Rethinking Generalized Beta family of distributions

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Abstract. We approach the Generalized Beta (GB) family of distributions using a mean-reverting stochastic differential equation (SDE) for a power of the variable, whose steady-state (stationary) probability density function (PDF) is a modified GB (mGB) distribution. The SDE approach allows for a lucid explanation of Generalized Beta Prime (GB2) and Generalized Beta (GB1) limits of GB distribution and, further down, of Generalized Inverse Gamma (GIGa) and Generalized Gamma (GGa) limits, as well as describe the transition between the latter two. We provide an alternative form to the “traditional” GB PDF to underscore that a great deal of usefulness of GB distribution lies in its allowing a long-range power-law behavior to be ultimately terminated at a finite value. We derive the cumulative distribution function (CDF) of the “traditional” GB, which belongs to the family generated by the regularized beta function and is crucial for analysis of the tails of the distribution. We analyze fifty years of historical data on realized market volatility, specifically for S&P500, as a case study of the use of GB/mGB distributions and show that its behavior is consistent with that of negative Dragon Kings.

1 Introduction

Generalized Beta family of distributions has been widely used in econometrics and econophysics, in particular for understanding of distributions of income and wealth [1–5]. Family members with scale-free, power-law tails may be of particular interest in this regard [7], which prompted development of stochastic models of economic exchange that produced power-law-tailed Inverse Gamma (IGa) [8], GIGa [9], and GB2 [10] as their steady-state distributions. These distributions have other numerous applications in various fields, such as physics [11–13], mathematical finance [14–20], and cognitive psychology [21], to name a few. Of particular importance to this work is that a very general yet simple model of stochastic volatility yields a GB2 steady-state distribution [22].

The first main objective of this work was to generalize GB distribution itself, relative to the one introduced in [2], by removing the correlations between the two scale parameters. The main purpose of such generalization is to obtain a distribution which has long power-law tails above the first scale, only to be eventually terminated, via a drop-off to zero, at the second scale. The need for such distribution was motivated by our looking into a possibility of Dragon Kings (DK) [23,24] in the realized volatility (RV) [25]. In particular, it is of great interest to glean into whether the most calamitous events in the stock market—Savings and Loan Crisis, Tech Bubble, Great Recession and Covid Pandemic—follow the Black Swan (BS) behavior and stay on the power-law tails or they deviate strongly higher and thus are DK. What we found instead [26] was behavior consistent with negative Dragon Kings (nDK) [27], which are a strong drop-offs of the power-law tails, and are well described by GB-type distribution functions, mGB.

It should be emphasized that one of the main attractions of the distributions that belong to the Generalized Beta family of distributions is that they emerge as steady-state distributions in a variety of underlying stochastic models, including those of economic exchange [8–10] and stochastic volatility [15–18,22]. Stochastic volatility is believed to be directly linked to market volatility, as seen in stock returns and RV [22]. Since the distribution of RV that we investigate here exhibits signature features of the GB distribution, it was, therefore, our second objective to derive a stochastic model that yields GB as its steady state. Another important advantage of attaining this objective is that the hierarchy of the GB family of distributions is far more clearly understood through parameters of the underlying stochastic models described by a class of stochastic differential equations.

While the most general description of stochastic dynamic models for Generalized Beta family of distributions can be found in [28], the model for the top GB—or for that matter the top Beta (B) distribution—
is notably absent. As it turns out, the “traditional” GB distribution—as defined in [2] and generalized here— does not appear to be a steady-state solution of an SDE. However, we did identify an SDE whose steady-state distribution—modified Beta (mB)—is very similar in properties to the Beta (B) distribution. We then constructed an mGB from mB using a change of variable to the power of the variable. We also derived an SDE from the mB CDF via the same change of variable, which yielded yet another mGB as its steady-state distribution. Both mGB distributions have key features of the traditional GB, and its generalization here, but seem to describe RV distribution better. There is very little numerical difference between the two mGB distributions, except that the former has a far simpler analytical form than the latter.

This paper is organized as follows. In Sect. 2, we discuss the generalization of the GB distribution that is cast in terms of two scale and three shape parameters, with one of the shape parameters representing a change of variable to the power of the variable, the power being the said shape parameter; this change of variable also signifies transition from B to GB distribution. We also derive CDF of GB distribution and show that it belongs to a class of incomplete-Beta-function-generated distributions [29–34]. In Sect. 3, we derive two novel SDE that produce mB and two mGB distributions and explore their properties relative to the “traditional” GB distribution. We also develop an SDE framework of producing GB family hierarchy and explain the nature of transition between power-law-distributed distributions and those with exponential-like tails. In Sect. 4 we investigate distributions of S&P realized volatility vis-a-vis fits by GB and mGB distributions and show that mGB provides a better fit and that termination of the distribution at a finite values is consistent with nDK behavior. We summarize our results in Sect. 5.

