Burstiness and aging in social temporal networks

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The presence of burstiness in temporal social networks, revealed by a power law form of the waiting time distribution of consecutive interactions, is expected to produce aging effects in the corresponding time-integrated network. Here we propose an analytically tractable model, in which interactions among the agents are ruled by a renewal process, and that is able to reproduce this aging behavior. We develop an analytic solution for the topological properties of the integrated network produced by the model, finding that the time translation invariance of the degree distribution is broken. We validate our predictions against numerical simulations, and we check for the presence of aging effects in a empirical temporal network, ruled by bursty social interactions.

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Our understanding of the structure and properties of social interactions has experienced a boost in recent years due to the new availability a large amounts of digital empirical data. This endeavor has found the necessary theoretical grounding in the newly established science of networks. A first round of network studies focused on a static network representation, in which nodes (standing for individuals) and edges (indicating social interactions) are constant and never change in time. From such static representation, a wealth of complex topological information was obtained, concerning e.g. the presence of scale-free, power-law degree distributions \( P(k) \sim k^{-\gamma} \), large clustering, positive degree correlations, or a distinct community structure. More recently, this framework has been challenged by the empirical observation of a temporal dimension in social (and other) networks, arising from the fact that social relationships are continuously created and terminated. From these temporal networks, a static representation is obtained by means of a temporal integration of the instantaneous interactions over a time window of width \( t \), and its associated topological properties, such as the degree distribution, \( P_t(k) \), are thus to be understood to depend on the integration time \( t \). The empirical study of the temporal aspects of social networks has unveiled an additional level of complexity, embodied in many statistical properties showing heavy-tailed distributions. Remarkable among them are the distribution \( \psi(\tau) \) of "inter-event or waiting times" between two consecutive social interactions, revealing the bursty nature of human dynamics, or the distribution \( F(a) \) of social activity, measuring the probability per unit time of establishing a new social relation, both approximately obeying power-laws decays of the form \( \psi(\tau) \sim \tau^{-\gamma} \) and \( F(a) \sim a^{-\gamma} \), respectively.

This twofold nature of social interactions naturally arises the issue of the relation between the temporal correlation properties of time-varying networks and the topological features of their static representations. Among others, Song et al. proposed an empirical scaling theory bridging the exponents of human dynamics patterns and social network architecture, while Perra et al. considered an activity driven model, built upon the idea of a constant social activity, defined as the probability per unit time that an agent becomes active and starts a social interaction. The activity driven model allows to show that the degree distribution \( P_t(k) \) of an integrated network is functionally related to the distribution of social activity by \( P_t(k) \sim t^{-\gamma} F \left( \frac{k}{t} - \langle a \rangle \right) \). Following this direction, in this paper we focus on a different property of social temporal networks, naturally expected in systems in which the addition of connections is ruled by a non-Poissonian, power-law distribution of interevent times: The presence of aging behavior, which in this context translates into a breaking of time translation invariance manifested in the dependence of the topological properties of the integrated network on the aging time \( t_\alpha \) at which the integration starts. In order to study this possibility from an analytic point of view, we propose and analyze a non-Poissonian activity driven (NoPad) model, in which the waiting time between consecutive agent activations follows an arbitrary form \( \psi(\tau, c) \), being a parameter quantifying the (possible) heterogeneity of waiting times in the population. We compute the topological properties of the ensuing integrated networks by applying the hidden variables formalism. We find that if \( \psi(\tau, c) \) is a power law distribution with exponent \( 1 + \alpha \), then the degree distribution exponent \( \gamma \) is simply related to \( \alpha \), mediated by the heterogeneity distribution \( \eta(c) \). In this model, effects of aging are clearly evident. We observe in particular that both the degree distribution \( P_{t_\alpha}(k) \) and the average degree \( \langle k \rangle_{t_\alpha} \), computed in a time window \([t_\alpha, t_\alpha + t]\) of width \( t \) depend explicitly on the initial integration time \( t_\alpha \). Evidence of this sort of aging is recovered in an empirical analysis of the temporal network defined by the scientific collaborations in the journal Physical Review Letters, published by the American Physical Society.

