Cryptographic security of quantum key distribution

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Abstract

This work is intended as an introduction to cryptographic security and a motivation for the widely used Quantum Key Distribution (QKD) security definition. We review the notion of security necessary for a protocol to be usable in a larger cryptographic context, i.e., for it to remain secure when composed with other secure protocols. We then derive the corresponding security criterion for QKD. We provide several examples of QKD composed in sequence and parallel with different cryptographic schemes to illustrate how the error of a composed protocol is the sum of the errors of the individual protocols. We also discuss the operational interpretations of the distance metric used to quantify these errors.

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1 Introduction

1.1 Background

The first Quantum Key Distribution (QKD) protocols were proposed independently by Bennett and Brassard [BB84] in 1984—inspired by early work on quantum money by Wiesner [Wie83]—and by Ekert [Eke91] in 1991. The original papers discussed security in the presence of an eavesdropper that could perform only limited operations on the quantum channel. The first security proofs that considered an unbounded adversary were given more than a decade later [May96, BBB+00, SP00, May01, BBB+06]. Another decade after the first such proof, König et al. [KRBM07] showed that the security criterion used was insufficient: even though it guarantees that an eavesdropper cannot guess the key, this only holds if the key is never used. If part of the key is revealed to the eavesdropper—for example, by using it to encrypt a message known to her—the rest becomes insecure. A new security criterion for QKD was introduced, along with a new proof of security for BB84 [RK05, BHL+05, Ren05]. It was argued that $\rho_{KE}$, the joint state of the final key ($K$) and quantum information gathered by an eavesdropper ($E$), must be close to an ideal key, $\tau_K$, that is perfectly uniform and independent from the adversary’s information $\rho_E$:

$$\left(1 - p_{\text{abort}}\right) D(\rho_{KE}, \tau_K \otimes \rho_E) \leq \varepsilon,$$

where $p_{\text{abort}}$ is the probability that the protocol aborts, $D(\cdot, \cdot)$ is the trace distance, and $\varepsilon \in [0, 1]$ is a (small) real number.

The type of security flaw suffered by the early QKD security criteria is well known in classical cryptography. It was addressed independently by Pfitzmann and Waidner [PW00, PW01, BPW04, BPW07] and Canetti [Can01, CDPW07, Can13], who introduced general frameworks to define cryptographic security, which they dubbed reactive simulatability and universal composability, respectively. These frameworks were adapted to quantum cryptography by Ben-Or and Mayers [BM04] and Unruh [Unr04, Unr10], and the security of QKD was discussed within these frameworks by Ben-Or et al. [BHL+05] and Müller-Quade and Renner [MQR09]. Recently, Maurer and Renner [MRT11] introduced a new cryptographic security framework, Abstract Cryptography (AC), which both simplifies and generalizes previous frameworks, and applies equally to the classical and quantum settings.

The core idea of all these security frameworks is to prove that the functionality constructed by the real protocol is indistinguishable from the func-

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1In [Ren05], Eq. (1) was introduced with a subnormalized state $\rho_{KE}$, with $\text{tr}(\rho_{KE}) = 1 - p_{\text{abort}}$, instead of explicitly writing the factor $(1 - p_{\text{abort}})$. The two formulations are however mathematically equivalent.

2This metric is defined and discussed in detail in Appendix A.

3Another formulation of this security criterion, $(1 - p_{\text{abort}}) \min_{\sigma_E} D(\rho_{KE}, \tau_K \otimes \sigma_E) \leq \varepsilon$, has also been proposed in the literature. We discuss this alternative in Appendix B.
tionality of an ideal resource that fulfills in a perfect way whatever task is expected of the cryptographic protocol — in the case of QKD, this ideal resource provides the two players with a perfect key, unknown to the adversary. If this ideal system is indistinguishable from the real one, then one can be substituted for the other in any context. Players who run a QKD protocol can thus treat the resulting key as if it were perfect, which trivially implies that it can be safely used and composed arbitrarily with other (secure) protocols.

1.2 Contributions
Since the security criterion of Eq. (1) provides the aforementioned compositional guarantees, it is widely used in the QKD literature and generally introduced as the correct security definition (see, e.g., the QKD review paper [SBPC+09]). A more detailed explanation as to why this is the case is however usually omitted due to the highly involved security frameworks. Even the technical works [RK05, BHL+05, Ren05, MQR09] that introduced and discuss Eq. (1) do not provide a self contained justification of this security notion. The current paper aims to fill in this gap by revisiting the security of QKD using the AC framework.

Our goals are twofold. Firstly, we provide an introduction to cryptographic security. We do not discuss the AC framework in detail, but explain the main ideas underlying cryptographic security and illustrate protocol composition with many examples. Secondly, we use this framework to show how Eq. (1) can be derived. We also provide in Appendix A an extensive discussion of the interpretation and operational meaning of the trace distance used in Eq. (1).

1.3 Abstract cryptography
The traditional approach to defining security [PW00, PW01, Can01] can be seen as bottom-up. One first defines (at a low level) a computational model (e.g., a Turing machine). One then defines how the machines communicate (e.g., by writing to and reading from shared tapes) and some form of scheduling. Next, one can define notions of complexity and efficiency. Finally, the security of a cryptosystem can be defined.

Abstract cryptography (AC) on the other hand uses a top-down approach. In order to state definitions and develop a theory, one starts from the other end, the highest possible level of abstraction — the composition of abstract systems — and proceeds downwards, introducing in each new lower level only the minimal necessary specializations. The (in)distinguishability of the real and ideal systems is defined as a metric on abstract systems, which, at a lower level, can be chosen to capture the distinguishing power of a computationally bounded or unbounded environment. The abstract
systems are instantiated with, e.g., a synchronous or asynchronous network of (abstract) machines. These machines can be instantiated with either classical or quantum processes.

One may give the analogous example of group theory, which is used to describe matrix multiplication. In the bottom-up approach, one would start explaining how matrices are multiplied, and then based on this find properties of the matrix multiplication. In contrast to this, the top-down approach would correspond to first defining the (abstract) multiplication group and prove theorems already on this level. The matrix multiplication would then be introduced as a special case of the multiplicative group. This simplifies greatly the framework by avoiding unnecessary specificities from lower levels, and does not hard code a computation or communication model (e.g., classical or quantum, synchronous or asynchronous) in the security framework.

1.4 Structure of this paper

In Section 2 we start by introducing a simplified version of the AC framework [MR11], which is sufficient for the specific adversarial structure relevant to QKD, namely honest Alice and Bob, and dishonest Eve. In Section 3 we model the real and ideal systems of a generic QKD protocol, and plug it in the AC security framework, obtaining a security definition for QKD. In Section 4 we then prove that this can be reduced to Eq. (1)\footnote{More precisely, the security definition of QKD is reduced to a combination of two criteria, secrecy (captured by Eq. (1)) and correctness.} In Section 5 we illustrate the composition of protocols in AC with examples of QKD composed in various settings. We emphasize that this section does not prove that the QKD security criterion is composable — the proof of this follows from the generic proof that the AC framework is composable [MR11] — but illustrates how the security of composed protocols results from the security of individual protocols and the triangle inequality. Further examples can be found in Appendix D, where we model the security of authentication and compose it with QKD, resulting in a key expansion protocol. We also provide a substantial review of the trace distance and its operational interpretations in Appendix A. In particular, we prove that it corresponds to the probability a distinguisher has of correctly guessing whether it is interacting with the real or ideal QKD system — the measure used in the AC framework — and discuss how to interpret this. An overview of the other appendices is given on page 31.
2 Cryptographic security

A central element in modeling security is that of resources — resources used in protocols and resources constructed by protocols. For example, a QKD protocol constructs a functionality which shares a secret key between two players. This functionality is a resource, which can be used by other protocols, e.g., to encrypt a message. To construct this secret key resource, a QKD protocol typically uses two other resources, an authentic classical channel and an insecure quantum channel. The authentic channel resource can in turn be constructed from an insecure channel resource and a password [RW03]. Composing the authentication protocol with the QKD protocol results in a scheme which constructs a secret key from a password and insecure channels. Part of the resulting secret key can be used in further rounds of authentication and QKD to produce even more secret key. This is illustrated in Figure 2.1.

For any cryptographic task one can define an ideal resource which fulfills this task in a perfect way. A protocol is then considered secure if the real resource actually constructed is indistinguishable from a system running the ideal resource. This notion of security based on distinguishing real and ideal systems is explained informally in Section 2.1. It is then illustrated with the one-time pad in Section 2.2. In Section 2.3 we give a formal security definition in the Abstract Cryptography (AC) framework for the special case of three party protocols with honest Alice and Bob, and dishonest Eve. Finally, in Section 2.4 we discuss how the metric used to quantify the (in)distinguishability between the real and ideal settings should be interpreted.

2.1 Real-world ideal-world paradigm

Cryptography aims at providing security guarantees in the presence of an adversary. And traditionally, security has been defined with respect to the information gathered by this adversary — but, as we shall see, this can be insufficient to achieve the desired security guarantees. A typical example of this is the security criterion used in early papers on QKD, e.g., [May96].
Figure 2.1 – A cryptographic protocol uses (weak) resources to construct other (stronger) resources. These resources are depicted in the boxes, and the arrows are protocols. Each box is a one-time-use resource, so the same resource appears in multiple boxes if different protocols require it. The long secret key resource in the center of the figure is split in three shorter keys, and each protocol uses one of these keys.
Let $K$ be the secret key produced by a run of a QKD protocol, and $Y$ be a random variable obtained by an adversary attacking the scheme and measuring her quantum system $E$. It can be argued that the key is unknown to the adversary if she gains only negligible information about it, i.e., if for all attacks and measurements of the resulting quantum system,

$$I(K;Y) \approx 0,$$  \hspace{1cm} (2)

where $I(K;Y)$ is the mutual information between $K$ and $Y$.

However, even if a key obtained from a protocol satisfying Eq. (2) is used in a perfectly secure encryption scheme like the one-time pad, it can leak information about the message. König et al. [KRBM07] give such an example: they find a quantum state $\rho_{KE}$ which satisfies Eq. (2), but which cannot be used to encrypt a message partly known to an adversary. They show that if the key is split in two, $K = K_1K_2$, and the adversary delays measuring her system $E$ until the first part, $K_1$, is revealed to her — e.g., because a known message was encrypted by the one-time pad with $K_1$ — she can obtain information about the rest of the key. More precisely, they prove that for this state $\rho_{K_1K_2E}$,

$$I(K_2;Y') \gg 0,$$

where $Y'$ is a random variable obtained by a measurement of the joint state $\rho_{K_1E}$ consisting of the partial key $K_1$ and the quantum information $E$ gathered during the QKD protocol. Even though the key obtained from the QKD protocol is approximately uniform and independent from the adversary’s information $Y$, it is unusable in a cryptographic context, and another approach than the adversarial viewpoint is necessary for defining cryptographic security.

This new approach was proposed independently by Canetti [Can01] and Pfitzmann and Waidner [PW00] for classical cryptography. The gist of their global security paradigm lies in measuring how well some real protocol can be distinguished from some ideal system that fulfills the task in an ideal way, and is often referred to as the “real-world ideal-world” paradigm.\footnote{As already noted in [Footnote 7], we use the notions real and ideal in a relative sense.}

To do this, the notion of an adversary is dropped in favor of a distinguisher. Apart from having the capabilities of the adversary, this distinguisher also encompasses any protocol that is run before, after, and during the protocol being analyzed. The role of the distinguisher is to capture “the rest of the world”, everything that exists outside of the honest players and the resources they share. A distinguisher is defined as an entity that can choose the inputs of the honest players (that might come from a

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\[ \text{BBB}^{+01}[\text{SP}00][\text{May}01]. \]
previously run protocol), receives their outputs (that could be used in a subsequent protocol), and simultaneously fulfills the role of the adversary, possibly eavesdropping on the communication channels and tampering with messages. This distinguisher is given a black box access to either the real or an ideal system, and must decide with which of the two it is interacting. A protocol is then considered secure if the real system constructed is indistinguishable from the ideal one. This is illustrated in Figure 2.2.

![Figure 2.2](image)

**Figure 2.2** – A distinguisher has a complete description of two systems, and is given a black-box access to one of the two. After interacting with the system, it must guess which one it is holding.

In the case of QKD, this means that the distinguisher does not only obtain the system $E$ of the eavesdropper, but also receives the final key $K$ generated by Alice and Bob. In the real world, this key is potentially correlated to $E$, and in an ideal system, $K$ is uniformly random and independent from $E$. The distinguisher can then run the attack of König et al. [KRB07] to distinguish between the real and ideal systems: if $Y'$, the result of the measurement of $K_1$ and $E$ is correlated to $K_2$, it knows that it was given the real system, otherwise it must have the ideal one. This specific attack is illustrated in more detail in Section 5.1.

### 2.2 Example: one-time pad

In this section, we illustrate with the one-time pad how security is defined in the real-world ideal-world paradigm. The one-time pad protocol uses a secret key $k$ to encrypt a message $x$ as $y := x \oplus k$. The ciphertext $y$ is then sent on an authentic channel to the receiver, who decrypts it, obtaining $x = y \oplus k$. $y$ is however also leaked to the adversary that is eavesdropping on the authentic channel. This is depicted in Figure 2.3.

![Figure 2.3](image)

The one-time pad protocol thus uses two resources, a secret key and an authentic channel. The resource we wish to construct with this encryption scheme is a secure channel: a resource which transmits a message $x$ from the sender to the receiver, and leaks only information about the message size $|x|$ at the adversary’s interface, but not the contents of the message. This
y = x ⊕ k
π_{otp}^{A}

Alice

x

y

x ⊕ k

Secret key

k

key

k

π_{otp}^{B}

Bob

x = y ⊕ k

Figure 2.3 – The real one-time pad system — Alice has access to the left interface, Bob to the right interface and Eve to the lower interface — consists of the one-time pad protocol (π_{otp}^{A}, π_{otp}^{B}), and the secret key and authentic channel resources. The combination of these resources and protocol constructs a system that takes a message $x$ at Alice’s interface, outputs a ciphertext $y$ at Eve’s interface and the original message $x$ at Bob’s interface.

is illustrated in Figure 2.4.

| x |

Alice

x

Bob

|x|↓

Eve

Figure 2.4 – A secure channel from Alice to Bob leaks only the message size at Eve’s interface.

Since an ideal resource “magically” solves the cryptographic task considered, e.g., by producing perfect secret keys or transmitting a message directly from Alice to Bob, the adversary’s interface of the ideal resource is usually quite different from her interface of the real system, which gives her access to the resources used. For the one-time pad, the real system from Figure 2.3 outputs a string $y$ at Eve’s interface, but the ideal secure channel from Figure 2.4 outputs an integer, $|x|$. To make the comparison between real and ideal systems possible, we define the ideal system to consist of the ideal resource as well as a simulator plugged into the adversary’s interface of the ideal resource, that recreates the communication occurring in the real system. For the one-time pad, this simulator must generate a ciphertext $y$ given the message length $|x|$. This is simply done by generating a random string of the appropriate length, as depicted in Figure 2.5. Note that putting such a simulator between the ideal resource and the adversary can only weaken her, since any operation performed by the simulator could equivalently be performed by an adversary connected directly to the interface of the ideal resource.
Figure 2.5 – The ideal one-time pad system — Alice has access to the left interface, Bob to the right interface and Eve to the lower interface — consists of the ideal secure channel and a simulator $\sigma_{otp}$ that generates a random string $y$ of length $|x|$.

