Convergence Analysis of H(div)-Conforming Finite Element Methods for a Nonlinear Poroelasticity Problem

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In this paper, we introduce and analyze H(div)-conforming finite element methods for a nonlinear model in poroelasticity. More precisely, the flow variables are discretized by H(div)-conforming mixed finite elements, while the elastic displacement is approximated by the H(div)-conforming finite element with the interior penalty discontinuous Galerkin formulation. Optimal a priori error estimates are derived for both semidiscrete and fully discrete schemes.

1. Introduction

The Biot poroelasticity model [1] describes the phenomena of coupled mechanics and flow in porous media, and it is of increasing importance today in a divergence of science and engineering applications. Applications of poroelasticity have been made to areas that include carbon sequestration in environment engineering, predictive ability in earthquake engineering, surface subsidence in field phenomena, logging technologies in reservoir engineering, and pathological condition in biomechanics.

In the context of linear poroelasticity systems, a large number of numerical methods have been developed in recent years. Among them, finite element methods (FEMs) are commonly used approaches for such coupled system. In [2–4], Taylor–Hood elements were applied for the approximation of the displacement field and fluid pressure. Recently, some techniques based on mixed formulations have been developed. A method coupling standard conforming FEM for the displacement with mixed FEMs for flow variables has been provided in [5, 6]. However, it is well known that a conforming FEM approximation of displacement may give rise to locking or nonphysical oscillations [7]. Some remedies of locking include nonconforming FEMs [8–10], mixed FEMs [11, 12], discontinuous Galerkin (DG) methods [13–15], hybrid high-order methods [16], and weak Galerkin methods [17–20]. We also refer the interested reader to [21–24] for further study of locking-free FEMs on linear poroelasticity. Moreover, some preconditioners that are robust with respect to the physical and discretization parameters have been developed in this direction (refer to [25–28] and the references therein).

Compared with linear model in poroelasticity, the numerical methods on nonlinear problem are still very rare. The goal of this paper is to develop H(div)-conforming finite element methods for a nonlinear poroelasticity model. The nonlinearity arises from the dependence of the hydraulic conductivity \( \kappa \) on the dilation \( \nabla \cdot \mathbf{u} \), that is, \( \kappa = \kappa(\nabla \cdot \mathbf{u}) \). Such model has been studied in [29], where the existence and uniqueness of the weak solution were proved. Moreover, the authors in [29] introduce a FEM approximation and give the optimal order error estimate. We also find that similar problem has been treated in [30], where a mixed method combining conforming and mixed FEMs was proposed and analyzed. Some other works of numerical methods for nonlinear poroelasticity model can be found in [31, 32]. In the current work, we will propose and analyze H(div)-conforming methods to solve the nonlinear problem studied in [29]. More precisely, the flow variables are discretized by H(div)-conforming mixed FEMs, while the displacement is
approximated by the H(div)-conforming FEM with interior penalty DG method. Combining H(div)-conforming finite elements with DG methods was initially proposed in [33, 34] (see also [35–37]); they mainly intended to solve Stokes and Navier–Stokes equations in fluid mechanics. Later, this method was extended to more complex Darcy–Stokes interface problems [38–40], Brinkman–problems [41], a magnetic induction model [42], and linear Biot model in poroelasticity [43]. We also refer the interested reader to [44] on robust numerical simulations of H(div)-conforming FEMs for Stokes, coupled Darcy–Stokes, and Brinkman problems. Our work is to extend H(div)-conforming FEMs of linear Biot model to the nonlinear case. With the help of the framework presented in [30], we will give a detailed a priori error analysis of the proposed numerical scheme. It is worth mentioning that we shall address some difficulties arising from the inherent nonlinearity (see the terms (43), (44), (68), and (69) below). The main advantage of our method is that we present a unified treatment for both flow variables and the displacement. In addition, we relax H1 conformity by using H(div) space to approximate the displacement, and thus the normal components are continuous and the tangential components are treated by interior penalty DG approach; this implies that our method is locally conservative.

The rest of this paper is organized as follows. In Section 2, we describe the nonlinear poroelastic model and present its mixed weak formulation. We propose a spatial semidiscrete scheme which is based on H(div)-conforming finite elements in Section 3, where a priori error estimate is also provided. In Section 4, a fully discrete scheme with the backward Euler time stepping is proposed and analyzed. Some conclusions are made in Section 5.

2. Preliminaries and Model Problem

2.1. The Model Problem. Let \( \Omega \subset \mathbb{R}^2 \) be a bounded convex polygonal domain, with Lipschitz boundary \( \partial \Omega \). We consider the following nonlinear poroelastic model in \( \Omega \) over a time interval \((0, T]\) (see [29]):

\[
\begin{align*}
\partial_t p + a \nabla \cdot \partial_t u - \nabla \cdot (\kappa (\nabla \cdot u) \nabla p) &= g, \quad \text{in } \Omega \times (0, T], \\
\nabla \cdot \sigma &= f, \quad \text{in } \Omega \times (0, T],
\end{align*}
\]

(1)

where

\[
\sigma = \lambda \text{tr}(\varepsilon(u)) I + 2 \mu \varepsilon(u) - a \rho \varepsilon(u), \quad \varepsilon(u) = \frac{1}{2} (\nabla u + (\nabla u)^T),
\]

(2)

in which \( u(x, t) \) is the displacement of the solid phase, \( p(x, t) \) is the fluid pressure, \( \sigma(x, t) \) is the total stress tensor, \( c_0 > 0 \) is the storage coefficient, \( 0 < \alpha \leq 1 \) is the Biot–Willis constant, and \( \lambda \) and \( \mu \) are Lamé constants; \( \kappa(\cdot) \) is the hydraulic conductivity that satisfies assumptions (6) and (7) below.

