Local behaviour of sequences of three-dimensional generalised monopoles

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Abstract. The purpose of this paper is to study the behaviour of sequences of generalised monopoles with a uniform bound on a certain $L^2$-norm. We focus on the case that the target hyperKähler manifolds are Swann bundles. In 3-dimensional case, suppose that there exists an open submanifold $Y'$ such that the hyperKähler potential along the monopoles has a uniform lower bound over $Y'$. Then we show that there exist convergent subsequences of generalised monopoles over any compact subset of $Y'$. Under similar assumptions, the same conclusion holds for the generalised harmonic spinors in dimension four.

1. Introduction and main results

Given a Riemannian manifold $(Z, g_Z)$ of dimension three or four, fix a Spin$^c$ structure $Q \to Z$. The Seiberg–Witten equations are non-linear PDE associated with this Spin$^c$ structure. A solution to the equations consists of a Spin$^c$ connection and a section of an associated bundle of $Q$. The latter is called a spinor. Note that the typical fiber of the associated bundle is the quaternionic space $\mathbb{H}$.

Taubes [15] and Pidstrygach [12] generalise these equations to more general settings. These generalisations are called generalised Seiberg–Witten equations. We refer to their solutions as (generalised) monopoles. The idea behind the generalisation is shown in the following two aspects. Firstly, the Spin$^c$ group is replaced by a more general Lie group called the Spin$^G$ group. Here $G$ can be any fixed compact Lie group. Secondly, the typical fiber $\mathbb{H}$ of the associated bundle is replaced by a hyperKähler manifold $(M, g_M, I_1, I_2, I_3)$ with certain symmetries. Such a manifold is called a target manifold. In particular, the spinor is a section of a fiber bundle and the fiber is not necessarily a vector space. Many well-known equations in gauge theory are just special cases of the generalised Seiberg–Witten equations (see Section 5.2 of [5]).

Compactness property is one of the central problems when we study the moduli space of non-linear equations. In the case that the target manifold is an $\mathbb{H}$-module, many compactness results have been achieved by many authors, e.g. [7,17,22]. Their methods are based on Taubes’s techniques in [16]. For the general cases, the generalised Seiberg–Witten equations are highly non-linear. Other aspects of the
generalised Seiberg–Witten equations have been studied by many mathematicians as well, eg. [3, 6, 18], etc.

The purpose of this paper is to study the behaviour of sequences of generalised monopoles with a uniform bound on a certain $L^2$-norm. Such a $L^2$-norm is an analogue of the usual $L^2$-norm of spinors. We focus on the cases that the target manifold belongs to a class of hyperKähler manifolds with hyperKähler potentials called Swann bundles [13]. Roughly speaking, a Swann bundle $\mathcal{U}(N)$ is a fiber bundle over a quaternionic Kähler manifold $N$ and the fiber is of the form $\mathbb{H}^K/\mathbb{Z}_2$. We assume that $N$ is compact; unless otherwise stated. Alternatively, a Swann bundle can be viewed as a metric cone of a 3-Sasaki manifold. Given a sequence of 3-dimensional generalised monopoles, suppose that the images of the spinor part of the monopoles stay away from the singularity of the metric cone, then we show that there exists a convergent subsequence. Besides, the methods can be applied to 4-dimensional cases equally well whenever the Lie group $G$ is zero-dimensional. In these cases, the generalised Seiberg–Witten equations are reduced to the generalised Dirac equation whose solutions are called harmonic spinors.

The main results of this paper are summarised as follows:

**Theorem 1.** Let $(Y, g_Y)$ be a Riemannian 3-manifold with a $\text{Spin}^G$ structure $\pi: Q \to Y$. Suppose that the target manifold $(M, g_M, I_1, I_2, I_3)$ is a Swann bundle $\mathcal{U}(N)$ over a compact quaternionic Kähler manifold $N$. Let $\rho_0$ be the unique hyperKähler potential defined in Sect. 2.2.2.

Let $\{(A_n, u_n)\}_{n=1}^{\infty}$ be a sequence of generalised monopoles (see Eq. (2.10)) defined by $\pi: Q \to Y$ and the Swann bundle. Suppose that $\{(A_n, u_n)\}_{n=1}^{\infty}$ has a uniform bound on the $L^2$-norm, i.e., $\int_Y \rho_0 \circ u_n \leq c_\circ$. Assume that $Y' \subset Y$ is an open submanifold of $Y$ such that $\lim_{n \to \infty} \inf_{Y'} \rho_0 \circ u_n \geq c_\circ^{-1}$ for some large constant $c_\circ > 0$. Then after gauge transformations, there exists a subsequence of $\{(A_n, u_n)\}_{n=1}^{\infty}$ that converges in $C^\infty_\text{loc}$ to a monopole $(A, u)$ over $Y'$.

**Theorem 2.** Let $(X, g_X)$ be a Riemannian 4-manifold with a $\text{Spin}^G$ structure $\pi: Q \to X$. Suppose that the Lie group $G$ is zero-dimensional. Let $(M, g_M, I_1, I_2, I_3)$ and $\rho_0$ be the Swann bundle and the hyperKähler potential respectively as in Theorem 1.

Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of harmonic spinors (see Definition 6) defined by $\pi: Q \to X$ and the Swann bundle. Suppose that $\int_X \rho_0 \circ u_n \leq c_\circ$ and $X' \subset X$ is an open submanifold of $X$ such that $\lim_{n \to \infty} \inf_{X'} \rho_0 \circ u_n \geq c_\circ^{-1}$ for some constant $c_\circ > 0$. Then there exists a subsequence of $\{u_n\}_{n=1}^{\infty}$ that converges in $C^\infty_\text{loc}$ to a harmonic spinor $u$ over $X'$.

**Remark 1.** Under assumptions $\int \rho_0 \circ u_n \leq c_\circ$ and $\lim_{n \to \infty} \inf_{Y'} \rho_0 \circ u_n \geq c_\circ^{-1}$, we can deduce a uniform bound on the $L^\infty$-norm of the curvatures and the $W^{1,2}$-norm of the spinors. One may apply the Uhlenbeck compactness to get a convergent subsequence of the connections. However, the standard elliptic bootstrapping argument cannot extract a convergent subsequence of the harmonic spinors, because the generalised Dirac operator is highly non-linear. The constants that appear in the elliptic estimates are not uniform (see pages 55–56 of [14]).
Our methods here are inspired by [21] and we also make use of the monotonicity property of the frequency function as in [7]. A brief summary of the proof is given in Sect. 2.8.

Remark 2. For an arbitrary sequence of monopoles \((A_n, u_n)_{n=1}^\infty\) there may be no \(Y' (X')\) as in Theorem 1 (2). For example, if \(\lim_{n \to \infty} \int \rho_0 \circ u_n = 0\), then we cannot find such submanifolds \(Y'\) and \(X'\).

The existence of the submanifolds \(Y'\) and \(X'\) is highly non-trivial to verify. Here is a possible way to detect the existence of \(Y'\) and \(X'\). One can define closed sets \(S_n, k = \{ x \in Y, |\rho_0 \circ u_n(x) \leq \frac{1}{k}\}\). Then for any fixed \(k\), \(\{S_n, k\}_{n=1}^\infty\) converges to a closed set \(S_k\) in Hausdorff distance. Define a set \(S := \cap_{k=1}^\infty S_k\). If \(S \neq Y\), then there exists \(k_0\) such that \(S_{k_0} \neq Y\). As a result, the closed set \(S_{k_0}^\epsilon = \{ x \in Y | dist(x, S_{k_0}) \leq \epsilon \}\) is not equal to \(Y\) for sufficiently small \(\epsilon > 0\). By the definition of the Hausdorff distance, any open submanifold \(Y' \subset Y - S_{k_0}\) lies inside \(Y - S_n, k_0\) for sufficiently large \(n\). In particular, \(\rho_0 \circ u_n(y) \geq k_0^{-1}\) for any \(y \in Y'\).

Another possible way is to follow the idea in [7,16,17]. One can define an \(L^\infty\)-function \(\rho\) by the rule that \(\rho(x) = \lim_{n \to \infty} \rho_0 \circ u_n(x)\). Similar to Proposition 2.1 of [16], \(\{\rho_0 \circ u_n\}_{n=1}^\infty\) converges weakly to \(\rho\) in the \(W^{1,2}\)-topology and strongly in the \(L^p\)-topology (1 \(\leq p < \infty\)). The proof of Proposition 2.1 of [16] only relies on a priori estimates and the Weitzenböck formula. The relevant estimates and the Weitzenböck formula are true in our setting (see Sect. 3). Thus the argument can be applied to our case equally well and we do not repeat them here. Suppose that \(c_\bullet^{-1} \leq \int_Y \rho_0 \circ u_n \leq c_\bullet\) for some constant \(c_\bullet > 0\). If one can show that \(\rho\) is continuous, then the \(L^p\)-convergence implies that \(Y - \rho^{-1}(0)\) is a non-empty open set.

Unfortunately, at present, we can’t find a sufficient condition to guarantee that \(S \neq Y\) or \(\rho\) is continuous. It is interesting to study the properties of \(\rho\) and \(S\) in the future.

2. Preliminaries

In this section, we review some essential definitions based on the papers [4,5,14].

2.1. Spin\(^G\) group and Spin\(^G\) structure

Let \(G\) be a compact Lie group and \(\varepsilon\) be a central element of \(G\) satisfying \(\varepsilon^2 = 1\). The Spin\(^G\) group is defined by

\[
\text{Spin}^G(\varepsilon) := \frac{\text{Spin}(n) \times G}{\langle (-1, \varepsilon) \rangle}.
\]

Here are some examples: \(\text{Spin}^{U(1)}(n) = \text{Spin}^c(n), \text{Spin}^{Z_2}(n) = \text{Spin}(n), \text{Spin}^1(n) = SO(n)\) and \(\text{Spin}^G(1) = SO(n) \times G\). Similar to the Spin\(^c\) case, we have the following exact sequence:

\[
1 \rightarrow \langle (1, \varepsilon) \rangle \rightarrow \text{Spin}^G(\varepsilon) \xrightarrow{\phi_0} SO(n) \times \bar{G} \rightarrow 1,
\]

(2.1)
where $\tilde{G} = \frac{G}{[1,\varepsilon]}$.

Let $Sp(1)$ denote the unit sphere in $\mathbb{H}$. We have isomorphisms $Spin(3) \cong Sp(1)$ and $Spin(4) \cong Sp(1) \times Sp(1)$. To distinguish the copies of $Sp(1)$ in $Spin(4)$, we denote the first one by $Sp(1)_+$ and the second one by $Sp(1)_-$. We focus on the cases that $n = 3$ or $4$ and denote the $Spin^G_\varepsilon(n)$ group by $H$ throughout.

**Definition 3.** Let $PSO(n)$ be the frame bundle of a Riemannian manifold $(Z, g_Z)$. Let $P_G \to Z$ be a principal $G$-bundle. A $Spin^G(n)$ structure is a principal $Spin^G(n)$-bundle $Q \to Z$ together with a covering $\phi : Q \to PSO(n) \times Z P_G$ such that $\phi(pg) = \phi(p)\phi_0(g)$ for any $p \in Q$ and $g \in Spin^G(n)$, where $\phi_0 : Spin^G(n) \to SO(n) \times G$ is the covering in (2.1).

