Multiple scattering Sunyaev–Zeldovich signal – I. Lowest order effect

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Accepted 2013 September 30. Received 2013 September 27; in original form 2013 August 26

ABSTRACT

Future high-resolution, high-sensitivity Sunyaev–Zeldovich (SZ) observations of individual clusters will allow us to study the dynamical state of the intracluster medium (ICM). To exploit the full potential of this new observational window, it is crucial to understand the origin of different contributions to the total SZ signal. Here, we investigate the signature caused by multiple scatterings at lowest order of the electron temperature. Previous analytic discussions of this problem used the isotropic scattering approximation, which even for the simplest cluster geometries is rather rough. We take a step forward and consistently treat the anisotropy of the ambient radiation field caused by the first scattering. We show that the multiple scattering SZ signal directly probes line-of-sight anisotropies of the ICM, thereby delivering a new set of SZ observables which could be used for 3D cluster-profile reconstruction. The multiple scattering signal should furthermore correlate spatially with the cluster’s X-ray and SZ polarization signals, an effect that could allow enhancing the detectability of this contribution.

Key words: galaxies: clusters: general – cosmic background radiation – cosmology: theory.

1 INTRODUCTION

A number of high-resolution Sunyaev–Zeldovich (SZ) experiments, including ALMA, CARMA, CCAT, and MUSTANG, are underway or planned, promising a dramatic increase in sensitivities, spatial resolution and spectral coverage over the next few years. These experiments will allow us to directly probe the dynamical state of the intracluster medium (ICM) of individual clusters, opening a new window to understanding their cosmological formation and evolution. To realize the full potential of these observations, it is important to understand the SZ signal and its dependence on the cluster atmosphere with high precision.

Since the first considerations of the thermal SZ (thSZ) effect (Zeldovich & Sunyaev 1969) and kinetic SZ (kSZ) effect (Sunyaev & Zeldovich 1980), several related signatures have been discussed in the literature. For instance, because the electrons residing inside clusters are very hot \(kT_e \simeq 5–10\text{ keV}\) (with \(k\) denoting the Boltzmann constant and \(T_e\) the electron temperature) relativistic temperature corrections need to be included (Wright 1979; Rephaeli 1995; Challinor & Lasenby 1998; Itoh, Kohyama & Nozawa 1998; Sazonov & Sunyaev 1998). Also, higher order kSZ terms, both caused by the cluster’s bulk motion (Nozawa, Itoh & Kohyama 1998; Sazonov & Sunyaev 1998; Challinor & Lasenby 1999) and the motion of the observer (Chluba, Hütsi & Sunyaev 2005; Nozawa, Itoh & Kohyama 2005) arise. Furthermore, relativistic electrons can create a non-thSZ signature indicative of past active galactic nucleus activity and merger shocks (Enßlin & Kaiser 2000; Colafrancesco, Marchegiani & Palladino 2003). Additional effects are related to details of the cluster’s atmosphere. For example, internal bulk and turbulent gas motions introduce spatially varying SZ signatures (e.g. Chluba & Mannheim 2002; Diego, Mazzotta & Silk 2003; Nagai, Kravtsov & Kosowsky 2003). Line-of-sight variations of the electron temperature cause a non-trivial SZ morphology which depends on moments of the cluster temperature and velocity profiles (Chluba et al. 2013, CNSN hereafter). Extra pressure support from large-scale magnetic field deforms the electron distribution inside clusters (Koch, Jetzer & Puy 2003). And finally, shocks and substructures in merging clusters leave distinct signatures in the SZ flux (e.g. Komatsu et al. 2001; Kitayama et al. 2004; Colafrancesco, Marchegiani & Buonanno 2011; Korngut et al. 2011; Menanteau et al. 2012; Mróczkowski et al. 2012; Prokhorov & Colafrancesco 2012; Zemcov et al. 2012).

In this work, we consider the effect of multiple scattering on the SZ signal (Sunyaev & Zeldovich 1980; Molnar & Birkinshaw 1999; Sazonov & Sunyaev 1999; Dolgov et al. 2001; Itoh et al. 2001; Colafrancesco et al. 2003; Shimon & Rephaeli 2004). As we show here, this signature depends on inhomogeneities in the cluster’s electron and temperature distribution. It therefore provides a new set of SZ observables that are valuable for the reconstruction of 3D

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1 Atacama Large Millimeter/submillimeter Array.

2 Combined Array for Research in Millimeter-wave Astronomy.

3 Cornell Caltech Atacama Telescope.

4 Multiplexed Squid TES Array at Ninety GHz.

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doi:10.1093/mnras/stt1861
cluster profiles, potentially allowing geometric degeneracies to be broken. We restrict our analysis to the lowest order in the electron temperature. Higher order corrections will be discussed in Chluba & Dai (2013, CD13 hereafter).

The main new physical ingredient is that for the second scattering we include the full anisotropy of the singly scattered radiation field. In previous studies (e.g. Itoh et al. 2001; Colafrancesco et al. 2003; Shimon & Rephaeli 2004), this aspect was omitted; however, even for the simplest cluster geometries this approximation is very rough (see Section 3). Some discussion of the scattering-induced anisotropy in the radiation field can be found in connection with SZ polarization effects (e.g. Sunyaev & Zeldovich 1980; Sazonov & Sunyaev 1999; Lavaux et al. 2004; Shimon et al. 2006) and accretion flows (e.g. Sunyaev & Titarchuk 1985), but the SZ signature in the frequency dependence of the specific intensity that we consider here was previously not studied. We determine a new set of SZ observables related to the second scattering correction (Section 3.4) and demonstrate how the signal depends on the cluster’s geometry (see Fig. 5). We also find that the spatial pattern of the second scattering SZ signal can be used to enhance its detectability by cross-correlating with X-ray and future SZ polarization data (Section 4.1.3).

For those readers less interested in the details of the derivation, we note that the central result of the paper is an expression, equation (12), for the total change in equation (12), for the total change

\[ n_c(x, r, \hat{\gamma}) \]

\[ = \frac{3}{8\pi} \int d^2\gamma' \left(1 + \mu_{sc}\right) n(x, r, \hat{\gamma}') - n(x, r, \hat{\gamma}) \]

\[ + \frac{3}{16\pi} \frac{\theta_e}{\gamma_{sc}} \int d^2\gamma' \left(2 - 4\mu_{sc} - 6\mu_{sc}^2 + 4\mu_{sc}^3\right) n(x, r, \hat{\gamma}') \]

\[ + \frac{3}{16\pi} \frac{\theta_e}{\gamma_{sc}} \int \frac{d^2\gamma'}{\partial x^2} \frac{\partial}{\partial x} \frac{d^2\gamma'}{\partial x^2} \int d^2\gamma' \left(1 + \mu_{sc}^2\right)(1 - \mu_{sc}) n(x, r, \hat{\gamma}') \]

2.1 Compton collision term at the lowest order in \( kT_e/m_e c^2 \)

The collision term \( C[n] \) in equation (1) accounts for Compton scattering of photons of frequency \( x \) and direction \( \hat{\gamma} \) out of the beam and for scattering of photons of other directions and frequencies into the beam. To linear order in the electron–temperature parameter \( \theta_e \equiv kT_e/m_e c^2 \) (where \( m_e \) is the electron mass), the collision operator for anisotropic incoming radiation, as shown in Appendix A, is (Chluba, Khatri & Sunyaev 2012a).

\[ C[n] \approx \frac{3}{16\pi} \int d^2\gamma' \left(1 + \mu_{sc}\right) n(x, r, \hat{\gamma}') - n(x, r, \hat{\gamma}) \]

\[ + \frac{3}{16\pi} \frac{\theta_e}{\gamma_{sc}} \int d^2\gamma' \left(2 - 4\mu_{sc} - 6\mu_{sc}^2 + 4\mu_{sc}^3\right) n(x, r, \hat{\gamma}') \]

\[ + \frac{3}{16\pi} \frac{\theta_e}{\gamma_{sc}} \int \frac{d^2\gamma'}{\partial x^2} \frac{\partial}{\partial x} \frac{d^2\gamma'}{\partial x^2} \int d^2\gamma' \left(1 + \mu_{sc}^2\right)(1 - \mu_{sc}) n(x, r, \hat{\gamma}') \]

where \( \mu_{sc} = \hat{n} \cdot \hat{\gamma}' \). Stimulated scattering and electron recoil were neglected, because for SZ clusters \( T_e \ll T_c \).

