Optimized Cramer’s rule in $WZ$ factorization and applications

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Abstract. In this paper, $WZ$ factorization is optimized with a proposed Cramer’s rule and compared with classical Cramer’s rule to solve the linear systems of the factorization technique. The matrix norms and performance time of $WZ$ factorization together with $LU$ factorization are analyzed using sparse matrices on MATLAB via AMD and Intel processor to deduce that the optimized Cramer’s rule in the factorization algorithm yields accurate results than $LU$ factorization and conventional $WZ$ factorization. In all, the matrix group and Schur complement for every $Z_{\text{system}}$ (2 \times 2 block triangular matrices from $Z$-matrix) are established.

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1. Introduction

Evans and Hatzopoulos [24] first posited $WZ$ factorization or quadrant interlocking factorization of nonsingular matrix. The factorization decomposes matrices into block forms which are then regrouped and solved as sub-blocks [32]. In $WZ$ factorization of nonsingular matrix $B$, $W$-matrix (bow-tie matrix) and $Z$-matrix (hourglass matrix) - which are also known as interlocking quadrant factors of $B$ - coexist in the form [6]

\[
W = \begin{bmatrix}
1 & * & * & * \\
* & 1 & * & * \\
* & * & 1 & * \\
* & * & * & 1
\end{bmatrix}
\quad \text{and} \quad
Z = \begin{bmatrix}
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & *
\end{bmatrix}
\]

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such that

$$B = WZ.$$  \hspace{1cm} (1)

The factorization exists for every nonsingular matrix due to its uniqueness, often with pivoting [33, 37]. Pivoting improves the numerical stability of WZ factorization [34]. Even without pivoting or reordering, WZ factorization will not fail if the matrix is real symmetric, positive definite or diagonally dominant, see [21, 43]. The factorization has been applied in scientific computing - especially in science and engineering - see also [10, 19, 21, 25, 39]. Other variations of WZ factorization are detailed in [15, 20, 23, 36, 38], but block WZ factorization (or its Z_{system}) is discussed in [7, 9, 26]. The newest and alternative form of WZ factorization with applications is the WH factorization, see [4, 6]. In addition, the numerical accuracy \((\frac{-log_{10}\|B-WZ\|}{n\|B\|})\) of WZ factorization depends on the matrix size but more on the matrix norms [11]. The matrix norm of WZ factorization is the Frobenius norm [28]. The Frobenius norm of WZ factorization from Equation (1) is given as

$$\|B-WZ\|_F = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |b_{i,j} - w_{i,j}z_{i,j}| \right)^{\frac{1}{2}}.$$  \hspace{1cm} (2)

Furthermore, WZ factorization proves to be better on Intel processors than on Advanced Micro Devices (AMD) processors [11]. Even though WZ factorization and LU factorization have similar computational complexity with LU factorization - \((\frac{2}{3}n^3 + \frac{1}{2}n^2 - \frac{7}{6}n)\) and WZ factorization - \((\frac{2}{3}n^3 - \frac{7}{6}n - 3)\), the WZ factorization still shown to be better than LU factorization (except block LU factorization) irrespective of the version of MATLAB or the number of processors used [22]. However, for a uniprocessor, WZ factorization does not exhibit any advantage over LU factorization since every step performed is in serial [32]. For sparse matrices, LU and WZ factorization generate approximately similar number of nonzero elements. LU factorization relies on leading principal submatrices, whereas WZ factorization relies on nonsingular central submatrices. WZ factorization simultaneously computes two matrix elements (two columns at a time), unlike LU factorization which computes one column at a time [12]. While LU factorization performs elimination in serial with \(n-1\) steps, WZ factorization executes components in parallel with \(\frac{n}{2}\) steps if \(n\) is even or \(\frac{n-1}{2}\) steps if \(n\) is odd. LU factorization is often known to be implemented in LAPACK library to exploit the standard software library architectures [17]. WZ factorization offers parallelization in solving both sparse and dense linear system to enhance performance using OpenMP, CUDA, BLAS or EDK HW/SW codesign architecture [1, 14]. Then, Yalamov [42] presented that WZ factorization is faster on computer with a parallel architecture than any other matrix factorization methods. Therefore, WZ factorization has the adaptability to solve linear systems on Single Instruction, Multiple Data (SIMD) or Multiple Instruction, Multiple Data (MIMD) shared memory parallel computers or mesh multiprocessors, see [3, 27, 30] and the references therein. The efficiency of WZ factorization depends on an efficacious use of the memory echelon because computational cost often relies not only on the total number of arithmetic operations used but also the data transferring time between different memory levels [9].

