BESICOVITCH TYPE PROPERTIES IN METRIC SPACES

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Abstract. We explore the relationship in metric spaces between different properties related to the Besicovitch covering theorem, with emphasis on geometrically doubling spaces.

1. Introduction

While the Besicovitch covering theorem has several important consequences, the spaces for which it holds have often been regarded as being rather special, cf. for example [He, pp. 7-8]. It is thus natural to explore different properties of Besicovitch type, see when they hold, study the relationship between them, and when these properties fail, try to approximate the metric by another metric with some such property, see for instance [LeRi], [LeRi2].

Recall that in the Besicovitch covering theorem one is given a set \( A \) covered by a collection \( C \) of centered balls, with uniformly bounded radii. The strong Besicovitch covering property takes the conclusion of the Besicovitch covering theorem, regarding the existence of a uniformly bounded number of disjoint subcollections whose union still covers \( A \), and makes it into a definition. The weaker notion of Besicovitch covering property, asks for the existence of a covering subcollection having uniformly bounded overlaps, so the intersections at any given point cannot exceed a certain constant. The Besicovitch intersection property is even weaker: given any collection of balls such that no ball contains the center of another, it is required that the overlap be uniformly bounded (there is no mention of any set to be covered). The precise statements for these notions appear in Definition 3.1.

We also consider two other weakenings of the strong Besicovitch covering property, first by requiring that the radii of balls in \( C \) not only be bounded away from infinity, but also form zero, and finally, by asking that all balls in \( C \) have the same radius. We call these properties the localized covering property and the equal radius covering property, cf. Definition 3.7. As far as I know, these properties are named here for the first time, but of course, within proofs they have been used before. In this article we collect some known facts regarding these properties, and prove some new results about the relationships between them.

It is shown in [LeRi2, Example 3.4] that the Besicovitch intersection property does not imply the Besicovitch covering property. We note here that the space in that example is ultrametric, cf. Theorem 3.4, so being ultrametric does not imply the Besicovitch covering property. Since all ultrametric spaces have the equal radius covering property with constant

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(Theorem 3.8), the equal radius covering property does not imply the Besicovitch covering property; the other direction also fails, cf. Theorem 3.9 so in particular, the Besicovitch covering property does not imply the strong Besicovitch covering property.

Consideration of weaker properties of Besicovitch type is interesting since more spaces satisfy them, they are easier to prove, and for certain purposes they may be sufficient. For instance, the Besicovitch intersection property implies (actually, it is equivalent to) the uniform weak type (1,1) of the centered maximal operator, cf. [Al].

After including a proof, for easy reference and later use, of the known fact that the localized Besicovitch covering property together with the Besicovitch intersection property entail the Besicovitch covering property, we center our interest in geometrically doubling metric spaces (spaces of homogeneous type in the terminology of [CoWe]) which are defined by the property that every ball of radius $r$ can be covered by at most $D$ balls of radius $r/2$.

We recall the fact that geometrically doubling metric spaces satisfy the localized covering property, and prove that in the presence of the approximate midpoint property, geometrically doubling, the localized covering property, and the equal radius covering property, are all equivalent. Geometrically doubling does not, by itself, imply the Besicovitch covering property: a well known example is given by the Heisenberg groups $\mathbb{H}^n$ with the Korányi metric: cf. [KoRe, pages 17-18] or [SaWh, Lemma 4.4]. But if $(X, d)$ is a geometrically doubling metric space with doubling constant $D$, and satisfies the Besicovitch intersection property with constant $L$, then $(X, d)$ has the Besicovitch covering property with constant $LD^2$, and if additionally $(X, d)$ satisfies the approximate midpoint property, then it has the strong Besicovitch covering property with constant $LD^5 + 1$, cf. Theorem 4.3.

2. SOME STANDARD DEFINITIONS

We will use $B^o(x, r) := \{ y \in X : d(x, y) < r \}$ to denote metrically open balls, and $B^{cl}(x, r) := \{ y \in X : d(x, y) \leq r \}$ to refer to metrically closed balls; open and closed will always be understood in the metric (not the topological) sense. If we do not want to specify whether balls are open or closed, we write $B(x, r)$. But when we utilize $B(x, r)$, all balls are taken to be of the same kind, i.e., all open or all closed. Also, whenever we speak of balls, we assume that suitable centers and radii have been chosen (recall that in general neither centers nor radii are unique).

Definition 2.1. Let $(X, d)$ be a metric space. A strict $r$-net (resp. non-strict $r$-net) in $X$ is a subset $S \subset X$ such that for any pair of distinct points $x, y \in S$, we have $d(x, y) > r$ (resp. $d(x, y) \geq r$).

We speak of an $r$-net if we do not want to specify whether it is strict or not. To ensure disjointness of the balls $B(x, r/2)$, $r$-nets are always taken to be strict when working with closed balls; otherwise, we assume $r$-nets are non-strict.

Definition 2.2. A metric space $X$ has the approximate midpoint property if for every $\varepsilon > 0$ and every pair of points $x, y \in X$, there exists a point $z$ such that $d(x, z), d(z, y) < \varepsilon + d(x, y)/2$. 

If in a metric space $X$ the standard triangle inequality is replaced by the stronger condition $d(x, y) \leq \max\{d(x, w), d(w, y)\}$, then $X$ is an ultrametric space.

Note that no ultrametric space $X$ has the approximate midpoint property: if $x, y \in X$ are distinct points, then there cannot exist a $w \in X$ such that $d(x, w), d(y, w) < d(x, y)$, since $d(x, y) = \max\{d(x, w), d(y, w)\}$.

It is obvious that if $S$ is an $r$-net in a metric space, the balls of radius $r/2$ centered at points of $S$ have disjoint interiors. Under the approximate midpoint property, the other direction also holds

**Proposition 2.3.** Let $X$ have the approximate midpoint property. Then $S \subset X$ is a non-strict $r$-net if and only if it is a collection of centers of balls of radius $r/2$ with disjoint interiors.

