INTEGRAL OBJECTS AND DELIGNE'S CATEGORY $\text{Rep}(S_t)$.  

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Abstract. We give negative answers to certain questions on abelian semisimple $\otimes$-categories raised by Kahn and Weibel in connection with the preprint of Kahn “On the multiplicities of a motive”. For the most interesting examples we used Deligne’s category $\text{Rep}(S_t, F)$ of representations of the “symmetric group $S_t$ with $t$ not an integer” with $F$ any algebraically closed field of characteristic zero.

Introduction  

In the preprint [Kah06] (now published as Parts I and IV of [Kah09]) Bruno Kahn, studying the rationality of certain zeta functions, semisimple rigid $k$-linear $\otimes$-category $\mathcal{A}$ and defined the notion of object (geometrically) of integral type. This is quickly reviewed, together with some basic terminology, in sections 1 and 2. His work raised some questions of general interest for $\otimes$-categories. In this note we give examples showing that:

1. there are abelian semisimple geometrically integral categories with non Schur-finite objects (see sections 3 and 4) answering a question of Weibel to Kahn ([Kah06] Remark 2.3 and [Kah09] Remark 2.3)),

2. the $\otimes$-product (even a tensor square) of geometrically integral objects in an abelian semisimple category need not be of integral type (see section 7), answering a question of Kahn to the author.

In section 3 we give a “toy example” based on the free rigid tensor category on one object. Then, in section 4, we review the main tool which is Deligne’s category $\text{Rep}(S_t, F)$ of representations of the “symmetric group $S_t$ with $t$ not an integer”, with $F$ any algebraically closed field of characteristic zero. This is a $\otimes$-category, “new” in some sense ([Del07], Introduction), with interesting properties and its study is of independent interest. In Proposition 6.1.1 we give two proofs of the fact that the canonical generator of Rep($S_t, F$) is not Schur-finite for $t \in F \setminus \mathbb{N}$ (that is, for such a $t$, the length of the tensor powers of its canonical $\otimes$-generator grows more than exponentially, see [Del02] Proposition 0.5 (i))). The first proof uses the universal property of Rep($S_t, F$), studied in section 5, while the second proof is more direct. This gives also an algebraic proof that for any such $t$ the category Rep($S_t, F$) is not $\otimes$-equivalent to a category of super-representation of a super group scheme (cf. [Del02] Théorème 0.6 and [Del07] Introduction), because there are not even $\otimes$-functors from it to the $\otimes$-category of supervector spaces (see Corollary 5.3.6).

We also incidentally note that Deligne’s category $\mathcal{A} = \text{Rep}(S_t, F)$ disproves the claims [Hai02, 4.4, 4.5], which was not previously noticed. As Professor Deligne kindly remarked in a letter to the author (22 mars 2007) also those in his [Del96], as well as [DMOS82, (1.27)], disprove the claims.
1. Terminology

1.1. \(*\)-categories and \(*\)-functors. In what follows \( \mathcal{A} \) denotes a \(*\)-category, by this I mean that \( \mathcal{A} \) is an additive pseudoabelian, i.e. Cauchy complete (see [Bor94 6.5] and [AK02 I.1]), (strict) monoidal category (as defined in [ML98 VII, p.162], see also [ML98 XI.3, Th. 1]) where the monoidal structure \(- \otimes - : \mathcal{A} \times \mathcal{A} \to \mathcal{A} \) is actually a commutative multifunctor. We denote by \( \mathbb{1} \) its tensor unit, so the (necessarily) commutative endomorphism monoid \( \mathcal{A}(\mathbb{1}) \) is actually a commutative unitary ring. If \( R \) is a commutative unitary ring, we say that \( \mathcal{A} \) is \( R \)-linear if \( \mathcal{A}(\mathbb{1}) \) is an \( R \)-algebra, in this case all \( \mathcal{A}(X,Y) \) are \( R \)-modules and \(- \otimes - : \mathcal{A} \times \mathcal{A} \to \mathcal{A} \) is in fact \( R \)-bilinear. Note that \( \mathcal{A} \) is always \( \mathcal{A}(\mathbb{1}) \)-linear.

A \(*\)-functor (called also strong monoidal functor by Mac Lane) \( F : \mathcal{A} \to \mathcal{B} \) between two such \(*\)-categories is a functor equipped with families of natural transformations expressing compatibility conditions such as \( F(\mathbb{1}_\mathcal{A}) \cong \mathbb{1}_\mathcal{B} \) and \( F(X \otimes Y) \cong F(X) \otimes F(Y) \) for any \( X,Y \). We refer to [ML98 XI.2], [DMOS82], or [SR72] for precise definitions and the needed commutative diagrams.

1.2. Symmetry. We assume, without explicit mention, that each \(*\)-category \( \mathcal{A} \) used in this paper is symmetric, that is we are also given a family of natural isomorphisms \( \tau_{X,Y} : X \otimes Y \to Y \otimes X \) such that \( \tau^{-1}_{X,Y} = \tau_{Y,X} \) for each \( X,Y \in \mathcal{A} \). Each \(*\)-functor is therefore also required to respect the symmetry (cf. [ML98 XI.2(10)]).

1.3. Rigidity. We say that such an \( \mathcal{A} \) is rigid if for each object \( X \) there are morphisms \( \varepsilon_X : X^\vee \otimes X \to \mathbb{1} \) (evaluation), \( \eta_X : \mathbb{1} \to X \otimes X^\vee \) (coevaluation) satisfying the following “triangular identities”

\[
\text{Id}_X = (X = \mathbb{1} \otimes X \xrightarrow{\eta_X \otimes X} X \otimes X^\vee \xrightarrow{X \otimes \varepsilon_X} X \otimes \mathbb{1} = X)
\]

and

\[
\text{Id}_{X^\vee} = (X^\vee = X^\vee \otimes \mathbb{1} \xrightarrow{X^\vee \otimes \eta_X} X^\vee \otimes X \xrightarrow{X^\vee \otimes \varepsilon_X} \mathbb{1} \otimes X^\vee = X^\vee).
\]

The object \( X^\vee \) is then called the (rigid) dual of \( X \) (see for example [AK02 II.6] or [JS93 pag. 72] for more details).

1.4. Trace and Euler characteristic. Rigid \(*\)-categories are (canonically) traced (as defined in [JSV96]) by \( \text{Tr} = \text{Tr}^{\mathcal{X}_{\mathcal{A}}} : \mathcal{A}(X) \to \mathcal{A}(\mathbb{1}) \), \( \text{Tr}(f) := \varepsilon_X \circ \tau_{X^\vee,X} \circ (f \otimes X^\vee) \circ \eta_X \). The Euler characteristic, \( \chi(X) \), of an object \( X \) (a.k.a. its rigid dimension) is then the categorical trace of its identity, that is \( \chi(X) := \text{Tr}(\text{Id}_X) \in \mathcal{A}(\mathbb{1}) \).

More generally the trace \( \text{Tr}^{\mathcal{A},B}_{f,U}(f) : A \to B \) of a morphism \( f : A \otimes U \to B \otimes U \) is defined as

\[
\text{Tr}^{\mathcal{A},B}_{f,U}(f) = (B \otimes (\varepsilon_U \circ \tau_{U,U^\vee}) \circ (f \otimes U^\vee) \circ (A \otimes \eta_U)).
\]

1.4.1. \(*\)-functor and traces. Rigidity is obviously preserved by any \(*\)-functor \( F : \mathcal{A} \to \mathcal{B} \), hence the canonical traces are also preserved by such a functor. In particular \( \chi(F(X)) = F(\chi(X)) \). If moreover \( F \) is \( \mathcal{A}(\mathbb{1}) \)-linear then \( \chi(X) \text{Id}_{\mathbb{1} \mathcal{S}} = F(\chi(X)) \).

The free traced monoidal category has been described in [JSV96] and [Abr05]. Thanks, for example, to [Abr05 Proposition 3] we have the following proposition.
1.4.2. Proposition. Let $\mathcal{A}$ be a traced monoidal category with zero object, $A$ an object of $\mathcal{A}$, $n \in \mathbb{N}_+$ and $\sigma \in \Sigma_n$.

1. $A^{\otimes n} = 0$ if and only if $A = 0$.
2. $\text{Tr}(\sigma_A, \ldots A): A^{\otimes n} \to A^{\otimes n}) = \chi(A)\text{cycles of } \sigma$.

