EXPONENTIAL ERGODICITY OF NON-LIPSCHITZ MULTIVALUED STOCHASTIC DIFFERENTIAL EQUATIONS*

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Abstract
We prove the exponential ergodicity of the transition probabilities of solutions to elliptic multivalued stochastic differential equations.

Résumé
On prouve l'ergodicité exponentielle des probabilités de transition des équations différentielles stochastiques elliptiques.

1. Introduction and Preliminaries

Consider the following stochastic differential equation:
\[ dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x, \quad \text{(1)} \]
where \( b : \mathbb{R}^d \to \mathbb{R}^d \) and \( \sigma : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^n \) are continuous functions, \((W_t)_{t \geq 0}\) is an \( n \)-dimensional standard Brownian motion defined on some complete probability space \((\Omega, \mathcal{F}, P)\). When \( \sigma \) is a uniformly elliptic square matrix and \( \sigma \) and \( b \) satisfy some regular conditions (more precisely, (H1), (H2) and (H4) below), it is recently proved in [6] that the solution is exponentially ergodic.

On the other hand, under the same uniform elliptic assumption and an additional one that \( \sigma \) and \( b \) are \( C^2 \), Cépa and Jacquot proved in [2] the ergodicity for the solution of the following stochastic variational inequality (SVI in short):
\[ dX_t + \partial \varphi(X_t) \ni b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x \in \text{Dom}(\varphi), \quad \text{(2)} \]
where \( \partial \varphi \) is the sub-differential of some convex function \( \varphi \) with a compact domain \( \text{Dom}(\varphi) = \{ x : \varphi(x) < \infty \} \).

A common drawback of the above two papers is the uniform elliptic assumption of the diffusion coefficients. The purpose of the present paper is to remove this assumption and instead assume only the ellipticity. Our main result as stated in Theorem 2.1 below unifies and improves the main results of both of [6] and [2]. In particular, our result applies to stochastic variational inequalities defined on non-compact domains. Furthermore, we do not need to assume that the diffusion matrix is square and our method even works for general multivalued stochastic differential equations (MSDEs in abbreviation):
\[ dX_t + A(X_t) \ni b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x \in D(A), \quad \text{(3)} \]
where \( A \) is a multivalued maximal monotone operator on \( \mathbb{R}^d \) with \( \text{Int}(D(A)) \neq \emptyset \).

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Now we introduce notions and notations. Given an operator $A$ from $\mathbb{R}^d$ to $2^{\mathbb{R}^d}$, define:

$$D(A) := \{ x \in \mathbb{R}^d : A(x) \neq \emptyset \},$$

$$\text{Gr}(A) := \{ (x, y) \in \mathbb{R}^{2d} : x \in \mathbb{R}^d, y \in A(x) \}.$$ 

Then $A$ is called monotone if $\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0$ for any $(x_1, y_1), (x_2, y_2) \in \text{Gr}(A)$, and $A$ is called maximal monotone if

$$(x_1, y_1) \in \text{Gr}(A) \Leftrightarrow \langle y_1 - y_2, x_1 - x_2 \rangle \geq 0, \quad \forall (x_2, y_2) \in \text{Gr}(A).$$

**Definition 1.1.** A pair of continuous and $\mathcal{F}_t$-adapted processes $(X, K)$ is called a solution of (3) if

(i) $X_0 = x_0, \ X_t \in \overline{D(A)}$ a.s.;

(ii) $K$ is of locally finite variation and $K_0 = 0$ a.s.;

(iii) $dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t - dK_t, \ 0 \leq t < \infty, \ a.s.;$

(iv) For any continuous and $\mathcal{F}_t$-adapted functions $(\alpha, \beta)$ with $(\alpha_t, \beta_t) \in \text{Gr}(A), \ \forall t \in [0, +\infty)$, the measure $\langle X_t - \alpha_t, dK_t - \beta_t dt \rangle$ is positive.