**Proof.** For the nontrivial direction, suppose $x, y \in S$ are distinct points with $d(x, y) < r$. By the approximate midpoint property, there is a $w \in X$ such that $d(x, w), d(y, w) < r/2$, so $w \in B^o(x, r/2) \cap B^o(y, r/2)$.

**Definition 2.4.** A metric space $X$ is a length space if for every $\varepsilon > 0$ and every pair of points $x, y$, there exists a curve with $x$ and $y$ as endpoints, such that its length is bounded above by $d(x, y) + \varepsilon$.

It is well known that for a complete metric space, having the approximate midpoint property is equivalent to being a length space.

3. Properties related to the Besicovitch covering theorem

By a disjoint collection of sets, we mean that all the sets in such a collection are disjoint. The strong Besicovitch covering property, as far as I know, appears for the first time in [Ri, Theorem 4.2 ii)]. The notion of Besicovitch covering property is taken from [LeRi], but the condition had been utilized before. The definition of Besicovitch intersection property also comes from [LeRi] (where it is called the weak Besicovitch covering property); note that unlike [LeRi], we do not require Besicovitch families to be finite.

**Definition 3.1.** A metric space $(X, d)$ has the Strong Besicovitch Covering Property if there exists a constant $L \geq 1$ such that for every $R > 0$, every $A \subset X$, and every cover $\mathcal{C}$ of $A$ given by

$$\mathcal{C} = \{B(x, r) : x \in A, 0 < r \leq R\},$$

for some $m \leq L$ there are disjoint subcollections $\mathcal{C}_1, \ldots, \mathcal{C}_m$ of $\mathcal{C}$ such that $A \subset \bigcup_{i=1}^m \cup \mathcal{C}_i$.

We say that $(X, d)$ has the Besicovitch Covering Property if there exists a constant $L \geq 1$ such that for every $R > 0$, every $A \subset X$, and every cover $\mathcal{C}$ of $A$ given by

$$\mathcal{C} = \{B(x, r) : x \in A, 0 < r \leq R\},$$

there is a subcollection $\mathcal{C}' \subset \mathcal{C}$ satisfying

$$1_A \leq \sum_{B(x, r) \in \mathcal{C}'} 1_{B(x, r)} \leq L.$$
A collection $\mathcal{C}$ of balls in a metric space $(X, d)$ is a Besicovitch family if for every pair of distinct balls $B(x, r), B(y, s) \in \mathcal{C}$, $x \notin B(y, s)$ and $y \notin B(x, r)$. The space $(X, d)$ has the Besicovitch Intersection Property with constant $L$, if there exists an integer $L \geq 1$ such that for every Besicovitch family $\mathcal{C}$, we have

$$\sum_{B(x, r) \in \mathcal{C}} 1_{B(x, r)} \leq L.$$ 

We call the lowest such $L = L(X, d)$ the Besicovitch constant of $(X, d)$.

**Remark 3.2.** We emphasize that when centers and radii are not unique, a ball is understood to come always with a previously chosen center, and a previously chosen radius (in this sense, a ball always comes with a “name”, and it might have many different names).

Note that a set of two balls $B_1$ and $B_2$ may be a Besicovitch family according to some choices of centers and radii, and fail to be one with some other choices. For a simple example, let $X = [0, 1] \cup [4, 5]$ with the distance inherited from the line, let $B_1 = [0, 1]$ and let $B_2 = (1/2, 1] \cup [4, 5] = B(9/2, 4)$. Then $B_1$ and $B_2$ form a Besicovitch family if we write $B_1 = B(1, 2)$, but not if we write $B_1 = B(0, 2)$. However, whether or not $X$ satisfies, for instance, the Besicovitch intersection property, does not depend on any particular choice of centers and radii, since the condition must hold for every possible Besicovitch family.

**Remark 3.3.** It is obvious that if $(X, d)$ has the strong Besicovitch covering property with constant $L$, then it has the Besicovitch covering property with constant $L$; and it is almost obvious that if $(X, d)$ has the Besicovitch covering property with constant $L$, then it has the Besicovitch intersection property with the same constant. The reason why the last assertion is not entirely obvious, is that radii are not assumed to be uniformly bounded in the definition of the Besicovitch intersection property. But this problem is easily avoided by passing to an intersecting finite subcollection: let $\mathcal{C}$ be an intersecting Besicovitch family of cardinality larger than $L$, and let $y \in \cap \mathcal{C}$. We may suppose, by throwing away some balls if needed, that $\mathcal{C} = \{B(x_1, r_1), \ldots, B(x_{L+1}, r_{L+1})\}$. Let $R := \max\{r_1, \ldots, r_{L+1}\}$, and let $A := \{x_1, \ldots, x_{L+1}\}$. Since all balls in $\mathcal{C}$ are needed to cover $A$, and they all intersect at $y$, we conclude that $(X, d)$ does not have the Besicovitch covering property with constant $L$.

Next we cite, with minor notational modifications, the clarifying Example 3.4 from [LeRi2], and prove that the space presented there is ultrametric (this is not shown in [LeRi2]). Theorem 3.4 contradicts [KlTo, Example 2.1], where it is asserted that every separable ultrametric space satisfies the Besicovitch covering property.

