STRUCTURE OF CORRELATIONS FOR THE BOLTZMANN-GRAD LIMIT OF HARD SPHERES

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ABSTRACT. We consider a gas of $N$ identical hard spheres in the whole space, and we enforce the Boltzmann-Grad scaling. We may suppose that the particles are essentially independent of each other at some initial time; even so, correlations will be created by the dynamics. We will prove a structure theorem for the correlations which develop at positive time. Our result generalizes a previous result which states that there are phase points where the three-particle marginal density factorizes into two-particle and one-particle parts, while further factorization is impossible. The result depends on uniform bounds which are known to hold on a small time interval, or globally in time when the mean free path is large.

1. Introduction

We are interested in the problem of deriving macroscopic evolutionary equations from a Newtonian gas of $N$ identical hard spheres, each having diameter $\varepsilon > 0$ and set in the spatial domain $\mathbb{R}^d$ for some $d \geq 2$. The formal scaling we will concern ourselves with is the Boltzmann-Grad scaling, which means that the mean free path for a particle of gas is of order one. Assuming that particles are initially independent of one another, the expected evolution equation in this scaling is Boltzmann’s equation with hard sphere collision kernel. Our goal is to refine known results on the propagation of chaos; we will study the structure of correlations on parts of the phase space where the pure factorization structure is necessarily destroyed. Note carefully that we are interested in the correlations between different particles’ configurations at a fixed time. We have proven, in our previous work, that on some parts of the reduced phase space, the marginal density for three particles factorizes into two-particle and one-particle contributions, while further factorization is impossible. Our aim is to generalize that result to correlations of $m - 1$ particles, for any finite $m$.

Dynamically-induced correlations in Newtonian hard sphere gases have received some attention in the recent literature; we will remark on two results in particular. Pulvirenti and Simonella analyzed the size of higher-order correlations in the context of Lanford’s theorem. [6, 7] Remarkably, the authors were able to quantify the correlations even among $\varepsilon^{-\alpha}$ particles for some $\alpha > 0$. This work required a sophisticated analysis of many-recollision
events, and employed a special representation formula to make clear the obstructions to factorization. The other work is a derivation of linear hydrodynamics by Bodineau, Gallagher and Saint-Raymond. This work relied on a perturbative expansion accounting for corrections to (linearized) factorization. Contrary to one’s naive expectation, the authors were able to quantitatively control corrections of all orders globally in time in a weighted $L^2$ norm.

Morally speaking, the results we prove in this work ought to show that, conditional on certain $L^\infty$ estimates and the factorization of the initial data, the $s$th marginal reduces to a tensor product of $(s-1)$st marginal and the first marginal, as long as the backwards trajectory of one particle is free. The reason we cannot remove the words “morally” and “ought to” is that the result requires the deletion of certain explicit sets upon which the convergence either may not or does not hold. In any case, if we view the first $(s-1)$ particle configurations as fixed and choose the $s$th particle’s configuration “randomly,” it is highly unlikely we will land on the exceptional set; this is true even if the first $(s-1)$ particles have very complicated backwards trajectories. For this reason, the deletion of an exceptional set does not alter significantly the interpretation of our results.

We have drawn particular inspiration from the methods of Pulvirenti and Simonella [7], and we have explicitly used their idea of employing an intermediate Boltzmann-Enskog-type hierarchy between the BBGKY and Boltzmann hierarchies. However, we have avoided using their special representation formula for correlations. Instead, we employ an unsymmetric Boltzmann-Enskog hierarchy (defined in our previous work [3]) which tracks correlations between just the first $m-1$ particles. This is convenient because we can show that the intermediate hierarchy propagates partial factorization in an exact sense. The theorem then follows by quantitatively comparing the BBGKY and Boltzmann-Enskog pseudo-dynamics; our primary concern is making such stability estimates precise.

We rely heavily on the developments of our previous work [3]; indeed, all we do is refine part (ii) of Theorem 2.1 from that work to include the case of arbitrarily many correlated particles (instead of just two correlated particles). For this reason, we will only briefly summarize the previous results that are relevant here, and fill in the missing ideas and estimates that are needed to prove our main theorem.

**Organization.** The notation and main theorem are given in Section 2. The BBGKY hierarchy, with basic results, is recalled in Section 3. In Section 4, we recall an unsymmetric Boltzmann-Enskog hierarchy from Appendix A of our previous work [3]. A crucial stability result is proven in Section 5; the remainder of the convergence proof proceeds as in the aforementioned work [3].
2. Notation and Main Results

We consider \( N \) identical hard spheres with diameter \( \varepsilon > 0 \), positions \( x_1, x_2, \ldots, x_N \in \mathbb{R}^d \), and velocities \( v_1, v_2, \ldots, v_N \in \mathbb{R}^d \). The tuple of all particle positions is written \( X_N = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^{dN} \), and the tuple of all particle velocities is written \( V_N = (v_1, v_2, \ldots, v_N) \in \mathbb{R}^{dN} \). We also write \( Z_N = (z_1, z_2, \ldots, z_N) = (X_N, V_N) \in \mathbb{R}^{2dN} \). The Boltzmann-Grad scaling \( N \varepsilon^{d-1} = \ell^{-1} \), for fixed \( \ell > 0 \), is assumed throughout. The \( N \)-particle phase space is the set

\[
D_N = \left\{ Z_N = (X_N, V_N) \in \mathbb{R}^{2dN} \left| \forall 1 \leq i < j \leq N, \ |x_i - x_j| > \varepsilon \right. \right\} \tag{1}
\]

As long as \( Z_N \in D_N \) we allow particles to move in straight lines with constant velocity: \( X_N = V_N, V_N = 0 \). Specular reflection is enforced at the boundary \( \partial D_N \); up to deletion of a zero measure set, all collisions are binary, non-grazing, and linearly ordered in time. If the \( i \)th and \( j \)th particles collide at time \( t_0 \) with \( x_i(t_0^-) = x_j(t_0^-) + \varepsilon \omega \) then the velocities transform according to the following rule:

\[
v_i(t_0^+) = v_i(t_0^-) + \omega \cdot (v_j(t_0^-) - v_i(t_0^-)) \]
\[
v_j(t_0^+) = v_j(t_0^-) - \omega \cdot (v_j(t_0^-) - v_i(t_0^-)) \tag{2}
\]

