Extensions of Ostrowski Type Inequalities via $h$-Integrals and $s$-Convexity

Khuram Ali Khan, Khalid Mahmood Awan, Allah Ditta, Ammara Nosheen, and Rostin Mabela Matendo

1. Department of Mathematics, University of Sargodha, Sargodha, Pakistan
2. Department of Mathematics, University of Jhang, Jhang, Pakistan
3. Department of Maths and Computer Science, Faculty of Science, University of Kinshasa, Kinshasa, Democratic Republic of the Congo

Correspondence should be addressed to Rostin Mabela Matendo; rostin.mabela@unikin.ac.cd

Received 15 September 2021; Accepted 15 October 2021; Published 11 November 2021

Academic Editor: Xiaolong Qin

Copyright © 2021 Khuram Ali Khan et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this paper, Hölder, Minkowski, and power mean inequalities are used to establish Ostrowski type inequalities for $s$-convex functions via $h$-calculus. The new inequalities are generalized versions of Ostrowski type inequalities available in literature.

1. Introduction

In mathematics, the quantum calculus is equivalent to usual infinitesimal calculus without depending upon the concept of limit. It has two major branches, $q$-calculus and the $h$-calculus. It is really the calculus of finite differences, but a more systematic analogy with classical calculus makes it additionally transparent. The definite $h$-integral is a Riemann sum so that the fundamental theorem of $h$-calculus allows one to evaluate finite sums and $h$-integration by parts which is simply the Abel transform. The theory of $h$-discrete calculus is the rapidly developing area of great interest both from theoretical and applied point of view. This calculus is the study of the definitions, properties, and applications of the related concepts, the fractional calculus and discrete fractional calculus.

1.1. $h$-Derivative [1]. The $h$-derivative is defined as follows: let $\phi: T_v \rightarrow \mathbb{R}, T_v = \{v, v + h, v + 2h, \ldots\}, h > 0$.

$$D_h\phi(\theta) = \frac{d_v\phi(\theta)}{d_h\theta} = \frac{\phi(\theta + h) - \phi(\theta)}{h},$$ (1)

where, classically, $\lim_{h \to 0} D_h\phi(\theta) = d\phi(\theta)/d\theta$. Also, $h$-differential is $d_v\phi(\theta) = \phi(\theta + h) - \phi(\theta)$, in particular $d_h\theta = h$.

1.2. $h$-Integral [1]. The $h$-integral is defined as follows: let $\phi: T_v \rightarrow \mathbb{R}, T_v = \{v, v + h, v + 2h, \ldots\}, h > 0$.

$$\int_v^\mu \phi(\theta)d_h\theta = \sum_{j=0}^{\mu-vh-1} \phi(v + jh)h,$$ (2)

where $v, \mu \in \mathbb{R}, v < \mu$.

1.3. Definition of $h$-Integral [1]. If $\mu - v \in h\mathbb{Z}$, we define $h$-integral to be
\[ \int_{v}^{\mu} \phi(\theta) d_{h} \theta = \begin{cases} h(\phi(v) + \phi(v + h) + \phi(v + 2h) + \cdots + \phi(\mu - h)), & \text{if } v < \mu, \\ 0 & \text{if } v = \mu, \\ -h(\phi(\mu) + \phi(\mu + h) + \phi(\mu + 2h) + \cdots + \phi(\mu - h)), & \text{if } v > \mu. \end{cases} \] (3)

With this definition, the definite \( h \)-integral is Riemann sum of \( \phi(\theta) \) on the interval \([v, \mu]\), which is proportioned to subintervals of equal width.

1.4. Formula of \( h \)-Integration by Parts [1]. Let \( \phi, g : [v, \mu] \rightarrow \mathbb{R} \) be the continuous functions and \( \theta \in [v, \mu] \), then the formula of \( h \)-integration by parts is stated as

\[ \int_{v}^{\mu} \phi(\theta) d_{h} g(\theta) = \phi(\mu) g(\mu) - \phi(v) g(v) - \int_{v}^{\mu} g(\theta + h) d_{h} \phi(\theta). \] (4)

1.5. Properties of \( h \)-Calculus [1]. The \( h \)-analogue of a binomial expansion \((\theta - v)^{n}\) is defined as

\[ (\theta - v)^{n} = (\theta - v)(\theta - v - h) \cdots (\theta - v - (n - 1)h). \] (5)

For \( n \geq 1 \) and \( (\theta - v)^{0} = 1 \),

\[ D_{h}(\theta - v)^{n} = n(\theta - v)^{n-1}, \]
\[ D_{h}(\theta - v)^{n} = -n(\theta - v)^{n-1}, \]
\[ D_{h} \frac{1}{(\theta - v)^{n}} = \frac{n}{(\theta + h - \theta)^{n+1}}, \]
\[ D_{h} \frac{1}{(\theta - v)^{n}} = \frac{n}{(\theta - v)^{n+1}}. \] (6)

Note that \( h \)-analogs of an integer \( n \) is still \( n \), and \( (\theta - 0)^{n}_{h} \neq 0 \).

