Solitons in the Presence of a Small, Slowly Varying Electric Field

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Abstract

We consider the perturbed sine-Gordon equation $\theta_{tt} - \theta_{xx} + \sin \theta = \varepsilon^2 f(\varepsilon x)$, where the external perturbation $\varepsilon^2 f(\varepsilon x)$ corresponds to a small, slowly varying electric field. We show that the initial value problem with an appropriate initial state close enough to the solitary manifold has a unique solution, which follows up to time $1/\varepsilon$ and errors of order $\varepsilon^{3/4}$ a trajectory on the solitary manifold. The trajectory on the solitary manifold is described by ODEs, which agree up to errors of order $\varepsilon^3$ with Hamilton equations for the restricted to the solitary manifold sine-Gordon Hamiltonian.

Key words: solitons, symplectic decomposition, modulation equations, Lyapunov function, sine-Gordon equation.

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1 Introduction

The perturbed sine-Gordon equation

$$\theta_{tt} - \theta_{xx} + \sin \theta = F(\varepsilon, x), \quad t, x \in \mathbb{R}, \quad \varepsilon \ll 1,$$

is a Hamiltonian evolution equation with Hamiltonian given by

$$H^\varepsilon(\theta, \psi) = \frac{1}{2} \int \psi^2 + \theta_x^2 + 2(1 - \cos \theta) - 2F(\varepsilon, x)\theta \, dx$$

and the symplectic form given by

$$\Omega \left( \left( \begin{array}{c} \theta' \\ \psi' \end{array} \right), \left( \begin{array}{c} \theta \\ \psi \end{array} \right) \right) = \left\langle \left( \begin{array}{c} \theta' \\ \psi' \end{array} \right), J \left( \begin{array}{c} \theta \\ \psi \end{array} \right) \right\rangle_{L^2(\mathbb{R}) \otimes L^2(\mathbb{R})} = \int_{\mathbb{R}} \psi'(x)\theta(x) - \theta'(x)\psi(x) \, dx,$$
\[
\mathbb{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

In first order formulation \((1)\) can be written as a system:

\[
\partial_t \begin{pmatrix} \theta \\ \psi \end{pmatrix} = \begin{pmatrix} \psi \\ \theta_{xx} - \sin \theta + F(\varepsilon, x) \end{pmatrix}.
\]

(4)

The unperturbed sine-Gordon equation \((F = 0)\), admits soliton solutions \( \begin{pmatrix} \theta_0(\xi(t), u(t), x) \\ \psi_0(\xi(t), u(t), x) \end{pmatrix} \), where

\[
\dot{\xi} = u, \quad \dot{u} = 0, \quad (\xi(0), u(0)) = (a, v) \in \mathbb{R} \times (-1, 1).
\]

Here the functions \( (\theta_0, \psi_0) \) are defined by

\[
\begin{pmatrix} \theta_0(\xi, u, x) \\ \psi_0(\xi, u, x) \end{pmatrix} := \begin{pmatrix} \theta_K(\gamma(u)(x - \xi)) \\ -u\gamma(u)\theta_K'(\gamma(u)(x - \xi)) \end{pmatrix}, u \in (-1, 1), \ \xi, x \in \mathbb{R},
\]

where

\[
\gamma(u) = \frac{1}{\sqrt{1 - u^2}}, \quad \theta_K(x) = 4 \arctan(e^x),
\]

and \( \theta_K \) satisfies \( \theta_K''(x) = \sin \theta_K(x) \) with boundary conditions \( \theta_K(x) \to \begin{pmatrix} 2\pi \\ 0 \end{pmatrix} \) as \( x \to \pm \infty \).

The states \( \begin{pmatrix} \theta_0(a, v, \cdot) \\ \psi_0(a, v, \cdot) \end{pmatrix} \) form the two-dimensional solitary manifold

\[
S_0 := \left\{ \begin{pmatrix} \theta_0(a, v, \cdot) \\ \psi_0(a, v, \cdot) \end{pmatrix} : v \in (-1, 1), \ a \in \mathbb{R} \right\}.
\]

(5)

The sine-Gordon equation arises in various physical phenomena such as dynamics of long Josephson junctions \([ZHQ95], [KM89]\), dislocations in crystals \([FK39]\), waves in ferromagnetic materials \([Mik78]\), etc. T. H. R. Skyrme \([Sky61]\) proposed the equation to model elementary particles. Dynamics of solitons under constant electric field were examined numerically, for instance, in \([IC79]\). In this paper, we investigate solitons in the presence of a time independent, small, slowly varying electric field of the type \( F(\varepsilon, x) = \varepsilon^2 f(\varepsilon x) \), which is physically relevant. Our main result is the following theorem.

**Theorem 1.1.** Let \( f \in H^3(\mathbb{R}), 0 < U < 1, (\xi_0, u_0) \in \mathbb{R} \times (-U, U) \) and \( m = \int (\theta_K'(Z))^2 \ dZ \).

We consider the Cauchy problem

\[
\partial_t \begin{pmatrix} \theta \\ \psi \end{pmatrix} = \begin{pmatrix} \psi \\ \theta_{xx} - \sin \theta + \varepsilon^2 f(\varepsilon x) \end{pmatrix} \begin{pmatrix} \theta(0, x) \\ \psi(0, x) \end{pmatrix} = \begin{pmatrix} \theta_0(\xi_0, u_0, x) \\ \psi_0(\xi_0, u_0, x) \end{pmatrix} + \begin{pmatrix} \varepsilon(0, x) \\ \w(0, x) \end{pmatrix},
\]

(6)

such that the following assumptions are satisfied:
(a) $\varepsilon$ is sufficiently small.

(b) $N(\theta(0, x), \psi(0, x), \xi, u_s) = 0$, where $N = (N_1, N_2) : L^\infty(\mathbb{R}) \times L^2(\mathbb{R}) \times \Sigma(2, U) \to \mathbb{R}^2$ is given by

$$N(\theta, \psi, \xi, u) := \begin{pmatrix}
\Omega \begin{pmatrix}
(\partial_\xi \theta_0(\xi, u, \cdot)) \\
(\partial_\psi \theta_0(\xi, u, \cdot))
\end{pmatrix},
(\theta(\cdot) - \theta_0(\xi, u, \cdot)) \\
(\psi(\cdot) - \psi_0(\xi, u, \cdot))
\end{pmatrix}
\end{pmatrix}
$$

with $V(l) := \frac{1}{l}$, $\Sigma(l, U) := \{(\xi, u) \in \mathbb{R} \times (-1, 1) : u \in (-U - V(l), U + V(l))\}$, and the symplectic form $\Omega$ given by [3].

(c) $|v(0, \cdot)|^2_{H^1(\mathbb{R})} + |w(0, \cdot)|^2_{L^2(\mathbb{R})} \leq \varepsilon^{\frac{1}{4}}$, where $(v(0, \cdot), w(0, \cdot))$ is given by [6].

The Cauchy problem has a unique solution on the time interval $0 \leq t \leq T$, where $T := T(\varepsilon) := \frac{1}{\varepsilon}$.

The solution may be written in the form

$$\begin{pmatrix}
\theta(t, x) \\
\psi(t, x)
\end{pmatrix} = \begin{pmatrix}
\theta_0(\xi(t), \bar{u}(t), x) \\
\psi_0(\xi(t), \bar{u}(t), x)
\end{pmatrix} + \begin{pmatrix}
v(t, x) \\
w(t, x)
\end{pmatrix},$$

where $v, w$ have regularity

$$(v(t), w(t)) \in C^1([0, T], H^1(\mathbb{R}) \oplus L^2(\mathbb{R})).$$

$\bar{\xi}, \bar{u}$ solve the following system of equations

$$\begin{align*}
\dot{\bar{\xi}}(t) = \bar{u}(t), \\
\dot{\bar{u}}(t) = -\varepsilon^2 \frac{f(\varepsilon \bar{\xi}(t))}{|\gamma(\bar{u}(t))|^3} \int \theta'_K(Z) dZ,
\end{align*}$$

with initial data $\bar{\xi}(0) = \xi_s$, $\bar{u}(0) = u_s$, and there exists a positive constant $c$ such that

$$|v|_{L^\infty([0, T], H^1(\mathbb{R}))} + |w|_{L^\infty([0, T], L^2(\mathbb{R}))} \leq c\varepsilon^{\frac{1}{4}}.$$

The constant $c$ depends on $f$. The parameters $\bar{\xi}, \bar{u}$ describe a fixed nontrivial perturbation of the uniform linear motion as $\varepsilon \to 0$ if condition $f(0) \neq 0$ is satisfied.

This result yields a fairly accurate description of the solution $(\theta, \psi)$ to the Cauchy problem, since we are able to control the dynamics of the transversal component $(v(t, \cdot), w(t, \cdot))$ by the upper bound on its norm [8] and the dynamics on the solitary manifold $S_0$ by the
ODEs (7). The time scale is nontrivial and the result provides a nontrivial dynamics on the solitary manifold \( S_0 \) as \( \varepsilon \to 0 \) if \( f(0) \neq 0 \).

Let us mention some related works. Orbital stability of soliton solutions under perturbations of the initial data has been proven for the unperturbed sine-Gordon equation (see [HPW82], [Stu12 Section 4]). D. M. Stuart [Stu92] considered also the perturbed sine-Gordon equation

\[
\theta_{TT} - \theta_{XX} + \sin \theta + \varepsilon g = 0,
\]

where the perturbation \( g = g(\theta) \) is a smooth function such that \( g_0(Z) = g(\theta_K(Z)) \in L^2(dZ) \) and \( \varepsilon \ll 1 \). He proved that there exists \( T_* = O(\varepsilon^2) \) such that the corresponding initial value problem with initial data

\[
(\hat{\theta}(0, X), \hat{\theta}_T(0, X)) \in H^1 \oplus L^2,
\]

has a unique solution of the form

\[
\theta(T, X) = \theta_K(Z) + \varepsilon \hat{\theta}(T, X), \quad Z = \frac{X - \int_T^T u - C(T)}{\sqrt{1 - u^2}},
\]

where \( \hat{\theta} \in C([0, T_*], H^1), \theta_T \in C([0, T_*], L^2) \) and

\[
C(T) = C_0(\varepsilon T) + \varepsilon \hat{C}, \quad u(T) = u_0(\varepsilon T) + \varepsilon \tilde{u}(T) \quad \Rightarrow \quad p = \frac{u}{\sqrt{1 - u^2}} = p_0(\varepsilon T) + \varepsilon \tilde{p}(T).
\]

Here \( \tilde{p}, \tilde{u}, \hat{C}, \frac{\partial p}{\partial T}, \frac{\partial u}{\partial T}, |\tilde{\theta}|_{H^1(\mathbb{R})} \) are bounded independent of \( \varepsilon \). The functions \( u_0, C_0 \) are solutions of certain explicitly given modulation equations. This result is also valid for perturbations \( g \) of the form

\[
g = g(\varepsilon T, \varepsilon X, \theta),
\]

if among others the following assumption is satisfied: There exists a time interval \( [0, t_*] \) and a constant \( A \) such that for all \( T \in [0, t_*] \):

\[
\left( \int g(\varepsilon T, \varepsilon X, \theta_K(Z))^2 dZ \right)^{\frac{1}{2}} \leq A, \quad dZ = \gamma(u) dX, \quad \gamma(u) = 1/\sqrt{1 - u^2}, \quad (9)
\]

where \( t_* \) and \( A \) are independent of \( \varepsilon \) (see [Stu92 p. 442]). The proof is based on an orthogonal decomposition of the solution into an oscillatory part and a one-dimensional ”zero-mode” term.

