A scaling study of the step scaling function in SU(3) gauge theory with improved gauge actions

S. Takeda¹, S. Aoki¹, M. Fukugita³, K-I. Ishikawa⁵, N. Ishizuka¹,², Y. Iwasaki¹,², T. Kanaya¹, T. Kaneko⁴, Y. Kuramashi¹,², M. Okawa⁵, Y. Taniguchi¹,², A. Ukawa¹,², T. Yoshi¹,²
(CP-PACS Collaboration)

¹Graduate School of Pure and Applied Sciences, University of Tsukuba, Tsukuba, 305-8571, Japan
²Center for Computational Sciences, University of Tsukuba, Tsukuba, 305-8577, Japan
³Institute for Cosmic Ray Research, University of Tokyo, Kashiwa 277-8582, Japan
⁴High Energy Accelerator Research Organization (KEK), Tsukuba, Ibaraki 305-0801, Japan
⁵Department of Physics, Hiroshima University, Higashi-Hiroshima, Hiroshima 739-8526, Japan
(November 5, 2018)

We study the scaling behavior of the step scaling function for SU(3) gauge theory, employing the renormalization-group improved Iwasaki gauge action and the perturbatively improved Lüscher-Weisz gauge action. We confirm that the step scaling functions from the improved gauge actions agree with that previously obtained from the plaquette action within errors in the continuum limit at both weak and strong coupling regions. We also investigate how different choices of boundary counter terms for the improved gauge actions affect the scaling behavior. In the extrapolation to the continuum limit, we observe that the cut off dependence becomes moderate for the Iwasaki action, if a perturbative reduction of scaling violations is applied to the simulation results. We also measure the low energy scale ratio with the Iwasaki action, and confirm its universality.

I. INTRODUCTION

The strong coupling constant is one of the fundamental parameters of QCD. The current world average leads to $\alpha_{\overline{MS}}(m_Z) = 0.1172(20)$ [1]. Lattice QCD calculations have a potential ability to determine the strong coupling constant from an experimental input at low energy scales. In practice, however, one must relate the high energy perturbative QCD scale to the low energy hadronic scale. Alpha Collaboration proposed the Schrödinger functional (SF) scheme as a vehicle for this purpose [2,3], and it has been successfully applied to lattice QCD in various aspects [4–7]. One of the most recent results related to our study is the running coupling constant of two massless flavor QCD reported in Ref. [8,9].

Recently CP-PACS and JLQCD Collaborations have started a project for $N_f = 3$ QCD simulations [10–14]. These simulations are essential to understand the low energy QCD dynamics for the real world in which three light quarks exist. One of the targets of the project is to evaluate the strong coupling constant $\alpha_{\overline{MS}}$ in $N_f = 3$ QCD using the SF scheme. In the project Iwasaki gauge action [15] is employed to avoid the strong lattice artifacts of the plaquette gauge action found in $N_f = 3$ simulations [11].

In a previous study [16], as our first step toward evaluation of $\alpha_{\overline{MS}}$ for $N_f = 3$, O($a$) boundary improvement coefficients in the SF scheme have been determined for various improved gauge actions up to one-loop order in perturbation theory. In addition the scaling violation in the step scaling function (SSF) for the coupling have been analyzed perturbatively. In the present paper, as the next step, we investigate the lattice cut off dependence of the SSF non-perturbatively in quenched lattice QCD simulations with improved gauge actions. The renormalization-group improved Iwasaki gauge action and the perturbatively improved Lüscher-Weisz gauge action are employed. We investigate the effect of various choices for boundary improvement coefficients in detail, to find the best choice, which will be used in our unquenched simulations in the future. We also confirm the universality of the SSF and the low energy scale ratio, by comparing our results with the previous ones obtained by ALPHA Collaboration [17,18].

The rest of this paper is organized as follows. In Sec. II, after a brief introduction of the SF scheme and its extension to improved gauge actions, we specify the action and the O($a$) boundary improvement coefficients used in our simulations. We then define the Schrödinger functional coupling constant, the step scaling function and the low energy scale ratio. In Sec. III, we give details of simulations and present our results with improved gauge actions for various choices for O($a$) improvement. In Sec. IV, we investigate the lattice cut off dependence of the step scaling function and the low energy scale ratio, and carefully take the continuum limit of these quantities, in order to
confirm their universality. Our conclusion is given in the last section, together with a discussion toward $N_t = 2$ and $3$ simulations.

II. PRELIMINARIES

A. Schrödinger functional

The SF scheme introduced by ALPHA Collaboration is a powerful tool to probe the energy evolution of physical quantities. In the SF scheme, the theory is defined on a finite box of size $L^3 \times T$ with the periodic boundary condition in the spatial directions and the Dirichlet boundary condition in the time direction. We set $T = L$ throughout this paper. In the pure SU(3) gauge theory with Wilson plaquette action $S[U]$, the Schrödinger functional is given by

$$ Z = \int D[U] e^{-S[U]}, \quad (II.1) $$

where the link variables $U(\mu, x)$ for the gauge fields satisfy the boundary conditions

$$ U(x, k)|_{x_0 = 0} = \exp\{aC\}, \quad U(x, k)|_{x_0 = L} = \exp\{aC'\}. \quad (II.2) $$

Here $a$ is lattice spacing, and $C$, $C'$ are spatially constant diagonal matrices, which depend on the background field parameters $\eta$ and $\nu$ [19].

An extension of the SF scheme to the improved gauge actions was first discussed by Klassen [20] in terms of a transfer matrix construction [21]. In this formulation, each boundary consists of two time slices, to achieve the tree-level $O(a^2)$ improvement.

