Reflection calculus and conservativity spectra

L. D. Beklemishev

Abstract. Strictly positive logics have recently attracted attention both in the description logic and in the provability logic communities for their combination of efficiency and sufficient expressivity. The language of Reflection Calculus, RC, consists of implications between formulas built up from propositional variables and the constant ‘true’ using only conjunction and diamond modalities which are interpreted in Peano arithmetic as restricted uniform reflection principles.

The language of RC is extended by another series of modalities representing the operators associating with a given arithmetical theory $T$ its fragment axiomatized by all theorems of $T$ of arithmetical complexity $\Pi_0^n$ for all $n > 0$. It is noted that such operators, in a strong sense, cannot be represented in the full language of modal logic.

A formal system $RC^\triangledown$ is formulated that extends RC and is sound and (it is conjectured) complete under this interpretation. It is shown that in this system one is able to express the iterations of reflection principles up to any ordinal $< \varepsilon_0$. Second, normal forms are provided for its variable-free fragment. This fragment is thereby shown to be algorithmically decidable and complete with respect to its natural arithmetical semantics.

In the last part of the paper the Lindenbaum–Tarski algebra of the variable-free fragment of $RC^\triangledown$ and its dual Kripke structure are characterized in several natural ways. Most importantly, elements of this algebra correspond to the sequences of proof-theoretic $\Pi_0^{n+1}$-ordinals of bounded fragments of Peano arithmetic called conservativity spectra, as well as to points of Ignatiev’s well-known Kripke model.

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1. Introduction

A system called Reflection Calculus and denoted by RC was introduced in [6] and, in a slightly different format, in [19]. From the point of view of modal logic, RC can be seen as a fragment of Japaridze’s polymodal provability logic GLP (see [35] and [17]) consisting of implications of the form $A \to B$, where $A$ and $B$ are formulas built up from $\top$ and propositional variables using just $\land$ and the ‘diamond’ modalities. We call such formulas $A$ and $B$ strictly positive.

Strictly positive modal logics, earlier and in a different guise, appeared in the work on description logic. They serve as a good compromise between the concerns of efficiency and sufficient expressivity in the knowledge base query answering. In particular, the strictly positive language corresponds to the OWL2EL profile of the OWL web ontology language, and is used in large ontology bases such as SNOMED CT.\(^1\)

From the algebraic standpoint strictly positive logics correspond to varieties of semilattices with monotone operators. In the abstract context such varieties were studied, in particular, for the closure operators in [34] by Jackson. The papers [37] and [38] undertake a general study of strictly positive logics and provide more references, especially in description logic and in the universal algebraic traditions.

Our concerns in the development of strictly positive provability logic are similar in a certain sense. The reflection calculus RC is much simpler than its modal companion GLP yet expressive enough for its main proof-theoretic applications. It was outlined in [6] that RC allows one to define a natural system of ordinal notations up to $\varepsilon_0$ and serves as a convenient basis for a proof-theoretic analysis of Peano arithmetic in the style of [3] and [4]. This includes a consistency proof for Peano arithmetic based on transfinite induction up to $\varepsilon_0$, a characterization of

\(^1\)See http://www.ihtsdo.org/snomed-ct.
its $\Pi^0_n$-consequences in terms of iterated reflection principles, a slowly terminating term rewriting system \cite{16}, and a combinatorial independence result \cite{5}.

An axiomatization of RC (as an equational calculus) was given by Dashkov in his paper \cite{19}, which initiated the study of strictly positive provability logics. Dashkov proved two further important facts about RC, which contrast sharply with the corresponding properties of GLP. First, RC is complete with respect to a natural class of finite Kripke frames. Second, RC is decidable in polynomial time, whereas most of the standard modal logics are PSPACE-complete, and the same holds for the variable-free fragment of GLP \cite{40}.

Another advantage of going to a strictly positive language is exploited in the present paper. Strictly positive modal formulas allow for more general arithmetical interpretations than those of the standard modal logic language. In particular, propositional formulas can now be interpreted as arithmetical theories rather than individual sentences. (Note that the ‘negation’ of a theory would not be well defined.)

As the first meaningful example for this framework we have analysed an extension of RC by a modality representing the full arithmetical uniform reflection principle \cite{7}. The corresponding strictly positive logic, though arithmetically complete, complete with respect to a nice class of finite Kripke models, and polytime decidable, turned out to be not equivalent to a fragment of any standard normal modal logic.\footnote{This was not noted in \cite{7}, however, it follows from Theorem 3 in \cite{11} saying that a strictly positive logic is a fragment of a normal modal logic if and only if it is Kripke frame complete. Modulo some reformulations this result is, in fact, equivalent to Theorem 1 in \cite{37}.}

More generally, any monotone operator acting on the semilattice of arithmetical theories can be considered as a modality in strictly positive logic. One such operation is particularly attractive from the point of view of proof-theoretic applications, namely, the map associating with a theory $T$ its fragment $\Pi_{n+1}(T)$ axiomatized by all theorems of $T$ of arithmetical complexity $\Pi^0_{n+1}$. Since the $\Pi^0_{n+1}$-conservativity relation of $T$ over $S$ can be expressed by $S \vDash \Pi_{n+1}(T)$, we call such operators $\Pi^0_{n+1}$-conservativity operators.

This relates our study to the fruitful tradition of research on conservativity and interpretability logics (for example, see \cite{45}, \cite{46}, \cite{21}, \cite{25}–\cite{27}, and \cite{32}). Our framework happens to be both weaker and stronger than the traditional one: in our system we are able to express the conservativity relations for each class $\Pi^0_{n+1}$ and are able to relate not only sentences but theories. However, in this framework the negation is lacking and the conservativity is not a binary modality and cannot be iterated. Yet we believe that the strictly positive language is both simpler and better tuned to the needs of proof-theoretic analysis of formal systems of arithmetic.

We introduce a strictly positive system $RC^\nabla$ with modalities $\&_n$ representing uniform reflection principles of arithmetical complexity $\Sigma_n$, and $\triangledown_n$ representing $\Pi^0_{n+1}$-conservativity operators. We provide an adequate semantics of $RC^\nabla$ in terms of the semilattice $G_{EA}$ of (numerated) arithmetical recursively enumerable theories extending elementary arithmetic $EA$. Further, we introduce transfinite iterations of monotone semi-idempotent operators acting on $G_{EA}$ along elementary well-orderings, somewhat generalizing the notion of a Turing–Feferman recursive progression of axiomatic systems but mainly following the same development as
in [5]. Our first result shows that \( \alpha \)-iterations of modalities \( \Diamond n \), for each \( n < \omega \) and all ordinals \( \alpha < \varepsilon_0 \), are expressible in the algebra \( \mathcal{G}_{EA} \). A variable-free strictly positive logic where such iterations are explicitly present in the language was introduced by Hermo Reyes and Joosten [29] and is thereby contained in \( RC^\nabla \). However, possible generalisations of their system to larger ordinal notation systems would be beyond the scope of \( RC^\nabla \).³

Then we turn to a purely syntactic study of the variable-free fragment of the system \( RC^\nabla \) and provide unique normal forms for its formulas. A corollary is that the variable-free fragment of \( RC^\nabla \) is decidable and arithmetically complete.

While the normal forms for variable-free formulas of \( RC \) correspond in a unique way to ordinals below \( \varepsilon_0 \), those of \( RC^\nabla \) are more general. It turns out that they are related in a canonical way to the collections of proof-theoretic \( \Pi^0_{n+1} \)-ordinals of (bounded) arithmetical theories for each complexity level \( \Pi^0_{n+1} \). The notion of \( \Pi^0_{n+1} \)-ordinal of a theory characterizes the strength of \( \Pi^0_{n+1} \)-sentences provable in a given theory [5].

Studying the collections of proof-theoretic ordinals corresponding to several levels of logical complexity as single objects seems to be a rather recent and interesting development. Such collections appeared for the first time in the paper [36] of Joosten under the name Turing–Taylor expansions. He established a one-to-one correspondence between such collections (for a certain class of theories) and the points of the Ignatiev universal model for the variable-free fragment of GLP. We call such collections conservativity spectra of arithmetical theories. Our results show that \( RC^\nabla \) provides a way to syntactically represent and conveniently handle such conservativity spectra.

The third part of our paper provides an algebraic model \( I \) for the variable-free fragment of \( RC^\nabla \). This model is obtained in a canonical way on the basis of the Ignatiev model. Our main theorem states the isomorphism of several representations of \( I \): the Lindenbaum–Tarski algebra of the variable-free fragment of \( RC^\nabla \); a constructive representation in terms of sequences of ordinals below \( \varepsilon_0 \); a representation in terms of the semilattice of bounded \( RC \)-theories and as the algebra of cones of the Ignatiev model. In § 10 we consider its dual relational structure \( I^* \), which is universal for the variable-free fragment of \( RC^\nabla \). We give a constructive characterization of this large Kripke frame in terms of sequences of ordinals.

Parts of this paper have appeared previously in the conference proceedings [9] and [12], though they have undergone a thorough revision here. Thanks are due to Albert Visser for suggesting many improvements, including Lemma 2.2, as well as to Ilya Shapirovsky, Joost Joosten, and Evgeny Kolmakov for comments and corrections.

2. The lattice of arithmetical theories

We define the intended arithmetical interpretation of the strictly positive modal language. Propositional variables (and strictly positive formulas) will now denote possibly infinite theories rather than individual sentences. We deal with recursively enumerable theories formulated in the language of elementary arithmetic EA and

³In the latest version of their paper Hermo Reyes and Joosten did in fact extend their setup to arbitrary ordinal notation systems.
containing the axioms of EA. The theory EA, also known as $I\Delta_0(\exp)$ or EFA, is formulated in the language of Peano arithmetic enriched by a symbol for exponentiation ($2^\bar{x}$). In addition to the standard quantifier-free defining axioms for all the symbols of the language, it contains the induction scheme for bounded formulas (see [28] and [4]). Bounded formulas in the language of EA are called elementary formulas, and the class of all such formulas is usually denoted by $\Delta_0(\exp)$.

To avoid well-known problems with the representation of theories in arithmetic, we assume that each theory $S$ comes equipped with an elementary numeration, that is, a bounded formula $\sigma(x)$ in the language of EA and defining the set of axioms of $S$ in the standard model of arithmetic $\mathbb{N}$.

Given such a $\sigma$, we have a standard arithmetical $\Sigma_1^0$-formula $\Box_\sigma(x)$ expressing the provability of the formula with G"{o}del number $x$ in $S$ (see [22]). We often write $\Box_\sigma\varphi$ for $\Box_\sigma(\check{\varphi})$. The expression $\bar{n}$ denotes the numeral $0''\ldots'' (n \text{ times})$. If $\varphi(v)$ contains a parameter $v$, then $\Box_\sigma\varphi(\bar{x})$ denotes a formula (with parameter $x$) expressing the provability of the sentence $\varphi(\bar{x}/v)$ in $S$.

Given two numerations $\sigma$ and $\tau$, we write $\sigma \leq_{\text{EA}} \tau$ if
\[ \text{EA} \vdash \forall x (\Box_\tau(x) \rightarrow \Box_\sigma(x)). \]

We will only consider the numerations $\sigma$ such that $\sigma \leq_{\text{EA}} \sigma_{\text{EA}}$, where $\sigma_{\text{EA}}$ is some standard numeration of EA. We call such numerated theories G"{o}delian extensions of EA.

The relation $\leq_{\text{EA}}$ defines a natural preordering on the set $\mathfrak{G}_{\text{EA}}$ of G"{o}delian extensions of EA. Let $\mathfrak{G}_{\text{EA}}$ denote the quotient by the associated equivalence relation $=_{\text{EA}}$, where by definition $\sigma =_{\text{EA}} \tau$ if and only if both $\sigma \leq_{\text{EA}} \tau$ and $\tau \leq_{\text{EA}} \sigma$. The set $\mathfrak{G}_{\text{EA}}$ is a lattice with $\wedge_{\text{EA}}$ corresponding to the union of theories and $\vee_{\text{EA}}$ to their intersection. These operations are defined on elementary numerations as follows:

\[ \sigma \wedge_{\text{EA}} \tau := \sigma(x) \lor \tau(x), \]
\[ \sigma \vee_{\text{EA}} \tau := \exists x_1, x_2 \leq x (\sigma(x_1) \land \tau(x_2) \land x = \text{disj}(x_1, x_2)), \]

where $\text{disj}(x_1, x_2)$ is an elementary term computing the G"{o}del number of the disjunction of the formulas given by G"{o}del numbers $x_1$ and $x_2$.

Our main concern here is the structure of the lower semilattice $(\mathfrak{G}_{\text{EA}}, \wedge_{\text{EA}}, 1_{\text{EA}})$ with top. Note that the top element $1_{\text{EA}}$ corresponds to (the equivalence class of) EA, whereas the bottom $0_{\text{EA}}$ is the class of all inconsistent G"{o}delian extensions of EA.

An operator $R: \mathfrak{G}_{\text{EA}} \rightarrow \mathfrak{G}_{\text{EA}}$ is said to be extensional if $\sigma =_{\text{EA}} \tau$ implies that $R(\sigma) =_{\text{EA}} R(\tau)$. Similarly, $R$ is said to be monotone if $\sigma \leq_{\text{EA}} \tau$ implies that $R(\sigma) \leq_{\text{EA}} R(\tau)$. Clearly, each monotone operator is extensional and each extensional operator correctly acts on the quotient lattice $\mathfrak{G}_{\text{EA}}$. An operator $R$ is said to be semi-idempotent if $R(R(\sigma)) \leq_{\text{EA}} R(\sigma)$, and $R$ is a closure operator if it is monotone, semi-idempotent, and, in addition, $\sigma \leq_{\text{EA}} R(\sigma)$. Operators considered in this paper will usually be at least monotone and semi-idempotent.

Meaningful monotone operators abound in arithmetic. Typical examples are the uniform $\Sigma_n$-reflection principles $R_n(\sigma)$ associating with $\sigma$ the extension of EA by the scheme
\[ \{ \forall x (\Box_\sigma \varphi(\bar{x}) \rightarrow \varphi(x)): \varphi \in \Pi_{n+1} \} \]
taken with its natural elementary numeration that we denote by $x \in R_n(\sigma)$. It is known that the theory $R_n(\sigma)$ is finitely axiomatizable. Moreover, $R_n(\sigma)$ is equivalent to Gödel’s consistency assertion $\text{Con}(\sigma)$ for $\sigma$.

Let $S$ be a Gödelian extension of EA numerated by $\sigma$, and let $x \in \Pi^0_n$ denote an elementary formula expressing that $x$ is the Gödel number of a $\Pi^0_n$-sentence. The following basic lemma will be useful in what follows.

**Lemma 2.1.** (i) *If $S$ extends EA by $\Pi^0_{n+1}$-axioms, then $R_n(\sigma)$ contains $S$.*

(ii) *If $EA \vdash \forall x(\sigma(x) \rightarrow x \in \Pi^0_{n+1})$, then $R_n(\sigma) \leq_{EA} \sigma$.***

**Proof.** The second claim is a straightforward formalization of the first. To prove (i) assume that $S \vdash \varphi$. Then there is a $\pi \in \Pi^0_{n+1}$ such that $EA \vdash \pi \rightarrow \varphi$ and $S \vdash \pi$. We have $EA \vdash \Box_\sigma\pi$ by $\Sigma_1$-completeness. Then $R_n(\sigma) \vdash \pi \vdash \varphi$. □

In this paper we study another series of monotone operators. Given a theory $S$ numerated by $\sigma$, let $\Pi_n(S)$ denote the extension of EA by all theorems of $S$ of complexity $\Pi^0_n$. The set $\Pi_n(S)$ is recursively enumerable but in general not elementary recursive. In order to comply with our definitions we apply a form of Craig’s trick that yields an elementary axiomatization of $\Pi_n(S)$.

Let $\Pi_n(\sigma)$ denote the elementary formula

$$\exists y, p \leq x (\text{Prf}_\sigma(y, p) \land y \in \Pi^0_n \land x = \text{disj}(y, "p \neq \bar{p}"))$$

and the theory numerated by this formula over EA. Here $\text{Prf}_\sigma(y, p)$ is an elementary formula expressing that $p$ is the Gödel number of a proof of $y$, so that $\exists p \text{Prf}_\sigma(y, p)$ is $\Box_\sigma(y)$. Then it is easy to see that the theory $\Pi_n(\sigma)$ is (externally) deductively equivalent to $\Pi_n(S)$.

We will implicitly rely on the following characterization.

**Lemma 2.2.** *It is provable in EA that*

$$\forall x \left( \Box_{\Pi_n(\sigma)}(x) \iff \exists \pi \in \Pi^0_n (\Box_\sigma(\pi) \land \Box_{\text{EA}}(\pi \rightarrow x)) \right).$$

**Proof.** The implication from right to left is easy; we sketch a proof of ($\rightarrow$). We reason within EA. Suppose that $p$ is a $\Pi_n(\sigma)$-proof of $x$. It is an EA-proof of $x$ from some assumptions $\pi'_1, \pi'_2, \ldots, \pi'_k$ such that each $\pi'_i$ has the form $\pi_i \lor \overline{\pi_i}$, where $\pi_i \in \Pi^0_n$ and $\text{Prf}_\sigma(\pi_i, p_i)$. Since $p$ contains witnesses for all the proofs $p_i$, from $p$ one can construct in an elementary way a sentence $\pi \in \Pi^0_n$ equivalent to $\pi_1 \land \cdots \land \pi_k$ together with its $\sigma$-proof and an EA-proof of $\pi \rightarrow x$, using a formalization of the deduction theorem in EA. A verification that it is indeed the required proof goes by an elementary induction on the length of $p$. □

Using Lemma 2.2, one can naturally infer that all the operators $R_n$ and $\Pi_n$ are monotone and semi-idempotent, and moreover, $\Pi_n$ is a closure operator. It is easy to see that EA can be replaced in all the previous considerations by any of its Gödelian extensions $T$. The main source of interest to us in this paper will be the structure of the semilattice with operators

$$\langle \emptyset_T; \land_T, 1_T, \{R_n, \Pi_{n+1}: n < \omega \} \rangle$$

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\(^4\) Over $\text{EA} + \text{BS}_1$ one can work with a natural recursively enumerable axiomatization of $\Pi_n(S)$.
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and its subsemilattice with operators $\mathfrak{S}^0_T$ generated by $1_T$. We call the former the RC$^\nabla$-algebra of Gödelian extensions of $T$. The term “RC$^\nabla$-algebra” will be explained below.

3. Strictly positive logics and reflection calculi

We refer the reader to the note [11] for a short introduction to strictly positive logics that is sufficient for the present paper, and to [38] for more information from the general algebraic perspective. For background on modal logic and provability logic we refer to the books [18], [43], and [17].

3.1. Normal strictly positive logics. Consider a modal language $\mathcal{L}_\Sigma$ with propositional variables $p, q, \ldots$, a constant $\top$, a conjunction $\land$, and a possibly infinite set of symbols $\Sigma = \{a_i : i \in J\}$ understood as ‘diamond’ modalities. The family $\Sigma$ is called the signature of the language $\mathcal{L}_\Sigma$. Strictly positive formulas (or simply formulas) are built up by the grammar:

$$A ::= p \mid \top \mid (A \land A) \mid aA, \quad \text{where } a \in \Sigma.$$  

Sequents are expressions of the form $A \vdash B$, where $A$ and $B$ are strictly positive formulas.

The basic sequent-style system, denoted by $K^+$, is given by the following axioms and rules:

1. $A \vdash A$; $A \vdash \top$; if $A \vdash B$ and $B \vdash C$, then $A \vdash C$;
2. $A \land B \vdash A$; $A \land B \vdash B$; if $A \vdash B$ and $A \vdash C$, then $A \vdash B \land C$;
3. $\text{if } A \vdash B, \text{ then } aA \vdash aB$ for each $a \in \Sigma$.

It is well known that $K^+$ axiomatizes the strictly positive fragment of a polymodal version of basic modal logic $K$ (see [11] and [38]). All our systems will also contain the following principle corresponding to the transitivity axiom in modal logic:

4. $aaA \vdash aA$.

The extension of $K^+$ by this axiom will be denoted by $K4^+$ [19].

Let $C[A/p]$ denote the result of replacing all occurrences of the variable $p$ in $C$ by $A$. A set $L$ of sequents is called a normal strictly positive logic if it contains the axioms and is closed under the rules of $K^+$ and under the following substitution rule:

$$\text{if } (A \vdash B) \in L \text{ then } (A[C/p] \vdash B[C/p]) \in L.$$  

We will only consider normal strictly positive logics below. We write $A \vdash_L B$ for the statement that $A \vdash B$ is provable in $L$ (or belongs to $L$). $A =_L B$ means that $A \vdash_L B$ and $B \vdash_L A$.

Any normal strictly positive logic $L$ satisfies the following simple positive replacement lemma, which we leave without proof.

**Lemma 3.1.** Suppose that $A \vdash_L B$; then $C[A/p] \vdash_L C[B/p]$ for any formula $C$. 

3.2. Algebraic semantics. Algebraic semantics for normal strictly positive logics is given by semilattices with monotone operators (SLO), that is, by structures of the form

\[ \mathcal{M} = (M; \wedge^M, 1^M, \{a^M : a \in \Sigma\}) , \]

where \((M, \wedge^M, 1^M)\) is a semilattice with top, and each \(a^M : M \to M\) is a monotone operator on \(M\): \(x \leq y\) implies that \(a^M(x) \leq a^M(y)\) for all \(x, y \in M\). Every strictly positive formula \(A\) in \(L_\Sigma\) represents a term \(A^M\) in \(M\). We say that \(A \vdash B\) holds in \(M\) (or that \(M\) satisfies \(A \vdash B\)) if

\[ M \models \forall \vec{x} A^M(\vec{x}) \leq B^M(\vec{x}) . \]

It is easy to see that \(A \vdash_{K^+} B\) if and only if \(A \vdash B\) holds in each SLO \(M\). The SLOs satisfying all the theorems of a normal strictly positive logic \(L\) are called \(L\)-algebras.

