We study discrete conjugate nets whose Laplace sequence is of period four. Corresponding points of opposite nets in this cyclic sequence have equal osculating planes in different net directions, that is, they correspond in an asymptotic transformation. We show that this implies that the connecting lines of corresponding points form a discrete W-congruence. We derive some properties of discrete Laplace cycles of period four and describe two explicit methods for their construction.

MSC 2010: 53A20 (primary), 51A20, 53A25.
Keywords: Discrete conjugate net, Laplace transform, asymptotic transform, discrete W-congruence, discrete projective differential geometry.

1. Introduction

In [12], H. Jonas established a couple of results pertaining to asymptotic transforms, W-congruences, conjugate nets and Laplace cycles of period four. This article explores discrete versions of Jonas’ theorems. It blends in with and extends more recent research on discrete conjugate nets, their Laplace transforms, discrete W-congruences and, in the limiting case, also discrete asymptotic nets. The original references to the discrete theory are [4–7, 14], an exposition of the current state of research can be found in [2].

Our contribution belongs to the field of discrete projective differential geometry. But we try to avoid the typical differential or difference equations and instead resort to synthetic reasoning. It is in the nature of our subject that we merely prove some incidence geometric results. This may seem like an old fashioned approach to an outdated topic. It is needless to say that we do not share this opinion, but it seems necessary to support our point of view by a few arguments:

- Arguably, the lack of ordering principles is one of the reasons why pure projective geometry is of little interest to today’s mathematicians. In contrast, a highly active research field, discrete differential geometry, has emerged from the desire to find discrete versions of differential geometric theorems [2]. We feel that differential geometry can serve as a guideline to single out “interesting” results from the wealth of projective incidence theorems.
- Our contributions generalize recent results on discrete conjugate nets and discrete asymptotic nets [4–7, 14]. Thus, our topic blends in with current research and should not be considered outdated.
• Relying on synthetic reasoning is of a particular appeal in the absence of metric structures. The reason is, that synthetic arguments clearly exhibit the few fundamental assumptions on which the theory is based, thus producing results that can be attributed to a very general type of projective geometry.

Referring to the last point, our results hold true for three-dimensional projective geometries over fields of characteristic \( \neq 2 \) and with sufficiently many elements (so that the mere formulation of certain results make sense). By Theorem 7 in [17, Section 50], commutativity of the underlying algebraic structure is equivalent to the validity of the fundamental theorem of projective geometry and the existence of doubly ruled surfaces [17, Section 103]. The last point is crucial for our considerations.

We continue this article by introducing some basic concepts and notations, like discrete conjugate nets and their Laplace transforms, in Section 2. The subsequent Section 3 features asymptotic transforms of (not necessarily conjugate) nets. The main result states that the discrete line congruence obtained by connecting corresponding points is a discrete W-congruence, see [5,6]. Theorem 13 shows how to construct asymptotic transforms on a given W-congruence. The results of this section comprise, as a limiting case, the theory of discrete asymptotic nets and their W-transform as laid out in [5,6,14]. Examples of asymptotically related nets can be obtained from a Laplace sequence of period four. They are studied in Section 4. Opposite nets of the cycle are asymptotically related so that their diagonal congruences are W-congruences. We conclude this article with two different methods for the construction of discrete Laplace cycles of period four.

Before continuing, we should mention that it is not difficult to obtain general conditions on conjugate nets that admit periodic Laplace cycles (in both, the smooth and the discrete case). Older references for the smooth case are [1,3,9], newer contributions include [10] and [16, Section 4.4]. The main problem is not the derivation of conditions but their geometric interpretation. The mentioned references provide some general results in this direction for the smooth case. This article is among the first contributions in the discrete setting.

2. Preliminaries

We denote by \( \mathbb{Z} \) the set of integers and by \( \mathbb{P}^3 \) the projective space of dimension three over the field \( \mathbb{K} \). We assume that the characteristic of \( \mathbb{K} \) is different from two.

**Definition 1.** A *discrete net* is a map

\[ f: \mathbb{Z}^2 \rightarrow \mathbb{P}^3, \quad (i, j) \mapsto f(i, j) =: f_i^j. \]

(Since we only deal with two-dimensional nets, we use the terser notation with upper and lower indices. The even short shift-notation of [2] seems not necessary for two-dimensional nets.) A *discrete conjugate net* is a discrete net such that every elementary quadrilateral \( f_i^j, f_i^{j+1}, f_{i+1}^j, f_{i+1}^{j+1} \) is planar.

The choice of \( \mathbb{Z}^2 \) as parameter space is only a matter of convenience. It allows us to ignore boundary conditions. We might as well admit sufficiently large sets of the shape

\[ \{(i, j) \in \mathbb{Z}^2 \mid i_0 < i < i_1 \text{ and } j_0 < j < j_1\} \]

with integers \( i_0 < i_1 \) and \( j_0 < j_1 \).
Throughout this paper, we make some regularity assumptions on the nets under consideration. Denote the span of projective subspaces by the symbol \( \vee \). We call a discrete net regular if for all \((i, j) \in \mathbb{Z}^2\) the osculating planes
\[
0_1 f^i_j := f^i_{j-1} \vee f^i_j \vee f^i_{j+1}, \quad \text{and} \quad 0_2 f^i_j := f^{i-1}_j \vee f^i_j \vee f^{i+1}_j,
\]
in the first and second net direction are well-defined, that is, of projective dimension two. For the regularity of a discrete conjugate net we require additionally, that the four points
\[
f^i_j, f^i_{j+1}, f^{i+1}_j, f^{i+1}_{j+1}
\]
are not collinear. Non-regular nets are called singular.