In this paper, however, we adopt an alternative formulation [22], which achieves the tree-level $O(a)$ improvement with only one time slice at each boundary. The dynamical variables to be integrated over are independent of the form of the action, whether plaquette or improved, and consist of the spatial link variables $U(k, x)$ with $x_0 = a, \cdots, L - a$ and temporal link variables $U(0, x)$ with $x_0 = 0, \cdots, L - a$ on the cylinder with volume $L^3 \times L$. This formulation is implemented more easily in numerical simulations.

B. Gauge action and $O(a)$ boundary improvement coefficients

The improved action we employ includes the plaquette and rectangle loops, and is given by

$$ S_{\text{imp}}[U] = \frac{1}{g_0^2} \sum_{C \in S_0} W_0(C, g_0^2) 2\mathcal{L}(C) + \frac{1}{g_0^2} \sum_{C \in S_1} W_1(C, g_0^2) 2\mathcal{L}(C), \quad (II.3) $$

with

$$ \mathcal{L}(C) = \Re \text{Tr}[I - U(C)], \quad (II.4) $$

where $W_i$ is a weight factor to be specified later and $U(C)$ is an ordered product of the link variables along a loop $C$ contained in a set $S_0$ (plaquette) or $S_1$ (rectangular). $S_0$ and $S_1$ consist of all loops of the given shape which can be drawn on the cylindrical lattice with the volume $L^3 \times L$. The loops involve the “dynamical links” in the sense specified above, and spatial links on the boundaries at $x_0 = 0$ and $x_0 = L$. In particular, rectangles protruding from the boundary of the cylinder are not included.

One needs to choose the weight factors appropriately to achieve the one-loop level $O(a)$ improvement. Among various possible choices, ours is given as follows.

$$ W_0(C, g_0^2) = \begin{cases} 
  c_0 c_5(g_0^2) & \text{for } C \in P_k : \text{Set of plaquettes that lie completely on one of the boundaries,} \\
  c_0 c_5^P(g_0^2) & \text{for } C \in P_t : \text{Set of plaquettes that just touch one of the boundaries,} \\
  c_0 & \text{for } C \in P_{\text{other}} : \text{otherwise},
\end{cases} \quad (II.5) $$

2
On the other hand, for the one-loop boundary terms, we can freely set a relation between improvement coefficients in eqs. (II.7) and (II.8) can uniquely be determined from two requirements that the tree-level action. The SF coupling is defined through the free energy \( \Gamma \) in eq. (II.12) where we impose the same boundary condition eq. (II.2) for the link variables as in the case of the Wilson plaquette.

where the coefficients \( c_0 \) and \( c_1 \) of the improved gauge action are normalized such that \( c_0 + 8c_1 = 1 \). In this paper we consider not only the Iwasaki action \( (c_1 = -0.331, c_2 = c_3 = 0) \) [15] but also the Lüscher-Weisz (LW) action \( (c_1 = -1/12, c_2 = c_3 = 0) \) [23] for comparison, since our perturbative analysis [16] shows that the LW action has a fairly small lattice artifact in the step scaling function. We call \( c_t^P(g_0^2) \) and \( c_t^R(g_0^2) \) O(\( a \)) boundary improvement coefficients. The assignments at the \( t = 0 \) boundary are shown in Fig. 1. The leading term of the O(\( a \)) boundary improvement coefficients in eqs. (II.7) and (II.8) can uniquely be determined from two requirements that the tree-level O(\( a \)) improvement is achieved and the lattice background field satisfies the equation of motion at the boundaries [22].

On the other hand, for the one-loop boundary terms, we can freely set a relation between \( c_t^P(1) \) and \( c_t^R(1) \), since there is only one requirement for the one-loop O(\( a \)) improvement.

Let us see how we specify the one-loop boundary terms. In Ref. [16], one finds the following relation to achieve the one-loop O(\( a \)) improvement:

\[
c_0 c_t^P(1) + 4c_1 c_t^R(1) = A_1/2,
\]

where \( A_1 \) is the coefficient of the \( a/L \) term in the one-loop correction \( m_t(0) \) (\( L/a \)) to the SF coupling. In our simulations we consider two choices: one called condition A is given by

\[
c_t^R(1) = 2c_t^P(1),
\]

and the other called condition B is specified by

\[
c_t^R(1) = 0.
\]

The difference between the conditions A and B is an O(\( a^3 \)) contribution in the one-loop correction to the SF coupling [16]. Although this difference is tiny at one-loop order, it may become larger at higher orders. The values of one-loop boundary terms for each condition and \( A_1 \) are given in Table I. For the LW action, the difference between the two conditions is small, so we do not carry out simulations with the condition B.

### C. Schrödinger functional coupling

The SF with the improved gauge action is given by

\[
Z = e^{-\Gamma} = \int D[U] e^{-S_{\text{SF}}[U]},
\]

where we impose the same boundary condition eq. (II.2) for the link variables as in the case of the Wilson plaquette action. The SF coupling is defined through the free energy \( \Gamma \) in eq. (II.12)

\[
g_{\text{SF}}^2(L) = k/\Gamma'|_{\eta=\nu=0} = k/\left\langle \frac{\partial S}{\partial \eta} \right\rangle_{\eta=\nu=0},
\]

where \( k \) is a normalization constant

\[
k = 12 \left( \frac{L}{a} \right)^2 [c_0 (\sin 2\gamma + \sin \gamma) + 4c_1 (\sin 4\gamma + \sin 2\gamma)],
\]

\[
\gamma = \left( \frac{a}{L} \right)^2 \left( \eta + \frac{\pi}{3} \right).
\]

The renormalized coupling \( g_{\text{SF}}^2(L) \) depends only on the scale determined by the box size \( L \). The \( \eta \)-derivative of the loop touching the boundaries (the left-most three loops in Fig. 1, for example), contributes to the observable \( \partial S/\partial \eta \) in eq. (II.13).
D. Step scaling function and Low energy scale ratio

The step scaling function (SSF) describes the evolution of the renormalized coupling under a finite rescaling factor $s$ (we take $s = 2$ in the following)

$$\sigma(2, u) = \tilde{g}^2(2L)\big|_{g^2(L) = u}.$$  (II.16)

By choosing the $n + 1$-th initial value of $\sigma(2, u_{n+1})$ such that $u_{n+1} = \sigma(2, u_n)$, the non-perturbative evolution of the running coupling can be constructed successively in order to cover a wide range of the energy scale.

