SCALING DISTANCES ON FINITELY RAMIFIED FRACTALS

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Abstract. In previous papers by A. Kameyama and by J. Kigami distances on fractals have been discussed having two different but similar properties. One property is that the maps defining the fractal are Lipschitz of prescribed constants less than 1, the other is that the diameters of the copies of the fractal are asymptotic to prescribed scaling factors. In this paper, on a large class of finitely ramified fractals, we prove that these two problems are equivalent and give a necessary and sufficient condition for the existence of such distances. Such a condition is expressed in terms of asymptotic behavior of the product of certain matrices associated to the fractal.

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1. Introduction

Fractals are very irregular mathematical objects. Over the last three decades, they have been investigated rather intensively. The notion of fractal is very general, and in this paper a specific but rather general class of fractals is considered. More ore less, we consider the so-called self-similar fractals, i.e., geometric objects having the property of containing copies of them at arbitrarily small scales. A concrete and relatively general class of self-similar fractals is the following: we are given finitely many contractive similarities \( \psi_1, \ldots, \psi_k \) in \( \mathbb{R}^n \), that is

\[
d(\psi_i(x), \psi_i(x')) = \bar{\alpha}_i d(x, x')
\]

for every \( x, x' \in K \), where \( \bar{\alpha}_i \) are constant lying in \( [0,1[ \). The self-similar fractal \( K \) (generated by the similarities \( \psi_i \)) is the set invariant with respect to such a set of similarities, more precisely, \( K \) is the only nonempty compact subset \( K \) of \( \mathbb{R}^n \) such that

\[
K = \bigcup_{i=1}^{k} \psi_i(K).
\]

More generally, we could define a self-similar fractal as a compact metric space, or also as a compact topological space satisfying (1.2) where
$\psi_i$ are maps satisfying certain conditions (see [3], [4]). Examples of self-similar fractals are the Cantor set, the Koch curve, the (Sierpinski) Gasket, the (Sierpinski) Carpet, the Vicsek Set, the (Lindstrom) Snowflake. See for example [5], [7] for the description of such fractals.

Here, following [3] and [4], we will say that $\alpha := (\alpha_1, ..., \alpha_k)$ is a polyratio if $\alpha_i \in [0, 1]$ for every $i = 1, ..., k$. When the self-similar fractal is defined by (1.2) in a metric space where $\psi_i$ satisfy (1.1), then, by definition the maps $\psi_i$ are $\tilde{\alpha}_i$-Lipschitz. More generally, one could ask for what polyratios $\alpha := (\alpha_1, ..., \alpha_k)$ there exists a distance $d$ on $K$ such that the maps $\psi_i$ are $\alpha_i$-Lipschitz with respect to $d$. This problem is discussed in [3] and in [4]. When this happens, following [3], [4], we will say that $\alpha$ is a metric polyratio (on $K$), and also, we will say that $d$ is an $\alpha$-self-similar distance.

In [4] a pseudodistance $D_\alpha$ is constructed on a general class of self-similar fractals for every polyratio, and it is proved that such if $\alpha$ is a metric polyratio, then such a pseudodistance is a distance, that is, if an $\alpha$-self-similar distance exists, then $D_\alpha$ is an $\alpha$-self-similar distance. Moreover, $D_\alpha$ induces on $K$ the same topology as the original one.

A similar problem has been studied in [6], in connection to problems related to analysis on fractals. Note that if $\alpha$ is a metric polyratio, then the diameter of the copy $\psi_{i_1} \circ \cdots \circ \psi_{i_m}(K)$ with respect to an $\alpha$-self-similar distance $d$ is not greater than $\alpha_{i_1} \cdots \alpha_{i_m} \text{diam}(K)$, $\text{diam}(K)$ denoting the diameter with respect to $d$. We will say that $\alpha$ is an asymptotic metric polyratio (on $K$) or shortly as. metric polyratio if there exist a distance $d$ on $K$ and positive constants $c_{1,\alpha}, c_{2,\alpha}$ such

$$c_{1,\alpha} \alpha_w \text{diam}_d(K) \leq \text{diam}_d(\psi_{i_1} \circ \cdots \circ \psi_{i_m}(K)) \leq c_{2,\alpha} \alpha_w \text{diam}_d(K) \quad (1.3)$$

where $\alpha_w = \alpha_{i_1} \cdots \alpha_{i_m}$, for every $i_1, ..., i_m = 1, ..., k$. We say in such a case that the distance $d$ is an $\alpha$-scaling distance.

The analysis on fractals deals with notions like Laplace operator, Dirichlet integral, Heat equation. The problem of defining such notions is nontrivial since we cannot define the derivative in the usual sense on fractals, as they have usually an empty interior. Analysis on fractals has been developed mainly on the finitely ramified fractals, and more specially on the P.C.F. self-similar sets. The Gasket, the Vicsek Set and the Snowflake are examples of P.C.F. self-similar sets, while the Carpet is not a P.C.F. self-similar sets and neither is finitely ramified. However, also on certain infinitely ramified fractals, e.g., the Carpet, analysis has been developed. Standard text-books on analysis on fractals are [5] and [7], where also the precise definition of P.C.F. self-similar sets is given.

In [6], the problem of what polyratios are as. metric polyratios is discussed as related to problems concerning Dirichlet forms and Heat Kernel. There, a rather general class of self-similar fractals (not only
Scaling distances on finitely ramified fractals is discussed, and some conditions are given. In particular, it is proved that in any case many polyratios are as. metric. However, the conditions given in [6] are not necessary and sufficient.

In this paper, I restrict the class of fractals (more or less I consider the connected P.C.F. self-similar sets). See Section 2 for the details. For such class of fractals, I prove that the notions of metric polyratio and as. metric polyratio are in fact the same. Moreover, I give a necessary and sufficient condition for a polyratio being metric (or as. metric). This condition is based on a finite set of special matrices which are described in Section 4. These matrices are related to the paths on the fractal. The condition is that this set of matrices satisfies a special property, which is strictly related to the notion of joint spectral radius, or better of joint spectral subradius. Joint spectral radius and joint spectral subradius are notions that generalize the notion of spectral radius to the case of a finite set of matrices. More precisely, given a finite set \( E \) of \( n \times n \) matrices, the spectral radius (resp. spectral subradius) is defined as

\[
\lim_{h \to +\infty} \max \{ ||A_{i_1} \cdots A_{i_h}|| : A_{i_1}, \ldots, A_{i_h} \in E \},
\]

(resp. \( \lim_{h \to +\infty} \min \{ ||A_{i_1} \cdots A_{i_h}|| : A_{i_1}, \ldots, A_{i_h} \in E \} \)).

A text-book on joint spectral radius and joint spectral subradius is [2].

Section 5 is devoted to prove the condition given here (Theorem 5.7). The condition is strictly related to the statement that such spectral objects are greater than or equal to 1. However, it is not equivalent. Namely, we require that a set \( E \) of matrices with nonnegative entries satisfies

\[
||A_{i_1} \cdots A_{i_h}(e_j)|| \geq c
\]

where \( c \) is a positive constant independent of \( h \), of the matrices \( A_{i_1}, \ldots, A_{i_h} \in E \) and of the vector \( e_j \) of the canonical basis. Note that usually, at least to my knowledge, it is difficult to evaluate joint spectral radius and joint spectral subradius of a finite set of matrices. So, I expect that in the general case an explicit and effective condition for a polyratio being metric on the given fractal could be hard to find. However, this can be done for some specific fractals. In Section 6, I give explicit necessary and sufficient conditions on the Gasket and on the Vicsek set. I expect that similar explicit conditions can be given for fractals having a simple structure, and for more complicated fractals, if the factors \( \alpha_i \) have some good symmetry properties.

2. Fractals

In this Section, we describe the construction of a fractal, following more or less the approach of [5]. The results of this section are standard
For every follows that $w$.

In fact, $w$ and the inclusion $\subseteq$.

If $w \in W^*$, let $|w^*| := m$ if $w^* \in W_m$, and we say that $w$ is a word and $m$ is the length of $w$. We equip $W$ with the product topology $\{1, ..., k\}^N$, where on $\{1, ..., k\}$ we put the discrete distance. Note that the unique word in $W_0$ is the empty word $\emptyset$.

Let $\sigma_i : W \to W$ be defined by $\sigma_i(w) = iw$, where if $w = (i_1, i_2, i_3, ...,)$, we set $iw = (i, i_1, i_2, i_3, ...)$. If $w \in W$, or $w \in W_m$ with $m \geq \overline{m}$, $w = (i_1, i_2, i_3, ...)$, let $w_{|(m)} \in W_{\overline{m}}$ be defined by $w_{|(m)} = (i_1, ..., i_m)$. If $w^* \in W^*$ and $m \leq |w^*|$, we say that $w_{|(m)}$ is a segment of $w^*$. We say that two words $w^*$ and $w^{*'}$ are incomparable if neither $w^*$ is a segment of $w^{*'}$ nor $w^{*'}$ is a segment of $w^*$. Denote by $i(m)$ the element of $W_m$ of the form $(i, ..., i)$ where $i$ is repeated $m$ times, for $m \in \mathbb{N} \cup \{\infty\}$.

Let $(K, \mathcal{D})$ be a compact metric space. We say that $K$ is a self-similar fractal if there exists a continuous map $\pi$ from $W$ onto $K$ and continuous one-to-one maps $\psi_i, i = 1, ..., k$ from $K$ into itself such that

$$\psi_i \circ \pi = \pi \circ \sigma_i \quad \forall i = 1, ..., k. \tag{2.1}$$

If $w^* = (i_1, ..., i_m)$, let $\sigma_{w^*} = \sigma_{i_1} \circ \cdots \circ \sigma_{i_m}$, $\psi_{w^*} = \psi_{i_1} \circ \cdots \circ \psi_{i_m}$. It follows that

$$\psi_{w^*} \circ \pi = \pi \circ \sigma_{w^*} \quad \forall w^* \in W^*. \tag{2.2}$$

For every $w \in W$ we have

$$\{\pi(w)\} = \bigcap_{m=1}^{\infty} \psi_{w_{|(m)}}(K). \tag{2.3}$$

In fact, $w = \sigma_{w_{|(m)}}(w')$ for some $w' \in W$, then, for every $m = 1, 2, ...$ we have:

$$\pi(w) = \pi(\sigma_{w_{|(m)}}(w')) = \psi_{w_{|(m)}}(\pi(w')) \in \psi_{w_{|(m)}}(K)$$

and the inclusion $\subseteq$ in (2.3) is proved. On the other hand, if $y \in \bigcap_{m=1}^{\infty} \psi_{w_{|(m)}}(K)$, then for every $m = 1, 2, ...$, there exists $w'_m \in W$ such that $y = \psi_{w_{|(m)}}(\pi(w'_m)) = \pi(\sigma_{w_{|(m)}}(w'_m))$. Since, by the definition of the topology on $W$, we have $\sigma_{w_{|(m)}}(w'_m) \xrightarrow{m \to +\infty} w$, in view of the continuity of $\pi$, we have $y = \pi(w)$, and (2.3) is completely proved.

Let now

$$E_w = \psi_w(E) \quad \forall w \in W^*, E \subseteq K.$$
By (2.1) we have
\[ \bigcup_{i=1}^{k} K_i = \bigcup_{i=1}^{k} \psi_i(\pi(W)) = \bigcup_{i=1}^{k} \pi(\sigma_i(W)) = \pi(\bigcup_{i=1}^{k} \sigma_i(W)) = \pi(W) = K. \]

More generally, for every \( m = 1, 2, 3, \ldots \) we have
\[ K = \bigcup_{w \in W_m} K_w. \]  

(2.4)

As a consequence, we have
\[ K_{i_1, \ldots, i_m} \supseteq K_{i_1, \ldots, i_m - 1} \quad \forall i_1, \ldots, i_m = 1, \ldots, k. \]  

(2.4')

For the following we will require additional properties on the fractal. We require that the fractal is a P.C.F. self-similar set with a little additional property similar to that required in [7]. Suppose \( \psi_i \) has a unique fixed point which we denote by \( P_i \), and let \( \hat{V} = \{ P_i, i = 1, \ldots, k \} \). Assume there exists a subset \( V = V^{(0)} = \{ P_1, \ldots, P_N \} \) of \( \hat{V} \), of \( N \) elements with \( 2 \leq N \leq k \).

The sets of the form \( V_{i_1, \ldots, i_m} \) will be called \( m \)-cells, and the sets of the form \( K_{i_1, \ldots, i_m} \) will be called \( m \)-copies. We will require that the intersection of two different \( m \)-cells amounts to the intersection of the corresponding \( m \)-copies. The Sierpinski Carpet is a fractal that does not fill such a requirement. More precisely, suppose that, when \( w^*, w'^* \in W^* \), \( w^* \neq w'^* \), \( |w^*| = |w'^*| \), then
\[ K_{w^*} \cap K_{w'^*} = V_{w^*} \cap V_{w'^*}, \]  

(2.7)

\[ \#(K_{w^*} \cap K_{w'^*}) \leq 1. \]  

(2.7')

Requirement (2.7) is more or less the finite ramification property. Requirement (2.7') is possibly not strictly necessary, but simplifies many arguments and is satisfied by almost all the finitely ramified fractals considered in the literature. Moreover, we require

If \( i = 1, \ldots, k, \ j, h = 1, \ldots, N \) and \( \psi_i(P_j) = P_h \) then \( i = j = h \).

(2.8)

Note that (2.8) in particular implies that \( P_j \neq P_{j'} \) if \( j \neq j' \). Let
\[ \hat{J} := \{ (j_1, j_2) : j_1, j_2 = 1, \ldots, N, j_1 \neq j_2 \}, \]

and note that \( \#(\hat{J}) = \hat{M} := N(N - 1) \). Let
\[ V^m = \bigcup_{i_1, \ldots, i_m=1}^k V_{i_1 \ldots i_m}, \quad V^{(\infty)} = \bigcup_{m=0}^\infty V^m. \]

Note that \( V^m \subseteq V^{m+1} \) for every positive integer \( m \). I now prove some lemmas useful in the sequel.

**Lemma 2.1.** For every \( w^* \in W^* \), the map \( \psi_{w^*} \) sends \( K \setminus V^{(0)} \) into itself. More precisely, if \( w^* \in W^m \), \( m > 0 \), and \( \psi_{w^*}(Q) = P_j \in V^{(0)} \), \( Q \in K \), then \( Q = P_j \) and \( w^* = j^{(m)} \).

