Sliding Window Polar Codes

Valerio Bioglio, Carlo Condo
Mathematical and Algorithmic Sciences Lab
Huawei Technologies France SASU
Email: {valerio.bioglio,carlo.condo}@huawei.com

Abstract—We propose a novel coupling technique for the design of polar codes of length $N$, making them decodable through a sliding window of size $M < N$. This feature allows to reduce the computational complexity of the decoder, an important possibility in wireless communication downlink scenarios. Our approach is based on the design of an ad-hoc kernel to be inserted in a multi-kernel polar code framework; this structure enables the sliding window decoding of the code. Simulation results show that the proposed sliding window polar codes outperform the independent blocks transmission in the proposed scenario, at the cost of a negligible decoding overhead.

Index Terms—Polar Codes, window decoding, information coupling

I. INTRODUCTION

Polar codes [1] are linear block codes relying on the channel polarization effect, that allows to create virtual bit-channels of different reliability. In a polar code of length $N$ and dimension $K$, the information is transmitted through the $K$ most reliable bit-channels, while the remaining bit-channels are “frozen” to a predetermined value, usually zero. Originally, polar codes were based on the polarization kernel $T_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, which limited code length $N$ to be a power of 2. This limitation has been overcome thanks to the discovery of polarization properties of larger kernels, for which kernels providing the best polarization properties have been identified [2]. In parallel, multi-kernel polar codes [3] have been proposed to create polar codes of virtually any length with efficient implementations and fast decoding [4], [5].

In this work, we focus on the communication between a transmitter and a receiver having different computational capabilities, namely when the receiver is less powerful than the transmitter, e.g. in the wireless communication downlink scenario depicted in Figure 1. In our model, the transmitter is able to create polar codewords of length $N$, while the receiver can handle only polar codewords of length $M < N$. The straightforward solution for this problem is to divide the information in $S = N/M$ blocks and transmit separately each block on a different codeword of length $M$. However, independent transmissions increase the block error rate (BLER) of the system, since transmitted information is recovered only if all the $S$ codewords are decoded correctly; a single error in one of the transmissions results in an overall decoding failure.

In this paper, we propose a novel polar code design for codes of length $N$ which can be decoded using a sliding window framework, i.e. considering $M < N$ received symbols per decoding step, and using the result of one step to facilitate the next. Recent works have already shown the benefits of inserting memory in the polar code encoding and decoding process [6], [7]. Moreover, windowed decoding has been proposed for the decoding of spatially coupled codes [8] for convolutional LDPC codes [9]; a similar approach has been proposed in [10] to construct partially information coupled polar codes. Our approach differs from the aforementioned techniques, as it is based on the definition of an ad-hoc kernel enabling the sliding window decoding of a particular multi-kernel polar code. Accordingly, the frozen set needs to be designed on the basis of the resulting transformation matrix. Finally, we show through both theoretical analysis and simulations that the proposed code design is able to outperform independent blocks transmission in the proposed scenario with negligible decoding overhead.

II. BACKGROUND

In this section, we review the basic concepts of polar codes design and decoding that will be useful for description and analysis of the proposed sliding window polar codes.

A. Basic channel transform

The basic combination of two independent binary discrete memoryless channels (IB-DMCs) $\mathcal{W}_0$ and $\mathcal{W}_1$ defined over the same alphabets $\mathcal{X} \rightarrow \mathcal{Y}$, with $\mathcal{X} = \{0, 1\}$, is depicted in Figure 2 and performed by the channel transformation matrix $T_2$. The transition probabilities of the split channels $\mathcal{W}^-$ and $\mathcal{W}^+$ are given by

$$W^-(y_0, y_1 | u_0) = \frac{1}{2} \sum_{x_1 \in \mathcal{X}} W_0(y_0 | u_0 \oplus u_1) W_1(y_1 | u_1) ,$$

$$W^+(y_0, y_1, u_0 | u_1) = W_0(y_0 | u_0 \oplus u_1) W_1(y_1 | u_1) .$$

Fig. 1: Wireless communication downlink scenario.
The one-step transformation defined by $T_2$ can be rewritten using operators $\boxplus$ and $\otimes$ introduced in [11] to degrade and enhance channel parameters as

