FRACTAL DIMENSION FOR A CLASS OF COMPLEX-VALUED 
FRACTAL INTERPOLATION FUNCTIONS

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Abstract. There are many research papers dealing with fractal dimension of 
real-valued fractal functions in the recent literature. The main focus of 
the present paper is to study fractal dimension of complex-valued functions. This 
paper also highlights the difference between dimensional results of the complex-
valued and real-valued fractal functions. In this paper, we study the fractal 
dimension of the graph of complex-valued function \( g(x) + ih(x) \), compare its 
fractal dimension with the graphs of functions \( g(x) + h(x) \) and \( (g(x), h(x)) \) 
and also obtain some bounds. Moreover, we study the fractal dimension of the 
graph of complex-valued fractal interpolation function associated with a germ 
function \( f \), base function \( b \) and scaling functions \( \alpha_k \).

1. INTRODUCTION

Fractal dimension is one of the major themes in Fractal Geometry. Estima-
tion of fractal dimension of sets and graphs has received a lot of attention in the 
literature \cite{8}. In 1986, Barnsley \cite{11} introduced the idea of fractal interpolation 
functions (FIF) and computed the Hausdorff dimension of affine FIF. Estimation 
of box dimension for a class of affine FIFs presented in \cite{13, 14, 3}. Several authors 
\cite{15, 12, 23, 17, 24} also calculated the fractal dimension of the graph of FIF. In 1991, 
Massopust \cite{18} estimated the box dimension of graph of vector-valued FIF. Later 
Hardin and Massopust \cite{4} constructed fractal interpolation functions from \( \mathbb{R}^n \) to 
\( \mathbb{R}^m \) and also given the formula for calculating the box dimension. We encour-
age the reader to see some recent works on fractal dimension of fractal functions defined 
on different domains such as Sierpinski gasket \cite{1, 25}, rectangular domain \cite{21} and 
interval \cite{23, 24}. To the best of our knowledge, we may say that there is no work 
available for dimension of complex-valued fractal functions. Here we give some ba-
sic results for complex-valued FIF and provide some results to convince the reader 
that there is some difference between dimensional result of the complex-valued and 
real-valued fractal functions.

In 1986, Mauldin and Williams \cite{20} were the pioneers who studied the problem 
of decomposition of the continuous functions in terms of fractal dimensions. They 
proved the existence of decomposition of any continuous function on \([0, 1]\) into 
sum of two continuous functions, where each have Hausdorff dimension one. Later 
in 2000, Wingren \cite{19} gave a technique to construct the above decomposition of 
Mauldin and Williams. Moreover, he proved the same type result of Mauldin and 
Williams for the lower box dimension. Bayart and Heurteaux \cite{5} also proved similar 
result for Hausdorff dimension \( \beta = 2 \), and raised the question for \( \beta \in [1, 2] \). Recently 
in 2013, Jia Liu and Jun Wu \cite{7} solved the question which was raised by Bayart and

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Haurteaux. More precisely they proved that, for any $\beta \in [1, 2]$, each continuous function on $[0, 1]$ can be decomposed into sum of two continuous functions, where each have Hausdorff dimension $\beta$. Falconer and Fraser [9] found an upper bound for upper box dimension of graph of sum of two continuous functions which depends on dimension of both of graphs.

In [7, 9], it is clear that the Hausdorff dimension of graph of $g + h$ does not depend on the Hausdorff dimension of graph of $g$ and $h$ whereas, the upper box dimension depends on both. Motivated from this we think about the behaviour of Hausdorff dimension of graph of $g + ih$, whether it depends on the Hausdorff dimension of graphs of $g$ and $h$ or not? We obtained an affirmative answer for this question. Also the upper box dimension of $g + ih$ depends on the upper box dimensions of $g$ and $h$ which is quite different from the upper box dimension of $g + h$. Finally, we studied some relations between fractal dimensions of the graphs of $g(x) + ih(x)$, $g(x) + h(x)$ and $(g(x), h(x))$.

The paper is organized as follows. In upcoming section 2, we give some preliminary results and required definition for next section. Section 3 consists of some dimensional results for complex-valued continuous functions and FIFs. In this section, first we establish some propositions and lemmas to form a relation between the fractal dimension of complex-valued and real-valued continuous functions. After that, we determine bound of Hausdorff dimension of $\alpha$-fractal function under some assumption. We also obtain some conditions under which $\alpha$-fractal function becomes a Hölder continuous function and bounded variation function, and calculate its fractal dimension.

2. PRELIMINARIES

Definition 2.1. Let $F$ be a subset of a metric space $(Y, d)$. The Hausdorff dimension of $F$ is defined as follows

$$\dim_H F = \inf \{\beta > 0 : \text{for every } \epsilon > 0, \text{there is a cover } \{V_i\} \text{ of } F \text{ with } \sum |V_i|^\beta < \epsilon \}$$

Definition 2.2. The box dimension of a non-empty bounded subset $F$ of a metric space $(Y, d)$ is defined as

$$\dim_B F = \lim_{\delta \to 0} \log N_\delta(F) / -\log \delta,$$

where $N_\delta(F)$ is the minimum number of sets of diameter $\delta > 0$ that can cover $F$. If this limit does not exist, then limsup and liminf is known as upper and lower box dimension respectively.

