Fundamental Scale Invariance

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Quantum field theories with fundamental scale invariance admit a scale-free formulation of the functional integral and effective action in terms of scale invariant fields. They correspond to exact scaling solutions of functional flow equations. Such theories are highly predictive since all relevant parameters for deviations from the exact scaling solution vanish. The non-linear restrictions for scaling solutions can explain properties that would seem to need fine tuning of parameters from a perturbative viewpoint.

The understanding of scale symmetry and its possible breaking is a central issue in quantum field theories. It has been discussed in the context of an understanding of the gauge hierarchy in particle physics [1–7], the cosmological constant [8, 9] or inflation [10–16]. Quantum scale symmetry [17] is a powerful symmetry that can render a model highly predictive. In this note we are motivated by three facts for which we develop a unified view.

1. A quantum scale symmetric standard model has been proposed in refs. [9, 18]. For such a model the quantum effective action does not contain any intrinsic parameter with dimension of length or mass. This distinguishes quantum scale symmetry from models with classical symmetry [19–24]. For a scale invariant classical action (classical scale symmetry) the running of couplings due to quantum fluctuations typically introduces explicit mass scales and violates scale symmetry. The quantum scale invariant standard model introduces an additional scalar singlet $\chi$. The running of couplings occurs now as functions of $q^2/\chi^2$ or $\hbar^2/\chi^2$, with $q^2$ the squared momentum and $\hbar$ the field for the Higgs doublet. Dimensionless ratios involving intrinsic mass scales are replaced by ratios of field values or ratios between momenta and fields. No intrinsic mass or length is present. This points to a fundamental theory without scales, where the running of effective couplings arises through their dependence on fields.

2. Scale symmetry is spontaneously broken whenever a scalar field takes a non-zero value. This is the case for the Higgs doublet or for the scalar field $\chi$ that replaces the Planck mass. In case of spontaneously broken scale symmetry one expects the presence of an exactly massless Goldstone boson. The proposal of dynamical dark energy [9] is based on a small dilatation anomaly. A scale invariant coupling to gravity replaces the Planck mass $M$ by a scalar field $\chi$, such that the curvature scalar $R$ appears in the effective action in the form

$$\Gamma = \int x \sqrt{g} \left\{ -\frac{1}{2} \chi^2 R + ck^4 \right\}. \quad (1)$$

For $c \neq 0$ scale symmetry is explicitly broken by the scale $k$ which has dimension of mass. The effective cosmological constant corresponds to the dimensionless ratio of scalar potential over the fourth power of the Planck mass, which is given for eq. (1) by

$$\lambda = \frac{ck^4}{\chi^4}. \quad (2)$$

For cosmological solutions $\chi$ is found to increase without bounds such that the cosmological constant vanishes asymptotically in the infinite future. At present, the Universe is old, but not infinitely old. The scalar field $\chi$ still has a finite value at the present time $t_0$, for which we may use the units $\chi(t_0) = M$. For a large ratio $\chi(t_0)/k$ one expects a small amount of dark energy, which is dynamical since $\chi$ increases with time. This early prediction of dynamical dark energy [9] seems to point towards a small explicit breaking of quantum scale symmetry by the scale $k$, which is of the order $10^{-3}$ eV for units with $\chi(t_0) = M$. The Goldstone boson becomes a pseudo-Goldstone boson – the cosmon. This is the almost massless field of dynamical dark energy. (In scale invariant unimodular gravity the term $ck^4$ arises as an integration constant rather than as an intrinsic parameter [10]. Since the predictions for observations are identical to the explicit breaking in the effective action (1), one finds again a pseudo-Goldstone boson. In view of the absence of an exact Goldstone boson the interpretation as a spontaneously broken exact global scale symmetry is not clear to us.)

3. Scaling solutions for flow equations [25–27] have been investigated for dilaton quantum gravity [28, 29]. This generalizes the effective action (1),

$$\Gamma = \int x \sqrt{g} \left\{ -\frac{1}{2} F(\chi)R + U(\chi) + \frac{1}{2} K(\chi)\partial^\mu \chi \partial_\mu \chi \right\}, \quad (3)$$

where the three functions $F$, $U$ and $K$ flow with $k$. The candidate scaling solutions found show indeed for large $\chi$ the behavior of the effective action (1). A general investigation of scaling solutions for effective potentials [30] finds scaling potentials that approach constants for large field values, with a limit (1). The scaling potentials depend on dimensionless field ratios as $\tilde{\rho} = \chi^2/(2k^2)$. They therefore involve a scale $k$. It has been observed [29, 30] that the scale $k$ disappears when the model is transformed by a Weyl scaling of the metric to the Einstein frame. This suggests that $k$ may actually not play the role of an intrinsic parameter with dimension of mass. Combined with a scale invariant standard model, for which all mass scales
are proportional to $\chi$, the quantum field equations derived from the effective action (3) are the ones of variable gravity [13]. Rather realistic cosmologies are obtained in this context [14, 16]. Thus the scaling solutions of flow equations may result in an acceptable cosmology and particle physics, without any need that the flow deviates from the scaling solutions due to some relevant parameters.

In the present note we develop a coherent picture of these three facets. They seem at first sight a bit contradictory. The quantum scale invariant standard model points towards exact scale symmetry, while dynamical dark energy seems to suggest a breaking by an intrinsic scale $k$. One point of view observes that $k \approx 10^{-3}$ eV is much smaller than the scales relevant for the standard model, such that the tiny scale anomaly is actually negligible except for the scales of present cosmology. For all other scales one may think that the small scale anomaly $\sim k^4$ plays no role, rendering models with $k \neq 0$ indistinguishable from models with exact scale symmetry and $k = 0$. This argument does not hold, however, for very early cosmology, as the inflationary epoch. For cosmon inflation [12–14, 16] the cosmon field $\chi$ is smaller than $k$ for the very early epochs of inflation. It is the scale $k$ that triggers the end of inflation once $\chi$ increases sufficiently beyond $k$. The same scale is also responsible for the small deviations from scale invariance of the primordial fluctuation spectrum. Thus again the scale $k$ plays a useful role. In this note we will develop a deeper view for which the scale $k$ is actually present, but does not correspond to an intrinsic parameter violating scale invariance explicitly.

We propose that a fundamental theory has no scale. More precisely, the quantum field theory does not involve any intrinsic parameter with dimension mass or length. The fields for the most basic constituents are dimensionless. Fields depend on spacetime coordinates, and one may decide to introduce a unit of length for distances between spacetime points. Correspondingly, derivatives of fields with respect to the spacetime coordinates or momenta carry dimension of inverse length or mass. (We use units $\hbar = c = 1$.) A metric field may arise as a composite or collective field constructed from derivatives of fundamental fields $\psi$

$$\tilde{g}_{\mu\nu} \sim f(\tilde{\psi}) \partial_{\mu} \tilde{\psi} \partial_{\nu} \tilde{\psi},$$ (4)

where we do not spell out other possible indices or the form of $f$. This metric has therefore dimension mass squared.