In WZ factorization, there are \(\sum_{k=1}^{\frac{n}{2}-1} (n-2k)\) of \(2 \times 2\) linear systems to be solved which account
for the elements in $W$-matrix and $Z$-matrix, for $k = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$. The direct solver of linear systems in $WZ$ factorization algorithm solely depends on a classical method called Cramer’s rule. Cramer’s rule solves the $2 \times 2$ linear systems of $WZ$ factorization under the nonsingularity constraint presumed for their determinants [8]. Though Cramer’s rule is assumed to be less practical due to its setbacks, many modifications have been made on Cramer’s rule to solve simple and large linear systems, see [5, 29, 41]. Due to round off errors which may become significant on problems with non-integer coefficients, Moler [35] then demonstrated that Cramer’s rule is inadequate even for $2 \times 2$ linear systems. However, Dunham [18] gave a counter example of $2 \times 2$ linear system to show that Cramer’s rule is sufficient. Linear systems, especially $2 \times 2$ linear equations, solved by Cramer’s rule can be forward stable or backward stable depending on the conditioning of the system [31, 40]. Cramer’s rule and Gaussian elimination requires about the same amount of arithmetic operations for finding the solution of $2 \times 2$ linear systems, but Cramer’s rule yields a highly accuracy and stability than Gaussian elimination even with pivoting [16, 29]. For this reason, Cramer’s rule has been applied to solve the linear systems in $WZ$ factorization for over three decades. Therefore, in Section 2, we proposed a method to optimize Cramer’s rule. While in Section 3, we apply the proposed method in $WZ$ factorization on sparse matrices via MATLAB R2017b and R2019b respectively. Then, the performance time and the matrix norm of optimized Cramer’s rule and classical Cramer’s rule in $WZ$ factorization and $LU$ factorization are compared on AMD Ryzen 5 1500X and Intel Core i5-7500 processor each having four cores and 16GB RAM with standard hardware. Furthermore, we relate Schur complement and matrix group to the partition of $Z$-matrix into $2 \times 2$ block triangular matrices.

2. Solving simple linear systems with optimized Cramer’s rule

A linear system is defined by

$$Bx = c,$$

(3)

where

$$\det(B) \neq 0, \ x = [x_1, x_2, \ldots, x_n]^T, \ c = [c_1, c_2, \ldots, c_n]^T, \ B \in \mathbb{R}^{n \times n}, \ x, c \in \mathbb{R}^n.$$

**Theorem 1.** [31][Cramer’s rule] Let $Bx = c$ be an $n \times n$ system of linear equation and $B$ an $n \times n$ nonsingular matrix, then the unique solution $x = [x_1, x_2, \ldots, x_n]^T$ to the linear system is given by

$$x_i = \frac{\det(B_{i|c})}{\det(B)}$$

(4)

where $B_{i|c}$ is the matrix obtained from coefficient matrix $B$ by substituting the column vector $c$ to the $i$th column of $B$, for $i = 1, 2, \ldots, n$.

Let $c_1$ be the row sum of matrix $B$. If the $i$th column of matrix $B$ is replaced with $c_1$ to obtain a new matrix $B_{i|c_1}$ with all other columns in $B$ and $B_{i|c_1}$ remain the same, for $i = 1, 2, \ldots, n$. Then,

$$\det(B) = \det(B_{i|c_1}).$$

(5)
It is a well-established theorem that if the $i$th column of matrix $B$ is the difference of the $i$th column of matrix $D_i$ and the $i$th column of matrix $E_i$, and all other columns in $D$ and $E$ are equal to the corresponding columns in $B$, for $i = 1, 2, ..., n$ [2]. Then

$$\det(B) = \det(D) - \det(E). \quad (6)$$

**Corollary 1.** Let $Bx = c$ be an $n \times n$ system of linear equation and $B$ an $n \times n$ nonsingular matrix of $x$, then the $i$th entry $x_i$ of the unique solution $x = [x_1, x_2, ..., x_n]^T$ to the linear system is given by

$$x_i = -\frac{\det(B_i \mid (c + c_1))}{\det(B)}, \quad (7)$$

where $B_i \mid (c + c_1)$ is the matrix obtained by subtracting the sum of column vector $c$ and $c_1$ the row sum of matrix $B$ for $i = 1, 2, ..., n$.

**Proof.** Let $c_2 = c + c_1$, where $c$ is the column vector and $c_1$ the row sum of matrix $B$. If $c_2$ is subtracted from the $i$th column of matrix $B$, then we can re-write Equation (6) as

$$\det(B_i \mid c_2) = \det(B) - \det(B_i \mid c). \quad (8)$$

But

$$\det(B_i \mid c_2) = \det(B_i \mid (c + c_1)) = \det(B_i \mid c) + \det(B_i \mid c_1). \quad (9)$$

Substitute Equation (5) in Equation (9) to get

$$\det(B_i \mid c_2) = \det(B_i \mid c) + \det(B). \quad (10)$$

Therefore,

$$\det(B_i \mid c_2) = \det(B) - (\det(B_i \mid c) + \det(B)).$$

Now,

$$x_i = -\frac{\det(B_i \mid c_2)}{\det(B)} = -\frac{\det(B_i \mid (c + c_1))}{\det(B)}. \quad (11)$$

The flowchart in Figure 1, the step by step in Algorithm 1, and the MATLAB code of the algorithm in Listing 1 show the computational steps of Corollary 1.
Let $B$ be coefficient matrix, $c$ the column vector and $x_i$ the set of linear solutions

Start

Is $B$ a square matrix?

