Exact renormalization group equation for lattice
Ginzburg-Landau models adapted to the solution in
the local potential approximation

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Abstract. The Wilson Green’s function approach and, alternatively, Feynman’s
diffusion equation and the Hori representation have been used to derive an exact
functional RG equation (EFRGE) that in the course of the RG flow interpolates
between the interaction part of the lattice Ginzburg-Landau Hamiltonian and the
logarithm of the generating functional of the S-matrix. Because the S-matrix vertices
are the amputated correlation functions of the fluctuating field, it has been suggested
that in the critical region the amputation of the long-range tails makes the S-matrix
functional more localized and thus more amenable to the local potential approximation
(LPA) than the renormalized free energy functional used in Wilson’s EFRGE.

By means of a functional Legendre transform the S-matrix EFRGE has been
converted into an EFRGE for the effective action (EA). It has been found that the
field-dependent part of EA predicted by the equation is the same as calculated within
the known EA EFRGE approaches but in addition it is accurately accounts for the
field-independent terms. These are indispensable in calculation of such important
quantities as the specific heat, the latent heat, etc.

With the use of the derived EFRGE a closed expression for the renormalization
counterterm has been obtained which when subtracted from the divergent solution of
the Wetterich equation would lead to a finite exact expression for the EA thus making
two approaches formally equivalent.

The S-matrix equation has been found to be simply connected with a generalized
functional Burgers’ equation which establishes a direct correspondence between the
first order phase transitions and the shock wave solutions of the RG equation.

The transparent semi-group structure of the S-matrix RG equation makes possible
the use of different RG techniques at different stages of the RG flow in order to improve
the LPA solution.

Keywords: exact renormalization group equations, local potential approximation, $n$-
vector spin-lattice models, Burgers’ equation, shock wave solutions, first order phase
transitions
1. Introduction

Exact functional renormalization group equations (EFRGEs) were introduced by Wilson [1] as a prospective method of dealing with problems that cannot be solved by other techniques. An important problem of this kind is the solution of models with strong coupling between the fluctuating fields for which rigorous solution methods similar to the perturbative techniques of the weak coupling case are unavailable. The EFRGE derived in [1], however, was too complicated to be solved beyond the perturbation theory. Therefore, using the flexibility of the RG approach, simpler EFRGEs were derived in [2, 3, 4, 5, 6, 7, 8, 9] and used in the solution of various field-theoretic problems (see review papers [10, 11, 12, 13] and references to earlier literature therein).

To the purposes of the present study the most important will be two kinds of EFRGEs: those dealing with the renormalization of the interaction functional suggested in [5, 4] and the equations introduced in [7, 8, 11, 9] which renormalize the effective action (EA). Our interest to these two approaches stems from the fact that they proved to be successful in application to the lattice models of the Ginzburg-Landau type [15, 16, 17, 18, 19]. Moreover, when applied to the same problem and solved in the local potential approximation (LPA) [20, 10, 11, 21, 13, 22, 15, 16, 6, 5, 17] EFRGEs of both kinds produce very similar results provided that the same cutoff in the momentum space in the form of the step function [5, 15] is being used in implementing the LPA. This suggests that both kinds of equations may be equivalent in some sense and so will give similar results in any approximation.

Another possible explanation is that LPA is very accurate and all EFRGEs would give similar solutions if solved within this approximation. This, however, does not seem to be plausible because when LPA is used with RG equations that do not belong to the above two categories, the results may disagree significantly as, e.g., between the solutions of the Blume-Capel model in [23] and in [19]. Besides, a rigorous assessment of the LPA accuracy meets with serious difficulties. The derivative expansion (DE) that is usually invoked as a way of justifying and correcting LPA (see extensive bibliography on the subject in [24] and [13]) is not a systematic approach [24]. For the purposes of the present study the most important conclusion of [24] was that DE can be valid only at moderate interaction strengths. This was confirmed in [16] by comparison of LPA solution using Wilson-type cutoff with the Monte Carlo (MC) simulations of [25] where the agreement did worsen with the growing interaction strength. But the spin-lattice models we will be interested in formally correspond to the case of infinitely strong coupling and though in the course of the RG flow it considerably weakens, in the transient region [1] the large values may persist long enough to make the resort to DE ungrounded. In view of this, the remarkable accuracy of the values of the phase transition temperatures obtained within LPA in [6, 15, 16, 23, 17, 18, 19] in strongly coupled lattice models requires explanation.

The aim of the present paper is to clarify the above issues and to develop techniques facilitating the use of nonperturbative RG methods in the solution of strongly coupled
Exact RG equation for lattice models

lattice models. Of special interest to us will be the spin-lattice case because of the prominent role that the Ising model (IM), the classical Heisenberg model, the Blume-Capel model and many others play in statistical physics.

Specifically, in this paper we will derive EFRGE for the renormalization of the interaction functional \([5, 4]\) using a general cutoff or, equivalently, the regulator function \([13]\) and will show that a fully renormalized interaction functional coincides with the generating functional of the logarithm of the S-matrix with minus sign. Therefore, RG equations of this kind will be called the S-matrix equations to distinguish them from the equations of \([7, 9, 11, 13]\) which will be called the EA equations. We will show that EFRGEs of the two kinds are connected by a functional Legendre transform (LT) and thus formally are equivalent, but with a caveat.

In Wilsonian RG approach it is habitual to neglect the field-independent multiplicative factors that contribute only to the normalization of the field probability distribution \([1]\) or, equivalently, to the field independent terms (f.i.t.) in the renormalized Hamiltonian. According to \([9]\), this practice was also accepted in the derivation of EA EFRGEs in \([7, 9, 11, 13]\) which means that only the field-dependent part of the EA can be assuredly correct when calculated with the use of the EA EFRGE schemes of the above references. In many cases this is completely sufficient which explains the successful use of the RG equation of \([7, 9, 11, 13]\) in quantitative description of the equation of state of magnetic systems \([26]\) and to the calculation of the magnetization and the critical temperatures in lattice systems \([15, 16]\). The reason was that only the field derivatives of EA were needed in the calculations and so the f.i.t. disappeared from all expressions.

However, in other physically important cases the knowledge of the f.i.t. is indispensable. For example, the Onsager solution of the 2D IM is a f.i.t.; the specific heat and the latent heat of the first order phase transitions (FOPT) (see, e.g., \([19]\)) both depend on f.i.t.

An advantage of the EA EFRGE derivation by the LT of the S-matrix equation is that no normalization factors have been omitted which has made possible to correct the RG approaches of \([7, 9, 11, 13]\) in such a way that they treated the f.i.t. exactly. This may have important consequences for the theories based on the Wetterich equation in which f.i.t. at the lower limit of integration is divergent \([7, 9, 11]\). To regularize the divergence various ways of choosing the ultraviolet cutoff have been suggested in \([27, 28, 29, 30, 31]\) which, however, relied on heuristic \textit{ad hoc} assumptions which are difficult to rigorously justify. To overcome this difficulty, by analogy with the renormalization in the relativistic quantum field theories a divergent counterterm will be inserted into the initial condition of the Wetterich equation which will render f.i.t. in the EA to be finite and unambiguously defined.

Besides the latent heat, discontinuities in the order parameter are important characteristics of FOPT. The analytic structure of the S-matrix EFRGE can be helpful in their description because when differentiated w.r.t. the field variable the equation may be qualitatively described as the generalized functional Burgers’ equation (BE)
In the LPA it can be used to describe FOPT as the shock wave solutions of the \( n \)-vector Ginzburg-Landau model for all natural \( n \) in contrast to the EA equation where the shock waves were found only in the \( n \to \infty \) limit \[33\]. The shock-wave picture puts on a firmer footing the mechanism of the suppression of the van der Waals loops plaguing the mean field (MF) theories discovered in the LPA RG approach in \[6\].

In the present study of major interest will be the RG description of non-universal quantities, such as the phase transition temperatures, because of their truly non-perturbative nature. Universal quantities, such as the critical exponents, though not accurately predicted by the LPA do not depend on the interaction strength and in the three-dimensional systems that we will consider in this paper can be efficiently calculated by perturbative techniques within simplified Hamiltonians characterized by a few parameters \[1, 34\]. In contrast, non-universal quantities may depend on an unlimited number of parameters and are particularly difficult to calculate beyond the perturbation theory.

Obviously, the universal quantities are also needed for a comprehensive description of critical phenomena in strongly interacting systems so it would be reasonable to accommodate the available rigorous results \[1, 34\] to the LPA solutions. To this end in the present paper a multi-step renormalization technique will be developed that will make possible the use of different renormalization methods at different stages of the RG flow. For example, it will be possible to use LPA within the transient region at an early stage of renormalization when the interactions are strong \[1\] and to switch to the perturbative treatment in the critical region where the conventional perturbation theory or the DE expansion become efficient \[24, 35, 36\].

The presentation of material will be as follows. After introducing the necessary notation in the next section, in section 3 we will derive the EFRGE for the interaction functional that underlies the LPA equations of \[5, 18, 17, 19\]. It will be shown that the RG flow interpolates between the initial local potential and the generating functional of the logarithm of the S-matrix. It will be argued that in conjunction with a self-consistency (SC) condition the S-matrix functional is more local than the FE functional of the Wilson approach \[1\] so the S-matrix EFRGE should be more accurately represented in the LPA than the Wilson one \[1\]. In Appendix B several forms of a LT will be introduced that connect the S-matrix and the EA RG equations both the exact and in the LPA form including the case of the multi-step renormalization. Because EFRGEs are the evolution equations, their solutions depend on the initial conditions. In sections 5.1, 5.2 and Appendix F it will be shown that for S-matrix EFRGE analytic initial conditions can be easily established even in the spin-lattice case despite that it formally corresponds to an infinitely strong coupling. The renormalization of the divergence in the Wetterich equation will be discussed in Appendix B.1.

In section 3.4 we will show that the Hori representation makes the (semi)group structure of the S-matrix EFRGE fully transparent which simplifies the stepwise renormalization needed in the preliminary exact renormalization in the spin models and also allows one to switch to a perturbative technique in the critical region.
Further, in section 4 qualitative arguments will be given in support of the standpoint that the use of the step function cutoff for the elimination of high momenta \[5, 15, 16, 18, 17, 19\] should be more accurate in the lattice case than in the continuum models. The FOPT will be discussed in section 6 and in Appendix D.

In final section 7 a brief summary will be given.

2. Definitions and notation

To simplify notation and to facilitate comparison with \[1\] all derivations in the present paper will be done for the Landau-Ginzburg model of the Ising universality class, that is, with the scalar field variable. Generalization to the \(n\)-vector models can be achieved along the same lines as in \[5, 18\] and in simple cases is straightforward (see section 5.2 below).

