Entanglement of stabilizer codewords

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Abstract

The geometric measure, the logarithmic robustness and the relative entropy of entanglement are proved to be equal for a stabilizer quantum codeword. The entanglement upper and lower bounds are determined with the generators of code. The entanglement of dual-containing CSS codes, Gottesman codes and the related codes are given. An iterative algorithm is developed to determine the exact value of the entanglement when the two bounds are not equal.

Index Terms: quantum code; Pauli measurement; multipartite entanglement

1 Introduction

A quantum code encodes logical qubits in physical qubits. A quantum code is usually a multipartite state if the physical qubits are owned by different parties. The quantification of multipartite entanglement is basically open even for a pure multipartite state until now. However, a variety of different entanglement measures have been proposed for multipartite setting. Among them are the (Global) Robustness of Entanglement [1], the Relative Entropy of Entanglement [2] [3], and the Geometric Measure [4]. The geometric measure measures the minimal noise (arbitrary state) that we need to add to make the state separable. The geometric measure is the distance of the state to its closest product state in terms of fidelity. The relative entropy of entanglement is a valid entanglement measure for a multipartite state, it is the relative entropy of the state to its closest separable state. The quantification of multipartite entanglement is usually very difficult as most measures are defined as the solutions to difficult variational problems. Even for pure multipartite states, the entanglement can only be obtained for some special scenarios. Fortunately, due to the inequality on the logarithmic robustness, relative entropy of entanglement and geometric measure of entanglement [5] [6] [7], these entanglement measures are all equal for stabilizer states [8]. A stabilizer state is a multiqubit pure state which is the unique simultaneous eigenvector of a complete set of commuting observables in the Pauli group, the latter consisting of all tensor products of Pauli matrices and the identity with an additional phase factor. A stabilizer state is the special case of a quantum code that encodes zero logical qubit. We may ask if the three entanglement measures are equal for a generic quantum codeword. The answer is true as shown in Section II. The three equal entanglement measures for a codeword then are simply called the entanglement of the codeword. Then the rest of the paper is organized as follows. In Section III, we derive the upper bound of the entanglement with Pauli measurements. In Section IV, the lower bound of the entanglement is obtained with the bipartition of the physical qubit system. Section V deals with the entanglement of the codewords of CSS codes. The entanglement of the family of Gottesman codes is the topic of Section VI. Section VII provides an iterative method for the possible exact value of the entanglement when the two bounds are not equal. Conclusions are drawn in Section VIII.

2 The entanglement measure for quantum codewords

The global robustness of entanglement $R(\rho)$ [1] is defined as

$$R(\rho) = \min t$$

such that there exists a state $\Delta$, satisfying

$$\sigma = (\rho + t\Delta)/(1 + t) \in Sep$$

where Sep is the set of separable states. The logarithmic robustness is

$$LR(\rho) = \log_2(1 + R(\rho)).$$

The relative entropy of entanglement is defined as the “distance” to the closest separable state in terms of relative entropy [3],

$$E_r(\rho) = \min_{\omega \in Sep} S(\rho \| \omega),$$

where $S(\rho \| \omega) = - S(\rho) - tr\{\rho \log_2 \omega\}$ is the relative entropy, $S(\rho)$ is the von Neumann entropy. The geometric measure of entanglement for pure state $|\psi\rangle$, is defined as

$$E_g(|\psi\rangle) = \min_{|\phi\rangle \in Pro} - \log_2 \langle \phi \mid \psi \rangle^2,$$

where $Pro$ is the set of product states. An extension of the definition of mixed state $\rho$ is also available, $E_g(\rho) = \min_{\omega \in Sep} - \log_2 tr(\rho \omega)$, however, $E_g$ is an entanglement monotone only for pure states $\rho = |\psi\rangle \langle \psi|$. It has been shown that the maximal number $N$ of pure states in the set $\{|\psi\rangle | i = 1,...,N\}$ that can be discriminated perfectly by LOCC is bounded by the amount of entanglement they contain [6]:

$$\log_2 N \leq n - LR(|\psi_1\rangle) \leq n - E_r(|\psi_1\rangle) \leq n - E_g(|\psi_1\rangle),$$
where \( n = \log_2 D_H \), \( D_H \) is the total dimension of the Hilbert space, and \( \bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i \) denotes the "average".

An \( n \)-qubit stabilizer state \(|S\rangle\) is defined as a simultaneous eigenvector with eigenvalue 1 of \( n \) commuting and independent Pauli group elements \( M_i \). The \( n \) eigenvalue equations \( M_i |S\rangle = |S\rangle \) define the state \(|S\rangle\) completely (up to an arbitrary phase). The group generated by the product of the \( n \) operators \( M_i \) is called the stabilizer \( S \), and \( M_i \) are generators of \( S \). A subgroup of \( S \) with \( n-k \) generators is also called a stabilizer \(^{12}\), denoted as \( M \). However, \( M \subset S \) stabilizes a \( 2^k \) dimensional space.

In principle, such a space is the coding space \(|\psi\rangle\), s.t. \( T |\psi\rangle = |\psi\rangle \), \( \forall T \in M \), corresponds to a stabilizer code encoding \( k \) into \( n \) qubits. In addition to the \( n-k \) stabilizer generators, a stabilizer code also has logical operators \( \bar{X}_1, \ldots, \bar{X}_k \) and \( \bar{Z}_1, \ldots, \bar{Z}_k \). We can take the basis codewords for this code to be

\[
|\overline{0}\rangle = \prod_{T \in M} T |0\rangle^\otimes n, \\
|\overline{1}\rangle = \prod_{i=1}^{k} \bar{X}_i |\overline{0}\rangle,
\]

where \( c = (c_1, \ldots, c_k) \) is a binary vector. \( \bar{Z}_i |\overline{0}\rangle = |\overline{0}\rangle \) for \( i = 1, \ldots, k \).

For a stabilizer state \(|S\rangle\), it has been shown that \(^8\)

\[
LR(|S\rangle) = E_r(|S\rangle) = E_g(|S\rangle).
\] (7)

This is also true for a stabilizer quantum codeword \(|\overline{c}\rangle\).

