Totally symmetric quasigroups of order 16

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Abstract
We present the number of totally symmetric quasigroups (equivalently, totally symmetric Latin squares) of order 16, as well as the number of isomorphism classes of such objects. Totally symmetric quasigroups of orders up to and including 16 that are (respectively) medial, idempotent, and unipotent are also enumerated.

KEYWORDS
Latin squares, medial, quasigroups, totally symmetric

INTRODUCTION

Suppose \( xy = z \) for some elements \( x, y, z \) in a quasigroup \( Q \) of order \( n \). We say that \( Q \) is totally symmetric if this implies that all six equations obtained by permuting the symbols \( x, y, \) and \( z \) in this equation hold: \( xy = z, xz = y, yx = z, yz = x, zx = y, \) and \( zy = x \).

In [1, 2] Rosemary Bailey enumerated the totally symmetric quasigroups of orders up to and including \( n = 10 \); these results were extended by Brendan McKay and Ian Wanless through \( n = 15 \) in [7]. The main purpose of this paper is to announce the results for \( n = 16 \), which are as follows:

**Theorem 1.** There are 91,361,407,076,595,590,705,971,200 totally symmetric quasigroups of order 16; these are divided into 4,366,600,209,354 isomorphism classes.

We give more detailed information in Table 1, which subdivides these totals based on the orders of the automorphism groups of the quasigroups in each class. Each entry in the table specifies an order of an automorphism group, the number of isomorphism classes of order 16 totally symmetric quasigroups having an automorphism group of that order, and the total number of quasigroups contained in those isomorphism classes.

A brief overview of the algorithm used to calculate these results is given in Section 2 of this document.

(We shall have more to say about the unique class with the largest automorphism group in Section 3.)
TABLE 1  Totally symmetric quasigroups of order 16.

| Aut | Isomorphism classes | Quasigroups |
|-----|---------------------|-------------|
| 1   | 4,366,595,344,552   | 91,361,356,919,980,461,490,176,000 |
| 2   | 4,706,496           | 49,236,513,458,356,224,000          |
| 3   | 60,959              | 425,144,116,260,864,000             |
| 4   | 92,028              | 481,370,626,953,216,000             |
| 5   | 78                  | 326,395,522,252,800                 |
| 6   | 564                 | 1,966,742,249,472,000               |
| 7   | 36                  | 107,602,919,424,000                 |
| 8   | 3986                | 10,424,780,061,696,000              |
| 12  | 84                  | 146,459,529,216,000                 |
| 16  | 363                 | 474,685,795,584,000                 |
| 20  | 2                   | 2,092,278,988,800                   |
| 21  | 3                   | 2,988,969,984,000                   |
| 24  | 76                  | 66,255,501,312,000                  |
| 32  | 92                  | 60,153,020,928,000                  |
| 36  | 1                   | 581,188,608,000                     |
| 48  | 7                   | 3,051,240,192,000                   |
| 60  | 1                   | 348,713,164,800                     |
| 64  | 6                   | 1,961,511,552,000                   |
| 96  | 8                   | 1,743,565,824,000                   |
| 168 | 1                   | 124,540,416,000                     |
| 192 | 8                   | 871,782,912,000                     |
| 288 | 1                   | 72,648,576,000                      |
| 1344| 1                   | 15,567,552,000                      |
| 20,160| 1                  | 1,037,836,800                      |

Returning to our totally symmetric quasigroup $Q$ of order $n$, if

$$w(x(yz)) = y(x(wz))$$

for all $w, x, y, z \in Q$, then $Q$ is said to be medial. This property can also be expressed as $(wx)(yz) = (wy)(xz)$, and has been referred to by various authors as abelian [3, 8, 10] or entropic [4]; medial appears to be the most modernly accepted term [5, 11, 13, 14], and so has been adopted for this paper.

The interest in medial totally symmetric quasigroups arises naturally in the study of elliptic curves, where one can define an addition for points on a curve by fixing a point $p$ (generally taken to be the projective point at infinity) and defining

$$x + y = p(xy).$$
Under this definition, if $Q$ is totally symmetric, then $p$ is the additive identity, $-x = px$, and the addition thus defined is associative \textit{if and only if} $Q$ is medial (see the beautiful exercise 1.11 in [12], which inspired the author's investigation into these objects). This addition enjoys the useful property that the sum of any two rational points on an elliptic curve $C$ is itself a rational point on $C$.

