Quasi-two-body decays $B_c \rightarrow D(s)[\rho(770), \rho(1450), \rho(1700) \rightarrow] \pi \pi$ in the perturbative QCD factorization approach

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In this paper, we studied the quasi-two-body $B_c \rightarrow D(s)[\rho(770), \rho(1450), \rho(1700) \rightarrow] \pi \pi$ decays by employing the perturbative QCD (PQCD) factorization approach. The two-pion distribution amplitudes $\Phi_{\pi\pi}$ are applied to include the final-state interactions between the pion pair, while the time-like form factors $F_{\pi}(w^2)$ associated with the $P$-wave resonant states $\rho(770)$, $\rho(1450)$ and $\rho(1700)$ are extracted from the experimental data of the $e^+e^-$ annihilation. We found that: (a) the PQCD predictions for the branching ratios of the quasi-two-body decays are in the order of $10^{-9}$ to $10^{-5}$ and the direct CP violations around $(10 - 40)\%$ in magnitude; (b) the two sets of the large hierarchy $R_{1a,1b,1c}$ and $R_{2a,2b,2c}$ for the ratios of the branching ratios of the considered decays are defined and can be understood in the PQCD factorization approach, while the self-consistency between the quasi-two-body and two-body framework for $B_c \rightarrow D(s)[\rho(770) \rightarrow] \pi \pi$ and $B_c \rightarrow D(s)\rho(770)$ decays are confirmed by our numerical results; (c) taking currently known $B(\rho(1450) \rightarrow \pi \pi)$ and $B(\rho(1700) \rightarrow \pi \pi)$ as input, we extracted the theoretical predictions for $B(B_c \rightarrow D\rho(1450))$ and $B(B_c \rightarrow D\rho(1700))$ from the PQCD predictions for the decay rates of the quasi-two-body decays $B_c \rightarrow D[\rho(1450), \rho(1700) \rightarrow] \pi \pi$. All the PQCD predictions will be tested in the future experiments.

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I. INTRODUCTION

In recent years, large amount of the three-body $B_{c(s)}$ decays have been measured [1, 2] and the large localized $CP$ asymmetries in several decay channels [3–5], have raised great interests. A number of works have been done by using rather different methods, for example, the QCD factorization (QCDF) approach [6–16], the perturbative QCD (PQCD) factorization approach [17–29], and the frameworks based on the symmetry principles [30–37]. Compared to $B_{c(s)}$ meson, $B_c$ meson is unique since it consists of two different heavy quarks: $\bar{b}$ and $c$ quark. With the flavor quantum numbers $B = -C = \pm 1$, the $B_c$ meson can not decay strongly but only weakly. Besides, it is heavier than $B_{c(s)}$ meson and more difficult to be produced unless in high energy hadron collisions. Fortunately, some $B_c$ events have been observed in the Tevatron and Large Hadron Collider (LHC) experiments [1, 2]. In recent works by the LHCb Collaboration, some three-body $B_c$ decays, for instance, $B_c^+ \to \{K K K, \pi \pi \pi, K K \pi, p p K, p p \pi\}$ have been measured [38, 39]. Meanwhile, more and more $B_c$ events will be collected with the continuous running of LHC. The research of the three-body $B_c$ decays could be an important topic for both experiment and theory in next few years.

As known, in three-body decays, one can measure the distribution of $CP$ asymmetry in the Dalitz plot [40] experimentally. However, from a theoretical point of view, it is too difficult to calculate $CP$ violation in the whole Dalitz plot but practical to analyze a process of quasi-two-body decay. Experimentally, three-body $B$ meson decays are known to be usually dominated by the low energy resonances on $\pi \pi$, $KK$ and $K\pi$ channels and most of the quasi-two-body decays are extracted from the Dalitz-plot analysis of three-body ones. In a quasi-two-body decay, the final-state interactions between the pair of mesons are considered while the rescattering between the third particle and the meson pair is usually ignored. In the views of PQCD [17, 18], a direct evaluation of the hard kernels which contain two virtual gluons at lowest order is not important, the dominant contributions come from the region where the two energetic light mesons are almost collimating to each other with an invariant mass below $O(\Lambda_{QCD})(\Lambda = m_q - m_b)$, and the two-meson distribution amplitudes [17, 18, 41–44] have been introduced to include both resonant and nonresonant contributions for the meson pair. In the previous work, the parameters in the $P$-wave two-pion distribution amplitudes were determined in PQCD approach [21]. Following Ref. [21], we have studied the quasi-two-body decays $B_{c(s)} \to P/D(\rho(770), \rho(1450), \rho(1700) \to \pi \pi)$ [25–28] where $P = \pi, K, \eta, \eta'$, and $D$ represents the charmed $D$ meson.

In the past several years, a series of semileptonic $B_c$ decays [45] and nonleptonic two-body $B_c$ decays [46–56] have been studied in the PQCD framework. End-point singularity is avoid by keeping the transverse momentum $k_T$ of the quarks, and the Sudakov formalism makes this approach more reliable. From those literatures, we know the following points which can be also helpful for us to study the three-body $B_c$ decays:

1. The size of annihilation contributions is a meaningful issue in $B_c$ physics since the two-body nonleptonic charmless decays $B_c \to h_1 h_2$ ($h_1$, $h_2$ represent the light pseudoscalar mesons, vector mesons, axial-vector mesons, scalar mesons and so on) occur through the weak annihilation diagrams only. As a feature of PQCD, the diagrams including factorizable, nonfactorizable and annihilation type are all calculable. From numerical calculation, the contribution from nonfactorizable and annihilation-type diagrams is also found to be of great importance in charmed decays $B_c \to D h$ ($D$ stands for charmed $D$ meson);

2. Since only tree operators are involved, the direct $CP$-violating asymmetries for those charmless $B_c$ decays are absent naturally, while there are both penguin and tree diagrams involved in $B_c \to D h$ decays and the possibly large direct $CP$ violations in some channels were predicted [53, 54].