For a given normal strictly positive logic \(L\) in a signature \(\Sigma\) and an alphabet of variables \(V\), its Lindenbaum–Tarski algebra is an SLO \(L_V^L\) whose domain consists of the equivalence classes of formulas of the language of \(L\) modulo \(=_L\). Let \([A]_L\) denote the equivalence class of \(A\). The operations are defined in the standard way as follows:

\[ [A]_L \wedge [B]_L := [A \wedge B]_L \quad \text{and} \quad a^L([A]_L) := [aA]_L \quad \text{for each} \ a \in \Sigma . \]

It is well known that \(A \vdash_L B\) if and only if \(A \vdash B\) holds in \(L_V^L\). Hence any normal strictly positive logic \(L\) is complete with respect to its algebraic semantics, that is, with respect to the class of all \(L\)-algebras.

The algebra \(L_V^L\) is also called the free \(V\)-generated \(L\)-algebra. In this paper we will be particularly interested in the algebras \(L_V^L\) where \(V\) is empty. In this case we denote the algebra \(L_V^L\) by \(L_0^L\).

3.3. The system RC. The Reflection calculus RC is a normal strictly positive logic formulated in the signature \(\{\diamond_n : n \in \omega\}\). It is obtained by adjoining to the axioms and rules of \(K4^+\) (stated for each \(\diamond_n\)) the following principles:

5. \(\diamond_n A \vdash \diamond_m A\) for all \(n > m\);
6. \(\diamond_n A \wedge \diamond_m B \vdash \diamond_n(A \wedge \diamond_m B)\) for all \(n > m\).

We note that RC proves the following polytransitivity principles:

\[ \diamond_n \diamond_m A \vdash \diamond_m A \quad \text{and} \quad \diamond_m \diamond_n A \vdash \diamond_m A \quad \text{for each} \ m \leq n . \]

Also, the converse of Axiom 6 is provable in RC, so that in fact we have

\[ \diamond_n(A \wedge \diamond_m B) \models_{RC} \diamond_n A \wedge \diamond_m B . \quad (1) \]

The system RC was introduced in an equational logic format by Dashkov [19]; the present formulation is from [6]. Dashkov showed that RC axiomatizes the set of all sequents \(A \vdash B\) such that the implication \(A \to B\) is provable in the polymodal logic GLP. Moreover, unlike GLP itself, RC is polytime decidable (whereas GLP is \(\text{PSpace}\)-complete [42]) and enjoys the finite frame property (whereas GLP is Kripke incomplete [17]).
We recall a correspondence between variable-free RC-formulas and ordinals [3]. Let $F$ denote the set of all variable-free RC-formulas, and let $F_n$ denote its restriction to the signature $\{\Diamond_i \colon i \geq n\}$, so that $F = F_0$. For each $n \in \omega$ we define binary relations $<_n$ on $F$ by

$$A <_n B \overset{\text{def}}{\iff} B \vdash_{RC} \Diamond_n A.$$ 

Obviously, $<_n$ is a transitive relation invariantly defined on the equivalence classes with respect to provable equivalence in RC, denoted by $=_{RC})$. Since RC is polytime decidable, so are both $=_{RC}$ and all of the $<_n$.

An RC-formula without variables and $\wedge$ is called a word (or a worm in some treatments). In fact, any such formula is syntactically a finite sequence of letters $\Diamond_i$ (followed by $\top$). If $A$ and $B$ are words, then $AB$ will denote $A[\top/B]$, that is, the word corresponding to the concatenation of these sequences. $A \overset{\wedge}{=} B$ denotes the graphical identity of formulas (words).

The set of all words will be denoted by $W$, and $W_n$ will denote its restriction to the signature $\{\Diamond_i \colon i \geq n\}$. The following facts are from [3] and [6]:

- every formula $A \in F_n$ is RC-equivalent to a word in $W_n$;
- $(W_n/\!=_{RC}<_n)$ is isomorphic to $(\varepsilon_0, <)$.

Here $\varepsilon_0$ is the first ordinal $\alpha$ such that $\omega^\alpha = \alpha$. Thus, the set $W_n/\!=_{RC}$ is well ordered by the relation $<_n$. The isomorphism can be established by an onto and order-preserving function $o_n \colon W_n \to \varepsilon_0$ such that, for all $A, B \in W_n$,

$$A \equiv_{RC} B \iff o_n(A) = o_n(B).$$

Then $o_n(A)$ is the order type of $\{B \in W_n \colon B <_n A\}/\!=_{RC}$.

The function $o(A) := o_0(A)$ can be inductively calculated as follows: if $A \overset{\wedge}{=} \Diamond_0^k \top$, then $o(A) = k$. If $A \overset{\wedge}{=} A_1 \Diamond_0 A_2 \Diamond_0 \cdots \Diamond_0 A_n$, where all $A_i \in W_1$ and not all of them are empty, then

$$o(A) = \omega^{o(A_1)} + \cdots + \omega^{o(A_n)}.$$ 

Here $B_-$ is obtained from $B \in W_1$ by replacing every $\Diamond_{m+1}$ by $\Diamond_{m}$. For $n > 0$ and $A \in W_n$ we let $o_n(A) = o_{n-1}(A_-)$.

### 3.4. The system RC$^\nabla$.

**Definition 3.2.** The signature of RC$^\nabla$ consists of the modalities $\Diamond_n$ and $\nabla_n$ for each $n < \omega$. The system RC$^\nabla$ is a normal strictly positive logic given by the following axioms and rules for all $m, n < \omega$:

1. RC for $\Diamond_n$, and RC for $\nabla_n$;
2. $A \vdash \nabla_n A$;
3. $\Diamond_n A \vdash \nabla_n A$;
4. $\Diamond_m \nabla_n A \vdash \Diamond_m A$ if $m \leq n$;
5. $\nabla_n \Diamond_m A \vdash \Diamond_m A$ if $m \leq n$.

As a basic syntactic fact about RC$^\nabla$, we mention the following useful lemma. For brevity we often write $\equiv$ instead of $=_{RC^\nabla}$ and $\vdash$ instead of $\vdash_{RC^\nabla}$.

**Lemma 3.3.** The following are theorems of RC$^\nabla$ for all $m < n$:

(i) $\Diamond_n (A \wedge \Diamond_m B) \equiv \Diamond_n A \wedge \Diamond_m B$;
(ii) $\nabla_n (A \wedge \Diamond_m B) \equiv \nabla_n A \wedge \Diamond_m B$. 
Proof. (i) Part ($\vdash$) follows from $\Diamond n \nabla_m B \vdash \Diamond_m B$. Part ($\vdash$) follows from $\Diamond_n A \wedge \Diamond_m B \vdash \Diamond_n (A \wedge \Diamond_m B)$ using positive replacement.

(ii) Part ($\vdash$) follows from $\nabla_n \Diamond_m B \vdash \Diamond_m B$. Part ($\vdash$) follows from $\nabla_n A \wedge \nabla_m \Diamond_m B \vdash \nabla_n (A \wedge \nabla_m \Diamond_m B)$ using Axiom 6 for $\nabla$ modalities, the fact that $\Diamond_m B = \nabla_m \Diamond_m B$, and positive replacement. □

A formula $A$ is said to be ordered if no modality with a smaller index $i$ (be it $\Diamond_i$ or $\nabla_i$) occurs in $A$ within the scope of a modality with a larger index $j > i$.

**Lemma 3.4.** Every formula $A$ of $\text{RC}^\nabla$ is equivalent to an ordered formula.

Proof. Apply equation (1) of RC for $\Diamond$ and for $\nabla$ modalities, and the identities of Lemma 3.3 from left to right, until the rules are not applicable to any of the subformulas of $A$. □

The intended arithmetical interpretation of $\text{RC}^\nabla$ maps strictly positive formulas to Gödelian theories in $\mathfrak{G}_T$ in such a way that $\top$ corresponds to $T$, $\wedge$ corresponds to the union of theories, $\Diamond_n$ corresponds to $R_n$, and $\nabla_n$ corresponds to $\Pi_{n+1}$, for each $n \in \omega$.

**Definition 3.5.** An arithmetical interpretation in $\mathfrak{G}_T$ is a map $*$ from the set of all formulas in the language of $\text{RC}^\nabla$ to $\mathfrak{G}_T$ that satisfies the following conditions for all $n \in \omega$:

- $\top^* = T$; $(A \wedge B)^* = (A^* \wedge_T B^*)$;
- $(\Diamond_n A)^* = R_n(A^*)$; $(\nabla_n A)^* = \Pi_{n+1}(A^*)$.

The following result shows, as expected, that every theorem of $\text{RC}^\nabla$ represents an identity of the structure $(\mathfrak{G}_T, \wedge_T, 1_T, \{R_n, \Pi_{n+1} : n < \omega\})$.

**Theorem 3.6.** For any formulas $A, B$ of $\text{RC}^\nabla$, if $A \vdash_{\text{RC}^\nabla} B$, then $A^* \leq_T B^*$ for all arithmetical interpretations $*$ in $\mathfrak{G}_T$.

Proof. The proof is routine by induction on the length of the derivation. For the axioms and rules of RC concerning the $\Diamond$-fragment the claim was carefully verified in [7]. The RC-axioms concerning the $\nabla$-fragment are obvious except for Axiom 6, that is, the principle

$$\nabla_n A \wedge \nabla_m B \vdash \nabla_n (A \wedge \nabla_m B).$$

Consider any arithmetical interpretation $*$, and let

$$S = A^* \quad \text{and} \quad U = B^*$$

be the corresponding Gödelian theories (with the associated numerations $\sigma$ and $\tau$, respectively). We rely on Lemma 2.2. The principle (2) is the formalization in EA of the following assertion: for any sentence $\pi \in \Pi^0_{n+1}$, if $S \cup \Pi_{m+1}(U) \vdash \pi$, then $\Pi_{n+1}(S) \cup \Pi_{m+1}(U) \vdash \pi$. Reasoning in EA, we consider a sentence $\varphi \in \Pi_{m+1}(U)$ such that $S, \varphi \vdash \pi$. Then $S \vdash \varphi \rightarrow \pi$ and, since $\varphi \rightarrow \pi$ is logically equivalent to a $\Pi^0_{n+1}$-sentence, we get that $\Pi_{n+1}(S) \vdash \varphi \rightarrow \pi$. Thus, $\Pi_{n+1}(S) \cup \Pi_{m+1}(U) \vdash \pi$.

Concerning the remaining axioms of $\text{RC}^\nabla$ we observe that Axiom 2 holds since the theory $\Pi_{n+1}(S)$ is (provably) contained in $S$. Axiom 3 is Lemma 2.1,(ii).

Axiom 4: assume that $R_m(\Pi_{n+1}(\sigma))$. In order to prove $R_m(\sigma)$ let $\varphi \in \Pi_{m+1}$ and $\Box \sigma \varphi$. Then clearly $\Box_{\Pi_{n+1}(\sigma)} \varphi$ since $m \leq n$, and hence $\varphi$, by $R_m(\Pi_{n+1}(\sigma))$. 

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Axiom 5 formalizes the fact that $R_m(\sigma)$ is an extension of $T$ by a $\Pi_{m+1}$-sentence.

Theorem 3.6, together with Gödel’s second incompleteness theorem, has as a corollary the following property of the logic $RC^\nabla$.

**Corollary 3.7.** For all $RC^\nabla$ formulas $A$,

$$A \nabla_{RC^\nabla} A.$$

**Proof.** Assume otherwise; then interpreting $RC^\nabla$ in $\mathcal{G}_{EA}$ gives us that $A^* \leq_{EA} R_n(A^*)$ by Theorem 3.6. Hence, by Gödel’s theorem the theory $A^*$ is inconsistent. This contradicts the soundness of $EA$. □

A similar fact is known for GLP and can also be proved by purely modal logic means (see [15] and [14]). An elementary argument for $RC^\nabla$ is given in Appendix A.

Fernández-Duque gives another proof for a generalization of $RC^\nabla$ with transfinitely many modalities. We will make use of Corollary 3.7 (for $RC^\nabla$) in the normal form theorems below. Whereas a reference to the given proof of Corollary 3.7 presupposes at least the soundness of $EA$, the elementary Kripke model argument for $RC^\nabla$ is formalizable in $EA$.

**Conjecture 1.** $RC^\nabla$ is arithmetically complete, that is, the converse of Theorem 3.6 also holds, provided that $T$ is arithmetically sound.

### 3.5. Kripke incompleteness of $RC^\nabla$

Kripke frames and models are understood in this paper in the usual sense. A **Kripke frame** $\mathcal{W}$ for the language of $RC^\nabla$ consists of a non-empty set $W$ equipped with a family of binary relations $\{R_n, S_n : n \in \omega\}$.

A **Kripke model** is a Kripke frame $\mathcal{W}$ together with a valuation $v : W \times \operatorname{Var} \to \{0, 1\}$ assigning a truth value to each propositional variable at every node of $\mathcal{W}$. As usual, we write $\mathcal{W}, x \models A$ to denote that a formula $A$ is true at a node $x$ of a model $\mathcal{W}$. This relation is inductively defined as follows:

- $\mathcal{W}, x \models p \iff v(x, p) = 1$ for each $p \in \operatorname{Var};$
- $\mathcal{W}, x \models \top;$
- $\mathcal{W}, x \models A \land B \iff (\mathcal{W}, x \models A \text{ and } \mathcal{W}, x \models B);$  
- $\mathcal{W}, x \models \bigcirc_n A \iff \exists y (xR_ny \text{ and } \mathcal{W}, y \models A);$  
- $\mathcal{W}, x \models \bigtriangledown_n A \iff \exists y (xS_ny \text{ and } \mathcal{W}, y \models A).$

A formula $A$ is **valid in a Kripke frame** $\mathcal{W}$ if $\mathcal{W}, x \models A$, for each $x \in W$ and each valuation $v$ on $\mathcal{W}$. The following lemma is standard and easy.

**Lemma 3.8.** A Kripke frame $\mathcal{W}$ validates all theorems of $RC^\nabla$ if and only if the following conditions hold for all $m, n < \omega$:

- (i) $R_n$ is transitive, $R_n \subseteq R_m$ if $m < n$, and $R_n^{-1}R_m \subseteq R_m$ if $m < n$;
- (ii) $S_n$ is transitive and reflexive, $S_n \subseteq S_m$ if $m < n$, and $S_n^{-1}S_m \subseteq S_m$ if $m < n$;
- (iii) $R_n \subseteq S_n$, $S_nR_m \subseteq R_m$, and $R_mS_n \subseteq R_m$ if $m \leq n$.

By the following proposition $RC^\nabla$ turns out to be incomplete with respect to its Kripke frames.
Proposition 3.9. The sequent
\[ \Diamond_1 A \land \nabla_0 B \vdash \Diamond_1 (A \land \nabla_0 B) \]  
(*)
is valid in every Kripke frame satisfying $\text{RC}^\nabla$. However, it is unprovable in $\text{RC}^\nabla$ 
(and arithmetically invalid).

Proof. 1) It is easy to see that the conditions $R_1 \subseteq S_1$ and $S_1^{-1}S_0 \subseteq S_0$ imply that $R_1^{-1}S_0 \subseteq S_0$. Therefore, (*) holds in each Kripke frame of $\text{RC}^\nabla$.

2) Take $\top$ as $A$ and $\Diamond_1 \top$ as $B$. The left-hand side is $\text{RC}^\nabla$-equivalent to $\Diamond_1 \top$. The right-hand side is equivalent to
\[ \Diamond_1 (\top \land \nabla_0 \Diamond_1 \top) =_{\text{RC}^\nabla} \Diamond_1 \top \land \nabla_0 \Diamond_1 \top, \]
by Lemma 3.3, (i). By Corollary 3.7, $\Diamond_1 \top \not\vdash_{\text{RC}^\nabla} \Diamond_0 \Diamond_1 \top$. Hence (*) is unprovable in $\text{RC}^\nabla$. \(\square\)

By Theorem 3 in [11] a normal strictly positive logic is a fragment of some normal modal logic if and only if it is Kripke complete. Hence we obtain the following.

Corollary 3.10. $\text{RC}^\nabla$ is not a strictly positive fragment of any normal modal logic.

4. The variable-free fragment of $\text{RC}^\nabla$

Let $\mathbb{F}_n^\nabla$ denote the set of all variable-free strictly positive formulas in the language of $\text{RC}^\nabla$ with the modalities \{\(\Diamond_i, \nabla_i: i \geq n\)\} only, and let $\mathbb{F}^\nabla$ denote $\mathbb{F}_0^\nabla$. We abbreviate $F \vdash_{\text{RC}^\nabla} \nabla_n G$ by $F \vdash \nabla_n G$ and $\nabla_n F =_{\text{RC}^\nabla} \nabla_n G$ by $F \equiv \nabla_n G$. First, we are going to establish the crucial result that every formula in $\mathbb{F}_n^\nabla$ is equivalent to a word in $\mathbb{W}_n$ modulo $\equiv_n$. From this fact we will infer a weak normal form theorem for the variable-free fragment of $\text{RC}^\nabla$. Second, we will obtain two different unique normal form theorems for the variable-free fragment by sharpening the weak normal forms.

4.1. Weak normal forms. We begin with a few auxiliary lemmas.

Lemma 4.1. (i) If $A \vdash \nabla_n B$ and $m < n$, then $A \land \Diamond_m C \vdash \nabla_n B \land \Diamond_m C$.

(ii) If $A \vdash \nabla_n B$ and $B \vdash \nabla_n C$, then $A \vdash \nabla_n C$.

(iii) If $A \vdash \nabla_n B$ and $B \vdash \Diamond_n C$, then $A \vdash \Diamond_n C$.

Proof. (i) $A \land \Diamond_m C \vdash \nabla_n B \land \Diamond_m C \vdash \nabla_n (B \land \Diamond_m C)$.

(ii) $A \vdash \nabla_n B \vdash \nabla_n \nabla_n C \vdash \nabla_n C$.

(iii) $A \vdash \nabla_n B \vdash \nabla_n \Diamond_n C \vdash \Diamond_n C$. \(\square\)

Lemma 4.2. (i) $\nabla_i A \land B = \nabla_i (\Diamond_i A \land B) \land B$.

(ii) $\nabla_i A \land B = \nabla_i (\nabla_i A \land B) \land B$.

Proof. In both (i) and (ii) the implication ($\vdash$) follows from the axiom $C \vdash \nabla_i C$. For
\(-\) we obtain $\nabla_i (\Diamond_i A \land B) \vdash \nabla_i \Diamond_i A = \Diamond_i A$ for (i), and similarly $\nabla_i (\nabla_i A \land B) \vdash \nabla_i \nabla_i A = \nabla_i A$ for (ii). \(\square\)

Lemma 4.3. The set of all formulas \{\(\Diamond_n F, \nabla_n G : F, G \in \mathbb{W}_n\)\} is linearly ordered by $\vdash_{\text{RC}^\nabla}$. 

Proof. For any \( F, G \in \mathbb{W}_n \) we know that \( F \vdash_{RC} \diamond_n G \), or \( G \vdash_{RC} \diamond_n F \), or \( F =_{RC} G \). In the first case we obtain provably in \( RC^\nabla \) that
\[
\diamond_n F \vdash \nabla_n F \vdash \diamond_n G.
\]
The second case is symmetrical. In the third case we obtain
\[
\diamond_n F = \diamond_n G \vdash \nabla_n F = \nabla_n G.
\]

Theorem 4.4. For each \( A \in \mathbb{F}^\nabla_n \) there is a word \( W \in \mathbb{W}_n \) such that \( A \equiv_n W \).

Proof. By Lemma 3.4 it is sufficient to prove the theorem for ordered formulas \( A \). The proof goes by induction on the length of the ordered \( A \). We can also assume that the minimal modality occurring in \( A \) is \( \diamond_n \) or \( \nabla_n \). (Otherwise, we prove it for the minimum \( m > n \) and infer that \( A \equiv_n W \) from \( A \equiv_m W \).) The base of induction is trivial; we consider the induction step.

Assume that the induction hypothesis holds for all formulas shorter than \( A \). Since \( A \) is ordered, \( A \) can be written in the form
\[
A = \diamond_n A_1 \wedge \cdots \wedge \diamond_n A_k \wedge \nabla_n B_1 \wedge \cdots \wedge \nabla_n B_l \wedge D,
\]
where \( D \in \mathbb{F}^\nabla_{n+1} \) and \( A_i, B_j \in \mathbb{F}^\nabla_n \). Since \( \diamond_n \) or \( \nabla_n \) must occur in \( A \), we know that \( D \) and each \( A_i \) and \( B_j \) are strictly shorter than \( A \). By the induction hypothesis and Lemma 4.3 we can delete from the conjunction all but one member of the form \( \diamond_n A_i, \nabla_n B_j \). Thus, \( A = D \wedge \diamond_n A' \) or \( A = D \wedge \nabla_n B' \) for some words \( A', B' \in \mathbb{W}_n \).

Now we apply the induction hypothesis to \( D \) and obtain a word \( V \in \mathbb{W}_{n+1} \) such that \( V \equiv_{n+1} D \). It follows that
\[
D \wedge \diamond_n A' \equiv_{n+1} V \wedge \diamond_n A' \quad \text{and} \quad D \wedge \nabla_n B' \equiv_{n+1} V \wedge \nabla_n B'
\]
by Lemma 4.1. Hence, it is sufficient to prove that \( V \wedge \diamond_n A' \equiv_{n} W \) for some \( W \in \mathbb{W}_n \), and similarly, \( V \wedge \nabla_n B' \equiv_{n} W \) for some \( W \in \mathbb{W}_n \).

In the first case we actually have \( V \wedge \diamond_n A' =_{RC} W \) for some word \( W \), which immediately yields the claim.

In the second case we write \( B' = B_1 \diamond_n B_2 \), where \( B_1 \in \mathbb{W}_{n+1} \). There are three cases to consider:

(a) \( B_1 \vdash \diamond_{n+1} V \), \quad (b) \( V \vdash \diamond_n B_1 \), \quad and \quad (c) \( V = B_1 \).