1.5. The monoidal ideal $\mathcal{N}$. In a traced category with zero object $\mathcal{A}$ we have the sets of morphisms universally of trace zero

$$\mathcal{N}(X, Y) := \{ g \in \mathcal{A}(X, Y) \mid \text{Tr}(f \circ g) = 0 \text{ for all } f \in \mathcal{A}(Y, X) \}.$$ 

Morphisms in $\mathcal{N}$ are often called also numerically trivial cycles. We refer to [AK02, 7.1.4, 7.1.5, 7.1.6, 7.4.2] for other properties of these sets. Here we just notice that it is not difficult to see the following.

1.5.1. Proposition. $\mathcal{N}$ is a monoidal ideal of $\mathcal{A}$. If $\mathcal{A}(1)$ is a field $\mathcal{N}$ is the biggest such ideal, and if moreover $\mathcal{A}$ is semisimple then $\mathcal{N} = 0$.

1.6. Isotypic Schur functors and finiteness conditions. Assume that $\mathcal{A}(1)$ contains $\mathbb{Q}$. The partitions $\lambda$ of an integer $|\lambda| = n$ give a complete set of mutually orthogonal central idempotents $d^{\lambda} := \frac{\dim V^{\lambda}_\lambda}{|\lambda|!} \sum_{\sigma \in \Sigma_n} \chi_{\lambda}(\sigma)\sigma \in \mathbb{Q}\Sigma_n$ in the group algebra $\mathbb{Q}\Sigma_n$ of the symmetric group on $n$ letters with $\mathbb{Q}$-coefficients (see [PH91]), where $\chi_{\lambda}$ is the character of the irreducible representation $V_{\lambda}$ of $\Sigma_n$ associated to the partition $\lambda$. For any $n \in \mathbb{N}$ and any object $X$ of $\mathcal{A}$, the group $\Sigma_n$ acts naturally on $X^{\otimes n}$ by means of the symmetry of the $\otimes$-category, we then have also a set of complete mutually orthogonal idempotents indexed by partitions $\lambda$ of $n$

$$d^{\lambda}_X := \frac{\dim V^{\lambda}_\lambda}{|\lambda|!} \sum_{\sigma \in \Sigma_n} \chi_{\lambda}(\sigma)\sigma \in \mathcal{A}(X^{\otimes n})$$

for each object $X$ of $\mathcal{A}$. Being $\mathcal{A}$ pseudoabelian, we thus define an endofunctor on $\mathcal{A}$ by setting $S_{\lambda}(X) = d^{\lambda}_X(X^{\otimes n})$. We call it the isotypic Schur functor; it is a multiple of the classical Schur functor $S_{\lambda}$ corresponding to $\lambda$ ([PH91], [Del02]). In particular, we set $\text{Sym}^n(X) = S_{(n)}(X)$ and $\Lambda^n(X) = S_{(1^n)}(X)$. Note also that $S_{(0)} = \mathbb{I}$ is the constant functor with value $\mathbb{I}$, and $S_{(1)} = \text{Id}_A$. An object $X$ of $\mathcal{A}$ is Schur-finite if there is a partition $\lambda$ such that $S_{\lambda}(X) = 0$. Schur-finiteness is stable under direct sums, tensor products, duals, and taking direct summands (see [Del02, AK02, Maz04, DPM05 and DPM09 for further reference]). For further reference we point out the following two propositions.

1.6.1. Proposition. Let $\lambda$ be a partition of a non negative integer $n$, let $X$ be an object of a $\otimes$-category $\mathcal{A}$, and let $F: \mathcal{A} \to \mathcal{B}$ be a $\otimes$-functor, then:

1. $\chi(S_{\lambda}(X)) = \frac{\dim V^{\lambda}_\lambda}{|\lambda|!} \sum_{\sigma \in \Sigma_n} \chi_{\lambda}(\sigma)\chi(A)^{\text{cycles of } \sigma} = \frac{(\dim V^{\lambda}_\lambda)^2}{|\lambda|!} \text{cp}_{\lambda}(\chi(X))$, where $\text{cp}_{\lambda}(T) := \prod_{(i,j) \in \lambda} (T + j - i) \in \mathbb{Z}[T]$ is the content polynomial of the partition

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1 In [Del09] pag. 324 and [Del07, 6.1] they are called “négligeables”.
2 Please note that this is not the Young idempotent defining $V_{\lambda}$; it is the central idempotent defining the isotypic component of $V_{\lambda}$ inside the regular representation $\mathbb{Q}\Sigma_n$.
3 Being $S_{\lambda}(X) = S_{\lambda}(X)^{\otimes \dim V^{\lambda}_\lambda}$ we have: $S_{\lambda}(X) = 0 \Leftrightarrow S_{\lambda}(X) = 0$. 
\( \lambda \). In particular, if \( S_\lambda(X) = 0 \) then \( \chi(X) \) is a root of \( cp_\lambda(T) \), hence it is an integer \( \chi(X) \in \{-|\text{columns of } \lambda| + 1, \ldots, |\text{rows of } \lambda| - 1\} \).

(2) \( F(S_\lambda(X)) = S_\lambda(F(X)) \).

Proof. Point (1) follows from part (1) of Proposition 1.4.2, which relies on the abstract computation of \( \text{Abr05} \), Proposition 3, together with the known properties of the content polynomial of the partition \( \lambda \) (see \( \text{Mac95} \) I.1, Example 11, I.3, Example 4 and the proof of I.7(7.6)). From the very definitions it’s clear that \( F(d_X^\lambda) = d_X^{F(X)} \) hence point (2).

1.6.2. Proposition. Let \( X = X_0|X_1 \) be a finitely generated supervector space over a characteristic zero field \( \mathbb{F} \), where \( X_0 \) is the even part and \( X_1 \) is the odd part of \( X \). Then \( S_\lambda(X) = 0 \) if and only if \( \lambda \supseteq ((1 + \dim_\mathbb{F} X_1)(1+\dim_\mathbb{F} X_0)) \), that is \( \lambda \) has at least \( \dim_\mathbb{F}(X_0) + 1 \) rows and \( \dim_\mathbb{F}(X_1) + 1 \) columns.

Proof. It’s not difficult to prove it directly, or read \( \text{Del02} \) Corollaire 1.9.

2. Objects (geometrically) of integral type

In order to keep prerequisites to the minimum I shall not review Kahn’s concept of “multiplicity”, instead I use a definition of being “(geometrically) of integral type” which only refers to the Euler characteristics of simple (actually \( \varepsilon \)-simple) objects. Moreover, although it is not always necessary, for the purposes of this note it is enough to work with \( \mathbb{F} \)-linear symmetric rigid \( \otimes \)-categories \( \mathcal{A} \) such that \( \mathcal{A}(1) = \mathbb{F} \) with \( \mathbb{F} \) an algebraically closed field of characteristic zero (in particular, the adverb “geometrically” is here pleonastic, see \( \text{Kah09} \) 2.1 d) and 2.2 d)).

2.1. Definition.

(1) An object \( X \) of \( \mathcal{A} \) is (geometrically) of integral type if

(i) \( \mathcal{A}(X) \) is a finite dimensional semisimple \( \mathbb{F} \)-algebra, and
(ii) \( \chi(X_i) = \text{Tr}(\text{Id}_{X_i}) \in \mathbb{Z} \) for each direct summand \( X_i \) of \( X \) with \( \mathcal{A}(X_i) \) a simple \( \mathbb{F} \)-algebra.

(2) The category \( \mathcal{A} \) is said to be (geometrically) of integral type if every object \( X \) of \( \mathcal{A} \) is such. The full subcategory of \( \mathcal{A} \) consisting of objects (geometrically) of integral type is denoted \( \mathcal{A}_{\text{int}} \).

2.2. Remarks.

(a) For a review of the notion of a “semisimple” category we refer to \( \text{AK02} \) 2.1.2 and Appendix A]. In \( \text{Kno07} \) 4.2], an object \( X \) is called \( \varepsilon \)-semisimple (resp. \( \varepsilon \)-simple) if \( \mathcal{A}(X) \) is semisimple (resp. simple) ring (without any finiteness assumption). If needed, see also \( \text{Kno07} \) 4.5 and the comment after \( \text{Kno07} \) 4.7 for an explanation of the decomposition of an \( \varepsilon \)-semisimple object into a finite direct sum of \( \varepsilon \)-simple objects.

