A finite goal set in the plane which is not a Winner

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September 20, 2007

Abstract

J. Beck has shown that if two players alternately select previously unchosen points from the plane, Player 1 can always build a congruent copy of any given finite goal set $G$, in spite of Player 2’s efforts to stop him [B]. We give a finite goal set $G$ (it has 5 points) which Player 1 cannot construct before Player 2 in this achievement game played in the plane.

1 Introduction

In the $G$-achievement game played in the plane, two players take turns choosing single points from the plane which have not already been chosen. A player achieves a weak win if he constructs a set congruent to the goal set $G \subset \mathbb{R}^2$ made up entirely of his own points, and achieves a strong win if he constructs such a set before the other player does so. (So a ‘win’ in usual terms, e.g., in Tic-Tac-Toe, corresponds to a strong win in our terminology.) This is a special case of a positional hypergraph game, where players take turns choosing unchosen points (vertices of the hypergraph) in the hopes of occupying a whole edge of the hypergraph with just their own points. [B, B96] contain results and background in this more general area.

The type of game we are considering here is the game-theoretic cousin of Euclidean Ramsey Theory (see [G] for a survey). Fixing some $r \in \mathbb{N}$ and some finite point set $G \subset \mathbb{R}^2$, the most basic type of question in Euclidean Ramsey Theory is to determine whether it is true that in every $r$-coloring of the plane, there is some monochromatic congruent copy of $G$.

Restricting ourselves to 2 colors, the game-theoretic analog asks when Player 1 has a ‘win’ in the achievement game with $G$ as a goal set. Though one can allow transfinite move numbers indexed by ordinals (see Question [H] in Section [B]), it is natural to restrict our attention to games of length $\omega$, in which moves are indexed by the natural numbers. In this case, a weak or strong winning

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strategy for a player is always a finite strategy (i.e., must always result in weak or strong win, respectively, in a finite, though possibly unbounded, number of moves) so long as the goal set $G$ is finite. J. Beck has shown [H] that both players have strategies which guarantee them a weak win in finitely many moves for any finite goal set—the proof is a potential function argument related to the classical Erdős-Selfridge theorem [ES]. The question of when the first player has a strong win—that is, whether he can construct a copy of $G$ first—seems in general to be a much harder problem. (A strategy stealing argument shows that the second player cannot have a strategy which ensures him a strong win: see Lemma 3.3.)

For some simple goal sets, it is easy to give a finite strong winning strategy for Player 1. This is the case for any goal set with at most 3 points, for example, or for the 4-point vertex-set of any parallelogram. We give a set $G$ of 5 points for which we prove that the first player cannot have a finite strong win in the $G$-achievement game (proving, for example, that such finite goal sets do in fact exist). This answers a question of Beck (oral communication).

Fix $\theta = t\pi$, where $t$ is irrational and $t < \frac{1}{9}$. Our set $G$ is a set of 5 points $g_i$, $1 \leq i \leq 5$, all lying on a unit circle $C$ with center $c \in \mathbb{R}^2$. For $1 \leq i \leq 3$, the angle from $g_i$ to $g_{i+1}$ is $\theta$. The point $g_5$ (the ‘middle point’) is the point on $C$ lying on the bisector of the angle $\angle g_2c g_3$. (See Figure 1) We call this set the irrational pentagon.

**Theorem 1.1.** There is no finite strong winning strategy for Player 1 in the $G$-achievement game when $G$ is the irrational pentagon.

**Idea:** Let $\theta^n_c(x)$ denote the image of $x \in \mathbb{R}^2$ under the rotation $n\theta$ about the point $c$. An important property of the irrational pentagon is that once a player has threatened to build a copy of it by selecting all the points $g_1, g_2, g_3, g_4$, he can give a new threat by choosing the point $\theta_c(g_1)$ or $\theta_c^{-1}(g_1)$. Furthermore, since $\theta$ is an irrational multiple of $\pi$, the player can continue to do this indefinitely, tying up his opponent (who must continuously block the new threats by selecting the corresponding middle points) while failing himself to construct a copy of $G$. If
Player 1 is playing for a finite strong win, he cannot let Player 2 indefinitely force in this manner. However, to deny Player 2 that possibility, we will see that Player 1’s only option is the same indefinite forcing, which leaves him no better. The rest of the rigorous proof is a case study.

2 The Proof

For the proof of Theorem 1.1, we will need the following lemma.

Lemma 2.1. There are no three unit circles \( C_1, C_2, C_3 \) so that the pairs \( C_i, C_j \) each intersect at 2 distinct points \( x_{ij} \) and \( y_{ij} \), so that the angles \( \angle x_{ij} c_i y_{jk} \) are less than \( \frac{\pi}{3} \) for all \( j \neq i \neq k \).