Thus, we will be interested in the calculation of the partition function

\[
Z(\vec{h}) = \int d\vec{s} e^{-H_0(\vec{s}) + \vec{h}^\dagger \vec{s}}
\]  

(1)

by means of the \(N\)-dimensional integral over the fluctuating scalar field which will be usually denoted as a column vector \(\vec{s} = [s_i]\) (similarly, \(\vec{h} = [h_i]\)) defined at sites \(i\) of a periodic lattice of size \(N\); \(\int d\vec{s} \equiv \prod_i \int_{-\infty}^{\infty} ds_i\). The factor \(1/k_B T\) will be assumed to be included in the parameters of the dimensionless Hamiltonian of the Landau-Ginzburg model

\[
H_0(\vec{s}) = \frac{1}{2} \sum_{ij} (\epsilon_{ij} + r \delta_{ij}) s_i s_j + U_0(\vec{s}) = \frac{1}{2} \vec{s}^\dagger (\hat{\epsilon} + r \hat{I}) \vec{s} + U_0(\vec{s})
\]  

(2)

where \(\hat{\epsilon} = [\epsilon_{ij}]\) is the matrix of the pair interactions which in the thermodynamic limit \(N \to \infty\) (which will be always implicitly assumed) is translationally invariant; the strength of the pair interactions will be characterized by a dimensionless constant \(K\) with \(\hat{\epsilon} \propto K\); \(\hat{I}\) is the unit matrix, \(U_0\) the interaction potential and \(\vec{h}\) is the source and/or the external field.

The dagger symbol used in (2) denotes Hermitian conjugation which for real fields means the transposition but for the Fourier transformed fields the complex conjugation should be additionally applied.

For convenience we will use The symmetric Fourier transform, e.g., in

\[
s_k = N^{-1/2} \sum_j e^{-ikj} s_j = \sum_j (\hat{F})_{kj} s_j;
\]  

(3)

the use of the unitary matrix \(\hat{F}\) \((\hat{F}^{-1} = \hat{F}^\dagger)\) will allow us to formally consider the real-space vector and its Fourier transform as the same abstract vector in two reference frames connected by a unitary rotation. This makes possible the use of the same symbol for vectors both with the lattice and with the momentum components.

The bare interaction \(U_0\) in (2) can in principle be any functional of \(\vec{s}\), but in the present paper we will assume the interaction functional to be the sum of site-local
Exact RG equation for lattice models

potentials

\[ U_0(s) = \sum_i u(s_i, 0) \]  

where the second argument of \( u \) is \( t \),—a scalar variable that will parametrise the RG flow. It can be chosen arbitrarily so we define it to be varying from \( t = 0 \) corresponding to the system in its initial or “bare” state to \( t = t^R \) in the fully renormalized state. The bare potential in (4) will be assumed to be analytic in the field variable but as we will see later the analyticity of the renormalized potential at \( t = t^R \) can be broken by phase transitions.

Here and below by superscript “R” we will mark all fully renormalized quantities that appear also in only partially renormalized form at arbitrary \( t \).

Further, because the separation of Hamiltonian in the quadratic and the interaction parts is not unique, the ambiguity has been used to define the first term in (2) in such a way that, first, the Fourier transform of \( \hat{\epsilon} \) had the small-momentum asymptotic

\[ \epsilon(k)_{k \to 0} \approx c k^2 \]  

where \( k = |k| \) and \( c \) is a constant proportional to \( K \). This can be achieved by subtracting an appropriate site-diagonal term in the quadratic part of (2) and adding it to \( U_0 \). Second, an arbitrary diagonal term \( r \hat{I} \) has been added to the quadratic part and subtracted from \( U_0 \). Obviously, neither \( H_0 \) nor the exact partition function (1) depend on \( r \) but in an approximate solution the independence may be broken and the arbitrary parameter can be used to improve the accuracy. This will be done below with the use of a SC condition (section 3.2).

The quantities that we are going to calculate in this paper will be the Helmholtz FE

\[ F^R(h) = -\ln Z(h) \]  

and its two derivatives: the magnetisation

\[ m_i \equiv \langle s_i \rangle = -\frac{\partial F^R(h)}{\partial h_i}, \]  

and the pair correlation function

\[ G^R_{ij} = -\frac{\partial F^R(h)}{\partial h_i \partial h_j} = \langle s_i s_j \rangle - m_i m_j. \]  

Throughout the paper the arrows will denote the \( N \)-dimensional lattice vectors; Fourier momenta and \( n \)-vectors will be boldface and for simplicity the same letters but without arrows or subscripts, such as \( h \) and \( m \), will denote the scalar values of the homogeneous external field and of the on-site magnetisation, respectively. In particular, according to (3) the homogeneous external field will have vector components

\[ h^h_k = \sqrt{N} \delta_{k,0} h. \]
3. Exact RG equations

In the derivation of S-matrix EFRGE we will use the Wilson Green’s function approach (see [1], ch. 11) which will be slightly modified in order to facilitate comparison with the EA approach as presented in review articles [11, 13]. To begin with let us consider an $N$-dimensional Gaussian kernel

$$G(\vec{s}, \vec{s}', t) = \det \left[ \hat{R}(t)/2\pi \right]^\frac{1}{2} \exp \left[ -\frac{1}{2} (\vec{s} - \vec{s}')^T \hat{R}(t) (\vec{s} - \vec{s}') \right]$$  \hspace{1cm} (10)

where $\hat{R}(t)$ is the regulator matrix [11,13] which is assumed to be a non-negative definite, symmetric, translationally invariant and that its site-diagonal matrix elements behave at $t \to 0$ as

$$\hat{R}(t \to 0) \to \infty$$  \hspace{1cm} (11)

in order to satisfy the conventional initial condition

$$G(\vec{s}, \vec{s}', t = 0) = \delta(\vec{s} - \vec{s}') \equiv \prod_i \delta(s_i - s_i').$$  \hspace{1cm} (12)

In the Wilson approach the regulator $\hat{R}$ in (10) should be chosen in such a way that the convolution of the Green function with the Boltzmann factor $e^{-\hat{H}_0 + \vec{h}^\dagger \vec{s}}$ at $t^R$ satisfied

$$Z(\vec{h}) = C \int d\vec{s}' G(\vec{s}, \vec{s}', t^R)e^{-\hat{H}_0(\vec{s}') + \vec{h}^\dagger \vec{s}'}$$  \hspace{1cm} (13)

with some proportionality coefficient $C$ which may depend on $\vec{s}, \vec{h}, t^R$, etc., with the only requirement that it could be calculated exactly.

Replacing $t^R$ in (13) by $t \in (0, t^R)$ one obtains an expression that interpolates between the Boltzmann factor and the fully renormalized partition function. Because the Green function satisfies a diffusion-type differential equation, by choosing a suitable form of the regulator one can derive a Wilsonian EFRGE in differential form, as shown in [1] with a particular choice of the regulator. In this paper we will derive along similar lines a different, S-matrix EFRGE, adopted for the use in conjunction with the LPA.

But before proceeding it is pertinent to point out that similar to the Wilson equation [1], the S-matrix EFRGE deals with the calculation of the Helmholtz FE. On the other hand, the EA in application to statistical-mechanical problems aims at calculating the Gibbs FE which necessitated the use of a modified LT in the derivation of EA EFRGE [13]. Of course, the two fully renormalized thermodynamic potentials are connected by the conventional LT [37].

3.1. S-matrix EFRGE

The advantage of S-matrix EFRGE derived below in comparison with the Wilson-type RG equations [1, 2, 3] and with EA EFRGEs of [7, 8, 14, 9] is that from a qualitative standpoint they are not suitable for the LPA in lattice models even with local interaction potentials. In both cases EFRGEs deal with renormalization of the total Hamiltonian.
which contains both the local potential and the non-local quadratic part. Because EFRGEs are essentially nonlinear, in the course of the RG flow the two terms are intermixed which, in particular, spreads the interaction potential over many lattice sites thus making the LPA along the whole RG trajectory dubious from a qualitative point of view.

In such cases LPA can be justified only formally within the DE [24, 13]. However, for DE to converge the effective expansion parameter should be at least smaller than unity. This is possible in the continuum models with a single cutoff parameter Λ. The lowest order correction in the dimensionless DE parameter in the isotropic systems will be of order \((k/Λ)^2 \leq 1\). In the lattice case, however, the cutoff is determined by the lattice constant \(a_{lat}\) but because in the corners of the Brillouin zone (BZ) of, e.g., sc lattice the squared momentum can be as large as \((ka_{lat})^2 \approx 3\pi^2 \gg 1\) the use of DE becomes meaningless at the early stages of the RG flow. (Henceforth we chose \(a_{lat}\) to be our length unit.)

Therefore, to justify LPA in lattice models at least qualitatively it seems reasonable to derive an EFRGE for the interaction part of the Hamiltonian and develop approximations that would keep the renormalized potential during the RG flow as local as possible. To this end let us consider functional

\[
S(\vec{s}, t) = e^{-U(\vec{s}, t)}
\]  

(14)

subject to the initial condition \(U(\vec{s}, t = 0) = U_0(\vec{s})\). In the Wilson Green’s function approach the RG evolution of \(S\) is described by the equation

\[
S(\vec{s}, t) = \int G(\vec{s}, \vec{s}', t) e^{-U_0(\vec{s}')} d\vec{s}'.
\]  

(15)

From explicit form of \(G\) (10) it is easy to surmise that the partition function can be obtained from the fully renormalized \(S^R\) or, equivalently, from \(U^R\) if instead of \((B.15)\) we impose on the regulator the following condition

\[
\hat{R}^R = \hat{R}(t^R) = \hat{\epsilon} + r \hat{I}.
\]  

(16)

With this regulator one gets at \(t = t^R\) after expanding the quadratic form in \((10)\)

\[
e^{-U^R(\vec{s})} = \sqrt{\det \left( \hat{R}^R/2\pi \right)} \exp \left( -\frac{1}{2} \vec{s} \hat{R}^R \vec{s} \right) Z(\hat{R}^R, \vec{s}).
\]  

(17)

Next choosing \(\vec{s}\) to satisfy

\[
\hat{R}^R \vec{s} = \vec{h}
\]  

(18)

we obtain the partition functional

\[
Z(\vec{h}) = \sqrt{\det \left( 2\pi \hat{G}^P \right)} \exp \left( \frac{1}{2} \vec{h} \hat{G}^P \vec{h} \right) e^{-U^R(\hat{G}^P \vec{h})}
\]  

(19)

where

\[
\hat{G}^P = \left( \hat{R}^R \right)^{-1} = \left( \hat{\epsilon} + r \hat{I} \right)^{-1}
\]  

(20)
is the bare propagator. The superscript “P” has been used instead of “R” to distinguish $\hat{G}^P$ from the fully renormalized pair correlation function $G^R_{ij}$ defined in \(8\).

