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Consistency of Oblique Decision Tree and its Boosting and Random Forest

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Classification and Regression Tree (CART), Random Forest (RF) and Gradient Boosting Tree (GBT) are probably the most popular set of statistical learning methods. However, their statistical consistency can only be proved under very restrictive assumptions on the underlying regression function. As an extension to standard CART, the oblique decision tree (ODT), which uses linear combinations of predictors as partitioning variables, has received much attention. ODT tends to perform numerically better than CART and requires fewer partitions. In this paper, we show that ODT is consistent for very general regression functions as long as they are continuous. Then, we prove the consistency of the ODT-based random forest (ODRF), whether fully grown or not. Finally, we propose an ensemble of GBT for regression by borrowing the technique of orthogonal matching pursuit and study its consistency under very mild conditions on the tree structure. After refining existing computer packages according to the established theory, extensive experiments on real data sets show that both our ensemble boosting trees and ODRF have noticeable overall improvements over RF and other forests.

Keywords: CART; gradient boosting tree; consistency; feature bagging; nonparametric regression; oblique decision tree; random forest

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1. Introduction

The classification and regression tree (Breiman, 1984, CART) is one of the most popular machine learning algorithms. It has apparent simplicity, visibility and interpretability and is therefore widely used in data mining and analysis. The random forest (Breiman, 2001a, RF), as an ensemble method of CART, is arguably a very efficient method for tabular data and is therefore one of the most popular methods in machine learning. Meanwhile, another improvement of CART, called gradient boosting tree (Friedman, 2001, GBT), is also an efficient method in machine learning.

However, it has been noticed by Breiman (1984) from the time when CART was first proposed that the use of marginal variables as the splitting variables could cause problems both in theory and in its numerical performance in classification and prediction. As a remedy for this disadvantage of CART, Breiman (1984) suggested using linear combinations of the predictors as the splitting variables. Later, the method became known as the oblique decision tree (Heath, Kasif and Salzberg, 1993, ODT) and has received much attention. ODT was expected to have better performance than CART, requiring fewer splits and therefore smaller trees than CART (see Kim and Loh (2001)), while the linear combination preserves interpretability in data analysis. There are also various oblique random forests (ODRF) in the literature, including Forest-RC of Breiman (2001a), Random Rotation Random Forest (RR-RF) of Blaser and Fryzlewicz (2016), Canonical Correlation Forests (CCF) of Rainforth and Wood (2015), Random Projection Forests (RPFs) of Lee, Yang and Oh (2015) and Sparse Projection Oblique Random Forests of Tomita et al. (2020). For these forests, the difference is on how to find the linear combination and can be regarded as different implementations of ODT or their random forests.

1.1. A review of studies on statistical consistency of decision trees

Despite the popularity of CART and RF, their statistical consistency has posed a big theoretical problem for statisticians for a long time and is still far from being fully resolved. The difficulty for statistical consistency of CART or RF can be attributed to two issues. The first one is the limitation of the method itself, i.e., the use of only marginal variables as partitioning variables when building the trees of CART or RF. The second issue is the mathematical complexity in analyzing the relationship between different layers of the tree.

Early works on the consistency of CART or RF were mainly for the simplified version of CART or RF. The most celebrated theoretical result is that of Breiman (2001b), which offers an upper bound on the generalization error of forests in terms of correlation and strength of the individual trees. This was followed by Breiman (2004) which focuses on a stylized version of the original algorithm. Lin and Jeon (2006) established lower bounds for non-adaptive forests (i.e., independent of the training set) via the connection between random forests and a particular class of nearest neighbor predictors; see also Biau and Devroye (2010). In the past ten years, various theoretical developments, for example, Biau, Devroye and Lugosi (2008), Ishwaran and Kogalur (2010), Biau (2012), Genuer (2012), and Zhu, Zeng and Kosorok (2015), have been made in analyzing the consistency of simplified models. Recent attempts toward narrowing the gap between the theory and practice are Denil, Matheson and Freitas (2013) who proves the first consistency result for online random forests, and Wager (2014) and Mentch and Hooker (2014) who study the asymptotic sampling distribution of forests.

The work of Scornet, Biau and Vert (2015) is a milestone. They proved that CART-based RF is consistent in the $L^2$ sense if the unknown regression function is additive of the marginal variables. Following their proofs, Syrgkanis and Zampetakis (2020), Klusowski (2021) and Chi et al. (2020) showed that RF is also consistent in the high-dimensional setting under different modelling assumptions. In particular, Klusowski (2021) found that there is a relationship between CART and greedy algorithms and
also gave a consistency rate \((\ln n)^{-1}\), where \(n\) is the sample size. Chi et al. (2020) improved this consistency rate under an additional assumption called sufficient impurity decrease (SID), which includes the additive model as a special case in a high-dimensional setting. However, all the above consistency results for CART or RF are based on very strong restrictions on \(m(x)\), such as the additive model or the SID condition.

On the other hand, consistency results for gradient boosting tree (GBT) are limited in the literature, and most papers in this area have focused only on how to improve its numerical performance. Bühlmann (2002) proved the \(L^2\) consistency of a binary classifier based on GBT. More recently, Zhou and Hooker (2022) estimated a varying coefficient regression model by using GBT and proved their consistency results. However, both of these papers proved the consistency of the gradient boosting tree by imposing additional technical assumptions on the proportions of data points in its terminal nodes.

### 1.2. Our contributions

Note that the existing consistency results for decision trees or their corresponding random forests are either proved under very strong assumptions on the unknown regression functions or are only for simplified versions of decision trees that are not practically used. In view of this, consistency results in this paper are novel. Our contributions are summarized as follows.

- We establish consistency results for ODT as well as ODRF, which is based on either fully grown trees or not fully grown trees, for general regression functions as long as they are continuous. The results include those of CART or RF as their special cases or corollaries.
- We introduce two methods of feature bagging to improve prediction performance and establish the consistency of the methods.
- We also refine the existing packages for ODRF according to the established theory. Extensive empirical studies have shown that both our ODRF and ensemble boosting trees tend to have superior performance over other methods, including standard RF and other ODRF implementations, which was not clearly demonstrated before using the existing packages; see for example Kim and Loh (2001).
- We establish an explicit relationship between neural networks and the gradient boosting tree of Friedman (2001). We then use this relationship to prove the consistency of the gradient boosting tree based on ODT. To our knowledge, the consistency rate is the fastest among all the tree-based regressions. Importantly, our results do not require any additional technical assumptions on the tree structure.

During our study of ODT, Cattaneo, Chandak and Klusowski (2022) also presented some consistency results about ODT. We summarize the three main differences between our results and theirs as follows. Firstly, oracle inequalities in Cattaneo, Chandak and Klusowski (2022) were intentionally designed for the high-dimensional regression and hold for functions whose Fourier transformations have finite first moments. Our study focuses on the consistency of ODT for fixed dimension regression and shows that ODT is consistent for more general regression functions, as long as they are continuous. Secondly, the number of layers \(K_n\), which is unknown before the construction of ODT, is employed to control the growth of the tree in Cattaneo, Chandak and Klusowski (2022), while the number of terminal leaves \(\ell_n\), which can be predefined in the algorithm, is used in our construction of ODT. In order to obtain the consistency of ODT Cattaneo, Chandak and Klusowski (2022) requires that \(K_n\) must be \(o(\log_2 n)\), which means that the depth of ODT cannot be large. In other words, Cattaneo, Chandak and Klusowski (2022) can only guarantee consistency for some special ODTs where \(K_n\) must be much smaller than \(\log_2 n\). Note that for a given set of data, no one knows what the maximum depth of its ODT might be.
before running the algorithm. Instead, this strong restriction is not required in our theory since we only need to assume \( t_n = o(n) \) and the non-random \( t_n \) can be indeed used as an input parameter to control the growth of ODT. Finally, but importantly, we also introduce boosting trees and various random forests based on ODT and study their asymptotic properties in different cases but these important parts are missing in Cattaneo, Chandak and Klusowski (2022). Especially, the proof of the consistency of ODRF with fully grown trees is completely different from that for a single tree in Klusowski (2021).

1.3. Organization of this paper

The rest of this paper is organised as follows. In Section 2, we introduce the notations used in the proofs and the algorithm to describe how to construct ODT. In Section 3, we first describe the idea of our proofs and then give our main results for the consistency of ODT. Section 4 presents consistency results for ODRFs based on trees that are either fully grown or not fully grown. In Section 5, we propose an ensemble of boosting trees for regression and analyze its statistical consistency. In Section 6, we explain the implementation of our two algorithms based on ODT and compare their numerical performance with RF and other decision forests.

2. Preliminaries

Suppose \( Y \) is the response and \( X \in [0, 1]^p \) is the predictor. Our interest is the estimation of the regression function \( m(x) := E(Y|X = x), x \in [0, 1]^p \). Denote by \( D_n = \{(X_i, Y_i)\}_{i=1}^n \) the independent samples of \((X, Y)\). Let \( \Theta^p = \{\theta: \theta \in \mathbb{R}^p \text{ and } ||\theta||_2 = 1\} \) be the unit sphere in \( \mathbb{R}^p \), and || \cdot ||_2 denotes the \( L^2 \) norm in \( \mathbb{R}^p \) space. Denote by \( A \) a node of ODT, which is a subset of \([0, 1]^p\), and denote its two daughters by \( A^+, A^- \). Note that either \( A^+ \) or \( A^- \) can be empty. Let \( Y = (Y_1, \ldots, Y_n)^T \in \mathbb{R}^n \) be the response vector. For any node \( A \), let \( N(A) := Card(\{X_i \in A\}) \) be the number of data points in \( A \). Let \( (\bar{Y}_A, \bar{Y} + f)_A := \frac{1}{N(A)} \sum_{X_i \in A} [(Y_i - \bar{Y}_A) \cdot (Y_i + f(X_i))] \) and \( ||Y - f||_A^2 := \frac{1}{N(A)} \sum_{X_i \in A} (Y_i - f(X_i))^2 \) for any \( f: [0, 1]^p \rightarrow \mathbb{R} \) and \( \bar{Y}_A := \frac{1}{N(A)} \sum_{X_i \in A} Y_i \), the sample mean for data in \( A \). Define the impurity gain in the regression problem (Breiman, 1984) by

\[
\Delta_A(\theta, s) = ||\bar{Y} - \bar{Y}_A||_{A,n}^2 - \left( P(A^+_{\theta, s}) ||\bar{Y} - \bar{Y}_{A^+_{\theta, s}}||_{A^+_{\theta, s},n}^2 + P(A^-_{\theta, s}) ||\bar{Y} - \bar{Y}_{A^-_{\theta, s}}||_{A^-_{\theta, s},n}^2 \right),
\]

where \( P(A^+_{\theta, s}) = N(A^+_{\theta, s})/N(A) \) and \( P(A^-_{\theta, s}) = N(A^-_{\theta, s})/N(A) \). Let \( t_n \) be the number of terminal leaves of an ODT, with \( 1 \leq t_n \leq n \). The total variation of any univariate function \( f(x), x \in [0, 1] \), is denoted by \( ||f||_{TV} := \sup_{\theta \geq 1} \sup_{0 \leq x_0 < \cdots < x_n \leq 1} \Sigma_{j=0}^{n-1} |f(x_{j+1}) - f(x_j)| \). If \( f' \in L^1[0, 1] \), where \( L^1[0, 1] \) denotes the \( L^1 \) space, we also have \( ||f||_{TV} = \int_0^1 |f'(x)|dx \). We refer Folland (1999) for more details about the total variation. For any linear combination of ridge functions \( g = \Sigma_{j=1}^T g_j(\theta_j^T x), x \in [0, 1]^p \), we define its total variation by \( ||g||_{TV} := \Sigma_{j=1}^T ||g_j||_{TV} \). The notation \( c \) is used in this paper to denote a positive constant that may change from line to line. The function \( c(a_1, \ldots, a_8) \) is employed to denote a positive constant that depends only on parameters \( a_1, \ldots, a_8 \).

According to Breiman (1984), the best splitting criteria of each node \( A \) that contains at least two data points is to choose \((\hat{\theta}_A, \hat{s}_A)\) by maximizing \( \Delta_A(\theta, s) \) over \( \mathbb{R} \times \Theta^p \). Based on the best splitting, the construction of ODT in regression problem is shown in Algorithm 1.

For a regression tree, we call the root, i.e. original data, as layer 0, and the last layer that contains only leaves, \( L \). Thus, the layer index \( \ell \) satisfies \( 0 \leq \ell \leq L \). Let \( k_\ell \) be the number of leaves at layer \( \ell \).
Consistency of Oblique Decision Tree

Figure 1. This figure shows an example of $T_{Dn,6,3}$, which has three layers $L = 3$ and 6 leaves. To be specific, we have root node $A_0^1$ in the layer 0, nodes $A_1^1$ and $A_2^1$ in the layer 1, nodes $A_1^2, A_2^2, A_2^3, A_2^4$ in the layer $\ell = 2$ and leaves $A_1^3, A_2^3, A_3^3, A_3^4, A_3^5, A_3^6, A_4^6$ in the layer 3. Note that in this case $A_2^3$ only contains one data point and can not be divided in further steps, which implies that $A_2^2 \equiv A_3^3$. It is also noteworthy that no matter how many data points in $A_2^4$ we have $A_2^4 \equiv A_3^6$ because $t_n$ is preset to be 6. Finally, we have estimators $m_{n,2}^j(x) = \sum_{j=1}^{4} \mathbb{I}(x \in A_j^1) \cdot P_{A_j^1}$ and $m_{n,3}(x) = \sum_{j=1}^{6} \mathbb{I}(x \in A_j^3) \cdot P_{A_j^3}$ given data $D_n$.

whose corresponding nodes are denoted by $\{A_j^k\}_{j=1}^{k_L}$. The estimator of $m(x)$ by using nodes at layer $\ell$ is defined by

$$m_{n,t_n,\ell}(x) = \sum_{j=1}^{k_L} \mathbb{I}(x \in A_j^\ell) \cdot \bar{Y}_{A_j^\ell}$$

for each $0 \leq \ell \leq L$. To simplify the notation, $m_{n,t_n,\ell}(x)$ is sometimes abbreviated to $m_{n,\ell}(x)$ if there is no confusion. With the above notations, the final estimator is $m_{n,L}(x)$ satisfying $k_L = t_n$. As an example of a regression tree, as shown in Figure 1, $L = 3$ and $t_n = 6$. Denote a regression tree with $t_n$ leaves by $T_{Dn,t_n}$, which can also be written as $T_{Dn,t_n,L}$. For the proof of the consistency, we also need to consider a truncated tree at any layer $\ell$, denoted by $T_{Dn,t_n,\ell}$. For example, tree $T_{Dn,t_n,2}$ in Figure 1 has leaves $\{A_1^1, A_2^2, A_3^3, A_4^4\}$. Note that leaves and nodes are relative, and a node on a fully grown tree can become a leaf on a truncated tree.

3. Consistency of ODT

Let us briefly describe our idea for the proof of the consistency of ODT, i.e. the consistency of $m_{n,L}(x)$ as an estimator of $m(x) = \mathbb{E}(Y|X=x), x \in [0,1]$. Three main facts are used in the proof. The first fact is the well-known universal approximation theorem; see, for example, Cybenko (1989). Under some mild conditions, function $m(x)$ can be approximated by a sequence of ridge functions with additive
Assume Theorem 3.1 (Consistency of ODT before pruning).

The second fact is that the directions searched in the ODT algorithm can indeed play the role of those in the universal approximation (3.1) or those directions in the additive model proved by Klusowski (2021) and that if \( \theta \) indeed has an additive structure. The details are given below.

Theorem 3.1 (Consistency of ODT before pruning). Assume \( X \in [0,1]^p \) and \( \mathbb{E}(e^{cY^2}) < \infty \) for some \( c > 0 \), and that \( m(x) \) is a continuous function. If \( t_n \to \infty \) and \( t_n = o \left( \frac{d}{\ln n} \right) \), we have

\[
\mathbb{E} \left( \int |m(x) - m(x)|^2 \, d\mu(x) \right) \to 0, \quad \text{as } n \to \infty.
\]
Remark 1. This theorem shows that ODT is mean squared consistent under very mild conditions. By comparing Theorem 3.1 with the consistency of CART given in Scornet, Biau and Vert (2015), we can conclude the advantages of ODT as follows. CART is consistent only when \( m(x) \) has an additive structure of the predictors, while Theorem 3.1 guarantees the consistency of ODT for any smooth functions. Furthermore, there is no restriction on the distribution of \( X \) and we only require \( Y \) to be a sub-Gaussian random variable in the theorem.

We need more notation in the proof of Theorem 3.1. Let \( \beta_n = \ln n \) and

\[ \hat{m}_{n,L}(x) = \max\{\min\{m_{n,L}(x), \beta_n\}, -\beta_n\}. \]

Then, \( \hat{m}_{n,L}(x) \) is a truncated version of \( m_{n,L}(x) \). More generally, let \( h : [0,1]^p \to \mathbb{R} \) be a function which is constant on each \( \mathcal{A}_j^{(L)}, j = 1, \ldots, t_n \) in Algorithm 1. Let \( \mathcal{H} \) be the collection of all functions \( h \) defined above, and \( \mathcal{H}^{\beta_n} = \{\max\{\min\{h, \beta_n\}, -\beta_n\} : h \in \mathcal{H}\} \). In statistical theory, the growth function and the covering number are two useful concepts, which are explained below.

**Definition 3.2 (Blumer et al. (1989)).** Let \( \mathcal{F} \) be a Boolean function class in which each \( f : \mathbb{Z} \to \{0, 1\} \) is binary-valued. The growth function of \( \mathcal{F} \) is defined by

\[ \Pi_{\mathcal{F}}(m) = \max_{z_1, \ldots, z_m} |\{(f(z_1), \ldots, f(z_m)) : f \in \mathcal{F}\}| \]

for each positive integer \( m \in \mathbb{Z}_+ \).

**Definition 3.3 (Györfi et al. (2002)).** Let \( z_1, \ldots, z_n \in \mathbb{R}^p \) and \( z^n = \{z_1, \ldots, z_n\} \). Let \( \mathcal{H} \) be a class of functions \( h : \mathbb{R}^p \to \mathbb{R} \). An \( L_q \) \( \varepsilon \)-cover of \( \mathcal{H} \) on \( z^n \) is a finite set of functions \( h_1, \ldots, h_N : \mathbb{R}^p \to \mathbb{R} \) satisfying

\[ \min_{1 \leq j \leq N} \left( \frac{1}{n} \sum_{i=1}^{n} |h(z_i) - h_j(z_i)|^q \right)^{1/q} < \varepsilon, \quad \forall h \in \mathcal{H}. \]

Then, the \( L_q \) \( \varepsilon \)-cover number of \( \mathcal{H} \) on \( z^n \), denoted by \( N_q(\varepsilon, \mathcal{H}, z^n) \), is the minimal size of an \( L_q \) \( \varepsilon \)-cover of \( \mathcal{H} \) on \( z^n \). If there exists no finite \( L_q \) \( \varepsilon \)-cover of \( \mathcal{H} \), then the above cover number is defined as \( N_q(\varepsilon, \mathcal{H}, z^n) = \infty \).

The proof of Theorem 3.1 also relies on Lemma 3.4 below.

**Lemma 3.4 (Bagirov, Clausen and Kohler (2009)).** Assume \( \mathbb{E}(e^{cY^2}) < \infty \) for some \( c > 0 \). Then, the truncated estimator \( \hat{m}_{n,L}(x) \) satisfies

\[
\mathbb{E}_{D_n} \int \left| \hat{m}_{n,L}(x) - m(x) \right|^2 d\mu(x) \leq 2\mathbb{E}_{D_n} \left( \frac{1}{n} \sum_{i=1}^{n} |m_{n,L}(X_i) - Y_i|^2 - \frac{1}{n} \sum_{i=1}^{n} |m(X_i) - Y_i|^2 \right) + \frac{c\ln^2 n}{n} \sup_{z^n} \left( N_1(1/(8n\beta_n), \mathcal{H}^{\beta_n} z^n) \right)
\]

for some \( c > 0 \), where \( N_1 \) is the cover number and \( \mu \) is the distribution of \( X \).
Next, we first use the technique of Klusowski (2021) to bound the first part of the RHS of (3.2), and then use Lemma 3.6 to bound the second term.

**Lemma 3.5.** Let $R(A) := \|Y - \overline{Y}_A\|^2_A - \|Y - g\|^2_A$, for any $g \in \mathcal{U}$, where $\mathcal{U}$ is a set of linear combinations of ridge functions:

$$\mathcal{U} = \sum_{j=1}^J g_j(\theta_j^T x) : \theta_j \in \Theta^P, g_j \in TV(\mathbb{R}), J \in \mathbb{Z}_+$$

and $TV(\mathbb{R})$ consists of functions defined on $\mathbb{R}$ with bounded total variation. Let $A$ be an internal node of tree $T_{D_n,t_n}$ which contains at least two data points. If $R(A) \geq 0$, then $\Delta_A(\hat{\theta}_A, \hat{s}_A)$ satisfies

$$\Delta_A(\hat{\theta}_A, \hat{s}_A) \geq \frac{R^2(A)}{\|g\|^2_{TV}}.$$ 

**Proof.** For each internal node $A$, partition it into two daughters $A^+_\theta,s = \{x \in A : \theta^T x \leq s\}$ and $A^-_{\theta,s} = \{x \in A : \theta^T x > s\}$. Let

$$\hat{Y}_A(x) := \frac{\mathbb{I}(x \in A^+_\theta,s)P(A^-_{\theta,s}) - \mathbb{I}(x \in A^-_{\theta,s})P(A^+_\theta,s)}{\sqrt{P(A^+_\theta,s)P(A^-_{\theta,s})}},$$

Following Lemma 3.1 in Klusowski (2021) we can show that

$$\Delta_A(\theta, s) = |\langle Y - \overline{Y}_A, \hat{Y}_A \rangle_A|^2$$

when $\theta$ is parallel to any axis and $P(A^+_{\theta,s})P(A^-_{\theta,s}) > 0$. In fact, it is not difficult to prove that (3.3) also holds for any $\theta \in \Theta^P$. Then, the remaining proof is similar to Lemma 7.1 in Klusowski (2021).