2.2. Target hyperKähler manifold

A hyperKähler manifold is a Riemannian manifold $(M, g_M)$ endowed with a triple of complex structures $\{I_1\}_{i=1}^3$ which are covariantly constant with respect to the Levi–Civita connection $\nabla$ and satisfy the quaternionic relations. Note that the tangent bundle $TM$ is a bundle of $\mathbb{H}$-module because we have the following ring homomorphism

$$I : \mathbb{H} \to End TM$$

where $\zeta = h_0 + h_1 i + h_2 j + h_3 k \to I_\zeta = h_0 I_1 T_M + h_1 I_2 + h_2 I_3 + h_3 I_3$. (2.2)

Observe that $I_\zeta$ is still a complex structure whenever $|\zeta| = 1$ and $\zeta \in \text{Im} \mathbb{H}$. Under the isomorphism $\mathfrak{sp}(1) \cong \text{Im} \mathbb{H}$, the hyperKähler form $\omega \in \Omega^2(M, \mathfrak{sp}(1)^\ast)$ is defined by $\omega_\zeta = g_M(\cdot, I_\zeta \cdot)$ for any $\zeta \in \mathfrak{sp}(1) \cong \text{Im} \mathbb{H}$.

2.2.1. Actions on hyperKähler manifolds

**Definition 4.** Let $(M, g_M, I_1, I_2, I_3)$ be a hyperKähler manifold with an isometric $Sp(1)$-action. The $Sp(1)$-action is called permuting if

$$dq I_\zeta dq^{-1} = I_{q \zeta q},$$

for any $q \in Sp(1)$, where $dq : T_x M \to T_{qx} M$ is the differential of this action and $\zeta \in \text{Im} \mathbb{H}$ with $|\zeta| = 1$.

Assume that $M$ admits a hyperKähler $G$-action, i.e., it preserves the metric and complex structures. The hyperKähler $G$-action is called hyperHamiltonian if it is Hamiltonian with respect to each $\omega_\zeta$. In other words, there is a $G$-equivariant map $\mu : M \to \mathfrak{sp}(1)^\ast \otimes g^\ast$ such that

$$\iota^G_{K_{M,G} \omega_\zeta} = \langle d\mu, \zeta \otimes \bar{\xi} \rangle,$$  

(2.3)

where $\zeta \otimes \bar{\xi} \in \mathfrak{sp}(1) \otimes g$ and $K_{M,G}^{M,G} |_m := \frac{d}{dt} \exp(t\bar{\xi}) \cdot m |_{t=0}$ are the fundamental vector fields. Such a map $\mu$ is called a hyperKähler moment map.

Assume that $(M, g_M, I_1, I_2, I_3)$ admits a permuting $Sp(1)$-action and a hyperKähler $G$-action. Suppose that the $Sp(1)$-action and the hyperKähler $G$-action satisfy the following assumptions:
1. The $G$-action commutes with the $Sp(1)$-action.
2. Let $\varepsilon$ be a central element of $G$ satisfying $\varepsilon^2 = 1$. We require that the element $(-1, \varepsilon) \in Sp(1) \times G$ acts trivially on $M$.

Under these assumptions, we have an $Sp(1) \times G$-action on $M$ and it descends to a $Spin_\varepsilon^G(3) = \frac{Sp(1) \times G}{\langle -1, \varepsilon \rangle}$-action. This action is called a permuting $Spin_\varepsilon^G(3)$-action. A $Spin_\varepsilon^G(4) = \frac{Sp(1)_+ \times Sp(1)_- \times G}{\langle 1, \varepsilon \rangle}$-action is called permuting if the $Sp(1)_+$-action is permuting while the $Sp(1)_- \times G$-action is hyperKähler.

### 2.2.2. HyperKähler potential

Let $(M, g_M, I_1, I_2, I_3)$ be a hyperKähler manifold. A function $\rho \in C^\infty(M, \mathbb{R})$ is called a hyperKähler potential if it satisfies $dI_\xi d\rho = 2\omega_\xi$ for any $\xi \in \mathfrak{sp}(1)$ with $|\xi| = 1$. Here $I_\xi$ acts on $d\rho$ by $I_\xi d\rho(\cdot) := d\rho(I_\xi \cdot)$.

Suppose that $(M, g_M, I_1, I_2, I_3)$ admits a permuting $Sp(1)$-action. By Pidstrygach [12] (also see Proposition 2.2.7 of [5]), the hyperKähler form $\omega$ is exact. Define a map $\chi : \mathfrak{sp}(1) \otimes \mathfrak{sp}(1) \to \Gamma(M, TM)$ by

$$\xi \otimes \xi' \mapsto -I_\xi R_{\xi Sp(1)},$$

where $R_{\xi Sp(1)}$ is the fundamental vector field of the $Sp(1)$-action. Decompose $\chi$ into $\chi_0 + \chi_1 + \chi_2$, where $\chi_0 = -\frac{1}{3} \sum_{i=1}^3 I_i K_{\xi Sp(1)}^i M$ is its diagonal, $\chi_1$ is its antisymmetric part and $\chi_2$ is its trace-free symmetric part. The following lemma gives a sufficient condition of the existence of a hyperKähler potential.

**Lemma 5.** (Proposition 5.5 of [13], Lemma 3.2.3 of [14]) Let $M$ be a hyperKähler manifold with a permuting $Sp(1)$-action. Suppose that $\chi_2 = 0$. Then there is a unique hyperKähler potential $\rho_0$ such that $\rho_0 = \frac{1}{2} g_M(\chi_0, \chi_0)$.

Conversely, if $M$ admits a hyperKähler potential, then $M$ admits a local permuting $Sp(1)$-action with $\chi_2 = 0$.

The choice of a hyperKähler potential is not unique. We fix the choice provided by the above lemma throughout.

### 2.3. Swann bundle

The hyperKähler manifolds admitting a permuting $Sp(1)$-action and a hyperKähler potential are the Swann bundles. This class of hyperKähler manifolds is constructed by Swann [13]. We briefly review the construction of Swann bundles and their properties.

Let $(N, g_N)$ be a quaternionic Kähler manifold of dimension $4n$, i.e., its holonomy is contained in $Sp(n)\times Sp(1) := (Sp(n) \times Sp(1))/\langle \pm 1 \rangle$. Let $F$ denote the $Sp(n)\times Sp(1)$-reduction of the $SO(4n)$ frame bundle of $N$. Then $\mathcal{C}(N) := F/Sp(n)$ is a principal $SO(3)$-bundle over $N$. The Swann bundle is defined by

$$\mathcal{U}(N) := \mathcal{C}(N) \times_{SO(3)} (\mathbb{H}^\times / \mathbb{Z}_2),$$

(2.4)
where $\mathbb{H}^\times := \mathbb{H} - \{0\}$. Suppose that the scalar curvature of $(N, g_N)$ is positive, then the Swann bundle is a hyperKähler manifold with a permuting $SO(3)$ or $Sp(1)$-action and vanishing $\chi_2$. The metric on $\mathcal{U}(N)$ is of the form

$$g_{\mathcal{U}(N)} = dr^2 + r^2 g_{\mathcal{C}(N)},$$

where $r$ is the radius coordinate of $\mathbb{H}^\times$.

What follows are two facts related to the desired properties in Sects. 2.2.1 and 2.2.2:

F.1 The Swann bundle $\mathcal{U}(N)$ is a hyperKähler manifold with $\chi_2 = 0$. In this case, the hyperKähler potential is $\rho_0 = \frac{1}{2} r^2$.

F.2 If a Lie group $G$ acts on $N$, preserving the quaternionic Kähler structure, then the action can be lifted to a hyperHamiltonian action of $G$ on $\mathcal{U}(N)$. Also, the action leaves $\mathcal{C}(N)$ invariant (see Proposition 4.2 and Theorem 4.6 of [13]).

For more details about the construction of $\mathcal{U}(N)$ and its properties, we refer the reader to Swann’s original paper [13].

We remark that a class of compact quaternionic Kähler manifolds with positive scalar curvature is the so-called Wolf spaces. The Wolf spaces are the only compact, homogeneous, quaternionic Kähler manifolds. They are classified by Wolf [19] and Alekseevskii [1,2]. Some examples of Wolf spaces are as follows:

$\mathcal{H}P^n = \frac{Sp(n+1)}{Sp(n) \times Sp(1)}$, $Gr_2(\mathbb{C}^n) = \frac{SU(n)}{S(U(n-2) \times U(2))}$,

$\tilde{Gr}_2(\mathbb{R}^n) = \frac{SO(n)}{S(SO(n-4) \times SO(4))}$, $G_2 = \frac{G_2}{SO(4)}$.

**Examples** Even we mainly use the properties F.1, F.2 of Swann bundles in this paper, it is still worth reviewing a class of examples $\mathcal{O}$ of Swann bundles. The Swann bundle $\mathcal{O}$ is obtained by P. B. Kronheimer as a moduli space of Nahm equations [10]. The hyperKähler structure on $\mathcal{O}$ is written down by P. Kobak and Swann explicitly in [11]. The description that follows of Swann bundles paraphrases what is presented in [11,13].

Let $G$ be a compact, simply connected, simple Lie group. Let $G^\mathbb{C}$ be its complexification. Let $\mathfrak{g}$ and $\mathfrak{g}^\mathbb{C}$ denote the Lie algebra of $G$ and $G^\mathbb{C}$ respectively. Fix a real structure $\sigma$ on $\mathfrak{g}^\mathbb{C}$ such that $\mathfrak{g}$ is the eigenspace to the eigenvalue 1. Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}^\mathbb{C}$. For example, we could take $G = SU(2)$, $G^\mathbb{C} = SL(2, \mathbb{C})$, and $\sigma$ to be the minus conjugate transpose of matrices. The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ has a Cartan basis

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Fix a system of roots $\Delta$ with positive roots $\Delta_+$. Let $\alpha \in \Delta_+$ be a highest root. Then $\alpha$ induces a Lie algebra embedding $\mathfrak{sl}(2, \mathbb{C}) \hookrightarrow \mathfrak{g}^\mathbb{C}$. The image of $\{E, H, F\}$ is denoted by $\{E_\alpha, H_\alpha, F_\alpha\}$. We choose the embedding such that it is compatible with $\sigma$, in the sense that $\sigma(E_\alpha) = -F_\alpha$ and $\sigma(H_\alpha) = -H_\alpha$. Let $\mathcal{O}$ be the orbit of $E_\alpha$ under the adjoint action of $G^\mathbb{C}$. The hyperKähler structure and the $Sp(1)$-action are given as follows:
The complex structure $I$ on $G^C$ descends to the orbit $O$. Fix a negative Killing form $<,>$ on $g^C$. Define a function $\eta : O \rightarrow \mathbb{R}$ by
$$\eta|_X := |X|^2 = <X, \sigma X>.$$ Note that $\eta$ is $G$-invariant. Let $\rho_0(X) := |E_\alpha|\sqrt{\eta}$. Then we can define a Kähler metric on $O$ by
$$\omega_I := \frac{1}{2}d(\rho'_0Id\eta)g(\xi_A, \xi_B)|_X = 2\text{Re}(\rho'_0 <\xi_A, \sigma \xi_B> + \rho''_0 <\xi_A, \sigma X> <\sigma \xi_B, X>).$$
where $A, B \in g^C, \xi_A = [A, X], \xi_B = [B, X]$ are the fundamental vector fields at $X$, and $\rho'_0 = \frac{d}{d\eta}\rho_0, \rho''_0 = \frac{d^2}{d\eta^2}\rho_0$.