Recent results of Benjamin Young [14] establish that the number of medial totally symmetric quasigroups of order $n$ is precisely the number of “labeled” abelian groups of order $n$ (i.e., counting isomorphic but nonidentical groups separately); this is sequence A034382 in the \textit{On-Line Encyclopedia of Integer Sequences} [9]. Furthermore, Schwenk [10] gives a formula for the number of isomorphism classes of medial totally symmetric quasigroups: If 3 does not divide $n$, it is precisely the number of isomorphism classes of abelian groups of order $n$; otherwise each abelian group of order $n$ contributes one more than the number of nonisomorphic cyclic 3-groups in its invariant factor decomposition (for $n \leq 16$ with $n$ divisible by 3, each abelian group of order $n$ has a single such factor, so for the quasigroups considered in this paper there are in this case twice as many isomorphism classes of medial totally symmetric quasigroups as there are isomorphism classes of abelian groups).

Data on totally symmetric quasigroups and medial totally symmetric quasigroups of orders 1–16 are presented in Table 2.

We include as well (in Table 3) the numbers of \textit{idempotent} and \textit{unipotent} totally symmetric quasigroups and classes, where a quasigroup is \textit{idempotent} if $xx = x$ for all $x \in Q$, and

| Order | Totally symmetric quasigroups | Medial |
|-------|-------------------------------|--------|
|       | Number | Classes | Number | Classes |
| 1     | 1      | 1       | 1      | 1       |
| 2     | 2      | 1       | 2      | 1       |
| 3     | 3      | 2       | 3      | 2       |
| 4     | 16     | 2       | 16     | 2       |
| 5     | 30     | 1       | 30     | 1       |
| 6     | 480    | 3       | 360    | 2       |
| 7     | 1290   | 3       | 840    | 1       |
| 8     | 163,200| 13      | 15,360 | 3       |
| 9     | 471,240| 12      | 68,040 | 4       |
| 10    | 386,400,000 | 139 | 907,200 | 1 |
| 11    | 2,269,270,080 | 65 | 3,991,680 | 1 |
| 12    | 12,238,171,545,600 | 25,894 | 159,667,200 | 4 |
| 13    | 149,648,961,369,600 | 24,316 | 518,918,400 | 1 |
| 14    | 8,089,070,513,113,497,600 | 92,798,256 | 14,529,715,200 | 1 |
| 15    | 160,650,421,233,958,656,000 | 122,859,802 | 163,459,296,000 | 2 |
| 16    | 91,361,407,076,595,590,705,971,200 | 4,366,600,209,354 | 4,250,979,532,800 | 5 |
unipotent if $xx = k$ for all $x \in Q$ and some fixed $k \in Q$. These properties are closely related; McKay and Wanless prove that there is a bijective correspondence between isomorphism classes of idempotent totally symmetric quasigroups of order $n$ and isomorphism classes of unipotent totally symmetric quasigroups of order $n + 1$ [7, Theorem 5.2], and that moreover the number of unipotent totally symmetric quasigroups of order $n + 1$ is precisely $n + 1$ times the number of idempotent totally symmetric quasigroups of order $n$ [7, Theorem 2.4(vi) and Lemma 3.1].

The results of Bailey, McKay and Wanless, Young, and Schwenk all provide independent confirmation of the correctness of the results reported in Theorem 1, as the software used to establish that theorem reproduces a great many of the results computed and predicted by these authors. Specifically, the software gives:

–the number of totally symmetric quasigroups, as well as the number of isomorphism classes of totally symmetric quasigroups, computed by McKay and Wanless for sets of order $n \leq 15$ (agreeing, of course, with Bailey’s results for $n \leq 10$);
–the number of medial totally symmetric quasigroups of order $n$ for all $n \leq 16$ predicted by Young;
–the number of isomorphism classes of medial totally symmetric quasigroups of order $n$ for all $n \leq 16$ predicted by Schwenk;