**Theorem 3.4.** There exists a separable ultrametric space without the Besicovitch covering property.

**Proof.** Let $X := \mathbb{N} \setminus \{0\}$ with the distance $d(i, i) := 0$, and for $i \neq j$, $d(i, j) := 1 - 1/\max\{i, j\}$. To see that $(X, d)$ is an ultrametric space, select three different points $i, j, k \geq 1$ and note that

$$d(i, j) := 1 - \frac{1}{\max\{i, j\}} \leq 1 - \frac{1}{\max\{i, j, k\}}$$
For the reader’s convenience we recall why \( X \) does not satisfy the Besicovitch covering property. Since the space is bounded, all sequences of balls are admissible covers of their sets of centers. As in [LeRi, Example 3.4], we consider metrically closed balls, but this is just for convenience. Cover \( X \setminus \{1\} \) with the collection \( \mathcal{C} := \{B^\text{cl}(i, 1-1/i) : i \geq 2\} \), and note that \( B^\text{cl}(i, 1-1/i) = \{1, \ldots, i\} \). It follows that any subcover \( \mathcal{C}' \) of \( X \setminus \{1\} \) must contain infinitely many balls. Since \( \{1, 2\} \subset \cap \mathcal{C}' \), the Besicovitch covering property fails. The argument can be easily adapted to open balls: just enlarge slightly the radii in the preceding example. Say, instead of \( B^\text{cl}(i, 1-1/i) \), use \( B^o(i, 1 - 1/(i + 2^{-i})) \). \hfill \Box

In an ultrametric space, every point in a ball is a center, and so, if two balls intersect, then one of them contains the other. Thus, the following is immediate:

**Proposition 3.5.** Any intersecting Besicovitch family in an ultrametric space contains exactly one ball.

**Proposition 3.6.** Let \((X, d)\) be an ultrametric space satisfying the Besicovitch covering property with constant \( L \). Then \((X, d)\) has both the Besicovitch covering property and the strong Besicovitch covering property with constant 1.

**Proof.** Let \((X, d)\) be ultrametric and satisfy the Besicovitch Covering Property with constant \( L \). Given \( R > 0 \), \( A \subset X \), and a centered cover \( \mathcal{C} = \{B(x, r) : x \in A, 0 < r \leq R\} \), by hypothesis there is a subcollection \( \mathcal{C}' \subset \mathcal{C} \) satisfying

\[
1_A \leq \sum_{B(x, r) \in \mathcal{C}'} 1_{B(x, r)} \leq L.
\]

Well order \( A \), so \( A = \{x_\alpha : \alpha < \Lambda\} \) for some cardinal \( \Lambda \). We generate a disjoint subcover \( \mathcal{C}'' \subset \mathcal{C}' \) as follows: from the finite collection of balls in \( \mathcal{C}' \) containing \( x_0 \) select one with maximal radius \( r_0 \), and disregard all the others. Let \( B_0 := B(x_0, r_0) \). Assuming \( B_0 \) has been chosen for all \( \alpha < \beta \), let \( x_{\beta_\gamma} \) be the first point in \( A \setminus \cup_{\alpha < \beta} B_\alpha \). Then set \( B_\beta := B(x_{\beta_\gamma}, r_{\beta_\gamma}) \), where \( r_{\beta_\gamma} \) is the largest radius of all balls in \( \mathcal{C}' \) containing \( x_{\beta_\gamma} \). Continue till all of \( A \) is covered, and note that all selected balls are disjoint: suppose among the chosen balls there exist \( \alpha < \beta \) such that \( B_\alpha \cap B_\beta \neq \emptyset \). Then the center of \( B_\beta \) does not belong to \( B_\alpha \) by construction, so it must be the case that the center of \( B_\alpha \) belongs to \( B_\beta \). So the radius of \( B_\beta \) is strictly larger than the radius of \( B_\alpha \), contradicting the choice of \( B_\alpha \) as a ball with largest radius containing its center. \hfill \Box

**Definition 3.7.** A metric space \((X, d)\) has the **Localized Covering Property** if for every pair \((r, R)\) with \( 0 < r \leq R < \infty \), there exists a constant \( L = L(R/r) \geq 1 \) (so \( L \) depends on the ratio \( R/r \)) such that for every \( A \subset X \), and every cover \( \mathcal{C} \) of \( A \) given by

\[
\mathcal{C} = \{B(x, s) : x \in A, r \leq s \leq R\},
\]

for some \( m \leq L \) there are disjoint subcollections \( \mathcal{C}_1, \ldots, \mathcal{C}_m \) of \( \mathcal{C} \) such that \( A \subset \bigcup_{i=1}^m \mathcal{C}_i \).
A metric space \((X, d)\) has the Equal Radius Covering Property if there exists an \(L \geq 1\) such that for every \(r\) with \(0 < r < \infty\), every \(A \subset X\), and every cover \(C\) of \(A\) given by
\[
C = \{B(x, r) : x \in A\},
\]
there are disjoint subcollections \(C_1, \ldots, C_m\) of \(C\) with \(m \leq L\), such that \(A \subset \bigcup_{i=1}^{m} C_i\).

Thus, the equal radius covering property is just the special case \(r = R\) of the localized covering property.

**Theorem 3.8.** Every ultrametric space has the equal radius covering property, with constant \(L = 1\).

**Proof.** Let \(r > 0\), let \(A \subset X\) be non-empty, and let \(C = \{B(x, r) : x \in A\}\). Well order \(A\), so \(A = \{x_{\alpha} : \alpha < \Lambda\}\). Let \(B_0 := B(x_0, r)\), and assuming \(B_\alpha\) has been chosen for all \(\alpha < \beta\), let \(x_{\beta}\) be the first point not in \(\bigcup_{\alpha < \beta} B_\alpha\). Then set \(B_\beta := B(x_{\beta}, r)\). Continue till no point of \(A\) is left to be covered, and note that all selected balls are disjoint, since no chosen ball contains a center of another, and they all have the same radius. \(\square\)

Thus, the ultrametric space from the proof of Theorem 3.4 shows that the equal radius covering property does not imply the Besicovitch covering property. Next we show that the Besicovitch covering property does not imply the equal radius covering property. It follows that the strong Besicovitch covering property entails both of the preceding properties, and is implied by neither.

