NONLINEAR OPERATIONS ON A CLASS OF MODULATION SPACES

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Abstract. We discuss when the nonlinear operation \( f \mapsto F(f) \) maps the modulation space \( M_{s}^{p,q}(\mathbb{R}^{n}) \) \((1 \leq p,q \leq \infty)\) to the same space again. It is known that \( M_{s}^{p,q}(\mathbb{R}^{n}) \) is a multiplication algebra when \( s > n - n/q \), hence it is true for this space if \( F \) is entire. We claim that it is still true for non-analytic \( F \) when \( q \geq 4/3 \).

1. Introduction

We discuss nonlinear operations \( f \mapsto F(f) \), that is, the composition of functions \( F \) and \( f \). Let \( X \) be a function space. Then does the nonlinear operation map \( X \) to the same space \( X \)? For the simplest case \( F(z) = z^{2} \), that is, \( F(f) = f^{2} \), the answer is yes when \( X \) is a multiplication algebra. From this observation, we immediately obtain the affirmative answer to this question for any entire functions \( F(z) \) and multiplication algebras \( X \). The typical examples of multiplication algebras are \( L^{p} \)-Sobolev spaces \( H_{s}^{p}(\mathbb{R}^{n}) \) \((1 < p < \infty)\) with \( s > n/p \) and Besov spaces \( B_{s}^{p,q}(\mathbb{R}^{n}) \) \((1 \leq p,q \leq \infty)\) with \( s > n/p \) (see Propositions 3.1 and 3.2).

When \( F \) fails to satisfy the analyticity, answering this question is not so straightforward. We however have an affirmative answer by virtue of the theory of paradifferential operators introduced by Bony [3] and developed by Meyer [9]. The main argument is to write the composition \( F(f) \) in the form of a linear operation

\[
F(f) = M_{F,f}(x,D)f
\]

(assuming that \( F(0) = 0 \), \( f \in H_{s}^{p}(\mathbb{R}^{n}) \) is real-valued, and \( s > n/p \) to be embedded in \( L^{\infty}(\mathbb{R}^{n}) \)), where \( M_{F,f}(x,D) \) is a pseudo-differential operator of the H"ormander class \( S_{1,1}^{0} \). Since pseudo-differential operators of this class are \( H_{s}^{p} \)-bounded for \( s > 0 \), we get the following result:

Theorem A ([9, Theorem 1]). Let \( 1 < p < \infty \) and \( s > n/p \). Assume that \( f : \mathbb{R}^{n} \rightarrow \mathbb{R} \), \( f \in H_{s}^{p}(\mathbb{R}^{n}) \), \( F \in C^{\infty}(\mathbb{R}) \) and \( F(0) = 0 \). Then, we have \( F(f) \in H_{s}^{p}(\mathbb{R}^{n}) \).

Remark 1.1. We can state Theorem A in an explicit form (see, e.g., Taylor [17, Section 3.1]):

\[
\|F(f)\|_{H_{s}^{p}} \leq C\|F\|_{C^{[\alpha]+1}(\Omega)} \left(1 + \|f\|_{L^{\infty}}^{[\alpha]+1}\right)\|f\|_{H_{s}^{p}},
\]

where \( \Omega = \{ t : |t| \leq C'\|f\|_{L^{\infty}} \} \), and the constants \( C \) and \( C' \) are universal.

By the same argument, we have a similar conclusion for Besov spaces (Runst [13]), and a result for complex-valued functions \( f : \mathbb{R}^{n} \rightarrow \mathbb{C} \) can be also stated considering the nonlinear operation \( f \mapsto F(\text{Re } f, \text{Im } f) \) with two-variable functions \( F(s,t) \), although in this paper we only consider real-valued functions \( f : \mathbb{R}^{n} \rightarrow \mathbb{R} \) just for the sake of simplicity.

The objective of this paper is to establish a similar result for modulation spaces \( M_{s}^{p,q}(\mathbb{R}^{n}) \). Modulation spaces are relatively new function spaces introduced by Feichtinger [4] in 1980’s to measure the decaying and regularity properties of a function or distribution in a way different from \( L^{p} \)-Sobolev spaces or Besov spaces. The main idea of modulation spaces is to consider the space variable and the variable of its Fourier transform simultaneously, while they are treated independently in \( L^{p} \)-Sobolev spaces and Besov spaces. Because of their nature, modulation spaces

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have several significant properties. For example, the Schrödinger propagator $e^{it\Delta}$ and the wave propagator $e^{itD}$ map the modulation space $M^{p,q}$ to the same space (Bényi-Gröchenig-Okoudjou-Rogers [1]), which means, we have no loss of regularity when we work on modulation spaces, while it is not true for $L^p$-Sobolev spaces $H^p$ or Besov spaces $B^s_{p,q}$ (Miyachi [10]). When we try to utilize this advantage for nonlinear analysis, it is indispensable to ask whether the nonlinear operation also maps $M^{p,q}$ to itself.

We know that modulation spaces $M^{p,q}(\mathbb{R}^n)$ ($1 \leq p, q \leq \infty$) with $s > n/q'$ (with $s = 0$ when $q = 1$) are multiplication algebras, where $1/q+1/q' = 1$ (Proposition 3.3), hence nonlinear operation $f \mapsto F(f)$ maps these spaces to themselves when $F(z)$ is entire. Then it is natural to expect the same conclusion for non-analytic $F$ as is the case for $L^p$-Sobolev spaces and Besov spaces. Unfortunately, it is not obvious because the argument of paradifferential operators does not work in this case because pseudo-differential operators of class $S^0_{1,\delta}$ with $\delta > 0$ have exotic mapping property and are not $M^{p,q}$-bounded (see [15]). Furthermore, if $F(z)$ is not necessarily analytic, a negative answer for $M_0^{p,1}$ is known. In fact, Bhimani-Ratnakumar [2] established that the nonlinear operation $f \mapsto F(\text{Re} f, \text{Im} f)$ is a mapping on $M_0^{1,1}(\mathbb{R}^n)$ if and only if $F$ is real analytic and $F(0,0) = 0$, and Kobayashi-Sato [8] generalized this result to the case $M_0^{p,1}$ with $1 \leq p < \infty$ although it is restricted to the case when $n = 1$. On the other hand, it is still possible for general $M^{p,q}_s(\mathbb{R}^n)$ with $1 < q \leq \infty$ and $s > n/q'$ when $F$ is not analytic. Our main result states that it is affirmative for $q$ in a range away from $q = 1$:

**Theorem 1.1.** Let $1 \leq p < \infty$, $4/3 \leq q < \infty$ (or $p = q = \infty$) and $s > n/q'$. Assume that $f : \mathbb{R}^n \to \mathbb{R}$, $f \in M^{p,q}_s(\mathbb{R}^n)$, $F \in C^\infty(\mathbb{R})$ and $F(0) = 0$. Then, we have $F(f) \in M^{p,q}_s(\mathbb{R}^n)$.

We remark that the condition $s > n/q'$ (with $s = 0$ when $q = 1$) is necessary for modulation spaces $M^{p,q}_s(\mathbb{R}^n)$ to be multiplication algebras (Guo-Fan-Wu-Zhao [6]). See also Appendix B. We also remark that Theorem 1.1 is reduced to the following result due to the local equivalence between the modulation spaces $M^{p,q}_s$ and the Fourier Lebesgue spaces $FL^q_s$:

**Theorem 1.2.** Let $4/3 \leq q < \infty$ and $s > n/q'$. Assume that $f : \mathbb{R}^n \to \mathbb{R}$, $f \in FL^q_s(\mathbb{R}^n)$, $F \in C^\infty(\mathbb{R})$ and $F(0) = 0$. Then, we have $F(f) \in FL^q_s(\mathbb{R}^n)$.

Finally, we just refer to the work by Reich-Reissig-Sickel [12] which also discusses the non-analytic nonlinear operations, but on modulation spaces with quasi-analytic regularity.

We explain the organization of this note. In Section 2, we introduce basic notations of function spaces and their properties which are used in this paper. In Section 3, we list examples of multiplication algebras as a starting point of our argument. In Section 4, we prove the theorem of nonlinear operation on Fourier Lebesgue spaces (Theorem 1.2). In Section 5, we lift it to the case of modulation spaces (Theorem 1.1) by using the local equivalence between Fourier Lebesgue spaces and modulation spaces. This equivalence is a well known fact but the proof is given in Appendix A for the sake of self-containedness. In Appendix B, necessity for Fourier Lebesgue spaces and modulation spaces to be multiplication algebras is considered.

### 2. Preliminaries

#### 2.1. Basic notations

We collect notations which will be used throughout this paper. We denote by $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{Z}_+$ the sets of reals, integers and non-negative integers, respectively. The notation $a \lesssim b$ means $a \leq Cb$ with a constant $C > 0$ which may be different in each occasion, and $a \sim b$ means $a \lesssim b$ and $b \lesssim a$. For $1 \leq p \leq \infty$, $p'$ is the dual number of $p$ and satisfies that $1/p + 1/p' = 1$. We write $(x) = (1 + |x|^2)^{1/2}$ for $x \in \mathbb{R}^n$ and $|s| = \max\{n \in \mathbb{Z} : n \leq s\}$ for $s \in \mathbb{R}$.

We denote the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}^n$ by $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ and its dual, the space of tempered distributions, by $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$. The Fourier transform and the
inverse Fourier transform of $f \in \mathcal{S}(\mathbb{R}^n)$ are given by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi \cdot x} f(x) dx$$

and

$$\mathcal{F}^{-1}f(x) = \check{f}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} f(\xi) d\xi,$$

respectively. For $m \in \mathcal{S}'(\mathbb{R}^n)$, the Fourier multiplier operator is given by

$$m(D)f = \mathcal{F}^{-1}[m \cdot \mathcal{F}f] = (\mathcal{F}^{-1}m) \ast f,$$

and for $s \in \mathbb{R}$ the Bessel potential by $(I - \Delta)^{s/2}f = \mathcal{F}^{-1}[(\cdot)^s \cdot \mathcal{F}f]$ for $f \in \mathcal{S}(\mathbb{R}^n)$.