1.6. \( h \)-Fractional Function [2]. Let \( t, \alpha \in \mathbb{R} \), then the \( h \)-fractional function \( t^{(\alpha)}_{h} \) is defined by

\[ t^{(\alpha)}_{h} = h^{\alpha} \Gamma(t/h + 1) / \Gamma(t/h + 1 - \alpha), \] (7)

where \( \Gamma \) is the Euler gamma function \( t/h \notin \{-1, -2, -3, -4 \ldots \} \). Note that \( \lim_{h \to 0} t^{(\alpha)}_{h} = t^{\alpha} \), hence we define

\[ \phi(\theta) - \frac{1}{\mu - v} \int_{v}^{\mu} \phi(u) du = \frac{(\theta - v)^{2}}{\mu - v} \int_{0}^{1} u \phi^{'}(u \theta + (1 - u)\mu) du - \frac{\mu - \theta^{2}}{\mu - v} \int_{0}^{1} u \phi^{'}(u \theta + (1 - u)\mu) du, \] (11)

for each \( \theta \in [v, \mu] \).

Using Lemma 1, Alomari et al. in [26] presented the following integral inequalities.

Theorem 1. Let \( \phi : I \rightarrow \mathbb{R} \), where \( I \subseteq \mathbb{R} \) is an interval, be a mapping differentiable in the interior \( I^{0} \) of \( I \), and let \( v, \mu \in I^{0} \) with \( v < \mu \). If \( |\phi^{'}(\theta)| \leq M \) for all \( \theta \in [v, \mu] \), then the following inequality holds:

\[ \left| \phi(\theta) - \frac{1}{\mu - v} \int_{v}^{\mu} \phi(u) du \right| \leq M \left( \frac{1}{4} + \frac{(\theta - v + \mu)2}{(\mu - v)^{2}} \right), \] (9)

for all \( \theta \in [v, \mu] \).

From the invention of (9), it is being studied extensively by many researchers (see [4–9]). Generalizations, extensions, and variants of this inequality exist in literature (see [10–17]) for different classes of convex functions. More on \( s \)-convex functions and on the conformable functions can be seen in [4, 18–24].

In [25], the class of functions which are called \( s \)-convex in the second sense has been introduced by Hudzik and Maligranda as follows:

1.7. \( s \)-Convex Function. A function \( \phi : \mathbb{R}^{+} \rightarrow \mathbb{R} \) is said to be \( s \)-convex in second sense if

\[ \phi(\Omega \theta + (1 - \Omega)\phi) \leq \Omega^{s} \phi(\theta) + (1 - \Omega)^{s} \phi(\phi), \] (10)

for each \( \theta, \phi \in \mathbb{R}^{+}, \Omega \in [0, 1] \) and for unique \( s \in (0, 1] \).

The integral equality is established by Alomari et al. in [26].

Lemma 1. Suppose that \( \phi : J \subset \mathbb{R} \rightarrow \mathbb{R} \) is a mapping such that \( \phi^{'} \in L[v, \mu] \), then

\[ \left| \phi(\theta) - \frac{1}{\mu - v} \int_{v}^{\mu} \phi(u) du \right| \leq M \left( \frac{1}{4} + \frac{(\theta - v + \mu)2}{(\mu - v)^{2}} \right), \] (11)

for each \( \theta \in [v, \mu] \).

Theorem 2. Consider the function \( \phi : J \subset \mathbb{R}^{+} \rightarrow \mathbb{R} \) such that \( \phi^{'} \in L[v, \mu] \) for \( v, \mu \in J \). If \( |\phi^{'}(\theta)| \leq M, \theta \in [v, \mu] \), the following result holds:
\[ \phi(\theta) - \frac{1}{\mu - \nu} \int_{\nu}^{\mu} \phi(u) \, du \leq \frac{M}{\mu - \nu} \left( \frac{(\theta - \nu)^2 + (\mu - \theta)^2}{s + 1} \right), \]

for each \( \theta \in [\nu, \mu] \).

