EMBEDDING CALCULUS KNOT INVARIANTS ARE OF FINITE TYPE

RYAN BUDNEY, JAMES CONANT, ROBIN KOYTCHEFF, AND DEV SINHA

ABSTRACT. We show that the map on components from the space of classical long knots to the \(n\)th stage of its Goodwillie–Weiss embedding calculus tower is a map of monoids whose target is an abelian group and which is invariant under clasper surgery. We deduce that this map on components is a finite type-\((n - 1)\) knot invariant. We also compute the \(E^2\)-page in total degree zero for the spectral sequence converging to the components of this tower as \(\mathbb{Z}\)-modules of primitive chord diagrams, providing evidence for the conjecture that the tower is a universal finite-type invariant over the integers. Key to these results is the development of a group structure on the tower compatible with connect-sum of knots, which in contrast with the corresponding results for the (weaker) homology tower requires novel techniques involving operad actions, evaluation maps, and cosimplicial and subcubical diagrams.

1. INTRODUCTION

In this paper we connect three current threads in studying knots and the moduli space of all knots: the Goodwillie–Weiss embedding calculus, Budney’s operad actions, and Vassiliev’s theory of finite-type invariants. This work also connects two fundamental results on commutativity which are over fifty years old. In 1949 H. Schubert [Sch49] established and applied the fact that connect-sum of knots is commutative. In 1947 Steenrod [Ste47] gave explicit formulae exhibiting commutativity of cup product, in the course of defining the cohomology operations which bear his name. We establish and use compatibility of these results as we develop a group structure on the components of the Goodwillie–Weiss tower.

One of our main results is that the Goodwillie–Weiss tower for classical knots yields invariants of finite type. Such a result is modest compared to the larger conjecture made in [BCSS05] that the tower gives all such invariants. We provide evidence for this conjecture through calculation of the \(E^2\)-term of the spectral sequence for the components of this tower. These results are meaningful first steps, and establishing them requires some new tools as well as combinations of tools in the areas of operad actions, clasper surgery, cosimplicial and cubical diagrams, and evaluation maps.

A large part of the work is “getting off the ground,” showing that the Goodwillie–Weiss tower yields abelian group-valued invariants. Because the knot invariants are defined as an induced map on \(\pi_0\), such invariants are a priori only set valued. Once the group structure is established, we use the Habiro surgery criterion for finite type to establish the main result that the \(n\)th stage in the tower defines a type-\((n - 1)\) invariant.

Our work is a natural sequel to two previous papers. The first is [BCSS05], in which we establish that the third stage of the tower is a universal type-two invariant. Here the group structure and the finite-type result were straightforward from ad hoc arguments. The more interesting aspect of this work was the development of geometry to explicitly distinguish components in the tower, yielding a new interpretation of the type two knot invariant by counting collinearities of points. This geometric approach was continued in [Flo13].

The second predecessor is Volic’s thesis [Vol06], which establishes that the Goodwillie–Weiss tower for the (space level) rational homology of knots is a universal rational finite-type invariant. The map to the homology tower factors through the tower we consider, so our tower encodes such a universal rational invariant as well. The corresponding question over the integers is open, and of considerable interest since Vassiliev invariants with values in arbitrary \(\mathbb{Z}\)-modules are not well understood. A universal invariant is
known to exist over the integers (see for example Habiro’s $\psi_n$ map from Theorem 6.1 of the current paper), but it is not computational in nature and does not directly relate to the combinatorics of weight systems. Identification of the tower as a universal invariant would likely resolve questions about the combinatorics and geometry of finite-type invariants, and in particular the open question of whether weight systems “integrate.”

Our techniques are distinct from those of Volic in three ways. In our result the type-$n$ invariants are potentially realized at the $(n + 1)$st stage, while in the rational homology tower they are realized precisely at stage $2n$. Volic uses the Vassiliev derivative approach to establish his finite-type result, while we use Habiro’s clasper surgery. Finally, Volic’s result proceeds by extending the theory of Bott-Taubes integration to the homology tower, while our approach through the standard (or “homotopy”) tower invites new techniques from geometric and algebraic topology.

We extend the abelian group structures to the spectral sequence level, which is crucial for analysis since studying components of an arbitrary tower of spaces can potentially lead to unending ad-hoc calculations. With group structure in hand, we immediately establish convergence and then use results of the second author [Con08] to identify the $E^2$ term in degree zero as the $\mathbb{Z}$-modules of primitive chord diagrams.

Based on this, we conjecture that the map from the knot space to the tower sends linear combinations of knots given by resolving a singular knot to the corresponding elements of $E^2$. We also conjecture that the spectral sequence collapses, and together these two conjectures would imply that weight systems over the integers all integrate to finite-type invariants. Unlike Vassiliev’s original spectral sequence, this spectral sequence does not involve a subtle limit process (see [Giu10]) but instead is simply the spectral sequence of a tower of fibrations. It thus is more amenable to standard tools from algebraic topology.

The paper is organized as follows. Section 2 gives needed general background on compactified configuration spaces and on cubical and cosimplicial diagrams. Section 3 recalls the resulting mapping space and cosimplicial models we prefer to use, interchangeably, for the $n$-th stage of the Taylor Tower for $\text{Emb}^{fr}(\mathbb{R}, \mathbb{R}^3)$. Some readers may prefer to only look back at these sections as needed.

In Section 4 we construct homotopy-commutative multiplications on the $n$-th stage of the Taylor tower for $\text{Emb}^{fr}(\mathbb{R}, \mathbb{R}^3)$. This shows that $\pi_0$ of each stage of the Taylor tower is an abelian monoid. Section 5 contains two main results, the first being that the projections in the tower are surjective on $\pi_0$. This is then used to show that $\pi_0$ of each stage is a group.

In Section 6 we show that $\pi_0$ of the map from the space of knots to its $(n + 1)$st Goodwillie–Weiss approximation is invariant under clasper surgery and thus of type $n$. In Section 7 we show that the homotopy spectral sequence for the tower is a spectral sequence of abelian groups (in particular in total degrees zero and one), and we recast previous calculations as calculations in this spectral sequence in order to identify the $E^2$ term.

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## Contents

1. Introduction
   1.1. Acknowledgments
2. Compactification of configuration spaces
3. The models
   3.1. Cosimplicial model
   3.2. A mapping space model
   3.3. Other variants of the mapping space model
   3.4. Evaluation maps
EMBEDDING CALCULUS KNOT INVARIANTS ARE OF FINITE TYPE

4. Little intervals actions and the evaluation map
4.1. The little intervals action on the knot space
4.2. The little intervals action on aligned maps
4.3. A multiplication on infinitesimal aligned maps
5. Projection maps, layers in the tower, and abelian group structure
5.1. Cosimplicial and cubical diagrams
5.2. Surjectivity on components of maps in the tower
5.3. Group structure
6. The homotopy tower is a finite-type invariant
7. The homotopy spectral sequence for the tower
References

2. Compactification of configuration spaces

2.0.1. Basic definition. We briefly review the simplicial compactification of configuration space, defined in [Sin09, Sin04]. For any manifold $M$ – not necessarily compact, and possibly with boundary – let $C_n(M)$ denote the configuration space of $n$ points in $M$. It is the space of distinct ordered $n$-tuples of points in $M$. For an $M$ embedded in some Euclidean space $\mathbb{R}^d$, let $C_n(M)$ denote the closure of the image of

$$C_n(M) \hookrightarrow M^n \times (S^{d-1})^n$$

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n), \left(\frac{x_i - x_j}{|x_i - x_j|}\right)_{1 < j}$$

As shown in [Sin04], this is independent of embedding and is a quotient of $C_n[M]$, the Fulton–MacPherson compactification of $C_n(M)$. If $M$ is noncompact, then $C_n(M)$ as defined above as well as the Fulton-MacPherson compactification are not compact but are more accurately described as completions. If $M$ is one-dimensional and connected, then $C_n(M)$ is in general not connected. For such $M$, by abuse we use $C_n(M)$ to denote the connected component where the $n$ points are in (cyclic) order.

2.0.2. Framings and tangent data. We need framed configurations. For any manifold $M$, define $C_n^fr(M)$ as the pullback

$$C_n^fr(M) \longrightarrow (FrM)^n$$

$$\downarrow$$

$$C_n(M) \longrightarrow M^n,$$

where $FrM \to M$ is the unit frame bundle of the tangent bundle of $M$. If $M$ is parallelizable and $d$-dimensional, then $C_n^fr(M)$ is homeomorphic to $C_n(M) \times O(d)^n$.

Let $C_n^t(M)$ be defined similarly, with the unit tangent bundle $STM$ in place of the unit frame bundle.

2.0.3. Distinguished boundary points. Suppose $M$ has two distinguished points $y_0, y_1$ in its boundary. Then, as in [BCSS05] and [Sin09], let $C_n(M, \partial)$ denote the subspace of $C_{n+2}(M)$ where the first and last points are located at $y_0$ and $y_1$, by abuse omitting dependence on these points from notation.

Further, if there are distinguished tangent vectors $v_0 \in TM|y_0$ and $v_1 \in TM|y_1$, let $C_n'(M, \partial)$ be the subspace of $C_{n+2}^t(M)$ where $(p, v_1)$ and $(p_{n+2}, v_{n+2})$ are $(y_0, v_0)$ and $(y_1, v_1)$ respectively. By fixing framings at $y_0$ and $y_1$ define $C_n^fr(M, \partial)$ similarly.

Define $C_n(\mathbb{I}, \partial)$ by taking the two endpoints to be the distinguished points. The fact that $C_n(\mathbb{I}, \partial)$ is the $n$-simplex is the main rationale for calling this compatification “simplicial.” Define $C_n'(\mathbb{I}^d)$ by taking
\(\{y_0, y_1\}\) to be \(\partial I \times \{(0, \ldots, 0)\}\) and \(v_0 = v_1 = (1, 0, \ldots, 0)\), and similarly define \(C^{fr}(I^d, \partial)\) by using the identity element in \(O(d)\) for framings at those boundary points (using the standard parallelization of \(I^d \subset \mathbb{R}^d\)).

2.0.4. Quotients by translation and scaling, and insertion maps. There are maps between (products of) \(C_n^{\mathbb{R}}(\mathbb{R}^d)\) and \(C_n^{\mathbb{I}}(I^d, \partial)\) defined by “inserting an infinitesimal configuration into a point of another configuration.” Let \(C_n^{\mathbb{R}}(\mathbb{R}^d) := C_n(\mathbb{R}^d)/(\mathbb{R}^d \times \mathbb{R}_+)\) be the quotient of configuration space by translations and positive scalings of all \(n\) points. We then define \(\tilde{C}_n^{\mathbb{R}}(\mathbb{R}^d)\) as the closure of the image of the map

\[
\tilde{C}_n^{\mathbb{R}}(\mathbb{R}^d) \to (S^{d-1})\left(\mathbb{R}\right) \\
(x_1, \ldots, x_n) \mapsto \prod_{i<j} \frac{x_i - x_j}{|x_i - x_j|},
\]

which is injective except on collinear configurations.

The similarly defined \(\tilde{C}_n^{\mathbb{I}}(I^d)\) is homeomorphic to \(\tilde{C}_n^{\mathbb{R}}(\mathbb{R}^d)\), so both \(C_n^{\mathbb{R}}(\mathbb{R}^d)\) and \(C_n^{\mathbb{I}}(I^d)\) naturally surject onto \(\tilde{C}_n^{\mathbb{R}}(\mathbb{R}^d)\).

We proceed directly to the framed setting in defining insertion maps. Let \(\tilde{C}_n^{fr}(\mathbb{R}^d) := \tilde{C}_n^{\mathbb{R}}(\mathbb{R}^d) \times (O(d))^n\), the framed version of the “infinitesimal configuration space.”