In [Mas16] we studied equation (14) with the perturbation \( F(\varepsilon, x) = \varepsilon f(\varepsilon x) \), where \( f \in H^1(\mathbb{R}) \). We proved that the corresponding Cauchy problem with initial data \( \varepsilon \)-close to the solitary manifold \( S_0 \) has a unique solution which follows up to time \( 1/\varepsilon^{\frac{1}{2}} \) and errors of
order $\varepsilon$ a trajectory on $S_0$. The trajectory on $S_0$ is described by parameters which satisfy ODEs for uniform linear motion. Notice that the perturbation $F(\varepsilon, x) = \varepsilon f(\varepsilon x)$ in [Mas16] is not comparable to the perturbations considered in [Stu92], since $F(\varepsilon, x) = \varepsilon f(\varepsilon x)$ does not depend on time and the condition (0) is not satisfied due to

$$|f(\varepsilon \cdot)|_{L^2(\mathbb{R})} = \varepsilon^{-\frac{1}{2}} |f(\cdot)|_{L^2(\mathbb{R})}.$$ 

The main result of the present paper yields richer dynamics on the solitary manifold than the mentioned result of [Mas16] in the following sense: In the present paper the parameters $\tilde{\xi}, \tilde{u}$ satisfy ODEs which describe nontrivial perturbation of the straight line unperturbed dynamics if $f(0) \neq 0$, whereas in [Mas16] the corresponding ODEs describe merely straight line unperturbed dynamics. So in this paper the influence of the electric field $F(\varepsilon, x)$ is identifiable in the dynamics on the solitary manifold in contrast to the mentioned result of [Mas16]. The richer dynamics can be captured, among others, due to the fact that we are able to reach the dynamically relevant time frame $\varepsilon^{-1}$ (see proof of Lemma 7.2).

In [Mas17] we considered the sine-Gordon equation with perturbations

$$F : (-1, 1) \to H^{1,1}(\mathbb{R}), \varepsilon \mapsto F(\varepsilon, \cdot),$$

of class $C^n((-1, 1), H^{1,1}(\mathbb{R}))$ whose first $k$ derivatives vanish at 0, i.e., $\partial^l_x F(0, \cdot) = 0$ for $0 \leq l \leq k$, where $k + 1 \leq n$ and $n \geq 1$. We constructed there, by successive deformation of the classical solitary manifold $S_0$, a virtual solitary manifold $S^\varepsilon_n$, which was obtained in $n$ iteration steps. The virtual solitary manifold $S^\varepsilon_n$ is defined by an implicitly given function $(\theta^\varepsilon_n(a, v, x), \psi^\varepsilon_n(a, v, x))$ analogous to (5) and it is adjusted to the perturbation $F$. The result of [Mas17], which was obtained by using the concept of virtual solitons and Lyapunov functional methods, is as follows: For $\xi_s \in \mathbb{R}, \varepsilon \ll 1$ the initial value problem

$$\partial_t \begin{pmatrix} \theta \\ \psi \end{pmatrix} = \begin{pmatrix} \psi \\ \partial^2_x \theta - \sin \theta + F(\varepsilon, x) \end{pmatrix}, \begin{pmatrix} \theta(0, x) \\ \psi(0, x) \end{pmatrix} = \begin{pmatrix} \theta^\varepsilon_n(\xi_s, u_s, x) \\ \psi^\varepsilon_n(\xi_s, u_s, x) \end{pmatrix} + \begin{pmatrix} v(0, x) \\ w(0, x) \end{pmatrix},$$

with appropriate initial data that is $\varepsilon^n$-close to $S^\varepsilon_n$, i.e., $|v(0, \cdot)|_{H^1(\mathbb{R})}^2 + |w(0, \cdot)|_{L^2(\mathbb{R})} \leq \varepsilon^{2n}$, with initial velocity that satisfies the smallness assumption $|u_s| \leq \tilde{C} \varepsilon^{\frac{k+1}{2}}$, has a unique solution $(\theta, \psi)$ which may be written up to time $1/((\tilde{C} \varepsilon^{\frac{k+1}{2}}))$ in the form

$$\begin{pmatrix} \theta(t, x) \\ \psi(t, x) \end{pmatrix} = \begin{pmatrix} \theta^\varepsilon_n(\xi(t), \tilde{u}(t), x) \\ \psi^\varepsilon_n(\xi(t), \tilde{u}(t), x) \end{pmatrix} + \begin{pmatrix} v(t, x) \\ w(t, x) \end{pmatrix}.$$ 

The solution remains $\varepsilon^n$-close to $S^\varepsilon_n$, i.e., $|v(t, \cdot)|_{H^1(\mathbb{R})}^2 + |w(t, \cdot)|_{L^2(\mathbb{R})} \leq \tilde{C} \varepsilon^{2n}$, and the dynamics on $S^\varepsilon_n$ is described precisely by the parameters $(\xi(t), \tilde{u}(t))$ which satisfy exactly the ODEs

$$\dot{\xi}(t) = \tilde{u}(t), \quad \dot{\tilde{u}}(t) = \lambda^\varepsilon_n(\xi(t), \tilde{u}(t)).$$
with initial data $\tilde{\xi}(0) = \xi_s$, $\tilde{u}(0) = u_s$, where the function $\lambda_n$ is given implicitly. The
parameters $\xi, u$ describe a fixed nontrivial perturbation of the uniform linear motion as $\varepsilon \to 0$ if the perturbation $F$ satisfies a specific condition.

The result of [Mas17] can be applied to perturbations of type $F(\varepsilon, x) = \varepsilon^2 f(\varepsilon x)$ with
appropriate $f$ by choosing $n = 1$ and $k = 0$. This yields a stability statement with a
shorter time scale than in the main result of the present paper and with an additional
smallness assumption on the initial velocity. In this paper we make use of the fact that the
perturbation $F(\varepsilon, x) = \varepsilon^2 f(\varepsilon x)$ is slowly varying (see for instance proof of Lemma 7.2),
which is one of the reasons for the differences in the statements of both results.

There exist also many results on stability of solitons for several other equations. For
instance, J. Holmer and M. Zworski considered in [HZ08] the Gross-Pitaevskii equation
$i\partial_t u + \frac{1}{2}\partial_x^2 u - V(x)u + u|u|^2 = 0$, with a slowly varying smooth potential $V(x) = W(\varepsilon x)$
and proved that up to time $\frac{\log(1/\varepsilon)}{\varepsilon}$ and errors of size $\varepsilon^2$ in $H^1$, the solution is a soliton
evolving according to the classical dynamics of a natural effective Hamiltonian.

Further long (but finite)-time results for different equations with external potentials can
be found in [FGJS04, JFGS06, HZ07, Hol11]. For results on orbital stability and long time
soliton asymptotics see for example [Wei86, Ben76, Bon75, MP12, SW90, BP92, IKV12,
KMM17, CMnPS16].

The main result of this paper is based on [Mas17, Part II], where many of the compu-
tations are presented in greater detail.

Let us comment on our techniques. The local solution of (6) exists due to the contrac-
tion mapping theorem. By the following approach we derive some estimates which imply
that the local solution is continuatable and that the bound stated in Theorem 1.1 is satisfied.
We decompose the solution of (6) into a point on the virtual solitary manifold
$S_0$ and a
transversal component which is symplectic orthogonal to the tangent space of
$S_0$ at the corresponding point:

$$
\begin{pmatrix}
\theta(t, x) \\
\psi(t, x)
\end{pmatrix}
= \begin{pmatrix}
\theta_0(\xi(t), u(t), x) \\
\psi_0(\xi(t), u(t), x)
\end{pmatrix}
+ \begin{pmatrix}
v(t, x) \\
w(t, x)
\end{pmatrix}.
$$

(10)

This symplectic decomposition is possible in a small uniform distance to the solitary man-
ifold due to the implicit function theorem. The energy

$$H(\theta, \psi) = \frac{1}{2} \int \psi^2 + \theta_x^2 + 2(1 - \cos \theta) \, dx$$

and the momentum

$$\Pi(\theta, \psi) = \int \psi \theta_x \, dx$$

are conserved quantities of the unperturbed sine-Gordon equation. We make use of this fact
and achieve control over the transversal component $(v, w)$ of the solution by introducing
an almost conserved Lyapunov function, given by

$$L = \int \frac{w^2}{2} + \frac{\partial_x v^2}{2} + \frac{\cos(\theta_0(\xi, u, \cdot))v^2}{2} + uw\partial_x v \, dx,$$
where \((v, w)\) and \((\xi, u)\) are such as in (10). \(L\) is the quadratic part of
\[
H(\theta_0 + v, \psi_0 + w) + u\Pi(\theta_0 + v, \psi_0 + w) - \left( H(\theta_0, \psi_0) + u\Pi(\theta_0, \psi_0) \right),
\]
where the linear part vanishes due to symplectic orthogonality in decomposition (10). The Lyapunov function is bounded from below in terms of 
\[
|v(t, \cdot)|^2_{H^1(\mathbb{R})} + |w(t, \cdot)|^2_{L^2(\mathbb{R})},
\]
which is a consequence of the symplectic orthogonality in the decomposition and of spectral properties of the operator \(-\partial_x^2 + \cos \theta_K(Z)\).

To obtain the ODEs (7) we follow the idea of [HZ08]. Namely, we compute the flow of the Hamiltonian \(H^\varepsilon\) restricted to the solitary manifold \(S_0\) and discard all terms of order \(\varepsilon^3\) or higher. The parameters \((\xi, u)\) from (10) satisfy the ODEs (7) up to errors of order \(\varepsilon^{11/4}\). This will be used in order to control from above the Lyapunov function and consequently also the norm of the transversal component \((v, w)\). Using Gronwall’s lemma we pass from the approximate equations for the parameters \((\xi, u)\) to the exact ODEs (7). It suffices to consider the flow of the restricted Hamiltonian \(H^\varepsilon\) without terms of order \(\varepsilon^3\) (or higher), since the fact that \((\xi, u)\) satisfies ODEs (7) up to mentioned orders enables us to carry out all computations leading to the main result.

The ODEs (7) can be rescaled in time by introducing 
\[
s = \varepsilon t, \quad \hat{\xi}(s) := \xi(s/\varepsilon), \quad \hat{u}(s) := \frac{1}{\varepsilon} \tilde{u}(s/\varepsilon).
\]
The corresponding transformed ODEs have the form
\[
\frac{d}{ds} \hat{\xi}(s) = \hat{u}(s), \quad \frac{d}{ds} \hat{u}(s) = -\frac{f(\varepsilon \hat{\xi}(s))}{\gamma(\varepsilon \hat{u}(s))^{3/2}} \int \theta'_K(Z) dZ
\]
and converge to ODEs that describe a fixed nontrivial perturbation of the uniform linear motion as \(\varepsilon \to 0\) if \(f(0) \neq 0\).