Proof. Let \( B_i \) denote the unit ball whose boundary is \( C_i \) for each \( i \), and choose \( C_i \) and \( C_j \) from \( \{ C_1, C_2, C_3 \} \) so that the area \( A(B_i \cap B_j) \) is maximal. In Figure 2, for any \( C_k \) intersecting the circle \( C_j \) at points \( x_{jk}, y_{jk} \) lying on \( C_j \) between \( r_1 \) and \( r_2 \), we would have \( A(B_i \cap B_k) > A(B_i \cap B_j) \), a contradiction. The maximum angle between the points \( r_1 \) and \( r_2 \) on \( C_j \) is \( \frac{2\pi}{3} \).

We are now ready to prove Theorem 1.1. Let \( G \) now denote the irrational pentagon.

It is clear that Player 2 can either play indefinitely or reach a point where it is his move, he has a point \( h_1 \) at least 10 units away from any of Player 1’s points, and Player 1 has no more than 2 points in any given (closed) ball of radius 10. (For example: on each turn until he has reached this point, Player 2 moves at least 30 units away from all of Player 1’s points.) Reaching this point, Player 2 begins to build a copy of \( G \); that is, he arbitrarily designates some ‘center point’ \( c \) at unit distance from the point \( h_1 \), and chooses as his move a point \( h_2 \) which is an angle \( \theta \) away from \( h_1 \) on the unit circle \( C \) centered at \( c \). In fact, \( h_1 \) and \( h_2 \) lie on two unit circles which are disjoint except at \( h_1, h_2 \), and so Player 1’s response can lie on only one of them; thus we assume without loss of generality that his response does not lie on the circle \( C \).
Following Player 1’s response, Player 2 will continue constructing his threat by choosing the point $h_3$ which lies on the circle $C$ and is separated from the points $h_1, h_2$ by angles $2\theta, \theta$, respectively. Thus regardless of Player 1’s choice of response, we see that Player 2 can reach the following situation:

\begin{itemize}
  \item[(*)] It is Player 1’s turn, Player 2 has points $h_1, h_2, h_3$ separated consecutively by angles $\theta$ on a unit circle $C$ centered at $c$, and Player 1 has at most 3 points in any unblocked copy of $G$. Finally, Player 1 does not control 4 points of any unblocked copy of $G$, and controls at most one point within 8 units of $c$.
  
  \item[(**) ] Moreover, there is in fact at most one unblocked copy of $G$ on which Player 1 has 3 points, and, if it exists, Player 1 controls no other points within (say) 5 units of those 3 points.
\end{itemize}

We classify the rest of the proof into Cases 1, 2, 3 based on Player 1’s move. The analysis in Cases 1 and 3 depend just on the conditions in paragraph (\(*)\), while Case 2 depends on the conditions in both paragraphs (\*) and (**).

**Case 1:** A natural response for Player 1 might be to play on the circle $C$, thus attempting to prevent Player 2 from building a significant threat. Since no point is a rotation of $h_1$ about the point $c$ by both positive and negative integer multiples of $\frac{\theta}{2}$, we may assume WLOG that Player 1 does not choose any rotations of $h_1$ about $c$ by positive integer multiples of $\frac{\theta}{2}$. Thus Player 2 responds by choosing the point $h_4$ on $C$ which is at an angle $\theta, 2\theta, 3\theta$ from the points $h_3, h_2, h_1$, respectively. Since Player 2 is now threatening to build a copy of $G$ on his next move and Player 1 is not (he has $\leq 3$ points on any unblocked copy of $G$), Player 1 must take the point on $C$ which together with $h_1, h_2, h_3, h_4$ complete a copy of $G$. Player 2’s response is naturally to choose the point $h_5$ on $C$ at angle $\theta, 2\theta, 3\theta$ from $h_4, h_3, h_2$, and we are in essentially the same situation: Player 1 has always at most 3 points in any unblocked congruent copy of $G$ (since he has only one point ‘near’ $C$ which is not on $C$, and any set congruent to $G$ and not on $C$ intersects $C$ in at most 2 points), and Player 2 can force indefinitely.