Denote by \( \Gamma_d \) (resp. \( \Gamma_t \)) the Dirichlet (resp. traction) boundary for the elastic variables. Similarly, denote by \( \Gamma_f \) (resp. \( \Gamma_f \)) the pressure Dirichlet (resp. fluid normal flux) boundary. We assume that \( \partial \Omega = \Gamma_d \cup \Gamma_t \) with \( |\Gamma_d| > 0 \), and \( \partial \Omega = [\Gamma_f] \cup \Gamma_f \). The boundary conditions and initial conditions for the above system read as

\[
\begin{align*}
u &= 0, \quad \text{on } \Gamma_d \times (0, T], \\
o &= 0, \quad \text{on } \Gamma_t \times (0, T], \\
p &= 0, \quad \text{on } \Gamma_f \times (0, T],
\end{align*}
\]

(3)

where \( \mathbf{n} \) denotes the outward unit normal vector.

Setting \( q = -\kappa(\nabla \cdot u) \nabla p \), we can reformulate equation (1) by

\[
\begin{align*}
c_0 \partial_t p + a \nabla \cdot \partial_t u + \nabla \cdot q &= g, \quad \text{in } \Omega \times (0, T], \\
\kappa^{-1}(\nabla \cdot u) q + \nabla p &= 0, \quad \text{in } \Omega \times (0, T],
\end{align*}
\]

(4)

2.2. Weak Formulation. We start by introducing some notation. Let \( H^s(\mathcal{D}) \) (\( s \geq 0 \)) be the standard Sobolev space, equipped with norm \( \| \cdot \|_{H^s(\mathcal{D})} \) and seminorm \( | \cdot |_{H^s(\mathcal{D})} \). If \( s = 0 \), \( H^0(\mathcal{D}) \) is understood as \( L^2(\mathcal{D}) \). Denote \( \langle w, v \rangle_{\mathcal{D}} = \int_\mathcal{D} wv dx \) and \( \langle w, v \rangle_{\partial \mathcal{D}} = \int_{\partial \mathcal{D}} wv ds \). When \( \mathcal{D} = \Omega \), we shall omit the index \( \partial \). For the space \( H^s(\mathcal{D}) \), its norm is still denoted by \( \| \cdot \|_{H^s(\mathcal{D})} \). A subspace of \( H^s(\Omega) \) with vanishing trace on \( \Gamma_d \) is given by \( H^s_{0,\Gamma_d}(\Omega) = \{ v \in H^s(\Omega) : v|_{\Gamma_d} = 0 \} \). Additionally, we define \( H(\text{div}; \Omega) = \{ v \in (L^2(\Omega))^2 : \nabla \cdot v \in L^2(\Omega) \} \) with its graph norm \( \| v \|_{\text{div}} = (\| v \|_{L^2(\Omega)}^2 + \| \nabla \cdot v \|_{L^2(\Omega)}^2)^{1/2} \). Two subspaces are then introduced by

\[
\begin{align*}
H_{0,\Gamma_d}(\text{div}; \Omega) &= \{ v \in H(\text{div}; \Omega) : \nabla \cdot v|_{\Gamma_d} = 0 \} \quad \text{and}
H_{0,\Gamma_d}(\text{div}; \Omega) &= \{ v \in H(\text{div}; \Omega) : \nabla \cdot v|_{\Gamma_d} = 0 \}. 
\end{align*}
\]

For convenience, we set \( \mathcal{P} = L^2(\Omega), \quad Q = H_{0,\Gamma_d}(\text{div}; \Omega) \) and \( \mathcal{V} = (H^1_{0,\Gamma_d}(\Omega))^2 \). Integrating by parts, we deduce that the mixed weak formulation for (4) reads as follows: find \((p, \sigma, u, w) \in \mathcal{P} \times Q \times \mathcal{V} \) for every \( t \in (0, T] \) such that

\[
\begin{align*}
c_0 (\partial_t p, w) + a(\nabla \cdot \partial_t u, w) + \langle \nabla \cdot q, w \rangle &= (g, w), \\
(\kappa^{-1}(\nabla \cdot u), z) - (p, \nabla \cdot z) &= 0, \\
(a(u, v) - \alpha(p, \nabla \cdot v), v) &= (f, v),
\end{align*}
\]

(5)

for any \((w, z, v) \in \mathcal{P} \times Q \times \mathcal{V} \), where \( a(u, v) = 2\mu(\varepsilon(u), \varepsilon(v)) + \lambda (\nabla \cdot u, \nabla \cdot v) \). Additionally, we assume that there exist positive \( k_d \) and \( k_f \) such that

\[
k_d \leq \kappa^{-1}(\theta) \leq k_f, \quad \forall \theta \in \mathbb{R},
\]

(6)

and \( \kappa^{-1} \) is Lipschitz continuous:

\[
|\kappa^{-1}(\theta) - \kappa^{-1}(\eta)| \leq L|\theta - \eta|, \quad \forall \theta, \eta \in \mathbb{R},
\]

(7)
To deal with functions of time and space, we introduce the standard Bochner space $L^p(0, T; H^s(\Omega))$, which consists of all functions $u: [0, T] \rightarrow H^s(\Omega)$ with norm

$$\|u\|_{L^p(0,T;H^s(\Omega))} = \left( \int_0^T \|u(t)\|_p^p \, dt \right)^{1/p},$$

for $1 \leq p < \infty$. When $p = \infty$, the norm is defined as

$$\|u\|_{L^\infty(0,T;H^s(\Omega))} = \sup_{0 \leq t \leq T} \|u(t)\|_s.$$  

(8)

(9)

3. Error Estimates for the Semidiscrete Scheme

In this section, we aim to present the semidiscrete numerical method which focuses on discretizing the spatial variables. Let $\mathcal{T}_h$ be a shape-regular decomposition of $\Omega$ into triangles $\{K\}$. $h_K$ denotes the diameter of $K$ and $h = \max_{K \in \mathcal{T}_h} h_K$. Denote by $\partial^i_K$ the set of interior edges of elements in $\mathcal{T}_h$, $\partial^e_K$ the set of boundary edges on $\Gamma_d$, and $\partial^b_K$ the set of boundary edges on $\Gamma$. Therefore, the set of all edges $\partial h = \partial^i h \cup \partial^e h \cup \partial^b h$. The length of the edge $e \in \partial h$ is denoted by $\ell_h$. For every edge $e \in \partial h$, we fix a unit normal $n_e$ such that for edges on the boundary $\partial h$, $n_e$ is the outward unit normal.