In this work, we will extend the previous studies as presented in Refs. [21, 25–28] to the quasi-two-body decays $B_c \to D_{c(s)}[\rho(770), \rho(1450), \rho(1700) \to \pi \pi]$, and give our predictions about the branching ratios and direct $CP$ violations of those decays. For simplicity, we generally use the abbreviation $\rho = \rho(770), \rho' = \rho(1450), \rho'' = \rho(1700)$ in the following sections. By now, the $B_c \to D_{c(s)}\rho$ decay has been studied in several frameworks, for examples, a relativistic constituent quark model based on the Bethe-Salpeter formalism [57], the QCD factorization approach with input from light-front quark model [58] and the PQCD approach [53]. But there are hardly any studies for $B_c$ decays with final states $\rho'$ and $\rho''$ since the structure of the excited states $\rho'$ and $\rho''$ is not yet completely clear [2]. There are a small number of theoretical studies for $\rho'$ and $\rho''$, for examples, the Refs. [59–63]. Experimentally, the observation of both $\rho'$ and $\rho''$ has been reported in study of the $\tau^{-} \to \pi^{-}\pi^{0} \nu_{\tau}$ decay by Belle [64] and $e^{-}e^{-} \to \pi^{+}\pi^{-}(\gamma)$ decay by BABAR [65]. Meanwhile, several quasi-two-body $B$ meson decays like $B^{0} \to K^{+}\rho^{-}, B^{-} \to \pi^{-}\rho^{0}$ and $B^{0} \to D^{0}\rho^{0}(\rho^{''})$ have been observed in experiments [66–68]. For the phenomenological analysis of $B_c \to D_{c(s)}\rho' (\rho'')$, we can not treat it as in Refs [46–56] due to the lack of the distribution amplitudes for $\rho'$ and $\rho''$. But in quasi-two-body framework [21], we here firstly attempt to study the $B_c \to D_{c(s)}[\rho' (\rho'')] \to \pi \pi$ decays by singling out the component of $\rho' (\rho'')$ in the two-pion distribution amplitudes. Then, the branching ratios of $B_c \to D_{c(s)}[\rho' (\rho'')] \to \pi \pi$ relying on a reliable estimation for the branching fraction $B(\rho' (\rho'') \to \pi \pi)$.

The paper is organized as follows. In Sec. II, we give a brief introduction for the theoretical framework and perturbative calculations for the considered decays. Then, the numerical values and phenomenological analysis are given in Sec. III. Finally, the last section contains a short summary.
II. THE THEORETICAL FRAMEWORK

In the PQCD approach based on $k_T$ factorization, one separates the hard and soft dynamics in a QCD process [17]. The hard part is calculable in the perturbation theory while the soft part is not calculable perturbatively but have to be treated as an universal input determined from experiments. The amplitude of the process, consequently, could be expressed as a convolution of a hard kernel $H$ with the hadron wave functions $\Phi(x,k_T)$ ($x$ means a longitudinal momentum fraction and $k_T$ represents a transverse momentum). For a quasi-two-body $B_c$-meson decay, its decay amplitude $\mathcal{A}$ in PQCD approach can then be written conceptually as the following convolution [17, 18]

$$\mathcal{A} = \Phi_{B_c} \otimes H \otimes \Phi_{h_1h_2} \otimes \Phi_{h_3}. \quad (1)$$

The symbols $\otimes$ means the convolution integrations over the parton kinematic variables and the specific calculation formula will be shown in the following subsections.

A. Coordinates and wave functions

In the light-cone coordinates, the $B_c$ meson momentum $p_B$, the momenta $p_1, p_2$ for each pion and the total momentum of the pion pair $p = p_1 + p_2$, and the $D$ meson momentum $p_3$ in the rest frame of $B_c$ meson are chosen as

$$p_B = \frac{m_{B_c}}{\sqrt{2}} (1, 1, 0_T), \quad p = \frac{m_{B_c}}{\sqrt{2}} (1 - r^2, \eta, 0_T), \quad p_3 = \frac{m_{B_c}}{\sqrt{2}} (r^2, 1 - \eta, 0_T),$$

$$p_1 = \frac{m_{B_c}}{\sqrt{2}} (\zeta (1 - r^2), (1 - \zeta) \eta, 0_T), \quad p_2 = \frac{m_{B_c}}{\sqrt{2}} ((1 - \zeta) (1 - r^2), \zeta \eta, 0_T), \quad (2)$$

where $\eta = w^2/[(1 - r^2) m_{B_c}^2]$ with the mass ratio $r = m_D/m_{B_c}$ and the invariant mass squared of the pion pair $w^2 = p^2 = m^2(\pi\pi)$, $\zeta$ is the momentum fraction for one of the pion pair. The momenta of the light quarks in the $B_c$ meson and the final state mesons are defined as $k_B$, $k$ and $k_3$ respectively

$$k_B = x_B p_B + (0, 0, k_{BT}), \quad k = z p^+ + (0, 0, k_T), \quad k_3 = x_3 p_3^+ + (0, 0, k_{3T}), \quad (3)$$

where the momentum fraction $x_B, z$ and $x_3$ run between zero and unity.

For $B_c$ meson, we use the same wave function as in Refs. [46–56]:

$$\Phi_{B_c}(x) = \frac{i}{\sqrt{6}} (P_B + m_{B_c}) \gamma_5 \phi_{B_c}(x, b), \quad (4)$$

with the distribution amplitude $\phi_{B_c}(x, b)$ [55]

$$\phi_{B_c}(x, b) = \frac{f_{B_c}}{2 \sqrt{6}} \delta \left( x - \frac{m_c}{m_{B_c}} \right) \exp \left[ - \frac{1}{2} \omega_B b^2 \right], \quad (5)$$

where the exponent term describes the $k_T$-dependence of $\phi_{B_c}(x, b)$; while the parameter $\omega_B = (0.60 \pm 0.05)$ GeV, $m_c$ is the charm quark mass, $m_{B_c}$ is the $B_c$ meson mass, and $f_{B_c}$ is the decay constant of $B_c$ meson.
For $D$ meson, the two-parton light-cone distribution amplitudes in the heavy quark limit can be written as [53–55, 69–72]

$$\langle D(p_3)|q_\alpha(z)|\bar{c}_\beta(0)|0\rangle = \frac{i}{\sqrt{6}} \int_0^1 dx e^{ixp_3^z} \left[ \gamma_5 (\bar{q} \gamma_3 m_D) \phi_D(x,b) \right]_{\alpha\beta},$$

with the distribution amplitude $\phi_D(x,b)$

$$\phi_D(x,b) = \frac{1}{2\sqrt{6}} f_D 6x(1-x) [1 + C_D(1-2x)] \exp\left[ -\frac{\omega^2 b^2}{2} \right],$$

where $C_D = 0.5 \pm 0.1$, $\omega = 0.1$ GeV and $f_D = 211.9$ MeV [2] for the $D$ meson, and $C_{D_s} = 0.4 \pm 0.1$, $\omega = 0.2$ GeV and $f_{D_s} = 249$ MeV [2] for $D_s$ meson.

