AROUND FINITE-DIMENSIONALITY IN
FUNCTIONAL ANALYSIS-
A PERSONAL PERSPECTIVE
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Abstract: As objects of study in functional analysis, Hilbert spaces stand out as special objects of study as do nuclear spaces (ala Grothendieck) in view of a rich geometrical structure they possess as Banach and Frechet spaces, respectively. On the other hand, there is the class of Banach spaces including certain function spaces and sequence spaces which are distinguished by a poor geometrical structure and are subsumed under the class of so-called Hilbert-Schmidt spaces. It turns out that these three classes of spaces are mutually disjoint in the sense that they intersect precisely in finite dimensional spaces. However, it is remarkable that despite this mutually exclusive character, there is an underlying commonality of approach to these disparate classes of objects in that they crop up in certain situations involving a single phenomenon—the phenomenon of finite dimensionality—which, by definition, is a generic term for those properties of Banach spaces which hold good in finite dimensional spaces but fail in infinite dimension.

1. Introduction

A major effort in functional analysis is devoted to examining the extent to which a given property enjoyed by finite dimensional (Banach) spaces can be extended to an infinite dimensional setting. Thus, given a property (P) valid in each finite dimensional Banach space, the desired extension to the infinite dimensional setting admits one of the following three (mutually exclusive) possibilities:
(a). (P) holds good in all infinite dimensional Banach spaces.
(b). (P) holds good in some infinite dimensional Banach spaces.
(c). (P) fails in each infinite dimensional Banach space.
Examples of (P) verifying (a) include, for instance, the validity of the inverse function theorem that one learns in an advanced course on calculus whereas a typical example of property (P) verifying (b) is provided by reflexivity or say, the Radon-Nikodym property of a

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Banach space. Our main focus will be on property (P) falling under (c)-henceforth to be designated as finite-dimensional properties-which will be seen to exhibit remarkable richness when considered in an infinite dimensional setting. This latter setting will naturally necessitate the consideration of an infinite dimensional framework beyond the world of Banach spaces in which these properties are sought to be salvaged, which in our case will be provided by the class of Frechet spaces. Interestingly, this leads to the identification of nuclear (Frechet) spaces as natural objects in which-at least some of the most important-finite-dimensional properties are valid in an infinite dimensional setting outside the framework of Banach spaces. On the other hand, as an important subclass of Banach spaces, there is the class of so-called Hilbert-Schmidt spaces which include certain important function spaces like \( C(K), L_1(\mu), L_\infty(\mu) \) or sequence spaces like \( c_0, \ell_\infty, \ell_1 \) besides many more. These spaces are characterized by the property that a bounded linear operator acting between Hilbert spaces and factoring over such a space is already a Hilbert Schmidt operator. In particular, no infinite dimensional Banach space can simultaneously be a Hilbert space as well as a Hilbert-Schmidt space. In a similar vein, a Hilbert space can never be nuclear unless it is finite dimensional. It turns out that, along with nuclear spaces, these three (disjoint) classes of spaces which are distinguished by properties that are mutually exclusive arise as different manifestations of this single phenomenon of finite dimensionality. The present article is devoted to a discussion of certain interesting aspects of this phenomenon and to point out the ways in which it leads to the consideration of nuclear spaces on the one hand and of Hilbert spaces and Hilbert-Schmidt spaces on the other.

2. Notation

Throughout this paper, we shall let \( X, Y, Z \) denote Banach spaces, unless otherwise stated. We shall use the symbol \( X^* \) for the dual of \( X \) whereas \( B_X \) shall be used for the closed unit ball of \( X \):

\[
B_X = \{ x \in X : \|x\| \leq 1 \}.
\]

We shall also make use of the following notation:

\( L(X,Y) \), Banach space of all bounded linear operators from \( X \) into \( Y \).

\( P(\mathbb{N}) \), set of all permutations on \( \mathbb{N} \).

\( c_0(X) = \{ (x_n) \subset X : x_n \to 0 \} \)

\( \ell_p(X) = \{ (x_n) \subset X : \sum_{n=1}^{\infty} |< x_n, f >|^p < \infty, \forall f \in X^* \} \)
\[ c_0(X) = \{(x_n) \subset X : x_n \to 0\} \]
\[ \ell_p[X] = \left\{(x_n) \subset X : \sum_{n=1}^{\infty} \|x_n\|^p < \infty \right\}. \]

Clearly, \( \ell_p\{X\} \subset \ell_p[X] \), \( \forall 1 \leq p < \infty \). Further, the indicated inclusion is continuous when these spaces are equipped with natural norms defined by:

\[ \varepsilon \left( (x_n) \right) = \sup_{f \in B_{X^*}} \sum_{n=1}^{\infty} |<x_n, f>|^p, (x_n) \in \ell_p[X]. \]

\[ \pi \left( (x_n) \right) = \left( \sum_{n=1}^{\infty} \|x_n\|^p \right)^{1/p}, (x_n) \in \ell_p\{X\}. \]

For a locally convex space \( X \), a locally convex topology on each of these sequence spaces can be defined exactly in the same fashion, with the norm being replaced by a family of seminorms generating its topology. For later use, let us note the following equivalent description of these spaces of sequences identifying them as certain classes of bounded linear maps.

**Theorem 2.1.** For \( 1 < p < \infty \) and \( q \) where \( \frac{1}{p} + \frac{1}{q} = 1 \) the correspondence \( T \to \{T(e_p)\} \) provides an isometric isomorphism of

(i) \( L(\ell_p, X) \) onto \( \ell_p[X] \)

(ii) \( \Pi_p(\ell_p, X) \) onto \( \ell_p\{X\} \).

Here \( \Pi_p \) denotes the class of \( p \)-summing maps defined below (see Definition 3). The proof of (i) is a straightforward consequence of the fact that \( ||T|| = ||T^*|| \) and that \( ||T^*|| \) is nothing but \( \varepsilon_p \left( (x_n) \right) \).

An outline of a proof of (ii) is provided in Theorem 3.17(a).

3. **Definitions and Examples**

Given a property \( (P) \), we say that \( (P) \) is a *finite-dimensional property* \((FD)-property, for short) if it holds good for all finite dimensional Banach spaces but fails for each infinite dimensional Banach space.

**Example 3.1.** (i) Heine-Borel Property (closed bounded subsets of \( X \) are compact).

(ii) \( X^* = X' \) (algebraic dual of \( X \)).

