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Abstract. In this paper we show that the classical definition and the associated characterizations of wide-sense Markov (WSM) signals are not valid for improper complex signals. For that, we propose an extension of the concept of WSM to a widely linear (WL) setting and the study of new characterizations. Specifically, we introduce a new class of signals, called widely linear Markov (WLM) signals, and we analyze some of their properties based either on second-order properties or on state-space models from a WL processing standpoint. The study is performed in both the forwards and backwards directions of time. Thus, we provide two forwards and backwards Markovian representations for WLM signals. Finally, different estimation recursive algorithms are obtained for these models.

1. Introduction

Next, we present the main definitions, notation and auxiliary results. We also present an example which motivates the necessity of the new concept introduced.

The proofs of the results given along the paper can be found in [1].

Bold capital letters will be used to refer to matrices and bold lower-case letters will be used to refer to vectors. The row \(j\) of any matrix \(A(\cdot)\) will be denoted by \(A[j](\cdot)\) and the \(n\)-vector of zeros by \(0_n\). Furthermore, the superscripts *, \(T\) and \(H\) represent the complex conjugate, transpose and complex transpose, respectively.

Let \(\{x_t, t \in \mathbb{Z}\}\) be a zero-mean complex random signal with correlation function \(r(t,s) = E[x_t x_s^*]\) and complementary correlation function \(c(t,s) = E[x_t x_s]\). The linear minimum-mean square error (MMSE) estimator of \(x_t\) based on the set of observations \(\{x_{t_1}, x_{t_2}, \ldots, x_{t_m}\}\) will be denoted by \(\hat{x}(t|t_1, t_2, \ldots, t_m)\) and we will refer to it as the strictly linear (SL) estimator.

The Markov condition on a signal \(\{x_t, t \in \mathbb{Z}\}\) establishes the following identity for the conditional probability:

\[
P(x_t \leq x|x_{t_1}, x_{t_2}, \ldots, x_{t_m}) = P(x_t \leq x|x_{t_1})
\]

for all \(x\) and \(t > t_1 > \ldots > t_m\). Doob [2] introduced a weaker concept based on the SL estimator which has received great attention in the literature. A signal \(x_t\) is called WSM if \(\hat{x}(t|\tau \leq s) = \hat{x}(t|s)\) for any \(s < t\). Such signals have remarkable properties. For example,
a signal \( x_t \) is WSM if, and only if, the function \( \tilde{k}(t, s) = r(t, s)r^{-1}(s, s) \) has the triangular property, i.e.
\[
\tilde{k}(t, s) = \tilde{k}(t, \tau)\tilde{k}(\tau, s), \quad t \geq \tau \geq s \tag{1}
\]

Another characterization in terms of so-called Markovian state-space models can be found in [2]. It is shown that a signal \( \{x_t, t \geq 0\} \) is WSM if, and only if, it has a state-space model of the form
\[
x_{t+1} = \tilde{k}(t + 1, t)x_t + u_t \tag{2}
\]
where \( u_t \) is a white noise uncorrelated with \( x_0 \). Doob’s definition was later generalized in [3] in the following sense: \( x_t \) is a WSM signal of order \( n \geq 1 \) if \( \tilde{x}(t|\tau \leq s) = \tilde{x}(t|s, s - 1, \ldots, s - n + 1) \) for any \( s < t \). The authors also studied the second-order properties of such signals.

All these studies have a common characteristic: the information supplied by the complementary correlation function is ignored, i.e., the results are derived assuming implicitly that the signal is proper (\( c(t, s) = 0 \)). Nowadays, the research activity in the field of the complex-valued signal is more and more focused on the better performing and less familiar WL processing. It has proved to be a more useful approach than SL processing when complex-valued random signals are in general improper (i.e., they are correlated with their complex conjugates).

In this setting the SL MMSE estimator is replaced by the WL MMSE estimator, denoted by \( \hat{x}^{WL}(t|t_1, t_2, \ldots, t_m) \), which uses the information of the augmented vector of observations \( [x_{t_1}, x_{t_1}^*, x_{t_2}, x_{t_2}^*, \ldots, x_{t_m}, x_{t_m}^*]^T \). The immediate question that arises is whether the classical concept of WSM signals remains valid in the WL processing approach. The following example gives us the answer.