**Lemma 3.6 (Györfi et al. (2002)).** Let $\mathcal{H}$ be a class of functions $h: \mathbb{R}^P \rightarrow [0,B]$ with finite VC dimension $VC(\mathcal{H}) \geq 2$. For any $B/4 > \varepsilon > 0$, the cover number in Definition 3.3 satisfies

$$N_1(\varepsilon, \mathcal{H}, z^n) \leq 3 \left( \frac{2eB}{\varepsilon} \ln \left( \frac{3eB}{\varepsilon} \right) \right)^{VC(\mathcal{H})},$$

for all $z^n = \{z_1, \ldots, z_n\}, z_i \in \mathbb{R}^P$.

**Proof of Theorem 3.1.** The outline of this proof is as follows. We first show that the truncated estimator $\hat{m}_{t_n}(x)$ defined above is mean squared consistent by considering the two terms on the right hand side (RHS) of Lemma 3.4 separately. The details are given in Part I and Part II below. We will then prove in Part III that the untruncated estimator $m_{t_n}(x)$ is also consistent.

**Part I:** Consider the first part of RHS of (3.2). For the theoretical analysis, we introduce the following class of linear combinations of ridge functions:

$$\text{Ridge}_J := \left\{ \sum_{j=1}^J c_j \sigma(\theta_j^T x + d_j) : \theta_j \in \Theta^P, c_j, d_j \in \mathbb{R}, \forall j \geq 1 \right\},$$

where $\sigma(v) = e^v/(1 + e^v)$ with $v \in \mathbb{R}$.
Recall the definition of $T_{\mathcal{D}_n,L}$, a truncated tree, and $m_{n,L}(x)$ is an estimator of $m(x)$ by taking averages of data in each terminal leaf of $T_{\mathcal{D}_n,L}$, namely

$$m_{n,L}(x) = \sum_{j=1}^{k_L} \mathbb{I}(x \in \mathcal{A}_L^j) \cdot \bar{y}_{\mathcal{A}_L^j}.$$ 

Let $L_0 := \lfloor \log_2 t_n \rfloor \leq L$. Note that $T_{\mathcal{D}_n,\ell}$ is fully grown for each $0 \leq \ell \leq L_0$, i.e. $T_{\mathcal{D}_n,\ell}$ is generated recursively by splitting all leaves of the previous $T_{\mathcal{D}_n,\ell-1}$ except those leaves containing only one data point.

For any given $g \in \mathcal{G}_J$, we prove

$$\|\mathbb{Y} - m_{n,L_\ell}(X)\|_n^2 - \|\mathbb{Y} - g(X)\|_n^2 \leq \frac{\|g\|^2_{TV}}{\log t_n + 4} \quad (3.4)$$

for any $t_n > 1$, where $\|g\|^2_{TV}$ is the total variation of $g$ in Lemma 3.5. Define the approximation error by $R_{\mathcal{D}_n,\ell} := \|\mathbb{Y} - m_{n,\ell}(X)\|_n^2 - \|\mathbb{Y} - g(X)\|_n^2$ for any $0 \leq \ell \leq L_0$. Without loss of generality, we can also assume $R_{\mathcal{D}_n,\ell-L_0} \geq 0$ since $R_{\mathcal{D}_n,\ell} \leq R_{\mathcal{D}_n,\ell-L_0}$ and (3.4) holds obviously if $R_{\mathcal{D}_n,\ell-L_0} < 0$. Similarly, define $R(A) := \|\mathbb{Y} - m_{n,\ell}(X)\|_n^2 - \|\mathbb{Y} - g(X)\|_n^2$ for each $A \in \mathcal{O}_{\ell,1} := \{\mathcal{A}_L^j\}_{j=1}^{k_L}$ which contains at least one data point. Then, for any $0 \leq \ell \leq L_0$ we have

$$R_{\mathcal{D}_n,\ell} = \sum_{A \in \mathcal{O}_{\ell,1}} w(A)R(A),$$

where $w(A) = \text{Card}(A)/n$ is the proportion of data within $A$. Note that

$$R_{\mathcal{D}_n,\ell-L_0} = R_{\mathcal{D}_n,\ell-L_0-1} - \sum_{A \in \mathcal{O}_{\ell-L_0-1,2}} w(A)\Delta A(\hat{\theta}_A, \hat{s}_A) \quad (3.5)$$

$$\leq R_{\mathcal{D}_n,\ell-L_0-1} - \sum_{A \in \mathcal{O}_{\ell-L_0-1,2}: R(A) > 0} w(A)\Delta A(\hat{\theta}_A, \hat{s}_A) \leq R_{\mathcal{D}_n,\ell-L_0-1} - \sum_{A \in \mathcal{O}_{\ell-L_0-1,2}: R(A) > 0} \frac{1}{\|g\|^2_{TV}} w(A)R^2(A), \quad (3.6)$$

where $\mathcal{O}_{\ell-L_0-1,2} \subseteq \mathcal{O}_{\ell-L_0-1,1}$ is a collection of nodes which must contain at least two data points, and (3.5) follows from the definition of impurity gain, and (3.6) follows from Lemma 3.5. Decompose $R_{\mathcal{D}_n,\ell-L_0-1}$ into two parts:

$$R_{\mathcal{D}_n,\ell-L_0-1}^+ := \sum_{A \in \mathcal{O}_{\ell-L_0-1,1}: R(A) > 0} w(A)R(A),$$

$$R_{\mathcal{D}_n,\ell-L_0-1}^- := \sum_{A \in \mathcal{O}_{\ell-L_0-1,1}: R(A) \leq 0} w(A)R(A)$$

satisfying $R_{\mathcal{D}_n,\ell-L_0-1} = R_{\mathcal{D}_n,\ell-L_0-1}^+ + R_{\mathcal{D}_n,\ell-L_0-1}^-$. By Jensen’s inequality and $R(A) \leq 0$ for any leaf $A$ of $T_{\mathcal{D}_n,\ell-L_0-1}$ which contains only one data point, we have

$$\sum_{A \in \mathcal{O}_{\ell-L_0-1,2}: R(A) > 0} w(A)R^2(A) \geq \left( \sum_{A \in \mathcal{O}_{\ell-L_0-1,2}: R(A) > 0} w(A)R(A) \right)^2 = (R_{\mathcal{D}_n,\ell-L_0-1}^+)^2. \quad (3.7)$$
In conclusion, (3.10) and (3.11) together imply that

\[ \left. \begin{array}{l}
R_{D_n, L_0} \leq R_{D_n, L_0-1} - \frac{1}{\|g\|^2_{TV}} (R^+_{D_n, L_0-1})^2 \\
\leq R_{D_n, L_0-1} - \frac{1}{\|g\|^2_{TV}} R^2_{D_n, L_0-1}.
\end{array} \right. \]  

(3.8)

where the last equation (3.8) holds because \( R^+_{D_n, L_0-1} > R_{D_n, L_0-1} \) and \( R_{D_n, L_0-1} \geq 0 \) by assumption. Again, using the same arguments above, (3.8) also implies

\[ R_{D_n, \ell} \leq R_{D_n, \ell-1} - \frac{1}{\|g\|^2_{TV}} R^2_{D_n, \ell-1} \]  

(3.9)

for any integer \( 1 \leq \ell \leq L_0 \). In conclusion, the fact that \( R_{D_n, 1} \leq R_{D_n, 0} - \frac{1}{\|g\|^2_{TV}} R^2_{D_n, 0} \leq \|g\|^2_{TV}/4 \) and (3.9) implies that (3.4) holds by mathematical induction.

Note that

\[ \|Y - m_{n, L}(X)\|_n^2 \leq \|Y - m_{n, L_0}(X)\|_n^2. \]

Therefore, the first part on RHS of (3.2) satisfies, for any \( g \in \text{Ridge}_f \),

\[ 2E_{D_n} \left( \frac{1}{n} \sum_{i=1}^{n} |m_{n, L}(X_i) - Y_i|^2 - \frac{1}{n} \sum_{i=1}^{n} |m(X_i) - Y_i|^2 \right) \leq 2E_{D_n} (\|Y - m_{n, L}(X)\|_n^2 - \|Y - g(X)\|_n^2) \]

\[ \hspace{5cm} + 2E_{D_n} (\|Y - g(X)\|_n^2 - \|Y - m(X)\|_n^2) \]

\[ \leq \frac{2\|g\|^2_{TV}}{\log_2 n} + 2E (g(X) - m(x))^2 \]

\[ \leq \frac{2\|g\|^2_{TV}}{\log_2 n} + 2 \max_x (g(x) - m(x))^2 . \]  

(3.10)

By Pinkus (1999), there is a series of \( g_j \in \text{Ridge}_f, J \geq 1 \) satisfying

\[ \lim_{J \to \infty} \max_{x} |g_j(x) - m(x)| = 0. \]  

(3.11)

In conclusion, (3.10) and (3.11) together imply that

\[ 2E_{D_n} \left( \frac{1}{n} \sum_{i=1}^{n} |m_{n, L}(X_i) - Y_i|^2 - \frac{1}{n} \sum_{i=1}^{n} |m(X_i) - Y_i|^2 \right) \to 0 \]  

(3.12)

as \( t_n \to \infty \), which completes the proof of Part I.

**Part II:** Now we consider the second part of the RHS of (3.2) by applying Lemma 3.6. Recall that \( \mathcal{H}_n \) consists of real functions \( h : [0, 1]^p \to \mathbb{R} \) that are constant on each \( A^f_{L_j}, j = 1, \ldots, t_n \) obtained in Algorithm 1. Define a Boolean class of functions:

\[ \mathcal{F}_n = \{ \text{sgn}(f(x, y)) : f(x, y) = h(x) - y, h \in \mathcal{H}_n \}. \]


Consistency of Oblique Decision Tree

where \( sgn(v) = 1 \) if \( v \geq 0 \) and \( sgn(v) = -1 \) otherwise and \( \mathcal{H}_{tn} \) is defined below Lemma 3.6. Recall that VC dimension of \( \mathcal{F}_{tn} \), denoted by \( VC(\mathcal{F}_{tn}) \), is the largest integer \( m \in \mathbb{Z}_+ \) satisfying \( 2^m \leq \Pi_{\mathcal{F}_{tn}}(m) \) (see, for example, Kosorok (2008)). Therefore, we focus on bounding \( \Pi_{\mathcal{F}_{tn}}(m) \) for each positive integer \( m \in \mathbb{Z}_+ \). Let \( z_1, \ldots, z_m \in \mathbb{R}^p \) be the series of points which maximize \( \Pi_{\mathcal{F}_{tn}}(m) \). Under the above notations, we have two observations as follows.

- For any \( h_{tn} \in \mathcal{H}_{tn} \) that is constant on each \( \mathbb{A}_L^j, j = 1, \ldots, t_n \), there is \( h_{tn-1} \in \mathcal{H}_{tn-1} \) and a leaf \( \mathbb{A}_L \) of corresponding \( T_{2m, t_n-1} \) such that \( \mathbb{A} = \mathbb{A}_L^k \cup \mathbb{A}_L^{k+1} \) for some \( k \) and \( h_{tn-1} \) is constant on \( \mathbb{A}_L^j, j = 1, \ldots, k-1, k+2, \ldots, t_n \) and \( \mathbb{A} \).
- All half-planes in \( \mathbb{R}^p \) pick out at most \( (me/(p+1))^{p+1} \) subsets from \( \{z_1, \ldots, z_m\} \) when \( m \geq p+1 \) (see, e.g., Kosorok (2008)), namely

\[
\text{Card}(\{z_1, \ldots, z_m\} \cap \{x \in \mathbb{R}^p : \theta^T x \leq s\} : \theta \in \Theta^p, s \in \mathbb{R}) \leq (me/(p+1))^{p+1}.
\]

Based on the above two facts, we can conclude

\[
\Pi_{\mathcal{F}_{tn}}(m) \leq \Pi_{\mathcal{F}_{tn-1}}(m) \cdot \left( \frac{me}{p+1} \right)^{p+1}.
\]

(3.13)

Then, combination of (3.13) and \( \Pi_{\mathcal{F}_1}(m) \leq \left( \frac{me}{p+1} \right)^{p+1} \) implies that

\[
\Pi_{\mathcal{F}_{tn}}(m) \leq \left( \frac{me}{p+1} \right)^{t_n \cdot p + t_n}.
\]

(3.14)

Solving the inequality

\[
2^m \leq \left( \frac{me}{p+1} \right)^{t_n \cdot p + t_n}
\]

by using the basic inequality \( \ln x \leq \gamma \cdot x - \ln \gamma - 1 \) with \( x, \gamma > 0 \) yields

\[
VC(\mathcal{H}_{tn}) \leq \frac{4}{\ln 2} \cdot p(t_n + 1) \ln (2p(t_n + 1)) \leq c(p) \cdot t_n \ln(t_n),
\]

(3.15)

where the constant \( c(p) \) depends on \( p \) only. Then, by Lemma 3.6 we have

\[
N_1(1/(80n\beta_n), \mathcal{H}_{tn}^{\beta_n}, z_1^n) \leq 3 \left( \frac{4e\beta_n}{1/(80n\beta_n)} \ln \left( \frac{6e\beta_n}{1/(80n\beta_n)} \right) \right)^{VC(\mathcal{H}_{tn}^{\beta_n})} \leq 3 \left( \frac{4e\beta_n}{1/(80n\beta_n)} \ln \left( \frac{6e\beta_n}{1/(80n\beta_n)} \right) \right)^{-t_n \ln(t_n)} \leq 3 \left( \frac{480e\beta_n^2}{1/(80n\beta_n)} \right)^{c \cdot t_n \ln(t_n)} \leq 3 \left( 480e\beta_n^2 \right)^{c \cdot t_n \ln(t_n)}
\]

(3.16)

for any \( z_1, \ldots, z_n \in \mathbb{R}^p \). Inequality (3.16) implies that the second part of RHS of (3.2) satisfies

\[
\frac{c \ln^2 n}{n} \sup_{z_1^n} \ln \left( N_1(1/(80n\beta_n), \mathcal{H}_{tn}^{\beta_n}, z_1^n) \right) \to 0, \text{ as } n \to \infty,
\]

(3.17)
In conclusion, combination of (3.19), (3.18) and (3.23) finishes arguments for Part II.

**Part III:** By Lemma 3.4, (3.12) and (3.17) imply that

\[
IV := 2E_{D_n} \int |\hat{m}_{n,L}(x) - m(x)|^2 d\mu(x) \to 0.
\]

(3.18)

Finally, we show that (3.18) also holds for the un-truncated estimator \( m_{n,L}(x) \). Note that

\[
E_{D_n} \int |m_{n,L}(x) - m(x)|^2 d\mu(x) \leq 2E \int |\hat{m}_{n,L}(x) - m_{n,L}(x)|^2 d\mu(x) + IV \\
\leq 2E \int |\hat{m}_{n,L}(x) - m_{n,L}(x)|^2 I(|m_{n,L}(x)| \geq \beta_n) d\mu(x) + IV \\
\leq 2E_{D_n} \left( E \left( |m_{n,L}(X)|^2 : I(|m_{n,L}(X)| > \beta_n) | D_n \right) \right) + IV \\
:= V + IV.
\]

(3.19)

Since \( \max_{x \in [0,1]} |m_{n,L}(x)| \leq \|m\|_\infty + \max_{1 \leq i \leq n} |\epsilon_i| \), where \( \epsilon_i = Y_i - m(X_i), 1 \leq i \leq n \) are i.i.d. and share a common sub-Gaussian distribution, we have

\[
V \leq 2E_{D_n} \left[ \left( 2\|m\|_\infty^2 + 2 \max_{1 \leq i \leq n} |\epsilon_i|^2 \right) I \left( \max_{1 \leq i \leq n} |\epsilon_i| \geq \beta_n - \|m\|_\infty \right) \right] \\
\leq 2E_{D_n} \left[ \left( 2\|m\|_\infty^2 + 2 \max_{1 \leq i \leq n} |\epsilon_i|^2 \right) I \left( \max_{1 \leq i \leq n} |\epsilon_i| \geq c \cdot \ln n \right) \right] \\
\leq 2\|m\|_\infty^2 \cdot P \left( \max_{1 \leq i \leq n} |\epsilon_i| \geq c \cdot \ln n \right) + 2 \left( E \left( \max_{1 \leq i \leq n} |\epsilon_i|^4 \right) \cdot P \left( \max_{1 \leq i \leq n} |\epsilon_i| \geq c \cdot \ln n \right) \right)^{\frac{1}{2}}.
\]

(3.20)

Note that

\[
P \left( \max_{1 \leq i \leq n} |\epsilon_i| > c \cdot \ln n \right) = 1 - P \left( \max_{1 \leq i \leq n} |\epsilon_i| \leq c \cdot \ln n \right) \\
= 1 - \left[ P \left( |\epsilon_1| \leq c \cdot \ln n \right) \right]^n \leq 1 - \left( 1 - e^{-c \cdot \ln^2 n} \right)^n \\
= 1 - e^{-n \ln(1 - e^{-c \cdot \ln^2 n})} \\
\leq -n \cdot \ln(1 - e^{-c \cdot \ln^2 n}) \\
\leq c \cdot n \cdot e^{-c \cdot \ln^2 n},
\]

(3.21)

where (3.21) is obtained from the fact \( 1 + v \leq e^v, v \in \mathbb{R} \); and (3.22) is due to the fact \( \lim_{v \to 0} \frac{\ln(1+v)}{v} = 1 \).

By (3.22) and \( E \left( \max_{1 \leq i \leq n} |\epsilon_i|^4 \right) \leq n \cdot E(\epsilon_i^4) \), we have

\[
V \to 0.
\]

(3.23)

In conclusion, combination of (3.19), (3.18) and (3.23) finishes arguments for Part III. \( \square \)
For binary classification problems, where \( Y \) only takes 0 or 1, we can still use ODT in which the Gini impurity is usually employed to divide each internal leaf defined by

\[
\Delta^c_A(\theta, s) = -2 \sum_{k=1}^{2} P^2(k|A) + P(A^+_\theta, s) \sum_{k=1}^{2} P^2(k|A^+_\theta, s) + P(A^-_{\theta, s}) \sum_{k=1}^{2} P^2(k|A^-_{\theta, s}),
\]

where \( P(k|A) \) denotes the proportion of class \( k \) in \( A \) and \( k \in \{1, 2\} \). For the oblique classification tree, we find Algorithm 1 still works after changing \( \Delta_A(\theta, s) \) in line 13 to \( \Delta^c_A(\theta, s) \). Then by voting, the estimated class of input \( x \in [0, 1]^p \) is given by

\[
\hat{C}_{n,t,n}(x) = \begin{cases} 
1, & m_{n,L}(x) \geq 0.5, \\
0, & m_{n,L}(x) < 0.5.
\end{cases}
\] (3.24)

After some calculations, we have

\[
\Delta^c_A(\theta, s) = 2 \Delta_A(\theta, s)
\]

if \( Y \) only takes 0 or 1. Therefore, we know \( \hat{C}_{n,t,n}(x) \) is Bayes-optimal in the asymptotical sense by using Theorem 3.1 and Theorem 1.1 in Györfi et al. (2002).

**Corollary 1.** The misclassification probability of classifier \( \hat{C}_{n,t,n}(x) \) in (3.24) satisfies

\[
P(\hat{C}_{n,t,n}(X) \neq Y) \rightarrow \inf_{f: [0,1]^p \rightarrow \{0,1\}} P(f(X) \neq Y) \rightarrow 0 \text{ as } n \rightarrow \infty
\]

under conditions in Theorem 3.1.

Pruning is an important step in reducing the complexity of the tree model and the variance of the tree estimator. See, for example, Breiman (1984). For a regression tree, pruning is to select the best number of leaves by balancing the squared loss function and a penalty:

\[
t^*_n = \arg \min_{1 \leq \tau \leq n} \frac{1}{n} \sum_{i=1}^{n} (Y_i - m_{n,\tau,L}(X_i))^2 + \alpha_n \cdot \tau,
\]

where \( \alpha_n > 0 \) is a penalty parameter. Then, the pruned estimator in the regression is given by

\[
m_{pru,n}(x) := m_{n,t^*_n,L}(x).
\]

For classification tree, pruning is usually to select the best number of leaves by balancing 0–1 loss and a penalty:

\[
t^*_n = \arg \min_{1 \leq \tau \leq n} \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}(Y_i \neq \hat{C}_{n,\tau}(X_i)) + \alpha_n \cdot \tau,
\]

where \( \alpha_n > 0 \) is a given penalty parameter and \( \mathbb{I}(\cdot) \) denotes the indicator function. Then, the classifier after pruning is given by

\[
\hat{C}_{pru,n}(x) := \hat{C}_{n,t^*_n}(x).
\]
Lemma 3.7. Let $\beta_n = \ln n$. Assume $E(e^{-Y^2}) < \infty$ for some $c > 0$. Then, for any $1 \leq \tau \leq n$ and $g_J \in G_J$ we have

$$\begin{align*}
\mathbb{E}_{\mathcal{D}_n} \left( \int |\hat{m}_{pru,n}(x) - m(x)|^2 d\mu(x) \cdot 1(A_n) \right) \leq & 2\mathbb{E}_{\mathcal{D}_n} \left( \left( \|m_\tau(X) - \mathbb{V}\|_n^2 - \|g_J(X) - \mathbb{V}\|_n^2 \right) \cdot 1(A_n) \right) \\
& + c \cdot E \left( m(X) - g_J(X) \right)^2 + 2\tau \cdot \alpha_n + c \cdot \frac{\ln^6 n}{\alpha_n \cdot n} + c \cdot \frac{\ln n}{n},
\end{align*}$$

(3.25)

where $\hat{m}_{pru,n}(x) = \max \{ \min \{ m_{pru,n}(x), \beta_n \}, -\beta_n \}$, and $A_n = \{ \max_{1 \leq i \leq n} |Y_i| \leq \beta_n \}, c > 0$.

