Multiloop $\Phi^3$ Amplitudes from Bosonic String Theory.

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Abstract

We show how the multiloop amplitudes of $\Phi^3$ theory (in the worldline formulation of Schmidt and Schubert) are recovered from the $N$-tachyon $(h+1)$-loop amplitude in bosonic string theory in the $\alpha' \to 0$ limit, if one keeps the tachyon coupling constant fixed. In this limit the integral over string moduli space receives contributions only from those corners where the world-sheet resembles a $\Phi^3$ particle diagram. Technical issues involved are a proper choice of local world-sheet coordinates, needed to take the string amplitudes off-shell, and a formal procedure for introducing a free mass parameter $M^2$ instead of the tachyonic value $-4/\alpha'$. 

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1 Introduction

In the limit of vanishing inverse string tension, $\alpha' \to 0$, any string theory reduces to an effective field theory. Since in this limit the string shrinks to a particle-like object, it is intuitively clear that the Polyakov path integral over possible world-sheet histories, which defines the string scattering amplitudes, must reduce to a path integral over world-line histories. Thus, by considering the $\alpha' \to 0$ limit of string theory amplitudes one arrives automatically at a world-line formulation of the particle theory amplitudes associated with the low-energy effective field theory. Sometimes, the world-line formulation thus obtained offers a considerable improvement in computational efficiency over ordinary Feynman diagram techniques. The Bern-Kosower rules for one-loop Yang-Mills theory [1, 2] constituted the first, and most spectacular, example of this.

World-line formulations of particle theory may also be obtained directly from field theory [3, 4, 5, 6, 7]. This approach has the obvious advantage of bypassing the complexities of string theory altogether. This is particularly appealing in the case of multiloop amplitudes. On the other hand, some aspects of the Bern-Kosower rules (such as the subtle combination of gauge choices and the integration-by-parts procedure) would have been quite hard to discover without the help of string theory, and it is therefore natural to expect that the $\alpha' \to 0$ limit of string theory can be helpful also in formulating multiloop extensions of the Bern-Kosower rules.

However, the $\alpha' \to 0$ limit of string multiloop gluon amplitudes is technically very intricate; many of the complications arise from the presence of contact terms. This problem can be avoided by considering instead (as a toy model) the case of multiloop $\Phi^3$ theory, as has been advocated already by Di Vecchia et al [8]: By taking the $\alpha' \to 0$ limit of $N$-tachyon $(h + 1)$-loop amplitudes, keeping the 3-tachyon coupling constant $g$ fixed, one expects to obtain the $N$-point $(h + 1)$-loop amplitudes of $\Phi^3$-theory, with a mass

$$M^2 = -\frac{4}{\alpha'} \to -\infty .$$

(1.1)

This is due to the fact that in string theory any contribution to the tree-level $N$-tachyon amplitude is proportional to $N - 2$ powers of the string coupling constant and hence to
$N - 2$ powers of the tachyon coupling constant $g$. Therefore, on dimensional grounds alone, any would-be $\Phi^N$ coupling ($N \geq 4$) in the effective tachyon field theory must be proportional to

$$g^{N-2}(\alpha')^{N-3}, \quad (1.2)$$

and hence vanishing in the $\alpha' \rightarrow 0$ and $g$ fixed limit.

As we shall see, surviving contributions to the $N$-tachyon $(h + 1)$-loop amplitudes in the $\alpha' \rightarrow 0$ and $g$ fixed limit come only from infinitesimal regions around the singular points on the boundary of moduli space, where the string world-sheet degenerates into a $\Phi^3$ particle diagram. In such a region (also called a $\Phi^3$-like corner of moduli space) the world-sheet consists of cylinders, very long in units of the diameter, joined together at $N + 2h$ vertices. These corners of moduli space were studied extensively in ref. [9], where it was shown how to map the world-sheet moduli into the Schwinger proper times (SPTs) of the corresponding $\Phi^3$ particle diagram.

In the Schwinger Proper Time (SPT) parametrization, the negative sign of $M^2$ gives rise to factors which are exponentially enhanced, rather than exponentially suppressed,

$$\exp{-M^2\tau} = \exp{\left\{\frac{4}{\alpha'}\tau\right\}}. \quad (1.3)$$

Therefore, the integral over the SPT $\tau$ diverges for large values of $\tau$. In the limit $\alpha' \rightarrow 0$ for fixed $\tau$ even the integrand becomes ill-defined. However, if one formally replaces the $4/\alpha'$ appearing in the exponent by $-M^2$ and takes $M^2$ to be a free positive parameter, then one obtains the multiloop amplitudes of $\Phi^3$ theory with an arbitrary mass. Since the string theory starting point is that $\alpha'M^2 = -4$, this procedure requires that one “knows” when to interpret the number $-4$ as $\alpha'M^2$ and when not to do so. This potential ambiguity is avoided because of the fact that, in $\Phi^3$-theory, the particle mass enters the off-shell amplitudes only through the propagators, i.e. in the SPT parametrization through factors such as (1.3), which are easily identified. If the amplitudes are on-shell, on the other hand, the particle mass enters also through the mass-shell condition $p_i^2 = -M^2$ for the external states, $i = 1, \ldots, N$.

It is well known that, when one considers off-shell string amplitudes, i.e. relaxes the
condition $\alpha' p_i^2 = 4$ for the $i$'th external tachyon state ($i = 1, \ldots, N$), the integrated vertex operator

$$\int d^2 z \ e^{i p_i \cdot X(z, \bar{z})}$$

(1.4)
develops a dependence on the choice of holomorphic coordinate, $z$. Let $w_i$ denote the coordinate chosen for the $i$'th vertex operator. This coordinate may depend in a complicated way on the moduli of the string world-sheet. In ref. $[9]$ a prescription was given for $w_i$ in the $\Phi^3$-like corners of moduli space.

In the present paper we show how this prescription for the local coordinates, together with the above procedure for introducing a free mass parameter $M^2$, gives rise, in the $\alpha' \to 0$ and $g$ fixed limit, to a worldline formula for $N$-point $(h + 1)$-loop $\Phi^3$ amplitudes that correctly reproduces the SPT integrand appearing in the formulae obtained from field theory in refs. $[5, 9]$, up to an overall normalization constant.

The paper is organized as follows: In section 2 we review the worldline formulation of $\Phi^3$-theory. In section 3 we consider the string theory formalism and argue that only $\Phi^3$-like corners of moduli space contribute to the $N$-tachyon amplitude in the $\alpha' \to 0$ and $g$ fixed limit. In section 4 we study a large class of $\Phi^3$-like corners in detail and show how the string modular integrand reduces to the SPT integrand of the particle worldline formula. Section 5 contains some remarks on the region of integration. Finally, in Appendix A we show how the symmetric SPT parametrization available in the two-loop case is contained in our general discussion, and in Appendix B we provide certain technical details needed for the comparison of the string and particle formulae.

2 Particle Theory Formalism

The world-line formulation of $\Phi^3$ theory has been developed by Schmidt and Schubert $[3]$. In their formalism it is possible to sum large classes of $N$-point $(h + 1)$-loop Feynman diagrams by means of a single SPT integral formula.

Included is any diagram that can be obtained by inserting $N$ external legs on a Schmidt-Schubert type vacuum diagram, that is, on a vacuum diagram built by connecting
Figure 1: A Schmidt-Schubert type vacuum diagram. The arrows indicate the direction of increasing Schwinger Proper Time. All possible orderings of the $2h$ vertices are allowed.

$2h$ points inserted on a circle with $h$ internal propagators (see Fig. 1). We let $\mathcal{C}_{N_0, N_1, \ldots, N_h}^{(h+1)}$ denote the class of $\Phi^3$-diagrams where $N_0$ external legs (with incoming momenta $p_n^{(0)}, n = 1, \ldots, N_0$) are inserted on the circle (also known as the fundamental loop) and $N_i$ external legs (with momenta $p_n^{(i)}, n = 1, \ldots, N_i$) are inserted on the $i$'th internal propagator, $i = 1, \ldots, h$.

