Models for representing limit states in geomechanics

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Abstract. A new paradigm of scientific knowledge enabling to display the ultimate stress-strain states in geomechanics is presented in the article. In this case, distortion serves as a natural science theory. The main results concerning the development of distortion theory in natural systems are presented. Geometrical models for representing limiting states and determining distortion invariance during the materials deformation are shown. The format for forming a “strength certificate” of organic mineral soils for solving power and energy problems in the design of executive working bodies of mining machines is given. The technique for representing deformation processes in a reduced square system is described. The proposed methodology and criteria for the stress-strain state are a further development of the E.I. Shemyakin’s synthetic theory of strength. Areas of mutual influence of the main parameters of the soils strength certificate are as follows: specific adhesion and angle of internal friction and their compliance with the experiments of Buisman. The universality of the models under consideration is shown. This enables to give a mathematical description of the limiting parameters and criteria of stress-strain states in an invariant form and a clear geometric mapping.

1. Introduction
The matter of distributing stresses and strains in soils and similar materials with different types of loading is of significant interest for various disciplines. In addition to construction, where the importance of this issue is undeniable, we can also name road construction, hydraulic engineering, mining, underground work, fortification and a number of other areas [1-5]. In [6] – [12], the importance of determining the invariants of limiting states is shown.

The main difficulty in studying experimental material is the difficulty of its comprehensive interpretation from a theoretical point of view. Physical relationships should be considered in two interrelated aspects, specifically, equations describing the relationship between stresses and strains (model of soil behavior) and strength criteria that establish the conditions for soils transition to an ultimate state [13].

In [10], the fundamentals of constructing a theory for estimating the invariants of limiting states in geomechanics are described. The accumulated theoretical and practical material enables to formulate a new scientific discovery in the field of natural science, continuum mechanics, information flows, economic and social phenomena being distortion, which is a universal method in assessing the possibility of limiting states invariants in natural environments and objects of artificial intelligence. In
this regard, a new paradigm of scientific knowledge was proposed due to the presence of a special probabilistic-statistical regularity outside the spatio-temporal regularity of various natural (structural) systems (including artificial intelligence) functioning in their limiting states associated with the maximum rate (density) of entropy change.

Distortion acts as a naturalistic theory [10]. At the same time, the naturalistic theory should describe (model) the behavior of ideal objects corresponding to certain real objects. Natural science theory is based on empirical material. That is why it becomes possible to evaluate this theory correctness by comparing its consequences with empirical data.

As a result of distortion theory development, the following has been achieved:

- Effective geometric models for representing limit states in a reduced unit square system using entropy coordinates, Langmuir models, Fermat theorems, plasticity ellipse in continuum mechanics, Mohr pie chart in modeling and analyzing linear, surface and volumetric problems with regards to the determination of the functional nonlinearity level of deterministic and stochastic laws have been proposed [8].
- An entropy criterion (invariant) is proposed for estimating (quantitative and qualitative) the limiting equilibrium state of a structural system (medium, material) characterizing the similarity of stress-strain states, which is invariant and is the ratio of two opposite principles: tension - compression, fracture - hardening, attraction - repulsion, heating - cooling, order - chaos, etc. This criterion is presented as an additional invariant in the E. Shemyakin’s synthetic theory of strength [11].
- A universal classification (normalization) of the limiting asymptotics of nonlinear processes was compiled, corresponding to the states of natural systems at critical points of the medium in a stress-strain field of “rest”, “limit cycle”, “sliding”, “golden section”, “rolling” and “spinning”, which from a physical point of view, similar to changes in the conditions of contact interaction of structural formations from the standpoint of their invariants of internal adhesion and friction with regards to the Coulomb-Moore law [6, 7].
- An information-energy approach has been applied to constructing the general theory of the invariants of limiting states, which provides a quantitative and qualitative approach to assessing the structural parameters of systems in natural environment objects [10].
- A classification table of limit states has been compiled. The links the basic laws of their manifestation in various natural systems, information flows, economic processes and social phenomena [10].

2. Diagram of stress-strain state and strength certificate
The fatigue strength of the structural system during deformation is determined only by the highest and lowest stresses of the cycle \( \sigma_{\text{min}}, \sigma_{\text{max}} \) and does not depend on the law of stress variation within the specified interval. Their ratio is expressed by the cycle asymmetry coefficient \( r = \sigma_{\text{min}}/\sigma_{\text{max}} \). Any cycle can be represented as a result of applying a constant voltage \( \sigma_m = (\sigma_{\text{max}} + \sigma_{\text{min}})/2 \) to a voltage that varies in a symmetrical cycle with an amplitude \( \sigma_a = (\sigma_{\text{max}} - \sigma_{\text{min}})/2 \). In continuum mechanics, a graphical representation of the stress-strain state of a material can be a model representing limit states being a stress diagram constructed by analogy with Mohr circles (Figure 1).