For $e \in \partial^i h$ shared by elements $K_+$ and $K_-$, let $v$ be a scalar or vector piecewise smooth function and set $v^e = v|_{\partial^i (K_+ \cup K_+)}$, and we define the average and jump by

$$[v] = \frac{1}{2} (v^+ + v^-) \quad \text{and} \quad [v] = v^+ - v^-.$$  

(10)

On a boundary edge, we set

$$[v] = v \quad \text{and} \quad [v] = v.$$  

(11)

Consider the following finite element spaces:

$$Q_h = \left\{ q \in H^1(\Omega); q|_K \in BDM_k(K) \right\},$$

(12)

$$V_h = \left\{ v \in H^1(\Omega); v|_K \in BDM_k(K) \right\},$$

(13)

$$\mathcal{P}_h = \left\{ w \in L^2(\Omega); w|_K \in P_{k-1}(K) \right\},$$

(14)

where BDM$_k$ ($k \geq 1$) is the $H^1$-conforming space introduced by Brezzi, Douglas, and Marini and $P_k (K)$ denotes the space of polynomials of degree less than or equal to $k$ on $K$.

Let $\Pi_h: Q(\text{resp.} V) \rightarrow Q_h(\text{resp.} V_h)$ be the BDM interpolation and $P_h: \mathcal{P} \rightarrow \mathcal{P}_h$ be the $L^2$ projection. It is well known that they satisfy the following properties [45, 46]:

$$\|z - P_h z\|_0 \leq C h^l \|z\|_{l,K}, \quad 0 \leq l \leq k,$$

(15)

$$\|z - P_h z\|_0 \leq \|z\|_{1,K}, \quad 0 \leq l \leq k,$$

(16)

$$\|\nabla \cdot (v - \Pi_h v)\|_0 \leq C h^{l-1} \|v\|_{l,K}, \quad s = 0, 1, 1 \leq l \leq k + 1.$$

(17)

(18)

$$\|\nabla \cdot (v - \Pi_h v)\|_0 \leq C h^l \|z\|_{l,K}, \quad 0 \leq l \leq k.$$

(19)

Here and throughout the paper, we utilize $C$ to represent a positive generic constant that is independent of $h$ and $\Delta t$ but may take different values at different occurrences.

3.1. Semidiscrete $H(\text{div})$-Conforming Methods with DG Formulation. The expression of $a_h(\cdot, \cdot)$ below is similar to the statement in Section 3.1 in the work [43]. For completeness, we derive its scheme as follows.

Multiplying the third equation in (4) by any $v \in V_h$ on each element $K$, integrating by parts, and then summing over all elements in $\mathcal{T}_h$, we arrive at

$$- \sum_{T \in \mathcal{T}_h} \langle \nabla \cdot \alpha, v \rangle_T = \sum_{T \in \mathcal{T}_h} \langle \alpha, \nabla v \rangle_T - \sum_{T \in \mathcal{T}_h} \langle \alpha n_T, v_{0T} \rangle = \langle f, v \rangle,$$

(20)

where $n_T$ is outward normal of $T$.

We first have

$$\sum_{T \in \mathcal{T}_h} \langle \alpha n_T, v_{0T} \rangle = \sum_{e \in \partial h} \langle \langle \alpha n_e \rangle, [v] \rangle_e + \sum_{e \in \partial h} \langle \{\alpha n_e\}, [v] \rangle_e$$

(21)

Here in the third line, we have used $\alpha n_e = 0$ in $\partial^i h$, and in the last line, we have used the fact that $\langle \{v \cdot n_e\}, [v] \rangle = 0$ on $e \in \partial h \cup \partial b h$ and the regularity of the exact solutions.

In addition, it is easy to check that

$$\sum_{T \in \mathcal{T}_h} \langle \alpha, \nabla v \rangle_T = 2\mu \sum_{e \in \partial h \cup \partial b h} \langle \{\epsilon(u)n_e\}, [v] \rangle_e + \lambda \sum_{e \in \partial h \cup \partial b h} \langle \{\nabla \cdot u\}n_e\}, [v] \rangle_e$$

(22)

Substituting (21) and (22) into (20) yields
Adding some stabilized terms as in [47], we propose the following DG method for the third equation in (4):

\[ a_h(u, v) - \alpha (p, \nabla \cdot v) = (f, v), \]

with

\[ a_h(u, v) = 2\mu \sum_{k \in T_h} \langle (\varepsilon(u)n_e), [v] \rangle_e - 2\mu \sum_{e \in \Gamma_h} \langle \langle \varepsilon(u)n_e \rangle, [v] \rangle_e \]

\[ + \lambda (\nabla \cdot u, \nabla \cdot v) - \alpha (\nabla \cdot v, p) = (f, v), \quad \forall v \in V_h. \]

Consequently, after integrating by parts, we find that the exact solutions of (4) satisfy

\[ c_0 (\partial_t p, w) + \alpha (\nabla \cdot \partial_t u, w) + (\nabla \cdot q, w) = (g, w), \]

\[ (\kappa^{-1} (\nabla \cdot u) q_h, z) - (p, \nabla \cdot z) = 0, \]

\[ a_h(u, v) - \alpha (p, \nabla \cdot v) = (f, v), \]

for any \((w, z, v) \in \mathcal{N}_h \times Q_h \times V_h\).