The SSF $\sigma(2, u)$ in the continuum theory is obtained by the continuum limit of a lattice SSF $\Sigma(2, u, a/L)$

$$\sigma(2, u) = \lim_{a/L \to 0} \Sigma(2, u, a/L).$$  (II.17)

In this paper, we study the SSF at a weak coupling $u = 0.9944$ and a strong coupling $u = 2.4484$, where our results can be compared with those of ALPHA Collaboration.

To fix the scale in a physical unit, one needs to relate the box size $L$ prescribed at a certain value $\tilde{g}^2_{\text{SP}}(L)$ to some reference scale. Following the conventional way, we set $L = L_{\text{max}}$ defined implicitly

$$\tilde{g}^2_{\text{SP}}(L_{\text{max}}) = 3.480,$$  (II.18)

and adopt Sommer’s scale $r_0$ [24] as the reference scale. Eventually this amounts to computing the ratio $L_{\text{max}}/r_0$ and extrapolating it to the continuum limit

$$L_{\text{max}}/r_0 = \lim_{a/L_{\text{max}} \to 0} (L_{\text{max}}/a) \times (a/r_0).$$  (II.19)

III. SIMULATION DETAILS AND RESULTS

We follow the calculation procedure of Ref. [19]. Simulations for the SSF on larger lattices are performed on CP-PACS using 4 partitions of 64 PU’s, while $r_0/a$ are calculated with 2 partitions of 512 PU’s.

As mentioned in Sec. II C, the SF coupling is obtained by calculating the observable $\partial S/\partial \eta$ for gauge configurations with the SF boundary condition. When the number of spatial lattice points, $L/a$, is a multiple of 4, the gauge configurations are generated by a combined five-hit pseudo-heat-bath algorithm and an over-relaxation algorithm (HB). The combination of one pseudo-heat-bath update sweep followed by $N_{\text{OR}} = L/2a$ over-relaxation sweeps is called an iteration. The measurement is implemented after each sweep, i.e. $(1 + L/2a)$ measurements are made per one iteration. Because of a restriction of the HB method optimized for the improved gauge action on CP-PACS, we employ the hybrid Monte Carlo (HMC) algorithm for $L/a$ being different from multiples of 4, for instance $L/a = 6$. The step size for the molecular dynamics is adjusted to achieve an acceptance rate in a range from 0.7 to 0.8. The measurement is made for every trajectory.

Our computations for the renormalized coupling $\tilde{g}^2_{\text{SP}}(L)$ are carried out on lattices $L/a = 4, 6, 8, 12$. In this calculation, a re-weighting technique is used for a tuning of $\beta$ [25] such that $\tilde{g}^2_{\text{SP}}(L)$ becomes a certain prescribed value $u$ for each $L/a$. And then, using the same $\beta$, a computation on a lattice with twice the linear size $2L/a$ gives $\tilde{g}^2_{\text{SP}}(2L)$. The results are summarized in Table II and III for the weak ($u = 0.9944$) and strong ($u = 2.4484$) couplings, respectively. Errors in both $\tilde{g}^2_{\text{SP}}(L)$ and $\tilde{g}^2_{\text{SP}}(2L)$ are estimated by a jack-knife method. The bin size for jack-knife errors is 100 iterations for the HB and 500 trajectories for HMC, respectively. The precision in $\tilde{g}^2_{\text{SP}}(L)$ is attained by accumulating around 120000 – 140000 iterations on the lattices $L/a = 4, 8, 12$, and 300000 trajectories on the lattices $L/a = 6$. As for $\tilde{g}^2_{\text{SP}}(2L)$, the number of iterations is around 40000 – 50000 to achieve their precision. Errors in $\tilde{g}^2_{\text{SP}}(L)$ are propagated into $\Sigma(2, u, a/L)$, the lattice SSF, where $u$ is the central value of $\tilde{g}^2_{\text{SP}}(L)$. A formulae of the error propagation using a perturbative expansion of the SSF can be found in Ref. [26].

We performed an additional set of simulations with the Iwasaki action to determine the low energy scale. The tuning of $\beta$ to the conventional point $\tilde{g}^2_{\text{SP}}(L_{\text{max}}) = 3.480$ and the error analysis are made in the same way as mentioned above. In Table IV we list the results, which will be used in Sec. IV B as the first factor on the right hand side of eq. (II.19). To complete the scale determination, one needs the second factor in eq. (II.19). In addition to the previous results of $r_0/a$ [27–29], we carried out simulations at $\beta = 3.00$ and 3.53 to cover the range of $\beta$ in Table IV. Analysis procedures for the static quark potential and extraction of $r_0/a$ parallel those in Ref. [29]. The simulation parameters and results in this work are shown in Table V. To avoid finite size effects, we followed a criterion [18] that the parameter $\beta$ and $L/a$ are chosen such that $L/r_0 \sim 3.3$. Following the above reference, the number of over-relaxation sweeps are taken to satisfy $N_{\text{OR}} \sim 1.5(r_0/a)$.
IV. CONTINUUM EXTRAPOLATION