Yes

Optimized Cramer’s rule cannot be used

Let $c_1 = \text{row sum of } B$

Compute $c_2 = c + c_1$

Let $D_i$ be the $i$th column of $B$, for $i = 1, 2, ..., n$

Compute $E_i = D_i - c_2$

Compute $x_i = \frac{\det(E_i)}{\det(B)}$

Display $x_i$

Stop

No

Figure 1: Flowchart of an optimized Cramer’s rule
Algorithm 1 An optimized Cramer’s rule

1: procedure
2: \( B \leftarrow n \times n \) coefficient matrix
3: \( c \leftarrow \) column vector
4: \( x_i \leftarrow \) solutions of linear system
5: for \( i \) do
6: \( c_1 \leftarrow \) row sum of \( B \)
7: \( c_2 \leftarrow c + c_1 \)
8: \( D_i \leftarrow \) \( i \)th row of \( B \)
9: \( E_i \leftarrow D_i - c_2 \)
10: \( \det(E) \leftarrow \) determinant of \( E_i \)
11: \( \det(B) \leftarrow \) determinant of \( B \)
12: \( x_i \leftarrow \frac{\det(E)}{\det(B)} \)
13: end for
14: end procedure

Listing 1: MATLAB code of optimized Cramer’s rule.

```matlab
function x=Optimized Cramer’s rule(B,c)
B=input(‘matrix B =’);
c=input(‘column vector =’);
n=size(B,1);
m=size(B,2);
if n˜=m
 Error(‘The matrix is not square.’);
x=[];
else
 detB=det(B);
 if det(B)˜=0
 x=zeros(n,1);
c1=sum(B,2);
c2=c+c1;
 for j=1:n
 if j˜=1 && j˜=n
 E=[B(:,1:j-1) B(:,j)−c2 B(:,j+1:n)];
 elseif j==1
 E=[B(:,1)−c2 B(:,2:n)];
 elseif j==n
 E=[B(:,1:n−1) B(:,n)−c2];
 end
 detE=det(E);
 x(j)=−(det(E)/detB);
 end
 else
 Error(‘Matrix B is singular.’);
x=[];
 end
end
```

**Proposition 1.** Let \( Bx = c \) be an \( n \times n \) system of linear equation where \( B \) is an \( n \times n \) non-singular matrix of \( x \) for the distinct solution of \( x = [x_1, x_2, \ldots, x_n]^T \) and \( c \) the column vector. If \( x_i = \frac{\det(B_{ij})}{\det(B)} \)
and $x_i = -\frac{\det(B_{i,(c_c^1)})}{\det(B)}$, then

$$-\frac{\det(B_{i,c+c_1})}{\det(B)} = \frac{\det(B_{i,c})}{\det(B)},$$

where $B_{i,c}$ is the matrix obtained from matrix $B$ by substituting the column vector $c$ to the $i$th column of $B$ and $B_{i,(c+c_1)}$ is the matrix obtained by subtracting the sum of column vector $c$ and $c_1$ the row sum of coefficient matrix from the $i$th column of $B$, for $i = 1, 2, \ldots, n$.

**Proof.** We begin by substituting Equation (8) to the numerator of Equation (11) to obtain

$$x_i = -\frac{\det(B) - \det(B_{i,c_2})}{\det(B)}$$

$$= -\frac{\det(B) - \det(B_{i,c_1+c})}{\det(B)}$$

$$= -\frac{\det(B) - (\det(B_{i,c}) + \det(B_{i,c}))}{\det(B)}$$

Recall that $\det(B_{i,c_1}) = \det(B)$. Thus,

$$x_i = \frac{\det(B_{i,c})}{\det(B)}.$$

Corollary 1 as well as Theorem 1 indicates if a system is inconsistent or indeterminate without completely solving the systems, unlike other direct solvers. Notwithstanding, the optimized Cramer’s rule use a few more arithmetic operations than classical Cramer’s rule for higher linear systems. However, based on our background analysis, the optimized Cramer’s rule, especially for examples of $2 \times 2$ well and ill-conditioned linear systems lower than the relative residual measurements of Cramer’s rule. This distinct advantage makes optimized Cramer’s rule suitable for solving the $2 \times 2$ linear systems of $WZ$ factorization.

### 3. Application of optimized Cramer’s rule in $WZ$ factorization

For the $WZ$ factorization algorithm, we obtain the the $i$th to the $(n - 1)$th element of the $(i - 1)$th and $(n - i + 1)$th column of $W$-matrix by computing $w_{i,k}^{(k)}$ and $w_{i,n-k+1}^{(k)}$ from

$$\begin{cases}
\varepsilon_{i,k}^{(k-1)} w_{i,k}^{(k)} + z_{n-k+1,k}^{(k-1)} w_{i,n-k+1}^{(k)} = -\varepsilon_{i,k}^{(k-1)} \\
\varepsilon_{i,n-k+1}^{(k-1)} w_{i,k}^{(k)} + z_{n-k, n-k+1}^{(k-1)} w_{i,n-k+1}^{(k)} = -\varepsilon_{i,n-k+1}^{(k-1)}
\end{cases}$$

which update the elements of $Z$-matrix from

$$z_{i,j}^{(k)} = z_{i,j}^{(k-1)} + w_{i,k}^{(k)} z_{k,j}^{(k-1)} + w_{i,n-k+1}^{(k)} z_{n-k+1,j}^{(k-1)}$$
and we then proceed similarly for the central submatrices of size \((n - 2k)\) and so on, where \(k = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\), \(i, j = k + 1, \ldots, n - k\) and \(z_i^{(k)} \in \mathbb{R}\), see [9]. We can now re-write Equation (12) in matrix form as