To prove that the one-time pad constructs a secure channel from an authentic channel and a secret key, we view the real and ideal one-time pad systems of Figure 2.3 and Figure 2.5 as black boxes, and need to show that no distinguisher can tell with which of the two it has been connected. For both black boxes, if the distinguisher inputs $x$ at Alice’s interface, the same string $x$ is output at Bob’s interface and a uniformly random string of length $|x|$ is output at Eve’s interface. The two systems are thus completely indistinguishable — if the distinguisher were to take a guess, it would be right with probability exactly $1/2$ — and we say that the one-time pad has perfect security.

If two systems are indistinguishable, they can be used interchangeably in any setting. For example, let some protocol $\pi'$ be proven secure if Alice and Bob are connected by a secure channel. Since the one-time pad constructs such a channel, it can be used in lieu of the secure channel, and composed with $\pi'$. Or equivalently, the contrapositive: if composing the one-time pad and $\pi'$ were to leak some vital information, which would not happen with a secure channel, a distinguisher that is either given the real or ideal system could run $\pi'$ internally and check whether this leak occurs to know with which of the two it is interacting.

2.3 General security definition

The previous sections introduced the concepts of resources, protocols and simulator in an informal manner. In the AC framework these elements are defined in an abstract way. For example, a resource is an abstract system that is shared between all players and provides each one with an interface that allows in- and outputs. AC does not define the internal workings of a resource. It postulates axioms that these abstract systems must fulfill — e.g. there must exist a metric and a parallel composition operator on the space of resources — and is valid for any instantiation which respects these axioms. In
the group theory analogy introduced in Section 1.3, these axioms correspond to the group axioms (closure, associativity, identity and invertibility). Any set and operation that respects these group axioms is an instantiation of a group, and any theorem proven for groups applies to this instantiation.

Thus, AC defines cryptographic security for abstract systems which fulfill certain basic properties. In the following we briefly sketch what these are. Note that examples — such as the model of the one-time pad given in Figures 2.3 and 2.5 — necessarily assume some instantiation of the abstract systems. Since we consider only simple examples in this work, we do not provide formal generic definitions of these lower levels, and refer to the discussions in [MRT11, Man12, DFPR14] on how this can be modeled.

Resource. An $I$-resource is an (abstract) system with interfaces specified by a set $I$ (e.g., $I = \{A, B, E\}$). Each interface $i \in I$ is accessible to a user $i$ and provides her or him with certain controls (the possibility of reading outputs and providing inputs). Resources are equipped with a parallel composition operator, $\parallel$, that maps two resources to another resource.

Converter. To transform one resource into another, we use converters. These are (abstract) systems with two interfaces, an inside interface and an outside interface. The inside interface connects to an interface of a resource, and the outside interface becomes the new interface of the constructed resource. We write either $\alpha_i R$ or $R \alpha_i$ to denote the new resource with the converter $\alpha$ connected to the interface $i$ of $R$, and $\alpha R$ or $R \alpha$ for a set of converters $\alpha = \{\alpha_i\}_i$, for which it is clear to which interface they connect.

A protocol is a set of converters (one for every honest player) and a simulator is also a converter. Another type of converter that we need is a filter, which we often denote by $\sharp$ or $\diamond$. When placed over a dishonest player’s interface, a filter prevents access to the corresponding controls and emulates an honest behavior.

Serial and parallel composition of converters is defined as follows:

$$ (\alpha \beta)_i R := \alpha_i (\beta_i R) \quad \text{and} \quad (\alpha \parallel \beta)_{i}(R \parallel S) := (\alpha_i R) \parallel (\beta_i S). \quad (3) $$

Filtered resource. A pair of a resource $R$ and a filter $\sharp$ together specify the (reactive) behavior of a system both when no adversary is present — with the filter plugged in the adversarial interface, $R_{\sharp E}$ — and in the case of a cheating player that removes the filter and has full access to her interface of $R$. We call such a pair $(R, \sharp)$ a filtered resource, and usually denote it by $R_{\sharp}$.

\footnote{There is no mathematical difference between $\alpha_i R$ and $R \alpha_i$. It sometimes simplifies the notation to have the converters for some players written on the right of the resource and the ones for other players on the left, instead of all on the same side, hence the two notations.}
Metric. There must exist a pseudo-metric $d(\cdot, \cdot)$ on the space of resources, i.e., for any three resources $R, S, T$, it satisfies the following conditions:

1. (identity) $d(R, R) = 0$, (4)
2. (symmetry) $d(R, S) = d(S, R)$, (5)
3. (triangle inequality) $d(R, S) \leq d(R, T) + d(T, S)$. (6)

Furthermore, this pseudo-metric must be non-increasing under composition with resources and converters: for any converter $\alpha$ and resources $R, S, T$, we require

$$d(\alpha R, \alpha S) \leq d(R, S) \quad \text{and} \quad d(R \parallel T, S \parallel T) \leq d(R, S).$$ (7)

We are now ready to define the security of a cryptographic protocol. We do so in the three player setting, for honest Alice and Bob, and dishonest Eve. Thus, in the following, all resources have three interfaces, denoted $A, B$ and $E$, and we only consider honest behaviors (given by a protocol $(\pi_A, \pi_B)$) at the $A$ and $B$-interfaces, but arbitrary behavior at the $E$-interface. We refer to [MR11] for the general case, when arbitrary players can be dishonest.

Definition 2.1 (Cryptographic security [MR11]). Let $\pi_{AB} = (\pi_A, \pi_B)$ be a protocol and $R\diamond = (R, \diamond)$ and $S\Diamond = (S, \Diamond)$ denote two filtered resources. We say that $\pi_{AB}$ constructs $S\Diamond$ from $R\diamond$ within $\varepsilon$, which we write $R\diamond \xrightarrow{\pi_{AB}} S\Diamond$, if the two following conditions hold:

i) We have $d(\pi_{AB}R\diamond_E, S\Diamond_E) \leq \varepsilon$.

ii) There exists a converter $\sigma_E$—which we call simulator—such that $d(\pi_{AB}R, S\sigma_E) \leq \varepsilon$.

If it is clear from the context what filtered resources $R\diamond$ and $S\Diamond$ are meant, we simply say that $\pi_{AB}$ is $\varepsilon$-secure.

The first of these two conditions measures how close the constructed resource is to the ideal resource in the case where no malicious player is intervening, which we call availability. The second condition captures security in the presence of an adversary. These two equations are illustrated in Figure 2.6.

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13If additionally $d(R, S) = 0 \implies R = S$, then $d$ is a metric.
14This is sometimes referred to as the correctness of the protocol in the cryptographic literature. But in QKD, correctness has another meaning—namely the probability that Alice and Bob end up with different keys when Eve is active. Instead, the term robustness is traditionally used to denote the performance of a QKD protocol under honest (noisy) conditions. We refer to Section 4.4 for a discussion of the relation between availability and robustness.
Condition (i) from Definition 2.1. If Eve’s interfaces are blocked by filters emulating honest behavior, the functionality constructed by the protocol should be indistinguishable from the ideal resource.

Condition (ii) from Definition 2.1. If Eve accesses her cheating interface of \( R \), the resulting system must be simulatable in the ideal world by a converter \( \sigma_E \) that only accesses Eve’s interface of the ideal resource \( S \).

Figure 2.6 – A protocol \((\pi_A, \pi_B)\) constructs \( S_\diamond \) from \( R_\# \) within \( \varepsilon \) if the two conditions illustrated in this figure hold. The sequences of arrows at the interfaces between the objects represent (arbitrary) rounds of communication.

It follows from the \( \text{AC} \) framework [MR11] that if two protocols \( \pi \) and \( \pi' \) are \( \varepsilon \) and \( \varepsilon' \)-secure, the composition of the two is \((\varepsilon + \varepsilon')\)-secure. We illustrate this with several examples in Section 5 and Appendix D, and sketch a generic proof in Appendix C.2.

2.4 The distinguishing metric

The usual pseudo-metric used to define security in the real-world ideal-world paradigm is the distinguishing advantage, defined as follows. If a distinguisher \( \mathcal{D} \) can guess correctly with probability \( p_{\text{distinguish}}^{\mathcal{D}}(R, S) \) with which of two systems \( R \) and \( S \) it is interacting, we define its advantage as

\[
d^{\mathcal{D}}(R, S) := 2p_{\text{distinguish}}^{\mathcal{D}}(R, S) - 1 .
\]

Changing the power of the distinguisher \( \mathcal{D} \) (e.g., computationally bounded or unbounded) results in different metrics and different levels of security. In this work we are interested only in information-theoretic security, we therefore consider only a computationally unbounded distinguisher, and drop the
We write
\[
d(R, S) \leq \varepsilon \quad \text{or} \quad R \approx_\varepsilon S,
\]
if two systems \(R\) and \(S\) can be distinguished with advantage at most \(\varepsilon\), and in the following, the distance between two resources always refers to the distinguishing advantage of an unbounded distinguisher. A more extensive discussion of distinguishers is given in Appendix C.1.

Although any pseudo-metric which satisfies the basic axioms can be used in Definition 2.1, the distinguishing advantage is of particular importance, because it has an operational definition — the advantage a distinguisher has in guessing whether it is interacting with the real or ideal system. If the distinguisher notices a difference between the two, then something in the real setting did not behave ideally. This can be loosely interpreted as a failure occurring. If the distinguisher can guess correctly with probability 1 with which system it is interacting, a failure must occur systematically. If it can only guess correctly with probability \(1/2\), no failure occurs at all. If it can guess correctly with probability \(p\), this can be seen as a failure occurring with probability \(\varepsilon = 2p - 1\). The distinguishing advantage can thus be interpreted as the probability that a failure occurs in the real protocol.\(^{15}\) And in any practical implementation, the value \(\varepsilon\) can be chosen accordingly.

A bound on the security of a protocol does however not tell us how “bad” this failure is. For example, a key distribution protocol which produces a perfectly uniform key, but with probability \(\varepsilon\) Alice and Bob end up with different keys, is \(\varepsilon\)-secure. Likewise, a protocol which gives 1 bit of the key to Eve with probability \(\varepsilon\), but is perfect otherwise, and another protocol which gives the entire key to Eve with probability \(\varepsilon\), but is perfect otherwise, are both \(\varepsilon\)-secure as well. One could argue that leaking the entire key is worse than leaking one bit, which is worse than not leaking anything but generating mismatching keys, and this should be reflected in the level of security of the protocol. However, leaking one bit can be as bad as leaking the entire key if only one bit of the message is vital, and this happens to be the bit obtained by Eve. Having mismatching keys and therefore misinterpreting a message could have more dire consequences than leaking the message to Eve. How bad a failure is depends on the use of the protocol, and since the purpose of cryptographic security is to make a security statement that is valid for all contexts, bounding the probability that a failure occurs is the best it can do.

Since such a security bound gives no idea of the gravity of a failure — a faulty QKD protocol might not only leak the current key, but all future keys as well if the current key is used to authenticate messages in future rounds — the probability \(\varepsilon\) of a failure occurring must be chosen small enough that the

\(^{15}\)A formal derivation of this interpretation is given in Appendix A.3 for the trace distance — the distinguishing advantage between two quantum states.
accumulation of all possible failure probabilities over a lifetime is still small enough. For example, if an implementation of a QKD protocol produces a key at a rate of 1 Mbit/s with a failure per bit of $10^{-24}$, then this protocol can be run for the age of the universe and still have an accumulated failure strictly less than 1.

3 Quantum key distribution

In order to apply the general AC security definition to QKD, we need to specify the ideal key filtered resource, which we do in Section 3.1. Likewise, we specify in Section 3.2 the real QKD system consisting of the protocol, an authentic classical channel and an insecure quantum channel. Plugging these systems in Definition 2.1, we obtain in Section 3.3 the security criteria for QKD.

3.1 Ideal key

The goal of a key distribution protocol is to generate a secret key shared between two players. One can represent such a resource by a box, one end of which is in Alice’s lab, and another in Bob’s. It provides each of them with a secret key of a given length, but does not give Eve any information about the key. This is illustrated in Figure 3.1a, and is the key resource we used in the one-time pad construction (Figure 2.3).

However, if we wish to realize such a functionality with QKD, there is a caveat: an eavesdropper can always prevent any real QKD protocol from generating a key by cutting or jumbling the communication lines between Alice and Bob, and this must be reflected in the definition of the ideal resource. This box thus also has an interface accessible to Eve, which provides her with a switch that, when pressed, prevents the box from generating this key. We depict this in Figure 3.1b.

If modeled with the secret key resource of Figure 3.1b, the one-time pad is trivially secure conditioned on Eve preventing a key from being distributed — in this case, Alice and Bob do not have a key and do not run the one-time pad. The security of the one-time pad is thus reduced to the case where a key is generated, which corresponds to Figure 3.1a and is the situation analyzed in Section 2.2.

If no adversary is present, a filter covers Eve’s interface of the resource, making it inaccessible to the distinguisher. This filter emulates the honest behavior that one expects in the case of a non-malicious noisy channel. For a protocol and noisy channel that together produce a key with probability $1 - \delta$, the filter should flip the switch on the $E$-interface of the ideal key with probability $\delta$. This is illustrated in Figure 3.1c and discussed in more detail in Section 4.4.
A resource that always gives a key $k$ to Alice and Bob, and nothing to Eve.

A resource that allows Eve to decide if Alice and Bob get a key $k$ or an error $\perp$.

The resource from Figure 3.1b with a filter $\Diamond_E$, modeling the case with no adversary.

The resource from Figure 3.1b with a simulator $\sigma_E$.

Figure 3.1 – Some depictions of shared secret key resources, with filter and simulator converters in the last two.

Remark 3.1 (Adaptive key length). For a protocol to construct the shared secret key resource of Figure 3.1b, it must either abort or produce a key of a fixed length. A more practical protocol could adapt the secret key length to the noise level on the quantum channel. This provides the adversary with the functionality to control the key length (not only whether it gets generated or not), and can be modeled by allowing the key length to be input at Eve’s interface of the ideal key resource.

3.2 Real protocol

To construct the secret key resource of Figure 3.1b, a QKD protocol uses some other resources: a two-way authentic classical channel and an insecure quantum channel. An authentic channel faithfully transmits messages between Alice and Bob, but provides Eve with a copy as well. An insecure channel is completely under the control of Eve, she can apply any operation allowed by physics to the message on the channel. If Eve does not intervene, some noise might still be present on the channel, which is modeled by a filter that prevents Eve from reading the message, but introduces honest noise.
instead. Since an authentic channel can be constructed from an insecure channel and a short shared secret key, QKD is sometimes referred to as a key expansion protocol.