\textbf{Proof.} The proof is similar to the proof of (6.23) in Lemma 1 of Zhan, Zhang and Xia (2022). \hfill \Box

Theorem 3.8 (Consistency of ODT after pruning). Assume $E(e^{-Y^2}) < \infty$ for some $c > 0$ and $m(x)$ is a continuous function. If $\alpha_n = o(1)$ and $\ln^6 n/(\alpha_n \cdot n) = o(1)$, we have

$$\mathbb{E}_{\mathcal{D}_n} \left( \int |m_{pru,n}(x) - m(x)|^2 d\mu(x) \right) \rightarrow 0, \text{ as } n \rightarrow \infty.$$  

\textbf{Proof.} Let $A_n = \{ \max_{1 \leq i \leq n} |Y_i| \leq \beta_n \}$ and $m_{pru}(x) = \max \{ \min \{ m_{pru}(x), \beta_n \}, -\beta_n \}$, where $\beta_n = \ln n$. Note that

$$\begin{align*}
\mathbb{E}_{\mathcal{D}_n} \left( \int |m_{pru,n}(x) - m(x)|^2 d\mu(x) \right) \leq & 2\mathbb{E}_{\mathcal{D}_n} \left( \int |m_{pru,n}(x) - \hat{m}_{pru,n}(x)|^2 d\mu(x) \right) \\
& + 2\mathbb{E}_{\mathcal{D}_n} \left( \int |\hat{m}_{pru,n}(x) - m(x)|^2 d\mu(x) \cdot 1(A_n) \right) \\
& + 2\mathbb{E}_{\mathcal{D}_n} \left( \int |\hat{m}_{pru,n}(x) - m(x)|^2 d\mu(x) \cdot 1(A_n^c) \right) \\
:= & I + II + III.
\end{align*}$$

(3.26)

By similar arguments in the proof of $V \rightarrow 0$ in Part III of the proof of Theorem 3.1, it is not difficult to show that

$$I \rightarrow 0.$$  

(3.27)

Next, we consider $II$. For any $\epsilon > 0$, (3.11) shows that for some large $J$ there exists $g_J \in \text{Ridge}_J$ such that

$$c \cdot E \left( m(X) - g_J(X) \right)^2 \leq \frac{\epsilon}{3}. \hspace{1cm} (3.28)$$

Fix $g_J$ above and let $\tau_n = 1/\sqrt{\alpha_n}$ for each $n \in \mathbb{Z}_+$. Then, by (3.4) we know that there is a $N_1 \in \mathbb{Z}_+$ such that

$$2\mathbb{E} \left( \frac{\|Y - m_{\tau_n}(X)\|_n^2 - \|Y - g(x)\|_n^2}{\|Y - g(x)\|_n^2} \cdot 1(A_n) \right) \leq \frac{2\|g_J\|_V^2}{\log_2 \tau_n + 4} \leq \frac{\epsilon}{3} \hspace{1cm} (3.29)$$

$$2\mathbb{E} \left( \frac{\|Y - m_{\tau_n}(X)\|_n^2 - \|Y - g(x)\|_n^2}{\|Y - g(x)\|_n^2} \cdot 1(A_n^c) \right) \leq \frac{2\|g_J\|_V^2}{\log_2 \tau_n + 4} \leq \frac{\epsilon}{3} \hspace{1cm} (3.29)$$
for all \( n \geq N_1 \). We can also find \( N_2 \in \mathbb{Z}_+ \) such that
\[
2\tau \cdot a_n + c \frac{\ln^3 n}{a_n \cdot n} + c \frac{\ln n}{n} \leq \varepsilon
\] (3.30)
for all \( n \geq N_2 \). By Lemma 3.7, the combination of (3.28), (3.29) and (3.30) shows that
\[
II \to 0.
\] (3.31)
Finally, by \( \|\hat{m}_{pru,n}\|_\infty \leq \beta_n \) and \( \|m\|_\infty < \infty \) and (3.22), it is easy to know
\[
III \to 0.
\] (3.32)
In conclusion, the combination of (3.26), (3.27), (3.31) and (3.32) completes the proof. \( \square \)

### 4. Consistency of ODRF

In this section, we study the ODT-based random forest, hereafter called ODRF. For ease of exposition, we introduce several notations that are often used in this part. Let \( q \) be a random number randomly chosen with equal probability from \( \{1, \ldots, p\} \). Given \( q \), we suppose \( S \) is uniformly chosen from \( \Omega_q = \{C \subseteq \Omega : \text{Card}(C) = q\} \). The law of \( S \) is given as follows
\[
P(S) = \frac{1}{p} \cdot \frac{1}{\text{Card}(S)}, \quad \forall S \subseteq \Omega. \tag{4.1}
\]
Denote by \( S_\tau, \tau = 1, 2, \ldots \), a sequence of independent copies of \( S \). Therefore, \( S_\tau, \tau \geq 1 \), can be regarded as a (random) sample of \( S \). Let \( X_\tau \) be a sub-vector of \( X \) consisting of coordinates indexed by elements of \( S \). Let \( \Xi_{t-1} = \{S_1, \ldots, S_{t-1}\} \) with \( t_0 \geq 2 \), which can be regarded as a random element in the product of probability spaces, \( \Omega \otimes \Omega_{t_0-1} \).

Now we construct a single random ODT based on data \( D_n(\mathfrak{G}) \subseteq \mathbb{D}_n \) as follows. First, resample data \( D_n(\mathfrak{G}) := \{(X_i, Y_i) : i \in \mathfrak{G}\} \) without replacement from \( \mathbb{D}_n \), where \( \mathfrak{G} \) is randomly chosen satisfying \( \text{Card}(\mathfrak{G}) = a_n \) and \( a_n \) is a predefined size of subsample. Then, replace lines 13-14 in Algorithm 1 with the following steps:

- In the \( \tau \)-th division (\( 1 \leq \tau \leq t_n - 1 \)) of node A, randomly choose \( S_\tau \in \Omega \) with \( \text{Card}(S_\tau) = q \);
- \( \{\hat{\theta}_A, \delta_A\} = \arg\max_{\theta_{S_\tau}, s} \Delta_{A,q}(\theta_{S_\tau}, s) \), where \( \Delta_{A,q} \) is defined in the same way as \( \Delta_A \) except that only \( \{(X_i, S_\tau, Y_i) : i \in \mathfrak{G}\} \) are used in the calculation of \( \Delta_{A,q} \);
- Partition the node A into \( A^+_{\hat{\theta}_A, \delta_A} = \{x \in A : \hat{\theta}_A^T x_{S_\tau} \leq \delta_A\} \) and \( A^-_{\hat{\theta}_A, \delta_A} = \{x \in A : \hat{\theta}_A^T x_{S_\tau} > \delta_A\} \).

Denote by \( m^r_{n,a_n,t_n,\ell}(x) \) the output estimator of Algorithm 1 of above randomization; the corresponding tree \( T_{D_n(\mathfrak{G}),a_n,t_n,\ell} \) with \( \ell \) layers and \( t_n \) leaves is named as random ODT. Note that \( \ell \) is a random variable depending on both \( \{(X_i, Y_i) : i \in \mathfrak{G}\} \) and \( \{S_{\tau}\}_{\tau=1}^{t_n-1} \). For the proof of the consistency, we also need to consider a truncated random tree at a particular layer \( \ell : 0 \leq \ell \leq \ell' \), denoted by \( T^r_{D_n(\mathfrak{G}), a_n,t_n,\ell} \). For simplicity, we abbreviate \( T^r_{D_n(\mathfrak{G}), a_n,t_n,\ell} \) as \( T^r_{D_n(\mathfrak{G})} \) for each layer \( \ell \) in the following context and similarly \( m^r_{n,\ell}(x) \) is an abbreviation of \( m^r_{n,a_n,t_n,\ell}(x) \) if there is no confusion.

Under the above preparations, we next introduce how to construct a random forest based on ODT. First, we independently construct \( B \in \mathbb{Z}_+ \) random ODT’s and their corresponding estimators are denoted...
by \( m_{n,a_n,t_n,L}^{r,b}(x) \), \( b = 1, \ldots, B \). Following the idea of the classical Random Forest Breiman (2001a), the ODT-based Random Forest (ODRF) in regression is given by

\[
m_{ODRF,B,n}(x) := \frac{1}{B} \sum_{b=1}^{B} m_{n,a_n,t_n,L}^{r,b}(x).
\] (4.2)

It is worth noting that ODRF is very similar to Forest-RC in Breiman (2001a), except that the latter only selects the best splitting plane with fixed \( q \) variables from randomly generated hyperplanes.

We study the consistency of ODRF in two different schemes. In the first one, each tree of the forest is not fully grown, i.e., we follow a rule to stop growing a tree. Thus, each leaf might contain more than one observation. The estimation error of ODRF is controlled by restricting the divergent rate of \( t_n \), which goes to infinity in order to reduce the approximation error. For the first scheme, the proof of the consistency of ODRF is similar to the proof of a single tree. For the second scheme, each tree of ODRF is fully grown and each leaf of random ODT contains only one data point. This makes the proof difficult and different from the traditional analysis for learning algorithms. The key requirement is to assume that \( a_n \) is not large. We will use the technique Scornet, Biau and Vert (2015) for the proof.

### 4.1. ODRF with trees of not fully grown

The main theorem of this section is given below, which shows that ODRF is consistent in the first scheme for general \( m(x) \) under the sub-Gaussian assumption of \( Y \). For comparison, recall that the classical Random Forest is known to be consistent only for additive models.

**Theorem 4.1 (Consistency of ODRF with not fully grown trees).** Assume \( E(e^{c Y^2}) < \infty \), where \( c > 0 \) and \( m(x), x \in [0, 1]^p \) is continuous. If \( a_n \to \infty \) and \( t_n \to \infty \) and \( t_n = o\left(\frac{a_n}{\ln a_n}\right) \), we have

\[
E_{D_n,\xi_{n-1}} \left( \int |m_{ODRF,B,n}(x) - m(x)|^2 \, d\mu(x) \right) \to 0, \text{ as } n \to \infty.
\]

The outline of the proof of Theorem 4.1 is as follows. Without loss of generality, we can assume \( Card(\mathcal{B}) = n \) because there is no difference between \( a_n \to \infty \) and \( n \to \infty \) in the first scheme. First, we will prove that the truncated estimator \( \hat{m}_{n,L}^{t_n}(x) = \max \{ \min \{ m_{n,L}^r(x), \beta_n \}, -\beta_n \} \), where \( \beta_n = \ln n \), is mean squared consistent under the conditions in Theorem 4.1. Second, we give the consistency of the untruncated estimator \( m_{n,L}^r(x) \).

Following Lemma 3.4, it is not difficult to show that

\[
E_{D_n,\xi_{n-1}} \left( \int |\hat{m}_{n,L}^{t_n}(x) - m(x)|^2 \, d\mu(x) \right) \leq 2 \cdot E_{D_n,\xi_{n-1}} \left( \|m_{n,L}(X) - Y\|_n^2 + \|m(X) - Y\|_n^2 \right) + \frac{c \ln^2 n}{n} \cdot \sup_{\xi_1^n} \left( N_1(1/(80n\beta_n), \mathcal{H}_{\xi_1^n}, \xi_1^n) \right)
\] (4.3)

for some \( c > 0 \), where \( \mathcal{H}_{\xi_1^n} = \{ \max \{ \min \{ h, \beta_n \}, -\beta_n \} : h \in \mathcal{H}_{\xi_1^n} \} \) and \( \xi_1^n \in (\mathbb{R}^p)^{\otimes n} \); \( \mathcal{H}_{\xi_1^n} \) is defined in Part II of the proof of Theorem 3.1; \( N_1 \) is the cover number in Definition 3.3; and \( \mu \) is the distribution of \( X \). We prove Theorem 4.1 by showing that each part on the RHS of (4.3) converges to 0 as \( n \) goes to \( \infty \), similar to Part I and Part II in the proof of Theorem 3.1. In the process of bounding the first part of
the RHS of (4.3), we only provide the proof of a key inequality:

\[ \mathbf{E}_{Z_{m,n}}(\|Y - m_{n,L}(X)\|^2_n - \|Y - g_p(X)\|^2_n) \leq c(p) \cdot \frac{\|g_p\|^2_{TV}}{\log_2 t_n} + 4 \]  

(4.4)

for any \( t_n > 1 \) and real function \( g_p \in \mathcal{G}_{1,p,J} \) presented in Lemma 4.2 and some \( c(p) > 0 \), which is analogous to (3.4). Employing the ideas in Klusowski (2021), we will see that the proof of (4.4) relies on Lemma 4.2 below.

**Lemma 4.2.** Let \( A \) be any internal node of random ODT, \( T^r_{D_n,L} \) in the \( \tau \)-th partition. Define \( R(A) := \|Y - \bar{Y}_A\|^2_n - \|Y - g_A(v)\|^2_n \) for any \( g_A(v) = \sum_{k=1}^{V} g_{k,q}(v) \) where \( g_{k,q,x}(x) \in \text{Ridge}_{k,q,J} \) with

\[ \text{Ridge}_{k,q,J} = \left\{ \sum_{j=1}^{J} c_{k,q,j} \cdot \sigma(\theta^T_{C_k,j} x + d_{k,q,j}) : C_{k,q,j} \in \Omega_q, \theta_{C_k,j} \in \Theta^q, c_{k,q,j}, d_{k,q,j} \in \mathbb{R} \right\} \]

and \( \sigma(v) = e^v / (1 + e^v), v \in \mathbb{R} \). If \( R(A) \geq 0 \) given \( q = q_0 \), then we have

\[ \mathbf{E}_{S} (\Delta_{A,q}(\hat{\theta}_A, \hat{s}_A)|D_n, q = q_0) \geq c(p, q_0) \cdot \frac{R^2(A)}{\|g_{q_0}\|^2_{TV}} \cdot V, \]

where the expectation is taken over the random index set \( S \) and \( c(p, q_0) > 0 \). \( \Box \)

**Proof.** For simplicity, notations \( A^+ \) and \( A^- \) are used to denote \( A^+_{\hat{\theta}_A, \hat{s}_A} \) and \( A^-_{\hat{\theta}_A, \hat{s}_A} \), respectively when \( q = q_0 \). For each \( 1 \leq k \leq L \), let \( \Delta_A(\hat{\theta}_A, \hat{s}_A, x_{C_k,q_0}) \) be the value of impurity gain in (2.1) when variables \( x_{C_k,q_0} \) are only used in the calculation of \( \Delta_A(\hat{\theta}_A, \hat{s}_A) \), namely only \( x_{C_k,q_0} \) are employed to divide the node \( A \). Then, we have

\[ \mathbf{E}_{S} (\Delta_{A,q}(\hat{\theta}_A, \hat{s}_A)|D_n, q = q_0) \geq p_{q_0} \cdot \sum_{k=1}^{V} \Delta_A(\hat{\theta}_A, \hat{s}_A, x_{C_k,q_0}) \geq p_{q_0} \cdot \max_{1 \leq k \leq V} \Delta_A(\hat{\theta}_A, \hat{s}_A, x_{C_k,q_0}) \]

(4.5)

where \( p_{q_0} = 1 / p \cdot 1 \left( \frac{P}{q_0} \right) \) is the probability of each \( C_{k,q_0} \). Let \( g_{q_0}(x) = \sum_{k=1}^{V} g_{k,q_0}(\theta^T_{C_k,q_0} x) \) be the hyperplane \( \theta^T_{C_k,q_0} x = s \) is used to divide \( A \). Note that \( w_{k,q_0} \geq 0 \) and \( \int_{-1}^{1} w_{k,q_0}(s) ds \leq 1 \). Following similar arguments on page 19 of Klusowski (2021), it is not difficult to show that

\[ \max_{1 \leq k \leq V} \Delta_A(\hat{\theta}_A, \hat{s}_A, x_{C_k,q_0}) \geq \frac{\left\langle (Y - \bar{Y}_A, g_{C_k,q_0}(\theta^T_{C_k,q_0} x))_A \right\rangle^2}{\left( \sum_{k=1}^{V} \int_{-1}^{1} |g'_{k,q_0}(s)| \sqrt{P(A^+|C_k,q_0)} \cdot P(A^-|C_k,q_0) ds \right)^2} \]

(4.6)
for each $1 \leq k \leq V$. By taking average on $|\langle \mathcal{Y} - \hat{Y}_A, g_{k,q_0}(\theta^\ell_{C_k,q_0}) \rangle_A|^2$, $1 \leq k \leq V$, (4.6) implies

$$
\max_{1 \leq k \leq V} \Delta_A(\hat{\theta}_A, \hat{s}_A, x_{C_k,q_0}) \geq \frac{|\langle \mathcal{Y} - \hat{Y}_A, g_{q_0} \rangle_A|^2}{V \cdot \left( \sum_{k=1}^{V} \int_{-1}^{1} |g_{k,q_0}(s)| \sqrt{P(A^+|C_k,q_0)P(A^-|C_k,q_0)} ds \right)^2}.
$$

(4.7)

Note that

$$
\langle \mathcal{Y} - \hat{Y}_A, g_{q_0} \rangle_A = \langle \mathcal{Y} - \hat{Y}_A, \mathcal{Y} \rangle_A - \langle \mathcal{Y} - \hat{Y}_A, \mathcal{Y} - g_{q_0} \rangle_A
\geq \|\mathcal{Y} - \hat{Y}_A\|_A^2 - \|\mathcal{Y} - \hat{Y}_A\|_A \|\mathcal{Y} - g_{q_0}\|_A
\geq \|\mathcal{Y} - \hat{Y}_A\|_A^2 - \frac{1}{2} \left( \|\mathcal{Y} - \hat{Y}_A\|_A^2 + \|\mathcal{Y} - g_{q_0}\|_A^2 \right),
$$

(4.8)

where the Cauchy-Schwarz inequality is used in the second line. Therefore, (4.7) and (4.8) and the fact that $R(A) \geq 0$ by the assumption of this lemma imply

$$
\max_{1 \leq k \leq V} \Delta_A(\hat{\theta}_A, \hat{s}_A, x_{C_k,q_0}) \geq \frac{R^2(A)}{4V \cdot \left( \sum_{k=1}^{V} \int_{-1}^{1} |g_{k,q_0}(s)| \sqrt{P(A^+|C_k,q_0)P(A^-|C_k,q_0)} ds \right)^2}.
$$

(4.9)

Next, we only need to consider how to bound the denominator of the RHS of (4.9). In fact, following similar arguments on page 20 of Klusowski (2021), it is easy to get a bound by using the total variation of each $g_{k,q_0}, 1 \leq k \leq V$:

$$
\sum_{k=1}^{V} \int_{-1}^{1} |g_{k,q_0}(s)| \sqrt{P(A^+|C_k,q_0)P(A^-|C_k,q_0)} ds \leq \frac{1}{2} \|g_{q_0}\|_{TV}.
$$

(4.10)

Therefore, combination of (4.5) and (4.9) and (4.10) completes the proof.

\[ \square \]

**Proof of Theorem 4.1.** Recall two facts. First, in the proof of (4.4), $D_n$ is given, and so the notation of the conditional expectation or probability is actually w.r.t. $D_n$, but we omit $D_n$ for simplicity. Second, for any internal node $A$ of $T_{D_n,\ell}^r$, $\Delta_{A,q}(\hat{\theta}_A, \hat{s}_A)$ only depends on $S_\ell$ for some $1 \leq \lambda \leq t_n - 1$ once data $D_n$ is given.

It is easy to observe the following two facts:

- Regardless of the choice of each $S_\tau, 1 \leq \tau \leq t_n - 1$, the level $L$ of any random ODT, $T_{D_n,\ell,n}^r$, must not be less than $L_0 = \lfloor \log_2 t_n \rfloor$;
- the tree $T_{D_n,\ell}^r$ is fully grown for each $0 \leq \ell \leq L_0$, namely $T_{D_n,\ell}^r$ is obtained recursively by splitting all leaves of $T_{D_n,\ell-1}^r$ until all leaves contain only one data point.

Recall that $L$ is a random variable depending on both $D_n$ and $S_\tau, 1 \leq \tau \leq t_n - 1$, and that $m_{n,\ell}^r(x)$ is the estimator obtained using the leaves of $T_{D_n,\ell}^r$. Given $D_n$, define the expectation of $\|\mathcal{Y} - m_{n,\ell}^r(X)\|_n^2$ over random $\theta$'s by

$$
E_{\ell} \left( \|\mathcal{Y} - m_{n,\ell}^r(X)\|_n^2 \right) := \sum_{T_{D_n,\ell}^r} P_{\theta}(T_{D_n,\ell}^r) \cdot \|\mathcal{Y} - m_{n,\ell}^r(X)\|_n^2.
$$

(4.11)

where $P_{\theta}(T_{D_n,\ell}^r)$ is the probability of a realisation of $S_\ell$, each corresponding to a partition of an internal node of $T_{D_n,\ell}^r$. Using the fact that $\|\mathcal{Y} - m_{n,\ell}^r(X)\|_n^2$ is almost certainly a decreasing sequence as
Theorem 3.1, we have $R_{D_n, \ell} = R_{D_n, \ell-1} - \sum_{A \in \mathcal{O}_{\ell-1,2}} w(A) \cdot \Delta_{A, q}(\hat{\theta}_A, \hat{s}_A)$, (4.14)

where $w(A) = \frac{1}{n} \cdot \text{Card}(\{X_i \in A : i = 1, \ldots, n\})$ as defined above. Given the tree $T_{D_n, \ell-1}$, define the conditional expectation on $Q_{\ell-2}$ by

$E_{Q_{\ell-2}} \left( R_{D_n, \ell} | T_{D_n, \ell-1} \right) = R_{D_n, \ell-1} - \sum_{A \in \mathcal{O}_{\ell-1,2}} w(A) \cdot E_{S^A} \left( \Delta_{A, q}(\hat{\theta}_A, \hat{s}_A) \right)$,

(4.16)

where random index $S^A \in \{S_r\}^n_{r=1}$ corresponds to the partition of $A$. Note that leaves $Q_{\ell-2}$ are not randomized once the tree $T_{D_n, \ell-1}$ is given.

