Topological Chern indices in molecular spectra

F. Faure\textsuperscript{1} and B. Zhilinskii\textsuperscript{2}
\textsuperset{1}{L.P.M.M.C. (Maison des Magistres Jean Perrin, CNRS) BP 166, 38042 Grenoble Cedex 9, France. faure@labs.polyufr-gre.fr}
\textsuperset{2}{Université du Littoral, MREID, 145 av. M. Schumann, 59140 Dunkerque, France zhilin@univ-littoral.fr}

Topological Chern indices are related to the number of rotational states in each molecular vibrational band. Modification of the indices is associated to the appearance of “band degeneracies”, and exchange of rotational states between two consecutive bands. The topological dynamical origin of these indices is proven through a semi-classical approach, and their values are computed in two examples. The relation with the integer quantum Hall effect is briefly discussed.

PACS numbers: 03.65.Sq; 02.40.-k; 31.15.Gy;

Topological numbers play an important role in many area of physics \cite{1}, but their appearance in molecular physics and especially in rovibrational problems has not been systematically appreciated so far. Simple molecular systems typically allow adiabatic separation of vibrational and rotational motion. For non-degenerate isolated electronic state (this is the case of ground state for most molecules) the rovibrational energy level system consists of vibrational bands, each associated with one or several degenerate vibrational states. If the rovibrational coupling is not too strong, further splitting of the rovibrational structure into sub-bands can be clearly seen. The well-known example is the splitting of the triply degenerate vibrational structure for a spherically symmetrical molecule into three sub-bands due to the first-order Coriolis interaction \cite{2,3}. Within each sub-band formed by $2j + 1 - C$ levels, with respectively $C = +2, 0, -2$, all energy levels are usually characterized by the quantum number $j$ of the total angular momentum, and by another quantum number $R = j + C/2$ which characterizes the coupling of $j$ with the vibrational angular momentum.

In this letter we show that the integer $C$ can be defined in much more general situation as an additional quantum number having a precise topological meaning, namely a Chern index, whose construction will be explained below. This index is defined in the classical limit of the rotational motion. It can be associated with any vibrational band presented in the energy level pattern of molecular systems. Theorem \cite{4} relates this topological index to the number of rotational states within the band. A modification of the index is associated with the formation of a contact (a degeneracy) between two consecutive vibrational bands, and is shown to generically imply an exchange of one rotational state between the two bands.

Such a relation was first conjectured in 1988 \cite{4} after the study of the simple model \cite{5}, and a number of effective Hamiltonians reconstructed from experimental data (see Ref. \cite{6} for the molecular examples: SiH\textsubscript{4}, CD\textsubscript{4}, SnH\textsubscript{4}, CF\textsubscript{4}, Mo(CO)\textsubscript{6}). The universal character of the redistribution phenomena and its relevance to integer Hall effect was discussed on several occasions \cite{7,8}.

To explain the physical phenomenon and to prepare the formulation of a rigorous statement let us consider a toy problem which involves two quantum angular momenta $\mathbf{J}$ and $\mathbf{S}$, with fixed modulus $J^2 = j(j + 1)$ and $S^2 = s(s+1)$ with $j, s$ integer or half-integer. $\mathbf{J}$ acts in the space $\mathcal{H}_j$ of dimension $(2j + 1)$ which is the irreducible representation space of the $SU(2)$ group. Similarly, $\mathbf{S}$ acts in the space $\mathcal{H}_s$ of dimension $(2s + 1)$. The total space is $\mathcal{H}_{\text{tot}} = \mathcal{H}_j \otimes \mathcal{H}_s$ with dimension $(2j + 1)(2s + 1)$.

The most general quantum Hamiltonian $\hat{H}(\mathbf{S}, \mathbf{J}/j)$ we will consider is a hermitian operator acting in $\mathcal{H}_{\text{tot}}$ and its action in space $\mathcal{H}_j$ is supposed to be expressed in terms of the operators $\mathbf{J}/j$. The factor $1/j$ is introduced here to ensure the existence of the classical limit for $j \to \infty$. An extremely simple form of $\hat{H}$ is

$$\hat{H} = (1 - t)S_z + \frac{t}{j}(\mathbf{J} \cdot \mathbf{S}), \quad (1)$$

with $j > s$, $t \in \mathbb{R}$, which was used initially in Ref. \cite{4} to study the redistribution phenomenon and further in Ref. \cite{5} to establish its relation with the classical monodromy. We use this Hamiltonian \cite{4} to illustrate the strict formulation of our result \cite{6}, but its validity extends to a general $\hat{H}(\mathbf{S}, \mathbf{J}/j)$.