We make the following assumptions:

**H1** (Monotonicity) There exists $\lambda_0 \in \mathbb{R}$ such that for all $x, y \in \mathbb{R}^d$

$$2\langle x - y, b(x) - b(y) \rangle + \| \sigma(x) - \sigma(y) \|_{\text{HS}}^2 \leq \lambda_0 |x - y|^2 (1 + \log |x - y|^{-1}).$$

**H2** (Growth of $\sigma$) There exists $\lambda_1 > 0$ such that for all $x \in \mathbb{R}^d$

$$\| \sigma(x) \|_{\text{HS}} \leq \lambda_1 (1 + |x|).$$

**H3** (Ellipticity of $\sigma$)

$$\sigma \sigma^*(x) > 0, \ \forall x \in \mathbb{R}^d.$$

**H4** (One side growth of $b$) There exist a $p \geq 2$ and constants $\lambda_3 > 0, \lambda_4 \geq 0$ such that for all $x \in \mathbb{R}^d$

$$2\langle x, b(x) \rangle + \| \sigma(x) \|_{\text{HS}}^2 \leq -\lambda_3 |x|^p + \lambda_4.$$

**Theorem 1.2.** Assume **H1** and **H2** hold. Then (3) has a unique strong solution. 

**Proof.** The existence of a weak solution is proved in [2] and the pathwise uniqueness can be proved in a more or less standard way using a version of Bihari inequality (see [5]). Finally by Yamada-Watanabe’s theorem the existence of a unique strong solution follows. \hfill \Box

Let $\{X_t(x), t \geq 0, x \in \mathbb{E}\}$ denote the unique solution to (3). It is obviously a Markow family and its transition semigroup and transition probability are defined respectively as:

$$P_t f(x_0) := \mathbb{E} f(X_t(x_0)), \ t > 0, \ f \in B_0(\mathbb{R}^d)$$

and

$$P_t(x_0, E) := \mathbb{P}(X_t(x_0) \in E),$$

where $x_0 \in \mathbb{E}$ and $B_0(\mathbb{R}^d)$ denotes the set of all bounded measurable functions on $\mathbb{R}^d$. For general notions (e.g., strong Feller property, irreducibility, ergodicity, etc) concerning Markov semigroups, we refer to [2, 6].
2. MAIN RESULT

Now we state the main result of the paper.

**Theorem 2.1.** Assume (H1)-(H3). Then the transition probability \( P_t \) of the solution to (3) is irreducible and strong Feller. If in addition, (H4) holds, then there exists a unique invariant probability measure \( \mu \) of \( P_t \) having full support in \( \overline{D(A)} \) such that

(i) If \( p \geq 2 \) in (H4), then for all \( t > 0 \) and \( x_0 \in \overline{D(A)} \), \( \mu \) is equivalent to \( P_t(x_0, \cdot) \), and

\[
\lim_{t \to \infty} \| P_t(x_0, \cdot) - \mu \|_{\text{Var}} = 0,
\]

where \( \| \cdot \|_{\text{Var}} \) denotes the total variation of a signed measure.

(ii) If \( p > 2 \) in (H4), then for some \( \alpha, C > 0 \) independent of \( x_0 \) and \( t \),

\[
\| P_t(x_0, \cdot) - \mu \|_{\text{Var}} \leq C \cdot e^{-\alpha t}.
\]

Moreover, for any \( q > 1 \) and each \( \varphi \in L^q(\overline{D(A)}, \mu) \)

\[
\| P_t \varphi - \mu(\varphi) \|_q \leq C_q \cdot e^{-\alpha t/q} \| \varphi \|_q, \quad \forall t > 0,
\]

where \( \alpha \) is the same as above and \( \mu(\varphi) := \int_{\overline{D(A)}} \varphi(x) \mu(dx) \). In particular, let \( L_q \) be the generator of \( P_t \) in \( L^q(\overline{D(A)}, \mu) \). Then \( L_q \) has a spectral gap (greater than \( \alpha/q \)) in \( L^q(\overline{D(A)}, \mu) \).

The proof consists in proving the irreducibility and strong Feller property.

