THE SHARP UPPER BOUNDS FOR THE FIRST POSITIVE EIGENVALUE OF THE KOHN-LAPLACIAN ON COMPACT STRICTLY PSEUDOCONVEX HYPERSURFACES

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Abstract. We give sharp and explicit upper bounds for the first positive eigenvalue $\lambda_1(\Box_b)$ of the Kohn-Laplacian on compact strictly pseudoconvex hypersurfaces in $\mathbb{C}^{n+1}$ in terms of their defining functions. As an application, we show that in the family of real ellipsoids, $\lambda_1(\Box_b)$ has a unique maximum value at the CR sphere.

1. Introduction

Let $(M^{2n+1}, \theta)$ be a compact strictly pseudoconvex pseudohermitian manifold of real dimension $2n + 1 \geq 3$. Let $\bar{\partial}_b : L^2(M) \to L^2_{0,1}(M)$ be the tangential Cauchy–Riemann operator and $\bar{\partial}_b^*$ the formal adjoint with respect to the volume measure $dv = \theta \wedge (d\theta)^n$. The Kohn-Laplacian acting on functions is given by $\Box_b = \bar{\partial}_b^* \bar{\partial}_b$ and the sub-Laplacian is given by $\Delta_b = 2\text{Re} \Box_b$.

There has been growing interest in the relation between the spectra of the sub-Laplacian and the Kohn-Laplacian and the geometric qualities of the underlying CR manifolds. We mention here, for example, the Lichnerowicz-type estimate for the first positive eigenvalue of the sub-Laplacian on compact manifolds with a lower bound on Ricci and torsion was studied in, e.g., [1, 2, 12, 17, 10]. The characterization of extremal case, the Obata-type problem, was studied in, e.g., [9, 20, 15]. In particular, X. Wang and the first author proved an Obata-type theorem in CR geometry for compact manifolds in [20] which characterizes the CR sphere (among compact manifolds) as the only extremal case in the Lichnerowicz-type estimate for the sub-Laplacian.

We refer the reader to the aforementioned papers and references therein for more details on these problems.

The eigenvalue problem for $\Box_b$ is more involved. It is well-known that on a non-embeddable compact strictly pseudoconvex manifold of three-dimension, $\text{Spec}(\Box_b)$ contains a sequence of “small” eigenvalues converging rapidly to zero. In this case, we can not define the first positive eigenvalue of $\Box_b$. In fact, by the theorems of Boutet de Monvel, Burns, and Kohn, zero is an
isolated eigenvalue of $\Box_b$ if and only if $M$ is embeddable in some complex space $\mathbb{C}^N$ \cite{5, 4, 14}; see also \cite{3}. Thus, for embedded manifolds, it makes sense to define and study the first positive eigenvalue $\lambda_1$ of $\Box_b$.

In \cite{7}, Chanillo, Chiu, and Yang proved a Lichnerowicz-type lower bound for $\lambda_1$ for three-dimensional manifolds (which are not assumed to be embedded a priori). Their method also gives the same estimate for five dimensional case. In a preprint \cite{8}, Chang and Wu gave a lower bound in general dimension and proved some partial results on characterizing the equality case.

In \cite{19}, X. Wang, the first, and the third author completely analyzed the equality case by establishing an Obata-type theorem for the Kohn-Laplacian; we refer to \cite{19} for more details.

In this paper, we shall give sharp upper bounds for $\lambda_1$ on compact strictly pseudoconvex CR manifolds embedded in $\mathbb{C}^{n+1}$. Suppose $\rho$ is a smooth strictly plurisubharmonic function on $\mathbb{C}^{n+1}$ and $\nu$ is a regular value of $\rho$ such that $M := \rho^{-1}(\nu)$ is compact. On $M$, consider the “usual” pseudohermitian structure $\theta$ “induced” by $\rho$:

$$\theta = \iota^*(i/2)(\overline{\partial}\rho - \partial\rho),$$

(1.1)

where $\iota : M \to \mathbb{C}^{n+1}$ is the usual embedding. This pseudohermitian structure gives rise to a volume form $du := \theta \wedge (d\theta)^n$ on $M$. Furthermore, $\rho$ induces a Kähler metric $\rho_{jk}dz^j\overline{dz}^k$ in a neighborhood $U$ of $M$. Let $[\rho^{jk}]$ be the inverse of $[\rho_{jk}]$. For a smooth function $u$ on $U$, the length of $\partial u$ in the Kähler metric is given by

$$|\partial u|_\rho^2 = \rho^{jk}u_j\overline{u}_k.$$

(1.2)

Here we use the usual summation convention: repeated Latin indices are summing from 1 to $n+1$. We also use $\rho^{jk}$ and $\rho_{jk}$ to raise and lower the indices, e.g., $u^k = \rho^{k\ell}u_\ell$, so that $|\partial u|_\rho^2 = \overline{u}_k u^k$. We define the following degenerate differential operator

$$\tilde{\Delta}_\rho = \left(|\partial \rho|_\rho^{-2}\rho^{jk}\rho_{jk} - \rho^{jk}\right)\partial_j\partial_{\overline{k}}.$$

(1.3)

Our first result in this paper is the following sharp upper bound for $\lambda_1$.

**Theorem 1.1.** Let $\rho$ be a smooth strictly plurisubharmonic function defined on an open set $U$ of $\mathbb{C}^{n+1}$, $M$ a compact connected regular level set of $\rho$, and $\lambda_1$ the first positive eigenvalue of $\Box_b$ on $M$. Assume that for some $j$,

$$\text{Re} \rho_j\tilde{\Delta}_\rho \rho_j + \frac{1}{n} |\partial \rho|_\rho^{-2} |\tilde{\Delta}_\rho \rho_j|^2 \leq 0 \text{ on } M.$$

(1.4)

Then

$$\lambda_1(M, \theta) \leq n \max_M |\partial \rho|_\rho^{-2}$$

(1.5)

and the equality holds only if $|\partial \rho|_\rho^2$ is constant along $M$.

