Normal forms for three-parametric families of area-preserving maps near an elliptic fixed point

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Abstract

We study dynamics of area-preserving maps in a neighbourhood of an elliptic fixed point. We describe simplified normal forms for a fixed point of co-dimension 3. We also construct normal forms for a generic three-parameter family which contains such degeneracy and use the normal form theory to describe generic bifurcations of periodic orbits in these families.

1 Introduction

In a Hamiltonian system small oscillations around a periodic orbit are often described using the normal form theory [1, 2] which provide an important tool for the study of local dynamics (see e.g. [1, 3, 4]). In the case of two degrees of freedom the Poincaré section is used to reduce the problem to studying a family of area-preserving maps in a neighbourhood of a fixed point. The Poincaré map depends on the energy level and possibly on other parameters involved in the problem. A sequence of coordinate changes is used to transform the map to a normal form. Our approach to the normal form of a map is similar to [5].

In the absence of resonances the normal form is a rotation of the plane, and the angle of the rotation depends on the amplitude. In a generic one-parameter family of area-preserving maps, the normal form provides a description for a chain of islands which is born from the origin when the multiplier of the fixed point crosses a resonant value [1, 2, 6, 7, 8].

In [9] unique normal forms for two-parametric families were constructed and used to analyse bifurcations of $n$-periodic orbits.
In the paper we study three-parametric families of APM with fixed point of elliptic type. In such families two possibilities of degenerations are possible:
(1) degenerations in the twist terms;
(2) degeneration in the leading resonance term.
Normal form in the case of (1) was constructed in [9] (for arbitrary number of parameters). Normal forms for (2) are constructed in the paper first for an individual map (Section 2) and then for families (Section 3). In Sections 4 and 5 the normal forms are used to investigate bifurcations.

1.1 Individual maps
Let $F_0 : \mathbb{R}^2 \to \mathbb{R}^2$ be an area-preserving map (APM) which also preserves orientation. Let the origin be a fixed point: $F_0(0) = 0$. Since $F_0$ is area-preserving $\det DF_0(0) = 1$. Therefore the two eigenvalues of the Jacobian matrix $DF_0(0)$ are $\lambda_0$ and $\lambda_0^{-1}$. We will consider an elliptic fixed point, i.e. the case of non real $\lambda_0$. As the map is real $\lambda_0^{-1} = \lambda_0^*$. Consequently the multipliers of an elliptic fixed point belong to the unit circle: $|\lambda_0| = 1$, i.e. $\lambda_0 = e^{i\alpha_0}$.

There is a linear area-preserving change of variables such that the Jacobian of $F_0$ takes the form of a rotation:

$$DF_0(0) = R_{\alpha_0}, \quad \text{where} \quad R_{\alpha_0} = \begin{pmatrix} \cos \alpha_0 & -\sin \alpha_0 \\ \sin \alpha_0 & \cos \alpha_0 \end{pmatrix}.$$  

(1)

It is well known that APM with elliptic fixed point can be represented in Birkhoff normal form [11], i.e. there is an area-preserving change of coordinates which transforms $F_0$ into the resonant normal form $N_0$ that commutes with the rotation: $N_0 \circ R_{\alpha_0} = R_{\alpha_0} \circ N_0$. The change of coordinates and the map $N_0$ are formal series. The linear part of a normal form $N_0$ is $R_{\alpha_0}$. Following the method suggested in [8] by Takens (see e.g. [10, 8]), we consider a formal series $H_0$ such that

$$N_0 = R_{\alpha_0} \circ \Phi_{H_0}^1,$$  

(2)

where $\Phi_{H_0}^t$ is a flow generated by the Hamiltonian $H_0$. The Hamiltonian has the Takens normal form, i.e. it is invariant with respect to the rotation: $H_0 \circ R_{\alpha_0} = H_0$.

Our goal is to transform formal series $H_0$ to the most simple form. We use changes of variables which commutate with the rotation $R_{\alpha_0}$. Then in
new variables the map is still in Birkhoff normal form and the corresponding Hamiltonian remains in the Takens normal form.

It is convenient to use complex variables defined by
\[ z = \frac{x + iy}{\sqrt{2}} \quad \text{and} \quad \bar{z} = \frac{x - iy}{\sqrt{2}}. \] (3)

A fixed point is called resonant if there exists \( n \in \mathbb{N} \) such that \( \lambda_0^n = 1 \). The least positive \( n \) is called the order of the resonance [1]. The rotation \( R_{\alpha_0} \) takes the form \((z, \bar{z}) \mapsto (e^{i\alpha_0}z, e^{-i\alpha_0}\bar{z})\). As the map \( N_0 \) commutes with \( R_{\alpha_0} \) it contains only resonant terms:
\[ N_0(z, \bar{z}) = \lambda_0 z + \sum_{k+l \geq 2, k-l \equiv 1 \pmod{n}} f_{kl} z^k \bar{z}^l. \]

Corresponding Hamiltonian has the Takens normal form [8]:
\[ H_0(z, \bar{z}) = \sum_{k+l \geq 3, k \equiv l \pmod{n}} h_{kl} z^k \bar{z}^l, \quad h_{kl} = h_{lk}^*. \] (4)

It was established in [8], [9] that if \( h_{n0} \neq 0 \) then Hamiltonian \( H_0 \) can be transformed to a normal form
\[ \tilde{H}_0(z, \bar{z}) = \sum_{k \geq 2} a_k z^k \bar{z}^k + (z^n + \bar{z}^n) \sum_{k \geq 0} b_k z^k \bar{z}^k. \]

In the paper we consider the case of \( h_{n0} = 0 \) but \( h_{22} \neq 0 \) and \( h_{n+1,1}^2 - 4h_{22}h_{2n,0} \neq 0 \).

In order to investigate Hamiltonian it is convenient to use the symplectic polar coordinates \((I, \varphi)\) given by
\[ \begin{cases} x = \sqrt{2I} \cos \varphi, \\ y = \sqrt{2I} \sin \varphi \end{cases} \quad \text{or} \quad \begin{cases} z = \sqrt{I} e^{i\varphi}, \\ \bar{z} = \sqrt{I} e^{-i\varphi}. \end{cases} \] (5)

The Hamiltonian in Takens normal form (4) takes the form:
\[ H_0(I, \varphi) = I^2 \sum_{k \geq 0} a_k I^k + \sum_{j \geq 1, k \geq 0} a_{jk} I^{j+n/2+k} \cos(jn \varphi + \beta_{jk}), \] (6)

where \( a_k, a_{jk} \) and \( \beta_{jk} \) are real coefficients.

Proposition 5 of Section 2 implies the following theorem.
Theorem 1 Let \( N_0 \) be BNF for some APM \( F_0 \) with resonance of order \( n \) at the origin. Let for corresponding Hamiltonian in TNF (6) \( a_{10} = 0, a_0 \neq 0 \) and \( a_{11}^2 e^{2i\beta_{11}} - 8a_0 a_{20} e^{i\beta_{20}} \neq 0 \). Then there is a formal canonical change of coordinates such that in new variables \( \tilde{N}_0 = R_{\alpha_0} \circ \Phi_{\tilde{H}_0} \) and

\[
\tilde{H}_0(I, \varphi) = I^2 \sum_{k \geq 0} a_k I^k + I^{n/2} \sum_{k \geq 1} b_k \cos(n\varphi + \psi_k) I^{2k} + I^n \cos 2n\varphi \sum_{k \geq 0} c_k I^{2k}.
\tag{7}
\]

The coefficients in the form (7) are not unique: the coefficients \( b_k \) are replaced by \( -b_k \) after rotation by \( \pi/n \).

There is also an alternative normal form (only \( a_0 \neq 0 \) is required), which contains fewer terms of low orders:

Theorem 2 Let \( N_0 \) be BNF for some APM \( F_0 \) with resonance of order \( n \) at the origin. Let for corresponding Hamiltonian in TNF (6) \( a_0 \neq 0 \). Then there is a formal canonical change of coordinates such that in new variables \( \tilde{N}_0 = R_{\alpha_0} \circ \Phi_{\tilde{H}_0} \) and

\[
\tilde{H}_0(I, \varphi) = I^2 \sum_{k \geq 0} a_k I^k + \sum_{k \geq 1} c_k I^{nk/2} \cos(kn\varphi + \psi_k).
\]

Corresponding Proposition 6 is proved in Section 2.