**Theorem 3.9.** There exists a separable metric space which satisfies the Besicovitch covering property, but for which the equal radius covering property fails, and hence, so does the strong Besicovitch covering property.

**Proof.** We will show that the Besicovitch covering property does not imply the equal radius covering property with \(r = 1\).

We take \(X := \mathbb{N}^2\) with the metric \(d\) defined next. For all \(x \in X\), we set \(d(x, x) = 0\). Suppose that \(x \neq y\). If \(x\) and \(y\) belong to the same horizontal or to the same vertical copy of \(\mathbb{N}\), that is, if either \(x = (x_1, y_1)\) and \(y = (x_2, y_1)\), or \(x = (x_1, y_1)\) and \(y = (x_1, y_2)\), we set \(d(x, y) = 1\); in all other cases, we set \(d(x, y) = 2\). In order to check that \(d\) is a metric, the only nontrivial statement we have to verify, is the triangle inequality when the three points \(x, y, z\) are different. But then \(d(x, y) \leq 2 \leq d(x, z) + d(z, y)\).

In this example it seems more natural to consider metrically closed balls, and we will do so, though of course there is no difference between, say, a metrically closed ball of radius 1 and an open ball of radius 3/2.

Let us say that a collection of points is in general position if no two of them have the same first coordinate, or the same second coordinate; thus, points in general position are at distance two from each other.

Given three unit balls \(B^d(x, 1)\), \(B^d(y, 1)\) and \(B^d(z, 1)\) with centers in general position, it is clear that \(B^d(x, 1) \cap B^d(y, 1) \cap B^d(z, 1) = \emptyset\). So, given any collection \(\{B^d(x_i, 1) : i \in I\}\) of unit balls with centers in general position (where \(I\) is an index set) we have \(\sum_{i \in I} 1_{B^d(x_i, 1)} \leq 2\).
Now it is easy to check that the Besicovitch covering property holds: for any \( A \subset X \) and any every cover \( \mathcal{C} \) of \( A \) given by \( \mathcal{C} = \{ B^d(x, r) : x \in A, 0 < r \leq 2 \} \), we define \( \mathcal{C}' \) as follows: if \( \mathcal{C} \) contains a ball of radius 2, say, \( B^d(z, 2) \), it covers the whole space, so just set \( \mathcal{C}' := \{ B^d(z, 2) \} \). Next, we assume that all balls in \( \mathcal{C} \) have radii bounded by 1. If all balls have radii strictly smaller than 1, then they are all disjoint, and we set \( \mathcal{C}' = \mathcal{C} \). If \( \mathcal{C} \) contains at least one ball of radius exactly 1, we construct \( \mathcal{C}' \) as follows: order lexicographically the centers of the balls of radius 1, so \((x_1, y_1) < (x_2, y_2)\) precisely when \(x_1 < x_2\) or when \(x_1 = x_2\) and \(y_1 < y_2\). Let \( \{ B^d(w_i, 1) : i \in I \subset \mathbb{N} \} \) be the collection of unit balls in \( \mathcal{C} \), ordered according to their centers. Choose \( B^d(w_{i_0}, 1) := B^d(w_0, 1) \), and supposing the balls \( B^d(w_{i_j}, 1) \) have been selected for \( 1 \leq j \leq k \), let \( B^d(w_{i_{k+1}}, 1) \) be the first ball in the list centered at a point of \( A \setminus \bigcup_{j=0}^{k} B^d(w_{i_j}, 1) \). Continue until all points in \( A \) that are centers of balls of radius 1 in \( \mathcal{C} \) have been covered. If the chosen family \( \{ B^d(w_{i_j}, 1) : j \in J \subset \mathbb{N} \} \) does not cover \( A \), for each \( y \in A \setminus \bigcup \{ B^d(w_{i_j}, 1) : j \in J \} \) choose \( B^d(y, r_y) \) with \( r_y < 1 \), and let \( \mathcal{C}' \) be \( \{ B^d(w_{i_j}, 1) : j \in J \} \cup \{ B^d(y, r_y) : y \in A \setminus \bigcup \{ B^d(w_{i_j}, 1) : j \in J \} \} \). Since the selected balls of radii < 1 are all disjoint, and do not intersect the chosen balls of radius 1 either, we have that

\[
1_A \leq \sum_{B^d(x,r) \in \mathcal{C}'} 1_{B^d(x,r)} \leq 2.
\]

However, \((X, d)\) does not satisfy the equal radius covering property with \( r = 1 \), since any two unit balls with centers in general position intersect at two points. We give more details: let \( A := \{(j, j) : j \in \mathbb{N}\} \) be the main diagonal of \( X \), and let \( \mathcal{C} = \{ B^d((j, j), 1) : j \in \mathbb{N}\} \). Since each point in \( A \) belongs to only one ball, any covering subcollection must be \( \mathcal{C} \) itself. But if \( i < j \), then \( B^d((i, i), 1) \cap B^d((j, j), 1) = \{(i, j), (j, i)\} \), so any disjoint subcollection can contain at most one ball. \( \square \)

The fact that the Besicovitch intersection property together with the localized covering property entail the Besicovitch covering property is known, and follows from standard arguments, as shown in [KlTo, Lemma 2.3] (with different notation and quantitative bounds) and in [LeR2, Proposition 3.7] (without stating quantitative bounds). We include the argument for the sake of readability and to keep track of the constants.

**Theorem 3.10.** Let \((X, d)\) be a metric space satisfying the Besicovitch intersection property with constant \( L \). Suppose that \((X, d)\) has the localized covering property with constant \( C \), for every pair \((r, 2r)\) with \( 0 < r < \infty \). Then \((X, d)\) has the Besicovitch covering property with constant \( LC \).

Thus, the Besicovitch intersection property together with the localized covering property, are strictly stronger than the Besicovitch covering property, by Theorems 3.9 and 3.10.

