Set-valued risk statistics with the time value of money

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Abstract The time value of money is a critical factor not only in risk analysis, but also in insurance and financial applications. In this paper, we consider a special class of set-valued risk statistics by introducing the time value of money. In fact, the risk statistics established by this method is closer to financial reality than traditional ones. Moreover, this new risk statistic can be used for the quantification of portfolio risk. By further developing the properties related to these risk statistics, we are able to derive representation results for such risk.

Keywords risk statistics · set-valued · portfolio · time value

1 Introduction

Research on risk is a popular topic in both quantitative and theoretical research, and risk models have attracted considerable attention. The quantitative calculation on risk involves two problems: choosing an appropriate risk model and allocating risk to individual institutions. This has led to further research on risk statistics.

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In traditional research of risk statistic, the property of translative invariance denies the time value of money. However, as pointed out by EL Karoui and Ravanel (2009), the translative invariance axiom may fail once there is any form of uncertainty about interest rates because the money has a time value. For example, when \( m \) dollars are added to a future position \( X \), the capital requirement at time \( t = 0 \) is reduced by less than \( m \) dollars because the value of the money may grow with time. Therefore, it is more appropriate to study risk statistics from the perspective of the time value of money.

Evaluating risk of a portfolio consisting of several financial positions. And the set-valued risk measures are more appropriate than scalar risk measures, especially in the case where several different kinds of currencies are involved when one is determining capital requirements for the portfolio. In fact, Hamel and Heyde (2010) pointed out that the basic question for quantifying portfolio risk is how to evaluate the risk of a multivariate random outcome in terms of more than one reference instrument, for example if the regulator accepts deposits in more than one currency. This is of particular importance if transaction costs have to be paid for each transaction between assets including the reference instruments. Other studies of set-valued risk measures include those of Hamel (2009), Hamel et al. (2011), Hamel et al. (2013), Labuschagne and Offwood-Le Roux (2014), Farkas et al. (2015), Molchanov and Cascos (2016), Ararat et al. (2017), Deng and Sun (2020), Sun and Dong (2021) and the references therein. A natural set-valued risk statistic can be considered as an empirical (or a data-based) version of a set-valued risk measure.

From the statistical point of view, the behaviour of a random variable can be characterized by its observations, the samples of the random variable. Heyde et al. (2007) and Kou et al. (2013) first introduced the class of natural risk statistics and the corresponding representation results are also derived. Later, Tian and Suo (2012) obtained representation results for convex risk statistics, and the corresponding results for quasiconvex risk statistics were obtained by Tian and Jiang (2015). However, all of these risk statistics are designed to quantify risk of a single financial position (i.e. a random variable) by its samples. A natural question is determining how to quantify risk of a portfolio by its samples, especially in the situation where different kinds of currencies are possibly involved in the portfolio.

The main focus of this paper is a new class of set-valued risk statistics with the time value of money, named cash sub-additive risk statistics with an axiomatic approach. By further developing the properties related to cash sub-additive risk statistics, we are able to derive representation results for such risk. This new class of risk statistics can be considered as an extension of those introduced by Sun and Hu (2019) from the empirical (or a data-based) version.
The remainder of this paper is organized as follows. In Sect. 2 we briefly introduce some preliminaries. In Sect. 3 we state the representation result of cash sub-additive risk statistics. In Sect. 4 we investigate the alternative data-based versions of cash sub-additive risk measures. Finally, Sect. 5 discusses the main proof in this paper.

