Counting toroidal binary arrays, II

S. N. Ethier* and Jiyeon Lee†

Abstract

We derive formulas for (i) the number of toroidal $n \times n$ binary arrays, allowing rotation of rows and/or columns as well as matrix transposition, and (ii) the number of toroidal $n \times n$ binary arrays, allowing rotation and/or reflection of rows and/or columns as well as matrix transposition.

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1 Introduction

A previous paper [1] found the number of (distinct) toroidal $m \times n$ binary arrays, allowing rotation of rows and/or columns, to be

$$a(m, n) := \frac{1}{mn} \sum_{c \mid m} \sum_{d \mid n} \varphi(c) \varphi(d) \frac{2^{mn/\text{lcm}(c,d)}}{\text{lcm}(c,d)},$$

where $\varphi$ is Euler’s phi function and lcm stands for least common multiple. This is A184271 in the On-Line Encyclopedia of Integer Sequences [2]. The main diagonal is A179043. It was also shown that, allowing rotation and/or reflection of rows and/or columns, the number becomes

$$b(m, n) := b_1(m, n) + b_2(m, n) + b_3(m, n) + b_4(m, n),$$

where

$$b_1(m, n) := \frac{1}{4mn} \sum_{c \mid m} \sum_{d \mid n} \varphi(c) \varphi(d) \frac{2^{mn/\text{lcm}(c,d)}}{\text{lcm}(c,d)},$$

**Department of Mathematics, University of Utah, 155 South 1400 East, Salt Lake City, UT 84112 USA. ethier@math.utah.edu. Partially supported by a grant from the Simons Foundation (209632).**

**Department of Statistics, Yeungnam University, 214-1 Daedong, Kyeongsan, Kyeongbuk 712-749, South Korea. leejy@yu.ac.kr. Supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT & Future Planning (No. 2013R1A1A3A04007670).**
\[
b_2(m, n) := \frac{1}{4n} \sum_{d | n} \varphi(d) 2^{mn/d}
\]
\[
+ \begin{cases}
(4n)^{-1} \sum' \varphi(d) (2^{(m+1)n/(2d)} - 2^{mn/d}), & \text{if } m \text{ is odd;}

(8n)^{-1} \sum' \varphi(d) (2^{mn/(2d)} + 2^{(m+2)n/(2d)} - 2 \cdot 2^{mn/d}), & \text{if } m \text{ is even},
\end{cases}
\]
with \(\sum' := \sum_{d | n: d \text{ is odd}}\).

\[
b_3(m, n) := b_2(n, m),
\]
and

\[
b_4(m, n) := \begin{cases}
2^{(mn-3)/2}, & \text{if } m \text{ and } n \text{ are odd;}

3 \cdot 2^{mn/2-3}, & \text{if } m \text{ and } n \text{ have opposite parity;}

7 \cdot 2^{mn/2-4}, & \text{if } m \text{ and } n \text{ are even.}
\end{cases}
\]

(The formula for \(b_2(m, n)\) given in [1] is simplified here.) This is A222188 in the OEIS [2]. The main diagonal is A209251.

Our aim here is to derive the corresponding formulas when \(m = n\) and we allow matrix transposition as well. More precisely, we show that the number of (distinct) toroidal \(n \times n\) binary arrays, allowing rotation of rows and/or columns as well as matrix transposition, is

\[
\alpha(n) = \frac{1}{2} a(n, n) + \frac{1}{2n} \sum_{d | n} \varphi(d) 2^{n(n+d-2\lfloor d/2 \rfloor)/(2d)}, \quad (3)
\]

where \(a(n, n)\) is from (1). When we allow rotation and/or reflection of rows and/or columns as well as matrix transposition, the number becomes

\[
\beta(n) = \frac{1}{2} b(n, n) + \frac{1}{4n} \sum_{d | n} \varphi(d) 2^{n(n+d-2\lfloor d/2 \rfloor)/(2d)}
\]
\[
+ \begin{cases}
2^{(n^2-5)/4}, & \text{if } n \text{ is odd;}

5 \cdot 2^{n^2/4-3}, & \text{if } n \text{ is even},
\end{cases} \quad (4)
\]

where \(b(n, n)\) is from (2). At the time of writing, sequences (3) and (4) were not in the OEIS.

