Percolation on a Feynman Diagram

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Abstract

In a recent paper we investigated Potts models on “thin” random graphs – generic Feynman diagrams, using the idea that such models may be expressed as the $N \to 1$ limit of a matrix model. The models displayed first order transitions for all $q > 2$, giving identical behaviour to the corresponding Bethe lattice.

We use here one of the results of namely a general saddle point solution for a $q$ state Potts model expressed as a function of $q$, to investigate some peculiar features of the percolative limit $q \to 1$ and compare the results with those on the Bethe lattice.
1 Introduction and \( q \geq 2 \) Solutions

In a series of papers \([1, 2, 3, 4, 5]\) we have exploited the observation made in \([6]\) that an ensemble of random graphs could be thought of as arising from the perturbative Feynman diagram expansion of a scalar integral to investigate spin models living on such graphs. The random graphs are of unrestricted topology as can be seen by thinking of the scalar model as the \( N \to 1 \) limit of an \( N \times N \) Hermitian matrix model. We consequently denote them as “thin” graphs, to distinguish them from the planar “fat” graphs which appear in the \( N \to \infty \) limit familiar from two dimensional quantum gravity.

The random graph spin models are of interest because they display mean field behaviour due to the tree-like local structure of the graphs \([7]\). Random graphs, which are closed, have an advantage over genuine tree-like structures such as the Bethe lattice for both numerical and analytical work because dominant boundary effects are absent. The numerical advantages of random graphs over the Bethe lattice have also been noted by other authors \([8]\). The equilibrium behaviour on random graphs of ferromagnetic Ising models \([3]\) and spin glasses \([4]\) was found to parallel that of the equivalent model on the appropriate Bethe lattice with the same number of neighbours \([9]\).

If we take the Hamiltonian for a \( q \)-state Potts model to be

\[
H = \beta \sum_{\langle ij \rangle} (\delta_{\sigma_i, \sigma_j} - 1),
\]

where the spins \( \sigma_i \) take on \( q \) values, the partition function on \( \phi^3 \) (i.e. 3-regular) random graphs with \( 2n \) vertices is

\[
Z_n(\beta) \times N_n = \frac{1}{2\pi i} \oint \frac{d\lambda}{\lambda^{2n+1}} \int \prod_{i=1}^{q} \frac{d\phi_i}{2\pi \sqrt{\det K}} \exp(-A),
\]

where the action \( A \) is given by

\[
A = \frac{1}{2} \sum_{i=1}^{q} \phi_i^2 - c \sum_{i<j} \phi_i \phi_j - \frac{\lambda}{3} \sum_{i=1}^{q} \phi_i^3,
\]

\( K \) is the inverse of the quadratic form in the action, the coupling \( c \) is related to the temperature as

\[
c = \frac{1}{(\exp(2\beta) + q - 2)}
\]

and \( N_n \) is the number of undecorated (no spin) \( \phi^3 \) graphs with \( 2n \) vertices

\[
N_n = \left( \frac{\sqrt{6}}{6} \right)^{2n} \frac{(6n-1)!!}{(2n)!!}.
\]

The role of the contour integration is to pick out graphs with \( 2n \) vertices and the prefactor of \( N_n \) disentangles the factorial growth in the number of graphs themselves from any non-analyticity due to a transition in the decorating spin model.

The model may be solved in a saddle point approximation, valid for large \( n \), and one finds a high temperature solution of the form \( \phi_i = 1 - (q - 1)c, \forall i \) and low temperature broken symmetry solutions \( \phi_i = \ldots \phi_{q-1} \neq \phi_q \). There are two low temperature branches, the first being

\[
\begin{align*}
\phi_{1\ldots q-1} &= \frac{1 - (q - 3)c - \sqrt{1 - 2(q - 1)c + (q - 5)(q - 1)c^2}}{2} \\
\phi_q &= \frac{1 + (q - 1)c + \sqrt{1 - 2(q - 1)c + (q - 5)(q - 1)c^2}}{2}
\end{align*}
\]