We will use some function spaces. The space of smooth functions with compact support on $\mathbb{R}^n$ is denoted by $C_0^\infty = C_0^\infty(\mathbb{R}^n)$. The Lebesgue space $L^p = L^p(\mathbb{R}^n)$ is equipped with the norm

$$\|f\|_{L^p} = \left( \int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p}$$

for $1 \leq p < \infty$. If $p = \infty$, $\|f\|_{L^\infty} = \sup_{x \in \mathbb{R}^n} |f(x)|$. Moreover, we denote the $L^p$-Sobolev space $H^p_s$ by $H^p_s(\mathbb{R}^n) = \{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{H^p_s} = \|(I - \Delta)^{s/2}f\|_{L^p} < \infty \}$ for $1 < p < \infty$ and $s \in \mathbb{R}$, and the (weighted) Fourier Lebesgue space $\mathcal{F}L^p_s(\mathbb{R}^n) = \{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{\mathcal{F}L^p_s} = \|\langle \cdot \rangle^s \mathcal{F}f\|_{L^p} < \infty \}$ for $1 \leq p < \infty$ and $s \in \mathbb{R}$. We remark that $H^2_s = \mathcal{F}L^2_s$. Moreover, by the Hölder inequality, we have $\mathcal{F}L^q_s(\mathbb{R}^n) \hookrightarrow H^2_s(\mathbb{R}^n)$ if $2 < q \leq \infty$ and $n(1/2 - 1/q) < s - \tilde{s}$. Note that the second condition is equivalent to $\tilde{s} < n/2 + (s - n/q)$. From this relation, we immediately see the following.

**Proposition 2.1.** Let $2 < q \leq \infty$, $s > n/q'$ and $n/2 < \tilde{s} < n/2 + (s - n/q')$. Then, we have

$$\mathcal{F}L^q_s(\mathbb{R}^n) \hookrightarrow H^2_s(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n).$$

For $1 \leq q \leq \infty$ and $s \in \mathbb{R}$, we denote by $\ell^q_s$ the set of all complex number sequences $\{a_k\}_{k \in \mathbb{Z}^n}$ such that

$$\|a_k\|_{\ell^q_s} = \left( \sum_{k \in \mathbb{Z}^n} |k|^s |a_k|^q \right)^{1/q} < \infty$$

if $q < \infty$, and $\|a_k\|_{\ell^\infty_s} = \sup_{k \in \mathbb{Z}^n} |k|^s |a_k| < \infty$ if $q = \infty$. For the sake of simplicity, we will write $\|a_k\|_{\ell^q_s}$ instead of the more correct notation $\|\{a_k\}_{k \in \mathbb{Z}^n}\|_{\ell^q_s}$.

We end this subsection by mentioning a key fact on the boundedness of Fourier multiplier operators invented by Hahn [7, Theorem 9].

**Proposition 2.2.** Let $2 \leq p < \infty$ and $s > n/p$. Then, if $2p/(p + 2) \leq q \leq 2p/(p - 2)$ when $p \neq 2$ or $1 \leq q < \infty$ when $p = 2$, we have

$$\|m(D)f\|_{L^q} \lesssim \|m\|_{H^2_s} \|f\|_{L^q}$$

for all $m \in H^2_s(\mathbb{R}^n)$ and all $f \in L^q(\mathbb{R}^n)$.

**Remark 2.1.** In Proposition 2.2, we excluded $q = \infty$ for the case $p = 2$. This comes from that $\mathcal{S}$ is not dense in $L^\infty$. In this case, we regard $m(D)f$ as the convolution $(\mathcal{F}^{-1}m) \ast f$. Then, this is well-defined since $H^2_s(\mathbb{R}^n) \hookrightarrow \mathcal{F}L^1_0(\mathbb{R}^n)$ for $s > n/2$, and thus Proposition 2.2 holds for $q = \infty$ and $p = 2$. In fact, if $s > n/2$,

$$\|m(D)f\|_{L^\infty} = \|(\mathcal{F}^{-1}m) \ast f\|_{L^\infty} \leq \|\mathcal{F}^{-1}m\|_{L^1} \|f\|_{L^\infty} \lesssim \|m\|_{H^2_s} \|f\|_{L^\infty}$$

holds for all $m \in H^2_s(\mathbb{R}^n)$ and all $f \in L^\infty(\mathbb{R}^n)$.
2.2. Modulation spaces. We give the definition of modulation spaces which were introduced by Feichtinger [4] (see also Gröchenig [5]). We fix a function (called a window function) \( g \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\} \) and denote the short-time Fourier transform of \( f \in \mathcal{S}'(\mathbb{R}^n) \) with respect to \( g \) by

\[
V_g f(x, \xi) = \int_{\mathbb{R}^n} e^{-i\xi^t} \overline{g(t - x)} f(t) dt.
\]

We will sometimes write \( V_g[f] \) when the form of \( f \) is complicated. For \( 1 \leq p, q \leq \infty \) and \( s \in \mathbb{R} \), the modulation space \( M_s^{p,q} \) is defined by

\[
M_s^{p,q}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{M_s^{p,q}} = \left\| \langle \xi \rangle^s V_g f(x, \xi) \right\|_{L^p(\mathbb{R}^n)} \right\}.
\]

We note that the definition of modulation spaces is independent of the choice of window functions. \( M_s^{p,q} \) are Banach spaces and \( \mathcal{S} \subset M_0^{p,q} \subset \mathcal{S}' \). In particular, \( \mathcal{S} \) is dense in \( M_0^{p,q} \) if \( 1 \leq p, q < \infty \). For \( 1 \leq p, q < \infty \), the dual space of \( M_0^{p,q} \) can be seen as \( (M_0^{p,q})' = M_0^{q',p'} \). Moreover, we have the following complex interpolation theorem. If \( 0 < \theta < 1 \), \( s = (1 - \theta)s_1 + \theta s_2 \), \( 1/p = (1 - \theta)/p_1 + \theta/p_2 \), and \( 1/q = (1 - \theta)/q_1 + \theta/q_2 \), we have \( (M_{s_1}^{p_1,q_1}, M_{s_2}^{p_2,q_2})_\theta = M_s^{p,q} \). As a further elementary property, we note the following embedding proved by Feichtinger [4, Proposition 6.5].

**Proposition 2.3.** Let \( 1 \leq p_1, p_2, q_1, q_2 \leq \infty \) and \( s_1, s_2 \in \mathbb{R} \). Then, we have \( M_{s_1}^{p_1,q_1} \hookrightarrow M_{s_2}^{p_2,q_2} \) for \( p_1 \leq p_2, q_1 \leq q_2 \) and \( s_1 \geq s_2 \).

2.3. Besov spaces. We here give the definition of Besov spaces (see also [18, Section 2.3]). Let \( \varphi \in \mathcal{S}(\mathbb{R}^n) \) satisfy that \( \varphi = 1 \) on \( \{ \xi : |\xi| \leq 1/2 \} \) and \( \text{supp} \varphi \subset \{ \xi : |\xi| \leq 1 \} \). We put \( \psi = \varphi(\cdot/2) - \varphi(\cdot) \), and then see that \( \text{supp} \psi \subset \{ \xi : 1/2 \leq |\xi| \leq 2 \} \). Moreover, we set \( \varphi_j = \varphi(\cdot/2^j) \) and \( \psi_j = \psi(\cdot/2^j) \) for \( j \in \mathbb{Z}_+ \), and denote the Fourier multiplier operators with respect to them by

\[
S_j f = \varphi_j(D)f \quad \text{and} \quad \Delta_j f = \psi_j(D)f.
\]

We remark that

\[
\varphi + \sum_{j=0}^{\infty} \psi_j = 1 \quad \text{and} \quad S_0 f + \sum_{j=0}^{\infty} \Delta_j f = f,
\]

and also that \( \Delta_j f = S_{j+1} f - S_j f \). By using these notations, for \( 1 \leq p, q \leq \infty \) and \( s \in \mathbb{R} \), the Besov space \( B_s^{p,q} \) is defined by

\[
B_s^{p,q}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{B_s^{p,q}} = \|S_0 f\|_{L^p} + \left( \sum_{j=0}^{\infty} 2^{jsq} \|\Delta_j f\|_{L^p}^q \right)^{1/q} \right\}.
\]

Note that the norm of the Besov space is read with the usual modification for \( q = \infty \). Besov spaces also have basic properties like modulation spaces, namely, completeness, density, duality and interpolation. However, we omit mentioning the details and refer the reader to [18, Section 2.3].

3. Multiplication algebras

In this section, we collect some properties called multiplication algebras. A function space \( X \) is said to be a multiplication algebra if for all \( f, g \in X \) the product \( f \cdot g \) exists and belongs to \( X \), and if the inequality \( \|f \cdot g\|_X \lesssim \|f\|_X \cdot \|g\|_X \) holds for all \( f, g \in X \). More precisely, see [18, Section 2.8]. The following results on \( L^p \)-Sobolev and Besov spaces are well-known (see, e.g., Strichartz [14, Chapter II, Theorem 2.1] and Triebel [18, Theorem 2.8.3]).
Proposition 3.1. Let $1 < p < \infty$ and $s > n/p$. Then, we have
\[ \|f \cdot g\|_{H^p_s} \lesssim \|f\|_{H^p_s} \cdot \|g\|_{H^p_s} \]
for all $f, g \in H^p_s(\mathbb{R}^n)$.

Proposition 3.2. Let $1 \leq p, q \leq \infty$ and $s > n/p$. Then, we have
\[ \|f \cdot g\|_{B^{p,q}_s} \lesssim \|f\|_{B^{p,q}_s} \cdot \|g\|_{B^{p,q}_s} \]
for all $f, g \in B^{p,q}_s(\mathbb{R}^n)$.

