CUBIC ERGODIC AVERAGES FOR ACTIONS OF AMENABLE GROUPS.

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Abstract. We unify and extend some previous results about multiparameter configurations in sets of positive density in products of amenable groups. We show that if \( G \) is an amenable group, and \( E \subset G^d \) has positive density with respect to some Følner sequence \( \{ \Phi_N \}_{N \in \mathbb{N}} \) in \( G^d \), then \( E \) contains many “cubic” configurations. This generalizes the author’s result for the case \( d = 2 \), and Q. Chu’s for the case \( G = \mathbb{Z} \).

1. Introduction

Khintchine [K] proved the following strengthening of Poincaré's recurrence theorem.

Theorem 1.1. (Khintchine’s Recurrence Theorem.) Let \((X, \mathcal{X}, \mu)\) be a probability space, and let \( T : X \to X \) be an invertible transformation which preserves \( \mu \). For all \( B \in \mathcal{X} \) and all \( c > 0 \), the set \( \{ n \in \mathbb{Z} : \mu(B \cap T^{-n}B) > \mu(B)^2 - c \} \) has bounded gaps.

One can deduce from this the fact that if \( A \subset \mathbb{Z} \) has positive density, that is if \( d^*(A) := \limsup_{N \to \infty} \frac{|A \cap [N, M]|}{M - N + 1} > 0 \), then for all \( c > 0 \), the set \( \{ n : d^*(A \cap (A - n)) > d^*(A)^2 - c \} \) has bounded gaps.

Bergelson established the following natural two-dimensional generalization of Khintchine’s recurrence theorem in [B].

Theorem 1.2. Let \((X, \mathcal{X}, \mu)\) be a probability space, and let \( T : X \to X \) be an invertible transformation which preserves \( \mu \). For all \( B \in \mathcal{X} \) and all \( c > 0 \), the set

\[
\{ (n, m) : \mu(B \cap T^{-n}B \cap T^{-m}(B \cap T^{-n}B)) > \mu(B)^4 - c \}
\]

meets every large enough square \([N_1, N_1 + M] \times [N_2, N_2 + M] \) in \( \mathbb{Z}^2 \).

Bergelson applied Theorem 1.2 together with the Furstenberg correspondence principle, to deduce

Corollary 1.3. If \( A \subset \mathbb{Z} \) with \( d^*(A) > 0 \), then

\[
\{ (n, m) : d^*(A \cap (A - n) \cap (A - n - m)) > \mu(A)^4 - c \}
\]

meets every large enough square \([N_1, N_1 + M] \times [N_2, N_2 + M] \) in \( \mathbb{Z}^2 \).
Roughly speaking, this says that if $A \subseteq \mathbb{Z}$ with $d^*(A) = \delta > 0$, then there are many $(n, m) \in \mathbb{Z}^2$ for which the quadruples \{a, a + n, a + m, a + n + m\} appear in $A$ (almost) as often as one would expect in a set $A$ generated randomly by selecting each $a \in \mathbb{Z}$ independently with probability $\delta$.

Higher-dimensional generalizations of Theorem 1.2 and Corollary 1.3 were obtained by Host and Kra in [HK].

One may wonder if results similar to Corollary 1.3 hold for subsets of $\mathbb{Z}^2$. Defining the upper Banach density of a subset $A \subset \mathbb{Z}^2$ as

$$d^*(A) := \limsup_{\min(M_1 - N_1, M_2 - N_2) \to \infty} \frac{|A \cap ([N_1, M_1] \times [N_2, M_2])|}{(M_1 - N_1 + 1)(M_2 - N_2 + 1)},$$

one may ask if $d^*(A \cap (A - (n, 0))) \cap (A - (0, m))) \cap (A - (n, m)) > d^*(A)^4 - c$ for many $n, m$. More generally, one may ask if similar results hold for subsets of an arbitrary group $G$.

Of course, the question implies that there is a notion of density on $G$ similar to that of $d^*$ on $\mathbb{Z}$ and $\mathbb{Z}^2$. As a substitute for sequences of intervals $[N, M]$, we use Følner sequences, which we define now.

If $G$ is a countable discrete group, a sequence $\{\Phi_N\}_{N \in \mathbb{N}}$ of finite subsets of $G$ is called a left (resp. right) Følner sequence if for all $g \in G$,

$$\lim_{N \to \infty} \frac{|\Phi_N \cap g\Phi_N|}{|\Phi_N|} = 1 \text{ (resp. } \lim_{N \to \infty} \frac{|\Phi_N \cap \Phi_N g|}{|\Phi_N|} = 1).$$

A Følner sequence is called two-sided if it is both a left- and a right Følner sequence. If a discrete group $G$ has a Følner sequence, then $G$ is called amenable. We will usually denote Følner sequences $\{\Phi_N\}_{N \in \mathbb{N}}$ without subscripts, as in “let $\Phi$ be a Følner sequence.”

If $\Phi$ is a left Følner sequence in a group $G$, we can define the upper density (with respect to $\Phi$) of a set $A \subset G$ with respect to $\Phi$ by $d_\Phi(A) = \limsup_{N \to \infty} \frac{|A \cap \Phi_N|}{|\Phi_N|}$. It is easy to check that $d_\Phi(A) = d_\Phi(gA)$ for all $g \in G$ and $A \subset G$. One can define density with respect to a right Følner sequence in the same way, so that $d_\Phi$ is invariant under right multiplication if $\Phi$ is a right Følner sequence.

