INEQUALITIES FOR THE POLAR DERIVATIVE OF A POLYNOMIAL

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Abstract

In this paper we obtain new results concerning maximum modulus of the polar derivative of a polynomial with restricted zeros. Our results generalize and refine upon the results of Aziz and Shah [An integral mean estimate for polynomial, Indian J. Pure Appl. Math. 28 (1997) 1413–1419] and Gardner, Govil and Weems [Some result concerning rate of growth of polynomials, East J. Approx. 10(2004) 301–312].

1. INTRODUCTION AND STATEMENT OF RESULTS

The problems in the analytic theory of polynomials concerning derivative of the polynomials have been frequently investigated. Over many decades, a large number of research papers, e.g., [1, 2, 3, 4, 5, 9, 11] have been published.

If \( p(z) = \sum_{m=0}^{n} a_m z^m \) is a polynomial of degree \( n \), then

\[
\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|.
\]

(1)

The above inequality, which is an immediate consequence of Bernstein’s inequality on the derivative of a trigonometric polynomial is best possible with equality holding for the polynomial \( p(z) = \lambda z^n \), \( \lambda \) being a complex number.

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It is noted that in (1.1) equality hold if and only if \( p(z) \) has all its zeros at the origin and so it is natural to seek improvements under appropriate assumptions on the zeros of \( p(z) \).

If \( p(z) \) having no zeros in \( |z| < 1 \), then the above inequality can be replaced by

\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|.
\]  

(2)

Inequality (2) was conjectured by Erdos and later proved by Lax [10]. On the other hand, it was shown by Turan [12] that if all the zeros of \( p(z) \) lie in \( |z| < 1 \), then

\[
\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \max_{|z|=1} |p(z)|.
\]  

(3)

The above inequality was generalized by Govil [3]. Who proved that if \( p(z) \) is a polynomial of degree \( n \) having all its zeros in \( |z| < k \), then for \( k \leq 1 \)

\[
\max_{|z|=1} |p'(z)| \geq \frac{n}{1 + k} \max_{|z|=1} |p(z)|,
\]  

(4)

and for \( k \geq 1 \)

\[
\max_{|z|=1} |p'(z)| \geq \frac{n}{1 + k^n} \max_{|z|=1} |p(z)|.
\]  

(5)

Both the above inequalities are best possible with equality in (4) holding for \( p(z) = (z + k)^n \), while in (5) the equality holds for the polynomial \( p(z) = z^n + k^n \). As an extension of (2) was shown by Malik [11] that, if \( p(z) \neq 0 \) in \( |z| < k, k \geq 1 \), then

\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{1 + k} \max_{|z|=1} |p(z)|.
\]  

(6)

Equality in (6) holds for \( p(z) = (z + k)^n \).

By considering a more general class of polynomials \( p(z) = a_0 + \sum_{\nu=\mu}^{n} a_{\nu}z^{\nu}, 1 \leq \mu \leq n \), not vanishing in \( |z| < k, k > 0 \), then for \( 0 < r \leq R \leq k \), inequality (6) is generalized by Aziz and Shah [4] by proving

\[
\max_{|z|=R} |p'(z)| \leq nR^{\mu-1}(R^\mu + k^\mu)^{\frac{n-1}{\mu}} \max_{|z|=r} |p(z)| - \min_{|z|=k} |p(z)|.
\]  

(7)

Equality in (7) holds for \( p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}} \) where \( n \) is a multiple of \( \mu \). On the other hand, for the class of polynomial \( p(z) = a_nz^n + \sum_{\nu=\mu}^{n} a_{n-\nu}z^{n-\nu}, 1 \leq \mu \leq n \), of degree \( n \) having all its zeros in \( |z| \leq k, k \leq 1 \), Aziz and Shah [2] proved

\[
\max_{|z|=1} |p'(z)| \geq \frac{n}{1 + k^\mu} \{ \max_{|z|=1} |p(z)| + \frac{1}{k^{n-\mu}} \min_{|z|=k} |p(z)| \}.
\]  

(8)
Let $D_\alpha\{p(z)\}$ denote the polar derivative of the polynomial $p(z)$ of degree $n$ with respect to the point $\alpha$, then

$$D_\alpha\{p(z)\} = np(z) + (\alpha - z)p'(z).$$

The polynomial $D_\alpha\{p(z)\}$ is of degree at most $n - 1$ and it generalized the ordinary derivative in the sense that

$$\lim_{\alpha \to \infty} \frac{D_\alpha\{p(z)\}}{\alpha} = p'(z).$$

Dewan [7] extended (8) to the polar derivative of a polynomial and proved the following.

