Abstract

In [12] Hausel introduced a commutative algebra – the multiplicity algebra – associated to a fixed point of the $\mathbb{C}^*$-action on the Higgs bundle moduli space. Here we describe this algebra for a fixed point consisting of a very stable rank 2 vector bundle and zero Higgs field for a curve of low genus. Geometrically, the relations in the algebra are described by a family of quadrics and we focus on the discriminant of this family, providing a new viewpoint on the moduli space of stable bundles. The discriminant in our examples demonstrates that as the bundle varies, we obtain a continuous variation in the isomorphism class of the algebra.

Dedicated to Oscar García-Prada on the occasion of his 60th birthday

1 Introduction

The integrable system introduced in [14] is now 35 years old but there are still unexplored features. A new issue arose in the recent paper [11] with Tamás Hausel and work which followed on from it [12]. It concerns fixed points $m$ of the $\mathbb{C}^*$-action on the moduli space $\mathcal{M}$ of Higgs bundles (these fixed point sets form a subject on which Oscar García-Prada has in particular made fundamental contributions e.g. [1, 2, 10]). Such a point is given by a Higgs bundle $(E, \Phi)$ where $\Phi$ is necessarily nilpotent and so the functions $h = (h_1, \ldots, h_n)$ defining the integrable system all vanish. Then one defines from the $h_i$ a commutative algebra associated to the fixed point $m$, which
is called the \textit{multiplicity algebra} \cite{12}. When the fixed point is \textit{very stable}, the algebra is finite-dimensional and the dimension gives the multiplicity of the component of the nilpotent cone containing \( m \), hence the name. An interesting observation is that for certain fixed points at the upper end of the nilpotent cone the multiplicity algebras are isomorphic to the cohomology of homogeneous spaces and it is this which suggests further research into their structure in more generality.

This paper is about the algebras defined at the \textit{lower} end of the nilpotent cone, namely the case of a very stable bundle, where \( \Phi = 0 \). Here we learn about the algebra by studying examples, but some underlying structure remains in general: in all cases the \( \mathbb{C}^* \)-action defines a grading and yields a Poincaré duality ring – a nondegenerate pairing on subspaces of complementary degree.

When \( E \) has rank 2 and fixed determinant the relations for the algebra are given by \( 3g - 3 \) homogeneous quadratic functions on a vector space of dimension \( 3g - 3 \), more invariantly given by the quadratic map \( \text{tr} \Phi^2 : H^0(C, \text{End}_0 E \otimes K) \to H^0(C, K^2) \). Considering \( H^0(C, \text{End}_0 E \otimes K) \) as the cotangent space at the point \([E]\) of the moduli space \( \mathcal{N} \) of stable bundles, this is the definition of the integrable system.

The particular issue we address here is to exhibit algebras which, up to equivalence, contain continuous parameters unlike the integral structure of cohomology. We use the discriminant: a class \( \alpha \in H^1(C, K^*) \) defines a homomorphism \( \alpha : H^0(C, \text{End}_0 E \otimes K) \to H^1(C, \text{End}_0 E) \) which by Serre duality is equivalent to the quadratic map above. Then \( \det \alpha = 0 \) describes a hypersurface of degree \( 3g - 3 \) in the projective space \( \mathbb{P}(H^1(C, K^*)) \) and the projective equivalence class of this is an invariant of the algebra, independent of any specific choice of generators and relations. In the terminology of classical algebraic geometry we have a family of quadrics and the discriminant is a familiar invariant.

In genus \( g = 2 \) we have three quadratic functions on a 3-dimensional vector space which geometrically is a net of conics. In \cite{15} we showed that if the degree of \( E \) is odd then there are just finitely many equivalence classes of algebras, the very stable ones having three generators \( \xi_i \) with \( \xi_i^2 = 0 \). This is a case where the algebra is indeed isomorphic to the cohomology \( H^*((\mathbb{C}P^1)^3, \mathbb{C}) \). In contrast we show in Section \textsection{3} that when \( E \) has even degree the discriminant is a cubic curve isomorphic to a plane section of a singular cubic surface, and the modulus of this elliptic curve varies as \( E \) varies in the moduli space of stable bundles. To do this we make essential use
of the description in [25] of the integrable system using a classical relationship between certain curves of genus 2, 3 and 5. The discriminant approach also gives a new viewpoint on the moduli space: the space of discriminants can be seen as the projection from a point on the Igusa quartic threefold, the quotient of the moduli space $P^3$ of stable bundles (as described in the foundational paper [18]) by the group $H^1(C, Z_2)$.

The geometry of these same three curves comes into play when we consider in Section 4 a family of bundles of degree zero on a general nonhyperelliptic curve of genus 3. The moduli space $\mathcal{N}$ for $\Lambda^2 E \cong O$ is well-known [19] to be a special singular quartic hypersurface in $P^7$, initially studied by A.Coble. We consider a two-dimensional family by choosing a line bundle $U$ of order 2 and looking at the fixed point set in $\mathcal{N}$ under the action $E \mapsto E \otimes U$. This is a quartic surface in $P^3 \subset P^7$, in fact the Kummer surface of the Jacobian for a genus 2 curve, part of the story of the previous section.

The results of Pauly [23] allow us to write down the six relations for the algebra and to determine the discriminant variety more geometrically. We find that for our chosen family the degree 6 hypersurface is reducible to a singular quadric and a quartic. The degeneracy subspace of the quadric is a projective plane and its intersection with the quartic hypersurface is a quartic curve. It is the modulus of this curve which varies as we vary $E$ in the family, but we use a genericity argument to show this rather than using the explicit formula. The geometry behind this introduces a quartic surface with 10 nodes associated to a pair of conics.

The original motivation for this study in [11] involves mirror symmetry for the Higgs bundle moduli space. The precise role of the multiplicity algebra in this context has yet to be determined.

2 The quadratic map

Let $C$ be a curve of genus $g > 1$, $E$ a rank 2 vector bundle over $C$ with $\Lambda^2 E$ fixed, and let $\text{End}_0 E$ denote the bundle of trace zero endomorphisms of $E$. For $\Phi \in H^0(C, \text{End}_0 E \otimes K)$ we consider $\text{tr} \Phi^2 \in H^0(C, K^2)$. When $E$ is stable, then the dimension of both these spaces is $3g - 3$ and we consider the map

$$\text{tr} \Phi^2 : H^0(C, \text{End}_0 E \otimes K) \to H^0(C, K^2)$$

which commutes with the scalar action of $\lambda \in \mathbb{C}^*$ on $\Phi$ and $\lambda^2$ on the right hand side. As $[E]$ varies over the moduli space $\mathcal{N}$ of stable bundles, linear
functions on $H^0(C, K^2)$ generate the integrable system of [14].

We can view the situation in different ways, one is to regard it as a family of quadrics in $P^{3g-4} \cong P(H^0(C, \text{End}_0 E \otimes K))$ parametrized by $P^{3g-4} \cong P(H^1(C, K^*))$, evaluating $\alpha \in H^1(C, K^*) = H^0(C, K^2)^*$ on $\text{tr} \Phi^2$. Thus in genus 2 it is a net of conics using the classical term. The locus of singular quadrics is a hypersurface, the discriminant, in $P(H^1(C, K^*))$.

