On the Sum-Capacity of Two-User Optical Intensity Multiple Access Channels

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Abstract—This paper investigates the sum-capacity of two-user optical intensity multiple access channels with a per-user peak- or average-intensity constraint. By leveraging tools on decomposition of random variables, we derive lower bounds on the sum-capacity. In the high signal-to-noise ratio (SNR) regime, they asymptotically match the sum-capacity. In particular, for the peak-intensity constrained channel, our result closes the high-SNR asymptotic sum-capacity gap in the existing works.

Index Terms—intensity modulation-direct detection, multiple access channel, sum-capacity

I. INTRODUCTION

With the abundance of the optical spectrum, optical wireless communications (OWC) has become a promising solution to the spectrum scarcity of conventional radio-frequencies (RF) communication. It also has many merits in comparison to RF communication, including higher data-rates, higher security or locality, and electromagnetic compatibility. Many current OWC systems use the intensity modulation and direct detection (IM-DD) transmission scheme. In such systems, information is carried on the modulated intensity of the emitted light, and receivers use photodetectors to measure incoming optical intensity. A widely adopted channel model for the IM-DD based OWC system is the Gaussian optical intensity channel, which captures key properties including nonnegativity of optical intensity, input-independent additive Gaussian noise, and practical constraints on limited average and/or peak optical intensity. Based on this model, there have been extensive studies on capacity analysis [1], [2], [3], [4], [5], [6], [7], [8] and coding and modulation constructions [9], [10], [11], [12], [13] in the literature.

In indoor environment, OWC can be used for serving multiple users (or devices) simultaneously. Multiuser OWC has received lots of research attention in recent years. Capacity region analysis of several multiuser OWC channels including multiple access channels (MACs) [14], [15], broadcast channels [16], [17] and interference channels [18], etc., have been considered. In this paper, we consider a two-user MAC with a per-user peak- or average-intensity constraint, and mainly focus on its sum-capacity. The existing works most related to ours are [14], [15]. In [14], inner and upper bounds on the capacity region of the MAC have been established where peak- and average-intensity constraints are imposed on each user. The low-SNR asymptotic capacity region is characterized, while there is still some gap between the bounds at high SNR. The authors in [15] consider the MAC with a per-user average-intensity constraint or a per-user peak-intensity constraint. For the average-power constrained case, asymptotically tight inner and outer bounds are derived at high SNR. For the peak-intensity constrained case, at high SNR the asymptotic capacity gap between the bounds is bounded within 0.09 bits. The key idea in these papers is utilizing capacity results of two additive noise channels where the noises obey certain maxentropic distributions.

Different with above mentioned works, in this paper we instead leverage tools on decomposition of random variables. We first establish that a uniformly distributed or an exponentially distributed random variable can be decomposed as a sum of two independent random variables while satisfying certain support or moment constraints. Then we apply these results to the two-user MAC to derive lower bounds on sum-capacity. In both considered cases the derived lower bounds asymptotically match the sum-capacity at high SNR, thus characterizing the high-SNR asymptotic sum-capacity. Specifically, for the peak-intensity case, our result close the high-SNR asymptotic capacity gap shown in [15].

II. CHANNEL MODEL

Consider a two-user MAC, and the channel output is given by

$$Y = X_1 + X_2 + Z,$$

where $X_j$, $j = 1, 2$, denotes the channel input from user $j$, and $Z$ denotes Gaussian noise with variance $\sigma^2$, i.e.,

$$Z \sim \mathcal{N}(0, \sigma^2).$$

Since $X_j$ is proportional to the optical intensity, its support must satisfy

$$\text{supp } X_j \subset \mathbb{R}^+, \quad j = 1, 2.$$  

We consider two types of input intensity constraints. One is the per-user peak-intensity constraint:

$$\Pr(X_j \leq A_j) = 1, \quad j = 1, 2,$$

where constant $A_j$ denotes the peak intensity constraint of user $j$. The other one is the per-user average-intensity constraint:

$$\mathbb{E}[X_j] \leq E_j, \quad j = 1, 2,$$
where constant $E_j$ denotes the average intensity constraint of user $j$. Without loss of generality, in this paper we assume $A_1 + A_2 = 1$, and $E_1 + E_2 = 1$. Then the signal-to-noise ratio (SNR) at the receiver side can be defined as

$$\text{SNR} = \frac{1}{\sigma}.\quad (6)$$

We mainly focus on the sum-capacity of the MAC in the paper. The sum-capacity is defined as the maximum of the sum of rate pairs in capacity region $C$, i.e.,

$$C_{\text{sum}} = \max \left\{ R_1 + R_2 : (R_1, R_2) \in C \right\}.\quad (7)$$