**Theorem A** Let $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}, 1 \leq \mu \leq n$, be a polynomial of degree $n$ having all its zeros in $|z| \leq k, k \leq 1$. Then for every real or complex number $\alpha$ with $|\alpha| \geq k^\mu$, we have

$$\max_{|z|=1} |D_\alpha p(z)| \geq \frac{n(|\alpha| - k^\mu)}{1 + k^\mu} \max_{|z|=1} |p(z)| + n\left(\frac{|\alpha| + 1}{k^{n-\mu}(1 + k^\mu)}\right)m$$

$$n\left(\frac{k^\mu - A^\mu}{1 + k^\mu}\right) \max_{|z|=1} |p(z)| + \frac{n(A^\mu - k^\mu)}{k^n(1 + k^\mu)} m,$$

where $m = \min_{|z|=k} |p(z)|$ and

$$A^\mu = \frac{n(|a_n| - \frac{m}{k^{\mu}})k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}}{n(|a_n| - \frac{m}{k^{\mu}})k^{\mu-1} + \mu|a_{n-\mu}|}.$$  

Dividing both sides of inequality (9) by $|\alpha|$ and let $|\alpha| \to \infty$ we get (8).

As an extension of (6) to the polar derivative of a polynomial, we have the following result due to Dewan [7].

**Theorem B** If $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}, 1 \leq \mu \leq n$, is a polynomial of degree $n$ having no zeros in $|z| < k, k \geq 1$, then for every real or complex number $\alpha$ with $|\alpha| \geq 1$,

$$\max_{|z|=1} |D_\alpha p(z)| \leq \frac{n}{1 + k^\mu} \left\{(|\alpha| + k^\mu) \max_{|z|=1} |p(z)| - (|\alpha| - 1)m\right\}$$

where $m = \min_{|z|=k} |p(z)|$.

The above theorem is an extension of a result of Aziz [1] and for $\mu = 1$, it reduces to a result of Aziz and Shah [2].

In this paper, we first obtain the following generalization of inequality (8) which is also a refinement of inequality (9).
Theorem 1. Let $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}, 1 \leq \mu \leq n$, be a polynomial of degree $n$ having all its zeros in $|z| \leq k, k \leq 1$ and $\alpha$ is any real or complex number with $|\alpha| \geq \frac{k^\mu}{R^{\nu-1}}$, then for $rR \geq k^2$ and $r \leq R$, we have

$$
\max_{|z|=R} |D_\alpha p(z)| \geq n(R^{\mu-1}|\alpha| - k^\mu) \left( \frac{R^\mu + k^\mu}{(r^\mu + k^\mu)} \right)^{\frac{n}{\mu}} \max_{|z|=r} |p(z)| \\
+ \frac{n(R^{\mu-1}|\alpha| + R^\mu)}{k^{n-\mu}(R^\mu + k^\mu)} \left( \min_{|z|=k} |p(z)| \right) \\
+ n(k^\mu - R^\mu A'_\mu) \frac{(R^\mu + k^\mu)^{\frac{n}{\mu}}}{(r^\mu + k^\mu)^{\frac{n}{\mu}}} \max_{|z|=r} |p(z)| \\
+ \frac{nR^n(R^\mu A'_\mu - k^\mu)}{k^\mu R^\mu + k^\mu} \left( \min_{|z|=k} |p(z)| \right) \\
+ \frac{nR^{\mu-1}}{(R^\mu + k^\mu)} (|\alpha| - RA'_\mu) \left( \frac{R}{r} \right)^n - \left( \frac{R^\mu + k^\mu}{(r^\mu + k^\mu)} \right)^{\frac{n}{\mu}} \min_{|z|=k} |p(z)|,
$$

(11)

where

$$
A'_\mu = \frac{n(|a_n| - \frac{m |a_n|}{k^\mu} k^{\mu-1} + \mu |a_{n-\mu}| k^{\mu-1})}{nR(|a_n| - \frac{m |a_n|}{k^\mu} k^{\mu-1} + \mu |a_{n-\mu}|)}
$$

and $m = \min_{|z|=k} |p(z)|$.

Remark 1. For $R = r = 1$ theorem 1 reduces to (9).