Along the fundamental loop we define a SPT $\tau^{(0)} \in [0, T]$ and along the $i$'th internal propagator we introduce a SPT $\tau^{(i)} \in [0, \bar{T}_i], i = 1, \ldots, h$. The points where the $i$'th internal propagator is joined to the fundamental loop are denoted by $\tau^{(0)} = \tau_{\alpha_i}$, corresponding to $\tau^{(i)} = 0$, and by $\tau^{(0)} = \tau_{\beta_i}$, corresponding to $\tau^{(i)} = \bar{T}_i$. The $N_0$ legs inserted on the fundamental loop have SPTs $\tau_n^{(0)} \in [0; T], n = 1, \ldots, N_0$, and the $N_i$ external legs on the $i$'th internal propagator have SPTs $\tau_n^{(i)} \in [0; \bar{T}_i], n = 1, \ldots, N_i$.

The corresponding amplitude, equal to the sum of all the diagrams in $\mathcal{C}_{N_0, \ldots, N_h}^{(h+1)}$, is given by

$$\Gamma_{N_0, \ldots, N_h}^{(h+1)} = \mathcal{N} g^{N+2h} \int_0^\infty \frac{dT}{T} T^{-D/2} \prod_{i=1}^h \int_0^{\bar{T}_i} d\tau_{\alpha_i} \int_0^T d\tau_{\beta_i}.$$
\[ \times \prod_{n=1}^{N_0} \int_0^T d\tau_n^{(0)} \cdot \prod_{i=1}^h \prod_{n=1}^{N_i} \int_0^{\bar{T}_i} d\tau_n^{(i)} \cdot (\text{det} A)^{-D/2} \exp \left[ -M^2 (T + \sum_{i=1}^h \bar{T}_i) \right] \]

\[ \times \exp \left[ \frac{1}{2} \sum_{i,j=0}^h \sum_{n=1}^{N_i} \sum_{n'=1}^{N_i} P_n^{(i)} P_{n'}^{(j)} G_{ij}^{(h)} (\tau_n^{(i)}, \tau_{n'}^{(j)}) \right], \quad (2.1) \]

where \( \mathcal{N} \) is a normalization constant (which we do not try to fix), \( g \) is the \( \Phi^3 \) coupling constant, and

\[ N = \sum_{i=0}^h N_i \quad (2.2) \]

is the total number of external legs.

The various Green functions appearing in (2.1) are given by [3]

\[ G_{00}^{(h)} (\tau_1^{(0)}, \tau_2^{(0)}) = G_B (\tau_1^{(0)}, \tau_2^{(0)}) - \sum_{k,l=1}^h X_k (\tau_1^{(0)}, \tau_2^{(0)}) A_{kl}^{-1} X_l (\tau_1^{(0)}, \tau_2^{(0)}) , \quad (2.3) \]

\[ G_{ii}^{(h)} (\tau_1^{(i)}, \tau_2^{(i)}) = |\tau_1^{(i)} - \tau_2^{(i)}| - (\tau_1^{(i)} - \tau_2^{(i)})^2 A_{ii}^{-1} , \quad i = 1, \ldots, h , \quad (2.4) \]

\[ G_{i0}^{(h)} (\tau_1^{(i)}, \tau_2^{(0)}) = \tau_1^{(i)} + G_B (\tau_{\alpha_i}, \tau_2^{(0)}) \]

\[ - \sum_{k,l=1}^h [-\tau_1^{(i)} \delta_{ik} + X_k (\tau_{\alpha_i}, \tau_2^{(0)})] A_{kl}^{-1} [-\tau_1^{(i)} \delta_{il} + X_l (\tau_{\alpha_i}, \tau_2^{(0)})] , \quad i = 1, \ldots, h , \quad (2.5) \]

\[ G_{ij}^{(h)} (\tau_1^{(i)}, \tau_2^{(j)}) = \tau_1^{(i)} + \tau_2^{(j)} + G_B (\tau_{\alpha_i}, \tau_{\alpha_j}) \]

\[ - \sum_{k,l=1}^h [-\tau_1^{(i)} \delta_{ik} + \tau_2^{(j)} \delta_{jk} + X_k (\tau_{\alpha_i}, \tau_{\alpha_j})] A_{kl}^{-1} [-\tau_1^{(i)} \delta_{il} + \tau_2^{(j)} \delta_{jl} + X_l (\tau_{\alpha_i}, \tau_{\alpha_j})] , \quad i, j = 1, \ldots, h , \quad i \neq j , \quad (2.6) \]

where \( A \) is the \( h \times h \) matrix defined by

\[ A_{ij} \equiv \bar{T}_i \delta_{ij} - X (\tau_{\alpha_i}, \tau_{\beta_i}; \tau_{\alpha_j}, \tau_{\beta_j}) , \quad (2.7) \]

\[ X (\tau_a, \tau_b; \tau_c, \tau_d) \equiv \frac{1}{2} \left[ G_B (\tau_a, \tau_c) - G_B (\tau_a, \tau_d) - G_B (\tau_b, \tau_c) + G_B (\tau_b, \tau_d) \right] , \quad (2.8) \]

\( G_B \) is the bosonic Green function on a loop of SPT length \( T \) [3]

\[ G_B (\tau_1, \tau_2) = |\tau_1 - \tau_2| - \frac{(\tau_1 - \tau_2)^2}{T} , \quad (2.9) \]

and we introduced the shorthand notation

\[ X_j (\tau_a, \tau_b) \equiv X (\tau_a, \tau_b; \tau_{\alpha_j}, \tau_{\beta_j}) . \quad (2.10) \]
It may be noted that the integrand in eq. (2.1) remains invariant if one adds the same constant, \( \Delta \tau \), to all the SPT variables defined on the fundamental loop, i.e. the variables \( \tau_{\alpha i}, \tau_{\beta i}, i = 1, \ldots, h \) and \( \tau_n^{(0)}, n = 1, \ldots, N_0 \). As argued in detail in Appendix B one may use this symmetry to remove all dependence on, say, \( \tau_{\alpha 1} \). The resulting empty integration can then be trivially performed to give just a factor of \( T \) and one obtains

\[
\Gamma_{N_0,\ldots,N_h}^{(h+1)} = \mathcal{N} g^{N+2h} \int_0^\infty dTT^{-D/2} \prod_{i=1}^h \int_0^T d\bar{T}_i \cdot \int_0^T d\tau_{\alpha i} \int_0^T d\tau_{\beta i} \times \prod_{n=1}^{N_0} \int_0^T d\tau_n^{(0)} \cdot \prod_{i=1}^h \prod_{n=1}^{N_i} \int_0^T d\tau_n^{(i)} \cdot (\det A)^{-D/2} \exp \left[ -M^2 (T + \sum_{i=1}^h \bar{T}_i) \right] \times \exp \left[ \frac{1}{2} \sum_{i,j=0}^h \sum_{n=1}^{N_i} \sum_{n'=1}^{N_j} p_n^{(i)} p_n^{(j)} G_{ij}^{(h)} (\tau_n^{(i)}, \tau_n^{(j)}) \right],
\]

(2.11)

where now everywhere \( \tau_{\alpha 1} = 0 \). This is the formula that we will now reproduce from string theory.

### 3 String Theory Formalism

In string theory the entire connected \( N \)-tachyon \((h + 1)\)-loop amplitude is given by the “master formula”

\[
\mathcal{T}^{(h+1)}(p_1, \ldots p_N) = C_h \left( \frac{\kappa}{\pi} \right)^N \int d^2 \mathcal{M} (\det 2\pi \text{Im} \tau)^{-D/2} \exp \left[ \frac{\alpha'}{2} \sum_{i \neq j}^N p_i \cdot p_j G_{\text{str}}^{(h)} (z_i, z_j) \right],
\]

(3.1)

where \( C_h \) is the normalization of the \((h + 1)\)-loop vacuum amplitude, given in terms of the gravitational coupling, \( \kappa \), by [10]

\[
C_{h+1} = \left( \frac{2\kappa^2}{\alpha'} \right)^{h} \left( \frac{1}{2\pi} \right)^{D(h+1)/2+3h} \left( \alpha' \right)^{-2+(h+1)(4-D)/2},
\]

(3.2)

and the \( N \) powers of \( \kappa/\pi \) are due to the normalization of the vertex operators [11]. The integral is over one copy of \( N \)-punctured genus \((h + 1)\) moduli space.