The $P$ wave two-pion distribution amplitudes are defined in the same way as in Ref. [21]:

$$\Phi_{\pi\pi}^{I=1} = \frac{1}{\sqrt{2N_c}} \left[ p_0^0(z,\zeta,w^2) + w\phi^0(z,\zeta,w^2) + \frac{\hat{p}_1\hat{p}_2 - \hat{p}_2\hat{p}_1}{w(2\zeta-1)} \phi^0(z,\zeta,w^2) \right],$$

where

$$\phi^0(z,\zeta,w^2) = \frac{3F_\pi(w^2)}{\sqrt{2N_c}} \left[ 1 + a_0^0C_2^{3/2}(t) \right] P_1(2\zeta - 1),$$

$$\phi^s(z,\zeta,w^2) = \frac{3F_s(w^2)}{\sqrt{2N_c}} \left[ 1 + a_0^s(1 - 10z + 10z^2) \right] P_1(2\zeta - 1),$$

$$\phi^t(z,\zeta,w^2) = \frac{3F_t(w^2)}{\sqrt{2N_c}} \left[ 1 - 2z \right] \left[ 1 + a_0^tC_2^{3/2}(t) \right] P_1(2\zeta - 1),$$

with $C_2^{3/2}(t) = \frac{3}{2}(5t^2 - 1)$, $t = 1 - 2z$ and the Legendre polynomial $P_1(2\zeta - 1) = 2\zeta - 1$. The Gegenbauer moments are chosen as $a_0^0 = 0.30 \pm 0.05$, $a_0^s = 0.70 \pm 0.20$ and $a_0^t = -0.40 \pm 0.10$ [26]. The time-like form factor $F_\pi$ which includes the strong interactions between the $P$-wave diagrams and the pion pair can be written in the form of [65]

$$F_\pi(w^2) = \frac{1}{1 + \sum_i c_i} \left\{ BW^{GS}_{\rho}(w^2,m_\rho,\Gamma_\rho) \frac{1 + c_{\rho}BW^{KS}_{\omega}(w^2,m_\omega,\Gamma_\omega)}{1 + c_{\omega}} + \sum_i c_i BW^{GS}_{\omega}(w^2,m_\omega,\Gamma_i) \right\},$$

where $i = (\rho', \rho''(2254))$. The explicit expressions of $BW^{GS}_{\rho,i}$, $BW^{KS}_{\omega}$, and relevant parameters can be also found for example in Ref. [65].

### B. Decay amplitudes

For the considered $B_c \to D_{(s)}[\rho, \rho', \rho''] \to \pi\pi$ decays, the effective Hamiltonian $H_{eff}$ [73] can be written as:

$$H_{eff} = \frac{G_F}{\sqrt{2}} \left\{ \sum_{q=u,c} V_{q}\bar{V}_{d(s)} \left[ C_1(\mu)O_1(\mu) + C_2(\mu)O_2(\mu) \right] - V_{ub}\bar{V}_{d(s)} \sum_{i=3}^{10} C_i(\mu)O_i(\mu) \right\},$$

where $G_F = 1.16639 \times 10^{-5}$ GeV$^{-2}$ is the Fermi coupling constant, $C_i(\mu)$ are the Wilson coefficients at the renormalization scale $\mu$, $O_i(\mu)$ are the effective four quark operators and $V_{ij}$ are the CKM matrix elements.

At the leading order, there are eight diagrams contributing to the considered decays as shown in the Fig. 1. The four diagrams in first line are the emission type diagrams while the diagrams in the second line are the four annihilation type diagrams. By making analytical evaluations for those Feynman diagrams in Fig. 1, we can obtain the total decay amplitudes of these considered decays.

For the three $\rho(770)$-related $B_c \to D_{(s)}[\rho] \to \pi\pi$ decays, their total decay amplitudes can be written in the following form

$$A(B_c^+ \to D^0[\rho^+] \to \pi^+\pi^0) = \frac{G_F}{\sqrt{2}} \left\{ V_{ub}\bar{V}_{ud} \left[ a_1 F^{LL}_{eD} + C_1 M^{LL}_{eD} \right] + V_{cb}\bar{V}_{cd} \left[ a_1 F^{LL}_{aD} + C_1 M^{LL}_{aD} \right] 
- V_{ub}\bar{V}_{ud} \left[ a_4 + a_{10} \right] \left( F^{LL}_{eD} + F^{LL}_{aD} \right) + \left( a_6 + a_8 \right) \left( F^{LP}_{eD} + F^{LP}_{aD} \right) + \left( C_3 + C_9 \right) \left( M^{LL}_{eD} + M^{LL}_{aD} \right) + \left( C_5 + C_7 \right) \left( M^{LP}_{eD} + M^{LP}_{aD} \right) \right\},$$

(12)
\[
\sqrt{2}A(B_c^+ \to D^+[\rho^0 \to \pi^+\pi^-]) = \frac{G_F}{\sqrt{2}} \left\{ V_{ub}^* V_{ud} \left[ a_2 F_{eD}^{LL} + C_2 M_{eD}^{LL} \right] - V_{ub}^* V_{cd} \left[ a_1 F_{eD}^{LL} + C_1 M_{eD}^{LL} \right] \right. \\
- V_{tb}^* V_{ts} \left( \left[ -a_4 + \frac{5}{3} C_9 + C_{10} \right] F_{eD}^{LL} + \frac{3}{2} a_7 F_{eD}^{LR} - \left( a_6 - \frac{1}{2} a_8 \right) F_{eD}^{SP} \right) \\
+ \left( -C_3 + \frac{3}{2} a_{10} \right) M_{eD}^{LL} - \left( C_5 - \frac{1}{2} C_7 \right) M_{eD}^{LR} + \frac{3}{2} C_8 M_{eD}^{SP} \\
- \left( a_4 + a_{10} \right) F_{eD}^{LL} - (a_6 + a_8) F_{eD}^{SP} - (C_3 + C_9) M_{eD}^{LL} - (C_5 + C_7) M_{eD}^{LR} \right\}, \quad (13)
\]

\[
\sqrt{2}A(B_c^+ \to D_s^+[\rho^0 \to \pi^+\pi^-]) = \frac{G_F}{\sqrt{2}} \left\{ V_{ub}^* V_{us} \left[ a_2 F_{eD}^{LL} + C_2 M_{eD}^{LL} \right] \\
- V_{ub}^* V_{ts} \left( 8 a_9 F_{eD}^{LL} + a_7 F_{eD}^{LR} + C_{10} M_{eD}^{LL} + C_8 M_{eD}^{SP} \right) \right\}, \quad (14)
\]

where \( a_i \) are the combinations of the Wilson coefficients \( C_i \),

\[
a_{1,2} = C_2 + C_1 \frac{1}{3}, \quad a_i = C_i + C_{i+1} \frac{3}{3}, \text{ for } i = (3, 5, 7, 9); \text{ or } i = (4, 6, 8, 10). \quad (15)
\]