The first integral in equation (2) describes the effect of Thomson scattering (no energy exchange or equivalently scattering of photons by resting electrons), leaving the photon spectrum of the incoming radiation field unaltered. If the incoming photon field is isotropic this term vanishes.

The second integral in equation (2) accounts for leading order temperature corrections to the total scattering cross-section. These again do not alter the shape of the photon spectrum (no energy exchange), and vanish for cold/resting electrons. For isotropic incoming photon field this term cancels [Klein-Nishina corrections, leading to contributions \( \propto \theta_e/\gamma_{sc} \) are neglected here], but for the radiation dipole through octupole a first-order temperature correction arises: although for resting electrons the Thomson cross-section only couples to the monopole and quadrupole anisotropy of the incoming photon field, once electrons are moving, due to light aberration (see e.g. Chluba 2011, for related discussion), also photons from the radiation dipole and octupole scatter into the line of sight at \( O(\theta_e^2) \). Higher order multipoles couple at order \( O(\theta_e^4) \).

The last integral in equation (2) takes the energy exchange between electrons and photons into account. It depends only on the radiation monopole through octupole. If the incoming radiation monopole through octupole have identical energy distribution, then their spectra are all altered in the same way by the scattering event. The difference is captured by an overall, frequency-independent scattering efficiency factor that depends on the multipole. This is the case relevant to the second scattering SZ correction, for which all multipoles of the singly scattered radiation field have a spectrum \( \propto Y_3(x) \), which represents the well-known thSZ spectrum (Zeldovich & Sunyaev 1969) defined in equation (7).

2.2 Rewriting the collision term using Legendre polynomials

The collision term can be further simplified by expanding the radiative field in terms of Legendre polynomials, \( P_l(x) \):

\[ n(x, r, \mu_{sc}) = \sum_l P_l(\mu_{sc}) n_l(x, r) \]

\[ n_l(x, r) = \frac{2l + 1}{4\pi} \int d^2\hat{\gamma}' P_l(\mu_{sc}) n(x, r, \hat{\gamma}') \]

Here, \( n(x, r, \mu_{sc}) \) denotes the azimuthally averaged occupation number of the photon field along the propagation direction \( \hat{\gamma} \).

\[ \text{See Morandi, Nagai & Cui (2013) for recent development based on future high-precision measurements of the thSZ effect using CCAT.} \]

\[ \text{The left-hand side (the Liouville operator) of the Boltzmann equation reads} \]

\[ c^{-1} \frac{dn}{dt} = c^{-1} \partial_x n_c + \hat{\gamma} \cdot \nabla n_c \] (c being the speed of light), thus “static” mean that changes in the medium are happening at time-scales much longer than the photon travel time through the IC. In this case, the solution for the photon field becomes stationary (no explicit dependence on time) and the time-derivative term can be neglected, such that \( c^{-1} \frac{dn}{dt} \approx \hat{\gamma} \cdot \nabla n_c \), where \( s \) parametrizes the distance along the photon path (cf. section 6.4 of Mihalas & Mihalas 1984). In the cluster frame, this condition is fulfilled even if the cluster is moving at constant velocity.
The Legendre coefficients can also be obtained from a spherical harmonic expansion of the radiation field with respect to a general system, \( n(x, r, \hat{\gamma}) = \sum_m Y_m(\hat{\gamma}) n_m(x, r) \), using the identity \( n_i(x, r) = \sum_m Y_m(\hat{\gamma}) n_m(x, r) \). Carrying out the integrals in equation (2), we find (compare equations C15 and C19 of Chluba et al. 2012a):

\[
C[n] = n_0(x, r) + \frac{1}{10} n_2(x, r) - n(x, r, \hat{\gamma})
\]

\[
- \theta_e \left[ \frac{2}{5} n_1(x, r) + \frac{3}{5} n_2(x, r) - \frac{6}{35} n_3(x, r) \right]
\]

\[
+ \frac{\theta_e}{x^2} \frac{\partial}{\partial x} x^4 \frac{\partial}{\partial x} \left[ n_0(x, r) - \frac{2}{5} n_1(x, r) + \frac{1}{10} n_2(x, r) \right]
\]

\[
- \frac{3}{70} n_3(x, r) \right].
\]

(4)

This expression captures all the relevant physics of the collission term, equation (2). Like equation (6), it can be directly obtained by computing the first and second moments of the frequency shift over the scattering kernel when accounting for the full angular dependence of the incoming photon field (see Chluba et al. 2012a, and CNSN). Only multipoles with \( \ell \leq 3 \) really matter for the scattering problem, greatly simplifying the next steps of the calculation. This expressions also clearly shows that only the azimuthally symmetric part of the incoming photon distribution with respect to \( \hat{\gamma} \) really matters, a property that due to symmetries of the scattering problem also holds at higher temperature (see CD13).

2.3 Born approximation for multiple scattering

In the optically thin limit (\( \tau \ll 1 \)), relevant for galaxy clusters, repeated Compton scattering only introduces a small correction to the radiation field. To solve equation (1), we can thus use a perturbative Ansatz (a ‘Born-series’), \( n(x, r, \hat{\gamma}) \approx n^{(0)} + \epsilon n^{(1)} + \epsilon^2 n^{(2)} + \ldots + \epsilon^5 n^{(5)} \), yielding the iteration scheme

\[
\frac{\partial n^{(k)}(x, r, \hat{\gamma})}{\partial \tau} \approx C\left[n^{(k-1)}(x, r, \hat{\gamma})\right].
\]

(5)

The Compton collision term is linear in \( n(x, r, \hat{\gamma}) \) since induced scattering terms cancel for \( T_e = T_c \). Without scattering nothing changes (\( \partial; n^{(0)}(x, r, \hat{\gamma}) = 0 \)), and the first scattering signal is given by

\[
\frac{\partial n^{(1)}(x, r, \hat{\gamma})}{\partial \tau} = C[n^{(0)}(x, r, \hat{\gamma})].
\]

For higher order scatterings, the local anisotropy of the radiation field becomes important, even if the unscattered radiation field was isotropic. Since the optical depth of typical clusters is \( \tau \lesssim 0.01 \), we can restrict ourselves to the first and second scattering effect.

3 MULTIPLE SCATTERING SZ CONTRIBUTIONS AT LOWEST ORDER IN ELECTRON TEMPERATURE

3.1 SZ signal after the first scattering

Assuming that the incoming, unscattered radiation field is given by the isotropic CMB blackbody, \( n^{(0)}(x, r, \hat{\gamma}) = n^{(0)}(x) \equiv 1/(e^{1} - 1) \), the only non-vanishing incoming-radiation multipole is the monopole (\( \ell = 0 \)). The collision term in this case simplifies to the well-known form (see Zeldovich & Sunyaev 1969),

\[
C_0[n] \approx \theta_e \frac{\partial}{\partial x} x^4 \frac{\partial}{\partial x} x^4 \frac{\partial}{\partial x} n(x) = \theta_e \left[ \frac{4x^4}{\partial x^4} + x \frac{\partial}{\partial x^2} \right] n(x),
\]

for an isotropic radiation field. The coefficients, \( \theta_e \) and \( \theta_c \), of the derivative terms are simply the first and second moments of the scattered photon’s relative frequency shift, \( \Delta_e \equiv (\nu' - \nu)/\nu \), over the scattering kernel (e.g. see Sazonov & Sunyaev 2000).