From a discrete conjugate net \( f \) we derive two further discrete conjugate nets by means of the Laplace transform \([1, 2]\):

**Definition 2.** The first and second Laplace transforms of a regular discrete conjugate net are the discrete conjugate nets
\[
\mathcal{L}_1 f: \mathbb{Z}^2 \to \mathbb{P}^3, \quad (i, j) \mapsto (O^i_j \cap f_{i+1}^{j+1}) \cap (O^i_j \cap f_{i+1}^{j+1}),
\]
\[
\mathcal{L}_2 f: \mathbb{Z}^2 \to \mathbb{P}^3, \quad (i, j) \mapsto (O^i_j \cap f_{i+1}^{j+1}) \cap (O^i_j \cap f_{i+1}^{j+1}).
\]
By regularity of \( f \), they are well-defined but not necessarily regular.

Note that we adopt the indexing convention of \([2\) pp. 76–77] since it is more convenient for our purposes than the convention used in of \([4, 7]\). An example is depicted in the lower right drawing of Figure 3, where \( \mathcal{L}_1 f := h \) and \( \mathcal{L}_2 f := h \). It is easy to see that \( \mathcal{L}_1 f \) and \( \mathcal{L}_2 f \) are indeed discrete conjugate nets due to
\[
\{ \mathcal{L}_k f^i_j, \mathcal{L}_k f^i_{j+1}, \mathcal{L}_k f^{i+1}_j, \mathcal{L}_k f^{i+1}_{j+1} \} \subset 0_k f^{i+1}_{j+1}, \quad k \in \{1, 2\}.
\]

Let us study repeated application of the Laplace transform. Because of \( \mathcal{L}_2 \mathcal{L}_1 f^i_j = \mathcal{L}_2 \mathcal{L}_1 f = f^{i+1}_{j+1} \), the composition of two Laplace transforms in different net directions effects only an index shift of \( f \). This is not interesting so that we focus on the compositions of Laplace transforms in the same net direction:

**Definition 3.** The \( i \)-th Laplace sequence to a discrete conjugate net \( f \) is the sequence \( l \mapsto \mathcal{L}_l^i f \) where \( \mathcal{L}_l^i f \) is recursively defined by \( \mathcal{L}_1^i f = \mathcal{L}_1 \mathcal{L}_1^{i-1} f \) and \( \mathcal{L}_0^i f = f \).

In general, both Laplace sequences
\[
\mathcal{L}_0^0 f, \mathcal{L}_1^1 f, \mathcal{L}_2^2 f, \ldots \quad \text{and} \quad \mathcal{L}_0^0 f, \mathcal{L}_1^1 f, \mathcal{L}_2^2 f, \ldots
\]
are infinite, at least if the field \( K \) is infinite. It is, however, possible that the iterated construction breaks down at some point due to net singularities. Another possibility is that the sequences become periodic. An instance of this is precisely the case we are interested in:

**Definition 4.** The Laplace sequences of a discrete conjugate net \( f \) are called a Laplace cycle of period four, if \( \mathcal{L}_1 f = \mathcal{L}_2 f \).

Indeed, in this case we have \( \mathcal{L}_1^{l+4} = \mathcal{L}_1^l \) and \( \mathcal{L}_2^{l+4} = \mathcal{L}_2^l \) for any positive integer \( l \). We suggest to think of a Laplace cycle of period four in terms of the cyclic sequence of four conjugate nets
\[
f, \quad h := \mathcal{L}_1 f, \quad g := \mathcal{L}_1^2 f = \mathcal{L}_2 f, \quad k := \mathcal{L}_2^2 f.
\]
The first or second Laplace sequence started from any of the nets \( f, h, g, \) and \( k \) are identical, up to indexing. Thus, the four nets can be treated on equal footing. We call the nets \( f \) and \( g \) (as well as \( h \) and \( k \)) opposite nets of the cycle. Examples are depicted in Figure 3 and Figure 4.
The necessity of our defining property is also established there. Its sufficiency is not difficult to see and, in fact, is implicitly used in later publications. A more elementary characterization of asymptotic transforms and W-congruences

In a Laplace cycle of period four, the osculating planes in different directions of opposite nets coincide, for example \( \mathcal{O}_1 f = \mathcal{O}_2 g \) and \( \mathcal{O}_2 f = \mathcal{O}_1 g \). Assume that corresponding points do not coincide \( (f_i^1 \neq g_i^1, h_i^1 \neq k_i^1) \) and that, at every vertex, the two osculating planes are different \( (\mathcal{O}_1 f_i^1 \neq \mathcal{O}_2 f_i^1, \mathcal{O}_1 h_i^1 \neq \mathcal{O}_2 h_i^1, \text{etc.}) \). Setting \( K_i^1 := f_i^1 \lor g_i^1 \) and \( L_i^1 := h_i^1 \lor k_i^1 \), we then have

\[
K_i^1 = \mathcal{O}_1 f_i^1 \cap \mathcal{O}_2 f_i^1 = \mathcal{O}_2 g_i^1 \cap \mathcal{O}_1 g_i^1, \\
L_i^1 = \mathcal{O}_1 h_i^1 \cap \mathcal{O}_2 h_i^1 = \mathcal{O}_2 k_i^1 \cap \mathcal{O}_1 k_i^1.
\]

In order to capture this relation between \( f \) and \( g \) (or \( h \) and \( k \)) in more generality, we study the axis congruence of a discrete net and its asymptotic transforms. Our terminology goes back to the smooth case and in particular to \([12]\) and \([18]\).