In case (c) by Lemma 4.2 we obtain:
\[
V \wedge \nabla_n B_1 \diamond_n B_2 = V \wedge \nabla_n (V \wedge \diamond_n B_2) = V \wedge \diamond_n B_2 = V \diamond_n B_2.
\]
In case (a) we show that \( \nabla_n (V \wedge \nabla_n B') = \nabla_n B' \). First,
\[
B' \vdash \diamond_{n+1} V \wedge \nabla_n B' \vdash \nabla_{n+1} V \wedge \nabla_n B' = \nabla_{n+1} (V \wedge \nabla_n B').
\]
Hence \( \nabla_n B' \vdash \nabla_n \nabla_{n+1} (V \wedge \nabla_n B') = \nabla_n (V \wedge \nabla_n B') \). On the other hand,
\[
\nabla_n (V \wedge \nabla_n B') \vdash \nabla_n \nabla_n B' \vdash \nabla_n B'.
\]
In case (b) we show that \( \nabla_n(V \land \nabla_n B') = \nabla_n(V \land \Diamond_n B_2) \), so that one can infer that \( \nabla_n(V \land \nabla_n B') = \nabla_n V \land \nabla_n B_2 \). On the one hand, we have
\[
\nabla_n B' = \nabla_n(B_1 \land \Diamond_n B_2) \vdash \nabla_n \Diamond_n B_2 = \Diamond_n B_2,
\]
which implies that \( \nabla_n(V \land \nabla_n B') \vdash \nabla_n(V \land \Diamond_n B_2) \). On the other hand,
\[
V \land \Diamond_n B_2 = V \land \Diamond_{n+1} B_1 \land \Diamond_n B_2 = V \land \Diamond_{n+1}(B_1 \land \Diamond_n B_2) = V \land \Diamond_{n+1} B' \vdash V \land \nabla_n B'.
\]
Hence \( \nabla_n(V \land \Diamond_n B_2) \vdash \nabla_n(V \land \nabla_n B'). \) □

From Theorem 4.4 we obtain the following strengthening of Lemma 4.3.

**Corollary 4.5.** The set of all formulas \( \{ \Diamond_n F, \nabla_n G : F, G \in \mathbb{F}_n^\nabla \} \) is linearly ordered by \( \vdash_{RC^\nabla} \).

**Corollary 4.6.** For all formulas \( A, B \in \mathbb{F}_n^\nabla \), \( A \vdash \Diamond_n B \), or \( B \vdash \Diamond_n A \), or \( A \equiv_n B \).

**Proof.** Consider the words \( A_1 \equiv_n A \) and \( B_1 \equiv_n B \). By the linearity property for words, \( A_1 \vdash \Diamond_n B_1 \), or \( B_1 \vdash \Diamond_n A_1 \), or \( A_1 = B_1 \). In the first case we obtain \( A \vdash \nabla_n A_1 \vdash \nabla_n \Diamond_n B_1 \vdash \Diamond_n B_1 \vdash \Diamond_n \nabla_n B \vdash \Diamond_n B \). The second case is symmetrical, and the third case immediately implies that \( A \equiv_n B \). □

**Corollary 4.7.** For all \( A, B \in \mathbb{F}_n^\nabla \), \( \Diamond_n A \vdash \Diamond_n B \) if and only if \( A \vdash \nabla_n B \).

**Proof.** Assume that \( \Diamond_n A \vdash \Diamond_n B \). By Corollary 4.6, \( A \vdash \Diamond_n B \), or \( B \vdash \Diamond_n A \), or \( A \equiv_n B \). In the first and third cases we immediately have \( A \vdash \nabla_n B \). In the second case we obtain \( \Diamond_n A \vdash \Diamond_n B \vdash \Diamond_n \Diamond_n A \), contradicting Corollary 3.7.

In the opposite direction, if \( A \vdash \nabla_n B \), then \( \Diamond_n A \vdash \Diamond_n \nabla_n B \vdash \Diamond_n B \). □

**Theorem 4.8** (weak normal forms). Every formula \( A \in \mathbb{F}_n^\nabla \) is equivalent in \( RC^\nabla \) to a formula of the form
\[
\nabla_n A_n \land \nabla_{n+1} A_{n+1} \land \cdots \land \nabla_{n+k} A_{n+k}
\]
for some \( k \), where \( A_i \in \mathbb{W}_i \) for all \( i = n, \ldots, n+k \).

**Proof.** Induction on the build-up of \( A \in \mathbb{F}_n^\nabla \). We consider the following cases.

1) \( A \equiv B \lor C \). The induction hypothesis is applicable to \( B \) and \( C \), so it is sufficient to prove that for any \( B_i, C_i \in \mathbb{W}_i \) there is a word \( A_i \in \mathbb{W}_i \) such that
\[
\nabla_i B_i \land \nabla_i C_i = \nabla_i A_i.
\]

By Lemma 4.3 we can take one of \( B_i \) or \( C_i \) as \( A_i \).

2) \( A \equiv \nabla_i B \) for some \( i \geq n \). Then we obtain
\[
\nabla_i B = \nabla_i(\nabla_n B_n \land \nabla_{n+1} B_{n+1} \land \cdots \land \nabla_{n+k} B_{n+k})
\]
\[
= \nabla_n B_n \land \cdots \land \nabla_{i-1} B_{i-1} \land \nabla_i(\nabla_i B_i \land \cdots \land \nabla_{n+k} B_{n+k})
\]
\[
= \nabla_n B_n \land \cdots \land \nabla_{i-1} B_{i-1} \land \nabla_i B'_i
\]
for some \( B'_i \in \mathbb{W}_i \), by Theorem 4.4.
3) \( A \equiv \Diamond_i B \) for some \( i \geq n \). Then using Lemma 3.3 we obtain
\[
\Diamond_i B = \Diamond_i (\nabla_n B_n \land \nabla_{n+1} B_{n+1} \land \cdots \land \nabla_{n+k} B_{n+k})
\]
(\( \forall n \in \mathbb{N}, i \leq n \land \Diamond_i (\nabla_n B_n \land \nabla_{n+1} B_{n+1} \land \cdots \land \nabla_{n+k} B_{n+k}) \))
\[
= \Diamond_n B_n \land \cdots \land \Diamond_{i-1} B_{i-1} \land \Diamond_i (\nabla_i B_i \land \cdots \land \nabla_{n+k} B_{n+k})
\]
(\( \forall n \in \mathbb{N}, i \leq n \land \Diamond_i (\nabla_i B_i \land \cdots \land \nabla_{n+k} B_{n+k}) \)))
\[
= \nabla_n \Diamond_n B_n \land \cdots \land \nabla_{i-1} \Diamond_{i-1} B_{i-1} \land \nabla_i \Diamond_i B_i'
\]
for some \( B'_i \in \mathbb{W}_i \), by Theorem 4.4. \( \square \)

Weak normal forms are in general non-unique. However, the following lemma and its corollary show that the ‘tails’ of the weak normal forms are invariant (up to equivalence in \( \text{RC}^\lor \)).

**Lemma 4.9.** Let \( A \equiv \nabla_n A_n \land \cdots \land \nabla_{n+k} A_k \) and \( B \equiv \nabla_n B_n \land \cdots \land \nabla_{n+k} B_{n+k} \) be weak normal forms, \( B_m \not\equiv \top \), and \( A \vdash B \). Then \( k \geq m \) and for all \( i \) with \( n \leq i \leq k \)

(i) \( \nabla_i A_i \land \cdots \land \nabla_k A_k \vdash_i \nabla_i B_i \land \cdots \land \nabla_m B_m \).

(ii) \( \nabla_i A_i \land \cdots \land \nabla_k A_k \vdash \nabla_i B_i \land \cdots \land \nabla_m B_m \).

**Proof.** By definition, (ii) implies (i), but we first prove (i) and then strengthen it to (ii). For \( i = n \) both claims are vacuous, so we assume that \( i > n \).

Let \( \overline{A_i} := \nabla_i A_i \land \cdots \land \nabla_k A_k \) and \( \overline{B_i} := \nabla_i B_i \land \cdots \land \nabla_m B_m \).

By Corollary 4.6, \( \overline{A_i} \vdash \Diamond_i \overline{B_i} \), or \( \overline{B_i} \vdash \Diamond_i \overline{A_i} \), or \( \overline{A_i} \equiv_i \overline{B_i} \). In the first and third cases we obviously have \( \overline{A_i} \vdash_i \overline{B_i} \), as required.

Assume that \( \overline{B_i} \vdash \Diamond_i \overline{A_i} \). Consider the formula
\[
C := \Diamond_n A_n \land \cdots \land \Diamond_{i-1} A_{i-1} \land \overline{B_i}.
\]
We show that \( C \vdash \Diamond_i C \), contradicting Corollary 3.7.

Using our assumption and Lemma 3.3, (i), we obtain
\[
C \vdash \Diamond_n A_n \land \cdots \land \Diamond_{i-1} A_{i-1} \land \Diamond_i \overline{A_i}
\]
\( \vdash \Diamond_i (\Diamond_n A_n \land \cdots \land \Diamond_{i-1} A_{i-1} \land \overline{A_i}) \)
\( \vdash \Diamond_n A_n \land \cdots \land \Diamond_{i-1} A_{i-1} \land \Diamond_i A \)
\( \vdash \Diamond_n A_n \land \cdots \land \Diamond_{i-1} A_{i-1} \land \Diamond_i B \)
\( \vdash \Diamond_i (\Diamond_n A_n \land \cdots \land \Diamond_{i-1} A_{i-1} \land B) \)
\( \vdash \Diamond_i C. \)

This proves (i).

To prove (ii) assume the contrary and consider the maximal number \( i \) for which \( \overline{A_i} \not\vdash \overline{B_i} \). Such an \( i \) exists since both \( A \) and \( B \) have finitely many terms. Thus we have \( \overline{A_{i+1}} \vdash \overline{B_{i+1}} \) and
\[
\nabla_i A_i \land \overline{A_{i+1}} \not\vdash \nabla_i B_i \land \overline{B_{i+1}}.
\]
It follows that \( \nabla_i A_i \land \overline{A_{i+1}} \not\vdash \nabla_i B_i \). Since \( \overline{B_i} \vdash \nabla_i B_i \), we obtain \( \overline{A_i} \not\vdash \overline{B_i} \), contradicting (i). \( \square \)

**Corollary 4.10.** Let \( \nabla_n A_n \land \cdots \land \nabla_{n+k} A_k \) be any weak normal form of a formula \( A \in \mathbb{F}_n \) with \( A_k \not\equiv \top \). Then \( k \) and each tail \( \nabla_i A_i \land \cdots \land \nabla_k A_k \) is uniquely defined up to equivalence in \( \text{RC}^\lor \).
There are two formats for graphically unique normal forms. We call them ‘fat’ and ‘thin’, because the former normal forms consist of larger expressions, whereas the latter are obtained by pruning certain parts of a given formula. Fat normal forms, presented below, have a natural proof-theoretic meaning and are closely related to collections of proof-theoretic ordinals called conservativity spectra or Turing–Taylor expansions [36].

4.2. Fat normal forms.

**Definition 4.11.** A formula $A \in \mathbb{F}^
abla_n$ is in the *fat normal form* for $\mathbb{F}^
abla_n$ if either $A \equiv \top$ or it has the form $\nabla_n A_n \land \nabla_{n+1} A_{n+1} \land \cdots \land \nabla_{n+k} A_{n+k}$, where $A_i \in \mathbb{W}_i$ for all $i = n, \ldots, n+k$, $A_{n+k} \not\equiv \top$, and

$$\nabla_i A_i \vdash \nabla_i (\nabla_i A_i \land \cdots \land \nabla_{n+k} A_{n+k}).$$

(3)

A variable-free formula $A$ is in the *fat normal form* if $A$ is in the fat normal form for $\mathbb{F}^
abla_0$.

**Remark 4.12.** In a fat normal form, $\nabla_i A_i =_{\text{RC}} \nabla_i (\nabla_i A_i \land \cdots \land \nabla_{n+k} A_{n+k})$ for each $i$ with $n \leq i \leq n+k$.

**Theorem 4.13.** (i) Every $A \in \mathbb{F}^
abla_n$ is equivalent to a formula in the fat normal form for $\mathbb{F}^
abla_n$.

(ii) For any $A \in \mathbb{F}^
abla_n$, the words $A_i$ in the fat normal form of $A$ for $\mathbb{F}^
abla_n$ are unique modulo equivalence in RC.

**Proof.** (i) First we apply Theorem 4.4. Then, using induction on $k$, we show that any formula $\nabla_n A_n \land \cdots \land \nabla_{n+k} A_{n+k}$ can be transformed into one satisfying (3).

For $k = 0$ the claim is trivial. Otherwise, by the induction hypothesis we can assume that (3) holds for $i = n+1, \ldots, n+k$. Then we argue using Lemma 4.2, as follows:

$$\nabla_n A_n \land \nabla_{n+1} A_{n+1} \land \cdots \land \nabla_{n+k} A_{n+k} = \nabla_n (\nabla_n A_n \land \nabla_{n+1} A_{n+1} \land \cdots \land \nabla_{n+k} A_{n+k}) \land \nabla_n A_n \land \cdots \land \nabla_{n+k} A_{n+k},$$

where $A' \in \mathbb{W}_n$ is obtained from Theorem 4.4. Note that

$$\nabla_n A'_n \vdash \nabla_n (\nabla_n A_n \land \nabla_{n+1} A_{n+1} \land \cdots \land \nabla_{n+k} A_{n+k}) \vdash \nabla_n (\nabla_n A'_n \land \nabla_{n+1} A_{n+1} \land \cdots \land \nabla_{n+k} A_{n+k}),$$

hence (3) holds for $i = n$. This proves (i).

(ii) To prove (ii) we apply Lemma 4.9. Assume that $A \vdash B$, where $A = \nabla_n A_n \land \cdots \land \nabla_{n+k} A_{n+k}$ is in the fat normal form and $B = \nabla_n B_n \land \cdots \land \nabla_{n+m} B_{n+m}$ is in a weak normal form. Then $k \geq m$ and $\nabla_i A_i \vdash \nabla_i B_i$ for all $i = n, \ldots, n+m$.

It follows that, if $A, B \in \mathbb{F}^
abla_n$ are both in the fat normal form and $A = B$ in RC$\nabla$, then $m = k$ and $\nabla_i A_i = \nabla_i B_i$, for $i = n, \ldots, n+k$. Since $\mathbb{W}_i$ is linearly preordered by $<_i$ in RC, the latter is only possible if $A_i =_{\text{RC}} B_i$. □
Remark 4.14. As stated in Theorem 4.13, fat normal forms are unique only modulo equivalence of the constituent words $A_i$ in RC. However, we know that words have graphically unique RC-normal forms [3]. Combining the two notions together yields graphically unique normal forms for $\text{RC}^\triangledown$.

Thus, we can test the equality of two variable-free formulas in $\text{RC}^\triangledown$ by graphically comparing their unique normal forms. Alternatively, we observe the following property.

Lemma 4.15. Let $A \equiv \nabla_n A_n \land \nabla_{n+1} A_{n+1} \land \cdots \land \nabla_k A_k$ and $B \equiv \nabla_n B_n \land \nabla_{n+1} B_{n+1} \land \cdots \land \nabla_m B_m$ be any fat normal forms. Then $A \vdash_{\text{RC}^\triangledown} B$ holds if and only if $k > m$ and $\diamondsuit_i A_i \vdash_{\text{RC}} \diamondsuit_i B_i$ for all $i$ with $n \leq i \leq m$.

Proof. By Lemma 4.9 and Remark 4.12, $A \vdash_{\text{RC}^\triangledown} B$ holds if and only if $\nabla_i A_i \vdash_{\text{RC}^\triangledown} \nabla_i B_i$ for all $i$ with $n \leq i \leq m$. However, the latter is equivalent to $\diamondsuit_i A_i \vdash_{\text{RC}} \diamondsuit_i B_i$ by Corollary 4.7. Since words are linearly preordered in RC, the latter is also equivalent to $\diamondsuit_i A_i \vdash_{\text{RC}} \diamondsuit_i B_i$. $\square$

The transformation of a variable-free formula to its fat normal form is computable. Hence we obtain the following.

Corollary 4.16. The set of variable-free sequents $A \vdash B$ provable in $\text{RC}^\triangledown$ is decidable.

From the uniqueness of normal forms we also obtain arithmetical completeness of the variable-free fragment of $\text{RC}^\triangledown$ in the standard way.

Corollary 4.17. Suppose that $A$ and $B$ are variable-free and $T$ is a sound Gödelian extension of EA. Then $A \vdash_{\text{RC}^\triangledown} B$ if and only if $A^* \leq_T B^*$ for all arithmetical interpretations $*$ in $\mathfrak{S}_T$.

Corollary 4.18. Suppose that $T$ is a sound Gödelian extension of EA. Then the algebra $\mathfrak{S}_T^{0\triangledown}$ is isomorphic to the Lindenbaum–Tarski algebra of the variable-free fragment of $\text{RC}^\triangledown$.

4.3. Thin normal forms. Let $A \equiv \nabla_0 A_0 \land \nabla_1 A_1 \land \cdots \land \nabla_k A_k$ be in a weak normal form. As before, we write

$$A_i \equiv \nabla_i A_i \land \cdots \land \nabla_k A_k.$$ 

Definition 4.19. $A$ is in a thin normal form if either $A \equiv \top$ or $A \equiv \nabla_0 A_0 \land \nabla_1 A_1 \land \cdots \land \nabla_k A_k$, where $A_k \neq \top$, $A_i \in \mathbb{W}_i$ for all $i < k$, and there is no $B_i \in \mathbb{W}_i$ such that $B_i <_i A_i$ and $A_i =_{\text{RC}^\triangledown} \nabla_i B_i \land A_{i+1}$.

This definition allows one to easily prove the existence and uniqueness of thin normal forms using the fact that words in $\mathbb{W}_i$ are pre-well-ordered by $<_i$.

Theorem 4.20. For each $A \in \mathbb{F}^\triangledown$ there is a thin normal form equivalent to $A$ in $\text{RC}^\triangledown$, and it is unique modulo equivalence of the constituent words $A_i$ in RC.

Proof. We recursively define the words $A_k, A_{k-1}, \ldots, A_0$. To determine $k$ and $A_k$ one takes any weak normal form for $A$ (observe that $\top$ is the $<_i$-minimum for each $i$). Once one has defined $A_k, \ldots, A_{i+1}$ one can define $A_i$ by considering all the
weak normal forms with the given \( \nabla_{i+1}A_{i+1}, \ldots, \nabla_kA_k \) and selecting the one with the \(<_i\)-minimal \( A_i \). By using induction on \( k - i \) it is also easy to see that all the words \( A_k, \ldots, A_0 \) are thus uniquely determined modulo RC. \( \square \)

The given proof, though short, is non-constructive. Now we will show that the thin normal form can be effectively computed. First we consider a particular case when the given weak normal form is \( \nabla_0A \land \nabla_1B \). Then we will reduce the general case to this one.

Let \( A \in \mathbb{W}_0, B \in \mathbb{W}_1, B \not\equiv \top \), and \( A = A_0 \diamond_0 A_1 \diamond_0 \cdots \diamond_0 A_n \) with \( A_i \in \mathbb{W}_1 \). If \( B \vdash \nabla_0A \), then \( \nabla_0A \land \nabla_1B = \nabla_0\top \land \nabla_1B \), which is its thin normal form. So we assume that \( B \nvdash \nabla_0A \).

We define

\[
B|A := A_i \diamond_0 \cdots \diamond_0 A_n,
\]

where \( i \) is the least integer such that \( B \leq A_i \). Such an \( i \) exists, for otherwise \( B \vdash \diamond_0A \vdash \nabla_0A \). Clearly, \( B|A \) can be found effectively from \( A \) and \( B \) by deleting an appropriate initial segment of \( A \). Also, note that \( B \land A =_{\text{RC}} B \land (B|A) \). We consider three cases.

Case 1: \( A_0 >_1 B \). We claim that \( \nabla_0A \land \nabla_1B \) is in a thin normal form.

Assume that \( A' <_0 A \); then

\[
A \vdash \diamond_1B \land \diamond_0A' = \diamond_1(B \land \diamond_0A') \vdash \diamond_1(\nabla_0A' \land \nabla_1B).
\]

Hence, if \( \nabla_0A' \land \nabla_1B \vdash \nabla_0A \), then

\[
A \vdash \diamond_1(\nabla_0A \land \nabla_1B) \vdash \diamond_1A,
\]

contradicting Corollary 3.7.

Case 2: \( A_0 <_1 B \). We claim that \( \nabla_0\diamond_0(B|A) \land \nabla_1B \) is the thin normal form of \( \nabla_0A \land \nabla_1B \). First we show that

\[
\diamond_0(B|A) \land \nabla_1B \vdash \nabla_0A.
\]

Using downwards induction on \( j := i \) to 0, we show that

\[
\diamond_0(B|A) \land \nabla_1B \vdash \diamond_0(A_j \diamond_0 \cdots \diamond_0 A_n).
\]

The base of induction holds since \( B|A = A_i \diamond_0 \cdots \diamond_0 A_n \). Assume that the claim holds for \( j \). Since \( \nabla_1B \vdash \nabla_1\diamond_{j-1}A_{j-1} = \diamond_1A_{j-1} \), we get that

\[
\diamond_0(B|A) \land \nabla_1B \vdash \diamond_0(A_j \diamond_0 \cdots \diamond_0 A_n) \land \nabla_1B
\]

\[
\vdash \diamond_1A_{j-1} \land \diamond_0(A_j \diamond_0 \cdots \diamond_0 A_n)
\]

\[
\vdash \diamond_1A_{j-1} \land \diamond_0A_j \diamond_0 \cdots \diamond_0 A_n \text{ (since } A_{j-1} \in \mathbb{W}_1) \]

\[
\vdash \diamond_0A_{j-1} \land \diamond_0A_j \diamond_0 \cdots \diamond_0 A_n.
\]

Hence the claim holds for \( j - 1 \) and by induction we conclude that

\[
\diamond_0(B|A) \land \nabla_1B \vdash \diamond_0A \vdash \nabla_0A.
\]
Now we need to show that for all $A' <_0 \diamond_0(B|A)$ one has $\nabla_0 A' \land \nabla_1 B \not\vdash \nabla_0 A$. If $\diamond_0(B|A) \vdash \diamond_0 A'$, then by Corollary 4.7

$$B|A \vdash \nabla_0 A'.$$

Also, $B|A \vdash A_i \vdash \nabla_1 B$ since we assume that $B \leq_0 A_i$. It follows that $B|A \vdash \nabla_0 A' \land \nabla_1 B$. On the other hand, $A \vdash \diamond_0(B|A)$ and $\nabla_0 A \vdash \diamond_0(B|A)$, whence $\nabla_0 A' \land \nabla_1 B \not\vdash \nabla_0 A$ by Corollary 3.7.