It is shown in [19] that the sum-capacity can be expressed as

$$C_{\text{sum}} = \max_{p_1(X_1)p_2(X_2)} I(X_1 + X_2; Y),\quad (8)$$

where the maximum is over the product of all feasible input distributions $p_j(X_j)$ of user $j$. We use $C_{\text{p-sum}}$ and $C_{\text{a-sum}}$ to denote the sum-capacity of the peak-intensity and average-intensity constrained MAC, respectively.

### III. DECOMPOSITION OF RANDOM VARIABLES

#### A. Decomposition of a uniform random variable

It is known that any real number $a \in [0, 1]$ can be uniquely written in the form$^2$

$$a = \sum_{j=1}^{\infty} \epsilon_a(j) \cdot 2^{-j},\quad (9)$$

where $\epsilon_a(j)$ is 0 or 1. We further define an index set:

$$I_a = \{ j \mid \epsilon_a(j) = 1, j \in \mathbb{N} \}.\quad (10)$$

Now we present the result on the decomposition of a uniformly distributed random variable.

**Lemma 1.** Given $a \in (0,1)$, consider a random variable $U$ uniformly distributed on interval $[0,1]$. Then $U$ can be decomposed as a sum of two independent random variables, i.e.,

$$U = U_1 + U_2,\quad (11)$$

where $U_1 = \sum_{j \in I_a} B_j \cdot 2^{-j}$, and $U_2 = \sum_{j \notin I_a} B_j \cdot 2^{-j}$, with $B_j$’s being independent and identically distributed Bernoulli random variables and $\Pr(B_j = 1) = \frac{1}{2}$.

**Proof:** We first use the well known fact that the binary expansion of $U$ can be written as

$$U = \sum_{j=1}^{\infty} B_j \cdot 2^{-j}.\quad (12)$$

Then by $I_a \cup I_0^c = \mathbb{N}_+$, we have

$$U = \sum_{j \in I_a} B_j \cdot 2^{-j} + \sum_{j \in I_0^c} B_j \cdot 2^{-j}\quad (13)$$

$$= U_1 + U_2.\quad (14)$$

The independence of $U_1$ and $U_2$ can be directly derived from the independence of $B_j$’s.

#### B. Decomposition of an exponential random variable

We first present the existing result on the dyadic expansion of an exponential random variable. To make the paper more self-contained, we give a proof of the following proposition based on characteristic functions (c.f.) of random variables in Appendix 1.

**Proposition 1** ([20]). Given $\lambda \in (0, \infty)$, and consider a random variable $U$ exponentially distributed with parameter $\lambda$, i.e.,

$$f_U(u) = \lambda e^{-\lambda u},\quad u \geq 0.\quad (15)$$

Then

$$U = \sum_{j = -\infty}^{\infty} B_j \cdot 2^j,\quad (16)$$

where $B_j$’s are independently distributed Bernoulli random variables and $\Pr(B_j = 1) = \frac{1}{1 + e^{\lambda}}$.

Now we present the result on the decomposition of an exponential random variable.

**Lemma 2.** Given $a \in (0,1)$, and consider a random variable $U$ exponentially distributed with parameter 1. There exists a set $\mathcal{I} \subset \mathbb{Z}$ such that $U$ can be decomposed as a sum of two independent random variables, i.e.,

$$U = U_1 + U_2,\quad (17)$$

where $U_1 = \sum_{j \in \mathcal{I}} B_j \cdot 2^{-j}$, and $U_2 = \sum_{j \notin \mathcal{I}} B_j \cdot 2^{-j}$, with $B_j$’s being independent and identically distributed Bernoulli random variables and $\Pr(B_j = 1) = \frac{1}{1 + e^{\lambda}}$.