(iii) \( w^* \)-convergence = norm convergence in \( X^* \) (Josefson-Nisenweig theorem)

(iv) Completeness of the weak-topology on \( X \).
(v) Equivalence of all norms on X.
(vi) Hahn-Banach property: For a given Banach space X, the following holds:
Each bounded linear operator defined on a subspace of X and taking values in an arbitrary Banach space Y can be extended to a bounded linear operator on X and each bounded linear operator defined on X can be extended to a bounded linear operator on any superspace of X.
(vii) Dvoretzky-Hanani Property: Given a null sequence \( x_n \) in X, there exist signs \( \epsilon_n = \pm 1, n \geq 1 \) such that \( \sum_{n=1}^{\infty} \epsilon_n x_n \) converges in X.
(viii) McArthur Property: A series \( \sum_{n=1}^{\infty} x_n \) in X such that \( \sum_{n=1}^{\infty} x_{\pi(n)} \) converges to the same sum for all permutations \( \pi \in P(\mathbb{N}) \) for which \( \sum_{n=1}^{\infty} x_{\pi(n)} \) converges is already unconditionally convergent.
(ix) Riemann-Rearrangement Property (RRP) (also called Dvoretzky-Rogers property (DRP)):
\[
uvw(X) = \ell_1[X] = \ell_1\{X\} = abc(x).
\]
Here \( uc(X) = \left\{ (x_n) \subset X : \sum_{n=1}^{\infty} x_{\pi(n)} \text{ converges in } X, \forall \pi \in P(\mathbb{N}) \right\} \)
and
\[
abc(X) = \left\{ (x_n) \subset X : \sum_{n=1}^{\infty} ||x_n|| < \infty \right\}.
\]
Clearly, \( abc(X) \subset uc(X) \). More generally, we have the equality:
\[
\ell_p\{X\} = \ell_p[X], 1 \leq p < \infty
\]
as an important (FD)-property. This is the p-analogue of the famous Dvoretzky-Rogers theorem referred to above.

**Definition 3.2. ([9], Chap. 2):** \( T \in L(X, Y) \) is said to be
(i) nuclear (\( T \in N(X, Y) \)) if there exist \( \{\lambda_n\} \subset \ell_1, \{f_n\} \subset X^* \) and \( \{f_n\} \subset Y \) bounded such that
\[
T(x) = \sum_{n=1}^{\infty} \lambda_n < x, f_n > y_n, x \in X
\]
The class \( N(X, Y) \) then becomes a Banach space when equipped with the nuclear norm defined by
\[
\nu(T) = \inf \left\{ \sum_{n=1}^{\infty} |\lambda_n| ||f_n|| ||y_n|| \right\},
\]
where infimum is taken over all representations of \( T \) as given above.
(ii) $p$- (absolutely) summing $\left( T \in \Pi_p(X, Y) \right)$ if $\forall \{x_n\} \in \ell_p[X]$, it follows that $\left\{ (T(x_n)) \right\} \in \ell_p\{X\}$. By the open mapping theorem, this translates into the finitary condition: $\exists c > 0$ s.t.

$$\left( \sum_{i=1}^{n} ||x_i||^p \right)^{1/p} \leq c \sup_{f \in B_{X^*}} \left\{ \left( \sum_{i=1}^{n} |<x_i,f>|^p \right)^{1/p} \right\},$$

$\forall \{x_i\}_{i=1}^{n} \subset X, n \geq 1$. The infimum of all such $c$'s appearing above and denoted by $\pi_p(T)$ is called the $p$-summing norm of $T$, making $\Pi_p(X, Y)$ into a Banach space.

Grothendieck’s theorem (see [24], Theorem 5.12) states that all operators on an $L_1$ space and taking values in a Hilbert space are absolutely summing whereas those acting on an $L_\infty$ space are always 2-summing.

**Remark 3.3.** (See [22], Chap.II).

(i) Nuclear maps are always compact (as the uniform limit of a sequence of finite rank operators)

(ii) $N(X, Y) \subset \Pi_p(X, Y) \subset \Pi_q(X, Y), \forall 1 \leq p \leq q$.

(iii) $\Pi_p(H_1, H_2) = HS(H_1, H_2)$, class of Hilbert-Schmidt maps acting between Hilbert spaces $H_1$ and $H_2$ and $1 \leq p < \infty$.

(iv) $T \in \Pi_p(X, Y)$ if and only if $TS \in \Pi_p(\ell_q, Y)$ for each $S \in L(\ell_q, X), \frac{1}{p} + \frac{1}{q} = 1$.

(v) A composite of 2-summing maps (and hence of absolutely summing maps) is always nuclear with $\nu(TS) \leq \pi_2(T)\pi_2(S)$, for all $S \in \Pi_2(X, Y), T \in \Pi_2(Y, Z)$.

**Three important features of (FD)-properties:**

As stated in the Introduction, the commonality of approach to the three classes of spaces, namely nuclear spaces, Hilbert spaces and Hilbert-Schmidt spaces through finitedimensional phenomena is based on three important features lurking in the shadows of these phenomena but which manifest themselves only in an infinite dimensional context. These three features involving a given finite dimensional property (P) derive from:

a. Frechet space analogue of (P).

b. Size of the set of objects failing (P).

c. Factorisation property of (P).

In view of its impact on the development of functional analysis via Dvoretzky-Rogers theorem and a host of other variants of this latter
theorem, in what follows we shall mainly discuss the (RRP) as an important (FD)-property and single it out as the main example to illustrate the aforementioned phenomena while making an effort to provide the necessary details in respect of other (FD)-properties, wherever possible. Before we proceed further, let us pause to argue why (RRP) is indeed an (FD)-property. To this end, we recall:

Dvoretzky-Rogers Lemma ([9], Lemma 1.3): Given $n \geq 1$ and a $2n$-dimensional normed space $X$, there exist $n$ vectors $x_1, x_2, \ldots, x_n \in B_X$ with $\|x_i\| \geq 1/2$ for $1 \leq n$ such that above lemma to assert the following $(\ast)$ which is a stronger statement than that (RRP) is an (FD) property.

$(\ast)$ Given an infinite dimensional Banach space $X$ and $(\lambda_i)_{i=1}^\infty \in \ell_2$, there exists $(x_i)_{i=1}^\infty \subset X$ such that $\sum_{n=1}^\infty x_n$ is unconditionally convergent in $X$ and $\|x_i\| = |\lambda_i|, i \geq 1$.