**Definition 5.** The axis congruence of a (not necessarily conjugate) net \( f: \mathbb{Z}^2 \to \mathbb{P}^3 \) is the map that sends a point \( (i, j) \in \mathbb{Z}^2 \) to the line \( \mathcal{O}_1 f_i^1 \cap \mathcal{O}_2 f_i^1 \). It is well-defined only for vertices where the two osculating planes are different.

**Definition 6.** Two discrete nets \( f \) and \( g \) are called asymptotically related or asymptotic transforms of each other, if \( \mathcal{O}_1 f = \mathcal{O}_2 g \) and \( \mathcal{O}_2 f = \mathcal{O}_1 g \).

Using these definitions, we may rephrase our findings of Section 2 as

**Proposition 7.** Opposite nets in a discrete Laplace cycle of period four have the same axis congruence and are asymptotically related.

Note that the concepts of osculating plane, axis congruence and asymptotic transform make perfect sense in the smooth setting and Proposition 7 holds true as well, see \([12]\).

For an important class of discrete nets the axis congruence is undefined at every vertex. These nets are characterized by \( \mathcal{O}_1 f = \mathcal{O}_2 f \) and are called asymptotic nets or \( A \)-nets, see \([5, 6, 14]\) or \([2, \text{Section 2.4}]\). We usually exclude them from our considerations. It should, however, be mentioned that asymptotic nets can be obtained from a general net \( f \) by a suitable passage to the limit that satisfies \( \lim_{i} \mathcal{O}_1 f = \lim_{i} \mathcal{O}_2 f \). In this case, the common limit of both osculating planes is the tangent plane \( T_f \) at the respective vertex. Two asymptotic nets \( f_i^1 \) and \( g_i^1 \) are said to be W-transforms of each other, if the lines \( f_i^1 \lor g_i^1 \) are contained in both tangent planes \( T_f^1 \) and \( T_g^1 \). Pairs of W-transforms appear as limit of pairs of asymptotic transforms. Indeed, our theory comprises the theory of discrete asymptotic nets and their W-transforms as limiting case. In Section 3.3 we explicitly describe how to set up this limiting process.

We denote the Grassmannian of lines in \( \mathbb{P}^3 \) by \( \mathbb{L}^3 \). Via the Klein map, a straight line is identified with a point on a quadric in \( \mathbb{P}^5 \), the Plücker quadric. Klein map and Plücker quadric in real projective three-space are described in detail in \([15, \text{Section 2.1}]\). There is no essential difference in projective spaces over commutative fields.

**Definition 8.** A map \( L: \mathbb{Z}^2 \to \mathbb{L}^3 \) is called a W-congruence if its Klein image on the Plücker quadric is a conjugate net.

W-congruences over real projective spaces have been introduced in \([5]\) as discrete line congruences that connect corresponding points of a discrete asymptotic net and its W-transform. The necessity of our defining property is also established their. Its sufficiency is not difficult to see and, in fact, is implicitly used in later publications.
W-congruences simply demands that the four lines $A_i^1, A_i^{j+1}, A_i^{+1}, A_i^i$ belong to a regulus [17, Section 103], that is, they are skew lines on a doubly ruled surface.

§ 3.1. W-congruences as axis congruence of asymptotic transforms. Our main result on asymptotic transforms is the discrete version of [12, p. 248].

**Theorem 9.** If $f$ and $g$ are asymptotic transforms of each other, their common axis congruence is a W-congruence.

**Proof.** Denote the lines of the common axis congruence of $f$ and $g$ by $A_i^1 = f_i^j \lor g_i^j$. Because of $0_1 f_i^j = 0_2 g_i^j$ and $0_2 f_i^{j+1} = 0_1 g_i^{j+1}$ the straight line $f_i^j \lor g_i^{j+1}$ intersects the four lines

$$A_i^1, A_i^{j+1}, A_i^{j+1}, A_i^{+1}.$$  

(1)

Similar reasoning shows that the same is true for the lines $f_i^{j+1} \lor g_i^{j+1}, f_i^{j+1} \lor g_i^j$, and $f_i^{j+1} \lor g_i^{j+1}$. Thus, the four lines (1) admit four transversal lines. This is only possible, if they are skew generators on a hyperboloid.

The proof of Theorem 9 allows a different interpretation. The point $g_i^{j+1}$ is the projection of $f_i^j$ onto $A_i^{j+1, j+1}$ from the center $A_i^{j+1}$ and, at the same time, from the center $A_i^i$. The W-congruence property guarantees that both projections always yield the same result. This projective relation between the ranges of points $f_i^j \in A_i^1$ and $g_i^{j+1} \in A_i^{j+1}$ is important enough to introduce a new notation.

**Definition 10.** Given three pairwise skew lines $A, B, C$ we denote by $C \uparrow B_A$ the projection of $A$ onto $B$ from the center $C$. More precisely, $C \uparrow B_A$ maps the point $a \in A$ to the point $(a \lor C) \cap B$.