The upper bound in (1.5) is sharp and the equality occurs on the sphere with the standard pseudohermitian structure. Moreover, in Example 4.3 below, we shall see that the condition (1.4) can not be relaxed.
Notice that condition (1.4) is satisfied if there exists $j$ such that $\rho_{jk} = 0$ for all $k$ and $l$ and hence we can easily construct examples for which Theorem 1.1 does apply. In particular, if $\rho_{jk} = \delta_{jk}$, then (1.4) holds. We shall show that in this case, we can improve the estimate by taking the average value of $|\partial \rho|^2$ instead of its maximum. Thus, we define $v(M) = \int_M \theta \wedge (d\theta)^n$ be the volume of $M$.

**Theorem 1.2.** Let $\rho$ be a smooth strictly plurisubharmonic function defined on an open set $U$ of $\mathbb{C}^{n+1}$, $M$ a compact connected regular level set of $\rho$, and $\lambda_1$ the first positive eigenvalue of $\Box_b$ on $M$. Suppose that $\rho_{jk} = \delta_{jk}$, then

$$
\lambda_1 \leq \frac{n}{v(M)} \int_M |\partial \rho|^{-2} \wedge (d\theta)^n. \tag{1.6}
$$

The equality occurs only if $|\partial \rho|^2$ is constant on $M$. If furthermore, $\rho$ is defined in the domain bounded by $M$, then $M$ must be a sphere.

The estimate (1.6) is a special case of a more general estimate in Theorem 4.1 below which provides a sharp upper bound for $\lambda_1$ in terms of the eigenvalues of the complex Hessian matrix $[\rho_{jk}]$.

Our main motivation for proving the upper bound in Theorem 1.2 comes from its application to the eigenvalue problems on the real ellipsoids, the compact regular level sets of a real plurisubharmonic quadratic polynomial. The ellipsoids were studied by Webster [23] who showed that an ellipsoid is not biholomorphic equivalent to the sphere unless it is complex linearly equivalent to the sphere. (It is now well-known that two generic ellipsoids are not biholomorphic equivalent). The eigenvalue problem on ellipsoids was also studied by Tran and the first author [21]. This paper provides an upper bound for the first positive eigenvalue of $\Delta_b$ on the real ellipsoids in $\mathbb{C}^2$. We shall show that on real ellipsoids, the upper bound in Theorem 1.2 can be computed explicitly.

**Corollary 1.3.** Let $\rho(Z)$ be a real-valued strictly plurisubharmonic homogeneous quadratic polynomial satisfying $\rho_{jk} = \delta_{jk}$. Suppose that $M = \rho^{-1}(\nu)$ ($\nu > 0$) is a compact connected regular level set of $\rho$. Then

$$
\lambda_1(M, \theta) \leq \lambda_1(\sqrt{\nu} S^{2n+1}, \theta_0) = n/\nu. \tag{1.7}
$$

The equality occurs if and only if $(M, \theta) = (\sqrt{\nu} S^{2n+1}, \theta_0)$.

Here, $\sqrt{\nu} S^{2n+1}$ is the sphere $\|Z\|^2 = \nu$ and $\theta_0 = \iota^\ast (i\bar{\partial}\|Z\|^2)$ is the “standard” pseudohermitian structure on the sphere.

The paper is organized as follows. In Section 2, we shall give two simple formulas for the Kohn-Laplacian on compact real hypersurfaces in complex manifolds. These formulas allow us to compute $\Box_b$ explicitly in terms of the defining function $\rho$; see Proposition 2.1. These formulas will be crucial for the latter sections. In Section 3, we shall prove a general estimate for $\lambda_1(\Box_b)$ and Theorem 1.1. In Section 4, we shall give a sharp upper bound for $\lambda_1$ in terms of the
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eigenvalues of the complex Hessian \([\rho_{j\bar{k}}]\), implying the estimate in Theorem 1.2 and prove the characterization of equality case. We also give a family of examples (beside the ellipsoids) where we can apply this bound. These examples also show that the condition (1.4) in Theorem 1.1 can not be relaxed. In Section 5, we shall compute the bound in Theorem 1.2 explicitly in the case of ellipsoids, proving Corollary 1.3.

2. The Kohn-Laplacian on compact real hypersurfaces

In this section, we shall give two formulas for \(\Box_b\) on a compact regular level set of a Kähler potential \(\rho\) in terms of \(\partial\rho\) and the metric \(\rho_{j\bar{k}}dz^jdz^k\). First, let us start with a compact real hypersurface in \(\mathbb{C}^{n+1}\) arising as a regular level set of a strictly plurisubharmonic function \(\rho\):

\[
M = \rho^{-1}(\nu) := \{Z \in U : \rho(Z) = \nu\}. \tag{2.1}
\]

Here \(\rho\) is smooth on a neighborhood \(U\) of \(M\) and \(d\rho \neq 0\) along \(M\). We assume that the complex Hessian \(H(\rho) := [\rho_{j\bar{k}}]\) is positive definite and thus \(\rho\) defines a Kähler metric \(\rho_{j\bar{k}}dz^jdz^k\) on \(U\).

Let \([\rho_{j\bar{k}}]^{-1}\) be the inverse of \(H(\rho)\). For a smooth function \(u\) on \(U\), the length of \(\partial u\) in the Kähler metric is then given by

\[
|\partial u|^2_\rho = \rho_{j\bar{k}}u_j\bar{u}_k. \tag{2.2}
\]

We shall always equip \(M\) with the pseudohermitian structure \(\theta\) “induced” by \(\rho\):

\[
\theta = \iota^*(i/2)(\bar{\partial}\rho - \partial\rho). \tag{2.3}
\]

For local computations, it is convenient to work in the local admissible holomorphic coframe \(\{\theta^\alpha : \alpha = 1, 2, \ldots, n\}\) on \(M\) given by

\[
\theta^\alpha = dz^\alpha - ih^\alpha \theta, \quad h^\alpha = |\partial\rho|^{-2} \rho^\alpha = |\partial\rho|^{-2} \rho_{j\bar{k}} \rho_j\rho_{\bar{k}}, \quad \alpha = 1, 2 \ldots n. \tag{2.4}
\]