The \( F_{eD}^{LL} \) \(^1\) and other individual amplitudes relevant with the eight sub-diagrams in Fig. 1 can be written in the following forms:

1. From the factorizable emission diagrams Fig. 1(a) and 1(b):
   \[
   F_{eD}^{LL} = 8 \pi C_F m_B^4 F_\pi \int_0^1 dx_B dx_3 \int_0^{1/\Lambda} b_B db_B b_3 db_3 \phi_B \phi_D \\
   \times \left\{ \left[ -\bar{\eta} |(1-x_3) + (r-2) r_b + x_3(1-2r) + r^2(x_3 - 2r_b) \right] E_c(t_a) h_c(\alpha, \beta, b, B) \right. \\
   + \left. \left[ \bar{\eta} [2r(x_3 - 1) + \eta x_B] + r^2(x_B - 1) \right] E_c(t_b) h_b(\alpha, \beta, b, B) \right\}, \quad (16)
   \]

\[
M_{eD}^{LL} = -32 \pi C_F m_B^4 / \sqrt{6} \int_0^1 dx_B dx_3 \int_0^{1/\Lambda} b_B db_B b db \phi_B \phi_D \phi^0 \\
\times \left\{ \left[ r |(1 + \eta)(1-x_B) - \bar{\eta} x_3 - \eta z \right] - (1 - \eta^2)(1-x_B - z) \right\} E_n(t_c) h_n(\alpha, \beta, b, b, b) \\
+ \left. \left[ \bar{\eta} (1-x_3) \eta + r [(1 + \eta)x_B - \eta z + \bar{\eta} (z - x_B)) \right] E_n(t_d) h_d(\alpha, \beta, b, b, b) \right\}, \quad (18)
\]

\[
M_{eD}^{LR} = 32 \pi C_F m_B^4 / \sqrt{6} \int_0^1 dx_B dx_3 \int_0^{1/\Lambda} b_B db_B b db \phi_B \phi_D \sqrt{\eta (1-r^2)} \\
\times \left\{ \left[ \bar{\eta} (1-x_B - z)(\phi^* + \phi) - r (\bar{\eta} x_3 - z)(\phi^* - \phi') + 2r(1-x_B - z)\phi^* \right] E_n(t_c) h_n(\alpha, \beta, b, b, b) \\
- \left[ r(\bar{\eta} - x_3 - z)(\phi^* + \phi') - \bar{\eta} (x_B - z)(\phi^* - \phi') + 2(z - x_B)\phi^* \right] E_n(t_d) h_d(\alpha, \beta, b, b, b) \right\}, \quad (19)
\]

\[
M_{eD}^{SP} = -32 \pi C_F m_B^4 / \sqrt{6} \int_0^1 dx_B dx_3 \int_0^{1/\Lambda} b_B db_B b db \phi_B \phi_D \phi^0 \\
\times \left\{ \left[ \bar{\eta} (1-x_3) + r [(1 + \eta)(1-x_B) - \eta z + \bar{\eta} (z - 2(1-x_B)) \right] E_n(t_c) h_n(\alpha, \beta, b, b, b) \\
- \left[ r(\bar{\eta} (1-x_3) + \eta (z - x_B) - x_B) + (1 - \eta^2)(z - x_B) \right] E_n(t_d) h_d(\alpha, \beta, b, b, b) \right\}. \quad (20)
\]

\(^1\) The subscripts LL, LR and SP correspond to the contributions from the \((V-A)(V-A), (V-A)(V+A)\) and \((S-P)(S+P)\) currents, respectively.
(3) From the factorizable annihilation diagrams Fig. 1(e) and 1(f):

\[
F_{aL}^{LL} = -8\pi C_F m_{B_c}^4 f_{B_c} \int_0^1 dz dx_3 \int_0^{1/\Lambda} d\bar{b} b_3 db_3 \phi_D \\
\times \left\{ \left[ \eta \left( 1 - r^2 \right) \bar{\eta} x_3 + \eta \phi^0 + 2r \sqrt{\eta (1 - r^2)} \left[ 1 + \eta + \eta x_3 \right] \phi^t \right] E_a(t_c) h_e(\alpha, \beta, b_3, b) S_t(x_3 \bar{\eta}) \\
+ \left[ 2r (1 + \eta) r r_c - \bar{\eta} z + r^2 (2r z - 1) \right] \phi^0 - \sqrt{\eta (1 - r^2)} \left[ 2r z (\phi^s + \phi^t) + (2r - r_c) \bar{\eta} (\phi^s - \phi^t) \right] \\
\times E_a(t_f) h_f(\alpha, \beta, b, b_3) S_t(z \bar{\eta}) \right\},
\]

\[\tag{21}\]

\[
F_{aD}^{SP} = 16\pi C_F m_{B_c}^4 f_{B_c} \int_0^1 dz dx_3 \int_0^{1/\Lambda} d\bar{b} b_3 db_3 \phi_D \\
\times \left\{ \left[ r(x_3 \bar{\eta} + 2\eta)\phi^0 + 2\sqrt{\eta (1 - r^2)} \bar{\eta} \phi^s \right] E_a(t_c) h_e(\alpha, \beta, b_3, b) S_t(x_3 \bar{\eta}) \\
+ \left[ 2r (1 - (1 - z) \eta) - \bar{\eta} r_c \phi^0 + \sqrt{\eta (1 - r^2)} \left[ \bar{\eta} z (\phi^s - \phi^t) - 4r r_c \phi^s \right] \right] E_a(t_f) h_f(\alpha, \beta, b_3) S_t(z \bar{\eta}) \right\}.
\]

\[\tag{22}\]