**Proof.** Fix \( R > 0 \) and \( A \subset X \). Let \( \mathcal{C} \) be a cover of \( A \) given by \( \mathcal{C} = \{ B(x, r) : x \in A, 0 < r \leq R \} \). Call the collection \( \mathcal{D}^n \) of balls \( B(x, r) \) with \( r(x) \in \left((1/2)^{n+1} R, (1/2)^n R \right] \), the \( n^{th} \) generation balls. Let \( E_i \) be the points in \( A \) that are centers of balls in the \( i^{th} \) generation. Consider the family \( \mathcal{C}' = \mathcal{D}^1 \) of first generation balls. From the localized covering property
we obtain $m_1$ disjoint subcollections $C_1^1, \ldots, C_{m_1}^1$ of $C^1$, which cover $E_1$, and where $m_1 \leq C$. If $E_1 = A$ the process stops, and we set $C_k^k = \emptyset$ for $k > 1$. Otherwise, we let $C^2$ be the collection of balls from $D^2$ with centers in $A_2 := E_2 \setminus \bigcup_{j=1}^{m_1} C_j^1$. Again we use the localized covering property with $A_2$ and $C^2$, to obtain $m_2$ disjoint subcollections $C_1^2, \ldots, C_{m_2}^2$ of $C^2$, which cover $A_2$, with $m_2 \leq C$. We keep repeating, so that at stage $k$, the set $A_k$ is covered with disjoint collections $C_1^k, \ldots, C_{m_k}^k$ of centered balls from the $k^{th}$ generation.

Set $C' := \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{m_k} C_i^k$. Since $C'$ is a cover of $A$, all we need to do is to prove that $\sum_{B(x,r) \in C'} 1_{B(x,r)} \leq LC$. Note that any finite collection of balls in $C'$ with nonempty intersection, such that no two of them belong to the same generation, is a Besicovitch family: the balls with larger radii cannot contain the center of balls with smaller radii, for otherwise the latter would not have been chosen, and thus, the balls with smaller radii do not contain the centers of balls with larger radii either. Now select any $y \in X$. Let $C_y$ be the collection of all balls in $C'$ that contain $y$. By the Besicovitch intersection property with constant $L$, $C_y$ can contain at most $L$ balls from different generations. Since any ball in $C_y$ can intersect at most $C - 1$ balls in $C_y$ of its own generation, and at most $C$ balls in $C_y$ belonging to other generations, it follows that the cardinality of $C_y$ is at most $LC$. □

4. Geometrically doubling metric spaces and Besicovitch type properties

**Definition 4.1.** A metric space is *geometrically doubling* if there exists a positive integer $D$ such that every ball of radius $r$ can be covered with no more than $D$ balls of radius $r/2$. We call the smallest such $D$ the *doubling constant* of the space.

Next we describe the main results of this section. The following corollary summarizes qualitatively (omitting explicit constants) Theorems 4.8 and 4.10.

**Corollary 4.2.** All geometrically doubling metric spaces satisfy the localized covering property, and hence, the equal radius covering property. In the presence of the approximate midpoint property, geometrically doubling, the localized covering property, and the equal radius covering property, are all equivalent.

The first implication of Theorem 4.3 is not new, but we include it for completeness; the fact that 2) implies 3) is new.

**Theorem 4.3.** Let $(X, d)$ be a a geometrically doubling metric space with doubling constant $D$. Each of the following statements implies the next:

1) The space $(X, d)$ has the Besicovitch intersection property with constant $L$.

2) The space $(X, d)$ has the Besicovitch covering property with constant $LD^3$.

3) If in addition $(X, d)$ satisfies the approximate midpoint property, then it has the strong Besicovitch covering property with constant $LD^6 + 1$.

**Remark 4.4.** Let $X$ be geometrically doubling with constant $D$, and let $M$ be the maximum size of an $r$-net in $B(x, r)$, taken over all $x \in X$ and all $r > 0$. Then $M \leq D$, by Lemma 4.6 below.
We use $D^o$ and $D^{cl}$ to refer to the doubling constants for open and for closed balls. It is easy to see, by enlarging balls slightly, that the geometrically doubling condition is satisfied for open balls if and only if it is satisfied for closed balls, but the constants will in general be different. By analogy with previous notation, we use $M^{cl}$ and $M^o$ for the maximum sizes of strict and non-strict $r$-nets respectively, in balls of radius $r$.

**Example 4.5.** To clarify the meaning of the doubling constant $D$, note first that if $D = 1$ then $X$ has only one point, a case excluded by assumption, so $D \geq 2$.

In the special case $X = \mathbb{R}^d$ with the euclidean distance, and $\mu = \lambda^d$, the $d$-dimensional Lebesgue measure, $2^d < D < 5^d$. Thus, $\log_2 D$ provides an upper bound for the dimension of the space.

The inequalities $2^d < D < 5^d$ follow from well known volumetric arguments. First, note that by translation and dilation invariance, it is enough to consider collections of translates of $B(0, 1)$, such that the said collections cover $B(0, 2)$. Since $\lambda^d(B(0, 2)) = 2^d\lambda^d B(0, 1)$, at least $2^d$ translates of $B(0, 1)$ are needed to cover $B(0, 2)$. And since there must be some overlap among the covering balls, we get $2^d < D$.

To see why $D < 5^d$, let $\{v_1, \ldots, v_N\}$ be a maximal $1$-net in $B(0, 2) \subset \mathbb{R}^d$. The maximality of $\{v_1, \ldots, v_N\}$ entails that $B(0, 2) \subset \bigcup_1^N B(v_i, 1)$, so $D \leq N$. Since the balls $B(v_i, 1/2)$ are disjoint, contained in $B(0, 5/2)$, and do not form a packing of $B(0, 5/2)$, we conclude that $N < (5/2)^d/(1/2)^d = 5^d$.