A QKD protocol typically has three phases: quantum state distribution, error estimation and classical post-processing (for a detailed review of QKD see [SBPC+09]). In the first, Alice sends some quantum states on the insecure channel to Bob, who measures them upon reception, obtaining a classical string. In the error estimation phase, they communicate on the (two-way) authentic classical channel to sample some bits at random positions in the string and estimate the noise on the quantum channel by comparing these values to what Bob should have obtained. If the noise level is above a certain threshold, they abort the protocol and output an error message. If the noise is low enough, they move on to the third phase, and make use of the authentic channel to perform error correction and privacy amplification on their respective strings, resulting in keys $k_A$ and $k_B$ (which, ideally, should be equal). We sketch this in Figure 3.2.

Remark 3.2 (Source of entanglement). In this work we use an insecure quantum channel from Alice to Bob to construct the shared secret key resource. An alternative resource that is frequently used in QKD instead of this insecure channel, is a source of entangled states under the control of Eve. The source sends half of an entangled state to Alice and another half to Bob. It can be modeled similarly to the insecure channel depicted in Figure 3.2, but with the first arrow reversed: the states are sent from Eve to Alice and from Eve to Bob.

3.3 Security

Let $(\pi_{qkd}^A, \pi_{qkd}^B)$ be the QKD protocol. Let $\mathcal{Q}$ and $\mathcal{A}$ be the insecure quantum channel and authentic classical channel, respectively, with their filters $\mathcal{E}_E$ and $\mathcal{F}_E$. Let $\mathcal{K}$ denote the secret key resource of Figure 3.1b and let $\mathcal{E}_E$ be its filter. Applying Definition 2.1, we find that $(\pi_{qkd}^A, \pi_{qkd}^B)$ constructs $\mathcal{K}\mathcal{E}_E$ from $\mathcal{Q}$ and $\mathcal{A}_{\mathcal{E}}$ within $\varepsilon$ if

$$\pi_{qkd}^A \pi_{qkd}^B (\mathcal{Q}\parallel\mathcal{A})(\mathcal{E}_E) \approx_\varepsilon \mathcal{K}\mathcal{E}_E \quad (9)$$

and

$$\exists \mathcal{E}_E, \quad \pi_{qkd}^A \pi_{qkd}^B (\mathcal{Q}\parallel\mathcal{A}) \approx_\varepsilon \mathcal{K}\mathcal{E}_E \quad (10)$$

The left- and right-hand sides of Eq. (9) are illustrated in Figures 3.2b and 3.1c and the left- and right-hand sides of Eq. (10) are illustrated in Figures 3.2a and 3.1d. These two conditions are decomposed into simpler criteria in Section 4.

\footnote{In fact, a short non-uniform key is sufficient for authentication [RW03], see Footnote 6.}

\footnote{We model QKD this way in Appendix D.3.}
(a) When Eve is present, her interface gives her complete control of the insecure channel and allows her to read the messages on the authentic channel.

(b) When no eavesdropper is present, filters forward Alice’s quantum messages to Bob and block the authentic channel’s output at the $E$-interface. The filter $\sharp_E$ might produce non-malicious noise that modifies $\rho$ and models a (honest) noisy channel.

Figure 3.2 – The real QKD system — Alice has access to the left interface, Bob to the right interface and Eve to the lower interface — consists of the protocol $(\pi_{qkd}^A, \pi_{qkd}^B)$, the insecure quantum channel $Q$ and two-way authentic classical channel $A$. Alice and Bob abort if the insecure channel is too noisy, i.e., if $\rho'$ is not similar enough to $\rho$ to obtain a secret key of the desired length. They run the classical post-processing over the authentic channel, obtaining keys $k_A$ and $k_B$. The message $t$ depicted on the two-way authentic channel represents the entire classical transcript of the classical post-processing.
4 Security reduction

By applying the general AC security definition to QKD, we obtained two criteria, Eqs. (9) and (10), capturing availability and security, respectively. In this section we derive Eq. (1), the trace distance criterion discussed in the introduction, from Eq. (10). We first show in Section 4.1 that the distinguishing advantage used in the previous sections reduces to the trace distance between the quantum states gathered by the distinguisher interacting with the real and ideal systems. Then in Section 4.2 we fix the simulator \( \sigma_E \) from the ideal system. In Section 4.3 we decompose the resulting security criterion into a combination of secrecy — Eq. (1) — and correctness — the probability that Alice’s and Bob’s keys differ. In the last section, 4.4, we consider the security condition of Eq. (9), which captures whether, in the absence of a malicious adversary, the protocol behaves as specified by the ideal resource and corresponding filter. We show how this condition can be used to model the robustness of the protocol — the probability that the protocol aborts with non-malicious noise.

4.1 Trace distance

The security criteria given in Eqs. (9) and (10) are defined in terms of the distinguishing advantage between resources. To simplify these equations, we rewrite them in terms of the trace distance, \( D(\cdot, \cdot) \). A formal definition of this metric is given in Appendix A.1 along with a discussion of how to interpret it in the rest of Appendix A. We start with the simpler case of Eq. (9) in the next paragraph, then deal with Eq. (10) after that.

The two resources on the left- and right-hand sides of Eq. (9) simply output classical strings (a key or error message) at Alice and Bob’s interfaces. Let these pairs of strings be given by the joint probability distributions \( P_{AB} \) and \( \tilde{P}_{AB} \). The distinguishing advantage between these systems is thus simply the distinguishing advantage between these probability distributions — a distinguisher is given a pair of strings sampled according to either \( P_{AB} \) or \( \tilde{P}_{AB} \) and has to guess from which distribution it was sampled — i.e.,

\[
D\left(\pi^{qkd}_A \pi^{qkd}_B (\mathcal{Q}||A)(\mathcal{E}||b_E), \mathcal{K} \right) = d(P_{AB}, \tilde{P}_{AB}).
\]

The distinguishing advantage between two probability distributions is equal to their total variation distance\(^{18}\) — which we prove in in Appendix A.2 — i.e.,

\[
d(P_{AB}, \tilde{P}_{AB}) = D(P_{AB}, \tilde{P}_{AB}).
\]

Putting the two together we get

\[
d\left(\pi^{qkd}_A \pi^{qkd}_B (\mathcal{Q}||A)(\mathcal{E}||b_E), \mathcal{K} \right) = D(P_{AB}, \tilde{P}_{AB}) .
\]

\(^{18}\)The total variation distance between two probability distributions is equivalent to the trace distance between the corresponding (diagonal) quantum states. We use the same notation for both metrics, \( D(\cdot, \cdot) \), since the former is a special case of the latter.
where \( P_{AB} \) and \( \hat{P}_{AB} \) are the distributions of the strings output by the real and ideal systems, respectively.

The resources on the left- and right-hand sides of Eq. (10) are slightly more complex. They first output a state \( \varphi_C \) at the \( E \)-interface, namely the quantum states prepared by Alice, which she sends on the insecure quantum channel. Without loss of generality, the distinguisher now applies any map \( \mathcal{E} : \mathcal{L}(\mathcal{H}_C) \rightarrow \mathcal{L}(\mathcal{H}_{CE'}) \) allowed by quantum physics to this state, obtaining \( \rho_{CE'} = \mathcal{E}(\varphi_C) \) and puts the \( C \) register back on the insecure channel for Bob, keeping the part in \( E' \). Finally, the systems output some keys (or error messages) at the \( A \) and \( B \)-interfaces, and a transcript of the post-processing at the \( E \)-interface. Let \( \rho_{ABE}^\mathcal{E} \) denote the tripartite state held by a distinguisher interacting with the real system, and let \( \tilde{\rho}_{ABE}^\mathcal{E} \) denote the state held after interacting with the ideal system, where the registers \( A \) and \( B \) contain the final keys or error messages, and the register \( E \) holds both the state \( \rho_{E'} \) obtained from tampering with the quantum channel and the post-processing transcript. Distinguishing between these two systems thus reduces to maximizing over the distinguisher strategies (the choice of \( \mathcal{E} \)) and distinguishing between the resulting states, \( \rho_{ABE}^\mathcal{E} \) and \( \tilde{\rho}_{ABE}^\mathcal{E} \):

\[
d(\sigma_{A}^{\text{qkd}}, \pi_{B}^{\text{qkd}}(Q\|A), \mathcal{K}\sigma_{E}) = \max_{\mathcal{E}} d(\rho_{ABE}^\mathcal{E}, \tilde{\rho}_{ABE}^\mathcal{E}) .
\]

The advantage a distinguisher has in guessing whether it holds the state \( \rho_{ABE}^\mathcal{E} \) or \( \tilde{\rho}_{ABE}^\mathcal{E} \) is given by the trace distance between these states, i.e.,

\[
d(\rho_{ABE}^\mathcal{E}, \tilde{\rho}_{ABE}^\mathcal{E}) = D(\rho_{ABE}^\mathcal{E}, \tilde{\rho}_{ABE}^\mathcal{E}) .
\]

This was first proven by Helstrom [Hel76]. For completeness, we provide a proof in Appendix A.2 Theorem A.5.

The distinguishing advantage between the real and ideal systems of Eq. (10) thus reduces to the trace distance between the quantum states gathered by the distinguisher. In the following, we usually omit \( \mathcal{E} \) where it is clear that we are maximizing over the distinguisher strategies, and simply express the security criterion as

\[
D(\rho_{ABE}, \tilde{\rho}_{ABE}) \leq \varepsilon ,
\]

where \( \rho_{ABE} \) and \( \tilde{\rho}_{ABE} \) are the quantum states gathered by the distinguisher interacting with the real and ideal systems, respectively.

### 4.2 Simulator

In the real setting (Figure 3.2a), Eve has full control over the quantum channel and obtains the entire classical transcript of the protocol. So for the real and ideal settings to be indistinguishable, a simulator \( \sigma_{E}^{\text{qkd}} \) must generate the same communication as in the real setting. This can be done
by internally running Alice’s and Bob’s protocol \((\pi_A^{\text{qkd}}, \pi_B^{\text{qkd}}))\), producing the same messages at Eve’s interface as the real system. However, instead of letting this (simulated) protocol decide the value of the key as in the real setting, the simulator only checks whether they actually produce a key or an error message, and presses the switch on the secret key resource accordingly. We illustrate this in Figure 4.1.

![Secret key resource](image)

**Figure 4.1** – The ideal QKD system — Alice has access to the left interface, Bob to the right interface and Eve to the lower interface — consists of the ideal secret key resource and a simulator \(\sigma_E^{\text{qkd}}\).

The security criterion from Eq. (11) can now be simplified by noting that with this simulator, the states of the ideal and real systems are identical when no key is produced. The outputs at Alice’s and Bob’s interfaces are classical, elements of the set \(\{\perp\} \cup \mathcal{K}\), where \(\perp\) symbolizes an error and \(\mathcal{K}\) is the set of possible keys. The states of the real and ideal systems can be written as

\[
\rho_{ABE} = p^\perp |\perp_A, \perp_B\rangle \langle \perp_A, \perp_B| \otimes \rho^\perp_E + \sum_{k_A,k_B \in \mathcal{K}} p_{k_A,k_B} |k_A, k_B\rangle \langle k_A, k_B| \otimes \rho_{E}^{k_A,k_B},
\]

\[
\tilde{\rho}_{ABE} = p^\perp |\perp_A, \perp_B\rangle \langle \perp_A, \perp_B| \otimes \rho^\perp_E + \frac{1}{|\mathcal{K}|} \sum_{k \in \mathcal{K}} |k,k\rangle \langle k,k| \otimes \sum_{k_A,k_B \in \mathcal{K}} p_{k_A,k_B} \rho_{E}^{k_A,k_B}.
\]

Plugging these in Eq. (11) we get

\[
D(\rho_{ABE}, \tilde{\rho}_{ABE}) = (1 - p^\perp)D\left(\rho_{ABE}^\top, \tau_{AB} \otimes \rho^\top_E\right) \leq \varepsilon, \tag{12}
\]

where

\[
\rho_{ABE}^\top := \frac{1}{1 - p^\perp} \sum_{k_A,k_B \in \mathcal{K}} p_{k_A,k_B} |k_A, k_B\rangle \langle k_A, k_B| \otimes \rho_{E}^{k_A,k_B}
\]

(13)

is the renormalized state of the system conditioned on not aborting and \(\tau_{AB} := \frac{1}{|\mathcal{K}|} \sum_{k \in \mathcal{K}} |k,k\rangle \langle k,k|\) is a perfectly uniform shared key.
4.3 Correctness & secrecy

We now break Eq. (12) down into two components, often referred to as correctness and secrecy, and recover the security definition for QKD introduced in [RK05, BHL+05, Ren05]. The correctness of a QKD protocol refers to the probability that Alice and Bob end up holding different keys. We say that a protocol is $\varepsilon_{\text{cor}}$-correct if for all adversarial strategies,

$$\Pr[K_A \neq K_B] \leq \varepsilon_{\text{cor}},$$

(14)

where $K_A$ and $K_B$ are random variables over the alphabet $\mathcal{K} \cup \{\bot\}$ describing Alice’s and Bob’s outputs.$^{19}$

The secrecy of a QKD protocol measures how close the final key is to a distribution that is uniform and independent of the adversary’s system. Let $p^{\bot}$ be the probability that the protocol aborts, and $\rho_{AE}^{\bot}$ be the resulting state of the $AE$ subsystems conditioned on not aborting. A protocol is $\varepsilon_{\text{sec}}$-secret if for all adversarial strategies,

$$(1 - p^{\bot})D(\rho_{AE}^{\bot}, \tau_A \otimes \rho_{E}^{\bot}) \leq \varepsilon_{\text{sec}},$$

(15)

where the distance $D(\cdot, \cdot)$ is the trace distance and $\tau_A$ is the fully mixed state.$^{20}$

**Theorem 4.1.** If a QKD protocol is $\varepsilon_{\text{cor}}$-correct and $\varepsilon_{\text{sec}}$-secret, then Eq. (10) is satisfied for $\varepsilon = \varepsilon_{\text{cor}} + \varepsilon_{\text{sec}}$.