Definition 2.3. For $r > 0$ and $t \geq 0$, let $P_t^r(F) = \sup \{ \sum_i |B_i|^t \}$, where $\{B_i\}$ is a collection of disjoint balls of radii at most $r$ with centres in $F$. As $r$ decreases, $P_t^r$ also decreases. Therefore the limit

$$P_0^t(F) = \lim_{r \to 0} P_t^r(F)$$

exists. We define

$$P_t^r(F) = \inf \left\{ \sum_i P_0^t(F_i) : F \subset \bigcup_{i=1}^{\infty} F_i \right\},$$
and it is known as the $t$-dimensional packing measure. Packing dimension is defined as follows:

$$\dim_p(F) = \inf \{ t \geq 0 : P^t(F) = 0 \} = \sup \{ t \geq 0 : P^t(F) = \infty \}.$$  

**Note**- We denote graph of function $f$ by $G(f)$ throughout this paper.

**Remark 2.4.** $f = f_1 + if_2 : [a, b] \to \mathbb{C}$ is a Hölder continuous function with Hölder exponent $\sigma$ if and only if $f_i$ is also Hölder continuous function with Hölder exponent $\sigma$ for every $i = 1, 2$.

**Theorem 2.5.** If $f : [0, 1] \to \mathbb{R}$ is a Hölder continuous function with the Hölder exponent $\sigma \in (0, 1)$. Then $\overline{\dim}_H(G(f)) \leq 2 - \sigma$.

2.1. **Iterated Function Systems.** Let $(Y, d)$ be a complete metric space, and we denote the family of all nonempty compact subsets of $Y$ by $H(Y)$. For any $A_1, A_2 \in H(Y)$, we define the Hausdorff metric by

$$h(A_1, A_2) = \inf \{ \delta > 0 : A_1 \subset A_2 \delta \text{ and } A_2 \subset A_1 \delta \},$$

where $A_1 \delta$ and $A_2 \delta$ denote the $\delta$-neighbourhood of sets $A_1$ and $A_2$, respectively. It is well-known that $(H(Y), h)$ is a complete metric space.

**Note**- A map $\theta : (Y, d) \to (Y, d)$ is called a contraction if there exists a constant $c < 1$ such that

$$d(\theta(a), \theta(b)) \leq c d(a, b), \ \forall \ a, b \in Y.$$

**Definition 2.6.** The system $\mathcal{I} = \{(Y, d); \theta_1, \theta_2, \ldots, \theta_N \}$ is called an iterated function system (IFS), if each $\theta_i$ is a contraction self-map on $Y$ for $i \in \{1, 2, \ldots, N\}$.

**Note**- Let $\mathcal{I} = \{(Y, d); \theta_1, \theta_2, \ldots, \theta_N \}$ be an IFS. We define a mapping $S$ from $H(Y)$ into $H(Y)$ given by

$$S(A) = \cup_{i=1}^N \theta_i(A).$$

The map $S$ is a contraction map under the Hausdorff metric $h$. If $(Y, d)$ is a complete metric space therefore, by Banach contraction principle, there exists a unique $E \in H(Y)$ such that $E = \cup_{i=1}^N \theta_i(E)$ and it is called the attractor of the IFS. We refer the reader to see [12, 8] for details.

**Definition 2.7.** We say that an IFS $\mathcal{I} = \{(Y, d); \theta_1, \theta_2, \ldots, \theta_N \}$ satisfies the open set condition(OSC) if there is a non-empty open set $O$ with $\theta_i(O) \subset O \ \forall \ i \in \{1, 2, \cdots, N\}$ and $\theta_i(O) \cap \theta_j(O) = \emptyset$ for $i \neq j$. Moreover, if $O \cap E \neq \emptyset$, where $E$ is the attractor of the $\mathcal{I}$, then we say that $\mathcal{I}$ satisfies the strong open set condition(SOSC). If $\theta_i(E) \cap \theta_j(E) = \emptyset$ for $i \neq j$, then we say that the IFS $\mathcal{I}$ satisfies the strong separation condition(SSC).

2.2. **Fractal Interpolation Functions.** Consider a set of data points $\{(x_i, y_i) \in \mathbb{R} \times \mathbb{C} : i = 1, 2, \ldots, N\}$ with $x_1 < x_2 < \cdots < x_N$. Set $T = \{1, 2, \ldots, N-1\}$ and $J = [x_1, x_N]$. For each $k \in T$, set $J_k = [x_k, x_{k+1}]$ and let $P_k : J \to J_k$ be a contractive homeomorphism satisfying

$$P_k(x_1) = x_k, \ P_k(x_N) = x_{k+1}.$$  

For each $k \in T$, let $\Psi_k : J \times \mathbb{C} \to \mathbb{C}$ be a continuous map such that

