On convex hulls of orbits of Coxeter groups and Weyl groups

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Abstract

The notion of a linear Coxeter system introduced by Vinberg generalizes the geometric representation of a Coxeter group. Our main theorem asserts that if \( v \) is an element of the Tits cone of a linear Coxeter system and \( W \) is the corresponding Coxeter group, then \( Wv \subseteq v - C_v \), where \( C_v \) is the convex cone generated by the coroots \( \check{\alpha} \), for which \( \alpha(v) > 0 \). This implies that the convex hull of \( Wv \) is completely determined by the image of \( v \) under the reflections in \( W \). We also apply an analogous result for convex hulls of \( W \)-orbits in the dual space, although this action need not correspond to a linear Coxeter system. Motivated by the applications in representation theory, we further extend these results to Weyl group orbits of locally finite and locally affine root systems. In the locally affine case, we also derive some applications on minimizing linear functionals on Weyl group orbits.

Keywords: linear Coxeter system, Coxeter group, Weyl group, Tits cone, convex hull.

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Introduction

The present paper is motivated by the unitary representation theory of locally finite, resp., locally affine Lie algebras and their analytic counterparts ([Ne98, Ne10, Ne12]). In the algebraic context, these Lie algebras \( g \) contain a maximal abelian subalgebra \( t \) for which the complexification \( g_C \) has a root decomposition \( g_C = t_C \oplus \bigoplus_{\alpha \in \Delta} g_{\alpha}^C \) and there is a distinguished class of unitary representations \( (\rho, V) \) of \( g \) on a pre-Hilbert space \( V \) for which the operators \( \rho(x), x \in g \), are skew-symmetric, on which \( \rho(t) \) is diagonalizable, and the corresponding weight set \( P_V \subseteq it^* \) has the form

\[
P_V = \text{conv}(W\lambda) \cap (\lambda + Q),
\]

where \( W \subseteq \text{GL}(t) \) is the Weyl group of the pair \((g, t)\), and \( Q \subseteq it^* \) is the root group. Then

\[
\text{Ext}(\text{conv}(P_V)) = W\lambda
\]

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is the set of extremal weights. For finite-dimensional compact Lie algebras \( \mathfrak{g} \) and the unitary (=compact) forms of Kac–Moody algebras, this is the well-known description of the weight set of unitary highest weight modules (\cite{Kac90}). For the generalization to the locally finite, resp., locally affine case we refer to \cite{Ne98}, resp., \cite{Ne10} for details. Motivated by these representation theoretic issues, the present paper addresses a better understanding of the convex hulls of Weyl group orbits in \( \mathfrak{t} \) and \( \mathfrak{t}^* \).

In the locally finite, resp., locally affine case, the Weyl group is a direct limit of finite, resp., affine Coxeter groups, and it is the geometry of linear actions of Coxeter groups that provides the key to the crucial information on convex hulls of orbits. Therefore this paper is divided into two part. The first part deals with linear Coxeter systems \((\text{\cite{Vin71}})\). These are realizations of Coxeter groups by groups generated by a finite set \( (r_s)_{s \in S} \) of reflections of a finite-dimensional vector space \( V \) satisfying the following conditions. We write \( r_s(v) = v - \alpha_s(v)\tilde{\alpha}_s \) with elements \( \alpha_s \in V^* \) and \( \tilde{\alpha}_s \in V \) and \( \text{cone}(M) \) for the convex cone generated by the subset \( M \) of a real vector space. Then the conditions for a linear Coxeter system are:

\begin{enumerate}[(LCS1)]
  \item The polyhedral cone \( K := \{ v \in V : (\forall s \in S) \alpha_s(v) \geq 0 \} \) has interior points.
  \item \( (\forall s \in S) \alpha_s \not\in \text{cone}(\{ \alpha_t : t \neq s \}) \).
  \item \( (\forall v \in W \setminus \{ 1 \}) vK^0 \cap K^0 = \emptyset \), where \( K^0 \) denotes the interior of \( K \).
\end{enumerate}

For any linear Coxeter system, the set \( T := WK \subseteq V \) is a convex cone, called the Tits cone. A reflection in \( W \) is an element conjugate to one of the \( r_s, s \in S \). Any reflection can be written as \( r_\alpha(v) = v - \alpha(v)\tilde{\alpha} \) with \( \alpha \in V^* \) and \( \tilde{\alpha} \in V \), where \( \alpha \) belongs to the set \( \Delta := W\{ \alpha_s : s \in S \} \) of roots and \( \tilde{\alpha} \in \tilde{\Delta} := W\{ \tilde{\alpha}_s : s \in S \} \) is a coroot. In these terms, our first main result (Theorem 2.5) asserts that

\[ \mathcal{W}v \subseteq v - C_v \quad \text{for} \quad C_v := \text{cone}\{ \tilde{\alpha} : \alpha(v) > 0 \} \quad \text{and} \quad v \in T = WK. \tag{1} \]

As an inspection of two-dimensional examples shows, the restriction to elements of the Tits cone is crucial and that the boundary \( \partial T \) may contain elements \( v \) for which \( \mathcal{W}v \) is not contained in \( v - C_v \).

For the applications to orbits of weights, we also need a corresponding result for elements in the dual space. Here a difficulty arises from the fact that the concept of a linear Coxeter system is not preserved by exchanging the role of \( V \) and \( V^* \), so that Theorem 2.5 cannot be applied directly. However, this can be overcome by reduction to the subspace \( U \subseteq V^* \) generated by the dual cone \( C^*_v \) of \( C_S := \text{cone}\{ \alpha_s : s \in S \} \). Then \( WC^*_v \) is the Tits cone for a linear Coxeter system on \( U \) (Theorem 2.10).

**Structure of this paper:** In Section 3 we recall some basics on linear Coxeter systems and generalize some results which are well-known for the geometric representation of a Coxeter group to linear Coxeter systems. Here the main results are Theorem 1.10 relating positivity conditions to the length function and Proposition 1.12 asserting that the stabilizer of an element in the fundamental chamber \( K \) is generated by reflections. Our main results, 1.10 and its dual version, are proved in Section 2. In Section 6 we turn to the applications to Weyl group orbits of linear functionals for locally finite and locally affine root systems.

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\footnote{The proof of Theorem 2.5 relies on various results from the unpublished diploma thesis of Georg Hofmann \cite{HoG99} which contains already a proof for the special case where \( v \in K^0 \), i.e., where the stabilizer of \( v \) is trivial.}
The two final subsections are also motivated by the representation theoretic problem to determine maximal and minimal eigenvalues of elements of the Cartesian subalgebra in extremal weight representations. In this context we provide in the locally affine case a complete classification of the set $P^+_d$ of those linear functionals $\lambda$ for which $\lambda(d) = \min(\hat{W}\lambda)(d)$ holds for a certain distinguished element $d$ and show that any orbit of the affine Weyl group $\hat{W}$ intersects $P^+_d$ in an orbit of the corresponding locally finite Weyl group $W$. Writing $\hat{W}$ as a semidirect product $\mathcal{N} \rtimes W$, where $\mathcal{N}$ acts by unipotent isometries of a Lorentzian form, the minimization of the $d$-value on a $\hat{W}$-orbit turns into a problem of minimizing a quadratic form on an infinite-dimensional analog of a lattice in a euclidean space. These issues are briefly discussed in Subsection 3.4.

The results of the present paper constitute a crucial ingredient in the classification of semibounded unitary representations of double extensions of loop groups with values in Hilbert–Lie groups carried out in [Ne12].

**Notation:** For a subset $E$ of the real vector space $V$ we write $\text{conv}(E)$ for its convex hull and $\text{cone}(E)$ for the convex cone generated by $E$. For a convex cone $C \subseteq V$ we write $H(C) := C \cap -C$ for the largest subspace contained in $C$. We also write

$$E^* := \{ \alpha \in V^*: (\forall v \in E) \alpha(v) \geq 0 \}$$

for its dual cone in $V^*$. For $E \subseteq V^*$ we define its dual cone by

$$E^* := \{ v \in V: (\forall \alpha \in E) \alpha(v) \geq 0 \}.$$

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1 Linear Coxeter systems

In this section we recall Vinberg’s concept of a linear Coxeter system, generalizing the geometric representation of a Coxeter group.

**Definition 1.1.** (a) Let $V$ be a real vector space. A reflection data on $V$ consists of a family $(\alpha_s)_{s \in S}$ of linear functionals on $V$ and a family $(\check{\alpha}_s)_{s \in S}$ of elements of $V$ satisfying

$$\alpha_s(\check{\alpha}_s) = 2 \quad \text{for} \quad s \in S.$$
Then \( r_s(v) := v - \alpha_s(v)\tilde{\alpha}_s \) is a reflection on \( V \). We write \( \mathcal{W} := \langle r_s : s \in S \rangle \subseteq \text{GL}(V) \) for the subgroup generated by these reflections and

\[
\text{co}(v) := \text{conv}(\mathcal{W}v)
\]

for the convex hull of a \( \mathcal{W} \)-orbit. We say that a reflection data is \textit{of finite type} if \( S \) is finite and \( \dim V < \infty \).

(b) We consider the following polyhedral cones in \( V \) resp. \( V^* \):

\[
C_S := \text{cone}\{\tilde{\alpha}_s : s \in S\} \subseteq V, \quad \tilde{C}_S := \text{cone}\{\alpha_s : s \in S\} \subseteq V^*;
\]

and the \textit{fundamental chamber}

\[
K := \{v \in V : (\forall s \in S) \alpha_s(v) \geq 0\} = (\tilde{C}_S)^*.
\]

(c) A reflection data of finite type is called a \textit{linear Coxeter system} (cf. [Vin71]) if

\begin{itemize}
  \item [(LCS1)] \( K \) has interior points, i.e., the cone \( \tilde{C}_S \subseteq V^* \) is pointed.
  \item [(LCS2)] \( (\forall s \in S) \alpha_s \not\in \text{cone}\{\alpha_t : t \neq s\} \).
  \item [(LCS3)] \( (\forall w \in W \setminus \{1\}) wK^0 \cap K^0 = \emptyset \).
\end{itemize}

Then \( T := WK \) is called the associated \textit{Tits cone}.