the second having the signs in front of the square roots reversed

\[
\begin{align*}
\phi_{1\ldots q-1} &= \frac{1 - (q - 3)c + \sqrt{1 - 2(q - 1)c + (q - 5)(q - 1)c^2}}{2} \\
\phi_q &= \frac{1 + (q - 1)c - \sqrt{1 - 2(q - 1)c + (q - 5)(q - 1)c^2}}{2}
\end{align*}
\]
The low temperature solutions can be recovered from the saddle point equations for an “effective action” $A$, namely

$$\frac{\partial A}{\partial \phi} = \frac{\partial A}{\partial \tilde{\phi}} = 0$$

where

$$A = \frac{1}{2} (q-1) [1 - c(q-2)] \phi^2 - \frac{1}{3} (q-1) \phi^3 + \frac{1}{2} \tilde{\phi}^2 - \frac{1}{3} \tilde{\phi}^3 - c(q-1) \phi \tilde{\phi}$$  \hspace{1cm} (8)

with $\phi = \phi_1...q_{-1}$, $\tilde{\phi} = \phi_q$. Similarly, at high temperature one can write another effective action

$$A_0 = \frac{q}{2} (1 - c(q-1)) \phi_0^2 - \frac{q}{3} \phi_0^3$$  \hspace{1cm} (9)

whose saddle point equation gives $\phi_0 = 1 - (q-1)c$. Both equ.(8,9) follow from imposing the expected symmetry on the $\phi_i$ in the original action in equ.(3).

The Potts magnetisation

$$m = \frac{\phi_q^3}{(\sum_{i=1}^q \phi_i)}$$  \hspace{1cm} (10)

is related to the standard order parameter by

$$M = \frac{q \max(m) - 1}{q - 1}.$$  \hspace{1cm} (11)

and gives a clear picture of the topology of the phase diagram. In Fig.1 we plot $m$ against $c$ for $q = 4$, this being representative of all the $q > 2$ models. The upper (solid) branch is given by $m$ for the solutions in equ.(6) where it gives a maximum and the lower (dashed) branch is given by $m$ for the solutions in equ.(7). The high temperature (dotted) line is at $m = \tilde{m} = 1/q$.

A first order transition occurs for $q > 2$ at

$$c(Q) = \frac{1 - (q-1)^{-1/3}}{q - 2}.$$  \hspace{1cm} (12)

when the low temperature and high temperature saddle point effective actions become equal. One finds a jump in the magnetisation at the first order transition point of

$$\Delta M = \frac{q - 2}{q - 1}.$$  \hspace{1cm} (13)

Spinodal points are present at $P$ where the high temperature solution joins the lower branch

$$c(P) = \frac{1}{2q - 1}.$$  \hspace{1cm} (14)

and $O$ where the low temperature branches first become real

$$c(O) = \frac{q - 1 - 2\sqrt{q - 1}}{(q - 1)(q - 5)}.$$  \hspace{1cm} (15)

For $q = 2$ $O,P,Q$ merge and one recovers the continuous mean field Ising transition. All the exponents, critical values and jumps are in agreement with the calculations in [10] for the Bethe lattice proper.

2 Percolation and the $q \to 1$ limit

As we have a solution and expressions for the critical points that are apparently valid for all $q$ we shall now attempt to emulate the work of [11] for the Bethe lattice and explore the percolative limit $q \to 1$. The solution of the Bethe lattice percolation problem is of course well known from elementary considerations [12, 13]. One finds that the percolation threshold on a Bethe lattice with co-ordination number $z$ is

$$p_c = 1/(z - 1)$$

and other cluster quantities are also explicitly calculable. Our objective here, as in [11]
for the Bethe lattice proper, is rather to shed light on the properties of the Potts model correspondence with percolation.