The FE functional \(6\) corresponding to \(19\) reads

\[
F^R(\bar{h}) = U^R(\hat{G}^P \bar{h}) - \frac{1}{2} \bar{h}^\dagger \hat{G}^P \bar{h} - \frac{1}{2} \text{Tr} \ln \left(2\pi \hat{G}^P\right). \tag{21}
\]

3.2. Self-consistency condition

Standard perturbative analysis (see, e.g., [1] [38]) shows that renormalized interaction vertices in $F^R(\bar{h})$ \(21\) are the connected correlation functions coupled to the source field $\bar{h}$ with the external legs in the diagrammatic representation corresponding to the exact pair correlation functions $\hat{G}^R$. Here by interaction vertices we mean the terms in $F^R$ of the third and higher orders in the source field which obviously are all contained in the renormalized potential $U^R$. But according to [1] (see, e.g., equation (7.45)), factors $\hat{G}^R$ describe the leading long-distance behaviour at the critical point when the nonlocality is the most pronounced. Thus, if instead of $\hat{G}^P$ in the argument of $U^R$ in \(21\) we had $\hat{G}^R$ then functional $U^R(\bar{s})$ would be more local than $F^R$ and so better representable by the LPA.

Obviously that with only one parameter $r$ at hand it is in general not possible to make $\hat{G}^P$ to be equal to $\hat{G}^R$ because in the exact pair correlation function the mass operator will not be local in nontrivial models. However, in ferromagnetic systems the long-distance behaviour is governed by the smallest momenta in the Fourier transformed pair correlation function so for our purposes it would be sufficient to satisfy a simpler requirement

\[
G^R(k \to 0) \approx G^P(k \to 0) = 1/r \tag{22}
\]

where according to \(20\)

\[
G^P(k) = \frac{1}{\epsilon(k) + r}. \tag{23}
\]

It should be pointed out that here we have neglected the modification of the long-distance behaviour due to the critical exponent $\eta$. It can be expected, however, that it is important only when $r \to 0$. Besides, the exponent is equal to zero in the LPA so anyway we could not account for it in this approximation. A possibility to correct this within our RG approach will be discussed in section 5.3.

Now substituting \(21\) into \(8\) one finds that condition \(22\) in terms of the fully renormalized potential reads

\[
\frac{\partial^2 U^R(\bar{s})}{\partial s_k \partial s_{-k}} \bigg|_{k \to 0, s_k = 0} = 0 \tag{24}
\]

where we assume that the solution has been obtained in a translationally-invariant system in homogeneous external field $h$ so that $\bar{s} = \hat{G}^P \bar{h}$ in \(21\) in this case should be calculated with the use of \(9\) and \(23\) as

\[
s_{k=0} = G^P(k) h^k_{k=0} = \sqrt{Nh/r} \tag{25}
\]
The SC condition (24) will be used everywhere below in the solutions of the LPA RG equations.

### 3.3. S-matrix EFRGE in differential form

Differentiating (15) w.r.t. \( t \), using in the derivative of \( G \) the commutativity of translationally-invariant matrices and Jacobi’s formula for invertible matrices one arrives at the exact linear RG equation

\[
\partial_t S(\vec{s}, t) = \frac{1}{2} \sum_{ij} \partial_t G_{ij} \frac{\partial^2 S(\vec{s}, t)}{\partial s_i \partial s_j}
\]

(26)

where \( \partial_t \equiv \partial/\partial t \) and

\[
\dot{G}(t) = [G_{ij}(t)] = \hat{R}^{-1}(t).
\]

(27)

Substituting \( S \) from (14) in (26) the non-linear EFRGE for the interaction potential in the lattice coordinates is obtained as

\[
\partial_t U = \frac{1}{2} \sum_{ij} \partial_t G^{(0)}_{ij} \left( \frac{\partial^2 U}{\partial s_i \partial s_j} - \frac{\partial U}{\partial s_i} \frac{\partial U}{\partial s_j} \right).
\]

(28)

But in implementation of the Wilsonian renormalization we will need it in the momentum representation \[5, 4\]

\[
\partial_t U(\vec{s}, t) + \frac{1}{2} \sum_k \partial_t G(\vec{k}, t) \frac{\partial U}{\partial s_{-k}} \frac{\partial U}{\partial s_k} = \frac{1}{2} \sum_k \partial_t G(\vec{k}, t) \frac{\partial^2 U}{\partial s_{-k} \partial s_k}.
\]

(29)

Here the terms have been rearranged in such a way that after differentiation w.r.t. \( s_\mathbf{q} \) acquired the structure of a \( \mathcal{N} \)-dimensional generalized BE \[32, 39\] for \( \vec{U}_\mathbf{q} = [\partial U/\partial s_\mathbf{q}] \).

It will greatly simplify under LPA and in this form will be useful in the description of FOPTs in section 6.

### 3.4. Feynman’s diffusion equation as an EFRGE

In our notation Feynman’s diffusion equation \[40\] is obtained from (26) by assuming that \( G_{ij}(t) = tG^p_{ij}, \ 0 \leq t \leq t^R = 1 \):

\[
\partial_t S(\vec{s}, t) = \frac{1}{2} \sum_{ij} G^p_{ij} \frac{\partial^2 S(\vec{s}, t)}{\partial s_i \partial s_j}.
\]

(30)

Thus, (30) is an EFRGE with a particular choice of the cutoff matrix \( \partial_t \dot{G}(t) \).

The Hori representation \[40, 38, 5\] can be obtained by formally integrating (30) or, more generally, (26) as

\[
S(\vec{s}, t) = \exp \left( \frac{1}{2} \sum_{ij} G_{ij}(t) \frac{\partial^2}{\partial s_i \partial s_j} \right) S_0(\vec{s}).
\]

(31)

It can also be obtained independently of Feynman’s equation \[38\] so the latter as well as the S-matrix EFRGE \[29\] can be straightforwardly derived from (31) by simple differentiation w.r.t. \( t \) \[5, 17\].
An important property of the Hori representation is that its semi-group RG structure becomes completely transparent if $\hat{G}(t)$ in the exponential is written as $\int_0^t \partial_t \hat{G}(t')dt'$. Now from the properties of the integral it is seen that the renormalization can be performed in several finite steps, for example, first from $t = 0$ to $t = t_0$ and then from $t_0$ to $t$. At the second step $S(\vec{s}, t_0)$ should be taken as the initial condition and $\hat{G}(t)$ in the exponential should be replaced by

$$\hat{\Delta}(t, t_0) = \hat{G}(t) - \hat{G}(t_0). \quad (32)$$

This possibility will be used in sections 4 and 5.3 below. The multi-step renormalization can also be obtained in the Green’s function approach (see Appendix A), though not as straightforwardly as in the Hori representation.

### 3.5. Legendre transform of S-matrix EFRGE

In [Appendix B] explicit LTs of the S-matrix equation have been performed both exactly and in the LPA. It has been shown that the EA equation (B.14) from [7, 9, 11, 13] can be obtained from (29) by neglecting f.i.t. in the exact equation. We use somewhat different notation in these equations because it is not clear whether our LT in Appendix B and the modified LT in [7, 9, 11, 13] are the same at intermediate stages of the RG flow. But the equations are mathematically equivalent so with the same initial conditions will lead to the same solutions.

The EA EFRGE (B.13) takes into account all contributions and so can be used in statistical-mechanical calculations. It is to be noted that because with condition (16) $G(k, t)$ in (B.13) is not divergent neither at $t = 0$ nor at $t^R$ which makes this EFRGE easier to deal with computationally than equations (B.14) and (B.26). It is this EA EFRGE in LPA that was used in all calculations in [17, 18, 19]. An equation similar to (B.13) was obtained in [9] but in the derivation the normalization factors and the vacuum contributions that correspond to f.i.t. in FE case have been dropped.

### 4. LPA

The step-function cutoff suggested in [5] for continuum models proved to be very accurate in application to LPA solutions of lattice models [6, 15, 16, 23, 17, 18, 19]. This success can be qualitatively understood within the Kadanoff picture of critical phenomena [41, 1] as follows.

Formally, LPA consists in assuming that functional $U(\vec{s}, t)$ preserves the local structure of $U_0$ (1) throughout the whole evolution from $t = 0$ to $t = t^R$:

$$U(\vec{s}, t)_{LPA} \approx N \sum_{l_i(k_i)} N^{-\frac{1}{2}} u_l(t) s_{k_1} s_{k_2} \ldots s_{k_l} \delta_{\vec{k_1} + \vec{k_2} + \ldots + \vec{k_l}}, \quad (33)$$

i.e., the Fourier-transformed coefficients of the expansion do not depend on the momenta apart from the lattice Kronecker symbols $\delta^L$. The latter differs from the conventional Kronecker delta in that the momenta may sum not only to zero but to any of the reciprocal lattice vectors, as can be seen from the formal definition (E.1).
Exact RG equation for lattice models

The LPA ansatz (33) allows one to establish a one-to-one correspondence between functional $U$ and (in the Ising universality class) a function of a real variable $x$

$$ u(x, t) = \sum_{l=0}^{\infty} u_l(t)x^l $$

(34)

which can be interpreted as the on-site potential. The general $n$-vector case is treated similarly with using $n$-vector $x = (x_1, x_2, \cdots, x_n)$ instead of $x$ [5].

The evolution equation for $u$ should be obtained by substituting (33) into (29). However, in general case the equation will be incompatible with the LPA ansatz so the task is to satisfy it approximately. The difficulty causes the momentum dependence of the cutoff function $\partial_t G(k, t)$. Indeed, if it was equal to unity

$$ \partial_t G = 1 $$

(35)

then equation (29) could be satisfied exactly because in the real space the inverse Fourier transform of unity is $\delta_{ij}$ which would make the RG equation separable, that is, representable as the sum over the sites of identical local potentials. Feynman’s diffusion equation (30) in this case would admit solution in quadratures which in terms of $u$ would read

$$ e^{-u(x, t)} = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} dx_0 e^{-(x-x_0)^2/2t} e^{u(x_0, 0)} $$

(36)

The layer-cake renormalization scheme [5, 17, 18] has been devised with the aim of emulating (35)–(36) with a maximum possible accuracy in the case of momentum-dependent cutoff function. To this end the latter was chosen as

$$ \partial_t G(k, t) = \theta[G^P(k) - t] $$

(37)

where $\theta$ is the step function that is equal to unity when the Fourier momentum $k$ is inside the region $\Omega(t)$ defined by the condition

$$ G^P(k \in \Omega(t)) \geq t $$

(38)

(see figure[1] and according to (37) $\partial_t G = 0$ outside of $\Omega$. For the purposes of the present paper it will be sufficient to keep in the fully renormalized quantities only the zero-momentum component $s_{k=0}$. Therefore, in the course of the RG flow the components outside $\Omega$ can be set equal to zero [11]: $s_k|_{k \notin \Omega} = 0$ where $\Omega = BZ \setminus \Omega$ is the part of BZ external to $\Omega$. Next one may rescale the momenta in (33) so that $\Omega$ regained the same volume as the original BZ and the rescaled system could be considered as describing the same lattice model approximately renormalized by the Kadanoff real-space procedure [41].