**Proof.** Let \( Q \in K \) and suppose and \( \psi_{w^*}(Q) = P_j \in V^{(0)} \). Then \( w^* = j^{(m)} \) and consequently \( Q = P_j \). In fact in the opposite case, as \( P_j = \psi_{j^{(m)}}(P_j) \), by (2.7) we have \( Q \in V^{(0)} \), and by (2.8) and an inductive argument we have \( w^* = j^{(m)} \), a contradiction.

Note that (2.7) holds under the hypothesis \( |w^*| = |w^*| \), and that (2.7) is no longer valid if \( w^*, w^{\prime} \) are two arbitrary different words. In fact, if \( w^{\prime} \) is a segment of \( w^* \), then \( K_{w^*} \cap K_{w^{\prime}} = K_{w^*} \) by (2.4). However, as we see now, this is the only case in which (2.7) does not hold.

**Lemma 2.2.** If \( w^* \) and \( w^{\prime} \) are two incomparable words, then \( K_{w^*} \cap K_{w^{\prime}} = V_{w^*} \cap V_{w^{\prime}} \).

**Proof.** Let \( w^* = (i_1, \ldots, i_m) \), \( w^{\prime} = (i_1', \ldots, i_m') \), and, since the case \( m = m' \) follows at once from (2.7), we can and do assume \( m' > m \).

As we have also assumed that \( w^* \) and \( w^{\prime} \) are incomparable, we have \( (i_1, \ldots, i_m) \neq (i_1', \ldots, i_m') \). Thus, if \( Q \in K_{w^*} \cap K_{w^{\prime}} \), then

\[ Q \in K_{i_1 \ldots i_m} \cap K_{i_1' \ldots i_m'} = V_{i_1 \ldots i_m} \cap V_{i_1' \ldots i_m'}. \]

It remains to prove that \( Q \in V_{i_1 \ldots i_m} \cap V_{i_1' \ldots i_m'} \). To see this, note that there exist \( Q' \in K \) and \( P_j \in V^{(0)} \) such that

\[ Q = \psi_{i_1 \ldots i_m}(\psi_{i_{m+1}' \ldots i_m'}(Q')) = \psi_{i_1' \ldots i_m'}(P_j). \]

Hence, by Lemma 2.1, we have \( Q' = P_j \) (and \( i_{m+1}' = \cdots = i_m' = j \)). Thus \( Q \in V_{i_1 \ldots i_m'} \).

**Lemma 2.3.** For every \( m' \geq m \) we have

\[ K_{i_1 \ldots i_m} \cap V^{(m)} \subseteq V_{i_1 \ldots i_m} \]

and hence \( K_{i_1 \ldots i_m} \cap V^{(m)} \) has at most one element if \( m' > m \).

**Proof.** Let \( Q \in K_{i_1 \ldots i_m} \cap V^{(m)} \) and let \( i_1', \ldots, i_m' \) be such that \( Q \in V_{i_1' \ldots i_m'} \). Then, if \( (i_1, \ldots, i_m) = (i_1', \ldots, i_m') \) we have \( Q \in V_{i_1 \ldots i_m} \). If, on the contrary, \( (i_1, \ldots, i_m) \neq (i_1', \ldots, i_m') \), we have

\[ Q \in K_{i_1 \ldots i_m} \cap K_{i_1' \ldots i_m'} \subseteq V_{i_1 \ldots i_m}, \]
that for every $Q, Q' \in V^m$ has the form $\psi_{i_1, \ldots, i_m}(Q) = \psi_{i_1, \ldots, i_m}(P_j)$ with $P_j \in V^0$, $Q \in K$, we have $\psi_{i_{m+1}, \ldots, i_m}(Q) = P_j$ thus by Lemma 2.1, $j = i_{m+1} = \cdots = i_m$, $Q = P_j = P_{i_{m+1}}$. □

Corollary 2.4. The set $V_w \cap V_{w'}$ has at most one point, whenever $w, w' \in W^*$, $w \neq w'$.

Proof. Let $m = |w|$, $m' = |w'|$. If $m = m'$ this follows from (2.7'). If for example $m' > m$, let $w = (i_1, \ldots, i_m)$, $w' = (i'_1, \ldots, i'_{m'})$. Then $V_w \cap V_{w'} \subseteq V_{i_1, \ldots, i_m} \cap V^{(m)}$ and we conclude by Lemma 2.3. □

Corollary 2.5. We have $\psi_i(V^{(m+1)} \setminus V^{(m)}) \subseteq V^{(m+2)} \setminus V^{(m+1)}$ for every $m \in \mathbb{N}$ and every $i = 1, \ldots, k$.

Proof. Let $Q \in V^{(m+1)} \setminus V^{(m)}$, namely $Q = \psi_{i_1, \ldots, i_{m+1}}(P_j)$. Then, clearly, $\psi_i(Q) \in V^{(m+2)}$. Also, if $\psi_i(Q) \in V^{(m+1)}$, then $\psi_i(Q) \in K_{i_1, \ldots, i_m} \cap V^{(m+1)} \subseteq V_{i_1, \ldots, i_m}$, by Lemma 2.3, and $\psi_i(Q) = \psi_i(\psi_{i_1, \ldots, i_m}(P_{j'}))$ for some $j' = 1, \ldots, N$, hence $Q = \psi_{i_1, \ldots, i_m}(P_{j'}) \in V^{(m)}$, a contradiction. □

In the sequel we will often use with no mention the following simple consequence of (2.7).

Lemma 2.6. i) $V^{(\infty)} \cap K_{w^*} = V_{w^*} \quad \forall w^* \in W^*$.

ii) $V_{w^*}^{(\infty)}$ is dense in $K_{w^*}$ for every $w^* \in W^*$.

Proof. i) To prove the $\subseteq$ part, note that if $Q \in V^{(\infty)} \cap K_{w^*}$ then $Q \in V_{w^*}^{(\infty)}$ for some $w^* \in W^*$ with $|w^*| = |w^*|$, thus, if $w^* \neq w^*$ by (2.7) we have $Q \in V_{w^*} \subseteq V_{w^*}^{(\infty)}$. The $\supseteq$ part is trivial.

ii) Let $Q \in K_{w^*}$. By (2.2), $Q = \pi(\sigma_{w^*}(w'))$ for some $w' \in W$. Next, note that

$$\sigma_{w^*} \circ \sigma_{w'(m)}(1^{(\infty)}) \overrightarrow{m \to +\infty} \sigma_{w^*}(w')$$

(2.10)

In fact, putting $\overline{m} = |w^*|$, we have

$$\left(\sigma_{w^*} \circ \sigma_{w'(m)}(1^{(\infty)})\right)(\overline{m}+m) = \left(\sigma_{w^*}(w')\right)(\overline{m}+m),$$

hence (2.10) follows from the definition of the topology on $W$. It follows from (2.10) that

$$\pi(\sigma_{w^*} \circ \sigma_{w'(m)}(1^{(\infty)})) \overrightarrow{m \to +\infty} \pi(\sigma_{w^*}(w')) = Q.$$ 

On the other hand,

$$\pi(\sigma_{w^*} \circ \sigma_{w'(m)}(1^{(\infty)})) = \psi_{w^*} \circ \psi_{w'(m)}(\pi(1^{(\infty)})),$$

(2.11)

and since $\psi_i(P_1) = P_1$, we have $\psi_{i(m)}(P_1) = P_1$, hence by (2.3), $\pi(1^{(\infty)}) = P_1$, thus, in view of (2.11), $\pi(\sigma_{w^*} \circ \sigma_{w'(m)}(1^{(\infty)})) \in V_{w^*}^{(\infty)}$. □

Finally, we require that the fractal is connected. By this we mean that for every $Q, Q' \in V^{(1)}$ there exist $Q_0, \ldots, Q_n \in V^{(1)}$ such that $Q_0 = Q$, $Q_n = Q'$, and for every $h = 1, \ldots, n$ there exists $i(h) = 1, \ldots, k$.
such that $Q_{h-1}, Q_h \in V_{i(h)}$. Note that for example the Cantor set is not connected. From now on, all fractals are meant to have all the properties required in this Section.

I now introduce the problems discussed in this paper. Following Introduction, we say that $\alpha := (\alpha_1, ..., \alpha_k)$ is a polyratio if $\alpha_i \in [0, 1]$ for every $i = 1, ..., k$. Here we put $\alpha_w := \alpha_{i_1} \cdots \alpha_{i_m}$, $\alpha_0 = 1$. Put $\alpha_{\min} = \min \{\alpha_i\}$, $\alpha_{\max} = \max \{\alpha_i\}$. Given a polyratio $\alpha$ as above we say that a distance $d$ on $K$ is $\alpha$-self-similar if the maps $\psi_i$, $i = 1, ..., k$, are $\alpha_i$-Lipschitz. If an $\alpha$-self-similar distance exists on $K$ we say that $\alpha$ is a metric polyratio (on $K$). We say that a distance $d$ is an $\alpha$-scaling distance if there exist positive constants $c_{1, \alpha}, c_{2, \alpha}$ such that for every $i_1, ..., i_m = 1, ..., k$ (1.3) holds. We say that $\alpha$ is an asymptotic metric polyratio (on $K$) or shortly as. metric polyratio if there exists an $\alpha$-scaling distance on $K$. The problems discussed in this paper are what polyratios are metric and what polyratios are as. metric. We will treat such problems in Section 5. Sections 3 and 4 are devoted to introduce preparatory notions.

3. Graphs on the Fractal

In this Section we first define a suitable graph on $V^{(\infty)}$, and then, based on it, the notion of a path on $V^{(\infty)}$. We define a graph on $V^{(\infty)}$ putting $Q \sim Q'$ for $Q, Q' \in V^{(\infty)}$ if $Q \neq Q'$ and there exist $P, P' \in V^{(0)}$ and $w^* \in W^*$ such that $Q = \psi_{w^*}(P)$, $Q' = \psi_{w^*}(P')$. Put also $Q \simeq Q'$ if either $Q \sim Q'$ or $Q = Q'$. Given $i = (j_1, j_2) \in \tilde{J}$, let

$$P_i := (P_{j_1}, P_{j_2}), \quad \psi_{w^*}(P_i) = (\psi_{w^*}(P_{j_1}), \psi_{w^*}(P_{j_2})).$$

Let $\tilde{Y}$ be the set of $(Q, Q') \in V^{(\infty)} \times V^{(\infty)}$ such that $Q \sim Q'$. The following function will be useful in the sequel: Let $\alpha_{\overline{w}} : \tilde{Y} \to \mathbb{R}$ be defined as

$$\alpha_{\overline{w}}(\psi_{w^*}(P), \psi_{w^*}(P')) = \alpha_{w^*}, \quad \forall P, P' \in V^{(0)} \forall w^* \in W^*.$$

Note that, in view of (2.7), such a definition is correct, i.e., the pair $(Q, Q')$ with $Q, Q' \in V^{(\infty)}$, $Q \neq Q'$ can be represented uniquely as $(\psi_{w}(P), \psi_{w}(P'))$. The following lemma is a consequence of the assumption on the fractal (in particular of (2.7)).

Lemma 3.1. If $Q, Q' \in V^{(\infty)}$, $Q \sim Q'$, and $Q \in K_{i_1, ..., i_m}$, $Q' \notin K_{i_1, ..., i_m}$, then $Q \in V_{i_1, ..., i_m}$.

Proof. We have $Q, Q' \in V_{i_1, ..., i_m^\prime}$ for some $i_1^\prime$, ..., $i_m^\prime$. Let $h = \min\{l, m\}$. If

$$(i_1^\prime, ..., i_h) = (i_1, ..., i_h),$$

then

$$Q' \in K_{i_1, ..., i_h} \setminus K_{i_1, ..., i_m}.$$
thus $m > h = l$ and
\[
Q \in V_{i_1,...,i_l} \cap K_{i_1,...,i_m} \subseteq V_{i_1,...,i_m},
\]
where the inclusion holds since, if
\[
Q = \psi_{i_1,...,i_l} \circ \psi_{i_{l+1},...,i_m}(\tilde{Q}) = \psi_{i_1,...,i_l}(P)
\]
with $\tilde{Q} \in K$, $P \in V^{(0)}$, we have $\psi_{i_{l+1},...,i_m}(\tilde{Q}) = P$ thus by Lemma 2.1, $\tilde{Q} \in V^{(0)}$.

If instead $(i'_1, ..., i'_h) \neq (i_1, ..., i_h)$, then either $l \geq m$ and $Q \in K_{i_1,...,i_m} \cap K_{i'_1,...,i'_m}$ or $l < m$ and $Q \in K_{i'_1,...,i'_m}$, thus, by (2.4), $Q \in K_{i'_1,...,i'_m}$ for some $i'_{l+1}, ..., i'_m$. In both cases $(i_1, ..., i_m) \neq (i'_1, ..., i'_m)$, thus in view of (2.7), $Q \in V_{i_1,...,i_m}$. □

A $V^{(\infty)}$-path (or simply path) $\Pi$ is a sequence of the form
\[
(Q_0, ..., Q_n) = (Q_{0,\Pi}, ..., Q_{n,\Pi})
\]
such that $Q_h \in V^{(\infty)}$ for every $h = 0, ..., n$ and, moreover, $Q_{h-1} \sim Q_h$ for every $h = 1, ..., n$. Thus, there exist $\tilde{w}(h, \Pi) \in W^*$, $\tilde{v}(h, \Pi) \in \tilde{J}$ such that
\[
(Q_{h-1,\Pi}, Q_h, \Pi) = \psi_{\tilde{w}(h, \Pi)}(P_{\tilde{v}(h, \Pi)}).
\]
More generally we say that a weak path is a sequence $(Q_0, ..., Q_n)$ such that $Q_h \in V^{(\infty)}$ for every $h = 0, ..., n$ and, moreover, $Q_{h-1} \simeq Q_h$ for every $h = 1, ..., n$.

Here, we say that $Q_{h,\Pi}$, $h = 0, ..., n(\Pi)$, are the vertices of $\Pi$, and that the pairs $(Q_{h-1,\Pi}, Q_h, \Pi)$, $h = 1, ..., n(\Pi)$, are the edges of $\Pi$. We say that $n(\Pi) + 1$ is the length of $\Pi$.