The paper is organized as follows. We prove in [Section 2] that in a uniform distance to the solitary manifold the decomposition into symplectically orthogonal components is possible. The existence of a local solution \((\theta, \psi)\) with initial state close to the solitary manifold is established in [Section 3]. In [Section 4] we show that the parameters \((\xi, u)\) from the symplectic decomposition (10) satisfy the modulation equations (7) up to certain errors, which are expressed in powers of \(\varepsilon\) and powers of norms of \(v, w\). We introduce a Lyapunov function and compute its time derivative in [Section 5]. A lower bound on the Lyapunov function is proved in [Section 6]. In [Section 7] we prove a version of [Theorem 1.1] with approximate equations for the parameters \((\xi, u)\). In [Section 8] we rescale in time the parameters \((\xi, u)\) and determine thereby up to what orders in \(\varepsilon\) they differ from exact solutions of ODEs (7). The proof of [Theorem 1.1] is completed in [Section 9].

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Notation Occasionally we drop the dependence of functions on certain variables. For a Hilbert space \(H\) its inner product is denoted by \(\langle \cdot, \cdot \rangle_H\).
2 Symplectic Orthogonal Decomposition

In this section, we show that if \((\theta, \psi) \in L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R})\) is close enough (in the \(L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R})\) norm) to the region

\[
S_0(U) := \left\{ \begin{pmatrix} \theta_0(\xi, u, \cdot) \\ \psi_0(\xi, u, \cdot) \end{pmatrix} : (\xi, u) \in \Sigma(4, U) \right\},
\]

of the solitary manifold \(S_0\), then there exists a unique \((\xi, u) \in \Sigma(2, U)\) such that we are able to decompose the solution in the following way. The solution can be written as the sum

\[
\begin{pmatrix} \theta(\cdot) \\ \psi(\cdot) \end{pmatrix} = \begin{pmatrix} \theta_0(\xi, u, \cdot) \\ \psi_0(\xi, u, \cdot) \end{pmatrix} + \begin{pmatrix} v(\cdot) \\ w(\cdot) \end{pmatrix},
\]

where \((\theta_0(\xi, u, \cdot), \psi_0(\xi, u, \cdot))\) is a point on the solitary manifold and \((v(\cdot), w(\cdot))\) is a transversal component, which is symplectic orthogonal to the tangent vectors \(\left( \frac{\partial \theta_0}{\partial \xi}(\xi, u, \cdot), \frac{\partial \psi_0}{\partial \xi}(\xi, u, \cdot) \right)\) and

\[
\begin{pmatrix} \frac{\partial \theta_0}{\partial \xi}(\xi, u, \cdot) \\ \frac{\partial \psi_0}{\partial \xi}(\xi, u, \cdot) \end{pmatrix}
\]

at the corresponding point of the solitary manifold \(S_0\), i.e., the orthogonality condition

\[
\mathcal{N}(\theta, \psi, \xi, u) = 0
\]

is satisfied. In the next lemma, we prove that the symplectic decomposition is possible in a small uniform distance to the solitary manifold \(S_0\).

**Lemma 2.1.** Let \(0 < U < 1\). Let

\[
\mathcal{O} = \mathcal{O}_{U,p} = \left\{ (\theta, \psi) \in L^\infty(\mathbb{R}) \times L^2(\mathbb{R}) : \inf_{(\xi, u) \in \Sigma(4, U)} \left\| \begin{pmatrix} \theta(\cdot) \\ \psi(\cdot) \end{pmatrix} - \begin{pmatrix} \theta_0(\xi, u, \cdot) \\ \psi_0(\xi, u, \cdot) \end{pmatrix} \right\|_{L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R})} < p \right\}.
\]

There exists \(r > 0\) such that if \(p \leq r\) then for any \((\theta, \psi) \in \mathcal{O}_{U,p}\) there exists a unique \((\xi, u) \in \Sigma(2, U)\) such that

\[
\mathcal{N}(\theta, \psi, \xi, u) = 0
\]

and the map

\[
(\theta, \psi) \mapsto (\xi(\theta, \psi), u(\theta, \psi))
\]

is in \(C^1(\mathcal{O}_{U,p}, \Sigma(2, U))\).

**Proof.** Notice that \(U(4) \leq U(3) \leq U(2)\) and \(\Sigma(4, U) \subset \Sigma(3, U) \subset \Sigma(2, U)\). We consider \((\xi_0, u_0) \in \Sigma(3, U)\). Since

\[
D_{\xi, u}\mathcal{N}(\theta_0(\xi_0, u_0, x), \psi_0(\xi_0, u_0, x), \xi_0, u_0) = \begin{pmatrix} 0 & \gamma^2(u_0)m \\ -\gamma^2(u_0)m & 0 \end{pmatrix},
\]

(11)
we obtain
\[
\det D_{\xi,u}N(\theta_0(\xi_0, u_0, x), \psi_0(\xi_0, u_0, x), \xi_0, u_0) \neq 0. \tag{12}
\]

We prove that there exist \( r > 0, \tilde{\delta} > 0 \) such that \( \forall (\xi_0, u_0) \in \Sigma(3, U) \) there exist balls
\[
B_r(\theta_0(\xi_0, u_0, \cdot), \psi_0(\xi_0, u_0, \cdot)) \subset L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R}), \quad B_{\tilde{\delta}}(\xi_0, u_0) \subset \Sigma(2, U)
\]
and a map
\[
T_{\xi_0,u_0} : B_r(\theta_0(\xi_0, u_0, \cdot), \psi_0(\xi_0, u_0, \cdot)) \to B_{\tilde{\delta}}(\xi_0, u_0)
\]
such that \( T_{\xi_0,u_0}(\theta_0(\xi_0, u_0, x), \psi_0(\xi_0, u_0, x)) = (\xi_0, u_0) \) and \( N(\theta, \psi, T_{\xi_0,u_0}(\theta, \psi)) = 0 \) on \( B_r(\theta_0(\xi_0, u_0, \cdot), \psi_0(\xi_0, u_0, \cdot)) \). Therefore we refer to [Dei85, Theorem 15.1] and check their proof of the implicit function theorem whereas we show that \( r \) and \( \tilde{\delta} \) do not depend on \( (\xi_0, u_0) \). We introduce
\[
\bar{N}_{\xi_0,u_0}(\theta, \psi, \xi, u) = N(\theta(\cdot) + \theta_0(\xi_0, u_0, \cdot), \psi(\cdot) + \psi_0(\xi_0, u_0, \cdot), \xi + \xi_0, u + u_0).
\]

It holds \( \bar{N}_{\xi_0,u_0}(0, 0, 0, 0) = (0, 0) \). We set \( K_{\xi_0,u_0} := D_{(\xi,u)} \bar{N}_{\xi_0,u_0}(0, 0, 0, 0) \) and
\[
S_{\xi_0,u_0}(\theta, \psi, \xi, u) = K_{\xi_0,u_0}^{-1} \bar{N}_{\xi_0,u_0}(\theta, \psi, \xi, u) - I(\xi, u),
\]
which is well defined due to [12]. There exists \( B > 0 \) such that
\[
\forall (\xi, u) \in \mathbb{R} \times [-U - V(2), U + V(2)], \quad \beta_1 + \beta_2 \leq 2, \quad p = 1, 2 :
\]
\[
\left| \partial_{\xi}^{\beta_1} \partial_u^{\beta_2} \theta_0(\xi, u, \cdot) \right|_{L^p(\mathbb{R})} \leq B, \quad \left| \partial_{\xi}^{\beta_1} \partial_u^{\beta_2} \psi_0(\xi, u, \cdot) \right|_{L^p(\mathbb{R})} \leq B.
\]

Notice that
\[
\forall (\xi, u) \in \Sigma(2, U) : \quad \frac{1}{|\gamma(u)|^3} \leq \frac{1}{c}.
\]

In this proof we denote by \( \| \cdot \| \) the maximum row sum norm of a \( 2 \times 2 \) matrix induced by the maximum norm \( |\cdot|_\infty \) in \( \mathbb{R}^2 \). We claim that \( \exists k \in (0, 1), \tilde{\delta} > 0, \forall (\xi_0, u_0) \in \Sigma(3) \)
\[
\forall (\theta, \psi, (\xi, u)) \in B_{\tilde{\delta}}(0) \times B_{\tilde{\delta}}(0) : \| D_{(\xi,u)} S_{\xi_0,u_0}(\theta, \psi, \xi, u) \| \leq k < 1.
\]

Due to (11) it holds that
\[
D_{(\xi,u)} S_{\xi_0,u_0}(\theta, \psi, \xi, u) = \frac{1}{|\gamma(u)|^3 m} \left( \begin{array}{cc}
-\partial_{\xi} \bar{N}^2_{\xi_0,u_0}(\theta, \psi, \xi, u) & -\partial_u \bar{N}^2_{\xi_0,u_0}(\theta, \psi, \xi, u) \\
\partial_{\xi} \bar{N}^1_{\xi_0,u_0}(\theta, \psi, \xi, u) & \partial_u \bar{N}^1_{\xi_0,u_0}(\theta, \psi, \xi, u) \\
\end{array} \right) - \left( \begin{array}{cc}
1 & 0 \\
0 & 1 \\
\end{array} \right).
\]
Similarly as above one shows that 

\[ \delta \]

make use of existence theory we consider the problem

In the following we argue similar to [Stu98, Proof of theorem 2.1]. In order to be able to

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The claim follows by estimating each entry of \( D_{(\xi,u)}S_{\xi_0,u_0}(\theta, \psi, \xi, u) \), for instance:

\[
| - \frac{1}{\gamma(u_0)^3 m} \partial_x \tilde{N}^2_{\xi_0,u_0}(\theta, \psi, \xi, u) - 1 |
\]

\[
\leq \frac{1}{\gamma(u_0)^3 m} \left( |\partial_x \partial_u \psi_0(\bar{\xi}, \bar{u}, x)|_{L^1_{\xi}(\mathbb{R})} |\theta(x) + \theta_0(\xi_0, u_0, x) - \theta_0(\bar{\xi}, \bar{u}, x)|_{L^\infty_{\xi}(\mathbb{R})} 
+ |\partial_x \partial_u \theta_0(\bar{\xi}, \bar{u}, x)|_{L^1_{\xi}(\mathbb{R})} |\psi(x)|_{L^2_{\xi}(\mathbb{R})} 
+ |\partial_x \partial_u \theta_0(\bar{\xi}, \bar{u}, x)|_{L^1_{\xi}(\mathbb{R})} |\psi_0(\xi_0, u_0, x) - \psi_0(\bar{\xi}, \bar{u}, x)|_{L^\infty_{\xi}(\mathbb{R})} \right) 
+ | - \frac{1}{\gamma(u_0)^3 m} \int -\partial_x \psi_0(\bar{\xi}, \bar{u}, x) \partial_x \theta_0(\bar{\xi}, \bar{u}, x) + \partial_x \theta_0(\bar{\xi}, \bar{u}, x) \partial_x \psi_0(\bar{\xi}, \bar{u}, x) dx - 1 |.
\]

Similarly as above one shows that \( \exists r \leq \delta \) \( \forall (\xi_0, u_0) \in \Sigma(3) \) \( \forall (\theta, \psi) \in B_r(0) : |S_{\xi_0,u_0}(\theta, \psi, 0, 0)|_\infty < \delta(1 - k) \), which completes the proof.

\[ \square \]

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In the following we argue similar to [Stu98, Proof of theorem 2.1]. In order to be able to make use of existence theory we consider the problem

\[
\begin{pmatrix}
\bar{v}(0, x) \\
\bar{w}(0, x)
\end{pmatrix} = \begin{pmatrix}
\theta(0, x) - \theta_0(\xi_s, u_s, x) \\
\psi(0, x) - \psi_0(\xi_s, u_s, x)
\end{pmatrix},
\]