Mohr’s circle is a pie chart that gives a visual representation of the stresses in various sections passing through a given point. Within a pie chart, all possible stress-strain states for a given structural object can be displayed.

Such a method is universal for determining the ultimate stress-strain states by the criterion (invariant) of the limiting equilibrium state of a structural system [3]. In this case, the radius of the main stress circle is \( R_{\text{min}}=\sigma_a \), and the radius of the limiting stress is \( R_{\text{max}}=\sigma_m \).

An ultimate stress state, or ultimate equilibrium, is condition for the onset of deformations of microdestructions that have not yet arisen in a soil.
Figure 1. Strain-stress state chart to describe the status of any structural system

The Coulomb-Mohr condition [9, 14] is most widely used as the main condition for the limiting state or strength condition for organomineral soils. Coulomb Shear Criterion is popular because it is simple and can be established by direct shear testing [15]. The Coulomb-Mohr strength criterion is a bilinear dependence of the tangential stresses of a material on the magnitude of the applied normal stresses, within which the envelope strength curve is determined by two values of the angle of internal friction and specific adhesion, as well as the normal stress in the shear plane at which the yield state occurs [16, 17]. This relationship is a strength certificate (Figure. 2)

$$\tau = C_0 + \sigma \tan \varphi,$$

where $\tau$ is shear stress invariant; $\sigma$ is normal stress invariant; $C_0$ is cohesive strength invariant.

The Coulomb-Mohr law is formulated as follows: soil resistance to shear is a function of the first degree normal pressure. The ratio of the main stresses is determined by the following dependence: $\sigma_3/\sigma_1 = \tan^2(\pi/4 - \varphi/2)$.

Tensile strength ratio $\sigma_3$ to compressive strength $\sigma_1$ is equal to the critical ratio of principal stresses $P_{KL} = \sigma_3/\sigma_1$. This ratio is independent of the grip value $C_0$, i.e. is a function of the angle of internal friction $\varphi$.

The resulting expression is similar to the two-dimensional limit state problem, where $\beta$, $\gamma$ are the angles of the bearing pads respectively; $\varphi$ is angle of internal friction. The advantage of this approach is the possibility of representing the stress-strain state for various materials based on the generalized Coulomb-Mohr diagram (Figure. 3).

Geometrically specific adhesion $C_0$ is represented by the segment cut off on the $\tau$ axis tangent to the limit point $M$ on the limit state diagram.

The provisions of the general theory of limiting states invariants [12] are a certain stage in the development of the synthetic strength theory proposed by E. Shemyakin [11].

Figure 2. Structure of strength certificate

Figure 3. Scheme of soils limiting state with regard of organic mineral soils
The following defining invariants describing the stress state and having a clear physical meaning were proposed:

\[ \tau_{\text{max}} = T = \frac{\sigma_1 - \sigma_3}{2}; \]
\[ \sigma_n = \frac{\sigma_1 + \sigma_3}{2}; \]
\[ \mu_\sigma = \frac{\sigma_2 - \sigma_n}{T}. \]

These invariants completely characterize the stress state in the elementary volume of the medium, i.e. the following statement is true: to characterize the stress state in a continuous medium element, it is necessary to introduce three invariants.

The first invariant describes the maximum tangential stress on a sloping ground and, accordingly, sliding (which divides the angles between the first and third main stresses in half) along it. It is realized in pure form, for example, upon compression along the first main direction of magnitude and tension along the third main direction of value \( T \).

The second invariant reflects the normal stress (which provides resistance to sliding) on a sloping ground. It does not depend on the directions of the main stress vectors. This is hydrostatic stress.

The third invariant is the Lode-Nadai parameter, which describes not only the type of stress state and the influence of the second principal stress, but also the type of inclined areas along which sliding occurs [9]. It supplements the first two invariants to an arbitrary one and gives an answer to the question of \( \sigma_2 \) (average principal stress) influence on material strength.