\[
\begin{bmatrix}
    z_{k,k}^{(k-1)} & z_{n-k+1,k}^{(k-1)} & \cdots & z_{n-k+1,n-k+1}^{(k-1)} \\
    z_{k,n-k+1}^{(k-1)} & z_{n-k+1,n-k+1}^{(k-1)} & \cdots & z_{n-k+1,n-k+1}^{(k-1)} \\
    \vdots & \vdots & \ddots & \vdots \\
    z_{k,n-k+1}^{(k-1)} & z_{n-k+1,k}^{(k-1)} & \cdots & z_{n-k+1,n-k+1}^{(k-1)} \\
\end{bmatrix}
\begin{bmatrix}
    w_{1,k}^{(k)} \\
    w_{2,k}^{(k)} \\
    \vdots \\
    w_{n-k+1,n-k+1}^{(k)} \\
\end{bmatrix}
\begin{bmatrix}
    c_{1,k} \\
    c_{2,k} \\
    \vdots \\
    c_{n-k+1,n-k+1} \\
\end{bmatrix}
\]

(14)

If we apply Theorem 1 to derive W-matrix by computing \(w_{i,k}^{(k)}\) and \(w_{i,n-k+1}^{(k)}\) (from \(Bw = c\)) with respect to first and second column of \(B\) from Equation (14), we will obtain

\[
w_{i,k}^{(k)} = \frac{\det(B_{1\mid i})}{\det(B)} \quad \text{and} \quad w_{i,n-k+1}^{(k)} = \frac{\det(B_{2\mid i})}{\det(B)}.
\]

(15)

The factorization obtained using Cramer’s rule when we grouped and ordered the scalar operations into matrix-vector operation is the vectorized \(WZ\) factorization (VWZ factorization), see [13] for its MATLAB code.

Furthermore, if Corollary 1 is applied to compute \(w_{i,k}^{(k)}\) and \(w_{i,n-k+1}^{(k)}\) in Equation (14). Then,

\[
\det(B) = \frac{\det(B_{1\mid i})}{\det(B)} - \frac{\det(B_{2\mid i})}{\det(B)}
\]

(16)

The \(W^\alpha Z^\alpha\) factorization is the factorization obtained from using Corollary 1, where the W-matrix obtained is referred to as \(W^\alpha\)-matrix and its Z-matrix as \(Z^\alpha\)-matrix. The complete MATLAB code of \(W^\alpha Z^\alpha\) factorization is given in Listing 2.

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**Listing 2: MATLAB code of \(W^\alpha Z^\alpha\) factorization.**

1. \texttt{function optimizedWZfactorization(B,W,Z)\n
2. \texttt{steps of elimination \textendash{} from B to Z\n
3. \texttt{end}**
B = input ('matrix B = ');

n = size(B, 1);
W = zeros(n);

for k = 1:ceil((n-1)/2)
    k2 = n - k + 1;
    determinant = B(k,k) * B(k2,k2) - B(k2,k) * B(k,k2);
    if determinant == 0
        exitflag = 0;
        for i1 = k:k2
            for i2 = i1:k2
                determinant = B(i1,k) * B(i2,k2) - B(i2,k) * B(i1,k2);
                if determinant #= 0
                    disp('input matrix cannot be factorized to Z-matrix')
                    tmp = B(i1,k:k2);
                    B(i1,k:k2) = B(k,k:k2);
                    tmp = B(i2,k:k2);
                    B(i2,k:k2) = B(k2,k:k2);
                    exitflag = 1;
                    break
                end
            end
        end
        if exitflag == 0
            Z = B;
            return
        end
    end
end

% finding elements of W

% To compute ith to the (n-1)th element of (i-1)th column of W
W(k+1:k2-1,k) = -(B(k2,k) * B(k+1:k2-1,k2) + B(k2,k2) * B(k+1:k2-1,k)) / determinant;

% To compute ith to the (n-1)th element of (n-i+1)th column of W
W(k+1:k2-1,k2) = -(B(k,k) * B(k+1:k2-1,k) + B(k,k2) * B(k+1:k2-1,k2)) / determinant;

for m=1:n
    W(m,m) = 1;
end

% updating B
B(k+1:k2-1,k) = 0;
B(k+1:k2-1,k2) = 0;
B(k+1:k2-1,k1:k2-1) = B(k+1:k2-1,k1:k2-1) + W(k+1:k2-1,k) * B(k,k1:k2-1)
    + W(k+1:k2-1,k2) * B(k2,k1:k2-1);

Z = B;
end

Besides, if there is no regrouping or ordering of scalar operations into matrix-vector operation then the factorization is a sequential WZ factorization. For the MATLAB code of WZ factorization, we replace line 32 to line 44 in Listing 2 with line 1 to line 9 of Listing 3.

Listing 3: MATLAB code of sequential WZ factorization.