**Proof.** Let us define $\gamma_{ABE}$ to be a state obtained from $\rho_{ABE}^{\bot}$ [Eq. (13)] by throwing away the $B$ system and replacing it with a copy of $A$, i.e.,

$$\gamma_{ABE} = \frac{1}{1 - p^{\bot}} \sum_{k_A, k_B \in \mathcal{K}} p_{k_A, k_B} |k_A, k_A\rangle \langle k_A, k_A| \otimes \rho_{E}^{k_A, k_B}.$$  

From the triangle inequality we get

$$D(\rho_{ABE}^{\bot}, \tau_{AB} \otimes \rho_{E}^{\bot}) \leq D(\rho_{ABE}^{\bot}, \gamma_{ABE}) + D(\gamma_{ABE}, \tau_{AB} \otimes \rho_{E}^{\bot}).$$

Since in the states $\gamma_{ABE}$ and $\tau_{AB} \otimes \rho_{E}^{\bot}$ the $B$ system is a copy of the $A$ system, it does not modify the distance. Furthermore, $\text{tr}_B(\gamma_{ABE}) = \text{tr}_B(\rho_{ABE}^{\bot})$. Hence

$$D(\gamma_{ABE}, \tau_{AB} \otimes \rho_{E}^{\bot}) = D(\gamma_{AE}, \tau_A \otimes \rho_{E}^{\bot}) = D(\rho_{AE}^{\bot}, \tau_A \otimes \rho_{E}^{\bot}).$$

\footnote{This can equivalently be written as $(1 - p^{\bot}) \Pr[K_A^{\bot} \neq K_B^{\bot}] \leq \varepsilon_{\text{cor}}$, where $p^{\bot}$ is the probability of aborting and $K_A^{\bot}$ and $K_B^{\bot}$ are Alice and Bob’s keys conditioned on not aborting.}

\footnote{Eq. (15) is a reformulation of Eq. (1)}
For the other term note that
\[
D(\rho_{ABE}, \gamma_{ABE}) \leq \sum_{k_A,k_B} \frac{p_{k_A,k_B}}{1-p^\perp} D\left(|k_A,k_B\rangle \langle k_A,k_B| \otimes \rho_{E}^{k_A,k_B}, |k_A,k_A\rangle \langle k_A,k_A| \otimes \rho_{E}^{k_A,k_B}\right)
= \sum_{k_A \neq k_B} \frac{p_{k_A,k_B}}{1-p^\perp} = \frac{1}{1-p^\perp} \Pr[K_A \neq K_B].
\]

Putting the above together with Eq. (12), we get
\[
D(\rho_{ABE}, \tilde{\rho}_{ABE}) = (1-p^\perp) D(\rho^{\top}_{ABE}, \tau_{AB} \otimes \tilde{\rho}_{E}) \leq \Pr[K_A \neq K_B] + (1-p^\perp) D(\rho^{\top}_{AE}, \tau_{A} \otimes \tilde{\rho}_{E}^\top).
\]

Remark 4.2 (Tightness of the security criteria). In Theorem 4.1 we prove a bound on the second security condition of Definition 2.1 for QKD in terms of the correctness and secrecy of the protocol. The converse can also be shown: if Eq. (10) holds for some \(\varepsilon\), then the corresponding QKD protocol is both \(\varepsilon\)-correct and \(2\varepsilon\)-secret. \(\Box\)

4.4 Robustness

So far in this section we have discussed the security of a QKD protocol with respect to a malicious Eve using the second condition from Definition 2.1 (Eq. (10)). A QKD protocol which always aborts without producing any key trivially satisfies Eq. (10) with \(\varepsilon = 0\), but is not a useful protocol at all! The real system must not only be indistinguishable from ideal when an adversary is present, but also when the adversarial interfaces are covered by filters emulating honest behavior. This is modeled by the first condition from Definition 2.1, namely Eq. (9) for QKD. If no adversary is tampering with the quantum channel — only natural non-malicious noise is present — we expect a secret key to be generated with high probability. This can be captured by designing the filter \(\diamond_E\) to allow a key to be produced with high probability: if the real system does not generate a key with the same probability, this immediately results in a gap noticeable by the distinguisher.

The probability of a key being generated depends on the noise introduced by the filter \(\sharp_E\) covering the adversarial interface of the insecure quantum channel \(Q\) in the real system (illustrated in Figure 3.2b). Suppose that this

\(\Box\) The factor 2 is a result of the existence of the simulator \(\sigma_E\) in the security definition. We cannot exclude that for some specific QKD protocol there exists a different simulator \(\hat{\sigma}_E\) — different from the one used in this proof — generating a state \(\hat{\rho}_E\) when interacting with the distinguisher, such that
\[
D(\rho^{\top}_{AE}, \tau_A \otimes \hat{\rho}_E) \leq D(\rho^{\top}_{AE}, \tau_A \otimes \hat{\rho}_E) \leq \frac{1}{2} D(\rho^{\top}_{AE}, \tau_A \otimes \hat{\rho}_E).
\]

Alternatively, we have that for any \(\rho_E\),
\[
D(\rho^{\top}_{AE}, \tau_A \otimes \rho_E) \leq \frac{1}{2} D(\rho^{\top}_{AE}, \tau_A \otimes \rho_E).
\]

Hence the failure \(\varepsilon\) of the generic simulator used in this proof is at most twice larger than optimal.
noise is parametrized by a value $q$, e.g., a depolarizing channel with probability $q$. For every $q$, the protocol has a probability of aborting, $\delta$, which is called the robustness. Let $\mathcal{E}^q$ denote a filter of the channel $\Omega$ that models this noise, and let $\mathcal{E}^q$ denote the filter of the ideal key resource $\mathcal{K}$, which flips the switch to prevent a key from being generated with corresponding probability $\delta$. Eq. (9) thus becomes

$$\pi_{qkd}^A \pi_{qkd}^B (Q\|A)(\mathcal{E}^q\|\mathcal{B}) \approx \varepsilon \mathcal{K}^q,$$

where varying $q$ and $\delta$ results in a family of real and ideal systems.

We now prove that in this case the failure $\varepsilon$ from Eq. (16) is bounded by $\varepsilon_{cor} + \varepsilon_{sec}$. Note that this statement is only useful if the probability of aborting, $\delta$, is small for reasonable noise models $q$.

**Lemma 4.3.** If the filters from Eq. (16) are parametrized such that $\mathcal{E}^q$ aborts with exactly the same probability as the protocol $(\pi_{qkd}^A, \pi_{qkd}^B)$ run on the noisy channel $\mathcal{E}^q$, then the availability of the protocol is bounded by the security, i.e.,

$$d\left(\pi_{qkd}^A \pi_{qkd}^B (Q\|A)(\mathcal{E}^q\|\mathcal{B}), \mathcal{K}^q\right) \leq d\left(\pi_{qkd}^A \pi_{qkd}^B (Q\|A), \mathcal{K}^q\right),$$

where the simulator $\sigma_{qkd}^E$ is the one used in the previous sections, introduced in Section 4.2, Figure 4.1.

**Proof.** Since $\mathcal{E}^q$ aborts with exactly the same probability as the real system and since $\sigma_{qkd}^E$ simulates the real system, we can substitute $\sigma_{qkd}^E(\mathcal{E}^q\|\mathcal{B})$ for $\mathcal{E}^q$. The result then follows, because the converter $\mathcal{E}^q\|\mathcal{B}$ on both the real and ideal systems can only decrease their distance (Eq. (7)).

## 5 Examples of composition

It is immediate from the AC framework [MR11] that the composition of two protocols satisfying Definition 2.1 is still secure.$^{22}$ In this section we attempt to provide a better feeling for protocol composition by illustrating it with several examples. We compose QKD in series and in parallel, and show that — as a result of the triangle inequality and the security of the individual protocols — the corresponding composed real systems are indistinguishable from the composed ideal systems.

In Section 5.1 we first look at a situation in which part of the key is known to the adversary. In Section 5.2 we compose QKD with a one-time pad. And in Section 5.3 we compose two runs of a QKD protocol in parallel. We provide a more extensive example of protocol composition in

$^{22}$See Appendix C.2 for a proof sketch.
Appendix D, where we model the security of authentication and compose it with QKD, resulting in a key expansion protocol.

To simplify the examples, we only consider security in the presence of an adversary and ignore the first condition from Definition 2.1. For the same reason, when writing up the security condition with the trace distance, we hard-code the simulator used in Section 4 in the security criterion. Furthermore, as shown in Section 4.2, conditioned on aborting, the real and ideal systems of QKD are identical, so the security criterion can be reduced to the case in which the QKD protocol terminates with a shared key between Alice and Bob, which happens with probability \(1 - p_{\text{abort}}\). With these simplifications, a QKD protocol is \(\varepsilon\)-secure if

\[
(1 - p_{\text{abort}}) D(\rho_{ABE}, \tau_{AB} \otimes \rho_E) \leq \varepsilon,
\]

where \(\tau_{AB}\) is a perfect shared key and \(\rho_{ABE}\) and \(\tau_{AB} \otimes \rho_E\) are the final states, conditioned on producing a key, that the distinguisher holds after interacting with the real and ideal systems, respectively.

5.1 Partially known key

The accessible information given in Eq. (2) is shown to be insufficient to define security for a QKD protocol by considering a setting in which part of the key \(K\) is available to Eve \[KRBM07\]. This allows her to guess the remaining bits of the key, which would not have been possible had the key been distributed using an ideal resource. We analyze exactly this setting here, and argue that this does not affect the security of a QKD scheme that satisfies Definition 2.1.

To model this partial knowledge of the key, let Alice run a protocol \(\pi'_A\) that receives part of the secret key—generated either by a QKD protocol or by an ideal resource—and sends it on a channel to Eve. Plugging this in the real and ideal QKD systems from Figures 3.2a and 4.1, we get Figure 5.1.

It is immediate from Figure 5.1 that \(\pi'_A\) cannot increase the distance between the real and ideal systems and therefore cannot compromise security: the systems in gray can be run internally by a distinguisher attempting to guess whether it is interacting with the real or ideal QKD system, so this case is already bounded by the security of QKD.

This reasoning is summed up in the following equation, which can be directly derived from Eq. (7):

\[
\pi_A^{qkl} \pi_B^{qkl} (Q||A) \approx_{\varepsilon} \mathcal{K} \sigma_E^{qkl} \implies \pi'_A \pi_A^{qkl} \pi_B^{qkl} (Q||A) \approx_{\varepsilon} \pi'_A \mathcal{K} \sigma_E^{qkl}.
\]

The same can be obtained from the properties of the trace distance if we write out explicitly the states gathered by the distinguisher. If the QKD protocol is \(\varepsilon\)-secure, we have from Eq. (17) that

\[
(1 - p_{\text{abort}}) D(\rho_{ABE}, \tau_{AB} \otimes \rho_E) \leq \varepsilon,
\]

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(a) The QKD protocol \((\pi_{qkd}^A, \pi_{qkd}^B)\) generates a pair of keys \((k_A, k_B)\). Alice then runs \(\pi'_{A}\), which provides the first part of \(k_A = k^1_A \parallel k^2_A\) to Eve. The drawing of the insecure quantum channel and authentic classical channel have been removed to simplify the figure.

(b) The ideal secret key resource generates a key \(k = k^1 \parallel k^2\), part of which is provided to Eve by \(\pi'_{A}\). A simulator \(\sigma_{qkd}^{E}\) pads the ideal key resource to generate the same communication as in the real setting.

**Figure 5.1** – Alice runs a protocol \(\pi'_{A}\) which reveals the first half of her key to Eve. In each figure, Alice and Bob have access to the left and right interfaces, and Eve to the lower interface. If we remove the parts in gray we recover the real and ideal systems of QKD.
where \( \rho_{ABE} \) is the state gathered by a distinguisher interacting with the real QKD system (Figure 3.2a) and \( \tau_{AB} \otimes \rho_E \) is the state gathered by interacting with the ideal system (Figure 4.1), conditioned on the protocol not aborting. A distinguisher interacting with either of the two systems from Figure 3.1 gets extra information at Eve’s interface, namely the first part of Alice’s key \( k^1 \), and only the second part of that key \( k^2 \) at Alice’s interface. The complete states gathered by interacting with Figure 5.1a and Figure 5.1b are given by \( \rho'_{A'BE'} = \rho_{A'2BA1E} \) and \( \tilde{\rho}'_{A'BE'} = \tau_{A'2BA1} \otimes \rho_E \), respectively, where the original system \( A = A_1A_2 \) containing Alice’s key is split in two, \( A' = A_2 \) and \( E' = A_1E \). These can be obtained from \( \rho_{ABE} \) and \( \tau_{AB} \otimes \rho_E \) by a unitary map which simply permutes the registers. Thus, the trace distance does not increase. So we have

\[
(1 - p_{\text{abort}})D(\rho_{A'BE'}, \tilde{\rho}'_{A'BE'}) = (1 - p_{\text{abort}})D(\rho_{ABE}, \tau_{AB} \otimes \rho_E) \leq \varepsilon .
\]

If we analyze the same situation from the perspective of an adversary that can access only the \( E \)-interface, composing QKD with a protocol that reveals \( k^1 \) results in a net gain of information for this adversary. But as shown above, for a distinguisher that also receives the outputs of the honest players—the generated secret keys—there is no gain.

5.2 Sequential composition of key distribution and one-time pad

If we compose a one-time pad (depicted in Figure 2.3) and a QKD protocol (depicted in Figure 3.2a), we obtain Figure 5.2a, where the secret key resource used by the one-time pad is replaced by the QKD protocol. We showed in Section 2.2 that a one-time pad constructs a secure channel (Figure 2.4), which provides Eve with only one functionality, learning the length of the message. However, this was if the one-time pad protocol had access to a secret key resource with a blank \( E \)-interface, as in Figure 3.1a. In reality, QKD constructs a resource that allows Eve to prevent a key from being generated, as in Figure 3.1b. It can easily be shown that with access to this resource, a one-time pad constructs a secure channel with two controls at Eve’s interface: one for preventing any message from being sent and a second for learning the length of the message if she did not activate the first. This resource is illustrated in Figure 5.2c along with the appropriate simulator for constructing this resource with a one-time pad and a QKD protocol: the combination of the two simulators used in the individual proofs of the one-time pad (Figure 2.5) and QKD (Figure 4.1).

We now wish to show that the combination of an \( \varepsilon \)-secure QKD protocol and a (perfect) one-time pad results in a combined scheme that constructs within \( \varepsilon \) a secure channel from authentic classical channels and an insecure...
(a) The composition of a one-time pad and a QKD protocol. The authentic channels and insecure quantum channels used have not been depicted as boxes to simplify the figure.

(b) A hybrid system consisting of a real one-time pad and ideal secret key resource with simulator.

(c) The ideal secure channel and corresponding composed simulator $\sigma_E^{\text{otp}} \sigma_E^{\text{qkd}}$.

Figure 5.2 — Steps in the security proof of the sequential composition of a one-time pad and QKD protocol. In each figure, Alice and Bob have access to the left and right interfaces, and Eve to the lower interface. If we remove the gray parts from Figures 5.2a and 5.2b we recover the real and ideal systems of QKD. If we remove the dashed parts from Figures 5.2b and 5.2c we recover the real and ideal systems of the one-time pad.
quantum channel. To do this, we look at an intermediary step consisting of the combination of an ideal secret key resource and a one-time pad, which we illustrate in Figure 5.2b. If we remove the gray parts from Figures 5.2a and 5.2b we recover the real and ideal systems of QKD. If the QKD protocol is $\varepsilon$-secure, then the distinguishing advantage between these two figures can also be at most $\varepsilon$. Likewise, if we remove the dashed parts from Figures 5.2b and 5.2c we recover the real and ideal systems of the one-time pad. Since the one-time pad is perfectly secure, the distinguishing advantage between these two figures must be 0. It follows from the triangle inequality that the composition of an $\varepsilon$-secure QKD protocol and a one-time pad is $\varepsilon$-secure.