**Proof:** Let $\lambda = 1$, and take expectation at both sides in (16) yielding

$$\sum_{j = -\infty}^{\infty} \frac{2^j}{1 + e^{\lambda}} = 1.\quad (18)$$

It is sufficient to show there exists a set $\mathcal{I} \subset \mathbb{Z}$ such that

$$\sum_{j \in \mathcal{I}} \frac{2^j}{1 + e^\lambda} = a.\quad (19)$$

To prove this, we will apply the Guthrie-Nymann theorem [21] to show the set of all subsums of the infinite series in (18) forms the closed interval $[0,1]$. To this end, we first rearrange and merge the sequence in (18) to form a new infinite series that satisfies required conditions on the theorem.
Denote the sequence in (18) by \(a_j = \frac{2^j}{1 + \epsilon^j} \) for \(j \in \mathbb{Z}\), and define a new sequence \((b_n)_{n \geq 0}\), where

\[
b_n = \begin{cases} 
a_0 & \text{if } n = 0, 
a_{n-1} & \text{if } n = 1, 
a_1 & \text{if } n = 2, 
a_{n-2} & \text{if } n = 3, 
a_2 & \text{if } n = 4, 
a_{n-4} + a_{n-2} & \text{if } n \geq 5.
\end{cases}
\]

Then by (18) it is direct to see

\[
\sum_{j=0}^{\infty} b_j = 1.
\]

We further define the \(n\)th “tail” of above series as

\[
r_n = \sum_{j=n+1}^{\infty} b_j.
\]

It is proved in Appendix B that sequence \((b_n)_{n \geq 0}\) is monotonically decreasing, and satisfies

\[
b_n \leq r_n, \quad \forall n \in \mathbb{N}.
\]

By the Guthrie-Nymann theorem, the set of subsums of the new infinite series in (21) indeed forms a closed interval. Since the least upper bound and greatest lower bound of the series are 0 and 1, respectively, the formed interval is \([0, 1]\). Hence there must exist an index set \(\mathcal{J} \subseteq \mathbb{N}\) such that

\[
\sum_{j \in \mathcal{J}} b_j = a. \quad (24)
\]

Then by (18) there exists a set \(\mathcal{I} \subseteq \mathbb{Z}\) satisfying (19). The proof is concluded.

**Remark 1.** The conclusion on the existence of set \(\mathcal{J}\) in above proof does not hold for general convergent series. In fact, the set of subsum of infinite convergent series are a closed interval, a finite union of closed intervals, or homeomorphic to Cantor set, or a mixed of these three (also known as Cantorval). The interested readers can be referred to [21] for more details.

**IV. CAPACITY RESULTS**

In this section we respectively present derived capacity bounds on the peak- and average-intensity constrained MAC based on the decomposition results in Section III-B.

**A. Peak-Intensity Constrained MAC**

The following upper bound of sum-capacity is based on the result of the single-user channel with peak-intensity constraint in [2], [15].

**Proposition 2** ([2], [15]). The sum-capacity of peak-intensity constrained MAC is upper bounded by

\[
C_{\text{p-sum}} \leq \min \left\{ \frac{1}{2} \log \left( 1 + \frac{1}{4\epsilon^2} \right), \log \left( 1 + \frac{1}{\sqrt{2\pi\epsilon}} \right) \right\}.
\]

Now we present our lower bound by applying the results in Lemma 1.

**Proposition 3.** The sum-capacity of peak-intensity constrained MAC is lower bounded by

\[
C_{\text{p-sum}} \geq \frac{1}{2} \log \left( 1 + \frac{1}{2\pi\epsilon^2} \right).
\]

**Proof:** Let \(a = A_1\) in Lemma 1. By the assumption \(A_1 + A_2 = 1\), it is direct to see the support of \(U_1\) and \(U_2\) are subsets of \([0, A_1]\) and \([0, A_2]\), respectively. Also, by Lemma 1, \(U_1\) and \(U_2\) are independent. Let \(X_1 = U_1\), and \(X_2 = U_2\), then \((X_1, X_2)\) forms an admissible input pair. By the Entropy Power Inequality (EPI), combined with the fact that \(X_1 + X_2\) is uniformly distributed on \([0, 1]\), we have

\[
I(X_1 + X_2; Y) \geq \frac{1}{2} \log \left( 1 + e^{\text{h}(X_1 + X_2) - 2\text{h}(Z)} \right)
\]

\[
= \frac{1}{2} \log \left( 1 + \frac{1}{2\pi\epsilon^2} \right).
\]

The proof is concluded.

Combined with (25) and (26), and let \(\epsilon \to 0\), we characterize the high-SNR asymptotic capacity.