We also need a notion that generalizes the idea of a subset $R$ of $\mathbb{Z}^2$ that meets every large enough square $[N_1, N_1 + M] \times [N_2, N_2 + M]$. We call a subset $S \subset G$ left- (resp. right-) syndetic if there exists finitely many $g_1, \ldots, g_k \in G$ such that $G = \bigcup_{i=1}^k g_i S$ (resp. $G = \bigcup_{i=1}^k S g_i$). One can easily check that $A \subset \mathbb{Z}$ is syndetic if and only if $A$ meets every long enough interval, and $A \subset \mathbb{Z}^2$ is syndetic if and only if $A$ meets every large enough square in $\mathbb{Z}^2$.

The following generalization of Corollary 1.3 appears in [Gr1].

**Theorem 1.4.** ([Gr], Corollary 1.6) Let $G$ be a countable amenable group with identity $e$. Let $E \subset G \times G$, and let $\Phi$ be a right Følner sequence in

\footnote{The exponents $-1$ do not appear in [Gr], but, they can be removed or added at will via a change of variables.}
G \times G with \lim \sup_{N \to \infty} \frac{|E \cap \Phi_N|}{|\Phi_N|} = \delta > 0. Then for all c > 0, the set

\left\{ (g,h) : \lim \sup_{N \to \infty} \frac{|E \cap E(g^{-1}, e) \cap E(e, h^{-1}) \cap E(g^{-1}, h^{-1}) \cap \Phi_N|}{|\Phi_N|} > \delta^4 - c \right\}

is both left- and right syndetic in G \times G.

This is deduced from the following theorem and corollary in [Gr]. (We say two actions T, S of G commute if S_g T_h = T_h S_g for all g, h \in G.)

**Theorem 1.5.** ([Gr], Theorem 1.4) Let G be a countable amenable group, let \( (X, \mathcal{B}, \mu) \) be a probability space with probability measure \( \mu \), and commuting measure preserving actions T, S of G on X. Then

1. For all \( f_1, f_2, f_3 \in L^\infty(\mu) \), and all two-sided Følner sequences \( \Phi, \Psi \) in G, the limit

\[
L = \lim_{N \to \infty} \frac{1}{|\Phi_N||\Psi_N|} \sum_{(g,h) \in \Phi_N \times \Psi_N} f_1(T_g x) f_2(S_h x) f_3(T_g S_h x)
\]

exists in \( L^2(\mu) \).

2. L is equal to a constant \( \mu \)-almost everywhere for all \( f_1, f_2, f_3 \in L^\infty(\mu) \) if and only if \( T \times T \) and \( S \times S \) are ergodic.

It is also shown in [Gr] that the limit L above is independent of the choice of Følner sequences \( \Phi, \Psi \).

**Corollary 1.6.** ([Gr], Corollary 1.5) Let \( (X, \mathcal{B}, \mu), T \) and S be as in Theorem 1.5. Suppose \( f \in L^\infty(\mu) \) is a nonnegative function. Then for all two-sided Følner sequences \( \Phi, \Psi \)

1. \[
\lim_{n \to \infty} \frac{1}{|\Phi_N||\Psi_N|} \sum_{(g,h) \in \Phi_N \times \Psi_N} \int f T_g f S_h f T_g S_h f d\mu \geq \left( \int f d\mu \right)^4.
\]

In this paper we generalize Theorem 1.5 and Theorem 1.3 to higher-dimensional cases. Specifically, we establish the following theorem and corollary.

**Theorem 1.7.** Let G be a countable amenable group, let \( (X, \mathcal{B}, \mu) \) be a probability space with probability measure \( \mu \), let \( d \in \mathbb{N} \), and let \( T^{(1)}, \ldots, T^{(d)} \) be commuting actions of G on X which preserve \( \mu \). Let \( \Phi^{(1)}, \ldots, \Phi^{(d)} \) be Følner sequences in G, and let \( F_N \) denote \( \Phi^{(1)} \times \cdots \times \Phi^{(d)} \). Let \( f_\varepsilon, \varepsilon \in \{0, 1\}^d \) be \( 2^d \) bounded, \( \mathcal{B} \)-measurable functions on X.

1. The limit

\[
L = \lim_{N \to \infty} \frac{1}{|F_N|} \sum_{g \in F_N} \prod_{\varepsilon \in \{0, 1\}^d} f_\varepsilon \circ R_g^\varepsilon
\]

exists in \( L^2(\mu) \), where \( R_g^\varepsilon := \prod_{i=1}^d (T^{(i)} g_i)^{\varepsilon_i} \) for \( g = (g_1, \ldots, g_d), \varepsilon = (\varepsilon_1, \ldots, \varepsilon_d) \).
(2) Furthermore, if each $f_\varepsilon = f$ is a nonnegative function, then

$$\int L \, d\mu \geq \left( \int f \, d\mu \right)^{2^d}.$$ 

**Corollary 1.8.** Let $G$ be a countable amenable group, let $G^d$ denote the $d$th cartesian power of $G$, and let $E \subset G^d$. For $g = (g_1, \ldots, g_d), h = (h_1, \ldots, h_d) \in G^d$, let $h.g := \{h_1^{\varepsilon_1}g_1, \ldots, h_d^{\varepsilon_d}g_d : \varepsilon \in \{0,1\}^d\}$. If there is a left Følner sequence $\Psi$ in $G^d$ with $d_\Psi(A) > 0$, then for all $c > 0$, the set

$$\{h \in G^d : d_\Psi(\{g : h.g \subset E\}) > d_\Psi(E)^{2^d} - c\}$$

is both left- and right- syndetic in $G^d$.