This reasoning is summed up in the following equation, which can be directly derived from Eqs. (3), (7) and the triangle inequality (Eq. (6)):

$$\pi_{qkd}^A \pi_{qkd}^B (Q \| A) \approx \varepsilon \sigma_{qkd}^E \pi_{otp}^A \pi_{otp}^B (K \| A) = S \sigma_{otp}^E \sigma_{qkd}^E.$$ 

The same can be obtained from the properties of the trace distance if we write out explicitly the states gathered by the distinguisher. After a run of an $\varepsilon$-secure QKD scheme, we know that

$$(1 - p_{abort}) D(\rho_{ABE}, \tau_{AB} \otimes \rho_E) \leq \varepsilon.$$ 

The encryption and decryption operations of the one-time pad, $(\pi_{otp}^A, \pi_{otp}^B)$, plugged into Figures 5.2a and 5.2b modify the states $\rho_{AB}$ and $\tau_{AB}$. They correspond to a unitary map $\mathcal{E}^{otp} : \mathcal{X} \times \mathcal{K} \times \mathcal{K} \to \mathcal{X} \times \mathcal{X} \times \mathcal{X}$ which takes the message $x_A$ and Alice’s and Bob’s keys $k_A, k_B$, and generates the ciphertext and Bob’s message while persevering Alice’s message,

$$\mathcal{E}^{otp} : (x_A, k_A, k_B) \mapsto (x_A, x_A \oplus k_A \oplus k_B, x_A \oplus k_A).$$

A unitary map does not change the trace distance, so for

$$\rho_{X_A X_B Y} = \mathcal{E}^{otp}(\rho_{X_A} \otimes \rho_{AB}) \quad \text{and} \quad \tau_{X_A X_B Y} = \mathcal{E}^{otp}(\rho_{X_A} \otimes \tau_{AB})$$

we have

$$(1 - p_{abort}) D(\rho_{X_A X_B Y E}, \tau_{X_A X_B Y} \otimes \rho_E) = (1 - p_{abort}) D(\rho_{ABE}, \tau_{AB} \otimes \rho_E) \leq \varepsilon,$$

where $\rho_{X_A X_B Y E}$ and $\tau_{X_A X_B Y} \otimes \rho_E$ are the states held be a distinguisher interacting with Figures 5.2a and 5.2b respectively.

We also know that the one-time pad perfectly constructs a secure channel from an authentic channel and a secret key, i.e., if we remove the simulator $\sigma_{qkd}^E$ from Figures 5.2b and 5.2c the corresponding systems are indistinguishable — a distinguisher interacting with them obtains two states needs only a single use one-way authentic channel. This distinction is however not relevant to the current argument, so we refer to both resources as “authentic channels” and use the same notation, $\mathcal{A}$, for each.
$\tau_{XAXBY}$ and $\tau'_{XAXBY}$ with $D(\tau_{XAXBY}, \tau'_{XAXBY}) = 0$. Plugging the simulator $\sigma^{qkd}_E$ in Eve’s interface simply results in the state $\rho_E$ being appended to $\tau$ and $\tau'$. The final state held by the distinguisher is thus $\tau_{XAXBY} \otimes \rho_E$ and $\tau'_{XAXBY} \otimes \rho_E$, respectively, with $D(\tau_{XAXBY} \otimes \rho_E, \tau'_{XAXBY} \otimes \rho_E) = 0$.

By the triangle inequality, the distance between Figures 5.2a and 5.2c is then

$$(1 - p_{\text{abort}})D(\rho_{XAXBYE}, \tau'_{XAXBY} \otimes \rho_E) \leq \varepsilon.$$ 

### 5.3 Parallel composition of key distribution with itself

If two QKD protocols are run in parallel, as illustrated in Figure 5.3a the adversary can entwine their respective messages as she plea ses, e.g., parts of the state $\rho$ sent on the insecure channel by the first protocol can be input into the insecure channel of the second protocol. We wish to show that even in this case, the combined protocol is still $2\varepsilon$-secure — i.e., indistinguishable from the parallel compositions of two ideal key resources and their individual simulators — if each QKD protocol is $\varepsilon$-secure. This ideal case is depicted in Figure 5.3b.

Like for serial composition, this follows from the triangle inequality. If the real QKD system is $\varepsilon$-close to the ideal QKD system, then two real QKD systems in parallel must be $\varepsilon$-close to an ideal and real QKD system composed in parallel, since otherwise a distinguisher could run a real QKD system internally in parallel to the system it is testing. Likewise, a real and ideal QKD system in parallel must be $\varepsilon$-close to two ideal QKD systems in parallel. And hence two parallel runs of an $\varepsilon$-secure QKD protocol is $2\varepsilon$-secure.

The trace distance notation does not lend itself to writing up parallel composition of protocols. So instead of using this notation as in the previous examples, we write up the reasoning from the paragraph above in more detail using the resource-converter formalism. If the real and ideal system of a QKD protocol are $\varepsilon$-close, then

$$\pi^{qkd}_A \pi^{qkd}_B(\mathcal{Q} \mid A) \approx_\varepsilon \mathcal{K} \sigma^{qkd}_E .$$

It follows immediately from this and [Eq. (7)] that

$$\left( \pi^{qkd}_A \pi^{qkd}_B(\mathcal{Q} \mid A) \right) \approx_\varepsilon \mathcal{K} \sigma^{qkd}_E$$

$$\left( \pi^{qkd}_A \pi^{qkd}_B(\mathcal{Q} \mid A) \right) \approx_\varepsilon \mathcal{K} \sigma^{qkd}_E.$$ 

From the triangle equality ([Eq. (6)]) we then have

$$\left( \pi^{qkd}_A \pi^{qkd}_B(\mathcal{Q} \mid A) \right) \approx_\varepsilon \mathcal{K} \sigma^{qkd}_E.$$
(a) Two QKD protocols and their respective resources run in parallel.

(b) Two secret key resources and two simulators run in parallel.

**Figure 5.3** – Real and ideal systems for two QKD protocols executed in parallel. In each figure, Alice and Bob have access to the left and right interfaces, and Eve to both the upper and lower interface.
Finally, using Eq. (3) to rearrange this expression, we get

\[
\left( \pi_{qkd}^A \pi_{qkd}^B \right) \left( \pi_{qkd}^{A'} \pi_{qkd}^{B'} \right) (Q \| A \| Q \| A) \approx 2 \varepsilon (\mathcal{K} \| \mathcal{K} \) \left( \sigma_{E} \| \sigma_{E'} \right),
\]

i.e., the parallel composition of two runs of a QKD protocol, \( \pi_{qkd}^A \pi_{qkd}^B \left( \pi_{qkd}^{A'} \pi_{qkd}^{B'} \right) \), run with authentic classical and insecure quantum channel resources, \( Q \| A \| Q \| A \), is 2\( \varepsilon \)-close to the parallel composition of two ideal key resources, \( \mathcal{K} \| \mathcal{K} \), and a simulator, \( \sigma_{E} \| \sigma_{E'} \).

### Appendices

In Appendix A, we formally define the trace distance and show that it corresponds to the distinguishing advantage between two quantum states. We also prove several lemmas that help interpret its meaning and how to choose a value in a practical implementation. In Appendix B, we discuss an alternative to the secrecy criterion of Eq. (1), which has appeared in the literature. In Appendix C, we provide some details on technical aspects of the Abstract Cryptography framework. In Appendix C.1, we discuss how to define a distinguisher so that the resulting distinguishing advantage is non-increasing under compositions. In Appendix C.2, we sketch a proof that the security definition from Definition 2.1 is composable. A complete proof of this can be found in \[MR11, Mau12\]. And finally, in Appendix D, we model the security of authentication with universal hashing [WC81, Sti94], then use this as a subprotocol of QKD to authenticate the classical post-processing. Since this type of authentication uses a short key (and an insecure classical channel) to construct an authentic channel and QKD uses an authentic channel (and an insecure quantum channel) to construct a long key, the composition of the two is a key expansion protocol, which constructs a long key from a short key (and insecure channels).

### A Trace distance

We have used several times in this work the well-known fact that the distinguishing advantage between two systems that output states \( \rho \) and \( \sigma \) is equivalent to the trace distance between these states. In this appendix, we prove this fact, along with several other theorems that help interpret the meaning of the trace distance.

In Appendix A.1, we first define the trace distance— as well as its classical counterpart, the total variation distance—and prove some basic lemmas that can also be found in textbooks such as [NC00]. In Appendix A.2, we then show the connection between trace distance and distinguishing advantage, which was originally proven by Helstrom [Hel76]. In Appendix A.3, we
prove that we can alternatively think of the trace distance between a real
and ideal system as a bound on the probability that a failure occurs in the
real system. Finally, in Appendix A.4 we bound two typical information
theory notions of secrecy — the conditional entropy of a key given the eaves-
dropper’s information and her probability of correctly guessing the key — in
terms of the trace distance. Although such measures of information are gen-
erally ill-suited for defining cryptographic security, they can help interpret
the notion of a key being $\varepsilon$-close to uniform.

A.1 Metric definitions

In the case of a classical system, statistical security is defined by the to-
tal variation (or statistical) distance between the probability distributions
describing the real and ideal settings, which is defined as follows.\footnote{We
employ the same notation $D(\cdot, \cdot)$ for both the total variation and trace
distance, since the former is a special case of the latter.}

\textbf{Definition A.1} (total variation distance). The total variation distance
between two probability distributions $P_Z$ and $\tilde{P}_Z$ over an alphabet $\mathcal{Z}$ is defined as

$$D(P_Z, \tilde{P}_Z) := \frac{1}{2} \sum_{z \in \mathcal{Z}} |P_Z(z) - \tilde{P}_Z(z)| .$$

Using the fact that $|a - b| = a + b - 2 \min(a, b)$, the total variation distance
can also be written as

$$D(P_Z, \tilde{P}_Z) = 1 - \sum_{z \in \mathcal{Z}} \min[P_Z(z), \tilde{P}_Z(z)] . \quad (18)$$

In the case of quantum states instead of classical random variables, the
total variation distance generalizes to the trace distance. More precisely,
the trace distance between two density operators that are diagonal in the
same orthonormal basis is equal to the total variation distance between the
probability distributions defined by their respective eigenvalues.

\textbf{Definition A.2} (trace distance). The trace distance between two quantum
states $\rho$ and $\sigma$ is defined as

$$D(\rho, \sigma) := \frac{1}{2} \tr |\rho - \sigma| .$$

We now introduce some technical lemmas involving the trace distance,
which help us derive the theorems in the next sections. Most of these proofs
are taken from \cite{NC00}.
Lemma A.3. For any two states \( \rho \) and \( \sigma \) and any operator \( 0 \leq M \leq I \), the two following inequalities hold:

\[
D(\rho, \sigma) \geq \text{tr}(M(\rho - \sigma)) ,
\]
\[
\text{tr}(M|\rho - \sigma|) \geq |\text{tr}(M(\rho - \sigma))| .
\]

Furthermore, each of these inequalities is tight for some values of \( M \).

The trace distance can thus alternatively be written as

\[
D(\rho, \sigma) = \max_M \text{tr}(M(\rho - \sigma)) .
\]

Proof. We start with the proof of Eq. (19). Let \( \{\lambda_x, |\psi_x\rangle\}_x \) be the eigenvalues and vectors of \( \rho - \sigma \), and define

\[
Q_+ := \sum_{x: \lambda_x \geq 0} \lambda_x |\psi_x\rangle \langle \psi_x| \quad \text{and} \quad Q_- := \sum_{x: \lambda_x < 0} -\lambda_x |\psi_x\rangle \langle \psi_x| .
\]

We have \( \rho - \sigma = Q_+ - Q_- \) and \( |\rho - \sigma| = Q_+ + Q_- \). Note that since \( \text{tr}(Q_+ - Q_-) = \text{tr}(\rho - \sigma) = 0 \), we have \( \text{tr} Q_+ = \text{tr} Q_- \), hence

\[
D(\rho, \sigma) = \frac{1}{2} \text{tr} |\rho - \sigma| = \frac{1}{2} (\text{tr} Q_+ + \text{tr} Q_-) = \text{tr} Q_+ .
\]

If we set \( \Gamma_+ := \sum_{x: \lambda_x \geq 0} |\psi_x\rangle \langle \psi_x| \), the projector on \( Q_+ \), we get

\[
\text{tr}(\Gamma_+ (\rho - \sigma)) = \text{tr} Q_+ = D(\rho, \sigma) .
\]

And for any operator \( 0 \leq M \leq I \),

\[
\text{tr}(M(\rho - \sigma)) = \text{tr}(M(Q_+ - Q_-)) \leq \text{tr}(MQ_+) \leq \text{tr} Q_+ = D(\rho, \sigma) .
\]

To prove that Eq. (20) holds, note that for any operator \( 0 \leq M \leq I \),

\[
|\text{tr}(M(\rho - \sigma))| = |\text{tr}(M(Q_+ - Q_-))| \leq \text{tr}(M(Q_+ + Q_-)) = \text{tr}(M|\rho - \sigma|) .
\]

Eq. (20) is tight for any operator \( M \) which satisfies

\[
|\text{tr}(M(Q_+ - Q_-))| = \text{tr}(M(Q_+ + Q_-)) ,
\]

i.e., any operator such that either \( 0 \leq M \leq \Gamma_+ \) or \( 0 \leq M \leq \Gamma_- \), where \( \Gamma_+ \) is defined as above and \( \Gamma_- := \sum_{x: \lambda_x \leq 0} |\psi_x\rangle \langle \psi_x| \).

Let \( \{\Gamma_x\}_x \) be a positive operator-valued measure (POVM) — a set of operators \( 0 \leq \Gamma_x \leq I \) such that \( \sum_x \Gamma_x = I \) — and let \( P_X \) denote the outcome of measuring a quantum state \( \rho \) with \( \{\Gamma_x\}_x \), i.e., \( P_X(x) = \text{tr}(\Gamma_x \rho) \). Our next lemma says that the trace distance between two states \( \rho \) and \( \sigma \) is equal to the total variation between the outcomes — \( P_X \) and \( Q_X \) — of an optimal measurement on the two states.
Lemma A.4. For any two states $\rho$ and $\sigma$,
\[
D(\rho, \sigma) = \max_{\{\Gamma_x\}_x} D(P_X, Q_X),
\]
where $P_X$ and $Q_X$ are the probability distributions resulting from measuring $\rho$ and $\sigma$ with a POVM $\{\Gamma_x\}_x$, respectively, and the maximization is over all POVMs. Furthermore, if the two states $\rho_{ZB}$ and $\sigma_{ZB}$ have a classical subsystem $Z$, then the measurement satisfying Eq. (22) leaves the classical subsystem unchanged, i.e., the maximum is reached for a POVM with elements
\[
\Gamma_x = \sum_z |z\rangle\langle z| \otimes M_z^x,
\]
where $\{|z\rangle\}_z$ is the classical orthonormal basis of $Z$.