A bundle $E$ is called very stable [16] if there is no nonzero nilpotent Higgs field $\Phi \in H^0(C, \text{End}_0 E \otimes K)$. In rank 2 nilpotency is equivalent to $\text{tr} \Phi^2 = 0$, or a non-empty base locus for the family of quadrics. The very stable bundles are important, not only because they constitute the generic situation, but also because then the quadratic map is proper [24], a crucial feature in the more general setting of [11].

One further point – our question just involves the endomorphism bundle and not $E$ itself and so is insensitive to the action $E \mapsto E \otimes L$ where $L$ is a line bundle (with $L^2$ trivial if we want to fix $\Lambda^2 E$). Equivalently, we are only concerned with the projective bundle $P(E)$.

3 Genus two curves

3.1 Odd degree

Let $C$ be a smooth curve of genus 2, then Atiyah [3] described the projective bundle associated to an indecomposable rank 2 vector bundle $E$ of odd degree. The construction represents the projective bundle by a vector bundle which is an extension $O \rightarrow E \rightarrow L$ where $\deg L = 1$.

The extension class $[\alpha]$ lies in $H^1(C, L^*)$ and since $\dim H^1(C, L^*) = 2$ it is uniquely determined up to a multiple by its annihilator, a section $s \in H^0(C, KL)$ with $[\alpha]s = 0 \in H^1(C, K)$. Then $s$ has divisor $p + q + r$.

Now $E$ has 4 degree zero subbundles, and the four divisors are

$$p + q + r, \quad p + \sigma(q) + \sigma(r), \quad q + \sigma(r) + \sigma(p), \quad r + \sigma(p) + \sigma(q)$$

where $\sigma : C \rightarrow C$ is the hyperelliptic involution. Let $\pi : C \rightarrow P^1$ denote the quotient map, then $(\pi(p), \pi(q), \pi(r)) = (a_1, a_2, a_3)$ and out of the eight inverse images, the four above give the same projective bundle. Hence we have Atiyah’s description of the moduli space as a branched double covering of the symmetric product $S^3P^1$. This symmetric product may be considered...
as \( P(V) \) where \( V \) is the 4-dimensional space of cubic polynomials \( p(x) \) with roots \( a_1, a_2, a_3 \). If \( x_1, \ldots, x_6 \in \mathbb{P}^1 \) are the images of the fixed points of \( \sigma \) (so that the curve has equation \( y^2 = (x-x_1) \ldots (x-x_6) \)) then the branch locus consists of the six planes \( p(x_i) = 0 \).

In [15] we calculated the three-dimensional space of sections of \( \text{End}_0 E \otimes K \) in terms of a \( C^\infty \) splitting of the extension: writing \( E = \mathcal{O} \oplus L \) with \( \partial \)-operator \( \partial E(f,g) = \partial f + \alpha g \) where \( \alpha \in \Omega^{0,1}(C, L^* \otimes K) \) represents the extension class. The condition \( [\alpha]s = 0 \) means we can write \( \alpha = \partial u/s \) where \( u \) is supported in a neighbourhood of the zeros \( p, q, r \) of \( s \) and in fact can be chosen to vanish at \( q, r \).

In concrete terms the space of Higgs fields has basis

\[
(x - a_2) \left( 1 - \frac{2}{s}(u - u_1(x-a_2)) \right) \frac{dx}{y}, \quad (x - a_3) \left( 1 - \frac{2}{s}(u - u_1(x-a_3)) \right) \frac{dx}{y},
\]

\[
\left( -u \frac{1}{s}(-u^2 + u_1^2 \frac{(x-a_2)(x-a_3)}{a_1-a_2}) \right) \frac{dx}{y}
\]

where \( u_1 = u(p) \).

**Remark:** Consider the map from \( \mathcal{O} \oplus K^* \) to \( E \) given by \((1, t) \mapsto (1 - ut, st)\) where \( t \) is a local holomorphic section of \( K^* \). Since \( \partial(1 - ut) + (\partial u/s)st = 0 \) this is holomorphic and the quotient sheaf is \( \mathcal{O}_{p+q+r}(L) \). So a Hecke transform on \( E \) at the points \( p, q, r \) gives the bundle \( \mathcal{O} \oplus K^* \). But the formulas above show that each Higgs field \( \Phi \) on \( E \) preserves the trivial subbundle at \( p, q, r \) and so they transform to the Higgs fields on \( \mathcal{O} \oplus K^* \). It follows that \( H^0(C, \text{End}_0 E \otimes K) \), considered as a Lagrangian submanifold of the Higgs bundle moduli space \( \mathcal{M} \), is a Hecke transform of the Hitchin section. Then the isomorphism of the multiplicity algebra with the cohomology of \( (\mathbb{CP}^1)^3 \) is a consequence of [12] Theorem 3.10, as well as the direct calculation in [15].

The quadratic form on \( H^0(C, \text{End}_0 E \otimes K) \) with values in \( H^0(C, K^2) \) takes the form

\[
f(a_1) \frac{(x-a_2)(x-a_3)}{(a_1-a_2)(a_1-a_3)} \xi_1^2 + f(a_2) \frac{(x-a_3)(x-a_1)}{(a_2-a_3)(a_2-a_1)} \xi_2^2 + f(a_3) \frac{(x-a_1)(x-a_2)}{(a_3-a_1)(a_3-a_2)} \xi_3^2
\]

where \( f(x) = (x-x_1) \ldots (x-x_6) \) and we identify \( H^0(C, K^2) \) with \( \pi^* H^0(P^1, \mathcal{O}(2)) \).
Our main focus in this article is the discriminant, and since the multiplicity algebra has relations $\xi_1^2 = \xi_2^2 = \xi_3^2 = 0$ this consists of three lines in $\mathbb{P}^2$. However, introducing the geometry of the curve itself we have three lines in $\mathbb{P}(H^1(C, K^*)) \cong \mathbb{P}^2$, the dual space of $H^0(C, K^2)$. The linear system $K^2$ maps $C$ to a conic $C_0$ in $\mathbb{P}(H^1(C, K^*))$ – evaluating $H^0(C, K^2)$ at a point $p \in C$ is the map. Considering the coefficients of $\xi_i^2$ in the above formula we see that the three lines form the sides of the triangle with vertices $a_1, a_2, a_3$ on the conic, isomorphic to $\mathbb{P}^1 = \pi(C)$. Thus, apart from the double covering, the discriminant reproduces Atiyah’s parametrization.