**Proposition 1** The entanglement measures of the logarithmic robustness, the relative entropy of entanglement and the geometric measure of entanglement are all equal for a stabilizer quantum codeword \(|\overline{c}\rangle\).

Proof: It is enough to prove the statement for \(|\overline{0}\rangle\), since \(|\overline{c}\rangle\) is locally equivalent to \(|\overline{0}\rangle\). Notice that \(|\overline{0}\rangle\) is stabilized by \( \{M_1, \ldots, M_{n-k}, \bar{Z}_1, \ldots, \bar{Z}_k\} \). The \( n-k \) generators and \( k \) logical operators commute with each other and are independent. Thus \(|\overline{0}\rangle\) is a stabilizer state, the three entanglement measures are equal according to Ref. \(^8\).

For a stabilizer quantum codeword, we will simply call these three quantities the entanglement, and denote as \( E(|\overline{c}\rangle) \).

### 3 Entanglement upper bound

A generator which is composed of \( Z \) operator or identity in each qubit and does not contain \( X \) or \( Y \) operators in any qubit is called a \( Z \)-type generator.

**Proposition 2** The entanglement of a codeword is upper bounded by the minimal number of stabilizer generators which are not \( Z \)-type generators.

Proof: The codeword \(|\overline{0}\rangle\) = \( N \prod_{i=1}^{n-k} (I + M_i) |0\rangle^\otimes n \), where \( N \) is the normalization factor. For a \( Z \)-type generator \( M_i \), we have \( (I + M_i) |0\rangle^\otimes n = 2 |0\rangle^\otimes n \). In the operator product \( \prod_i (I + M_i) \), we may move the factor \( (I + M_i) \) to the rightmost. The number of terms \( R \) in the linear decompositions of \(|\overline{0}\rangle\) into product states is upper bounded by \( 2^n \), with \( r' \) the number of non-\( Z \)-type generators. Let \( r = \min r' \) be the minimal number of non-\( Z \)-type generators. Notice that we can artificially increase the number of non-\( Z \)-type generators by replacing a \( Z \)-type generator with the product of the \( Z \)-type generator and a non-\( Z \)-type generator. Hence we count the minimal number of non-\( Z \)-type generators. It follows that the Schmidt measure \(^9\)

\[
E_s = \min \log_2 R
\]

is upper bounded by \( r \). The geometric measure is upper bounded by Schmidt measure \(^{10}\), hence the theorem follows.

For a stabilizer group \( M \), up to over all phase \( \pm 1, \pm i \), each generator \( M_i = X^{a_i} Z^{b_i} \), with \( a_i = \bigotimes_j X^{a_{ij}} Z^{b_{ij}} \), where \( a_i \) and \( b_i \) are the binary vectors \( (a_{11}, a_{12}, \ldots, a_{in}) \) and \( (b_{11}, b_{12}, \ldots, b_{in}) \), respectively. An alternative representation of the generator \( M_i \) is \( (a_i | b_i) \). We may use generator matrix ( follow Ref. \(^{11}\) ), it is called stabilizer matrix in Ref. \(^{12}\) ) \((A | B)\) to represent the stabilizer group \( M \), where \( A \) and \( B \) are \( (n-k) \times n \) matrices with elements \( A_{ij} = a_{ij}, B_{ij} = b_{ij} \). It is always possible to arrange \((A | B)\) in the form of (see e.g. \(^{12}\) Ch.4)

\[
\begin{pmatrix}
I & D & F & G \\
0 & 0 & J & K
\end{pmatrix}
\] (8)

by the permutations of the qubits and replacements of the generators with other elements in the stabilizer group. Here \( I \) is an \( r \times r \) identity matrix, with \( r \) the \( F_2 \) rank of \( A \), where \( F_2 \) denotes the integer field \([0, 1]\) with addition and multiplication modulo 2. With the standard form \(^8\) of the generator matrix, we may improve the upper bound of the entanglement. We then investigate the effect of Pauli measurements on codewords. We work on the generators of \( M \) in the first \( r \) lines of \(^8\) and neglect the other \( n-k-r \) \( Z \)-type generators. We have \( M_i = X \otimes M'_i \) or \( M_i = Y \otimes M'_i \), \( M_{i+1} = Z \otimes M'_{i+1} \) or \( M_{i+1} = I \otimes M'_{i+1} \) (\( i = 1, \ldots, r-1 \)) for the standard form of the generators. Denote \(|\overline{0}_{n-1}\rangle\) = \( N' \prod_{i=1}^{r-1} (I + M'_i) |0\rangle^\otimes (n-1) \), where \( N' \) is the normalization factor, then Pauli \( Z \) measurement on the first qubit will project the codeword \(|\overline{0}\rangle\) to

\[
P_{z+}(|\overline{0}\rangle) = |0\rangle \otimes |\overline{0}_{n-1}\rangle,
\]

\[
P_{z-}(|\overline{0}\rangle) = |1\rangle \otimes M'_i |\overline{0}_{n-1}\rangle.
\] (9, 10)

The projection operators on the \( q \)-th qubit are \( P_{z+} = \frac{1}{2} (I + Z_q) \). The two measurement results \( \pm 1 \) are equally probable. Similarly, the \( X \) or \( Y \) measurements on the first qubit also project the codeword to two equally probable states corresponding to the two measurement results \( \pm 1 \), except for possible special case of \( X \) or \( Y \) measurements with only one result. When \( M_1 = X \otimes M'_1 \), the Pauli \( X \) or \( Y \) measurements on the first qubit will project the codeword \(|\overline{0}\rangle\) to \( P_{z+}(|\overline{0}\rangle) = \frac{1}{2} (|0\rangle \pm |1\rangle) \otimes (I + \)
\( M_1 \) \( [0] \), \( P_{z \pm} \) \( [0] \) is \( \frac{1}{2} \) \((0 \pm i \{1]\)) \( (I \dagger \pm i M'_1) \) \( [0] \).