It indicates the influence of extreme shear stresses:

\[ T_{23} = \frac{\sigma_1 - \sigma_3}{2}; \quad T_{12} = \frac{\sigma_1 - \sigma_2}{2}. \]

Despite numerous studies of the stress-strain state of various materials, the researchers were not able to construct a mathematical model of elastoplastic deformation adequate to the physical processes that occur in the structure of a material during its loading.

The introduction of the three invariants \( T, \sigma_n, \mu_\sigma \) enables to describe anisotropy of a material shear resistance and, thus, construct a mathematical model of the deformation of solids. This anisotropy occurs as a result of irreversible deformations and fracture.

Within the framework of the proposed model, one can confidently trace the behavior of the material in the transcendental state, specifically, after the tangential stress reaches the maximum value for the material, a new material, whose behavior requires a different description, is actually formed. When the limit value for a given body is reached, this shift value in the medium element is preserved, and the growth of the area of irreversible deformations occurs only due to an increase in their number. The beginning of other extreme sliding platforms operation determines the residual strength of the material.

In our opinion, to analyze the structural strength of the material at different levels of the stress-strain state, it is necessary to take into account the scale factor, namely, the ratio of two extreme tangential stresses

\[ P_{K(i)} = \frac{T_{23}}{T_{12}} = \frac{\sigma_1 - \sigma_3}{\sigma_1 - \sigma_2}. \]

The position of the point \( M \) (the tangency point of the straight line OM) in the diagram (Figure 3) corresponds to the equivalent value of the normal voltage \( \sigma_M \), which is determined by the condition of the limiting state, according to E. Shemyakin’s synthetic strength theory.

The introduction of the Lode parameter into a strength certificate [11] is an essential element in the description of an arbitrary triaxial stress state. The advantage of this approach is the possibility of representing the stress-strain state for various materials based on the generalized Mohr diagram.
The Lode-Nadai coefficient $\mu_\sigma$, which evaluates the stress state by the relative value of the average principal stress, is an indicator of a stress deviator [18]. This will definitely determine only the ratio of the difference between the diameters of Mohrs small and large circles.

The main invariants of the synthetic theory of strength can be supplemented by the criterion (invariant) of the limiting state $K_P = \mu_\sigma P_{K(L)}$, which we proposed as the fourth complex invariant for a qualitative assessment of the stress state.

3. Technique of representing deformation processes in the given square

It is more convenient to evaluate the laws of deformation processes in an invariant form. For this, experimental deformation diagrams are presented in the reduced coordinate system [12] (Figure 4).

The diagram of the given square (Figure. 4) shows the geometric relationship of the angle of internal friction with the position of the criterion point $A [X_A, Y_A]$ being a parameter for assessing the level of functional nonlinearity, which allows us to establish the following:

$$\varphi = 2arctg(1 - 2X_A) = 2arctg(P_{K(L)}) .$$

Here it is possible to set the ratio of the sliding pads angles $\varphi = \gamma - \beta$. 

This leads to the following conclusion about the physical nature of the angle of internal friction: the angle of internal friction $\varphi$ is a parameter of the functional nonlinearity of materials [9]. The sliding surfaces form an angle $\beta$ with the direction of the larger main stress, regardless of the adhesion value.

![Diagram](image)

**Figure 4.** Sliding platforms angles in the given square

The fact that the position of the sliding surfaces does not depend on the value of the comprehensive internal pressure ($C_0$ is adhesion) provides another way to estimate the value of $\varphi$ being the angle of internal friction of the material using the compression and tensile tests.

4. Results and discussion

Determining the parameters of the internal friction angle is associated with the formulation of experimental studies during the organomineral soils deformation. Specifically, in order to obtain the value of $\varphi$ being the angle of internal friction for a certain type of sand, it is necessary to introduce a binder between the sand grains, which itself has zero internal friction. Such a binder is water. Its binding force $C_0$ is in any case very small and is determined by capillary pressure. Cohesion is caused by the gravitational interaction between soil particles, and the water content is the main influencing factor for a particular soil mass [19].

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Thus, we come to the experiments of Buisman [20, 21]. Buisman found that for wet sand prisms with varying degrees of moisture, the following is approximate \( P_{K(L)} = 0.25 \), and we have

\[
\sin \varphi = \frac{1 - P_{K(L)}}{1 + P_{K(L)}} = \frac{1 - 0.25}{1 + 0.25} = 0.6.
\]

The value of the angle of internal friction is equal to \( \varphi = 36.87^\circ \), which leads to a sliding angle \( \beta = 26.56^\circ \). The angle formed by the plane of break with the axis of prism should therefore be \( \gamma = 63.43^\circ \). Approximately the same angle of lean was established experimentally [10].