With this notation, we have for example

$$A_i^1 \uparrow^{A_i^{j+1}} A_i^{j+1} = A_i^{j+1} \uparrow^{A_i^1} A_i^{j+1} \text{ or } A_i^1 \uparrow^{A_i^{j+1}} A_i^j = A_i^{j+1} \uparrow^{A_i^1} A_i^j.$$  

When range and image line stem from the same line congruence, we make this notation a little more readable by writing only the upper and lower index:

$$A_i^1 \uparrow^{i, j+1} A_i^{j+1} = A_i^{j+1} \uparrow^{i, j+1} A_i^{j+1} \text{ or } A_i^1 \uparrow^{i+1, j} A_i^{j+1} = A_i^{j+1} \uparrow^{i+1, j} A_i^{j+1}.$$  

(2)

A diagram of diverse projections between the lines of the congruence $\Lambda$ is depicted in Figure 1. Equation (2) justifies the drawing of diagonal arrows without indicating the projection center. For example, the arrow between $A_i^1$ and $A_i^{j+1}$ denotes projection between these two lines from either $A_i^{j+1}$ or $A_i^{j+1}$. 

§ 3.2. Asymptotic transforms on a given W-congruence. As a natural next step we investigate the question whether it is possible to construct a pair $f, g$ of asymptotically related nets whose vertices lie on a given W-congruence $\Lambda$. This turns out to be feasible in multiple ways. Essentially, we are allowed to choose the values of $f$ (or $g$) on an elementary quadrilateral. Theorem 13 below is a little more general. Its proof requires an auxiliary result.

**Lemma 11.** $A_i^{j+1, j} \uparrow^{i, j+1} A_i^{l, j+1} = (A_i^{l+1, j+1} \uparrow^{i, j+1} A_i^{l+1, j}) \circ (A_i^{j+1, j} \uparrow^{i, j+1} A_i^{l, j+1})$ and $A_i^{j+1, j} \uparrow^{i+1, j} A_i^{l+1, j} = (A_i^{j+1, j} \uparrow^{i+1, j} A_i^{l+1, j}) \circ (A_i^{j+1, j} \uparrow^{i, j+1} A_i^{l+1, j})$. 

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### Proof.
Consider a point $x \in A_1^i$. It defines the plane $\xi = x \vee A_{i+1}^j$. Clearly, $y := A_{i+1}^{j+1} \uparrow_{i,j+1}^i(x) = A_{i+1}^i \cap \xi$, and thus also $A_{i+1}^{j+2} \uparrow_{i+1,j+1}^1(y) = A_{i+2}^j \cap \xi = A_{i+1}^{j+1} \uparrow_{i,j}^1(x)$. The second claim is obtained from the first by interchanging the two net directions. 

The interpretation of Lemma 11 in the diagram of Figure 1 is that the projection paths along the sides of the rectangle can be decomposed into the paths from start point to center and from center to end-point.

#### Definition 12.
We call an index pair $(i, j)$ even-even if both $i$ and $j$ are even. Similarly, we speak of odd-odd, even-odd, or odd-even index pairs. We say that two index pairs are of the same parity if both are even-even, even-odd, odd-even, or odd-odd. A black vertex is even-even or odd-odd and a white vertex is either even-odd or odd-even.

#### Theorem 13.
On a given W-congruence we can find a four-parametric set of pairs of asymptotic transforms. Every such pair $f$, $g$ is uniquely determined by the values of $f$ at four vertices of pairwise different parity.

#### Proof.
Assume that the vertex $f_{1}^i$ is given. Clearly, we have

$$f_{i+2}^j = A_{i+1}^{j+1} \downarrow_{i,j}^1(f_{1}^i), \quad f_{i-2}^j = A_{i-1}^{j-1} \uparrow_{i,j}^1(f_{1}^i),$$

$$f_{i}^{j+2} = A_{i}^{j+1} \uparrow_{i,j}^1(f_{1}^i), \quad f_{i}^{j-2} = A_{i}^{j-1} \downarrow_{i,j}^1(f_{1}^i).$$

Proceeding in like manner, we can construct the values of $f$ on all vertices of the same parity as $(i, j)$. Thus, if there exists a net $f$ with axis congruence $A$, it is uniquely determined by the initial data. Since the net $f$ also determines all osculating planes $\partial_1 f_{1}^i$, $\partial_2 f_{1}^i$, the net $g$ is determined as well, for example $g_{i+1}^j = A_{i+1}^j \cap \partial_1 f_{1}^i$. Because of $A_{i+1}^{j+1} \downarrow_{i,j}^1 = A_{i+1}^{j+1} \uparrow_{i,j}^1$, the construction of $g$ is not ambiguous. Moreover, $f$ and $g$ are asymptotic transforms with axis congruence $A$.

We still have to show existence of $f$. This is necessary because the construction of the vertices of $f$ might produce a contradiction. The prototype case of this is the construction of $f_{i+2,j+2}^i$ in two ways according to

$$f_{i}^i \xrightarrow{A_{i+1}^{j+1} \rightarrow f_{i+2}^i} A_{i+2}^{j+1} \xrightarrow{f_{i+2}^i} A_{i+1}^{j+2} \xrightarrow{f_{i+2}^i} A_{i+2}^{j+2} \xrightarrow{f_{i+2}^i} f_{i+2}^i. \quad (3)$$