Regarding the difference between $D^o$ and $D^{cl}$, note that on $\mathbb{R}$, $D^{cl} = 2$ while $D^o = 3$, since $(-1, 1) = \{0\} \cup (-1, 0) \cup (0, 1)$, so $(-1, 1)$ can be covered with 3 balls of radius 1/2, and no less than 3. Likewise, if we consider the $\ell^\infty$ norm on $\mathbb{R}^d$, so balls are cubes with sides parallel to the axes, it is clear that $D^{cl} = 2^d$, and not difficult to see (for instance, by an induction argument) that $D^o = 3^d$.

The next lemma is essentially Lemma 2.3 of [Hy] (slightly rewritten so that the constant $N$ has the same value throughout the lemma). The ceiling function $[t]$ is defined as the least integer $n$ satisfying $t \leq n$, and the floor function $\lfloor t \rfloor$, as the largest integer $n$ satisfying $t \geq n$.

**Lemma 4.6.** Let $(X, d)$ be a metric space and let $N$ be a fixed positive integer. The following two statements are equivalent:

1) Every ball $B(x, r) \subset X$ can be covered with at most $N$ balls of radius $r/2$.

2) For all $t \in (0, 1]$, every ball $B(x, r) \subset X$ can be covered with at most $N^{[−\log_2 t]}$ balls of radius $tr$.

Furthermore, the following statement implies the last one, and both are implied by the two above.

3) For all $t \in (0, 1]$, every ball $B(x, r) \subset X$ can contain the centers of at most $N^{[−\log_2 (t/2)]}$ disjoint balls of radius $tr/2$.

4) For all $t \in (0, 1]$, every ball $B(x, r) \subset X$ can be covered with at most $N^{[−\log_2 (t/2)]}$ balls of radius $tr$.

**Proof.** Since 1) is a special case of 2), it is enough to deduce the latter from the former to obtain their equivalence. If $t = 1$ the result is obvious. For $0 < t < 1$, let $k$ be the
positive integer that satisfies $2^{-k} < t < 2^{-k+1}$. By an inductive argument, any $B(x, r)$ can be covered with at most $N^k$ balls of radius $2^{-k}r \leq tr$. Since $k \geq -\log_2 t > k - 1$, we have that $k = \lceil -\log_2 t \rceil$.

To see that 2) implies 3), let $S := \{y_1, \ldots, y_m\} \subset B(x, r)$ be a collection of centers of disjoint balls of radius $tr/2$. Cover $B(x, r)$ with at most $N^{-\lceil -\log_2 (tr/2) \rceil}$ balls of radius $tr/2$. Now every point in $S$ is contained in some such ball, and each ball $B(x_k, tr/2)$ in the cover contains at most one point from $S$, since if, say, $y_i, y_j \in B(x_k, tr/2)$, then $x_k \in B(y_i, tr/2) \cap B(y_j, tr/2)$, contradicting disjointness. Thus, the number $m$ of points in $S$ is bounded by the number of balls in the cover.

Regarding the implication from 3) to 4), just note that given any maximal disjoint collection of balls $\{B(x_i, tr/2) : x_i \in B(x, r)\}$, the collection $\{B(x_i, tr) : x_i \in B(x, r)\}$ covers $B(x, r)$: if there is a $w \in B(x, y)$ which for every $i$ does not belong to $B(x_i, tr)$, then for every $i$ we have $B(w, tr/2) \cap B(x_i, tr/2) = \emptyset$, contradicting maximality. \hfill \square

**Example 4.7.** Observe that 4) is quantitatively weaker than 2). This is necessarily the case, since in 2) we are considering coverings of the smallest possible cardinality, while in 4) we deal with coverings that come from centers of disjoint balls with half the radius. To see the difference, let $X = (-1, 1)$ with the usual distance. Then $D = 3$, as seen in Example 4.6. Next, consider the maximal 1/2-net $\{-6/10, -1/10, 4/10, 9/10\}$. The balls of radius 1/2 centered at these points give us a cover for which the minimal cardinality is not obtained.

The next “localized” Besicovitch covering theorem has essentially appeared as part of the proof of [DiGuLa, Lemma 3.1]. We isolate this result, which in our terminology states that geometrically doubling metric spaces have the localized covering property. For completeness and to keep track of the constants, we include the proof; the exposition is inspired by [Lo, Appendix B].

**Theorem 4.8.** Let $(X, d, \mu)$ be a geometrically doubling metric measure space with doubling constant $D$, let $A \subset X$ be arbitrary, let $r > 0$, and let $t \geq 1$. For each $x \in A$, select at least one ball $B(x, s) \subset B(x, r)$ with $r \leq s \leq tr$, and denote by $\mathcal{C}$ the collection of all such balls. Then for some $m \leq \max\{D^2, \lceil \log_2 t \rceil D^3\}$, there are subcollections $\mathcal{C}_1, \ldots, \mathcal{C}_m$ of $\mathcal{C}$ such that $A \subset \cup_{i=1}^m \cup \mathcal{C}_i$ and for each $i, 1 \leq i \leq m$, the balls in $\mathcal{C}_i$ are disjoint.