2 Preliminaries

In this section, we briefly introduce some preliminaries that are used throughout this paper. Let $d \geq 1$ be a fixed positive integer. The space $\mathbb{R}^{d \times n}$ represents the set of financial risk positions. An element $z$ of $\mathbb{R}^d$ is denoted by $z := (z_1, \cdots, z_d)$. An element $X$ of $\mathbb{R}^{d \times n}$ is denoted by $X := (X_1, \cdots, X_d) := (x_{1,1}, \cdots, x_{1,n_1}, \cdots, x_{d,1}, \cdots, x_{d,n_l})$. The $d \times n$ dimensional financial positions in $\mathbb{R}^{d \times n}$ have a strong realistic interpretation. This is indeed the case if we consider realistic situations where investors have access to different markets and form multi-asset portfolios in the presence of frictions such as transaction costs, liquidity problems, irreversible transfers, etc. With positive values of $X \in \mathbb{R}^{d \times n}$ we denote the gains while the negative denote the losses. The behavior of the $d$-dimensional random vector $D = (X_1, \cdots, X_d)$ in different scenarios is represented by different sets of data observed or generated in those scenarios because specifying accurate models for $D$ is usually very difficult. Here, we suppose that there always exist $l$ scenarios. Let $n_j$ be the sample size of $D$ in the $j^{th}$ scenario, $j = 1, \cdots, l$. Let
\[ n := n_1 + \cdots + n_l. \] More precisely, suppose that the behavior of \( D \) is represented by a collection of data \( X = (X_1, \ldots, X_d) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n \), where \( X_1 = (X_1^{11}, \ldots, X_1^{1d}) \in \mathbb{R}^n \), \( X_i^{ij} = (x_i^{1j}, \ldots, x_i^{nj}) \in \mathbb{R}^{nj} \) is the data subset that corresponds to the \( j \)th scenario with respect to \( X_i \). For each \( j = 1, \ldots, l \), \( h = 1, \ldots, n_j \), \( X_h^{ij} := (x_h^{1j}, x_h^{2j}, \ldots, x_h^{dj}) \) is the data subset that corresponds to the \( h \)th observation of \( D \) in the \( j \)th scenario, and can be based on historical observations, hypothetical samples simulated according to a model, or a mixture of observations and simulated samples.

Let \( K \) be a closed convex polyhedral cone of \( \mathbb{R}^d \) where \( K \supset \mathbb{R}_+^d \) where \( \mathbb{R}_+^d := \{(x_1, \ldots, x_d) \in \mathbb{R}^d; x_i > 0, 1 \leq i \leq d \} \) and \( K \cap \mathbb{R}_-^d = \emptyset \) where \( \mathbb{R}_-^d := \{(x_1, \ldots, x_d) \in \mathbb{R}^d; x_i < 0, 1 \leq i \leq d \} \). Let \( K \) be the positive dual cone of \( K \), that is \( K^+ := \{u \in \mathbb{R}^d : u^T v \geq 0 \text{ for any } v \in K\} \), where \( u^T \) means the transpose of \( u \). For any \( X = (X_1, \ldots, X_d), Y = (Y_1, \ldots, Y_d) \in \mathbb{R}^{d \times n}, X + Y \) stands for \( (X_1 + Y_1, \ldots, X_d + Y_d) \) and \( aX \) stands for \( a(X_1, \ldots, X_d) \) for \( a \in \mathbb{R} \). For any \( z := (z_1, \ldots, z_d) \in \mathbb{R}^d \), denote \( K_{1n} := \{(z_11_n, z_21_n, \ldots, z_d1_n) : z \in K\} \) and \( z1_n := \{(z, z, \ldots, z) : z \in \mathbb{R}\} \in \mathbb{R}^n \) where \( 1_n := (1, \ldots, 1) \in \mathbb{R}^n \). By \((K_{1n})^+ := \{w \in \mathbb{R}^{dn} : w^T \geq 0 \text{ for any } z \in K\} \). The partial order respect to \( K \) is defined as \( a \leq_K b \).

Therefore, \( b - a \in K \) where \( a, b \in \mathbb{R}^d \) and \( X \subseteq K_{1n} \) \( Y \) means \( Y - X \in K_{1n} \) where \( X, Y \in \mathbb{R}^{d \times n} \). Let \( M := \mathbb{R}^m \times \{0\}^{d-m} \) be the linear subspace of \( \mathbb{R}^d \) for \( 1 \leq m \leq d \). The introduction of \( M \) was considered by Hamel (2009). Denote \( M_+ := M \cap \mathbb{R}_+^d \) where \( \mathbb{R}_+^d := \{(x_1, \ldots, x_d) \in \mathbb{R}^d; x_i \geq 0, 1 \leq i \leq d \} \). Therefore, a regulator can only accept security deposits in the first \( m \) reference instruments. Denote \( K_M := K \cap M \) by the closed convex polyhedral cone in \( M \), \( intK_M \) the interior of \( K_M \) in \( M \). We denote \( T_M := \{A \subset M : A = clco(A + K_M)\} \) and \( T_{M^+} := \{A \subset K_M : A = clco(A + K_M)\} \), where the \( clco(A) \) represents the closed convex hull of \( A \).

