Blowup dynamics for smooth equivariant solutions to energy critical Landau-Lifschitz flow

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Abstract
In this paper, we study the energy critical 1-equivariant Landau-Lifschitz flow mapping \( \mathbb{R}^2 \) to \( S^2 \) with arbitrary given coefficients \( \rho_1 \in \mathbb{R}, \rho_2 > 0 \). We prove that there exists a codimension one smooth well-localized set of initial data arbitrarily close to the ground state which generates type-II finite-time blowup solutions, and give a precise description of the corresponding singularity formation. In our proof, both the Schrödinger part and the heat part play important roles in the construction of approximate solutions and the mixed energy/Morawetz functional. However, the blowup rate is independent of the coefficients.

1 Introduction

1.1 Setting of the problem
We consider the energy-critical Landau-Lifschitz equation

\[
\begin{aligned}
\left\{
\begin{array}{l}
u_t = \rho_1 u \wedge \Delta u - \rho_2 u \wedge (u \wedge \Delta u), \\
u|_{t=0} = u_0 \in \dot{H}^1,
\end{array}
\right. \\
(t,x) \in \mathbb{R} \times \mathbb{R}^2, \ u(t,x) \in S^2,
\end{aligned}
\] (1.1)

where the exchange constant \( \rho_1 \in \mathbb{R} \) and the Gilbert damping constant \( \rho_2 > 0 \). This equation was proposed by Landau and Lifshitz [25] in studying the dispersive theory of magnetization of ferromagnets. It describes the evolution of magnetization in classical ferromagnet, and thus the study on (1.1) is of fundamental importance for the understanding of nonequilibrium magnetism, see [1] for more details. If \( \rho_1 = 0 \), (1.1) reduces to the harmonic map heat flow (1.5), a model in nematic liquid flow [8]. If \( \rho_2 = 0 \), (1.1) becomes the conservative Schrödinger map flow (1.6), which is of much fundamental interest in differential geometry [16].

For equation (1.1), the associated Dirichlet energy is given by

\[
E(u) = \int_{\mathbb{R}^2} |\nabla u|^2 dx,
\] (1.2)
which is dissipative along the flow

\[
\frac{d}{dt} \left( \int_{\mathbb{R}^2} |\nabla u|^2 \right) = -2\rho_2 \int_{\mathbb{R}^2} |u \wedge \Delta u|^2.
\]  

(1.3)

Moreover, the energy (1.2) is invariant under the mixed symmetric transformations of the scaling and the rotation

\[ u(t, x) \rightarrow u_{\lambda, O}(t, x) = O u \left( \frac{t}{\lambda}, \frac{x}{\lambda^2} \right), \quad (\lambda, O) \in \mathbb{R}_+^* \times O(3). \]

A remarkable feature of (1.1) is that smooth solutions preserve the magnitude \(|u(t, x)| = 1\) for all \((t, x) \in \mathbb{R}_+^* \times \mathbb{R}^2\), once we fix \(|u_0| = 1\) initially. In particular, there is a specific class of solutions with an additional symmetry, called the \(k\)-equivariant maps, which take the form

\[
u(t, x) = e^{k\theta R} \begin{bmatrix} u_1(t, r) \\ u_2(t, r) \\ u_3(t, r) \end{bmatrix}, \quad \text{with} \quad R = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]  

(1.4)

where \((r, \theta)\) is the polar coordinate on \(\mathbb{R}^2\), and \(k \in \mathbb{Z}^*\) is the homotopy degree given by

\[ k = \frac{1}{4\pi} \int_{\mathbb{R}^2} (\partial_1 u \wedge \partial_2 u) \cdot u. \]

A typical stationary solution of (1.1) is the \(k\)-equivariant harmonic map

\[ Q_k(r, \theta) = \frac{e^{k\theta R} \begin{bmatrix} 2r^k \\ 0 \\ 1 - r^{2k} \end{bmatrix} }{1 + r^{2k}}, \quad k \in \mathbb{Z}. \]

According to the Bogomol’nyi’s factorization [9], \(Q_k\) is the minimizers of the Dirichlet energy (1.2) in homotopy-\(k\) class with

\[ E(Q_k) = 4\pi|k|. \]

In other words, \(Q_k\) is the ground state of (1.1). Since this paper is mainly concerned with the 1-equivariant solutions, we use the convention \(Q = Q_1\) for the 1-equivariant ground state.

1.2 Related geometric flows

In the past decades, great progress has been made on both the harmonic heat flow problem and the Schrödinger map problem. For the harmonic heat flow,

\[
(\text{Heat flow}) \quad \begin{cases} u_t = \Delta u + |\nabla u|^2 u, \\ u_{|t=0} = u_0, \end{cases} \quad (t, x) \in \mathbb{R} \times \mathbb{R}^2, \quad u(t, x) \in \mathbb{S}^2, \quad (1.5)
\]
we refer to [11, 14, 19, 29] for existence and uniqueness results. Since 2-dimensional heat flow is energy critical, singularity formation by energy concentration is possible. It is known that concentration implies the bubbling off of a nontrivial harmonic map at a finite number of blowup points, see Struwe [42], Ding and Tian [17], Qing and Tian [39], Topping [44]. For k-equivariant case, blowup near $Q_k$ for $k \geq 3$ has been ruled out in [23], where the harmonic map is proved to be asymptotically stable. Chang, Ding and Ye [12] found the first example of the finite-time blowup solutions of heat flow. For $D^2$ initial manifold and $S^2$ target, van den Berg, Hulshof, and King [6] implemented a formal analysis based on the matched asymptotics techniques and predicted the existence of blow-up solutions with quantized rates

$$\lambda_L(t) \approx \frac{C|T - t|^L}{|\log(T - t)|^{\frac{7L}{2L - 1}}}, \quad L \in \mathbb{N}^*.$$  

For $\mathbb{R}^2$ initial manifold, Raphaël and Schweyer [36, 37] exhibited a set of initial data arbitrarily close to the least energy harmonic map in the energy-critical topology such that the corresponding solutions blow up in finite time with the quantized blow-up rate $\lambda_L(t)$ for any $L \geq 1$. The case $L = 1$ corresponds to the stable regime. Without equivariant assumption, Davila, del Pino and Wei [15] constructed a solution which blows up precisely at finite number of given points if the starting manifold is a bounded domain in $\mathbb{R}^2$. The profile around each point is close to an asymptotically singular scaling of a 1-corotational harmonic map with blowup rate $\lambda_L(t)$, $L = 1$. This rate was expected to be generic for 1-corotational heat flow, see [6]. For more results on harmonic map heat flow, see [30] and the references therein.

For the Schrödinger map problem

\[
\begin{align*}
\text{(Schrödinger map)} \quad & \begin{cases} u_t = u \wedge \Delta u, \\ u|_{t=0} = u_0, \end{cases} \quad (t, x) \in \mathbb{R} \times \mathbb{R}^2, \quad u(t, x) \in S^2, \quad (1.6)
\end{align*}
\]

there are quite many works. The local well-posedness of Schrödinger map was established in [18, 32, 43]. When the target is $S^2$, the global well-posedness for small data in critical spaces was proved by Bejenaru, Ionescu, Kenig and Tataru [3]. Their result was extended to the target of Kähler manifolds by Li [26, 27]. The stationary solutions of Schrödinger flow are harmonic maps. When the energy is less than $4\pi$, 1-equivariant solutions are global in time and scatter, see [4]. Gustafson, Kang, Tsai, Nakanishi [21–23] showed that harmonic maps are asymptotically stable in high equivariant classes ($k \leq 3$), which precludes the blowup solutions near the harmonic maps in those cases. However, in 1-equivariant class, Bejenaru and Tataru [5] showed that harmonic maps are stable under smooth well-localized perturbations, but unstable under $\dot{H}^1$ topology. Merle, Raphaël and Rodnianski [34] proved the existence of a codimension one set of smooth well-localized initial data arbitrarily close to $Q$ which generates type-II blowup solutions, and they figured out the detailed asymptotic behavior of these solutions near the blowup time. Perelman [38] presented another type-II blowup solution with different singularity formation.
There are a lot of works devoted to the study of Landau-Lifshitz flow. The global existence of weak solutions or partial regular solutions have been shown in [2,10,20,24,33,45]. However, the dynamical behavior is much less studied. The asymptotic stability of ground state harmonic maps in high equivariant classes were proved by Gustafson, Nakanishi and Tsai [23]. In 1-equivariant class, the solutions with energy less than $4\pi$ were proved to converge to a constant map in energy space by Li and Zhao [28]. The equivariant blowup solution was constructed in [7] by formal asymptotics and was verified by numerical experiments. However, as the authors said in [7] ‘mathematically rigorous justification is required’. As far as we know, the problem of blowup dynamics of the Landau-Lifschitz equation near $Q$ remains open until now.

1.3 Statement of the result

Our work is in continuation of the investigation of the Schrödinger map [34], the wave map [35], the harmonic heat flow [37]. We establish the existence of smooth 1-equivariant type-II blowup solutions to (1.1), and gives a sharp description on the asymptotic behavior on its singularity formation.

**Theorem 1.1 (Existence and description of blowup LL flow)** There exists a set of smooth well-localized 1-equivariant initial data with its elements arbitrarily close to the ground state $Q$ in $\dot{H}^1$ topology, such that the corresponding solution to (1.1) blows up in finite time. The singularity formation corresponds to the concentration of the universal bubble of energy in the scale invariant energy space:

$$u - e^{\Theta(t)}RQ\left(\frac{x}{\lambda(t)}\right) \rightarrow u^* \in \dot{H}^1 \text{ as } t \rightarrow T,$$

(1.7)

for geometrical parameters $(\Theta(t), \lambda(t)) \in C^1([0,T), \mathbb{R} \times \mathbb{R}_+)$ with their asymptotic behaviors near the blowup time $T$ given by

$$\lambda(t) = C(u_0)(1 + o(1))\frac{(T - t)}{|\log(T - t)|^2}, \quad C(u_0) > 0,$$

(1.8)

$$\Theta(t) \rightarrow \Theta(u_0) \in \mathbb{R} \text{ as } t \rightarrow T.$$

(1.9)

Moreover, there holds the propagation of regularity:

$$\Delta u^* \in L^2.$$

(1.10)

**Comments on the result** 1. **On the blowup asymptotics**: The overall blowup behavior is similar to that of Schrödinger map problem [34]. One of the major distinctions is that, due to the additional heat flow term (also referred to as the Lifschitz dissipation), the approximate solution is completely different from that of Schrödinger map problem. It behaves like a combination of the approximate solution of (1.5) and (1.6). However, the appearance of the heat flow term does not deteriorate the error estimate.
2. **On the Morawetz estimate**: As in [34], an unsigned quadratic term induced by the commutator \([\partial_t, H]\) appears in the plain energy identity. It cannot be controlled via the energy bounds directly (see (5.12)), and thus requires an extra Morawetz functional to create cancellation. The construction of a suitable functional is the core of the analysis. However, the presence of the heat flow term makes our situation very complicated, which is the main difficulty of this work. The key observation we have made is the intrinsic structure of the operator \(A, A^*\) gathered in Lemma 5.3, which enables us to capture the competition of the Schrödinger part and the heat flow part, and formulate the uncontrollable term. In our construction, the ratio of \(\rho_1, \rho_2\) plays a crucial role, based on which the exquisitely-designed coefficients (5.20) are eventually responsible for the distinct controls of various errors coming from the Morawetz estimate (5.22).

3. **On the universality of the blowup rate**: The blowup rate (1.8) is independent of the coefficients \(\rho_1, \rho_2\), in spite of the evident influence of the latter. In fact, the subdued contribution of the coefficients has been observed by van den Berg and Williams through formal asymptotics [7]. Indeed, we shall see in Lemma 3.1 that the coupling coefficients involving \(\rho_1, \rho_2\) in front of the approximate profiles are consistent with the expressions in the flux computation (3.15). This correspondence produces cancellations in computing the modulation equations, which gives the identical dynamic system for the modulation parameters (4.34) as in [34, 36]. That explains, to some extend, the reason why the blowup rate is irrelevant to the coefficients.

4. **On the codimension one instability**: In our construction, the initial data of the blowup solutions are characterized by specifying the modulation parameters \(a, b\) and the radiation term \(w\). The choice of \(b_0\) and \(w_0\) resist small perturbations, while \(a_0\), representing the time derivative of the phase, is unstable and thus selected accordingly afterward. As shown in Section 4. This regime ensures, in some weak sense, that the solutions evolve from a codimension one set of smooth initial data will blow up in the way we describe in Theorem 1.1.

**Notions** We use the polar coordinates \((r, \theta)\) and \((y, \theta)\) on \(\mathbb{R}^2\), where by the anticipated scaling transform, we will set \(y = r/\lambda\). We use the convention

\[
\partial_r = \frac{1}{y} \partial_\theta, \quad \Lambda f = y \cdot \nabla_y f.
\]

For any given parameter \(b\), we introduce the scales

\[
B_0 = \frac{1}{\sqrt{b}}, \quad B_1 = \frac{|\log b|}{\sqrt{b}}.
\]

The cut-off function \(\chi\) is a smooth radially symmetric function defined by

\[
\chi(x) = \begin{cases} 
1, & |x| \leq 1, \\
0, & |x| \geq 2,
\end{cases}
\]
with its scaling given by
\[ \chi_M(x) = \chi \left( \frac{x}{M} \right), \]
provided any large constant \( M > 0 \).

### 1.4 Strategy of the proof

Let us briefly sketch our approach for proving the Theorem 1.1, which follows from the strategy developed in [34–36].

**Step 1. Renormalization.** We look for the 1-equivariant solution with its energy slightly higher than the ground state \( Q \) which takes the form
\[ u(t) = e^{\Theta(t)R}(Q + \hat{v}) \left( t, \frac{r}{\lambda(t)} \right), \tag{1.12} \]
where \( \hat{v} \) is a perturbation with small enough Sobolev norm
\[ \|\hat{v}\|_{\dot{H}^1} \ll 1. \]

The blowup mechanism suggests \( \lambda \to 0 \) as \( t \to T \), and thus it is suitable to consider the renormalized function \( v(s, y) \) under the self-similar transformation
\[ u(t) = e^{\Theta R}v(s, y), \quad \frac{ds}{dt} = \frac{1}{\lambda(t)^2}, \quad \frac{y}{r} = \frac{1}{\lambda(t)}. \tag{1.13} \]

Then from (1.1) we obtain the renormalized equation for \( v \):
\[ \partial_s v - \frac{\lambda_s}{\lambda} \Delta v + \Theta_s R v = \rho_1 v \wedge v - \rho_2 v \wedge (v \wedge \Delta v). \]

To understand the solution’s behavior in the vicinity of the ground state, we apply the Frenet basis associated to \( Q \), that is \([e_\tau, e_r, Q]\) with
\[ e_r = \frac{\partial_y Q}{|\partial_y Q|}, \quad e_\tau = \frac{\partial_\theta Q}{|\partial_\theta Q|}, \quad Q(y, \theta) = e^{\theta R} \begin{bmatrix} \Lambda \phi(y) \\ 0 \\ Z(y) \end{bmatrix}, \]
where \( \phi, \Lambda \phi, Z \) are given by (2.2), (2.3). Then replacing \( v \) by \( Q + \hat{v} \), we reformulate (1.1) and encounter the following repeated linear pattern in \( \hat{v} \):
\[ \Delta \hat{v} + |\nabla Q|^2 \hat{v}, \]
which gives rise to the linearized Hamiltonian/Schrödinger operator
\[ H = -\Delta + \frac{V(y)}{y^2}, \quad \text{with} \quad V(y) = \frac{y^4 - 6y^2 + 1}{(1 + y^2)^2}. \]
and also its vectorial version $\mathbb{H}$ (2.19), (2.26). Finally, we obtain the following component
equations of $\hat{v}$, which is equivalent to (1.1). (lot stands for the lower order errors)

$$
\begin{align*}
\partial_s \hat{\alpha} - \frac{\lambda_s}{\Lambda} \Lambda \hat{\alpha} &= \rho_1 H \hat{\beta} - \rho_2 \hat{H} \hat{\alpha} + \frac{\lambda_s}{\Lambda} \Lambda \phi + \text{lot}, \\
\partial_s \hat{\beta} - \frac{\lambda_s}{\Lambda} \Lambda \hat{\beta} &= -\rho_1 H \hat{\alpha} - \rho_2 \hat{H} \hat{\beta} - \Theta_s Z \hat{\alpha} + \text{lot}, \\
\partial_s \hat{\gamma} - \frac{\lambda_s}{\Lambda} \Lambda \hat{\gamma} &= -\rho_1 \hat{\alpha} H \hat{\beta} + \rho_1 \hat{\beta} H \hat{\alpha} + \rho_2 \hat{\alpha} H \hat{\beta} - \frac{\lambda_s}{\Lambda} \Lambda \phi \hat{\alpha} + \text{lot},
\end{align*}
$$

(1.14)

Under the convention $\hat{v} = [e_r, e_\tau, Q] \hat{w}$, (1.14) can be rewritten as the frequently used
vectorial form

$$
\partial_s \hat{w} - \frac{\lambda_s}{\Lambda} \Lambda \hat{w} + \Theta_s Z \hat{R} \hat{w} + \hat{\mathcal{J}} \left( \rho_1 \hat{H} \hat{w} - \rho_2 \hat{J} \hat{H} \hat{w} + \hat{p} \Lambda \phi \right) = 0.
$$

(1.15)

**Step 2. Construction of the approximate solution.** To characterize the evolution of
$\lambda_s, \Theta_s$ appearing in (1.14), we define two more modulation parameters $a, b$, and claim
the following slow modulated ansatz:

$$
a \approx -\Theta_s, \quad b \approx -\frac{\lambda_s}{\Lambda}, \quad a_s \approx 0, \quad b_s \approx -(b^2 + a^2).
$$

Then we seek for the approximate solution of (1.15) whose leading part is

$$
\hat{w}_0 = a \Phi_{1,0} + b \Phi_{0,1} + b^2 S_{0,2},
$$

(1.16)

where $\Phi_{1,0}, \Phi_{0,1}$ are the first order profiles responsible for the cancellation of the expressions inside the big brace in (1.15), while $S_{0,2}$ is the second order profile dominating the third component $\hat{\gamma}$, which is chosen to obey the constraint

$$
\hat{\alpha}^2 + \hat{\beta}^2 + (1 + \hat{\gamma})^2 = 1.
$$

Moreover, we set the higher order profiles $\Phi_{i,j}$ (3.5) to improve the precision of the approximate solution. In general, the constructions of these profiles are mainly concerned
with solving a typical type of elliptic equations

$$
\begin{align*}
H \Phi_{i,j} &= E_{i,j}(\Lambda \phi, \Phi_{k,l}), \quad \text{for } i + j = 1, \\
H \Phi_{i,j} &= E_{i,j}(\Lambda \phi, \Phi_{k,l}, S_{k,l}), \quad \text{for } 2 \leq i + j \leq 3,
\end{align*}
$$

(1.17)

with $k \leq i, l \leq j, k + l < i + j$. Its solution can be obtained by the method of variation
of constants with the known resonance $\Lambda \phi$ of the Hamiltonian $H$. The constructions
of $\Phi_{1,0}, \Phi_{0,1}$ (3.22) are quite straightforward, while the higher order profiles $\Phi_{i,j}$ require
further manipulations, owing to their fast growing tails as $y \to +\infty$ induced by $T_1 - \Lambda T_1$
where

$$
\begin{align*}
T_1 &= -y \log y + y + O \left( \frac{(\log y)^2}{y} \right), \\
\Lambda T_1 &= -y \log y + O \left( \frac{(\log y)^2}{y} \right).
\end{align*}
$$

(1.18)
To handle this, a radiation $\Sigma_b$ (3.29) has been applied to modify the recursive system (1.17), as illustrated in (3.34). Furthermore, the presence of $\Sigma_b$ functions have profound influence on the error $\Psi_0$ of the approximate solution, the coupling formulas derived from which eventually leads to the forthcoming modulation equations

$$
\begin{align*}
    a_s &= -\frac{2ab}{|\log b|} + \text{lot}, \\
    b_s &= -b^2 \left( 1 + \frac{2}{|\log b|} \right) + \text{lot},
\end{align*}
$$

(1.19)

as well as $a = -\Theta_s + \text{lot}$, $b = -\lambda_s/\lambda + \text{lot}$. These identities constitute an ODE system, which determines the asymptotics for the modulation parameters. By refined computations on the ODE system, we see given $\lambda_0 > 0$, the scaling parameter $\lambda(t)$ change its sign in finite time $T$, which yields the finite-time blowup.

Step 3. Control of the radiation. Based on the approximate solution $\tilde{w}_0 = \chi_{B_1} w_0$ localized at the scale $B_1$, we seek for an actual solution, by showing the existence of the radiation $w$ in the decomposition

$$
\tilde{w} = \tilde{w}_0 + w.
$$

Recall (1.12), it remains to prove the existence of $w$ in suitable Sobolev space that permits the modulation equations (1.19), but this requires the estimate of the form

$$
\sum_{i=0}^{4} \int \frac{|\partial_y^i w|^2}{1 + y^{8-2i}} \lesssim \frac{b^4}{|\log b|^2},
$$

(1.20)

which is the key to finish the analysis. This is accomplished by the energy method together with the bootstrap argument. To be specific, we introduce the energies of $w$, namely $E_1, E_2, E_4$, at the level of $H^1, H^2, H^4$ respectively, and then claim the bootstrap bounds of the energies (4.14)–(4.16) on a small initial interval. By proving that refined bounds (4.19)-(4.21) hold, we know the radiation $w$ is trapped in the bootstrap regime, and then the estimate (1.20) follows. To show the refined bounds, we derive the following differential inequality for $E_4$:

$$
\frac{d}{dt} \left( \frac{E_4}{\lambda^6} \right) \leq \frac{b}{\lambda^8} \left[ 2(1 - d_2 + o(1))E_4 + O \left( \frac{b^4}{|\log b|^2} \right) \right],
$$

(1.21)

which is the core of our analysis. The main difficulty is the appearance of the unsigned quadratic terms emerging therefrom:

$$
\int \mathbb{R} \mathbb{H}_2^0 \cdot \left[ -\mathbb{H}^\perp_\lambda (GW^\perp) + \mathbb{G}^\perp_2 W^0_2 \right].
$$

(1.22)

This term, however can not be handled using the same Morawetz estimate in [34], because of its intrinsic connection with the coefficients $\rho_1, \rho_2$. Therefore we construct a new Morawetz term $M(t)$ using the factorization of $\mathbb{H}^\perp_\lambda$ (2.37), with its coefficients involving $\rho_2/\rho_1$, to cancel (1.22). Analysis behind this cancellation splits into two cases: $|\rho_2/\rho_1| \leq 1$ and $|\rho_2/\rho_1| > 1$, and in both cases the interaction of (1.22) and Morawetz term will
This fact suggests the invariance of the blowup speed against different $\rho_1, \rho_2$. Finally, we apply (1.21) to close the bootstrap arguments, and the description of the blowup speed as well as the behavior of the residue term $w$ are direct consequences of (1.20), (1.19), which yield Theorem 1.1.

Structure of the paper In Section 2, we introduce the Frenet basis, renormalized variables and the linear Hamiltonian $H$, and convert the problem to the vectorial form. In Section 3, we construct an approximate solution $w_0$ to the renormalized equation, and also its localized version $\tilde{w}_0$, then give estimates on corresponding error terms. In Section 4, we are aimed at seeking for an actual solution, so we consider the correction term $w$, set up the trapped regime, introduce the definitions of energies $E_1, E_2, E_4$, and obtain the modulation equations. In Section 5, we compute the mixed energy identity/Morawetz formula, and close the bootstrap bounds. In Section 6, we exhibit the that the finite-time blowup, and give a precise description of the blowup speed as well as the behavior of the correction term $w$. In Appendix A, we list some results from [34] on the interpolation bounds of the correction term $w$, which are frequently used in Section 5. In Appendix B, we give a proof of the Lemma 5.4, which is also a crucial component in Section 5.

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2 1-equivariant flow in the Frenet basis

2.1 The ground state and Frenet basis

Let us introduce the geometric set up of the 1-equivariant solution $u$ of the Landau-Lifschitz equation (1.1). Maps with values in $S^2$ will be viewed as maps into $\mathbb{R}^3$ with image parameterized by Euler angle $(\phi, \theta)$. The ground state $Q$ is a harmonic map of homotopy degree $k = 1$ satisfying

$$\Delta Q = -|\nabla Q|^2 Q. \quad (2.1)$$

Its explicit formula is

$$Q(r, \theta) = \begin{bmatrix} \sin(\phi(r)) \cos(\theta) \\ \sin(\phi(r)) \sin(\theta) \\ \cos(\phi(r)) \end{bmatrix}, \quad \text{with} \quad \phi(r) = 2 \arctan(r). \quad (2.2)$$
For the ease of notations, we define the following functions
\[
\begin{align*}
\Lambda \phi (r) &= r \partial_r (2 \arctan (r)) = \frac{2r}{1 + r^2} = \sin (\phi (r)), \\
Z (r) &= \frac{1 - r^2}{1 + r^2} = \cos (\phi (r)),
\end{align*}
\tag{2.3}
\]
according to which (2.2) is rewritten as
\[
Q (r, \theta) = \begin{bmatrix}
\Lambda \phi (r) \cos (\theta) \\
\Lambda \phi (r) \sin (\theta) \\
Z (r)
\end{bmatrix} = e^{\theta R} \begin{bmatrix}
\Lambda \phi (r) \\
0 \\
Z (r)
\end{bmatrix}.
\]

In order to study the flow in the vicinity of \( Q \), we choose a suitable gauge, namely Frenet basis \([e_r, e_\tau, Q]\), with
\[
e_r = \frac{\partial_r Q}{|\partial_r Q|} = e^{\theta R} \begin{bmatrix}
Z \\
0 \\
- \Lambda \phi
\end{bmatrix}, \quad e_\tau = \frac{\partial_\tau Q}{|\partial_\tau Q|} = e^{\theta R} \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}.
\tag{2.4}
\]
The action of derivatives and rotations in the Frenet basis are given by the following lemma. The proof follows from direct computations.

**Lemma 2.1 (Derivation and rotation of Frenet basis [34])** There holds:

(i) **Action of derivatives:**
\[
\begin{align*}
\partial_r e_r &= -(1 + Z) Q, & \Lambda e_r &= - \Lambda \phi Q, \\
\partial_r e_\tau &= 0, & \Lambda e_\tau &= 0, \\
\partial_r Q &= (1 + Z) e_r, & \Lambda Q &= \Lambda \phi e_r, \\
\partial_\tau e_r &= \frac{Z}{r} e_\tau, & \Delta e_r &= - \frac{1}{r^2} e_r - \frac{2(1 + Z)}{r} Q, \\
\partial_\tau e_\tau &= - \frac{Z}{r} e_r - (1 + Z) Q, & \Delta e_\tau &= - \frac{1}{r^2} e_\tau, \\
\partial_\tau Q &= (1 + Z) e_\tau, & \Delta Q &= -2(1 + Z)^2 Q.
\end{align*}
\tag{2.5}
\]

(ii) **Action of rotations:**
\[
Re_r = Z e_r, \quad Re_\tau = - Ze_\tau - \Lambda \phi Q, \quad RQ = \Lambda \phi e_r.
\]

Moreover, the scaling and rotation symmetries yield the two parameters family of the harmonic map
\[
Q_{\Theta, \lambda} (r) = e^{\Theta R} Q \left( \frac{\lambda}{\lambda} \right), \quad (\Theta, \lambda) \in \mathbb{R} \times \mathbb{R}^*_+,
\]
with the infinitesimal generators:
\[
\frac{d}{d\lambda} (Q_{\Theta, \lambda}) \bigg|_{\lambda=1, \Theta=0} = - \Lambda \phi e_r, \quad \frac{d}{d\Theta} (Q_{\Theta, \lambda}) \bigg|_{\lambda=1, \Theta=0} = - \Lambda \phi e_\tau.
\]
2.2 The component and vectorial formula

Let us introduce two time-dependent geometrical parameters $\Theta(t), \lambda(t)$, and look for solutions of a specific form

$$u = S(Q + \hat{v}) = e^{\Theta(t)R}(Q + \hat{v}) \left( t, \frac{r}{\lambda(t)} \right),$$

(2.6)

where we denote $S$ the mixed transformation corresponding to scaling and rotation induced by $\lambda, \Theta$, and the map $\hat{v}$ is a small perturbation near $Q$. We express $\hat{v}$ under component form with respect to the Frenet basis

$$\hat{v} = [e_r, e_T, Q], \quad \hat{w} = [\hat{\alpha}, \hat{\beta}, \hat{\gamma}]^T.$$ 

(2.7)

As the image of $u$ lies in $S^2$, the component functions $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ obey the constraint

$$\hat{\alpha}^2 + \hat{\beta}^2 + (1 + \hat{\gamma})^2 = 1.$$ 

(2.8)

The LL map problem is equivalent to the existence of the component functions $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$, coupling with the dynamical behavior of the geometrical parameters $\lambda, \Theta$. In this subsection we derive the equation for $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ and its vectorial form. We apply the commonly used self-similar transformation:

$$\frac{ds}{dt} = 1, \quad y = \frac{1}{\lambda},$$

(2.9)

which actually defines a renormalized time $s$ going from certain starting time $s_0$, eventually to $+\infty$, as the original time $t$ runs from 0 to the lifespan $T$ of $u$. Accordingly, for any given function $v(t, r)$, we use the convention

$$v_\lambda(t, r) = v \left( t, \frac{r}{\lambda} \right) = v(s, y),$$

(2.10)

which leads to

$$\partial_r v_\lambda = \frac{1}{\lambda} (\partial_r v)_\lambda, \quad \Delta v_\lambda = \frac{1}{\lambda^2} (\Delta v)_\lambda.$$ 

Also, for the transformation $S$, there holds

$$S v = e^{\Theta R} v_\lambda, \quad S(v_1 \wedge v_2) = S v_1 \wedge S v_2.$$ 

Invoking these with (2.1), we compute each term in (1.1) for solution (2.6):

$$\begin{cases} 
  u_t = \partial_t (S Q) + \partial_t (S \hat{v}), \\
  \rho_1 u \wedge \Delta u = \frac{\rho_1}{\lambda^2} S \left[ Q \wedge (\Delta \hat{v} + |\nabla Q|^2 \hat{v} + \hat{v} \wedge \Delta \hat{v}) \right], \\
  -\rho_2 u \wedge (u \wedge \Delta u) = -\frac{\rho_2}{\lambda^2} S \left\{ (Q \wedge (\Delta \hat{v} + |\nabla Q|^2 \hat{v} + \hat{v} \wedge \Delta \hat{v}) \wedge \left[ Q \wedge (\Delta \hat{v} + |\nabla Q|^2 \hat{v} + \hat{v} \wedge \Delta \hat{v}) \right] \right\} \\
  \quad = \frac{\rho_2}{\lambda^2} S \left\{ (e_r \cdot (\Delta \hat{v} + |\nabla Q|^2 \hat{v})) e_r + (e_T \cdot (\Delta \hat{v} + |\nabla Q|^2 \hat{v})) e_T \\
  \quad \quad - \hat{v} \wedge (Q \wedge (\Delta \hat{v} + |\nabla Q|^2 \hat{v})) - (Q + \hat{v}) \wedge (\hat{v} \wedge \Delta \hat{v}) \right\}. 
\end{cases}$$

(2.11)
Considering any given function $v$, from (2.9), (2.10) and the definition of $S$, we have

$$\partial_t(v_\lambda) = \frac{1}{\lambda^2} \left( \partial_x v - \frac{\lambda_s}{\lambda} Av \right)_\lambda, \quad \partial_t(Sv) = \frac{1}{\lambda^2} S \left( \partial_x v + \Theta_s Rv - \frac{\lambda_s}{\lambda} Av \right).$$

In particular, by Lemma 2.1, we have

$$\partial_t(S e_r) = \frac{1}{\lambda^2} S \left( \Theta_s Z e_r + \frac{\lambda_s}{\lambda} A \phi Q \right),$$

$$\partial_t(S e_\tau) = \frac{1}{\lambda^2} S \left( - \Theta_s Z e_\tau - \Theta_s A \phi Q \right),$$

$$\partial_t(S Q) = \frac{1}{\lambda^2} S \left( \Theta_s A \phi e_\tau - \frac{\lambda_s}{\lambda} A \phi e_r \right).$$

Thus the time derivative of $S \hat{v}$ is

$$\partial_t(S \hat{v}) = \partial_t(\hat{v}_\lambda S e_r) + \partial_t(\hat{v}_\lambda S e_\tau) + \partial_t(\hat{v}_\lambda S Q)$$

$$= \frac{1}{\lambda^2} \left\{ \left[ \partial_x \hat{v} - \frac{\lambda_s}{\lambda} A \hat{v} - \Theta_s \hat{v} Z - \frac{\lambda_s}{\lambda} \hat{v} \Lambda \phi \right] S e_r ight. + \left[ \partial_x \hat{v} - \frac{\lambda_s}{\lambda} A \hat{v} + \Theta_s \hat{v} Z + \Theta_s \hat{v} A \phi \right] S e_\tau 
$$

$$+ \left[ \partial_x \hat{v} - \frac{\lambda_s}{\lambda} A \hat{v} - \Theta_s \hat{v} A \phi - \Theta_s \hat{v} \Lambda \phi \right] S Q \right\}.$$

Next, the linear terms in the forthcoming $\hat{v}$ equation, as exhibited in (2.11), are

$$\rho_1 Q \wedge (\Delta \hat{v} + |\nabla Q|^2 \hat{v})$$

$$+ \rho_2 \left[ (e_r \cdot (\Delta \hat{v} + |\nabla Q|^2 \hat{v})) e_r + (e_r \cdot (\Delta \hat{v} + |\nabla Q|^2 \hat{v})) e_r \right].$$

Similarly, by $\hat{v} \wedge \hat{v} = 0$, the nonlinear terms in (2.11) is reformulated to

$$\rho_1 \hat{v} \wedge \Delta \hat{v} - \rho_2 \hat{v} \wedge \left( Q \wedge (\Delta \hat{v} + |\nabla Q|^2 \hat{v}) \right) - \rho_2 (Q + \hat{v}) \wedge (\hat{v} \wedge \Delta \hat{v})$$

$$= \rho_1 \hat{v} \wedge (\Delta \hat{v} + |\nabla Q|^2 \hat{v}) - \rho_2 \hat{v} \wedge \left( Q \wedge (\Delta \hat{v} + |\nabla Q|^2 \hat{v}) \right)$$

$$- \rho_2 (Q + \hat{v}) \wedge (\hat{v} \wedge (\Delta \hat{v} + |\nabla Q|^2 \hat{v})).$$

Notice here there are repeated linear patterns on $\hat{v}$, namely $\Delta \hat{v} + |\nabla Q|^2 \hat{v}$, in both (2.14) and (2.15). From the property of the ground state (2.1), there holds $|\nabla Q|^2 = -\Delta Q/Q = \ldots$
$2(1 + Z)^2$. Using the derivation of the Frenet basis (2.5), we compute:

$$\Delta \hat{v} + |\nabla Q|^2 \hat{v}$$

$$= \Delta \hat{a} e_r + 2 \partial_r \hat{a} \partial_r e_r + \Delta e_r + \Delta \hat{b} e_r + 2 \partial_r \hat{b} \partial_r e_r + \hat{b} \Delta e_r$$

$$+ \Delta \hat{c} Q + 2 \partial_r \hat{c} \partial_c Q + \Delta \hat{d} + 2(1 + Z)^2 (\hat{a} e_r + \hat{b} e_r + \hat{c} Q)$$

$$= \left\{ \Delta \hat{a} + \left( 2(1 + Z)^2 - \frac{1}{r^2} \right) \hat{a} + 2(1 + Z) \partial_r \hat{c} \right\} e_r + \left\{ \Delta \hat{b} + \left( 2(1 + Z)^2 - \frac{1}{r^2} \right) \hat{b} \right\} e_r$$

$$+ \left\{ \Delta \hat{c} - 2(1 + Z) \partial_r \hat{a} - \frac{2Z(1 + Z)}{r} \hat{a} \right\} Q.$$  (2.16)

The first and second components on the RHS of (2.16) inspire us to define the Hamiltonian/Schrödinger operator

$$H = -\Delta + \frac{V(r)}{r^2},$$  (2.17)

where the potential is

$$V(r) = \frac{1}{r^2} - 2(1 + Z)^2 = \Lambda Z + Z^2 = \frac{r^4 - 6r^2 + 1}{(1 + r^2)^2}.$$  (2.18)

To make the formulas brief, we further introduce the following operators, each of which maps the vector $\hat{w}$ to a scalar function of its components:

$$\begin{align*}
\mathbb{H}^{(1)}(\hat{w}) &= H \hat{a} - 2(1 + Z) \partial_r \hat{c}, \\
\mathbb{H}^{(2)}(\hat{w}) &= H \hat{b}, \\
\mathbb{H}^{(3)}(\hat{w}) &= -\Delta \hat{c} + 2(1 + Z) \partial_r \hat{a} + \frac{2Z(1 + Z)}{r} \hat{a}.
\end{align*}$$  (2.19)

Then (2.16) is actually

$$\Delta \hat{v} + |\nabla Q|^2 \hat{v} = - \left( \mathbb{H}^{(1)}(\hat{w}) e_r + \mathbb{H}^{(2)}(\hat{w}) e_r + \mathbb{H}^{(3)}(\hat{w}) Q \right).$$

Using (2.16), (2.17) to treat (2.14), we have

$$\rho_1 Q \wedge (\Delta \hat{v} + |\nabla Q|^2 \hat{v})$$

$$+ \rho_2 \left[ \left( e_r \cdot (\Delta \hat{v} + |\nabla Q|^2 \hat{v}) \right) e_r + \left( e_r \cdot (\Delta \hat{v} + |\nabla Q|^2 \hat{v}) \right) e_r \right]$$

$$= \left( \rho_1 \mathbb{H}^{(1)}(\hat{w}) - \rho_2 \mathbb{H}^{(1)}(\hat{w}) \right) e_r + \left( - \rho_1 \mathbb{H}^{(1)}(\hat{w}) - \rho_2 \mathbb{H}^{(1)}(\hat{w}) \right) e_r.$$  (2.20)

Finally the nonlinear terms (2.15) are

$$\rho_1 \hat{v} \wedge (\Delta \hat{v} - \rho_2 \hat{v} \wedge (Q \wedge (\Delta \hat{v} + |\nabla Q|^2 \hat{v})) - \rho_2 (Q + \hat{v}) \wedge (\hat{v} \wedge \Delta \hat{v})$$

$$= \left\{ \rho_1 \left( \hat{c} \mathbb{H}^{(2)}(\hat{w}) - \hat{b} \mathbb{H}^{(3)}(\hat{w}) \right) \right\} e_r$$

$$+ \rho_2 \left( - (\hat{b}^2 + \hat{c}^2 + 2\hat{c}) \mathbb{H}^{(1)}(\hat{w}) + \hat{a} \mathbb{H}^{(2)}(\hat{w}) + \hat{b} (1 + \hat{c}) \mathbb{H}^{(3)}(\hat{w}) \right) e_r.$$
we obtain the component equations of $\hat{v}$.