(4) From the nonfactorizable annihilation diagrams Fig. 1(g) and 1(h):

\[
M_{aL}^{LL} = 32\pi C_F m_{B_c}^4 / \sqrt{6} \int_0^1 dx_B d\bar{b} d\bar{b}_3 \int_0^{1/\Lambda} b_B b_{B_3} \phi_B \phi_B \\
\times \left\{ \left[ \bar{\eta} (1 + \eta) (1 - x_B - z) - r_B \phi^0 + r \sqrt{\eta (1 - r^2)} [-z (\phi^s + \phi^t) - \bar{\eta} x_3 (\phi^s - \phi^t)] \\
+ 2r \left[ x_B - 2r_B \phi^t \right] \right] E_n(t_c) h_g(\alpha, \beta, b, b_B) \\
+ \left[ \bar{\eta} (\bar{\eta} x_3 - (1 + \eta) x_B + r_c + \eta z) \phi^0 + r \sqrt{\eta (1 - r^2)} \bar{\eta} x_3 (\phi^s + \phi^t) + z (\phi^s - \phi^t) \\
+ 2(2r_c - x_B) \phi^s \right] \right\} E_n(t_h) h_h(\alpha, \beta, b, b_B)
\]

\[\tag{23}\]

\[
M_{aD}^{LL} = -32\pi C_F m_{B_c}^4 / \sqrt{6} \int_0^1 dx_B d\bar{b} d\bar{b}_3 \int_0^{1/\Lambda} b_B b_{B_3} \phi_B \phi_B \\
\times \left\{ \left[ (1 + \eta) (1 + r_c - x_B - \bar{\eta} x_3 - \eta z) \phi^0 - \bar{\eta} \sqrt{\eta (1 - r^2)} (1 + r_c + x_B - x_3 - z) (\phi^s + \phi^t) \right] \right\} \\
\times E_n(t_c) h_g(\alpha, \beta, b, b_B) \\
- \left[ r \left[ (1 + \eta) (x_B + r_c - \bar{\eta} x_3 - \eta z) \phi^0 - \bar{\eta} \sqrt{\eta (1 - r^2)} (r_c + x_B - x_3 - z) (\phi^s + \phi^t) \right] \right\} \\
\times E_n(t_h) h_h(\alpha, \beta, b, b_B)
\]

\[\tag{24}\]

where $\bar{\eta} = 1 - \eta$, $C_F = 4/3$ is the color factor. The explicit expressions of the hard functions ($h_a, \cdots, h_h$), the hard scales ($t_a, \cdots, t_h$), the evolution factors ($E_a, \cdots, E_n$) and the threshold resummation factor $S_t(x_1)$ will be given in Appendix A.

For the decays involving $\rho'$ and $\rho''$ mesons, one can get the relevant expressions for the corresponding decay amplitudes by simple replacements of $\phi^{0,s,t}$ for $\rho$ meson to the ones for $\rho'$ or $\rho''$, respectively.

III. NUMERICAL RESULTS

Besides those specified in previous sections, the following input parameters will also be used in our numerical calculations (the masses, decay constants and QCD scale are in units of GeV) [2]:

\[
\begin{align*}
\Lambda^{(f=4)}_{MS} &= 0.25, & m_{B_c} &= 6.275, & m_{D^+} &= 1.870, & m_{D^0} &= 1.865, & m_{D_s^+} &= 1.968, \\
m_{s\pi^\pm} &= 0.140, & m_{s\pi^0} &= 0.135, & m_b &= 4.8, & f_{B_c} &= 0.489, & \tau_{B_c} &= 0.507 \text{ ps}.
\end{align*}
\]

\[\tag{25}\]

For the Wolfenstein parameters ($\lambda, \bar{\rho}, \bar{\eta}$) of the CKM mixing matrix, we use $\lambda = 0.2512 \pm 0.0005, \bar{\rho} = 0.124 \pm 0.019, \bar{\eta} = 0.356 \pm 0.011$. 


TABLE I. The PQCD predictions for the $CP$ averaged branching ratios and the direct $CP$ asymmetries of the $B_c \to D(\rho, \rho', \rho'' \to \pi \pi)$ decays.

| Mode | Results |
|------|---------|
| $B_c^+ \to D^0[\rho^+ \to \pi^+ \pi^0]$ | $B \left(10^{-5}\right) = 1.64 \pm 0.33 \pm 0.21 \pm 0.03 \pm 0.09 \pm 0.01 \left(C_D(r)\right)$ |
| $A_{CP}$ | $0.20 \pm 0.07 \pm 0.14 \pm 0.07 \pm 0.08 \pm 0.03 \left(C_D(r)\right)$ |
| $B_c^+ \to D^+[\rho^0 \to \pi^+ \pi^0]$ | $B \left(10^{-5}\right) = 6.61 \pm 0.39 \pm 0.09 \pm 0.13 \pm 0.31 \left(C_D(r)\right)$ |
| $A_{CP}$ | $-0.33 \pm 0.08 \pm 0.06 \pm 0.01 \pm 0.04 \pm 0.03 \left(C_D(r)\right)$ |
| $B_c^+ \to D_s^+[\rho^0 \to \pi^+ \pi^0]$ | $B \left(10^{-7}\right) = 2.63 \pm 0.33 \pm 0.08 \pm 0.08 \pm 0.02 \pm 0.01 \left(C_D(r)\right)$ |
| $A_{CP}$ | $0.42 \pm 0.03 \pm 0.06 \pm 0.02 \pm 0.00 \pm 0.04 \left(C_D(r)\right)$ |

For the considered $B_c \to D(\rho, \rho', \rho'' \to \pi \pi)$ decays, the differential decay rate can be written in the following form

$$\frac{dB}{dw^2} = \tau_{B_c} \frac{\left|\vec{p}_\pi \cdot \vec{p}_D\right|}{32\pi^3 m_{B_c}^3} |A|^2,$$  \hspace{1cm} (26)

where $\tau_{B_c}$ is the mean lifetime of $B_c$ meson, $|\vec{p}_\pi|$ and $|\vec{p}_D|$ denote the magnitudes of the $\pi$ and $D$ momenta in the center-of-mass frame of the pion pair,

$$|\vec{p}_\pi| = \frac{1}{2} \sqrt{w^2 - 4m_\pi^2},$$

$$|\vec{p}_D| = \frac{1}{2} \sqrt{(m_{B_c}^2 - m_D^2)^2 - 2(m_{B_c}^2 + m_D^2)w^2 + w^4}/w^2.$$  \hspace{1cm} (27)

Based on the decay amplitudes as given in Eqs. (12-24) and the differential decay rate in Eq. (26), we obtain the PQCD predictions for the $CP$-averaged branching ratios ($B$) and the direct $CP$-violating asymmetries ($A_{CP}$) of the $B_c \to D(\rho, \rho', \rho'' \to \pi \pi)$ decays, and list the numerical results in Table I. The first error of these PQCD predictions comes from $\omega_B = (0.60 \pm 0.05)$ GeV for $B_c$ meson, the following three errors are from the Gegenbauer coefficients in the two-pion distribution amplitudes: $a_3^0 = 0.30 \pm 0.05, a_2^0 = 0.70 \pm 0.20, a_2^0 = -0.40 \pm 0.10$ and the last error is from $C_D = 0.5 \pm 0.1$ ( $C_{D_s} = 0.4 \pm 0.1$) in $D$ ($D_s$) meson wave function. The total theoretical error is about 10% to 30% of the central values.