By Chen and Hu (2017), a set-valued convex risk statistic is any map \( \rho : \mathbb{R}^{d \times n} \to T_M \) are organized as follows,

A0 Normalized: \( K_M \subset \rho(0) \) and \( \rho(0) \cap -intK_M = \emptyset \);
A1 Monotonicity: for any \( X, Y \in \mathbb{R}^{d \times n}, X - Y \in K_{1n} \) implies that \( \rho(X) \supseteq \rho(Y) \);
A2 Translative invariance: for any \( X \in \mathbb{R}^{d \times n} \) and \( z \in M \), \( \rho(X - z1_n) = \rho(X) + z \);
A3 Convexity: for any \( X, Y \in \mathbb{R}^{d \times n} \) and \( \lambda \in [0, 1] \), \( \rho(\lambda(X) + (1 - \lambda)Y) \supseteq \lambda\rho(X) + (1 - \lambda)\rho(Y) \).

Let \( D := (D_1, \ldots, D_d) \in \mathbb{R}^{d \times n} \) be the stochastic discount factor for certain currency \( X \in \mathbb{R}^{d \times n} \), i.e. each element of \( D \) belongs to \([0, 1]\). A function \( \rho : \mathbb{R}^{d \times n} \to T_M \) is called proper if \( dom\rho := \{X \in \mathbb{R}^{d \times n} : \rho(X) \neq \emptyset\} \neq \emptyset \) and \( \rho(X) \neq M \) for all \( X \in dom\rho \). \( \rho \) is said to be closed if graph\( \rho \) is a closed set.
with respect to the product topology in $\mathbb{R}^{d \times n} \times M$.

In fact, for the common convex risk statistics the transitive invariance axiom has been largely accepted by academics and practitioners. However, as it pointed out by EL Karoui and Ravanelli (2009), while regulators and financial institutions determine and collect today the reserve amounts to cover future risky positions, the transitive invariance axiom requires that risky positions and reserve amounts are expressed in the same numéraire. Unfortunately, when the interest rates are stochastic this procedure is unreasonable. Implicitly, since the time value of money, the cash sub-additivity turn out to be more suitable than the transitive invariance axiom. In this paper, we will derive a new class of set-valued risk statistics with the time value of money.

3 Empirical versions of set-valued cash sub-additive risk measures

Since the time value of money is a critical factor in risk analysis, we will consider a special class of set-valued risk statistics, named cash sub-additive risk statistics. Note that these set-valued risk statistics are the empirical versions of set-valued cash sub-additive risk measures introduced by Sun and Hu (2019).

In this section, we state the representation results of cash sub-additive risk statistics. However, our viewpoint is not the same as Chen and Hu (2017). Instead, we start from the relation between set-valued convex risk statistics and set-valued cash sub-additive risk statistics.

We begin with the axioms related to cash sub-additive risk statistics.

Definition 31 A set-valued cash sub-additive risk statistic is a set-valued function $R : \mathbb{R}^{d \times n} \to \mathbb{T}_M$ that satisfies $A_0, A_1, A_3$ and the following property.

$A_4$ Cash sub-additivity: for any $X \in \mathbb{R}^{d \times n}$ and $z \in K_M$,

$$R(X + z1_n) \subseteq R(X) - z \quad \text{or} \quad R(X - z1_n) \supseteq R(X) + z.$$ 

In fact, the cash sub-additivity of set-valued cash sub-additive risk statistic is derived from the idea of the time value of money. A basic reason is that while regulators and financial institutions determine and collect today the reserve amounts to cover future risky positions, the cash additivity requires that risky positions and reserve amounts are expressed in the same numéraire. Implicitly, when $m$ dollars are added to a future position $X$, the capital requirement at time $t = 0$ is reduced by less than $m$ dollars because the value of the money may grow with time. In this context, we define a set-valued risk statistic in the discounted position.