Inserting (2.12), (2.13), (2.20), (2.21) into (2.11), and projecting them onto the spatial variable used here is $y$

Accordingly, modifications should be made by replacing $r$

reasonable in view of the constraint $\lambda$

An essential feature of our analysis is to keep track of the geometric structure of (1.1), so

$(2.19)$ actually yield the vectorial Hamiltonian/ Schrödinger operator:

\begin{align}
\frac{\partial}{\partial s} \hat{\alpha} &= \rho_1 (1 + \hat{\gamma}) \mathbb{H}^{(2)}(\hat{w}) - \rho_2 (1 + \hat{\gamma})^2 \mathbb{H}^{(1)}(\hat{w}) + \frac{\lambda_s}{\lambda} (1 + \hat{\gamma}) \mathcal{A}\phi + \Theta_s \hat{\beta} Z \\
\frac{\partial}{\partial s} \hat{\beta} &= -\rho_1 (1 + \hat{\gamma}) \mathbb{H}^{(3)}(\hat{w}) - \rho_2 (1 + \hat{\gamma})^2 \mathbb{H}^{(2)}(\hat{w}) - \Theta_s \hat{\alpha} Z - \Theta_s (1 + \hat{\gamma}) \Delta \phi \\
\frac{\partial}{\partial s} \hat{\gamma} &= \rho_1 \hat{Z} \mathbb{H}^{(1)}(\hat{w}) - \rho_2 (1 + \hat{\gamma}) \mathbb{H}^{(2)}(\hat{w}) - \frac{\lambda_s}{\lambda} \hat{\alpha} \Delta \phi + \Theta_s \hat{\beta} \Lambda \phi \\
&+ \rho_2 (1 + \hat{\gamma}) \mathbb{H}^{(1)}(\hat{w}) + \rho_2 (1 + \hat{\gamma}) \mathbb{H}^{(2)}(\hat{w}) - \rho_2 (1 + \hat{\gamma}) \mathbb{H}^{(3)}(\hat{w}) - \rho_2 \hat{\beta}^2 \mathbb{H}^{(3)}(\hat{w}).
\end{align}

These component equations, especially (2.22), (2.23), reveal the structure of a combination of the quasilinear Schrödinger equation and the quasilinear heat equation.

**Remark 2.1** (i) The linear term on the RHS of (2.24) is comparatively small owing to the presence of $\lambda_s/\lambda$ and $\Theta_s$, which eventually implies the smallness of $\hat{\gamma}$. This is actually reasonable in view of the constraint (2.8), and it will be confirmed in Section 3. (ii) The spatial variable used here is $y = r/\lambda$ instead of $r$, due to the action of $S$ in (2.11). Accordingly, modifications should be made by replacing $r$ with $y$ in the definitions of (2.17), (2.19), for example, $H_y = -\Delta_y + V(y)/y^2$. However, since the potential risk of ambiguity is low, we will still use the original notations.

An essential feature of our analysis is to keep track of the geometric structure of (1.1), so we rewrite the system into vectorial form. We set $e_z = [0, 0, 1]^T$, and define the rotation related to $\hat{w}$:

\[
\dot{J} = (e_z + \hat{w}) \wedge,
\]

where we note that $e_z \wedge = R$ is just the usual rotation (1.4). The operators defined in (2.19) actually yield the vectorial Hamiltonian/ Schrödinger operator:

\[
\mathbb{H} \hat{w} = \begin{bmatrix}
\mathbb{H}^{(1)}(\hat{w}) \\
\mathbb{H}^{(2)}(\hat{w}) \\
\mathbb{H}^{(3)}(\hat{w})
\end{bmatrix} = \begin{bmatrix}
H \hat{\alpha} \\
H \hat{\beta} \\
-\Delta \hat{\gamma}
\end{bmatrix} + 2(1 + Z) \begin{bmatrix}
\partial_y \hat{\gamma} \\
0 \\
\partial_y \hat{\alpha} + Z \hat{\alpha}
\end{bmatrix}.
\]
Moreover, we define the following vector
\[ \hat{p} = \begin{bmatrix} \Theta_s \\ \lambda_s / \lambda \\ 0 \end{bmatrix}. \] (2.27)

Therefore (2.22)–(2.24) can be rewritten as the following vectorial form
\[ \partial_s \hat{w} - \frac{\lambda_s}{\lambda} A \hat{w} + \Theta_s Z R \hat{w} + \hat{J} \left( \rho_1 \mathbb{H} \hat{w} - \rho_2 \hat{J} \mathbb{H} \hat{w} + \hat{p} \Lambda \phi \right) = 0. \] (2.28)

### 2.3 The linearized Hamiltonian

The linearized Hamiltonian/Schrödinger operator \( H \) (2.17) and \( \mathbb{H} \) (2.26) naturally arise when computing the Landau-Lifschitz flow near \( SQ \) (2.16). In this subsection, we collect some of their properties, which will be of essential importance in the derivation of Laypunov mononicity.

The linearized Hamiltonian \( H \) admits the following factorization:
\[ H = A^* A, \] (2.29)

with
\[ A = -\partial_y + \frac{Z}{y}, \quad A^* = \partial_y + \frac{1 + Z}{y}, \]

where \( A^* \) is the adjoint of \( A \). Given any radially symmetric function \( f(y) \), these operators can be reformulated as
\[ Af = -A \phi \partial_y \left( \frac{f}{A \phi} \right), \quad A^* f = \frac{1}{y A \phi} \partial_y (y A \phi f). \] (2.30)

Thus, their kernels on \( \mathbb{R}^*_+ \) are explicit:
\[
\begin{cases}
Af = 0 & \text{iff } f \in \text{span}(A \phi), \\
A^* f = 0 & \text{iff } f \in \text{span}\left(\frac{1}{y A \phi}\right).
\end{cases}
\]

Hence considering \( Hf = 0 \) on \( \mathbb{R}^*_+ \), we deduce either \( f = C A \phi \) or \( A f = C \frac{y}{y A \phi} \) for any constant \( C \in \mathbb{R} \). In other words, we have
\[ Hf = 0 \text{ iff } f \in \text{span}(A \phi, \Gamma), \]

where \( A \phi \) is a resonance of \( H \) at the origin induced by energy critical scaling invariance, and
\[ \Gamma(y) = A \phi(y) \int_1^y \frac{dx}{x (A \phi(x))^2} = \frac{1}{4(1 + y^2)} \left( y^3 + 4y \log y - \frac{1}{y} \right). \]
The asymptotics for $\Lambda \phi, \Gamma$ are

$$
\Lambda \phi(y) = \begin{cases} 
2y + O(y^3) & \text{as } y \to 0, \\
\frac{2}{y} + O\left(\frac{1}{y^2}\right) & \text{as } y \to +\infty,
\end{cases}
$$

(2.31)

and

$$
\Gamma(y) = \begin{cases} 
-\frac{1}{4y} + O(y \log y) & \text{as } y \to 0, \\
y \log \frac{y}{4} + O\left(\frac{\log y}{y}\right) & \text{as } y \to +\infty.
\end{cases}
$$

(2.32)

Owing to the Wronskian of $\Lambda \phi, \Gamma$ given by

$$
\Lambda \phi' \Gamma - \Gamma' \Lambda \phi = -\frac{1}{y},
$$

the variation of constants formula yields a regular solution to the inhomogeneous equation

$$
Hu = f,
$$

which is given by

$$
u(y) = \Lambda \phi(y) \int_0^y f(x) \Gamma(x) x dx - \Gamma(y) \int_0^y f(x) \Lambda \phi(x) x dx.
$$

(2.33)

In particular, if $f = \Lambda \phi$, the solution is

$$
T_1(y) = \frac{(1 - y^4) \log(1 + y^2) + 2y^4 - y^2 - 4y^2 \int_0^y \frac{\log(1 + x^2)}{x} dx}{2y(1 + y^2)},
$$

(2.34)

with the asymptotics

$$
T_1(y) = \begin{cases} 
-\frac{y^4}{4} + O(y^5) & \text{as } y \to 0, \\
-y \log y + y + O\left(\frac{(\log y)^2}{y}\right) & \text{as } y \to +\infty.
\end{cases}
$$

(2.35)

To deal with the evolution of the forthcoming radiation (4.3), we extract the dominating part of the vectorial Hamiltonian $H$, which is defined by

$$
H_{\perp} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} H \alpha \\ H \beta \\ 0 \end{bmatrix}.
$$

(2.36)

The given factorization (2.29) implies the corresponding factorization of $H_{\perp}$, which is $H_{\perp} = A^* A$ with

$$
A \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} A \alpha \\ A \beta \\ 0 \end{bmatrix}, \quad A^* \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} A^* \alpha \\ A^* \beta \\ 0 \end{bmatrix}.
$$

(2.37)

Moreover, there holds the relation of $H_{\perp}$ with (1.4):

$$
R H_{\perp} = H_{\perp} R, \quad R H R = -H_{\perp}.
$$

(2.38)
3 Construction of the approximate solution

This section is devoted to the construction of the approximate self-similar solution to the vectorial equation (2.28) with controllable growth in spacial variable \( y \). The blowup construction heavily relys on the behavior of the modulation parameters \( \lambda, \Theta \), and also their derivatives \( \lambda_s, \Theta_s \), which emerged in (2.28). To study them, we introduce two more parameters, as well as their anticipated dynamics

\[
a \approx -\Theta_s, \quad b \approx -\frac{\lambda_s}{\lambda}, \quad a_s \approx 0, \quad b_s \approx -(b^2 + a^2). \tag{3.1}
\]

This approximation will be justified in subsection (4.3). The layout of our approximate solution resembles a series expansion of \( a, b \). More precisely, we have the following lemma.

**Lemma 3.1 (The approximate solution)** Let \( M \) be a large enough universal constant, then there exists a universal small constant \( b^* = b^*(M) > 0 \), such that for any \( C^1 \) maps

\[
a, b : [s_0, +\infty) \mapsto (-b^*, b^*) \tag{3.2}
\]

with the prior bound

\[
|a| \lesssim \frac{b}{|\log b|}. \tag{3.3}
\]

and the varying scale \( B_0, B_1 \) given by (1.11), the following holds: There exist smooth radially symmetric profiles \( \Phi_{1,0}(y), \Phi_{0,1}(y), \Phi_{i,j}(b, y) \) and \( S_{0,2}(y) \sim |T_1(y)|^2 \) admitting the asymptotics

\[
\Phi_{1,0}, \Phi_{0,1} = \begin{cases} O(y^3) & \text{for } y \leq 1, \\ O(y \log y) & \text{for } 1 \leq y \leq 2B_1, \end{cases} \tag{3.4}
\]

and

\[
|\Phi_{i,j}| \lesssim \begin{cases} y^{2(i+j)+1} & \text{for } y \leq 1, \\ 1 + y^{2(i+j)-3} & \text{for } 1 \leq y \leq 2B_1, \\ \frac{1}{b |\log b|} & \text{for } 2 \leq y \leq 3B_1, \end{cases} \tag{3.5}
\]

for non-negative integers \( i, j \) in the range of \( 2 \leq i + j \leq 3 \), such that

\[
w_0 = a\Phi_{1,0} + b\Phi_{0,1} + \sum_{2 \leq i+j \leq 3} a^ib^j\Phi_{i,j} + b^2S_{0,2} \tag{3.6}
\]

is an approximate self-similar solution to (2.28) in the following sense. Let

\[
\text{Mod} + \Psi_0 = \partial_s w_0 - \frac{\lambda_s}{\lambda} \Lambda w_0 + \Theta_s ZR w_0 + \hat{J} \left( \rho_1 \mathbb{H} w_0 - \rho_2 \mathbb{H} w_0 + \hat{p} A \phi \right), \tag{3.7}
\]
where Mod is the modulation vector given by (The summations are for $2 \leq i + j \leq 3$

\[
\text{Mod}(t) = a_s \left( \Phi_{1,0} + \sum i a^{i-1} b^j \Phi_{i,j} \right)
+ (b_s + b^2 + a^2) \left( \Phi_{0,1} + 2bS_{0,2} + \sum j a^i b^j \Phi_{i,j} + \sum a^i b^j \partial_y \Phi_{i,j} \right)
- \left( \frac{\lambda_s}{\lambda} + b \right) \left( A\phi(e_x + O(w_0)) + Aw_0 \right)
+ \left( \Theta_s + a \right) \left( A\phi(e_y + O(w_0)) + ZRW_0 \right),
\]

(3.8)

and $\Psi_0$ is the error satisfying the estimates:

(i) Weighted bounds:

\[
\int_{y \leq 2B_1} \frac{|\partial_y \Psi_0^{(1)}|^2 + |\partial_y \Psi_0^{(2)}|^2}{y^{2-2i}} \lesssim \frac{b^4}{|\log b|^2}, \quad 0 \leq i \leq 3,
\]

(3.9)

\[
\int_{y \leq 2B_1} \frac{|\partial_y \Psi_0^{(3)}|^2}{y^{8-2i}} \lesssim \frac{b^6}{|\log b|^2}, \quad 0 \leq i \leq 4,
\]

(3.10)

\[
\int_{y \leq 2B_1} |H\Psi_0^{(1)}|^2 + |H\Psi_0^{(2)}|^2 \lesssim b^4 |\log b|^2,
\]

(3.11)

\[
\int_{y \leq 2B_1} \frac{|\partial_y \Psi_0^{(1)}|^2 + |\partial_y \Psi_0^{(2)}|^2}{y^{2-2i}} \lesssim \frac{b^4}{|\log b|^2}, \quad 0 \leq i \leq 1,
\]

(3.12)

\[
\int_{y \leq 2B_1} |AH\Psi_0^{(1)}|^2 + |AH\Psi_0^{(2)}|^2 \lesssim b^5,
\]

(3.13)

\[
\int_{y \leq 2B_1} |H^2\Psi_0^{(1)}|^2 + |H^2\Psi_0^{(2)}|^2 \lesssim \frac{b^6}{|\log b|^2}.
\]

(3.14)

(ii) Flux computation: Let $\Phi_M$ be defined by (4.5), then

\[
\left\{ \begin{array}{l}
\frac{\langle H\Psi_0^{(1)}, \Phi_M \rangle}{\langle A\phi, \Phi_M \rangle} = \frac{2(\rho_1 ab - \rho_2 b^2)}{\rho_1^2 + \rho_2^2} \left( 1 + O\left( \frac{1}{|\log b|} \right) \right), \\
\frac{\langle H\Psi_0^{(2)}, \Phi_M \rangle}{\langle A\phi, \Phi_M \rangle} = \frac{2(\rho_1 b^2 + \rho_2 ab)}{\rho_1^2 + \rho_2^2} \left( 1 + O\left( \frac{1}{|\log b|} \right) \right).
\end{array} \right.
\]

(3.15)

**Proof of Lemma 3.1.**

We proceed the proof by first constructing the profiles $\Phi_{1,0}, \Phi_{0,1}, S_{0,2}$, deriving the modulation vector Mod, and then eliminating the growing tails induced by the $AT_1 - T_1$, constructing the higher order profiles $\Phi_{i,j}$ ($2 \leq i + j \leq 3$), and finally estimating the error $\Psi_0$.

**Step 1. Construction of $\Phi_{1,0}, \Phi_{0,1}$.** Setting $\dot{w} = w_0$ in (2.28) gives the RHS of (3.7). Assuming the smallness of $a, b$, we see the expression

\[
\dot{J} \left( \rho_1 \mathbb{H} w_0 - \rho_2 \mathbb{J} \mathbb{H} w_0 + \mathbb{J} A\phi \right)
\]

(3.16)
is the major error to be eliminated in the first place. From the definition of (2.25), (2.27), we have

\[
\hat{J}\left(\rho_1 \mathbb{H}w_0 - \rho_2 \mathbb{H}w_0 + \hat{p} \Lambda \phi\right) = \hat{J}\left((\rho_1 - \rho_2 R) \mathbb{H}w_0 - \rho_2 w_0 \wedge \mathbb{H}w_0 - \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \Lambda \phi\right) - \left(\frac{\lambda_s}{\lambda} + b\right) \Lambda \phi (e_x + O(w_0)) + (\Theta_s + a) \Lambda \phi (e_y + O(w_0)).
\]

We decompose \(w_0\) by

\[
w_0 = w_0^1 + w_0^2 + b^2 S_{0,2}, \quad \text{with} \quad \begin{cases} w_0^1 = a \Phi_{1,0} + b \Phi_{0,1}, \\ w_0^2 = \sum_{2 \leq i + j \leq 3} a^i b^j \Phi_{i,j}, \end{cases}
\]

and further assume the profiles are of the form

\[
S_{0,2} = S_{0,2}^{(3)} e_z, \quad \Phi_{i,j}^{(3)} = 0 \quad \text{for} \quad 1 \leq i + j \leq 3.
\]

Thus the leading profiles of \(w_0\) should be \(w_0^1 + b^2 S_{0,2}\), where \(w_0^1\) is mainly responsible for the first and second components, while \(b^2 S_{0,2}\) for the third one. From these we compute:

\[
(\rho_1 - \rho_2 R) \mathbb{H}w_0 = (\rho_1 - \rho_2 R) \begin{bmatrix} \mathbb{H}^{(1)}(w_0^1) \\ \mathbb{H}^{(2)}(w_0^1) \\ 0 \end{bmatrix} + (\rho_1 - \rho_2 R) \begin{bmatrix} \mathbb{H}^{(1)}(w_0^2) \\ \mathbb{H}^{(2)}(w_0^2) \\ 0 \end{bmatrix} - 2b^2 (1 + Z) \partial_y S_{0,2}^{(3)} (\rho_1 e_x - \rho_2 e_y) - \rho_1 b^2 \Delta S_{0,2}^{(3)} e_z + 2\rho_1 (1 + Z) \sum_{i=1,2} \left(\partial_y + \frac{Z}{y}\right) (w_0^i)^{(1)} e_z.
\]

Injecting (3.20) into (3.17), we let

\[
(\rho_1 - \rho_2 R) \begin{bmatrix} \mathbb{H}^{(1)}(w_0^1) \\ \mathbb{H}^{(2)}(w_0^1) \\ 0 \end{bmatrix} - \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \Lambda \phi = 0,
\]

which yields the equations for the first-order profiles \(\Phi_{1,0}, \Phi_{0,1}:\)

\[
(\rho_1 - \rho_2 R) \mathbb{H}^{-1} \Phi_{1,0} = \Lambda \phi e_x, \quad (\rho_1 - \rho_2 R) \mathbb{H}^{-1} \Phi_{0,1} = \Lambda \phi e_y.
\]

Using (2.34), we have the following smooth solution

\[
\Phi_{1,0} = \frac{1}{\rho_1^2 + \rho_2^2} \begin{bmatrix} \rho_1 \\ \rho_2 \\ 0 \end{bmatrix} T_1, \quad \Phi_{0,1} = \frac{1}{\rho_1^2 + \rho_2^2} \begin{bmatrix} -\rho_2 \\ \rho_1 \\ 0 \end{bmatrix} T_1.
\]

Their asymptotics (3.4) follows directly from (2.35). Moreover, there holds

\[
\Phi_{0,1} = R \Phi_{1,0}, \quad \Phi_{1,0} = -R \Phi_{0,1}.
\]
Step 2. Construction of $S_{0,2}$. In view of the constraint (2.8), the anticipated approximate solution should meet

$$(w_0^{(1)})^2 + (w_0^{(2)})^2 + (1 + w_0^{(3)})^2 \approx 0,$$  \hspace{1cm} (3.24)

with sufficient precision. From the decomposition (3.18), we see

$$w_0^{(i)} \approx (w_0^1)^{(i)} = a\Phi_0^{(i)} + b\Phi_{0,1}^{(i)}, \quad \text{for} \quad i = 1, 2,$$

$$w_0^{(3)} = b^2 S_{0,2}.$$

In consideration of (3.3), the profile $a\Phi_{1,0}$ is negligible compared with $b\Phi_{0,1}$, so that

$$(w_0^{(1)})^2 + (w_0^{(2)})^2 \approx b^2 \left[ (\Phi_0^{(1)})^2 + (\Phi_0^{(2)})^2 \right].$$

In order to be compatible with (3.24), we let

$$S_{0,2}^{(3)} = -\frac{1}{2(\rho_1^2 + \rho_2^2)}(T_1)^2,$$  \hspace{1cm} (3.25)

which admits the following asymptotics from (2.35):

$$S_{0,2}(y) = \begin{cases} O(y^6) & \text{for} \quad y \leq 1, \\ O(y^2(\log y)) & \text{for} \quad 1 \leq y \leq 2B_1. \end{cases}$$  \hspace{1cm} (3.26)

Step 3. Derivation of the modulation vector. In view of (3.7), we have

$$\partial_s w_0 - \frac{\lambda_s}{\lambda} A w_0 + \Theta Z R w_0$$

$$= a_s \Phi_{1,0} + b_s \Phi_{0,1} + \sum \partial_s \left( a^{i}b^{j}\right) \Phi_{i,j} + \sum \left( a^{i}b^{j}b_{s}\partial_{s} \Phi_{i,j} + 2bb_{s}S_{0,2} \right)$$

$$- \left( \frac{\lambda_s}{\lambda} + b \right) A w_0 + ab \Lambda \Phi_{1,0} + b^2 \Lambda \Phi_{0,1} + \sum a^{i}b^{j+1} \Lambda \Phi_{i,j} + b^3 \Lambda S_{0,2}$$

$$+ (\Theta_s + a) Z R w_0 - a^2 Z R \Phi_{1,0} - ab Z R \Phi_{0,1} - \sum a^{i+1}b^{j} Z R \Phi_{i,j} - ab^2 Z R \Phi_{i,j}$$

$$= -\left( \frac{\lambda_s}{\lambda} + b \right) A w_0 + (\Theta_s + a) Z R w_0 + a_s \left( \Phi_{1,0} + \sum i a^{i-1}b^{j} \Phi_{i,j} \right)$$

$$+ (\partial_s b + a^2 + b^2) \left( \Phi_{0,1} + 2b S_{0,2} + \sum j a^{i}b^{j-1} \Phi_{i,j} + \sum a^{i}b^{j} \partial_{s} \Phi_{i,j} \right)$$

$$+ ab(\Lambda \Phi_{1,0} - \Phi_{1,0}) + b^2(\Lambda \Phi_{0,1} - \Phi_{0,1}) + b^3(\Lambda S_{0,2} - 2S_{0,2})$$

$$+ ab(1 + Z) \Phi_{1,0} - a^2(1 + Z) \Phi_{0,1} - 2a^2 b S_{0,2}$$

$$+ \sum a^{i}b^{j} \left( b \Lambda \Phi_{i,j} - a Z R \Phi_{i,j} \right) - (a^2 + b^2) \sum \left( j a^{i}b^{j-1} \Phi_{i,j} + a^{i}b^{j} \partial_{s} \Phi_{i,j} \right),$$

where (3.23) has been applied. The above summations run over the indices $i, j$ for $2 \leq i+j \leq 3$. By collecting the terms with coefficients $(\Theta_s+a), (\lambda_s/\lambda+b), a_s, (b_s+a^2+b^2)$
where the coefficients are \( \leq \)

From (2.31), (2.32), there holds the asymptotics: for \( y \)

Applying (2.33), we see \( \Sigma \)

Step 4. The \( \Sigma_b \) function. To create cancellations on the forthcoming growing tails of the second order profiles (see (3.33), (3.34)), we introduce the radiation \( \Sigma_b \) defined by

where the coefficients are

Applying (2.33), we see

which implies

From (2.31), (2.32), there holds the asymptotics: for \( y \geq 6B_0 \),

while for \( 1 \leq y \leq 6B_0 \),

\[
\Sigma_b(y) = -y + O\left( \frac{\log y}{y} \right) .
\]
Furthermore, the derivatives of \( c_b, d_b \) on \( b \) are

\[
\partial_b c_b = O\left( \frac{1}{b |\log b|^2} \right), \quad \partial_b d_b = O\left( \frac{1}{b^2 |\log b|} \right),
\]

and thus

\[
\partial_b \Sigma_b = \partial_b c_b T_1 1_{y \leq \frac{y_0}{2} \log b} + O\left( \frac{1}{b^2 y |\log b|} \right) 1_{\frac{y_0}{4} \leq y \leq 6b_0} \\
= O\left( \frac{\frac{y^3}{b |\log b|^2} 1_{y \leq 1} + \frac{y}{b |\log b|} 1_{1 \leq y \leq 6b_0}}{1} \right). \tag{3.32}
\]

**Step 5. Manipulation on the growing tails.** Due to the explicit construction (3.22), (3.25) based on \( T_1 \), we observe the pattern \( \Lambda T_1 - T_1 \) appears in each term in the third line on the RHS of (3.28). By (2.35), the asymptotics of \( \Lambda T_1 - T_1 \) at the scale of \( y \sim 2b_1 \) is

\[
\Lambda T_1 - T_1 = -y + O\left( \frac{(\log y)^2}{y} \right), \tag{3.33}
\]

suggesting the potential fast growing tails of the second order profiles (see (3.45), (3.46)), which will make the approximate solution out of control. To fix this, we use \( \Sigma_b \) to create the following cancellation for \( 6B_0 \leq y \leq 2b_1 \):

\[
\Lambda T_1 - T_1 - \Sigma_b = O\left( \frac{(\log y)^2}{y} \right). \tag{3.34}
\]

For \( 1 \leq y \leq 6B_0 \), there holds

\[
\Lambda T_1 - T_1 - \Sigma_b = -y \left( 1 - \frac{\int y^y \chi_{\frac{y_0}{2}} (A\phi)^2 dx}{\int \chi_{\frac{y_0}{2}} (A\phi)^2 dx} \right) + O\left( \frac{(\log y)^2}{y} \right) + O\left( \frac{1 + y}{|\log b|} \right) \\
\lesssim \frac{y}{|\log b|} \left( 1 + |\log(y\sqrt{b})| \right).
\]

These with (2.33) yield the bound

\[
|H^{-1}(\Lambda T_1 - T_1 - \Sigma_b)| = O\left( y^5 1_{y \leq 1} + \frac{1 + y}{b |\log b|} 1_{1 \leq y \leq 2b_1} \right) \lesssim \frac{1 + \frac{y}{b |\log b|}}{b |\log b|}. \tag{3.35}
\]

Therefore we define the vectorial functions

\[
\Sigma_{1,0} = \frac{1}{\rho_1^2 + \rho_2^2} \begin{bmatrix} \rho_1 \\ \rho_2 \\ 0 \end{bmatrix} \Sigma_b, \quad \Sigma_{0,1} = \frac{1}{\rho_1^2 + \rho_2^2} \begin{bmatrix} -\rho_2 \\ \rho_1 \\ 0 \end{bmatrix} \Sigma_b, \quad \Sigma_{0,2} = \frac{1}{\rho_1^2 + \rho_2^2} T_1 \Sigma_b e_z,
\]

corresponding to the cancellations of \( \Lambda \Phi_{1,0} - \Phi_{1,0}, \Lambda \Phi_{0,1} - \Phi_{0,1}, \Lambda S_{0,2} - S_{0,2} \) respectively. Similar to (3.23), we have

\[
\Sigma_{0,1} = R \Sigma_{1,0}, \quad \Sigma_{1,0} = -R \Sigma_{0,1}.
\]
Then (3.28) can be rewritten as

\[
\psi_0 = \hat{J} \left\{ (\rho_1 - \rho_2 R) \begin{bmatrix} \mathbb{H}(w_0^1) \\ \mathbb{H}(w_0^2) \\ 0 \end{bmatrix} - 2b^2 (1+Z) \partial_y S_{0,2}^{(3)} (\rho_1 e_x - \rho_2 e_y) \\
- \rho_2 w_0 \wedge \mathbb{H} w_0 - \rho_1 b^2 \Delta S_{0,2}^{(3)} e_z + 2\rho_1 (1+Z) \sum_{i=1,2} (\partial_x + \frac{Z}{y}) (w_0^i)^{(1)} e_z \right\}
\]

(3.36)

\[
+ \sum_{2 \leq i+j \leq 4} a^i b^j E_{i,j} + b^3 (4S_{0,2} - 2S_{0,2} + \Sigma_{0,2}) \\
+ ab \Sigma_{1,0} + b^2 \Sigma_{0,1} - 2a^2 b S_{0,2} - (a^2 + b^2) \sum_{2 \leq i+j \leq 3} a^i b^j \partial_i \Phi_{i,j}.
\]

where

\[
\sum_{2 \leq i+j \leq 4} a^i b^j E_{i,j} = ab(1+Z) \Phi_{1,0} - a^2 (1+Z) \Phi_{0,1} \\
+ ab (\Delta \Phi_{1,0} - \Phi_{1,0} - \Sigma_{1,0}) + b^2 (\Delta \Phi_{0,1} - \Phi_{0,1} - \Sigma_{0,1}) \\
+ \sum_{2 \leq i+j \leq 3} a^i b^j (b \Delta \Phi_{i,j} - a Z \Phi_{i,j}) - (a^2 + b^2) \sum_{2 \leq i+j \leq 3} ja^i b^{j-1} \Phi_{i,j}.
\]

(3.37)

**Step 6. The \( \hat{J} \) structure.** From the definition (2.25), we have the identity

\[
\hat{J} e_z = (e_z + w_0) \wedge e_z = -(e_z + w_0) \wedge w_0 = -\hat{J} w_0.
\]

(3.38)

It implies the smallness of the third component under the action of \( \hat{J} \), which is helpful in analyzing the terms inside the big brace in (3.36). First, we compute the nonlinear term in (3.18): By the decomposition (3.18):

\[
w_0 \wedge \mathbb{H} w_0 = w_0^1 \wedge \mathbb{H} w_0^1 + w_0^1 \wedge \mathbb{H} w_0^2 + b^2 w_0^1 \wedge \mathbb{H} S_{0,2} \\
+ w_0^2 \wedge \mathbb{H} w_0^1 + w_0^2 \wedge \mathbb{H} w_0^2 + b^2 w_0^2 \wedge \mathbb{H} S_{0,2} \\
+ b^2 S_{0,2} \wedge \mathbb{H} w_0^1 + b^2 S_{0,2} \wedge \mathbb{H} w_0^2 + b^4 S_{0,2} \wedge \mathbb{H} S_{0,2}.
\]

(3.39)

From (3.22), (2.26), we see

\[
w_0^1 \wedge \mathbb{H} w_0^1 = w_0^1 \wedge \begin{bmatrix} \mathbb{H}(w_0^1) \\ \mathbb{H}(w_0^2) \\ 0 \end{bmatrix} + w_0^1 \wedge \begin{bmatrix} 0 \\ 0 \\ \mathbb{H}(w_0^1) \end{bmatrix} = (a \Phi_{1,0} + b \Phi_{0,1}) \wedge \mathbb{H}^1 (a \Phi_{1,0} + b \Phi_{0,1}) - \mathbb{H}^{(3)} (w_0^1) (e_z \wedge w_0^1)
\]

\[
= -2(1+Z) (\partial_y + \frac{Z}{y}) (w_0^1)^{(1)} R w_0^1.
\]

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Besides, (3.19) implies \((w_0^2)^{(3)} = 0\), which together with (3.38) yields

\[
\dot{J}(w_0^1 \land \mathbb{H}w_0^1) = \dot{J}\left\{ w_0^1 \land \left[ \mathbb{H}^{(1)}(w_0^2) \begin{bmatrix} 0 \\ w_0^1 \end{bmatrix} \right] \right\} = 0,
\]

An analogous identity switching the position of \(w_0^1\) and \(w_0^2\) holds for \(w_0^2 \land \mathbb{H}w_0^1\). Again from (3.19), \(S_{0,2} = S_{0,2}^{(3)} e_z\), and then direct computations give

\[
\dot{J}\left( b^2 w_0^1 \land \mathbb{H}S_{0,2} + b^2 S_{0,2} \land \mathbb{H}w_0^1 \right) = \dot{J}\left( b^2 \Delta S_{0,2}^{(3)} R w_0^1 + 2b^2 (1+Z) \partial_\eta S_{0,2}^{(3)} (w_0^1)^{(2)} w_0 + b^2 S_{0,2}^{(3)} R \mathbb{H}w_0^1 \right).
\]