| Order | Idempotent Number | Classes | Unipotent Number | Classes |
|-------|------------------|---------|------------------|---------|
| 1     | 1                | 1       | 1                | 1       |
| 2     | 0                | 0       | 2                | 1       |
| 3     | 1                | 1       | 0                | 0       |
| 4     | 0                | 0       | 4                | 1       |
| 5     | 0                | 0       | 0                | 0       |
| 6     | 0                | 0       | 0                | 0       |
| 7     | 30               | 1       | 0                | 0       |
| 8     | 0                | 0       | 240              | 1       |
| 9     | 840              | 1       | 0                | 0       |
| 10    | 0                | 0       | 8400             | 1       |
| 11    | 0                | 0       | 0                | 0       |
| 12    | 0                | 0       | 0                | 0       |
| 13    | 1,197,504,000    | 2       | 0                | 0       |
| 14    | 0                | 0       | 16,765,056,000   | 2       |
| 15    | 60,281,712,691,200| 80     | 0                | 0       |
| 16    | 0                | 0       | 964,507,403,059,200| 80    |
the number of idempotent totally symmetric quasigroups, as well as the number of isomorphism classes of idempotent totally symmetric quasigroups computed by McKay and Wanless for those orders \( n < 16 \) that have such objects \((n = 1, 3, 7, 9, 13, 15)\);

-the number of unipotent totally symmetric quasigroups, as well as the number of isomorphism classes of unipotent totally symmetric quasigroups computed by McKay and Wanless for those orders \( n \leq 16 \) that have such objects \((n = 1, 2, 4, 8, 10, 14, 16)\).

## 2 PROCEDURAL SUMMARY

Viewing a totally symmetric quasigroup as a Latin square, or, equivalently, as being represented by its Cayley table, the entries along the main diagonal are of precisely two types—those for which \( xx = x \) (i.e., idempotent elements) and those for which this is not the case. In [1], Rosemary Bailey presented criteria constraining the quantity of each type of diagonal entry based on the order \( n \) of the quasigroup, and further described a procedure for constructing directed graphs for each permissible configuration of the diagonal. The software developed to enumerate quasigroups for this paper begins by constructing these “Bailey graphs”; for \( n = 16 \) there are 980 distinct, nonisomorphic Bailey graphs: 901 with one idempotent element, 77 with four idempotent elements, and two with seven idempotent elements. These correspond to 980 starting configurations, each with a distinct, nonisomorphic arrangement of the elements on the diagonal, and—in deference to the magnitude of the problem—each is processed separately, with the results recorded and ultimately combined.

The fundamental process amounts to constructing totally symmetric quasigroups (by building their Cayley tables) and then checking completed instances for isomorphism to quasigroups already constructed. In [7], McKay and Wanless describe a technique for extending a Bailey graph to a directed graph that completely describes the quasigroup; we employ this technique, allowing us to use the nauty graph automorphism software [6] to test for isomorphism.

A great deal of technical sleight-of-hand is required to make the algorithm workable in practice. Although there is a wide range of complexity among the starting configurations, a fairly typical example might have approximately 25 billion isomorphism classes and 500 sextillion quasigroups. Maintaining data structures for individual quasigroups is thankfully not necessary, but maintaining 25 billion isomorphism classes—with enough information to distinguish them from each other—is a daunting task, and in fact cannot be done within any reasonable limit on the amount of computer memory. The McKay–Wanless graph for \( n = 16 \) requires 408 bytes to represent in nauty; using compression to approximately halve this amount still results in an isomorphism class structure that requires roughly 240 bytes. This amounts to over 6 terabytes (TB) of memory just to store the isomorphism classes; naturally there are other memory requirements as well, making the prospect unfeasible for even the most powerful of today’s readily available consumer computers. This limitation is addressed by maintaining a large hash table of isomorphism classes, and, when an appropriate memory threshold is reached, flushing sections of the hash table to disk for storage and later processing. As the process is ongoing, this then requires further flushing of newly constructed isomorphism classes to disk, should they happen to belong to already flushed sections of the hash table. In some of the more challenging configurations the available disk space proves to be insufficient for the task, leaving no option but to discard some of the flushed sections of the hash table;
when this occurs subsequent iterations through the entire process are required to recover the abandoned data.¹

3 | SAMPLE RESULTS

We include in this section further details pertaining to a sampling of some of the more symmetric starting configurations—in general the more symmetric configurations are easier to compute, as they lend themselves to a higher degree of isomorph detection early on in the process. These examples would be good candidates for independent verification of the results presented herein, without committing to the full computation.