**Proof.** Let $\mathcal{C}_1 \subset \mathcal{C}$ be a maximal disjoint collection of balls. Since $X$ is separable, $\mathcal{C}_1$ is either finite or countably infinite. To simplify notation, we write $\mathcal{C}_1 = \{B(x_{1,n}, s_{1,n}) : n < L_1 \leq \infty\}$. Next, repeat the argument, but with the set $B_1 := A \setminus \cup \mathcal{C}_1$ instead of $A$, to obtain a second maximal disjoint collection $\mathcal{C}_2 = \{B(x_{2,n}, s_{2,n}) : n < L_2 \leq \infty\}$, and assuming that $\mathcal{C}_1, \ldots, \mathcal{C}_k$ have been chosen, we let $\mathcal{C}_{k+1} = \{B(x_{k+1,n}, s_{k+1,n}) : n < L_{k+1} \leq \infty\}$ be a maximal disjoint subcollection of $\mathcal{C}$ with $x_{k+1,n} \in A \setminus \cup_{i=1}^k \cup \mathcal{C}_i$. We show that for some $k \geq \lceil \log_2 4t \rceil$, $A \setminus \cup_{i=1}^k \cup \mathcal{C}_i = \emptyset$. Towards a contradiction, suppose that $k \geq \lceil \log_2 4t \rceil$ and $w \in A \setminus \cup_{i=1}^k \cup \mathcal{C}_i$. Then there is an $s \in [r, tr]$ such that $B(w, s) \subset \mathcal{C}$. Since $B(w, s)$ has not previously been chosen, for each $i = 1, \ldots, k$ there exist a ball $B(y_{i,s_i}) \subset \mathcal{C}_i$ and a $w_i \in X$ with $w_i \in B(w, s) \cap B(y_{i,s_i})$. Then the set $\{w, w_1, \ldots, w_k\}$ is an $r$-net in $B(w, s+tr) \subset B(w, 2tr)$ of cardinality $k + 1 > \lceil \log_2 4t \rceil$, contradicting part 3) of Lemma 4.6.
Suppose next that \( t > 2 \); the constant \( D^{[\log_2 4t]} \) can be improved by repeated application of the preceding result, splitting the range of radii into smaller subintervals, and then adding up the constants from different scales, as in \([StSt]\) Lemma. More precisely, writing \( 2^{L-1} < t \leq 2^L, L \in \mathbb{N} \setminus \{0\} \), we see that \( L = [\log_2 t] \). Applying the bound \( m \leq D^3 \) to the collections of balls \( B(x, s) \) with \( s \in [2^{j-1}r, 2^j r] \), for \( j = 1, \ldots, L \) and adding up the numbers of disjoint collections, we get the result. Note that it is immaterial whether or not we start selecting the balls with radii in the longer subintervals.

\[ \square \]

**Remark 4.9.** Recall that any metric space \( X \) that admits a doubling measure is geometrically doubling; hence, all its balls are totally bounded by Lemma 4.6 and thus, if \( X \) is complete, then all its closed balls are compact; in particular, \( X \) is locally compact. This needs not be the case for spaces satisfying the Besicovitch intersection property, so in a certain sense, they can be less special than those with doubling measures. Given a prime \( p \), denote by \( \mathbb{Q}_p \) the field of \( p \)-adic numbers, and by \( \mathbb{C}_p \) the completion of its algebraic closure. Then \( \mathbb{C}_p \) is a complete separable ultrametric space that fails to be locally compact, cf. \([Sc]\) Corollary 17.2 (iii)]. By ultrametricity we have that the Besicovitch constant \( L(\mathbb{C}_p, | \cdot |_p) = 1 \).

**Theorem 4.10.** Suppose the metric space \( (X, d) \) has the approximate midpoint property. If there exists an \( L > 0 \) such that \( (X, d) \) satisfies the equal radius covering property with constant \( L \), then \( (X, d) \) is geometrically doubling with constant \( L^2 \).

**Proof.** Fix \( B(x, r) \subset X \), and let \( C = \{B(y, r/2) : y \in B(x, r)\} \). By hypothesis, for some \( m \leq L \) there are subcollections \( C_1, \ldots, C_m \) of \( C \) such that \( B(x, r) \subset \bigcup_{i=1}^{m} \bigcup C_i \) and for each \( i, 1 \leq i \leq m \), the balls in \( C_i \) are disjoint. By the approximate midpoint property, the collections of centers of balls in each \( C_i \) are \( r \)-nets. Given \( B(y_j, r/2) \subset C_i \), we have that \( d(y_j, x) < r \), so \( x \in B(y_j, r) \), and hence \( C'_i := \{B(y, r) : B(y, r/2) \subset C_i \} \) is a family of Besicovitch balls. Thus, its cardinality is bounded by \( L \), and hence, \( B(x, r) \) can be covered with at most \( mL \leq L^2 \) balls of radius \( r/2 \). \(

\[ \square \]

**Lemma 4.11.** Let \( X \) have the approximate midpoint property, and let \( y \in B^\alpha(x, r) \subset X \). Then for every \( t \in (0, r) \), there is a \( z \in X \) such that \( y \in B^\alpha(z, t) \subset B^\alpha(x, r) \).

**Proof.** If \( d(x, y) < t \) we just set \( z = x \) and the result is obvious, so suppose \( 0 < t \leq d(x, y) \). Denote by \((\hat{X}, \hat{d})\) the completion of \( (X, d) \); then \( \hat{X} \) is a length space, since it has the approximate midpoint property. Let \( 0 < \varepsilon < r - d(x, y) \), and let \( \Gamma : [0, 1] \to \hat{X} \) be a curve with \( \Gamma(0) = y, \Gamma(1) = x \), and length \( \ell(\Gamma) < d(x, y) + \varepsilon/2 \). Let \( t_0 \) be the first time such that \( \hat{d}(y, \Gamma(t_0)) = t \), and notice that

\[ \hat{d}(\Gamma(t_0), x) \leq \ell(\Gamma|_{[0, 1]}) \leq d(x, y) - \ell(\Gamma|_{[0, t_0]}) + \varepsilon/2 \leq d(x, y) - t + \varepsilon/2. \]

By the density of \( X \) in \( \hat{X} \), there exists a \( z \in X \) such that \( z \in B^\alpha(\Gamma(t_0), \varepsilon/2) \cap B^\alpha(y, t) \). Trivially \( y \in B^\alpha(z, t) \), so all we need to do is to check that \( B^\alpha(z, t) \subset B^\alpha(x, r) \). Let \( w \in X \) be such that \( w \in B^\alpha(z, t) \). Then

\[ d(w, x) \leq d(w, z) + d(z, x) < t + \hat{d}(z, \Gamma(t_0)) + \hat{d}(\Gamma(t_0), x) \]
< t + \varepsilon/2 + d(x, y) - t + \varepsilon/2 = d(x, y) + \varepsilon < r.