For every \(m, n,\) and \(i \in \{1, \ldots, n\}\), we define a map \(\phi_i\) which, informally, inserts a configuration of \(m\) points (with framings) into the \(i\)th point of a configuration of \(n\) points (with framings). In the resulting configuration of \(m + n - 1\) points, the \(m\) points form an “infinitesimal configuration.” Precisely, in coordinates we define

\[
\phi_i : C_n^{fr}(\mathbb{I}^d, \partial) \times C_m^{fr}(\mathbb{I}^d) \to C_{m+n-1}(I^d, \partial)
\]
as follows. First, suppressing the dependence on \(i\), we let

\[
\tilde{j} = \begin{cases} 
  j & \text{if } j \leq i \\
  i & \text{if } i \leq j \leq i + m - 1 \\
  j + m - 1 & \text{if } j > i + m - 1.
\end{cases}
\]

Now define \(\phi_i\) as sending

\[
\left((x_j)_{j=1}^m, (u_{jk})_{j<k}, (\alpha_j)_{j=1}^m\right) \times \left((y_{j})_{j=1}^m, (v_{jk})_{j<k}, (\beta_j)_{j=1}^m\right) \mapsto \left((z_j)_{j=1}^{m+n-1}, (w_{jk})_{j<k}, (\gamma_j)_{j=1}^{m+n-1}\right),
\]

where \(z_j = x_j\), where

\[
w_{jk} = \begin{cases} 
  \alpha_i v_{(j-i+1)(k-i+1)} & \text{if } i \leq j, k \leq i + m - 1 \\
  \alpha_{i-1} & \text{otherwise},
\end{cases}
\]

and where

\[
\gamma_j = \begin{cases} 
  \alpha_i \beta_{j-i+1} & \text{if } i \leq j \leq i + m - 1 \\
  \alpha_{j-1} & \text{otherwise}.
\end{cases}
\]

One must check that such maps send subspaces of \((\mathbb{R}^d)^n \times (S^{d-1})\left(\mathbb{R}\right)\) to each other appropriately. Here we only cite a similar check, namely [Sim04, Proposition 6.6], using the description of the \(C_n^{\mathbb{R}}(\mathbb{R}^d)\) as a subspace of \((\mathbb{R}^d)^n \times (S^{d-1})\left(\mathbb{R}\right)\) given in [Sim04, Theorem 5.14]. The results are not dependent on the \(y_j\) coordinates of \(C_m^{fr}(\mathbb{I}^d)\), which means that these insertion maps factor through the projection to \(\tilde{C}_m^{fr}(\mathbb{R}^d)\) on that factor.
3. The models

We study the space of framed knots $\text{Emb}^f_r(\mathbb{R}, \mathbb{R}^3)$. A framed knot is a smooth embedding of $\mathbb{R}$ into $\mathbb{R}^3$ together with a smooth map $\mathbb{R} \to O(3)$ whose first (column) vector is the unit derivative map. Embeddings take $I = [-1,1] \subset \mathbb{R}$ into $\mathbb{R}^2 \subset \mathbb{R}^3$, and on $\mathbb{R} \setminus I$ are standard, given by $t \mapsto (t,0,0)$. The framing is required to be constant at the identity on $\mathbb{R} \setminus I$.

We primarily use a mapping space (or homotopy limit) model for the $n$-th stage of the Goodwillie–Weiss (Taylor) tower for the space of framed knots $\text{Emb}^f_r(\mathbb{R}, \mathbb{R}^d)$. But for both spectral sequence calculations and for clarity through comparison, we use a cosimplicial model as well. For each of these, there are a few variants depending on the choice of compactifications of configuration spaces $C_n(M)$. The Fulton–MacPherson (Axelrod–Singer) compactification $C_n[M]$ has the advantage of being a smooth manifold with corners. In this paper, we use instead the simplicial compactification $C_n(M)$, developed in the previous section. It is not a manifold with corners, but it has the advantage that (one component of) $C_n(I)$ is the $n$-simplex. As a result, there is a cosimplicial model, and the corresponding mapping space model is defined in terms of the face poset of the simplex rather than the associahedron.

3.1. Cosimplicial model. A cosimplicial space is a functor to $\mathcal{C}^\text{op}$ from the category $\Delta$ with one object for each ordered set $[n] = (0,\ldots,n)$ and morphisms given by order-preserving maps. For each $n$ and for each $i = 0,\ldots,n$, there is a morphism $[n-1] \to [n]$ given by “skipping” $i$ and a morphism $[n+1] \to [n]$ given by “merging” $i$ and $i+1$. These morphisms respectively give rise to coface maps $d^i$ and codegeneracy maps $s^i$ in the cosimplicial space. We let $\Delta_n$ be the full sub-category containing the first $n+1$ objects.

The spaces $C^f_r(I^n, \partial)$ can be made into a cosimplicial space, similar to the unframed version defined in [Sim09 Corollary 4.22] by using the first vector in the framing as the “doubling direction.” This is also related to a framed version studied by Salvatore in [Sal06 Section 3]. There one uses the language of operads with multiplication, fixing a two-point infinitesimal configuration $\mu \in \hat{C}^f_r(\mathbb{R}^d)$, say $\mu = ((1,0,\ldots,0),id)$, and using the $\circ_i$ with this multiplication element to define cosimplicial structure maps following Gerstenhaber–Voronov and McClure–Smith [MS02].

The relation of $C^f_r(I^n, \partial)$ to the knot space via the evaluation map is slightly more direct than that of $\tilde{C}^f_r(\mathbb{R}^d)$.

In more detail, we have the following.

**Definition 3.1.** The cosimplicial space $C^f_r(I^n, \partial)$ has $n$th entry $\text{Emb}^f_r(\mathbb{R}, \mathbb{R}^d)$. The codegeneracy $s_i : \text{Emb}^f_r(\mathbb{R}, \mathbb{R}^d) \to \text{Emb}^f_r(\mathbb{R}, \mathbb{R}^d)$ is the extension to compactifications of the projection which forgets the $i$th point.

For $1 \leq i \leq n$, the coface $d^i$ is given by doubling the $i$th point by inserting into its position the infinitesimal two-point configuration $\mu$ rotated by the $i$th framing $\alpha$: $d^i(x) = x \circ_i \mu$. The coface $d^0$ doubles the left basepoint $(y_0)$: $d^0(x) = \mu \circ_2 x$. The coface $d^{n+1}$ doubles the right basepoint $(y_1)$: $d^{n+1}(x) = \mu \circ_1 x$.

The fourth author showed [Sim09 Theorem 6.1] that the homotopy-invariant totalization of these cosimplicial spaces gives a model for the Goodwillie–Weiss tower. Because these cosimplicial spaces are not fibrant, one must either replace the standard cosimplicial space $\Delta^*$ or the configuration spaces, which means losing some control of geometry and combinatorics. By sacrificing some symmetry, a standard approach to cosimplicial spaces through (sub)cubical diagrams developed in the next section retains both combinatorics and geometry, and in particular is compatible with an evaluation map (also known as a Gauss map) from the knot space.

3.2. A mapping space model. Our mapping space models are defined as homotopy limits of subcubical diagrams of compactified configuration spaces. A subcubical diagram is a functor from $\mathcal{P}_n(n + 1)$, the poset of nonempty subsets of $[n] := \{1,\ldots,n+1\}$ (or equivalently nonempty subsets of $[n] := \{0,\ldots,n\}$). Subcubical diagrams encode the face relations of simplices. We define $AM^f_r$ to be the homotopy limit of
the subcubical diagram given on objects by $\mathcal{A}M(S) = C_{fr,n}^3(\mathbb{I}^3, \partial)$. On maps $\mathcal{A}M$ sends an inclusion $S \hookrightarrow S \cup \{j\}$ to the doubling map $d^i$ that is the corresponding coface map in the cosimplicial space $C_{fr,n}^3(\mathbb{I}^3, \partial)$.

Because the cosimplicial and subcubical diagram categories both involve ordered sets, there is an immediate relationship between them. As in [Sin09] and [MV], the functor $G_n : \mathcal{P}_n(n + 1) \to \Delta_n$ or by abuse the composite to $\Delta$ – is given by sending $S \subset n$ to $[\# S - 1]$ and the inclusion $S \subset T$ to $[\# S - 1] \cong S \to T \cong [\# T - 1]$, where the isomorphisms are canonical since $S, T$ are canonically ordered.

Our subcubical diagram $\mathcal{A}M_n$ is the pull-back under the functor $G_n$ of the cosimplicial space $C_{fr,n}^3(\mathbb{I}^3, \partial)$. That is, $AM_{fr,n}$ is the homotopy limit of the composite functor $C_{fr,n}^3(\mathbb{I}^3, \partial) \circ G_n : \mathcal{P}_n(n + 1) \to \mathbb{T} \mathcal{O}p$.

A main technical result is the following.

**Theorem 3.2** (Theorem 6.7 in [Sin09]). $G_n$ is left cofinal.

So if $X^\bullet$ is a cosimplicial space then the homotopy limit of the subcubical diagram $X^\bullet \circ G_n$ is equivalent to the homotopy limit of the restriction of $X^\bullet$ to $\Delta_n$. By work of Bousfield and Kan [BK72], this is homotopy equivalent to $\mathbb{T} \mathcal{O}l^r X^\bullet$, the $n$th stage in the homotopy invariant totalization tower.

Using the definition of homotopy limit through categories, the homotopy limit of a subcubical diagram is given by a collection of maps from simplices. Since the structure maps $d^i$ are injective, an element $\varphi$ of $AM_{fr,n}$ is determined by a map $\Delta^n \to C_{fr,n}^3(\mathbb{I}^3, \partial)$, so $AM_{fr,n}$ is a subset of $\text{Map}(\Delta^n, C_{fr,n}^3(\mathbb{I}^3, \partial))$. Because the faces of the simplex map to configurations which are degenerate in an “aligned” manner we sometimes refer to this as the subspace of aligned maps. Explicitly, if some (consecutive) collection of $t_i$ in $\varphi(t_i)$ are equal, then the corresponding points in the configuration $\varphi(t_i)$ have collided in $\mathbb{I}^3$, their framings ($\alpha_i \in O(3)$) are all equal, and (since our standard two-point configuration $\mu \in \mathcal{C}_{fr,n}^3(\mathbb{I}^3, \partial)$ has vector $(1, 0, 0)$) the first column of $\alpha_i$ is the direction of collision of these points.

3.3. **Other variants of the mapping space model.** We can similarly define a mapping space $\tilde{A}M_{fr,n}$. This is done by replacing $C_{fr,n}^3(\mathbb{I}^3, \partial)$ by $\mathcal{C}_{fr,n}^3(\mathbb{I}^3)\Simeq C_{fr,n}^3(\mathbb{I}^3, \partial)$ and appropriately replacing the doubling maps $d^i$ in the definition of $AM_{fr,n}$.

The mapping space $AM_n$ of [BCSS05] is defined similarly to $AM_{fr,n}$, but with tangent vectors instead of frames. It is the homotopy limit of a diagram $C_{fr,n}^3(\mathbb{I}^3, \partial) \circ G_n$, where $C_{fr,n}^3(\mathbb{I}^3, \partial)$ is precisely the cosimplicial space in [Sin09] Corollary 4.22. One of the main results of [Sin09] is that the collection of $AM_n$ is equivalent to the Taylor tower for $\text{Emb}(\mathbb{R}, \mathbb{R}^3)$. Moreover, the map from the embedding space to the tower is homotopic to the evaluation map as defined below. As in the framed setting, there is a similarly defined model $\tilde{A}M_n$ which one can show also models the Taylor tower.