The collective flow of \( N \) identical hard spheres of diameter \( \varepsilon > 0 \) defines a measurable map \( \psi_N^t : D_N \to D_N \) preserving the Lebesgue measure on \( D_N \). We define a measurable involution \( Z_N \mapsto Z_N^* \) on \( \partial D_N \) which is defined almost everywhere by the following properties:

\[
\left( \text{a.e. } Z_N \in \partial D_N \right) \quad \lim_{t \to 0^+} \psi_N^t Z_N = \psi_N^t Z_N^* = \lim_{t \to 0^+} \psi_N^t Z_N
\]

\[
\left( \text{a.e. } Z_N \in \partial D_N \right) \quad \lim_{t \to 0^-} \psi_N^t Z_N = \psi_N^t Z_N^* = \lim_{t \to 0^-} \psi_N^t Z_N
\]

We introduce a probability measure \( f_N(0) \) on \( D_N \) and define \( f_N(t) \) to be the pushforward of \( f_N(0) \) under \( \psi_N^t \). Since \( \psi_N^t \) preserves the Lebesgue measure on \( D_N \), this says that

\[
f_N(t, Z_N) = f_N(0, \psi_N^{-t} Z_N) \tag{3}
\]

We assume that \( f_N(0) \) is symmetric under interchange of particle indices; since the particles are identical, it follows that \( f_N(t) \) is symmetric as well. We extend \( f_N(t) \) by zero to be defined on all of \( \mathbb{R}^{2dN} \).

We define the marginals \( f_N^{(s)}(t) \), \( 1 \leq s \leq N \), by the formula

\[
f_N^{(s)}(t, Z_s) = \int_{\mathbb{R}^{2d(N-s)}} f_N(t, Z_N) dz_{s+1} \ldots dz_N \tag{4}
\]

Then the support of \( f_N^{(s)}(t) \) is contained in the closure of \( D_s \), where

\[
D_s = \left\{ Z_s = (X_s, V_s) \in \mathbb{R}^{2ds} \left| \forall 1 \leq i < j \leq s, \ |x_i - x_j| > \varepsilon \right. \right\} \tag{5}
\]
Note carefully that $\mathcal{D}_s$ depends on $\epsilon$ whenever $s \geq 2$; however, this dependence is suppressed in our notation. The flow of $s$ identical hard spheres of diameter $\epsilon$ is written $\psi_s^\epsilon : \mathcal{D}_s \to \mathcal{D}_s$; again, the implicit dependence on $\epsilon$ is suppressed in the notation. We also define $E_s(Z_s) = \frac{1}{2} \sum_{i=1}^s |v_i|^2$, which is the total energy of $s$ particles.

In order to state our main results, we will require a notion of deletion of particles; this will be helpful in defining the exceptional set where convergence may fail. For any $1 \leq k \leq s$ we define

$$r_k Z_s = (z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_s)$$

(6)

In other words, if $Z_s$ is any ordered list of particle configurations, then $r_k Z_s$ is the same list with the $k$th entry removed. For example,

$$r_4 r_3 Z_5 = r_4 (z_1, z_2, z_4, z_5) = (z_1, z_2, z_4)$$

(7)

In this example, $r_4$ deletes the fourth particle in the list, not the particle with initial label $z_4$.

**Remark.** An alternative notation for particle deletion would be possible if we chose to associate with each particle a label, so that $Z_2$ really denotes $\{(z_1, "1"), (z_2, "2")\}$. Then we could define $r_2$ to be the deletion of the particle with label equal to 2. However, such notation is not needed here, so instead $Z_s$ is to be viewed simply as an ordered list of points in $\mathbb{R}^{2d}$.

Now for any $Z_s \in \mathcal{D}_s$ we define the set of points $\mathcal{J}_s Z_s \subset \mathbb{R}^{2d}$ as follows:

$$\mathcal{J}_s Z_s = \bigcup_{\tau_2, \ldots, \tau_s \geq 0} \psi_1^{\tau_2 + \cdots + \tau_s} r_{k_2} \psi_2^{\tau_2} r_{k_3} \psi_3^{\tau_3} \cdots r_{k_s} \psi_s^{s - \tau_s} Z_s$$

(8)

Note that $\mathcal{J}_s Z_s$ is a finite set, and its cardinality can even be controlled in terms of $s$. $\mathcal{J}_s Z_s$ We want to view the first $m - 1$ particles as “interacting” and the remaining $s - m + 1$ particles as “free.” Hence we define:

$$\mathcal{G}_{s|m} = \left\{ Z_s \in \mathcal{D}_s \left| \begin{array}{l} \forall m \leq i \leq s, \forall \tau > 0, (x^0, v^0) \in \mathcal{J}_{m-1} \mathcal{Z}_{m-1}, \\
|v_i - v^0| - \tau \geq \epsilon \\
\text{and} \forall m \leq i \neq j \leq s, \forall \tau > 0, \\
|v_i - v_j| \geq \epsilon \end{array} \right. \right\}$$

(9)

The condition $Z_s \in \mathcal{G}_{s|m}$ means that the last $s - m + 1$ particles are free under the backwards flow no matter the history of the first $m-1$ particles, including the possibility that some of the first $m-1$ particles may be “removed” from the interaction at arbitrary intermediate times. We warn the reader that this is only a heuristic explanation and the true definition is given by (10).

We will also need a condition which forces particles to disperse from one another. Hence for any $\eta > 0$ we define:

$$\mathcal{U}_{s}^{\eta} = \left\{ Z_s \in \overline{\mathcal{D}}_s \left| \begin{array}{l} \forall (x^0, v^0), (x^1, v^1) \in \mathcal{J}_s Z_s : (x^0, v^0) \neq (x^1, v^1), \\
|v^0 - v^1| > \eta \end{array} \right. \right\}$$

(10)
We also define
\[ K_s = \{ Z_s \in \overline{D_s} \mid \forall \tau > 0, \ \psi_s^{-\tau} Z_s = (X_s - V_s \tau, V_s) \} \]  
(11)
\[ \mathcal{U}_s^I = \left\{ Z_s \in \overline{D_s} \mid \inf_{1 \leq i < j \leq s} |v_i - v_j| > \eta \right\} \]  
(12)