**Corollary 1.** The high-SNR asymptotic capacity of peak-intensity constrained MAC is given by

\[
\lim_{\epsilon \to 0} \{C_{\text{p-sum}} + \log \epsilon\} = -\frac{1}{2} \log(2\pi\epsilon).
\]

Fig. 1 depicts derived sum-capacity bounds for the peak-intensity constrained MAC by one example. At high SNR, we can see the closed-form lower bound asymptotically match the capacity. The lower bound in [15] is close to the derived bound here, but there still exists a nonvanishing capacity gap between them at high SNR. It should be noted that there may exist better lower bounds at low SNR [1], [3], but here we mainly focus on the high-SNR regime.
B. Average-Intensity Constrained MAC

Similar with Section IV-A, the following upper bound of the sum-capacity is based on the result of single-user channel with average-intensity constraint in [1], [15].

**Proposition 4** ([1], [15]). The sum-capacity of average-intensity constrained MAC is upper bounded by

$$C_{a\text{-sum}} \leq \frac{1}{2} \log \left( \frac{e}{2\pi} \left( \frac{1}{\sigma} + 2 \right)^2 \right).$$  \hspace{1cm} (30)

Now we present our lower bound by applying the results in Lemma 2.

**Proposition 5.** The sum-capacity of average-intensity constrained MAC is lower bounded by

$$C_{a\text{-sum}} \geq \frac{1}{2} \log \left( 1 + \frac{e}{2\pi \sigma^2} \right).$$  \hspace{1cm} (31)

**Proof:** Let $a = A_1$ in Lemma 2, and let $X_1 = U_1$, and $X_2 = U_2$. Then the proof can be completed by following the similar lines as in the proof of Proposition 3.

Combined with (30) and (31), and let $\sigma \to 0$, we characterize the high-SNR asymptotic capacity.

**Corollary 2.** The high-SNR asymptotic capacity of average-intensity constrained MAC is given by

$$\lim_{\sigma \to 0} \{C_{a\text{-sum}} + \log \sigma\} = \frac{1}{2} \log \frac{e}{2\pi}. \hspace{1cm} (32)$$

Fig. 2 plots the sum-capacity bounds for the average-intensity constrained MAC by one example. At high SNR, we can see the derived lower bound asymptotically matches the upper bound.

V. CONCLUSION

In this paper we propose methods on decomposing a uniformly or exponentially distributed random variable as a sum of two independent random variables based on its binary expansion. With the decomposition results, we further derive capacity lower bounds on the sum-capacity of two-user MAC when the input is subject to a peak- or average-intensity power constraint. These lower bounds are asymptotically tight at high SNR, thus characterizing high-SNR asymptotic capacities.

APPENDIX

A. Proof of Eq. (16)

The c.f. of $2^j \cdot B_j$ is

$$\phi_{2^j \cdot B_j}(t) = E[e^{it2^j \cdot B_j}] = \frac{1 + e^{-(\lambda-it)2^j}}{1 + e^{-\lambda2^j}}. \hspace{1cm} (33)$$

Then the c.f. of $U$ is

$$\phi_U(t) = \prod_{j=-\infty}^{\infty} \frac{1 + e^{-(\lambda-it)2^j}}{1 + e^{-\lambda2^j}}. \hspace{1cm} (34)$$

Using the relation

$$\prod_{j=0}^{n} \frac{1 + e^{2^jz}}{1 + e^{2^j}} = \frac{1 - e^{-2^{n+1}z}}{1 - e^{-2z}}, \hspace{1cm} (35)$$

we have

$$\prod_{j=0}^{n} \frac{1 + e^{-(\lambda-it)2^j}}{1 + e^{-\lambda2^j}} = \frac{1 - e^{-(\lambda-it)2^{n+1}}}{1 - e^{-(\lambda-it)2}}, \hspace{1cm} (36)$$

Following above similar arguments, we can derive

$$\prod_{j=-1}^{-n} \frac{1 + e^{-(\lambda-it)2^j}}{1 + e^{-\lambda2^j}} = \frac{1 - e^{-\lambda2^{-n}}}{1 - e^{-(\lambda-it)2^{-n}}}. \hspace{1cm} (37)$$

Plug (36) and (37) into (34), and let $n \to \infty$, we have

$$\phi_U(t) = \frac{\lambda}{\lambda - it}, \hspace{1cm} (38)$$

which is the c.f. of the exponential distribution with parameter $\lambda$. The proof is concluded.