Each point in this space corresponds to an \( N \)-punctured genus \((h + 1)\) Riemann surface. In the Schottky parametrization [12] the Riemann surface is identified with the
sphere, which has a global complex coordinate \( z \) (defined up to an overall projective transformation), modulo the action of the discrete group of projective transformations generated by \( h + 1 \) generators; moduli space is parametrized by the multiplier \( k_\mu \) and the fixed points \( z_\alpha_\mu \) and \( z_\beta_\mu \) of these generators \( (\mu = 0, 1, \ldots, h) \) and by the Koba-Nielsen variables \( z_i, i = 1, \ldots, N \), specifying the positions of the vertex operators.

The SPT length of the \( \mu \)'th closed loop is given by

\[
T_\mu = -\frac{\alpha'}{2} \ln |k_\mu| , \quad (3.3)
\]

and in the limit where we take \( \alpha' \to 0 \), \( T_\mu \) will vanish unless, at the same time, \( k_\mu \) becomes very small. It is important to notice that even in theories such as Yang-Mills theory, where contact term contributions can arise from internal propagators of vanishing SPT length, the SPT length \( T_\mu \) of an entire closed loop must remain finite if all \( h + 1 \) loops are to survive in the \( \alpha' \to 0 \) limit. For this reason we may always expand in powers of \( k_\mu \) and \( \bar{k}_\mu \). As explained in ref. \[10\] it follows directly from the sewing procedure that the contribution to the modular integrand due to tachyons circulating in the \( \mu \)'th loop is obtained as the leading term in this expansion, proportional to \( d^2k_\mu/|k_\mu|^4 \). Ignoring all contributions of higher order in \( k_\mu \) and \( \bar{k}_\mu \) (which account for the exchange of massless states and of states with positive \( M^2 \)) one finds \[15\]

\[
\mathcal{M} = \frac{1}{d^2V_{abc}} \prod_{\mu=0}^{h} \frac{d^2k_\mu}{|k_\mu|^4} \frac{d^2z_\alpha_\mu}{|z_\alpha_\mu - z_\beta_\mu|^4} \prod_{i=1}^{N} \frac{d^2z_i}{|V'_i(0)|^2} , \quad (3.4)
\]

where

\[
d^2V_{abc} = \frac{d^2z_\alpha d^2z_\beta d^2z_c}{|(z_a - z_b)(z_a - z_c)(z_b - z_c)|^2} , \quad (3.5)
\]

and it is understood that projective invariance is used to fix three of the \( 2h + N + 2 \) points \( \{z_\alpha_\mu, z_\beta_\mu, z_i\} \) at definite values \( z_a, z_b \), and \( z_c \).

By definition the factor

\[
V'_i(0) = \left. \frac{dz}{dw_i} \right|_{w_i=0} , \quad (3.6)
\]

relates the global coordinate \( z \), which assumes the value \( z_i \) at the point where the \( i \)'th vertex operator is inserted, to the local coordinate \( w_i \). (It is conventional to take \( w_i = 0 \)
at the point \( z = z_i \). For on-shell tachyons \((\alpha' p^2 = +4)\) the factors (3.6) in \( d^2 M \) cancel against similar factors appearing in the bosonic Green function [16],

\[
G_{\text{str}}^{(h)}(z_1, z_2) = \ln \left| \frac{E(z_1, z_2)}{(V_1'(0)V_2'(0))^{1/2}} \right| - \frac{1}{2} \sum_{\mu,\nu=0}^{h} \Omega_\mu (2\pi \text{Im} \tau)_{\mu\nu} \Omega_\nu ,
\]

where, to leading order in the multipliers, the prime form, period matrix and Abelian integrals are given by [12]

\[
E(z_1, z_2) = z_1 - z_2 , \quad (2\pi \text{Im} \tau)_{\mu\nu} = -\delta_{\mu\nu} \ln |k_\mu| - (1 - \delta_{\mu\nu}) \ln \left| \frac{z_\alpha - z_\beta}{z_\mu - z_\nu} \right| ,
\]

\[
\Omega_\mu = \ln \left| \frac{z_\mu - z_\beta}{z_1 - z_\beta} \frac{z_\mu - z_\alpha}{z_2 - z_\alpha} \right| .
\]

For off-shell tachyons the amplitude depends on the choice of \(|V_1'(0)|\), which we choose in accordance with the prescription of ref. [9]: In any \( \Phi^3 \)-like corner of moduli space each vertex operator insertion point \( z_i \) represents an external leg inserted on some very long cylinder, and we choose

\[
|V_1'(0)|^{-1} = |\omega(z_i)| ,
\]

where \( \omega \) is any one-differential which is holomorphic throughout the cylinder and has period \( 2\pi i \) along any contour encircling the cylinder (if we change the orientation of the contour, \( V_1'(0) \) changes sign; but that is not important since we are only interested in \(|V_1'(0)|\)). As one approaches the singular point on the boundary of moduli space, where the world-sheet degenerates into the \( \Phi^3 \)-diagram, the cylinder becomes infinitely long (in units of the width) and \( \omega \) becomes unique.

If we start by evaluating eq. (3.1) in the case \( h + 1 = 0 \) and \( N = 3 \), we find the precise relation between the gravitational coupling \( \kappa \) and the tachyon coupling constant, \( g \):

\[
T^{(0)}(p_1, p_2, p_3) = \frac{4\kappa}{\alpha'} \overset{\text{def}}{=} g .
\]

The expression (3.12) for \( T^{(0)}(p_1, p_2, p_3) \) holds both on and off the mass-shell, provided that one chooses

\[
|V_1'(0)|^{-1} = \left| \frac{(z_{i-1} - z_{i+1})}{(z_i - z_{i-1})(z_i - z_{i+1})} \right| ,
\]
where we adopted a cyclic notation (i.e. $z_{i+3} = z_i$). This is in fact the well-known Lovelace choice \cite{17}. It can be obtained from our general prescription (3.11) in the following sense: The world-sheet (there is only one, since there are no moduli for a three-punctured sphere) consists of three semi-infinite cylinders joined together at a vertex. The string state propagating along the $i$’th cylinder is represented as a vertex operator inserted at the point $z_i$. From the point of view of this vertex operator, the world-sheet looks like an infinite cylinder, represented as the complex sphere with the points $z = z_{i-1}$ and $z = z_{i+1}$ removed. The unique holomorphic one-differential of period $2\pi i$ is given by

$$\omega(z) = \frac{z_{i-1} - z_{i+1}}{(z - z_{i-1})(z - z_{i+1})},$$

from which eq. (3.13) is immediately obtained, if we take $z = z_i$ as in eq. (3.11).