% finding elements of W
% To compute ith to the (n-1)th element of (i-1)th column of W
for i=k+1:k2-1
    W(i,k) = (B(k2,k) * B(i,k) - B(k2,k) * B(i,k2)) / determinant;
% To compute ith to the (n-1)th element of (n-i+1)th column of W
W(i,k2) = (B(k,k) * B(i,k2) - B(k,k2) * B(i,k)) / determinant;
% updating B
\[ \text{for } j = k + 1 : k^2 - 1 \]
\[ B(i, j) = B(i, j) + W(i, k) \times B(k, j) + W(i, k^2) \times B(k^2, j); \]

For the computation and analysis, the square sparse matrices used to investigate \( LU \), \( WZ \), \( VWZ \) and \( W^oZ^o \) factorization, in Table 1, 2 and 3, are obtained from \textit{The SuiteSparse Matrix Collection}. Table 1 gives the basic information about the sparse matrices, Table 2 and Table 3 illustrate the performance time and matrix norm of \( LU \), \( WZ \), \( VWZ \) and \( W^oZ^o \) factorization on Intel and AMD processor via MATLAB R2017b and R2019b respectively.

| Matrix name  | Matrix size | Nonzero entries | Group           | Year   | kind                          |
|--------------|-------------|-----------------|-----------------|--------|-------------------------------|
| Trefethen200 | 500         | 3,996           | JGD_Trefethen   | 2008   | Combinatorial problem         |
| tub1000      | 1000        | 97,645          | Bai             | 1994   | Computational fluid dynamic problem |
| consol       | 1500        | 7,996           | Langemyr        | 2002   | Structural problem            |
| olm2000      | 2000        | 12,349          | Bai             | 1994   | Computational fluid dynamic problem |
| cryg2500     | 2500        | 174,296         | Bai             | 1996   | Materials problem             |
| nasa2910     | 2910        | 66,528          | Nasa            | 1995   | Structural problem            |
| thermal      | 3456        | 28,505          | Brunetiere      | 2000   | Thermal problem               |
| ACTIVSg2000  | 4000        | 219,024         | TAMU_SmartGridCenter | 2018 | Power network problem         |
| bcsstk28     | 4410        | 29,600          | HB              | 1984   | Structural problem            |
| rdb5000      | 5000        | 262,943         | Bai             | 1994   | Computational fluid dynamic problem |
| s3rmq4mr1    | 5489        | 54,471          | Cylshell        | 1997   | Structural problem            |
| C – 32       | 5975        | 51,480          | Schenk_JBMNA    | 2006   | Optimization problem          |
| n36e – b7    | 6435        | 340,200         | JGD_Homology    | 2008   | Combinatorial problem         |
| Kuu          | 7102        | 834,226         | MathWorks       | 2006   | Structural problem            |
| fp           | 7548        | 834,226         | MKS             | 2006   | Electromagnetics problem      |
| bcsstk38     | 8032        | 355,460         | Boeing          | 1995   | Structural problem            |
| Kaufhold     | 8765        | 42,471          | MathWorks       | 2006   | Counter example problem       |
| nd3k         | 9000        | 3,279,690       | ND              | 2003   | 2D, 3D problem                |
| nemeth19     | 9506        | 818,302         | Nemeth          | 1999   | Quantum chemistry problem     |
| cryg10000    | 10000       | 818,302         | Bai             | 1996   | Materials problem             |
| bundle1      | 10581       | 818,302         | Lourakis        | 2006   | Computer graphics problem     |
| wing nodal   | 10937       | 15,0796         | DIMACS10        | 2000   | Undirected graph              |
Table 2: Performance time of LU, WZ, VWZ and W"Z" factorization on Intel and on AMD processor.

| Matrix Name   | MATLAB R2019b Intel | AMD | MATLAB R2019b AMD | Intel | AMD |
|---------------|---------------------|-----|-------------------|-------|-----|
| testfellow200 | 4.55                | 4.55| 9.28              | 9.28  | 9.28|
| convec| 6.34                | 6.34| 10.68             | 10.68 | 10.68|
| ams2000| 5.62                | 5.62| 9.48              | 9.48  | 9.48|
| cscr2000| 5.32                | 5.32| 9.28              | 9.28  | 9.28|
| matrix2910| 4.55                | 4.55| 8.28              | 8.28  | 8.28|
| realcd| 5.62                | 5.62| 10.68             | 10.68 | 10.68|
| ACTF5/3000| 6.34                | 6.34| 10.68             | 10.68 | 10.68|
| level3| 5.62                | 5.62| 9.48              | 9.48  | 9.48|
| Frobenius| 4.55                | 4.55| 9.28              | 9.28  | 9.28|
| level2| 5.62                | 5.62| 9.48              | 9.48  | 9.48|
| level1| 4.55                | 4.55| 9.28              | 9.28  | 9.28|
| level0| 5.62                | 5.62| 9.48              | 9.48  | 9.48|
| level-1| 4.55                | 4.55| 9.28              | 9.28  | 9.28|
| level-2| 5.62                | 5.62| 9.48              | 9.48  | 9.48|
| level-3| 4.55                | 4.55| 9.28              | 9.28  | 9.28|
| level-4| 5.62                | 5.62| 9.48              | 9.48  | 9.48|
| level-5| 4.55                | 4.55| 9.28              | 9.28  | 9.28|
| level-6| 5.62                | 5.62| 9.48              | 9.48  | 9.48|
| level-7| 4.55                | 4.55| 9.28              | 9.28  | 9.28|
| level-8| 5.62                | 5.62| 9.48              | 9.48  | 9.48|
| level-9| 4.55                | 4.55| 9.28              | 9.28  | 9.28|
| level-10| 5.62                | 5.62| 9.48              | 9.48  | 9.48|
| level-11| 4.55                | 4.55| 9.28              | 9.28  | 9.28|
| level-12| 5.62                | 5.62| 9.48              | 9.48  | 9.48|
| level-13| 4.55                | 4.55| 9.28              | 9.28  | 9.28|
| level-14| 5.62                | 5.62| 9.48              | 9.48  | 9.48|
| level-15| 4.55                | 4.55| 9.28              | 9.28  | 9.28|
| level-16| 5.62                | 5.62| 9.48              | 9.48  | 9.48|
| level-17| 4.55                | 4.55| 9.28              | 9.28  | 9.28|
| level-18| 5.62                | 5.62| 9.48              | 9.48  | 9.48|
| level-19| 4.55                | 4.55| 9.28              | 9.28  | 9.28|
| level-20| 5.62                | 5.62| 9.48              | 9.48  | 9.48|