Geometry is usually constructed with a dimensionless metric. For this purpose one introduces a scale $k$ with dimension mass,

$$g_{\mu\nu} = k^{-2} \tilde{g}_{\mu\nu}.$$ (5)

Scalars are either fundamental fields, or composites of fundamental fields. Involving no derivatives, scalar fields $\tilde{\chi}$ are dimensionless on a fundamental level. We may decide to use a different “canonical” normalization

$$\chi = k \tilde{\chi},$$ (6)

such that $\chi$ carries dimension of mass and a diffeomorphism invariant kinetic term can be canonical. Obviously, the scale $k$ has no physical meaning and is not an intrinsic parameter. It is introduced only for convenience. The model could be formulated in terms of fields as $\psi$, $\tilde{\chi}$, $\tilde{g}_{\mu\nu}$, for which the scale $k$ never appears. The quantum effective action, formulated in terms of fields as $\psi$, $\tilde{\chi}$, $\tilde{g}_{\mu\nu}$, is trivially independent of $k$. We call the fields $\psi$, $\tilde{\chi}$, $\tilde{g}_{\mu\nu}$ “scale invariant fields”, since they are associated to a formulation for which no scale appears. The rescaled fields as $g_{\mu\nu}$ or $\chi$ may be called “canonical fields”.

Expressed in terms of the canonical fields the effective action will generically depend on

$$k \partial_k \Gamma_k[\varphi] = \zeta_k[\varphi],$$ (7)

with $\varphi$ standing collectively for canonical fields as $g_{\mu\nu}$ and $\chi$. The flow generator $\zeta_k[\varphi]$ does not vanish, and the flow equation (7) describes the dependence of the effective action on the scale $k$. On the other hand, we know that for fixed scale invariant fields $\tilde{\varphi}$ the effective action does not involve $k$,

$$k \partial_k \Gamma_k[\tilde{\varphi}] = 0.$$ (8)

The solutions of eq. (7) therefore include a particular scaling solution for which eq. (8) holds once the canonical fields are expressed in terms of the scale invariant fields. It is precisely this scaling solution that defines the theory. It expresses the fact that the dependence on $k$ is introduced into the model only by a redefinition of fields (5), (6). So far these statements seem almost trivial, related to field rescalings rather than running couplings. We will see that they continue to hold for situations where the flow generator $\zeta_k$ describes the physical effects of running couplings.

Dilatation transformations or global scale transformations are rescalings of the canonical fields $\varphi$ at fixed $k$. The possible scale symmetry associated to these transformations can be violated for situations for which the variation of scale invariant fields $\tilde{\varphi}$ plays a role. The effective action $\Gamma_k[\varphi]$ is not independent of $k$, and not invariant under rescalings of canonical fields $\varphi$ at fixed $k$. It is only invariant under simultaneous rescalings of $\varphi$ and $k$. To be specific, let us consider scalar fields $\chi$ with scale invariant scalar fields $\tilde{\chi} = \chi/k$. The effective action is a functional of $\tilde{\chi}$,

$$\Gamma_k[\tilde{\chi}] = \Gamma_k[\chi/k].$$ (9)

It is invariant under the simultaneous rescaling $\chi \rightarrow \alpha \chi$, $k \rightarrow \alpha k$, but not under dilatations $\chi \rightarrow \alpha \chi$ at fixed $k$. An example is the effective potential for a real scalar field $\chi$,

$$\sqrt{g} U = \sqrt{\tilde{g}} u = \frac{1}{8} \sqrt{\tilde{g}} \lambda (\tilde{\chi}^2 - \kappa^2) = \frac{1}{8} \sqrt{\tilde{g}} \lambda \left( \frac{\chi}{k} \right) (\chi^2 - 2\kappa k^2 \chi^2 + \kappa^2 k^2).$$ (10)

The minimum occurs for $\chi^2 = \kappa k^2$, and the mass term $m^2 = \partial^2 U/\partial \chi^2$ does not vanish. There is no Goldstone boson despite the fact that no intrinsic scale is present. We have to distinguish between scale invariance, which means
the absence of intrinsic mass scales, and scale symmetry, which means invariance under rescaling of canonical fields at fixed $k$.

There are particular limits for which quantum scale symmetry becomes exact for scale invariant theories. For these limits the effective action becomes invariant under global scalings of the canonical fields. In particular, they concern the limiting behavior for $\tilde{\chi} \to 0$ or $\tilde{\chi} \to \infty$. If for $\tilde{\chi} \to 0$ the effective action $\Gamma[\tilde{\chi}]$ reaches a well defined limit, the dimensionless couplings $g(\tilde{\chi})$ reach limits $g_*$ that do no longer depend on $\tilde{\chi}$. At the same time, they do no depend on $\chi$ and on $k$. Their flow with $k$ stops – the couplings approach a fixed point. At fixed $\chi$ the limit $\tilde{\chi} \to 0$ corresponds to a diverging “renormalization scale” $k \to \infty$. This limit is an ultraviolet (UV) fixed point.

If in the limit $\tilde{\chi} \to \infty$ the effective action also reaches a well defined limit, dimensionless couplings become again independent of $\tilde{\chi}$. This corresponds to an infrared (IR) fixed point, since for fixed $\chi$ the renormalization scale $k$ reaches zero. The UV- and IR-fixed points are in the first instance fixed points in the dependence of couplings on the scale invariant field $\tilde{\chi}$. This translates to the independence of $k$ and global scale symmetry. For the IR-fixed point one has $\chi \to \infty$ at fixed $k$, such that the exact scale symmetry is spontaneously broken. Particles can be massive with masses $\sim \chi$. A massless Goldstone boson is predicted. For the UV-fixed point fixed $k$ corresponds to $\chi \to 0$. The global scale symmetry is not spontaneously broken and all particle masses go to zero. Cosmology can be described by a crossover [14, 17], where $\chi$ increases from zero in the infinite past to infinity in the infinite future. Inflation is the early period near the UV-fixed point, while the present cosmological epoch is already close to the IR-fixed point with very large $\tilde{\chi}$. The pseudo-Goldstone boson is the cosmon, which is responsible for dynamical dark energy. According to eq. (2), the present dark energy density in units of the Planck mass is tiny, $\lambda \sim \tilde{\chi}^{-4}$, for large values of the dimensionless scale invariant field $\tilde{\chi}$.

Quantum field theories without scale

Let us consider some well-defined (regularized) quantum field theory involving dimensionless fields $\tilde{\sigma}_i(x)$. Here $x$ may be the sites of a discrete lattice or the space-time points of a continuous manifold. Examples are lattice gauge theories with $\tilde{\sigma}$ the link variables, or lattice spinor gravity [31] with Grassmann variables $\tilde{\sigma}$ describing “fundamental fermions”. A given quantum field theory is specified by a functional integral over the fields $\tilde{\sigma}$, with an action $S[\tilde{\sigma}]$ being a functional of these fields.