Some of modulation spaces are also multiplication algebras (see, e.g., Feichtinger [4, Remark 6.4 and Proposition 6.9] and Sugimoto-Tomita-Wang [16, Proposition 3.2]).

Proposition 3.3. Let $1 \leq p, q \leq \infty$ and $s > n/q'$. Then, we have
\[ \|f \cdot g\|_{M^{p,q}_s} \lesssim \|f\|_{M^{p,q}_s} \cdot \|g\|_{M^{p,q}_s} \]
for all $f, g \in M^{p,q}_s(\mathbb{R}^n)$, and
\[ \|f \cdot g\|_{M^{p,1}_s} \lesssim \|f\|_{M^{p,1}_s} \cdot \|g\|_{M^{p,1}_s} \]
for all $f, g \in M^{p,1}_s(\mathbb{R}^n)$.

Finally, we give the following counterpart for Fourier Lebesgue spaces.

Proposition 3.4. Let $1 \leq q \leq \infty$ and $s > n/q'$. Then, we have
\[ \|f \cdot g\|_{\mathcal{F}L^q_s} \lesssim \|f\|_{\mathcal{F}L^q_s} \cdot \|g\|_{\mathcal{F}L^q_s} \]
for all $f, g \in \mathcal{F}L^q_s(\mathbb{R}^n)$, and
\[ \|f \cdot g\|_{\mathcal{F}L^q_0} \lesssim \|f\|_{\mathcal{F}L^q_0} \cdot \|g\|_{\mathcal{F}L^q_0} \]
for all $f, g \in \mathcal{F}L^q_0(\mathbb{R}^n)$.

Proof of Proposition 3.4. From the inequality $\langle \xi \rangle^s \lesssim \langle \xi - \eta \rangle^s + \langle \eta \rangle^s$ for any $\xi, \eta \in \mathbb{R}^n$ and $s \geq 0$, we have
\[
\|f \cdot g\|_{\mathcal{F}L^q_s} \simeq \left\| \langle \xi \rangle^s \int_{\mathbb{R}^n} \hat{f}(\xi - \eta) \cdot \hat{g}(\eta) d\eta \right\|_{L^q(\mathbb{R}^n)} \\
\lesssim \left\| \int_{\mathbb{R}^n} \langle \xi - \eta \rangle^s \left| \hat{f}(\xi - \eta) \right| \cdot \left| \hat{g}(\eta) \right| d\eta \right\|_{L^q(\mathbb{R}^n)} + \left\| \int_{\mathbb{R}^n} \langle \xi - \eta \rangle^s \cdot \left| \hat{g}(\eta) \right| d\eta \right\|_{L^q(\mathbb{R}^n)}.
\]
Then, we have by the Young and Hölder inequalities
\[
\|f \cdot g\|_{\mathcal{F}L^q_s} \lesssim \left\| \langle \cdot \rangle^s \hat{f} \right\|_{L^q} \cdot \|g\|_{L^1} + \left\| \hat{f} \right\|_{L^q} \cdot \left\| \langle \cdot \rangle^s \hat{g} \right\|_{L^q} \\
\leq \|f\|_{\mathcal{F}L^q_s} \cdot \|g\|_{L^q} + \|f\|_{\mathcal{F}L^q_s} \cdot \|g\|_{\mathcal{F}L^q_s},
\]
which yields from the assumption $s > n/q'$ that $\|f \cdot g\|_{\mathcal{F}L^q_s} \lesssim \|f\|_{\mathcal{F}L^q_s} \cdot \|g\|_{\mathcal{F}L^q_s}$. Here, we remark that, in the case $q = 1$, $\|\cdot\|_{L^{q'}}$ is finite even if $s = 0$, which gives the conclusion for $q = 1$ and $s = 0$. \(\square\)
4. Proof of Theorem 1.2

We begin this section with an observation which will be used in the proof of Theorem 1.2. Put

\[ G(t) = F(t) - \sum_{k=1}^{N} \frac{F^{(k)}(0) t^k}{k!} \]

for any \( N \in \mathbb{N} \), where \( F \in C^\infty(\mathbb{R}) \) and \( F(0) = 0 \). Then, we see that \( G(0) = G^{(1)}(0) = \cdots = G^{(N)}(0) = 0 \), and have

\[ F(f) = G(f) + \sum_{k=1}^{N} F^{(k)}(0) \frac{f^k}{k!}. \]

In order to obtain Theorem 1.2, we will prove that the right hand side of (4.2) belongs to \( \mathcal{F}L^4 \). However, it is trivial that the second term belongs to \( \mathcal{F}L^4 \), since \( \mathcal{F}L^4(\mathbb{R}^n) \) with \( s > n/q' \) is a multiplication algebra (see Proposition 3.4). Hence, Theorem 1.2 is reduced to the following statement.

Proposition 4.1. Let \( 4/3 \leq q \leq \infty \) and \( s > n/q' \). Assume that \( f : \mathbb{R}^n \to \mathbb{R}, f \in \mathcal{F}L^4(\mathbb{R}^n), G \in C^\infty(\mathbb{R}) \) and \( G(0) = G^{(1)}(0) = \cdots = G^{([s]+2)}(0) = 0 \). Then, we have \( G(f) \in \mathcal{F}L^4(\mathbb{R}^n) \).

Before starting the proof of Proposition 4.1, we transform \( G(f) \) to a more manageable alternative expression, which was provided by Meyer [9, Section 2]. We first remark that \( f \in \mathcal{F}L^4_b(\mathbb{R}^n) \) if and only if \( f \in \mathcal{F}L^4(\mathbb{R}^n) \) for \( q \geq 2 \) and \( f \in \mathcal{F}L^4(\mathbb{R}^n) \) for \( q > 2 \). Then, we prove that \( f \) belonging to \( B^{\infty,1}_0(\mathbb{R}^n) \), hence to \( L^{\infty}(\mathbb{R}^n) \), and so \( f \) is a bounded uniformly continuous function. Then \( S_j f \) converges uniformly to \( f \) as \( j \to \infty \), and \( G(f) = G(\lim_{j \to \infty} S_j f) = \lim_{j \to \infty} G(S_j f) \). By the mean value theorem and the fact \( S_{j+1} f = S_j f + \Delta_j f \), we have

\[ G(f) = G(S_0 f) + \sum_{j=0}^{\infty} [G(S_{j+1} f) - G(S_j f)] \]

\[ = G(S_0 f) + \sum_{j=0}^{\infty} \int_0^1 G^{(1)}(S_j f + t\Delta_j f) dt \cdot \Delta_j f = G(S_0 f) + \sum_{j=0}^{\infty} m_j \cdot \Delta_j f, \]

where we set

\[ m_j = \int_0^1 G^{(1)}(S_j f + t\Delta_j f) dt. \]

Moreover, we decompose \( m_j \) into the low and high frequency parts. Recall from Section 2.3 that \( \varphi(\xi) + \sum_{m=0}^{\infty} \psi(2^{-m} \xi) = 1 \) for any \( \xi \in \mathbb{R}^n \). Then, it follows that

\[ \varphi \left( \frac{\xi}{C \cdot 2^j} \right) + \sum_{m=0}^{\infty} \psi \left( \frac{\xi}{C \cdot 2^{j+m}} \right) = 1 \]

for any \( \xi \in \mathbb{R}^n \), where \( C \) is a sufficiently large constant. Using this decomposition, we have

\[ m_j = \varphi \left( \frac{D}{C \cdot 2^j} \right) m_j + \sum_{m=0}^{\infty} \psi \left( \frac{D}{C \cdot 2^{j+m}} \right) m_j = q_j + \sum_{m=0}^{\infty} p_{j,m}, \]

where we set

\[ q_j = \varphi \left( \frac{D}{C \cdot 2^j} \right) m_j \quad \text{and} \quad p_{j,m} = \psi \left( \frac{D}{C \cdot 2^{j+m}} \right) m_j. \]
Therefore, $G(f)$ is expressed in the following form:

\[ G(f) = G(S_0 f) + \sum_{j=0}^{\infty} q_j \cdot \Delta_j f + \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} p_{j,m} \cdot \Delta_j f. \]

From now on, we give estimates for each terms of the expression (4.5) without specifying constants explicitly. (We however remark that these implicit constants may depend on $\|f\|_{F(L^1_2)}$) We start by stating two lemmas. The first one is for Lemma 4.1.

**Lemma 4.1.** Let $1 < q \leq \infty$, $s > n/q'$ and $n/2 < \tilde{s} < n/2 + (s - n/q')$. Suppose that $f \in F(L^q_2(\mathbb{R}^n)$ and all the assumptions of $G$ are the same as in Proposition 4.1. Then, we have

\[ \|q_j\|_{H^0_q} \leq 1 \quad \text{if} \quad 1 < q \leq 2; \]
\[ \|q_j\|_{H^s_q} \leq 1 \quad \text{if} \quad 2 < q \leq \infty \]

for any $j \in \mathbb{Z}_+$. Here, the implicit constants are independent of $j \in \mathbb{Z}_+$.