This admissible coframe is valid when \(\rho_{n+1} \neq 0\). It is shown by Luk and the first author [18, p. 679] that at the point \(p\) with \(\rho_{n+1} \neq 0\),

\[
d\theta = ih_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}}, \tag{2.5}
\]

where the Levi matrix \([h_{\alpha\bar{\beta}}]\) is given explicitly:

\[
h_{\alpha\bar{\beta}} = \rho_{\alpha\bar{\beta}} - \rho_{\alpha} \partial_{\bar{\beta}} \log \rho_{n+1} - \rho_{\bar{\beta}} \partial_{\alpha} \log \rho_{n+1} + \rho_{n+1} \frac{\rho_{n+1} \rho_{\alpha} \rho_{\bar{\beta}}}{|\rho_{n+1}|^2}. \tag{2.6}
\]

We can check directly that the inverse \([h^{\gamma\bar{\beta}}]\) of the Levi matrix is given by

\[
h^{\gamma\bar{\beta}} = \rho^{\gamma\bar{\beta}} - \frac{\rho^{\gamma\bar{\beta}}}{|\partial\rho|^2} \rho^\gamma = \sum_{k=1}^{n+1} \rho_{k\bar{k}} \rho^{\gamma\bar{k}}. \tag{2.7}
\]

We use the Levi matrix and its inverse to lower and raise the Greek indices; repeated Greek indices are summing from 1 to \(n\). The Tanaka-Webster covariant derivatives are given by

\[
\nabla_\alpha \nabla_\beta f = Z_\alpha Z_\beta f - \omega^\gamma_\beta(Z_\alpha)Z_\alpha f \tag{2.8}
\]
where \( \{Z_\alpha\} \) is the holomorphic frame dual to \( \{\theta^\alpha\} \) and \( \omega^\beta_\alpha \) are the connection forms. More precisely,

\[
Z_\alpha = \frac{\partial}{\partial z^\alpha} - \frac{\rho_\alpha}{\rho_{n+1} \partial z_{n+1}},
\]

and the Tanaka-Webster connection forms are computed in [18]; see also [23].

\[
\omega^\beta_\alpha = (Z^\gamma h^\alpha_\beta - h^\beta h^\gamma_\alpha)\theta^\gamma + h^\alpha h^\gamma Z^\beta \theta^\gamma, \quad h_\alpha = h^\alpha_\beta h^\beta.
\]

Also, the Reeb vector field is given by

\[
T = i \sum_{j=1}^{n+1} \left( h_j \frac{\partial}{\partial z^j} - h^j \frac{\partial}{\partial \bar{z}^j} \right), \quad h^j = \rho_j |\partial \rho|^2 \rho^j.
\]

The formula (2.12) below, expressing \( \Box_b \) in terms of \( \rho \), will be crucial for our analysis.

**Proposition 2.1.** Let \( U \) be an open set in a Kähler manifold \( X \) and \( \rho \) a Kähler potential on \( U \). Let \( M \) be a smooth, compact, connected, regular level set of \( \rho \), \( \theta = i^{n+1} (\bar{\partial} \rho - \partial \rho) \), and \( \Box_b \) the Kohn-Laplacian defined on \( M \) with respect to \( dv = \theta \wedge (d\theta)^n \).

(i) If \( f \) is a smooth function on \( U \), then the following identity holds on \( M \).

\[
\Box_b f = -\text{trace}(i\partial \bar{\partial} f) + |\partial \rho|^{-2} \partial \rho \wedge \bar{\partial} \rho + n |\partial \rho|^{-2} (\partial \rho, \bar{\partial} \rho),
\]

(ii) Suppose that \( (z^1, z^2, \ldots, z^{n+1}) \) is a local coordinate system on an open set \( V \). Define the vector fields

\[
X^j = \rho_j \partial_j - \rho_j \partial^j, \quad X^\bar{j} = \bar{X}^j.
\]

Then the following holds on \( M \cap V \).

\[
\Box_b f = -\frac{1}{2} |\partial \rho|^{-2} \rho^p \rho^q X_{pq} X^j f.
\]

**Remark 2.2.** (a) The trace operator is taken with respect to the Kähler form and thus \( -\text{trace}(i\partial \bar{\partial} f) \) is the Laplace-Beltrami operator acting on \( f \). In local coordinates, (2.12) can be written as

\[
\Box_b f = \left( |\partial \rho|^{-2} \rho^p \rho^q - \rho^p \bar{\rho}^q \right) f_{jk} + n |\partial \rho|^{-2} \rho^p f_k.
\]

(b) Formulas (2.14) and (2.12) are generalizations of two formulas for the Kohn-Laplacian on the sphere appeared in [11]. This paper also studies the Kohn-Laplacian for forms on the sphere (with volume element induced from \( \mathbb{C}^{n+1} \)). Notice that the fields \( X^j \) are tangential Cauchy-Riemann vector fields on \( M \) generating \( T^{1,0} \) at each point.

**Proof.** We first prove (i). It is well-known [16] that the Kohn Laplacian acting on function can be given locally by

\[
-\Box_b f = h^\beta_\alpha \nabla_\alpha \nabla_\beta f.
\]
Thus, we can work in a local coordinate \((z^1, z^2, \ldots, z^n, w = z^{n+1})\) on \(X\) and assume that \(\rho_w = \partial_w \rho \neq 0\). Choose the local frame and coframe as in (2.4). Notice that
\[
Z^\alpha = h^{\alpha \beta} Z_\alpha = h^{\alpha \beta} \partial_\alpha - h^{\alpha \beta} \frac{\rho_\alpha}{\rho_{n+1}} \partial_{n+1} = \rho^k \partial_k - \frac{\rho^2}{|\partial \rho|^2} \rho^k \partial_k. 
\] (2.17)
Therefore,
\[
-\Box_b f = Z^\beta Z_\beta f - nh^\beta f_\beta
\]
\[
= \left[ \rho^k \partial_k - |\partial \rho|^{-2} \rho^k \partial_k \right] f_\beta - \rho_\beta \frac{\rho^k}{\rho_\beta} f_\beta - n h^\beta f_\beta
\]
\[
= \rho^k f_\beta - \rho_\beta \frac{\rho^k}{\rho_\beta} f_\beta - n h^\beta f_\beta
\]
\[
- \frac{\rho^k}{|\partial \rho|^2} f_\beta + \frac{n f_\beta}{|\partial \rho|^2}.
\] (2.18)
Here we use summation convention: \(k\) runs from 1 to \(n+1\) and \(\beta\) runs 1 to \(n\). Simplifying the right hand side, we easily obtain
\[
-\Box_b f = \left( \rho^k - |\partial \rho|^{-2} \rho^k \right) f_\beta - n |\partial \rho|^2 f_\beta, 
\] (2.19)
which is clearly equivalent to (2.12).