Previous modeling efforts have shown that the concept of memory can induce non-Poissonian interevent time distributions in temporal networks. Here we propose a model joining the activity driven framework with the empirically observed bursty nature of social interactions,
which allows for a simple mathematical treatment. The NoPad model is defined as follows: Each agent \(i\) in a network of size \(N\) is endowed with a *time-dependent* activity \(a_i(t)\), which represents the probability per unit time that agent \(i\) becomes active for the first time after a time \(t\) from its last activation. When an agent becomes active, it generates an edge that is connected to another agent chosen uniformly at random. Edges last for a period of time which we assume to be infinitesimally small. The activation of each individual follows a renewal process \([21]\) with a *failure rate* given by the agent’s activity \(a_i(t)\). The interevent time distribution for the node activation \(N\) which allows for a simple mathematical treatment. The variable model resides in computing the probability \(\Pi\) that a vertex with hidden variable \(\vec{h}\) is excited by the reception of a connection emitted by other active agent.

Shifting from the activity (failure rate) to the equivalent waiting time distribution. An explicitly time dependent activity rate leads to a non-Poissonian activity pattern, where the degree distribution is given, for \(k \gg (c_0t)^\alpha\), by \(P_i(k) \sim (c_0t)^\beta (k - \langle r \rangle t)^{1 - \frac{\beta}{\alpha}}\). Following Ref. \([15]\), this probability can be written as \(P_i(k) \equiv \Pi_i(r_i, r_j) = 1 - (1 - \frac{1}{N})^r_i (1 - \frac{1}{N})^r_j\), where \(r_i\) is the number of times node \(i\) has become active up to time \(t\). This number of activations is itself a random variable with distribution \(\chi_i(r|c)\), depending on the agent’s heterogeneity \(c\) and time \(t\) \([21]\). The mapping to a hidden variables network is now clear: The hidden variables are the vector \(\vec{h} \rightarrow (r, c)\), with a probability distribution \(\rho(\vec{h}) \rightarrow \rho(r, c) \equiv \eta(c)\chi_i(r|c)\), and the connection probability takes the form \(\Pi(\vec{h}, \vec{h'}) \rightarrow \Pi_i(r, r') \sim (r + r')/N\), independent of \(c\) and \(c'\), in the limit \(N \gg 1\) and \(N \gg r, r'\).

Applying this mapping on Eq. \([2]\) in the limit \(N \gg r, r'\), we obtain a generating function \(\tilde{g}(z| r, c)\) with an exponential form. The resulting propagator is a Poisson distribution, sharply peaked at its average value \(r + \langle r \rangle_1\), where \(\langle r \rangle_1 = \sum_c \eta(c) \sum_r r\chi_i(r|c)\) is the average number of activation events at time \(t\). This leads, through Eq. \([1]\), to an approximate expression for the degree distribution of the integrated network at time \(t\)

\[
P_i(k) \approx \sum_c \eta(c)\chi_i(k - \langle r \rangle_1|c)\quad (3)
\]

The remaining element to close the calculation is the probability \(\chi_i(r|c)\), whose expression can be easily worked out in Laplace space \([21]\). For the empirically relevant case of heavy-tailed waiting time distributions, of the form

\[
\psi(t, c) = ac(c + 1)^{-(\alpha + 1)}\quad 0 < \alpha < 1
\]



\[
\frac{\chi_i(r|c)}{\chi_i(r|c) + \eta(c)\chi_i(r|c)} \sim 1 / (c + 1)\quad (4)
\]

corresponding to a time dependent activity \(a(t,c) = ac/(1 + ct)\), we can use the approximation developed in Ref. \([22]\), valid for large \(r/(ct)^\alpha\), namely

\[
\chi_i(r|c) \sim \frac{1}{\langle c \rangle} e^{\xi_0 / (\langle c \rangle_1)^{\beta / (1 - \alpha)}}\quad (5)
\]

where \(\xi_0 = -[1 - \alpha]([\alpha / u]^{\alpha} \Gamma(1 - \alpha))^{1/(1 - \alpha)}\).