$$|\Psi_k(t, z_1) - \Psi_k(t, z_2)| \leq \tau_k |z_1 - z_2|,$$

$$\Psi_k(x_1, y_1) = y_k, \Psi_k(x_N, y_N) = y_{k+1},$$
where \((t, z_1), (t, z_2) \in J \times \mathbb{C}\) and \(0 \leq \tau_k < 1\). In particular, we can take for each \(k \in T\),
\[
P_k(t) = a_k t + d_k, \quad \Psi_k(t, y) = \alpha_k y + q_k(t).
\]
Constants \(a_k\) and \(d_k\) are uniquely determined by the condition \(P_k(x_1) = x_k, P_k(x_N) = x_{k+1}\). The multiplier \(\alpha_k\) is called the scaling factor, which satisfies \(-1 < \alpha_k < 1\) and \(q_k : J \to \mathbb{C}\) is a continuous function such that \(q_k(x_1) = y_k - \alpha_k y_1\) and \(q_k(x_N) = y_{k+1} - \alpha_k y_N\). Now for each \(k \in T\), we define functions \(W_k : J \times \mathbb{C} \to J \times \mathbb{C}\) by
\[
W_k(t, y) = \left( P_k(t), \Psi_k(t, y) \right).
\]
Then the IFS \(\mathcal{J} := \{ J \times \mathbb{C}; W_1, W_2, \ldots, W_{N-1} \}\) has a unique attractor, \([11, \text{Theorem 1}]\), which is the graph of a function \(h\) which satisfying the self-referential equation:
\[
h(t) = \alpha_k h\left( P_k^{-1}(t) \right) + q_k\left( P_k^{-1}(t) \right), \quad t \in J, k \in T.
\]
The above function \(h\) is known as the fractal interpolation function (FIF).

2.3. \(\alpha\)-Fractal Functions. To obtain a class of fractal functions with respect to a given continuous function on a compact interval in \(\mathbb{R}\), we can adapt the idea of construction of FIF. The space of all complex-valued continuous functions defined on \(J = [x_1, x_N]\) in \(\mathbb{R}\) is denoted by \(C(J)\), with the sup norm. Let \(f\) be a given function in \(C(J)\), known as the germ function. For constructing the IFS, we consider the following assumptions

1. Let \(\Delta := \{x_1, x_2, \ldots, x_N : x_1 < x_2 < \cdots < x_N\}\) be a partition of \(J = [x_1, x_N]\).
2. Let \(\alpha_k : J \to \mathbb{C}\) be continuous functions with \(\|\alpha_k\|_\infty = \max\{|\alpha_k(t)| : t \in J\} < 1\), for all \(k \in T\). These \(\alpha_k\) are called the scaling functions and \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{N-1}) \in (C(J))^{N-1}\) is called the scaling vector.
3. Let \(b : J \to \mathbb{C}\) be a continuous function such that \(b \neq f\) and \(b(x_1) = f(x_1), b(x_N) = f(x_N)\), named as the base function.

Motivated by \([11, 12]\), Navascués \([10]\) considered the following set of functions

\[
P_k(t) = a_k t + d_k, \quad \Psi_k(t, y) = \alpha_k(t)y + f\left( P_k(t) \right) - \alpha_k(t)b(t).
\]
Then the corresponding IFS \(\mathcal{J} := \{ J \times \mathbb{C}; W_1, W_2, \ldots, W_{N-1} \}\), where
\[
W_k(t, y) = \left( P_k(t), \Psi_k(t, y) \right),
\]
has a unique attractor, which is the graph of a continuous function \(f^\alpha_{\Delta, b} : J \to \mathbb{C}\) such that \(f^\alpha_{\Delta, b}(x_n) = f(x_n), n = 1, 2, \ldots, N\). For simplicity, we denote \(f^\alpha_{\Delta, b}\) by \(f^\alpha\). The real valued \(f^\alpha\) is widely known as \(\alpha\)-fractal function, see, for instance, \([24, 23, 25, 21, 22]\). Moreover, \(f^\alpha\) satisfies the following equation

\[
f^\alpha(t) = f(t) + (f^\alpha - b)(P_k^{-1}(t)) \quad \forall t \in J, k \in T.
\]
We can also treat \(f^\alpha\) as a "fractal perturbation" of \(f\).
3. Main Theorems

In the following lemma, we provide a relationship between Hausdorff dimension of complex-valued continuous function and Hausdorff dimension of its real and imaginary part.

Lemma 3.1. Suppose \( f : [a, b] \to \mathbb{C} \) is a continuous function and \( g, h : [a, b] \to \mathbb{R} \) is real and imaginary part of \( f \) respectively, that is, \( f = g + ih \). Then we have

(1) \( \dim_H(G(g + ih)) \geq \max\{\dim_H(G(g)), \dim_H(G(h))\} \).

(2) \( \dim_H(G(g + ih)) = \dim_H(G(g)) \), provided the imaginary part \( h \) is Lipschitz.

Proof.