\textbf{Definition 1.2.} Let \( V \) be a real vector space endowed with a symmetric bilinear form \( \beta \). A \textit{symmetric reflection data on} \( V \) consists of a family \( (\tilde{\alpha}_s)_{s \in S} \) of non-isotropic elements of \( V \) for which the linear functionals

\[
\alpha_s(v) := \frac{2\beta(v, \tilde{\alpha}_s)}{\beta(\tilde{\alpha}_s, \tilde{\alpha}_s)}
\]

define a reflection data on \( V \). Then the reflections

\[
r_s(v) = v - \alpha_s(v)\tilde{\alpha}_s = v - 2\frac{\beta(v, \tilde{\alpha}_s)}{\beta(\tilde{\alpha}_s, \tilde{\alpha}_s)}\tilde{\alpha}_s
\]
preserve \( \beta \), so that \( \mathcal{W} \subseteq \text{O}(V, \beta) \).

\textbf{Remark 1.3.} (a) Condition \( (\text{LCS2}) \) means that, for each \( s \in S \), the dual cone of \( \text{cone}\{\alpha_t : t \neq s\} \) is strictly larger than \( K \), i.e., there exists an element \( v \in K \cap \ker \alpha_s \) with \( \alpha_t(v) > 0 \) for \( t \neq s \). Then \( K \cap \ker \alpha_s \) is a codimension 1 face of the polyhedral cone \( K \).

(b) Typical examples of linear Coxeter systems arise from the geometric representation of a Coxeter system \( (\mathcal{W}, S) \) (cf. [Hu92, §5.3], [Vin71]).

The following criterion for the recognition of linear Coxeter systems will be convenient in many situations because conditions \( (\text{C1}) \) and \( (\text{C2}) \) are preserved by the passage to the dual reflection data obtained by exchanging \( V \) and \( V^* \).

\textbf{Proposition 1.4.} Let \( (V, (\alpha_s)_{s \in S}, (\tilde{\alpha}_s)_{s \in S}) \) be a reflection data of finite type satisfying \( (\text{LCS1}) \). Then it defines a \textit{linear Coxeter system} if and only if the following conditions are satisfied for \( s \neq t \in S \):

\[
\text{co}(v) := \text{conv}(\mathcal{W}v)
\]
(C1) $\alpha_s(\tilde{a}_t)$ and $\alpha_t(\tilde{a}_s)$ are either both negative or both zero.

(C2) $\alpha_s(\tilde{a}_t)\alpha_t(\tilde{a}_s) \geq 4$ or $\alpha_s(\tilde{a}_t)\alpha_t(\tilde{a}_s) = 4\cos^2 \frac{\pi}{k}$ for some natural number $k \geq 2$.

In this case $(W, (r_s)_{s \in S})$ is a Coxeter system.

**Proof.** According to [Vin71, p. 1085], for a reflection data of finite type satisfying (LCS1/2), condition (LCS3) is equivalent to both conditions (C1/2).

Therefore it remains to show that (C1) implies (LCS2). So we assume that $\alpha_s = \sum_{t \neq s} \lambda_t a_t$ with finitely many non-zero $\lambda_t \geq 0$. Then (C1) leads to the contradiction

$$2 = \alpha_s(\tilde{a}_s) = \sum_{t \neq s} \lambda_t a_t(\tilde{a}_s) \leq 0.$$ 

Therefore (C1) implies (LCS2). That $(W, (r_s)_{s \in S})$ is a Coxeter system follows from [Vin71, Thm. 2(6)].

**Remark 1.5.** As a consequence of the preceding theorem, we obtain for every subset $S_0 \subseteq S$ and every linear Coxeter system $(V, (\alpha_s)_{s \in S}, (\tilde{a}_s)_{s \in S})$ a linear Coxeter system

$$(V, (\alpha_s)_{s \in S_0}, (\tilde{a}_s)_{s \in S_0}).$$

**Remark 1.6.** (a) If $(V, (\alpha_s)_{s \in S}, (\tilde{a}_s)_{s \in S})$ is a reflection data, then we also consider the elements $\tilde{a}_s$ as linear functionals on $V^*$. We thus obtain a reflection data $(V^*, (\tilde{a}_s)_{s \in S}, (\alpha_s)_{s \in S})$. Suppose that $(V, (\alpha_s)_{s \in S}, (\tilde{a}_s)_{s \in S})$ is a linear Coxeter system. Then (C1/2) also hold for the dual reflection data $(V^*, (\tilde{a}_s)_{s \in S}, (\alpha_s)_{s \in S})$. Hence it is a linear Coxeter system if and only if (LCS1) holds, i.e., if the convex cone $C_S$ is pointed, i.e., $H(C_S) = \{0\}$.

(b) If $C_S$ is not pointed, then we still obtain a linear Coxeter system by replacing $V^*$ by the smaller subspace

$$U := \text{span}(C^*_S) = H(C_S)^\perp \subseteq V^*.$$ 

Let $q \colon V \to V/U^\perp \cong U^\ast$ denote the canonical projection. We put

$$\tilde{S} := \{s \in S \colon \tilde{a}_s \not\in U^\perp = H(C_S)\}.$$ 

Proposition 1.4 implies that $(U, (q(\tilde{a}_s))_{s \in \tilde{S}}, (\alpha_s)_{s \in S})$ is a linear Coxeter system because the convex cone $\tilde{C}_S \subseteq \tilde{C}_S$ is pointed. We write

$$W_U := \langle r_s \colon s \in \tilde{S} \rangle \subseteq \text{GL}(U)$$

for the corresponding reflection group.

To relate this group to $W$, we first claim that $U$ is $W$-invariant. For $s \in S \setminus \tilde{S}$ the relation $\tilde{a}_s \in U^\perp$ implies that the reflection $r_s$ acts trivially on the subspace $U \subseteq V^*$. Next we observe that, if an element $\sum_{s \in S} \lambda_s \tilde{a}_s \in C_S$ with $\lambda_s \geq 0$ is contained in $H(C_S)$, then $\lambda_s > 0$ implies that $\tilde{a}_s \in H(C_S)$, i.e., $s \in S \setminus \tilde{S}$. We conclude that

$$H(C_S) = \text{cone}\{\tilde{a}_s \colon s \in S \setminus \tilde{S}\} = C_S \setminus \tilde{S}.$$
For \( s \in \tilde{S} \) we now derive from (C1) that \( \alpha_s \in -(C_{S \setminus \tilde{S}})^* = -H(C_S)^* = U. \) Therefore \( U \) is also invariant under \( r_s \). Hence \( U \) is \( \mathcal{W} \)-invariant. As the reflections \( r_s, s \in S \setminus \tilde{S} \), act trivially on \( U \), we see that the restriction map
\[
R: \mathcal{W} \to \mathcal{W}_U, \quad w \mapsto w|_U
\]
is a surjective homomorphism.

**Lemma 1.7.** Let \( a \geq 2 \) and
\[
\begin{align*}
    r_1 &:= \begin{pmatrix} -1 & a \\ 0 & 1 \end{pmatrix}, \\
    r_2 &:= \begin{pmatrix} 1 & 0 \\ a & -1 \end{pmatrix} \in \text{GL}_2(\mathbb{R}).
\end{align*}
\]
If \( e_1 \) and \( e_2 \) denote the standard basis of \( \mathbb{R}^2 \), then
\[
(r_1 r_2)^n e_1, r_2 (r_1 r_2)^n e_1 \in \text{cone}\{e_1, e_2\} \quad \text{for} \quad n \in \mathbb{N}_0.
\]

**Proof.** We define a linear functional on \( \mathbb{R}^2 \) by \( \beta(x) := x_1 - \frac{1}{2}ax_2 \). First we show by induction on \( n \) that
\[
(r_1 r_2)^n e_1 \in \text{cone}\{e_1, e_2\} \quad \text{and} \quad \beta((r_1 r_2)^n e_1) \geq 0 \quad \text{hold for any} \quad n \in \mathbb{N}_0.
\]
For \( n = 0 \) this is trivial. So let \( v := (r_1 r_2)^n e_1 \) and assume that the assertion holds for some \( n \in \mathbb{N}_0 \). With
\[
    r_1 r_2 = \begin{pmatrix} a^2 - 1 & -a \\ a & -1 \end{pmatrix}
\]
we then obtain
\[
    \beta(r_1 r_2 v) = (a^2 - 1)v_1 - av_2 - \frac{1}{2}a(au_1 - v_2) = (\frac{a^2}{2} - 1)v_1 - \frac{a}{2}v_2
\]
\[
\geq \frac{a^2}{2} - 2v_1 + \beta(v) \geq 0
\]
We also have
\[
(a^2 - 1)v_1 - av_2 = (a^2 - 3)v_1 + 2v_1 - \frac{a}{2}v_2 \geq v_1 + 2\beta(v) \geq 0
\]
and
\[
av_1 - v_2 = \frac{1}{a}((a^2 - 2)v_1 + 2v_1 - \frac{a}{2}v_2) \geq \frac{1}{a}(2v_1 + 2\beta(v)) \geq 0.
\]
(3)
This proves that \( r_1 r_2 v \in \text{cone}\{e_1, e_2\} \) and completes our induction.