1.2 Families

Let us consider a three-parametric family \( F_\mu \) of APM with fixed point of an elliptic type at the origin with \( \lambda_\mu = e^{i\alpha_\mu}, \mu = (\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3 \). We assume that for \( \mu = (0, 0, 0) \) the function \( F_0 \) has resonance of order \( n \) (i.e. \( \lambda_0 = e^{i\alpha_0}, \lambda_0^n = 1 \)).

After a linear change of coordinates the map \( F_\mu \) takes the form \( F_\mu = R_{\alpha_\mu} \circ \Phi \), where \( \Phi \) is a tangent-to-identity APM. In complex variables (3)

\[
F_\mu = (f_\mu, \bar{f}_\mu),
\]

\[
f_\mu(z, \bar{z}) = \lambda_\mu z + \sum_{k+l \geq 2} f_{kl}(\mu) z^k \bar{z}^l.
\]

It is natural to use the value

\[
\varepsilon = \alpha_0 - \alpha_\mu.
\tag{8}
as one of parameters. Let $\mu_3$ be expressed in terms of $(\varepsilon, \mu_1, \mu_2)$. Then $f_\mu(z, \bar{z})$ can be presented as a series in three variables $(z, \bar{z}, \varepsilon)$ with coefficients depending on $(\mu_1, \mu_2)$:

\[
f_\mu(z, \bar{z}) = \lambda_0 z + \sum_{k+l+m \geq 2} f_{klm}(\mu_1, \mu_2) z^k \bar{z}^l \varepsilon^m.
\]

After an appropriate change of coordinates the map $f_\mu$ can be written in Birkhoff normal form which contains only resonant terms:

\[
f_\mu(z, \bar{z}) = \lambda_0 z + \sum_{k+l+m \geq 2, k-l=1 \mod n} f_{klm}(\mu_1, \mu_2) z^k \bar{z}^l \varepsilon^m.
\]

Interpolation theorem for families (see for example [8]) gives $F_\mu = R_{\alpha_0} \circ \Phi_1^{H_\mu}$, where Hamiltonian $H_\mu$ in complex variables has the form

\[
H_\mu(z, \bar{z}) = \sum_{k+l+m \geq 3, k \equiv l \mod n} h_{klm}(\mu_1, \mu_2) z^k \bar{z}^l \varepsilon^m,
\]

where $h_{111}(\mu_1, \mu_2) = 1$ as $\varepsilon$ is determined by (8). Or, in the symplectic coordinates (5)

\[
H_\mu(I, \phi) = \varepsilon I + I^2 \sum_{k \geq 0} a_{km}(\mu_1, \mu_2) I^k \varepsilon^m + \sum_{j \geq 1} \sum_{k,m \geq 0} b_{jkm}(\mu_1, \mu_2) I^{k+j/2} \varepsilon^m \cos(jn\phi + \psi_{jkm}(\mu_1, \mu_2)).
\]

The form (11) is not unique. Rotations and time-one shifts $\Phi_1^\chi$ with resonant Hamiltonian $\chi$ preserve its structure. Our goal is to derive the most simple form of $H_\mu$, i.e. to eliminate as many terms of $H_\mu$ as possible. If $a_{100}(\mu_1, \mu_2) = 2|h_{000}(\mu_1, \mu_2)| \neq 0$ for all values of $(\mu_1, \mu_2)$ then Hamiltonian can be transformed to the form [?]:

\[
\tilde{H}_\mu(z, \bar{z}) = \varepsilon z \bar{z} + \sum_{k \geq 2, m \geq 0} a_{km}(\mu_1, \mu_2) z^k \bar{z}^k \varepsilon^m + (\varepsilon^n + \bar{z}^n) \sum_{k, m \geq 0} b_{km}(\mu_1, \mu_2) z^k \bar{z}^k \varepsilon^m
\]

or, in coordinates $(I, \phi)$:

\[
\tilde{H}_\mu(I, \phi) = \varepsilon I + \sum_{k \geq 2, m \geq 0} a_{km}(\mu_1, \mu_2) I^k \varepsilon^m + I^{n/2} \cos n\phi \sum_{k, m \geq 0} b_{km}(\mu_1, \mu_2) I^k \varepsilon^m.
\]
In this paper the main attention is paid to the case of Hamiltonian (10) when the following conditions are valid:

\[
h_{n00}(0, 0) = 0, \quad h_{220}(0, 0) \neq 0, \quad h_{n+1,1,0}(0, 0) - 4h_{220}(0, 0)h_{2n,0,0}(0, 0) \neq 0.
\]

(13)

Note that these conditions correspond to a family which is a three-parametric unfolding of \(F_0\) from Theorem 1. We show that there is such canonical change of variables that Hamiltonian has the form (11) with the last sum containing only two harmonics, i.e. terms with \(j = 1\) and \(j = 2\). Moreover \(a_{1,2l+1,m} = 0, a_{2,2l+1,m} = 0\) and \(\psi_{2l,m} = 0\) for all \(l\) and \(m\).

For the sake of receiving \(\psi_{200}(\mu_1, \mu_2) = 0\), as it will be shown in Section 3, it is necessary to make a rotation by an angle \(\phi = -\frac{1}{2n} \arg(h_{2n,0,0} - \frac{h_{n+1,1,0}}{4h_{220}})\).

After the rotation the small coefficient of \(z^n\) in (10) is

\[
h_{n00}(\mu_1, \mu_2) \exp\left(-\frac{i}{2} \arg\left(h_{2n,0,0}(\mu_1, \mu_2) - \frac{h_{n+1,1,0}(\mu_1, \mu_2)}{4h_{220}(\mu_1, \mu_2)}\right)\right) = \nu(\mu_1, \mu_2),
\]

(14)

\[
\nu(\mu_1, \mu_2) = \nu_1(\mu_1, \mu_2) + i\nu_2(\mu_1, \mu_2) = \gamma(\mu_1, \mu_2)e^{i\beta(\mu_1, \mu_2)}.
\]

Let \(\mu_1\) and \(\mu_2\) be expressed by \(\nu_1\) and \(\nu_2\): \(\mu_1 = \mu_1(\nu_1, \nu_2), \mu_2 = \mu_2(\nu_1, \nu_2)\). Then we can consider \((\nu_1, \nu_2)\) as new parameters instead of \((\mu_1, \mu_2)\).

Below we use the following notations: \(\mu = (\mu_1, \mu_2, \mu_3), \Upsilon = (\varepsilon, \nu_1, \nu_2), \mathbf{m} = (m_1, m_2, m_3), |\mathbf{m}| = m_1 + m_2 + m_3, \Upsilon^\mathbf{m} = \varepsilon^{m_1}\nu_1^{m_2}\nu_2^{m_3}\).

The following theorem gives a simplification of the Hamiltonian to a normal form.

**Theorem 3** Let \(F_\mu\) be a smooth (\(C^\infty\) or analytic) family of area preserving maps with fixed point of elliptic type at the origin such that

1. \(F_0\) has a resonant of order \(n\) at the origin: \(\lambda_0 = e^{i\alpha_0}, \lambda^n = 1\);
2. coefficients of Hamiltonian \(H_\mu\) in the TNF (10) satisfy conditions (13);
3. parameters \(\mu = (\mu_1, \mu_2, \mu_3)\) can be expressed by \((\varepsilon, \nu_1, \nu_2)\) defining by (8) and (14).