Table 3: Matrix norms of LU, WZ, VWZ and W"Z" factorization on MATLAB R2019b.

| Matrix Name   | Performing norm | MATLAB R2019b Intel | AMD | MATLAB R2019b AMD | Intel | AMD |
|---------------|----------------|---------------------|-----|-------------------|-------|-----|
| testfellow200 | Frobenius norm | 5.82                | 5.82| 9.28              | 9.28  | 9.28|
| convec| 6.34                | 6.34| 10.68             | 10.68 | 10.68|
| ams2000| 5.62                | 5.62| 9.48              | 9.48  | 9.48|
| cscr2000| 5.32                | 5.32| 9.28              | 9.28  | 9.28|
| matrix2910| 4.55                | 4.55| 9.28              | 9.28  | 9.28|
| realcd| 5.62                | 5.62| 9.48              | 9.48  | 9.48|
| ACTF5/3000| 6.34                | 6.34| 10.68             | 10.68 | 10.68|
| level3| 5.62                | 5.62| 9.48              | 9.48  | 9.48|
| Frobenius| 4.55                | 4.55| 9.28              | 9.28  | 9.28|
| level2| 5.62                | 5.62| 9.48              | 9.48  | 9.48|
| level1| 4.55                | 4.55| 9.28              | 9.28  | 9.28|
| level0| 5.62                | 5.62| 9.48              | 9.48  | 9.48|
| level-1| 4.55                | 4.55| 9.28              | 9.28  | 9.28|
| level-2| 5.62                | 5.62| 9.48              | 9.48  | 9.48|
| level-3| 4.55                | 4.55| 9.28              | 9.28  | 9.28|
| level-4| 5.62                | 5.62| 9.48              | 9.48  | 9.48|
| level-5| 4.55                | 4.55| 9.28              | 9.28  | 9.28|
| level-6| 5.62                | 5.62| 9.48              | 9.48  | 9.48|
| level-7| 4.55                | 4.55| 9.28              | 9.28  | 9.28|
| level-8| 5.62                | 5.62| 9.48              | 9.48  | 9.48|
| level-9| 4.55                | 4.55| 9.28              | 9.28  | 9.28|
| level-10| 5.62                | 5.62| 9.48              | 9.48  | 9.48|
| level-11| 4.55                | 4.55| 9.28              | 9.28  | 9.28|
| level-12| 5.62                | 5.62| 9.48              | 9.48  | 9.48|
| level-13| 4.55                | 4.55| 9.28              | 9.28  | 9.28|
| level-14| 5.62                | 5.62| 9.48              | 9.48  | 9.48|
| level-15| 4.55                | 4.55| 9.28              | 9.28  | 9.28|
| level-16| 5.62                | 5.62| 9.48              | 9.48  | 9.48|
| level-17| 4.55                | 4.55| 9.28              | 9.28  | 9.28|
| level-18| 5.62                | 5.62| 9.48              | 9.48  | 9.48|
| level-19| 4.55                | 4.55| 9.28              | 9.28  | 9.28|
| level-20| 5.62                | 5.62| 9.48              | 9.48  | 9.48|
MATLAB R2017b on AMD processor.

MATLAB R2019b on AMD processor.

MATLAB R2017b on Intel processor.

MATLAB R2019b on Intel processor.

Figure 2: Performance time of $LU$, $WZ$, $VWZ$ and $WoZo$ factorization on AMD and Intel processor via MATLAB R2017b and R2019b respectively.
LU on AMD and Intel processor.

WZ on AMD and Intel processor.

VWZ on AMD and Intel processor.

WoZ on AMD and Intel processor.