In the two extremes limits $t = 0$ (no “spin-orbit” coupling), and $t = 1$ (“spin-orbit” coupling), the energy level spectrum of \cite{5} shows different patterns of energy levels into bands indexed by $g$. For $t = 0$ all energies $E_g = g$, $g \in \{-s, ..., +s\}$ appear with the same multiplicities $N_g = (2j + 1)$. For $t = 1$ the spectrum is split into degenerate multiplets characterized by different eigenvalues of the coupled angular momentum $N^2 = (\mathbf{J} + \mathbf{S})^2$. As in the case of standard spin-orbit coupling with $j > s$ there are $2s + 1$ different levels $E_g = \{n(n + 1) - j(j + 1) - s(s + 1)/(2j), g = n - j \in \{-s, ..., +s\}\}$, with different multiplicities $N_g = (2j + 1) + 2g$. The two different limiting cases for the structures of the $(2s + 1)$ bands suggest to introduce
a new quantum number $C_g$ associated with the value of $N_g$ within each band.

To define $C_g$ for a general Hamiltonian $\tilde{H}(S, J/j)$ we assume that $j \gg s$, so that it is physically reasonable to consider $J_{cl} = J$ as classical, whereas $S$ remains quantum. The classical dynamics for $J_{cl}$ can be defined through the $SU(2)$ coherent states $|J_{cl}\rangle$ [1]. The classical phase space for $J_{cl}$ is the sphere $S_{2j}^2$. From $dJ_{cl}/dt = \partial J_{cl} H_s \times J_{cl}$ and because of the factor $1/j$ in Eq. (1), $J_{cl}$ corresponds to a slow dynamical variable compared to $S$, and the Born-Oppenheimer approximation suggests to consider for each $J_{cl}$, the Hermitian operator $\hat{H}_s(J_{cl}) = \langle J_{cl} | \hat{H} | J_{cl} \rangle$ acting on $\mathcal{H}_s$, with spectrum $\hat{H}_s(J_{cl}) |\psi_{g,J_{cl}}\rangle = E_{g,J_{cl}} |\psi_{g,J_{cl}}\rangle$, $g \in \{-s, \ldots, +s\}$. We suppose that for each $J_{cl}$, the $(2s + 1)$ eigenvalues are isolated: $E_{-s,J_{cl}} < E_{-s+1,J_{cl}} < \cdots < E_{+s,J_{cl}}$. This is the generic situation, because degeneracies are of codimension 3, and $J_{cl}$ is only two-dimensional.

For each level $g$, let us note $\langle \psi_{g,J_{cl}} |$ the eigenvector defined up to a multiplication by a phase $e^{i\alpha}$. The application $J_{cl} \rightarrow \langle \psi_{g,J_{cl}} |$ defines then a $U(1)$ fiber bundle over the sphere $S_{2j}^2$, which is the set of all the possible phases $\alpha$ for every values of $J_{cl}$. The topology of this bundle is characterized by a Chern number $C_g \in \mathbb{Z}$ [3]. $C_g$ reveals the possible global twist of the fiber of phases $\alpha$ over the sphere $S_{2j}^2$, in the same way the well known Möbius strip is the real line fiber bundle over the circle $S^1$ with a global twist $+1$.

![FIG. 1. The Chern index $C_g$ expresses the twist made by the circles for the phases $\alpha$ of the eigenvectors $e^{i\alpha} |\psi_{g,J_{cl}}\rangle$ over the sphere of spin $J_{cl}$, in the same way the Möbius strip has a twist made by the lines over a circle.](image)

Note that

\[
\sum_{g=-s}^{+s} C_g = 0, \tag{2}
\]

just because of the additivity of Chern indices, and because the eigenvectors $\langle \psi_{g,J_{cl}} |$ span the space $\mathcal{H}_s$ which does not depend on $J_{cl}$.