$$W^- = W_0 \boxplus W_1,$$
$$W^+ = W_0 \otimes W_1.$$  

The Bhattacharyya parameter of channel $W_i$ is defined as

$$Z(W_i) = \sum_{y \in Y} \sqrt{W_i(y|0)W_i(y|1)}$$

and can be used to evaluate the channel reliability, since it provides an upper bound on the probability that an error occurs under ML decoding [11]. The Bhattacharyya parameter of the transformed channels can be calculated on the basis of that of the original channels as

$$Z(W_0 \boxplus W_1) \leq 1 - (1 - Z(W_0)) \cdot (1 - Z(W_1)),$$
$$Z(W_0 \otimes W_1) = Z(W_0)Z(W_1).$$

**B. Polar codes**

Polar codes rely on the polarization acceleration enabled by the $n$-fold Kronecker product of the basic channel transformation kernel $T_2$: for a polar code length $N = 2^n$, its channel transformation matrix is defined as $T_N = T_2^\otimes n$. As the code length goes toward infinity, bit-channels become completely noisy or completely noiseless, and the fraction of noiseless bit-channels approaches the channel capacity. In case of finite code lengths, however, the polarization of bit-channels is incomplete, generating bit-channels that are partially noisy. The Bhattacharyya parameter of these intermediary bit-channels can be tracked throughout the polarization stages to estimate their reliabilities. For an $(N,K)$ polar code, the indices of the $K$ most reliable bit-channels are selected to form the information set $I$. The input vector $u = [u_0, u_1, \ldots, u_{N-1}]$ is then created by assigning the $K$ message bits to the entries of $u$ whose indices are listed in $I$; the remaining entries of $u$, forming the frozen set $F$, are set to zero. The codeword $x$ is finally calculated as $x = u \cdot T_N$.

**C. Successive-Cancellation Decoding**

In [11], the native decoding algorithm for polar codes, called successive cancellation (SC), was proposed as well. The decoding process is portrayed in Figure 3 as a depth-first binary tree search, with priority given to the left branch. Given a node at stage $t$, let us define as $\alpha$ the set of $2^t$ likelihoods received from its parent at stage $t+1$, initialized as the channel information at the root node. The node computes $2^{t-1}$ soft values composing $\alpha^l$, that is then sent to the left child at stage $2^{t-1}$. From the left child, $2^{t-1}$ partial sums composing $\beta^l$ are received, allowing for the calculation of likelihood vector $\alpha^r$ to be sent to the right child, that returns the $\beta^r$ partial sum vector. The $2^t$-element partial sum vector $\beta$ can be finally computed and sent to parent node at stage $t+1$. At $t = 0$, the decoded bit $\hat{u}_i$ is estimated by taking hard decisions on the basis of received likelihoods.

To improve the error-correction performance of SC at moderate code lengths, SC list (SCL) decoding has been proposed in [12]. It relies on $L$ parallel SC decoders working on different paths. Every time an information bit is estimated, the paths are doubled, with $L$ decoders considering it a 0, and $L$ a 1. To each path is associated a metric, such that the $L$ paths with the largest path metric are discarded, and the decoding proceeds until last information bit has been decoded.

### III. Sliding Window Polar Codes

Here we describe how to design, encode and decode a polar code of length $N = 2^n$ and dimension $K$ such that it is decodable through a sliding window of size $M = 2^m$.

**A. Sliding window kernel**

The sliding window kernel $W_S$ is the cornerstone of the proposed construction. It is defined by the full binary lower triangular matrix of size $S$, i.e. the square matrix of size $S \times S$ having ones on and below the diagonal, and zeros above the diagonal, as depicted in Figure 3. The matrix $W_S$ imposes a channel transformation that can be described as a cascade of basic channel transformations, as represented in Figure 4.

