On p-adic limits of topological invariants

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ABSTRACT

The purpose of this article is to define and study new invariants of topological spaces: the p-adic Betti numbers and the p-adic torsion. These invariants take values in the p-adic numbers and are constructed from a virtual pro-p completion of the fundamental group. The key result of the article is an approximation theorem which shows that the p-adic invariants are limits of their classical analogues. This is reminiscent of Lück’s approximation theorem for $L^2$-Betti numbers.

After an investigation of basic properties and examples, we discuss the p-adic analogue of the Atiyah conjecture: When do the p-adic Betti numbers take integer values? We establish this for a class of spaces and discuss applications to cohomology growth.

1. Introduction

Fix a prime number $p$. Let $X$ be a finite CW-complex with fundamental group $\Gamma = \pi_1(X, x_0)$. The purpose of this article is to construct and study p-adic valued invariants of $X$ using virtual pro-p completions of its fundamental group. A virtual pro-p completion of $\Gamma$ is a homomorphism $\varphi: \Gamma \to G$ into a profinite group $G$ which has an open pro-p subgroup (cf. §5.1). The first family of invariants to be defined are the p-adic Betti numbers. The $j$th p-adic Betti number of $X$ is a $p$-adic integer

$$b_j^{[p]}(X; \varphi, k) \in \mathbb{Z}_p,$$

which depends on a virtual pro-p completion $\varphi$ and a field $k$ which is either the field of rational numbers $\mathbb{Q}$ or a finite field of characteristic $\ell \neq p$. The p-adic Betti numbers are homotopy invariants in the natural sense: If $f: Y \to X$ is a homotopy equivalence, then $b_j^{[p]}(X; \varphi, k) = b_j^{[p]}(Y; \varphi \circ f_*, k)$ where $f_*: \pi_1(Y, y_0) \to \pi_1(X, x_0)$ is the induced isomorphism of fundamental groups. The second invariant to be defined here is the p-adic torsion of $X$. The $j$th p-adic torsion of $X$ is a $p$-adic integer (or $\infty$)

$$t_j^{[p]}(X; \varphi, R) \in \mathbb{Z}_p \cup \{\infty\},$$

which depends on a virtual pro-p completion $\varphi$ and a commutative ring $R$ in which $p$ is invertible. Just as p-adic Betti numbers, the p-adic torsion is a homotopy invariant. The definition of p-adic invariants does not depend on the cellular structure and generalizes to connected, locally path-connected and semilocally simply connected spaces.

The approximation theorem. The central result in this article is an approximation theorem which shows that the p-adic invariants are p-adic limits of their classical analogues. Recall that the fundamental group $\Gamma$ acts on the universal covering $\tilde{X}$ of $X$ via deck transformations such...
that $X \cong \tilde{X}/\Gamma$ and every finite index subgroup $\Delta \leq f.i. \Gamma$ gives rise to a finite sheeted covering space $\tilde{X}/\Delta$ of $X$.

**Theorem 1.1.** Let $X$ be a finite CW-complex and let $\varphi: \Gamma \to G$ be a virtual pro-$p$ completion of its fundamental group. For every chain $G_1 \supseteq G_2 \supseteq G_3 \ldots$ of open normal subgroups in $G$ which satisfies $\bigcap_{n \in \mathbb{N}} G_n = \{1\}$, the following assertions hold:

(i) \[ \lim_{n \to \infty} b_j(\tilde{X}/\varphi^{-1}(G_n);k) = b_j^p(X;\varphi,k), \]

(ii) \[ \lim_{n \to \infty} \left| \text{tors} H^j(\tilde{X}/\varphi^{-1}(G_n);R) \right| = t_j^p(X;\varphi,R), \]

where the limits are taken in $\mathbb{Z}_p \cup \{\infty\}$. Here $|\text{tors} H^j(Y;R)|$ denotes cardinality of the $R$-torsion part in the $j$th cohomology.

This result is a special case of Theorem 5.11 below, which is significantly more general. It is worth noting that the theorem, in particular, includes the statement that the limits exist and, for this introduction, it is receivable to consider this the definition of the $p$-adic invariants.

**Comparison with other Betti numbers.** The approximation theorem allows to transfer some properties of ordinary Betti numbers to $p$-adic Betti numbers. For instance, straightforward applications of the Approximation Theorem yield a Künneth formula (see §5.12) and a form of Poincaré duality (see Proposition 5.13).

In other aspects, however, the nature of $p$-adic Betti numbers is quite different from the nature of ordinary Betti numbers. For example, the $p$-adic Betti numbers are invariant under passing to finite sheeted covering spaces (see §5.9) and the $p$-adic Euler characteristic always vanishes if $\varphi$ has an infinite image (see §5.14).

It seems more appropriate to consider $p$-adic invariants to be analogues of $L^2$-invariants. From this perspective, the $p$-adic Approximation Theorem is the analogue of Lück’s approximation theorem for $L^2$-Betti numbers which states that

\[ \lim_{n \to \infty} \frac{b_j(\tilde{X}/\Gamma_n;\mathbb{Q})}{|\Gamma:\Gamma_n|} = b_j^{(2)}(X) \]

for every chain of finite index normal subgroups $\Gamma_n \leq f.i. \Gamma$ with trivial intersection; see [21]. To a certain extend our results on $p$-adic approximation, go beyond the known $L^2$-approximation results. For instance, it is currently unknown whether an analogue of Lück’s theorem holds for Betti numbers with coefficients in a finite field (see [23, Conjecture 3.4]), whereas the $p$-adic Approximation Theorem is valid for $k = \mathbb{F}_\ell$ for all primes $\ell \neq p$. Similarly, the $p$-adic Approximation Theorem applies to torsion cohomology, whereas the approximation of $L^2$-torsion by torsion homology is a fundamental open problem; see [23, Conjecture 8.9].

Anton Claußnitzer discussed another approach towards an $p$-adic form of Lück’s approximation theorem in his PhD thesis [10]. His methods are based on $p$-adic operator algebras but seem to be too restrictive to apply to the topological setting.

In spite of the analogy, the $p$-adic Betti numbers behave very much like antagonists of the $L^2$-Betti numbers. For example, the $L^2$-Betti numbers distinguish free groups of different rank, whereas the $p$-adic Betti numbers of the free group $F_r$ do not depend on the rank $r$; more precisely,

\[ b_0^p(F_r;\varphi,k) = b_1^p(F_r;\varphi,k) = 1, \]

whenever $\varphi$ has an infinite image; see §6.3. Similarly, the $p$-adic Betti numbers do not see the genus of a surface; see §6.1. On the other hand, the $p$-adic Betti numbers of free abelian groups
coincide with the ordinary Betti numbers (see §6.2), whereas the $L^2$-Betti numbers of every infinite amenable group vanish; see [9, Theorem 0.2].

The Approximation Theorem establishes a relationship between the $j$th $p$-adic Betti number of $X$ and the $j$th virtual Betti number of $X$, which is defined as the supremum of rational Betti numbers $vb_j(X) = \sup_Y b_j(Y; \mathbb{Q})$ over all finite sheeted covering spaces $Y$ of $X$. In fact, suppose that $vb_j(X)$ is finite, then the sequence approximating $b^{[p]}_j(X; \varphi, k)$ has to stabilize at some natural number between 0 and $vb_j(X)$. Even though the virtual Betti numbers often take the value $\infty$, there are interesting families of groups and spaces with finite virtual Betti numbers, for example, finitely presented nilpotent-by-abelian groups have finite first virtual Betti numbers; see [6].

The values of $p$-adic Betti numbers. In all our examples the $p$-adic Betti numbers are in fact integers. This comes as a surprise, since the construction of $p$-adic invariants really uses $p$-adic analysis. In fact, the $p$-adic torsion can take transcendental values; see §6.8. The following question is intriguing.

**Question 1.2.** Under which conditions on $X$ and $\varphi$ are the $p$-adic Betti numbers $b^{[p]}_j(X; \varphi, k)$ in $\mathbb{Z}$?

Our list of examples is simply too short to propose a conjectural answer. We will refer to this question as the $p$-adic Atiyah question, since it is reminiscent of the Atiyah conjecture for $L^2$-Betti numbers.

**Conjecture.** Let $X$ be a finite connected CW-complex with fundamental group $\Gamma$. If $\Gamma$ is torsion-free, then $b^{(2)}_j(X) \in \mathbb{Z}$.

This conjecture goes back to a question of Atiyah concerning the values of $L^2$-Betti numbers of compact Riemannian manifolds; see [3]. The formulation given here is only a special case of a more general algebraic formulation of the conjecture; for the precise statement, we refer to [22, Conjecture 10.2]. The Atiyah conjecture is known for certain classes of torsion-free groups, but is open in general. Recently, A. Jaikin-Zapirain obtained significant new results concerning the general conjecture; see [17].

We will study the $p$-adic Atiyah question in Section 7. As for the Atiyah conjecture, a purely algebraic formulation will play an important role. We shall prove the following results.

**Theorem 1.3.** Let $X$ be a connected CW-complex with finite $(j + 1)$-skeleton. Let $(\varphi, G)$ be a virtual pro-$p$ completion of $\Gamma = \pi_1(X)$. If $G$ is virtually abelian, then

$$b^{[p]}_j(X; \varphi, k) \in \mathbb{Z}.$$

**Theorem 1.4.** Let $X = X_1 \vee X_2 \vee \cdots \vee X_n$ be a CW-complex, which is a wedge sum of finite connected CW-complexes $X_i$ with virtually abelian fundamental groups. Then all $p$-adic Betti numbers of $X$ are integers.

**Applications.** Applications to cohomology growth provide another incentive to investigate the $p$-adic Atiyah question. In fact, if the $p$-adic Betti numbers are integers, then there is a useful dichotomy concerning the possible growth rates of ordinary Betti numbers in towers of finite sheeted covering spaces. The sequence of Betti numbers either stabilizes at the $p$-adic Betti number or grows relatively fast. Our most general result in this direction is Theorem 7.7 below.
We illustrate this for the special case of $p$-adic analytic towers. Let $X$ be a finite connected CW-complex $X$ with $\pi_1(X) = \Gamma$. For every homomorphism $\varphi : \Gamma \to GL_m(\mathbb{Z}_p)$, we obtain an associated chain of principal congruence subgroups

$$\Gamma_n = \varphi^{-1} (\ker (GL_m(\mathbb{Z}_p) \to GL_m(\mathbb{Z}/p^n\mathbb{Z}))).$$

Note that the closure $G = \overline{\varphi(\Gamma)}$ of the image of $\varphi$ is a $p$-adic Lie group.

**Theorem 1.5.** Assume that $b_j^{[p]}(X; \varphi, k) \in \mathbb{Z}$. Then for the principal congruence chain $(\Gamma_n)_n$, either the sequence $b_j(\tilde{X}/\Gamma_n; k)$ stabilizes at $b_j^{[p]}(X; \varphi, k)$ or

$$b_j(\tilde{X}/\Gamma_n; k) \geqslant k|\Gamma : \Gamma_n|^{1/d}$$

for all sufficiently large $n$ and some constant $\kappa > 0$ where $d = \dim(G)$.

Once again, there is an $L^2$-analogue which reveals the antagonistic behaviour of the $p$-adic invariants. Assume for simplicity that $\varphi : \Gamma \to GL_m(\mathbb{Z}_p)$ is injective, then a theorem of Bergeron, Linnell, Lück and Sauer (see [5], Theorem 1) yields

$$b_j(\tilde{X}/\Gamma_n; \mathbb{Q}) = b_j^{(2)}(X)|\Gamma : \Gamma_n| + O(|\Gamma : \Gamma_n|^{-1/d})$$

as $n$ tends to infinity where $d = \dim(G)$. Their result is based on Iwasawa theory and a similar result holds for homology with coefficients in $\mathbb{F}_p$-coefficients. Remarkably, their methods do not apply to homology with coefficients in $\mathbb{F}_\ell$ for primes $\ell \neq p$, whereas our methods fail for $k = \mathbb{F}_p$. Similar Iwasawa methods have been applied by Calegari and Emerton to prove asymptotic upper bounds for multiplicities of cohomological automorphic representations; see [8]. We hope that a better understanding of the $p$-adic Atiyah question provides corresponding asymptotic lower bounds.

1.6. **Outlook and open problems**

The $p$-adic invariants appear here for the first time and there are several loose ends and open questions. Here is a short list of issues, beyond the $p$-adic Atiyah question, which might deserve attention in future research.

1.6.1. **More examples.** The list of examples for which we can actually compute $p$-adic invariants is quite small. At the moment, our main method for computing $p$-adic invariants is the Approximation Theorem which, however, seems difficult to apply in more complicated examples. For instance, it would be highly interesting to study the $p$-adic Betti numbers of hyperbolic three-manifolds. Since closed hyperbolic three-manifolds virtually fibre over the circle [2], this amounts to an investigation of mapping tori.

1.6.2. **Dependence on the completion.** Is there a connected finite aspherical CW-complex $X$ and two injective completions $(\varphi_1, G_1)$ and $(\varphi_2, G_2)$ of its fundamental group such that $b_j^{[p]}(X; \varphi_1, k) \neq b_j^{[p]}(X; \varphi_2, k)$ for some $j$? The $p$-adic Betti numbers of $X$ with respect to two different virtual pro-$p$ completions $\varphi_1$ and $\varphi_2$ are distinct in general. It is easy to find such examples if $\ker(\varphi_1) \neq \ker(\varphi_2)$; see §6.4. Even if $\varphi_1$ and $\varphi_2$ are both injective, the $p$-adic Betti numbers can be distinct; see §6.5. However, we are not aware of an aspherical example.

1.6.3. **Formulation in terms of $p$-adic operator algebras.** Our construction of $p$-adic Betti numbers uses only a small amount of $p$-adic analysis and does not involve a $p$-adic companion of the group von Neumann algebra, which plays the central role in the $L^2$-theory. Is there a definition of the $p$-adic Betti numbers in terms of $p$-adic operator algebras?
1.6.4. Generalizing $p$-adic approximation. Lück’s approximation theorem has been generalized in several directions [1, 4, 14, 18] and it would be interesting to know whether there are $p$-adic analogues of these generalizations. For instance, is there a $p$-adic Farber condition which implies an approximation result like Theorem 1.1 for chains of open subgroups $G \supseteq G_1 \supseteq G_2 \supseteq \ldots$ which are not necessarily normal in $G$?

Organization of the article

Section 2: preliminaries and notation. We define and review basic notions. We recall the notion of free profinite $\mathbb{Z}_p$-modules and define smooth admissible representations of profinite groups.

Section 3: the $p$-adic dimension function. In this section, we define the $K$-group $K_0^{adm}(k[G])$ of smooth admissible $k$-representations of a pro-$p$ group $G$. We construct the $p$-adic dimension function

$$p\text{-dim}_k^G : K_0^{adm}(k[G]) \to \mathbb{Z}_p,$$

which plays the key role in defining the $p$-adic Betti numbers. We study the behaviour of the $p$-adic dimension with respect to restriction, inflation and induction of representations.

Section 4: the $p$-adic cardinality function. We introduce the admissible Burnside ring $A_p^{adm}(G)$ of a virtually pro-$p$ group $G$. We construct a $p$-adic cardinality function $\#_p^G : A_p^{adm}(G) \to \mathbb{Z}_p$, which will be used to define the $p$-adic torsion. We study the effect of restriction and inflation in the admissible Burnside ring.

Section 5: $p$-adic topological invariants. This is the main part of the article. We define the $p$-adic Betti numbers and the $p$-adic torsion and we verify that these are homotopy invariants. The main result is the Approximation Theorem 5.11. It will be used to establish a Künneth formula, Poincaré duality, a formula for wedge sums and formula for the $p$-adic Euler characteristic.

Section 6: examples. Here we discuss a number of examples: tori, surfaces, free groups and the infinite cyclic coverings of knot complements. We illustrate that the $p$-adic Betti numbers depend on the chosen completion and we study the $p$-adic torsion of (free abelian)-by-cyclic groups to show that the torsion can take transcendental values.

Section 7: the $p$-adic Atiyah question. In the last section, we discuss the $p$-adic Atiyah question (as explained in the introduction) and we explain the applications concerning the growth of Betti numbers.

2. Preliminaries and notation

2.1

The symbols $\mathbb{N}, \mathbb{N}_0, \mathbb{Z}$ and $\mathbb{Q}$ denote the set of natural numbers, the set of non-negative integers, the ring of integers and the field of rational numbers, respectively. For a prime number $p$, the ring of $p$-adic integers is denoted by $\mathbb{Z}_p$ and the field of $p$-adic numbers by $\mathbb{Q}_p$. The $p$-adic absolute value of $x \in \mathbb{Q}_p$ is $|x|_p = p^{-\nu_p(x)}$ where $\nu_p$ denotes the $p$-adic valuation. All rings considered below are assumed to be associative and to have a unit.
2.2. Profinite groups

A profinite group $G$ is a compact totally disconnected Hausdorff topological group. Every profinite group has a neighbourhood base of open normal subgroups and is isomorphic to an inverse limit of finite groups. A profinite group $G$ is a pro-$p$ group if it is isomorphic to an inverse limit of finite $p$-groups. A profinite group is called finitely generated if it has a finitely generated dense subgroup.

Let $G$ be a profinite group and let $W$ be a set. The space of locally constant $W$-valued functions on $G$ will be denoted by $C^\infty(G,W)$. This notation will also be used for finite groups $G$.

**Definition 2.3.** Let $G$ be a profinite group. An exhaustive chain $(N_n)_{n \in \mathbb{N}}$ is a decreasing sequence $N_1 \supseteq N_2 \supseteq N_3 \ldots$ of closed normal subgroups $N_n \trianglelefteq_c G$ such that $\bigcap_{n \in \mathbb{N}} N_n = \{1\}$. We note that every finitely generated profinite group has an exhaustive chain consisting of open subgroups.

2.4. Virtually pro-$p$ groups and the $p$-adic index

Let $p$ be a prime number and let $G$ be a profinite group which is virtually a pro-$p$ group, this means that $G$ has an open pro-$p$ subgroup. We sometimes need to interpret the order of $|G|$ or, more generally, the index $|G : K|$ of a closed subgroup $K \trianglelefteq_c G$ as a $p$-adic number. We define $\|G : K\| \in \mathbb{Z}_p$ as

$$\|G : K\| = \begin{cases} |G : K| & \text{if } K \text{ is open in } G \\ 0 & \text{otherwise.} \end{cases}$$

We also write $\|G\| = \|G : \{1\}\|$.