The quantum effective action $\Gamma[\tilde{\varphi}]$ is defined by a functional differential equation (“background field identity”)

$$\exp(-\Gamma[\tilde{\varphi}]) = \int \mathcal{D}\tilde{\chi} \exp \left\{ -S[\tilde{\varphi} + \tilde{\chi}] + \int_x \frac{\partial \Gamma[\tilde{\varphi}]}{\partial \tilde{\varphi}} \tilde{\chi} \right\}. \quad (11)$$

The effective action is a functional of the multicomponent macroscopic fields $\tilde{\varphi}_i(x)$, treated here as a vector $\tilde{\varphi}$. It involves the euclidean action $S[\tilde{\varphi} + \tilde{\chi}]$, which is a functional of the microscopic fields $\tilde{\sigma} = \tilde{\varphi} + \tilde{\chi}$. The functional integration is shifted to an integral over the fluctuation fields $\tilde{\chi}$. The first functional derivatives of $\Gamma$ are called sources

$$\tilde{J}_i(x) = \frac{\partial \Gamma}{\partial \tilde{\varphi}_i(x)}, \quad \tilde{J} = \frac{\partial \Gamma}{\partial \tilde{\varphi}}. \quad (12)$$

and $\int_x \tilde{J}_\chi$ is the scalar product of the source vector and the fluctuation vector. For fermions $\tilde{\varphi}$ and $\tilde{\chi}$ are Grassmann variables. For a continuum formulation of local gauge theories one adds a gauge fixing term and the associated Faddeev-Popov determinant or ghost term. A gauge invariant effective action can be obtained by a “physical gauge fixing” [32].

The effective action is the generating functional for the one-particle irreducible Green’s functions. All information relevant for observations can be extracted from its functional derivatives. The first derivative yields the field equations in the presence of the sources (12), and the second derivative $\Gamma^{(2)}$ defines the inverse propagator. Evaluating the propagator on a solution of the field equations yields the fluctuation spectrum. For example, the primordial fluctuation spectrum in inflationary cosmology can be directly extracted from $\Gamma^{(2)}$ [33].

Let us focus on a discretized theory, formulated on a lattice. We denote a typical distance between lattice points by $a$, and consider physical phenomena involving distances $l$ of many lattice points, $l/a \gg 1$. The units for $a$ or $l$ do not matter, what only counts is the ratio $l/a$. For example, we could choose $a = 1$ or define a length unit by some multiple of $a$. We are interested in the continuum limit $l/a \to \infty$. Theories with fundamental scale invariance are those for which the effective action $\Gamma[\tilde{\varphi}]$ remains well defined in the continuum limit, without any divergences for quantities relevant for observation. This is a highly non-trivial property. For the example of lattice-QCD for strong interactions this requirement is not met. On the other hand, any finite theory obeys this condition. We can keep $l$ fixed and move $a \to 0$. For a finite theory all quantities relevant for observation remain finite in this limit.

For the more general case the effective action may still remain well defined in the continuum limit if one employs renormalized fields $\varphi_{R,i}(x)$. They are related to the original scale invariant fields $\tilde{\varphi}_i(x)$ by use of a renormalization scale $k$,

$$\varphi_{R,i}(x) = k^{d_i} f_i(k) \tilde{\varphi}_i(x), \quad (13)$$

with $d_i$ defining the canonical dimensions of $\varphi_{R,i}(x)$ and $f_i(k)$ some possible dimensionless functions of $k$ which gives rise to so called anomalous dimensions. The renormalization scale $k$ has dimension of $a^{-1}$, typically mass or inverse length. Any non-constant $f_i(k)$ needs to involve some other mass scale, and we take $f_i(ka)$. For renormalizable theories $\Gamma[\varphi_{R}]$ remains finite in the continuum limit taken at fixed $\varphi_R$. The divergences at fixed $\tilde{\varphi}$ are then connected to the relation (13) between $\varphi_R$ and $\tilde{\varphi}$. This is particularly
apparent if we make the renormalized fields dimensionless by multiplication with an appropriate power of $a$,
\[
\hat{\phi}_{R,i}(x) = a^{d_i} \phi_{R,i}(x) = (ka)^{d_i} f_i(ka) \hat{\phi}_i(x).
\] (14)

Divergences can appear for $ka \to 0$. In addition, the existence of a continuum limit may require the selection of certain dimensionless parameters in the action $S[\tilde{\phi}]$. (This corresponds to ultraviolet fixed points of renormalized dimensionless couplings.)

Theories with fundamental scale invariance are renormalizable theories with the additional property that no renormalization scale $k$ needs to be introduced. This is the case for finite theories, but the class of theories with well-defined effective action $\Gamma[\tilde{\phi}]$ may be larger. We can define theories with fundamental scale symmetry by the property
\[
\partial_k \Gamma[\tilde{\phi}] = 0.
\] (15)

The absence of a dependence on $k$ is trivial if no renormalization scale is introduced. The non-trivial part consists in the statement that $\Gamma[\tilde{\phi}]$ is well defined in the continuum limit.

Eq. (15) is easily translated to the $k$-dependence of the effective action at fixed renormalized fields $\phi_{R,i}(x)$,
\[
\partial_k \Gamma[\phi_{R}] + \int \sum_i \frac{\partial \Gamma}{\partial \phi_{R,i}(x)} \partial_k \phi_{R,i}(x)|_{\tilde{\phi}} = 0,
\] (16)
or
\[
k \partial_k \Gamma[\phi_{R}] = -\int \sum_i \left( d_i + \frac{\partial \ln f_i}{\partial \ln k} \right) \phi_{R,i}(x) \frac{\partial \Gamma}{\partial \phi_{R,i}(x)}.
\] (17)

The $k$-dependence or the “flow” of the effective action at fixed renormalized fields can be non-trivial for theories with fundamental scale invariance. Even for dimensionless fields ($d_i = 0$) there is a contribution from the anomalous dimension $\sim \partial \ln f_i/\partial \ln k$. Solutions of flow equations obeying eq. (17) are “scaling solutions”.

The introduction of renormalized fields is not mandatory for theories with fundamental scale invariance. It is, however, often very convenient. An example is a metric that arises as a composite field
\[
\hat{g}_{\mu\nu}(x) = \partial_{\mu} \hat{H}_a(x) \partial_{\nu} \hat{H}_b(x) G^{ab},
\] (18)
with $\hat{H}_a(x)$ some combinations of dimensionless fields $\tilde{\phi}_i(x)$, and summations over double indices implied. Due to the derivatives, this metric has dimension mass squared. For a description of geometry one would like to introduce a dimensionless metric, and may do so by using $g_{\mu\nu}(x) = \hat{g}_{\mu\nu}(x)/k^2$. In this case the renormalized field is the canonical field. Since the canonical dimensions for geometric quantities are one of the main motivations for the use of canonical fields, we will next describe the scale invariant fields and the notion of fundamental scale invariance in some more detail for quantum gravity, starting from a formulation with canonical fields.

