REDUCING MOD $p$ COMPLEX REPRESENTATIONS
OF FINITE REDUCTIVE GROUPS

G. LUSZTIG

Dedicated to the memory of Jim Humphreys

ABSTRACT. We state a conjecture on the reduction modulo the defining characteristic of a unipotent representation of a finite reductive group.

INTRODUCTION

0.1. Let $k$ be an algebraic closure of the finite field with $p$ elements ($p$ is a prime number). Let $\mu$ be the group of roots of 1 in $C$. We fix a surjective homomorphism $\phi: \mu \rightarrow k^*$ whose kernel is the set of roots of 1 of order a power of $p$. Let $\Gamma$ be a finite group. Let $R\Gamma$ (resp. $R_p\Gamma$) be the Grothendieck group of virtual (finite dimensional) representations of $\Gamma$ over $C$ (resp. over $k$) and let $R^+\Gamma$ (resp. $R^+_p\Gamma$) be the subset of $R\Gamma$ (resp. $R_p\Gamma$) given by actual representations of $\Gamma$ over $C$ (resp. over $k$). Following Brauer and Nesbitt [BN] there is a well defined map $\rho \mapsto \underline{\rho}$ from $R^+\Gamma$ to $R^+_p\Gamma$ characterized by the following property: for any $g \in \Gamma$ the eigenvalues of $g$ on $\rho$ are obtained by applying $\phi$ to the eigenvalues of $g$ on $\rho$. We say that $\rho$ is the reduction modulo $p$ of $\rho$. In the remainder of this paper we assume that $\Gamma = G(F_p)$ is the group of $F_p$-rational points of an almost simple simply connected linear algebraic group $G$ over $k$ with a given split $F_p$-structure with $p$ sufficiently large. Our goal is to present some remarks on the map $\rho \mapsto \underline{\rho}$ in this case.

0.2. Assume that $G = SL_2(k)$. Assume that $\rho \in R^+\Gamma$ is irreducible. If $\rho$ has dimension 1, $p$, $(p+1)/2$ or $(p-1)/2$, then $\underline{\rho}$ is irreducible. If dim $\rho = p + 1$, then $\underline{\rho}$ has two composition factors, of dimension $c, p+1-c$ with $2 \leq c \leq (p-1)/2$ (and any such $c$ occurs). If dim $\rho = p - 1$, then either $\underline{\rho}$ has two composition factors, of dimension $c, p-1-c$ with $2 \leq c \leq (p-3)/2$ (and any such $c$ occurs) or $\rho$ is irreducible. These results can be found in the paper [BN] of Brauer and Nesbitt (they actually consider the group $PSL_2(F_p)$ instead of $SL_2(F_p)$ but their method applies also to $SL_2(F_p)$).

0.3. Assume that $G = SL_3(k)$. In the case where $\rho$ is an irreducible representation in $R^+\Gamma$ which has a line stable under the upper triangular subgroup, the complete description of the composition factors of $\rho$ was given in [CL] (written in 1973). For one of the cuspidal irreducible representations $\rho$ of $\Gamma$, $\underline{\rho}$ has exactly two composition factors (except when $p = 2$ when $\rho$ is irreducible), as stated in [L1] (where the case $p = 2$ was overlooked); one has dimension $p(p-1)(2p-1)/2$ and the other (when $p > 2$) has dimension $(p-1)(p-2)/2$. This is analogous to the cuspidal irreducible...
representation of $SL_2(F_p)$ for which $\rho$ is irreducible. In the case where $\rho$ is in one of the three main series of irreducible representations of $\Gamma$, a description of $\rho$ was given by Humphreys in [11], [12].

0.4. For general $G$ let $\mathfrak{U}$ be the set of unipotent representations of $\Gamma$ (up to isomorphism). A study of the map $\rho \mapsto \rho$ in the case where $\rho$ is one of the irreducible representations of $\Gamma$ attached in [12] 1.9 to a generic character of a “maximal torus” of $\Gamma$ appears in Jantzen’s paper [11]; a study of the map $\rho \mapsto \rho$ in the case where $\rho \in \mathfrak{U}$ appears in Jantzen’s paper [12]. (The notion of unipotent representation of $\Gamma$ is defined in [12] 7.8.)