Indeed, we can choose an increasing sequence $(n_k)_{k=1}^\infty$ of positive integers such that for $m, n$ and $n_k \leq N \leq n_{k+1}$, we have

$$\sum_{i=m}^n |\lambda_i|^2 \leq 2^{-2k}$$

Applying the above Dvoretzky-Rogers lemma to each block $[n_k, n_{k+1})$ yields a sequence $(y_i)_{i=1}^\infty \subset B_X$ with $\|y_i\| \geq 1/2$ for $i \geq 1$ such that for all choices of scalars $\alpha_i$ and $n_k \leq N \leq n_{k+1}$, we have

$$\|\sum_{i=n_k}^N \alpha_i y_i\| \leq \left( \sum_{i=n_k}^N |\alpha_i|^2 \right)^{1/2}$$

Setting $x_i = \lambda_i y_i / \|y_i\|$ and taking $\epsilon_i = \pm 1$ for $i \geq 1$ gives

$$\|\sum_{i=n_k}^N \epsilon_i x_i\| \leq \left( \sum_{i=n_k}^N \frac{|\lambda_i|^2}{\|y_i\|^2} \right)^{1/2} \leq 2^{-k+1}, \ k \geq 1.$$ 

In other words, the series $\sum_{i=n_k}^N \epsilon_i x_i$ has Cauchy partial sums and is, therefore, convergent. Equivalently, the series $\sum_{i=1}^\infty x_n$ is unconditionally convergent $\|x_i\| = |\lambda_i|$ for all $i \geq 1$.

Comments: Besides the proof of the Dvoretzky-Rogers (DR) - theorem given above, there are many more, exploiting techniques ranging from local theory of Banach spaces to topological tensor products. Instead, we shall briefly include details of yet another proof of this important statement which is based on a theorem of the author proved in [27]. It characterizes the finite dimensionality of a Banach space $X$ in terms of the equality of certain operator ideals:
Theorem 3.4. The following statements for a Banach space $X$ are equivalent:

(i) $\Pi_2(X, \ell_2) = N(X, \ell_2)$.

(ii) $\dim X \leq \infty$.

To see how the proof of the (DR)-theorem based on Theorem 2 works, we note that the given condition in (DR)-theorem, namely: $\ell_1[X] = \ell_1\{X\}$ leads to the equality: $\ell_2[X] = \ell_2\{X\}$ which, by virtue of Theorem 2.1 translates into: $L(\ell_2, X) = \Pi_2(\ell_2, X)$. Thus there exists $c > 0$ such that

$$\pi_2(T) \leq c \| T \|, \forall T \in L(\ell_2, X)$$

Now a trace duality argument applied to the above equality gives: $\Pi_2(\ell_2, X) = N(X, \ell_2)$, which is the same thing as $X$ being finite dimensional, by Theorem 3.4. Indeed, given $T \in \Pi_2(\ell_2, X)$ and finite dimensional spaces $E$ and $F$ with $u \in L(E, X), v \in L(\ell_2, F), w \in L(F, E)$, we note that $uvw \in L(\ell_2, X)$, and so, applying (3.1) to $uvw$ and combining the resulting estimate with Remark 3.3 (v) gives:

$$\nu(uwvT) \leq \pi_2(uwv)\pi_2(T) \leq c \| uvw \| \pi_2(T)$$

This together with an application of [9], Lemma 6.14 yields:

$$\text{trace}(uwvTu) = \text{trace}(uvwT)$$

This gives:

$$i(T) \leq c\pi_2(T)$$

where $i$ denotes the integral norm. In other words, $T$ is an integral operator taking its values in a reflexive space, and so has to be nuclear. It follows that $\Pi_2(X, \ell_2) = N(X, \ell_2)$ and, therefore, $X$ is finite dimensional by Theorem 3.4.

(a) Frechet-space setting:

When suitably formulated in the setting of Frechet spaces $X$, it turns out that in most of the cases, there exist infinite dimensional Frechet spaces in which an (FD)-property holds. It also turns out that, at least in most cases of interest, the class of Frechet spaces in which this holds coincides with the class of nuclear spaces in the sense of Grothendieck [11]. To define a nuclear space, we shall assume- in the interest of technical simplifications- that the topology of locally convex spaces in question will be generated by a system of norms as opposed to a family of seminorms.

Definition 3.5. A locally convex space (lcs, for short) $X$ is said to be nuclear if for each continuous norm $p$ on $X$, there exists a continuous norm $q$ on $X$, $q \geq p$ such that the identity map $i:(X,q)\rightarrow(X,p)$ is nuclear. It is easily verified that this is equivalent to requiring that for each Banach space $Z$, each continuous linear map $T: X\rightarrow Z$ is nuclear.
Example 3.6. The following are well known examples of nuclear spaces;

(1) All finite dimensional spaces.
(2) $\omega$ countable product of the line.
(3) $H(C)$, space of entire functions(on the plane).
(4) $H(D)$, space of holomorphic functions on the unit disc D.
(5) $D(\Omega)$, space of test functions on an open set $\Omega$ in $\mathbb{R}^n$.
(6) $D'(\Omega)$, space of distributions on an open set $\Omega$ in $\mathbb{R}^n$.

Remark 3.7. Banach+Nuclear = Finite Dimensional.

Example 3.8. The Frechet- space analogue of the following (FD)-properties are valid exactly when the Frechet space in question is nuclear.

(i) (RRP): Unconditionally convergent series in X are absolutely convergent.

(ii) Levy-Steinitz Property: Given a convergent series $\sum_{n=1}^{\infty} x_n$ in X, then $DS\left(\sum_{n=1}^{\infty} x_n\right)$ is an affine space. more precisely,

$$DS\left(\sum_{n=1}^{\infty} x_n\right) = \Gamma\left(\sum_{n=1}^{\infty} x_n\right) + \sum_{n=1}^{\infty} x_n$$

where $\Gamma\left(\sum_{n=1}^{\infty} x_n\right) = \{ x \in X; f(x) = 0, \forall f \in X^* s.t. \sum_{n=1}^{\infty} | <x_n, f > | <\infty \}$ and $DS\left(\sum_{n=1}^{\infty} x_n\right)$, the domain of sums of the series $\sum_{n=1}^{\infty} x_n$ is defined by:

$$DS\left(\sum_{n=1}^{\infty} x_n\right) = \{ x \in X; \exists \pi P(\mathbb{N}) s.t \sum_{n=1}^{\infty} x_{\pi(n)} = x \}$$

(iii) $D(C(K, X), Y) = \Pi_1((C(K, X), Y)$. Here, $D(C(K, X), Y)$ stands for the class of dominated operators: $T \in D(C(K, X), Y)$ if there exists a Borel measure $\mu$ on K such that

$$\|Tf\| \leq \int_k \|f(t)\|d\mu(t), \forall f \in C(K, X)$$

(iv) Bochner Property(BP): Positive definite functions f on X arise as Fourier transforms of regular Borel measures on $X^*$:

$$f(x) = \int_{X^*} e^{-i<x, x^*>}d\mu, x \in X$$

(v) Weakly closed subgroups of a Frechet space X coincide with closed subgroups of X.
(vi) Equality involving vector measures:
(a) $M(X) = M_{bv}(X)$.
Here, $M(X)$ and $M_{bv}(X)$ denote the spaces of $X$-valued measures:

$M(X) = \left\{ \mu : A \to X : \mu \left( \bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} \mu(A_n), A_m \cap A_n = \emptyset, \forall m \neq n \right\}$

$M_{bv}(X) = \left\{ \mu \in M(X) : \sup_{(A_n) \subseteq A} \sum_{n=1}^{\infty} \| \mu(A_n) \| < \infty, A_m \cap A_n = \emptyset, \forall m \neq n \right\}$

(b) $c_0(X) \subset R_{bv}(X)$
where the symbols involved have the following meanings:

$R(X) = \{ (x_n) \subset X : \exists \mu \in M(X), (x_n) \subset rg(\mu) \}$

$R_{bv}(X) = \{ (x_n) \subset X : \exists \mu \in M_{bv}(X), (x_n) \subset rg(\mu) \}$

$rg(\mu) = \{ \mu(A) : A \in \mathcal{A} \}$

Comments: Let us begin by saying that each of the above properties which are valid in nuclear Frechet spaces provide yet further evidence that as opposed to Hilbert spaces, nuclear spaces are more suited to be looked upon as infinite dimensional analogues of finite dimensional spaces. Further, the proof for the equivalence of each of the above properties to nuclearity is different in each case, drawing upon techniques from different areas of analysis, depending upon the nature of the property. We shall, however, settle for a sketch of proof of the equivalence of (1) with nuclearity, using by now the standard techniques from the theory of operator ideals proposed by Pietsch as opposed to the complicated approach via tensor products which was originally given by Grothendieck in his thesis [11].

Proof of (i): Let us say that operator ideals $A$ and $B$ are equivalent if some power of $A$ is contained in $B$ and some suitable power of $A$ is contained in $B$. The definition of nuclear spaces given earlier shows that the class of locally convex spaces determined by an operator ideal that is equivalent to the ideal of nuclear operators in the sense just described coincides with the class of nuclear spaces. Now, (ii) and (iv) of Remark 2(a) shows that the ideals $N$ and $A$ are equivalent. Combining this fact with the definition of an absolutely summing map completes the proof. Indeed, assume that each unconditionally convergent series in the Frechet space $X$ is absolutely convergent. This yields that the inclusion $i : \ell_1\{X\} \to \ell_1[X]$ is a well-defined bijective continuous (linear) map. Further, the completeness of $X$ yields that each of these spaces is complete w.r.t. the metrisable locally convex topologies defined in (I). Thus, the inverse mapping theorem applies to yield that the inverse
map \( i = i^- : \ell_1[X] \to \ell_1\{X\} \) is continuous. In other words, for each continuous norm \( p \) on \( X \), there exists a continuous norm \( q \) on \( X \), \( q \geq p \) and \( c > 0 \) such that

\[
\sum_{i=1}^{n} p(x_i) \leq c \sup_{f \in B_{X_q^*}} \left( \sum_{i=1}^{n} |< x_n, f >| \right), \forall (x_i)_{i=1}^{n} \subset X, n \geq 1.
\]

By 3.2(ii), this means that the identity map \( i : (X, q) \to (X, p) \) is absolutely summing and this completes the argument. Converse is a straightforward consequence of 3.3(i).

(ii) The first complete proof of this statement was given by W. Banaszczyk [5] in 1990 which makes use of certain combinatorial lemmas now known as the “Rearrangement Lemma” and the “Lemma on Rounding off coefficients” involving a finite set of vectors in a (metrisable) nuclear space. A further strengthening of this statement valid in all complete (DF) - spaces was proposed by Bonet and Defant [7] a little later. A unified treatment of the (LS)-theorem which subsumes the finite dimensional as well as the nuclear analogue besides certain instances of its validity in a Banach space setting was given by author in [30].

(iii) The result is folklore for \( X \), a 1-dimensional space and generalizes easily to the case of \( X \) being finite dimensional. C. Swartz [32] proved the converse of this latter statement, i.e., the equality \( D(C(K, X), Y) \) for each Banach space \( Y \) is an (FD)-property for \( X \). The Frechet space analogue of the equality was established by M. Nakamura 19 who showed the equivalence of this equality to the nuclearity of \( X \). Here, the necessity part is a direct consequence of the definition of a nuclear space whereas sufficiency can be proved by using (i) above which consists in showing that each unconditionally convergent series in \( X \) is absolutely convergent. Thus, let \( \sum_{n=1}^{\infty} x_n \) be an unconditionally convergent series in \( X \) and fix \( t \) in \( K \). Consider the map \( T : C(K, X) \to X \) defined by \( T(f) = f(t) \). Letting \( \{p_n; n \geq 1\} \) denote a generating family of (semi)norms for the topology of \( X \) and \( \delta_t \) the Dirac measure at \( t \), we see that for each \( n \geq 1, p_n(T(f)) = < p_n(f), \delta_t > \) which shows that \( T \) is 1-dominated, and hence absolutely summing by the given hypothesis. Let \( g \in C(K) \) be the function: \( g(s) = 1 \), for all \( s \) in \( K \). By identifying the dual of \( C(K, X) \) with \( X^* \) - valued regular (c.a.) Borel measures on \( K \), it follows that the sequence \( \{g(x_n; n \geq 1\} \) is weakly absolutely summable in \( C(K, X) \), and so \( \{T(g(x_n; n \geq 1\} = \{x_n; n \geq 1\} \) is weakly absolutely summable in \( X \), i.e., the series \( \sum_{n=1}^{\infty} x_n \) absolutely
(iv) The necessity part of Bochner’s theorem in the setting of lcs was
proved by Minlos [17] who showed its validity in metrisable nuclear
spaces. The converse that the validity of Bochner’s theorem in a metris-
able lcs $X$ implies nuclearity of $X$ is a deep result of D. Muschtari [18].