Combining these, the nonlinear term is actually given by

\[
\dot{J}(\rho_2 w_0 \land \mathbb{H}w_0) = \dot{J}\left\{ 2\rho_2 (1+Z) \sum_{i+j \leq 3} \left( \partial_\eta + \frac{Z}{y} \right) (w_0^1)^{(1)} R w_0^1 \right. \\
- \rho_2 b^2 \Delta S_{0,2}^{(3)} R w_0^1 - \rho_2 b^2 S_{0,2}^{(3)} R \mathbb{H}w_0^1 + \sum_{4 \leq i + j \leq 7} a^i b^j R_{i,j} \right\},
\]

where \(R_{i,j}\) consists of functions with coefficient \(a^i b^j\) for \(4 \leq i + j \leq 7\):

\[
\sum_{4 \leq i + j \leq 7} a^i b^j R_{i,j} = -\rho_2 \left\{ 2b^2 (1+Z) \partial_\eta S_{0,2}^{(3)} (w_0^1)^{(2)} w_0 + w_0^2 \land \mathbb{H}w_0^2 \right. \\
+ \left. b^2 w_0^2 \land \mathbb{H}S_{0,2} + b^2 S_{0,2} \land \mathbb{H}w_0^2 + b^4 S_{0,2} \land \mathbb{H}S_{0,2} \right. \\
+ \left. O\left( w_0^1 \land \mathbb{H}w_0^1 \mid w_0^1 \right) + O\left( w_0^2 \land \mathbb{H}w_0^2 \mid w_0 \right) \right\}.
\]

Next, for the last two terms in the brace in (3.36), using (3.38) directly, we have

\[
\dot{J}\left\{ -\rho_1 b^2 \Delta S_{0,2}^{(3)} e_z + 2\rho_1 (1+Z) \sum_{i=1,2} \left( \partial_\eta + \frac{Z}{y} \right) (w_0^i)^{(1)} e_z \right\} \\
= \dot{J}\left\{ \rho_1 b^2 \Delta S_{0,2}^{(3)} w_0 - 2\rho_1 (1+Z) \sum_{i=1,2} \left( \partial_\eta + \frac{Z}{y} \right) (w_0^i)^{(1)} w_0 \right\}.
\]

Furthermore, for any vector \(v\) with \(v^{(3)} = 0\), we observe that

\[
v = \dot{J}( -Rv ) + (w_0 \cdot v) e_z - w_0^{(3)} v.
\]

In particular, take \(v = E_{i,j}\), and there holds

\[
\sum_{2 \leq i + j \leq 4} a^i b^j E_{i,j} = \sum_{2 \leq i + j \leq 4} a^i b^j \left\{ \dot{J}( -RE_{i,j} ) + (w_0 \cdot E_{i,j}) e_z - b^2 S_{0,2}^{(3)} E_{i,j} \right\},
\]

(3.42)
where we note that $E^{(3)}_{i,j} = 0$ follows from $\Phi^{(3)}_{i,j} = \Sigma^{(3)}_{i,0} = \Sigma^{(3)}_{0,i} = 0$. Now we use (3.42) to place $E_{i,j}$ into the big brace in (3.36), and apply (3.40), (3.41). Then the error becomes

$$
\Psi_0 = \tilde{J} \left\{ (\rho_1 - \rho_2 R) \begin{bmatrix} \mathcal{H}^{(1)}(w_0^2) \\ \mathcal{H}^{(2)}(w_0^2) \\ 0 \end{bmatrix} + \sum_{2 \leq i+j \leq 7} a^i b^j \tilde{E}_{i,j} \right\} + ab \Sigma_{1,0} + b^2 \Sigma_{0,1} - b^3 \Sigma_{0,2} - 2a^2 b S_{0,2} - (a^2 + b^2) \sum_{2 \leq i+j \leq 3} a^i b^j \partial_b \Phi_{i,j} \quad (3.43)
$$

$$
+ \sum_{2 \leq i+j \leq 4} a^i b^j 
\left[ (w_0 \cdot E_{i,j}) e_z - b^2 S^{(3)}_{0,2} E_{i,j} \right] + b^3 \left( AS_{0,2} - 2 S_{0,2} + \Sigma_{0,2} \right),
$$

where

$$
\sum_{2 \leq i+j \leq 7} a^i b^j \tilde{E}_{i,j} = - \sum_{2 \leq i+j \leq 4} a^i b^j R E_{i,j} - 2b^2 (1 + Z) \partial_y S^{(3)}_{0,2} (\rho_1 e_x - \rho_2 e_y) - 2(1 + Z) \sum_{i+j \leq 3} \left( \partial_y + \frac{Z}{y} \right) (w_0^{(1)}) (\rho_1 - \rho_2 R) w_0^1 - 2 \rho_1 b^2 S^{(3)}_{0,2} R \mathcal{H} w_0^1 
$$

$$
+b^2 \Delta S^{(3)}_{0,2} (\rho_1 - \rho_2 R) w_0^1 + 2 \rho_1 b^2 (1 + Z) \left( \partial_y + \frac{Z}{y} \right) (w_0^{(1)}) S^{(3)}_{0,2} w_0 + \rho_1 \left[ b^2 \Delta S^{(3)}_{0,2} - 2(1 + Z) \left( \partial_y + \frac{Z}{y} \right) (w_0^{(2)}) \right] (w_0^2 + b^2 S_{0,2}) + \sum_{4 \leq i+j \leq 7} a^i b^j R_{i,j}. \quad (3.44)
$$

Step 7. Construction of $\Phi_{i,j}$. We choose suitable $\Phi_{i,j}$ by eliminating the error (3.44). More precisely, we let

$$
(\rho_1 - \rho_2 R) \begin{bmatrix} \mathcal{H}^{(1)}(w_0^2) \\ \mathcal{H}^{(2)}(w_0^2) \\ 0 \end{bmatrix} + \sum_{2 \leq i+j \leq 3} a^i b^j \tilde{E}_{i,j} = 0,
$$

which, in view of (3.19), yields the following linear system

$$
\begin{bmatrix} \rho_1 & -\rho_2 \\ -\rho_2 & \rho_1 \end{bmatrix} \begin{bmatrix} H\Phi^{(1)}_{i,j} \\ H\Phi^{(2)}_{i,j} \end{bmatrix} = \begin{bmatrix} \tilde{E}^{(1)}_{i,j} \\ \tilde{E}^{(2)}_{i,j} \end{bmatrix}, \quad \text{for } 2 \leq i+j \leq 3. \quad (3.45)
$$

It can also be solved using (2.33), and then the constructions of $\Phi_{i,j}$ are obtained. We
consider first the case $i + j = 2$. From (3.37), (3.44), we see

$$
- \sum_{i+j=2} a^i b^j \hat{E}_{i,j} = a^2 \left\{ (1+Z)\Phi_{1,0} + 2 \rho_1 (1+Z) \left( \partial_y + \frac{Z}{y} \right) \Phi_{1,0}^{(1)} (\rho_1 - \rho_2 R) \Phi_{1,0} \right\}
$$

$$
+ ab \left\{ (1+Z) \Phi_{0,1} + (A \Phi_{0,1} - \Phi_{0,1} - \Sigma_{0,1}) + 2 \rho_1 (1+Z) \sum_{i+j=1}^{i+j=1} \left( \partial_y + \frac{Z}{y} \right) \Phi_{i,j}^{(1)} (\rho_1 - \rho_2 R) \Phi_{k,l} \right\} \quad (3.46)
$$

$$
+ b^2 \left\{ 2(1+Z) \partial_y S^{(3)}_{0,2} (\rho_1 e_x - \rho_2 e_y) - (A \Phi_{1,0} - \Phi_{1,0} - \Sigma_{1,0}) + 2 \rho_1 (1+Z) \left( \partial_y + \frac{Z}{y} \right) \Phi_{0,1}^{(1)} (\rho_1 - \rho_2 R) \Phi_{1,0} \right\}.
$$

This together with the asymptotics for $\Phi_{1,0}, \Phi_{0,1}, S_{0,2}$ and (3.35) implies the bound

$$
\Phi_{i,j} = O \left( y^5 1_{y \leq 1} + \frac{1 + y}{b |\log b|} 1_{y \geq 1} \right) \lesssim \frac{1 + y}{b |\log b|}, \quad (3.47)
$$

and also the crude bound

$$
|\Phi_{i,j}| \lesssim 1 + y^3.
$$

Moreover, from (3.32), we have the estimate for the derivatives on $b$:

$$
\partial_b \Phi_{i,j} = O \left( \frac{y^5}{b |\log b|^2} 1_{y \leq 1} + \frac{y^3}{b |\log b|} 1_{1 \leq y \leq 6 B_0} \right).
$$

Next for $i + j = 3$, from (3.37), (3.44) again, we have

$$
- \sum_{i+j=3} a^i b^j \hat{E}_{i,j} = \sum_{i+j=2} a^i b^j \left( b \mathcal{R} \Phi_{i,j} + a Z \Phi_{i,j} \right) + (a^2 + b^2) \sum_{i+j=2} j a^i b^{j-1} R \Phi_{i,j}
$$

$$
- 2(1+Z) \sum_{i+j=3} \left( \partial_y + \frac{Z}{y} \right) (w_0^{(1)})^{(1)} (\rho_1 - \rho_2 R) w_0^{(1)}
$$

$$
- \rho_2 b^2 \Phi_{0,1} S^{(3)}_{0,2} R \mathcal{H} w_0^{(1)} + b^2 \Delta S^{(3)}_{0,2} (\rho_1 - \rho_2 R) w_0^{(1)}.
$$

This together with (3.22), (3.25), (3.47) gives for $i + j = 3$ that

$$
\Phi_{i,j} = O \left( y^7 1_{y \leq 1} + \frac{1 + y^3}{b |\log b|} 1_{y \geq 1} \right) \lesssim \frac{1 + y^3}{b |\log b|}, \quad (3.48)
$$

the crude bound

$$
|\Phi_{i,j}| \lesssim 1 + y^5,
$$

and the derivatives

$$
\partial_b \Phi_{i,j} = O \left( \frac{y^7}{b |\log b|^2} 1_{y \leq 1} + \frac{y^5}{b |\log b|} 1_{1 \leq y \leq 6 B_0} \right).
$$
Step 8. Estimates on the error. Due to the choice of \( \Phi_{i,j} \), the error (3.43) boils down to

\[
\psi_0 = ab \Sigma_{1,0} + b^2 \Sigma_{0,1} - b^3 \Sigma_{0,2} - 2a^2 b S_{0,2} - (a^2 + b^2) \sum_{2 \leq i+j \leq 3} a^i b^j \partial_k \Phi_{i,j} \\
+ \sum_{2 \leq i+j \leq 4} a^i b^j (w_0 \cdot E_{i,j}) e_z + b^3 (AS_{0,2} - 2S_{0,2} + \Sigma_{0,2}) \\
- b^2 S^{(3)}_{0,2} \sum_{2 \leq i+j \leq 4} a^i b^j E_{i,j} + \hat{J} \left( \sum_{4 \leq i+j \leq 7} a^i b^j \tilde{E}_{i,j} \right). \tag{3.49}
\]

For each line of the RHS of (3.49), we estimate them in the region \([0, 2B_1] \) by previous asymptotics. From (3.31), (3.26), there holds

\[
ab \Sigma_{1,0} + b^2 \Sigma_{0,1} - b^3 \Sigma_{0,2} - 2a^2 b S_{0,2} \\
= O(b^2 \Sigma_{0,1}) (e_x + e_y) + O(b^3 T_1 \Sigma_{0,2}) e_z + O(ab^2 S_{0,2}) \\
= b^2 O \left( \frac{y^3}{|\log b|} 1_{y \leq 1} + \frac{y \log y}{|\log b|} 1_{1 \leq y \leq \frac{n_0}{4}} + y 1_{y \geq \frac{n_0}{4}} \right) (e_x + e_y) \\
+ b^2 O \left( \frac{y^6}{|\log b|} 1_{y \leq 1} + \frac{y^2 (\log y)^2}{|\log b|} 1_{1 \leq y \leq \frac{n_0}{4}} + y^2 \log y 1_{y \geq \frac{n_0}{4}} \right) e_z \\
+ a^2 b O \left( y^2 (\log y)^2 1_{y \geq 1} \right). \tag{3.50}
\]

By the \( b \) derivatives estimates, we have

\[
(a^2 + b^2) \sum_{2 \leq i+j \leq 3} a^i b^j \partial_k \Phi_{i,j} = O \left( \frac{b^2 y^5}{|\log b|} 1_{y \leq 1} + \frac{b^2 y^3}{|\log b|} 1_{1 \leq y \leq 6B_0} \right).
\]

For the second line, (3.25) induces the cancellation:

\[
\left( b(\Lambda \Phi_{0,1} - \Phi_{0,1} - \Sigma_{0,1}) \cdot b \Phi_{0,1} \right) e_z + b^3 (AS_{0,2} - 2S_{0,2} + \Sigma_{0,2}) = 0,
\]

which together with explicit formula (3.37) gives

\[
\sum_{2 \leq i+j \leq 4} a^i b^j (w_0 \cdot E_{i,j}) e_z + b^3 (AS_{0,2} - 2S_{0,2} + \Sigma_{0,2}) \\
= ab^2 O \left( y^6 1_{y \leq 1} + \frac{y^2 \log y}{|\log b|} \left( 1 + |\log (y \sqrt{b})| \right) 1_{1 \leq y \leq 6B_0} + \frac{y^2 \log y}{|\log b|} 1_{y \geq 6B_0} \right) e_z.
\]

For the third line, from (3.37), (3.44), and the crude bounds of \( \Phi_{i,j} \), we have

\[
|b^2 S^{(3)}_{0,2} \sum_{2 \leq i+j \leq 4} a^i b^j E_{i,j}| + \left| \sum_{4 \leq i+j \leq 7} a^i b^j \tilde{E}_{i,j} \right| \lesssim b^4 \left( y^9 1_{y \leq 1} + y^5 (\log y)^2 1_{y \geq 1} \right).
\]

Collecting these estimates, we see (3.50) is the dominate part of \( \psi_0 \), which yields the bound

\[
\int_{y \leq 2B_1} |\psi_0^{(1)}|^2 + |\psi_0^{(2)}|^2 \lesssim \frac{b^4}{|\log b|^2}, \quad \int_{y \leq 2B_1} |\psi_0^{(3)}|^2 \lesssim \frac{b^6}{|\log b|^7}. \tag{3.51}
\]
This yields (3.9), (3.10) with $i = 0$. The cases $i \geq 1$ are similar. After the action of $H$, we have

$$a b H \Sigma_{1,0} + b^2 H \Sigma_{0,1} = O\left(b^2 \Sigma_b\right)(e_x + e_y)$$

$$= \frac{b^2}{|\log b|} \left(\int_{y \leq 1} + \int_{1 \leq y \leq 6B_0} \right) (e_x + e_y),$$

which still leads to the same bounds as (3.51). So do most of the other terms, except the $b^3 R A \Phi_{0,3}$ coming from the last term of (3.44), which, from (3.48), can be bounded by

$$\int_{y \leq 2B_1} |H(b^4 R A \Phi_{0,3})|^2 \lesssim b^8 \int_{y \leq 1} |H(y^7)|^2 + \int_{1 \leq y \leq 2B_1} \frac{|H(y^3)|^2}{b^2 |\log b|^2} \lesssim b^4 |\log b|^2,$$

and thus (3.11) follows. The estimates (3.12)–(3.14) are obtained similarly. The details are left to readers. Finally, for the flux computations, we observe from (3.29) that

$$(\mathbb{H}^+ \psi_0, \Phi_M) = (ab \mathbb{H}^+ \Sigma_{1,0} + b^2 \mathbb{H}^+ \Sigma_{0,1}, \Phi_M) + C(M) \frac{b^6}{|\log b|^2}$$

$$= \frac{abc b}{\rho_1^2 + \rho_2^2} \left[\rho_1 \rho_2 \rho_1 \rho_2 \rho_1 \rho_2 \rho_1 \rho_2 \right] + C(M) \frac{b^6}{|\log b|^2},$$

which together with (3.30) yields the desired (3.15). This concludes the proof. \qed

Remark 3.1 The construction of $S_{0,2}$ (3.25) may seem to be unnatural, but it actually corresponds to the constraint (2.8), ensuring that LL flow maps from $\mathbb{R}^2$ to $\mathbb{S}^2$. On the contrary, if we simply let $w_0^{(3)} = S_{0,2}^{(3)} = 0$, the error arising therefrom will make $w^{(3)} = \gamma$ (see (4.2), (4.3)) out of control, and destroy the corresponding bounds in Lemma A.4.

3.1 Localization of the profiles

The aim of this subsection is to localize the approximate solution constructed in Lemma 3.1 within the spacial scale $y \leq 2B_1$.

Lemma 3.2 (Localization) Under the assumption of Lemma 3.1, we define the localized profile

$$\tilde{w}_0 = \chi_{B_1} w_0,$$  \hspace{1cm} (3.52)

and $\tilde{\Phi}_{i,j} = \chi_{B_1} \Phi_{i,j}$ accordingly. Then (3.52) is an approximate solution to (2.28) in the sense that

$$\tilde{\psi}_0 + \tilde{\text{Mod}}(t) = \partial_t \tilde{w}_0 - \frac{\lambda_s}{\Lambda} \Lambda \tilde{w}_0 + \Theta_s Z R \tilde{w}_0 + \tilde{J} \left( \rho_1 \tilde{H} \tilde{w}_0 - \rho_2 \tilde{H} \tilde{w}_0 + \tilde{p} \Lambda \phi \right),$$  \hspace{1cm} (3.53)

where $\tilde{J} := (e_z + \tilde{w}_0) \Lambda$. The localized modulation vector $\tilde{\text{Mod}}$ is given by

$$\tilde{\text{Mod}} = \chi_{B_1} \text{Mod} + (b\dot{s} + b^2 + \Lambda^2) \frac{O\left(\tilde{w}_0\right)}{b} \tilde{1}_{y \sim B_1} - \left(\frac{\lambda_s}{\Lambda} + b\right) O(\tilde{w}_0) \tilde{1}_{y \sim B_1}$$

$$+ (\Theta_s + a) \Lambda \phi \left( e_x + O(\tilde{w}_0) \right) \tilde{1}_{y \geq B_1} - \left(\frac{\lambda_s}{\Lambda} + b\right) \Lambda \phi \left( e_x + O(\tilde{w}_0) \right) \tilde{1}_{y \geq B_1},$$  \hspace{1cm} (3.54)
while the localized error $\tilde{\psi}_0$ satisfies the following estimates:

$$
\int_{y \leq 2B_1} \frac{|\partial_y^i \tilde{\psi}_0^{(1)}|^2 + |\partial_y^i \tilde{\psi}_0^{(2)}|^2}{y^{s-2i}} \leq \frac{b^4}{|\log b|^2}, \quad 0 \leq i \leq 3,
$$

$$
\int_{y \leq 2B_1} \frac{|\partial_y^i \tilde{\psi}_0^{(3)}|^2}{y^{s-2i}} \leq \frac{b^6}{|\log b|^2}, \quad 0 \leq i \leq 4,
$$

$$
\int_{y \leq 2B_1} |H\tilde{\psi}_0^{(1)}|^2 + |H\tilde{\psi}_0^{(2)}|^2 \leq b^4 |\log b|^2,
$$

$$
\int_{y \leq 2B_1} \frac{|\partial_y^i H\tilde{\psi}_0^{(1)}|^2 + |\partial_y^i H\tilde{\psi}_0^{(2)}|^2}{y^{2-2i}} \leq \frac{b^4}{|\log b|^2}, \quad 0 \leq i \leq 1,
$$

$$
\int_{y \leq 2B_1} |AH\tilde{\psi}_0^{(1)}|^2 + |AH\tilde{\psi}_0^{(2)}|^2 \leq b^5,
$$

$$
\int_{y \leq 2B_1} |H^2\tilde{\psi}_0^{(1)}|^2 + |H^2\tilde{\psi}_0^{(2)}|^2 \leq \frac{b^6}{|\log b|^2}.
$$

\begin{align*}
\frac{(H\tilde{\psi}_0^{(1)}, \Phi_M)}{(A\phi, \Phi_M)} &= \frac{2(\rho_1 b - \rho_2 b^2)}{(\rho_1^2 + \rho_2^2) |\log b|} \left( 1 + O\left( \frac{1}{|\log b|} \right) \right), \\
\frac{(H\tilde{\psi}_0^{(2)}, \Phi_M)}{(A\phi, \Phi_M)} &= \frac{2(\rho_2 b^2 + \rho_2 ab)}{(\rho_1^2 + \rho_2^2) |\log b|} \left( 1 + O\left( \frac{1}{|\log b|} \right) \right).
\end{align*}

**Proof of Lemma 3.2.**

From the localization (3.52) and the error of the approximate solution (3.7), we compute:

$$
\partial_s \tilde{w}_0 - \frac{\lambda_s}{\lambda} \Lambda \tilde{w}_0 + \Theta_s ZR \tilde{w}_0 + \tilde{J} \left( \rho_1 \mathbb{H} \tilde{w}_0 - \rho_2 \tilde{\mathbb{H}} \tilde{w}_0 + \tilde{\rho} \Lambda \phi \right) \\
= \partial_s \chi_{B_1} \tilde{w}_0 - \frac{\lambda_s}{\lambda} \left( \Lambda \chi \right)_{B_1} \tilde{w}_0 + \chi_{B_1} \left( \partial_s \tilde{w}_0 - \frac{\lambda_s}{\lambda} \Lambda \tilde{w}_0 + \Theta_s Z R \tilde{w}_0 \right) \\
+ \left( (e_z + w_0) - (1 - \chi_{B_1}) \tilde{w}_0 \right) \wedge \left( \rho_1 \mathbb{H} (\chi_{B_1} \tilde{w}_0) - \rho_2 (e_z + \tilde{w}_0) \wedge \mathbb{H} (\chi_{B_1} \tilde{w}_0) + \tilde{\rho} \Lambda \phi \right) \\
= \chi_{B_1} (\Psi_0 + \text{Mod}) \\
+ \partial_s \chi_{B_1} \tilde{w}_0 - \frac{\lambda_s}{\lambda} \left( \Lambda \chi \right)_{B_1} \tilde{w}_0 + (1 - \chi_{B_1}) \tilde{J} (\tilde{\rho} \Lambda \phi) \\
- \chi_{B_1} (1 - \chi_{B_1}) \tilde{w}_0 \wedge \left( \rho_1 \mathbb{H} \tilde{w}_0 - \rho_2 (e_z + \tilde{w}_0) \wedge \mathbb{H} \tilde{w}_0 + \tilde{\rho} \Lambda \phi \right) \\
+ \rho_2 \chi_{B_1} (1 - \chi_{B_1}) \tilde{J} (w_0 \wedge \mathbb{H} \tilde{w}_0) \\
- \tilde{J} \left( (\rho_1 - \rho_2 (e_z + \tilde{w}_0)) \wedge \left( 2 \partial_y \chi_{B_1} \partial_y \tilde{w}_0 + \Delta_y \tilde{w}_0 + 2(1 + Z) \partial_y \chi_{B_1} e_y \wedge \tilde{w}_0 \right) \right).
$$

For the second line of RHS of (3.63), we reformulate them according to the anticipated
modulation equations

\[ \partial_t \chi_{B_1} w_0 - \frac{\lambda_s}{\chi} (A \chi)_{B_1} w_0 + (1 - \chi_{B_1}) \tilde{J}(\tilde{\rho} \Lambda \phi) \]

\[ = (b_s + b^2 + a^2) O \left( \frac{w_0}{b} \right) 1_{y \sim B_1} - \left( \frac{\lambda_s}{\chi} + b \right) O(w_0) 1_{y \sim B_1} \]

\[ + O(b^2 y \log y) (e_x + e_y) 1_{y \sim B_1} + O(b^3 y^2 (\log y)^2) \epsilon_2 1_{y \sim B_1} + O(by^{-1}) (e_x + e_y) 1_{y \geq B_1} \]

\[ + (\Theta_s + a) \Lambda \phi (e_x + O(\tilde{\omega})) 1_{y \geq B_1} - \left( \frac{\lambda_s}{\chi} + b \right) \Lambda \phi (e_x + O(\tilde{\omega})) 1_{y \geq B_1}. \]

To treat the third line of the RHS of (3.63), from the cancellation (3.21), which is

\[ (\rho_1 - \rho_2 R) \begin{bmatrix} \mathbb{H}^{(1)}(w_0) \\ \mathbb{H}^{(2)}(w_0) \end{bmatrix} - \begin{bmatrix} a \\ b \end{bmatrix} \Lambda \phi = 0, \]

we have the estimate

\[ - \chi_{B_1} (1 - \chi_{B_1}) w_0 \wedge \left( \rho_1 \mathbb{H} w_0 - \rho_2 (e_x + w_0) \wedge \mathbb{H} w_0 + \Lambda \phi \tilde{p} \right) \]

\[ = O \left( |w_0| \cdot (|\mathbb{H}(w_0^2)| + b^2 |\mathbb{H}S_{0,2}| + |w_0| |\mathbb{H} w_0|) \right) 1_{y \sim B_1} \]

\[ - (\Theta_s + a) \Lambda \phi (e_x \wedge w_0) 1_{y \sim B_1} - \left( \frac{\lambda_s}{\chi} + b \right) \Lambda \phi (e_y \wedge w_0) 1_{y \sim B_1} \]

\[ = O(b^3 y^2 \log y) 1_{y \sim B_1} - (\Theta_s + a) \Lambda \phi O(w_0) 1_{y \sim B_1} - \left( \frac{\lambda_s}{\chi} + b \right) \Lambda \phi O(w_0) 1_{y \sim B_1}. \]

Moreover, the fourth line of (3.63) is in fact

\[ \rho_2 \chi_{B_1} (1 - \chi_{B_1}) \tilde{J}(w_0 \wedge \mathbb{H} w_0) \]

\[ = O \left( |w_0| |\mathbb{H} w_0| + |w_0|^2 |\mathbb{H} w_0| \right) 1_{y \sim B_1} = O(b^2 (\log y)^2) 1_{y \sim B_1}. \]

Finally for the last line of (3.63), we estimate it by brute force

\[ - \tilde{J} \left\{ (\rho_1 - \rho_2 (e_x + \tilde{\omega}) \wedge) \left( 2 \partial_y \chi_{B_1} \partial_y w_0 + \Delta_y w_0 + 2(1 + Z) \partial_y \chi_{B_1} e_y \wedge w_0 \right) \right\} \]

\[ = O \left( \frac{1}{B_1^2} |\partial_y w_0| + \frac{1}{B_1^2} |w_0| + \frac{1}{B_1 (1 + y^2)} |w_0| \right) 1_{y \sim B_1} = O \left( \frac{b^3 \log y}{|\log b|} \right) 1_{y \sim B_1}. \]

Injecting these computations into (3.63), we obtain the localized error

\[ \psi_0 = \chi_{B_1} \psi_0 + O(b^3 y \log y) (e_x + e_y) 1_{y \sim B_1} \]

\[ + O(b^3 y^2 (\log y)^2) e_x 1_{y \sim B_1} + O(by^{-1}) (e_x + e_y) 1_{y \geq B_1} \]

\[ + O(b^3 y^2 \log y) 1_{y \sim B_1} + O(b^2 (\log y)^2) 1_{y \sim B_1} + O \left( \frac{b^3 \log y}{|\log b|} \right) 1_{y \sim B_1} \]

\[ = \chi_{B_1} \psi_0 + O(b^3 y \log y) (e_x + e_y) 1_{y \sim B_1} \]

\[ + O(b^3 y^2 (\log y)^2) e_x 1_{y \sim B_1} + O(by^{-1}) (e_x + e_y) 1_{y \geq B_1}. \]
and also the explicit formula of the localized modulation vector (3.54). Note from (3.64) that most of the additional errors are supported in the region \([B_1, 2B_1]\) (labelled by \(1_{y-B_1}\)), and thus we can easily check they do not perturb the estimate in Lemma 3.1, in particular thanks to the choice \(B_1\). The details are left to readers.

\[\square\]

4 The Trapped Regime

We now aim at seeking for an actual blowup solution to (1.1). To this end, we in this section study the solution of the form (4.2). More precisely, we set up the bootstrap regime, give the orthogonality condition, and compute the modulation equations of the geometrical parameters.

4.1 Bootstrap setup and orthogonality conditions

In this subsection, we describe the set of initial data leading to the blowup scenario of Theorem 1.1, and give the corresponding bootstrap statement. For any initial data \(u_0 \in \dot{H}^1\) with

\[
\|\nabla u_0 - \nabla Q\|_{L^2} \ll 1, \quad (4.1)
\]

we may decompose the corresponding solution \(u(t)\) in a small time interval \([0, t_1]\) by

\[
u(t) = e^{\Theta R}(Q + \hat{v})\lambda, \quad \text{with} \quad \begin{cases} 
\hat{v} = [e_r, e_r, Q] \hat{w}, \\
\hat{w} = \hat{w}_0 + w,
\end{cases} \quad (4.2)
\]

where \(\hat{w}_0\) is the localized approximate solution, and \(w\) is a correction term called the radiation. The existence of this decomposition at any fixed time follows from the variational characterization of \(Q\), see for example [31,36]. Then by the standard modulation theory, the solution can be modulated such that there exists the geometrical parameter maps \(\lambda, \Theta, a, b \in C^1([0, t_1], \mathbb{R})\), and the radiation

\[
w = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \quad (4.3)
\]

satisfying the orthogonality conditions

\[
(\alpha, \Phi_M) = (\alpha, H\Phi_M) = (\beta, \Phi_M) = (\beta, H\Phi_M) = 0. \quad (4.4)
\]

Here \(\Phi_M\) is a localized replacement of \(A\phi\) defined, for sufficiently large universal constant \(M \gg 1\), by

\[
\Phi_M = \chi_M A\phi - c_M H(\chi_M A\phi), \quad (4.5)
\]

with

\[
c_M = \frac{\langle \chi_M A\phi, T_1 \rangle}{\langle H(\chi_M A\phi), T_1 \rangle} \sim c_\chi M^2(1 + o(1)).
\]
It admits the following non-degenerate and orthogonal properties

\[
\begin{align*}
\|\Phi_M\|_{L^2} &= 4 \log M(1 + o(1)), \\
(L\phi, \Phi_M) &= (T_1, H\Phi_M) = 4 \log M(1 + o(1)), \\
(T_1, \Phi_M) &= (L\phi, H\Phi_M) = 0.
\end{align*}
\]

(4.6)

Indeed, to study the orthogonality (4.4), we define the vectorial function

\[
F(\lambda, \Theta, a, b, u) = \left[ (\alpha, \Phi_M), (\beta, \Phi_M), (\alpha, H\Phi_M), (\beta, H\Phi_M) \right].
\]

Using the explicit construction of \(\tilde{w}_0\) in Lemma 3.2, we compute the following derivatives at point \(P\) where \((\lambda, \Theta, a, b, u) = (1, 0, 0, 0, Q)\)

\[
\frac{\partial}{\partial \lambda} \bigg|_P \left( e^{-\Theta R} u_{1,1} \right) = L\phi e_r, \\
\frac{\partial}{\partial \Theta} \bigg|_P \left( e^{-\Theta R} u_{1,1} \right) = -L\phi e_r,
\]

from which follows the non-degeneracy of the Jacobian of \(F\) at \(P\):

\[
\begin{vmatrix}
(L\phi, \Phi_M) & 0 & (L\phi, H\Phi_M) & 0 \\
0 & (L\phi, \Phi_M) & 0 & (L\phi, H\Phi_M) \\
(\Phi_{1,0}^{(1)}, \Phi_M) & (\Phi_{1,0}^{(2)}, \Phi_M) & (\Phi_{1,0}^{(1)}, H\Phi_M) & (\Phi_{1,0}^{(2)}, H\Phi_M) \\
(\Phi_{0,1}^{(1)}, \Phi_M) & (\Phi_{0,1}^{(2)}, \Phi_M) & (\Phi_{0,1}^{(1)}, H\Phi_M) & (\Phi_{0,1}^{(2)}, H\Phi_M)
\end{vmatrix}
= \frac{(1 + o(1))}{\rho_1^2 + \rho_2^2} (L\phi, \Phi_M)^4 \neq 0.
\]

According to the implicit function theorem, there exist a constant \(\delta > 0\), a neighborhood \(V_P\) of the point \(P\), and a unique \(C^1\) geometrical parameter map

\[
(\lambda, \Theta, a, b) : \{ u \in \dot{H}^1 : \|u - Q\|_{\dot{H}^1} < \delta \} \to V_P,
\]

such that \(F(\lambda, \Theta, a, b, u) = 0\). In view of (4.2), this ensures the orthogonality of the radiation (4.4). Moreover, by the regularity of LL flow, the map \((\lambda, \Theta, a, b)\) is \(C^1\) function of \(t\), concluding our claim. Then we may measure the regularity of the \(w\) by the following Sobolev norms associated to the linear operator (2.36): the energy norm

\[
\mathcal{E}_1 = \int |\nabla w|^2 + \frac{|w|^2}{|y|}, \quad (4.7)
\]

and higher-order energy norm

\[
\mathcal{E}_2 = \int |H^1 w|^2, \quad \mathcal{E}_4 = \int |(H^1)^2 w|^2. \quad (4.8)
\]

We now make the following assumptions on the initial data \(u_0\), which describe a codimension one set (see Proposition 4.1 and its proof in subsection 5.4) close to the ground state \(Q\) in \(\dot{H}^1\):
• Initial bound on the modulation parameters:

\[ 0 < b(0) < b^*(M) \ll 1, \quad a(0) \leq \frac{b(0)}{4|\log b(0)|}. \]  

(4.9)

• Initial data of the scaling parameter: (Up to a fixed scaling, we can always assume this; thus this assumption is not compulsory)

\[ \lambda(0) = 1. \]  

(4.10)

• Initial energy bounds:

\[
\begin{cases} 
0 < \mathcal{E}_1(0) < \delta(b^*) \ll 1, \\
0 < \mathcal{E}_2(0) + \mathcal{E}_4(0) < b(0)^{10}.
\end{cases}
\]  

(4.11)

where \( \delta(b^*) \) denotes a generic constant related to \( b^* \) which satisfies

\[ \delta(b^*) \to 0, \text{ as } b^* \to 0. \]  

(4.12)

The propagation of regularity of \( u(t) \) ensures that these bounds can be preserved in a small time interval \([0, t_1]\). This suggests given a universal large constant \( K \), independent of \( M \) in Lemma 3.1, we may assume on \([0, t_1]\) the following bounds:

• Pointwise bounds on the modulation parameters:

\[ 0 < b(t) \leq K b^*(M), \quad a(t) \leq \frac{b(t)}{|\log b(t)|}. \]  

(4.13)

• Pointwise energy bounds:

\[
\begin{align*}
\mathcal{E}_1(t) &\leq K \delta(b^*), \\
\mathcal{E}_2(t) &\leq K b(t)^2 |\log b(t)|^6, \\
\mathcal{E}_4(t) &\leq K \frac{b(t)^4}{|\log b(t)|^2}.
\end{align*}
\]  

(4.14) \hspace{1cm} (4.15) \hspace{1cm} (4.16)

The core of our analysis is the following proposition, which yields the contraction of the blowup regime.

**Proposition 4.1 (Trapped regime)** Assume that \( K \) in (4.13)–(4.16) has been chosen large enough, independent of \( M \). Then for any large enough rescaled initial time \( s_0 \), and any initial data \((w, \lambda, \Theta, b)(0)\) satisfying (4.9), (4.10), (4.11), there exists

\[ a(0) = a\left(b(0), w(0)\right) \in \left[-\frac{b(0)}{4|\log b(0)|}, \frac{b(0)}{4|\log b(0)|}\right], \]  

(4.17)

such that (4.2) the corresponding solution to (1.1) satisfies for \( t \in [0, t_1] \) that:
Refined bounds on the modulation parameters:

\[ 0 < b(t) \leq \frac{K}{2} b^*(M), \quad a(t) \leq \frac{b(t)}{2 \| \log b(t) \|}. \]  \hfill (4.18)

Refined energy bounds:

\[ \mathcal{E}_1(t) \leq \frac{K}{2} \delta(b^*), \]  \hfill (4.19)
\[ \mathcal{E}_2(t) \leq \frac{K}{2} b(t)^2 | \log b(t) |^6, \]  \hfill (4.20)
\[ \mathcal{E}_4(t) \leq K (1 - \eta) \frac{b(t)^4}{| \log b(t) |^2}, \]  \hfill (4.21)

where \( \eta \in (0, 1) \) is a universal constant independent of \( M \).