To prove (ii), we notice that
\[
X_{jk} f = \rho_k f_j - \rho_j f_k. 
\] (2.20)
Therefore,
\[
X_{pq} X_{jk} f = \rho_p \rho_k f_{jp} + \rho_q \rho_{kp} f_j - \rho_q \rho_{jp} f_k - \rho_p \rho_j f_{kp}
\]
\[
- \rho_p \rho_k f_{jq} - \rho_q \rho_{kp} f_j + \rho_p \rho_j f_{kq} + \rho_q \rho_{jq} f_k. 
\] (2.21)
Contracting both sides with \(\rho^k \rho^\beta\), using (i), we easily obtain (ii). \(\square\)

3. An estimate for eigenvalues and proof of Theorem 1.1

We denote by \(S\): \(L^2(M) \to \ker \Box_b (= \ker \Box_{\bar{b}})\) the Szegő orthogonal projection with respect to the volume measure \(dv := \theta \wedge (d\theta)^n\). It is well-known that if \(M\) is embeddable, then \(\text{Spec}(\Box_{\bar{b}})\) consists of zero and a sequence of point eigenvalues \(\{\lambda_k\}\) increasing to infinity. The positive eigenvalues of \(\Box_b\) are of finite multiplicity and eigenfunctions are smooth [3, 6]. Furthermore, we have the following orthogonal decomposition:
\[
L^2(M, dv) = \bigoplus_{k=0}^{\infty} E_k, \quad E_0 = \ker \Box_b. 
\] (3.1)
Note that \(E_0\) is of infinite dimension.
**Theorem 3.1.** Let \((M, \theta)\) be an embedded compact strictly pseudoconvex pseudohermitian manifold and \(0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots\) the eigenvalues for \(\Box_b\). Define
\[
m(a) = \inf \left\{ \left| a - \frac{1}{\lambda_k} \right|^2 : k \in \mathbb{N} \right\}, \quad M(a) = \sup \left\{ \left| a - \frac{1}{\lambda_k} \right|^2 : k \in \mathbb{N} \right\}. \tag{3.2}
\]
Then for any \(a \in \mathbb{R}\), any function \(u \not\in \ker \Box_b\),
\[
(m(a) - a^2)\|\Box_b u\|^2 \leq \|u - S(u)\|^2 - \int_M |\bar{\partial}_b u|^2 \leq (M(a) - a^2)\|\Box_b u\|^2. \tag{3.3}
\]

**Proof.** Let \(E_k\) be the eigenspace of \(\Box_b\) associated to the eigenvalue \(\lambda_k\). Then \(m_k := \dim(E_k) < \infty\). Let \(\{f_{k,j}\}_{j=1}^{m_k}\) be an orthonormal basis for \(E_k\). For any \(k, \ell\), using integration by parts, we obtain
\[
\int_M (\Box_b u - \lambda_k u)f_{k,\ell} = \int_M (u\Box_b f_{k,\ell} - \lambda_k u f_{k,\ell}) = \int_M u(\Box_b f_{k,\ell} - \lambda_k f_{k,\ell}) = 0. \tag{3.4}
\]
This implies that for any real number \(a\),
\[
\langle u - a\Box_b u, f_{k,\ell} \rangle = -(a - 1/\lambda_k)\langle \Box_b u, f_{k,\ell} \rangle. \tag{3.5}
\]
Therefore, since \(\Box_b u \in (\ker \Box_b)^\perp\),
\[
M(a)\|\Box_b u\|^2 = \sum_{k=1}^{\infty} \sum_{\ell=1}^{m_k} M(a) |\langle \Box_b u, f_{k,\ell} \rangle|^2 \tag{3.6}
\]
\[
\geq \sum_{k=1}^{\infty} \sum_{\ell=1}^{m_k} \left| a - \frac{1}{\lambda_k} \right|^2 |\langle \Box_b u, f_{k,\ell} \rangle|^2 \tag{3.7}
\]
\[
= \sum_{k=1}^{\infty} \sum_{\ell=1}^{m_k} |\langle u - a\Box_b u, f_{k,\ell} \rangle|^2 \tag{3.8}
\]
\[
= \|u - a\Box_b u\|^2 - \|S(u - a\Box_b u)\|^2 \tag{3.9}
\]
\[
= \|u\|^2 + a^2\|\Box_b u\|^2 - 2a \int_M \bar{u}\Box_b u - \|S(u)\|^2. \tag{3.10}
\]
Here we have used \(\|S(u - a\Box_b u)\|^2 = \|S(u)\|^2\). We conclude that
\[
(M(a) - a^2)\|\Box_b u\|^2 \geq \|u - S(u)\|^2 - 2a \int_M |\bar{\partial}_b u|^2. \tag{3.11}
\]
This proves the second inequality. The first inequality can be proved similarly. \(\Box\)

The following two corollaries are undoubtedly known, but we can not find in the literature.