In Fig. 2, we show the PQCD predictions for the differential decay rate $dB/dw$ (Fig. 2(a)) and for the $CP$-violating asymmetry $A_{CP}$ (Fig. 2(b)) for the considered $B_c^+ \to D^0[\rho^+ \to \pi^+ \pi^0]$ decay and its charged-conjugation $B_c^- \to D^0[\rho^- \to \pi^- \pi^0]$ decay. In Fig. 3, we show the same kinds of PQCD predictions as in Fig. 2 but for the $B_c^+ \to D_s^+(\rho^0 \to \pi^+ \pi^- \pi^0)$ decay and its charged-conjugation $B_c^- \to D_s^-(\rho^0 \to \pi^- \pi^+ \pi^0)$ decay.

From the Figs. (2,3) and the numerical results as listed in Table I, we have the following observations:

1. For the considered quasi-two-body decays, the PQCD predictions are in the order of $10^{-9}$ to $10^{-5}$ for the $CP$-averaged branching ratios, and around $(10 - 40)\%$ in size for the direct $CP$ violations. The $B_c^+ \to D^0[\rho^+ \to \pi^+ \pi^0]$ decay has the largest branching ratio, $\sim 1.64 \times 10^{-5}$, and to be measured in LHCb experiment.
of the decay amplitude in Eq. (responsible for the ratio $R_{BA}$ dominant contribution comes from the term proportional to $1/a_\pi^0$ decay, besides the strong suppression due to $ho^0\to\pi^+\pi^0$ decays.

For the special $B_c^+ \to D^0[\rho^0 \to \pi^+\pi^-]$ decay, the factorizable emission diagram (i.e. the term proportional to $a_1 F^{LL}_{cD}$ of the decay amplitude in Eq. (12)) provide the dominant contribution. For $B_c^+ \to D^+[\rho^0 \to \pi^+\pi^-]$ decay, however, the dominant contribution comes from the term proportional to $a_2 F^{LL}_{cD}$ of the decay amplitude in Eq. (13). The small ratio $R_{1a} \approx 0.04$ can be understood basically by the strong suppression due to the ratio $|a_2/a_1|^2 \sim 0.04$. For $B_c^+ \to D_s^+[\rho^0 \to \pi^+\pi^-]$ decay, besides the strong suppression due to $|a_2/a_1|^2$, a new suppression factor $|V_{us}/V_{ud}|^2 \sim \lambda^2$ may also be responsible for the ratio $R_{1c}$.

For the decay modes with the same pion pair final states but involving the different intermediate resonant state $\rho, \rho'$ or $\rho''$, there exists the second hierarchy between the PQCD predictions for their decay rates. Taking $B_c^+ \to D^0[\rho, \rho', \rho'' \to \pi^+\pi^0$.
decays as a example, we can define the new ratios $R_{2a,2b,2c}$:

$$R_{2a} = \frac{B(B_c^+ \to D^0[\rho^+ \to \pi^+\pi^0])}{B(B_c^+ \to D^0[\rho^+ \to \pi^+\pi^0])} \approx 8.3 \times 10^{-2},$$  \tag{31}

$$R_{2b} = \frac{B(B_c^+ \to D^0[\rho''^+ \to \pi^+\pi^0])}{B(B_c^+ \to D^0[\rho^+ \to \pi^+\pi^0])} \approx 3.8 \times 10^{-2},$$  \tag{32}

$$R_{2c} = \frac{B(B_c^+ \to D^0[\rho''^+ \to \pi^+\pi^0])}{B(B_c^+ \to D^0[\rho''^+ \to \pi^+\pi^0])} \approx 0.46.\tag{33}$$

Here the main reason for the hierarchy as shown by above ratios $R_{2a,2b}$ and $R_{2c}$ is the difference between the pion pair form factor $F_\pi$ for different intermediate resonance $\rho$, $\rho'$ and $\rho''$. For other two sets of decay modes, we find the similar hierarchy. From the three curves as shown in Fig. 4(a), one can see directly the large difference between the differential decay rates $d\mathcal{B}/dw$ for $B_c^+ \to D_s^+[\rho,\rho',\rho'' \to \pi^+\pi^-]$ decays.

(4) By using the following well-known relation of the decay rates between the quasi-two-body and the corresponding two-body decay modes,

$$B(B_c \to D_{(s)}[\rho(\rho',\rho'' \to \pi\pi]) = B(B_c \to D_{(s)}\rho(\rho',\rho'')) \cdot B(\rho(\rho',\rho'') \to \pi\pi),$$  \tag{34}

one can extract the theoretical predictions for $B(B_c \to D_{(s)}\rho(\rho',\rho''))$ from those for relevant quasi-two-body decays, if the decay rates $B(\rho(\rho',\rho'') \to \pi\pi)$ can be determined by employing other theoretical methods or measured directly by experiments.

For the decays involving $\rho$ meson, for example, since $B(\rho \to \pi\pi) \approx 100\%$, we therefore do expect a similar branching ratio for the two-body $B_c \to D_{(s)}\rho$ decay and the corresponding quasi-two-body one. In order to examine this general expectation, we do the calculations for $B(B_c \to D_{(s)}\rho)$ directly by employing the PQCD approach in the same way as Ref. [53]. We used the same formulae as in Ref. [53], but with the updated input parameters and the new wave functions. In the second and third column of Table II, we list our PQCD predictions obtained in the framework of the “Quasi-two-body” and two-body decay. In the forth, fifth and sixth columns of the Table II, as a comparison, we also show the relevant PQCD predictions as given in Ref. [53], and the theoretical predictions obtained by using the relativistic constituent quark model (RCQM) [57], or by using the light-front quark model (LFQM) [58].