Observe that whenever $(N_n)_{n \in \mathbb{N}}$ is an exhaustive chain of open subgroups in $G$, then $(|G : N_nK|)_{n \in \mathbb{N}}$ is a convergent sequence of $p$-adic numbers and

$$\lim_{n \to \infty} |G : N_nK| = \|G : K\| \in \mathbb{Z}_p.$$  \hfill (2.1)

2.5. Profinite $\mathbb{Z}_p$-modules and algebras

Let $p$ be a prime number. A profinite $\mathbb{Z}_p$-module $M$ is a compact Hausdorff topological $\mathbb{Z}_p$-module which admits a neighbourhood base of 0 consisting of open submodules. Equivalently, $M$ is isomorphic to an inverse limit of finite $\mathbb{Z}_p$-modules. A profinite $\mathbb{Z}_p$-algebra $A$ is a commutative compact Hausdorff topological $\mathbb{Z}_p$-algebra which admits a neighbourhood base of 0 which consists of open ideals; equivalently, $A$ is an inverse limit of finite $\mathbb{Z}_p$-algebras.

2.6. Free profinite $\mathbb{Z}_p$-modules

Let $I$ be a countable set and $M$ a profinite $\mathbb{Z}_p$-module. A map $f : I \to M$ is called 0-convergent if for every open submodule $U \trianglelefteq_M M$, all but finitely many $i \in I$ satisfy $f(i) \in U$.

Let $\mathcal{M}(I, \mathbb{Z}_p)$ denote the set of functions from $I$ to $\mathbb{Z}_p$. Equipped with the topology of pointwise convergence $\mathcal{M}(I, \mathbb{Z}_p)$ is a profinite $\mathbb{Z}_p$-module.

For convenience, we fix the following notation. Let $F \subseteq I$ be a finite set and let $\varepsilon > 0$. We define $\mathcal{U}(F, \varepsilon)$ to be the set of all $\eta$ such that $|\eta(i)|_p < \varepsilon$ for all $i \in F$. The set $\mathcal{U}(F, \varepsilon)$ is an open submodule of $\mathcal{M}(I, \mathbb{Z}_p)$ and the collection of these sets is a neighbourhood base for 0 in $\mathcal{M}(I, \mathbb{Z}_p)$.

Write $\delta_i \in \mathcal{M}(I, \mathbb{Z}_p)$ for the indicator function of $i \in I$. In fact, $\mathcal{M}(I, \mathbb{Z}_p)$ is the free profinite $\mathbb{Z}_p$-module over $I$. In other words, $\mathcal{M}(I, \mathbb{Z}_p)$ has the following universal property.
Lemma. For every profinite $\mathbb{Z}_p$-module $M$ and every 0-convergent map $f: I \to M$, there exists a unique homomorphism $\tilde{f}: \mathcal{M}(I, \mathbb{Z}_p) \to M$ of profinite $\mathbb{Z}_p$-modules such that $\tilde{f}(\delta_i) = f(i)$ for all $i \in I$.

Proof. The $\mathbb{Z}_p$-submodule generated by $\{\delta_i | i \in I\}$ is the space of finitely supported functions on $I$. Since this submodule is dense in $\mathcal{M}(I, \mathbb{Z}_p)$ in the topology of pointwise convergence, the uniqueness is obvious.

In order to construct $\tilde{f}$, we define, for every open submodule $U \subseteq M$,

$$\tilde{f}_U(\alpha) = \sum_{i \in I} \alpha(i)f(i) + U \in M/U$$

for all $\alpha \in \mathcal{M}(I, \mathbb{Z}_p)$. The maps $\tilde{f}_U$ form a compatible family of linear maps into the finite quotients of $M$. Since $M \cong \varprojlim_{U \subseteq M} M/U$, we can use the universal property of the inverse limit to obtain the desired map $\tilde{f}$. \qed

2.7. Smooth representations of profinite groups

Let $R$ be a commutative ring and let $G$ be a profinite group. A smooth representation $(\rho, V)$ (or simply $V$) of $G$ over $R$ is a representation $\rho: G \to \text{Aut}_R(V)$ of $G$ on an $R$-module $V$ such that the stabilizer of every element $v \in V$ is open in $G$. We say that $V$ is admissible, if for every open subgroup $H \subseteq G$, the space $V^H$ of $H$-invariant vectors is finitely generated.

For example, the $R$-module of locally constant functions $C^\infty(G, R)$ carries two commuting smooth admissible representations of $G$. The left regular representation $l$ of $G$ on $C^\infty(G, R)$ is defined as

$$l(g)f(x) = f(g^{-1}x)$$

for all $g, x \in G$ and all $f \in C^\infty(G, R)$. The right regular representation $r$ is defined by

$$r(g)f(x) = f(xg)$$

for all $g, x \in G$ and all $f \in C^\infty(G, R)$.

2.8. Induction of smooth representations

Let $G$ be a profinite group and let $H \subseteq G$ be a closed subgroup. Let $(\sigma, W)$ be a smooth representation of $H$ over $R$. The representation (smoothly) induced by $\sigma$ from $H$ to $G$ is the representation of $G$ on the space

$$\text{ind}^G_H(W) = \{f \in C^\infty(G, W) | (\forall h \in H, x \in G) f(xh) = \sigma(h^{-1})f(x)\}$$

defined by the left regular action

$$l(g)f(x) = f(g^{-1}x).$$

Since every locally constant function $f \in C^\infty(G, W)$ is invariant under some open normal subgroup $N \subseteq G$, it follows that $\text{ind}^G_H(W)$ is smooth.

Lemma. (a) Let $N \subseteq G$ be an open normal subgroup. Then the space of $N$-invariants in $\text{ind}^G_H(W)$ is canonically isomorphic to $\text{ind}^{G/N}_{H/N}(W^{H \cap N})$ where $W^{H \cap N}$ denotes the representation of $H/N \cong H/(H \cap N)$ on the space of $(H \cap N)$-invariants in $W$.

(b) If $(\sigma, W)$ is admissible, then so is $\text{ind}^G_H(W)$.
(c) For every exhaustive chain $(N_n)_{n \in \mathbb{N}}$ of open normal subgroups in $G$, we have

$$\text{ind}_H^G(W) = \lim_{n \to \infty} \left( \text{ind}_{H\cap N_n}^{G/N_n}(W^{H\cap N_n}) \right).$$

Proof. Assertion (a) follows from the observation that for every $N$-invariant $f \in \text{ind}_H^G(W)$, the equality

$$f(x) = f(xn) = \sigma(n^{-1})f(x)$$

holds for all $n \in H \cap N$ and all $x \in G$.

Suppose that $(\sigma, W)$ is admissible, then $W^{H\cap N}$ is a finitely generated $R$-module for every $N \trianglelefteq_p G$. By (a) the $R$-module $\text{ind}_H^G(W)^N$ is isomorphic to a direct sum of $[G : HN]$ copies of $W^{N\cap H}$ and is therefore finitely generated. Let $(N_n)_{n \in \mathbb{N}}$ be an exhaustive chain of open normal subgroups in $G$. Then every locally constant function $f \in C^\infty(G, W)$ is invariant under $N_n$ for some $n \in \mathbb{N}$. In particular, we see that

$$\text{ind}_H^G(W) = \lim_{n \to \infty} \text{ind}_H^G(W)^{N_n} \cong \lim_{n \to \infty} \left( \text{ind}_{H\cap N_n}^{G/N_n}(W^{H\cap N_n}) \right). \quad \square$$

3. The $p$-adic dimension function

3.1

Fix a prime number $p$. Throughout the letter $k$ will denote either $\mathbb{Q}$ or a finite field of characteristic $\text{char}(k) = \ell$ with $\ell \neq p$. The purpose of this section is to define a $p$-adic-valued dimension function for smooth admissible representations of finitely generated virtually pro-$p$ groups.

3.2

The set of isomorphism classes of smooth irreducible $k$-representations of a profinite group $G$ will be denoted by $\text{Irr}_k(G)$. Every smooth irreducible $k$-representation is finite dimensional and the action of $G$ factors over some finite continuous quotient. The class of $S$ will be denoted by $[S]$.

3.3

Let $G$ be a pro-$p$ group. Recall that $p$ does not divide the characteristic of $k$. Every smooth admissible $k$-representation $V$ of $G$ decomposes as a direct sum

$$V \cong \bigoplus_{[S] \in \text{Irr}_k(G)} m(V, S)[S],$$

where each $[S] \in \text{Irr}_k(G)$ occurs with a finite multiplicity $m(V, S) \in \mathbb{N}_0$. The isomorphism type of $V$ is determined uniquely by the multiplicities. In particular, the isomorphism classes of smooth admissible $k$-representations of $G$ form a set $\text{adm}_k(G)$. In fact, $\text{adm}_k(G)$ is a monoid using the direct sum of representations. The class of a representation $V$ in $\text{adm}_k(G)$ will be denoted by $[V]$.

**Definition 3.4.** Let $G$ be a pro-$p$ group. The $K$-group of smooth admissible $k$-representations $\mathcal{K}_0^\text{adm}(k[G])$ is the free profinite $\mathbb{Z}_p$-module $\mathcal{M}([\text{Irr}_k(G)], \mathbb{Z}_p)$ on $\text{Irr}_k(G)$.

The multiplicity map $m \colon \text{adm}_k(G) \to \mathcal{K}_0^\text{adm}(k[G])$ sends $[V]$ to $m([V]) \in \mathcal{K}_0^\text{adm}(k[G])$, where $m([V])$ is a function on $\text{Irr}_k(G)$ to $m(V, S)$. Since $m(V \oplus W, S) = m(V, S) + m(W, S)$, it
is an injective monoid homomorphism. We use this injective map to actually identify \( \text{adm}_k(G) \) with its image in the admissible \( K \)-group.

**Lemma 3.5.** The set of classes of smooth finite dimensional representations of \( G \) is dense in \( \mathcal{K}_0^{\text{adm}}(k[G]) \).

**Proof.** Let \( \eta \in \mathcal{K}_0^{\text{adm}}(k[G]) \) and \( F \subseteq \text{Irr}_k(G) \) be a finite set. We consider the open neighbourhood \( \eta + U(F, \varepsilon) \) as in §2.6. For every \([S] \in F\) we find, using that \( \mathbb{N}_0 \) is dense in \( \mathbb{Z}_p \), a number \( n_S \in \mathbb{N}_0 \) such that \( |n_S - \eta([S])_p| < \varepsilon \). Hence, the class of the finite-dimensional smooth \( k \)-representation

\[
V = \bigoplus_{[S] \in F} n_SS
\]

is contained in \( \eta + U(F, \varepsilon) \).

**Theorem 3.6.** Let \( p \) be a prime and \( G \) a finitely generated pro-\( p \) group. Recall that \( k \) denotes either \( \mathbb{Q} \) or a finite field with \( \text{char}(k) \neq p \). There is a unique continuous \( \mathbb{Z}_p \)-linear map

\[
p^{\text{-dim}}_k^G : \mathcal{K}_0^{\text{adm}}(k[G]) \to \mathbb{Z}_p
\]

such that \( p^{\text{-dim}}_k^G V = \dim_k V \), whenever \( V \) is finite dimensional over \( k \).

**Proof.** We use that \( \mathcal{K}_0^{\text{adm}}(k[G]) \) is the free profinite \( \mathbb{Z}_p \)-module over \( \text{Irr}_k(G) \). Lemma 3.7 shows that the map \( \dim_k : [S] \mapsto \dim_k(S) \) is \( 0 \)-convergent. Therefore, we can deduce the existence of a continuous \( \mathbb{Z}_p \)-linear map \( p^{\text{-dim}}_k^G \) which satisfies \( p^{\text{-dim}}_k^G(S) = \dim_k S \) for all \([S] \in \text{Irr}_k(G)\) from §2.6. Since every finite-dimensional smooth representation is the direct sum of a finite number of irreducible ones, we conclude that \( p^{\text{-dim}}_k^G \) has the desired property.

**Lemma 3.7.** Let \( G \) be a finitely generated pro-\( p \) group. For every \( \nu \in \mathbb{N} \), all but finitely many \([S] \in \text{Irr}_k(G)\) satisfy \( p^\nu \mid \dim_k(S) \).

**Proof.** Step 1: There are only finitely many conjugacy classes of continuous homomorphisms \( \psi : G \to \text{GL}_n(k) \).

Suppose that \( k \) is a finite field. By assumption, \( G \) is finite generated and since \( \text{GL}_n(k) \) is a finite group, there are at most \( |\text{GL}_n(k)|^{d(G)} \) many homomorphisms \( \psi \) as above.

Now assume that \( k = \mathbb{Q} \). By a well-known theorem of Jordan (1880) and Minkowski (1887), there are only finitely many conjugacy classes of finite subgroups in \( \text{GL}_n(\mathbb{Q}) \). Choose representatives \( F_1, \ldots, F_j \) for the conjugacy classes. As above, for each \( i \), there are only finitely many continuous homomorphisms from \( G \) to \( F_i \).

Step 2: Every irreducible smooth \( \overline{k} \)-representation of \( G \), where \( \overline{k} \) denotes the algebraic closure of \( k \), has dimension \( p^j \) for some \( j \in \mathbb{N}_0 \).

Since every smooth representation factors over a continuous finite quotient, we may assume that \( G \) is a finite \( p \)-group. Suppose that \( k \) is a finite field. Since \( \text{char}(k) = \ell \neq p \), the dimensions of irreducible representations of \( G \) over \( \overline{k} \) coincide with dimensions of irreducible representations of \( G \) over \( \mathbb{Q} \). Indeed, this follows since Brauer characters and ordinary characters coincide in this case; see [12, Corollary (18.11)]. Hence, it suffices to consider the case \( k = \mathbb{Q} \). In this case, \( \overline{k} \) is an algebraically closed field of characteristic zero and the assertion follows from [16, Theorem (3.11)].

Step 3: The dimension of a smooth irreducible \( k \)-representation of \( G \) divides \( (p - 1)p^j \) for some \( j \in \mathbb{N} \).
As before we may assume that $G$ is a finite $p$-group. Let $S$ be an irreducible $k$-representation of $G$. The $\overline{k}$-representation $\overline{k} \otimes_k S$ decomposes into $\overline{k}$-irreducible representations

$$\overline{k} \otimes_k S = \bigoplus_{i=1}^{f} mS_i,$$

where each occurs with the same multiplicity $m$, and moreover, the $S_i$ form one orbit under the action of the Galois group $\text{Gal}(\overline{k}/k)$; see [16, (9.21)]. Let $k(p^\infty)$ be the subfield of $k$ obtained by adjoining all $p$-power roots of unity to $k$. Since $k(p^\infty)$ is a splitting field for $G$, the representations $S_i$ are $\text{Gal}(\overline{k}/k(p^\infty))$ invariant. We deduce that $f$ is the degree of a finite field extension $k'/k$ with $k' \subseteq k(p^\infty)$ and thus divides $[k(\zeta) : k]$ where $\zeta$ is a $p^m$th root of unity for some $m \in \mathbb{N}$.

Assume that $k = \mathbb{Q}$. By a theorem of Roquette $m = 1$ or $m = 2$ where the latter can only occur if $p = 2$; see [16, (10.14)]. Moreover, $[\mathbb{Q}(\zeta) : \mathbb{Q}] = \varphi(p^n) = (p-1)p^{n-1}$ and we deduce with Step 2 that $\dim_k(S) = f m \dim_{\overline{k}}(S_1)$ divides $(p-1)p^j$ for some $j$. Finally, assume that $k$ is a finite field. In this case, $m = 1$ [16, (9.21)] and $[k(\zeta) : k]$ divides $\varphi(p^n)$, and thus, we may conclude as above.

Now we can complete the proof. Let $\nu \in \mathbb{N}$ be given. We deduce from Step 1 that for each $n \in \mathbb{N}$, the number $r_k(n)$ of isomorphism classes of $n$-dimensional smooth irreducible $k$-representations of $G$ is finite. In particular, there is only a finite set $X \subseteq \text{Irr}_k(G)$ of representations whose dimension divides $(p-1)p^{\nu-1}$. Let $[S] \in \text{Irr}_k(G) \setminus X$. By Step 3, $\dim_k(S)|(p-1)p^j$ for some $j \in \mathbb{N}$. However, since $\dim_k(S) \nmid (p-1)p^{\nu-1}$, we conclude that $p^\nu \nmid \dim_k(S)$. $\square$

3.8. Restriction to open subgroups

Let $H \leq_\sigma G$ be an open subgroup of a finitely generated pro-$p$ group $G$. The restriction $V|_H$ to $H$ of a smooth admissible $k$-representation $V$ of $G$ is a smooth admissible representation of $H$. The next result extends this operation to a continuous map between the admissible $\mathcal{K}$-groups.

**Proposition.** Let $G$ be a finitely generated pro-$p$ group and let $H \leq_\sigma G$. There is a unique continuous $\mathbb{Z}_p$-linear map

$$\text{res}_H^G : \mathcal{K}_0^{\text{adm}}(k[G]) \to \mathcal{K}_0^{\text{adm}}(k[H])$$

such that $\text{res}_H^G([V]) = [V|_H]$ for every $[V] \in \text{adm}_k(G)$.

Restriction preserves the $p$-adic dimension, that is,

$$p \cdot \dim_k^H(\text{res}_H^G(\eta)) = p \cdot \dim_k^G(\eta)$$

for all $\eta \in \mathcal{K}_0^{\text{adm}}(k[G])$.

**Proof.** Assume for a moment that $\text{res}_H^G$ exists, then the map $p \cdot \dim_k^H \circ \text{res}_H^G$ is a continuous $\mathbb{Z}_p$-linear map from $\mathcal{K}_0^{\text{adm}}(k[G])$ to $\mathbb{Z}_p$ and satisfies

$$p \cdot \dim_k^H(\text{res}_H^G([V])) = p \cdot \dim_k^H([V|_H]) = \dim_k V|_H = \dim_k V$$

for every finite-dimensional smooth $k$-representation $V$ of $G$. The uniqueness statement in Theorem 3.6 yields $p \cdot \dim_k^H \circ \text{res}_H^G = p \cdot \dim_k^G$.

We will construct $\text{res}_H^G$ using the universal property explained in §2.6. This amounts to showing that the map $\text{Irr}_k(G) \to \mathcal{K}_0^{\text{adm}}(k[H])$ which sends $[S]$ to $[S|_H]$ is 0-convergent. Let $F \subseteq \text{Irr}_k(H)$ be a finite set and let $\varepsilon > 0$. Then $[S|_H]$ does not lie in the open submodule $U(F, \varepsilon) \subseteq \mathcal{K}_0^{\text{adm}}(k[G])$ only if $m(S|_H, T) \neq 0$ for some $[T] \in F$.

It remains to show that for every $[T] \in \text{Irr}_k(H)$, there are only finitely many $[S] \in \text{Irr}_k(G)$ such that $m(S|_H, T) \neq 0$. Indeed, $m(S|_H, T) \neq 0$ exactly if there is a non-zero $H$-equivariant
homomorphism from $T$ to $S$. By Frobenius reciprocity, the later holds if and only if there is a non-zero homomorphism of $G$-representations from the induced representation $\text{ind}_H^G(T)$ to $S$; see [12, (10.8)]. However, a non-trivial homomorphism from $\text{ind}_H^G(T)$ to the irreducible representation $S$ exists precisely when $S$ occurs as a direct summand of $\text{ind}_H^G(T)$. Since the representation $\text{ind}_H^G(T)$ is finite dimensional, its dimension is $|G:H| \dim_K(T)$, there are at most finitely many such $[S] \in \text{Irr}_k(G)$. 