The Born Oppenheimer approach tells that the total spectrum $\hat{H} |\phi_i\rangle = E_i |\phi_i\rangle$, $i \in \{1, \ldots, (2s + 1)(2j + 1)\}$ in $\mathcal{H}_{tot}$, can be represented as formed in $2s + 1$ groups of levels (bands) with eigen-functions of each group spanning vector spaces: $L_g \subset \mathcal{H}_{tot}$, $g \in \{-s, \ldots, +s\}$. We precise now the number of levels in each group:

**Theorem.**

For a general Hamiltonian $\hat{H}(S, J/j)$, and for $j$ large enough then $\dim L_g = \dim \mathcal{H}_j - C_g$, so

\[
N_g = (2j + 1) - C_g. \tag{3}
\]

The number $N_g$ of states $|\phi_i\rangle$ in each band $L_g$ of $\mathcal{H}_{tot}$ is given by the quantum number $j$ and an additional quantum number $C_g$, namely the topological Chern index.

A few remarks are in order here. Since $2j + 1$ is the number of quanta in the classical phase space $S_{2j}^2$ for spin $J_{cl}$, Eq. (3) looks like a Weyl formula with a correction. The index $C_g$ has been defined and can be computed in “a semi-quantal” approach where $J_{cl}$ is considered as a classical variable and $S$ quantum. Nevertheless Eq. (3) provides an information on the full quantum problem: the spectrum of $\hat{H}$. Finally the topological nature of $C_g$ reveals a qualitative and robust property of the spectrum of $\hat{H}$, stable under perturbations, provided no degeneracy appears between consecutive bands. One can say that $C_g$ expresses a topological coupling between the dynamical variables $J$ and $S$.

Two approaches to the computation of the Chern indices of different bands will be suggested. The first one we use below is algebraic. The second one uses the Berry’s connection, and is based on the curvature formula [13]. This last formula could be useful for numerical computations.

The algebraic calculation is based on the geometric interpretation of the Chern index $C_g$ as the total intersection number between the one-dimensional curve $\langle \psi_{g,J_{cl}} |$, $\mathcal{H}_s$ with the hyperplane $(|\psi_0\rangle)^\perp = \{\varphi\}$ such that $\langle \varphi |\psi_0\rangle = 0$. Here $|\psi_0\rangle \in \mathcal{H}_s$ is arbitrary. Each intersection has number $\sigma = +1 (-1)$ if the curve orientation is compatible (incompatible) with the orientation of the hyperplane. $C_g$ is the sum of these intersection numbers [1].

The application of this algebraic method of calculation to the two limiting cases of Hamiltonian (4) is immediate. For $t = 0$ the Hamiltonian $\hat{H}_s(J_{cl}) = S_z$ does not depend on $J_{cl}$, the application $J_{cl} \rightarrow \langle \psi_{g,J_{cl}} |$ is constant, the topology of the bundle is trivial with zero Chern index: $C_g = 0$, in accordance with Eq. (3). For $t = 1$ we have

\[
\hat{H}_s(J_{cl}) = \langle J_{cl} | \hat{H} | J_{cl} \rangle = \frac{1}{j} (J_{cl} \cdot S),
\]

with $J_{cl}$ a vector on the sphere $S_{2j}^2$, and $S$ a vectorial operator in $\mathcal{H}_s$. The eigenvector $|\psi_{g,J_{cl}}\rangle$ is easily obtained from the eigenvector $|m_s = g\rangle \propto S_{s+g}^z |0\rangle$ of $S_z$ by a rotation which transforms the $z$ axis to the $J_{cl}$ axis on the sphere. It is convenient to choose a coherent state $|S_{0,cl}\rangle$ as the reference state $|\psi_0\rangle$, where $S_{0,cl}$ is an arbitrary classical spin $S$. An intersection of the curve $\langle \psi_{g,J_{cl}} |$, with the hyperplane $(|\psi_0\rangle)^\perp$ is then given by the equation $\langle S_{0,cl} |\psi_{g,J_{cl}}\rangle = 0$ which has a very simple
interpretation: the point $S_{0,cl}$ is a zero of the Husimi representation Hus$(S_{cl}) = |\langle S_{cl} | \psi_{g,J,cl} \rangle|^2$ of the state $|\psi_{g,J,cl} \rangle$ on the sphere $S^2_s$ (the classical phase space of $S$), shown in figure 2. The Husimi representation has $(s-g)$ zeros inside the (oriented) trajectory, and $(s+g)$ zeros outside $S_{0,cl}$. When the axis $J_i$ moves with direct orientation on the whole sphere, the $(s-g)$ zeros pass with positive orientation over the fixed point $S_{0,cl}$, giving $\sigma_1 = s - g$, whereas the $s + g$ zeros pass with negative orientation over $S_{0,cl}$, giving $\sigma_2 = -(s + g)$. So $C_g = \sigma_1 + \sigma_2 = -2g$ in accordence with Eq.(3).

