A note on a separating system of rational invariants for finite dimensional generic algebras

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Abstract. The paper deals with a construction of a separating system of rational invariants for finite dimensional generic algebras. In the process of dealing an approach to a rough classification of finite dimensional algebras is offered by attaching them some quadratic forms.

1. Introduction
In [1] we have offered an approach to classification problem of finite dimensional algebras with respect to basis changes. It has been shown that if there is a special map with certain properties then it is possible to classify (to list canonical representatives) of all algebras who’s set of structural constants do not nullify a certain polynomial. In this case one can provide a separating system of rational invariants for the algebras considered. The approach was successfully applied in [2] to get a complete classification of all 2-dimensional algebras over algebraically closed fields.

Unfortunately, so far we have no example of such a map in 3-dimensional case. Therefore in the current paper we deal with a weaker problem, namely with a construction of separating system of rational invariants for finite dimensional generic algebras. The existence such of system of invariants is known [3]. By generic algebras we mean the set of all algebras who’s system of structural constants does not nullify a fixed nonzero polynomial in structural variables, over the basic field $F$. In process of dealing with the problem we show a way for a rough classification of finite dimensional algebras by attaching them some quadratic forms.

2. Main results
Further, whenever $A = (a_{ij}) \in Mat(p \times q, F)$, $B \in Mat(p' \times q', F)$ we use $A \otimes B$ for the matrix
\[
\begin{pmatrix}
a_{11}B & a_{12}B & \ldots & a_{1q}B \\
a_{21}B & a_{22}B & \ldots & a_{2q}B \\
& \ddots & \ddots & \ddots \\
a_{p1}B & a_{p2}B & \ldots & a_{pq}B
\end{pmatrix},
\]
where $F$ is a field of characteristic not 2.

Let us consider an $m$-dimensional algebra $A$ with multiplication $\cdot$ given by a bilinear map $(u, v) \mapsto u \cdot v$. If $e = (e_1, e_2, \ldots, e_m)$ is a basis for $A$ then one can represent the bilinear map by a matrix
\[
A_e = (A_{e_{jk}})_{i,j,k=1,2,\ldots,m} \in Mat(m \times m^2; F),
\]
where $e_j \cdot e_k = e_1 A_{e_{1jk}} + e_2 A_{e_{2jk}} + \ldots + e_m A_{e_{mk}}$, $j, k = 1, 2, \ldots, m$, such that $u \cdot v = e A_e (u \otimes v)$ for any $u = eu, v = ev$, where $u = (u_1, u_2, \ldots, u_m), v = (v_1, v_2, \ldots, v_m)$ are column vectors. So
the algebra $A$ is presented by the matrix $A_\tau \in Mat(m \times m^2; F)$ called the matrix of structure constants (MSC) of $A$ with respect to the basis $e$.

If $e' = (e'_1, e'_2, \ldots, e'_m)$ is also a basis for $A$, $g \in GL(m, F)$, $e'g = e$ then it is well known that

$$A_{e'} = gA_{e}(g^{-1})^\otimes 2.$$  

Further a basis $e$ is fixed and therefore instead of $A_\tau$ we use $A$, we do not make difference between $A$ and its matrix $A$. Let $X = (X_{jk})_{i,j,k=1,2,\ldots,m}$ stand for a variable matrix and $Tr_1(X)$, $Tr_2(X)$ stand for the row vectors

$$\left( \sum_{i=1}^{m} X_{i1}^i, \sum_{i=1}^{m} X_{i2}^i, \ldots, \sum_{i=1}^{m} X_{im}^i \right), \quad \left( \sum_{i=1}^{m} X_{1i}^i, \sum_{i=1}^{m} X_{2i}^i, \ldots, \sum_{i=1}^{m} X_{mi}^i \right),$$

respectively.

We use $\tau$ for the representation of $GL(m, F)$ on the $m^3$-dimensional vector space $V = Mat(m \times m^2; F)$ defined by $\tau : (g, A) \mapsto B = gA(g^{-1} \otimes g^{-1})$.