2.1. Irreducibility.

**Lemma 2.2.** Suppose \( y_0 \in \text{Int}(D(A)) \), \( m > 0 \), and \( Y_t \) is the solution to the following MSDE:

\[
dY_t + A(Y_t)dt \ni -m(Y_t - y_0)dt + \sigma(Y_t)dW_t, \quad Y_0 = x_0,
\]

where \( \sigma \) is the diffusion coefficient of (3). Then under (H1) and (H2) we have

\[
E|Y_t - y_0|^2 \leq e^{-C(m)t}|x_0 - y_0|^2 + \frac{C_0}{C(m)},
\]

where \( C(m) = 2(m - 2\lambda_1^2 - 1/2) \) and \( C_0 = 2\lambda_1^2(1 + 2|y_0|^2) + |A^\circ(y_0)|^2 \). Here \( A^\circ \) is the minimal section of \( A \) and \( |A^\circ(y_0)| < +\infty \) because \( y_0 \in \text{Int}(D(A)) \) (see [1]).

**Proof.** The proof is adapted from [2]. Consider the solution \( Y_t^n \) to the following equation:

\[
dY_t^n + A_n(Y_t^n)dt = -m(Y_t^n - y_0)dt + \sigma(Y_t^n)dW_t, \quad Y_0^n = x_0
\]

where \( A_n \) is the Yosida approximation of \( A \). From [1] we know that \( A_n \) is monotone, single-valued and \( |A_n(x)| \nearrow |A^\circ(x)| \) if \( x \in D(A) \), where \( A^\circ \) is the minimal section of \( A \). Moreover, since the law of \( Y_t^n \) converges to that of \( Y_t \), it is enough to prove the inequality for \( Y_t^n \). Hence by (H2)

\[
-2m|x - y_0|^2 + \| \sigma(x) \|_{\text{HS}}^2 - 2\langle A_n(x), x - y_0 \rangle \\
\leq -2m|x - y_0|^2 + \lambda_1^2(1 + |x|^2) - 2\langle A_n(x) - A_n(y_0), x - y_0 \rangle - 2\langle A_n(y_0), x - y_0 \rangle \\
\leq -2m|x - y_0|^2 + \lambda_1^2(1 + |x|^2) + |x - y_0|^2 + |A^\circ(y_0)|^2 \\
\leq -2m|x - y_0|^2 + 2\lambda_1^2(1 + 2|x - y_0|^2 + 2|y_0|^2) + |x - y_0|^2 + |A^\circ(y_0)|^2 \\
= -C(m)|x - y_0|^2 + C_0.
\]

Thus, by Itô’s formula we have

\[
\frac{d}{dt}E|Y_t^n - y_0|^2 = -2E(\langle Y_t^n - y_0, A_n(Y_t^n) \rangle) + E[\text{Tr}(\sigma\sigma^*(Y_t^n))] - 2mE|Y_t^n - y_0|^2
\]

\[
\leq -2mE|Y_t^n - y_0|^2 + \frac{C_0}{C(m)}.
\]
By (6) we have
\[ E|Y_t^n - y_0|^2 \leq e^{-C(m)t}|x_0 - y_0|^2 + \frac{C_0}{C(m)}. \]

Therefore
\[ E|Y_t^n - y_0|^2 \leq e^{-C(m)t}|x_0 - y_0|^2 + \frac{C_0}{C(m)}. \]

\[ \Box \]

**Proposition 2.3.** Under (H1)-(H3), the transition probability \( P_t \) is irreducible.

*Proof.* It suffices to prove that for any \( x_0 \in \overline{D(A)} \), \( T > 0 \), \( y_0 \in \text{Int}(D(A)) \) and \( a > 0 \),

\[ P_T(x_0, B(y_0, a)) = P(X_T(x_0) \in B(y_0, a)) = P(|X_T(x_0) - y_0| \leq a) > 0, \]

or equivalently:
\[ P(|X_T(x_0) - y_0| > a) < 1. \]