**Definition 3.9.** Let $G$ be a finitely generated virtually pro-$p$ group and let $V$ be a smooth admissible $k$-representation of $G$. We define the $p$-adic dimension of $V$ as

$$p\dim_k^G V = p \dim_K^H ([V_H])$$

where $H \leq_G G$ is some open pro-$p$ subgroup. The previous proposition shows that this definition is independent of the choice of the subgroup $H$. In addition, with this definition, any open subgroup $H \leq_G G$ satisfies the identity (3.1).

3.10. **Inflation**

Let $\pi: G \to H$ be a surjective homomorphism of finitely generated pro-$p$ groups. Then every smooth admissible $k$-representation $V$ of $H$ becomes a smooth admissible $k$-representation of $G$ via $\pi$. This new representation $\text{infl}_H^G(V)$ is called the inflation of $V$ via $\pi$.

**Proposition.** There is a unique continuous $\mathbb{Z}_p$-linear map $\text{infl}_H^G: K_0^\text{adm}(k[H]) \to K_0^\text{adm}(k[G])$ such that $\text{infl}_H^G[V] = [\text{infl}_H^G(V)]$ for every smooth admissible $k$-representation of $H$. Inflation preserves the $p$-adic dimension, that is,

$$p \dim_k^G (\text{infl}_H^G(\eta)) = p \dim_K^H (\eta).$$

**Proof.** Again, uniqueness and (3.2) follow immediately once the existence is established. Let $[S] \in \text{Irr}_k(H)$, then the inflation $\text{infl}_H^G(S)$ is irreducible. This yields an injective map from $\text{Irr}_k(H)$ into $\text{Irr}_k(G)$. In addition, a class $[T] \in \text{Irr}_k(G)$ can be represented by an inflated representation if and only if $T$ factors over the kernel of $\pi$. Hence, for $\eta \in K_0^\text{adm}(k[H])$, we define

$$\text{infl}_H^G(\eta)([T]) = \eta([S])$$

if $T \cong \text{infl}_H^G(S)$ for some $S$ and we impose $\text{infl}_H^G(\eta)([T]) = 0$ otherwise. It is easily verified that this defines a 0-convergent map a $\text{Irr}_k(H) \to K_0^\text{adm}(k[G])$. 

**Lemma 3.11 (Approximation lemma).** Let $G$ be a finitely generated virtually pro-$p$ group and let $V$ be a smooth admissible $k$-representation. Every exhaustive chain $(N_n)_{n \in \mathbb{N}}$ in $G$ satisfies

$$\lim_{n \to \infty} p \dim_k^{G/N_n} V^{N_n} = p \dim_k^G V.$$

In particular, if each $N_n$ is open in $G$, then

$$\lim_{n \to \infty} \dim_k V^{N_n} = p \dim_k^G V \in \mathbb{Z}_p.$$

**Proof.** Let $H \leq_G G$ be an open normal subgroup. We claim that there is some $e \in \mathbb{N}$ such that $N_n \subseteq H$ for all $n \geq e$. Indeed, the intersection $\bigcap_{n \in \mathbb{N}} (N_n \setminus H) = \emptyset$ and, since the sets $N_n \setminus H$ are compact and nested, this implies that $N_e \setminus H$ is empty for some $e \in \mathbb{N}$. Applying this to an open normal pro-$p$ subgroup reduces the proof to the case where $G$ is a pro-$p$ group.
Due to Section 3.10, we have the equality $p$-$\dim_k^{G} \inf_{G/N_n}^{G} (V^{N_n}) = p$-$\dim_k^{G} V^{N_n}$. It remains to show that the sequence $(\inf_{G/N_n}^{G} (V^{N_n}))_{n \in \mathbb{N}}$ converges to $[V]$ in $\mathcal{K}_{\mathcal{O}}^\mathrm{adm}$($k[G]$), and then the continuity of the $p$-adic dimension function completes the proof. Let $A \subseteq \text{Irr}_{k}(G)$ be a finite subset, then there is an open normal subgroup $H \triangleleft_{o} G$ such that $H$ acts trivially on $S$ for all $[S] \in A$. We have seen that $N_n \subseteq H$ for all sufficiently large $n \in \mathbb{N}$, and thus $m(V,S) = m(\inf_{G/N_n}^{G} (V^{N_n}), S)$ for all $[S] \in A$. This finishes the proof. □

\textbf{Proposition 3.12.} Let $H \trianglelefteq G$ be a finitely generated closed subgroup of a finitely generated virtually pro-$p$ group $G$. The $p$-adic dimension satisfies

$$p$-$\dim_k^{G} (\text{ind}_H^{G}(W)) = \|G : H\| p$-$\dim_k^{H}(W).$$

for every smooth admissible $k$-representation $W$ of $H$. Here, we use the notation of §2.4.

\textbf{Proof.} Let $(N_n)_{n \in \mathbb{N}}$ be an exhaustive chain of open subgroups in $G$. We observe that $(H \cap N_n)_{n \in \mathbb{N}}$ is an exhaustive chain in $H$. We combine Lemma 2.8 (a), the Approximation Lemma 3.11 and equation (2.1) to obtain

$$p$-$\dim_k^{G} (\text{ind}_H^{G}(W)) = \lim_{n \to \infty} p$-$\dim_k^{G} N_n / H_{N_n} (W^{H \cap N_n})$$

$$= \lim_{n \to \infty} |G : H N_n| \dim_k(W^{H \cap N_n}) = \|G : H\| p$-$\dim_k^{H}(W).$$

□

4. The $p$-adic cardinality function

4.1

Throughout $p$ denotes a fixed prime number and $G$ a finitely generated virtually pro-$p$ group. We will define the notion of admissible $G$-set and define a corresponding admissible Burnside ring. On this ring, there is a continuous $p$-adic valued notion of cardinality.

\textbf{Definition 4.2.} A set $X$ with a $G$-action is \textit{admissible} if every $x \in X$ has an open stabilizer in $G$ and for every open subgroup $H \trianglelefteq G$, the set of $H$-invariants $X^H$ is finite.

4.3

Let $X$ be an admissible $G$-set. It follows from the definition of admissibility that every $G$-orbit in $X$ is finite and, moreover, for every open subgroup $H \trianglelefteq G$, there are only finitely many orbits isomorphic to $G/H$ contained in $X$. Let $\text{Or}(G)$ denote the set of isomorphism classes of finite transitive $G$-sets; it is in bijection with the set of conjugacy classes of open subgroups of $G$. Now we can write

$$X \cong \bigsqcup_{Z \in \text{Or}(G)} \iota(X, Z) Z$$

as a disjoint union of finite orbits, where each orbit type $Z$ occurs with a finite multiplicity $\iota(X, Z) \in \mathbb{N}_0$. Clearly, the isomorphism type of $X$ is uniquely determined by the multiplicities $\iota(X, Z)$. Finite disjoint unions and finite direct products of admissible $G$-sets are admissible.
4.4. The admissible Burnside ring

Let $\mathcal{A}^{adm}_p(G) = \mathcal{M}(\text{Or}(G),\mathbb{Z}_p)$ denote the free profinite $\mathbb{Z}_p$-module on $\text{Or}(G)$; compare § 2.6. Here we should not think of the elements of $\mathcal{A}^{adm}_p(G)$ as functions; instead, it is more convenient to denote $\alpha \in \mathcal{A}^{adm}_p(G)$ as a formal series

$$\alpha = \sum_{Z \in \text{Or}(G)} \alpha_Z Z$$

with $\mathbb{Z}_p$-coefficients. For $\alpha, \beta \in \mathcal{A}^{adm}_p(G)$, we define

$$\alpha \cdot \beta = \sum_{Z \in \text{Or}(G)} \left( \sum_{X,Y \in \text{Or}(G)} \iota(X \times Y, Z) \alpha_X \beta_Y \right) Z.$$

Note that for each $Z \in \text{Or}(G)$, there are only finitely many $X,Y \in \text{Or}(G)$ such that $X \times Y$ contains an orbit isomorphic to $Z$. This defines an associative, commutative multiplication on $\mathcal{A}^{adm}_p(G)$ which becomes thus a profinite $\mathbb{Z}_p$-algebra.

We will call $\mathcal{A}^{adm}_p(G)$ the admissible Burnside ring of $G$. Let $X$ be an admissible $G$-set, then $\iota(X) \in \mathcal{A}^{adm}_p(G)$ will denote the element

$$\iota(X) = \sum_{Z \in \text{Or}(G)} \iota(X,Z) Z.$$

We note that $\iota(X) + \iota(Y) = \iota(X \sqcup Y)$ and $\iota(X) \cdot \iota(Y) = \iota(X \times Y)$.

**Lemma 4.5** (Universal property). Let $A$ be a profinite $\mathbb{Z}_p$-algebra. Let $f$ be a function which assigns to every finite $G$-set $X$ an element $f(X) \in A$ in such a way that

(i) $f|_{\text{Or}(G)}$ is 0-convergent,
(ii) $f(X \sqcup Y) = f(X) + f(Y)$ and
(iii) $f(X \times Y) = f(X)f(Y)$ for all finite $G$-sets $X$ and $Y$.

Then there is a unique homomorphism $\tilde{f} : \mathcal{A}^{adm}_p(G) \to A$ of profinite $\mathbb{Z}_p$-algebras with $\tilde{f}(\iota(X)) = f(X)$ for all finite $G$-sets.

**Proof.** Since $\mathcal{A}^{adm}_p(G)$ is the free profinite $\mathbb{Z}_p$-module on $\text{Or}(G)$ assumptions, (i) and (ii) imply the existence of a unique homomorphism $f : \mathcal{A}^{adm}_p(G) \to A$ of profinite $\mathbb{Z}_p$-modules with $\tilde{f} \circ \iota = f$. It remains to show that $\tilde{f}$ is multiplicative. We observe that, since $\mathbb{N}_0$ is dense in $\mathbb{Z}_p$, the image under $\iota$ of the set of finite $G$-sets is dense in $\mathcal{A}^{adm}_p(G)$; compare with the proof of Lemma 3.5. Now continuity of $\tilde{f}$ and assumption (iii) yield the claim. □

**Theorem 4.6.** Let $G$ be a finitely generated virtually pro-$p$ group. There is a unique continuous homomorphism $\#^G_p : \mathcal{A}^{adm}_p(G) \to \mathbb{Z}_p$ of $\mathbb{Z}_p$-algebras such that $\#^G_p(\iota(X)) = |X|$ for every finite $G$-set $X$.

**Proof.** We use the universal property of the admissible Burnside ring. The function $| \cdot |$ is additive and multiplicative; therefore, we only need to show that $| \cdot | : \text{Or}(G) \to \mathbb{Z}_p$ is 0-convergent. Indeed, fix some open normal pro-$p$ subgroup $H \leq_o G$. The cardinality $|Z|$ of an orbit type is the index of the stabilizer $K \leq_o G$ of some point in $Z$. Since $G$ is finitely generated, it has only finitely many open subgroups of a given index. Further, $|G : K|$ is of the form $dp^\nu$ for some $\nu \in \mathbb{N}_0$ and $d$ a divisor of the index $|G : H|$. We deduce that the map is 0-convergent and this completes the proof. □
DEFINITION 4.7. Let $X$ be an admissible $G$-set, then $\#_p^G(X) := \#_p^G(\iota(X))$ will be called the $p$-adic cardinality of $X$.

4.8. Restriction to open subgroups

Let $G$ be a finitely generated virtually pro-$p$ group and $H \leq_G G$ be an open subgroup. Given an admissible $G$-set $X$, we may restrict the action to $H$ in order to obtain an admissible $H$-set $X|_H$.

PROPOSITION. There is a unique homomorphism of profinite $\mathbb{Z}_p$-algebras

$$\text{res}_H^G: \mathcal{A}_p^{\text{adm}}(G) \to \mathcal{A}_p^{\text{adm}}(H)$$

such that $\text{res}_H^G(\iota(X)) = \iota(X|_H)$ for every admissible $G$-set $X$. Moreover, $\#_p^G(\alpha) = \#_p^H(\text{res}_H^G(\alpha))$ for all $\alpha \in \mathcal{A}_p^{\text{adm}}(G)$.

Proof. Existence and uniqueness of the restriction homomorphism follow from the universal property of the admissible Burnside ring. It is clear that restriction is additive and multiplicative; thus, it remains to show that $\text{res}_H^G: \text{Or}(G) \to \mathcal{A}_p^{\text{adm}}(H)$ is 0-convergent. This follows from the observation that for each fixed $Y \in \text{Or}(H)$, there are only finitely many $Z \in \text{Or}(G)$ such that $Y$ occurs as an $H$-orbit in $Z|_H$.

The equality $\#_p^G(\alpha) = \#_p^H(\text{res}_H^G(\alpha))$ is a direct consequence of the uniqueness of cardinality and restriction homomorphisms. □

4.9. Inflation

Let $\pi: G \to H$ be a surjective homomorphism of finitely generated virtually pro-$p$ groups. Every admissible $H$-set $Y$ of $H$ becomes an admissible $G$-set via $\pi$; it will be denoted by $\text{infl}_H^G(Y)$ and is called the inflation of $Y$ via $\pi$.

PROPOSITION. There is a unique homomorphism of profinite $\mathbb{Z}_p$-algebra

$$\text{infl}_H^G: \mathcal{A}_p^{\text{adm}}(H) \to \mathcal{A}_p^{\text{adm}}(G)$$

such that $\text{infl}_H^G(\iota(Y)) = \iota(\text{infl}_H^G(Y))$ for every admissible $H$-set $Y$. Furthermore, $\#_p^G(\text{infl}_H^G(\alpha)) = \#_p^H(\alpha)$ for all $\alpha \in \mathcal{A}_p^{\text{adm}}(H)$.

Proof. As before this is an immediate consequence of the universal property of the admissible Burnside ring. □

LEMMA 4.10 (Approximation lemma). Let $G$ be a finitely generated virtually pro-$p$ group and let $X$ be an admissible $G$-set. Every exhaustive chain $(N_n)_{n \in \mathbb{N}}$ in $G$ satisfies

$$\lim_{n \to \infty} \#_p^{G/N_n} X^{N_n} = \#_p^G X.$$  

In particular, if every $N_n$ is open in $G$, then

$$\lim_{n \to \infty} |X^{N_n}| = \#_p^G X.$$  

Proof. We proceed as in the proof of Lemma 3.11. Since $\#_p^G(\text{infl}_{G/N_n}^G(X^{N_n})) = \#_p^{G/N_n} X^{N_n}$ and the $p$-adic cardinality is continuous, it is sufficient to check that the sequence $\text{infl}_{G/N_n}^G(X^{N_n})$ converges to $X$ in $\mathcal{A}_p^{\text{adm}}(G)$. However, for every orbit type $Z \in \text{Or}(G)$, the multiplicity $\iota(\text{infl}_{G/N_n}^G(X^{N_n}), Z)$ equals $\iota(X, Z)$ as soon as the action of $G$ on $Z$ factors through $G/N_n$. □
5. \textit{p-Adic topological invariants} \\
Let $X$ be a connected, locally path-connected and semilocally simply connected topological space. The fundamental group of $X$ at some fixed basepoint $x_0 \in X$ will be denoted by $\Gamma = \pi_1(X, x_0)$. We write $q: \tilde{X} \to X$ to denote the universal covering space of $X$. The choice of a base point $\tilde{x}_0 \in \tilde{X}$ with $q(\tilde{x}_0) = x_0$ yields an isomorphism between $\Gamma$ and the group of deck transformations of $\tilde{X}$. In this way, the universal covering carries a left action of $\Gamma$ such that $\tilde{X}/\Gamma \cong X$. Throughout we fix a prime number $p$. The purpose of this section is to construct $p$-adic-valued invariants of $X$ for every virtual pro-$p$ completion of $\Gamma$.

**Definition 5.1.** A virtual pro-$p$ completion $(\varphi, G)$ of $\Gamma$ is a homomorphism $\varphi: \Gamma \to G$ with dense image to a finitely generated virtually pro-$p$ group $G$.

5.2. \textit{Direct limits of cohomology groups} \\
Let $(\varphi, G)$ be a virtual pro-$p$ completion of $\Gamma$ and let $R$ be a commutative ring.

The inverse image $\Gamma_K = \varphi^{-1}(K)$ of an open subgroup $K \leq_o G$ is a finite index subgroup of $\Gamma$. Moreover, for open subgroups $K_1 \subseteq K_2 \leq_o G$, we obtain a corresponding finite sheeted covering map

$$q_{K_1, K_2}: \tilde{X}/\Gamma_{K_1} \to \tilde{X}/\Gamma_{K_2}.$$ 

For every $j \in \mathbb{N}_0$, the maps $q_{N_1, N_2}^j$ provide a directed system of cohomology $R$-modules $(H^j(\tilde{X}/\Gamma_N, R))_{N \leq_o G}$ indexed by the open normal subgroups of $G$. The direct limit will be denoted by

$$\tilde{H}^j(X; \varphi, R) = \lim_{N \leq_o G} H^j(\tilde{X}/\Gamma_N; R).$$

More generally, let $A \subseteq X$ be a subset. For brevity, we put $A_K = q_{K,G}^{-1}(A) \subseteq \tilde{X}/\Gamma_K$ for any open subgroup $K \leq_o G$. We define

$$\tilde{H}^j(X, A; \varphi, R) = \lim_{N \leq_o G} H^j(\tilde{X}/\Gamma_N, A_N; R).$$

The canonical map $H^j(\tilde{X}/\Gamma_N, A_K; R) \to H^j(X, A; \varphi, R)$ will be denoted by $i_N$. The group $\Gamma/\Gamma_N$ acts $R$-linearly on $H^j(\tilde{X}/\Gamma_N, A_N; R)$. Since, moreover, by assumption, the image of $\varphi$ is dense, the homomorphism $\varphi$ yields an isomorphism $\Gamma/\Gamma_N \cong G/N$. Using this isomorphism, we obtain an $R$-linear representation of $G/N$ on $H^j(\tilde{X}/\Gamma_N, A_N; R)$. In the limit, this provides an $R$-linear representation of $G$ on $\tilde{H}^j(X, A; \varphi, R)$. The action of $G$ is smooth, that is, the stabilizer of every element is open in $G$.

**Lemma 5.3.** Let $A \subseteq X$ be a subspace. Let $R$ be a commutative ring such that $p$ is invertible in $R$.

(a) Let $K \leq_o G$ be an open pro-$p$ subgroup. The space $\tilde{H}^j(X, A; \varphi, R)^K$ of $K$-invariants is canonically isomorphic to $H^j(\tilde{X}/\Gamma_N, A_K; R)$ as $R$-module.