**Corollary 3.2.** Let \((M, \theta)\) be as in Theorem 3.1, then
\[
\lambda_1 = \inf \left\{ \|\Box_b u\|^2 : \int_M |\bar{\partial}_b u|^2 = 1 \right\} = \inf \left\{ \int_M |\bar{\partial}_b u|^2 : \|u - S(u)\|^2 = 1 \right\}. \tag{3.12}
\]
Proof. For any \( a > \frac{1}{\lambda_1} \), we have
\[
m(a) = \left| a - \frac{1}{\lambda_1} \right|^2.
\] (3.13)
From Theorem 3.1, we have for any \( u \) with \( \int_M |\bar{\partial}_b u|^2 = 1 \),
\[
\left[ \left| a - \frac{1}{\lambda_1} \right|^2 - a^2 \right] \| \Box_b u \|^2 \leq \| u - S(u) \|^2 - 2a.
\] (3.14)
This is equivalent to
\[
\left( \frac{1}{\lambda_1} \right)^2 - 2a \left( \frac{1}{\lambda_1} - \frac{1}{\| \Box_b u \|^2} \right) \leq \frac{\| u - S(u) \|^2}{\| \Box_b u \|^2}.
\] (3.15)
Letting \( a \to +\infty \), we easily obtain
\[
\lambda_1 \leq \| \Box_b u \|^2.
\] (3.16)
Since \( u \) is arbitrary, we conclude that
\[
\lambda_1 \leq \inf \left\{ \| \Box_b u \|^2 : \int_M |\bar{\partial}_b u|^2 = 1 \right\}.
\] (3.17)
The reverse inequality is trivial.
To prove the second we take \( a = \frac{1}{2} \left( \frac{1}{\lambda_k} + \frac{1}{\lambda_{k+1}} \right) \). Clearly,
\[
m(a_k) = \left| a_k - \frac{1}{\lambda_k} \right|^2 = \left| a_k - \frac{1}{\lambda_{k+1}} \right|^2
\] (3.21)
By Theorem 3.1, we have
\[
\left[ \left| a_k - \frac{1}{\lambda_k} \right|^2 - a_k^2 \right] \| \Box_b u \|^2 \leq \| u - S(u) \|^2 - 2a_k.
\] (3.22)
By direct calculation, we have that
\[
\left( \frac{1}{\lambda_k} - \frac{1}{\| \Box_b u \|^2} \right) \left( \frac{1}{\lambda_{k+1}} - \frac{1}{\| \Box_b u \|^2} \right) \geq \frac{1}{\| \Box_b u \|^4} - \frac{\| u - S(u) \|^2}{\| \Box_b u \|^2}.
\] (3.23)
By Corollary (3.2) \( \lambda_1 \leq \|q_u\|^2 \). Moreover, \( \lambda_k \to \infty \) as \( k \to \infty \). We deduce that there exists \( k_0 \) such that
\[
\lambda_{k_0} \leq \|q_u\|^2 < \lambda_{k_0+1}.
\] (3.24)

Therefore, (3.23) with \( k = k_0 \) implies that
\[
\frac{1}{\|q_u\|^4} - \frac{\|u - S(u)\|^2}{\|q_u\|^2} \leq 0.
\] (3.25)

This completes the proof. □

**Proposition 3.4.** Let \((M, \theta)\) be a compact strictly pseudoconvex pseudohermitian manifold. If there is a smooth non-CR function \( f \) on \( M \) such that \( |q_f|^2 \leq B(z) \text{Re} \, q_f \) for some non-negative function \( B \) on \( M \), then
\[
\lambda_1 \leq \max_M B(z).
\] (3.26)

If the equality holds, then \( B \) must be a constant.

**Proof.** Since \( |q_f|^2 \leq B(z) \text{Re} \, q_f \), by Corollary (3.2)
\[
\lambda_1 \int_M q_f \leq \int_M |q_f|^2 \leq \int_M B(z) \text{Re} (q_f).
\] (3.27)

By the Mean Value Theorem of the integral, there is \( z_0 \in M \) such that
\[
0 \leq \int_M (B - \lambda_1) \text{Re} (q_f) = (B(z_0) - \lambda_1) \int_M q_f = (B(z_0) - \lambda_1) \int_M |q_f|^2.
\] (3.28)

This implies
\[
\lambda_1 \leq B(z_0) \leq \max_M B(z).
\] (3.29)

It is clear that if \( \lambda_1 = \max_M B \) then \( B \) is a constant. □

We end this section by proving the Theorem 1.1.

**Proof of Theorem 1.1.** By the condition (1.4) and the expression for the Kohn-Laplacian given by (2.12), we have
\[
q_f = \tilde{\Delta} q_f + n |q_f|^2 \rho^2 q_f = \tilde{\Delta} q_f + n |q_f|^2 \rho^2 q_f.
\] (3.30)

Then
\[
|q_f|^2 = \frac{n}{|q_f|^2} \text{Re} \left(q_f q_{\rho} + \frac{1}{n} |q_f|^2 \tilde{\Delta} q_f + \frac{1}{n} |q_f|^2 \rho^2 q_f \right) \leq \frac{n}{|q_f|^2} \text{Re} \left(q_f q_{\rho} \right).
\] (3.31)

Applying Proposition 3.3 with \( B(z) = n |q_f|^2 \), we obtain
\[
\lambda_1 \leq n \max_M |q_f|^2.
\] (3.32)

The equality holds only if \( |q_f|^2 \) is a constant on \( M \). The proof of Theorem 1.1 is complete. □
4. Proof of Theorem 1.2

The following theorem gives a sharp upper bound for \( \lambda_1(\Box_b) \) in terms of the eigenvalues of the complex Hessian matrix \([\rho_{jk}]\) and the length of \( \partial \rho \). This theorem implies the estimate in Theorem 1.2.

**Theorem 4.1.** Let \( \rho \) be a smooth strictly plurisubharmonic function defined on an open set \( U \) of \( \mathbb{C}^{n+1} \), \( M \) a compact connected regular level set of \( \rho \), and \( \lambda_1 \) the first positive eigenvalue of \( \Box_b \) on \( M \). Let \( r(z) \) be the spectral radius of the matrix \([\rho_{jk}(z)]\) and \( s(z) = \text{trace} [\rho_{jk}] - r(z) \). Then

\[
\lambda_1 \leq n^2 \frac{\int_M r(z)|\partial \rho|^2}{\int_M s(z)}. \tag{4.1}
\]

Here the spectral radius of a square matrix is the maximum of the moduli of its eigenvalues.