Using eqs. (3.2) and (3.12) we may rewrite the overall numerical constant appearing in the “master formula” (3.1) as follows

$$C_{h+1} \left(\frac{\kappa}{\pi}\right)^N g^{N+2h} (\alpha')^{N+3h-(h+1)D/2} \left(\frac{1}{2\pi}\right)^{N+D(h+1)/2+3h} \left(\frac{1}{2}\right)^{3h+N}.$$  

(3.15)

To properly count the powers of $\alpha'$ one must recall the relation (3.3) between the diagonal elements of the matrix $(2\pi \text{Im} \tau)_{\mu\nu}$ and the SPT lengths $T_\mu$, $\mu = 0, 1, \ldots, h$, of the $h+1$ closed loops. Since $T_\mu$ must remain finite in the $\alpha' \to 0$ limit, $(\det 2\pi \text{Im} \tau)^{-D/2}$ scales like $(\alpha')^{(h+1)D/2}$; therefore, for fixed $T_\mu$

$$C_{h+1} \left(\frac{\kappa}{\pi}\right)^N (\det 2\pi \text{Im} \tau)^{-D/2} \propto (\alpha')^{N+3h}.$$ 

(3.16)

In conclusion, we obtain a finite amplitude from the string “master formula” (3.1) in the $\alpha' \to 0$ and $g$ fixed limit if and only if the measure scales like

$$\text{d}^2\mathcal{M} \propto (\alpha')^{-(N+3h)}.$$ 

(3.17)

Now, whenever we change integration variables in $\text{d}^2\mathcal{M}$ from a complex modular parameter to a SPT which is kept finite in the $\alpha' \to 0$ limit we obtain a factor of $1/\alpha'$. As an
example of this we may consider the change of variables from \( k_\mu = |k_\mu| e^{i\phi_\mu} \) to \( T_\mu \):

\[
\frac{d^2 k_\mu}{|k_\mu|^4} = \frac{d|k_\mu|}{|k_\mu|} |k_\mu|^{-2} d\phi_\mu = \frac{2}{\alpha'} dT_\mu e^{-2\log|k_\mu|} d\phi_\mu = \frac{2}{\alpha'} dT_\mu d\phi_\mu e^{+\frac{4}{\alpha'} T_\mu} = \frac{2}{\alpha'} dT_\mu d\phi_\mu e^{-M^2 T_\mu},
\]

where we introduced the tachyon mass \( M \) like in eq. (1.3).

We see that in order to obtain the scaling behaviour (3.17) we have to trade all the \( 3h + N \) complex moduli for a finite SPT, plus a phase. In other words, we have to approach the various \( \Phi^3 \)-like corners of moduli space. In each \( \Phi^3 \)-like corner of moduli space there is a one-to-one map of the \( 3h + N \) complex string moduli into the SPTs of the corresponding \( \Phi^3 \) diagram (plus \( 3h + N \) phases that just give rise to empty integrations), and when one takes the limit \( \alpha' \to 0 \), keeping all the \( 3h + N \) SPTs fixed, one approaches the singular point on the boundary of moduli space where the world-sheet degenerates into the \( \Phi^3 \) particle diagram in question \[9\].

Thus, instead of a single integral over all of moduli space, as in eq. (3.1), we obtain a sum of contributions from all \( \Phi^3 \)-like corners, each contribution being on the form of a \( 3h + N \) real-dimensional SPT integral.

### 4 Particle Theory from String Theory

Having made these general observations we would now like to consider in detail how the particle formula (2.11) can be obtained from the string “master formula” (3.1) precisely by restricting ourselves to corners of \( N \)-punctured genus \((h + 1)\) moduli space, where the world sheet resembles a \( \Phi^3 \)-diagram constructed by inserting \( N \) external legs on a Schmidt-Schubert type vacuum diagram.

It was shown in ref. [9] how, in any such corner of \( N \)-punctured genus \((h + 1)\) moduli space, where \( z_i \) describes an external line inserted on the \( \mu \)’th internal propagator with SPT \( \tau_{\alpha}^{(\mu)} \) and \( z_j \) a line inserted on the \( \nu \)’th internal propagator with SPT \( \tau_{\alpha'}^{(\nu)} \) (\( \mu, \nu =\)
0, 1, . . . , h) one finds
\begin{equation}
G_{\text{str}}^{(h)}(z_i, z_j) \xrightarrow{\alpha' \to 0} \frac{1}{\alpha'} G^{(h)}_{\mu\nu}(\tau_{n}^{(\mu)}, \tau_{n'}^{(\nu)})
\end{equation}
in the limit $\alpha' \to 0$, when all the SPTs are kept fixed, provided that the coordinate-dependent factors $|V'_i(0)|$ are chosen in accordance with the prescription already described.

Thus, with the proper choice of $V'_i(0)$ we find
\begin{equation}
\exp \left[ \frac{\alpha'}{2} \sum_{i \neq j=1}^{N} p_i \cdot p_j G_{\text{str}}^{(h)}(z_i, z_j) \right] \xrightarrow{\alpha' \to 0} \exp \left[ \frac{1}{2} \sum_{i,j=0}^{h} \sum_{n=1}^{N_i} \sum_{n'=1}^{N_j} p_n^{(i)} p_{n'}^{(j)} G_{ij}^{(h)}(\tau_{n}^{(i)}, \tau_{n'}^{(j)}) \right].
\end{equation}
(Strictly speaking one obtains the exponent of the particle formula (2.11) except for the diagonal terms, where $i = j$ and $n = n'$. But these vanish anyway, due to the explicit form of the Green functions (2.3) and (2.4).) It remains to investigate the behaviour of the period matrix determinant and of the integration measure.

We choose to represent the fundamental loop by the 0'th string loop and we use projective invariance on the sphere to fix
\begin{equation}
z_{\alpha_0} = \infty \quad \text{and} \quad z_{\beta_0} = 0.
\end{equation}
If we introduce the notation
\begin{align}
(2\pi \text{Im}\tau)_{00} &= \Delta, \\
(2\pi \text{Im}\tau)_{i0} &= (2\pi \text{Im}\tau)_{0i} = \ln \left| \frac{z_{\alpha_i}}{z_{\beta_i}} \right| = x_i \quad \text{for} \quad i = 1, \ldots, h,
\end{align}
the determinant is obtained as follows:
\begin{equation}
\det(2\pi \text{Im}\tau) = \begin{vmatrix}
\Delta & x_1 & x_2 & \cdots & x_h \\
x_1 & \vdots & \ddots & \ddots & \vdots \\
x_2 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
x_h & \ddots & \ddots & \ddots & \Delta
\end{vmatrix}.
\end{equation}

\footnote{The 0'th internal propagator is the fundamental loop.}
Multiply the $i$-th row by $\Delta$, and then subtract the 0-th row multiplied by $x_i$ from the $i$-th row. Repeating this for $i = 1, 2, \cdots, h$, we obtain

$$
\det(2\pi \text{Im} \tau) = \Delta^{1-h} \det \left[ \Delta A_{ij}^{\text{str}} \right],
$$

where the $h \times h$ matrix $A_{ij}^{\text{str}}$ is given by

$$
A_{ij}^{\text{str}} = (2\pi \text{Im} \tau)_{ij} - x_i x_j \Delta^{-1}.
$$

This matrix is closely related to the matrix $A$ appearing in the particle formula (2.1). Indeed, as was shown in ref. [9], if we take the limit $\alpha' \to 0$, keeping the SPTs fixed, we have

$$
A_{ij}^{\text{str}} \xrightarrow{\alpha' \to 0} \frac{2}{\alpha'} A_{ij}
$$

and thus

$$
\det(2\pi \text{Im} \tau) = \Delta \det A_{ij}^{\text{str}} \xrightarrow{\alpha' \to 0} \left( \frac{2}{\alpha'} \right)^{h+1} T \det A,
$$

where we also used the relation

$$
T = \frac{\alpha'}{2} \Delta
$$

between $\Delta$ and the SPT $T$ of the fundamental loop, q.v. eqs. (3.3), (3.9) and (4.4).

Consider finally the integration measure.