**Remark 1.** One may make the analogous conclusion about right Følner sequences if one replaces $h.g$ with $g_*h := \{(g_1h_1^{\varepsilon_1}, g_2h_2^{\varepsilon_2}, \ldots, g_dh_d^{\varepsilon_d}) : \varepsilon \in \{0,1\}^d\}$.

Note that $h.g \subseteq E$ if and only if $g \in \bigcap_{\varepsilon \in \{0,1\}^d}(h_1^{\varepsilon_1}, \ldots, h_d^{\varepsilon_d})E$, so the conclusion is equivalent to concluding that

$$\{h \in G^d : d_\Psi \left( \bigcap_{\varepsilon \in \{0,1\}^d}(h_1^{\varepsilon_1}, \ldots, h_d^{\varepsilon_d})E \right) > d_\Psi(E)^{2^d} - c\}$$

is both left- and right syndetic in $G^d$.

Theorem 1.7 and Corollary 1.8 were established for the case $G = \mathbb{Z}$ in [C], using results from [H]. Theorem 1.7 for the case $G = \mathbb{Z}$ is Theorem 1.1 of [C]. Although Theorem 1.7 was established for the case $G = \mathbb{Z}$ in [A], the methods of [H] and [C] provide additional insight.

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2. Preparation the Proof of Theorem 1.7

The papers [C] and [H] together constitute a self-contained proof of Theorem 1.7 in the case $G = \mathbb{Z}$. A self-contained proof of Theorem 1.7 for an arbitrary amenable group $G$ would be virtually identical to the papers [C] and [H], except that the proofs of Lemma 2 in [H] and Theorem 4 of [H] must be changed, and some routine facts about Følner sequences must be applied.

Before we present those changes, we recall some background.

2.1. Background. By a standard probability space, we mean a measure space $(X, \mathcal{X}, \mu)$ which is measure-theoretically isomorphic to a measure space $(K, \mathcal{B}, \nu)$ where $K$ is a compact metric space, $\mathcal{B}$ is the $\sigma$-algebra of Borel subsets of $K$, and $\nu$ is a regular Borel measure on $\mathcal{B}$ with $\nu(K) = 1$. The relevant probability spaces for deducing Corollary 1.8 have compact
metric topologies on the underlying spaces, and the general case of Theorem 1.7 can be deduced from the special case where \((X, \mathcal{X}, \mu)\) is a standard probability space by an application of Theorem 2.15 of [G].

Let \((X, \mathcal{X}, \mu)\) be a standard probability measure space, let \(G\) be a countable group, and let \(T\) be an action of \(G\) on \(X\) which preserves \(\mathcal{X}\) and \(\mu\). We say that \(X = (X, \mathcal{X}, \mu, T)\) is a measure preserving \(G\)-system (or \(G\)-system).

If \(k \in \mathbb{N}\), a \(k\)-fold self-joining of \(X\) is a \(G\)-system \((X^k, \mathcal{X}^\otimes k, \lambda, T^\otimes k)\), where \(X^k\) is the \(k\)th cartesian power of \(X\), \(\mathcal{X}^\otimes k\) is the \(k\)-fold product \(\sigma\)-algebra, \(T^\otimes k\) is the action given by \(g \mapsto T^\otimes k_g\), where \(T^\otimes k\) is the \(k\)th cartesian power of \(T\), and \(\lambda\) is a \(T^\otimes k\)-invariant probability measure on \((X^k, \mathcal{X}^\otimes k)\) satisfying the following condition: if \(A \in \mathcal{X}\), then \(\lambda(\prod_{i=1}^{k-1} X \times A \times \prod_{i=j+1}^{k} X) = \mu(A)\), for \(1 \leq j \leq k\).

If \(Y = (Y, \mathcal{Y}, \nu, S)\) is another \(G\)-system, we say that \(Y\) is a factor of \(X\) if there is a map \(\pi: X \to Y\) with \(\pi^{-1}(\mathcal{Y}) \subseteq \mathcal{X}\), \(\mu(\pi^{-1}(A)) = \nu(A)\) for all \(A \in \mathcal{Y}\), and \(\pi \circ T_g = S_g \circ \pi\) for all \(g \in G\).

If \(D \subseteq \mathcal{X}\) is any countably generated, \(T\)-invariant sub \(\sigma\)-algebra, one can always realize \(D\) as a \(\sigma\)-algebra of the form \(\pi^{-1}(\mathcal{Y})\), where \((Y, \mathcal{Y}, \nu, S)\) is a factor of \(X\) and \((Y, \mathcal{Y}, \nu)\) is a standard probability space. In this way, we have a correspondence between factors of \(X\) and \(T\)-invariant sub \(\sigma\)-algebras of \(\mathcal{X}\).

If \(D \subseteq \mathcal{X}\) is a countably generated sub \(\sigma\)-algebra of \(\mathcal{X}\), and \(f \in L^2(\mu)\), we let \(E(\mu)(f|D)\) denote the conditional expectation of \(f\) on \(D\), which agrees with the orthogonal projection of \(f\) onto the closed subspace of \(L^2(\mu)\) consisting of the \(D\)-measurable functions. If there is no ambiguity, we write \(E(f|D)\) for \(E(\mu)(f|D)\).

If \(Y\) is a factor of \(X\), the relatively independent product of \(X\) with itself over \(Y\) is the 2-fold self joining \((X \times X, \mathcal{X} \otimes \mathcal{X}, \mu \times \gamma \mu, T \times T)\), where \(\mu \times \gamma \mu\) is given by \(\int f \otimes g \, d\mu \times \gamma \mu = \int E(f|\mathcal{Y})E(g|\mathcal{Y}) \, d\mu\), for \(f, g \in L^\infty(\mu)\).