(1) Let us define a mapping \( \Phi : G(f) \to G(g) \) as follows

\[
\Phi(x, g(x) + ih(x)) = (x, g(x)).
\]

We aim to show that \( \Phi \) is a Lipschitz mapping. Using simple properties of norm, it follows that

\[
\|\Phi(x_1, g(x_1) + ih(x_1)) - \Phi(x_2, g(x_2) + ih(x_2))\|^2 \\
= \|(x_1, g(x_1)) - (x_2, g(x_2))\|^2 \\
= |x_1 - x_2|^2 + |g(x_1) - g(x_2)|^2 \\
\leq |x_1 - x_2|^2 + |g(x_1) - g(x_2)|^2 + |h(x_1) - h(x_2)|^2 \\
= \|(x_1, g(x_1) + ih(x_1)) - (x_2, g(x_2) + ih(x_2))\|^2.
\]

That is, \( \Phi \) is a Lipschitz map. Now, Lipschitz invariance property of Hausdorff dimension yields

\[
\dim_H(G(f)) \geq \dim_H(G(g)).
\]

On similar lines, we obtain

\[
\dim_H(G(f)) \geq \dim_H(G(h)).
\]

Combining both of the above inequalities, we get

\[
\dim_H(G(f)) \geq \max\{\dim_H(G(g)), \dim_H(G(h))\},
\]

completing the proof of item (i).

(2) In this part, we continue our proof with the same mapping \( \Phi : G(f) \to G(g) \), defined by

\[
\Phi(x, g(x) + ih(x)) = (x, g(x)).
\]

Here our aim is to show that \( \Phi \) is a bi-Lipschitz map. From Part (1) of Lemma 3.1, it is obvious that \( \Phi \) is a Lipschitz map. And

\[
\|(x_1, g(x_1) + ih(x_1)) - (x_2, g(x_2) + ih(x_2))\|^2 \\
= |x_1 - x_2|^2 + |g(x_1) - g(x_2)|^2 + |h(x_1) - h(x_2)|^2 \\
\leq |x_1 - x_2|^2 + C_2^2 |x_1 - x_2|^2 + |g(x_1) - g(x_2)|^2 \\
\leq (1 + C_2^2)\{(x_1 - x_2)^2 + |g(x_1) - g(x_2)|^2\} \\
= (1 + C_2^2)\|(x_1, g(x_1)) - (x_2, g(x_2))\|^2 \\
= (1 + C_2^2)\|\Phi(x_1, g(x_1) + ih(x_1)) - \Phi(x_2, g(x_2) + ih(x_2))\|^2.
\]
Therefore, \( \Phi \) is a bi-Lipschitz map. In the light of the bi-Lipschitz invariance property of Hausdorff dimension, we get
\[
\dim_H(G(f)) = \dim_H(G(g)),
\]
this completes the proof.
\( \square \)

Now, we will present some results similar to the above in terms of other dimensions.

**Proposition 3.2.** Suppose \( f : [a, b] \to \mathbb{C} \) is a continuous function and \( g, h : [a, b] \to \mathbb{R} \) is real and imaginary part of \( f \) respectively, that is, \( f = g + ih \). Then we have
\[
\dim_P(G(g + ih)) \geq \max\{\dim_P(G(g)), \dim_P(G(h))\},
\]
\[
\dim_B(G(g + ih)) \geq \max\{\dim_B(G(g)), \dim_B(G(h))\},
\]
\[
\dim_B(G(g + ih)) \geq \max\{\dim_B(G(g)), \dim_B(G(h))\}.
\]

**Proof.** The proof is similar to part (1) of Lemma 3.1, hence we omit. \( \square \)

**Proposition 3.3.** Suppose \( f : [a, b] \to \mathbb{C} \) is a continuous function and \( g, h : [a, b] \to \mathbb{R} \) is real and imaginary part of \( f \) respectively, that is \( f = g + ih \). If \( h \) is a Lipschitz function, then we have
\[
\dim_P(G(f)) = \dim_P(G(g)), \overline{\dim_B}(G(f)) = \overline{\dim_B}(G(g)),
\]
and
\[
\dim_B(G(f)) = \dim_B(G(g)).
\]

**Proof.** The proof is similar to part (2) of Lemma 3.1 hence omitted. \( \square \)

**Lemma 3.4.** Suppose \( f : [a, b] \to \mathbb{C} \) is a continuous function and \( g, h : [a, b] \to \mathbb{R} \) is real and imaginary part of \( f \) respectively, that is, \( f = g + ih \). If \( h \) is a Lipschitz function on \([0, 1]\), then
\[
\dim_H(G(g + ih)) = \dim_H(G(g + h)) = \dim_H(G(g, h)) = \dim_H(G(g)),
\]
\[
\overline{\dim_B}(G(g + ih)) = \overline{\dim_B}(G(g + h)) = \overline{\dim_B}(G(g, h)) = \overline{\dim_B}(G(g)),
\]
\[
\dim_P(G(g + ih)) = \dim_P(G(g + h)) = \dim_P(G(g, h)) = \dim_P(G(g)).
\]