A. Step scaling function

In this subsection we investigate the cut off dependence of the SSF and perform the continuum extrapolation

\[ \sigma(2, u) = \lim_{a/L \rightarrow 0} \Sigma(2, u, a/L). \]  

(IV.1)

The lattice SSF \( \Sigma(2, u, a/L) \) as a function of \( a/L \) at the weak coupling \( u = 0.9944 \) is shown in Fig. 2 (a) and (c) for the Iwasaki action and LW action, respectively. For the Iwasaki action, even after the one-loop \( O(a) \) improvement with either the condition A or B, the scaling violation is still rather large, which makes the extrapolation to the continuum limit difficult. To improve the scaling behaviour of the SSF, we apply a perturbative removal of the lattice artifacts suggested in Ref. [30] given by

\[ \Sigma_1^{(k)}(2, u, a/L) = \frac{\Sigma^{(k)}(2, u, a/L)}{1 + \delta_1^{(k)}(a/L)u}, \]

(IV.2)

where \( \Sigma^{(k)}(2, u, a/L) \) is the SSF (simulation raw data) with the “\( k \)”-level \( O(a) \) improvement coefficient (e.g., \( k = 0 \): tree level \( O(a) \) improvement case, \( k = 1A \): one-loop \( O(a) \) improvement with condition A case, etc.). \( \delta_1^{(k)}(a/L) \) is the one-loop relative deviation, given by

\[ \delta^{(k)}(2, u, a/L) = \frac{\Sigma^{(k)}(2, u, a/L) - \sigma(2, u)}{\sigma(2, u)} = \delta_1^{(k)}(a/L)u + O(a^2), \]

(IV.3)

whose numerical values are given in Table VII. This method eliminates not only \( O(a) \) but also \( O(a^n) \) with \( n > 1 \) lattice artifacts at one-loop order. Figure 2 (b) shows the cut off dependence of \( \Sigma_1^{(k)}(2, u, a/L) \). Indeed the scaling violations are much reduced by this method, so that we can reliably take the continuum extrapolation linearly in \( a \) as

\[ \Sigma_1^{(k)}(2, u, a/L) = \sigma^{(k)}(2, u) + \omega_1^{(k)}(u)a/L, \]

(IV.4)

where \( \sigma^{(k)}(2, u) \) and \( \omega_1^{(k)}(u) \) are fit parameters. In Table VIII we quote the extrapolated value for the Iwasaki action, which is obtained by a simultaneous fit for \( k = 0, 1A \) and \( 1B \) data with the constraint that they agree in the continuum limit \( \sigma^{(k)}(2, u) = \sigma(2, u) \). The fit has a good \( \chi^2/N_{\text{dof}} \) as listed in Table VIII.

As shown in Fig. 2 (c), the scaling violations are quite small for the LW action. This is consistent with the fact found in Ref. [16] that the lattice artifacts are quite small at one loop. Moreover, the difference between the tree level and one-loop \( O(a) \) improvement is invisible in this precision, as a result of the smallness of the improvement coefficients \( c_t^{P(1)} \) and \( c_t^{R(1)} \). As shown in Fig. 2 (d), the perturbative removal of lattice artifacts has almost no effect except for \( L/a = 4 \), since \( \delta_1^{(k)}(2, a/L) \) with \( L/a = 6, 8, 12 \) is quite small. In Table VIII we quote the extrapolated value for the LW action, obtained by a linear fit to data of the one-loop \( O(a) \) improved action with the perturbative removal of lattice artifacts. For comparison results of ALPHA Collaboration are also included [17].

Results at the strong coupling \( u = 2.4484 \) are plotted in Fig. 3. For the Iwasaki action, the one-loop \( O(a) \) improvement shows large lattice artifacts for both conditions A and B, particularly for the coarse lattice (see Fig. 3 (a)). As shown in Fig. 3 (b), the perturbative removal of lattice artifacts well reduces the scaling violation in the case of the condition B, but it still remains rather large for the condition A. Therefore, we include a quadratic term in the fitting form,

\[ \Sigma_1^{(k)}(2, u, a/L) = \sigma^{(k)}(2, u) + \omega_1^{(k)}(u)a/L + \omega_2^{(k)}(u)(a/L)^2, \]

(IV.5)

for the data of \( k = 1A \) and \( 1B \), while we use the linear fitting form eq. (IV.4) for the data of \( k = 0 \). The extrapolated values obtained with the constraint \( \sigma^{(k)}(2, u) = \sigma(2, u) \) for a unique continuum value is listed in Table VIII. We note that \( \omega_2^{(1A)}(u)/\omega_1^{(1A)}(u) \approx O(10) \) in the fit for the condition A. This suggests that the condition A accidentally enhances the coefficient of \( O(a^2) \) term. It does not necessarily mean, however, that the one-loop \( O(a) \) improvement itself is inefficient at this coupling constant. Indeed the one-loop \( O(a) \) improvement with the condition B shows good scaling behavior. As for the LW action, Fig. 3 (c) and (d) show that neither the one-loop \( O(a) \) improvement or the perturbative removal works effectively. Concerning the extrapolation we simply use the same procedure as in the
weak coupling case, i.e. the linear fitting form to the perturbative removal data for one-loop $O(a)$ improvement. The result is given in Table VIII.