2 Generalized Beta distribution

Employing a “dimensionless” variable, we use the following form of the GB PDF:

\[
f_{GB}(x; \alpha, \beta_1, \beta_2, p, q) = \frac{\alpha}{\beta_1 B(p, q)} \left(1 + \left(\frac{x}{\beta_2}\right)^\beta_1\right)^{-\alpha - 1}\left(1 + \left(\frac{x}{\beta_2}\right)^\beta_2\right)^{-\alpha - 1},
\]

(1)

where \(\beta_1\) and \(\beta_2\) are scale parameters and \(\alpha, p\) and \(q\) are shape parameters, all positive. \(B(p, q)\) is the beta function and \(x \leq \beta_1\). This transforms into the form used by McDonald [2] with the substitution \(\beta_1 = \beta/(1-c)^{1/\alpha}\) and \(\beta_2 = \beta/c^{1/\alpha}\), \(0 \leq c \leq 1\); however, we do not find it necessary to impose such correlations between \(\beta_1\) and \(\beta_2\). We derive the following expressions for GB CDF and 1-CDF, the latter being important in investigating power-law dependencies:

\[
F_{GB}(x; \alpha, \beta_1, \beta_2, p, q) = I \left(\left(\frac{x}{\beta_1}\right)^\alpha + \left(\frac{x}{\beta_2}\right)^\alpha; p, q\right),
\]

(2)

\[
1 - F_{GB}(x; \alpha, \beta_1, \beta_2, p, q) = I \left(1 - \left(\frac{x}{\beta_1}\right)^\alpha; q, p\right),
\]

(3)

where \(I(y; p, q) = B(y; p, q)/B(p, q)\) and \(B(y; p, q)\) are, respectively, the regularized and incomplete beta functions [35].

Equations (1)–(3) reproduce well-known GB hierarchies [1–3,28]: GB1 and GB2 for \(\beta_2 \rightarrow +\infty\) and \(\beta_1 \rightarrow +\infty\), respectively, and further B1 and B2 (Beta Prime) for \(\alpha = 1\). We arrive at the latter two also by taking the limits in reverse order: first \(\alpha = 1\), which yields Beta (B) distribution and then \(\beta_2 \rightarrow +\infty\) or \(\beta_1 \rightarrow +\infty\). The further down limits of GGa and GIGa are best understood from analysis of an SDE that produces all the above distributions and will be discussed in Sect. 3. We also observe the special nature of the shape parameter \(\alpha\) in that GB, and all its family members, are easily generated form the B family \((\alpha = 1)\) by a simple change of variable

\[
f_{B}(y; \beta_1, \beta_2, p, q) dy = f_{GB}(x; \alpha, \beta_1, \beta_2, p, q) dx,
\]

(4)

where \(y = x^\alpha\) and

\[
f_{B}(x; \beta_1, \beta_2, p, q) = \frac{1}{\beta_1 B(p, q)} \left(1 + \left(\frac{x}{\beta_2}\right)^\beta_1\right)^{-\alpha - 1}\left(1 + \left(\frac{x}{\beta_2}\right)^\beta_2\right)^{-\alpha - 1} \times \left(1 - \frac{x}{\beta_1}\right)^{-p-1} \left(1 + \frac{x}{\beta_2}\right)^{-p-q},
\]

(5)

with the appropriate rescaling of \(\beta_1\) and \(\beta_2\). The same holds true also for the corresponding SDE in Sect. 3, where the mean-reversion SDE for variable \(x\) produces B family of steady-state distributions, while the same mean-reversion SDE for variable \(x^\alpha\) produces GB family. More precisely, the steady-state distribution of the corresponding SDE for the variable \(x\) is a modified Beta distribution, mB, which is very similar to B, while the steady-state distribution for variable \(x^\alpha\) is a modified GB, mGB. One can also obtain a second modified GB distribution via the change of variable (4) in mB. Again, both mGB are very similar to GB. It should be also emphasized that below B/GB level all members of the respective families can be written in a traditional form—see footnote 2 in Sect. 3 below.

