ON THE DISTRIBUTION OF THE FIRST POINT OF COALESCENCE FOR SOME COLLATZ TRAJECTORIES

ROY BURSON

Abstract. This paper is a numerical evaluation of some trajectories of the Collatz function. Specifically, I assess the coalescence points of each integer \( n \equiv 0 \pmod{2} \) and \( n \equiv 2 \pmod{3} \) through a sophisticated algorithm that has been developed to test on any different modulus classes. The data discovered illustrate that the distribution of the first point of coalescence is closely related to the solutions of some exponential diophantine equation. Afterwards, I show that the first point of coalescence of the integers \( n \) and \( 3n + 2 \) appear to tend to an expected value of \( \frac{4}{5}n \). When the algorithm was pushed to its peak estimation it has been discovered that the expected value begins to deviate from the initial estimation of \( \frac{4}{5}n \). The first point of coalescence of the integers \( n \) and \( 3n + 2 \) appear eradicate from a “step by step” point of view but from a topological point of view seem to be localized around the diophantine solution of some particular functions.

Introduction

This paper is a numerical evaluation of Burson earlier work \([4]\) regarding Collatz conjecture. In this prior work Burson established a conditional proof of the Collatz conjecture under the assumption that that integers \( n \) and \( 3n + 2 \) eventually cycle together. More precisely stated, Burson showed that in order to prove the Collatz conjecture it suffices to show that the residues \([0]_3, [2]_3 \in \mathbb{Z}_3\) share the same cycle or are in the same forward orbit. For the proof of this result refer back to Burson \([4]\) as it will not be represented in this work. On the other hand, I provide the necessary definitions that are needed to grasp the material presented herein. I give a well defined description of the first point of coalescence of two whole numbers and model its distribution through a python algorithm and present the results via Matlab. This numerical assessment consist of localizing the first point of coalescence of the trajectories \( O^+(n) \) and \( O^+(3n+2) \) for values of \( n \) in the interval \([1, 100], [1, 500], [1, 1000], [1, 10000], \) and \([1, 100000]\). Due to algorithmic data structures and time complexity the interval \([1, 100000]\) is not exceeded in the computational process. The algorithms used in this work is not provided here. However, one of the algorithms is publicly available at [https://github.com/rsb29592/Collatz-Trajectories](https://github.com/rsb29592/Collatz-Trajectories).

The evidence provided herein indicates that the first point of coalescence of the trajectories \( O^+(n) \) and \( O^+(3n + 2) \) are closely related to the integer solutions of the functions \( f: \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) and \( g: \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) defined by

\[
    f(x, y) = \left( \frac{3}{2} \right)^y x, \quad g(x, y) = \left( \frac{2}{3} \right)^y x
\]

Date: May 17, 2020.

Key words and phrases. Collatz function.
This observation will be investigated further in a subsequent paper. Perhaps even if a plausible explanation can be developed to explain why the coalescence points studied herein seem to be localized around these equations the development of an adequate approximation will demand much more effort and may even be indeterminable. It is also apparent that these results can be generalized by altering the relationship between the input data tested. That is, if \( a, b \in \mathbb{N} \) then instead of considering when the trajectories \( O^+(n) \) and \( O^+(3n + 2) \) merge together we can instead consider when the trajectories \( O^+(n) \) and \( O^+(an + b) \) merge together. Although this appears to be a much more difficult problem than that of which has already been presented the algorithm that has been used in this work has been extended to incorporated such a scenario in case it leads to further results. The results that were found for any other trajectories will not be discussed in this work as they are not apparently needed at this point of my investigation.

1. Notation

In this section I provide the notation and terminology that is used throughout the work. A detailed explanation of some of these definitions can also be found in [4, 1, 2].

Definition 1. The Trajectory or Forward Orbit of a positive integer \( n \) is the set
\[
O^+(n) = \{ n, T(n), T^2(n), \cdots \}
\]
were \( T : \mathbb{N} \to \mathbb{N} \) is the Collatz function defined by \( n \mapsto \frac{n}{2} \) if \( n \) is even and \( n \mapsto 3n + 1 \) if \( n \) is odd, and \( T^k \) is the function \( T^k : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) defined by the rule
\[
(n, k) \mapsto (T \circ T \circ \cdots \circ T)^{\text{\textregistered} \text{k-times}}(n)
\]

Definition 2. Given two integers \( n, m \in \mathbb{N} \) define the relation \( n \sim m \) if and only if the two trajectories \( O^+(n) \) and \( O^+(m) \) coalesce. That is,
\[
n \sim m \iff O^+(n) \cap O^+(m) \neq \emptyset
\]

Definition 3. Given two integers \( n, m \) such that \( n \sim m \), the Merging Height of \( n \) or \( m \), denoted \( \varphi(n, m) \), is defined by the rule
\[
\varphi(n, m) = \min\{k_1, k_2 \in \mathbb{N} : T^{k_1}(n) = T^{k_2}(m)\}
\]
If \( \varphi(n, m) \) does not exist then we decree that \( \varphi(n, m) = \infty \).