An $E$-path $\Pi$ is a path whose elements belong to $E$ whenever $E \subseteq V^{(\infty)}$. In particular we will use the terms $V^{(1)}$-paths, $V^{(0)}$-paths. So, $\Pi$ is a $V^{(1)}$-path if for every $h = 1, ..., n(\Pi)$, $Q_{h,\Pi} \neq Q_{h-1,\Pi}$ and either $Q_{h,\Pi}, Q_{h-1,\Pi}$ lie in a common 1-cell, or both lie in $V^{(0)}$ (see Lemma 2.3).

A path $\Pi$ is strong if $\tilde{w}(h, \Pi) \neq \emptyset$ for every $h = 1, ..., n(\Pi)$, in other words if two consecutive vertices of the path are not both in $V^{(0)}$. We say that $\Pi$ is a $\iota$-path if $P_\iota = (Q_{0,\Pi}, Q_{n(\Pi),\Pi})$. If $E \subseteq V^{(\infty)}$, we say that $\Pi$ is a $(\iota, E)$-path if it is both a $E$-path and a $\iota$-path. We say that a path is a strong $\iota$-path if it is both a strong path and $\iota$-path. We say that $\Pi$ connects $Q$ to $Q'$ if $Q_{0,\Pi} = Q$ and $Q_{n(\Pi),\Pi} = Q'$. We say that $\Pi$ is a strict path if $Q_{h,\Pi} \neq Q_{h',\Pi}$ when $h \neq h'$.

When $\Pi$ is a path and $w^* \in W^*$, we define the path $w^*(\Pi) = (\psi_{w^*}(Q_{0,\Pi}), ..., \psi_{w^*}(Q_{n(\Pi),\Pi}))$. If $\Pi = (Q_0, ..., Q_n)$ and $\Pi' = (Q'_0, ..., Q'_n)$ with $Q_n = Q'_0$ we put
\[
\Pi \circ \Pi' = (Q_0, ..., Q_n, Q'_1, ..., Q'_n).
A subpath of a path $\Pi$ is a path of the form

$$\Pi' := (Q_{n_1, \Pi}, Q_{n_1+1, \Pi}, \ldots, Q_{n_2-1, \Pi}, Q_{n_2, \Pi})$$

with $0 \leq n_1 \leq n_2 \leq n(\Pi)$, and put $n_1 =: n_1(\Pi, \Pi')$, $n_2 =: n_2(\Pi, \Pi')$.

Note the following simple properties of the graphs.

**Lemma 3.2.**

i) If $Q, Q' \in K$, $w^* \in W^*$, then $Q \sim Q'$ if and only if $\psi_w^*(Q) \sim \psi_{w^*}(Q')$.

ii) If $w^* \in W^*$, then the sequence $(Q_0, \ldots, Q_n)$ is a path if and only if $(\psi_{w^*}(Q_0), \ldots, \psi_{w^*}(Q_n))$ is a path.

**Proof.**

i) The $\Rightarrow$ part is trivial. Prove $\Leftarrow$. Suppose $\psi_w^*(Q) \sim \psi_{w^*}(Q')$, so that there exist $w_1^* \in W^*$ and $P_j, P'_j \in V(0)$ such that $\psi_w^*(Q) = \psi_{w_1^*}(P_j)$, $\psi_{w^*}(Q') = \psi_{w^*}(P'_j)$. Let $w_1^* := (i_1, \ldots, i_m)$, $w^* := (i'_1, \ldots, i'_{m'})$. Let $\tilde{Q} = \psi_w^*(Q)$, $\tilde{Q}' = \psi_{w^*}(Q')$. We have $\tilde{Q} \neq \tilde{Q}'$ and

$$\tilde{Q}, \tilde{Q}' \in K_{i_1', \ldots, i'_m} \cap K_{i_1, \ldots, i_m} \cap V(m),$$

thus, on one hand, by Lemma 2.3 we have $m' \leq m$, on the other by (2.7) and (2.4), $(i_1, \ldots, i_m) = (i'_1, \ldots, i'_{m'})$. It follows that

$$\psi_{i_1', \ldots, i'_m}(Q) = \psi_{i_1', \ldots, i'_m}(\psi_{i_{m'+1}, \ldots, i_m}(P_j)),$$

$$\psi_{i'_1, \ldots, i'_{m'}}(Q') = \psi_{i'_1, \ldots, i'_{m'}}(\psi_{i_{m'+1}, \ldots, i_m}(P'_j)),$$

and $Q = \psi_{i_{m'+1}, \ldots, i_m}(P_j) \sim \psi_{i_{m'+1}, \ldots, i_m}(P'_j) = Q'$. Thus i) is proved, and ii) is a simple consequence of i). 

In the next lemmas we investigate some properties of the graphs connecting points lying in some specific subsets of $V(\infty)$. In particular, the statement of Lemma 3.3 corresponds to the intuitive idea that $V_{i_1, \ldots, i_m}$ is a sort of boundary of $K_{i_1, \ldots, i_m}$. Hence, a path connecting a point of $K_{i_1, \ldots, i_m}$ to a point not in $K_{i_1, \ldots, i_m}$ necessarily passes through $V_{i_1, \ldots, i_m}$.

**Lemma 3.3.** If $\Pi$ is a path connecting $Q \notin K_{i_1, \ldots, i_m}$ to $Q' \in K_{i_1, \ldots, i_m}$, then there exists $\overline{n} \leq n(\Pi)$ such that $Q_{n, \Pi} \notin K_{i_1, \ldots, i_m}$ for every $n \leq \overline{n}$ and $Q_{\overline{n}+1, \Pi} \in V_{i_1, \ldots, i_m}$.

**Proof.** Let $\overline{n}$ be the maximum $n$ such that

$$Q_{n', \Pi} \notin K_{i_1, \ldots, i_m} \quad \forall n' = 0, \ldots, n.$$

Then $Q_{\overline{n}, \Pi} \notin K_{i_1, \ldots, i_m}$, $Q_{\overline{n}+1, \Pi} \in K_{i_1, \ldots, i_m}$, and $Q_{\overline{n}, \Pi} \sim Q_{\overline{n}+1, \Pi}$. Thus, by Lemma 3.1, $Q_{\overline{n}+1, \Pi} \in V_{i_1, \ldots, i_m}$. 

**Lemma 3.4.** Suppose $\Pi$ is a path connecting $Q \in K_{i_1, \ldots, i_m} \setminus V(m)$ to $Q' \in V(m)$. Then there exists $\overline{n}$ such that $Q_{\overline{n}, \Pi} \in V_{i_1, \ldots, i_m}$ and $Q_{n, \Pi} \in K_{i_1, \ldots, i_m}$ for every $n \leq \overline{n}$.
Proof. The proof is similar to that of Lemma 3.3 but a bit more complicated. We have $Q_{0,\Pi} = Q$, $Q_{n(\Pi),\Pi} = Q'$. Let $\pi$ be the maximum $n \in [0, n(\Pi)]$ such that

$$Q_{n',\Pi} \in K_{i_1,...,i_m} \forall n' = 0, ..., n.$$  

We will prove that $Q_{\pi,\Pi} \in V_{i_1,...,i_m}$. If $\pi = n(\Pi)$, then, in view of Lemma 2.3, $Q_{\pi,\Pi} \in K_{i_1,...,i_m} \cap V^{(m)} \subseteq V_{i_1,...,i_m}$.

If $\pi < n(\Pi)$ then we have $Q_{\pi,\Pi} \sim Q_{\pi+1,\Pi}$ by the definition of a path, and $Q_{\pi,\Pi} \in K_{i_1,...,i_m}$, $Q_{\pi+1,\Pi} \notin K_{i_1,...,i_m}$, by the definition of $\pi$. Thus, we have $Q_{\pi,\Pi} \in V_{i_1,...,i_m}$ by Lemma 3.1. \qed

Note that, given a path $(Q_0,...,Q_n)$ connecting $Q_0$ to $Q_n$, we can associate to it a sort of reverse path connecting $Q_n$ to $Q_0$, that is $(Q_n,...,Q_0)$. Thus, we can use Lemmas 3.3 and 3.4 (and other similar lemmas) also in the reverse direction. This is what we will do in the proof of Lemma 3.5.

**Lemma 3.5.** If $\Pi$ is a path connecting two points in $K_{i_1,...,i_m}$ and $\Pi$ is not entirely contained in $K_{i_1,...,i_m}$, then there exist $l < m$ and a subpath of $\Pi$ of length greater than 2, connecting two points in $K_{i_l,...,i_{l+1}}$, entirely contained in $K_{i_1,...,i_l}$.

**Proof.** Since every point of $\Pi$ lies in $K = K_{01}$, by our hypothesis there exists a natural $l < m$ (possibly 0) such that there exists a point $Q_{\pi,\Pi} \in K_{i_1,...,i_l} \setminus K_{i_1,...,i_{l+1}}$. We can assume $l$ is the minimum natural number satisfying such a property. By Lemma 3.3 there exist $n_1$ and $n_2$ with $n_1 < \pi < n_2$ such that $Q_{n,\Pi} \notin K_{i_1,...,i_{l+1}}$ for every $n = n_1 + 1, ..., n_2 - 1$, and $Q_{n_1,\Pi}, Q_{n_2,\Pi} \in V_{i_1,...,i_{l+1}} \subseteq K_{i_1,...,i_l}$. But by the definition of $l$ we then have $Q_{n,\Pi} \in K_{i_1,...,i_l}$ for every $n = n_1 + 1, ..., n_2 - 1$. \qed

Suppose we are given a path $\Pi$. We now describe a way to construct a longer path by inserting new paths between any pair of consecutive vertices of $\Pi$, and in Lemma 3.6 we will prove that any path connecting two given points $Q$ and $Q'$ can be obtained by repeating this process starting from a path connecting $Q$ and $Q'$ lying in the same level $V^{(m)}$ of the fractal as the one where $Q$ and $Q'$ stay.

More formally, let $\Gamma$ be the set of the functions $\gamma$ from $\tilde{J}$ to the set of the strict paths such that $\gamma(t)$ is a $(t, V^{(1)})$-path. Let $\overline{\Gamma}$ be the set of the functions $\overline{\gamma}$ from $\tilde{J} \times W^*$ to the set of the strict paths such that $\overline{\gamma}(t, w)$ is a $(t, V^{(1)})$-path. For $\overline{\gamma} \in \overline{\Gamma}$, let

$$\overline{D}(\overline{\gamma})(\psi_w(P_{j_1}), \psi_w(P_{j_2})) = \psi_w(\overline{\gamma}(j_1, j_2), w),$$

$$\overline{D}(\overline{\gamma})(Q_0, ..., Q_n) = \overline{D}(\overline{\gamma})(Q_0, Q_1) \circ \cdots \circ \overline{D}(\overline{\gamma})(Q_{n-1}, Q_n),$$

when $\Pi := (Q_0, ..., Q_n)$ is a path. When this happens, clearly, $\overline{D}(\overline{\gamma})(\Pi)$ is a path as well. We will say that $\overline{D}(\overline{\gamma})(\Pi)$ is the $\overline{\gamma}$-insertion of $\Pi$. 


Moreover, note that if \( Q_{0,\Pi} = Q_{0,\overline{D}(\gamma)(\Pi)}, Q_{n,\Pi} = Q_{n,\overline{D}(\gamma)(\Pi)} \) in other words, \( \Pi \) and \( \overline{D}(\gamma)(\Pi) \) have the same end-points.

**Lemma 3.6.** If \( \Pi' \) is a strict path connecting two points \( Q \) and \( Q' \) of \( V^{(m)} \), then there exist a strict path \( \Pi \) lying in \( V^{(m)} \) connecting \( Q \) and \( Q' \) and \( \gamma_1, ..., \gamma_r \in \Gamma \) such that

\[
\Pi' = \overline{D}(\gamma_1) \circ \cdots \circ \overline{D}(\gamma_r)(\Pi).
\]

**Proof.** We proceed by induction on the number of points \( s(\Pi') \) of \( \Pi' \) not lying in \( V^{(m)} \). The lemma is trivial if \( s(\Pi') = 0 \). Suppose \( s(\Pi') > 0 \) and let \( Q_{\gamma,\Pi'} \notin V^{(m)} \). Thus \( Q_{\gamma,\Pi'} \in V^{(m)} \setminus V^{(m-1)} \) for some \( m > m \). We can assume that \( m \) is the maximum index with such a property, in other words

\[
Q_{n,\Pi'} \in V^{(m)} \quad \forall n = 0, ..., n(\Pi').
\] (3.1)

Since \( Q, Q' \in V^{(m)} \subseteq V^{(m-1)} \), we have \( 0 < m < n(\Pi') \). Moreover, there exist \( \gamma_1, ..., \gamma_{m-1} \) such that \( Q_{\gamma,\Pi'} \in K_{\gamma_1, ..., \gamma_{m-1}} \), thus \( Q_{\gamma,\Pi'} \in K_{\gamma_1, ..., \gamma_{m-1}} \setminus V^{(m-1)} \). By Lemma 3.4 there exist \( n_1, n_2 \) such that \( n_1 < m < n_2 \), and

\[
Q_{n_1,\Pi'}, Q_{n_2,\Pi'} \in V_{\gamma_1, ..., \gamma_{m-1}}
\]

and

\[
Q_{n,\Pi'} \in K_{\gamma_1, ..., \gamma_{m-1}} \quad \text{if} \quad n_1 \leq n \leq n_2.
\] (3.2)

Let

\[
\Pi'' = (Q_{0,\Pi'}, ..., Q_{n_1,\Pi'}, Q_{n_2,\Pi'}, ..., Q_{n(\Pi'),\Pi'}).
\]

Then, \( \Pi'' \) is a strict path and \( s(\Pi'') < s(\Pi') \). By the inductive hypothesis there exist a strict path \( \Pi \) lying in \( V^{(m)} \) connecting \( Q \) and \( Q' \) and \( \gamma_1, ..., \gamma_r \in \Gamma \) such that

\[
\Pi'' = \overline{D}(\gamma_1) \circ \cdots \circ \overline{D}(\gamma_r)(\Pi).
\]