### Scale invariant fields for quantum gravity

Consider an effective action for the metric $g_{\mu\nu}$ and a scalar field $\chi$ of the form
\[
\Gamma = \int_x \sqrt{\hat{g}} \left\{ -\frac{F(\chi)}{2} R + \frac{1}{2} K(\chi) \partial_{\mu} \chi \partial_{\nu} \chi g^{\mu\nu} + U(\chi) \right\},
\] (19)
with $R$ the curvature scalar formed from the metric $g_{\mu\nu}$, $g = \text{det}(g_{\mu\nu})$, and effective potential
\[
U(\chi) = \frac{b^2}{2} \chi^2 + \frac{1}{8} (\chi(\chi))^4.
\] (20)
The scale invariant metric and scalar field are given by
\[
\hat{g}_{\mu\nu} = k^2 g_{\mu\nu}, \quad \hat{\chi} = \frac{\chi}{k}, \quad \hat{\rho} = \frac{1}{2} \chi^2.
\] (21)

In terms of these fields the effective action (19) reads
\[
\Gamma = \int_x \sqrt{\hat{g}} \left\{ -\hat{w} \hat{R} + \frac{1}{2} K \partial_{\mu} \hat{\chi} \partial_{\nu} \hat{\chi} \hat{g}^{\mu\nu} + \hat{u} \right\},
\] (22)
with
\[
w = \frac{F}{2k^2}, \quad u = \frac{U}{k^4},
\] (23)
and $\hat{R}$ the curvature scalar for the metric $\hat{g}_{\mu\nu}$.

This effective action is independent of $k$ if $w$, $u$ and $K$ only depend on $\hat{\chi}$ or the invariant $\hat{\rho}$. This is precisely the case for the scaling solution of flow equations. In general, the requirement of independence of $k$ constitutes a strong restriction. If $F$ contains an intrinsic mass scale as the Planck mass $M$, for example $F = M^2 + 2 w_0 k^2 + \xi \chi^2 / 2$, the function $w$ involves the ratio $M^2/k^2$ and therefore depends on $k$
\[
w = \frac{M^2}{2k^2} + w_0 + \frac{\xi}{2} \hat{\rho}.
\] (24)

Only for $M^2 = 0$ the effective action for the scale invariant fields does not involve $k$. Similarly, for
\[
u = \frac{\mu^2}{k^2} \hat{\rho} + \frac{\lambda}{2} \hat{\rho}^2,
\] (25)
the independence of $k$ requires $\mu^2 = 0$ and $\lambda$ to depend only on $\hat{\rho}$. From the point of view of flow equations the parameters as $M^2$ or $\mu^2$ denote deviations from the scaling solution due to relevant parameters. The condition that the effective action is independent of $k$, once it is expressed in terms of the scale invariant fields, requires that it corresponds precisely to the scaling solution of flow equations. This amounts to quantum scale invariance.

We observe that the scale invariant fields are, in general, not dimensionless. For $\chi$ and $k$ with dimension of mass, and $g_{\mu\nu}$ dimensionless, one finds that $\hat{\chi}$ is dimensionless, while $\hat{g}_{\mu\nu}$ carries the dimension of mass squared. If we combine the scale transformations (global dilatation transformations),
\[
\chi \to \alpha \chi, \quad g_{\mu\nu} \to \alpha^{-2} g_{\mu\nu},
\] (26)
with a rescaling of $k$

$$k \rightarrow \alpha k,$$  

(27)

the scale invariant fields remain indeed unchanged. We emphasize that the notion of scale invariant fields refers to the combined scaling (26), (27), while with respect to the scaling (26) alone neither $\tilde{g}_{\mu\nu}$ nor $\tilde{\chi}$ are invariant.

We can extend the notion of scale invariant fields to other fields. For example, gauge fields $A_\mu^\xi$ are scale invariant fields. Their kinetic term, constructed from the scale invariant field strength $F^\mu_\nu$, reads,

$$\Gamma_{\text{kin},A} = \frac{1}{4} \int_x \sqrt{g} Z_F (\chi) F^\mu_\nu F^\rho_\sigma g^{\mu\rho} g^{\nu\sigma} = \frac{1}{4} \int_x \sqrt{g} Z_F (\chi) F^\mu_\nu F^\rho_\sigma g^{\mu\rho} g^{\nu\sigma}.$$  

(28)

Independence of $k$ requires that $Z_F$ depends only on $\tilde{\rho}$ or similar scale invariant quantities.

Since gauge fields carry dimension of mass, this is another example that the scale invariant fields $A_\mu = A_\mu^\xi$ are not dimensionless. The fact that some of the scale invariant fields are not dimensionless has an important conceptual consequence. The scale invariance of the effective action does not correspond to a simple change of units for length or mass. Expressing the effective action in terms of dimensionless quantities, it is rather trivial that it remains invariant under a change of length or mass units. This is not the topic here.

Formulating gravity with fermions involves the vierbein $e^m_\mu$ and the spin connection $\omega_{\mu\nu mn}$. The kinetic term for a Dirac fermion $\psi$ reads

$$\Gamma_{\text{kin}, \psi} = \frac{i}{2} \int_x e Z_F \bar{\psi} \gamma^m D_\mu \psi e^\mu_m + \text{h.c.}, \quad \bar{\psi} = \psi^\dagger \gamma^0,$$  

(29)

with the inverse vierbein $e^m_\mu$ and $e = \text{det}(e^m_\mu)$ replacing $\sqrt{g}$,

$$e^m_\mu e^n_\nu = \delta_m^n, \quad e^m_\mu e^\mu_n = \delta^n_m, \quad e = \text{det}(e^m_\mu).$$  

(30)

The covariant derivative involves the spin connection

$$D_\mu = \partial_\mu - \frac{1}{2} \omega_{\mu\nu mn} \Sigma^m_\nu, \quad \Sigma^m_\nu = -\frac{1}{4} \{\gamma^m, \gamma^\nu\}.$$  

(31)

The Dirac matrices obey the usual anticommutation relations

$$\{\gamma^m, \gamma^n\} = 2\eta^{mn}, \quad \eta^{mn} = \eta_{mn} = \text{diag} (-1, 1, 1, 1),$$  

(32)

and Lorentz indices $m$ are raised and lowered with $\eta^{mn}$ or $\eta_{mn}$.

The spin connection is a scale invariant field, while the scale invariant vierbein and fermion field are given by

$$e^m_\mu = k e^m_\mu, \quad \bar{\psi} = k^{-3/2} \psi, \quad \omega_{\mu\nu mn} = \omega_{\mu\nu mn}.$$  

(33)

The kinetic term (29) is indeed independent of $k$,

$$\Gamma_{\text{kin}, \psi} = \frac{i}{2} \int_x e Z_F \bar{\psi} \gamma^\mu D_\mu \psi e^\mu_m + \text{h.c.},$$  

(34)

provided that $Z_\psi$ is a function of scale invariant fields. This extends in a straightforward way to Weyl fermions. The scale invariant fermion field $\bar{\psi}$ is dimensionless, while the scale invariant vierbein carries dimension of mass.