(b) Let $K \leq_o G$ be a closed normal pro-$p$ subgroup and let $f: G \to G/K$ denote the factor map. There is a canonical isomorphism

$$\tilde{H}^j(X, A; f \circ \varphi, R) \cong \tilde{H}^j(X, A; \varphi, R)^K$$

of $R$-modules with $G/K$-action.
Proof. (a) For open normal subgroups \( N \trianglelefteq_o G \) with \( N \subseteq K \), the maps
\[
q_{N,K}^*: H^j(\tilde{X}/\Gamma_N, A_N; R) \to H^j(\tilde{X}/\Gamma_N, A_N; R)
\]
form a compatible system and induce a canonical map \( i_K \) from the relative cohomology \( H^j(\tilde{X}/\Gamma_N, A_N; R) \) to \( H^j(X, A; \varphi, R) \) such that \( i_K = i_N \circ q_{N,K}^* \) for all \( N \trianglelefteq_o G \) with \( N \subseteq K \).

Since \( K \) is a pro-\( p \) group and \( p \) is invertible in \( R \), it follows from the argument used in [15, Proposition 3G.1] that \( q_{N,K}^* \) is injective and the image is exactly the space of \( K \)-invariants in \( H^j(\tilde{X}/\Gamma_N, A_N; R) \).

As observed above, \( \alpha \) lies in the image of \( q_{N,K}^* \) and therefore \( \alpha \) lies in the image of \( i_N \circ q_{N,K}^* = i_K \).

(b) The module \( \tilde{H}^j(X, A; f \circ \varphi, R) \) can be constructed as the direct limit of the \( H^j(\tilde{X}/\Gamma_N, A_N; R) \) over all open normal subgroups \( N \trianglelefteq_o G \) which contain \( K \). This is a subsystem of the directed system used to define \( \tilde{H}^j(X, A; \varphi, R) \) and the inclusion of directed systems yields a canonical map
\[
i_K: \tilde{H}^j(X, A; f \circ \varphi, R) \to \tilde{H}^j(X, A; \varphi, R),
\]
which satisfies \( i_K \circ i_{N/K} = i_N \) for all \( K \subseteq N \subseteq_o G \). The map \( i_K \) is \( G \)-equivariant and thus takes values in the subspace of \( K \)-invariants. Let \( \alpha \in \tilde{H}^j(X, A; \varphi, R)^K \). Since the action of \( G \) is smooth, there is some \( N \subseteq_o G \) such that \( \alpha \) is \( N \)-stable. We may assume that \( N \) contains \( K \) and that \( N \) is pro-\( p \). By (a) we know that \( \alpha \) lies in the image of \( i_N \) and \( i_K \circ i_{N/K} = i_N \) yields that \( \alpha \) lies in the image of \( i_K \). As in (a), the map \( i_K \) is injective since all the maps \( q_{N_1,N_2}^* \) are injective; it provides the required isomorphism onto the space of \( K \)-invariants. \( \square \)

5.4. Homotopy invariance

Let \( f: X \to Y \) be a homotopy equivalence of connected, locally path-connected and semilocally simply connected spaces. We choose base points \( x_0 \in X \) and \( y_0 \in Y \) in such a way that \( f(x_0) = y_0 \). The fundamental groups will be denoted by \( \Gamma = \pi_1(X, x_0) \) and \( \Delta = \pi_1(Y, y_0) \).

Let \( \varphi: \Gamma \to G \) and \( \psi: \Delta \to G \) be virtual pro-\( p \) completions such that the following diagram commutes.

\[
\begin{array}{ccc}
\Gamma = \pi_1(X, x_0) & \xrightarrow{f_*} & \pi_1(Y, y_0) = \Delta \\
\varphi \downarrow & & \downarrow \psi \\
G & \xleftarrow{\psi} & G \\
\end{array}
\]

(5.1)

**Lemma.** In the situation above, the homotopy equivalence \( f \) induces an \( R \)-linear isomorphism of smooth representations of \( G \)
\[
f^*: H^j(Y; \psi, R) \to H^j(X; \varphi, R)
\]
for every commutative ring \( R \).

**Proof.** Recall that we fixed base points \( \tilde{x}_0 \in \tilde{X} \) and \( \tilde{y}_0 \in \tilde{Y} \) in order to identify the groups of deck transformations with the fundamental groups. Let \( \tilde{f}: \tilde{X} \to \tilde{Y} \) the unique lift of \( f \) such that \( \tilde{f}(\tilde{x}_0) = \tilde{y}_0 \). Then \( \tilde{f} \) is a homotopy equivalence which intertwines the actions of \( \Gamma \) and \( \Delta \), that is,
\[
\tilde{f}(\gamma x) = f_*(\gamma)\tilde{f}(x)
\]
for all \( x \in \tilde{X} \) and \( \gamma \in \Gamma \).
For every open normal subgroup \( N \leq \Gamma \), the subgroups \( \Gamma_N = \varphi^{-1}(N) \) and \( \Delta_N = \varphi^{-1}(N) \) satisfy \( f_* (\Gamma_N) = \Delta_N \) due to (5.1). In particular, \( \tilde{f} \) induces a homotopy equivalence
\[
\tilde{f}_N : \tilde{X}/\Gamma_N \to \tilde{Y}/\Delta_N,
\]
which intertwines the actions of \( \Gamma/\Gamma_N \cong G/N \cong \Delta/\Delta_N \). Thus, the map \( \tilde{f}_N^* \) induced in the cohomology is an \( R \)-linear \( G/N \)-equivariant map. The maps \( (\tilde{f}_N^*)_{N \leq \Gamma} \) are compatible with the directed system and \( \tilde{f}^* = \lim_{N \leq \Gamma} \tilde{f}_N^* \) is the desired isomorphism. \( \square \)

5.5. The cochain complex

Here, we give a description of a cochain complex which computes \( H^j(X; \varphi, R) \) in the case where \( X \) is a connected CW-complex with finite \( d \)-skeleton for some \( d > j \).

Let \( (C_*, \partial_*) \) be the cellular chain complex of \( X \) (where \( X \) is equipped the cellular structure lifted from \( X \)). Recall that \( C_\ast \) is a chain complex of free \( \mathbb{Z}[\Gamma] \)-modules. Since \( X \) has a finite \( d \)-skeleton, \( C_j \) is a finitely generated free \( \mathbb{Z}[\Gamma] \)-module of, say, rank \( e_j \) for all \( j \leq d \). After choosing bases \( x_1^{(j)}, \ldots, x_{e_j}^{(j)} \in C_j \) for the chain modules, the \( \mathbb{Z}[\Gamma] \)-equivariant boundary map \( \partial_{j+1} : \mathbb{Z}[\Gamma]^{e_{j+1}} \to \mathbb{Z}[\Gamma]^{e_j} \) is given by right multiplication with a matrix \( A_{j+1} \in M_{e_{j+1}, e_j}(\mathbb{Z}[\Gamma]) \) on row vectors for all \( j < d \), that is,
\[
\partial_{j+1}(x_i^{(j+1)}) = \sum_{m=1}^{e_j} a_{i,m} x_m^{(j)}.
\]

**Lemma.** Let \( R \) be a commutative ring and let \( (\varphi, G) \) be a virtual pro-\( p \) completion of \( \Gamma \). Under the assumptions above, the cochain complex
\[
\tilde{C}_* : 0 \to C^\infty(G, R)^{\Gamma_0} \xrightarrow{\tilde{\partial}^0} C^\infty(G, R)^{\Gamma_1} \xrightarrow{\tilde{\partial}^1} \cdots \xrightarrow{\tilde{\partial}^{d-1}} C^\infty(G, R)^{\Gamma_d} \to 0
\]
computes the homology \( H^j(X; \varphi, R) \) for all \( j \leq d - 1 \) where the coboundary map \( \tilde{\partial}^j \) is multiplication with the matrix \( A_{j+1} \) via the right regular representation (the elements of \( C^\infty(G, R)^{\Gamma_d} \) are considered to be column vectors).

**Proof.** Let \( \Delta \leq f_* \Gamma \), \( \Gamma \) be a finite index normal subgroup of \( \Gamma \). Recall that the (cellular) cohomology of \( \tilde{X}/\Delta \) is computed by the cochain complex \( \text{Hom}_{\mathbb{Z}[\Gamma]}(C_\ast, R) \). However, for every \( \mathbb{Z}[\Gamma] \)-module \( M \), there is a canonical isomorphism
\[
\Psi : \text{Hom}_{\mathbb{Z}[\Gamma]}(M, C^\infty(\Gamma/\Delta, R)) \to \text{Hom}_{\mathbb{Z}[\Delta]}(M, R)
\]
of \( R \)-modules where \( C^\infty(\Gamma/\Delta, R) \) is a \( \mathbb{Z}[\Gamma] \)-module via the right regular action. To be more precise, the isomorphism \( \Psi \) is defined as \( \Psi(\beta)(x) = \beta(x)(1_{\Gamma/\Delta}) \) for all \( x \in M \). In addition, \( \text{Hom}_{\mathbb{Z}[\Gamma]}(M, C^\infty(\Gamma/\Delta, R)) \) is an \( R[\Gamma/\Delta] \)-module, using the action of \( \Gamma/\Delta \) on \( C^\infty(\Gamma/\Delta, R) \) via the left regular representation; see §2.7. The \( R \)-module \( \text{Hom}_{\mathbb{Z}[\Gamma]}(M, R) \) also carries an \( R[\Gamma/\Delta] \)-module structure by imposing \( (\gamma \Delta \cdot \alpha)(x) = \alpha(\gamma^{-1} x) \) for all \( \gamma \in \Gamma, \alpha \in \text{Hom}_{\mathbb{Z}[\Delta]}(M, R) \) and all \( x \in M \). Now it is easy to see that \( \Psi \) is in fact an isomorphism of \( R[\Gamma/\Delta] \)-modules.

Taking direct limits is exact; therefore, the cochain complex
\[
\lim_{N \leq \Gamma} \text{Hom}_{\mathbb{Z}[\Gamma]}(C_\ast, C^\infty(\Gamma/\Gamma_N, R))
\]
computes the cohomology \( H^*(X; \varphi, R) \). We note that \( \Gamma/\Gamma_N \cong G/N \). By assumption, \( C_j \) is a finitely generated (free) \( \mathbb{Z}[\Gamma] \)-module for \( j \leq d \), and hence, we may exchange Hom and the direct limit, that is,
\[
\lim_{N \leq \Gamma} \text{Hom}_{\mathbb{Z}[\Gamma]}(C_j, C^\infty(G/N, R)) \cong \text{Hom}_{\mathbb{Z}[\Gamma]}(C_j, C^\infty(G, R)) \cong C^\infty(G, R)^{\Gamma_j}.
\]
The isomorphism $F$ is constructed using our chosen bases, which means that $F: \alpha \mapsto (\alpha(x_1^{(j)}), \ldots, \alpha(x_{\ell}^{(j)}))^T$.

Finally, we describe the coboundary maps in coordinates. Let $f \in C^\infty(G,R)^{e_j}$ with coordinates $(f_1, \ldots, f_{e_j})^T$ and write $f = F(\alpha)$. Then $\bar{\partial}^j(F(\alpha)) = F(\alpha \circ \partial_{j+1})$ and

$$\alpha(\partial_{j+1}(x_i^{(j+1)})) = \sum_{m=1}^{e_j} \alpha(a_{i,m}x_m^{(j)}) = \sum_{m=1}^{e_j} r(a_{i,m})f_m.$$ 

In other words, $\bar{\partial}^j(f) = r(A_{j+1})f$.

**Remark.** Here it is worth noting that $C^\infty(G,R)$ carries two commuting representations of $G$: the left and the right regular representation; see §2.7. In particular, if we use a matrix $A \in M_{\ell,m}(\mathbb{Z}[\Gamma])$ to act on $C^\infty(G,R)^{m}$ using the right regular representation, then kernel and image are smooth representations of $G$ via the left regular action.

5.6. **Definition of $p$-adic Betti numbers**

Let $k$ be either $\mathbb{Q}$ or a finite field of characteristic $\ell \neq p$. Let $(\varphi, G)$ be a virtual pro-$p$ completion of $\Gamma$.

It follows from Lemma 5.3 that the $k$-representation of $G$ on $\bar{H}^j(X;\varphi,k)$ is admissible, if $H^j(\bar{X}/\Gamma_N; k)$ is finite dimensional for all $N \leq_o G$. In this case, the $p$-adic dimension

$$b_j^p(X;\varphi,k) = p\text{-dim}_G \bar{H}^j(X;\varphi,k) \in \mathbb{Z}_p$$

will be called the $j$th $p$-adic Betti number of $X$ with respect to $\varphi$ and with coefficients in $k$. If $\bar{H}^j(X;\varphi,k)$ is not admissible, then we impose $b_j^p(X;\varphi,k) = \infty$.

5.7. **Definition of $p$-adic torsion**

Let $(\varphi, G)$ be a virtual pro-$p$ completion of $\Gamma$. Now we consider cohomology with coefficients in a commutative ring $R$. The submodule of $R$-torsion elements $\text{tors } \bar{H}^j(X;\varphi,R)$ is stable under the $G$-action and will be considered as a $G$-set. If tors $\bar{H}^j(X;\varphi,R)$ is an admissible $G$-set (see §4.2), then the $p$-adic cardinality

$$t_j^p(X;\varphi,R) = \#_p\left(\text{tors } \bar{H}^j(X;\varphi,R)\right) \in \mathbb{Z}_p$$

will be called the $j$th $p$-adic torsion of $X$ with respect to $\varphi$ and with coefficients in $R$. As for the Betti numbers, we put $t_j^p(X;\varphi,R) = \infty$ whenever tors $\bar{H}^j(X;\varphi,R)$ is not admissible.

**Remark 5.8.** (a) The $p$-adic Betti numbers and the $p$-adic torsion are homotopy invariants in the following sense. Let $f: X \to Y$ be a homotopy equivalence as in §5.4. It follows from Lemma 5.4 that are equalities $b_j^p(X;\varphi,k) = b_j^p(Y;\psi,k)$ and $t_j^p(X;\varphi,R) = t_j^p(Y;\psi,R)$ for virtual pro-$p$ completions $\varphi$ and $\psi$ as in (5.1).

(b) For a closed subset $A \subseteq X$, we define the relative $p$-adic Betti numbers and $p$-adic torsion using the relative cohomology groups $\bar{H}^j(X,A;\varphi,R)$.

(c) As a case of special interest, we stress that if $X$ is a connected CW-complex with finite $j$-skeleton, then $\bar{H}^j(X;\varphi,k)$ and the torsion part tors $\bar{H}^j(X,\varphi,\mathbb{Z}_{(p)})$ are always admissible. In particular, the $j$th $p$-adic Betti number and $j$th $p$-adic torsion are finite.

(d) The $p$-adic Betti numbers and $p$-adic torsion take values in $\mathbb{Z}_p \cup \{\infty\}$, which will be considered as a topologically disjoint union. In particular, a sequence converges to $\infty$ exactly

---

1 An element $\alpha$ in an $R$-module is a torsion element if there is a regular element $r \in R$ with $r\alpha = 0$. 
if it eventually takes the value $\infty$. Moreover, we extend the addition by imposing $x + \infty = \infty + x = \infty$ for all $x \in \mathbb{Z}_p \cup \{\infty\}$.

(e) As usual we define the $p$-adic invariants of a group $\Gamma$ to be the $p$-adic invariants of a $K(\Gamma,1)$-space.

5.9. Virtual invariance

Let $(\varphi,G)$ be a virtual pro-$p$ completion of $\Gamma$. For every open subgroup $K \leq_o G$, the group $\Gamma_K = \varphi^{-1}(K)$ is a finite index subgroup of $\Gamma$ and $Y = \bar{X}/\Gamma_K$ is a finite sheeted covering space of $X$ with fundamental group $\Gamma_K$. In addition, $(\varphi|_{\Gamma_K}, K)$ is a virtual pro-$p$ completion of $\Gamma_K$.

**Lemma.** In the situation described above, we have
\[
\bar{b}^p_j(X; \varphi, k) = \bar{b}^p_j(Y; \varphi|_{\Gamma_K}, k)
\]
and
\[
\bar{t}^p_j(X; \varphi, R) = \bar{t}^p_j(Y; \varphi|_{\Gamma_K}, R).
\]

**Proof.** We observe that
\[
\bar{H}^j(Y; \varphi|_{\Gamma_K}, k) \cong \bar{H}^j(X; \varphi, k)|_K,
\]
and (provided that these representations are admissible) we deduce from §3.8 and Definition 3.9 that
\[
\bar{b}^p_j(Y; \varphi|_{\Gamma_K}, k) = p \cdot \dim^K_k \res^G_K \bar{H}^j(X; \varphi, k) = p \cdot \dim^G_k \bar{H}^j(X; \varphi, k).
\]
This proves the claim on Betti numbers since $\bar{H}^j(X; \varphi, k)$ is admissible, exactly if its restriction to $K$ is admissible. It follows from §4.8 that essentially the same argument applies in the case of $p$-adic torsion. \qed

**Example 5.10.** We consider a topological space $X$ with $\Gamma = \pi_1(X)$ and a subset $A \subseteq X$. Let $G$ be a finite group and let $\varphi : \Gamma \to G$ be a surjective homomorphism. Then $(\varphi,G)$ is a virtual pro-$p$ completion for every prime $p$. The kernel $\Gamma_1 = \ker(\varphi)$ is a finite index subgroup of $\Gamma$. Recall that $A_1$ denotes the inverse image of $A$ in $\bar{X}/\Gamma_1$. In this case, $\bar{H}^j(X, A; \varphi, k) = \bar{H}^j(\bar{X}/\Gamma_1, A_1; k)$ is the $j$th relative cohomology of the finite sheeted covering of $X$ corresponding to $\Gamma_1$. It is equipped with the usual action of $\Gamma/\Gamma_1 \cong G$. Hence, the $j$th relative $p$-adic Betti number with respect to $\varphi$ is
\[
b^p_j(X, A; \varphi, k) = b_j(\bar{X}/\Gamma_1, A_1; k).
\]
Moreover, for every commutative ring $R$, there is a canonical isomorphism $\bar{H}^j(X, A; \varphi, R) \cong \bar{H}^j(\bar{X}/\Gamma_1, A_1; R)$, and thus,
\[
t^p_j(X, A; \varphi, R) = | \text{tors } \bar{H}^j(\bar{X}/\Gamma_1, A_1; R)|.
\]

**Theorem 5.11.** Let $X$ be a connected, locally path-connected and semilocally simply connected topological space and let $A \subseteq X$ be a subset. Let $k$ be either $\mathbb{Q}$ or a finite field of characteristic $\ell \neq p$ and let $R$ be a commutative ring in which $p$ is invertible.