Further details on the rescaling can be found in [5] but in this paper the RG LPA equation of [5] in the scaling form will be used only for the calculation of critical exponents. For the calculation of non-universal quantities, however, the rescaling has been omitted because it introduces a large but trivial Lyapunov exponent equal to the spatial dimension of the system which makes the RG equation numerically unstable. Instead, we simply substitute (33) in (29) and observe that the second derivative on the
r.h.s. makes all terms in (33) independent of \( \mathbf{k} \) so they all acquire the common factor \[ p(t) = \frac{1}{N} \sum_k \theta [G(\mathbf{k}) - t] = \int_{t_0}^{t_{-1}} dE \rho(E) \] (39)

where \( \rho(E) \) is the density of states corresponding to the dispersion \( \epsilon(\mathbf{k}) \) [15, 18, 17].

In the second term on the l.h.s. of (29), however, the summation over \( \mathbf{k} \) is lifted by one of the two Kronecker deltas so the dependence on the momentum remains in the cutoff function (37). In the LPA we neglect it by assuming that in the critical region the most important are small momenta for which (35) holds. In Appendix E it has been argued that this approximation should be better fulfilled in the lattice case than in the continuum Landau-Ginzburg model [5]. Thus, the LPA RG equation for the local potential (34) in the Ising universality class reads

\[ u_t + \frac{1}{2} u_{xx} = 2 p(t) u_{xx} \] (40)

To make notation more compact, here we have made use of the fact that in the Ising universality class in the LPA there is no need for the vector component subscripts so henceforth they will be used to denote partial derivatives. In this notation the free energy (21) per site in LPA reads

\[ f(h) = u^R(h/r) - h^2/2r - \frac{1}{2N} \text{Tr} \ln \left( 2\pi \hat{G}^P \right) \] (41)

where use has been made of (19) and the argument of \( u^R \) has been found by substituting (25) in (33) at the end of renormalization when only \( \mathbf{k} = 0 \) component contributes to

---

**Figure 1.** Illustration of the layer-cake renormalization scheme. The stacked narrow rectangles are cross-sections of \( dG = \theta[G^P(\mathbf{k}) - t]dt \) along the diagonal of the Brillouin zone on which \( G^P(\mathbf{k}) \) reaches its minimum value \( t_0 \) defined in (17); \( \bar{t} = t - t_0 \); \( \Lambda \) is an approximate cutoff momentum in the region where \( G^P(\mathbf{k}) \) is almost isotropic (for further explanations see the text).
Exact RG equation for lattice models

$U^R$ which means that at $t^R$ the argument of $u^R$ is

$$x = h/r.$$  \hfill (42)

The equation of state is obtained as

$$m(h) = -\frac{df}{dh} = \frac{h}{r} - \frac{1}{r}u^R_x |_{x=h/r}$$  \hfill (43)

and the SC condition (24) in the LPA takes the form

$$u^R_{xx} |_{x=h/r} = 0.$$  \hfill (44)

The general $n$-vector case can be treated similarly \cite{5} and in fully $O(n)$-symmetric case the LPA equation simplifies to \cite{18}

$$u_t + \frac{1}{2}(\nabla u)^2 = \frac{1}{2}p(t)\nabla^2 u$$ \hfill (45)

where the differential operators act in the space of $n$-vectors $x$.

5. Illustrative calculations

5.1. Critical temperatures in $\varphi^4$ model

The first example we consider is the $\varphi^4$ model on sc lattice \cite{25,16}. In notation of section 2 the interaction potential reads

$$U_0(\vec{s}) = \sum_i \left[ (1 - 3K - r/2)s_i^2 + \lambda(s_i^2 - 1)^2 \right]$$ \hfill (46)

(cf. equation (2) in \cite{25}). The LPA solution has been obtained as follows. First $U_0$ in (46) has been mapped onto $u(x,0)$ by replacing $s_i$ with $x$ in the summand. Then equation (40) has been solved numerically for $h = 0$ (the critical value of the field) and iterated to satisfy the SC condition (44) (for details of the procedure see \cite{17}) to determine $r$ corresponding to chosen parameters $K$ and $\lambda$. The critical value $K_c$ corresponds to $r = 0$. Calculations within the method of lines have been performed for 11 values of $\lambda$ in the range 0.1—2.5 that were used in MC simulations in \cite{25}. The calculated $K_c$ agreed with the MC values with accuracy better than 0.25% similar to the LPA calculations in \cite{16} based on EA EFRGE \cite{7,9,11,13} and the step function cutoff. Moreover, even the systematic change with $\lambda$ of the discrepancy sign has been similar in both calculations, presumably, as a consequence of equivalence of the exact equations and of the same cutoff used in the LPA in \cite{16} and in the present paper.

5.2. LPA for spin-lattice models

The $O(n)$-symmetric spin-lattice models in principle can be solved in the same way as the $\varphi^4$ model, only using equations (45) instead of (40). To this end it would be sufficient to replace in (16) $s_i$ by $n$-dimensional vectors $\vec{s}_i$ and to let $\lambda \to \infty$ to suppress fluctuations of the vector length. However, numerical solution of (45) with infinite initial condition would be technically very difficult. But in the partition function (11) the $\lambda \to \infty$ limit
Table 1. Dimensionless inverse critical temperatures of the $n$-vector spin models on cubic lattices calculated in the LPA. The errors have been estimated by comparison with the MC simulations data \cite{43} for $n = 1$ (IM) and with the high temperature expansion for $n = 2, 3$ and $4$ \cite{42}.

|   | Lattice | $K_c$  | Error |
|---|---------|--------|--------|
| 1 | fcc     | 0.1023 | 0.2%   |
| 1 | bcc     | 0.1579 | 0.3%   |
| 1 | sc      | 0.2235 | 0.8%   |
| 2 | bcc     | 0.3225 | 0.6%   |
| 2 | sc      | 0.4597 | 1.2%   |
| 3 | bcc     | 0.4905 | 0.8%   |
| 3 | sc      | 0.7025 | 1.4%   |
| 4 | bcc     | 0.6608 | 0.8%   |
| 4 | sc      | 0.9488 | 1.4%   |

amounts to appearance in the integrand of the factor $\prod_i \delta(s_i^2 - 1)$ \cite{42} which is site-local and despite being singular it can be exactly renormalized with the use of \cite{36} which has been generalized to the $O(n)$ case in Appendix F. The exact renormalization has been done in the interval (see figure 1)

$$0 \leq t \leq t_0 = \min_k G^P(k) = \frac{1}{r + \max_k \epsilon(k)}$$

where \cite{35} is satisfied exactly because the integral in \cite{39} saturates to unity above the upper limit of the dispersion. Explicit expressions for $u^{(n)}(x, t_0)$ from \cite{F.5} can be used as the initial conditions for LPA RG equation \cite{45}. In the symmetric phase the equation depends only on the radial coordinate and its numerical integration for $n > 1$ has been as straightforward as in $n = 1$ case \cite{18}. The critical temperatures in table \cite{1} have been found as the points where the SC values of $r$ interpolated to zero. The accuracy similar to that in table \cite{1} was also obtained in RG calculations for IM \cite{23} and additionally for $XY$- and the Heisenberg models on the cubic lattice in \cite{15}. In the later case, however, it was found to be necessary to resort to a non-functional technique and to introduce heuristic modifications in the partially renormalized EA which makes the good accuracy obtained less convincing than in the present approach. As is seen, the values of critical temperatures have been calculated with the accuracy better than 2% which should be sufficient for most practical purposes because in realistic lattice models microscopic Hamiltonians are rarely known with better accuracy.

5.3. Critical exponents

The critical exponents have been calculated with the use of the scaling form of \cite{45} \cite{5}. Because in the LPA there is only one independent exponent, for comparison purposes exponent $\nu$ has been used. The values found were 0.65(0.63) for $n = 1$, 0.71(0.67) for $n = 2$, 0.76(0.71) for $n = 3$ and 0.80(0.75) for $n = 4$; in parentheses are shown the
rounded values from precise calculations taken from [12]. The LPA values were closest to those calculated in [11] in the lowest order of the derivative expansion, though being systematically larger on $\sim 0.01$. This apparently is a consequence of a similar non-perturbative RG approach used by the authors while the difference could be attributed to the fact that $\eta$ in the calculations in the above reference was not equal to zero but obtained from RG equations.

Though not very large, the errors in the LPA values of critical exponents can be seen in the discrepancy between experimental and theoretical curves in the disordered phase in [17] and can be even larger in other experiments, e.g., on the specific heat where exponent $\alpha$ is small and the LPA error is about 100% in the Ising universality class.

Going beyond LPA within lattice nonperturbative RG, however, is not an easy task mainly because of the large values of the momenta within BZ which precludes the use of DE; seemingly more appropriate expansion in circular harmonics, on the other hand, worsens the accuracy of critical temperatures which is undesirable [44, 15]. From a practical standpoint it seems reasonable to exploit the fact that in the critical region the interactions in the course of the RG flow diminish to the point where the perturbative treatment becomes justifiable [1]. This makes possible to invoke the multistep renormalization technique to switch from LPA to a perturbative treatment in a consistent manner at an appropriately chosen late-stage point $t_1 \lesssim t^R$ (see figure 1). Using the Hori representation (31) the usual diagrammatic techniques can be used in the calculations with the only modification that the conventional propagator that behaves at small $k$ as $G^P(k) \simeq 1/(ck^2 + r)$ (see (3)) should be replaced according to (32) by

$$
\Delta(k, t^R, t_1) \simeq \frac{1}{ck^2 + r} - \frac{1}{c\Lambda^2 + r},
$$

where $\Lambda$ is the momentum cutoff corresponding to $t_1$ (see discussion after equation (E.2)). As is seen, when $r, k \to 0$ the conventional first term in (48) dominates so the leading terms in the perturbative expansion will coincide with the standard theory. The diagrams containing the second term will differ from the leading contribution in that the propagator will be replaced by a constant and so the diagram should be easier to calculate than the main contribution. It may be hoped that when $t_1$ is sufficiently close to $t^R$ the corrections to the critical temperature will be small and good precision of $K_c$ can be preserved. The critical exponents in this approach will be the same as those calculated within the chosen perturbative technique [1, 34, 35, 36].

6. First order phase transitions

For simplicity we restrict our discussion of FOPTs to the ferromagnetic IM where transitions occur below the critical temperature when the external magnetic field $h$ changes between infinitesimally small values $0^{-}$ and $0^{+}$. During the change the spontaneous magnetization jumps from $-m_0$ to $m_0 > 0$ or vice versa so the susceptibility $dm/dh$ is infinite inside the coexistence region but in the immediate vicinity of $h = 0$
it should be finite. This behaviour means that the free energy per site \( f(h) \) contains a singular contribution of the form \( f_{\text{sing}} \approx m_0|h| \) \(^{(45, 46)}\) with the respective term in the local potential

\[
u_R^{\text{sing}} = -r m_0 |x|.
\]

Though mathematically simple, this term is not straightforward to obtain as a solution of RG equations because the local potential \((34)\) is assumed to be an analytic function of the field variable.