On the orbit $O$, the complex symplectic form $\omega_c$ of Kirillov, Kostant and Souriau is given by
$$\omega_c(\xi_A, \xi_B)|_X := <X, [A, B]>.$$ Write $\omega_c = \omega_J + i\omega_K$. The 2-forms $\omega_J, \omega_K$ are expected to serve as the other two Kähler forms. Therefore, we get
$$J\xi_A|_X = -2\rho'_0[X, \sigma \xi_A] - 2\rho''_0 <\sigma \xi_A, X][X, \sigma X]$$ from the relation $g(\xi_A, \xi_B) = \omega_J(J\xi_A, \xi_B)$. We can deduce a formula for $K$ similarly. They satisfy the quaternionic relation $IJ = K$. Also, the computation in Theorem 5.2 of [11] shows that $J, K$ are almost complex structures. By Lemma 2.2 of [8] (on page 64), $(O, g, I, J, K)$ is a hyperKähler manifold. Proposition 5.6 in [13] implies that $\rho_0$ is a hyperKähler potential.

Define an action of $\mathbb{H}^\times$ on $O$ that is generated by vector fields $\nabla \rho_0, J\nabla \rho_0, K\nabla \rho_0$ and $K\nabla \rho_0$. The argument in Proposition 5.5 of [13] shows that this $Sp(1) \subset \mathbb{H}^\times$-action is permuting. After mod out the $\mathbb{H}^\times$-action, the quotient is the Wolf space.

2.4. Clifford multiplication

Recall that we set $H = Spin^G_{\epsilon}(n)$. Let $C^\infty(X, Y)^H$ denote the space of $H$-equivariant maps from $X$ to $Y$. The Clifford multiplication in dimension three is a $Spin^G_{\epsilon}(3)$-equivariant homomorphism
$$c_3 : C^\infty(Q, (\mathbb{R}^3)^* \otimes TM)^H \rightarrow C^\infty(Q, TM)^H.$$ Under the isomorphism $(\mathbb{R}^3)^* \cong Im\mathbb{H}$, the Clifford multiplication is defined by $c_3(h \otimes v) = I_h v$.

In 4-dimensional case, the scalar multiplication
$$Im\mathbb{H} \rightarrow \text{End}(TM), \ h \rightarrow I_h$$ extends to an $H$-equivariant map $Cl_3 \rightarrow \text{End}(TM)$, i.e., $TM$ is a $Cl_3$-module. Identify $Cl^0_4$ with $Cl_3$. Define
$$E := Cl_4 \otimes_{Cl^0_4} (TM, I_1).$$
The splitting $Cl_4 = Cl_4^0 \oplus Cl_4^1$ induces a decomposition $E = E^+ \oplus E^-$, where $E^+ = Cl_4^0 \otimes Cl_4^0 (TM, I_1)$ and $E^- = Cl_4^1 \otimes Cl_4^0 (TM, I_1)$. They are respectively analogues of the $Spin^c$ bundles $S_+$ and $S_-$. Each of them is a copy of $TM$ while admitting different $Spin^c$ $(4)$-actions. The Clifford multiplication is a $Spin^c$ $(4)$-equivariant map $c_4 : C^\infty(Q, (\mathbb{R}^4)^*)^H \to \text{End}(E^+ \oplus E^-)$ defined by

$$
c_4(e_0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad c_4(e_l) = \begin{pmatrix} 0 & -l_l \\ -l_l & 0 \end{pmatrix},
$$

(2.5)

where $\{e_0, \{e_l\}_{l=1}^3\}$ is the standard basis of $(\mathbb{R}^4)^*$.

### 2.5. Generalised Dirac operator

Let $\mathcal{N} = C^\infty(Q, M)^H$ denote the space of $H$-equivariant maps from $Q$ to $M$. Note that this is nothing but just the space of sections of the associated bundle $\mathcal{M} := Q \times_H M$. An element of $\mathcal{N}$ is referred to as a spinor.

Note that the Lie algebra of the $Spin^G$ group can be decomposed into $\mathfrak{lie} H = \mathfrak{so}(n) \oplus \mathfrak{g}$. Let $\mathcal{A}$ denote the space of connections on $\pi : Q \to Z$ whose $\mathfrak{so}(n)$-component is induced by the Levi–Civita connection, i.e.,

$$\mathcal{A} = \{ A \in \mathcal{A}(Q) | pr_{\mathfrak{so}(n)} \circ A = \pi^*_G A_Z \},$$

where $A_Z$ denotes the Levi–Civita connection of $P_{\mathfrak{so}(n)} \to Z$. Then $\mathcal{A}$ is an affine space over $\Omega^1(Q, \mathfrak{g})^H_{\text{hor}} = \Omega^1(Z, \mathfrak{g}_Q)$, where $\mathfrak{g}_Q$ is the associated bundle $Q \times_{Ad \mathfrak{g}}$.

Fix a connection $A \in \mathcal{A}$. For any $u \in \mathcal{N}$, define a covariant derivative $d_A : \mathcal{N} \to C^\infty(Q, (\mathbb{R}^n)^* \otimes u^*TM)^H$ by

$$d_A u := du + K_{A,M,H}^M |_u,$$

(2.6)

where $K_{A,M,H}^M |_u$ is defined by

$$K_{A,M,H}^M |_u(v) := K_{A,v}^M |_u$$

for any vector field $v \in \Gamma(TM)$.

We also define another covariant derivative on $C^\infty(Q, TM)^H$. Before that, let us recall the definition of the connector. Let $\pi : E \to B$ be a vector bundle. For any $(v, w) \in E \times_B E$, its vertical lift is defined by

$$vl_E(v, w) := \frac{d}{dt}(v + tw)|_{t=0} \in TE^{vert},$$

where $TE^{vert} := \ker \pi_*$ is the vertical bundle. The vertical lift $vl_E$ gives an isomorphism $E \times_B E \cong TE^{vert}$. Given any connection $\Phi : TE \to TE$, the corresponding connector $K_\Phi$ is defined by

$$K_\Phi := pr_2 \circ vl_E^{-1} \circ \Phi,$$

where $pr_2 : E \times_B E \to E$ is the projection onto the second factor. The covariant derivative $\nabla^\Phi$ of $\Phi$ can be expressed as $\nabla^\Phi s = K_\Phi \circ ds$ for any section $s \in \Gamma(E)$. For more details, please see [9].
Take $E = TM$ and $\Phi$ to be the Levi–Civita connection of $(M, g_M)$. The corresponding connector is denoted by $K$. We define another covariant derivative by the following formula:

$$\nabla_{TM}^A : C^\infty(Q, TM)^H \rightarrow C^\infty(Q, (\mathbb{R}^n)^* \otimes TM)^H$$

$$\nabla_{TM}^A V := K \circ d_{TM}^A V,$$  \hspace{1cm} (2.7)

where $d_{TM}^A V := (dV + K_{TM}^{A,H})|_V$ is defined in the same way as in (2.6). Alternatively, $d_{TM}^A V = \frac{d}{dt} V(\sigma(t))|_{t=0}$, where $\sigma$ is a horizontal path in $Q$ such that $\sigma'(0)$ is a horizontal lift of $w \in TZ$ with respect to $A$. When we restrict our attention to a spinor $u$, then $\nabla_{TM}^A$ descends to a covariant derivative (still denoted by $\nabla_{TM}^A$) on the vector bundle $\pi u^*TM := u^*TM/H$ in the usual sense (see Remark 4.3.3 of [5].)

Similar to the usual case, the generalised Dirac operator is defined to be the composition of the Clifford multiplication $c_n$ and the covariant derivative, i.e.,

$$D_A u := c_n \circ d_A u.$$  \hspace{1cm} (2.8)

**Definition 6.** A spinor $u \in \mathcal{N}$ such that $D_A u = 0$ is called a harmonic spinor.

Since the Dirac operator $D_A$ is highly non-linear, it is useful to consider its linearization $D_{lin, u}^A : C^\infty(Q, u^*TM)^H \rightarrow C^\infty(Q, u^*TM)^H$. By Lemma 3.6.8 of [4], we know that the linearization of $D_A$ at $u$ is

$$D_{lin, u}^A v = \sum_i c_n(dx^i)\nabla_{A,i}^T M v.$$  \hspace{1cm} (2.9)

### 2.6. Generalised Seiberg–Witten equations

Let $(M, g_M, I_1, I_2, I_3)$ be a hyperKähler manifold with a permuting $Spin^G$-action. Let $\mu : M \rightarrow \mathfrak{sp}(1)^* \otimes g^*$ be the moment map.

Let $(Y, g_Y)$ be a closed Riemannian 3-manifold and $Q \rightarrow Y$ be a $Spin^G$ structure. Let $\{e_i\}_{i=1}^3$ and $\{\zeta_i\}_{i=1}^3$ be respectively orthonormal bases of $(\mathbb{R}^3)^*$ and $\mathfrak{sp}(1)^*$. Then we define an isomorphism $(\mathbb{R}^3)^* \cong \mathfrak{sp}(1)^*$ by identifying $e_i$ with $\zeta_i$. Under this isomorphism, the moment map induces a map $\mu : \mathcal{N} \rightarrow C^\infty(Q, (\mathbb{R}^3)^* \otimes g)^H$. Here we abuse the same notation to denote the induced map. As a result, $\mu(u)$ is a $g$-valued 1-form on $Y$ for $u \in \mathcal{N}$.

The 3-dimensional generalised Seiberg–Witten equations ask that a pair $(A, u) \in A \times \mathcal{N}$ obey

$$\begin{cases} D_A u = 0, \\ \#F_a + \mu(u) = 0, \end{cases}$$  \hspace{1cm} (2.10)

where $a$ is the $g$-component of $A$ and $F_a$ is its curvature.

For 4-dimensional case, let $(X, g_X)$ be a closed Riemannian 4-manifold with a $Spin^G$ structure $Q \rightarrow X$. Similar to the 3-dimensional case, we fix orthonormal
bases $\{\eta_l\}_{l=1}^3$ and $\{\zeta_l\}_{l=1}^3$ for $\Lambda^2^+ (\mathbb{R}^4)^* \otimes \mathfrak{sp}(1)^*$ respectively. Then we get an isomorphism $\Lambda^2^+ (\mathbb{R}^4)^* \cong \mathfrak{sp}(1)^*$ via identifying these bases. Under this isomorphism, we regard $\mu$ as a map $\mu : \mathcal{N} \to C^\infty (Q, \Lambda^2^+ (\mathbb{R}^4)^* \otimes g)^H$. Therefore, $\mu(u)$ is a $g$-valued self-dual 2-form on $X$ for $u \in \mathcal{N}$. The 4-dimensional generalised Seiberg–Witten equations are

$$
\begin{cases}
D_A u = 0, \\
F_a^+ + \mu(u) = 0,
\end{cases}
$$

(2.11)

where $F_a^+ = \frac{1}{2} (F_a + \ast F_a)$ is the self-dual part of the curvature.