### 3.2 Even degree

Now consider $E$ a stable bundle over $C$ with $\Lambda^2 E$ trivial. The moduli space of (S-equivalence classes of) semi-stable bundles in this case is well-known to be $\mathbb{P}^3$ [18], defined by considering the line bundles $L$ of degree 1 such that $E \otimes L$ has a non-zero section: for a given $E$, $L$ varies in a $2\Theta$-divisor in $\text{Pic}^1(C)$. There are explicit formulas in [13] for the three quadratic functions:

\[
\begin{align*}
    h_1 &= rst[\xi_0(u_0^2 - 1) + \xi_1(u_0u_1 + u_2) + \xi_2(u_2u_0 + u_1)]^2 - \nonumber \\
    &\quad \quad \quad \quad st[\xi_0(u_0u_1 - u_2) + \xi_1(u_1^2 + 1) + \xi_2(u_1u_2 + u_0)]^2 + \nonumber \\
    &\quad \quad \quad \quad 4rs(\xi_0u_0 + \xi_1u)^2 - rt[\xi_0(u_0^2 + 1) + \xi_0(u_0u_1 + u_2) + \xi_2(u_2u_0 - u_1)]^2 \nonumber \\
    h_2 &= \quad t(u_0^2 + u_1^2 + u_2^2 + 1)[(\xi_0^2 + \xi_1^2 + \xi_2^2) + (\xi_0u_0 + \xi_1u_1 + \xi_2u_2)^2] + \nonumber \\
    &\quad \quad \quad \quad st(u_0^2 - u_1^2 + u_2^2 - 1)[(\xi_0^2 - \xi_1^2 + \xi_2^2) - (\xi_0u_0 + \xi_1u_1 + \xi_2u_2)^2] + \nonumber \\
    &\quad \quad \quad \quad 4r(u_0u_2 - u_1)[\xi_0\xi_2 + (\xi_0u_0 + \xi_1u_1 + \xi_2u_2)\xi_1] + \nonumber \\
    &\quad \quad \quad \quad 4s(u_0u_2 + u_1)[\xi_0\xi_2 - (\xi_0u_0 + \xi_1u_1 + \xi_2u_2)\xi_1] + \nonumber \\
    &\quad \quad \quad \quad 4t(u_0u_1 + u_2)[\xi_0\xi_2 - (\xi_0u_0 + \xi_1u_1 + \xi_2u_2)\xi_2] \nonumber \\
    h_3 &= \quad s[\xi_0(u_2u_0 + u_1) + \xi_1(u_1u_2 + u_0) + \xi_2(u_2^2 - 1)]^2 - \nonumber \\
    &\quad \quad \quad \quad \xi_0(u_2u_0 - u_1) + \xi_1(u_1u_2 + u_0) + \xi_2(u_2^2 + 1)]^2 - \nonumber \\
    &\quad \quad \quad \quad t[\xi_0(u_0u_1 + u_2) + \xi_2(u_1u_2 - u_0) + \xi_1(u_2^2 + 1)]^2 + 4r(\xi_1u_1 + \xi_2u_2)^2. \nonumber
\end{align*}
\]

Here $(u_0, u_1, u_2)$ are affine coordinates on $\mathbb{P}^3$, and $(\xi_0, \xi_1, \xi_2)$ the corresponding coordinates on the cotangent space. The genus 2 curve $C$ is the double cover of $\mathbb{P}^2$ branched over the six points $0, 1, \infty, r, s, t$. Then $E$ is determined by fixing $u_i$ and the relations in the algebra are given by the vanishing of the $h_i$. This makes it difficult to draw conclusions about the structure of the multiplicity
algebra from this mass of formulae so here we adopt a more geometric approach, drawing on the paper of van Geemen and Previato [25], which was the first investigation into explicit formulae for the integrable system. The genus 2/3/5 story below appears in various contexts (see [7], [9], [17], [26]).

If $E \otimes L$ has a non-zero section, then by Riemann-Roch, Serre duality and the isomorphism $E^* \cong E \otimes \Lambda^2 E^* \cong E$, we have $H^0(C, E \otimes KL^*) \neq 0$. It follows that each $2\Theta$-divisor is symmetric with respect to the involution $L \mapsto KL^*$ on Pic$_1(C)$. If the divisor is smooth, it is a curve $C_5$ of genus 5 and the involution acts freely with quotient $C_3$ of genus 3 (we use the notation of [25]). This is a plane section of a Kummer quartic surface, and if it is nonsingular then the bundle is certainly very stable, since, as in [22], the complement – the so-called “wobbly” locus – consists of either the strictly semistable bundles or the 16 hyperplanes which meet the Kummer surface in a double conic.

The double covering is defined by a line bundle $U$ on $C_3$ with $U^2$ trivial and we consider the 2-dimensional space $H^0(C_3, K_3 U)$. The tangent space at $[E] \in N$ is $H^1(C, \text{End}_0 E)$ but in terms of the corresponding $2\Theta$-divisor $C_5$ it is identified with a subspace of sections of the normal bundle in Pic$_1(C)$. The normal bundle is the canonical bundle so there is a map from global sections of the tangent bundle of Pic$_1(C)$ to sections of the normal bundle i.e. $H^1(C, \mathcal{O}) \to H^0(C_5, K_5)$. Deformations of $E$ are given by curves $C_5$ which are divisors of the $2\Theta$ line bundle and so are transverse to translations on Pic$_1(C)$, which is the image of $H^1(C, \mathcal{O})$. They are also symmetric with respect to the involution. It follows that $H^1(C, \text{End}_0 E)$ is isomorphic to the even sections of $K_5$ and $H^1(C, \mathcal{O})$ to the odd ones. In terms of the genus 3 curve we have

$$H^1(C, \text{End}_0 E) \cong H^0(C_3, K_3), \quad H^1(C, \mathcal{O}) \cong H^0(C_3, U K_3)$$

and taking duals $H^0(C, \text{End}_0 E \otimes K) \cong H^1(C_3, \mathcal{O})$.

If $s,t$ form a basis of $H^0(C_3, U K_3)$ then we have sections $q_1 = s^2, q_2 = st, q_3 = t^2$ of $K_3^2$. If $C_3$ is nonhyperelliptic then every quadratic differential $q_i$ is uniquely a quadratic form $Q_i$ in elements of $H^0(C_3, K_3)$. Since $q_1 q_3 = s^2 t^2 = (st)^2 = q_2^2$ the homogeneous quartic equation $Q_1 Q_3 - Q_2^2 = 0$ defines $C_3$ in its canonical embedding: $C_3 \subset \mathbb{P}(H^1(C_3, \mathcal{O}))$.

The key result is:

**Proposition 1.** [25] Under the isomorphism $H^0(C, \text{End}_0 E \otimes K) \cong H^0(C_3, K)^*$ the net of conics is spanned by $Q_1, Q_2, Q_3$. 

7
Remark:
1. The three quadratic forms correspond to basis elements of $S^2 H^0(C, UK_3) \cong S^2 H^1(C, O)$ and since each quadratic differential in genus 2 is a quadratic in sections of $K$, this is $H^1(C, K^*) \cong H^0(C, K^2)^*$. This is the invariant form for the map but it is more convenient from our point of view to think of the multiplicity algebra as the 8-dimensional algebra generated by $1, x, y, z$ with relations $Q_i(x, y, z) = 0$.

2. The essential input for the authors of [25] to prove the isomorphisms in (1) involves the section $s$ of $E \otimes L$ and $s'$ of $E \otimes KL^*$. Then $\Phi = s \otimes s' - \langle s, s' \rangle / 2$ is a trace zero Higgs field.