Definition 4. Given two integers \( n, m \) such that \( n \sim m \), the Coalescence Point
\( C : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) of the integers \( n \) and \( m \) is defined by the rule \( (n, m) \mapsto T^{\varphi(n, m)}(n) \) or \( (n, m) \mapsto T^{\varphi(n, m)}(m) \) since they are equivalent.

Definition 5. Let \( k \in \mathbb{N} \) and \( f : \mathbb{N} \to \mathbb{N} \) be a function. The average coalescence point \( \zeta : \mathbb{N} \to \mathbb{N} \) with respect to \( f \) is defined as the mapping
\[
k \mapsto \frac{1}{k} \sum_{n \leq k} C(n, f(n)) = \frac{1}{k} \sum_{n \leq k} T^{\varphi(n, f(n))}(n)
\]
ON THE DISTRIBUTION OF THE FIRST POINT OF COALESCENCE FOR SOME COLLATZ TRAJECTORIES

2. THE FIRST POINT OF COALESCENCE FOR INTEGERS $0(\text{mod}3)$, AND $2(\text{mod}3)$

In this section I discuss the empirical results that have been obtained from the algorithm designed for this work. After discussing the empirical results obtained I elaborate on the dispersion algorithm that utilizes the spatial data in order to identify the equations which localize the behavior of the first point of coalescence of these specific trajectories. I will demonstrate that the first coalescence point of $n$ and $3n + 2$ can be approximated near specific regions or lines in $\mathbb{R}^2$ and then after make a couple observations from the data obtained. First, as discussed prior, the algorithm used in this work has been developed to handle a broader scenario than actually need be. That is if $a, b \in \mathbb{N}$ then the algorithm is designed to compute the first point of coalescence of the to integers $n$ and $an + b$ up to any value that the user specifies. By letting $a = 3$ and $b = 2$ then one obtains the important trajectory relationship shown in [4]. The first test (for these settings) deployed accounted for all of the integers $k \leq 100$. The scatter plot of these results are depicted in Figure 1. Even for such small values of $k$ the spatial data exhibits some linear behaviors. Figure 2 illustrates the first employment of the behavior of the average value of the first point of coalescence for the trajectories $O^+(n)$ and $O^+(3n + 2)$. For these specific trajectories the average value of the first point of coalescence is defined by the summation

$$\varsigma(k) = \frac{1}{k} \sum_{n \leq k} C(n, 3n + 2) = \frac{1}{k} \sum_{n \leq k} T^{\varphi(n,3n+2)}(n)$$

As seen in figure 2 it appears that the average value can be estimated by a simple linear approximation. Figure 3 and 4 illustrate the behavior of $T^{\varphi(n,m)}(m)$ and $\varsigma(k)$ for values $m, k \in [1, 1000]$. By extending the domain from $k \leq 100$ to $k \leq 1000$ we see dramatic changes and observe even sharper linear properties are observed. The linear properties of the functions $T^{\varphi(n,m)}(m)$ and $\varsigma$ become clearer as we enlarge the domain. Figures 5-8 show the behavior of $T^{\varphi(n,m)}(m)$ and $\varsigma(k)$ in the interval $\in [1, 10000]$ and $[1, 100000]$. One can easily see that majority of the data seem to be located near the solutions to specific diophantine equations. The surprising result is the algorithm indicates that $\varsigma(n) \approx \frac{4}{5}n$ for large values of $n$. From here it seems reasonable to be tempted to immediately try to find a direct formulation that guarantees this type of behavior. However, since this was not the main purpose of my investigation this observation will be postponed for the time being and it will be investigated in a subsequent paper. It is thus of most interest to find the set of functions

$$\{ f : f : \mathbb{N} \to \mathbb{N}, f(n) = an + b = T^{\varphi(n,3n+2)}(n), a, b \in \mathbb{Z}\}$$