Case 3: $A_0 = B$. Let $C := A_1 \diamond_0 \cdots \diamond_0 A_n$; thus $A \equiv B \diamond_0 C$. We claim that $\nabla_0 \diamond_0 C \land \nabla_1 B$ is the thin normal form of $\nabla_0 A \land \nabla_1 B$.

First, $\nabla_1 B \land \diamond_0 C \vdash \nabla_1 (B \land \diamond_0 C) \vdash \nabla_0 (B \diamond_0 C) = \nabla_0 A$. Hence $\nabla_0 \diamond_0 C \land \nabla_1 B = \nabla_0 A \land \nabla_1 B$.

Second, we show that if $A' <_0 \diamond_0 C$, then $\nabla_0 A' \land \nabla_1 B \not\vdash \nabla_0 A$. Assume that $A' <_0 \diamond_0 C$. By Corollary 4.7 we have $C \vdash \nabla_0 A'$. Also, since $A_1 \geq_1 B$, we have $C \vdash A_1 \vdash \nabla_1 B$ by Corollary 4.7. It follows that $C \vdash \nabla_0 A' \land \nabla_1 B$. On the other hand, $A = B \land \diamond_0 C \vdash \diamond_0 C$, hence $\nabla_0 A \vdash \nabla_0 \diamond_0 C \vdash \diamond_0 C$. Therefore, $\nabla_0 A' \land \nabla_1 B \not\vdash \nabla_0 A$ by Corollary 3.7.

In all the three cases we have explicitly constructed the thin normal form. Hence we obtain the following theorem.

**Theorem 4.21.** For any variable-free formula of $\text{RC}^{\nabla}$ its unique thin normal form can be effectively constructed.

**Proof.** Let a formula $A \equiv \nabla_0 A_0 \land \nabla_1 A_1 \land \cdots \land \nabla_k A_k$ in a weak normal form be given. We argue by induction on $k$. For $k = 0$ the claim is obvious. Consider $k > 0$; by the induction hypothesis we may assume that $\overline{A_1} := \nabla_1 A_1 \land \cdots \land \nabla_k A_k$ is in a thin normal form. (To apply the induction hypothesis formally, one should consider the formula obtained from $\overline{A_1}$ by decreasing all indices of modalities by 1.) By Theorem 4.4 there is a word $B \in W_1$ such that $\nabla_1 B \equiv \nabla_1 A_1 \land \cdots \land \nabla_k A_k$.

Consider the formula $\nabla_0 A_0 \land \nabla_1 B$ and bring it to a thin normal form, that is, find a $<_0$-minimal $A_0' \in W_0$ such that

$$\nabla_0 A_0' \land \nabla_1 B = \nabla_0 A_0 \land \nabla_1 B.$$

We claim that $A' := \nabla_0 A_0' \land \nabla_1 A_1 \land \cdots \land \nabla_k A_k$ is equivalent to $A$ and is in a thin normal form.

First, $A' \vdash \nabla_0 A_0' \land \nabla_1 B \vdash \nabla_0 A_0$, hence $A' \vdash A$. On the other hand, $A \vdash \nabla_0 A_0 \land \nabla_1 B = \nabla_0 A_0'$, hence $A \vdash A'$.

Second, assume that there is an $A'' <_0 A_0'$ such that $\nabla_0 A'' \land \overline{A_1} \vdash \nabla_0 A_0'$. By Lemma 4.1,

$$\nabla_0 A'' \land \overline{A_1} \equiv \nabla_0 A'' \land \nabla_1 B.$$

Hence $\nabla_0 A'' \land \nabla_1 B \vdash \nabla_0 A_0'$, contradicting the $<_0$-minimality of $A_0'$. □

5. Iterating monotone operators on $\mathfrak{S}_{\text{EA}}$

Transfinite iterations of reflection principles play an important role in proof theory, starting from the papers [44] of Turing and [23] of Feferman on recursive progressions. Here we present a general result on defining iterations of monotone semi-idempotent operators in $\mathfrak{S}_{\text{EA}}$. 
An operator \( R : \mathcal{G}_{EA} \to \mathcal{G}_{EA} \) is said to be \emph{computable} if the function \( \overline{\sigma} \mapsto \overline{R(\sigma)} \) is computable. By extension of terminology we also say that any operator \( R' \) such that
\[
\forall \sigma \in \mathcal{G}_{EA} \quad R'(\sigma) =_{EA} R(\sigma)
\]
for some computable \( R \) is computable.

Bounded formulas in the language of \( EA \) will henceforth be said to be \emph{elementary}. An operator \( R : \mathcal{G}_{EA} \to \mathcal{G}_{EA} \) is said to be \emph{uniformly definable} if there is an elementary formula \( Ax_R(x, y) \) such that:

(i) \( R(\sigma) =_{EA} Ax_R(x, \overline{\sigma}) \) for each \( \sigma \in \mathcal{G}_{EA} \);

(ii) \( EA \vdash \forall x, y (Ax_R(x, y) \to x \geq y) \).

The operators \( R_n \) and \( \Pi_{n+1} \) are uniformly definable in a very special way. For example, the formula \( R_n(\sigma) \) is obtained by substituting \( \sigma(x) \) for \( X(x) \) into a fixed elementary formula containing a single positive occurrence of a predicate variable \( X \). More generally, it can be shown that an operator \( R : \mathcal{G}_{EA} \to \mathcal{G}_{EA} \) is uniformly definable if and only if \( R \) is computable. A proof of this fact is given in Appendix B.

**Definition 5.1.** A uniformly definable operator \( R \) is said to be:

- \emph{provably monotone} if
  \[
  EA \vdash \forall \sigma, \tau (\overline{\sigma} \leq_{EA} \overline{\tau} \to \overline{R(\tau)} \leq_{EA} \overline{R(\sigma)})
  \]

- \emph{reflexively monotone} if
  \[
  EA \vdash \forall \sigma, \tau (\overline{\sigma} \leq_{EA} \overline{\tau} \to \overline{R(\tau)} \leq \overline{R(\sigma)})
  \]

Here \( \sigma \) and \( \tau \) range over the Gödel numbers of elementary formulas in one free variable, \( \overline{\sigma} \leq_{EA} \overline{\tau} \) abbreviates the formula
\[
\Box_{EA} \forall x (\Box_\sigma(x) \to \Box_\tau(x)),
\]
and \( \overline{R(\tau)} \leq \overline{R(\sigma)} \) stands for
\[
\forall x (\Box_{Ax_R(\cdot, \sigma)}(x) \to \Box_{Ax_R(\cdot, \tau)}(x)).
\]

Reflexive monotonicity here refers to the fact that \( \overline{R(\tau)} \leq \overline{R(\sigma)} \) is the statement of inclusion of theories rather than provable inclusion. Since the formula \( \overline{\sigma} \leq_{EA} \overline{\tau} \) implies its own provability in \( EA \), reflexively monotone operators are (provably) monotone but not necessarily vice versa. It is also easy to see that the operators \( R_n \) (along with all the usual reflection principles) are reflexively monotone.

Next we turn to iterations of operators along ordinal notation systems. In this paper, ordinal notation systems will be \emph{pre-well-orderings}, that is, reflexive and transitive binary relations whose quotient ordering is a well-ordering. An \emph{elementary pre-well-ordering} is a pair of bounded formulas \( D(x) \) and \( x \leq y \) and a constant \( 0 \) such that the relation \( \leq \) is, provably in \( EA \), a linear preordering on \( D \) with the least element \( 0 \), and is a pre-well-ordering of \( D \) in the standard model of arithmetic. Given an elementary well-ordering \((D, \preceq, 0)\), we will denote its elements by
Greek letters and will identify them with an initial segment of the ordinals. We write

\[ x \approx y \iff (x \leq y \land y \leq x), \]
\[ x < y \iff (x \leq y \land y \neq x). \]

Let \( R \) be a uniformly definable monotone operator. The \( \alpha \text{th iterate of } R \text{ along } (D, \preceq, 0) \) is a map associating with any numeration \( \sigma \) the Gödelian extension of EA numerated by an elementary formula \( \rho(\bar{\alpha}, x) \) such that, provably in EA,

\[ \rho(\alpha, x) \leftrightarrow ((\alpha \approx 0 \land \sigma(x)) \lor \exists \beta < \alpha \, \text{Ax}_R(x, \, \gamma \rho(\bar{\beta}, x) \gamma)). \]  

(4)

A natural Gödel numbering of formulas and terms should satisfy the inequalities \( \gamma \rho(\bar{\beta}, x) \gamma \geq \gamma \bar{\beta} \gamma \geq \beta \). Hence the quantifier on \( \beta \) in equation (4) can be bounded by \( x \). Thus, some elementary formula \( \rho(\alpha, x) \) satisfying (4) can be constructed by the fixed point lemma.

The parametrized family of theories numerated by \( \rho(\alpha, x) \) will be denoted by \( R^\alpha(\sigma) \) and the formula \( \rho(\alpha, x) \) will be more suggestively written as \( x \in R^\alpha(\sigma) \). Then (4) can be interpreted as saying that \( R^0(\sigma) \doteq EA \sigma \) and, if \( \alpha > 0 \), then

\[ R^\alpha(\sigma) =_{EA} \bigcup \{R(R^\beta(\sigma)) : \beta < \alpha\}. \]

Lemma 5.2. Suppose that \( R \) is uniformly definable.

(i) If \( 0 < \alpha \leq \beta \), then \( R^\beta(\sigma) \leq_{EA} R^\alpha(\sigma) \).

(ii) \( EA \vdash \forall \alpha, \beta \, (0 < \alpha < \beta \rightarrow \text{“} R^\beta(\sigma) \leq R^\alpha(\sigma) \text{”}) \).

Proof. Obviously, (i) follows from (ii). For the latter we unwind the definition of \( \rho(\alpha, x) \) and prove that within EA

\[ \forall \alpha, \beta \, (0 < \alpha < \beta \rightarrow \forall x \, (\rho(\alpha, x) \rightarrow \rho(\beta, x))). \]  

(5)

This is sufficient to obtain from the same premise

\[ \forall \alpha, \beta \, (0 < \alpha < \beta \rightarrow \forall x \, (\square_{\rho(\alpha, \cdot)}(x) \rightarrow \square_{\rho(\beta, \cdot)}(x))). \]  

(6)

For a proof of (5) we reason within EA. If \( \rho(\alpha, x) \) and \( \alpha \neq 0 \), then by (4) there is a \( \gamma < \alpha \) such that \( \text{Ax}_R(x, \, \gamma \rho(\bar{\gamma}, x) \gamma) \). By the provable transitivity of \( < \), from \( \alpha < \beta \) we obtain \( \gamma < \beta \), and hence also \( \rho(\beta, x) \). \( \square \)

Lemma 5.3. Suppose that \( R \) is reflexively monotone. If \( \tau \leq_{EA} \sigma \), then \( R^\alpha(\tau) \leq_{EA} R^\alpha(\sigma) \) and, moreover, \( EA \vdash \forall \alpha \, \text{“} R^\alpha(\tau) \leq R^\alpha(\sigma) \text{”} \).

Proof. We argue by reflexive induction as in [2], that is, we prove in EA that

\[ \forall \beta \prec \alpha \, \exists \, \square_{EA} \forall x \, (\square_{R^\sigma(\tau)}(x) \rightarrow \square_{R^\tau(\tau)}(x)) \rightarrow \forall x \, (\square_{R^\sigma(\tau)}(x) \rightarrow \square_{R^\tau(\tau)}(x)), \]  

(7)

and then apply Löb’s theorem in EA. Assume that \( \tau \leq_{EA} \sigma \), and reason within EA.

If \( \square_{R^\sigma(\tau)}(x) \), then either \( \alpha \approx 0 \land \square_{\sigma}(x) \), or there is a \( \beta < \alpha \) such that \( \square_{R(\gamma \rho(\bar{\gamma}, x) \gamma)}(x) \).

In the first case we obtain \( \square_{\tau}(x) \) by the external assumption that \( \tau \leq_{EA} \sigma \), and we are done. In the second case, by the premise and the reflexive monotonicity of \( R \) we obtain \( \square_{R(\gamma \rho(\bar{\gamma}, x) \gamma)}(x) \), which implies \( \square_{R^\tau(\tau)}(x) \). \( \square \)
Corollary 5.4. The iteration of $R$ along $(D, \prec)$ is uniquely defined, that is, for a given $\sigma$ equation (4) has a unique solution modulo $=_{\text{EA}}$.

Lemma 5.5. Suppose that $R$ is reflexively monotone and semi-idempotent. Then:

(i) if $0 \prec \alpha$, then $R(R^\alpha(\sigma)) \leq_{\text{EA}} R^\alpha(\sigma)$;

(ii) $\text{EA} \vdash \forall \alpha (0 \prec \alpha \rightarrow \text{"} R(R^\alpha(\sigma)) \leq R^\alpha(\sigma) \text{"})$.

Proof. The assertion (i) follows from (ii). For the latter it is sufficient to prove the claim within $\text{EA} + B\Sigma_1$ and refer to the $\Pi^0_2$-conservativity of $B\Sigma_1$ over $\text{EA}$ (see [28]). By the $\Sigma_1$-collection axiom $B\Sigma_1$ it is sufficient to prove that each axiom of $R^\alpha(\sigma)$ is provable in $R(R^\alpha(\sigma))$.

We reason in $\text{EA} + B\Sigma_1$. If $0 \prec \alpha$ and $x \in R^\alpha(\sigma)$, then there is a $\beta \prec \alpha$ such that $x \in R(R^\beta(\sigma))$. We consider two cases. If $0 \prec \beta$, then since (provably) $\beta \prec \alpha$, we get from Lemma 5.2 that $R^\alpha(\sigma) \leq_{\text{EA}} R^\beta(\sigma)$. By the reflexive monotonicity of $R$ we obtain $R(R^\alpha(\sigma)) \leq R(R^\beta(\sigma))$. Hence $\square_{R(R^\alpha(\sigma))}(x)$, and we are done.

If $\beta \approx 0$, then by definition $R^\beta(\sigma) =_{\text{EA}} \sigma$. Hence, by the reflexive monotonicity of $R$, $R(\sigma) \leq R(R^\beta(\sigma))$. Since $0 \prec \alpha$, $R^\alpha(\sigma) \leq_{\text{EA}} R(R^0(\sigma)) \leq_{\text{EA}} R(\sigma)$ by definition. It follows that $R(R^\alpha(\sigma)) \leq R(R(\sigma)) \leq R(\sigma)$ and therefore $\square_{R(R^\alpha(\sigma))}(x)$.

Thus, using $B\Sigma_1$ we may conclude that $0 \prec \alpha$ implies that
\[ \forall x \left( \square_{R^\alpha(\sigma)}(x) \rightarrow \square_{R(R^\alpha(\sigma))}(x) \right), \]
as required. \( \Box \)

The following lemma is most naturally stated for elementary pre-well-orderings equipped with elementary formulas $\text{Suc}(\alpha, \beta)$ expressing "$\beta$ is a successor of $\alpha$" and $\text{Lim}(\alpha)$ expressing "$\alpha$ is a limit" that provably in $\text{EA}$ satisfy their defining properties:
\[ \forall \alpha, \beta \left( \text{Suc}(\alpha, \beta) \leftrightarrow (\alpha \prec \beta \land \forall \gamma (\gamma \prec \beta \rightarrow \gamma \preceq \alpha)) \right) \]
and
\[ \forall \alpha \left( \text{Lim}(\alpha) \leftrightarrow \neg \alpha \approx 0 \land \forall \beta (\beta \prec \alpha \rightarrow \exists \gamma (\beta \prec \gamma \land \gamma \preceq \alpha)) \right). \]

Lemma 5.6. Suppose that $R$ is reflexively monotone and semi-idempotent. Then:

(i) $R^\alpha(\sigma) =_{\text{EA}} \sigma$ if $\alpha \approx 0$;

(ii) $R^\beta(\sigma) =_{\text{EA}} R(R^\alpha(\sigma))$ if $\text{Suc}(\alpha, \beta)$;

(iii) $R^\lambda(\sigma) =_{\text{EA}} \bigwedge \{ R^\alpha(\sigma) : 0 \prec \alpha \prec \lambda \}$ if $\text{Lim}(\lambda)$.

Here $\bigwedge \{ R^\alpha(\sigma) : 0 \prec \alpha \prec \lambda \}$ denotes the Gödelian theory numeredated by $\exists \alpha (0 \prec \alpha \prec \bigwedge \forall x \in R^\alpha(\sigma))$.

Proof. The assertion (i) is easy. For (ii) assume that $\text{Suc}(\alpha, \beta)$. The implication $R^\beta(\sigma) \leq_{\text{EA}} R(R^\alpha(\sigma))$ is easy, since $\alpha \prec \beta$ and this fact is provable in $\text{EA}$. For the opposite implication it is sufficient to prove in $\text{EA} + B\Sigma_1$ that
\[ \forall x (x \in R^\beta(\sigma) \rightarrow \square_{R(R^\alpha(\sigma))}(x)). \]

Then one will be able to conclude using $B\Sigma_1$ that
\[ \forall x (\square_{R^\beta(\sigma)}(x) \rightarrow \square_{R(R^\alpha(\sigma))}(x)) \]
and then appeal to the $\Pi^0_2$-conservativity of $B\Sigma_1$ over $\text{EA}$. 
We reason in $\Sigma_1 + BΣ$. Assume that $x \in R^\beta(\sigma)$; then (since $\beta \neq 0$) there is a $\gamma \prec \beta$ such that $x \in R(R^\gamma(\sigma))$. If $\gamma \approx \alpha$, then $x \in R(R^\alpha(\sigma))$, and we are done. Otherwise $\gamma \prec \alpha$, and one has $R^\alpha(\sigma) \subseteq R(R^\gamma(\sigma))$. On the other hand, by Lemma 5.5, $R(R^\alpha(\sigma)) \subseteq R^\alpha(\sigma)$. Hence $R(R^\alpha(\sigma)) \subseteq R(R^\gamma(\sigma))$, and therefore $\Box R(R^\alpha(\sigma))(x)$, as required.

To prove (iii) we argue in a similar manner. Assume that $\text{Lim}(\lambda)$; then this fact is also provable in $\Sigma_1$. To prove the implication from left to right we reason in $\Sigma_1 + BΣ$. Assume that $x \in R^\lambda(\sigma)$. Since $\lambda \neq 0$, there is a $\beta \prec \lambda$ such that $x \in R(R^\beta(\sigma))$. Since $\text{Lim}(\lambda)$, there is an $\alpha$ such that $\beta \prec \alpha \prec \lambda$. Then $R^\alpha(\sigma) \subseteq EA R(R^\beta(\sigma))$ and hence $\Box R^\alpha(\sigma)(x)$.

From right to left we reason in $\Sigma_1 + BΣ$. Assume that $0 \prec \alpha \prec \lambda$ and $x \in R^\alpha(\sigma)$. Since $\alpha \prec \lambda$, we have $R^\lambda(\sigma) \subseteq R(R^\alpha(\sigma))$ by definition. On the other hand, since $\alpha > 0$, we have $R(R^\alpha(\sigma)) \subseteq R^\alpha(\sigma)$ by Lemma 5.5. Then $R^\lambda(\sigma) \subseteq R^\alpha(\sigma)$ and $\Box R^\lambda(\sigma)(x)$, as required. □

6. Expressibility of iterated reflection

In this section we confuse the arithmetical and reflection calculus notation: we write $\Diamond_n$ for $R_n$ and $\nabla_n$ for $\Pi_{n+1}$. Our goal is to show that the iterated operators $\Diamond_\alpha$ for natural ordinal notations $\alpha \prec \varepsilon_0$ are expressible in the language of $\text{RC}^\nabla$. We will rely on the so-called reduction property (see [3]; the present version is somewhat more general and follows from [2], Theorem 2; also, cf. [10]).

Let $\Sigma_1 + BΣ$ denote the theory $R_1(\Sigma_1)$, which is known to be equivalent to the extension of $\Sigma_1$ by the axiom asserting the totality of the iterated exponential function [28]. Theories in this and the following section will be Gödelian extensions of $\Sigma_1 + BΣ$. We could have worked more generally over $\Sigma_1$ at the cost of replacing the reflection and conservativity operators of $\text{RC}_\Sigma$ by their analogues stated for cut-free provability (see [2], Appendix C). Taking the cut-free version of $\Sigma_1$ as our base Gödelian theory seems to be a better choice for proof-theoretic applications. However, for simplicity we prefer to strengthen our base theory to $\Sigma_1 + BΣ$, as done in some previous papers we would like to refer to.

Working in $\text{RC}_\Sigma$, we write $\Diamond_n(\tau)$ for $\Diamond_n(\sigma \land \tau)$. Obviously, $\Diamond_n(\sigma)$ is a monotone semi-idempotent operator for each $\sigma$. Also, 1 will stand for $1_{\Sigma_1 + BΣ}$.