Then there is a formal Hamiltonian \(\tilde{H}_\Upsilon\) and formal canonical change of variables which conjugates \(F_\mu\) with \(R_{\alpha_0} \circ \Phi_{\tilde{H}_\Upsilon}^1\). Moreover, \(\tilde{H}_\Upsilon\) in coordinates
$(I, \varphi)$ has the following form:

$$
\tilde{H}_T(I, \varphi) = \varepsilon I + \gamma I^{n/2} \cos(n \varphi + \beta) + I^2 \sum_{k, |m| \geq 0} a_{km} I^{k} \Upsilon^{m}
+ I^{n/2} \sum_{k, |m| \geq 0, \quad k + m_1 \geq 1} b_{km} I^{2k} \Upsilon^{m} \cos(n \varphi + \psi_{km})
+ I^n \sum_{k, |m| \geq 0} c_{km} I^{2k} \Upsilon^{m} \cos(2n \varphi).
$$

The theorem follows from Proposition 8 of Section 3.

From Proposition 9 follows an alternative variant, useful for study of bifurcations:

**Theorem 4** If $F_\mu$ is a smooth ($C^\infty$ or analytic) family of area preserving maps such that $F_0$ has a resonant elliptic fixed point at the origin and in (11) $a_{00}(0, 0) \neq 0$, then there is a formal Hamiltonian $\tilde{H}_T$ and formal canonical change of variables which conjugates $F_\mu$ with $R_{\alpha_0} \circ \Phi^1_{\tilde{H}_T}$. Moreover, $\tilde{H}_T$ has the following form:

$$
\tilde{H}_T(I, \varphi) = \varepsilon I + \gamma I^{n/2} \cos(n \varphi + \beta) + I^2 \sum_{k, |m| \geq 0} a_{km} I^{k} \Upsilon^{m}
+ I^{n/2} \sum_{k + m_1 \geq 1, \quad m_2, m_3 \geq 0} c_{km} I^{kn/2} \Upsilon^{m} \cos((k + 2)n \varphi + \psi_{km}).
$$

Then we use the normal forms (12) and (15) to study bifurcations. We keep lower order terms and skip those, which do not change the picture qualitatively. The detailed discussion of typical level sets and their bifurcations for the case of degeneracy in the main resonant term are presented in Section 4.

In Section 5 there is a brief description of some possible bifurcations for three-parametric families when in (12) $a_{20}(0, 0) = a_{30}(0, 0) = 0$, $b_{00}(\mu_1, \mu_2) \neq 0$. This case is different from two-parametric families considered in [9] only in tiny domain in the space of parameters.

2 **Simplification of a formal Hamiltonian with $h_{n,0} = 0$ for individual map**

In this section we construct two degenerate resonant normal forms (17) and (27). The first one provides particularly simple form of Hamiltonian in the
symplectic polar coordinates: it contains only two lowest harmonics of the angle variable, namely $n\varphi$ and $2n\varphi$. An alternative normal form (27) contains fewer terms of low orders.

**Proposition 5** If

$$H(z, \bar{z}) = \sum_{k+l \geq 4, k,l \geq 0, \, k = l \, (\text{mod } n)} h_{kl} z^k \bar{z}^l$$  \hspace{1cm} (16)

is a formal series such that $h_{kl} = h_{lk}$, $h_{n0} = 0$, $h_{22} \neq 0$ and $h_{n+1,1}^2 - 4h_{22}h_{2n,0} \neq 0$ then there exists a formal canonical change of variables which transforms the Hamiltonian $H$ into

$$\tilde{H}(z, \bar{z}) = \sum_{k \geq 0} a_k (z \bar{z})^{k+2} + \sum_{k \geq 1} (b_k z^n + b_k^* \bar{z}^n)(z \bar{z})^{2k} + \sum_{k \geq 0} c_k (z \bar{z})^{2k} (z^{2n} + \bar{z}^{2n}),$$  \hspace{1cm} (17)

where $a_k, c_k \in \mathbb{R}$, $b_k \in \mathbb{C}$. Moreover, $a_0 = h_{22}$ and $c_0 = \frac{h_{2n,0}}{4h_{22}} - \frac{h_{n+1,1}^2}{2h_{22}}$.

**Proof.** The key idea behind the proof is based on studying terms of formal power series in the order of their $\delta$-degree. For a resonant monomial $(z^k \bar{z}^l, \, k = l \, \text{mod } n)$ we define its $\delta$-degree by

$$\delta(z^k \bar{z}^l) = \left| \frac{k-l}{n} \right| + \min \{k, l\} = \frac{1}{2}(k+l) - \frac{n-2}{2n} |k-l|.$$  \hspace{1cm} (18)

Grouping in (16) terms of the same $\delta$-degree we get

$$H(z, \bar{z}) = \sum_{m \geq 2} h_m(z, \bar{z}),$$

where $h_m(z, \bar{z})$ is a homogeneous resonant polynomial of $\delta$-degree $m$.

The term of the lowest $\delta$-degree is

$$h_2(z, \bar{z}) = h_{22}z^2\bar{z}^2 + h_{n+1,1}z^{n+1}\bar{z} + h_{1,n+1}z\bar{z}^{n+1} + h_{2n,0}z^{2n} + h_{0,2n}\bar{z}^{2n}.$$

Let $\chi$ be a resonant polynomial. After the substitution $(z, \bar{z}) \to \Phi_\chi^1(z, \bar{z})$ the Hamiltonian takes the form

$$\tilde{H} = H + L_\chi H + \sum_{k \geq 2} \frac{1}{k!} \overline{L_\chi^k} H,$$  \hspace{1cm} (19)
where
\[ L_\chi H = -i\{H, \chi\}. \]

The δ-degrees of monomials in Poisson brackets
\[ \{z^{k_1}z^{l_1}, z^{k_2}z^{l_2}\} = (k_1 l_2 - k_2 l_1)z^{k_1+k_2-1}z^{l_1+l_2-1} \]
satisfy the following relation:
\[ \delta(z^{k_1+k_2-1}z^{l_1+l_2-1}) \geq \delta(z^{k_1}z^{l_1}) + \delta(z^{k_2}z^{l_2}) - 1. \] (20)

Indeed,
\[
\begin{align*}
\delta(z^{k_1+k_2-1}z^{l_1+l_2-1}) &= \frac{1}{2}(k_1 + k_2 + l_1 + l_2 - 2) - \frac{n-2}{2n} \left| k_1 - l_1 + k_2 - l_2 \right| \\
&\geq \frac{1}{2}(k_1 + l_1) - \frac{n-2}{2n} |k_1 - l_1| + \frac{1}{2}(k_2 + l_2) - \frac{n-2}{2n} |k_2 - l_2| - 1.
\end{align*}
\]

Let \( \chi = \alpha z^n + \alpha^*\bar{z}^n \) with \( \alpha = -\frac{h_{n+1,1}}{2m h_{22}} \). It is not difficult to see that after the substitution \((z, \bar{z}) \mapsto \Phi_1^l(z, \bar{z})\) the coefficients of Hamiltonian \( \tilde{H} = \sum_{k+l \geq 4} \tilde{h}_{kl}z^k\bar{z}^l \) of δ-degree 2 are
\[
\begin{cases}
\tilde{h}_{22} = h_{22}, \\
\tilde{h}_{n+1,1} = h_{n+1,1} + 2i\alpha h_{22} = 0, \\
\tilde{h}_{2n,0} = h_{2n,0} + i\alpha h_{n+1,1} - n^2\alpha^2 h_{22} = h_{2n,0} - \frac{h_{n+1,1}^2}{4h_{22}}.
\end{cases}
\]

After rotation \( z \mapsto e^{-i\frac{\arg h_{2n,0}}{2n}} z \) the coefficient \( \tilde{h}_{2n,0} \mapsto c_0 = |\tilde{h}_{2n,0}| \). Then in new variables
\[ h_2(z, \bar{z}) = a_0 z^2\bar{z}^2 + c_0(z^{2n} + \bar{z}^{2n}), \] (21)
where \( a_0 = h_{22}, \ c_0 = \left| h_{2n,0} - \frac{h_{n+1,1}^2}{4h_{22}} \right| \). So the terms of δ-degree 2 has the declared form.