Fix \( a, T \) and \( y_0 \). By Lemma 2.2 and Chebyshev’s inequality, we can choose an \( m \) large enough such that, denoting by \( (Y_t, \tilde{K}_t) \) the unique solution to

\[ dY_t + A(Y_t)dt \equiv -m(Y_t - y_0)dt + \sigma(Y_t)dW_t, \quad Y_0 = x_0 \in \overline{D(A)}, \tag{4} \]

we have
\[ P(|Y_T(x_0) - y_0| > a) \leq \left( e^{-C(m)T}|x_0 - y_0|^2 + \frac{C_0}{C(m)} \right)/a^2 < 1. \tag{5} \]

Set
\[ \tau_N := \inf\{t : |Y_t| \geq N\}. \]

Note that by [2]
\[ E \left[ \sup_{t \in [0,T]} |Y_t(x_0)| \right] \leq C \]

for some constant \( C \) depending on \( x_0, y_0, \lambda_1, m \) and \( T \). Thus we may fix an \( N \) so that
\[ P(\tau_N \leq T) + P(|Y_T(x_0) - y_0| > a) < 1. \tag{6} \]

Define
\[ U_t := \sigma(Y_t)^*[\sigma(Y_t)\sigma(Y_t)^*]^{-1}(-m(Y_t - y_0) - b(Y_t)) \]

and
\[ Z_T = \exp \left( \int_0^{T \wedge \tau_N} U_s dW_s - \frac{1}{2} \int_0^{T \wedge \tau_N} |U_s|^2 ds \right). \]

Since \(|U_{t \wedge \tau_N}|^2\) is bounded, \( E[Z_T] = 1 \) by Novikov’s criteria.

By Girsanov’s theorem, \( W_t^* := W_t + V_t \) is a \( Q \)-Brownian motion, where
\[ V_t := \int_0^{t \wedge \tau_N} U_s ds, \quad Q := Z_T P. \]

By (5) we have
\[ Q(\{\tau_N \leq T\} \cup \{|Y_T(x_0) - y_0| > a\}) < 1. \tag{7} \]

Note that the solution \( (Y_t, \tilde{K}_t) \) of (4) also solves the MSDE below
\[ Y_t + \int_0^t A(Y_s)ds \equiv \int_0^t \sigma(Y_s)dW_s^* + \int_0^{t \wedge \tau_N} b(Y_s)ds - \int_0^{t \wedge \tau_N} m(Y_s - y_0)ds. \]

Set
\[ \theta_N := \inf\{t : |X_t| \geq N\}. \]
Then the uniqueness in distribution for (H) yields that the law of \( \{ (X_t \mathbf{1}_{\{\theta_N \geq T\}})_{t \in [0,T]}, \theta_N \} \) under \( P \) is the same as that of \( \{ (Y_t \mathbf{1}_{\{\tau_N \geq T\}})_{t \in [0,T]}, \tau_N \} \) under \( Q \). Hence

\[
P(|X_T(x_0) - y_0| > a) \leq P(\{ \theta_N \leq T \} \cup \{ \theta_N > T, |X_T(x_0) - y_0| > a \})
\]
\[
= Q(\{ \tau_N \leq T \} \cup \{ \tau_N > T, |Y_T(x_0) - y_0| > a \})
\]
\[
\leq Q(\{ \tau_N \leq T \} \cup \{ |Y_T(x_0) - y_0| > a \}) < 1.
\]

\[\square\]

2.2. **Strong Feller Property.** The proof of the following lemma is plain by using Kolmogorov’s lemma on path regularity of stochastic processes.

**Lemma 2.4.** Denote by \((X_t(x), K_t(x))\) the solution of (3) with initial value \( x \). Then for any \( p > d \), there exists \( t_p > 0 \) such that for all \( r > 0 \)

\[
E \left[ \sup_{x \in D_r, s \leq t_p} |X_s(x)|^p \right] < \infty,
\]

where \( D_r := \overline{D(A)} \cap \{ |x| \leq r \} \).