Definition 32 Let $D \in \mathbb{R}^{d \times n}$ be the discount factor. A set-valued discount risk statistic, say $R_D$, is a set-valued convex risk statistic which is defined in the discounted factor $DX$ where $X \in \mathbb{R}^{d \times n}$.
By set-valued discount risk statistic $\overline{R}$, we can define a set-valued convex function by
\[
\overline{R}(X) := \overline{R}(D_X).
\]
As $\overline{R}$ is a set-valued convex risk statistic, for any $z \in K_M$, we have
\[
R(X + z1_n) = \overline{R}(DX + Dz1_n) \subseteq \overline{R}(DX + z1_n) = \overline{R}(DX) - z = R(X) - z.
\]
This property of $R$ is due to the time value of money. That is to say that $R$ is expressed in terms of the current numéraire but defined in the future financial positions with the future numéraire.

We now introduce a special case of set-valued cash sub-additive risk statistics that is said to be set-valued convex loss-based risk statistics, see Sun et al. (2018). Note that the scalar case of convex loss-based risk measures was studied by Chen et al. (2018) and Cont et al. (2013). The following definition is derived from Sun et al. (2018).

**Definition 33** A set-valued loss-based risk statistic is a mapping $\varrho : \mathbb{R}^{d \times n} \to \mathbb{T}_M$, that satisfies the following properties.

- **B0 Normalization:** $K_M \subseteq \varrho(0)$ and $\varrho(0) \cap -\text{int}K_M = \emptyset$;
- **B1 Cash losses:** for any $z \in K_M$, $z \in \varrho(0 - z1_n)$;
- **B2 Monotonicity:** for any $X, Y \in \mathbb{R}^{d \times n}$, $X - Y \in K_{1_n}$ implies $\varrho(X) \supseteq \varrho(Y)$;
- **B3 Loss-dependence:** for any $X \in \mathbb{R}^{d \times n}$, $\varrho(X) = \varrho(X \wedge_{K_{1_n}} 0)$, where
  \[
  X \wedge_{K_{1_n}} 0 := \begin{cases} X, & X \notin K_{1_n}, \\ 0, & X \in K_{1_n}. \end{cases}
  \]
- **B4 Convexity:** for any $\lambda \in [0, 1]$ and $X, Y \in \mathbb{R}^{d \times n}$, $\varrho(\lambda X + (1 - \lambda)Y) \supseteq \lambda\varrho(X) + (1 - \lambda)\varrho(Y)$.

**Remark 31** The set-valued loss-based risk statistics start from the point of regulators. Namely, the regulators almost only focus on the loss of investment rather than revenue. Especially, the axiom of translation invariance in coherent and convex risk statistics will definitely fail when we only deal with the loss-based risk. Therefore, the loss-based risk is particularly interesting.

We claim that the set-valued loss-based risk statistics are the special cases of set-valued cash sub-additive risk statistics. Indeed, for any $X \in \mathbb{R}^{d \times n}$, $z \in K_M$ and $\varepsilon \in (0, 1)$, we have
\[
\varrho((1 - \varepsilon)X - z1_n) = \varrho((1 - \varepsilon)X + \varepsilon(-\frac{z}{\varepsilon})1_n)
\]
\[
\supseteq (1 - \varepsilon)\varrho(X) + \varepsilon\varrho(-\frac{z}{\varepsilon}1_n)
\]
\[
\supseteq (1 - \varepsilon)\varrho(X) + z,
\]
where the last inclusion is due to the property of cash losses. Therefore, by the arbitrariness of $\varepsilon$, we conclude that
\[
\varrho(X - z1_n) \supseteq \varrho(X) + z.
\]
This indicates that \( q \) satisfies the property \( A4 \). Therefore, \( q \) is cash sub-additive.

Next, we introduce an example of set-valued loss-based risk statistic called \( \text{AV}@R^{\text{loss}} \). Note that, Hamel et al. (2013) first introduced set-valued \( \text{AV}@R \), where they also provided the representation result and proved it is a set-valued coherent risk measure.