We proceed by induction. Let \( h_m(z, \bar{z}) \) has the declared form for \( m < p \), namely
\[
\begin{align*}
h_{2j+2}(z, \bar{z}) &= a_{2j}(z\bar{z})^{2j+2} + c_j(z\bar{z})^{2j}(z^{2n} + \bar{z}^{2n}), \quad \text{for } 0 \leq j \leq \left\lfloor \frac{p-3}{2} \right\rfloor, \quad (22) \\
h_{2j+1}(z, \bar{z}) &= a_{2j-1}(z\bar{z})^{2j+1} + (z\bar{z})^{2j}(b_j z^n + b_j^* \bar{z}^n), \quad \text{for } 1 \leq j \leq \left\lfloor \frac{p-2}{2} \right\rfloor. \quad (23)
\end{align*}
\]
For the sake of receiving such form for the term of $\delta$-degree $p$ we use several changes of variables consequently. Any homogeneous polynomial $\chi$ of $\delta$-degree $p - 1 \geq 2$ generates the change of variables $(z, \bar{z}) \mapsto \Phi^1_\chi(z, \bar{z})$. Formulae (19) and (20) imply

$$h_m = h_m \quad \text{for} \quad m \leq p - 1$$

and for $m = p$

$$h_p = h_p + L(\chi).$$

Here $L(\chi)$ is the homological operator:

$$L(\chi) = [L\chi h_2]_p,$$  \hspace{1cm} (24)

where $h_2$ is determined by (21) and $[\cdot]_p$ denotes terms of $\delta$-degree $p$.

Let $p$ be $\delta$-degree of resonant monomial $z^k \bar{z}^l$ and $j = \frac{k-l}{n}$. Let

$$Q_{p,0}(z, \bar{z}) = z^p \bar{z}^p$$

and for $1 \leq j \leq p$

$$Q_{p,j}(z, \bar{z}) = z^{p+j(n-1)} \bar{z}^{p-j}, \quad Q_{p,-j}(z, \bar{z}) = z^{p-j} \bar{z}^{p+j(n-1)}.$$

The homological operator (24) acts on monomial $Q_{p-1,j}(z, \bar{z})$ of $\delta$-degree $p-1$ with $j > 0$ by

$$L(Q_{p-1,j}) = 2ia_0 nj Q_{p,j} - 2inc_0(p - 1 - j)Q_{p,j+2}$$

and for $j = 0$:

$$L(Q_{p-1,0}) = -2inc_0(p - 1)(Q_{p,2} - Q_{p,-2}).$$

We denote coefficients of resonant polynomials $h_p(z, \bar{z})$ and $\tilde{h}_p(z, \bar{z})$ by $g_j$ and $\tilde{g}_j$ respectively:

$$h_p = g_0 Q_{p,0} + \sum_{1 \leq j \leq p} (g_j Q_{p,j} + g_j^* Q_{p,-j}), \quad \tilde{h}_p = \tilde{g}_0 Q_{p,0} + \sum_{1 \leq j \leq p} (\tilde{g}_j Q_{p,j} + \tilde{g}_j^* Q_{p,-j}).$$

The change of variables generated by resonant polynomial $\chi_k$ of $\delta$-degree $p - 1$

$$\chi_k = \alpha_k Q_{p-1,k} + \alpha_k^* Q_{p-1,-k}, \quad 1 \leq k \leq p - 1,$$

$$\chi_0 = \alpha_0 Q_{p-1,0}$$

(25)
transforms coefficients \( g_j \) by the following way:

\[
\begin{align*}
\tilde{g}_j &= g_j, \quad j \notin \{k, k+2\}, \\
\tilde{g}_k &= g_k + 2ia_0nk\alpha_k, \\
\tilde{g}_{k+2} &= g_{k+2} - 2inc_0(p - 1 - k)\alpha_k.
\end{align*}
\]

Below we describe the order of this transformations wich provides the declared form of the terms of \( \delta \)-degree \( p \).

Let \( k_0 = 1 \) for even \( p \) and \( k_0 = 2 \) for odd \( p \). Change of variables generated by \( \chi_k \) \((25)\) with \( \alpha_k = \frac{-g_k}{2ia_0kn} \) eliminates \( g_k \), changes \( g_{k+2} \) and does not affect any other terms of \( \delta \)-degree \( p \):

\[
\begin{align*}
\tilde{g}_j &= g_j, \quad j \notin \{k, k+2\}, \\
\tilde{g}_k &= 0, \\
\tilde{g}_{k+2} &= g_{k+2} + \frac{c_0(p-1-k)}{ka_0}g_k.
\end{align*}
\]

Starting with \( k = k_0 \), then \( k = k_0 + 2 \) etc. proceeding up to \( k = p - 1 \) one get corresponding terms \( \tilde{g}_k = 0 \) for all \( k \geq k_0 \) for which \( k - k_0 \) is even.

To eliminate terms with odd \( k - k_0 \) we use changes of variables generated by \( \chi_k \) \((25)\) with

\[
\begin{align*}
\alpha_k &= \frac{g_{k+2}}{2inc_0(p-k-1)}, \quad 1 \leq k \leq p - 2, \\
\alpha_0 &= \frac{3g_2}{2inc_0(p-1)}.
\end{align*}
\]

Each of these substitutions eliminates \( g_{k+2} \) (for \( k = 0 \) only image part of \( g_2 \) eliminates), changes \( g_k \) and does not affect any other terms of \( \delta \)-degree \( p \):

\[
\begin{align*}
\tilde{g}_j &= g_j, \quad j \notin \{k, k+2\}, \\
\tilde{g}_{k+2} &= 0, \quad (\text{for } k = 0 \quad \Im \tilde{g}_2 = 0), \\
\tilde{g}_k &= g_k + \frac{ka_0g_{k+2}}{c_0(p-k-1)}.
\end{align*}
\]

Starting from \( k = p - 2 \), then \( k = p - 4 \) etc. down to \( k = k_0 - 1 \) we get corresponding \( \tilde{g}_{k+2} = 0 \) and \( \tilde{g}_2 \in \mathbb{R} \).

**Remark about uniqueness.** The kernel of the homological operator is empty for odd \( p \) and one-dimensional for even \( p \). So only \([h^k_2]_{2k} \in \ker L \) which implies uniqueness of the coefficients \( a_k, c_k, |b_k|, \arg b_k \) (mod \( \pi \)) in formula \((17)\). But rotation by \( \pi/n \) change \( \arg b_k \mapsto \arg b_k + \pi \).

An alternative normal form is derived in the next proposition.
Proposition 6 If
\[ H(z, \bar{z}) = a_0 z^{2} \bar{z}^{2} + \sum_{k+l \geq 5, k,l \geq 0, \ k=l \ (\text{mod} \ n)} h_{kl} z^{k} \bar{z}^{l} \] (26)
is a formal series such that \( h_{kl} = h_{lk}^{*} \) and \( a_0 \neq 0 \), then there exists a formal tangent-to-identity canonical change of variables which transforms the Hamiltonian \( H \) into
\[ \tilde{H}(z, \bar{z}) = (z \bar{z})^{2} \sum_{k \geq 0} a_{k} (z \bar{z})^{k} + \sum_{k \geq 1} (c_{k} z^{nk} + c_{k}^{*} \bar{z}^{nk}), \] (27)
where \( a_{k} \in \mathbb{R}, c_{k} \in \mathbb{C} \).

Proof. Let \( \chi(z, \bar{z}) = \alpha z^{k} \bar{z}^{l} + \alpha^{*} \bar{z}^{k} z^{l}, k \neq l \). After the substitution \((z, \bar{z}) \rightarrow \Phi_{1}(z, \bar{z})\) the Hamiltonian takes the form (19), where
\[ L_{\chi}H = -i\{H, \chi\} = 2ia_{0} \alpha(k-l) z^{k+1} \bar{z}^{l+1} - 2ia_{0} \alpha^{*}(k-l) \bar{z}^{l+1} z^{k+1} + O_{k+l+3}, \]
where \( O_{k+l+3} \) denotes terms of degree \( k+l+3 \) and higher. Then \( \tilde{h}_{st} = h_{st} \) for \( 4 \leq s + t \leq k + l + 1 \) and for \( s + t = k + l + 2 \) if \( (s, t) \neq (k + 1, l + 1) \). And
\[ \tilde{h}_{k+1,l+1} = h_{k+1,l+1} + 2ia_{0} \alpha(k-l). \]
Let \( \alpha = -\frac{h_{k+1,l+1}}{2ia_{0}(k-l)} \). Then \( \tilde{h}_{k+1,l+1} = 0 \).