(v) It is a well-known theorem of S. Mazur that the weak closure of
a convex set in a normed space coincides with its norm-closure. In
particular, a (linear) subspace of a normed space $X$ is closed if and
if it is weakly closed. However, this equivalence does not carry over
to subgroups of an infinite dimensional normed space $X$. In fact, the
existence of such groups is important from the viewpoint of unitary
representations of topological groups as it leads to easy examples of
topological groups which may not even admit weakly continuous unitary
representations. The existence of closed subgroups which are not
weakly closed was proved by S. J. Sidney [24] in the setting
of infinite dimensional Banach spaces admitting a separable infinite di-
dimensional quotient whereas the general case for an infinite dimensional
normed space was settled by Banaszczyk [2]. Also, the validity of this
property in nuclear (metrizable) spaces is again due to Banaszczyk [3]
and the equivalence of this property with the nuclearity of a metrisable
lcs is due to M. Banaszczyk and W. Banaszczyk [1]. As a consequence, it
follows that, as in the case of locally compact groups, nuclear Frechet
spaces admit unitary representations on a Hilbert space which is even
faithful if the topology of the space is given by a sequence of norms.

(vi) In the theory of vector measures, It is known that each vector
measure taking its values in a finite dimensional space is of bounded
variation. A simple application of Dvoretzky-Rogers theorem yields
that there are vector measure taking values in an infinite dimensional
Banach space which are not of bounded variation. Same is true of
(b) which says that the property involving the containment of null se-
quences from a Banach space $X$ inside the range of $X$-valued measures
is an (FD)-property. Duchon [10] showed that nuclear spaces are the
only Frechet spaces $X$ for which each vector measure taking values in $X$
is of bounded variation whereas the equivalence of (b) with the nucle-
arity of $X$ was proved by Bonet and Madrigal [8]. As a strengthening
of the latter property, it was shown by the author [28] that this equiv-
ance remains valid for the smaller sequence space $\ell_p\{X\}$ in place of
$c_0(X)$.
The above discussion serves to bring home the view that, as opposed to Hilbert spaces, nuclear spaces provide the most convenient infinite dimensional framework for the validity of certain important (FD)-properties which, by their very definition, fail in each infinite dimensional Banach space.

(b). Size of the set of objects failing (FD)
1. Given an (FD)- property (P), it turns out that for a given infinite-dimensional Banach space X, the set of objects in X failing (P) is usually very big: it could be topologically big(dense) algebraically big(contains an infinite-dimensional space) big in the sense of category(non-meagre), big in the sense of functional analysis (contains an infinite-dimensional closed subspace).

2. Examples:
   (i). For an infinite-dimensional Banach space X, the difference set uc(X)/abc(X) contains a c-dimensional subspace.
   (ii) M(X)/Mbv(X) is non-meagre.
   (iii) D(C(K), X)/Π(C(K), X) contains an infinite-dimensional space.
   (iv) M([0, 1], X)/B([0, 1], X) contains an infinite-dimensional space.
   Here, M and B stand, respectively, for the class of McShane and Bochner integrable functions on [0, 1] taking values in X.
   (v) DS(∑∞n=1 the domain of sums in the Levy-Steinitz theorem is far from being convex.

Comments: As in (a) above, let us see how to prove (i) which measures the extent of failure of the Riemann Rearrangement Theorem in an infinite dimensional Banach space.

Proof of (i): We begin by considering an uncountable almost disjoint family \( \{A_\alpha\}_{\alpha \in \Lambda} \) infinite subsets of \( \mathbb{N} \): \( A_\alpha \cap A_\beta \) is a finite set for \( \alpha \neq \beta \). An easy way to produce such a family is by writing \( \Lambda = [0, 1] \) and letting \( \{r_n\}_{k=1}^\infty \) denote the rationals in \([0, 1]\). For \( \alpha \in \Lambda \), choose a subsequence \( \{r_{n_k}\}_{k=1}^\infty \) of \( \{r_n\}_{n=1}^\infty \) such that \( r_{n_k} \xrightarrow{k} \alpha \) and define \( A_\alpha = \{n_k : n \geq 1\} \).

By construction, the family \( \{A_\alpha\}_{\alpha \in \Lambda} \) has the desired properties. By Dvoretzky-Rogers theorem, we can choose a series \( \sum_{n=1}^\infty x_n \) in X which is unconditionally convergent but which does not converge absolutely. For every \( \alpha \in \Lambda \) define a sequence \( x^\alpha = (x^\alpha_i)_{i=1}^\infty \) in X where \( x^\alpha = x_n \) if \( i = nth \) element of \( A_\alpha \) and \( x^\alpha = 0 \) otherwise. Clearly, the
series $\sum_{n=1}^{\infty} x_n^a$ is unconditionally convergent. However, it is not absolutely convergent as it has a subseries which is not absolutely convergent. By virtue of almost disjointness of $A'_\alpha$'s, it follows that the family $\{x^\alpha : \alpha \in \Lambda\}$ is linearly independent. Thus $E = \text{span}\{x^\alpha : \alpha \in \Lambda\}$ is $c$-dimensional ($c =$cardinality of continuum). Noting that each element of $E$ is unconditionally summable, the proof will be completed by showing that each element of $E$ is absolutely non-summable. To this end, let $\{\alpha_1, \alpha_2, \cdots, \alpha_n\} \subset \Lambda$ and $\lambda_1, \lambda_2, \cdots, \lambda_n$ be (non-zero) scalars. Then, $z = \lambda_1 x^{\alpha_1}, \lambda_2 x^{\alpha_2}, \cdots, \lambda_n x^{\alpha_n}$ is not absolutely summable. Indeed, we can choose an infinite set $A \subset A_{a_1}$ such that $A_{a_1} - A$ is finite, $A \cap (U_{i=2}^{n} A_{a_i}) = \phi$ and, therefore, $\sum_{i \in A} z_i = \sum_{i \in A} \lambda_i x_{a_i}$ which contains all but finitely many terms of a non-absolutely convergent series.

It follows that $\sum_{i=1}^{\infty} z_i$ is a non-absolutely convergent series. Property (ii) was proved by R. Anantharaman and K.M. Garg [1]. However, it is not known if this set (together with 0) also contains an infinite dimensional space. In his report, one of the referees has suggested a strategy drawing on some recent techniques in the area of spaceability that would yield that the set under reference contains a large infinite dimensional space! Also (iii) is a recent result of F.J.G. Pacheco and Puglisi [20] whereas the proof of (iv) is part of joint work of the author with F.J.G. Pacheco [21].

(v) The first example demonstrating the failure of Levy-Steinitz theorem in infinite dimensional Hilbert space was given by Marcinkiewicz which can be used, via Dvoretzkys spherical sections theorem, to show that such examples can also be constructed in each infinite dimensional Banach space. The strategy involved in Marcinkiewicz example consists in showing that the constructed series has a nonconvex domain of sums and, therefore, such a series fails the Levy-Steinitz theorem. Although in the case under reference, it is not clear what the size of the set of objects failing the Levy-Steinitz should mean, it is possible to quantify the extent of failure of this theorem in terms of the degree of non-convexity of the domain of sums. In fact, it is possible to produce, in each infinite dimensional Banach space, counterexamples to the Levy-Steinitz theorem for which the domain of sums is highly non-convex in the sense that it consists precisely of two distinct points!