**Example 31** (Loss-based average value at risk) For any \( X \in \mathbb{R}^{d \times n} \) and \( 0 < \alpha < 1 \), we define \( \text{AV}@R^{\text{loss}} \) as

\[
\text{AV}@R^{\text{loss}}(X) = \inf_{z \in \mathbb{R}^d} \left\{ \frac{1}{\alpha} (-(X \wedge_K 1_n) \uparrow_M + z) - z \right\} + \mathbb{R}^m.
\]

It is clear that \( \text{AV}@R^{\text{loss}} \) satisfies all the properties of Definition 33. So \( \text{AV}@R^{\text{loss}} \) is a set-valued loss-based risk statistic, and hence it is also cash sub-additive. In fact, the \( \text{AV}@R^{\text{loss}} \) has been widely used in many fields, including bank brokers. And the \( \text{AV}@R^{\text{loss}} \) is just one of his practical applications of set-valued cash sub-additive risk statistics.

We now need to derive the representation results of cash sub-additive risk statistics. Any pair \( (X, \mu) \), where \( X \in \mathbb{R}^{d \times n} \) and \( \mu \in M \), can be viewed as the coordinates of a \( X \in \mathbb{R}^{d \times n} \) in \( T := \{0, 1\} \) with the element \( \theta \),

\[
\overrightarrow{X}(\theta) := X I_{\{1\}}(\theta) + \mu 1_n I_{\{0\}}(\theta). \tag{3.1}
\]

For any \( \overrightarrow{X}, \overrightarrow{Y} \in \mathcal{X} \) with \( \overrightarrow{X} = X I_{\{1\}} + \mu_1 1_n I_{\{0\}} \) and \( \overrightarrow{Y} = Y I_{\{1\}} + \mu_2 1_n I_{\{0\}} \), where \( X, Y \in \mathbb{R}^{d \times n}, \mu_1, \mu_2 \in M \), we define the order as \( \overrightarrow{X} \leq_{K1_n} \overrightarrow{Y} \in K1_n \) in the case of \( Y \leq_{K1_n} X \) and \( \mu_2 \leq_{K} \mu_1 \).

We begin with stating the relation between a set-valued convex risk statistic and a set-valued cash sub-additive risk statistic.

**Proposition 31** Given a set-valued cash sub-additive risk statistic \( R \) in \( \mathbb{R}^{d \times n} \) with \( 0 \in R(0) \), we define a set-valued risk statistic \( \rho \) as follows. For any \( \overrightarrow{X} \) defined as (3.1), with \( X \in \mathbb{R}^{d \times n}, \mu \in M \),

\[
\rho(\overrightarrow{X}) := R(X - \mu 1_n) - \mu. \tag{3.2}
\]

Then \( \rho \) is a set-valued convex risk statistic with \( \rho(0) = 0 \) and \( \rho(X I_{\{1\}}) = R(X) \).

Before we state the representation results of cash sub-additive risk statistics, the representation results of set-valued convex risk statistics should be recalled. The set-valued convex risk statistics were studied by Chen and Hu (2017). We now only state their main results and omit their proofs.
Proposition 32 \( \rho : \mathbb{R}^{d \times n} \rightarrow T_M \) is a set-valued proper closed convex risk statistic in the case of there is a function \( -\alpha : (K_{1,n})^+ \cap \mathbb{R}^{d \times n}_{+} \rightarrow T_M \) such that for any \( \overline{X} \in \mathbb{R}^{d \times n} \),
\[
\rho(\overline{X}) = \bigcap_{v \in (K_{1,n})^+ \cap \mathbb{R}^{d \times n}_{+}} \left\{ -\alpha(v) + S_v(-\overline{X}) \right\},
\]
where
\[
S_v(-\overline{X}) := \{ u \in M : v^\text{tr}(\overline{X} + u 1_n) \geq 0 \}.
\]
In particular, (3.3) is satisfied with \( -\alpha \) replaced by \( -\alpha_{\text{min}} \) with
\[
-\alpha_{\text{min}}(v) := \text{cl} \bigcup_{Z \in \{X \in \mathbb{R}^{d \times n} : 0 \in \rho(X)\}} S_v(Z).
\]

Using Propositions 32, we are able to derive the representation results of set-valued cash sub-additive risk statistics.