Figure 3: Combined performance time of LU, WZ, VWZ and WoZ factorization on AMD and Intel processor via MATLAB R2017b and R2019b.
Figure 4: Matrix norms of $LU$, $WZ$, $VWZ$ and $W^{\infty}Z^{\infty}$ factorization on AMD and Intel processor via MATLAB R2017b and R2019b respectively.
In Figure 2, the sequential $WZ$ factorization, on average for both MATLAB R2017b and 2019b, is about 22% faster than $LU$ factorization on Intel processor and about 17% times on AMD processor. The most preferred factorization algorithm according to the performance time is $VWZ$ factorization while $LU$ factorization is the worst. However, $W^0Z^0$ factorization in general is about 28% faster than $WZ$ factorization and 41% than $LU$ factorization. The performance time of $W^0Z^0$ factorization approaches $WZ$ factorization as the version of MATLAB improves. The performance time of all the factorization algorithms increase exponentially with increase in matrix size. The version of MATLAB has minimal influence on the algorithms but the performance time significantly depends on the size of the matrix and architecture of the algorithm. Nevertheless, the higher the version of MATLAB the better the result on performance time. $Kuu$ and $n3c6−b7$ have the highest matrix dimension difference of 667. Even though $Kaufhold$ and $nd3k$ have the least matrix dimension difference of 235, $Kaufhold$ has 1.3% nonzero elements of $nd3k$. The surge in performance time of $nd3k$ is due to the number of nonzero elements in the matrix for the factorization to utilize. $nd3k$ has more than 4% of nonzero elements while other sparse matrices in Table 1 have less than 2% nonzero elements.

Now, Figure 3 shows that the improved version of MATLAB contributes to better performance time of each algorithm. The algorithms on MATLAB R2017b spend more time in execution than on MATLAB R2019b irrespective of the type of processor used. The figure also shows that the time to execute the algorithms via MATLAB R2017b and MATLAB R2019b on AMD processor is longer than on Intel processor.

Figure 4 displays the matrix norms for AMD and Intel on MATLAB R2017b and R2019b respectively. Our background analysis shows that the matrix norms of $LU$, $WZ$, $VWZ$ and $W^0Z^0$ factorization are influenced by the architecture of the algorithm used. Due to minimal round-off error, the matrix norms of $W^0Z^0$ factorization are better than $LU$, $WZ$ and $VWZ$ factorization. The $LU$ factorization has the worst algorithm for matrix norm. The matrix norms of all the factorization algorithms increase as the size of their matrices increase. Furthermore, the accuracy of our algorithms based on the relative residual depends more on the Frobenius norm than the matrix size. In Table 3, $comsol$, $thermal$, $n3c6−b7$, $nemeth19$ and $wing$ have their Frobenius norms below 500 and their numerical accuracy below 25. $Kaufhold$ with 0.06% nonzero entries has the highest Frobenius norm among the given matrices.

**Proposition 2.** Schur complement exists for every $Z$-system.

**Proof.** For the existence of $Z$-matrix, the necessary and sufficient condition for $WZ$ factorization is that matrix $B$ must be centro-nonsingular (see [37]). First, let $Z$-matrix of even order being
factorized from nonsingular matrix $B$ be

$$
Z = \begin{bmatrix}
\alpha_{k,k} & \cdots & \alpha_{k,2} & \cdots & \beta_{k,2} & \cdots & \beta_{k,n} \\
\cdot & Z_{1,1} & \cdots & \cdots & Z_{1,2} & \cdot \\
\cdot & \cdots & \cdots & \cdots & \cdots & \cdots \\
\alpha_{k,k} & \cdots & \beta_{k,l} \\
\cdot & \cdots & \cdots & \cdots & \cdots & \cdots \\
\gamma_{k,k} & \cdots & \delta_{k,l} \\
\cdot & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdot & Z_{1,1} & \cdots & \cdots & Z_{1,2} & \cdot \\
\cdot & \cdots & \cdots & \cdots & \cdots & \cdots \\
\gamma_{n,k} & \cdots & \gamma_{n,2} & \cdots & \delta_{n,2} & \cdots & \delta_{n,n} \\
\end{bmatrix}
$$

where $k = 1, 2, \ldots, \frac{n}{2}; l = n - k + 1$. Then, the determinant of $Z$-matrix is

$$
det(Z) = det \begin{bmatrix}
\alpha_{k,k} & \cdots & \beta_{k,l} \\
\cdot & \cdots & \cdots \\
\gamma_{k,k} & \cdots & \delta_{k,l} \\
\cdot & \cdots & \cdots \\
\cdot & Z_{1,1} & \cdots & \cdots \\
\cdot & \cdots & \cdots & \cdots \\
\gamma_{n,k} & \cdots & \gamma_{n,2} & \cdots & \delta_{n,2} & \cdots & \delta_{n,n} \\
\end{bmatrix}_{1 \leq k \leq \frac{n}{2}; l = n - k + 1}
$$

$$
= \prod_{k=1}^{\frac{n}{2}} (\delta_{k,l} \alpha_{k,l} - \gamma_{k,l} \beta_{k,l})_{l = n - k + 1} \neq 0. \tag{18}
$$

Next, partition Equation (17) into $Z_{system}$ of $2 \times 2$ triangular block matrices $\left(\left[Z_{i,j}\right]_{i,j=1}^{2}\right)$ with each block containing $\frac{n}{2} \times \frac{n}{2}$ matrix to have

$$
Z_{system} = \begin{bmatrix}
Z_{1,1} & Z_{1,2} \\
Z_{2,1} & Z_{2,2} \\
\end{bmatrix}
$$