In the framed setting, $AM_{fr,n}$ and $\tilde{AM}_{fr,n}$ are both equivalent to the $n$-th stage of the Taylor tower for $\text{Emb}^{fr}(\mathbb{R}, \mathbb{R}^3)$. For $AM_{fr,n}$, one can see this using an easy adaptation of the arguments in [Sin09] to the framed setting. For $\tilde{AM}_{fr,n}$ the paper [Sal06] contains a short proof of this statement.

To summarize, the entries in Taylor tower for $\text{Emb}(\mathbb{R}, \mathbb{R}^3)$ can be modeled by either $AM_n$ or $\tilde{AM}_n$, and those for the Taylor tower for $\text{Emb}^{fr}(\mathbb{R}, \mathbb{R}^3)$ can be modeled by either $AM_{fr,n}$ or $\tilde{A}M_{fr,n}$. We will use all of these models, guided by convenience.

3.4. **Evaluation maps.** Our main results make use of evaluation maps. We start with the simpler setting of $AM_{fr,n}$, where one just evaluates a knot along the unit interval. By functoriality for embeddings of compactified configuration spaces [Sim04], we have that an embedding $f : \mathbb{I} \to \mathbb{I}^3$ will extend to a map from $\Delta^n = C_n(\mathbb{I}, \partial) \to C_n(\mathbb{I}^3, \partial)$. For a framed embedding with framing $\alpha \in \text{Map}(\mathbb{I}, O(3))$ to go along with the embedding $f$, this map is given by
\[
\begin{align*}
ev_n : \text{Emb}^{fr}(\mathbb{R}, \mathbb{R}^3) & \to AM_{fr,n} \subset \text{Map}(\Delta^n, C_n(\mathbb{I}^3, \partial) \times O(3)^n) \\
(f, \alpha) & \mapsto ((-1 \leq t_1 \leq ... \leq t_n \leq 1) \mapsto (f(t_1), ..., f(t_n), \alpha(t_1), ..., \alpha(t_n))
\end{align*}
\]
Intuitively, \(ev_n\) samples the knot and its framing at \(n\) points in the domain.

For \(\tilde{AM}_n\), the evaluation map is slightly more complicated. Since the points added by \(d_0\) and \(d_n\) are “at infinity” from the perspective of the other points, we need to evaluate at times throughout \(\mathbb{R}\), rather than just at times in \(I\). Let \(h\) be a monotone homeomorphism that takes \((-1, 1)\) to \(\mathbb{R}\). Define the map

\[
ev_n : \text{Emb}(\mathbb{R}, \mathbb{R}^3) \to \tilde{AM}_n^f \subset \text{Map}(\Delta^n, \tilde{C}_n(\mathbb{I}^3, \partial) \times O(3)^n)
\]
on the interior of the simplex where times are distinct by setting

\[
(f, \alpha) \mapsto ((-1 < t_1 < ... < t_n < 1) \mapsto (f(h(t_1)), ..., f(h(t_n)), \alpha(h(t_1)), ..., \alpha(h(t_n)))
\]
and extending continuously to the full simplex. Above \(f(h(t_1)), ..., f(h(t_n))\) denotes the image of these points under the projection \(C_n(\mathbb{R}^3) \to \tilde{C}_n(\mathbb{R}^3) = \tilde{C}_n(\mathbb{I}^3)\). In extending to the full simplex, we again appeal to the functoriality of the compactifications when some \(t_i = t_{i+1}\) etc. For the case where \(t_i \to \pm 1\) (so that \(h(t_i) \to \pm \infty\), we use that the embedding \(f\) is standard at infinity. Thus, while the limit of such configurations in \(C_n(\mathbb{R}^3)\) (as defined above) does not exist, the limit in \(\tilde{C}_n(\mathbb{R}^3)\) does exist and is an “infinitesimal configuration” where all vectors involving \(f(h(t_i))\) point along the long axis.

One of the main results of [Sin09], namely Theorem 5.4, immediately extends to the framed setting to establish that \(ev_n\) agrees in the homotopy category with the canonical map from the embedding space \(\text{Emb}^f(\mathbb{R}, \mathbb{R}^2)\) to the \(n\)-th stage of the Taylor tower \(T_n \text{Emb}\). Through homotopy equivalence between \(AM_n^f\) and \(\tilde{AM}_n^f\) induced by \(h\) the same is true for \(\ev_n\).

**Remark 3.3.** In dimensions \(d > 3\) it is known that its connectivity increases with \(n\) and hence the tower converges to the embedding space. Thus, the homotopy theory of evaluation maps captures the nature of knots in these dimensions. For \(d = 3\), it is not known if \(ev_n\) is even 0-connected (that is surjective) much less injective in the limit. We show below that on components \(ev_n\) defines abelian-group-valued invariants of finite type \((n - 1)\).

### 4. Little intervals actions and the evaluation map

In this section we first define an action of the (nonsymmetric) little intervals operad \(C_1\) on \(AM_n\). This action is compatible via the evaluation map \(ev_n\) with a natural \(C_1\)-action on \(\text{Emb}(\mathbb{R}, \mathbb{R}^3)\). The same is true in the framed case of \(AM_n^f\) and \(\text{Emb}^f(\mathbb{R}, \mathbb{R}^3)\). Then we define an action of \(C_1\) on \(\tilde{AM}_n\) (and \(\tilde{AM}_n^f\)).

**4.1. The little intervals action on the knot space.** The little intervals operad \(C_1\) has as its \(n\)th entry \(C_1(n)\) the space of \(n\) disjoint subintervals of \(I\), which we topologize as a subspace of \(I^{2n}\) through the endpoints of the intervals. Given an subinterval \(L \subseteq \mathbb{I}\) we by abuse let \(L\) also denote the orientation-preserving affine-linear transformation which sends \(I\) to \(L\). Moreover, we let \(\bar{L}\) denote the map \(\mathbb{I}^3 \to \mathbb{I}^3\) which applies \(L\) to the first coordinate, and then shrinks the second and third coordinates by the same scaling factor (but doesn’t translate them).

We first specify the \(C_1\)-action on the knot space. If \(\mathcal{L} = \bigcup L_i\) is a union of \(k\) disjoint little intervals, its action on a \(k\)-tuple of embeddings \(f_i : \mathbb{R} \to \mathbb{R}^3\) yields the embedding which at time \(t\) has value

\[
\mathcal{L} \cdot (f_1, \ldots, f_k)(t) = \begin{cases} \bar{L}_i \circ f_i \circ L_i^{-1}(t) & t \in \text{some } L_i \\ (t, 0, 0) & \text{otherwise}. \end{cases}
\]
That is, the embeddings are “shrunk and placed in succession” according to \(\mathcal{L}\). The action on \(\text{Emb}^f(\mathbb{R}, \mathbb{R}^3)\) is similar, with the framings unchanged by the shrinking. We will view this action as a case of “insertion
into the trivial embedding, with standard framing”, and generalize to insertion into an arbitrary knot with framing below.

4.2. The little intervals action on aligned maps. Defining an action on $AM_n$ is more involved. For the evaluation map $ev_n$ to be compatible with the action, we want to take a configuration in the interval (that is, a point $\tilde{t}$ in the $n$-simplex) and evaluate the various maps which are being acted on at various subsets of the configuration. As these maps in $AM_n$ must each take $n$ points in the interval as input, we adjust accordingly.

First, define $L^{-1}$ on a configuration in the interval by applying to each point the piecewise-linear map which is $L^{-1}$ on the image of $L$ and sends points to the left of $L$ to $-1$ and points to the right of $L$ to $1$.

Next define the restriction of some $\varphi \in AM_n$ to $L$, denoted $\varphi|_L$, by applying $\varphi$ to $L^{-1}(\tilde{t})$ and then applying projection maps to forget all of the points in the resulting configurations whose indices $j$ do not correspond to a $t_j \in L$. This is not continuous, as the different projection maps produce points in different configuration spaces, but is a useful auxiliary construction.

Finally, define the action of a union of little intervals $L = \bigcup L_i$ on a collection of maps $\varphi_i \in AM_n$ as the map in $AM_n$ which takes $\tilde{t} \in \Delta^n$ to the union of all of the $L_i \circ \varphi_i|_{L_i}$, along with a point in the configuration at $(t_j,0,0)$ for each $t_j$ which is not in any $L_i$.

Continuity, the fact that this is an action of $C_1$, and the fact that the evaluation map is compatible with this action and that on the knot space are all straightforward. The $C_1$-action on $AM_n^{fr}$ is defined similarly, where the framings are unchanged because each interval shrinks an aligned map in all directions.

4.3. A multiplication on infinitesimal aligned maps. Now we define a multiplication on $\tilde{AM}_n^{fr}$ determined by a choice of two intervals $L_1, L_2 \in C_1(1)$. Since $L_1, L_2$ will not be required to be disjoint, this product will be homotopy-commutative.

4.3.1. “Acting” by one interval on an infinitesimal aligned map. For a little interval $L$, let $L(0)$ be its midpoint. Given an interval $L$ and $\varphi \in \tilde{AM}_n^{fr}$, we will first define an element $L \cdot \varphi \in AM_n^{fr}$, which can be thought of roughly as an “action of $L$ on $\varphi$.” This aligned map $L \cdot \varphi$ will produce an infinitesimal configuration at the point $(L(0),0,0)$ together with some points along $\mathbb{I} \times (0,0)$. For continuity, we need points near the boundary (and, say, inside) of $L$ to be pulled towards $L(0)$.

Explicitly, let $L^o = L([-\frac{1}{2},\frac{1}{2}])$, which is the “core” of $L$. Let $e_L : \mathbb{I} \to \mathbb{I}$ be a monotone continuous map which sends $L^o$ to the point $L(0)$ and which is the identity outside of $L$. We will define the map $L \cdot \varphi$ piecewise on $\Delta^n$ according to which $t_i$ are in $L^o$. That is, we have a partition of $\Delta^n$ determined by $L^o$. Let $i = i(L^o, \tilde{t}), j = j(L^o, \tilde{t})$ be the indices of the leftmost and rightmost $t_k$ in $L^o$. For an element $\varphi \in \tilde{AM}_n^{fr}$, define $\varphi|_L$ as in the previous section, though this now produces equivalence classes of configurations with framings. Now define

\[(L \cdot \varphi)(\tilde{t}) = ((e_L(t_1),...,e_L(t_{i-1}), L(0), e_L(t_{j+1}),...,e_L(t_n) \times (0,0)) \circ_{(L^o,\tilde{t})} \varphi|_{L^o}(\tilde{t}).\]

For a fixed $L$, it is clear that the output varies continuously with $\varphi$ and with all $\tilde{t}$ in this piece of $\Delta^n$. To check that the various pieces fit together to a continuous function, suppose that, say, $t_i$ is equal to $L(-\frac{1}{2})$, the left endpoint of $L^o$. The formulae on the two pieces of $\Delta^n$ that meet at $t_i = L(-\frac{1}{2})$ are (by expanding the definition of $\varphi|_{L^o}$)

\[((e_L(t_1),...,e_L(t_{i-1}), L(0), e_L(t_{j+1}),...,e_L(t_n) \times (0,0)) \circ_{(L^o,\tilde{t})} \varphi((L^o)^{-1}(t_i,t_{i+1},...,t_j))\]

and (using that $e_L(t_i) = e_L(L(-\frac{1}{2})) = L(0)$)

\[((e_L(t_1),...,e_L(t_{i-1}), L(0), e_L(t_{j+1}),...,e_L(t_n) \times (0,0)) \circ_{i+1} \varphi((L^o)^{-1}(t_{i+1},...,t_j)).\]
Thus two such maps can be composed, and we denote the composition by \( \circ \).