Proof. Using Eq. (20) from Lemma A.3 we get
\[
D(P_X, Q_X) = \frac{1}{2} \sum_x |\text{tr}(\Gamma_x(\rho - \sigma))| \\
\leq \frac{1}{2} \sum_x \text{tr}(\Gamma_x|\rho - \sigma|) \\
= \frac{1}{2} \text{tr}|\rho - \sigma| = D(\rho, \sigma).
\]
The conditions for equality are given at the end of the proof of Lemma A.3, e.g., a measurement with $\Gamma_x = |\psi_x\rangle\langle \psi_x|$, where $\{|\psi_x\rangle\}_x$ are the eigenvectors of $\rho - \sigma$. If $\rho_{ZB} = \sum_z p_z |z\rangle\langle z| \otimes \rho^z_B$ and $\sigma_{ZB} = \sum_z q_z |z\rangle\langle z| \otimes \sigma^z_B$, then
\[
\rho - \sigma = \sum_z |z\rangle\langle z| \otimes (p_z \rho^z_B - q_z \sigma^z_B),
\]
and the eigenvectors of $\rho - \sigma$ have the form $|\psi_{z,x}\rangle = |z\rangle \otimes |\varphi^z_x\rangle_B$, where $|\varphi^z_x\rangle_B$ is an eigenvector of $p_z \rho^z_B - q_z \sigma^z_B$. So the optimal measurement, $\Gamma_{z,x} = |z\rangle\langle z| \otimes |\varphi^z_x\rangle\langle \varphi^z_x|_B$, satisfies Eq. (23). \hfill \QED

A.2 Distinguishing advantage

Helstrom [Hel76] proved that the advantage a distinguisher has in guessing whether it was provided with one of two states, $\rho$ or $\sigma$, is given by the trace distance between the two, $D(\rho, \sigma)$.

We first sketch the classical case, then prove the quantum version.

Let a distinguisher be given a value sampled according to probability distributions $P_Z$ or $\bar{P}_Z$, where $P_Z$ and $\bar{P}_Z$ are each chosen with probability $1/2$.

\[\text{Actually, Helstrom [Hel76] solved a more general problem, in which the states $\rho$ and $\sigma$ are picked with apriori probabilities $p$ and $1 - p$, respectively, instead of 1/2 as in the definition of the distinguishing advantage.}\]
Suppose the value received by the distinguisher is \( z \in \mathcal{Z} \). If \( P_Z(z) > P_{\tilde{Z}}(z) \), its best guess is that the value was sampled according to \( P_Z \). Otherwise, it should guess that it was \( P_{\tilde{Z}} \). Let \( \mathcal{Z}' := \{ z \in \mathcal{Z} : P_Z(z) > P_{\tilde{Z}}(z) \} \) and \( \mathcal{Z}'' := \{ z \in \mathcal{Z} : P_Z(z) \leq P_{\tilde{Z}}(z) \} \). There are a total of \( 2|\mathcal{Z}| \) possible events: the sample is chosen according to \( P_Z \) or \( P_{\tilde{Z}} \) and takes the value \( z \in \mathcal{Z} \).

These events have probabilities \( \frac{P_Z(z)}{2} \) and \( \frac{P_{\tilde{Z}}(z)}{2} \). Conditioned on \( P_Z \) being chosen and \( z \) being the sampled value, the distinguisher has probability 1 of guessing correctly with the strategy outlined above if \( z \in \mathcal{Z}' \), and 0 otherwise. Likewise, if \( P_{\tilde{Z}} \) was selected, it has probability 1 of guessing correctly if \( z \in \mathcal{Z}'' \) and 0 otherwise. The probability of correctly guessing whether it was given a value sampled according to \( P_Z \) or \( P_{\tilde{Z}} \), which we denote \( p_{\text{distinguish}}(P_Z, P_{\tilde{Z}}) \), is obtained by summing over all possible events weighted by their probabilities. Hence

\[
p_{\text{distinguish}}(P_Z, P_{\tilde{Z}}) = \sum_{z \in \mathcal{Z}'} \frac{P_Z(z)}{2} + \sum_{z \in \mathcal{Z}''} \frac{P_{\tilde{Z}}(z)}{2}
\]

\[
= \frac{1}{2} \left( 1 - \sum_{z \in \mathcal{Z}''} P_Z(z) \right) + \frac{1}{2} \left( 1 - \sum_{z \in \mathcal{Z}'} P_{\tilde{Z}}(z) \right)
\]

\[
= 1 - \frac{1}{2} \sum_{z \in \mathcal{Z}} \min[P_Z(z), P_{\tilde{Z}}(z)]
\]

\[
= \frac{1}{2} + \frac{1}{2} D(P_Z, P_{\tilde{Z}}),
\]

where in the last equality we used the alternative formulation of the total variation distance from Eq. (18).

We now generalize the argument above to quantum states.

**Theorem A.5.** For any states \( \rho \) and \( \sigma \), we have

\[
p_{\text{distinguish}}(\rho, \sigma) = \frac{1}{2} + \frac{1}{2} D(\rho, \sigma).
\]

**Proof.** If a distinguisher is given one of two states \( \rho \) or \( \sigma \), each with probability \( 1/2 \), its probability of guessing which one it holds is given by a maximization of all possible measurements it may do: it chooses some POVM \( \{\Gamma_0, \Gamma_1\} \), where \( \Gamma_0 \) and \( \Gamma_1 \) are positive operators with \( \Gamma_0 + \Gamma_1 = I \), and measures the state it holds. If it gets the outcome 0, it guesses that it holds \( \rho \) and if it gets the outcome 1, it guesses that it holds \( \sigma \). The probability of guessing correctly is given by

\[
p_{\text{distinguish}}(\rho, \sigma) = \max_{\Gamma_0, \Gamma_1} \left[ \frac{1}{2} \text{tr}(\Gamma_0 \rho) + \frac{1}{2} \text{tr}(\Gamma_1 \sigma) \right]
\]

\[
= \frac{1}{2} \max_{\Gamma_0} [\text{tr}(\Gamma_0 \rho) + \text{tr}((I - \Gamma_0)\sigma)]
\]

\[
= \frac{1}{2} + \frac{1}{2} \max_{\Gamma_0} \text{tr}(\Gamma_0 (\rho - \sigma))
\]

(24)
The proof concludes by plugging Eq. (21) in Eq. (24).

### A.3 Probability of a failure

The trace distance is used as the security definition of QKD, because the relevant measure for cryptographic security is the distinguishing advantage (as discussed in Section 2), and as proven in Theorem A.5 the distinguishing advantage between two quantum states corresponds to their trace distance. This operational interpretation of the trace distance involves two worlds, an ideal one and a real one, and the distance measure is the (renormalized) difference between the probabilities of the distinguisher correctly guessing to which world it is connected.

In this section we describe a different interpretation of the total variation and trace distances. Instead of having two different worlds, we consider one world in which the outcomes of interacting with the real and ideal systems co-exist. And instead of these distance measures being a difference between probability distributions, they become the probability that any (classical) value occurring in one of the systems does not simultaneously occur in the other. We call such an event a failure—since one system is ideal, if the other behaves differently, it must have failed—and the trace distance becomes the probability of a failure occurring.

Given two random variables $Z$ and $\tilde{Z}$ with probability distributions $P_Z$ and $P_{\tilde{Z}}$, any distribution $P_{Z\tilde{Z}}$ with marginals given by $P_Z$ and $P_{\tilde{Z}}$ is called a coupling of $P_Z$ and $P_{\tilde{Z}}$. The interpretation of the trace distance treated in this section uses one specific coupling, known as a maximal coupling in probability theory [Tho00].

**Theorem A.6 (maximal coupling).** Let $P_Z$ and $P_{\tilde{Z}}$ be two probability distributions over the same alphabet $Z$. Then there exists a probability distribution $P_{Z\tilde{Z}}$ on $Z \times Z$ such that

$$\Pr[Z = \tilde{Z}] := \sum_z P_{Z\tilde{Z}}(z,z) \geq 1 - D(P_Z, P_{\tilde{Z}}) \quad (25)$$

and such that $P_Z$ and $P_{\tilde{Z}}$ are the marginals of $P_{Z\tilde{Z}}$, i.e.,

$$P_Z(z) = \sum_{\tilde{z}} P_{Z\tilde{Z}}(z,\tilde{z}) \quad (\forall z \in Z) \quad (26)$$

$$P_{\tilde{Z}}(\tilde{z}) = \sum_z P_{Z\tilde{Z}}(z,\tilde{z}) \quad (\forall \tilde{z} \in \tilde{Z}) \quad (27)$$

It turns out that the inequality in Eq. (25) is tight, i.e., one can also show that for any distribution $P_{Z\tilde{Z}}$, $\Pr[Z = \tilde{Z}] \leq 1 - D(P_Z, P_{\tilde{Z}})$. We will however not use this fact here.

Consider now a real system that outputs values given by $Z$ and an ideal system that outputs values according to $\tilde{Z}$. Theorem A.6 tells us that there
exists a coupling of these distributions such that the probability of the real system producing a different value from the ideal system is bounded by the total variation distance between $P_Z$ and $P_{\tilde{Z}}$. Thus, the real system behaves ideally except with probability $D(P_Z, P_{\tilde{Z}})$.

We first prove this theorem, then in Corollary A.7 here below we apply it to quantum systems.

Proof of Theorem A.6. Let $Q_{Z,\tilde{Z}}$ be the real function on $\mathcal{Z} \times \mathcal{Z}$ defined by

$$Q_{Z,\tilde{Z}}(z, \tilde{z}) = \begin{cases} \min[P_Z(z), P_{\tilde{Z}}(\tilde{z})] & \text{if } z = \tilde{z} \\ 0 & \text{otherwise} \end{cases}$$

(for all $z, \tilde{z} \in \mathcal{Z}$). Furthermore, let $R_Z$ and $R_{\tilde{Z}}$ be the real functions on $\mathcal{Z}$ defined by

$$R_Z(z) = P_Z(z) - Q_{Z,\tilde{Z}}(z,z) ,$$

$$R_{\tilde{Z}}(\tilde{z}) = P_{\tilde{Z}}(\tilde{z}) - Q_{Z,\tilde{Z}}(\tilde{z},\tilde{z}) .$$

We then define $P_{Z,\tilde{Z}}$ by

$$P_{Z,\tilde{Z}}(z, \tilde{z}) = Q_{Z,\tilde{Z}}(z, \tilde{z}) + \frac{1}{D(P_Z, P_{\tilde{Z}})} R_Z(z) R_{\tilde{Z}}(\tilde{z}) .$$

We now show that $P_{Z,\tilde{Z}}$ satisfies the conditions of the theorem. For this, we note that for any $z \in \mathcal{Z}$

$$R_Z(z) = P_Z(z) - \min[P_Z(z), P_{\tilde{Z}}(z)] \geq 0 ,$$

i.e., $R_Z$ and, likewise, $R_{\tilde{Z}}$, are nonnegative. Since $Q_{Z,\tilde{Z}}$ is by definition also nonnegative, we have that $P_{Z,\tilde{Z}}$ is nonnegative, too. From [Eq. (26)] or [Eq. (27)], which we will prove below, it follows that $P_{Z,\tilde{Z}}$ is also normalized. Hence, $P_{Z,\tilde{Z}}$ is a valid probability distribution.

To show [Eq. (25)] we use again the non-negativity of $R_Z$ and $R_{\tilde{Z}}$, which implies

$$\sum_{z} P_{Z,\tilde{Z}}(z, z) \geq \sum_{z} Q_{Z,\tilde{Z}}(z, z) = \sum_{z} \min[P_Z(z), P_{\tilde{Z}}(z)] = 1 - D(P_Z, P_{\tilde{Z}}) ,$$

where in the last equality we used the alternative formulation of the total variation distance from [Eq. (18)].

To prove [Eq. (26)] we first note that

$$\sum_{\tilde{z}} R_{\tilde{Z}}(\tilde{z}) = \sum_{\tilde{z}} P_{\tilde{Z}}(\tilde{z}) - \sum_{\tilde{z}} Q_{Z,\tilde{Z}}(\tilde{z}, \tilde{z})$$

$$= 1 - \sum_{\tilde{z}} \min[P_Z(\tilde{z}), P_{\tilde{Z}}(\tilde{z})] = D(P_Z, P_{\tilde{Z}}) .$$
Using this we find that for any \( z \in \mathcal{Z} \)

\[
\sum_{\tilde{z}} P_{Z,Z}(z, \tilde{z}) = \sum_{\tilde{z}} Q_{Z,Z}(z, \tilde{z}) + R_{Z}(z) \frac{1}{D(P_{Z}, P_{\tilde{Z}})} \sum_{\tilde{z}} R_{Z}(\tilde{z})
\]

\[
= Q_{Z,Z}(z, z) + R_{Z}(z) = P_{Z}(z) .
\]

By symmetry, this also proves Eq. (27). □

In the case of quantum states, Theorem A.6 can be used to couple the outcomes of any observable applied to the quantum systems.

**Corollary A.7.** For any states \( \rho \) and \( \sigma \) with trace distance \( D(\rho, \sigma) \leq \varepsilon \), and any measurement given by its POVM operators \( \{\Gamma_{w}\} \) with outcome probabilities \( P_{W}(w) = \text{tr}(\Gamma_{w}\rho) \) and \( \tilde{P}_{W}(w) = \text{tr}(\Gamma_{w}\sigma) \), there exists a coupling of \( P_{W} \) and \( \tilde{P}_{W} \) such that

\[
\Pr[W \neq \tilde{W}] \leq D(\rho, \sigma) .
\]

**Proof.** Immediate by combining Lemma A.4 and Theorem A.6 □

**Corollary A.7** tells us that if two systems produce states \( \rho \) and \( \sigma \), then for any observations made on those systems there exists a coupling for which the values of each measurement will differ with probability at most \( D(\rho, \sigma) \). It is instructive to remember that this operational meaning is not essential to the security notion or part of the framework in any way. It is an intuitive way of understanding the trace distance, so as to better choose a suitable value. It allows this distance to be thought of as a maximum failure probability, and the value for \( \varepsilon \) to be chosen accordingly.

**A.4 Measures of uncertainty**

Non-composable security models often use measures of uncertainty to quantify how much information an adversary might have about a secret, e.g., entropy as used by Shannon to prove the security of the one-time pad [Sha49]. These measures are often weaker than what one obtains using a global distinguisher, and in general do not provide good security definitions. They are however quite intuitive and in order to further illustrate the quantitative value of the distinguishing advantage, we derive bounds on two of these measures of uncertainty in terms of the trace distance, namely on the probability of guessing the secret key in Appendix A.4.1 and on the von Neumann entropy of the secret key in Appendix A.4.2.