The discriminant in $P^2 = P(H^1(C, K^*))$ is the cubic curve $D$ defined by $\det(\sum_{i=1}^3 y_i Q_i) = 0$. The curve $C$, as before, is mapped to a conic $C_0 \subset P(H^1(C, K^*))$ by the linear system $K^2$ with six distinguished points $x_1, \ldots, x_6$ the branch points of $\pi : C \to P^1$. We shall use the following

**Proposition 2.** The six points $x_1, \ldots, x_6$ lie on the discriminant.

**Proof.** A fixed point $\tilde{x}_i$ of $\sigma$ is the divisor for the unique section $s$ of a square root $K^{1/2}$ of the canonical bundle. Hence $\dim H^0(C, K^{1/2})$ is odd. If $E$ is a rank 2 bundle of degree zero, and hence topologically trivial, then since $\text{End}_0 E$ is odd-dimensional and has an orthogonal structure given by $\text{tr} \phi^2$ it follows from the mod 2 index theorem [4] that $\dim H^0(C, \text{End}_0 E \otimes K^{1/2})$ is also odd (this is part of a more general story [20]) and hence contains a non-zero section $\Psi$. So $s\Psi \in H^0(C, \text{End}_0 E \otimes K)$ vanishes at $\tilde{x}_i$ and hence so does $\text{tr}(s\Psi \Phi)$ for all $\Phi$. This means that evaluation of $H^0(C, K^2)$ at $\tilde{x}_i$, i.e. its image of $x_i$ in $P(H^1(C, K^*))$ under the bicanonical map, gives a degenerate quadratic form, and hence lies on the discriminant. □

**Remark:** For higher genus $\dim H^0(C, \text{End}_0 \otimes K) = 3g - 3 > 3 = \text{rk}(\text{End}_0 \otimes K)$ and so for any point $x \in C$ there exists a Higgs field vanishing at $x$. Consequently the bicanonical image of $C$ always lies in the discriminant hypersurface.

To describe the discriminant curve for a bundle $E$, from Proposition 2 we need to understand the cubics through the six points $x_i$ on $C_0$. Blow up $P^2$ at these points, then the conic $C_0$ becomes a $-2$ curve. The linear system of cubics through the six points is then equivalent to $3H - E_1 - \cdots - E_6$. 

8
where $E_i$ are the exceptional curves and $H$ the class of a line. This maps the surface to $S \subset \mathbb{P}^3$, a cubic surface with a node from the collapse of the $-2$ curve. Then any cubic curve through the $x_i, i = 1, .., 6$, is defined by a plane section of $S$. We shall show next that conversely a generic section arises from a stable bundle $E$.

**Proposition 3.** To each genus 2 curve $C$ we associate a singular cubic surface $S$ as above. Then a generic plane section is the discriminant of the net of conics defined by a very stable rank 2 bundle $E$ on $C$.

**Proof.** We start with the classical fact (see [6]) that the choice of a non-trivial line bundle of order 2 on a nonsingular plane cubic curve $D$ provides an equation of $D$ as $\det(\sum_{i=1}^{3} y_i Q_i) = 0$ where $Q_1, Q_2, Q_3$ are symmetric $3 \times 3$ matrices. Parametrize the conic $C_0 \cong \mathbb{P}^1$ by $(y_1, y_2, y_3) = (1, 2u, u^2)$ then the six points of intersection $b_i \in D \cap C_0$ are the roots of $\det(Q_1 + 2uvQ_2 + v^2Q_3) = 0$.

Each $Q_i$ defines a conic $Q_i(x, y, z) = 0$ in a projective plane $\mathbb{P}^2$ and $Q_1Q_3 - Q_2^2 = 0$ a quartic curve $C_3$ which for generic $D$ will be smooth. Since $O(1)$ on $\mathbb{P}^2$ is the canonical bundle $K_3$ on $C_3$ the equation says that $Q_1$ on $C_3$ is a section of $K_3^2$ with double zeros, that is $s^2$ for a section $s$ of $K_3U$ for some line bundle $U$ with $U^2$ trivial. But on $C_3$

$$(uQ_1 + vQ_2)^2 = u^2Q_1^2 + 2uvQ_1Q_2 + v^2Q_2^2 = Q_1(u^2Q_1 + 2uvQ_2 + v^2Q_3)$$

so as $u, v$ vary the bundle $U$ is constant and we have $Q_1 = s^2$, $Q_3 = t^2$, $Q_2 = st$ for a basis $s, t$ of sections of $K_3U$.

The line bundle $U$ defines an unramified covering, a curve $C_5$ of genus 5 with an involution $\tau$. The Prym variety $P(C_5, C_3)$ is a 2-dimensional abelian variety consisting of the divisor classes in $\text{Pic}^0(C_5)$ which are anti-invariant under $\tau$. It has two components and $x - \tau(x)$ embeds $C_5$ (if it is not hyperelliptic) in the non-trivial component. Then (see [26] or the other references above) each component is isomorphic to the Jacobian of the genus 2 curve $y^2 = \det(Q_1 + 2uQ_2 + u^2Q_3)$ and $C_5$ is a symmetric 2Theta-divisor with $C_3$ a plane section of the Kummer surface. This defines a point in the moduli space which, thanks to Proposition 1, has the given cubic $D$ as discriminant.

The isomorphism of the two abelian surfaces can be standardized up to elements of order two [25], for at the branch point $x_i$ the conic is a pair of lines each of which is a bitangent to $C_3$, that is sections $s_1, s_2$ of $K_3^{1/2}, K_3^{1/2}U$. Then pulled back to $C_5$ this gives a line bundle $K_5^{1/2}$ with $H^0(C_5, K_5^{1/2}) = \ldots$
2. So $b_i \in \text{Pic}^1(C)$ corresponds to $K_5/2 \in \text{Pic}^4(C_5)$ and $K_5/2 + a - \tau(a)$ identifies with the Prym variety. Since we are concerned with $\text{End}_0 E \otimes K$, the choice of line bundle of order 2 is irrelevant.

**Corollary 4.** For a fixed genus 2 curve, the isomorphism class of the multiplicity algebra for a very stable rank 2 bundle $E$ of even degree varies continuously as $E$ varies.

**Proof.** From [5], apart from quadrics, smooth plane sections of a surface vary the modulus of the curve non-trivially. In our case each section is a cubic curve isomorphic to the discriminant of the net of conics associated to $E$.

**Remark:** Passing from the discriminant to the bundle $\text{End}_0 E$ involved a degree 3 covering of the space $\mathbb{P}^3$ of plane sections of the singular cubic surface. This fits into a classical situation: two threefolds, dual to each other, the Igusa quartic $B$ in $\mathbb{P}^4$ and the Segre cubic in the dual projective space. The quartic has an interpretation as the GIT quotient of six points on a conic and a point $x$ on it represents by duality a hyperplane section of the cubic. The six points $x_i$ give us the genus 2 curve $C$ and the singular cubic surface $S$ is constructed, as in the proposition, by blowing up the points. Then the plane sections of $S$ correspond by duality to the lines through $x$. Projection from $x \in B$ is then a threefold covering of $\mathbb{P}^3$, the space of discriminants. But the Igusa quartic is also the quotient of $\mathbb{P}^3$ by the action of $H^1(C, \mathbb{Z}_2)$, or equivalently the moduli space of projective bundles $\mathbb{P}(E)$, so we have an analogue of Atiyah’s description of the moduli space as a branched cover of $\mathbb{P}^3$, at least in the very stable situation.