A Lorentz invariant kinetic term for the spin connection,

$$\Gamma_{\text{kin}, \omega} = \frac{1}{8} \int x e Z_F F_{\mu\nu mn} F_{\rho\sigma pq} g^{\mu\rho} g^{\nu\sigma} \eta^{mp} \eta^{nq},$$  

(35)

involves the scale invariant field strength

$$F_{\mu\nu mn} = \partial_\mu \omega_{\nu mn} - \partial_\nu \omega_{\mu mn} + \omega_{\mu mn} \partial_\rho \omega_{\nu pq} - \omega_{\nu pq} \partial_\rho \omega_{\mu mn}.$$  

(36)

Here the metric is a bilinear in the vierbein

$$g_{\mu\nu} = e^n_\mu e^n_\nu, \quad \bar{g}_{\mu\nu} = e^n_\mu e^n_\nu.$$  

(37)

Similar to the case of other gauge fields (28) the kinetic term (35) does not depend on $k$ once expressed in terms of scale invariant fields. This holds provided that the dimensionless function $Z_\omega$ only depends on $\tilde{\rho}$ or similar scale invariant quantities.

Lorentz symmetry also allows for a term linear in $F_{\mu\nu mn}$, 

$$\Gamma_R = \frac{1}{8} \int x F (\chi) F_{\mu\nu mn} e^{\rho n} e^{\sigma p} e^{\mu\rho\sigma} \varepsilon^{mp} \varepsilon^{nq},$$  

(38)

Expressed in terms of scale invariant fields and using the dimensionless function $w$ in eq. (23) this term becomes

$$\Gamma_R = \frac{1}{4} \int w F_{\mu\nu mn} e^{\rho n} e^{\sigma p} e^{\mu\rho\sigma} \varepsilon^{mn} \varepsilon^{pq}.$$  

(39)

It is independent of $k$ only if $w$ is a function involving only scale invariant combinations as $\tilde{\rho}$.

Finally, a Lorentz invariant kinetic term for the vierbein is constructed from its covariant derivative, the tensor $U_{\mu\nu}^m$,

$$U_{\mu\nu}^m = D_\mu e^m_\nu - \partial_\mu e^m_\nu - \Gamma_{\mu\nu}^\lambda (e) e^\lambda_m + \omega_{\mu n} e^\nu_n,$$  

(40)

where $\Gamma_{\mu\nu}^\lambda$ depends on the vierbein via eq. (37)

$$\Gamma_{\mu\nu}^\lambda (e) = \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}).$$  

(41)

The kinetic term involves functions $Z(\chi)$ and $Y(\chi)$ with dimension mass squared,

$$\Gamma_{\text{kin}, e} = \frac{1}{2} \int_x e U_{\mu\nu}^m U_{\rho\sigma}^n \eta_{mn} (Z g^{\mu\rho} g^{\nu\sigma} + Y g^{\mu\nu} g^{\rho\sigma}).$$  

(42)

In terms of the scale invariant vierbein it reads

$$\Gamma_{\text{kin}, e} = \frac{1}{2} \int_x \bar{e} U_{\mu\nu}^m U_{\rho\sigma}^n \eta_{mn} (Z g^{\mu\rho} g^{\nu\sigma} + Y g^{\mu\nu} g^{\rho\sigma}).$$  

(43)

with dimensionless functions

$$z = \frac{Z}{k^2}, \quad y = \frac{Y}{k^2}.$$  

(44)
Independence of $k$ follows if $z$ and $y$ are functions of scale invariant fields.

All $k$-independent terms in the effective action for the scale invariant fields involve dimensionless functions as $u$, $w$, $K$, $Z_f$, $Z_\omega$, $Z_\varphi$, $z$ or $y$, that only depend on dimensionless fields or invariants as $\tilde{\rho}$. As a consequence, no intrinsic length or mass scale is present in the effective action. This is an expression of the close connection between quantum scale invariance and the absence of parameters with dimension of length or mass in the effective action [17]. In terms of the canonical fields $g_{\mu\nu}$, $\chi$, $e_\mu$, $\omega_{\mu mn}$, $A_\mu$, $\psi$ the only mass scale appearing in the effective action is the “renormalization scale” $k$. It is only introduced by the transition from the scale invariant fields to canonical fields.

One can construct construct a Levi-Civita-connection $\Gamma_{\mu\nu}^\rho(e)$ from the vierbein by inserting into eq. (41) the relation (37) for the metric. From this a curvature tensor $R_{\mu\nu\rho\sigma}(e)$ is defined as a function of the vierbein in the standard way. We can also construct from the vierbein an object transforming as a spin connection

$$\omega(e)_{\mu np} = - e^m_\mu (\Omega_{mnnp}(e) - \Omega_{nmpm}(e) + \Omega_{pmnm}(e)),$$

$$\Omega_{nmpm}(e) = - \frac{1}{2} (e^m_\mu e^n_\nu - e^m_\mu e^n_\sigma)\partial_\mu e_{\nu p}.$$  

The field strength $F_{\mu\nu mn}(e)$ constructed from $\omega(e)_{\mu np}$ is related to $R_{\mu\nu\rho\sigma}(e)$ by

$$e^m_\mu e^n_\nu F_{\mu\nu mn}(e) = R_{\mu\nu\rho\sigma}(e).$$  

More generally, $F_{\mu\nu\rho\sigma}$ and $R_{\mu\nu\rho\sigma}$ are related by the commutator of covariant derivatives of the vierbein [34]

$$[D_\mu, D_\nu] e^m_\rho = F^m_{\mu \nu} e_\rho - R^m_\mu \sigma (e) e^m_\sigma = D_\mu U_{\nu\rho}^m - D_\nu U_{\mu\rho}^m = V_{\mu\nu\rho}^m.$$  

If the tensor $V_{\mu\nu\rho}^m$ vanishes the field strength for the spin connection can be identified with the curvature tensor similar to eq. (46). In this case the term (38) equals the term proportional to the curvature scalar $R(e)$ in eq. (19). This is the way how standard Riemannian geometry with the Einstein-Hilbert action can be recovered as a low energy limit. Due to the quadratic term (42) the field equations obey for low momenta to the approximate solution $U_{\mu\nu}^m = 0$, and therefore $V_{\mu\nu\rho}^m = 0$.

For vanishing $V_{\mu\nu\rho}^m$ the term (35) involves the squared Riemann tensor. This invariant contains four derivatives of the vierbein. The higher order derivatives only appear through the identification (46). Similar to the curvature tensor, there are other possible contractions of the field strength. We define

$$F_{\mu m} = F_{\mu\nu mn} e^{\nu m}, \quad F = F_{\mu m} e^{m\mu},$$

and generalize the term (35) to

$$\Gamma_{\text{kin}, \omega} = \int x \left\{ \frac{Z_\omega}{8} F_{\mu\nu mn} F^{\mu\nu mn} + \frac{A}{2} F_{\mu m} F^{\mu m} + \frac{B}{2} F^2 \right\}.$$  

For $V_{\mu\nu\rho}^m = 0$ this generates corresponding four-derivative invariants formed from the Riemann tensor $R_{\mu\nu\rho\sigma}(e)$.