Now, we are ready to prove (4.15). By (4.11) and (4.15), it is easy to check

$E_{\ell} \left( R_{D_n, \ell} \right) = \sum_{T_{D_n, \ell}} P_{\theta} \left( T_{D_n, \ell} \right) \cdot E_{Q_{\ell-2}} \left( R_{D_n, \ell} | T_{D_n, \ell-1} \right)$

$= E_{\mathbb{E}_n} \left( E_{Q_{\ell-2}} \left( R_{D_n, \ell} | T_{D_n, \ell-1} \right) \right)$,

(4.17)

which is similar to the law of iterated expectations. We highlight the fact that both $T_{D_n, \ell-1}$ and $Q_{\ell-2}$ are random, so (4.17) is not a trivial result from the classical law of iterated expectations. Then, from (4.16) and Lemma 4.2, we have

$E_{Q_{\ell-2}} \left( R_{D_n, \ell} | T_{D_n, \ell-1} \right) = R_{D_n, \ell-1} - \sum_{A \in \mathcal{O}_{\ell-1,2}} w(A) \cdot E_{S^A} \left( \Delta_{A, q}(\hat{\theta}_A, \hat{s}_A) \right)$

$\leq R_{D_n, \ell-1} - \sum_{A \in \mathcal{O}_{\ell-1,2}} \frac{w(A)}{p} \cdot E_{S^A} \left( \Delta_{A, q}(\hat{\theta}_A, \hat{s}_A) \right) | q = p$
where \( R(A) = \| y - \tilde{f}_A \|^2_A - \| y - g_p \|^2_A \) and \( c(p) > 0 \). Decompose \( R_{D_n, \ell-1} \) into two parts
\[
R^+_{D_n, \ell-1} := \sum_{A \in O_{\ell-1}; R(A) > 0} w(A) R(A) \quad \text{and} \quad R^-_{D_n, \ell-1} := \sum_{A \in O_{\ell-1}; R(A) \leq 0} w(A) R(A)
\]
satisfying \( R_{D_n, \ell-1} = R^+_{D_n, \ell-1} + R^-_{D_n, \ell-1} \). By Jensen’s inequality and the fact that \( R(A) \leq 0 \) for any leaf \( A \) of \( T_{D_n, \ell-1} \) which contains only one data point, we have
\[
\sum_{A \in O_{\ell-1}; R(A) > 0} w(A) \cdot R^2(A) \geq \left( \sum_{A \in O_{\ell-1}; R(A) > 0} w(A) \cdot R(A) \right)^2 = (R^+_{D_n, \ell-1})^2. \tag{4.19}
\]
Therefore, if \( R_{D_n, \ell-1} \geq 0 \) then the combination of (4.18) and (4.19) implies that
\[
E_{Q_{\ell-1,2}} \left( R_{D_n, \ell| T^p_{D_n, \ell-1}} \right) \leq R_{D_n, \ell-1} - \frac{c(p)}{\| g_p \|^2_T} \cdot (R^+_{D_n, \ell-1})^2 \leq R_{D_n, \ell-1} - \frac{c(p)}{\| g_p \|^2_T} \cdot R^2_{D_n, \ell-1}, \tag{4.20}
\]
where (4.20) is from \( R^+_{D_n, \ell-1} > R_{D_n, \ell-1} \geq 0 \). In conclusion, (4.20) implies that
\[
E_{Q_{\ell-1,2}} \left( R_{D_n, \ell| T^p_{D_n, \ell-1}} \right) \leq R_{D_n, \ell-1} - \frac{c(p)}{\| g_p \|^2_T} \cdot \max \{ R_{D_n, \ell-1}, 0 \}^2 \tag{4.21}
\]
because \( E_{Q_{\ell-1,2}} \left( R_{D_n, \ell| T^p_{D_n, \ell-1}} \right) \leq R_{D_n, \ell-1} \) holds no matter what the sign of \( R_{D_n, \ell-1} \) is. By (4.17) and Jensen’s inequality again, taking expectation over \( \Xi_n \) on both sides of (4.21) yields
\[
E_{\ell}(R_{D_n, \ell}) \leq E_{\ell-1}(R_{D_n, \ell}) - \frac{c(p)}{\| g_p \|^2_T} \cdot E_{\ell}(\max \{ R_{D_n, \ell-1}, 0 \})^2 \leq E_{\ell-1}(R_{D_n, \ell}) - \frac{c(p)}{\| g_p \|^2_T} \cdot \left[ E_{\ell}(\max \{ R_{D_n, \ell-1}, 0 \}) \right]^2 \tag{4.22}
\]
for each \( 1 \leq \ell \leq \mathcal{L}_0 \). Finally, we complete the proof of (4.13) by considering the sign of \( E_{\ell-1}(R_{D_n, \ell-1}) \) in the following two cases.

**Case 1:** There is an \( \ell_0 \), with \( 1 \leq \ell_0 \leq \mathcal{L}_0 \), such that \( E_{\ell_0-1}(R_{D_n, \ell_0-1}) \leq 0 \). By checking (4.15), it is easy to know that
\[
E_{Q_{\ell-1,2}} \left( R_{D_n, \ell| T^p_{D_n, \ell-1}} \right) \leq R_{D_n, \ell-1}. \tag{4.23}
\]
By using (4.17), taking expectation over \( \Xi_n \) on both sides of (4.23) implies that
\[
E_{\ell}(R_{D_n, \ell}) \leq E_{\ell-1}(R_{D_n, \ell-1})
\]
for all $1 \leq \ell \leq L_0$. Therefore, we have

$$E_{\mathcal{L}_0}(R_{\mathcal{D}_n, \mathcal{L}_0}) \leq E_{\ell_0-1}(R_{\mathcal{D}_n, \ell_0-1}) \leq c(p) \cdot \frac{\|g_p\|^2_{TV}}{\log 2n + 4} \quad (4.24)$$

by the assumption of this case.

**Case 2:** For each $1 \leq \ell \leq L_0$, we have $E_{\ell-1}(R_{\mathcal{D}_n, \ell-1}) > 0$. In this case, (4.22) implies that

$$E_{\ell}(R_{\mathcal{D}_n, \ell}) \leq E_{\ell-1}(R_{\mathcal{D}_n, \ell-1}) - \frac{c(p)}{\|g_p\|^2_{TV}} \cdot [E_{\ell-1}(R_{\mathcal{D}_n, \ell-1})]^2 \quad (4.25)$$

for all $1 \leq \ell \leq L_0$ because $E_{\ell_0} (\max \{R_{\mathcal{D}_n, \ell-1}, 0\}) \geq E_{\ell-1}(R_{\mathcal{D}_n, \ell-1}) > 0$. When $\ell = 1$, by (4.25) it is easy to know that $E_1(R_{\mathcal{D}_n, 1}) \leq c(p) \cdot \frac{\|g_p\|^2_{TV}}{4}$. Using the above initial condition and (4.25) again, we also have, by mathematical induction,

$$E_{\mathcal{L}_0}(R_{\mathcal{D}_n, \mathcal{L}_0}) \leq c(p) \cdot \frac{\|g_p\|^2_{TV}}{\log 2n + 4}. \quad (4.26)$$

In conclusion, (4.13) follows (4.24) and (4.26), which implies that inequality (4.4) is also true. By (4.3) and following Parts I and II in the proof of Theorem 4.1, it is easy to know that $\hat{m}^r_{\ell, n}(x)$ is mean squared consistent. Then following the same arguments in Part III of that proof, we can also prove

$$E_{\mathcal{D}_n, \mathcal{L}_{n-1}} \int |m_{\mathcal{D}_n, \ell}(x) - m(x)|^2 d\mu(x) \to 0 \quad \text{as } n \to 0. \quad (4.27)$$

Finally, by the Jensen’s inequality, we have

$$E_{\mathcal{D}_n, \mathcal{L}_{n-1}} \int |m_{\mathcal{O}D\mathcal{R}F, B, n}(x) - m(x)|^2 d\mu(x) \leq E_{\mathcal{D}_n, \mathcal{L}_{n-1}} \int |m_{\mathcal{D}_n, \ell}(x) - m(x)|^2 d\mu(x)$$

shows that (4.27) also holds for the estimator $m_{\mathcal{O}D\mathcal{R}F, B, n}(x), x \in [0, 1]^p$ obtained by ODRF.

In Theorem 4.1, the number of features in the linear combination is randomly selected from 1 to $p$. Next, we introduce a simplified ODRF with fixed $q$ (ODRF$_q$), namely $\theta_r$ is only uniformly chosen from $\boldsymbol{\Omega}_q$ each time. Its corresponding tree is similarly called random ODT$_q$. In this case, we show below that estimator $m_{\mathcal{O}D\mathcal{R}F_q, n}$ corresponding to ODRF$_q$ is consistent for an extended additive model:

$$m(x) = \sum_{\tau=1}^V m_\tau(x_{C_\tau}), \quad (4.28)$$

where $x_{C_\tau}$ is defined similarly to $x_{A_\tau}$ defined above and each $m_\tau$ is a function of only $q$ variables $x_{C_\tau}$, with $\text{Card}(C_\tau) = q$. Note that $C_\tau, 1 \leq \tau \leq V$, are fixed indexes and the maximum terms $V$ in model (4.28) is $\binom{p}{q}$.

**Corollary 2 (Consistency of ODRF$_q$).** Assume $E(e^{cY^2}) < \infty$, where $c > 0$ and $m(x), x \in [0, 1]^p$, follows model (4.28) with each continuous $m_\tau(x), 1 \leq \tau \leq V$. If $a_n \to \infty$, $t_n \to \infty$ and $\tau_n = o\left(\frac{\ln a_n}{a_n}\right)$,

we have

$$E_{\mathcal{D}_n, \mathcal{L}_{n-1}} \int |m_{\mathcal{O}D\mathcal{R}F_q, B, n}(x) - m(x)|^2 d\mu(x) \to 0, \quad \text{as } n \to \infty.$$
As a summary of Theorem 4.1 and Corollary 2, if the subset size of the features $q$ is randomly selected from 1 to $p$ in each splitting, then the estimator is consistent for any continuous underlying regression function; if $q$ is fixed, the estimator is consistent only if the underlying regression function has a special structure. As a special case, $q = 1$ corresponds to the traditional RF, the estimator is consistent for additive models.

4.2. ODRF with fully grown trees

Note that in the ODRF with fully grown trees, each leaf only contains one data point. Using the notations in Scornet, Biau and Vert (2015), $a_n$ is the number of bootstrapped data points used to construct each tree and $\Theta_1, \ldots, \Theta_B$ are independent random variables sharing with the same distribution. In fact, $\Theta_j(j = 1, \ldots, B)$ is designed to resample $a_n$ data points in the construction of the $j$-th tree and select $q$ variables in each node splitting of that tree. To save space, we only consider the first way for feature bagging, where $q$ is random and (4.1) is applied. Let $\Theta$ be the population version of $\Theta_1, \ldots, \Theta_B$. Let $Z_i = \mathbb{1}_{X \in X_i}$ be the indicator that both $X_i$ and $X$ are in the same partition in the random ODT generated by $D_n$ and $\Theta$. Following the same way, we have $Z'_i = \mathbb{1}_{X \in X'_i}$, where $X'$ is an independent copy of $X$.

We also define

$$
\psi_{i,j}(Y_i, Y_j) = \mathbb{E}(Z_i Z'_j | X, \Theta, \Theta', X_1, \ldots, X_n, Y_i, Y_j)
$$

and

$$
\psi_{i,j} = \mathbb{E}(Z_i Z'_j | X, \Theta, \Theta', X_1, \ldots, X_n).
$$

Finally, for any random variables $W_1, W_2, Z$, notation $\text{Corr}(W_1 W_2 | Z)$ denotes the conditional correlation coefficient. Similar to Scornet, Biau and Vert (2015), we need the following assumptions in the consistency analysis of ODRF with fully grown trees. Define $Z_{i,j} = (Z_i, Z'_j)$ and assume that one of the two assumptions below is satisfied.

A.1 There exist $0 < \delta < 1$ and $c > 0$ such that,

$$
\mathbb{E}^{\frac{1}{2}}(\max_{i,j \neq j} |\psi_{i,j}(Y_i, Y_j) - \psi_{i,j}|^2) \leq c \left( \frac{1}{a_n} \right)^{2+\delta}.
$$

A.2 There is a constant $c > 0$ and a sequence $\gamma_n \rightarrow 0$ such that,

$$
\max_{t_1, t_2 = 0, 1} |\text{Corr}(Y_i - m(X_i), \mathbb{1}_{Z_{i,j} = (t_1, t_2)} | X_i, X_j, Y_j)| \leq \gamma_n
$$

and

$$
\max_{t_1 = 0, 1} |\text{Corr}((Y_i - m(X_i))^2, \mathbb{1}_{Z_i = t_1} | X_i)| \leq c.
$$

Assumption A.2 is similar to (H2.2) in Scornet, Biau and Vert (2015), while the difference between assumption A.1 and (H2.1) in Scornet, Biau and Vert (2015) is that we further assume the convergence rate for $\max_{i \neq j} |\psi_{i,j}(Y_i, Y_j) - \psi_{i,j}|$. This change is due to the observation that ODRF has more complicated partitions than those of RF in Breiman (2001a). Therefore, a slightly stronger assumption is required when dealing with the estimation error for ODRF with fully grown trees. Especially, in the case where partitions are independent of $Y_1, Y_2, \ldots, Y_n$, our assumption A.1 is satisfied.
In order to analyze the consistency of ODRF, the condition \( B \to \infty \) is necessary in the second scheme. If \( B \) is finite as \( n \to \infty \), the consistency is hardly to be guaranteed. For example, we can set \( B = 1 \), which is just the case of a single tree with full growth. In the last section, we show that \( t_n = o(a_n) \) is an important condition for the consistency of random ODT. As pointed out by Györfi et al. (2002), standard conditions for tree consistency also require that the leaf contains data points that go to infinity as \( n \to \infty \). Therefore, in this section we allow \( B \) to depend on \( n \), written by \( B_n \). In fact, this second scheme is more interesting because we will show that the ensemble of \( B_n \) inconsistent trees can indeed estimate \( m(x), x \in [0, 1]^p \) well.

**Theorem 4.3 (Consistency of ODRF with fully grown trees).** Assume that \( X \) follows uniform distribution in \([0, 1]^p\) and \( ε = Y - m(X) \) follows standard normal distribution and \( X, ε \) are independent. Suppose \( m(x), x \in [0, 1]^p \) is continuous and assumption A.1 or A.2 holds. If \( t_n = a_n, t_n \to \infty, a_n \ln n/n \to 0 \) and \( B_n/\ln n \to \infty \), then

\[
\lim_{n \to \infty} E(m_{ODRF, B_n,n}(X) - m(X))^2 = 0.
\]

In the rest of this section, we prove Theorem 4.3 based on the inequality below:

\[
E(m_{ODRF, B_n,n}(X) - m(X))^2 = E(m_{ODRF, B_n,n}(X) - E_{θ}(m_{θ,a_n}(X))) + E_{θ}(m_{θ,a_n}(X)) - m(X))^2
\]

\[
\leq 2E(m_{ODRF, B_n,n}(X) - E_{θ}(m_{θ,a_n}(X)))^2
\]

\[
+ 2E(E_{θ}(m_{θ,a_n}(X)) - m(X))^2 := 2(\text{Part}_{n,1} + \text{Part}_{n,2}),
\]

where \( m_{θ,a_n}(X) := m_{θ,a_n,a_n,L}(X) \) is the abbreviation.

The term \( \text{Part}_{n,1} \) is bounded as follows. Let \( Z_{i,n} := m_{θ,a_n,a_n,L}(X) \). Then, we have two observations below. Firstly, \( Z_{i,n}, i = 1, \ldots, B_n \) are i.i.d given \( D_n \) and \( X \). Secondly, \( |Z_{i,n}| \leq \max_{1 \leq i \leq n} |ε_i| + \|m\|_∞ \). The above two pieces of observations lead that

\[
\text{Part}_{n,1} = E E[(m_{ODRF, B_n,n}(X) - E_{θ}(m_{θ,a_n}(X)))^2 |D_n, X]
\]

\[
= EE \left[ \frac{1}{B_n} \sum_{b=1}^{B_n} (Z_{b,n} - E_{θ_b}(Z_{b,n}))^2 |D_n, X \right]
\]

\[
= \frac{1}{B_n} \cdot E E[(Z_{1,n} - E_{θ}(Z_{1,n}))^2 |D_n, X]
\]

\[
\leq \frac{4}{B_n} \cdot \left( E(\max_{1 \leq i \leq n} |ε_i|^2) + \|m\|_∞^2 \right).
\]

(4.29)

By \( E(\max_{1 \leq i \leq n} |ε_i|^2) \leq c \cdot \ln n \) and (4.29), we know

\[
\text{Part}_{n,1} \leq \frac{4}{B_n} \cdot (\ln n + \|m\|_∞^2),
\]

which goes to 0 as \( n \to \infty \) under our assumptions.

Therefore, it is sufficient to prove the \( L^2 \) consistency for \( E_{θ}(m_{θ,a_n}(X)) \), namely \( \text{Part}_{n,2} \to 0 \). Here, we follow the path of Theorem 2 in Scornet, Biau and Vert (2015). For any node \( A \), define the theoretical
Lemma 4.4. Let function $m$ that follows an additive model here and $m$ is a probability measure for node $A$ which takes the form \( \{ x \in A : \theta^T x \leq s \} \). Such calculation is based on the condition that $\Delta_A(\theta, s) \equiv 0$ for any $\theta$ and $s$.

**Proof.** Without loss of generality, we assume that $m(x)$ is always positive on $A$. Otherwise, one can replace $m(x)$ by the function $m(x) + \sup_{x \in [0, 1]^p} |m(x)| + 1$. For any Lebesgue measurable set $A \subseteq A$, we define a function

$$G(A) := \frac{\int_A m(x)dx}{\int_A m(x)dx}.$$  

Since $\int_A m(x)dx > 0$, $G(A)$ is well defined. It is easy to check that $G(A)$ is a probability measure on the measurable space \((A, \mathcal{B}(A))\), where $\mathcal{B}(A)$ is the class of Lebesgue sets in $A$. Next we calculate $\int_A m(x)dx$ for node $A$ which satisfies $\Delta_A^* \equiv 0$ for any $\theta$ and $s$.

Let $X_A$ be the restriction of $X$ on $A$. Namely, we have

$$P(X_A \in A) = \frac{P(X \in A)}{P(X \in A)} \quad \forall A \in \mathcal{B}(A).$$

By some simple analysis, we can verify

$$\text{Var}(Y|X \in A) = \text{Var}(m(X)|X \in A) = \text{Var}(X_A),$$

$$P(\theta^T X \leq s|X \in A) = P(\theta^T X_A \leq s),$$

$$\text{Var}(Y|X \in A, \theta^T X_A \leq s) = \text{Var}(m(X)|X \in A, \theta^T X_A \leq s) = \text{Var}(m(X_A)|\theta^T X_A \leq s).$$
Using the above equations, the rule of population CART can be calculated in the following way.

\[
\Delta_{A_{*}}(\theta, s) := \text{Var}(Y|X \in A) - P(\theta^T X \leq s|X \in A)\text{Var}(Y|X \in A, \theta^T X \leq s)
\]

\[
= P(\theta^T X > s|X \in A)\text{Var}(Y|X \in A, \theta^T X > s)
\]

\[
= \text{Var}(m(X_{A_{*}})) - P(\theta^T X_{A_{*}} \leq s)\text{Var}(m(X_{A_{*}})|\theta^T X_{A_{*}} \leq s) - P(\theta^T X_{A_{*}} > s)\text{Var}(m(X_{A_{*}})|\theta^T X_{A_{*}} > s)
\]

\[
= \text{Var}(m(X_{A_{*}})) - E(\text{Var}(m(X_{A_{*}})|\theta^T X_{A_{*}} \leq s))
\]

where in the last line we use the law of total variance. Note that \(E(m(X_{A_{*}})|\theta^T X_{A_{*}} \leq s)\) is a Bernoulli random variable, whose p.d.f is

\[
E(m(X_{A_{*}})|\theta^T X_{A_{*}} \leq s) = \begin{cases} 
1/S((A_{*})_{\theta,s}^{\theta}) \int_{(A_{*})_{\theta,s}^{\theta}} m(x)dx, & P(\theta^T X_{A_{*}} \leq s) \\
1/S((A_{*})_{\theta,s}^{-\theta}) \int_{(A_{*})_{\theta,s}^{-\theta}} m(x)dx, & P(\theta^T X_{A_{*}} > s), 
\end{cases}
\]

where \(S(A)\) denotes the Lebesgue measure of any \(A \in B(A_{*})\) and \((A_{*})^{\theta}_{\theta,s}, (A_{*})^{-\theta}_{\theta,s}\) are two daughters of \(A_{*}\) (see the beginning of Section 2). First, we consider the case where \(0 < P(\theta^T X_{A_{*}} \leq s) < 1\). Recall our condition that \(\Delta_{A_{*}}(\theta, s) \equiv 0\) for any \(\theta\) and \(s\). Such a condition implies that in this case, we have

\[
\frac{1}{S((A_{*})_{\theta,s}^{\theta})} \int_{(A_{*})_{\theta,s}^{\theta}} m(x)dx = \frac{1}{S((A_{*})_{\theta,s}^{-\theta})} \int_{(A_{*})_{\theta,s}^{-\theta}} m(x)dx,
\]

which implies

\[
\int_{(A_{*})_{\theta,s}^{\theta}} m(x)dx = S((A_{*})_{\theta,s}^{\theta}) \bar{X}_{A_{*}}, \quad \bar{X}_{A_{*}} := \frac{1}{S(A_{*})} \int_{A_{*}} m(x)dx, \tag{4.31}
\]

where \(\bar{X}_{A_{*}}\) denotes the average of \(m(x)\) in \(A_{*}\). If \(P(\theta^T X_{A_{*}} \leq s) = 1\), then \((A_{*})^{\theta}_{\theta,s} = A_{*}\) almost surely in Lebesgue measure. It is obvious that (4.31) holds in this second case. With the similar argument, we know (4.31) is also true if \(P(\theta^T X_{A_{*}} \leq s) = 0\). In conclusion, (4.31) holds for whatever value of \(P(\theta^T X_{A_{*}} \leq s)\).