Repeating this substitutions one gets by induction \( \tilde{h}_{kl} = 0 \) with the exception of \( \tilde{h}_{kk} \) and \( \tilde{h}_{nk,0} = h_{0,nk}^{*} \).

\[ \blacksquare \]

3 Simplification of a formal Hamiltonian for families with small \( h_{11} \) and \( h_{n0} \)

Now we consider the three-parametric unfolding of Hamiltonian (16):
\[ H(z, \bar{z}; \varepsilon, \mu_{1}, \mu_{2}) = \varepsilon z \bar{z} + \sum_{k+l \geq 3, m \geq 0} h_{klm}(\mu_{1}, \mu_{2}) z^{k} \bar{z}^{l} \varepsilon^{m}, \] (28)
where \( h_{klm}(\mu_1, \mu_2) = h_{iklm}^*(\mu_1, \mu_2) \), \( h_{n00}(0,0) = 0 \), \( h_{220}(\mu_1, \mu_2) \neq 0 \),
\[
h_{n+1,1,0}^2(\mu_1, \mu_2) - 4h_{220}(\mu_1, \mu_2)h_{2n,0,0}(\mu_1, \mu_2) \neq 0.
\]

The purpose of the section is to prove that after some canonical formal transformations Hamiltonian \([28]\) takes the form \([34]\). An alternative form \([37]\) is also derived.

As in Section 2, \( \delta \)-degree is defined by \([18]\). Then
\[
H(z, \bar{z}; \varepsilon, \mu_1, \mu_2) = \sum_{p \geq 1, m \geq 0} h_{(p)m}(z, \bar{z}; \mu_1, \mu_2)\varepsilon^m,
\]
(29)

where \( h_{(p)m} \) is homogeneous resonant polynomial on \((z, \bar{z})\) of \( \delta \)-degree \( p \) with coefficients depending on \((\mu_1, \mu_2)\). The terms of \( \delta \)-degree 1 are
\[
\begin{align*}
h_{(1)1}(z, \bar{z}; \mu_1, \mu_2) &= z\bar{z} + h_{n01}(\mu_1, \mu_2)z^n + h_{n01}^*(\mu_1, \mu_2)\bar{z}^n, \\
h_{(1)m}(z, \bar{z}; \mu_1, \mu_2) &= h_{nm0}(\mu_1, \mu_2)z^n + h_{nm0}^*(\mu_1, \mu_2)\bar{z}^n, \quad m \neq 1.
\end{align*}
\]
(30)

The all transformations described below preserve the form of \( h_{(1)m} \) (i.e. \( h_{110} = 1 \) and \( h_{11m} = 0 \) for \( m \neq 1 \)) although coefficients \( h_{nm0} \) are changed.

As for individual map, normalisation of terms of \( \delta \)-degree \( p = 2 \) is different from \( p \geq 3 \). So it is convenient to consider the case of \( p = 2 \) separately.

**Lemma 7** Let \( H(z, \bar{z}; \varepsilon, \mu_1, \mu_2) \) be a formal series \([29]\) with \( h_{(1)m}(z, \bar{z}; \mu_1, \mu_2) \) as in \([30]\) and
\[
\begin{align*}
h_{(2)m}(z, \bar{z}; \mu_1, \mu_2) &= h_{2m}(\mu_1, \mu_2)z^2\bar{z}^2 + h_{n+1,1,m}(\mu_1, \mu_2)z^{n+1}\bar{z} \\
&+ h_{n+1,1,m}^*(\mu_1, \mu_2)\bar{z}^{n+1} + h_{2n,0,m}(\mu_1, \mu_2)z^{2n} + h_{2n,0,m}^*(\mu_1, \mu_2)\bar{z}^{2n}.
\end{align*}
\]

If \( h_{220}(\mu_1, \mu_2) \neq 0 \) and \( h_{n+1,1,0}^2(\mu_1, \mu_2) - 4h_{220}(\mu_1, \mu_2)h_{2n,0,0}(\mu_1, \mu_2) \neq 0 \) then there exists a formal canonical change of variables which transforms the Hamiltonian \( H \) into \( \tilde{H} = \sum_{p \geq 1, m \geq 0} \tilde{h}_{(p)m}\varepsilon^m \) with
\[
\begin{align*}
\tilde{h}_{n00}(\mu_1, \mu_2) &= h_{n00}(\mu_1, \mu_2) \exp \left(-\frac{i}{2} \arg \left( h_{2n,0,0}(\mu_1, \mu_2) - \frac{h_{n+1,1,0}^2(\mu_1, \mu_2)}{4h_{220}(\mu_1, \mu_2)} \right) \right), \\
\tilde{h}_{11m}(\mu_1, \mu_2) &= h_{11m}(\mu_1, \mu_2) = 0 \text{ for } m \neq 1, \quad \tilde{h}_{111}(\mu_1, \mu_2) = h_{111}(\mu_1, \mu_2) = 1, \\
\tilde{h}_{(2)m}(z, \bar{z}; \mu_1, \mu_2) &= \tilde{h}_{2m}(\mu_1, \mu_2)z^2\bar{z}^2 + \bar{h}_{2n,0,m}(\mu_1, \mu_2)(z^{2n} + \bar{z}^{2n}),
\end{align*}
\]
(31)

where \( \tilde{h}_{2n,0,m}(\mu_1, \mu_2) \in \mathbb{R} \) for all \( m \geq 0 \),
\[
\tilde{h}_{2n,0,0}(\mu_1, \mu_2) = \left| h_{2n,0,0}(\mu_1, \mu_2) - \frac{h_{n+1,1,0}^2(\mu_1, \mu_2)}{4h_{220}(\mu_1, \mu_2)} \right|.
\]
(32)
Proof. The lemma is proved by induction. For \( m = 0 \) we use change of variables \((z, \bar{z}) \mapsto \Phi^1(z, \bar{z})\) with \( \chi = \beta z^n + \beta^* \bar{z}^n \), \( \beta = -\frac{h_{n+1,1,0}}{2\nu h_{220}} \). From (19) we obtain coefficients of terms of \( \delta \)-degree 1:

\[
\begin{align*}
\tilde{h}_{n0m} &= h_{n0m} \quad \text{for } m = 0 \text{ and } m \geq 2, \\
\tilde{h}_{n01} &= h_{n01} + in\beta = h_{n01} - \frac{h_{n+1,1,0}}{2h_{220}}.
\end{align*}
\]

and of \( \delta \)-degree 2, \( m = 0 \):

\[
\begin{align*}
\tilde{h}_{220} &= h_{220} \quad \text{for } n \geq 4, \\
\tilde{h}_{220} &= h_{220} - in^2\beta^* h_{300} \quad \text{for } n = 3, \\
\tilde{h}_{n+1,1,0} &= h_{n+1,1,0} + 2in\beta h_{220} = 0, \\
\tilde{h}_{2n,0,0} &= h_{2n,0,0} + in\beta h_{n+1,1,0} - n^2\beta^2 h_{220} = h_{2n,0,0} - \frac{h^2_{n+1,1,0}}{4h_{220}} \neq 0.
\end{align*}
\]

Note that \( \tilde{h}_{220} \) is changed only for the case of resonance of the order \( n = 3 \) but \( \tilde{h}_{220}(0,0) = h_{220}(0,0) = h_{300}(0,0) = 0 \).

After rotation \( z \mapsto z \exp\left(-i\frac{2\pi h_{2n,0,0}}{2n}\right) \), the coefficient \( \tilde{h}_{2n,0,0} \mapsto |\tilde{h}_{2n,0,0}| \).

Thus \( h_{(2)0} \) has the form (32).