In natural conditions for hard rocks, the angle of internal friction \( \varphi = 37^\circ .... 36^\circ \) corresponds to the limit value of the adhesion invariant. As the rock strength decreases, the value of the angle of internal friction \( \varphi = 33 ... 30^\circ \) also decreases.

For clay soils of the quaternary deposits, the following is relevant \( \varphi = 30^\circ .... 17^\circ \) depending on their consistency. For peat of a natural structure, the following is relevant \( \varphi = 25^\circ .... 15^\circ \) depending on the type of peat and the initial moisture content.

The angle of internal friction \( \varphi \) for various types of non-rocky soils varies in the following ranges: sandy soil \( 25^\circ .... 43^\circ \); silt-loam soil is \( 7^\circ .... 30^\circ \). The angle of internal friction of the soil also depends on the coefficient of porosity. The porosity coefficient varies from zero (no pores) to infinity. The lower the porosity coefficient is, the greater the angle of internal friction is.

The angle of internal friction characterizes the friction between soil particles and is more dependent on the magnitude of the vertical pressure on the soil. The strength of soil specimen texture was affected when the sample was compacted [22].

Cohesion \( C_0 \) is seen as the resistance of structural bonds to any movement of soil particles bound by them. Adhesion is inherent in silt-loam soils. The shear resistance of non-rocky soils is determined by the forces of friction and adhesion, whose values depend on the type and moisture of a soil [23].

To build an approximate strength certificate, the following two numerical characteristics of conditional strength are introduced: the angle of internal friction \( \varphi \) and the adhesion \( C_0 \). At the same time, with an increase in loads, the strength certificate [9] indicates an increasing effect of friction during rock movement and a decreasing effect of adhesion \( C_0 \) (Figure. 5).

![Figure 5. Area of mutual influence of stress-strain state determining factors](image)

Under the influence of an external load, at individual points in the soil mass, effective stresses can exceed internal bonds between soil particles, which causes sliding or shifts at sites. This reflects the manifestations of friction, which either prevents the separation of the body into parts or characterizes additional strength.

In cohesive soils, both viscous and water-colloidal bonds take place. Separating these resistances is often not possible.

If \( \varphi = 0 \), the Coulomb-Mohr strength criterion turns into a Tresca criterion.

If \( \varphi = 90^\circ \), then the Coulomb-Mohr strength criterion corresponds to a model of a viscous medium.
Strength passport establishes the relationship between the determining parameters of stress-strain states being $\sigma$ and $\tau$ and arising at a point in a medium during the interaction of the operative parts of mining machines with soils and rocks [24] with regards to their physical and mechanical characteristics being $C_0$ and $\varphi$.

5. Conclusion

Thus, the strength certificate establishes the relationship between the determining parameters of stress-strain states arising at a point in the medium during the interaction of the working bodies of mining machines with soils with regards to their physical and mechanical characteristics.

On the basis of the strength certificate the main tasks are solved when designing the executive working bodies of mining machines for the development of soft soils. These tasks are as follows:

- energy task establishes an estimate of energy consumption for a given productivity of a mining machine;
- power task enables to determine the effort on the elements of operative parts necessary for their strength calculations with regards to the static dynamic modes of operation;
- the proposed model extends to the entire diversity of deformation mode formation regardless of the absolute scale of a particular deformation object. This is the models universality, which lies in the fact that it provides an opportunity to give a mathematical description of the limiting parameters and criteria in an invariant form and visual geometric description of stress-strain states.

References

[1] Demenkov P A, Karasev M A, Petrov D N 2017 Predicting land-surface deformations during the construction of underground facilities of complex spatial configuration International Journal of Civil Engineering and Technology 8(11) pp. 1161-1171 ISSN: 09766308

[2] Protosenya A G, Iovlev G A 2020 Prediction of spatial stress-strain behavior of physically nonlinear soil mass in tunnel face area Mining Informational and Analytical Bulletin 4 pp. 128-139 DOI: 10.25018/0236-1493-2020-4-0-33-43.pdf

[3] Zuev B Y, Zubov V P, Fedorov A S 2019 Application prospects for models of equivalent materials in studies of geomechanical processes in underground mining of solid minerals Eurasian Mining 1 pp. 8-12 DOI: 10.17580/em.2019.01.02