This then leads to the well-known thSZ effect (Zeldovich & Sunyaev 1969), with spectral shape,

\[
\Delta I_s/I_o \approx \frac{y}{(e^y - 1)^2} \left[ \frac{x e^y + 1}{x e^y - 1} - 4 \right] \equiv y x^2 Y_0(x).
\]

(7)

Here, the Compton y parameter, \( y(\hat{\gamma}) = \int (k T_e/m_c e^2) \sigma_T N_e d l' \), for a distant observer is computed along the line of sight through the ICM, and \( I_s = (2h/c^2)(kT_e/h^3) \approx 270 \text{ MJy sr}^{-1} \). We also assumed that the cluster is at rest in the CMB rest frame and internal motions are negligible (no kSZ).

3.2 Second scattering correction

3.2.1 Simple illustration for the scattering-induced anisotropy

While the derivation given in this section applies to general cluster geometry, to illustrate the physics of the multiple scattering problem, let us consider a spherical, isothermal cloud of free electrons at rest in the CMB rest frame with constant density, \( N_e \), and radius \( R \) as a simple example. At any location, \( r \), inside the cluster we can compute the singly scattered radiation field in different directions using equation (7), with isothermal \( y \) parameter, \( y(r, \hat{\gamma}) = \theta_e (r, \hat{\gamma}) \). Here, the optical depth, \( \tau(r, \hat{\gamma}) = \int \sigma_T N_e d l' \), is computed from \( r \) in the direction \( \hat{\gamma} \) through the ICM, where \( s \) parametrizes the length along photon path. At the cluster centre, the incident radiation field is fully isotropic, even in the limit of infinite number of scatterings; however, due to variations of the scattering optical depth in different directions, after one scattering the local photon field becomes anisotropic elsewhere (see Fig. 1). This scattering-induced anisotropy, generally being sourced by both electron temperature and density variations, has to be taken into account when calculating the second scattering correction to the SZ signal.
3.2.2 Explicit form of the second scattering SZ correction

We now give the explicit expression of the second scattering correction to the thSZ effect. After the first scattering, the correction to the radiation field at any position $r$ inside the cluster is given by

$$n^{(2)}(x, r, \hat{y}) \approx y(r, \hat{y})Y_0(x),$$  

where the spatial dependence is introduced only by the variation of the $y$ parameter in different directions $\hat{y}$, with the scatterer centred at position $r$. This expression shows that for the second scattering SZ signal, spectral and spatial parts of the radiation field separate. Since all multipoles of the singly scattered radiation have the same spectrum, they also have the same after the second scattering. When including higher order temperature corrections and variations of the electron temperature along different lines of sight, this separation becomes slightly more complicated and the spectra of each multipole differ after the second scattering (see CD13).

Inserting equation (8) into equation (4), and carrying out the line-of-sight average, we find the second scattering correction to the thSZ signal:

$$\Delta I^{(2), T}(x, \hat{y}) = \Delta I^{(2), T}_{I_0}(x, \hat{y}) + \Delta I^{(2), \sigma}_{I_0}(x, \hat{y}) + \Delta I^{(2), E}_{I_0}(x, \hat{y}),$$  

$$\Delta I^{(2), T}_{I_0}(x, \hat{y}) \approx x^3 Y_0(x) \left[ (\gamma y_0) + \frac{1}{10} (\gamma y_2) - (\gamma y(r, \hat{y})) \right],$$  

$$\Delta I^{(2), \sigma}_{I_0}(x, \hat{y}) \approx x^3 Y_0(x) \left[ - \frac{2}{5} (\hat{y} \gamma_{y_0}) - \frac{3}{10} (\hat{y} \gamma_{y_2}) + \frac{6}{35} (\hat{y} \gamma_{y_3}) \right],$$  

$$\Delta I^{(2), E}_{I_0}(x, \hat{y}) \approx x^3 Y_0^\prime(x) \left[ (\hat{y} \gamma_{y_0}) - \frac{2}{5} (\hat{y} \gamma_{y_2}) + \frac{1}{10} (\hat{y} \gamma_{y_3}) - \frac{3}{70} (\hat{y} \gamma_{y_5}) \right],$$  

$$Y_0^\prime(x) = x^{-2} \delta_x x^2 \delta_x x_0,$$

$$= 120 G(x) + 34 Y_0(x) + 12 x G(x) Y_0(x),$$  

$$+ 2x G(x) \frac{\left[ x \sinh(x) - 12 \cosh(x) \right]}{x \sinh(x)},$$  

$$(X) = \int X(r) \sigma_T N_e(r) \, dz,$$  

with $G(x) = xe^{x}/[e^x - 1]^2$. Here, $z$ parameterizes the photon path along the line-of-sight $\hat{y}$. We also introduced the Legendre coefficients, $y$, of $y(r, \hat{y})$, which are defined like in equation (3).

Equation (9) describes the full second scattering correction at lowest order in the electron temperature. The first two lines in equation (9b) provide corrections, $\Delta I^{(2), T}$ and $\Delta I^{(2), \sigma}$, to the amplitude of the usual thSZ distortion, equation (7). The second is smaller by a factor $\theta_e \ll 1$ compared with the first, and so we neglect it below.

The third provides the new frequency dependence, proportional to $x^3 Y_0^\prime(x)$, induced in the second scattering. The results indicate that only radiation multipoles up to $l = 3$ (the octupole) are required to describe the radiation field obtained in second scattering.

Since the energy-exchange correction, $\Delta I^{(2), E}$, has a frequency dependence that differs from that of the usual lowest order thSZ effect, measurement of its amplitude can provide information about anisotropies in the electron temperature and density profiles. As shown in Fig. 2, the function $Y_0^\prime(x)$ numerically reaches values $\approx 10$ times larger than the maximum of $Y_0(x)$. This makes this second-scattering correction roughly an order of magnitude larger than would be guessed from its $\sigma_e \approx 10^{-2}$ suppression relative to the lowest order SZ signal.

To calculate the observable emergent specific intensity, we only need to compute the line-of-sight averages of the $y$ parameter multipoles, weighted by the Thomson optical depth, $\exp(-\tau_{\text{th}})$, and the $y$ parameter itself, $\exp(-\tau_{\text{th}})$. These moments and

$$\langle y(r, \hat{y}) \rangle = \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \theta_x(r') \sigma_T N_e(r') \, dz' \right) \sigma_T N_e(r) \, dz'$$  

$$= \int_{0}^{\infty} \left( \int_{-\infty}^{\infty} \theta_x(r') \, dr' \right) \, dr''$$

can be obtained directly from cluster simulations. This greatly simplifies the computation, since spatial and spectral terms have been separated by our method. Below we give some simple examples assuming spherical symmetry.