This proposition implies the solution \( u \) is trapped in the regime (4.18), (4.19)–(4.21), and thus the bounds can be maintained within the lifespan of \( u \). The proof is given in subsection 5.4.

4.2 Equation for the radiation

Recall from (4.2) that we have the decomposition

\[ \hat{w} = \tilde{\omega}_0 + w, \]

Injecting this into the vectorial formula (2.28), and applying (3.53), we obtain the equation for the radiation

\[ \partial_s w - \frac{\lambda_s}{\lambda} \Lambda w + \rho_1 \hat{J}_H w - \rho_2 \hat{J}^2 H w + f = 0, \]  \hfill (4.22)

where

\[ f = \tilde{\text{Mod}} + \tilde{\Psi}_0 + \mathcal{R}. \]  \hfill (4.23)

The modulation vector \( \tilde{\text{Mod}} \), the error \( \tilde{\Psi}_0 \) are given by (3.53), and the term \( \mathcal{R} \) contains the residue involving the phase derivative and cross terms of \( \tilde{\omega}_0 \) and \( w \):

\[ \mathcal{R} = \Theta_s Z \mathcal{R} w - \rho_2 \hat{J} (w \wedge \hat{H} \tilde{u}_0) + w \wedge \left( \rho_1 \hat{H} \tilde{u}_0 - \rho_2 \hat{J} \hat{H} \tilde{u}_0 + \Lambda \phi \right). \]  \hfill (4.24)

We rewrite \( w \) in terms of the original coordinate \((r, t)\) by

\[ W(t, r) = w(s, y), \]  \hfill (4.25)

and then (4.22) is equivalent to

\[ \partial_t W + \rho_1 \hat{J}_\lambda \hat{H}_\lambda W - \rho_2 \hat{J}^2 \hat{H}_\lambda W + F = 0, \]  \hfill (4.26)
where
\[
\hat{J}_\lambda = (e_z + \hat{W}) \wedge, \quad F = \frac{1}{\lambda^2} f_\lambda.
\] (4.27)

Here the rescaled vectorial Hamiltonian \( H_\lambda \) is defined by
\[
H_\lambda \begin{bmatrix} \alpha_\lambda \\ \beta_\lambda \\ \gamma_\lambda \end{bmatrix} = \begin{bmatrix} H_\lambda \alpha_\lambda \\ H_\lambda \beta_\lambda \\ -\Delta_r \gamma_\lambda \end{bmatrix} + \frac{2(1 + Z_\lambda)}{\lambda} \begin{bmatrix} -\partial_r \gamma_\lambda \\ 0 \\ \partial_r \alpha_\lambda + \frac{\lambda}{\delta} \alpha_\lambda \end{bmatrix},
\] (4.28)

where the rescaled scalar Hamiltonian is
\[
H_\lambda = -\Delta_r + \frac{V_\lambda}{r^2}.
\]
The notation of \( \lambda \) subscript is given by (2.10), from which \( Z_\lambda(r) = Z(r/\lambda) \), \( \alpha_\lambda(t, r) = \alpha(t, r/\lambda) \), and so forth. Similar to (2.29), \( H_\lambda \) admits the factorization
\[
H_\lambda = A_\lambda^* A_\lambda
\] (4.29)

In the forthcoming analysis, we define the dominating part of \( H_\lambda \) by
\[
H_\perp \lambda \begin{bmatrix} \alpha_\lambda \\ \beta_\lambda \\ \gamma_\lambda \end{bmatrix} = \begin{bmatrix} H_\lambda \alpha_\lambda \\ H_\lambda \beta_\lambda \\ 0 \end{bmatrix},
\] (4.30)

which, from (4.29), admits the factorization \( H_\perp \lambda = A_\lambda^* A_\lambda \) with
\[
A_\lambda = -\partial_r + \frac{Z_\lambda}{r}, \quad A_\lambda^* = \partial_r + 1 + \frac{Z_\lambda}{r}.
\] (4.31)

In view of the scaling \( y/r = 1/\lambda \), there holds
\[
Hw = \lambda^2 H_\lambda W, \quad H_\perp w = \lambda^2 H_\perp \lambda W,
\] (4.32)

and from (2.38), we have
\[
R H_\perp \lambda = H_\perp \lambda R, \quad RH_\lambda = -H_\perp \lambda.
\] (4.33)

### 4.3 Modulation equations

This subsection is devoted to derivation of the modulation equations of \((\lambda, \Theta, a, b)\), which constitute a ODE system that determines the blowup dynamics. We will use the equation (4.22) and the orthogonality conditions (4.4).

**Proposition 4.2 (Modulation equations)** Assume (4.13)–(4.16). Then there hold the following estimates on the modulation parameters:

\[
\begin{align*}
|a_s + 2ab \log|b|| + |b_s + b^2 (1 + \frac{2}{\log|b|})| & \lesssim \frac{1}{\sqrt{\log M}} \left( \sqrt{\mathcal{E}_a + \frac{b^2}{\log|b|}} \right), \\
|a + \Theta| + |b + \frac{\lambda a}{\lambda}| & \lesssim C(M) b^3.
\end{align*}
\] (4.34)
Remark 4.1 (i) The expression in companion with $b_s$ in the first identity is not $a^2 + b^2$, different from what is suggested by $\tilde{\text{Mod}}$. This is a direct result the smallness of $a$ (4.13) and the flux computations (3.61), (3.62). (ii) The $\sqrt{\log M}$ on the denominator is crucial in the following analysis, especially when closing the bootstrap bound of $E_4$ and specifying an appropriate initial data of $a$.

Proof of Proposition 4.2.
We project (4.22) onto $\{e_x, e_y\}$ to obtain the equations for $\alpha, \beta$, and then take their $L^2$ inner product with $H\Phi_M$, $\Phi_M$ defined by (4.6) respectively. Computing each term in the resulting formulas by the interpolation bounds in Appendix A, (4.13)–(4.16), and also (3.54), (3.61), (3.62), we obtain

\[
\begin{align*}
\rho_1 a_s - \rho_2 (b_s + b^2 + a^2) &+ \frac{2}{|\log b|} (\rho_1 ab - \rho_2 b^2) \lesssim \frac{1}{\sqrt{\log M}} \left( \sqrt{E_4} + \frac{b^2}{|\log b|^2} \right), \\
\rho_2 a_s + \rho_1 (b_s + b^2 + a^2) &+ \frac{2}{|\log b|} (\rho_1 ab + \rho_2 b^2) \lesssim \frac{1}{\sqrt{\log M}} \left( \sqrt{E_4} + \frac{b^2}{|\log b|^2} \right),
\end{align*}
\]

(4.35)

Then (4.34) follows by simple cancellations. For more details, reader can refer to [34]. Here we only compute the first line in (4.35). We first define

\[
U(t) = |a_s| + |b_s + b^2 + a^2| + |a + \Theta_s| + |b + \frac{\lambda_s}{\Lambda}|.
\]

(4.36)

Then taking the inner product of the first component of (4.22) with $H\Phi_M$, we obtain the system

\[
0 = (\partial_s \alpha, H\Phi_M) - \frac{\lambda_s}{\Lambda} (\Lambda \alpha, H\Phi_M)
\]

\[
+ \rho_1 \left( (\hat{J} H w^{(1)}) \cdot H\Phi_M \right) - \rho_2 \left( (\hat{J}^2 H w^{(1)}) \cdot H\Phi_M \right)
\]

\[
+ (R, H\Phi_M) + (\tilde{\text{Mod}}, H\Phi_M) + (\tilde{\Psi}_0, H\Phi_M).
\]

(4.37)

The time derivative term vanishes thanks to (4.4). For the second term, by commuting $H, \Lambda$, we split it into

\[
(\Lambda \alpha, H\Phi_M) = (\Lambda H \alpha, \Phi_M) + 2(H \alpha, \Phi_M) - \left( \frac{AV}{y^2} \alpha, \Phi_M \right). 
\]

(4.38)

Again using (4.4), the second term vanishes. From (A.3), (4.16), we have

\[
| (\Lambda H \alpha, \Phi_M) | \lesssim C(M) \left( \int \frac{|\partial_y H \alpha|^2}{1 + y^2} \right)^{\frac{1}{2}} \lesssim C(M) \sqrt{E_4},
\]

36
where \(C(M)\) is a universal constant appearing in (A.3) induced by the coercivity (A.2). The constants in the following formulas are similar. For the last term in (4.38), an analogous estimate holds, giving the bound
\[
\left| \frac{\lambda_s}{\lambda} \right| |(A\alpha, H\Phi_M)| \lesssim C(M)b\sqrt{E_4}. \tag{4.39}
\]

For the third term in (4.37), from (2.26),
\[
\left| \left( (\hat{J} H w)^{(1)}, H\Phi_M \right) \right| \lesssim \left| \left( (1 + \hat{\gamma}) H^2 \beta, \Phi_M \right) \right|
+ \left| \left( \hat{\beta} \left( -\Delta \gamma + 2(1 + Z) \left( \partial_y + \frac{Z}{y} \right) \alpha \right), H\Phi_M \right) \right|, \tag{4.40}
\]
where by commuting \(H\), we split the first one into
\[
\left| \left( (1 + \hat{\gamma}) H^2 \beta, \Phi_M \right) \right| \lesssim \left| \left( (1 + \hat{\gamma}) H^2 \beta, \Phi_M \right) \right|
+ \left| \left( \partial_y \hat{\gamma} \partial_y H^2 \beta, \Phi_M \right) \right| + \left| \left( \Delta \gamma H^2 \beta, \Phi_M \right) \right|, \tag{4.41}
\]
Using (A.20), (A.22), we have
\[
\left| \left( (1 + \hat{\gamma}) H^2 \beta, \Phi_M \right) \right| \lesssim \|1 + \hat{\gamma}\|_{L\infty} \|H^2 \beta\|_{L^2} \|\Phi_M\|_{L^2} \lesssim \sqrt{\log M} \sqrt{E_4}.
\]

For the remaining, by (A.3), (A.16), there holds the better bound
\[
\left| \left( \partial_y \hat{\gamma} \partial_y H^2 \beta, \Phi_M \right) \right| + \left| \left( \Delta \gamma H^2 \beta, \Phi_M \right) \right| \lesssim C(M) \|\Theta_s ZRw\|_{L^2(y \leq 2M)} + \|\tilde{J} (w \wedge \tilde{H} \tilde{w}_0)\|_{L^2(y \leq 2M)} + \|w \wedge \left( \rho_1 \tilde{H} \tilde{w}_0 - \rho_2 \tilde{J} \tilde{H} \tilde{w}_0 + \Lambda \phi \tilde{p} \right)\|_{L^2(y \leq 2M)}. \tag{4.42}
\]

By (A.3), the phase derivative term is bounded by
\[
\|\Theta_s ZRw\|_{L^2(y \leq 2M)} \lesssim C(M) \left( b + U(t) \right) \sqrt{E_4}.
\]

Similarly, from the construction of $\tilde{w}_0$, the rest of the terms in (4.42) are bounded by

$$\left\| \tilde{J}(w \wedge \mathcal{H}\tilde{w}_0) \right\|_{L^2(y \leq 2M)} + \left\| w \wedge \left( \rho_1 \mathcal{H}\tilde{w}_0 - \rho_2 \tilde{J}\mathcal{H}\tilde{w}_0 + A\phi \hat{p} \right) \right\|_{L^2(y \leq 2M)} \lesssim C(M) \sqrt{\mathcal{E}_4} \left( \left\| \mathcal{H}\tilde{w}_0 \right\|_{L^\infty(y \leq 2M)} + \left\| \rho_1 \mathcal{H}\tilde{w}_0 - \rho_2 \tilde{J}\mathcal{H}\tilde{w}_0 + A\phi \hat{p} \right\|_{L^\infty(y \leq 2M)} \right) \lesssim C(M) (b + U(t)) \sqrt{\mathcal{E}_4},$$

and hence

$$\left| (\mathcal{R}, H\Phi_M) \right| \lesssim C(M)(b + U(t)) \sqrt{\mathcal{E}_4}. \quad (4.43)$$

It remains to consider the contributions of $\widehat{\text{Mod}}$ and $\Psi_0$ in (4.37). From (3.54), we observe on the support of $\Phi_M$, which is $\{y \leq 2M\}$, that $\text{Mod}$ is given by

$$\text{Mod} = a_s \Phi_{1,0} + (b_s + a^2 + b^2) \Phi_{0,1} - \left( b + \frac{\lambda_s}{\chi} \right) A\phi e_x + (a + \Theta_s) A\phi e_y + C(M)b U(t),$$

which together with (4.6) yields that

$$(\widehat{\text{Mod}}^{(1)}, H\Phi_M) = a_s (\Phi_{1,0}^{(1)}, H\Phi_M) + (b_s + a^2 + b^2)(\Phi_{0,1}^{(1)}, H\Phi_M) + C(M)b U(t) = \frac{1}{\rho_1^2 + \rho_2^2} \left[ \rho_1 a_s - \rho_2 (b_s + a^2 + b^2) \right] (A\phi, \Phi_M) + C(M)b U(t). \quad (4.44)$$

Injecting (4.39), (4.40), (4.41), (4.43), (4.44) into (4.37), applying the flux computation (3.15) and also the non-degeneracy (4.6), we obtain

$$\frac{1}{\rho_1^2 + \rho_2^2} \left[ \rho_1 a_s - \rho_2 (b_s + a^2 + b^2) + \frac{2}{\left| \log b \right|} (\rho_1 ab - \rho_2 b^2) \right] = \frac{1}{(A\phi, \Phi_M)} \left[ \left( \sqrt{\log M} + C(M)(b + U(t)) \right) \sqrt{\mathcal{E}_4} + \frac{b^2}{\left| \log b \right|^2} \right],$$

where the term involving $b+U(t)$ is negligible due to the smallness of $b$ and the absorption to the LHS. This yields the desired first line in (4.35). The other two identities can be obtained by similar computations.

**Remark 4.2** The modulation equations (4.34) build up a closed ODE system for the geometrical parameters $(\lambda, \Theta, a, b)$, determining their asymptotic behavior as the rescaled time $s \to +\infty$, see (5.89). However, a direct integration on (4.34) can only gives

$$b(s) \sim \frac{1}{s}, \quad a(s) \sim (\log s)^2 + C_1, \quad \Theta(s) \leq \int_{s_0}^s |a| \, d\sigma + C_2, \quad (4.45)$$

insufficient to ensure the smallness of $a(s)$ and the convergence of the phase (1.9). To overcome this, a refined bound for $a(s)$ will be derived in Section 6.
5 Energy method

In this section, we consider the fourth order energy $E_4$ of $w$, and derive the mixed energy identity/Morawetz estimate, which is the core of our analysis. It actually helps us close the refined bounds in Proposition 4.1. The overall strategy is called the energy method, see [34] for related arguments.

5.1 The energy identity

In this subsection, we study the evolution of $E_4$ by investigating its equivalent $\int |J_\lambda H_\lambda W_2|^2$ (see (5.10)). First, we define the suitable second derivative of $W$:

$$W_2 = \hat{J}_\lambda H_\lambda W.$$  \hfill (5.1)

Recalling (4.25), we let

$$w_2 = \hat{J}H_\lambda w = \lambda^2 W_2$$  \hfill (5.2)

be its equivalent acting on the rescaled variable $(s, y)$. Moreover, to ease the notation, we denote the rotation with respect to $\hat{w}$ by

$$R_{\hat{w}} \hat{w} = \hat{w} \wedge = \hat{W} \wedge,$$  \hfill (5.3)

and as a consequence,

$$\hat{J}_\lambda = R + R_{\hat{w}} = \hat{J}.$$  \hfill (5.4)

Using (5.4), (4.26), we compute the equation for $W_2$:

$$\begin{cases}
\partial_t W_2 = -\rho_1 J_\lambda H_\lambda W_2 + \rho_2 J_\lambda H_\lambda J_\lambda W_2 - \hat{J}_\lambda H_\lambda F + R[\partial_t, H_\lambda] W + Q_1, \\
Q_1 = \partial_t \hat{W} \wedge H_\lambda W + R_{\hat{w}}[\partial_t, H_\lambda] W.
\end{cases}$$  \hfill (5.5)

In consideration of the quadratic nature of the third component, the dominate part of $W$ is actually

$$W^\perp = -R^2 W = [\alpha_\lambda, \beta_\lambda, 0]^T, \quad W^{(3)} = W - W^\perp = \gamma_\lambda \epsilon_2.$$  

Similarly, we decompose $W_2$ into

$$W_2^0 = R H_\lambda W^\perp, \quad W_2^1 = W_2 - W_2^0,$$  \hfill (5.6)

and also $w_2^0 = R H w^\perp, w_2^1 = w_2 - w_2^0$ accordingly. From (4.26), $W^\perp$ satisfies the equation

$$\begin{cases}
\partial_t W^\perp = -\rho_1 W_2^0 + \rho_2 R W_2^0 - F^\perp + Q_2^1, \\
Q_2^1 = \rho_1 R^2 W_2^1 - \rho_2 R^2 R_{\hat{w}} W_2^0 - \rho_2 R^2 J_\lambda W_2^1,
\end{cases}$$  \hfill (5.7)

while $W_2^0$ satisfies the equation

$$\begin{cases}
\partial_t W_2^0 = -\rho_1 R H_\lambda W_2^0 - \rho_2 H_\lambda W_2^0 - R H_\lambda F^\perp + Q_2^2, \\
Q_2^2 = R[\partial_t, H_\lambda] W^\perp + R H_\lambda Q_2^1.
\end{cases}$$  \hfill (5.8)
We define two more functions, whose behaviors play importance roles on the forthcoming energy estimate:
\[
G(t, r) = \frac{b(AV)_\lambda}{\lambda^2 r^2}, \quad L(t, r) = \frac{b(AZ)_\lambda}{\lambda^2 r}.
\] (5.9)
As we shall see later, \( G \) appears in an unsigned quadratic term, and \( L \), containing the information on the structure of \( \Lambda, \Lambda^* \), will be used to obtain the mixed energy identity. We now derive the plain energy identity at the level of \( \mathcal{E}_4 \).

**Lemma 5.1 (Plain energy identity)** Under the above definitions, there holds
\[
\frac{1}{2} \frac{d}{dt} \int |\dot{J}_\lambda \mathbb{H}_\lambda W_2|^2 = -\rho_1 \int \dot{J}_\lambda \mathbb{H}_\lambda W_2 \cdot (\dot{J}_\lambda \mathbb{H}_\lambda)^2 W_2
+ \rho_2 \int \dot{J}_\lambda \mathbb{H}_\lambda W_2 \cdot (\dot{J}_\lambda \mathbb{H}_\lambda)^2 J_\lambda W_2
+ \int \dot{J}_\lambda \mathbb{H}_\lambda W_2 \cdot (\dot{J}_\lambda \mathbb{H}_\lambda) Q_1 - \int \dot{J}_\lambda \mathbb{H}_\lambda W_2 \cdot (\dot{J}_\lambda \mathbb{H}_\lambda)^2 F
+ Q_3 + \int R\mathbb{H}_\lambda W_2^0 \cdot \left[ -\mathbb{H}_\lambda^\perp (GW^\perp) + GRW_2^0 \right],
\] (5.10)
where
\[
Q_3 = -\left( b + \frac{\lambda^4}{\lambda^2} \right) \int R\mathbb{H}_\lambda W_2^0 \cdot \left[ -\mathbb{H}_\lambda^\perp \left( \frac{(AV)_\lambda}{\lambda^2 r^2} W^\perp \right) + \left( \frac{(AV)_\lambda}{\lambda^2 r^2} R W_2^0 \right) \right]
+ \int R\mathbb{H}_\lambda W_2^0 \cdot \left( R\mathbb{H}_\lambda \left[ \partial_t, \mathbb{H}_\lambda \right] \gamma \epsilon \varepsilon + R\mathbb{H}_\lambda \left[ \partial_t, \mathbb{H}_\lambda \right] W \right)
+ \partial_t W^\perp \mathbb{H}_\lambda W_2^0 + R\mathbb{H}_\lambda \left[ \partial_t, \mathbb{H}_\lambda \right] W_2^0 + \left[ \partial_t, \dot{J}_\lambda \mathbb{H}_\lambda \right] W_2^1
+ \int R\mathbb{H}_\lambda W_2^1 \cdot \left( \dot{J}_\lambda \mathbb{H}_\lambda \left[ \partial_t, \mathbb{H}_\lambda \right] W + \left[ \partial_t, \dot{J}_\lambda \mathbb{H}_\lambda \right] W_2 \right)
+ \int R\mathbb{H}_\lambda W_2 \cdot \left( \dot{J}_\lambda \mathbb{H}_\lambda \left[ \partial_t, \mathbb{H}_\lambda \right] W + \left[ \partial_t, \dot{J}_\lambda \mathbb{H}_\lambda \right] W_2 \right).
\] (5.11)

**Remark 5.1** (i) \( Q_3 \) will be proved to be small errors in Proposition 5.1. (ii) The majority of the RHS of (5.10) can be estimated by the interpolation bounds listed in Lemma 5.1 directly, but the last quadratic term is unsigned, and can only be controlled via (5.39), (A.3), (A.29), (A.31), by the following rough bound
\[
\int R\mathbb{H}_\lambda W_2^0 \cdot \left[ -\mathbb{H}_\lambda^\perp (GW^\perp) + GRW_2^0 \right]
\lesssim \frac{b}{\lambda^8} \left( \int |R\mathbb{H}_\lambda w_2^0|^2 \right)^{\frac{1}{2}} \left( \int |H_\lambda^\perp \left( \frac{w_1}{1+y^4} \right) |^2 + \frac{|w_2|^2}{1+y^8} \right)^{\frac{1}{2}} \lesssim \frac{b}{\lambda^8} \left( \mathcal{E}_4 + \frac{b^4}{|\log b|^2} \right).
\] (5.12)
It is of the same size as the leading terms in (5.10) (the first two lines on the RHS), and the implicit constant in (5.12) will deteriorate the forthcoming energy estimate. To treat this, we introduce the Morawetz type formula in the next subsection.
**Proof of Lemma 5.1.**

Injecting (5.5) into the time derivative of \( \int |\bar{J}_\lambda \mathbb{H}_\lambda W_2|^2 \), we obtain an analogue of the RHS of (5.10). The last line of the latter is given by (5.13) as below, different from that of (5.10). Thus it remains to do some algebraic computation. Using (5.4), (5.6), we compute

\[
\begin{align*}
\int \bar{J}_\lambda \mathbb{H}_\lambda W_2 \cdot \left( \bar{J}_\lambda \mathbb{H}_\lambda R[\partial_t, \mathbb{H}_\lambda]W + [\partial_t, \bar{J}_\lambda \mathbb{H}_\lambda]W_2 \right) \\
= \int R \mathbb{H}_\lambda W_2^0 \cdot \left( \bar{J}_\lambda \mathbb{H}_\lambda R[\partial_t, \mathbb{H}_\lambda]W + [\partial_t, \bar{J}_\lambda \mathbb{H}_\lambda]W_2 \right) \\
+ \int R \mathbb{H}_\lambda W_2^1 \cdot \left( \bar{J}_\lambda \mathbb{H}_\lambda R[\partial_t, \mathbb{H}_\lambda]W + [\partial_t, \bar{J}_\lambda \mathbb{H}_\lambda]W_2 \right) \\
+ \int R \mathbb{H}_\lambda W_2 \cdot \left( \bar{J}_\lambda \mathbb{H}_\lambda R[\partial_t, \mathbb{H}_\lambda]W + [\partial_t, \bar{J}_\lambda \mathbb{H}_\lambda]W_2 \right) \\
\end{align*}
\]

(5.13)

The first term on the RHS of (5.13) can be further splited into

\[
\begin{align*}
\int R \mathbb{H}_\lambda W_2^0 \cdot \left( \bar{J}_\lambda \mathbb{H}_\lambda R[\partial_t, \mathbb{H}_\lambda]W + [\partial_t, \bar{J}_\lambda \mathbb{H}_\lambda]W_2 \right) \\
= \int R \mathbb{H}_\lambda W_2^0 \cdot \left( R \mathbb{H}_\lambda R[\partial_t, \mathbb{H}_\lambda]W + [\partial_t, \bar{J}_\lambda \mathbb{H}_\lambda]W_2^0 \right) \\
+ \int R \mathbb{H}_\lambda W_2^0 \cdot \left( R \mathbb{H}_\lambda R[\partial_t, \mathbb{H}_\lambda]W + [\partial_t, \bar{J}_\lambda \mathbb{H}_\lambda]W_2^1 \right), \\
\end{align*}
\]

(5.14)

where the involved commutators are

\[
[\partial_t, \bar{J}_\lambda \mathbb{H}_\lambda]W = R[\partial_t, \mathbb{H}_\lambda]W + \partial_t \bar{W} \wedge \mathbb{H}_\lambda W + R \mathbb{H}_\lambda^* [\partial_t, \mathbb{H}_\lambda]W,
\]

and

\[
[\partial_t, \mathbb{H}_\lambda]W = \frac{\lambda_s}{\lambda^3} \left\{ \left( \frac{(AV)\lambda}{r^2} \alpha_\lambda - \frac{2}{\lambda} \left( 1 + Z + AZ \right)_\lambda \partial_t \gamma \right) e_x + \left[ \frac{(AV)\lambda}{r^2} \beta_\lambda \right] e_y \\
+ \left[ \frac{2}{\lambda} \left( 1 + Z + AZ \right)_\lambda \partial_t \alpha_\lambda + \frac{2}{\lambda^2} (Z + Z^2 + AZ(1 + 2Z))_\lambda \alpha \right] e_z \right\},
\]

(5.15)

From (4.33), (5.15), we have

\[
\begin{align*}
R \mathbb{H}_\lambda R[\partial_t, \mathbb{H}_\lambda]W_2^0 = \frac{\lambda_s}{\lambda} \mathbb{H}_\lambda^+ \left( \frac{(AV)\lambda}{r^2} W_2^0 \right) \approx -\mathbb{H}_\lambda^+ (GW_2^0), \\
R [\partial_t, \mathbb{H}_\lambda]W_2^0 = -\frac{\lambda_s}{\lambda} \frac{(AV)\lambda}{r^2} RW_2^0 \approx GRW_2^0.
\end{align*}
\]

(5.16)
Using these, we rewrite the first term on the RHS of (5.14) as
\[ \int \mathcal{R}_\lambda W^0 \cdot \left( \mathcal{R}_\lambda R \left[ \partial_t, \mathcal{H}_\lambda \right] W + \left[ \partial_t, \mathcal{J}_\lambda \right] W^0 \right) \]
\[ = \int \mathcal{R}_\lambda W^0 \cdot \left[ - \mathcal{H}_\lambda (GW^\perp) + GRW^0 \right] \]
\[ - \left( b + \frac{\lambda_s}{\lambda} \right) \int \mathcal{R}_\lambda W^0 \cdot \left[ - \mathcal{H}_\lambda \left( \frac{(AV)_\lambda}{\lambda^2 r^2} W^\perp \right) + \frac{(AV)_\lambda}{\lambda^2 r^2} RW^0_2 \right] \]
\[ + \int \mathcal{R}_\lambda W^0 \cdot \left( \mathcal{R}_\lambda R \left[ \partial_t, \mathcal{H}_\lambda \right] \gamma e_z + \partial_t \hat{e} \hat{W} \wedge \mathcal{H}_2 W^0 \right). \] (5.17)

Inserting this into (5.14) yields the last line of (5.10). By collecting the rest, we obtain (5.11), and thus complete the proof. □

5.2 Morewatz correction

The aim of this subsection is to modify the plain energy (5.10), by adding an extra Morawetz term to the energy functional. The resulting formula is called the mixed energy/Morawetz identity, which, containing no uncontrollable terms, is appropriate to close the bootstrap. It is notable that the newly-added Morawetz term heavily rely on the coefficients \( \rho_1, \rho_2 \) in (1.1).

Indeed, to characterize the relative size of the absolute values of \( \rho_1, \rho_2 \), we define their ratio by
\[ \rho = \left( \frac{\rho_1}{\rho_2} \right)^2. \]

Recall that \( \rho_1 \in \mathbb{R} \) and \( \rho_2 > 0 \), we see \( \rho \) is well-defined and takes value in \([0, +\infty)\). We further introduce the following conditional parameters depending on \( \rho \):
\[ k_1(\rho) = \begin{cases} 1, & \rho \geq 1, \\ 0, & \rho < 1, \end{cases} \]
\[ k_2(\rho) = 1 - k_1(\rho), \quad \Delta k(\rho) = k_1(\rho) - k_2(\rho). \] (5.18)

It is easy to see \( k_2 \in \{0, 1\} \) and \( \Delta k \in \{-1, 1\} \). Now we introduce the extra term, and derive the Morawetz type formula. The results are collected in the following lemma.

**Lemma 5.2 (Morawetz correction)** The desired Morawetz term is
\[ M(t) = c_1 \int \mathcal{H}_\lambda W^0_2 \cdot GW^\perp + c_2 \int R\mathcal{H}_\lambda W^0_2 \cdot LW^0_2 \]
\[ + c_3 \int \mathcal{R}\mathcal{H}_\lambda W^0_2 \cdot GW^\perp - c_4 \int \mathcal{H}_\lambda W^0_2 \cdot LW^0_2, \] (5.19)

where the coefficient in front of the integrals are
\[ (c_1, c_2, c_3, c_4) = \left( \frac{\rho_1}{\rho_1^2 + \rho_2^2}, \frac{2\rho_1(\rho_1^2 - \rho_2^2)}{(\rho_1^2 + \rho_2^2)(\Delta k^2 + \rho_2^2)}, \frac{\rho_2}{\rho_1^2 + \rho_2^2}, \frac{2\rho_1^2\rho_2(1 + \Delta k)}{(\rho_1^2 + \rho_2^2)(\Delta k^2 + \rho_2^2)} \right). \] (5.20)
Moreover, there holds the boundedness
\[ |M(t)| \lesssim \frac{\delta(b^*)}{\lambda^8} \left( E_4 + \frac{b^4}{|\log b|^2} \right). \] (5.21)

Its time derivative (Morawetz type identity) is given by
\[ \frac{d}{dt} M(t) = \int RH_{\lambda} W_2^0 \cdot \left[ -H_{\lambda}^1 (GW^\perp) + GRW_2^0 \right] + b^4 \frac{E_4}{\sqrt{\log M}} + \frac{b^4}{|\log b|^2} \] (5.22)
\[ -2\rho_1 c_2 k_1 \int H_{\lambda} W_2^0 \cdot L A_{\lambda} W_2^0 + 2\rho_1 c_2 k_2 \int A_{\lambda} H_{\lambda} W_2^0 \cdot L W_2^0. \]

**Remark 5.2**
(i) The first term on the RHS of (5.22) is just the uncontrollable quadratic term (5.12). (ii) The coefficients \( c_i \) in (5.20) are well-defined. Indeed, by the positivity of the Gilbert damping \( \rho_2 > 0 \), we see \( \rho_1^2 + \rho_2^2 > 0 \). Moreover, the vanishing \( \Delta k \rho_1^2 + \rho_2^2 = 0 \) implies \( \Delta k = -1 \) and \( \rho_1^2 = \rho_2^2 \), from which \( \rho = \rho_1^2 / \rho_2^2 = 1 \), and thus \( \Delta k = 1 \), contradiction. This ensures the denominators in the definitions of \( c_i \) are non-zero.

Assume Lemma 5.2 holds for now. We subtract (5.22) from the plain energy (5.10), and obtain the following mixed energy/Morawetz identity
\[ \frac{d}{dt} \left\{ \frac{1}{2} \int |J_{\lambda} H_{\lambda} W_2|^2 - M(t) \right\} \]
\[ = -\rho_1 \int J_{\lambda} H_{\lambda} W_2 \cdot (J_{\lambda} H_{\lambda})^2 W_2 + \rho_2 \int J_{\lambda} H_{\lambda} W_2 \cdot (J_{\lambda} H_{\lambda})^2 J_{\lambda} W_2 + 2\rho_1 c_2 k_1 \int H_{\lambda} W_2^0 \cdot L A_{\lambda} W_2^0 - 2\rho_1 c_2 k_2 \int A_{\lambda} H_{\lambda} W_2^0 \cdot L W_2^0 \] (5.23)
\[ + Q_3 + \int J_{\lambda} H_{\lambda} W_2 \cdot (J_{\lambda} H_{\lambda}) Q_1 - \int J_{\lambda} H_{\lambda} W_2 \cdot (J_{\lambda} H_{\lambda})^2 F + b^4 \frac{E_4}{\sqrt{\log M}} + \frac{b^4}{|\log b|^2} \].

Note that the uncontrollable term has been cancelled. The identity (5.23) is the key to derive the mixed energy/Morawetz estimate. Now before giving a proof for Lemma 5.2, we introduce the following brief lemma on the structure of \( A, A^* \), and their relation with the functions \( G, L \).

**Lemma 5.3** (**Structure of \( A, A^* \)) There holds the identities
\[ \int A_{\lambda} H_{\lambda} W_2^0 \cdot L W_2^0 + \int H_{\lambda} W_2^0 \cdot L A_{\lambda} W_2^0 = \int H_{\lambda} W_2^0 \cdot GW_2^0, \] (5.24)
\[ \int R A_{\lambda} H_{\lambda} W_2^0 \cdot L W_2^0 = \int R H_{\lambda} W_2^0 \cdot GW_2^0, \] (5.25)
\[ \int R H_{\lambda} W_2^0 \cdot L A_{\lambda} W_2^0 = 0. \] (5.26)
and the non-positivity
\[ \int \mathbb{H}_\lambda W_2^0 \cdot L \mathbb{A}_\lambda W_2^0 \leq 0. \]  
(5.27)

**Proof of Lemma 5.3.**

Recalling the function \( V(2.18) \), we see \( \mathcal{A}V = \lambda (AZ + Z^2) \). Applying (4.31), we compute for any radially symmetric function \( f(r) \) with \( r = \lambda y \) that
\[
\mathcal{A}_\lambda^*(Lf) + L \mathcal{A}_\lambda f = \left( \partial_r L + \frac{2Z + 1}{r} L \right) f
= \frac{b}{\lambda^4} \left[ \partial_y \left( \frac{AZ}{y} \right) + \frac{(2Z + 1)AZ}{y^2} \right] f
= \frac{b(A(AZ + Z^2))(y)}{\lambda^4 y^2} f = Gf.
\]
(5.28)

This together with the commutativity \( \mathcal{A}_\lambda R = R \mathcal{A}_\lambda \) leads to
\[
\int \mathcal{A}_\lambda \mathbb{H}_\lambda W_2^0 \cdot LW_2^0 = - \int \mathbb{H}_\lambda W_2^0 \cdot L \mathcal{A}_\lambda W_2^0 + \int \mathbb{H}_\lambda W_2^0 \cdot GW_2^0, 
\]
(5.29)
\[
\int R \mathcal{A}_\lambda \mathbb{H}_\lambda W_2^0 \cdot LW_2^0 = - \int R \mathbb{H}_\lambda W_2^0 \cdot L \mathcal{A}_\lambda W_2^0 + \int R \mathbb{H}_\lambda W_2^0 \cdot GW_2^0, 
\]
(5.30)

where (5.29) is just (5.24). Moreover, by a similar computation on \( \mathcal{A}_\lambda (Lf) + L \mathcal{A}_\lambda^* f \) as (5.28), and the fact \( L(t, r) < 0 \) for any \( r > 0 \), we have
\[
\int \mathbb{H}_\lambda W_2^0 \cdot L \mathcal{A}_\lambda W_2^0 = \int \mathcal{A}_\lambda W_2^0 \cdot \mathcal{A}_\lambda (L \mathcal{A}_\lambda W_2^0)
= \int \mathcal{A}_\lambda W_2^0 \left[ - \mathcal{A}_\lambda^* \mathcal{A}_\lambda W_2^0 + \left( \frac{2Z + 1}{r} L - \partial_r L \right) \mathcal{A}_\lambda W_2^0 \right]
= \int \mathcal{A}_\lambda W_2^0 \left[ - L \mathcal{H}_\lambda W_2^0 + \frac{2L}{r} \mathcal{A}_\lambda W_2^0 \right] = \int \frac{L}{r} | \mathcal{A}_\lambda W_2^0 |^2 \leq 0,
\]
(5.31)

which is the non-positivity (5.27). Note here we have used \( \mathbb{H}_\lambda W_2^0 \cdot X = \mathbb{H}_\lambda^* W_2^0 \cdot X \), for any vector \( X \) under Frenet basis with the third component \( X^{(3)} = 0 \). An analogous argument can be formulated to show the vanishing
\[
\int R \mathbb{H}_\lambda W_2^0 \cdot L \mathcal{A}_\lambda W_2^0 = \int \frac{L}{r} R \mathcal{A}_\lambda W_2^0 \cdot \mathcal{A}_\lambda W_2^0 = 0,
\]
which is (5.26). Combining this with (5.30) yields (5.25). This completes the proof. □

The structural bonus illustrated in Lemma 5.3 will be found essential for the following analysis. We now use it to prove Lemma 5.2.