Theorem 31 Any set-valued proper closed cash sub-additive risk statistic \( R \) in \( \mathbb{R}^{d \times n} \) has the following form. For any \( X \in \mathbb{R}^{d \times n} \),
\[
R(X) = \bigcap_{v \in (K_{1,n})^+ \cap \mathbb{R}^{d \times n}_{+}} \left\{ -\gamma(v) + \mathcal{G}_v(-X) \right\},
\]
where
\[
-\gamma : (K_{1,n})^+ \cap \mathbb{R}^{d \times n}_{+} \rightarrow T_M
\]
and
\[
\mathcal{G}_v(-X) := \{ u \in M : v^\text{tr}(XI_{1} + u 1_n) \geq 0 \}.
\]
In particular, (3.4) is satisfied with \( -\gamma \) replaced by \( -\gamma_{\text{min}} \) with
\[
-\gamma_{\text{min}}(v) := \text{cl} \bigcup_{Z \in \{X \in \mathbb{R}^{d \times n} : 0 \in R(X)\}} \mathcal{G}_v(Z).
\]

4 Alternative data-based versions of set-valued cash sub-additive risk measures

In this section, we develop another framework of data-based versions of cash sub-additive risk measures. This framework is a little different from the previous one. However, almost all the arguments are the same as those in the previous section. Therefore, we only state the corresponding notations and results, and omit all the proofs and relevant explanations.

We replace \( M \) by \( \tilde{M} \in \mathbb{R}^{d \times n} \) that is a linear subspace of \( \mathbb{R}^{d \times n} \). We also replace \( K \) by \( \tilde{K} \in \mathbb{R}^{d \times n} \) that is a closed convex polyhedral cone where \( \tilde{K} \supseteq \mathbb{R}^{d \times n}_{+} \). The partial order respect to \( \tilde{K} \) is defined as \( X \preceq_{\tilde{K}} Y \), which
Proposition 41
Given a set-valued data-based cash sub-additive risk statistic \( \tilde{\rho} : \mathbb{R}^{d \times n} \to \mathbb{T}_{\tilde{M}} \) that satisfies the following properties,

C0 Normalization: \( \tilde{K}_{\tilde{M}} \subseteq \tilde{\rho}(0) \) and \( \tilde{\rho}(0) \cap -\text{int} \tilde{K}_{\tilde{M}} = \emptyset \);
C1 Monotonicity: for any \( X, X_2 \in \mathbb{R}^{d \times n} \), \( X_1 - X_2 \in \mathbb{R}^{d \times n} \cap \tilde{K} \) implies that \( \tilde{\rho}(X_1) \subseteq \tilde{\rho}(X_2) \);
C2 Translative invariance: for any \( X \in \mathbb{R}^{d \times n} \) and \( z \in \tilde{M} \), \( \tilde{\rho}(X + z) = \tilde{\rho}(X) + z \);
C3 Convexity: for any \( X, Y \in \mathbb{R}^{d \times n} \), \( \lambda \in [0, 1] \), \( \tilde{\rho}(\lambda(X) + (1 - \lambda)Y) \supseteq \lambda \tilde{\rho}(X) + (1 - \lambda)\tilde{\rho}(Y) \).

We now define the set-valued data-based cash sub-additive risk statistics.

Definition 42 A set-valued data-based convex risk statistic is a function \( \tilde{\rho} : \mathbb{R}^{d \times n} \to \mathbb{T}_{\tilde{M}} \) that satisfies C0, C1, C3 and the following property.