Remark 2. Dividing the two sides of (11) by $|\alpha|$, letting $|\alpha| \to \infty$, we get

$$
\max_{|z|=R} |p'(z)| \geq \frac{nR^{\mu-1}(R^\mu + k^\mu)^{\frac{n}{\mu}}}{(r^\mu + k^\mu)^{\frac{n}{\mu}}} \max_{|z|=r} |p(z)| + \frac{nR^{\mu-1}}{k^{n-\mu}(R^\mu + k^\mu)} \left( \min_{|z|=k} |p(z)| \right) \\
+ \frac{nR^{\mu-1}}{(R^\mu + k^\mu)} \left( \frac{R}{r} \right)^n - \left( \frac{R^\mu + k^\mu}{(r^\mu + k^\mu)} \right)^{\frac{n}{\mu}} \min_{|z|=k} |p(z)|.
$$

(12)

This includes inequality (8) as special case. The following result immediately follows by taking $k = 1$ in theorem 1.

Corollary 1. If $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}, 1 \leq \mu \leq n$, be a polynomial of degree $n$ having all its zeros in $|z| \leq 1$ and $\alpha$ is any real or complex number $\alpha$ with
Equality in (14) holds for $p$ where $m$ and $\alpha$ are such that

\[
\max_{|z|=r} |D_\alpha p(z)| \geq \frac{n(R^{\mu-1}|\alpha| + R^\mu)}{R^\mu + 1} \min_{|z|=1} |p(z)|
\]

\[+ n(R^{\mu-1}|\alpha| - 1)(R^\mu + 1)\frac{R^\mu - 1}{(\mu + 1)^{\mu - 1}} \max_{|z|=r} |p(z)|
\]

\[+ n(1 - R^\mu A''_\mu)\frac{(R^\mu + 1)\frac{R^\mu}{\mu - 1}}{(\mu + 1)^{\mu - 1}} \max_{|z|=r} |p(z)|
\]

\[+ \frac{nR^n(R^\mu A''_\mu - 1)}{(R^\mu + 1)} \min_{|z|=1} |p(z)|
\]

\[+ \frac{nR^{n-1}}{(R^\mu + 1)} (|\alpha| - RA''_\mu)[(\frac{R}{r})^n - (\frac{R^\mu + 1}{r^\mu + 1})\frac{R^\mu}{\mu - 1}] \min_{|z|=1} |p(z)|.
\]

(13)

where

\[A''_\mu = \frac{\nu R^n(|a_n| - m) + \nu R^{n-1}|a_{n-\mu}|}{nR(|a_n| - m) + \mu|a_{n-\mu}|}
\]

and $m = \min_{|z|=1} |p(z)|$.

We next prove the following result which is a generalization of the inequality (7) and result due to Gardner, Govil and Weems [3].

**Theorem 2.** If $p(z) = a_0 + \sum_{\nu=m}^n a_\nu z^\nu$, $1 \leq m \leq n$, be a polynomial of degree $n$ having no zeros in $|z| < k$, $k \geq 1$ for $0 \leq r \leq R \leq k$, then for every real or complex number $\alpha$ with $|\alpha| \geq R$,

\[
\max_{|z|=R} |D_\alpha p(z)| \leq \frac{n(R^{m-1}|\alpha| + k^m)}{R^m + k^m} \left((\frac{R^m + k^m}{r^m + k^m})\frac{R^\mu}{\mu - 1} \max_{|z|=r} |p(z)|
\]

\[+ \frac{(R^m + k^m)(\frac{R^\mu}{\mu - 1})}{(R^m + k^m)} - 1 \min_{|z|=k} |p(z)|
\]

\[+ \frac{n(R^{m-1}|\alpha| - R^m)}{R^m + k^m} \min_{|z|=k} |p(z)|.
\]

(14)

Equality in (14) holds for $p(z) = (z^m + k^m)^\frac{R^\mu}{\mu - 1}$.

**Remark 3.** For $R = r = 1$ theorem 2 reduces to theorem B.

**Remark 4.** Dividing the two sides of (14) by $|\alpha|$ and letting $|\alpha| \to \infty$, we have the following inequality, which is an improvement as well as a generalization of a result proved by Bidkham and Dewan [3]

\[
\max_{|z|=R} |p'(z)| \leq \frac{nR^{m-1}(R^m + k^m)(\frac{R^\mu}{\mu - 1})}{(R^m + k^m)} \left\{ \max_{|z|=r} |p(z)| - \min_{|z|=k} |p(z)| \right\}.
\]
The result is best possible and equality holds for the polynomial \( p(z) = (z^m + k^m)^{\frac{n}{m}} \) where \( n \) is a multiple of \( m \).

2. LEMMAS

For the proofs of these theorems we needs the following lemmas.