In both 3-dimensional and 4-dimensional cases, a solution $(A, u)$ of the generalised Seiberg–Witten equations is referred to as a (generalised) monopole. Let $\mathcal{G} := C^\infty (Q, G)^H$ be the gauge group. For $g \in \mathcal{G}$, the gauge action on $A \times \mathcal{N}$ is given by

$$
g \cdot (A, u) = (g^* A, g^{-1} u) = (\text{Ad}_{g^{-1}}(A) + g^* \eta, g^{-1} u),
$$

where $\eta \in \Omega^1 (G, g)^G$ is the left-invariant Maurer–Cartan form on $G$. By Proposition 4.2.8 of [4], the generalised Seiberg–Witten equations are gauge-invariant.

2.7. Sobolev norm

For a pair $(A, u) \in A \times \mathcal{N}$, we can define its Sobolev norm as follows. Fix a smooth reference connection $A_0 \in A$. Then we can identify $A$ with $\Omega^1 (Z, g_Q)$. The Sobolev norm on $A$ is just defined as usual. For $\mathcal{N}$, we use $A_0$ to define the Sobolev norm as follows. Consider an equivariant embedding $\iota : M \hookrightarrow \mathbb{R}^N$ for some positive integer $N$. For $u \in \mathcal{N}$, $\iota \circ u$ becomes an equivariant map from $Q$ to $\mathbb{R}^N$. Then the Sobolev norm of $u$ is defined to be the usual Sobolev norm of $\iota \circ u$. By the same trick, we can define the Sobolev norm for the gauge group as well. For more details, we refer the reader to Appendix B of [20].

We assume that $kp > \text{dim} Z$ throughout. We can extend the definition of generalised Seiberg–Witten equations (2.10), (2.11), and monopoles to the Sobolev completion of $A \times \mathcal{N}$. The monopoles are not necessarily smooth by the definition. But we can always find a smooth one after a suitable gauge transformation (see Corollary 5.3.3 of [14]).

2.8. Summary of the proof

As the proof of Theorems 1 and 2 are basically the same, we focus on the 3-dimensional case from now on. We will indicate the corresponding changes for the case of Theorem 2 in Sect. 5.

Here we summarise the idea of the proof. To this end, let us introduce several functions as follows: Let $\delta_0 > 0$ be the injectivity radius of $(Y, g_Y)$. Given a monopole $(A, u)$ and $y \in Y$, for $r \leq \delta_0$, we define

$$
F_y(r) := \frac{1}{r} \int_{B_r(y)} |d_A u|^2 + 2|\mu(u)|^2 \quad \text{and} \quad f_y(r) := \int_{\partial B_r(y)} |\chi_0 \circ u|^2. \quad (2.12)
$$
The frequency function is defined by $N_y(r) := \frac{r^2 F_y(r)}{f_y(r)}$. The proof of the theorem is based on the following observations:

1. The Weitzenböck formula leads to a uniform upper bound on $\rho_0 \circ u$ (see Lemma 11). In particular, the function $F_y$ is controlled by $N_y$.

2. Under the assumption $\rho_0 \circ u \geq c^{-1}_\circ$, we show that $F_y$ is almost monotone with respect to $r$. The Heinz trick and the monotonicity of $F_y$ yields the following: If $F_y$ is sufficiently small for some $r$, then we obtain a bound on $|d_A u|^2 + |F_u|^2$ over a ball $B_r(y)$. Moreover, such a bound only depends on $c_\circ$, $c_\circ$, $r$, and some geometric data. The bound ensures that we can extract a convergent subsequence of the monopoles.

3. To ensure that $F_y$ is small enough for some $r$, we show that $N_y$ also satisfies certain monotonicity property. This property implies that $N_y$ is controlled by $\frac{1}{\rho_0 u}$. As a consequence, we can get a uniform bound for $|d_A u|^2 + |F_u|^2$ on the region $\{ y \in Y | \rho_0 \circ u(y) \geq c^{-1}_\circ \}$.

Many computations and arguments here are modelled on [7,21,22].

**Remark 3.** The proof here cannot be applied to 4-dimensional case when $\text{dim} G \neq 0$. Apart the more complicated computation, the reason is that the frequency function doesn’t satisfy the monotonicity property (see [17]).

**Remark 4.** In some cases, the moment map can be written as $\mu = 2\rho_0 \nu$, where $\nu : C(N) \to \mathfrak{sp}(1)^* \otimes \mathfrak{g}^*$ is a 3-Sasaki moment map on $C(N)$. If $\inf_{C(N)} |\nu| \geq c_0^{-1}$, then one can use the same argument in Theorem 5.4.1 of [14] to obtain a uniform upper bound on $\rho_0 \circ u$ without the assumption $\int_\gamma \rho_0 \circ u \leq c_\circ$. In these cases, the assumption $\int_Y \rho_0 \circ u \leq c_\circ$ in Theorem 1 can be removed. An example that fulfills the condition $\inf_{C(N)} |\nu| \geq c_0^{-1}$ is given on page 56 of [14].

### 3. A priori estimates

Let $(A, u)$ be a generalised monopole. According to Corollary 5.3.3 of [14], $(A, u)$ is gauge equivalent to a smooth monopole. Therefore, we assume that the monopoles under consideration are smooth throughout.

Recall that the Swann bundle satisfies $\chi_2 = 0$. Under this assumption, keep in mind that we have

$$d_A u = \nabla^TM A \chi_0 \circ u$$

(3.1)

for any spinor $u \in N$. This is an important property that helps us to reduce the non-linear differential to a linear one. Equation (3.1) is proved in Corollary 4.6.2 of [14]. Since (3.1) plays a crucial role in the later analysis, we prove it again in Lemma 8. We fix a $G$-invariant metric $g_\mathfrak{g}$ over the Lie algebra $\mathfrak{g}$ throughout.

**Definition 7.** Recall that $K^M,G u$ is the fundamental vector field along $u$. The $C^k$-norm of $K^M,G u$ is defined by

$$|K^M,G u|_{C^k} := \sup \sum_{\rho=0}^k (|\nabla^M \rho K^M,G u|, u).$$
where $\nabla^M$ is the Levi–Civita connection of $(M, g_M)$. The definition is similar for $K^M, Sp(1)|_u$.

Recall that the moment map is an equivariant map $\mu : M \to \mathfrak{sp}(1)^* \otimes g^*$. We define the Hessian of $\mu$ by the Levi–Civita connection $\nabla^M$ and the trivial connections on $\mathfrak{sp}(1)^*$ and $g^*$, denoted by $Hess\mu$.

3.1. Some identities

Before we estimate the monopole, let us deduce some useful identities (Lemmas 8, 9 and 10) which are used for the later computation.

Lemma 8. Let $(A, u) \in A \times \mathcal{N}$. Then we have $d_Au = \nabla^A_{TM} A \chi_0 \circ u$.

Proof. By Lemma 2.2.29 of [5], $\chi_0 = grad\rho_0$. By Proposition 5.6 of [13], we have $g_M = Hess\rho_0$. For any $v, w \in TM$, we have

$$g_M(\nabla^M_v \chi_0, w) = v(g_M(\chi_0, w)) - g_M(\chi_0, \nabla^M_v w)$$

$$= vw\rho - (\nabla^M_v w)\rho_0$$

$$= (Hess\rho_0)(v, w) = g_M(v, w).$$

Therefore, $\nabla^M_v \chi_0 = v$ for any $v \in TM$.

Let $v$ be a vector filed on $Y$ and $\tilde{v}$ be a horizontal lift of $v$ with respect to $A$. In particular, $A(\tilde{v}) = 0$. By definition and the above observation, we have

$$\nabla^A_{TM} (\chi_0 \circ u) = K \circ d^A_{TM} (\chi_0 \circ u) = K \circ d(\chi_0 \circ u)(\tilde{v})$$

$$= K \circ dA \chi_0 \circ du(\tilde{v})$$

$$= \nabla^M_{du(\tilde{v})} \chi_0 = du(\tilde{v}) = d_A \chi_0.$$

□

Lemma 9. Let $(A, u) \in A \times \mathcal{N}$ and $v \in C^\infty(Y, TY)$. We have the following identities:

1. $\nabla_A v \mu(u) = du \mu(d_A, v u)$;
2. $\nabla_A w \nabla_A, w \mu(u) = Hess\mu|_u (d_A, v u, A, d_A, w u) + du \mu(\nabla^M_{A, v} d_A, w u)$. 

Proof. Under the identification $C^\infty(Q, (\mathbb{R}^3)^* \otimes g)^H \cong C^\infty(Y, T Y^* \otimes g_Q)$, we have

$$\nabla_A v \mu(u)|_{\pi(p)} = \frac{d}{dt} |_{t=0} \mu(u_{\gamma(t)})(t) = du(p) \mu \circ d_p u(\gamma'(0)) = du(p) \mu(d_A, v u),$$

where $\gamma$ is a horizontal path in $Q$ such that $\gamma(0) = p$ and $\gamma'(0) = \tilde{v}(p)$ and $\tilde{v}$ is a horizontal lift of $v$ with respect to $A$. Then we get the first assertion of the lemma.
To prove the second statement, let $\sigma$ be a horizontal path in $Q$ such that $\sigma(0) = p$ and $\sigma'(0) = \tilde{v}(p)$ as before, where $\tilde{v}$ is a horizontal lift of $v$ with respect to $A$. Let $d_{A,w}$ denote a vector field on $M$ such that $d_{A,w}|_{u(\sigma(t))} = d_{A,w}u(\sigma(t))$. Then

$$
(Hess\mu|_{u(p)})(d_{A,w}u, d_{A,w}u) = \frac{d}{dt}d_{u(\sigma(t))}\mu(d_{A,w}u(\sigma(t)))|_{t=0} - d_{u(p)}\mu(\nabla_{d_{A,w}u}d_{A,w})
$$

$$
= d_{(u(p),d_{A,w}u)|p}(d\mu)(d^{TM}_{A,v}(d_{A,w}u)) - d_{u(p)}\mu\left(\mathcal{K}\circ\frac{d}{dt}d_{A,w}u(\sigma(t))|_{t=0}\right)
$$

$$
= d_{(u(p),d_{A,w}u)|p}(d\mu)(d^{TM}_{A,v}(d_{A,w}u)) - d_{u(p)}\mu(\nabla^{TM}_{A,v}d_{A,w}u).
$$

On the other hand, we have

$$
\nabla_{A,v}\nabla_{A,w}\mu(u)|_{\pi(p)} = \frac{d}{dt}d_{u(\sigma(t))}\mu(d_{A,v}u(\sigma(t)))|_{t=0}
$$

$$
= d_{(u(p),d_{A,w}u)|p}(d\mu)(d^{TM}_{A,v}d_{A,w}u).
$$

Combine Eqs. (3.2) and (3.3); then we get the second conclusion. □

Fix $u \in \mathcal{N}$. Define an operator $\mathcal{Y}_u : \Omega^1(Y, g_Q) \rightarrow \Gamma(u^*TM)$ along $u$ by

$$
\mathcal{Y}_u(\eta) := \sum_k l_k K_{<\eta,e_k>}|_u,
$$

where $\{e_k\}_{k=1}^3$ is an orthonormal basis of $TY^*$ and $\eta \in \Omega^1(Y, g_Q)$.