When \( M_1 = Y \otimes M'_1 \), the Pauli \( X,Y \) measurements on the first qubit will project the codeword \( [0] \) to \( P_{z \pm} \) \( [0] \) is \( \frac{1}{2} \) \((0 \pm i \{1]\)) \( (I \dagger \pm i M'_1) \) \( [0] \). The minimal number of local Pauli measurements to completely disentangles the state vector \( |\psi\rangle \) in each of the measurement results, we obtain the upper bound

\[
E_S(|\psi\rangle) \leq \log_2(N_{necq}),
\]

where \( N_{necq} \) is the number of measurement results with non-zero probability.

The minimal number of local Pauli measurements to disentangle a stabilizer quantum codeword can be called its "Pauli persistency".

**Proposition 3** The entanglement is upper bounded by "Pauli persistency" for a stabilizer quantum codeword \( [0] \).

**Proof:** With the formula (11), the fact that different measurement results of codeword are obtained with probability 1/2, and geometric measure is upper bounded by Schmidt measure, the statement follows.

**Example.**—Consider \([8,1,3]\) code \([13]\) with stabilizer in the standard form

\[
\begin{align*}
X & Z Z Z Z Z \ Y \\
I & X I Z I Z I Z \\
I & I X Z I I I Z \\
Z & Z Z X I I I I \\
Z & Z Z Y Z X Z \\
Z & Z I I Z X I I \\
Z & Z I I I X Y
\end{align*}
\]

the Pauli \( Z \) measures are applied at the 1, 5, 7 qubits. For each qubit measured, the corresponding row and column are deleted. What left for the remain qubits is the stabilizer

\[
\begin{align*}
X & I Z Z Z \\
I & X Z I Z \\
I & Z X I I \\
I & Z I I X
\end{align*}
\]

The \([0]\) generated by stabilizer \([12]\) is a product of graph state \( |G_4\rangle \) with \([0]\) \( = |G_4\rangle \otimes |0\rangle \). The graph state \( |G_4\rangle \) is generated by a new stabilizer obtained with deleting the last column of \([12] \). "Pauli persistency" of graph state \( |G_4\rangle \) is 2, so "Pauli persistency" for codeword of \([8,1,3]\) code in Grassl code-table \([13]\) is 5.

We will use proposition \([8]\) to obtain the entanglement upper bound of codeword in the following except Table 1, where proposition \([8]\) is used for tighter upper bounds.

**4 Entanglement lower bound**

The index of physical qubits is denoted as \( \mathcal{I} = \{1,2,\ldots,n\} \), for a bipartition, we may assign \( m \) qubits to \( A \), and the remain \( n-m \) qubits to \( B \). The index sets of \( A \) and \( B \) are \( \mathcal{I}_A \) and \( \mathcal{I}_B = \mathcal{I} - \mathcal{I}_A \), respectively.

The reduced state of the codeword \([0]\) then should be \( \rho_B = Tr_A [0] \rangle \langle 0 | \). The bipartite entanglement of the bipartition \( \{\mathcal{I}_A,\mathcal{I}_B\} \) then will be \( -Tr\rho_B \log_2 \rho_B \), the entropy of \( \rho_B \).

**Proposition 4** The entanglement of a codeword is lower bounded by any bipartite entanglement of the codeword, 

\[
E \geq -Tr\rho_B \log_2 \rho_B.
\]

**Proof:** If we define \( E_{\rho_B} \) as the relative entropy of entanglement with respect to some bipartition, we have that \( E_r \geq E_{\rho_B} \) since the set of fully separable states is a subset of the bipartite separable states. Notice that \( E_r = E \) and \( E_{\rho_B} \) is equal to the bipartite entanglement \( E_{\rho_B} = -Tr\rho_B \log_2 \rho_B \) for pure state, the statement then follows.

We will obtain the entropy of \( \rho_B \) by diagonalizing \( \rho_B \) and at last the entropy can be expressed with the code stabilizer. The entanglement of the codeword is lower bounded by the maximal bipartite entanglement among all bipartitions.

Since a \( Z \)-type generator does not contribute new items to codeword \([0]\) , we simply ignore \( Z \)-type generators. Thus we take \((A |B) = (I D |E F)\) in the following. \( A, B \) are \( r \times n \) binary matrices with \( r \) the number of non-\( Z \)-type generators and \( r \leq n - k \).

The codeword

\[
[0] = N \sum_{\mu} (-1)^{\alpha(\mu)} X^{\mu A} |0\rangle \otimes \langle 0|^n
\]

where the summation on \( r \) dimensional binary vector \( \mu \) is from \( (0,0,\ldots,0) \) to \( (1,1,\ldots,1) \) \( (\mu |A_i) \) is the \( j \)-th component of the binary vector \( \mu A \), \( N \) is the normalization factor, and \( \alpha(\mu) = \sum_{i<j} (\mu_i a_{ij}) (\mu_j b^T_{ij}) \). We may rewrite \( \alpha(\mu) \) as

\[
\alpha(\mu) = \frac{1}{2} [\mu \Gamma \mu^T - \text{Tr} (\Lambda \Gamma \Lambda)] = \frac{1}{2} \mu \Gamma_1 \mu^T,
\]

with \( \Lambda = \text{diag} \{\mu_1,\ldots,\mu_r\} \), and \( \Gamma_1 \) is the matrix \( \Gamma \) with diagonal elements nullified, where

\[
\Gamma = AB^T = F^T + DG^T.
\]