**Lemma 2.1.** If \( p(z) = a_0 + \sum_{\nu=m}^{n} a_{\nu} z^\nu, 1 \leq m \leq n \), is a polynomial of degree \( n \) such that \( p(z) \neq 0 \) in \( |z| < k, k > 0 \), then for \( 0 \leq r \leq R \leq k \),
\[
\max_{|z|=r} |p(z)| \geq \left( \frac{R^m + k^m}{R^m + k^m} \right)^{\frac{n}{m}} \max_{|z|=R} |p(z)| + \left[ 1 - \left( \frac{R^m + k^m}{R^m + k^m} \right)^{\frac{n}{m}} \right] \min_{|z|=k} |p(z)|. \tag{15}
\]
Here the result is best possible and equality hold for the polynomial \( p(z) = (z^m + k^m)^{\frac{n}{m}} \) where \( n \) is multiple of \( m \). This result is due to Dewan [7].

**Lemma 2.2.** If \( p(z) = a_0 + \sum_{\nu=\mu}^{n} a_{\nu} z^\nu, 1 \leq \mu \leq n \), is a polynomial of degree \( n \) having no zeros in \( |z| \leq k, k \geq 1 \), then for every real or complex number \( \alpha \) with \( |\alpha| \geq 1 \)
\[
\max_{|z|=1} |D\alpha p(z)| \leq \frac{n}{1 + k^\mu} \left\{ (|\alpha| + k^\mu) \max_{|z|=1} |p(z)| - (|\alpha| - 1)m \right\},
\]
where \( m = \min_{|z|=k} |p(z)| \).
This result is due to [6].

3. PROOF OF THE THEOREMS

**Proof of theorem 1.** By hypothesis the polynomial \( p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{\nu-n} z^{n-\nu}, 1 \leq \mu \leq n \), has all its zeros in \( |z| \leq k \), where \( k \leq 1 \), therefore it follows that \( F(z) = P(Rz) \) has all its zeros in \( |z| \leq \frac{k}{R} \), where \( \frac{k}{R} \leq 1 \) and \( |\alpha| \geq \left( \frac{k}{R} \right)^\mu \). Applying inequality (9) to the polynomial \( F(z) \) we get
\[
\max_{|z|=1} |D\alpha F(z)| \geq \frac{n(|\alpha| - \frac{k^\mu}{R^n})}{1 + \frac{k^\mu}{R^n}} \max_{|z|=1} |F(z)| \nonumber
\]
\[
+ n\left( \frac{k^\mu}{R^n} \right)^{n-\mu} (1 + \frac{k^\mu}{R^n}) \min_{|z|=\frac{k}{R}} |F(z)| \nonumber
\]
\[
+ n\left( \frac{k^\mu}{R^n} - A^\prime_\mu \right) \frac{1}{(1 + \frac{k^\mu}{R^n})^{\mu}} \max_{|z|=1} |F(z)| \nonumber
\]
\[
+ \frac{n(A^\prime_\mu - \frac{k^\mu}{R^n})}{R^n (1 + \frac{k^\mu}{R^n})^{\mu}} \min_{|z|=\frac{k}{R}} |F(z)|, \tag{16}
\]
where
\[
A^\prime_\mu = \frac{n(|a_n| - \frac{m}{k^n}) \frac{k^\mu}{R^n} + \mu |a_{n-\mu}| \frac{k^{\mu-1}}{R^{n-1}}}{nR(|a_n| - \frac{m}{k^n})k^{\mu-1} + \mu |a_{n-\mu}|}.
\]
and \( m = \min_{|z|=k} |p(z)| \),
which is equivalent to

\[
\max_{|z|=1} |np(Rz) + (\alpha - z)Rp'(Rz)| \geq \frac{n(|\alpha| R^\mu - k^\mu)}{R^\mu + k^\mu} \max_{|z|=1} |p(Rz)| \\
+ \frac{nR^n(|\alpha| + 1)}{K^{n-\mu}(R^\mu + K^\mu)} \min_{|z|=k/R} |p(Rz)| + n \frac{(K^\mu - R^\mu A'_\mu)}{(R^\mu + k^\mu)} \max_{|z|=1} |p(Rz)| \\
+ \frac{nR^n(R^\mu A'_\mu - k^\mu)}{k^n(R^\mu + k^\mu)} \min_{|z|=k/R} |p(Rz)|,
\]

which gives

\[
\max_{|z|=R} |np(z) + (\alpha R - z)p'(z)| \geq \frac{n(|\alpha R| R^{\mu-1} - k^\mu)}{R^\mu + k^\mu} \max_{|z|=R} |p(z)| \\
+ \frac{n(|\alpha R| R^{\mu-1} + R^n)}{K^{n-\mu}(R^\mu + K^\mu)} \min_{|z|=1/R} |p(z)| + n \frac{(K^\mu - R^\mu A'_\mu)}{(R^\mu + k^\mu)} \max_{|z|=R} |p(z)| \\
+ \frac{nR^n(R^\mu A'_\mu - k^\mu)}{k^n(R^\mu + k^\mu)} \min_{|z|=k} |p(z)|. \tag{17}
\]