**Lemma 10.** Let $(A, u) \in \mathcal{A} \times \mathcal{N}$. For any $\xi \in C^\infty(Q, g)^H$, we have the following identities:

1. If $(A, u)$ is a monopole, then

$$
<d^*_A F_A, \xi> = -g_M\left(K_{\xi}^{M,G}|_u, d_A u\right);
$$

2. $<\mathcal{Y}_u(\nabla_A \mu(u)), d_A u> = -|d_A \mu(d_A u)|^2$.

**Proof.** Let $\{x^i\}_{i=0}^3$ be the normal coordinates at $y \in Y$ and $\xi \in C^\infty(Q, g)^H$. We may assume that $d_{a}\xi = 0$ at $y$. Write $<\mu(u), \xi> = \sum_{i=1}^3 b_i dx^i$. Then

$$
<d_A \mu(u), \xi> = (b_{2,1} - b_{1,2}) dx^3 + (b_{3,2} - b_{2,3}) dx^1 + (b_{1,3} - b_{3,1}) dx^2.
$$

By Lemma 9 and the Dirac equation (2.8), we have

$$
b_{2,1} - b_{1,2} = <d_A \mu(d_{A,1} u), d x^2 \otimes \xi>_{g_Y \otimes g_\delta} - <d_A \mu(d_{A,2} u), d x^1 \otimes \xi>_{g_Y \otimes g_\delta}
$$

$$
= \omega_2(K_{\xi}^{M,G}|_u, d_{A,1} u) - \omega_1(K_{\xi}^{M,G}|_u, d_{A,2} u)
$$

$$
= g_M(K_{\xi}^{M,G}|_u, I_2 d_{A,1} u - I_1 d_{A,2} u)
$$

$$
= g_M(I_3 K_{\xi}^{M,G}|_u, -I_1 d_{A,1} u - I_2 d_{A,2} u)
$$

$$
= g_M(K_{\xi}^{M,G}|_u, d_{A,3} u).
$$

(3.4)
The computation is similar for the other two terms; the details are left to the reader. Therefore, we have
\[ \langle d^* F_a, \xi \rangle = - \sum_i g_M \left( K^M_G |_{u}, d_{A,i} d x^i \right) = - g_M \left( K^M_G |_{u}, d_{A} u \right). \]

Now we prove the second equality. By definition and Lemma 9, we have
\[ \langle Y_u (\nabla_A \mu(u)), d_{A} u \rangle = \langle I_k K^M_G <\nabla_{A,j}(\mu(u)), d_{A,j} u \rangle, d_{A} u \rangle \]
\[ = - \omega_k \left( K^M_G <d_{A,j}(d_{A,j} u), dx^k>, d_{A,j} u \right) \]
\[ = - g_{\mathcal{B}} \left( <d_{A,j}(d_{A,j} u), dx^k>, <d_{A,j}(d_{A,j} u), dx^k> \right) = - |d_{A,j}(d_{A,j} u) |^2. \]

\[ \square \]

3.2. Estimates

Now we begin to estimate the monopoles. From now on, we use \( c_0 > 0 \) to denote a large uniform constant which only depends on the uniform bound \( \int_Y \rho_0 \circ u \leq c_\infty \) and the geometric data. It may be different from line to line.

First of all, let us recall the Weitzenböck formula in our setting.

**Theorem 3.1.** (Theorem 6.2.1 of [5]) The Weitzenböck formula in dimension three is as follows:
\[ D^\text{lin,u}*_{A} D_{A} u = \nabla^T_{A} d_{A} u + \frac{S_Y}{4} \chi_0 \circ u + \mathcal{Y}_u (*F_{A}), \] (3.5)
where \( D^\text{lin,u}*_{A} \) and \( \nabla^T_{A} \) are the \( L^2 \)-adjoint of the linearization (2.9) and \( \nabla^T_{A} \) respectively, and \( s_Y \) is the scalar curvature of \( (Y, g_Y) \).

**Lemma 11.** Let \( (A, u) \) be a monopole. Assume that \( \int_Y |\chi_0 \circ u|^2 \leq c_\infty \). Then we have \( \int_Y |d_{A} u|^2 + 2|\mu(u)|^2 \leq c_0 \) and \( |\chi_0 \circ u|^2 \leq c_0 \).

**Proof.** By Eqs. (2.10) and (3.5), we have
\[ 0 = \nabla^T_{A} \chi_0 \circ u + \frac{S_Y}{4} \chi_0 \circ u + \mathcal{Y}_u (*F_{A}). \] (3.6)
According to Proposition 3.2.6 of [4], \( <\mu, \zeta \otimes \xi> = - \frac{1}{2} \omega_\xi (\chi_0, K^M_G). \) Hence,
\[ \langle \mathcal{Y}_u (*F_{A}), \chi_0 \circ u \rangle = \langle \chi_0 \circ u, I_k K^M_G <\nabla_{F_{A,e_k}} |_{u} \rangle \]
\[ = \omega_k (\chi_0 \circ u, K^M_G <\nabla_{F_{A,e_k}} |_{u} \rangle \]
\[ = - 2 <\mu(u), \zeta_k \otimes < * F_{A}, e_k >> \]
\[ = 2 g_{\mathcal{B}} (<\mu(u), e_k>, < - * F_{A}, e_k>) = 2|\mu(u)|^2. \] (3.7)
Take inner product of Eq. (3.6) with \( \chi_0 \circ u \); then we have
\[ \frac{1}{2} d^* d |\chi_0 \circ u|^2 + |d_{A} u|^2 + \frac{S_Y}{4} |\chi_0 \circ u|^2 + 2|\mu(u)|^2 = 0. \] (3.8)
Integrating (3.8) we obtain the bound \( \int_Y |d_Au|^2 + 2|\mu(u)|^2 \leq c_0. \)

Assume that \( |\chi_0 \circ u|^2 \) attains its maximum at \( p \). Let \( G_p \) be the Green function of \( d^*d \) with pole at \( p \). Then we have

\[
|\chi_0 \circ u|^2(p) \leq c_0 \int_{B_r(p)} G_p|\chi_0 \circ u|^2 + c_0 \int_{Y - B_r(p)} G_p|\chi_0 \circ u|^2 + \frac{1}{vol(Y)} \int_Y |\chi_0 \circ u|^2
\]

\[
\leq c_0 r^2 |\chi_0 \circ u|^2(p) + \left( \frac{c_0}{r} + \frac{1}{vol(Y)} \right) \int_Y |\chi_0 \circ u|^2.
\]

Take \( r > 0 \) such that \( c_0 r^2 = \frac{1}{2} \); then we get the bound on the sup-norm of \( |\chi_0 \circ u| \).

\( \Box \)

**Remark 5.** Let \( Y' \) be the open submanifold in Theorem 1 such that \( \inf_{Y'} \rho_0 \circ u_n \geq c_0^{-1} \). According to the assumption that \( N \) is compact, (2.4) and Lemma 11, the condition \( \inf_{Y'} \rho_0 \circ u_n \geq c_0^{-1} \) implies that the images of \( \{u_n|y'=\infty \}_{n=1} \) are contained in a compact subset of \( M \).

The consequence of the above observation is that we can find constants \( c_k > 0 \) satisfying

\[
|\mathrm{Hess}\mu|_{u_n|Y'}|C^k| + |Rm_M|_{u_n|Y'}|C^k| + |K^{M,G}|_{u_n|Y'}|C^k| + |K^{M,Sp(1)}|_{u_n|Y'}|C^k| \leq c_k,
\]

where \( Rm_M \) is the Riemannian curvature of \( g_M \). As tensors on \( M \), the norms for \( \mathrm{Hess}\mu \) and \( Rm_M \) are the usual tenor norms defined by the metric \( g_M \) and the Levi–Civita connection. If we restrict \( \mathrm{Hess}\mu \) and \( Rm_M \) on a compact subset \( K \) of \( M \), then we can get the uniform bound in (3.9) (The bound depends on the compact set \( K \)). Note that the fundamental vector field can be regarded as a map \( K^{M,G}: M \to TM \otimes g^* \). Hence, if we restrict \( K^{M,G} \) to a compact set of \( M \), then we can get a \( C^k \)-bound on \( K^{M,G} \) for any fixed \( k \) as well. It is similar for \( K^{M,Sp(1)} \).

In the rest part of this paper, we assume that \( \int_Y |\chi_0 \circ u|^2 \leq c_0 \); unless otherwise stated.

**Lemma 12.** Let \((A, u)\) be a monopole. We have the following identity:

\[
\frac{1}{2} d^* d |F_a|^2 + |\nabla_A (*F_a)|^2 + |\nabla_A^*(F_a)|^2
\]

\[
= -\frac{s_y}{4} g_M(\chi_0 \circ u, \nabla_A^*(F_a)) \leq <\text{tr}_{g_M} \text{Hess}\mu|_u(d_Au, d_Au), \mu(u)>.
\]

In particular, we have

\[
\frac{1}{2} d^* d |F_a|^2 + |\nabla_A (*F_a)|^2 + |\nabla_A^*(F_a)|^2 + \frac{1}{2} |\nabla_A^*(F_a)|^2 \leq c_0 |\chi_0 \circ u|^2 + |\text{Hess}\mu|_u|d_Au|^2 |F_a|.
\]

**Proof.** Let \( \{x^i\}_{i=0}^3 \) be the normal coordinates at \( y \). Then by Lemma 9 and the Weitzenböck formula (3.6), we have

\[
\nabla_{A,i} \nabla_{A,i} \mu(u)|_y = \text{Hess}\mu|_u(d_{A,i}u, d_{A,i}u) + d_{u}\mu(\nabla^{TM}_{A,i} d_{A,i}u)
\]

\[
= \text{Hess}\mu|_u(d_{A,i}u, d_{A,i}u) + d_{u}\mu \left( \frac{s_y}{4} \chi_0 \circ u + \nabla_A^*(F_a) \right).
\]

(3.10)
For any vector field $V \in C^\infty(Q, u^*TM)^H$, we have

\[
<d_u \mu(V), *F_a>_g_Y @ g_b = <d_u \mu(V), \zeta_i \otimes <*F_a, dx^l>> = \omega_l(K_{<*F_a, dx^l>})|u, V)
\]

(3.11)

Take inner product of Eq. (3.10) with $*F_a$. Then Eq. (3.11) implies that

\[
\frac{1}{2}d^*|F_a|^2 + |\nabla A(*F_a)|^2 = \nabla A^*, \nabla A(*F_a), *F_a>
\]

\[
= \text{Hess} \mu|u(d_{A,i}u, d_{A,i}u), *F_a| - |\gamma_u(*F_a)|^2 - \frac{s_Y}{4}g_M(\chi_0 \circ u, \gamma_u(*F_a)).
\]

Lemma 13. Let $(A, u)$ be a monopole. Then we have

\[
\frac{1}{2}d^*|d_Au|^2 + |\nabla_A^TM d_Au|^2 + |d_u \mu(d_Au)|^2 + |d^*_A F_a|^2 \leq c_0 |F_a|^2 + (|\text{Rm}_M|u| + |K_{M,G}|^2_{C^1})|d_Au|^4 + c_0(1 + |K_{M,Sp(1)}|^4_{C^1}).
\]