Proof. The map \( L \cdot \varphi \) is a map \( C_n(\mathbb{I}, \partial) \to C_n^{fr}(\mathbb{I}^3, \partial) \). The image of this map lies in the subspace \( C_n^{fr}(\mathbb{I} \times (0, 0), \partial) \) of configurations whose projections to \( (\mathbb{I}^3)^n \) lie in \( (\mathbb{I} \times (0, 0))^n \). We can view \( C_n(\mathbb{I}, \partial) \) as a subspace of \( C_n^{fr}(\mathbb{I} \times (0, 0), \partial) \) in an obvious way with identity framings at every point. We will now extend the domain of \( L \cdot \varphi \) to all of \( C_n^{fr}(\mathbb{I} \times (0, 0), \partial) \).

Let \( p : C_n^{fr}(\mathbb{I}^3, \partial) \to (\mathbb{I}^3)^n \) be the projection. For any \( c \in C_n^{fr}(\mathbb{I} \times (0, 0), \partial) \), \( p(c) \in (\mathbb{I} \times (0, 0))^n \cong \mathbb{I}^n \). Let \( \vec{t}(c) = (t_1 < ... < t_m) \) be the set of distinct points in \( p(c) \) (so \( m \leq n \)). Then \( c \) can be written uniquely as

\[
c = (...((\vec{t}(c) \times (0, 0)) \circ_{m_1} c_1) \circ_{m_2} ... \circ_{m_k} c_k
\]

for some \( c_i \in C_n(\mathbb{I}^3, \partial) \) if we require that \( m_i \geq m_{i-1} + n_{i-1} \).

**Definition 4.1.** Define the extension of \( L \cdot \varphi \) to \( C_n^{fr}(\mathbb{I} \times (0, 0), \partial) \) using the definition of \( L \cdot \varphi \) on points in \( C_n(\mathbb{I}, \partial) \cong \Delta^n \):

\[
(L \cdot \varphi)(c) = (...(((L \cdot \varphi)(\vec{t}(c))) \circ_{m_1} c_1) \circ_{m_2} ... \circ_{m_k} c_k.
\]

The key point now is checking continuity when points enter or exit the infinitesimal configurations \( c_i \). The argument is similar to the one for continuity of the map \( \mathbb{I} \) but with \((1, 0, 0)\) replaced by the tangent vector (i.e., first vector in the framing) at \( \varphi(t_{m_i}) \).

Now both the input and output of the map \( L \cdot \varphi \) can be regarded as elements in \( C_n^{fr}(\mathbb{I} \times (0, 0), \partial) \). Thus two such maps can be composed, and we denote the composition by \( \circ \). Given two little intervals \( L_1, L_2 \in C_1(1) \), we define a multiplication of elements \( \varphi, \psi \in \widetilde{AM}_n^{fr} \) as below, where we take the equivalence class in \( C_n^{fr}(\mathbb{I}^3, \partial) \) of the right-hand side:

\[
((L_1, L_2) \cdot (\varphi, \psi)) : \vec{t} \mapsto ((L_2 \cdot \psi) \circ (L_1 \cdot \varphi))(\vec{t})
\]

**Proposition 4.2.** Let \( L_- = [-1, 0], L_+ = [0, 1] \). Define

\[
\varphi \cdot \psi = (L_-, L_+) \cdot (\varphi, \psi) = (L_+ \cdot \psi) \circ (L_- \cdot \varphi).
\]

Then \( \varphi \cdot \psi \) is homotopic to \( \psi \cdot \varphi \).

**Proof.** The map \((L, \varphi) \mapsto (L \cdot \varphi)\) varies continuously with \( L \), so any two multiplications induced by choices of \((L_1, L_2)\) are homotopic.

It remains to show that the multiplication \( \varphi \cdot \psi \) is compatible with the evaluation map. We first present a slightly different multiplication, which is homotopic to the one defined above, but easier to describe. In particular, it avoids the use of the maps \( c_L \) which pull points towards \( L \). We need these maps for homotopy-commutativity, but not for defining the multiplication itself.

**Definition 4.3.** Let \( \mu_0 \) be the multiplication

\[
\mu_0 : \widetilde{AM}_n^{fr} \times \widetilde{AM}_n^{fr} \to \widetilde{AM}_n^{fr}
\]

defined by

\[
(\mu_0(\varphi, \psi))(\vec{t}) = (\mu \circ_2 \psi|_{[0,1]}(\vec{t})) \circ_1 \varphi|_{[-1,0]}(\vec{t}),
\]
where $\mu$ on the right-hand side is the standard 2-point configuration along the long axis. More generally, let $\mu_1$ be the multiplication

$$\mu_1 : C_1(2) \times (\tilde{AM}_n^{fr})^2 \to \tilde{AM}_n^{fr}$$

defined by

$$(\mu_1(L_1, L_2, \varphi, \psi))(\vec{t}) = ((t_1, ..., t_{i-1}, L_1(0), t_{j+1}, ..., t_{k-1}, L_2(0), t_{l+1}, ..., t_n) \times (0, 0)) \circ_i \varphi|_{L_1}(\vec{t}) \circ_k \varphi|_{L_2}(\vec{t})$$

where $t_i, ..., t_j$ are the points in $L_1$, and $t_k, ..., t_l$ are the points in $L_2$.

Note that $\mu_1(L_-, L_+, \varphi, \psi) = \mu_0(\varphi, \psi)$.

**Proposition 4.4.** The multiplication $\varphi \cdot \psi$ is homotopic to the multiplication $\mu_0$. More generally, suppose $(L_1, L_2) \in C_1(2)$. Then the multiplication $(L_1, L_2) \cdot (\varphi, \psi)$ is homotopic to the multiplication $\mu_1$.

**Proof.** For the first, consider a homotopy parametrized by $s \in [0, 1]$ which at time $s = 0$ starts with $(L_- \cdot \cdot \psi) \circ (L_+ \cdot \varphi)$ where each of the two terms is given by equation (1). As $s$ increases, we simultaneously modify these two terms by “growing” each $L_{\pm}$ to $L_{\pm}^s$: $L_{\pm}^s(s) = [-\frac{1}{2} \pm \frac{s+1}{4}, \frac{1}{2} \pm \frac{s+1}{4}]$. This affects the term $\varphi|_{L_1}(\vec{t})$ and the index $i(L_+, \vec{t})$. We can leave the map $e_L$ fixed over the homotopy. Even though each of the two maps we modify will not produce continuously varying configurations in $C_n^{fr}(\mathbb{R}^3, \partial)$, the resulting configuration in $\tilde{C}_n^{fr}(\mathbb{R}^3, \partial)$ will vary continuously. This is because the intervals $L_i^s(s)$ have disjoint interiors and because in $\tilde{C}_n^{fr}(\mathbb{R}^3, \partial)$, the only thing we record about the image of a point outside of $L_i^s(s)$ is that all direction vectors between it and any other point lie along the long axis.

The second half of the proposition follows from similar arguments. \hfill \Box

From now on, we will take the multiplication(s) on $\tilde{AM}_n^{fr}$ to be the one(s) defined in Definition 4.3.

**Remark 4.5.** The multiplication $\mu_1$ generalizes to an action of the little intervals operad on $\tilde{AM}_n^{fr}$. However for homotopy-commutativity we need the multiplication of Definition 4.3 which does not extend to a little intervals action.

Now we compare this multiplication on $\tilde{AM}_n^{fr}$ to the one on the knot space given by connect-sum. These two multiplications are not compatible on the nose. In the product $\varphi \cdot \psi$ of two aligned maps, the vector between any two configuration points coming from different aligned maps points along the long axis, while this is not true for the evaluation of a connect-sum. In other words, the operations of connect-sum and “shrinking to infinitesimal size” do not strictly commute. But they commute up to homotopy via a homotopy that pushes factors in a connect-sum farther and farther apart.

**Proposition 4.6.** The multiplication on $\tilde{AM}_n^{fr}$ is compatible up to homotopy with connect-sum (stacking) on the knot space via the evaluation map $\tilde{ev}_n$.

**Proof.** A choice of two little intervals $(L_1, L_2) \in C_1(2)$ determines a multiplication, both on $\tilde{AM}_n^{fr}$ and on the knot space itself. For either space, any two choices yield homotopic multiplications, so we fix a convenient choice. Assume that $L_1(1) < L_2(-1)$ so that the connect-sum is “separated.” By Proposition 4.4 we may use $\mu_1$ as the multiplication on $\tilde{AM}_n^{fr}$.

Consider the map $\tilde{ev}_n$ that sends an embedding $f$ to the map

$$(t_1, ..., t_n) \mapsto (f(h(t_1)), ..., f(h(t_n)), \alpha(t_1), ..., \alpha(t_n)).$$

where $h$ is a homeomorphism from $[0, 1]$ to $[0, \infty)$. Let $EM_n^{fr}(\mathbb{R})$ be the image of $\tilde{ev}_n$, a subspace of $\text{Map}(\Lambda^n, C_n^{fr}(\mathbb{R}^3, \partial))$, where $C_n^{fr}(\mathbb{R}^3, \partial)$ is $C_n^{fr}(\mathbb{R}^3)$ compactified at infinity in the directions of the long axis. It consists of evaluation maps, and is homeomorphic to the knot space. Then the evaluation map $\tilde{ev}_n$ factors as

$$\text{Emb}^{fr}(\mathbb{R}, \mathbb{R}^3) \to EM_n^{fr}(\mathbb{R}) \to \tilde{AM}_n^{fr}$$
where the right-hand map is induced by the quotient of configurations in \( \mathbb{R}^3 \) by translation and positive scaling. We can precompose with the multiplication on the knot space determined by \((L_1, L_2)\):

\[
(5) \quad \text{Emb}^{fr}(\mathbb{R}, \mathbb{R}^3) \times \text{Emb}^{fr}(\mathbb{R}, \mathbb{R}^3) \to \text{Emb}^{fr}(\mathbb{R}, \mathbb{R}^3) \to EM_n^{fr}(\mathbb{R}) \to \tilde{A}M_n^{fr}.
\]

We construct a homotopy of the left-hand map, parametrized by \( t \in (0,1] \) such that the limit of the composite as \( t \to 0 \) is equal to the composite

\[
(6) \quad \text{Emb}^{fr}(\mathbb{R}, \mathbb{R}^3) \times \text{Emb}^{fr}(\mathbb{R}, \mathbb{R}^3) \xrightarrow{c_t \times c_t} \tilde{A}M_n^{fr} \times \tilde{A}M_n^{fr} \xrightarrow{q} \tilde{A}M_n^{fr},
\]

where the right-hand map is the multiplication on \( \tilde{A}M_n^{fr} \) determined by \((L_1, L_2)\).

For \( t \in (0,1] \), let \( H_t : \mathbb{R} \to \mathbb{R} \) be the piecewise-linear homeomorphism that scales the interval \([L_1(1), L_2(-1)]\) by a factor of \(1/t\) and preserves the length of each \( L_i \). Then the homeomorphism \( H_t \times id_{\mathbb{R}^2} \) of \( \mathbb{R}^3 \) induces a map on \( C_n(\mathbb{R}^3, \partial) \) and on \( C_n^{fr}(\mathbb{R}^3, \partial) \) by acting trivially on the framing. We thus consider the composite

\[
\text{Emb}^{fr}(\mathbb{R}, \mathbb{R}^3) \times \text{Emb}^{fr}(\mathbb{R}, \mathbb{R}^3) \xrightarrow{\text{Map}([\Delta^n, C_n^{fr}(\mathbb{R}^3, \partial)])} \text{Map}([\Delta^n, C_n^{fr}(\mathbb{R}^3, \partial)]) \xrightarrow{q} \tilde{A}M_n^{fr}.
\]

As \( t \) varies from 1 to 0, we get a homotopy of the above composite, which at \( t = 1 \) is the map \([5]\) given by “connect-sum, then evaluate.” The map \( H_0 \) is not defined, but the limit of the configurations in \( \tilde{C}_n^{fr}(\mathbb{R}^3, \partial) = \tilde{C}_n^{fr}(\mathbb{R}^3, \partial) \) as \( t \to 0 \) does exist. Thus the limit of the composite exists, and it is equal to the map \([6]\) given by “evaluate, then connect-sum.”