**Proof of Lemma 5.2.**

We first show the boundedness (5.21), and then prove (5.22) by computing the time derivatives of each term in (5.19), formulating the uncontrollable term, and finally estimating the errors via Appendix A.
Step 1. Control of $\mathbf{M}(t)$. From the lossy logarithmic bound (A.5), we have

$$|\mathbf{M}(t)| \lesssim \frac{b}{\lambda^6} \left( \int |\mathbb{H}w_2^0| \frac{|w^+|}{1 + y^2} + \int |A_w^0| \frac{|w^0_2|}{1 + y^2} \right)$$

$$\lesssim \frac{b}{\lambda^6} \left( \int |\mathbb{H}w_2^0|^2 + \int \frac{|A_w^0|^2}{y^2(1 + y^2)} \right)^{\frac{1}{2}} \left( \int \frac{|w^+|^2}{1 + y^8} + \int \frac{1 + |\log y|^2}{1 + y^4} |w^0_2|^2 \right)^{\frac{1}{2}}$$

$$\lesssim \frac{b}{\lambda^6} |\log b|^C \left( \mathcal{E}_4 + \frac{b^4}{|\log b|^2} \right) \lesssim \delta(b^*) \left( \mathcal{E}_4 + \frac{b^4}{|\log b|^2} \right).$$

Step 2. Computations on time derivatives. For the first integral in (5.19), we apply the equations (5.7), (5.8) to compute

$$\partial_t \int \mathbb{H}_\lambda W_2^0 \cdot GW^\perp = \partial_t \int W_2^0 \cdot \mathbb{H}_\lambda (GW^\perp)$$

$$= -\rho_1 \int \mathbb{H}_\lambda W_2^0 \cdot \mathbb{H}_\lambda (GW^\perp) - \rho_2 \int \mathbb{H}_\lambda W_2^0 \cdot \mathbb{H}_\lambda (GW^\perp)$$

$$- \rho_1 \int \mathbb{H}_\lambda W_2^0 \cdot GW_2^0 - \rho_2 \int \mathbb{H}_\lambda W_2^0 \cdot GW_2^0 + \int W_2^0 \cdot [\partial_t, \mathbb{H}_\lambda] (GW^\perp) \quad (5.32)$$

$$+ \int \mathbb{H}_\lambda W_2^0 \cdot (\partial_t G) W^\perp + \int \mathbb{H}_\lambda Q_2^1 \cdot GW^\perp + \int \mathbb{H}_\lambda W_2^0 \cdot G Q_2^1$$

$$- \int \mathbb{H}_\lambda F^\perp \cdot \mathbb{H}_\lambda (GW^\perp) - \int \mathbb{H}_\lambda W_2^0 \cdot GF^\perp.$$
from (5.7), (5.8) give

\[
\partial_t \int R A_\lambda W^0_2 \cdot L W^0_2 = \rho_1 \left( \int A_\lambda \mathbb{H}_\lambda W^0_2 \cdot L W^0_2 - \int \mathbb{H}_\lambda W^0_2 \cdot L A_\lambda W^0_2 \right) - \rho_2 \int R \mathbb{H}_\lambda W^0_2 \cdot G W^0_2
\]

\[
= \rho_1 \left( \int A_\lambda \mathbb{H}_\lambda W^0_2 \cdot L W^0_2 - \int \mathbb{H}_\lambda W^0_2 \cdot L A_\lambda W^0_2 \right) - \rho_2 \int R \mathbb{H}_\lambda W^0_2 \cdot G W^0_2 + 2 \rho_1 k_2 \int A_\lambda \mathbb{H}_\lambda W^0_2 \cdot L W^0_2 - \int A_\lambda \mathbb{H}_\lambda F \cdot L W^0_2 - \int A_\lambda \mathbb{H}_\lambda W^0_2 \cdot L Q^2 \].

(5.34)

where from the vanishing (5.26) and the identity (5.25), the first two lines on the RHS of (5.34) are actually

\[
\rho_1 \left( \int A_\lambda \mathbb{H}_\lambda W^0_2 \cdot L W^0_2 - \int \mathbb{H}_\lambda W^0_2 \cdot L A_\lambda W^0_2 \right) - \rho_2 \int R \mathbb{H}_\lambda W^0_2 \cdot G W^0_2
\]

We rewrite the expression in the big parenthesis by \(1 = k_1 + k_2\), separating it into two identical formulas, and then apply (5.24) twice on each of them:

\[
\rho_1 \left( \int A_\lambda \mathbb{H}_\lambda W^0_2 \cdot L W^0_2 - \int \mathbb{H}_\lambda W^0_2 \cdot L A_\lambda W^0_2 \right) = \rho_1 k_1 \left( \int \mathbb{H}_\lambda W^0_2 \cdot G W^0_2 - 2 \int \mathbb{H}_\lambda W^0_2 \cdot L A_\lambda W^0_2 \right)
\]

\[
+ \rho_1 k_2 \left( - \int \mathbb{H}_\lambda W^0_2 \cdot G W^0_2 + 2 \int A_\lambda \mathbb{H}_\lambda W^0_2 \cdot L W^0_2 \right).
\]

Now by \(\Delta k = k_1 - k_2\), we gather the above integrals to get

\[
\partial_t \int R A_\lambda W^0_2 \cdot L W^0_2 = \Delta k \rho_1 \int \mathbb{H}_\lambda W^0_2 \cdot G W^0_2 - \rho_2 \int R \mathbb{H}_\lambda W^0_2 \cdot G W^0_2 + 2 \rho_1 k_2 \int A_\lambda \mathbb{H}_\lambda W^0_2 \cdot L W^0_2 - 2 \rho_1 k_1 \int \mathbb{H}_\lambda W^0_2 \cdot L A_\lambda W^0_2 + \int R [\partial_t, A_\lambda] W^0_2 \cdot L W^0_2 \]

\[
+ \int A_\lambda \mathbb{H}_\lambda W^0_2 \cdot (\partial_t L) W^0_2 + \int A_\lambda Q^2 \cdot L W^0_2 + \int A_\lambda W^0_2 \cdot L Q^2 \]

\[
+ \int A_\lambda \mathbb{H}_\lambda F \cdot L W^0_2 - \int \mathbb{H}_\lambda F \cdot L A_\lambda W^0_2.
\]

(5.35)
Finally, for the last term of (5.19), we repeat the analysis done for (5.35) using Lemma 5.3 to obtain

$$\partial_t \int A_\lambda W_2^0 \cdot LW_2^0 = -\rho_1 \int R\mathbb{H}_\lambda W_2^0 \cdot GW_2^0 - \rho_2 \int \mathbb{H}_\lambda W_2^0 \cdot GW_2^0$$

$$+ \int [\partial_t, A_\lambda] W_2^0 \cdot LW_2^0 + \int A_\lambda W_2^0 \cdot (\partial_t L) W_2^0 + \int A_\lambda Q_2^0 \cdot LW_2^0$$

$$+ \int A_\lambda W_2^0 \cdot LQ_2^0 - \int R\mathbb{A}_\lambda \mathbb{H}_\lambda F^\perp \cdot LW_2^0 - \int R\mathbb{H}_\lambda F^\perp \cdot LA_\lambda W_2^0.$$  \hspace{1cm} (5.36)

Step 3. Numerology on the coefficients. Injecting (5.32), (5.33), (5.35), (5.36) into (5.19), we obtain the formula

$$\frac{d}{dt} M(t) = C_1 \int R\mathbb{H}_\lambda W_2^0 \cdot \mathbb{H}_\lambda^\perp (GW^\perp) + C_2 \int \mathbb{H}_\lambda W_2^0 \cdot \mathbb{H}_\lambda^\perp (GW^\perp)$$

$$+ C_3 \int R\mathbb{H}_\lambda W_2^0 \cdot GW_2^0 + C_4 \int \mathbb{H}_\lambda W_2^0 \cdot GW_2^0$$

$$- 2\rho_1 c_1 k_1 \int \mathbb{H}_\lambda W_2^0 \cdot LA_\lambda W_2^0 + 2\rho_1 c_2 k_2 \int A_\lambda \mathbb{H}_\lambda W_2^0 \cdot LW_2^0$$

$$+ Q_4 + Q_5 + Q_F,$$

where the coefficients $C_i$ for $1 \leq i \leq 4$ are given by

$$\begin{cases} C_1 = -(\rho_1 c_1 + \rho_2 c_3) = -1, \\ C_2 = \rho_1 c_3 - \rho_2 c_1 = 0, \\ C_3 = -\rho_1 c_3 + \rho_1 c_4 - \rho_2 c_1 - \rho_2 c_2 = 0, \\ C_4 = -\rho_1 c_1 + \rho_1 c_2 \Delta k + \rho_2 c_3 + \rho_2 c_4 = 1. \end{cases}$$

In view of the first two lines of (5.37), this formulates the desired uncontrollable term, thanks to the very choice of $c_i$ (5.20). Moreover, the $Q$ terms on the last line of (5.37) are given explicitly by (5.38), (5.42), (5.47). We estimate them respectively in the rest of this proof.

Step 4. Estimating $Q_4$. By above computation, the $Q_4$ term is given by

$$Q_4 = c_1 \int W_2^0 \cdot [\partial_t, \mathbb{H}_\lambda] (GW^\perp) + c_3 \int R W_2^0 \cdot [\partial_t, \mathbb{H}_\lambda] (GW^\perp)$$

$$+ c_2 \int R [\partial_t, A_\lambda] W_2^0 \cdot LW_2^0 - c_4 \int [\partial_t, A_\lambda] W_2^0 \cdot LW_2^0$$

$$+ c_1 \int \mathbb{H}_\lambda W_2^0 \cdot (\partial_t G) W^\perp + c_3 \int R\mathbb{H}_\lambda W_2^0 \cdot (\partial_t G) W^\perp$$

$$+ c_2 \int R\mathbb{A}_\lambda W_2^0 \cdot (\partial_t L) W_2^0 - c_4 \int A_\lambda W_2^0 \cdot (\partial_t L) W_2^0.$$  \hspace{1cm} (5.38)
which contains the commutators and $\partial_t G, \partial_t L$. From the explicit formulas of $Z, V$, there holds

$$
\left| \frac{\Lambda Z}{y} \right| \lesssim \frac{1}{1+y^3}, \quad \left| \frac{\Lambda V}{y^2} \right| \lesssim \frac{1}{1+y^4},
$$

which together with (5.16) implies

$$
\left| \Lambda Z \right| \lesssim 1 + y^3, \quad \left| \Lambda V \right| \lesssim 1 + y^4,
$$

(5.39)

which together with (5.16) implies

$$
\begin{cases}
R[\partial_t, \mathbb{H}_\lambda](GW^\perp) = \frac{b^2}{\lambda^8} O\left(\frac{1}{1+y^8}\right) R^\perp \\
L[\partial_t, A_\lambda] W_2^0 = \frac{b^2}{\lambda^8} O\left(\frac{1}{1+y^8}\right) w_2^0.
\end{cases}
$$

(5.40)

Using these with (A.3), (A.29), we can estimate the first term in (5.38) by Cauchy-Schwartz

$$
\left| \int W_2^0 \cdot \left[ \partial_t, \mathbb{H}_\lambda \right] (GW^\perp) \right| \lesssim \frac{b^2}{\lambda^8} \int \left| w_2^0 \right| \frac{1}{1+y^4} \cdot \frac{1}{1+y^4}
$$

$$
\lesssim \frac{b^2}{\lambda^8} \left( \int \frac{|w_2^0|^2}{(1+y^4)(1+|\log y|^2)} \right)^{\frac{1}{2}} \left( \int \frac{|w_{-1}|^2}{(1+y^8)(1+|\log y|^2)} \right)^{\frac{1}{2}}
$$

$$
\lesssim \frac{b \delta(b^*)}{\lambda^8} \left( \mathcal{E}_4 + \frac{b^4}{|\log b|^2} \right).
$$

The extra $b^2$ given by (5.40) ensures the same bounds for the other terms on the first and second lines of (5.38). For the rest two lines, by the definition (5.9) we see

$$
\begin{cases}
\partial_t G = \frac{b^2}{\lambda^8} r^2 \left( O(\Lambda V) + O(A^2 V) \right)(y) = \frac{b^2}{\lambda^6} O\left(\frac{1}{1+y^4}\right), \\
\partial_t L = \frac{b^2}{\lambda^8} r \left( O(\Lambda Z) + O(A^2 Z) \right)(y) = \frac{b^2}{\lambda^6} O\left(\frac{1}{1+y^3}\right),
\end{cases}
$$

which also bring $b^2$ as the above one, so the estimates are very similar. Applying the interpolation bounds in Lemma A few times more, we obtain

$$
|\mathcal{Q}_4| \lesssim \frac{b \delta(b^*)}{\lambda^8} \left( \mathcal{E}_4 + \frac{b^4}{|\log b|^2} \right).
$$

(5.41)

Step 5. Estimating $\mathcal{Q}_5$. We consider the $\mathcal{Q}_5$ term

$$
\mathcal{Q}_5 = c_1 \int \mathbb{H}_\lambda W_2^0 \cdot G \mathcal{Q}_2^1 + c_3 \int R \mathbb{H}_\lambda W_2^0 \cdot G \mathcal{Q}_2^1 + c_2 \int R A_\lambda \mathcal{Q}_2^2 \cdot LW_2^0 \\
- c_4 \int A_\lambda \mathcal{Q}_2^2 \cdot LW_2^0 + c_1 \int \mathbb{H}_\lambda \mathcal{Q}_2^2 \cdot GW^\perp + c_3 \int R \mathbb{H}_\lambda \mathcal{Q}_2^2 \cdot GW^\perp \\
+ c_2 \int R A_\lambda W_2^0 \cdot L \mathcal{Q}_2^2 - c_4 \int A_\lambda W_2^0 \cdot L \mathcal{Q}_2^2,
$$

(5.42)
Observing from the definition (5.7), (5.8) and the commutators (5.16), we have the brute identities
\[
\begin{align*}
\mathcal{Q}_2^1 &= \frac{1}{\lambda^2} \left( O(w^2_2) + O(\dot{w} \wedge w^0_2) \right), \\
\mathcal{Q}_2^2 &= \frac{1}{\lambda^3} \left[ O\left( \frac{b}{1 + y^4} \right) R w^\perp + O(\mathbb{H} w^1_2) + O\left( \mathbb{H}(\dot{w} \wedge w^0_2) \right) \right].
\end{align*}
\] (5.43)

From (A.8), (A.20), there holds the smallness of \( \dot{w} \):
\[
\| \dot{w} \|_{L^\infty} \leq \| \dot{w}_0 \|_{L^\infty} + \| w^\perp \|_{L^\infty} + \| \gamma \|_{L^\infty} \lesssim \delta(b^*) \tag{5.44}
\]

Using this and the estimates on \( w^1_2 \) (A.30), we have
\[
\int \mathbb{H}_\lambda W^0_2 \cdot G \mathcal{Q}_2^1 \lesssim \frac{b}{\lambda^8} \int \mathbb{H} w^0_2 \left( \frac{1}{1 + y^4} |w^2_2| + |\dot{w} \wedge w^0_2| \right) \\
\lesssim b \delta(b^*) \lambda^8 \left( \mathcal{E}_4 + \frac{b^4}{\log b^4} \right) \tag{5.45}
\]

For the rest terms involving \( \mathcal{Q}_2^2 \) in (5.42), we estimate last term on the first line for instance. From (5.28), there holds
\[
\int R \mathcal{A}_\lambda \mathcal{Q}_2^2 \cdot LW^0_2 = \int R \mathcal{Q}_2^2 \cdot \mathcal{A}_\lambda^* (LW^0_2) = - \int R \mathcal{Q}_2^2 \cdot L \mathcal{A}_\lambda W^0_2 + \int R \mathcal{Q}_2^2 \cdot GW^0_2,
\]
and thus it can be controlled using (5.43), (A.3), (A.29), (A.33), we obtain
\[
\int \left| R \mathcal{A}_\lambda \mathcal{Q}_2^2 \cdot LW^0_2 \right| \lesssim \int |R \mathcal{Q}_2^2| \cdot \left( |L \mathcal{A}_\lambda W^0_2| + |GW^0_2| \right) \\
\lesssim \frac{b}{\lambda^8} \int \left( \frac{|b| w^\perp}{1 + y^4} + |\mathbb{H} w^1_2| + \left| \mathbb{H}(\dot{w} \wedge w^0_2) \right| \right) \cdot \left( \frac{|\mathcal{A}_\lambda w^0_2|}{1 + y^4} + \frac{|w^0_2|}{1 + y^4} \right) \\
\lesssim \frac{b \delta(b^*)}{\lambda^8} \left( \mathcal{E}_4 + \frac{b^4}{\log b^4} \right).
\]
The terms involving \( \mathbb{H}_\lambda \mathcal{Q}_2^2 \) can be estimated using the self-adjointness of \( \mathbb{H}_\lambda \) and Lemma A. Therefore we obtain
\[
|\mathcal{Q}_3| \lesssim \frac{b \delta(b^*)}{\lambda^8} \left( \mathcal{E}_4 + \frac{b^4}{\log b^4} \right) \tag{5.46}
\]

**Step 6. Estimating \( \mathcal{Q}_F \).** The \( \mathcal{Q}_F \) term is given by
\[
\mathcal{Q}_F = - c_1 \int R \mathbb{H}_\lambda F^\perp \cdot \mathbb{H}_\lambda (GW^\perp) + c_3 \int \mathbb{H}_\lambda F^\perp \cdot \mathbb{H}_\lambda (GW^\perp) \\
+ c_2 \int \mathcal{A}_\lambda \mathbb{H}_\lambda F^\perp \cdot LW^0_2 + c_4 \int R \mathcal{A}_\lambda \mathbb{H}_\lambda F^\perp \cdot LW^0_2 \\
- c_1 \int \mathbb{H}_\lambda W^0_2 \cdot GF^\perp - c_3 \int R \mathbb{H}_\lambda W^0_2 \cdot GF^\perp \\
- c_2 \int \mathbb{H}_\lambda F^\perp \cdot L \mathcal{A}_\lambda W^0_2 - c_4 \int R \mathbb{H}_\lambda F^\perp \cdot L \mathcal{A}_\lambda W^0_2. \tag{5.47}
\]
We claim the estimate:

\[
\int \frac{|f^\perp|^2}{1 + y^8} + \int \frac{1 + |\log y|^2 |\mathbb{H} f^\perp|^2}{1 + y^4} + \int \frac{1 + |\log y|^2 |\mathbb{A} \mathbb{H} f^\perp|^2}{1 + y^2} \lesssim \frac{E_4}{\log M} + \frac{b^4}{|\log b|^2},
\]

and then from (5.39), and (A.3), (A.29), (A.31), it follows that

\[
|Q_F| \lesssim \frac{b}{\lambda^8} \left( \frac{1}{(1 + y^8)(1 + |\log y|^2)} + \frac{1}{(1 + y^4)(1 + |\log y|^2)} \right)
\]

\[
+ \int |w_0|^2 + \int \frac{|\mathbb{A} w_0|^2}{(1 + y^2)(1 + |\log y|^2)} \left( \frac{E_4}{\sqrt{\log M}} + \frac{b^4}{|\log b|^2} \right)
\]

\[
\lesssim \frac{b}{\lambda^8} \left( \frac{E_4}{\sqrt{\log M}} + \frac{b^4}{|\log b|^2} \right),
\]

which is the desired bound for \(Q_F\). This together with (5.41), (5.46) concludes the proof. Recall (4.23), \(f\) consists of three separate terms, namely \(\tilde{\Psi}_0\), Mod, \(\mathcal{R}(t)\), and thus it remains to prove (5.48) for each one of them. The estimates for \(\tilde{\Psi}_0\) follow directly from (3.55), (3.58), (3.59), and we are left to handle the other two.

**Step 7. Contribution of \(\tilde{\text{Mod}}\).** From (4.34), we see

\[
U(t) \lesssim \frac{1}{\sqrt{\log M}} \left( \frac{E_4}{\sqrt{\log M}} + \frac{b^4}{|\log b|^2} \right),
\]

which together with (3.8), (3.54) yields

\[
\tilde{\text{Mod}}^\perp = U(t) O \left( \frac{\tilde{w}_0}{b} + by^5 1_{y \leq 1} + \frac{by^3}{|\log b|} 1_{1 \leq y \leq 6B_0} + \frac{1}{y} 1_{y \geq B_1} \right).
\]

and also

\[
\mathbb{H} \tilde{\text{Mod}} = \chi_{B_1} \left( \frac{a_s}{\rho_1^2 + \rho_2^2} \begin{bmatrix} \rho_1 \\ \rho_2 \\ 0 \end{bmatrix} \Lambda \phi + \frac{b_s + b^2 + a^2}{\rho_1^2 + \rho_2^2} \begin{bmatrix} -\rho_2 \\ \rho_1 \\ 0 \end{bmatrix} \Lambda \phi \right)
\]

\[
+ U(t) O \left( by^3 1_{y \leq 1} + \frac{by}{|\log b|} 1_{1 \leq y \leq 6B_0} + \frac{\log y}{y} 1_{1 \leq y \leq 2B_1} + \frac{1}{y^3} 1_{y \geq B_1} \right)
\]

\[
+ U(t) O \left( 1_{y \leq 1} + \frac{\log y}{y^2} 1_{1 \leq y \leq 2B_1} \right) \mathbb{E} \zeta.
\]

Therefore we have

\[
\int \frac{\tilde{\text{Mod}}^\perp|^2}{1 + y^3} + \int \frac{1 + |\log y|^2 |\mathbb{H} \tilde{\text{Mod}}|^2}{1 + y^4} \lesssim \frac{E_4}{\log M} + \frac{b^4}{|\log b|^2}.
\]
Moreover, the cancellation

\[ AHT_1 = A\Lambda \phi = 0 \]

makes the first profiles in \( \tilde{w}_0/b \) vanish, implying the refined bound

\[
\int \frac{1 + |\log y|^2}{1 + y^2} |\Lambda \overline{\text{Mod}_1}|^2 \lesssim \delta(b^*) \left( \mathcal{E}_4 + \frac{b^4}{|\log b|^2} \right),
\]

which concludes (5.48) for \( \overline{\text{Mod}} \).

**Step 8. Contribution of \( \mathcal{R} \).** By the definition (4.24), we separate \( \mathcal{R} \) into

\[
\mathcal{R} = \mathcal{R}_1 + \mathcal{R}_2, \quad \text{where}\quad \begin{cases} 
\mathcal{R}_1 = w \wedge \left( \rho_1 \mathbb{H} \tilde{w}_0 - \rho_2 \mathbb{H} \tilde{w}_0 + \Lambda \phi \hat{p} \right), \\
\mathcal{R}_2 = - \rho_2 \hat{J} \left( w \wedge \mathbb{H} \tilde{w}_0 \right) - \rho_2 w \wedge \left( \tilde{w}_0 \wedge \mathbb{H} \tilde{w}_0 \right) - \rho_2 w \wedge \left( w \wedge \mathbb{H} \tilde{w}_0 \right) + \Theta \hat{z} \Lambda \phi \theta w.
\end{cases}
\]

By the given approximate solution (3.4), (3.5), (3.18), we compute

\[
\rho_1 \mathbb{H} \tilde{w}_0 - \rho_2 \mathbb{H} \tilde{w}_0 + \Lambda \phi \hat{p}
= \chi_{B_1} \left( \rho_1 - \rho_2 \right) \mathbb{H} w_0 + \left[ \frac{a + \Theta_s}{b + \frac{A}{x}} \right] \Lambda \phi \\
+ \left( -2 \partial_y \chi_{B_1} \partial_y w_0 - \Delta \chi_{B_1} w_0 - 2(1 + Z) \partial_y \chi_{B_1} (e_y \wedge w_0) + (1 - \chi_{B_1}) \hat{p} \Lambda \phi \right)
= O \left( b^2 y^3 1_{y \leq 1} + \frac{b^3}{|\log b|} 1_{y \sim B_1} + \frac{b}{y} 1_{y \gtrsim B_1} \right),
\]

which gives the brute force estimate for \( \mathbb{H} \mathcal{R}_1 \):

\[
\mathbb{H} \left[ w \wedge \left( \rho_1 \mathbb{H} \tilde{w}_0 - \rho_2 \mathbb{H} \tilde{w}_0 + \Lambda \phi \hat{p} \right) \right]
\lesssim \left( |H w| + |\Delta y| + \left( \frac{\partial_y + \frac{b}{y}}{1 + y^2} \right) \right) \left( b^2 y^3 1_{y \leq 1} + \frac{b^3}{|\log b|} 1_{y \sim B_1} + \frac{b}{y} 1_{y \gtrsim B_1} \right) + \left( |\partial_y w| + \frac{|w|}{y} \right) \left( b^2 y^2 1_{y \leq 1} + \frac{b^3}{y |\log b|} 1_{y \sim B_1} + \frac{b}{y^2} 1_{y \gtrsim B_1} \right).
\]

This together with interpolation bounds in Appendix A yields

\[
\int \frac{|\mathcal{R}_1|^2}{1 + y^8} + \int \frac{1 + |\log y|^2}{1 + y^4} |\mathbb{H} \mathcal{R}_1|^2 + \int \frac{1 + |\log y|^2}{1 + y^2} |A \mathbb{H} \mathcal{R}_1|^2 \lesssim b^2 \left( \mathcal{E}_4 + \frac{b^4}{|\log b|^2} \right).
\]

Now we treat the terms involving \( \mathcal{R}_2 \). By the construction of \( \tilde{w}_0 \) in Lemma 3.2, we have

\[
\begin{align*}
\tilde{w}_0 &= bO \left( y^3 1_{y \leq 1} + y \log y 1_{1 \leq y \leq 2B_1} \right), \\
\mathbb{H} \tilde{w}_0 &= bO \left( 1_{y \leq 1} + \frac{\log y}{y} 1_{1 \leq y \leq 2B_1} \right).
\end{align*}
\]

(5.53)
The bounds of the first two terms in $R_2$ (5.51) easily follows using Lemma A. To control the third one, we apply (5.44) to get
\[ \int |w \wedge (w \wedge \mathbb{H} \tilde{w}_0)|^2 \lesssim \delta(b^*) \left( E_4 + \frac{b^4}{|\log b|^2} \right). \]
Moreover, we compute by brute force that for $y \geq 1$,
\[ |\mathbb{H}(w \wedge (w \wedge \mathbb{H} \tilde{w}_0))| \lesssim b \left( \frac{\log y}{y} \left( |\partial_\nu^2 w| |w| + |\partial_\nu w|^2 + \frac{|w|^2}{y^2} \right) \right) 1_{y \leq 2B_1}, \]
which together with (A.3), (A.14), (A.16), (A.21) implies
\[ \int_{y \geq 1} \frac{1 + |\log y|^2}{1 + y^4} |\mathbb{H}(w \wedge (w \wedge \mathbb{H} \tilde{w}_0))|^2 \lesssim b \delta(b^*) \left( E_4 + \frac{b^4}{|\log b|^2} \right). \]
The $y \leq 1$ case and the estimate for $A\mathbb{H}(w \wedge (w \wedge \mathbb{H} \tilde{w}_0))$ can be obtained in the same way. For the last term in $R_2$ involving $\Theta_s$, by (4.34), we have
\[ |\Theta_s ZRw| \lesssim (|a| + U(t)) |w^1|, \]
\[ |\mathbb{H}(\Theta_s ZRw)| \lesssim (|a| + U(t)) \left( |\mathbb{H}^1 w| + \frac{|\partial_\nu w^1|}{1 + y^2} + \frac{|w^1|}{y(1 + y^2)} \right), \]
\[ |A\mathbb{H}(\Theta_s ZRw)| \lesssim (|a| + U(t)) \left( |A\mathbb{H}^1 w| + \frac{|\partial_\nu^2 w^1|}{1 + y^2} + \frac{|\partial_\nu w^1|}{y(1 + y^2)} + \frac{|w^1|}{y^2(1 + y^2)} \right), \]
together with (5.49), (A.3) yields the bounds for $\Theta_s ZRw$. In summary, we obtain
\[ \int |R_2^\perp|^2 + \int \frac{1 + |\log y|^2}{1 + y^4} |\mathbb{H} R_2^\perp|^2 + \int \frac{1 + |\log y|^2}{1 + y^4} |A\mathbb{H} R_2^\perp|^2 \lesssim b \delta(b^*) \left( E_4 + \frac{b^4}{|\log b|^2} \right). \]
This concludes the proof. \[\square\]

5.3 Energy estimate

The aim of this subsection is to prove the following proposition, which is the heart of our analysis.

**Proposition 5.1 (Mixed energy/Morawetz estimate)** Assume Lemma 5.2 and Appendix A. Then there exists a universal constant $d_2 \in (0, 1)$ independent of $M$, such that the following differential inequality holds
\[ \frac{d}{dt} \left\{ \frac{1}{\lambda^6} \left[ E_4 + \delta(b^*) \left( E_4 + \frac{b^4}{|\log b|^2} \right) \right] \right\} \leq \frac{b}{\lambda^8} \left[ 2 \left( 1 - d_2 + \frac{C}{\sqrt{\log M}} \right) E_4 + O \left( \frac{b^4}{|\log b|^2} \right) \right], \]
where $\delta(b^*)$ is the infinitesimal defined by (4.12).
Proof of Proposition 5.1.
This proof is based on the mixed energy/Morawetz identity (5.23). We proceed by first controlling the mixed energy functional, and then estimating each term on the RHS of (5.23) respectively.

Step 1. Control of the functional. Recall (2.38), (5.4), (5.44), there holds
\[ J_H w_2 = (R + R_w) H(R + R_w) H w \]
\[ = -(1 + \delta(b^*)) (H^{rac{1}{2}})^2 w^1 + O\left( H\left(\frac{\partial_\gamma}{1 + y^2}\right)\right) + O\left(RH w_0(H w)\right). \]  
(5.56)

From (A.16), (A.19), the second term on the RHS of (5.56) is \( L^2 \) bounded by
\[ \int \left|O\left( H\left(\frac{\partial_\gamma}{1 + y^2}\right)\right)\right|^2 \leq \int \frac{|\partial_\gamma^2|}{1 + y^4} + \int \frac{(\partial_\gamma^2 - \frac{\partial_\gamma^2}{y^2})}{y^2(1 + y^4)} + \int \frac{|\partial_\gamma|}{1 + y^6} + \int \frac{|\partial_\gamma|}{1 + y^8} \]
\[ \leq \sum_{0 \leq i \leq 3} \int \frac{|\partial_\gamma^2|}{y^2(1 + y^{6-2})(1 + |log y|^2)} + \int \frac{|A \partial_\gamma|}{y^4(1 + |log y|^2)} \]
\[ \leq \delta(b^*) \left[ E_4 + \frac{b^4}{|log b|^2} \right]. \]

The same \( L^2 \) bound holds also for the last term in (5.56) by (A.3), (A.7), (A.11), (A.21), (A.22), (A.24). In view of the definition (4.8), we have
\[ \int |\hat{J}_\lambda H_A W_2|^2 = \frac{1}{\lambda^6} \int |\hat{J} H w_2|^2 = \frac{1}{\lambda^6} \left[ E_4 + \delta(b^*) \left( E_4 + \frac{b^4}{|log b|^2} \right) \right]. \]  
(5.57)

Moreover, by (5.21), the morawetz term does not perturb the bound
\[ \frac{1}{2} \int |\hat{J}_\lambda H_A W_2|^2 - M(t) = \frac{1}{2\lambda^6} \left[ E_4 + \delta(b^*) \left( E_4 + \frac{b^4}{|log b|^2} \right) \right]. \]  
(5.58)

Step 2. The quasilinear terms with coefficients \( \rho_1, \rho_2 \). We consider the first two lines of the RHS of (5.23):
\[ - \rho_1 \int \hat{J}_\lambda H_A W_2 \cdot (\hat{J}_\lambda H_A)^2 W_2 + \rho_2 \int \hat{J}_\lambda H_A W_2 \cdot (\hat{J}_\lambda H_A)^2 \hat{J}_\lambda W_2 \]
\[ + 2\rho_1 c_2 k_1 \int H_A W_{20} \cdot L A_\lambda W_{20} - 2\rho_1 c_2 k_2 \int A_\lambda H_A W_{20} \cdot LW_{20}. \]  
(5.59)

By the scaling (4.32) and definition (5.2), we see
\[ - \rho_1 \int \hat{J}_\lambda H_A W_2 \cdot (\hat{J}_\lambda H_A)^2 W_2 + \rho_2 \int \hat{J}_\lambda H_A W_2 \cdot (\hat{J}_\lambda H_A)^2 \hat{J}_\lambda W_2 \]
\[ = \frac{1}{\lambda^8} \left[ - \rho_1 \int \hat{J} H w_2 \cdot (\hat{J})^2 w_2 + \rho_2 \int \hat{J}^2 H w_2 \cdot (\hat{J})^2 \hat{J} w_2 \right]. \]  
(5.60)

To treat the first term of (5.59), we introduce the following lemma, whose proof is given in Appendix B.
Lemma 5.4 (Gain of two derivatives) Assume Appendix A. There exists a constant $d_1 \in (0, 1)$ such that for any vector $\Gamma$ under the Frenet basis $[e_r, e_\tau, Q]$, there holds the inequality
\[
- \int \mathcal{J} \mathcal{H} \Gamma \cdot (\mathcal{J} \mathcal{H})^2 \Gamma \leq \frac{b(1 - d_1)(|\rho_1| + |\rho_2|)}{2(\rho_1^2 + \rho_2^2)} \|\mathcal{J} \mathcal{H} \Gamma\|_{L^2}^2 + b \delta(b^*) \|\mathcal{H} \Gamma\|_{L^2}^2,
\] (5.61)
where $\delta(b^*)$ is the infinitesimal defined by (4.12).