Thus, the corresponding semidiscrete H(div)-conforming FEMs for (4) can be designed as follows: find \((p_h, q_h, u_h) \in \mathcal{N}_h \times Q_h \times V_h\) such that

\[ c_0 (\partial_t p_h, w) + \alpha (\nabla \cdot \partial_t u_h, w) + (\nabla \cdot q_h, w) = (g, w), \]

\[ (\kappa^{-1} (\nabla \cdot u_h) q_h, z) - (p_h, \nabla \cdot z) = 0, \]

\[ a_h(u_h, v) - \alpha (p_h, \nabla \cdot v) = (f, v), \]

for any \((w, z, v) \in \mathcal{N}_h \times Q_h \times V_h\), with the initial conditions given by

\[ p_h(0) = P_hP_0, \quad u_h(0) = \Pi_h u_0. \]

\[ a_h(x, y) \leq C_a \|x\|_h \|y\|_h, \quad \forall x, y \in V(h). \]

Moreover, for sufficiently large penalty parameter \(\beta\), it holds that

Lemma 1. There exists a constant \(C_a > 0\) such that

\[ \|v\|_h = \left( \sum_{k \in T_h} 2\mu \|\varepsilon(v)\|_{0,k}^2 + \sum_{e \in \Gamma_h} 2\mu h_e^{-1} \|\varepsilon(v)\|_{0,e}^2 + \sum_{e \in \Gamma_h} 2\mu h_e \|\varepsilon(v)n_e\|_{0,e}^2 + \lambda \|\nabla \cdot v\|_h^2 \right)^{1/2}. \]
\[ a_h(v,v) \geq C_h \|v\|_{H_0^1}^2, \quad \forall v \in V_h. \]  

(35)

We are now in a position to state the following error estimate, which is the main result of this section.

**Theorem 1.** Let \((p, q, u) \in \mathcal{P} \times \mathcal{Q} \times \mathcal{V} \) and \((p_h, q_h, u_h) \in \mathcal{P}_h \times \mathcal{Q}_h \times \mathcal{V}_h \) be the solutions of (4) and (31), respectively. Then, the following holds:

\[
\begin{align*}
    &c_0(\partial_t (p - p_h), w) + \alpha (\nabla \cdot \partial_t (u - u_h), w) + (\nabla \cdot (q - q_h), w) = 0, \\
    &\left( \kappa^{-1} (\nabla \cdot u) q - \kappa^{-1} (\nabla \cdot u_h) q_h, z \right) - (p - p_h, \nabla \cdot z) = 0, \\
    &a_h(u - u_h, v) - a(p - p_h, \nabla \cdot v) = 0,
\end{align*}
\]

for any \((w, z, v) \in \mathcal{P}_h \times \mathcal{Q}_h \times \mathcal{V}_h\).

Proof. Subtracting (30) from (31) yields

\[
\begin{align*}
    &\frac{1}{2} \frac{d}{dt} \| \theta_p \|_{L^2(0,T;\mathcal{P})}^2 + \frac{\epsilon}{2} \| \theta_p \|_{L^2(0,T;\mathcal{P})}^2 \\
    &+ \int_0^t \left( \kappa^{-1} (\nabla \cdot u_h) \theta_q(s), \theta_q(s) \right) ds = \lambda_1 + \lambda_2 + \lambda_3,
\end{align*}
\]

with

\[
\begin{align*}
    &\lambda_1 = - \int_0^t \left( \kappa^{-1} (\nabla \cdot u) \xi_q(s), \theta_q(s) \right) ds, \\
    &\lambda_2 = - \int_0^t \left( \kappa^{-1} (\nabla \cdot u) - \kappa^{-1} (\nabla \cdot u_h) \right) \Pi_h q(s), \theta_q(s) ds, \\
    &\lambda_3 = - \int_0^t a_h(\xi_u(s), \partial_t \theta_u(s)) ds.
\end{align*}
\]

For the first term \(\lambda_1\), using (6) and Cauchy–Schwarz and Young’s inequalities, we find that

\[
\lambda_1 \leq \int_0^t \left( \kappa^{-1} (\nabla \cdot u) \xi_q(s), \theta_q(s) \right) ds \
\leq \frac{1}{4\epsilon_1} \int_0^t \| \kappa^{-1} (\nabla \cdot u) \xi_q(s) \|_{L^2(0,T;\mathcal{P})}^2 ds + \epsilon_1 \int_0^t \| \theta_q(s) \|_{L^2(0,T;\mathcal{P})}^2 ds,
\]

(42)

where \(\epsilon_1\) is an arbitrarily small number.
For the second term $\mathcal{A}_2$, with the aid of (6) and (7) and Cauchy–Schwarz and Young’s inequalities, we deduce that

\[\mathcal{A}_2 = - \int_0^t \left( \kappa^{-1} (\nabla \cdot \mathbf{u}) - \kappa^{-1} (\nabla \cdot \mathbf{u}_h) \right) \Pi_h \mathbf{q}(s) \cdot \theta_q(s) \, ds\]

\[\leq \int_0^t \left\| \kappa^{-1} (\nabla \cdot \mathbf{u}) - \kappa^{-1} (\nabla \cdot \mathbf{u}_h) \right\|_0 \left\| \Pi_h \mathbf{q}(s) \right\|_0 \left\| \theta_q(s) \right\|_0 \, ds\]

\[\leq \frac{1}{4\epsilon_2} \int_0^t \left( \kappa^{-1} (\nabla \cdot \mathbf{u}) - \kappa^{-1} (\nabla \cdot \mathbf{u}_h) \right) \Pi_h \mathbf{q}(s) \cdot ds + \epsilon_2 \int_0^t \left\| \theta_q(s) \right\|_0^2 \, ds\]

\[\leq \frac{1}{4\epsilon_2} \int_0^t \left( \kappa^{-1} (\nabla \cdot \mathbf{u}) - \kappa^{-1} (\nabla \cdot \mathbf{u}_h) \right) \Pi_h \mathbf{q}(s) \cdot ds + \epsilon_2 \int_0^t \left\| \theta_q(s) \right\|_0^2 \, ds\]

\[\leq \frac{L C_1}{4\epsilon_2} \int_0^t \left\| \mathbf{u}(s) - \Pi_h \mathbf{u}(s) \right\|_0^2 \, ds + \epsilon_2 \int_0^t \left\| \theta_q(s) \right\|_0^2 \, ds\]

