Einstein-Podolsky-Rosen Steering in Quantum Phase Transition

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We investigate the Einstein-Podolsky-Rosen (EPR) steering and its criticality in quantum phase transition. It is found that the EPR steerability function of the ground state of XY spin chain exhibits nonanalytic feature in the vicinity of a quantum phase transition by showing that its derivative with respect to anisotropy parameter diverges at the critical point. We then verify the universality of the critical phenomena of the EPR steerability function in the system. We also use two-qubit EPR-steering inequality to explore the relation between EPR steering and quantum phase transition.

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Einstein et al. presented the so-called Einstein-Podolsky-Rosen (EPR) paradox to question the completeness of quantum mechanics based on locality and realism in 1935 [1]. Soon after, Schrödinger [2] introduced the term of entanglement to describe the correlations between two particles. Entanglement that a quantum state which cannot be separated, is the first type of nonlocal effect identified. In 1964, Bell [3] presented a method in the form of Bell inequality to describe quantum nonlocal property based on the assumptions of locality and realism. The violation of Bell inequality rules out local hidden variable theories to complete quantum mechanics as well as implies the existence of the so-called Bell nonlocality [4–7], that is the second type of nonlocal phenomenon arising from the EPR paradox. For Bell nonlocality, entanglement is necessary but not sufficient. Recently, EPR steering has been shown to be the third type of nonlocal property [8–14]. EPR steering, like entanglement, was originated from Schrödinger’s reply to the EPR paradox to reflect the inconsistency between quantum mechanics and local realism. For a pure entangled state held by two separated observers Alice and Bob, Bob’s qubit can be “steered” into different states although Alice has no access to the qubit. The EPR steering has been experimentally observed by violation of EPR-steering inequalities and the violation demonstrates the impossibility of using local hidden state theories to describe quantum mechanics [11]. Within the hierarchy of nonlocality, Bell nonlocality is the strongest, followed by EPR steering, while entanglement is the weakest [15, 16].

In spite of the essential role played by quantum nonlocality in quantum information and computation, Literatures [15, 18] have shown that both entanglement and Bell nonlocality can be used as a tool to reveal quantum phase transition (QPT) in many-body quantum systems. For entanglement, one needs to find out the exact form of the reduced density matrix of the ground state and utilize entanglement measure to signal QPTs. While for Bell nonlocality, which is the most stringent to test experimentally, the reduced density matrix of the ground state may not violate Bell inequality and hence one does not have Bell nonlocality, but the Bell function value still can be used to capture QPTs. EPR steering lies strictly intermediate between Bell nonlocality and entanglement [15, 16] and in principle, EPR steering should be easier observed than Bell nonlocality due to the asymmetry description between two observers Alice and Bob. It has been shown experimentally that some Bell-local states exhibit EPR steering and the EPR-steering inequality allowing for multi-setting measurements for two parties has been presented [11]. Although many efforts have been devoted to the investigations of EPR steering, the EPR-steering inequalities in the literatures are not strong enough for two-qubit systems. Therefore, it is not possible to observe the EPR steering for some states, especially for mixed states. Very recently, a criterion for EPR steering of two qubits has presented and it has been numerically proved to be a strong condition to witness steerability [19]. This offers an effective way to detect EPR steering for two qubits. An interesting question is whether EPR steering can be used to investigate the behavior of condensed matter systems.

In this work, we investigate the XY spin chain to establish the relation between EPR steering and QPTs. Utilizing the EPR-steering criterion proposed in our recent work [19], we find the EPR steerability function \( S \) as defined in Eq. (6) of reduced density matrix of the ground state. The function \( S \) indicates the existence of EPR steering when it is smaller than 0. We analyze the function \( S \) and its nonanalytic behavior at the transition point. Our results show that the quantum criticality in the XY model can be captured by the EPR steerability function of the ground state, and this enables conveniently testable quantum nonlocality to signal the QPT. We also explore the relation between EPR steering and QPT by utilizing two-qubit EPR-steering inequality.

Consider one-dimensional XY spin chain with Hamiltonian
given by
\[ H = \sum_{-M}^{M} [(1 + \gamma)\sigma_z^i\sigma_z^{i+1} + (1 - \gamma)\sigma_x^i\sigma_x^{i+1} + h\sigma_z^i], \]  
where \( M = (N - 1)/2 \) for the spin number \( N \) odd, \( \gamma \) is an anisotropy parameter, \( h \) is the strength of magnetic field, and \( \sigma_x^i, \sigma_y^i, \sigma_z^i \) are Pauli operators associated with local spin at site \( i \). The system undergoes a QPT at the critical point \( h_c \) which is differing XX-like phase \( \gamma = 0 \) from Ising-like phase \( 0 < \gamma \leq 1 \).