\square

Proof of Theorem 4.3. The implication 1) \Rightarrow 2) follows from Theorem 4.8 with t = 2, together with Theorem 3.10 with C = D^3.

And 2) \Rightarrow 3) follows from Lemma 4.11. Let \( C \) be a cover of \( A \) given by \( C = \{ B(x, r) : x \in A, 0 < r \leq R \} \), and let \( C' \subset C \) satisfy

\[
1_A \leq \sum_{B(x, r) \in C'} 1_{B(x, r)} \leq LD^3.
\]

We rearrange the balls in \( C' \) into \( m \leq LD^6 + 1 \) disjoint collections as follows. Using the same terminology as in the proof of Theorem 3.10 we select a maximal disjoint collection of first generation balls; to those we add a maximal disjoint collection of second generation balls subject to the condition that the second generation balls do not intersect any of the previously chosen balls. We keep doing this for every generation, always subject to not intersecting balls already chosen and belonging to previous generations. Then we let \( C_1 \) be the union of these disjoint balls coming from all generations. To define \( C_2, C_3, \ldots \) we repeat the process above with the balls left over in \( C' \), after the preceding disjoint subcollections have been defined. Suppose we already have \( C_1, \ldots, C_k \) and there is a \( B(w, s) \in C' \setminus \bigcup_{i=1}^{k} C_i \), so we need a new collection containing it. We must show that \( k \leq LD^6 \).

For \( 1 \leq j \leq k \), select \( B(x_j, r_j) \in C_j \) such that \( B(w, s) \cap B(x_j, r_j) \neq \emptyset \) and \( B(x_j, r_j) \) belongs to the same generation as \( B(w, s) \), or to an earlier one. We can always do this because otherwise \( B(w, s) \) would have been chosen. It thus follows that for \( 1 \leq j \leq k \), \( r_j > s/2 \). Using Lemma 4.11 for each \( j = 1, \ldots, k \) we select \( B(w_j, s/2) \subset B(x_j, r_j) \) such that \( B(w, s) \cap B(w_j, s/2) \neq \emptyset \). Then

\[
\sum_{j=1}^{k} 1_{B(w_j, s/2)} \leq \sum_{B(x, r) \in C} 1_{B(x, r)} \leq LD^3,
\]

and every \( w_j \in B(w, 3s/2) \). Next, let \( w'_1 := w_1 \); remove from the list \( \{w_1, \ldots, w_k\} \) all centers of balls \( B(w_j, s/2) \) containing \( w'_1 \), and note that at most \( LD^3 \) centers have been eliminated. Then let \( w'_2 \) be the first \( w_{a_2} \) belonging to the reduced list. Note that \( \{w'_1, w'_2\} \) is an \( s/2 \) net. From the reduced list, remove all centers of balls \( B(w_j, s/2) \) containing \( w'_2 \), and let \( w'_3 \) be the first center appearing in the twice reduced list. Keep repeating until the process stops, which must happen after at most \( D^3 \) steps, since this is the maximal size of an \( s/2 \) net in \( B(w, 3s/2) \), by Lemma 4.6.3 with \( t = 1/3 \), applied to the disjoint balls \( B(w'_1, s/4) \).

\square

Note that while the geometrically doubling condition and the Besicovitch covering properties considered in Theorem 4.3 are inherited by subsets, the approximate midpoint property is not. This suggests the following

**Open question:** Can the assumption of the approximate midpoint property in part 3) of Theorem 4.3 be relaxed, or even removed?

From Propositions 3.5, 3.6 and Theorem 4.3 we obtain the following

**Corollary 4.12.** Every geometrically doubling ultrametric space with doubling constant \( D \), satisfies the strong Besicovitch covering property with constant 1.
Corollary 4.13. For every prime \( p \), the field \( \mathbb{Q}_p \) of \( p \)-adic numbers satisfies the Besicovitch covering property.

Proof. Recalling that \( \mathbb{Q}_p \) is a locally compact ultrametric group, the geometrically doubling condition follows. \qed

Example 4.14. Being geometrically doubling metric and satisfying the Besicovitch covering property are not comparable conditions. As noted above, the Heisenberg groups with the Korányi metric are geometrically doubling and do not have the Besicovitch intersection property. For an example of an ultrametric space that satisfies the strong Besicovitch covering property but is not geometrically doubling, just take \( (X, d) \) to be any infinite set with the \( 0−1 \) metric: for all \( x, y \in X \), \( d(x, x) = 0 \), and if \( x \neq y \), \( d(x, y) = 1 \). Given any collection of open balls, if one of them has radius > 1, then it covers the whole space, while if all radii are \( \leq 1 \), then all balls are disjoint. The same example, but with \( X \) finite, say, of cardinality \( n \), shows that there are geometrically doubling ultrametric spaces with \( D \) arbitrarily large: clearly \( D = n \).

Remark 4.15. It is shown in [LeRi, Theorem 1.6] that Besicovitch covering properties behave poorly under bilipschitz functions. The example from Theorem [3.4] shows that a bilipschitz function (the identity) can preserve the Besicovitch intersection property while failing to preserve the Besicovitch covering property: consider the \( 0−1 \) metric \( d \) and the ultrametric \( d_u \) on \( \mathbb{N} \setminus \{0\} \), and note that \( 2^{-1}d \leq d_u \leq d \).

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