From Eq. \([3]\), the degree distribution of the integrated network up to time \(t\) is given, in the continuous \(c\) limit, by

\[
P_i(k) \sim \langle k - \langle r \rangle_1 \rangle^\beta / t \int du \eta \left(\frac{u}{t} - \langle r \rangle_1\right)^{\beta / (1 - \alpha)} e^{\xi_0 / u^\alpha} / u^\alpha \quad (6)
\]

As we will argue below, a reasonable form for the heterogeneity distribution is a power-law one,

\[
\eta(c) = \frac{\beta}{\gamma} e^{\beta / (c_0 t)^\alpha} \quad (7)
\]

with \(\beta > \alpha\). From here, we obtain

\[
\langle r \rangle_1 \sim \frac{\sum_c \eta(c) \sin(\pi \alpha) / (ct)^\alpha}{\pi \alpha} = \frac{\beta \sin(\pi \alpha)}{(\beta - \gamma \alpha) \pi \alpha} (c_0 t)^\alpha
\]

while the degree distribution is given, for \(k \gg (c_0 t)^\alpha\), by

\[
P_i(k) \sim (c_0 t)^\beta (k - \langle r \rangle_1)^{1 - \gamma / \alpha} \quad (8)
\]

\[1\] We assume that the time between activation events is not affected by the reception of a connection emitted by other active agent.
Eq. (8) establishes the relation between the exponent of the power-law degree distribution $P(k) \sim k^{-\gamma}$ and the exponent of the long-tailed waiting time distribution, $\psi(t) \sim t^{-1-\alpha}$, namely

$$\gamma = 1 + \beta/\alpha. \quad (9)$$

This relation, mediated through the exponent $\beta$, manifests the relevance of the assumed distribution of heterogeneity $\eta(c)$. We can motivate the form assumed by relating it with the empirical activity measurements performed in Ref. [13]. There, it was actually measured the average activity of an individual $i$ over a time interval, $\bar{a}_i(\Delta t)$, defined as the ratio between the number of social acts performed by individual $i$ in the time interval $\Delta t$, and the total number of social acts in the system in that interval. In the NoPad model, the number of social acts of an individual with heterogeneity $c$ in an interval $\Delta t$ is determined by the number of times $n$ has become active in that interval, which from Eq. (7) is given by $\bar{r}_\Delta(c) \sim c^{\gamma} (\Delta t)^{-\delta}$. Therefore, we have $\bar{a}_i(c) \sim c^{\alpha}$, independent of $\Delta t$. A simple transformation between probability distribution allows to write $\eta(c) \sim F(\bar{a}(c)) \frac{d\bar{a}(c)}{d(c)} \sim c^{1-\gamma(\delta-1)}$. From here, we recover the postulated heterogeneity distribution, Eq. (9), with an exponent $\beta = \alpha(\delta-1)$. Most remarkably, for this value of $\beta$, the integrated network shows a degree exponent $\gamma = 1 + \beta/\alpha = \delta$, i.e. we recover the main result of the activity driven model, stating the equivalence between degree and activity distributions [14].

In order to check our analytic predictions, we have performed numerical simulations of the NoPad model with the waiting time and heterogeneity distributions Eqs. (4) and (6), respectively. The integrated network at time $t$ is generated as follows: To each node $i$ is assigned a heterogeneity $c_i$ extracted from the distribution $\eta(c)$. Then, we generate the number $r_i$ of times that each node becomes active up to time $t$, according to the distribution Eq. (4). Finally, each node $i$ is connected to $r_i$ neighbors chosen at random, avoiding multiple and self connections. In Fig. 1 we show the degree distribution $P(k)$ for different values of the exponents $\alpha$ and $\beta$ of the waiting time and heterogeneity distributions. As one can see, the scaling relation of Eq. (9) is fulfilled remarkably well. In the same Figure we validate the scaling of the $P_t(k)$ with the integration time $t$, Eq. (5), showing the collapse of the degree distribution for different $t$.

The dependence of the topological properties of the NoPad model on the distribution of renewal events $\chi(r,c)$ readily suggests that the model will be affected by aging effects when the waiting time distributions have the power-law form Eq. (4) with $\alpha < 1$ [17]. We check these effects in Fig. 2 (inset), where we plot the aged degree distribution $P_{t_a,t}(k)$ obtained from networks integrated for a time interval $t$, started after waiting for an aging time $t_a$. This Figure shows that while the asymptotic shape of the $P_{t_a,t}(k)$ remains constant for large $k$, its peak shifts to smaller values of $k$ as increasing $t_a$.