---

Figure 1: Diagram of projections
If we can show that both routes in (3) yield the same value, the validity of our construction is guaranteed. In other words, we have to show
\[
(A_{i+2} \uparrow^2 \downarrow^2_{i,j}) \circ (A_{i+1} \uparrow^2_{i,j}) = (A_{i+1} \uparrow^2_{i,j}) \circ (A_{i+1} \uparrow^2_{i,j+1}).
\]
A glance at Figure 7 immediately confirms this: The projection paths along two adjacent sides of the rectangle equals the diagonal projection path. A formal argument uses Lemma 11 and the identities (2). We have
\[
(A_{i+2} \uparrow^2 \downarrow^2_{i,j}) \circ (A_{i+1} \uparrow^2_{i,j}) = (A_{i+1} \uparrow^2_{i,j+1}) \circ (A_{i+1} \uparrow^2_{i,j+1})
\]
\[\text{identity on } A_{i+1} \uparrow^2_{i,j}.\]
By interchanging the two net directions, we arrive at
\[
(A_{i+2} \uparrow^2 \downarrow^2_{i,j}) \circ (A_{i+1} \uparrow^2_{i,j+1}) = (A_{i+1} \uparrow^2_{i,j+1}) \circ (A_{i+1} \uparrow^2_{i,j+1}).
\]  
Equation (2) shows that the expressions after the last equal sign in (4) and (5) are equal. \(\Box\)

It is interesting to compare Theorem 13 with the results of [12]. While Theorem 13 shows existence of a four-parametric set of asymptotically related nets on the discrete W-congruence \(A\), there only exists a one-parametric set of such surfaces on a smooth W-congruence.

§ 3.3. Asymptotic nets as limiting case. We already mentioned that asymptotic nets related by a W-transform can be seen as limiting case of pairs of asymptotically related nets. Here, we provide more details on this remark.

Consider a W-congruence \(A\). By Theorem 13, we can construct pairs \(f, g\) of asymptotic transforms on \(A\) by prescribing suitable values of \(f\) on an elementary quadrilateral. The value of \(f\) on an even-even vertex determines the values of \(0_1 f = O_2 g\) on all odd-even vertices, the values of \(0_2 f = O_1 g\) on all even-odd vertices, and the values of \(g\) on all odd-odd vertices. Alternatively, we can also prescribe the values of \(0_1 f \) and \(0_2 f\) on a black and a white vertex. If we make the additional requirement \(0_1 f = 0_2 f = O_1 g = O_2 g\), a similar construction is possible but only one plane on a black and one plane on a white vertex can be chosen. The resulting nets \(f\) and \(g\) will then form a pair of asymptotic nets, related by a W-transform.

Consider now four sequences
\[
(0_1 f^0_1)_n, \quad (0_2 f^0_2)_n, \quad (0_1 f^0_1)_n, \quad (0_2 f^0_2)_n
\]
of planes such that for each \(n\)
\[
A_0^0 \subset (0_1 f^0_1)_n, \quad A_0^0 \subset (0_2 f^0_2)_n, \quad A_1^0 \subset (0_1 f^0_1)_n, \quad A_1^0 \subset (0_2 f^0_2)_n,
\]
and
\[
\lim_{n \to \infty} (0_1 f^0_1)_n = \lim_{n \to \infty} (0_2 f^0_2)_n, \quad \lim_{n \to \infty} (0_1 f^0_1)_n = \lim_{n \to \infty} (0_2 f^0_2)_n.
\]
For each \(n\), the planes in (6) define asymptotically related nets \(f_n\) and \(g_n\) and the point-wise limits \(f = \lim_{n \to \infty} f_n\) and \(g = \lim_{n \to \infty} g_n\) exist. By construction, the nets \(f\) and \(g\) are asymptotic nets related by a W-transform.
§ 3.4. Two further results. We conclude this section with two simple but curious observations on the cross-ratio of asymptotic transforms on the same W-congruence and on their local configuration.

Corollary 14. Consider two pairs \( f, g \) and \( f', g' \) of asymptotically related conjugate nets on the same W-congruence \( A \). Then the cross-ratio

\[
CR(f_{i,j}, f'_{i,j}; g_{i,j}, g'_{i,j})
\]

is constant on all black and on all white vertices, respectively.

**Proof.** Because vertices of the same parity are obtained from the vertices on \( A_{i,j} \) by a series of projections, they give rise to the same cross-ratio \( [7] \). But also the vertices \( g_{i+1,j+1} \) and \( g'_{i+1,j+1} \) are the projections of \( f_{i,j} \) and \( f'_{i,j} \) from the center \( A_{i+1,j} \) (or from the center \( A_{i,j+1} \)). Thus,

\[
CR(f_{i,j}, f'_{i,j}; g_{i,j}, g'_{i,j}) = CR(g_{i+1,j+1}, g'_{i+1,j+1}; f_{i+1,j+1}, f'_{i+1,j+1}) = CR(f_{i+1,j+1}, f'_{i+1,j+1}; g_{i+1,j+1}, g'_{i+1,j+1}).
\]

This is precisely what had to be shown. \( \square \)

In [12], H. Jonas proved that for each line of a smooth W-congruence \( A \), the vertices of two asymptotic transforms \( f \) and \( g \) on \( A \) are harmonic with respect to the two asymptotically parametrized focal nets of \( A \). In view of Theorem [13] and Section 3.3, this theorem cannot have a discrete version and Corollary 14 is the closest result we can get.

Our next result studies the local geometry of an elementary quadrilateral in asymptotically related nets. Recall that two quadruples of points \( (a_0,a_1,a_2,a_3) \) and \( (b_0,b_1,b_2,b_3) \) form a pair of Möbius tetrahedra if

\[
a_i \in b_j \lor b_k \lor b_l \quad \text{and} \quad b_i \in a_j \lor a_k \lor a_l
\]

for any choice of pairwise different \( i,j,k,l \in \{1,2,3,4\} \). In other words, the vertices of one tetrahedron lie in the face planes of the other.