Let \( h_{(2)m} \) has the form (32) for all \( m \leq M - 1 \). In order to normalise the term \( h_{(2)M} \) we use successively two substitutions of the type \((z, \bar{z}) \mapsto \Phi^M(z, \bar{z})\). Then

\[
\widetilde{H} = H + \varepsilon^M L_{\chi} H + \sum_{k \geq 2} \frac{\varepsilon^k M}{k!} L_{\chi}^k H.
\]

First we assume \( \chi = \beta z^n + \beta^* \bar{z}^n \) and then \( \chi = \alpha z \bar{z} \). Choosing appropriate value of \( \beta \) we eliminate \( h_{n+1,1,M} \) and we get \( \Im h_{2n,0,M} = 0 \) by choosing \( \alpha \).

After the first substitution with \( \beta = -\frac{h_{n+1,1,M}}{2\nu h_{220}} \) the coefficients of terms of \( \delta \)-degree 1 are:

\[
\begin{align*}
\tilde{h}_{n0m} &= h_{n0m} \quad \text{for } m \neq M + 1, \\
\tilde{h}_{n,0,M+1} &= h_{n,0,M+1} + in\beta = h_{n,0,M+1} - \frac{h_{n+1,1,M}}{2h_{220}}.
\end{align*}
\]

and for terms of \( \delta \)-degree 2 for \( m \leq M - 1 \) the coefficients are not changed and for \( m = M \):

\[
\begin{align*}
\tilde{h}_{22M} &= h_{22M} \quad \text{for } n \geq 4, \\
\tilde{h}_{22M} &= h_{22M} - in^2\beta^* h_{300} \quad \text{for } n = 3, \\
\tilde{h}_{n+1,1,M} &= h_{n+1,1,M} + 2in\beta h_{220} = 0, \\
\tilde{h}_{2n,0,M} &= h_{2n,0,M} + in\beta h_{n+1,1,0} = h_{2n,0,M} - \frac{h_{n+1,1,M} h_{n+1,1,0}}{2h_{220}}.
\end{align*}
\]

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The change \((z, \bar{z}) \mapsto \Phi^M_\chi(z, \bar{z})\) with \(\chi = \alpha z \bar{z}\) leads to \(\tilde{h}_{klm} = h_{klm}\) for \(m \leq M - 1\) and

\[
\tilde{h}_{klM} = h_{klM} - i\alpha(k - l)h_{kl0}\quad \text{for} \quad k > l.
\]

So,

\[
\tilde{h}_{n,0,M} = h_{n,0,M} - in\alpha h_{n00}
\]

and \(\tilde{h}_{22M} = h_{22M}, \quad \tilde{h}_{n+1,1,M} = h_{n+1,1,M} = 0, \quad \tilde{h}_{2n,0,M} = h_{2n,0,M} - 2in\alpha h_{2n,0,0}\).

Choosing \(\alpha = \frac{3h_{2n,0,M}}{2ih_{2n,0,0}}\) we get \(\tilde{h}_{2n,0,M} \in \mathbb{R}\). \(\blacksquare\)

Now we introduce new parameters \((\nu_1, \nu_2)\) instead of \((\mu_1, \mu_2)\). Let

\[
\tilde{h}_{n00}(\mu_1, \mu_2) = \nu = \nu_1 + i\nu_2,
\]

where \(\tilde{h}_{n00}(\mu_1, \mu_2)\) is determined by the formula (31). Let \(\mu_1\) and \(\mu_2\) be expressed in terms of \((\nu_1, \nu_2)\) as power series. Then we get Hamiltonian \(H\) in the form of the series in five variables \((z, \bar{z}; \varepsilon, \nu_1, \nu_2)\).

Let \(\Upsilon = (\varepsilon, \nu_1, \nu_2), \quad m = (m_1, m_2, m_3), \quad |m| = m_1 + m_2 + m_3, \quad \Upsilon^m = \varepsilon^{m_1} \nu_1^{m_2} \nu_2^{m_3}\). After collecting terms of the same \(\delta\)-degree Hamiltonian \(H\) takes the form:

\[
H = \varepsilon z \bar{z} + \nu z^n + \nu^* \bar{z}^n + \sum_{m_1 \geq 1 \atop m_2, m_3 \geq 0} (b_{0m} z^n + b_{0m}^* \bar{z}^n) \Upsilon^m
\]

\[
+ az^2 \bar{z}^2 + c(z^{2n} + \bar{z}^{2n}) + \sum_{|m| \geq 1} (a_{0m} z^2 \bar{z}^2 + c_{0m}(z^{2n} + \bar{z}^{2n})) \Upsilon^m
\]

\[
+ \sum_{p \geq 3, |m| \geq 0} h_{(p)m}(z, \bar{z}) \Upsilon^m.
\]

Here the terms of \(\delta\)-degree 1 are in the form (30) and the terms of \(\delta\)-degree 2 are already in the form (32), \(a = h_{220}(0,0), \quad c = \frac{h_{2n,0,0}(0,0) - h_{2n,0,0}^2(0,0)}{4h_{220}(0,0)}\).

The next proposition completes transformation of Hamiltonian to the normal form.
Proposition 8 Let \((\cdot)_{m}\) be a formal series where \(h(\cdot)_{m}\) are real-valued resonant polynomials on \((z, \bar{z})\) of \(\delta\)-degree \(p\) and \(ac \neq 0\). There exists a formal canonical change of variables which transforms the Hamiltonian \(H\) into

\[
\widetilde{H}(z, \bar{z}; \Upsilon) = \varepsilon z\bar{z} + \nu z^n + \nu^* \bar{z}^n + a(z\bar{z})^2 + c(z^{2n} + \bar{z}^{2n}) + (z\bar{z})^2 \sum_{k+|m| \geq 1} a_{km}(z\bar{z})^{k} \Upsilon^{|m|} \tag{34}
\]

\[
\sum_{k+|m| \geq 1} (z\bar{z})^{2k} (b_{km} z^n + b_{km}^* \bar{z}^n) \Upsilon^{|m|} \sum_{k+|m| \geq 1} c_{km}(z\bar{z})^{2k} \Upsilon^{|m|},
\]

where \(a_{km}, c_{km} \in \mathbb{R}\), \(b_{km} \in \mathbb{C}\).

**Proof.** Let \(\delta\)-degree of monomial \(z^k \bar{z}^l\) is defined by (18) as before. Now we introduce a new \(\delta\)-degree for monomial on five variables:

\[
\delta(\Upsilon) (z^k \bar{z}^l) = \delta(z^k \bar{z}^l) + 2|\Upsilon| \tag{35}
\]

Then

\[
H(z, \bar{z}; \Upsilon) = \sum_{s \geq 2} h_s(z, \bar{z}; \Upsilon),
\]

where \(h_s\) is a homogeneous polynomial of \(\delta\)-degree \(s\). The main term of \(\delta\)-degree \(s = 2\) is already in the normal form:

\[
h_2(z, \bar{z}; \Upsilon) = a(z\bar{z})^2 + c(z^{2n} + \bar{z}^{2n}).
\]

We proceed by induction. Let \(S \geq 3\) and \(h_s(z, \bar{z}; \Upsilon)\) has the declared form for \(s \leq S - 1\).

The term of \(\delta\)-degree \(S\) has the form:

\[
h_S(z, \bar{z}; \Upsilon) = \sum_{0 \leq |M| \leq [\frac{S-1}{2}]} h_{(S-2|M|)}(z, \bar{z}) \Upsilon^M, \tag{36}
\]

where \(h_{(S-2|M|)}(z, \bar{z})\) is a polynomial of \(\delta\)-degree \(S-2|M|\). The terms of \(\delta\)-degree 1 and 2 are already in the normal form. So we consider \(S - 2|M| \geq 3\), i.e. \(0 \leq |M| \leq [\frac{S-3}{2}]\).

