The status of our understanding of colour confinement is reviewed.

1 Introduction

Confinement of colour is a fundamental problem in quantum field theory. Understanding the mechanism of confinement can also suggest observable predictions for heavy ion collisions.

Phenomenology strongly indicates that quarks exist as fundamental particles of hadronic matter. However quarks have never been observed as free particles. Upper limits to their production cross section, \( \sigma_q \), are reported by the particle data group. For reactions initiated by protons \( \sigma_q \leq 10^{-40} \text{ cm}^2 \). The ratio to the total cross section \( \sigma_{tot} \approx 10^{-25} \text{ cm}^2 \) has then the bound \( \sigma_q/\sigma_{tot} \leq 10^{-15} \). Similarly the expected abundance of relic quarks in the standard cosmological model as compared to the abundance of nucleons is expected to be \( n_q/n_p \geq 10^{-12} \).

The experimental upper limit produced by Millikan like experiments looking for fractionally charged particles is \( n_q/n_p \leq 10^{-27} \), corresponding to zero quarks observed in \( \sim 1 \text{ g of matter} \). Again \( n_{q,obs}/n_{q,exp} \leq 10^{-15} \). A non zero value of the above ratios, smaller than \( 10^{-15} \) would have no natural theoretical explanation. The most natural possibility is that they are equal to zero, and that confinement is an absolute property to be explained in terms of symmetry.

Perturbative quantization is not adequate to describe long distance physics: a non perturbative formulation like lattice can instead provide answers from first principles. Lattice does indeed indicate that colour is confined below a transition temperature \( T \sim 150 \text{ MeV} \), at which a deconfining transition takes place to a phase of quark gluon plasma. The parameters used to detect confinement are either the string tension \( \sigma \), which enters in the long range part of the \( q\bar{q} \) potential: \( V(R) \sim R \rightarrow \infty \sigma R \), and is extracted from the numerical determinations of Wilson loops; or the expectation value \( \langle L \rangle \) of the Polyakov line \( L(x) \) (the parallel transport from \( t = -\infty \) to
\[ t = \infty \text{ along the time axis. } \langle L \rangle \sim \exp(-\zeta/T), \zeta \text{ being the chemical potential of a quark in the vacuum: } \langle L \rangle = 0 \text{ means } \zeta = \infty, \text{ i.e. confinement. Lattice simulations at finite temperature are made on a lattice } N_T \times N_S^3 (N_S \gg N_T), \text{ the temperature } T \text{ being related to the lattice spacing as } T = 1/aN_T. \] 

Since 
\[
a \sim \frac{1}{\beta \rightarrow \infty} \exp(-b_0\beta)
\] 

with \( \beta = 2N_c/g^2 \) and \( b_0 > 0 \) (asymmetric freedom), low temperature (confined phase) corresponds to strong coupling. Explaining confinement in terms of symmetry, means then studying the symmetry of the strong coupling (disordered) phase. The key word for that is duality. 

2 Duality.

Duality is a deep concept in statistical mechanics and in quantum field theory. It applies to systems admitting non local excitations with non trivial topology. Such systems admit two complementary descriptions. A direct description in terms of fields \( \Phi \), which is suitable in the weak coupling regime, \( g \ll 1 \) (ordered phase). Their v.e.v. \( \langle \Phi \rangle \) are called order parameters: in this description there exist non local excitations \( \mu \). A dual description, suitable in the strong coupling regime (disordered phase), in which the excitations \( \mu \) become local, and the fields \( \Phi \) non local. The symmetry is described in terms of \( \langle \mu \rangle \), the disordered parameters. The "dual" effective coupling \( g_D \) is related to the direct \( g \) by the relation \( g_D \sim 1/g \): duality maps the strong coupling regime of one description into the weak coupling regime of the other.

Examples of system with duality are:

1) The 2d Ising model, where \( \Phi \) is the spin, \( \mu \) are the kinks.
2) The \( N = 2 \) SUSY QCD, where \( \Phi \) are the gauge superfields, \( \mu \) monopoles.
3) The 3d Heisenberg magnet, where \( \Phi \) is the magnetization, \( \mu \) are the Weiss domains.
4) The 3d XY model (liquid \( He_4 \)) where \( \Phi \) are the velocities, \( \mu \) the vortices.
5) Compact \( U(1) \) gauge theory, where \( \Phi \) is the e.m. field, \( \mu \) are monopoles.

The problem for \( QCD \) is to find the dual excitations \( \mu \), which condense in the confined phase, and are weakly interacting. A less ambitious task is to understand the symmetry of the confining vacuum. The keywords in this game are vortices and monopoles.