**A.4.1 Probability of guessing**

Let \( \rho_{KE} = \sum_{k \in K} p_{k} |k\rangle \langle k|_{K} \otimes \rho_{E}^{k} \) be the joint state of a secret key in the \( K \) subsystem and Eve’s information in the \( E \) subsystem. To guess the value of
the key, Eve can pick a POVM \( \{ \Gamma_k \}_{k \in K} \), measure her system, and output the result of the measurement. Given that the key is \( k \), her probability of having guessed correctly is \( \text{tr}(\Gamma_k \rho_k^E) \). The average probability of guessing correctly for this measurement is then given by the sum over all \( k \), weighted by their respective probabilities \( p_k \). And Eve’s probability of correctly guessing the key is defined by taking the maximum over all measurements,\n
\[
\begin{align*}
p_{\text{guess}}(K|E)_\rho &:= \max_{\{ \Gamma_k \}_{k \in K}} \sum_{k \in K} p_k \text{tr}(\Gamma_k \rho_k^E) \, , \\
\text{(28)}
\end{align*}
\]

**Lemma A.8.** For any bipartite state \( \rho_{KE} \) with classical \( K \),\n
\[
p_{\text{guess}}(K|E)_\rho \leq \frac{1}{|K|} + D(\rho_{KE}, \tau_K \otimes \rho_E) \, ,
\]

where \( \tau_K \) is the fully mixed state.

**Proof.** Note that for \( M := \sum_k |k\rangle \langle k| \otimes \Gamma_k \), where \( \{ \Gamma_k \} \) maximizes Eq. (28), the guessing probability can equivalently be written\n
\[
p_{\text{guess}}(K|E)_\rho = \text{tr}(M \rho_{KE}) \, .
\]

Furthermore,\n
\[
\text{tr}[M(\tau_K \otimes \rho_E)] = \frac{1}{|K|} \, .
\]

In **Lemma A.3** we proved that for any operator \( 0 \leq M \leq I \),\n
\[
\text{tr}(M(\rho - \sigma)) \leq D(\rho, \sigma) \, .
\]

Setting \( \rho = \rho_{KE} \) and \( \sigma = \tau_K \otimes \rho_E \) in the above inequality, we finish the proof:

\[
\text{tr}(M \rho_{KE}) \leq \text{tr}(M(\tau_K \otimes \rho_E)) + D(\rho_{KE}, \tau_K \otimes \rho_E) \, ,
\]

\[
\implies p_{\text{guess}}(K|E)_\rho \leq \frac{1}{|K|} + D(\rho_{KE}, \tau_K \otimes \rho_E) \, .
\]

**A.4.2 Entropy**

Let \( \rho_{KE} = \sum_{k \in K} p_k |k\rangle \langle k| \otimes \rho_k^E \) be the joint state of a secret key in the \( K \) subsystem and Eve’s information in the \( E \) subsystem. We wish to bound the von Neumann entropy of \( K \) given \( E \) — \( S(K|E)_\rho = S(\rho_{KE}) - S(\rho_E) \), where \( S(\rho) := -\text{tr}(\rho \log \rho) \) — in terms of the trace distance \( D(\rho_{KE}, \tau_K \otimes \rho_E) \). We first derive a lower bound on the von Neumann entropy, using the following theorem from Alicki and Fannes [AF04].

**Theorem A.9 (From [AF04]).** For any bipartite states \( \rho_{AB} \) and \( \sigma_{AB} \) with trace distance \( D(\rho, \sigma) = \varepsilon \leq 1/4 \text{ and dim } H_A = d_A \), we have

\[
|S(A|B)_\rho - S(A|B)_\sigma| \leq 8\varepsilon \log d_A + 2h(2\varepsilon) \, ,
\]

where \( h(p) = -p \log p - (1-p) \log (1-p) \) is the binary entropy.
Corollary A.10. For any state $\rho_{KE}$ with $D(\rho_{KE}, \tau_K \otimes \rho_E) = \varepsilon \leq 1/4$, where $\tau_K$ is the fully mixed state, we have

$$S(K|E)_\rho \geq (1 - 8\varepsilon) \log |K| - 2h(2\varepsilon).$$

Proof. Immediate by plugging $\rho_{KE}$ and $\tau_K \otimes \rho_E$ in Corollary A.9. \qed

Given the von Neumann entropy of $K$ conditioned on $E$, $S(K|E)_\rho$, one can also upper bound the trace distance of $\rho_{KE}$ from $\tau_K \otimes \rho_E$ by relating $S(K|E)_\rho$ to the relative entropy of $\rho_{KE}$ to $\tau_K \otimes \rho_E$ — the relative entropy of $\rho$ to $\sigma$ is defined as $S(\rho || \sigma) := \text{tr} (\rho \log \rho) - \text{tr}(\rho \log \sigma)$.

Lemma A.11. For any quantum state $\rho_{KE}$,

$$D(\rho_{KE}, \tau_K \otimes \rho_E) \leq \sqrt{1/2\big(\log |K| - S(K|E)_\rho\big)}.$$

Proof. From the definitions of the relative and von Neumann entropies we have

$$S(\rho_{KE} || \tau_K \otimes \rho_E) = \log |K| + S(\rho_{KE} || \text{id}_K \otimes \rho_E) = \log |K| - S(K|E)_\rho,$$

where $\text{id}_K$ is the identity matrix. We then use the following bound on the relative entropy [OP93, Theorem 1.15] to conclude the proof:

$$S(\rho || \sigma) \geq 2(D(\rho, \sigma))^2.$$ \qed

Corollary A.10 and Lemma A.11 can be written together in one equation, upper and lower bounding the conditional von Neumann entropy:

$$(1 - 8\varepsilon) \log |K| - 2h(2\varepsilon) \leq S(K|E)_\rho \leq \log |K| - 2\varepsilon^2,$$

where $\varepsilon = D(\rho_{KE}, \tau_K \otimes \rho_E)$.

**B Alternative secrecy criterion**

In Section 4, we derived two conditions — secrecy and correctness — that together imply that a real QKD system is indistinguishable from the ideal one. An alternative definition for $\varepsilon$-secrecy was proposed in the literature [TSSR10, TLGR12]:

$$(1 - p_{\text{abort}}) \min_{\sigma_E} D(\rho_{KE}, \tau_K \otimes \sigma_E) \leq \varepsilon . \quad (29)$$

This alternative notion is equivalent to the standard definition of secrecy [Eq. (1)] up to a factor 2, as can be seen by the following calculation. Let $\varphi_E$ be the state for which the minimum in Eq. (29) is achieved. Then,

$$D(\rho_{KE}, \tau_K \otimes \rho_E) \leq D(\rho_{KE}, \tau_K \otimes \varphi_E) + D(\tau_K \otimes \varphi_E, \tau_K \otimes \rho_E) \leq 2D(\rho_{KE}, \tau_K \otimes \varphi_E).$$
We thus have
\[ D(\rho_{KE}, \tau_K \otimes \rho_E) \leq 2 \min_{\sigma_E} D(\rho_{KE}, \tau_K \otimes \sigma_E) \leq 2D(\rho_{KE}, \tau_K \otimes \rho_E). \]

This means that any QKD scheme proven secure with one definition is still secure according to the other, with a minor adjustment of the failure parameter \( \varepsilon \).

However, we do not know how to derive this alternative notion from a composable framework. In particular, it is not clear if the failure \( \varepsilon \) from Eq. (29) is additive under parallel composition. For example, the concatenation of two keys that each, individually, satisfy Eq. (29) could possibly have distance from uniform greater than 2\( \varepsilon \). For this reason, the arXiv version of [TLGR12] was updated to use Eq. (1) instead.

C More Abstract Cryptography

C.1 Distinguishing metric

A distinguisher has been introduced as a single entity that has to guess which of two systems it is holding. Mathematically, it is more convenient to model a distinguisher as a set \( \mathcal{D} \). Each element \( D \in \mathcal{D} \) is a system with \( n+1 \) interfaces. \( n \) of them connect to the \( n \) interfaces of a resource \( R \) or \( S \) and the last interface outputs a bit, as illustrated in Figure 2.2 on page 7. Thus, for any \( D \in \mathcal{D} \) and any compatible system \( \mathcal{R} \), \( D(\mathcal{R}) \) is a binary random variable. The distinguishing advantage can be rewritten as
\[ d^D(\mathcal{R}, \mathcal{S}) = \max_{D \in \mathcal{D}} \Pr[D(\mathcal{R}) = 1] - \Pr[D(\mathcal{S}) = 1]. \]

For a set \( \mathcal{D} \) to be a valid distinguisher, it has to be closed under composition with all resources and converters. For a converter \( \alpha \) and a resource \( \mathcal{T} \), define
\[ D(\alpha \cdot): \mathcal{R} \mapsto D(\alpha \mathcal{R}) \]
\[ D(\cdot \parallel \mathcal{T}): \mathcal{R} \mapsto D(\mathcal{R} \parallel \mathcal{T}). \]

A distinguisher \( \mathcal{D} \) is closed under composition with a set of converters \( \Sigma \) and a set of resources \( \Gamma \)\(^26\) if for all \( D \in \mathcal{D} \), all \( \alpha \in \Sigma \) and all \( \mathcal{T} \in \Gamma \),
\[ D(\alpha \cdot) \in \mathcal{D} \quad \text{and} \quad D(\cdot \parallel \mathcal{T}) \in \mathcal{D}. \quad (30) \]

\(^{26}\)For a set of converters \( \Sigma \) and a set of resources \( \Gamma \) to be valid, they also have to be closed under composition, i.e., for all \( \alpha, \beta \in \Sigma \) and all \( R, S \in \Gamma \),
\[ \alpha \beta \in \Sigma, \quad \alpha \| \beta \in \Sigma, \quad \alpha \mathcal{R} \in \Gamma, \quad \text{and} \quad \mathcal{R} \| \mathcal{S} \in \Gamma. \]
For example, the set of all possible distinguishers is closed under composition with the sets of all possible converters and resources, and is used for information-theoretic security. The set of all efficient distinguishers is closed under composition with the sets of all efficient converters and resources, and is used for computational security. The fact that the distinguishing advantage is non-increasing under composition (see Eq. (7) on page 11) follows directly from the closure of the distinguisher, Eq. (30).

C.2 Generic protocol composition

In this section we briefly sketch why the security criteria of Definition 2.1 guarantee that the composition of two secure protocols is also secure. We write up the argument in the case where an adversary is present (Eq. (ii) from Definition 2.1). The case with no adversary follows similarly with the simulator $\sigma$ removed and a filter connected to Eve’s interface of every resource. Proofs of this can be found in [MR11,Mau12].

C.2.1 Sequential composition

Let protocols $\pi$ and $\pi'$ construct $S_0$ from $R_\sharp$ and $T_\square$ from $S_0$ within $\varepsilon$ and $\varepsilon'$, respectively, i.e.,

$$R_\sharp \xrightarrow{\pi,\varepsilon} S_0 \quad \text{and} \quad S_0 \xrightarrow{\pi',\varepsilon'} T_\square .$$

It then follows from the triangle inequality of the distinguishing metric that $\pi'\pi$ constructs $T_\square$ from $R_\sharp$ within $\varepsilon + \varepsilon'$,

$$R_\sharp \xrightarrow{\pi'\pi,\varepsilon+\varepsilon'} T_\square .$$

To see why this holds when an adversary is present, note that since $\pi R$ cannot be distinguished from $S_\sigma$ with advantage greater than $\varepsilon$, by Eq. (7) a distinguisher running $\pi'$ in particular cannot distinguish them. Hence

$$\pi'\pi R \approx_{\varepsilon} \pi' S_\sigma .$$

Likewise, a distinguisher running $\sigma$ does not know if it is interacting with $\pi' S$ or $T_\sigma'$, i.e.,

$$\pi' S_\sigma \approx_{\varepsilon'} T_\sigma' \sigma .$$

Combining the two equations above, we get

$$\pi'\pi R \approx_{\varepsilon+\varepsilon'} T_\sigma' \sigma .$$

So there exists a simulator, namely $\sigma', \sigma$, such that the real and ideal systems cannot be distinguished with advantage greater than $\varepsilon + \varepsilon'$. 

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C.2.2 Parallel composition

The argument for parallel composition is similar to that of sequential composition. Let \( \pi \) and \( \pi' \) construct \( S_\Box \) and \( S'_\Box \) from \( R_\flat \) and \( R'_\flat \) within \( \varepsilon \) and \( \varepsilon' \), respectively, i.e.,

\[
R_\flat \xrightarrow{\pi,\varepsilon} S_\Box \quad \text{and} \quad R'_\flat \xrightarrow{\pi',\varepsilon'} S'_\Box.
\]

If these resources and protocols are composed in parallel, we find that \( \pi \parallel \pi' \) constructs \( S_\Box \parallel S'_\Box \) from \( R_\flat \parallel R'_\flat \) within \( \varepsilon + \varepsilon' \),

\[
R_\flat \parallel R'_\flat \xrightarrow{\pi \parallel \pi',\varepsilon + \varepsilon'} S_\Box \parallel S'_\Box,
\]

where \( R_\flat \parallel R'_\flat := (R_\parallel R'_\flat, \sharp \parallel \flat) \) is the filtered resource consisting of the parallel composition of the resources and filters from \( R_\flat \) and \( R'_\flat \).

For the case where an adversary is present, this can be proven as follows. From the definition of parallel composition of converters in Eq. (3) we have

\[
(\pi \parallel \pi')(R \parallel R') = (\pi R)(\pi' R').
\]

Since for some \( \sigma \), \( \pi R \) cannot be distinguished from \( S \sigma \) with advantage greater than \( \varepsilon \), by Eq. (7) running \( \pi' R' \) in parallel cannot help the distinguisher, hence

\[
\pi R \parallel \pi' R' \approx_{\varepsilon} S \sigma \parallel \pi' R'.
\]

For the same reason we also have

\[
S \sigma \parallel \pi' R' \approx_{\varepsilon'} S \sigma \parallel S' \sigma'.
\]

Combining the equations above and one more use of Eq. (3) we obtain

\[
(\pi \parallel \pi')(R \parallel R') \approx_{\varepsilon + \varepsilon'} (S \parallel S') (\sigma \parallel \sigma').
\]

Thus, there exists a simulator, namely \( \sigma \parallel \sigma' \), such that the real and ideal systems cannot be distinguished with advantage greater than \( \varepsilon + \varepsilon' \).

D Composition of authentication and QKD

In this section we model recursive composition of an authentication protocol and QKD. Starting with a short (uniform) key, we construct an authentic channel, which is then used in a QKD protocol to obtain a long key. Part of this new key is then consumed in another round of authentication, which is used by QKD, resulting in more shared secret key. This can be repeated indefinitely. From the composable models of the security definitions we immediately have that the total failure is bounded by the sum of the individual failures of (each run of) each protocol.
The goal of this section is to write this out explicitly: we show that
\[
D(\rho_{A^n B^n E}, \tilde{\rho}_{A^n B^n E}) \leq n(\varepsilon_{\text{auth}} + \varepsilon_{\text{qkd}}),
\]
where \(A^n\) and \(B^n\) are registers containing all shared, unused secret keys generated in \(n\) rounds of authentication and QKD, \(E\) contains the adversary’s information, \(\rho_{A^n B^n E}\) and \(\tilde{\rho}_{A^n B^n E}\) are the states obtained by a distinguisher interacting with the real and ideal systems\(^{27}\) and \(\varepsilon_{\text{auth}}\) and \(\varepsilon_{\text{qkd}}\) are the (probabilities of) failure of the authentication and QKD protocols in each round.\(^{28}\)

For this recursive construction it is not necessary to use the (interactive) authentication protocol of Renner and Wolf\(^{29}\) which only requires a password. Instead we use the simpler universal hashing of Wegman and Carter\(^{30}\), which appends a tag\(^{31}\) to the message that is sent, but requires an (almost) uniform key. A more detailed analysis of this authentication method, including key recycling and a proof that strong universal hashing meets the corresponding security definition can be found in \cite{Por14}.