**Proposition 2.5.** Under (H1)-(H3), the semigroup \( P_t \) is strong Feller.

**Proof.** We divide the proof into two steps.

Step 1: Assume that there exists a \( \lambda_2 > 0 \) such that \( ||\sigma^* \sigma||_{\text{HS}} \leq \lambda_2 \). Consider the following drift transformed MSDE:

\[
\begin{cases}
    dY_t + A(Y_t)dt \geq b(Y_t)dt + \sigma(Y_t)dw_t + |x_0 - y_0|^{\alpha} \frac{X_t - Y_t}{|X_t - Y_t|} \cdot 1_{\{X_t \neq Y_t\}} \cdot 1_{\{t < \tau\}} dt, \\
    Y_0 = y_0 \in D(A),
\end{cases}
\]

where \( \alpha \in (0, 1) \), \( X_t \) is the solution to (3) and \( \tau \) is the coupling time given by

\[
\tau := \inf \{ t > 0 : |X_t - Y_t| = 0 \}.
\]

An argument similar to [6] allows to prove it admits a unique solution.

For \( T > 0 \) define

\[
U_T := \exp \left[ \int_0^{T\wedge \tau} \langle dW_s, H(X_s, Y_s) \rangle - \frac{1}{2} \int_0^{T\wedge \tau} |H(X_s, Y_s)|^2 ds \right]
\]

and

\[
\bar{W}_t := W_t + \int_0^{t\wedge \tau} H(X_s, Y_s) ds,
\]

where

\[
H(x, y) := |x_0 - y_0|^\alpha \cdot \sigma^*(y) [\sigma \sigma^*(y)]^{-1} \frac{x - y}{|x - y|}.
\]

Since \( ||\sigma^* \sigma||_{\text{HS}} \leq \lambda_2 \), we have

\[
|H(x, y)|^2 \leq \lambda_2 \cdot |x_0 - y_0|^{2\alpha}.
\]

Thus,

\[
EU_T = 1 \quad \text{and} \quad EU_T^2 \leq \exp \left[ \lambda_2 T \cdot |x_0 - y_0|^{2\alpha} \right].
\]

By the elementary inequality \( e^r - 1 \leq re^r \) for \( r \geq 0 \), we have for any \( |x_0 - y_0| \leq \eta \),

\[
(E|1 - U_T|)^2 \leq E|1 - U_T|^2 = EU_T^2 - 1 \leq \exp \left[ \lambda_2 T \cdot |x_0 - y_0|^{2\alpha} \right] - 1 \leq C_{T, \lambda_2, \eta} \cdot |x_0 - y_0|^{2\alpha}
\]

(9)
and
\[
(\mathbb{E} [(1 + U_T)1_{\{\tau \geq T\}}])^2 \leq (3 + \mathbb{E} U_T^2) \cdot \mathbb{P}(\tau \geq T) \\
\leq C_{T, \lambda_2, \eta} \cdot \mathbb{P}((2T) \land \tau \geq T) \\
\leq C_{T, \lambda_2, \eta} \cdot \mathbb{E}((2T) \land \tau)/T .
\]  
(10)

First applying Itô’s formula to \(\sqrt{Z_{t\land\tau}}^2 + \varepsilon\) where \(Z_s := X_s - Y_s\), then letting \(\varepsilon \downarrow 0\), and finally taking expectation, we have by \((H1)\),
\[
\mathbb{E}[X_{t\land\tau} - Y_{t\land\tau}] \leq |x_0 - y_0| - |x_0 - y_0|^{\alpha} \cdot \mathbb{E}(t \land \tau) + \frac{\lambda_0}{2} \int_0^t \rho_\eta(\mathbb{E}|X_{s\land\tau} - Y_{s\land\tau}|)\,ds ,
\]
which implies by Bihari inequality that for any \(t > 0\) and \(|x_0 - y_0| < \eta\)
\[
\mathbb{E}[X_{t\land\tau} - Y_{t\land\tau}] \leq |x_0 - y_0|^{\exp(-\lambda_0 t/2)}
\]
and thus
\[
\mathbb{E}(t \land \tau) \leq |x_0 - y_0|^{1-\alpha} + \frac{\lambda_0 t}{2} \rho_\eta(|x_0 - y_0|^{\exp(-\lambda_0 t/2)}) \cdot |x_0 - y_0|^{-\alpha} .
\]  
(11)