The convention for binary addition is mod 2. \( \Gamma \) is symmetric for any two generators should commute with each other, namely, \( AB^T + B^T A = 0 \). The reduced state \( \rho_B = \sum_{\mu,\mu'} \prod_{j \in \mathcal{I}_B} \delta(\mu_{A_j} \mu'_{A_j}) (-1)^{\alpha(\mu) + \alpha(\mu')} \otimes \prod_{j' \in \mathcal{I}_B} X^{\mu A}_{j'} |0\rangle \otimes (m-n) X^{\mu A}_{j'} \), disregarding normalization, which is

\[
\rho_B = \sum_{\mu,\mu'} \prod_{j \in \mathcal{I}_A} \delta(\mu_{A_j} \mu'_{A_j}) (-1)^{\alpha(\mu) + \alpha(\mu')} \langle (\mu A) | (\mu A) \rangle_B.
\]

with \( \langle (\mu A) | B \rangle = \langle (\mu A) | \mathcal{I}_{m+1,\ldots,n} (\mu A) \rangle_A \).
We then consider to diagonalize $\rho_B$ in order to obtain its entropy. Without loss of generality, let $I_1 = 1$, $I_2 = 2$, $I_m = m \leq r$, and denote $\mu = (\nu, \tau)$, with $\nu = (\mu_1, \ldots, \mu_m)$, $\tau = (\mu_{m+1}, \ldots, \mu_r)$. Then $|\langle \mu A \rangle_B| = |\langle \mu D \rangle_{n-r}| = |\tau, \mu D|$. Denote $|\Psi(\nu)\rangle = \sum_{\tau}(-1)^{\alpha(\nu)}|\langle \mu A \rangle_B|$, then
\[
\rho_B = \sum_{\nu', \nu', \tau', \tau} \delta_{\nu', \nu'}(-1)^{\alpha(\nu)+\alpha(\nu')}|\langle \mu A \rangle_B||\langle \mu A \rangle_B|
= \sum_{\nu', \nu'} \delta_{\nu', \nu'} |\Psi(\nu)\rangle \langle \Psi(\nu')\rangle = \sum_{\nu} |\Psi(\nu)\rangle \langle \Psi(\nu)|.
\]
For the orthogonality of $|\Psi(\nu)\rangle$, we turn to
\[
\langle \Psi(\nu') | \Psi(\nu) \rangle = \sum_{\tau, \tau'} (-1)^{\alpha(\nu)+\alpha(\nu')} \delta_{\nu', \nu} \delta_{\mu D, \mu' D}
= \sum_{\tau} (-1)^{\alpha(\nu, \tau)+\alpha(\nu', \tau)} \delta_{\nu', \nu} \delta_{\mu D, \mu' D} \delta_{\nu', \nu} \delta_{\mu D, \mu' D}.
\]
(15)
The bipartite entanglement of the codeword is at least $m$ when all $|\Psi(\nu)\rangle$ are orthogonal with each other. This is obviously from the factor that
\[
\rho_B = \frac{1}{2^n} \sum_{\nu = (0,0,\ldots,0)}^{1,1,\ldots,1} |\Psi(\nu)\rangle \langle \Psi(\nu)|,
\]
where $|\Psi(\nu)\rangle$ are orthonormal and the normalization factor is retained. The bipartite entanglement may be less than $m$ only when the non-orthogonality of the ensemble $|\Psi(\nu)\rangle$ is found. The conditions for nonzero $\langle \Psi(\nu') | \Psi(\nu) \rangle$ are
\[
(\nu + \nu', 0) D = 0, \quad (\nu' + \nu) \Gamma_3 = 0,
\]
(16)
where $0$ stands for the $r - m$ dimensional zero vector $(0, 0, \ldots, 0)$. $\Gamma_3$ is produced by deleting the first $m$ columns and the last $r - m$ rows of the $r \times r$ matrix $\Gamma_1$, so $\Gamma_3$ is a $m \times (r - m)$ submatrix of $\Gamma_1$. More explicitly, we may write $\Gamma_1$ as
\[
\Gamma_1 = \begin{bmatrix} \Gamma_2 & \Gamma_3 \\ \Gamma_3^T & \Gamma_4 \end{bmatrix}.
\]
Then $\alpha(\nu, \tau) + \alpha(\nu', \tau) = \frac{1}{2} \left( (\nu \Gamma_2 \nu' T + \nu' \Gamma_2 \nu' T) + (\nu + \nu') \Gamma_3 T + \tau \Gamma_4 T \right)$. The $\Gamma_4$ term always contributes a $+1$ factor in the summation of $\rho_B$ for $\Gamma_4$ is symmetric and with nullified diagonal elements. The $\Gamma_2$ term contributes a constant factor in the summation of $\rho_B$. Then $\rho_B$ follows. Let $D'$ be the matrix produced by deleting the last $r - m$ rows and preserving the first $m$ rows of $D$, the rank of the $m \times (n - m)$ matrix $Q(A, B) = (\Gamma_3, D')$ gives the number of independent vectors $|\Psi(\nu)\rangle$ for a specific bipartition of first $m$ qubits for $A$ with respect to last $n - m$ qubits for $B$. Maximizing with respect to all bipartitions except the $m > r$ cases, we hence obtain the maximal of bipartite entanglement as the lower bound of the entanglement
\[
E_i = \max_{\text{partitions}} \text{rank}_{\mathbb{F}_2} Q(A, B).
\]
Since $Q(A, B)$ is a $m \times (n - m)$ matrix, its rank must not exceed $\min\{m, n - m\} \leq \lfloor \frac{n}{2} \rfloor$, so we have $E_i \leq \lfloor \frac{n}{2} \rfloor$.

One of the special case that should be notified is when $m = r$. It follows that $\Gamma_3$ is an $r \times r$ matrix and does not exist at all. So Eq. (17) disappears, and we only need to consider Eq. (19). If $\text{rank}_{\mathbb{F}_2} D = r$, then Eq. (19) fulfills only when $\nu = \nu'$, hence $E_i = r$.

5 Entanglement of CSS codes

5.1 Dual-containing CSS codes

An important class of quantum codes, constructed from classical codes, invented by Calderbank, Shor [14] & Steane [15], has the generator matrix of the form (e.g. [16])
\[
A|B = \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix},
\]
(19)
where $U$ and $V$ are $l \times n$ matrices. Requiring $UV^T = 0$ ensures that the generators commute with each other. As there are $2l$ stabilizer conditions applying to $n$ qubit states, $k = n - 2l$ qubits are encoded in $n$ qubits. We may write the classical parity check matrix $U$ in a systematic way
\[
U = \begin{bmatrix} I & D \end{bmatrix}.
\]
(20)
Since $AB^T = 0$, we get $\alpha(\mu) = 0$ for all binary vectors $\mu$. Consider the case of $m = l$, we have $\nu = \mu$. Then $\langle \Psi(\nu') | \Psi(\nu) \rangle = \delta_{\nu', \nu} \delta_{\mu D, \mu' D}$. From $\nu D = \nu' D$, we have $(\nu + \nu') D = 0$. Thus the condition for the orthogonality of $|\Psi(\nu)\rangle$ is
\[
\nu D = 0 \Rightarrow \nu = 0,
\]
(21)
for all binary vector $\nu$. The lower bound of the entanglement of the codeword is $l$ when the condition (21) is fulfilled. For dual-containing code, we have $V = U$, thus $UU^T = 0$, so that $DD^T = I$, the condition (21) is fulfilled. The lower bound of entanglement is $E_i = l = \frac{n - k}{2}$.