If \((X, \mathcal{X}, \mu, T)\) is a \(G\)-system, we let \(\mathcal{I}_T\) denote the \(\sigma\)-algebra of \(T\)-invariant \(\mathcal{X}\)-measurable sets.

2.1.1. The mean ergodic theorem. The mean ergodic theorem for amenable groups follows.

**Theorem 2.1.** Let \((X, \mathcal{X}, \mu)\) be a probability space, let \(G\) be an amenable group, \(T: X \to X\) be a \(\mu\)-preserving action of \(G\) on \(X\). Let \(\Phi\) be a left Følner sequence in \(G\). Then for all \(f \in L^2(\mu)\),

\[
\lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{g \in \Phi_N} f \circ T_g = Pf
\]

in \(L^2(\mu)\), where \(P\) is the orthogonal projection of \(f\) onto the space of \(T\)-invariant functions.

We will need a slight variation, which follows immediately from Theorem 2.1.
Proposition 2.2. Let $\langle X, \mathcal{X}, \mu \rangle$ be a probability space, let $G$ be an amenable group, $T : X \to X$ be a $\mu$-preserving action of $G$ on $X$. Let $\Phi$ be a right Følner sequence in $G$. Then for all $f \in L^2(\mu)$,
\[ \lim_{N \to \infty} \frac{1}{|\Phi_N|^2} \sum_{g,h \in \Phi_N} f \circ T_{gh^{-1}} = Pf \]
in $L^2(\mu)$, where $P$ is the orthogonal projection of $f$ onto the space of $T$-invariant functions.

To prove this, write $Pf = \frac{1}{|\Phi_N|} \sum_{h \in \Phi_N} Pf \circ T_h$, so that
\[ \frac{1}{|\Phi_N|^2} \sum_{g,h \in \Phi_N} f \circ T_{gh^{-1}} - Pf = \frac{1}{|\Phi_N|} \sum_{h \in \Phi_N} \frac{1}{|\Phi_N|} \sum_{g \in \Phi_N} (f \circ T_g - Pf) \circ T_h^{-1}. \]
Taking norms, we have
\[ \left\| \frac{1}{|\Phi_N|^2} \sum_{g,h \in \Phi_N} f \circ T_{gh^{-1}} - Pf \right\|_{L^2(\mu)} \leq \frac{1}{|\Phi_N|} \left\| \sum_{h \in \Phi_N} \frac{1}{|\Phi_N|} \sum_{g \in \Phi_N} (f \circ T_g - Pf) \circ T_h \right\| = \left\| \frac{1}{|\Phi_N|} \sum_{g \in \Phi_N} (f \circ T_g - Pf) \right\|. \]
Now apply Theorem 2.1.

2. Følner sequences and syndeticity. To derive Corollary 1.8 from Theorem 1.7, we need the following lemma, whose proof is similar to the proof of Lemma 4.7 of [Gr] that we omit it.

Lemma 2.3. Let $G$ be an amenable group, $d \in \mathbb{N}$, and let $S \subset G^d$. Then $S$ is both left and right syndetic if and only if for all two-side $d$ Følner sequences $\Phi(i), i = 1, \ldots, d$ in $G$, there exists $N_1, \ldots, N_d \in \mathbb{N}$ such that $S \cap \Phi_{N_1} \times \cdots \times \Phi_{N_d} \neq \emptyset$.

3. Proof of Theorem 1.7.

We will present the constructions of [H], adapted to the setting of our Theorem 1.7 and then explain how these are used in [C] to establish Theorem 1.1 of [C].

3.1. The cubes joinings. Throughout this section, we fix a standard probability space $\langle X, \mathcal{X}, \mu \rangle$, an amenable group $G$, and commuting, $\mu$-preserving actions of $G$ on $\langle X, \mathcal{X} \rangle$, denoted by $T^{(1)}, \ldots, T^{(d)}$. Let $I_i$ denote the $\sigma$-algebra consisting of $T^{(i)}$-invariant sets.
Unless otherwise stated, for \( k > 0 \), we consider the set \( \{0,1\}^k \) to have the reverse-lexicographic order, and we write \( 0 \) for the element of \( \{0,1\}^k \) all of whose coordinates are 0.

For each sequence \( P = (i_1, i_2, \ldots, i_k) \), of distinct integers \( i_j \leq d \), we define a joining \( \mu^P \) on \( X^{2^k} \) as follows. For \( P = \emptyset, \mu^P = \mu \). If \( k > 0, P = (i_1, \ldots, i_k) \) and \( \mu^Q \) is defined for all \( Q \) of length \( k-1 \), let \( T_{Q,j} \) denote the \((T(j))^{\otimes 2^{k-1}}\)-invariant \( \sigma \)-algebra of the system \((X^{2^{k-1}}, \mu^Q, (T(j))^{\otimes 2^{k-1}})\), let \( Q = (i_1, \ldots, i_{k-1}) \), and define \( \mu^P \) on \( X^{2^k} \) by

\[
\int f \otimes g \, d\mu^P = \int \mathbb{E}(f|T_{Q,k})\mathbb{E}(g|T_{Q,k}) \, d\mu^Q,
\]

for \( f, g \in L^\infty(X^{2^{k-1}}) \), so that \( \mu^P \) is the relatively independent product of \( \mu^Q \) with itself over \( T_{Q,k} \).

