An exact expression for the image error in a catadioptric sensor

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Abstract

A catadioptric sensor induces a projection between a given object surface and an image plane. The prescribed projection problem is the problem of finding a catadioptric sensor that realizes a given projection. Here we present a functional that describes the image error induced by a given mirror when compared with a given projection. This expression can be minimized to find solutions to the prescribed projection problem. We present an example of this approach, by finding the optimum spherical mirror that serves as a passenger side mirror on a motor vehicle.
1 Introduction

A catadioptric sensor generally consists of a camera pointed at a convex mirror. Catadioptric sensors of this type tend to have large fields of view, and hence their most common application is panoramic imaging.

We will assume a simple model of the dioptic component, known as the pinhole model, which realizes the perspective projection. A limiting case of perspective projection is the orthographic projection, in which the rays that impinge upon the image plane are parallel. Such systems can be thought of as very narrow field perspective devices.

Almost all work on catadioptric sensor design refers to sensors employing rotationally symmetric mirrors, since these mirrors are the simplest to make and to mathematically model. The design of such mirrors reduces to solving an ordinary differential equation.

The earliest example of a camera employing a mirror is due to A.S. Wolcott [15] and appears in the 1840 patent “Method of taking Likenesses by Means of a Concave Reflector and Plates so Prepared as that Luminous or other Rays with act Thereon.” Remarkably, this device, which is designed for use with a daguerreotype plate, appears just a few years after the invention of photography. Since this patent, an enormous number of catadioptric cameras and projection devices have appeared, many of which are documented on the webpage [11], which is a historical resource on catadioptric sensor design created by the author. The first use of differential methods for the design of a mirror shape in a catadioptric sensor appears in the 1945 patent of Benford [2]. Other works that make use of differential methods include [3], [4], [1], [8], [12], [5], [6], [10], [7], [13]. Early applications to robotics include [16] and [17]. A heuristic approach to image error is discussed in [14].

2 Statement of the Prescribed Projection Problem

In this section we state the prescribed projection problem, which is our fundamental problem of interest. Suppose one is given a fixed surface, \( S \), in \( \mathbb{R}^3 \), which we will call the object surface and a camera with image plane \( I \), also fixed in \( \mathbb{R}^3 \). A given mirrored surface \( M \) induces a transformation \( T_M \) from some subset of \( I \) to \( S \) by following a ray (determined by the camera model) from a point \( q \in I \) until it intersects the mirror at a point \( r \). The ray is then reflected according to the usual law that the angle of incidence is equal the angle of reflection and intersects \( S \) at a point \( s \). We then define \( T_M(q) = s \).

The prescribed projection problem for systems containing a single mirror is:

Given \( G : I \rightarrow S \), find \( M \) such that \( T_M = G \). If no such \( M \) exists, then find \( M \) such that \( T_M \) is a good approximation to \( G \).

We will refer to \( G(q) \) as the target point. If an exact solution to the problem exists, then there are several ways to calculate it. Otherwise, there are numerous ways to formulate and solve the approximation problem.

Notice that for a given \( M \), with \( q, r \) and \( s \) as above, that the vector \( \frac{q-r}{|q-r|} + \frac{s-r}{|s-r|} \) is normal to \( M \) at \( r \). This suggests a method of constructing a vector field \( W \) on \( \mathbb{R}^3 \) that will be normal to the solution: for each \( r \in \mathbb{R}^3 \) lying on a ray that enters the camera, define

\[
W(r) = \frac{q(r) - r}{|q(r) - r|} + G(q(r)) - r \quad \frac{G(q(r)) - r}{|G(q(r)) - r|}
\]

where \( q(r) \) is the projection of \( r \) to \( I \) along the ray. We refer to this construction as the vector field method. Thus our problem is solved if we find a surface whose gradient is parallel to \( W \).
3 Image Error and Projection Error

A disadvantage of some of previous approaches to the problem is that they do not directly address the error in the image, i.e., the goal should be to minimize the distortion error in the resulting image.