Let $(G, \varphi)$ be a virtual pro-$p$ completion of $\Gamma = \pi_1(X)$. Let $(N_n)_{n \in \mathbb{N}}$ be an exhaustive chain in $G$. The following identities hold in $\mathbb{Z}_p \cup \{\infty\}$:

(i) \[
\lim_{n \to \infty} \bar{b}^p_j(X, A; f_n \circ \varphi, k) = \bar{b}^p_j(X, A; \varphi, k),
\]
(ii) \[
\lim_{n \to \infty} \bar{t}^p_j(X, A; f_n \circ \varphi, R) = \bar{t}^p_j(X, A; \varphi, R),
\]
for all \( j \in \mathbb{N}_0 \) where \( f_n : G \to G/N_n \) is the factor homomorphism. In particular, if all \( N_n \) are open subgroups, then

\[
\begin{align*}
(i') \lim_{n \to \infty} b_j(\bar{X}/\Gamma_{N_n}; A_{N_n}; k) &= b_j^{[p]}(X, A; \varphi, k), \\
(ii') \lim_{n \to \infty} |\text{tors} H^j(\bar{X}/\Gamma_{N_n}, A_{N_n}; R)| &= t_j^{[p]}(X, A; \varphi, R).
\end{align*}
\]

Proof. The group \( N_n \) is a pro-\( p \) group for all sufficiently large \( n \in \mathbb{N} \); this follows from the argument given in the proof of Lemma 3.11. Therefore, by Lemma 5.3 (b), there is a canonical isomorphism \( H^j(X, A; f_n \circ \varphi, R) \cong H^j(X, A; \varphi, R)^{N_n} \). Thus, if the representations \( H^j(X, A; f_n \circ \varphi, k) \) (respectively, the torsion submodules \( \text{tors} H^j(X, A; \varphi, R) \)) are admissible for all \( n \geq n \), then the theorem is a direct consequence of Lemma 3.11 (respectively, Lemma 4.10).

Suppose now that there is some \( e \in \mathbb{N} \) such that \( N_e \) is a pro-\( p \) group and \( H^j(X, A; f_n \circ \varphi, k) \) is not admissible. In this case, \( \bar{H}^j(X, A; \varphi, k) \) and \( H^j(X, A; f_n \circ \varphi, k) \) are not admissible for all \( n \geq e \). Indeed, there is an open subgroup \( N_n \subseteq K \subseteq G \) such that \( H^j(X, A; f_n \circ \varphi, k)^{K/N_n} \) is infinite dimensional. Without loss of generality, we may assume that \( K \) is pro-\( p \) and by Lemma 5.3, we obtain

\[
\bar{H}^j(X, A; f_n \circ \varphi, k)^{K/N_n} \cong \bar{H}^j(X, A; \varphi, k)^K \cong H^j(X, A; f_n \circ \varphi, k)^{K/N_n}
\]

for all \( n \geq e \). We conclude that \( \bar{H}^j(X, A; \varphi, k) \) and \( H^j(X, A; f_n \circ \varphi, k) \) are not admissible. The same argument applies to the torsion part for cohomology with coefficients in \( R \).

Assertions (i') and (ii') follow from the discussion in Example 5.10. \( \square \)

5.12. Künneth formula

Let \( X_1, X_2 \) be two connected, locally path-connected and semilocally simply connected topological spaces and let \( \Gamma_i = \pi_1(X_i) \) denote the corresponding fundamental group. Given virtual pro-\( p \) completions \( \varphi_i : \Gamma_i \to G_i \) for \( i \in \{1, 2\} \), then the direct product

\[
\varphi_1 \times \varphi_2 : \Gamma_1 \times \Gamma_2 \to G_1 \times G_2
\]

is a virtual pro-\( p \) completion of \( \Gamma_1 \times \Gamma_2 \).

Proposition. Let \( k \) be either \( \mathbb{Q} \) or a finite field of characteristic \( \ell \neq p \). Assume that \( b_j(X_i; \varphi_i, k) < \infty \) for all \( 0 \leq j \leq n \) and \( i \in \{1, 2\} \). Then the following Künneth formula holds:

\[
b_n^{[p]}(X_1 \times X_2; \varphi_1 \times \varphi_2, k) = \sum_{i+j=n} b_i^{[p]}(X_1; \varphi_1, k) b_j^{[p]}(X_2; \varphi_2, k).
\]

Proof. We fix an exhaustive chain \( (N_n)_{n \in \mathbb{N}} \) of the form \( N_n = N_n^{(1)} \times N_n^{(2)} \) where \( (N_n^{(i)})_{n \in \mathbb{N}} \) is an exhaustive chain in \( G_i \). We apply the Approximation Theorem 5.11 with the chain \( (N_n)_{n \in \mathbb{N}} \). Since all involved Betti numbers are finite, the assertion follows from the Künneth formula for ordinary Betti numbers. \( \square \)

Proposition 5.13 (Poincaré duality). Let \( M \) be a compact, connected, orientable \( n \)-manifold with boundary \( \partial M \). Decompose the boundary \( \partial M = A \cup B \) as a disjoint union of two closed \((n-1)\)-manifolds (where \( A \) and \( B \) may be empty). Let \((\varphi, G)\) be a virtual pro-\( p \) completion of \( \Gamma = \pi_1(M) \). For all \( j \in \{0, \ldots, n\} \), the equality

\[
b_j^{[p]}(M; A; \varphi, k) = b_{n-j}^{[p]}(M; B; \varphi, k)
\]
holds. Moreover, for every principle ideal domain $R$ in which $p$ is invertible, we have

$$t_j^{[p]}(M, A; \varphi, R) = t_{n-j+1}^{[p]}(M, B; \varphi, R).$$

**Proof.** This is a consequence of the Approximation Theorem and the Poincaré duality isomorphism

$$H^j(\widetilde{M}/\Gamma_N, A_N; R) \cong H_{n-j}(\widetilde{M}/\Gamma_N, B_N; R)$$

for the finite sheeted coverings $\widetilde{M}/\Gamma_N$ of $M$ (where $N \leq_o G$); see [15, Theorem 3.43].

For the $p$-adic torsion, one needs to verify in addition that there is an isomorphism

$$\text{tors} H_j^{[p]}(\widetilde{M}/\Gamma_N, B_N; R) \cong \text{tors} H_j(\widetilde{M}/\Gamma_N, B_N; R)$$

for all $j \in \mathbb{N}$. Indeed, observe that the universal coefficient theorem yields an isomorphism

$$\text{tors} H_j^{[p]}(\widetilde{M}/\Gamma_N, B_N; R) \cong \text{Ext}^1_R(H_j(\widetilde{M}/\Gamma_N, B_N; R), R).$$

The homology of a compact manifold is finitely generated (see [15, Corollary A.9]) and since $R$ is a principle ideal domain, we deduce

$$H_j(\widetilde{M}/\Gamma_N, B_N; R) \cong R^d \oplus R/a_1 R \oplus \ldots R/a_t R$$

for certain $a_1, \ldots, a_t \in R \setminus \{0\}$. The claim follows since $\text{Ext}^1_R(R/a_i R, R) \cong R/a_i R$ and $\text{Ext}^1_R(R, R) = 0$. □

5.14. The $p$-adic Euler characteristic

Suppose that $X$ is a finite CW-complex; in particular, its Euler characteristic $\chi(X)$ is well defined. Since all of its $p$-adic Betti numbers are finite, we may also define its $p$-adic Euler characteristic

$$\chi^{[p]}(X; \varphi) = \sum_{j=0}^{\dim(X)} (-1)^j t_j^{[p]}(X; \varphi, k),$$

where $k$ is either $\mathbb{Q}$ or a finite field of characteristic $\ell \neq p$. A priori this definition depends on the chosen field $k$; however, as for the ordinary Euler characteristic, we will see that the choice of $k$ is inessential.

Let $(N_n)_{n \in \mathbb{N}}$ be an exhaustive chain of open subgroups $N_n \leq_o G$ (which exists, since $G$ is finitely generated). From Theorem 5.11, we deduce that

$$\chi^{[p]}(X; \varphi) = \lim_{n \to \infty} \chi(\widetilde{X}/\Gamma_{N_n}) = \lim_{n \to \infty} |G : N_n| \chi(X) \in \mathbb{Z}_p.$$ 

The following result is an immediate consequence of equation (2.1).

**Proposition.** Let $X$ be a connected finite CW-complex and let $(\varphi, G)$ be a virtual pro-$p$ completion. Then

$$\chi^{[p]}(X; \varphi) = \|G\| \chi(X)$$

with the notation introduced in § 2.4.

5.15. Wedge sums

Let $X_1, X_2$ be two connected CW-complexes with finitely generated fundamental groups. For instance, this assumption is satisfied if $X_1$ and $X_2$ have a finite 1-skeleton. Let $Y = X_1 \vee X_2$ be the wedge sum subject to chosen base points $x_i \in X_i$. The new base point will be denoted as $y_0 \in Y$. 

By the theorem of Seifert–van Kampen, the fundamental group \( \Gamma = \pi_1(Y, y_0) \) is isomorphic to the free product \( \Gamma_1 \ast \Gamma_2 \) of \( \Gamma_1 = \pi_1(X_1, x_1) \) and \( \Gamma_2 = \pi_1(X_2, x_2) \). Let \((\varphi, G)\) be a virtual pro-
\( \Gamma \)-completion of \( \Gamma = \Gamma_1 \ast \Gamma_2 \). We denote by \( K_i \) the closure of \( \varphi(\Gamma_i) \) in \( G \). Note that \((\varphi|\Gamma_i, K_i)\) is a virtual pro-pro-
completion since \( \Gamma_i \) is finitely generated by assumption.

**Proposition.** Let \( X_1, X_2 \) be two connected CW-complexes with finitely generated fundamental groups. Let \( R \) be a commutative ring. There is an exact sequence of \( R \)-modules with \( G \)-action

\[
0 \rightarrow R \rightarrow C^\infty(G, R) \rightarrow I(X_1, X_2) \rightarrow \tilde{H}^1(X_1 \vee X_2; \varphi, R) \rightarrow 0,
\]

where

\[
I(X_1, X_2) = \text{ind}_{K_i}^G \tilde{H}^1(X_1, \{x_1\}; \varphi|\Gamma_1, R) \oplus \text{ind}_{K_2}^G \tilde{H}^1(X_2, \{x_2\}; \varphi|\Gamma_2, R).
\]

Moreover, for every \( j \geq 2 \), there is an isomorphism

\[
\tilde{H}^j(X_1 \vee X_2; \varphi, R) \cong \text{ind}_{K_1}^G \tilde{H}^j(X_1; \varphi|\Gamma_1, R) \oplus \text{ind}_{K_2}^G \tilde{H}^j(X_2; \varphi|\Gamma_2, R).
\]

In particular, if \( R = k \) is a field which is either \( \mathbb{Q} \) or a finite field of characteristic \( \ell \neq p \), then the following equalities of \( p \)-adic Betti numbers hold whenever all involved Betti numbers are finite:

\[
b_1^{[p]}(X_1 \vee X_2; \varphi, k) = 1 + \|G\| - \|G : K_1\| - \|G : K_2\|
\]

\[
+ \|G : K_1\| b_1^{[p]}(X_1; \varphi|\Gamma_1, k) + \|G : K_2\| b_1^{[p]}(X_2; \varphi|\Gamma_2, k)
\]

\[
b_j^{[p]}(X_1 \vee X_2; \varphi, k) = \|G : K_1\| b_j^{[p]}(X_1; \varphi|\Gamma_1, k) + \|G : K_2\| b_j^{[p]}(X_2; \varphi|\Gamma_2, k)
\]

for all \( j \geq 2 \). Here we use the notation explained in \( \S \, 2.4 \).

**Proof.** We reduce to the case where \( G \) is a finite group. Indeed, taking direct limits is exact and induced representations of profinite groups are defined the direct limit of induced representations of its finite quotients; see \( \S \, 2.8 \). Similarly, the formula for Betti numbers will follow from the Approximation Theorem 5.11 and equation (2.1).

Let \( q: Z \rightarrow Y = X_1 \vee X_2 \) be a finite connected \( G \)-covering. The restriction of \( q \) to the inverse image \( q^{-1}(X_i) = Z_i \) is a finite \( G \)-covering of \( X_i \) which, however, is not necessarily connected. Let \( Z_i^0 \) be a connected component and let \( K_i \trianglelefteq G \) be its setwise stabilizer. We observe that

\[
Z_i = \bigsqcup_{g \in G/K_i} gZ_i^0;
\]

and deduce that \( H^j(Z_i, q_i^{-1}(x_i); R) \cong \text{ind}_{K_i}^G H^j(Z_i^0, q_i^{-1}(x_i); R) \). In fact, an isomorphism is given by the map \( \alpha \mapsto f_\alpha \) with \( f_\alpha(g) = \iota^* g^*(\alpha) \), for all \( g \in G \) where \( \iota: Z_i^0 \rightarrow Z_i \) denotes the inclusion.

There are arbitrarily small open neighbourhoods \( U_i \) of \( x_i \) in \( X_i \) such that \( x_i \) is a strong deformation retract of \( U_i \); see the proof of Proposition A.4 in [15]. Now the open subset \( V = U_1 \cup U_2 \subseteq Y \) is contractible and we may choose it so small that it is regularly covered, that is, \( q^{-1}(V) \cong \bigsqcup_{g \in G} gV_0 \) where \( V_0 \) is a connected component of the fibre and \( q|V_0 \) is a homeomorphism onto \( V \). It follows from the long exact sequence of the triple \((Z, q^{-1}(V), q^{-1}(y_0))\) that the inclusion map induces an isomorphism

\[
H^j(Z, q^{-1}(y_0); R) \xrightarrow{\cong} H^j(Z, q^{-1}(V); R).
\]
We apply excision twice to obtain isomorphisms

\[ H^j(Z, q^{-1}(V); R) \cong H^j(Z \setminus q^{-1}(y_0), q^{-1}(V) \setminus q^{-1}(y_0); R) \]

\[ \cong H^j(Z_1, q_{Z_1}^{-1}(U_1); R) \oplus H^j(Z_2, q_{Z_2}^{-1}(U_2); R). \]

We obtain an isomorphism

\[ H^j(Z, q^{-1}(y_0); R) \xrightarrow{\sim} H^j(Z_1, q_{Z_1}^{-1}(x_1); R) \oplus H^j(Z_2, q_{Z_2}^{-1}(x_2); R) \] (5.2)

for every \( j \) and any commutative ring \( R \). Since \( q_{Z_i}^{-1}(x_i) \) is just a finite discrete set of points, the statements for \( j \geq 2 \) follow readily from the long exact sequences of the pairs \( (Z_i, q^{-1}(x_i)) \).

Finally, we consider the case \( j = 1 \). The initial part of the long exact sequence of the pair \( (Z, q^{-1}(y_0)) \) reads as follows:

\[ 0 \to H^0(Z; R) \to H^0(q^{-1}(y_0); R) \to H^1(Z, q^{-1}(y_0); R) \to H^1(Z; R) \to 0. \]

Since \( Z \) is connected, we have \( H^0(Z; R) \cong R \). In addition, \( q^{-1}(y_0) \) is a finite discrete set of points on which \( G \) acts simply transitively, that is, \( H^0(q^{-1}(y_0); R) \cong C^\infty(G; R) \). We deduce the claimed exact sequence using (5.2).

Let \( R = k \) be a field. The exact sequence yields

\[ b_1(Z; k) = 1 - |G| + \sum_{i=1}^{2} |G : K_i|b_1(Z_i^0, q_{Z_i^0}^{-1}(x_i); k). \] (5.3)

Finally, we apply the long exact sequences of the pairs \( (Z_i^0, q_{Z_i^0}^{-1}(x_i)) \) to see that

\[ b_1(Z_i^0, q_{Z_i^0}^{-1}(x_i); k) = |K_i| + b_1(Z_i^0; k) - 1. \]

Substituting into formula (5.3) yields the desired identity. \( \Box \)

6. Examples

Here we discuss a number of examples. Throughout \( p \) denotes a prime number and \( k \) a field which is either \( \mathbb{Q} \) of a finite field of characteristic \( \ell \neq p \).

6.1. Surfaces

Let \( \Sigma_g \) be a closed oriented surface of genus \( g > 0 \). For every infinite virtual pro-\( p \) completion \((\varphi, G)\) of \( \Gamma = \pi_1(\Sigma_g) \), we have

\[ b_2^{|p|}(\Sigma_g, \varphi; k) = 1, \]

\[ b_1^{|p|}(\Sigma_g, \varphi; k) = 2, \]

where the first line follows from Poincaré duality (Proposition 5.13) and the second from the formula for the \( p \)-adic Euler characteristic § 5.14.

6.2. Tori

Let \( T^d \) be the \( d \)-dimensional torus and let \((\varphi, G)\) be any virtual pro-\( p \) completion of \( \pi_1(T^d) \cong \mathbb{Z}^d \). Since every finite sheeted covering of \( T^d \) is homeomorphic to \( T^d \), the Approximation Theorem 5.11 implies

\[ b_j^{|p|}(T^d, \varphi; k) = \binom{d}{j} \]

for all \( j \in \{0, \ldots, d\} \).
6.3. Free groups

Let $F_r$ be the free group of rank $r \in \mathbb{N}$ and let $(\varphi, G)$ be any infinite virtual pro-$p$ completion. Then we find that

$$b^p_1(F_r, \varphi; k) = 1$$

is independent of the rank $r$. This can be deduced either from the Nielson–Schreier formula and the Approximation Theorem 5.11 or from the formula for the $p$-adic Euler characteristic § 5.14.

6.4. Betti numbers depend on the completion (I)

The $p$-adic Betti numbers depend on the virtual pro-$p$ completion $\varphi$. This is not surprising, if one considers completions $(\varphi_1, G_1)$ and $(\varphi_2, G_2)$ with $\ker(\varphi_1) \neq \ker(\varphi_2)$.

Consider $X = S^1 \vee S^1$ with fundamental group $\Gamma = F_2$. Let $\varphi_1 : \Gamma \to \hat{F}_2^p$ be the pro-$p$ completion of $F_2$ and let $\varphi_2 : \Gamma \to \{1\}$ be the trivial completion. In this case, $b^p_1(X; \varphi_1, k) = 1$ (see § 6.3), whereas

$$b^p_1(X; \varphi_2, k) = b_1(X; k) = 2$$

by Example 5.10.

6.5. Betti numbers depend on the completion (II)

In this example, we will see that the $p$-adic Betti numbers of a space $X$ with respect to pro-$p$ completions $\varphi$ and $\psi$ can be distinct even if $\ker(\varphi) = \ker(\psi)$.

Let $X = S^1 \times (S^1 \vee S^3)$. The space $X$ is a finite CW-complex. The fundamental group of $X$ is a free abelian group $\Gamma = \langle s, t \rangle \cong \mathbb{Z}^2$ of rank 2, where $s$ is the class of a simple loop in the first factor and $t$ is the class of a simple loop in the second.

We consider two distinct injective virtual pro-$p$ completions of $\Gamma$. The first is simply the pro-$p$ completion $\varphi : \Gamma \to \mathbb{Z}_p^\infty$ with $\varphi(s) = (1, 0)$ and $\varphi(t) = (0, 1)$. The K"unneth formula and the formula for wedge sums yield

$$b^p_4(X; \varphi, k) = 0,$$

see § 5.12 and § 5.15.