Suppose \(Z\) is a random vector defined on \(A_{*}\), whose law is \(G(A), A \in B(A_{*})\). For any \(\theta \in \mathbb{R}^p\), \(\theta^T Z\) is a random variable with the distribution function

\[
F_{\theta^T Z}(s) = P\{Z \in A_{*} : \theta^T Z \leq s\} = \frac{S((A_{*})_{\theta,s}^{\theta})}{S(A_{*})}
\]

by using inequality (4.31). Suppose \(U\) is another random vector defined in \(A_{*}\), which has a uniform distribution. Then, \(\theta^T U\) is a random variable with the distribution function

\[
F_{\theta^T U}(s) = P\{U \in A_{*} : \theta^T U \leq s\} = \frac{S((A_{*})_{\theta,s}^{\theta})}{S(A_{*})}
\]

for any \(\theta \in \mathbb{R}^p\) and \(s \in \mathbb{R}\). Therefore, we have

\[
E(e^{i\theta^T U}) = E(e^{i\theta^T Z}), \forall \theta \in \mathbb{R}^p.
\]
In other words, the characteristic functions of $U$ and $Z$ are the same in $\mathbb{R}^p$, indicating that $Z$ has the same distribution with $U$, see for example Theorem 1.6 in Shao (2003). Thus, for any $A \in \mathcal{B}(A_*)$, 

$$G(A) = P(U \in A) = \frac{S(A)}{S(A_*)} = \int_A \frac{m(x)dx}{\int_{A_*} m(x)dx}.$$  

The above inequality indicates that $\int_{A_*} m(x)dx \cdot m(x), x \in A_*$ is a density function of $U$. In conclusion, 

$$m(x) = \int_{A_*} m(x)dx \cdot \frac{1}{S(A_*), \forall x \in A_*, a.s.} \tag{4.32}$$

Since $m(x)$ is assumed to be continuous in $[0,1]^p$, (4.32) also holds for any $x \in A_*$. 

The second step is to show that the cuts based on $\Delta^*_n(\theta, s)$ and $\Delta_n(\theta, s)$ are close to each other. We refer to pages 12 & 14 in Scornet, Biau and Vert (2015) for the notations below. In the following analysis, $d = (\theta, s)$ is used to denote a cut corresponding with the hyperplane $\theta^T x = s$, where $\theta \in \Theta^p$, and $s \in [-\sqrt{p}, \sqrt{p}]$. For any $x \in [0,1]^p$, we call $\mathcal{A}_k(x)$ the class of all possible $k \geq 1$ consecutive cuts used to construct the convex polytope containing $x$. In other words, the above mentioned polytope is obtained after a sequence of cuts $d_k = (d_1, \ldots, d_k)$. For any $d_k \in \mathcal{A}_k(x)$, let $A(x, d_k)$ be the polytope containing $x$ that is built by using $d_k$. Then, the distance between two cuts $d_k$ and $d_k'$ is defined by 

$$\|d_k - d_k'\|_\infty := \underset{1 \leq j \leq k}{\sup} \max\{||\theta_j - \theta'_j||_2, |s - s'|\}.$$ 

For any $x \in [0,1]^p$ and $d_k \in \mathcal{A}_k(x)$, we define rule 

$$\Delta_{n,k}(x, d_k) := \frac{1}{N(A(x, d_{k-1}))} \sum_{i=1}^n (Y_i - \bar{Y}_A(x,d_{k-1}))^2 \mathbb{I}(X_i \in A(x, d_{k-1}))$$

$$- \frac{1}{N(A(x, d_{k-1})))} \sum_{i=1}^n (Y_i - \bar{Y}_{A_L}(x,d_{k-1}))^2 \mathbb{I}(X_i \in A_L(x, d_{k-1}))$$

$$- \frac{1}{N(A(x, d_{k-1})))} \sum_{i=1}^n (Y_i - \bar{Y}_{A_R}(x,d_{k-1}))^2 \mathbb{I}(X_i \in A_R(x, d_{k-1}))$$

where $A_L(x, d_{k-1}) := A(x, d_{k-1}) \cap \{z : \theta_0^T z \leq s_k\}$ and $A_R(x, d_{k-1}) := A(x, d_{k-1}) \cap \{z : \theta_0^T z > s_k\}$ are two daughters of $A(x, d_{k-1})$. Actually, $\Delta_{n,k}(x, d_k)$ is the CART rule which will be used to find the best cut $d_k$ in the polytope $A(x, d_{k-1})$. Similar to Scornet, Biau and Vert (2015), $A^\xi(x, d_{k+1}) \subseteq \mathcal{A}_{k+1}(x)$ denotes the set of all $d_{k+1}$ such that $A(x, d_{k+1})$ contains a hypercube of edge length $\xi > 0$. Finally, define $\bar{A}^\xi(x) := \{d_k : d_{k-1} \in A^\xi(x, d_{k-1})\}$ that is equipped with the norm $\|d_k\|_\infty$. 

Lemma 4.5. Fix $x \in [0,1]^p$, $k \in \mathbb{Z}_+$ and suppose $\xi > 0$. Then $\Delta_{n,k}(x, \cdot)$ is stochastically equicontinuous on $\bar{A}^\xi(x)$; that is, for all $\alpha, \rho > 0$, there exists $\delta > 0$ such that 

$$\lim_{n \to \infty} P\left( \sup_{\|d_k - d'_k\|_\infty \leq \delta} |\Delta_{n,k}(x, d_k) - \Delta_{n,k}(x, d'_k)| > \alpha \right) \leq \rho. \tag{4.33}$$
Proof. We will first consider a simple case with \( k = 1 \) and \( p = 2 \). Then, we will find that the arguments for the other cases are similar to this simple case. Our goal is to choose a \( \delta > 0 \) such that (4.33) holds. Now we have two cuts denoted by \( \mathbf{d}_1 = (\theta_1, s_1) \) and \( \mathbf{d}_1' = (\theta_2, s_2) \) satisfying \( \max \{ ||\theta_1 - \theta_2||, |s_1 - s_2| \} \leq \delta \). Denote by \( \mathcal{R}(\mathbf{d}_1, \mathbf{d}_1', \delta) \) the rectangle with the smallest area which contains points lying on \( \theta_1' z = s_1, z \in [0, 1] \) or \( \theta_2' z = s_2, z \in [0, 1] \). Then, it is not difficult to see that

\[
\text{Area}(\delta) := \sup_{\mathbf{d}_1, \mathbf{d}_1', \|\mathbf{d}_1 - \mathbf{d}_1'\|_\infty \leq \delta} S(\mathcal{R}(\mathbf{d}_1, \mathbf{d}_1', \delta)) \to 0
\]  

as \( \delta \to 0 \). In this simple case, \( \mathcal{A}^\xi(x) = [0, 1]^p \) for any \( x \in \mathcal{A}^\xi_1(x) \). To prepare our arguments, we need the following three preliminary results.

First, there exists \( N_1 \in \mathbb{Z}_+ \) and \( c(\rho) > 0 \) such that for all \( n > N_1 \),

\[
\max_{1 \leq i \leq n} |\varepsilon_i| \leq c(\rho) \sqrt{n} \tag{4.35}
\]

holds with probability at least \( 1 - \rho \). Second, denote \( \mathcal{F} \) by the class of all subsets of \([0, 1]^p\). Note that there are at most \( n^2 \) sets taking the form \( \{ i : X_i \in F \} \) for \( F \in \mathcal{F} \). Therefore, for any \( \delta > 0 \) there exists \( N_2(\delta) \in \mathbb{Z}_+ \) such that for all \( n > N_2(\delta) \) and \( F \in \mathcal{F} \) satisfying \( N(F) > \sqrt{n} \) such that

\[
\left| \frac{1}{N(F)} \sum_{i : X_i \in F} \varepsilon_i \right| \leq \frac{\alpha}{4} \sqrt{\text{Area}(\delta)} \tag{4.36}
\]

holds with probability at least \( 1 - \rho \). Third, we prove a uniform error bound for approximating \( \mathbf{P}(X \in A(x, \mathbf{d}_k)) \) through the empirical method. Define a class of sets

\[
\mathcal{F}_{k_1, k_2} := \left\{ [0, 1]^p \bigcap_{j=1}^{k_1} \{ z : \theta_{j, 1}^T z \leq s_{j, 1} \} \bigcap_{j=1}^{k_2} \{ z : \theta_{j, 2}^T z > s_{j, 2} \} : \theta_{j, 1}, \theta_{j, 2} \in \mathbb{R}^P, s_{j, 1}, s_{j, 2} \in \mathbb{R} \right\}
\]

where \( k_1, k_2 \in \mathbb{Z}_+ \cup \{ 0 \} \). Then, we know

\[
A(x, \mathbf{d}_k) \in \mathcal{F}_k := \bigcup_{k_1, k_2 \in \mathbb{Z}_+ \cup \{ 0 \}, k_1 + k_2 = k} \mathcal{F}_{k_1, k_2}
\]

for any \( x \in [0, 1]^P \) and any cut \( \mathbf{d}_k \). Next we bound the uniform error of approximating expectation of the indicator functions in \( \mathcal{F}_k := \{ \{ z \in F \} : F \in \mathcal{F}_k \} \). In the first step, we show \( \mathcal{F}_k \) is a VC class. By Lemma 9.12 (i) in Kosorok (2008), we know that either \( \mathcal{F}_{1, 0} \) or \( \mathcal{F}_{0, 1} \) has VC dimension \( p + 1 \). According to Lemma 9.7 in Kosorok (2008), we know \( \mathcal{F}_k \) is a VC class with dimension no larger than \( k(k + 1)(p + 1) \). Then, by following the standard arguments in Example 4.8 in Sen (2018), we know

\[
\mathbb{E} \left( \sup_{g \in \mathcal{F}_k} |(\mathbb{P}_n - \mathbb{P})g| \right) \leq c \sqrt{\frac{k(k + 1)(p + 1)}{n}}, \tag{4.37}
\]

where \( \mathbb{P}_n(g) = \frac{1}{n} \sum_{j=1}^{n} g(X_j) \) and \( P(g) = \mathbb{E}(g(X)) \) stand for the operators of empirical and population expectation respectively. On the other hand, it is easy to check that \( \sup_{g \in \mathcal{F}_k} |(\mathbb{P}_n - \mathbb{P})g| \) is a function of \( (X_1, \ldots, X_n) \) and has the bounded differences property for constants \( 1/n \)'s (see page 56 in Boucheron,
Lugosi and Massart (2013)). Therefore, the application of McDiarmid’s inequality leads that for any $t > 0$,
\[
P\left( \sup_{g \in \mathcal{G}_k} |(\mathbb{P}_n - P)g| - \mathbb{E}\left( \sup_{g \in \mathcal{G}_k} |(\mathbb{P}_n - P)g| \right) > t \right) \leq e^{-2nt^2}.
\]  

(4.38)

Finally, the combination of (4.37) and (4.38) implies that with probability larger than $1 - \rho$,
\[
\sup_{g \in \mathcal{G}_k} |(\mathbb{P}_n - P)g| \leq c_1 \sqrt{\frac{k(k+1)(p+1)}{n}} + c_2 \sqrt{\frac{1}{2n} \frac{\log 1}{\rho}}
\]

for some $c_1, c_2 > 0$. In conclusion, for any $\delta > 0$ there exists $N_3(\delta) \in \mathbb{Z}_+$ such that for all $n > N_3(\delta)$ and all $A(x, d_k)$,
\[
\langle S(A(x, d_k)) - \text{Area}^2(\delta) \rangle n \leq N(A(x, d_k)) \leq \langle S(A(x, d_k)) + \text{Area}^2(\delta) \rangle n
\]  

(4.39)

holds with probability at least $1 - \rho$.

The following analysis of the simple case is carried out on the event where equations (4.35), (4.36) and (4.39) all hold. We divide all cases of $d_1$ (or $d_1'$) into two groups:
\[
\text{Area}_1([0, 1]^P) := \{ (\theta, s) : S(A^+_{\theta, s}) \leq \text{Area}(\delta) \text{ or } S(A^-_{\theta, s}) \leq \text{Area}(\delta) \}
\]
\[
\text{Area}_2([0, 1]^P) := \{ (\theta, s) : S(A^+_{\theta, s}) > \text{Area}(\delta) \text{ and } S(A^-_{\theta, s}) > \text{Area}(\delta) \},
\]

where $\text{Area}(\delta)$ is defined in (4.34). Let $d_1 = (\theta_1, s_1)$ and $d' = (\theta_2, s_2)$. We have three cases of locations of $d_1, d'$: (i) $d_1, d' \in \text{Area}_2([0, 1]^P)$; (ii) $d_1, d' \in \text{Area}_1([0, 1]^P)$; (iii) One of them is in $\text{Area}_1([0, 1]^P)$ and the other is in $\text{Area}_2([0, 1]^P)$. We only study the first case in the proof and the arguments of the other two cases are similar to the first one. In fact, there are two different situations in the case 1: (a). $d_1$ does not intersect with $d'_1$ in $[0, 1]^P$; (b). $d_1$ intersects with $d'_1$ in $[0, 1]^P$. The situation (a) is similar to the first case in the proof of Lemma 2 in Scornet, Biau and Vert (2015). So, we only give the proof for situation (b).

As illustrated in Figure 2, we can rewrite $\Delta_{n, 1}(\theta_1, s_1) - \Delta_{n, 1}(\theta_2, s_2)$ in the following way:
\[
\Delta_{n, 1}(\theta_1, s_1) - \Delta_{n, 1}(\theta_2, s_2) = \frac{1}{n} \sum_{i : X_i \in \mathcal{A}_{L, 1}} (Y_i - \bar{Y}_{A_{L, 1}})^2 + \frac{1}{n} \sum_{i : X_i \in \mathcal{A}_{R, 1}} (Y_i - \bar{Y}_{A_{R, 1}})^2
\]
\[
- \frac{1}{n} \sum_{i : X_i \in \mathcal{A}_{L, 2}} (Y_i - \bar{Y}_{A_{L, 2}})^2 - \frac{1}{n} \sum_{i : X_i \in \mathcal{A}_{R, 2}} (Y_i - \bar{Y}_{A_{R, 2}})^2
\]
\[
= \left[ \frac{1}{n} \sum_{i : X_i \in I} (Y_i - \bar{Y}_{A_{L, 1}})^2 - \frac{1}{n} \sum_{i : X_i \in I} (Y_i - \bar{Y}_{A_{R, 1}})^2 \right]
\]
\[
+ \left[ \frac{1}{n} \sum_{i : X_i \in II} (Y_i - \bar{Y}_{A_{L, 1}})^2 - \frac{1}{n} \sum_{i : X_i \in II} (Y_i - \bar{Y}_{A_{R, 1}})^2 \right]
\]
\[
+ \left[ \frac{1}{n} \sum_{i : X_i \in IV} (Y_i - \bar{Y}_{A_{L, 1}})^2 - \frac{1}{n} \sum_{i : X_i \in IV} (Y_i - \bar{Y}_{A_{R, 1}})^2 \right]
\]
Figure 2. This is an example of the situation (b). Here, \( d_1 = (\theta_1, s_1) \) divides \([0, 1]^p\) into two parts where the left part \( A_{L,1} = I \cup IV \) and the right part \( A_{R,1} = II \cup III \), while cut \( d'_1 = (\theta_2, s_2) \) divides \([0, 1]^p\) into another two parts where the left one \( A_{L,2} = I \cup III \) and the right one \( A_{R,2} = II \cup IV \).

\[
\begin{align*}
\|J_1\| &= \frac{1}{n} \sum_{i : X_i \in III} (Y_i - \bar{Y}_{A_{L,2}})^2 - \frac{1}{n} \sum_{i : X_i \in IV} (Y_i - \bar{Y}_{A_{R,2}})^2 \\
&= (\bar{Y}_{A_{L,1}} - \bar{Y}_{A_{L,2}}) \cdot \frac{2}{n} \sum_{i : X_i \in I} \left( Y_i - \frac{\bar{Y}_{A_{L,1}} + \bar{Y}_{A_{L,2}}}{2} \right) \\
&= \|\bar{Y}_{A_{L,1}} - \bar{Y}_{A_{L,2}}\| \cdot \frac{2}{n} \sum_{i : X_i \in I} \left( Y_i - \frac{\bar{Y}_{A_{L,1}} + \bar{Y}_{A_{L,2}}}{2} \right) \\
&:= J_4 \times J_5.
\end{align*}
\]

First, we bound \(|J_1|\) in the following way.

\[
|J_1| := \frac{1}{n} \sum_{i : X_i \in III} (Y_i - \bar{Y}_{A_{L,2}})^2 - \frac{1}{n} \sum_{i : X_i \in IV} (Y_i - \bar{Y}_{A_{R,2}})^2 \\
= (\bar{Y}_{A_{L,1}} - \bar{Y}_{A_{L,2}}) \cdot \frac{2}{n} \sum_{i : X_i \in I} \left( Y_i - \frac{\bar{Y}_{A_{L,1}} + \bar{Y}_{A_{L,2}}}{2} \right) \\
= \|\bar{Y}_{A_{L,1}} - \bar{Y}_{A_{L,2}}\| \cdot \frac{2}{n} \sum_{i : X_i \in I} \left( Y_i - \frac{\bar{Y}_{A_{L,1}} + \bar{Y}_{A_{L,2}}}{2} \right) \\
:= J_4 \times J_5.
\]

Let us find an upper bound for \( J_4 \). Note that

\[
|J_4| = \frac{1}{N(A_{L,1})} \sum_{i : X_i \in A_{L,1}} Y_i - \frac{1}{N(A_{L,2})} \sum_{i : X_i \in A_{L,2}} Y_i \\
= \frac{1}{N(A_{L,1})} \sum_{i : X_i \in I} Y_i + \frac{1}{N(A_{L,1})} \sum_{i : X_i \in IV} Y_i - \frac{1}{N(A_{L,2})} \sum_{i : X_i \in I} Y_i - \frac{1}{N(A_{L,2})} \sum_{i : X_i \in IV} Y_i
\]
We bound $J_6$ by observing that $III \cup IV$ is contained in the rectangle $R(d_1,d'_1,\delta)$, whose area is no larger than $\text{Area}(\delta) := \sup_{d_1,d'_1 \|d_1-d'_1\|_\infty \leq \delta} S(R(d_1,d'_1,\delta))$. Meanwhile, we can always have $0 < \text{Area}(\delta) < 1$ by choosing $\delta$ small enough. Applying (4.39) for $R(d_1,d'_1,\delta)$, the above argument implies that

$$1 - \frac{N(A_{L,1})}{N(A_{L,2})} \leq \frac{\text{Area}(\delta) + \text{Area}^2(\delta)}{\sqrt{\text{Area}(\delta) - \text{Area}^2(\delta)}} \leq 4\sqrt{\text{Area}(\delta)}.$$  

(4.40)

On the other hand, we also have

$$N(I) = N(A_{L,1}) - N(IV) \geq n\sqrt{\text{Area}(\delta)} - n(\text{Area}(\delta) + \text{Area}^2(\delta)) = n(\sqrt{\text{Area}(\delta)} - \text{Area}^2(\delta)) \geq \sqrt{n},$$  

(4.41)

where the last line holds if we choose $\delta > 0$ small enough. Therefore, the combination of (4.32) and (4.41) implies that

$$J_6 = \left| 1 - \frac{N(A_{L,1})}{N(A_{L,2})} \right| \sum_{i: X_i \in I} Y_i = \left| 1 - \frac{N(A_{L,1})}{N(A_{L,2})} \right| \frac{N(I)}{N(A_{L,1})} \frac{1}{N(A_{L,2})} \sum_{i: X_i \in I} Y_i \leq 4\sqrt{\text{Area}(\delta)} \cdot \left( \frac{1}{N(I)} \sum_{i: X_i \in I} e_i + \|m\|_\infty \right) \leq 4\sqrt{\text{Area}(\delta)} (\alpha + \|m\|_\infty),$$  

(4.42)

where the last line follows from (4.36).