**Proposition 15.** Assume that \( f \) and \( g \) is a pair of asymptotically related nets. Then the two quadruples

\[
(a_0,a_1,a_2,a_3) := (f_{0,0}, f_{1,0}, g_{0,1}, g_{1,1}) \quad \text{and} \quad (b_0,b_1,b_2,b_3) := (f_{1,1}, f_{0,1}, g_{1,0}, g_{0,0})
\]

form the vertices of a pair of Möbius tetrahedra.

**Proof.** The proof is simply a matter of comparing the Möbius conditions \( [8] \) with the conditions on the asymptotic transform. For example \( a_0 \in b_1 \lor b_2 \lor b_3 \) is true because of \( O_2 f_{0,0} = O_1 g_{0,0} \). \( \square \)

Proposition 15 yields a different possibility for proving Theorem 9 by means of \( [8] \) Equation (1.0)] and a straightforward calculation. The classic result of Möbius in [13] states that seven of the eight incidence conditions \( [8] \) imply the eighth. This means that the conditions on the asymptotic relation between two discrete nets \( f \) and \( g \) are not independent either.

4. Periodic Laplace cycles

In this section we specialize the results of Section 3 to opposite nets \( f, g \) (or \( h, k \)) in a Laplace cycle \( f, h, g, k \) of period four.
§ 4.1. General results. Call the set of connecting lines of corresponding points of one pair of opposite nets a diagonal congruence of the cycle. As an immediate consequence of Theorem 9 and the fact that opposite nets in a Laplace cycle of period four are asymptotically related we have

**Corollary 16.** The two diagonal congruences of a discrete Laplace cycle of period four are W-congruences.

A given W-congruence $A$ is, in general, not the diagonal congruence of a Laplace cycle of period four. In other words, it is impossible to find conjugate nets $f$ and $g$ such that $f_i, g_i \in A_i$ for all indices $(i, j)$. But the observation at the beginning of [12, Section 3] also holds in the discrete setting:

**Theorem 17.** If a W-congruence $A$ appears as the axis congruence of a discrete conjugate net $f$, the net $f$ gives rise to a Laplace cycle of period four.

This theorem is an immediate consequence of Lemma 19, below.

**Lemma 18.** Consider two spatial quadrilaterals $(a_0, a_1, a_2, a_3)$ and $(b_0, b_1, b_2, b_3)$ such that for $i \in \{0, 1, 2, 3\}$ the lines $a_i \lor a_{i+1}$ and $b_i \lor b_{i+1}$ intersect in a point $f_i$ and span a plane $\varphi_i$ (indices modulo four). Then the four points $f_0, f_1, f_2,$ and $f_3$ lie in a plane $\varphi$ if and only if the four planes $\varphi_0, \varphi_1, \varphi_2,$ and $\varphi_3$ contain a point $f$. In this case, the quadrilaterals correspond in a perspective collineation [17, Section 29] with center $f$ and plane of perspectivity $\varphi$ (Figure 2).

**Proof.** The statement is self-dual. Thus, we only have to prove one implication. Assume that the four planes $\varphi_0, \varphi_1, \varphi_2, \varphi_3$ intersect in a point $f$ and consider the central perspectivity with center $f$ and axis $\overline{f_0} \lor f_1 \lor f_2$ that transforms $a_0$ to $b_0$. Clearly, the image of all points $a_i$ is the corresponding point $b_i$ for $i \in \{0, 1, 2, 3\}$. Thus, the point $f_3$ also lies in $\overline{f}$.

**Lemma 19.** Given are four skew lines $A_0, A_1, A_2, A_3$ of a regulus and a plane $\varphi$ not incident with either of these lines. For $i \in \{0, 1, 2, 3\}$ set

$$f_i := \varphi \cap A_i \quad \text{and} \quad g_{i+2} := A_{i+2} \cap (f_i \lor A_{i+1})$$
(indices modulo four). Then the four points \( g_0, g_1, g_2, g_3 \) lie in a plane as well \( (\text{Figure 2}) \).

**Proof.** For \( i \in \{0, 1, 2, 3\} \) denote by \( B_i \) the hyperboloid’s second generator (different from \( A_i \)) through \( f_i \) so that \( g_{i+2} = A_{i+2} \cap B_i \) (indices modulo four). Consider now the two spatial quadrilaterals whose edges are the lines \( A_0, B_1, A_2, B_3 \) and \( B_0, A_1, B_2, A_3 \). Since their intersection points \( f_0, f_1, f_2, f_3 \) are coplanar, we can apply Lemma 18 and see that the planes \( \phi_i := A_i \cap B_i \) intersect in a point \( f \). Moreover, the two spatial quadrilaterals correspond in a central perspective with center \( f \) and plane of perspectivity \( \phi \). This implies that the point \( f \) lies on \( g_0 \lor g_2 \) and \( g_1 \lor g_3 \). In particular, the four points \( g_0, g_1, g_2 \) and \( g_3 \) are coplanar. \( \square \)

Theorem 17 gives us a simple means to test whether a conjugate net \( f \) has a Laplace sequence of period four. Admittedly, an equally simple test consists of the construction of the Laplace transforms. A more urgent characterization is that of the diagonal congruences of a discrete Laplace cycle of period four. They are necessarily W-congruences but this is not sufficient.