For simplicity instead of $\tau$-equivalent, $\tau$-invariant we use the terms equivalent and invariant, respectively.

We represent each MSC $A$ as a row vector with entries from $Mat(m, F)$ by parting it consequently into elements of $Mat(m, F)$:

$$A = (A_1, A_2, \ldots, A_m), \quad A_1, A_2, \ldots, A_m \in Mat(m, F).$$

If $C$ is a block matrix with blocks from $Mat(m, F)$ we use notation $C^\tau$, where $\ast$ is the tensor product or transpose operation, to mean that the operation $\ast$ with $C$ is done “$Mat(m, F)$” (not over $F$), for example for the above presented “row” matrix $A$ the notation $A^\tau$ means “column” vector over $Mat(m, F)$ with the entries $A_1, A_2, \ldots, A_m$.

$$A^\otimes 2 = (A_1^2, A_1 A_2, A_1 A_3, A_2 A_1, A_2^2, A_2 A_3, A_3 A_1, A_3 A_2, A_3^2).$$

One can see that the equality $B = gA(g^{-1} \otimes g^{-1})$ can be represented as

$$B = (B_1, B_2, \ldots, B_m) = gA(g^{-1})^\otimes 2 = (gA_1 g^{-1}, gA_2 g^{-1}, \ldots, gA_m g^{-1})(g^{-1} \otimes I),$$

where $I$ is $m \times m$ identity matrix. Moreover, for any matrices $C$ and $D$ the equality

$$(C \otimes I) \otimes (D \otimes I) = (C \otimes D) \otimes I$$

holds true. Therefore, the following equalities hold true

$$(B_1, B_2, \ldots, B_m)^\otimes k = (gA_1 g^{-1}, gA_2 g^{-1}, \ldots, gA_m g^{-1})^\otimes k((g^{-1})^\otimes k \otimes I),$$

$$\left( \begin{array}{c} B_1 \\ B_2 \\ \vdots \\ B_m \end{array} \right)^\otimes k = \left( \begin{array}{cccc} B_1^2 & B_1 B_2 & \cdots & B_1 B_m \\ B_2 B_1 & B_2^2 & \cdots & B_2 B_m \\ \vdots & \vdots & \ddots & \vdots \\ B_m B_1 & B_m B_2 & \cdots & B_m^2 \end{array} \right)^\otimes k.$$
\((((g^t)^{-1})^\otimes k \otimes I)\left( \begin{array}{ccc}
 gA_1^2g^{-1} & gA_1A_2g^{-1} & \cdots & gA_1A_mg^{-1} \\
 gA_2A_1g^{-1} & gA_2^2g^{-1} & \cdots & gA_2A_mg^{-1} \\
 \vdots & \vdots & \ddots & \vdots \\
 gA_mA_1g^{-1} & gA_mA_2g^{-1} & \cdots & gA_m^2g^{-1}
\end{array} \right) \overline{k} \right) ((g^{-1})^\otimes k \otimes I).\)

Component-wise application of trace to this equality, which is denoted by \(\tilde{Tr}\) results in

\[
\tilde{Tr}\left( \left( \begin{array}{ccc}
 B_1^2 & B_1B_2 & \cdots & B_1B_m \\
 B_2B_1 & B_2^2 & \cdots & B_2B_m \\
 \vdots & \vdots & \ddots & \vdots \\
 B_mB_1 & B_mB_2 & \cdots & B_m^2
\end{array} \right) \overline{k} \right) = ((g^{-1})^\otimes k) \tilde{Tr}\left( \left( \begin{array}{ccc}
 gA_1^2g^{-1} & gA_1A_2g^{-1} & \cdots & gA_1A_mg^{-1} \\
 gA_2A_1g^{-1} & gA_2^2g^{-1} & \cdots & gA_2A_mg^{-1} \\
 \vdots & \vdots & \ddots & \vdots \\
 gA_mA_1g^{-1} & gA_mA_2g^{-1} & \cdots & gA_m^2g^{-1}
\end{array} \right) \overline{k} \right) (g^{-1})^\otimes k,
\]

as far as for any matrices \(C, D\) and \(E\), where \(D\) is a block matrix with blocks from \(Mat(m, F)\) and \((C \otimes I)D(E \otimes I)\) has a meaning, the equality

\[
\tilde{Tr}((C \otimes I)D(E \otimes I)) = CT\tilde{r}(D)E
\]

is valid. One can represent the above obtained matrix equality in a compact form as follows

\[
\tilde{Tr}((B^T B)\overline{k}) = ((g^{-1})^\otimes k)^t \tilde{Tr}((A^T A)\overline{k}) (g^{-1})^\otimes k.
\]