By (3.1) and (3.2) and (2.4'), for every \( n \) such that \( n_1 \leq n \leq n_2 \) there exists \( \gamma_{m-1} = 1, ..., k \) (depending on \( n \)) such that we have

\[
Q_{n,\Pi'} \in K_{\gamma_1, ..., \gamma_{m-1}} \cap V^{(m)} \subseteq V_{\gamma_1, ..., \gamma_{m-1}} \subseteq V^{(1)}_{\gamma_1, ..., \gamma_{m-1}}
\]

where the first inclusion follows from Lemma 2.3. Hence, in view of Lemma 3.2 ii),

\[
\overline{\Pi} := (\psi_{\gamma_1, ..., \gamma_{m-1}}^{-1}(Q_{n_1,\Pi'}), ..., \psi_{\gamma_1, ..., \gamma_{m-1}}^{-1}(Q_{n_2,\Pi'}))
\]

is a strict \((\gamma, V^{(1)})\)-path for some \( \gamma \in \widehat{J} \). Therefore, \( (Q_{n_1,\Pi'}, Q_{n_2,\Pi'}) = (\psi_{\gamma_1, ..., \gamma_{m-1}}^{-1}(P_2)) \), and let \( \gamma \in \Gamma \) be defined by

\[
\gamma(t, w^*) = \begin{cases} 
\overline{\Pi} & \text{if } (t, w^*) = (\gamma_1, ..., \gamma_{m-1}) \\
P_t & \text{otherwise}. 
\end{cases}
\]
Note that
\[
D(\gamma)(Q_{n_1,\Pi^v}, Q_{n_2,\Pi^v}) = D(\gamma)\left(\psi_{i_1, \ldots, i_{m-1}}(P_i)\right) = \\
\psi_{i_1, \ldots, i_{m-1}}(\Pi) = \left(Q_{n_1,\Pi^v}, \ldots, Q_{n_2,\Pi^v}\right)
\]
so that
\[
D(\gamma)(Q_{h,\Pi^v}, Q_{h+1,\Pi^v}) = \\
\begin{cases} 
\tau(Q_{n_1,\Pi^v}, Q_{n_2,\Pi^v}) = (Q_{n_1,\Pi^v}, \ldots, Q_{n_2,\Pi^v}) & \text{if } h = n_1 \\
(Q_{h,\Pi^v}, Q_{h+1,\Pi^v}) & \text{otherwise}
\end{cases}
\]
and, putting \(Q_n\) short for \(Q_{n,\Pi^v}\), we have
\[
D(\tau)(\Pi^v) = \\
D(\tau)(Q_0, Q_1) \circ \cdots \circ D(\tau)(Q_{n_1, Q_{n_2}}) \circ \cdots \circ D(\tau)(Q_{n_1, \Pi^v - 1}, Q_{n_2, \Pi^v}) = (Q_{n_1, \Pi^v}, \ldots, Q_{n_2, \Pi^v}) = \Pi^v,
\]
so that
\[
D(\tau) \circ D(\tau_1) \circ \cdots \circ D(\tau_r) = \Pi^v.
\]

4. Linear operators related to the paths

In this Section we associate some special linear operators (or equivalently matrices) to every element of \(\Gamma\) or \(\Gamma\). Basically, we interpret the edges of the paths as the basis of a linear space. Recall that we have defined the function \(\alpha_{w}\) on \(\tilde{Y}\) with real values. Then, we associate to the given path the vector having as the \((Q, Q')\)-component the value \(\alpha_{w}(Q, Q')\). More precisely, let \(e_i\) be the elements of the canonical basis in \(\mathbb{R}^\hat{J}\), that is \((e_i)_{i' \in \hat{J}} = \delta_{i, i'}\). For every \(\gamma \in \Gamma\) we define a linear operator \(T^{(\gamma)}_{\alpha}\) from \(\mathbb{R}^\hat{J}\) to \(\mathbb{R}^\hat{J}\) by
\[
T^{(\gamma)}_{\alpha}(e_i) = \sum_{i' \in \hat{J}} \tilde{\alpha}_{\gamma, i, i'} e_{i'},
\]
or in other words
\[
T^{(\gamma)}_{\alpha}(e_i) = \sum_{h: \tilde{\alpha}(h, \gamma(i)) = i'} \alpha_{\tilde{\alpha}(h, \gamma(i))}.\]
Now, we introduce similar notions related to \(\tilde{\Gamma}\) instead of \(\Gamma\). The reason for which we introduce these notions both in \(\Gamma\) and in \(\tilde{\Gamma}\) is that on one hand we have a closer relationship between \(\mathcal{D}(\tau)\) and the linear operators related to \(\Gamma\), on the other the linear operator related to \(\tilde{\Gamma}\) are simpler to handle. However, for our purposes the two notions are
equivalent, more precisely, the asymptotic behaviors of the composition of linear operators are in some sense the same in both situations (Corollary 4.4).

Let \( e_{i,w^*} \) be the elements of the canonical basis of \( \mathcal{Z} \) where \( \mathcal{Z} \) is the set of elements \( z \) of \( \mathbb{R}^{\hat{J} \times W^*} \) such that \( z_{i,w^*} = 0 \) for almost all \( (i,w^*) \) (i.e., for all \( (i,w^*) \) but finitely many). That is, \( (e_{i,w^*})_{i',w'^*} = \delta_{(i,w^*),(i',w'^*)} \). Let \( \mathcal{Z}_m \) be the subset of \( \mathcal{Z} \) of the elements \( z \) of \( \mathbb{R}^{\hat{J} \times W^*} \) such that \( z_{i,w^*} = 0 \) for every \( w^* \notin \hat{W}_m \). We also set \( e_i := e_{i,\emptyset} \). For every \( \gamma \in \Gamma \) we define a linear operator \( T^{(\alpha)}_\gamma \) from \( \mathcal{Z} \) into itself by

\[
T^{(\alpha)}_\gamma (e_{i,w^*}) = \sum_{h=1}^{n(\gamma)} \alpha(\gamma)_{i,w^*}(h) e_{i',w'^*}(h) \epsilon_{i',w'^*}(h,\gamma),
\]

Note that for every \( m \in \mathbb{N} \), \( T^{(\alpha)}_\gamma \) maps \( \mathcal{Z}_m \) into \( \mathcal{Z}_{m+1} \). Let \( H : \mathbb{R}^\hat{J} \rightarrow \mathbb{R} \) be defined by

\[
H(x) = \sum_{i \in \hat{J}} x_i, \quad \overline{H}(x) = \sum_{(i,w^*) \in \hat{J} \times W^*} x_{i,w^*}.
\]

Let \( f : \hat{Y} \rightarrow \mathbb{R} \). We now define the sum operators along a path, that is, some kind of sum of \( f \) along a given path. We will use such notions in the sequel specially when \( f = \alpha_{\hat{Y}} \), but also other \( f \) will be considered in Section 5. Namely, if \( \Pi \) is a path, we define

\[
\mathcal{\Sigma}_{\Pi}^1(f) = \sum_{h=1}^{n(\Pi)} f(Q_{h-1,\Pi}, Q_{h,\Pi}) e_{i,h,\Pi}, \overline{\mathcal{\Sigma}}_{\Pi}^1(f) \in \mathcal{Z},
\]

\[
\mathcal{\Sigma}_{\Pi}^{\hat{J}}(f) = \sum_{h=1}^{n(\Pi)} f(Q_{h-1,\Pi}, Q_{h,\Pi}) e_{i,h,\Pi} \in \mathbb{R}^\hat{J},
\]

\[
\Sigma_{\Pi}(f) = \sum_{h=1}^{n(\Pi)} f(Q_{h-1,\Pi}, Q_{h,\Pi}).
\]

We will occasionally use such definitions (more specifically, the definition of \( \Sigma_{\Pi}(f) \)) also when \( \Pi \) is a weak path, with the convention \( f(Q, Q) = 0 \). We will use the following simple properties in the sequel without mention for paths \( \Pi, \Pi' \):

\[
\Sigma_{\Pi}(f) = \overline{H}(\mathcal{\Sigma}_{\Pi}^1(f)) = H(\mathcal{\Sigma}_{\Pi}^{\hat{J}}(f)).
\]

\[
\mathcal{\Sigma}_{\Pi \circ \Pi'}^1(f) = \mathcal{\Sigma}_{\Pi}^1(f) + \mathcal{\Sigma}_{\Pi'}^1(f), \quad \mathcal{\Sigma}_{\Pi \circ \Pi'}^{\hat{J}}(f) = \mathcal{\Sigma}_{\Pi}^{\hat{J}}(f) + \mathcal{\Sigma}_{\Pi'}^{\hat{J}}(f).
\]

Recall that, given \( \iota = (j_1,j_2) \in \hat{J} \), the symbol \( P_{\iota} \) denotes the pair \( (P_{j_1}, P_{j_2}) \), which, of course can be interpreted as a path of length 1. We immediately have:
\[ \tilde{\Sigma}_{P_i}(\alpha_{\mathcal{W}}) = e_i, \quad \tilde{\Sigma}_{P_i}(\alpha_{\mathcal{W}}) = e_i, \quad (4.1) \]

**Remark 4.1.** When \( \Pi \) is a path, the following formulas are immediate consequences of the definition of \( \tilde{w}(h, \Pi) \).

\[ \alpha_{\mathcal{W}}(Q_{h-1, \Pi}, Q_{h, \Pi}) = \alpha_{\tilde{w}(h, \Pi)}, \quad (4.2) \]
\[ \Sigma_{\Pi}(\alpha_{\mathcal{W}}) = \sum_{h=1}^{n(\Pi)} \alpha_{\tilde{w}(h, \Pi)}. \quad (4.2') \]

Note that (4.2) and (4.2') are also valid when \( \Pi \) is a weak path if we use the convention \( \alpha_{\tilde{w}(h, \Pi)} = 0 \) when \( Q_{h-1} = Q_h \).

For every \( \gamma \in \Gamma \) let \( I(\gamma) \in \Gamma \) defined by \( I(\gamma)(i, w^*) = \gamma(i) \). Let \( D(\gamma) = \overline{D(I(\gamma))} \). The following properties will be useful in the sequel.

For every \( i, i' \in \tilde{J} \) and every \( w^* \in W^* \) we have

\[ (T_{I(\gamma)}(e_i))^\prime_{i'} = \tilde{\alpha}_{\gamma,i,i'} = \sum_{w^* \in W^*} (T_{I(\gamma)}(e_i)^\prime_{i',w^*}), \quad (4.3) \]
\[ H(T_{I(\gamma)}(e_i))_{i'} = \sum_{i=1}^{n(I)} \alpha_{\tilde{w}(\gamma,i)} = H(T_{I(\gamma)}(e_i)^\prime_{i,w^*}). \quad (4.4) \]

Next lemma shows that the linear operators \( T_{I(\gamma)}^{(\alpha)} \) and \( \overline{T}_{I(\gamma)}^{(\alpha)} \) allows us to evaluate the sum operators along insertion paths.

**Lemma 4.2.** For every path \( \Pi \) we have

i) \( \tilde{\Sigma}_{D(\gamma_1) \circ \cdots \circ D(\gamma_n)}(\alpha_{\mathcal{W}}) = T_{\gamma_1}^{(\alpha)} \circ \cdots \circ T_{\gamma_n}^{(\alpha)}(\tilde{\Sigma}_{\Pi}(\alpha_{\mathcal{W}})) \quad \forall \gamma_1, \ldots, \gamma_n \in \Gamma. \)

ii) \( \tilde{\Sigma}_{D(\gamma_1) \circ \cdots \circ D(\gamma_n)}(\alpha_{\mathcal{W}}) = T_{\gamma_1}^{(\alpha)} \circ \cdots \circ T_{\gamma_n}^{(\alpha)}(\tilde{\Sigma}_{\Pi}(\alpha_{\mathcal{W}})) \quad \forall \gamma_1, \ldots, \gamma_n \in \Gamma. \)

**Proof.** it suffices to prove the Lemma when \( n = 1 \). Note that for every \( i \in \tilde{J} \), every \( w^* \in W^* \) and every \( \gamma \in \Gamma \) we have

\[ \tilde{\Sigma}_{\psi_{w^*}(\gamma,i,w^*)}(\alpha_{\mathcal{W}}) = \sum_{h=1}^{n(\gamma,i,w^*)} \alpha_{w^* \tilde{w}(h,\gamma,i,w^*)} e_{\tilde{w}(h,\gamma,i,w^*)} (e_{i,w^*}). \]

Let \( \gamma \in \Gamma \). We thus have

\[ \tilde{\Sigma}_{D(\gamma)}(\alpha_{\mathcal{W}}) = \sum_{h=1}^{n(\Pi)} \tilde{\Sigma}_{D(\gamma)}(Q_{h-1,\Pi}, Q_{h,\Pi})(\alpha_{\mathcal{W}}) = \]
\[ \sum_{h=1}^{n(\Pi)} \tilde{\Sigma}_{D(\gamma)}(\psi_{w}(h,\Pi, P_{(h,\Pi)}))(\alpha_{\mathcal{W}}) = \]

\[ \sum_{h=1}^{n(\Pi)} \tilde{\Sigma}_{D(\gamma)}(\psi_{w}(h,\Pi, P_{(h,\Pi)}))(\alpha_{\mathcal{W}}) = \]

\[ \sum_{h=1}^{n(\Pi)} \tilde{\Sigma}_{D(\gamma)}(\psi_{w}(h,\Pi, P_{(h,\Pi)}))(\alpha_{\mathcal{W}}) = \]

\[ \sum_{h=1}^{n(\Pi)} \tilde{\Sigma}_{D(\gamma)}(\psi_{w}(h,\Pi, P_{(h,\Pi)}))(\alpha_{\mathcal{W}}) = \]
$\sum_{h=1}^{n(\Pi)} \psi_{\bar{v}(h,\Pi)}(\tilde{\tau}(h,\Pi),\tilde{\omega}(h,\Pi))^{(\alpha)} = \sum_{h=1}^{n(\Pi)} \alpha_{\bar{v}(h,\Pi)}(\tilde{t}(h,\Pi),\tilde{w}(h,\Pi))^{(\alpha)}$

This proves i), and ii) can be proved similarly.

We are now going to introduce a notion for polyratios which will turn out to be equivalent to those of metric and of as. metric.