By repeated application of the mean ergodic theorem, the joining \( \mu^P \) can also be defined as follows. For \( f_\varepsilon \in L^\infty(X), \varepsilon \in \{0,1\}^k \), the integral

\[
\int \prod_{\varepsilon \in \{0,1\}^k} f_\varepsilon \, d\mu^P
\]

is equal to the iterated limit

\[
\lim_{N_1 \to \infty} \cdots \lim_{N_k \to \infty} \frac{1}{|\Phi_{N_1}| \cdots |\Phi_{N_k}|} \sum_{g \in \Phi_k} \cdots \sum_{g \in \Phi_k} \prod_{\varepsilon \in \{0,1\}^k} f_\varepsilon \circ \prod_{j=1}^k (T(j)_\varepsilon^j) \, d\mu.
\]

The next lemma appears as Theorem 4 in [H] for the case \( G = \mathbb{Z} \), and is essential to the proof of Theorem 1.1 in [C].

**Lemma 3.1.** The joining \( \mu^P \) for \( P = (i_1, \ldots, i_k) \) depends only on the indices \( \{i_1, \ldots, i_k\} \) and not on the order.

**Proof.** We argue as in Proposition 3 of [H]. For the case \( k = 2 \), this was shown in [GG]. We prove the general statement by induction on \( k \). The induction hypothesis is “For \( r < k \), the joining \( \mu^{(i_1,\ldots,i_r)} \) is independent of the order of the \( (i_1,\ldots,i_r) \).” It follows that the joining \( \mu^{(i_1,\ldots,i_k)} \) is independent of the order of the first \( r \) coordinates. We then show that for \( P = (i_1, \ldots, i_k), Q = (i_2, i_1, \ldots, i_{k-1}, i_k), \mu^P = \mu^Q \).

Write \( U = T(i_{k-1}) \), \( V = T(i_k) \) and let \( U', V' \) denote the product actions \((U)^{\otimes 2^{k-2}}, (V)^{\otimes 2^{k-2}}\), respectively. Then \( \mu^P \) and \( \mu^Q \) are the joinings \((\mu^{i_1,\ldots,i_{k-2}})^{U', V'}, (\mu^{i_2,\ldots,i_{k-2}})^{U', V'} \), and equality follows from the case \( k = 2 \), so \( \mu^{(i_1,\ldots,i_k)} \) does not depend on the order of the indices \( i_1, \ldots, i_k \). \( \square \)

Now we can define, unambiguously, \( \mu^\eta \) for a subset \( \eta = \{i_1, \ldots, i_k\} \) of \( \{1, \ldots, d\} \), as the joining \( \mu^{(i_1,\ldots,i_k)} \).

Given \( \eta \subset \{1, \ldots, k\} \), we can define a seminorm \( \|f\|_\eta^k \) on \( L^\infty(X) \) by

\[
\|f\|_\eta^k = \int \bigotimes_{\varepsilon \in \{0,1\}^k} f(x_\varepsilon) \, d\mu^\eta(x).
\]

One checks that this is a seminorm, and that \( \int \bigotimes_{\varepsilon \in \{0,1\}^k} f_\varepsilon \, d\mu^\eta \leq \min_{\varepsilon \in \{0,1\}^k} \|f_\varepsilon\|_\eta \), just as in the proof of Proposition 2 of [H].
The next lemma is Lemma 2 of [1], adapted to the setting of actions of amenable groups.

We will use the main inequality in the proof of the van der Corput trick from [BMZ], encoded as the following observation. (See also [BR].)

**Observation.** Let $\Phi$ and $\Psi$ left Følner sequences in $G$, and let $c > 0$. For each $m \in \mathbb{N}$, Let $N(m)$ be large enough that $|\Phi_N(m)\triangle g\Phi_N(m)| < c|\Phi_N(m)|$ for all $g \in \Psi_N(m)$. If $\{x_g\}_{g \in G}$ is a sequence of vectors in a Hilbert space $\mathcal{H}$ with $\|x_g\| \leq 1$ for all $g$, we then have

$$\left\| \frac{1}{|\Phi_N(m)|} \sum_{g \in \Phi_N(m)} x_g - \frac{1}{|\Psi_m|} \sum_{h \in \Psi_m} \frac{1}{|\Phi_N(m)|} \sum_{g \in \Phi_N(m)} x_{hg} \right\| < 2c.$$  

**Lemma 3.2.** Let $\eta \subset \{1, \ldots, d\}, |\eta| = k$. Let $\Phi^{(1)}, \ldots, \Phi^{(k)}$ be Følner sequences in $G$, and let $f_\epsilon \in L^1(X), \epsilon \in \{0,1\}^k$ be uniformly bounded by 1. Then for all $\delta > 0$, there exists $N_0$ such that whenever $N_i > N_0$,

$$\left| \frac{1}{|\Phi^{(1)}_{N_1}|} \cdots \frac{1}{|\Phi^{(k)}_{N_k}|} \sum_{g_i \in \Phi_{\eta_i}} \int \prod_{\epsilon \in \{0,1\}^k} f_\epsilon \circ \prod_{\eta \in \eta} (T_{g_\epsilon(i)})^{\epsilon_i} d\mu \right| \leq \|f_0\|_\eta + \delta.$$  

**Proof.** Let

$$J = \frac{1}{|\Phi^{(1)}_{N_1}|} \cdots \frac{1}{|\Phi^{(k)}_{N_k}|} \sum_{g_i \in \Phi_{\eta_i}} \int \prod_{\epsilon \in \{0,1\}^k} f_\epsilon \circ \prod_{\eta \in \eta} (T_{g_\epsilon(i)})^{\epsilon_i} d\mu.$$  

Let $\Psi$ be a Følner sequence in $G$, let $\alpha > 0$, and for each $m$, choose $N(m)$ so that $|\Phi^{(i)}_N \triangle g\Phi^{(i)}_N|, |\Phi^{(i)}_N \triangle g\Phi^{(i)}_N| < \alpha|\Phi^{(i)}_N|$, for all $i \leq k$, and all $g \in \Psi_m$, whenever $N > N(m)$.