3.3 Second scattering SZ correction in the isotropic scattering approximation (ISA)

In the ISA (e.g. Itoh et al. 2001; Colafrancesco et al. 2003; Shimon & Rephaeli 2004), the singly scattered photon field is assumed to be isotropic and the same as in the direction along the line of sight defined by $\hat{y}$. Explicitly this means $y(r, \hat{y}) \approx y(r, \hat{y})$, where $\hat{y}$ is the direction of the incoming photon. Therefore, none of the anisotropies are taken into account and even the variation of the local monopole is neglected, although generally $\int y(r, \hat{y}) \, dr' \neq \int y(r, \hat{y}) \, dr'$. At lowest order in $\theta_e$, with equation (9) we therefore have

$$\Delta I^{(2), \text{iso}}(x, \hat{y}) \approx \Delta I^{(2), E, \text{iso}}(x, \hat{y}) \approx I_0 x^3 Y_0^\prime(x) \frac{y(\hat{y})^2}{2},$$

where we used the identity $(\theta_x y(r, \hat{y})) \equiv y(\hat{y})^2/2$, which can be deduced using equation (10). This expression neglects geometric form factors. In addition, no Thomson correction, $\Delta I^{(2), T}(x, \hat{y})$, is obtained in the ISA, although it usually is comparable to the energy-exchange correction, $\Delta I^{(2), E}(x, \hat{y})$, in equation (9) (see Fig. 5).

3.4 Total SZ signal with second scattering corrections

The second scattering SZ signal gives rise to new observables that depend on the geometry of the cluster. The required line-of-sight moments scale like $\langle y \rangle \propto \tau(\hat{y}) y(\hat{y})/2$, $\langle y \hat{y}_x \rangle \propto y^2(\hat{y})/2$ and

$$\langle y(\hat{y}) \rangle \propto \tau(\hat{y}) y(\hat{y})/2.$$  

With equations (7) and (9), it is therefore
useful to rewrite the total SZ signal as

\[
\Delta I(x, \hat{\nu})/I_0 \approx x^3 y_0(x) y(\hat{\nu}) \left(1 + \frac{\tau(\hat{\nu})}{2} \right)
\]

\[
+ x^3 y_0^2(x) y(\hat{\nu}) \frac{\sigma^2(\hat{\nu})}{2}
\]

\[
y_\ell(\hat{\nu}) = \frac{2}{\sqrt{y(\hat{\nu})}} \left[\langle \theta_2 \rangle + \frac{\langle \gamma_z \rangle}{10} - \langle y(r, \hat{\nu}) \rangle\right]
\]

\[
y_\ell(\hat{\nu}) = \frac{2}{\sqrt{y(\hat{\nu})}} \left[\langle \theta_2 \rangle \gamma_0 + \frac{\langle \gamma_z \rangle y(\hat{\nu})}{10} - \frac{3}{70} \langle \theta_3 \gamma_0 \rangle\right].
\]  

The correction due to Thomson scattering is determined by \( y_\ell(\hat{\nu}) \), while the energy-exchange correction depends on \( y_{\ell 0}(\hat{\nu}) \). For the ISA we have \( y_\ell(\hat{\nu}) = 0 \) and \( y_{\ell 0}(\hat{\nu}) = 1 \).

Since the total SZ signal depends on two independent spectral functions, \( y_0(x) \) and \( y_\ell(x) \) (see Fig. 2), from the observational point of view it is possible to directly measure \( A(\hat{\nu}) = y(\hat{\nu})[1 + \gamma(\hat{\nu}) y_\ell(\hat{\nu})]/2 \) and \( B(\hat{\nu}) = y_{\ell 0}(\hat{\nu}) y^2(\hat{\nu})/2 \) for different lines of sight. These observables depend on both the temperature and density profiles of the electron distribution, and thus allow probing properties of the cluster atmosphere.

## 4 SPHERICALLY SYMMETRIC SYSTEMS

For spherically symmetric systems, the computations of the required \( y \)-parameter moments can be further simplified. Due to the symmetry of the problem, the \( y \) parameter in any direction \( \hat{\nu} \) around location \( r \) is given by

\[
y(r, \hat{\nu}) = y(r, \mu_r) \equiv \int_0^\infty \sigma_T \theta_2(r^*) N_e(r^*) \, d\ell
\]

with \( r^*(s, \mu_r) = \sqrt{r^2 + 2s \mu_r + s^2} \) and \( \mu_r = \hat{\nu} \cdot \hat{r} \), where \( \hat{r} \) defines the radial direction. To compute the Legendre coefficients of \( y(r, \hat{\nu}) \), we can use the addition theorem for spherical harmonics to simplify the calculation. This gives

\[
y_\ell(\hat{\nu}) = \frac{2 \ell + 1}{4 \pi} \int d^2 \hat{\nu} \, P_\ell(\hat{\nu} \cdot \hat{r}) \, y(r, \hat{\nu})
\]

\[
= \sum_m \int d^2 \hat{\nu} \, Y_{\ell m}(\hat{\nu}) \overline{Y}_{\ell m}(\hat{\nu}) \, y(r, \mu_r)
\]

\[
= \frac{2 \ell + 1}{2} \int d\mu_r \, P_\ell(\hat{\nu} \cdot \hat{r}) \, P_\ell(\mu_r) \, y(r, \mu_r)
\]

\[
= P_\ell(z/r) \, y^*_\ell(r) \equiv y_\ell(b, z).
\]  

where we introduced the impact parameter \( b \) of the line-of-sight \( \hat{\nu} \) from the cluster centre at the position \( z \), so that \( r = \sqrt{b^2 + z^2} \). The Legendre coefficient \( y^*_\ell(r) = y_\ell(b = 0, z) \) describes the simplest case, i.e., the line of sight through the centre, and the factor \( P_\ell(z/r) \) provides the geometric transformation to the case \( b \neq 0 \). From the practical point of view, this means that for spherically symmetric systems we can compute the required line-of-sight moments \( \langle y_\ell \rangle \) and \( \langle \theta_2 \gamma_0 \rangle \) in two independent steps, first giving \( y^*_\ell(r) \) and then averaging over \( z \). Since \( y_\ell(b = 0, z) = (-1)^\ell y_\ell(b, z) \), all odd moments vanish after averaging over the line of sight. For general cluster geometry this is not necessarily true but it greatly simplifies the calculation for (quasi) spherically systems.

Performing the required integrals, the parameters \( y_\ell(\hat{\nu}) \) and \( y_{\ell 0}(\hat{\nu}) \) in equation (12), as a function of impact parameter \( b \) become (see Appendix B for more details)

\[
y_\ell(b) = \frac{\sigma_T}{m_e \epsilon^2} \frac{2}{[y_{\ell 0}(\hat{\nu})]^2} \sum_{\ell = 0}^{\infty} \kappa_\ell \int_0^{R^2 - b^2} dz \, P_\ell(r) P_\ell \left(\frac{z}{r}\right)
\]

\[
\times \int_0^1 \! d\mu \, P_\ell(\mu) \, y_{\ell 0}(r \sqrt{1 - \mu^2})
\]

\[
y_{\ell 0}(b) = \frac{2 \sigma_T}{y_{\ell 0}(\hat{\nu})} \sum_{\ell = 0}^{\infty} \kappa_\ell \int_0^{R^2 - b^2} dz \, N_e(r) P_\ell \left(\frac{z}{r}\right)
\]

\[
- 2 \int_0^{R^2 - b^2} dz \, N_e(r) \, y(r, \frac{z}{r})
\]  

in terms of \( y_{\ell 0}(b) \), the parameter observed along a line of sight that passes an impact parameter \( b \) from the centre. In these expressions, \( r = \sqrt{z^2 + b^2} \), and \( \kappa_0 = 1 \) and \( \kappa_2 = 1/2 \). We also assumed that the cluster extends only to a maximal radius \( R \) and introduced the pressure profile \( P_\ell(r) = k T(r) N_e(r) \). To simplify the computation, the innermost integrals over \( d\mu \) can be tabulated as a function of \( r \).