0.5. In unpublished notes written in 1978, the author gave a conjectural description (on the level of dimensions only) of $\rho$ as an explicit linear combination of Weyl modules (see 1.1) in the case where $\rho \in \mathfrak{U}$ and $G$ has type $B_2, G_2, A_3, A_4$; for types $A_1, A_2$ this was known earlier, see 0.2, 0.3. Later the author found that this description has been proved to be correct when $G$ has type $B_2$ by Jantzen [13] or type $G_2$ by Mertens [11]. (A copy of [11] was provided to the author by J. Humphreys.) Recently the author understood that the conjectural description in 1978 can be partly explained by a surprising (conjectural) general pattern which will be described in this paper. Namely, there should exist a family of objects $M_w \in R^+_p \Gamma$ indexed by the “near involutions” (see 1.2) $w$ in $W$ such that for any $\rho \in \mathfrak{U}$, $\rho$ is an explicit linear combination of $M_w$ with $w$ near involutions in the two-sided cell determined by $\rho$; the coefficients are natural numbers whose definition involves among other things the character table of the $J$-ring associated to the Weyl group.

1. RECOLLECTIONS

1.1. Let $B$ be a Borel subgroup of $G$ defined over $F_p$; let $T$ be a maximal torus of $B$ defined and split over $F_p$. Let $X$ be the group of characters $T \to k^*$ with group operation written as addition. For any $\lambda \in X$ there is (up to isomorphism) at most one irreducible rational $G$-module $L(\lambda)$ (over $k$) such that $T$ acts on some $B$-stable line in $L(\lambda)$ through the character $\lambda$; this is uniquely defined up to isomorphism. Let $X^+$ be the set of all $\lambda \in X$ for which $L(\lambda)$ is defined. There is a unique $\mathbb{Z}$-basis $\{\varpi_i; i \in I\}$ of $X$ such that $X^+ = \sum_{i \in I} N \varpi_i$. For $I' \subset I$ we set $\lambda_{I'} = \sum_{i \in I'} \varpi_i \in X^+$.

For $\lambda \in X^+$ let $V(\lambda)$ be a rational $G$-module (over $k$) whose character (an element of the group ring $\mathbb{Z}[X]$) is the same as that of the characteristic 0 analogue of $L(\lambda)$; it is given by the Weyl character formula. Note that $V(\lambda)$ is well defined up to rearrangement of its composition factors. Let $X^+_p$ be the set of all $\lambda \in X^+$ of the form $\sum_{i \in I} n_i \varpi_i$ with $0 \leq n_i \leq p - 1$ for all $i$. For $\lambda \in X^+_p$ we denote by $V(\lambda) \in R^+_p \Gamma$ and $L(\lambda) \in R^+_p \Gamma$ the restriction of $V(\lambda)$ and $L(\lambda)$ to $\Gamma = G(F_p)$.

1.2. Let $W \subset Aut(X)$ be the Weyl group of $G$. For any $i \in I$ there is a unique element $s_i \in W$ such that $s_i \neq 1$ and $s_i(\varpi_j) = \varpi_j$ for any $j \in I - \{i\}$. Recall that $W$ is a Coxeter group on the generators $\{s_i; i \in I\}$. Let $w \mapsto l(w)$ be the length function of this Coxeter group. Let $w_0$ be the longest element of $W$; let $\nu$ be its length. For any $w \in W$ let $\mathcal{L}(w) = \{i \in I; l(s_iw) < l(w)\}$.

Let $u^{1/2}$ be an indeterminate and let $H$ be the free $Q[u^{1/2}, u^{-1/2}]$-module with basis $\{T_w; w \in W\}$ and with an algebra structure as in [13] 3.3. Let $\hat{W}$ be the set
of all irreducible $W$-module $E$ over $\mathbb{Q}$ (up to isomorphism). For $E \in \hat{W}$ let $E(u)$ be an $H$-module (free as a $\mathbb{Q}[u^{1/2}, u^{-1/2}]$-module) associated to $E$ as in [L2, 1.1]. There is a well defined integer $a_E \geq 0$ such that for $w \in W$ we have
\[
\text{tr}(u^{-l(w)/2}T_w, E(u)) = (-1)^{l(w)}c_w,E u^{-a_E/2} \mod u^{(-a_E+1)/2}\mathbb{Z}[u^{1/2}]
\]
where $c_w,E \in \mathbb{Z}$ for all $w$ and $c_w,E \neq 0$ for some $w \in W$. (One can interpret $c_w,E$ in terms of the character of the irreducible representation associated to $E$ of the $J$-ring of $W$ at the basis element of the $J$-ring corresponding to $w$, see [L4, 3.5(b)].) For $w \in W$ we set $\alpha_w = \sum_{E \in W} c_w,E E$, a virtual representation of $W$. Let $\mathcal{J}$ be the set of “near involutions” of $W$ that is the set of all $w \in W$ such that $w, w^{-1}$ are in the same left cell of $W$. (If $W$ is of classical type, $\mathcal{J}$ is exactly the set of involutions in $W$.) According to [L4, 3.5] for $w \in W$ we have
\[
w \in \mathcal{J} \text{ if and only if } \alpha_w \neq 0.
\]
For $w \in W$ let $R_w \in \mathbb{R}\Gamma$ be as in [L2, 1.5]. By [L3, 6.17], for $w \in W$ there is a well defined object $R_{\alpha_w} \in \mathbb{R}^+\Gamma$ such that
\[
\sharp(W) R_{\alpha_w} = \sum_{E \in W, y \in W} \text{tr}(y, E)c_w,E R_y
\]
in $\mathbb{R}\Gamma$. Note that $R_{\alpha_w}$ is zero unless $w \in \mathcal{J}$.