(13)

\[
\partial_t \begin{pmatrix}
\bar{v}(t, x) \\
\bar{w}(t, x)
\end{pmatrix} = \begin{pmatrix}
\bar{w}(t, x) - \psi_0(\xi_s, u_s, x) \\
[\bar{w}(t, x) + \theta_0(\xi_s, u_s, x)]_{xx} - \sin(\bar{v}(t, x) + \theta_0(\xi_s, u_s, x)) + \varepsilon^2 f(\varepsilon x)
\end{pmatrix},
\]

(14)

By [Mar76, Theorem VIII 2.1, Theorem VIII 3.2] there exists a local solution (see also [Stu98, Proof of theorem 2.1], [Stu92, p.434]), where

\[
(\bar{v}, \bar{w}) \in C^1([0, T_{loc}], H^1(\mathbb{R}) \oplus L^2(\mathbb{R})).
\]

The function \((\theta, \psi)\) given by \( \theta(t, x) = \bar{v}(t, x) + \theta_0(\xi_s, u_s, x) \) and \( \psi(t, x) = \bar{w}(t, x) + \psi_0(\xi_s, u_s, x) \) solves obviously locally the Cauchy problem \((6)\) and \((\theta, \psi) \in C^1([0, T_{loc}], L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R}))\) due to Morrey’s embedding theorem. We are going to obtain a bound in Section 7 which will imply that the local solution is indeed continuable. So from now we assume that \((\bar{v}, \bar{w}) \in C^1([0, \overline{T}], H^1(\mathbb{R}) \oplus L^2(\mathbb{R}))\) is a solution of \((13)-(14)\) and \((\theta, \psi)\) is a solution of \((6)\) such that \((\theta, \psi) \in C^1([0, \overline{T}], L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R}))\), where \( \overline{T} > 0 \).
Given \((\theta, \psi)\) we choose the parameters \((\xi(t), u(t))\) according to Lemma 2.1 and define \((v, w)\) as follows:

\[
v(t, x) = \theta(t, x) - \theta_0(\xi(t), u(t), x), \tag{15}
\]

\[
w(t, x) = \psi(t, x) - \psi_0(\xi(t), u(t), x). \tag{16}
\]

\((v(t, x), w(t, x))\) is well defined for \(t \geq 0\) so small that \(|v(t)|_{L^\infty(\mathbb{R})} + |w(t)|_{L^2(\mathbb{R})} \leq r\) and \((\xi(t), u(t)) \in \Sigma(4, U)\), where \(r\) and \(U\) are from Lemma 2.1. We formalize this in the following definition.

**Definition 3.1.** Let \(t^*\) be the "exit time":

\[
t^* := \sup \left\{ T > 0 : |v|_{L^\infty(\mathbb{R})} + |w|_{L^2(\mathbb{R})} \leq r, \right. \\
\left. (\xi(t), u(t)) \in \Sigma(4, U), \ 0 \leq t \leq T \right\},
\]

where \(r\) and \(U\) are from Lemma 2.1.

Notice that \((\xi_s, u_s) = (\xi(0), u(0)) \in \Sigma(4, U)\). We will choose \(\varepsilon\) such that, among others,

\[
|v(0)|_{L^\infty(\mathbb{R})} + |w(0)|_{L^2(\mathbb{R})} \leq \frac{r}{2},
\]

where \((v(0), w(0))\) is given by (15). Thus \((v(t, x), w(t, x))\) is well defined for \(0 \leq t \leq t^*\). In the following lemma we obtain more information on \((v, w)\).

**Lemma 3.2.** Let \(T = \min\{t^*, T\}\) and let \((v, w)\) be defined by (15)-(16). Then \((v, w) \in C^1([0, T], H^1(\mathbb{R}) \oplus L^2(\mathbb{R}))\).

**Proof.** This follows by using (15)-(16) and the fact that \((\tilde{v}, \tilde{w}) \in C^1([0, T], H^1(\mathbb{R}) \oplus L^2(\mathbb{R}))\), since the difference \((\theta_K(\gamma(u_0)(\cdot - \xi_0)) - \theta_K(\gamma(\tilde{u})(\cdot - \xi_0)))\) is in \(L^2(\mathbb{R})\) for all \((\xi_0, u_0), (\xi, \tilde{u}) \in \mathbb{R} \times (-1, 1)\). \(\Box\)

We compute the time derivatives of \(v\) and \(w\), which will be needed in the following sections.

**Lemma 3.3.** The equations for \((v, w)\), defined by (15)-(16), are

\[
\dot{v}(x) = w(x) - \dot{\xi} \partial_x \theta_0(\xi, u, x) - \dot{u} \partial_u \theta_0(\xi, u, x) + u \partial_x \theta_0(\xi, u, x),
\]

\[
\dot{w}(x) = \partial^2_x v(x) - \cos \theta_0(\xi, u, x) v(x) + \varepsilon^2 f(\varepsilon x) + \frac{\sin \theta_0(\xi, u, x) v^2(x)}{2} + \tilde{R}(v)(x) + u \partial_x \psi_0(\xi, u, x) - \dot{\xi} \partial_x \psi_0(\xi, u, x) - \dot{u} \partial_u \psi_0(\xi, u, x),
\]

for times \(t \in [0, t^*]\), where \(\tilde{R}(v) = O(|v|^3_{H^3(\mathbb{R})})\).
Proof. By taking the time derivatives of \((v, w)\) and using (15)-(16), (9) we obtain

\[
\dot{v}(x) = w(x) + \dot{\psi}_0(\xi, u, x) - \dot{\xi} \partial_\xi \theta_0(\xi, u, x) - \dot{u} \partial_u \theta_0(\xi, u, x)
\]

\[
= w(x) - \dot{\xi} \partial_\xi \theta_0(\xi, u, x) - \dot{u} \partial_u \theta_0(\xi, u, x) + \partial_\xi \dot{\theta}_0(\xi, u, x)
\]

and

\[
\dot{w}(x) = \partial_\xi^2 \theta(\xi, u, x) - \sin(\theta(x) + \varepsilon^2 f(\varepsilon x)) - \dot{\xi} \partial_\xi \psi_0(\xi, u, x) - \dot{u} \partial_u \psi_0(\xi, u, x)
\]

\[
= \partial_\xi^2 \theta_0(\xi, u, x) + \partial_\xi^2 v(x) - \sin \theta_0(\xi, u, x) - \cos \theta_0(\xi, u, x)v(x) + \frac{\sin \theta_0(\xi, u, x)v^2(x)}{2}
\]

\[
+ \tilde{R}(v)(x) + \varepsilon^2 f(\varepsilon x) - \dot{\xi} \partial_\xi \psi_0(\xi, u, x) - \dot{u} \partial_u \psi_0(\xi, u, x)
\]

\[
+ u \partial_x \psi_0(\xi, u, x) - \partial_x \psi_0(\xi, u, x)
\]

\[
= \partial_\xi^2 v(x) - \cos \theta_0(\xi, u, x)v(x) + \varepsilon^2 f(\varepsilon x) + \frac{\sin \theta_0(\xi, u, x)v^2(x)}{2} + \tilde{R}(v)(x)
\]

\[
+ u \partial_x \psi_0(\xi, u, x) - \dot{\xi} \partial_\xi \psi_0(\xi, u, x) - \dot{u} \partial_u \psi_0(\xi, u, x),
\]

where we have expanded the term \(\sin(\theta(\xi, u, x) + v(x))\). \(\square\)

4 Modulation Equations

The restriction of the Hamiltonian \(H^\varepsilon\) given by (2), with \(F(\varepsilon, x) = \varepsilon^2 f(\varepsilon x)\), to the solitary manifold \(S_0\), can be expressed in the form

\[
H^\varepsilon(\theta_0(\xi, u, x), \psi_0(\xi, u, x)) = m\gamma(u) - \int \varepsilon^2 f(\varepsilon(y + \xi))\theta_K(\gamma(u)y) \, dx
\]

for an appropriate \(f\). The derivatives with respect to \(u\) and \(\xi\) are given by

\[
\partial_u H^\varepsilon(\theta_0(\xi, u, x), \psi_0(\xi, u, x)) = mu[\gamma(u)]^3 - u[\gamma(u)]^3 \int \varepsilon^2 f(\varepsilon(y + \xi))y\theta'_K(\gamma(u)y) \, dy,
\]

\[
\partial_\xi H^\varepsilon(\theta_0(\xi, u, x), \psi_0(\xi, u, x)) = \gamma(u) \int \varepsilon^2 f(\varepsilon(y + \xi))\theta'_K(\gamma(u)y) \, dx.
\]

Thus we obtain for the restricted Hamiltonian \((\xi, u) \mapsto H^\varepsilon(\theta_0(\xi, u, x), \psi_0(\xi, u, x))\) the following Hamiltonian equations of motion (with respect to the corresponding restricted symplectic form)

\[
\dot{\xi} = u - \varepsilon^3 \frac{f'((\varepsilon\xi)u)}{[\gamma(u)]^3} \int Z^2 \theta'_K(Z) \, dZ + O(\varepsilon^5),
\]

\[
-\dot{u} = \varepsilon^2 \frac{f((\varepsilon\xi))}{[\gamma(u)]^3} \int \theta'_K(Z) \, dZ + O(\varepsilon^4),
\]

12
where we expanded $f(\varepsilon(y + \xi))$ around $y = 0$ and used that $Z\theta'_K(Z)$ is an odd function.

In the following we derive estimates which show how far the parameters $(\xi(t), u(t))$ satisfy the equation above. Let us start with a definition.

**Definition 4.1.** Let $\varepsilon > 0$, $(\xi, u) \in \mathbb{R} \times (-1, 1)$. We set

$$W(\varepsilon, \xi, u) := \frac{\varepsilon^2 f(\varepsilon \xi) \int \theta'_K(Z) dZ}{|\gamma(u)|^3 m}.$$ 

**Lemma 4.2.** There exists an $\varepsilon_0 > 0$ such that the following statement holds. Let $\varepsilon \in (0, \varepsilon_0]$ and $(v, w)$ be given by (15)-(10), with $(\xi, u)$ obtained from [Definition 2.1]. Let

$$|v|_{L^\infty([0,t^*],H^1(\mathbb{R}))}, |u|_{L^\infty([0,t^*],L^2(\mathbb{R}))} \leq \varepsilon_0,$$

where $t^*$ is from *Definition 3.1*. Then

$$|\dot{\xi} - u| \leq C |v|_{H^1(\mathbb{R})} + |w|_{L^2(\mathbb{R})}|\varepsilon|^2 + C |v|_{H^1(\mathbb{R})}^2,$$

$$|\dot{u} + W(\varepsilon, \xi, u)| \leq C |v|_{H^1(\mathbb{R})} + |w|_{L^2(\mathbb{R})}|\varepsilon|^2 + C |v|_{H^1(\mathbb{R})}^2,$$

for $0 \leq t \leq t^*$, where $C$ depends on $f$.