**Definition 2.1.** Let us be given, for each \( N \in \mathbb{N} \), a sequence of densities \( F_N = \{ f_N^{(s)} \}_{1 \leq s \leq N} \), with each \( f_N^{(s)} \) defined on \( D_s \) and symmetric with respect to particle interchange. Then \( \{ F_N \}_{N \in \mathbb{N}} \) is \((m - 1)\)-nonuniformly \( f \)-chaotic for some density \( f(z) \) if, for some \( \kappa \in (0, 1) \), there holds for each integer \( 3 \leq m' \leq m \), every \( s \geq m' - 1 \), and all \( R > 0 \) that
\[ \lim_{N \to \infty} \left\| f_N^{(s)}(Z_s) - f_N^{(m'-1)} \otimes f^{\otimes(s-m'+1)}(Z_s) 1_{Z_s \in \mathcal{G}_{|m'|}^\kappa \cap U_s^\kappa} \right\|_{L^\infty_{\overline{Z_s}}} = 0 \]  
(13)
and, for each integer \( s \geq 1 \) and all \( R > 0 \),
\[ \lim_{N \to \infty} \left\| f_N^{(s)}(Z_s) - f^{\otimes s}(Z_s) \right\|_{L^\infty_{\overline{Z_s}}} = 0 \]  
(14)
where \( \eta(\varepsilon) = \varepsilon^\kappa \). If \( \{ F_N \}_{N \in \mathbb{N}} \) is \((m - 1)\)-nonuniformly \( f \)-chaotic for every \( 3 \leq m \in \mathbb{N} \) then we say that \( \{ F_N \}_{N \in \mathbb{N}} \) is \( \infty \)-nonuniformly \( f \)-chaotic.

**Remark.** Note that \( L^\infty_{\overline{Z_s}} \) refers to the essential supremum norm, since the marginals are only defined up to sets of measure zero.

**Remark.** The definition of \((m - 1)\)-nonuniform chaoticity is not exactly the same as the notion of \(2\)-nonuniform chaoticity we have introduced previously for \( m = 3 \). Nevertheless, the two notions are almost the same, in terms of the complexity of sets involved in the definition.

Recall the Boltzmann equation for hard spheres,
\[ (\partial_t + v \cdot \nabla_x) f(t) = \ell^{-1} Q(f(t), f(t)) \]  
(15)
where
\[ Q(f, f) = \int_{\mathbb{R}^d \times \mathbb{R}^d} [\omega \cdot (v_1 - v)]_+ (f(x, v^*) f(x, v^*_1) - f(x, v)f(x, v_1)) \, d\omega dv_1 \]  
(16)

We are now ready to state our main result.

**Theorem 2.1.** For each \( N \in \mathbb{N} \), let \( \{ f_N^{(s)}(t) \}_{1 \leq s \leq N} \) solve the hard sphere BBGKY hierarchy, enforcing the Boltzmann-Grad scaling \( N \varepsilon^{d-1} = \ell^{-1} \). Assume that each \( f_N^{(s)}(t) \) is symmetric with respect to particle interchange. Let \( f(t, x, v) \) solve the Boltzmann equation (15) for \( 0 \leq t \leq T \);
Furthermore, assume that $f(t) \in W^{1,\infty}(\mathbb{R}^d \times \mathbb{R}^d)$, $f(t) \geq 0$, $\int f(t) dx dv = 1$, and that there exists $\beta_T > 0$ such that
\[
\sup_{0 \leq t \leq T} \sup_{x,v} e^{\frac{1}{2} \beta_T |v|^2} f(t, x, v) < \infty
\]  
(17)

Further suppose that there exists $\tilde{\beta}_T > 0$, $\tilde{\mu}_T \in \mathbb{R}$ such that
\[
\sup_{N \in \mathbb{N}} \sup_{1 \leq s \leq N} \sup_{0 \leq t \leq T} e^{\tilde{\beta}_T E_s(Z_s)} e^{\tilde{\mu}_T s} |f_N^{(s)}(t, Z_s)| < \infty
\]  
(18)

Then we have the following:
(i) If $\{F_N(0)\}_{N \in \mathbb{N}}$ is $(m-1)$-nonuniformly $f(0)$-chaotic, then for each $t \in [0, T]$, $\{F_N(t)\}_{N \in \mathbb{N}}$ is $(m-1)$-nonuniformly $f(t)$-chaotic (with the same $\kappa$).
(ii) If $\{F_N(0)\}_{N \in \mathbb{N}}$ is $\infty$-nonuniformly $f(0)$-chaotic, then for each $t \in [0, T]$, $\{F_N(t)\}_{N \in \mathbb{N}}$ is $\infty$-nonuniformly $f(t)$-chaotic (with the same $\kappa$).

Remark. We have stated Theorem 2.1 without any explicit error estimates for simplicity. However, it is not hard to extract quantitative estimates from the proof. Note that good error estimates cannot be expected even for $s \approx \log N$ due to our reliance on wildly divergent bounds on the number of collisions of hard spheres. [2]

3. THE BBGKY HIERARCHY

The marginals $f_N^{(s)}(t)$ solve a set of equations called the BBGKY hierarchy (Bogoliubov-Born-Green-Kirkwood-Yvon). The hierarchy is written as follows, for $1 \leq s < N$ (the $s = N$ component just obeys Liouville’s equation):
\[
(\partial_t + V_s \cdot \nabla x_s) f_N^{(s)}(t, Z_s) = (N-s) \varepsilon^{d-1} C_{s+1} f_N^{(s+1)}(t, Z_s)
\]  
(19)

The specular boundary condition $f_N^{(s)}(t, Z^*_s) = f_N^{(s)}(t, Z_s)$ is enforced along $\partial D_s$; it actually holds for a.e. $(t, Z_s) \in [0, T] \times \partial D_s$ ($T > 0$ is arbitrary). The collision operator $C_{s+1}$ is written
\[
C_{s+1} f_N^{(s+1)}(Z_s) = C_{i,s+1} f_N^{(s+1)}(Z_s) - C_{i,s+1}^- f_N^{(s+1)}(Z_s)
\]  
(20)

\[
C_{i,s+1}^\pm = \sum_{i=1}^{s} C_{i,s+1}^\pm
\]  
(21)

\[
C_{i,s+1}^- f_N^{(s+1)}(t, Z_s) = \int_{\mathbb{R}^d \times S^{d-1}} dv_{s+1} \left[ \omega \cdot (v_{s+1} - v_i) \right]_- \times
\]  
(22)

\[
\times f_N^{(s+1)}(t, Z_s, x_i + \varepsilon \omega, v_{s+1})
\]

\[
C_{i,s+1}^+ f_N^{(s+1)}(t, Z_s) = \int_{\mathbb{R}^d \times S^{d-1}} dv_{s+1} \left[ \omega \cdot (v_{s+1} - v_i) \right]_+ \times
\]  
(23)

\[
\times f_N^{(s+1)}(t, x_i, v_i^*, x_i + \varepsilon \omega, v_{s+1}^*)
\]