Theorem 4. Consider the function \( \phi: J \subset \mathbb{R}^+ \rightarrow \mathbb{R} \) such that \( \phi^t \in L[\nu, \mu] \) for \( \nu, \mu \in J \). If \( |\phi'|^m \) is s-convex in second sense on \([\nu, \mu] \), for some static \( s \in (0, 1], m \geq 1 \) and \( |\phi^t(\theta)| \leq M, \theta \in [\nu, \mu] \), the following integral inequality holds:

\[ \phi(\theta) - \frac{1}{\mu - \nu} \int_{\nu}^{\mu} \phi(u) \, du \leq \frac{M}{(1 + n)^{1/m}} \left( \frac{(\theta - \nu)^2 + (\mu - \theta)^2}{2(\mu - \nu)} \right), \]

for each \( \theta \in [\nu, \mu] \).

2. Main Results

Initially, we establish the following identity.

Lemma 2. Suppose \( \phi: J \rightarrow \mathbb{R} \) be \( h \)-differentiable mapping on interior of interval \( J \) in which \( \nu, \mu \in J \) and \( \nu < \mu \). If \( D_h \phi \in L[\nu, \mu] \), then the following \( h \)-integral equality is valid:

\[ \phi(\theta) + \frac{h}{\mu - \nu} \phi(\nu) + \frac{(\mu - \theta)\phi(\mu)}{\mu - \nu} - \frac{1}{\mu - \nu} \int_{\nu}^{\mu} \phi(u) \, du \]

\[ = \frac{(\theta - \nu)^2}{\mu - \nu} \int_{0}^{1} uD_h \phi(u\theta + (1 - u)\nu) \, du - \frac{(\mu - \theta)^2}{\mu - \nu} \int_{0}^{1} uD_h \phi(u\theta + (1 - u)\mu) \, du, \]

for each \( \theta \in [\nu, \mu] \).

Proof. By formula (4) of \( h \)-integration by parts, the first term of right hand side of (15) becomes
From (16) and (17),

\[
\frac{(\mu - \theta)^2}{\mu - \nu} \int_0^1 u D_h \phi(u \theta + (1-u)\mu) du
\]

\[
= \frac{(\mu - \theta)^2}{\mu - \nu} \left( u \left( \phi(u \theta + (1-u)\mu) \right) + \int_0^1 \phi(\theta(u + h) + (1-(u+h))\mu) \frac{d\mu}{\mu - \theta} \right)
\]

\[
= \frac{\mu - \theta}{\mu - \nu} \phi(\theta) + \frac{\mu - \theta}{\mu - \nu} \left( h \sum_{j=0}^{1/\theta - 1} \phi(\theta(jh + h) + (1-(jh + h))\mu) \right)
\]

\[
= \frac{\theta - \nu}{\mu - \nu} \phi(\theta) + \frac{\theta - \nu}{\mu - \nu} \left( h \sum_{j=1}^{1/\theta - 1} \phi(\theta(jh) + (1-jh)\mu) \right)
\]

\[
= \frac{\theta - \nu}{\mu - \nu} \phi(\theta) + \frac{\theta - \nu}{\mu - \nu} \left( h \sum_{j=1}^{1/\theta - 1} \phi(\theta(jh) + (1-jh)\mu) \right)
\]

\[
= \frac{\theta - \nu}{\mu - \nu} \phi(\theta) + \frac{\theta - \nu}{\mu - \nu} \left( h \sum_{j=1}^{1/\theta - 1} \phi(\theta(jh) + (1-jh)\mu) \right)
\]

and the second term of right hand side of (15) becomes

\[
\frac{(\mu - \theta)^2}{\mu - \nu} \int_0^1 u D_h \phi(u \theta + (1-u)\mu) du
\]

\[
= \frac{(\mu - \theta)^2}{\mu - \nu} \left( u \left( \phi(u \theta + (1-u)\mu) \right) + \int_0^1 \phi(\theta(u + h) + (1-(u+h))\mu) \frac{d\mu}{\mu - \theta} \right)
\]

\[
= \frac{\mu - \theta}{\mu - \nu} \phi(\theta) + \frac{\mu - \theta}{\mu - \nu} \left( h \sum_{j=0}^{1/\theta - 1} \phi(\theta(jh + h) + (1-(jh + h))\mu) \right)
\]