This structure imposes to decode input bit $u_i$ using only two channel likelihood out of $S$, namely $\hat{x}_i$ and $\hat{x}_{i+1}$, the former being modified by the hard decisions taken on bits $0 \leq j < i$. We will see how to leverage on this property to construct multi-kernel polar code mixing $W_S$ with classical polar kernel $T_2^\otimes m$ to enable a sliding window decoding mechanism.

If the encoded bits are transmitted over channel $W$, the virtual channel $W_t$ experienced by input bit $u_i$ transmitted through channel transformation matrix $W_S$ undergoes the transformation

$$W_t = W \boxplus (W \boxplus W \boxplus \ldots \boxplus W),$$

where $i+1$ times.
For the AWGN channel, the equations tracking the likelihood where
\[
\phi_i = \frac{1}{2} \left( 1 + \frac{1}{x_i} \right)
\]
be calculated as

\[Z(W_{i-1}) \leq \begin{cases} 
1 - (1 - Z(W)) \cdot (1 - Z(W)^i) & \text{if } i < S, \\
Z(W)^S & \text{if } i = S.
\end{cases}
\]

The Bhattacharyya parameter of these virtual channels, depicted in Figure 4b, can hence be calculated as

\[Z(W_{i-1}) \leq \begin{cases} 
1 - (1 - Z(W)) \cdot (1 - Z(W)^i) & \text{if } i < S, \\
Z(W)^S & \text{if } i = S.
\end{cases}
\]

The Bhattacharyya parameter can be used to evaluate the bit-channel reliability for various channel models. For binary erasure channels (BECs), the bit error probabilities can be directly calculated from them, while under additive white Gaussian noise (AWGN) channels, density evolution under Gaussian approximation (DE/GA) technique can be implemented on their basis [13]; the latter algorithm estimates the likelihood distribution of the polarized channels by tracking their mean at each stage of the SC decoding tree. Given the block representation of kernel \(W_S\) depicted in Figure 4c, the bit error probability \(\delta_i\) of bit \(u_i\) under BEC transmission can be calculated as

\[\delta_i = \begin{cases} 
1 - (1 - \delta) \cdot (1 - \delta^S) & \text{if } i < S, \\
\delta^S & \text{if } i = S,
\end{cases}
\]

where \(\delta\) is the bit erasure probability of the original BEC. For the AWGN channel, the equations tracking the likelihood mean \(\mu_i\) of input bit \(u_i\) for \(W_S\) are given by

\[\mu_{i-1} = \begin{cases} 
\phi^{-1} (1 - (1 - \phi(\mu)) \cdot (1 - \phi(\mu))) & \text{if } i < S, \\
\mu_i & \text{if } i = S,
\end{cases}
\]

where \(\phi(\cdot)\) is defined as

\[\phi(x) = \begin{cases} 
1 - \frac{x}{2 \sqrt{\pi x}} \int_{-\infty}^{x} \tanh \left( \frac{z}{2} \right) \, dz & \text{if } x > 0; \\
1 & \text{if } x = 0,
\end{cases}
\]

and can be approximated as described in [13].

**B. Code design**

As for classical polar codes, the design of a sliding window polar code entails the selection of its transformation matrix \(T\) and frozen set \(F\). Given \(S = N/M\), the transformation matrix of the code is defined as \(T = W_S \otimes T_M\), where \(T_M = T_2^{\otimes m}\) is the transformation matrix of a polar code of length \(M\).

A sliding window polar code can hence be described as a particular multi-channel polar code [13] constructed by placing kernel \(W_S\) before the transformation matrix of a classical polar code of length \(M\). Note that the position of \(W_S\) with respect to the other kernels is not interchangeable. The Tanner graph of the resulting transformation matrix \(T\) is depicted in Figure 5.