An irreducible representation $\rho$ of $\Gamma$ (over $\mathbb{C}$) is in $\mathfrak{U}$ if and only if the multiplicity $(\rho : R_{\alpha_w})$ is nonzero for some $w \in \mathcal{J}$.

1.3. In the examples below (types $A_1, A_2, B_2, G_2, A_3$) we write $I = \{1, 2, \ldots \}$ where the notation is such that

(type $A_1$) if $\lambda = (a-1)\varpi_1$ with $a \geq 1$ then $\dim V(\lambda) = a$;
(type $A_2$) if $\lambda = (a-1)\varpi_1 + (b-1)\varpi_2$ with $a \geq 1, b \geq 1$ then $\dim V(\lambda) = ab(a+b)/2$;
(type $B_2$) if $\lambda = (a-1)\varpi_1 + (b-1)\varpi_2$ with $a \geq 1, b \geq 1$ then $\dim V(\lambda) = ab(a+b)/(a+2b)/6$;
(type $G_2$) if $\lambda = (a-1)\varpi_1 + (b-1)\varpi_2$ with $a \geq 1, b \geq 1$ then $\dim V(\lambda) = ab(a+b)(a+2b)/(a+3b)(2a+3b)/120$;
(type $A_3$) if $\lambda = (a-1)\varpi_1 + (b-1)\varpi_2 + (c-1)\varpi_3$ with $a \geq 1, b \geq 1, c \geq 1$ then $\dim V(\lambda) = abc(a+b)(b+c)(a+b+c)/12$.

For a sequence $i_1, i_2, \ldots$ in $I$ we often write $w = i_1 i_2 \ldots$ instead of $w = s_{i_1} s_{i_2} \ldots$; we write $\emptyset$ instead of $w$ where $w$ is the unit element of $W$. We now describe the elements $R_{\alpha_w}$ in several examples. For $\rho \in \mathfrak{U}$ we write $d(\rho) = \dim \rho$.

Type $A_1$, $I = \{1\}$. We have $\mathcal{J} = \emptyset, 1$, $\mathfrak{U} = \{1, S\}$ where $d(1) = 1, d(S) = p$ and
\[
R_{\alpha_0} = 1, R_{\alpha_1} = S.
\]

Type $A_2$, $I = \{1, 2\}$. We have $\mathcal{J} = \emptyset, 1, 2, 121$, $\mathfrak{U} = \{1, r, S\}$ where $d(1) = 1, d(r) = p^2 + p, d(S) = p^3$ and
\[
R_{\alpha_0} = 1, R_{\alpha_1} = R_{\alpha_2} = r, R_{\alpha_{121}} = S.
\]