**Proof of Lemma 4.1.** We first consider the estimate with $1 < q \leq 2$. Set $f_{j,t} = S_j f + t\Delta_j f$. Recalling the definition of $m_j$ from (4.3), we have

\[ \|q_j\|_{H^0_q} \lesssim \|m_j\|_{H^0_q} \leq \int_0^1 \left\| G^{(1)}(f_{j,t}) \right\|_{H^0_q} dt. \]

Observe that

\[ \|f_{j,t}\|_{H^0_q} \lesssim (\|F^{-1} \varphi_j\|_{L^1} + t\|F^{-1} \psi_j\|_{L^1}) \|f\|_{H^0_q} \]
\[ \lesssim (\|F^{-1} \varphi\|_{L^1} + t\|F^{-1} \psi\|_{L^1}) \|f\|_{F(L^1_2)} \lesssim \|f\|_{F(L^1_2)}, \]

which means that $f_{j,t} \in H^0_q$ for any $j \in \mathbb{Z}_+$ and any $t \in [0,1]$. Then, using Theorem A and Remark 1.1 together with the assumptions $G \in C^{\infty}(\mathbb{R})$ and $G^{(1)}(0) = 0$, we have

\[ \left\| G^{(1)}(f_{j,t}) \right\|_{H^0_q} \lesssim \left\| G^{(2)}(C^{[s]+1}(\Omega)) \left( 1 + \|f_{j,t}\|_{L^\infty}^{[s]+1} \right) \right\|_{H^0_q} \]
\[ \lesssim \left\| G(C^{[s]+3}(\Omega)) \left( 1 + \|f\|_{L^\infty}^{[s]+1} \right) \right\|_{F(L^1_2)}, \]

where $\Omega = \{ t : |t| \lesssim \|f\|_{L^\infty} \}$. Note that the last quantity is finite since $f \in F(L^q_2(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ for $s > n/q'$ and the smooth function $G \in C^{\infty}(\mathbb{R})$ is measured by $C^{[s]+3}$ on the closed and bounded domain $\Omega$. Therefore, we have $\|q_j\|_{H^0_q} \lesssim 1$ for $1 < q \leq 2$.

We next consider the estimate with $2 < q \leq \infty$. This is, however, immediately given by the same argument as above. In fact, since we already know from Proposition 2.1 that $f \in F(L^q_2(\mathbb{R}^n) \hookrightarrow H^2_{\tilde{s}}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$, we have by Theorem A and Remark 1.1

\[ \left\| G^{(1)}(f_{j,t}) \right\|_{H^s_q} \lesssim \left\| G^{(2)}(C^{[s]+1}(\Omega)) \left( 1 + \|f_{j,t}\|_{L^\infty}^{[s]+1} \right) \right\|_{H^s_q} \]
\[ \lesssim \left\| G(C^{[s]+3}(\Omega)) \left( 1 + \|f\|_{L^\infty}^{[s]+1} \right) \right\|_{F(L^1_2)}. \]

Note that the last quantity is finite. Hence, we obtain $\|q_j\|_{H^s_q} \lesssim 1$ for $2 < q \leq \infty$. \hfill \Box

The second one is concerning $p_{j,m}$ in (4.4).

**Lemma 4.2.** Let $1 < q \leq \infty$, $s > n/q'$ and $n/2 < \tilde{s} < n/2 + (s - n/q')$. Suppose that $f \in F(L^q_2(\mathbb{R}^n)$ and all the assumptions of $G$ are the same as in Proposition 4.1. Then, we have

\[ \|p_{j,m}\|_{H^0_q} \lesssim 2^{-m([s]+1)} \quad \text{if} \quad 1 < q \leq 2; \]
\[ \|p_{j,m}\|_{H^s_q} \lesssim 2^{-m([s]+1)} \quad \text{if} \quad 2 < q \leq \infty \]

for any $j, m \in \mathbb{Z}_+$. Here, the implicit constants are independent of $j, m \in \mathbb{Z}_+$. \hfill \Box
To prove Lemma 4.2, we prepare the following:

**Lemma 4.3.** Let $1 < q \leq \infty$, $s > n/q'$ and $n/2 < \tilde{s} < n/2 + (s - n/q')$, and let $\alpha \in \mathbb{Z}^n_+$ satisfy that $|\alpha| = [s] + 1$. Suppose that $f \in \mathcal{F}L^q_s(\mathbb{R}^n)$ and all the assumptions of $G$ are the same as in Proposition 4.1. Then, we have

$$
\|\partial^{\alpha} m_j\|_{H^q_s} \lesssim 2^{j([s]+1)} \quad \text{if} \quad 1 < q \leq 2;
$$

$$
\|\partial^{\alpha} m_j\|_{H^q_s} \lesssim 2^{j([s]+1)} \quad \text{if} \quad 2 < q \leq \infty
$$

for any $j \in \mathbb{Z}_+$. Here, the implicit constants are independent of $j \in \mathbb{Z}_+$.

**Proof of Lemma 4.3.** We first consider the case $1 < q \leq 2$. Set $f_{j,t} = S_j f + t \Delta_j f$. Then we have by Proposition 3.1

$$
\|\partial^{\alpha} m_j\|_{H^q_s} \lesssim \int_0^1 \left\|\partial^{\alpha}[G^{(1)}(f_{j,t})]\right\|_{H^q_s} dt
$$

$$
\lesssim \sum_{\mu=1}^{[s]+1} \sum_{\alpha_{1}+\cdots+\alpha_{\mu}=\alpha} \int_0^1 \left\|G^{(\mu+1)}(f_{j,t})\right\|_{H^q_s} \cdot \left\|\partial^{\alpha_{1}} f_{j,t}\right\|_{H^q_s} \cdots \left\|\partial^{\alpha_{\mu}} f_{j,t}\right\|_{H^q_s} dt,
$$

where $|\alpha| = [s] + 1$. Observe that for $\beta \in \mathbb{Z}^n_+$

$$
\left\|\partial^{\beta} f_{j,t}\right\|_{H^q_s} \lesssim \left(\|F^{-1}[\xi^\beta \cdot \varphi_j]\|_{L^1} + t\|F^{-1}[\xi^\beta \cdot \psi_j]\|_{L^1}\right) \|f\|_{H^q_s} \lesssim 2^{j|\beta|} \|f\|_{\mathcal{F}L^q_s},
$$

which also means that $f_{j,t} \in H^q_s$ for any $j \in \mathbb{Z}_+$ and any $t \in [0, 1]$. Therefore, by using Theorem A and Remark 1.1 together with the assumptions $G \in C^\infty(\mathbb{R})$ and $G^{(2)}(0) = \cdots = G^{([s]+2)}(0) = 0$, we have for $\mu = 1, \cdots, [s] + 1$

$$
\left\|G^{(\mu+1)}(f_{j,t})\right\|_{H^q_s} \lesssim \|G^{(\mu+2)}\|_{C^{[\mu]+1}(\Omega)} \left(1 + \|f\|_{L^\infty}^{[s]+1}\right) \|f\|_{H^q_s}
$$

$$
\lesssim \|G\|_{C^{[\mu]+3}(\Omega)} \left(1 + \|f\|_{L^\infty}^{[s]+1}\right) \|f\|_{\mathcal{F}L^q_s},
$$

where $\Omega = \{t : |t| \lesssim \|f\|_{L^\infty}\}$. Note that the last quantity makes sense surely since $f \in \mathcal{F}L^q_s(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ for $s > n/q'$ and $G \in C^\infty(\mathbb{R})$ is considered on the closed and bounded domain $\Omega$. Hence, we obtain

$$
\|\partial^{\alpha} m_j\|_{H^q_s} \lesssim \sum_{\mu=1}^{[s]+1} \sum_{\alpha_{1}+\cdots+\alpha_{\mu}=\alpha} \left(2^{j[\alpha_{1}]} \|f\|_{\mathcal{F}L^q_s}\right) \cdots \left(2^{j[\alpha_{\mu}]} \|f\|_{\mathcal{F}L^q_s}\right) \lesssim 2^{j([s]+1)},
$$

which completes the proof for $1 < q \leq 2$.

We next consider the case $2 < q \leq \infty$. Repeating the same lines as above, since we already know from Proposition 2.1 that $f \in \mathcal{F}L^q_s(\mathbb{R}^n) \hookrightarrow H^2_s(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$, we have for $\beta \in \mathbb{Z}^n_+$

$$
\left\|\partial^{\beta} f_{j,t}\right\|_{H^q_s} \lesssim \left(\|F^{-1}[\xi^\beta \cdot \varphi_j]\|_{L^1} + t\|F^{-1}[\xi^\beta \cdot \psi_j]\|_{L^1}\right) \|f\|_{H^q_s} \lesssim 2^{j|\beta|} \|f\|_{\mathcal{F}L^q_s},
$$

and by Theorem A and Remark 1.1 for $\mu = 1, \cdots, [s] + 1$

$$
\left\|G^{(\mu+1)}(f_{j,t})\right\|_{H^q_s} \lesssim \|G^{(\mu+2)}\|_{C^{[\mu]+1}(\Omega)} \left(1 + \|f\|_{L^\infty}^{[s]+1}\right) \|f\|_{H^q_s}
$$

$$
\lesssim \|G\|_{C^{[\mu]+3}(\Omega)} \left(1 + \|f\|_{L^\infty}^{[s]+1}\right) \|f\|_{\mathcal{F}L^q_s}.
$$

Hence, we obtain $\|\partial^{\alpha} m_j\|_{H^q_s} \lesssim 2^{j([s]+1)}$ for $2 < q \leq \infty$. \qed
Proof of Lemma 4.2. By the moment condition of $\psi$ and a Taylor expansion, we have
\[
p_{j,m}(x) = C^n \cdot 2^{(j+m)n} \int_{\mathbb{R}^n} \tilde{\psi}(C \cdot 2^{j+m}y) \cdot m_j(x-y) dy
\]
\[
= C^n \cdot 2^{(j+m)n} \int_{\mathbb{R}^n} \tilde{\psi}(C \cdot 2^{j+m}y) \left\{ m_j(x-y) - \sum_{|\alpha|<M} \frac{(-y)^\alpha}{\alpha!} (\partial^\alpha m_j)(x) \right\} dy
\]
\[
= C^n \cdot 2^{(j+m)n} \int_{\mathbb{R}^n} \tilde{\psi}(C \cdot 2^{j+m}y) \left\{ M \sum_{|\alpha|=M} \frac{(-y)^\alpha}{\alpha!} \int_0^1 (1-t)^{M-1} \cdot (\partial^\alpha m_j)(x-ty) dt \right\} dy,
\]
where $M = [s] + 1$. Taking the $H'_2$-norm to the both sides, we have
\[
\|p_{j,m}\|_{H'_2} \lesssim 2^{(j+m)n} \int_{\mathbb{R}^n} |\tilde{\psi}(C \cdot 2^{j+m}y)| \cdot |y|^{[s]+1} \left\{ \sum_{|\alpha|=[s]+1} \int_0^1 \| (\partial^\alpha m_j)(x-ty) \|_{H'_2}^\alpha dt \right\} dy
\]
\[
\sim 2^{-(j+m)([s]+1)} \left( \int_{\mathbb{R}^n} |\tilde{\psi}(y)| \cdot |y|^{[s]+1} dy \right) \sum_{|\alpha|=[s]+1} \| \partial^\alpha m_j \|_{H'_2}
\]
\[
\sim 2^{-(j+m)([s]+1)} \sum_{|\alpha|=[s]+1} \| \partial^\alpha m_j \|_{H'_2}.
\]
Since we have $\| \partial^\alpha m_j \|_{H'_2} \lesssim 2^{2([s]+1)}$ for $1 < q \leq 2$ by Lemma 4.3, we obtain $\|p_{j,m}\|_{H'_2} \lesssim 2^{-m([s]+1)}$.