### 4 Genus 3 curves

#### 4.1 The vector bundles

Let $C$ be a nonhyperelliptic curve of genus 3 – a nonsingular plane quartic in the canonical embedding. The moduli space $\mathcal{N}$ of (semi)stable rank two bundles $E$ with $\Lambda^2 E$ trivial is isomorphic to the Coble quartic hypersurface in $\mathbb{P}^7$ [19]. The decomposable bundles $E = L \oplus L^*$ describe a three dimensional Kummer variety, the quotient of the Jacobian $\text{Jac}(C)$ by $x \mapsto -x$, which is the singular locus. The Coble quartic is characterized by this property together with invariance by the action on $\mathbb{C}^5$ of a finite Heisenberg
group which corresponds to $E \mapsto E \otimes U$ for $U \in H^1(C, \mathbb{Z}_2)$, a line bundle with $U^2$ trivial.

The discriminant here is a degree 6 hypersurface in $P(H^0(C, K^*)) = P^5$ which is quite complicated – as noted above the bicanonical embedding of $C$ lies in it, and the rank of the quadratic form there is at most 3, so this is a curve of singularities. We shall consider instead the family of bundles which are defined by the fixed points in $N$ of one element $U$. This is the intersection of a $P^3 \subset P^7$ with the quartic hypersurface, hence a quartic surface, and we have a 2-dimensional family.

A fixed point in the moduli space means an isomorphism $\psi : E \rightarrow E \otimes U$. We can view this as a Higgs bundle twisted with $U$ rather than the usual $K$, which implies that we have a genus 5 spectral curve $\pi : \tilde{C} \rightarrow C$, the unramified covering corresponding to $U$, and $E$ is the direct image $\pi_*(L \otimes \pi^*U^{1/2})$ where $\sigma^*L \cong L^*$ i.e. $L$ lies in the Prym variety. The bundle $E$ is strictly semistable if $L^2$ is trivial.

A section $e$ of $E = \pi_*(L \otimes \pi^*U^{1/2})$ on an open set $V \subset C$ is by definition of the direct image a section $s \in H^0(\pi^{-1}(V), L \otimes \pi^*U^{1/2})$ and then $s\sigma^*s \in H^0(V, U)$ is a nondegenerate $U$-valued quadratic form $(e, e)$ on $E$. In the presence of this orthogonal structure, the trace zero endomorphisms $\text{End}_0 E$ split into symmetric and skew symmetric components, so we have

$$\text{End}_0 E \cong U \oplus E'.$$  \hspace{1cm} (2)

The isomorphism $\psi : E \cong E \otimes U$ gives an involution on $\text{End}_0 E$ and the above summands are the eigenspaces. Moreover the rank 2 bundle $E'$ has an orthogonal structure, this time with values in the trivial bundle. If we consider $\psi$ as a $U$-valued Higgs field on $E$, its eigenspaces in the adjoint representation are squares of the eigenspaces on $E$. This means that $E' = \pi_*L^2$.

The expression $\text{End}_0 E \cong U \oplus E'$ is an orthogonal decomposition with respect to the quadratic form $\text{tr } \phi^2$ so the map

$$\text{tr } \Phi^2 : H^0(C, \text{End}_0E \otimes K) \rightarrow H^0(C, K^2)$$

in this case takes $(u, e) \in H^0(C, UK) \oplus H^0(C, E' \otimes K)$ to a multiple of $u^2 + (e, e)$.

We met the first term $H^0(C, UK)$ with $C = C_3$ in the previous section, giving quadratic forms $Q_1, Q_2, Q_3$. These will appear next in a different form.
4.2 The net of quadrics

In [23], Pauly associates to a rank 2 stable bundle $E$ with trivial determinant a bundle $F$ with $\Lambda^2 F \cong K$ such that $\dim H^0(C, E \otimes F) = 4$. It arises as follows: $E$ is defined as an extension $L^* \to E \to L$ where $\deg L = 1$ and the extension class lies in $H^1(C, L^{-2})$ which has dimension 4. This class annihilates a 3-dimensional subspace $W \subset H^0(C, L^2 K)$. Then $(F \otimes L)^*$ is defined as the kernel of the evaluation map $ev : C \times W \to L^2 K$. In the long exact sequence

$$0 \to H^0(C, F \otimes L^*) \to H^0(C, F \otimes E) \to H^0(C, F \otimes L) \xrightarrow{\delta} H^1(C, F \otimes L^*) \quad (3)$$

consider first $H^0(C, F \otimes L^*)$. We have $(F \otimes L)^* \subset C \times W$ so since $F \cong F^* \otimes K$, $F \otimes L^* K^* \subset C \times W$. Tensoring with $K$ gives

$$H^0(C, F \otimes L^*) \to H^0(C, K) \otimes W \to H^0(C, K^2 L^2)$$

and by Riemann-Roch $\dim H^0(C, F \otimes L^*) \geq 9 - 8 = 1$. The connecting homomorphism $\delta$ in the sequence (3) is zero on $W^* \subset H^0(C, F \otimes L)$ and, if the bundle is not a so-called exceptional one, $\dim H^0(C, F \otimes L^*) = 1$ and this gives the four dimensions for $H^0(C, E \otimes F)$. Importantly, it turns out that $F$ is independent of the choice of $L^*$, which is a maximal subbundle – there are generically eight of these. As shown in [23], if $E$ is stable then $F$ is stable unless $E$ is defined by a point on the Coble quartic which lies on a trisecant of the Kummer variety.

The isomorphism $\Lambda^2 E \cong \mathcal{O}$ defines a skew form on $E$ and $\Lambda^2 F \cong K$ gives a skew form on $F$ with values in $K$ so the tensor product has a $K$-valued symmetric form. Then the 4-dimensional space $V = H^0(C, E \otimes F)$ has a symmetric bilinear form $(,)$ with values in $H^0(C, K)$, geometrically a net of quadrics in $\mathbb{P}^3 = \mathbb{P}(V)$.

4.3 The multiplicity algebra

Following on from this approach, in [15] the author made use of the natural map

$$\Lambda^2(E \otimes F) \to S^2 E \otimes \Lambda^2 F \cong \text{End}_0 E \otimes K$$

which induces one from $\Lambda^2 H^0(C, E \otimes F)$ to $H^0(C, \text{End}_0 E \otimes K)$. Both spaces have dimension 6 and it was shown that this is an isomorphism.
Explicitly, if we choose a basis \( v_1, \ldots, v_4 \) of \( V \) then the quadratic form \( \text{tr} \Phi_1 \Phi_2 \) with values in \( H^0(C, K^2) \) is given up to a scale by

\[
(v_i \wedge v_j, v_i \wedge v_k) = (v_i, v_i)(v_j, v_k) - (v_i, v_k)(v_j, v_i)
\]

\[
(v_1 \wedge v_2, v_3 \wedge v_4) = (v_1, v_3)(v_2, v_4) - (v_1, v_4)(v_2, v_3) + \sqrt{\det(v_i, v_j)}.
\]

The square root means that the equation of the quartic is \( \det(v_i, v_j) - Q^2 \) for some quadratic \( Q(x, y, z) \). Thus \( \det(v_i, v_j) = 0 \) is a quartic curve meeting \( C \) tangentially at 8 points, and Pauly describes in more detail the rational map from the moduli space of projective bundles \( P(E) \) to the space of tangential quartics.