\[\leq \frac{L C_1 C_2^2}{2\epsilon_2} \int_0^t \left\| \mathbf{u}(s) - \Pi_h \mathbf{u}(s) \right\|_0^2 \, ds + \frac{L C_1 C_2^2}{2\epsilon_2} \int_0^t \left\| \Pi_h \mathbf{u}(s) - \mathbf{u}_h(s) \right\|_0^2 \, ds + \epsilon_2 \int_0^t \left\| \theta_q(s) \right\|_0^2 \, ds\]

\[= \frac{L C_1 C_2^2}{2\epsilon_2} \int_0^t \left\| \mathbf{u}(s) - \Pi_h \mathbf{u}(s) \right\|_0^2 \, ds + \frac{L C_1 C_2^2}{2\epsilon_2} \int_0^t \left\| \Pi_h \mathbf{u}(s) - \mathbf{u}_h(s) \right\|_0^2 \, ds + \epsilon_2 \int_0^t \left\| \theta_q(s) \right\|_0^2 \, ds\]

where $\epsilon_2$ is an arbitrarily small number and $C_q = \max_{\theta \in \Theta_\Omega} \left\| \Pi_h \mathbf{q}(s) \right\|_{\infty}$.

To estimate the third term $\mathcal{A}_3$, integrating by parts and noting that $\theta_u(0) = 0$, we first obtain

\[\mathcal{A}_3 = \int_0^t a_h(\partial_t \xi_u(s), \theta_u(s)) \, ds - a_h(\xi_u(t), \theta_u(t)),\]

which combined with (34) and Young’s inequality implies that

\[\mathcal{A}_3 \leq C_a \int_0^t \left( \left\| \partial_t \xi_u(s) \right\|_h^2 + \left\| \theta_u(s) \right\|_h^2 \right) \, ds + \frac{C_a}{4\epsilon_2} \left\| \xi_u(t) \right\|_h^2\]

\[+ \epsilon_2 \left\| \theta_u(t) \right\|_h^2,\]

where $\epsilon_3$ is an arbitrarily small number.

Combining the bounds in (41)–(46) and using (6) and (35), we have

\[\left( C_a + \frac{C_b}{2} \right) \left\| \theta_u(t) \right\|_h^2 + \frac{C_b}{2} \left\| \theta_p(t) \right\|_0^2 + (k_a - \epsilon_1 - \epsilon_2) \cdot \int_0^t \left\| \theta_q(s) \right\|_0^2 \, ds\]

\[\leq \frac{L C_1 C_2^2}{2\epsilon_2} \int_0^t \left\| \theta_u(s) \right\|_h^2 \, ds + \frac{L C_1 C_2^2}{2\epsilon_2} \int_0^t \left\| \Pi_h \mathbf{u}(s) - \mathbf{u}_h(s) \right\|_0^2 \, ds + \epsilon_2 \int_0^t \left\| \theta_q(s) \right\|_0^2 \, ds\]

\[\leq \frac{C_b}{2} \left( \epsilon_1 + \frac{C_b}{2} \right) \left\| \theta_u(t) \right\|_h^2 + \frac{C_b}{2} \left\| \theta_p(t) \right\|_0^2 + (k_a - \epsilon_1 - \epsilon_2) \cdot \int_0^t \left\| \theta_q(s) \right\|_0^2 \, ds\]

\[\leq \frac{C_b}{2} \left( \epsilon_1 + \frac{C_b}{2} \right) \left\| \theta_u(t) \right\|_h^2 + \frac{C_b}{2} \left\| \theta_p(t) \right\|_0^2 + \epsilon_2 \int_0^t \left\| \theta_q(s) \right\|_0^2 \, ds\]

Now, we choose appropriate $\epsilon_i (i = 1, 2, 3)$ such that $C_{\min} = \min \{ (C_a/2) - \epsilon_1, (C_b/2), k_a - \epsilon_1 - \epsilon_2 \} > 0$. Consequently, the above inequality still holds if we replace its left-hand side term by $C_{\min} \left( \left\| \theta_u(t) \right\|_h^2 + \left\| \theta_p(t) \right\|_0^2 + \int_0^t \left\| \theta_q(s) \right\|_0^2 \, ds \right)$.
Then, dividing both sides of the above inequality by $C_{\min}$ and using Gronwall’s inequality, we have
\[
\|\theta_n(t)\|_h^2 + \|\theta_p(t)\|_h^2 + \int_0^t \|\theta_q(s)\|_h^2 ds \\
\leq C\left(\int_0^t \left(\|\xi_q(s)\|_h^2 + \|\partial_t \xi_n(s)\|_h^2 + \|\xi_u(s)\|_h^2\right) ds + \|\xi_u(t)\|_h^2\right).
\]

Noting that the above estimate holds for all $0 \leq t \leq T$ and using appropriate approximation properties stated in (16), (18), and (19), we obtain
\[
\sup_{0 \leq s \leq T} \|\theta_n(s)\|_h^2 + \sup_{0 \leq s \leq T} \|\theta_p(s)\|_h^2 + \int_0^T \|\theta_q(s)\|_h^2 ds \\
\leq C(h^{2k} \left(\int_0^T \|q(s)\|_h^2 + \|\partial_t u(s)\|_{h+1}^2 + \|u(s)\|_{h+1}^2 \right) + h^{2k} \sup_{0 \leq s \leq T} \|u(s)\|_{h+1}^2).
\]

This can be stated by the following equivalent formulation:
\[
\|\theta_n(s)\|_{L^2(0,T;L^p)}^2 + \|\theta_p(s)\|_{L^2(0,T;L^2)}^2 + \|\theta_q(s)\|_{L^2(0,T;L^2)}^2 \\
\leq C(h^{4k} \left(\int_0^T \|q(s)\|_h^2 + \|\partial_t u(s)\|_{h+1}^2 + \|u(s)\|_{h+1}^2 \right) + \sup_{0 \leq s \leq T} \|u(s)\|_{h+1}^2).
\]

This together with the standard interpolation error estimates for $\phi_n$, $\xi_n$, and $\xi_u$ and the triangle inequality yields the desired estimate (36). \qed