**Example 1** Consider a signal \( \{x_t, t \geq 0\} \) with correlation function \( r(t, s) = \frac{1}{2}(e^{3|t-s|} + e^{|t-s|}) \) and complementary correlation function \( c(t, s) = \frac{1}{2}(e^{3|t-s|} - e^{|t-s|}) \). It is easy to check that \( r(t, s) \) does not satisfy the triangular property (1) and then, the signal cannot be modeled by a representation of the form (2). However, as we will show later, it is possible to find a state-space representation for such a signal given by (5). Thus, the classical WSM condition is clearly insufficient in the improper case to find a state-space representation for the signal involved.

2. WLM Signals

From Example 1 we extract the following consequence: the classical definition of a WSM signal must be extended to deal with improper signals, this new concept must be characterized adequately to avoid the drawback shown in this example and new results about modeling are necessary to exploit the information available in both \( x_t \) and \( x_t^* \). Next, we introduce such a definition in a WL processing setting.

**Definition 1** A complex-valued signal \( \{x_t, t \in \mathbb{Z}\} \) is said to be WLM of order \( n \geq 1 \), briefly a \( \text{WLM}(n) \) signal, if the following condition holds
\[
\hat{x}^{WL}(t|\tau \leq s) = \hat{x}^{WL}(t|s, s - 1, \ldots, s - n + 1)
\]
for any \( s < t \).

Notice that this concept extends both the classical notion of WSM introduced by [2] and the later generalization given in [3].

In the rest of the section we provide different characterizations of WLM\((n)\) signals. For that, we need to introduce some additional notation. Denote the augmented forwards vector of order \( n \geq 1 \) of \( x_t \) as the 2n-vector
\[
x_t = [x_t, x_t^*, x_{t-1}, x_{t-1}^*, \ldots, x_{t-n+1}, x_{t-n+1}^*]^T
\]
and its correlation function by $R(t, s) = E[x_t x_s^H]$. From now on, we assume that det $\{R_t\} \neq 0$ with $R_t := R(t, t)$. Moreover, we define the normalized correlation function as

$$K(t, s) = R(t, s)R_s^{-1}$$

Similarly, we define the augmented backwards vector of order $n \geq 1$ of $x_t$ as the $2n$-vector

$$x^b_t = [x_{t+n-1}, x^*_{t+n-1}, x_{t+n-2}, x^*_{t+n-2}, \ldots, x_t, x^*_t]^T$$

The following results establish the relation between the signals $x_t$ and their augmented forwards and backwards versions. We start first with the augmented forwards vector and we give a test similar to (1) for a signal being WLM($n$).

**Theorem 1** The following statements are equivalent:

(i) $\{x_t, t \in Z\}$ is a WLM($n$) signal.

(ii) For $s < t$, the WL MMSE estimator of $x_t$ on the basis of the set $\{x_r, x^*_r, \tau \leq s\}$ is of the form

$$\hat{x}^{WL}(t|\tau \leq s) = K(t, s)x_s$$

(iii) For $t \geq \tau \geq s$,

$$K(t, s) = K(t, \tau)K(\tau, s)$$

Now, we suggest a characterization based on the augmented backwards vector. This result also shows the independence from the time direction of the Markov condition.

**Theorem 2** The following statements are equivalent:

(i) $\{x_t, t \in Z\}$ is a WLM($n$) signal.

(ii) $\hat{x}^{WL}(t|\tau \geq s) = \hat{x}^{WL}(t|s, s+1, \ldots, s+n-1)$ for any $s > t$.

(iii) For $s > t$, the WL MMSE estimator of $x^b_t$ on the basis of the set $\{x^b_r, x^*_r, \tau \geq s\}$ is of the form

$$\hat{x}^{bWL}(t|\tau \geq s) = K(t+n-1, s+n-1)x^b_s$$

### 3. Modeling of WLM Signals

We aim to provide different ways of modeling for WLM($n$) signals. We present a new characterization in which the equivalence between a WLM signal of order $n$ and their forwards and backwards representations is given. Such representations show that a WLM($n$) signal depends only on the $n$ preceding or subsequent states and their conjugates.