(3.12)

Proof. By the curvature formulas in Lemmas 2.4.1 and 2.4.2 of [14], we have

\[
\nabla_{A,j}^{TM} \nabla_{A,i}^{TM} d_{A,i} u
\]

\[
= \nabla_{A,j}^{TM} \nabla_{A,i}^{TM} d_{A,i} u + \mathcal{K} \circ K_{F_A(\partial_j, \partial_i)}^{TM, H} + \text{Rm}_M(d_{A,j}u, d_{A,i}u)d_{A,i} u
\]

\[
= \nabla_{A,i}^{TM} (\nabla_{A,i}^{TM} d_{A,j} u + K_{F_A(\partial_j, \partial_i)}|u|) + \mathcal{K} \circ K_{F_A(\partial_j, \partial_i)}^{TM, H}
\]

\[
+ \text{Rm}_M(d_{A,j}u, d_{A,i}u)d_{A,i} u
\]

\[
= - \nabla_{A,i}^{TM} \nabla_{A,j}^{TM} d_{A,j} u + \nabla_{A,i}^{TM} (K_{F_A(\partial_j, \partial_i)}|u|)
\]

\[
+ \mathcal{K} \circ K_{F_A(\partial_j, \partial_i)}^{TM, H} d_{A,i} u + \text{Rm}_M(d_{A,j}u, d_{A,i}u)d_{A,i} u
\]

Lemma 2.4.3 of [14] implies that $\mathcal{K} \circ K_{F_A(\partial_j, \partial_i)}^{TM, H} d_{A,i} u = \nabla_{A,i}^{TM} (K_{F_A(\partial_j, \partial_i)}^{TM, H}|u)$. By definition, we have

\[
\nabla_{A,i}^{TM} (K_{F_A(\partial_j, \partial_i)}|u|) = \nabla_{d_{A,i,u}}^{TM} (K_{F_A(\partial_j, \partial_i)}|u|) + \nabla_{A,i}^{TM} F_A(\partial_j, \partial_i)|u|
\]

\[
= \nabla_{d_{A,i,u}}^{TM} (K_{F_A(\partial_j, \partial_i)}|u|) + K_{V_{A,i} F_A(\partial_j, \partial_i)}^{TM, H} + K_{V_{A,i} F_A(\partial_j, \partial_i)}^{TM, Sp(1)}
\]

\[
= \nabla_{d_{A,i,u}}^{TM} (K_{F_A(\partial_j, \partial_i)}|u|) + K_{V_{A,i} F_A(\partial_j, \partial_i)}^{TM, Sp(1)} + K_{d_{A,i,u} F_A(\partial_j, \partial_i)}^{TM, Sp(1)} + K_{d_{A,i,u} F_A(\partial_j, \partial_i)}^{TM, G}
\]

Therefore,

\[
\nabla_{A,j}^{TM} \nabla_{A,i}^{TM} d_{A,j} u = \nabla_{A,j}^{TM} (\nabla_{A,i}^{TM} d_{A,j} u) + 2\nabla_{d_{A,i,u}}^{TM} (K_{F_A(\partial_j, \partial_i)}|u|
\]

\[
+ K_{V_{A,i} F_A(\partial_j, \partial_i)}^{TM, H} + K_{d_{A,i,u} F_A(\partial_j, \partial_i)}^{TM, Sp(1)} + K_{d_{A,i,u} F_A(\partial_j, \partial_i)}^{TM, G} + \text{Rm}_M(d_{A,j}u, d_{A,i}u)d_{A,i} u.
\]
By the Weitzenböck formula (3.6), we get
\[
\nabla_A^{TM} \ast \nabla_A^{TM} d_A, j u = -\frac{1}{4} (sy)_j \chi_{0} \circ u - \frac{SY}{4} d_A, j u - \nabla_A^{TM} \mathcal{Y}_u \ast F_a
\]
\[+ 2 \nabla_{d_A, i u} (K_{F_A(\partial_j, \partial_i)}|_u) + Rm_M (d_A, j u, d_A, i u) d_A, i u
\]
\[+ K_{d_A \ast F_A(\partial_j)} |_u + K_{d_A \ast F_A(\partial_i)} |_u
\]
\[= -\frac{1}{4} (sy)_j \chi_{0} \circ u - \frac{SY}{4} d_A, j u - (\nabla_{d_A, i u} \mathcal{Y}_u) \ast F_a - \nabla_{A, j} \ast F_a
\]
\[+ 2 \nabla_{d_A, i u} (K_{F_A(\partial_j, \partial_i)}|_u) + Rm_M (d_A, j u, d_A, i u) d_A, i u
\]
\[+ K_{d_A \ast F_A(\partial_j)} |_u + K_{d_A \ast F_A(\partial_i)} |_u
\]
Take inner product of the above equation with $d_A u$. By Lemma 10, we get
\[
\frac{1}{2} d_A^* d_A |d_A u|^2 + |\nabla_A^{TM} d_A u|^2 + |d_A \mu(d_A u)|^2 + |d_A \ast F_a|^2
\]
\[\leq c_0 (1 + |\nabla^M K_{F_{A, Y}}|_u|^2 + |K_{d_A \ast F_{A, Y}}|_u|^2)|d_A u|^2
\]
\[+ c_0 |F_a|^2 + (|Rm_M|_u + |\nabla^M K_{M, G}|_u|^2)|d_A u|^4 + c_0 \chi_0 \circ u|^2 + c_0.
\]
\]

\[\square\]

**Lemma 14.** Let $(A, u)$ be a monopole. Recall that the function $F_y$ is defined by $F_y(r) = r^{-1} \left( \int_{B_{r}(y)} |d_A u|^2 + 2 |\mu(u)|^2 \right)$ for $r \leq \delta_0$, where $\delta_0$ is the injectivity radius of $(y, g_y)$. Then we have
\[
\frac{\partial F_y}{\partial r} \geq \frac{2}{r} \int_{\partial B_{r}(y)} \left( |d_A u|^2 + |F_a(\partial_r, \cdot)|^2 \right) - c_0 \left( |K_{F_{A, Y}}|_u|_C^0 + 1 \right)^2 F_y(r) - c_0 r^2.
\]
(3.13)

Moreover, if $|K_{F_{A, Y}}|_u|_C^0 \leq c_0$, then there exists constants $c_1, c_2 > 0$ (independent of $y$) such that $e^{c_1 r} F_y(r) + c_2 r^3$ is non-decreasing.

**Proof.** By the definition of $F$, we have
\[
\frac{\partial F}{\partial r} = -\frac{1}{r^2} \int_{B_{r}} (|d_A u|^2 + 2 |\mu(u)|^2) + \frac{1}{r} \int_{\partial B_{r}} (|d_A u|^2 + 2 |\mu(u)|^2).
\]
(3.14)

Define symmetric 2-tensors $S = S_{ij} dx^i \otimes dx^j$ and $R = R_{ij} dx^i \otimes dx^j$ respectively by
\[
S_{ij} := <d_A, i u, d_A, j u> - \frac{1}{2} (gy)_i j |d_A u|^2,
\]
(3.15)
\[
R_{ij} := <(F_a)_{ik}, (F_a)_{jk}> - (gy)_i j |F_a|^2.
\]

Let $\{x^i\}$ be the normal coordinates at $y$. The divergence of $S$ at $y$ is
\[
S_{ij} := \nabla_{A, j} d_A, i u, d_A, j u > + d_A, i u, \nabla_{A, j} d_A, j u > - \delta_{ij} \nabla_{A, k} d_A, k u, d_A, k u
\]
\[= K_{F_A(\partial_j, \partial_i)} |_u, d_A, j u > - \nabla_{A, j} d_A, k u, d_A, k u
\]
\[= K_{F_A(\partial_j, \partial_i)} |_u, d_A, j u > + \frac{S}{4} \chi_0 \circ u + \mathcal{Y}_u \ast F_a, d_A, i u >.
\]
(3.16)
Note that
\[
\langle \mathcal{V}_a(F_a), d_{A,i}u \rangle = \langle I_k K_{(F_a)_k}^M, d_{A,i}u \rangle = -\omega_k(K_{(F_a)_k}^M, d_{A,i}u) = -\langle d_A \mu(d_{A,i}u), dx^k \otimes (F_a)_k \rangle = -\langle \nabla_{A,i} \mu(u), * F_a \rangle = \frac{1}{2} \partial_i |F_a|^2.
\]  
(3.17)

To compute \(R_{ij;j}\), note that \(R_{ij} = -\langle (F_a)_i, (F_a)_j \rangle\) for \(i \neq j\). By a direct computation, we get
\[
R_{1j;j} = -\partial_1 |F_a|^2 + \partial_1 ((F_a)_1)^2 + R_{12;2} + R_{13;3} = -\frac{1}{2} \partial_1 |F_a|^2 + \langle (F_a)_j, (F_a);_1 - (F_a)_1;_j \rangle - \langle (F_a)_j;_j, (F_a)_1 \rangle.
\]
Since \(d^*_a(F_a) = -(F_a)_j;_j = 0\), the last term vanishes. For any \(\xi \in C^\infty(Q, g)^H\), by the same computation as in (3.4), we have
\[
\langle (F_a)_2;_1 - (F_a)_1;_2, (F_a)_2 \rangle = g_M(K_{(F_a)_2}^M, -d_{A,3}u) = g_M(K_{(F_a)_2}^M, -d_{A,3}u).
\]
The computation is similar for the other terms. In sum, we have
\[
R_{ij;j} = -\frac{1}{2} \partial_i |F_a|^2 + g_M(K_{(F_a)_j}^M, (F_a)_j;_j, d_{A,j}u). \tag{3.18}
\]

Let \(T = S + R\). Combine Eqs. (3.16), (3.17) and (3.18); then we have
\[
T_{ij;j} = \langle K_{(F_a)_j}^M, Sp(1)_{(F_a)_j} \rangle u, d_{A,j}u \rangle + \frac{S_Y}{4} \chi_0 \circ u, d_{A,i}u \rangle \tag{3.19}
\]
and \(|div T| = c_0(|\chi_0 \circ u| + (K_{(F_a)_j}^M, Sp(1)_{(F_a)_j}) |d_A u|)|d_A u|\).