For an alternative description, we could define a group action on the set of \(n \times n\) binary arrays, which has \(2^{n^2}\) elements. If the group is generated by \(\sigma\) (row rotation) and \(\tau\) (column rotation), then the number of orbits is given by \(a(n, n)\); see [1]. If the group is generated by \(\sigma, \tau, \text{ and } \zeta\) (matrix transposition), then the number of orbits is given by \(\alpha(n)\); see Theorem 1 below. If the group is generated by \(\sigma, \tau, \rho\) (row reflection), and \(\theta\) (column reflection), then the number of orbits is given by \(b(n, n)\); see [1]. If the group is generated by \(\sigma, \tau, \rho, \theta, \text{ and } \zeta\), then the number of orbits is given by \(\beta(n)\); see Theorem 2 below.
Both theorems are proved using Pólya’s enumeration theorem (actually, the simplified unweighted version; see, e.g., van Lint and Wilson [3, Theorem 37.1, p. 524]).

To help clarify the distinction between the various group actions, we consider the case of $3 \times 3$ binary arrays as in [1]. When the group is generated by $\sigma$ and $\tau$ (allowing rotation of rows and/or columns), there are 64 orbits, which were listed in [1]. When the group is generated by $\sigma$, $\tau$, and $\zeta$ (allowing rotation of rows and/or columns as well as matrix transposition), there are 44 orbits, which are listed in Table 1 below. When the group is generated by $\sigma$, $\tau$, $\rho$, and $\theta$ (allowing rotation and/or reflection of rows and/or columns), there are 36 orbits, which were listed in [1]. When the group is generated by $\sigma$, $\tau$, $\rho$, $\theta$, and $\zeta$ (allowing rotation and/or reflection of rows and/or columns as well as matrix transposition), there are 26 orbits, which are listed in Table 2 below.

Table 1: A list of the 44 orbits of the group action in which the group generated by $\sigma$, $\tau$, and $\zeta$ acts on the set of $3 \times 3$ binary arrays. (Rows and/or columns can be rotated and matrices can be transposed.) Each orbit is represented by its minimal element in 9-bit binary form. Subscripts indicate orbit size. Bars separate different numbers of 1s.

\[
\begin{array}{cccccccc}
000 & 000 & 000 & 000 & 000 & 000 & 000 & 000 \\
000 & 000 & 000 & 000 & 000 & 000 & 000 & 000 \\
111 & 111 & 111 & 111 & 111 & 111 & 111 & 111 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
000 & 000 & 000 & 000 & 000 & 000 & 000 & 000 \\
000 & 000 & 000 & 000 & 000 & 000 & 000 & 000 \\
000 & 000 & 000 & 000 & 000 & 000 & 000 & 000 \\
000 & 000 & 000 & 000 & 000 & 000 & 000 & 000 \\
000 & 000 & 000 & 000 & 000 & 000 & 000 & 000 \\
000 & 000 & 000 & 000 & 000 & 000 & 000 & 000 \\
000 & 000 & 000 & 000 & 000 & 000 & 000 & 000 \\
000 & 000 & 000 & 000 & 000 & 000 & 000 & 000 \\
\end{array}
\]
Table 2: A list of the 26 orbits of the group action in which the group generated by $\sigma$, $\tau$, $\rho$, $\theta$, and $\zeta$ acts on the set of $3 \times 3$ binary arrays. (Rows and/or columns can be rotated and/or reflected and matrices can be transposed.) Each orbit is represented by its minimal element in 9-bit binary form. Subscripts indicate orbit size. Bars separate different numbers of 1s.

Table 3 provides numerical values for $\alpha(n)$ and $\beta(n)$ for small $n$.