**Proof.** The technique we use is similar to that in the proof of [IKV12, Lemma 6.2]. We start with some definitions and set

$$\Omega(u) := \begin{pmatrix} \Omega(t_1(\xi, u, \cdot), t_2(\xi, u, \cdot)) & \Omega(t_2(\xi, u, \cdot), t_2(\xi, u, \cdot)) \\ \Omega(t_2(\xi, u, \cdot), t_1(\xi, u, \cdot)) & \Omega(t_2(\xi, u, \cdot), t_2(\xi, u, \cdot)) \end{pmatrix} = \gamma(u)^3 m \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$ 

Now we consider for any $(\bar{\xi}, \bar{u}) \in \mathbb{R} \times [-U - V(2), U + V(2)]$, $(\bar{v}, \bar{w}) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ the matrix:

$$M(\bar{\xi}, \bar{u}, \bar{v}, \bar{w}) = \begin{pmatrix} \langle \partial_2^2 \psi_0(\bar{\xi}, \bar{u}, \cdot), \bar{v}(\cdot) \rangle_{L^2(\mathbb{R}) \oplus L^2(\mathbb{R})} & \langle \partial_\xi \partial_2 \psi_0(\bar{\xi}, \bar{u}, \cdot), \bar{w}(\cdot) \rangle_{L^2(\mathbb{R}) \oplus L^2(\mathbb{R})} \\ \langle -\partial_2^2 \psi_0(\bar{\xi}, \bar{u}, \cdot), \bar{v}(\cdot) \rangle_{L^2(\mathbb{R}) \oplus L^2(\mathbb{R})} & \langle -\partial_\xi \partial_2 \psi_0(\bar{\xi}, \bar{u}, \cdot), \bar{w}(\cdot) \rangle_{L^2(\mathbb{R}) \oplus L^2(\mathbb{R})} \end{pmatrix}.$$ 

It holds for all $(\bar{\xi}, \bar{u}) \in \mathbb{R} \times [-U - V(2), U + V(2)]$, $(\bar{v}, \bar{w}) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$:

$$\| [\Omega(\bar{u})]^{-1} M(\bar{\xi}, \bar{u}, \bar{v}, \bar{w}) \| \leq C (|\bar{v}|_{H^1(\mathbb{R})} + |\bar{w}|_{L^2(\mathbb{R})}),$$

where we denote by $\| \cdot \|$ a matrix norm. Let $I = I_2$ be the identity matrix of dimension 2. Due to (17) we are able to choose $\varepsilon_0 > 0$ such that if $|\bar{v}|_{H^1(\mathbb{R})}, |\bar{w}|_{L^2(\mathbb{R})} \leq \varepsilon_0$ then the matrix

$$I + [\Omega(\bar{u})]^{-1} M(\bar{\xi}, \bar{u}, \bar{v}, \bar{w})$$

is invertible.
is invertible by von Neumann’s theorem. Using (15)-(16) we express the orthogonality condition $N(\theta, \psi, \xi, u) = 0$ from Lemma 2.1 in terms of $(v, w, \xi, u)$ and take its derivative with respect to $t$. For simplicity of notation, we drop $(\theta, \psi, \xi, u)$ and obtain using Lemma 3.3 in matrix form:

$$0 = \frac{d}{dt} \left( \begin{array}{c} N_1 \\ N_2 \end{array} \right) = \Omega \left( \begin{array}{c} \dot{\xi} - u \\ \dot{\xi} + W(\varepsilon, \xi, u) \end{array} \right) + M \left( \begin{array}{c} \dot{\xi} - u \\ \dot{\xi} + W(\varepsilon, \xi, u) \end{array} \right) + \left( \begin{array}{c} P_1 \\ P_2 \end{array} \right),$$

where $M = M(\xi, u, v, w)$, $\Omega = \Omega(u)$, $P_1 = P_1(\xi, u, v, w)$, $P_2 = P_2(\xi, u, v, w)$,

$$P_1(\xi, u, v, w) = \left\langle \begin{array}{c} -u \partial_x v(\cdot) - w(\cdot) \\ -\partial_x^2 v(\cdot) + \cos(\theta_0(\xi, u, \cdot))v(\cdot) - u \partial_x w(\cdot) \end{array} \right| \begin{array}{c} -\partial_\xi \psi_0(\xi, u, \cdot) \\ \partial_\xi \theta_0(\xi, u, \cdot) \end{array} \right\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})}$$

and

$$P_2(\xi, u, v, w) = \left\langle \begin{array}{c} -u \partial_x v(\cdot) - w(\cdot) \\ -\partial_x^2 v(\cdot) + \cos(\theta_0(\xi, u, \cdot))v(\cdot) - u \partial_x w(\cdot) \end{array} \right| \begin{array}{c} -\partial_\xi \psi_0(\xi, u, \cdot) \\ \partial_\xi \theta_0(\xi, u, \cdot) \end{array} \right\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})}$$

If $|v|_{H^1(\mathbb{R})}, |w|_{L^2(\mathbb{R})} \leq \varepsilon_0$ then we obtain as mentioned above by von Neumann’s theorem that

$$\left( \begin{array}{c} \dot{\xi} - u \\ \dot{\xi} + W(\varepsilon, \xi, u) \end{array} \right) = - \left( I + \Omega^{-1} M \right)^{-1} \left[ \Omega^{-1} \left( \begin{array}{c} P_1 \\ P_2 \end{array} \right) \right].$$

Now we take a closer look at the terms that occur in $P_1$ and $P_2$. Integration by parts and
symplectic orthogonality yield that
\[
\left\langle \begin{pmatrix} -u \partial_x v(\cdot) - w(\cdot) \\ -\partial_x^2 v(\cdot) \cos(\theta_0(\xi, u, \cdot)) v(\cdot) - u \partial_x w(\cdot) \end{pmatrix}, \begin{pmatrix} -\partial_x \psi_0(\xi, u, \cdot) \\ -\partial_x \theta_0(\xi, u, \cdot) \end{pmatrix} \right\rangle_{L^2(\mathbb{R}) \oplus L^2(\mathbb{R})} = 0,
\]
\[
\left\langle \begin{pmatrix} -u \partial_x v(\cdot) - w(\cdot) \\ -\partial_x^2 v(\cdot) \cos(\theta_0(\xi, u, \cdot)) v(\cdot) - u \partial_x w(\cdot) \end{pmatrix}, \begin{pmatrix} -\partial_u \psi_0(\xi, u, \cdot) \\ -\partial_u \theta_0(\xi, u, \cdot) \end{pmatrix} \right\rangle_{L^2(\mathbb{R}) \oplus L^2(\mathbb{R})} = 0,
\]
which can also be deduced from [Mas17, Lemma A.5]. It holds that
\[
-\int \partial_x \theta_0(\xi, u, x) \varepsilon^2 f(\varepsilon x) \, dx - W(\varepsilon, \xi, u) \gamma(u)^2 m
= \int \theta'_K(Z) \varepsilon^2 f(\varepsilon(\frac{Z}{\gamma(u)} + \xi)) \, dZ - \varepsilon^2 f(\varepsilon \xi) \int \theta'_K(Z) \, dZ
= \int \theta'_K(Z) \varepsilon^2 \left[ f(\varepsilon \xi) + \frac{\varepsilon}{\gamma(u)} f'(\varepsilon \xi) Z + \ldots \right] \, dZ - \varepsilon^2 f(\varepsilon \xi) \int \theta'_K(Z) \, dZ
\]
and
\[
-\int \partial_u \theta_0(\xi, u, x) \varepsilon^2 f(\varepsilon x) \, dx
= -u \gamma(u) \int Z \theta'_K(Z) \varepsilon^2 f(\varepsilon(\frac{Z}{\gamma(u)} + \xi)) \, dZ
= -u \gamma(u) \int Z \theta'_K(Z) \varepsilon^2 \left[ f(\varepsilon \xi) + \frac{\varepsilon}{\gamma(u)} f'(\varepsilon \xi) Z + \ldots \right] \, dZ.
\]
Since \( Z \theta'_K(Z) \) is an odd function we obtain
\[
|P_1| \leq C[|v|_{H^1(\mathbb{R})} + |w|_{L^2(\mathbb{R})}] \varepsilon^2 + C \varepsilon^4 + C |v|^2_{H^1(\mathbb{R})},
|P_2| \leq C[|v|_{H^1(\mathbb{R})} + |w|_{L^2(\mathbb{R})}] \varepsilon^2 + C \varepsilon^3 + C |v|^2_{H^1(\mathbb{R})}.
\]

\[
\bbox{\Box}
\]

5 Lyapunov Function

In this section we introduce the Lyapunov function and compute its time derivative.

**Definition 5.1.** Let \((v, w)\) be given by (15)-(16), with \((\xi, u)\) obtained from Lemma 2.1. We define the Lyapunov function \(L\) by
\[
L = \int \frac{w^2(x)}{2} + \frac{\partial_x v^2(x)}{2} + \frac{\cos(\theta_K(\gamma(u)(x - \xi))) v^2(x)}{2} + uw(x) \partial_x v(x) \, dx. \tag{18}
\]
Lemma 5.2. It holds for times $t \in [0, t^\ast]$ that

$$
\frac{d}{dt} L = \int w(x) \left[ \frac{\sin(\theta_0(\xi, u, x))v^2(x)}{2} + \tilde{R}(v)(x) \right] \\
+ u\partial_x v(x) \left[ \frac{\sin(\theta_0(\xi, u, x))v^2(x)}{2} + \tilde{R}(v)(x) \right] dx \\
- \dot{u} \int \frac{\sin(\theta_0(\xi, u, x))}{2} \partial_u \theta_0(\xi, u, x)v^2(x) dx \\
+ (\xi - u) \int \cos(\theta_0(\xi, u, x))v(x)\partial_x v(x) dx + \dot{u} \int w(x)\partial_x v(x) dx \\
+ \varepsilon^2 \int \dot{v}(x)f(\varepsilon x) dx - u\varepsilon \gamma(u)\varepsilon^2 \int (x - \xi)\theta_0'(\gamma(u)(x - \xi))f(\varepsilon x) dx \\
+ (u - \dot{\xi})\gamma(u)\varepsilon^2 \int \theta_0'(\gamma(u)(x - \xi))f(\varepsilon x) dx - u\varepsilon^3 \int v\theta_0'(\varepsilon x) dx.
$$