The frozen set can be designed according to the general multi-channel polar code approach [13] using equations described in Section IIIA. Given the structure of \(T\), however, the calculation of channel bit reliabilities can be simplified as follows. The Tanner graph depicted in Figure 5 shows that the channel transformation imposed by \(T\) can be seen as the juxtaposition of \(S\) polar code transformations of length \(M\), each one altering a different channel whose reliability depends on \(W_S\). Given the transmission channel \(W\), it is thus possible to initially calculate the Bhattacharyya parameter of the virtual channel \(W_s\) experienced by the \(s\)-th polar code \(P_s\) defined by \(T_M\) using (10); then, classical polar codes design equations can be used to evaluate the bit-channel reliabilities of input bits \(u_{(s-1)M}, u_{(s-1)M+1}, \ldots, u_{sM-1}\). With reference to Figure 5, the Bhattacharyya parameters at the left of each \(T_M\) block would be independently calculated using the inputs at their right, that have already undergone one further polarization step through \(W_S\). Finally, all the bit-channels of input vector \(u\) are sorted in order of reliability, where the indices of the \(N - K\) least reliable bit-channels form the frozen set \(F\), and the indices of the remaining \(K\) bit-channels form the information set \(I\) of the code.

**C. Encoding**

The straightforward encoding process for the proposed sliding window polar codes follows the standard polar code technique. The \(K\) message bits are inserted in the input vector \(u\) according to the information set previously calculated, namely storing their values in the indices listed in \(I\), while the remaining bits of \(u\) are set to zero. Codeword \(x\) can be calculated as \(x = u \cdot T\), where \(T\) is the transformation matrix of the code calculated as previously described. Codeword \(x\) is then transmitted through the channel.

However, the particular structure of \(W_S\) allows for an alternative encoding algorithm based on previously described set of \(S\) polar codes \(P_1, \ldots, P_S\) of size \(M\). The information
set $\mathcal{I}_s$ of polar code $\mathcal{P}_s$ can be extracted from the global information set $\mathcal{I}$ as the set of entries of $\mathcal{I}$ comprised between $(s-1) \cdot M$ and $s \cdot M - 1$, for $s = 1, \ldots, S$. Similarly, partial input vectors $u_s$ for $1 \leq s \leq S$ are extracted from $u$ or can be created on the basis of the message bits. Each partial input vector can be encoded independently through polar encoding of length $M$, obtaining $S$ partial codewords $t^{(1)}, \ldots, t^{(S)}$. Finally, codeword $x$ is obtained by backward accumulation of the partial codewords, starting from the last one, i.e., $x = [t^{(1)} \oplus \ldots \oplus t^{(S)}], t^{(2)} \oplus \ldots \oplus t^{(S)}, \ldots, t^{(S-1)} \oplus t^{(S)}]$. This encoding strategy, depicted in Figure 5, follows the structure of sliding window kernel $W_S$ and permits to maintain the classical polar encoding structure, however slightly increasing the encoding latency. In fact, each parallel encoder requires $\log_2 M$ steps to encode the $M$-length polar codes, then $S$ steps to obtain $x$, for a total of $\log_2 M + S$ steps, i.e. slightly larger than the $\log_2 N = \log_2 M + \log_2 S$ steps required for the encoding of a standard polar code of length $N$.

D. Decoding

The decoding of the proposed sliding window polar code design is performed in $S$ SC decoding steps, each one employing $M$ likelihoods. Each SC decoding step outputs $M$ bits, representing the estimated input vector $\hat{u}$; each SC decoding step takes $M$ received symbols as input, conveniently modified by the $M$ estimated input bits decoded at the previous step. In the following, logarithmic likelihood ratios (LLRs) are used in SC decoding, as proposed in [13].

The decoding procedure is detailed in Algorithm 1. Let us suppose the $N$ channel LLRs to be stored in vector $y$. This vector is initially split into $S$ sub-vectors of size $M$ as $y = [y^{(1)} | y^{(2)} | \ldots | y^{(S)}]$. The LLR buffer $l$ is initialized with the first $M$ entries of $y$, namely fixing $l = y^{(1)}$. At decoding step $s$, the decoder performs the SC decoding of the $(M, K_s)$ polar code $\mathcal{P}_s$ defined as follows. The information set $\mathcal{I}_s$ is calculated as the set of entries of $\mathcal{I}$ comprised between $(s-1) \cdot M$ and $s \cdot M - 1$, while the transformation matrix is given by $T_M$. The estimated input vector $\hat{u}^{(s)}$ is obtained through SC decoding of code $\mathcal{P}_s$, using vector $v = l \oplus y^{(s+1)}$ as channel LLRs, where $y^{(s)}$ is the sub-vector of $y$ containing its entries from $s \cdot M$ to $(s+1) \cdot M - 1$ as $y^{(s)}(i) = y((s-1) \cdot M + i)$ and