3 Monopoles

The natural topology in 3 dimensions comes from a mapping of the sphere \( S_2 \) at infinity on a group. A mapping of \( S_2 \) on \( SO(3)/U(1) \), or on \( SU(N)/SU(p) \otimes SU(N - p) \otimes U(1) \), \( (p = 1 \ldots N - 1) \) has monopoles as topological excitations. If the dual excitations \( \mu \) have nonzero magnetic charge, \( \langle \mu \rangle \neq 0 \) signals dual superconductivity. Then confinement is produced by the squeezing of the chromoelectric field of a \( q\bar{q} \) pair into Abrikosov like flux tubes. The energy is proportional to the length, so that \( V = \sigma R \). A conserved magnetic charge can be defined in a gauge theory by a procedure known as abelian projection. Choosing \( SU(2) \) for simplicity of notation, consider a field \( \Phi(x) \) belonging to the adjoint representation, i.e. a vector in colour space. Define \( \Phi(x) \equiv \Phi(x)/|\Phi(x)| \): this can be done everywhere except at zeros of \( \Phi(x) \). Define a field strength tensor
\[
F_{\mu\nu} = \hat{\Phi} \cdot \vec{G}_{\mu\nu} - \frac{1}{g} \hat{\Phi} \cdot (D_\mu \hat{\Phi} \wedge D_\nu \hat{\Phi})
\]

Here \( \vec{G}_{\mu\nu} = \partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu + g \vec{A}_\mu \wedge \vec{A}_\nu \), \( D_\mu = \partial_\mu + g \vec{A}_\mu \wedge \) is the covariant derivative.
$F_{\mu\nu}$ is a colour singlet and gauge invariant. In fact both terms in eq.(2) are separately gauge invariant: the choice of the coefficients is such that bilinear terms in $A_\mu A_\nu$ cancel. By simple algebra

$$F_{\mu\nu} = \Phi \cdot (\partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu) - \frac{1}{g} \Phi (\partial_\mu \Phi \wedge \partial_\nu \Phi)$$

(3)

$\hat{\Phi}(x)$ can be made a constant vector by a gauge transformation: the second term in eq.(3) then drops and $F_{\mu\nu}$ becomes an abelian field

$$F_{\mu\nu} = \partial_\mu (\hat{\Phi} \vec{A}_\nu) - \partial_\nu (\hat{\Phi} \vec{A}_\mu)$$

(4)

Such a gauge transformation is called abelian projection of $\hat{\Phi}$: it is singular at the zeros of $\vec{\Phi}(x)$, where $\hat{\Phi}$ is not defined. The dual tensor $F^*_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}$ defines a magnetic current $j_\mu = \partial_\nu F^*_{\nu\mu}$ which is identically conserved, being $F^*_{\mu\nu}$ antisymmetric. $j_\mu$ is usually zero (Bianchi identities): it can be non zero in the compact formulation of the theory. Monopoles sit then at the zeros of $\vec{\Phi}(x)$.

There exists a conserved magnetic charge for any choice of the field $\vec{\Phi}(x)$.

If an operator $\mu$ carrying nonzero magnetic charge has non zero v.e.v., $\langle \mu \rangle \neq 0$, the $U(1)$ magnetic simmetry is broken à la Higgs, and this implies dual superconductivity.

A magnetically charged operator $\mu$ can be constructed\(^\text{12}\). The construction is the adaptation to a compact theory of the translation operator $e^{ipa}$

$$e^{ipa}|x\rangle = |x + a\rangle$$

(5)

In the gauge theory $x$ is the field, $p$ the conjugate momentum, $a$ a classical monopole configuration: the translation operator adds a classical monopole to any field configuration\(^\text{12,13}\).

$\langle \mu \rangle$ is then determined by numerical simulations as a function of the temperature.

If the idea of dual superconductivity as a mechanism for confinement is correct, $\langle \mu \rangle$ should be different from zero in the confined phase, and go to zero in the deconfined phase. Of course this is strictly true in the infinite volume limit, when a phase transition can exist.

This is exactly what happens: $\langle \mu \rangle$ is determined at different sizes of the spatial size of the lattice, and the extrapolation to infinite volume is done by finite size scaling analysis.

The result is that $\langle \mu \rangle \neq 0 \text{ } T < T_c, \langle \mu \rangle = 0 \text{ } T > T_c$. Moreover $\langle \mu \rangle \simeq (1 - T/T_c)^\delta, \delta = 0.5 \pm 1$.

The result is independent of the abelian projection. Dual superconductivity is the mechanism of colour confinement. The same analysis can be made in the presence of quarks: contrary to the string tension, which is not defined in this case, because the string breaks producing $q\bar{q}$ pairs, or to $\langle L \rangle$ which also is not defined in this case because quarks break the $Z_N$ symmetry, dual superconductivity is a property of the vacuum which can exist. The study of $\langle \mu \rangle$ can shed light on the relation between chiral phase transition and deconfinement transition.

4 Vortices.

Vortices in $3 + 1$ dimensions are defects associated to closed lines, $C$, which are a kind of “dual Wilson loops”. They obey the relation with Wilson loop $W(C')$

$$B(C)W(C') = W(C')B(C) \exp(i \frac{\hbar \mu_{CC'}}{N} 2\pi)$$

(6)

In the same way as $\langle W(C') \rangle \neq 0, \langle B(C) \rangle \neq 0$ has no special meaning. $B(C)$ does not carry any conserved quantum number. What follows from eq.(3) is that if $\langle B(C) \rangle$ obey the area law $\langle W(C) \rangle$ obey the perimeter law and viceversa.

This has been checked on the lattice\(^\text{12}\).
5 Concluding remarks.

The QCD vacuum in the confined phase is a dual superconductor for the $U(1)$ magnetic symmetry defined by any abelian projection. This result is for sure a step in the direction of understanding confinement. It says that the dual excitations have magnetic charge in all the abelian projections. The details of these excitations are, however, still unknown.

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