If each $2 \times 2$ triangular block matrix is singular (i.e $Z_{1,1}Z_{2,2} = Z_{1,2}Z_{2,1}$), then $Z_{system}$ is not invertible which contradicts Equation (18). Hence, there exists at least two nonsingular triangular block matrices in $Z_{system}$. If $Z_{1,1}$ is invertible as well as $Z_{2,2}$, then the Schur complement of the block $Z_{1,1}$ in $Z_{system}$ is given as

$$
Z_{2,2} - Z_{2,1}Z_{1,1}^{-1}Z_{1,2}. \tag{19}
$$

The determinant of Equation (19) is nonsingular because $Z_{2,2} - Z_{2,1}Z_{1,1}^{-1}Z_{1,2}$ is a lower triangular invertible matrix (see [9]) and

$$
\frac{det(Z_{2,2} - Z_{2,1}Z_{1,1}^{-1}Z_{1,2})}{det(Z_{1,1})} \neq 0.
$$

This implies

$$
det(Z_{system}) = det(Z_{1,1})det(Z_{2,2} - Z_{2,1}Z_{1,1}^{-1}Z_{1,2}).
$$
Hence, the Schur complement of $Z_{\text{system}}$ depends on the existence of nonsingular $Z$-matrix.

**Corollary 2.** $Z_{\text{system}}$ is a matrix group of degree 2 over $\mathbb{R}$.  

**Proof.** Let $GL(n, \mathbb{R})$ be the matrix group of order $n$ over $\mathbb{R}$ satisfying matrix multiplication and $M_n(\mathbb{R})$ the size of the matrix. Let the matrix group of $Z_{\text{system}}$ be $GL_Z(2, \mathbb{R})$ of degree 2 over $\mathbb{R}$ defined as

$$GL_Z(2, \mathbb{R}) = \left\{ Z_0 = \begin{bmatrix} Z_{1,1} & Z_{1,2} \\ Z_{2,1} & Z_{2,2} \end{bmatrix} : \det(Z_0) = Z_{1,1}Z_{2,2} - Z_{1,2}Z_{2,1} \neq 0 \right\}$$

Since $GL_Z(2, \mathbb{R})$ is an invertible matrix based on Proposition 2, then there exists an inverse such that its identity is $I_2$ (that is $I_2Z_0I_2 = Z_0$). To see that $GL_Z(2, \mathbb{R})$ is closed under matrix multiplication, we let $Z_{s(k)}, Z_{s(m)}, Z_{s(n)} \in M_2(\mathbb{R}) = Z_0$ such that $Z_{s(k)} = \{k_{i,j}\}$, $Z_{s(m)} = \{m_{i,j}\}$ and $Z_{s(n)} = \{n_{i,j}\}$. Then the associativity holds as

$$(Z_{s(k)} \ast Z_{s(m)}) \ast Z_{s(n)} = ((k_{i,j}) \ast (m_{i,j})) \ast (n_{i,j})$$

$$= \left( \sum_{r=1}^{2} k_{i,r} m_{r,j} \right) \ast (n_{i,j})$$

$$= \left( \sum_{x=1}^{2} \left( \sum_{r=1}^{2} k_{i,r} m_{r,x} \right) \ast (n_{x,j}) \right)$$

$$= \left( \sum_{x=1}^{2} k_{i,x} \ast \left( \sum_{r=1}^{2} m_{s,r} \ast n_{r,j} \right) \right)$$

$$= (k_{i,j}) \ast ((m_{i,j}) \ast (n_{i,j}))$$

$$= Z_{s(k)} \ast (Z_{s(m)} \ast Z_{s(n)}).$$

**Corollary 3.** If $GL_Z(2, \mathbb{R})$ is the matrix group of $Z_{\text{system}}$ with degree 2 over $\mathbb{R}$, then $GL_Z(n, \mathbb{R})$ is the matrix group of $Z$-matrix with degree $n$ over $\mathbb{R}$.

**Proof.** Let $GL_Z(n, \mathbb{R})$ be a matrix group of $Z$-matrix of order $n$ over $\mathbb{R}$ and $GL_Z(2, \mathbb{R})$ be the matrix group of $Z_{\text{system}}$ of order 2 over $\mathbb{R}$. From Proposition 2, $Z_{\text{system}}$ is the $2 \times 2$ triangular block matrices partitioned from $Z$-matrix of order $n$. Since $Z_{\text{system}}$ is a matrix group, based on Corollary 2, in which $Z_{\text{system}}$ is a subset $Z$-matrix. Conspicuously $Z$-matrix has axioms of a matrix group which is invertible and closed under matrix multiplication with property of associativity.

**4. Conclusions**

The advantage of optimized Cramer’s rule over classical Cramer’s rule to solve $2 \times 2$ linear systems in $WZ$ factorization is to obtain good floating points and to minimize round-off error without loss of generality in the coefficient matrix of linear systems. Although, the optimized
Cramer’s rule has high performance time than VWZ factorization and low performance time than WZ factorization, the method produces better matrix norms than all other factorization algorithms, irrespective of the processors used. We passionately advocate that W"Z" factorization should be compared with LU, WZ and VWZ factorization on shared memory parallel computers or mesh multiprocessors.

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