### 4.1 Optical depth moments for illustrative cases

The second scattering SZ signal, equation (12), directly depends on the geometry of the cluster atmosphere. General cases can be studied using cluster simulations, but to illustrate the main effects we consider an isothermal, constant density sphere and an isothermal \( \beta \)-model (see Appendix C for definitions). Since for this case \( \langle y_\ell \rangle = \theta_2(\tau_\ell), \langle \theta_2 \gamma_0 \rangle = \theta_2(\tau_\ell) \) and \( \langle y(r, \hat{\nu}) \rangle = \theta_2(\tau_\ell)^2/2 \), we only have to compute line-of-sight moments of the optical depth field, \( \langle \tau_\ell \rangle \). Some details of the calculation for \( b = 0 \) are given in Appendix C, and the general case is obtained by averaging \( y_\ell(b, z) = P_\ell(z/r) y^*_\ell(r) \) along the line of sight.

The dependence of \( \langle \tau_\ell \rangle \) on impact parameter \( b \) for the two cases is illustrated in Figs 3 and 4. For the constant density sphere, the moments show significant variation with \( b \). They drop rapidly with \( \ell \), and in all cases the largest values of \( \langle \tau_\ell \rangle \) are found for the line of sight directly through the centre. This behaviour depends critically on the geometry and density profile of the cloud, and for an isothermal \( \beta \)-profile the situations is already very different. In contrast to the constant density sphere, the profiles \( \langle \tau_\ell \rangle \) are monotonic, with higher order terms decreasing more gradually. It is evident that the amplitude and spatial morphology of the second scattering SZ correction differs in the two cases. Consequently, the second scattering correction directly probes the geometry of the scattering medium, an effect that is not captured by the ISA.

#### 4.1.1 Total second scattering signal and comparison to the ISA

We now compute the total correction caused by second scattering terms. In Fig. 5, we show a comparison of the signal for the constant density, isothermal sphere and the isothermal \( \beta \)-model. The overall signal is very small, in both cases reaching no more than \( \lesssim 0.2 \) per cent of \( \Delta F^{(1)} \). However, as Fig. 5 illustrates, the second scattering signal directly depends on the geometry of the scattering electron distribution, an effect that is absent in the ISA.
Comparing the different contributions in equation (12) also shows that the dominant correction is caused in the Thomson limit, $\Delta I^{(2),T}$. For the isothermal sphere and impact parameter $b = 0$, we find $\Delta I^{(2),T}(x) \approx -0.06 \tau_c \Delta I^{(1)}(x)$. For some clusters one encounters central optical depth $\tau_c \approx 0.01$, so that this effect gives rise to a correction $\approx -0.06$ per cent relative to the singly scattered SZ signal. For an isothermal $\beta$-profile ($\beta = 2/3$), we obtain a slightly larger effect, giving $\Delta I^{(2),T}(x) \approx -0.15 \tau_c \Delta I^{(1)}(x)$ or a correction $\approx -0.15$ per cent relative to $\Delta I^{(1)}(x)$ for $\tau_c = 0.01$. Comparing the energy exchange correction with the ISA, we find $\Delta I^{(2),E,\text{iso}} \approx 1.12 \Delta I^{(2),E}$ and $\Delta I^{(2),E,\text{iso}} \approx 1.4 \Delta I^{(2),E}$ for the isothermal sphere and $\beta$-model, respectively. Hence, this part of the correction is overestimated when using the ISA. The total correction ($\Delta I^{(2),T} + \Delta I^{(2),E}$) for the isothermal $\beta$-model is $\approx 3-4$ times larger than obtained with the ISA.

4.1.2 Effect on the crossover frequency

Because the frequency dependence of $\Delta I^{(2),E}$ differs from $\Delta I^{(1)}$, the position of the crossover frequency (null of the SZ signal at $v \approx 217$ GHz or $\chi_c \approx 3.83$) is affected by the second scattering contribution. From equation (12), we find

$$\Delta \chi_c \approx 3.1 \gamma E \gamma \beta \approx 0.01$$

which for the isothermal $\beta$-model ($\gamma E \approx 0.71$ at $b = 0$) with total $\gamma$ parameter $y \approx 10^{-4}$ means $\Delta \chi_c \approx 2.2 \times 10^{-4}$. For the constant density sphere ($\gamma E \approx 0.89$ at $b = 0$), we find $\Delta \chi_c \approx 2.7 \times 10^{-4}$, while the ISA gives $\Delta \chi_c \approx 3.1 \times 10^{-4}$.

For comparison, at $T_e \lesssim 30$ keV the shift in the position of the crossover frequency caused by higher order temperature corrections is $\Delta \chi_c \approx 0.0421 T_e - 3.29 \times 10^{-3} T_e^2$ with $T_e = T_e/5$ keV. For $\theta_e = 0.01(=T_e \approx 1)$, this means $\Delta \chi_c \approx 0.0418$, which is roughly two orders of magnitudes larger. Similarly, the shift introduced by line-of-sight variations of the electron temperature is roughly given by $\Delta \chi_c \approx 0.0421 T_e [1 - 0.069 T_e/\omega^{(1)}]$, where $\omega^{(1)}$ parametrizes the line-of-sight temperature variance (see CSNN). For $\theta_e = 0.01$ and typical temperature-dispersion $\omega^{(1)} \approx 0.1$ this means $\Delta \chi_c \approx 3.9 \times 10^{-3}$, which is about one order of magnitude larger than the effect caused by multiple scattering. Note that at lowest order in $T_e$, equation (16) agrees with the expressions given by Dolgov et al. (2001) when setting $\gamma E \equiv 1$, i.e., ignoring scattering-induced anisotropies of the radiation.

4.1.3 Spatial dependence of the total second scattering signal

One interesting aspects of the second scattering correction is that it introduces both a new spatial and spectral dependence. At lowest order in temperature, the singly scattered signal has a spectrum $\propto \chi E Y_0(x)$ and spatial variation $y \approx \theta \phi$. In the ISA, the second scattering contribution [cf. equation (11)] has spectrum $\propto \chi E Y_0(x)$ and spatial dependence $\sim \theta^2/2$. On the other hand, when including the anisotropy of the singly scattered radiation field, two new contributions arise, each with individual spatial and spectral dependence [see equation (9)]. The Thomson correction terms scale like $\kappa_T \approx \gamma T (\theta \phi^2/2) \approx \theta \phi \langle \tau_0 \rangle + \langle \tau_2 \rangle / 10 - \tau^2 / 2$ with a spectrum $\propto \chi E Y_0(x)$, while the energy-exchange term has...
spectrum \( \propto x^3 Y_0(x) \) and spatial dependence \( \kappa_0 = y_0 (\theta_0^2 r^2/2) \approx \theta_0^2 [r_0 + (2\ell)/10]/10 \) at \( b = 0 \) by symmetry.

In Fig. 6, we illustrate the spatial dependence of different terms for an isothermal \( \beta \)-profile with \( \beta = 2/3 \). The second scattering correction is most important at \( b \approx 0 \), exhibiting an \( N_0^2 \)-scaling like X-rays. Thus, the detectability of the second scattering signal may be enhanced by cross-correlating with X-ray maps, as the X-ray emission also scales with \( N_0^2 \). Similarly, part of the SZ polarization signature (e.g. Sunyaev & Zeldovich 1980; Sazonov & Sunyaev 1999; Lavaux et al. 2004; Shimin et al. 2006) caused by the second scattering should correlate spatially with the SZ intensity signal.