For the second completion, we fix any irrational element $\omega \in \mathbb{Z}_p$ and define $\psi = \psi_\omega : \Gamma \to \mathbb{Z}_p$ with $\psi(s) = \omega$ and $\psi(t) = 1$. Note that the irrationality of $\omega$ implies that $\psi$ is injective. The universal covering $\tilde{X}$ has one $\Gamma$-orbit of 3-cells and one $\Gamma$-orbit of 4-cells. We can choose $\mathbb{Z}[\Gamma]$-bases of $C^\text{cell}_3(\tilde{X})$ and $C^\text{cell}_4(\tilde{X})$ such that the boundary map $\partial_3$ is given by multiplication with $1 - s \in \mathbb{Z}[\Gamma]$.

Let $\Gamma_n = \psi^{-1}(p^n\mathbb{Z}_p)$ and let $X_n = \tilde{X}/\Gamma_n$ be the associated finite sheeted covering of $X$. We conclude that

$$H_4(X_n; k) \cong k[\mathbb{Z}/p^n\mathbb{Z}]/(1 - s)k[\mathbb{Z}/p^n\mathbb{Z}],$$

where $s$ acts like addition with $\omega + p^n\mathbb{Z}_p$ on $\mathbb{Z}/p^n\mathbb{Z}$. A short calculation shows that $b_4(X_n; k) = p^{\min(n, \nu_p(\omega))}$ and we deduce

$$b^p_4(X; \psi, k) = p^{\nu_p(\omega)}.$$

It follows, in particular, that $b^p_4(X; \psi, k)$ can take the value $p^m$ for every $m \geq 0$. 


6.6. Infinite cyclic coverings

Let $X$ be a connected finite CW-complex with fundamental group $\Gamma = \pi_1(X)$. Let $\varphi': \Gamma \to \mathbb{Z}$ be an epimorphism to the infinite cyclic group. For every integer $m \in \mathbb{N}$ which is coprime to $p$, there is a virtual pro-$p$ completion $\varphi_m': \Gamma \to \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}_p$ obtained from $\varphi'$ and the completion $\tau_m: \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}_p$. In the following, the profinite group $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}_p$ is denoted by $G$ and the generator $\tau_m(1)$ by $t$.

We consider the associated infinite cyclic covering $Y = \tilde{X} / \ker(\varphi')$ of $X$. The cellular chain complex $C_\ast(Y;k)$ of $Y$ with coefficients in $k$ is a chain complex of finitely generated free modules over the Laurent polynomial ring $k[t,t^{-1}]$.

$$C_\ast(Y;k): \cdots \rightarrow C_{j+1}(Y;k) \xrightarrow{\partial_{j+1}} C_j(Y;k) \xrightarrow{\partial_j} C_{j-1}(Y;k) \rightarrow \cdots$$

We write $e_j$ for the rank of $C_j(Y;k)$. Since $k[t,t^{-1}]$ is a principal ideal domain, the elementary divisor theorem shows that we can choose bases of the chain modules $C_{j-1}$, $C_j$ and $C_{j+1}$ such that the boundary maps are given by right multiplication with diagonal matrices $A_j \in M_{e_j,e_{j-1}}(k[t,t^{-1}])$. Let $A_{j+1} \in M_{e_{j+1},e_j}(k[t,t^{-1}])$ of the form

$$A_j = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ g_1 & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & \vdots \\ 0 & \cdots & g_u \end{pmatrix} \quad A_{j+1} = \begin{pmatrix} f_1 \\ \vdots \\ f_v \\ 0 \end{pmatrix}$$

with $g_u | g_{u-1} | \cdots | g_1$ and $f_v | f_{v-1} | \cdots | f_1$ for certain non-zero $g_1, \ldots, g_u, f_1, \ldots, f_v \in k[t,t^{-1}]$ with $u+v \leq e_j$. The polynomials are uniquely determined up to units and are called the invariant factors of $A_j$, respectively, $A_{j+1}$.

**Lemma.** In the situation described above, assume that the characteristic of $k$ does not divide $m$. Then the $j$th $p$-adic Betti number with respect to $\varphi_m$ is

$$b_j^{[p]}(X;\varphi_m,k) = \sum_{i=1}^{\nu} |V(f_i) \cap \mu(mp^\infty)| + \sum_{i=1}^{m} |V(g_i) \cap \mu(mp^\infty)|,$$

where $V(f_i), V(g_i) \subseteq \bar{k}^\times$ denote the vanishing sets of the Laurent polynomials in the algebraic closure $\bar{k}$ of $k$ and

$$\mu(mp^\infty) = \{ \zeta \in \bar{k}^\times \mid \zeta^{mp^n} = 1 \text{ for some } n \in \mathbb{N}_0 \}.$$

**Proof.** In view of the description of the cochain complex discussed in §5.5 we see that the cochain complex

$$\cdots \rightarrow C^\infty(G,k)^{e_{j-1}} \xrightarrow{r(A_{j-1})} C^\infty(G,k)^{e_j} \xrightarrow{r(A_j)} C^\infty(G,k)^{e_{j+1}} \rightarrow \cdots$$

computes the cohomology $\check{H}^j(X;\varphi,k)$. We recall that $p\text{-dim}^G_k(C^\infty(G,k)) = 0$. In particular, we obtain the formula

$$b_j^{[p]}(X;\varphi,k) = \sum_{i=1}^{\nu} p\text{-dim}^G_k \ker(r(f_i)) - \sum_{i=1}^{m} p\text{-dim}^G_k \text{im}(r(g_i)),$$

where $r(f_i)$ is the linear map obtained from the right regular representation on $C^\infty(G,k)$. Due to the relation $p\text{-dim}^G_k(\ker(g_i)) + p\text{-dim}^G_k(\text{im}(g_i)) = p\text{-dim}^G_k(C^\infty(G,k)) = 0$, it is sufficient to understand the $p$-adic dimensions of kernels.

The irreducible smooth representations of $G$ over the algebraic closure $\bar{k}$ are one-dimensional and are parametrized by the roots of unity in $\mu(mp^\infty) \subseteq \bar{k}^\times$. More precisely, for every $\zeta \in \mu(mp^\infty)$, there is a unique 1-dimensional irreducible representation $S_\zeta$ such that $t \in G$
acts by $\zeta$. In addition, every irreducible representation $S_\zeta$ occurs with multiplicity one in the right regular representation $C^\infty(G, \bar{k}) \cong \bar{k} \otimes_k C^\infty(G, k)$. Indeed, $S_\zeta$ is the space spanned by the character $\omega_\zeta \colon G \to \bar{k}^\times$ with $t \mapsto \zeta$.

Let $0 \neq f \in k[t, t^{-1}]$ and write $f = \sum a_n t^n$ where almost all coefficients $\lambda_a$ vanish. How does $f$ act on an irreducible representation $S_\zeta$? A calculation shows that

$$r(f) \omega_\zeta(x) = \sum_{a \in \mathbb{Z}} \lambda_a \omega_\zeta(x + t^a) = \sum_{a \in \mathbb{Z}} \lambda_a \zeta^a \omega_\zeta(x) = f(\zeta) \omega_\zeta(x)$$

for all $x \in G$. In particular, the kernel of $r(f)$ consists exactly of those $S_\zeta$ for which $\zeta$ is a root of the Laurent polynomial $f$. In particular, the kernel is finite dimensional and its ($p$-adic) dimension is exactly $|V(f) \cap \mu(mp^n)|$.

\[\square\]

**Corollary.** For all sufficiently large $n \in \mathbb{N}$, we have

$$b_j(Y/mp^n\mathbb{Z}; \mathbb{Q}) = (mp^n)^{j(2)}(Y, \mathbb{N}\mathbb{Z}) + b_j^{[p]}(X; \varphi_m, \mathbb{Q}), \tag{6.1}$$

where $b_j^{(2)}(Y, \mathbb{N}\mathbb{Z})$ denotes the $L^2$-Betti number of the infinite cyclic covering $Y$ with respect to the action of the infinite cyclic group.

**Proof.** In the situation described above, the $L^2$-Betti number of $Y$ with respect to the action by deck transformations is

$$b_j^{(2)}(Y, \mathbb{N}\mathbb{Z}) = \dim_{\mathbb{Q}(t)} \mathbb{Q}(t) \otimes_{\mathbb{Q}[t, t^{-1}]} H_j(Y, \mathbb{Q}) = e_j - u - v;$$

see [22, Lemma 1.34]. The sets $V(f_i) \cap \mu(mp^n)$ and $V(g_i) \cap \mu(mp^n)$ are finite; thus, there is some $n_0 \in \mathbb{N}$ such that all elements are $mp^n$th roots of unity. We leave it as an exercise to check that the $j$th Betti number of the finite covering $Y/mp^n\mathbb{Z}$ is

$$b_j(Y/p^n\mathbb{Z}; \mathbb{Q}) = (mp^n)^{j-1} - u - v + \sum_{i=1}^v |V(f_i) \cap \mu(mp^n)| + \sum_{i=1}^u |V(g_i) \cap \mu(mp^n)|.$$

We conclude that formula (6.1) holds for all $n \geq n_0$. \[\square\]

### 6.7. Knot complements and $p$-adic Betti numbers

Let $K \subset S^3$ be a knot and let $X = S^3 \setminus \nu(K)$ be the complement of an open tubular neighbourhood. The fundamental group $\pi_1(X) = \Gamma$, which is called the knot group of $K$, has an infinite cyclic abelianization; see [11, p. 112]. The infinite cyclic covering of $X$ will be denoted by $Y$. We write $\varphi' \colon \Gamma \to \mathbb{Z}$ to denote the abelianization map. As in §6.6 we obtain a virtual pro-$p$ completion $\varphi_m \colon \Gamma \to \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}_p$ for every natural number $m$ which is coprime to $p$.

**Proposition.** Let $X = S^3 \setminus \nu(K)$ be a knot complement with $\pi_1(X) = \Gamma$. Let $p$ be a prime number, $m \in \mathbb{N}$ coprime to $p$ and $\varphi_m \colon \Gamma \to \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}_p$ be the associated virtual pro-$p$ completion. Then

$$b_1^{[p]}(X; \varphi_m, \mathbb{Q}) = 1 + \sum_{i=1}^\infty |V(\Delta_i) \cap \mu(mp^n)|, \tag{6.2}$$

where $V(\Delta_i) \subseteq \mathbb{C}^\times$ is the vanishing set of the $i$th knot polynomial $\Delta_i$ of $K$; see [11, Chapter VIII]. For $m = 1$, the first $p$-adic Betti number $b_1^{[p]}(X; \varphi_1, \mathbb{Q})$ equals 1.

**Proof.** The knot complement $X$ is aspherical and is a $K(\Gamma, 1)$-space; see [7, 3.30]. We fix a finite presentation $\Gamma = \langle \gamma_1, \ldots, \gamma_s \mid w_1, \ldots, w_r \rangle$. Since we are only interested in the first
is the limit of the ordinary Betti numbers of the
for all
The cellular chain complex
C
X cohomology of
524
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Z
where the action on
A
by
ϕ
is defined via the matrix
N
for a suitable
p
root of unity. Then the
ϕ
acts on
p
equivariant Alexander matrix
A
where
A
is a root of unity and
A
is the sixth cyclotomic polynomial and we deduce that
1
Δ
i
is the sixth cyclotomic polynomial and we deduce that
1
Δ
i
has integral coefficients. Assume that
Δ
1
vanishes at some
p
th root of unity. Then the
p
th cyclotomic polynomial
Φ
p
divides
Δ
1
in
Q[t, t^{-1}]
and by Gauß’ lemma even in
Z[t, t^{-1}]. However, this yields a contradiction since
|Δ
1
(1)| = 1 (see [11, p. 135]) and
Φ
1
(1) = 0 and
Φ
p
(1) = p for all
p
≥ 1. We deduce that
V(Δ
i
) T
Q
for all
i
. By definition,
Δ
1
= 1 for all
i
> 1. We recall that
f
v
| (f
v
− 1) | ··· | (f
v
- 1)
and hence
V(Δ
i
) = V(f
i
). Formula (6.2) follows from the lemma in § 6.6. The knot polynomials
Δ
i
have integral coefficients. Assume that
Δ
1
vanishes at some
p
th root of unity. Then the
p
th cyclotomic polynomial
Φ
p
divides
Δ
1
in
Q[t, t^{-1}]
and by Gauß’ lemma even in
Z[t, t^{-1}]. However, this yields a contradiction since
|Δ
1
(1)| = 1 (see [11, p. 135]) and
Φ
1
(1) = 0 and
Φ
p
(1) = p for all
p
≥ 1. We deduce that
V(Δ
i
) T
Q
for all
i
. By definition, Δ
1
= 1 for all i > 1. We recall that Δ
1
= t - 1. The invariant factors
f
1
, . . . ,
f
v
of the Alexander matrix
A
2 can be normalized to satisfy
Δ
i
= f
i
· f
i+1 · · · f
v
for all
i
< v. By definition, Δ
i
= 1 for all
i
> 1. We recall that
f
v
| (f
v
− 1) | ··· | (f
v
- 1)
and hence
V(Δ
i
) = V(f
i
). Formula (6.2) follows from the lemma in § 6.6.

The proposition and Theorem 8.21 in [7] show that the first
p
-adic Betti number minus one is the limit of the ordinary Betti numbers of the
mp
-fold branched coverings of
X
as
n
tends to
∞.

Example. The trefoil knot
K
has knot polynomials
Δ
1
= t^2 - t + 1 and Δ
i
= 1 for all
i
≥ 2. We observe that Δ
1
is the sixth cyclotomic polynomial and we deduce that
b
i
[X; \varphi_m, Q] = \begin{cases} 1 & \text{if } 6 \nmid mp \\ 3 & \text{if } 6 \nmid mp \end{cases}.

6.8. The
p
-adic torsion of (free abelian)-by-cyclic groups

In this example, we compute the
p
-adic torsion
i
[X; \varphi_m, Q] of certain (free abelian)-by-cyclic groups. For instance, the fundamental groups of 3-dimensional solvmanifold fall into this class.

Let
A
∈ GL
N
(Z). We study the semidirect product group
Γ
= \mathbb{Z}^N \rtimes \mathbb{Z},
where the action on \mathbb{Z}^N is defined via the matrix
A
. More precisely, the multiplication is given by
(v, i)(w, j) = (v + A^i w, i + j).

We fix a prime number
p
and make the following assumptions:

(A1) No eigenvalue of
A
is a root of unity and
(A2) \ A \equiv 1 \mod p.

Assumption (A2) is convenient, but not really necessary. We can achieve (A2) replacing
A
by
A^e
such that
A^e \equiv 1 \mod p
for a suitable
e
∈ \mathbb{N}. This amounts to passing to a finite index subgroup of
Γ
; by § 5.9 this is not essential for the computation of
p
-adic torsion. Assumption (A2) implies that
Γ
is residually a finite
p
-group. Indeed, we use (A2) to consider
A
as an element in the first principal congruence subgroup
GL^1_N(\mathbb{Z}_p)
. Since
GL^1_N(\mathbb{Z}_p)
is a pro-
 p
-group, we can define
A^λ \in GL^1_N(\mathbb{Z}_p)
for every
λ \in \mathbb{Z}_p
cf. [27, Lemma 4.1.1]). It is easy to verify that the group
G = \mathbb{Z}_p^N \rtimes \mathbb{Z}_p,
where \lambda \in \mathbb{Z}_p acts on \mathbb{Z}_p^N as
A^\lambda \in GL^1_N(\mathbb{Z}_p)
is the pro-
 p
-completion of
Γ
. Let \varphi : \Gamma \rightarrow \hat{G}
be the completion map.
The $p$-adic logarithm. Below we shall make use of the $p$-adic logarithm and exponential map. The power series

$$\log(1 + B) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} B^j$$

converges for all $B \in pM_N(\mathbb{Z}_p)$ and hence defines an analytic function $\log: 1 + pM_N(\mathbb{Z}_p) \to pM_N(\mathbb{Z}_p)$. If $p$ is odd, then the $p$-adic logarithm is a homeomorphism and its inverse is given by the $p$-adic exponential function, which is defined using the exponential series. For $p = 2$, the exponential series converges only on $4M_N(\mathbb{Z}_2)$ and the logarithm is only a homeomorphism when restricted to $1 + 4M_N(\mathbb{Z}_2)$.

**Proposition.** Under the assumptions (A1) and (A2) above, we have

$$t_{2}^{[p]}(\Gamma; \varphi, \mathbb{Z}[1/p]) = \varepsilon \det(\log(A))_{(p')},$$

where $\varepsilon = \text{sign } \det(A - 1)$ if $p$ is odd and $\varepsilon = \text{sign } \det(A^2 - 1)$ if $p = 2$. Here $x_{(p')} = x p^{-\nu_p(x)}$ is the $p'$-part of a non-zero $p$-adic number $x$.

**Corollary.** The $p$-adic torsion takes transcendental values.

**Proof of the Corollary.** We only discuss the case $p > 2$ but similar examples exist also for $p = 2$. By Hensel’s lemma, there is an element $\xi \in \mathbb{Z}_p$ with $\xi^2 = p^2 + 4$. The matrix

$$A_p = \begin{pmatrix} 1 + p^2 & p \\ p & 1 \end{pmatrix}$$

has two distinct eigenvalues $\lambda_{\pm} = 1 + \frac{p^2 \pm \sqrt{p}}{2} \in \mathbb{Z}_p$. In particular, $A_p$ satisfies the assumptions (A1) and (A2) above. Since $\lambda_+ \lambda_- = 1$, we have $0 = \log(1) = \log(\lambda_+) + \log(\lambda_-)$. We diagonalize $A_p$ to see that $\det(\log(A_p)) = \log(\lambda_+) \log(\lambda_-) = -\log(\lambda_+)^2$. It is a result of Mahler that at most one of the $p$-adic numbers $z$ and $\exp(z)$ is algebraic (if $\exp$ converges at $z$; see [24]). Since $\lambda_+$ is algebraic, we deduce that $\det(\log(A_p)) = -\log(\lambda_+)^2$ is transcendental.

Clearly, the $p'$-part of a transcendental number is transcendental, and thus, $t_{2}^{[p]}(\Gamma; \varphi, \mathbb{Z}[1/p])$ is transcendental.

**Proof of the Proposition.** A short calculation shows that the commutator of $(v, e), (w, f) \in \Gamma$ is given by

$$[(v, e), (w, f)] = (v, e)(w, f)(-A^{-e}v, -e)(-A^{-f}w, -f)$$

$$= ((A^e - 1)w - (A^f - 1)v, 0). \quad (6.3)$$

We deduce that the subgroup $\Gamma_n = p^n \mathbb{Z}_p \times p^n \mathbb{Z}$ is normal in $\Gamma$ since $[\Gamma_n, \Gamma] \subseteq \Gamma_n$, where we use (A2) and the consequence $A_p^{p^n} \equiv 1 \text{ mod } p^{n+1}$. In fact, $\Gamma_n = \varphi^{-1}(G_n)$ where $G_n = p^n \mathbb{Z}_p \times p^n \mathbb{Z}_p$.