Fixing $z_a = z_{a_0} = \infty$, $z_b = z_{\beta_0} = 0$ and $z_c = z_{a_1} = 1$ in (3.5), the integration measure (3.4) becomes

$$
d^2 \mathcal{M} = d^2 \mathcal{M}_1 d^2 \mathcal{M}_2,
$$

where

$$
d^2 \mathcal{M}_1 = \frac{d^2 k_0}{|k_0|^2} \cdot \prod_{i=1}^h \frac{d^2 k_i}{|k_i|^2} \cdot \frac{|z_{a_1}|^2 d^2 z_{\beta_1}}{|z_{a_1} - z_{\beta_1}|^4} \cdot \prod_{i=2}^h \frac{d^2 z_{a_i} d^2 z_{\beta_i}}{|z_{a_i} - z_{\beta_i}|^4},
$$

$$
d^2 \mathcal{M}_2 = \prod_{i=1}^N \frac{d^2 z_i}{|V_i(0)|^2}.
$$
Let us consider piece by piece. The relations between SPTs and string moduli for the Schmidt-Schubert type vacuum diagram are given by

\[ \tau_{\alpha_i} = -\frac{\alpha'}{2} \ln |z_{\alpha_i}| \quad \text{for} \quad i = 1, \ldots, h, \quad (4.15) \]

\[ \tau_{\beta_i} = -\frac{\alpha'}{2} \ln |z_{\beta_i}| \quad \text{for} \quad i = 1, \ldots, h, \]

\[ T = -\frac{\alpha'}{2} \ln |k_0| \]

\[ T_i = -\frac{\alpha'}{2} \ln |k_i| \quad \text{for} \quad i = 1, \ldots, h. \]

Here \( T_i \) is the SPT of the entire \( i \)'th loop. The SPT of the \( i \)'th internal propagator is given by

\[ \bar{T}_i = T_i - |\tau_{\alpha_i} - \tau_{\beta_i}| = -\frac{\alpha'}{2} \ln \left| \frac{k_i(z_{\alpha_i} - z_{\beta_i})^2}{z_{\alpha_i}z_{\beta_i}} \right| \quad \text{for} \quad i = 1, \ldots, h, \quad (4.16) \]

where one should keep in mind that \( \tau_{\alpha_i} > \tau_{\beta_i} \) (\( \tau_{\alpha_i} < \tau_{\beta_i} \)) corresponds to \( |z_{\alpha_i}| \ll |z_{\beta_i}| \) \((|z_{\alpha_i}| \gg |z_{\beta_i}|)\) in the \( \alpha' \to 0 \) limit. Using these equations we find

\[ d^2 \mathcal{M}_1 \overset{\alpha' \to 0}{\longrightarrow} \left( \frac{4\pi}{\alpha'} \right)^{3h} dT \left( \prod_{i=1}^h dT_i \right) d\tau_{\beta_1} \left( \prod_{i=2}^h d\tau_{\alpha_i} d\tau_{\beta_i} \right) \exp \left[ \frac{4}{\alpha'} (T + \sum_{i=1}^h T_i) \right]. \quad (4.17) \]

Since at this point the integrand no longer depends on the phases of the modular parameters, the corresponding integrations become trivial and produce the factor \( (2\pi)^{3h} \) appearing above.

Now let us consider \( d^2 \mathcal{M}_2 \). If \( z_i \) describes a leg inserted on the fundamental loop, then, as shown in ref. [9], the corresponding SPT \( \tau^{(0)} \) and the local coordinate factor \( V_i'(0) \) are given by

\[ \tau^{(0)} = -\frac{\alpha'}{2} \ln |z_i|, \quad V_i'(0) = z_i, \quad (4.18) \]

and we find

\[ \frac{d^2 z_i}{|V_i'(0)|^2} = \frac{d^2 z_i}{|z_i|^2} \overset{\alpha' \to 0}{\longrightarrow} \frac{4\pi}{\alpha'} d\tau^{(0)}, \quad (4.19) \]

where again the factor of \( 2\pi \) is obtained from the empty phase integral.

If instead \( z_i \) is inserted on the \( j \)-th internal propagator, \( j = 1, \ldots, h \), we have

\[ \tau^{(j)} = -\frac{\alpha'}{2} \ln \left| \frac{z_i - z_{\alpha_j}}{z_{\alpha_j}} \right| = -\frac{\alpha'}{2} \ln |z_i - z_{\alpha_j}| - \tau_{\alpha_j}, \quad V_i'(0) = |z_i - z_{\alpha_j}|, \quad (4.20) \]
in which case
\[
\frac{d^2z_i}{|V_i'(0)|^2} = \frac{d^2z_i}{|z_i - z_{\alpha j}|^2} = \frac{d^2(z_i - z_{\alpha j})}{|z_i - z_{\alpha j}|^2} \xrightarrow{\alpha' \to 0} \frac{4\pi}{\alpha'} d(\tau^{(j)} + \tau_{\alpha j}) = \frac{4\pi}{\alpha'} d\tau^{(j)}.
\] (4.21)

Here we effectively treated \(z_{\alpha j}\) and \(\tau_{\alpha j}\) as constants. For \(j = 1\) they actually are constants, since we have fixed \(z_{\alpha 1} = 1\), corresponding to \(\tau_{\alpha 1} = 0\). For \(j = 2, \ldots, h\) our procedure is justified by the fact that one should really consider (4.21) as a part of the total integration measure \(d^2M\), and the changes of variables \((z_i, z_{\alpha j}) \to (z_i - z_{\alpha j}, z_{\alpha j})\) and \((\tau^{(j)}, \tau_{\alpha j}) \to (\tau^{(j)} - \tau_{\alpha j}, \tau_{\alpha j})\) have unit Jacobian.

In summary,
\[
d^2M_2 \xrightarrow{\alpha' \to 0} \left(\frac{4\pi}{\alpha'}\right)^N h \prod_{\mu=0}^{N} \prod_{n=1}^{N} d\tau_n^{(\mu)},
\] (4.22)

and by combining eqs. (4.17) and (4.22) the total measure is therefore reduced to
\[
d^2M \xrightarrow{\alpha' \to 0} \left(\frac{4\pi}{\alpha'}\right)^{3h+N} dT \left(\prod_{i=1}^{h} dT_i\right) d\tau_{\alpha 1} \left(\prod_{i=2}^{h} d\tau_{\alpha i}, d\tau_{\beta i}\right) \exp \left[\frac{4}{\alpha'}(T + \sum_{i=1}^{h} \bar{T}_i)\right] \prod_{\mu=0}^{N} \prod_{n=1}^{N} d\tau_n^{(\mu)},
\] (4.23)
after we integrate out the \(3h + N\) phases.

5 Summary

By combining eqs. (3.15), (4.2), (4.10) and (4.23) we find that in any \(\Phi^3\)-like corner of moduli space that corresponds to a diagram of the class \(C^\alpha_{N_0, \ldots, N_h}\), the integrand of the string formula (3.1) reduces in the \(\alpha' \to 0\) and \(g\) fixed limit to the following expression

\[
N' \ g^{N+2h} dTT^{-D/2} \cdot \prod_{i=1}^{h} d\bar{T}_i \cdot d\tau_{\beta 1} \cdot \prod_{i=2}^{h} d\tau_{\alpha i} d\tau_{\beta i} \\
\times \prod_{\mu=0}^{N} \prod_{n=1}^{N} d\tau_n^{(\mu)}. \ (\det A)^{-D/2} \ \exp \left[\frac{4}{\alpha'}(T + \sum_{i=1}^{h} \bar{T}_i)\right] \\
\times \exp \left[\frac{1}{2} \sum_{i,j=0}^{h} \sum_{n=1}^{N_i} \sum_{n'=1}^{N_j} p_n^{(i)} p_n^{(j)} G^{(h)}_{ij}(\tau_n^{(i)}, \tau_n^{(j)})\right],
\] (5.1)

where
\[
N' = \left(\frac{1}{4\pi}\right)^{D(h+1)/2}.
\] (5.2)
If we make the substitution $4/\alpha' \to -M^2$ in the exponent, the expression (5.1) clearly reproduces the integrand appearing in the particle formula (2.11), except for the unspecified overall normalization $\mathcal{N}$.