4. Error Estimates for the Fully Discrete Scheme

4.1. Fully Discrete Finite Element Method. Let $N$ be a positive integer and set $\Delta t = T/N$ and $t^n = n\Delta t$ ($1 \leq n \leq N$). For the function $f(t,x)$, set $f^n = f(x,t^n)$, $\forall x \in \Omega$. The fully discrete mixed element method with back Euler time stepping reads as follows: at each time $t = t^n$, find $(p^n_h, q^n_h, u^n_h) \in \mathcal{P}_h \times \mathcal{Q}_h \times \mathcal{V}_h$ such that
\[
c_0(\partial_t p^n_h, w) + a(\nabla \cdot \partial_t u^n_h, w) + (\nabla \cdot q^n, w) = (g^n, w),
\]
\[
(\kappa^{-1}(\nabla \cdot u^n)q^n, z) - (p^n, \nabla \cdot z) = 0,
\]
\[
a_h(u^n, v) - a(p^n, \nabla \cdot v) = (f^n, v),
\]
for any $(w, z, v) \in \mathcal{P}_h \times \mathcal{Q}_h \times \mathcal{V}_h$, where $\partial_t p^n_h = p^n - p^n_{h-1}/\Delta t$ and $\partial_t u^n_h = u^n_{h-1} - u^n_h/\Delta t$ with initial condition
\[
\begin{align*}
p^n_0 &= P_h p_0, \\
u^n_0 &= \Pi_h u_0.
\end{align*}
\]

4.2. Error Estimates. The standard Taylor expansion leads to
\[
\frac{f^n - f^{n-1}}{\Delta t} = \partial_t f^n + \frac{1}{\Delta t} \int_{t^{n-1}}^{t^n} (t^n - s) \partial_{tt} f(s) ds,
\]
where $\partial_t f^n = \partial_t f(t^n)$.

Now, we prove the following error estimate which is the main result of this section.

**Theorem 2.** Let $(p, q, u) \in \mathcal{P} \times \mathcal{Q} \times \mathcal{V}$ and $(p^n_h, q^n_h, u^n_h) \in \mathcal{P}_h \times \mathcal{Q}_h \times \mathcal{V}_h$ be the solutions of (4) and (51), respectively. Then, the following finite element error estimate holds:
\[
\max_{1 \leq n \leq N} \|u^n - u^n_h\|_0 + \max_{1 \leq n \leq N} \|p^n - p^n_h\|_0 \\
+ \Delta t \sum_{n=1}^N \|q^n - q^n_h\|_0 \leq C(h^k + \Delta t),
\]
where $C$ depends on the regularity of $u$, $u_0$, $u_1$, $p$, $p_1$, $p_{tt}$, and $q$.

**Proof.** Since (30) holds for the exact solution $u$ at any time $t = t^n$, this together with Taylor expansion (53) yields
\[
\begin{align*}
c_0(\partial_{tt} p^n_h, w) + a(\nabla \cdot \partial_t u^n_h, w) + (\nabla \cdot q^n, w) &= (g^n, w) + c_0 \left(\int_{t^{n-1}}^{t^n} (t^n - s) \partial_{tt} p(s) ds, w\right), \\
(\kappa^{-1}(\nabla \cdot u^n)q^n, z) - (p^n, \nabla \cdot z) &= 0, \\
a_h(u^n, v) - a(p^n, \nabla \cdot v) &= (f^n, v),
\end{align*}
\]
for every \((w, z, v) \in \mathcal{P}_h \times \mathbf{Q}_h \times \mathbf{V}_h\).

Subtracting (51) from (55), we obtain

\[
\frac{c_0}{\Delta t} \left( \frac{p^n - p^n_h}{\Delta t} - (p^{n-1} - p_{h_1}^{n-1}) \right), w + \alpha \left( \nabla \cdot \left( \frac{u^n - u^n_h}{\Delta t} - (u^{n-1} - u_{h_1}^{n-1}) \right), w \right) + (\nabla \cdot (q^n - q^n_h), w)
\]

\[
= \frac{c_0}{\Delta t} \left( \int_{t_{n-1}}^t (t^{n-1} - s) \partial_t p(s) \text{ds}, w \right) + \frac{\alpha}{\Delta t} \left( \int_{t_{n-1}}^t (t^{n-1} - s) \nabla \cdot \partial_t u(s) \text{ds}, w \right),
\]

\[
\left( \kappa^{-1} (\nabla \cdot u^n) q^n - \kappa^{-1} (\nabla \cdot u^n_0) q^n_0, z \right) - (p^n - p^n_h, \nabla \cdot z) = 0,
\]

\[
a_h(u^n - u^n_h, v) - \alpha (p^n - p^n_h, \nabla \cdot v) = 0,
\]

for every \((w, z, v) \in \mathcal{P}_h \times \mathbf{Q}_h \times \mathbf{V}_h\).

We then split the error \(p^n - p^n_h = \xi^n_p + \theta^n_p\), with \(\xi^n_p = p^n - P_h p^n\) and \(\theta^n_p = P_h p^n - p^n_h\). Similarly, \(q^n - q^n_h = \xi^n_q + \theta^n_q\) with \(\xi^n_q = q^n - P_h q^n\) and \(\theta^n_q = P_h q^n - q^n_h\). \(u^n - u^n_h = \Xi^n_u + \theta^n_u\) with \(\Xi^n_u = u^n - \Pi_h u^n\) and \(\theta^n_u = \Pi_h u^n - u^n_h\). Since the estimates for \(\xi^n_p, \xi^n_q, \text{ and } \Xi^n_u\) can be derived by the interpolation error bounds, it leaves us to estimate \(\theta^n_p, \theta^n_q, \text{ and } \theta^n_u\). To this end, using (15) and (17), we can rewrite (56) by

\[
\frac{c_0}{\Delta t} \left( \frac{\theta^n_p - \theta^{n-1}_p}{\Delta t}, w \right) + \alpha \left( \nabla \cdot \left( \frac{\theta^n_q - \theta^{n-1}_q}{\Delta t} \right), w \right) + (\nabla \cdot \theta^n_q, w)
\]