B. Proof of Eq. (23)

We first prove sequence $(b_n)_{n\geq1}$ is monotonically decreasing. Using the fact that function $f(x) = \frac{x}{x+1}$ is monotonically increasing on $[0, 1]$, and decreasing on $[2, +\infty)$, then $\forall n \in \mathbb{N}$, $a_{-n+1} > a_{-n}$, and $a_{n-1} > a_{n}$. Hence when $n \geq 5$,

$$b_n = a_{-n+2} + a_{n-2} > b_{n+1} = a_{-n+1} + a_{n+1}. \hspace{1cm} (39)$$

When $n \leq 4$, we can numerically verify

$$b_n > b_{n+1}. \hspace{1cm} (40)$$
Now we prove Eq. (23) holds. When \( n \geq 7, \)

\[
\begin{align*}
    r_n &= \sum_{i=1}^{\infty} \left( \frac{2^{-i-2}}{1 + e^{2^{-i-2}}} + \frac{2^{-i-2}}{1 + e^{2^{-i-2}}} \right) \\
    &> \sum_{i=1}^{\infty} \frac{2^{-i-2}}{1 + e^{2^{-i-2}}} \\
    &= \sum_{i=1}^{\infty} \frac{2^{-i-2}}{1 + e^{2^{-i-2}}} \cdot \frac{1}{1 + e^{2^{-i-2}}} \\
    &= \sum_{i=1}^{\infty} \frac{2^{-i-2}(e^{2^{-i-2}} - 1)}{1 + e^{2^{-i-2}}} \\
    &= \sum_{i=1}^{\infty} \frac{2^{-i-2}}{1 + e^{2^{-i-2}}}. \\
\end{align*}
\]

(41)

(42)

(43)

(44)

(45)

(46)

(47)

(48)

where (45) follows from the fact \( e^x - 1 > x, \) and \( \frac{x}{1 + x^2} > \frac{1}{2}, \) \( \forall x > 1, \) and (47) from the following

\[
    a_{n-2} \leq \frac{2^{n-2}}{1 + e^{2^{n-2}}} < \frac{2^{n-2}}{2^{2(n-2)}}, \]

(49)

(50)

(51)

(52)

When \( n \leq 6, \) we can numerically verify

\[
    b_n < b_{n+1} + b_{n+2} < \sum_{i=n+1}^{\infty} b_i = r_n.
\]

(53)

The proof is concluded.

REFERENCES

[1] A. Lapidoth, S. M. Moser, and M. Wigger, “On the capacity of free-space optical intensity channels,” IEEE Trans. Inf. Theory, vol. 55, no. 10, pp. 4449–4461, Oct. 2009.

[2] A. L. McKellips, “Simple tight bounds on capacity for the peak-limited discrete-time channel,” in Proc. IEEE Intl. Symp. Inf. Theory, Chicago, IL, USA, Jun. 27 – Jul. 2, 2004, p. 348.

[3] A. Chaaban, Z. Rezki, and M. S. Alouini, “Fundamental limits of parallel optical wireless channels: capacity results and outage formulation,” IEEE Trans. Commun., vol. 65, no. 1, pp. 296–311, Jan. 2017.

[4] A. Chaaban, Z. Rezki, and M. S. Alouini, “Capacity bounds and high-SNR capacity of MIMO intensity-modulation optical channels,” IEEE Trans. Wireless Commun., vol. 17, no. 5, pp. 3003–3017, May 2018.

[5] L. Li, S. M. Moser, L. Wang, and M. Wigger, “On the capacity of MIMO optical wireless channels,” IEEE Trans. Inf. Theory, vol. 66, no. 9, pp. 5660–5682, Oct. 2020.

[6] S. M. Moser, L. Wang, and M. Wigger, “Capacity results on multiple-input single-output wireless optical channels,” IEEE Trans. Inf. Theory, vol. 64, no. 11, pp. 6954–6966, Nov. 2018.

[7] R. H. Chen, L. Li, J. Zhang, W. Zhang, and J. Zhou, “On the capacity of MISO optical intensity channels with per-antenna intensity constraints,” IEEE Trans. Inf. Theory, vol. 68, no. 6, pp. 3920 – 3941, Jun. 2022.