Applying (5.61) with $\Gamma = w_2$, and also (A.31), we have for some constant $d_2 = d_2(b^*) \in (0, 1)$ independent of $W_2$ that
\[
- \rho_1 \int \mathcal{J} \mathcal{H} w_2 \cdot (\mathcal{J} \mathcal{H})^2 w_2 \leq b(1 - d_1) \|\mathcal{J} \mathcal{H} w_2\|_{L^2}^2 + b \delta(b^*) \|\mathcal{H} w_2\|_{L^2}^2
\leq b(1 - d_2) \mathcal{E}_4 + b \delta(b^*) \left( \mathcal{E}_4 + \frac{b^4}{|\log b|^2} \right),
\] (5.62)
where we used $\rho_1(|\rho_1| + |\rho_2|) \leq 2(\rho_1^2 + \rho_2^2)$. Then for the second term (5.59), we use (5.4) to separate it into
\[
- \int \mathcal{J} \mathcal{H} w_2 \cdot \mathcal{H}(\mathcal{J} \mathcal{H} \mathcal{J} w_2)
= - \int R^2 \mathcal{H} w_2 \cdot \mathcal{H}[(\mathcal{R} \mathcal{H} R) w_2]
- \int R^2 \mathcal{H} w_2 \cdot \mathcal{H} \left[ \left( \mathcal{R} \mathcal{H} R \mathcal{w}_0 + R_0 \mathcal{H} R + R_0 \mathcal{H} R \mathcal{w}_0 \right) w_2 \right]
- \int \left( \mathcal{R} \mathcal{R} \mathcal{w}_0 + R_0 \mathcal{R} R + R_0^2 \mathcal{w}_2 \right) \mathcal{H} w_2 \cdot \mathcal{H} \left[ \left( \mathcal{R} \mathcal{H} R \mathcal{H} R \mathcal{w}_0 + R_0 \mathcal{H} R + R_0 \mathcal{H} R \mathcal{w}_0 \right) w_2 \right].
\] (5.63)

By (2.38), (2.37), the first term on the RHS is actually
\[
- \int R^2 \mathcal{H} w_2 \cdot \mathcal{H}[(\mathcal{R} \mathcal{H} R) w_2] = \int R^2 \mathcal{H} w_2 \cdot \mathcal{H}(\mathcal{H}^\perp w_2)
= \int \left( - \mathcal{H}^\perp w_2 + 2(1 + Z) \partial_y w_2^{(3)} e_x \right) \cdot \left[ (\mathcal{H}^\perp)^2 w_2 + 2(1 + Z) \left( \partial_y + \frac{Z}{y} \right) H w_2^{(1)} e_x \right]
= - \int |A \mathcal{H}^\perp w_2|^2 + \int 2(1 + Z) \partial_y w_2^{(3)} H^2 w_2^{(1)}.
\] (5.64)

We claim the negative integral $- \int |A \mathcal{H}^\perp w_2|^2$ is the leading term of (5.63), and the other terms in (5.63) are small errors in the sense that they can be bounded by $- \int |A \mathcal{H}^2 w^2|^2$ plus $b^5 \delta(b^*) |\log b|^{-2}$. Indeed, in view of factorization (2.37), we observe the LHS of (5.63) can be reformulated as
\[
- \int \mathcal{J} \mathcal{H} w_2 \cdot \mathcal{H}(\mathcal{J} \mathcal{H} \mathcal{J} w_2)
= \int A(\mathcal{J} \mathcal{H} w_2) \cdot A(\mathcal{J} \mathcal{H} \mathcal{J} w_2) - \int (\mathcal{J} \mathcal{H}^2 w_2)^{(1)} \cdot 2(1 + Z) \partial_y (\mathcal{J} \mathcal{H} \mathcal{J} w_2)^{(3)}
+ \int \partial_y (\mathcal{J} \mathcal{H}^2 w_2)^{(3)} \cdot \partial_y (\mathcal{J} \mathcal{H} \mathcal{J} w_2)^{(3)} + \int (\mathcal{J} \mathcal{H}^2 w_2)^{(3)} \cdot 2(1 + Z) \left( \partial_y + \frac{Z}{y} \right) (\mathcal{J} \mathcal{H} \mathcal{J} w_2)^{(1)}.
\]
The second order spacial derivative from $\mathbb{H}$ can act on either the second derivative of $w_2$ ($\mathbb{H}w_2$ and $\mathbb{H}\hat{w}_2$), or $\hat{w}$ coming from $\hat{J}$ (recall $\hat{J} = (e_z + \hat{w})\wedge$). According to this, the estimates can be split into different types. The most involved situation is that one single derivative acts on $\hat{w}$, and the other acts on $\mathbb{H}w_2$ (or $\mathbb{H}\hat{w}_2$), where the whole integrand is basically given by $\partial_y \hat{w}$ times $\mathbb{H}w_2$ (or $\mathbb{H}\hat{w}_2$) times a third derivative of $w_2$. A key observation here is that the third derivative of $w_2$ actually formulates $\Delta \mathbb{H}w_2$, so that a straight application of Cauchy-Schwartz finishes the estimate. To see this, we estimate for instance the second term on the RHS of (5.64). Recalling (5.2), we see

$$w_2^{(3)} = \hat{J}w = \hat{\alpha}H\beta - \hat{\beta}H\alpha.$$ 

This together with (A.3), (A.5) (A.11), (A.15), and the construction of the approximation solution implies

$$\int |2(1+Z)\partial_y w_2^{(3)} H^2 w_2^{(1)}|$$

$$= \int \left| A \left[ O\left(\frac{4}{1+y^2}\right) \partial_y \left(\hat{\alpha}H\beta - \hat{\beta}H\alpha\right)\right] \cdot |AHw_2^{(3)}| \right.$$ 

$$\leq C \left( \|\partial_y^2 \hat{w}^\perp\|_{L^\infty} + \|\partial_y \hat{w}^\perp\|_{L^\infty} + \|\partial_y \hat{w}^\perp\|_{L^\infty} \right) \sum_{i=0}^4 \left| \frac{\partial_y^i w_1^\perp}{1+y^4-i} \cdot |AH\hat{w}_2| \right.$$ 

$$\leq b \log b[C_1 \sum_{i=0}^4 \left| \frac{\partial_y^i w_1^\perp}{1+y^4-i} \cdot |AH\hat{w}_2| \right]$$ 

$$\leq b \delta(b^*) \frac{b^4}{|\log b|^2} + o(1) \int |AH\hat{w}_2|^2,$$

where the $o(1)$ is a positive constant that can be chosen sufficiently small. In other situations, we can control the terms directly by either $\|\partial_y \hat{w}\|_{L^\infty} b^4 |\log b|^{-2}$ or $\|\hat{w}\|_{L^\infty} \int |AH\hat{w}_2|^2$, and thus the claim follows. Therefore we obtain

$$\rho_2 \int \mathbb{H}w_2 \cdot (\hat{J})^2 \hat{J}w_2 \leq \frac{-\rho_2^2}{2} \int |AH\hat{w}_2|^2 + b \delta(b^*) \left( E_4 + \frac{b^4}{|\log b|^2} \right).$$

(5.65)

Now for the second line of (5.59), from the definition of $k_i$ (5.18), the coefficients satisfy $k_1(\rho_1^2 - \rho_2^2) \geq 0, k_2(\rho_2^2 - \rho_1^2) \geq 0$, and thus

$$\left\{ \begin{array}{ll}
2\rho_1 c_2 k_1 &= \frac{k_1(\rho_1^2 - \rho_2^2)}{\Delta k \rho_1^2 + \rho_2^2} \cdot \frac{4\rho_1^2}{\rho_1^2 + \rho_2^2} \in [0, 4], \\
-2\rho_1 c_2 k_2 &= \frac{k_2(\rho_2^2 - \rho_1^2)}{\Delta k \rho_1^2 + \rho_2^2} \cdot \frac{4\rho_2^2}{\rho_1^2 + \rho_2^2} \in [0, 4].
\end{array} \right.$$

These together with (5.27) ensure the negativity

$$2\rho_1 c_2 k_1 \int \mathbb{H}_\lambda W_2^0 \cdot L\mathbb{H}_\lambda W_2^0 \leq 0.$$ 

(5.66)
In addition, from the decomposition of $W_2^0$ (5.6), we have
\[ \int \mathcal{A}_\lambda \mathcal{H}_\lambda W_2^0 \cdot L W_2^0 = -\frac{4b}{\lambda^8} \left( - \int \mathcal{A}_\lambda^+ w_2^1 \cdot \frac{y w_2^0}{(1+y^2)^2} + \int \mathcal{A}_\lambda^+ w_2^2 \cdot \frac{y w_2^0}{(1+y^2)^2} \right), \]
where by (A.29), (A.33), the first integral in the parenthesis is bounded by
\[ \left| \int \mathcal{A}_\lambda^+ w_2^1 \cdot \frac{y w_2^0}{(1+y^2)^2} \right| \leq \int |\mathcal{H}_\lambda^+ w_2^1| \cdot \mathcal{A}_\lambda^+ \left( \frac{y w_2^0}{(1+y^2)^2} \right) \lesssim \int |\mathcal{R}(R^2 w_2^1)| \cdot \left( \left| \frac{\partial_y w_2^0}{1+y^3} + \frac{|w_2^0|}{y(1+y^3)} \right| \lesssim \delta(b^*) \left( \mathcal{E}_4 + \frac{b^4}{|\log b|^2} \right). \]
Moreover, from the positivity of $\rho_2$ and $b$, the second integral is bounded by
\[ \int \mathcal{A}_\lambda^+ w_2^2 \cdot \frac{y w_2^0}{(1+y^2)^2} \leq \frac{\rho_2}{16b} \int \left| \mathcal{A}_\lambda^+ w_2^2 \right|^2 + O(b) \left( \mathcal{E}_4 + \frac{b^4}{|\log b|^2} \right). \]
Putting these bounds together, we see
\[ -2\rho_1 c_2 k_2 \int \mathcal{A}_\lambda \mathcal{H}_\lambda W_2^0 \cdot L W_2^0 \leq \frac{1}{\lambda^8} \left[ \frac{\rho_2}{4} \int \left| \mathcal{A}_\lambda^+ w_2^2 \right|^2 + b \delta(b^*) \left( \mathcal{E}_4 + \frac{b^4}{|\log b|^2} \right) \right]. \quad (5.67) \]
Finally, collecting together (5.60), (5.62), (5.65), (5.66), (5.67), we obtain the control of the quadratic terms in (5.23): There holds
\[ -\rho_1 \int \mathcal{J}_\lambda \mathcal{H}_\lambda W_2 \cdot (J_\lambda H_\lambda)^2 W_2 + \rho_2 \int \mathcal{J}_\lambda \mathcal{H}_\lambda W_2 \cdot (J_\lambda H_\lambda)^2 J_\lambda W_2 \]
\[ + 2\rho_1 c_2 k_1 \int \mathcal{H}_\lambda W_2^0 \cdot L \mathcal{H}_\lambda W_2^0 - 2\rho_1 c_2 k_2 \int \mathcal{A}_\lambda \mathcal{H}_\lambda W_2^0 \cdot L W_2^0 \leq \frac{b(1-d_2)}{\lambda^8} \left( \mathcal{E}_4 + \frac{b^4}{|\log b|^2} \right), \]
for some constant $d_2 \in (0, 1)$. Here we note that the integral $\int \left| \mathcal{A}_\lambda^+ w_2^2 \right|^2$ in (5.67) has been absorbed by the one in (5.65) with negative sign.

**Step 3. The $Q_3$ term.** According to the definition (5.11), we separate $Q_3$ into four parts
\[ Q_3 = Q_3^1 + Q_3^2 + Q_3^3 + Q_3^4, \]
where
\[ Q_3^1 = -\left( b + \frac{\lambda^4}{\lambda} \right) \int \mathcal{R} \mathcal{H}_\lambda W_2^0 \cdot \left[ - \mathcal{H}_\lambda \left( \frac{\lambda^4}{\lambda^2 r^2} W \right) + \frac{\lambda^4}{\lambda^2 r^2} \mathcal{R} W_2^0 \right], \]
\[ Q_3^2 = \int \mathcal{R} \mathcal{H}_\lambda W_2^0 \cdot \left( \mathcal{R} \mathcal{H}_\lambda \mathcal{R} \left[ \partial_\lambda, \mathcal{H}_\lambda \right] \gamma e_z + \mathcal{R} \mathcal{H}_\lambda \mathcal{R} \left[ \partial_\lambda, \mathcal{H}_\lambda \right] W \right. \]
\[ + \partial_\lambda \mathcal{W} \wedge \mathcal{H}_\lambda W_2^0 + \mathcal{R} \mathcal{H}_\lambda \mathcal{H}_\lambda W_2^0 + \left[ \partial_\lambda, \mathcal{J}_\lambda H_\lambda \right] W_2^0 \right), \quad (5.69) \]
\[ Q_3^3 = \int \mathcal{R} \mathcal{H}_\lambda W_2^0 \cdot \left( \mathcal{J}_\lambda H_\lambda \mathcal{R} \left[ \partial_\lambda, \mathcal{H}_\lambda \right] W + \left[ \partial_\lambda, \mathcal{J}_\lambda H_\lambda \right] W_2 \right), \]
\[ Q_3^4 = \int \mathcal{R} \mathcal{H}_\lambda W_2^0 \cdot \left( \mathcal{J}_\lambda H_\lambda \mathcal{R} \left[ \partial_\lambda, \mathcal{H}_\lambda \right] W + \left[ \partial_\lambda, \mathcal{J}_\lambda H_\lambda \right] W_2 \right). \]
We estimate each one of these respectively. From the modulation equations (4.34), and the rough bound of the uncontrollable term (5.12), we have

$$|Q_3^1| \lesssim \frac{O(b^3)}{\lambda^8} \left( E_4 + \frac{b^4}{\|\log b\|^2} \right) \lesssim \frac{b \delta(b^*)}{\lambda^8} \left( E_4 + \frac{b^4}{\|\log b\|^2} \right). \tag{5.70}$$

To treat $Q_3^2$, we apply (5.15), (4.34) to get the brute estimate

$$\left[ \partial_t, \mathbb{H}_\lambda \right] W = \frac{b}{\lambda^4} O \left( \frac{w^\perp}{1 + y^4} + \frac{\partial_y \gamma e_x}{1 + y^2} + \frac{1}{1 + y^2} \left( \partial_y \alpha + \frac{\alpha}{y} e_z \right) \right). \tag{5.71}$$

Now we consider the $L^2$ bound of each term in the big parenthesis. By (5.71), (A.16), the first term in the brace of $Q_3^2$ is

$$\int \left| \mathbb{H}_\lambda \mathbb{R}_1 \left[ \partial_t, \mathbb{H}_\lambda \right] \right|^2 \lesssim \int \left| \mathbb{H}_\lambda \left( \frac{w^\perp}{1 + y^4} \right) \right|^2 \lesssim \frac{b^2 \delta(b^*)}{\lambda^{10}} \left( \frac{E_4}{1 + y^2} + \frac{b^4}{\|\log b\|^2} \right).$$

Similarly from (5.15), (5.44), (A.3) the second term in the brace is bounded by

$$\int \left| \mathbb{R}_2 \mathbb{H}_\lambda \mathbb{R}_1 \left[ \partial_t, \mathbb{H}_\lambda \right] \right|^2 \lesssim \frac{b^2 \delta(b^*)}{\lambda^{10}} \left( \frac{E_4}{1 + y^2} + \frac{b^4}{\|\log b\|^2} \right).$$

The rest of the terms in the parenthesis can be controlled in a similar fashion. Thus we have

$$|Q_3^2| \lesssim \left\| \mathbb{R}_2 \mathbb{H}_\lambda \mathbb{W}_0^0 \right\|_{L^2} \left[ \frac{b^2 \delta(b^*)}{\lambda^{10}} \left( \frac{E_4}{1 + y^2} + \frac{b^4}{\|\log b\|^2} \right) \right] \lesssim \frac{b \delta(b^*)}{\lambda^8} \left( \frac{E_4}{1 + y^2} + \frac{b^4}{\|\log b\|^2} \right). \tag{5.72}$$

For $Q_3^3$, we compare it with the LHS of (5.14). The latter, according to the proof of Lemma 5.1 (see (5.14), (5.17)), has been split into three separate parts in (5.69). Specifically,

$$\int \mathbb{R}_2 \mathbb{H}_\lambda \mathbb{W}_0^0 \cdot \left( \mathbb{J}_\lambda \mathbb{H}_\lambda \mathbb{R}_2 \left[ \partial_t, \mathbb{H}_\lambda \right] \right) \mathbb{W} + \left[ \partial_t, \mathbb{J}_\lambda \mathbb{H}_\lambda \right] \mathbb{W}_2 \right) = \int \mathbb{R}_2 \mathbb{H}_\lambda \mathbb{W}_2^0 \cdot \left[ - \frac{G W^0}{\lambda^2} (GW^0) + GW^0 \right] + Q_3^1 + Q_3^2. \tag{5.73}$$

In view of the rough bound (5.12) and the above computations on $Q_3^1, Q_3^2$, we have already estimated the integrals in (5.73) using Cauchy-Schwartz and Appendix A. Note that the first factor of their integrands are always $\mathbb{R}_2 \mathbb{H}_\lambda \mathbb{W}_2^0$, bounded by

$$\int \left| \mathbb{R}_2 \mathbb{H}_\lambda \mathbb{W}_2^0 \right|^2 = \frac{1}{\lambda^6} \int \left| \mathbb{R}_2 \mathbb{H}_\lambda \mathbb{W}_2^0 \right|^2 \lesssim \frac{1}{\lambda^6} \left( E_4 + \frac{b^4}{\|\log b\|^2} \right).$$
On the contrary, we review (5.69) and see the first factor of the integrand of $Q_3^{3}$ is $R_\lambda W_2^3$ instead of $R_\lambda W_2^0$. This, by (A.33), actually gives a better bounded than $R_\lambda W_2^0$:

$$
\int |R_\lambda W_2^3|^2 \leq \frac{1}{\lambda^6} \int |\hat{\mathbb{H}} w_2|^2 \lesssim \frac{\delta(b^*)}{\lambda^6} \left( \mathcal{E}_4 + \frac{b^4}{|\log b|^2} \right).
$$

The extra infinitesimal $\delta(b^*)$ bonus appearing here with the previous estimates yields the desired bound

$$
|Q_3^3| \lesssim \delta(b^*) \left( \left| \int R_\lambda W_2^0 \left[ -\mathbb{H} (GW^{-1}) + GRW_2^0 \right] \right| + |Q_4^1| + |Q_3^2| \right) 
\lesssim \frac{b \delta(b^*)}{\lambda^8} \left( \mathcal{E}_4 + \frac{b^4}{|\log b|^2} \right). 
$$

(5.74)

Finally, for $Q_3^3$, the smallness of $\|\hat{w}\|_{L^\infty}$ (5.44) implies the same bound as (5.74) via similar analysis. Thus we have

$$
|Q_3| \lesssim \frac{b \delta(b^*)}{\lambda^8} \left( \mathcal{E}_4 + \frac{b^4}{|\log b|^2} \right). 
$$

(5.75)

**Step 4. The term involving $Q_1$.** Recall the definition of $Q_1$ (5.5), we see the term involving $Q_1$ in (5.23) is actually given by

$$
\begin{aligned}
\int \hat{J}_\lambda \mathbb{H}_\lambda W_2 \cdot \hat{J}_\lambda \mathbb{H}_\lambda Q_1 \\
= \int \hat{J}_\lambda \mathbb{H}_\lambda W_2 \cdot \hat{J}_\lambda \mathbb{H}_\lambda \left( \partial_t \hat{W} \wedge \mathbb{H}_\lambda W + R_{\mathbb{H}_\lambda} [\partial_t, \mathbb{H}_\lambda] W \right) \\
= \frac{1}{\lambda^8} \int \hat{J}_\lambda \mathbb{H}_\lambda w_2 \cdot \hat{J}_\lambda \mathbb{H}_\lambda (\partial_s \hat{w}_0 \wedge \mathbb{H}_\lambda W) + \frac{1}{\lambda^8} \int \hat{J}_\lambda \mathbb{H}_\lambda w_2 \cdot \hat{J}_\lambda \mathbb{H}_\lambda \left[ (-\rho_1 w_2 + \rho_2 \hat{w}_2 + f) \wedge \mathbb{H}_\lambda W \right] \\
+ \int \hat{J}_\lambda \mathbb{H}_\lambda W_2 \cdot \hat{J}_\lambda \mathbb{H}_\lambda (\hat{W} \wedge [\partial_t, \mathbb{H}_\lambda] W).
\end{aligned}
$$

(5.76)

By the explicit formula of $w_0$ given by Lemma 3.1, and the modulation equations (4.34), we have

$$
\partial_s \hat{w}_0 = (A\chi)_{B_1} \left( - \frac{\partial_s B_1}{B_1} \right) w_0 + \chi_{B_i} \partial_s w_0 \\
= O(b w_0 1_{y^\beta < B_1}) + (\partial_s w_0 1_{y^\beta < B_1}) = b^2 O(y^{3} 1_{y^\beta < B_1} + y \log y 1_{y^\beta < B_1} + y^\beta 1_{1 < y^\beta < B_1}).
$$

(5.77)

Applying this with (5.77), (5.57), (A.6), (A.18), we obtain

$$
\begin{aligned}
\left| \int \hat{J}_\lambda \mathbb{H}_\lambda w_2 \cdot \hat{J}_\lambda \mathbb{H}_\lambda (\partial_s \hat{w}_0 \wedge \mathbb{H}_\lambda W) \right|
\lesssim b^2 \left( \int |\hat{J}_\lambda \mathbb{H}_\lambda w_2|^2 \right)^{1/2} \left( \sum_{i=0}^{4} \int |\partial_s^i w|^2 |\log y|^2 y^{2i} (1 + y^{1-2i}) \right)^{1/2} 
\lesssim b \delta(b^*) \left( \mathcal{E}_4 + \frac{b^4}{|\log b|^2} \right),
\end{aligned}
$$

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Also, by the double wedge formula, there holds
\[ w_2 \wedge Hw = (e_z + \hat{w}) \wedge Hw \]
\[ = (e_z \cdot Hw) Hw - ||Hw||^2 e_z + O(||\hat{w}||^2). \]

We apply (A.3), (A.16), (A.18), (A.24), (A.25) and it follows that
\[
\left| \int \hat{J} H w_2 \cdot H (w_2 \wedge H w) \right| \lesssim b \delta (b^*) \left( \mathcal{E}_4 + \frac{b^4}{||\log b||^2} \right).
\]

The term involving \( \rho_2 \hat{J} w_2 \) in (5.76) can be estimated similarly, and the one with \( f \) can be treated as in Lemma 5.2. Now for last line of (5.76), we use (5.71) with (A.3), (A.16) to get
\[
\left| \int \hat{J}_\lambda H W_2 \cdot \hat{J}_\lambda H \left( \hat{W} \wedge [\partial_t, H] W \right) \right| \lesssim \frac{b \delta (b^*)}{\lambda^8} \left( \mathcal{E}_4 + \frac{b^4}{||\log b||^2} \right).
\]

These yield that
\[
\left| \int \hat{J}_\lambda H W_2 \cdot \hat{J}_\lambda H \lambda W_1 \right| \lesssim \frac{b \delta (b^*)}{\lambda^8} \left( \mathcal{E}_4 + \frac{b^4}{||\log b||^2} \right). \quad (5.78)
\]

**Step 5. The integral involving \( F \).** From the definition (4.27), there holds the estimate:
\[
\int \hat{J}_\lambda H W_2 \cdot (\hat{J}_\lambda H)^2 F \lesssim \frac{1}{\lambda^8} \left( \mathcal{E}_4 + \frac{b^4}{||\log b||^2} \right)^{\frac{1}{2}} \left( \int |(\hat{J} H)^2 f|^2 \right)^{\frac{1}{2}}. \quad (5.79)
\]

We claim the following bound:
\[
\int |(\hat{J} H)^2 \hat{\text{Mod}}|^2 + \int |(\hat{J} H)^2 \hat{\Psi}_0|^2 + \int |(\hat{J} H)^2 \mathcal{R}|^2 \lesssim b \delta (b^*) \left( \mathcal{E}_4 + \frac{b^4}{||\log b||^2} \right). \quad (5.80)
\]

In view of (4.23), we inject (5.80) into (5.79) and then the desired bound for the \( F \) integral follows. Collecting this with (5.58), (5.68), (5.78), (5.75), we conclude the proof of Proposition 5.1. Now it remains to show (5.80).

**Step 6. Contribution of \( \hat{\text{Mod}} \).** We see from (5.50) that
\[
\hat{J} H \hat{\text{Mod}} = \chi_{B_1} \left( \frac{a_s}{\rho_1^2 + \rho_2^2} \left[ \begin{array}{c} -\rho_2 \\ \rho_1 \\ 0 \end{array} \right] A \phi - \frac{b_s + b^2 + a^2}{\rho_1^2 + \rho_2^2} \left[ \begin{array}{c} \rho_1 \\ \rho_2 \\ 0 \end{array} \right] A \phi \right)
\]
\[ + U(t) \left( by^2 1_{y \leq 1} + \frac{by}{||\log b||} 1_{1 \leq y \leq 6B_0} + \frac{\log y}{y} 1_{y \sim B_1} + \frac{1}{y^4} 1_{y \geq B_1} \right) \]
\[ + U(t) \left[ \hat{w} \wedge O \left( 1_{y \leq 1} + \frac{\log y}{y} 1_{1 \leq y \leq 2B_1} + \frac{1}{y^4} 1_{y \geq B_1} \right) \right]. \]
Applying \( \tilde{J}_\mathbb{H} \) again, we find in the region of \( 0 \leq y \leq B_1 \) that the first line on the above RHS vanishes, and hence

\[
(\tilde{J}_\mathbb{H})^2 \tilde{\text{Mod}} = U(t) \left( by1_{y \leq 1} + \frac{b}{y |\log b|}1_{1 \leq y \leq 6B_0} + \frac{\log y}{y^3}1_{y \sim B_1} + \frac{1}{y^3}1_{y \geq B_1} \right)
+ U(t) \tilde{J}_\mathbb{H} \left[ \hat{w} \land O \left( 1_{y \leq 1} + \frac{\log y}{y}1_{1 \leq y \leq 2B_1} + \frac{1}{y^3}1_{y \geq B_1} \right) \right].
\]

For the first part above, a straightforward computation with (5.49) yields

\[
U(t)^2 \int \left( by1_{y \leq 1} + \frac{b}{y |\log b|}1_{1 \leq y \leq 6B_0} + \frac{\log y}{y^3}1_{y \sim B_1} + \frac{1}{y^3}1_{y \geq B_1} \right)^2 \lesssim b^2 \left( \frac{\mathcal{E}_4}{|\log M|} + \frac{b^4}{|\log b|^2} \right),
\]

while the estimate for the second part requires a separation by \( \hat{w} = \hat{w}_0 + w^\perp + \gamma e_z \) and further applications of the interpolation bounds in Appendix A. The resulting bound is identical. The details are left to readers. This yields the estimate for Mod.

**Step 7. Contribution of \( \tilde{\Psi}_0 \).** According to (5.4), we let

\[
\tilde{J}_\mathbb{H}\tilde{\Psi}_0 = \mathcal{P}_1 + \mathcal{P}_2,
\]

where

\[
\mathcal{P}_1 = \mathbb{R}_\mathbb{H}\tilde{\Psi}_0,
\]

\[
\mathcal{P}_2 = \mathbb{R}_{\tilde{\mathbb{H}}\tilde{\mathbb{H}}}\tilde{\Psi}_0.
\]

By (2.19), the explicit expressions are

\[
\mathcal{P}_1 = \begin{bmatrix}
-\hat{H}\tilde{\Psi}_0^{(2)} \\
\hat{H}\tilde{\Psi}_0^{(1)} \\
0
\end{bmatrix} - 2(1 + Z)\partial_y \tilde{\Psi}_0^{(3)} e_y,
\]

\[
\mathcal{P}_2 = \left\{ \begin{array}{l}
\hat{\beta} \left[ -\Delta \tilde{\Psi}_0^{(3)} + 2(1 + Z) \left( \partial_y + Z \frac{y}{2} \right) \tilde{\Psi}_0^{(1)} \right] - \hat{\gamma} \hat{H}\tilde{\Psi}_0^{(2)} \\
+ \left\{ \hat{\gamma} \hat{H}\tilde{\Psi}_0^{(1)} - 2(1 + Z)\partial_y \tilde{\Psi}_0^{(3)} - \hat{\alpha} \left[ -\Delta \tilde{\Psi}_0^{(3)} + 2(1 + Z) \left( \partial_y + Z \frac{y}{2} \right) \tilde{\Psi}_0^{(1)} \right] \right\} e_y \\
+ \left\{ \hat{\alpha} \hat{H}\tilde{\Psi}_0^{(2)} - \hat{\beta} \left[ \hat{H}\tilde{\Psi}_0^{(1)} - 2(1 + Z)\partial_y \tilde{\Psi}_0^{(3)} \right] \right\} e_z
\end{array} \right\} e_x.
\]

Using (3.56), (3.58), (3.60), we have

\[
\int |\mathbb{H}\mathcal{P}_1|^2 \lesssim \int |\hat{H}^2\tilde{\Psi}_0^{(1)}|^2 + \int |\hat{H}^2\tilde{\Psi}_0^{(2)}|^2
+ \int \left| \left( \partial_y + Z \frac{y}{2} \right) \hat{\Psi}_0^{(2)} \right|^2 + \sum_{1 \leq i \leq 3} \int \frac{|\partial_y \hat{\Psi}_0^{(3)}|^2}{(1 + y^2)^{6-2i}} \lesssim \frac{b^6}{|\log b|^2},
\]

which together with (5.44) yields

\[
\int |\tilde{J}_\mathbb{H}\mathcal{P}_1|^2 \lesssim (1 + \|\hat{w}\|_{L^\infty}) \int |\mathbb{H}\mathcal{P}_1|^2 \lesssim \frac{b^6}{|\log b|^2}.
\]
Similarly, (5.44) implies
\[ \int |\dot{H} \mathbb{P}_2|^2 \lesssim \int |\mathbb{H} \mathbb{P}_2|^2. \] (5.83)

Thus it suffices to control \( \mathbb{H} \mathbb{P}_2 \) directly. For this, we compute its components by (5.82), and estimate them respectively. For instance, let us consider for \( y \leq 1 \) the following term coming from the first component of \( \mathbb{H} \mathbb{P}_2 \):

\[ H \left\{ \bar{\beta} \left[ - \Delta \bar{\psi}_0^{(3)} + 2(1 + Z) \left( \partial_y + \frac{Z}{y} \right) \bar{\psi}_0^{(1)} \right] \right\} \]

\[ \lesssim |H(\bar{\beta} \Delta \bar{\psi}_0^{(3)})| + |H \left[ \bar{\beta} (1 + O(y^2)) \left( \partial_y + \frac{Z}{y} \right) \bar{\psi}_0^{(1)} \right] | \]

\[ \lesssim \frac{y |H \bar{\beta}|}{1 + |\log y|} \left( \sum_{1 \leq i \leq 2} \frac{\left| \partial_y \bar{\psi}_0^{(3)} \right|^2}{y^{4-i}} + \sum_{0 \leq i \leq 1} \frac{\left| \partial_y \bar{\psi}_0^{(1)} \right|^2}{y^{3-i}} \right) + \frac{y \left| \partial_y \bar{\beta} \right| + \left| \frac{\bar{\beta}}{y} \right|}{1 + \left| \log y \right|} \sum_{0 \leq i \leq 3} \frac{\left| \partial_y \bar{\psi}_0^{(1)} \right|^2}{y^{3-i}}. \] (5.84)

Using (3.55), (3.56), (A.11), (A.13), we have

\[ \int_{y \leq 1} |H \left\{ \bar{\beta} \left[ - \Delta \bar{\psi}_0^{(3)} + 2(1 + Z) \left( \partial_y + \frac{Z}{y} \right) \bar{\psi}_0^{(1)} \right] \right\} |^2 \]

\[ \lesssim \left\| \frac{H \bar{\beta}}{y(1 + |\log y|)} \right\|_{L^\infty(y \leq 1)}^2 \left( \sum_{1 \leq i \leq 2} \int \frac{\left| \partial_y \bar{\psi}_0^{(3)} \right|^2}{y^{8-2i}} + \sum_{0 \leq i \leq 1} \int \frac{\left| \partial_y \bar{\psi}_0^{(1)} \right|^2}{y^{8-2i}} \right) \]

\[ + \left( \left\| \partial_y \bar{\beta} \right\|_{L^\infty(y \leq 1)} + \left\| \frac{\bar{\beta}}{y} \right\|_{L^\infty(y \leq 1)} \right) \sum_{0 \leq i \leq 3} \int \frac{\left| \partial_y \bar{\psi}_0^{(1)} \right|^2}{y^{6-2i}} \lesssim \frac{b^6}{|\log b|^2}. \]

The estimates of (5.84) for \( y \geq 1 \) can be derived in a similar fashion. Other terms in \( \mathbb{H} \mathbb{P}_2 \) can be also treated in the same way. Therefore we obtain

\[ \int |\mathbb{H} \mathbb{P}_2|^2 \lesssim \frac{b^6}{|\log b|^2}, \]

which together with (5.83) yields the bound for \( \mathbb{P}_2 \). This concludes the estimate for \( \bar{\psi}_0 \).