Table 3: The values of $\alpha(n)$ and $\beta(n)$ for $n = 1, 2, \ldots, 12$.

| n   | $\alpha(n)$ | $\beta(n)$ |
|-----|-------------|-------------|
| 1   | 2           | 2           |
| 2   | 6           | 6           |
| 3   | 44          | 26          |
| 4   | 2200        | 805         |
| 5   | 674384      | 172112      |
| 6   | 954625404   | 239123150   |
| 7   | 5744406453840 | 1436120190288 |
| 8   | 14415102471496836 | 36028817512382026 |
| 9   | 1492250101286519583840 | 3731252531904344833632 |
| 10  | 633825300114296535353838471200 | 1584563250089172460150272 |
| 11  | 10985353370542379175011389992368 | 27463083426334757146923113956672 |
| 12  | 774314305635525874186817081357314024 | 19358285762613388352671214587814834041520 |

We take this opportunity to correct a small gap in the proof of Theorem 2 in [1]. The proof assumed implicitly that $m, n \geq 3$. The theorem is correct as stated for $m, n \geq 1$, so the proof is incomplete if $m$ or $n$ is 1 or 2. Following the proof of Theorem 2 below, we supply the missing steps.
2 Rotation of rows and columns, and matrix transposition

Let \( X_n := \{0, 1\}^{[0, 1, \ldots, n-1]} \) be the set of \( n \times n \) matrices of 0s and 1s, which has \( 2^n \) elements. Let \( \alpha(n) \) denote the number of orbits of the group action on \( X_n \) by the group of order \( 2n^2 \) generated by \( \sigma \) (row rotation), \( \tau \) (column rotation), and \( \zeta \) (matrix transposition). (Exception: If \( n = 1 \), the group is of order 1.)

Informally, \( \alpha(n) \) is the number of (distinct) toroidal \( n \times n \) binary arrays, allowing rotation of rows and/or columns as well as matrix transposition.

Theorem 1. With \( a(n, n) \) defined using (1), \( \alpha(n) \) is given by (3).

Proof. Let us assume that \( n \geq 2 \). By Pólya’s enumeration theorem,

\[
\alpha(n) = \frac{1}{2n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (2^{A_{ij}} + 2^{E_{ij}}),
\]

(5)

where \( A_{ij} \) (resp., \( E_{ij} \)) is the number of cycles in the permutation \( \sigma^i \tau^j \) (resp., \( \sigma^i \tau^j \zeta \)); here \( \sigma \) rotates the rows (row 0 becomes row 1, row 1 becomes row 2, \ldots, row \( n-1 \) becomes row 0), \( \tau \) rotates the columns, and \( \zeta \) transposes the matrix.

We know from [1] that

\[
a(n, n) = \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} 2^{A_{ij}},
\]

(6)

so it remains to find \( E_{ij} \). The permutation \( \zeta \) has \( n \) fixed points and \( \binom{n}{2} \) transpositions, so \( E_{00} = n(n + 1)/2 \).

Notice that \( \sigma \) and \( \tau \) commute, whereas \( \sigma \zeta = \zeta \tau \) and \( \tau \zeta = \zeta \sigma \). Let \( (i, j) \in \{0, 1, \ldots, n - 1\}^2 - \{(0, 0)\} \) be arbitrary. Then

\[
(\sigma^i \tau^j \zeta)^2 = (\sigma^i \tau^j \zeta)(\zeta \tau^i \sigma^j) = \sigma^{i+j} \tau^{i+j},
\]

hence

\[
(\sigma^i \tau^j \zeta)^{2d} = \sigma^{(i+j)d} \tau^{(i+j)d} = ((\sigma \tau)^{i+j})^d,
\]

\[
(\sigma^i \tau^j \zeta)^{2d+1} = \sigma^{(i+j)d+i+j} \tau^{(i+j)d+j} \zeta.
\]

Clearly, \( (\sigma^i \tau^j \zeta)^{2d+1} \) cannot be the identity permutation, so \( \sigma^i \tau^j \zeta \) is of even order. Using the fact that, in the cyclic group \( \{a, a^2, \ldots, a^{n-1}, a^n = e\} \) of order \( n \), \( a^k \) is of order \( n / \gcd(k, n) \), we find that the permutation \( \sigma^i \tau^j \zeta \) is of order 2d, where \( d := n / \gcd(i + j, n) \). Therefore, every cycle of this permutation must have length that divides 2d.