### 4.2 Angle-averaged SZ signal

For spherical systems with finite radius \( R \), we can also compute the angle-averaged SZ signal, assuming that the cluster is filling a circular beam with radius \( R \). It is well known, that the average single scattering signal depends on the volume integral over the electron pressure profile, \( P_e(r) = k T_e(r) N_e(r) \)

\[
\langle y \rangle = \frac{1}{\pi R^2} \int b \, db \, d\varphi \int \theta_e \left( \sqrt{z^2 + b^2} \right) N_e \left( \sqrt{z^2 + b^2} \right) \sigma_T \, dz
\]

\[
= \frac{1}{\pi R^2} \frac{\sigma_T}{m_e c^2} \int P_e(r) \, dV.
\]

For the second scattering signal, geometric form factors appear, which depend on the cluster atmosphere. This can be seen from

\[
\langle y \rangle = \frac{1}{\pi R^2} \int b \, db \, d\varphi \int P_e(z/r) \gamma_e \gamma_r \sigma_T \, dz
\]

\[
= \frac{\sigma_T}{\pi R^2} \int P_e(\mu) \gamma_e \gamma_r N_e(r) \, d\varphi \, d\mu \, r^2 \, dr
\]

\[
= \delta_0 \frac{\sigma_T}{\pi R^2} \left( \frac{1}{4\pi} \int \theta_e \left( r^+ \right) N_e \left( r^+ \right) \sigma_T \, ds \, d\varphi \prime \, d\mu \prime \right) N_e(r) \, dV
\]

\[
= \frac{\delta_{\rho 0}}{\pi R^2} \frac{\sigma_T}{m_e c^2} \int \left( \int \frac{P_e \left( r + r' \right)}{4\pi r'^{2}} \, dV' \right) N_e(r) \, dV
\]

\[
= \frac{\delta_{\rho 0}}{\pi R^2} \frac{\sigma_T}{m_e c^2} \int \int P_e \left( r' \right) N_e(r) \, dV' \, dV
\]

where we used \( r^+ = \sqrt{r^2 + 2r y + y^2} = |r + r'| \). In the ISA, we have \( \langle y \rangle_{\text{ISA}} = 0 \) and \( \langle (\theta y) \rangle_{\text{ISA}} = \frac{y^2}{2} \). Similar to equation (12), it thus makes sense to define the scattering form factors

\[
\bar{y}_T = \frac{\langle y \rangle_{\text{ISA}}}{\langle y \rangle_{\text{ISA}}} \quad \text{and} \quad \bar{y}_e = \frac{\langle (\theta y) \rangle_{\text{ISA}}}{\langle y \rangle_{\text{ISA}}^2}.
\]

With equations (7) and (9), the total average SZ signal thus reads

\[
\Delta T(x)/T_0 \approx \frac{x^3 Y_0(x)}{y^3/2} + x^3 Y_0(x) \bar{y}_T \bar{y}_e \frac{y^2}{2}.
\]

The above expressions show that the average SZ signal just depends on the overall geometry of the electron pressure and density distribution, but local anisotropies, that were visible for a resolved cluster, no longer contribute.

The geometric form factors have to be computed on a case-by-case basis. For constant pressure and electron density profiles, we find \( \bar{y} = (2/3) y_0 \) and \( \langle (\theta y) \rangle = (1/4) y_0^2 = (9/16) y_0^2 \). This implies \( \bar{y}_T + 1 = \bar{y}_e = 1.25 \). Assuming that the pressure and electron density profiles scale like 1/r, we similarly find \( \bar{y} = y_0/R \) and \( \langle (\theta y) \rangle = \ln(2)/R^2 \). This implies \( \bar{y}_T + 1 = \bar{y}_e = 1.25 \approx 1.4 \). This shows that depending on the geometry of the cluster profile the ISA scattering approximation only provides a rough approximation for the second scattering signal. In particular it underestimate the average SZ signal related to \( Y_0^2(x) \). Although the ISA overestimated the second scattering SZ signal for small impact parameters, at larger radii, which contribute strongly to the averaged signal, it underestimates the signal, explaining this net total effect.

### 5 CONCLUSION

We analysed the effect of multiple scattering on the SZ signal demonstrating how this small correction (order \( \approx 0.1 \) per cent) depends on the geometry of the ICM. Previous studies applied the ISA which does not capture the effects discussed here, giving very different results for the second scattering correction (see Fig. 5).

The difference has two main sources: (i) due to the anisotropy of the singly scattered radiation field, the second scattering already introduces a modification in the Thomson limit which is absent in the ISA; (ii) the effective monopole of the radiation field along the line of sight differs from the one used in ISA.

Although, very small in comparison to the singly scattered signal, our analysis shows that the second scattering SZ signal delivers a new set of observables (see Section 3.4) which in the future might be useful for 3D cluster-profile reconstruction, allowing us to break geometric degeneracies. Since at lowest order in \( k T_e/m_e c^2 \), the second scattering correction for isothermal clusters roughly scales as \( r^2 \propto N_0^2 \), it introduces an additional spatial dependence of the SZ signal, similar to the one of X-rays.

Cross-correlating high-resolution X-ray maps with future SZ maps might thus allow enhancing the detectability of the second scattering
signal. Similarly, combining with high-resolution SZ polarization measurements should allow extracting this contribution, potentially opening a path for more detailed studies of the clumpiness of the ICM.

The method developed here can in principle be directly applied to simulated clusters. This can be used to give more realistic estimates for the significance of the second scattering signal. The computation is greatly simplified with our approach, since spatial averages over the ICM can be performed independently of the spectral integrals, accelerating the calculation.

For the examples discussed here, we assumed spherical symmetry. In this case, all odd moments of the optical depth field vanish; however, in general this is an oversimplification. More realistic cluster atmospheres might exhibit larger differences in the second scattering signal and could thus provide a way to directly probe the asphericity and clumpiness of the ICM along the line of sight. From the observational point of view this is very intriguing, but given the low amplitude of the effect this certainly remains challenging. For our estimates we also assumed isothermality. Variations of the electron temperature cause additional anisotropies in the singly scattered radiation which might even exceed those introduced by optical depth effects. The second scattering SZ contribution therefore could be used to directly probe the cluster’s temperature field. These aspects should, however, be addressed using realistic cluster simulations.

We mention that a similar analysis can be carried out for SZ polarization effects. In particular, the scattering-induced quadrupole anisotropy of the radiation field sources an SZ polarization effect, which is directly related to line-of-sight average of the Compton y parameter’s quadrupole components. In this case, the projections perpendicular to the line of sight are most important (i.e., moments like $\langle y_2 x_2 \rangle$), defining both amplitude and direction of the polarization signature. Our separation of spatial and spectral dependence should allow simplifying the calculation of the SZ polarization effect. In particular, the scattering-induced quadrupole anisotropy of the radiation field is expected to be small compared to the asphericity and clumpiness of the ICM along the line of sight. From the observational point of view this is very intriguing, but given the low amplitude of the effect this certainly remains challenging. For our estimates we also assumed isothermality. Variations of the electron temperature cause additional anisotropies in the singly scattered radiation which might even exceed those introduced by optical depth effects. The second scattering SZ contribution therefore could be used to directly probe the cluster’s temperature field. These aspects should, however, be addressed using realistic cluster simulations.

We plan to explore these possibilities in the future.

ACKNOWLEDGEMENTS

JC thanks Donghui Jeong for stimulating discussions on the problem. The authors are also grateful to Anthony Challinor, Daisuke Nagai and Rashid Sunyaev for useful comments on the manuscript. This work was supported by the grants DoE SC-0008108 and NASA NNX12AE86G.