Theorem 4.8. Each space \( \tilde{A}M_n^{fr} \), or equivalently on the \( n \)-th stage of the Taylor tower, has a homotopy-commutative multiplication. This multiplication is compatible up to homotopy (via the evaluation map \( c_t \)) with the multiplication on \( \text{Emb}^{fr}(\mathbb{R}, \mathbb{R}^3) \) given by stacking long knots.

Corollary 4.9. For each \( n \), \( \pi_0 \) of the \( n \)-th stage \( T_n \text{Emb}^{fr}(\mathbb{R}, \mathbb{R}^3) \) of the Taylor tower is an abelian monoid. Moreover \( \pi_0(\text{Emb}^{fr}(\mathbb{R}, \mathbb{R}^3) \to T_n \text{Emb}^{fr}(\mathbb{R}, \mathbb{R}^3)) \) is a homomorphism of abelian monoids.
We have not used that the ambient dimension is three, so the proofs of Theorem 4.8 and Corollary 4.9 yield analogues exactly as above, but with \( \mathbb{R}^3 \) replaced by \( \mathbb{R}^d \), \( d \geq 3 \). A statement similar to Theorem 4.8 was proven in [Sin06] for knots modulo immersions, but only for the limit of the tower, not for each stage. Recently, Turchin has established a version of this theorem for the stages in the tower in the cosimplicial model [Tur], but it is not obvious that the structure studied in that paper is compatible with the evaluation map, as needed to connect with finite-type theory.

**Remark 4.10.** At this point we can explain the connection between Schubert’s elementary geometric result that connect sum of knots is commutative [Sch49] and Steenrod’s deep, formal work on commutativity of cup product and operations in cohomology [Ste47]. This connection was implied to us by the work of McClure and Smith [MS03, MS04], whose product structures on totalizations of cosimplicial spaces are related to Steenrod’s formulae for higher cup products [MS03] and equivalent to ours on \( AM_n \). (We will not prove this here.)

The product \( \mu_0(f, g) \) of two aligned maps \( f \) and \( g \) is

\[
(t_1, \ldots, t_n) \mapsto \bigcup_{t_i \leq 0 \leq t_{i+1}} \hat{f}(t_1, \ldots, t_i) \ast \hat{g}(t_{i+1}, \ldots, t_n).
\]

Here the union refers to a decomposition of the domain, the \( n \)-simplex; \( \hat{f} \) and \( \hat{g} \) are obtained from \( f \) and \( g \) by appending times and rescaling (as we did regularly this section); and \( \ast \) indicates a “stacking product” of configurations in \( \mathbb{R}^3 \) (or more generally \( \mathbb{R}^d \)).

This is formally similar to the standard formula for cup product

\[
\varphi \cup \psi(\sigma : [v_1, \ldots, v_n] \to X) = \sum \varphi(\sigma|[v_1, \ldots, v_i]) \cdot \psi(\sigma|[v_{i+1}, \ldots, v_n]).
\]

Here \( \varphi \) and \( \psi \) are cochains, \( \sigma \) is a chain, and brackets \([,]\) around some variables refers to the simplex given as convex linear combinations of those variables. This sum is zero but for one term, but as McClure and Smith show, it is the correct sum to write down for purposes of generalization.

Finally, recall that Proposition 4.4 gives a homotopy from \( \mu_0(f, g) \) to the multiplication defined by (3) with \( (L_1, L_2) = ([−1, 0], [0, 1]) \). By choosing a path in \( C_1(1)^2 \) from \( ([−1, 0], [0, 1]) \) to \( ([0, 1], [−1, 0]) \) (and again applying a homotopy from Proposition 4.4) we ultimately get a homotopy from \( \mu_0(f, g) \) to \( \mu_0(g, f) \).

We choose the following path in \( C_1(1)^2 \). (A reader who is familiar with the overlapping intervals operad \( C_1' \) can think of this as a path in the \( C_1'(2) \), where the first interval always lies above the second.) Start with \( ([−1, 0], [0, 1]) \); grow the second interval to obtain \( ([−1, 0], [−1, 1]) \); then translate \( [−1, 0] \) to \([0, 1]; \) finally shrink \([−1, 1]\) to \([−1, 0]\) to obtain the pair \(([0, 1], [−1, 0])\).

If we apply the formula (3) for the products of \( f \) and \( g \) governed by this path of 1-disks, we see a formal analogue for Steenrod’s formula for cup-one, namely

\[
\varphi \cup_1 \psi(\sigma) = \sum_{i<j} \varphi(\sigma|[v_1, \ldots, v_i, v_j, \ldots, v_n]) \cdot \psi(\sigma|[v_i, \ldots, v_j]).
\]

The main difference is that the product rather than using an underlying commutative ring uses operad insertion maps. McClure and Smith show this to be an appropriate extension of Steenrod’s formula.

5. **Projection maps, layers in the tower, and abelian group structure**

Our goal is to show that each stage of the Goodwillie–Weiss tower for knots has an abelian group structure compatible with connected sum. We use two models for the maps in the totalization tower of a cosimplicial space using subcubical diagrams, the second of which to our knowledge has not been used before in the literature.
5.1. **Cosimplicial and cubical diagrams.** Following Goodwillie, we find it fruitful to use cosimplicial diagrams and cubical diagrams interchangeably. Some of this material was used in [Sin09] and is treated clearly in the forthcoming volume [MV]. We recall some of the basic ideas before using variants of them for our main arguments.

To begin, just as the standard map $\text{Tot}^n X \to \text{Tot}^{n-1} X$ can be defined by inclusions of the indexing categories $\Delta_{n-1} \subset \Delta_n$, through the equivalence of Theorem 3.2 this map is also defined by the canonical inclusion of $\mathcal{P}_\nu[n-1] \subset \mathcal{P}_\nu[n]$. We will analyze the difference — that is, the layers in the totalization tower — in two different ways this section. First we use a sort of Mayer–Vietoris decomposition of the category $\mathcal{P}_\nu[n]$.

**Definition 5.1.**

- Let $\mathcal{P}_{\neq n}[n] \subset \mathcal{P}_\nu[n]$ be the full subcategory given by all nonempty subsets of $[n]$ except the singleton $\{n\}$.
- Let $\mathcal{P}_n[n]$, be the (cubical) poset of all subsets of $[n]$ containing $n$.
- Let $\mathcal{P}_{n+}[n]$ be the (subcubical) poset of subsets of $[n]$ containing $n$ and at least one other element.

The inclusion $\iota : \mathcal{P}_\nu[n-1] \to \mathcal{P}_{\neq n}[n]$ is left cofinal, so the map induced on homotopy limits is an equivalence. Thus if $X^\bullet$ is a cosimplicial space, we can replace the maps $\text{holim}_{\mathcal{P}_\nu[n]} X^\bullet \circ \mathcal{G}_n \to \text{holim}_{\mathcal{P}_\nu[n-1]} X^\bullet \circ \mathcal{G}_n$ by an alternate model for the maps in the tower, namely

$$\text{holim}_{\mathcal{P}_\nu[n]} X^\bullet \circ \mathcal{G}_n \to \text{holim}_{\mathcal{P}_{\neq n}[n]} X^\bullet \circ \mathcal{G}_n. $$

The poset $\mathcal{P}_\nu[n]$ can be written as the union of $\mathcal{P}_{\neq n}[n]$ and $\mathcal{P}_n[n]$. The union is along the poset $\mathcal{P}_{n+}[n]$.

Applying $\text{holim}(\_ \circ \mathcal{G}_n)$ to the diagram above, we get a pullback square (of fibrations) [Goo92 Proposition 0.2]. Thus, to study the fiber(s) of the map from $\widetilde{\text{Tot}}^{n} \to \widetilde{\text{Tot}}^{n-1}$, which up to homotopy is the left-hand column of the induced map of homotopy limits of this square, it suffices to study the right-hand column. We say fiber(s) because in our application we study disconnected spaces.

In the based case one can say more, as we use for spectral sequence calculations below. Since $\mathcal{P}_{n+}[n]$ is just the cube $\mathcal{P}_n[n]$ with its initial object removed, the map on homotopy limits induced by the right vertical arrow is just the map from the initial object at $\{n\}$ to the homotopy limit of the rest of the diagram, which is subcubical. Thus, in the based setting the fiber is just the total fiber of the cube $\mathcal{P}_n[n]$, and the original map is $k$-connected if and only if the $n$-cube $X^\bullet \circ \mathcal{G}_n(\mathcal{P}_n[n])$ is $k$-cartesian.

Because $X^\bullet$ is a cosimplicial space, the codegeneracy maps define retractions of cubes which allow us to give an alternate description of this total fiber as an $n$-fold loop space. For example, the fiber of either coface $X^0 \to X^1$ is loops on the fiber of the codegeneracy $X^1 \to X^0$.

To this end, we consider a diagram whose shape is the category $(0 \to 1 \to 2)^n$. Define $\bar{\mathcal{P}}_n[n]$ to be the category with the same objects as $\mathcal{P}_n[n]$ (namely, subsets $S \subset [n]$ which contain the element $n$) but whose morphisms are all order-preserving maps. Consider the functor $\mathcal{H}_n : (0 \to 1 \to 2)^n \to \bar{\mathcal{P}}_n[n]$ which sends $(0 \to 1)^n$ to the cube $\mathcal{P}_n[n]$, so that every composition along a coordinate axis is the identity. (In many cases, there are two choices of maps, but either choice is sufficient.)
This $\mathcal{H}_n$ can be viewed as a “retract of $n$-cubes.” For example, for $n = 2$, this diagram is

\begin{equation}
\begin{array}{ccc}
\{2\} & \to & \{0, 2\} \\
\downarrow & & \downarrow \\
\{1, 2\} & \to & \{0, 1, 2\} \\
\downarrow & & \downarrow \\
\{2\} & \to & \{0, 2\}
\end{array}
\end{equation}

where the upper-left square is $\mathcal{P}_2[2]$.

Let $\tilde{\mathcal{G}}_n : \tilde{\mathcal{P}}[n] \to \Delta$ be the following extension of $\mathcal{G}_n$: on objects $\tilde{\mathcal{G}}_n$ sends $S$ to $[\#S - 1]$, and on morphisms it sends $S \to T$ to $[\#S - 1] \cong T \cong [\#T - 1]$, where the isomorphisms are determined by the ordering of $S, T$. The diagram $\tilde{\mathcal{G}}_n \circ \mathcal{H}_n : (0 \to 1 \to 2)^n \to \Delta$ is again a retract of cubes, as is the diagram $X^* \circ \tilde{\mathcal{G}}_n \circ \mathcal{H}_n$.

**Definition 5.2.** Let $\mathcal{P}(n)$ denote the category of all subsets of $n$ with inclusions as morphisms. Let $\mathcal{G}_n' : \mathcal{P}(n) \to \Delta_n$ be the functor which on objects sends $S$ to $[n - \#S]$. On morphisms, it sends the inclusion $S \subset S'$ to the composite

$$[n - \#S] \cong ([n] - S) \xrightarrow{p} ([n] - S') \cong [n - \#S'],$$

where the first and last maps are the order-preserving isomorphisms and where $p$ sends $i \in [n] - S$ to the largest element of $[n] - S'$ which is less than or equal to $i$.

If we identify $(1 \to 2)^n$ with $\mathcal{P}(n)$, we see that the restriction of $\tilde{\mathcal{G}}_n \circ \mathcal{H}_n$ to $(1 \to 2)^n$ is naturally equivalent to $\mathcal{G}_n'$. The following is [MV, Proposition 5.5.7].