C4 Cash sub-additivity: for any \( X \in \mathbb{R}^{d \times n} \) and \( z \in \tilde{K}_{\tilde{M}} \),

\[
\tilde{R}(X + z) \subseteq \tilde{R}(X) - z \quad \text{or} \quad \tilde{R}(X - z) \supseteq \tilde{R}(X) + z.
\]

We need more notions. Any pair \((X, \tilde{\mu})\), where \( X \in \mathbb{R}^{d \times n} \) and \( \tilde{\mu} \in \tilde{M} \), can be viewed as the coordinates of a \( \tilde{X} \in \mathbb{R}^{d \times n} \) in \( \mathcal{T} := \{0, 1\} \) with the element \( \theta \),

\[
\tilde{X}(\theta) := XI_{\{1\}}(\theta) + \tilde{\mu}I_{\{0\}}(\theta) \tag{4.1}
\]

Proposition 41 Given a set-valued data-based cash sub-additive risk statistic \( \tilde{R} \) in \( \mathbb{R}^{d \times n} \) with \( 0 \in \tilde{R}(0) \), we define a set-valued data-based risk statistic \( \tilde{\rho} \) as follows. For any \( \tilde{X} \) defined as (4.1), where \( X \in \mathbb{R}^{d \times n} \), \( \tilde{\mu} \in \tilde{M} \),

\[
\tilde{\rho}(\tilde{X}) := \tilde{R}(X - \tilde{\mu}) - \tilde{\mu} \tag{4.2}
\]

Then \( \tilde{\rho} \) is a set-valued data-based convex risk statistic with \( \tilde{\rho}(0) = 0 \) and \( \tilde{\rho}(XI_{\{1\}}) = \tilde{R}(X) \).

The representation results of set-valued data-based convex risk statistics were studied in Chen and Hu (2017). We now state the main results of this section.

Theorem 41 Any set-valued data-based proper closed cash sub-additive risk statistic \( \tilde{R} \) in \( \mathbb{R}^{d \times n} \) has the following form. For any \( X \in \mathbb{R}^{d \times n} \),

\[
\tilde{R}(X) = \bigcap_{\tilde{v} \in \tilde{K} + \text{int} \mathbb{R}_{+}^{d \times n}} \{ -\tilde{\gamma}(\tilde{v}) + \tilde{\mathcal{E}}(-X) \} \tag{4.3}
\]
where 
\[-\tilde{\gamma} : \tilde{K}^+ \cap \mathbb{R}^{d \times n} \to T_{\tilde{M}}\]
and 
\[\tilde{G}_\psi(-X) := \{ \tilde{u} \in \tilde{M} : \tilde{\psi}(X I_{\{1\}} + \tilde{u}) \geq 0 \} \]
In particular, (4.3) is satisfied with 
\[-\tilde{\gamma}_{\min}(\tilde{b}) := \text{cl} \bigcup_{Z \in \{ X \in \mathbb{R}^{d \times n} : 0 \in \tilde{r}(X) \}} \tilde{G}_\psi(Z) \]

5 Proofs of main results

In this section, we provide all the proofs of the results stated in Sect. 3.

**Proof of Proposition 31.** It is easy to check that \( \rho(0) = 0, \rho(X I_{\{1\}}) = R(X) \)
and \( \rho \) satisfies the property of **A0**. Next, we derive that \( \rho \) satisfies properties of **A1, A2** and **A3**.

**A1.** Monotonicity: for any \( \tilde{X} = X I_{\{1\}} + \mu_1 1_n I_{\{0\}} \) and \( \tilde{Y} = Y I_{\{1\}} + \mu_2 1_n I_{\{0\}} \),
where \( X, Y \in \mathbb{R}^{d \times n}, \mu_1, \mu_2 \in \tilde{M} \) with \( \tilde{X} - \tilde{Y} \in K 1_n \), then
\[ \rho(\tilde{X}) = R(X - \mu_1 1_n) - \mu_1 \supseteq R(X - \mu_2 1_n) - \mu_2 \supseteq R(Y - \mu_2 1_n) - \mu_2 = \tilde{r}(\tilde{Y}), \]
which shows that \( \rho \) is monotone.

**A2.** Translative invariance: for any \( b \in \tilde{M}, \tilde{X} = X I_{\{1\}} + \mu_1 1_n I_{\{0\}} \) where \( X \in \mathbb{R}^{d \times n} \) and \( \mu \in \tilde{M} \),
\[ \rho(\tilde{X} + b 1_n) = \rho \left( (X + b 1_n) I_{\{1\}} + (\mu_1 + b 1_n) I_{\{0\}} \right) \]
\[ = R \left( X + b 1_n - (\mu + b) 1_n \right) - \mu - b \]
\[ = R(X - \mu 1_n) - \mu - b \]
\[ = \rho(\tilde{X}) - b, \]
which shows that \( \rho \) satisfies the translative invariance.