It follows from the universal coefficient theorem that $\text{tors } H^2(\Gamma_n, \mathbb{Z}[1/p]) \cong \text{tors } H_1(\Gamma_n, \mathbb{Z}[1/p])$; see the proof of Proposition 5.13.

From (6.3), one sees that $[\Gamma_n, \Gamma_n] = (A^{p^n} - 1)p^n \mathbb{Z}_p \times \{0\}$ and we deduce $H_1(\Gamma_n, \mathbb{Z}) \cong \mathbb{Z}^{N}/(A^{p^n} - 1)\mathbb{Z}^{N} \oplus \mathbb{Z}$. By assumption (A1), the matrix $A^{p^n} - 1$ has full rank and therefore the elementary divisor theorem yields $|\text{tors } H_1(\Gamma_n, \mathbb{Z})| = |\det(A^{p^n} - 1)|$. Since we are interested in the homology with coefficients in $\mathbb{Z}[1/p]$, we need to take the $p'$-part. Now an application of the Approximation Theorem 5.11 shows that

$$t_{2}^{[p]}(\Gamma; \varphi, \mathbb{Z}[1/p]) = \lim_{n \to \infty} |\det(A^{p^n} - 1)|_{(p')}.$$ \quad (6.4)
Claim. \( \lim_{n \to \infty} p^{-n}(A^{p^n} - 1) = \log(A) \).

We write \( A = 1 + B \) with \( B \in pM_N(\mathbb{Z}_p) \). Then
\[
p^{-n}(A^{p^n} - 1) = p^{-n} \sum_{j=1}^{p^n} \left( \binom{p^n}{j} \right) B^j = \sum_{j=1}^{p^n} \left( \frac{p^n - 1}{j} \right) B^j
\]
\[
= \sum_{j=1}^{2n} \left( \frac{p^n - 1}{j} \right) B^j + O(p^n)
\]
\[
= \sum_{j=1}^{2n} \left( \frac{(-1) \cdot (-2) \cdot (-3) \cdots (-j+1)}{1 \cdot 2 \cdots (j-1)} \right) B^j + O(p^n)
\]
\[
= \sum_{j=1}^{2n} \left( \frac{(-1)^{j-1}}{j} \right) B^j + O(p^n) = \log(A) + O(p^n),
\]
where we use the notation \( O(p^n) \) to denote error terms in \( p^nM_N(\mathbb{Z}_p) \). The claim follows taking the limit \( n \to \infty \).

In addition, this means that the sequence \( \det(p^{-n}(A^{p^n} - 1)) \) converges to \( \det(\log(A)) \) as \( n \) tends to \( \infty \). In order to deduce that \( \lim_{n \to \infty} \det(A^{p^n} - 1)(p') = \det(\log(A))(p') \), we show that \( \det(\log(A)) \neq 0 \). Assume that \( p \) is odd and suppose that \( \det(\log(A)) = 0 \). Then there is a non-zero vector \( v \in \mathbb{Z}_p^N \) such that \( \log(A)v = 0 \). We deduce that \( Av = \exp(\log(A))v = v \); this means that \( A \) has the eigenvalue \( 1 \), which is excluded by assumption (A1). If \( p = 2 \), the same argument applies to \( A^2 \) using that \( \log(A^2) = 2 \log(A) \).

In order to plug \( \det(\log(A)) \) into formula (6.4), we need to determine the sign of \( \det(A^{p^n} - 1) \). Let \( \lambda_1, \ldots, \lambda_N \in \mathbb{C} \) be the eigenvalues (occurring with algebraic multiplicities) of \( A \). If \( \lambda_j = \overline{\lambda}_{j+1} \) is a pair of complex conjugate eigenvalues, then \( (\lambda_j^{p^n} - 1)(\lambda_{j+1}^{p^n} - 1) = |\lambda_j^{p^n} - 1|^2 > 0 \). Therefore, the complex eigenvalues of \( A \) which are not real do not contribute to the sign of \( \det(A^{p^n} - 1) \). Suppose now that \( \lambda_j \) is a real eigenvalue. If \( p \) is odd, then \( \lambda_j^{p^n} - 1 \) has the same sign as \( \lambda_j - 1 \) and so \( \text{sign}(\det(A - 1)) = \text{sign}(\det(A^{p^n} - 1)) \). Similarly, if \( p = 2 \), then \( \lambda_j^{2^n} - 1 \) has the same sign as \( \lambda_j^2 - 1 \) and we conclude that \( \text{sign}(\det(A^2 - 1)) = \text{sign}(\det(A^{2^n} - 1)) \).

Finally, we use (6.4) and obtain
\[
t_2^{[p]}(\Gamma; \varphi, \mathbb{Z}[1/p]) = \lim_{n \to \infty} \varepsilon \det(A^{p^n} - 1)(p') = \varepsilon \det(\log(A))(p'). \quad \Box
\]

7. The \( p \)-adic Atiyah question

Fix a prime number \( p \). Throughout this section, \( k \) denotes either the field \( \mathbb{Q} \) of rational numbers or a finite field of characteristic \( \ell \neq p \).

7.1

As discussed in the introduction, it is natural to wonder which values \( p \)-adic Betti numbers can actually take. Is every \( p \)-adic number a \( p \)-adic Betti number? In fact, we are not aware of any example where the \( p \)-adic Betti numbers take a finite value which is not an integer. Here we discuss the following \( p \)-adic analogue of the Atiyah conjecture.

**Question 7.2.** Let \( X \) be a connected finite CW-complex and \( (\varphi, G) \) a virtually pro-\( p \) completion of \( \Gamma = \pi_1(X) \). Under which assumptions is \( b_j^{[p]}(X; \varphi, k) \in \mathbb{Z} \)?
7.3. A dichotomy

The integrality of p-adic Betti numbers has a strong influence on the possible growth rates of Betti numbers in towers of finite sheeted covering spaces. Let \((N_n)_{n\in\mathbb{N}}\) be an exhaustive chain in \(G\) and write \(\Gamma_n = \varphi^{-1}(N_n)\). Informally, there is the following dichotomy:

Suppose that \(b_j^p(X; \varphi, k) \in \mathbb{Z}\) then the sequence \((b_j(\tilde{X}/\Gamma_n; k))_{n\in\mathbb{N}}\) either

1. stabilizes at \(b_j^p(X; \varphi, k)\) or
2. grows faster than a universal lower bound.

The universal bound will depend on the prime and the field \(k\). We will make this precise below under the assumption that the chain \((N_n)_{n\in\mathbb{N}}\) is obtained from the Frattini series in a pro-\(p\) group \(G\).

7.4. The Frattini series

Let \(G\) be a profinite group. The Frattini subgroup \(\Phi(G)\) is the intersection of all maximal open subgroups of \(G\). If \(G\) is a pro-\(p\) group, then the Frattini subgroup can be described as

\[\Phi(G) = \overline{G^p[G, G]}\]

that is, \(\Phi(G)\) is the smallest closed subgroup such that \(G/\Phi(G)\) is an elementary abelian \(p\)-group; see [13, 1.13]. The Frattini series is the normal series of \(G\) defined as

\[\Phi^0(G) = G \quad \text{and} \quad \Phi^{n+1}(G) = \Phi(\Phi^n(G)).\]

For a pro-\(p\) group, the Frattini series defines an exhaustive chain of open subgroups. Moreover, for a finite \(p\)-group \(G\), the Frattini series eventually reaches the trivial subgroup and we define the Frattini length of \(G\) by

\[\mathcal{F}(G) = \min\{n \in \mathbb{N}_0 \mid \Phi^n(G) = \{1\}\}\]

**Lemma 7.5.** Let \(G\) and \(H\) be finite \(p\)-groups. Then

1. \(\mathcal{F}(\mathbb{Z}/p^r\mathbb{Z}) = r\);
2. \(\mathcal{F}(G) \leq \mathcal{F}(N) + \mathcal{F}(G/N)\) for every normal subgroup \(N \leq G\);
3. \(\mathcal{F}(G \times H) = \max(\mathcal{F}(G), \mathcal{F}(H))\);
4. if \(H \leq G\), then \(\mathcal{F}(G) \leq \mathcal{F}(H) + \nu_p([G : H])\).

**Proof.** (i) follows immediately from \(\Phi(\mathbb{Z}/p^r\mathbb{Z}) \cong \mathbb{Z}/p^{r-1}\mathbb{Z}\). For (ii) let \(n = \mathcal{F}(G/N)\) and \(m = \mathcal{F}(N)\). Then \(\Phi^n(G) \subseteq N\) and thus \(\Phi^{n+m}(G) \subseteq \Phi^m(N) = \{1\}\). Statement (iii) follows immediately from the observation \(\Phi(G \times H) = \Phi(G) \times \Phi(H)\). The last assertion can be obtained by induction on \([G : H]\). If \([G : H] = p\), then \(H\) is maximal and \(\Phi(G) \subseteq H\). Thus, \(\mathcal{F}(H) \supseteq \mathcal{F}(\Phi(G)) = \mathcal{F}(G) - \nu_p([G : H])\). Let \([G : H] \geq p^2\). Take a maximal subgroup \(M \leq G\) which contains \(H\). Then the induction hypothesis yields

\[\mathcal{F}(H) \geq \mathcal{F}(M) - \nu_p([M : H]) \geq \mathcal{F}(G) - \nu_p([G : H])\]

**Proposition 7.6.** Let \(p\) be a prime and let \(G\) be a finite \(p\)-group. If \(\rho : G \to \text{GL}_N(k)\) is a faithful irreducible representation, then

\[\mathcal{F}(G) \leq \nu_p(N) + c_{k,p}\]

with constants \(c_{q,p} = 1\) and \(c_{p,q} = p \log_p(q)\) (where \(q = |k|\)).

**Proof.** We first reduce to the case where \(\rho\) is primitive, which means not induced from a proper subgroup. Indeed, suppose that \(\rho\) is induced from an irreducible representation of
dimension \( N \) from a subgroup \( H \leq G \), then \( N = N' | G : H | \). Assume that (7.1) holds for \( H \), then Lemma 7.5(iv) implies

\[
\mathcal{F}(G) \leq \mathcal{F}(H) + \nu_p(|G : H|) \leq \nu_p(N') + c_{k,p} + \nu_p(|G : H|) = \nu_p(N) + c_{k,p}.
\]

Since enlarging the group can only increase the Frattini length, we may further assume that \( p(G) \) is a maximal \( p \)-subgroup of \( \text{GL}_N(k) \). The maximal primitive \( p \)-subgroups of \( \text{GL}_N(k) \) are well understood going back to work of Vol’vacev. Here, we use the complete description contained in [20].

Assume that \( p > 2 \). Let \( k(\zeta_p) \) be the field obtained from adjoining a primitive \( p \)-th root of unity to \( k \). By [20, (II.4)], we have \( N = [k(\zeta_p) : k] \) and \( G \) is a cyclic group of order \( p^a \) where

\[
\alpha = \max \{ i \mid k(\zeta_p) \text{ contains a primitive } p^i \text{-th root of unity} \}.
\]

Using \( N < p \) and Lemma 7.5(i), we see that \( \mathcal{F}(G) = \alpha \leq \nu_p(N) + \alpha \). For \( k = \mathbb{Q} \) we obtain \( N = p - 1 \) and \( \alpha = 1 = c_{Q,p} \). If \( k = \mathbb{F}_q \) is a finite field, then \( \alpha = \nu_p(q^N - 1) \leq N \log_p(q) \leq p \log_p(q) = c_{q,p} \).

Assume that \( p = 2 \) and \( k = \mathbb{Q} \). By [20, (IV.4)] , we have \( N = 1 \) and \( G \) is the group with 2-elements. In particular, \( \mathcal{F}(G) = 1 \leq \nu_2(1) + 1 \).

Assume that \( p = 2 \) and \( k = \mathbb{F}_q \). Suppose first that \( q \equiv 1 \mod 4 \). By [20, (III.4)(i)], this implies that \( N = 1 \) and \( G \) is a cyclic group of order \( \nu_2(q - 1) \). Again Lemma 7.5(i) yields \( \mathcal{F}(G) = \nu_2(1) + \nu_2(q - 1) = \nu_2(1) + 2 \log_2(q) \).

Suppose now that \( q \equiv 3 \mod 4 \). By [20, (III.4)(ii)], there are two possibilities. Either \( N = 1 \) and \( |G| = 2 \) (and the assertion is obvious) or \( N = 2 \) and \( G \) is a semidihedral group of order \( 2^{\gamma + 1} \) where \( \gamma = \nu_2(q^2 - 1) \). In the latter case, we obtain

\[ \mathcal{F}(G) \leq \gamma + 1 = \nu_2(2) + \gamma \leq \nu_2(2) + \log_2(q^2) = \nu_2(2) + c_{q,p} \]

\[ \Box \]

Theorem 7.7. Let \( X \) be a connected, locally path-connected and semilocally simply connected space with \( \pi_1(X) = \Gamma \). Let \( \varphi : \Gamma \rightarrow G \) be a completion where \( G \) is a finitely generated pro-p group. We consider \( \Gamma_n = \varphi^{-1}(\Phi^n(G)) \); see §7.4.

Assume that \( b_j^{|p|}(X; \varphi, k) \in \mathbb{Z} \). Then either \( b_j(\tilde{X} / \Gamma_n; k) = b_j^{|p|}(X; \varphi, k) \) for all sufficiently large \( n \) or

\[ b_j(\tilde{X} / \Gamma_n; k) \geq p^{n + 1 - c_{k,p}} + b_j^{|p|}(X; \varphi, k) \]

for all sufficiently large \( n \) with the constants \( c_{k,p} \) as in Proposition 7.6.

Proof. Assume that the sequence of \( j \)-th Betti numbers \( b_j(n) = b_j(\tilde{X} / \Gamma_n; k) \) does not eventually stabilize at \( b_j^{|p|} = b_j^{|p|}(X; \varphi, k) \). Since the sequence is increasing, its values will eventually exceed the \( p \)-adic Betti number; that is, there is some \( n_0 \in \mathbb{N} \) such that \( b_j(n) > b_j^{|p|} \) for all \( n \geq n_0 \). The numbers \( b_j(n) - b_j^{|p|} \) are positive integers (for \( n \geq n_0 \)) and the product formula [25, p. 108] implies

\[ b_j(n) - b_j^{|p|} = [b_j(n) - b_j^{|p|}]^{-1} \prod_{q \neq p} |b_j(n) - b_j^{|p|}|_q^{-1}, \]

where the product runs over all primes \( q \neq p \). Note that almost all terms in the product equal 1. Since \( b_j(n) - b_j^{|p|} \) is an integer, we obtain \( |b_j(n) - b_j^{|p|}|_q^{-1} \geq 1 \) for every prime \( q \neq p \).

Recall that by 5.3 (a), there is an isomorphism \( H^j(\tilde{X} / \Gamma_n; k) \cong H^j(X; \varphi, k)^{\Phi^n(G)} \). Decompose \( H^j(X; \varphi, k) \) into irreducible constituents

\[ H^j(X; \varphi, k) = \bigoplus_{[S] \in \text{irr}_k(G)} m_S S \]
with certain multiplicities $m_S \in \mathbb{N}_0$. The representations occurring in the subrepresentation $H^j(X; \varphi, k)^{\Phi^n(G)}$ are exactly those that factor over $\Phi^n(G)$. We deduce that the difference $b_j(n) - b_j^{[p]}$ is divisible by $p^{e(n)}$ where

\[ e(n) = \min \{ \nu_p(N) \mid \rho: G \to \text{GL}_n(k) \text{ irreducible with } \Phi^n(G) \not\subset \ker(\rho) \}. \]

Let $\rho: G \to \text{GL}_n(k)$ be an irreducible representation which does not factor over $\Phi^n(G)$. The Frattini length of $\rho(G)$ is at least $n + 1$. An application of Proposition 7.6 yields

\[ n + 1 \leq \mathcal{F}(\rho(G)) \leq \nu_p(N) + c_{k,p}. \]

We conclude $|b_j(n) - b_j^{[p]}|_p \leq p^{-e(n)} \leq p^{c_{k,p} - (n+1)}$ and this completes the proof. \hfill $\square$

7.8. Proof of Theorem 1.5

We recall that $\varphi: \Gamma \to \text{GL}_m(\mathbb{Z}_p)$ is a homomorphism of groups and the closure $G = \overline{\varphi(\Gamma)}$ is a compact $p$-adic Lie group of dimension $d = \dim(G)$ say. We use the notation $\text{GL}_m^n(\mathbb{Z}_p) = \ker(\text{GL}_m(\mathbb{Z}_p) \to \text{GL}_m(\mathbb{Z}/p^nu\mathbb{Z}))$ for the principal congruence subgroup of level $n$. We study the chain

\[ \Gamma_n = \varphi^{-1}(\text{GL}_m^n(\mathbb{Z}_p)) \]

of principal congruence subgroups in $\Gamma$. The following argument is based on the theory of uniform pro-$p$ groups and their Lie algebras as developed in [13, Chapter 4]. For example, the principal congruence subgroups $\text{GL}_m^n(\mathbb{Z}_p)$ are uniform pro-$p$ groups for all $n \geq 2$.

Let $U \leq_o G$ be an open normal uniform pro-$p$ subgroup with Lie $\mathbb{Z}_p$-algebra $u = \log(U)$; for the existence, we refer to [13, Theorem 8.1]. We can choose $U$ so small that $U \subseteq \text{GL}_m^n(\mathbb{Z}_p)$. Pick a natural number $r \geq 2$ such that $G \cap \text{GL}_m^n(\mathbb{Z}_p) \subseteq U^r = \exp(p^r u)$. This means that the Lie algebra $u$ satisfies $u \cap p^r \text{gl}_m(\mathbb{Z}_p) \subseteq p^ru$. We deduce

\[ p^{n+r}u \subseteq u \cap p^{n+r} \text{gl}_m(\mathbb{Z}_p) \subseteq p^{n+1}u \]

for every $n \in \mathbb{N}_0$. By [13, Lemma 3.4], the Frattini series of $U$ is $\Phi^n(U) = U^{p^n} = \exp(p^nu)$; cf. [13, Lemma 4.14]. In particular, we conclude

\[ \Phi^{n+r}(U) \subseteq U \cap \text{GL}_m^{n+r}(\mathbb{Z}_p) = G \cap \text{GL}_m^n(\mathbb{Z}_p) \subseteq \Phi^{n+1}(U) \]

For $\Delta_n = \varphi^{-1}(\Phi^n(U))$, we observe that

\[ \Delta_{n+r} \leq \Gamma_{n+r} \leq \Delta_{n+1} \]

for all $n \in \mathbb{N}_0$. It should be noted that the groups $\Phi^i(U)$ and $G \cap \text{GL}_m^{n+r}(\mathbb{Z}_p)$ are in fact pro-$p$ groups; this allows us to deduce the inequalities

\[ b_j(\tilde{X}/\Gamma_{r+n}; k) \geq b_j(\tilde{X}/\Gamma_{r+n}; k) \geq b_j(\tilde{X}/\Delta_{n+1}; k) \]

from Lemma 5.3 (a). We see, in particular, that the sequence $(b_j(\tilde{X}/\Delta_n; k))_{n \in \mathbb{N}}$ stabilizes at the $p$-adic Betti number exactly if the sequence $(b_j(\tilde{X}/\Gamma_n; k))_{n \in \mathbb{N}}$ does.