**Proposition 5.3.** In this situation of a retract of cubes, the total fiber of the $(0 \to 1)^n$ cube is $n$-fold loops on the total fiber of the $(1 \to 2)^n$ cube.

Putting the above work all together gives the following, which is [MV, Proposition 9.4.10].

**Theorem 5.4.** If $X^*$ is a based cosimplicial space, the fiber of the $n$th map in the Tot tower (that is, the layer) is homotopy equivalent to $n$-fold loops applied to the total fiber of $X^* \circ \mathcal{G}_n'$.

5.2. **Surjectivity on components of maps in the tower.** In order to inductively establish a group structure on components of stages in the tower, we need a surjectivity result.

**Theorem 5.5.** The restriction map $\tilde{\mathcal{M}}_n \to \tilde{\mathcal{M}}_{n-1}$ induces a surjection on $\pi_0$.

We prove this unframed version of the theorem first and then use it to prove the desired framed version, as the end of the proof we give here breaks down in the framed setting.

**Proof of Theorem 5.5.** We extend and apply cubical techniques as developed in the previous subsection. Recall from Definition 3.1 and Section 3.3 the cosimplicial space $C_\partial([3, \partial])$, which we abbreviate here as $C$. This functor defines both our cosimplicial model and, through pullback by the functors $\mathcal{G}_n$, our mapping space models.

We use the square of posets (8) above. We are led to consider the map the map on homotopy limits induced by $\mathcal{P}_{n+1}[n] \to \mathcal{P}_n[n]$, which if surjective on components implies the same for the map $\tilde{\mathcal{M}}_n \to \tilde{\mathcal{M}}_{n-1}$, the left-vertical map in the square (8).

The initial object in $\mathcal{P}_n[n]$ is $C(\{n\}) = C_\partial([3, \partial]) = \ast$, a point. Thus the map on homotopy limits induced by $\mathcal{P}_{n+1}[n] \to \mathcal{P}_n[n]$ is surjective on components if and only if the homotopy limit over $\mathcal{P}_{n+1}[n]$ is
connected. By reindexing, we can describe this as the homotopy limit over the $n$-cubical poset $P_n[1-n]$ of a functor $D$ given on objects by $D(S) = C^\times_{\text{i.d}}([3], \partial)$ and on morphisms by doubling as usual.

We will prove its connectedness by induction on $n$. The base case $n = 1$ is immediate, as $C^\times_1([3]) = S^2$ is connected. For the induction step, we exhibit $P_n[1-n]$ as a fibration over a connected space with a connected fiber. Consider the reindexed pushout square $\square$, but with $n$ replaced by $n - 1$ everywhere. Again, the inclusion $P_n[1-n-1] \hookrightarrow P_{n-1}[1-n-1]$ is left cofinal, so the induced map of homotopy limits is an equivalence. Thus the left-hand column of the reindexed $\square$ gives a fibration whose base, the homotopy limit of $D$ over $P_n[1-n-1]$, is connected by induction.

The square of holim’s induced by $\square$ is a pullback, so it suffices to establish connectedness of the fiber of the map induced by the right-hand column, taken over the component that the connected space $\text{holim}_{P_{n-1}[1-n-1]} D$ maps to. Here we choose basepoints for $D$ (as well as the original $C$) as forced by choosing the basepoint $(1, 0, 0)$ in $S^2 = C^\times_1([3], \partial)$.

Now that this induced square of holim’s is of based spaces, we can describe the fiber of this map as the fiber of the map induced by the right-hand column, taken over the component that the connected space $\text{holim}_{P_{n-1}[1-n-1]} D$ maps to. Here we choose basepoints for $D$ (as well as the original $C$) as forced by choosing the basepoint $(1, 0, 0)$ in $S^2 = C^\times_1([3], \partial)$.

To show that $\Omega_n^{-1} \text{tfib} D^1$ is connected or, equivalently, that $D^1$ is $(n-1)$-Cartesian, we use a Blakers-Massey theorem. This is an $(n-1)$-cube of spaces $C^\times_i([3], \partial)$, $i \leq n - 1$, of configurations in $[3]$ up to scaling and translation, with a tangent vector at each point. The maps forget points and corresponding tangent vectors. Up to homotopy, we can replace this by a cube of spaces $C^\times_i([3], \partial)$ of honest configurations in $[3]$ with a tangent vector at each point. Every map in this cube is a fibration. So as in the end of the proof of Theorem 7.1 of [Sin09], we can take the fiber in one direction. The resulting $(n-2)$-cube which we call $\varphi D^1$ has entries $[3] - f([i])$, where the deleted points are images of a fixed embedding $f : [n] \hookrightarrow [3]$. The maps in the cube are inclusions of codimension zero open submanifolds and are thus cofibrations. Moreover, $\varphi D^1$ is a push-out cube, so it is strongly co-Cartesian.

Each inclusion $[3] - f([i+1]) \hookrightarrow [3] - f([i])$ is a 2-connected map. The Blakers-Massey theorem (as stated in Theorem 2.3 of [Goo92], see also [MV Theorem 6.2.1]) applies to give that the total fiber of $\varphi D^1$ is $n$-connected, so that its $n$th loop space is connected, which yields the result.

We will say more about this total fiber – and thus the layers in the tower – both below in this section and in Section 6 when we make spectral sequence calculations.

In the proof just given, the analogous $(n-1)$-cube with frames instead of tangent vectors is not $(n-1)$-Cartesian, which is why we prove the framed version separately now.

**Theorem 5.6.** In the framed setting, the map $\widetilde{AM}_n^\text{fr} \to \widetilde{AM}_{n-1}^\text{fr}$ induces a surjection on $\pi_0$.

**Proof.** As $\widetilde{AM}_n^\text{fr}$ is a subspace of $\text{Map}(\Delta^n, C_n([3], \partial) \times O(3)^n)$, we consider the diagram

$$
\begin{array}{ccc}
\Delta^n & \to & C_n([3], \partial) \times O(3)^n \\
\downarrow & & \downarrow \\
\Delta^{n-1} & \to & C_{n-1}([3], \partial) \times O(3)^{n-1}.
\end{array}
$$

Suppose we have $\Phi \in AM_{n-1}^\text{fr}$. Let $\varphi$ be the image of $\Phi$ under the map $AM_{n-1}^\text{fr} \to AM_{n-1}$, which essentially composes a map to $C_n([3], \partial) \times O(3)^n$ with the projection onto $C_n([3], \partial) \times (S^2)^n$, using the first vector in each frame. By Theorem 5.5 there is a $\psi \in AM_n$ whose image in $AM_{n-1}$ is in the same component as $\varphi$. We lift $\psi$ to a map $\Delta^n \to C_n([3], \partial) \times O(3)^{n-1}$, using $\Phi$ to define the map to the $O(3)^{n-1}$ factor.

It remains to lift this map to one additional factor of $O(3)$. The map $\Delta^n \to C_n([3], \partial) \times O(3)^n$ is prescribed on two faces of $\Delta^n$, namely the faces on which the additional factor must agree with another
factor. Away from these faces, there are no constraints on the map to the additional factor of $O(3)$. Thus topologically the problem is to extend this map from $D^{n-1} \subset \partial D^n$ to $D^n$, which is immediate. □

5.3. Group structure. For purposes of establishing group structure, we first analyze the layers in the tower in the mapping space model using configurations modulo translation and scaling. That is, we analyze the fibers $\mathcal{L}_n$ of the maps $\tilde{AM}_n \to \tilde{AM}_{n-1}$ over the “identity” base points, along with the maps from the layers to the stages in the tower.

**Lemma 5.7.** The $n$th layer $\mathcal{L}_n$ is a grouplike space, and its map to $\tilde{AM}_n$ is a map of $H$-spaces.

Here we are using the the little intervals action for the $H$-space structure on $\tilde{AM}_n$. A similar statements is true for the variant $\tilde{AM}_n^{fr}$.

We know that $\mathcal{L}_n$ is grouplike from a basic understanding of Tot-towers, and in particular the description of the fiber of a cubical diagram defined using codegeneracies. However, we need to see the monoid structure on $\mathcal{L}_n$ in a different (but presumably homotopic) way which will be more obviously compatible with that on $\tilde{AM}_n$.

We start with another model for the stages and layers in the totalization tower for a cosimplicial space $XX^\bullet$, again building on the diagram $X^\bullet \circ \mathcal{G}_n$, whose homotopy limit is $\tilde{Tot}^n X^\bullet$. While the previous section used a sort of Mayer–Vietoris decomposition of $P_\nu[n]$, in this one we “fiber $P_\nu[n]$ over $P_\nu[n-1]$.” Consider the functor $i_n$ from $P_\nu[n]$ to $P_\nu[n-1]$ which modifies a subset $S$ by identifying $(n-1)$ and $n$ – that is, changing occurrences of $n$ in $S$ to $n-1$.

**Lemma 5.8.** The homotopy limit of $X^\bullet \circ \mathcal{G}_n \circ i_n$ is homotopy equivalent to $\tilde{Tot}^{n-1} X^\bullet$.

The $n$th degeneracy $s^n : X^n \to X^{n-1}$ extends to a map

$s^n : X^\bullet \circ \mathcal{G}_n \to X^\bullet \circ \mathcal{G}_n \circ i_n$.

On homotopy limits, the map induced by $s^n$ agrees with the standard map from $\tilde{Tot}^n X^\bullet \to \tilde{Tot}^{n-1} X^\bullet$.

**Proof.** For the first statement, it suffices to show that the functor $i_n$ is left cofinal [BK72 XI.9.2]. In other words, we have to show that the geometric realization of the fiber-product category $\prod \nu[n]_\nu \times \nu[n-1]_\nu \downarrow S$ is contractible for every $S \in P_\nu[n-1]$. But this category has a final object, namely the object corresponding to $S \cup n \in P_\nu[n]$. Thus the homotopy limit of $X^\bullet \circ \mathcal{G}_n \circ i_n$ is equivalently $\tilde{Tot}^{n-1} X^\bullet$.

It follows from the (co)simplicial identities that $s^n$ induces a natural transformation of functors $s^n : \mathcal{G}_n \circ X^\bullet \to i_n \circ \mathcal{G}_n \circ X^\bullet$, where the map on each object is given by either its last degeneracy or the identity map.

The standard map $\tilde{Tot}^n X^\bullet \to \tilde{Tot}^{n-1} X^\bullet$ is induced by the inclusion functor $j_n : P_\nu[n-1] \to P_\nu[n]$. Note that the composite $i_n \circ j_n$ is the identity. Consider the following diagram to complete the proof.

\[
\begin{array}{ccc}
\text{holim}(X^\bullet \circ \mathcal{G}_n) & \xrightarrow{- \circ j_n} & \text{holim}(X^\bullet \circ \mathcal{G}_n) \\
\downarrow & \downarrow & \downarrow \\
\text{holim}(X^\bullet \circ \mathcal{G}_n-1) & \xrightarrow{- \circ j_n} & \text{holim}(X \circ \mathcal{G}_n-1 \circ i_n)
\end{array}
\]

The left-hand vertical map is the map $\tilde{Tot}^n X^\bullet \to \tilde{Tot}^{n-1} X^\bullet$ (since $X^\bullet \circ \mathcal{G}_n \circ j_n = X^\bullet \circ \mathcal{G}_n-1 \circ i_n$). Precomposing $X^\bullet \circ \mathcal{G}_n-1 \circ i_n$ with $j_n$ induces the horizontal map because $i_n \circ j_n = id$. This horizontal map is an equivalence because on these homotopy limits, “precomposing with $j_n$” is the right-inverse to the equivalence given by “precomposing with $i_n$.” □
Proof of Lemma 5.7. First we apply Lemma 5.8 to our cosimplicial space $\tilde{C}_\bullet(\mathbb{R}^3, \partial)$, recalling that the homotopy limit of $C_\bullet(\mathbb{R}^3, \partial) \circ G_n$ is $\tilde{AM}_n$.