The two-spin reduced density matrix at sites \( i \) and \( j \) of the ground state of the spin chain is of the form,
\[ \rho_{ij} = \frac{1}{4} \left( I + \langle \sigma_z^i \rangle \sigma_z^j \otimes 1 + \langle \sigma_z^j \rangle 1 \otimes \sigma_z^i + \langle \sigma_x^i \sigma_x^j \rangle \sigma_z^i \otimes \sigma_z^j \right) \sum_{XY=x,y} \langle \sigma_Y^i \sigma_Y^j \rangle \sigma_X^i \otimes \sigma_X^j. \]  
The nonzero correlations \( \langle \sigma_z^i \rangle \) and \( \langle \sigma_x^i \sigma_x^j \rangle \) for the XY model are given by [20]
\[ \langle \sigma_z^i \rangle = -G_0, \langle \sigma_x^i \sigma_x^j \rangle = G_0^2 - G_{i,j}G_{j-i}, \]  
where
\[ G_r = \frac{1}{M} \sum_{m=1}^{M} (-\cos \frac{2\pi k}{N}) \cos(r\frac{2\pi k}{N})/\Lambda_k \]  
\[ + \frac{\gamma}{M} \sum_{m=1}^{M} \sin\frac{2\pi k}{N} \sin(r\frac{2\pi k}{N})/\Lambda_k, \]  
with \( \Lambda_k = \sqrt{(\gamma \sin \frac{2\pi k}{N})^2 + \cos^2 \frac{2\pi k}{N}} \). We need to explore quantum nonanalytic property in thermodynamic limit when \( N \to \infty \), then the sums in the expectation values are replaced by integrals,
\[ G_r = \frac{1}{\pi} \int_0^{\pi} d\phi (-\cos \phi) \cos(r\phi)/\Lambda_\phi \]  
\[ + \frac{\gamma}{\pi} \int_0^{\pi} d\phi \sin \phi \sin(r\phi)/\Lambda_\phi, \]  
and \( \Lambda_\phi = \sqrt{(\gamma \sin \phi)^2 + \cos^2 \phi} \).

The very recent proposed criterion for EPR steering [13] is obtained from the constraints on the eigenvalues of partial transpose of 2-qubit density operator \( \rho_{ij} \). Let \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \) be four eigenvalues of \( \rho_{ij}^T \) listed in ascending order, the experience condition for \( \rho_{ij} \) bearing EPR steering is
\[ S = \lambda_1 + \lambda_2 - (\lambda_1 - \lambda_2)^2 < 0. \]  
If the above eigenvalue combination \( S \) is negative, the EPR steering of \( \rho_{ij} \) can be certified. To demonstrate the relation between EPR steering and QPTs, we plot EPR steerability function \( S \) of \( \rho_{ij} \) and its derivative with respect to the field strength \( h \) when \( \gamma = 0.6 \) for different values of \( N \). Fig.1(a) shows the variance of \( S \) with increasing \( h \) when \( \gamma \) is fixed to be 0.6. It is obvious that \( S < S_c < \) and this tells us that \( \rho_{ij} \) exhibits EPR steering. We plot \( dS/dh \) versus \( h \) in Fig.1(b) for different spin size \( N \). The nonanalytic property of the EPR steerability function in the XY model is clearly shown at the critical point \( h_c = 1 \). Thus, \( S \) of the ground state is a witness of QPT.

We next explore the scaling behavior of \( S \) by the finite size scaling approach [21] to further understand the relation between EPR steering and QPT. In QPT, the critical feature can be characterized by a universal quantity whose behavior at criticality is entirely described by a critical exponent \( \nu \) in the form of \( \xi \sim |\lambda - \lambda_c|^{-\nu} \). To study quantum criticality in XY model, one needs to distinguish two universality classes depending on the anisotropy parameter \( \gamma \). For any value of \( \gamma \), quantum critical behavior occurs at the transition point \( h_c = 1 \). For \( \gamma = 0 \) the XY model belongs to XX universality class and critical exponent is \( \nu = 1 \), and for \( 0 < \gamma < 1 \) the model belongs to Ising universality class and critical exponent is \( \nu = 1/2 \).

Let us consider the derivative of steerability function with respect to \( h \) as a function \( h \) for different spin size \( N \) as shown in Fig.1(b). For finite spin size, the curves do not have divergence, but show obvious humps near critical point \( h_c = 1 \). With increasing spin size \( N \), the peak of each curve becomes sharper. Each curve approaches to its maximal value at pseudocritical point \( h_m \) which changes towards the critical point \( h_c \) when \( N \) increases. In the thermodynamic limit, when \( N \to \infty \), the singular behavior of \( dS/dh \) is clear in the vicinity of the quantum critical point, and it can be analyzed as,
\[ \frac{dS}{dh} \approx \kappa_1 |h - h_c| + \text{const}, \]  
where \( \kappa_1 = 0.2356 \). We then plot the value of \( dS/dh \) at \( h_m \) versus spin size \( N \) in Fig.2 which shows the relation,
\[ \frac{dS}{dh} \bigg|_{h_m} \approx \kappa_2 \ln N + \text{const}, \]  
with \( \kappa_2 = 0.2355 \). According to the scaling ansatz in the case of logarithmic divergence [21], the ratio \( \kappa_1/\kappa_2 \) gives the exponent \( \nu \) of \( S \). By our numerical results, \( \nu = 1 \) is approximately obtained for the XY model when \( \gamma = 0.6 \).
Therefore, our results show that the EPR steerability function of the ground state can signal the quantum criticality in the XY model.