We observe in Table VIII that the three values obtained with the Iwasaki and LW action in the present work and that of ALPHA Collaboration [17] at the weak coupling are consistent within 1σ. At the strong coupling, the value for the LW action undershoots relative to the others by 1.5–2σ. We think that the latter disagreement is caused by a large lattice artifact for the LW action, which makes the choice of the fitting form difficult. For example, if we assume that $O(a)$ errors for the LW action are negligible, we can obtain a result consistent with the values of the other actions within 1σ, by using a purely quadratic fitting form. Further investigation are needed to clarify this point.

B. Low energy scale ratio

We now combine $L_{\text{max}}/a$ and $a/r_0$ for the Iwasaki action to form the ratio $L_{\text{max}}/r_0$, and extrapolate it to the continuum limit:

$$L_{\text{max}}/r_0 = \lim_{a/L_{\text{max}} \to 0} (L_{\text{max}}/a) \times (a/r_0). \tag{IV.6}$$

In the fourth column of Table IX, we give the first factor, which is taken from Table IV; the error of $L_{\text{max}}/a$ is estimated by propagating that of $\bar{g}^2_{\text{SF}}(L_{\text{max}})$.

The second factor $r_0/a$ is given in the third column of Table IX. This is obtained by an interpolation of the results for $r_0/a$ in Table VI using a polynomial [31]

$$\ln(a/r_0) = c_1 + c_2(\beta - 3) + c_3(\beta - 3)^2. \tag{IV.7}$$

The fit, plotted in Fig. 4, gives

$$c_1 = -2.193(6), \quad c_2 = -1.344(7), \quad c_3 = 0.191(24), \tag{IV.8}$$

with $\chi^2/N_{\text{dof}} = 4.10/6$ in the range $2.456 \leq \beta \leq 3.53$. The error of $r_0/a$ in Table IX includes both statistical and systematic ones. We take the central value of $r_0/a$ from the result of the fit above, and estimate the systematic error from the difference of $r_0/a$ between the central value and the result of another fit including a $c_4(\beta - 3)^3$ term.

The combination of the two factors for various $\beta$ values are listed in the fifth column in Table IX. The $\beta$ dependence of $L_{\text{max}}/r_0$ can be considered as lattice cut off effects. For an extrapolation to the continuum limit, we use a fit form

$$a/r_0 = b_1 a/L_{\text{max}} + b_2 (a/L_{\text{max}})^2 + b_3 (a/L_{\text{max}})^3, \tag{IV.9}$$

where $b_i (i = 1, 2, 3)$ are fit parameters and $b_1$ is the continuum value of the low energy scale ratio, rather than fitting $L_{\text{max}}/r_0$ as a function of $a/r_0$. In this way one can avoid the correlation of errors which complicates the latter fit\(^4\).

We apply eq.(IV.9) to three sets of data, i.e., data for the tree level $O(a)$ improvement and those of the one-loop $O(a)$ improvement with the conditions A and B. For the first set of data, we set $b_3 = 0$ (a linear fit) and exclude the point at $a/L = 1/4$. An alternative fit including that point and allowing a non-zero $b_3$ yields a consistent value for $b_1$ within errors. However, we think that the fit is not so reliable since $\chi^2/N_{\text{dof}}$ is too small and the linear terms are rather large. Therefore we exclude the point and use the linear fit for the remaining 3 points. In Fig. 5 $L_{\text{max}}/r_0$ is plotted as a function of $a/L_{\text{max}}$. Dashed lines are the fit curves eq.(IV.9) divided by $a/L_{\text{max}}$, and the point at $a/L_{\text{max}} = 0$ shows the extrapolated value $b_1$.

The extrapolated value is given in Table X, together with the previous result for the standard Wilson plaquette action [18]. While rather large lattice artifacts are observed for both standard Wilson plaquette action and Iwasaki action, the extrapolated values agree within errors.

\(^4\)Since the error on $L_{\text{max}}/a$ is small, one may perform a continuum extrapolation of $L_{\text{max}}/r_0$ as a function of $a/L_{\text{max}}$ neglecting the error on the x-axis. We observe consistency between the two methods.
V. CONCLUSIONS AND DISCUSSIONS

In this paper, we have calculated the step scaling function (SSF) at the weak and the strong couplings for both Iwasaki and LW actions with the one-loop $O(a)$ improved as well as the tree level $O(a)$ improved boundary terms. We have also calculated the low energy scale ratio for the Iwasaki action with both tree level and one-loop $O(a)$ improvements. The extrapolated values of the SSF at the weak and strong coupling for various gauge actions are consistent within $1\sigma$ and $2.3\sigma$, respectively. The low energy scale ratio is also consistent between the Iwasaki and plaquette actions within $1\sigma$. In conclusion, we have confirmed the universality of both quantities.

We have investigated lattice cut off effects in some detail. In the extrapolation procedure, the perturbative removal of lattice artifacts reduces the scaling violation of the SSF for the Iwasaki action with the tree level $O(a)$ improvement and the one-loop $O(a)$ improvement with the condition B. Indeed, at the strong coupling at the coarsest lattice $L/a = 4$, cut off effects are of order $1\%$ and $3\%$, respectively, if one compares the extrapolated value obtained by the constrained fit. At the weak coupling, they are roughly $1\%$ for both cases. We conclude that for the Iwasaki gauge action these combinations of improvements are the good choice for controlling lattice artifacts. This conclusion is also supported by the fact that an individual extrapolation to the continuum limit with the linear fitting form for the data set with the tree level $O(a)$ improvement or the one-loop $O(a)$ improvement with the condition B gives a result consistent with the extrapolated value estimated from the constrained fit within errors, at both weak and strong couplings.