Note that \(\tilde{Tr}((A^T A)\overline{k})\) is a symmetric matrix. The obtained equality yields the following theorem.

**Theorem 1.** Invariants of the quadratic form given by the matrix \(\tilde{Tr}((X^T X)\overline{k})\) are invariants of the \(m\)-dimensional algebra defined by MSC \(X\).

This result can be used for a rough classification of finite dimensional algebras: Two \(m\)-dimensional algebras \(A, B\) are roughly equivalent if the quadratic forms given by matrices

\[
\tilde{Tr}(A^T A) = \left( \begin{array}{ccc}
 Tr(A_1^2) & Tr(A_1A_2) & \cdots & Tr(A_1A_m) \\
 Tr(A_2A_1) & Tr(A_2^2) & \cdots & Tr(A_2A_m) \\
 \vdots & \vdots & \ddots & \vdots \\
 Tr(A_mA_1) & Tr(A_mA_2) & \cdots & Tr(A_m^2)
\end{array} \right),
\]

\[
\tilde{Tr}(B^T B) = \left( \begin{array}{ccc}
 Tr(B_1^2) & Tr(B_1B_2) & \cdots & Tr(B_1B_m) \\
 Tr(B_2B_1) & Tr(B_2^2) & \cdots & Tr(B_2B_m) \\
 \vdots & \vdots & \ddots & \vdots \\
 Tr(B_mB_1) & Tr(B_mB_2) & \cdots & Tr(B_m^2)
\end{array} \right).
\]
are equivalent.

It is clear that entries of $T_r(X^T X)$ are polynomials in components of $X$ and there exists nonsingular matrix $Q(X^T X)$ with rational entries in $X$ such that the matrix

$$T_r(X^T X) = (Q(X^T X)^{-1})^T T_r(X^T X) Q(X^T X)^{-1} = D(X)$$

is a diagonal matrix and $Q(g) = I$, whenever $g$ is a nonsingular diagonal matrix and $X = \tau(Q(X^T X), X)$.

Therefore in algebraically closed field $F$ case one can define a nonempty invariant open subset $V_0 \subset V$ such that $T_r(\overline{A}) = D(A)$ and $D(A)$ is nonsingular whenever $A \in V_0$.

**Theorem 2.** Two algebras $A, B \in V_0$ are isomorphic if and only if

$$\mathcal{B} = \tau(g_0, \overline{A})$$

for some $g_0 \in GL(m, F)$ such that $g_0^T D(B) g_0 = D(A)$.

**Proof.** If $B = \tau(g, A)$ then $\mathcal{B} = \tau(Q(B^T B), B) = \tau(Q(B^T B), \tau(g, A)) = \tau(Q(B^T B)g) Q(A^T A)^{-1}, \tau(Q(A^T A), A) = \tau(Q(B^T B)g Q(A^T A)^{-1}, \overline{A})$, and for $g_0 = Q(B^T B)g Q(A^T A)^{-1}$ one has

$$g_0^T D(B) g_0 = (Q(B^T B)g Q(A^T A)^{-1})^T D(B) Q(B^T B) g Q(A^T A)^{-1} =
\quad (Q(A^T A)^{-1})^T (g^T (Q(B^T B)^T D(B) Q(B^T B)) g) Q(A^T A)^{-1} =
\quad (Q(A^T A)^{-1})^T (g^T (T_r(B^T B)) g) Q(A^T A)^{-1} =
\quad (Q(A^T A)^{-1})^T T_r(A^T A) Q(A^T A)^{-1} = D(A).$$