\[
= \frac{\theta - \nu}{\mu - \nu} \phi(\theta) + \frac{\theta - \nu}{\mu - \nu} \left( h \sum_{j=0}^{1/\theta - 1} \phi(\theta(jh + h) + (1-(jh + h))\mu) \right)
\]

\[
= \frac{\theta - \nu}{\mu - \nu} \phi(\theta) + \frac{\theta - \nu}{\mu - \nu} \left( h \sum_{j=0}^{1/\theta - 1} \phi(\theta(jh + h) + (1-(jh + h))\mu) \right)
\]

\[
= \frac{\theta - \nu}{\mu - \nu} \phi(\theta) + \frac{\theta - \nu}{\mu - \nu} \left( h \sum_{j=0}^{1/\theta - 1} \phi(\theta(jh + h) + (1-(jh + h))\mu) \right)
\]

From (16) and (17),

\[
\frac{\theta - \nu}{\mu - \nu} \phi(\theta) + \frac{\mu - \theta}{\mu - \nu} \left( h \sum_{j=1}^{1/\theta - 1} \phi(\theta(jh) + (1-jh)\mu) \right)
\]

\[
= \phi(\theta) + \frac{h(\theta - \nu)\phi(\nu) + h(\mu - \theta)\phi(\mu)}{\mu - \nu} \int_0^\mu \phi(u) du - \frac{\mu - \theta}{\mu - \nu} \int_0^\mu \phi(u) du,
\]
which is the required result.

Using Lemma 2, we prove the following results.

\[ \phi(\theta) + h \frac{\theta - \nu}{\mu - \nu} \phi(\nu) + (\mu - \theta) \phi(\mu) - \frac{1}{\mu - \nu} \int_\nu^\mu \phi(u) d_h u \]

\[ \leq M \frac{(\theta - \nu)^2 + (\mu - \theta)^2}{\mu - \nu} \left[ \frac{h^{s+1}}{s+1} + \frac{1}{s+1} \left( \frac{-h^{s+2} + (1 + h)^{s+2}}{s+2} \right) + \frac{1}{s+2} \right], \]  

(19)

for each \( \theta \in [\nu, \mu] \).

\[ \phi(\theta) + h \frac{\theta - \nu}{\mu - \nu} \phi(\nu) + (\mu - \theta) \phi(\mu) - \frac{1}{\mu - \nu} \int_\nu^\mu \phi(u) d_h u \]

\[ \leq \frac{(\theta - \nu)^2}{\mu - \nu} \int_0^1 u |D_h \phi(u \theta + (1 - u)\nu)| d_h u + \frac{(\mu - \theta)^2}{\mu - \nu} \int_0^1 u |D_h \phi(u \theta + (1 - u)\mu)| d_h u \]

\[ \leq \frac{(\theta - \nu)^2}{\mu - \nu} \left[ \int_0^{\nu_{h(\nu+\theta)}} |D_h \phi(\theta)| d_h u + \int_0^{\nu_{h(\nu+\theta)}} u (1 - u) |D_h \phi(\nu)| d_h u \right] + \frac{(\mu - \theta)^2}{\mu - \nu} \left[ \int_0^{\nu_{h(\nu+\theta)}} |D_h \phi(\theta)| d_h u + \int_0^{\nu_{h(\nu+\theta)}} u (1 - u) |D_h \phi(\mu)| d_h u \right] \]

\[ \leq \frac{(\theta - \nu)^2}{\mu - \nu} \left[ \int_0^{\nu_{h(\nu+\theta)}} d_h u + \int_0^{\nu_{h(\nu+\theta)}} u (1 - u) d_h u \right] + \frac{(\mu - \theta)^2}{\mu - \nu} \left[ \int_0^{\nu_{h(\nu+\theta)}} d_h u + \int_0^{\nu_{h(\nu+\theta)}} u (1 - u) d_h u \right] \]

\[ = \frac{M (\theta - \nu)^2 + (\mu - \theta)^2}{\mu - \nu} \left[ \int_0^{\nu_{h(\nu+\theta)}} d_h u + \int_0^{\nu_{h(\nu+\theta)}} u (1 - u) d_h u \right] \]

\[ \int_0^{\nu_{h(\nu+\theta)}} d_h u = \frac{1}{s+2} \]