In this case, because the projection maps $C_\bullet(\mathbb{R}^3, \partial) \to \tilde{C}_\bullet(\mathbb{R}^3, \partial)$ are fibrations, as are identity maps, $\tilde{s}^n$ will induce a fibration on homotopy limits. (See [BCSS05] Lemma 3.5 for an explicit proof in this case of a standard result about enriched model structures on diagram categories in general.)

We choose as a basepoint in the homotopy limit of $\tilde{C}_\bullet(\mathbb{R}^3, \partial) \circ G_{n-1} \circ t_n$ the evaluation map of the standard unknot, which because of the quotienting by translation and scaling is a constant map. Taking the fiber over this point now defines our desired model for the $n$th layer of the Goodwillie–Weiss tower in this case. Explicitly, $\tilde{L}_n$ is the subspace of $\tilde{AM}_n$ of maps $\Delta^n \to \tilde{C}_n(\mathbb{R}^3, \partial)$ whose projection by forgetting the last point in the configuration yields the chosen constant map. This implies that such maps are themselves constant on the $t_n = t_{n-1}$ and $t_n = t_{n+1} (= 1)$ faces of $\Delta^n$.

We perform a straightforward check that the product $\mu_0$ on $\tilde{AM}$ from Definition 4.3 restricts to a product on $\tilde{L}_n$. Recall that $\mu_0(\varphi, \psi)$ evaluated at $(t_1, \ldots, t_n)$ is the “stacked product”, by insertion into the standard two-point configuration $\mu$ (along the $x$-axis), of the two configurations given by taking $\varphi(2t_1 + 1, 2t_2 + 1, \ldots, 2t_i + 1, 1, \ldots, 1)$ and $\psi(0, 0, \ldots, 2t_{i+1} - 1, \ldots, 2t_n - 1)$ and forgetting points whose indices are greater than $i$ for $\varphi$ or less than or equal to $i$ for $\psi$. Here $t_i \leq 0 \leq t_{i+1}$. If $t_n$ is greater than or equal to zero, the configuration resulting from $\varphi$ must be the standard one, and the one resulting from $\psi$ is standard after forgetting the last point. When stacked the result is a configuration which is standard after forgetting the last point. If $t_n$ is less than zero, then the result is just $\varphi$. In either case, the product is in $\tilde{L}_n$.

We now show that this multiplication is homotopic to one with inverses up to homotopy, which is essentially a “loop sum on the $t_n$ coordinate.” The restriction of a map in $\tilde{L}_n$ to a segment in $\Delta^n$ for which all $t_i$ for $i < n$ are fixed and $t_n$ varies between $t_{n-1}$ and $t_{n+1} = 1$ is a loop in the configuration space $\tilde{C}_n(\mathbb{R}^3, \partial)$, based at the standard point. (Here it is necessary to be using the configuration space modulo translations and scalings, so that these loops begin and end at the exact same configuration.) Loop concatenation as the $t_i$ vary will be continuous, and we use it to define the second multiplication on $\tilde{L}_n$.

This clearly has inverses.

To show that these two products are homotopic, we first recall Proposition 4.7 shows that the product $\mu_0$ is homotopic to a “decoupled” version. In the decoupled version, the product of $\varphi$ and $\psi$ is given by $\varphi$ composed with a linear scaling on $\Delta^n$ for $\ell \in \Delta^n$ whose average is less than or equal to zero and $\psi$ on those whose average is greater than or equal to zero. But our loop sum product applies $\varphi$ and $\psi$ independently to rescaled times as well, according to whether $t_n$ is greater than or less than $\frac{1}{2}(t_{n-1} + 1)$. A homotopy between these domains of definition for $\varphi$ and $\psi$ as well as the rescalings needed yields the result.

\[ \Box \]

Theorem 5.9. $\pi_0(\tilde{AM}^f_n)$ is an abelian group, compatible with the action of the little intervals operad.

Proof. By Corollary 4.9 $\pi_0(\tilde{AM}^f_n)$ is an abelian monoid. We prove that is a group by induction on $n$.

The basis case of $n = 0$ is clear, since $\tilde{AM}^f_0 = *$. So suppose $n \geq 1$, and suppose we know that $\tilde{AM}^f_{n-1}$ is a group. From Lemma 5.7 we know that for $n \geq 1$, $\pi_0$ of the fiber $\tilde{L}_n$ of $\tilde{AM}^f_n \to \tilde{AM}^f_{n-1}$ is a group and that $\pi_0(\tilde{L}_n) \to \pi_0(\tilde{AM}_n)$ is a map of monoids. Thus applying $\pi_0$ to the fiber sequence $\tilde{L}_n \to \tilde{AM}^f_n \to \tilde{AM}^f_{n-1}$ gives an exact sequence of monoids. By Theorem 5.6 $\pi_0(\tilde{AM}^f_n \to \tilde{AM}^f_{n-1})$ is surjective. The proof of the lemma is thus reduced to the elementary fact that if $G \to H \to K \to 0$ is an exact sequence of monoids with $G, K$ groups, then $H$ is a group.

\[ \Box \]

Recall that Lemma 5.7 held for the unframed version $AM_n$ as well as the framed version.
Corollary 5.10. The homotopy fibers of $AM_n$ over varying components of $AM_{n-1}$ are homotopy equivalent.

6. The homotopy tower is a finite-type invariant

In this section we show that $\pi_0(ev_n): \pi_0\Emb^{fr}(\mathbb{R}, \mathbb{R}^3) \to \pi_0AM^{fr}_n$, which we now know to be an abelian group valued invariant compatible with connect-sum of knots, is a finite-type invariant of type $(n-1)$. The main tool is a theorem of Habiro [Hab00], which states that two classical knots share finite-type invariants of degree $\leq n - 1$, if and only if they differ by a series of $C_n$-moves.

We describe these moves now in a way that will facilitate our proof. Let $E_2$ be a copy of $D^2 \times I$, with two properly embedded subarcs which clasp in the center as in Figure 1. Iteratively form $E_n$ from $E_{n+1}$ by replacing a regular neighborhood of the top left arc of $E_n$ by a copy of $E_2$.

![Figure 1. The basic clasp $E_2$ and the iteratively constructed $E_4$. The ambient cylinders are not drawn.](image)

Now a basic $C_n$-move on a knot $K$ is given by finding an embedding of $E_n$ into $\mathbb{R}^3$ which meets the knot $K$ as the given collection of arcs, and sliding another subarc of $K$ across the central disk $D^2$ of the embedded $E_n$ as in Figure 2.

The basic tool we need is the following theorem, which follows directly from work of Habiro [Hab00].

Theorem 6.1. Suppose that $\nu$ is an additive invariant of (unframed) knots. If it is invariant under $C_n$-moves then it is a finite-type invariant of degree $n-1$.

Proof. The $C_n$-moves constructed here are an alternate formulation of clasper surgery, and in fact are very close to Habiro’s original formulation in his master’s thesis. If we had allowed $E_2$ to replace an arbitrary arc in iterating the construction (as opposed to the upper left), we would get an arbitrary capped clasper surgery. The ones constructed here correspond to “linear claspers,” where all nodes are directly adjacent to a leaf. The topological IHX relation allows us to reduce to this case. (See Theorem 13 of [CT04].) So the $C_n$-moves introduced here are equivalent to the $C_n$-moves in [Hab00]. The argument in Theorem 6.1 works just as well if we allow $E_2$ to replace an arbitrary subarc at each stage, so one could dispense with this subtlety.

The monoid of knots modulo the equivalence relation of $n$-equivalence is a finitely generated abelian group [Gon98, Hab00]. Theorem 6.17 of [Hab00] states that the natural projection $\psi_{n-1}$ from knots to this abelian group is a universal additive finite-type invariant of degree $n-1$. So if $\nu$ is an additive invariant of knots which also is invariant under $C_n$ moves, it induces a homomorphism on the group of knots modulo $n$-equivalence. It thus factors as a composition of a degree $n-1$ invariant with a group homomorphism. It is therefore a degree $n-1$ invariant itself. □
In order to move to the framed case, we need the following lemma. For any integer $k$, let $fr_k$ be the map from unframed knots to framed knots which adds a $k$-framing.

**Lemma 6.2.** Let $U_1$ represent the +1 framed unknot. Let $\nu$ be an additive framed knot invariant taking values in an abelian group. Then

$$\nu \circ fr_k = k\nu(U_1) + \nu \circ fr_0.$$

**Proof.** One can push the twisting of the framing onto a standard subarc of the knot to see that $fr_k(K) = U_k \# fr_0(K)$, where $U_k$ is a $k$ framed unknot. Then one separates each of the twists, and uses the fact that $\nu$ is additive. \qed

**Corollary 6.3.** Suppose that $\nu$ is an additive invariant of framed knots. If it is invariant under $C_n$-moves then it is a finite-type invariant of degree $n - 1$ for $n \geq 2$.

**Proof.** Note that $\nu \circ fr_0$ is an additive invariant of unframed knots, and that it is invariant under $C_n$-moves, since $\nu$ is. Therefore $\nu \circ fr_0$ is finite-type of type $n - 1$ by Theorem 6.1. On the other hand, by Lemma 6.2 $\nu(K) = fr(K)\nu(U_1) + \nu \circ fr_0(K)$, where $fr(K)$ is the framing number. The invariant $fr$ is known to be type 1, so we have a linear combination of a type $n - 1$ and type 1 invariant, which is therefore of type $n - 1$. \qed

This corollary gives us the main tool we need to show that $\pi_0(ev_n)$ is of finite type.

**Theorem 6.4.** The map $\pi_0(ev_n)$ is invariant under $C_n$-moves, and therefore is a type $n - 1$ invariant.

**Proof.** Let $K$ be a framed knot, and let $K'$ be the knot after the $C_n$-move has been applied. Our strategy is to find a family of isotopies on $K$ or actually on the subarcs in $E_n$ which depend continuously on $n$-point configurations in $I$. This will give rise to a new element $\overline{K}$ which is homotopic to $ev_n(K)$ and a new element $\overline{K}'$ which is homotopic to $ev_n(K')$. This family will have the property that whenever there are strictly less than $n$ configuration points on the subarcs of $E_n$, then those configuration points never meet the central disk of $E_n$. This implies that $\overline{K}$ and $\overline{K}'$ are homotopic since we can just push the arc across the central disk of $E_n$ without ever introducing collisions of configuration points. This is by design when there are $n - 1$
points or less inside $E_n$, and if there are $n$, that means there is no point left over to lie in the exterior arc that we are homotoping.

Let us describe the family of reembeddings of $E_n$ that does the job. Consider a nested copy of $E_i$ inside $E_n$. It consists of an arc, $\alpha_i$ clasping with a copy of $E_{i-1}$. Now we slide the central clasp along the axis of the cylinder in the following way. A set of configuration points in the arc $\alpha_i$ will pull the clasp toward that side of the cylinder in a manner that increases as the minimum distance of these points to the midpoint of the arc $\alpha_i$ decreases. Configuration points in the copy of $E_{i-1}$ exert a similar tug to their end of $E_i$ in a manner which increases as they get closer to the midpoints of their arcs. However the tug of a point is halved in power when you pass to $E_{i-1}$. This has been set up so that a point at the midpoint of $\alpha_i$ will always exert a tug that equals or exceeds the collective tug of $E_{i-1}$.