Here $v_i^* = v_i + \omega \cdot (v_{s+1} - v_i)$ and $v_{s+1}^* = v_{s+1} - \omega \cdot (v_{s+1} - v_i)$.
The BBGKY hierarchy is well-posed locally in time in suitable $L^\infty$ norms. Roughly speaking, as long as the initial data \( \{ f_N^{(s)}(0) \} \) satisfies the following bound, for some $\beta_0 > 0$, $\mu_0 \in \mathbb{R}$,\[\sup_{1 \leq s \leq N} \sup_{Z_s \in D_s} \left| f_N^{(s)}(0, Z_s) \right| e^{\beta_0 E_s(Z_s)} e^{\mu_0 s} \leq 1 \quad (24)\] then, in the Boltzmann-Grad scaling $N \varepsilon^{d-1} = \ell^{-1}$, on a small time interval $T_L < C_d \ell e^{\mu_0 \beta_0 \frac{d+1}{2}}$, the BBGKY hierarchy has a unique solution satisfying the bound\[\sup_{0 \leq t \leq T_L} \sup_{1 \leq s \leq N} \sup_{Z_s \in D_s} \left| f_N^{(s)}(t, Z_s) \right| e^{\frac{1}{2} \beta_0 E_s(Z_s) e^{(\mu_0 - 1)s}} \leq 1 \quad (25)\] The well-posedness statement can be made more precise by using time-dependent weights but this is not necessary for any of our results. In fact the estimate (25) is sufficient to guarantee that the formal series we write are bounded, uniformly in $N$, for a short time. Our arguments can be iterated in time for as long as uniform bounds are available; this is how we ultimately deduce Theorem 2.1.

Let us define the operators $T_s(t)$ which act on functions $f^{(s)} : D_s \to \mathbb{R}$ as follows:\[\left( T_s(t)f^{(s)} \right)(Z_s) = f^{(s)}(\psi_s^{-t}Z_s) \quad (26)\] Then the functions $f_N^{(s)}(t)$ solve the following mild form of the BBGKY hierarchy:\[f_N^{(s)}(t) = T_s(t)f_N^{(s)}(0) + (N - s) \varepsilon^{d-1} \int_0^t T_s(t - t_1)C_{s+1}f_N^{(s+1)}(t_1)dt_1 \quad (27)\] Iterating this formula in the standard way \[4,6\], we express $f_N^{(s)}(t)$ as a finite sum of terms involving only the initial data:\[f_N^{(s)}(t) = \sum_{k=0}^{N-s} a_{N,k,s} \times \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} T_s(t - t_1)C_{s+1} \cdots T_{s+k}(t_k)f_N^{(s+k)}(0)dt_k \cdots dt_1 \quad (28)\] where\[a_{N,k,s} = \frac{(N - s)!}{(N - s - k)!} \varepsilon^{k(d-1)} \quad (29)\] Following the arguments of Lanford \[3,4,6\], we can define pseudo-trajectories which encode the possible (re)collision sequences which contribute to the solution $f_N^{(s)}(t)$ of the BBGKY hierarchy. Given a final state $Z_s \in D_s$, along with times $0 \leq t_k \leq \cdots \leq t_1 \leq t$, velocities $v_{s+1}, \ldots, v_{s+k} \in \mathbb{R}^d$, impact
parameters $\omega_1, \ldots, \omega_k \in S^{d-1}$, and indices $i_j \in \{1, \ldots, s + j - 1\}$, we define the point

$$Z_{s,s+k}[Z_s; t_1, \ldots, t_k; v_{s+1}, \ldots, v_{s+k}; \omega_1, \ldots, \omega_k; i_1, \ldots, i_k] \in D_{s+k}$$  \hspace{1cm} (30)

We think of $Z_{s,s+k}$ as being the image of $Z_s$ under a sequence of $k$ particle creations at times $t_j$. We evolve backwards the point $Z_s$ under the hard sphere flow for a time $t - t_1$; then, we create a particle adjacent to particle $i_1$, so $x_{s+1} = x_{i_1} + \varepsilon \omega_1$; we force a collisional change of variables, if needed, to place all particles in a \textit{pre-collisional} state. We then continue the backwards flow of $s + 1$ particles for a time $t_1 - t_2$, then create another particle, and so forth.

We define iterated collision kernels

$$b_{s,s+k}[Z_s, t; \{t_j, v_{s+j}, \omega_j, i_j\}]_{j=1}^k$$  \hspace{1cm} (31)

which simply records the accumulated product of impact parameters, e.g. $\omega \cdot (v_1 - v_2)$ for a collision involving the particles 1 and 2; finally, the iterated Duhamel series (28) becomes

$$f_{N}(t, Z_s) = \sum_{k=0}^{N-s} a_{N,k,s} \times$$

$$\times \sum_{i_1=1}^{s} \cdots \sum_{i_k=1}^{s+k-1} \int_0^t \cdots \int_0^{t_{k-1}} \int_{\mathbb{R}^{2s}} \int_{(S^{d-1})^k} \left( \prod_{m=1}^{k} d\omega_m dv_{s+m} dt_m \right) \times$$

$$\times \left( b_{s,s+k}[0, Z_{s,s+k}[]] \right) \left[ Z_s, t; \{t_j, v_{s+j}, \omega_j, i_j\} \right]_{j=1}^k$$

We refer to Section 7 of our previous work [3] for further details.

4. The Unsymmetric Boltzmann-Enskog Hierarchy

We summarize the developments of Appendix A, sections A.1 and A.2, of our previous work. [3] We will not quote precise results; the reader may refer to Appendix A of that work for theorems and proofs.