Starting with a W-congruence and an undermined face plane of an elementary quadrilateral we can compute the intersection points of the face plane with the corresponding axes and neighbouring vertices (by means of considerations that already appear in our proof of Theorem 13). Exploiting the planarity conditions, we arrive at a system of algebraic equations that have no solutions in general. Theorem 17 makes us conjecture that in special cases this system can be reduced to a quadratic equation, thus giving rise to a pair of two discrete conjugate nets on the given congruence. Maybe even a continuum of solutions is feasible in non-trivial cases.

§ 4.2. Construction of Laplace cycles of period four. Now we are going to present two methods for constructing a Laplace cycle \( f, h, g, k \) of period four. The first method works on the level of the nets while the second method requires as input partial information on the net \( f \) and the diagonal congruence \( A \) through \( f \).

The first construction is illustrated in \( \text{Figure 3} \). The top-left picture shows the initialization. We arbitrarily prescribe \( f_0^0, g_0^0, h_0^0 \), and \( k_0^0 \). Then we have four degrees of freedom to choose

\[
f_1^0 \in f_0^0 \land h_0^0, \quad f_0^1 \in f_0^0 \land k_0^0, \quad g_1^0 \in g_0^0 \land k_0^0, \quad g_0^1 \in g_0^0 \land h_0^0.
\]

These points define

\[
f_1^1 = (f_0^0 \land k_0^0) \cap (f_0^0 \land h_0^0) \quad \text{and} \quad g_1^1 = (g_0^0 \land h_0^0) \cap (g_0^0 \land k_0^0).
\]

Next, we have four more degrees of freedom to choose points

\[
h_1^0 \in h_0^0 \land g_1^0, \quad h_0^1 \in h_0^0 \land f_1^0, \quad k_1^0 \in k_0^0 \land f_1^0, \quad k_0^1 \in k_0^0 \land g_1^0
\]

(\text{Figure 3} top-right). They define

\[
h_1^1 = (h_0^0 \land f_1^0) \cap (h_0^1 \land g_1^0) \quad \text{and} \quad k_1^1 = (k_0^0 \land g_1^0) \cap (k_0^1 \land f_1^0).
\]

The same steps can be repeated for obtaining adjacent quadrilaterals in the first or second net direction. The respective steps in the first and second net direction are depicted in the middle row of \( \text{Figure 3} \). The only difference is that certain input points are already prescribed so that only four degrees of freedom per face remain. Once all points \( f_i^j, g_i^j, h_i^j, k_i^j \) with \( i \in \{0, 1\} \) and \( j \in \mathbb{Z} \) or \( j \in \{0, 1\} \) and \( i \in \mathbb{Z} \) are found, the remaining points of all four nets are uniquely

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Figure 3: Construction of a Laplace chain of period four
determined (Figure 3, bottom row). A Laplace cycle of period four in three-space is shown in Figure 4.

The presented construction simultaneously builds up the nets $f$, $h$, $g$, and $k$. But it is also possible to directly determine only one of the nets, say $f$, and its axis congruence $A$. The construction is described in the proof of Theorem 22 below. It requires some auxiliary concepts and results.

**Definition 20 (projection between conics).** Consider two conics $C$, $D \subset \mathbb{P}^3$ with a common point $z$ and a straight line $Z$ through $z$ that does not lie in the conics’ planes. Denote by $\gamma$ the plane spanned by $Z$ and the tangent of $C$ in $z$ and by $\delta$ the plane spanned by $Z$ and the tangent of $D$ in $z$. If $\gamma \neq \delta$, set $c = (C \cap \delta) \setminus z$ and $d = (D \cap \gamma) \setminus z$. Otherwise, set $c = d = z$. The projection $Z^D_C$ of $C$ onto $D$ from the center $Z$ is the map

$$x \in C \mapsto \begin{cases} 
  z & \text{if } x = c, \\
  d & \text{if } x = z, \\
  \{D \cap (x \vee Z)\} \setminus z & \text{else.}
\end{cases} \quad (9)$$

This definition also makes sense if one or two of the conics consists of a pair of intersecting lines. In this case, any line in the conic’s plane through the intersection point $s$ of the two lines is regarded as tangent. Moreover, we require $z \neq s$.

We do not expect any confusion arising from the use of the same notation for the projections between conics and between lines. In fact, the projection between two lines $K$ and $L$ can be seen as a degenerate case of the projection between the conics $K \cup M$ and $L \cup M$ where $M$ is a transversal of $K$ and $L$. With this understanding, we need not distinguish regular and degenerate conics in the following.
The union of all points on the lines of a regulus is called a **quadric surface** or simply a **quadric** [17, p. 301]. The intersection of a quadric with a plane is a (possibly degenerate) conic.

**Lemma 21.** Consider five lines \( A_i^1 \) and five points \( f_i^1 \in A_i^1 \) with \( (i, j) \in \{(0, 0), (±1, 0), (0, ±1)\} \). For \( (i, j) = (±1, ±1) \) denote by \( Q^1_i \) the quadric through \( A^0_0, A^0_1 \) and \( A^0_{-1} \) and by \( C^1_i \) the (possibly degenerate) conic \( Q^1_i \cap \{f_0^1 \lor f_0^1 \lor f_0^1\} \). Then the composition of projections

\[
(A_{0}^{-1} \uparrow C_{-1}^{-1}) \circ (A_{0}^{1} \uparrow C_{1}^{1}) \circ (A_{0}^{-1} \uparrow C_{-1}^{1})
\]

is the identity on \( C_{-1}^{-1} \).