**A3.** Convexity: for any \( \lambda \in (0, 1), \tilde{X} = X I_{\{1\}} + \mu_1 1_n I_{\{0\}} \) and \( \tilde{Y} = Y I_{\{1\}} + \mu_2 1_n I_{\{0\}} \),
where \( X, Y \in \mathbb{R}^{d \times n}, \mu_1, \mu_2 \in \tilde{M} \),
\[ \rho(\lambda \tilde{X} + (1 - \lambda) \tilde{Y}) \]
\[ = \rho \left( (\lambda X + (1 - \lambda) Y) I_{\{1\}} + (\lambda \mu_1 + (1 - \lambda) \mu_2) 1_n I_{\{0\}} \right) \]
\[ = R \left( (\lambda X + (1 - \lambda) Y) - (\lambda \mu_1 + (1 - \lambda) \mu_2) 1_n \right) - \lambda \mu_1 - (1 - \lambda) \mu_2 \]
\[ = R \left( \lambda (X - \mu_1 1_n) + (1 - \lambda) (Y - \mu_2 1_n) \right) - \lambda \mu_1 - (1 - \lambda) \mu_2 \]
\[ \supseteq \lambda R(X - \mu_1 1_n) + (1 - \lambda) R(Y - \mu_2 1_n) - \lambda \mu_1 - (1 - \lambda) \mu_2 \]
\[ = \lambda \rho(\tilde{X}) + (1 - \lambda) \rho(\tilde{Y}), \]
which shows that \( \rho \) is convex.
Proof of Proposition 32. See Chen and Hu (2017).

Proof of Theorem 31. From Proposition 31, we can define a set-valued convex risk statistic $\rho$ in $\mathbb{R}^{d \times n}$ by

$$R(X) = \rho(X_{I(1)}) = \bigcap_{v \in (K1_n)^+ \cap \mathbb{R}_+^{d \times n}} \{ -\alpha(v) + S_v(-XI_{I(1)}) \},$$

where $-\alpha : (K1_n)^+ \cap \mathbb{R}_+^{d \times n} \to T_M$ and

$$S_v(-X) := \{ u \in M : v^T (X + u1_n) \geq 0 \}.$$

In particular, the function $-\alpha$ can be replaced by $-\alpha_{\text{min}}$ where

$$-\alpha_{\text{min}}(v) := \text{cl} \bigcup_{Z \in \{ X \in \mathbb{R}^{d \times n} : 0 \in \rho(X_{I(1)}) \}} S_v(Z_{I(1)}).$$

We now denote

$$\mathcal{S}_v(X) := \{ u \in M : v^T (-XI_{I(1)} + u1_n) \geq 0 \}.$$

Therefore, the set-valued cash sub-additive risk statistic $R$ can be expressed as

$$R(X) = \bigcap_{v \in (K1_n)^+ \cap \mathbb{R}_+^{d \times n}} \{ -\gamma(v) + \mathcal{S}_v(-X) \},$$

where $-\gamma : (K1_n)^+ \cap \mathbb{R}_+^{d \times n} \to T_M$ and can replaced by $-\gamma_{\text{min}}$ where

$$-\gamma_{\text{min}}(v) := \text{cl} \bigcup_{Z \in \{ X \in \mathbb{R}^{d \times n} : 0 \in R(X) \}} \mathcal{S}_v(Z).$$

Conclusions

In fact, the time value of money is a critical factor in insurance and financial applications. However, in traditional research of risk statistic, the time value of money is always ignored, that is, when $m$ dollars are added to a future position, the capital requirement at time $t = 0$ is still $m$ dollars. This is obviously inconsistent with reality. Thus, we derive a new class of risk statistics, named set-valued cash sub-additive risk statistic. Yet, we do not conduct theoretical analysis on exponential dispersion models like Shushi and Yao (2020). Our results provide the macro models from the perspective of the time value of money.
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