(b) Note that \( \varepsilon \)-semisimpleness is quite a mild property, for example if \( \mathcal{A} \) is the (not semisimple) rigid \( \mathbb{C} \)-linear \( \otimes \)-category of locally free sheaves (of finite rank) over the complex projective line \( \mathbb{P}_1 \) (or any other geometrically connected scheme with at least two points) then \( \mathbb{1} = \mathcal{O}_{\mathbb{P}_1} \) and \( \mathcal{A}(\mathcal{O}_{\mathbb{P}_1}(n)) = \mathcal{A}(\mathcal{O}_{\mathbb{P}_1}) = \mathcal{O}_{\mathbb{P}_1}(\mathbb{P}_1) = \mathbb{C} \) for any \( n \in \mathbb{Z} \), and the objects \( \mathcal{O}_{\mathbb{P}_1}(n) \) are of course
indecomposable but far from being “simple” (i.e. with no proper subobjects, in the categorical sense): they are not even Artinian!

(c) Being \( A(\mathbb{F}) = \mathbb{F} \), the (here undefined) “multiplicity” \( \mu(X) \) of an \( \varepsilon \)-simple object \( X \) is exactly the endomorphism \( \mu(X) \in A(X) \) such that \( \mu(X) = \chi(X) \cdot \text{Id}_X \) (see [Kah09, 1.3 b), 2.2 d]).

**2.3. Proposition (Stability properties).** The subcategory \( A_{\text{int}} \) is closed under direct sums, direct summands, and duality; moreover it contains the Schur-finite objects.

**Proof.** The stability properties are proved in [Kah09, 2.2]. To see that \( A_{\text{int}} \) contains the Schur-finite objects it’s enough to note that each direct summand of a Schur-finite object \( X \) is still such and that, using part (1) of Proposition 1.6.1, the Euler characteristic \( \chi(X) \) of a Schur-finite object \( X \), say \( S_{\lambda}(X) = 0 \), has to be a root of the content polynomial of the partition \( \lambda \), and hence in particular \( \chi(X) \in \mathbb{Z} \). This is also proved in [Kah09, Proposition 2.2 c)] by means of the main, and deep, result of [Del02]. □

**2.4. Two questions.** The following questions arise naturally:

a) (Weibel) Does being of integral type imply being Schur-finite?

This question is very interesting in that, although Deligne characterized Schur-finite objects in abelian \( \otimes \)-categories \( A \), with \( A(\mathbb{F}) \) an algebraically closed field of characteristic zero, as those whose length of \( \otimes \)-powers grows at most exponentially (cf. [Del02, Proposition 0.5.(i)]), there seems to be no analog of [Del90, Théorème 7.1] for Schur-finiteness.

b) (Kahn, question to the author) Is \( A_{\text{int}} \) closed under \( \otimes \) for every \( A \)?

In this note we answer, in the negative, to these questions as follows.

a) In section 3, we show that for each \( n \in \mathbb{Z} \) there is a not semisimple \( \mathbb{C} \)-linear rigid, symmetric \( \otimes \)-categories \( T_n \), with hom-sets of finite dimension over \( \mathbb{C} \), “freely generated” by an \( \varepsilon \)-simple object \( X \) with Euler characteristic \( \chi(X) = n \). Whence \( X \) is of integral type, but it is easy to see that \( X \) is not Schur-finite. This could appear not so definitive in that the \( \otimes \)-category \( T_n \) in which \( X \) sits is not semisimple and not of integral type as a whole.

In order to give a fully satisfactory example we quickly review Deligne’s construction in section 4, and we study, in section 5, a suitable concept of “étale algebras” in \( \otimes \)-categories to fully employ its universal property. In

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4It says the following: In an abelian \( \otimes \)-categories \( A \), with \( A(\mathbb{F}) \) an algebraically closed field of characteristic zero we have: for every \( X \) there is an \( n \in \mathbb{N} \) such that \( \wedge^n X = 0 \) if and only if \( \chi(X) \in \mathbb{N} \) for every object \( X \). This is also equivalent to \( A \) being tannakian.

5Note that, by [Del02 Proposition 0.5(ii)] (but it is also easy to see directly), any semisimple symmetric rigid \( \mathbb{C} \)-linear \( \otimes \)-category (without assuming of integral type) having only finitely many simple objects is necessarily Schur-finite, and therefore its pseudoabelian hull (=abelian hull) is of integral type and of homological origin in the sense of [Kah09 Definition 5.1.b)] by [Del02 Théorème 0.6]. This kind of categories (also not symmetric), called “fusion categories”, are well known and deeply studied in different contexts (see, e.g., [ENO05]), but they are of no use here.

Therefore, if we look for a semisimple \( A \) of integral type, but not Schur-finite, \( A \) must have infinitely many simple objects, and, in view of Deligne’s Theorem [Del02 Proposition 0.5.(i)], the length of their tensor powers must have a “more than exponential” growth.
Proposition 6.1.1 of section 6 we give two proofs of the fact that the \( \otimes \)-generator \( X = [1] \) of Deligne’s category \( \text{Rep}(S_t, \mathbb{C}) \), defined in \([\text{Del07}]\), is not Schur-finite (for all \( t \in \mathbb{C} \setminus \mathbb{N} \)). The first proof, in the style of the proof of Proposition 5.3, relies on the universal property of \( \text{Rep}(S_t, \mathbb{C}) \): the key result is Proposition 5.3.6. The second proof comes from an explicit lower bound on the rate of growth of \( \text{length}_{\text{Rep}(S_t, \mathbb{C})}([1]^n) \), based on two results of Deligne (\([\text{Del07}, \text{Proposition 5.1, Lemme 5.2}]\)), showing that the length of the tensor powers of \([1]\) grows “more than exponentially". The conclusion then follows from \([\text{Del02}, \text{Proposition 0.5.(i)}]\).

In Proposition 6.2.1 we eventually show that \( \text{Rep}(S_t, \mathbb{C}) \) is (geometrically) of integral type if \( t \in \mathbb{Z} \setminus \mathbb{N} \).

b) In section 7, again working with \( \text{Rep}(S_t, \mathbb{C}) \), we show that \( \mathcal{A}_{\text{int}} \) is not always a \( \otimes \)-subcategory of \( \mathcal{A} \): for suitably chosen \( t \in \mathbb{C} \setminus \mathbb{Z} \) and simple object \( \{\lambda\} \in \text{Rep}(S_t, \mathbb{C}) \) we show that \( \{\lambda\} \) is of integral type but \( \{\lambda\} \otimes \{\lambda\} \) is not such. This is achieved by means of Deligne’s description of the Grothendieck ring of the semisimple category \( \text{Rep}(S_t, \mathbb{C}) \).

### 3. Free rigid \( \otimes \)-categories on one object

In \([\text{DMOS82}, \text{Examples (1.26)}]\) there is a construction of the free rigid additive \( \otimes \)-category on one object \( (\mathcal{T}, X_\mathcal{T}) \). After recalling what is meant by this we introduce the free rigid \( \otimes \)-category on one object \( \mathcal{T}_t \) with prescribed Euler characteristic \( t \) and use it to give a first negative answer to Weibel’s question 2.4 a).

Let \( R \) be any commutative ring with identity.

**3.1. Definition.** A free rigid \( R \)-linear \( \otimes \)-category on one object is a pair \( (\mathcal{T}, X_\mathcal{T}) \) such that

(i) \( \mathcal{T} \) is a rigid \( R \)-linear \( \otimes \)-category, and

(ii) for any object \( B \) in any rigid \( R \)-linear \( \otimes \)-category \( \mathcal{B} \) ther is, up to \( \otimes \)-isomorphism, a unique \( \otimes \)-functor \( F: \mathcal{T} \rightarrow \mathcal{B} \) such that \( F(X_\mathcal{T}) = B \).

**3.2. Remarks.**

a) Such a pair \( (\mathcal{T}, X_\mathcal{T}) \) is clearly unique up to \( \otimes \)-equivalences. The existence of such a gadget can be achieved, for example, either by “abstract non-sense”, in the vein of \([\text{Day77}]\), or “constructively”, as in \([\text{DMOS82}, (1.26)]\).

b) It follows that \( X_\mathcal{T} \) is a \( \otimes \)-generator of \( \mathcal{T} \) (i.e., any object is a direct summand of a finite direct sum of tensor powers of \( X_\mathcal{T} \) and its dual \( X_\mathcal{T}^\ast \)) and that \( \mathcal{T}(\mathbb{1}) = R[T] \) with \( T = \chi(X_\mathcal{T}) \) algebraically independent over \( R \).

c) Note that the free \( R \)-linear \( \otimes \)-category on one object and the free rigid \( R \)-linear \( \otimes \)-category \( \mathcal{T} \) are related but quite different: the endomorphisms of \( \mathbb{1} \) of the first are reduced to \( R \), while \( \mathcal{T}(\mathbb{1}) \) must contain an algebraically independent element over \( R \) as remarked above.