We claim that all cycles have length \( d \) or \( 2d \). Accepting that for now, let us determine how many cycles have length \( d \). A cycle that includes entry \((k, l)\) has length \( d \) if \((k, l)\) is a fixed point of \( (\sigma^i \tau^j \zeta)^d \). For this to hold we must have

\[
(k, l) = (k, k + d),
\]

where \( 1 \leq k \leq n-1 \), \( 0 \leq d \leq n-1 \), and \( d = 0 \) or \( d \equiv 0 \pmod{n} \) if \( i + j \equiv 0 \pmod{n} \).
$d$ odd (otherwise there would be no fixed points because we have excluded the case $i = j = 0$ and $(i + j)d/2 = \text{lcm}(i, j, n)/2$ is not a multiple of $n$). Since

$$(\sigma^i\tau^j\zeta)^d = \sigma^i(i+j)(d-1)/2+i\tau(j)(d-1)/2+j\zeta,$$

we must also have

$$(k, l) = ([l + (i + j)(d - 1)/2 + j], [k + (i + j)(d - 1)/2 + i]), \quad (7)$$

where $d := n/\gcd(i + j, n)$ and, for simplicity, $[r] := (r \mod n) \in \{0, 1, \ldots, n - 1\}$. For each $k \in \{0, 1, \ldots, n - 1\}$, there is a unique $l$ (namely, $l := [k + (i + j)(d - 1)/2 + i]$) such that $(7)$ holds; indeed,

$$
\begin{align*}
[l + (i + j)(d - 1)/2 + j] &= [k + (i + j)(d - 1)/2 + i] + (i + j)(d - 1)/2 + j] \\
&= [k + (i + j)(d - 1)/2 + i + (i + j)(d - 1)/2 + j] \\
&= [k + (i + j)d] \\
&= [k + (i + j)(n/\gcd(i + j, n))] \\
&= [k + \text{lcm}(i + j, n)] \\
&= k.
\end{align*}
$$

This shows that there are $n$ fixed points of $(\sigma^i\tau^j\zeta)^d$. Each cycle of length $d$ of $\sigma^i\tau^j\zeta$ will account for $d$ such fixed points, hence there are $n/d$ such cycles. All remaining cycles will have length $2d$, and so there are $n(n - 1)/(2d)$ of these. The total number of cycles is therefore $n(n + 1)/(2d)$.

The other possibility is that $d$ is even and all cycles have the same length, $2d$, so there are $n^2/(2d)$ of them. Notice that $d$ is a divisor of $n$, so the contribution to

$$
\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} 2^{E_{ij}}
$$

from odd $d$ is

$$
\sum_{d \mid n: \text{d is odd}} n\varphi(d)2^{n(n+1)/(2d)} \quad \text{(8)}
$$

and from even $d$ is

$$
\sum_{d \mid n: \text{d is even}} n\varphi(d)2^{n^2/(2d)}. \quad \text{(9)}
$$

The reason for the coefficient $n\varphi(d)$ is that, if $d \mid n$, then the number of elements of the cyclic group $\{e, \sigma, (\sigma\tau)^2, \ldots, (\sigma\tau)^{n-1}\}$ that are of order $d$ is $\varphi(d)$. And for a given $(i, j) \in \{0, 1, \ldots, n - 1\}^2$, there are $n$ pairs $(k, l) \in \{0, 1, \ldots, n - 1\}^2$ such that $[k + l] = [i + j]$. Putting (8) and (9) together, we obtain

$$
\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} 2^{E_{ij}} = \sum_{d \mid n} n\varphi(d)2^{n(n+d-2[d/2])/(2d)}, \quad \text{(10)}
$$

which, together with (5) and (6), yields (3). \qed
It remains to prove our claim that, for \((i, j) \in \{0, 1, \ldots, n - 1\}^2 - \{(0, 0)\}\), the permutation \(\sigma^i \tau^j \zeta\) cannot have any cycles whose length is a proper divisor of \(d := n / \gcd(i + j, n)\). Let \(c \mid d\) with \(1 \leq c < d\). We must show that \((\sigma^i \tau^j \zeta)^c\) has no fixed points. We can argue as above with \(c\) in place of \(d\). For \((k, l)\) to be a fixed point of \((\sigma^i \tau^j \zeta)^c\) we must have \((i + j)c\) a multiple of \(n\). But \(d := n / \gcd(i + j, n)\) is the smallest integer \(c\) such that \((i + j)c\) is a multiple of \(n\) because \((i + j)n / \gcd(i + j, n) = \text{lcm}(i + j, n)\).