It should be noted that since the variable of the regularized beta changes between 0 and 1, GB CDF belongs to a class of distributions generated by a seed CDF [29–
34], see also [36], which in this case is given by

$$F(x; \alpha, \beta_1, \beta_2) = \frac{\left( \frac{x}{\beta_1} \right)^{\alpha} + \left( \frac{x}{\beta_2} \right)^{\alpha}}{1 + \left( \frac{x}{\beta_2} \right)^{\alpha}}.$$ (6)

In the limiting cases of GB1, \( \beta_2 \to \infty \), and GB2, \( \beta_1 \to \infty \), it reproduces the results of [37]. With the corresponding PDF \( f(x; \alpha, \beta_1, \beta_2) = F'(x; \alpha, \beta_1, \beta_2) \), GB PDF can be rewritten as [29]

$$f_{GB}(x; \alpha, \beta_1, \beta_2, p, q) = \frac{1}{\hat{B}(p, q)} F^{p-1} f(1 - F)^{q-1},$$ (7)

which produces an alternative form of GB PDF,

$$f_{GB}(x; \alpha, \beta_1, \beta_2, p, q) = \frac{\alpha}{x \hat{B}(p, q)} \left( \left( \frac{x}{\beta_1} \right)^{\alpha} + \left( \frac{x}{\beta_2} \right)^{\alpha} \right)^p \times \left( 1 - \left( \frac{x}{\beta_1} \right)^{\alpha} \right)^{q-1} \left( 1 + \left( \frac{x}{\beta_2} \right)^{-p-q} \right),$$ (8)

equivalent to (1). In particular, it should be mentioned that for \( \alpha = 1 \) the seed distribution (6) contains only scale parameters \( \beta_1 \) and \( \beta_2 \), so the effect of generating B distribution via (2) and (8) is the attainment of the shape parameters \( p \) and \( q \).

In what follows, we will be specifically interested in the \( \beta_2 \ll \beta_1 \) circumstance since for \( \beta_2 \ll x \ll \beta_1 \) GB exhibits a power-law dependence,

$$f_{GB} \propto \left( \frac{x}{\beta_2} \right)^{-q-1}, \quad 1 - F_{GB} \propto \left( \frac{x}{\beta_2} \right)^{-p},$$ (9)

which is also the power-law tail of GB2, but GB PDF eventually terminates at \( \beta_1 \), which may be the case in some of the possible negative Dragon Kings (nDK) phenomena [27], such as realized market volatility, which we discuss in Sect. 4.

3 Generalized Beta family as steady-state distributions of stochastic differential equation

Invoking now an SDE approach to the GB family of distributions, we point out that—with the exception of GB (and B) themselves—all members of GB family of distributions where obtained as steady-state distributions of SDE by Hertzler [28]. We, however, provide our own, considerably simplified, version of those results and, in addition, obtain the SDE yielding mGB.

First, to underscore the significance of the change of variable \( y = x^\alpha \) in generalizing from B to GB family of distributions that was mentioned in Sect. 2, we first consider the following SDE, which combines the multiplicative [14,16,19,20] and Heston (Cox-Ingersoll-Ross) [15,17] models of stochastic volatility [22]:

$$dy = -\gamma (y - \theta) dt + \sqrt{\kappa^2 y + \kappa_2^2 y^2} dW_t,$$ (10)

where \( dW_t \) is the Wiener process, \( dW_t \sim N(0, dt) \). Its steady-state distribution is a modified B2 [22], which can be easily derived using the standard Fokker–Planck formalism [38,39]:

$$f_{mB2}(x; \beta_2, p, q) = \frac{(1 + \frac{x}{\beta_2})^{-p-q-1}(\frac{x}{\beta_2})^{-1+p}}{\beta_2 \hat{B}(p, q + 1)} \times (p + q)(1 + \frac{x}{\beta_2})^{-p-q-1}(\frac{x}{\beta_2})^{-1+p},$$ (11)

where

$$\beta_2 = \frac{k^2}{\kappa^2},$$ (12)

$$p = \frac{2\gamma \theta}{\kappa^2},$$ (13)

and

$$q = \frac{2\gamma}{\kappa^2}.$$ (14)

This PDF is normalizable for \( q > 0 \) and \( p > 0 \).

If we now consider the same mean-reverting process (10) for \( y = x^\alpha \), and define \( \gamma' = \gamma / \alpha \), \( k' = k / \alpha \) and \( k_2' = k_2 / \alpha \), we obtain a stochastic process given by

$$dx = -\gamma' (x - \theta x^{1-\alpha}) dt + \sqrt{\kappa'^2 x^{2-\alpha} + \kappa_2'^2 x^{1-2\alpha}} dW_t,$$ (16)

whose steady-state distribution is a GB2, given by (compare with GB2 in [10,22,25])

$$f_{mGB2}(x; \alpha, \beta_2, p, q) = \frac{\alpha(p + q)(1 + (\frac{x}{\beta_2})^\alpha)^{p-q-1}(\frac{x}{\beta_2})^{-1+p}}{q\beta_2 \hat{B}(p, q)},$$ (17)