**Theorem 6.1** (reduction property). For all $\sigma \in \text{RC}_{\Sigma_1 + BΣ}$ and $n \in \omega$,

$$\Diamond_\omega_{n,\sigma}(1) =_{\Sigma_1 + BΣ} \nabla_n \Diamond_{n+1}(\sigma).$$

We also observe that the theory $\Diamond_\omega_{n,\sigma}(1)$ is equivalent to the theory axiomatized over $\Sigma_1$ by the union of theories $\{Q_{n,k}(\sigma) : k < \omega\}$, where $Q_{n,0}(\sigma) := \Diamond_n(\sigma)$ and $Q_{n,k+1}(\sigma) := \Diamond_n(\sigma \land Q_{n,k}(\sigma))$ are defined by formulas in one variable of $\text{RC}$. The corresponding Gödelian theory taken with its natural numeration will be denoted by $\bigwedge_{k<\omega} Q_{n,k}(\sigma)$.

Concerning these formulas, we note three well-known facts.

**Lemma 6.2.** Provably in $\Sigma_1$:

(a) $\forall B \in \mathbb{W}_n \forall k \ Q_{n,k+1}(B) \vdash_{\text{RC}} Q_{n,k}(B) \land \Diamond_n Q_{n,k}(B)$;
(b) $\forall B \in \mathbb{W}_n \forall k \ Q_{n,k}(B) \vdash_{n} \Diamond_{n+1} B$;
(c) $\forall B \in \mathbb{W}_n \exists A \in \mathbb{W}_n \ Q_{n,k}(B) =_{\text{RC}} A$. 

The first two of these claims are proved by easy induction on $k$. The third is a consequence of a more general theorem that any variable-free formula of RC is equivalent to a word. An explicit rule for calculating such an $A$ is also well known and related to the so-called worm sequence (see [4], Lemma 5.9).

We consider the strictly pre-well-ordered set of words $(\mathcal{W}_n, <_n, \top)$ modulo equivalence in RC, together with its natural representation in EA, as an elementary pre-well-ordering. Recall that for each $A \in \mathcal{W}_n$, $o_n(A)$ denotes the order type of $\{B <_n A : B \in \mathcal{W}_n\}$ modulo $=_{RC}$. In a formalized context, the ordinal $o_n(A)$ is represented by its notation, the word $A$; however, we still write $o_n(A)$, since it reminds us that $A$ must be viewed as an ordinal and indicates which system of ordinal notation is being considered. From the reduction property we obtain the following theorem, which was stated as Theorem 6 in [3] in a somewhat different way. We provide a proof for the reader’s convenience, though it is nearly the same as in [3].

For a word $A$ and a G"odelian theory $\sigma \in \mathcal{G}_{EA^+}$ let $A^*(\sigma)$ denote the interpretation of the formula $A[p/\top]$ in $\mathcal{G}_{EA^+}$ sending $p$ to $\sigma$.

**Theorem 6.3.** For all words $A \in \mathcal{W}_n \setminus \{\top\}$,

$$\nabla_n A^*(\sigma) =_{EA^+} \diamond_n o_n(A)(\sigma)$$

holds in $\mathcal{G}_{EA^+}$.

**Proof.** We argue by reflexive induction in $EA^+$ and prove that, for all $\sigma \in \mathcal{G}_{EA^+}$ and all $n < \omega$,

$$EA^+ \vdash \forall B <_n A “\nabla_n B^*(\sigma) =_{EA^+} \diamond_n o_n(B)(\sigma)” \rightarrow “\nabla_n A^*(\sigma) = \diamond_n o_n(A)(\sigma)”.$$  

Arguing within $EA^+$, we will omit the quotation marks and read the expressions $\tau \leq \nu$ as $\forall x (\square_\tau(x) \rightarrow \square_\nu(x))$ and $\tau = \nu$ as $\forall x (\square_\tau(x) \leftrightarrow \square_\nu(x))$.

If $A \equiv \diamond_n B$, then $o_n(A) = o_n(B) + 1$. If $B \equiv \top$, then the claim follows since $\nabla_n A^*(\sigma) = \nabla_n \diamond_n \sigma = \diamond_n \sigma$. If $B \not\equiv \top$, then $\nabla_n B^*(\sigma) =_{EA^+} \diamond_n o_n(B)(\sigma)$ by the reflexive induction hypothesis. It follows that

$$\diamond_n (\diamond_n o_n(B)(\sigma)) = \diamond_n \nabla_n B^*(\sigma) = \diamond_n B^*(\sigma).$$

Therefore, we obtain

$$\diamond_n o_n(A)(\sigma) = \diamond_n (\diamond_n o_n(B)(\sigma)) = \diamond_n B^*(\sigma) = A^*(\sigma) = \nabla_n A^*(\sigma).$$

If $A \equiv \diamond_{m+1} B$ with $m \geq n$, then $\nabla_n A^*(\sigma) = \nabla_n \diamond_m \diamond_{m+1} B^*(\sigma)$. By the reduction property, $\diamond_m \diamond_{m+1} B^*(\sigma) = \bigwedge_{k < \omega} Q^k_m(B^*(\sigma))$. Moreover, by Lemma 6.2, (a), if a sentence is provable in $\bigwedge_{k < \omega} Q^k_m(B^*(\sigma))$, then it must be provable in $Q^k_m(B^*(\sigma))$ for some $k < \omega$. Hence we can infer that

$$\nabla_n A^*(\sigma) = \nabla_n \bigwedge_{k < \omega} Q^k_m(B^*(\sigma)) = \bigwedge_{k < \omega} \nabla_n Q^k_m(B^*(\sigma)) = \bigwedge_{k < \omega} \diamond_n Q^k_m(B^*(\sigma)).$$

By Lemma 6.2, (b) and (c), each of the formulas $Q^k_m(B)$ is $<_n$-below $A \equiv \diamond_{m+1} B$ and is equivalent to a word in $\mathcal{W}_n$. Hence

$$\bigwedge_{C <_n A} \diamond_n C^*(\sigma) \leq \bigwedge_{k < \omega} \diamond_n Q^k_m(B^*(\sigma)).$$
By the reflexive induction hypothesis, for each \( C <_n A \) we have

\[
\Diamond_n C^*(\sigma) = \Diamond_n \nabla_n C^*(\sigma) = \Diamond_n \Diamond_n^{\alpha_n(C)}(\sigma).
\]

(If \( C = \top \), then the claim holds trivially.) It follows that

\[
\Diamond_n^{\alpha_n(A)}(\sigma) = \bigwedge_{C <_n A} \Diamond_n \Diamond_n^{\alpha_n(C)}(\sigma) = \bigwedge_{C <_n A} \Diamond_n C^*(\sigma)
\]

\[
\leq \bigwedge_{k < \omega} \Diamond_n Q^k_n(B^*(\sigma)) = \nabla_n A^*(\sigma).
\]

On the other hand, if \( C <_n A \), then

\[
A^*(\sigma) \leq \Diamond_n C^*(\sigma) \quad \text{and} \quad \nabla_n A^*(\sigma) \leq \nabla_n \Diamond_n C^*(\sigma) \leq \Diamond_n C^*(\sigma).
\]

Consequently,

\[
\nabla_n A^*(\sigma) \leq \bigwedge_{C <_n A} \Diamond_n C^*(\sigma) = \Diamond_n^{\alpha_n(A)}(\sigma).
\]

Thus, we have proved that \( \nabla_n A^*(\sigma) = \Diamond_n^{\alpha_n(A)}(\sigma) \), as required. \( \square \)

For ordinals \( \alpha < \varepsilon_0 \), let \( A^n_\alpha \in \mathbb{W}_n \) denote a canonical notation for \( \alpha \) in the system \((\mathbb{W}_n, <_n, \top)\). Thus, \( o_n(A^n_\alpha) = \alpha \). We are going to show that the operations \( \Diamond_n^{\alpha} \) are expressible in \( \mathbf{RC}^\forall \) in the following sense.

**Theorem 6.4.** For each \( n < \omega \) and \( 0 < \alpha < \varepsilon_0 \) there is an \( \mathbf{RC} \)-formula \( A(p) \) such that

\[
\forall \sigma \in \mathfrak{G}^{\alpha}_{\mathbf{EA}^+} \Diamond_n^{\alpha}(\sigma)_{\mathbf{EA}^+} \nabla_n A^*(\sigma).
\]

**Proof.** Take \( A(p) := A^n_\alpha[p/\top] \) and apply Theorem 6.3. \( \square \)

7. Proof-theoretic \( \Pi_{n+1}^0 \)-ordinals and conservativity spectra

Let \( S \) be a Gödelian extension of \( \mathbf{EA}^+ \) and \((\Omega, \preceq, 0)\) a fixed elementary recursive well-ordering, that is, an elementary pre-well-ordering for which it is provable in \( \mathbf{EA} \) that \( \preceq \) is antisymmetric. In this section we additionally assume that \( \Omega \) is an epsilon-number and is equipped with elementary terms representing the ordinal constants and functions \( 0, 1, +, \cdot, \omega^x \). These functions should provably in \( \mathbf{EA} \) satisfy certain minimal natural axioms \( \text{NWO} \) listed in [1]. We call such well-orderings *nice*. Recall the following definitions from [3] (writing 1 for \( 1_{\mathbf{EA}^+} \)):

- the \( \Pi_{n+1}^0 \)-*ordinal* of \( S \), denoted by \( \text{ord}_n(S) \), is the supremum of all \( \alpha \in \Omega \) such that \( S \vdash R_n^\alpha(1) \);
- \( S \) is \( \Pi_{n+1}^0 \)-*regular* if \( S \) is \( \Pi_{n+1}^0 \)-conservative over \( R_n^\alpha(1) \) for some \( \alpha \in \Omega \).

The following basic proposition states that \( \Pi_{n+1}^0 \)-ordinals are insensitive to \( \Pi_{n+1}^0 \)-conservative extensions and to extensions by consistent \( \Sigma_{n+1}^0 \)-axioms.

**Proposition 7.1.** For any theories \( S \) and \( T \) and a nice well-ordering \( \Omega \), and for all \( n \in \omega \),

(i) if \( S \vdash \Pi_{n+1}(T) \), then \( \text{ord}_n(S) \geq \text{ord}_n(T) \),

(ii) if \( T \) is axiomatized by \( \Sigma_{n+1}^0 \)-sentences and \( S \cup T \) is consistent, then \( \text{ord}_n(S \cup T) = \text{ord}_n(S) \).
Proof. The first claim follows from the fact that $R_n^\alpha(1)$ is a $\Pi^{0}_{n+1}$-axiomatized theory. The second follows from the well-known result by Kreisel and Lévy [39] that $R_n(U)$ is not contained in any consistent $\Sigma^{0}_{n+1}$-axiomatized extension of $U$. \qed

We refer the reader to [3] or [8] for an extended discussion of proof-theoretic $\Pi^{0}_{n+1}$-ordinals. In this paper we consider the sequences of $\Pi^{0}_{n+1}$-ordinals associated with a given system. Such sequences as objects of study appeared first in the paper [36] by Joosten. For theories between $\text{EA}^+$ and a given system. Such sequences as objects of study appeared first in the paper [36] by Joosten. For theories between $\text{EA}^+$ and $\text{PA}$ he showed that their conservativity spectra correspond to decreasing sequences of ordinals below $\varepsilon_0$ of a certain kind, that is, to points in the so-called Ignatiev frame. We reproduce this interesting characterization here in a slightly more general and streamlined way and also show its close relationship with fat normal forms for $\text{RC}^\nabla$.

Definition 7.2. The conservativity spectrum of $S$ is the sequence $(\alpha_0, \alpha_1, \alpha_2, \ldots)$ with

$$\alpha_i = \text{ord}_i(S) \quad \text{for all } i \in \omega.$$  

Here are some examples of theories and their spectra (the results are either well known and/or can be found in [2]):

1) $I\Sigma_1$: $(\omega^\omega, \omega, 1, 0, 0, \ldots)$, $\text{PRA}$: $(\omega^\omega, \omega, 0, 0, 0, \ldots)$;
2) $\text{PA}$: $(\varepsilon_0, \varepsilon_0, \varepsilon_0, \ldots)$, $\text{PA} + \text{Con}(\text{PA})$: $(\varepsilon_0 \cdot 2, \varepsilon_0, \varepsilon_0, \ldots)$;
3) $\text{PA} + R_1(\text{PA})$: $(\varepsilon^2_0, \varepsilon_0 \cdot 2, \varepsilon_0, \varepsilon_0, \ldots)$.

We will need the following auxiliary lemma concerning iterations of the reflection operators $R_n$ on $\mathcal{G}_T$, for any Gödelian extension $T$ of $\text{EA}$.

Lemma 7.3. If $(D, \leq, 0)$ is an elementary pre-well-ordering, then for $\sigma_1, \sigma_2 \in \mathcal{G}_T$

$$\forall \alpha > 0 \quad R^\alpha_{n+1}(\sigma_1) \land_T R_n(\sigma_2) =_T R^\alpha_{n+1}(\sigma_1 \land_T R_n(\sigma_2)).$$

Proof. The proof is routine by reflexive induction on $\alpha$ using the $\text{RC}$-identity

$$R_{n+1}(\sigma_1) \land_T R_n(\sigma_2) =_T R_{n+1}(\sigma_1 \land_T R_n(\sigma_2)). \quad \Box$$

The following proposition provides a necessary condition for a sequence of ordinals to be a conservativity spectrum.

Proposition 7.4. For any theory $S$, a nice well-ordering $\Omega$, and all $n \in \omega$,

(i) $\text{ord}_{n+1}(S) \leq \ell(\text{ord}_n(S))$,

(ii) if $S$ is $\Pi^{0}_{n+1}$-regular and $n > 0$, then

$$\forall i < n \quad \text{ord}_i(S) = \omega^{\text{ord}_{i+1}(S)}.$$

Proof. (i) Let $\bar{\alpha}$ denote the conservativity spectrum of $S$ and assume that $\alpha_{n+1} > \ell(\alpha_n)$. Select a $\gamma$ such that $\alpha_n = \gamma + \omega^{\ell(\alpha_n)}$. Note that

$$S \vdash R^\alpha_{n+1}(1) \land_{\text{EA}^+} R^\gamma_{n+1}(1).$$

Then by Lemma 7.3

$$R^\alpha_{n+1}(1) \land_{\text{EA}^+} R^\gamma_{n+1}(1) =_{\text{EA}^+} R^\alpha_{n+1}(R^\gamma_{n+1}(1)).$$
By Theorem 3 in [3],
\[ \forall \beta \in \Omega \ \Pi_{n+1} R_{n+1}^\beta (\sigma) =_{EA^+} R_n^\beta (\sigma) \]
for all \( \Pi_{n+1} \)-axiomatized extensions \( \sigma \) of \( \text{EA}^+ \). Hence
\[ S \vdash R_{n+1}^{\alpha_n+1} (\gamma+1) \vdash R_n^{\gamma+\omega^{n+1}} (1). \]
It follows that \( \alpha_n = \text{ord}_n(S) \geq \gamma + \omega^{n+1} \). On the other hand, by our assumption
\( \gamma + \omega^{\alpha_n+1} > \gamma + \omega^{i(\alpha_n)} = \alpha_n \), a contradiction.

(ii) Since a \( \Pi_m \)-regular theory is \( \Pi_i \)-regular for all \( i < m \), it is sufficient to prove the claim for \( i = n - 1 \). If \( S \) is \( \Pi_0^{n+1} \)-regular, then \( \Pi_{n+1}(S) = R_n^a(1) \), where \( a = \text{ord}_n(S) \). It follows that
\[ \Pi_n(S) = \Pi_n(\Pi_{n+1}(S)) = \Pi_n(R_n^a(1)) = R_n^{\omega^a}(1). \quad \Box \]

We consider the ordering of words \( (W_i, <_i) \) modulo equivalence in RC as a nice well-ordering of length \( \varepsilon_0 \). As before, the order type of a word \( A_i \in W_i \) is denoted by \( o_i(A_i) \). A direct correspondence between fat normal forms and conservativity spectra is expressed by the following theorem.

**Theorem 7.5.** Let \( A \equiv \bigwedge_{0} \forall n \bigwedge_{1} A_1 \land \cdots \land \forall k A_k \) for some \( k \), where \( A_n \in W_n \) for all \( n \leq k \), be in the fat normal form. Let \( A^* \) denote the interpretation of \( A \) in \( \mathfrak{G}_{EA^+} \). Then \( A_n \) represents the \( \Pi_0^{n+1} \)-ordinal of \( A^* \):
\[ o_n(A_n) = \text{ord}_n(A^*). \]
Moreover, \( A^* \) is equivalent to the union of progressions, that is, in \( \mathfrak{G}_{EA^+} \)
\[ A^* =_{EA^+} R_0^{o_n(A_0)}(1) \land R_1^{o_1(A_1)}(1) \land \cdots \land R_k^{o_k(A_k)}(1). \quad (8) \]

**Proof.** First, applying Proposition 7.1 we observe that
\[ \text{ord}_n(A^*) = \text{ord}_n((\bigwedge_n A_n \land \bigwedge_{n+1} A_{n+1} \land \cdots \land \bigwedge_k A_k)^*) = \text{ord}_n(A_n^*). \]
The first equality holds because the deleted part of the normal form of \( A \) is interpreted as a true \( \Pi_0^n \)-theory. The second equality holds since the remaining part of the fat normal form of \( A \) is \( \Pi_0^{n+1} \)-conservative over \( A_n^* \):
\[ \bigwedge_n A_n =_{RC} \bigwedge_n (\bigwedge_n A_n \land \bigwedge_{n+1} A_{n+1} \land \cdots \land \bigwedge_k A_k). \]
Then by Theorem 6.3,
\[ \Pi_{n+1}(A_n^*) =_{EA^+} R_n^{o_n(A_n)}(1), \quad (9) \]
and hence \( A_n^* \) is \( \Pi_0^{n+1} \)-regular and \( \text{ord}_n(A_n^*) = o_n(A_n) \). Moreover, equation (9) also yields the representation (8) of \( A^* \) as a union of progressions. \( \Box \)

Joosten [36] calls the representations of theories as unions of Turing progressions Turing–Taylor expansions. Thus, the fat normal form of \( A \) represents the Turing–Taylor expansion of \( A^* \) by way of (8).

We note also that Theorem 7.5 gives another way of showing that the fat normal form of \( A \) is unique. We will come back to the topic of conservativity spectra after we discuss the Ignatiev frame and its associated RC\( ^\nabla \)-algebra.
8. Ignatiev frame and Ignatiev $\text{RC}^\triangledown$-algebra

In this and the following section we characterize the Lindenbaum–Tarski algebra of the variable-free fragment of $\text{RC}^\triangledown$ in several ways. It turns out that this structure is closely related to the so-called Ignatiev’s Kripke frame. This frame, denoted here by $\mathcal{I}$, was introduced by Ignatiev [33] as a universal frame for the variable-free fragment of Japaridze’s logic GLP. Later this frame was slightly modified and studied in more detail in [15] and [31]. In particular, Icard established a detailed relationship between $\mathcal{I}$ and the canonical frame for the variable-free fragment of $\text{GLP}$ and used it to define a complete topological semantics for this fragment. Fernández-Duque and Joosten [24] generalized $\mathcal{I}$ to a version of GLP with transfinitely many modalities. Ignatiev’s frame is defined constructively (‘coordinatewise’) as follows.

Let $\bar{\alpha}$ denote the set of all $\omega$-sequences of ordinals $\bar{\alpha} = (\alpha_0, \alpha_1, \ldots)$ such that $\alpha_i \leq \varepsilon_0$ and $\alpha_{i+1} \leq \ell(\alpha_i)$ for all $i \in \omega$. Here the function $\ell$ is defined by: $\ell(\beta) = 0$ if $\beta = 0$, and $\ell(\beta) = \gamma$ if $\beta = \delta + \omega^\gamma$ for some $\delta$ and $\gamma$. Thus, all sequences in $\bar{\alpha}$, with the exception of the one identically equal to $\varepsilon_0$, are eventually zero. Elements of $\bar{\alpha}$ will also be called $\ell$-sequences.

The relations $R_n$ on $\bar{\alpha}$ are defined by

$$\bar{\alpha}R_n\bar{\beta} \iff (\forall i < n \alpha_i = \beta_i \text{ and } \alpha_n > \beta_n).$$

It is easy to see that all the relations $R_n$ are upwards well-founded strict partial orderings. The structure $\mathcal{I} = (\bar{\alpha}, (R_n)_{n \in \omega})$ is called the extended Ignatiev frame (see [31]). The Ignatiev frame is its restriction to the subset $\bar{\alpha} \in \bar{\alpha}$ such that $\alpha_i < \varepsilon_0$ for all $i \in \omega$. This subset is upwards closed with respect to all the relations $R_n$, hence the evaluations of the variable-free $\text{RC}$-formulas (and $\text{GLP}$-formulas) in $\bar{\alpha}$ and in $\bar{\alpha}$ coincide.

We denote by $\mathcal{M}$, $\bar{\alpha} \models \varphi$ the truth of a GLP-formula $\varphi$ at a node $\bar{\alpha}$ of the Kripke model $\mathcal{M}$. If $\varphi$ is variable-free, then the set $\{\bar{\alpha} : \mathcal{I}, \bar{\alpha} \models \varphi\}$ will be denoted by $v(\varphi)$. In particular, for variable-free formulas $\varphi$ we denote by $v(\varphi)$ the truth set of $\varphi$ in the given Kripke frame.

The following important theorem is a consequence of Ignatiev’s results but, in fact, has an easier direct proof (which we omit for brevity).

**Proposition 8.1.** For any variable-free formulas $A$ and $B$ of $\text{RC}$,

$$A \vdash_{\text{RC}} B$$

if and only if $\mathcal{I}, \bar{\alpha} \models A \rightarrow B$ for all $\bar{\alpha} \in \bar{\alpha}$.