\[ \int_0^{\nu_{h(\nu+\theta)}} u (1 - u) d_h u = \left[ \frac{-(1 - u + h)^{s+1}}{s+1} \right]_0^1 + \frac{1}{s+1} \int_0^1 (1 - u + h) d_h u \]

\[ \leq \frac{h^{s+1}}{s+1} + \frac{1}{s+1} \int_0^1 (1 - u)^{s+1} d_h u \]

\[ = \frac{h^{s+1}}{s+1} + \frac{1}{s+1} \left[ \frac{1}{s+2} (1 + h - u)^{s+1} \right]_0^1 \]

\[ = \frac{h^{s+1}}{s+1} + \frac{1}{s+1} \left( \frac{-h^{s+2} + (1 + h)^{s+2}}{s+2} \right) \]

\[ \leq M \frac{(\theta - \nu)^2 + (\mu - \theta)^2}{\mu - \nu} \left[ \frac{h^{s+1}}{s+1} + \frac{1}{s+1} \left( \frac{-h^{s+2} + (1 + h)^{s+2}}{s+2} \right) + \frac{1}{s+2} \right]. \]

(20)
Theorem 6. Suppose $\phi: J \subset \mathbb{R}^n \rightarrow \mathbb{R}$ be a $h$-differentiable mapping on $J$ in such a way that $D_h \phi \in L[\nu, \mu]$, in which $\nu, \mu \in J$ for $\nu < \mu$. If $|D_h \phi|^{m}$ is $s$-convex $[\nu, \mu]$ for some static $s \in (0, 1]$, $m > m/m - 1$, and $|D_h \phi(\theta)| \leq M$, $\theta \in [\nu, \mu]$, then we have the $h$-integral inequality in discrete calculus:

$$
\left| \phi(\theta) + h (\theta - \nu) \phi(\nu) + (\mu - \theta) \phi(\mu) \right| \leq \frac{1}{\mu - \nu} \int_0^\mu \phi(u) \, du.
$$

(21)

Proof. From Lemma 2 and keeping the familiar Hölder inequality, we have

$$
\left| \phi(\theta) + h (\theta - \nu) \phi(\nu) + (\mu - \theta) \phi(\mu) \right| \leq \frac{1}{\mu - \nu} \int_0^\mu \phi(u) \, du.
$$

(22)

Proof. From Lemma 2 and keeping the familiar Hölder inequality, we have

$$
\left| \phi(\theta) + h (\theta - \nu) \phi(\nu) + (\mu - \theta) \phi(\mu) \right| \leq \frac{1}{\mu - \nu} \int_0^\mu \phi(u) \, du.
$$

(23)
which completes the proof.

**Theorem 7.** Suppose \( \phi : J \rightarrow \mathbb{R} \) be a \( h \)-differentiable mapping on interior of a positive interval \( J \) in such a way that

\[
D_h \phi \in L[\nu, \mu], \text{ for } \nu, \mu \in J. \text{ If } |D_h \phi|^m \text{ is a } s\text{-convex in second sense on } [\nu, \mu] \text{ for some static } s \in (0, 1], \text{ and } |D_h \phi(\theta)| \leq M, \theta \in [\nu, \mu], \text{ then we have the following } h\text{-integral inequality:}
\]

\[
|\phi(\theta) + h(\theta - \nu)\phi(\nu) + (\mu - \theta)\phi(\mu) - \frac{1}{\mu - \nu} \int_{\nu}^{\mu} \phi(u) d_h u| \leq M \left[ \frac{(\theta - \nu)^2 + (\mu - \theta)^2}{\mu - \nu} \left[ \frac{1}{2} \right]^{1 - 1/m} \right]^{1/m} \left[ \frac{h^{s+1}}{s+1} + \frac{1}{s+1} \left( \frac{-h^{s+2} + (1+h)^{s+2}}{s+2} \right) + \frac{1}{s+2} \right]^{1/m},
\]

for each \( \theta \in [\nu, \mu] \).