Proof. We use a similar technique as in the proof of [KSK97, Lemma 2.1]. We can assume that the initial data of our problem has compact support. This allows us to do the following computations (integration by parts etc.). The claim for non-compactly supported initial data follows by density arguments. Since $\int \partial_x v(x)\partial_x^2 v(x) + w(x)\partial_x w(x) dx = 0$ and

$$
\int \frac{\partial}{\partial u} \left[ \frac{\cos(\theta_0(\xi, u, x))}{2} \right] v^2(x) dx \\
= \int \xi \cos(\theta_0(\xi, u, x))v(x)\partial_x v(x) - \dot{u} \frac{\sin(\theta_0(\xi, u, x))}{2} \partial_u \theta_0(\xi, u, x)v^2(x) dx,
$$

we obtain by taking the time derivative of the Lyapunov function [18] and by using Lemma 3.3

$$
\dot{L} = (u - \dot{\xi}) \begin{pmatrix}
-u\partial_x v(\cdot) - w(\cdot) \\
-u\partial_x v(\cdot) - w(\cdot)
\end{pmatrix} \begin{pmatrix}
\partial_x^2 v(\cdot) + \cos(\theta_0(\xi, u, \cdot))v(\cdot) - u\partial_x w(\cdot) \\
\partial_x^2 v(\cdot) + \cos(\theta_0(\xi, u, \cdot))v(\cdot) - u\partial_x w(\cdot)
\end{pmatrix} \begin{pmatrix}
-\partial_\xi \psi_0(\xi, u, \cdot) \\
-\partial_\xi \psi_0(\xi, u, \cdot)
\end{pmatrix} \\
+ w(x) \left[ \frac{\sin(\theta_0(\xi, u, x))v^2(x)}{2} + \tilde{R}(v)(x) \right] + u\partial_x v(x) \left[ \frac{\sin(\theta_0(\xi, u, x))v^2(x)}{2} + \tilde{R}(v)(x) \right]
- \dot{u} \int \frac{\sin(\theta_0(\xi, u, x))}{2} \partial_u \theta_0(\xi, u, x)v^2(x) dx + (\dot{\xi} - u) \int \cos(\theta_0(\xi, u, x))v(x)\partial_x v(x) dx \\
+ \dot{u} \int w(x)\partial_x v(x) dx + \int w(x)\varepsilon^2 f(\varepsilon x) dx + \int u\partial_x v(x)\varepsilon^2 f(\varepsilon x) dx.
$$
The first two terms vanish by the same argument as in the proof of Lemma 4.2. Using the identity
\[ \gamma(u)u\theta_K(\gamma(u)(x-\xi)) = -\partial_t[\theta_K(\gamma(u)(x-\xi)) + \left[-u\dot{u}\gamma(u)^3(x-\xi) + (u-\dot{\xi})\gamma(u)\right]\theta_K(\gamma(u)(x-\xi)), \]

it follows from (6) and (15)-(16) that
\[
\int w\varepsilon^2 f(\varepsilon x) dx = \varepsilon^2 \int \dot{v}(x)f(\varepsilon x) dx - u\varepsilon^2 \int (x-\xi)\theta_K'(\gamma(u)(x-\xi))f(\varepsilon x) dx
\]
\[
+ (u-\dot{\xi})\gamma(u)\varepsilon^2 \int \theta_K'(\gamma(u)(x-\xi))f(\varepsilon x) dx.
\]
The claim follows, since
\[
\int u\partial_x v(x)\varepsilon^2 f(\varepsilon x) dx = -u\varepsilon^3 \int v(x)f'(\varepsilon x) dx.
\]

6 Lower Bound

We introduce a functional \(\mathcal{E}\) and prove a lower bound on \(\mathcal{E}\) by using symplectic orthogonality combined with functional analytic arguments. This will imply a lower bound on the Lyapunov function \(L\), which will play a key role in the proof of the main result.

Definition 6.1. For \((v,w) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})\), \((\xi,u) \in \mathbb{R} \times (-1,1)\) we set
\[ \mathcal{E}(v,w,\xi,u) := \frac{1}{2} \int (w(x) + u\partial_x v(x))^2 + v_Z^2(x) + \cos(\theta_K(Z))v^2(x) dx, \]
where \(Z = \gamma(u)(x-\xi)\) and \(v_Z(x) = \partial_z v\left(\frac{z}{\gamma(u)} + \xi\right) = \frac{1}{\gamma(u)}\partial_x v(x).\)

A straightforward computation yields the following lemma.

Lemma 6.2. For \((v,w) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})\), \((\xi,u) \in \mathbb{R} \times (-1,1)\) it holds that
\[ \mathcal{E}(v,w,\xi,u) = \int \frac{w^2(x)}{2} + \frac{(\partial_x v(x))^2}{2} + \frac{\cos(\theta_K(\gamma(u)(x-\xi)))v^2(x)}{2} + uw(x)\partial_x v(x) dx. \]

Recalling the relations (15)-(16) we introduce a notation in order to be able to express the orthogonality conditions in terms of the variables \((v,w,\xi,u)\) instead of the variables \((\theta,\psi,\xi,u)\).
Definition 6.3. For \((v, w) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})\), \((\xi, u) \in \mathbb{R} \times (-1, 1)\) we set

\[
\mathcal{N}_1(v, w, \xi, u) = \int \partial_x \psi_0(\xi, u, x)v(x) - \partial_x \psi_0(\xi, u, x)w(x) \, dx,
\]

\[
\mathcal{N}_2(v, w, \xi, u) = \int \partial_u \psi_0(\xi, u, x)v(x) - \partial_u \psi_0(\xi, u, x)w(x) \, dx.
\]

Now we prove a lower bound on the functional \(\mathcal{E}\).

Lemma 6.4 (Stuart). There exists \(c > 0\) such that if \((\xi, u) \in \mathbb{R} \times [-U - V(2), U + V(2)] \subset \mathbb{R} \times (-1, 1)\) and \((v, w) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})\) satisfies

\[
\mathcal{N}_2(v, w, \xi, u) = 0
\]

then

\[
\mathcal{E}(v, w, \xi, u) \geq c(|v|_{H^1(\mathbb{R})}^2 + |w|_{L^2(\mathbb{R})}^2).
\]

Proof. We follow closely \[Stu12\] and \[Stu98\]. This proof is a slight modification of the proof of \[Stu12\] Lemma 4.3. Notice that the operator \(-\partial_x^2 + \cos \theta_K(Z)\) is nonnegative. It has (see \[Stu92\]) an one dimensional null space spanned by \(\theta\) and the essential spectrum \([1, \infty)\). We argue by contradiction and assume first: \(\exists \xi \in \mathbb{R} \\forall j \in \mathbb{N} \exists u_j \in [-U - V(2), U + V(2)] \exists (\bar{v}_j, \bar{w}_j) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})\):

\[
\mathcal{N}_2(\bar{v}_j, \bar{w}_j, \xi, u_j) = 0, \quad \mathcal{E}(\bar{v}_j, \bar{w}_j, \xi, u_j) < \frac{1}{j}(|\bar{v}_j|_{H^1(\mathbb{R})}^2 + |\bar{w}_j|_{L^2(\mathbb{R})}^2).
\] (19)

This statement is also true for the sequences \(v_j := \bar{v}_j(|\bar{v}_j|_{H^1(\mathbb{R})}^2 + |\bar{w}_j|_{L^2(\mathbb{R})}^2)^{-\frac{1}{2}}\) and \(w_j := \bar{w}_j(|\bar{v}_j|_{H^1(\mathbb{R})}^2 + |\bar{w}_j|_{L^2(\mathbb{R})}^2)^{-\frac{1}{2}}\). Assuming that \(|v_j|_{L^2(\mathbb{R})} \xrightarrow{j \to \infty} 0\) we obtain \(|(v_j)|_{L^2(\mathbb{R})} \xrightarrow{j \to \infty} 0\) and \(|w_j|_{L^2(\mathbb{R})} \xrightarrow{j \to \infty} 0\). This is a contradiction to the fact that \(|v_j|_{H^1(\mathbb{R})}^2 + |w_j|_{L^2(\mathbb{R})}^2 = 1 \forall j \in \mathbb{N}\). By passing to a subsequence we may assume (without loss of generality) that there exists \(\delta > 0\) such that

\[
|v_j|_{L^2(\mathbb{R})} \geq \delta \forall j \in \mathbb{N}.
\] (20)

Since \((v_j, w_j)\) is bounded in \(H^1(\mathbb{R}) \times L^2(\mathbb{R})\) we may assume that \(v_j \xrightarrow{H^1(\mathbb{R})} v\) and \(w_j \xrightarrow{L^2(\mathbb{R})} w\) by taking subsequences. Due to Rellich’s theorem we may assume by passing to subsequences again that \(v_j \xrightarrow{L^2(\Omega)} v\), where \(\Omega \subset \mathbb{R}\) is bounded and open. Passing to a further subsequence we may assume almost everywhere convergence and also \(u_j \xrightarrow{e} u\). Due to the fact that

\[
\exists \, r > 0 \quad \text{s.t.} \quad |\cos(\theta_K(Z))| > \frac{1}{2} \quad \text{for} \quad |Z| > r
\] (21)

and that \(-\partial_x^2 + \cos \theta_K(Z)\) is a nonnegative operator we obtain the estimate

\[
\mathcal{E}(v_j, w_j, \xi, u_j) \geq \frac{1}{4} \int_{-\infty}^{\xi} (u_j + \xi) v_j^2(x) \, dx + \frac{1}{4} \int_{\xi}^{\infty} (u_j + \xi) v_j^2(x) \, dx,
\]

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where we used integration by parts and substitution. Hence (19) implies that
\[ \int_{\{x \in \mathbb{R} : |x| \geq \tilde{r}\}} v_j^2(x) \, dx \overset{j \to \infty}{\to} 0 \]
for a sufficiently large \( \tilde{r} \). As a consequence (20) and the strong convergence on bounded domains yield
\[ \int_{\{x \in \mathbb{R} : |x| \leq \tilde{r}\}} v^2(x) \, dx \geq \bar{\delta}, \]
from which it follows that \( v \not\equiv 0 \). Weak convergence implies using the triangle inequality that
\[ \tilde{N}_2(v, w, \xi, u) = 0 \quad (22) \]
and
\[ \frac{1}{2} \int (w(x) + uv'(x))^2 \, dx \leq \liminf_{j \to \infty} \frac{1}{2} \int (w_j(x) + u_jv'_j(x))^2 \, dx, \quad (23) \]
\[ \frac{1}{2} \int \left( \frac{1}{\gamma(u)} v'(x) \right)^2 \, dx \leq \liminf_{j \to \infty} \frac{1}{2} \int \left( \frac{1}{\gamma(u_j)} (v'_j(x))^2 \right) \, dx. \quad (24) \]
Due to (21) we are able to apply Fatou’s lemma for a sufficiently large \( \tilde{r} \) and obtain
\[ \frac{1}{2} \int_{\{x \in \mathbb{R} : |x| > \tilde{r}\}} \cos(\theta_K(\gamma(u)(x - \xi))) v^2(x) \, dx \]
\[ \leq \liminf_{j \to \infty} \frac{1}{2} \int_{\{x \in \mathbb{R} : |x| > \tilde{r}\}} \cos(\theta_K(\gamma(u_j)(x - \xi))) v_j^2(x) \, dx, \quad (25) \]
where we have used that \((v_j)\) converges almost everywhere. The dominated convergence theorem yields
\[ \frac{1}{2} \int_{\{x \in \mathbb{R} : |x| \leq \tilde{r}\}} \cos(\theta_K(\gamma(u)(x - \xi))) v^2(x) \, dx \]
\[ = \lim_{j \to \infty} \frac{1}{2} \int_{\{x \in \mathbb{R} : |x| \leq \tilde{r}\}} \cos(\theta_K(\gamma(u_j)(x - \xi))) v_j^2(x) \, dx. \quad (26) \]
(19) together with (23)-(26) imply that \( E(v, w, \xi, u) = 0 \). It follows that \((v(x), w(x)) = \alpha(\theta_K(\gamma(u)(x - \xi)), -w(\gamma(u)\theta'_K(\gamma(u)(x - \xi))))\) for some \( \alpha \neq 0 \), since \( v \not\equiv 0 \). This is a contradiction to (22). The constant \( c \) does not depend on \( \xi \), since \( \hat{N}_2(v, w, \xi, u) = \hat{N}_2(v(\cdot + \xi), w(\cdot + \xi), 0, u) = 0 \) implies that \( E(v, w, \xi, u) = E(v(\cdot + \xi), w(\cdot + \xi), 0, u) \geq c(0)(|v|^2_{H^1(\mathbb{R})} + |w|^2_{L^2(\mathbb{R})}) \).

\[ \square \]

**Remark 6.5.** Let \((v, w)\) be given by (15)-(16), with \((\xi, u)\) obtained from **Lemma 2.1**. It holds that \( L(t) = E(v(t), w(t), \xi(t), u(t)) \) for times \( t \in [0, t^*] \).
7 Description of the Dynamics with Approximate Equations for the Parameters $$(\xi, u)$$

We prove first a version of Theorem 1.1 with approximate equations for the parameters $$(\xi, u)$$.