Let \(\chi\) be a homogeneous polynomial on \((z, \bar{z})\) of \(\delta\)-degree \(P - 1 = S - 2|M| - 1\). After the substitution \((z, \bar{z}) \mapsto \Phi^M_\chi(z, \bar{z})\) the Hamiltonian takes the form

\[
\widetilde{H} = H + \Upsilon^M L_\chi H + \sum_{j \geq 2} \frac{\Upsilon^{|M|}}{j!} L_\chi^j H.
\]
Then
\[ \tilde{h}_s(z, \bar{z}; \Upsilon) = h_s(z, \bar{z}; \Upsilon) \quad \text{for} \quad s \leq S - 1 \]
and
\[ \tilde{h}_S(z, \bar{z}; \Upsilon) = h_S(z, \bar{z}; \Upsilon) + \Upsilon^M L(\chi), \]
where \( L \) is the homological operator (24) from Section 2. So
\[ \tilde{h}_m = h_m \quad \text{for} \quad m \neq M \]
and
\[ \tilde{h}_M = h_M + M(\chi). \]

It was been shown in Section 2 that for odd \( P \) there exists such \( \chi \) that
\[ \tilde{h}_M = a_P M zP \bar{z}P + (z \bar{z})^P (b_P M z^n + b_P M \bar{z}^n) \]
and for even \( P \)
\[ \tilde{h}_M = a_P M zP \bar{z}P + c_P M (z \bar{z})^P (z^{2n} + \bar{z}^{2n}), \]
where \( a_P, c_P \in \mathbb{R}, b_P \in \mathbb{C} \). So, taking sequentially all sets of \((m_1, m_2, m_3)\) such that \(0 \leq m_1 + m_2 + m_3 \leq \left\lfloor \frac{S-3}{2} \right\rfloor\) we transform all terms of \( \delta \Upsilon \)-degree \( S \) to the declared form. \( \blacksquare \)

The following proposition establishes the normal form of Hamiltonian useful for study of bifurcations.

**Proposition 9** Let (33) be a formal series where \( a \neq 0 \) and \( h_m \) are real-valued resonant polynomials of \( \delta \)-degree \( p \). There exists a formal canonical change of variables which transforms the Hamiltonian \( H \) into
\[ \tilde{H}(z, \bar{z}; \Upsilon) = \varepsilon z \bar{z} + a(z \bar{z})^2 + (z \bar{z})^2 \sum_{k+|m| \geq 1} a_{km}(z \bar{z})^k \Upsilon^m + \nu z^n + \nu^* \bar{z}^n + \sum_{k \geq 1, k+m \geq 2, |m| \geq 0} (c_{km} z^{kn} + c_{km} \bar{z}^{kn}) \Upsilon^m \]  \hspace{1cm} (37)
where \( a_{km} \in \mathbb{R}, c_{km} \in \mathbb{C}, c_{20} = c \).

**Proof.** We use \( \delta \Upsilon \)-degree introduced by
\[ \delta \Upsilon(z^k \bar{z}^l \Upsilon^m) = k + l + 2|m| \]
and normalized order by order as in Proposition 8 but now we choose the complement to the image of the homological operator as in the proof of Proposition 6. \( \blacksquare \)
4 Bifurcations for the case of degeneracy in the leading resonant term

We discuss typical level sets of Hamiltonian \( [37] \) for \( 0 < |\varepsilon| + |\nu| \leq \varepsilon_0 \) with sufficiently small \( \varepsilon_0 \). In order to investigate bifurcations we keep lower order terms and skip those, which do not change the picture qualitatively:

\[
H(z, \bar{z}; \varepsilon, \tilde{\nu}) = \varepsilon z\bar{z} + a(z\bar{z})^2 + \tilde{\nu}z^n + \tilde{\nu}^*\bar{z}^n + c(z^{2n} + \bar{z}^{2n}),
\]

where \( \tilde{\nu} = \nu + \varepsilon c_{1100} + \cdots + \varepsilon^k c_{k100}, \quad k = \left\lfloor \frac{n}{2} \right\rfloor. \)

Applying the symplectic polar coordinates \( [5] \) and assuming \( \tilde{\nu} = \frac{1}{2}\gamma e^{i\beta} \) we get the model Hamiltonian in the form

\[
H(I, \varphi; \varepsilon, \gamma, \beta) = \varepsilon I + I^2 + \gamma I^{n/2} \cos(n\varphi + \beta) + I^n \cos 2n\varphi. \tag{38}
\]

Here the coefficients \( a \) and \( c \) are normalized to unity with the help of a scaling applied to the variable \( I \), the parameters \( \varepsilon \) and \( \gamma \), and the Hamiltonian function \( H \).

The case of \( n = 3 \) is different from the case of \( n \geq 4 \).

4.1 Typical level sets of the model Hamiltonian for \( n \geq 4 \)

Since

\[
\partial_t H = \varepsilon + 2I + O(\gamma I^{n/2-1}) + O(I^{n-1}), \tag{39}
\]

when \( \varepsilon > 0 \) for \( n \geq 4 \) Hamiltonian \( H \) has not any critical points near the origin and level sets are closed curves.

Typical level sets for \( \varepsilon < 0 \) are depending on value of parameters \( \varepsilon \) and \( \gamma \). We consider separately the case of \( \gamma \gg |\varepsilon|^{n/2} \) and \( \gamma = O(|\varepsilon|^{n/2}). \)

If \( |\varepsilon|^{n/2} = o(\gamma) \) then the last term in (38) can be omitted. Then critical points of Hamiltonian are located near \( I = -\varepsilon^2, \cos(n\varphi + \beta) = \pm 1 \). There is a chain of \( n \) islands on the distance of order \( |\varepsilon|^{1/2} \) from the origin. The size of islands is of order \( \gamma^{1/2}|\varepsilon|^{n/4} \).

For \( \gamma = O\left(|\varepsilon|^{n/2}\right) \) the last term in (38) is also essential. In order to study bifurcations we introduce the scaling:

\[
\gamma = 4\left(-\varepsilon^2\right)^{n/2}b, \quad I = -\varepsilon^2 + \left(-\varepsilon^2\right)^{n/2}J, \quad \psi = n\varphi, \quad H = \left(-\varepsilon^2\right)^n \tilde{H}. \tag{40}
\]
Figure 1: Bifurcation diagram on the complex plane of \( \nu = \gamma e^{i\beta} \) for \( \varepsilon < 0 \). The domains \( D_2 \) and \( D'_2 \) are separated by the vertical line segment.

After this scaling the Hamiltonian takes the form (scipping the constant term) \( \bar{H}(J, \psi) = \bar{H}_0(J, \psi) + O \left( \left( -\frac{\varepsilon}{2} \right)^{n/2-1} \right) \), where

\[
\bar{H}_0(J, \psi) = J^2 + 4b \cos(\psi + \beta) + \cos 2\psi.
\] (41)

Equilibrium points of \( \bar{H}_0(J, \psi) \) locate at the points for which \( J = 0 \) and

\[
b \sin(\psi + \beta) + \sin \psi \cos \psi = 0.
\] (42)

To figure out how many solutions has this equation we denote \( \sin \psi = x \) and \( \cos \psi = y \). Then equation (42) is equivalent to the system

\[
\begin{cases}
xb \cos \beta + yb \sin \beta + xy = 0 \\
x^2 + y^2 = 1
\end{cases}
\]

First equation corresponds to hyperbole (or 2 straight lines if \( \cos \beta = 0 \), or \( \sin \beta = 0 \), or \( b = 0 \)). One branch of the hyperbole (or one of stright lines) passes through the origin. So at least two solutions exist for arbitrary \( b \) and \( \beta \).

There are two more solutions if second branch of hyperbole (or second straight line) crosses the unit circle. Tangency condition for the unit circle and the second brunch of the hyperbole is

\[
(b \cos \beta)^{2/3} + (b \sin \beta)^{2/3} = 1.
\]
Figure 2: Critical level sets of the model Hamiltonian for \( \varepsilon < 0 \) depending on value of \( \nu \) indicated on bifurcation diagram on Fig. 1: (a) \( \nu \in D_2 \), (b) \( \nu \in D_2, \beta = 0 \), (c) \( \nu \) is in the boundary between \( D_2 \) and \( D_2' (\beta = \pm \pi/2) \), (d) \( \nu \) is in the boundary between \( D_2 \) and \( D_1 \).