The known EPR-steering inequalities can also be used to demonstrate the fact that EPR steering is able to capture quantum criticality in XY model. Here we consider the $N$-setting EPR-steering inequalities proposed in Ref. [11] which is based on the assumption that observer Alice’s measurement result is described by the random variable $A_k = \pm 1 \ (k = 1, \ldots, N)$ and Bob’s $k$th measurement is defined by Pauli observables $\bar{\sigma}_k^B$ along some axis $\hat{n}_k$, and the two qubit EPR-steering inequality is of the form

$$S_N = \frac{1}{N} \sum_{k=1}^{N} \langle A_k \bar{\sigma}_k^B \rangle \leq C_N,$$

where $C_N$ is the limit imposed by local hidden state theories. When $N = 2$, $C_2 = 1/\sqrt{2}$; when $N = 3$, $C_3 = 1/\sqrt{3}$; and when $N = 10$, $C_{10} = 0.5236$. It is obvious that the more the number of measurement settings, the stronger the two-qubit EPR-steering inequality is. We utilize 10-setting EPR-steering inequality to investigate the EPR steering of XY spin chain and plot quantum prediction of $S_{10}$ and its derivative with respect to $h$ in thermodynamic limit when $N \to \infty$ in Fig. 3. From Fig. 3(a), we find that for some values of $h$, the quantum predictions do not exceed $C_{10}$ and the 10-setting EPR-steering inequality cannot identify the EPR steering of the ground state in the vicinity of critical point. Even if no violation is found, the derivative of quantum prediction of $S_{10}$ still exhibits singular property at the critical point $h_c$, as shown in Fig. 3(b). In a word, $S_{10}$ is able to signal the nonanalytic features in the XY spin chain when quantum prediction of $S_{10}$ is smaller than $C_{10}$, or no EPR steering identified by the inequality. The result is similar to that for Bell’s inequality in QPT [18] that Bell function value can capture QPTs although Bell’s inequality is not violated. On the other hand, according to the EPR steering criterion $S$, it is found that the density matrix of the ground state has EPR steering. This suggests that the 10-setting EPR-steering inequalities are not strong enough to detect EPR steering in the XY model and so it is not a tight inequality for EPR steering. We expect more effective EPR-steering inequalities which can enable to detect EPR steering in QPT, and this will make it convenient to demonstrate experimentally the connection between EPR steering and QPT.

To summarize, we have investigated the relation between EPR steering and QPT in the anisotropic spin-1/2 XY model by using the 2-qubit EPR steering criterion. The EPR steerability function $S$ shows the existence of the EPR steering of the ground state of the model. As the spin number goes to infinity, the system undergoes QPT between the spin-fluid and the Ising-like phases, which can be captured by $S$. By studying the nonanalytic behavior of $S$ in the vicinity of transition point $h_c = 1$, we find that $S$ is a universal quantity to describe QPT in the XY model, and this makes it possible to demonstrate experimentally the connection between EPR steering and QPT. The result that EPR steerability function is able to signal QPT can be understood as follows. The function $S$ is the combination of eigenvalues of partial transpose of $\rho_{ij}$ which changes dramatically at the transition point, and the information of the critical change is obviously contained in the eigenvalues of $\rho_{ij}$ or its partial transpose. Thus, the EPR steerability function $S$ can reflect the critical feature in QPT. We believe that the result is applicable to other quantum many-body systems. We also discuss the relation between EPR steering and QPT by resorting to 10-setting EPR-steering inequality. Although the EPR-steering inequality is not violated near the critical point, quantum prediction of left-hand-side of the inequality still exhibits singular behavior. This suggests two particular points: (1) Quantum prediction of left-hand-side of EPR-steering inequalities plays a interesting role to capture QPT no matter whether the inequalities are violated or not, just like the case for Bell’s inequalities in QPT [18]; (2) The present two-qubit EPR-steering inequalities are not strong enough to detect EPR steering in the XY model. Our results from EPR steering criterion show that the reduced density matrix of the ground state of XY spin chain bears EPR steering and it indeed signals the QPT. We expect more effective EPR-steering inequalities which can enable to detect EPR steering in QPT.

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