As mentioned in the introduction, this work is the second step toward $N_f = 2$ and 3 simulations. The present study shows that we should use the tree level $O(a)$ improved action or the one-loop $O(a)$ improved action with the condition B in future simulations with dynamical quarks.

ACKNOWLEDGMENTS

This work is supported in part by the Grant-in-Aid of the Ministry of Education (Nos. 13135204, 13640260, 14046202, 14740173, 15204015, 15540251, 15540279, 15740134, 16028201, 16540228, 16740147, 16-11968). S.T. is supported by the JSPS Research Fellowship.

![FIG. 1. The assignments of the weight factor for loops near the boundary $t = 0$ eq. (II.5, II.6).](image-url)
FIG. 2. Results of SSF at the weak coupling $\mu = 0.9944$ with the Iwasaki action (upper) and LW action (lower). The simulation raw data (left-hand side) and the data with “perturbative removal of the lattice artifacts” (right-hand side) are shown. The points at $a/L = 0$ in (b) and (d) represent the extrapolated values obtained by the constrained fit, that the dotted lines indicate the fitting function, for the Iwasaki action and by the linear fit, whose fitting function is shown as dotted line, for the data of 1-loop $O(a)$ improved action for the LW action respectively.
FIG. 3. Results of SSF at the strong coupling $u = 2.4484$ with the various gauge actions. Concerning the symbols and the lines, the same explanation as the weak coupling case is followed.

FIG. 4. Interpolation of $r_0/a$ with phenomenological representation (eq. (IV.7)) in the range $2.456 \leq \beta \leq 3.53$. 

9
FIG. 5. Cut off dependence of $L_{\text{max}}/r_0$ for some choices of the boundary counter terms with the Iwasa action. The error on x-axis are invisible in this scale. The constrained fit is represented by dotted lines. In the fit the point at $a/L_{\text{max}} = 1/4$ of the tree level O($a$) improvement are not included.
\begin{table}[h]
\centering
\begin{tabular}{|l|c|c|c|}
\hline
action & $c_{(1)}^{P}$ with condition A & $c_{(1)}^{P}$ with condition B & $A_{1}$ \\
\hline
Iwasaki & 0.1518 & 0.04161 & 0.3036 \\
LW & -0.00297 & & -0.00594 \\
\hline
\end{tabular}
\caption{The values of $c_{(1)}^{P}$ and $A_{1}$ for the improved gauge actions with each condition.}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|l|c|c|c|c|c|}
\hline
action & degree of O($\alpha$) impr. & $L/a$ & $\beta$ & $\bar{g}_{SF}^{2}(L)$ & $\bar{g}_{SF}^{2}(2L)$ & $\Sigma(2, \ u, a/L)$ \\
\hline
Iwasaki & tree & 4 & 6.5447 & 0.9944(5) & 1.0953(14) & 1.0953(16) \\
& & 6 & 6.8485 & 0.9944(8) & 1.0915(19) & 1.0915(21) \\
& & 8 & 7.0733 & 0.9944(12) & 1.0973(27) & 1.0973(31) \\
& & 12 & 7.3765 & 0.9944(18) & 1.0989(53) & 1.0989(57) \\
& one-loop & 4 & 6.1467 & 0.9944(5) & 1.1305(16) & 1.1305(17) \\
& condition A & 6 & 6.5930 & 0.9944(8) & 1.1230(21) & 1.1230(24) \\
& & 8 & 6.8799 & 0.9944(13) & 1.1192(29) & 1.1192(33) \\
& & 12 & 7.2547 & 0.9944(14) & 1.1132(40) & 1.1132(43) \\
& one-loop & 4 & 6.2258 & 0.9944(5) & 1.1284(15) & 1.1284(16) \\
& condition B & 6 & 6.6358 & 0.9944(9) & 1.1147(22) & 1.1147(24) \\
& & 8 & 6.9010 & 0.9944(9) & 1.1133(30) & 1.1133(31) \\
& & 12 & 7.2722 & 0.9944(15) & 1.1134(56) & 1.1134(58) \\
LW & tree & 4 & 8.2189 & 0.9944(4) & 1.1177(12) & 1.1177(13) \\
& & 6 & 8.5889 & 0.9944(7) & 1.1128(17) & 1.1128(19) \\
& & 8 & 8.8479 & 0.9944(10) & 1.1136(27) & 1.1136(30) \\
& & 12 & 9.2017 & 0.9944(16) & 1.1122(32) & 1.1122(37) \\
& one-loop & 4 & 8.2199 & 0.9944(4) & 1.1158(12) & 1.1158(13) \\
& condition A & 6 & 8.5957 & 0.9944(7) & 1.1115(17) & 1.1115(19) \\
& & 8 & 8.8406 & 0.9944(10) & 1.1153(27) & 1.1153(30) \\
& & 12 & 9.2060 & 0.9944(16) & 1.1085(47) & 1.1085(51) \\
\hline
\end{tabular}
\caption{Results of SSF at the weak coupling $u = 0.9944$.}
\end{table}
TABLE III. Results of SSF at the strong coupling $u = 2.4484$.