The origin of the singularity can be elucidated by differentiating LPA equation \((40)\) w.r.t. \(x\) to arrive at the generalized BE \(^{(32)}\)

\[
\mu_t + \mu \mu_x = (p/2) \mu_{xx},
\]

where

\[
\mu(x, t) = u_x(x, t).
\]

According to \((19)\), \(\mu^R(x)\) should be discontinuous at \(x = 0\). As is known \(^{(47)}\), the discontinuous shock waves appear in the inviscid BE which in our case would mean vanishing \(p\). According to \((39)\), \(p = 0\) at the end of renormalization when \(t^R = 1/r\).

It is important to note that in models studied in \(17, 19\) and in the present paper \(p(t)\) additionally satisfied the condition \((dp/dt)_\mu = 0\) due to \(\rho(E = 0) = 0\) in \((39)\). It seems that the latter condition should also be satisfied for the presence of FOPTs because the calculations with a model which did not satisfy it the discontinuous solutions could not be found.

To describe FOPTs with the use of the equation of state \((43)\) it would be sufficient to solve \((50)\) and \((51)\) using the method of lines \(^{(48)}\). However, because \(\mu^R(x)\) is discontinuous, its first and the second derivatives entering \((50)\) will be difficult to deal with numerically. Therefore, in the actual calculations in \(17, 19\) the difficulties were alleviated by using the integrated form of \((50)\), that is, the original LPA equation \((10)\).

Furthermore, additional regularization has been found to be useful that can be achieved with the use of the partial LT \(\text{C.1} - \text{C.2}\). By solving the transformed equation \((\text{C.3})\) instead of \((40)\) it can be seen from \(\text{C.1}\) that \(x = h/r\) at \(t^R\) so the jump in \(u_x\) at \(x = 0\) is transformed in a finite interval of \(y\) values. Thus, the only singularities in \(v^R(y)\) will be two kinks at points \(y_0^\pm = -t^R u_x^R|_{x=0}^\pm\) while inside the interval \(\mu^R(y) = v^R_y(y)\) will be a linear function of \(y\), as can be seen from \(\text{C.5}\) for \(h = 0\). So the strongest singularities in equation \(\text{C.3}\) will be two discontinuities in the second derivative at the kink points corresponding to the jump of the susceptibility at the boundary separating the ordered phase and the coexistence region.

The mechanism of appearance of the linear in \(y\) segment in \(\mu(y, t)\) can be qualitatively understood from the equation obtained by differentiating \(\text{C.3}\) w.r.t. \(y\):

\[
\mu_t = \frac{p(t) \mu_{yy}}{2(1 + t \mu_y)^2}.
\]

As is seen, when the term in parentheses vanishes the r.h.s. becomes singular. Because \(t \geq 0\) this may happen only for negative values of \(\mu_y\) which appear when \(v(y, t)\) has
a negative curvature in some region of $y$ values. The latter appears in the initial potentials (F.3) $u$ or $v$ which are equal due to (C.2) at sufficiently low temperatures. Further, the coefficient before $\mu_{yy}$ in (52) is always positive and so can be considered as a space and time dependent diffusivity. At vanishing denominator the singularity becomes non-integrable so the maximum negative slope tolerated by equation (52) is $-1/\bar{t}^R$; steeper slopes will be smeared out by the diffusion because the diffusivity will diverge in such cases. In numerical calculations in [18, 17, 19] and in the exactly solvable model below it was found that the fully renormalized solution exhibited the universal behaviour $\mu^R(y) = -y/\bar{t}^R + (f.i.t.)$ corresponding to the maximum allowable steepness. Substitution in (C.5) shows that the change of $\mu^R(x)$ within the coexistence region is confined to a single point $x = 0$, as expected.

The above qualitative reasoning can be illustrated with an exactly solvable example provided by the infinite-range IM (IRIM) also known as the Husimi-Temperley [49] and the Curie-Weiss model [50] which is often considered to be exactly solvable in the MF approximation [50]. A straightforward application of the MF theory, however, leads to unphysical van der Waals loops in the FOPT region which suggests that the MF solution below $T_c$ is flawed. In [49, 51] it was shown that the loops are replaced by the shock wave solutions of the BE that can be derived for that particular model without resort to RG.

In Appendix D it has been shown that exactly the same BE as in [49, 51] is obtained for this model as the S-matrix RG equation in the LPA. In the IRIM case $p(t)$ in (40) becomes equal to $1/N$ so that the generalized BE (50) turns into the conventional BE with constant viscosity $1/2N$. To our purposes the most important is the fact that the jumps in $\mu^R(x)$ as the external field crosses zero are between the MF spontaneous magnetisation values $\pm m_0$. Thus, though $\mu = u_x$ in (50) and $m(x)$ in [49] do not coincide, from (43) it can be seen that at $h = 0$ the functions in both cases are the same. Now taking into account that our LPA RG solution in Appendix D coincides with the MF solution [50] outside the coexistence region and unifying it with the rigorous treatment in [49] of discontinuities in the solution of BE which is equally valid for our $h = 0$ case we conclude that the LPA RG equation solves exactly both the IRIM and the problem of the van der Waals loops. Finally, in order to check whether the numerical solution by the method of lines used in [18, 17, 19] gives in the IRIM case the same solution as the analytic approach, (C.3) has been solved with $p = 1/2N$ for $N = 1000$ and the exact solution has been reproduced with the accuracy $O(10^{-3})$.

In connection with the numerical solutions it is pertinent to note that though (C.3) in the FOPT region has been found to be easier to deal with than with equation (40) [17, 19], it may still be worthwhile to adopt the techniques of [48] to the case of (40). The reason is that the phase transitions in $u^R(x)$ take place at a single value of argument $x$ while in (C.3) $x$ is mapped on an interval of $y$ values of length $O(1)$. But $v^R(y)$ within the interval behaves linearly and in the symmetric case can be fully characterized by a single parameter, the slope. However, for good numerical accuracy the discretization step has to be chosen to be $O(10^{-3})$ or smaller [22, 17]. Thus, in the method of lines
O(10^3) points carry essentially the same physical information. Because in practice the maximally possible number of lines is restricted, the use of (40) may prove to be more suitable for achieving better accuracy.

7. Conclusion

In the extensive review of the nonperturbative RG approach [13] the authors motivated the need for the modern implementation of Wilson’s RG suggested in [7, 8, 11, 2] by the complexity of previously derived EFRGEs [1, 2, 3] which impeded the development of reliable approximate computational schemes. The results of the present paper can be considered as further advance in this direction in the case of the classical lattice models.

The new advancements include such important achievement as the derivation of the S-matrix EFRGE and of its Legendre transformed form which account for all contributions to the free energy and EA functionals, not only for the field-dependent terms as is conventional in the Wilsonian RG [1, 9]. This has made possible the RG calculation of all thermodynamic quantities in lattice systems. Moreover, by LT of the S-matrix EFRGE it has been possible to identify the cause of the divergence inherent in the conventional formulation of the Wetterich equation [7, 9, 11, 13]. An explicit expression for a divergent contrerterm has been derived which having being subtracted from the fully renormalized solution leads to the exact finite expression for the Gibbs FE. This allows one to obtain with the use of the Wetterich equation the same results as with the S-matrix EFRGE.

Further, within the S-matrix RG approach a thermodynamic description of the FOPT of general kind may be achieved. By the latter are meant the FOPT not only due to the symmetry breaking as in the IM below $T_c$ but also those that are defined by the crossing of the FE curves, such as in the Blume-Capel model below the tricritical point [19]. The description is further facilitated by the structure of the S-matrix EFRGE, especially in the LPA, which can be straightforwardly connected with the generalized BE through its differentiation w.r.t. the field variable. This has made possible to generalize on the nonperturbative RG the observation made in connection with the MF solution of IRIM in [49, 51] that FOPT correspond to the shock-wave solutions of RG equations. In the case of EA EFRGEs this could be approximately justified only in the large-$n$ limit [33] which excludes the majority of physically important small-$n$ models, such as the IM, the Blume-Capel model, etc.. The connection of S-matrix EFRGE with the generalized Burgers’ equation has made possible to apply in the solution of RG equations the concepts and techniques developed in the field of nonlinear differential equations [47, 32, 39].

Some useful improvements in the S-matrix approach to the lattice models in comparison with the approach of [15, 16] can be mentioned. For example, the problem of establishing the initial condition does not arise in the S-matrix case; the need to calculate improper integrals is absent because the momenta much larger than the inverse lattice spacing do not appear in the formalism. All necessary LTs have been found in
explicit analytic form and the need for their numerical calculation do not arise. These improvements are important in the solution of the evolution equations of RG type because the positive Lyapunov exponents inherent in RG in the critical region may enhance small errors in the initial condition resulting in degraded accuracy.

The absence of large unphysical momenta in the S-matrix EFRGE is also important from the point of view of qualitative soundness of the approximate solutions obtained. In the absence of rigorous methods of solution of strongly coupled field-theoretic models, qualitative arguments play a significant role. Therefore, the use in the S-matrix approach of the SC condition and of the amputated correlation functions in the renormalized interaction potential makes LPA a natural approximation from a qualitative standpoint. Furthermore, the use of the cutoff in the form of the step function in the spin-lattice case leads to the exact renormalization at the earliest stage of the RG flow when the Fourier momenta are not yet eliminated. In the present paper it has been argued that at an early stage of the momenta elimination the gradual loss of LPA accuracy caused by the widening gaps in the periodic zone scheme is accompanied by the weakening of interactions. So that at the late stage of the RG flow they may become sufficiently weak for LPA to be justifiable by DE. This qualitative reasoning agrees with the results of where the use of the step function cutoff in the strong coupling case resulted in considerably more accurate values of critical temperatures than with the Wilson-type cutoff. The values of the critical temperatures calculated in within the EA LPA in \( \varphi^4 \) model are very similar to those of the present paper based on the S-matrix equation. Apparently this is a consequence of the equivalence of the EA and the S-matrix equations also in the LPA. The high accuracy of the obtained results further supports the qualitative correctness of the picture of the RG flow described by the S-matrix EFRGE.

Further, the possibility to easily implement a multi-step renormalization within the S-matrix formalism makes possible to use in the critical region the accurate perturbative results obtained, for example, with the use of DE. Such an approach may be useful in correcting the deviations of the LPA predictions from the experimental data caused by errors in the LPA critical exponents.