(c) . Factorization of (FD)-properties
1. As opposed to Hilbert spaces which possess a rich geometrical structure, the class of those Banach spaces which lie at the other end of the spectrum is distinguished by a relatively poor geometrical structure as is testified by spaces like $c_0, \ell_\infty, \ell_1, C(K), L_\infty(\Omega)$ etc. The latter class
of Banach spaces is subsumed under the class of so-called Hilbert-Schmidt spaces which arise as spaces with a particular factorisation property. To put this definition into perspective, let us recall that a trace class(nuclear) operator $T : \ell_2 \to \ell_2$ admits a factorization over each infinite-dimensional Banach space $X$: there exist bounded linear maps $T_1 : \ell_2 \to X$, $T_2 : X \to \ell_2$ such that $T = T_2 T_1$. Even more, this statement also holds for the more general class of Hilbert-Schmidt maps. Conversely, a Hilbert space operator factoring over each infinite dimensional Banach space is necessarily of the Hilbert-Schmidt type. For this, it is enough to choose a test space for $X$ which may be taken to be any of the spaces listed above: $c_0, \ell_\infty, \ell_1, C(K), L_\infty(\Omega)$ besides many more that include the disc algebra ($\mathcal{D}$). This motivates the following definition, due originally to H.Jarchow [13].

**Definition 3.9.** $X$ is said to be a Hilbert-Schmidt space if each bounded linear operator acting between Hilbert spaces and factoring over $X$ is already a Hilbert Schmidt map.

**Remark 3.10.** Recalling the well-known extension property of 2-summing maps, it is possible to prove the following characterization of Hilbert-Schmidt spaces:

**Theorem 3.11.** For a Banach space $X$, the following are equivalent:

(i) $X$ is a Hilbert-Schmidt space

(ii) $L(X, \ell_2) = \Pi_2(X, \ell_2)$.

Proof: The simple proof of $(i) \Rightarrow (ii)$ is a consequence of the definition of a Hilbert-Schmidt space combined with the fact that a map $T \in L(X, Y)$ is p-summing if and only if $T S$ is p-summing for each $S \in L(\ell_p, X)$. The other implication follows from Remark 3.3(iii).

**Example 3.12.** (i) Hahn-Banach Extension Property: For an extremely disconnected space $K$ and an arbitrarily given Banach space $X$, every bounded linear operator on $C(K)$ into $X$ extends to a bounded linear operator on any superspace containing $C(K)$. (Converse is also true). Something similar is true for all the spaces listed above: $c_0, \ell_\infty, \ell_1, C(K), A(D), L_\infty(\Omega)$: a bounded linear operator defined on any of these spaces and taking values in $\ell_2$ extends to a bounded linear operator on any Banach space containing the given space. In fact, we have the following theorem which is a consequence of Theorem c.3 combined with the extension property of 2-summing maps. The reverse implication follows by noting that every Banach space embeds into a $C(K)$ space and that by Grothendiecks theorem (see Section II), an $\ell_2$-valued bounded linear map on a $C(K)$-space is already 2-summing.
Theorem 3.13. For a given Banach space $X$, TFAE:

(i) $X$ is a Hilbert-Schmidt space.

(ii) A bounded linear map on $X$ into $\ell_2$ extends to a bounded linear map on any superspace of $X$.

Let us note a dual counterpart to this result:

Theorem 3.14. For a Banach space $X$ with $\dim > 2$, TFAE:

(i) A bounded linear map defined on a subspace of $X$ and taking values in an arbitrary Banach space $Z$ extends to a bounded linear map on $X$.

(ii) $X$ is a Hilbert space.

The proof of $(ii) \Rightarrow (i)$ follows from the projection theorem in Hilbert spaces whereas $(i) \Rightarrow (ii)$ is a consequence of the famous Lindenstrauss-Tzafriri theorem [16] to the effect that projection theorem holds good precisely when the space in question is Hilbertian.

Remark 3.15. The (FD)-property involving a multivariate analogue of the Hahn-Banach extension property lends itself to a similar factorisation scheme as has been witnessed in the case of the classical Hahn-Banach in Theorems 3.13 and 3.14 above. More precisely, we have the following theorem on the extension of bilinear forms on a Banach space:

Theorem 3.16. A Banach space $X$ is a Hilbert space if and only if for each 2-dimensional subspace $Y$ of $X$, every continuous bilinear functional on $Y \times Y^*$ extends to a continuous bilinear functional on $X \times X^*$ Theorem(b)[14]. A Banach space $X$ such that for each Banach space $Z$ containing $X$ isometrically, each continuous bilinear form on $X \times X$ extends to a continuous bilinear form on $Z \times Z$ is a Hilbert-Schmidt space.

The fact that a Hilbert space can never be Hilbert-Schmidt space unless it is finite dimensional motivates the following problem:

Factorization Problem: Given an (FD)-property $(P)$, whether it is possible to write $(P) = (Q) \land (R)$ for some properties $(Q)$ and $(R)$ such that $X$ verifies $(Q)$ iff $X$ is Hilbertian and $X$ verifies $(R)$ iff $X$ is Hilbert-Schmidt.

As noted above, the property $(P)$ defined by the “Hahn-Banach extension property on subspaces and superspaces”, admits such a factorization. In what follows, we shall come across a number of (FD) - properties which admit such a factorization, some of which are listed below.

(i) $\ell_2\{X\} = \ell_2[X]$

(ii) Bochner Property (BP)
(iii) $c_0(X) \subset R_{tu}(X)$
(iv) $\Pi_2(X,.) = \Pi^{d_2}(X,.)$
(v) $\Pi_2(X, \ell_2) = N(X, \ell_2)$

**Missing Link:** There is a 'missing link' involved in each of the above properties (P) which when inserted between the objects involved gives rise to properties (Q) and (R) with $(P) = (Q) \land (R)$. Let us see what this missing link looks like in case of (i) above.