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APPENDIX A: LOWEST ORDER FOKKER-PLANCK EXPANSION OF THE COLLISION TERM

For hot electrons, $T_e \gg T_0$, recoil and stimulated effects do not contribute and the Boltzmann collision term takes the simple form (section 2.1 of CNSN)

$$\mathcal{C}[n] = \int \frac{f_*(p)}{N_e \sigma_T} \, d\Omega \frac{d^3 \hat{n} (x', r, \hat{r}) - d^3 \hat{n} (x, r, \hat{r})}{2} \, d^3 \hat{p} \, d^3 \hat{p} \, d^3 p. \quad (A1)$$

Here, $f_*(p) = N_e e^{-\sqrt{\pi \eta}/(4\pi \eta^2)} \int [\delta_{1} / \delta_{2}]$ is the relativistic Maxwell–Boltzmann distribution with $\eta = p/m_e c$. In the required, limit the Compton scattering cross-section reads (e.g. see...
with Lorentz factor $\gamma = \sqrt{1 + \beta^2} = \frac{\eta}{\beta}$, $\kappa = 1 - \beta\mu$ and $\alpha_{sc} = 1 - \mu_{sc}$, where $\mu_{sc} = \beta \cdot \gamma$ is the cosine of the scattering angle between the incoming and outgoing photon. Furthermore, $\mu = \beta \cdot \gamma$ and $\mu' = \beta \cdot \gamma'$ are the direction cosines of the angle between the scattering electron and the incoming and outgoing photon, respectively. The Thomson scattering cross-section is denoted by $\sigma_T \approx 6.65 \times 10^{-25}$ cm$^2$.

Here, we are only interested in contributions at lowest order of the electron temperature, $k T_e / m_e c^2 \ll 1$. We thus perform a Fokker-Planck expansion of the Boltzmann collision term in the small frequency shift $\Delta_\nu = (\nu' - \nu)/\nu$, replacing $n(x', r, \gamma') \approx n(x, r, \gamma') + x \delta_\nu n(x, r, \gamma') \Delta_\nu + \frac{1}{2} x^2 \delta^2_\nu n(x, r, \gamma') \Delta_\nu^2$ in equation (A1). This gives

$$C[n] \approx \langle n(x, r, \gamma') \rangle - n(x, r, \gamma') + D_\nu \langle n(x, r, \gamma') \Delta_\nu \rangle$$

$$+ \frac{1}{2} D_\nu^2 \langle n(x, r, \gamma') \Delta_\nu^2 \rangle,$$

(A3)

with the differential operator $D^m = x^m \delta^m_\nu$. Higher order derivative terms are of higher order in temperature and thus were neglected.

We now compute the moments of the frequency shift, $\langle n(x, r, \gamma') \Delta_\nu \rangle$. Neglecting the tiny recoil effect we have

$$\Delta_\nu \approx \frac{\beta(\mu' - \mu)}{1 - \beta \mu'} \approx \eta(\mu' - \mu) \left[1 + \eta \mu' \right] + O(\eta^3).$$

(A4)

Similarly, up to second order of $\eta$, the scattering cross-section takes the form

$$\frac{d\sigma}{d\Omega} \approx \frac{3 \sigma_T}{16 \pi} \left[1 + \mu_{sc}^2\right] + \frac{3 \sigma_T}{16 \pi} \left[2 \mu' \left(1 - \mu_{sc} + 2 \mu_{sc}^2\right)ight]$$

$$- \mu \left(1 + 2 \mu_{sc} - \mu_{sc}^2\right) \eta - \frac{3 \sigma_T}{16 \pi} \left[1 - 2 \mu_{sc} + 3 \mu_{sc}^2\right]$$

$$- \mu^2(1 - \mu_{sc})^2 - 2 \mu^2 (2 - 4 \mu_{sc} + 5 \mu_{sc}^2)$$

$$+ 4 \mu_{sc}(2 - \mu_{sc}) \mu \mu' \eta^2 + O(\eta^3).$$

(A5)

The integrals over electron momenta $p^2 dp = (m_e c)^3 \eta^2 d\eta$ give

$$\int \frac{4\pi N_e}{N_e} f_\nu(\eta) \eta^2 d\eta = 1,$$

$$\int \frac{4\pi N_e}{N_e} f_\nu(\eta) \eta d\eta \approx 2 \sqrt{2} e / \pi + O\left(\theta_e / \sqrt{2}\right)$$

$$\int \frac{4\pi N_e}{N_e} f_\nu(\eta) \eta^4 d\eta \approx 3 \theta_e + O\left(\theta_e^2\right).$$

(A6)

We will see that the second integral does not contribute at the end, since the integral over $\mu$ cancels the corresponding terms by symmetry. We now consider the integrals over $\psi$, defining the azimuthal angle of the scattering electron with respect to $\gamma$. Only $\mu' = \mu_{sc} \mu + \cos(\psi - \psi_{sc})(1 - \mu^2) / (1 - \mu_{sc}^2) / (1 - \mu_{sc}^2) \mu' \mu$ depends on $\psi$. At order $\theta_e$, we encounter integrals of the type

$$\int \mu^d d\psi = 2\pi,$$

$$\int \mu' d\psi = 2\pi \mu \mu_{sc}$$

$$\int \mu'' d\psi = \pi \left[1 - \mu_{sc}^2 - \mu^2 \left(1 - 3 \mu_{sc}^2\right)\right].$$

(A7)

As these expressions show, there is no remaining dependence of the collision term on the azimuthal angle of the scattered photon, $\psi_{sc}$. We can thus also carry out the integral over $\phi_{sc}$ for the incoming photon and replace $n(x, r, \gamma') \rightarrow \frac{1}{2\pi} \int n(x, r, \gamma') d\phi_{sc}$. This gives

$$\int \frac{d\sigma}{d\Omega} \frac{d\phi_{sc}}{4\pi \sigma_T} \approx \frac{3}{16} \left[1 + \mu_{sc}^2\right] - \frac{3}{16} \left[1 + \mu_{sc}^2 - 4 \mu_{sc}^4\right] \eta \mu$$

$$+ \frac{3}{16} \left[1 - 2 \mu_{sc} + 4 \mu_{sc}^3 - 5 \mu_{sc}^4\right]$$

$$- \mu^2 (1 - 2 \mu_{sc} + 6 \mu_{sc}^2 + 8 \mu_{sc}^4 - 15 \mu_{sc}^6) \eta^2,$$

$$\int \frac{d\sigma}{d\Omega} \Delta_\nu \frac{d\phi_{sc}}{4\pi \sigma_T} \approx \frac{3}{32} (1 - \mu_{sc}) \left[1 + \mu_{sc}^2\right] \eta \mu$$

$$+ \frac{3}{32} (1 - \mu_{sc}^2) \left[3 - 2 \mu_{sc} + 5 \mu_{sc}^2\right] \eta^2$$

$$- \frac{3}{32} (1 - \mu_{sc}) \left[1 + 3 \mu_{sc} + 2 \mu_{sc}^2 + 15 \mu_{sc}^4\right] \eta^2 \mu^2$$

$$\times \left[1 + \mu_{sc} + (1 - 3 \mu_{sc}^2) \eta^2\right].$$

(A8)

