Copper oxides become superconductors rapidly upon doping with electron holes, suggesting a fundamental pairing instability. The Cooper mechanism explains normal superconductivity as an instability of a fermi-liquid state, but high-temperature superconductors derive from a Mott-insulator normal state, not a fermi liquid. We show that precocity to pair condensation with doping is a natural property of competing antiferromagnetism and \(d\)-wave superconductivity on a singly-occupied lattice, thus generalizing the Cooper instability to doped Mott insulators, with significant implications for the high-temperature superconducting mechanism.

**Keywords** Cooper-pair instability, high-temperature superconductivity, \(SU(4)\) model

**PACS numbers** 74.72.-h, 74.72.Kf, 74.20.-z, 74.20.Mn

Understanding cuprate high-temperature superconductors is complicated by unusual properties of the normal state and how this state becomes superconducting with doping [1]. Band theory suggests that cuprates at half lattice filling should be metals, but they are instead insulators with antiferromagnetic (AF) properties. This behavior is thought to result from a Mott-insulator normal state, where the insulator properties follow from strong on-site Coulomb repulsion rather than band-filling properties. Upon doping the normal states with electron holes, there is a rapid transition to a superconducting (SC) state, with evidence for a pairing gap at zero temperature typically appearing for about 3%-5% hole density per copper site in the copper–oxygen plane. In addition, there is strong evidence at low to intermediate doping for a partial energy gap at temperatures above the SC transition temperature \(T_c\) that is termed a pseudogap (PG), with the size of the SC gap and PG having opposite doping dependence at low doping [2, 3].

Parent states of normal superconductors are fermi liquids (strongly interacting systems having excitations in one-to-one correspondence with the excitations of a non-interacting fermi gas). Normal superconductors are described by Bardeen–Cooper–Schrieffer (BCS) theory [4], and result from condensation of zero-spin, zero-momentum fermion pairs into a new collective state with long-range coherence of the wavefunction. The key to understanding normal superconductivity was the demonstration by Cooper [5] that normal fermi liquids possess a fundamental instability: an electron pair above a filled fermi sea can form a bound state for vanishingly small attractive interaction. In normal superconductors, the attraction is provided by interactions with lattice phonons, which bind weakly over a limited frequency range because electrons and the lattice have different response times. However, it is the Cooper instability, not the microscopic origin of the attractive interaction, that is most fundamental: a weak electron–electron interaction alone cannot produce a superconducting state, but the Cooper instability can (in principle) produce a superconducting state for any weakly attractive interaction.

The rapid onset of superconductivity in high-\(T_c\) compounds with hole doping suggests a fundamental instability against pair condensation, but it is difficult to understand this (and the appearance of PG states) within the standard BCS framework because the superconductor appears to derive from a Mott insulator, not a normal fermi liquid. Just as for normal superconductivity, we believe that the key to understanding high-temperature superconductivity is not the attractive interaction leading to pair binding (as important as that is), but rather the nature of the instability that produces the superconducting state. Since at larger doping the high-temperature superconducting state exhibits many...
properties of a normal BCS superconductor (but with d-wave pairs), this instability must reduce to the Cooper instability at larger doping, but evolve into something more complex at lower doping, where the normal state approaches a Mott insulator, and a PG exists above the SC transition temperature.

To account for rapid onset of superconductivity with hole-doping, Laughlin [6] (see also Refs. [7, 8]) proposed a modified Hamiltonian with an attractive term that partially overcomes the on-site repulsion. Then the insulator at half filling is actually a “thin, ghostly superconductor”, which fails to superconduct only because its long-range order is disrupted at very low doping, ostensibly by fluctuations due to low superfluid density. This proposed new state is termed a gossamer superconductor. This idea might provide a justification for the resonating valence bond (RVB) state [9], which assumes implicitly that quantum antiferromagnets should exhibit superconductivity, even though cuprate ground states at half-filling appear to be best described as an insulating state with long-range AF order and no superconductivity [10–14]. In the RVB model, it is usually assumed that the long-range Néel order of the ground state at exactly half filling is replaced quickly by the RVB spin-liquid ground state upon hole-doping, with details lacking. Laughlin [6] contends that the real issue for validity of the RVB picture is not whether all quantum antiferromagnets are secretly superconductors, but whether some are. The gossamer superconductor is then proposed as a second kind of antiferromagnetism—distinguished by a small background superfluid density—that is the true normal state in the cuprates, and is the harbinger of a spin-liquid RVB ground state for low hole-doping.