Given the notation that $T_M$ is the transformation induced from the image plane to the object surface by the mirror $M$, then the goal is to find a solution to the equations

$$T_M(x) = G(x)$$

which is a system of partial differential equations that is generally inconsistent (an example of such a system is equations [8] and [9]). We then define the projection error induced by a mirror $M$ as

$$P_e(M) = \frac{1}{\text{Area}(U)} \int_U \left| T_M(x) - G(x) \right|^2 dA,$$

where $U$ is the domain in the image plane over which the surface $M$ is a graph. For example, in the approach to the blindspot problem described below, the projection error of a mirror described as a graph $x = f(y, z)$ over $[-1,1] \times [1,1]$ is

$$\frac{1}{4} \int_{-1}^{1} \int_{-1}^{1} \left( \frac{(1 - f_y^2 - f_z^2)}{2f_y} (y + k) + f(y, z) - \alpha y \right)^2 + \left( \frac{(y + k) f_z}{f_y} + z - \alpha z \right)^2 dydz,$$

where the prescribed transformation $G$ is $[x_0, y, z] \mapsto [\alpha y, -k, \alpha z]$. (Here we are not taking the surface to be at infinity.) For example, if $\alpha = 1$ then the mirror $x = y$ makes the projection error functional 0.

The projection error compares the image formed by projecting the domain $U$ from the image plane to the object surface, whereas we are interested in the error formed by the projection from the object plane to the image plane via $M$. Thus we define the image error as the quantity

$$I_e(M) = \frac{1}{\text{Area}(V)} \int_V \left| G^{-1}(y) - T_M^{-1}(y) \right|^2 dA,$$

where $V$ is the image of $U$ under $T_M$. In this form it is not possible to directly minimize the image error because computing $T_M^{-1}$ is intractable. While it is possible to compute the projection for a generic $M$, it does not appear possible to compute $T_M^{-1}$ because the structure of the bundle of rays from the object plane to $M$ is unknown until $M$ is given. Nevertheless, performing a change of variables on the integral [10] by taking $y = T_M(x)$ gives

$$I_e = \frac{1}{\text{Area}(V)} \int_V \left| G^{-1}(y) - T_M^{-1}(y) \right|^2 dA = \frac{1}{\text{Area}(V)} \int_U \left| G^{-1}(T_M(x)) - x \right|^2 \left| \det(dT_M(x)) \right| dA$$

This functional is amenable to numerical minimization. One great advantage of this functional is that any solution $M$ derived by any one of the previous methods may be improved by minimizing $I_e$ with $M$ as an initial condition.

4 An application to design

The main purpose of deriving an expression for the image error is so that it can be minimized over some appropriate family of surfaces and hence provide an answer to the prescribed projection problem.
In [9], the authors consider the equations for a sideview mirror. It is shown that the projection $T_M$ induced by a surface $x = f(y, z)$ viewed orthographically along the x-axis (the image plane is $x = x_0$) to the plane $y = -k, k > 0$ is

$$[x_0, y, z] \rightarrow [g_1(y, z), -k, g_2(y, z)], \quad (7)$$

where

$$g_1 = \frac{(1 - f_y^2 - f_z^2)(y + k)}{2f_y} + f(y, z) \quad (8)$$

$$g_2 = -\frac{(y + k)f_z}{f_y} + z. \quad (9)$$

The desired projection, $G$ is

$$[x_0, y, z] \rightarrow [\alpha y, -k, \alpha z] \quad (10)$$

Since $T_M$ and $G$ are known, we may minimize the image error over some class of surfaces. Ideally one would perform this minimization over a large space, such as polynomials, trigonometric functions, or spline functions. Here we answer the question “What is the best spherical sideview mirror ?”.

We consider a spherical mirror which goes through the origin and with a center in the plane $z = 0$, which has the general form

$$f(y, z) = -a + \sqrt{a^2 - z^2 - y^2 + 2yb}. \quad (11)$$

Thus the parameters that are free for minimization are $a$ and $b$. For this problem, if the required field of view is 45 degrees (this determines $\alpha$ and assuming that the image plane is a unit square of side length 2, the optimal result, (using a gradient descendent algorithm) is

$$a \sim 2.83 \quad (12)$$

$$b \sim 2.38. \quad (13)$$

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