**Proof.** First, we define

\[
C_j = \int_M |\rho_j|^2 |\partial \rho|^2, \quad D_j = \int_M \left( \rho_{ij} - \frac{|\rho_j|^2}{|\partial \rho|^2} \right).
\tag{4.2}
\]

From Proposition 2.1, we can compute

\[
\Box_b \bar{z}_j = n |\partial \rho|^2 \rho_j.
\tag{4.3}
\]

Therefore,

\[
\|\Box_b \bar{z}_j\|^2 = n^2 \int_M |\rho_j|^2 |\partial \rho|^2 = n^2 C_j.
\tag{4.4}
\]

We can also compute

\[
|\partial_b \bar{z}_j|^2 = \delta_{ja\beta} \delta_{\bar{b}j} \left( \rho^a \bar{\rho}^\beta - \frac{\rho^a \rho^\beta}{|\partial \rho|^2} \right) = \rho_{ij} - \frac{|\rho_j|^2}{|\partial \rho|^2}. \tag{4.5}
\]

Here without lost of generality, we assume \( j \neq n + 1 \). Therefore,

\[
\int_M |\partial_b \bar{z}_j|^2 = D_j. \tag{4.6}
\]

Thus, from Corollary 3.2 above, we obtain for all \( j \),

\[
\lambda_1 \leq n^2 C_j / D_j. \tag{4.7}
\]

Next, observe that \( 1/r(z) \) is the smallest eigenvalue of the Hermitian matrix \([\rho_{j\bar{k}}(z)]\), and thus, for all \((n+1)\)-vector \( v^j \),

\[
\frac{1}{r(z)} \sum_{j=1}^{n+1} |v^j|^2 \leq v^j \rho_{j\bar{k}} v^\bar{k}. \tag{4.8}
\]

Plugging \( v^j = \rho^j \) into the inequality, we easily obtain \( \sum_{j=1}^{n+1} |\rho_j|^2 \leq r(z)|\partial \rho|^2 \). Consequently

\[
\sum_j C_j = \sum_{j=1}^{n+1} \int_M |\rho_j|^2 |\partial \rho|^2 \leq \int_M r(z)|\partial \rho|^2, \tag{4.9}
\]
and therefore,
\[ \sum_j D_j = \sum_{j=1}^{n+1} \int_M \left( \rho^{ij} - \frac{|\rho|^2}{|\partial \rho|^2} \right) \geq \int_M \left[ \text{trace}[\rho^{ik}] - r(z) \right] = \int_M s(z). \] (4.10)

Thus, from (4.7), (4.9), and (4.10), we obtain
\[ \lambda_1 \leq n^2 \min_j \left( C_j / D_j \right) \leq \frac{n^2 \sum_j C_j}{\sum_j D_j} = \frac{n^2 \int_M r(z) |\partial \rho|^2}{\int_M s(z)}. \] (4.11)

The proof is complete. \[ \Box \]

Proof of Theorem 1.2. Since \( \rho^j_k = \delta_{jk} \), we have \( r(z) = 1 \) and \( s(z) = n \). Therefore, by Theorem 4.1,
\[ \lambda_1 \leq \frac{n^2 \int_M r(z) |\partial \rho|^2}{\int_M s(z)} = \frac{n}{v(M)} \int_M |\partial \rho|^2. \] (4.12)

which proves the inequality.

Next we suppose that \( \lambda_1 = \frac{n}{v(M)} \int_M |\partial \rho|^2 \). We shall show that \( |\partial \rho|^2 \) is constant along \( M \). Put
\[ b_j = n^{-1} \Box_b \bar{z}^j = |\partial \rho|^2 \rho_j. \] (4.13)

Then by inspecting the proof of Theorem 4.1 above, in particular, the estimate (3.6), we have for all \( j \),
\[ \langle b_j, f_k,\ell \rangle = 0, \quad \text{for all } \ell, \text{ for all } k \neq 1. \] (4.14)

Thus, \( b_j \perp \ker \Box_b \) and (4.13) imply that \( b_j \in E_1 \) (the eigenspace corresponding to \( \lambda_1 \)). Therefore,
\[ \Box_b b_j = \lambda_1 b_j. \] (4.15)

Recall that \( \Box_b \bar{z}^j = nb_j \). We then deduce that
\[ \Box_b \left[ \bar{z}^j - \frac{n}{\lambda_1} \frac{\rho_j}{\partial \rho} \right] = 0. \] (4.16)

Hence, \( \bar{z}^j - n \rho^j / (\lambda_1 |\partial \rho|^2) \) restricted to \( M \) is a CR function. Since \( X_{\bar{k}} \) is a tangential CR vector fields on \( M \), we have
\[ X_{\bar{k}} \left[ \bar{z}^j - \frac{n}{\lambda_1} \frac{\rho_j}{\partial \rho} \right] = 0. \] (4.17)

By direct calculation, this is equivalent to
\[ \frac{n}{\lambda_1} \rho_j X_{\bar{k}}(\partial \rho^2) / |\partial \rho|^2 = \left( 1 - \frac{n}{\lambda_1 |\partial \rho|^2} \right) (\rho_j \delta_{jk} - \rho_k \delta_{jl}). \] (4.18)

Since \( M \) is compact, there exists point \( x \in M \) such that
\[ |\partial \rho(x)|^2 = \max_M |\partial \rho|^2. \] (4.19)
At the maximum point \( x \), we also have \( X_j^{k}\partial\rho|^4_\rho = 0 \). Thus,
\[
1 - \frac{n}{\lambda_1 |\partial\rho|^2_\rho} \left( \rho_l \delta_{jk} - \rho_k \delta_{jl} \right) = 0 \quad \text{at } x. \tag{4.20}
\]
Since \( \partial\rho(x) \neq 0 \), we can assume that \( \rho_l(x) \neq 0 \). Taking \( j = k = 2 \), we have at \( x \),
\[
1 - \frac{n}{\lambda_1 |\partial\rho|^2_\rho} = 0. \tag{4.21}
\]
Therefore,
\[
\min |\partial\rho|^{-2}_\rho = |\partial\rho(x)|^{-2}_\rho = \frac{\lambda_1}{n} = \frac{1}{v(M)} \int_M |\partial\rho|^{-2}_\rho. \tag{4.22}
\]
As the right most term is the average of \( |\partial\rho|^{-2}_\rho \) on \( M \), we deduce from above that \( |\partial\rho|^{-2}_\rho \) must be constant on \( M \).