In Appendix D.1 we sketch how to construct an authentic channel from a shared secret key and insecure channel resource, and provide a generic simulator. In Appendix D.2 we compose multiple authentication protocols in parallel so that we may have multiple use authentic channels in QKD. In Appendix D.3 we compose such a construction with QKD, obtaining a key expansion protocol. And finally in Appendix D.4 we iteratively compose a key expansion protocol with itself, resulting in a continuous stream of secret key bits.

\section*{D.1 Authentication}

The QKD and one-time pad protocols discussed in this work make use of authentic channels as depicted in Figure D.1a, which always deliver the correct message to the receiver. This is however impossible to construct from an insecure channel, since Eve can always cut the communication between Alice and Bob, and prevent any message from being transmitted. What can be constructed, is a channel which guarantees that Bob does not receive a corrupted message. He either receives the correct message sent by Alice, or an error, which symbolizes an attempt by Eve to change the message. This can be modeled by giving Eve’s idealized interface two controls: the first provides her with Alice’s message, the second allows her to input one bit that prevents Alice’s message from being delivered to Bob and produces an error instead. We illustrate this in Figure D.1b.

\footnote{Unlike the examples from Section 5 we cannot simplify the state \(\tilde{\rho}_{A^n B^n E}\) by conditioning on obtaining a key and writing it as \(\tau_{A^n B^n} \otimes \rho_E\), because the \(n\) repetitions of the protocol lead to \(n + 1\) events: it never aborted, aborted after one round, two rounds, etc.}

\footnote{One can also use different parameters in each round, e.g., so that the failure in round \(i\) is half of that in round \(i - 1\), and the sum \(\sum_i (\varepsilon_{\text{auth}}^i + \varepsilon_{\text{qkd}}^i)\) is bounded for all \(n\).}

\footnote{This tag is often called a message authentication code (MAC) in the literature.}
(a) A resource that always transmits Alice’s message to Bob, and provides Eve with a copy.

(b) A resource that allows Eve to decide if Bob gets Alice’s message or not, but does not allow the message to be manipulated.

Figure D.1 – The authentic channel on the right can be constructed by an authentication protocol. The one on the left is a simplification as used in the one-time pad [Figure 2.3] or QKD [Figure 3.2] constructions.

A construction of this authentic channel resource from an insecure channel and a shared secret key resource is typically accomplished by computing the hash $h_k(x)$ of the message $x$, and sending the string $x||h_k(x)$ to Bob, where $k$ is the shared secret key and $\{h_k\}_{k \in \mathcal{K}}$ a family of hash functions [Sti94]. Alice’s part of the authentication protocol $\pi_A^{\text{auth}}$ thus gets a key $k$ from an ideal key resource, a message $x$ from Alice, and sends $x||h_k(x)$ down the insecure channel. When Bob receives a string $x'||y'$, he needs to check whether $y' = h_k(x')$. His part of the protocol gets a key $k$ from an ideal key resource, a message $x'||y'$ from the channel, and outputs $x'$ if $y' = h_k(x')$, otherwise an error $\bot$. If the ideal key resource used by both players produces an error instead of a key, Alice and Bob abort, and the protocol is trivially secure. So for simplicity we omit this possibility in the following, and assume that they always get a shared secret key. This is depicted in [Figure D.2]

Figure D.2 – The real authentication system — Alice has access to the left interface, Bob to the right interface and Eve to the lower interface — consists of the authentication protocol $(\pi_A^{\text{auth}},\pi_B^{\text{auth}})$, and the secret key and insecure channel resources, $\mathcal{K}$ and $\mathcal{C}$.
In the case where no adversary is present and filters cover Eve’s interfaces of Figures D.1b and D.2, the real and ideal systems are indistinguishable as they are both identity channels which faithfully transmit $x$ from Alice to Bob. So in the following we only consider the case where an adversary is present, condition [ii] in Definition 2.1.

In the ideal setting, the authentic channel (Figure D.1b) has the same interface on Alice’s and Bob’s sides as the real setting (Figure D.2): Alice can input a message, and Bob receives either a message or an error. However, Eve’s interface looks quite different: in the real setting she can modify the transmission on the insecure channel, whereas in the ideal setting the adversarial interface provides only controls to read the message and interrupt the transmission. From Definition 2.1 we have that an authentication protocol constructs the authentic channel if there exists a simulator $\sigma_{E}^{\text{auth}}$ that can recreate the real interface while accessing just the idealized one.

An obvious choice for the simulator is to first generate its own key $k$ and output $x \parallel h_{k}(x)$. Then upon receiving $x' \parallel y'$, it checks if $x' \parallel y' = x \parallel h_{k}(x)$ and cuts the transmission on the authentic channel if this does not hold. We illustrate this in Figure D.3.

![Figure D.3 – The ideal authentication system — Alice has access to the left interface, Bob to the right interface and Eve to the lower interface — consists of the ideal authentication resource and a simulator $\sigma_{E}^{\text{auth}}$.](image)

An authentication protocol is then $\varepsilon$-secure if Figures D.2 and D.3 are $\varepsilon$-close, i.e.,

$$\pi_{A}^{\text{auth}} \pi_{B}^{\text{auth}}(K \parallel C) \approx_{\varepsilon} A \sigma_{E}^{\text{auth}} .$$  \hspace{1cm} (32)

Portmann [Por14] showed that Eq. (32) is satisfied if the hash functions used are $\varepsilon$-almost strongly universal.\footnote{Unlike the case of QKD, here we are not interested in a filter which introduces (honest) noise on the channel, as this can be removed by encoding the communication with an appropriate error correcting code. Therefore, the filter on the (real) insecure channel faithfully forwards the message, and the filter on the (ideal) authentic channel allows the message to be transmitted.}

\footnote{A family of functions is said to be almost strongly universal if any two different messages are almost uniformly mapped to all pairs of tags.}
D.2 Parallel composition of authentication

In Appendix D.1 we modeled one run of an authentication protocol, that constructs a one-time use authentic channel. In general, QKD protocols require multiple rounds of authenticated communication. This is achieved by running the same protocol in parallel multiple times with new keys. It is straightforward from Eqs. (3), (6), (7) and (32) that \( \ell \) parallel repetitions of the authentication protocol are \( \ell \varepsilon \)-close to \( \ell \) ideal authentic channels and simulators in parallel,

\[
\left( \pi_A^{\text{auth}} \pi_B^{\text{auth}} \| \cdots \| \pi_A^{\text{auth}} \pi_B^{\text{auth}} \right) (\mathcal{K} \| \mathcal{C} \| \cdots \| \mathcal{K} \| \mathcal{C} ) \\
= \left( \pi_A^{\text{auth}} \pi_B^{\text{auth}} (\mathcal{K} \| \mathcal{C} ) \right) \| \cdots \| \left( \pi_A^{\text{auth}} \pi_B^{\text{auth}} (\mathcal{K} \| \mathcal{C} ) \right) \\
\approx_{\ell \varepsilon} \left( A \sigma_E^{\text{auth}} \right) \| \cdots \| \left( A \sigma_E^{\text{auth}} \right) = (A \| \cdots \| A) \left( \sigma_E^{\text{auth}} \| \cdots \| \sigma_E^{\text{auth}} \right). 
\]

In the following, when we speak of the authentication used in QKD we always refer to parallel repetitions of the protocol that construct multiple use authenticated channels \( A \| \cdots \| A \). For simplicity, we use the same notation for multiple authentic channels as we have for single channels — we denote the resulting multiple use authentic channel by \( A \), as well as \( \mathcal{C} \) for the multiple use insecure channel, \( \mathcal{K} \) for a key sufficiently long for authenticating every message and \( \varepsilon^{\text{auth}} \) for the accumulated failure of all parallel repetitions in one round of QKD.

D.3 Sequential composition of authentication and key distribution

Let \( \pi^{\text{auth}} \) be an authentication protocol which constructs with failure \( \varepsilon^{\text{auth}} \) a (multiple use) authentic channel \( A \) from a short secret key of length \( \ell \), \( \mathcal{K}^{\ell} \), and an insecure classical channel \( \mathcal{C} \),

\[
\mathcal{K}^{\ell} \| \mathcal{C} \xrightarrow{\pi^{\text{auth}}, \varepsilon^{\text{auth}}} A. 
\]

Let \( \pi^{\text{qkd}} \) be a QKD protocol which constructs with failure \( \varepsilon^{\text{qkd}} \) a long secret key of length \( m \), \( \mathcal{K}^{m} \), from an authentic channel \( A \) and an insecure quantum channel \( \mathcal{Q} \),

\[
A \| \mathcal{Q} \xrightarrow{\pi^{\text{qkd}}, \varepsilon^{\text{qkd}}} \mathcal{K}^{m}. 
\]

By sequentially composing the two protocols, we immediately have that \( \pi^{\text{qkd}} \pi^{\text{auth}} \) constructs a long secret key \( \mathcal{K}^{m} \) from a short secret key \( \mathcal{K}^{\ell} \) and insecure channels \( \mathcal{C} \) and \( \mathcal{Q} \), with failure \( \varepsilon^{\text{auth}} + \varepsilon^{\text{qkd}} \),

\[
\mathcal{K}^{\ell} \| \mathcal{C} \| \mathcal{Q} \xrightarrow{\pi^{\text{qkd}}, \pi^{\text{auth}}, \varepsilon^{\text{auth}} + \varepsilon^{\text{qkd}}} \mathcal{K}^{m}. 
\]
(a) The composition of a QKD and authentication protocols. The insecure channels have not been depicted as boxes to simplify the figure.

(b) A hybrid system consisting of a QKD protocol and two-way authentic channels with simulator.

(c) The ideal secret key and corresponding composed simulator $\sigma_E^{\text{qkd}}\sigma_E^{\text{auth}}$. This ideal key resource has two switches preventing the key from being generated: one to capture an abort from the authentication protocol and one from the QKD protocol.

Figure D.4 – Steps in the security proof of the composition of QKD and an authentication protocol. If we remove the gray parts from Figures D.4a and D.4b we recover the real and ideal systems of authentication. If we remove the dashed parts from Figures D.4b and D.4c we recover the real and ideal systems of QKD.
The generic argument for sequential composition is given in Appendix C.2.1. Here we illustrate it in the special case of authentication and QKD, and draw it in Figure D.4.

Figure D.4a depicts the real world: the two protocols are composed in sequence and run using the short key and insecure channel resources. In Figure D.4b we have a system consisting of the real QKD protocol and the ideal authentic channel and simulator. We know that the black parts of Figures D.4a and D.4b are $\varepsilon_{\text{auth}}$-close, so adding the QKD protocol in gray can only reduce the distance. Figure D.4c depicts the ideal secret key resource and simulators. By removing the dashed simulator $\sigma_E^{\text{auth}}$ from Figures D.4b and D.4c we recover the real and ideal QKD systems from Figures 3.2a and 4.1—with an extra switch on the authentic channel in the real system and on the secret key resource in the ideal system. Since these are $\varepsilon_{\text{qkd}}$-close, so are Figures D.4b and D.4c. Putting the two statements together with the triangle inequality finishes the argument.

This reasoning is summed up in the following equation, which can be directly derived from Eqs. (3), (6) and (7):

$$\pi_A^{\text{auth}} \pi_B^{\text{auth}} \left( K^\ell \| C \right) \approx_{\varepsilon_{\text{auth}}} \pi_A^{\text{qkd}} \pi_B^{\text{qkd}} \left( A \| Q \right) \approx_{\varepsilon_{\text{qkd}}} \pi_A^{\text{auth}} \pi_B^{\text{auth}} \left( K^\ell \| C \| Q \right) \approx_{(\varepsilon_{\text{auth}} + \varepsilon_{\text{qkd}})} \pi_A^{\text{qkd}} \pi_B^{\text{qkd}} \pi_A^{\text{auth}} \pi_B^{\text{auth}} \approx_{\varepsilon_{\text{qkd}}} \varepsilon_{\text{auth}}^{\text{qkd}} \varepsilon_{\text{auth}}^{\text{auth}}.$$

Let $\rho_{ABE}$ be the state gathered by a distinguisher interacting with the real system from Figure D.4a and $\tilde{\rho}_{ABE}$ be the state gathered by the distinguisher interacting with the ideal system from Figure D.4c. By the argument above we have

$$D(\rho_{ABE}, \tilde{\rho}_{ABE}) \leq \varepsilon_{\text{auth}} + \varepsilon_{\text{qkd}}.$$

### D.4 Iterated key expansion

In Appendix D.3 we show that the composition of QKD and authentication—i.e., key expansion—constructs a long key from a short key and insecure channels. To show that this can be done recursively, we need to argue that part of the long key can be kept for the next round of key expansion. So far the secret keys have been treated as blocks, entirely consumed by a protocol, which is not convenient for the analysis of a protocol that uses only part of a key. Instead, we should think of these key resources—e.g., $K^\ell$ and $K^m$ in Figures D.4a and D.4c—as parallel composition of resources that produce a single bit of key, i.e., $K^\ell = K^1_1 \| \cdots \| K^1_i$, where $K^1_i$ is the $i$th instance of a resource $K^1$ that produces one bit of key (or an error message) at Alice and Bob’s interfaces, and has a switch at Eve’s interface that decides if it produces the key or error.
Then, a proof that
\[ K^\ell \parallel Q \xrightarrow{\pi^\text{qkd},\pi^\text{auth},\varepsilon^\text{auth} + \varepsilon^\text{qkd}} K^m, \]
is immediately also a proof that
\[ K^{\ell'} \parallel Q \xrightarrow{\pi^\text{qkd},\pi^\text{auth},\varepsilon^\text{auth} + \varepsilon^\text{qkd}} K^{m+\ell'-\ell}, \]
for any \( \ell' \geq \ell \). Iterating the protocol \( n \) times we get
\[ K^{\ell} \parallel Q^n \xrightarrow{(\pi^\text{qkd},\pi^\text{auth})^n, n(\varepsilon^\text{auth} + \varepsilon^\text{qkd})} K^{nm-(n-1)\ell}, \]
where \( \mathcal{E}^n \) and \( Q^n \) are \( n \) instances of the resources \( \mathcal{E} \) and \( Q \) in parallel, and \( (\pi^\text{qkd},\pi^\text{auth})^n \) is \( n \) times the sequential composition of \( \pi^\text{qkd},\pi^\text{auth} \). [Eq. (31)]
follows immediately from this.

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