We say that the polynario $\alpha$ is $(T,\Gamma)$-uniformly positive (or short $(T,\Gamma)$-u.p.) if there exists $c_{3,\alpha} > 0$ such that for every $n \in \mathbb{N}$, every $\gamma_1, \ldots, \gamma_n \in \Gamma$ and every $\iota \in \hat{J}$, we have

$H(T^{(\alpha)}_{\gamma_1} \circ \cdots \circ T^{(\alpha)}_{\gamma_n}(e_\iota)) \geq c_{3,\alpha}.$ (4.5)

We say that the polynario $\alpha$ is $(\tilde{T},\tilde{\Gamma})$-uniformly positive (or short $(\tilde{T},\tilde{\Gamma})$-u.p.) if there exists $c_{3,\alpha} > 0$ such that for every $m \in \mathbb{N}$, every $\gamma_1, \ldots, \gamma_m \in \tilde{\Gamma}$ and every $\iota \in \hat{J}$, we have

$\overline{H}(T^{(\alpha)}_{\gamma_1} \circ \cdots \circ T^{(\alpha)}_{\gamma_m}(e_\iota)) \geq c_{3,\alpha}.$ (4.5')

Note that, since $\hat{J}$ and $V^{(1)}$ are finite sets, and the elements of $\Gamma$ are by definition strict $V^{(1)}$-paths, then $\Gamma$ is a finite set as well. Thus, as hinted in Introduction, the notion of $(T,\Gamma)$-u.p. is related to that of joint spectral radius and to that of joint spectral subradius of a finite set of matrices. On the contrary, $\tilde{T}$ is an infinite set, thus the property of being $(\tilde{T},\tilde{\Gamma})$-u.p. is more complicated to verify. However, as we will see in Corollary 4.4, the two properties are equivalent. We need a Lemma.

**Lemma 4.3.** i) For every $\gamma_1, \ldots, \gamma_m+1 \in \Gamma$, we have

$H(T^{(\alpha)}_{\gamma_1} \circ \cdots \circ T^{(\alpha)}_{\gamma_m+1}(e_\iota)) \geq \sum_{\iota' \in \hat{J}} T^{(\alpha)}_{\gamma_{m+1}+1}(e_{\iota'}) H(T^{(\alpha)}_{\gamma_1} \circ \cdots \circ T^{(\alpha)}_{\gamma_m}(e_{\iota'})).$

ii) For every $\gamma_1, \ldots, \gamma_{m+1} \in \tilde{\Gamma}$, we have

$\overline{H}(T^{(\alpha)}_{\gamma_1} \circ \cdots \circ T^{(\alpha)}_{\gamma_{m+1}}(e_{\iota})) = \sum_{(\iota',w^*)} T^{(\alpha)}_{\gamma_{m+1}}(e_{\iota',w^*}) \overline{H}(T^{(\alpha)}_{\gamma_1} \circ \cdots \circ T^{(\alpha)}_{\gamma_m}(e_{\iota',w^*})).$
\[ n(\gamma_{m+1}(l, w^r)) \]
\[ \sum_{h=1}^{n(\gamma_{m+1}(l, w^r))} \tilde{\alpha}_h H(T^{(a)}_{\gamma_1} \circ \cdots \circ T^{(a)}_{\gamma_m}(e_h \gamma_{m+1}(l, w^r)), w^r \gamma_{m+1}(l, w^r)) \],
\[ \tilde{\alpha}_h := \alpha(\gamma_{m+1}(l, w^r)). \]

iii) Given \( \gamma_1 \in \Gamma, l = 1, 2, \ldots, m \), for every \( \iota \in \bar{J} \) and \( w^r \in W^* \) we have
\[ H(T^{(a)}_{\gamma_1} \circ \cdots \circ T^{(a)}_{\gamma_m}(e_{\iota})) = H(T^{(a)}_{I(\gamma_1)} \circ \cdots \circ T^{(a)}_{I(\gamma_m)}(e_{\iota}, w^r)). \]

iv) Given \( \gamma_1 \in \bar{\Gamma}, l = 1, 2, \ldots, m \), there exist \( \gamma_1 \in \Gamma \) such that for every \( \iota \in \bar{J} \) we have
\[ H(T^{(a)}_{\gamma_1} \circ \cdots \circ T^{(a)}_{\gamma_m}(e_{\iota})) \leq H(T^{(a)}_{\gamma_1} \circ \cdots \circ T^{(a)}_{\gamma_m}(e_{\iota})). \quad (4.6) \]

Proof. i) Note that
\[ H(T^{(a)}_{\gamma_1} \circ \cdots \circ T^{(a)}_{\gamma_m}(e_{\iota})) = \]
\[ \sum_{\iota'' \in J} \left( (T^{(a)}_{\gamma_1} \circ \cdots \circ T^{(a)}_{\gamma_m} \circ T^{(a)}_{\gamma_{m+1}}(e_{\iota}))_{\iota''} \right) = \]
\[ \sum_{\iota'' \in J} (T^{(a)}_{\gamma_1} \circ \cdots \circ T^{(a)}_{\gamma_m}(e_{\iota})_{\iota''} = \]
\[ \sum_{\iota'' \in J} \left( \sum_{\iota' \in \bar{J}} T^{(a)}_{\gamma_{m+1}}(e_{\iota})_{\iota'} \right)_{\iota''} = \]
\[ \sum_{\iota'' \in J} T^{(a)}_{\gamma_{m+1}}(e_{\iota})_{\iota''} = \]
\[ \sum_{\iota'' \in J} T^{(a)}_{\gamma_{m+1}}(e_{\iota})_{\iota''} H(T^{(a)}_{\gamma_1} \circ \cdots \circ T^{(a)}_{\gamma_m}(e_{\iota'})) \]

and i) is proved. ii) the proof of the first equality is very similar and is omitted. The second equality follows from the definition of \( T^{(a)}_{\gamma_{m+1}} \).

iii) The case \( m = 1 \) follows from (4.4). The general case follows by induction. In fact, if iii) holds for \( m \), by i), ii), (4.3) and the definitions of \( T(\gamma) \) and \( \overline{T(\gamma)} \), \( H \) and \( \overline{H} \), we have.
\[ H(T^{(a)}_{\gamma_1} \circ \cdots \circ T^{(a)}_{\gamma_m}(e_{\iota})) = \]
\[ \sum_{\iota' \in \bar{J}} T^{(a)}_{\gamma_{m+1}}(e_{\iota})_{\iota'} H(T^{(a)}_{\gamma_1} \circ \cdots \circ T^{(a)}_{\gamma_m}(e_{\iota'})) \]
\[ = \sum_{\iota' \in \bar{J}} \left( \sum_{w^r \in W^*} T^{(a)}_{I(\gamma_{m+1})}(e_{\iota', w^r})_{\iota', w^r} \overline{H}(T^{(a)}_{I(\gamma_1)} \circ \cdots \circ T^{(a)}_{I(\gamma_m)}(e_{\iota', w^r})) \right) \]
\[ = \sum_{(\iota', w^r) \in \bar{J} \times W^*} \overline{T^{(a)}_{I(\gamma_{m+1})}(e_{\iota', w^r})_{\iota', w^r} H(T^{(a)}_{I(\gamma_1)} \circ \cdots \circ T^{(a)}_{I(\gamma_m)}(e_{\iota', w^r}))} \]
and iii) holds for \( m + 1 \).

iv) Note that for every \( t, t' \in \hat{J} \), every \( w^* \in \tilde{W}_m \) and every \( \gamma \in \Gamma \) we have the following simple analog of (4.3)

\[
\sum_{w^* \in \tilde{W}_m} \left( T_{\gamma}^{(a)}(e_{t, w^*}) \right)_{t', w^*} = \sum_{w^* \in \tilde{W}_{m+1}} \left( T_{\gamma}^{(a)}(e_{t, w^*}) \right)_{t', w^*} = \tilde{\alpha}_{\gamma, t', t, w^*},
\]

\[\text{(4.7)}\]

We define \( \gamma_l \) by induction. Suppose we have defined \( \gamma_1, \ldots, \gamma_l, l < m \), satisfying

\[
H \left( T_{\gamma_1}^{(a)} \circ \cdots \circ T_{\gamma_l}^{(a)}(e_t) \right) \leq \mathcal{H} \left( T_{\gamma_1}^{(a)} \circ \cdots \circ T_{\gamma_l}^{(a)}(e_{t'}) \right) \quad \forall w^* \in \tilde{W}_{m-l},
\]

and define \( \gamma_{l+1} \) in this way. Let

\[
\theta_{t, t', w^*} := \sum_{t' \in \hat{J}} \tilde{\alpha}_{\gamma_{l+1}, t', t, w^*} H \left( T_{\gamma_1}^{(a)} \circ \cdots \circ T_{\gamma_l}^{(a)}(e_{t'}) \right),
\]

\[
\gamma_{l+1}(t) = \gamma_{l+1}(t, w^*(t))
\]

where for every given \( t \in \hat{J} \), \( w^*(t) \) is an element of \( \tilde{W}_{m-l-1} \) minimizing \( \theta_{t, t', w^*} \), i.e., such that

\[
\theta_{t, t', w^*} \leq \theta_{t, w^*} \quad \forall w^* \in \tilde{W}_{m-l-1}.
\]

Note that, in view of (4.3) and (4.7) we have

\[
\sum_{w^* \in \tilde{W}_{m-l}} T_{\gamma_{l+1}}^{(a)}(e_{t, w^*(t)})_{t', w^*} = T_{\gamma_{l+1}}^{(a)}(e_{t'})_{t'}
\]

\[\text{(4.9)}\]

for every \( t, t' \in \hat{J} \). Thus, for every \( w^* \in \tilde{W}_{m-l-1} \) we have

\[
\mathcal{H} \left( T_{\gamma_1}^{(a)} \circ \cdots \circ T_{\gamma_l}^{(a)}(e_{t, w^*}) \right) = \sum_{(t', w^*) \in \hat{J} \times \tilde{W}_m} T_{\gamma_{l+1}}^{(a)}(e_{t, w^*(t)})_{t', w^*} H \left( T_{\gamma_1}^{(a)} \circ \cdots \circ T_{\gamma_l}^{(a)}(e_{t', w^*}) \right)
\]

\[\text{(by ii)}\]

\[
= \sum_{t' \in \hat{J}} \sum_{w^* \in \tilde{W}_{m-l}} T_{\gamma_{l+1}}^{(a)}(e_{t, w^*(t)})_{t', w^*} \mathcal{H} \left( T_{\gamma_1}^{(a)} \circ \cdots \circ T_{\gamma_l}^{(a)}(e_{t', w^*}) \right)
\]

\[\text{(by (4.8))}\]

\[
\geq \sum_{t' \in \hat{J}} \sum_{w^* \in \tilde{W}_{m-l}} T_{\gamma_{l+1}}^{(a)}(e_{t, w^*(t)})_{t', w^*} H \left( T_{\gamma_1}^{(a)} \circ \cdots \circ T_{\gamma_l}^{(a)}(e_{t'}) \right)
\]

\[
= \theta_{i}(w^*) \geq \theta_{i}(w^*(t))
\]
\[
\sum_{\iota' \in \hat{J}} \sum_{w^* \in \hat{W}_{m-l}} T_{\iota_{l+1}}^{(\alpha)}(e_{\iota, w^*(\iota)}),
\]  
\[
= \sum_{\iota' \in \hat{J}} T_{\iota_{l+1}}^{(\alpha)}(e_{\iota}),
\]

\[
= H \left( T_{\iota_{l+1}}^{(\alpha)}(e_{\iota}) \right) \quad \text{(by (4.9))}
\]

and (4.8) holds for \( l + 1 \), and the inductive step is completed. Now, iv) follows from the case \( l = m \). \( \square \)

**Corollary 4.4.** \( \alpha \) is \((T, \Gamma)\)-u.p. if and only if it is \((\hat{T}, \hat{\Gamma})\)-u.p.

**Proof.** This immediately follows from Lemma 4.3 iii) and iv). \( \square \)

## 5. Distances on the Fractal

In this Section, we prove that for a polyratio \( \alpha \) the following are equivalent

i) \( \alpha \) is a metric polyratio

ii) \( \alpha \) is an as. metric polyratio

iii) \( \alpha \) is \((T, \Gamma)\)-u.p.

In the sequel we will always use implicitly Corollary 4.4. First, we prove that both i) and ii) imply iii).

**Lemma 5.1.** If \( \alpha \) is either a metric polyratio or an as. metric polyratio on \( K \), then \( \alpha \) is \((T, \Gamma)\)-u.p.

**Proof.** If \( \alpha \) is either a metric polyratio or an as. metric polyratio, then there exists a distance \( d \) on \( K \) such that for some \( c_{2, \alpha} > 0 \) the second inequality in (1.3) holds. Observe that for every path \( \Pi \) connecting \( P_{j_1} \) to \( P_{j_2} \) for every \( h = 1, \ldots, n(\Pi) \) we have

\[
d(P_{j_1}, P_{j_2}) \leq \sum_{h=1}^{n(\Pi)} d(Q_{h-1, \Pi}, Q_{h, \Pi}) = \Sigma_\Pi(d). \quad (5.1)
\]

Moreover, for some \( j_1, j_2 \) (depending on \( h \)) we have

\[
d(Q_{h-1, \Pi}, Q_{h, \Pi}) \leq d(\psi_{\bar{\iota}(h, \Pi)}(P_{j_1}), \psi_{\bar{\iota}(h, \Pi)}(P_{j_2}))
\]

\[
\leq \text{diam}_d K_{\bar{\iota}(h, \Pi)} \leq c_{2, \alpha} \alpha_{\bar{\iota}(h, \Pi)} \text{diam}_d(K)
\]

\[
= c_{2, \alpha} \text{diam}_d(K) \alpha_{\bar{\iota}}(Q_{h-1, \Pi}, Q_{h, \Pi}),
\]

for every \( h = 1, \ldots, n(\Pi) \), thus, summing the previous inequalities, we obtain

\[
\Sigma_\Pi(d) = \sum_{h=1}^{n(\Pi)} d(Q_{h-1, \Pi}, Q_{h, \Pi})
\]

\[
\leq c_{2, \alpha} \text{diam}_d(K) \sum_{h=1}^{n(\Pi)} \alpha_{\bar{\iota}}(Q_{h-1, \Pi}, Q_{h, \Pi}) = c_{2, \alpha} \text{diam}_d(K) \Sigma_\Pi(\alpha_{\bar{\iota}}),
\]
and, in view of (5.1),

\[ d(P_{j_1}, P_{j_2}) \leq c_{2,\alpha} \text{diam}_d(K) \Sigma_{\Pi}(\alpha_{w}). \]  

(5.2)