\(^2\) Notice, that in [22] we used the standard B2 PDF,

$$f_{B2}(y; \beta_2, p, q) = \frac{(1 + \frac{y}{\beta_2})^{-p-q-1}(\frac{y}{\beta_2})^{-1+p}}{\beta_2 \hat{B}(p, q)},$$ (15)

whence \( q = 1 + \frac{2\gamma}{\kappa^2} \). The reason behind this ambiguity in the definition of \( q \) stems from the fact that \( p \) and \( q \) are independent at the B2/GB2 level in the steady-state PDF of the respective SDE, which is not the case for B/GB as will be shown below. Similar ambiguity with respect to the value of \( q \) exists for B1/GB1 for the same reason as for B2/GB2—that \( p \) and \( q \) are independent at that level and, consequently, B1/GB1 PDF can be written either in standard or modified version.
where

\[ \beta_2^\alpha = \frac{\kappa'^2}{\kappa^2}, \quad (18) \]

\[ p\alpha = -1 + \alpha + \frac{2\gamma'\theta}{\kappa'^2}, \quad (19) \]

and

\[ q = -1 + \alpha + \frac{2\gamma'}{\kappa'^2}. \quad (20) \]

Notice the obvious renormalization of \( p \) and \( q \) in going from mB2 to mGB2 (or B2 to GB2 [22]), unlike undergoing the change of variable in accordance with (4). Nonetheless, this analysis confirms that, for simplicity, it is sufficient to obtain an SDE for mB, given that the SDE for mGB will then automatically follow via aforementioned change of variable \( x \to x^\alpha \). Analytical renormalization of \( p \) and \( q \) is irrelevant for numerical analysis, since they are obtained from fitting.

The SDE formalism also makes analysis of limiting cases physically appealing. For instance, at the current GB2 level of hierarchy, substituting \( \kappa = 0 \) directly into (16) yields a GIGa (IGa for \( \alpha = 1 \) [8]) steady-state distribution [9,20,22], while substituting \( \kappa_2 = 0 \) yields GGa (Ga for \( \alpha = 1 \) [18]). (19) and (20) then also explain that those two cases correspond to \( p \to \infty \) and \( q \to \infty \) [1–3,28] respectively, when starting from (17). However, the explicit form of the steady-state distributions indicates that this just corresponds to the exponential-like distribution [9,20,22], while substituting (23) yields a GIGa (IGa for \( \alpha = 1 \) [8]) steady-state distribution [9,20,22].

But, as before, the SDE given by (21) allows us to easily trace B-hierarchy: \( \kappa_1 = 0 \) yields the B2 family discussed above, with further \( \kappa = 0 \) and \( \kappa_2 = 0 \) yielding IGa and Ga, respectively; conversely, \( \kappa_2 = 0 \) yields the B1 family, with further \( \kappa_1 = 0 \) yielding Ga. As discussed above, this can be immediately upgraded to the GB hierarchy via the \( x \to x^\alpha \) change of variable, with the result summarized as follows:

\[ mGB \xrightarrow{\kappa_1=0} GB2 \xrightarrow{\kappa_2=0} GIGa, \]

\[ mGB \xrightarrow{\kappa_1=0} GB2 \xrightarrow{\kappa_2=0} GGa, \]

\[ mGB \xrightarrow{\kappa_2=0} GB1 \xrightarrow{\kappa_1=0} GGa. \]

As was mentioned before, GB2/GB1 distributions can be written either in its standard or modified form, depending on the definition of \( q \).

It is obvious form (24) that (G)Ga is a tying link between (G)B2 and and (G)B1. This can be easily seen by considering the following SDE:

\[ dx = -\gamma(x-\theta)dt + \sqrt{\kappa'^2x + (2c-1)\kappa'^2x}dW_t, \quad (25) \]

where \( 0 \leq c \leq 1 \). For \( c = 0 \) we have \( \kappa = \kappa_1 \) and B1 above; for \( c = 1/2 \) we have G; for \( c = 1 \) we have \( \kappa_2 = \kappa_1 \) and B2 above. This simply means that as \( c \) changes we observe a transition from a PDF defined on a finite interval, (G)B, to a PDF with a power-law tail, (G)B2, via a distribution with an exponential-like tail.