| Action       | degree of $O(a)$ impr. | $L/a$ | $\beta$ | $\bar{g}_F^2(L)$ | $\bar{g}_F^2(2L)$ | $\Sigma(2, u, a/L)$ |
|--------------|------------------------|-------|---------|-----------------|-----------------|---------------------|
| Iwasaki      | tree                   | 4     | 3.2663  | 2.4484(28)      | 3.332(11)       | 3.332(12)           |
|              |                        | 6     | 3.5754  | 2.4484(52)      | 3.352(16)       | 3.352(18)           |
|              |                        | 8     | 3.7872  | 2.4484(40)      | 3.361(24)       | 3.361(25)           |
|              |                        | 12    | 4.0096  | 2.4484(65)      | 3.396(43)       | 3.396(44)           |
|              | one-loop               | 4     | 2.9628  | 2.4484(29)      | 4.008(21)       | 4.008(21)           |
|              | condition A            | 6     | 3.3803  | 2.4484(60)      | 3.692(21)       | 3.692(23)           |
|              |                        | 8     | 3.6544  | 2.4484(42)      | 3.625(27)       | 3.625(28)           |
|              |                        | 12    | 4.0091  | 2.4484(68)      | 3.512(45)       | 3.512(46)           |
|              | one-loop               | 4     | 3.0624  | 2.4484(29)      | 3.712(15)       | 3.712(16)           |
|              | condition B            | 6     | 3.4395  | 2.4484(56)      | 3.573(20)       | 3.573(21)           |
|              |                        | 8     | 3.6908  | 2.4484(47)      | 3.519(28)       | 3.519(29)           |
|              |                        | 12    | 4.0283  | 2.4484(64)      | 3.518(38)       | 3.518(39)           |
| LW I         | tree                   | 4     | 4.8992  | 2.4484(24)      | 3.619(13)       | 3.619(14)           |
|              |                        | 6     | 5.2786  | 2.4484(49)      | 3.540(18)       | 3.540(20)           |
|              |                        | 8     | 5.5325  | 2.4484(42)      | 3.486(23)       | 3.486(24)           |
|              |                        | 12    | 5.8878  | 2.4484(62)      | 3.468(32)       | 3.468(33)           |
|              | one-loop               | 4     | 4.9055  | 2.4484(25)      | 3.601(13)       | 3.601(13)           |
|              | condition A            | 6     | 5.2784  | 2.4484(51)      | 3.530(18)       | 3.530(19)           |
|              |                        | 8     | 5.5332  | 2.4484(40)      | 3.499(26)       | 3.499(26)           |
|              |                        | 12    | 5.8867  | 2.4484(58)      | 3.463(25)       | 3.463(27)           |

TABLE IV. Tuning of $\beta$ at $u = 3.480$ for Iwasaki action.

| Degree of $O(a)$ impr. | $L_{max}/a$ | $\beta$  | $\bar{g}_F^2(L_{max})$ |
|------------------------|-------------|-----------|------------------------|
| tree                   | 4           | 2.7000    | 3.480(6)               |
|                        | 6           | 3.0057    | 3.480(11)              |
|                        | 8           | 3.2154    | 3.480(11)              |
|                        | 12          | 3.5219    | 3.480(13)              |
| one-loop               | 4           | 2.4594    | 3.480(7)               |
| condition A            | 6           | 2.8556    | 3.480(14)              |
|                        | 8           | 3.1047    | 3.480(11)              |
|                        | 12          | 3.4496    | 3.480(16)              |
| one-loop               | 4           | 2.5382    | 3.480(6)               |
| condition B            | 6           | 2.8021    | 3.480(11)              |
|                        | 8           | 3.1376    | 3.480(12)              |
|                        | 12          | 3.4734    | 3.480(14)              |

TABLE V. Simulation parameters and results performed in this work for $r_0/a$ with the Iwasaki action. $N_{OR}$ and $N_{conf}$ indicate the number of over-relaxation sweep and configuration respectively.
| $\beta$   | $r_0/\alpha$ | reference |
|----------|--------------|-----------|
| 2.456    | 4.080(16)    | [29]      |
| 2.461    | 4.089(14)    | [28]      |
| 2.487    | 4.286(15)    | [29]      |
| 2.528    | 4.570(21)    | [29]      |
| 2.575    | 4.887(16)    | [29]      |
| 2.659    | 5.556(30)    | [28]      |
| 3.000    | 8.88(13)     | in this work |
| 3.200    | 11.53(15)    | [27]      |
| 3.530    | 17.35(13)    | in this work |

TABLE VI. $\beta$ versus $r_0/\alpha$ with the Iwasaki action. The value of $r_0/\alpha$ are taken from the references quoted in the last column.

| $L/\alpha$ | $\delta_1^{(0)}$ | $\delta_1^{(1A)}$ | $\delta_1^{(1B)}$ | $\delta_1^{(0)}$ | $\delta_1^{(1A)}$ |
|------------|------------------|-------------------|-------------------|------------------|-------------------|
| 4          | -0.02096         | 0.01700           | 0.01577           | 0.003278         | 0.002536          |
| 6          | -0.01922         | 0.00608           | 0.00592           | 0.000911         | 0.000417          |
| 8          | -0.01499         | 0.00399           | 0.00395           | 0.000527         | 0.000156          |
| 12         | -0.01064         | 0.00201           | 0.00200           | 0.000296         | 0.000049          |

TABLE VII. 1-loop deviations for various gauge actions and with $k$-level $O(a)$ improvement.

| action     | $\sigma(2, u = 0.9944)$ | $\chi^2/N_{dof}$ | $\sigma(2, u = 2.4484)$ | $\chi^2/N_{dof}$ |
|------------|--------------------------|------------------|--------------------------|------------------|
| Iwasaki    | 1.107(3)                 | 1.5/8            | 3.485(34)                | 1.9/6            |
| LW         | 1.111(4)                 | 2.0/2            | 3.410(30)                | 0.1/2            |
| plaquette [17] | 1.110(11)            | 3.3/2            | 3.464(40)                | 0.9/4            |

TABLE VIII. The extrapolated values of the SSF with various gauge actions for both the weak and strong coupling. The values in the case of the plaquette action are quoted from the reference as indicated.