**Flow equation**

The independence of $\Gamma[\varphi]$ of $k$ does not imply that there are no running couplings. This running or flow is, however, of a particular type. Any dependence on $k$ is accompanied by a dependence on fields. For example, a running gauge coupling in a scale invariant Yang-Mills theory can occur in the presence of a scalar singlet field $\chi$. It is due to the function $Z_F(\tilde{\rho})$ in eq. (28). The effective running gauge coupling $g(k)$ is related to $Z_F$ in one loop order by

$$Z_F = \frac{1}{g^2(k)} = \frac{1}{g^2} \frac{11N}{48\pi^2} \ln \tilde{\rho},$$  

where we have taken an SU($N$)-Yang-Mills theory and $\tilde{\rho} = \chi^2/(2k^2)$. At fixed $\chi$ the running with $k$ obeys the standard one loop formula

$$k \partial_k \frac{1}{g^2} = \frac{11N}{24\pi^2}.$$  

In the limit $k \to 0$ the scale in the running is effectively replaced by momentum, $k^2 \to q^2$. In this limit quantum scale symmetry becomes exact [9, 17]. For this version of scale symmetric QCD the UV-cutoff $A_{\text{UV}}$ is replaced by a scalar field $\chi$, such that also the confinement scale $\Lambda_{\text{QCD}}$ is proportional to $\chi$ [9, 18].

A convenient method for the investigation of running couplings in theories with fundamental scale invariance is functional renormalization for the effective average action [25]. We define the effective average action or flowing action $\Gamma_k[\varphi]$ similarly to eq. (11) by adding an infrared cutoff term $\Delta_k[\chi]$ [32],

$$\Gamma_k[\varphi] = -\ln (Z_k[\varphi]) - C_k[\varphi],$$  

where the $k$-dependence partition function reads

$$Z_k[\varphi] = \int \mathcal{D} \chi \exp \left\{ -S[\varphi + \chi] - \Delta_k[\chi; \varphi] \right. \left. + \int_x \left( \frac{\partial \Gamma_k}{\partial \varphi} + L_k[\varphi] \right) \chi \right\}.$$  

The cutoff term is bilinear in the fluctuation fields $\chi$ and may depend on the macroscopic fields $\varphi$,

$$\Delta_k[\chi; \varphi] = \frac{1}{2} \int x \chi_i(x) R_{k_{ij}}(-D^2; \varphi) \chi_j(x).$$  

The covariant Laplacian $D^2$ is formed with the macroscopic fields $\varphi$, such that the cutoff (54) can be made invariant under local gauge transformations. The functionals $C_k[\varphi]$ and $L_{k_{ij}}[\varphi]$ can be used for optimization and will be discussed later.

The dimension of the cutoff function $R_k$ is dictated by the dimension of the fields $\chi_i(x)$. For the example of scalars the dimension of $R_k$ is mass squared, and we introduce a dimensionless function $r_k$

$$R_k = ek^2 r_k \left( -\frac{D^2 \varphi}{k^2} \right).$$
This can be generalized to fields with other dimensions. We require that \( R_k \) vanishes for \( k \to 0 \), such that we recover the effective action (11) in this limit. For \( k \neq 0 \) the quadratic term (54) acts as an infrared cutoff. For high momenta, corresponding to large values of \(-D^2/k^2\), the cutoff function is chosen to vanish rapidly, such that the functional integral over fluctuations with momenta much larger than \( k \) is not affected. For \( k \to \infty \) the quadratic term \( \sim \Delta_k \) dominates the functional integral which becomes effectively Gaussian. In this limit one typically has \( \lim_{k \to \infty} \Gamma_k[\varphi] = S[\varphi] \). The effective average action interpolates between the classical action for \( k \to \infty \) and the quantum effective action for \( k \to 0 \). The fluctuation effects that map \( S[\varphi] \) to \( \Gamma[\varphi] \) are taken into account in continuous steps.

We consider cutoff functions that depend on the macroscopic fields [32]. This permits us to maintain local gauge symmetries by employing covariant derivatives constructed from the connection which involves the microscopic vierbein (or metric) or macroscopic gauge fields. In particular, diffeomorphism symmetry requires that \( R_k \) is proportional to \( e = \sqrt{g} \). The price to pay for maintaining gauge symmetry are corrections to the flow equation that may be minimized by suitable optimization functionals \( C_k[\varphi], L_k[\varphi] \). The optimization functionals vanish for \( R_k = 0 \) and therefore for \( k \to 0 \), while for \( k \to \infty \) one has vanishing \( C_k \) and finite \( L_k \).

Expectation values are computed in the presence of \( \Delta_k \),

\[
\langle A \rangle = Z_k^{-1} \int \mathcal{D}\chi A[\chi] \exp \left\{ -S[\varphi + \chi] - \Delta_k[\chi] \right\} + \int_x \left( \frac{\partial \Gamma_k}{\partial \varphi} + L_k \right) \chi, \tag{56}
\]

They therefore depend on \( k \). We may consider the family of effective average actions \( \Gamma_k[\varphi] \) for different \( k \) as a family of different models, labeled by \( k \). The models apparently differ by their infrared cutoffs and have the same high momentum behavior.

We can shift the integration to \( \sigma = \varphi + \chi \),

\[
\langle A \rangle = Z_k^{-1} \int \mathcal{D}\sigma A[\sigma] \exp \left\{ -S[\sigma] - \Delta_k[\sigma - \varphi] \right\} + \int_x \left( \frac{\partial \Gamma_k}{\partial \varphi} + L_k \right) (\sigma - \varphi), \tag{57}
\]

where

\[
Z_k = \int \mathcal{D}\sigma \exp \left\{ -S[\sigma] - \Delta_k[\sigma - \varphi] \right\} + \int_x \left( \frac{\partial \Gamma_k}{\partial \varphi} + L_k \right) (\sigma - \varphi). \tag{58}
\]

Taking the functional derivative of \( \Gamma_k[\varphi] \),

\[
\frac{\partial \Gamma_k}{\partial \varphi_i(x)} = \left( \frac{\partial}{\partial \varphi_i(x)} \Delta_k[\sigma - \varphi; \varphi] \right) - \int_y \left( \frac{\partial^2 \Gamma_k}{\partial \varphi_i(x) \partial \varphi_j(y)} + \frac{\partial L_{k,j}}{\partial \varphi_i(x)} \right) (\sigma_j(y) - \varphi_j(y)), \tag{59}
\]

relates the macroscopic field \( \varphi_i(x) \) to the expectation value of the microscopic field \( \sigma_i(x) = \langle \sigma_i(x) \rangle \). Here the \( \varphi \)-derivative of \( \Delta_k \) in the expectation value (first term on r.h.s.) is performed under the integral at fixed \( \sigma \).