As described in Section 2, each starting configuration includes precisely 1, 4, or 7 idempotent elements; the highest degree of symmetry is found in some of the configurations with one idempotent element, and these have the added advantage of having more of the Cayley table specified from the start (the idempotent elements contribute no additional information to the configuration, whereas total symmetry implies that if \( xx = y \) for some \( x \neq y \), then \( xy = x \) and \( yx = x \), yielding two additional table entries). Thus the simplest configurations to compute are those that are highly symmetric with a single idempotent element. We present four such configurations (those with the largest automorphism groups—i.e., the most symmetric), as well as the most symmetric configuration with four idempotent elements, and the more symmetric of the two configurations with seven idempotent elements. For each we specify the initial configuration of the main diagonal as a 16-tuple, with the \( i \)th coordinate corresponding to the \( i \)th diagonal entry of the Cayley table (the result of the product of \( i \) with itself). We follow the convention prevalent in Latin squares of using the symbols 1–16.

**Example 1** (The unipotent case—one idempotent element).
Diagonal: \((1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)\).
Order of the automorphism group of the starting configuration: 1,307,674,368,000.

| Aut | Isomorphism classes | Quasigroups |
|-----|---------------------|------------|
| 1   | 36                  | 753,220,435,968,000 |
| 2   | 6                   | 62,768,369,664,000  |
| 3   | 12                  | 83,691,159,552,000  |
| 4   | 8                   | 41,845,579,776,000  |

¹One of the referees has suggested a very clever optimization during the review: Essentially, the size of the automorphism group of a given quasigroup dictates how many isomorphic quasigroups one can expect to encounter; once these are all accounted for, there is no longer any need to maintain a data structure for the isomorphism class in memory—a counter can be incremented, and the memory freed. In many cases this is sufficient to keep the memory requirements under a reasonable limit, thereby precluding the need to utilize disk storage, and avoiding the drastic measures required when that storage is exhausted. In practice this optimization sometimes produces dramatic performance improvements, but sometimes does not, and in the worst cases devolves to the situation as described above, where the available memory and disk space are both entirely consumed by the process.
| |Aut| Isomorphism classes | Quasigroups |
|---|---|---|
|5 | 1 | 4,184,557,977,600 |
|6 | 1 | 3,487,131,648,000 |
|8 | 2 | 5,230,697,472,000 |
|12 | 3 | 5,230,697,472,000 |
|21 | 1 | 996,323,328,000 |
|24 | 2 | 1,743,565,824,000 |
|32 | 1 | 653,837,184,000 |
|36 | 1 | 581,188,608,000 |
|60 | 1 | 348,713,164,800 |
|96 | 1 | 217,945,728,000 |
|168 | 1 | 124,540,416,000 |
|192 | 1 | 108,972,864,000 |
|288 | 1 | 72,648,576,000 |
|20,160 | 1 | 1,037,836,800 |

Total number of isomorphism classes: 80.
Total number of quasigroups: 964,507,403,059,200.

Note that the last class, containing 1,037,836,800 quasigroups, is medial as well as unipotent, is the only class with both of these properties, and has the largest order automorphism group of any class (see Table 1).

(This is, in fact, the elementary abelian 2-group of order 16, the only totally symmetric quasigroup that is also a group. As described earlier, the five isomorphism classes of abelian groups of order 16 correspond precisely to the five isomorphism classes of medial totally symmetric quasigroups of order 16; the other four are isotopic but not isomorphic to their corresponding quasigroups.)

**Example 2** (One idempotent element).
Diagonal: (1 1 2 2 2 2 2 2 2 2 2 2 2 2 2).
Order of the automorphism group of the starting configuration: 87,178,291,200.
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Example 5 (Four idempotent elements).
Diagonal: (1 1 3 3 5 5 7 7 7 7 7 7 7 7 7).
Order of the automorphism group of the starting configuration: 2,177,280.

|Aut| Isomorphism classes | Quasigroups |
|---|---------------------|-------------|
| 1 | 1,487,202           | 31,116,414,967,013,376,000 |
| 2 | 840                 | 8,787,571,752,960,000 |
| 3 | 242                 | 1,687,771,717,632,000 |
| 4 | 698                 | 3,651,026,835,456,000 |
| 8 | 38                  | 99,383,251,968,000 |
| 12| 6                   | 10,461,394,944,000 |
| 24| 6                   | 5,230,697,472,000 |

Total number of isomorphism classes: 1,489,032.
Total number of quasigroups: 31,130,656,412,663,808,000.