The upper bound of the entanglement $E_u$ of the codeword (0) is the number of $X$ generators now, which is $l$. Thus the entanglement of dual-containing CSS codeword is
\[
E = \frac{n - k}{2}.
\]
(22)
For a CSS code that is not dual-containing, the upper bound $E_u$ is still $l$, the number of $X$ generators. The lower bound is the binary rank of $D$.

5.2 The graph state of a CSS code

A stabilizer code with stabilizer generators $M_1, \ldots, M_{n-k}$ and logical operations $\overline{X}_1, \ldots, \overline{X}_k$ and $\overline{Z}_1, \ldots, \overline{Z}_k$, is equivalent to the codeword stabilizer (CWS) code defined by codeword stabilizer $\{M_1, \ldots, M_{n-k}, \overline{Z}_1, \ldots, \overline{Z}_k\}$ and word operators which are products of $\overline{X}_i$. Any CWS code is locally Clifford-equivalent to a standard form of CWS code with a graph-state stabilizer and word operators consisting only of $Z$ operators. The standard codeword stabilizer is generated by $X_r \overline{Z}_s$. The set of $r,s$ forms the
adjacency matrix of the graph [17]. Hence, given a quantum stabilizer error-correcting code, we can always find the corresponding graph state. The entanglement of the codeword of the quantum stabilizer code and the graph state should be the same, since they are locally Clifford-equivalent. A CSS code has a generator matrix [19] and \( \mathbf{U} \) can further written in the form of [20]. We now construct the graph state stabilizer. The generator matrix of \( \{ M_1, \ldots, M_{n-k}, Z_1, \ldots, Z_k \} \) is

\[
\begin{pmatrix}
U & 0 \\
0 & V \\
0 & W
\end{pmatrix},
\]

where \( \langle 0 | W \rangle \) is the generator matrix for logical operations \( Z_1, \ldots, Z_k \). With elementary row transformation we have transformed \( U \) into the systematical form \( \begin{bmatrix} I & D \end{bmatrix} \). We want show that it is always possible to transform \( \begin{bmatrix} V \\ W \end{bmatrix} \) into the form of \( \begin{bmatrix} D' & I \end{bmatrix} \), where \( I \) is an \((n-l) \times (n-l)\) identity matrix. We first transform \( \begin{bmatrix} V \\ W \end{bmatrix} \) into \( \begin{bmatrix} R & P \end{bmatrix} \), where \( P \) is an upper triangle square matrix, namely \( P_{ij} = 0 \) for \( i > j \). There is the case that \( P_{jj} = 0 \), we then interchange the \( j \)-th qubit with some later qubit such that \( P_{jj} = 1 \). This is always possible since the elements of the \( j \)-th line of \( P \) can not be all zeros, otherwise the elements of \( j \)-th line of \( \begin{bmatrix} R & P \end{bmatrix} \) should be all zeros due to mutual commutation of the generators, namely, \( U V T = 0 \) and \( U W T = 0 \). That is

\[
IR^T + DP^T = 0.
\]

From which we can deduce that if the \( j \)-th line of \( P \) are all zeros, we have \( R_{ji} = 0 \), for all \( i \leq l \). An all zero line in \( \begin{bmatrix} R & P \end{bmatrix} \) means the generator is the identity, this is not the case. It is easy to transform \( \begin{bmatrix} R & P \end{bmatrix} \) to \( \begin{bmatrix} D' & I \end{bmatrix} \), then Eq. (24) will be

\[
D' = D^T.
\]

Performing Hadamard transformation to the last \( n-l \) qubits, the generator matrix undergoes the transformation

\[
\begin{bmatrix}
I & D \\
0 & 0 \\
D^T & I
\end{bmatrix} \Rightarrow \begin{bmatrix}
I & 0 \\
0 & I \\
0 & D^T
\end{bmatrix}.
\]

Thus the adjacency matrix of the locally Clifford-equivalent graph state of CSS codeword is

\[
\gamma = \begin{bmatrix}
0 & D^T \\
D^T & 0
\end{bmatrix}.
\]

A graph state with adjacency matrix of (25) is two-colorable, we can simply assign the first \( l \) qubits with one color and the remain \( n-l \) qubits with another. The entanglement upper bound should be [18] \( E_u = n - (n-l) = l \). The lower bipartite bound is [9] \( E_l = \frac{1}{2} \text{rank}_{\mathbb{F}_2} \gamma = \text{rank}_{\mathbb{F}_2} D \).

For a dual-containing CSS code, the corresponding graph state is further characterized by \( DD^T = I \) in addition to two-colorable. The \( D \) matrix has a full rank and the upper and lower bounds of entanglement coincide. The entanglement of the codeword is also \( E = l \) with the theory of graph state and formula (25).