| degree of $O(a)$ impr. | $\beta$   | $r_0/\alpha$ | $L_{max}/\alpha$ | $L_{max}/r_0$ |
|------------------------|-----------|--------------|------------------|---------------|
| tree                   | 2.700     | 5.886(26)    | 4.000(11)        | 0.680(4)      |
|                        | 3.0057    | 9.03(12)     | 6.000(35)        | 0.664(10)     |
|                        | 3.2154    | 11.87(19)    | 8.000(41)        | 0.674(12)     |
|                        | 3.5219    | 17.16(14)    | 12.00(8)         | 0.699(7)      |
| one-loop               | 2.4594    | 4.098(12)    | 4.000(9)         | 0.976(4)      |
| condition A            | 2.8556    | 7.352(59)    | 6.000(33)        | 0.816(8)      |
|                        | 3.1047    | 10.29(16)    | 8.000(31)        | 0.777(12)     |
|                        | 3.4496    | 15.78(16)    | 12.00(7)         | 0.761(9)      |
| one-loop               | 2.5382    | 4.625(12)    | 4.000(8)         | 0.865(3)      |
| condition B            | 2.8921    | 7.735(73)    | 6.000(28)        | 0.776(8)      |
|                        | 3.1376    | 10.74(17)    | 8.000(38)        | 0.745(12)     |
|                        | 3.4734    | 16.22(15)    | 12.00(7)         | 0.740(8)      |

TABLE IX. The low energy scale at various values of $\beta$ for Iwasaki action.
TABLE X. The extrapolated values of the low energy scale ratio with various gauge actions. The values in the case of the plaquette action are quoted from the reference as indicated.

| Action      | $L_{\text{max}}/r_0$ | $\chi^2/N_{\text{dof}}$ |
|-------------|-----------------------|--------------------------|
| Iwasaki     | 0.749(14)             | 3.93/ 5                  |
| plaquette [18]| 0.738(16)          |                          |

[1] K. Hagiwara et al. (Particle Data Group Collaboration), Phys. Rev. D66 (2002) 010001
[2] M. Lüscher, R. Narayanan, P. Weisz and U. Wolff, Nucl. Phys. B384 (1992) 168.
[3] S. Sint, Nucl. Phys. B421 (1994) 135.
[4] M. Lüscher, S. Sint, R. Sommer, P. Weisz and U. Wolff, Nucl. Phys. B491 (1997) 323.
[5] K. Jansen, R. Sommer, Nucl. Phys. B530 (1998) 185.
[6] M. Lüscher, S. Sint, R. Sommer, H. Wittig, Nucl. Phys. B491 (1997) 344.
[7] M. Guagnelli, J. Heitger, R. Sommer, H. Wittig, Nucl. Phys. B560 (1999) 465.
[8] A. Bode et al. (ALPHA Collaboration), Phys. Lett. B515 (2001) 49.
[9] M. Della Morte et al. (ALPHA Collaboration), Nucl. Phys. Proc. Suppl. 119 (2003) 439.
[10] S. Aoki et al. (JLQCD Collaboration), Nucl. Phys. Proc. Suppl. 106 (2002) 1079.
[11] S. Aoki et al. (JLQCD Collaboration), Nucl. Phys. Proc. Suppl. 106 (2002) 263.
[12] S. Aoki et al. (CP-PACS Collaboration), Nucl. Phys. Proc. Suppl. 119 (2003) 433.
[13] K-I. Ishikawa et al. (CP-PACS/JLQCD Collaboration), Nucl. Phys. Proc. Suppl. 129 (2004) 444.
[14] T. Kaneko et al. (CP-PACS/JLQCD Collaboration), Nucl. Phys. Proc. Suppl. 129 (2004) 188.
[15] Y. Iwasaki, Nucl. Phys. B258 (1985) 141; Univ. of Tsukuba report UTHEP-118 (1983), unpublished.
[16] S. Takeda, S. Aoki and K. Ide, Phys. Rev. D68 (2003) 014505
[17] S. Capitani et al. (ALPHA Collaboration), Nucl. Phys. B544 (1999) 669.
[18] S. Necco and R. Sommer, Nucl. Phys. B622 (2002) 328;
[19] M. Lüscher, R. Sommer, P. Weisz and U. Wolff, Nucl. Phys. B413 (1994) 481.
[20] T. Klassen, Nucl.Phys. B509 (1998) 391.
[21] M. Lüscher, P. Weisz, Nucl. Phys. B240 (1984) 349.
[22] S. Aoki, R. Frezzotti, and P. Weisz, Nucl.Phys. B540 (1999) 501.
[23] M. Lüscher, P. Weisz, Commun. Math. Phys. 97 (1985) 59; erratum-ibid.98 (1985) 433; Phys. Lett. 158B (1985) 250.
[24] R. Sommer, Nucl. Phys. B411 (1994) 839.
[25] M. Lüscher, P. Weisz and U. Wolff, Nucl. Phys. B359 (1991) 221.
[26] B. Gehrmann, Ph. D. thesis, Humboldt University hep-lat/0207016.
[27] M. Okamoto et al. (CP-PACS Collaboration), Phys. Rev. D60, 094510 (1999).
[28] A. Ali Khan et al. (CP-PACS Collaboration), Phys. Rev. D64, 114501 (2001).
[29] A. Ali Khan et al. (CP-PACS Collaboration), Phys. Rev. D65, 054505 (2002).
[30] G. de Divitiis et al. (ALPHA Collaboration), Nucl. Phys. B437 (1995) 447.
[31] M. Guagnelli, R. Sommer and H. Wittig (ALPHA Collaboration), Nucl. Phys. B535 (1998) 389.