The PDF and the CDF of the steady-steady distribution obtained from (23) are given, respectively, by

\[ f_{mB}(x; \beta_1, \beta_2, p, q) = \frac{(p+q) \left( 1 + \frac{\beta_1}{\beta_2} \right)^{p+1} \left( \frac{x}{\beta_1} \right)^{p-1} \left( 1 - \frac{x}{\beta_1} \right)^{q-1} \left( 1 + \frac{x}{\beta_2} \right)^{-p-q-1}}{\beta_1 \left( 1 + \frac{\beta_1}{\beta_2} \right) q B(p, q)}, \quad (26) \]

and

\[ F_{mB}(x; \beta_1, \beta_2, p, q) = \frac{1}{B(p, q)} \left( \frac{\beta_2}{\beta_1} \right)^p \left( \frac{x}{\beta_1} \right)^q \left( \frac{x}{\beta_2} \right)^p, \quad (27) \]

where

\[ p = 2\gamma\theta, \quad (28) \]

and

\[ q = 2\gamma(\beta_1 - \theta) \left( 1 + \frac{\beta_1}{\beta_2} \right)^{-1}. \quad (29) \]

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One version of the mGB PDF and the CDF related to GB can be obtained via the change of variable (4) applied to (26) and (27) and are given, respectively, by

\[ f_{mGB}(x; \alpha, \beta_1, \beta_2, p, q) = \frac{\alpha (p+q) \left( 1 + \left( \frac{\beta_1}{\beta_2} \right)^\alpha \right)^{p+1} \left( \frac{x}{\beta_1} \right)^{\alpha p - 1} \left( 1 - \left( \frac{x}{\beta_1} \right)^\alpha \right)^{q-1} \left( 1 + \left( \frac{x}{\beta_2} \right)^\alpha \right)^{-p-q-1}}{\beta_1 \left( p + 1 + \left( \frac{\beta_1}{\beta_2} \right)^\alpha \right)^p B(p, q)} \]  

(30)

and

\[ F_{mGB}(x; \alpha, \beta_1, \beta_2, p, q) = I \left( \left( \frac{x}{\beta_1} \right)^\alpha + \left( \frac{x}{\beta_2} \right)^\alpha \right)^q 1 + \left( \frac{x}{\beta_2} \right)^\alpha \left[ q + \left( \frac{\beta_2}{\beta_1} \right)^\alpha \right] \left( 1 + \left( \frac{x}{\beta_2} \right)^\alpha \right)^q \times \left( \frac{1 + \left( \frac{\beta_1}{\beta_2} \right)^\alpha \left( \frac{x}{\beta_1} \right)^\alpha \left( \frac{x}{\beta_2} \right)^\alpha}{1 + \left( \frac{x}{\beta_2} \right)^\alpha} \right)^p \right) \]  

(31)

The second term in (31) is the difference between \( F_{mGB} \) and \( F_{GB} \) in (2) and is zero at zero and at \( \beta_1 \). Due to this difference, the asymptotic behaviors of \( F_{GB} \) and \( F_{mGB} \) on approach to \( \beta_1 \), \( x \to \beta_1 \), develop a considerable contrast in the limit \( \beta_2 \ll \beta_1 \) of interest here:

\[ 1 - F_{mGB} \approx \frac{1}{q B(p, q)} \left( \frac{1 - \left( \frac{x}{\beta_1} \right)^\alpha}{1 + \left( \frac{x}{\beta_2} \right)^\alpha} \right)^q \]  

(33)

\[ 1 - F_{mGB} \approx \frac{1 + \frac{p}{q} \left( \frac{\beta_2}{\beta_1} \right)^\alpha}{q B(p, q)} \left( \frac{1 - \left( \frac{x}{\beta_1} \right)^\alpha}{1 + \left( \frac{x}{\beta_2} \right)^\alpha} \right)^q \left( \frac{\beta_2}{\beta_1} \right)^\alpha \]  

(34)

that is \( 1 - F_{mGB} \) drops off to zero (\( F_{mGB} \) saturates to unity) faster than \( 1 - F_{GB} \) due to the factor \( \left( \frac{\beta_2}{\beta_1} \right)^\alpha \).

Another mGB distribution is obtained from the SDE for B with \( x \) replaced by \( x^\alpha \):

\[ dx = -\gamma(x - \theta x^{1-\alpha})dt + \sqrt{x^{2-\alpha}} \left( 1 - \left( \frac{x}{\beta_1} \right)^\alpha \right) \left( 1 + \left( \frac{x}{\beta_2} \right)^\alpha \right) dW_t. \]  

(35)

PDF and CDF of the steady-state solution are given, respectively, by

\[ \hat{f}_{mGB}(x; \alpha, \beta_1, \beta_2, p, q) = \frac{\alpha \left( \frac{x}{\beta_1} \right)^{\alpha p - 1} \left( 1 - \left( \frac{x}{\beta_1} \right)^\alpha \right)^{\frac{1}{q-1}} \left( 1 + \left( \frac{x}{\beta_2} \right)^\alpha \right)^{-p-q-1}}{\beta_1 B(p, q - \frac{1}{\alpha} + 1) \quad gf_1(p, p + q + 1; p + q - \frac{1}{\alpha} + 1; -\left( \frac{x}{\beta_2} \right)^\alpha)}, \]  