Visa versa if $\mathcal{B} = \tau(g_0, \overline{A})$ for some $g_0$ such that $g_0^T D(B) g_0 = D(A)$ then for $g = Q(B^T B)^{-1} g_0 Q(A^T A)$ one has

$$\tau(g, A) = \tau(Q(B^T B)^{-1} g_0 Q(A^T A), A) = \tau(Q(B^T B)^{-1} g_0, \tau(Q(A^T A), A) =
\tau(Q(B^T B)^{-1} g_0, \overline{A}) = \tau(Q(B^T B)^{-1}, \tau(g_0, \overline{A})) = \tau(Q(B^T B)^{-1}, \overline{B}) = B.$$

Assume that there exists a matrix $P(X)$, with entries consisting of rational functions of components $X$, such that $P(\overline{A})$ is nonsingular for any $A \in V_0$ and the equality

$$P(\tau(g, A)) = P(\overline{A}) g^{-1}$$

holds true whenever $g^T D(\tau(g, A)) g = D(A). \quad (1)$

**Theorem 3.** For $A, B \in V_0$ there exists $g_0 \in GL(m, F)$ such that $g_0^T D(B) g_0 = D(A)$ and $\mathcal{B} = \tau(g_0, \overline{A})$ if and only if

$$\tau(P(\overline{B}), \overline{B}) = \tau(P(\overline{A}), \overline{A}), \quad (P(\overline{B})^{-1})^T D(B) P(\overline{B})^{-1} = (P(\overline{A})^{-1})^T D(A) P(\overline{A})^{-1}.$$

**Proof.** If $\mathcal{B} = \tau(g_0, \overline{A})$ and $g_0^T D(B) g_0 = D(A)$ then

$$\tau(P(\overline{B}), \overline{B}) = \tau(P(\tau(g_0, \overline{A})), \tau(g_0, \overline{A})) = \tau(P(\overline{A}) g_0^{-1}, \tau(g_0, \overline{A})) = \tau(P(\overline{A}), \overline{A})$$

and

$$((P(\overline{A}) g_0^{-1})^{-1})^T D(B) (P(\overline{A}) g_0^{-1})^{-1} = ((P(\overline{A})^{-1})^{-1})^T D(A) (P(\overline{A})^{-1})^{-1} =
\quad ((P(\overline{A})^{-1})^{-1})^T g_0^T D(B) g_0 P(\overline{A})^{-1} = (P(\overline{A})^{-1})^T D(A) P(\overline{A})^{-1}.$$
Visa versa, if equalities
\[ \tau(P(\overline{B}), \overline{B}) = \tau(P(\overline{A}), \overline{A}) \text{, } (P(\overline{B})^{-1})^t D(B)P(\overline{B})^{-1} = (P(\overline{A})^{-1})^t D(A)P(\overline{A})^{-1} \]
are valid then for \( g_0 = P(\overline{B})^{-1}P(\overline{A}) \) one has \( g_0^t D(B)g_0 = D(A) \) and
\[ \tau(g_0, \overline{A}) = \tau(P(\overline{B})^{-1} P(\overline{A}), \overline{A}) = \tau(P(\overline{B})^{-1}, \tau(P(\overline{A}), \overline{A})) = \tau(P(\overline{B})^{-1}, \tau(P(\overline{B}), \overline{B})) = \overline{B}. \]

So Theorems 2 and 3 imply that the system of entries of matrices
\[ \tau(P(\overline{X}), \overline{X}), \text{ } (P(\overline{X})^{-1})^t \tilde{T}_r(\overline{X}^t \overline{X})P(\overline{X})^{-1} \]
is a separating system of rational invariants for algebras from \( V_0 \).

The above presented results show the importance of construction of matrix \( P(X) \) with property (1). Further we discuss a construction of such matrix by the use of rows \( r(\overline{A}) \) for which the equality \( r(\tau(g, \overline{A})) = r(\overline{A})g^{-1} \) is valid, whenever \( g^t D(\tau(g, A))g = D(A) \). To construct such rows one can use the following approach.