The mapping between the Potts model and a percolation problem was first written down by Fortuin and Kasteleyn [14] and is the basis of the cluster algorithms that have been so successful in combating critical slowing down in simulations [15]. The q-state Potts model partition function may be written as a bond partition function in terms of spin clusters

\[ Z = \sum_{\text{configs}} p^o(1-p)^u q^{N_c} \] (16)

where \( o = \sharp \text{occupied bonds} \), \( u = \sharp \text{unoccupied bonds} \) and \( N_c = \sharp \text{clusters} \). The factor \( p \), which converts the “geometrical” spin clusters into critical spin clusters is \( 1 - \exp(-2\beta) \) and the sum over configurations, in distinction to standard lattices, includes a sum over different Feynman diagrams. From equ.(16) one can see that the mean number of connected clusters per vertex \( N_o \) (averaged over the ensemble of Feynman diagrams) is given by

\[ N_o = \lim_{n \to \infty} \lim_{q \to 1} \frac{-\log(Z_n)}{n(q-1)} \] (17)

where \( Z_n \) is the partition function in equ.(3). Other thermodynamic quantities in the Potts model are related to percolative quantities, for instance the magnetisation \( m \) to the percolation probability \( P \)

\[ P = \lim_{q \to 1} \frac{q m - 1}{q - 1}, \] (18)

and the magnetic susceptibility \( \chi \) to the mean size of finite clusters \( S \)

\[ S (1 - P) = \lim_{q \to 1} \frac{\chi}{q - 1}. \] (19)

In principle the above relations apply per site, but with the uniform random graphs we consider it is possible to consider spatially averaged values and drop any site indices.

To get some understanding of the limit \( q \to 1 \) that is required in the above let us first look at the region \( 1 < q < 2 \). In Fig.2 we have plotted the values of \( c(O), c(P) \) and \( c(Q) \) where we can see that they fan out again from equality at \( q = 2 \) in the same order as for \( q > 2 \) (i.e. \( Q \) is sandwiched between an upper value at \( O \) and a lower value at \( P \)). We can also see that \( c(O) \) and \( c(Q) \) diverge as \( q \to 1 \). The sign of the jump in the magnetisation is reversed with respect to the \( q > 2 \) solutions as can be seen in Fig.3, where we have taken \( q = 1.2 \) as an illustrative example. On reducing \( c \) (i.e. \( T \)) a first order transition occurs at \( Q \) from the horizontal high temperature solution to the lower dashed branch of the low temperature curve. The portions \( QP \) and the upper branch to the left of \( P \) are metastable, whereas the portion \( OP \) of the lower branch and the remainder of the high temperature line to the left of \( P \) are unstable.

As we have noted, as \( q \to 1, c(Q) \to \infty \) or \( \beta_{\text{crit}} \to 0 \). The first order transition is therefore clearly not that associated with the percolation problem as

\[ p = 1 - \exp(-2\beta) \] (20)

and the known Bethe lattice solution (for \( z = 3 \)) is \( p_{\text{crit}} = 1/2 \). We must look rather at the spinodal point \( P \) where one has \( c(P) = 1/(2q - 1) \), or \( \exp(2\beta(P)) = q + 1 \). In the limit \( q \to 1 \), one thus has \( \exp(2\beta(P)) \to 2 \) and \( p_{\text{crit}} \to 1/2 \), as expected. This result is not a surprise, given the discussion of [14] for percolation on the Bethe lattice, where an exactly analogous situation occurs. In the limit \( q \to 1 \) the percolation transition is thus associated not with the first order transition at \( Q \) to a stable state, but rather to the spinodal point \( P \) where the high temperature phase joins a metastable branch.

It is also possible to directly calculate various percolative quantities in the saddle point approximation. For example, to find the percolation probability \( P \) we note \( \phi \to c, \phi \to 1 \) as \( q \to 1 \), so we have from equs.(10,13) \( P = 1 - c^3 \), which using the relation between \( c \) and \( p \) for \( q = 1 \), namely \( c = (1-p)/p \), gives

\[ P = 1 - \left( \frac{1-p}{p} \right)^3. \] (21)
This reproduces the standard result for $P$ on a Bethe lattice with 3 neighbours. The probability of a point being connected to infinity by occupied sites is $pP$ with our conventions.