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\[ \text{For } t \in \mathbb{N} \text{ the category } \text{Rep}(S_t, \mathbb{C}) \text{ is not semisimple.} \]
3.3. Definition. Let \( t \in R \). A free rigid \( R \)-linear \( \otimes \)-category on one object with Euler characteristic \( t \) is a pair \((T, X_t)\) such that

(i) \( T_t \) is a rigid \( R \)-linear \( \otimes \)-category,
(ii) \( \chi(X_t) = t \), and
(iii) for any object \( B \) with \( \chi(B) = t \) in any rigid \( R \)-linear \( \otimes \)-category \( B \) there is, up to \( \otimes \)-isomorphism, a unique \( \otimes \)-functor \( F: T_t \to B \) such that \( F(X_t) = B \).

As above, it follows that \( X_t \) is a \( \otimes \)-generator of \( T_t \) and that \( T_t(1) = \mathbb{R} \).

3.4. Lemma. For any commutative ring with identity and any \( t \in R \) there exists a free rigid \( R \)-linear \( \otimes \)-category on one object with Euler characteristic \( t \), and any two such are \( \otimes \)-equivalent.

Proof. Let \( T \) be the free rigid \( R \)-linear \( \otimes \)-category on one object, and let \( \mathcal{I}_t \) be the \( \otimes \)-ideal \( \mathcal{I}_t \) of \( T \) generated by the morphism \( T - t \in T(1) \), i.e. \( \mathcal{I}_t(U, V) = (T - t) \mathcal{T}(U, V) \). The pseudo-abelian envelope \( T_t \) of \( T / \mathcal{I}_t \) is a rigid \( R \)-linear \( \otimes \)-category satisfying the required universal property. The last part is clear.

Applying the previous construction to \( R = \mathbb{C} \) we get the first example answering question 2.4 a).

3.5. Proposition. Let \( T_z \) to be the free rigid \( \mathbb{C} \)-linear \( \otimes \)-category on one object \( X_z \) with Euler characteristic \( z \in \mathbb{C} \). Then:

(1) \( X_z \) is not Schur-finite for any \( z \in \mathbb{C} \).
(2) \( X_z \) is of integral type if and only if \( z \in \mathbb{Z} \).

Proof. (1) If \( z \in \mathbb{C} \setminus \mathbb{Z} \) this is obvious, indeed if \( S_{\lambda}(X_z) = 0 \) for some partition \( \lambda \) then \( z = \chi(X_z) \) is a root of the content polynomial of \( \lambda \) by (1) of Proposition 1.6.1 and these roots are integers by definition of content polynomial.

In case \( z \in \mathbb{Z} \), we see that \( X_z \) cannot be Schur-finite by universality. Assume by contradiction that \( S_{\lambda}(X_z) = 0 \) for some partition \( \lambda \). Now, for any \( n \in \mathbb{N} \), by “definition” (i.e. the universal property), there exists a (unique) \( \otimes \)-functor \( F_n: T_z \to s\mathcal{V}_{\mathbb{C}} \) sending the \( \otimes \)-generator \( X_z \) to the super vector space \( F_n(X_z) \) taken to be \( \mathbb{C}^{n+1}|\mathbb{C}^n \) if \( z \geq 0 \) or \( \mathbb{C}^n|\mathbb{C}^{n+z} \) if \( z \leq 0 \). But by (2) of Proposition 1.6.1 we would also have \( 0 = F_n(S_{\lambda}(X_z)) = S_{\lambda}(F_n(X_z)) \). Taking \( n > \max\{|\text{rows of } \lambda|, |\text{columns of } \lambda|\} \), we get in any case a contradiction with Proposition 1.6.2 hence \( S_{\lambda}(X_z) \neq 0 \) for any partition \( \lambda \).

(2) \( T_z(X_z) = \mathbb{C} \) hence \( X_z \) is \( \varepsilon \)-simple with \( \chi(X_z) = z \).

3.6. Remark. Let \( z = n \in \mathbb{Z} \). If \( n = 0 \), then \( X_0 \) is a phantom in \( T_0 \), that is \( \text{Id}_{X_0} \in \mathcal{N} \). Indeed \( \text{Tr}(f) = \text{Tr}(f \cdot \text{Id}_T) = f \cdot \text{Tr}(\text{Id}_T) = 0 \) for any \( f \in T_0(X_0) = \mathbb{C} \).

More generally, all the objects of \( T_0 \), but those in the subcategory generated by \( 1 \), are phantoms. For \( n \in \mathbb{Z} \setminus \{0\} \), the object \( X_n \) is not a phantom. But all the objects of \( T_n \) represented by a partition (i.e. \( S_{\lambda}(X_n) \)) with at least \( n + 1 \) rows if \( n > 0 \), or at least \( n + 1 \) columns if \( n < 0 \), but those in the subcategory generated by \( 1 \), are phantoms. Indeed these objects are in the kernel of the \( \otimes \)-functor sending \( X_n \) to \( \mathbb{C}^n|0 \) for \( n > 0 \) and to \( 0|\mathbb{C}^n \) if \( n < 0 \) (hence the idempotents defining these objects are all universally of trace zero). Note that, in particular, \( X_n \) is even or odd in \( T_n / \mathcal{N}_n \) according to \( n \geq 0 \) or \( n \leq 0 \). Hence one cannot deduce a semisimple example from \( T_n \) in this way.
The categories $\mathcal{T}_n$, with $n \in \mathbb{Z}$, are clearly \textit{neutrally of homological type}, as any $\mathbb{C}$-super vector space of Euler characteristic $n$ gives a $\otimes$-functor (realization), but they are not of \textit{homological origin} in the sense of [Kali09 Definition 5.1.] (as we see also from $\mathcal{N}_n \neq 0$ for all $n \in \mathbb{Z}$).

This “singular” behaviour of “free constructions” under certain specializations has been already explicitly observed and studied by Deligne (cf. [Del96] and [Del07]).

4. Semisimple examples: Deligne’s construction $\text{Rep}(S_t, R)$

In [Del07] (a preprint was available since 2004) Deligne has constructed several “interpolating” families of abelian rigid $\otimes$-categories for categories of representations associated to the series of symmetric, orthogonal, and linear groups. His work has been generalized by F. Knop ([Kno07]). More specifically, for any commutative ring with identity $R$ and any element $t \in R$, Deligne has constructed a rigid $R$-linear pseudo-abelian $\otimes$-category $\text{Rep}(S_t, R)$ satisfying a suitable universal property. If $R = \mathbb{F}$ is a field of characteristic zero and $t \in \mathbb{F} \setminus \mathbb{N}$ then $\text{Rep}(S_t, R)$ is abelian semisimple, while for $t \in \mathbb{N}$ then $\text{Rep}(S_t, R)/\mathcal{N}$ is $\otimes$-equivalent to the category of (finite dimensional) $\mathbb{F}$-linear representations of the symmetric group $\Sigma_t$ on $t$ letters.

In this section I will not really enter in either Deligne’s nor Knop’s construction, I shall only give some hints in the construction stressing the properties of $\text{Rep}(S_t, R)$ which I will use.

4.1. Sketch of Deligne’s construction $\text{Rep}(S_t, R)$. Let $R$ be any commutative ring with identity and let $t \in R$ an element of it. The rigid $R$-linear pseudo-abelian $\otimes$-category $\text{Rep}(S_t, R)$, whose hom-sets are finitely generated projective $R$-module, is obtained in three steps:

$$\text{Rep}_0(S_T) \longrightarrow \text{Rep}_1(S_T) \longrightarrow \text{Rep}(S_t, R),$$

where $\text{Rep}(S_t, R)$ is defined as the pseudo-abelian envelope of the specialization to $T \mapsto t$ of the rigid $\mathbb{Z}[T]$-linear $\otimes$-category $\text{Rep}_1(S_T)$ ([Del07] 2.16), which is the additive envelope of the $\mathbb{Z}[T]$-category $\text{Rep}_0(S_T)$ ([Del07] 2.12]).