Next, we consider $J_7$. Since $N(A_{L,1}), N(A_{L,2}) \geq \sqrt{n}$, we only consider its first term without loss of generality:

$$J_{7,1} := \left| \frac{1}{N(A_{L,1})} \right| \sum_{i: X_i \in III \cup IV} Y_i.$$
By \( N(A_{L,1}) \geq \sqrt{\text{Area}(\delta)} \cdot n \) and \( N(IV) \leq (\text{Area}(\delta) + \text{Area}^2(\delta)) \cdot n \), we have

\[
J_{7,1} \leq \frac{1}{N(A_{L,1})} \left| \sum_{i:X_i \in IV} m(X_i) \right| + \frac{1}{N(A_{L,1})} \left| \sum_{i:X_i \in IV} \varepsilon_i \right|
\leq \frac{N(IV)}{N(A_{L,1})} \| m \|_\infty + \frac{1}{N(A_{L,1})} \left| \sum_{i:X_i \in IV} \varepsilon_i \right|
\leq 2\sqrt{\text{Area}(\delta)} \cdot \| m \|_\infty + \frac{1}{\sqrt{\text{Area}(\delta)}} \cdot \frac{1}{n} \sum_{i:X_i \in IV} \varepsilon_i.
\]

(4.43)

If \( N(IV) \geq \sqrt{n} \), by (4.36) we have

\[
\frac{1}{n} \sum_{i:X_i \in IV} \varepsilon_i \leq 1 \cdot \frac{\alpha}{\sqrt{\text{Area}(\delta)}}.
\]

(4.44)

If \( N(IV) < \sqrt{n} \), by (4.35) we have

\[
\frac{1}{n} \sum_{i:X_i \in IV} \varepsilon_i \leq c(\rho) \sqrt{\frac{\log n}{n}}.
\]

(4.45)

Therefore, by (4.43), (4.44) and (4.45), there exists \( \delta(\alpha) > 0 \) and \( N_4(\delta(\alpha)) \in \mathbb{Z}_+ \) such that for all \( n > N_4(\delta(\alpha)) \), we have

\[
J_7 \leq 4\sqrt{\text{Area}(\delta(\alpha))} \cdot \| m \|_\infty + \frac{2}{\sqrt{\text{Area}(\delta(\alpha))}} \cdot \frac{1}{n} \left| \sum_{i:X_i \in IV} \varepsilon_i \right| \leq \alpha.
\]

(4.46)

Next, we bound \( J_5 \). Note that

\[
J_5 := \frac{2}{n} \sum_{i:X_i \in I} \left| Y_i - \bar{Y}_{A_{L,1}} - \bar{Y}_{A_{L,2}} \right|
\leq \frac{2}{n} \sum_{i:X_i \in I} (Y_i - \bar{Y}_{A_{L,1}}) + \frac{2}{n} \sum_{i:X_i \in I} (Y_i - \bar{Y}_{A_{L,2}})
:= J_{5,1} + J_{5,2}.
\]

Since \( J_{5,1} \) is similar to \( J_{5,2} \), we only need to analyze \( J_{5,1} \). By some calculations, we have

\[
J_{5,1} = \frac{2}{n} \left| \sum_{i:X_i \in I} (Y_i - \bar{Y}_{A_{L,1}}) + \sum_{i:X_i \in IV} (Y_i - \bar{Y}_{A_{L,1}}) - \sum_{i:X_i \in IV} (Y_i - \bar{Y}_{A_{L,1}}) \right|
= \frac{2}{n} \left| \sum_{i:X_i \in A_{L,1}} (Y_i - \bar{Y}_{A_{L,1}}) - \sum_{i:X_i \in IV} (Y_i - \bar{Y}_{A_{L,1}}) \right|
\]
With the similar argument, we also know there exists $\delta > 0$ such that for all $n > N_6(\delta)$,

$$J_5 \leq \frac{\alpha}{4}. \tag{4.47}$$

Next, we bound $J_{5,1,2}$. Since $S(A_{L,1}) > Area(\delta)$, $N(A_{L,1}) \geq (Area(\delta) - Area^2(\delta))n \geq \sqrt{n}$ whenever $n$ is larger than some $N_6(\delta) \in \mathbb{Z}_+$. Therefore, when $n > N_6(\delta)$, we have $|\bar{V}_{A_{L,1}}| \leq ||m||_\infty + \alpha$. By using this result, if $n > N_6(\delta)$,

$$J_{5,1,2} = \frac{N(IV)}{n} |\bar{V}_{A_{L,1}}| \leq \frac{N(IV)}{n} (||m||_\infty + \alpha) \leq Area(\delta) \cdot (||m||_\infty + \alpha). \tag{4.48}$$

By (4.47) and (4.48), there exists $\delta(\alpha) > 0$ with $N_7(\delta(\alpha)) \in \mathbb{Z}_+$ such that for all $n > N_7(\delta(\alpha))$, we have

$$J_5 \leq \alpha. \tag{4.49}$$

Finally, the combination of (4.42), (4.46) and (4.49) leads that there exists $\delta(\alpha) > 0$ with $N_8(\delta) \in \mathbb{Z}_+$ such that for all $n > N_8(\delta(\alpha))$,

$$|J_1| \leq (4\sqrt{Area(\delta(\alpha))(\alpha + ||m||_\infty)} + \alpha)\alpha \leq \frac{\alpha}{3}. \tag{4.50}$$

With the similar argument, we also know there exits $\delta > 0$ with $N_9(\delta) \in \mathbb{Z}_+$ such that for all $n > N_9(\delta(\alpha))$, we have

$$|J_2| \leq \frac{\alpha}{3}. \tag{4.51}$$

Now we bound the last term $J_3$ by decomposing it into two parts:

$$J_3 = \left[ \frac{1}{n} \sum_{i : X_i \in IV} (Y_i - \bar{V}_{A_{L,1}})^2 - \frac{1}{n} \sum_{i : X_i \in IV} (Y_i - \bar{V}_{A_{R,1}})^2 \right]$$

$$+ \left[ \frac{1}{n} \sum_{i : X_i \in III} (Y_i - \bar{V}_{A_{L,1}})^2 - \frac{1}{n} \sum_{i : X_i \in III} (Y_i - \bar{V}_{A_{R,1}})^2 \right]$$

$$:= J_{3,1} + J_{3,2}.$$ 

By symmetry of $J_{3,1}$ and $J_{3,2}$, we only need to bound $J_{3,1}$. By some calculations, we know

$$|J_{3,1}| := \left| \frac{1}{n} \sum_{i : X_i \in IV} (Y_i - \bar{V}_{A_{L,1}})^2 - \frac{1}{n} \sum_{i : X_i \in IV} (Y_i - \bar{V}_{A_{R,1}})^2 \right|$$
Meanwhile, by the similarity of the corresponding cuts and the fact that the corresponding partitions are the same, we have

\[ \sum_{i : X_i \in IV} (Y_i - \bar{Y}_{A_{L,i}}) = \sum_{i : X_i \in IV} (Y_i - \bar{Y}_{A_{R,i}}) \]

Therefore, we have

\[ \frac{1}{n} \sum_{i : X_i \in IV} (Y_i - \bar{Y}_{A_{L,i}}) = \frac{1}{n} \sum_{i : X_i \in IV} (Y_i - \bar{Y}_{A_{R,i}}) \]

Recall that if \( |\bar{Y}_{A_{L,i}}| \leq |m|_\infty + \alpha \) if \( n > N_0(\delta) \) and \( \frac{1}{n} |\sum_{i : X_i \in IV} Y_i| \leq \frac{\alpha}{4} \) and \( \frac{N(IV)}{n} \leq Area(\delta) \). Therefore, there exists \( \delta(\alpha) > 0 \) with \( N_{10}(\delta(\alpha)) \in \mathbb{Z}_+ \) such that for all \( n > N_{10}(\delta(\alpha)) \), \( |J_{3,1}| \leq \frac{\alpha}{6} \) and

\[ |J_3| \leq \frac{\alpha}{3}. \quad (4.52) \]

Finally, the combination of (4.50), (4.51) and (4.52) implies that there exists \( \delta(\alpha) > 0, N_{11} \in \mathbb{Z}_+ \) such that if \( n > N_{11} \),

\[ |\Delta_{n,1}(\theta_1, s_1) - \Delta_{n,1}(\theta_2, s_2)| \leq |J_1| + |J_2| + |J_3| \leq \alpha \]

with probability larger than \( 1 - 3\rho \). This finishes the proof for the simple case \( k = 1 \) and \( p = 2 \).

Next, we consider the case where \( k > 1 \). Without loss of generality, we prove the equicontinuity of \( \Delta_{n,k}(\cdot, \cdot) \) only for \( k = 4, p = 2 \). Consider the decomposition of \( \Delta_{n,4}(d_4) - \Delta_{n,4}(d'_4) \):

\[ \Delta_{n,4}(d_4) - \Delta_{n,4}(d'_4) = \Delta_{n,4}(d_1, d_2, d_3, d_4) - \Delta_{n,4}(d_1, d_2, d_3, d'_4) + \Delta_{n,4}(d_1, d_2, d'_3, d'_4) - \Delta_{n,4}(d_1, d_2, d'_3, d_4) + \Delta_{n,4}(d_1, d'_2, d'_3, d'_4) - \Delta_{n,4}(d_1, d'_2, d'_3, d_4) + \Delta_{n,4}(d'_1, d'_2, d'_3, d'_4) - \Delta_{n,4}(d'_1, d'_2, d'_3, d_4)
\]

\[ := \sum_{j=1}^{4} Part_j. \]

Then, \( Part_j, j = 1, \ldots, 4 \) can be divided into two groups depending on whether the fourth coordinates are the same. In \( Part_1 \) all corresponding cuts are same except \( d_4 \neq d'_4 \), while \( d_4 = d'_4 \) in each \( Part_j, j = 2, 3, 4 \). Then, we can use the same method to bound \( Part_1 \), which was used in the case \( k = 1, p = 2 \). Meanwhile, by the similarity of \( Part_j, j = 2, 3, 4 \), we only analyze \( Part_2 \). Without loss of generality, we just consider a case of \( Part_2 \) shown in Fig 3. To simplify notations, let \( A_b := \triangle ABC \) and \( A_a := \triangle ADE \). Therefore, we have \( \triangle ABC_L = A_{b,L}, \triangle ABC_R = A_{b,R} \) and \( \triangle ADE_L = A_{a,L}, \triangle ADE_R = A_{a,R} \).

Using the above notations, we have

\[ \Delta_{n,4}(d_1, d_2, d_3, d_4) - \Delta_{n,4}(d_1, d_2, d'_3, d'_4)
\]

\[ = \frac{1}{N(A_b)} \sum_{i : X_i \in A_b} (Y_i - \bar{Y}_{A_b})^2 - \frac{1}{N(A_a)} \sum_{i : X_i \in A_a} (Y_i - \bar{Y}_{A_a})^2
\]

\[ + \frac{1}{N(A_a)} \sum_{i : X_i \in A_{a,R}} (Y_i - \bar{Y}_{A_{a,R}})^2 - \frac{1}{N(A_b)} \sum_{i : X_i \in A_{b,R}} (Y_i - \bar{Y}_{A_{b,R}})^2
\]

\[ + \frac{1}{N(A_a)} \sum_{i : X_i \in A_{a,L}} (Y_i - \bar{Y}_{A_{a,L}})^2 - \frac{1}{N(A_b)} \sum_{i : X_i \in A_{b,L}} (Y_i - \bar{Y}_{A_{b,L}})^2
\]

\[ := T_1 + T_2 + T_3. \quad (4.53) \]
where $\text{T}$ and $\text{I}$ denote the pentagon $\triangle ABC$ and $\triangle ADE$. The cut $d'_4$ divides $\triangle ABC$ into two daughters, where $\triangle ABC_L = \square AFGC$ and $\triangle ABC_R = \triangle GFB$. Meanwhile, the cut $d'_4$ divides $\triangle ADE$ into two daughters, where $\triangle ABC_L = \square ADHO$ and $\triangle ABC_R = \triangle HOE$.

Here, we assume $S(A_{b,L}) > 2\text{Area}(\delta)$. Otherwise, it is easy to prove the trivial case $S(A_{b,L}) \leq 2\text{Area}(\delta)$. First, we bound $T_1$. By the assumption $d_4 \in \bar{A}_k^\xi(x)$, we know from (4.39) that $N(A_b) \geq 0.5\xi^2 \cdot n$ with probability larger than $1 - \rho$ when $n$ is large enough. This means we can write $N(A_b) = n$ in order to simplify notation. With the same argument, we can also write $N(A_a) = n$. By symmetry of $\frac{1}{N(A_b)} \sum_{i: X_i \in A_b} (Y_i - \bar{Y}_{A_b})^2$ and $\frac{1}{N(A_a)} \sum_{i: X_i \in A_a} (Y_i - \bar{Y}_{A_a})^2$, we only need to consider the case $T_1 \geq 0$ and further assume $N(A_b) \geq N(A_a)$. Therefore,

$$T_1 \leq \frac{1}{N(A_a)} \left[ \sum_{i: X_i \in I} (Y_i - \bar{Y}_{A_b})^2 - \sum_{i: X_i \in I} (Y_i - \bar{Y}_{A_a})^2 \right]$$

$$+ \frac{1}{N(A_a)} \left[ \sum_{i: X_i \in A_a \setminus I} (Y_i - \bar{Y}_{A_b})^2 - \sum_{i: X_i \in A_a \setminus I} (Y_i - \bar{Y}_{A_a})^2 \right]$$

$$:= T_{1,1} + T_{1,2},$$

where $I$ denotes the pentagon $\triangle ADHC$. Since $BC$ and $DE$ are contained in $\Re(d_3, d'_3, \delta)$, we have $S(\triangle ADE) > \text{Area}(\delta)$. Note that

$$T_{1,1} = \frac{1}{n} \sum_{i: X_i \in I} (\bar{Y}_{A_a} - \bar{Y}_{A_b}) \left( Y_i - \bar{Y}_{A_a} + \bar{Y}_{A_b} \right)$$

and $S(I) > \text{Area}(\delta)$. Thus, we can use the method which bounds $|J_1|$ above to find an upper bound of $T_{1,1}$. Next, we focus on $T_{1,2}$. By some calculations,

$$T_{1,2} \leq \frac{2}{n} \sum_{i: X_i \in A_b \setminus I} Y_i^2 + \frac{2}{n} \sum_{i: X_i \in A_b \setminus I} \bar{Y}_{A_b}^2$$

$$\leq \frac{2}{n} \sum_{i: X_i \in A_b \setminus I} (m(X_i) + \epsilon_i)^2 + 2 \frac{N(A_b \setminus I)}{n} (\|m\|_\infty + \alpha)^2.$$
At this point, we need three observations. By (4.39), the first one is

$$N(A_b \setminus I)/n \leq 2 \text{Area}(\delta) + \text{Area}^2(\delta)$$

(4.55)

with probability larger than $1 - \rho$ when $n$ is large. The second one relates to the chi-squared distribution. Recall that $\mathcal{F}$ is the class of all subsets of $[0, 1]^p$ and there are at most $n^2$ sets taking the form $\{i : X_i \in F\}$ for $F \in \mathcal{F}$. Since $P(\chi^2(n) \geq 5n) \leq \exp(-n)$, there exists $N_{12} \in \mathbb{Z}_+$ such that for any $F \in \mathcal{F}$ satisfying $N(F) \geq \sqrt{n}$ and $n > N_{12},$

$$\frac{1}{N(F)} \sum_{i : X_i \in F} \varepsilon_i^2 \leq 5,$$

with probability larger than $1 - \rho$. The third observation is like (4.35). There exists $N_1 \in \mathbb{Z}_+, c(\rho) > 0$ such that for all $n > N_1,$

$$\max_{1 \leq i \leq n} |\varepsilon_i|^2 \leq c(\rho) \ln n$$

with probability at least $1 - \rho.$ With the second and third observations, there exists $N_{13} \in \mathbb{Z}_+$ such that for all $n > N_{13},$

$$\frac{4}{n} \cdot \sum_{i : X_i \in A_b \setminus I} \varepsilon_i^2 \leq \frac{\alpha}{6}$$

(4.56)

with probability larger than $1 - \rho.$ By (4.54), (4.55) and (4.56), there exits $\delta(\alpha) > 0, N_{14} \in \mathbb{Z}_+$ such that if $n > N_{14},$

$$T_1 \leq \alpha$$

(4.57)

with probability larger than $1 - \rho.$ With the same argument, there also exits $\delta(\alpha) > 0, N_{15} \in \mathbb{Z}_+$ such that if $n > N_{15},$

$$T_2 \leq \alpha$$

(4.58)

with probability larger than $1 - \rho.$ Finally, consider $T_3$ by using the decomposition below.

\[
T_3 := \frac{1}{N(A_a)} \sum_{i : X_i \in A_{a,L}} (Y_i - \bar{Y}_{a,L})^2 - \frac{1}{N(A_b)} \sum_{i : X_i \in A_{b,L}} (Y_i - \bar{Y}_{b,L})^2 \\
= \frac{1}{N(A_a)} \left[ \sum_{i : X_i \in I} (Y_i - \bar{Y}_{a,L})^2 + \sum_{i : X_i \in A_{a,L} \setminus I} (Y_i - \bar{Y}_{a,L})^2 \right] \\
- \frac{1}{N(A_b)} \left[ \sum_{i : X_i \in I} (Y_i - \bar{Y}_{b,L})^2 + \sum_{i : X_i \in A_{b,L} \setminus I} (Y_i - \bar{Y}_{b,L})^2 \right] \\
= \left[ \frac{1}{N(A_a)} \sum_{i : X_i \in I} (Y_i - \bar{Y}_{a,L})^2 - \frac{1}{N(A_b)} \sum_{i : X_i \in I} (Y_i - \bar{Y}_{b,L})^2 \right] \\
\]
Then, \( \mathcal{I} \) for ODRF may not be sparse anymore. Meanwhile, more careful calculations are required in order to any \( i \) in Scornet, Biau and Vert (2015) can not be applied. For example, consider a simple case where in Scornet, Biau and Vert (2015) can not be applied. For example, consider a simple case where \( X_i = 0, 0, \ldots, 0 \) and \( X = (1, 1, \ldots, 1) \) locate in the same partition. When \( n \geq 3 \), \( X_i \) can never be the layered nearest neighbor of \( X \) with probability 1. Therefore, it is possible that for any \( i = 1, \ldots, n \), \( X \) is located in the partition that contains point \( X_i \). This indicates summands in \( I'_n \) for ODRF may not be sparse anymore. Meanwhile, more careful calculations are required in order to bound \( I'_n \). Let \( \mathcal{D}_n(\Theta) \subseteq \{X_1, X_2, \ldots, X_n\} \) be the sampled data given \( \Theta \) and \( e_i = Y_i - m(X_i) \), \( i = 1, \ldots, n \). Then,

\[
I'_n = \mathbb{E} \left[ \sum_{i,j} I_{X \Theta X_i \Theta X_j} I_{X_i \in \mathcal{D}_n(\Theta)} I_{X_j \in \mathcal{D}_n(\Theta')} e_i e_j \right]
\]

\[
= \mathbb{E} \left[ \sum_{i,j} I_{X_i \in \mathcal{D}_n(\Theta)} I_{X_j \in \mathcal{D}_n(\Theta')} e_i e_j \mathbb{E}(I_{X \Theta X_i \Theta X_j} | X, \Theta, \Theta', X_1, \ldots, X_n, Y_i, Y_j) \right]
\]

\[
= \mathbb{E} \left[ \sum_{i,j} I_{X_i \in \mathcal{D}_n(\Theta)} I_{X_j \in \mathcal{D}_n(\Theta')} e_i e_j \psi_{i,j}(Y_i, Y_j) \right]
\]
Then, we focus on bounding the two conditional expectations above. Given realizations given variables. Let $w_{i,j} := \psi_{i,j}(i, j) - \psi_{i,j}$. By the law of total expectation and Jensen’s inequality,

$$\left| I_{n,1}' \right|^2 \leq \mathbb{E} \left( \sum_{i \neq j} \mathbb{E} \left( \sum_{i \neq j} \epsilon_i \epsilon_j \left| \Theta, \Theta' \right. \right) \cdot \mathbb{E} \left( \sum_{i \neq j} w_{i,j}^2 \left| \Theta, \Theta' \right. \right) \right).$$

Then, we focus on bounding the two conditional expectations above. Given realizations $\Theta = \vartheta$ and $\Theta' = \vartheta'$, it is known that both $\mathcal{D}(\vartheta)$ and $\mathcal{D}(\vartheta')$ are determined. Without loss of generality, we can assume $\mathcal{D}(\vartheta) = \mathcal{D}(\vartheta') = \{ X_1, \ldots, X_{a_n} \}$ below. Note that

$$\mathbb{E} \left( \sum_{i \neq j} \epsilon_i \epsilon_j \left| \Theta = \vartheta, \Theta' = \vartheta' \right. \right) = \mathbb{E} \left( \sum_{i \neq j} \epsilon_i \epsilon_j \right) \leq a_n^2.$$

By (4.60) and (4.61),

$$\left| I_{n,2}' \right|^2 \leq a_n^2 \cdot \mathbb{E}_{\Theta, \Theta'} \mathbb{E} \left( \sum_{i \neq j} w_{i,j}^2 \left| \Theta, \Theta' \right. \right) = a_n^2 \cdot \mathbb{E} \left( \sum_{i \neq j} w_{i,j}^2 \right).$$

For any realizations $\vartheta$ and $\vartheta'$, the number of pairs in $\{(i, j) : i \neq j, i \in \mathcal{D}(\vartheta), j \in \mathcal{D}(\vartheta')\}$ does not exceed $a_n^2$. According to this fact and assumption A.1, (4.62) implies

$$\left| I_{n,2}' \right|^2 \leq a_n^4 \cdot \mathbb{E} \left( \max_{i, j} w_{i,j}^2 \right) \leq \frac{c^2}{a_n^2 b},$$

which converges to 0 as $a_n \to \infty$. This completes the proof. \hfill \qed

5. Consistency of gradient boosting tree and its bagging

The algorithm of gradient boosting tree was first developed by Friedman (2001), and has been popularly used. To reduce redundancy, we call it boosting tree later. In a boosting process, trees are constructed in a sequential manner where the $k$–th tree is trained by using predictors and residuals from previous trees. Then, the estimator of $m(x)$ in step $k$ is a linear combination of previous $k$ trees. Details of boosting tree can be found in Friedman (2001). In this section, we use ODT as the basic tree model (base learner). To improve the performance of a boosting tree, we apply the feature bagging in this process, in the same way as the random forest; see also in Section 4. Our final estimator of $m(x)$, which is called the ensemble of ODT-based boosting trees, denoted by ODBT, is the average of many boosting trees.