Finally, suppose that \( |\partial\rho|^2_\rho \) is constant along \( M \) and \( \rho \) extends to the domain bounded by \( M \) and satisfies \( \rho_{jk} = \delta_{jk} \) on the domain. We shall show in the lemma below that \( M \) must be a sphere and complete the proof of Theorem 1.2. □

**Lemma 4.2.** Let \( M \) be a compact connected regular level set of \( \rho \) which bounds a domain \( D \). Suppose that \( \rho_{jk} = \delta_{jk} \) on \( D \). If \( |\partial\rho|^2_\rho \) is constant on \( M \), then \( M \) must be a sphere.

**Proof of Lemma 4.2.** The proof is an application of Serrin’s theorem [22, Theorem 1]. Let \( D \) be the domain with \( M \) is its boundary. Define \( u = \rho - \nu \) on a neighborhood of \( \overline{D} \). Since \( M \) is smooth and the function \( u \) satisfies \( \Delta u = -4(n + 1) \) in \( D \), \( u = 0 \) on \( \partial D \), and the normal derivative \( \partial u/\partial n = 2|\partial\rho|_\rho \) is a constant on \( \partial D \) by assumption, we can apply the Serrin’s theorem to conclude that \( M \) is a standard sphere. □

We end this section by the following example which gives a sharp upper bound on the family of compact level sets of Kähler potentials of Fubini-Study metric. This example also shows that the condition (1.4) in Theorem 1.1 can not be relaxed.

**Example 4.3.** Let \( \rho \) be a strictly plurisubharmonic function of the form
\[
\rho(Z) = \log(1 + \|Z\|^2) + \psi(Z, \bar{Z}), \tag{4.23}
\]
where \( \psi \) is a real-valued pluriharmonic function. We suppose that \( \rho \) is defined and proper in some domain \( U \subset \mathbb{C}^{n+1} \) (e.g., if \( \psi = -\log |z_1| \), then \( \rho \) is defined and proper on \( (\mathbb{C} \setminus \{0\}) \times \mathbb{C}^n \)).

Observe that
\[
\rho_{jk} = \frac{1}{1 + \|Z\|^2} \left( \delta_{jk} - \frac{\bar{z}^j z^k}{1 + \|Z\|^2} \right), \quad \rho^{jk} = (1 + \|Z\|^2) \left( \delta_{jk} + z^k z^j \right), \tag{4.24}
\]
By a routine calculation, we see that the characteristic polynomial of \([\rho^{jk}]\) is
\[
P_{[\rho^{jk}]}(\lambda) = (1 + \|Z\|^2 - \lambda)^n \left[ (1 + \|Z\|^2)^2 - \lambda \right]. \tag{4.25}
\]
Thus, the spectral radius of $|\rho^{jk}|$ is $r(Z) = (1 + \|Z\|^2)^2$ and $s(Z) = \text{trace}[\rho^{jk}] - r(Z) = n(1 + \|Z\|^2)$. By Theorem 4.1, if $M$ is a compact, connected, regular level set of $\rho$, then

$$\lambda_1 \leq n \frac{\int_M (1 + \|Z\|^2)|\partial \rho|^{-2}}{\int_M (1 + \|Z\|^2)} \leq n \max_M (1 + \|Z\|^2) |\partial \rho|^{-2}.$$ (4.26)

Notice that if $\psi = 0$ and then $M_\nu := \rho^{-1}(\nu)$ is the sphere $\|Z\|^2 = e^{\nu} - 1$ with

$$\theta = i e^{-\nu} \sum_{j=1}^{n+1} (z^j d\bar{z}^j - \bar{z}^j dz^j).$$ (4.27)

Moreover, $|\partial \rho|^{-2} = e^{\nu} - 1$ on $M_\nu$ and $\lambda_1 = n e^{\nu}/(e^{\nu} - 1)$. Therefore, the condition (1.4) in Theorem 1.1 can not be relaxed.

5. THE REAL ELLIPSOIDS: PROOF OF COROLLARY 1.3

The proof of Theorem 1.3 follows from Theorem 4.1 and the proposition below.

**Proposition 5.1.** Let $Q(Z)$ be a quadratic polynomial and let $M_\nu = \rho^{-1}(\nu)$ be a compact regular level set of $\rho$, where $\rho$ is given by

$$\rho(Z) = \sum_{k=1}^{n+1} |z^k|^2 + 2\text{Re} Q(Z)$$ (5.1)

Then

$$C_\nu := \frac{1}{v(M_\nu)} \int_{M_\nu} |\partial \rho|^{-2} = \frac{1}{\nu}. \quad (5.2)$$

**Proof.** We observe that

$$\text{Re} \sum_{j=1}^{n+1} z^j \rho^j = \sum_{j=1}^{n+1} |z^j|^2 + \text{Re} \sum_{j=1}^{n+1} z^j Q_j = \nu - 2\text{Re} Q + \text{Re} \sum_{j=1}^{n+1} z^j Q_j. \quad (5.3)$$

As $Q$ is a quadratic polynomial, we can check that $\sum_{j=1}^{n+1} z^j Q_j = 2Q$. Hence

$$\text{Re} \sum_{j=1}^{n+1} z^j \rho^j = \nu \quad \text{on } M_\nu. \quad (5.4)$$

Therefore,

$$\int_{M_\nu} |\partial \rho|^{-2} = \text{Re} \frac{1}{\nu} \sum_{j=1}^{n+1} \int_{M_\nu} z^j \rho_j = \text{Re} \frac{1}{\nu} \sum_{j=1}^{n+1} \int_{M_\nu} \frac{(z^j + Q_j)\rho_j}{|\partial \rho|^2} = \frac{v(M)}{\nu}, \quad (5.5)$$

Here, we use

$$\int_{M_\nu} \frac{Q_j\rho_j}{|\partial \rho|^2} = \frac{1}{n} \int_{M_\nu} \overline{Q_j} \Box_{\nu} z^j = \frac{1}{n} \int_{M_\nu} z^j \Box_{\nu} Q_j = 0. \quad (5.6)$$