To sum up, the main advantage of the RG approach based on the S-matrix EFRGE is that it provides conceptually simpler and computationally easier techniques of solution of spin-lattice models than the alternative methods while preserving the same accuracy in the calculation of the critical temperatures as was achieved in. Besides, the possibility to calculate correct values of the f.i.t. in FE within the S-matrix formalism allows for its use in the description of FOPT of any kind.

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Appendix A. Stepwise renormalization in the Green’s function formalism

Renormalization in several finite steps can be performed recursively by adding steps one at a time so only a two-step case need be considered. This can be done with the use of the identity satisfied by the Gaussian kernel

\[
\int_{-\infty}^{\infty} d\vec{s}_0 \det \left( \frac{\hat{X}}{2\pi} \right)^{\frac{1}{2}} e^{-\frac{1}{2} (\vec{s} - \vec{s}_0)^{\dagger} \hat{X} (\vec{s} - \vec{s}_0)} \det \left( \frac{\hat{Y}}{2\pi} \right)^{\frac{1}{2}} e^{-\frac{1}{2} (\vec{s}_0 - \vec{s}^{\prime})^{\dagger} \hat{Y} (\vec{s}_0 - \vec{s}^{\prime})} = \det \left( \frac{\hat{Y}}{2\pi (\hat{X} + \hat{Y})} \right)^{\frac{1}{2}} e^{-\frac{1}{2} (\vec{s} - \vec{s}^{\prime})^{\dagger} \left( \hat{X} + \hat{Y} \right)^{-1} (\vec{s} - \vec{s}^{\prime})}
\]

where all matrices are considered to be translationally-invariant, hence, commuting with each other.

Now by assuming that the second kernel on the first line is \( \mathcal{G}(\vec{s}_0, \vec{s}^{\prime}, t_0) \) from (10) with \( \hat{Y} = \hat{R}(t_0) \) and that on the second line we want to obtain (10), we conclude that

\[
\frac{\hat{X} \hat{R}(t_0)}{\hat{X} + \hat{R}(t_0)} = \hat{R}(t).
\]

Solving this one finds that the regulator matrix for the second step is

\[
\hat{X}(t, t_0) = \left[ \hat{R}^{-1}(t) - \hat{R}^{-1}(t_0) \right]^{-1} = \left[ \hat{G}(t) - \hat{G}(t_0) \right]^{-1}
\]

As is seen, \( \hat{X} \) satisfies the initial condition for Green’s functions \( \hat{X}(t \to t_0, t_0) \to \infty \); besides, because \( \hat{R}(0) = \infty \), \( \hat{X}(t, 0) = \hat{R}(t) \) so \( \text{(A.3)} \) is valid for all \( t \) and \( t_0 \leq t \).

Appendix B. Legendre transforms

By analogy with the LTs of [52, 18] we introduce a new fluctuating field \( \vec{\phi} \) and a new interaction potential \( W \) as

\[
\phi_k(\vec{s}, t) = s_k - G(k, t) \frac{\partial U(\vec{s}, t)}{\partial s_k}
\]

\[
W(\vec{\phi}, t) = U(\vec{s}, t) - \frac{1}{2} \sum_q \frac{\partial U(\vec{s}, t)}{\partial s_q} G(q, t) \frac{\partial U(\vec{s}, t)}{\partial s_q}
\]

where we assume \( G(k, t) \) to be a non-negative even function of \( k \) satisfying \( G(k, 0) = 0 \).

To transform the exact RG equation (29) to the new variables we first take the derivatives of (B.1) and (B.2) w.r.t. \( s_k' \):

\[
\frac{\partial \phi_k}{\partial s_{k'}} = \delta_{kk'} - G(k, t) \frac{\partial^2 U}{\partial s_{-k} \partial s_k'}
\]

\[
\sum_k \frac{\partial W}{\partial \phi_k} \frac{\partial \phi_k}{\partial s_{k'}} = \frac{\partial U}{\partial s_{k'}} - \sum_q \frac{\partial U(\vec{s}, t)}{\partial s_q} G(q, t) \frac{\partial^2 U}{\partial s_{-q} \partial s_{k'}}
\]
and observe that by substituting (B.3) in (B.4) and changing the summation subscript from $k$ to $q$ one arrives at the system of linear equations for the row vector $(\partial W / \partial \phi_q - \partial U / \partial s_q)^\dagger$

$$
\sum_q \left( \frac{\partial W}{\partial \phi_q} - \frac{\partial U}{\partial s_q} \right)^\dagger \left( \delta_{qk'} - G(q,t) \frac{\partial^2 U}{\partial s_q \partial s_{k'}} \right) = 0. \quad (B.5)
$$

Because $G(q,t)$ can be arbitrary, the matrix of the system in general is not degenerate which means that the solution should be trivial and the following useful equality should hold

$$
\frac{\partial W}{\partial \phi_q} = \frac{\partial U}{\partial s_q}. \quad (B.6)
$$

In particular, with its use transformations (B.1) and (B.2) are easily reversed. For example, from (B.1) one gets

$$
s_k(\phi, t) = \phi_k + G(k, t) \frac{\partial W}{\partial \phi_{-k}}(\phi, t) \quad (B.7)
$$

The next step in transforming the RG equation is to differentiate (B.1) and (B.2) w.r.t. $t$:

$$
\partial_t \phi_k = -\partial_t G(k, t) \frac{\partial U}{\partial s_{-k}} - G(k, t) \frac{\partial^2 U}{\partial s_{-k} \partial t} \quad (B.8)
$$

$$
\partial_t W + \sum_k \frac{\partial W}{\partial \phi_k} \partial_t \phi_k = \partial_t U - \sum_q \frac{\partial U}{\partial s_q} G(q, t) \frac{\partial^2 U}{\partial s_q \partial t} - \frac{1}{2} \sum_q \frac{\partial U}{\partial s_{-q}} \partial_t G(q, t) \frac{\partial U}{\partial s_q} \quad (B.9)
$$

By substituting (B.8) in (B.9) and using (B.6) one can express the terms on the l.h.s. in (29) in terms of $W(\phi, t)$ as

$$
\partial_t U + \frac{1}{2} \sum_q \frac{\partial U}{\partial s_{-q}} \partial_t G(q, t) \frac{\partial U}{\partial s_q} = \partial_t W. \quad (B.10)
$$

The r.h.s. in (29) can be transformed by differentiating (B.6) and (B.7) w.r.t. $\phi_{-k'}$ as

$$
\sum_q \frac{\partial^2 U}{\partial s_k \partial s_{-q}} \left[ \delta_{q,k'} + G(q,t) \frac{\partial^2 W}{\partial \phi_q \partial \phi_{-k'}} \right] = \frac{\partial^2 W}{\partial \phi_k \partial \phi_{-k'}}. \quad (B.11)
$$

Now denoting the matrix in the brackets by $\tilde{B} = [B_{q,k'}]$ (B.11) can be solved as

$$
\frac{\partial^2 U}{\partial s_k \partial s_{-k}} = \sum_{k'} \frac{\partial^2 W}{\partial \phi_k \partial \phi_{-k'}} (\tilde{B}^{-1})_{k',k}. \quad (B.12)
$$

This expression can be substituted in (29) to give the exact RG equation in terms of the Legendre-transformed quantities $\tilde{\phi}$ and $W$

$$
\partial_t W = \sum_{k',k} \partial_t G(k, t) \frac{\partial^2 W}{\partial \phi_k \partial \phi_{-k'}} (\tilde{B}^{-1})_{k,k'} \quad (B.13)
$$

which structure is very similar (but not identical) to that of equation (3.31) in [9].
Appendix B.1. The Wetterich equation

The EA EFRGE derived in [7, 11] (the Wetterich equation) reads
\[
\partial_t \tilde{\Gamma}(\vec{\varphi}, t) = \frac{1}{2} \text{Tr} \left\{ \partial_t \hat{R}^{(0)} \left( \hat{\Gamma}_{\varphi \varphi} + \hat{R}^{(0)} \right)^{-1} \right\}
\]
(B.14)
where \(\hat{R}^{(0)}\) in addition to the requirement to all regulators (11) also satisfies the condition
\[
\hat{R}^{(0)}(t = t^R) = 0
\]
(B.15)
(the superscript (0) is used to distinguish such regulators). The RG flow in (B.14) is initiated by
\[
\tilde{\Gamma}(\vec{\varphi}, t = 0) = H_0(\vec{\varphi}).
\]
(B.16)

To establish its connection with (B.13) we first derive an expression for the Gibbs FE that is equal to the fully renormalized EA. To this end using Fourier transformed (7) and (21) one finds
\[
\bar{m}_k = G^P(k)h_k - G^P(k) \frac{\partial U_R(\bar{s})}{\partial s_{-k}} \bigg|_{s_k = G^P(k)h_k}.
\]
(B.17)
Solving this w.r.t. the derivative
\[
\frac{\partial U_R(\bar{s})}{\partial s_{-k}} \bigg|_{s_k = G^P(k)h_k} = h_k - [G^P(k)]^{-1} m_k,
\]
(B.18)
substituting it in (B.2) at \(t^R\) where \(\phi_k = m_k\) and using (20) one gets in the vector notation
\[
W^R(\vec{m}) + \frac{1}{2} \vec{m}^\dagger(\hat{\epsilon} + r\hat{I})\vec{m} = U^R(\hat{G}^P \bar{h}) - \frac{1}{2} \hat{h}^\dagger \hat{G}^P \bar{h} + \vec{m}^\dagger \bar{h}.
\]
(B.19)
Comparing this with (21) and adding the last term in (21) to both sides
\[
W^R(\vec{m}) + \frac{1}{2} \vec{m}^\dagger(\hat{\epsilon} + r\hat{I})\vec{m} - \frac{1}{2} \text{Tr} \ln \left( 2\pi \hat{G}^P \right) = F^R(\bar{h}) + \vec{m}^\dagger \bar{h}
\]
(B.20)
one sees that the r.h.s. is the \(\bar{h} - \vec{m}\) Legendre transform of the Helmholtz FE, so, by definition, it is equal to the Gibbs FE which we will denote \(\Gamma^R(\vec{m})\). The expression is naturally generalized at arbitrary \(t\) as
\[
\Gamma(\vec{\phi}, t) = W(\vec{\phi}, t) + \frac{1}{2} \vec{\phi}^\dagger(\hat{\epsilon} + r\hat{I})\vec{\phi} - \frac{1}{2} \text{Tr} \ln \left( 2\pi \hat{G}^P \right).
\]
(B.21)
Further, because at \(t = 0\) the LT (B.11)-(B.2) is trivial, the initial condition for (B.21) is obtained by simply changing the variable from \(\bar{s}\) to \(\vec{\varphi}\) so taking into account (2) one gets
\[
\Gamma(\vec{\varphi}, t = 0) = H_0(\vec{\varphi}) - \frac{1}{2} \text{Tr} \ln \left( 2\pi \hat{G}^P \right)
\]
(B.22)
which differs from the condition (B.16) of EA approach by the second term. Because the latter is constant in \(\vec{\varphi}\) and \(t\), it remains unaltered in the course of the RG flow. Therefore, the constant term instead of the initial condition (B.22) can be added to the
solution at the end of the flow which below will be used in the renormalization of $\tilde{\Gamma}^R$ in the Wetterich equation.