### 5.3 Toric Codes

Toric code is proposed to encode quantum information in topological structure [19], it is a kind of quantum LDPC code [16]. The toric code is based on a \( k \times k \) square lattice on the torus. Each edge of the lattice is attached with a qubit, so there are \( n = 2k^2 \) qubits. For each vertex \( s \) and each face \( p \), operators of the following form are defined:

\[
A_s = \prod_{j \in \text{star}(s)} X_j, \quad B_p = \prod_{j \in \text{boundary}(p)} Z_j.
\]

These operators commute with each other. Due to \( \prod_s A_s = 1 \) and \( \prod_p B_p = 1 \), there are \( m = 2k^2 - 2 \) independent operators constitute the stabilizer of the toric code. The code encodes \( n-m = 2 \) qubits. Toric code is a kind of CSS code from its definition. We will show that it is not a dual containing code, the upper and lower bound of entanglement may not coincide. By proper numbering the edges, the generator matrix can be written in the following form

\[
U = \begin{bmatrix}
I & I & \Omega & \Omega \\
I & I & \Omega & \Omega' \\
\Omega^T & I & I & I \\
\Omega'^T & I & I & I
\end{bmatrix},
\]

\[
V = \begin{bmatrix}
\Omega & I & I & I \\
\Omega & I & I & I \\
\Omega & I & I & I \\
\Omega & I & I & I
\end{bmatrix},
\]

where \( I \) is the \( k \times k \) identity matrix, \( \Omega \) is a \( k \times k \) matrix with 2\( k \) nonzero entries,

\[
\Omega = \begin{bmatrix}
1 & 1 \\
1 & 1 \\
\cdots & \cdots \\
1 & 1
\end{bmatrix},
\]

where \( I', \Omega' \) and \( \Omega'^T \) are the matrices obtained by deleting the last line of \( I, \Omega \) and \( \Omega^T \), respectively. In order to show that the \( D \) matrix has not a full rank in general, we transform it into the following form by elementary row transformation:

\[
\begin{bmatrix}
I & \Omega \\
\Omega & I \\
\Omega & I \\
0 & \Delta \\
\cdots & \cdots \\
\Delta & \Delta & \Delta & 1
\end{bmatrix},
\]

where \( \mathbf{1} = (1, 1, \ldots, 1)^T \) is a \( k-1 \) dimensional column vector, \( \Delta = [I_{k-1}, \mathbf{1}] \). The \( (k^2 - 1) \times (k^2 + 1) \) matrix \( D \)
is apparently not a full rank matrix, for the rank of Ω is \( k - 1 \), we have the lower bound of entanglement

\[
E_l = k^2 - k + 1.
\]

While the upper bound of entanglement is \( E_u = k^2 - 1 \). Hence the upper bound is no longer equal to the lower bound unless \( k = 2 \).

The codeword of a toric code is a highly entangled state for large \( k \), the entanglement scales as \( E \sim \frac{k}{2} \). It seems that the area law [20] does not work for toric code. Area law usually means that the entanglement is proportional to the boundary area when the bulk of qubits is cut to two parts. The number of bulk qubits now is \( n = 2k^2 \), so according to area law, the (bipartite) entanglement should be proportional to the cutting length \( L \), which is now proportional to \( k \) in most cases. However, the largest bipartite entanglement is proportional to \( k^2 \), corresponding to a sophisticated cutting with length \( L \propto k^2 \), despite that a random cutting of toric qubits into two parts usually yields a boundary length \( L \propto \sqrt{n} \). Hence the area law still works, but the boundary may be very long. So care should be taken when we talk about the area law of multipartite entanglement and the largest bipartite entanglement.

6 Entanglement of Gottesman codes and the related codes

6.1 Gottesman codes

A serial quantum codes \([2^m, 2^m - m - 2, 3]\) (\( m \geq 3 \)) that fulfill quantum Hamming bound had been proposed by Gottesman [21]. By construction, the first two generators of the stabilizer are \( X_1 \cdots X_{2^m} \) and \( Z_1 \cdots Z_{2^m} \). An explicit construction of the remaining \( m \) generators is given by the matrix \((H|CH)\), where \( H = [h_0, h_1, \ldots, h_{2^m - 1}] \) with the \((k + 1)^{th}\) column \( h_k \) being the binary vector representing integer \( k = 0, 1, \ldots, 2^m - 1 \) and \( C \) is any invertible and fixed point free \( m \times m \) matrix, i.e., \( Cs \neq 0 \) and \( Cs \neq s \) for all \( s \in \mathbb{F}_2^m \) except \( s = 0 \). The generator \( Z_1 \cdots Z_{2^m} \) is a \( Z \)-type generator and omitted hereafter regarding the entanglement of codewords. We may arrange \( H = [H_0, H_1, H_2, \ldots, H_m] \), with \( H_0 = h_0 = [0, 0, 0, \ldots, 0]^T \), \( H_1 = [h_{2^m - 1}, h_{2^{m-1}}, \ldots, h_2, h_1] = I_{m \times m} \), and \( H_j \) is an \( m \times (m_j) \) matrix whose column vector has weight \( j \). The generator matrix of the last \( m \) generators in the standard form will be \((I|D|F|G)\), with \( D = [H_2, H_3, \ldots, H_m], F = C, G = CD \),

\[
F^T + DG^T = 0. \tag{26}
\]

where we have used the facts that any two rows of matrix \( H \) are orthogonal, and each row vector of \( H \) has even weight, so \( \sum_{i=1}^{2^m} H_i H_i^T = 0 \), and \( \sum_{i=1}^m H_i H_i^T = H_1 H_1^T = I \). Eq. (26) and the fact that the generator \( X_1 \cdots X_{2^m} \) does not contain any \( Z \) operator leads to

\[
\alpha(\mu) = 0,
\]

for all \( m + 1 \) dimensional binary vectors \( \mu \). To obtain the lower bound of entanglement for Gottesman codewords, we should verify if condition (21) is satisfied or not. When the generator \( X_1 \cdots X_{2^m} \) is considered, the whole \( D \) matrix for \( m + 1 \) generators is

\[
D = \begin{bmatrix}
1 & 0 & \cdots & (m - 1)_{F_2}
\end{bmatrix},
\]

where \( 1 \) and \( 0 \) are vectors \((1, 1, \ldots, 1) \) and \((0, 0, \ldots, 0)\) with proper dimensions, respectively. We have

\[
DD^T = \begin{bmatrix}
\sum_{i=1}^{m'} (m'_{i(2^m)}) & \sum_{i=1}^{m'} 1H_{2i}^T \\
\sum_{i=1}^{m'} H_{2i}1^T & \sum_{i=1}^{m'} H_{2i}H_{2i}^T
\end{bmatrix},
\]