In our case \( E \cong E \otimes U \) and from the construction of the bundle \( F \) it follows that \( F \cong F \otimes U \) which means the composition of the isomorphisms defines an involution \( \tau \) on \( V = H^0(C, E \otimes F) \) which is orthogonal with respect to the \( H^0(C, K) \)-valued inner product. We also have the symmetric form on \( E \) with values in \( U \) which gives a skew form on \( V \) with values in \( H^0(C, UK) \) (this defines a homomorphism from \( \Lambda^2(E \otimes F) \) to \( UK \) which gives the decomposition (2)).

**Remark:** The analogy was drawn in [15] with the differential geometry of four dimensions expressed in terms of the two spinor bundles \( S_+, S_- \) with \( E, F \) playing similar roles here. In pursuit of this analogy the situation we have here is parallel to Kähler geometry in two complex dimensions with the involution being parallel to the complex structure and the skew form to the Kähler form.

The involution \( \tau \) on \( V \) induces one on \( \Lambda^2 V \cong H^0(C, \text{End}_0 V \otimes K) \cong H^0(C, UK) \oplus H^0(C, E' \otimes K) \). The first summand is a 2-dimensional +1 eigenspace, with a 4-dimensional \(-1\) eigenspace as orthogonal complement. It follows that \( \tau \) splits \( V = V^+ \oplus V^- \) into two orthogonal 2-dimensional subspaces spanned by \( v_1, v_2 \) and \( v_3, v_4 \), the \( \pm 1 \) eigenspaces of \( \tau \), and each has a quadratic form \( A^+, A^- \). Then \( \det(v_i, v_j) = \det A^+ \det A^- \) so the equation of the quartic is \( \det A^+ \det A^- - Q^2 = 0 \) – the familiar expression \( Q_1 Q_2 = Q^2 \) associated to \( U \) as in the first section.

In Pauly’s description of the moduli space of projective bundles on \( C \) using tangential quartic curves, the case we are considering is where the tangential quartic is a pair of conics, defined by two sections of \( KU \).
4.4 Explicit forms

Take a basis \(v_1, v_2, v_3, v_4\) as above and write \(b_{ij} = (v_i, v_j)\), sections of \(K\), and take a corresponding basis \(v_{23}, v_{31}, v_{12}, v_{41}, v_{42}, v_{43}\) where \(v_{ij} = v_i \wedge v_j\) for the six-dimensional exterior product \(\Lambda^2 V \cong H^0(C, \text{End}_0 E \otimes K)\). The involution acts as 1 on \(v_{12}, v_{34}\) and \(-1\) on the others. Then the matrix of the quadratic form \(\text{tr} \Phi^2\) with values in \(H^0(C, K^2)\) is

\[
\begin{bmatrix}
  b_{22}b_{33} & -b_{12}b_{33} & 0 & Q - b_{12}b_{34} & -b_{22}b_{34} & 0 \\
  -b_{12}b_{33} & b_{11}b_{33} & 0 & b_{11}b_{34} & Q + b_{12}b_{34} & 0 \\
  0 & 0 & Q_1 & 0 & 0 & Q \\
  Q - b_{12}b_{34} & b_{11}b_{34} & 0 & b_{11}b_{44} & b_{12}b_{44} & 0 \\
  -b_{22}b_{34} & Q + b_{12}b_{34} & 0 & b_{12}b_{44} & b_{22}b_{44} & 0 \\
  0 & 0 & Q & 0 & 0 & Q_2
\end{bmatrix} \tag{4}
\]

where \(Q_1 = \det A^+ = b_{11}b_{22} - b_{12}^2, Q_2 = \det A^- = b_{33}b_{44} - b_{34}^2\). The discriminant is given by the vanishing of this determinant.

Remark: To explain the roles of quadratic forms in this picture, recall that the discriminant lies in \(P(H^1(C, K^*))\) and by Serre duality \(H^1(C, K^*)\) consists of linear functions on \(H^0(C, K^2)\) and the latter are quadratic expressions in \(x, y, z \in H^0(C, K)\). Then \(x^2, y^2, z^2, xy, \ldots\) as sections of \(H^0(C, K^2)\) are to be regarded as linear functions on its dual \(H^1(C, K^*)\).

It is clear from (4) that \(Q_1 Q_2 - Q^2\) is a quadratic factor of the determinant and this gives a singular quadric whose degeneracy subspace is the \(P^2\) defined by the linear forms \(Q_1 = Q_2 = Q = 0\). The other component is a quartic hypersurface \(X\). Then \(X \cap P^2\) is a quartic curve whose modulus is an invariant of the algebra.

We look at this slightly differently. The intersection of the hypersurface \(X\) with the \(P^3\) defined by \(Q_1 = Q_2 = 0\) is a surface given by the determinant of the \(4 \times 4\) matrix obtained from (4) by deleting rows and columns 3 and 6. Its intersection with \(Q = 0\) is the determinant of this matrix where we set \(Q = 0\) also. This is

\[
\begin{bmatrix}
  b_{22}b_{33} & -b_{12}b_{33} & -b_{12}b_{34} & -b_{22}b_{34} \\
  b_{12}b_{33} & b_{11}b_{33} & 0 & b_{11}b_{34} \\
  -b_{12}b_{34} & b_{11}b_{34} & b_{11}b_{44} & b_{12}b_{44} \\
  -b_{22}b_{34} & b_{12}b_{34} & b_{12}b_{44} & b_{22}b_{44}
\end{bmatrix} \tag{5}
\]
which now depends only on the two conics and not on the genus 3 curve \( C \). It is the inner product on the 4-dimensional anti-invariant part of \( \Lambda^2 V \) induced from the inner product on \( V \). Its determinant defines a quartic surface \( S \subset \mathbb{P}^3 \) and the curve \( X \cap \mathbb{P}^2 \) we are seeking is its intersection with the plane \( Q = 0 \).

**Remark:** The determinant of this is to be understood as a polynomial in the quadratics in \( x, y, z \) without imposing the relations \((xy)(yz) = (xz)(y^2)\) etc. Those relations define a Veronese surface in \( \mathbb{P}^5 \) and the determinant is then \((Q_1 Q_2)^2\).

To obtain a formula we take a generic pair \( Q_1 = x^2 - y^2 - z^2, Q_2 = a^2 x^2 - b^2 y^2 - z^2 \) or equivalently we take the quadratic form on \( V \cong \mathbb{C}^4 \) to be

\[
\begin{bmatrix}
A^+ & 0 \\
0 & A^-
\end{bmatrix} = \begin{bmatrix}
x + y & z & 0 & 0 \\
z & x - y & 0 & 0 \\
0 & 0 & ax + by & z \\
0 & 0 & z & ax - by
\end{bmatrix}
\]

(6)

From this point of view the six parameters in the choice of \( Q(x, y, z) \) provide the \( 3g - 3 = 6 \) degrees of freedom for the genus 3 curve \( C: (x^2 - y^2 - z^2)(a^2 x^2 - b^2 y^2 - z^2) = Q^2 \). The extra parameters \( a, b \) yield the 2-dimensional family of vector bundles but, from the equation of the curve, they are interrelated.