Let now \( \gamma_1, \ldots, \gamma_l \in \Gamma \) and let \( \{j_1, j_2\} \in \widehat{J} \). Set \( \Pi := (P_{j_1}, P_{j_2}) \), so that \( \Sigma_{\Pi}(\alpha_{w}) = e_{\{j_1, j_2\}} \) (see 4.1). Recall that hence also the path \( \overline{D}(\gamma_{\Pi}) \circ \cdots \circ \overline{D}(\gamma_1)(\Pi) \) connects \( P_{j_1} \) to \( P_{j_2} \), so that we can use (5.2) with this path in place of \( \Pi \), and, in view also of Lemma 4.2 i), we have

\[ \min \{d(P_{h_1}, P_{h_2}) : (h_1, h_2) \in \widehat{J}\} \leq d(P_{j_1}, P_{j_2}) \leq c_{2,\alpha} \text{diam}_d(K) \Sigma_{\Pi}(\alpha_{w}) = c_{2,\alpha} \text{diam}_d(K) \overline{H}(\overline{T}_{\gamma_1}^{(a)} \circ \cdots \circ \overline{T}_{\gamma_l}^{(a)}(e_{\{j_1, j_2\}})). \]

Hence, (4.5)' is satisfied with

\[ c_{3,\alpha} = \frac{\min \{d(P_{h_1}, P_{h_2}) : (h_1, h_2) \in \widehat{J}\}}{c_{2,\alpha} \text{diam}_d(K)}. \]

Recall the definition of the pseudodistance given in [4]. If \( Q, Q' \in K \) and \( w_1^*, \ldots, w_n^* \in W^* \), we say that \( (w_1^*, \ldots, w_n^*) \) is a prechain if \( K_{w_1^*} \cap K_{w_n^*+1} \neq \emptyset \) for every \( h = 1, \ldots, n - 1 \). We say that \( (w_1^*, \ldots, w_n^*) \) is between \( Q \) and \( Q' \) if \( Q \in K_{w_i^*} \) and \( Q' \in K_{w_{i+1}^*} \). We say that a prechain \( (w_1^*, \ldots, w_n^*) \) is a chain if \( w_1^*, \ldots, w_n^* \) are pairwise incomparable. Denote by \( G'(Q, Q') \) the set of prechains between \( Q \) and \( Q' \) and by \( G(Q, Q') \) the set of chains between \( Q \) and \( Q' \). Let \( \alpha = (\alpha_1, \ldots, \alpha_k) \) be a polyratio and let

\[ A(\mathcal{C}) = \sum_{h=1}^{n} \alpha_{w_{h}^{*}}, \quad \text{if} \ \mathcal{C} \in G'(Q, Q'), \]

\[ D_{\alpha}(Q, Q') := \inf \{A(\mathcal{C}) : \mathcal{C} \in G'(Q, Q')\} = \inf \{A(\mathcal{C}) : \mathcal{C} \in G'(Q, Q')\}. \]

Note that a standard argument shows that \( D_{\alpha} \) is in any case a pseudodistance, in the sense that satisfies all properties of a distance, except possibly for the fact that \( D_{\alpha}(Q, Q') = 0 \) could occur also when \( Q \neq Q' \).

**Lemma 5.2.** i) For every polyratio \( \alpha, D_{\alpha} \) is a pseudodistance, and, if \( D_{\alpha}(Q, Q') > 0 \), for every \( Q, Q' \in V^{(\infty)} \) with \( Q \neq Q' \), \( D_{\alpha} \) is an \( \alpha \)-self-similar distance.

ii) If \( \alpha \) is a metric polyratio, then \( D_{\alpha} \) is an \( \alpha \)-self-similar distance, which induces the same topology as the original distance.

**Proof.** i) This follows from Theorem 1.33 and Prop. 1.12 in [4]. ii) See Prop. 1.13 and 1.11 in [4]. Note that, according to the definitions in [4], which are equivalent to the definitions here, but slightly differ from them, in the hypothesis of Prop. 1.13 of [4] it is assumed that \( d \)
induces the same topology on $K$ as the original distance, but the proof does not use this fact.

Now, we are going to prove the converse of Lemma 5.1, i.e., if $\alpha$ is $(T, \Gamma)$-u.p., then $D_{\alpha}$ is at the same time an $\alpha$-self-similar distance and an $\alpha$-scaling distance on $K$. In order to do this, we will introduce a pseudodistance $\tilde{d}_{\alpha}$ on $V^{(\infty)}$ that is more strictly related to the notion of $(T, \Gamma)$-u.p. than $D_{\alpha}$. However, in Lemma 5.6 we will prove that $D_{\alpha}$ and $\tilde{d}_{\alpha}$ are equivalent on $V^{(\infty)}$. A similar distance in discussed in [4], Theorem 1.34. Clearly, for every path $\Pi$ and for every $w^* \in W^*$ we have

$$\Sigma_{V^{(\infty)}}(\alpha_w) = \alpha_w \cdot \Sigma_{V^{(\infty)}}(\alpha_w).$$

(5.3)

Let $(\mathcal{P}a)_{Q,Q'}^\prime$ be the set of the weak paths connecting $Q$ to $Q'$, let $(\mathcal{P}a)_{Q,Q'}^\prime$ be the set of the paths connecting $Q$ to $Q'$, and let $(\mathcal{P}a)_{Q,Q'}^\prime$ be the set of the strict paths connecting $Q$ to $Q'$. We define now the function $\tilde{d}_{\alpha}$ on $V^{(\infty)}$ by

$$\tilde{d}_{\alpha}(Q, Q') = \inf \{ \Sigma_{\Pi}(\alpha_w) : \Pi \in (\mathcal{P}a)_{Q,Q'} \} = \inf \{ \Sigma_{\Pi}(\alpha_w) : \Pi \in (\mathcal{P}a)_{Q,Q'}^\prime \}. $$

It can be easily proved that $\tilde{d}_{\alpha}$ is a pseudodistance on $V^{(\infty)}$. The only nontrivial point for proving this, is that for every $Q, Q' \in V^{(\infty)}$ we have $(\mathcal{P}a)_{Q,Q'} \neq \emptyset$, so that $\tilde{d}_{\alpha}(Q, Q') < +\infty$. This will follow from the next lemma.

**Lemma 5.3.** There exists a constant $C_{1,\alpha} \geq 1$ such that

$$\text{diam}_{\tilde{d}_{\alpha}}(V^{(\infty)}_{w^*}) \leq C_{1,\alpha} \alpha_w. $$

for every $w^* \in W^*$. Thus, in particular, $\text{diam}_{\tilde{d}_{\alpha}}(V^{(\infty)}_{w^*})$ is finite, for every $w^* \in W^*$.

**Proof.** Since the fractal is connected, there exists $C_{\alpha} \geq 1$ such that for every $Q, Q' \in V^{(1)}$ there exists a path $\Pi$ connecting them such that $\Sigma_{\Pi}(\alpha_w) \leq C_{\alpha}$. Thus, for every $w^* \in W^*$, in view of (5.3), the path $\Pi' := \psi_{w^*}(\Pi)$ satisfies $\Sigma_{\Pi'}(\alpha_w) \leq C_{\alpha} \alpha_w$. Since $\Pi'$ connects $\psi_{w^*}(Q)$ to $\psi_{w^*}(Q')$, we then have

$$\tilde{d}_{\alpha}(\psi_{w^*}(Q), \psi_{w^*}(Q')) \leq C_{\alpha} \alpha_w. $$

(5.4)

Now, suppose $w^* = (i_1, ..., i_m)$. If $Q \in V^{(\infty)}_{w^*}$, there exist $i_{m+1}, ..., i_{m'}$ and $P \in V^{(0)}$ such that $Q = \psi_{i_1, ..., i_m, i_{m+1}, ..., i_{m'}}(P)$. Let $\Pi = (Q_{m}, ..., Q_{m'})$, where $Q_h = \psi_{i_1, ..., i_m, i_{m+1}}(P)$. Then $\Pi$ is a path that connects $Q := \psi_{i_1, ..., i_m}(P) \in V_{i_1, ..., i_m}$ to $Q$. Therefore,
\[ \tilde{d}_\alpha(Q, \tilde{Q}) \leq \sum_{h=m+1}^{m'} \tilde{d}_\alpha(Q_{h-1}, Q_h) \]

\[ = \sum_{h=m+1}^{m'} \tilde{d}_\alpha\left(\psi_1, \ldots, i_{m-1}, \psi_1, \ldots, i_{h-1}(P), \psi_1, \ldots, i_{m-1}, \psi_1, \ldots, i_{h-1}(\psi_1(P)) \right) \]

\[ \leq C_\alpha \sum_{h=m+1}^{m'} \alpha_{i_1, \ldots, i_{m-1}} \quad \text{(by (5.4))} \]

\[ = C_\alpha \alpha_{w^*} \sum_{h=m+1}^{m'} \alpha_{i_{m+1}} \cdots \alpha_{i_{h-1}} \]

\[ \leq C_\alpha \alpha_{w^*} \sum_{h=m+1}^{m'} \alpha_{\max}^{-1} \]

\[ \leq \frac{C_\alpha}{1 - \alpha_{\max}} \alpha_{w^*} \]

Now, given \( Q, Q' \in V_{w^*}^{(\infty)} \), let \( \tilde{Q}, \tilde{Q}' \) be as above. Then, we have

\[ \tilde{d}_\alpha(Q, Q') \leq \tilde{d}_\alpha(Q, \tilde{Q}) + \tilde{d}_\alpha(\tilde{Q}, \tilde{Q}') + \tilde{d}_\alpha(\tilde{Q}', Q') \]

\[ \leq \left( C + \frac{2}{1 - \alpha_{\max}} \right) \alpha_{w^*}, \]

thus the Lemma is proved with \( C_{1, \alpha} = C_\alpha + 2 \frac{C_\alpha}{1 - \alpha_{\max}} \).

\[ \boxed{\text{Lemma 5.4. If } \alpha \text{ is } (T, \Gamma) \text{ u.p., and } \Pi \text{ is a strict path connecting two different points } Q \text{ and } Q' \text{ of } V^{(m)}, \text{ then} \}

\[ \Sigma_\Pi(\alpha_{w^*}) \geq c_{3, \alpha}(\overline{\alpha_{\min}})^m \] (5.5)

\[ \text{and consequently, } \tilde{d}_\alpha \text{ is a distance on } V^{(\infty)}. \]

\[ \text{Proof. In view of Lemma 3.6, there exist a strict path } \Pi' \text{ lying in } V^{(m)} \text{ connecting } Q \text{ and } Q' \text{ and } \overline{T}_1, \ldots, \overline{T}_m \in \overline{\Gamma} \text{ such that} \]

\[ \Pi = D(\overline{T}_1) \circ \cdots \circ D(\overline{T}_m)(\Pi'). \]

\[ \text{Note that, by Lemma 2.3, since } \Pi' \text{ lies in } V^{(m)}, \text{ then } |w(Q_{h-1, \Pi'}, Q_{h, \Pi'})| \leq m, \text{ thus} \]

\[ \alpha_{w^*}(Q_{h-1, \Pi'}, Q_{h, \Pi'}) \geq (\overline{\alpha_{\min}})^m \quad \forall h = 1, \ldots, n(\Pi'). \] (5.6)

\[ \text{Therefore, by Lemma 4.2 we have} \]

\[ \Sigma_\Pi(\alpha_{w^*}) = \Sigma_{D(\overline{T}_1) \circ \cdots \circ D(\overline{T}_m)(\Pi')} \left( \alpha_{w^*} \right) = \]

\[ \overline{H} \left( \overline{T}^{(\alpha)}_{\overline{T}_1} \circ \cdots \circ \overline{T}^{(\alpha)}_{\overline{T}_m}(\Delta_{\Pi'}(\alpha_{w^*})) \right) = \]
\[
= \sum_{h=1}^{n(P)} \alpha_{\overline{\Pi}}(Q_{h-1,\Pi'}, Q_{h,\Pi'}) \overline{T}(\overline{T}_{\gamma_1}^{(a)} \circ \cdots \circ \overline{T}_{\gamma_m}^{(a)} (\epsilon_{\overline{\gamma}(h,\Pi'), \overline{d}(h,\Pi')})) \\
\geq \alpha_{\overline{\Pi}}(Q_{0,\Pi'}, Q_{1,\Pi'}) \overline{H}(\overline{T}_{\gamma_1}^{(a)} \circ \cdots \circ \overline{T}_{\gamma_m}^{(a)} (\epsilon_{\overline{\gamma}(1,\Pi'), \overline{d}(1,\Pi')})) \\
\geq c_{3,a}(\overline{d}_{\text{min}})^m
\]
where we have used (5.6) and the definitions of (\overline{T}, \overline{\Gamma})-u.p. and of \(\Sigma_{\Pi'}(\alpha_{\overline{\Pi}})\). Therefore, (5.5) holds. In order to prove that \(d_{\alpha}\) is a distance, the only nontrivial fact to prove is that if \(Q, Q' \in V^{(\infty)}\), \(Q \neq Q'\), then \(\overline{d}_{\alpha}(Q, Q') > 0\). But, since for some natural \(m\) we have \(Q, Q' \in V^{(m)}\), by (5.5) we have \(\overline{d}_{\alpha}(Q, Q') \geq c_{3,a}(\overline{d}_{\text{min}})^m > 0\). \(\square\)

**Lemma 5.5.** If \(\alpha\) is \((T, \Gamma)\) u.p. there exists \(c_{4,\alpha} > 0\) such that for every \(w^* \in W^*\)
\[
c_{4,\alpha} \alpha_{w^*} \text{diam}_{\overline{d}_{\alpha}}(V^{(\infty)}) \leq \text{diam}_{\overline{d}_{\alpha}}(V_{w^*}^{(\infty)}) \leq \alpha_{w^*} \text{diam}_{\overline{d}_{\alpha}}(V^{(\infty)}). \tag{5.7}
\]