\[
= \frac{c_0}{\Delta t} \left( \int_{t_{n-1}}^t (t^{n-1} - s) \partial_t p(s) \text{ds}, w \right) + \frac{\alpha}{\Delta t} \left( \int_{t_{n-1}}^t (t^{n-1} - s) \nabla \cdot \partial_t u(s) \text{ds}, w \right),
\]

\[
\left( \kappa^{-1} (\nabla \cdot u^n) q^n - \kappa^{-1} (\nabla \cdot u^n_0) q^n_0, z \right) - (\theta^n_p, \nabla \cdot z) = 0,
\]

\[
a_h(\theta^n_q, v) - \alpha (\theta^n_p, \nabla \cdot v) = -a_h(\xi^n_u, v),
\]

for every \((w, z, v) \in \mathcal{P}_h \times \mathbf{Q}_h \times \mathbf{V}_h\).

Setting \(w = \theta^n_p, z = \theta^n_q, \text{ and } v = (\theta^n_u - \theta^{n-1}_u)/\Delta t\) in the above equations, adding these equations, and using

\[
\kappa^{-1} (\nabla \cdot u^n) q^n - \kappa^{-1} (\nabla \cdot u^n_0) q^n_0 = \kappa^{-1} (\nabla \cdot u^n) (q^n - \Pi_h q^n)
\]

\[
+ \left( \kappa^{-1} (\nabla \cdot u^n) - \kappa^{-1} (\nabla \cdot u^n_0) \right) \Pi_h q^n
\]

\[
+ \kappa^{-1} (\nabla \cdot u^n_0) (\Pi_h q^n - q^n_0),
\]

we infer that

\[
a_h(\theta^n_q, \theta^n_u) + c_0 \left\| \theta^n_q \right\|^2 + \Delta t \left( \kappa^{-1} (\nabla \cdot u^n) \theta^n_q, \theta^n_q \right)
\]

\[
= a_h(\theta^n_u, \theta^n_u) + c_0 \left( \theta^n_u, \theta^n_u \right) + \Delta t \left( \kappa^{-1} (\nabla \cdot u^n) \theta^n_q, \theta^n_q \right)
\]

\[
+ c_0 \left( \int_{t_{n-1}}^t (t^{n-1} - s) \partial_t p(s) \text{ds}, \theta^n_p \right)
\]

\[
+ \alpha \left( \int_{t_{n-1}}^t (t^{n-1} - s) \nabla \cdot \partial_t u(s) \text{ds}, \theta^n_p \right)
\]

\[
- \Delta t \left( \kappa^{-1} (\nabla \cdot u^n_0) \xi^n_u, \theta^n_q \right)
\]

\[
- \Delta t \left( \kappa^{-1} (\nabla \cdot u^n) - \kappa^{-1} (\nabla \cdot u^n_0) \Pi_h q^n, \theta^n_q \right)
\]

\[
- a_h(\xi^n_u, \theta^n_u - \theta^{n-1}_u) .
\]
To give bounds for the above error equation, we introduce the following useful inequalities:

\[ a_h(\theta^{(n)}_p, \theta^{(n-1)}_p) \leq \frac{1}{2} \left( a_h(\theta^{(n)}_p, \theta^{(n-1)}_u) + a_h(\theta^{(n)}_u, \theta^{(n)}_p) \right), \]  \hspace{1cm} (60)

\[ c_0(\theta^{(n)}_p, \theta^{(n)}_u) \leq \frac{c_0}{2} \left( \|\theta^{(n)}_p\|_d^2 + \|\theta^{(n)}_u\|_d^2 \right). \]  \hspace{1cm} (61)

Summing (59) from 1 to \( m \leq N \), applying (60) and (61), and noting that \( \theta^{(0)}_u = 0 \) and \( \theta^{(0)}_p = 0 \), we obtain

\[ \frac{1}{2} \left( a_h(\theta^{(n)}_u, \theta^{(n)}_m) + c_0\|\theta^{(n)}_m\|_0^2 \right) + \Delta t \sum_{n=1}^m \left( \kappa^{-1}(\nabla \cdot \mathbf{u})\theta^{(n)}_q, \theta^{(n)}_q \right) \leq B_1 + B_2 + B_3 + B_4 + B_5, \]  \hspace{1cm} (62)

where

\[ B_1 = c_0 \sum_{n=1}^m \left( \int_{p=1}^{p=n} (t^{(n-1)} - s) \partial_u P(s) ds \right), \]

\[ B_2 = \alpha \sum_{n=1}^m \left( \int_{p=1}^{p=n} (t^{(n-1)} - s) \nabla \cdot \partial_u \mathbf{u}(s) ds \right), \]

\[ B_3 = -\sum_{n=1}^m \Delta t \left( \kappa^{-1}(\nabla \cdot \mathbf{u}^m) \xi^{(n)}_q, \xi^{(n)}_q \right), \]

\[ B_4 = -\sum_{n=1}^m \Delta t \left( \kappa^{-1}(\nabla \cdot \mathbf{u}^m) - \kappa^{-1}(\nabla \cdot \mathbf{u}^n) \right) \Pi_h \mathbf{q}^n, \mathbf{q}^n, \]

\[ B_5 = -\sum_{n=1}^m a_h(\xi^{(n)}_u, \xi^{(n)}_u - \xi^{(n-1)}_u). \]

The first term \( B_1 \) can be bounded by

\[ B_1 = c_0 \sum_{n=1}^m \left( \int_{p=1}^{p=n} (t^{(n-1)} - s) \partial_u P(s) ds \right), \]

\[ \leq c_0 \sum_{n=1}^m \left( \int_{p=1}^{p=n} (t^{(n-1)} - s) \partial_u P(s) ds \right) \|\theta^{(n)}_p\|_0^2. \]  \hspace{1cm} (64)