The set of sequences $\bar{\alpha} \in \bar{\alpha}$ such that $\alpha_{i+1} = \ell(\alpha_i)$ for all $i < \omega$ is called the main axis of $\mathcal{I}$ and is denoted by $O$. Obviously, a sequence $\bar{\alpha} \in O$ is uniquely determined by its initial element $\alpha_0$, hence $O$ naturally corresponds to the ordinals up to $\varepsilon_0$. We can also associate with every word $A \in \mathcal{W}$ an element $\iota(A) \in O$ by letting

$$\iota(A) := (o(A), \ell(o(A)), \ldots, \ell^{(n)}(o(A)), \ldots).$$

The following lemma, explicitly stated by Icard (see [31], Lemma 3.8, and also another argument in [20], Lemma 10.2), describes all the subsets of $\mathcal{I}$ definable by words (and hence by all variable-free formulas of $\text{RC}$).
Lemma 8.2. Suppose that $A \in \mathcal{W}$ and $\vec{\alpha} = \iota(A)$. Then for all $\vec{\beta} \in \mathcal{F}$

$$\mathcal{F}, \vec{\beta} \models A$$

if and only if $\alpha_i \leq \beta_i$ for all $i \in \omega$.

Our goal is to transform $\mathcal{I}$ into an $\mathcal{RC}^\triangledown$-algebra $\mathcal{J}$ with the same domain $\mathcal{I}$, that is, into an SLO satisfying $\mathcal{RC}^\triangledown$. We consider the set $\mathcal{I}$ equipped with the partial ordering

$$\vec{\alpha} \leq_\mathcal{I} \vec{\beta} \iff \forall n \in \omega \alpha_n \geq \beta_n.$$ 

The structure $(\mathcal{I}, \leq_\mathcal{I})$ can be seen as a subordering of the product ordering on the set of all $\omega$-sequences of ordinals $\leq \varepsilon_0$, which we denote by $\mathcal{E}$.

A cone in $\mathcal{E}$ is the set of points

$$E_{\vec{\alpha}} := \{ \vec{\beta} \in \mathcal{E} : \vec{\beta} \leq_\mathcal{I} \vec{\alpha} \}$$

for some $\vec{\alpha} \in \mathcal{E}$. A sequence $\vec{\alpha} \in \mathcal{E}$ is said to be bounded if $\alpha_i < \varepsilon_0$ for all $i \in \omega$ and $\alpha_i \neq 0$ for only finitely many $i \in \omega$. Obviously, each $\vec{\alpha} \in \mathcal{I}$ is bounded.

Lemma 8.3. Suppose that $\vec{\alpha} \in \mathcal{E}$ is bounded. Then $E_{\vec{\alpha}} \cap \mathcal{I}$ is non-empty and has a greatest point $\vec{\beta}$ with respect to $\leq_\mathcal{I}$.

Proof. Let $n \in \omega$ be the largest number such that $\alpha_n \neq 0$. Consider the sequence $\vec{\beta}$ such that $\beta_i = 0$ for all $i > n$, $\beta_n := \alpha_n$, and for all $i < n$

$$\beta_i := \begin{cases} 
\alpha_i & \text{if } \ell(\alpha_i) \geq \beta_{i+1}, \\
\alpha_i + \omega \beta_{i+1} & \text{otherwise.}
\end{cases}$$

It is easy to see that $\vec{\beta}$ is the greatest point of $E_{\vec{\alpha}} \cap \mathcal{I}$. Also, note that $\vec{\beta}$ can be effectively computed from $\vec{\alpha}$. $\square$

Corollary 8.4. $(\mathcal{I}, \leq_\mathcal{I})$ is a lower semilattice with top.

Proof. Let $\vec{\alpha}, \vec{\beta} \in \mathcal{I}$. The sequence

$$\vec{\gamma} := (\max(\alpha_i, \beta_i))_{i<\omega}$$

is the g.l.b. of $\vec{\alpha}$ and $\vec{\beta}$ in $\mathcal{E}$ and is bounded. By Lemma 8.3, $E_{\vec{\gamma}} \cap \mathcal{I}$ has a greatest point, which has to be the g.l.b. of $\vec{\alpha}$ and $\vec{\beta}$ in $\mathcal{I}$. The top of the semilattice is the sequence identically equal to 0. $\square$

We denote by $\wedge_\mathcal{I}$ the meet operation of this semilattice. A non-empty set $C_{\vec{\alpha}} := E_{\vec{\alpha}} \cap \mathcal{I}$ is called a cone in $\mathcal{I}$. The set of all cones in $\mathcal{I}$ ordered by inclusion is denoted by $C(\mathcal{I})$. The orderings $(C(\mathcal{I}), \subseteq)$ and $(\mathcal{I}, \leq_\mathcal{I})$ are isomorphic by means of the map $\vec{\alpha} \mapsto C_{\vec{\alpha}}$.

Corollary 8.5. For all $\vec{\alpha}, \vec{\beta} \in \mathcal{I}$,

$$C_{\vec{\alpha} \wedge_\mathcal{I} \vec{\beta}} = C_{\vec{\alpha}} \cap C_{\vec{\beta}}.$$

Let $C(O)$ denote the set $\{C_\bar{x}: \bar{x} \in O\}$ of all cones in $\mathcal{I}$ generated by the points of the main axis. For all $X \subseteq I$ define

$$R_n^{-1}(X) := \{y \in X: \exists x \in X \ yR_n x\}.$$  

We claim that the operations $\cap$ and $R_n^{-1}$ map cones of $C(O)$ to cones of $C(O)$. Let $\mathfrak{C}(O)$ denote the algebra $(C(O); \cap, \{R_n^{-1}: n \in \omega\})$. Then the following proposition holds.$^5$

**Proposition 8.6.** The algebra $\mathfrak{C}(O)$ is isomorphic to the Lindenbaum–Tarski algebra $\mathfrak{L}^0_{RC}$.

*Proof.* Let $v: \mathcal{F} \to \mathcal{P}(I)$ denote the map associating with each variable-free formula $A$ of $RC$ the set $v(A)$ of all points where this formula is true in $\mathcal{I}$. By the soundness and completeness of $RC$ with respect to the Ignatiev model, we have $v(A) = v(B)$ if and only if $A =_{RC} B$. Moreover, by Lemma 8.2 the range of $v$ consists of all the cones of $C(O)$. Thus, $v$ factors into a bijective map $\bar{v}: \mathfrak{L}^0_{RC} \to C(O)$. The operations $\cap$ and $R_n^{-1}$ correspond to the definition of truth in a Kripke model, hence $C(O)$ is closed under these operations, and $\bar{v}$ is an isomorphism of the respective algebras. $\square$

We remark that the paper [41] of Pakhomov in fact shows that the elementary theory of the algebra $\mathfrak{L}^0_{RC}$ is undecidable. We now define the structure of an $RC^\triangledown$-algebra on $I$.

**Definition 8.7.** For all $n \in \omega$ we define the functions $\triangledown^2_n, \diamond^2_n: I \to I$ as follows. For each element $\bar{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots) \in I$ let

$$\triangledown^2_n(\bar{\alpha}) := (\alpha_0, \alpha_1, \ldots, \alpha_n, 0, \ldots)$$

and

$$\diamond^2_n(\bar{\alpha}) := (\beta_0, \beta_1, \ldots, \beta_n, 0, \ldots),$$

where $\beta_{n+1} := 0$ and $\beta_i := \alpha_i + \omega^{\beta_{i+1}}$ for all $i \leq n$.

The algebra $\mathfrak{I} = (I, \land, \{\diamond^2_n, \triangledown^2_n: n \in \omega\})$ is called the Ignatiev $RC^\triangledown$-algebra.

The definition of the operations $\diamond^2_n$ is motivated by the following lemma and its corollary.

**Lemma 8.8.** Suppose that $\bar{\alpha} \in I$ and $\bar{\beta} = \diamond^2_n(\bar{\alpha})$. Then $\bar{\beta} \in O$ and

(i) $C_\bar{\beta} = \bigcap_{i \leq n} R_i^{-1}(C_\bar{\alpha})$,

(ii) if $\bar{\alpha} \in O$, then $C_\bar{\beta} = R_n^{-1}(C_\bar{\alpha})$.

*Proof.* (i) It is easy to see that each set $R_i^{-1}(C_\bar{\alpha})$, where $i \leq n$, is a cone in $\mathcal{I}$ generated by the bounded sequence $(\alpha_0, \ldots, \alpha_{i-1}, \alpha_i + 1, 0, \ldots)$ in $\mathcal{I}'$. Hence the intersection of these cones is the cone generated by $(\alpha_0 + 1, \ldots, \alpha_{n-1} + 1, \alpha_n + 1, 0, \ldots)$. Its greatest element in $I$ obviously coincides with $\diamond^2_n(\bar{\alpha})$.

(ii) Clearly, $\bar{\beta} \in R_n^{-1}(C_\bar{\alpha})$ since $\bar{\beta}' := (\beta_0, \beta_1, \ldots, \beta_{n-1}, \alpha_n, \alpha_{n+1}, \ldots)$ satisfies $\bar{\beta}' R_n \bar{\beta}'$ and $\bar{\beta}' \leq_\alpha \bar{\alpha}$. In the opposite direction, we show by downward induction on $i \leq n$ that if $\bar{\gamma} \in R_n^{-1}(C_\bar{\alpha})$, then $\gamma_{i+1} = \beta_i$. For $i = n$ the claim is obvious. If $i < n$, then $\gamma_i = \alpha_i$. Since $\ell(\gamma_i) \geq \gamma_{i+1} \geq \beta_{i+1}$ and $\ell(\alpha_i) = \alpha_{i+1} < \beta_{i+1}$, we must also have $\gamma_i \geq \alpha_i + \omega^{\beta_{i+1}} = \beta_i$. $\square$

$^5$We do not distinguish notationally between an operation on a set and its restriction to a subset.
Corollary 8.9. $\mathcal{C}(O)$ is isomorphic to the algebra
\[ \mathfrak{O} = (O, \land, \{ \Diamond_n^3 : n \in \omega \}). \]

Proof. Consider the bijection $c : \vec{\alpha} \mapsto C\vec{\alpha}$ from $O$ to $\mathcal{C}(O)$. By Corollary 8.5 this map preserves the meet, and by Lemma 8.8 it preserves the diamond modalities.

We summarize the previous results in the following theorem characterizing the Lindenbaum–Tarski algebra of the variable-free fragment of RC.

Theorem 8.10. The algebras $L^0_{RC}$, $\mathcal{C}(O)$, and $\mathfrak{O}$ are naturally isomorphic by the following maps:

(i) $\bar{v} : L^0_{RC} \to \mathcal{C}(O)$;

(ii) $c : \mathfrak{O} \to \mathcal{C}(O)$;

(iii) $\bar{v} : L^0_{RC} \to \mathfrak{O}$.

Here, for any $A \in \mathbb{F}$,

$\bar{v}([A]_{RC}) := v(A')$,

where $A' \in \mathbb{W}$ is a word such that $A \equiv_{RC} A'$. This definition is invariant, since, for any words $A'$ and $A''$, if $A' \equiv_{RC} A''$, then $o(A') = o(A'')$ and hence $\bar{v}(A') = \bar{v}(A'')$.

For a proof that $\bar{v}$ is an isomorphism it is sufficient to observe that $v(A) = c(v(A))$, for each $A \in \mathbb{W}$, by Lemma 8.2.

Our next goal is to show that $\mathfrak{I}$ is isomorphic to the Lindenbaum–Tarski algebra of $\RC^\forall$. First we need an auxiliary lemma.

Lemma 8.11. For every $\vec{\alpha} \in I$ and $n \in \omega$, there is an $\vec{\alpha}' \in O$ such that $\vec{\alpha}' \leq_\forall \vec{\alpha}$ and $\Diamond_n^3(\vec{\alpha}) = \Diamond_n^3(\vec{\alpha}')$.

Proof. Let

$\alpha'_n := \alpha_n, \quad \forall i \geq n \quad \alpha'_{i+1} := \ell(\alpha'_i), \quad \forall i < n \quad \alpha'_i := \alpha_i + \omega^{\alpha'_{i+1}}.$

It is easy to check that $\vec{\alpha}'$ is as required. □

Let $A^3$ denote the value of a variable-free $\RC^\forall$-formula $A$ in $\mathfrak{I}$. The following lemma shows that $\mathfrak{I}$ is a model of the variable-free fragment of $\RC^\forall$.

Lemma 8.12. For any $A, B \in \mathbb{F}^\forall$, $A \vdash_{RC^\forall} B$ implies that $A^3 \leq_{\ forall} B^3$.

Proof. We argue by induction on the length of the $\RC^\forall$-derivation. In almost all the cases the proof is routine. We consider only the non-trivial case of the axiom

$\Diamond_n A \land \Diamond_m B \vdash \Diamond_n (A \land \Diamond_m B)$

for $m < n$. Let $\vec{\alpha} = A^3$ and $\vec{\beta} = B^3$. Using Lemma 8.11, we obtain $\vec{\alpha}', \vec{\beta}' \in O$ such that

$\vec{\alpha}' \leq_\forall \vec{\alpha}, \quad \vec{\beta}' \leq_\forall \vec{\beta}$ and $\Diamond_n^3 \vec{\alpha} = \Diamond_n^3 \vec{\alpha}', \quad \Diamond_m^3 \vec{\beta} = \Diamond_m^3 \vec{\beta}'$.

By Theorem 8.10 the algebra $\mathfrak{O}$ satisfies $\RC$, hence

$\Diamond_n^3 \vec{\alpha}' \land_\forall \Diamond_m^3 \vec{\beta'} \leq_\forall \Diamond_n^3 (\vec{\alpha}' \land_\forall \Diamond_m^3 \vec{\beta}')$.

Therefore, $\Diamond_n^3 \vec{\alpha} \land_\forall \Diamond_m^3 \vec{\beta} \leq_\forall \Diamond_n^3 (\vec{\alpha}' \land_\forall \Diamond_m^3 \vec{\beta}') \leq_\forall \Diamond_n^3 (\vec{\alpha} \land_\forall \Diamond_m^3 \vec{\beta})$. The second inequality holds by the monotonicity of $\land_\forall$ and $\Diamond_n^3$. □
Lemma 8.13. Suppose that \( A \models \nabla_0 A_0 \land \nabla_1 A_1 \land \cdots \land \nabla_n A_n \) is in the fat normal form. Then

\[
A^3 = (o_0(A_0), o_1(A_1), \ldots, o_n(A_n), 0, \ldots).
\]

Proof. First, since each \( A_i \in \mathbb{W}_i \), we get from Theorem 8.10 that

\[
(A_i)^3 = \iota(A_i) = (\omega_i(o_i(A_i)), \omega_i-1(o_i(A_i)), \ldots, \omega_i^{\alpha_i}(A_i), o_i(A_i), \ell(o_i(A_i)), \ldots),
\]

where \( \omega_0(\alpha) = \alpha \) and \( \omega_{i+1}(\alpha) = \omega_\omega(\alpha) \) by definition. Hence,

\[
(\nabla_i A_i)^3 = (\omega_i(o_i(A_i)), \omega_i-1(o_i(A_i)), \ldots, \omega_i^{\alpha_i}(A_i), o_i(A_i), 0, \ldots).
\]

Let \( \overline{A}_i := \nabla_i A_i \land \nabla_{i+1} A_{i+1} \land \cdots \land \nabla_n A_n \). Using downwards induction on \( i \leq n \), we show that \( (\overline{A}_i)^3 \) equals

\[
(\omega_i(o_i(A_i)), \omega_i-1(o_i(A_i)), \ldots, o_i(A_i), o_{i+1}(A_{i+1}), \ldots, o_n(A_n), 0, \ldots). \tag{10}
\]

For \( i = n \) the claim follows from the above. Assume that \( i < n \) and that the claim holds for \( i + 1 \). Since in a fat normal form

\[
\nabla_i A_i \vdash_{RC} \nabla_i(\nabla_i A_i \land \nabla_{i+1} A_{i+1}),
\]

the sequence \( (\nabla_i A_i)^3 \) majorizes the sequence \( (\nabla_i(\nabla_i A_i \land \nabla_{i+1} A_{i+1}))^3 \) coordinate-wise in view of Lemma 8.12. The former has the ordinal \( o_i(A_i) \) at the \( i \)th position, and the latter has the least ordinal \( \alpha \) such that \( \alpha \geq o_i(A_i), \omega_i^{\alpha_i+1}(A_{i+1}) \), and \( \ell(\alpha) \geq o_{i+1}(A_{i+1}) \) at the same place. Therefore, \( o_i(A_i) = \alpha \) and \( \ell(o_i(A_i)) \geq o_{i+1}(A_{i+1}) \).

Now consider the sequence \( (\overline{A}_i)^3 = (\nabla_i A_i \land \overline{A}_{i+1})^3 \). By the induction hypothesis its tail coincides with that of \( (10) \) starting from position \( i + 1 \). Since \( \ell(o_i(A_i)) \geq o_{i+1}(A_{i+1}) \), the ordinal \( o_i(A_i) \) occurs in it at the \( i \)th position. Also, for each \( k < i \) we have \( \omega_k(o_i(A_i)) \geq \omega_k(\omega_i^{\alpha_i+1}(A_{i+1})) \). It follows that the sequence \( (\overline{A}_i)^3 \) coincides with \( (10) \). □

The following corollary will be useful later.

Corollary 8.14. For any \( A, B \in \mathbb{W} \) and \( n \in \omega \), if \( \mathcal{I} \models \nabla_n A = \nabla_n B \), then \( A =_{RC} B \).

Proof. First, we infer that \( \mathcal{J} \models \nabla_0 A = \nabla_0 \nabla_n A = \nabla_0 \nabla_n B = \nabla_0 B \). By Lemma 8.13 we conclude that \( o(A) = o(B) \), and therefore \( A =_{RC} B \). □

Theorem 8.15. For all \( A, B \in \mathbb{P}, A \vdash_{RC} B \) if and only if \( A^3 \leq_3 B^3 \).

Proof. We only need to prove the ‘only if’ part. Moreover, it is sufficient to prove it for fat normal forms \( A \models \nabla_0 A_0 \land \nabla_1 A_1 \land \cdots \land \nabla_n A_n \) and \( B \models \nabla_0 B_0 \land \nabla_1 B_1 \land \cdots \land \nabla_m B_m \). If \( A^3 \leq_3 B^3 \), then by Lemma 8.13 we have \( n \geq m \) and \( o_i(A_i) \geq o_i(B_i) \) for each \( i \leq m \). Since \( A_i, B_i \in \mathbb{W}_i \), this means that \( A_i \vdash_{RC} \triangle_i B_i \) or \( A_i =_{RC} B_i \). In either case we can infer that \( \nabla_i A_i \vdash_{RC} \nabla_i B_i \) for each \( i \leq m \). It follows that \( A \vdash_{RC} B \). □

Theorem 8.15 essentially implies the following.

Corollary 8.16. The Ignatiev RC\(^V\)-algebra \( \mathcal{I} \) is isomorphic to the Lindenbaum–Tarski algebra of the variable-free fragment of RC\(^V\).
9. \( \mathcal{I} \) as the algebra of variable-free RC-theories

Another, perhaps even more natural, view of the Ignatiev RC\( ^\nabla \)-algebra is via an interpretation of points of \( \mathcal{I} \) as variable-free RC-theories. It nicely agrees with the arithmetical interpretation in that we can also view such a theory as an arithmetical theory (every variable-free RC-formula corresponds to an arithmetical sentence). In this section we will presuppose that the language of RC\( ^\nabla \) is variable-free, and we will only consider variable-free formulas and theories.

A set \( T \) of strictly positive formulas is called an RC\( T \)-theory if \( \top \in T \) and \( B \in T \) whenever there are \( A_1, \ldots, A_n \in T \) such that \( A_1 \land \cdots \land A_n \vdash_{\text{RC}} B \). A theory \( T \) is said to be improper if \( T \) coincides with the set of all strictly positive formulas, otherwise it is said to be proper.\(^6\) A theory is said to be bounded if there is an RC-formula \( A \) such that \( T \subseteq \{ B : A \vdash_{\text{RC}} B \} \). We will use the following basic fact.

The set \( \mathcal{I} \) has a natural topology generated as a subbase by the set of all cones in \( \mathcal{I} \) and their complements. By [31], Theorem 3.12, this topology coincides with the product topology of \( \mathcal{E} \) induced on \( \mathcal{I} \). Obviously, for each RC-formula \( A \) the set \( v(A) \) is open-and-closed. Moreover, this topology is compact and totally disconnected on \( \mathcal{I} \), since \( \mathcal{I} \) is closed in \( \mathcal{E} \) and \( \mathcal{E} \) is compact by the Tychonoff theorem. As a consequence, we obtain the following strong completeness result.

For each RC-theory \( T \) we define \( v(T) := \{ \bar{\alpha} \in \mathcal{I} : \mathcal{I}, \bar{\alpha} \models T \} \).

Proposition 9.1. Let \( T \) be an RC-theory and \( A \) an RC-formula.