**Proof.** The mapping \( \Phi : G(g + h) \to G(g) \) defined by
\[
\Phi(x, (g(x) + h(x))) = (x, g(x))
\]
is a bi-Lipschitz map, see part (1) of Lemma 3.1. Now bi-Lipschitz invariance property of Hausdorff dimension, gives
\[
(3.1) \quad \dim_H(G(g + h)) = \dim_H(G(g)).
\]
Again we can show \( \Phi : G(g, h) \to G(g) \) defined by
\[
\Phi(x, (g(x), h(x))) = (x, g(x))
\]
is a bi-Lipschitz map, see part (2) of Lemma 3.1. By using bi-Lipschitz invariance property of Hausdorff dimension, we get
\[
(3.2) \quad \dim_H(G(g, h)) = \dim_H(G(g)).
\]
Further, by Lemma 3.1, Equation (3.1) and Equation (3.2) we get
\[
\dim_H(G(g + ih)) = \dim_H(G(g + h)) = \dim_H(G(g, h)) = \dim_H(G(g)).
\]
Remark 3.6. The Peano space filling curve $g : [0, 1] \to \mathbb{R}$ is $\frac{1}{3}$-Hölder continuous, see details [17]. The component functions satisfy $\dim_H G(g_1) = \dim_H G(g_2) = 1.5$. On the other hand, we have $\dim_H G(g) \geq 2$. Now, consider a complex-valued mapping $f(x) = g_1(x) + ig_2(x)$, and using Lemma 3.5, $\dim_H G(f) \geq 2$. From this example, we say that the upper bound of the Hausdorff dimension for the graph of a complex-valued function cannot be expressed in terms of its Hölder exponent as we do for real-valued function.

From above, it is clear that, dimensional results for complex-valued function and real-valued function are different. Now, we are ready to give some dimensional results for complex-valued fractal interpolation function.

Define a metric $D$ on $J \times \mathbb{C}$ by

$$D((t_1, z_1), (t_2, z_2)) = |t_1 - t_2| + |z_1 - z_2| \quad \forall \; (t_1, z_1), (t_2, z_2) \in J \times \mathbb{C}.$$ 

Then $(J \times \mathbb{C}, D)$ is a complete metric space.

Theorem 3.7. Let $\mathcal{I} := \{J \times \mathbb{C}; W_1, W_2, \ldots, W_{N-1}\}$ be the IFS defined in the construction of $f^*$ such that

$$c_k D((t_1, z_1), (t_2, z_2)) \leq D(W_k(t_1, z_1), W_k(t_2, z_2)) \leq C_k D((t_1, z_1), (t_2, z_2)),$$

where $(t_1, z_1), (t_2, z_2) \in J \times \mathbb{C}$ and $0 < c_k \leq C_k < 1 \quad \forall \; k \in T$. Then $r \leq \dim_H(G(f^*)) \leq R$, where $r$ and $R$ are given by $\sum_{k=1}^{N-1} c_k^r = 1$ and $\sum_{k=1}^{N-1} C_k^R = 1$ respectively.
Proof. For upper bound of \( \dim_H(G(f^n)) \), follow Proposition 9.6 in [3]. For the lower bound of \( \dim_H(G(f^n)) \) we proceed as follows.

Let \( V = (x_1, x_N) \times \mathbb{C} \). Then

\[
W_i(V) \cap W_j(V) = \emptyset,
\]
for each \( i \neq j \in T \).

Because

\[
P_i((x_1, x_N)) \cap P_j((x_1, x_N)) = \emptyset, \quad \forall \ i \neq j \in T.
\]

We can observe that \( V \cap G(f^n) \neq \emptyset \), this implies that IFS \( \mathcal{I} \) satisfies the SOSC. Then there exists an index \( i \in T^* \) such that \( W_i(G(f^n)) \subset V \), where \( T^* := \cup_{n \in \mathbb{N}} \{1, 2, \ldots, N - 1\}^n \). We denote \( W_i(G(f^n)) \) by \( (G(f^n))_i \), for any \( i \in T^* \). Now, it is obvious that for each \( n \in \mathbb{N} \), the sets \( \{(G(f^n))_j : j \in T^n \} \) is disjoint.

Then, for each \( n \in \mathbb{N} \), IFS \( \mathcal{L}_n = \{W_{j_i} : j \in T^n \} \) satisfies the hypothesis of Proposition 9.7 in [3]. Hence, by Proposition 9.7 in [3] if \( A^*_n \) is an attractor of the IFS \( \mathcal{L}_n \), then \( r_n \leq \dim_H(A^*_n) \), where \( r_n \) is given by \( \sum_{j \in T^n} c_{j_i, j_i}' = 1 \). Then \( r_n \leq \dim_H(A^*_n) \leq \dim_H(G(f^n)) \) because \( A^*_n \subset G(f^n) \). Suppose that \( \dim_H(G(f^n)) < r \). This implies that \( r_n < r \). Let \( c_{\max} = \max\{c_1, c_2, \ldots, c_{N-1}\} \).

We have

\[
c_i^{-r_n} = \sum_{j \in T^n} c_j'^r \geq \sum_{j \in T^n} c_j'^r c_{j_i}' \dim_H(G(f^n))^{-r} \geq \sum_{j \in T^n} c_j'^r c_{\max}^{-r} \dim_H(G(f^n))^{-r}.
\]

This implies that

\[
c_i^{-r} \geq c_{\max}^{-r} \dim_H(G(f^n))^{-r}.
\]

We have a contradiction for large value of \( n \in \mathbb{N} \). Therefore, we get \( \dim_H(G(f^n)) \geq r \), proving the assertion.

\[\square\]

Remark 3.8. In [16], Roychowdhury estimated the Hausdorff and box dimension of attractor of the hyperbolic recurrent iterated function system consisting of bi-Lipschitz mappings under the open set condition using Bowen’s pressure function and volume argument. Note that recurrent iterated function is a generalization of the iterated function, hence so is Roychowdhury’s result. We should emphasize on the fact that in the above we provide a proof without using pressure function and volume argument. Our proof can be generalized to general complete metric spaces.