4.1.1. The $\mathbb{Z}[T]$-category $\text{Rep}_0(S_T)$. The \textit{objects} of $\text{Rep}_0(S_T)$ are finite sets. The symbol $[U]$ denotes the object corresponding to the finite set $U$, if $U = \{1, \ldots, n\}$ one writes $[n]$ for $[U]$. \textit{Morphisms} between $U, V$ (finite sets) are given by (the $\mathbb{Z}[T]$-free module generated by) \textit{glueing data} on $U$ and $V$ (“donnée de recollement sur $U, V$”), i.e. equivalence relations (equivalently, partitions) $R$ on $U \coprod V$ inducing the discrete equivalence on $U$ and $V$. In particular the endomorphism ring of the object $[\emptyset]$ is $\text{Rep}_0(S_T)([\emptyset]) = \mathbb{Z}[T]$. Such morphisms are \textit{composed} according to suitable \textit{universal polynomial rules} (described in [Del07] 2.10]) which are products of linear factors of the form $T - i$ with $i \in \mathbb{N}$ ([Del07] 2.10.2]). In this way $\text{Rep}_0(S_T)$ is a category enriched over the category of $\mathbb{Z}[T]$-modules: actually its hom-sets $\text{Rep}_0(S_T)([U], [V])$ are finitely generated free $\mathbb{Z}[T]$-modules for any pair of finite sets $U, V$, and composition is $\mathbb{Z}[T]$-bilinear. Note that $\text{Rep}_0(S_T)$ is not an additive category: it lacks products and it is not even pointed (\textit{i.e.} there is no zero object).
4.1.2. The rigid $\mathbb{Z}[T]$-linear $\otimes$-category $\text{Rep}_1(S_T)$. As remarked above the category $\text{Rep}_0(S_T)$ is not even additive, nor monoidal. It turns out that making it additive allows one to define a structure of rigid $\otimes$-category on the resulting $\mathbb{Z}[T]$-linear category. The category $\text{Rep}_1(S_T)$ is defined as the additive envelope of $\text{Rep}_0(S_T)$, it is therefore a $\mathbb{Z}[T]$-linear category. Its objects are $n$-tuples of objects of the former category, for $n \in \mathbb{N}$, and morphisms between them are just matrices of morphisms from $\text{Rep}_0(S_T)$, composed accordingly. The $\otimes$-structure is defined biadditively to $\text{Rep}_1(S_T) \times \text{Rep}_1(S_T)$ the bifunctor

$$\otimes : \text{Rep}_0(S_T) \times \text{Rep}_0(S_T) \rightarrow \text{Rep}_1(S_T)$$

given by $[U] \otimes [V] := \oplus [C]$ where the (formal) direct sum is extended over all glueing data on $U$ and $V$. In this way $1 = [0]$ is the $\otimes$-unit and, for example, $[1] \otimes t$ is the direct sum of $[U/R]$ over all equivalence relations (equivalently, partitions) $R$ on $U = \{1, \ldots, n\}$, in particular $[n]$ is a direct summand of $[1] \otimes n$.

A closer inspection to the definitions (cf. [Del07, 2.16], or [Kno07, (3.15)]) shows that each object $[U]$ of $\text{Rep}_0(S_T)$ (and hence of $\text{Rep}_1(S_T)$) is actually self-dual: $[U] = [U]$. Hence the object $[1]$ is a $\otimes$-generator of $\text{Rep}_1(S_T)$.

4.1.3. The rigid $R$-linear pseudo-abelian $\otimes$-category $\text{Rep}(S_t, R)$. The category $\text{Rep}(S_t, R)$ is defined as the pseudo-abelian envelope of the specialization $\text{Rep}_1(S_T) \otimes_{T-a} R$. If $U$ is a finite set its corresponding object in $\text{Rep}(S_t, R)$ is $[U]_t$, and it will be denoted simply $[U]$ if $t$ is clear from the context. By construction it is a rigid $R$-linear pseudo-abelian $\otimes$-category, and it is not difficult to check that its hom-sets are finitely generated projective $R$-modules. As for $\text{Rep}_1(S_T)$ the unit object of $\text{Rep}(S_t, R)$ is $1 = [0]_t$, with $\text{Rep}(S_t, R)(1) = R$, and every object $[U]$ ($U$ a finite set) is self dual and endowed with a natural structure of “ACU” algebra (i.e. commutative monoid) in $\text{Rep}(S_t, R)$ (see [Del07 1.2, 2.5, 2.16]). The object $[1]$ is a self dual algebra with $\chi([1]) = t$ such that $\text{Rep}(S_t, R)$ is the pseudo-abelian hull of the full subcategory on the objects $[1] \otimes n$ (with $n \in \mathbb{N}$), hence $[1]$ is still a $\otimes$-generator of $\text{Rep}(S_t, R)$.

4.1.4. “Partition objects” and their Euler characteristic. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq 0)$ be a partition (of some integer $|\lambda| = \sum \lambda_i$). Assume that $R$ is a $\mathbb{Q}$-algebra and that $t \in R$ is such that $t - n \in R$ is a unit for any $n \in \mathbb{N} \cap [0, 2|\lambda| - 2]$, then by [Del07, Remarque 5.6] there is, functorially w.r.t. $R$, an object $[\lambda]_t$ of $\mathcal{A} = \text{Rep}(S_t, R)$ such that:

a) $\mathcal{A}([\lambda]_t) = R$,

b) if $\lambda, \mu$ are two distinct partitions, and $t - n \in R$ is a unit for any $n \in \mathbb{N} \cap [0, 2(\lambda \lor \mu) - 2]$, then $\mathcal{A}([\lambda]_t, [\mu]_t) = 0$,

c) if $t - n \in R$ is a unit for any $n \in \mathbb{N}$, then for every object $Y$ of $\mathcal{A}$ we have:

$$\bigoplus_{\lambda} \mathcal{A}([\lambda]_t, Y) \otimes_R [\lambda]_t \cong Y,$$

where $\mathcal{A}([\lambda]_t, Y) \otimes_R [\lambda]_t$ is the object of $\mathcal{A}$ representing the functor (cf. [Del07 3.7])

$$Z \mapsto \text{Hom}_R(\mathcal{A}([\lambda]_t, Y), \mathcal{A}([\lambda]_t, Z)).$$

If $R$ is a PID then $\mathcal{A}([\lambda]_t, Y) \otimes_R [\lambda]_t$ is just a sum of copies of $[\lambda]_t$.

\footnote{ACU stands for “associatif, commutatif, à unité” as in [SR72] and [Del07 1.9]}.\footnote{ACU stands for “associatif, commutatif, à unité” as in [SR72] and [Del07 1.9]}
For example, taking $R = \mathbb{C}$ and $t \in \mathbb{C} \setminus \mathbb{N}$ one has: $1 = |0| = [0]$, while $[1] = 1 \oplus [1]_t$ (see [Del07], 5.1, 5.5).

Moreover, by [Del07] Lemme 7.3, 7.4, for any partition $\lambda$ there is a universal polynomial

$$Q_{\lambda}(T) := \frac{\dim V_{\lambda}}{|\lambda|!} \prod_{a=1}^{\dim V_{\lambda}} (T - (|\lambda| + \lambda_a - a)) \in \mathbb{Q}[T]$$

such that the Euler characteristic of the object $[\lambda]$ of $\text{Rep}(S_t, \mathbb{F})$ corresponding to $\lambda$ is given by $\chi([\lambda]) = Q_{\lambda}(t)$ for any $t \in R \setminus \mathbb{N} \cup [0, 2|\lambda| - 2]$.

4.2. Theorem (Deligne). If $R = \mathbb{F}$ is a field of characteristic zero then $\text{Rep}(S_t, \mathbb{F})$ is a semisimple category (hence also abelian) if and only if $t \in \mathbb{F} \setminus \mathbb{N}$; the simple objects of $\text{Rep}(S_t, \mathbb{F})$ are given by partitions and they are absolutely simple.