The \( \mathbb{P}^3 \) is given by \( Q_1 = 0 = Q_2 \) so we use the relations \( x^2 - y^2 - z^2 = 0, a^2 x^2 - b^2 y^2 - z^2 = 0 \) to give homogeneous coordinates \( w_1 = yz, w_2 = zx, w_3 = xy, z^2 = (a^2 - b^2)w_0 \) and then the matrix (5) is (with \( c = (a - b)(1 + ab), d = (a + b)(1 - ab)\))

\[
M = \begin{bmatrix}
cw_0 - (a - b)w_3 & -bw_1 - aw_2 & (b^2 - a^2)w_0 & w_1 - w_2 \\
-bw_1 - aw_2 & dw_0 + (a + b)w_3 & w_1 + w_2 & (a^2 - b^2)w_0 \\
(b^2 - a^2)w_0 & w_1 + w_2 & cw_0 + (a - b)w_3 & -bw_1 + aw_2 \\
w_1 - w_2 & (a^2 - b^2)w_0 & -bw_1 + aw_2 & dw_0 - (a + b)w_3
\end{bmatrix}
\]

(7)

Incorporating terms from the third and sixth rows and columns of (6) this yields the following explicit formulas for the algebra. It is generated by \( 1, \xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3 \) subject to the following relations, where \( \xi \cdot \eta = \xi_1 \eta_1 + \xi_2 \eta_2 + \xi_3 \eta_3 \):

15
\[(a-b)(1+ab)(\xi_1^2 + \eta_1^2) + (a+b)(1-ab)(\xi_2^2 + \eta_2^2) + 2(a^2-b^2)(\xi_1 \eta_1 - \xi_2 \eta_2) + a_1 \xi \eta \]
\[(\xi_1 \eta_2 - \xi_2 \eta_1) + a(\xi_1 \xi_2 - \eta_1 \eta_2) + a_2 \xi \eta, \quad (\xi_1 \eta_2 + \xi_2 \eta_1) - b(\xi_1 \xi_2 + \eta_1 \eta_2) + a_3 \xi \eta \]
\[(a + b)(\xi_3^2 - \eta_3^2) - (a - b)(\xi_1^2 - \eta_1^2) + a_4 \xi \eta, \quad \xi_3^2 + a_5 \xi \eta, \quad \eta_3^2 + a_6 \xi \eta.\]

Just as the formulas for genus 2 in Section 3.2 were of little use in determining the isomorphism class of the algebra, in this case we consider a more degenerate version and observe that our family consists of a continuous deformation of it.

### 4.5 A special case

The first algebra to be calculated using this method is in [15] where the relations have the simple form

\[
\xi_i^2 = a_i(\xi_1 \eta_1 + \xi_2 \eta_2 + \xi_3 \eta_3), \quad \eta_i^2 = b_i(\xi_1 \eta_1 + \xi_2 \eta_2 + \xi_3 \eta_3) \quad (8)
\]

This was based on taking \(Q_1 = xy, Q_2 = z(x + y + z)\) two degenerate quadratic forms. The discriminant is then given by three singular quadrics and the algebra may be regarded as a deformation of the relations \(\xi_1^2 = \eta_1^2 = 0\) for the cohomology of \(H^*(\text{CP}^1)^6, \text{C}\). This is a continuous variation in the isomorphism class of the algebra, but the parameters describe a variation of the curve \(C\) rather than the bundle \(E\). In fact, if we take the approach of Section 4.1 then the bundle \(E'\) here corresponds to taking \(L^2 = \pi^*U_1\) where \(U_1^2\) is trivial, hence for a given curve there are only finitely many bundles of this type.

In this case we have \(\text{End}_0 E \cong U \oplus UU_1 \oplus U_1\). We can check if \(E\) is very stable by looking for base points of the family of quadrics, that is non-trivial solutions to \(8\), but these only exist if \(\sqrt{a_1 b_1} + \sqrt{a_2 b_2} + \sqrt{a_3 b_3} = 1\). So for a generic curve \(C\) a bundle of this type is very stable.

Our family of vector bundles is obtained by direct image from the Prym variety by deforming \(L\) away from \(U_1\) and so we can say already that a generic member will be very stable.

**Remark:** In [22] the authors show that the “wobbly” bundles are cut out by a hypersurface in \(\text{P}^7\) of degree 48.
For this special case the quartic curve $X \cap P^2$ is a pair of conics. To show a variation in the modulus it is enough to show that as $a, b$ vary we get an irreducible curve. But by analyzing the surface $S$ we get rather more.

5 The quartic surface $S$

5.1 Plane sections

A variety such as $S$ (det $M = 0$ from [7]) which is defined by the determinant of a symmetric matrix of linear forms in $n$ variables has singularities when $n > 3$. These hypersurfaces are classically known as symmetroids (see e.g. [8]). For a quartic surface the generic symmetroid has 10 nodes.

To find the singular locus in our case first note that the plane section $w_0 = 0$ gives a quartic curve

$$(b^2 - 1)^2 w_1^4 + ((a^2 - 1)w_2^2 + (a^2 - b^2)w_3^2)^2 - 2(b^2 - 1)w_1^2((a^2 - 1)w_2^2 + (b^2 - a^2)w_3^2) = 0$$

which is reducible to a pair of conics:

$$(b^2 - 1)w_1^2 = (a^2 - 1)w_2^2 - (a^2 - b^2)w_3^2 \pm 2\sqrt{(1 - a^2)(a^2 - b^2)}w_2w_3.$$

These meet where $w_2 = 0$ or $w_3 = 0$.

If $w_0 = w_3 = 0 \sqrt{b^2 - 1}w_1 = \pm \sqrt{a^2 - 1}w_2$ is a singular point of this curve. The matrix is then

$$\begin{bmatrix}
0 & -bw_1 - aw_2 & 0 & w_1 - w_2 \\
-bw_1 - aw_2 & 0 & w_1 + w_2 & 0 \\
0 & w_1 + w_2 & 0 & -bw_1 + aw_2 \\
w_1 - w_2 & 0 & -bw_1 + aw_2 & 0 
\end{bmatrix}$$

and two pairs of rows are linearly dependent so the matrix has rank $\leq 2$. This is a singular point of $S$.

Calculating the Hessian of the quartic function det $M$ at this point one sees that unless $a^2 = b^2$ or $(a^2 - 1)(b^2 - 1) = 1$ the singularity is a node, and similarly at the other point $x_0 = x_2 = 0$. Moreover, at no other points on the curve $w_0 = 0$ is the rank of the matrix $\leq 2$, so this means that $S$ has no codimension 1 singularities. Then a general plane section is a smooth quartic curve.
Proposition 5. Let $N$ be the moduli space of rank 2 semi-stable bundles $E$ with trivial determinant on a non-hyperelliptic genus 3 curve $C$ and $U$ a line bundle with $U^2$ trivial. Then a generic bundle with $E \cong E \otimes U$ is very stable, and the isomorphism class of the multiplicity algebra varies non-trivially in this two-dimensional family, for a generic curve $C$.