**Proof.** From Lemma 2 and keeping in view the familiar power mean inequality, we get

\[
\left| \phi(\theta) + h(\theta - \nu)\phi(\nu) + (\mu - \theta)\phi(\mu) - \frac{1}{\mu - \nu} \int_{\nu}^{\mu} \phi(u) d_h u \right| \\
\leq \frac{1}{\mu - \nu} \int_{0}^{1} u |D_h \phi(u \theta + (1-u)\nu)| d_h u + \frac{(\mu - \theta)^2}{\mu - \nu} \left[ \int_{0}^{1} |D_h \phi(u \theta + (1-u)\nu)|^m d_h u \right]^{1/m} \\
\leq \frac{1}{\mu - \nu} \left( \int_{0}^{1} |D_h \phi(u \theta + (1-u)\nu)|^m d_h u \right) \left( \int_{0}^{1} u(u \theta + (1-u)\nu)^{s+1} d_h u \right)^{1/m} \\
+ \frac{(\mu - \theta)^2}{\mu - \nu} \left( \int_{0}^{1} u(u \theta + (1-u)\nu)^{s+1} d_h u \right)^{1/m} \\
\cdot \int_{0}^{1} |D_h \phi(u \theta + (1-u)\nu)|^m d_h u \leq \int_{0}^{1} u(u \theta + (1-u)\nu)^{s+1} d_h u + \int_{0}^{1} u(u \theta + (1-u)\nu)^{s+1} d_h u
\]

\[
\leq M \left[ \frac{1}{s+1} \right]^{1/m} \int_{0}^{1} u(u \theta + (1-u)\nu)^{s+1} d_h u
\]

\[
\cdot \int_{0}^{1} u(u \theta + (1-u)\nu)^{s+1} d_h u = \left[ \frac{1}{s+1} \right]^{1/m} \int_{0}^{1} u(u \theta + (1-u)\nu)^{s+1} d_h u
\]

\[
= \frac{h^{s+1}}{s+1} + \frac{1}{s+1} \int_{0}^{1} (1-u)^{s+1} d_h u
\]

\[
= \frac{h^{s+1}}{s+1} + \frac{1}{s+1} \left[ \frac{1}{s+2} (1+h)^{s+2} \right]_{0}^{1}
\]

\[
= \frac{h^{s+1}}{s+1} + \frac{1}{s+1} \left( \frac{h^{s+2} + (1+h)^{s+2}}{s+2} \right),
\]

\[
\leq M \left[ \frac{1}{s+1} \right]^{1/m} \int_{0}^{1} u(u \theta + (1-u)\nu)^{s+1} d_h u
\]
\[
\int_0^1 |D_{\phi} (u\theta + (1-u)\mu)|^m d\mu u \leq M^m \left( \int_0^1 u^{(s+1)} |D_{\phi} (\theta)|^m d\mu u + \int_0^1 u (1-u)^2 |D_{\phi} (\mu)|^m d\mu u \right) \\
\leq M^m \left( \int_0^1 u^{(s+1)} d\mu u + \int_0^1 u (1-u)^2 d\mu u \right) \\
\leq M^m \left( \frac{h^{(s+1)}}{s+1} + \frac{1}{s+1} \left( \frac{h^{(s+2)}}{s+2} + \frac{1}{s+2} \right) \right) \\
\leq M \left[ \frac{(\theta - \nu)^2 + (\mu - \nu)^2}{\mu - \nu} \right]^{1-1/m} \left[ \frac{h^{(s+1)}}{s+1} + \frac{1}{s+1} \left( \frac{-h^{(s+2)} + (1+h)^{s+2}}{s+2} \right) + \frac{1}{s+2} \right]^{1/m}.
\]

**Remark 1.**

(a) In Theorem 5, if we take \( h = 0 \), then (19) diminishes the inequality (12) of Theorem 2

(b) In Theorem 6, if we take \( h = 0 \), then (21) diminishes the inequality (13) of Theorem 3

(c) In Theorem 7, if we take \( h = 0 \), then (24) diminishes the inequality (14) of Theorem 4

In [27], if \( \tau = \mathbb{Z} \),

\[
\phi(\theta) + \frac{(\theta - \nu)\phi(\nu) + (\mu - \theta)\phi(\mu)}{\mu - \nu} - \frac{1}{\mu - \nu} \int_{\nu}^\mu \phi(u) d_1 u \\
= \frac{(\theta - \nu)^2}{\mu - \nu} \int_0^1 u D_1 \phi (u\theta + (1-u)\nu) d_1 u - \frac{(\mu - \theta)^2}{\mu - \nu} \int_0^1 u D_1 \phi (u\theta + (1-u)\nu) d_1 u.
\]