(36)
and

\[
\tilde{F}_{mGB}(x; \alpha, \beta_1, \beta_2, p, q) = \frac{\left( \frac{x}{p_1} \right)^{\alpha p} \left( \frac{1}{p} F_1 \left( p; \frac{1}{\alpha} - q, p + q; p + 1; \left( \frac{x}{p_1} \right)^{\alpha}, -\left( \frac{x}{p_1} \right)^{\alpha} \right) - \frac{1}{p_2} \left( \frac{x}{p_2} \right)^{\alpha} F_1 \left( p + 1; \frac{1}{\alpha} - q, p + q + 1; p + 2; \left( \frac{x}{p_2} \right)^{\alpha}, -\left( \frac{x}{p_2} \right)^{\alpha} \right) \right)}{B \left( p, q - \frac{1}{\alpha} + 1 \right) 2F_1 \left( p, p + q + 1; p + q - \frac{1}{\alpha} + 1; -\left( \frac{x}{p_2} \right)^{\alpha} \right)},
\]

(37)

where

\[
p\alpha = \alpha - 1 + 2\gamma \theta, \tag{38}
\]

\[
q\alpha = 1 - \alpha + 2\gamma (\beta_1 - \theta) \left( 1 + \beta_1 \beta_2 \right)^{-1}, \tag{39}
\]

and \( 2F_1 \) and \( F_1 \) are hypergeometric and Appell hypergeometric functions, respectively [35].

4 Market volatility as possible example of application of Generalized Beta distribution

Realized volatility \( RV \) is the square root of realized variance, which is defined as follows

\[
RV^2 = 100^2 \times \frac{252}{n} \sum_{i=1}^{n} r_i^2, \tag{40}
\]

where

\[
r_i = \ln \frac{S_i}{S_{i-1}} \tag{41}
\]

are daily returns and \( S_i \) is the reference (closing) price on day \( i \). This is an annualized value, where 252 represents the number of trading days in a year. In particular, \( n = 1 \) are the daily returns and \( n = 21 \) are monthly returns (typical number of trading days in a month). Here, we present results for \( n = 1, 2, 3, 5, 7, 9, 13, 17, 21 \).

We fitted distributions of RV for S&P index from 1970 through May of 2021, which covers four major financial upheavals: Savings and Loan Crisis, Tech Bubble, Great Recession, and Covid Pandemic. In a companion manuscript [26], we undertake a far more detailed numerical analysis, which includes actual time-series of RV identifying the far-end-tail points of the distribution, linear tail fitting of RV distribution on a log–log scale meant to test conformity to power-law behavior, and p-value test for DK [27]. Here, we limit our discussion to fitting RV distribution using (1)–(2) and (30)–(31), including the confidence intervals (CI) of these fits [40].

Fitting was conducted using Bayesian for PDF\(^3\) and Gradient Descent for CDF and we find very small differences in the parameters of the distribution between the two techniques—at most 8% for GB (1)–(2) and at most 3% for mGB (30)–(31). Additionally, the two-sample KS statistic between (31) and (37) is 0.0015 ≪ 0.0169, the latter being the standard value for the number of points in our sample, as per [41] with alpha value being 0.05. Given such negligible difference, relative simplicity of (30)–(31) vis-a-vis the actual solution (36)–(37) of the SDE (35), qualifies the former as a near exact solution of the SDE and explains its choice for fitting.

Tables 1 and 2 list the parameters of GB and mGB distributions obtained from fitting the RV distribution, the corresponding Kolmogorov-Smirnov (KS) statistic, and the goodness-of-fit KS values for our RV sample size, as per table in [42] with alpha value being 0.05. Figs. 1 and 2 show plots of KS statistic and scale parameters \( \beta_1 \) and \( \beta_2 \) from the tables as a function of \( n \). Figures 3, 4, 5, 6, 7, 8, 9, 10, 11 show GB and mGB fits of RV data—1-CDF, or complimentary CDF (ccdf)—with the corresponding 95% CI, on a log-log scale.

One noteworthy feature in those plots is that a relatively small subset of data points moves up from the straight line (on a log-log scale) portion of the GB and mGB tails, (9), and outside their CI, which can be viewed as “potential DK.” This feature is further analyzed in [26] using the p-value test [27]. However, the data invariably falls back in and indicates termination at final values, as intended to be explained by GB and mGB.