On the complex plane of \( \gamma e^{i \beta} = \nu = \nu_1 + i \nu_2 \) corresponding line is given by astroid:

\[
\nu_1^{2/3} + \nu_2^{2/3} = 2^{4/3} \left( -\frac{\varepsilon}{2} \right)^{n/3}
\]

and is presented on Fig. 1. Typical critical level sets of Hamiltonian corresponding different values of parameter \( \nu \) (provided \( \varepsilon < 0 \)) are shown on Fig. 2. Fragments on the figure are repeated \( n \) times. On the boundary between \( D_2 \) and \( D_2' \) (vertical line segment) there is a chain of \( 2n \) islands (Fig. 2(c)). Outside a domain bounded by the astroid there is a chain of \( n \) islands.

4.2 Typical level sets of model Hamiltonian for \( n = 3 \)

For \( \varepsilon < 0 \) the case of \( n = 3 \) essentially differs from the case of \( n \geq 4 \) as the third term in (39) cannot be omitted.

The first three terms in (39) are of the same order if \( I \) and \( \varepsilon \) are of the order of \( \gamma^2 \). So we use the following scaling in (38)

\[
\varepsilon = a \gamma^2, \quad I = \gamma^2 J, \quad H = \gamma^4 \bar{H}
\]

and get \( \bar{H} = \bar{H}_0 + O(\gamma^2) \), where

\[
\bar{H}_0 = a J + J^{3/2} \cos(3 \varphi + \beta) + J^2.
\]
The critical points of $\bar{H}_0$ are located at the points for which $\cos(3\phi + \beta) = \sigma = \pm 1$, and
\[ a + \frac{3\sigma}{2} J^{1/2} + 2J = 0. \]
The last equation can be solved explicitly. There are no real solutions for $a > a_0$, where $a_0 = \frac{9}{32}$. If $a \in (0; a_0)$ there are two solutions for $\sigma = -1$ and no one for $\sigma = 1$. And if $a < 0$ there is one solution for $\sigma = -1$ and one for $\sigma = 1$.

The line
\[ \varepsilon = \frac{9}{32} \gamma^2. \]
on the plane $(\gamma, \varepsilon)$ bounded the domain in which level sets of the Hamiltonian $\bar{H}_0$ are closed curves (the domain $D_0$ on Fig. 3). The critical level sets of the Hamiltonian $\bar{H}_0$ are illustrated on Figure 4.
For $\varepsilon < 0$ the case of $n = 3$ is not different from $n \geq 4$ (see the analysis in the previous subsection). In particular, the boundary between $D_1$ and $D_2$ is given by (43). On the Fig. 3 we assume $\nu_1 = \nu = \gamma$, $\nu_2 = 0$ which correspond to $\beta = 0$ ($\nu = \gamma e^{i\beta}$). Then the boundary between $D_1$ and $D_2$ is given by

$$
\varepsilon = -2^{-1/3} \gamma^{2/3}.
$$

For $(\gamma, \varepsilon) \in D_2$ level sets of the model Hamiltonian are presented on Fig. 2.

## 5 Families with twist degeneracy

The theorem about normal forms for families with twist degeneracy can be found in [9]. For three-parametric families it takes the following form.

**Proposition 10** Let $\Upsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$, $m = (m_1, m_2, m_3)$, $\Upsilon^m = \varepsilon_1^{m_1} \varepsilon_2^{m_2} \varepsilon_3^{m_3}$. If

$$
H(z, \bar{z}; \Upsilon) = \varepsilon_1 z \bar{z} + \varepsilon_2 (z \bar{z})^2 + \varepsilon_3 (z \bar{z})^3 + \sum_{k+l>2, k,l,|m|\geq 0} h_{klm} z^k \bar{z}^l \Upsilon^m \quad (46)
$$

is a formal series such that $h_{klm} = h^{*}_{lkm}$, $h_{22m} = h_{33m} = 0$ and $h_{440} h_{000} \neq 0$, then there exists a formal tangent-to-identity canonical change of variables which transforms the Hamiltonian $H$ into

$$
\tilde{H}(z, \bar{z}; \Upsilon) = \varepsilon_1 z \bar{z} + \varepsilon_2 (z \bar{z})^2 + \varepsilon_3 (z \bar{z})^3 + (z \bar{z})^4 \sum_{k,|m|\geq 0} a_{km} (z \bar{z})^k \Upsilon^m
$$

$$
+ (z^n + \bar{z}^n) \sum_{k,|m|\geq 0} b_{km} (z \bar{z})^k \Upsilon^m,
$$

where $a_{km}, b_{km} \in \mathbb{R}$, $b_{00} = |h_{000}|$. Moreover

- if $3 \leq n \leq 8$
  \[ a_{km} = 0 \text{ for } k = n - 5 \text{ (mod } n) \text{ and } b_{km} = 0 \text{ for } k = n - 1 \text{ (mod } n) \]
- if $n \geq 8$
  \[ b_{km} = 0 \text{ for } k = 3 \text{ (mod } 4) \]
The coefficients $a_{km}$ and $b_{km}$ are defined uniquely.

In the symplectic polar coordinates (5) the model Hamiltonian is

$$H(I, \varphi; \Upsilon) = \varepsilon_1 I + \varepsilon_2 I^2 + \varepsilon_3 I^3 + I^4 + b_0 I^{n/2} \cos n\varphi.$$  \hspace{1cm} (47)

Its typical level sets depend on which of two last terms is the main. If the term $I^4$ is smaller then the last term ($n \leq 7$ or $n = 8$ and $|b_0| > 1$) then for $\Upsilon = 0$ the origin is not stable. If $n \geq 9$ or $n = 8$ and $|b_0| < 1$ then the origin is stable for $\Upsilon = 0$. Bifurcations in stable and unstable cases are briefly considered below.

### 5.1 Stable case

Let $n \geq 9$. For $0 \leq I \ll 1$, the Hamiltonian (47) can be considered as a small perturbation of $H_0 = \varepsilon_1 I + \varepsilon_2 I^2 + \varepsilon_3 I^3 + I^4$. The level sets of $H_0$ are circles for all values of the parameters. The Hamiltonian (47) does not posses this symmetry. The implicit function theorem implies that the last term can affect on the shape of level sets of $H$ only near zeroes of $\partial_I H_0$. The equation

$$\partial_I H_0 = \varepsilon_1 + 2\varepsilon_2 I + 3\varepsilon_3 I^2 + 4I^3 = 0$$  \hspace{1cm} (48)

has from 0 to 3 solutions depending on $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$.

Let $D_k$ be a domain in the parameter space such that equation (48) has $k$ roots, i.e. $H_0$ has $2kn$ stationary points. These domains on the unit sphere in the space $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ are shown on the Figure 5.
Figure 6: The typical critical level sets for the Hamiltonian (47) in the stable case if parameters \((\varepsilon_1, \varepsilon_2, \varepsilon_3) \in D_3\).

Figure 7: Bifurcation diagram for unstable case on the unit sphere in the space of parameters \((\varepsilon_1, \varepsilon_2, \varepsilon_3)\).

In \(D_0\) all level sets of \(H\) are closed curves around the origin. In \(D_1\) there is one chain of islands. In \(D_2\) typical level sets are similar to ones in two-parametric families [9].

In neighbourhood of the point \(\varepsilon_1 = 0, \varepsilon_2 = 0, \varepsilon_3 = -1\) there is a tiny domain \(D_3\). When parameters are in \(D_3\) the equation (48) have 3 roots. They correspond to three sets of hyperbolic and three sets of elliptic stationary points of the model Hamiltonian (47). Some of the possible corresponding critical level sets are shown on Figure 6. The fragment shown on the figure is repeated \(n\) times around the origin.
Figure 8: Some critical level sets for $n = 8$, $b_0 = 2$.

5.2 Unstable case

For $n \leq 7$ or $n = 8$ and $|b_0| > 1$ the bifurcation diagram on the unit sphere in the space of parameters $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ is shown in Figure 7. In $D_1$ and $D'_1$ typical level sets are the same as in one-parametric families. In $D_2$ and $D'_2$ typical level sets and their bifurcations are essentially the same as in two-parametric families. In a neighbourhood of the points $\varepsilon_1 = \varepsilon_2 = 0$, $\varepsilon_3 = \pm 1$ there are tiny domains with additional sets of stationary points. Some possible critical level sets for this case are shown on Figure 8.

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