By taking \( h = 1 \), in Theorem 5, we have

\[
\left| \phi(\theta) + \frac{(\theta - \nu)\phi(\nu) + (\mu - \theta)\phi(\mu)}{\mu - \nu} - \frac{1}{\mu - \nu} \int_{\nu}^\mu \phi(u) d_1 u \right| \\
\leq M \frac{(\theta - \nu)^2 + (\mu - \theta)^2}{\mu - \nu} \left[ \frac{1}{s+1} + \left( \frac{-1 + (2)^{s+2}}{(s+1)(s+2)} \right) \right]^{1/m}.
\]

By taking \( h = 1 \), in Theorem 6, we have

\[
\left| \phi(\theta) + \frac{(\theta - \nu)\phi(\nu) + (\mu - \theta)\phi(\mu)}{\mu - \nu} - \frac{1}{\mu - \nu} \int_{\nu}^\mu \phi(u) d_1 u \right| \\
\leq M \left[ \frac{(\theta - \nu)^2 + (\mu - \theta)^2}{\mu - \nu} \right]^{1-1/m} \left[ \frac{1}{s+1} + \left( \frac{-1 + (2)^{s+2}}{(s+1)(s+2)} \right) \right]^{1/m}.
\]
By taking \( h = 1 \) and \( s = 1 \) in Theorem 5, we have

\[
\phi(\theta) + \frac{(\theta - \nu)\phi(\nu) + (\mu - \theta)\phi(\mu)}{\mu - \nu} - \frac{1}{\mu - \nu} \int_{\nu}^{\mu} \phi(u) \, du_1 \nu \leq M \left[ \frac{(\theta - \nu)^2 + (\mu - \theta)^2}{\mu - \nu} \right].
\] (32)

By taking \( h = 1 \) and \( s = 1 \) in Theorem 6, we have

\[
\phi(\theta) + \frac{(\theta - \nu)\phi(\nu) + (\mu - \theta)\phi(\mu)}{\mu - \nu} - \frac{1}{\mu - \nu} \int_{\nu}^{\mu} \phi(u) \, du_1 \nu \leq M \left[ \frac{(\theta - \nu)^2 + (\mu - \theta)^2}{\mu - \nu} \right] \left( \frac{1}{n+1} \right)^{1/n}.
\] (33)

By taking \( h = 1 \) and \( s = 1 \) in Theorem 7, we have

\[
\phi(\theta) + \frac{(\theta - \nu)\phi(\nu) + (\mu - \theta)\phi(\mu)}{\mu - \nu} - \frac{1}{\mu - \nu} \int_{\nu}^{\mu} \phi(u) \, du_1 \nu \leq M \left[ \frac{(\theta - \nu)^2 + (\mu - \theta)^2}{\mu - \nu} \right]^{1 - 1/m} \frac{1}{2}
\] (34)

3. Conclusion

Our results extend and generalize the results of Alomari et al. In this work, some important Ostrowski type inequalities are established in the context of \( h \)-calculus. The derived results constitute contributions to the theory of \( h \)-integral and can be specialized to yield numerous interesting integral inequalities including some known results. An interesting feature of our results is that they provide new estimates and best approximation on Ostrowski type of inequalities for \( h \)-integral. If we take limit \( h \to 0 \), Ostrowski type of \( h \)-integral inequalities reduces to simple inequalities present in [26]. The presented results motivate scientists to stimulate more work in such directions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