For \( v' := r_2 v = r_2 (r_1 r_2)^n e_1 \), we finally obtain with (3)
\[
v'_1 = v_1 \geq 0 \quad \text{and} \quad v'_2 = av_1 - v_2 \geq 0,
\]
hence that \( v' \in \text{cone}\{e_1, e_2\} \).

\[\square\]
Lemma 1.8. Let \((V, (\alpha_s, \alpha_t), (\check{\alpha}_s, \check{\alpha}_t))\) be a reflection data satisfying (C1/2). Then the subgroup \(\Gamma \subseteq \text{GL}(V)\) generated by the reflections \(r_s\) and \(r_t\) is a Coxeter group. Its length function \(\ell\) satisfies for \(g \in \mathcal{W}\):

\[
\ell(gr_s) \geq \ell(g) \quad \Rightarrow \quad g\check{\alpha}_s \in C_S = \text{cone}\{\check{\alpha}_s, \check{\alpha}_t\}.
\]

Proof. In view of \(\ell(gr_s) \geq \ell(g)\), every reduced expression for \(g\) is an alternating product of \(r_s\) and \(r_t\) ending in \(r_t\). After normalizing the pair \((\alpha_s, \check{\alpha}_s)\) suitably, we may assume that \(\alpha_s(\check{\alpha}_t) = \alpha_t(\check{\alpha}_s)\). Let

\[a := -\alpha_s(\check{\alpha}_t) \geq 0 \quad \text{and} \quad U := \text{span}\{\check{\alpha}_s, \check{\alpha}_t\}.
\]

Then

\[
\det \begin{pmatrix} \alpha_s(\check{\alpha}_s) & \alpha_s(\check{\alpha}_t) \\ \alpha_t(\check{\alpha}_s) & \alpha_t(\check{\alpha}_t) \end{pmatrix} = 4 - a^2
\]

shows that, if \(a^2 \neq 4\), then \(\check{\alpha}_s\) and \(\check{\alpha}_t\) are linearly independent and the same holds for the restrictions of \(\alpha_s\) and \(\alpha_t\) to \(U\).

If \(\check{\alpha}_s\) and \(\check{\alpha}_t\) are linearly independent, then the corresponding matrices of the restrictions of \(r_s\) and \(r_t\) to \(U\) are given by

\[
r_1 := \begin{pmatrix} -1 & a \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad r_2 := \begin{pmatrix} 1 & 0 \\ a & -1 \end{pmatrix}.
\]

Moreover, \(r_1\) and \(r_2\) are orthogonal with respect to the symmetric bilinear form \((\cdot, \cdot)\) defined with respect to the basis \((\check{\alpha}_s, \check{\alpha}_t)\) by the matrix

\[
\begin{pmatrix} 1 & -a/2 \\ -a/2 & 1 \end{pmatrix}
\]

because \(\alpha_s(v) = 2(\check{\alpha}_s, v)\) and \(\alpha_t(v) = 2(\check{\alpha}_t, v)\) hold for each \(v \in U\).

Case 1: \(a^2 < 4\). In this case the bilinear form on \(U\) is positive definite and the assertion follows from the argument in [Hu92, p. 112].

Case 2: \(a^2 > 4\). In this case the bilinear form on \(U\) is indefinite and the assertion follows immediately from Lemma [1.7]

Case 3: \(a^2 = 4\). Here we have to distinguish two cases. If \(\check{\alpha}_s\) and \(\check{\alpha}_t\) are linearly dependent, then \(C_S = U = \mathbb{R}\check{\alpha}_s\) follows from \(\alpha_t(\check{\alpha}_s) = -a < 0\) and \(\alpha_t(\check{\alpha}_t) = 2 > 0\). In this case the assertion follows from the invariance of \(U\) under \(\mathcal{W}\). If \(\check{\alpha}_s\) and \(\check{\alpha}_t\) are linearly independent, then the assertion follows from Lemma [1.7].

The following proposition generalizes the corresponding well known result for the geometric representation of a Coxeter group to arbitrary linear Coxeter systems (cf. [Hu92, Thm. 5.4]).

Proposition 1.9. Let \((V, (\alpha_s)_{s \in S}, (\check{\alpha}_s)_{s \in S})\) be a reflection data of finite type satisfying (C1/2). Assume that the subgroup \(\mathcal{W} \subseteq \text{GL}(V)\) generated by the reflections \((r_s)_{s \in S}\) is a Coxeter group. Let \(\ell: \mathcal{W} \to \mathbb{N}_0\) denote its length function and \(g \in \mathcal{W}, s \in S\).

(i) If \(\ell(gr_s) > \ell(g)\), then \(g\check{\alpha}_s \in C_S\).
(ii) If \( \ell(gr_s) < \ell(g) \), then \( g\hat{\alpha}_s \in -C_S \).

**Proof.** First we note that (ii) is a consequence of (i), applied to the element \( g' = gr_s \) and using \( g\hat{\alpha}_s = -g'\hat{\alpha}_s \).

We prove (i) by induction on the length of \( g \). The case \( \ell(g) = 0 \), i.e., \( g = 1 \) is trivial. If \( \ell(g) > 0 \), then there exists a \( t \in S \) with \( \ell(gr_t) = \ell(g) - 1 \). Then \( t \neq s \) follows from \( \ell(gr_s) > \ell(g) \). Let \( S := \{s, t\} \) and consider the subgroup \( \widetilde{W} := \langle r_s, r_t \rangle \subseteq W \) with the length function \( \tilde{\ell} \). Let

\[
A := \{f \in W : f \in g\widetilde{W}, \ell(g) = \ell(f) + \tilde{\ell}(f^{-1}g)\}.
\]

Obviously \( g \in A \), so that \( A \) is not empty. Pick \( f \in A \) with minimal length \( \ell(f) \) and put \( f' := f^{-1}g \in \widetilde{W} \). Then \( g = ff' \) with \( \ell(g) = \ell(f) + \tilde{\ell}(f') \).

We also note that \( gr_t \in A \) follows from \( \ell(g) = \ell(gr_t) + 1 = \ell(gr_t) + \tilde{\ell}(r_t) \). Hence the choice of \( f \) implies that \( \ell(f) \leq \ell(gr_t) = \ell(g) - 1 \). We now want to apply the induction hypothesis to the pair \( (f, r_s) \). To this end, we have to compare the length or \( f \) and \( fr_s \). If \( \ell(fr_s) < \ell(f) \), then \( \ell(fr_s) = \ell(f) - 1 \) and we have

\[
\ell(g) \leq \ell(fr_s) + \ell(r_s f^{-1}g) \leq \ell(fr_s) + \tilde{\ell}(r_s f^{-1}g)
\]

\[
\leq \ell(f) + 1 + \tilde{\ell}(f^{-1}g) + 1 = \ell(f) + \tilde{\ell}(f^{-1}g) = \ell(g).
\]

We conclude that \( \ell(g) = \ell(fr_s) + \tilde{\ell}(r_s f^{-1}g) \), so that \( fr_s \in A \), contradicting the minimality of \( \ell(f) \). This implies that \( \ell(fr_s) > \ell(f) \) because \( \ell(fr_s) \neq \ell(f) \) ([Hu92, p. 108]), so that the induction hypothesis leads to \( f\hat{\alpha}_s \in C_S \). By the same argument we obtain \( \ell(fr_t) > \ell(f) \) and therefore \( f\hat{\alpha}_s \in C_S \).

From

\[
\ell(f) + \tilde{\ell}(f') = \ell(g) < \ell(gr_s) = \ell(ff'r_s) \leq \ell(f) + \ell(f'r_s) \leq \ell(f) + \tilde{\ell}(f'r_s)
\]

we further derive that \( \tilde{\ell}(f') \leq \tilde{\ell}(f'r_s) \). Now Lemma 1.8 implies that

\[
f'\hat{\alpha}_s \in C_S = \text{cone}\{\hat{\alpha}_s, \hat{\alpha}_t\}
\]

and thus \( g\hat{\alpha}_s = ff'\hat{\alpha}_s \in fC_S \subseteq C_S \).

**Theorem 1.10.** Let \( (V, (\alpha_s)_{s \in S}, (\hat{\alpha}_s)_{s \in S}) \) be a linear Coxeter system and \( \ell : W \rightarrow \mathbb{N}_0 \) be the length function with respect to the generating set \( \{r_s : s \in S\} \). Then the following assertions hold for \( g \in W \):

(i) If \( \ell(gr_s) > \ell(g) \), then \( g\hat{\alpha}_s \in C_S \) and \( g\alpha_s \in \hat{C}_S \).

(ii) If \( \ell(gr_s) < \ell(g) \), then \( g\hat{\alpha}_s \in -C_S \) and \( g\alpha_s \in -\hat{C}_S \).

**Proof.** In view of Proposition 1.4 the linear Coxeter system \( (V, (\alpha_s)_{s \in S}, (\hat{\alpha}_s)_{s \in S}) \) and its dual \( (V^*, (\alpha_s)_{s \in S}, (\hat{\alpha}_s)_{s \in S}) \) satisfy the assumptions of Proposition 1.4 because the fact that \( (W, (r_s)_{s \in S}) \) is a Coxeter system implies that the adjoints \( r_s^{\ast} \in \text{GL}(V^*) \) define a Coxeter system in the subgroup \( W \cong \langle r_s^{\ast} : s \in S \rangle \subseteq \text{GL}(V^*) \). This implies the assertion.

**Remark 1.11.** For \( s \in S \) and \( g \in W \) the condition \( g\alpha_s \in \hat{C}_S = K^* \) is equivalent to the linear functional \( g\alpha_s \) taking non-negative values on \( K \). Therefore Theorem 1.10 implies in particular that each linear functional \( \alpha \in W(\alpha_s : s \in S) \) either is positive or negative on \( K^0 \).
Proposition 1.12. If \((V, (\alpha_s)_{s \in S},(\check{\alpha}_s)_{s \in S})\) is a linear Coxeter system, \(v \in K\) and
\[
I := \{ s \in S: \alpha_s(v) = 0 \},
\]
then the subgroup \(W_I \subseteq W\) generated by the reflections \(\{r_s: s \in I\}\) coincides with the stabilizer \(W_v = \{ w \in W: wv = v \}\).

**Proof.** Clearly \(W_I \subseteq W_v\) because the generators of \(W_I\) fix \(v\). If \(W_I\) is strictly smaller than \(W_v\), there exists an element \(g \in W_v \setminus W_I\) of minimal positive length. Then \(\ell(gr_s) > \ell(g)\) for every \(s \in I\) (recall that \(\ell(gr_s) \neq \ell(g)\) by [Hn92, p. 108]) implies that \(g\alpha_s \in \check{C}_S\) (Theorem 1.10). If \(s \notin I\), then \(\alpha_s(v) > 0\) implies that \((g\alpha_s)(v) = \alpha_s(g^{-1}v) = \alpha_s(v) > 0\). Therefore \(g\alpha_s\) takes positive values on \(K^0\) which entails \(g\check{\alpha}_S \in \check{C}_S\) (Remark 1.11). We thus arrive at \(g\check{C}_S \subseteq \check{C}_S\). As the element \(g^{-1} \in W_v \setminus W_I\) has the same length, we also obtain \(g^{-1}\check{C}_S \subseteq \check{C}_S\), and thus \(g\check{C}_S = \check{C}_S\). Now \(K = (\check{C}_S)^*\) leads to \(gK = K\), and by (LCS3) further to \(g = 1\), contradicting \(g \notin W_I\). \(\square\)

2 A convexity theorem for linear Coxeter systems

Before we come to our main theorem in this section, we have to define the roots and coroots of a linear Coxeter system. This is crucial to obtain a formulation of the theorem which does not depend on the generating system. This will be essential for the infinite-dimensional generalization where roots and coroots still make sense but \(W\) need not be a Coxeter group.