**Case 2:** Another response for Player 1 which may be possible is to play within the vicinity of his previously chosen points such that he controls 4 points of an unblocked copy of $G$. By (\**) Player 1 has only one 4-point threat, and so Player 2 can choose the corresponding fifth point to avoid losing. Now, Player 1 may be able to continue to make threats on his subsequent moves, but it is easy to check using the conditions of (\**) that his moves will have to stay on a single unit circle $C_1$ to do so, and that he will never be able to generate more than one threat, and thus never be able end his indefinite forcing with a win. On the other hand, each time it is Player 1’s move, the conditions in paragraph (\*) are still satisfied, and so any move other than a continuation of the forcing will allow the analysis from Cases 1 and 3 to apply.

**Case 3:** Finally, we consider the case where Player 1 does ‘none of the above’: that is, he chooses a point not on the circle $C$, but which nevertheless does not increase to 4 the number of points he controls in some congruent copy of $G$. This is the case where we make use of Lemma 2.1.

Player 1 now has as many as two points within 8 units distance of the
By choosing successively points $h_4, h_5, h_6,$ etc., as in Case 1, Player 2 hopes to successively force Player 1 to take the corresponding fifth point of each congruent copy of $G$ that Player 2 threatens to build at each step. The only snag is this: it is conceivable that Player 1, in taking these corresponding ‘fifth’ points, builds his own threat. He already has two points in the vicinity, and it is possible that they lie on a congruent copy of $G$ which intersects the circle $C$ in two points which Player 1 will eventually be forced to take by Player 2’s moves. In this case, Player 2 would have to respond and could conceivably end up losing the game if Player 1 is able to break is forcing sequence.

Of course, this is only truly a problem if Player 1 is threatening this in ‘both directions’—that is, regardless of whether $h_4$ is at angles $\theta, 2\theta, 3\theta$ to the points $h_3, h_2, h_1,$ respectively, or to the points $h_1, h_2, h_3,$ respectively. However, such a double threat is immediately ruled out by Lemma 2.1, since this would require two sets $S_1, S_2 \cong G$ (each a subset of a $3\theta$-arc of a unit circle) intersecting each other in two points (previously chosen by Player 1) and each also each intersecting $C$ in two places. This completes the proof.

3 Further Questions

1. Our (rather crude) methods do not appear suited to much larger goal sets. So we ask: are there arbitrarily large goal sets $G$ for which Player 1 cannot force a finite strong win in the $G$-achievement game played in $\mathbb{R}^2$?

2. We have examples of 4-point sets for which Player 1 has strong winning strategies, and we have given here a 5-point example where Player 2 has a drawing strategy. Are there 4-point sets where Player 2 has a drawing strategy?

3. Player 1 can easily be shown to have strong winning strategies for any goal set of size at most 3, and any 4-point goal set which consists of the vertices of a parallelogram. It is not difficult to give a 5 point goal set for which Player 1 can be shown to have a strong winning strategy. Are there arbitrarily large goal sets $G$ for which Player 1 has a strong winning strategy?

4. We restricted our attention here to the first $\omega$ moves, and indeed, our proof does not show that Player 1 can’t force a strong win if transfinite move numbers are allowed. So we ask: are there finite sets $G$ for which Player 1 cannot force a strong win, when the players make a move for each successor ordinal?

5. In the general achievement game played on a hypergraph (in which the two players select vertices, and the goal sets are the edges) we define some stronger win types for Player 1:

Definition 3.1. In the achievement game played on a hypergraph $\mathcal{H}$, Player 1 has a fair win if he builds some $e \in E(\mathcal{H})$ on a turn which comes before any turn on which Player 2 builds some $f \in E(\mathcal{H})$.

Each ‘turn’ of the game consists of a move by Player 1 followed by a move by Player 2. Definition 3.1 requires simply that Player 1 builds a goal set in fewer turns than it takes Player 2 to do the same (if Player 2 can at all).
Figure 3: The hypergraph $\mathcal{H}_T$, in the case where $T$ is the balanced binary directed tree of depth 2.

**Definition 3.2.** In the achievement game played on a hypergraph $\mathcal{H}$, Player 1 has an *early win* if he builds some $e \in E(\mathcal{H})$, say in $n$ moves, such that there is no $m \leq n$ for which Player 2 had $|e| - 1$ points of a set $e \in E(\mathcal{H})$ on his $m$th turn, and on which Player 1 had no point on his $m$th turn.