Let \(r(p) = dist(p, y)\) denote the distance function. The divergence theorem implies that
\[
\int_{B_r} div T \left(\frac{1}{2} \nabla r^2\right) = r \int_{\partial B_r} T_{ij;j} r j - \int_{B_r} <T, \frac{1}{2} \nabla^2 r^2 >.
\]
Note that \(\frac{1}{2} \nabla^2 r^2 = g_Y + O(r)\). Hence
\[
\int_{B_r} div T \left(\frac{1}{2} \nabla r^2\right) = r \int_{\partial B_r} T(\partial_r, \partial_r) - \int_{B_r} tr_{g_Y} T + O(r) \int_{B_r} |T| = r \int_{\partial B_r} \left( (|d_A r u|^2 + |F_a(\partial_r, \cdot)|^2) - \frac{1}{2} (|d_A u|^2 + 2|\mu(u)|^2) \right) + \frac{1}{2} \int_{B_r} (|d_A u|^2 + 2|\mu(u)|^2) + O(r) \int_{B_r} (|d_A u|^2 + 2|\mu(u)|^2). 
\]
Therefore, we obtain
\[
\frac{\partial F}{\partial r} = -\frac{2}{r^2} \int_{B_r(x)} d\nu T \left( \frac{1}{2} \nabla r^2 \right) + \frac{2}{r} \int_{\partial B_r} (|d_{A_r} u|^2 + |F_a(\partial_r, \cdot)|^2) + O(1) F(r)
\]
\[
\geq \frac{2}{r} \int_{\partial B_r} (|d_{A_r} u|^2 + |F_a(\partial_r, \cdot)|^2) - c_0 \left( |K_{F_{\text{M.Data}}(1)}|_{u|c_0} + 1 \right)^2 F(r) - c_0 r^2.
\]
(3.20)

Assume that \( |K_{F_{\text{M.Data}}(1)}|_{u|c_0} \leq c_0 \). Then we find constants \( c_1, c'_1 > 0 \) such that
\[
\frac{\partial}{\partial r} F(r) \geq -c_1 F(r) - c'_1 r^2.
\]
(3.21)

By the differential inequality (3.21), we have
\[
\left( e^{c_1 r} F(r) \right)' \geq -c'_1 e^{c_1 r} r^2
\]
\[
\geq -c'_1 e^{c_1 \delta_0 r^2} = \left( -\frac{1}{3} e^{c_1 \delta_0 c'_1 r^3} \right)'.
\]

Therefore, \( \left( e^{c_1 r} F + \frac{1}{3} e^{c_1 \delta_0 c'_1 r^3} \right)' \geq 0 \). Take \( c_2 = \frac{1}{3} e^{c_1 \delta_0 c'_1} \); then \( e^{c_1 r} F + c_2 r^3 \) is non-decreasing.
\( \square \)

**Proposition 15.** (Heinz trick, see Appendix A of [21]) Let \( U \subset Y \) be an open subset and \( f : U \to [0, \infty) \) be a non-negative function. Suppose that there exists a constant \( c > 0 \) such that \( f \) satisfies the following properties:

1. \( d^* df \leq c (f^2 + 1) \).
2. If \( B_r(y) \subset B_{\frac{r}{2}}(x) \), then \( s^{-1} \int_{B_r(y)} f \leq cr^{-1} \int_{B_{\frac{r}{2}}(x)} f + cr^2 \).

Then there exist constants \( c_0 > 0, \epsilon_0 > 0 \) and \( \delta_1 > 0 \) depending on the \( c \) and the geometric data such that for all \( r \leq \delta_1 \) and \( B_r(x) \subset U \) with

\[
\epsilon = r^{-1} \int_{B_r(x)} f \leq \epsilon_0,
\]
we have
\[
\sup_{B_{\frac{r}{2}}(x)} f \leq c_0 r^{-2} \epsilon + c_0 r^2.
\]

To simplify the notation, we assume that \( \delta_1 = \delta_0 \) all time.

*From now on, we assume that*
\[
|\text{Hess} \mu|_{c_0} + |\text{Rm}_M|_{u|c_0} + |K_{M,Sp(1)}|_{u|c_1} + |K_{M,G}|_{u|c_1} \leq c_0.
\]
(3.22)

As pointed out in Remark 5, (3.22) is true over the submanifold \( Y' \).

**Corollary 3.2.** There exists a constant \( \epsilon_0 > 0 \) depending only on the geometric data and the bound \( c_0 \) in (3.22) with the following significant: Let \( (A, u) \) be a monopole satisfying (3.22). Fix \( y \in Y \). If \( F_y(r) \leq \epsilon_0 \), then we have
\[
|d_{A u}|^2 + 2|F_{\text{d}}|^2 \leq c_0 r^{-2} \epsilon_0 + c_0 r^2
\]
over the ball \( B_{\frac{r}{4}}(y) \).
Proof. Take \( f = |dAu|^2 + 2|F_a|^2 \) in Proposition 15. The first condition of Proposition 15 follows from Lemmas 11, 12, and 13. To verify the second condition, note that \( B_r(y) \subseteq B_{\frac{r}{2}}(x) \) implies that \( s \leq \frac{r}{2} \) and \( B_{\frac{r}{2}}(y) \subseteq B_r(x) \). By Lemma 14, we have
\[
 s^{-1} \int_{B_{\frac{r}{2}}(y)} f \leq e^{c_1 s} F_y(s) + c_2 s^3 \\
 \leq e^{c_1 \frac{s}{2}} F_y \left( \frac{r}{2} \right) + c_2 \frac{r^3}{8} \\
 \leq 2e^{c_1 \frac{s}{2}} \left( r^{-1} \int_{B_r(x)} f \right) + c_2 \frac{r^3}{8}.
\]
Thus, we can apply Proposition 15 and then obtain the result. \( \square \)

4. Frequency function

In this subsection, we follow the techniques in [7, 22] to show that a sequence of monopoles \( \{(A_n, u_n)\}_{n=1}^{\infty} \) has a convergent subsequence if the assumption (3.22) holds. Now the universal constant \( c_0 \) also dependent on (3.22).

Definition 16. Let \( \epsilon_0 > 0 \) be the constant given by Corollary 3.2. For any \( x \in Y \), define a number at \( x \) by
\[
 r(x) := \sup \{ r \in (0, \delta_0) \mid \frac{1}{r} \int_{B_r(x)} (|dAu|^2 + 2|F_a|^2) \leq \epsilon_0 \}. \tag{4.1}
\]
Corollary 3.2 implies that for any \( r \leq r(x) \), we have a uniform upper bound on \( |dAu|^2 + |F_a|^2 \) over a ball \( B_{\frac{r}{2}}(x) \).

Let \( (A, u) \) be a monopole. Reintroduce the functions \( f_x, F_x \) defined in (2.12) for \( (A, u) \). The frequency function is \( N_x(r) = \frac{r^2 F_x(r)}{f_x(r)} \). We study their properties in this section.

Lemma 17. The function \( f_x(r) = \int_{\partial B_r(x)} |\chi_0 \circ u|^2 \) satisfies the following equation:
\[
f'_x(r) = \frac{2}{r} f_x(r) + 2r F_x(r) + 2 \int_{B_r(x)} \left( \frac{Sy}{4} |\chi_0 \circ u|^2 \right) + r_x(r),
\]
where \( |r_x(r)| \leq c_0 rf_x(r) \).

Proof. By a straightforward computation, we have
\[
f'(r) = \frac{2}{r} f(r) + \int_{\partial B_r} \partial_r |\chi_0 \circ u|^2 + r(r).
\]
The term \( r(r) \) comes from the non-flatness of the metric. It satisfies \( |r(r)| \leq c_0 rf(r) \). By the divergence theorem, we obtain
\[
\int_{\partial B_r} \partial_r |\chi_0 \circ u|^2 = \int_{B_r} div(\nabla(|\chi_0 \circ u|^2)) = -\int_{B_r} d^* d|\chi_0 \circ u|^2.
\]
According to Eq. (3.8), we get
\[ \int_{\partial B_r} \partial_r |\chi_0 \circ u|^2 = 2r F + 2 \int_{B_r} \frac{SY}{4} |\chi_0 \circ u|^2. \]

\[ \square \]

**Corollary 4.1.** For any \(0 < s < r \leq \delta_0\), we have
\[ e^{c_0 s^2} \frac{f(s)}{s^2} \leq e^{c_0 r^2} \frac{f(r)}{r^2}. \]  

(4.2)

Also, we have \(\int_{B_r} |\chi_0 \circ u|^2 \leq c_0 r f(r)\).

**Proof.** By the fact that \(\int_{B_r} h^2 \leq c_0 (r^2 \int_{B_r} |dh|^2 + r \int_{\partial B_r} h^2)\), Kato’s inequality, and Lemma 17, we have
\[ f' \geq \frac{2}{r} f - c_0 r f. \]  

(4.3)

Then we get (4.2) by integrating Inequality (4.3). Using this monotonicity property, we have
\[ \int_{B_r} |\chi_0 \circ u|^2 \leq c_0 r f(r). \]

\[ \square \]

Define a non-negative function \(\kappa_x\) by the relation \(\kappa_x^2 = e^{-2\sigma_x r^2} f_x\), where
\[ \sigma_x(r) := \int_0^r \frac{1}{f_x(s)} \left( \int_{B_x(s)} \left( \frac{SY}{4} |\chi_0 \circ u|^2 \right) + \frac{1}{2} \varphi_x(s) \right) ds. \]

Note that Corollary 4.1 implies that \(|\sigma(r)| \leq c_0 r^2\) and \(|\sigma'(r)| \leq c_0 r\). By Lemma 17, it is straightforward to check that \(\kappa\) satisfies the differential equality
\[ \frac{d\kappa}{dr} = \frac{1}{r} N \kappa. \]  

(4.4)

**Lemma 18.** We have the following inequality:
\[ N'(r) \geq \frac{2e^{-2\sigma}}{r^2 \kappa^2} \int_{\partial B_r} \left( |d_{A,r} u - \frac{N}{r} \chi_0 \circ u|^2 + |F_a(\partial_r, \cdot)|^2 \right) \]
\[ - c_0 \left( |K_{F_{A_Y}}^M S^p(1)| u \right|_{C_0}^2 + 1 \right) N - c_0 \frac{r^2 e^{-2\sigma}}{\kappa^2}. \]

**Proof.** By Eq. (4.4) and Lemma 14, we have
\[ N'(r) = \frac{F'}{\kappa^2} e^{-2\sigma} - 2 N \sigma' - 2 \frac{N^2}{r} \]
\[ \geq \frac{2e^{-2\sigma}}{r \kappa^2} \int_{\partial B_r} \left( |d_{A,r} u|^2 + |F_a(\partial_r, \cdot)|^2 \right) - c_0 \left( |K_{F_{A_Y}}^M S^p(1)| u \right|_{C_0}^2 + 1 \right)^2 N \]
\[ - c_0 \frac{r^2 e^{-2\sigma}}{\kappa^2} - 2 N c_0 r - 2 \frac{N^2}{r}. \]  

(4.5)
Using integration by part and the Weitzenböck formula, we get

$$N = \frac{e^{-2\sigma}}{r^2k^2} \int_{B_r} (|d_Au|^2 + 2|\mu(u)|^2)$$

$$= \frac{e^{-2\sigma}}{r^2k^2} \int_{B_r} 2|\mu(u)|^2 - \frac{e^{-2\sigma}}{r^2k^2} \int_{\partial B_r} <\chi_0 \circ u, d_{A,r} u>$$

$$= \frac{e^{-2\sigma}}{r^2k^2} \int_{B_r} \left( \frac{SY}{4} |\chi_0 \circ u|^2 + <\chi_0 \circ u, \mathcal{Y}_a(\ast F_a)> \right)$$

$$= \frac{e^{-2\sigma}}{r^2k^2} \int_{\partial B_r} <\chi_0 \circ u, d_{A,r} u> + O(r^2).$$

The term $-2\frac{N^2}{r^2}$ in Inequality (4.5) is equal to $-4\frac{N^2}{r} + 2\frac{N^2}{r}$. Replace $-4\frac{N^2}{r}$ by $-4\frac{e^{-2\sigma}}{r^2k^2} \int_{\partial B_r} <\xi \circ u, d_{A,r} u> + O(r)N$. Then we get

$$N'(r) \geq \frac{2e^{-2\sigma}}{r^2k^2} \int_{\partial B_r} \left( |d_{A,r} u - \frac{N}{r} \chi_0 \circ u|^2 + 2|F_a(\partial_r, \cdot)|^2 \right)$$

$$- c_0(\kappa_{F_{A,Y}}^{M,S\mathcal{P}(1)}|_u|_{C^0} + 1)^2 N - c_0 \frac{r^2e^{-2\sigma}}{k^2} - c_0N.$$

\[\square\]

**Lemma 19.** Let $x \in Y$ such that $\rho_0 \circ u(x) \neq 0$. For any $\epsilon_1 > 0$, there exists a constant $c_{\epsilon_1} > 0$ such that we have $N(x) \leq \epsilon_1$ for any $r \leq \min\{\sqrt{\epsilon_1}, c_{\epsilon_1} \rho_0 \circ u(x)\}$.