Type $B_2$, $I = \{1, 2\}$. We have $\mathcal{J} = \emptyset, 1, 2, 121, 212, 1212$, $\mathfrak{U} = \{1, r, e_1, e_2, \theta, S\}$ where
\[
d(1) = 1, d(r) = p(p+1)^2/2, d(e_1) = d(e_2) = p(p^2+1)/2, d(\theta) = p(p-1)^2/2, \\
d(S) = p^4
\]
\[
R_{\alpha_0} = 1, R_{\alpha_1} = r + e_1, R_{\alpha_2} = r + e_2, R_{\alpha_{121}} = \theta + e_2, R_{\alpha_{212}} = \theta + e_1, R_{\alpha_{1212}} = S.
\]
Type $G_2$, $I = \{1, 2\}$. We have $J = \{\emptyset, 1, 2, 121, 212, 12121, 21212, 121212\}$, $\mathcal{M} = \{1, r, r', e_1, e_2, e', f, g, h, S\}$ where
\[
\begin{align*}
d(1) &= 1, 
d(r) = p(p + 1)^2(p^2 + p + 1)/6, 
d(r') = p(p + 1)^2(p^2 - p + 1)/2, 
d(e_1) = d(e_2) = p^4 + p^2 + 1)/3, 
d(e') = p(p - 1)^2(p^2 - p + 1)/6, 
d(f) = p(p - 1)^2(p^2 + p + 1)/2, 
d(g) = d(h) = p(p^2 - 1)^2/3, 
d(S) = p^6,
\end{align*}
\]
$R_{\alpha_0} = 1, R_{\alpha_1} = r + r' + e_1, R_{\alpha_2} = r + r' + e_2,$
\[
\begin{align*}
R_{\alpha_{121}} &= r' + e_2 + f + g + h, 
R_{\alpha_{212}} = r' + e_1 + f + g + h, 
R_{\alpha_{1212}} = e_1 + e' + f,
R_{\alpha_{21212}} = e_2 + e' + f, 
R_{\alpha_{121212}} = S.
\end{align*}
\]
Type $A_3$, $I = \{1, 2, 3\}$. We have $J = \{\emptyset, 1, 2, 3, 121, 212, 232, 2132, 13231, 121321\}$, $\mathcal{M} = \{1, r, r', r'', S\}$ where
\[
\begin{align*}
d(1) &= 1, 
d(r) : p^3 + p^2 + p, 
d(r') : p^4 + p^2, 
d(r'') : p^5 + p^4 + p^3, 
d(S) : p^6,
\end{align*}
\]
$R_{\alpha_0} = 1, R_{\alpha_1} = R_{\alpha_2} = R_{\alpha_3} = r, R_{\alpha_{13}} = R_{\alpha_{212}} = r', 
R_{\alpha_{121}} = R_{\alpha_{1321}} = R_{\alpha_{22}} = r'', R_{\alpha_{121212}} = S.$

2. The elements $M_w \in \mathcal{R}_p^+ \Gamma$ for $w \in J$

2.1. In each of the examples in $[13]$ and for any $w \in J$ we define a virtual representation $M_w \in \mathcal{R}_p \Gamma$ as a certain integer combination of objects $V(\lambda)$. If $I = \{1, 2, \ldots, s\}$ we write $V_{n_1, n_2, \ldots, n_s}$ instead of $V(\lambda)$ where $\lambda = n_1 \varpi_1 + n_2 \varpi_2 + \cdots + n_s \varpi_s$. We set $\delta(w) = \dim(M_w)$.

Type $A_1$: $M_\emptyset = V_0$, $M_1 = V_{p-1}$; $\delta(\emptyset) = 1, \delta(1) = p$.

Type $A_2$: $M_\emptyset = V_{0,0}$, $M_1 = V_{p-1,0}$, $M_2 = V_{0,p-1}$, $M_{121} = V_{p-1,p-1}$; $\delta(\emptyset) = 1, \delta(1) = \delta(2) = p(p + 1)/2, d(121) = p^3$.

Type $B_2$: $M_\emptyset = V_{0,0}$, $M_1 = V_{p-1,0}$, $M_2 = V_{0,p-1}$, $M_{121} = V_{p-3,0}$, $M_{212} = V_{0,p-2}$, $M_{1212} = V_{p-1,p-1}$; $\delta(\emptyset) = 1$.

Type $G_2$: $M_\emptyset = V_{0,0}$, $M_1 = V_{p-1,0}$, $M_2 = V_{0,p-1}$, $M_{121} = V_{p-4,1}$, $M_{212} = V_{1,p-2}$, $M_{1212} = V_{p-4,0}$, $M_{21212} = V_{0,p-2}$, $M_{121212} = V_{p-1,p-1}$; $\delta(\emptyset) = 1$.

Type $A_3$: $M_\emptyset = V_{0,0,0}$,
$M_1 = V_{p-1,0,0}$,
$M_2 = V_{0,p-1,0} - V_{0,p-3,0}$. 