The integrals over $d\mu$ yield $\int d\mu = 2$, $\int \mu \mu'd\mu = 0$ and $\int \mu^2 d\mu = 2/3$, so that finally one is left with

$$\langle n(x, r, \gamma') \rangle \approx \frac{3}{16 \pi} \int \frac{1}{2} \delta_\nu \left(1 + \mu_{sc}^2\right) n(x, r, \gamma')$$

$$+ \frac{3}{16 \pi} \theta_e \int \frac{1}{2} \delta_\nu \left[2 - 4 \mu_{sc} - 6 \mu_{sc}^2 + 4 \mu_{sc}^4\right] n(x, r, \gamma')$$

$$\langle n(x, r, \gamma') \Delta_\nu \rangle \approx \frac{3}{4 \pi} \theta_e \int \frac{1}{2} \delta_\nu \left(1 + \mu_{sc}^2\right) (1 - \mu_{sc}) n(x, r, \gamma')$$

$$\langle n(x, r, \gamma') \Delta_\nu^2 \rangle \approx \frac{1}{2} \langle n(x, r, \gamma') \Delta_\nu \rangle.$$
where $s$ parametrizes the integration along the direction $\hat{r}'$ with $\mu_s = \hat{r} \cdot \hat{r}'$ (see Fig. B1 for illustration) and $P_s(r) = kT_e(r)N_e(r)$ is the electron pressure profile. Defining $r^*(s, \mu_s) = \sqrt{r^2 + s^2 + 2\mu_s r s}$, equation (B1) can be further expressed by
\begin{align}
\gamma_e^*(r) &= \frac{2\ell + 1}{2} \int_0^1 P_s(\mu_s) \frac{\sigma_T}{m_e c^2} \int_0^\infty P_e \left( r^*(s, \mu_s) \right) \\
& \quad + P_e \left( r^*(s, -\mu_s) \right) \right) \, ds \, d\mu_s \\
& = \frac{2\ell + 1}{2} \int_0^1 P_s(\mu_s) \left( \frac{\sigma_T}{m_e c^2} \int_0^\infty P_e \left( r^*(s, \mu_s) \right) \right) \, ds \, d\mu_s,
\end{align}
where we used that $\ell$ is even. Geometrically, the inner integral in parenthesis is simply the observed $y$ parameter, $y_{\text{obs}}(b)$, as a function the effective impact parameter $b = r(1 - \mu_s^2)^{1/2}$ from the cluster centre. Therefore, for even $\ell$ all $\gamma_e^*(r)$ can be directly computed using $y_{\text{obs}}(b)$.

**(APPENDIX C: $\langle \tau_e^* \rangle$ FOR SIMPLE CASES)**

**C1 Evaluation of $\langle \tau_e^* \rangle$ for isothermal, constant density sphere**

For illustration, we compute $\langle \tau_e^* \rangle = \langle \tau_e \rangle(b = 0)$ for an isothermal, constant density sphere of electrons and line of sight directly through the centre of the sphere (impact parameter $b = 0$). In this case, the optical depth is azimuthally symmetric around the $z$-axis which we align with $\hat{r}$. The optical depth in the direction $\hat{r}'$ at location $z_0$ ($z_0 = 0$ at the centre of the sphere) is then given by $\tau(z_0, \hat{r}') = N_e \sigma_T r(z_0, \hat{r}')$, where the length through the scattering medium from the boundary of the sphere to $z_0$ is determined by $r^2 = R^2 - z_0^2 - 2Rz_0\mu_R \hat{r} \cdot \hat{r}'$. Here, $R$ is the radius of the sphere, and $\mu_R$ is the direction cosine towards the point on the surface of the sphere that is hit by the chosen line of sight (see Fig. B1). We need the Legendre coefficients, $r_5(z_0) = \frac{2\ell + 1}{2} \int_0^1 r(z_0, \hat{r}') P_5(\mu_s) d\mu_s$, where $\mu_s$ is the direction cosine for $\hat{r}'$ relative to a coordinate system which is centred at $z_0$. Therefore, we have to express $\mu_R$ in terms of $\mu_s$. Using $R_{LR} = r\mu_R + z_0$, this eventually yields $r(\mu_s) = [R^2 - z_0^2(1 - \mu_s^2)]^{1/2} - z_0 \mu_s$. For $\ell = 0$ and $\ell = 2$, we find
\begin{align}
\tau_5^*(z_0) &= \frac{R}{2} \left[ 1 + \left( \frac{1}{z_0} - z_0 \right) \arcsinh \left( \frac{z_0}{\sqrt{1 - z_0^2}} \right) \right] \\
\tau_7^*(z_0) &= -\frac{3R}{16} \left[ 1 - 3 \left( \frac{1}{z_0} - z_0 \right) \arcsinh \left( \frac{z_0}{\sqrt{1 - z_0^2}} \right) \right],
\end{align}
with $z_0 = z_0/R$. To obtain $\langle \tau_e^* \rangle$, we have to calculate $\langle \tau_e^* \rangle = \int_0^R r_5(z_0) \, dz_0 = R^2 \int_0^1 r_5^*(z_0) \, dz_0$. The integrals can be carried out numerically. Defining $r_e = 2RN_e\sigma_T$ we obtain
\begin{align}
\langle \tau_e^* \rangle = (2\ell + 1)I_e \frac{\tau_e^*}{2},
\end{align}
with $I_0 \approx 0.867, I_2 = 2.92 \times 10^{-2}, I_4 = -1.80 \times 10^{-3}$ and $I_6 = 3.43 \times 10^{-4}$. The non-vanishing average optical depth multipoles drop rapidly with $\ell$. This behaviour depends critically on the geometry and density profile of the scattering cloud, and for an isothermal $\beta$-profile the situations is already different (see Section C2). For general impact parameter see Fig. 3.

**C2 Evaluation of $\langle \tau_e^* \rangle$ for isothermal $\beta$-profile**

As a second simple example, we consider an isothermal $\beta$-profile (Cavaliere & Fusco-Femiano 1978): $N_e(r) = N_{e,0}(1 + (r/r_c)^2)^{-3\beta/2}$ and $T_e \equiv \text{const.}$
\begin{align}
N_e(r) = N_{e,0}(1 + (r/r_c)^2)^{-3\beta/2} \text{ and } T_e \equiv \text{const.}
\end{align}

Here, $N_{e,0} \approx 10^{-3} \text{cm}^{-3}$ is the typical central number density of free electrons, $r_c \approx 100 \text{kpc}$ is the typical core radius of clusters, and $\beta \approx 2/3$ (e.g. see Reese et al. 2002).

The line-of-sight optical depth at impact parameter $b$ is
\begin{align}
\tau(b) &= \frac{N_{e,0}\sigma_T r_c}{[1 + (b/r_c)^2]^{9\beta/2 - 1/2}} \frac{\sqrt{\pi\Gamma(3\beta/2 - 1/2)}}{\Gamma(3\beta/2)},
\end{align}
which implies $\tau(b) \approx 3.142 N_{e,0}\sigma_T r_c [1 + (b/r_c)^2]^{-1/2}$ for $\beta = 2/3$. With the same arguments and definitions as above we find
\begin{align}
\tau_e^*(z_0) &= \frac{2\ell + 1}{2} \int_0^1 \int_0^\infty N_e \left( \sqrt{z_0^2 + s^2 + 2\mu_s z_0} \right) \\
& \quad \times \sigma_T P_5(\mu_s) \, ds \, d\mu_s.
\end{align}

For general value of $\beta$, the required integrals are best carried out numerically. For $\beta = 2/3$, $\ell = 0$ and $\ell = 2$, we have
\begin{align}
\tau_5^*(z_0)/N_{e,0}\sigma_T r_c &= \frac{\pi \arctan z_0}{\zeta_0} \\
\tau_7^*(z_0)/N_{e,0}\sigma_T r_c &= \frac{3\pi(3 + \zeta_0^2) \arctan z_0 - 3\zeta_0}{\zeta_0^3}
\end{align}
with $\zeta_0 = z_0/r_c$. Averaging over $\zeta_0$ (impact parameter $b = 0$), we find $\langle \tau_5^* \rangle = \ln 2 \times (\tau_e^2/2) \approx 0.693(\tau_e^2/2)$, $\langle \tau_7^* \rangle = 5(3 - \ln 16)/8 \times (\tau_e^2/2) \approx 0.142(\tau_e^2/2)$, $\langle \tau_2^* \rangle = 9(48 \ln 4 - 65)/256 \times (\tau_e^2/2) \approx 0.0542(\tau_e^2/2)$, where $\tau = (b = 0)$ is the optical depth through the centre of the cluster. Fig. 4 shows the non-vanishing coefficients for $\ell \leq 6$, $\beta = 2/3$ and general impact parameter.

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