**Proof.** Let \(Q, Q' \in V_{w^*}, Q = \psi_{w^*}(P), Q' = \psi_{w^*}(P')\) with \(P, P' \in V^{(\infty)}\). We have
\[
\overline{d}_{\alpha}(Q, Q') = \min\{A, B\},
\]
\[
A := \inf_{\Pi \in P} \Sigma_{\Pi}(\alpha_{\overline{\Pi}}), \quad P := \psi_{w^*}((\overline{P})_{P,P'}),
\]
\[
B := \inf_{\Pi \in P'} \Sigma_{\Pi}(\alpha_{\overline{\Pi}}), \quad P' = (\overline{P})_{Q,Q'} \setminus P.
\]
Let \(w^* = (i_1 \ldots i_m)\). Since, in view of (5.3),
\[
A = \alpha_{w^*} \overline{d}_{\alpha}(P, P') \tag{5.8}
\]
the second inequality in (5.7) follows at once. By Lemma 3.5, if \(\Pi \in P\), then there exist \(l < m\) and a subpath \(\Pi'\) of \(\Pi\) connecting different points of \(V_{i_1 \ldots i_{l+1}}\) entirely contained in \(V_{i_1 \ldots i_l}^{(\infty)}\). Thus, \(\Pi' = \psi_{i_1 \ldots i_l}(\Pi'')\), where \(\Pi''\) is a path in \(V^{(\infty)}\) connecting two different points in \(V^{(1)}\). By Lemma 5.4, \(\Sigma_{\Pi''}(\alpha_{\overline{\Pi}}) \geq c_{3,a}(\overline{d}_{\text{min}})\). Hence,
\[
\Sigma_{\Pi}(\alpha_{\overline{\Pi}}) \geq \Sigma_{\Pi'}(\alpha_{\overline{\Pi}}) = \alpha_{i_1 \ldots i_l} \Sigma_{\Pi''}(\alpha_{\overline{\Pi}}) \\
\geq \alpha_{w^*} c_{3,a}(\overline{d}_{\text{min}}) \geq c_{4,\alpha} \alpha_{w^*} \text{diam}_{\overline{d}_{\alpha}}(V^{(\infty)})
\]
if \(c_{4,\alpha} \leq \frac{c_{3,a}(\overline{d}_{\text{min}})}{\text{diam}_{\overline{d}_{\alpha}}(V^{(\infty)})}\). If moreover, \(c_{4,\alpha} < 1\), in view of (5.8), the first inequality in (5.7) holds. \(\square\)

**Lemma 5.6.** For every polyratio \(\alpha\)
\[
D_{\alpha}(Q, Q') \leq \overline{d}_{\alpha}(Q, Q') \leq C_1 \alpha D_{\alpha}(Q, Q') \quad \forall Q, Q' \in V^{(\infty)}. \tag{5.9}
\]
Proof. We can and do assume \( Q \neq Q' \). Let \( \Pi \in (\mathcal{P}a)_{Q,Q'} \). Then for every \( h = 1, \ldots, n(\Pi) - 1 \) we have \( Q_{h,\Pi} \in K_{\tilde{w}(h,\Pi)} \cap K_{\tilde{w}(h+1,\Pi)} \). Thus, \( \overline{\mathcal{C}} := (\tilde{w}(1,\Pi), \ldots, \tilde{w}(n(\Pi),\Pi)) \in \mathcal{G}'(Q,Q') \). Moreover,

\[
A(\overline{\mathcal{C}}) = \sum_{h=1}^{n(\Pi)} \alpha_{\tilde{w}(h,\Pi)} = \Sigma_{\Pi}(\alpha_{\mathcal{C}})
\]

so that the first inequality in (5.9) follows. Let next \( \overline{\mathcal{C}} \in \mathcal{G}(Q,Q') \), and \( \overline{\mathcal{C}} = (w_1^*, \ldots, w_n^*) \). Suppose for the moment \( n > 1 \). For \( h = 0, \ldots, n-2 \) let \( Q_h \) be an element of \( K_{w_{h+1}^*} \cap K_{w_{h+2}^*} \), thus, \( w_{h+1}^* \) and \( w_{h+2}^* \) being incomparable, by Lemma 2.2 we have \( Q_h \in V_{w_{h+1}^*} \cap V_{w_{h+2}^*} \); it follows that \( \Pi \), defined by

\[
\Pi := (Q_0, \ldots, Q_{n-2})
\]

is a weak path, not necessarily a path. In fact, for every \( h = 1, \ldots, n-2 \) \( Q_{h-1}, Q_h \in V_{w_{h+1}^*} \), so that either \( Q_{h-1} \neq Q_h \) and \( \tilde{w}(h,\Pi) = w_{h+1}^* \), or \( Q_{h-1} = Q_h \), and, thanks to the convention \( \alpha_{\tilde{w}(h,\Pi)} = 0 \) (see Remark 4.1), we have

\[
\tilde{d}_\alpha(Q_0, Q_{n-2}) \leq \Sigma_{\Pi}(\alpha_{\mathcal{C}}) = \sum_{h=1}^{n-2} \alpha_{\tilde{w}(h,\Pi)} \leq \sum_{h=2}^{n-1} \alpha_{w_h^*}.
\]

Moreover, since \( Q \in K_{w_1^*} \cap V^{(\infty)} = V_{w_1^*}^{(\infty)} \), we have \( Q, Q_0 \in V_{w_1^*}^{(\infty)} \), hence by Lemma 5.3 \( \tilde{d}_\alpha(Q, Q_0) \leq C_{1,\alpha} \alpha_{w_1^*} \). Similarly, \( \tilde{d}_\alpha(Q_{n-2}, Q') \leq C_{1,\alpha} \alpha_{w_n^*} \). In conclusion

\[
\tilde{d}_\alpha(Q, Q') \leq \tilde{d}_\alpha(Q, Q_0) + \tilde{d}_\alpha(Q_0, Q_{n-2}) + \tilde{d}_\alpha(Q_{n-2}, Q')
\]

\[
\leq C_{1,\alpha} \alpha_{w_1^*} + \sum_{h=2}^{n-1} \alpha_{w_h^*} + C_{1,\alpha} \alpha_{w_n^*}
\]

\[
\leq C_{1,\alpha} \left( \alpha_{w_1^*} + \sum_{h=2}^{n-1} \alpha_{w_h^*} + \alpha_{w_n^*} \right) = C_{1,\alpha} \sum_{h=1}^{n} \alpha_{w_h^*} = C_{1,\alpha} A(\overline{\mathcal{C}}).
\]

We have proved the inequality

\[
\tilde{d}_\alpha(Q, Q') \leq C_{1,\alpha} A(\overline{\mathcal{C}})
\]

(5.10) when \( n > 1 \). However, (5.10) also holds in the case \( n = 1 \). In fact, in this case, \( Q, Q' \in V^{(\infty)} \cap K_{w_1^*} = V_{w_1^*}^{(\infty)} \). Thus,

\[
\tilde{d}_\alpha(Q, Q') \leq \text{diam}_{\tilde{d}_\alpha}(V_{w_1^*}^{(\infty)}) \leq C_{1,\alpha} \alpha_{w_1^*} = C_{1,\alpha} A(\overline{\mathcal{C}}).
\]

Since (5.10) holds for every \( \overline{\mathcal{C}} \in \mathcal{G}(Q,Q') \), the second inequality in (5.9) easily follows. \( \square \)
Theorem 5.7. If the polyratio \( \alpha \) is \((T, \Gamma)-u.p.\), then \( D_\alpha \) is both \( \alpha \)-self-similar and \( \alpha \)-scaling. Thus, \( \alpha \) is both metric and as. metric. Moreover, \( D_\alpha \) induces on \( K \) the same topology as the original distance.

Proof. By Lemma 5.6 and Lemma 5.4, if \( Q, Q' \in V^{(\infty)} \), \( Q \neq Q' \), then \( D_\alpha(Q, Q') \geq \frac{1}{\delta_{Q,Q'}^\alpha} \tilde{d}_\alpha(Q, Q') > 0 \). Thus, by Lemma 5.2, \( D_\alpha \) is an \( \alpha \)-scaling distance on \( K \) and induces on \( K \) the same topology as the original distance. Therefore, in view of Lemma 2.6 ii) \( V^{(\infty)} \) is dense in \( K_w^* \) for every \( w^* \in W^* \) also with respect to \( D_\alpha \). Now, \( \alpha \) is as. metric by Lemmas 5.5 and 5.6. \( \square \)

Remark 5.8. Note that it follows from Lemma 4.3 i) that, if we have \( H(T^{(\alpha)}(e_i)) \geq 1 \) \( \forall \ell \in J \) \( \forall \gamma \in \Gamma \), then \( \alpha \) is \((T, \Gamma)-u.p.\). More precisely, we can prove by induction that

\[
H(T^{(\alpha)}(e_i)) \geq 1 \quad \forall \ell \in J \quad \forall \gamma_1, ..., \gamma_m \in \Gamma. \tag{5.11}
\]

In fact, (5.11) is trivial for \( m = 0 \). Also, if it holds for \( m \), then

\[
H(T^{(\alpha)}(e_i)) = \sum_{\ell' \in J} T^{(\alpha)}_{\gamma_{m+1}}(e_{\ell'}) H(T^{(\alpha)}_{\gamma_1} \circ \cdots \circ T^{(\alpha)}_{\gamma_m}(e_{\ell'})) \geq \sum_{\ell' \in J} T^{(\alpha)}_{\gamma_{m+1}}(e_{\ell'}) \text{ by (5.11)}
\]

where in the last equality we have used Lemma 4.3 i) with \( m = 0 \), since \( H(e_{\ell'}) = 1 \), and (5.11) holds for \( m + 1 \). As a consequence of (5.11), and of Corollary 4.4, if \( \alpha_i \geq \frac{1}{2} \) for every \( i = 1, ..., k \), then there exists an \( \alpha \)-scaling distance on \( K \).

In fact, if \( \gamma(i) \) is not a strong \( V^{(1)} \)-path, then \( \tilde{w}(h, \gamma(i)) = \emptyset \), thus \( \alpha_i h, \gamma(i) = 1 \) for at least one \( h = 1, ..., n(\gamma(i)) \). Therefore, in view of (4.4), (5.11) holds. If on the contrary, \( \gamma(i) \) is a strong \( V^{(1)} \)-path, as it connects two different points of \( V^{(0)} \), the sum in (4.4) has at least two summands. Indeed, \( \gamma(i) \) connects two different points of \( V^{(0)} \), and two consecutive vertices of \( \gamma(i) \), by definition of a strong \( V^{(1)} \)-path, lie in a same 1-cell, which, by (2.8), cannot contain more than one point of \( V^{(0)} \). Moreover, in (4.4) we have \( \tilde{w}(h, \gamma(i)) = i_h \) for some \( i_h = 1, ..., k \), and, by our assumptions, such summands are not less than \( \frac{1}{2} \), thus (5.11) holds in any case.

6. Examples

In view of the results of Section 5, the fact whether a given polyratio \( \alpha \) is metric (or as. metric, which is the same) on \( K \) is reduced to the problem whether \( \alpha \) is \((T, \Gamma)-u.p.\). In turns, such a problem is strictly related to the notion of joint spectral radius or better joint spectral subradius. More precisely, it is related to the fact that the joint spectral
subradius is greater than or equal to 1. However, the notion of \((T, \Gamma)\)-u.p. in general is not perfectly equivalent to having the joint spectral subradius greater than or equal to 1. Note that the determination of the joint spectral radius, as well as of the joint spectral subradius is in general difficult. In this Section, we will discuss some explicit necessary and sufficient conditions for \(\alpha\) being \((T, \Gamma)\)-u.p.. However, such conditions require the existence of some special paths, and this occurs only on some fractals having a rather simple structure.

If \(\Pi = (\Pi(1), ..., \Pi(l))\) is a finite sequence of paths we put

\[
\Sigma_\Pi(\alpha_w) = \sum_{\beta=1}^{l} \Sigma_{\Pi_\beta}(\alpha_w).
\]

We say that the subpaths \(\Pi(1), ..., \Pi(l)\) of \(\Pi\) are \emph{separated}, if they have length greater than 1, and moreover, the intervals

\[
[n_1(\Pi, \Pi(\beta)), n_2(\Pi, \Pi(\beta))], \quad \beta = 1, ..., l,
\]

are mutually disjoint, roughly speaking this means that they have no common edge. In this setting we say that \(\tilde{\Pi} := (\Pi(1), ..., \Pi(l))\) is a \emph{multisubpath} or shortly multisp. of \(\Pi\). We say that a sequence \(\Pi' = (Q_{h_0, \Pi}, ..., Q_{h_s, \Pi})\) is a \emph{refinement} of \(\Pi\) if \(0 = h_0 < h_1 < \cdots < h_s = n(\Pi)\), and moreover \(Q_{h_{l+1}, \Pi} = Q_{h_{l+1}, \Pi}\) for every \(l = 0, ..., s-1\). Thus, \(\tilde{\Pi}\) is a path. Note that if \(\Pi\) is a \emph{strict} path, then every refinement of \(\Pi\) amounts to \(\Pi\) itself.

Suppose now there exist finitely many \(V^{(1)}\) paths \(\Pi_1, ..., \Pi_r\) such that

(i) For every \(s = 1, ..., r\) there exists \(\iota_s \in \hat{J}\) such that \(\Pi_s\) is a strict and strong \((\iota_s, V^{(1)})\)-path, and also \(\iota(h, \Pi_s) = \iota_s\) for every \(h\).

(ii) There exists \(\tilde{m} \in \mathbb{N}\) such that every path \(\Pi\) connecting two different points of \(V^{(0)}\) and such that \(|\tilde{w}(h, \Pi)| \geq \tilde{m}\) for every \(h = 1, ..., n(\Pi)\), also contains a subpath of the form \(\psi_{w^*(\Pi)}\) where \(\Pi\) is a \(\iota_s\)-path for some \(s = 1, ..., r\), and \(w^* \in W_{\tilde{m}}\).

(iii) For every \(s = 1, ..., r\) and every \(\iota_s\)-path \(\Pi\), there exist \(s' = 1, ..., r\) and a multisp. \(\hat{\Pi} := (\hat{\Pi}_1, ..., \hat{\Pi}_{n(\Pi_{s'})})\) of \(\Pi\) with refinements \(\hat{\Pi}_\beta\) of \(\hat{\Pi}_\beta\), \(s(\beta) = 1, ..., r\) such that \(\hat{\Pi}_\beta = \psi_{w(\beta, \Pi_{s'})}(\hat{\Pi}_{\beta}')\) where \(\hat{\Pi}_{\beta}'\) is a \(\iota_s(\beta)\)-path, and we put \(\hat{\Pi}' = (\hat{\Pi}_1', ..., \hat{\Pi}_{n(\Pi_{s'})}')\).