[4] Mazurov B T, Mustafin M G, Panzhih A A 2019 Estimation Method for Vector Field Divergence of Earth Crust Deformations in the Process of Mineral Deposits Development Journal of Mining Institute 238 p. 376-382 DOI: 10.31897/pmi.2019.4.376

[5] Protosenya A G, Verbilo P E 2017 Research of compression strength of fissured rock mass Journal of Mining Institute 223 pp. 51-57 DOI:10.18454/PMI.2017.1.51

[6] Mironov V A, Zyuzin B F and Lotov V N 1995 Distortnost' v mekhanike gornyh porod (Tver: Tver state technical University)

[7] Mironov V A, Zyuzin B F, Lotov V N and Terent'ev A A 1997 Distortion in natural systems Monografiya (Minsk: Belarusskaya navuka)

[8] Bogatov B A, Mironov V A, Zyuzin B F and Lotov V N 2000 Prediction of limit States in nonlinear geomechanics (Minsk: Belorusskaya gornaya akademiya)

[9] Iovlev D D, Maksimova L A, Nepershin R I, Radaev YU N, Senashov S I and SHemyakin E I 2008 Limit state of deformable bodies and rocks. (Moskva, Fizmatlit)

[10] Zyuzin B F and Mironov V A 2019 Distortion - a natural science theory. Monografiya. (Tver': TvGTU)

[11] Shemyakin E I 2007 On the invariants of the stressed and deformed state in mathematical models of continuum mechanics Modern problems of strength, plasticity and stability: collection of articles for the 75th anniversary of V. G. Zubchaninov's birth. (Tver': TvGTU)

[12] Mironov V A and Zyuzin B F 2015 Invariants of distortion. Monografiya. (Tver': TvGTU)

[13] Coulomb C A 1776 Essai sur une application des regles de maximis et minimis quelques
problèmes de statique, relatifs à l’architecture *Memoires de Mathematique de l’Academie Royale de Science* 7 343-378

[14] Azzam R 2016 Shear Failure Criterion and Constant Volume Ring Shear Testing Method for Clayey Soil *Engineering* 8 545-560. DOI: 10.4236/eng.2016.88051.

[15] Krahn J 2004 *Stability modeling with SLOPE/W. An engineering methodology*: 1st ed., revision 1. (Calgary, Alberta: GEO-SLOPE International Ltd.)

[16] Duncan J M, Wright S G and Brandon T L 2014 *Soil Strength and Slope Stability*: 2nd ed. (New York: John Wiley and Sons)

[17] Feng J, Li L, Jin J, Dai J and Luo P 2018 An improved geomechanical model for the prediction of fracture generation and distribution in brittle reservoirs *PLoS ONE* 13(11) e0205958.

[18] Arpád Nádai, A M Wahl 1933 Plasticity: a Mechanics of the Plastic State of Matter. *Nature* 131, 383 https://doi.org/10.1038/131383c0

[19] Han Z, Li J, Gao P, Huang B, Ni J and Wei C 2020 Determining the Shear Strength and Permeability of Soils for Engineering of New Paddy Field Construction in a Hilly Mountainous Region of Southwestern China *International journal of environmental research and public health*, 17(5) 1555 https://doi.org/10.3390/ijerph17051555

[20] Buisman A S K 1936 Results of long duration settlement tests. *Proceedings 1st International Conference on Soil Mechanics and Foundation Engineering, Cambridge, Mass* I 103–106.

[21] Koppejan A W 1948 A formula combining the Terzaghi load compression relationship and the Buisman secular time effect. *Proc. 2th International Conference on Soil Mechanics and Foundation Engineering, Rotterdam*, vol. 3 32-37

[22] Yusoff S A N M, Bakar I, Wijayesekera D C, Zainorabidin A and Madun A 2015 Comparison of Geotechnical Properties of Laterite, Kaolin and Peat *Applied mechanics and material* 773 1438-1442.

[23] Azhar ATS, Norhaliz W, Ismail B, Ezree A M and Nizam Z M 2017 Effect of Different Peat Size and Pre-Consolidation Pressure of Reconstituted Peat on Effective Undrained Shear Strength Properties. *IOP Conf. Series: Materials Science and Engineering* 226 012076 doi:10.1088/1757-899X/226/1/012076

[24] A V Mikhailov et al 2017 Excavating and loading equipment for peat mining. *IOP Conf. Ser.: Earth Environ. Sci.* 87 022014 doi :10.1088/1755-1315/87/2/022014