An analytical treatment of these aging effects is in principle possible, using the results reported in Ref. [23]. We can however easily understand them at the level of the aged average degree $\langle k \rangle_{t_a,t}$. Since the average degree is two times the average number of activation event, see Eq. (3), if we consider a network integrated from time $t_a$ to $t_a+t$, we obviously have $\langle k \rangle_{t_a,t} = 2(\langle r \rangle_{t_a+t} - \langle r \rangle_{t_a})$. By applying Eq. (7) one can obtain

$$\langle k \rangle_{t_a,t} \sim \langle (t_a + t)^\alpha - t_a^\alpha \rangle \equiv t_a^{\alpha} F \left( t_a^{-1} \right). \quad (10)$$

Thus, $\langle k \rangle_{t_a,t}$ exhibits a generic scaling behavior depending on $t_a$. For $t \gg t_a$, the average degree is independent of $t_a$, $\langle k \rangle_{t,t_a} \sim t^\alpha = \langle k \rangle_t$, and aging effects are negligible. On the other hand, for $t \ll t_a$, the average degree decays with $t_a$ as $\langle k \rangle_{t,t_a} \sim t_a^{-\alpha-1}$, and aging effects induce an anomalous behavior depending on $\alpha$ [17]. In Fig. 2 we check that the Eq. (10) correctly reproduces the behavior of the NoPad model.

We explore the possibility of aging effects in empirical temporal networks by considering the scientific collaboration network in the journal Physical Review Letters (PRL), published since 1958 [19]. In this network, two authors are connected by a link if they co-authored a
paper published in PRL. Since PRL is weekly edited, time is measured in units of weeks. In order to avoid spurious effects due to effective aging of the population considered, and single out the role of the heavy-tailed waiting distribution, we select only those authors who published at least one paper in any APS journal before and after an interval of 30 years, spanning from 1968 up to 1998. We then reconstruct the temporal network of the \( N = 677 \) resulting authors, by considering those papers co-written by two authors in this interval, and drawing an instantaneous edge between the authors at the date of the paper’s publication. In Fig. 3 we check that the waiting time distribution between two consecutive publications of the same author has a clear heavy-tailed form, approximately compatible with a power-law decay \( \psi(\tau) \sim \tau^{-1-\alpha} \), with exponent \( \alpha \simeq 0.3 \). We then proceed to construct the integrated networks, varying the aging time \( t_a \) between 0 and 30 years and fixing the integration time as \( t = 1, 2 \) and 3 years. In this construction it is important to realize that the actual aging time of the network is in principle unknown. Each author \( i \), indeed, starts his academic life at some time \( T_i^0 \), included in the observational time window between 1958 and 1968. Aging effects are thus observed in networks integrated over a time window explicitly dependent on \( T_i^0 \) of each author considered. This point makes extremely difficult to detect aging effects in the degree distribution \( P_{t_a,t}(k) \), also because the low activation ratio \( a_i(t) \) yields a very sparse network, with small degree values. Nevertheless, we are able to observe aging behavior in the average degree \( \langle k \rangle_{t_a,t} = \sum_k k P_{t_a,t}(k) \). Fig. 3 (inset) shows the aged average degree \( \langle k \rangle_{t_a,t} \) of the empirical data, plotted in the rescaled form suggested by the NoPad model, Eq. (10). As one can see, the data are compatible with the theoretical prediction, particularly in the limit of large \( t_a \), where we expect the actual aging time \( T_i^0 \) to become small with respect to \( t_a \).

To sum up, in this paper we addressed the aging effects observed in time-integrated networks produced by bursty social interactions. We proposed a mathematically tractable model, the NoPad model, aimed to combine the non-Poissonian form of the waiting time distribution with the activity-driven framework, and we developed an analytic solution for its topological properties, through the hidden variables formalism. Aging effects are clear in the NoPad model, as demonstrated by the dependence of the degree distribution \( P_{t_a,t}(k) \) not only on the integration time window \( t \), but also on the aging time \( t_a \) at which we start the integration. Inspired by the results obtained in the model, we checked that aging behavior can also be observed in real temporal networks. At this respect, it is important to notice that, in real systems, the effects purely derived by a heavy-tailed interevent time distribution can be mixed with, and masked by, other features, such finite-size effects, population fluctuations, s actual aging of the individuals, memory effects, clustering or community partitioning. The elucidation of the contribution of all these effects in the physical aging of temporal networks remains an open issue, deserving further empirical and theoretical effort.

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