2.1 Roots and coroots

**Definition 2.1.** Let \((V, (\alpha_s)_{s \in S},(\check{\alpha}_s)_{s \in S})\) be a linear Coxeter system.

(a) We define the set of roots by
\[
\Delta := \mathcal{W}\{\alpha_s: s \in S\} \subseteq V^* \quad \text{and put} \quad \Delta^\pm := \Delta \cap \pm K^* = \Delta \cap \pm \check{C}_S.
\]
Roots in \(\Delta^+\) are called **positive** and roots in \(\Delta^-\) are called **negative**. Remark 1.11 shows that
\[
\Delta = \Delta^+ \cup \Delta^-.
\]
We likewise define the corresponding sets of coroots
\[
\check{\Delta} := \mathcal{W}\{\check{\alpha}_s: s \in S\} \subseteq V.
\]

(b) The subset
\[
\mathcal{R} := \{wr_s w^{-1}: w \in W, s \in S\} \subseteq \mathcal{W}
\]
is called the set of reflections in \(W\). If \(r = wr_s w^{-1}\) is a reflection, \(\alpha := w\alpha_s\) and \(\check{\alpha} := w\check{\alpha}_s\), then
\[
r(v) = v - \alpha(v)\check{\alpha} \quad \text{for} \quad v \in V \quad (4)
\]
and \(\text{Fix}(r) = \ker \alpha = w \ker \alpha_s\).
Remark 2.2. We claim that a reflection \( r \in W \) is uniquely determined by its hyperplane of fixed points. So let \( r \in \mathcal{R} \). Then the hyperplane \( \text{Fix}(r) \) intersects the interior \( T^0 \) of the Tits cone \( T = WK \) (cf. Remark 1.3(a)). Since each \( W \)-orbit in \( T^0 \) meets \( K \) exactly once \(([\text{Vin74}] \text{ Thm. 2})\), there exists a unique face \( F \subseteq K \) of codimension one and some \( w \in W \) with \( \text{Fix}(w^{-1}rw) \cap K = F \). Then there exists a uniquely determined \( s \in S \) with \( F = \ker \alpha_s \cap K \) (Remark 1.3). Now \( w^{-1}rw \) and \( r_s \) are two reflections in the same hyperplane \( \ker \alpha_s \), hence fixing \( F \) pointwise. Therefore \( w^{-1}rw(K^0) \cap r_s(K^0) \neq \emptyset \) leads to \( r_s = w^{-1}rw \), so that \( r = wrw^{-1} \).

Next we note that, if \( \alpha = wa_s = w' \alpha_t \) for some \( w, w' \in W \) and \( s, t \in S \), then \( wrw^{-1} \) and \( w'r_t(w')^{-1} \) are both reflections with the same sets of fixed points, so that the preceding argument implies that they are equal: \( wrw^{-1} = w'r_t(w')^{-1} \). This in turn shows that \( wa_s \otimes w\alpha_s = w'\alpha_t \otimes w'\alpha_t = wa_s \otimes w'\alpha_t \), and hence that \( w\alpha_s = w'\alpha_t \).

Definition 2.3. In view of the preceding remark, we can associate to each root \( \alpha \in \Delta \) a well-defined coroot \( \check{\alpha} \in \check{\Delta} \) such that the map \( \Gamma : \Delta \to \check{\Delta} \), \( \alpha \mapsto \check{\alpha} \) is \( W \)-equivariant and the reflections in \( W \) have the form \( [\check{\mathsf{I}}] \).

Remark 2.4. Since the roots \( \alpha_s, s \in S \), are positive by definition, Theorem 1.10 implies that

\[
\text{cone}(\Delta^+) = \text{cone}\{\alpha_s : s \in S\} = \hat{C}_S,
\]

and hence that

\[
K = (\hat{C}_S)^{\ast} = \{v \in V : (\forall \alpha \in \Delta^+) \alpha(v) \geq 0\}.
\]

2.2 Convex hulls of orbits in the Tits cone

The following theorem strengthens the corresponding assertion for elements \( v \in K^0 \) in [Hu92] p. 20] substantially because it provides also sharp information if the stabilizer \( W_v \) is non-trivial.

Theorem 2.5. Let \( (V, (\alpha_s)_{s \in S}, (\hat{\alpha}_s)_{s \in S}) \) be a linear Coxeter system and \( T = WK \subseteq V \) be its Tits cone. For \( v \in T \) we have

\[
Wv \subseteq v - C_v, \quad \text{where} \quad C_v := \text{cone}\{\hat{\alpha} \in \hat{\Delta} : \alpha(v) > 0\}.
\]

Proof. As \( T = WK \) and \( C_{gv} = gC_v \), we may w.l.o.g. assume that \( v \in K \). We put \( I := \{s \in S : \alpha_s(v) = 0\} \) and recall from Proposition 1.12 that the corresponding parabolic subgroup \( W_I \subseteq W \) coincides with the stabilizer \( W_v \) of \( v \). Let

\[
W^I := \{g \in W : (\forall s \in I) \ell(gr_s) > \ell(g)\},
\]

so that \( W = W^I W_I \) by [Hu92] p. 123].

We now show \( gv - v \in -C_v \) by induction on the length \( \ell(g) \) of \( g \). The assertion is trivial for \( g = 1 \), i.e., \( \ell(g) = 0 \). Suppose that \( \ell(g) > 0 \). Then \( g^{-1} = h^{-1}g_I \) with \( h^{-1} \in W^I \) and \( g_I \in W_I \) satisfying

\[
\ell(g) = \ell(g^{-1}) = \ell(h^{-1}) + \ell(g_I) = \ell(h) + \ell(g_I)
\]

([Hu92] p. 123]). If \( g_I \neq 1 \), then \( \ell(h) < \ell(g) \) and our induction hypothesis leads to

\[
 gv - v = g_I^{-1}hv - v = g_I^{-1}(hv - v) \in -g_I^{-1}C_v = -C_v.
\]
when do we have equality?

that it suffices to verify that \((v)\) implies for \(v\) by the induction hypothesis. We may therefore assume that \(\alpha_s(v) > 0\). Then our induction hypothesis implies

\[
gv - v = (gr_s)v - v = (gr_s)(v - \alpha_s(v)\hat{\alpha}_s) - v = (gr_s)v - v - \alpha_s(v)(gr_s)\hat{\alpha}_s \in -C_v + \alpha_s(v)g\hat{\alpha}_s.
\]

In view of \(\alpha_s(v) > 0\), it remains to see that \(g\hat{\alpha}_s = (g\alpha_s)\hat{\alpha}_s \in -C_v\) (cf. Definition 2.1(c)), so that it suffices to verify that \((g\alpha_s)(v) < 0\). As \((g\alpha_s)(v) < 0\). Then our induction hypothesis implies

\[
-\alpha_s \in g^{-1}\{\beta \in \Delta^+ : \beta(v) = 0\} = g^{-1}\Delta^+_s \subset \Delta^+
\]

by construction of \(g\); but this leads to the contradiction \(-\alpha_s \in \Delta^+\). This proves that \((g\alpha_s)(v) < 0\), which completes the proof.

\[\tag*{\Box}\]

**Corollary 2.6.** For \(v \in T\), the following assertions hold:

(i) \(\text{cone}(Wv - v) = -C_v\).

(ii) \(v\) is an extreme point of \(\text{co}(v)\) if and only if the cone \(C_v\) is pointed.

(iii) For \(\lambda \in V^*\),

\[
\lambda(v) = \min \lambda(Wv) \iff \lambda \in -C_v^*.
\]

Proof. (i) For any \(v \in V\) with \(\alpha(v) > 0\) the relation \(r_\alpha(v) = v - \alpha(v)\hat{\alpha}\) implies that

\[
-\hat{\alpha} \in \mathbb{R}_+(Wv - v),
\]

so that Theorem 2.5 implies for \(v \in T\) that \(\text{cone}(Wv - v) = -C_v\).

(ii) and (iii) follow immediately from (i).

\[\tag*{\Box}\]

**Corollary 2.7.** The following conditions are equivalent

(i) The cone \(C_S\) is pointed, i.e., \((V^*, (\hat{\alpha}_s)_{s \in S}, (\alpha_s)_{s \in S})\) is a linear Coxeter system.

(ii) There exists a \(v \in K^0\) which is an extreme point of \(\text{co}(v)\).

(iii) Each \(v \in K^0\) is an extreme point of \(\text{co}(v)\).

Proof. This is immediate from Corollary 2.6 and the fact that \(C_S = C_v\) for \(v \in K^0\).

\[\tag*{\Box}\]

**Problem 2.8.** From Theorem 2.5 we obtain for \(v \in T\) the relation

\[
\text{co}(v) \subseteq \bigcap_{w \in W} w(v - C_v).
\]

When do we have equality?
Example 2.9. We take a closer look at linear Coxeter systems with a 2-element set $S = \{ s, t \}$ and $\dim V = 2$.

First we show that $\alpha_s$ and $\alpha_t$ are linearly independent. If this is not the case, then $\alpha_s(\check{a}_t) \leq 0$ implies that $\alpha_s = \lambda \alpha_t$ for some $\lambda < 0$, but this leads to the contradiction $K = \emptyset$. Therefore $\alpha_s$ and $\alpha_t$ are linearly independent.