Motivated by the spirit in Friedman (2001), the ODT-based boosting tree is constructed as follows.
Proposition 1. The boosting tree with \( \sigma \) denoted by \( m_{r,t} \) is the ODT.

Theorem 5.1 (Consistency of the ensemble of boosting trees). Assume \( \beta = \ln n, k_n \to \infty \) and \( k_n = o \left( \frac{n}{\ln n} \right) \), for any \( t \geq 2 \) we have

\[
\mathbb{E} \left( \int |\hat{m}_{k_n,\text{boost}}(x) - m(x)|^2 d\mu(x) \right) \to 0, \quad \text{as } n \to \infty.
\]

The proof of Theorem 5.1 is based on the following proposition.

Proposition 1. The boosting tree with \( k \) ODT’s equals to a neural network with the Heaviside activation \( \sigma_0(v) = 1(v \geq 0), v \in \mathbb{R} \). Meanwhile, this neural network has three layers with at most \( 2t^2 k \) neurons in the first hidden layer and at most \( tk \) neurons in the second hidden layer.
Proof. Without loss of generality, in (5.1) we can assume that all leaves of the first tree $T^r_{D^i_{\lambda,t}}$ be $t$ leaves of $T^r_{D^i_{\lambda,t}}$. Then, we know each $A_j$ forms after performing $L_j$ cuts in $[0,1]^P$ and $L_j \leq t$. Therefore,

$$A_j = \tilde{A}_{j,1} \cap \cdots \cap \tilde{A}_{j,L_j},$$

where $\tilde{A}_{j,k} = \{x \in [0,1]^P : \theta^T_{j,k} x > s_k\}$ or $\tilde{A}_{j,k} = \{x \in [0,1]^P : \theta^T_{j,k} x \leq s_k\}$. Meanwhile, the following equation holds

$$I(x \in A_j) = \sigma_0 \left( \sum_{\ell=1}^{L_j} \sigma_0(s_{\ell} - \theta^T_{j,\ell} x) - L_j \right)$$

if

$$A_j = \{x \in [0,1]^P : \theta^T_{j,1} x \leq s_1\} \cap \cdots \cap \{x \in [0,1]^P : \theta^T_{j,L_j} x \leq s_{L_j}\}.$$  

Since $I(\{x \in [0,1]^P : \theta^T x > s\}) = \sigma_0(0) - \sigma_0(s - \theta^T x)$, we can assume (5.3) holds without loss of generality. This is because that if $\theta^T_{j,\ell} x > s$ we only need to replace $\sigma_0(s_{\ell} - \theta^T_{j,\ell} x)$ by $\sigma_0(0) - \sigma_0(s - \theta^T x)$ in (5.3). Recall that $\hat{Y}_{A_j}$ is the output of $T^r_{D^i_{\lambda,t}}$ if $x \in A_j$. Therefore, we have

$$T^r_{D^i_{\lambda,t}} = \sum_{j=1}^{t} \hat{Y}_{A_j} \sigma_0 \left( \sum_{\ell=1}^{L_j} \sigma_0(s_{\ell} - \theta^T_{j,\ell} x) - L_j \right),$$

which is a neural network with three layers. Thus $T^r_{D^i_{\lambda,t}}$ can be regarded as a neural network with at most $\sum_{j=1}^{t} L_j$ neurons in the first hidden layer and at most $t$ neurons in the second hidden layer. Since feed-forward neural networks have additive structures, we can easily rewrite the boosting tree with $k$ ODT’s as a larger neural network. This completes the proof. 

Proposition 1 suggests that a boosting tree can be regarded as a neural network and thus techniques in neural networks can be applied to analyze the boosting tree. The Conclusion and Future Work in Cattaneo, Chandak and Klusowski (2022) suggests that it is likely to study multi-layer networks by ODT. Here, we do not follow their proof but consider the converse way. We will show below that theories about trees can be indeed studied by those of neural networks. Let us come back to the proof of Theorem 5.1 that is based on Lemma 3.4. Firstly, we bound the second term of the RHS of (3.2), the variance part of $\hat{y}^1_{k_n,boost}$, by using Proposition 1. Secondly, techniques in neural networks are applied to bound the first term that relates to the training error of $\hat{y}^1_{k_n,boost}$.

Variance part of $\hat{y}^1_{k_n,boost}$. Let $\mathcal{N}_{3,2t^2,t}$ be the class of neural networks with the Heaviside activation $\sigma_0(v) = I(v \geq 0)$ where each network has three layers and there are at most $2t^2$ neurons in the first hidden layer and at most $t$ neurons in the second hidden layer. Define the $k_n$ sum of several $\mathcal{N}_{3,2t^2,t}$ by

$$\omega_{k_n} \mathcal{N}_{3,2t^2,t} := \sum_{f_j \in \mathcal{N}_{3,2t^2,t}} \sum_{j=1}^{k_n} f_j.$$  




where $k_n \in \mathbb{Z}_+$. By the proof of Proposition 1, any $m^{1}_{k_n,\text{boost}}$ is in the class $\oplus^{k_n} \mathcal{N} \mathcal{N}(3.2)^{t}$, thus, it is sufficient to bound the covering number $N_{t}(1/(80\beta_n), \oplus^{k_n} \mathcal{N} \mathcal{N}(3.2)^{t}, z''_1^n)$, where $z_1, \ldots, z_n \in \mathbb{R}^p$ and $z''_1^n = \{z_1, \ldots, z_n\}$. This can be done by calculating the VC dimension of $\oplus^{k_n} \mathcal{N} \mathcal{N}(3.2)^{t}$. By the additivity property of neural networks, $\oplus^{k_n} \mathcal{N} \mathcal{N}(3.2)^{t}$ can also be regarded as a class of neural networks with the number of parameters $W = (p + 1) \cdot 2^t k_n + (t + 1) \cdot tk_n + tk_n + 1$. By Bartlett et al. (2019), we know

\[ VC(\oplus^{k_n} \mathcal{N} \mathcal{N}(3.2)^{t}) = O(W \ln W), \]

where $VC(\cdot)$ denotes the VC dimension defined in Part II of proof of Theorem 3.1. Since both $p$ and $t$ are fixed, thus we also have

\[ VC(\oplus^{k_n} \mathcal{N} \mathcal{N}(3.2)^{t}) = O(k_n \ln k_n). \]  \hspace{1cm} (5.6)

Following the arguments in (3.16), (5.6) implies

\[ N_{t}(1/(80\beta_n), \oplus^{k_n} \mathcal{N} \mathcal{N}(3.2)^{t}, z''_1^n) \leq 3 \left(480e\beta_n^3\right)^{c \cdot k_n \ln(k_n)} \]  \hspace{1cm} (5.7)

for any $z''_1^n = \{z_1, \ldots, z_n\}$ with $z_1, \ldots, z_n \in \mathbb{R}^p$. This completes our first part.

\textbf{Training error of $\hat{m}^{1}_{k_n,\text{boost}}$.} In the second part, we bound the training error of $\hat{m}^{1}_{k_n,\text{boost}}$ by using techniques in neural networks. Here, we prove an oracle inequality related to $m^{1}_{k_n,\text{boost}}(x), x \in [0, 1]^p$. For any $1 \leq j \leq k$, let $S_{j,1}, S_{j,2}, \ldots, S_{j,t-1} \subseteq \{1, 2, \ldots, p\}$ be a sequence of random indexes chosen for the CART splitting in the tree $T^{D_{k_n,t}}$. Recall that $S_{j,1}$ relates to the splitting of the root node $[0, 1]^p$ of $T^{D_{k_n,t}}$. Let $\Phi_k = (S_{1,1}, S_{1,2}, \ldots, S_{1,t-1}, \ldots, S_{k,1}, S_{k,2}, \ldots, S_{k,t-1})$ be the collection of random indexes. Then, we define a class of shallow neural networks:

\[ \mathcal{G} := \left\{ \sum_{j=1}^{\infty} a_j \sigma_0(\theta_j^T x + s_j) : \sum_{j=1}^{\infty} |a_j| < \infty, \theta_j \in \mathbb{R}^p, s_j \in \mathbb{R} \right\} , \]  \hspace{1cm} (5.8)

where $\sigma_0(v) = \mathbb{I}(v \geq 0), v \in \mathbb{R}$ is the Heaviside activation.

\textbf{Lemma 5.2 (Oracle inequality for boosting tree).} For any $h(x) = \sum_{j=1}^{\infty} a_j \sigma_0(\theta_j^T x + s_j) \in \mathcal{G}$, we have

\[ E_{\Phi_k} \Vert Y - m_{k_n,\text{boost}}(X) \Vert_n^2 \leq \Vert Y - h \Vert_n^2 + c(p) \left(\sum_{j=1}^{\infty} |a_j| \right)^2 \cdot \frac{1}{k_n + 1} , \]

where $Y := (Y_1, Y_2, \ldots, Y_n)^T \in \mathbb{R}^n$ and the constant $c(p) > 0$ only depends on $p$.

\textbf{Proof.} For the activation function $\sigma_0$, we can assume $\theta_j \in \Theta^p$ since $\sigma_0(\theta_j^T x + s_j) = \sigma_0(\theta_j^T x) \Vert \theta_j \Vert_2 + s_j / \Vert \theta_j \Vert_2$ for any $x \in [0, 1]^p$. Rewrite the function $h(x)$ as

\[ h(x) = \sum_{j=1}^{\infty} a_j \Vert \sigma_0(\theta_j^T x + s_j) \Vert_n \cdot \frac{\sigma_0(\theta_j^T x + s_j)}{\Vert \sigma_0(\theta_j^T x + s_j) \Vert_n} \in \mathcal{G} \]

and define

\[ \|h\|_{\mathbb{E}L} := \sum_{j=1}^{\infty} |a_j| \Vert \sigma_0(\theta_j^T x + s_j) \Vert_n \leq \sum_{j=1}^{\infty} |a_j| < \infty . \]
since \( \| \sigma_0(\theta_j^T x + s_j) \|_n \leq 1 \) and \( h(x) \in \mathcal{G} \). Then, define the dictionary

\[
\mathcal{D}ic_n = \left\{ \frac{\sigma_0(\theta^T x + s)}{\| \sigma_0(\theta^T x + s) \|_n} : \theta \in \Theta^p, s \in \mathbb{R} \right\}.
\]

Denote by \( r_{k_n} = \mathcal{Y} - (m_{k_n, \text{boost}}(X_1), \ldots, m_{k_n, \text{boost}}(X_n))^\top \in \mathbb{R}^n \) the vector of training errors in the \( k_n \)-th step for each \( k_n \geq 0 \). Firstly, we have

\[
\| r_{k_n-1} \|_n^2 = (r_{k_n-1}, h + m - h)_n \leq \| h \|_{EL,1} \cdot \sup_{g \in \mathcal{D}ic_n} | \langle r_{k_n-1}, g \rangle |_n + \| r_{k_n-1} \|_n \| m - h \|_n \tag{5.9}
\]

where in the first equation, we use the fact that \( r_{k_n-1} \) is orthogonal to \( m_{k_n, \text{boost}} \) in \( \mathbb{R}^n \) (recall the projection step (5.2) in our boosting process); and in the last inequality, we use Cauchy-Schwarz inequality in \( \mathbb{R}^n \). Note that \( \| r_{k_n-1} \|_n \| \mathcal{Y} - h \|_n \leq \frac{1}{2} (\| r_{k_n-1} \|_n^2 + \| \mathcal{Y} - h \|_n^2) \). Therefore, (5.9) implies that

\[
\sup_{g \in \mathcal{D}ic_n} | \langle r_{k_n-1}, g \rangle |_n \geq \frac{\| r_{k_n-1} \|_n^2 - \| \mathcal{Y} - h \|_n^2}{2 \cdot \| h \|_{EL,1}}. \tag{5.10}
\]

Secondly, we study the relationship between \( \| r_{k_n-1} \|_n^2 \) and \( \mathbb{E}_{\Phi_{k_n}} | \phi_{k_n-1} | \| r_{k_n} \|_n^2 \). Recall the estimator corresponding with the boosting tree is \( m_{k_n, \text{boost}}(x) := \sum_{j=1}^{k_n} a_{k_n,j}^* m_{r,j}(x) \). From (5.2), we see that \( r_{k_n} \in \mathbb{R}^n \) is the residual vector of projecting \( \mathcal{Y} \) onto the linear space \( \text{span}\{v_j, j = 1, \ldots, k_n\} \) and the vector \( v_j := (m_{r,j}(X_1), m_{r,j}(X_2), \ldots, m_{r,j}(X_n))^\top \in \mathbb{R}^n \). Therefore, for any \( a, b, s \in \mathbb{R} \) and \( \theta \in \Theta^p \),

\[
\mathbb{E}_{\Phi_{k_n}} | \phi_{k_n-1} | \| r_{k_n} \|_n^2 \leq \mathbb{E}_{\Phi_{k_n}} | \phi_{k_n-1} | \| r_{k_n-1} - m_{r,k_n} \|_n^2 \leq | \mathbb{E}_{\Phi_{k_n}} | \phi_{k_n-1} | \| r_{k_n-1} - a \cdot (\theta^T x s_{k_n-1} \leq s) - b \cdot (\theta^T x s_{k_n-1} > s) \|_n^2 \leq \mathbb{E}_{\Phi_{k_n}} | \phi_{k_n-1} | \| r_{k_n-1} - a \cdot \sigma_0(\theta^T x s_{k_n-1} + s) \|_n^2 \leq \mathbb{E}_{\Phi_{k_n}} | \phi_{k_n-1} | \left( \| r_{k_n-1} \|_n^2 - \langle r_{k_n-1}, \frac{\sigma_0(\theta^T x s_{k_n-1} + s)}{\| \sigma_0(\theta^T x s_{k_n-1} + s) \|_n} \rangle_n \right)^2 \right) \leq \| r_{k_n-1} \|_n^2 - c(p) \cdot \langle r_{k_n-1}, \frac{\sigma_0(\theta^T x + s)}{\| \sigma_0(\theta^T x + s) \|_n} \rangle_n^2 \tag{5.11}
\]

where \( c(p) > 0 \); and (5.11) holds since the training error obtained by the last layer of \( T_{\mathcal{D}ic_n}^{\theta,s} \) is smaller than the training error of the first layer; and (5.12) holds because we have a positive probability to choose all features in the cut of root node of the \( k_n \)-th random ODT. Since (5.12) is true for arbitrary \( \theta \in \Theta^p \) and \( s \in \mathbb{R} \), thus we have

\[
\| r_{k-1} \|_n^2 - \mathbb{E}_{\Phi_{k_n}} | \phi_{k_n-1} | \| r_{k_n} \|_n^2 \geq c(p) \cdot \sup_{g \in \mathcal{D}ic_n} | \langle r_{k_n-1}, g \rangle |_n^2. \tag{5.13}
\]

In conclusion, the combination of (5.10) and (5.13) implies that

\[
\| r_{k_n-1} \|_n^2 - \mathbb{E}_{\Phi_{k_n}} | \phi_{k_n-1} | \| r_{k_n} \|_n^2 \geq \max \left( \frac{\| r_{k_n-1} \|_n^2 - \| m - h \|_n^2, 0} {4 \cdot \left( \| h \|_{EL,1}/\sqrt{c(p)} \right)^2} \right). \tag{5.14}
\]
Taking expectation w.r.t. $\Xi_{k_n-1}$ on both sides of (5.14) yields

$$E_{\Phi_k} \|r_{k_n}\|_n^2 \leq E_{\Phi_{k_n-1}} \|r_{k_n-1}\|_n^2 - \left( E_{\Phi_{k_n-1}} \max \left( \|r_{k_n-1}\|_n^2 - \|m - h\|_n^2, 0 \right) \right)^2 - 4 \cdot \left( \|h\|_{EL1} / \sqrt{c(p)} \right)^2. \quad (5.15)$$

where we use the law of iterated expectation and Jensen’s inequality.

If $E_{\Phi_{k_n-1}} \left( \|r_{k_n-1}\|_n^2 - \|m - h\|_n^2 \right) \leq 0$, then the inequality $E_{\Phi_k} \left( \|r_{k_n}\|_n^2 - \|m - h\|_n^2 \right) \leq 0 also holds; and the proof is already completed. This argument is verified because $E_{\Phi_{k_n-1}} \|r_{k_n}\|_n^2 \leq \|r_{k_n-1}\|_n^2$ almost surely and $E_{\Phi_{k_n}} \left( \|r_{k_n}\|_n^2 \right) = E_{\Phi_{k_n-1}} \|r_{k_n}\|_n^2$. Therefore, without loss of generality, we can assume that for each $1 \leq j \leq k_n$, the inequality $E_{\Phi_{j-1}} \left( \|r_{j-1}\|_n^2 - \|m - h\|_n^2 \right) \geq 0$ holds. Under this assumption, from (5.15) the inequality

$$a_j \leq a_{j-1} \left( 1 - \frac{a_{j-1}}{M} \right) \quad (5.16)$$

holds for each $1 \leq j \leq k_n$, where $a_j = E_{\Xi_j} \left( \|r_j\|_n^2 - \|m - h\|_n^2 \right) \geq 0$ and $M = 4 \left( \|h\|_{EL1} / \sqrt{c(p)} \right)^2$. If $a_0 \leq M$, we know that $a_{k_n} \leq \frac{M}{k_n+1}$ holds by using mathematical induction and (5.16). Otherwise $a_1 < 0$, which implies that $a_{k_n} < 0$ for all $k_n \geq 1$ because $(a_j)_{j=0}^\infty$ is decreasing. Therefore, the result of this lemma is always true, regardless of the sign of $a_0$.

**Proof of Theorem 5.1.** Lemma 3.4 shows that

$$E \left( \int |m_{1,boost}^1(X) - m(x)|^2 d\mu(x) \right) \leq 2E_{D_n,\Xi_n} \left( \|m_{1,boost}^1(X) - \mathbb{V}\|_n^2 - \|m(X) - \mathbb{V}\|_n^2 \right) + c \frac{\ln^2 n}{n} \cdot \sup_{\varepsilon^2_n} \left( \mathcal{N}(1/(80nt_n), \Theta_{kn}\mathbb{N}_{3,2\varepsilon^2_n}, \varepsilon^2_n) \right), \quad (5.17)$$

where $\mathbb{V} = (Y_1, \ldots, Y_n)^T \in \mathbb{R}^n$ and the covering number of $\Theta_{kn}\mathbb{N}_{3,2\varepsilon^2_n}$ defined in (5.5) is denoted by $\mathcal{N}(\cdot, \cdot, \cdot)$. To give an upper bound for the first part on the RHS of (5.17), let

$$I = \|m_{1,boost}^1(X) - \mathbb{V}\|_n^2 - \|m(X) - \mathbb{V}\|_n^2$$

$$= (\|\mathbb{V} - m_{1,boost}^1\|_n^2 - \|\mathbb{V} - h_n\|_n^2) + (\|\mathbb{V} - h_n\|_n^2 - \|\mathbb{V} - m\|_n^2) := II + III,$$

where $h = \sum_{j=1}^\infty a_j \sigma_0 (\theta_{n,j}^T x + s_{n,j}) \in \mathcal{G}$. For II, by Lemma 5.2 we have

$$E_{\Phi_{k_n}}(II) \leq c(p) \left( \sum_{j=1}^\infty |a_j| \right)^2 \cdot \frac{1}{k_n + 1}. \quad (5.18)$$

Taking expectation w.r.t. $D_n$ on both sides of (5.18) implies that

$$E_{\Phi_{k_n}, D_n}(II) \leq c(p) \left( \sum_{j=1}^\infty |a_j| \right)^2 \cdot \frac{1}{k_n + 1}. \quad (5.19)$$
It is easy to see that III is independent of $\Phi_k$, thus we further have
\[
E_{\Phi_k, D_n} (\text{III}) = E \left( (Y - h_n(X))^2 - (Y - m(X))^2 \right) = E (m(X) - h_n(X))^2. \tag{5.20}
\]
The combination of (5.19) and (5.20) yields
\[
E_{\Phi_k, D_n} (\text{I}) \leq c(p) \left( \sum_{j=1}^{\infty} |a_j| \right)^2 \cdot \frac{1}{k_n + 1} + E (m(X) - h(X))^2. \tag{5.21}
\]
For the second part on the RHS of (5.17), we can use (5.7). Therefore, it follows from (5.21) and (5.7) that
\[
E \left| \tilde{m}_{\text{ens}}(x) - m(x) \right|^2 d\mu(x) \leq c(p) \left( \sum_{j=1}^{\infty} |a_j| \right)^2 \cdot \frac{1}{k_n + 1} + E (m(X) - h(X))^2. \tag{5.22}
\]
Next, we finish our arguments by showing the RHS of (5.22) goes to 0 as $n \to \infty$. Let $\epsilon > 0$ be any given number. Since $\sigma_0(v), v \in \mathbb{R}$ is a squashing function, by Lemma 16.1 in Györfi et al. (2002) there is $h_\epsilon = \sum_{j=1}^{\infty} a_{\epsilon,j} \sigma_0(\theta_{\epsilon,j}^T x + s_{\epsilon,j}) \in \mathcal{G}$ such that
\[
E (m(X) - h(X))^2 \leq \frac{\epsilon}{2}. \tag{5.23}
\]
Replace $h$ and $\sum_{j=1}^{\infty} |a_j|$ in (5.22) by $h_\epsilon$ and $\sum_{j=1}^{\infty} |a_{\epsilon,j}|$ respectively. By the assumptions in Theorem 5.1, there exists $N \in \mathbb{Z}_+$ such that
\[
c(p) \frac{\ln^2 n}{n} k_n \ln(k_n) \ln(n \beta_n) + c(p) \left( \sum_{j=1}^{\infty} |a_{\epsilon,j}| \right)^2 \cdot \frac{1}{k_n + 1} \leq \frac{\epsilon}{2}
\]
for all $n \geq N$. The combination of (5.23) and (5.24) guarantees the consistency of $\hat{m}_{\text{ens}}^1(x)$. Finally, the consistency of $\hat{m}_{\text{ens}}^{\text{boost}}(x)$ follows from the Jensen’s inequality.