We are going to write down an unsymmetric Boltzmann-Enskog-type hierarchy which tracks correlations between the first $m - 1$ particles. Let us define the unsymmetric $s$-particle phase space, where $m \geq 2$ is fixed and $s \geq m - 1$:

$$\tilde{D}_s = \left\{ Z_s = (X_s, V_s) \in \mathbb{R}^{2ds} \bigg| \forall 1 \leq i < j \leq m - 1, |x_i - x_j| > \varepsilon \right\}$$  \hspace{1cm} (33)

Furthermore, define the collision operators

$$\tilde{C}_{s+1} = \sum_{i=1}^{s} \left( \tilde{C}_{i,s+1}^+ - \tilde{C}_{i,s+1}^- \right)$$  \hspace{1cm} (34)
where
\[ \tilde{C}_{i,s+1}^{g_{\varepsilon}(s+1)}(t, Z_s) = \int_{\mathbb{R}^d \times \mathbb{R}^{d-1}} d\omega dv_{s+1} \left[ \omega \cdot (v_{s+1} - v_i) \right] \times \]
\[ \times g_{\varepsilon}(s+1)(t, \ldots, x_i, v_i, \ldots, x_i + \varepsilon \omega, v_{s+1}) \]
\[ \tilde{C}_{i,s+1}^{+g_{\varepsilon}(s+1)}(t, Z_s) = \int_{\mathbb{R}^d \times \mathbb{R}^{d-1}} d\omega dv_{s+1} \left[ \omega \cdot (v_{s+1} - v_i) \right] \times \]
\[ \times g_{\varepsilon}(s+1)(t, \ldots, x_i, v_i^*, \ldots, x_i + \varepsilon \omega, v_{s+1}^*) \]
with \( v_i^* = v_i + \omega \cdot (v_{s+1} - v_i) \) and \( v_{s+1}^* = v_{s+1} - \omega \cdot (v_{s+1} - v_i) \). The unsymmetric Boltzmann-Enskog hierarchy is then written, for \( Z_s \in \mathcal{D}_s \), \( s \geq m - 1 \),
\[ (\partial_t + V_s \cdot \nabla x_s) g_{\varepsilon}(s) = \ell^{-1} \tilde{C}_{s+1}^{g_{\varepsilon}(s+1)}(t) \] (if \( s \geq m - 1 \) \( ) \) (37)
with boundary condition
\[ g_{\varepsilon}(s)(t, Z_s^*) = g_{\varepsilon}(s)(t, Z_s) \quad \text{a.e.} \ (t, Z_s) \in [0, T] \times \partial \mathcal{D}_s \] (38)
and initial conditions \( g_{\varepsilon}(s)(0, Z_s) \) given for \( s \geq m - 1 \) and \( Z_s \in \mathcal{D}_s \). We also define the function \( g_{\varepsilon}(t, x, v), t \in [0, T], \ x, v \in \mathbb{R}^d \), to be the solution of the equation
\[ (\partial_t + v \cdot \nabla x) g_{\varepsilon}(t) = \ell^{-1} \tilde{C}_2 (g_{\varepsilon}(t) \otimes g_{\varepsilon}(t)) \] (39)
with given initial data \( g_{\varepsilon}(0) \).

The unsymmetric Boltzmann-Enskog hierarchy \( \text{(37)} \) and the Boltzmann-Enskog equation \( \text{(39)} \) are locally well-posed when the data is in appropriate weighted \( L^\infty \) spaces, just like the BBGKY hierarchy. The proof proceeds as in the proof of Lanford’s theorem. Another important result is that the unsymmetric Boltzmann-Enskog hierarchy propagates partial factorization, in the following sense: Suppose that for all \( s \geq m - 1 \) we have
\[ g_{\varepsilon}(s)(0) = g_{\varepsilon}^{(m-1)}(0) \otimes g_{\varepsilon}(0)^{(s-m+1)} \] (40)
along with weighted \( L^\infty \) bounds at the initial time. Then on a small time interval \( 0 \leq t \leq T \) we also have
\[ g_{\varepsilon}(s)(t) = g_{\varepsilon}^{(m-1)}(t) \otimes g_{\varepsilon}(t)^{(s-m+1)} \] (41)
for all \( s \geq m - 1 \). To prove this, one constructs a solution which satisfies the partial factorization \( ansatz \), then the conclusion follows by uniqueness. This is similar to the proof that the Boltzmann hierarchy propagates factorization.

To conclude, we remark that the unsymmetric Boltzmann-Enskog hierarchy has associated pseudo-trajectories just like the BBGKY hierarchy, which we denote
\[ \tilde{Z}_{s,s+k} \left[ Z_s, t; \{ t_j, v_{s+j}, \omega_j, i_j \}_{j=1}^k \right] \] (42)
We denote the associated iterated collision kernel by
\[ \tilde{b}_{s,s+k} \left[ Z_s, t; \{ t_j, v_{s+j}, \omega_j, i_j \}_{j=1}^k \right] \] (43)
Remark. Pseudo-trajectories for the unsymmetric Boltzmann-Enskog hierarchy are similar to pseudo-trajectories for the BBGKY or Boltzmann hierarchies. In fact for the unsymmetric case we allow recollisions involving the first $m-1$ particles among each other. On the other hand, if two particles “collide” and at least one of them has index $j \geq m$ then the two particles simply pass through each other without interacting. Thus, to the extent that the first $m-1$ particles are isolated from the remaining particles, it follows that the unsymmetric Boltzmann-Enskog hierarchy tracks correlations for a cluster of $m-1$ particles.

The solution $g^s_\varepsilon(t, s \geq m-1)$ of the unsymmetric Boltzmann-Enskog hierarchy has the following representation in terms of the data:

$$g^s_\varepsilon(t, Z_s) = \sum_{k=0}^{\infty} \ell^{-k} \times \sum_{i_1=1}^{s} \cdots \sum_{i_k=1}^{s+k-1} \int_0^t \cdots \int_0^t \int_{\mathbb{R}^d} \int_{(S^{d-1})^k} \left( \prod_{m=1}^{k} d\omega_m dv_{s+m} dt_m \right) \times \left( \bar{b}_{s,s+k}^{(s+k)}(0, \hat{Z}_{s,s+k}^s[\cdot]) \right) \left[ Z_s, t; \{ t_j, v_{s+j}, \omega_j, t_j \}_{j=1}^k \right]$$

(44)

5. Stability of Pseudo-Trajectories

We finally turn to the main new estimate which allows us to conclude Theorem 2.1. Both the statement and the proof largely follow Proposition 8.3 and Proposition A.3 of our previous work [3]. Once we have Proposition 5.3, to be proven momentarily, it is a simple matter to prove Theorem 2.1 by estimating errors pointwise, as in Section 12 of our previous work. [3] Note that Proposition 5.3 really only holds for hard spheres because it relies on certain bounds on the number of collisions. [2]

We recall two useful lemmas from a previous work. [3] Combining these two lemmas, one deduces that pathological collision events (recollisions) occur with small probability. These lemmas are related to the collisional change of variables $Z_s \mapsto Z_s^*$, which is interpreted as a reflection in a suitable frame of reference.