(a) Suppose that $\alpha_s(\check{a}_t) = \alpha_t(\check{a}_s) = -2$. Then $\alpha_s$ and $\alpha_t$ vanish on $\check{a}_s + \check{a}_t$, and since $V^*$ is spanned by $\alpha_s$ and $\alpha_t$, it follows that $\check{a}_t = -\check{a}_s$. In particular, the cone $C_S = \mathbb{R} \check{a}_s$ is not pointed.

This implies that the action of $W$ on $V$ leaves all affine subspaces of the form $v + C_S$ invariant. If $\alpha_s^*, \alpha_t^* \in V$ is the dual basis of $\alpha_s, \alpha_t$, then

$$K = \mathbb{R}_+ \alpha_s^* + \mathbb{R}_+ \alpha_t^* \quad \text{and} \quad \check{a}_s = -\check{a}_t = 2(\alpha_s^* - \alpha_t^*).$$

The linear map $r_tr_s$ fixes the line $C_S$ pointwise and induces on $V/C_S$ the identity, hence is unipotent. Moreover,

$$r_tr_s(\alpha_s^*) = r_t(\alpha_s^* - \check{a}_s) = r_t(-\alpha_s^* + 2\alpha_t^*) = -\alpha_s^* + 2\alpha_t^* - 2\check{a}_t = -5\alpha_s^* + 6\alpha_t^* = \alpha_s^* - 3\check{a}_s,$$

so that the convexity of the Tits cone implies that

$$T^0 = C_S + \mathbb{R}_+ \alpha_s^*$$

is an open half plane and $T = T^0 \cup \{ 0 \}$. We further have

$$\co(v) = v + C_S \quad \text{for} \quad v \in T^0$$

and

$$\check{\Delta} = \{ \check{a}_s, \check{a}_t \} = \{ \pm \check{a}_s \},$$

whereas $\check{\Delta} = W\{ \alpha_s, \alpha_t \}$ is infinite. More precisely, we have

$$r_tr_s \alpha_s = r_t(-\alpha_s) = -\alpha_s + \alpha_s(\check{a}_t) \alpha_t = -\alpha_s - 2\alpha_t = \alpha_s - 2(\alpha_s + \alpha_t),$$

and since $W = \{ r_s(r_t r_s)^n, (r_t r_s)^n : n \in \mathbb{Z} \}$, it follows that

$$\Delta = \{ \pm \alpha_s, \pm \alpha_t \} + 2\mathbb{Z}(\alpha_s + \alpha_t).$$

The two coroots $\check{a}_s$ and $\check{a}_t$ lie in the boundary of the Tits cone and $W \check{a}_s = \{ \pm \check{a}_s \}$. Moreover,

$$\{ \alpha \in \Delta : \alpha(\check{a}_s) > 0 \} = \{ \alpha_s, -\alpha_t \} + 2\mathbb{Z}(\alpha_s + \alpha_t),$$

so that

$$C_{\check{a}_s} = \cone \{ \check{a} : \alpha(\check{a}_s) > 0 \} = \mathbb{R}_+ \check{a}_s$$

has the property that

$$W \check{a}_s \subseteq \check{a}_s - C_{\check{a}_s}.$$
(b) Suppose that \( \alpha_s(\alpha_t) = \alpha_t(\alpha_s) = -3 \). Then

\[
\det \begin{pmatrix}
\alpha_s(\alpha_s) & \alpha_s(\alpha_t) \\
\alpha_t(\alpha_s) & \alpha_t(\alpha_t)
\end{pmatrix} = 4 - 9 < 0
\]

implies that \( \check{\alpha}_s \) and \( \check{\alpha}_t \) are linearly independent. In this case the symmetric bilinear form \( (\cdot, \cdot) \) on \( V \) represented by the matrix

\[
A = \begin{pmatrix}
1 & -\frac{3}{5} \\
-\frac{3}{5} & 1
\end{pmatrix}
\]

with respect to the basis \( \check{\alpha}_s, \check{\alpha}_t \) is \( \mathcal{W} \)-invariant. If \( \alpha^*_s, \alpha^*_t \in V \) is the dual basis of \( \alpha_s, \alpha_t \), then

\[
K = \mathbb{R}_+ \alpha^*_s + \mathbb{R}_+ \alpha^*_t \quad \text{and} \quad \alpha^*_s = -\frac{2}{5} \check{\alpha}_s - \frac{3}{5} \check{\alpha}_t, \quad \alpha^*_t = -\frac{3}{5} \check{\alpha}_s - \frac{2}{5} \check{\alpha}_t.
\]

Now \( (\alpha^*_s, \alpha^*_t) = (\alpha^*_t, \alpha^*_s) < 0 \) implies that

\[
T = \mathcal{W}K \subseteq \{ v \in V : (v, v) \leq 0, (v, \alpha^*_s) \leq 0 \}.
\]

In particular, \( \check{\alpha}_s \) is not contained in \( \pm T \). For this element we have

\[
r_s r_t (\check{\alpha}_s) = r_s (\check{\alpha}_s + 3 \check{\alpha}_t) = -\check{\alpha}_s + 3(\check{\alpha}_s + 3 \check{\alpha}_s) = 8 \check{\alpha}_s + 3 \check{\alpha}_t
\]

and

\[
r_t r_s (\check{\alpha}_s) = r_t (-\check{\alpha}_s) = -\check{\alpha}_s - 3 \check{\alpha}_t,
\]

so that

\[
\frac{7}{2} \check{\alpha}_s = \frac{1}{2} (r_t r_s (\check{\alpha}_s) + r_s r_t (\check{\alpha}_s)) \in \text{co}(\check{\alpha}_s)
\]

shows that \( \check{\alpha}_s \) is an interior point of \( \text{co}(\check{\alpha}_s) \).

On the other hand, the cone

\[
C_{\check{\alpha}_s} = \text{cone}\{ \check{\alpha} : \alpha(\check{\alpha}_s) > 0 \} = \text{cone}\{ \check{\alpha} : (\check{\alpha}, \check{\alpha}_s) > 0 \}
\]

is proper, so that \( \mathcal{W} \check{\alpha}_s \not\subseteq \check{\alpha}_s - C_{\check{\alpha}_s} \).

In general the dual of a linear Coxeter system is not a linear Coxeter system. However, we have seen in Remark 1.6(b) that we always obtain a linear Coxeter system on the subspace \( U = \text{span}(C^*_S) \subseteq V^* \). For orbits in the corresponding Tits cone, we have the following variant of Theorem 2.5.

**Theorem 2.10.** (Convexity Theorem for \( V^* \)) Let \( (V, (\alpha_s)_{s \in S}, (\check{\alpha}_s)_{s \in S}) \) be a linear Coxeter system and \( \tilde{T} = WC^*_S \subseteq V^* \). For any \( \lambda \in \tilde{T} \) we then have

\[
\mathcal{W} \lambda \subseteq \lambda - C_{\lambda} \quad \text{for} \quad C_{\lambda} := \text{cone}\{ \alpha \in \Delta : \lambda(\check{\alpha}) > 0 \}.
\]

**Proof.** We may w.l.o.g. assume that \( \lambda \in C^*_S \). Consider the subspace

\[
U := \text{span}(C^*_S) = H(C_S)^\perp \subseteq V^*
\]

13
and recall from Remark 1.6(b) that \((U, \langle q(\bar{a}_s) \rangle)_{s \in \widetilde{S}}, (\alpha_s)_{s \in \widetilde{S}})\) is a linear Coxeter system with fundamental chamber \(C_{S}^*\) for which we have the surjective restriction map

\[ R: \mathcal{W} \to \mathcal{W}_U, \quad w \mapsto w|_U. \]

It follows in particular that, for \(\lambda \in C_{S}^* \subseteq U\), we have \(\mathcal{W}_\lambda = \mathcal{W}_U \lambda\). Applying Theorem 2.3, we obtain

\[ \mathcal{W}_\lambda = \mathcal{W}_U \lambda \subseteq \lambda - C_{\lambda}^U, \quad \text{where} \quad C_{\lambda}^U = \text{cone}\{\alpha \in \bar{\Delta}_U : \lambda(\bar{\alpha}) > 0\}. \]  

(5)

On the other hand \(\mathcal{W}|_U = \mathcal{W}_U\) implies

\[ \bar{\Delta}_U = \mathcal{W}_U\{\alpha_s : s \in \widetilde{S}\} = \mathcal{W}\{\alpha_s : s \in \widetilde{S}\} \subseteq U. \]

For \(s \in S \setminus \widetilde{S}\) we have \(\bar{\alpha}_s \in U^\perp\), hence also \(W\bar{\alpha}_s \in U^\perp\) because \(U\) is \(\mathcal{W}\)-invariant and thus \(\lambda(W\bar{\alpha}_s) = \{0\}\). This shows that

\[ C_\lambda = \text{cone}\{\alpha \in \bar{\Delta} : \lambda(\bar{\alpha}) > 0\} = C_{\lambda}^U, \]

and by 14, the proof is complete.

The following proposition extends Proposition 1.12 to stabilizers of elements in \(C_{S}^*\).

**Proposition 2.11.** If \(\lambda \in C_{S}^*\), then \(\mathcal{W}_\lambda = \langle r_s : \lambda(\bar{\alpha}_s) = 0 \rangle\).

**Proof.** We recall the subspace \(U := (\bar{C}_S)^* - (\bar{C}_S)^* \subseteq V^*\) and the related objects also used in the preceding proof. We then have a surjective homomorphism \(R: \mathcal{W} \to \mathcal{W}_U, w \mapsto w|_U\), and \(\lambda \in U\) implies that \(\text{ker}\ R \subseteq \mathcal{W}_\lambda\).

For \(s \in S \setminus \widetilde{S}\) we have \(\bar{\alpha}_s \in H(C_S) \subseteq \text{ker}\ \lambda\), which leads to \(r_s \in \text{ker}\ R \subseteq \mathcal{W}_\lambda\). Therefore

\[ S_\lambda := \{s \in S : \lambda(\bar{\alpha}_s) = 0\} \supseteq S \setminus \widetilde{S}. \]

Let \(\widetilde{S}_\lambda := \widetilde{S} \cap S_\lambda\). Since \(C_{\lambda}^*\) is the fundamental chamber of the linear Coxeter system in \(U\), Proposition 1.12 yields

\[ \mathcal{W}_{U,\lambda} = \langle r_s : s \in \widetilde{S}_\lambda, \lambda(\bar{\alpha}_s) = 0 \rangle. \]

This implies that \(\mathcal{W}_\lambda \subseteq \langle r_s : s \in \widetilde{S}_\lambda \rangle \text{ ker } R\).