Finally, we excluded the case \(n = 1\) at the beginning of the proof, but we notice that the formula (3) gives \(\alpha(1) = 2\), which is correct.

3 Rotation and reflection of rows and columns, and matrix transposition

Let \(X_n := \{0, 1\}^{0, 1, \ldots, n - 1} \times n\) be the set of \(n \times n\) matrices of 0s and 1s, which has \(2^{n^2}\) elements. Let \(\beta(n)\) denote the number of orbits of the group action on \(X_n\) by the group of order \(8n^2\) generated by \(\sigma\) (row rotation), \(\tau\) (column rotation), \(\rho\) (row reflection), \(\theta\) (column reflection), and \(\zeta\) (matrix transposition).

(Exceptions: If \(n = 2\), the group is of order 8; if \(n = 1\), the group is of order 1.)

Informally, \(\beta(n)\) is the number of (distinct) toroidal \(n \times n\) binary arrays, allowing rotation and/or reflection of rows and/or columns as well as matrix transposition.

**Theorem 2.** With \(b(n, n)\) defined using (2), \(\beta(n)\) is given by (4).

**Proof.** Let us assume that \(n \geq 3\). (We will treat the cases \(n = 1\) and \(n = 2\) later.) By Pólya’s enumeration theorem,

\[
\beta(n) = \frac{1}{8n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (2^{A_{ij}} + 2^{B_{ij}} + 2^{C_{ij}} + 2^{D_{ij}} + 2^{E_{ij}} + 2^{F_{ij}} + 2^{G_{ij}} + 2^{H_{ij}}),
\]

where \(A_{ij}\) (resp., \(B_{ij}, C_{ij}, D_{ij}, E_{ij}, F_{ij}, G_{ij}, H_{ij}\)) is the number of cycles in the permutation \(\sigma^i \tau^j\) (resp., \(\sigma^i \tau^j \rho, \sigma^i \tau^j \theta, \sigma^i \tau^j \rho \theta, \sigma^i \tau^j \zeta, \sigma^i \tau^j \rho \zeta, \sigma^i \tau^j \theta \zeta, \sigma^i \tau^j \rho \theta \zeta\)); here \(\sigma\) rotates the rows (row 0 becomes row 1, row 1 becomes row 2, \ldots, row \(n - 1\) becomes row 0), \(\tau\) rotates the columns, \(\rho\) reflects the rows (rows 0 and \(n - 1\) are interchanged, rows 1 and \(n - 2\) are interchanged, \ldots, rows \([n/2] - 1\) and \(n - [n/2]\) are interchanged), \(\theta\) reflects the columns, and \(\zeta\) transposes the matrix. The order of the group generated by \(\sigma, \tau, \rho, \theta,\) and \(\zeta\) is \(8n^2\), using the assumption that \(n \geq 3\).

We have already evaluated

\[
\alpha(n, n) = \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} 2^{A_{ij}},
\]

\[
\alpha(n) = \frac{1}{2n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (2^{A_{ij}} + 2^{E_{ij}}),
\]

\[
\text{polv}(n, n) = \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} 2^{D_{ij}}.
\]
and
\[ b(n, n) = \frac{1}{4n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (2^{A_{ij}} + 2^{B_{ij}} + 2^{C_{ij}} + 2^{D_{ij}}), \]
so
\[ \beta(n) = \frac{1}{2} b(n, n) + \frac{1}{4} (\alpha(n) - \frac{1}{2} a(n, n)) + \frac{1}{8n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (2^{E_{ij}} + 2^{F_{ij}} + 2^{G_{ij}} + 2^{H_{ij}}). \quad (11) \]