with \( m' = \lfloor m/2 \rfloor \). Notice that the \( l^{th} \) element of the \( m \)-dimensional vector \( H_{2i}1^T \) is the weight of \( l^{th} \) line of \( H_{2i} \), each line of \( H_{2i} \) has the same weight \( t_i \) by the definition of \( H \), each column of \( H_{2i} \) has the same weight \( 2i \), so \( mt_i = 2i (m'_{i(2^m)}) \) is the total weight of the matrix \( H_{2i} \), thus \( t_i = (m'_{i(2^m)}) \), and \( \sum_{i=1}^{m'} 1H_{2i}^T = \sum_{i=1}^{m'} t_i = \sum_{i=1}^{m'} (m'_{i(2^m)}) = 2^{m-2} \), which is 0 in \( F_2 \) for \( m \geq 3 \), so that \( \sum_{i=1}^{m'} H_{2i}1^T = 0 \). Meanwhile \( \sum_{i=1}^{m'} (m'_{i(2^m)}) - 1 = 2^{m-1} - 1 \), which is 1 in \( F_2 \) for \( m \geq 2 \). We have

\[
DD^T = I.
\]

The condition (21) is fulfilled. The entanglement lower bound of Gottesman codeword is \( m + 1 \). The number of the generators which are not \( Z \)-type is \( m + 1 \), so the upper bound of the codeword is \( m + 1 \). We conclude that the entanglement of codewords is \( m + 1 \) for Gottesman code \([2^m, 2^m - m - 2, 3]\) (\( m \geq 3 \)). Written with the length of the code \( n = 2^m \), the entanglement of the codewords is

\[
E = \log_2 n + 1. \tag{27}
\]

6.2 Family of 8m codes

The family of codes with parameters \([8m, 8m - l_m - 5, 3]\) with \( l_m = \lceil \log_2 m \rceil \) was constructed [22]. Alternative generator matrices of the codes were given [24] based on Gottesman codes. The number of generators is \( l_m + 5 \). There is one \( Z \)-type generator in the stabilizer. Thus the upper bound of entanglement (might not be tight) of codewords is \( E_u = l_m + 4 \). To obtain the lower bound of entanglement, we will utilize the generator matrices of [23] directly instead of transforming them into the standard form of Eq. (25). The code can be divided into \( m \) blocks, each block has 8 qubits. The generator matrices can be written as

\[
(A|B) = (A_1, A_2, \ldots, A_m | B_1, B_2, \ldots, B_m) \tag{28}
\]

where \( A_i \) and \( B_i \) are \((l_m + 5) \times 8\) binary matrices, and every line of \((A_i | B_i)\) is either a line from the generator matrix of Gottesman \([8,3,3]\) code or corresponds to \( X^{g_8}, Y^{g_8}, Y^{g_8} \) or \( Z^{g_8} \). It is observed that \( A_i A_i^T = 0, A_i B_i^T = 0, B_i B_i^T = 0 \), for all \( i, j \). Thus we have

\[
\Gamma = AB^T = \sum_i A_i B_i^T = 0. \tag{29}
\]
Hence $\alpha(\mu) = 0$ for arbitrary binary vector $\mu$. Meanwhile, we have $\sum_{i=1} A_iA_i^T = 0$, thus
\[
AA^T = 0.
\] (30)

Notice that elementary row transformation of $A$ keeps Eq. (30). After elementary row transformation, $A$ can be transformed to standard form
\[
A \mapsto A' = \begin{bmatrix} I & D \\ 0 & 0 \end{bmatrix}.
\] (31)

Hence
\[
A'A^T = \begin{bmatrix} I + DD^T & 0 \\ 0 & 0 \end{bmatrix} = 0,
\]
and
\[
DD^T = I.
\] (32)

What is crucial is the dimension of the identity matrix in Eq. (31). There is one obvious Z-type generator. Besides this one, it is always possible to work out a full rank identity matrix from $A$ by elementary row transformation, thus the rank of the identity matrix in Eq. (31) as well as in Eq. (29) is $l_m + 4$. Based on Eq. (29) and Eq. (32), the lower bound of the entanglement is $E_l = l_m + 4$ which coincides with the upper bound. Hence the entanglement of the codewords of length $n = 8m$ code is
\[
E = [\log_2 n] + 1.
\] (33)

Notice that Eq. (27) for the entanglement of Gottesman codes can be merged into Eq. (33).

6.3 Pasted codes

The $[[n, 7, 3]]$ code is obtained \cite{24} by pasting Gottesman $[[8, 3, 3]]$ code and cyclic $[[5, 1, 3]]$ code. For the entanglement of the codewords of $[[13, 7, 3]]$ code, there is one Z-type generator among the 6 generators, the upper bound should be $E_u = 5$. A direct calculation shows that the lower bound $E_l$ is also 5. So we have the entanglement $E = 5$, which fulfill Eq. (33).

The family of perfect codes $[[n, n_m - 2m, 3]]$ with $n_m = (4^m - 1)/3$ for $m \geq 3$ is successively constructed by pasting Gottesman $2^{2m+1}$ code \cite{24, 25}. The entanglement upper bound should be $E_u = 2m + 2$ which is determined by the number of non Z-type generators of Gottesman $2^{2m+1}$ code according to the structure of $8n_m - 1$ code. The entanglement lower bound is $E_l = 2m + 2$. Since the code is a pasting of several Gottesman codes, the entanglement lower bound can be obtained to be $E_l = 2m + 2$ by a technic similar to that of Subsection 6.2. The entanglement is $E = 2m + 2$ and can also be written in the form of Eq. (33) with the code length $n = 8n_m$.