More difficult is the calculation of the susceptibility, which requires the solution of the saddle point equations for the effective action in an external field $H$

$$A = \frac{1}{2} (q - 1) [1 - c (q - 2)] \phi^2 - \frac{1}{3} (q - 1) \phi^3 + \frac{1}{2} \tilde{\phi}^2 - \frac{H}{3} \tilde{\phi}^3 - c (q - 1) \phi \tilde{\phi}$$

(note the $H$ in front of $\tilde{\phi}^3$), as we have

$$\lim_{q \to 1} \lim_{H \to 1} \frac{\chi}{q - 1} = \lim_{q \to 1} \lim_{H \to 1} \frac{1}{q - 1} \frac{dm}{dH} = \lim_{q \to 1} \lim_{H \to 1} \left( 3 \frac{\phi^2}{\phi^4} \left[ \phi \frac{\partial \phi}{\partial H} - \tilde{\phi} \frac{\partial \phi}{\partial H} \right] + \frac{\phi^3}{\phi^3} \right)$$

(22)

The first two terms in equ. (23) come from the implicit dependence of the solutions on $H$ and the final term is $\partial m / \partial H$. In calculating this one uses the expression for the magnetisation in the presence of a field

$$m = \frac{H \tilde{\phi}^3}{(q - 1) \phi^3 + H \tilde{\phi}^3}$$

(24)

The full solutions for $\phi$, $\tilde{\phi}$ in the presence of a field are not particularly illuminating so we do not reproduce them here, but we find the simple limits

$$\lim_{q \to 1} \lim_{H \to 1} \phi = c$$

$$\lim_{q \to 1} \lim_{H \to 1} \frac{\partial \phi}{\partial H} = \frac{c}{c - 1}$$

$$\lim_{q \to 1} \lim_{H \to 1} \tilde{\phi} = 1$$

$$\lim_{q \to 1} \lim_{H \to 1} \frac{\partial \tilde{\phi}}{\partial H} = -1$$

(25)

giving

$$\lim_{q \to 1} \frac{\chi}{q - 1} = S (1 - P) = c^3 \frac{1 + 2c}{1 - c},$$

(26)

which again reproduces the Bethe lattice result for $S$.

3 Discussion

The somewhat peculiar features of the correspondence between the $q \to 1$ limit of the Potts model and percolation on the Bethe lattice in which the spinodal point is tied to the percolative transition are clearly preserved on Feynman diagrams. Using the relations between Potts model observables and those associated with the percolation problem, we obtain the standard Bethe lattice results for quantities such as $P$ and $S$. The inventory of results in which the (ferromagnetic) transitions of spin models on random graphs are identical to those on corresponding Bethe lattices is thus extended to percolation and Potts models analytically continued to real values of $q$.

It is also interesting to note that the Ising spin glass transition on random graphs, which appears in the $k \to 0$ limit of a $k$ Ising replica model, displays some very similar features to the percolative limit discussed here. In such models the spin glass transition in the (quenched) $k \to 0$ limit is associated with a continuous transition which appears $\forall k \geq 2$ between the high temperature phase and a phase that is metastable when $k > 2$ [3]. The “true” transition in the $k$ Ising replica model is first order for $k > 2$ [10, 12] and is not that associated with the $k \to 0$ limit. One might hope to be able to analyse the multi-Ising models in a similar fashion to the Potts models by writing down an effective action for various $k$ and seeing if a general formula could be extracted to allow analytical continuation in $k$. The superficial similarity with the percolative limit is intriguing and certainly merits further investigation to see if there is any deeper connection.
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Figure 1: The magnetisation $m$ for a 4 state Potts model as calculated from the saddle point solutions. The high temperature branch is shown dotted, the upper low temperature branch solid and the lower low temperature branch dashed.
Figure 2: $c(Q)$ (solid middle line), $c(O)$ (upper dotted line) and $c(P)$ (lower dashed line) plotted against $q$ for $1 < q < 2$. $c(O)$ and $c(Q)$ diverge as $q \to 1$. 
Figure 3: The magnetisation $m$ for a $q = 1.2$ state Potts model as calculated from the saddle point solutions. The key is as for Fig.1, but note that the first order transition at $Q$ is now to the lower branch.