Assume now that the sequences of Betti numbers do not stabilize. Then we deduce from Theorem 7.7 that

\[ b_j(\tilde{X}/\Gamma_{n+r-1}; k) \geq b_j(\tilde{X}/\Delta_n; k) \geq c_1p^n + b_j^{[p]}(X; \varphi, k) \]

for some constant $c_1 > 0$. Moreover, we observe that for all $n \geq 1$

\[ |\Gamma : \Gamma_{n+r-1}| = |\Gamma : \Gamma_r| |\Gamma_r : \Gamma_{n+r-1}| \leq |\Gamma : \Gamma_r| |\Delta_1 : \Delta_{n+r-1}| \leq |\Gamma : \Gamma_r| |U : \Phi^{n+r}(U)| = |\Gamma : \Gamma_r| p^{d(n+r)}, \]
where the last step uses [13, Proposition 4.4] and the fact that the lower $p$-series agrees with the Frattini series in uniform pro-$p$ groups. This shows that $|\Gamma : \Gamma_n| \leq c_2 p^n$ for some constant $c_2 > 0$ and the proof of Theorem 1.5 is complete.

7.9. An algebraic reformulation of Theorem 1.3

Consider the situation of Theorem 1.3. By the results of §5.5, there is an isomorphism $H^i(X; \varphi, k) \cong \ker(\bar{\partial}^i)/\im(\bar{\partial}^{i-1})$ where

$$C^\infty(G, k)^{\xi_{i-1}} \xrightarrow{\partial^i} C^\infty(G, k)^{\xi_i} \xrightarrow{\partial^{i-1}} C^\infty(G, k)^{\xi_{i+1}},$$

and $\bar{\partial}^i = r(A_i)$ is defined as multiplication with a matrix $A_i \in M_{e_{i+1}, e_i}(\mathbb{Z}[\Gamma])$ via the right regular representation. We obtain the following purely algebraic reformulation of the $p$-adic Atiyah question.

Let $\Gamma$ be a finitely generated group, let $A \in M_{n,m}(\mathbb{Z}[\Gamma])$ and let $(\varphi, G)$ be a virtual pro-$p$ completion. Consider the multiplication map

$$r(A) : C^\infty(G, k)^m \to C^\infty(G, k)^n$$

defined by $A$ using the right regular representation of $G$ on $C^\infty(G, k)$. The kernel of $r(A)$ is a smooth admissible representation using the left regular representation of $G$. Under which assumptions is $p$-$\dim_k \ker(r(A)) \in \mathbb{Z}$? Theorem 1.3 is an immediate consequence of the next result.

**Theorem 7.10.** Let $\Gamma$ be a group and let $\varphi : \Gamma \to G$ be a homomorphism into a finitely generated, virtually pro-$p$ group $G$ which is virtually abelian. For every matrix $A \in M_{m,n}(k[\Gamma])$, the kernel of $r(A) : C^\infty(G, k)^n \to C^\infty(G, k)^m$ satisfies

$$p$-$\dim_k \ker(r(A)) \in \mathbb{Z}.$$

7.11. Reduction to the case $G = \mathbb{Z}_p^d$

In this section, we explain why it is sufficient to prove Theorem 7.10 for the group $G = \mathbb{Z}_p^d$. Since $G$ is finitely generated, virtually pro-$p$ and virtually abelian, we can find an open normal subgroup $G_0 \trianglelefteq G$ which is a finitely generated abelian pro-$p$ group; this means $G_0 \cong F \times \mathbb{Z}_p^d$ for some $d \in \mathbb{N}_0$ and a finite abelian $p$-group $F$. Choosing $G_0$ smaller if necessary, we may assume that $G_0 \cong \mathbb{Z}_p^d$.

By Proposition 3.8 and Definition 3.9, we have

$$p$-$\dim_k \ker(r(A)) = p$-$\dim_k \ker(r(A)|_{G_0}) + p$-$\dim_k (\ker(r(A))_0 \cong \ker(r(B)).$$

Put $\Gamma_0 = \varphi^{-1}(G_0)$. Let $u = |G : G_0| = |\Gamma : \Gamma_0|$. To complete the reduction step, we show that there is a matrix $B \in M_{um, un}(k[\Gamma_0])$ such that $\ker(r(A))_0 \cong \ker(r(B))$.

For simplicity, we consider the case $n = m = 1$, that is, $A \in k[\Gamma]$. Fix a set of representatives $\gamma_1, \ldots, \gamma_u \in \Gamma$ for the cosets of $\Gamma_0$. For $i, j \in \{1, \ldots, u\}$, we write $\gamma_i \gamma_j = y_{i,j} \gamma_{\sigma(i)(j)}$ with $y_{i,j} \in \Gamma_0$ and $\sigma(i)$ is a permutation of $\{1, \ldots, u\}$. The coset representatives provide an isomorphism of (left regular) representations

$$\Theta: \ker(r(A))_0 \cong \ker(r(A)).$$

defined as $\Theta$: $f \mapsto \{(r(\gamma_1) f)_{G_0}, \ldots, (r(\gamma_u) f)_{G_0}\}^T$.

We write $A = \sum_{j=1}^u x_j \gamma_j$ for certain $x_j \in k[\Gamma_0]$, then the ith entry of $\Theta(P(A) f)$ satisfies

$$\Theta_i(P(A) f) = \Theta_i \left( \sum_{j=1}^u r(x_j \gamma_j) f \right) = \left( \sum_{j=1}^u r(\gamma_i) r(x_j \gamma_j) f \right)|_{G_0}.$$
We write $b_{i,s} = \gamma_i x_{\sigma(i)^{-1}(s)} y_{i,\sigma(i)^{-1}(s)} \in k[\Gamma_0]$, then the matrix $B = (b_{i,s}) \in M_{u,u}(k[\Gamma_0])$ satisfies $r(B) \Theta(f) = \Theta(r(A) f)$ for all $f \in C^\infty(G, k)$.

Before we complete the proof of Theorem 7.10, we collect the following lemma.

**Lemma 7.12.** Let $\overline{k}$ be the algebraic closure of $k$.

(a) Let $x_1, \ldots, x_m \in \overline{k}^\times$ and let $E_1, \ldots, E_m \subset (\overline{k}^\times)^d$ be subgroups of the $d$-dimensional algebraic torus. Then the $p$-adic limit

$$\lim_{j \to \infty} \mu(p^j)^d \cap \bigcap_{i=1}^m x_i E_i \in \mathbb{Z}_p$$

exists and lies in $\mathbb{Z}$.

(b) Let $V \subset (\overline{k}^\times)^d$ be a Zariski closed subset and let $E \subset (\overline{k}^\times)^d$ be a subgroup. Then the $p$-adic limit

$$\lim_{j \to \infty} \mu(p^j)^d \cap V \cap E \in \mathbb{Z}_p$$

exists and is a rational integer.

**Proof.** (a) It is sufficient to consider those cosets $x_i E_i$ which contain some element from $\mu(p^\infty)^d$; in this case, we may assume that $x_i \in \mu(p^\infty)^d$. We proceed by induction on $m$. Let $m = 1$ and assume $x_1 \in \mu(p^j_0)^d$, then $\mu(p^j)^d \cap x_1 E_1 = \mu(p^j)^d \cap E_1$ for all $j \geq j_0$. Note that $E_1 \cap \mu(p^\infty)^d$ is a subgroup of the $p$-torsion group $\mu(p^\infty)^d$. In particular, if $E_1 \cap \mu(p^\infty)^d$ is infinite, then

$$\lim_{j \to \infty} |\mu(p^j)^d \cap E_1| = 0 \in \mathbb{Z}_p.$$ 

On the other hand, if $E_1 \cap \mu(p^\infty)^d$ is finite, then the limit is the order $|E_1 \cap \mu(p^\infty)^d| \in \mathbb{N}$.

Let $m \geq 2$. For the induction step, we use the principle of inclusion–exclusion to see that $|\mu(p^j)^d \cap \bigcup_{i=1}^m x_i E_i|$ equals

$$|\mu(p^j)^d \cap \bigcup_{i=1}^{m-1} x_i E_i| + |\mu(p^j)^d \cap x_m E_m| - |\mu(p^j)^d \cap \bigcup_{i=1}^{m-1} (x_i E_i \cap x_m E_m)|.$$

This completes the proof, since $x_i E_i \cap x_m E_m$ is either empty or of the form $x_i'(E_i \cap E_m)$.

(b) By a result of Poizat–Gramain, the intersection $V \cap \mu(p^\infty)^d = \bigcup_{i=1}^m x_i E_i$ is a finite union of cosets of subgroups $E_1, \ldots, E_m \subset (\overline{k}^\times)^d$; see [26, Theorems 1 and 5]. For fields of characteristic zero, this has been conjectured by Lang and has been proven by Laurent [19]; for $\overline{k} = \mathbb{C}$, a different proof can be found in [28]. Assertion (b) follows from (a), since $V \cap E \cap \mu(p^\infty)^d = \bigcup_{i=1}^m x_i E_i \cap E$ and $x_i E_i \cap E$ is either empty or of the form $x_i'(E_i \cap E)$. □

### 7.13. Proof of Theorem 7.10

Since the matrix $A$ has finitely many entries all of which have finite support in $\Gamma$, we may assume that $\Gamma$ is finitely generated. By §7.11, we may assume that $G = \mathbb{Z}_p^d$. In addition, we can reduce to the case where $\Gamma$ is a free abelian group. Indeed, the group $\overline{\Gamma} = \Gamma / \ker(\varphi)$ is a
free abelian group and the action of \( r(A) \) only depends on the image of \( A \) under the reduction homomorphism \( M_{m,n}(k[\Gamma]) \to M_{m,n}(k[\overline{\Gamma}]) \).

Assume that \( \Gamma \) is free abelian. Observe that \( \Gamma \) has rank \( d + s \) for some \( s \in \mathbb{N}_0 \) since the image \( \varphi(\Gamma) \) is dense in \( G = \mathbb{Z}_p^d \). We pick a basis \( t_1, \ldots, t_{d + s} \) of \( \Gamma \). The group ring \( k[\Gamma] \) is isomorphic to the Laurent polynomial ring \( k[t_1^{\pm 1}, \ldots, t_{d + s}^{\pm 1}] \). Since the set \( \{ \varphi(t_1), \ldots, \varphi(t_{d + s}) \} \) generates \( \mathbb{Z}_p^d \) as \( \mathbb{Z}_p \)-module, there is \( d \)-element subset which is a \( \mathbb{Z}_p \)-basis of \( \mathbb{Z}_p^d \). Without loss of generality, \( \varphi(t_1), \ldots, \varphi(t_d) \) is the standard basis of \( \mathbb{Z}_p^d \).

We observe that for every subgroup \( G_N = p^NG \), there is an isomorphism \( \overline{k} \otimes_k C^\infty(G/G_N, k) \cong C^\infty(G/G_N, \overline{k}) \), and moreover, the dimension of the kernel of \( r(A_N) \) acting on acting \( C^\infty(G/G_N, k) \) does not change if we extend the scalars to \( \overline{k} \).

How can we describe the kernel of \( r(A) \) acting on \( C^\infty(G, \overline{k})^n \)? The right regular representation \( C^\infty(G, \overline{k}) \) decomposes as a direct sum over irreducible representations:

\[
C^\infty(G, \overline{k}) = \bigoplus_{S \in \text{Irr}(G)} S
\]

each occurring with multiplicity one since \( G \) is abelian. The irreducible representations of \( G = \mathbb{Z}_p^d \) can be parametrized by the set \( \mu(p^\infty)^d \) of \( d \)-tuples of \( p \)-power roots of unity in \( \overline{k} \). For \( \zeta = (\zeta_1, \ldots, \zeta_d) \), the irreducible representation \( (\rho_\zeta, S_\zeta) \) is the unique 1-dimensional \( \overline{k} \)-representation on which the \( i \)-th basis element \( \varphi(t_i) \) of \( \mathbb{Z}_p^d \) acts like \( \zeta_i \).

We need to take a closer look in order to describe the action of \( \Gamma \) on \( S_\zeta \). Since the action of \( \Gamma \) is defined via the action of \( G \) using the homomorphism \( \varphi : \Gamma \to G \), we get \( \rho_\zeta(t_i) = \zeta_i \in \overline{k}^* \) for all \( i \in \{1, \ldots, d\} \). We can write \( \varphi(t_{d+i}) = \sum_{j=1}^d \lambda_{i,j} \varphi(t_j) \) for certain \( \lambda_{i,j} \in \mathbb{Z}_p \). Therefore, \( t_{d+i} \) acts as \( \prod_{j=1}^d \zeta_j^{\lambda_{i,j}} \) on \( S_\zeta \). The map

\[
\varepsilon : \mu(p^\infty) \to E = \left\{ \zeta \in \mu(p^\infty)^{d+s} \mid \zeta_{d+i} = \prod_{j=1}^d \zeta_j^{\lambda_{i,j}} \text{ for all } 1 \leq i \leq s \right\}
\]

sending \( \zeta \) to \( (\zeta_1, \ldots, \zeta_d, \prod_{j=1}^d \zeta_j^{\lambda_{1,j}}, \cdots, \prod_{j=1}^d \zeta_j^{\lambda_{s,j}}) \) is an isomorphism of groups and a Laurent polynomial \( P \in k[t_1^{\pm 1}, \ldots, t_{d+s}^{\pm 1}] \) acts on \( S_\zeta \) as multiplication by \( \rho_\zeta(P) = \varepsilon(\zeta) \); same argument as in §6.6.

Now we consider a matrix \( A \in M_{m,n}(k[t_1^{\pm 1}, \ldots, t_{d+s}^{\pm 1}]) \). What is the dimension of the kernel of \( \rho_\zeta(A) \) acting on \( S_\zeta^n \)? Let \( J_i \subseteq k[t_1^{\pm 1}, \ldots, t_{d+s}^{\pm 1}] \) be the ideal generated by all \( (i \times i) \)-minors of \( A \). The rank of \( \rho_\zeta(A) \) is the maximal \( i \) such that \( \varepsilon(\zeta) \notin V(J_i) \). We deduce that

\[
\dim_k \ker \left( \rho_\zeta(A) \right) = \sum_{i=1}^n |V(J_i) \cap \{ \varepsilon(\zeta) \}|.
\]

Observe that the irreducible constituents of \( C^\infty(G/G_N, \overline{k}) \) are exactly those representations \( S_\zeta \) with \( \zeta \in \mu(p^N)^d \), that is,

\[
C^\infty(G/G_N, \overline{k}) \cong \bigoplus_{\zeta \in \mu(p^N)^d} S_\zeta.
\]

Hence we get the formula

\[
\dim_k \ker (r(A_N)) = \sum_{i=1}^s |V(J_i) \cap \varepsilon(\mu(p^N)^d)|
\]
for the dimension of the kernel of \( r(A) \) acting on \( C^\infty(G, k)^G_N = C^\infty(G/G_N, k) \). We note that 
\[ E \cap \mu(p^N)^{d+s} = \varepsilon(\mu(p^N)^d) \] 
Now the proof is complete using the Approximation Lemma 3.11
\[ \lim_{N \to \infty} \dim_k \ker(r(A_N)) = p \cdot \dim_k^G(\ker(r(A))) \]
and Lemma 7.12 (b).

**Definition 7.14.** Let \( X \) be a finite connected CW-complex. We say that all \( p \)-adic Betti numbers of \( X \) are integers if \( b_j^{[p]}(X; \varphi, k) \in \mathbb{Z} \) for all \( j \in \mathbb{N}_0 \) and every virtual pro-\( p \) completion \( \varphi \) of \( \Gamma = \pi_1(X) \).

**Lemma 7.15.** Let \( X_1 \) and \( X_2 \) be finite connected CW-complexes. All \( p \)-adic Betti numbers of \( X_1 \) and \( X_2 \) are integers if and only if all \( p \)-adic Betti numbers of \( X_1 \lor X_2 \) are integers.

**Proof.** Let \( \Gamma_1 = \pi_1(X_1) \). Assume that all \( p \)-adic Betti numbers of \( X_1 \) and \( X_2 \) are integers. The \( p \)-adic Betti numbers of a finite CW-complex are finite; hence, we may apply the formula for \( p \)-adic Betti numbers of wedge sums in \( \S 5.15 \) to deduce that all \( p \)-adic Betti numbers of \( X_1 \lor X_2 \) are integers.

Conversely, assume that all \( p \)-adic Betti numbers of \( X_1 \lor X_2 \) are finite. Let \( \varphi : \Gamma_1 \lor \Gamma_2 \rightarrow G \) be a virtual pro-\( p \) completion. We have to show that \( b_j^{[p]}(X_1 \lor X_2; \varphi, k) \) is an integer. If \( G_1 \) is finite, there is nothing to do (see Example 5.10). Assume that \( G_1 \) is infinite. Then \( \varphi : \Gamma_1 \lor \Gamma_2 \rightarrow G \) with \( \varphi|_{\Gamma_1} = \varphi_1 \) and \( \varphi|_{\Gamma_2} \equiv 1 \) is a virtual pro-\( p \) completion of \( \Gamma_1 \lor \Gamma_2 = \pi_1(X_1 \lor X_2) \). The formula for wedge sums in \( \S 5.15 \) yields
\[ b_j^{[p]}(X_1 \lor X_2; \varphi, k) = b_j^{[p]}(X_1; \varphi_1, k) \]
for all \( j \in \mathbb{N}_0 \). Hence all \( p \)-adic Betti numbers of \( X_1 \) are integers. The assertion for \( X_2 \) follows in the same way. \( \square \)

**7.16. Proof of Theorem 1.4**

Let \( X = X_1 \lor \cdots \lor X_n \) be a wedge sum of finite connected CW-complexes. We assume that the fundamental groups \( \Gamma_i = \pi_1(X_i) \) are virtually abelian. We want to show that all \( p \)-adic Betti numbers of \( X \) are integers. In view of Lemma 7.15, it is sufficient to show that all \( p \)-adic Betti numbers of \( X_i \) are integers. However, \( \Gamma_i \) is virtually abelian, and hence, in every virtual pro-\( p \) completion \( \varphi : \Gamma_i \rightarrow G \), the group \( G \) is virtually abelian. An application of Theorem 1.3 completes the proof.

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