7 Iteration Algorithm

Denote $f = \langle 0 | \Phi_S \rangle$, the closest product state $|\Phi_S\rangle = \bigotimes (x_j|0\rangle + y_j|1\rangle)$ with $|x_j|^2 + |y_j|^2 = 1$. Using Lagrange multiplier method, we have $L = [f]^2 - \sum_j \lambda_j (|x_j|^2 + |y_j|^2 - 1)$, where $\lambda_j$ are the multipliers. The extremal equations should be $\frac{\partial L}{\partial x_j} f - \lambda_j x_j^* = 0$, $\frac{\partial L}{\partial y_j} f - \lambda_j y_j^* = 0$. Let $z_j = y_j/x_j$, we have
\[
z_j^* = \frac{\partial f/\partial y_j}{\partial f/\partial x_j}.
\] (35)

Thus the group element $M^{\mu_1}_1 M^{\mu_2}_2 \ldots M^{\mu_{n-k}}_{n-k}$ is isomorphic to $\left( \begin{array}{cccc} \sum_{i=1}^{n-k} \mu_i a_i \\ \sum_{i=1}^{n-k} \mu_i b_i \end{array} \right) = (\mu A | \mu B)$, where $\mu = (\mu_1, \mu_2, \ldots, \mu_{n-k})$ is the binary vector. $\mu A$ and $\mu B$ are binary vectors of length $n$. So that
\[
f = N \langle 0 | \otimes_{j=1}^{n-k} (I + L_j) \bigotimes (x_j|0\rangle + y_j|1\rangle) \rangle
\]
\[
= N \sum_{\mu=0}^{1} \prod_{i=1}^{n-k} Z^{\mu B} X^{\mu A} (-1)^{\alpha(\mu)} (-i)^{\mu g} \bigotimes (x_j|0\rangle + y_j|1\rangle)
\]
\[
= N \sum_{\mu=0}^{1} (-1)^{\alpha(\mu)} (-i)^{\mu g} \langle 0 | \otimes_{j=1}^{n} X^{\mu A} \bigotimes (x_j|0\rangle + y_j|1\rangle) \rangle
\]
\[
= N \sum_{\mu=0}^{1} (-1)^{\alpha(\mu)} (-i)^{\mu g} \prod_{j=1}^{n} x_j^{(-\mu A)j} y_j^{(\mu A)j}. (36)
\]

where $g = (g_1, \ldots, g_{n-k})$, and $g_i$ is the number of $Y$ operator in $L_i$. From (35), the iteration equation for $z_j$ is
\[
z_j^* = \sum_{\mu(\mu A)_j = 1} (-1)^{\alpha(\mu)} (-i)^{\mu g} \prod_{m \neq j} x_j^{(-\mu A)_m} y_j^{(\mu A)_m}. (37)
\]

Notice that the iteration may sometimes fail to reach the global maximum of $\langle f | f \rangle$. So, if the ultimate iteration result of the separable state $|\Phi_S\rangle$ is the closest product state for $|0\rangle$, the entanglement of a quantum code from
Table 1 The entanglement and the bounds

| [n,k,d] | E  | E_n | E_j |
|--------|----|-----|-----|
| [4,1,2]| 2  | 2   | 2   |
| [4,2,2]| 2  | 2   | 2   |
| [5,1,3]| 2.9275 | 3  | 2   |
| [5,2,2]| 2  | 2   | 2   |
| [6,1,3]| 2.9275 | 3  | 2   |
| [6,2,2]| 3  | 3   | 3   |
| [6,3,2]| 2  | 2   | 2   |
| [6,4,2]| 2  | 2   | 2   |
| [7,1,3]| 3  | 3   | 3   |
| [7,2,2]| 4  | 4   | 3   |
| [7,3,2]| 4  | 4   | 3   |
| [7,4,2]| 3  | 3   | 3   |
| [8,1,3]| 5  | 5   | 4   |
| [8,2,3]| 4.8549 | 5  | 4   |
| [8,3,3]| 5  | 5   | 4   |
| [8,4,2]| 4  | 4   | 4   |
| [8,5,2]| 3  | 3   | 3   |
| [8,6,2]| 2  | 2   | 2   |
| [9,1,3]| 5  | 5   | 4   |
| [9,2,3]| 5  | 5   | 4   |
| [9,3,3]| 5  | 5   | 4   |
| [9,4,2]| 4  | 4   | 4   |
| [9,5,2]| 3  | 3   | 3   |
| [9,6,2]| 2  | 2   | 2   |

The iteration method will be

\[ E = -\log_2 |f_x|^2 = n - k - n_s \]

\[ -2 \log_2 \left| \sum_{\mu=0}^{1} (-1)^{\alpha(\mu)} \prod_{j=1}^{n} x_{j}^{1-(\mu A_j)} y_{j}^{(\mu A_j)} \right| \]

where \( n_s \) is the number of Z-type generators, \( f_x, x_j, \) and \( y_j \) are the extremal values of \( f_x, x_j \) and \( y_j \), respectively.

The entanglement for some quantum codes with listed generators by Grassl [3] is as Table 1. The calculation is based on the iterative algorithm for unequal upper and lower bounds of entanglement.

8 Conclusions

The three entanglement measures (the geometric measure, the logarithmic robustness and the relative entropy of entanglement) are proved to be equal for quantum stabilizer codeword. The entanglement upper bound of a stabilizer codeword can be the minimal number of non Z-type generators. A Z-type generator is the tensor product of identity and/or Pauli \( Z \) operators. Further tight upper bound is the "Pauli persistency", the minimal number of Pauli measurements to resolve the entanglement. The entanglement lower bound based on bipartite entanglement is reduced to a formula of calculating the maximal rank of some matrices. The matrices are derived from the generator matrix of the code stabilizer. The entanglement of a self-dual CSS code is proved to be the number of \( X \) generators regardless of the detail structure of the stabilizer. We also derive the adjacency matrix of the corresponding graph state of CSS code. Upper and lower bounds of entanglement are given for toric codes, the entanglement is about half of the code length. Comments are given on the area law of entanglement for toric codes. The entanglement values of the Gottesman codes \([2^n, 2^n - m - 2, 3]\) (\( m \geq 3 \)), 8m codes and Gottesman pasting codes are equal to their minimal numbers of non Z-type generators. The entanglement \( E \) of the Gottesman codes and the related codes scales with the code length \( n \) as \( E = [\log_2 n] + 1 \) or \( E = [\log_2 n] + 1 \). An iterative algorithm is developed to obtain the entanglement of the codeword as precisely as possible.

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