From Table II, one can see that the PQCD predictions as listed in the column two and three agree very well with each other. This is a new confirmation for the self-consistency between the quasi-two-body and two-body framework of the PQCD approach for the considered $B_c$ meson decays. Although we used the same two-body framework and the decay amplitudes as in Ref. [53], but one can see that the PQCD predictions obtained in this work (column three) are much larger than those (column four) as given in Ref. [53], since we here used different distribution amplitude $\phi_{B_c}(x,b)$, different wave function $\phi_{D_{(s)}}(x,b)$ for $D_{(s)}$ meson and the updated Gegenbauer moments, masses and decay constants as well. In Ref. [53], we set $\phi_{B_c}(x,b) = \delta(x-m_{B_c})$. In this paper, however, we take $\phi_{B_c}(x,b) = \delta(x-m_{B_c}) \cdot \exp[-\omega_{B_c}^2 b^2/2]$ as given in Eq. (7).

The wave function $\phi_{D}(x,b) = N_D[x(1-x)]^2 \cdot \exp\left(-\frac{x^2 m_D^2}{2 \omega_D^2} - \frac{\omega_D^2 b^2}{2}\right)$ used in Ref. [53] also be very different from the
TABLE II. In the framework of the quasi-two-body or two-body decays, we list the PQCD predictions for the $CP$ averaged branching ratios $B(B_c \to D_{s(0)}[\rho \to \pi\pi])$ decays. As a comparison, we also list the theoretical predictions as given in Refs. [53, 57, 58].

| Decays                  | Quasi-two-body | Two-body | PQCD [53] | RCQM [57] | LFQM [58] |
|-------------------------|----------------|----------|-----------|-----------|-----------|
| $B(B_c^+ \to D^0[\rho^+ \to \pi^+\pi^0])$ | $1.64^{+0.49}_{-0.41}$ | $1.59^{+0.18}_{-0.17}$ | 0.662 | 0.60 | 0.13 |
| $B(B_c^+ \to D^+[\rho^0 \to \pi^+\pi^-])$ | $6.61^{+1.64}_{-0.99}$ | $6.28^{+1.17}_{-0.48}$ | 1.4 | 3.9 | 0.2 |
| $B(B_c^+ \to D_{s}^+[\rho^0 \to \pi^+\pi^-])$ | $2.63^{+0.35}_{-0.35}$ | $2.62^{+0.34}_{-0.32}$ | 0.95 | $-$ | 0.02 |

one as given in Eq. [7] of this paper. By direct examination, we find that the dominant changes of the PQCD predictions are induced by the difference between the wave function $\phi_{D_{s(0)}}(x, b)$ used here and the one used in Ref. [53]. More studies for the structure of the heavy mesons, such as $B_c$, $D$ and $D_s$, are clearly required. Precise experimental measurements for more $B_c$ meson decays can also test our predictions and help us to improve the theoretical framework itself.

(5) Due to the lack of the distribution amplitudes for $\rho'$ and $\rho''$, we can not calculate the branching ratios of the two-body decays $B_c \to D\rho'$ and $B_c \to D\rho''$ by using the traditional way in the PQCD approach. In the framework of the quasi-two-body decays, fortunately, we can extract the PQCD predictions for the branching ratios of the two-body decays $B_c \to D\rho'$ and $B_c \to D\rho''$ from the PQCD predictions for the branching ratios of the quasi-two-body decays $B_c \to D[\rho', \rho'' \to \pi\pi]$ if we take previously determined decay rates $B(\rho' \to \pi\pi) = 10.04^{+5.24}_{-2.61}\%$ and $B(\rho'' \to \pi\pi) = 8.11^{+2.42}_{-1.47}\%$ [21, 27] as input. Based on the relation as given in Eq. (34) and the numerical results as listed in Table I, we can then extract the PQCD predictions for the following two-body $B_c$ meson decays:

$$B(B_c^+ \to D^0\rho'^+) = 1.36^{+0.36}_{-0.23} \times 10^{-5},$$

$$B(B_c^+ \to D^+\rho'^0) = 1.17^{+0.22}_{-0.13} \times 10^{-6},$$

$$B(B_c^+ \to D^+\rho'^0) = 1.91^{+0.29}_{-0.23} \times 10^{-7},$$

$$B(B_c^+ \to D^0\rho''^+) = 7.77^{+1.94}_{-0.76} \times 10^{-6},$$

$$B(B_c^+ \to D^+\rho''^0) = 7.40^{+1.43}_{-0.96} \times 10^{-7},$$

$$B(B_c^+ \to D_s^+\rho''^0) = 1.15^{+0.12}_{-0.14} \times 10^{-7},$$

where the individual errors have been added in quadrature. These PQCD predictions will be tested at the future LHCb experiments.

(6) In Fig. 4(b), we show the total differential decay rate after the inclusion of the contributions from all three resonant states $\rho$, $\rho'$ and $\rho''$. From the magnitude and the shape of the curve as illustrated in 4(b), one can see clearly the strong destructive interference near 1.6 GeV: a clear dip at $w \approx 1.6$ GeV, similar with the one as shown in Fig. 45 of the Ref. [65], where the pion form-factor squared $|F_{\pi}(x)|^2$ measured by BABAR are illustrated as a function of the invariant mass of the pion pair in the range from 0.3 to 3 GeV. In our work, the same dip is induced by the strong destructive interference between $\rho'$ and $\rho''$, as shown in Fig. 4(b). Numerically, the PQCD predictions for the individual decay rate of $\rho'$ and $\rho''$ and the interference term between them are the following:

$$B(B_c^+ \to D_s^+[\rho' \to \pi^+\pi^-]) \approx 1.92 \times 10^{-8},$$

$$B(B_c^+ \to D_s^+[\rho'' \to \pi^+\pi^-]) \approx 9.29 \times 10^{-9},$$

$$\text{interf. term}_{\rho' - \rho''} \approx -1.75 \times 10^{-8}.$$ (37)

It is easy to see that the interference term is indeed large and negative when compared with other two individual contributions.

IV. SUMMARY

In this paper, we studied the quasi-two-body $B_c \to D_{s(0)}[\rho, \rho', \rho'' \to \pi\pi]$ decays in PQCD factorization approach. The two-pion distribution amplitudes have been applied to include the final-state interactions between the pion pair. The contributions from the $\rho$, $\rho'$ and $\rho''$ intermediate resonant states were estimated by introducing the time-like form factor $F_{\pi}$ involved in the $P$-wave two-pion distribution amplitudes. The PQCD predictions for the $CP$-averaged branching ratios and direct $CP$-violating asymmetries of the considered quasi-two-body decays are obtained and listed in Table I and II. Based on the relation
as given in Eq. (34), we extract the theoretical predictions for the branching ratios of the two-body decays $B_c \to D_{(s),X}$ with $X = (\rho, \rho', \rho'')$ from those PQCD predictions for $\mathcal{B}(B_c \to D_{(s)}[\rho, \rho', \rho'' \to \pi \pi])$ and those previously determined decay rates $\mathcal{B}(\rho' \to \pi \pi)$ and $\mathcal{B}(\rho'' \to \pi \pi)$.