$M_3 = V_{0,0,p-1}$,
$M_{13} = V_{p-1,0,p-1} - V_{p-2,0,p-2}$,
$M_{2132} = V_{0,p-1,0} + V_{0,p-3,0}$,
$M_{121} = V_{p-1,p-1,0}$,
$M_{13231} = V_{p-1,0,p-1} + V_{p-2,0,p-2}$,
$M_{232} = V_{0,p-1,p-1}$,
$M_{21321} = V_{p-1,p-1,p-1}$;
$\delta(\emptyset) = 1$,
$\delta(1) = \delta(3) = p(p+1)(p+2)/6$,
$\delta(2) = p(p+1)^2(p+2)/12 - p(p-1)^2(p-2)/12 = p(2p^2+1)/3$,
$\delta(13) = p^2(p+1)(2p+1)/12 - p^2(p-1)(2p-1)/12 = p^2(3p^2+1)/6$,
$\delta(2132) = p(p+1)^2(p+2)/12 + p(p-1)^2(p-2)/12 = p^2(p+2)/3$,
$\delta(121) = \delta(232) = p^3(p+1)(2p+1)/6$,
$\delta(13231) = p^2(p+1)(2p+1)/12 + p^2(p-1)(2p-1)/12 = p^2(3p^2+1)/3$.

Note that in each of the cases above we have $M_w \in \mathcal{R}_p \Gamma$. (This is obvious except for type $A_3$ and $w = 2$ or $w = 13$ where it can be verified directly.)

2.2. For any $\rho \in \mathcal{U}$ we write $\rho$ (with $\rho \in \mathcal{U}$) as an $\mathbf{N}$-linear combination of $M_w$ (in $\mathcal{R}_p \Gamma$) in each case in 1.3.

Type $A_1$: $\frac{1}{2} = M_0, S = M_1$.

Type $A_2$: $\frac{1}{2} = M_0, r = M_1 + M_2, S = M_{121}$.

Type $B_2$: $\frac{1}{2} = M_0, r = M_1 + M_2, e_1 = M_1 + M_{212}, e_2 = M_{121} + M_2, g = M_{121} + M_{212}, \bar{S} = M_{1212}$.

Type $G_2$: $\frac{1}{2} = M_0, r = M_1 + M_2, r' = M_1 + M_{121} + M_2 + M_{212}, e_1 = M_1 + M_{1212} + M_{212}, e_2 = M_{121} + M_2 + M_{212}, \bar{r}' = M_{1212} + M_{212}, \bar{e}' = M_{1212} + M_{212}, \bar{g} = \bar{h} = M_{121} + M_{212}, \bar{S} = M_{1212}$.

Type $A_3$: $\frac{1}{2} = M_0, r = M_1 + M_2 + M_3, r' = M_{13} + M_{13231} + M_{332}, \bar{S} = M_{12321}$.

We return to a general $G$. We state the following

**Conjecture 2.3.** There exist nonzero objects $M_w \in \mathcal{R}_p \Gamma$, ($w \in \mathcal{J}$) such that for any $\rho \in \mathcal{U}$ we have

(a) $\rho = \sum_{w \in \mathcal{J}} (\rho : R_{\alpha_w}) M_w$.

Moreover, we can assume that the following properties hold.

(i) For any $w \in \mathcal{J}$, $M_w$ is a $\mathbf{Z}$-linear combination of $V_\lambda$ with $\lambda$ very close to $(p-1)\lambda_{\mathcal{L}(w)}$ (see [1,1]).

(ii) We have $\dim(M_w) = \pi_w(p)$ where $\pi_w(t) \in \mathbb{Q}[t]$ ($t$ an indeterminate) is independent of $p$. There exists an involution $w \leftrightarrow \tilde{w}$ of $\mathcal{J}$ such that $t^w \pi_w(1/t) = \pm \pi_{\tilde{w}}(t)$ and $\mathcal{L}(\tilde{w}) = I - \mathcal{L}(w)$ for all $w \in \mathcal{J}$.

(iii) For $w \in \mathcal{J}$ we write $\pi_w(t) \in t^{c(w)} \mathbb{Q}[t], \pi_w(t) \notin t^{c(w)-1} \mathbb{Q}[t]$ where $c(w) \in \mathbb{N}$ is well defined. Then $c(w)$ depends only on the two-sided cell of $W$ containing $w$; it is the value of the $a$-function (see [14, 3.1]) of $W$ on that two-sided cell.

A similar statement can be made when $F_p$ is replaced by the finite field $F_{p^n}$ with $p^n$ elements for some $n \geq 2$.

The conjecture does not say what the $M_w$ are explicitly.