On other hand, since \( p(z) \) has all its zeros in \(|z| < k, k > 0 \) therefore it follows that

\[ q(z) = z^n p\left(\frac{1}{z}\right) \neq 0 \text{ for } |z| < 1/k. \]

Applying inequality (15) to \( q(z) \), we get

\[
\max_{|z|=1/R} |q(z)| \geq \left(\frac{1}{k}\right)^{|\mu|} - \left(\frac{1}{r}\right)^{|\mu|} \max_{|z|=1/R} |q(z)| + \left[1 - \left(\frac{1}{k}\right)^{|\mu|} - \left(\frac{1}{r}\right)^{|\mu|}\right] \min_{|z|=1/k} |q(z)|
\],

which gives

\[
\max_{|z|=R} |p(z)| \geq \left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu}\right)^\frac{n}{|\mu|} \max_{|z|=r} |p(z)| + \left[\left(\frac{R}{r}\right)^n - \left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu}\right)^\frac{n}{|\mu|}\right] \min_{|z|=k} |p(z)|. \tag{18}
\]

From (17) and (18), we have

\[
\max_{|z|=R} |D_{R\alpha} p(z)| \geq n(|\alpha R| R^{\mu-1} - k^\mu) \frac{(R^\mu + k^\mu)^\frac{n}{|\mu|} - 1}{(r^\mu + k^\mu)^\frac{n}{|\mu|}} \max_{|z|=R} |p(z)| \\
+ \frac{n(R^{n-1}|\alpha R| + R^n)}{k^{n-\mu}(R^\mu + k^\mu)} \min_{|z|=k} |p(z)| \\
+ n(k^\mu - R^\mu A'_\mu) \frac{(R^\mu + k^\mu)^\frac{n}{|\mu|} - 1}{(r^\mu + k^\mu)^\frac{n}{|\mu|}} \max_{|z|=r} |p(z)| \\
+ \frac{nR^n(R^\mu A'_\mu - k^\mu)}{k^n(R^\mu + k^\mu)} \min_{|z|=k} |p(z)| \\
+ \frac{nR^{\mu-1}(|\alpha R| - RA'_\mu) \left[\left(\frac{R}{r}\right)^n - \left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu}\right)^\frac{n}{|\mu|}\right] \min_{|z|=R} |p(z)|,
\]

which is equivalent to (11).
Proof of theorem 2. By hypothesis the polynomial $p(z) = a_0 + \sum_{\nu=m}^{n} a_{\nu} z^\nu$, $1 \leq m \leq n$, having no zeros in $|z| < k$, where $k \geq 1$, therefore it follows that $F(z) = p(Rz)$ having no zeros in $|z| \leq \frac{k}{R}$, where $\frac{k}{R} \geq 1$. Hence using lemma 2 for $|\alpha| \geq 1$, we have

$$\max_{|z|=1} |D_\alpha F(z)| \leq \frac{n}{1 + (\frac{k}{R})^m} \{(|\alpha| + (\frac{k}{R})^m) \max_{|z|=1} |F(z)| - (|\alpha| - 1) \min_{|z|=k/R} |F(z)|\},$$

which is equivalent to

$$\max_{|z|=1} |np(Rz) + (\alpha - z) R p'(Rz)| \leq \frac{n}{1 + (\frac{k}{R})^m} \{(|\alpha| + (\frac{k}{R})^m) \max_{|z|=1} |p(Rz)| - (|\alpha| - 1) \min_{|z|=k/R} |p(Rz)|\}.$$ 

Replacing $Rz$ by $z$ we get

$$\max_{|z|=R} |D_\alpha p(z)| \leq \frac{n(R^{m-1} |\alpha R| + k^m)}{R^m + k^m} \max_{|z|=R} |p(z)|$$

$$- \frac{n(R^{m-1} |\alpha| - R^m)}{(R^m + k^m)} \min_{|z|=k} |p(z)|$$

for $R \geq 1$

or

$$\max_{|z|=R} |D_\alpha p(z)| \leq \frac{n(R^{m-1} |\alpha| + k^m)}{R^m + k^m} \max_{|z|=R} |p(z)|$$

$$- \frac{n(R^{m-1} |\alpha| - R^m)}{(R^m + k^m)} \min_{|z|=k} |p(z)|.$$

(19)

By combining (19) and (15) we get (11).

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