$$A \oplus B = 2 \tanh^{-1}(\tanh(A/2) \cdot \tanh(B/2)) \quad \approx \text{sgn}(A) \cdot \text{sgn}(B) \cdot \min(A, B).$$

Vector $x^{(s)} = \hat{u}^{(s)} \cdot T_M$ is then calculated to be used as partial sum, namely to update the LLR buffer as $l = (1)^{x^{(s)}} \cdot l + y^{(s+1)}$. The decoding of last sub input vector $\hat{u}^{(S)}$ constitutes an exception since it is obtained using directly

\[ \text{Fig. 5: Tanner graph of transformation matrix } T \text{ of sliding window polar code.} \]

Algorithm 1 Sliding Window Successive Cancellation

1: Load information set $\mathcal{I}$
2: Load channel LLRs $y$
3: Initialize buffer $l = y^{(1)}$
4: for $s = 1 \ldots S - 1$ do
5:   $\mathcal{I}_s = \{i - (s-1) \cdot M \mid i \in \mathcal{I}, (s-1) \cdot M \leq i < s \cdot M\}$
6:   $\hat{u}^{(s)} = \text{Decode}(l \oplus y^{(s+1)}, \mathcal{I}_s)$
7:   $x^{(s)} = \hat{u}^{(s)} \cdot T_M$
8:   $l = (1)^{x^{(s)}} \cdot l + y^{(s+1)}$
9: $\mathcal{I}_S = \{i - (S-1) \cdot M \mid i \in \mathcal{I}, (S-1) \cdot M \leq i < S \cdot M\}$
10: State $\hat{u}^{(S)} = \text{Decode}(l, \mathcal{I}_S)$
11: return $\hat{u} = [\hat{u}^{(1)}, \ldots, \hat{u}^{(S)}]$
Fig. 6: Encoding scheme of sliding window polar codes.

the content of the LLR buffer \( l \), i.e. imposing \( v = l \). Finally, the input vector estimation \( \hat{u} \) is obtained juxtaposing the sub input vectors as \( \hat{u} = [\hat{u}^{(1)}, \ldots, \hat{u}^{(S)}] \). This decoding strategy can be run on-the-fly during the reception of channel signals. This procedure is depicted in Figure 7. An SC list decoder \[12\] for the proposed sliding window polar codes can be easily implemented on the basis of the described decoding algorithm.

E. Performance analysis

In this section, we analyze the performance of the proposed scheme both in terms of latency and BLER.

Under AWGN channels it is possible to exploit the results of the DE/GA algorithm to approximate the BLER of SC decoding of polar codes. Considering binary phase-shift keying (BPSK) modulation, the mean values of the channel LLRs are given by \( -10 \gamma / 10 \rangle / N \), where \( \gamma \) represents the channel \( E_b / N_0 \) in dB. Under the Gaussian assumption, the bit error probability \( P_e(u_i) \) is related to the LLR mean value \( \mu_i \) as \( P_e(u_i) = Q(\sqrt{\mu_i/2}) \), where \( Q(\cdot) \) denotes the tail probability of the standard Gaussian distribution. The block error probability under SC decoding can hence be approximated by

\[
P^{SC}_e \sim \sum_{i \in \mathbb{I}} Q\left(\sqrt{\frac{\mu_i}{2}}\right).
\]

Equation (16) can be used directly for sliding window polar codes designed under DE/GA. An estimate of the BLER under SC in case of the independent transmission of \( S \) uncorrelated polar codewords with the same length \( M \) is given by \( P^{SC}_e \sim 1 - (1 - P^{SC}_e)^S \), where \( P^{SC}_e \) is the expected BLER of a single codeword transmission, calculated according to (16).