So every early win is a fair win, and every fair win is a strong win. In general, none of the win types we have defined are the same, and they all occur for Player 1 for some hypergraph: Already for $K_4$, Player 1 has a strong win but not a fair win. On the hypergraph $\mathcal{H}_T$, whose vertices are the vertices of some balanced binary directed tree $T$, and whose edges are the vertex-sets of longest directed paths in $T$ (Figure 3), Player 1 has a fair win and an early win. Finally, let the hypergraph $\mathcal{F}_n$ have vertex set $[n] \times \{0, 1\}$. Edges are of two types: Type 1 edges are the $n$-subsets $S \subset [n] \times \{0, 1\}$ for which the $\pi_1(S) = [n]$ and $(1, 0) \in S$, and Type 2 edges are all the pairs $\{(m, 0), (m, 1)\}$ where $m \in [n]$ (see Figure 4). Player 1 has a fair win in $\mathcal{F}_n$ for $n \geq 2$, but not an early win. Probably, however, the situation is not so rich in the plane:

**Conjecture 3.3.** There is no finite point set $G \subset \mathbb{R}^2$ for which Player 1 has a strategy which ensures a fair win in the $G$-achievement game played in the plane.

The conjecture may seem painfully obvious. If we play the achievement game in $\mathbb{R} \setminus \{c\}$ for any point $c \in \mathbb{R}^2$, for example, Player 2 can prevent a fair win by always choosing the point which is the central reflection across $c$ of Player 1’s last move. Annoyingly, even proving that Player 1 cannot have an early win for any $G$ when playing in $\mathbb{R}^2$ may be very difficult.

For the sake of completeness, we note the situation on the hypergraph $\mathcal{H}_T$ is in some way the worst possible for Player 2. It is easy to see that although Player 2 never occupies all but one vertex of an unblocked edge when playing on $\mathcal{H}_T$, it is easy for him to occupy all but one vertex of some edge which may be blocked. The natural strengthening of the ‘early win’ suggested here never occurs for Player 1:

**Definition 3.4.** In the achievement game played on a hypergraph $\mathcal{H}$, Player 1 has a *humiliating win* if he occupies some $e \in E(\mathcal{H})$ before Player 2 occupies all but one vertex of some edge $f \in E(\mathcal{H})$. 

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Figure 4: The hypergraph $F_3$. There are four (in general $2^{n-1}$) Type 1 edges, and three (in general $n$) Type 2 edges. (The vertex $(1,0)$ is marked with $\times$.)

(So every humiliating win is an early win.) The fact that Player 1 never has a humiliating win will follow from the strategy stealing argument; we include the proof for completeness.

**Lemma 3.5 (Strategy Stealing).** On any hypergraph $\mathcal{H}$, a second player cannot have a strategy which ensures strong win in the achievement game.

**Proof.** The proof of Lemma 3.5 is the strategy stealing argument; we include the proof for completeness. We argue by contradiction: if the second player has a strong win strategy $\sigma$ (a function from game positions to vertices), the first player makes an arbitrary first move $g$ (his ghost move). Now on each move, the first player mimics the second player’s strategy by ignoring his ghost move: formally, let $G_n$ denote the game’s position on the $n$th move, and let $G_n \setminus x$ denote the game position modified so that the vertex $x$ is unchosen. Then on each turn, the first player chooses the point $\sigma(G_n \setminus g)$ if it is not equal to $g$ (and thus must be unoccupied, since $\sigma$ is a valid strategy), or, if $\sigma(G_n \setminus g) = g$, the first player chooses an arbitrary point $x \in V(\mathcal{H})$ and sets $g := x$. The fact that $\sigma$ was a ‘strong win’ strategy for the second player implies that the first player will occupy all of an edge $e \in E(\mathcal{H})$ (even requiring $e \notin g$) before the second player occupies all some some edge $f \in E(\mathcal{H})$. In particular, the first player has a strong win, a contradiction.

**Fact 3.6.** On any hypergraph $\mathcal{H}$, Player 2 can prevent Player 1 from achieving a humiliating win.

**Proof.** Denote by $x$ the vertex Player 1 chooses on his first move. The hypergraph $\mathcal{H} \setminus x$ is the hypergraph with vertex-set $V \setminus \{x\}$ and edges $e \setminus \{x\}$ for each $e \in E(H)$. We see that Player 1 has a humiliating win on $\mathcal{H}$ only if he has a strong win on $\mathcal{H} \setminus \{x\}$ as a second player, and we are done by Lemma 3.5.

Lemma 3.5 is deceptive in its simplicity. Of course we emphasize that the strategy stealing argument shows only the existence of a strategy for a first player to prevent a second player strong win. In general, we have no better way to find such a strategy than the naïve ‘backwards labeling’ method, which runs on the whole game tree. Thus, though Fact 3.6 tells us that Player 2 should
never fall more than one behind Player 1 (in the sense of Definition 3.4), it is quite possible for this to happen in actual play between good (yet imperfect) players.

Acknowledgment

I’d like to thank József Beck for discussing with me the questions I consider here, and for helpful suggestions regarding the presentation.

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