By the same manner as above, we also have $\|p_{j,m}\|_{H'_2} \lesssim 2^{-m([s]+1)}$ for $2 < q \leq \infty$. \hfill \Box

We are now ready to prove Proposition 4.1.

Proof of Proposition 4.1. We recall the alternative form of $G(f)$ given in (4.5), that is,
\[
G(f) = G(S_0 f) + \sum_{j=0}^{\infty} q_j \cdot \Delta_j f + \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} p_{j,m} \cdot \Delta_j f,
\]
and prove that the function $G(f)$ belongs to $\mathcal{F}L^q_c$, which will be archived by three steps. In the first and second steps, we consider the second and third summations, and then consider $G(S_0 f)$ in the last step.

**Step 1:** We first consider the case $q < \infty$. Taking the $\mathcal{F}L^q_c$-norm to the second summation in (4.5), we have
\[
\left( \sum_{\ell=0}^{\infty} \int_{\Omega_\ell} |q_j \cdot \Delta_j f|^q \right)^{1/q} \Rightarrow \left( \sum_{\ell=0}^{\infty} \int_{\Omega_\ell} \langle \xi \rangle^{sq} |q_j \cdot \Delta_j f|^q \right)^{1/q} d\xi,
\]
where $\Omega_\ell = \{ \xi : 2\ell < |\xi| \leq 2^{\ell+1} \}$ if $\ell \neq 0$ and $\Omega_0 = \{ \xi : |\xi| \leq 2 \}$. We remark that
\[
\supp \mathcal{F} [q_j \cdot \Delta_j f] \subset \{ \xi : |\xi| \leq C \cdot 2^{j+1} \},
\]
since $\mathcal{F} [q_j \cdot \Delta_j f] = [\varphi(k_\ell \cdot \vec{m}) \ast [\tilde{\psi}_j \vec{f}]]$. This means that on the domain $\Omega_\ell$, $\mathcal{F} [q_j \cdot \Delta_j f]$ always vanishes unless $j \geq \ell - N$ ($j \geq 0$ if $\ell = 0, \cdots, N$), where $N$ is a constant which depends only on $C \gg 1$ (roughly, $2^N \sim C$). Hence, the right hand side of (4.6) is equal to
\[
\left( \sum_{\ell=0}^{\infty} \int_{\Omega_\ell} \langle \xi \rangle^{sq} \right) \left( \sum_{j=\ell-N}^{\infty} \int_{\Omega_\ell} |q_j \cdot \Delta_j f|^q d\xi \right)^{1/q},
\]

(4.6)
where the inner summation should be read as $\sum_{j=0}^{\infty}$ if $\ell = 0, \cdots, N$. Then, using the Hölder inequality to the inner summation, we have

$$
(4.7) \lesssim \left( \sum_{\ell=0}^{\infty} \int_{\Omega_\ell} 2^{j sq} \left( \sum_{j=\ell-N}^{\infty} 2^{jsq} |F[q_j \cdot \Delta_j f](\xi)|^q \right) \right)^{\frac{q}{q'}} \left( \sum_{j=\ell-N}^{\infty} 2^{-jsq'} d\xi \right)^{1/q} 
$$

$$
(4.8) \lesssim \left( \sum_{\ell=0}^{\infty} \int_{\Omega_\ell} \sum_{j=\ell-N}^{\infty} 2^{jsq} |F[q_j \cdot \Delta_j f](\xi)|^q d\xi \right)^{1/q}
$$

$$
\lesssim \left( \sum_{j=0}^{\infty} 2^{jsq} \int_{\mathbb{R}^n} |F[q_j \cdot \Delta_j f](\xi)|^q d\xi \right)^{1/q}.
$$

Here, in the last inequality, we used the fact that $\mathbb{R}^n = \bigcup_{\ell=0}^{\infty} \Omega_\ell$. Now, we observe that

$$
\|F[q_j \cdot \Delta_j f]\|_{L^q} = \|\tilde{q}_j(D) \left[ \psi_j \cdot \hat{f} \right]\|_{L^q},
$$

where $\tilde{q}_j(x) = q_j(-x)$. Then, we see that the last quantity of (4.8) is equal to

$$
(4.9) \left( \sum_{j=0}^{\infty} 2^{jsq} \left\| \tilde{q}_j(D) \left[ \psi_j \cdot \hat{f} \right]\right\|_{L^q}^{q} \right)^{1/q}.
$$

Apply to (4.9) Proposition 2.2 with $p = q'$ for $4/3 \leq q \leq 2$ and with $p = 2$ for $2 < q < \infty$. Here, we note that the assumption $4/3 \leq q \leq 2$ is used to assure the conditions $2q'/(q' + 2) \leq q \leq 2q'/(q' - 2)$ and $q' \geq 2$ in Proposition 2.2. Then, we have

$$
(4.9) \lesssim \begin{cases} \left( \sum_{j=0}^{\infty} 2^{jsq} \left\| q_j \right\|_{H^s}^q \left\| \psi_j \cdot \hat{f} \right\|_{L^q}^q \right)^{1/q} & \text{if } 4/3 \leq q \leq 2; \\ \left( \sum_{j=0}^{\infty} 2^{jsq} \left\| q_j \right\|_{H^s}^q \left\| \psi_j \cdot \hat{f} \right\|_{L^q}^q \right)^{1/q} & \text{if } 2 < q < \infty, \end{cases}
$$

where $s$ is the number satisfying that $n/2 < s < n/2 + (s - n/q')$. Thus, we obtain from Lemma 4.1

$$
(4.9) \lesssim \left( \sum_{j=0}^{\infty} 2^{jsq} \left\| \psi_j \cdot \hat{f} \right\|_{L^q}^{q} \right)^{1/q}.
$$

Since it follows that $\sum_{j=0}^{\infty} |\psi_j|^q \lesssim 1$ (if $q < \infty$) and $2^j \sim (\xi)$ on the support of $\psi_j$, we realize that

$$
\left( \sum_{j=0}^{\infty} 2^{jsq} \left\| \psi_j \cdot \hat{f} \right\|_{L^q}^{q} \right)^{1/q} \sim \left( \sum_{j=0}^{\infty} \left\| \psi_j \cdot (\xi)^{s} \hat{f} \right\|_{L^q}^{q} \right)^{1/q} \lesssim \|f\|_{F^s_{L^q}}
$$

for $4/3 \leq q < \infty$, which gives the desired result for the case $4/3 \leq q < \infty$.

We next consider the case $q = \infty$. However, this case is obtained similarly to the above. In fact, we have

$$
(4.10) \sup_{\xi \in \mathbb{R}^n} \left| (\xi)^s \sum_{j=0}^{\infty} F[q_j \cdot \Delta_j f](\xi) \right| \lesssim \sup_{\ell \in \mathbb{Z}^+} \left( \sup_{\xi \in \Omega_\ell_N} \left( \sum_{j=\ell-N}^{\infty} 2^{\ell s} \sum_{j=0}^{\infty} |F[q_j \cdot \Delta_j f](\xi)| \right) \right).
$$
Then, we have by the Fubini-Tonelli theorem
\[ |F[q_j \cdot \Delta_j f](\xi)| \lesssim \|q_j\|_{H^2} \left\| \psi_j \cdot \langle \cdot \rangle^{-s} \cdot \langle \cdot \rangle^s f \right\|_{L^\infty} \lesssim 2^{-js} \|f\|_{F^{s\infty}_L} .\]
Hence, we obtain
\[ 2^{\ell s} \sum_{j=\ell-N}^\infty |F[q_j \cdot \Delta_j f](\xi)| \lesssim 2^{\ell s} \sum_{j=\ell-N}^\infty 2^{-js} \|f\|_{F^{s\infty}_L} \lesssim \|f\|_{F^{s\infty}_L} \]
for any \( \ell \in \mathbb{Z}_+ \), where all the implicit constants above are independent of \( \ell \in \mathbb{Z}_+ \). Substituting this estimate into (4.10), we have the desired result for the case \( q = \infty \).
Combining all the calculations above, we obtain for \( 4/3 \leq q \leq \infty \)
\[ (4.11) \quad \left\| \sum_{j=0}^\infty q_j \cdot \Delta_j f \right\|_{F^{s\infty}_L} < \infty. \]