Let us begin with
\[ \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} 2^{H_{ij}}. \]

Here we are concerned with the permutations \( \sigma^i \tau^j \rho \theta \zeta \) for \( (i, j) \in \{0, 1, \ldots, n - 1\}^2 \). We will need some multiplication rules for the permutations \( \sigma, \tau, \rho, \theta, \) and \( \zeta \), specifically
\[ \sigma \tau = \tau \sigma, \quad \sigma \theta = \theta \sigma, \quad \tau \rho = \rho \tau, \quad \rho \theta = \theta \rho, \quad \sigma \rho = \rho \sigma^{-1}, \quad \tau \theta = \theta \tau^{-1}, \]
and
\[ \sigma \zeta = \zeta \sigma, \quad \tau \zeta = \zeta \tau, \quad \rho \zeta = \zeta \rho, \quad \theta \zeta = \zeta \theta. \]

It follows that (with \( \tau^{-i} := (\tau^{-1})^i \))
\[ \sigma^i \tau^j \rho \theta \zeta = \xi^i \tau^j \zeta \theta \rho = \xi \tau^i \theta \zeta \rho = \xi \theta \rho \tau^{-i} \sigma^{-j}, \]
and hence
\[ (\sigma^i \tau^j \rho \theta \zeta)^2 = (\sigma^i \tau^j \rho \theta \zeta)(\xi \rho \tau^{-i} \sigma^{-j}) = \sigma^{i-j} \tau^{i+j} = (\sigma \tau^{-1})^{i-j} = (\sigma^{-1} \tau)^{-i-j}. \quad (12) \]

In particular, if \( i \in \{0, 1, \ldots, n - 1\} \), then the permutation \( \sigma^i \tau^j \rho \theta \zeta \) is of order 2. Furthermore, under this permutation, the entry in position \((k, l)\) moves to position \((n - 1 - [l + i], n - 1 - [k + i])\), where, as before, \( [r] := (r \mod n) \in \{0, 1, \ldots, n - 1\} \). Thus, \((k, l)\) is a fixed point if and only if
\[ (k, l) = (n - 1 - [l + i], n - 1 - [k + i]). \quad (13) \]

For each \( k \in \{0, 1, \ldots, n - 1\} \) there is a unique \( l \in \{0, 1, \ldots, n - 1\} \) (namely \( l := n - 1 - [k + i] \)) such that (13) holds; indeed,
\[ n - 1 - [l + i] = n - 1 - [n - 1 - [k + i] + i] = n - 1 - [n - 1 - (k + i) + i] = n - 1 - [n - 1 - k] = n - 1 - (n - 1 - k) = k. \]

Thus, \( \sigma^i \tau^j \rho \theta \zeta \) with \( i \in \{0, 1, \ldots, n - 1\} \) is of order 2 and has exactly \( n \) fixed points, hence \( \binom{n}{2} \) transpositions. This implies that \( H_{ii} = n(n + 1)/2 \) for such \( i \).

Now we let \((i, j) \in \{0, 1, \ldots, n - 1\}^2\) be arbitrary but with \( i \neq j \). Let us generalize (12) to
\[ (\sigma^i \tau^j \rho \theta \zeta)^{2d} = \sigma^{(i-j)d} \tau^{(i+j)d} = ((\sigma \tau^{-1})^{i-j})^d = ((\sigma^{-1} \tau)^{i+j})^d, \]
The proof proceeds much like the proof of Theorem 1. Specifically, \( \sigma^i \tau^j \rho \theta \zeta \) is of order \( 2d \), where \( d := n / \gcd(|i - j|, n) \). All cycles have length \( d \) or \( 2d \). In fact, if \( d \) is odd, there are \( n/d \) cycles of length \( d \) and \( n(n - 1)/(2d) \) cycles of length \( 2d \). If \( d \) is even, there are \( n^2/(2d) \) cycles, all of length \( 2d \). And for a given \( (i, j) \in \{0, 1, \ldots, n - 1\}^2 \), there are \( n \) pairs \( (k, l) \in \{0, 1, \ldots, n - 1\}^2 \) such that \( |k - l| = |i - j| \).