Fix some $M > 0$, and fix $N_1, \ldots, N_k > \max_{m \leq M} N(m)$.

Write $\prod_{\epsilon \in \{0,1\}^k} f_\epsilon \circ \prod_{\eta \in \eta} (T_{g_\epsilon(i)})^{\epsilon_i}$ as

$$\prod_{\epsilon \in \{0,1\}^k, \epsilon_k = 0} f_\epsilon \circ \prod_{\eta \in \eta} (T_{g_\epsilon(i)})^{\epsilon_i} \cdot \prod_{\epsilon \in \{0,1\}^k, \epsilon_k = 1} f_\epsilon \circ \prod_{\eta \in \eta} (T_{g_\epsilon(i)})^{\epsilon_i}.$$  

Writing $F_{g_1, \ldots, g_{k-1}}$ for the first factor in [2], and $F'_{g_1, \ldots, g_k} \circ T^{(k)}_{g_k}$ for the second factor, we have

$$J = \frac{1}{|\Phi^{(1)}_{N_1}| \cdots |\Phi^{(k-1)}_{N_{k-1}}|} \sum_{g_i \in \Phi_{\eta_i}} \int \frac{1}{|\Phi_k|} \sum_{g_k \in \Phi_k} F_{g_1, \ldots, g_{k-1}} \cdot F'_{g_1, \ldots, g_{k-1}} \circ T^{(k)}_{g_k} d\mu.$$  

Applying the Cauchy-Schwartz inequality, we find

$$|J|^2 \leq \frac{1}{|\Phi^{(1)}_{N_1}| \cdots |\Phi^{(k-1)}_{N_{k-1}}|} \sum_{g_i \in \Phi_{\eta_i}} \left| \int \frac{1}{|\Phi_k|} \sum_{g_k \in \Phi_k} F'_{g_1, \ldots, g_{k-1}} \circ T^{(k)}_{g_k} d\mu \right|^2_{L^2(\mu)}.$$
By the observation above, replacing the average inside the norm by

\[
\frac{1}{|\psi_m|} \sum_{h_k \in \psi_m} \int \frac{1}{|\phi_k|} \sum_{g_k \in |\phi_k|} F_{g_1, \ldots, g_{k-1}} \circ T_{h_k, g_k}^{(k)} \, d\mu
\]

introduces an error of at most $2\alpha$, so we estimate the norm of (3). The square of the norm is at most

\[
\frac{1}{|\psi_m|^2} \sum_{g_k, h_k \in \psi_m} \int F_{g_1, \ldots, g_{k-1}} \circ T_{h_k, g_k}^{(k)} \cdot F_{g_1, \ldots, g_{k-1}} \circ T_{h_k, g_k}^{(k)} \, d\mu.
\]

Applying $T_{g_k, j_k}^{(k)}$ to the integrand, this becomes

\[
\frac{1}{|\psi_m|^2} \sum_{g_k, h_k \in \psi_m} \int F_{g_1, \ldots, g_{k-1}} \circ T_{h_k, j_k}^{(k)} \cdot F_{g_1, \ldots, g_{k-1}} \, d\mu.
\]

Then $|J|^2$ is bounded by

\[
\alpha + \frac{1}{|\psi_m|} \sum_{j_k \in \psi_m} \sum_{g_k \in \psi_m} \int F_{g_1, \ldots, g_{k-1}} \circ T_{h_k, j_k}^{(k)} \cdot F_{g_1, \ldots, g_{k-1}} \, d\mu.
\]

Repeating this argument $k$ times produces the inequality

\[
|J|^2 < C\alpha + \frac{1}{|\psi_m|} \sum_{j_k \in \psi_m} \int \prod_{i=1}^{k} f_0 \circ \prod_{i=1}^{k} (T_{h_k, j_k}^{(k)})^{\epsilon_i} \, d\mu,
\]

where $C$ depends only on $k$.

Now for a given $\delta > 0$, by Proposition 2.2 and the definition of $\|\cdot\|_\eta$, there exists $M$ so that there are choices $m_1, \ldots, m_k < M$ making the average on the right-hand side of (4) differ from $\|f_0\|_\eta^{2k}$ by at most $\frac{1}{2}\delta$. Having chosen $\alpha = \frac{1}{2C}\delta$, we get $J < \|f_0\|_\eta^{2k} + \delta$. □

3.2. The magic extension and proof of part (1) of Theorem 1.7

In this section we use the joining $\mu^{(1, \ldots, d)}$ to construct an extension of the system $X = (X, \mathcal{F}, \mu, T^{(1)}, \ldots, T^{(d)})$ with some nice properties.

Let $X^* = X^{\{0,1\}^d}$, $\mathcal{F}^* = \mathcal{F} \otimes 2^d$, $\mu^* = \mu^{(1, \ldots, d)}$, and for $1 \leq i \leq d$, define $T^{(i)*}: G \times X^* \to X^*$ by

\[
T_g^{(i)*} x = \begin{cases} 
T_g^{(i)} x_{\epsilon_i} & \text{if } \epsilon_i = 0; \\
x_{\epsilon_i} & \text{if } \epsilon_i = 1.
\end{cases}
\]

One can check from the definition of $\mu^*$ that it is invariant under each $T^{(i)}$. The system $X^* = (X^*, \mathcal{F}^*, \mu^*, T^{(1)*}, \ldots, T^{(d)*})$ is called the magic extension of $X$. 
Furthermore, $X$ is a factor of $X^*$, with the factor map defined as projection on the first coordinate.