The EFRGE for $\Gamma$ can be obtained as follows. First, in (B.13) we represent matrix $\hat{B}$ as the product $\hat{G}\hat{A}$ with $\hat{G} = [G(k,t)\delta_{kk'}]$ and

$$\hat{A} = \left[ W_{\phi_k\phi_{-k'}} + R(k,t)\delta_{kk'} \right] \equiv \hat{W}_{\phi\phi} + \hat{R}. \tag{B.23}$$

We remind that according to (27) $\hat{R} = \hat{G}^{-1}$, hence, in (B.13) $\partial_t \hat{G} = -\hat{R}^{-1}\partial_t \hat{R}\hat{R}^{-1}$.

Next, in the matrix identity

$$\hat{W}_{\phi\phi} \hat{B}^{-1} = (\hat{A} - \hat{R})\hat{A}^{-1}\hat{R} = \hat{R} - \hat{R}\hat{A}^{-1}\hat{R}. \tag{B.24}$$

the regulator function can be represented in the form

$$\hat{R} = \hat{\epsilon} + r\hat{I} + \hat{R}^{(0)} \tag{B.25}$$

compatible with both (16) and (B.15). Now substituting (B.24) in (B.13), using $t$-independence of the last two terms in (B.21) and the invariance of the trace under the cyclic permutations we arrive at an EFRGE

$$\partial_t \Gamma(\vec{\phi}, t) = \frac{1}{2} \text{Tr} \left[ \partial_t \hat{R}^{(0)} \left( \hat{\Gamma}_{\phi\phi} + \hat{R}^{(0)} \right)^{-1} \right] - \frac{1}{2} \partial_t \ln \det \left( \hat{\epsilon} + r\hat{I} + \hat{R}^{(0)} \right) \tag{B.26}$$

where $\hat{\Gamma}_{\phi\phi} = \left[ \Gamma_{\phi_k\phi_{-k'}} \right]$.

As can be seen, (B.26) differs from the Wetterich equation (B.14) only by the last term which, however, plays an significant role by regularizing the equation at $t \to 0$ when $\hat{R}^{(0)} \to \infty$. Because in this limit $\Gamma$ in (B.22) is finite and so is negligible in comparison with the regulator, the two terms on the r.h.s. in (B.26) cancel to zero at $t = 0$ so unlike (B.14) the equation is integrable near $t = 0$ as well as everywhere along the RG trajectory. This ensures the use of (B.26) in determining the exact Gibbs FE.

Because the cancellation of singularities at $t = 0$, is very important for the consistency of (B.26), it cannot be rearranged in the WE form through a simple transfer of the last term on the r.h.s. on the l.h.s.. To overcome this difficulty we first regularize the equation in the conventional way by starting the evolution from small positive value of $t = t_0$ (which is equivalent to starting from large finite value of $\Lambda = \Lambda_0$ or $k$ in the momentum parametrization [7, 9, 11, 13]). Now it is easily seen, that (B.26) can be cast in the form of WE (B.14) for a modified EA

$$\tilde{\Gamma}(\vec{\phi}, t, t_0) = \Gamma(\vec{\phi}, t) + \frac{1}{2} \text{Tr} \ln \left( 2\pi \hat{G}^{t_0} \right) + \frac{1}{2} \ln \det \left( \hat{\epsilon} + r\hat{I} + \hat{R}^{(0)}(t') \right) \bigg|_{t'=t_0}' \tag{B.27}$$

where the second term on the r.h.s. has been added in order that instead of (B.22) $\tilde{\Gamma}$ satisfied at $t_0 \to 0$ the initial condition (B.16).

At the end of the evolution at $t = t^R$ after some rearrangement the fully renormalized Gibbs FE can be found with the use of the WE solution (B.27) as

$$\Gamma^R(\vec{m}) = \lim_{t_0 \to 0} \left\{ \tilde{\Gamma}(\vec{m}, t^R, t_0) - \frac{1}{2} \ln \det [2\pi \hat{G}(t_0)] \right\} \tag{B.28}$$

where the second term in the braces is the counterterm with $\hat{G}$ equal to the inverse of the regulator (B.25).
It is to be pointed out that when differentiated w.r.t. $\phi$ the equations (B.14) and (B.26) and their respective initial conditions (B.16) and (B.22) becomes identical which proves that both the equations and their solutions differ only by f.i.t.. It is this differentiated form of WE that has been used by Wetterich et al. in concrete RG calculations [53, 26].

Thus, the Wetterich equation (B.14) solved with the conventional initial condition (B.16) will give the exact solution of the EA EFRGE both for the field-dependent part and for f.i.t. provided the divergence in the EA will be removed according to (B.28).

Appendix B.2. LT in a two-step renormalization

As could be noted, the only property of $G(k, t)$ that was used in the derivation of RG equation (B.13) was that $G$ had the same time derivative as in (29). But from (32) one can see that $\partial_t G = \partial_t \Delta$. So for the second-step renormalization one can derive RG equation similar to (B.13) by using the LT (B.1) and (B.2) with $G$ replaced by $\Delta$

$$y_k(\vec{s}, t) = s_k - \Delta(k, t, t_0) \frac{\partial U(\vec{s}, t)}{\partial s_{-k}}$$

$$V(\vec{y}, t) = U(\vec{s}, t) - \frac{1}{2} \sum_q \frac{\partial U(\vec{s}, t)}{\partial s_{-q}} \Delta(q, t, t_0) \frac{\partial U(\vec{s}, t)}{\partial s_q}.$$  \hspace{1cm} (B.29)

However, unless $t_0 = 0$, functional $V$ and variables $\vec{y}$ do not acquire immediate physical meaning at the end of renormalization (for $t_0 = 0$ they will coincide with $W$ and $\vec{\phi}$). To see this we note that the equation for $V$ similar to (B.13) has as the initial condition $V(\vec{y}, t_0) = U(\vec{y}, t_0)$, as follows from (B.29)–(B.30) with $\Delta(t_0, t_0) = 0$ (see (32)). But according to (29) $U(\vec{y}, t_0)$ depends on the arbitrary values of $G(k, t)$ in the interval $0 \leq t < t_0$ while in the RG equation for $V$ and in the LT (B.29)–(B.30) only $G(k, t)$ for $t_0 \leq t < t_R$ contribute. Because of this arbitrariness, both $V^R$ and $\vec{y}$ cannot be given a physical meaning at $t_R$. However, similar to (B.6) the equality $\partial V / \partial y_q = \partial U / \partial s_q$ holds so the transform (B.29)–(B.30) can be reversed and the physical quantities $U^R$ and $\vec{s}$ in (18) can be expressed in parametric form in terms of $V^R$ and $\vec{y}$. If needed, the Gibbs FE can be found with the use of (B.20).

Appendix C. Legendre transforms in LPA

The formalism just described considerably simplifies in the LPA. First we note that in the layer-cake renormalization scheme of section 4 visualized in figure 1 the fluctuating field “sees” only the flat part of $G(k, t) = t$ [38] because beyond the cutoff the field has been set to zero. Thus, $\Delta^{LPA}(t, t_0) = t - t_0 = \bar{t}$ (see figure 1). In terms of the local potential $v(y, t)$ corresponding to $V$ which now depends on a single scalar variable $y$ the LT (B.29)–(B.30) simplifies to

$$y = x - \bar{t}u_x$$

$$v = u - \bar{t}u_x^2 / 2$$  \hspace{1cm} (C.1)

$$v = u - \bar{t}u_x^2 / 2$$  \hspace{1cm} (C.2)
Exact RG equation for lattice models

where for brevity the arguments of $y(x, t), v(y, t)$ and $u(x, t)$ have been omitted and the subscript notation for partial derivative has been used.

The exact equation for $V$ is obtained from (B.13) via substitutions $W \rightarrow V$, $\vec{\phi}$ to $y$ and $G \rightarrow \Delta$. In the LPA it simplifies to

$$v_t = \frac{p(t)v_{yy}}{2(1 + \dot{t}v_{yy})}.$$  \hfill (C.3)

It is possible also to derive it directly from (40) by using (C.1)–(C.2) and repeating the steps from [Appendix B][18].

Similar to (B.6) in the exact case, the equality

$$v_y(y, t) = u_x(x, t) \hfill (C.4)$$

holds also in LPA so at the end of renormalization this can be used to find the equation of state in parametric form. Thus, from (C.1) one gets at $t_R$ where according to (42)

$$\frac{h}{r} = y + \dot{t}v^R_y.$$  \hfill (C.5)

The expression for the magnetisation can be expressed through $y$ by replacing in (43) $\frac{h}{r}$ by the r.h.s. of (C.5) which gives

$$m(y) = y - t_0v^R_y.$$  \hfill (C.6)

As is seen, $y = m$ only when $t_0 = 0$ in which case the equation of state is obtained by replacing $y$ in (C.5) by $m$. When $t_0 \neq 0$ (C.5) and (C.6) define the equation of state parametrically.

Appendix D. LPA solution of IRIM

In the IRIM all spin pairs interact with the same dimensionless strength $K/N$ so the sites can be numbered in an arbitrary order which means that the model is structureless. Because the use of the conventional lattice Fourier transform is not warranted in this case, all calculations will be carried out in the space of the site numbers.

As is easy to see, the pair interaction matrix in IRIM can be cast in the form

$$\hat{\epsilon} = K(\hat{I} - \hat{E}) \hfill (D.1)$$

where the idempotent matrix $\hat{E}$, $\hat{E}^2 = \hat{E}$, has matrix elements $E_{ij} = N^{-1}$ and, as a consequence, matrix $\hat{I} - \hat{E}$ is also idempotent. The spectrum of idempotent matrices consists of only two points: 0 and 1, so the spectrum of $\hat{\epsilon}$ in (D.1) is $(0, K)$ so similar to the lattice case the lowest eigenvalue of $\hat{\epsilon}$ is zero.