Of course, in order to reproduce the particle formula completely one should also consider the region of integration for the SPTs, not only the SPT integrand. It should be possible to derive the region of integration from string theory, but this is not trivial. First of all it requires a careful analysis of the region of integration of the string moduli in each $\Phi^3$-like corner of moduli space. The well-known mapping into SPTs allows one to translate the region of integration for the string moduli into a region of integration for the SPTs. Next, one should sum the contributions from all the $\Phi^3$-like corners that correspond to diagrams in $C_{N_0,\ldots,N_h}^{(h+1)}$ and one should analyze how the SPT integrations of the various corners combine into a single larger region of integration. But even at this point one cannot expect to recover the region of integration of the particle formula (2.11). The point is that the integral in the string formula (3.1) is only over one copy of moduli space, whereas the region of integration of the SPTs in the particle formula corresponds to a region of integration for the string moduli that covers several copies of moduli space. For example, the SPT configurations related by an interchange of $\tau_{\alpha_i}$ and $\tau_{\beta_i}$ (for any $i = 1, \ldots, h$) are both included in the SPT integral (2.1) even though they certainly correspond to the same Riemann surface, i.e. the same point in moduli space, inasmuch as they are related by the modular transformation $(z_{\alpha_i}, z_{\beta_i}) \to (z_{\beta_i}, z_{\alpha_i})$ that replaces the $i$'th Schottky generator by its inverse. These questions are obviously of importance for any practical application of the string multiloop techniques, not only to $\Phi^3$-theory but also to Yang-Mills theory, and we hope to address them in a future publication.

In conclusion, we have argued in this paper that the integral over moduli space defining the $N$-tachyon $(h + 1)$-loop amplitude in the closed bosonic string reduces in the $\alpha' \to 0$ limit (for fixed tachyon coupling constant) to a sum of contributions from all $\phi^3$-like corners of moduli space, with each contribution being in the form of a SPT integral. We have explicitly shown how the modular integrand of the string formula (3.1) reduces to the SPT integrand of the $\Phi^3$ particle theory formula (2.11) in any corner corresponding
to a $\Phi^3$-diagram belonging to the class $\mathcal{C}_{N_0,N_1,...,N_h}$.

The main simplifying feature of the $\Phi^3$-theory toy model was the absence of contact terms, which was reflected by the fact that the string modular integrand $\mathrm{d}^2\mathcal{M}$ had to scale like $(\alpha')^{-(N+3h)}$, q.v. eq. (3.17). In the physically more interesting case of the $N$-gluon $(h+1)$-loop amplitude, where one takes $\alpha' \to 0$ for fixed Yang-Mills coupling constant, the $\alpha'$ power counting is different and surviving contributions in the $\alpha' \to 0$ limit are no longer compelled to appear only from $\Phi^3$-like corners of moduli space. There will also be other contributions. At the one-loop level these extra contributions (called Type II) were studied in great detail in ref. [18]. They may be removed by an integration-by-parts procedure (as shown in ref. [19]) but this does not seem possible at more than one loop. Recent work by Magnea and Russo [20] in the specific case of the two-loop vacuum amplitude constitutes a successful first step towards a full understanding of these extra contributions at the multiloop level.

Appendix A. The symmetric parametrization of the two-loop case

In this appendix we consider the special case of two loops ($h = 1$). The unique Schmidt-Schubert type vacuum diagram consists of three propagators, joined together at an “upper” and a “lower” 3-point vertex, and there exists a symmetric world-line parametrization [3, 21] which clearly displays the symmetries of this diagram: One defines a SPT

\[ \tau_{\text{sym}}^{(i)}, \ i = 1, 2, 3, \]

along each of the propagators, taking $\tau_{\text{sym}}^{(i)} = 0$ at the “upper” vertex and $\tau_{\text{sym}}^{(i)} = t_i$ at the “lower” vertex (see Fig. 2).

The relation between the symmetric parametrization and the one underlying eq. (2.11) is given by

\[
\begin{align*}
t_1 &= \tau_{\beta_1}, \\
t_2 &= T - \tau_{\beta_1} \\
t_3 &= \bar{T}_1, \\
\end{align*}
\]

(A.1)
Figure 2: The symmetric SPT parametrization of the two-loop Schmidt-Schubert vacuum diagram.

where we used translational invariance along the fundamental loop to choose the “upper” vertex to be at \( \tau_{\alpha_1} = 0 \).

In the symmetric parametrization a leg on the fundamental loop with SPT \( \tau^{(0)} \) is considered to sit on the first propagator, with \( \tau^{(1)}_{\text{sym}} = \tau^{(0)} \), if \( \tau^{(0)} \in [0; \tau_{\beta_1}] \), and to sit on the second propagator, with \( \tau^{(2)}_{\text{sym}} = T - \tau^{(0)} = t_1 + t_2 - \tau^{(0)} \), if \( \tau^{(0)} \in [\tau_{\beta_1}; T] \). Whereas along the internal propagator we simply have \( \tau^{(3)}_{\text{sym}} = \tau^{(1)} \).

If we rewrite eq. (2.11) for \( h = 1 \) in terms of the symmetric parametrization and restrict ourselves to the contribution where \( N_i \) legs are inserted on the \( i \)’th propagator, \( i = 1, 2, 3 \), we obtain

\[
\Gamma_{N_1,N_2,N_3} = \mathcal{N} g^{N+2} \prod_{a=1}^{3} \int_{0}^{\infty} dt_a e^{-M^2 t_a} \cdot (t_1 t_2 + t_2 t_3 + t_3 t_1)^{-D/2} \\
\prod_{n=1}^{N_1} \int_{0}^{t_1} d\tau^{(1)}_n \cdot \prod_{n=1}^{N_2} \int_{0}^{t_2} d\tau^{(2)}_n \cdot \prod_{n=1}^{N_3} \int_{0}^{t_3} d\tau^{(3)}_n \\
\times \exp \left[ \frac{1}{2} \sum_{a=1}^{3} \sum_{j,k=1}^{N_a} p^{(a)}_j p^{(a)}_k g^{\text{sym}}_{aa} (\tau^{(a)}_j, \tau^{(a)}_k) + \sum_{a=1}^{3} \sum_{j=1}^{N_a} \sum_{k=1}^{N_{a+1}} p^{(a)}_j p^{(a+1)}_k g^{\text{sym}}_{aa+1} (\tau^{(a)}_j, \tau^{(a+1)}_k) \right],
\]
where we adopted a cyclic notation, e.g. $t_{a+3} = t_a$ and dropped the label “sym” on the SPTs. The symmetric Green functions appearing in eq. (A.2),

$$G^\text{sym}_{aa}(x, y) = |x - y| - \frac{t_{a+1} + t_{a+2}}{t_1 t_2 + t_2 t_3 + t_3 t_1} (x - y)^2,$$

(A.3)

$$G^\text{sym}_{aa+1}(x, y) = x + y - \frac{x^2 t_{a+1} + y^2 t_a + (x + y)^2 t_{a+2}}{t_1 t_2 + t_2 t_3 + t_3 t_1},$$

(A.4)

can be obtained from the Green functions $G^{(1)}_{00}$ and $G^{(1)}_{10}$ by making the appropriate changes of SPT variables. In view of this it should be clear that the string theory derivation of the Green functions (2.3), (2.4), (2.5) and (2.6) given in ref. [9] include also the symmetric Green functions (A.3) and (A.4).