that resemble the linear behavior of the first point of coalescence of the trajectories that are being studied herein. This means solving for the corresponding coefficients $a$ and $b$ so that the equality holds $f(n) = an + b = T^{\varphi(n,3n+2)}(n)$ holds valid. In order to find the values of $a$ and $b$ for each localized trajectory I have developed a dispersion algorithm that measures the disbursement and runs a Nearest Neighbors Algorithm (NNA) and finds the nearest points to a specific trajectory to form the best linear approximations possible. The precise results of this algorithm will be discussed in the next section.
The time complexity of the algorithm is quite interesting. The algorithm took 5.49ms to process all of the integers up to 100, 7.381ms for integers up to 1000, 22.297ms for integers up to 10000, and a massive 338.165ms for all the integers up to 100000. These values were produced by timing the algorithm with the built-in timing mechanism and functions available. These results have been documented and are given in the Table 1. The left column gives the input interval and the right column denotes how much time has elapsed from the beginning to the need of the algorithm during each test. Figure 9 illustrates the behavior of the time complexity for these specific trajectories that have been analyzed. Amazingly, it seems reminiscent of that of the average value of the first point of coal sense. That is because the time complexity is a linear function similarly to the approximation. A small cusp is also observed near the lower proportional of the time complexity curve. The graphs indicate that average value is roughly ten times larger than the value of the time complexity on all test runs. Using this time complexity curve we can estimate the time it will take to run any test.

3. Dispersion Testing of Spatial Data

In this section I discuss the dispersion test on the spatial data that has been obtained from the algorithm developed to depict the first point of coalescence \( \varphi(n, m) \) of \( n \) and \( m \). The dispersion test consist of encrypting the spatial data and finding the suitable linear approximations to the trajectories that are being analyzed. The dispersion algorithm that has been developed throughout this work is not publicly available because this particular algorithm has multiple purposes, which examining the Collatz trajectories is only one of its purposes. The algorithm indicates that the distribution of the coalescence points are closely related to the integer solutions of the equations

\[
f(x, y) = \left(\frac{3}{2}\right)^y x, \quad g(x, y) = \left(\frac{2}{3}\right)^y x.
\]

Majority of the coalescence points seem to reside on these curves but not all of them are. As we can see from the figures provided there are some isolated points of interest. The purpose of this dispersion algorithm was to find these specific coefficients and intercepts in order to investigate the properties of the first point of coalescence further.

The experimental evidence is crude and does not explain a lot. Henceforth, in future investigations it would be of particular interest to study the relationship between the exponent \( y \) and the \( x \) component of these two functions which are being modeled here (that is the \( f \) and the \( g \) functions). in order to analyze these trajectories more thoroughly one needs to try to relate the value \( y \) and \( x \) through the value \( n \). Given a particular value \( n \leq 100000 \) the algorithm proves that the trajectories first merge near the diophantine solutions to the equations \( f(x, y) = \left(\frac{3}{2}\right)^y x \) and \( g(x, y) = \left(\frac{2}{3}\right)^y x \). These equations have integral solutions \((x, y)\) whenever \( 2^y|x \) or \( 3^y|x \). These are precisely the integers divisible by a power of 2 or 3. In this regard these empirical results indicate that the Collatz function is shuffling through the powers of 2 and 3. This "shuffling" that occurs between powers of 2 and 3 seems to be the next most important observation to investigate.

4. Figures and Tables
Fig 1: Assessment of the first point of coalescence $C(n, 3n + 2)$ for $n \in [1, 100]$.

Fig 2: Assessment of $\varsigma(k)$ for $k \in [1, 100]$. 

$\varsigma(k) = \frac{4}{5k}$
Fig 3: Assessment of the first point of coalescence $C(n, 3n + 2)$ for $n$ and $3n + 2$ for $n \in [1, 1000]$.

Fig 4: Assessment of $\varsigma(k)$ for $k \in [1, 1000]$. 
Fig 5: Assessment of the first point of coalescence $C(n, 3n + 2)$ for $n \in [1, 10000]$.

Fig 6: Assessment of $\varsigma(k)$ for $k \in [1, 10000]$. 
Fig 7: Assessment of the first point of coalescence
$C(n, 3n + 2)$ for $n \in [1, 100000]$.

Fig 8: Assessment of $\varsigma(k)$ for $k \in [1, 100000]$. 
ON THE DISTRIBUTION OF THE FIRST POINT OF COALESCENCE FOR SOME COLLATZ TRAJECTORIES

| Interval | Time elapsed    |
|----------|-----------------|
| [1, 100] | 5.493256600 ms  |
| [1, 1000] | 7.3807948 ms    |
| [1, 10000] | 22.9695312 ms  |
| [1, 100000] | 338.1653204 ms |

Table 1. Time complexity for specified Intervals

Fig 7: Time complexity Curve for $k \leq 100000$
References

1. Lagarias, J. C. "The 3x+1 problem and its generalization" Amer Math. Monthly 3-23, 1985.
2. Lagarias, J. C. "The 3x+1 problem: An Annotated Bibliography" Amer Math. Monthly 3-23, 2012.
3. J. C. Lagarias, The Ultimate Challenge: The 3x+1 Problem, American Mathematical Society, 2010
4. Burson, Roy. "Integer Representations and Trajectories of the 3x + 1 Problem. arxiv preprint arXiv:1906.10566

Department of Mathematics, California State University Northridge, California, 91330
E-mail address: roy.burson.618@my.csun.edu.