(i) The (FD)-property in question involves the equality of two $X$-valued sequence spaces for which one sided inclusion is always true: $\ell_2\{X\} \subset \ell_2[X]$, regardless of the space $X$. The missing object in the factorization of the equality in (i) entails the search for an $X$-valued sequence space $\lambda(X)$ for which it always holds that $\ell_2\{X\} \subset \lambda(X) \subset \ell_2[X]$ and that strengthening these inclusions to equalities results in $X$ being (isomorphically) a Hilbert space and a Hilbert-Schmidt space, respectively. In the case under study, the right candidate for $\lambda(X)$ turns out to be the space:

$$\ell_2\{X_0\} = \left\{(x_n) \subset X : \sum_{n=1}^{\infty} \|T(x_n)\|^2 < \infty, \forall T \in L(X, \ell_2)\right\}$$

for which it holds that $\ell_2\{X\} \subset \ell_2\{X_0\} \subset \ell_2[X]$. Now, the desired factorization of the equality in (i) is given in the following theorem:

**Theorem 3.17.** For a Banach space $X$, the following statements are true:

(a) $\ell_2\{X\} = \ell_2\{X_0\}$ if and only if $X$ is Hilbertian.
(b) $\ell_2\{X_0\} = \ell_2[X]$ if and only if $X$ is a Hilbert-Schmidt space.

**Proof (a):** The proof is accomplished by noting that the vector-valued sequence spaces appearing here can be identified with the space $\Pi_2(\ell_2)$ of 2-summing maps and the space $\Pi_d^2(\ell_2, X)$ of dual 2-summing maps, respectively. The desired correspondence is provided by the map $T \rightarrow \{T(e_n)\}$ which sets up an isometric isomorphism between

(i) $\Pi_2(\ell_2, X)$ and $\ell_2\{X\}$ and between
(ii) $\Pi_d^2(\ell_2, X)$ and $\ell_2\{X_0\}$

To see why it is so, let us begin by remarking that the proof of (i) as given below can be suitably modified to prove the p-analogue of (i) as stated in Theorem 2.1. Now a simple consequence of the Hahn-Banach theorem yields that $x = (x_n) \in \ell_2\{X\}$ if and only if there exists $a = (a_n) \in \ell_2$ such that $|<x^*, x_n>| \leq a_n$, $\forall n \geq 1$ and $\forall x^* \in B_X$. We use it to show that $T \in L(\ell_2, X)$ given by $T(e_n) = x_n$, $n \geq 1$, is
2-summing whenever \((x_n) \in \ell_2\{X\}\). To this end, let \((\alpha^{(n)})\) be a weakly 2-summable sequence in \(\ell_2\). Then for \(x^* \in B_{X^*}\) and \(n \geq 1\), we see that

\[
| < x^*, T(\alpha^{(n)}) > | = | < T^*(x^*), \alpha^{(n)} > | = | < x^*, x_i >_1, \alpha^{(n)} |
\]

\[
= \sum_{i=1}^{\infty} | x^*, x_i > \alpha^{(n)}_i | \leq \sum_{i=1}^{\infty} | \alpha_i \alpha^{(n)}_i | = c_n, \text{ say.}
\]

Finally, weak 2-summability of \((\alpha^{(n)})\) in \(\ell_2\) yields that \(c = (c_n) \in \ell_2\) and this completes the argument. Regarding the proof of (ii), we note that \((x_n) \in \ell_2\{X_0\}\) if and only if \(ST\) is a Hilbert-Schmidt map for each \(S \in L(X, H)\) where \(H\) is a Hilbert space, or equivalently, \((ST)^* = T^*S^*\) is a Hilbert-Schmidt map. Since \(S\) was chosen arbitrarily, Remark 3.3(iv) yields that \(T^*\) is 2-summing.

Now the identifications set up in (i) and (ii) above yield that the equality of sequence spaces in (a) amounts to the operator-ideal equality: \(\Pi^d_2(\ell_2, X) = \Pi_2(\ell_2, X)\). However, it is well-known that the latter equation holds precisely when \(X\) is Hilbertian (See [9], Theorem 4.19). An alternative argument based on the “Eigenvalue Theorem” of Johnson et al [15] proceeds as follows. Let \(T : X \to X\) be a nuclear map on \(X\). Then \(T\) can be factored as \(T = T_2D_2D_1T_1\) where \(T_1 \in L(X, \ell_\infty), T_2 \in L(\ell_1, X), D_1 \in \Pi_2(\ell_\infty, \ell_2), D_2 \in \Pi^d_2(\ell_2, \ell_1)\). The ideal property of \(\Pi^d_2\) combined with the above equation gives \(T_2D_2 \in \Pi_2(\ell_2, X)\). Thus as a composite of two 2-summing maps, \(T\) has absolutely summable eigenvalues by [22], Proposition3.4.5 and Theorem 3.7.1, and so \(X\) is isomorphic to a Hilbert space, by the ‘Eigenvalue Theorem’. See [31] for further applications of the Eigenvalue Theorem in the context of factorization of certain operator ideal equations.

**Proof (b):** It is easily seen that the equality \(\ell_2\{X_0\} = \ell_2[X]\) involving sequence spaces translates into the equality \(L(X, \ell_2) = \Pi_2(X, \ell_2)\) involving operator ideals which, by virtue of Theorem III.c.3, holds exactly when \(X\) is a Hilbert Schmidt space.

(ii). The (FD)- property in question says that there exist positive definite functions on each infinite dimensional Banach space which do not arise as the Fourier transform of a regular Borel measure on the dual of \(X\). However, it turns out that if \(X\) is a Hilbert space, there always exists a locally convex topology on \(X\) (the so-called Sazonov topology) such that each positive definite function on \(X\) which is continuous in this topology is already a Fourier transform. Remarkably, it turns out that as soon as such a topology exists on a Banach space \(X\), then \(X\)
is a Hilbert space. Thus, the locally convex topology \( \tau \) would provide the missing link if the following statement were true:

\[
(*) \text{ Every positive definite function on } X \text{ is } -\text{continuous if and only if } X \text{ is a Hilbert-Schmidt space.}
\]

However, it is not known if \((*)\) is true.