Indeed, we can obtain the Green functions (A.3) and (A.4) directly from the string theory Green function as follows:

First we define the SPTs pertaining to the fundamental loop

$$T = -\frac{\alpha'}{2} \ln |k_0|, \quad \tau_{\alpha_1} = -\frac{\alpha'}{2} \ln |z_{\alpha_1}|, \quad \tau_{\beta_1} = -\frac{\alpha'}{2} \ln |z_{\beta_1}|,$$

(A.5)

and then we transform these according to [21], q.v. eq. (A.1) above

$$t_2 = T - |\tau_{\alpha_1} - \tau_{\beta_1}| = -\frac{\alpha'}{2} \ln \left| k_0 \frac{(z_{\alpha_1} - z_{\beta_1})^2}{z_{\alpha_1} z_{\beta_1}} \right|,$$

(A.6)

$$t_1 = |\tau_{\alpha_1} - \tau_{\beta_1}| = \frac{\alpha'}{2} \ln \left| \frac{(z_{\alpha_1} - z_{\beta_1})^2}{z_{\alpha_1} z_{\beta_1}} \right|.$$  

(A.7)

The proper time for the internal propagator is given by (q.v. eq. (4.16))

$$t_3 = -\frac{\alpha'}{2} \ln \left| k_1 \frac{(z_{\alpha_1} - z_{\beta_1})^2}{z_{\alpha_1} z_{\beta_1}} \right|.$$  

(A.8)

We may use symmetry of the world-sheet Green function under the exchange of $z_{\alpha_1}$ and $z_{\beta_1}$ to arrange for $\tau_{\beta_1} > \tau_{\alpha_1}$ or in other words, $|z_{\beta_1}| \ll |z_{\alpha_1}|$. Then, in terms of the parameters $t_1$, $t_2$ and $t_3$ the period matrix (3.9) becomes

$$(2\pi \text{Im} \tau)_{\mu\nu} = \frac{2}{\alpha'} \begin{pmatrix} t_1 + t_2 & t_1 \\ t_1 & t_1 + t_3 \end{pmatrix},$$

(A.9)
and the world-sheet Green function (3.7) assumes the form

\[ G_{\text{str}}^{(1)}(z_1, z_2) = \ln \left| \frac{z_1 - z_2}{(V_1'(0))^{1/2}(V_2'(0))^{1/2}} \right| - \frac{\alpha'}{4} \left( t_1 t_2 + t_2 t_3 + t_3 t_1 \right)^{-1} \]

\[ \times \left[ (t_1 + t_3) \Omega_0^2 - 2t_1 \Omega_0 \Omega_1 + (t_1 + t_2) \Omega_1^2 \right] . \]

We notice how the determinant of the period matrix (A.9) gives rise to the factor \( t_1 t_2 + t_2 t_3 + t_3 t_1 \) appearing in the worldline integral (A.2). To recover the various Green functions (A.3) and (A.4) from (A.10) we have to specify the appropriate pinching limits. This can be done by following the rules given in ref. [9].

If \( z_i \) describes a leg inserted on the internal propagator, we may consider the pinching limit \( |z_i - z_{\alpha 1}| \ll |z_{\alpha 1}| \) and the corresponding SPT is (q.v. eq. (4.20))

\[ \tau_i^{(3)} = -\frac{\alpha'}{2} \ln \left| \frac{z_i - z_{\alpha 1}}{z_{\alpha 1}} \right| . \]

(A.11)

If instead \( z_i \) describes an external leg located on the fundamental loop, we have

\[ \tau_i^{(1)} = -\frac{\alpha'}{2} \ln \left| \frac{z_i}{z_{\alpha 1}} \right| \quad \text{for} \quad |z_{\beta 1}| \ll |z_i| \ll |z_{\alpha 1}| , \]

(A.12)

\[ \tau_i^{(2)} = -\frac{\alpha'}{2} \ln \left| k_0 \frac{z_{\alpha 1}}{z_i} \right| \quad \text{for} \quad |k_0 z_{\alpha 1}| \ll |z_i| \ll |z_{\beta 1}| . \]

It is sufficient to consider the two cases \( G_{31}^{\text{sym}} \) and \( G_{33}^{\text{sym}} \), because of cyclic symmetry.

In the case of \( G_{31}^{\text{sym}} (\tau^{(3)}, \tau^{(1)}) \) we have the diagram shown in Fig. 3, which according to the rules of ref. [9] corresponds to the pinching limit

\[ |z_{\beta 1}| \ll |z_2| \ll |z_{\alpha 1}| \simeq |z_1| \quad \text{and} \quad |z_1 - z_{\alpha 1}| \ll |z_{\alpha 1}| , \]

(A.13)

and the choice of local coordinates

\[ \begin{align*}
V_1'(0) &= |z_1 - z_{\alpha 1}| \\
V_2'(0) &= |z_2| .
\end{align*} \]

(A.14)

In the pinching limit (A.13) we find

\[ \Omega_0 = \ln \left| \frac{z_2}{z_1} \right| \simeq \ln \left| \frac{z_2}{z_{\alpha 1}} \right| = -\frac{2}{\alpha'} \tau^{(1)} , \]

\[ \Omega_1 = \ln \left| \frac{z_{\beta 1} - z_{\alpha 1}}{z_{\beta 1} - z_1} \right| \simeq \ln \left| \frac{z_2 (z_{\alpha 1} - z_1)}{(z_{\alpha 1})^2} \right| = -\frac{2}{\alpha'} (\tau^{(3)} + \tau^{(1)}) . \]

(A.15)
Figure 3: The $\Phi^3$ diagram relevant for the Green function $G_{31}^{\text{sym}}$.

\[
\ln \left| \frac{z_1 - z_2}{(V_1'(0))^{1/2}(V_2'(0))^{1/2}} \right| \simeq \frac{1}{2} \ln \left| \frac{(z_{\alpha_1})^2}{z_2(z_1 - z_{\alpha_1})} \right| = \frac{1}{\alpha'}(\tau^{(1)} + \tau^{(3)}),
\]
and thus, by inserting (A.15), (A.16) and (A.17) into eq. (A.10),

\[
G_{\text{str}}^{(1)}(z_1, z_2) \xrightarrow{\alpha' \to 0} \frac{1}{\alpha'} G_{31}^{\text{sym}}(\tau^{(3)}, \tau^{(1)}).
\]

In the case of $G_{33}^{\text{sym}}(\tau_1^{(3)}, \tau_2^{(3)})$ we consider instead the pinching limit (q.v. Fig.4)

\[
|z_2 - z_{\alpha_1}|, |z_1 - z_{\alpha_1}| \ll |z_{\alpha_1}| \quad \text{and} \quad |z_{\beta_1}| \ll |z_{\alpha_1}|,
\]
with

\[
\begin{cases}
V_1'(0) = |z_1 - z_{\alpha_1}| \\
V_2'(0) = |z_2 - z_{\alpha_1}|
\end{cases}
\]
and we find

\[
\Omega_0 \simeq 0,
\]
\[
\Omega_1 \simeq \ln \left| \frac{z_1 - z_{\alpha_1}}{z_2 - z_{\alpha_1}} \right| = -\frac{2}{\alpha'}(\tau_1^{(3)} - \tau_2^{(3)}),
\]
\[
\ln \left| \frac{z_1 - z_2}{(V_1'(0))^{1/2}(V_2'(0))^{1/2}} \right| \simeq \frac{1}{\alpha'}|\tau_1^{(3)} - \tau_2^{(3)}|,
\]
and thus

\[
G_{\text{str}}^{(1)}(z_1, z_2) \xrightarrow{\alpha' \to 0} \frac{1}{\alpha'} G_{33}^{\text{sym}}(\tau_1^{(3)}, \tau_2^{(3)}).
\]
It is now easy to see that if we evaluate the two-loop version of the string “master formula” (3.1) in the relevant pinching limits, mapping the string moduli directly into the SPTs of the symmetric parametrization, we recover the entire SPT integrand of the particle formula (A.2), except for the overall numerical constant.

In conclusion, we have shown explicitly how the procedure for extracting $\Phi^3$ particle theory from the bosonic string leads to the correct two-loop particle formulae, regardless of which of the two SPT parametrizations we choose. This should in fact be clear a priori, since the process by which we extract contributions from the $\Phi^3$-like corners of moduli space, including the choice of local coordinates, is entirely geometrical and does not rely on any specific choice of SPT parametrization.
Appendix B. Translational invariance along the fundamental loop

In this appendix we show how to obtain the worldline formula (2.11) from the more standard expression (2.1) by using the invariance of the integrand under the transformation

\[
\begin{align*}
\tau_{\alpha_i} &\rightarrow \tau_{\alpha_i} + c \\
\tau_{\beta_i} &\rightarrow \tau_{\beta_i} + c \\
\tau_{n(0)} &\rightarrow \tau_{n(0)} + c
\end{align*}
\]  

for \( i = 1, \ldots, h \) and \( n = 1, \ldots, N_0 \) (B.1)

that translates all SPTs pertaining to the fundamental loop by the same constant, \( c \).