**Theorem 7.1.** Suppose that the assumptions of Theorem 1.1 are satisfied. Then the Cauchy problem defined by (6) has a unique solution on the time interval

$$0 \leq t \leq T$$ where $T = T(\varepsilon) = \frac{1}{\varepsilon}$.

The solution may be written in the form

$$\begin{pmatrix} \theta(t, x) \\ \psi(t, x) \end{pmatrix} = \begin{pmatrix} \theta_0(\xi(t), u(t), x) \\ \psi_0(\xi(t), u(t), x) \end{pmatrix} + \begin{pmatrix} v(t, x) \\ w(t, x) \end{pmatrix}$$

where $v, w, u, \xi$ have regularity $$(\xi(t), u(t)) \in C^1([0, T], \mathbb{R} \times (-1, 1))$$, $(v(t), w(t)) \in C^1([0, T], H^1(\mathbb{R}) \times L^2(\mathbb{R}))$ such that the orthogonality condition

$$N(\theta(t, x), \psi(t, x), \xi(t), u(t)) = 0$$

is satisfied. There exist positive constants $c, C$ such that

$$\left| \dot{\xi} - u \right| \leq C\varepsilon^{\frac{1}{4}}, \quad \left| \dot{u} + W(\varepsilon, \xi, u) \right| \leq C\varepsilon^{\frac{1}{4}},$$

and

$$|v|_{L^\infty([0, T], H^1(\mathbb{R}))}^2 + |w|_{L^\infty([0, T], L^2(\mathbb{R}))}^2 \leq c\varepsilon^{\frac{1}{4}}.$$

The constants $c, C$ depend on $f$.

Theorem 7.1 yields approximate equations for the parameters $$(\xi, u)$$ whereas Theorem 1.1 provides ODEs (7) which are exactly satisfied. The bound for the transversal component $(v, w)$ in Theorem 7.1 is better than in Theorem 1.1. Notice further that in Theorem 7.1 the orthogonality conditions are satisfied which do not have to hold in Theorem 1.1.

The proof of Theorem 7.1 needs some preparation. Now we suppose that (6) has a solution and we make some assumptions on $(v, w)$ given by (15)-(16) and on $(\xi, u)$ obtained from Lemma 2.1. Then the following lemma yields us more accurate information about $(v, w)$ and $(\xi, u)$.

**Lemma 7.2.** Suppose that the assumptions of Theorem 1.1 on $f$ and $$(\xi_s, u_s)$$ are satisfied and let $\varepsilon$ be sufficiently small. Assume that (6) has a solution $$(\theta, \psi)$$ on $[0, T]$ such that

$$(\theta, \psi) \in C^1([0, T], L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R})).$$
Suppose that $0 \leq T \leq t^\ast \leq \overline{T}$ with $t^\ast$ from Definition 3.1. Let $(v, w)$ be given by (15)–(16), with $(\xi, u)$ obtained from Lemma 2.1 such that
\[
|v|^2_{L^\infty([0,T],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,T],L^2(\mathbb{R}))} \leq \varepsilon^{5/2}.
\]
Then, provided $0 \leq T \leq \varepsilon^{-1}$, it holds that
(a) $\forall t \in [0, T]: (\xi(t), u(t)) \in \Sigma(5, U)$;
(b) $|v|^2_{L^\infty([0,T],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,T],L^2(\mathbb{R}))} \leq \overline{C}(L(0) + \varepsilon^{11/4})$, where $\overline{C}$ depends on $f$ (and on $c$ from Lemma 6.4).

Proof. Lemma 4.2 yields for times $t \in [0, T]$:

\[
|\dot{\xi} - u| \leq \overline{C}|v|_{H^1(\mathbb{R})} + |w|_{L^2(\mathbb{R})}\varepsilon^2 + \overline{C}\varepsilon^3 + C|v|^2_{H^1(\mathbb{R})} \leq C\varepsilon^{5/2},
\]
\[
|\dot{u} + W(\varepsilon, \xi, u)| \leq \overline{C}|v|_{H^1(\mathbb{R})} + |w|_{L^2(\mathbb{R})}\varepsilon^2 + \overline{C}\varepsilon^4 + C|v|^2_{H^1(\mathbb{R})} \leq C\varepsilon^{5/2}.
\]

Thus we obtain for times $t \in [0, T]$: $|u(t) - u(0)| \leq \int_0^t |\dot{u}(s)| \, ds \leq C\varepsilon^2 t$, which implies $|u(t)| \leq C\varepsilon^2 t + |u(0)|$. It follows (a) due to $|u_s| < U$ and the smallness assumption on $\varepsilon$. Using Lemma 5.4 and Lemma 6.4 we obtain for times $0 \leq t \leq T \leq \varepsilon^{-1}$, the following estimate,

\[
c\left(|v(t)|^2_{H^1(\mathbb{R})} + |w(t)|^2_{L^2(\mathbb{R})}\right) \leq L(t) = L(0) + \int_0^t L(t) \, dt
\]
\[
\leq L(0) + \int_0^t w(x)[\frac{\sin \theta_0(\xi, u, x)v^2(x)}{2} + \tilde{R}(v)(x)] + u\partial_x v(x)[\frac{\sin \theta_0(\xi, u, x)v^2(x)}{2} + \tilde{R}(v)(x)]
\]
\[
- \dot{u}\int \frac{\sin(\theta_0(\xi, u, x))}{2}\partial_x \theta_0(\xi, u, x)v^2(x) \, dx
\]
\[
+ (\dot{\xi} - u)\int \cos(\theta_0(\xi, u, x))v(x)\partial_x v(x) \, dx + \dot{u}\int w(x)\partial_x v(x) \, dx
\]
\[
- w\dot{u}\gamma(u)^3\varepsilon^2 \int (x - \xi)\theta'_K(\gamma(u)(x - \xi)) f(\varepsilon x) \, dx
\]
\[
+ (u - \dot{\xi})\gamma(u)\varepsilon^2 \int \theta'_K(\gamma(u)(x - \xi)) f(\varepsilon x) \, dx - w\varepsilon^3 \int v(x)f'(\varepsilon x) \, dx dt
\]
\[
+ |v(t)|_{H^1(\mathbb{R})}\varepsilon^2 |f|_{L^2(\mathbb{R})} + |v(0)|_{H^1(\mathbb{R})}\varepsilon^2 |f|_{L^2(\mathbb{R})}
\]
\[
\leq L(0) + C \int_0^t \varepsilon^{2+\frac{3}{8}+\frac{3}{4}} dt + \frac{C}{8} |v(t)|_{H^1(\mathbb{R})}^2 + \frac{C}{8} |v(0)|_{H^1(\mathbb{R})}^2 + \frac{4}{C} |f|_{L^2(\mathbb{R})}^2 \varepsilon^3,
\]
since

\[-u\varepsilon^3 \int v(x)f'(\varepsilon x) \, dx \leq |u|\varepsilon^{2 + \frac{1}{4}} |v|_{H^1(\mathbb{R})} |f'|_{L^2(\mathbb{R})} \, . \]

After bringing two terms on the left hand side we obtain

\[\tilde{c}(|v|^2_{L^\infty([0,T],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,T],L^2(\mathbb{R}))}) \leq L(0) + C \int_0^t \varepsilon^{\frac{4}{5}} \, dt + \frac{4}{\varepsilon^3} |f|^2_{L^2(\mathbb{R})} \varepsilon^3. \]

\[\blacksquare\]

**Theorem 7.3.** Suppose that the assumptions of Theorem 1.1 are satisfied. Assume that (6) has a solution \((\theta,\psi)\) on \([0,T]\) such that

\[(\theta,\psi) \in C^1([0,T],L^\infty(\mathbb{R}) \oplus L^2(\mathbb{R})) \, . \]

Suppose that \(0 \leq T \leq T\). Then, provided \(0 \leq T \leq \varepsilon^{-1}\), it holds that \((v,w)\) given by (15)-(16) is well defined for times \([0,T]\) and there exists a constant \(\tilde{c}\) such that

\[(a) \ |v|^2_{L^\infty([0,T],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,T],L^2(\mathbb{R}))} \leq \tilde{c} \varepsilon^{\frac{1}{4}} \, , \]

\[(b) \ \forall t \in [0,T] : (\xi(t),u(t)) \in \Sigma(5,U) \, . \]

**Proof.** Notice that \(\Sigma(5,U) \subset \Sigma(4,U)\). We define an exit time

\[t_* := \sup \{ T > 0 : |v|^2_{L^\infty([0,T],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,T],L^2(\mathbb{R}))} \leq 2\tilde{C}(L(0) + \varepsilon^{\frac{11}{4}}), \]

\[(\xi(t),u(t)) \in \Sigma(5,U), \ 0 \leq t \leq T \} \, . \]

Suppose \(t_* \leq \varepsilon^{-1}\). Then there exists a time \(\hat{t}\) such that \(\frac{1}{\varepsilon} > \hat{t} > t_*\), with

\[\forall t \in [0,\hat{t}] : (\xi(t),u(t)) \in \Sigma(4,U), \ (\xi(\hat{t}),u(\hat{t})) \notin \Sigma(5,U) \]

or

\[\tilde{C}(L(0) + \varepsilon^{\frac{11}{4}}) < 2\tilde{C}(L(0) + \varepsilon^{\frac{11}{4}}) < |v|^2_{L^\infty([0,\hat{t}],H^1(\mathbb{R}))} + |w|^2_{L^\infty([0,\hat{t}],L^2(\mathbb{R}))} \leq \varepsilon^{\frac{1}{2}}. \]

This leads a contradiction to Lemma 7.2. \(\blacksquare\)

The previous theorem implies that the local solution of (6) discussed in Section 3 is indeed continuable up to times \(\varepsilon^{-1}\). Theorem 7.3 and Lemma 4.2 yield the approximate equations for the parameters \((\xi,u)\). This concludes the claim of Theorem 7.1. \(\blacksquare\)
8 ODE Analysis

In this section we lay the groundwork for passing from the approximate equations for the
parameters \((\xi, u)\) in Theorem 7.1 to the ODEs given by (7). We start with a preparing
lemma.