**Lemma 5.1.** Fix $v \in \mathbb{R}^d \setminus \{0\}$ ($d \geq 2$) and let $S^{d-1} = \{ w \in \mathbb{R}^d \| w \| = 1 \}$. For any $\omega \in S^{d-1}$ define

$$u_\omega = |v|^{-1} (2\omega \cdot v - v) \in S^{d-1}$$

(45)

If $S^{d-1}_v = \{ \omega \in S^{d-1} \| \omega \cdot v > 0 \}$ then the map $\omega \mapsto u_\omega$ restricts to a diffeomorphism $S^{d-1}_v \to S^{d-1} \setminus \{-|v|^{-1}v\}$.

**Lemma 5.2.** Let $L \subset \mathbb{R}^d$ ($d \geq 2$) be a line, and let $S^{d-1} = \{ w \in \mathbb{R}^d \| w \| = 1 \}$. For $\rho > 0$ define the solid cylinder

$$C_\rho = \{ u \in \mathbb{R}^d \| \text{dist}(u, L) \leq \rho \}$$

(46)
Then
\[
\int_{\mathbb{S}^{d-1}} 1_{\omega \in C_{\rho}} d\omega \leq C_d \rho^{(d-1)/2}
\] (47)
for some constant \( C_d \) depending only on the ambient dimension \( d \) (but not depending on \( \rho \) or \( L \)).

**Proposition 5.3.** There is a constant \( c_d > 0 \) such that all the following holds: Assume that
\[
Z_{s,s+k} [Z_s, t; t_1, \ldots, t_k; v_{s+1}, \ldots, v_{s+k}; \omega_1, \ldots, \omega_k; i_1, \ldots, i_k] = \\
= (X'_{s+k}, V'_{s+k}) \in \mathcal{G}_{(s+k)|m} \cap \hat{U}^\eta_{s+k}
\] (48)
and \( E_{s+k}(Z'_{s+k}) \leq 2R^2 \) with \( \eta < R \); then,

(i) for all \( \tau \geq 0 \) we have
\[
Z_{s,s+k} [Z_s, t + \tau; t_1 + \tau, \ldots, t_k + \tau; v_{s+1}, \ldots, v_{s+k}; \omega_1, \ldots, \omega_k; i_1, \ldots, i_k] = \\
= (X'_{s+k}, V'_{s+k}) \in \mathcal{G}_{(s+k)|m} \cap \hat{U}^\eta_{s+k}
\] (49)

(ii) for any \( i_{k+1} \in \{1, 2, \ldots, s + k\} \), and for any \( \alpha, y > 0 \) and \( \theta \in (0, \frac{\pi}{2}) \) such that \( \sin \theta > c_d y^{-1} \varepsilon \), there exists a measurable set \( \mathcal{B} \subset [0, \infty) \times \mathbb{R}^d \times \mathbb{S}^{d-1} \), which may depend on \( Z_s, t, \) and \{\( t_j, v_{s+j}, \omega_j, i_j \)\}_{j=1}^{k} \), such that
\[
\forall T > 0, \\
\int_0^T \int_{B_{2R}^d} \int_{\mathbb{S}^{d-1}} 1_{(\tau, v_{s+k+1}; \omega_{k+1}) \in \mathcal{B}} d\omega_{k+1} dv_{s+k+1} d\tau \leq \\
\leq C_{d,s,k} TR^d \left[ \alpha + \frac{y}{\eta T} + C_{d,\alpha} \left( \frac{\eta}{R} \right)^{d-1} + C_{d,\alpha} \theta^{(d-1)/2} \right]
\] (50)
and
\[
Z_{s,s+k+1} [Z_s, t + \tau; t_1 + \tau, \ldots, t_k + \tau, 0; v_{s+1}, \ldots, v_{s+k}, v_{s+k+1}; \\
\omega_1, \ldots, \omega_k, \omega_{k+1}; i_1, \ldots, i_k, i_{k+1}] \\
\in \mathcal{G}_{(s+k+1)|m} \cap \hat{U}^\eta_{s+k+1}
\] (51)
whenever \( (\tau, v_{s+k+1}, \omega_{k+1}) \in ([0, \infty) \times \mathbb{R}^d \times \mathbb{S}^{d-1}) \setminus \mathcal{B} \).

**Remark.** The smallness of the error estimate comes from setting \( \eta = \varepsilon^{\kappa} \) (recall \( \kappa \in (0, 1) \) is fixed), \( y = \varepsilon^{(1+\kappa)/2} \) and \( \theta \sim \varepsilon^{(1-\kappa)/4} \) to satisfy the constraint \( \sin \theta \geq c_d y^{-1} \varepsilon \). We regard \( \alpha, R \) as fixed as \( \varepsilon \to 0 \); it is then found that \( \alpha \to 0 \) and \( R \to \infty \) result in a vanishingly small overall error. Obviously these choices are not uniquely determined but they suffice for obtaining the convergence. The implicit dependence on \( \alpha \) could be quantified by writing a quantitative version of Lemma 5.7, accounting for the size of the Jacobian for the mapping \( \omega \to u_\omega \) (this is a \((d-1) \times (d-1)\) determinant) on the set \( \{\omega \cdot v \geq |v| \sin \alpha\} \).