In the absence of the momentum representation the density of states can be found according to the formula $\rho(E) = -\pi^{-1} \Im \Tr(E + i\varepsilon - \hat{\epsilon})^{-1}$. Now using the idempotence of $\hat{\epsilon}$ it is easy to find that

$$\left(\frac{1}{z - \hat{\epsilon}}\right)_{ii} = \frac{1}{Nz} - \left(1 - \frac{1}{N}\right)\frac{1}{z - K}.$$  \hfill (D.2)

With $z = E + i\varepsilon$ one gets

$$\rho(E) = N^{-1}\delta(E) + (1 - N^{-1})\delta(E - K)$$  \hfill (D.3)
so that according to (39)

\[ p(t) = N^{-1} + (1 - N^{-1})\theta(t^{-1} - r - K). \]  

(D.4)

As is seen, when

\[ 0 \leq t \leq t_0 = 1/(r + K) \]  

(D.5)
p(t) = 1 and in this range the RG equation can be solved by (36) with \( u(x, t_0) \) for IM (which includes IRIM as a special case) calculated explicitly in (F.6). By reminding that the LT (C.1)–(C.2) at \( t_0 \) is trivial, on the basis of (F.6) we obtain the explicit expression

\[ v(y, t_0) = \frac{y^2}{2t_0} - \ln \left( \cosh \frac{y}{t_0} \right) + \frac{1}{2} \ln(2\pi t_0). \]  

(D.6)

For \( t > t_0 \) \( p(t) = 1/N \) in (D.4) so the RG equation (50) for IRIM becomes the conventional BE with the constant viscosity \( 1/2N \) and in the thermodynamic limit \( N \to \infty \) it reduces to the inviscid BE with the discontinuous shock wave solutions corresponding to FOPTs [49, 51]. The transformed equation (C.3) acquires a simple form in this limit

\[ v_R |_{N \to \infty} = 0. \]  

(D.7)

In this form, however, it produces the MF solution exhibiting the undesirable van der Waals loops in the coexistence region. To see this we note that (D.7) implies that the initial condition does not change with \( t \) so at \( t_R = 1/r \) the renormalized \( v^R(y) = v(y, t_0) \). Substituting it in (C.5) and (C.6) on gets after some rearrangement the MF equation of state in parametric form

\[ h = \frac{y}{t_0} - K \tanh(y/t_0) \]  

(D.8)

\[ m = \tanh(y/t_0) \]  

(D.9)

from which immediately follows the exact IRIM MF equation [50]

\[ m = \tanh(Km + h). \]  

(D.10)

The MF free energy of IRIM [50]

\[ f = Km^2/2 - \ln \left[ 2 \cosh(Km + h) \right]. \]  

(D.11)

is obtained from (I1) with the use of inverted transform (C.2), (D.6), (D.9) and noticing that the last term in (D.6) is cancelled by the last term in (I1) as is easily calculated with the use of definitions of \( \hat{G} \) (20) and (D.1). As is seen, the arbitrary parameter \( r \) completely disappears from the solution given by equations (D.10) and (D.11). The MF solution, however, is not fully consistent because FE (D.11) is not convex and as a consequence the van der Waals loops will appear in the equation of state. As explained in the main text, these deficiencies can be avoided if the thermodynamic limit in (D.7) is taken more carefully.
Appendix E. Particularity of the lattice LPA

The LPA ansatz (33) in general case does not satisfy the S-matrix EFRGE (29) because of the second term on the r.h.s.. However, the continuum and the lattice cases differ from each other due to the difference between the Kronecker symbols in (33) which on the lattice takes the form

$$\delta^L_{k_1+k_2+...+k_l} = \delta_{k_1+k_2+...+k_l,0} + \sum_{K \neq 0} \delta_{k_1+k_2+...+k_l,K}$$

(E.1)

where on the r.h.s. the deltas are the conventional Kronecker symbols and in the summation over the reciprocal lattice vectors $K$ we singled out term $K = 0$ which is the only one that is present in the continuum theory [1, 5].

Substituting (33) in the nonlinear term in (29) and using the first Kronecker delta to lift the summation over $k$ and the definition (E.1) one arrives at the sum of quadratic in $u_l$ contributions of the form

$$u_l(t)u_{l'}(t) \sum_k \theta [G(k, t) - t] \delta^L_{\sum_{j=1}^{l-1} k_j, -k + \sum_{j'=1}^{l'-1} k'_{j'}} \delta^L_{\sum_{j=1}^{l-1} k_j, \sum_{j'=1}^{l'-1} k'_{j'}}$$

(E.2)

where $\Lambda$ in the argument of the step function on the second line should be determined from equation (38) defining the momentum cutoff. In the lattice case it depends on the crystal anisotropy, i.e., on the momentum direction $k|k|$. In the isotropic case $\Lambda$ would coincide with the conventional momentum cutoff [1, 5].

If the step functions in (E.2) were equal to unity (the LPA) than the terms (E.2) would sum up to the second term on the l.h.s. in (40). But this would be valid only if the arguments of the step functions were always positive. In the isotropic continuum theory when only $K = 0$ contributes to (E.2) this would mean that the sum of $k_j$ is smaller in absolute value than $\Lambda$. But this can be guaranteed only for $l \leq 2$ because, by our definition of the renormalization procedure, all individual field momenta reside within $\Omega(t)$ defined in (38). However, for $l > 2$ the absolute value of the sum may exceed $\Lambda$ in which case the step functions will dependent on $k_j$. As a result, many contribution in the momenta space will be lost from contributions $l, l' > 2$ and the errors introduced by the LPA will be enhanced in the strong coupling case when $u_{l>2}$ are large.

The problem alleviates in the lattice models when all $K$ contribute in (E.2). This can be visualized by considering the step functions in the periodic zone scheme where each $K$ becomes the centre of a region $\Omega_K$ which is the same $\Omega$ as in (38) only shifted on vector $K$. Now the step function in (E.2) is equal to unity within all $\Omega_K$ and zero at the outside. In the course of the renormalization this will create a muffin-tin structure where at the early stages of RG flow there will much more volume in momentum space where the LPA conjecture is fulfilled. For example, in the region $0 \leq t \leq t_0$ in figure $\Omega = BZ$ so in the periodic zone scheme the whole momentum space becomes covered and the step function is unity everywhere. This makes LPA exact which has allowed us
Exact RG equation for lattice models

To perform the exact renormalization in (36) and (F.1). As $t$ grows above $t_0$ all $\Omega_K$ start to shrink but close to $t_0$ only narrow gaps between $\Omega_K$ will appear so the violation of the LPA assumption $\theta = 1$ in (E.2) will be relatively small and can be roughly assessed by the value of $\kappa(t) = \text{vol}[\Omega(t)]/\text{vol}[\Omega(t)]$. We note that the estimate is not based on the smallness of the momentum, as in DE, or on the values of $u_l$. In fact, it relies on the closeness of $\Omega \simeq$ to BZ while $|k|$ may be as large as $\sim \sqrt{3}\pi$ on sc lattice. As $t$ advances farther toward $t_R$ $\Omega$ shrinks and $\kappa(t)$ grows to large values which signals the breakdown of the LPA assumption (35) in a large part of BZ. However, because under Wilsonian renormalization $u_l$ with large $l$ fast attenuate towards small values [1], it is expected that the terms quadratic in $u_l$ violating the LPA assumption will become small. Besides, closer to $\Omega(t)$ tends to be almost spherical, the lattice nature of the system smears out and $k/\Lambda$ becomes an acceptable DE parameter to justify LPA on the late stages of renormalization. It is to be stressed that the above reasoning relied on the shape of the cutoff function similar to the step function and is not applicable to the Wilson-type momenta elimination in large-$k$ shells [1]. This seems to be confirmed by the RG calculations in [16] where the use of the Wilson-type cutoff led to poor agreement with the MC simulations at large interaction strengths while the use of the step function resulted in a perfect agreement with the MC data.

Appendix F. Initial condition in the spin models

In $O(n)$-symmetric case the $n$-dimensional integral in the exact initial renormalization (36) takes the form

$$e^{-u(x,t_0)} = \frac{1}{(2\pi t_0)^{n/2}} \int d\mathbf{x}_0 \exp \left( -\frac{(\mathbf{x} - \mathbf{x}_0)^2}{2t_0} \right) e^{-u(x_0,0)}. \tag{F.1}$$

For spin-lattice models we assume that for all $n$ the spin length is equal to unity

$$e^{-u(x_0,0)} = \delta(x_0^2 - 1). \tag{F.2}$$

The integration in (F.1) is conveniently carried out in hyperspherical coordinates [54] in which the integration over $|\mathbf{x}_0|$ is trivial due to the delta-function in (F.2) and with the choice of the direction of $\mathbf{x}$ along the first axis: $\mathbf{x} = (x \cos \vartheta, 0, 0, \ldots, 0)$, $x = |\mathbf{x}|$ (F.1) reduces to

$$e^{-u^{(n)}(x,t_0)} = C_n e^{-\frac{t_0a^2}{2}} \int_0^\pi e^{\frac{x_0}{t_0} \cos \vartheta} \sin^{n-2} \vartheta d\vartheta$$

$$= C_n e^{-\frac{t_0a^2}{2}} \int_{-1}^{1} e^{az} (1 - z^2)^{n-3} \frac{1}{2} dz \equiv C_n e^{-\frac{t_0a^2}{2}} b(n, a) \tag{F.3}$$

where in $C_n$ are gathered all $\mathbf{x}$-independent factors from (F.1) and from the spherical volume [54] which can be easily recovered if needed; $a = x/t_0$, the integral on the second line is obtained by the change of variables $z = \cos \vartheta$. Functions $b(n, a)$ have been introduced in order to make use of a recursion relation they satisfy. The latter is obtained by integrating by parts twice in (F.3) with respect to $d(e^{az})$ to get

$$b(n, a) = \frac{n-3}{a^2} \left[ (n-5)b(n-4, a) - (n-4)b(n-2, a) \right]. \tag{F.4}$$
As is seen, functions $b(n, a)$ can be calculated recursively for all $n \geq 6$ if they are known for $n = 2 - 5$. The latter are given by the terms in brackets in the expressions calculated directly from (F.3):

\begin{align*}
    e^{-u^{(2)}(x, t_0)} &= C_2 e^{-\frac{t_0 a^2}{2}} \left[ \pi I_0 (a) \right] \\
    e^{-u^{(3)}(x, t_0)} &= C_3 e^{-\frac{t_0 a}{2}} \left[ \frac{2}{\alpha} \sinh a \right] \\
    e^{-u^{(4)}(x, t_0)} &= C_4 e^{-\frac{t_0 a}{2}} \left[ \frac{\pi}{\alpha} I_1 (a) \right] \\
    e^{-u^{(5)}(x, t_0)} &= C_5 e^{-\frac{t_0 a^2}{2}} \left[ \frac{4}{\alpha^3} (a \cosh a - \sinh a) \right]
\end{align*}

where $I_{0,1}$ are the modified Bessel functions of the first kind.

Case $n = 1$ is not covered by (F.3) but using (F.2) multiplied by 2 to accord with the conventional definition of IM, the partially renormalized potential can be straightforwardly calculated as

\[ u^{(1)}(x, t_0) = \frac{t_0 a^2}{2} - \ln(2 \cosh a) + \frac{1}{2} \ln(2\pi t_0) \]  

where we took into account field-independent terms needed in Appendix D.

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Exact RG equation for lattice models

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