We arrive at the conclusion that

\[
\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} 2^{H_{ij}} = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} 2^{E_{ij}},
\]

where the equality holds by symmetry. We consider the permutations \( \sigma^i \tau^j \rho \zeta \) for \( (i, j) \in \{0, 1, \ldots, n - 1\}^2 \). From the multiplication rules, it follows that

\[
\sigma^i \tau^j \rho \zeta = \zeta \theta \tau^{-i} \sigma^j
\]

and hence

\[
(\sigma^i \tau^j \rho \zeta)^2 = (\sigma^i \tau^j \rho \zeta)(\zeta \theta \tau^{-i} \sigma^j) = \sigma^i \tau^j \rho \theta \tau^{-i} \sigma^j = \sigma^{i-j} \tau^{i+j} \rho \theta = \theta \rho \tau^{-i-j} \sigma^{-i-j},
\]

which implies

\[
(\sigma^i \tau^j \rho \zeta)^4 = (\sigma^{i-j} \tau^{i+j} \rho \theta)(\theta \rho \tau^{-i-j} \sigma^{-i-j}) = e.
\]

So the permutation \( \sigma^i \tau^j \rho \zeta \) is of order 4. The entry in position \( (k, l) \) moves to position \( ([l + j], n - 1 - [k + i]) \) under this permutation. Thus, \( (k, l) \in \{0, 1, \ldots, n - 1\}^2 \) is a fixed point of \( \sigma^i \tau^j \rho \zeta \) if and only if

\[
(k, l) = ([l + j], n - 1 - [k + i]).
\]

There is a solution \( (k, l) \) if and only if there exists \( l \in \{0, 1, \ldots, n - 1\} \) such that, with \( k := [l + j] \), we have \( n - 1 - [k + i] = l \) or, equivalently,

\[
[l + i + j] = n - 1 - l. \tag{17}
\]

When \( i + j \leq n - 1 \), (17) is equivalent to

\[
l + i + j = n - 1 - l \quad \text{or} \quad l + i + j - n = n - 1 - l
\]

or to

\[
l = (n - 1 - i - j)/2 \quad \text{or} \quad l = (2n - 1 - i - j)/2.
\]

If \( n \) is odd and \( i + j \) is odd, then there is one fixed point, \( (k, l) = ([2n - 1 - i + j]/2, [(2n - 1 - i - j)/2]) \). If \( n \) is odd and \( i + j \) is even, then there is one fixed
point, \((k, l) = ([n-1-i+j]/2), ([n-1-i-j]/2])\). If \(n\) is even and \(i+j\) is odd, then there are two fixed points, namely

\[
(\kappa, l) = ([n-1-i+j]/2), ([n-1-i-j]/2]), \\
(\kappa, l) = ([2n-1-i+j]/2), ([2n-1-i-j]/2)).
\]

Finally, if \(n\) is even and \(i+j\) is even, then there is no fixed point.

When \(i+j \geq n\), (17) is equivalent to

\[
l + i + j - n = n - 1 - l \quad \text{or} \quad l + i + j - 2n = n - 1 - l
\]

or to

\[
l = (2n-1 - i - j)/2 \quad \text{or} \quad l = (3n-1 - i - j)/2.
\]

If \(n\) is odd and \(i+j\) is odd, then there is one fixed point, \((\kappa, l) = ([2n-1-i+j]/2), ([2n-1-i-j]/2)). If \(n\) is odd and \(i+j\) is even, then there is one fixed point, \((\kappa, l) = ([n-1-i+j]/2), ([3n-1-i-j]/2)) = ([n-1-i+j]/2), ([n-1-i-j]/2)). If \(n\) is even and \(i+j\) is odd, then there are two fixed points, namely

\[
(\kappa, l) = ([2n-1-i+j]/2), ([n-1-i-j]/2)), \\
(\kappa, l) = ([n-1-i+j]/2), ([n-1-i-j]/2)).
\]

Finally, if \(n\) is even and \(i+j\) is even, then there is no fixed point. Notice that the results are the same for \(i+j \geq n\) as for \(i+j \leq n-1\).