**Proof.** Given a point \( f_1^1 \in C_1^{-1} \), there exists a unique line \( B_1^1 \) on \( Q_i^1 \) that intersects \( A_0^0 \) in a point \( b_1 \). Conversely, any points \( b_1 \in A_0^0 \) gives rise to a point \( f_1 \in C_1^{-1} \). Similarly, the points of \( C_{-1}^{-1} \) and \( C_{-1}^{-1} \) are related to the points of \( A_0^0 \). The crucial observation is now that \( f_{-1}^1 \) and \( f_1^1 \) correspond in the projection \( A_0^1 \uparrow C_0^1 \) if and only if their corresponding points on \( A_0^0 \) are equal. This is true due to

\[
b_{-1} = (f_{-1}^1 \lor A_0^1) \cap A_0^0 = (f_1^1 \lor A_0^1) \cap A_0^0 = b_1.
\]

Thus, starting from \( f_1^1 \), we can construct the remaining points \( f_{-1}^1, f_{-1}^1, f_{-1}^1 \) and \( f_1^1 \), and they all correspond to the same point \( b_1 = b_{-1} = b_1 = b_{-1} \in A_0^0 \). We infer that after four successive projections, we obtain the initial point \( f_1^1 \). This finishes the proof. \( \square \)

**Theorem 22.** Consider a Laplace cycle \( f, h, g, k \) of period four and denote its diagonal congruence by \( A \). The Laplace cycle is uniquely determined by

- suitable values of \( f \) and \( A \) on all vertices \((i, 0)\) and \((0, i)\) with \( i \in \mathbb{Z} \) and
- the value of \( f_1^1 \) in the intersection of the plane \( f_0^0 \lor f_0^1 \lor f_1^1 \) and the quadric spanned by the three lines \( A^0_0, A^0_1, \) and \( A^0_{-1} \).

Here, the points \( f_0^0, f_1^1 \), and lines \( A^0_0, A^0_1 \) are said to be in “suitable position”, if for any \( i \in \mathbb{Z} \) the conditions

- \( A_i^0 \cap A_{i+1}^0 = \emptyset, A_i^1 \cap A_{i+1}^1 = \emptyset, \)
- \( A_i^0 \subset f_0^0 \lor f_0^1 \lor f_1^1, A_i^1 \subset f_{-1}^1 \lor f_0^1 \lor f_{-1}^1, \)
- \( f_i^1 \subset A_i^1, f_i^0 \subset A_i^0, \)

are fulfilled.

**Proof.** We observe at first, that the given data determines the nets \( f \) and \( A \) uniquely. Since \( f_1^1 \) lies on the quadric of \( A^0_0, A^0_1 \) and \( A^0_{-1} \), the line \( A^1_1 \) is well-defined. The vertex \( f_{-1}^1 \) necessarily lies in the planes

\[
f_1^1 \lor A_0^1 \quad \text{and} \quad f_0^0 \lor f_0^1 \lor f_0^1
\]

and on the quadric \( Q \) through \( A^0_0, A^0_1, A^0_{-1} \). Thus, it is the projection of \( f_1^1 \) from the conic \( C_1^1 = Q_1 \cap \{f_0^0 \lor f_0^1 \lor f_1^1\} \) onto the conic \( C_{-1}^1 = Q_{-1} \cap \{f_0^0 \lor f_0^1 \lor f_1^1\} \). Once \( f_{-1}^1 \) is found, the line \( A_{-1}^1 \) is determined as well. Proceeding in like manner, we can inductively construct \( f \) as a discrete conjugate net and \( A \) as a discrete W-congruence. This shows uniqueness of \( f \) and \( A \) and, by Theorem [17] also of the complete Laplace cycle.
As to existence, we have to consider all possibilities to run into a contradiction. For example, we have to guarantee that the values for constructing $f_{-1}$ via the routes

$$f^1_0 \xleftarrow{c_1} f^1_{-1} \xrightarrow{c_1} f^1_{-1} \xrightarrow{c_1} f^1_{-1} \xrightarrow{c_1} f^1_{-1}$$

yield the same result. This is indeed the case by virtue of Lemma 21. This lemma also shows that, given the vertices $f^1_i$ for all $i, j \in \{-1, 0, 1\}$ plus the vertex $f^2_0$, the vertices $f^1_2$ and $f^1_2$ can be constructed without contradiction. The same is true for all vertices $f^1_i, f^1_{-1}$ by induction and for all vertices $f^1_i, f^1_{-1}$ by analogy. Finally, Lemma 21 is also responsible for the fact that given the vertices $f^1_i$ for $i, j \in \{0, 1, 2\}$ but different from $i = j = 2$, the two possibilities for constructing $f^1_2$ coincide. This allows the contradiction-free construction of all remaining vertices so that existence of the conjugate net $f$ is shown as well.

5. Conclusion

We have shown that many results of [12] on asymptotically related nets and, in particular, conjugate nets also hold in a discrete setting, thus adding some new insight into the already established geometry of discrete conjugate and asymptotic nets and their transformations. It turned out that some of Jonas’ results admit similar but not identical discrete interpretations, for example, Theorem 13 and Corollary 14.

The subject of our research is not yet exhausted. Besides the characterization of diagonal congruences of Laplace cycles of period four mentioned in Section 4.1, a worthy topic of future research seem to be the considerations of [11] on transformations of Laplace cycles of period four. A proper discrete transformation theory could integrate our topic into the “consistency as integrability” paradigm of [2]. We will attempt this in a future publication.

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