**Theorem 3** A signal $\{x_t, 0 \leq t \leq m\}$ is a WLM($n$) if, and only if, it has the forwards and backwards representations

$$x_{t+1} = k^T_t x_t + w_t, \quad t \geq n - 1$$

$$x_t = k^{bT}_{t+1} x^b_{t+1} + w^b_t, \quad t \leq m - n + 1$$

where $k^b_t, k^T_t$ are $2n$-vectors, and $w_t, w^b_t$ are doubly white noises such that

$$E[w_t x_{n-1}] = 0_{2n}, \quad t \geq n - 1$$

$$E[w^b_t x^b_{m-n+1}] = 0_{2n}, \quad t \leq m - n + 1$$

**Example 1 (continued)** It is not difficult to check that $x_t$ is a WLM(1) signal by using property (3). Hence, applying the theorem above, it has the state-space representation

$$x_{t+1} = \frac{1}{2}(e^3 + e)x_t + \frac{1}{2}(e^3 - e)x^*_t + w_t$$

with $w_t$ a doubly white noise uncorrelated with $x_0$ and $x^*_0$.  

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4. Estimation Problem of WLM Signals

Once the modeling problem has been solved for WLM(n) signals, we address the MMSE estimation problem of such signals under a WL processing approach. The forwards and backwards representations given in Theorem 3 notably simplify the design of different recursive estimation algorithms. To this end, we use the Kalman recursions on the forwards representation to provide the solution for the prediction and filtering problems.

Suppose that we observe a WLM(n) signal \{x_t, 0 \leq t \leq m\} via the process

\[ y_t = h_t x_t + v_t, \quad 0 \leq t \leq m \]

with \( v_t \) a doubly white noise such that \( E[v_t v_t^*] = n_{1,t} \) and \( E[v_t v_t] = n_{2,t} \) with \( n_{1,t} > |n_{2,t}| \). Moreover, we assume that \( v_t \) is uncorrelated with \( x_s \) and \( x_s^* \) for all \( t, s \).

Consider the 2-vector \( y_t = [y_t, y_t^*]^T \), the 2 × 2n matrix

\[ H_t = \begin{bmatrix} h_t & 0 & 0 & \cdots & 0 \\ 0 & h_t^* & 0 & \cdots & 0 \end{bmatrix} \]

the 2n × 2n matrix \( Q_t = R_{t+1} - K_t R_t K_t^H \) and the 2 × 2 matrix

\[ N_t = \begin{bmatrix} n_{1,t} & n_{2,t} \\ n_{2,t}^* & n_{1,t}^* \end{bmatrix} \]

Denote the WL filtered estimator of \( x_t \) by \( \hat{x}_t^W \) and the one-step-ahead predictor of \( x_{t+1} \) by \( \hat{x}_{t+1}^W \), both obtained on the basis of the information provided by the set \( \{y_0, y_0^*, \ldots, y_t, y_t^*\} \), and consider their associated errors \( p_t = E[|x_t - \hat{x}_t^W|^2] \) and \( p_{t+1|t} = E[|x_{t+1} - \hat{x}_{t+1}^W|^2] \). Also denote the estimate of \( x_{n-1} \) obtained from the information provided by \( [y_{n-1}, y_{n-1}^*, \ldots, y_0, y_0^*]^T \) by \( \hat{x}_{n-1} \) and its associated error by \( P_{n-1} \). By combining the forwards representation (4) and the classical Kalman filter we present the following Algorithm.

**Algorithm 1** WL filter and prediction

Require: \( y_t, H_t, N_t, K_t, Q_t, g = [1, 0, \ldots, 0]^T \), \( \hat{x}_{n-1} \), and \( P_{n-1} \)

Ensure: \( \hat{x}_{t+1|t}^W, \hat{x}_{t+1}^W, p_{t+1|t}, \) and \( p_{t+1|t+1} \)

1: for \( t \geq n-1 \) do
2: \( \hat{x}_{t+1|t} = K_t \hat{x}_t \)
3: \( P_{t+1|t} = K_t P_t K_t^H + Q_t \)
4: \( F_{t+1} = P_{t+1|t} H_{t+1|t} [H_{t+1|t} P_{t+1|t} + H_{t+1|t}^H N_{t+1|t}]^{-1} \)
5: \( \hat{x}_{t+1} = \hat{x}_{t+1|t} + F_{t+1} (y_{t+1} - H_{t+1|t} \hat{x}_{t+1|t}) \)
6: \( P_{t+1|t+1} = P_{t+1|t} - F_{t+1} [H_{t+1|t} P_{t+1|t}] F_{t+1}^T \)
7: \( \hat{x}_{t+1}^W = g^T \hat{x}_{t+1|t} \)
8: \( \hat{x}_{t+1}^W = g^T \hat{x}_{t+1}^W \)
9: \( p_{t+1|t} = g^T P_{t+1|t} g \)
10: \( p_{t+1|t} = g^T P_{t+1|t} g \)
11: end for

5. References

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