Taking \(\alpha = \exp\{-(\lambda_0 T)/2\}\), there exists an \(0 < \eta' < \eta\) such that for any \(|x_0 - y_0| < \eta'\)
\[
\mathbb{E}((2T) \land \tau) \leq C_{T, \lambda_0, \eta'} \cdot |x_0 - y_0|^{\exp(-\lambda_0 T)/2} .
\]  
(12)

But by Girsanov’s theorem, \((\tilde{W}_t)_{t\in[0,T]}\) is still a \(n\)-dimensional Brownian motion under the new probability measure \(U_T \cdot \mathbb{P}\). Note that \((Y_t, \tilde{K}_t)\) also solves
\[
dY_t + A(Y_t)\,dt \equiv b(Y_t)\,dt + \sigma(Y_t)\,d\tilde{W}_t, \quad Y_0 = y_0 .
\]
So, the law of \(X_T(y_0)\) under \(\mathbb{P}\) is the same as that of \(Y_T(y_0)\) under \(U_T \cdot \mathbb{P}\). Thus by \((9)\), \((10)\) and \((12)\), for any \(f \in B_b(\mathbb{R}^d)\),
\[
|P_T f(x_0) - P_T f(y_0)| = |\mathbb{E}(f(X_T(x_0)) - U_T \cdot f(Y_T(y_0)))| \\
\leq \mathbb{E}(|(1 - U_T) \cdot f(X_T(x_0)) \cdot 1_{\{\tau \leq T\}}| + \mathbb{E}(|f(X_T(x_0)) - U_T \cdot f(Y_T(y_0))) \cdot 1_{\{\tau > T\}}|) \\
\leq ||f||_0 \cdot \mathbb{E}|1 - U_T| + ||f||_0 \cdot \mathbb{E}[(1 + U_T)1_{\{\tau > T\}}] \\
\leq C_{T, \lambda_0, \lambda_2, \eta} \cdot ||f||_0 \cdot |x_0 - y_0|^{\exp(-\lambda_0 T)/4} .
\]

Step 2: Now we prove the proposition under \((H3)\). By the Markov property of the solution, we only need to prove that for every \(f \in B_b(\mathbb{R}^d)\), \(x \mapsto P_t f(x)\) is continuous on \(D_r\) for all \(t \leq t_p\), \(p > d\) where \(p\) and \(t_p\) are specified in Lemma \(2.4\).

Set
\[
c_0 := ||f||_\infty
\]
and
\[
\tau := \inf \left\{ t > 0 : \sup_{x \in D_r} |X_t(x)| > N \right\} .
\]

Let \(\varepsilon > 0\) be given. For \(t \leq t_p\), by Lemma \(2.4\) and Chebyshev inequality, there exists \(N > r\) such that
\[
\mathbb{P}(\tau \leq T) = \mathbb{P}\left(\sup_{x \in D_r, t \leq t_p} |X_t(x)| > N\right) \leq \mathbb{E}\left[\sup_{x \in D_r, t \leq t_p} |X_t(x)|^p\right] / N^p < \varepsilon .
\]  
(13)
Define
\[
\tilde{\sigma}(x) := \sigma(x), \quad \forall |x| \leq N .
\]
Extend \( \bar{\sigma} \) to the whole \( \mathbb{R}^d \) such that it satisfies the condition \((H1)\) to \((H3)\). Denote by \( \tilde{X}_t(x) \) the solution to \((3)\) with \( \sigma \) replaced by \( \bar{\sigma} \). By Step 1, there exists a \( \delta > 0 \) such that if \( |x - y| < \delta \) and \( x, y \in D_\varepsilon \),

\[
|\mathbb{E}[f(\tilde{X}_t(x))] - \mathbb{E}[f(\tilde{X}_t(y))]| < \varepsilon.
\] (14)