Proof. In Section 4.4 we showed using [23] that such a bundle $E$ is defined up to tensoring by a line bundle by a pair of tangential conics. A generic pair can be simultaneously diagonalized and hence defined by equations $Q_1(x, y, z) = x^2 - y^2 - z^2 = 0, Q_2(x, y, z) = a^2x^2 - b^2y^2 - z^2 = 0$. We calculated the discriminant of the family of quadrics to be a singular quadric and a quartic hypersurface in $\mathbb{P}^5$. The projective equivalence class of the curve of intersection of the degeneracy plane of the quadric with the quartic is an invariant of the isomorphism class of the multiplicity algebra.

We identified the quartic curve as the intersection of a quartic surface $S$ with a hyperplane in $\mathbb{P}^5$ defined by the conic $Q$ which gives $Q_1Q_2 - Q^2 = 0$ as the equation of $C$. A generic section is a smooth quartic curve, but the special bundles in Section 4.5, which are very stable for generic $C$, belong to the connected family of bundles $E$ considered. This family thus defines quartic curves which are both smooth and reducible and hence vary in modulus. 

5.2 Ten singularities

Going back to the quartic surface $S$, by rescaling, the roles of $x, y, z$ interchange, and since $w_0$ corresponds to the $z^2$ term and $w_3$ to $xy$, we obtain nodes by the method above for the pairs $(y^2, zx)$ and $(x^2, yz)$. This yields six singular points.

To find the other four for the symmetroid, consider the intersection of the $\mathbb{P}^3$ given by $Q_1 = 0 = Q_2$ with the Veronese surface. As an element of the dual space of quadratic polynomials in $x, y, z$, a point on the Veronese corresponds to evaluation at some point $(a, b, c) \in \mathbb{C}^3$. So the intersection with $\mathbb{P}^3$ is in this case the four points of intersection $a_1, a_2, a_3, a_4$ of the conics $x^2 - y^2 - z^2 = 0, a_2^2x^2 - b_2^2y^2 - z^2 = 0$. In homogeneous coordinates these are $(x, y, z) = (\pm \sqrt{1 - b^2}, \pm \sqrt{1 - a^2}, \pm \sqrt{a^2 - b^2})$. The symmetric matrix on $V$ given by (20) then has rank 2 and correspondingly the inner product on the anti-invariant part of $\Lambda^2 V$ (which is the matrix (11)) has rank $\leq 2$. This is thus a singular point of the determinantal surface $S$.

The line in $\mathbb{P}^3$ joining evaluation at $a_1 = (\sqrt{1 - b^2}, \sqrt{1 - a^2}, \sqrt{a^2 - b^2})$ to
evaluation at \( a_2 = (\sqrt{1 - b^2}, \sqrt{1 - a^2}, -\sqrt{a^2 - b^2}) \) meets the plane \( z^2 = 0 \) in the singularity evaluated in the previous section, so we see that the ten nodes are labelled by the four points of intersection of two conics and the six lines joining them.

**Acknowledgments**

The author wishes to thank Tamás Hausel for raising this question and ICMAT for support, and in particular Oscar García-Prada for his continuing contributions to research into the geometry of vector bundles on curves.

**References**

[1] L.Alvarez-Cónsul & O.García-Prada, *Dimensional reduction, SL(2,C)-equivariant bundles and stable holomorphic chains*, International J. Math., 12 (2001) 159 – 201.

[2] L.Alvarez-Cónsul, O.García-Prada & A.Schmitt, *On the geometry of moduli spaces of holomorphic chains over compact Riemann surfaces*, International Mathematics Research Papers, (2006) 1– 82.

[3] M.F.Atiyah, *Complex fibre bundles and ruled surfaces*, Proc. Lond. Math. Soc. 5 (1955) 407–434.

[4] M.F.Atiyah & I.M.Singer, *The index of elliptic operators V*, Ann. of Math. 93 (1971) 139–149.

[5] A.Beauville, *Sur les hypersurfaces dont les sections hyperplanes sont à module constant*, Progr. Math., 86, The Grothendieck Festschrift, Vol. I, 121 – 133, Birkhäuser Boston, Boston, MA, (1990).

[6] A.Beauville, *Determinantal hypersurfaces*, Michigan Math. J. 48 (2000), 39 – 64.

[7] N.Bruin, *The arithmetic of Prym varieties in genus 3*, Compositio Math. 144 (2008) 317–338.

[8] I.Dolgachev, “Classical algebraic geometry”, Cambridge University Press (2012).
[9] L. Ducrohet, *The Frobenius action on rank 2 vector bundles over curves in small genus and small characteristic*, Ann. Inst. Fourier, Grenoble 59 (2009) 1641–1669.

[10] O. García-Prada, J. Heinloth & A. Schmitt, *On the motives of moduli of chains and Higgs bundles*, Journal of the European Mathematical Society, 16 (2014) 2617 – 2668.

[11] T. Hausel & N. Hitchin, *Very stable Higgs bundles, equivariant multiplicity and mirror symmetry*, Invent. math. 228 (2022) 893–989.

[12] T. Hausel, *Enhanced mirror symmetry for Langlands dual Hitchin systems*, arXiv:2112.09455v2.

[13] V. Heu & F. Loray, *Hitchin Hamiltonians in genus 2*, “Analytic and Algebraic Geometry” (A. Aryasomayajula et al. eds.), Springer Nature Singapore Pte Ltd. and Hindustan Book Agency, (2017) 153–172.

[14] N. Hitchin, *Stable bundles and integrable systems*, Duke Math. J., 54 (1987) 91–114.

[15] N. Hitchin, *Spinors, twistors and classical geometry*, SIGMA 17 (2021) 090.

[16] G. Laumon, *Un analogue global du cône nilpotent*, Duke Math. J. 57 (1988) 647–671.

[17] L. Masiewicki, *Universal properties of Prym varieties with an application to algebraic curves of genus 5*, Trans. Amer. Math. Soc. 222 (1976) 221–240.

[18] M. S. Narasimhan & S. Ramanan, *Moduli of vector bundles on a compact Riemann surface*, Ann. of Math. 89 (1969) 19–51.

[19] M. S. Narasimhan & S. Ramanan, *2Θ linear systems on abelian varieties*, Vector bundles and algebraic varieties (Bombay, 1984), 415–427, Oxford University Press, (1987).

[20] W. Oxbury, *Stable bundles and branched coverings over Riemann surfaces*, DPhil thesis, University of Oxford (1987).

[21] W. Oxbury, C. Pauly & E. Previato, *Subvarieties of SU_C(2) and 2Θ-divisors in the Jacobian*, Trans. Amer. Math. Soc. 350 (1998) 3587–3614.
[22] S.Pal & C.Pauly, *The wobbly divisors of the moduli space of rank 2 vector bundles*, Advances in Geometry 21 (2021) 473–482.

[23] C.Pauly, *Self-duality of Coble’s quartic hypersurface and applications*, Michigan Math. J. 50 (2002) 551-574.

[24] C.Pauly & A.Peón-Nieto, *Very stable bundles and properness of the Hitchin map*, Geometriae Dedicata, 198 (2019) 143 –145.

[25] B.van Geemen & E.Previato, *On the Hitchin system*, Duke Math. J., 85 (1996) 659–683.

[26] A. Verra, *The fibre of the Prym map in genus three*, Math. Ann. 276 (1987) 433–448.

Mathematical Institute, Woodstock Rd, Oxford OX2 6GG, UK