References

[1] V. Kac and P. Cheung, Quantum Calculus, Springer, Berlin, Germany, 2002.
[2] S. Shaimardan, Hardy-type Inequalities Quantum Calculus, Diva-Portal.Org, Luleå University of Technology, Luleå, Sweden, 2018.
[3] M. A. Ostrowski, "Über die Absolutabweichung einer differentierbaren funktion von ihren Integralmittelwert," Commentarii Mathematici Helvetici, vol. 10, pp. 226-227, 1938.
[4] Y. Khurshid, M. A. Khan, and Y. M. Chu, "Ostrowski type inequalities involving conformable integrals via preinvex functions," AIP Advances, vol. 10, Article ID 055204, 2020.
[5] S. Erden, H. Budak, M. Zeki Sarikaya, S. Iftikhar, and P. Kumam, "Fractional Ostrowski type inequalities for bounded functions," Journal of Inequalities and Applications, vol. 2020, no. 1, Article ID 123, 2020.
[6] B. Bayraktar and M. Gurbuz, "On some integral inequalities for \( (s,m) \) convex functions," TWMS Journal of Applied and Engineering Mathematics, vol. 10, pp. 288–295, 2020.
[7] Z. Sha, Y. Gouju, G. Ye, D. Zhao, and W. Liu, "On interval-valued K-Riemann integral and Hermite-Hadamard type inequalities," AIMS Mathematics, vol. 6, no. 2, pp. 1276–1295, 2021.
[8] Z. A. Zafar, N. Ali, G. Zaman, P. Thounthong, and C. Tung, "Analysis and numerical simulations of fractional order Vallis system," Alexandria Engineering Journal, vol. 59, no. 4, pp. 2591–2605, 2020.
[9] T. Lou, G. Ye, D. Zhao, and W. Liu, "Iq-Calculus and Iq-Hermite Hadamard inequalities for interval-valued functions," Advances in Difference Equations, vol. 2020, no. 1, Article ID 446, 2020.
[10] D. S. Mitrinovic, J. E. Pecaric, and A. M. Fink, Inequalities Involving Functions and Their Integrals and Derivatives, Kluw. Acad. Publ., Dort, The Netherlands, 1991.
[11] S. S. Dragomir, "On the Ostrowski’s integral inequality for mappings with bounded variation and applications," Journal of Mathematical Inequalities, vol. 1, no. 2, 1938.
[12] S. S. Dragomir and S. Wang, "A new inequality of Ostrowski’s type in \( L_1 \)-norm and applications to some special means and to some numerical quadrature rules," Tamkang Journal of Mathematics, vol. 28, no. 3, pp. 239–244, 1997.
[13] Z. Liu, "Some companions of an Ostrowski type inequality and application," Journal of Inequalities in Pure and Applied Mathematics, vol. 10, no. 2, 2009.
[14] B. G. Pachatte, "On an inequality of Ostrowski type in three independent variables," Journal of Mathematical Analysis and Applications, vol. 249, pp. 583–591, 2000.
[15] B. G. Pachpatte, "On a new Ostrowski type inequality in two independent variables," Tamkang Journal of Mathematics, vol. 32, no. 1, pp. 45–49, 2001.
[16] A. Rafiq, N. A. Mir, and F. Ahmad, "Weighted Cebyshev-Ostrowski type inequalities," Applied Mathematics and Mechanics, vol. 28, no. 7, pp. 901–906, 2007.
[17] M. Z. Sarikaya, "On the Ostrowski type integral inequality," Acta Mathematica Universitatis Comenianae, vol. LXXIX, no. 1, pp. 129–134, 2010.
[18] M. A. Khan, S. Begum, and Y. Khurshid, "Ostrowski type inequalities involving conformable fractional integrals," Journal of Inequalities and Applications, vol. 70, 2018.
[19] M. A. Khan, Y. Chu, T. U. Khan, and J. Khan, "Some new inequalities of Hermite-Hadamard type for \( s \)-convex functions with applications," Open Mathematics, vol. 15, no. 1, pp. 1414–1430, 2017.
[20] M. A. Khan, M. Hanif, and A. H. Khan, "Association of Jensen’s inequality for \( s \)-convex function with Csiszar
divergence,” Journal of Inequalities and Applications, vol. 2019, pp. 1–14, Article ID 162, 2019.

[21] S. I. Butt, M. Nadeem, and G. Farid, “On caputo fractional derivatives via exponential s-convex functions,” Turkish Journal of Science, vol. 5, no. 2, pp. 140–146, 2020.

[22] N. T. An, P. D. Dong, and X. Qin, “Robust feature selection via nonconvex sparsity-based methods,” Journal of Nonlinear and Variational Analysis, vol. 5, no. 1, pp. 59–77, 2021.

[23] A. R. Khan and J. Pecaric, “Positivity of general linear inequalities for n-Convex functions via the taylor formula using new green functions,” Communications in Optimization Theory, Article ID 5, 2019.

[24] H. Wang, “Certain integral inequalities related to-Lipschitzian mappings and generalized h-Convexity on fractal sets,” Nonlinear Functional Analysis, Article ID 12, 2021.

[25] H. Hudzik and L. Maligranda, “Ome remarks on s-convex functions,” Aequationes Mathematicae, vol. 48, no. 1, pp. 100–111, 1994.

[26] M. Alomari, M. Darus, S. S. Dragomir, and P. Cerone, “Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense,” Applied Mathematics Letters, vol. 23, no. 9, pp. 1071–1076, 2010.

[27] M. Bohner and A. Peterson, Dynamic Equations on Time Scales, Birkhäuser, Basel, Switzerland, 2001.