**Lemma 6.1.** Under assumption (iii), if \(\Pi\) is a strong \(\iota_s\)-path, \(s = 1, ..., r\), and \(\Sigma_{\Pi_{s'}}(\alpha_w) \geq 1\) for every \(\beta = 1, ..., n(\Pi_{s'})\), then

\[
\Sigma_{\Pi}(\alpha_w) \geq \Sigma_{\Pi_{s'}}(\alpha_w).
\]

**Proof.** Note that, if \(\Pi'\) is a refinement of \(\Pi\) we have

\[
\Sigma_{\Pi'}(\alpha_w) \leq \Sigma_{\Pi}(\alpha_w). \quad (6.1)
\]
In fact, using the previous notation we have

$$\sum_{l=1}^{s} \sum_{l_1=1}^{s} \alpha_{w}(Q_{h_{l-1}, \Pi}, Q_{h_l, \Pi})$$

$$= \sum_{l=1}^{s} \alpha_{w}(Q_{h_{l-1}, \Pi}, Q_{h_{l-1}+1, \Pi}) \leq \sum_{h=1}^{n(\Pi)} \alpha_{w}(Q_{h-1, \Pi}, Q_{h, \Pi}) = \Sigma_{\Pi}(\alpha_{w}).$$

Moreover,

$$\Sigma_{\hat{\Pi}}(\alpha_{w}) \leq \Sigma_{\Pi}(\alpha_{w}). \tag{6.2}$$

In fact, on one hand, by (6.1) we have

$$\Sigma_{\hat{\Pi}}(\alpha_{w}) = \sum_{\beta=1}^{n(\Pi, \hat{\Pi})} \Sigma_{\hat{\Pi}_{\beta}'}(\alpha_{w}) \leq \sum_{\beta=1}^{n(\Pi, \hat{\Pi})} \Sigma_{\hat{\Pi}_{\beta}'}(\alpha_{w}) = \Sigma_{\hat{\Pi}}(\alpha_{w}).$$

On the other, by (4.2) we have

$$\Sigma_{\hat{\Pi}}(\alpha_{w}) = \sum_{\beta=1}^{n(\Pi, \hat{\Pi})} \Sigma_{\hat{\Pi}_{\beta}'}(\alpha_{w})$$

$$= \sum_{\beta=1}^{n(\Pi, \hat{\Pi})} \sum_{h=1}^{n(\Pi, \hat{\Pi})} \alpha_{w}(h, \hat{\Pi}_{\beta})$$

$$= \sum_{\beta=1}^{n(\Pi, \hat{\Pi})} \sum_{h=1}^{n(\Pi, \hat{\Pi})} \alpha_{w}(h, \hat{\Pi}_{\beta}) \leq \sum_{h=1}^{n(\Pi)} \alpha_{w}(h, \Pi) = \Sigma_{\Pi}(\alpha_{w}).$$

In fact we can easily verify that

$$n(\hat{\Pi}_{\beta}) = n_2(\Pi, \hat{\Pi}_{\beta}) - n_1(\Pi, \hat{\Pi}_{\beta}),$$

and for every $h = 1, ..., n(\hat{\Pi}_{\beta})$ we have

$$\hat{w}(h, \hat{\Pi}_{\beta}) = \hat{w}(h + n_1(\Pi, \hat{\Pi}_{\beta}), \Pi),$$

$h + n_1(\Pi, \hat{\Pi}_{\beta}) \in [n_1(\Pi, \hat{\Pi}_{\beta}), n_2(\Pi, \hat{\Pi}_{\beta})]$, and moreover, the intervals $[n_1(\Pi, \hat{\Pi}_{\beta}), n_2(\Pi, \Pi)]$ are mutually disjoint. Thus, (6.2) is proved.

Finally, it is easy to verify that for every path $\Pi$ and every $w^* \in W^*$ we have

$$\Sigma_{w^*(\Pi)}(\alpha_{w}) = \alpha_{w^*} \Sigma_{\Pi}(\alpha_{w}). \tag{6.3}$$

By (6.2) and (6.3), we have

$$\Sigma_{\Pi}(\alpha_{w}) \geq \Sigma_{\hat{\Pi}}(\alpha_{w})$$

$$= \sum_{\beta=1}^{n(\Pi, \hat{\Pi})} \Sigma_{\hat{\Pi}_{\beta}'}(\alpha_{w})$$

$$= \sum_{\beta=1}^{n(\Pi, \hat{\Pi})} \alpha_{w}(h, \Pi) \Sigma_{\hat{\Pi}_{\beta}'}(\alpha_{w})$$
Lemma 6.2. Under the previous assumptions, suppose moreover that for every $s' = 1, ..., r$ we have $\Sigma_{\Pi'}(\omega) \geq 1$. Let $\Pi$ be a path connecting two different points of $V^{(0)}$. Then

$$\Sigma_{\Pi}(\omega) \geq \overline{\alpha}.$$  \hspace{1cm} (6.4)

Proof. Suppose for the moment $\Pi$ is a $\ell_s$-path for some $s = 1, ..., r$, and prove

$$\Sigma_{\Pi}(\omega) \geq 1. \hspace{1cm} (6.5)$$

Let $\overline{m} = \overline{m}(\Pi)$ be the maximum $m$ such that there exists a point in $\Pi$ that lies in $V^{(m)} \setminus V^{(m-1)}$ (here $V^{(-1)} := \emptyset$). Note that we have $Q_{h,\Pi} \in V^{(\overline{m})}$ for every $h$ by the definition of $\overline{m}$.

We prove (6.5) by induction on $\overline{m}$. If $\overline{m} = 0$, then $\Pi$ is a $V^{(0)}$-path, and thus $\alpha_{\omega}(Q_{h-1,\Pi}, Q_{h,\Pi}) = 1$ for every $h = 1, ..., n(\Pi)$, and (6.5) is trivial.

Suppose now (6.5) holds for $\overline{m} \leq m_1 \in \mathbb{N}$ and prove it holds also for $m_1 + 1$. Suppose $\overline{m} = m_1 + 1$. If $\Pi$ is not strong, then there exists $\tilde{h} = 1, ..., n(\Pi)$ such that $Q_{\tilde{h},\Pi}, Q_{\Pi,\Pi} \in V^{(0)}$, thus $\alpha_{\omega}(Q_{\tilde{h}-1,\Pi}, Q_{\Pi,\Pi}) = 1$ and (6.5) is trivial. So, suppose $\Pi$ is strong. For every $\beta = 1, ..., n(\Pi')$, and $h = 1, ..., n(\hat{\Pi})$ let $m_{\beta, \hat{h}} \in \mathbb{N}$ be such that

$$Q_{h,\hat{\Pi}} \in V^{(m_{\beta, \hat{h}})} \setminus V^{(m_{\beta, \hat{h}}-1)}.$$ 

Thus if $m_{\beta, \hat{h}} > 0$, we have

$$Q_{h,\hat{\Pi}} = \psi^\beta_{\omega}(\ell_{\beta, \Pi}, \hat{\Pi})(Q_{h,\hat{\Pi}}) \in V^{(m_{\beta, \hat{h}}+1)} \setminus V^{(m_{\beta, \hat{h}})}.$$ 

In fact, since $\Pi'$ is a strong $V^{(1)}$-path, then $\tilde{w}(\beta, \Pi')$ has length 1, and we can use Corollary 2.5. Now, since $Q_{h,\hat{\Pi}}$ is also a vertex of $\Pi'$, by definition we have $m_{\beta, \hat{h}} + 1 \leq \overline{m} = m_1 + 1$, thus $m_{\beta, \hat{h}} \leq m_1$, and this is trivially valid also if $m_{\beta, \hat{h}} = 0$. By definition, $\overline{m}(\hat{\Pi}) \leq m_1$, and by our inductive hypothesis, $\hat{\Pi}$ satisfies (6.5). By Lemma 6.1, $\Sigma_{\Pi} \alpha_{\omega} \geq \Sigma_{\Pi'}(\omega) \geq 1$, thus $\Pi$ satisfies (6.5).

Now, we prove (6.4). If there exists $\tilde{h} = 1, ..., n(\Pi)$ such that $|\tilde{w}(\tilde{h}, \Pi)| < \overline{m}$, then $\Sigma_{\Pi}(\omega) \geq \alpha_{\ell}(\Pi, \Pi) \geq \overline{\alpha}_{\tilde{m}}$, and (6.4) holds. In the opposite case, by assumption (ii), using notation thereof, we have $\alpha_{\omega} \geq \overline{\alpha}_{\tilde{m}}$, and $\Sigma_{\Pi}(\omega) \geq 1$ by (6.5) since $\Pi$ is a $\ell_s$-path. Also using (6.3), we obtain

$$\Sigma_{\Pi}(\omega) \geq \Sigma_{\psi_{w}^{\omega}}(\Pi)(\omega) = \alpha_{w} \Sigma_{\Pi}(\omega) \geq \overline{\alpha}_{\tilde{m}}.$$
Theorem 6.3. Under the previous assumptions, $\alpha$ is a metric polyratio on $K$ if and only if \( \Sigma_{\Pi_s}(\alpha) \geq 1 \) for every \( s = 1, \ldots, r \).

Proof. We use Corollary 5.8. Suppose there exists an $\alpha$-scaling distance on $K$, hence $\alpha$ is $(T, \Gamma)$-u.p.. Let $\gamma \in \Gamma$ such that $\gamma(t_s) = \Pi_s$. Putting $\Pi = D(\gamma)(P_{i_s})$ and $t_s = (j_1, j_2)$, we obtain

\[
\Pi = D(\gamma)(P_{i_s}) = D(\gamma)(P_{j_1}, P_{j_2}) = \gamma(j_1, j_2) = \Pi_s.
\]

Hence, in view of (4.1) and Lemma 4.2 we get

\[
T^{(\alpha)}_\gamma(e_{i_s}) = T^{(\alpha)}_\gamma(\hat{\Sigma}_{i_s}(\alpha))
\]

\[
= \hat{\Sigma}_{D(\gamma)(P_{i_s})}(\alpha)
\]

\[
= \sum_{h=1}^{n(\Pi)} \alpha(\Pi)(Q_{h-1, \Pi}, Q_{h, \Pi})e_{\hat{\gamma}(h, \Pi)} = 
\]

\[
\sum_{h=1}^{n(\Pi)} \alpha(\Pi)(Q_{h-1, \Pi}, Q_{h, \Pi})e_{\hat{\gamma}(h, \Pi)} = 
\]

\[
\left( \sum_{h=1}^{n(\Pi)} \alpha(\Pi)(Q_{h-1, \Pi}, Q_{h, \Pi}) \right)e_{i_s} = \left( \Sigma_{\Pi_s}(\alpha) \right)e_{i_s}.
\]

Consequently,

\[
(T^{(\alpha)}_\gamma)^n(e_{i_s}) = \left( \Sigma_{\Pi_s}(\alpha) \right)^n e_{i_s}
\]

for every positive integer $n$, thus, since $\alpha$ is $(T, \Gamma)$-u.p., we must have $\Sigma_{\Pi_s}(\alpha) \geq 1$. For the converse, suppose $\Sigma_{\Pi_s}(\alpha) \geq 1$ for every $s = 1, \ldots, r$, and prove that $\alpha$ is $(T, \Gamma)$-u.p. Let $\gamma_1, \ldots, \gamma_n \in \Gamma$ and let $t \in \hat{J}$. Of course, the path $D(\gamma_1) \circ \cdots \circ D(\gamma_n)(P)$ connects two points of $V^{(0)}$. Thus, by Lemma 6.2, we have

\[
\Sigma_{D(\gamma_1) \circ \cdots \circ D(\gamma_n)(P)}(\alpha) \geq \bar{\alpha}_m.
\]

By Lemma 4.2 ii) we have

\[
H\left( T^{(\alpha)}_{\gamma_1} \circ \cdots \circ T^{(\alpha)}_{\gamma_n}(e_t) \right)
\]

\[
= H\left( T^{(\alpha)}_{\gamma_1} \circ \cdots \circ T^{(\alpha)}_{\gamma_n}(\hat{\Sigma}_{P}(\alpha)) \right) = 
\]

\[
H\left( \hat{\Sigma}_{D(\gamma_1) \circ \cdots \circ D(\gamma_n)(P)}(\alpha) \right) = 
\]

\[
\Sigma_{D(\gamma_1) \circ \cdots \circ D(\gamma_n)(P)}(\alpha) \geq \bar{\alpha}_m
\]

and thus $\alpha$ is $(T, \Gamma)$-u.p..
We now apply Theorem 6.3 to two examples. For the Gasket with three 1-cells $V_1, V_2, V_3$, let $P_1, P_2, P_3$ be the three fixed points of the maps, let $Q_{j_1,j_2} = Q_{j_2,j_1} := \psi_{j_1}(P_{j_2}) = \psi_{j_2}(P_{j_1})$ when $j_1, j_2 = 1, 2, 3, j_1 \neq j_2$.

It is simple to verify that the six paths of the form $(P_{j_1}, Q_{j_1,j_2}, P_{j_2})$, $j_1, j_2 = 1, 2, 3$, satisfy (i), (ii), (iii), with $\tilde{m} = 0$, thus, in view of Theorem 6.3, $\alpha$ is a metric polyratio on the Gasket if and only if we have

$$\alpha_1 + \alpha_2 \geq 1, \quad \alpha_1 + \alpha_3 \geq 1, \quad \alpha_2 + \alpha_3 \geq 1.$$  \hfill (6.6)

In the Vicsek set, let $\psi_i, i = 1, 2, 3, 4, 5$ be the contractions defining it, and we order them in such a way that $\psi_5$ the contraction that fixes the center. Let $P_j$ be the fixed points of $V_j$ for $j = 1, 2, 3, 4$. Choose the order so that $P_1$ is opposite to $P_3$ and $P_2$ is opposite to $P_4$. Let $Q_j$ the only point in $V_j \cap V_5$. Now we take the four paths of the form $\Pi_j := (P_j, Q_j, Q_j', P_j')$ when $j = 1, 2, 3, 4$ and $P_j'$ is opposite to $P_j$. It is simple to verify that such paths satisfy (i), (ii), (iii), with $\tilde{m} = 1$.

Thus, by Theorem 6.3, $\alpha$ is a metric polyratio on the Vicsek set if and only if we have

$$\alpha_1 + \alpha_3 + \alpha_5 \geq 1, \quad \alpha_2 + \alpha_4 + \alpha_5 \geq 1.$$  \hfill (6.7)

Considerations similar could be extended to other fractals. However, in order to have simple conditions like (6.6) or (6.7) the structure of the fractal should be simple. In most cases, I expect that it could be hard to give simple necessary and sufficient conditions for having an $\alpha$-scaling distance.

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