Turning to the interpretation of numerical results, we observe that the realized variance \( RV^2 \), given by (40), is the mean of daily variances \( r_i^2 \), which explains the monotonous decrease of the largest value of RV with \( n \) in Figs. 3, 4, 5, 6, 7, 8, 9, 10, 11. On the other hand, “potential DK,” mentioned above, become more pronounced with larger \( n \) because RV remains highly elevated for longer periods of time during financial calamities. In this regard, it is of note that for all \( n \), the largest values of RV are found during the Savings and Loan Crisis, followed by COVID Pandemic and the Financial Crisis of 2008 [26].

It should be also noted that market upheavals are usually referenced by the longevity of the downturns, which can last much longer than 21 days, upon which

\(^3\) This can also be done, albeit less efficiently, using Maximum Likelihood Estimation (MLE).
Fig. 1  $k_s$ vs $n$ for GB and mGB fits; horizontal line shows the minimal value of goodness-of-fit $k_s$ from Tables 1 and 2

Table 1  Parameters of GB fit of RV distribution, its KS statistic $k_s$, and goodness-of-fit value of KS for our RV sample sizes

| $n$ | $GB(x; \alpha, \beta_1, \beta_2, p, q)$ | $k_s$ | $k_s$-table |
|-----|----------------------------------|------|-------------|
| 1   | $(1.5457, 398.8160, 27.4217, 0.6648, 2.7871)$ | 0.0102 | 0.0119 |
| 2   | $(2.0163, 316.3938, 16.6113, 0.8805, 1.8097)$ | 0.0044 | 0.0119 |
| 3   | $(2.1444, 254.1085, 13.2608, 1.2549, 1.6824)$ | 0.0074 | 0.0119 |
| 5   | $(2.2971, 196.7883, 10.8962, 1.7834, 1.5348)$ | 0.0043 | 0.0119 |
| 7   | $(2.4789, 179.8124, 9.7236, 2.1369, 1.3815)$ | 0.0060 | 0.0120 |
| 9   | $(2.4734, 169.5618, 9.0164, 2.5880, 1.3855)$ | 0.0048 | 0.0120 |
| 13  | $(2.4317, 137.6122, 7.6590, 3.8712, 1.4172)$ | 0.0075 | 0.0120 |
| 17  | $(2.2842, 117.9511, 6.3396, 6.1014, 1.5241)$ | 0.0063 | 0.0120 |
| 21  | $(2.3979, 106.5157, 6.2021, 6.5453, 1.4415)$ | 0.0068 | 0.0120 |

Table 2  Parameters of mGB fit of RV distribution, its KS statistic $k_s$, and goodness-of-fit value of KS for our RV sample sizes

| $n$ | $mGB(x; \alpha, \beta_1, \beta_2, p, q)$ | $k_s$ | $k_s$-table |
|-----|----------------------------------|------|-------------|
| 1   | $(1.5500, 399.9009, 27.4233, 0.6519, 1.7828)$ | 0.0087 | 0.0119 |
| 2   | $(1.9541, 302.8331, 16.2974, 0.9384, 0.8642)$ | 0.0052 | 0.0119 |
| 3   | $(2.1195, 254.8331, 13.2632, 1.2561, 0.6836)$ | 0.0053 | 0.0119 |
| 5   | $(2.3708, 200.5519, 10.7210, 1.7255, 0.4456)$ | 0.0054 | 0.0119 |
| 7   | $(2.4744, 180.8711, 9.7136, 2.1430, 0.3848)$ | 0.0051 | 0.0120 |
| 9   | $(2.5239, 160.224, 8.9839, 2.5856, 0.3582)$ | 0.0064 | 0.0120 |
| 13  | $(2.4506, 167.4719, 7.4788, 3.7661, 0.4092)$ | 0.0062 | 0.0120 |
| 17  | $(2.3026, 120.1110, 6.3561, 6.1121, 0.5403)$ | 0.0074 | 0.0120 |
| 21  | $(2.4016, 104.9925, 6.3853, 6.3429, 0.5043)$ | 0.0067 | 0.0120 |

the market-standard monthly volatility is calculated. More importantly, a different measure, rather than the market-standard RV, seems more appropriate for such circumstances. Namely, instead of daily returns, as in \((41)\) one should examine the accumulated (multi-day) returns \(r_i^{(n)} = \ln(\frac{S_i}{S_{i-n}})\) and the corresponding RV. We hope to address this point in a future work.

A very important point in applying GB/mGB to fit RV data is that only the daily RV is strictly speaking amenable to stochastic modeling given that stochastic variance, which feeds directly into realized variance\(^5\) since realizes variance is a square realized volatility, if one is described by an GB/mGB so will be the other—with renormalized parameters.