**Step 2:** We first consider the case \( q < \infty \). As in Step 1, we take the \( F^{s\infty}_L \)-norm to the third summation in (4.5) and decompose the \( L^q \)-norm by the dyadic decomposition. Then, we have
\[ (4.12) \quad \left\| \langle \xi \rangle^s \sum_{j=0}^\infty F \left[ \sum_{m=0}^\infty p_{j,m} \cdot \Delta_j f \right] \right\|_{L^q} \lesssim \sum_{m=0}^\infty \left( \sum_{j=0}^\infty \int_{\Omega_\ell} 2^{\ell s q} \left| F[p_{j,m} \cdot \Delta_j f](\xi) \right| d\xi \right)^{1/q}, \]
where \( \Omega_\ell = \{ \xi : 2^\ell < |\xi| \leq 2^{\ell+1} \} \) if \( \ell \neq 0 \) and \( \Omega_0 = \{ \xi : |\xi| \leq 2 \} \). Considering the support of \( F[p_{j,m} \cdot \Delta_j f] \), since we have
\[ F[p_{j,m}] \subset \{ \xi : C \cdot 2^{j(m-1)} \leq |\xi| \leq C \cdot 2^{j(m+1)} \} \quad \text{and} \quad F[\Delta_j f] \subset \{ \xi : 2^{j-1} \leq |\xi| \leq 2^{j+1} \}, \]
we see that
\[ \text{supp} \ F[p_{j,m} \cdot \Delta_j f] \subset \{ \xi : C \cdot 2^{j(m-2)} \leq |\xi| \leq C \cdot 2^{j(m+2)} \}. \]
This implies that on the domain \( \Omega_\ell \), the function \( F[p_{j,m} \cdot \Delta_j f] \) always vanishes unless \( j, \ell, m \in \mathbb{Z}_+ \) satisfy that \( j + m + N - 2 \leq \ell \leq j + m + N + 1 \), where \( N \) is the constant which depends only on \( C \gg 1 \). Put \( \Lambda = \{ j \in \mathbb{Z}_+ : \ell - m - N - 1 \leq j \leq \ell - m - N + 2 \} \), where this set is read as \( \Lambda = \emptyset \) if \( \ell - m - N + 2 < 0 \). Then, \( 0 \leq \# \Lambda \leq 4 \). Hence, the right hand side of (4.12) is equivalent to
\[ (4.13) \quad \sum_{m=0}^\infty \left( \sum_{j=0}^\infty \int_{\Omega_\ell} \sum_{j \in \Lambda} 2^{(j+m)s q} \left| F[p_{j,m} \cdot \Delta_j f](\xi) \right|^q d\xi \right)^{1/q}. \]
Then, we have by the Fubini-Tonelli theorem
\[ (4.13) \leq \sum_{m=0}^\infty 2^{m s} \left( \int_{\mathbb{R}^n} \sum_{j=0}^\infty 2^{j s q} \left| F[p_{j,m} \cdot \Delta_j f](\xi) \right|^q d\xi \right)^{1/q} \]
\[ = \sum_{m=0}^\infty 2^{m s} \left( \sum_{j=0}^\infty 2^{j s q} \left\| F[p_{j,m} \cdot \Delta_j f] \right\|_{L^q}^q \right)^{1/q} . \]
Using the identity \( \left\| F[p_{j,m} \cdot \Delta_j f] \right\|_{L^q} = \left\| \tilde{p}_{j,m}(D) [\psi_j \cdot \hat{f}] \right\|_{L^q} \), where \( \tilde{p}_{j,m}(x) = p_{j,m}(-x) \), we see that the last quantity of (4.14) is equal to
\[ (4.15) \quad \sum_{m=0}^\infty 2^{m s} \left( \sum_{j=0}^\infty 2^{j s q} \left\| \tilde{p}_{j,m}(D) [\psi_j \cdot \hat{f}] \right\|_{L^q}^q \right)^{1/q} . \]
As in Step 1, we have

\[
(4.9) \lesssim \begin{cases} 
\sum_{m=0}^{\infty} 2^{ms} \left( \sum_{j=0}^{\infty} 2^{j\frac{m}{q}} \| \psi_j \cdot \tilde{f} \|_{L^q}^{\frac{1}{q}} \right)^{1/q} & \text{if } 4/3 \leq q \leq 2; \\
\sum_{m=0}^{\infty} 2^{ms} \left( \sum_{j=0}^{\infty} 2^{j\frac{m}{q}} \| \psi_j \cdot \tilde{f} \|_{L^q}^{\frac{1}{q}} \right)^{1/q} & \text{if } 2 < q < \infty,
\end{cases}
\]

for \( n/2 < s < n/2 + (s - n/q') \). Hence, recalling the properties that \( \sum_{j=0}^{\infty} |\psi_j|^q \lesssim 1 \) (if \( q < \infty \)) and \( 2^j \sim \langle \xi \rangle \) on \( \text{supp} \psi_j \), we have by Lemma 4.2

\[
(4.15) \lesssim \sum_{m=0}^{\infty} 2^{ms} \cdot 2^{-m(s)+1} \left( \sum_{j=0}^{\infty} 2^{j\frac{m}{q}} \| \psi_j \cdot \tilde{f} \|_{L^q}^{\frac{1}{q}} \right)^{1/q} \lesssim \| f \|_{\mathcal{F}L^q_s}
\]

for \( 4/3 \leq q < \infty \), which gives the desired result for the case \( 4/3 \leq q < \infty \).

We next consider the case \( q = \infty \), which is obtained similarly to the above. In fact, we have

\[
\sup_{\xi \in \mathbb{R}^n} \left| \langle \xi \rangle^s \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \mathcal{F} \left[ \psi_j \cdot \Delta_j f \right] (\xi) \right| \lesssim \sup_{\ell \in \mathbb{Z}_+} \left( \sum_{m=0}^{\infty} 2^{\ell s} \sum_{j \in \Lambda} \mathcal{F} \left[ p_{j,m} \cdot \Delta_j f \right] (\xi) \right)
\]

(see above for the definition of the sets \( \Omega_\ell \) and \( \Lambda \)). Recalling from Lemma 4.2 that \( \| p_{j,m} \|_{H^2_s} \lesssim 2^{-m(s)+1} \) holds independently of \( j, m \in \mathbb{Z}_+ \) and following the same lines as in Step 1, we have

\[
| \mathcal{F} \left[ p_{j,m} \cdot \Delta_j f \right] (\xi) | \lesssim \| p_{j,m} \|_{H^2_s} \| \psi_j \cdot \tilde{f} \|_{L^\infty} \lesssim 2^{-m(s)+1} \cdot 2^{-j s} \| f \|_{\mathcal{F}L^\infty_s}.
\]

Hence, we obtain

\[
\sum_{m=0}^{\infty} 2^{\ell s} \sum_{j \in \Lambda} | \mathcal{F} \left[ p_{j,m} \cdot \Delta_j f \right] (\xi) | \lesssim \sum_{m=0}^{\infty} 2^{-m(s)+1} \cdot 2^{\ell s} \sum_{j \in \Lambda} 2^{-j s} \| f \|_{\mathcal{F}L^\infty_s} \sim \| f \|_{\mathcal{F}L^\infty_s}
\]

for any \( \ell \in \mathbb{Z}_+ \). This gives the desired result for the case \( q = \infty \).

Combining all the calculations above, we obtain for \( 4/3 \leq q \leq \infty \)

\[
(4.16) \quad \left\| \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} p_{j,m} \cdot \Delta_j f \right\|_{\mathcal{F}L^q_s} < \infty.
\]

**Step 3:** Lastly, we prove that \( G(S_0 f) \in \mathcal{F}L^q_s \). Observe that

\[
G(S_0 f) = \int_0^1 G^{(1)}(t \cdot S_0 f) dt \cdot S_0 f = m_f \cdot S_0 f,
\]

where \( m_f = \int_0^1 G^{(1)}(t \cdot S_0 f) dt \). Then, since \( \mathcal{F}L^q_s \hookrightarrow \mathcal{F}L^s_r \) for \( r \geq s \) and \( \langle \xi \rangle^r \lesssim 1 + |\xi_1|^r \cdots + |\xi_n|^r \) for \( r \geq 0 \), we have by Proposition 2.2 for \( 4/3 \leq q \leq 2 \)

\[
\| \langle \xi \rangle^s \mathcal{F} \left[ G(S_0 f) \right] \|_{L^q} \lesssim \| \mathcal{F} \left[ m_f \cdot S_0 f \right] \|_{L^q} + \sum_{\ell=1}^n \| \mathcal{F} \left[ \partial^{s+1}_\ell (m_f \cdot S_0 f) \right] \|_{L^q}
\]

\[
\lesssim \| m_f \|_{H^s_q} \| \phi \cdot \tilde{f} \|_{L^q} + \sum_{\ell=1}^n \sum_{\mu=0}^{n} \| \partial_\ell^\mu m_f \|_{H^s_q} \| \xi^{s+1-\mu} \phi \cdot \tilde{f} \|_{L^q}
\]

for any \( s \geq 0 \).
Proof of Theorem 1.1. By Steps 1-3, we conclude that $G(\xi) = 0$. Hence, in this section, we mainly prove that $G(f)$ belongs to $H^s_f \cap L^q$ if $f \in H^s_f \cap L^q$. In particular, if $s > n/q$, we have $G(f) \in H^s_f \cap L^q$. The second one is shown by Proposition 3.4. In fact, since $G(f) \in H^s_f \cap L^q$ if $f \in H^s_f \cap L^q$. Hence, we obtain $G(f) \in H^s_f \cap L^q$. 

Now, all the preparations are completed, so that we give the proof of Theorem 1.2.