Since

\[ \left\| \int_{p=1}^{p=n} (t^{(n-1)} - s) \rho_u(s) ds \right\|_0 \leq \left( \Delta t \right)^{1/2} \left( \int_{p=1}^{p=n} \|\partial_u P(s)\|^2 ds \right)^{1/2}, \]  \hspace{1cm} (65)

then we further obtain

\[ B_1 \leq C \left( \Delta t \sum_{n=1}^m \|\theta^{(n)}_p\|_0^2 + (\Delta t)^2 \int_{p=1}^{p=m} \|\partial_u P(s)\|^2 ds \right). \]  \hspace{1cm} (66)

Similarly, the second term \( B_2 \) can be estimated by

\[ B_2 \leq C \left( \Delta t \sum_{n=1}^m \|\partial_u P\|_0^2 + (\Delta t)^2 \int_{p=1}^{p=m} \|\nabla \cdot \partial_u \mathbf{u}(s)\|^2_0 ds \right). \]  \hspace{1cm} (67)

Combining (6) and Cauchy–Schwarz and Young’s inequalities, we have

\[ B_3 = -\sum_{n=1}^m \Delta t \left( \kappa^{-1}(\nabla \cdot \mathbf{u}^m) \xi^{(n)}_q, \xi^{(n)}_q \right) \]

\[ \leq \sum_{n=1}^m \Delta t \|\xi^{(n)}_q\|_0 \|\kappa^{-1}(\nabla \cdot \mathbf{u}^m) \xi^{(n)}_q\|_0 \]

\[ \leq \varepsilon_4 \Delta t \sum_{n=1}^m \|\xi^{(n)}_q\|_0^2 + \frac{1}{4\varepsilon_4} \Delta t \sum_{n=1}^m \|\kappa^{-1}(\nabla \cdot \mathbf{u}^m) \xi^{(n)}_q\|_0^2 \]

\[ \leq \varepsilon_4 \Delta t \sum_{n=1}^m \|\xi^{(n)}_q\|_0^2 + \frac{\kappa^2}{4\varepsilon_4} \Delta t \sum_{n=1}^m \|\xi^{(n)}_q\|_0^2, \]  \hspace{1cm} (68)

where \( \varepsilon_4 \) is an arbitrarily small number.

Similarly, we can bound \( B_4 \) by

\[ B_4 = -\sum_{n=1}^m \Delta t \left( \kappa^{-1}(\nabla \cdot \mathbf{u}) - \kappa^{-1}(\nabla \cdot \mathbf{u}^n) \right) \Pi_h \mathbf{q}^n, \mathbf{q}^n \]

\[ \leq \sum_{n=1}^m \Delta t \|\kappa^{-1}(\nabla \cdot \mathbf{u}) - \kappa^{-1}(\nabla \cdot \mathbf{u}^n)\|_0 \|\Pi_h \mathbf{q}^n\|_0 \]

\[ \leq \frac{1}{4\varepsilon_5} \Delta t \sum_{n=1}^m \|\kappa^{-1}(\nabla \cdot \mathbf{u}) - \kappa^{-1}(\nabla \cdot \mathbf{u}^n)\|_0^2 + \varepsilon_5 \Delta t \sum_{n=1}^m \|\xi^{(n)}_q\|_0^2 \]

\[ \leq \frac{1}{4\varepsilon_5} \Delta t \sum_{n=1}^m \|\kappa^{-1}(\nabla \cdot \mathbf{u}) - \kappa^{-1}(\nabla \cdot \mathbf{u}^n)\|_0^2 + \varepsilon_5 \Delta t \sum_{n=1}^m \|\xi^{(n)}_q\|_0^2 \]

\[ \leq \frac{1}{4\varepsilon_5} \Delta t \sum_{n=1}^m \|\kappa^{-1}(\nabla \cdot \mathbf{u}) - \kappa^{-1}(\nabla \cdot \mathbf{u}^n)\|_0^2 + \varepsilon_5 \Delta t \sum_{n=1}^m \|\xi^{(n)}_q\|_0^2 \]

\[ \leq \frac{1}{4\varepsilon_5} \Delta t \sum_{n=1}^m \|\kappa^{-1}(\nabla \cdot \mathbf{u}) - \kappa^{-1}(\nabla \cdot \mathbf{u}^n)\|_0^2 + \varepsilon_5 \Delta t \sum_{n=1}^m \|\xi^{(n)}_q\|_0^2 \]

\[ \leq \frac{1}{4\varepsilon_5} \Delta t \sum_{n=1}^m \|\kappa^{-1}(\nabla \cdot \mathbf{u}) - \kappa^{-1}(\nabla \cdot \mathbf{u}^n)\|_0^2 + \varepsilon_5 \Delta t \sum_{n=1}^m \|\xi^{(n)}_q\|_0^2, \]  \hspace{1cm} (69)

where \( \varepsilon_5 \) is an arbitrarily small number and

\[ C_m = \max_{1 \leq m \leq n} \|\Pi_h \mathbf{q}^n\|_{\infty}. \]
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5. Conclusion and Future Work

Combining the above estimate with the interpolation error estimates for $\tilde{\varphi}_h$ and $\tilde{\varphi}$ and using the triangle inequality, we obtain the desired assertion in (64).

Choose small enough $\varepsilon (\varepsilon = 4.5, 6)$ to make discrete Gronwall inequality and some approximation hold for any $1 \leq m \leq N'$.

Combining the bounds above and using (6) and (3), we have
\[
\sum_{m=1}^{\infty} \left( \frac{1}{m} \right)^2 = \sum_{m=1}^{\infty} \frac{1}{m^2} < \infty.
\]

To bound the last term $B_5$, we need the following inequalities:
\[
B_5 = \sum_{m=1}^{\infty} \left( \frac{1}{m} \right)^2 < \infty.
\]

Here and in the following, $a_k^{-1}, a_k^{-1}, \xi_k, \eta_k$ and Young's inequality hold.

Combining the bounds above and using (6) and (3), we have
\[
\sum_{m=1}^{\infty} \left( \frac{1}{m} \right)^2 = \sum_{m=1}^{\infty} \frac{1}{m^2} < \infty.
\]
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