Using (16), under the permutation \((\sigma^i \tau^j \rho \zeta)^2\), the entry in position \((\kappa, l)\) moves to position \((n-1-[k+i-j], n-1-[l+i+j])\). Thus, \((\kappa, l) \in \{0, 1, \ldots, n-1\}^2\) is a fixed point of \((\sigma^i \tau^j \rho \zeta)^2\) if and only if

\[
(\kappa, l) = (n-1-[k+i-j], n-1-[l+i+j]).
\]

A necessary and sufficient condition on \((\kappa, l)\) is (17) together with \([k+i-j] = n-1-k\). Solutions have \(l\) as before. On the other hand, \(k\) must satisfy

\[
k + i - j - n = n - 1 - k, \quad k + i - j = n - 1 - k, \quad \text{or} \quad k + i - j + n = n - 1 - k,
\]

or equivalently,

\[
k = ([n-1-i+j]/2) \quad \text{or} \quad k = ([2n-1-i+j]/2).
\]

If \(n\) is odd, the only fixed points of \((\sigma^i \tau^j \rho \zeta)^2\) are those already shown to be fixed points of \(\sigma^i \tau^j \rho \zeta\). If \(n\) is even and \(i+j\) is odd, there are two fixed points of \((\sigma^i \tau^j \rho \zeta)^2\) that are not fixed points of \(\sigma^i \tau^j \rho \zeta\), namely

\[
(\kappa, l) = ([n-1-i+j]/2), ([2n-1-i-j]/2)), \\
(\kappa, l) = ([2n-1-i+j]/2), ([n-1-i-j]/2)).
\]

Finally, there are no fixed points when \(n\) is even and \(i+j\) is even.
Consequently, if \( n \) is odd, then the permutation \( \sigma^i \tau^j \rho \zeta \), which is of order 4, has only one fixed point. Therefore, it has one cycle of length 1 and \((n^2 - 1)/4\) cycles of length 4. Thus,

\[
\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} 2^{F_{ij}} = n^2 2^{(n^2+3)/4}.
\]

For even \( n \), if \( i + j \) is odd, then the permutation \( \sigma^i \tau^j \rho \zeta \) has two cycles of length 1 and one cycle of length 2, and the remaining cycles are of length 4. If \( i + j \) is even, then all cycles of the permutation \( \sigma^i \tau^j \rho \zeta \) are of length 4, hence there are \( n^2/4 \) of them. Thus,

\[
\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} 2^{F_{ij}} = \frac{1}{2} n^2 2^{(n^2-4)/4+3} + \frac{1}{2} n^2 2^{n^2/4} = 5n^2 2^{n^2/4-1}.
\]

These results, together with (3), (10), (11), (14), and (15), yield (4).

Finally, recall that we have assumed that \( n \geq 3 \). We notice that the formula (4) gives \( \beta(1) = 2 \) and \( \beta(2) = 6 \), which are correct, as we can see by direct enumeration.

In the derivation of (2) in [1], the proof requires \( m, n \geq 3 \) because the group \( D_m \times D_n \) used in the application of Pólya’s enumeration theorem (\( D_m \) being the dihedral group of order \( 2m \)), is incorrect if \( m \) or \( n \) is 1 or 2. If \( m = 2 \), row rotation and row reflection are the same, so the latter is redundant. Thus, \( D_2 \) should be replaced by \( C_2 \), the cyclic group of order 2. The reason (2) is still valid is that \( b_1(2, n) = b_2(2, n) \) and \( b_3(2, n) = b_4(2, n) \), as is easily verified. If \( m = 1 \), again row reflection is redundant, so \( D_1 \) should be replaced by \( C_1 \). Here (2) remains valid because \( b_1(1, n) = b_2(1, n) \) and \( b_3(1, n) = b_4(1, n) \). A similar remark applies to \( n = 2 \) and \( n = 1 \), except that here \( b_1(m, 2) = b_3(m, 2) \), \( b_2(m, 2) = b_4(m, 2) \), \( b_1(m, 1) = b_3(m, 1) \), and \( b_2(m, 1) = b_4(m, 1) \).

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