The gossamer state has desirable properties but is created by hand: a strong attractive term is added to the Hamiltonian, which justifies modifying Gutzwiller projectors such that they only partially suppress double occupancy [7]. We shall show that a model implementing competition of d-wave pairing with antiferromagnetic correlations on a lattice with no double occupancy has Mott insulator properties at half filling, but is unstable toward developing a finite singlet pairing gap under infinitesimal hole-doping. Thus, we shall argue that many features motivating the idea of gossamer superconductivity are natural consequences of AF and SC competition on a lattice having strict no double occupancy at half filling. We shall argue further that a pseudogap with correct properties is a natural consequence of the same theory, thus accounting for both the precocious onset of a pairing gap and the appearance of pseudogaps at low doping in the cuprates. Finally, we shall discuss the implications of these results for gossamer superconductivity and for the RVB model.

We wish to solve for the doping and temperature dependence of observables in a theory that incorporates an equal footing d-wave superconductivity and antiferromagnetism. To do so, we shall employ the tools of Lie algebras, Lie groups, and generalized coherent states [15–22]. To construct a Hamiltonian embodying these degrees of freedom and expected conservation laws for charge and spin in the many-body wavefunction, we require at a minimum three staggered magnetization operators $\mathbf{Q}$ to describe AF, creation and annihilation operators $D^\dagger$ and $D$ for d-wave singlet pairs and a charge operator $M$ to describe superconductivity, and three spin operators $\mathbf{S}$ to impose spin conservation.

However, this set of 9 operators is physically incomplete since scattering of singlet pairs (antiparallel spins on adjacent sites) from the AF particle–hole degrees of freedom can produce triplet pairs (parallel spins on adjacent sites), which are not part of the operator set. The mathematical statement of this incompleteness is that the operator set \{\mathbf{Q}, D^\dagger, D, M, \mathbf{S}\} does not close a Lie algebra under commutation. As demonstrated in Refs. [15–17, 19, 20], a (minimally) complete operator set results if we add to these operators the six triplet pair operators $\pi^\dagger$ and $\pi$. Then the set of 15 operators \{\mathbf{Q}, D^\dagger, D, \pi^\dagger, \pi, M, \mathbf{S}\} closes the Lie algebra $SU(4)$. The explicit forms for these operators in both momentum and coordinate space, and the corresponding $SU(4)$ commutation algebra, may be found in Refs. [15–17, 19, 20].

A critical feature of this symmetry structure is that the $SU(4)$ algebra closes only if the 2-dimensional lattice on which the generators are defined has no doubly occupied sites [18]. Thus, the $SU(4)$ algebra embodies the minimal theory that describes AF and d-wave SC competition through a many-body wavefunction that conserves charge and spin, and that has no components corresponding to double site occupancy on the lattice. The Hamiltonian restricted to one-body and two-body terms is unique, with the general form

$$H = H_0 - G_0 D^\dagger D - G_1 \pi^\dagger \cdot \pi - \chi \mathbf{Q} \cdot \mathbf{Q} + \kappa \mathbf{S} \cdot \mathbf{S}$$

where $G_0$, $G_1$, $\chi$, and $\kappa$ are effective interaction strengths, and $H_0$ is the single-particle energy. The $T = 0$ ground state corresponds to a superposition of singlet and triplet fermion pairs.

We shall solve for the ground-state properties of this theory using generalized coherent states [22]. Our immediate interest is the ground-state total energy surface, which is the expectation value of the Hamiltonian in the ground coherent state (with total spin $S = 0$). The formalism for constructing the $SU(4)$ coherent state and the ground state energy surface has been developed extensively in Refs. [15–17, 19, 20], to which we refer for details. The overall $SU(4)$ symmetry may be used to eliminate the $\pi^\dagger \cdot \pi$ term from the Hamiltonian (1), leaving an energy surface that is a function of order parameters for singlet pairing and antiferromagnetism, with the