Step 8. Contribution of \( \mathcal{R} \). From the decomposition (5.51), and (5.44), we see

\[ |(\dot{J} \mathbb{H})^2 \mathcal{R}| \lesssim |\mathbb{H} \mathbb{R} \mathbb{H} \mathcal{R} | + |\mathbb{R}_w \mathbb{H} \mathbb{R} | + |\mathbb{H} \mathbb{R} \mathbb{R} \mathcal{R} | + |\mathbb{R}_w \mathbb{H} \mathbb{R} |. \] (5.85)

Using the estimate for \( \mathbb{H} \mathbb{R} \mathbb{R} \mathcal{R} \) (5.52), we compute by brute force that

\[ \mathbb{H} \mathbb{R} \mathbb{R} \mathcal{R} = O(H(\mathbb{H} \mathbb{R}_1)) + O \left( \frac{\partial_y \mathbb{H} \mathbb{R}_1(1 + y^2)}{y(1 + y^2)} \right) + O \left( \frac{\mathbb{H} \mathbb{R}_1}{y(1 + y^2)} \right) \]

\[ \lesssim \left[ \sum_{0 \leq i \leq 2} \left( \frac{\left| \partial_y H_{w^1} \right|}{y^{2-i}} + \frac{\left| \partial_y \Delta \gamma_1 \right|}{y^{2-i}} \right) + \sum_{0 \leq i \leq 3} \frac{\left| \partial_y w \right|}{y^{3-i}(1 + y^2)} \right] \]

\[ \times \left( b^2 y^3 1_{y \leq 1} + \frac{b^2}{|\log b|} 1_{y \sim B_1} + \frac{y}{y} 1_{y \geq B_1} \right). \]
where \( b^2 y^3 1_{y \leq 1} \) eliminates the possible singularity at the origin caused by \( |\partial_y^i w|/y^{2-i} \). From (A.3), (A.5), (A.16), all these terms can be controlled, and thus

\[
\int |\mathbb{H} R \mathbb{H} \mathcal{R}_1|^2 \lesssim b \delta(b^*) \left( \mathcal{E}_4 + \frac{b^4}{|\log b|^2} \right).
\]

The estimate for \( \mathbb{H} \hat{w} \mathbb{H} \mathcal{R}_1 \) is more involved, but in view of the smallness of derivatives of \( \hat{w} \) indicated by (A.15), (A.21), (A.22), (A.24), it could be handled similarly. Therefore we have

\[
\int |(\mathbb{H})^2 \mathcal{R}_1|^2 \lesssim b \delta(b^*) \left( \mathcal{E}_4 + \frac{b^4}{|\log b|^2} \right).
\]

Then for the terms involving \( \mathcal{R}_2 \) in (5.85), we recall from (5.51) that

\[
\mathcal{R}_2 = -\rho_2 \hat{J}(w \wedge \mathbb{H} \hat{w}_0) - \rho_2 w \wedge (\hat{w}_0 \wedge \mathbb{H} \hat{w}_0)
- \rho_2 w \wedge (w \wedge \mathbb{H} \hat{w}_0) + \Theta \mathcal{Z} \mathcal{R} w.
\]

The first two terms in \( \mathcal{R}_2 \) are concerned with \( \mathbb{H} \hat{w}_0 \), and thus easy to treat. Indeed, by (5.53), \( \mathbb{H} \hat{w}_0 \) would eventually become an extra \( b^2 \) bonus before the \( \mathcal{E}_4 \) bound, which results in the bound \( b^5 \delta(b^*)/|\log b|^2 \). Now we treat the third term in \( \mathcal{R}_2 \). By brute computations, and (A.3), (A.11), (A.13), (A.16), (A.22), we have

\[
\int |\mathbb{H} R \mathbb{H} (w \wedge (w \wedge \mathbb{H} \hat{w}_0))|^2 \lesssim \sum_{0 \leq i_1 + i_2 + i_3 \leq 4} \int \frac{|\partial_{i_1}^1 w|^2 |\partial_{i_2}^2 w|^2 |\partial_{i_3}^3 \mathbb{H} \hat{w}_0|^2}{y^{2(4-i_1+i_2+i_3)}} \lesssim b^2 \sum_{0 \leq i_1 + i_2 \leq 4} \int \frac{|\partial_{i_1}^1 w|^2 |\partial_{i_2}^2 w|^2}{y^{2(3-i_1-i_2)}} \left( 1_{y \leq 1} + \frac{\log y}{y^2} 1_{y \leq 2B_1} \right)^2
\]

\[
\lesssim b^2 |\log b|^4 \left( \| \partial_y w \|_{L^\infty} + \| w \|_{L^\infty} \right) \sum_{0 \leq i_1 \leq 4} \int \frac{|\partial_{i_1}^1 w|^2}{y^{2(2-i_1)}(1 + y^2)(1 + |\log y|^2)} + b^2 |\log b|^4 \int \frac{|\partial_{i_2}^2 w|^4}{(1 + y^2)(1 + |\log y|^2)} \lesssim b^3 \left( \mathcal{E}_4 + \frac{b^4}{|\log b|^2} \right) \left( \mathcal{E}_4 + \frac{b^4}{|\log b|^2} \right).
\]

Finally for the phase term \( \mathbb{H} R \mathbb{H} (\mathcal{Z} \mathcal{R} w) \) in \( \mathcal{R}_2 \), to deal with the possible singularity at the origin, we split it into

\[
\mathbb{H} R \mathbb{H} (\mathcal{Z} \mathcal{R} w) = \mathbb{H} R \mathbb{H} ((Z-1)\mathcal{R} w) + \mathbb{H} R \mathbb{H} (\mathcal{R} w)
= -\mathbb{H} \mathbb{H}^\perp ((Z-1)w) - \mathbb{H} \mathbb{H}^\perp w.
\] (5.86)

For the first term in (5.86), we see

\[
\mathbb{H} \mathbb{H}^\perp ((Z-1)w) = H^2((Z-1)\alpha) e_x + H^2((Z-1)\beta) e_y
+ 2(1+Z)\left( \partial_y + \frac{Z}{y} \right) H((Z-1)\alpha) e_z,
\]

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where the singularity can only arise from the first two components, so we compute using
the asymptotics
\[ Z - 1 = -2y^2 + O(y^4) \] for \( y \leq 1 \) that
\[
H^2((Z - 1)\alpha) \sim H^2(y^2\alpha) + O(H^2(y^4\alpha))
\]
\[
y^2H^2\alpha - 4y(\partial_y H\alpha + H\partial_y \alpha)
\]
\[
- 8H\alpha + 8\partial^2_y \alpha + \frac{4\partial_y \alpha}{y} + O(H^2(y^4\alpha)).
\]
The singular terms are actually
\[
-4y(\partial_y H\alpha + H\partial_y \alpha) + \frac{4\partial_y \alpha}{y} = -4y \left[ 2H\partial_y \alpha + \partial_y \left( \frac{V}{y^2} \right) \right]
\]
\[
= 8y \left( \partial^2_y \alpha + \frac{\partial^2_y \alpha}{y} \right) - 4y \left[ \frac{2V}{y^2} \partial_y \alpha + \partial_y \left( \frac{V}{y^2} \right) \alpha \right],
\]
where by \( V - 1 = 4y^2 + O(y^4) \), we have
\[
\frac{2V}{y^2} \partial_y \alpha + \partial_y \left( \frac{V}{y^2} \right) \alpha = \frac{2(V - 1)}{y^2} \partial_y \alpha - \partial_y \left( \frac{V - 1}{y^2} \right) \alpha + \frac{2(Z - 1)}{y^2} \alpha - \frac{2A\alpha}{y^2}
\]
\[
= O(\partial_y \alpha) + O(\alpha) - \frac{2A\alpha}{y^2}.
\]
From (A.10), the last term here admits estimate
\[
\int_{y \leq 1} \frac{|A\alpha|^2}{y^4} \lesssim C(M)\mathcal{E}_4, \tag{5.87}
\]
which helps control the singularity. For second term in (5.86), we can treat it in the
same way, and the singularity may come from the third component, that is
\[
\left( \partial_y + \frac{Z}{y} \right) H\alpha = -\partial^2_y \alpha - \frac{1 + Z}{y} \partial^2_y \alpha + \frac{V - 1}{y^2} \partial_y \alpha - \frac{Z - 1}{y^2} \partial_y \alpha + \partial_y \left( \frac{V - 1}{y^2} \right) \alpha
\]
\[
+ \frac{(Z - 1)(V - 1)}{y^3} \alpha + \frac{2(Z - 1)}{y^3} \alpha + \frac{V - 1}{y^3} \alpha + \frac{\partial_y \alpha}{y^2} - \frac{Z}{y^3} \alpha.
\]
The only singular terms are the last two terms, which still constitute \(-A\alpha/y^2\), and thus
are bounded by (5.87) again. Other non-singular terms in (5.86) can be controlled by
\( C(M)\mathcal{E}_4 \) using (A.3). Then by the modulation equations (4.34), we have
\[
\int \left| H\mathbb{R}(\Theta S Z \mathcal{R}w) \right|^2 \lesssim \left( |a| + U(t) \right)^2 \left( \mathcal{E}_4 + \frac{b^4}{|\log b|^2} \right) \lesssim b^2 \left( \mathcal{E}_4 + \frac{b^4}{|\log b|^2} \right).
\]
The last term in (5.85) share the same bound as above due the smallness of \( \hat{w} \). We omit
the details. Therefore we obtain
\[
\int \left| (\mathcal{J} \mathbb{H})^2 \mathcal{R} \right|^2 \lesssim b \delta(b^*) \left( \mathcal{E}_4 + \frac{b^4}{|\log b|^2} \right), \tag{5.88}
\]
which concludes the estimate for \( \mathcal{R} \), and ends the proof. \( \square \)
5.4 Closing the bootstrap bounds

In this subsection, we close the bootstrap argument by proving Proposition 4.1. In particular, we apply Proposition 5.1 to show the refined bounds (4.18)–(4.21).

Proof of Proposition 4.1.

Step 1. The refined bound for \( b \). The modulation equations (4.34) implies the asymptotic

\[
 b(s) = \frac{1}{s} + O\left(\frac{1}{s \log s}\right). \tag{5.89}
\]

This implies for \( s_0 \gg 1 \) chosen sufficiently large, \( b(s) \) decays over the rescaled time. In view of the self-similar transformation (2.9), we see for \( t \geq 0 \) that

\[
 0 < b(t) \leq b(0) < b^*.
\]

Thus the refined bound (4.18) for \( b(t) \) holds for \( K > 2 \).

Step 2. The refined bound for \( a \). We define the function

\[
 \kappa(s) = \frac{2a(s) |\log b(s)|}{b(s)}.
\]

In view of the assumption (4.9), we see \( \kappa(s_0) \in \mathcal{I} = [-\frac{1}{2}, \frac{1}{2}] \) (\( t = 0 \) when \( s = s_0 \)), and the refined bound (4.18) for \( a \) is equivalent to \( \kappa(s) \in \mathcal{I}^* = [-1, 1] \) for all \( s \in [s_0, +\infty) \).

To show the latter, we compute the equation for \( \kappa \) using (4.34):

\[
 \frac{d}{ds} \kappa(s) = \frac{2a_s |\log b|}{b} - \frac{2ab_s |\log b|}{b^2} \left(1 + \frac{1}{|\log b|}\right)
 + \frac{2a |\log b|}{b^2} \left(1 + \frac{1}{|\log b|}\right) \left[b^2 \left(1 + \frac{2}{|\log b|}\right) + O\left(\frac{b^2}{\sqrt{\log M |\log b|}}\right)\right]
 = \kappa b(1 + o(1)) + O\left(\frac{b}{\sqrt{\log M}}\right). \tag{5.90}
\]

From the largeness of \( M \), we see \( \kappa \) is monotone increasing/decreasing near \( \kappa = 1, -1 \), implying that if \( \kappa \) approaches the boundary of \( \mathcal{I}^* \), it will escape from \( \mathcal{I}^* \). We denote the moment when \( \kappa \) first leave by \( s^* > 0 \), called the exit time. And the desired bound (4.18) holds if we prove \( s^* = +\infty \). Let us consider the following subsets of \( \mathcal{I} \):

\[
 \mathcal{I}_+ = \{ \kappa(s_0) \in \mathcal{I} : s^* \in (s_0, +\infty) \text{ such that } \kappa(s^*) = 1 \},
\]

\[
 \mathcal{I}_- = \{ \kappa(s_0) \in \mathcal{I} : s^* \in (s_0, +\infty) \text{ such that } \kappa(s^*) = -1 \}.
\]

Now the problem can be categorized as three cases: \( \mathcal{I}_+ \) empty, \( \mathcal{I}_- \) empty, and both of them being nonempty. In the first case, for any \( \kappa(s_0) \in \mathcal{I} \), we have \( \kappa(s) < 1 \) for all \( s \in [s_0, +\infty) \). We choose

\[
 \kappa(s_0) > \frac{C}{\sqrt{\log M}}, \tag{5.91}
\]

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for $C > 0$ sufficiently large. Then by (5.90), $\kappa$ is monotone increasing, so that $\kappa(s) \geq \kappa(s_0) > -1$, and thus $s^* = +\infty$. An analogous statement holds for the second case. In the last case, by the $C^1$ dependence of the solutions on the initial data, we know $\mathcal{I}_+, \mathcal{I}_-$ are open subsets of $\mathcal{I}$, and obviously disjoint. This implies, from an elementary topological argument on the connectedness, that $\mathcal{I} \neq \mathcal{I}_+ \cup \mathcal{I}_-$. Consequently,

$$\exists \kappa(s_0) \in \mathcal{I} \setminus (\mathcal{I}_+ \cup \mathcal{I}_-) \neq \emptyset,$$

(5.92)

satisfying $s^* = +\infty$, which is what we need. In summary, we can always find suitable initial data $a(t = 0)$ satisfying (4.9) such that the refined bound (4.18) for $a(t)$ holds, concluding the proof of (4.18).

**Step 3. The refined bound for $\mathcal{E}_1$.** For any given vector under the Frenet basis

$$z = \hat{\alpha} e_r + \hat{\beta} e_\tau + (1 + \hat{\gamma}) Q,$$

satisfying the constraint (2.8), the corresponding Dirichlet energy is given by

$$\mathcal{E}(z) = \int |\nabla z|^2 = -\int z \cdot (\Delta z + |\nabla Q|^2 z) + \int |\nabla Q|^2 |z|^2
= (\hat{\alpha}, H\hat{\alpha}) + (\hat{\beta}, H\hat{\beta}) + (-\Delta \hat{\gamma}, \hat{\gamma})
+ 2 \int (1 + Z) \left( -\hat{\alpha} \partial_y \hat{\gamma} + \hat{\gamma} \left( \partial_y + \frac{Z}{y} \right) \hat{\alpha} \right) + \mathcal{E}(Q),$$

where we have used (2.16). By Lemma 3.2, straightforward computations show for any $s \in [s_0, +\infty)$ that

$$|\mathcal{E}(Q + \tilde{w}_0) - \mathcal{E}(Q)|(s) \lesssim \sqrt{b}.$$  

Then, in view of (2.19), the Dirichlet energy of the solution $u$ is

$$\mathcal{E}(u)(s) = \mathcal{E}(Q + \tilde{w}_0) + (w, H\tilde{w}_0) + (\tilde{w}_0, Hw) + (w, Hw)
= \mathcal{E}(Q) + (\alpha, H\alpha) + (\beta, H\beta) + O(\sqrt{b}),$$

where the cross terms are controlled by $O(\sqrt{b})$ via the bootstrap bounds. Also, the dissipative property (1.3) of the Dirichlet energy implies

$$(\alpha, H\alpha) + (\beta, H\beta) \leq (\alpha, H\alpha)(s_0) + (\beta, H\beta)(s_0) + O(\sqrt{b}).$$

(5.93)

Moreover, owing to the orthogonality (4.4), we have the coercivity estimate

$$(\alpha, H\alpha) + (\beta, H\beta) \geq C(M)\mathcal{E}_1(s),$$

(5.94)

for some universal constant $C(M) > 0$ independent of $K$, and for any $s \in [s_0, +\infty)$. Its opposite direction follows directly from integration by parts

$$(\alpha, H\alpha) + (\beta, H\beta) \leq \mathcal{E}_1(s).$$

(5.95)
Now putting together (5.93), (5.94), (5.95), we obtain

\[ E_1(s) \lesssim (\alpha, H\alpha) + (\beta, H\beta) \lesssim E_1(0) + O(\sqrt{b}) \lesssim \delta(b^*). \]

Note that the implicit constant in this inequality does not depend on \( K \). Therefore by choosing \( K \) large enough, the refined \( E_1 \) bound (4.19) is proved.

Step 4. The refined bound of \( E_4 \). A direct integration of (5.55) with respect to \( t \) shows there exists constant \( d_2 \in (0, 1) \) and some constant \( C > 0 \) independent of \( M \) such that

\[ E_4(t) \leq \frac{\lambda(t)^6}{\lambda(0)^6} E_4(0) + (2(1 - d_2)K + C) \lambda(t)^6 \int_0^t \frac{b^5}{\lambda^8 |\log b|^2} d\sigma. \tag{5.96} \]

To treat the first part on the RHS, we let \( C_1, C_2 > 0 \) be two universal constants large enough compared with the implicit constant in (4.34), and define

\[ \eta_1 = 2 - \frac{C_1}{\log M}, \quad \eta_2 = 2 + \frac{C_2}{\log M}. \]

Using the modulation equations (4.34), we compute the following derivative:

\[
\frac{d}{ds} \left( \frac{b |\log b| |\log b|^{\eta_i}}{\lambda} \right) = \frac{|\log b|^{\eta_i}}{\lambda} \left( b_s - \frac{b \lambda_s}{\lambda} + \frac{\eta_i b_s}{|\log b|} \right)
= \left( 1 - \frac{\eta_i}{|\log b|} \right) \frac{|\log b|^{\eta_i}}{\lambda} \left[ b_s + \left( 1 + \frac{\eta_i}{|\log b|} + O\left( \frac{1}{|\log b|^2} \right) \right) b^2 \right] \begin{cases} \leq 0, & i = 1, \\ \geq 0, & i = 2, \end{cases}
\]

where we see the choice of \( \eta_i \) is decisive to the sign. By direct integrations, it implies

\[
\frac{b(0)}{\lambda(0)} \left| \frac{\log b(t)}{\log b(t)} \right|^{\eta_2} \leq \frac{b(t)}{\lambda(t)} \leq \frac{b(0)}{\lambda(0)} \left| \frac{\log b(t)}{|\log b(t)|} \right|^{\eta_1}, \tag{5.97}
\]

from which we have

\[
\frac{\lambda(t)^6}{\lambda(0)^6} E_4(0) \leq \frac{b(t)^6 |\log b(t)|^{6\eta_2}}{b(0)^6 |\log b(0)|^{6\eta_2}} E_4(0) \leq b(t)^5 \leq \frac{b(t)^4}{|\log b(t)|^2}. \tag{5.98}
\]

For the second part on the RHS of (5.96), it follows from (4.34) that

\[
\begin{cases}
  b = -\frac{\lambda_s}{\lambda} + O(b^2) \leq -(1 + 2b(0)^2) \lambda t, \\
  -b_t = -\frac{b_s}{\lambda^2} \leq \left( 1 + \frac{2}{|\log b(0)|} \right) \frac{b^2}{\lambda^2}, \\
  4 + \frac{1}{|\log b|} \leq 4 + \frac{1}{|\log b(0)|}.
\end{cases}
\]

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using which we compute
\[
\int_0^t \frac{b^5}{\lambda^8 |\log b|^2} d\sigma \leq -\left(1 + 2b(0)^2\right) \int_0^t \frac{\lambda_t}{\lambda^7 |\log b|^2} b^4 d\sigma \\
= \frac{1}{6} \left(1 + 2b(0)^2\right) \left[ \frac{b^4}{\lambda^6 |\log b|^2} \int_0^t - \int_0^t \frac{b^3}{\lambda^6 |\log b|^2} \left(4 + \frac{1}{|\log b|}\right) d\sigma \right] \\
\leq \frac{1}{6} \left(1 + 2b(0)^2\right) \left[ \frac{b^4}{\lambda^6 |\log b|^2} \right]_0^t \\
+ \left(1 + \frac{2}{|\log b(0)|}\right) \left(4 + \frac{1}{|\log b(0)|}\right) \int_0^t \frac{b^5}{\lambda^8 |\log b|^2} d\sigma,
\]
and hence
\[
\int_0^t \frac{b^5}{\lambda^8 |\log b|^2} d\sigma \leq \left(\frac{1}{2} + O\left(\frac{1}{|\log b(0)|}\right)\right) \frac{b^4}{\lambda^6 |\log b|^2} \left(4 + \frac{1}{|\log b(0)|}\right) \int_0^t \frac{b^5}{\lambda^8 |\log b|^2} d\sigma,
\]

(5.99)

Injecting (5.98), (5.99) into (5.96), we obtain
\[
\mathcal{E}_4(t) \leq \left(1 - d_2 K + C\right) \frac{b(t)^4}{|\log b(t)|^2},
\]
for some constant \( C > 0 \) independent of \( K, M \). By choosing \( K > 0 \) large enough, we see there exist \( 0 < \eta < 1 \) such that \( (1 - d_2 K + C) \leq K(1 - \eta) \), and thus the desired bound (4.21) follows.

Step 5. The refined bound of \( \mathcal{E}_2 \). A direct interpolation between the bounds of \( \mathcal{E}_1, \mathcal{E}_4 \) is insufficient to show (4.20). Thus we treat it attentively. By (2.11), (2.16), we see
\[
u \wedge \Delta u = u \wedge (\Delta u + |\nabla Q|^2 u) = \hat{J} \hat{H} \hat{w},
\]
(5.100)
from which we compute:
\[
\frac{1}{2} \frac{d}{dt} \int \left| u \wedge \Delta u \right|^2 = -\rho_2 \int (u \wedge \Delta u) \cdot [u \wedge (u \wedge \Delta u)] \wedge \Delta u \\
+ \rho_1 \int (u \wedge \Delta u) \cdot u \wedge \Delta (u \wedge \Delta u) \\
- \rho_2 \int (u \wedge \Delta u) \cdot u \wedge \Delta [u \wedge (u \wedge \Delta u)] \\
= \frac{1}{\lambda^4} \left\{ \rho_2 \int \hat{J} \hat{H} \hat{w} \cdot (\hat{J}^2 \hat{H} \hat{w}) \wedge \hat{H} \hat{w} \\
- \rho_1 \int \hat{J} \hat{H} \hat{w} \cdot (\hat{J} \hat{H})^2 \hat{w} + \rho_2 \int \hat{J} \hat{H} \hat{w} \cdot \hat{J} \hat{H} (\hat{J}^2 \hat{H} \hat{w}) \right\}.
\]

(5.101)
Using Lemma 5.4, (A.31), (A.34), we estimate the second one

\[ \lambda^2 \int |u \wedge \Delta u|^2 = \int |\tilde{J} \tilde{H} \tilde{w}|^2 = \int |\tilde{J} \tilde{H} \tilde{w}|^2 + O \left( \int |\tilde{J} \tilde{H} \tilde{w}_0|^2 \right) = \mathcal{E}_2 + O(b^2 |\log b|^2). \]  

(5.102)

In addition, from (4.15) and Appendix A, we can estimate each term on the RHS of (5.101). Indeed, by (5.53), (A.11), (A.15), (A.21), the first term is bounded by

\[ \int \tilde{J} \tilde{H} \tilde{w} \cdot (\tilde{J}^2 \tilde{H} \tilde{w}) = \int \tilde{H} \tilde{w} \cdot (\tilde{J} \tilde{H} \tilde{w} \wedge \tilde{J}^2 \tilde{H} \tilde{w}) = \int (\tilde{H} \tilde{w} \cdot (Q + \tilde{w})) |\tilde{J} \tilde{H} \tilde{w}|^2 \]

\[ \leq \| \tilde{H} \tilde{w} \|_{L^\infty} \left( \| \tilde{J} \tilde{H} \tilde{w}_0 \|_{L^2}^2 + \| w_2 \|_{L^2}^2 \right) \]

\[ \lesssim b \left( b^2 |\log b|^2 + Kb^2 |\log b|^6 \right) \lesssim Kb^3 |\log b|^6. \]

Using Lemma 5.4, (A.31), (A.34), we estimate the second one

\[ -\int \tilde{J} \tilde{H} \tilde{w} \cdot (\tilde{J} \tilde{H})^2 \tilde{w} = -\int \tilde{J} \tilde{H} \tilde{w}_0 \cdot (\tilde{J} \tilde{H})^2 \tilde{w}_0 - \int w_2 \cdot (\tilde{J} \tilde{H})^2 \tilde{w}_0 \]

\[ -\int \tilde{J} \tilde{H} \tilde{w}_0 \cdot \tilde{J} \tilde{H} w_2 - \int w_2 \cdot \tilde{J} \tilde{H} w_2 \]

\[ \leq b \| \tilde{H} \tilde{w}_0 \|_{L^2}^2 + \| \tilde{J} \tilde{H} w_2 \|_{L^2} \| \tilde{J} \tilde{H} \tilde{w}_0 \|_{L^2} \]

\[ + \| \tilde{H} \tilde{w}_0 \|_{L^2} \| \tilde{H} w_2 \|_{L^2} + \| w_2 \|_{L^2} \| \tilde{J} \tilde{H} w_2 \|_{L^2} \]

\[ \lesssim b^3 |\log b|^2 + 2\sqrt{K}b^3 + Kb^3 |\log b|^2 \lesssim Kb^3 |\log b|^2. \]

Similarly, from (A.27), (A.34), the third term is bounded by

\[ \int \tilde{J} \tilde{H} \tilde{w} \cdot \tilde{J} \tilde{H} (\tilde{J}^2 \tilde{H} \tilde{w}) \leq \left( \| \tilde{H} \tilde{w}_0 \|_{L^2} + \| \tilde{H} w_2 \|_{L^2} \right) \left( \| \tilde{H} (\tilde{J}^2 \tilde{H} \tilde{w}_0) \|_{L^2} + \| \tilde{H} (\tilde{J} w_2) \|_{L^2} \right) \]

\[ \lesssim \left( b |\log b| + \sqrt{\mathcal{E}_2} \right) \frac{\sqrt{K}b^3}{|\log b|} \lesssim Kb^3 |\log b|^2. \]

We insert these bounds into (5.101) to obtain

\[ \frac{d}{dt} \int |u \wedge \Delta u|^2 \lesssim \frac{Kb^3 |\log b|^2}{\lambda^4}. \]

Integrating this in time from 0 to \( t \) and applying (5.102), we have

\[ \mathcal{E}_2(t) \lesssim b(t)^2 |\log b(t)|^2 + \lambda(t)^2 b(0)^{10} + K\lambda(t)^2 \int_0^t \frac{b(\sigma)^3 |\log b(\sigma)|^3}{\lambda(\sigma)^4} d\sigma. \]  

(5.103)

From (5.97), there holds

\[ \lambda(t)^2 b(0)^{10} \leq b(0)^{10} \left( b(t) \frac{b(0)}{b(0)} \left| \frac{\log b(t)}{\log b(0)} \right| \right)^{\eta_2} \lesssim b(t)^2 |\log b(t)|^5. \]

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Moreover, from (4.34), we have \( b^2 \lesssim -b_s \), which together with (5.97) gives
\[
\int_0^t \frac{b^3 \log b^3}{\lambda^4} \, d\sigma \lesssim - \int_0^t b b_s \frac{\log b^3}{\lambda^4} \, d\sigma \lesssim - \frac{b(0)^2 \log b(0)^2}{\lambda(0)^2} \int_0^t b \frac{b}{\log b^{2n_2-3}} \, d\sigma \lesssim \frac{b^2 \log b^{2n_2}}{\lambda^2} \lesssim \frac{b^2 \log b^5}{\lambda^2}.
\]
Inserting these into (5.103), we obtain
\[
E_2 \lesssim K b^2 |\log b|^5 \leq \frac{K}{2} b^2 |\log b|^6,
\]
where, in view of (5.89), the last inequality holds if \( b(0) \) is chosen small enough. This is the refined bound (4.20), and thus ends the proof. □

**Remark 5.3** (i) In the above proof we have actually specified the constants \( M, K \). To be explicit, we first let \( M \gg 1 \) be a large constant compared with those implicit constants from earlier estimates. Next the constant \( b^* = b^*(M) \) is chosen to be a small upper bound of \( b(t) \), making \( b(t) \) an higher-order infinitesimal relative to \( \frac{1}{\log M} \). Then we set \( K \gg 1 \) in comparison with other unnamed constants, such that the bootstrap bounds (4.14)–(4.16) and (4.13) for \( b(t) \) hold. (ii) We note that the initial bounds (4.9), (4.11) imply an open initial data set of \( b, w \) stable under small perturbation, while \( a(0) \) requires precise selection relying on \( b(0), w(0) \), through a topological argument (see (5.91), (5.92)). This actually yields a codimension one initial data set of \( u \), as mentioned in Theorem 1.1.

### 6 Description on the singularity formation

We are now ready to prove the Theorem 1.1. In this subsection, we prove the finite time blowup, show the convergence of the phase \( \Theta \), give a sharp description on the blowup speed \( \lambda \), and prove the strong convergence of the excess energy.

**Proof of Theorem 1.1.**

By Proposition 4.1, \( w \) has been trapped in the regime of (4.13), (4.14)–(4.16). Thus \( u \) satisfying (4.2) exists for \( s \in [s_0, +\infty) \). Now we suppose the lifespan of \( u \) is \( T \leq +\infty \), and study the corresponding asymptotic behavior.

**Step 1. Finite time blowup.** We apply (5.97) and \( \lambda_s/\lambda \lesssim -b \) to obtain
\[
\frac{d}{dt} \sqrt{\lambda} = \frac{\lambda_s}{2 \lambda^2 \sqrt{\lambda}} \lesssim \frac{-b}{\lambda \sqrt{\lambda}} \lesssim - \frac{1}{\sqrt{b}} \left( \frac{b(0)}{\lambda(0)} \left| \log b(0)^{n_2} \right| \right)^{\frac{1}{2}}.
\]
By the smallness and monotonicity of \( b \) (5.89), there exists constant \( C(u_0) > 0 \) such that
\[
\frac{d}{dt} \sqrt{\lambda} < -C(u_0) < 0,
\]

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which together with (5.97) implies
\[ T < +\infty, \quad \text{with} \quad \lambda(T) = b(T) = 0. \] (6.1)
Thus the scaling \( \lambda(t) \to 0 \) as \( t \to T \), and the solution \( u \) blows up at \( T \).

**Step 2. Convergence of the phase.** As illustrated in Remark 4.2, the modulation equations (4.34) on parameter \( a \) is too crude to show the convergence of the phase. We claim the following refined bound with additional logarithmic gain
\[ |a(t)| \lesssim C(\delta) \frac{b(t)}{|\log b(t)|^{\frac{3}{2}}}, \] (6.2)
for some universal small constant \( \delta > 0 \) and a large constant \( C(\delta) \). Applying (6.2) with (5.89), we obtain
\[ \lim_{s \to +\infty} \Theta(s) - \Theta(s_0) \leq \int_{s_0}^{+\infty} |a(s)| \, ds \leq \int_{s_0}^{+\infty} \frac{ds}{s(\log s)^{\frac{3}{2}}} < +\infty, \]
which implies the convergence of the phase
\[ \Theta(t) \to \Theta(T) \in \mathbb{R}, \quad \text{as} \quad t \to T. \]

It remains to prove (6.2). To this end, we choose the varying scale
\[ B_\delta = b^{-\delta}, \]
for sufficiently small constant \( \delta > 0 \), and repeat the computations as in Proposition 4.2. More precisely, we define the direction
\[ \Phi_\delta = \chi_{B_\delta} \Lambda \phi \left[ \begin{array}{c} \rho_1 \\ \rho_2 \\ 0 \end{array} \right], \quad \text{with} \quad \left\{ \begin{array}{l} \|\Phi_\delta\|_{L^2} \sim \sqrt{|\log b|}, \\ (\Phi_{1,0}, \mathbb{H}^+ \Phi_\delta) \sim |\log b|, \\ (\Phi_{0,1}, \mathbb{H}^+ \Phi_\delta) = 0. \end{array} \right. \] (6.3)

We take the scalar products of (4.22) with \( \mathbb{H}^+ \Phi_\delta \), and the resulting equation is
\[ 0 = (\partial_s w, \mathbb{H}^+ \Phi_\delta) - \frac{\lambda_s}{\lambda} (Aw, \mathbb{H}^+ \Phi_\delta) + (\rho_1 \hat{J} \mathbb{H} w, \mathbb{H}^+ \Phi_\delta) + (\rho_2 \hat{J}^2 \mathbb{H} w, \mathbb{H}^+ \Phi_\delta) + (\text{Mod}, \mathbb{H}^+ \Phi_\delta) + (\mathcal{R}, \mathbb{H}^+ \Phi_\delta). \]

Using Appendix A, we have
\[ \left| \frac{\lambda_s}{\lambda} \right| \left| (Aw, \mathbb{H}^+ \Phi_\delta) \right| \lesssim b^{1-C\delta} \left( \int \frac{|\mathbb{H}^+ (Aw)|^2}{(1 + y^4)(1 + |\log y|^4)} \right)^{\frac{1}{2}} \lesssim b^{1-C\delta} \sqrt{\mathcal{E}_4}. \]

Recalling (4.40), (4.41), we see the linear terms are bounded by
\[ |(\hat{J} \mathbb{H} w, \mathbb{H}^+ \Phi_\delta)| + |(\hat{J}^2 \mathbb{H} w, \mathbb{H}^+ \Phi_\delta)| \lesssim \sqrt{|\log b|} \sqrt{\mathcal{E}_4}, \]

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where the $\sqrt{|\log b|}$ comes from the $L^2$ norm of $\Phi_\delta$. Moreover, from (3.54), (4.34), and also the orthogonality of $\Phi_{0,1}$ and $\Phi_\delta$ (6.3), we observe

$$\langle \text{Mod}, \mathbb{H}^\perp \Phi_\delta \rangle = \langle \text{Mod}, \mathbb{H}^\perp \Phi_\delta \rangle = a_s(\Phi_{1,0}, \mathbb{H}^\perp \Phi_\delta) + O(b^{1-C\delta}U(t)).$$

The estimate for $\tilde{\Psi}_0$ follows from (3.57), and the bound for $\mathcal{R}$ can be obtained by simply repeating the computation (4.42), (4.43), with an extra $b^{1+C\delta}$ smallness from the $L^2$ norm of $\mathbb{H}^\perp \Phi_\delta$. Combining these bounds, we obtain

$$a_s(\Phi_{1,0}, \mathbb{H}^\perp \Phi_\delta) + \langle \partial_s w, \mathbb{H}^\perp \Phi_\delta \rangle \lesssim \frac{b^2}{|\log b|^2}. \quad (6.4)$$

Now we let

$$\tilde{a} = a + \frac{(w, \mathbb{H}^\perp \Phi_\delta)}{\langle \Phi_{1,0}, \mathbb{H}^\perp \Phi_\delta \rangle} = a + O\left( b^{-C\delta} \frac{b^2}{|\log b|} \right).$$

and its derivative follows from (6.4):

$$\tilde{a}_s = a_s + \frac{\langle \partial_s w, \mathbb{H}^\perp \Phi_\delta \rangle}{\langle \Phi_{1,0}, \mathbb{H}^\perp \Phi_\delta \rangle} + O\left( \frac{b^{3-C\delta}}{|\log b|^2} \right) \lesssim \frac{b^2}{|\log b|^2}.$$  

Integrating this from $s = +\infty$ where $\tilde{a} = 0$ to the present time $s$ using (5.89), we obtain the bound

$$|\tilde{a}| \lesssim \frac{b}{|\log b|^2}.$$  

This yields the corresponding bound for $a$ (6.2).

**Step 3. Derivation of the blowup speed.** We recall the modulation equations (4.34) that

$$b_s = -b^2 \left( 1 + \frac{2}{|\log b|} \right) + O\left( \frac{b^2}{\sqrt{\log M} |\log b|} \right),$$

From the rough asymptotics (5.89), we assume $b = 1/s + f/s^2$ with $|f| \ll s$, and $s$ sufficiently large to derive the refined estimate

$$b(s) = \frac{1}{s} - \frac{2}{s \log s} + O\left( \frac{1}{s (\log s)^2} \right).$$

Moreover, from $b + \lambda_s/\lambda = O(b^3)$, we see the concentration scale $\lambda$ admits

$$\frac{\lambda_s}{\lambda} = -\frac{1}{s} + \frac{2}{s \log s} + O\left( \frac{1}{s (\log s)^2} \right), \quad (6.5)$$

from which we have

$$\frac{1}{s} = C(u_0)(1 + o(1)) \frac{\lambda}{|\log \lambda|^2}.$$  

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Using $-1/s \sim \lambda_t$ by (6.5), we get
\[
\lambda_t = -C(u_0)(1 + o(1)) \frac{1}{|\log \lambda|^2}.
\] (6.6)

Integrating this from $t$ to $T$, the blowup speed (1.8) follows
\[
\lambda(t) = C(u_0)(1 + o(1)) \frac{(T - t)}{|\log(T - t)|^2}.
\] (6.7)

Combining (6.6) with (6.7), and applying $b + \lambda_s/\lambda = O(b^3)$ again, we obtain
\[
b(t) = C(u_0)(1 + o(1)) \frac{(T - t)}{|\log(T - t)|^2}.
\] (6.8)

**Step 4. Convergence of the excess energy.** From (4.2) we have the decomposition
\[u = e^{\Theta R}Q_\lambda + \tilde{u},\] where $\tilde{u} := e^{\Theta R} \hat{v}_\lambda$.

Owing to the dissipative Dirichlet energy (1.3) and the energy-critical scaling, the $\dot{H}^1$ norm of $\tilde{u}$ is bounded
\[
\|\nabla \tilde{u}\|_{L^2} \leq \|\nabla u\|_{L^2} + \|\nabla (e^{\Theta R} Q_\lambda)\|_{L^2} \leq \|\nabla u_0\|_{L^2} + \|\nabla Q\|_{L^2} \lesssim C(u_0).
\]

Moreover, we have the $\dot{H}^2$ bound
\[
\|\Delta \tilde{u}\|_{L^2} \lesssim C(u_0).
\] (6.9)

Indeed, from (2.16), (2.26), (A.27), we have
\[
\|\Delta \tilde{u}\|_{L^2} = \frac{1}{\lambda^2} \int |\Delta \hat{v}|^2 \lesssim \frac{1}{\lambda^2} \left( \int |\mathbb{H} \hat{w}|^2 + \int \frac{|\hat{w}|^2}{1 + y^8} \right) \lesssim \frac{1}{\lambda^2} (\mathcal{E}_2 + b^2 |\log b|^2).
\] (6.10)

The inequality (5.103) with the explicit asymptotics (6.7), (6.8) gives
\[
\mathcal{E}_2 \lesssim b^2 |\log b|^2 + \lambda^2 b(0)^{10} + \lambda^2 \int_0^t \frac{dt}{(T - t)|\log(T - t)|^2} \lesssim b^2 |\log b|^2 + \lambda^2.
\]

Injecting this into (6.10) with an application of (5.97) yields the desired $\dot{H}^2$ boundedness. By a simple localization process, this boundedness yields a strong convergence of $\nabla \tilde{u}$ outside the blowup point (the origin). More precisely, there exists $u^* \in \dot{H}^1$, such that for any $R > 0$ there holds
\[
\|\nabla u - \nabla u^*\|_{L^2(|x|>R)} \sim \|\nabla \tilde{u} - \nabla u^*\|_{L^2(|x|>R)} \to 0, \quad \text{as} \quad t \to T,
\]

which yields (1.7). Now the convergence (1.7) together with the $\dot{H}^2$ bound (6.9) gives (1.10). This concludes the proof of Theorem 1.1. \qed
A Coercivity estimates and interpolation estimates

In this appendix, we list some results on the coercivity of the Hamiltonians $H, H^2$, and also the interpolation estimates used in the proof of Proposition 4.2, 5.1. The complete proof can be found in [34].

