $\overline{X_n}_Q$ by adding rational boundary components of $X_n$.

For every $n$, the embedding $X_n \hookrightarrow X_{n+1}$ induces an embedding

$$\overline{SL(n, \mathbb{Z}) \backslash X_n}^S \hookrightarrow \overline{SL(n+1, \mathbb{Z}) \backslash X_{n+1}}^S.$$ 

Similarly, we can construct a completion $\overline{X_\infty}_Q$ of $X_\infty$ which is invariant under $SL(\infty, \mathbb{Z})$ and hence a completion $\overline{SL(\infty, \mathbb{Z}) \backslash X_\infty}^S$ of $SL(\infty, \mathbb{Z}) \backslash X_\infty$ such that for every $n$,

$$\overline{SL(n, \mathbb{Z}) \backslash X_n}^S \hookrightarrow \overline{SL(\infty, \mathbb{Z}) \backslash X_\infty}^S,$$

and

$$\overline{SL(\infty, \mathbb{Z}) \backslash X_\infty}^S = \cup_{n \geq 1} \overline{SL(n, \mathbb{Z}) \backslash X_n}^S = \lim_{n \to \infty} \overline{SL(n, \mathbb{Z}) \backslash X_n}^S.$$ 

18
It is reasonable to expect that this same construction works for other series of classical simple algebraic groups.

These spaces should be related to the stability of cohomology of arithmetic groups in [Bo]. In fact, the analogy of the inclusion $X_n \hookrightarrow X_{n+1}$ was used in [Bo] to formulate and prove the stability result there.

8 Some other reasons for constructing universal moduli spaces

The stability results for homology groups of the mapping class groups of surfaces, or equivalently, the homology groups of $\mathcal{M}_{g,n}$, motivated Mumford conjecture on stable cohomology of $\mathcal{M}_{g,n}$ [Mum] [MW]. The proof depends on realizing them as the cohomology groups of universal classifying spaces. See [Wa] for a summary. These classifying spaces are topological objects and well-defined up to homotopy equivalence.

We note that $\mathcal{M}_g$ is a classifying space for the mapping class group of a compact surface $S_g$ of genus $g$ for rational coefficients. One natural question is to construct a universal moduli space which is an infinite dimensional algebraic variety and enjoys some good algebraic geometry properties and whose cohomology groups realize the stable cohomology groups as conjectured by Mumford. We don’t know if our space $\mathcal{M}_\infty$ and its completion $\overline{\mathcal{M}}_{\infty}^{SBB}$ might be helpful to this purpose.

We also note that for classical families of compact Lie groups, their limits and limits of their classifying spaces are universal groups and universal spaces, and they are important in characteristic classes. See [Mi, §5].

In the famous Bott periodicity theorem [Bot], limits of increasing sequences of classical compact Lie groups and associated spaces also appear naturally.

All these results explain that, besides the applications to string theory and the theory of minimal surfaces, it is a natural and important problem to consider universal moduli spaces of Riemann surfaces and universal (or rather infinite dimensional) symmetric spaces of noncompact type and their quotients by arithmetic groups.

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