One finds

\[
\int_y \left( \frac{\partial^2 \Gamma_k}{\partial \varphi_i(x) \partial \varphi_j(y)} + \frac{\partial L_{k,j}(y)}{\partial \varphi_i(x)} \right) (\sigma_j(y) - \varphi_j(y)) = -K_i(x), \tag{60}
\]

where

\[
K_i(x) = \left( \frac{\partial}{\partial \varphi_i(x)} \Delta_k[\sigma - \varphi; \varphi] \right) + L_{k,i}(x) - \frac{\partial C_k}{\partial \varphi_i(x)}. \tag{61}
\]

vanishes for \( k = 0, \Delta_k = 0 \). For \( k = 0 \) the macroscopic field equals the expectation value of the microscopic field \( \varphi = \bar{\sigma} \), as for the usual construction of the effective action by a Legendre transform of the Schwinger functional.

For \( k \neq 0 \) a non-zero \( K_i(x) \) can arise from a possible dependence of the cutoff function \( eR_{k,ij} \) on \( \varphi \),

\[
K_i(x) = \frac{1}{2} \operatorname{tr} \left\{ \frac{\partial R_k}{\partial \varphi_i(x)} \right\} - (R_k)_{ij} \chi_j(x) + L_{k,i}(x) - \frac{\partial C_k}{\partial \varphi_i(x)}, \tag{62}
\]

Here \( G \) is the matrix of two-point functions

\[
G_{ij}(x,y) = \langle \chi_j(x) \chi_i(y) \rangle. \tag{63}
\]

For \( \bar{\chi} = 0 \) it is the connected two-point function or the propagator for \( \sigma \). In the trace in eq. (62) we consider the IR-cutoff function as a matrix

\[
R_{k,ij}(x,y) = \delta(x-y)R_{k,ij}(-D^2_y; \varphi(y)), \tag{64}
\]

and we have assumed for simplicity that \( R_k \) is symmetric. For \( k \neq 0 \) the first term on the r.h.s. of eq. (62) does not vanish. The optimization terms \( L_k \) and \( C_k \) are used to cancel this term. For any choice of \( C_k \) this defines the functional \( L_{k,i}(x) \). With this choice one has \( \langle \sigma_i(x) \rangle = \varphi_i(x), \chi_i(x) = 0 \) for all \( k \), despite the dependence of \( R_k \) on the macroscopic fields.

The exact flow equation for the effective average action obtains by taking a \( k \)-derivative of eqs. (52), (53)

\[
\partial_k \Gamma_k[\varphi] = \frac{1}{2} \operatorname{tr} \{ (\partial_k R_k) G \} - \partial_k C_k[\varphi]. \tag{65}
\]

We employ

\[
G = \left( \Gamma^{(2)}_k + R_k \right)^{-1} + \Delta_k G, \tag{66}
\]
with $\Gamma_k^{(2)}$ the matrix of second functional derivatives of $\Gamma_k$. The correction term $\Delta_k G$ can be computed as in ref. [32]. It vanishes if $R_k$ is independent of the macroscopic fields $\varphi$. We arrive at the flow equation

$$\partial_k \Gamma_k[\varphi] = \frac{1}{2} \text{tr} \left\{ (\partial_h R_k)(\Gamma_k^{(2)} + R_k)^{-1} \right\} + B_k[\varphi]. \quad (67)$$

The correction term,

$$B_k[\varphi] = \frac{1}{2} \text{tr} \left\{ (\partial_h R_k)\Delta_k G \right\} - \partial_h C_k,$n

vanishes for a suitable choice of $C_k$. For $R_k$ independent of the macroscopic fields one has $C_k = 0$.

In summary, by a suitable choice of $L_k$ and $C_k$ in the definition of $\Gamma_k$ we arrive at an effective average action that obeys the standard exact flow equation [25]. Furthermore, the macroscopic field $\varphi$ equals the expectation value of the microscopic field ($\sigma$) for all $k$. The exact form of the flow equation will actually not be crucial for our discussion. What is important is the existence of a flow equation that can account for running couplings.

**Effective average action for fundamental scale invariance**

In the preceding discussion we have defined the effective average action (53), (53) as a functional of the canonical fields. The flow equation describes the variation with the scale $k$ for fixed canonical fields $\varphi$. Let us now express $\Gamma_k$ as a functional of the scale invariant fields. For the classical action $S$ this has been discussed previously. With $\varphi$ and $\chi$ scaling in the same way as $\sigma$ the action $S[\tilde{\varphi} + \tilde{\chi}]$ of a theory with fundamental scale invariance does not depend on the scale $k$. Since the relation between $\tilde{\chi}_i$ and $\chi_i$ is only a $k$-dependent factor, the functional measures $\int D\chi$ and $\int D\tilde{\chi}$ differ only by a $k$-dependent but field-independent factor. This only results in an irrelevant additive constant for $\Gamma_k$.

For the infrared cutoff $\Delta_k$ we choose the same $k$ for the transition to scale invariant fields as the one that appears in $R_k$. As a result the cutoff term becomes independent of $k$ once it is expressed in terms of scale invariant fields. This may be demonstrated by the cutoff (55) for scalar fields. With $\epsilon = k^{-l} \bar{\epsilon}$ one has

$$R_k = k^{-2} \bar{\epsilon}_k(\tilde{D}^2; \tilde{\varphi}). \quad (69)$$

Here we use

$$-D^2 = -g^{\mu\nu} D_\mu D_\nu = -k^2 \tilde{g}^{\mu\nu} D_\mu D_\nu = -k^2 \tilde{D}^2, \quad -\tilde{D}^2 = -\tilde{D}^2, \quad \frac{\varphi}{k} = \tilde{\varphi}. \quad (70)$$

The factor $k^{-2}$ in eq. (69) is canceled by $\chi^2 = k^2 \tilde{\chi}^2$. This holds similarly for fields with other scaling dimensions if the prefactor multiplying $R_k$ involves besides $\epsilon$ only powers of $k$, multiplied by possible functions of scale invariant fields. Finally, $\partial_k \Gamma_k/\partial \chi_i(x)$ scales inversely to $\chi_i(x)$ and similar for $L_{k,i}(x)$. Up to an irrelevant multiplicative factor one finds

$$Z_k[\varphi] = \int D\tilde{\chi} \exp \left\{ -S[\tilde{\varphi} + \tilde{\chi}] - \Delta_k[\tilde{\chi}; \tilde{\varphi}] \right\} + \int_x \left( \frac{\partial \Gamma_k}{\partial \tilde{\varphi}} + L_k \right) \tilde{\chi}. \quad (71)$$

Since $\Delta_k[\tilde{\chi}; \tilde{\varphi}]$ no longer involves the scale $k$, one finds $Z_k[\tilde{\varphi}]$ and $\Gamma_k[\tilde{\varphi}]$ independent of $k$. This requires the optimization functionals $L_k[\tilde{\varphi}]$ and $C_k[\tilde{\varphi}]$ to be independent of $k$ once expressed in terms of scale invariant fields. This is self-consistent.