(iii). In the theory of vector measures, the so-called localisation problem deals with the issue of enclosing sequences from certain distinguished sequence spaces \( \lambda(X) \) from a Banach space \( X \) inside ranges of vector measures (with or without bounded variation) taking values in \( X \). For \( \lambda(X) = c_0(X) \), it turns out that Banach spaces \( X \) for which \( \lambda(X) \subset R(X) \) are precisely those for which \( X^* \) is a subspace of an \( L_1 \) space. Here the missing object in the desired factorization is a sequence space modeled on \( X \) which lies between \( R(X) \) and \( R_{bv}(X) \) and yields properties \((Q)\) and \((R)\) for which \((P) = (Q) \land (R) \). We define

\[
R_{vbu}(X) = \{(x_n) \subset X : \exists \text{ Banach space } Z \supset X \text{ and } \mu \in M_{bv}(Z) \text{ s.t. } (x_n) \subset \text{rg } (\mu)\}
\]

It can be proved that \( R_{vbu}(X) \subset R_{vbo}(X) \subset R(X) \). Now the inclusion relation in (iii) is equivalent to: (a) \( c_0(X) \subset R_{vbo}(X) \) and (b) \( R_{vbu}(X) \subset R_{vbo}(X) \). It was proved by Pineiro [23] (see also [27] for an alternative proof) that (a) holds exactly if the underlying space is Hilbertian whereas (b) holds if and only if \( N \) is a (GT)-space (i.e. satisfies Grothendieck’s theorem: \( L(X, \ell_2) = \Pi_1(X, \ell_2) \), a property which is obviously stronger than that of being a Hilbert-Schmidt space). In particular, \( X \) is a Hilbert-Schmidt space. A far reaching refinement of (iii) was proved by the author [29], replacing the space \( c_0(X) \) by the smaller sequence space \( \ell_p\{X\} \) for \( p > 2 \).

(iv). It is well known that an operator acting between Hilbert spaces is Hilbert-Schmidt if and only if its adjoint is. A suitable analogue of this result in the Banach space setting shall make sense once it is clear what the Banach space analogue of a Hilbert-Schmidt map ought to look like. We have already noted (see Remark 3.3 (iii)) that p-summing maps coincide with Hilbert-Schmidt maps on Hilbert spaces, with the equivalence of the p-summing norm and the Hilbert-Schmidt norm. However, considering that the natural norm on the ideal of 2-summing maps even coincides with the Hilbert-Schmidt norm, the class of 2-summing maps stands out as the most appropriate candidate for the Banach space analogue of Hilbert-Schmidt maps. In view of this, it is natural to ask if the above stated result is valid for 2-summing maps in the Banach space setting. It turns out that finite dimensional spaces are the only Banach spaces \( X \) for which this property holds, i.e. such
that (iv) holds. The desired factorization of (iv), therefore, amounts to the inclusions:

(a) \( \Pi_2(X, Z) \subset \Pi^{d_2}(X, Z) \forall \) Banach spaces \( Z \).
(b) \( \Pi^{d_2}(X, Z) \subset \Pi_2(X, Z) \forall \) Banach spaces \( Z \).

As has been seen to be the case in respect of the (FD)-properties encountered earlier, the above inclusions are valid precisely when the Banach space \( X \) is Hilbert and Hilbert-Schmidt, respectively. The proof makes use of Grothendieck’s theorem quoted earlier combined with some important estimates involving p-summing maps. For details, see [9], Chapter 4.

(v). The fact that this condition on \( X \) is an (FD)-property was proved by the author in [27] where similar other results are proved in connection with some problems arising in the theory of vector measures. The missing object in the above factorization entails the search for an ideal of operators which can be ’sandwiched’ between \( \Pi_2 \) and \( N \) such that the resulting equations characterize \((Q)\) and \((R)\), respectively. More precisely, we shall ’invent’ an operator ideal \( A \) such that

(i) \( \Pi_2(X, \ell_2) = A(X, \ell_2) \) iff \( X \) is Hilbertian.
(ii) \( A(X, \ell_2) = N(X, \ell_2) \) iff \( X \) is Hilbert-Schmidt.

Unfortunately, the natural choice for \( A \), namely the ideal \( \Pi_1 \) of absolutely summing maps that suggests itself for effecting the desired factorization does not work! In fact, it was proved in [27] that the class of Banach spaces \( X \) satisfying the equation \( \Pi_2(X, \ell_2) = N(X, \ell_2) \) are precisely those for which \( X^* \) has the Gordon-Lewis property and satisfies the Grothendieck theorem. However, It can be shown (see [31]) that we can choose the ’missing link’ to be \( A = \Pi_{HS} \) in order to achieve the desired factorization of the (FD)-property \((P)\) given in Theorem(8) above as \((P) = (Q) \land (R)\) where the properties \((Q)\) and \((R)\) are defined by the above equations (i) and (ii). The operator ideal \( \Pi_{HS} \) is defined by:

\[
\Pi_{HS}(X, Y) = \{ T \in L(X, Y); \exists T_1 \in L(X, \ell_2), S \in HS(\ell_2, \ell_2), T_2 \in L(\ell_2, Y) \ s.t. \ T = T_2ST_1 \}
\]

Remark 3.18. By modifying the technique employed in the above factorization, it is possible to factorise the (FD)-properties involving the following operator-ideal equations.

(a) \( K(\ell_2, X) = N(\ell_2, X) \),
(b) \( S_2^{(c)}(\ell_2, X) = N(\ell_2, X) \).
Here, $K$ denotes the class of compact maps whereas the symbol $S_2^{(e)}(\ell_2, X)$ is used for those maps for which $\{T(x_n)\}$ (absolutely) 2-summable in $X$ for some weakly 2-summable sequence $\{x_n\}$ in $\ell_2$. In each case, it turns out that the ideal $\Pi_2$ of 2-summing maps provides the desired missing link! (See [26]).

**Theorem 3.19.** For a Banach space, the following statements each in (A) and (B) are equivalent:
(A)(i) $X$ is a Hilbert space.
   (ii) $K(\ell_2, X) = \Pi_2(\ell_2, X)$.
   (iii) $S_2^{(e)}(\ell_2, X) = \Pi_2(\ell_2, X)$.
(B)(i) $X$ is a Hilbert-Schmidt space.
   (ii) $\Pi_2(\ell_2, X) = N(\ell_2, X)$.

**Concluding Remarks:** The three important features of (FD)-properties discussed as the main theme of this paper have been illustrated with a number of examples drawn from different situations witnessing finite dimensionality in functional analysis. However, notwithstanding the preponderance of examples cropping up in our discussion of these phenomena, it should be emphasized that not all (FD)-properties that one encounters in analysis lend themselves to a suitable analogue under each of these categories. For example, the set of objects failing the (FD)-property: $K(X) = L(X)$ for an infinite dimensional Banach space $X$ is not necessarily 'big', at least in certain pathological situations. In fact, it turns out that for the Argyros-Haydon example of a Banach space admitting a small space of operators, the set $L(X)/K(X)$ does not even contain a 2-dimensional space! In a similar vein, the Heine-Borel property or the property: $X^* = X$ does not characterize nuclearity of the Frechet space $X$. This motivates the search for a set of theorems identifying a given (FD)-property as being amenable to yield itself to one of the several features of finite dimensionality as spelled out in the previous sections. The search for such theorems promises to be a fruitful line of research in this circle of ideas.

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