Let $\mu^{**}$ be the measure constructed from $X^*$ the way $\mu^*$ was constructed from $X$. For $\eta \subseteq \{1, \ldots, d\}$, let $\| \cdot \|_\eta$ be the seminorm on $L^\infty(\mu^*)$ constructed from $X^*$ the way $\| \cdot \|_\eta$ was constructed from $X$.

For $\eta \subseteq \{1, \ldots, d\}$, let $Z_\eta$ denote the $\sigma$-algebra spanned by the $T(i)$-invariant sets for $i \in \eta$. That is,

\[ Z_\eta := \bigvee_{i \in \eta} I_i. \]

Exactly as in Theorem 3.2 of [C], we have

**Theorem 3.3.** For every $\varepsilon \subseteq \{1, \ldots, d\}, \varepsilon \neq \emptyset$, and every function $f \in L^\infty(\mu^*)$, if $E_{\mu^*}(f|Z_\varepsilon) = 0$ then $\|f\|_\varepsilon^* = 0$.

The proof is identical to the proof in [C], so we omit it.

We may proceed as in the proof of Proposition 3.3 in [C] to prove that

**Proposition 3.4.** Let $f_{\varepsilon, \varepsilon} \in \{0, 1\}^d$ be functions on $X^{(0,1)^d}$ with $\|f_{\varepsilon}\|_{L^\infty(\mu^*)} \leq 1$ for every $\varepsilon$. Let $\Phi^{(1)}, \ldots, \Phi^{(d)}$ be Følner sequences in $G$. Then the limit

\[ \lim_{\min(N_1, \ldots, N_d) \to \infty} \frac{1}{|\Phi^{(1)}_{N_1}| \cdots |\Phi^{(d)}_{N_d}|} \sum_{g \in \Phi^{(1)}_{N_1} \times \cdots \times \Phi^{(d)}_{N_d}} \prod_{i=1}^d f_{\varepsilon} \circ (T(i)^{\varepsilon})_i \]  

exists in $L^2(\mu^*)$.

Since $X$ is a factor of $X^*$, we can consider $L^\infty(\mu)$ as a subalgebra of $L^\infty(\mu^*)$, and conclude that the convergence asserted in part (1) of Theorem [1.7] holds.

**Remark 2.** For applications, it is useful to know that the limit is independent of the choice of Følner sequences $\Phi^{(i)}$. This can be accomplished by interpolating Følner sequences: given two collections of Følner sequences $\Phi^{(i)}$, $\Psi^{(i)}$, $i = 1, \ldots, d$ let $\Theta^{(i)}_{2N} = \Phi^{(i)}_N$, $\Theta^{(i)}_{2N+1} = \Psi^{(i)}_N$. Then the limit in (5) involving the Følner sequence $\Psi^{(i)}$ must agree with the limit involving $\Theta^{(i)}$ and the limit involving $\Phi^{(i)}$.

**3.3. Proof of the inequality in Theorem [1.7]** In Section 4 of [Gr], it is shown that if $\{u_{g,h}\}_{g,h \in G}$ is a bounded sequence indexed by $G \times G$ such that for all two-sided Følner sequences $\Phi$, $\Psi$, the limits

\[ L = \lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{(g,h) \in \Phi_N \times \Psi_N} u_{g,h} \]

\[ L' = \lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{g \in \Phi_N} \lim_{M \to \infty} \frac{1}{|\Psi_M|} \sum_{h \in \Psi_M} u_{g,h} \]
exist and are independent of the choice of Følner sequences $\Phi, \Psi$, then $L = L'$. By an identical argument, we have that given a sequence $\{u_g\}_{g \in G^d}$, indexed by $G^d$, if the limits

$$J = \lim_{N \to \infty} \frac{1}{|\Phi_N^{(1)}| \cdot \cdots \cdot |\Phi_N^{(d)}|} \sum_{g \in \Phi_N^{(1)} \times \cdots \times \Phi_N^{(d)}} u_g$$

$$J' = \lim_{N_1 \to \infty} \frac{1}{|\Phi_N^{(1)}|} \sum_{g_1 \in \Phi_N^{(i)}} \cdots \lim_{N_d \to \infty} \frac{1}{|\Phi_N^{(d)}|} \sum_{g_d \in \Phi_N^{(d)}} u_{(g_1, \ldots, g_d)}$$

exist and are independent of the choice of Følner sequences involved, then $J = J'$.

Proof of Theorem 1.7, part (2).

The inequality in Theorem 1.7 is proved inductively in the case $d = 2$ in [Gr], and the induction may be continued to achieve a proof for arbitrary $d$.