Proof. This is [Del07] Théorème 2.18, Proposition 5.1, Théorème 6.2. The simple objects are those described in [1, 1.3] which are absolutely simple as $\text{Rep}(S_t, \mathbb{F})([\lambda]) = \mathbb{F}$ for any characteristic zero field $\mathbb{F}$, any partition $\lambda$ and any $t \in \mathbb{F} \setminus \mathbb{N}$. □

5. Universal property of $\text{Rep}(S_t, R)$: étale algebras in rigid $\otimes$-categories

We would like to show that $\text{Rep}(S_t, \mathbb{C})$ is not Schur-finite for $t \in \mathbb{C} \setminus \mathbb{N}$. This is clear if $t \in \mathbb{C} \setminus \mathbb{Z}$, for $[1]$ Schur-finite would implies $t = \chi([1]) \in \mathbb{Z}$, as already remarked in the proof of Proposition 2.3. It remains the case $t \in \mathbb{Z} \setminus \mathbb{N}$. A way to show that $\text{Rep}(S_t, \mathbb{C})$ is not Schur-finite in this case, in analogy with the strategy of section 3, is by means of its universal property which says what are the possible $\otimes$-functors from $\text{Rep}(S_t, R)$. It is also possible to avoid the use of the universal property, as we will see in the two proofs of Proposition 6.1.1 nonetheless we found of some interest the following results.

5.1. The universal property of $\text{Rep}(S_t, R)$. Following [Del07] 8, 8.1, 8.2, 8.3 the category $\text{Rep}(S_t, R)$ is characterized by the following universal property: $R$-linear $\otimes$-functors $F: \text{Rep}(S_t, R) \rightarrow \mathcal{A}$ from it to any other $R$-linear pseudo-abelian $\otimes$-category $\mathcal{A}$ correspond bijectively, via $A = F([1])$, with the “ACU” algebras (i.e., commutative monoids) $(A, m, u)$ of $\mathcal{A}$ such that $\chi(A) = t$ and $A \otimes A \xrightarrow{m} A \xrightarrow{\tau_{A,A^*}} A^* \otimes A \xrightarrow{\varepsilon} 1$ is the self duality of $A$ where $T_m: A \rightarrow 1$ is the composite

8 More generally, any morphism $m: A \otimes M \rightarrow M$ (think of $M$ as an “$A$-module” as in [ML65, 18]) in a rigid $A$ induces a “trace”

$$T_m := (A = A \otimes 1 \xrightarrow{A \otimes m} A \otimes M \otimes M^* \xrightarrow{m \otimes M^*} M \otimes M^* \xrightarrow{\tau_{M,M^*}} M^* \otimes M \xrightarrow{\varepsilon} 1)).$$

5.1.1. Remark. From a categorical perspective, the algebras as above are all obviously (commutative) Frobenius algebras (Str04, FS08, Lam99), but the kind of algebras one can get as $F([1])$ is even more restricted.

For example, in the rigid $\mathbb{F}$-linear $\otimes$-category $\mathcal{A} = \mathcal{V}_\mathbb{F}$ of (finitely generated) vector spaces over the field $\mathbb{F}$ it is easy to see that the monoid objects $(A, m, u)$ with the properties stated above are exactly the étale algebras over $\mathbb{F}$ (see [Bon90], V, p.48,
Prop. 1), equivalently $A \cong \mathbb{K}_1 \times \cdots \times \mathbb{K}_n$ where $\mathbb{K}_i$ is any separable finite extension of the field $\mathbb{F}$. Note in particular that such an $A$ has to be a reduced ring (i.e., it has no non zero nilpotent elements).

5.2. Étale algebras in rigid $\otimes$-categories. In order to further investigate the kind of algebras on can get as $F([1])$ in a general rigid $R$-linear $\otimes$-category $\mathcal{A}$, with $F: \text{Rep}(S_t, R) \to \mathcal{A}$ a $\otimes$-functor, let us fix some terminology allowing us to do some “multilinear algebra” in a $\otimes$-category.

5.2.1. Definition. Let $\mathcal{A}$ be a $\otimes$-category. A bilinear pairing in a $\otimes$-category $\mathcal{A}$ is any morphism of the form $b: X \otimes Y \to \mathbb{1}$. We say that $b$ is a bilinear form if $Y = X$. If $\mathcal{A}$ is symmetric, we say that a bilinear form $b: X \otimes X \to \mathbb{1}$ is a symmetric [antisymmetric] if $b \circ \tau_{X,X} = b$ [$b \circ \tau_{X,X} = -b$]. If $\mathcal{A}$ is rigid, we say that $b: X \otimes Y \to \mathbb{1}$ is non degenerate, or a perfect pairing, if the induced morphism $X = X \otimes \mathbb{1} \xrightarrow{\eta} X \otimes Y \xrightarrow{b \otimes Y^\vee} Y \otimes Y^\vee \xrightarrow{T_m \otimes \mathbb{1}} Y \otimes A^\vee$ is an isomorphism.

5.2.2. Remark. The reader will have no difficulties in checking that in the rigid abelian $\otimes$-category $\mathcal{A} = \mathcal{V}_F$ of (finitely generated) vector spaces over the field $F$ all the previous notions coincide with the usual ones.

5.2.3. Definition. An étale algebra in a rigid $\otimes$-category $\mathcal{A}$ is a monoid object $(A, m, u)$ in $\mathcal{A}$ such that $m$ is commutative and the symmetric bilinear form $A \otimes A \xrightarrow{m} A \xrightarrow{T_m} 1,$

where $T_m: A \to 1$ is the composite

$$A = A \otimes \mathbb{1} \xrightarrow{A \otimes \eta_A} A \otimes A \otimes A^\vee \xrightarrow{m \otimes A^\vee} A \otimes A^\vee \xrightarrow{T_m \otimes \mathbb{1}} A^\vee \otimes A \xrightarrow{\epsilon_A} 1,$$

is non degenerate, i.e.

$$((T_m \circ m) \otimes \text{Id}_{A^\vee}) \circ (\text{Id}_A \otimes \eta_A): A = A \otimes \mathbb{1} \xrightarrow{A \otimes \eta_A} A \otimes A \otimes A^\vee \xrightarrow{(T_m \circ m) \otimes \mathbb{1}} A^\vee$$

is an isomorphism $A \cong A^\vee$.

5.2.4. Remark. We want to stress, following [JSV96] 2 and [Abr05] 4.3] that given any $m: A \otimes M \to M$ its “trace”, as defined above, is nothing but

$$T_m = \text{Tr}^{\mathcal{V}_F}_{A, A}(m: A \otimes M \to M \cong 1 \otimes M).$$

For example, if $A$ is an algebra of finite dimension over a field and $M$ is a finitely generated $A$-module then $T_m = \chi_M$ is the usual character of the module $M$.

We already mentioned that in the rigid $\otimes$-category $\mathcal{V}_F$ of finitely generated vector spaces over $\mathbb{F}$ an étale algebra in this categorical sense is nothing but an étale algebra in the classical sense. Let us see what are the étale algebras in the rigid $\otimes$-category $\mathcal{V}_F$ of super vector spaces.
5.3. Étale algebras in supercategories. The super category of a $\otimes$-category $\mathcal{A}$ is the $\otimes$-category $s_{\mathcal{A}}$ whose objects and morphisms are as in the product category $\mathcal{A} \times \mathcal{A}$, but the tensor product is defined on objects as

$$X \otimes s Y := X_0 \otimes Y_0 \oplus X_1 \otimes Y_1 \mid X_0 \otimes Y_1 \oplus X_1 \otimes Y_0$$

if $X = X_0 \mid X_1$ and $Y = Y_0 \mid Y_1$; on morphisms $f = f_0 \mid f_1 : X \rightarrow Y$ and $g = g_0 \mid g_1 : A \rightarrow B$ we have

$$f \otimes g := \left( \begin{array}{cc} f_0 \otimes g_0 & 0 \\ 0 & f_1 \otimes g_1 \end{array} \right) \mid \left( \begin{array}{cc} f_0 \otimes g_1 & 0 \\ 0 & f_1 \otimes g_0 \end{array} \right) : X \otimes s A \rightarrow Y \otimes s B.$$ 

In this way $1_s = 1 \mid 0$ and its endomorphism ring is $s_{\mathcal{A}}(1_s) = \mathcal{A}(1)$.