This invariance follows from the similar property of the bosonic Green function (2.9),

\[ G_B(\tau_a, \tau_b) = G_B(\tau_a + c, \tau_b + c) , \]  

(B.2)

once we notice that the SPTs of the fundamental loop enter eq. (2.1) only as arguments of \( G_B \) and that every \( G_B \) appearing has two SPTs of the fundamental loop as arguments.

If we introduce a simplified notation, where \( x_0, x_1, \ldots, x_n \) denote the \( n+1 = 2h + N_0 \) SPTs of the fundamental loop, and the integrand of the particle formula (2.1) is called \( f(x_0, x_1, \ldots, x_n) \), then the invariance (B.1) allows us to rewrite the amplitude (2.1) as

\[ A = \int_0^T dx_0 \int_0^T dx_1 \cdots \int_0^T dx_n \ f(x_0, x_1, \ldots, x_n) \]  

(B.3)

\[ = \int_0^T dx_0 \int_0^T dx_1 \cdots \int_0^T dx_n \ f(0, x_1 - x_0, \ldots, x_n - x_0) \]  

\[ = \int_0^T dx_0 \int_{-x_0}^{T-x_0} dx_1 \cdots \int_{-x_0}^{T-x_0} dx_n \ f(0, x_1, \ldots, x_n) . \]

At this point the dependence on \( x_0 \) has disappeared from the integrand but remains in the limits of integration.

If \( f \) was periodic with period \( T \) in each variable, we could just replace the integration region

\[ \int_{-x_0}^{T-x_0} dx_i \rightarrow \int_0^T dx_i \]  

(B.4)
for each $i = 1, \ldots, n$. But things are not so straightforward, because the Green function $G_B$, from which $f$ is constructed, is not periodic. Rather, it satisfies

\begin{align}
G_B(\tau_1 + T, \tau_2) &= G_B(\tau_1, \tau_2) \quad \text{if} \quad \tau_1 < \tau_2 \\
G_B(\tau_1 - T, \tau_2) &= G_B(\tau_1, \tau_2) \quad \text{if} \quad \tau_1 > \tau_2 ,
\end{align}

(B.5)

(B.6)

provided that $|\tau_1 - \tau_2| < T$.

Fortunately, this “restricted periodicity” turns out to be sufficient to justify the shift (B.4) of the integration region.

To demonstrate the underlying mechanism, consider first the case $n = 1$ (the two-loop vacuum diagram). Then we have

\begin{align}
\mathcal{A} &= \int_0^T dx_0 \int_{-x_0}^{x_0 + T} dx_1 f(0, x_1) \\
&= \int_0^T dx_0 \left( \int_0^{-x_0} dx_1 f(0, x_1) + \int_{-x_0}^{x_0 + T} dx_1 f(0, x_1) \right) .
\end{align}

(B.7)

In the first term, $x_1$ is the smallest of the two SPT arguments, since $x_1 < 0$ and the other is zero. In view of eq. (B.3) we may then write

\begin{equation}
f(0, x_1) = f(0, x_1 + T) ,
\end{equation}

(B.8)

inasmuch as $f$ is constructed entirely from $G_B$. If we then change integration variables from $x_1$ to $x'_1 = x_1 + T$ we obtain

\begin{align}
\mathcal{A} &= \int_0^T dx_0 \left( \int_{-x_0}^{x_0 + T} dx'_1 f(0, x'_1) + \int_{-x_0}^{x_0 + T} dx_1 f(0, x_1) \right) \\
&= \int_0^T dx_0 \int_{-x_0}^{x_0 + T} dx_1 f(0, x_1) = T \int_0^T dx_1 f(0, x_1) .
\end{align}

(B.9)

In the general case we break each of the $n$ $x_i$ integrations into two regions,

\begin{equation}
\int_{-x_0}^{-x_0 + T} dx_i = \int_{-x_0}^{0} dx_i + \int_{0}^{x_0 + T} dx_i .
\end{equation}

(B.10)

Then $\mathcal{A}$ becomes a sum of $2^n$ terms, where in a “typical” term we have $n_1$ integrations that run over negative values (from $-x_0$ to 0) and $n - n_1$ integrations that run over positive values (from 0 to $-x_0 + T$), $n_1 = 0, 1, \ldots, n$. 

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In each “typical” term we may break the integration region for the \( n_1 \) variables that run over negative values into \( n_1 \) smaller integration regions, each corresponding to a possible ordering of the \( n_1 \) negative variables.

Let us assume for notational simplicity that the \( n_1 \) negative variables are \( x_1, \ldots, x_{n_1} \). Then we rewrite the “typical” term as

\[
\int_{T_0}^{T} dx_0 \int_{-x_0}^{0} dx_1 \cdots \int_{-x_0}^{0} dx_{n_1} \int_{-x_0 + T}^{-x_0 + T} dx_{n_1 + 1} \cdots \int_{-x_0 + T}^{-x_0 + T} dx_n \ f(0, x_1, \ldots, x_n) = \sum_{\sigma \in S_{n_1}} \int_{T_0}^{T} dx_0 \int_{-x_0 + T}^{-x_0 + T} dx_{n_1 + 1} \cdots \int_{-x_0 + T}^{-x_0 + T} dx_n \ f(0, x_1, \ldots, x_n), \tag{B.11}
\]

where the sum is over all permutations of the integers \( \{1, 2, \ldots, n_1\} \). Since

\[
x_{\sigma(1)} < x_{\sigma(2)} < \ldots < x_{\sigma(n_1)} < 0 < x_{n_1 + 1}, \ldots, x_n, \tag{B.12}
\]

we may use eq. (B.3) to replace \( x_{\sigma(1)} \) by \( x_{\sigma(1)} + T = x'_{\sigma(1)} \) in the integrand \( f \) and then change variables to \( x'_{\sigma(1)} \). Since \( x'_{\sigma(1)} \) runs from \( -x_0 + T \) to \( T \), i.e. is positive, it is now \( x_{\sigma(2)} \) which is the smallest SPT variable, and we may repeat the above step. By continued use of this trick, the “typical” term (B.11) becomes

\[
\sum_{\sigma \in S_{n_1}} \int_{T_0}^{T} dx_0 \int_{-x_0 + T}^{-x_0 + T} dx'_{\sigma(1)} \int_{x'_{\sigma(1)}}^{T} dx'_{\sigma(2)} \cdots \int_{x'_{\sigma(n_1 - 1)}}^{T} dx'_{\sigma(n_1)} \int_{-x_0 + T}^{-x_0 + T} dx_n \ f(0, x'_{1}, \ldots, x'_{n_1}, x_{n_1 + 1}, \ldots, x_n) = \int_{T_0}^{T} dx_0 \int_{-x_0 + T}^{-x_0 + T} dx'_{n_1} \cdots \int_{-x_0 + T}^{-x_0 + T} dx_n \ f(0, x'_{1}, \ldots, x'_{n_1}, x_{n_1 + 1}, \ldots, x_n). \tag{B.13}
\]

Effectively, all integrations over negative SPTs have been shifted to positive values and when all the \( 2^n \) “typical” terms are recombined, the \( x_i \) integration now becomes

\[
\int_{-x_0 + T}^{T} dx_i + \int_{0}^{-x_0 + T} dx_i = \int_{T}^{T} dx_i. \tag{B.14}
\]

In conclusion, we arrive at the expression (2.11) for the amplitude,

\[
\mathcal{A} = \int_{0}^{T} dx_0 \int_{0}^{T} dx_1 \cdots \int_{0}^{T} dx_n \ f(0, x_1, \ldots, x_n) = T \int_{0}^{T} dx_1 \cdots \int_{0}^{T} dx_n \ f(0, x_1, \ldots, x_n). \tag{B.15}
\]
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