Hence

\[
|\mathbb{E}[f(X_t(x))] - \mathbb{E}[f(X_t(y))]| \\
\leq |\mathbb{E}[(f(X_t(x)) - f(X_t(y)))1_{(\tau > T)}]| + |\mathbb{E}[(f(X_t(x)) - f(X_t(y)))1_{(\tau \leq T)}]| \\
\leq |\mathbb{E}[(f(\tilde{X}_t(x)) - f(\tilde{X}_t(y)))1_{(\tau > T)}]| + 2c_0\varepsilon \\
\leq |\mathbb{E}[f(\tilde{X}_t(x)) - f(\tilde{X}_t(y))]| + |\mathbb{E}[(f(\tilde{X}_t(x)) - f(\tilde{X}_t(y)))1_{(\tau \leq T)}]| + 2c_0\varepsilon \\
\leq (1 + 4c_0)\varepsilon.
\]

Now we are in a position to complete the proof of Theorem 2.1.

**Proof.** (i) By Itô’s formula and \((H4)\), we get

\[
\mathbb{E}|X_t|^2 = |x_0|^2 + 2 \int_0^t \mathbb{E}\langle X_s, b(X_s) \rangle ds - 2 \int_0^t \mathbb{E}\langle X_s, dK_s \rangle + \int_0^t \mathbb{E}||\sigma(X_s)||_{HS}^2 ds \\
\leq |x_0|^2 + \int_0^t \mathbb{E}(-\lambda_3|X_s|^p + \lambda_4) ds.
\]

Taking derivatives with respect to \( t \) and using Hölder’s inequality give

\[
\frac{d\mathbb{E}|X_t|^2}{dt} \leq -\lambda_3\mathbb{E}|X_t|^p + \lambda_4 \leq -\lambda_3(\mathbb{E}|X_t|^2)^{p/2} + \lambda_4.
\]

Since \( \lambda_3 > 0 \) we have for all \( t > 0 \),

\[
\frac{1}{t} \int_0^t \mathbb{E}|X_s|^2 ds \leq \frac{\lambda_4}{\lambda_3}.
\]

Therefore by Krylov-Bogoliubov’s method (see \([3]\)), there exists an invariant probability measure \( \mu \). As we have just proved, \( P_t \) is strong Feller and irreducible. Then, again by \([3]\), \( \mu \) is equivalent to each \( P_t(x, \cdot) \) with \( x \in \overline{D(A)} \), \( t > 0 \) and consequently (i) holds.

(ii) If \( p > 2 \), consider the following ODE:

\[
f'(x) = -\lambda_3f(x)^{p/2} + \lambda_4, \quad f(0) = |x_0|^2.
\]

By the comparison theorem (cf. \([3]\)), there exists some \( C > 0 \) such that

\[
\mathbb{E}|X_t|^2 \leq f(t) \leq C(1 + t^{2/(2-p)}).
\]

We also have

\[
\inf_{x_0 \in B(0,r)} P_t(x_0, B(0, a)) > 0, \quad \forall r, a > 0, \quad t > 0
\]

because of the strong Feller property and irreducibility. Therefore (ii) holds due to Theorem 2.5 (b) and Theorem 2.7 in \([3]\). \( \square \)
References

[1] Cépa, E.: Équations différentielles stochastiques multivoques. Lect. Notes in Math. Sém. Prob. XXIX (1995) 86-107.

[2] Cépa, E. et Jacquot, S.: Ergodicité D’inégalités Variationnelles Stochastiques. Stochastics and Stochastics Reports. Vol. 63 (1997) pp. 41-64.

[3] Cerrai, S.: Second order PDE’s in finite and infinite dimension. A probabilistic approach. Lecture Notes in Mathematics, 1762. Springer-Verlag, Berlin, 2001. x+330 pp.

[4] Goldys, B. and Maslowski, B.: Exponential ergodicity for stochastic reaction-diffusion equations. Stochastic partial differential equations and applications—VII, 115–131, Lect. Notes Pure Appl. Math., 245, Chapman Hall/CRC, Boca Raton, FL, 2006.

[5] Ren, J. and Zhang, X.: Stochastic flows for SDEs with non-Lipschitz coefficients, Bull. Sci. Math., Vol. 127 (2003), 739-754.

[6] Zhang, X.: Exponential ergodicity of non-Lipschitz stochastic differential equations. Proc. Amer. Math. Soc. 137 (2009), 329-337.