Next we observe that, for \(s \in \widetilde{S}\) and \(t \in S \setminus \widetilde{S}\) we have \(\alpha_s \in U\) and \(\bar{\alpha}_t \in H(C_S) = U^\perp\), so that \(\alpha_s(\bar{\alpha}_t) = 0\). From (C1) we now also obtain \(\alpha_t(\bar{\alpha}_s) = 0\) and this implies that \(r_s r_t = r_t r_s\):

\[ r_s r_t(v) = r_s(v - \alpha_t(v)\bar{\alpha}_t) = v - \alpha_t(v)\bar{\alpha}_t - \alpha_s(v)\bar{\alpha}_s + \alpha_t(v)\alpha_s(\bar{\alpha}_t)\bar{\alpha}_s = v - \alpha_t(v)\bar{\alpha}_t - \alpha_s(v)\bar{\alpha}_s + \alpha_s(v)\alpha_t(\bar{\alpha}_s)\bar{\alpha}_t = r_t r_s(v). \]

Therefore

\[ \mathcal{W} = \langle r_s : s \in S \rangle = \langle r_s : s \in \widetilde{S} \rangle \langle r_t : t \in S \setminus \widetilde{S} \rangle = \mathcal{W}_S \mathcal{W}_{S \setminus \widetilde{S}} \]

is a product of two commuting subgroups. Since the subgroup \(\mathcal{W}_S\) of \(\mathcal{W}\) is a Coxeter group with Coxeter system \(\{r_s : s \in \widetilde{S}\}\), the restriction homomorphism \(R: \mathcal{W} \to \mathcal{W}_U\) maps \(\mathcal{W}_S\) bijectively onto \(\mathcal{W}_U\). On the other hand, \(\mathcal{W}_{S \setminus \widetilde{S}} \subseteq \text{ker}\ R\), so that \(\text{ker}\ R \cap \mathcal{W}_S = \{1\}\) leads to \(\text{ker}\ R = \mathcal{W}_{S \setminus \widetilde{S}}\). We finally arrive at

\[ \mathcal{W}_\lambda = \mathcal{W}_{\widetilde{S}_\lambda} \text{ ker } R = \mathcal{W}_{\widetilde{S}_\lambda} \mathcal{W}_{S \setminus \widetilde{S}} = \mathcal{W}_{\lambda}. \]

\[ \square \]
The following proposition describes the subset of a $W$-orbit in $T$ on which a linear functional $\lambda \in C^*_S$ takes its maximal values as the orbit of the stabilizer of $W_\lambda$ and we also provide a dual version.

**Proposition 2.12.** Let $\lambda \in C^*_S$ and $v \in K = (\check{C}_S)^*$, so that $\lambda(v) = \max \lambda(Wv)$.

If $g \in W$ satisfies $\lambda(gv) = \max \lambda(Wv)$, then $gv \in W_\lambda v$ and $g^{-1}\lambda \in W_\lambda \lambda$.

**Proof.** (a) First we show that $gv \in W_\lambda v$. In Proposition 2.11 we have seen that $W_\lambda = W_{S_\lambda}$ is a parabolic subgroup of $W$. Let

$$W^\lambda := \{ w \in W : (\forall s \in S_\lambda) \ell(r_s w) > \ell(w) \},$$

so that $W = W_\lambda W^\lambda$ by [Ha92, p. 123].

If there exists a $g \in W$ with $gv \notin W_\lambda v$ and $\lambda(gv) = \max \lambda(Wv)$, we choose such an element $g$ with minimal length. Then $g \in W^\lambda$ with $\ell(g) > 0$. We pick an $s \in S$ with $
abla \ell(g^{-1}r_s) = \ell(r_s g) < \ell(g)$ and observe that this implies that $s \notin S_\lambda$, i.e., $\lambda(\alpha_s) \neq 0$ and therefore $\lambda(\tilde{\alpha}_s) > 0$ because $\lambda \in C^*_S$. Then $g^{-1}\alpha_s \in -\check{C}_S$ by Theorem 1.10, which leads to $0 \geq (g^{-1}\alpha_s)(v) = \alpha_s(gv)$. We thus arrive at

$$\lambda(r_s gv) = \lambda(gv) - \alpha_s(gv) \lambda(\tilde{\alpha}_s) \geq \lambda(gv) \geq \lambda(r_s gv),$$

where the last inequality follows from the maximality of $\lambda(gv)$. We conclude that $\alpha_s(gv) = 0$, so that $r_s gv = gv \notin W_\lambda v$. This contradicts the minimality of the length of $g$.

(b) Now we show that $g^{-1}v \in W_v \lambda$. In Proposition 1.12 we have seen that $W_v$ is a parabolic subgroup of $W$ generated by the reflections $r_{\alpha_s}$ with

$$s \in S_v := \{ s \in S : \alpha_s(v) = 0 \}.$$

Let

$$W^v := \{ w \in W : (\forall s \in S_v) \ell(wr_s) > \ell(w) \},$$

so that $W = W^v W_v$ by [Ha92, p. 123].

If there exists a $g \in W$ with $g^{-1}v \notin W_v \lambda$ and $\lambda(gv) = \max \lambda(Wv)$, we choose such an element $g \in W$ of minimal length. Then $g \in W^v$ with $\ell(g) > 0$. We pick an $s \in S$ with $\ell(gr_s) < \ell(g)$ and observe that this implies that $s \notin S_v$, i.e., $\alpha_s(v) \neq 0$ and therefore $\alpha_s(v) > 0$ because $v \in K$. We further obtain $g\alpha_s \in -C_S$ from Proposition 1.9, which leads to $0 \geq \lambda(g\tilde{\alpha}_s) = (g^{-1}\lambda)(\tilde{\alpha}_s)$. We thus arrive at

$$\lambda(gr_s v) = \lambda(gv) - \alpha_s(v) \lambda(\tilde{\alpha}_s) \geq \lambda(gv) \geq \lambda(gr_s v),$$

where the last inequality follows from the maximality of $\lambda(gv)$. We conclude that $\lambda(g\tilde{\alpha}_s) = 0$, so that $r_s (g^{-1}\lambda) = g^{-1}\lambda \notin W_v \lambda$. This contradicts the minimality of the length of $g$. \qed

**Problem 2.13.** For $v \in K$ and $\alpha \in \Delta$, the relation $\alpha(v) > 0$ implies $\alpha \in \Delta^+ \subseteq \check{C}_S$, so that $C_v \subseteq C_S$ and $C^*_S \subseteq C^*_S$. Is the conclusion of Proposition 2.12 still valid under the weaker assumption $\lambda \in C^*_v$?
3 Orbits of locally finite and locally affine Weyl groups

Now we turn to the applications to Weyl group orbits of linear functionals for locally finite and locally affine root systems (Theorems 2.5 and 2.10).

3.1 Locally finite root systems

First we describe the irreducible locally finite root systems of infinite rank (cf. LN04 §8, NS01). Let $J$ be a set and $V := \mathbb{R}^J$ denote the free vector space over $J$, endowed with the canonical scalar product, given by

$$(x, y) := \sum_{j \in J} x_j y_j.$$ 

We write $(e_j)_{j \in J}$ for the canonical orthonormal basis and we realize the root systems in the dual space $V^* \cong \mathbb{R}^J$ which contains the linearly independent system $\varepsilon_j := e^*_j$, defined by $\varepsilon_j^*(e_k) = \delta_{jk}$. On $\text{span}\{\varepsilon_j : j \in J\}$ we also have a positive definite scalar product defined by $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$ for which the canonical inclusion $V \hookrightarrow V^*$ is isometric. The infinite irreducible locally finite root systems are given by

$$A_J := \{\varepsilon_j - \varepsilon_k : j, k \in J, j \neq k\},$$

$$B_J := \{\pm \varepsilon_j, \pm \varepsilon_j \pm \varepsilon_k : j, k \in J, j \neq k\},$$

$$C_J := \{\pm 2 \varepsilon_j, \pm \varepsilon_j \pm \varepsilon_k : j, k \in J, j \neq k\},$$

$$D_J := \{\pm \varepsilon_j \pm \varepsilon_k : j, k \in J, j \neq k\},$$

$$BC_J := \{\pm \varepsilon_j, \pm 2 \varepsilon_j, \pm \varepsilon_j \pm \varepsilon_k : j, k \in J, j \neq k\}.$$ 

Let $\Delta \subseteq V^* \cong \mathbb{R}^J$ be a locally finite root system of type $X_J$ with $X \in \{A, B, C, D, BC\}$. For $\alpha \in \text{span} \Delta$, we write $\alpha^\sharp \in V$ for the unique element determined by $\alpha(v) = (v, \alpha^\sharp)$ for $v \in V$.

For $\alpha \in \Delta$ we define its coroot by

$$\hat{\alpha} = \frac{2}{(\alpha, \alpha)} \alpha^\sharp.$$ 

This leads to a reflection system $(V, \Delta, \hat{\Delta})$ and a corresponding group $W$, called in this context the Weyl group.

**Theorem 3.1.** For $\lambda \in V^*$ we have

$$W\lambda \subseteq \lambda - C_\lambda \quad \text{for} \quad C_\lambda := \text{cone}\{\alpha \in \Delta : \lambda(\hat{\alpha}) > 0\}.$$ 

**Proof.** Let $w \in W$ and observe that $w$ is a finite product of reflections $r_{\alpha_1}, \ldots, r_{\alpha_n}$. We pick a finite dimensional subset $F \subseteq J$ such that $\alpha_j \in \mathbb{R}^F$ for $j = 1, \ldots, n$. Accordingly, we have an orthogonal direct sum

$$V = V_0 \oplus V_1 \quad \text{with} \quad V_0 = \mathbb{R}^F \quad \text{and} \quad V_1 := V_0^\perp = \mathbb{R}^{J \setminus F}.$$
which is invariant under $w$.