**Proof.** Claim (i) follows immediately from the definitions of \( \mathcal{G}_{(s+k)|m} \) and \( \hat{U}^\eta_{s+k} \), so we turn to Claim (ii). We begin by deleting creation times where
two particles may be too close to each other; this is dangerous because if particles are concentrated near the created particle at the time of particle creation then we will not be able to prove that the recollision probability is small. We define

\[ B_I = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in [0, \infty) \times \mathbb{R}^d \times S^{d-1} \mid \exists (x^0, v^0), (x^1, v^1) \in J_{s+k}Z'_{s+k} \text{ such that} \right. \]

\[ (x^0, v^0) \neq (x^1, v^1) \]

\[ \left. \text{and } |(x^0 - x^1) - (v^0 - v^1)| \leq y \right\} \]

(52)

We can estimate the measure of \( B_I \) because there is a known bound on the number of collisions of \( s + k \) hard spheres which is independent of the initial configuration. [2] Indeed each pair of distinct points in \( J_{s+k}Z'_{s+k} \) contributes \( O(\eta^{-1}y) \) to the measure of \( B_I \) because two particles can only stay within a distance \( y \) for a time of order \( \eta^{-1}y \); here we are using the time integrals explicitly, and we are also using the fact that \( Z'_{s+k} \in \mathcal{U}_{s+k} \). (Note that we have to account for possible deletions of particles when estimating the cardinality of \( J_{s+k}Z'_{s+k} \), but there are only \( s + k \) deletions and the dynamics between deletions is the usual hard sphere dynamics; hence, the total number of collisions is still finite and quantitatively bounded.) We obtain

\[ \int_0^T \int_{B_{2R}} \int_{S^{d-1}} 1_{(\tau, v_{s+k+1}, \omega_{k+1}) \in B_I} d\omega dv_{s+k+1} d\tau \leq \]

\[ \leq C_{d,s,k} R^d \eta^{-1}y \]

(53)

**Remark.** \( C_{d,s,k} \) may grow rapidly with \( s, k \) in accordance with known bounds on the number of collisions of hard spheres. [2]

We define \( Z'_{s+k}(\tau) = \psi^{-\tau}_{s+k}Z'_{s+k} \). Let us delete particle creations which are too close to grazing:

\[ B_{II} = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in [0, \infty) \times \mathbb{R}^d \times S^{d-1} \text{ such that} \right. \]

\[ \left. \omega_{k+1} \cdot (v_{s+k+1} - v'_{i_{k+1}}(\tau)) \leq (\sin \alpha) \left| v_{s+k+1} - v'_{i_{k+1}}(\tau) \right| \right\} \]

(54)

We have

\[ \int_0^T \int_{B_{2R}^d} \int_{S^{d-1}} 1_{(\tau, v_{s+k+1}, \omega_{k+1}) \in B_{II}} d\omega_{k+1} dv_{s+k+1} d\tau \leq C_{d} TR^d \alpha \]

(55)

The remainder of the proof is split between pre-collisional and post-collisional cases. Here pre-collisional means that \( \omega_{k+1} \cdot (v_{s+k+1} - v'_{i_{k+1}}(\tau)) \leq 0 \), and post-collisional means that \( \omega_{k+1} \cdot (v_{s+k+1} - v'_{i_{k+1}}(\tau)) > 0 \). For convenience we define

\[ A^+ = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in [0, \infty) \times \mathbb{R}^d \times S^{d-1} \text{ such that} \right. \]

\[ \omega \cdot (v_{s+k+1} - v'_{i_{k+1}}(\tau)) > 0 \]

(56)
particle creation. We define
\[ \mathcal{A}^- = \{ (\tau, v_{s+k+1}, \omega_{k+1}) \in [0, \infty) \times \mathbb{R}^d \times S^{d-1} \text{ such that } \omega \cdot (v_{s+k+1} - v'_{i_{k+1}}(\tau)) \leq 0 \} \] (57)

**Pre-collisional case.** This is the easier case. We first make sure that the
\((s + k + 1)\)-particle state is in \(\hat{U}_{s+k+1}^0\) at the time the particle is created. Note that the existing particles are at least \(y > c_d \varepsilon\) apart at the time of particle creation. We define
\[ \mathcal{B}_{III}^- = \{ (\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{A}^- \setminus (\mathcal{B}_I \cup \mathcal{B}_{II}) \text{ such that } \exists (x^0, v^0) \in J_{s+k}(Z'_{s+k}(\tau)) : |v^0 - v_{s+k+1}| \leq \eta \} \] (58)

We have
\[ \int_0^T \int_{B_{2R}^d} \int_{S^{d-1}} 1_{(\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{B}_{III}^-} \, d\omega dv_{s+k+1} \, d\tau \leq C_{d,s,k} T \eta^d \] (59)

The constant \(C_{d,s,k}\) depends on bounds on the number of collisions of hard spheres. \[2\]

The final estimate is to control recollisions under the backwards flow. We define
\[ \mathcal{B}_{IV}^- = \{ (\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{A}^- \setminus (\mathcal{B}_I \cup \mathcal{B}_{II}) \text{ such that } \exists (x^0, v^0) \in J_{s+k}(Z'_{s+k}(\tau)) : \\
\begin{align*}
&\left( x'_{i_{k+1}}(\tau) + \varepsilon \omega - x^0 \right) \cdot (v_{s+k+1} - v^0) \\
&|x'_{i_{k+1}}(\tau) + \varepsilon \omega - x^0| |v_{s+k+1} - v^0| \geq \cos \theta
\end{align*} \] (60)

Then we have
\[ \int_0^T \int_{B_{2R}^d} \int_{S^{d-1}} 1_{(\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{B}_{IV}^-} \, d\omega dv_{s+k+1} \, d\tau \leq C_{d,s,k} T R^d \theta^{d-1} \] (61)

As usual the constant \(C_{d,s,k}\) could be large.