2.4. By the results in 2.1, 2.2 the conjecture holds for types $A_1, A_2, B_2, G_2, A_3$. For these types, the involution $w \leftrightarrow \tilde{w}$ in 2.3 ii) is given as follows:
Type $A_1$: $\emptyset \leftrightarrow 1$;
Type $A_2$: $\emptyset \leftrightarrow 121$, $1 \leftrightarrow 2$;
Type $B_2$: $\emptyset \leftrightarrow 1212$, $1 \leftrightarrow 2$, $12 \leftrightarrow 212$;
Type $G_2$: $\emptyset \leftrightarrow 121212$, $1 \leftrightarrow 212$, $121 \leftrightarrow 212$;
Type $A_3$: $\emptyset \leftrightarrow 121321$, $1 \leftrightarrow 232$, $2 \leftrightarrow 13231$, $3 \leftrightarrow 121$, $13 \leftrightarrow 2132$;

Similar evidence exists for type $A_4$.

2.5. Here is the simplest nontrivial example of objects $M_w$ in 2.3. Assume that $V$ is a three dimensional $F_p$-vector space and $\Gamma = SL(V)$. Let $Z_1$ be the set of lines in $V$. Let $Z_2$ be the set of planes in $V$. Let $F_1$ be the vector space of functions $Z_1 \to k$ with sum of values equal to 0. Let $F_2$ be the vector space of functions $Z_2 \to k$ with sum of values equal to 0. Note that $F_1, F_2$ are naturally $\Gamma$-modules; they both represent $\rho$ where $\rho \in \mathfrak{t}$ has dimension $p^2 + p$. Define $\tau : F_1 \to F_2$ by $(\tau(f))(P) = \sum_{L \subseteq Z_1 : L \subseteq P} f(L)$ where $f \in F_1, P \in Z_2$. Define $\tau' : F_2 \to F_1$ by $(\tau'(f'))(L) = \sum_{P \subseteq Z_2 : P \subseteq L} f'(P)$ where $f' \in F_2, L \in Z_1$. Note that $\tau, \tau'$ are well defined $\Gamma$-linear maps. Let $M$ be the kernel of $\tau$ (it is also the image of $\tau'$). Let $M'$ be the kernel of $\tau'$ (it is also the image of $\tau$). Then $M, M'$ are the objects $M_w$ attached to $\rho$ in 2.3.

2.6. If in the sum 2.3(a) we replace each $M_w$ by the basis element $t_w$ of the $J$-ring of $W$ (see [L4, 3.5]), the resulting element of the $J$-ring is contained in the centre of that ring.

2.7. A statement similar to 2.3(a) can be made for any, not necessarily unipotent, irreducible representation $\rho$ of $\Gamma$. (We replace the $J$-ring of $W$ and the left cells considered in [L5, 1.9] in terms of an extended Weyl group.) We illustrate this in an example.

Let $O$ be an orbit for the obvious $W$-action on $X/(p - 1)X$ such that the stabilizer of any element of $O$ is trivial. For any $\zeta \in O$ there is a unique element $\tilde{\zeta} \in X_p^+$ whose image under $X \to X/(p - 1)X$ equals $\zeta$. Let $\zeta_0 \in O$. Let $\tilde{\zeta}_0'$ be the composition $T : k^* \to k^*$ where $\phi'$ is the homomorphism such that $\phi'(x) = x$ for any $x \in k^*$ (as in [B1]). We can restrict $\tilde{\zeta}_0'$ to $T(F_p)$ and we regard this restriction as a homomorphism $B(F_p) \to k^*$ trivial on the Sylow $p$-subgroup of $B(F_p)$. This last homomorphism can be induced to a representation $\rho$ of $\Gamma$ over $C$ which is in fact irreducible and depends only on $\mathfrak{t}$, not on $\zeta_0'$. From the results of [CL], $\rho$ has each of $L(\zeta)$ ($\zeta \in O$) as a composition factor (but it may also have other composition factors). We expect that

(a) $\rho = \sum_{\zeta \in O} M_\zeta$

where $M_\zeta \in R_p^+ \Gamma$ is a $Z$-linear combination of various $V(\lambda)$ with $\lambda \in X_p^+$ very close to $\zeta$.

In the case where $G = SL_3(k)$ such a statement can be deduced from [CL] (in this case we have $M_\zeta = V(\zeta)$ for each $\zeta \in O$); in the case where $G = Sp_4(k)$, a statement like (a) can be deduced from [J3].

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Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

Email address: gyuri@mit.edu