Assume that the equalities \( \overline{B} = gA(g^{-1})^2 \), \( \tilde{C} = gCg^t \) are true, where \( C^t = C \) and \( C \) is a nonsingular matrix. In this case
\[ \tilde{C}^\otimes 2 = g^\otimes 2 C^\otimes 2 (g^{-2})^t, \text{ } \overline{B} \tilde{C}^\otimes 2 = g\overline{A}C^\otimes 2 (g^{-2})^t, \text{ } \tilde{C}^\otimes 2 \overline{B}^t = g^\otimes 2 C^\otimes 2 \overline{A}^t g^t \]
and \( \overline{B} \tilde{C}^\otimes 2 \overline{B}^t = g\overline{A}C^\otimes 2 \overline{A}^t g^t \).

On induction it is easy to see that for any natural \( k \) the equality
\[ \overline{B}^\otimes 2^0 B^\otimes 2^1 \ldots B^\otimes 2^k - 1 \tilde{C}^\otimes 2^k (B^\otimes 2^0 \overline{B}^\otimes 2^1 \ldots B^\otimes 2^k - 1)^t = g\overline{A}^\otimes 2^0 \overline{A}^\otimes 2^1 \ldots \overline{A}^\otimes 2^k - 1 C^\otimes 2^k (\overline{A}^\otimes 2^0 \overline{A}^\otimes 2^1 \ldots \overline{A}^\otimes 2^k - 1)^t g^t \]
holds true. Therefore due to the equalities
\[ g\overline{A}^\otimes 2^0 \overline{A}^\otimes 2^1 \ldots \overline{A}^\otimes 2^k - 1 C^\otimes 2^k (\overline{A}^\otimes 2^0 \overline{A}^\otimes 2^1 \ldots \overline{A}^\otimes 2^k - 1)^t g^{-1}, \]
and \( B^\otimes 2^0 B^\otimes 2^1 \ldots B^\otimes 2^k - 1 \tilde{C}^\otimes 2^k (B^\otimes 2^0 \overline{B}^\otimes 2^1 \ldots B^\otimes 2^k - 1)^t \tilde{C}^{-1} \) one has
\[ Tr_{i}(B^\otimes 2^0 B^\otimes 2^1 \ldots B^\otimes 2^k - 1 \tilde{C}^\otimes 2^k (B^\otimes 2^0 \overline{B}^\otimes 2^1 \ldots B^\otimes 2^k - 1)^t \tilde{C}^{-1} B) = \]
\[ Tr_{i}(\overline{A}^\otimes 2^0 \overline{A}^\otimes 2^1 \ldots \overline{A}^\otimes 2^k - 1 C^\otimes 2^k (\overline{A}^\otimes 2^0 \overline{A}^\otimes 2^1 \ldots \overline{A}^\otimes 2^k - 1)^t C^{-1} \overline{A}) g^{-1}, \text{ } i = 1, 2. \]

The last equality shows that in our algebra case on can try to construct the needed matrix \( P(X) \) by the use of rows
\[ Tr_{i}(X^\otimes 2^0 X^\otimes 2^1 \ldots X^\otimes 2^{(k-1)} ((\tilde{T}_r(X^t X^{-1}))^{-1})^\otimes 2^k (X^\otimes 2^0 X^\otimes 2^1 \ldots X^\otimes 2^{(k-1)})^t \tilde{T}_r(X^t X) X), \]
where \( i = 1, 2, \text{ } k = 0, 1, 2, \ldots \) What is left here unjustified is that one should justify the existence, in general, of a linear independent system consisting of \( m \) such rows.

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References
[1] Bekbaev U 2017, *IOP Conf. Series: Journal of Physics: Conf. Series* 819, pp. 2-9.
[2] H. Ahmed, U. Bekbaev, I. Rakhimov 2017, *AIP Conference Proceedings*, 1830, 070016; doi: 10.1063/1.4980965.
[3] V. Popov 2014, arXiv: 1411.6570v2/math.AG, pp. 1-20.