(i) \( T \nvdash_{\text{RC}} A \) if and only if there is an \( \bar{\alpha} \in \mathcal{I} \) such that

\[ \mathcal{I}, \bar{\alpha} \models T \quad \text{and} \quad \mathcal{I}, \bar{\alpha} \nvdash A. \]

(ii) If \( T \) is bounded, then \( T \nvdash_{\text{RC}} A \) if and only if there is an \( \bar{\alpha} \in \mathcal{I} \) such that

\[ \mathcal{I}, \bar{\alpha} \models T \quad \text{and} \quad \mathcal{I}, \bar{\alpha} \nvdash A. \]

Proof. (i) The non-trivial implication is from left to right. Assume that \( T \nvdash_{\text{RC}} A \). There is an increasing sequence of finitely axiomatized theories \( (T_n)_{n \in \omega} \) such that \( T = \bigcup_{n \in \omega} T_n \). By the completeness of the variable-free fragment of RC with respect to \( \mathcal{I} \) each set \( v(T_n) \setminus v(A) \) is non-empty and open-and-closed. By the compactness of \( \mathcal{I} \) there is a point

\[ \bar{\alpha} \in \bigcap_{n \in \omega} v(T_n) \setminus v(A) = v(T) \setminus v(A). \]

(ii) In case \( T \) is bounded we have \( v(T) \supseteq v(B) \) for some word \( B \). There is a bounded sequence \( \bar{\beta} \in \mathcal{E} \) such that \( v(T) = E_{\bar{\beta}} \cap \mathcal{I} \): consider the pointwise supremum of the generating points of the cones \( v(T_n) \) in \( \mathcal{I} \), each of which is pointwise majorized by the greatest element \( B^3 \) of \( v(B) \). By Lemma 8.3 the set \( v(T) \) has a greatest point, say \( \bar{\gamma} \in \mathcal{I} \). Since \( \bar{\alpha} \in v(T) \), we have \( \bar{\alpha} \leq_{\mathcal{E}} \bar{\gamma} \), and hence \( \mathcal{I}, \bar{\gamma} \nvdash A \). \( \square \)

For any RC-theories \( T \) and \( S \) define \( T \leq_{\text{RC}} S \) if and only if \( T \supseteq S \). The g.l.b. of \( T \) and \( S \) in this ordering, denoted by \( T \land_{\text{RC}} S \), is the theory generated

\(^6\)We avoid the term ‘consistent’, for even the improper theory corresponds to a consistent set of arithmetical sentences.
by the union $T \cup S$. Thus, the set $\mathcal{X}^0_{RC}$ of all bounded variable-free RC-theories is a semilattice (it is in fact a lattice with $T \cap S$ being the l.u.b. of $T$ and $S$). The set \{ $A \in \mathcal{F}$: $T \vdash_{RC} A$ \} corresponds to the top of this lattice and is denoted by $\top_{RC}$.

For each $\vec{\alpha} \in \mathcal{I}$ we define an RC-theory $[\vec{\alpha}] := \{ A: \mathcal{I}, \vec{\alpha} \vdash A \}$. It is easy to see that $[\vec{\alpha}]$ is bounded if $\vec{\alpha} \in I$ (consider the point $\vec{\beta}$ on the main axis of $\mathcal{I}$ such that $\vec{\beta} \leq_3 \vec{\alpha}$ and a word $B$ such that $\nu(B) = \vec{\beta}$).

**Proposition 9.2.** (i) The map $\vec{\alpha} \mapsto [\vec{\alpha}]$ is an isomorphism between $(\mathcal{I}, \leq_3)$ and the ordered set $\mathcal{X}^0_{RC}$ of bounded RC-theories.

(ii) The map $v$ is an isomorphism between $\mathcal{X}^0_{RC}$ and the ordered set $(C(\mathcal{I}), \subseteq)$ of cones in $\mathcal{I}$.

**Proof.** It is sufficient to prove that:

(a) the maps $\vec{\alpha} \mapsto [\vec{\alpha}]$ and $T \mapsto v(T)$ are order-preserving;

(b) $v([\vec{\alpha}]) = C_{\vec{\alpha}}$ for all $\vec{\alpha} \in I$;

(c) if $v(T) = C_{\vec{\alpha}}$, then $T = [\vec{\alpha}]$.

Item (a) is obvious. For (b) we observe that

$$\vec{\beta} \in v([\vec{\alpha}]) \iff \forall A \in \mathcal{I}, \vec{\alpha} \vdash A \Rightarrow \mathcal{I}, \vec{\beta} \vdash A.$$ 

The right-hand side is equivalent to $\vec{\beta} \leq_3 \vec{\alpha}$: if $\vec{\beta} \leq_3 \vec{\alpha}$ and $\mathcal{I}, \vec{\alpha} \vdash A$, then $\mathcal{I}, \vec{\beta} \vdash A$ by Lemma 8.2. If $\vec{\beta} \not\leq_3 \vec{\alpha}$, then there is a word $A$ such that $\mathcal{I}, \vec{\alpha} \vdash A$ and $\mathcal{I}, \vec{\beta} \not\vdash A$, by Corollary 3.9 in [31]. Hence $\vec{\beta} \in v([\vec{\alpha}])$ if and only if $\vec{\beta} \in C_{\vec{\alpha}}$.

For (c) we use Proposition 9.1. Suppose that $\vec{\alpha} \in I$ and $v(T) = C_{\vec{\alpha}}$. Then $\mathcal{I}, \vec{\alpha} \vdash T$ and thus $T \subseteq [\vec{\alpha}]$. For the opposite inclusion assume that $A \in [\vec{\alpha}]$ and $A \not\in T$. By Proposition 9.1 there is a node $\vec{\beta} \in I$ such that $\mathcal{I}, \vec{\beta} \vdash T$ and $\mathcal{I}, \vec{\beta} \not\vdash A$. Thus, $\vec{\beta} \in v(T)$ and, since $v(A)$ is downwards closed, $\vec{\beta} \not\leq_3 \vec{\alpha}$. It follows that $v(T) \not\in C_{\vec{\alpha}}$. □

The operations of the Ignatiev RC$^\forall$-algebra can be interpreted in terms of the semilattice of bounded theories as follows. For each $T \in \mathcal{X}^0_{RC}$ let $\nabla_n^{RC} T$ denote the RC-theory axiomatized by the set

$$\{ \Diamond_m A: \Diamond_m A \in T \text{ and } m \leq n \}.$$

**Lemma 9.3.** For all $\vec{\alpha} \in \mathcal{I}$,

$$\nabla_n^{RC}([\vec{\alpha}]) = [\Diamond_n^{\mathcal{I}} \vec{\alpha}].$$

**Proof.** For the inclusion \((\subseteq)\) we need to show that if $m \leq n$ and $\Diamond_n A \in [\vec{\alpha}]$, then $\Diamond_m A \in [\Diamond_n^{\mathcal{I}} \vec{\alpha}]$. If $\Diamond_m A \in [\vec{\alpha}]$, then $\mathcal{I}, \vec{\alpha} \vdash \Diamond_m A$, and hence there is a $\vec{\beta}$ such that $\vec{\alpha} \Diamond_m \vec{\beta}$ and $\mathcal{I}, \vec{\beta} \vdash A$. Therefore, $\alpha_i = \beta_i$ for all $i < m$, and $\alpha_m > \beta_m$. Since $m \leq n$, the node $\Diamond_n^{\mathcal{I}} \vec{\alpha}$ has the same coordinates as $\vec{\alpha}$ for all $i \leq m$. Therefore, $(\Diamond_m^{\mathcal{I}} \vec{\alpha}) \Diamond_m \vec{\beta}$ and $\mathcal{I}, (\Diamond_m^{\mathcal{I}} \vec{\alpha}) \vdash \Diamond_m A$.

For the inclusion \((\supseteq)\) we consider any node $\vec{\gamma} \in I$ such that $\mathcal{I}, \vec{\gamma} \vdash \nabla_n^{RC} [\vec{\alpha}]$, and we show that $\mathcal{I}, \vec{\gamma} \vdash [\Diamond_n^{\mathcal{I}} \vec{\alpha}]$. This means that $v(\nabla_n^{RC} [\vec{\alpha}]) \subseteq v([\Diamond_n^{\mathcal{I}} \vec{\alpha}])$, and hence $\nabla_n^{RC}([\vec{\alpha}]) \supseteq [\Diamond_n^{\mathcal{I}} \vec{\alpha}]$ by Proposition 9.2.
Assume that $\mathcal{I}, \vec{\gamma} \not\vDash [\bigvee_n^3 \vec{\alpha}]$. Since $v(\bigvee_n^3 \vec{\alpha}) = C_{\bigvee_n^3 \vec{\alpha}}$, we have $\vec{\gamma} \not\vDash C_{\bigvee_n^3 \vec{\alpha}}$, and hence there is an $m \leq n$ such that $\gamma_m < \alpha_m$. Consider a word $A \in \mathbb{W}_m$ such that $o_m(A) = \gamma_m$. Recall that the point on the main axis corresponding to $A$ is

$$\iota(A) = (\omega_m(\gamma_m), \ldots, \omega^\gamma_m, \gamma_m, \ell(\gamma_m), \ldots).$$

We claim that $\mathcal{I}, \vec{\alpha} \vDash \diamond A$, whereas $\mathcal{I}, \vec{\gamma} \nvdash \diamond A$. The former holds because $\delta_m < \gamma_m$ for all $\delta$ such that $\vec{\gamma} R_m \vec{\delta}$; hence $\vec{\delta} \not\vDash \iota(A)$ and $\mathcal{I}, \vec{\delta} \nvdash A$. On the other hand, $\mathcal{I}, \vec{\alpha} \vDash \diamond A$, since there is a sequence $\vec{\alpha}' := (\alpha_0, \ldots, \alpha_m-1, \gamma_m, \gamma_{m+1}, \ldots)$ such that $\vec{\alpha}' R_m \vec{\alpha}'$ and $\mathcal{I}, \vec{\alpha}' \vDash A$.

To show that $\vec{\alpha}' \leq \iota(A)$ we prove that

$$\forall i \leq m \ \omega_{m-i}(\gamma_m) \leq \alpha_i,$$

by downward induction on $i \leq m$. Assume that the claim holds for some $i$ with $0 < i < m$. Then $\alpha_{i-1} > \omega(\alpha_{i-1}) > \omega_\alpha > \omega^\alpha = \gamma_{i-1}$. □

In order to define the operations $\diamond_n^{RC}$ on the set of bounded RC-theories, we need a few definitions. An RC-theory $T$ is of level $n$ if $T$ is generated by a (non-empty) set of formulas $\diamond_n A$ such that $A \in \mathbb{W}_n$. A theory $T$ is of level at least $n$ if it is generated by a (non-empty) subset of $\mathbb{W}_n \setminus \{\top\}$.

**Lemma 9.4.** Every bounded RC-theory $T$ is representable in the form

$$T = T_0 \wedge_{RC} T_1 \wedge_{RC} \cdots \wedge_{RC} T_n,$$

where each $T_i$ is of level $i$ or $T_i = \top_{RC}$.

**Proof.** Recall that every RC-formula is RC-equivalent to an ordered formula. Moreover, every variable-free RC-formula in which only the modalities $\diamond_i$ with $i \geq m$ occur is equivalent to a word in $\mathbb{W}_m$. Hence every formula is equivalent to a conjunction of formulas of the form $\diamond_i A$ with $A \in \mathbb{W}_i$. Since $T$ is bounded, the set of indices of modalities occurring in the axioms of $T$ is bounded, say by $n$. Therefore, each axiom of $T$ can be replaced by a finite set of formulas of various levels below $n$, and one can partition the union of all these sets into disjoint subsets of formulas of the same level. □

**Lemma 9.5.** For each $\vec{\alpha} \in I$ such that $\alpha_n > 0$, the theory generated by $[\vec{\alpha}] \cap \mathbb{W}_n$ corresponds to the sequence

$$\vec{\alpha}' := (\omega_n(\alpha_n), \ldots, \omega^\alpha_n, \alpha_n, \alpha_{n+1}, \ldots).$$

We remark that if $\alpha_n = 0$, then the theory generated by $[\vec{\alpha}] \cap \mathbb{W}_n$ is $\top_{RC}$.

**Proof.** Let $T$ be the theory generated by $[\vec{\alpha}] \cap \mathbb{W}_n$. We consider a $\vec{\beta} \in I$ such that $[\vec{\beta}] = T$ and show that $\vec{\beta} = \vec{\alpha}'$. It is easy to see that $\vec{\alpha} \leq \vec{\alpha}'$ and that the submodel of $\mathcal{I}$ generated from $\vec{\alpha}$ by the relations $R_k$ for all $k \geq n$ is isomorphic to the submodel generated by these relations from $\vec{\alpha}'$. Therefore, if $B$ is a formula in which only the modalities $\diamond_k$ with $k \geq n$ occur, then $\mathcal{I}, \vec{\alpha} \vDash B$ holds if and only if $\mathcal{I}, \vec{\alpha}' \vDash B$. It follows that $[\vec{\alpha}] \cap \mathbb{W}_n \subseteq [\vec{\alpha}']$, that is, $\vec{\alpha}' \leq \vec{\beta}$. □
Now assume that $\alpha' <_\prec \beta$, so there is a $k \in \omega$ such that $\beta_k < \alpha'_k$. If $k < n$, then $\beta_k < \omega_{n-k}(\alpha_n)$. For all ordinals $\gamma$ and $\delta$, if $\gamma < \omega^\delta$, then $\ell(\gamma) < \delta$. Then by induction, for all $i = k, \ldots, n$ we obtain $\beta_i < \omega_{n-i}(\alpha_n)$. Therefore, $\beta_n < \alpha_n$.

Thus, we may assume that $k \geq n$. In this case consider a word $B \in \mathbb{W}_k$ such that $\sigma_k(B) = \beta_k + 1$. Then

$$\iota(B) = (\omega_k(\beta_k + 1), \ldots, \omega_1(\beta_k + 1), \beta_k + 1, 0, \ldots).$$

We have $\mathcal{J}, \vec{\beta} \not\models B$, since $\beta_k + 1 > \beta_k$. On the other hand,

$$\forall i \leq k \quad \omega_i(\beta_k + 1) \leq \alpha_{k-i},$$

which is easy to see using induction on $i$. It follows that $\mathcal{J}, \vec{\alpha} \not\models B$, and therefore $[\vec{\beta}] \neq T$, a contradiction. □

**Corollary 9.6.** For each $\vec{\alpha} \in \mathcal{J}$ the theory $[\vec{\alpha}]$ is of level at least $n$ if and only if $\alpha_n > 0$ and

$$\forall i < n \quad \alpha_i = \omega_{n-i}(\alpha_n).$$

(11)

**Lemma 9.7.** For each bounded RC-theory $T$ of level at least $n$ there is a word $A \in \mathbb{W}_n$ such that $\nabla_n^{RC} A = \nabla_n^{RC} T$ in $\Sigma_{RC}^0$.

**Proof.** Suppose that $T = [\vec{\alpha}]$ is of level at least $n$. Let $A \in \mathbb{W}_n$ be such that $\sigma_n(A) = \alpha_n > 0$. Then by Lemma 9.3

$$\nabla_n^{RC} (T) = \nabla_n^{RC} ([\vec{\alpha}]) = [\nabla_n^{\vec{\alpha}}].$$

By (11)

$$\nabla_n^{\vec{\alpha}} = (\omega_n(\alpha_n), \omega_{n-1}(\alpha_n), \ldots, \alpha_n, 0, \ldots).$$

On the other hand, $\iota(A) = (\omega_n(\alpha_n), \omega_{n-1}(\alpha_n), \ldots, \alpha_n, \ell(\alpha_n), \ldots)$, and we get that $\nabla_n^{RC} A = [\nabla_n^{\vec{\alpha}}(\iota(A))] = [(\omega_n(\alpha_n), \omega_{n-1}(\alpha_n), \ldots, \alpha_n, 0, \ldots)]$. Proposition 9.2 yields the result. □

Now we can give the following definition of the theory $\diamondsuit_n^{RC} T$, for each bounded RC-theory $T$.

If $T$ is of level at least $n$ or $T = \top_{RC}$, we let $\diamondsuit_n^{RC} T$ be the theory generated by the formula $\diamondsuit_n A$, where $A \in \mathbb{W}_n$ is such that $\nabla_n^{RC} A = \nabla_n^{RC} T$ in $\Sigma_{RC}^0$. (Note that this definition is correct because any two words $A_1$ and $A_2$ satisfying $\nabla_n^{RC} A_1 = \nabla_n^{RC} A_2$ in $\Sigma_{RC}^0$ also satisfy $\diamondsuit_n A_1 = \diamondsuit_n A_2$ by Corollary 8.14.)

For each $i \leq n$ let $T_i$ denote the theory generated by $T \cap \mathbb{W}_i$. We define

$$\diamondsuit_n^{RC} (T) := \diamondsuit_0^{RC} (T_0) \land_{RC} \diamondsuit_1^{RC} (T_1) \land_{RC} \cdots \land_{RC} \diamondsuit_n^{RC} (T_n).$$

The following lemma shows that this definition agrees with the operations on the Ignatiev algebra.

**Lemma 9.8.** For all $\vec{\alpha} \in \mathcal{J}$,

$$\diamondsuit_n^{RC} ([\vec{\alpha}]) = [\diamondsuit_n^{\vec{\alpha}}].$$
Proof. If \( T = [\vec{\alpha}] \), then by Lemma 9.5, for each \( i \leq n \) either the theory \( T_i := T \cap W_i \) is \( \Gamma_{RC} \) or it corresponds to the sequence
\[
\vec{\alpha}' := (\omega_i(\alpha_i), \ldots, \omega^{\alpha_i}, \alpha_i, \alpha_{i+1}, \ldots), \quad \alpha_i > 0.
\]
If \( T_i = \Gamma_{RC} \), we have \( \diamond_{i}^{RC} T_i = \diamond_i \top \). Otherwise, \( \diamond_{i}^{RC} T_i = \diamond_i A_i \), where \( A_i \) corresponds to the sequence \( (\omega_i(\alpha_i), \ldots, \omega^{\alpha_i}, \alpha_i, \ell(\alpha_i), \ldots) \). In both cases
\[
\diamond_{i}^{RC} T_i = [(\omega_i(\alpha_i + 1), \ldots, \omega^{\alpha_i+1}, \alpha_i + 1, 0, \ldots)].
\]
Then we observe that \( \diamond_{n}^{RC}(T) = \diamond_{0}^{RC}(T_0) \land_{RC} \diamond_{1}^{RC}(T_1) \land_{RC} \cdots \land_{RC} \diamond_{n}^{RC}(T_n) \) corresponds to the cone generated by \( (\alpha_0 + 1, \alpha_1 + 1, \ldots, \alpha_n + 1, 0, \ldots) \) in \( \mathcal{S} \), which coincides with the cone of \( \diamond_{n}^{3}(\vec{\alpha}) \) (cf. Lemma 8.8).

Using Lemma 8.8, we can also isomorphically represent \( \mathcal{I} \) as an algebra of cones in \( \mathcal{S} \). Given a cone \( C \in C(\mathcal{S}) \), let
\[
\diamond_{n}^{\mathcal{S}}(C) := \bigcap_{i \leq n} R_{i}^{-1}(C).
\]
We also define
\[
\nabla_{n}^{\mathcal{S}}(C) := \bigcap \{R_{i}^{-1}(D) : D \in \mathcal{C}(\mathcal{S}), i \leq n, R_{i}^{-1}(D) \supseteq C\}.
\]

We summarize the main results of this paper in the next theorem.

**Theorem 9.9.** The following structures are isomorphic:

(i) \( \overline{\mathcal{S}}_{T}^{0} \), for any sound Gödelian extension \( T \) of \( EA \);

(ii) \( \mathcal{L}_{RC}^{0} \), the Lindenbaum–Tarski algebra of the variable-free fragment of \( \text{RC}^{\nabla} \);

(iii) \( \mathcal{I} = (I, \land_{\mathcal{I}}, \{\diamond_{n}^{\mathcal{S}}, \nabla_{n}^{\mathcal{S}} : n \in \omega\}) \);

(iv) \( (\mathcal{T}_{RC}^{0}, \land_{RC}, \{\diamond_{n}^{RC}, \nabla_{n}^{RC} : n \in \omega\}) \);

(v) \( \mathcal{C}(\mathcal{S}) = (C(\mathcal{S}), \cap, \{\diamond_{n}^{\mathcal{S}}, \nabla_{n}^{\mathcal{S}} : n \in \omega\}) \).

**Proof.** We only need to prove the isomorphism of (v) with either (iii) or (iv). Proposition 9.2 provides the isomorphisms of the semilattice reducts. Further, \( \diamond_{n}^{\mathcal{S}}(C_{\vec{\alpha}}) = C_{\diamond_{n}^{3}(\vec{\alpha})} \) for all \( \vec{\alpha} \in I \) by Lemma 8.8, (i). Consequently, \( \diamond_{n}^{\mathcal{S}} \) corresponds to the operation \( \diamond_{n}^{3} \) of the algebra (iii). On the other hand, \( \nabla_{n}^{\mathcal{S}}(C_{\vec{\alpha}}) = v(\nabla_{n}^{RC}(\vec{\alpha})) \). Hence, \( \nabla_{n}^{\mathcal{S}} \) corresponds to \( \nabla_{n}^{RC} \) of the algebra (iv).

We remark that the algebra \( \mathcal{C}(\mathcal{S}) \) has rather simple definitions of \( \land \) and \( \diamond_{n} \), but a somewhat convoluted definition of \( \nabla_{n} \). In contrast, \( \mathcal{T}_{RC}^{0} \) has simple \( \land \) and \( \nabla_{n} \) but a somewhat convoluted \( \diamond_{n} \). The algebra \( \mathcal{I} \), perhaps the most elegant of all three, has a more complicated meet operation (though the order relation \( \leq_{\mathcal{I}} \) is simple).

Finally, we briefly return to the subject of conservativity spectra and look at them from the point of view of the established isomorphisms.

We call a theory \( S \) in the language of \( PA \) bounded if \( S \) is contained in a consistent finitely axiomatizable theory. The unboundedness theorem by Kreisel and Lévy [39] implies that \( \text{ord}_{n}(S) = 0 \) for all sufficiently large \( n \in \omega \) whenever \( S \) is bounded. We need to restrict ourselves to bounded subtheories of \( PA \) if we want to establish a bijection between their conservativity spectra and the Ignatiev algebra.

Let \( \text{sp}(T) \) denote the conservativity spectrum of \( T \).
Theorem 9.10. (i) If $T$ is a Gödelian extension of $\text{EA}^+$ and $\text{PA} \vdash T$, then $\text{sp}(T) \in \mathcal{I}$. If, in addition, $T$ is bounded, then $\text{sp}(T) \in I$.

(ii) Let $\bar{\alpha} \in \mathcal{J}$, let $A$ be a variable-free $\text{RC}^\nabla$-formula corresponding to $\bar{\alpha}$ via the isomorphism, and let $A^* \in \mathfrak{S}_{\text{EA}^+}^0$ be its arithmetical interpretation. Then $A^*$ is a bounded subtheory of $\text{PA}$ and $\bar{\alpha} = \text{sp}(A^*)$.