Next we observe that $\Delta_0 := \Delta \cap \mathbb{R}^F$ is a finite root system of type $X_F$, which implies that the finite reflection system $(V_0, \Delta_0, \hat{\Delta}_0)$ comes from a finite Coxeter system with finite Coxeter group $W_0$ containing $w$. In this case the Tits cones in $V_0$ and $V_0^*$ coincide with the whole space, so that

$$W_0 \lambda_0 \subseteq \lambda_0 - C^0_{\lambda_0}$$

holds for $\lambda_0 := \lambda|_{V_0}$ and

$$C^0_{\lambda_0} := \text{cone}\{\alpha \in \Delta_0 : \lambda_0(\hat{\alpha}) = \lambda(\hat{\alpha}) > 0\} \subseteq C_\lambda.$$

Writing $\lambda = \lambda_0 \oplus \lambda_1$ according to the decomposition $V = V_0 \oplus V_1$, we now obtain $w \lambda = w \lambda_0 \oplus \lambda_1 \in (\lambda_0 - C^0_{\lambda_0}) \oplus \lambda_1 = \lambda - C^0_{\lambda_0} \subseteq \lambda - C_\lambda$. 

**Corollary 3.2.** For $d \in V$ and $\lambda \in V^*$, the following are equivalent

(i) $\lambda(d) = \min \langle W \lambda, d \rangle$.

(ii) $d \in -C^*_\lambda$.

(iii) $(\forall \alpha \in \Delta) \lambda(\hat{\alpha}) > 0 \Rightarrow \alpha(d) \leq 0$.

**Remark 3.3.** Since the canonical inclusion $V = \mathbb{R}^{(J)} \hookrightarrow V^* \cong \mathbb{R}^J$ is $W$-equivariant, Theorem 3.1 implies the corresponding result for $W$-orbits in $V$ itself:

$$Wv \subseteq v - C_v \quad \text{for} \quad C_v = \text{cone}\{\hat{\alpha} : \alpha(v) > 0\}.$$

### 3.2 Locally affine root systems

Let $V = \mathbb{R}^{(J)}$ be as above and $\Delta \subseteq V^* \cong \mathbb{R}^J$ be a locally finite root system of type $X_J$. We put

$$\hat{\Delta} := \mathbb{R} \times V \times \mathbb{R} \quad \text{and} \quad \Delta^{(1)} := \{0\} \times \Delta \times \mathbb{Z} \subseteq \mathbb{R} \times V^* \times \mathbb{R} \cong \hat{\Delta}^*.$$

We also define a Lorentzian form on $\hat{\Delta}$ by

$$((z, x, t), (z', x', t')) := (x, x') - zt' - z't.$$

Suppressing the first component, we have Yoshii’s classification ([YY10 Cor. 13]):

**Proposition 3.4.** The irreducible reduced locally affine root systems of infinite rank are the following, where $J$ is an infinite set: $A_J^{(1)}$, $B_J^{(1)}$, $C_J^{(1)}$, $D_J^{(1)}$, or

$$B_J^{(2)} := (B_J \times 2\mathbb{Z}) \cup \{\pm \varepsilon_j : j \in J\} \times (2\mathbb{Z} + 1)),
C_J^{(2)} := (C_J \times 2\mathbb{Z}) \cup (D_J \times (2\mathbb{Z} + 1))
(BC_J)^{(2)} := (B_J \times 2\mathbb{Z}) \cup (BC_J \times (2\mathbb{Z} + 1)).$$
Let \( \hat{\Delta} \subseteq \hat{V}^* \) be one of these locally affine root systems. We write
\[
\Delta_n := \{ \alpha \in \Delta : (0, \alpha, n) \in \hat{\Delta} \},
\]
so that
\[
\hat{\Delta} = \{(0) \times \Delta_0 \times Z \} \cup \{(0) \times \Delta_1 \times (Z + 1) \}.
\]
A quick inspection shows that all reflections corresponding to roots in \( \Delta_1 \) from \( \Delta_0 \) so that
\[
\hat{\Delta} \supseteq \{ \alpha \in \Delta : (0, \alpha, n) \in \hat{\Delta} \}.
\]
Therefore we obtain an injection
\[
\nu \colon \hat{W} \cong \langle (0, \alpha, 0) : \alpha \in \Delta_0 \rangle_{\text{grp}} \hookrightarrow \hat{W}.
\]

Since \( \hat{\Delta} \) consists of non-isotropic vectors for the Lorentzian form, we can also define for
\[
\alpha = (0, \alpha, n) \in \hat{\Delta}
\]
the coroot by
\[
\hat{\Delta} = \frac{2}{\langle \alpha, \alpha \rangle} \alpha^\vee = \frac{2}{\langle \alpha, \alpha \rangle} (n - \alpha^\vee, 0) = \left( \frac{-2n}{\langle \alpha, \alpha \rangle}, \hat{\alpha}, 0 \right) \tag{6}
\]
and obtain a reflection system \((\hat{V}, \hat{\Delta}, \hat{\Delta})\) and a corresponding (affine) Weyl group \( \hat{W} \). In the following we write
\[
c := (1, 0, 0) \quad \text{and} \quad d := (0, 0, 1)
\]
for these two distinguished elements of \( \hat{V} \).

**Theorem 3.5.** For \( \lambda \in \hat{V}^* \) with \( \lambda(c) \neq 0 \) the following assertions hold:

(i) \( \hat{W} \lambda \subseteq \lambda - C_\lambda \) for \( C_\lambda := \text{cone}\{ \alpha \in \hat{\Delta} : \lambda(\hat{\alpha}) > 0 \} \).

(ii) If \( \lambda(d) = \min(\hat{W} \lambda)(d) \), then any \( \mu \in \hat{W} \lambda \) with \( \mu(d) = \lambda(d) \) is contained in the orbit of \( \hat{W} \cong \hat{W}_d \).

**Proof.** (i) Since we can argue as in the locally finite case, it suffices to show that, if \( J \) is finite and \( \hat{\Delta} \) of type \( X_{t_f}^{(l)} \) for \( t = 1, 2 \), then \( \lambda \in \hat{V}^* \) is contained in the Tits cone, so that Theorem 2.10 applies. To this end, we have to recall the description of affine root systems with respect to a simple system of roots.

Since \( \lambda(c) \neq 0 \), the \( \hat{W} \)-orbit contains an element which is either dominant or antidominant (Ne10 Prop. 4.9; see also Ka90 Prop. 6.6 and MP95 Thm. 16), i.e., there exists a simple system of roots \( \Pi \subseteq \hat{\Delta} \) such that \( \lambda \) is dominant. This means that \( \lambda \) is contained in the corresponding fundamental chamber, hence in particular in the Tits cone. Now the assertion follows from Theorem 2.10.

(ii) With the same argument as before, it suffices it suffices to prove the assertion for the case where \( J \) is finite and \( \lambda \) is antidominant with respect to a given simple system \( \Pi \) of roots. So assume that \( \hat{\Delta} \) is of type \( X_{t_f}^{(l)} \) for some finite set \( J \). We write \( \Pi = \{ \alpha_0, \ldots, \alpha_r \} \) for a set of simple roots, where \( \alpha_1, \ldots, \alpha_r \) are simple roots of the corresponding finite root system \( \Delta \) and \( \alpha_0 = (-\theta, 1) \), where \( \theta \) is the “highest weight” in \( \Delta_1 \) with respect to the positive system defined by \( \{ \alpha_1, \ldots, \alpha_r \} \) (cf. Ka90).

Now \( \alpha_j(d) = 0 \) for \( j = 1, \ldots, r \) and \( \alpha_0(d) = 1 \) imply that \( d \in K = (C_S)^* \). Further, the antidominance of \( \lambda \) with respect to \( \Pi \) means that \( \lambda \in -C_S \). If \( \mu = w^{-1}\lambda \) satisfies \( \mu(d) = \lambda(wd) = \lambda(d) = \min(\hat{W} \lambda)(d) \), we obtain \( \mu \in \hat{W}_d \lambda \) from Proposition 2.12.
Finally, we note that the stabilizer group $\hat{W}_d$ is a parabolic subgroup of $\hat{W}$ generated by the fundamental reflections $r_{\alpha_j}$ fixing $d$, which is the case for $j > 0$. Therefore $\hat{W}_d \cong W$ is the Weyl group of the corresponding finite root system $X_J$.

\textbf{Corollary 3.6.} For $d = (0, 0, 1)$ and $\lambda \in \hat{V}^*$, the following are equivalent:

(i) $\lambda(d) = \min (\hat{W}\lambda, d)$.

(ii) $d \in -C_\lambda^*$.

(iii) $(\forall \alpha \in \hat{A}) \lambda(\hat{\alpha}) > 0 \Rightarrow \alpha(d) \leq 0$.

(iv) $(\forall \alpha = (0, \alpha, n) \in \hat{A}) \ n > 0 \Rightarrow \frac{\langle \alpha, \alpha \rangle}{2n} \lambda(\hat{\alpha}) \leq \lambda(c)$.

\textit{Proof.} The equivalence of (i) and (ii) follows from Theorem 3.5(i), and (iii) is a reformulation of (ii). The equivalence of (iii) and (iv) follows by negating the implication, inserting the formula for the coroot and using $\alpha(d) = n$.

\textbf{Definition 3.7.} Linear functionals $\lambda \in \hat{V}^*$ satisfying the equivalent conditions in Corollary 3.6 are called $d$-minimal.

\textbf{3.3 Characterization of $d$-minimal weights}

For $\lambda \in \hat{V}^*$, let $\lambda_c := \lambda(c)$.

\textbf{Lemma 3.8.} If $(\hat{W}\lambda)(d)$ is bounded from below, then $\lambda_c \geq 0$. If, in addition, $\lambda_c = 0$, then $\lambda$ is fixed by $\hat{W}$.

\textit{Proof.} If $(0, \alpha, n) \in \hat{A}$, then also $(0, \alpha, n + 2k) \in \hat{A}$ for every $k \in \mathbb{N}$, so that Corollary 3.6(iv) implies $\lambda_c \geq 0$.