From the analytical analysis and numerical calculations, we found the following points:

1. The PQCD predictions for the branching ratios of the quasi-two-body $B_c \to D_{(s)}[\rho, \rho', \rho'' \to \pi \pi]$ decays are in the order of $10^{-9}$ to $10^{-5}$, the direct CP violations are around $(10 - 40)\%$ in magnitude. The decay mode $B_c^+ \to D^0[\rho^+, \rho'' \to \pi^+ \pi^0]$ has a large branching ratio, $\sim 1.64 \times 10^{-5}$, and could be measured in the future LHCb experiment.

2. The two sets of the large hierarchy $R_{1a,1b,1c}$ for the ratios between the branching ratios $\mathcal{B}(B_c \to D_{(s)}[\rho \to \pi \pi])$ and $R_{2a,2b,2c}$ among the branching ratios $\mathcal{B}(B_c^+ \to D^0[\rho^+, \rho'' \to \pi^+ \pi^0])$ are defined and can be understood in the PQCD factorization approach. The self-consistency between the quasi-two-body and two-body framework for $B_c \to D_{(s)}[\rho \to \pi \pi]$ and $B_c \to D_{(s)}[\rho \to \pi \pi]$ decays are confirmed by our numerical results.

3. Taking previously determined decay rates $\mathcal{B}(\rho' \to \pi \pi) \approx 10\%$ and $\mathcal{B}(\rho'' \to \pi \pi) \approx 8.1\%$ as input, we extract the theoretical predictions for branching ratios $\mathcal{B}(B_c \to D[\rho'])$ and $\mathcal{B}(B_c \to D[\rho''])$ from the PQCD predictions for the branching ratios of the quasi-two-body decays $B_c \to D[\rho' \to \pi \pi]$ and $B_c \to D[\rho'' \to \pi \pi]$.

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Appendix A: Some relevant functions

The explicit expressions of the evolution factors $E_i(t)$, $E_a(t)$ and $E_n(t)$ and the threshold resummation factor $S_i(x)$ can be found, for example, in Refs. [25, 74]. We here show the explicit expressions of the hard functions $h_i$ with $i = (a, \cdots, h)$ which are obtained from the Fourier transform of the hard kernels:

$$h_i(\alpha, \beta, b_1, b_2) = h_1(\beta, b_2) \times h_2(\alpha, b_1, b_2),$$

$$h_1(\beta, b_2) = \begin{cases} K_0(\beta b_2), & \beta > 0 \\ K_0(i \sqrt{-\beta} b_2), & \beta < 0 \end{cases}$$

$$h_2(\alpha, b_1, b_2) = \begin{cases} \theta(b_2 - b_1) I_0(\sqrt{\alpha b_1}) K_0(\sqrt{\alpha b_2}) + (b_1 \leftrightarrow b_2), & \alpha > 0; \\ \theta(b_2 - b_1) I_0(\sqrt{-\alpha b_1}) K_0(i \sqrt{-\alpha b_2}) + (b_1 \leftrightarrow b_2), & \alpha < 0; \end{cases}$$

where $K_0$ and $I_0$ are modified Bessel functions with $K_0(ix) = \frac{x}{\pi}(-N_0(x) + iJ_0(x))$ and $J_0$ is the Bessel function. The hard scale $t_i$ is chosen as the maximum of the virtuality of the internal momentum transition in the hard amplitudes:

$$t_a = \max\{m_B \sqrt{|\alpha_a|}, m_B \sqrt{|\beta_a|}, 1/b_3, 1/b_B\}, \quad t_b = \max\{m_B \sqrt{|\alpha_b|}, m_B \sqrt{|\beta_b|}, 1/b_B, 1/b_3\};$$

$$t_c = \max\{m_B \sqrt{|\alpha_c|}, m_B \sqrt{|\beta_c|}, 1/b_3, 1/b_B\}, \quad t_d = \max\{m_B \sqrt{|\alpha_d|}, m_B \sqrt{|\beta_d|}, 1/b_B, 1/b\};$$

$$t_e = \max\{m_B \sqrt{|\alpha_e|}, m_B \sqrt{|\beta_e|}, 1/b_3, 1/b_B\}, \quad t_f = \max\{m_B \sqrt{|\alpha_f|}, m_B \sqrt{|\beta_f|}, 1/b, 1/b_3\};$$

$$t_g = \max\{m_B \sqrt{|\alpha_g|}, m_B \sqrt{|\beta_g|}, 1/b, 1/b_B\}, \quad t_h = \max\{m_B \sqrt{|\alpha_h|}, m_B \sqrt{|\beta_h|}, 1/b, 1/b_B\}.$$
where
\[ \alpha_a = r_b^2 + (1 - r^2)[(\eta - 1)x_3 - \eta], \quad \beta_a = (r^2 - x_B)[(1 - \eta)(x_3 - 1) + x_B]; \]
\[ \alpha_b = (r^2 - x_B)(x_B + \eta - 1), \quad \beta_b = \alpha_a; \]
\[ \alpha_c = \beta_a, \quad \beta_c = [1 - x_B - (1 - r^2)z][(1 - \eta)x_3 + x_B - 1]; \]
\[ \alpha_d = \beta_a, \quad \beta_d = [(1 - z)r^2 - x_B + z][(1 - \eta)(x_3 - 1) + x_B]; \]
\[ \alpha_e = (1 - r^2)[(\eta - 1)x_3 - \eta], \quad \beta_e = (1 - \eta)(r^2 - 1)x_3z; \]
\[ \alpha_f = \beta_c, \quad \beta_f = \beta_c; \]
\[ \alpha_g = \beta_e, \quad \beta_g = r_b^2 - [(1 - r^2)z + x_B - 1][(1 - \eta)x_3 + x_B - 1]; \]
\[ \alpha_h = \beta_c, \quad \beta_h = r_c^2 - [(1 - r^2)z + x_B][(1 - \eta)x_3 + x_B]. \] (A3)

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