(iii) Under these conditions, if $T$ is a Gödelian extension of $\text{EA}^+$ and $\text{sp}(T) = \bar{\alpha}$, then $T \vdash A^*$. 

Proof. (i) In view of Proposition 7.4, for the first claim it is sufficient to prove that $\alpha_n \leq \varepsilon_0$ for all $n \in \omega$. Because $\text{PA}$ contains $T$, this follows from Proposition 7.1, (i).

Since $\text{PA}$ is equivalent to the union of theories $\{R_n(1) : n \in \omega\}$, any finitely axiomatizable subtheory of $\text{PA}$ is contained in a theory of the form $R_n(1)$ for some $n \in \omega$ (we write 1 instead of $1_{\text{EA}^+}$). Consequently, its conservativity spectrum majorizes that of $R_n(1)$ in the sense of $\leq_{\mathcal{I}}$, that is, it belongs to $\mathcal{I}$.

(ii) That $A^*$ is a bounded subtheory of $\text{PA}$ easily follows by induction on the build-up of $A$. The equality $\text{sp}(A^*) = \bar{\alpha}$ follows from Theorem 7.5.

(iii) This follows from the fact that any theory $T$ such that $\text{sp}(T) \leq_{\mathcal{J}} \bar{\alpha}$ must contain the union of progressions

$$\text{R}^{\alpha_0}(1) \land \text{R}^{\alpha_2}(1) \land \cdots \land \text{R}^{\alpha_k}(1),$$

which is equivalent to $A^*$ by Theorem 7.5. □

Let $\text{th}: \mathcal{J} \to \mathcal{E}_{\text{EA}^+}$ denote the natural isomorphic embedding of $\text{RC}^\nabla$-algebras. As we have already noted, $\text{th}(\bar{\alpha})$ is a bounded subtheory of $\text{PA}$ for each $\bar{\alpha}$.

Corollary 9.11. The maps $\text{th}$ and $\text{sp}$ form a Galois connection: for each bounded subtheory $S$ of $\text{PA}$,

$$\text{sp}(S) \leq_{\mathcal{J}} \bar{\alpha} \iff S \leq_{\text{EA}^+} \text{th}(\bar{\alpha}).$$

Remark 9.12. The map $\text{sp}$ is order-preserving, but it is not a semilattice homomorphism, even when restricted to bounded subtheories of $\text{PA}$. For example, it is well known that $\text{ord}_1(I\Sigma_1) = \omega = \text{ord}_1(I\Pi_2^-)$ and both theories are $\Pi_2$-regular:

$$\text{sp}(I\Pi_2^-) = (\omega^\omega, \omega, 0, \ldots),$$

$$\text{sp}(I\Sigma_1) = (\omega^\omega, \omega, 1, 0, \ldots).$$

On the other hand, $\text{ord}_1(I\Sigma_1 \land_{\text{EA}} I\Pi_2^-) = \omega^2 > \omega$ and

$$\text{sp}(I\Sigma_1 \land_{\text{EA}} I\Pi_2^-) = (\omega^{\omega^2}, \omega^2, 1, 0, \ldots).$$

10. A universal Kripke frame for the variable-free fragment of $\text{RC}^\nabla$

In view of Theorem 9.9 it is natural to ask whether one can describe a convenient universal Kripke frame for the variable-free fragment of $\text{RC}^\nabla$. There are two known general constructions associating with an SLO $\mathfrak{S} = (B, \land_{\mathfrak{S}}, \{a_{\mathfrak{S}} : a \in \Sigma\})$ its ‘dual’ Kripke frame in such a way that $\mathfrak{S}$ is embeddable into the algebra of subsets of that frame (see [37], §4.1). One construction is similar to the construction of the canonical model of a strictly positive logic $L$ from its Lindenbaum–Tarski algebra.
and goes from \( \mathfrak{B} \) to the set of all filters on \( \mathfrak{B} \) equipped with binary relations \( \{ R_a : a \in \Sigma \} \) such that, for all filters \( F \) and \( G \),

\[
FR_a G \overset{\text{def}}{\iff} \forall x \in G \ a^{\mathfrak{B}}(x) \in F.
\]

The corresponding frame for the \( \mathcal{RC}^\nabla \)-algebra \( \mathfrak{I} \) is constructively described in [13] in terms of appropriate sequences of ordinals. However, the relations of the frame look rather complicated, so that one would really want a simpler construction for practical use.

Another approach (see [34] and [37]) is to consider the set \( \mathfrak{B} \) itself as a dual space and to specify binary relations on \( \mathfrak{B} \) by

\[
xR_a y \overset{\text{def}}{\iff} x \not\leq_{\mathfrak{B}} a^{\mathfrak{B}}(y).
\]

Let \( \mathfrak{B}^* \) denote the Kripke frame \( (\mathfrak{B}, \{ R_a : a \in \Sigma \}) \) together with the canonical valuation \( v : \mathfrak{B} \to \mathcal{P}(\mathfrak{B}) \), where \( v(x) := \{ y \in \mathfrak{B} : y \leq_{\mathfrak{B}} x \} \).

**Lemma 10.1.** For all \( x, y \in \mathfrak{B} \) and \( a \in \Sigma \) the following relations hold in the model \( \mathfrak{B}^* \):

(i) \[ v(x \wedge_{\mathfrak{B}} y) = v(x) \cap v(y); \]

(ii) \[ R_a^{-1}(v(x)) = v(a^{\mathfrak{B}}(x)). \]

**Proof.** The assertion (i) is just the fact that \( x \wedge_{\mathfrak{B}} y \) is the g.l.b. of \( x \) and \( y \). To prove (ii) we argue as follows: \( z \in R_a^{-1}(v(x)) \) means there is a \( u \leq_{\mathfrak{B}} x \) such that \( zR_a u \), that is, \( z \leq_{\mathfrak{B}} a^{\mathfrak{B}}(u) \). Thus, if \( z \in R_a^{-1}(v(x)) \), then we have \( a^{\mathfrak{B}}(u) \leq_{\mathfrak{B}} a^{\mathfrak{B}}(x) \) by monotonicity, and therefore \( z \leq_{\mathfrak{B}} a^{\mathfrak{B}}(x) \).

If \( z \leq_{\mathfrak{B}} a^{\mathfrak{B}}(x) \), then we take \( x \) for \( u \) and observe that \( u \leq_{\mathfrak{B}} x \) and \( z \leq_{\mathfrak{B}} a^{\mathfrak{B}}(u) \), hence \( z \in R_a^{-1}(v(x)) \). \( \square \)

We obtain the following corollaries.

**Proposition 10.2.** (i) The map \( v : \mathfrak{B} \to \mathcal{P}(\mathfrak{B}) \) is an embedding of \( \mathfrak{B} \) into the algebra \( (\mathcal{P}(\mathfrak{B}), \cap, \{ R_a^{-1} : a \in \Sigma \}) \).

(ii) If \( A \) and \( B \) in \( L_{\Sigma} \) are variable-free, then \( A \vdash B \) holds in \( \mathfrak{B} \) if and only if \( \mathfrak{B}^*, x \models A \to B \) for all \( x \in B \).

**Corollary 10.3.** The variable-free fragment of \( \mathcal{RC}^\nabla \) is complete with respect to \( \mathfrak{I}^* \).

The Kripke frame \( \mathfrak{I}^* \) has a simple constructive characterization. We know that its domain is the set \( I \) of all sequences of ordinals \( \vec{\alpha} = (\alpha_0, \alpha_1, \ldots) \) such that, for all \( n \in \omega \),

\[
\alpha_n < \varepsilon_0 \quad \text{and} \quad \alpha_{n+1} \leq \ell(\alpha_n).
\]

Our task is to characterize the relations \( R_n^* \) and \( S_n^* \) on \( I \) corresponding to \( \Diamond_n \) and \( \nabla_n \), respectively, for each \( n \in \omega \), where

\[
\vec{\alpha} R_n^* \vec{\beta} \overset{\text{def}}{\iff} \vec{\alpha} \leq I_n \vec{\beta},
\]

\[
\vec{\alpha} S_n^* \vec{\beta} \overset{\text{def}}{\iff} \vec{\alpha} \leq I_n^\nabla \vec{\beta}.
\]

The answer is given by the following proposition.
Proposition 10.4. For all $\bar{\alpha}, \bar{\beta} \in I$,
(i) $\bar{\alpha}R_n^*\bar{\beta} \iff \forall i \leq n \alpha_i > \beta_i$,
(ii) $\bar{\alpha}S_n^*\bar{\beta} \iff \forall i \leq n \alpha_i \geq \beta_i$.

Proof. The assertion (ii) is obvious since
$$\nabla^2_n\bar{\beta} = (\beta_0, \beta_1, \ldots, \beta_n, 0, \ldots).$$
To prove (i) recall that
$$\diamondsuit^2_n\bar{\beta} = (\beta'_0, \beta'_1, \ldots, \beta'_n, 0, \ldots),$$
where $\beta'_i = 0$ for $i > n$ and $\beta'_i = \beta_i + \omega^{\beta_{i+1}}$ for $i \leq n$. Clearly, $\beta'_i > \beta_i$ for all $i \leq n$. Hence, the ‘only if’ part of the claim is obvious.

To prove the ‘if’ part, we assume that $\alpha_i > \beta_i$ for all $i \geq n$ and prove that $\alpha_i \geq \beta'_i$ for all $i \geq n$ by downwards induction on $i \leq n$. If $i = n$, then $\beta'_i = \beta_i + 1$ and the claim is clear. If $i < n$, then $\alpha_i > \beta_i$ and by the induction hypothesis $\alpha_{i+1} \geq \beta'_{i+1}$. Since $\alpha \in I$, we have $\ell(\alpha_i) \geq \alpha_{i+1} \geq \beta'_{i+1}$. At this point we need an auxiliary lemma.

Lemma 10.5. For any ordinals $\alpha$, $\beta$, and $\gamma$, if $\alpha > \beta$ and $\ell(\alpha) \geq \gamma$, then $\alpha \geq \beta + \omega^\gamma$.

Proof. We can write $\alpha = \beta + \nu$ with $\nu > 0$. Then $\ell(\nu) = \ell(\alpha) \geq \gamma$; hence $\nu \geq \omega^\gamma$ and $\alpha = \beta + \nu \geq \beta + \omega^\gamma$. □

From this lemma we conclude that $\alpha_i \geq \beta_i + \omega^{\beta_{i+1}} = \beta'_i$, and the induction step is complete. □

Looking at the frame $\mathcal{I}^*$ as a dual of the Lindenbaum–Tarski algebra of the variable-free fragment of $RC^\nabla$, we observe that, for any $A, B \in \mathcal{F}^\nabla$, $AR_n^*B$ holds if and only if $A \vdash_{RC^\nabla} \nabla_n B$. Hence $R_n^*$ is the same as the previously considered relation $<_n$ on words (now extended to all variable-free formulas of $RC^\nabla$).

On the other hand, $AS_n^*B$ holds if and only if $A \vdash_{RC^\nabla} \diamondsuit_n B$. Therefore, $S_n^*$ is the same as the $\Pi^0_{n+1}$-conservativity relation previously denoted by $\vdash_n$ (cf. §4).

Instead of the Lindenbaum–Tarski algebra of the variable-free fragment of $RC^\nabla$ we can also work directly with its isomorphic arithmetical counterpart, the SLO $\mathcal{G}_T^0$ for a sound Gödelian extention $T$ of $EA$. Then $\sigma R_n^*\nu$ means that the Gödelian theory $\sigma$ proves $R_n(\nu)$, and $\sigma S_n^*\nu$ means that $\nu$ is $\Pi^0_{n+1}$-conservative over $\sigma$.

Remark 10.6. The same definitions also apply to a much larger Kripke frame $\mathcal{G}_T^*$ that is dual to the $RC^\nabla$-algebra of all Gödelian extensions of $T$, $\mathcal{G}_T$.

Remark 10.7. A recent paper by Hermo Reyes and Joosten [30] introduces a universal Kripke frame for the so-called Turing–Schmerl Calculus. This model turns out to be very similar to $\mathcal{I}^*$. The differences amount to the following two aspects. First, their relations $R_n$ can be defined as $R_n^* \cap \leq_3$. This reflects the fact that all their modalities satisfy the principle $\diamond A \vdash A$. Second, their models lack the $S_n$ relations, but allow the $\alpha$-iterations of the relations $R_n$.
Appendix A. Irreflexivity of \(<_0\) in RC

We work in (the variable-free fragment of) the reflection calculus RC. We will use the techniques of Kripke models for RC. The notions of the canonical tree for a formula \(A\), its RC-closure \(\text{RC}[A]\), and an RC-model were defined in [7]. We recall only that \(\text{RC}[A]\) is an (in a sense, minimal) RC-model satisfying \(A\) at the root. Its valuation will be empty if \(A\) is variable-free.

The following lemma is easily obtained from Lemma 3.4 by taking into account that words in \(\mathbb{W}_n\) are linearly preordered by the relation \(<_n\).

**Lemma A.1.** Any variable-free formula of RC is equivalent to \(\top\) or to a formula of the form \(A \equiv \bigwedge_{i \leq k} \Diamond_{m_i} A_i\), where:

(i) \(A_i \in \mathbb{F}_{m_i}\) for each \(i\);
(ii) \(m_0 > m_1 > \cdots > m_k\);
(iii) \(\Diamond_{m_i} A_i \not\prec_{\text{RC}} \Diamond_{m_j} A_j\) for all \(j > i\).

Such formulas are said to be properly ordered. If \(A\) is properly ordered, then \(\text{RC}[A]\) can be characterized as follows.

If \(A \equiv \top\), then \(\text{RC}[A]\) is the irreflexive singleton frame. If \(A \equiv \bigwedge_{i \leq k} \Diamond_{m_i} A_i\), then \(\text{RC}[A]\) consists of the disjoint union of the frames \(\text{RC}[A_i]\) for all \(i \leq k\), augmented by a new root \(a\). In addition to all the relations inherited from the frames \(\text{RC}[A_i]\), the following relations are postulated:

1. \(aR_n x\) for each \(i \leq k\), \(n \leq m_i\), and \(x \in \text{RC}[A_i]\);
2. \(xR_n y\) for each \(i \leq k\), \(n < m_i\), and \(x, y \in \bigcup_{j \leq i} \text{RC}[A_j]\);
3. \(xR_n y\) for each \(i \leq k\), \(n \leq m_i\), \(y \in \text{RC}[A_i]\), and \(x \in \bigcup_{j < i} \text{RC}[A_j]\).

The following lemma is routine.

**Lemma A.2.** \(\text{RC}[A]\) thus described is an RC-frame.

**Theorem A.3.** For any formula \(A\) of RC,

\[A \not\prec_{\text{RC}} \Diamond_0 A.\]

**Proof.** It is sufficient to prove the claim for variable-free and properly ordered \(A\). For such an \(A\) we argue by induction on the length of \(A\). The base is trivial. Suppose that \(A = \bigwedge_{i \leq k} \Diamond_{m_i} A_i\). If \(A \vdash \Diamond_0 A\), then there is a homomorphism \(f\) of \(\text{RC}[A]\) into itself such that \(f(o) \in \text{RC}[A_i]\).

Let \(X\) denote the subset of \(\text{RC}[A]\) corresponding to \(\text{RC}[A_i]\). Consider any \(n \geq m_i\) and an \(R_n\)-arrow whose source is in \(X\). By the construction of \(\text{RC}[A]\), this arrow can only be an old arrow from the frame \(\text{RC}[A_i]\). Hence the target of the arrow will also be in \(X\). Since \(A_i \in \mathbb{F}_{m_i}\), it follows that \(f(X \cup \{a\}) \subseteq X\). The subset \(X \cup \{a\}\) together with all the inherited relations can be regarded as a submodel of \(\text{RC}[A]\) isomorphic to \(\text{RC}[\Diamond_{m_i} A_i]\). Hence \(f\) induces a homomorphism

\[f: \text{RC}[\Diamond_{m_i} A_i] \to \text{RC}[A_i].\]

This implies that either \(A_i \vdash_{\text{RC}} \Diamond_{m_i} A_i\) (if \(f(o)\) is the root of \(\text{RC}[A_i]\)) or \(A_i \vdash_{\text{RC}} \Diamond_{m_i} A_i \vdash_{\text{RC}} \Diamond_{m_i} A_i\) (if \(f(o)\) is strictly above the root). In any case \(A_i \vdash_{\text{RC}} \Diamond_{m_i} A_i \vdash_{\text{RC}} \Diamond_0 A_i\), contradicting the induction hypothesis. \(\square\)
Appendix B. Uniform definability of computable operators

Theorem B.1. An operator \( R : \mathcal{G}_{EA} \to \mathcal{G}_{EA} \) is uniformly definable if and only if \( R \) is computable.

Proof. The main point is to show that computable operators \( R \) are uniformly definable. Let \( R \) be computable, so that there is an elementary \( \Sigma_1^0 \)-formula \( \text{Ax}_R(x, y) \) such that \( \text{Ax}_R(x, \overline{\sigma}) \) numerates the theory \( R(\sigma) \) for each \( \sigma \). Note that \( R(\sigma) \) is an elementary formula for each \( \sigma \). We claim that one can select \( \text{Ax}_R \) in such a way that for each \( \sigma \) there is an elementary numeration \( \delta \) such that

\[
\text{EA} \vdash \forall x \ (\text{Ax}_R(x, \overline{\sigma})) \leftrightarrow \delta(x)).
\]

Let \( \text{Sat}_{\Delta_0}(e, x) \) be a \( \Sigma_1^0 \)-truth-definition for elementary formulas that can be represented in the form

\[
\text{Sat}_{\Delta_0}(e, x) \leftrightarrow \exists q \leq 2^n_{d(e)} T(e, x, q),
\]

where \( T(e, x, q) \) is an elementary formula expressing that \( q \) is the protocol of a computation verifying that the elementary formula \( e \) holds on the assignment \( x \). For each specific formula \( e \), the size of \( q \) is bounded by a \( d \)-fold iterate of the exponential function in \( x \), where \( d \) depends elementarily on \( e \). Whereas in \( \text{EA} \) one cannot prove that \( 2^n_{d(e)} \) is defined for all \( e \) and \( x \), it is known that for each specific \( n \) there is an \( \text{EA} \)-proof of the formula \( \forall x \exists y 2^n_x = y \). So for each specific formula \( \sigma \) there is a number \( n = d(\overline{\sigma}) \) such that, provably in \( \text{EA} \),

\[
\forall x \ (\text{Sat}_{\Delta_0}(\overline{\sigma}, x) \leftrightarrow \exists q \leq 2^n_{d(e)} T(\overline{\sigma}, x, q)).
\]

Now if \( F_R(x, y) \) is a \( \Sigma_1^0 \)-formula strongly representing the map \( R : \overline{\sigma} \mapsto \overline{R(\sigma)} \), then we can define

\[
\text{Ax}_R(x, y) \overset{\text{def}}{=} \exists e \ (F_R(y, e) \land \text{Sat}_{\Delta_0}(e, x)).
\]

Thus, for each \( \sigma \) there is a provably unique \( \tau = R(\sigma) \) such that \( \text{EA} \vdash F_R(\overline{\sigma}, \overline{\tau}) \). Hence, \( \text{Ax}_R(x, \overline{\sigma}) \) is provably equivalent to \( \text{Sat}_{\Delta_0}(\overline{\tau}, x) \), and this is equivalent to an elementary formula by (13). This proves (12).

To provide a uniform definition of \( R \) we apply a version of Craig’s trick and let

\[
\text{Ax}_{R'}(x, y) \overset{\text{def}}{=} \exists z, p \leq x \ (x = \text{disj}(z, \overline{p \neq \overline{p}}) \land W_R(z, y, p)),
\]

where \( W_R(z, y, p) \) is an elementary formula expressing that \( p \) witnesses \( \text{Ax}_R(z, y) \). Here we may assume that

\[
\text{EA} \vdash W_R(z, y, p) \rightarrow z \leq p.
\]

Clearly, \( \text{Ax}_{R'}(x, y) \) is elementary, and the condition (ii) of uniform definability is satisfied. Externally, it numerates the same family of theories as \( \text{Ax}_R(x, y) \).

We show that for each \( \sigma \),

\[
\text{EA} \vdash \forall x \ (\Box_R(\sigma)(x) \rightarrow \Box_{R'}(\sigma)(x)).
\]

First we obtain an elementary numeration \( \delta \) such that

\[
\text{EA} \vdash \forall x \ (\text{Ax}_R(x, \overline{\sigma}) \leftrightarrow \delta(x)).
\]
It follows that $\text{EA} \vdash \forall x (\square_{R}(\sigma)(x) \leftrightarrow \square_{\delta}(x))$. Thus, using $\Pi_{2}^{0}$-conservativity of $\text{B} \Sigma_{1}$ over $\text{EA}$, we see that it suffices to prove that

$$\text{EA} + \text{B} \Sigma_{1} \vdash \forall x (\square_{\delta}(x) \rightarrow \square_{R}(\sigma)(x)).$$

By using $\text{B} \Sigma_{1}$ it suffices to prove that $\text{EA} \vdash \forall x (\delta(x) \rightarrow \square_{R}(\sigma)(x))$. We reason in $\text{EA}$. Assume that $\delta(x)$; then $\text{Ax}_{R}(x, \overline{\sigma^{1}})$. Consequently, there is a witness $p$ such that $W_{R}(x, \overline{\sigma^{1}}, p)$. Then for $u := \text{disj}(x, \overline{p}, \overline{q})$ we have $\text{Ax}_{R}(u, \overline{\sigma^{1}})$, and from $p$ we obtain a proof of $\overline{p} \neq \overline{p}$ and hence a proof of $x$ from the hypothesis $u$ in an elementary way. Therefore, $\square_{R}(\sigma)(x)$. □

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Lev D. Beklemishev
Steklov Mathematical Institute
of Russian Academy of Sciences, Moscow
E-mail: bekl@mi-ras.ru

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