**Lemma 8.1.** Let \(\tilde{\xi} = \tilde{\xi}(s), \tilde{u} = \tilde{u}(s), \epsilon_1 = \epsilon_1(s), \epsilon_2 = \epsilon_2(s)\) be \(C^1\) real-valued functions. Suppose that \(f \in H^3(\mathbb{R})\) and that
\[
|\epsilon_j| \leq \bar{c}\varepsilon^{3/4}
\]
on \([0, T]\) for \(j = 1, 2\). Assume that on \([0, T]\),
\[
\frac{d}{ds}\tilde{\xi}(s) = \tilde{u}(s) + \epsilon_1(s), \quad \tilde{\xi}(0) = \tilde{\xi}_0
\]
\[
\frac{d}{ds}\tilde{u}(s) = -\frac{f(\varepsilon\tilde{\xi}(s))}{[\gamma(\varepsilon\tilde{u}(s))]^{3/2}m} \int \theta_K(Z) dZ + \epsilon_2(s), \quad \tilde{u}(0) = \tilde{u}_0.
\]
Let \(\hat{\xi} = \hat{\xi}(s)\) and \(\hat{u} = \hat{u}(s)\) be \(C^1\) real-valued functions which satisfy the exact equations
\[
\frac{d}{ds}\hat{\xi}(s) = \hat{u}(s), \quad \hat{\xi}(0) = \hat{\xi}_0
\]
\[
\frac{d}{ds}\hat{u}(s) = -\frac{f(\varepsilon\hat{\xi}(s))}{[\gamma(\varepsilon\hat{u}(s))]^{3/2}m} \int \theta_K(Z) dZ, \quad \hat{u}(0) = \hat{u}_0.
\]
Then provided \(T \leq 1\), there exists \(c > 0\) such that the estimates
\[
|\tilde{\xi} - \hat{\xi}| \leq c\varepsilon^{3/4}, \quad |\tilde{u} - \hat{u}| \leq c\varepsilon^{3/4}.
\]

**Proof.** In the following proof we follow very closely [HZ08, Lemma 6.1]. Let \(x = x(t)\) and \(y = y(t)\) be \(C^1\) real-valued functions, \(C \geq 1\), and let \((x, y)\) satisfy the differential inequalities:
\[
\begin{cases}
|\dot{x}| \leq |y|, & x(0) = x_0 \\
|\dot{y}| \leq C|x| + C|y|, & y(0) = y_0,
\end{cases}
\]
We are going to apply the Gronwall lemma. Let \(z(t) = x^2 + y^2\). Then
\[
|\dot{z}| = |2x\dot{x} + 2y\dot{y}| \leq 2|x||\dot{y}| + 2C|x||y| + 2C|y||\dot{y}| \leq 4C(x^2 + y^2) = 4Cz
\]
and hence \(z(t) \leq z(0)e^{4Ct}\). Thus
\[
|\dot{x}| \leq \sqrt{2}\max(|x_0|, |y_0|) \exp(2Ct), \quad |\dot{y}| \leq \sqrt{2}\max(|x_0|, |y_0|) \exp(2Ct).
\]

(27)
Now we recall the Duhamel’s formula. Let \( X(s) : \mathbb{R} \to \mathbb{R}^2 \) be a two-vector function, \( X_0 \in \mathbb{R}^2 \) a two-vector, and \( A(s) : \mathbb{R} \to (2 \times 2 \text{ matrices}) \) a \( 2 \times 2 \) matrix function. We consider the ODE system

\[
\dot{X}(s) = A(s)X(s), \quad X(s') = X_0
\]

and denote its solution by \( X(s) = S(s, s')X_0 \) such that

\[
\frac{d}{ds}S(s, s')X_0 = A(s)S(s, s')X_0, \quad S(s', s')X_0 = X_0.
\]

Let \( F(s) : \mathbb{R} \to \mathbb{R}^2 \) be a two-vector function. We can express the solution to the inhomogeneous ODE system

\[
\dot{X}(s) = A(s)X(s) + F(s)
\]

with initial condition \( X(0) = 0 \) by the Duhamel’s formula

\[
X(s) = \int_0^s S(s, s')F(s')ds'.
\]

Let \( U = \hat{u} - \tilde{u} \) and \( \Xi = \hat{\xi} - \tilde{\xi} \). These functions satisfy

\[
\frac{d}{ds} \Xi(s) = U(s) + \epsilon_1(s),
\]

\[
\frac{d}{ds} U(s) = \left[ \frac{f(\hat{\xi}(s))}{[\gamma(\hat{\xi}(s))]^3} - \frac{f(\tilde{\xi}(s))}{[\gamma(\tilde{\xi}(s))]^3} \right] \int \theta K(Z) dZ + \epsilon_2(s).
\]

Notice that

\[
\frac{f(\hat{\xi}(s))}{[\gamma(\hat{\xi}(s))]^3} - \frac{f(\tilde{\xi}(s))}{[\gamma(\tilde{\xi}(s))]^3} = \frac{f(\hat{\xi}(s)) - f(\tilde{\xi}(s))}{[\gamma(\hat{\xi}(s))]^3} - \frac{f(\tilde{\xi}(s))}{[\gamma(\tilde{\xi}(s))]^3} = \frac{1}{[\gamma(\hat{\xi}(s))]^3} \frac{f(\hat{\xi}(s)) - f(\tilde{\xi}(s))}{\xi(s) - \hat{\xi}(s)}[\hat{\xi}(s) - \tilde{\xi}(s)]
\]

\[
+ \frac{f(\tilde{\xi}(s))}{[\gamma(\tilde{\xi}(s))]^3} \frac{[\gamma(\tilde{\xi}(s))]^3 - [\gamma(\tilde{\xi}(s))]^3}{\tilde{\xi}(s) - \tilde{\xi}(s)} \left[ \hat{u}(s) - \tilde{u}(s) \right].
\]
Let
\[
g^\varepsilon(s) = \begin{cases} 
\frac{1}{[\gamma(\varepsilon \tilde{u}(s))]^3 m} \int \theta'_K(Z) dZ \frac{f'(\varepsilon \tilde{\xi}(s)) - f'(\varepsilon \tilde{\xi}(s))}{(\tilde{\xi}(s) - \xi(s))} & \text{if } \tilde{\xi}(s) \neq \xi(s) \\
\frac{\varepsilon f'(\varepsilon \tilde{\xi}(s))}{[\gamma(\varepsilon \tilde{u}(s))]^3 m} \int \theta'_K(Z) dZ & \text{if } \tilde{\xi}(s) = \xi(s)
\end{cases}
\]

and obtain by Duhamel’s formula:
\[
h^\varepsilon(s) = \begin{cases} 
\frac{f(\varepsilon \tilde{\xi}(s))}{[\gamma(\varepsilon \tilde{u}(s))]^3 [\gamma(\varepsilon \tilde{u}(s))]^3} \int \theta'_K(Z) dZ \frac{[\gamma(\varepsilon \tilde{u}(s))]^3 - [\gamma(\varepsilon \tilde{u}(s))]^3}{\tilde{u}(s) - \tilde{u}(s)} & \text{if } \tilde{u}(s) \neq \tilde{u}(s) \\
\frac{3\varepsilon[\gamma(\varepsilon \tilde{u}(s))]^3 \tilde{u}(s)f(\varepsilon \tilde{\xi}(s))}{[\gamma(\varepsilon \tilde{u}(s))]^3 [\gamma(\varepsilon \tilde{u}(s))]^3 m} \int \theta'_K(Z) dZ & \text{if } \tilde{u}(s) = \tilde{u}(s)
\end{cases}
\]

We set
\[
A^\varepsilon(s) = \begin{bmatrix} 0 & 1 \\
g^\varepsilon(s) & h^\varepsilon(s)
\end{bmatrix}, \quad F(s) = \begin{bmatrix} \epsilon_1(s) \\
\epsilon_2(s)
\end{bmatrix}, \quad X(s) = \begin{bmatrix} \Xi(s) \\
U(s)
\end{bmatrix}
\]
and obtain by Duhamel’s formula:
\[
\begin{bmatrix} \Xi(s) \\
U(s)
\end{bmatrix} = \int_0^s \begin{bmatrix} \epsilon_1(t') \\
\epsilon_2(t')
\end{bmatrix} dt'.
\]

Applying (27) with
\[
\begin{bmatrix} x(s) \\
y(s)
\end{bmatrix} = S^\varepsilon(s + t', t') \begin{bmatrix} \epsilon_1(t') \\
\epsilon_2(t')
\end{bmatrix}, \quad \begin{bmatrix} x_0 \\
y_0
\end{bmatrix} = \begin{bmatrix} \epsilon_1(t') \\
\epsilon_2(t')
\end{bmatrix}
\]

yields
\[
S^\varepsilon(s, t') \begin{bmatrix} \epsilon_1(t') \\
\epsilon_2(t')
\end{bmatrix} \leq \sqrt{2} \begin{bmatrix} \exp(2C(s - t')) \\
\exp(2C(s - t'))
\end{bmatrix} \max(|\epsilon_1(t')|, |\epsilon_2(t')|).
\]

Using (28) we obtain that on \([0, T]\)
\[
|\Xi(s)| \leq \sqrt{2} T \exp(2CT) \sup_{0 \leq t \leq T} \max(|\epsilon_1(t)|, |\epsilon_2(t)|),
\]

\[
|U(s)| \leq \sqrt{2} T \exp(2CT) \sup_{0 \leq t \leq T} \max(|\epsilon_1(t)|, |\epsilon_2(t)|),
\]

which proves the claim.

In the following we show the relation between the parameters \((\xi, u)\) selected by the implicit function theorem according to Lemma 2.1 and the solutions \((\tilde{\xi}, \tilde{u})\) of the exact ODEs from the previous lemma.
Lemma 8.2. Suppose that the assumptions of Theorem 1.1 are satisfied. Let $\varepsilon$ be sufficiently small, $s = \varepsilon t$, where

$$0 \leq s \leq 1, \quad 0 \leq t \leq \frac{1}{\varepsilon}.$$ 

Let $(\xi, u)$ be the parameters selected according to Lemma 2.1 and $(\hat{\xi}, \hat{u})$ from Lemma 8.1. Then there exists $c > 0$ such that

$$|\xi(t) - \hat{\xi}(\varepsilon t)| \leq c \varepsilon^{\frac{3}{4}}, \quad |u(t) - \varepsilon \hat{u}(\varepsilon t)| \leq c \varepsilon^{\frac{7}{4}}.$$ 

Proof. We set

$$\tilde{\xi}(s) = \frac{\xi}{\varepsilon}(s), \quad \tilde{u}(s) = \frac{1}{\varepsilon} u(s).$$

For times $0 \leq t \leq \varepsilon^{-1}$, Theorem 7.1 yields that

$$|\dot{\xi} - u| \leq C \varepsilon^{\frac{11}{4}}, \quad |\ddot{u} + W(\varepsilon, \xi, u)| \leq C \varepsilon^{\frac{11}{4}}.$$ 

Thus $(\tilde{\xi}, \tilde{u})$ satisfy the assumptions of Lemma 8.1, which implies that

$$|\xi(t) - \hat{\xi}(\varepsilon t)| = |\tilde{\xi}(s) - \hat{\xi}(s)| \leq c \varepsilon^{\frac{3}{4}}, \quad |\varepsilon^{-1} u(t) - \hat{u}(\varepsilon t)| = |\tilde{u}(s) - \hat{u}(s)| \leq c \varepsilon^{\frac{7}{4}}.$$ 

9 Completion of the Proof of Theorem 1.1

Theorem 7.1 yields the dynamics with the parameters $(\xi, u)$ selected by the implicit function theorem according to Lemma 2.1 on the time interval $0 \leq t \leq \varepsilon^{-1}$. Using Lemma 8.2 and the triangle inequality we can replace $(\xi(t), u(t))$ with $(\tilde{\xi}(\varepsilon t), \tilde{u}(\varepsilon t) := (\xi(\varepsilon t), \varepsilon \tilde{u}(\varepsilon t)))$. We have to replace $\varepsilon^{\frac{11}{4}}$ with $\varepsilon^{\frac{3}{4}}$ in the upper bound on the squared norm of the transversal component $(v, w)$ since the difference of the parameters $|\xi(t) - \hat{\xi}(\varepsilon t)|$ in Lemma 8.2 is of order $\varepsilon^{\frac{3}{4}}$.

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