Remark 3.9. This theorem can be compared with Theorem(2.4) in [23].

The Hölder space is defined as follows:

\[
\mathcal{H}^\sigma(J) := \{h : J \rightarrow \mathbb{C} : h \text{ is Hölder continuous with exponent } \sigma\}.
\]

Note that \( (\mathcal{H}^\sigma(J), \|\cdot\|_\mathcal{H}) \) is a Banach space, where \( \|h\|_\mathcal{H} := \|h\|_\infty + \|h\|_\sigma \) and

\[
[h]_\sigma = \sup_{t_1 \neq t_2} \frac{|h(t_1) - h(t_2)|}{|t_1 - t_2|^\sigma}.
\]

Theorem 3.10. Let \( f, b, \alpha \in \mathcal{H}^\sigma(J) \) such that \( b(x_1) = f(x_1) \) and \( b(x_N) = f(x_N) \). Set \( c := \min\{a_k : k \in T\} \). If \( \frac{\|f\|_\mathcal{H}}{c} < 1 \), then \( f^\alpha \) is Hölder continuous with exponent \( \sigma \).

Proof. Let us define \( \mathcal{H}^\sigma_g(J) := \{h \in \mathcal{H}^\sigma(J) : h(x_1) = f(x_1), h(x_N) = f(x_N)\} \).

By basic real analysis technique, we may see that \( \mathcal{H}^\sigma_g(J) \) is a closed subset of \( \mathcal{H}^\sigma(J) \). Since \( (\mathcal{H}^\sigma(J), \|\cdot\|_\mathcal{H}) \) is a Banach space, it implies that \( \mathcal{H}^\sigma_g(J) \) will be a
complete metric space with respect to metric induced by $\|\cdot\|_{H}$. We define a map $S : \mathcal{H}_{f}^{\sigma}(J) \to \mathcal{H}_{f}^{\sigma}(J)$ by

$$(Sh)(t) = f(t) + \alpha_{k}(P_{k}^{-1}(t)) (h-b)(P_{k}^{-1}(t))$$

$\forall \ t \in J_{k}$ where $k \in T$. We shall show that $S$ is well-defined and contraction map on $\mathcal{H}_{f}^{\sigma}(J)$.

$$[Sh]_{\sigma} = \max_{k \in T} \sup_{t_{1} \neq t_{2}, t_{1}, t_{2} \in J_{k}} \frac{|Sh(t_{1}) - Sh(t_{2})|}{|t_{1} - t_{2}|^{\sigma}}$$

$$\leq \max_{k \in T} \sup_{t_{1} \neq t_{2}, t_{1}, t_{2} \in J_{k}} \frac{|f(t_{1}) - f(t_{2})|}{|t_{1} - t_{2}|^{\sigma}}$$

$$+ \sup_{t_{1} \neq t_{2}, t_{1}, t_{2} \in J_{k}} |\alpha_{k}(P_{k}^{-1}(t_{1}))|(h-b)(P_{k}^{-1}(t_{1})) - (h-b)(P_{k}^{-1}(t_{2}))|$$

$$\leq [f]_{\sigma} + \frac{||\alpha||_{\infty}}{c^{\sigma}} ([h]_{\sigma} + |b|_{\sigma}) + \frac{||h-b||_{\infty}}{c^{\sigma}} [\alpha]_{\sigma},$$

where $[\alpha]_{\sigma} = \max_{k \in T} \sup_{t_{1}, t_{2} \in J_{k}} \frac{|\alpha_{k}(t_{1}) - \alpha_{k}(t_{2})|}{|t_{1} - t_{2}|^{\sigma}}$. Let $g, h \in \mathcal{H}_{f}^{\sigma}(J)$, we have

$$\|Sg - Sh\|_{H} = \|Sg - Sh\|_{\infty} + [Sg - Sh]_{\sigma}$$

$$\leq ||\alpha||_{\infty} \|g-h\|_{\infty} + \frac{||\alpha||_{\infty}}{c^{\sigma}} |g-h|_{\sigma} + \frac{||g-h||_{\infty}}{c^{\sigma}} [\alpha]_{\sigma}$$

$$\leq \frac{||\alpha||_{H}}{c^{\sigma}} \|g - h\|_{H}.$$

This implies that $S$ is well-defined map on $\mathcal{H}_{f}^{\sigma}(J)$. Since $\frac{||\alpha||_{H}}{c^{\sigma}} < 1$, $S$ is a contraction map. By the application of Banach contraction mapping theorem, $S$ has a unique fixed point $f^{\alpha} \in \mathcal{H}_{f}^{\sigma}(J)$. Hence we are done. $\square$