**Proof.** Let $0 < r \leq r_0$ and $\rho = \rho_0 \circ u(x)$. By Corollary 4.1, we know that $\kappa^2(r) \geq c_0 \rho$. Also, $\kappa^2 \leq c_0$ because of Lemma 11. By (4.4), we have

$$\int_r^{r_0} \frac{N(t)}{t} dt = \log \frac{\kappa^2(r)}{\kappa^2(r_0)} \leq \log(c_0\rho^{-1}).$$

(4.6)

By Lemma 18 and $\kappa^2 \geq c_0 \rho$, we have

$$N(t) \geq c_0^{-1} N(r) - c_0 \rho^{-1} r_0^2,$$

(4.7)

for $r \leq t \leq r_0$. Combine (4.6) and (4.7); then we get

$$N(r) \leq c_0 \frac{\log(c_0\rho^{-1})}{\log(\frac{r_0}{r})} + c_0 r_0^2 \rho^{-1}.$$

(4.8)

If $c_0 \rho^{-1} \leq 1$, then we can take $r_0 = \sqrt{\epsilon_1}$. If $c_0 \rho^{-1} > 1$, we take $r_0$ to be square root of $\frac{\epsilon_1}{2c_0 \rho}$. Then the right hand side of (4.8) is less than $\epsilon_1$ if $c_0 \frac{\log(c_0\rho^{-1})}{\log(\frac{r_0^2}{\rho})} \leq \frac{1}{4} \epsilon_1$.

Note this is equivalent to require that $r \leq c_{\epsilon_1} \rho$.

In sum, if $r \leq \min\{\sqrt{\epsilon_1}, c_{\epsilon_1} \rho\}$, then $N(r) \leq \epsilon_1$.  \[\square\]

**Corollary 4.2.** Let $x \in Y$ such that $\rho_0 \circ u(x) \neq 0$. Then the quantity $r(x)$ defined in Definition 16 satisfies $r(x) \geq \min\{\sqrt{\epsilon_1}, c_{\epsilon_1} \rho_0 \circ u(x)\}$. 
Proof. By the uniform bound on $|\chi_0 \circ u|$ in Lemma 11, we have $F_\epsilon(r) = N_\epsilon(r) \frac{f}{r^2} \leq c_0 N_\epsilon(r)$. Take $\epsilon_1 = \frac{\epsilon_0}{c_0}$. Then Lemma 19 implies that $r(x) \geq \min \{ \sqrt{\epsilon_1}, c_\epsilon \rho_0 \circ u(x) \}$.

Proof of Theorem 1. Let $\{(A_n, u_n)\}_{n=1}^{\infty}$ be the sequence of monopoles in Theorem 1. After gauge transformations, we can assume that $\{(A_n, u_n)\}_{n=1}^{\infty}$ are smooth. Note that the functions $\rho_0 \circ u_n$ are preserved under the gauge transformations.

Suppose that there is an open submanifold $Y'$ such that $\lim_{n \to \infty} \inf_{Y'} |\rho_0 \circ u_n| \geq c^{-1}_\diamond$ for some constant $c_\diamond > 0$.

After passing to a subsequence, this condition implies that the images of $\{u_n|_{Y'}\}_{n=1}^{\infty}$ are contained in a compact subset of $M$. Therefore, we have

$$|Hess \mu|_{u_n|_{Y'}} + |Rm_M|_{u_n|_{Y'}} + |K_{M,G}|_{u_n|_{Y'}} |_{C^1} + |K_{M,S^p}|_{u_n|_{Y'}} |_{C^1} \leq c_0.$$ (See Remark 5.) As Corollary 3.2 and the analysis in Sect. 4 are local, we can apply them to the Seiberg–Witten equations over $Y'$.

For any compact subset $K \subset Y'$, Lemma 19 implies that $r_n(x) \geq c^{-1}_K$ over $K$, where $r_n(x)$ is the $u_n$-version of the number defined in Definition 16. Then Corollary 3.2 deduces a uniform bound $|d_{A_n} u_n|(x) + |F_{a_n}|(x) \leq C_K$ for any $x \in K$.

By Theorem B and the patching argument of [20], for any $p > 3$, we can find a sequence of gauge transformations $g_n \in G^{2,p}$ such that the sequence $\{g_n^* (A_n - A_0)\}$ has a uniform bound on the $W^{1,p}$-norm and $\{g_n^* A_n\}$ converges in $W^{1,p}$ weakly to $A \in A^{1,p}$. Write $g_n^* A_n = A_0 + \alpha_n$. Here we take $A_0$ to be a smooth approximation of $A$. By Theorem 8.1 of [20], we can assume that $d_{A_0}^* \alpha_n = 0$ and $* \alpha_n \rho|_{\partial K} = 0$ after gauge transformations. By the Sobolev embedding theorem, we have $|g_n^* A_n - A_0| \leq c_0$.

By Lemma 3.6.10 of [4], $d_{g_n^* A_n} (g_n^{-1} u_n) = d_{A_n} u_n |_{p_g(p)}$. Thus $|d_{g_n^* A_n} (g_n^{-1} u_n)|$ are still uniformly bounded. Hence, we have

$$|d_{A_0} (g_n^{-1} u_n)| \leq |d_{g_n^* A_n} (g_n^{-1} u_n)| + |K_{g_n^* A_n - A_0} u_n| \leq c_0.$$ (4.9)

By a direct computation, we have

$$d_{A_0}^* d_{A_0} \alpha_n = d_{g_n^* A_n}^* F_{g_n^* a_n} - d_{A_0}^* F_{a_0} - \frac{1}{2} d_{A_0}^* \ast (\alpha_n \land \alpha_n) - \frac{1}{2} \ast (\alpha_n \land F_{g_n^* a_n}).$$ (4.10)

Let $K' \subset K$ be a slightly smaller compact subset. Using Eq. (4.10), Lemma 10, and the Sobolev embedding theorem, we have

$$|\alpha_n|_{W^{2,p}(K')} \leq c_0 |d_{A_0}^* d_{A_0} \alpha_n|_{L^p(K)} + c_0 |\alpha_n|_{W^{1,p}(K)}$$

$$\leq c_0 |d_{g_n^* A_n}^* F_{g_n^* a_n} |_{L^p(K)} + |\alpha_n|_{L^p(K)} |F_{a_n}|_{L^\infty(K)} + |\alpha_n|_{W^{1,p}(K)} |\alpha_n|_{W^{1,p}(K)} + c_0$$

$$\leq c_0 |K_{g_n^* A_n} u_n|_{L^\infty(K)} |d_{A_n} u_n|_{L^\infty(K)} + |\alpha_n|_{L^p(K)} |F_{a_n}|_{L^\infty(K)} + c_0 |\alpha_n|_{W^{1,p}(K)} + c_0$$

$$\leq c_0.$$
The constant $c_0$ here also depends on $A_0$ and $K'$. By the Rellich’s lemma, $(g_n^*A_n)_{n=1}^\infty$ converges to $A$ strongly in the $W^{1,p}$-norm over $K'$.

Inequality (4.9) implies that $\{(g_n^*A_n, g_n^{-1}u_n)\}_{n=1}^\infty$ converges weakly in $W^{1,p}$-norm and converges in $C^0$ to $(A, u)$. The pair $(A, u)$ satisfies the generalised Seiberg–Witten equations over $K'$. After a gauge transformation, we assume that $(A, u)$ is smooth in a slightly smaller compact subset (still call it $K'$) (see Corollary 5.3.3 of [14]). For sufficiently large $n$, we can write

$$g_n \cdot (A_n, u_n) = (A + a_n, \exp_u(v_n)),$$

where $(a_n, v_n) \in \Omega^1(Y, g_Q)^{2,p} \oplus W^{1,p}(Y, \pi_1u^*TM)$. Moreover, $\{(a_n, v_n)\}_{n=1}^\infty$ converges in $C^0$ to zero, $(a_n)_{n=1}^\infty$ converges in $W^{1,p}$ to zero and $|\nabla_A^TM v_n| \leq c_0$.

After gauge transformations, we can further assume that $d_A^*a_n = 0$ and $|a_n|_{\beta K'} = 0$ and the above estimates are still true. Therefore, the generalised Seiberg–Witten equations can be written as

$$D_A^{lin, u_n} v_n = Q_0(\nabla_A^TM v_n, v_n) + Q_1(d_A u, v_n) + Q_2(v_n, a_n),$$

$$d_A^* + d_A a_n = *d_A \mu(v_n) + Q_3(v_n, v_n) + Q_4(a_n, a_n),$$

where $\{Q_i\}_{i=0}^4$ are certain quadratic operators in the sense that

$$|Q_i(x, y)|_{W^{k,p}} \leq c_0 |x|_{W^{k,p}} |y|_{W^{k,p}}$$

for $k \rho > 3$ or $k = 0$ and $p = \infty$. The standard elliptic bootstrapping argument implies that $(a_n, v_n)$ converges in $C^\infty$ to zero over a slightly smaller compact subset (see Theorem 5.3.2 of [14]). \qed

5. 4-dimensional case

First of all, observe that the 4-dimensional generalised Seiberg–Witten equations (2.11) are reduced to the Dirac equation $D_A u = 0$ when $dim G = 0$. We still have the Weitzenböck formula in dimension four (see [14]). Moreover, it becomes

$$D_A^{lin, u_n} D_A u = \nabla_A^TM^* d_A u + \frac{s_X}{4} \chi_0 \circ u$$

(5.1)

when $dim G = 0$. Using the Weitzenböck formula (5.1) and the assumption $\int_X \rho_0 \circ u \leq c_0$, one can deduce a uniform bound on $\rho_0 \circ u$ as in Lemma 11. The functions $F_x$ and $N_x$ are replaced by

$$F_x (r) = \frac{1}{r^2} \int_{B_r(x)} |d_A u|^2$$

and

$$N_x (r) = \frac{r^3 F_x (r)}{f_x (r)}.$$

The number in Definition 16 at $x \in X$ is replaced by

$$r (x) = \sup \{r \in (0, \delta_0] | \frac{1}{r^2} \int_{B_r(x)} |d_A u|^2 \leq \epsilon_0 \}.$$

The same argument can show that the functions $N_x (r)$ and $F_x (x)$ satisfy the monotonicity properties in Lemmas 14 and 18. The computation now is simpler due to the lack of curvature terms.
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