If the regression function $m(x)$ belongs to $\mathcal{G}$ defined in (5.8), we can further show that the ensemble boosting tree has a fast convergence rate $\ln^4 n/\sqrt{n}$ as in the following theorem.

**Theorem 5.3 (Fast consistency rate of boosting tree).** Assume $E(e^{cY^2}) < \infty$ for some $c > 0$ and that $m(x) \in \mathcal{G}$ defined in (5.8). If $\beta_n = \ln n$ and $k_n = \sqrt{n}$, for any $t \geq 2$ we have
\[
E \left( \int \left| \hat{m}_{\text{ens}}^{\text{boost}}(x) - m(x) \right|^2 d\mu(x) \right) = O \left( \frac{\ln^4 n}{\sqrt{n}} \right).
\]

The proof of Theorem 5.3 is similar to that for Theorem 5.1 but with no need to approximate $m(x)$ by using functions in $\mathcal{G}$.
Remark 3. Recall the definition of $\mathcal{G}$ in (5.8). At first glance, we might think that $\mathcal{G}$ is quite too simple because it only contains some step functions defined on $[0,1]^p$. But it is not difficult to prove the closure of $\mathcal{G}$ is exactly equal to the $L^2$ space with respect to $X$, $L^2(X) := \{ f(x) : \mathbf{E}(f^2(X)) < \infty \}$, namely

$$\mathcal{G} = L^2(X).$$

Therefore, the function class $\mathcal{G}$ in Theorem 5.3, which $m(x)$ is assumed to lie in, is really large.

In Theorem 5.3, we obtain the consistency rate $\ln^4 n/\sqrt{n}$ for the boosting tree provided that $m(x) \in \mathcal{G}$. Next, we compare Theorem 5.3 with Theorem 3.2 in Cattaneo, Chandak and Klusowski (2022) that gives a consistency rate of ODT. These comparisons are in three-fold. Firstly, their rate $n^{-(2/(2+q))}$ is slower than our rate $\ln^4 n/\sqrt{n}$, where $q > 2$ is not clearly specified in their Assumption 3. Secondly, our rate $\ln^4 n/\sqrt{n}$ is free from the curse of dimensionality, while Cattaneo, Chandak and Klusowski (2022) cannot show that $q$ does not depend on the dimension. Thirdly, in order to get fast consistency rate for ODT, two additional technical conditions, namely Assumption 3 & 4, are required in Cattaneo, Chandak and Klusowski (2022) and hard to verify. According to the comparisons above, it is highly possible that the boosting tree based on ODT is more efficient than the original ODT. Meanwhile, numerous real data performances in Section 6 also show the superiority of our ensemble boosting tree over many popular machine learning methods.

Finally, we end this section by giving the lower bound of boosting tree. Consider the truncation version of $\mathcal{G}$:

$$\mathcal{G}_1 := \left\{ \sum_{j=1}^{\infty} a_j \sigma_0(\theta_j^T x + s_j) : x \in [0,1]^p, \sum_{j=1}^{\infty} |a_j| \leq 1, \theta_j \in \mathbb{R}^p, s_j \in \mathbb{R} \right\}.$$

Theorem 5.3 implies that the upper bound of boosting tree satisfies

$$\sup_{m(x) \in \mathcal{G}_1} \mathbf{E} \left( \int |\hat{m}_{\text{ens,boost}}(x) - m(x)|^2 d\mu(x) \right) \leq c \cdot \frac{\ln^4 n}{\sqrt{n}} \quad (5.24)$$

for some constant $c > 0$. The following theorem shows that the consistency rate in (5.24) is nearly minimax optimal except for the logarithmic term when $p$ is large.

Theorem 5.4 (Lower bound of boosting tree). Suppose $X \sim U[0,1]^p$ and $\varepsilon = Y - m(X) \sim N(0,1)$. Then there exists a constant $c > 0$ such that

$$\inf_{\hat{m}} \sup_{m(x) \in \mathcal{G}_1} \mathbf{E} \left( \int |\hat{m}(x) - m(x)|^2 dx \right) \geq c \cdot \left( \frac{1}{n} \right)^{2p+2 \over 2p+2},$$

where the infimum is taken over all estimators.

Proof. First, we assume $\theta_j \in \Theta^p$ since $\sigma_0(\theta_j^T x + s_j) = \sigma_0(\theta_j^T x/\|\theta_j\|_2 + s_j/\|\theta_j\|_2)$ for any $x \in [0,1]^p$. Since $x$ is restricted on $[0,1]^p$, we can also assume $s_j \in [-M, M]$ for some large $M > 0$. Therefore, $\mathcal{G}_1$ can be rewritten as

$$\mathcal{G}_1 := \left\{ \sum_{j=1}^{\infty} a_j \sigma_0(\theta_j^T x + s_j) : x \in [0,1]^p, \sum_{j=1}^{\infty} |a_j| \leq 1, \theta_j \in \Theta^p, s_j \in [-M, M] \right\}.$$
The proof of this theorem relies on Theorem 1 of Yang and Barron (1999) below.

**Lemma 5.5 (Yang and Barron (1999)).** Consider the regression model

\[ Y_i = m(X_i) + \varepsilon_i, \quad i = 1, \ldots, n, \]

where \( X_i \sim U[0, 1]^p \) and \( \varepsilon_i \sim N(0, 1) \) are independent Gaussian noises and \( m \in \mathcal{M} \) for some function class \( \mathcal{M} \). Suppose that there exist \( \delta, \xi > 0 \) such that

\[ \ln \mathcal{N}(\xi, \mathcal{M}) \leq \frac{n\xi^2}{2}, \quad \ln \mathcal{P}(\delta, \mathcal{M}) \geq 2n\xi^2 + \ln 2, \quad (5.25) \]

where \( \mathcal{N}(\xi, \mathcal{M}), \mathcal{P}(\delta, \mathcal{M}) \) denotes the cardinality of a minimal \( \xi \) (\( \delta \))-covering (packing) for the class \( \mathcal{M} \) under the Lebesgue measure respectively. Then we have

\[ \inf_{\hat{m}} \sup_{m(x) \in \mathcal{M}} \mathbb{E} \left( \int |\hat{m}(x) - m(x)|^2 \, dx \right) \geq \frac{\delta^2}{8}. \quad (5.26) \]

Let \( \mathcal{M} = \mathcal{G}_1 \) in above lemma. By Siegel and Xu (2022), we know

\[ \ln \mathcal{N}(\xi, \mathcal{G}_1) \leq c \cdot \left( \frac{1}{\xi} \right)^{\frac{2p}{p+1}}. \]

for some \( c > 0 \). Then, by Siegel and Xu (2022), we have the result about the covering (or packing) number:

\[ \ln \mathcal{P}(\delta, \mathcal{G}_1) \geq \ln \mathcal{P}(2\delta, \tilde{\mathcal{G}}_1) \geq \ln \mathcal{N}(2\delta, \tilde{\mathcal{G}}_1) \geq c \cdot \left( \frac{1}{\delta} \right)^{\frac{2p}{p+1}}, \]

where \( \tilde{\mathcal{G}}_1 \) denotes the closure of \( \mathcal{G}_1 \) w.r.t. \( L^2 \) norm. Choosing \( \xi = \delta = c \cdot (1/n)^{\frac{p+1}{p}} \) for some \( c > 0 \), the two equations in (5.25) can be satisfied. Therefore, the result in (5.26) gives the lower bound as desired. This completes the proof.

\[ \square \]

### 6. Numerical Performance in Real Data

Note that the main difficulty in implementing ODT or ODRF is the estimation of the coefficient, \( \theta \), for the linear combinations, which is also one of the main differences amongst all the existing packages. The estimation methods of \( \theta \) include random projection, logistic regression, dimension reduction and many others. However, our experiments suggest that these estimations actually make little difference in the results. In our calculation, instead of using one single projection or linear combination, we provide a number of \( \theta \)'s, each of which is for the projection of a set of randomly selected \( q \) predictors, and then use Gini impurity or residuals sum of squares to choose one combination as splitting variable and splitting point. This is similar to Menze et al. (2011), where a number of random projections are provided from which one is selected. Because of the low estimation efficiency of \( \theta \) as dimension \( q \) increases, we select \( q \) randomly from 1 to \( \min(\lfloor n^{0.5} \rfloor, p) \) when splitting each node. This selection of \( q \) satisfies the requirements of Theorem 4.1 or Theorem 4.3 for the consistency. Our ODT and ODRF are implemented using our "ODRF" package in R via link [https://cran.r-project.org/web/packages/ODRF/](https://cran.r-project.org/web/packages/ODRF/).
In our calculation, the logistic regression function is used to find $\theta$ for each combination of $q$ predictors, but other alternatives are also provided in the package. We also scale the predictors individually before the computation, and while this makes no difference in theory, it sometimes makes the calculation more stable.

Our ODRF and ODBT are compared with the following methods or packages: the Random Rotation Random Forest (RotRF) of Blaser and Fryzlewicz (2016) which randomly rotates the data prior to inducing each tree, the Sparse Projection Oblique Random Forests (SPORF) of Tomita et al. (2020) which simply uses the random projection method and other methods that also use linear combinations as splitting variables including the method of Silva, Cook and Lee (2021), denoted by PPF, and the method of Menze et al. (2011), denoted by ORF. The comparison is also made with three axis-aligned popular methods, including Random Forest (RF) of Breiman (2001a), Generalized Random Forest (GRF) of Athey, Tibshirani and Wager (2019) and Reinforcement Learning Trees (RLT) of Zhu, Zeng and Kosorok (2015), and three popular boosting methods, including the extreme gradient boosting (XGB) of Chen and Guestrin (2016), the Boosted Regression Forest (BRF) of Athey, Tibshirani and Wager (2019), and Generalized Boosted Regression Models (GBM) of Friedman (2002).

Table 1. Dataset summaries for and regression and classification experiments.

| Dataset with continuous responses | Dataset with binary categorical response (0 and 1) |
|-----------------------------------|-----------------------------------------------|
| data.1 Servo                      | data.21 MAGIC                                 |
| data.2 Strike                     | data.22 EEG eye state                         |
| data.3 Auto MPG                   | data.23 Diabetic retinopathy                  |
| data.4 Low birth weight           | data.24 Parkinson multiple                    |
| data.5 Pharynx                    | data.25 Pistachio                             |
| data.6 Body fat                   | data.26 Breast cancer                         |
| data.7 Paris housing price        | data.27 Waveform (2)                          |
| data.8 Parkinsons                 | data.28 QSAR biodegradation                   |
| data.9 Auto 93                    | data.29 Spambase                              |
| data.10 Auto horsepower           | data.30 Mice protein expression               |
| data.11 Wave energy               | data.31 Ozone level detection                 |
| data.12 Baseball player statistics| data.32 Company bankruptcy                    |
| data.13 Year prediction MSD       | data.33 Hill valley                           |
| data.14 Residential building-Sales| data.34 Hill valley noisy                     |
| data.15 Residential building-Cost | data.35 Musk                                  |
| data.16 Geographical original-Latitude| data.36 ECG heartbeat                         |
| data.17 Geographical original-Latitude| data.37 Arrhythmia                           |
| data.18 Credit score              | data.38 Financial indicators of US stocks     |
| data.19 CT slices                 | data.39 Madelon                               |
| data.20 UJIndoor-Longitude        | data.40 Human activity                        |

The following functions and packages in R are used for the calculations: Axis-aligned methods including randomForest for RF, regression_forest and Classification_forest in package grf for GRF, and RLT for RLT. Oblique methods including rotationForest for RotRF.
### Table 2: Regression: average RPE based on 100 random partitions of each data set into training and test sets

| data     | RF   | GRF  | RLT  | RotRF | SPORF | ODRF | XGB   | BRF   | GBM   | ODBT |
|----------|------|------|------|-------|-------|------|-------|-------|-------|-------|
| data.1   | 0.287| 0.268| 0.310| 0.377 | 0.246 | 0.179| 0.108 | 0.385 | 0.308 | 0.127 |
| data.2   | 0.844| 0.793| 0.822| 0.797 | 0.774 | 0.778| 1.197 | 0.817 | 0.862 | 0.820 |
| data.3   | 0.132| 0.149| 0.135| 0.141 | 0.143 | 0.130| 0.152 | 0.139 | 0.144 | 0.117 |
| data.4   | 0.397| 0.365| 0.391| 0.403 | 0.415 | 0.369| 0.504 | 0.367 | 0.387 | 0.395 |
| data.5   | 0.398| 0.388| 0.307| 0.541 | 0.503 | 0.321| 0.353 | 0.374 | 0.444 | 0.356 |
| data.6   | 0.078| 0.039| 0.028| 0.163 | 0.139 | 0.034| 0.039 | 0.039 | 0.041 | 0.037 |
| data.7   | 0.011| 0.000| 0.000| 0.265 | 0.114 | 0.000| 0.000 | 0.000 | 0.001 | 0.001 |
| data.8   | 0.240| 0.202| 0.049| 0.456 | 0.362 | 0.292| 0.068 | 0.117 | 0.675 | 0.189 |
| data.9   | 0.402| 0.460| 0.435| 0.438 | 0.421 | 0.353| 0.491 | 0.445 | 0.423 | 0.349 |
| data.10  | 0.114| 0.198| 0.150| 0.216 | 0.166 | 0.100| 0.147 | 0.141 | 0.138 | 0.085 |
| data.11  | 0.244| 0.359| 0.289| 0.284 | 0.274 | 0.246| 0.263 | 0.237 | 0.647 | 0.179 |
| data.12  | 0.012| 0.016| 0.001| 0.293 | 0.165 | 0.001| 0.001 | 0.005 | 0.003 | 0.006 |
| data.13  | 0.820| 0.873| 0.788| 0.817 | 0.814 | 0.763| 0.927 | 0.848 | 0.829 | 0.763 |
| data.14  | 0.046| 0.081| 0.021| 0.196 | 0.157 | 0.019| 0.018 | 0.045 | 0.026 | 0.019 |
| data.15  | 0.064| 0.101| 0.050| 0.138 | 0.118 | 0.045| 0.050 | 0.050 | 0.051 | 0.040 |
| data.16  | 0.737| 0.831| 0.776| 0.792 | 0.773 | 0.753| 0.822 | 0.805 | 0.798 | 0.756 |
| data.17  | 0.529| 0.582| 0.527| 0.766 | 0.570 | 0.541| 0.629 | 0.560 | 0.566 | 0.652 |
| data.18  | 0.111| 0.151| 0.119| 0.251 | 0.124 | 0.102| 0.113 | 0.105 | 0.169 | 0.137 |
| data.19  | 0.049| 0.123| 0.048| 0.129 | 0.080 | 0.050| 0.061 | 0.039 | 0.224 | 0.059 |
| data.20  | 0.012| 0.027| 0.022| 0.044 | 0.015 | 0.010| 0.024 | 0.015 | 0.130 | 0.018 |
| Average  | 0.276| 0.300| 0.263| 0.375 | 0.319 | 0.254| 0.298 | 0.277 | 0.343 | 0.255 |
| no. of bests | 1 | 2 | 5 | 0 | 1 | 6 | 4 | 2 | 0 | 6 |

RerF in package rerf for SPORF, obliqueRF for ORF, and PPforest for PPF. boosting methods including xgboost for XGB, boostedRegressionForest in package grf for BRF, and gbm for GBM. We used the default tuning parameter values for all packages, but we used 100 trees for the ensemble methods. Note that because PPF and ORF cannot be used for regression, we only report their classification results.

We use 20 real data sets with continuous responses and 20 data sets with binary categorical response (0 and 1) to demonstrate the performance of the above methods. The data are available at one of the following websites (A) https://archive.ics.uci.edu/ml/datasets, (B) https://github.com/twgr/ccfs/ and (C) https://www.kaggle.com. If there are any missing values in the data, the corresponding samples are removed from the data. In the calculation, each predictor is scaled to [0, 1]. The specific information about all the data is summarized in Table 1.

For each data set, we randomly partition it into the training set and the testing set. The training set consists of \( n = \min([2N/3], 2000) \) randomly selected observations, where \( N \) is the number of observations in the original data sets, and the remaining observations form the test set. For regression, the relative prediction error, defined as

\[
RPE = \frac{\sum_{i \in \text{test set}} (\hat{y}_i - y_i)^2}{\sum_{i \in \text{test set}} (\bar{y}_{\text{train}} - y_i)^2},
\]
### Table 3: Classification: average MR (%) based on 100 random partitions of each data set into training and test sets

| Data        | RF   | GRF  | RLT  | RotRF | SPORF | PPF  | ORF  | ODRF | XGB  | BRF  | GBM  | ODBT |
|-------------|------|------|------|-------|-------|------|------|------|------|------|------|------|
| data.21     | 6.88 | 5.50 | 6.34 | 6.64  | 6.27  | 9.51 | 7.56 | 4.99 | 11.23| 5.70 | 1.87 | 6.16 |
| data.22     | 12.85| 18.46| 17.57| 14.14 | 12.14 | 47.49| 8.03 | 9.03 | 24.15| 34.85| 33.19| 23.95|
| data.23     | 32.56| 33.45| 30.20| 31.93 | 28.34 | 33.36| 24.43| 24.15| 24.15| 34.85| 33.19| 34.71|
| data.24     | 27.91| 30.29| 30.09| 27.96 | 27.22 | 34.96| 26.23| 27.21| 30.83| 29.77| 34.56| 25.00|
| data.25     | 10.49| 11.90| 11.04| 9.19  | 9.26  | 10.85| 7.39 | 6.88 | 11.02| 10.81| 10.83| 6.08 |
| data.26     | 4.28 | 5.87 | 5.38 | 2.85  | 3.30  | 4.20 | 2.88 | 6.11 | 5.11 | 4.20 | 3.26 |
| data.27     | 15.20| 17.03| 93.62| 14.02 | 13.77 | 19.75| 40.62| 14.44| 14.61| 12.85| 14.54| 11.63|
| data.28     | 13.37| 15.36| 13.68| 13.22 | 13.03 | 17.48| 12.94| 12.77| 15.79| 14.65| 17.11| 12.19|
| data.29     | 5.41 | 7.45 | 5.43 | 7.54  | 5.19  | 10.60| 6.06 | 6.05 | 7.23 | 6.75 | 6.71 | 6.40 |
| data.30     | 3.08 | 3.21 | 3.09 | 3.19  | 3.13  | 3.11 | 3.14 | 3.14 | 3.62 | 3.18 | 3.13 | 3.19 |
| data.31     | 41.32| 46.01| 42.38| 41.21 | 39.34 | 76.97| 76.95| 99.81| 38.11| 45.98| 50.51| 0.00 |
| data.32     | 3.97 | 6.91 | 5.53 | 5.33  | 6.55  | 19.69| 6.06 | 6.05 | 7.23 | 6.75 | 6.71 | 6.40 |
| data.33     | 11.95| 19.85| 11.98| 18.41 | 9.39  | 21.68| 17.53| 10.52| 14.15| 10.66| 19.69| 18.90|
| data.34     | 19.66| 24.21| 22.54| 21.07 | 14.47 | 29.38| 4.70 | 4.34 | 76.97| 76.95| 99.81| 3.64 |
| data.35     | 1.82 | 1.30 | 0.19 | 4.31  | 2.40  | 1.98 | 12.11| 0.10 | 0.06 | 0.30 | 0.06 | 0.06 |
| data.36     | 32.10| 36.23| 29.22| 41.25 | 38.29 | 41.38| 43.99| 39.93| 29.29| 36.46| 41.11| 46.85|
| data.37     | 0.06 | 0.14 | 0.12 | 0.05  | 0.06  | 0.08 | 0.00 | 0.06 | 0.00 | 0.13 | 0.00 | 0.07 |

| Average     | 14.73| 17.29| 20.57| 13.70 | 11.02 | 18.52| 12.65| 10.00| 17.68| 16.91| 21.19| 10.56|
| no. of bests| 3    | 0    | 1    | 0     | 1     | 0    | 4    | 2    | 2    | 0    | 3    | 10   |

where $\bar{y}_{\text{train}}$ is naive predictions based on the average of $y$ in the training sets, is used to evaluate the performance of a method. For classification, the misclassification rate, defined as

$$MR = \frac{\sum_{i \in \text{test set}} 1(\hat{y}_i \neq y_i)}{(N - n)},$$

is used to assess the performance. For each data set, the random partition is repeated 100 times, and averages of the RPEs or MRs are calculated to compare different methods. The calculation results are listed in Table 2 and Table 3. The smallest RPE or MR for each data set is highlighted in **bold** font.

By comparing the prediction errors, either in terms of RPE for regression or MR for classification, our ODRF is generally smaller than the other methods. Our ODRF is quite stable and achieves the smallest RPE and MR in most datasets as listed in Table 2 and Table 3. The advantages of ODRF are also confirmed by the fact that it has the smallest average RPE (or MS) across all datasets of all methods. The number of data sets for which a method is the best among all competitors, denoted by no. of bests, also suggests the superiority of ODRF over others, including both marginal-based forests and those with linear combinations as partitioning variables. On the other hand, our ODBT has similar performance to our ODRF, as shown in Table 2 and Table 3, and the number of bests for ODBT in Table 3 is even far superior to that of ODRF and other methods.

Finally, we make a brief conclusion for the above experiments. Although many computer programs are developed for ODRF, but they don't show consistently better performance over RF and are not commonly received; see for example Majumder (2020). We attribute this lack of improvement to the
programming details and the choice of linear combinations for the splitting. After refining these issues and redesigning the bagging, all of our experiments, including many not reported here, can indeed produce a more significant overall improvement than RF. With this numerical improvement and theoretical guarantee of consistency, ODRF or e.ODBT is expected to become more popular in the future.

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