**Theorem 3.11.** Let germ function $f$, base function $b$ and scaling function $\alpha_{j}$ be complex-valued functions such that

$$|f(t_{1}) - f(t_{2})| \leq l_{f}|t_{1} - t_{2}|^{\sigma},$$

$$|b(t_{1}) - b(t_{2})| \leq l_{b}|t_{1} - t_{2}|^{\sigma},$$

$$|\alpha_{j}(t_{1}) - \alpha_{j}(t_{2})| \leq l_{\alpha}|t_{1} - t_{2}|^{\sigma},$$

for each $t_{1}, t_{2} \in J, j \in T$, and for some $l_{f}, l_{b}, l_{\alpha} > 0, \sigma \in (0, 1]$. Let $f_{1}, f_{2}$ be component of $f$, $b_{1}, b_{2}$ be component of $b, \alpha_{1}^{j}, \alpha_{2}^{j}$ be component of $\alpha_{j}$ and $f_{1}^{\alpha}, f_{2}^{\alpha}$ be component of $f^{\alpha}$. Also, consider constants $l_{f_{i}}, \delta_{0} > 0$ such that for all $t_{1} \in J$ and $\delta < \delta_{0}$, there exists $t_{2} \in J$ with $|t_{1} - t_{2}| \leq \delta$ and

$$|f_{i}(t_{1}) - f_{i}(t_{2})| \geq l_{f_{i}}|t_{1} - t_{2}|^{\sigma} \quad \text{for} \ i \in \{1, 2\}.$$

If $||\alpha||_{H} < c^{\sigma}$ min \(\{1, \frac{t_{f_{1}} - 2(||b||_{\infty} + M)l_{\alpha}c^{-\sigma}}{2(k_{f_{1}, b_{1}, a} + l_{b})}, \frac{t_{f_{2}} - 2(||b||_{\infty} + M)l_{\alpha}c^{-\sigma}}{2(k_{f_{2}, b_{1}, a} + l_{b})}\}$, then we have

$$1 \leq \dim_{H} (G(f_{i}^{\alpha})) \leq \dim_{B} (G(f_{i}^{\alpha})) = 2 - \sigma \quad \text{for} \ i = 1, 2.$$

Moreover, $1 \leq \dim_{H} (G(f^{\alpha})) \leq \dim_{B} (G(f^{\alpha})) \geq 2 - \sigma.$
Proof. Since $\|\alpha\|_H < c^\sigma$, Theorem (3.11) yields that the fractal version $f^\alpha$ of $f$ is Hölder continuous with the exponent $\sigma$. Consequently,

$$|f_1^\alpha(t_1) - f_i^\alpha(t_2)| \leq |f_1^\alpha(t_1) - f_i^\alpha(t_2)| \leq k_{f,b,\alpha}|t_1 - t_2|^\sigma$$

for some $k_{f,b,\alpha} > 0$ and for $i = 1, 2$. Firstly, we try to give an upper bound for upper box dimension of $G(f_i^\alpha)$ for $i = 1, 2$ as follows: For $\delta \in (0, 1)$, let $m$ be the smallest natural number greater than or equal to $\frac{1}{\delta}$ and $N_\delta(G(f_i^\alpha))$ be the number of $\delta$-mesh that can intersect with $G(f_i^\alpha)$

$$N_\delta(G(f_i^\alpha)) \leq 2m + \sum_{r=0}^{m-1} \left( \frac{R_{f_i^\alpha}[(r\delta, (r+1)\delta)]}{\delta} \right)$$

(3.4)

$$\leq 2\left( \frac{1}{\delta} + 1 \right) + \sum_{r=0}^{m-1} \frac{R_{f_i^\alpha}[(r\delta, (r+1)\delta)]}{\delta}$$

$$\leq 2\left( \frac{1}{\delta} + 1 \right) + \sum_{r=0}^{m-1} k_{f,b,\alpha}\delta^{\sigma-1}.$$  

From this, we conclude that

$$\dim_B(G(f_i^\alpha)) = \lim_{\delta \to 0} \frac{\log N_\delta(G(f_i^\alpha))}{-\log \delta} \leq 2 - \sigma \forall i = 1, 2.$$  

Next, we will prove that $\dim_B(G(f_i^\alpha)) \geq 2 - \sigma \forall i = 1, 2$. For this, using the self-referential equation, we can write

$$f_1^\alpha(t) = f_1(t) + \alpha_1^k(P_k^{-1}(t))[f_1^\alpha(P_k^{-1}(t)) - b_1(P_k^{-1}(t))]$$

(3.5)

$$- \alpha_2^k(P_k^{-1}(t))[f_2^\alpha(P_k^{-1}(t)) - b_2(P_k^{-1}(t))]

$$

for every $t \in J_k$ and $k \in T$. Let $t_1, t_2 \in J_k$ such that $|t_1 - t_2| \leq \delta$. From Equation (3.5) we have

$$|f_1^\alpha(t_1) - f_i^\alpha(t_2)| \geq \left| f_1(t_1) - f_1(t_2) \right| + \alpha_1^k(P_k^{-1}(t_1))[f_1^\alpha(P_k^{-1}(t_1)) - \alpha_1^k(P_k^{-1}(t_2))]$$

$$- \alpha_1^k(P_k^{-1}(t_1)) b_1(P_k^{-1}(t_1)) + \alpha_1^k(P_k^{-1}(t_2)) \left| b_1(P_k^{-1}(t_2)) \right|$$

$$- \alpha_2^k(P_k^{-1}(t_1))[f_2^\alpha(P_k^{-1}(t_1)) - \alpha_2^k(P_k^{-1}(t_2))]$$