![FIG. 2. The Husimi distribution for the state $|\psi_{g,J,cl} \rangle$ on the sphere $S^2_s$ for spin $S$. $S_{0,cl}$ is a reference point.](image)

In this last example, the variation of Chern indices $\Delta C_g = -2g$ occurs at $t = 1/2$, with a degeneracy between the bands at $J_i = (0,0,-j)$, giving $\hat{H}_s(J_i) = 0$. In the case of two bands ($s = 1/2$, $g = \pm 1/2$) then $\Delta C_g = \mp 1$, and the two bands have a conical contact at the degeneracy. In the vicinity of the degeneracy, and for $j \to \infty$, we observe that $|J_-,J_s\rangle = -2J_s \approx 2j$, so $a = J_-/\sqrt{2j}$, $a^+ = J_+ / \sqrt{2j}$ fulfilled the harmonic oscillator commutation relations. In the basis $|\pm \rangle = |m_s = \pm 1/2\rangle$ of $H_s$, the expression of $\hat{H}$ can be simplified and gives

$$\hat{H} = \frac{1}{\sqrt{2j}} \begin{pmatrix} -\tilde{t} & a \\ a^+ & \tilde{t} \end{pmatrix}, \quad (4)$$

with $\tilde{t} = (2t - 1) \sqrt{2j}$. We also scales the energy with $\tilde{E} = E \sqrt{2j}$, and note $|n\rangle \propto a^n |0\rangle$. The stationary equation $H|\phi\rangle = E|\phi\rangle$ can easily be solved, giving for $n = 1, 2, \ldots$, $|\phi_n^\pm \rangle = |n\rangle \propto |n\rangle - \sqrt{n+1}/(\tilde{E}_n^+ + \tilde{t})|n-1\rangle + |n\rangle + \sqrt{n}/(\tilde{E}_n^- + \tilde{t})|n+1\rangle$ with $\tilde{E}_n^\pm = \pm \sqrt{n+1} + \tilde{t}$, and one single state $|\phi_0^+ \rangle = |0\rangle \propto |0\rangle - |1\rangle$ with $\tilde{E}_0 = \tilde{t}$. Figure 3 shows this spectrum with the simplified expressions of $|\phi_n^\pm \rangle$ obtained for large $|\tilde{t}|$. These simplified expressions involving the states $|n\rangle$ of the harmonic oscillator express the quantized modes for small oscillators near the extrema of the two bands $[6]$. We clearly observe the exchange of one state in the spectrum, giving $\Delta N_g = \pm 1$. As a consequence, $\Delta N_g + \Delta C_g = 0$. This gives for each band a conserved quantity, namely $(N_g + C_g)$. The variation $\Delta C_g = -1$ can also be considered as “the topological charge” associated to the degeneracy of the 2 x 2 matrix $[4]$ where $a_{cl} = x_{cl} - ip_{cl}$ is a classical variable. $|\Delta C_+ = -1\rangle$ is also obtained from Eq. (4), for a little sphere in space $(x_{cl}, p_{cl}, \tilde{t})$ centered at the degeneracy $(0,0,0)$.

![FIG. 3. Spectrum of rotational states for the Hamiltonian $\hat{H}(S, J/j)$ the very simple model Eq.(4) provides the general mechanism for the exchange of one state in the vicinity of every degeneracy between two consecutive bands. In the trivial case $\hat{H}_0 = S_z$, we have computed the value of the conserved quantity: $N_g + C_g = 2j + 1$. We deduce that it is still correct when $\hat{H}_0$ is deformed to $\hat{H}(S, J/j)$, proving formula (4).