**Proof of Theorem 1.2.** As is stated at the beginning of this section, $F(f)$ with $F \in C^\infty(\mathbb{R})$ and $F(0) = 0$ is given by

$$F(f) = G(f) + \sum_{k=1}^{N} F^{(k)}(0) \frac{f^k}{k!},$$

for any $N \geq 0$, where $G \in C^\infty(\mathbb{R})$ and $G(0) = G^{(1)}(0) = \cdots = G^{(N)}(0) = 0$. Choosing $N = [s] + 2$, we obtain from Proposition 4.1 that $G(f) \in F^s_L \cap L^q$ if $f \in F^s_L$. The second one is shown by Proposition 3.4. In fact, since $F \in C^\infty(\mathbb{R})$, we have $|F^{(k)}(0)| \lesssim 1$, so that it follows that

$$\left\| \sum_{k=1}^{N} F^{(k)}(0) \frac{f^k}{k!} \right\|_{F^s_L} \lesssim \sum_{k=1}^{N} \|f\|_{F^s_L}^k < \infty,$$

if $f \in F^s_L$. Hence, we obtain that $F(f) \in F^s_L$ if $f \in F^s_L$. 

5. **Proof of Theorem 1.1**

As in Section 4, $F(f)$ is expressed in the following form:

$$F(f) = G(f) + \sum_{k=1}^{N} F^{(k)}(0) \frac{f^k}{k!},$$

for any $N \in \mathbb{N}$, where $G(0) = G^{(1)}(0) = \cdots = G^{(N)}(0) = 0$. Applying a Taylor expansion to $G$, we have

$$G(f) = f^N \cdot H(f), \quad H(f) = \frac{1}{(N-1)!} \int_{0}^{1} (1 - \theta)^{N-1} G^{(N)}(\theta f) d\theta.$$  

Note that $H \in C^\infty(\mathbb{R})$ and $H(0) = 0$. Hence, in this section, we mainly prove that $G(f)$ in (5.2) belongs to $M^p_{s,q}$ if $f \in M^p_{s,q}$. In order to prove this, we prepare the following lemma:
Lemma 5.1. Let $4/3 \leq q \leq \infty$ and $s > n/q'$, and let $N$ be an arbitrary natural number. Suppose that $G$ is the function in (5.2), $f \in M_s^{p,q}$ and real-valued functions $\phi, \tilde{\phi} \in C_0^\infty(\mathbb{R}^n)$ satisfy that $\tilde{\phi} \equiv 1 \text{ on } \text{supp } \phi$. Then, we have
\[
\|\langle \xi \rangle^s V_{\phi} [G(f)](x,\xi)\|_{L^q(\mathbb{R}^n_\xi)} \lesssim \|\langle \xi \rangle^s V_{\tilde{\phi}} f(x,\xi)\|_{L^q(\mathbb{R}^n_\xi)}^N
\]
for any $x \in \mathbb{R}^n$. Here, the implicit constant is independent of $x \in \mathbb{R}^n$.

Proof of Lemma 5.1. We first observe from (5.2) and the assumption $\tilde{\phi}(\cdot - x) \equiv 1 \text{ on } \text{supp } \phi(\cdot - x)$ that
\[
V_{\phi} [G(f)](x,\xi) = \int_{\mathbb{R}^n} e^{-i\xi \cdot t} \phi(t - x) \cdot G(\tilde{\phi}(t - x) f(t)) \, dt
\]
\[
= \int_{\mathbb{R}^n} e^{-i\xi \cdot t} \phi(t - x) \cdot (\tilde{\phi}(t - x) f(t))^N \cdot H(\tilde{\phi}(t - x) f(t)) \, dt
\]
\[
= \mathcal{F} \left[ \phi(\cdot - x) \cdot (\tilde{\phi}(\cdot - x) f)^N \cdot H(\tilde{\phi}(\cdot - x) f) \right](\xi).
\]
Multiplying the weight $\langle \xi \rangle^s$ to both sides and taking the $L^q$-norm with respect to the $\xi$-variable, we have by Proposition 3.4
\[
\|\langle \xi \rangle^s V_{\phi} [G(f)](x,\xi)\|_{L^q(\mathbb{R}^n_\xi)} = \|\langle \xi \rangle^s \mathcal{F} \left[ \phi(\cdot - x) \cdot (\tilde{\phi}(\cdot - x) f)^N \cdot H(\tilde{\phi}(\cdot - x) f) \right](\xi)\|_{L^q(\mathbb{R}^n_\xi)}
\]
\[
\lesssim \|\phi(\cdot - x)\|_{\mathcal{F}L^q_\xi} \cdot \|\tilde{\phi}(\cdot - x) f\|^N_{\mathcal{F}L^q_\xi} \cdot \|H(\tilde{\phi}(\cdot - x) f)\|_{\mathcal{F}L^q_\xi}.
\]
It obviously follows that $\|\phi(\cdot - x)\|_{\mathcal{F}L^q_\xi} \sim 1$ and $\|\tilde{\phi}(\cdot - x) f\|_{\mathcal{F}L^q_\xi} = \|\langle \xi \rangle^s V_{\tilde{\phi}} f(x,\xi)\|_{L^q(\mathbb{R}^n_\xi)}$. We only consider $\|H(\tilde{\phi}(\cdot - x) f)\|_{\mathcal{F}L^q_\xi}$ to obtain the conclusion. By Lemma A.1 and Proposition 3.3, we have
\[
\|\tilde{\phi}(\cdot - x) f\|_{\mathcal{F}L^q_\xi} \sim \|\tilde{\phi}(\cdot - x) f\|_{M^{p,q}_\xi} \lesssim \|\tilde{\phi}\|_{M^{p,q}_\xi} \cdot \|f\|_{M^{p,q}_\xi} < \infty,
\]
where the implicit constants are both independent of $x \in \mathbb{R}^n$. Then, recalling that $H \in C_0^\infty(\mathbb{R})$ and $H(0) = 0$, we have $\sup_{x \in \mathbb{R}^n} \|H(\tilde{\phi}(\cdot - x) f)\|_{\mathcal{F}L^q_\xi} < \infty$ by Theorem 1.2 if $4/3 \leq q \leq \infty$. Hence, we obtain
\[
\|\langle \xi \rangle^s V_{\phi} [G(f)](x,\xi)\|_{L^q(\mathbb{R}^n_\xi)} \lesssim \|\langle \xi \rangle^s V_{\tilde{\phi}} f(x,\xi)\|_{L^q(\mathbb{R}^n_\xi)}^N.
\]
Here, recalling all the proofs in Section 4, we see that $\|H(\tilde{\phi}(\cdot - x) f)\|_{\mathcal{F}L^q_\xi}$ can be estimated by a polynomial of $\|\langle \xi \rangle^s V_{\tilde{\phi}} f(x,\xi)\|_{L^q(\mathbb{R}^n_\xi)}$. This implies that the explicit order of the power in the right hand side can be actually taken larger than $N$. However, the explicit expression is not important, since it is sufficient to understand that the order can be chosen arbitrarily large as we want. Hence, we here omitted the details.

Now, we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. We recall the expressions (5.1) and have by Proposition 3.3
\[
\|F(f)\|_{M^{p,q}_\xi} \lesssim \|G(f)\|_{M^{p,q}_\xi} + \sum_{k=1}^N \|f\|_{M^{p,q}_\xi}^k.
\]
Here, we choose $N \in \mathbb{N}$ such that $N \geq \lfloor \max(p/q, q/p) \rfloor + 1$, and it should be remarked that we exclude the cases $p = \infty$ and $q < \infty$, or $p < \infty$ and $q = \infty$ in Theorem 1.1, since such $N$ cannot be taken in those cases.
We first consider \( \|G(f)\|_{M^{p,q}_φ} \) for the case \( p \leq q \). Let real-valued functions \( φ, \tilde{φ} \in C_0^∞(\mathbb{R}^n) \) satisfy that \( \tilde{φ} \equiv 1 \) on \( \text{supp} \ φ \). Then, we have by the Minkowski inequality for integrals and Lemma 5.1
\[
\|G(f)\|_{M^{p,q}_φ} \lesssim \|\langle ξ \rangle^s V_φ |G(f)| (x, ξ)\|_{L^q(\mathbb{R}^n_ξ)} \|_{L^p(\mathbb{R}^n_x)}
\lesssim \left\| \|\langle ξ \rangle^s V_φ f(x, ξ)\|_{L^q(\mathbb{R}^n_ξ)} \right\|^N = \left\| \|\langle ξ \rangle^s V_φ f(x, ξ)\|_{L^q(\mathbb{R}^n_ξ)} \right\|^N \|L^{Np}(\mathbb{R}^n_x)\).
\]
Since \( Np > q \geq p \), we have by Proposition 2.3
\[
\|G(f)\|_{M^{p,q}_φ} \lesssim \left\| \|\langle ξ \rangle^s V_φ f(x, ξ)\|_{L^{Np}(\mathbb{R}^n_ξ)} \right\|^N \|L^{Nq}(\mathbb{R}^n_x)\).
\]
We next assume that \( q < p < \infty \). As above, Proposition 2.3 and Lemma 5.1 yield that
\[
\|G(f)\|_{M^{p,q}_φ} \lesssim \|G(f)\|_{M^{q,q}_φ} \lesssim \left\| \|\langle ξ \rangle^s V_φ f(x, ξ)\|_{L^q(\mathbb{R}^n_ξ)} \right\|^N \|L^{Nq}(\mathbb{R}^n_x)\).
\]
Since \( Nq > p > q \), we use Proposition 2.3 again and obtain
\[
\|G(f)\|_{M^{p,q}_φ} \lesssim \|f\|_{M^{p,q}_φ} \|L^q(\mathbb{R}^n_x)\) \sim \|f\|_{M^{p,q}_φ} \|L^q(\mathbb{R}^n_ξ)\).
\]
Therefore, for \( 1 \leq p < \infty \) and \( 4/3 \leq q < \infty \) (or \( p = q = \infty \)), we have \( \|G(f)\|_{M^{p,q}_φ} \lesssim \|f\|_{M^{p,q}_φ} \).