Lemma A.1 (Coercivity of $H$ [34]) Let $M \geq 1$ be a large enough universal constant. Let $\Phi_M$ be given by (4.5). Then there exists a universal constant $C(M) > 0$ such that for all radially symmetric function $u \in H^1$ satisfying

$$\int \frac{|u|^2}{y^4(1+|\log y|)^2} + \int |\partial_y (Au)|^2 < +\infty,$$

and the orthogonality conditions

$$(u, \Phi_M) = 0,$$

there holds

$$\int_{y \geq 1} \frac{|\partial_y^2 u|^2}{1 + |\log y|^2} + \int \frac{|\partial_y u|^2}{y^2(1+|\log y|)^2} + \int \frac{|u|^2}{y^4(1+|\log y|)^2} \leq C(M) \int |Hu|^2.$$  \hspace{1cm} (A.1)

Lemma A.2 (Coercivity of $H^2$ [34]) Assume the conditions in Lemma A.1. Then there exists a universal constant $C(M) > 0$ such that for all radially symmetric function $u$ satisfying

$$\int |\partial_y Ahu|^2 + \int \frac{|Ah u|^2}{y^2(1+y^2)} + \int \frac{|Hu|^2}{y^4(1+|\log y|)^2} + \int \frac{(\partial_y u)^2}{y^4(1+y^4)(1 + |\log y|)^2} < +\infty,$$

and the orthogonality conditions

$$(u, \Phi_M) = 0, \quad (Hu, \Phi_M) = 0,$$

there holds

$$\int \frac{|Hu|^2}{y^4(1+|\log y|)^2} + \int \frac{|\partial_y Hu|^2}{y^2(1+|\log y|)^2} + \int \frac{|\partial_y^2 u|^2}{(1 + |\log y|)^2} + \int \frac{|\partial_y^2 u|^2}{y^2(1+|\log y|)^2} + \int \frac{|\partial_y u|^2}{y^2(1+y^4)(1 + |\log y|)^2} + \int \frac{|u|^2}{y^4(1+y^4)(1 + |\log y|)^2} \leq C(M) \int |H^2 u|^2.$$  \hspace{1cm} (A.2)
We recall the notations
\[ w = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = w^\perp + \gamma \epsilon_z, \quad w_2 = \tilde{J}H w^\perp = w^\perp_0 + w^\perp_1. \]
and assume the bootstrap bounds (4.14)–(4.16), then the following interpolation bounds are the consequences of the coercivity estimate (A.1), (A.2) and the regularity of \( w \) at the origin ensured by the smoothness of the LL flow \( u \).

**Lemma A.3 (Interpolation bounds for \( w^\perp \) [34])** There holds:

\[ \begin{align*}
\int \frac{|w^\perp|^2}{y^4(1+y^4)(1+|\log y|^2)} + \int \frac{|\partial^i_y w^\perp|^2}{y^2(1+y^6-2i)(1+|\log y|^2)} &\lesssim C(M)E_4, \quad 1 \leq i \leq 3, \quad (A.3) \\
\int_{|y| \geq 1} \frac{|\partial^i_y w^\perp|^2}{(1+y^4-2i)(1+|\log y|^2)} &\lesssim C(M)E_2, \quad 1 \leq i \leq 2, \quad (A.4) \\
\int_{|y| \geq 1} \frac{1+|\log y|^C}{y^2(1+|\log y|^2)(1+y^6-2i)} |\partial^i_y w^\perp|^2 &\lesssim b^4 |\log b|^{C_1(C)}, \quad 0 \leq i \leq 3, \quad (A.5) \\
\int_{|y| \geq 1} \frac{1+|\log y|^C}{y^2(1+|\log y|^2)(1+y^6-2i)} |\partial^i_y w^\perp|^2 &\lesssim b^3 |\log b|^{C_1(C)}, \quad 0 \leq i \leq 2, \quad (A.6) \\
\int_{|y| \geq 1} |\partial^i_y H w^\perp|^2 &\lesssim b^3 |\log b|^6, \quad (A.7) \\
\|w^\perp\|_{L^\infty} &\lesssim \delta(b^*), \quad (A.8) \\
\|A w^\perp\|^2_{L^\infty} &\lesssim b^2 |\log b|^9, \quad (A.9) \\
\int_{y \leq 1} \frac{|A w^\perp|^2}{y^6(1+|\log y|^2)} &\lesssim C(M)E_4, \quad (A.10) \\
\left\| \frac{A w^\perp}{y^2(1+|\log y|)} \right\|^2_{L^\infty(y \leq 1)} + \left\| \frac{\Delta A w^\perp}{1+|\log y|} \right\|^2_{L^\infty(y \leq 1)} + \left\| \frac{H w^\perp}{y(1+|\log y|)} \right\|^2_{L^\infty(y \leq 1)} &\lesssim b^4, \quad (A.11) \\
\left\| \frac{\partial^i_y w^\perp}{y(1+|\log y|)} \right\|^2_{L^\infty(y \leq 1)} &\lesssim b^4, \quad (A.12) \\
\left\| \frac{w^\perp}{y} \right\|^2_{L^\infty(y \leq 1)} + \left\| \partial^i_y w^\perp \right\|^2_{L^\infty(y \leq 1)} &\lesssim b^4, \quad (A.13) \\
\left\| \frac{w^\perp}{y} \right\|^2_{L^\infty(y \geq 1)} + \left\| \partial^i_y w^\perp \right\|^2_{L^\infty(y \geq 1)} &\lesssim b^2 |\log b|^8, \quad (A.14) \\
\left\| \frac{w^\perp}{1+y^2} \right\|^2_{L^\infty} + \left\| \partial^i_y w^\perp \right\|^2_{L^\infty(y \geq 1)} &\lesssim C(M) b^2 |\log b|^2. \quad (A.15)
\end{align*} \]
Lemma A.4 (Interpolation bounds for $\gamma$ [34]) There holds:

\[
\begin{align*}
\int \frac{|\gamma|^2}{y^2(1 + y^2)(1 + |\log y|^2)} + \int \frac{|\partial_y \gamma|^2}{y^4(1 + y^4)(1 + |\log y|^2)} \\
+ \int \frac{|\partial_y^2 \gamma|^2}{y^2(1 + y^6)(1 + |\log y|^2)} \lesssim \delta(b^*) \left( E_4 + \frac{b^4}{|\log b|^2} \right), \quad 2 \leq i \leq 3, \tag{A.16}
\end{align*}
\]

\[
\int_{y \geq 1} \frac{1 + |\log y|^2}{y^4(1 + |\log y|^2)} |\partial_y \gamma|^2 \lesssim b^4 |\log b|^{C_1(C)}, \quad 0 \leq i \leq 2, \tag{A.17}
\]

\[
\int_{y \geq 1} \frac{1 + |\log y|^2}{y^6(1 + |\log y|^2)} |\partial_y^2 \gamma|^2 \lesssim b^3 |\log b|^{C_1(C)}, \quad 0 \leq i \leq 2, \tag{A.18}
\]

\[
\left\| \frac{1 + |y|^2}{y^2} \right\|_{L^\infty} \lesssim \delta(b^*), \tag{A.20}
\]

\[
\left\| \frac{1 + |\log y|^2}{y^2} \right\|^2_{L^\infty} + \|\partial_y \gamma\|^2_{L^\infty} \lesssim b^2 |\log b|^8, \tag{A.21}
\]

\[
\left\| \frac{\gamma}{y(1 + y)} \right\|^2_{L^\infty} + \left\| \frac{\partial_y \gamma}{y} \right\|^2_{L^\infty} \lesssim C(M) b^3 |\log b|^2. \tag{A.22}
\]

\[
\int |\Delta \gamma|^2 \lesssim \delta(b^*) E_2 + b^2 |\log b|^2, \tag{A.23}
\]

\[
\|\Delta \gamma\|^2_{L^\infty(y \geq 1)} \lesssim b^3 |\log b|^8, \tag{A.24}
\]

\[
\int |w^+|^2 |\Delta \gamma|^2 + \int_{y \geq 1} |\Delta^2 \gamma|^2 \lesssim \delta(b^*) \left( E_4 + \frac{b^4}{|\log b|^2} \right). \tag{A.25}
\]

Lemma A.5 (Interpolation bounds for $w_2$ [34]) There holds:

\[
\int |w_2|^2 = E_2 + O(b^2 |\log b|^2 + \delta(b^*) E_2), \tag{A.26}
\]

\[
\|\mathcal{R} w\|^2 \lesssim E_2 + b^2 |\log b|^2, \tag{A.27}
\]

\[
\frac{||w||^2}{(1 + y^4)(1 + |\log y|^2)} \lesssim C(M) E_4, \tag{A.28}
\]

\[
\frac{|w_2|^2}{(1 + y^4)(1 + |\log y|^2)} \lesssim C(M) E_4, \tag{A.29}
\]

\[
\frac{|w_2^2|^2}{(1 + y^4)(1 + |\log y|^2)} \lesssim \delta(b^*) \left( E_4 + \frac{b^4}{|\log b|^2} \right). \tag{A.30}
\]
\[ \int |\mathbb{H}w_2| \lesssim C(M) \left( E + \frac{b^4}{|\log b|^2} \right), \quad \text{(A.31)} \]
\[ \int |\mathbb{H} w_2^2| \lesssim \mathcal{E}_4 + \frac{b^4}{|\log b|^2}, \quad \text{(A.32)} \]
\[ \int |\mathbb{H} w_2^4| + \int |\mathbb{H} R^2 w_2^2| \lesssim \delta(b^*) \left( \mathcal{E}_4 + \frac{b^4}{|\log b|^2} \right), \quad \text{(A.33)} \]
\[ \int |\mathbb{H} \hat{J} w_2^2| \lesssim C(M) \frac{b^4}{|\log b|^2}. \quad \text{(A.34)} \]

### B Proof of Lemma 5.4

This appendix is devoted to prove Lemma 5.4. The proof is basically algebraic. The key here is to explore the structure of the LHS of (5.61).

**Proof of Lemma 5.4.**

**Step 1. Gain of two derivatives.** Let

\[ a = \alpha e_r + \beta e_\tau + \gamma Q, \quad \Gamma = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}, \]

be a decomposition of the vector \( a \) under the Frenet basis, with its components \((\alpha, \beta, \gamma)\) functions of the radial variable \( y \) satisfying the constraint

\[ \alpha^2 + \beta^2 + (1 + \gamma)^2 = 1. \]

To ease the notations, we introduce the vectors

\[ Z_1 = (1 + Z)e_y, \quad Z_2 = \frac{Z}{y} e_z - (1 + Z)e_x, \]

based on which we apply Lemma 2.1 and obtain

\[ \partial_y a = \partial_y \Gamma + Z_1 \wedge \Gamma, \quad \partial_\tau a = Z_2 \wedge \Gamma. \quad \text{(B.1)} \]

We also recall the double wedge formula

\[ a \wedge (b \wedge c) = (a \cdot c)b - (a \cdot b)c. \]

In view of (2.16), (2.19), we compute

\[ -\Gamma \cdot \mathbb{H} \mathbb{H} \Gamma = -\Gamma \cdot (e_z + \hat{w}) \wedge \mathbb{H} \Gamma = a \cdot \hat{v} \wedge (\Delta a + |\nabla Q|^2 a) = a \cdot (\hat{v} \wedge \Delta a). \quad \text{(B.2)} \]

which together with the action of the Laplacian in the Frenet basis gives

\[ \int a \cdot (\hat{v} \wedge \Delta a) = \int \Delta a \cdot (a \wedge \hat{v}) = \int a \cdot \Delta (a \wedge \hat{v}) = \int a \cdot (\Delta a \wedge \hat{v} + 2 \nabla a \wedge \nabla \hat{v}), \]

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and thus
\[ \int a \cdot (\hat{v} \wedge \Delta a) = \int a \cdot (\nabla a \wedge \nabla \hat{v}). \] (B.3)

Combining (B.1), (B.2), (B.3), we let \( \Gamma = \hat{\mathcal{H}} \Gamma \), and compute the LHS of (5.61):
\[
- \int \hat{\mathcal{H}} \Gamma \cdot (\hat{\mathcal{H}} \Gamma)^2 = \int \hat{\mathcal{H}} \Gamma \cdot \partial_y \hat{\mathcal{H}} \Gamma \wedge \partial_y \hat{w} + \int \hat{\mathcal{H}} \Gamma \cdot (Z_1 \wedge \hat{\mathcal{H}} \Gamma) \wedge \partial_y \hat{w} + \int \hat{\mathcal{H}} \Gamma \cdot \partial_y \hat{\mathcal{H}} \Gamma \wedge (Z_1 \wedge (e_z + \hat{w})) + \int \hat{\mathcal{H}} \Gamma \cdot (Z_1 \wedge \hat{\mathcal{H}} \Gamma) \wedge (Z_1 \wedge (e_z + \hat{w})) + \int \hat{\mathcal{H}} \Gamma \cdot (Z_2 \wedge \hat{\mathcal{H}} \Gamma) \wedge (Z_2 \wedge (e_z + \hat{w})),
\] (B.4)

which gives a two-derivatives gain. The normalization of \((e_z + \hat{w})\) and the structures of \(Z_1, Z_2\) produce some cancellations in the RHS of (B.4). To see this, we let
\[ \Gamma_2 = \mathcal{H} \Gamma, \] (B.5)
then the first term on the RHS of (B.4) can be simplified to
\[
\int \hat{J}_2 \cdot \partial_y \hat{J}_2 \wedge \partial_y \hat{w} = \int \hat{J}_2 \cdot [\partial_y \hat{w} \wedge \hat{J}_2 + (e_z + \hat{w}) \wedge \partial_y \hat{J}_2] \wedge \partial_y \hat{w} = \int (\hat{J}_2 \cdot \hat{J}_2) (\partial_y \hat{w} \cdot \partial_y \hat{w}) - \int (\hat{J}_2 \cdot \partial_y \hat{w}) (\hat{J}_2 \cdot \partial_y \hat{w}) + \int (\hat{J}_2 \cdot \partial_y \hat{w}) ((e_z + \hat{w}) \cdot \partial_y \hat{w}) - \int (\hat{J}_2 \cdot (e_z + \hat{w})) (\partial_y \hat{J}_2 \cdot \partial_y \hat{w}) = -\int (\hat{J}_2 \cdot \partial_y \hat{w}) (\hat{J}_2 \cdot \partial_y \hat{w}).
\]

The second term on the RHS of (B.4) is
\[
\int \hat{J}_2 \cdot (Z_1 \wedge \hat{J}_2) \wedge \partial_y \hat{w} = \int \hat{J}_2 \cdot [\hat{J}_2 (Z_1 \cdot \partial_y \hat{w}) - Z_1 (\hat{J}_2 \cdot \partial_y \hat{w})] = \int (1 + Z) \hat{J}_2 \hat{J}_2 (e_y \cdot \partial_y \hat{w}) - \int (1 + Z) (e_y \cdot \hat{J}_2 \hat{J}_2) (\partial_y \hat{w} \cdot \hat{J}_2).
\]
Moreover, the third term is
\[
\int \hat{J} \gamma_2 \cdot \partial_y \hat{J} \gamma_2 \wedge (Z_1 \wedge (e_z + \hat{w}))
\]
\[
= \int (\mathcal{H} \hat{J} \gamma \cdot Z_1) \left( \partial_y \hat{J} \gamma_2 \cdot (e_z + \hat{w}) \right) - (\mathcal{H} \hat{J} \gamma \cdot (e_z + \hat{w})) \left( \partial_y \hat{J} \gamma_2 \cdot e_y \right)
\]
\[
= \int (1 + Z) \left( \hat{J} \gamma_2 \cdot e_y \right) \left[ \left( \partial_y \hat{w} \wedge \hat{J} \gamma_2 + \hat{J} \partial_y \hat{J} \gamma_2 \right) \cdot (e_z + \hat{w}) \right]
\]
\[
= - \int (1 + Z) \left( e_y \cdot \hat{J} \gamma_2 \right) \left( \partial_y \hat{w} \cdot \hat{J} \gamma_2 \right).
\]

We observe
\[
Z_1 \wedge e_z = (1 + Z)e_x, \quad Z_2 \wedge e_z = (1 + Z)e_y, \quad \text{ (B.6)}
\]
then the last two terms of (B.4) can be reformulated as
\[
\int \hat{J} \gamma_2 \cdot (Z_1 \wedge \hat{J} \gamma_2) \wedge (Z_1 \wedge (e_z + \hat{w})) + \int \hat{J} \gamma_2 \cdot (Z_2 \wedge \hat{J} \gamma_2) \wedge (Z_2 \wedge (e_z + \hat{w}))
\]
\[
= - \int (\hat{J} \gamma_2 \cdot Z_1) (\hat{J} \gamma_2 \cdot Z_1 \wedge e_z) + \int \hat{J} \gamma_2 \cdot (Z_1 \wedge \hat{J} \gamma_2) \wedge (Z_1 \wedge \hat{w})
\]
\[
- \int (\hat{J} \gamma_2 \cdot Z_2) (\hat{J} \gamma_2 \cdot Z_2 \wedge e_z) + \int \hat{J} \gamma_2 \cdot (Z_2 \wedge \hat{J} \gamma_2) \wedge (Z_2 \wedge \hat{w})
\]
\[
= - \int \frac{Z(1 + Z)}{y} \left( e_z \cdot \hat{J} \gamma_2 \right) \left( e_y \cdot \hat{J} \gamma_2 \right)
\]
\[
+ \int \hat{J} \gamma_2 \cdot (Z_1 \wedge \hat{J} \gamma_2) \wedge (Z_1 \wedge \hat{w}) + \int \hat{J} \gamma_2 \cdot (Z_2 \wedge \hat{J} \gamma_2) \wedge (Z_2 \wedge \hat{w}).
\]

Injecting these computations into (B.4), we have the following refined formula
\[
- \int \mathcal{H} \hat{J} \gamma \cdot (\mathcal{H} \hat{J} \gamma)^2 \hat{J} \gamma
\]
\[
= - \int (\mathcal{H} \hat{J} \gamma \cdot \partial_y \hat{w}) (\mathcal{H} \hat{J} \gamma \cdot \partial_y \hat{w}) + \int (1 + Z) (e_y \cdot \partial_y \hat{w}) |\mathcal{H} \hat{J} \gamma|^2
\]
\[
- 2 \int (1 + Z) (e_y \cdot \mathcal{H} \hat{J} \gamma) (\partial_y \hat{w} \cdot \mathcal{H} \hat{J} \gamma) - \int \frac{Z(1 + Z)}{y} \left( e_z \cdot \mathcal{H} \hat{J} \gamma \right) \left( e_y \cdot \mathcal{H} \hat{J} \gamma \right)
\]
\[
+ \int \mathcal{H} \hat{J} \gamma \cdot (Z_1 \wedge \mathcal{H} \hat{J} \gamma) \wedge (Z_1 \wedge \hat{w}) + \int \mathcal{H} \hat{J} \gamma \cdot (Z_2 \wedge \mathcal{H} \hat{J} \gamma) \wedge (Z_2 \wedge \hat{w}). \quad \text{ (B.7)}
\]

Step 2. Extraction of the leading terms. To derive precise bounds for the RHS of (B.7), let us introduce the following decomposition of \( \hat{w}, \hat{w}_0 \) (This is different from decomposition (3.18)): \[
\hat{w} = \hat{w}_0^1 + \hat{w}_0^2 + w,
\]
with \[
\begin{cases}
\hat{w}_0^1 = b\Phi_{0,1} = b\chi_{B_1} \Phi_{0,1}, \\
\hat{w}_0^2 = \hat{w}_0 - \hat{w}_0^1.
\end{cases} \quad \text{ (B.8)}
\]

By (A.13), (A.21), the first term on the RHS of (B.7) is bounded by
\[
\left| \int (\mathcal{H} \hat{J} \gamma \cdot \partial_y \hat{w}) (\mathcal{H} \hat{J} \gamma \cdot \partial_y \hat{w}) \right| \leq \| \partial_y \hat{w} \|^2_{L^\infty} \| \mathcal{H} \hat{J} \gamma \|^2_{L^2} \lesssim b \delta(b^*) \| \mathcal{H} \hat{J} \gamma \|^2_{L^2}.
\]
According to (B.8), we apply (A.15), (A.22), and \( |a| \leq b \log b^{-1} \) to compute the second term
\[
\int (1+Z)(e_y \cdot \partial_y \tilde{w}) |\hat{\mathbb{H}} \Gamma|^2
\]
\[
= b \int (1+Z) \partial_y \tilde{\Phi}_{0,1}^{(2)} |\hat{\mathbb{H}} \Gamma|^2 + \int (1+Z) \left( e_y \cdot \partial_y (\tilde{w}_0^2 + w) \right) |\hat{\mathbb{H}} \Gamma|^2
\]
\[
= b \int (1+Z) \partial_y \tilde{\Phi}_{0,1}^{(2)} |\hat{\mathbb{H}} \Gamma|^2 + O \left( b \delta(b^*) \| \hat{\mathbb{H}} \Gamma \|_{L^2}^2 \right).
\]
Similarly, the third term on the RHS of (B.7) is
\[
-2 \int (1+Z) (e_y \cdot \hat{\mathbb{H}} \Gamma) (\partial_y \tilde{w} \cdot \hat{\mathbb{H}} \Gamma)
\]
\[
= -2b \int (1+Z) \left( e_y \cdot \hat{\mathbb{H}} \Gamma \right) \left( \partial_y \tilde{\Phi}_{0,1} \cdot \hat{\mathbb{H}} \Gamma \right)
\]
\[
-2 \int (1+Z) \left( e_y \cdot \hat{\mathbb{H}} \Gamma \right) \left( \partial_y (\tilde{w}_0^2 + w) \cdot \hat{\mathbb{H}} \Gamma \right)
\]
\[
\leq -2b \int (1+Z) \partial_y \tilde{\Phi}_{0,1}^{(1)} \left( e_x \cdot \hat{\mathbb{H}} \Gamma \right) \left( e_y \cdot \hat{\mathbb{H}} \Gamma \right)
\]
\[
-2b \int (1+Z) \partial_y \tilde{\Phi}_{0,1}^{(2)} \left( e_y \cdot \hat{\mathbb{H}} \Gamma \right)^2 + O \left( b \delta(b^*) \| \hat{\mathbb{H}} \Gamma \|_{L^2}^2 \right).
\]
Moreover, from (2.25) we note the simple identities
\[
e_x \cdot \hat{\mathbb{H}} \Gamma = e_x \cdot (e_x + \tilde{w}) \wedge \mathbb{H} \Gamma = e_x \cdot \tilde{w} \wedge \mathbb{H} \Gamma,
\]
\[
e_x \cdot \mathbb{H} \Gamma = e_y \cdot (e_x \wedge \mathbb{H} \Gamma) = e_y \cdot \hat{\mathbb{H}} \Gamma - e_y \cdot \mathbb{R}_\tilde{w} \mathbb{H} \Gamma,
\]
\[
e_y \cdot \mathbb{H} \Gamma = -e_x \cdot (e_x \wedge \mathbb{H} \Gamma) = -e_x \cdot \hat{\mathbb{H}} \Gamma + e_y \cdot \mathbb{R}_\tilde{w} \mathbb{H} \Gamma,
\]
from which the fourth term on the RHS of (B.7) is bounded by
\[
- \int \frac{Z(1+Z)}{y} \left( e_x \cdot \hat{\mathbb{H}} \Gamma \right) \left( e_y \cdot \hat{\mathbb{H}} \Gamma \right)
\]
\[
\leq -b \int \frac{Z(1+Z)}{y} \left( e_x \cdot \tilde{\Phi}_{0,1} \wedge \mathbb{H} \Gamma \right) \left( e_y \cdot \hat{\mathbb{H}} \Gamma \right) + \int O \left( \frac{|\tilde{w}_0^2| + |w|}{y(1+y^2)} \right) |\hat{\mathbb{H}} \Gamma|^2
\]
\[
\leq b \int \frac{Z(1+Z)}{y} \tilde{\Phi}_{0,1}^{(1)} \left( e_x \cdot \hat{\mathbb{H}} \Gamma \right) \left( e_y \cdot \hat{\mathbb{H}} \Gamma \right) + b \int \frac{Z(1+Z)}{y} \tilde{\Phi}_{0,1}^{(2)} \left( e_y \cdot \hat{\mathbb{H}} \Gamma \right)^2
\]
\[
+ O \left( \frac{\| \chi B \cdot T \|_{L^\infty}}{y(1+y^2)} \| \tilde{w} \|_{L^\infty} \right) \| \hat{\mathbb{H}} \Gamma \|_{L^2}^2 + O \left( \frac{\| \tilde{w}_0^2 \| + |w|}{y(1+y^2)} \| \hat{\mathbb{H}} \Gamma \|_{L^\infty} \right) \| \hat{\mathbb{H}} \Gamma \|_{L^2}^2
\]
\[
\leq b \int \frac{Z(1+Z)}{y} \tilde{\Phi}_{0,1}^{(1)} \left( e_x \cdot \hat{\mathbb{H}} \Gamma \right) \left( e_y \cdot \hat{\mathbb{H}} \Gamma \right)
\]
\[
+ b \int \frac{Z(1+Z)}{y} \tilde{\Phi}_{0,1}^{(2)} \left( e_y \cdot \hat{\mathbb{H}} \Gamma \right)^2 + O \left( b \delta(b^*) \| \hat{\mathbb{H}} \Gamma \|_{L^2}^2 \right).
\]
Furthermore, we compute the fifth term on the RHS of (B.7):

\[
\int \mathcal{J} \mathcal{H} \Gamma \cdot (Z_1 \wedge \mathcal{J} \mathcal{H} \Gamma) \wedge (Z_1 \wedge \hat{w}) = \int (Z_1 \cdot \mathcal{J} \mathcal{H} \Gamma) (\hat{w} \cdot Z_1 \wedge \mathcal{J} \mathcal{H} \Gamma) \\
= b \int (1 + Z)^2 \left( \hat{\phi}_{0,1} \cdot e_y \wedge \mathcal{J} \mathcal{H} \Gamma \right) (e_y \cdot \mathcal{J} \mathcal{H} \Gamma) \\
+ b \int (1 + Z)^2 \left( (\hat{w}_0^2 + w) \cdot e_y \wedge \mathcal{J} \mathcal{H} \Gamma \right) (e_y \cdot \mathcal{J} \mathcal{H} \Gamma) \\
\leq b \int (1 + Z)^2 \hat{\phi}_{0,1}^{(1)} (e_y \cdot \mathcal{J} \mathcal{H} \Gamma) (e_y \cdot \mathcal{J} \mathcal{H} \Gamma) + O \left( b \delta(b^*) \|\mathcal{J} \mathcal{H} \Gamma\|^2_{L^2} \right).
\]

Finally, for the last term on the RHS of (B.7), we have

\[
\int \mathcal{J} \mathcal{H} \Gamma \cdot (Z_2 \wedge \mathcal{J} \mathcal{H} \Gamma) \wedge (Z_2 \wedge \hat{w}) = \int (Z_2 \cdot \mathcal{J} \mathcal{H} \Gamma) (\hat{w} \cdot Z_2 \wedge \mathcal{J} \mathcal{H} \Gamma) \\
= b \int (Z_2 \cdot \mathcal{J} \mathcal{H} \Gamma) \left( (\hat{\phi}_{0,1} \wedge Z_2) \cdot \mathcal{J} \mathcal{H} \Gamma \right) + \int (Z_2 \cdot \mathcal{J} \mathcal{H} \Gamma) \left( (\hat{w}_0^2 + w) \wedge Z_2 \right) \cdot \mathcal{J} \mathcal{H} \Gamma \\
\leq b \int \frac{Z(1 + Z)}{y} \hat{\phi}_{0,1}^{(1)} (e_x \cdot \mathcal{J} \mathcal{H} \Gamma) (e_y \cdot \mathcal{J} \mathcal{H} \Gamma) \\
- b \int \frac{Z(1 + Z)}{y} \hat{\phi}_{0,1}^{(2)} (e_x \cdot \mathcal{J} \mathcal{H} \Gamma)^2 + \mathcal{R}_G + O \left( b \delta(b^*) \|\mathcal{J} \mathcal{H} \Gamma\|^2_{L^2} \right),
\]

where \( \mathcal{R}_G \) consists of the terms involving \( (e_x \cdot \mathcal{J} \mathcal{H} \Gamma) \):

\[
\mathcal{R}_G = -b \int \left( \frac{Z}{y} \right)^2 \hat{\phi}_{0,1}^{(1)} (e_x \cdot \mathcal{J} \mathcal{H} \Gamma) (e_y \cdot \mathcal{J} \mathcal{H} \Gamma) + b \int \frac{Z(1 + Z)}{y} \hat{\phi}_{0,1}^{(2)} (e_x \cdot \mathcal{J} \mathcal{H} \Gamma)^2 \\
+ b \int \left( \frac{Z}{y} \right)^2 \hat{\phi}_{0,1}^{(2)} (e_x \cdot \mathcal{J} \mathcal{H} \Gamma) (e_x \cdot \mathcal{J} \mathcal{H} \Gamma) - b \int (1 + Z)^2 \hat{\phi}_{0,1}^{(1)} (e_x \cdot \mathcal{J} \mathcal{H} \Gamma) (e_x \cdot \mathcal{J} \mathcal{H} \Gamma).
\]

Combing these computations with (B.7), and applying the definition of function \( Z \) (2.3) the operator \( A \) (2.29), we obtain the following inequality

\[
- \int \mathcal{J} \mathcal{H} \Gamma \cdot (\mathcal{J} \mathcal{H} \Gamma)^2 \Gamma \\
\leq b \int (1 + Z) A \hat{\phi}_{0,1}^{(2)} \left[ (e_y \cdot \mathcal{J} \mathcal{H} \Gamma)^2 - (e_x \cdot \mathcal{J} \mathcal{H} \Gamma)^2 - (e_x \cdot \mathcal{J} \mathcal{H} \Gamma)^2 \right] \\
+ 2b \int (1 + Z) A \hat{\phi}_{0,1}^{(1)} (e_x \cdot \mathcal{J} \mathcal{H} \Gamma) (e_x \cdot \mathcal{J} \mathcal{H} \Gamma) \\
+ b \int \frac{V}{y^2} \hat{\phi}_{0,1}^{(1)} (e_x \cdot \mathcal{J} \mathcal{H} \Gamma) (e_x \cdot \mathcal{J} \mathcal{H} \Gamma) - b \int \frac{V}{y^2} \hat{\phi}_{0,1}^{(1)} (e_y \cdot \mathcal{J} \mathcal{H} \Gamma) (e_x \cdot \mathcal{J} \mathcal{H} \Gamma). \tag{B.9}
\]
Step 3. **Upper bound on the quadratic terms.** Recalling (2.29), for $\tilde{T}_1 = \chi_{B_1} T_1$, we claim that there exists a constant $d_1 \in (0, 1)$ such that
\[
\forall y > 0, \quad 0 \leq (1 + Z) A \tilde{T}_1 \leq \frac{1}{2} (1 - d_1).
\] (B.10)

We prove this inequality for $T_1$, but the same result follows for $\tilde{T}_1$ immediately. First we note from (2.30) that
\[
\frac{1}{y \Lambda \phi} \partial_y (y \Lambda \phi A T_1) = A^* (A T_1) = \Lambda \phi,
\]
which yields the explicit expression of $(1 + Z) AT_1$:
\[
(1 + Z) AT_1 = \frac{1 + Z}{y \Lambda \phi} \int_0^y \tau \Lambda \phi^2 (\tau) d\tau = \frac{2 \log(1 + y^2)}{y^2} - \frac{2}{1 + y^2}.
\] (B.11)

To show (B.10), we define the functions
\[
f_1(y) = 2 \log(1 + y) - \frac{2y}{1 + y}, \quad f_2(y) = f_1(y) - \frac{y}{2}.
\]
By direct computations, we see
\[
f_1(0) = f_2(0) = 0, \quad \forall y > 0, \quad f_1'(y) > 0 \geq f_2'(y),
\]
where $f_2'(y) = 0$ holds for $y = 1$ only. This actually implies
\[
\forall y > 0, \quad 0 < \frac{2 \log(1 + y^2)}{y^2} - \frac{2}{1 + y^2} < \frac{1}{2},
\]
which together with (B.11) yields (B.10). Consequently, the quadratic terms in (B.9) can be estimated. By the explicit formula of $\Phi_{0,1}$ (3.22), we have
\[
b \int (1 + Z) A \tilde{T}_1 \left[ (e_y \cdot \tilde{J} \tilde{H} \Gamma)^2 - (e_x \cdot \tilde{J} \tilde{H} \Gamma)^2 - (e_z \cdot \tilde{J} \tilde{H} \Gamma)^2 \right] \leq \frac{b |\rho_1|}{\rho_1^2 + \rho_2^2} \| (1 + Z) A \tilde{T}_1 \|_{L^\infty} \int |\tilde{J} \tilde{H} \Gamma|^2 \leq \frac{b(1 - d_1) |\rho_1|}{2(\rho_1^2 + \rho_2^2)} \| \tilde{J} \tilde{H} \Gamma \|_{L^2}^2,
\] (B.12)
and also
\[
2b \int (1 + Z) A \tilde{T}_1 (e_x \cdot \tilde{J} \tilde{H} \Gamma) (e_y \cdot \tilde{J} \tilde{H} \Gamma) \leq \frac{2b |\rho_2|}{\rho_1^2 + \rho_2^2} \| (1 + Z) A \tilde{T}_1 \|_{L^\infty} \int |\tilde{J} \tilde{H} \Gamma|^2 \leq \frac{b(1 - d_1) |\rho_2|}{2(\rho_1^2 + \rho_2^2)} \| \tilde{J} \tilde{H} \Gamma \|_{L^2}^2.
\] (B.13)

For the last line of (B.9), from the smallness of $w$ (A.8), (A.20), we observe
\[
|e_z \cdot \tilde{J} \tilde{H} \Gamma| = |e_z \cdot \hat{w} \wedge \tilde{H} \Gamma| \leq |\hat{w}| |\tilde{H} \Gamma| \lesssim \delta(b^*) |\tilde{H} \Gamma|,
\]
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and hence
\[
\begin{align*}
\int V y^2 \Phi^{(2)}(e_z \cdot J^H \Gamma)(e_z \cdot J^H \Gamma) - b \int V y^2 \Phi^{(1)}(e_y \cdot J^H \Gamma)(e_z \cdot J^H \Gamma) \\
\lesssim b \delta(b^*) \left\| \frac{V}{y^2} T_1 \right\|_{L^\infty} \int |J^H \Gamma| \cdot |H \Gamma| \lesssim b \delta(b^*) \left\| H \Gamma \right\|_{L^2}^2.
\end{align*}
\] (B.14)

Eventually, we inject (B.12), (B.13), (B.14) into (B.9), and then (5.61) follows. \qed

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