The second example, addresses the Chern indices of the two-state model corresponding to $s = 1/2$, i.e. to $\dim H_s = 2$. In a given fixed basis, say $|m_s = \pm 1/2\rangle$, the matrix of $\hat{H}_s(J_i)$ has the form:

$$\hat{H}_s = \begin{pmatrix} h_{11}(J_i) & h_{12}(J_i) \\ h_{12}(J_i) & h_{22}(J_i) \end{pmatrix}. \quad (5)$$

This matrix will give two vibrational sub-bands with Chern indices $C_-$ and $C_+$. In Ref. [17], it is shown that the Chern indices have the following property: Let $J^*$ be the zeros of $h_{12}(J_{cl})$, and $\lambda(J^*)$, their degree defined as follows: take a small direct circle around $J^*$; its image by $h_{12}(J_{cl})$ is a closed curve around 0 with $\sigma(J^*)$ turns. Define the set $S^+ = \{J_{cl} \in S^2_j \text{ such that } (h_{22}(J_{cl}) - h_{11}(J_{cl})) > 0\}$. Then

$$C_+ = \sum_{J^* \in S^+} \sigma(J^*) = -C_- \quad (6)$$

A direct consequence of this property is that a change of Chern index $\lambda(J^*)$ can only occur when simultaneously $h_{12}(J^*) = h_{22}(J^*) - h_{11}(J^*) = 0$. This corresponds to a degeneracy in the spectrum of $\hat{H}_s$, and a conical contact of the two bands.

Formula (4) can be applied to a Hamiltonian describing the rotational structure of the doubly degenerated
vibrational state of a tetrahedral (or octahedral) spherical top molecule [16]. The most general Hamiltonian, taken up to the third degree in \( \mathbf{J} \), has the form [8] with

\[
\begin{align*}
    h_{12}(\mathbf{J}) &= \frac{(J_x^2 - J_y^2)}{j^2} + iXJ_xJ_yJ_z/j^3, \\
    h_{22}(\mathbf{J}) &= h_{11}(\mathbf{J}) = \frac{(3J_z^2 - j(j + 1))}{j^2},
\end{align*}
\]

with parameter \( X \in \mathbb{R} \). In this case the set \( S^+ \) includes all points around north and south pole for which \( J_z > \sqrt{3} \sqrt{j(j + 1)} \) or \( J_z < -\sqrt{3} \sqrt{j(j + 1)} \). As \( h_{12}(\mathbf{J}) = 0 \) for simultaneously \( J_x = \pm J_y \), and \( J_xJ_yJ_z = 0 \) there are two points \( \mathbf{J}^* \in S^+ \): the north pole \( J_z = j \) and the south pole \( J_z = -j \). We consider \( \mathbf{J} \) going through a closed path surrounding each \( \mathbf{J}^* \) to calculate \( \sigma(\mathbf{J}^*) \). This gives for north and for the south poles \( \sigma_{\text{north}} = \sigma_{\text{south}} = 2 \text{sign}(X) \), so that \( C_+ = 4 \text{sign}(X) \) and \( C_- = -C_+ \).

This calculation explains why the rotational structure of doubly degenerate vibrational state is generally split into two sub-bands with respectively \( 2j + 5 \) and \( 2j - 3 \) levels [13][14]. In our current approach the appearance of two bands with Chern indices \( \pm 4 \) for the Hamiltonian [8] is due to the formation of eight degeneracies [equivalent by symmetry] between the two vibrational bands at \( X = 0 \) parameter value. More generally the characterization of the rovibrational structure of molecules and its possible modification under the variation of some physical parameters like total angular momentum can be done systematically by using Chern indices as topological quantum numbers. In Ref. [8] similar effect was discussed without explicit introduction of Chern indices.