Let \(\mathcal{B}^- = \mathcal{B}_I \cup \mathcal{B}_{II} \cup \mathcal{B}_{III}^- \cup \mathcal{B}_{IV}^-\); then we have
\[ \int_0^T \int_{B_{2R}^d} \int_{S^{d-1}} 1_{(\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{B}^-} \, d\omega dv_{s+k+1} \, d\tau \leq C_{d,s,k} T R^d \left[ \alpha + \frac{y}{\eta T} + \left( \frac{\eta}{R} \right)^d + \theta^{d-1} \right] \] (62)

But by assumption we have \(\sin \theta > c_d y^{-1} \varepsilon\); we can choose \(c_d\) large enough that
\[ Z_{s,s+k+1}[Z_s, t + \tau; t_1 + \tau, \ldots, t_k + \tau, 0; v_{s+1}, \ldots, v_{s+k}, v_{s+k+1};
\omega_1, \ldots, \omega_k, \omega_{k+1}, i_1, \ldots, i_k, i_{k+1}] \] (63)

\[ \in \mathcal{G}_{(s+k+1)|m} \cap \mathcal{U}_{s+k+1}^\eta \]

whenever \((\tau, v_{s+k+1}, \omega_{k+1}) \in \mathcal{A}^- \setminus \mathcal{B}^-\).
Post-collisional case. Let us define
\[ v_{s+k+1}^* = v_{s+k+1} - \omega_{k+1} \omega_{k+1} \cdot \left( v_{s+k+1} - v'_{k+1}(\tau) \right) \]  
(64)
\[ v'_{k+1}(\tau) = v'_{k+1}(\tau) + \omega_{k+1} \omega_{k+1} \cdot \left( v_{s+k+1} - v'_{k+1}(\tau) \right) \]  
(65)
We have to be sure that both the \( s + k + 1 \) particle and the \( i_{k+1} \) particle do not recollide with other particles under the backwards flow. First let us define
\[ B^+_{III} = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in A^+ \setminus (B_I \cup B_{II}) \text{ such that } \right\} \]
\[ \exists (x^0, v^0) \in J_{s+k} \left( Z'_{s+k}(\tau) \right) : |v^0 - v_{s+k+1}^*| \leq \eta \]  
(66)
\[ B^+_{IV} = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in A^+ \setminus (B_I \cup B_{II}) \text{ such that } \right\} \]
\[ \exists (x^0, v^0) \in J_{s+k} \left( Z'_{s+k}(\tau) \right) : |v^0 - v'_{k+1}(\tau)| \leq \eta \]  
(67)
\[ B^+_{V} = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in A^+ \setminus (B_I \cup B_{II}) \text{ such that } \right\} \]
\[ |v_{s+k+1} - v'_{k+1}(\tau)| \leq \eta \]  
(68)
Then using Lemma 5.1 and Lemma 5.2 and bounds on the number of collisions of hard spheres [2], we have
\[ \int_0^T \int_{B_{2R}^d} \int_{S^{d-1}} 1_{(\tau,v_{s+k+1},\omega_{k+1}) \in B^+_{III}} d\omega dv_{s+k+1} d\tau \leq C_{d,s,k} C_{d,\alpha} T R \eta^{d-1} \]  
(69)
\[ \int_0^T \int_{B_{2R}^d} \int_{S^{d-1}} 1_{(\tau,v_{s+k+1},\omega_{k+1}) \in B^+_{IV}} d\omega dv_{s+k+1} d\tau \leq C_{d,s,k} C_{d,\alpha} T R \eta^{d-1} \]  
(70)
\[ \int_0^T \int_{B_{2R}^d} \int_{S^{d-1}} 1_{(\tau,v_{s+k+1},\omega_{k+1}) \in B^+_{V}} d\omega dv_{s+k+1} d\tau \leq C_d T \eta^d \]  
(71)
Now we define
\[ B^+_{VI} = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in A^+ \setminus (B_I \cup B_{II}) \text{ such that } \right\} \]
\[ \exists (x^0, v^0) \in J_{s+k} \left( Z'_{s+k}(\tau) \right) : (x^0, v^0) \neq (x'_{i_{k+1}}(\tau), v'_{i_{k+1}}(\tau)) \]
\[ \text{and } \frac{x'_{i_{k+1}}(\tau) + \varepsilon \omega - x^0}{|x'_{i_{k+1}}(\tau) + \varepsilon \omega - x^0|} \cdot \left| v'_{s+k+1} - v^0 \right| \geq \cos \theta \]  
(72)
\[ B^+_{VII} = \left\{ (\tau, v_{s+k+1}, \omega_{k+1}) \in A^+ \setminus (B_I \cup B_{II}) \text{ such that } \right\} \]
\[ \exists (x^0, v^0) \in J_{s+k} \left( Z'_{s+k}(\tau) \right) : (x^0, v^0) \neq (x'_{i_{k+1}}(\tau), v'_{i_{k+1}}(\tau)) \]
\[ \text{and } \frac{x'_{i_{k+1}}(\tau) - x^0}{|x'_{i_{k+1}}(\tau) - x^0|} \cdot \left| v'_{s+k+1} - v^0 \right| \geq \cos \theta \]  
(73)
Using Lemma $5.1$ and Lemma $5.2$ and bounds on the number of collisions of hard spheres $[2]$, we have

$$
\int_0^T \int_{B_{2R}^d} \int_{S^{d-1}} 1_{(\tau, v_{s+k+1}, \omega_{k+1}) \in B_{V I}^+} d\omega dv_{s+k+1} d\tau \leq C_{d,s,k} T R^d \theta^{(d-1)/2}
$$

$$
\int_0^T \int_{B_{2R}^d} \int_{S^{d-1}} 1_{(\tau, v_{s+k+1}, \omega_{k+1}) \in B_{V I}^+} d\omega dv_{s+k+1} d\tau \leq C_{d,s,k} T R^d \theta^{(d-1)/2}
$$

Let $B^+ = B_I \cup B_{II} \cup B_{III}^+ \cup B_{IV}^+ \cup B_{V I}^+ \cup B_{V I I}^+$; then we have

$$
\int_0^T \int_{B_{2R}^d} \int_{S^{d-1}} 1_{(\tau, v_{s+k+1}, \omega_{k+1}) \in B^+} d\omega dv_{s+k+1} d\tau \leq C_{d,s,k} T R^d \left[ \alpha + \frac{y}{\eta T} + C_{d,\alpha} \left( \frac{\eta}{R} \right)^{d-1} + C_{d,\alpha} \theta^{(d-1)/2} \right]
$$

By assumption, $\sin \theta > c_d y^{-1} \varepsilon$; as long as $c_d$ is chosen large enough, we have

$$
Z_{s,s+k+1}[Z_s, t + \tau; t_1 + \tau, \ldots, t_k + \tau, \emptyset; v_{s+1}, \ldots, v_{s+k}, v_{s+k+1};
\omega_1, \ldots, \omega_k, \omega_{k+1}; i_1, \ldots, i_k, i_{k+1}] 
\in \mathcal{G}_{(s+k+1)|m} \cap \mathcal{U}_{s+k+1}^0
$$

whenever $(\tau, v_{s+k+1}, \omega_{k+1}) \in A^+ \setminus B^+$. □

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