Suppose that the inequality holds for $d-1$ rather than $d$. Writing $F_{g_1, \ldots, g_{d-1}}$ for $\prod_{\varepsilon \in \{0,1\}} T_{g_1}^{(\varepsilon_1)} \cdots T_{g_{d-1}}^{(\varepsilon_{d-1})}$, we have

$$\prod_{\varepsilon \in \{0,1\}^d} f \circ \prod_{i=1}^d (T_{g_i}^{(\varepsilon_i)}) = F_{g_1, \ldots, g_{d-1}} \cdot F_{g_1, \ldots, g_{d-1}} \circ T_{g_d}^{(d)}.$$ Integrating and averaging over $\Phi^{(d)}$, we have

$$\lim_{N \to \infty} \frac{1}{|\Phi_N^{(d)}|} \sum_{g \in \Phi_N^{(d)}} \int F_{g_1, \ldots, g_{d-1}} \cdot F_{g_1, \ldots, g_{d-1}} \circ T_{g_d}^{(d)} \, d\mu$$

$$= \int E(F_{g_1, \ldots, g_{d-1}} | I_d)^2 \, d\mu$$

$$\geq \left( \int E(F_{g_1, \ldots, g_{d-1}} | I_d) \, d\mu \right)^2$$

$$= \left( \int F_{g_1, \ldots, g_{d-1}} \, d\mu \right)^2.$$ Averaging over $\Phi^{(1)} \times \cdots \times \Phi^{(d-1)}$, and applying the induction hypothesis, we find

$$\frac{1}{|\Phi_N^{(1)}| \cdot \cdots \cdot |\Phi_N^{(d-1)}|} \sum_{g_i \in \Phi_N^{(i)}} \left( \int F_{g_1, \ldots, g_{d-1}} \, d\mu \right)^2 \geq$$

$$\left( \frac{1}{|\Phi_N^{(1)}| \cdot \cdots \cdot |\Phi_N^{(d-1)}|} \sum_{g_i \in \Phi_N^{(i)}} \int F_{g_1, \ldots, g_{d-1}} \, d\mu \right)^2.$$ Averaging and applying the induction hypothesis yields the desired inequality. $\square$
Observation. Part (2) of Theorem 1.7 and Lemma 2.3 imply that for all \( c > 0 \)
\[
R := \{ (g_1, \ldots, g_d) \in G^d : \int \prod_{\xi \in \{0,1\}^d} f \circ \prod_{i=1}^d (T_{g_i}^{(i)})^{\xi_i} \, d\mu > \left( \int f \, d\mu \right)^{2d} - c \}
\]
is both left- and right syndetic, as \( R \) must meet at least one element of each sequence of the form \( \{ \Phi_N^{(1)} \times \cdots \times \Phi_N^{(d)} \}_{N \in \mathbb{N}} \), where each \( \Phi^{(i)} \) is a Følner sequence in \( G \).

To prove Corollary 1.8 we apply a variation of the Furstenberg correspondence principle as stated in [BMZ]. Let \( G \) be a countable amenable and consider the shift space \( \{0,1\}^G \). Let \( T^{(i)} \) denote the left shift in \( i \)-th coordinate of \( G^d \), so that \((T^{(i)}_g \xi)(g_1, \ldots, g_i, \ldots, g_d) = (\xi(g_1, \ldots, g^{-1}g_i, \ldots, g_d)) \).

**Proposition 3.5.** Let \( \Psi \) be a left Følner sequence in \( G^d \). Suppose \( E \subseteq G^d \) with \( d_\Psi(E) > 0 \). Let \( X = \{ \prod_{i=1}^d T^{(i)}_{g_i}1_E : g_i \in G \} \) be the orbit closure of \( 1_E \) in \( \{0,1\}^G \).

If
\[
d_\Psi(E) = \limsup_{n \to \infty} \frac{|E \cap \Psi_N|}{|\Psi_N|} > 0,
\]
then there exists a probability measure \( \mu \) on \( X \), invariant under each \( T^{(i)} \), such that
\[
\mu(\{ \xi \in X : \xi(e, \ldots, e) = 1 \}) \geq d_\Psi(E).
\]
Let \( A := \{ \xi \in X : \xi(e, \ldots, e) = 1 \} \). For all collections \( \{g_{i,j} : 1 \leq i \leq d, 1 \leq j \leq n \} \) of elements of \( G \), the inequality
\[
d_\Psi\left( \bigcap_{j=1}^n (g_{1,j}, \ldots, g_{d,j})E \right) \geq \mu\left( \bigcap_{j=1}^n \prod_{i=1}^d T^{(i)}_{g_{i,j}}A \right)
\]
holds.

(This is stated in [BMZ] for the case \( d = 2 \), but a similar proof works for arbitrary \( d \). Furthermore, the inequality \( \mu(\{ \xi \in X : \xi(e, \ldots, e) = 1 \}) > 0 \) is stated in [BMZ], but the inequality \( \mu(\{ \xi \in X : \xi(e, \ldots, e) = 1 \}) \geq d_\Psi(E) \) is actually proved. Also, the right shift, rather than the left shift, is used in [BMZ].)

**Proof of Corollary 1.8.** Let \( \Psi \) be a left Følner sequence in \( G^d \), and let \( E \subseteq G \) with \( d_\Psi(E) = \delta > 0 \). Let \( T^{(i)}_i = 1, \ldots, d, A \) and \( \mu \) be as in Proposition 3.5 so that \( \int 1_A \, d\mu = \delta \).

Let \( c > 0 \). By Theorem 1.7 the set
\[
\left\{ (g_1, \ldots, g_d) : \int \prod_{\xi \in \{0,1\}^d} 1_A \circ \prod_{i=1}^d (T_{g_i}^{(i)})^{\xi_i} \, d\mu > \delta^{2d} - c \right\}
\]
is both left- and right syndetic in $G^d$. By Proposition 3.5 this implies that the set of $(g_1, \ldots, g_d)$ such that \( d_f \left( \bigcap_{\varepsilon \in \{0,1\}^d} (g_1^{-\varepsilon_1}, \ldots, g_d^{-\varepsilon_d}) E \right) > \delta^{2d} - c \) is both left- and right syndetic in $G^d$. □

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