5.3.1. Writing convention. In what follows it will be helpful to adopt the usual convention about morphisms between finite biproducts in an additive category:

$$(X_1 \oplus \cdots \oplus X_n \rightarrow Y_1 \oplus \cdots \oplus Y_m) \mapsto \begin{pmatrix} F_1 & \cdots & F_m \\ \vdots & \ddots & \vdots \\ F_m & \cdots & F_m \end{pmatrix}, \text{ with } X_i \rightarrow Y_j.$$

Note the following special cases in $s_{\mathcal{A}}$:

$$(A \otimes s X \rightarrow f B \otimes s Y) \mapsto \left( \begin{array}{cc} f_{00}^{00} & f_{11}^{11} \\ f_{10}^{01} & f_{10}^{01} \end{array} \right) \mid \left( \begin{array}{cc} f_{01}^{01} & f_{10}^{10} \\ f_{10}^{01} & f_{10}^{01} \end{array} \right) \text{ with } A_i \otimes X_j \xrightarrow{f_{ij}} B_h \otimes Y_k,$$

$$(A \otimes s B \rightarrow m C) \mapsto \left( \begin{array}{cc} m_0^{00} & m_0^{11} \\ m_1^{01} & m_1^{10} \end{array} \right), \text{ with } A_i \otimes B_j \xrightarrow{m_{ij}} C_k,$$

and

$$(A \rightarrow n B \otimes s C) \mapsto \left( \begin{array}{c} n_{00}^0 \\ n_{11}^1 \end{array} \right) \mid \left( \begin{array}{c} n_{01}^1 \\ n_{10}^0 \end{array} \right) \text{ with } A_i \xrightarrow{n_{ik}} B_j \otimes C_k.$$

5.3.2. Associativity. It’s worth to point out explicitly the structure of the ternary associativity isomorphisms

$$\alpha^3 : (X \otimes s Y) \otimes s Z \rightarrow X \otimes s (Y \otimes s Z).$$

Writing only the parity symbols, the structure of $\alpha^3$ is as follows

$$\left( \alpha^3 \right)_0 : (00)0 + (11)0 + (01)1 + (10)1 \rightarrow 0(00) + 0(11) + 1(01) + 1(10)$$

$$\left( \alpha^3 \right)_1 : (00)1 + (11)1 + (01)0 + (10)0 \rightarrow 0(01) + 0(10) + 1(00) + 1(11)$$

Hence $(\alpha^3)_i : [(X \otimes s Y) \otimes s Z]_i \rightarrow [X \otimes s (Y \otimes s Z)]_i$ is obtained by “composing” the associativity isomorphisms of $\mathcal{A}$ with the cyclic permutation $(243)$ whose matrix representation is

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \text{ with inverse } P^{-1} = {^tP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Note that writing down the matrix representation of a morphisms of the form

$$f : A \otimes s B \otimes s C \rightarrow X \otimes s Y \otimes s Z$$

one should take into account the choice of the brackets.
5.3.3. Simmetry. It is crucial to recall that the simmetry of \( sA \) is defined by the Koszul rule
\[
\tau_{X,Y}^* = \begin{pmatrix} \tau_{X_0,Y_0} & 0 \\ 0 & -\tau_{X_1,Y_1} \end{pmatrix} \quad : \quad X \otimes Y \cong Y \otimes X.
\]

5.3.4. Rigidity. \( A \) is rigid if and only if \( sA \) is such and \( (X_0 \mid X_1)^* = X_0^* \mid X_1^* \) with evaluation and coevaluation given by
\[
\varepsilon_X = (\varepsilon_{X_0} \mid \varepsilon_{X_1}) \quad \mid \quad 0: X^* \otimes X \rightarrow 1, \quad \eta_X = \begin{pmatrix} \eta_{X_0} \\ \eta_{X_1} \end{pmatrix} \quad \mid \quad 1 \rightarrow X \otimes X^*.
\]

5.3.5. Remark. Let \( m: A \otimes_z M \rightarrow M \) be a morphism in \( sA \) described as
\[
m = (m_0^{00} \mid m_1^{01} \mid m_1^{10}) \quad \text{with} \quad m_{ij}^k: A_i \otimes M_j \rightarrow M_k.
\]
The trace induced by \( m: A \otimes_z M \rightarrow M \cong \mathbb{I}_A \otimes_z M \) is
\[
^sT_m = ^sT_{A,A}^{M \otimes_z A}(m) = \begin{pmatrix} \text{Tr}_{A_0}^{M_0} - \text{Tr}_{A_0}^{M_1} \end{pmatrix} \mid 0 = T_{m_0^{00}} - T_{m_1^{01}} \mid 0: A_0 \rightarrow 1 \mid 0.
\]
In case \( M = A \) then \( m: A \otimes_z A \rightarrow A \) and requiring such an \( m \) to be commutative in \( sA \) means \( m \circ \tau_{A,A}^* = m \). Since
\[
m \circ \tau_{A,A}^* = (m_0^{00} \circ \tau_{A_0,A_0} - m_1^{11} \circ \tau_{A_1,A_1} \mid (m_1^{01} \circ \tau_{A_0,A_1} m_1^{10} \circ \tau_{A_1,A_0})
\]
this last condition in \( sA \) is equivalent to the following conditions in \( A \): \( m_0^{00} \) is commutative, \( m_1^{11} \) is anticommutative and \( m_1^{10} \circ \tau_{A_1,A_0} = m_1^{10} \).

5.3.6. Proposition. The only étale algebras \( A \) in \( s\mathcal{V}_F \) are the étale algebras of \( \mathcal{V}_F \) thought of as (purely) even superobjects, that is in particular \( A_1 = 0 \), and hence \( \chi(A) > 0 \). In particular, if \( t \in F \setminus N \) there are no \( \otimes \)-functor \( F: \operatorname{Rep}(S_t,F) \rightarrow s\mathcal{V}_F \).

Proof. Let \( A = \mathcal{V}_F \) be the rigid \( \otimes \)-category of (finitely generated) vector spaces over the field \( F \), and let \( A \) be an étale algebra in \( sA \). Then \( \text{Tr}_{m_0^{00}} \circ m: A \otimes_z A \rightarrow A \cong \mathbb{I}_A \otimes_z A \) in \( sA \) induces not degenerate symmetric bilinear forms \( (\text{Tr}_{m_0^{00}} - \text{Tr}_{m_1^{01}}) \circ m_{ij}^k: A_i \otimes A_j \rightarrow F \) in \( A \), where \( \text{Tr}_{m_0^{00}}: A_0 \rightarrow F \) is the trace map induced by \( m_0^{00}: A_0 \otimes A_0 \rightarrow A_0 \) but, by supercommutativity, the image of \( m_1^{11}: A_1 \otimes A_1 \rightarrow A_0 \) would be made of elements with square zero, having therefore zero traces. Hence \( A_1 = 0 \).

Let us assume now by contradiction that there is a \( \otimes \)-functor \( F: \operatorname{Rep}(S_t,F) \rightarrow s\mathcal{V}_F \). Then \( F([1]) \) would be an étale superalgebra of superdimension \( \chi(F([1])) = \chi([1]) \) if \( t < 0 \), i.e. \( F([1]) \mid t \neq 0 \) which is impossible. \( \square \)

6. \( \operatorname{Rep}(S_t,\mathbb{C}) \) is an abelian semisimple not Schur-finite category of integral type for any \( t \in \mathbb{Z} \setminus N \)

6.1. We show, with two different proofs, that the \( \otimes \)-generator \([1]\) of \( \operatorname{Rep}(S_t,F) \) is not Schur-finite for any \( t \in F \setminus N \). Note that in case \( t \notin \mathbb{Z} \) this follows easily from part (1) of Proposition 5.6.1 for \( t = \chi([1]) \).

6.1.1. Proposition. Let \( F \) be an algebraically closed field of characteristic zero, and let \( t \in F \). If \( t \notin N \) then the object \([1]\) is not Schur-finite in \( \operatorname{Rep}(S_t,F) \).