Collecting all the estimates above, we obtain \( \|F(f)\|_{M^{p,q}_φ} < \infty \). This is the desired conclusion. □

**Appendix A. Local equivalence between modulation and Fourier Lebesgue spaces**

In this section, we state that modulation spaces are locally equivalent to Fourier Lebesgue spaces. The corresponding result for \( s = 0 \) was already proved by Okoudjou [11, Lemma 1], and the weighted case is obtained by following the same argument. However, for the reader’s convenience, we give a proof.

**Lemma A.1.** Let \( 1 \leq p, q \leq \infty \) and \( s \in \mathbb{R} \). Suppose that \( χ \in S(\mathbb{R}^n) \setminus \{0\} \) satisfies that \( \text{supp} \ χ \subset \{x : |x - x_0| \leq R\} \). Then, we have \( \|χ \cdot f\|_{M^{p,q}_φ} \sim \|χ \cdot f\|_{F^{p,q}_L} \). Here, the implicit constant is independent of \( x_0 \in \mathbb{R}^n \), but depends on \( R > 0 \).

**Proof of Lemma A.1.** Put \( f_χ = χ \cdot f \). We first prove the \( \lesssim \) part. Choose \( φ \in S(\mathbb{R}^n) \setminus \{0\} \) satisfying that \( \text{supp} \ φ \subset \{x : |x| \leq R\} \). Then, we see that \( V_φ[f_χ](x, ξ) \) always vanishes unless \( x \in \mathbb{R}^n \) satisfies that \( |x - x_0| \leq 2R \). Using the identity \( |V_φ[f_χ](x, ξ)| = |\tilde{φ}(D - ξ)f_χ(x)| \), we have by the Hölder and Hausdorff-Young inequalities
\[
\|V_φ[f_χ](x, ξ)\|_{L^p(\mathbb{R}^n_ξ)} = \|χ_{B_{2R}(x_0)}(x) \cdot V_φ[f_χ](x, ξ)\|_{L^p(\mathbb{R}^n_ξ)} \lesssim R^{n/p} \|\tilde{φ}(D - ξ)f_χ(·)\|_{L^∞} \lesssim R^{n/p} \|\tilde{φ}(t - ξ) \cdot F[f_χ](t)\|_{L^1(\mathbb{R}^n_ξ)}\).
\]
Multiplying the weight \( \langle ξ \rangle^s \) to the both sides, using the inequality \( |\langle ξ \rangle^s| \lesssim |t|^s(t - ξ)^{|s|} \) and taking the \( L^q \)-norm with respect to the \( ξ \)-variable, we have by the Young inequality
\[
\|\|\langle ξ \rangle^s V_φ[f_χ](x, ξ)\|_{L^p(\mathbb{R}^n_ξ)}\|_{L^q(\mathbb{R}^n_x)} \lesssim R^{n/p} \|\langle t - ξ \rangle |\tilde{φ}(t - ξ) \cdot F[f_χ](t)\|_{L^1(\mathbb{R}^n_ξ)}\|_{L^q(\mathbb{R}^n_x)} = R^{n/p} \|\langle |·|^{|s|} \tilde{φ}\rangle \ast (\langle · \rangle^s F[f_χ]|_{L^q})\|_{L^q(\mathbb{R}^n_x)} \lesssim R^{n/p} \|\langle · \rangle^s F[f_χ]|_{L^q}\|_{L^q}.
\]
We next prove the \( \geq \) part. Choose \( \phi \in \mathcal{S}(\mathbb{R}^n) \) satisfying that \( \text{supp} \phi \equiv 1 \) on \( \{x : |x| \leq 2R\} \). Then, \( \phi(\cdot - x) \equiv 1 \) on \( \text{supp} \chi \) if \( \chi \in \mathbb{R}^n \) satisfies that \( |x - x_0| \leq R \). Hence, it follows that

\[
R^{n/p} |\mathcal{F}[f_x](\xi)| \sim \left\| \chi_{B_R(x_0)}(x) \cdot \mathcal{F}[f_x](\xi) \right\|_{L^p(\mathbb{R}^n)} \\
= \left\| \chi_{B_R(x_0)}(x) \cdot \int_{\mathbb{R}^n} e^{-i\xi \cdot t} \phi(t - x) \cdot \chi(t)f(t)dt \right\|_{L^p(\mathbb{R}^n)} \leq \|\phi\|_{L^q(\mathbb{R}^n)} \left\| f_x \right\|_{L^p(\mathbb{R}^n)}.
\]

Multiplying the weight \( \langle \xi \rangle^s \) to the both sides and taking the \( L^q \)-norm with respect to the \( \xi \)-variable, we have

\[
\|\langle \cdot \rangle^s \mathcal{F}[f_x]\|_{L^q} \lesssim R^{-n/p} \left\| \langle \xi \rangle^s \mathcal{F}[f_x](x, \xi) \right\|_{L^q(\mathbb{R}^n)}.
\]

Therefore, recalling the property that the modulation space norm is independent of the choice of window functions, we obtain \( \|f_x\|_{M^{p,q}_s} \sim \|f\|_{F_{L^q}} \). \( \square \)

APPENDIX B. CONDITIONS FOR MODULATION SPACES AND FOURIER LEBESGUE SPACES TO BE MULTIPLICATION ALGEBRAS

In this section, we first consider necessary and sufficient conditions for modulation spaces to be multiplication algebras, that is, for the estimate

\[
\|f \cdot g\|_{M^{p,q}_s} \lesssim \|f\|_{M^{p,q}_s} \cdot \|g\|_{M^{p,q}_s},
\]

to hold. They are given as follows.

**Proposition B.1.** Let \( 1 \leq p \leq \infty, 1 < q \leq \infty \) and \( s \in \mathbb{R} \). Then, the modulation space \( M^{p,q}_s(\mathbb{R}^n) \) is a multiplication algebra if and only if the condition \( s > n/q' \) is satisfied.

Actually, this proposition is immediately obtained from [6, Theorem 1.5]. In fact, in [6], necessary and sufficient conditions for the more general estimate

\[
\|f \cdot g\|_{M^{p,q}_s} \lesssim \|f\|_{M^{p,q}_s} \cdot \|g\|_{M^{p,q}_s}
\]

were established, so that Proposition B.1 is given by setting \( p = p_1 = p_2, q = q_1 = q_2 \) and \( s = s_1 = s_2 \). (We remark that, although only the case \( q > 1 \) is considered in Proposition B.1, the whole case \( q \geq 1 \) is treated in [6].) However, for reader’s convenience, we give a proof of Proposition B.1 where the following two lemmas are essential:

**Lemma B.1** ([6, Proposition 5.1]). Let \( 1 \leq p, q \leq \infty \) and \( s \in \mathbb{R} \). Then, if the modulation space \( M^{p,q}_s(\mathbb{R}^n) \) is a multiplication algebra, we have \( \ell^q_s \ast \ell^q_s \to \ell^q_s \).

**Lemma B.2.** Let \( 1 < q \leq \infty \) and \( s \in \mathbb{R} \). Then, if \( \ell^q_s \ast \ell^q_s \to \ell^q_s \) holds, we have \( s > n/q' \).

**Proof of Lemma B.2.** We assume towards a contradiction that \( s \leq n/q' \). Since \( q > 1 \), we can take \( \varepsilon > 0 \) such that \( 1 - 1/q - \varepsilon > 0 \). For this \( \varepsilon > 0 \), we define the sequences

\[
a_{k,N} = \begin{cases} 
\langle k \rangle^{-n/q-s} (C + \log \langle k \rangle)^{-1/q-\varepsilon}, & \text{if } |k| \leq N, \\
0, & \text{otherwise};
\end{cases}
\]

\[
b_{k,N} = \begin{cases} 
1, & \text{if } N \leq |k| \leq 5N, \\
0, & \text{otherwise},
\end{cases}
\]

in \( k \in \mathbb{Z}^n \), where \( N > 0 \) is a sufficiently large integer and \( C > 1 \) is a suitable constant which depends only on the dimension \( n \).

We first estimate each sequence on \( \ell^q_s \). For the case \( q < \infty \), the spherical coordinate transform yields that

\[
\|a_{k,N}\|_{\ell^q_s}^q = \sum_{|k| \leq N} \langle k \rangle^{-n} (C + \log \langle k \rangle)^{-1-\varepsilon q}
\]
This contradicts to the assumption \( l \). The "IF" part is given by Proposition 3.3, and the "ONLY IF" part is
Proof of Proposition B.1.

We also have a similar optimality for Fourier Lebesgue spaces:

**Proposition B.2.** Let \( 1 < q \leq \infty \) and \( s \in \mathbb{R} \). Then, the Fourier Lebesgue space \( FL^q_s(\mathbb{R}^n) \) is a multiplication algebra if and only if the condition \( s > n/q' \) is satisfied.

For the proof of Proposition B.2, we use the following lemma instead of Lemma B.1:
Lemma B.3 ([6, Proposition 4.1]). Let $1 \leq q \leq \infty$ and $s \in \mathbb{R}$. Then, if the estimate
\[
\|\langle \cdot \rangle^s (f \ast g)\|_{L^q} \lesssim \|\langle \cdot \rangle^s f\|_{L^q} \cdot \|\langle \cdot \rangle^s g\|_{L^q}
\]
holds, we have $\ell_q^s \ast \ell_q^s \hookrightarrow \ell_q^s$.

Proof of Proposition B.2. The “IF” part is given by Proposition 3.4. The “ONLY IF” part is an immediate conclusion of Lemmas B.2 and B.3 if we notice the equivalence
\[
\|f \cdot g\|_{F \ell_q^s} \lesssim \|f\|_{F \ell_q^s} \cdot \|g\|_{F \ell_q^s} \iff \|\langle \cdot \rangle^s (\hat{f} \ast \hat{g})\|_{L^q} \lesssim \|\langle \cdot \rangle^s \hat{f}\|_{L^q} \cdot \|\langle \cdot \rangle^s \hat{g}\|_{L^q}.
\]

\[\square\]

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