We arrive at an important conclusion: The effective average action does no longer involve the scale $k$ if it is written as a functional of the scale invariant fields. The whole family of apparently different effective average actions for different $k$ describes actually the same model. The difference between the different members of the family is only due to the use of different canonical fields, all corresponding to the same scale invariant fields, but using different $k$ for the scaling. As an immediate consequence, the flow with $k$, evaluated for fixed scale invariant fields, vanishes

$$\partial_k \Gamma_k[\tilde{\varphi}] = 0. \quad (72)$$

This is precisely the setting (15) for a theory with fundamental scale invariance. The introduction of the infrared cutoff has not changed this.

In our case the rescaling to the canonical fields only involves the canonical dimension in eq. (13),

$$\varphi_i(x) = k^{d_i} \tilde{\varphi}_i(x). \quad (73)$$

In consequence, one obtains for the flow equation (17) for fixed canonical fields

$$k \partial_k \Gamma_k[\varphi] = - \int_x \sum_i d_i \tilde{\varphi}_i(x) \frac{\partial \Gamma_k}{\partial \tilde{\varphi}_i(x)} \quad (74)$$

The left hand side obeys the exact flow equation (65), (67), and we conclude that $\Gamma_k$ has to obey the condition

$$\frac{1}{2} \text{tr} \left\{ k \partial_k R_k(\Gamma_k^{(2)} + R_k)^{-1} \right\} + \int_x \sum_i d_i \tilde{\varphi}_i(x) \frac{\partial \Gamma_k}{\partial \tilde{\varphi}_i(x)} = 0. \quad (75)$$

This condition precisely defines the scaling solution for the flow equation.

One may compare to theories as QCD for which the renormalized fields need a non-zero anomalous dimension. In this case the cutoff function for the gauge fields contains a wave function renormalization

$$R_k^{\mu\nu} \sim ek^2 g^{\mu\nu} Z_F(ka)r_k(D_{\mu}[A]). \quad (76)$$

(This form is symbolic since $R_k^{\mu\nu}$ may involve other combinations where $k^2 g^{\mu\nu}$ is replaced by $g^{\mu\sigma} g^{\nu\tau} D_{\mu}D_{\nu}k^2/D^2$. They have the same scaling properties.) After translation
to scale invariant fields the factor $Z_F$ in $R_k^{\mu\nu}$ is absorbed by employing eq. (13)

$$A_\mu(x) = f(k)\tilde{A}_\mu(x) = Z_F^{-1/2}(ka)\tilde{A}_\mu(x).$$

A dependence on $ka$ remains, however, through the covariant derivative $D_\mu = \partial_\mu - i\tilde{A}_\mu^a T_a = \partial_\mu - iZ_F^{-1/2}\tilde{A}_\mu^a T_a$. This reflects the scale dependence of the gauge coupling. As a result, the effective average action depends on $k$ even once it is expressed in terms of scale invariant fields. Eq. (72) is not valid. This feature is different for scale invariant QCD in the presence of an additional scalar field. Now $Z_F$ depends on the scale invariant variable $\tilde{\rho}$, cf. eq. (50), instead of a dependence on $ka$. As a result, $\Gamma_k[\tilde{\varphi}]$ is now independent of $k$.

How can the effective average action for scale invariant fields describe a running of couplings despite the fact that no scale $k$ appears anymore? The functional integral still contains an infrared cutoff term. It is now a fixed term, corresponding to setting $k = 1$. The flow occurs now in field space. Changing the value of $\tilde{\rho}$ indeed amounts for fixed $\varphi$ to a change in $k$. The average effective action is a fixed functional, and the flow equation describes what happens if we rescale the field values according to the appropriate dimension. For scalar fields, flowing towards the infrared corresponds to an increase of $\tilde{\rho}$.

**Discussion**

We have investigated theories with fundamental scale invariance. In the formulation of the effective average action they contain an effective infrared cutoff. The ultraviolet scale can be given by some inverse lattice distance $a^{-1}$ or similar. We consider theories for which a continuum limit $ka \to 0$ exists. For theories with fundamental scale invariance the effective average action as a functional of the scale invariant fields remains well defined in the continuum limit.

For theories with fundamental scale invariance there is a close connection to quantum scale symmetry. If an infrared fixed point is reached for $k \to 0$ at fixed canonical fields, one recovers exact quantum scale symmetry in this limit. For a given scaling solution, and a given cosmological solution of the field equations derived from the corresponding effective action, one can infer the value of $k$ in standard particle physics units. For our example it turned out to be $k \approx 10^{-3}$ eV. For many observations there are physical cutoffs that stop effectively the flow of couplings. We may associate such physical cutoffs with some squared momentum $q^2$. For the example of quantum electrodynamics the flow stops for scales below the electron mass. In this case the limit $k \to 0$ can be taken since there is effectively no difference between $k = 10^{-3}$ eV and $k = 0$. In the scaling solutions one can effectively replace $k^2$ by $q^2$. With this replacement the effective action corresponding to the scaling solution exhibits exact quantum scale symmetry.

Theories with fundamental scale invariance have a very high predictive power, much stronger than arbitrary renormalizable theories. General renormalizable theories, both asymptotically free or asymptotically safe, have free parameters corresponding to the so called relevant parameters for small deviations of the flow from an ultraviolet fixed point. Theories with fundamental scale invariance correspond to exact scaling solutions of the flow equations. All relevant parameters are set to zero, and therefore not available as free parameters for an interpretation of observations. If there is a unique scaling solution, theories with fundamental scale symmetry contain no free parameters. Free parameters can only arise if there are families of scaling solutions, with parameters distinguishing between different members of such families.

The requirement of the existence of scaling solutions is already highly non-trivial. It guarantees that a theory is “renormalizable” or “ultraviolet complete”. These scaling solutions are all what is needed for theories with fundamental scale invariance. In contrast to general renormalizable theories no deviations from the scaling solution due to relevant parameters need to be studied. In the presence of quantum gravity the scaling solutions often have properties that are not familiar in perturbation theory for particle physics. For example, the effective scalar potential may reach a constant for large values of the fields [30]. Together with an effective Planck mass increasing $\sim \chi$ this solves the cosmological constant problem asymptotically, without any tuning of parameters. What is usually a tuning of parameters becomes the statement that for scaling solutions the effective potential becomes constant for large $\chi$ instead of increasing $\sim \chi^2$. It is well conceivable that other perturbative tuning problems as the gauge hierarchy could find a solution by properties of scaling solutions. It is highly unlikely that families of scaling solutions with twenty or more parameters exist. As a consequence, many renormalizable couplings of the standard model of particle physics would become predictable. Scaling solutions could be sufficiently restrictive to become candidates for the quest of a unified fundamental theory of physics.

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