Figure 8 shows the theoretical \( E_b / N_0 \) required by each code construction to reach the target BLER of \( 10^{-3} \) as a function of dimension \( K \) for different window sizes \( M \). Independent transmission (IND) can be seen as the state-of-the-art in the envisaged scenario: the transmitter divides the \( K \) message bits into \( S = N / M \) messages of \( K' = K / S \) bits, that are encoded and transmitted independently using \( S \) polar codes of length \( M \) and dimension \( K' \). The transmission is successful if all \( S \) blocks are decoded correctly. In case of full polar code (FULL) transmission, the transmitter ignores the limitations at the receiver and transmits a codeword obtained using the full \( (N, K) \) polar code designed according to the

best frozen set, which is decoded smoothly at the receiver. Finally, in the proposed sliding window decoding (SW) the transmitter designs and encodes a polar codeword using the described technique. The receiver uses the proposed sliding window decoder to decode the received signal. We can see that the proposed SW technique outperforms IND for all rates at equal window dimension, with a substantial gain of 1 dB for some of the rates. Moreover, for \( M = 256 \) sliding window polar codes show a gap of less than 0.5dB from the optimal FULL. Unfortunately, theoretical bounds on the BLER performance of polar codes under SCL are unknown, so it is not possible to perform a similar analysis for list decoders.

Regarding the SC decoding latency, for the first \( S - 1 \) phases the sliding window decoder requires \( 2M \) time steps to decode a length-\( M \) polar code and update the buffer \( l \), while \( 2M - 2 \) time steps are sufficient for the last decoding, leading to a total latency of \( 2MS - 2 = 2N - 2 \) time steps. This is equal to the latency of an SC decoder for a non-systematic code of length \( N \) under the same assumption. However, the implementation complexity of the proposed sliding window decoder is that of a polar decoder of length \( M \), plus \( M \) memory elements to store buffer \( l \) and the logic to update it. We can thus conservatively identify as a complexity upper bound that of a decoder for a code of length \( 2M \); consequently, if \( N \gg 2M \), the proposed decoder always yields a lower implementation cost than a length-\( N \) decoder. A similar analysis can be done for SCL decoding, obtaining the same outcome.

IV. Simulation results

In this section we present the BLER performance of the proposed sliding window design and decoding of polar codes,
comparing to state-of-the-art independent block transmissions and optimal full polar code transmission. We study a scenario where the transmitter sends $K$ bits to the receiver at a code rate $R = K/N$, but the receiver has limited decoding capabilities and can handle only $M < N$ bit.

Figure 11 shows the performance of the IND, FULL and SW strategies for codes of rate $R = K/N = 1/4$. The FULL case is used as a benchmark of the best possible achievable BLER. The codes are decoded using either the SC or SCL algorithms with list size $L = 8$. Black curves correspond to the SC bound calculated according to \[16\], while black markers depict simulation results obtained through SC decoding. We can see that the bounds perfectly match the simulations, hence they can be considered as reliable approximations of the SC decoding curves for the proposed code lengths. Red curves with red markers correspond to SCL decoding simulations. The proposed solution outperforms IND under both SC and SCL decoding, while approaching the optimal performance represented by FULL. The entity of the gain provided by SW with respect to IND is significant, corresponding to 1.5dB.

V. CONCLUSION

In this work, we presented a novel multi-kernel construction that allows a sliding window decoding approach, thanks to which only a fraction of the codeword bits need to be received before bits can be decoded. This feature is extremely useful in scenarios where the receiver has lower computational capabilities than the transmitter, e.g. downlink in wireless communications.

For this reason, we envisage the application of the proposed technique in future wireless 5G+ networks, where multiple devices with different computational capabilities need to be served concurrently. The performance of the proposed construction and decoding outperforms the state-of-the-art solution with little additional complexity, and can approach the achievable performance with lower complexity and no cost in terms of latency.

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