Hence, $C_\nu = \frac{1}{\nu}$. \qed
Proof of Corollary 1.3. From Theorem 1.2 and Proposition 5.1, we have
\[
\lambda_1 \leq nC_\nu = \lambda_1(\sqrt{\nu}S^{2n+1}). \tag{5.7}
\]
Also from Theorem 1.2, we see that the equality occurs if and only if \(M\) is the sphere and hence the proof is complete. However, we provide an elementary proof of this last step below. Notice that
\[
Q(Z) = \sum_{k,j=1}^{n} q_{jk} z_k z_j \tag{5.8}
\]
and \(Q = [q_{jk}]\) is \(n \times n\) symmetric matrix. By a well-known factorization theorem (see [13, Section 3.5]), we can write \(Q = U^T \Lambda U\), where \(U\) is a unitary matrix and \(\Lambda = \text{Diag}(A_1, \cdots, A_{n+1})\) is a diagonal matrix with \(A_j \geq 0\). We make a holomorphic unitary change of variables \(W = UZ\), then
\[
\rho(Z) = \|W\|^2 + \text{Re} \sum_{j=1}^{n+1} A_j w_j^2. \tag{5.9}
\]
Without loss of generality, one may assume that \(\rho(Z) = \|Z\|^2 + \text{Re} \sum_{j=1}^{n+1} |z_j|^2\). Since \(M_\nu\) is bounded, it is easy to see that \(A_j < 1\) for \(1 \leq j \leq n + 1\). Notice that on \(M_\nu\),
\[
c = |\partial \rho|^2 = \sum_{j=1}^{n+1} A_j^2 |z_j|^2 + \text{Re} \sum_{j=1}^{n+1} A_j z_j^2 = 2\nu - |Z|^2 + \sum_{j=1}^{n+1} A_j |z_j|^2. \tag{5.10}
\]
If the equality occurs, then \(|\partial \rho|^2\) is a constant along \(M_\nu\). Restricting \(z = \lambda e_j \in M_\nu\), one has \(|z_j|^2\) must be a constant. This can not be true unless \(A_j = 0\). This proves \(\rho(Z) = \|Z\|^2\) and \(M_\nu\) is a sphere centered at 0 with radius \(\sqrt{\nu}\). \(\square\)

References

[1] A. Aribi; S. Dragomir; and A. El Soufi, A lower bound on the spectrum of the sublaplacian. Journal of Geometric Analysis, 25.3 (2015) 1492-1519.
[2] E. Barletta; S. Dragomir, On the spectrum of a strictly pseudoconvex CR Manifold. Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg (1997) 67: 33.
[3] R. Beals and P. Greiner, Calculus on Heisenberg manifolds. Ann. of Math. Stud., vol. 119, Princeton Univ. Press, New Jersey, 1988.
[4] L. Boutet de Monvel, Intégration des équations de Cauchy-Riemann induites formelles, Séminaire Goulaouic-Lions-Schwartz, Exposé IX (1974-1975)
[5] D. Burns. Global behavior of some tangential Cauchy-Riemann equations. Partial Differential Equations and Geometry (Proc. Conf., Park City, Utah), Marcel Dekker, New York 1979.
[6] D. Burns; C. Epstein, Embeddability for Three-dimensional CR Manifolds, J. Amer. Math. Soc. 4 (1990), 809-840.
[7] S. Chanillo; H.-L. Chiu; P. Yang, Embeddability for 3-dimensional Cauchy-Riemann manifolds and CR Yamabe invariants. Duke Math. J. 161 (2012), no. 15, 2909-2921.
[8] S.-C. Chang; C.-T. Wu, On the CR Obata Theorem for Kohn Laplacian in a Closed Pseudohermitian Hypersurface in \(\mathbb{C}^{n+1}\). Preprint, 2012.
[9] S.-C. Chang, H.-L. Chiu, On the CR analogue of Obata’s theorem in a pseudohermitian 3-manifold. Math. Ann. 345 (2009), no. 1, 3351.
[10] H.-L. Chiu, The sharp lower bound for the first positive eigenvalue of the sub-Laplacian on a pseudohermitian 3-manifold. Ann. Global Anal. Geom. 30 (2006), no. 1, 81–96.
[11] D. Geller, The Laplacian and the Kohn Laplacian for the sphere. Journal of Differential Geometry. 1980;15(3):417-35.
[12] A. Greenleaf, The first eigenvalue of a sub-Laplacian on a pseudohermitian manifold. Communications in Partial Differential Equations, 10 (1985), no. 2, 191–217.
[13] L. K. Hua, Harmonic Analysis of Functions of Several Complex Variables in the classical Domains, Volume 6, Translations of Mathematical Monographs, AMS, Providence, Rhode Island, 1963
[14] J. J. Kohn, Boundaries of Complex Manifolds, Proc. Conf. Complex Manifolds (Minneapolis, 1964), Springer-Verlag, New York, 81-94, 1965.
[15] S. Ivanov; D. Vassilev. An Obata type result for the first eigenvalue of the sub-Laplacian on a CR manifold with a divergence-free torsion. Journal of Geometry 103, no. 3 (2012): 475-504.
[16] J. M. Lee, The Fefferman metric and pseudohermitian invariants. Trans. Amer. Math. Soc., 296(1), 411–429.
[17] S.-Y. Li; H-S Luk, The Sharp lower bound for the first positive eigenvalues of sub-Laplacian on the pseudo-hermitian manifold, Proc. of AMS, 132 (2004), 789–798.
[18] S.-Y. Li; H-S Luk, An explicit formula for the Webster pseudo-Ricci curvature on real hypersurfaces and its application for characterizing balls in $C^n$. Communications in Analysis and Geometry, 14(4), 673–701.
[19] S.-Y. Li; D. N. Son; X-D. Wang, A New Characterization of the CR Sphere and the sharp eigenvalue estimate for the Kohn Laplacian. Advances in Math., 281 (2015), 1285–1305.
[20] S.-Y. Li; X. Wang, An Obata-type Theorem in CR Geometry, Journal of Differential Geometry, 95(2013), no. 3, 483–502.
[21] S.-Y. Li; M.-A. Tran, On the CR-Obata theorem and some extremal problems associated to pseudoscalar curvature on the real ellipsoids in $C^{n+1}$. Transactions of the American Mathematical Society 363, no. 8 (2011): 4027-4042.
[22] J. Serrin, A symmetry problem in potential theory, Arch. Rational Mech. Anal. 43 (1971), 304-318.
[23] S. M. Webster, Pseudo-Hermitian structures on a real hypersurface. Journal of Differential Geometry 13, no. 1 (1978): 25–41.

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