Proof 1. Let \( t \in F \setminus N \), if \( [1] \) be Schur-finite then by Theorem 3.8.1 there is a \( \otimes \)-functor \( F: \operatorname{Rep}(S_t,F) \rightarrow s\mathcal{V}_F \), but this is impossible by Corollary 5.3.6. \( \square \)
Proof 2. As promised, we give also an alternate proof of the fact that $[1]$ is not Schur-finite in $\text{Rep}(S_t, F)$ with $t \in F \setminus \mathbb{N}$ without any reference to the universal property. By [Del07, Théorème 2.18] the category $\mathcal{A} := \text{Rep}(S_t, F)$ is abelian semisimple with simple objects which are absolutely simple. Hence for any object $A$ of $\mathcal{A}$ we have the lower bound: $\text{length}_\mathcal{A}(A) \geq \sqrt{\dim F(A)}$. Let now $A = [1]^n$ with $n \in \mathbb{N}$. By the very definition of the $\otimes$-product of $\text{Rep}_1(S_T)$, the object $[1]^{\otimes n}$ has $[n]$ as a direct summand. Moreover, by Deligne’s [Del07, Proposition 5.1], inside the object $[n]$ there is another direct summand: the object $[n]^*$, which has the property that $\mathcal{A}([n]^*) \cong \mathcal{F}_{\Sigma_n}$ as algebras, as proved in [Del07, Lemme 5.2]. Therefore we have:

$$\text{length}_\mathcal{A}([1]^{\otimes n}) \geq \text{length}_\mathcal{A}([n]) \geq \text{length}_\mathcal{A}([n]^*) \geq \sqrt{\dim F([n]^*)} = \sqrt{n}!$$

hence $[1]$ can’t be Schur-finite because $\text{length}_\mathcal{A}([1]^{\otimes n})$ grows faster than exponentially (cf. [Del02, Proposition 0.5. (i)]) in the category $\mathcal{A}$ which is a “catégorie $F$-tensorielle” in the sense of [Del02]. \qed

6.1.2. Remark. We note that this disproves the claims [Hai02, 4.4, 4.5].

6.2. It remains to show that $\text{Rep}(S_t, F)$ is (an abelian semisimple category of integral type for any field of characteristic zero $F$ and any $t \in F$ such that $t \in \mathbb{F} \setminus \mathbb{N}$). As recalled above (see 4.1.4 and Theorem 4.2) the simple objects of $\text{Rep}(S_t, F)$ are given by partitions and they are absolutely simple, hence to show they are geometrically of integral type it’s enough to show they have integer Euler characteristic.

6.2.1. Proposition. Let $F$ be an algebraically closed field of characteristic zero, and let $t \in F \setminus \mathbb{N}$. Then $\text{Rep}(S_t, F)$ is of integral type if and only if $t \in \mathbb{Z} \setminus \mathbb{N}$.

Proof. As recalled in 4.1.4 for the simple object $[\lambda]_t$ associated to the partition $\lambda$, Deligne has given the explicit expression

$$\chi([\lambda]_t) = Q_\lambda(t) \quad \text{where} \quad Q_\lambda(T) = \frac{\dim V_\lambda}{|\lambda|!} \prod_{a=1}^{|\lambda|} (T - (|\lambda| + \lambda_a - a)) \in \mathbb{Q}[T].$$

We claim that $Q_\lambda(F) \subseteq \mathbb{Z}$. By [Del07, 6.4, (4.1)], $Q_\lambda(n) = \dim V_\lambda(n) \in \mathbb{N}$ for all $n \geq 2|\lambda| + 1$, where $\{\lambda\}_m$ is a partition attached to $\lambda$ for each $m \geq |\lambda| + \lambda_1$ ([Del07, (6.3.1)]). Then the claim follows by the elementary fact recalled in Lemma 6.2.2 (I am certain it is well known, but I do not know any reference). Therefore all simple objects of $\text{Rep}(S_t, F)$ have integer Euler characteristic and $\text{Rep}(S_t, F)$ is geometrically of integral type. \qed

6.2.2. Lemma. Let $p(T) \in \mathbb{Q}[T]$ be a polynomial of degree $d$. Then $p(F) \subseteq \mathbb{Z}$ if and only if $p$ takes integer values on $d + 1$ consecutive integer points.

Proof. By induction on the degree $d$ of $p$. The assertion is true if $d = 0$. Let $d > 0$ and assume the result on all polynomials $q(T) \in \mathbb{Q}[T]$ of degree $d - 1$. By hypothesis there is an $n \in \mathbb{F}$ such that $p(n + i) \in \mathbb{Z}$ for $i \in \{0, \ldots, d\}$. Then, to low the degree, note that $\Delta p(T) := p(T) - p(T - 1) \in \mathbb{Q}[T]$ is a polynomial of degree $d - 1$ such that:

$p(F) \subseteq \mathbb{Z}$ if and only if $p$ takes an integer value in at least one integer and $\Delta p(F) \subseteq \mathbb{Z}$. Indeed clearly $p(F) \subseteq \mathbb{Z}$ implies $\Delta p(F) \subseteq \mathbb{Z}$; conversely, assume $\Delta p(F) \subseteq \mathbb{Z}$ and that there is a $n \in \mathbb{Z}$ such that $p(n) \in \mathbb{Z}$. Then $p(n + i) = p(n) + \sum_{j=1}^{i} \Delta p(n + j) \in \mathbb{Z}$ for each $i \in \mathbb{N}_+$, and $p(n - i) = p(n) - \sum_{j=1}^{i} \Delta p(n - j) \in \mathbb{Z}$ for each $i \in \mathbb{N}$. Whence
p(ℤ) ⊆ ℤ. We conclude the induction as follows: by hypothesis on p we have p(n) ∈ ℤ and Δp(T) assumes integer values on \([n + 1, \ldots, n + d] ∩ ℤ\), hence Δp(ℤ) ⊆ ℤ by the inductive hypothesis. □

7. Tensor powers of objects of integral type need not be such

The tensor powers of a geometrically integral (even absolutely simple) object are not, in general, still such. We can look for an example as follows.

We know that \(\text{Rep}(S_t, ℂ)\), with \(t ∈ ℂ \setminus ℍ\), is an abelian semisimple rigid ℂ-linear category, and we also have a complete description of its (absolutely) simple objects and their Euler characteristic. By [Del07, 5.10], simple objects \(|μ|_t\) are given by partitions \(μ\), and \(χ(|μ|_t) = \frac{\dim V_μ}{|μ|} \prod_{a=1}^{|μ|} (t - |μ| - μ_a + a)\). Moreover, the Grothendieck ring of such a category is also described in [Del07] by means of classical representation theory of symmetric groups, in particular we can effectively check if a simple summand \(|λ|_t\) appears in objects like \(|λ|_t \otimes |ν|_t\).

Hence, the idea is to take an absolutely simple object \(|λ|_t\) in a suitable specialization of \(\text{Rep}_1(S_T)\), with \(T → t ∈ ℂ \setminus ℤ\), in such a way that:

(a) \(χ(|λ|_t) ∈ ℤ\), so that our simple object \(|λ|_t\) is geometrically of integral type, but

(b) \(|λ|_t \otimes |λ|_t\) is not geometrically of integral type, i.e. some simple object \(|μ|_t\) with non integer Euler characteristic appears in its decomposition.

Let us work out the details.

(a) Take \(λ := (2, 1)\). Then \(χ(|λ|_t) = \frac{1}{4}(t - 4)(t - 2)t\) for any \(t ∈ ℂ \setminus ℤ\). As \(χ(|λ|_{t-1}) = -5\), then \(χ(|λ|_r) = -5\) also for \(r ∈ \{2 + iy/11, 2 - iy/11\}\), i.e. the other roots of the polynomial \(\frac{1}{3}(t - 4)(t - 2)t + 5 = \frac{1}{3}(t + 1)(t^2 - 7t + 15)\). Let me fix \(r := \frac{7 + iy/11}{2}\), from now on I work in the abelian semisimple rigid ℂ-linear category \(A := \text{Rep}(S_{7+iy/11}, ℂ)\). Hence \(|λ|_r\) is a geometrically integral object of \(A\).

(b) Consider the second tensor power \(|λ|_r \otimes |λ|_r\) of \(|λ|_r\). By [Del07, 5.11] and the known properties of Littelwood-Richardson coefficients, we know that among the simple objects \(|μ|_r\) appearing in the decomposition of \(|λ|_r \otimes |λ|_r\) there are some \(|μ|_r\) with \(|μ| = 6\). Hence it is enough to check that \(χ(|μ|_r) ∉ ℤ\) for at least one such \(|μ|_r\), which is indeed the case: take \(μ := (3, 2, 1)\) then \(\text{Ind}^{S_{3+iy/11}}_{S_3 \times S_1} (V_3 \otimes V_1) : V_6 \mapsto 2\), so the simple object \(|μ|_r\) is a direct summand of \(|λ|_r \otimes |λ|_r\). Now \(χ(|μ|_r)\) is given by the evaluation of the polynomial \(p := \frac{1}{10}(T - 8)(T - 6)(T - 2)(T - 1)T\) at \(r = 7 + iy/11\). Hence it is enough to compute the remainder of the division of \(p\) by the minimum polynomial of \(r \mod ℤ\), that is \(T^2 - 7T + 15\), which is \(7/10(135T - 1080)\). Therefore \(χ(|μ|_r) = \frac{1}{10}(135r - 1080) ∈ ℂ \setminus ℤ\).

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