The molecular application studied in this Letter was mainly inspired by the role played by topological Chern indices in the integer quantum Hall effect [20]. In this context, Chern indices describe the topology of Floquet bands \( (k) \rightarrow [\langle \psi_k(k) \rangle] \) where \( k \) is the Bloch wave vector, and give a quantum Hall conductance \( \sigma_g = (e^2/h)C_g \) under the hypothesis of adiabatic motion of \( k \) when a weak electrical field is applied. Contrary to Eq. [20], their sum for a given Landau level is \( \sum_g C_g = +1 \), because of the non trivial topology of the quantum space [20]. Many properties of the Chern indices are similar: a change \( \Delta C_g = \pm 1 \) is related to a conical degeneracy between consecutive bands. The application of semi-classical calculations of \( C_g \) done for the Hall conductance in Ref. [20], will be the subject of future work.

In summary we have discussed the role of Chern quantum numbers to molecular spectroscopy. The interpretation of good integer quantum numbers associated with rotational structure of different vibrational bands in terms of topological Chern numbers has naturally a wide applicability. These indices can be introduced any time when the adiabatic separation of variables enables one to split the global structure into bands associated with the “fast motion” and their internal structure described by a “slow motion” on a compact phase space. We have considered here only the problem when the dimension of the classical phase space formed by the slow variable \( \mathbf{J}_d \) is 2. Only the first Chern class appears in this case. Extension to higher dimension requires more delicate physical interpretation and more sophisticated mathematical tools in relation with the index theorem of Atiyah-Singer. In molecular spectroscopy many problems with intra-molecular dynamics are known in great detail. They can be used to test the applicability of this new concept.

We gratefully acknowledge Y. Colin de Verdiere for stimulating discussions and Bart Van-Tiggelen for helpful comments.

[1] D. J. Thouless. Topological quantum numbers in non relativistic physics. World Scientific, Singapore, 1998.
[2] L. D. Landau and E. M. Lifshitz Quantum mechanics. Pergamon Press, 1965. §105.
[3] L. C. Biedenharn and J. D. Louck Angular Momentum in Quantum Physics. Theory and Applications. Addison-Wesley, N.Y. 1981. §7.10.
[4] V. B. Pavlov-Verevkin, D. A. Sadovskii, and B. I. Zhirinskii, Europhys. Lett. 6, 573 (1988).
[5] D. A. Sadovskii, B. I. Zhirinskii, J. P. Champion, and G. Pierre, J. Chem. Phys. 92, 1523 (1990); V. M. Krivtsun, D. A. Sadovskii, and B. I. Zhirinskii J. Mol. Spectrosc. 139, 126 (1990); G. Dhont, D.A. Sadovskii, B. Zhirinskii, and V. Boudon, ibid, in press.
[6] B. I. Zhirinskii and S. Brodersen, J.Mol. Spectrosc. 163, 326 (1994).
[7] S. Brodersen and B. I. Zhirinskii, J.Mol. Spectrosc. 169, 1 (1995); 172, 303 (1995).
[8] B. I. Zhirinskii, Spectrochim. Acta A, 52, 881 (1996).
[9] J. Bellissard private communication.
[10] S. Zelditch private communication.
[11] D. A. Sadovskii and B. I. Zhirinskii, Phys. Lett. A 256, 235 (1999).
[12] W. M. Zhang, D. H. Feng and R. Gilmore. Rev. Mod. Phys., 62, 867 (1990).
[13] M. Nakahara. Geometry, topology and physics.Adam Hilger IOP Publishing Bristol, 1990.
[14] P. Griffiths and J. Harris. Principles of algebraic geometry. A Wiley-Interscience Publication. New York, 1978.
[15] P. Leboeuf and A. Voros. J. Phys. A, Math. Gen., 23, 1765 (1990).
[16] Harmonic oscillator quantum numbers \( n \) coincide with cluster indices used in [20] to describe the redistribution of energy levels between rovibrational bands.
[17] F. Faure. J. Phys. A, Math. Gen., 27, 7519 (1994).
[18] D.A. Sadovskii and B.I. Zhirinskii Mol. Phys., 65, 109 (1988).
[19] F. Michelot, J.Mol. Spectrosc. 63, 227 (1976).
[20] J. E. Avron, R. Seiler, and B. Simon. Phys. Rev. Lett., 51, 51 (1983).
[21] F. Faure. J. Phys. A, Math. Gen. in press.