Example 6 (Seven idempotent elements).
Diagonal: (1 1 3 3 5 5 7 7 9 9 11 11 13 13 13 13).
Order of the automorphism group of the starting configuration: 4320.

|Aut| Isomorphism Classes | Quasigroups |
|---|---------------------|-------------|
| 1 | 3,055,198,131       | 63,923,268,561,123,299,328,000 |
| 2 | 63,312              | 662,331,836,694,528,000 |
| 3 | 3233                | 22,547,793,235,968,000 |
| 4 | 372                 | 1,945,819,459,584,000 |
| 6 | 104                 | 362,661,691,392,000 |
| 12| 12                  | 20,922,789,888,000 |

Total number of isomorphism classes: 3,055,265,164.
Total number of quasigroups: 63,923,955,770,157,170,688,000.

4 | HARDWARE

The project was completed on a pair of Dell Precision 5810 workstations (running Fedora Linux at runlevel 3); both computers were equipped with 256 GB of RAM and 12 TB of hard drive space (comprised of three 4 TB drives, configured in a RAID0 array). One of the computers had an additional 1 TB solid state drive; that computer was outfitted with an Intel Xeon E5-2699A v4 22-core 2.4 GHz processor; the other had an Intel Xeon E5-2697A v4 16-core 2.6 GHz processor. (These specifications reflect the ultimate configurations of
the two systems; there was initially only one, and much upgrading of components—including central processing units, memory, and disk drives—was performed while the project was ongoing.) The final results reported herein were obtained after approximately 12 months of computing.  

5  FEASIBILITY OF COMPUTING RESULTS FOR ORDER 17

Whereas there are 980 nonisomorphic starting configurations for totally symmetric quasigroups of order 16, there are only 35 such configurations for order 17 (18 with one idempotent element, nine with four idempotent elements, five with seven idempotent elements, two with 10 idempotent elements, and one with 13 idempotent elements). This relative paucity of cases makes attempting the enumeration of totally symmetric quasigroups of order 17 a seemingly attractive proposition. Preliminary investigations, however, indicate that the problem is more difficult than it might initially seem.

The most symmetric of the order 17 starting configurations having one idempotent element (which should in principle be the easiest configuration to compute) has an automorphism group of order 7776. This configuration was completed in 17.68 h.  By comparison, a less symmetric configuration, with $\text{Aut} = 30$, took 319.25 h—more than 13 days. Most of the order 17 starting configurations with one idempotent have significantly smaller automorphism groups (making their computation more complex), and, as the correspondence between automorphism group size and computation time is not linear, it is difficult to estimate how long the computation for all of the single idempotent starting configurations would take, except to say that it is almost certainly well in excess of a year (on a single workstation comparable to those described in Section 4). Initial testing indicates that these single idempotent computations require only a small percentage of the available memory, and so avoid the complications of flushing to disk—the complexity of their computation is solely a combinatorial matter, and any increase in processing power should bring about a corresponding decrease in processing time.

The situation for those starting configurations with four idempotent elements is quite different; the most symmetric of these (with $\text{Aut} = 15,552$) uses all of the available memory and all of the available disk space, and ultimately requires four passes to complete, with a total computation time of 551 h (23 days). As with the single idempotent configurations, one would expect the remaining starting configurations, with smaller automorphism groups, to take much longer.

The starting configurations with 7, 10, and 13 idempotents are likely to be even more difficult to compute, though the complexity added by the larger number of idempotent elements may be offset at least somewhat by a corresponding increase in the symmetry of these configurations (i.e., they have comparatively large automorphism groups).

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2The optimization suggested by one of the referees (described in a previous footnote) would certainly have reduced the amount of time required. Without completely repeating the computation, it is difficult to say by how much. However, as the hardware improved over the course of the computation, and the software improved after its completion, it is not unreasonable to speculate that the computation could have been completed in something on the order of 3 or 4 months.

3The results cited in this section were obtained with the code optimized as described in the footnote to Section 2.
Ultimately, it seems justified to speculate that the enumeration of totally symmetric quasigroups of order 17 is feasible, but will likely require more powerful computing resources than that were utilized for the order 16 computation—a sizable cluster (or a supercomputer) could conceivably complete the task in a reasonable timeframe.

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DATA AVAILABILITY STATEMENT
The data that support the findings of this study are available from the corresponding author upon reasonable request.

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