We also set things up so that if $E_{i-1}$ has fewer than $i-1$ configuration points on it, then configuration points in $\alpha_i$ can never get more than $\varepsilon$ away from their end disk $D^2 \times 0$. The point is that any configuration points in $\alpha_i$ will have a greater tug on the clasp than $E_{i-1}$ when one is at the midpoint of $\alpha_i$. We just ensure that this tug is strong enough to pull $\alpha_i$ $\varepsilon$-close to its end disk. (We can come up with a uniform $\varepsilon$ in this way since there is a discrete gap between the maximal tug that $E_{i-1}$ can exert and the tug it exerts at less than full occupancy.)

Similarly arrange that if the arc $\alpha_i$ has no configuration points, then no configuration points in $E_{i-1}$ can get more than $\varepsilon$ away from $D^2 \times 1$ inside $E_i$. A general reembedding of $E_n$ is obtained by first reembedding the deepest copy of $E_2$, then the copy of $E_3$ containing it, etc. (Actually this process is rather commutative, the actual order of applying the reembeddings shouldn’t matter.)

So, with such a family of reembeddings, we claim that no configuration point ever passes through the $D^2 \times 1/2$ disk of $E_n$, provided that strictly less than $n$ configuration points are present inside $E_n$. Let the arcs in $E_n$ be called $\alpha_1, \alpha_2, \ldots$ arranged in order of increasing depth.

We know that there is some arc $\alpha_i$ which is not occupied by a configuration point. The points in $E_{i-1}$ stay within $\varepsilon$ of the end disk and cannot cross the center disk of $E_n$. Thus any points that do cross the center must lie on one of the arcs $\alpha_1, \ldots, \alpha_{i-1}$. Now consider such an arc $\alpha_j$, $j < i$. It links with $E_{j-1}$ which has less than full occupancy. Points in $\alpha_j$ cannot get further than $\varepsilon$ away from their end disk, and cannot cross the center disk of $E_n$, so we are done! \hfill \Box

7. The homotopy spectral sequence for the tower

In this section we further develop the spectral sequence for the homotopy groups and in particular the components of the stages in the Goodwillie–Weiss tower for classical knots (the cases of knots in higher-dimensional Euclidean space being covered in [Sin09]). We see that at the $E^2$ stage they are exactly what one would expect if the tower is to serve as a universal finite-type invariant. We in particular see similar structures to what Goodwillie and Weiss [GW99 Section 5] originally saw in higher dimensions, but can also compare that to newer results on the combinatorics of finite-type invariants [Con08]

A priori the spectral sequence of a totalization tower, or any other tower of fibrations, is difficult to discern in degree zero. Not only is $\pi_0$ only a set-valued functor, but homotopy groups can differ over different components. We saw in Section 5 however that this tower has additional algebraic structure, which leads to the following.

**Theorem 7.1.** The spectral sequence for the homotopy groups, and in particular components, of $AM_n$ as a stage in the Goodwillie–Weiss tower is a spectral sequence of abelian groups which converges.

**Proof.** By Lemma 5.7 and Theorem 5.9 the fiber sequence

$$L_n \to AM_{n}^{fr} \to AM_{n-1}^{fr}$$

is a fibration sequence of group-like spaces. It is thus loops on the fibration sequence defined on their classifying spaces. Its long exact sequence in homotopy groups is then a degree shift of that for the
classifying spaces, starting with \( \pi_1 \) of the classifying spaces being \( \pi_0 \) of these spaces. These exact sequences can be spliced in the usual way to obtain a spectral sequence, which by Corollary 4.9 is one of abelian groups.

Convergence follows from the fact that this tower is finite when we consider only \( AM_n^{tr} \), and the groups are finitely generated by Proposition 7.2 below along with the fact that homotopy groups of spheres are finitely generated.

We can now analyze the spectral sequence in further detail. By Theorem 5.3, \( L_n \) is equivalent to \( \Omega^n \) of the total fiber of the \( n \)-cube \( C_n^{tr}(I^n, \partial) \circ G_n^{tr} \). Unraveling definitions, this is an \( n \)-cube whose entries are spaces of configurations of at most \( n \) points, together with at most \( n \) frames, where each map in the cube forgets a point and a frame. We can express this cube as an entry-wise product of the cubes \( S \mapsto C_{\Omega^n}(I^n, \partial) \) and \( S \mapsto O(3)|^{\uparrow n} \). The total fiber of the product is the product of the total fibers, and for \( n \geq 2 \), the total fiber of the cube of \( O(3) \)'s is a point. Thus it suffices to consider the total fiber of \( S \mapsto C_{\Omega^n}(I^n, \partial) \). Furthermore, we can switch to open configuration spaces, for which these forgetting maps are well known to be fibrations. We take fibers in one direction and consider the resulting \( (n - 1) \)-cube instead. Here we see the entries as \( I^3 \) with a finite set of points removed, and maps which are inclusions. Up to homotopy, this is a cube of spaces \( \bigvee_T S^2 \) indexed by subsets \( T \subset n - 1 \), where each map projects off a wedge factor. Call this cube \( P(n-1) \bigvee S^2 \).

By Hilton’s Theorem [Hil55], the homotopy groups of a wedge \( \bigvee_n S^2 \) is a direct sum of homotopy groups of higher-dimensional spheres. Let \( L_n \) be the free graded Lie algebra (working over the integers for the rest of this section) on \( n \) odd-graded generators in degree one. Let \( B_n \) be a basis for \( L_n \), choosing these consistently as \( n \) varies. For example, we could use a graded version of Hall bases.

More specifically, Hilton’s Theorem states that \( \pi_*(\bigvee_n S^2) \) is a direct sum \( \bigoplus_{W \in B_n} \pi_* S^{|W| - 1} \), where \( W \) is the degree or word length of \( W \). The theorem is functorial if we use bases for free Lie algebras of different ranks which extend one another, since the Whitehead products used to define the elements of homotopy are functorial. Because the projection maps between wedge products of \( S^2 \) are split, these different bases split off. An immediate inductive calculation of the homotopy groups of the total fiber (as an iterated fiber of fibers) shows that what is left for homotopy groups is indexed by a basis \( B_n \) of the submodule of the \( L_n \) spanned by brackets in which all generators occur.

**Proposition 7.2.** The spectral sequence for the homotopy groups (including \( \pi_0 \)) of \( AM_n \) has as \( E^2_{-p,-q} \) the module \( \bigoplus_{W \in B_{p-1}} \pi_* S^{|W| - 1} \).

We now focus on total degree zero. Let \( A_n^p \) be the \( \mathbb{Z} \)-module of chord diagrams on a line segment with \( n \) chords, modulo the usual four-term relation \( 4T \) and the relation \( SEP \), which sets every separated (i.e. non-primitive) diagram to zero. Alternatively, \( A_n^p \) is the \( \mathbb{Z} \)-module of trivalent diagrams, modulo antisymmetry, the IHX relation, and \( SEP \). See [Cor08] for more details.

**Theorem 7.3.** The group \( E^2_{-n,n} \) is isomorphic to \( A_n^{n-1} \).

**Proof.** By Proposition 7.2 the group \( E^1_{-n,n} \) is isomorphic to the submodule of the free Lie algebra on \( n - 1 \) generators generated by \((n-1)\)-fold brackets where each generator appears exactly once. This module is \( \mathcal{L}ie(n-1) \), the \((n-1)\)-st space in the Lie operad, which well known to be free of rank \((n-1)!\).

Next, we consider the 1-line of the \( E^1 \) page. Under the identification of Proposition 7.2 these groups decompose into a free summand and two-torsion. The free summands are indexed by \( n \)-fold brackets in the free Lie algebra on \( n - 1 \) generators, again in which all generators occur. The two-torsion summands occur as composites of \( S^n \stackrel{\partial}{\to} S^{n-1} \) with \((n-1)\)-fold Whitehead products from \( S^{n-1} \to \mathcal{P}(n-1) \bigvee S^2 \). This summand is isomorphic to \( \mathcal{L}ie(n-1) \otimes \mathbb{Z}/2 \).

The differential \( d_1 \) must be zero on the torsion summand. We claim that on the free summand the differential is the integral version of the differential defined in [SS02]. There in Theorem 2.1, through the
tower of fibrations

\[ C_n(\mathbb{R}^d) \to C_{n-1}(\mathbb{R}^d) \to \cdots \to C_0(\mathbb{R}^d) \]

the well-known rational homotopy Lie algebra of the configuration space \( C_n(\mathbb{R}^d) \) is calculated as generated by classes \( b_{ij} \) in degree \( d - 1 \). Under the map from the total fiber of \( P(n - 1) (\vee S^2) \) to \( C_n(\mathbb{R}^3) \), the basis for the free Lie algebra on generators, say \( x_i \), sends a bracket to a corresponding bracket in the \( b_{ij} \)'s. Because the projections in the tower (11) are split, these brackets are integral generators of free summands as well. (In fact, one can use the splitting of the tower (11) and the Hilton–Milnor theorem to express homotopy groups of configuration spaces as a direct sum of homotopy groups of spheres.) The formulas for the differential given in [SS02] are given in terms of these integral generators, so they hold for the spectral sequence over the integers as well.

In [Con08] the cokernel of the rational \( d^1 \) is computed to be \( \mathcal{A}_{n-1} \otimes \mathbb{Q} \). While the result is stated rationally (which is where the conjecture was made), all of the calculations involve only integer coefficients.

At the \( E_2 \)-level the components of \( \text{AM}_n \) thus look like they should receive a universal finite type-\((n-1)\) invariant over the integers. Because the map Goodwillie-Weiss tower factors the map to the tower Volic considers in [Vol06], we already know it defines a universal finite type-\( \frac{d}{2} \) invariant. Proposition [7.3] implies that the invariant \( \pi_0(\text{ev}_n) \) is of type \((n-1)\), rather than \( \frac{d}{2} \). Standard finite-type theory then gives a map from \( \mathcal{A}_{n-1} \) to \( \pi_0(\text{AM}_n) \), namely by taking alternating sums of values on resolutions of knots with singularities described by a given chord diagram. We conjecture that this is an isomorphism at \( E^2 \), which collapses to \( E^\infty \). This would imply by Theorem [7.3] that all weight systems lift to finite-type invariants over the integers. That is, it would establish \( \pi_0(\text{ev}_\infty) \) as a refinement of the Kontsevich integral, defined over the integers.

One key step towards establishing this conjecture would be the collapse of the spectral sequence, which is now of a tower of fibrations amenable to tools from algebraic topology. The limiting process of Vassiliev’s spectral sequence is still not well understood. (See [Giu10] for this limiting process in a piecewise-linear setting.) This tower is built from maps of spaces, in particular the sequences of (10) and the diagrams which define \( \text{AM}_n \), which have been analyzed to great effect for knots in higher dimensions [LTV10].

Another key intermediate step would be the surjectivity on components of \( \text{ev}_n \). This statement would follow from deep connectivity results of Goodwillie–Klein [GK08] if those applied in codimension two, but it may be approachable more directly in this case.

We suspect, however, that direct analysis of the invariants which arise from \( \pi_0(\text{ev}_n) \) will be most fruitful. They have already led to new geometric insight in degrees three and four [BCSS05, Flo13]. An application of Sinha–Walter’s Hopf invariants [SW13] (which can fully be applied by the calculations of Proposition 7.2) seems to lead to Goussarov–Polyak–Viro formulae, which is a promising sign.

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E-mail address: ryan.budney@gmail.com

Department of Mathematics, University of Victoria, Victoria, BC, Canada

E-mail address: jim.conant@gmail.com

Department of Mathematics, University of Tennessee, Knoxville, TN, USA

E-mail address: rmjk@uvic.ca

Department of Mathematics, University of Victoria, Victoria, BC, Canada

E-mail address: dps@uoregon.edu

Department of Mathematics, University of Oregon, Eugene, OR, USA