\(^5\) Since realizes variance is a square realized volatility, if one is described by an GB/mGB so will be the other—with renormalized parameters.
Fig. 2 $\beta_1$ and $\beta_2$ vs $n$ for GB and mGB fits

Fig. 3 GB and mGB fits of RV, with respective CI, for $n = 1$
Fig. 4 GB and mGB fits of RV, with respective CI, for $n = 2$

Fig. 5 GB and mGB fits of RV, with respective CI, for $n = 3$

[22], can be modeled by an SDE, such as (35). So insofar as the multi-day market-standard RV is concerned, which is defined as a square root of the average of daily realized variances (40), there is no first-principle argument for it to be described by GB/mGB and so the latter are used for fitting solely for the reason that they reflect the main trends of empirical RV. An indication to this effect is seen in Figs. 3, 4, 5, 6, 7, 8, 9, 10, 11, where $n = 1$ case is more reminiscent of BS behavior and is better fitted by GB/mGB, while $n \geq 2$ progressively develop more pronounced “potential DK” and nDK behaviors.

Another very important point in the latter regard is that the empirically observed phenomena, such as “potential DK” and nDK mentioned above, are unlikely explainable by a steady-state process and a closed-form, if multi-parameter, analytical distribution function. While there exists no well-developed theory of DK/nDK phenomena, it is believed that some of the cases that fall in that category may be ascribed to

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6 An alternative approach to modeling intra-day and daily returns is by jump processes [43], rather than continuous models, as considered here.
dynamical effects—in the present case observed perturbations to the steady-state description. It is clearly seen in Figs. 3, 4, 5, 6, 7, 8, 9, 10, 11 that GB/mGB fall short of precisely describing “potential DK” and nDK parts of the data, with the former running above CI of GB/mGB while the latter drops more abruptly than GB/mGB. In this regard one should consider this Section more as a “proof of concept” of application of the theory developed in Secs. 2–3. While fitting well the entire curve and capturing its main features, it is clearly insufficient in addressing more intricate details of a small subset of the end points where we observe “potential DK” and nDK effects.

5 Summary

Main results of this article can be summarized as follows:

- We introduced an alternative form of Generalized Beta PDF (1) and evaluated its CDF (2), the latter belonging to a class of Incomplete Beta Function generated distributions.
- We identified an SDE (35), which produces a modified Generalized Beta distribution as its steady-state solution, whose PDF and CDF are given, respectively, by (36) and (37).
A similar yet simpler modified Generalized Beta distribution (30)–(31) was constructed via a change of variable (4) from a modified Beta distribution (26)–(27), which is a steady-state distribution of the $\alpha = 1$ SDE, (23).

We showed that an expanded version of (23), (21) allows for a physically appealing explanation of the Generalized Beta hierarchy, (24), as well as of natural link between (Generalized) Beta 1 and (Generalized) Beta 2 from (25).

We argued that Generalized Beta distribution is particularly useful in the circumstance when the distribution is characterized by a long power-law tail followed by a drop-off at a finite value of the variable. This has been applied to the distribution of realized volatility of the S&P500 index, which exhibits behavior characteristic of “negative Dragon Kings.” Specifically, we found that power-law tails, leading potentially to “Black Swan” effects, are abruptly terminated, capping realized volatility at very large but finite values. One interesting nuance to the above observation is that a small subset of data in the tails showed upward deviation from the linear part of the tails (on the log–log scale) and outside the confidence intervals of GB and mGB fits (as well as that of linear fit, all confirmed using
p-value test [26]), which can be viewed as potential to develop “Dragon Kings.” However, it is always followed by a rapid dropdown, as seen in Figs. 3, 4, 5, 6, 7, 8, 9, 10, 11. The largest values of realized volatility of S&P500 were, as expected, due to the most virulent market calamities: Savings and Loan crisis, Tech Bubble, Financial Crisis and Covid Pandemic. However, at least for S&P 500—perhaps due to the broad spectrum of the index and the size and strength of companies it represents—market mechanism seem to limit its price from below and thus to disallow emergence of “Dragon Kings” or even development of “Black Swans” beyond some limiting value (which is not to state that the upper limit could not move upward in the future). To reiterate, Generalized Beta seems to be well suited for such situations and in the particular case of realized volatility provides a very good overall fit of its distribution.

In the future, we will further examine applicability of Generalized Beta to phenomena bounded from above yet exhibiting a well-established power-law tail over the large portion of the distribution.
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Author contributions

Authors contributed equally to the paper, with Jiong Liu lead on numerical and R.A. Serota on analytical part.

Data availability statement We obtained S&P500 data at Yahoo! Finance. Our datasets are available upon request.

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