$$+ \alpha_2^k(P_k^{-1}(t_1)) b_2(P_k^{-1}(t_1))$$

$$- \alpha_2^k(P_k^{-1}(t_2)) \left| b_2(P_k^{-1}(t_2)) \right|$$

$$\geq \left| f_1(t_1) - f_1(t_2) \right| - \|\alpha\|_{\infty} \left| f_1^\alpha(P_k^{-1}(t_1)) - f_1^\alpha(P_k^{-1}(t_2)) \right|$$

$$- \|\alpha\|_{\infty} \left| b_1(P_k^{-1}(t_1)) - b_1(P_k^{-1}(t_2)) \right| - \left( \|b\|_{\infty} + \|f^\alpha\|_{\infty} \right)$$

$$\left| \alpha_1^k(P_k^{-1}(t_1)) - \alpha_1^k(P_k^{-1}(t_2)) \right| - \|\alpha\|_{\infty} \left| f_2^\alpha(P_k^{-1}(t_1)) - f_2^\alpha(P_k^{-1}(t_2)) \right|$$

$$- \|\alpha\|_{\infty} \left| b_2(P_k^{-1}(t_1)) - b_2(P_k^{-1}(t_2)) \right| - \left( \|b\|_{\infty} + \|f^\alpha\|_{\infty} \right) \left| \alpha_2^k(P_k^{-1}(t_1)) - \alpha_2^k(P_k^{-1}(t_2)) \right|.$$
With the help of Equation (3.3), we get
\[ |f^n_i(t_1) - f^n_i(t_2)| \geq \|f_i|||k_{f,b,\sigma}||P^{-1}_k(t_1) - P^{-1}_k(t_2)|^{\sigma - 2}\|f_i||P^{-1}_k(t_1) - P^{-1}_k(t_2)|^{\sigma - 2}\\]
\[ \geq l_{f_i}||k_{f,b,\sigma}||t_1 - t_2|^{\sigma - 2}\|f_i||P^{-1}_k(t_1) - P^{-1}_k(t_2)|^{\sigma - 2}\\]
\[ \geq l_{f_i}||k_{f,b,\sigma}||t_1 - t_2|^{\sigma - 2}\|f_i||P^{-1}_k(t_1) - P^{-1}_k(t_2)|^{\sigma - 2}\\]
\[ = \left( l_{f_i} - 2k_{f,b,\sigma} + l_b \right)\|f_i||P^{-1}_k(t_1) - P^{-1}_k(t_2)|^{\sigma - 2}\\]
Set \( L := l_{f_i} - 2k_{f,b,\sigma} + l_b \). Then by the given condition \( L > 0 \). Let \( \delta = c^n \) for \( n \in \mathbb{N} \) and \( w \) be the smallest natural number greater than or equal to \( \frac{1}{\log c} \). We estimate
\[ N_d(G(f^n_i)) \geq \sum_{r=0}^{w} \max \left\{ 1, \left( c^{-n} R_{f_i} |r\delta, (r+1)\delta| \right) \right\} \]
\[ \geq \sum_{r=0}^{w} \left( c^{-n} R_{f_i} |r\delta, (r+1)\delta| \right) \]
\[ \geq \sum_{r=0}^{w} Le^{-n c^{\alpha \sigma}} \]
\[ = Le^{n(\sigma - 2)}. \]
By using the above inequality for \( N_d(G(f^n_i)) \), we obtain
\[ \dim_B(G(f^n_i)) = \lim_{\delta \to 0} \frac{\log \left( N_d(G(f^n_i)) \right)}{-\log(\delta)} \geq \lim_{n \to \infty} \frac{\log \left( Le^{n(\sigma - 2)} \right)}{-n \log c} \]
\[ = 2 - \alpha, \]
Similarly, we get
\[ \dim_B(G(f^n_2)) \geq 2 - \sigma, \]
establishing the result.

**Definition 3.12.** A complex-valued function \( g : J \to \mathbb{C} \) is said to be of bounded variation if the total variation \( V(g, J) \) of \( g \) defined by
\[ V(g, J) = \sup_{Q=(y_0, y_1,...,y_m)} \sum_{k=1}^{m} |g(y_i) - g(y_{i-1})|, \]
is finite. The space of all bounded variation functions on \( J \), denoted by \( BV(J, \mathbb{C}) \), forms a Banach space with respect to the norm \( \|g\|_{BV} := |g(y_0)| + V(g, J) \).

**Theorem 3.13.** If \( f, b \in C(J, \mathbb{C}) \cap BV(J, \mathbb{C}) \) and \( \alpha_k \in C(J, \mathbb{C}) \cap BV(J, \mathbb{C}) \) \( \forall k \in T \) such that \( \|\alpha\|_{BV} < \frac{1}{2(N-1)} \). Then \( f^{\alpha} \in C(J, \mathbb{C}) \cap BV(J, \mathbb{C}) \). Moreover, \( \dim_H(G(f^{\alpha})) = \dim_B(G(f^{\alpha})) = 1 \).

**Proof.** Following Theorem (3.11) and \( [23 \text{ Theorem 3.24 }] \), one may complete the proof. \( \square \)
Remark 3.14. The above theorem will reduce to [23, Theorem 3.24] when all functions \( f, b \) and \( \alpha_k \) are real-valued.

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