If $\lambda_c = 0$, then $(0, \alpha, n) \in \hat{A}$ for some $n \neq 0$ implies the additional condition $\lambda(\hat{\alpha}) = 0$. Hence $\lambda_c = 0$ leads to $\lambda(\hat{\alpha}) = 0$ for every $\alpha \in \hat{A}$, so that $\lambda$ is fixed by $\hat{W}$.

\textbf{Proposition 3.9.} Suppose that $\hat{A}$ is one of the 7 irreducible locally affine root systems. For $\lambda \in \hat{V}^*$ with $\lambda_c > 0$, the following are equivalent:

(i) $\lambda$ is $d$-minimal.

(ii) $(\forall \alpha \in \Delta, n = 1, 2) \ (0, \alpha, n) \in \hat{A} \Rightarrow |\lambda(\hat{\alpha})| \frac{|\alpha, \alpha|}{2n} \leq \lambda_c$.

\textit{Proof.} That (ii) follows from (i) is a consequence of Corollary 3.6(iv) and the observation that $(0, \alpha, n) \in \hat{A}$ implies $(0, -\alpha, n) \in \hat{A}$.

If, conversely, (ii) holds, then the 2-periodic structure of the root system implies the condition in Corollary 3.6(iv).

\textbf{Theorem 3.10.} For the seven irreducible locally affine root systems $X_{J(r)} = \hat{A}$ of infinite rank, a linear functional $\lambda = (\lambda_c, \lambda, \lambda_d) \in \hat{V}^*$ with $\lambda_c > 0$ is $d$-minimal if and only if the following conditions are satisfied by the corresponding function $\bar{\lambda}: J \to \mathbb{R}, j \mapsto \lambda_j$:

$(A_j^{(i)}) \ \max \bar{\lambda} - \min \bar{\lambda} \leq \lambda_c$. 

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\[(B_1^{(1)}) \quad |\lambda_j| + |\lambda_k| \leq \lambda_c \text{ for } j \neq k.\]
\[(C_1^{(1)}) \quad |\lambda_j| \leq \lambda_c/2 \text{ for } j \in J.\]
\[(D_1^{(1)}) \quad |\lambda_j| + |\lambda_k| \leq \lambda_c \text{ for } j \neq k.\]
\[(B_2^{(2)}) \quad |\lambda_j| \leq \lambda_c \text{ for } j \in J.\]
\[(C_2^{(2)}) \quad |\lambda_j| + |\lambda_k| \leq \lambda_c \text{ for } j \neq k.\]
\[(BC_1^{(2)}) \quad |\lambda_j| \leq \lambda_c/2 \text{ for } j \in J.\]

**Proof.** In the untwisted case \( r = 1 \) we have \( \hat{\Delta} = \{0\} \times \Delta \times Z = X_1^{(1)} \), so that Proposition 3.9 asserts that \( \lambda \) is \( d \)-minimal if and only if \( |\lambda(\hat{a})| (\text{say}) \leq \lambda_c \) for \( \alpha \in \Delta \).

\( A_1^{(1)} \): For the root system \( \Delta = A_J \), all roots satisfy \( (\alpha, \alpha) = 2 \), so that the \( d \)-minimality condition on \( \lambda \) is
\[\lambda_j - \lambda_k \leq \lambda_c \quad \text{ for } j \neq k \in J.\]
This can also be written as \( \max \hat{x} - \min \hat{x} \leq \lambda_c \).

\( B_1^{(1)} \): For \( \Delta = B_J \), the roots \( \varepsilon_j \) satisfy \( (\varepsilon_j, \varepsilon_j) = 1 \) and \( \varepsilon_j = 2\varepsilon_j \). This leads to the \( d \)-minimality conditions
\[|\lambda_j| \leq \lambda_c \quad \text{ and } |\lambda_j + \lambda_k| \leq \lambda_c\]
which is equivalent to \( |\lambda_j| + |\lambda_k| \leq \lambda_c \) for \( j \neq k \).

\( C_1^{(1)} \): For the root system \( C_J \), the roots \( 2\varepsilon_j \) satisfy \( (2\varepsilon_j, 2\varepsilon_j) = 4 \). The \( d \)-minimality thus implies \( |\lambda_j| \leq \lambda_c/2 \). This also implies that \( |\lambda_j + \lambda_k| \leq \lambda_c \) for \( j \neq k \in J \), so that it characterizes the \( d \)-minimal weights.

\( D_1^{(1)} \): For the root system \( D_J \), we find the conditions \( |\lambda_j| + |\lambda_k| \leq \lambda_c \) which are equivalent to \( |\lambda_j| + |\lambda_k| \leq \lambda_c \) for \( j \neq k \in J \).

\( B_2^{(2)} \): In this case \( \Delta_0 = B_J \) and \( \Delta_1 = \{\pm \varepsilon_j : j \in J\} \) with \( \|\varepsilon_j\| = 1 \). In view of \( \varepsilon_j = 2\varepsilon_j \), we obtain from the roots in \( \Delta_1 \) the condition \( |\lambda_j| = 1/2|2\lambda_j| \leq \lambda_c \). For the roots \( \varepsilon_j \pm \varepsilon_k \in \Delta_0 \) we obtain the additional condition \( |\lambda_j + \lambda_k| \leq 2\lambda_c \) which is redundant.

\( C_2^{(2)} \): In this case \( \Delta_1 = D_J \) and \( \Delta_0 = C_J \) with \( \|2\varepsilon_j\|^2 = 4 \) lead to the conditions
\[|\lambda_j + \lambda_k| \leq \lambda_c \quad \text{ and } |\lambda_j| \leq \lambda_c,\]
which is equivalent to \( |\lambda_j| + |\lambda_k| \leq \lambda_c \) for \( j \neq k \in J \).

\( BC_2^{(2)} \): Here \( \Delta_1 = BC_J \) and \( \Delta_0 = B_J \) with \( \|2\varepsilon_j\|^2 = 4 \) lead to the conditions \( |\lambda_j| \leq \lambda_c/2 \) for the roots \( \alpha = \pm 2\varepsilon_j \), and the roots \( \alpha = \pm \varepsilon_j \) provide no additional restriction. For the roots \( \alpha = \varepsilon_j \pm \varepsilon_k \) we obtain \( |\lambda_j \pm \lambda_j| \leq \lambda_c \), which also is redundant.

**Remark 3.11.** (a) The preceding theorem implies that \( d \)-minimal weights \( \lambda \in \hat{V}^* \) define bounded functions \( \hat{\lambda} \colon J \to \mathbb{R} \) and, moreover, that the boundedness of \( \hat{\lambda} \) is equivalent to the existence of a \( \lambda_c > 0 \) such that \( \lambda = (\lambda_c, \hat{\lambda}, \lambda_d) \in \hat{V}^* \) is \( d \)-minimal.

(b) If \( \lambda \in V^* \) satisfies \( \lambda(\hat{a}) \in \mathbb{Z} \) for each \( \alpha \in \hat{\Delta} \), then the subset \( \lambda + \hat{\mathcal{Q}} \subseteq V^* \), where \( \hat{\mathcal{Q}} = (\hat{\Delta})_{\text{grp}} \) is the root group, is invariant under the Weyl group \( \hat{W} \). Therefore \( (\hat{W}\lambda)(d) \subseteq \lambda(d) + \mathbb{Z} \). If \( (\hat{\mathcal{W}}\lambda)(d) \) is bounded from below, we thus obtain the existence of a \( d \)-minimal element in \( \hat{W}\lambda \).
For general functionals which are not integral weights, the situation is more complicated, as Example 3.13 below shows.

3.4 The affine Weyl group

Recall the inclusion
\[ \iota_W: W \cong \langle r_{(0,\alpha,0)}: \alpha \in \Delta_0 \rangle_{\text{grp}} \hookrightarrow \hat{W} \]
of the locally finite Weyl group \( W \) into \( \hat{W} \) and note that it provides a section of the quotient homomorphism \( q: \hat{W} \rightarrow W \) corresponding to the passage from \( \hat{V} \) to \( V \). For \( n \in \mathbb{Z} \) and \( (0, \alpha, n) \in \hat{\Delta} \), the elements \( r_{(0,\alpha,0)}r_{(0,\alpha,n)} \) generate the normal subgroup \( \mathcal{N} := \ker q \).

To make the structure of \( \mathcal{N} \) more explicit, we consider for \( x \in V \) the endomorphism \( \tau_x = \tau(x) \) of \( \hat{V} \), defined by
\[ \tau_x(z, y, t) := \left( z + \langle y, x \rangle + \frac{t\|x\|^2}{2}, y + tx, t \right). \]
The maps \( \tau_x \) are isometries with respect to the Lorentzian form and an easy calculation shows that \( \tau_x = \tau(\xi_x) \) for \( x \in V \). For \( n \in \mathbb{Z} \) and \( (0, \alpha, n) \in \hat{\Delta} \), this leads to
\[ \mathcal{N} = \tau(\mathcal{T}) \quad \text{for} \quad \mathcal{T} := \langle n\hat{\alpha}: \alpha \in \Delta_n, n \in \mathbb{N} \rangle_{\text{grp}}. \]

Proposition 3.12. For the untwisted root systems of type \( X^{(1)}_J \), the group \( \mathcal{T} \) coincides with the group \( \hat{\mathcal{R}} := \langle \hat{\Delta} \rangle_{\text{grp}} \) of coroots. For the three twisted cases, it is given in \( V \) in terms of the canonical basis elements \( (e_j)_{j \in J} \) by:

(i) \( \mathcal{T} = 2\mathbb{Z}^{(J)} \) for \( B_J^{(2)} \).

(ii) \( \mathcal{T} = \{ \sum_{j \in J} n_j e_j : \sum_{j \in J} n_j \in 2\mathbb{Z} \} \) for \( C_J^{(2)} \).

(iii) \( \mathcal{T} = \sum_{j \in J} \mathbb{Z} e_j \cong \mathbb{Z}^{(J)} \) for \( BC_J^{(2)} \).

Proof. (i) For \( B_J^{(2)} \) we derive from \( \Delta_1 = \{ \pm \varepsilon_j : j \in J \} \), \( \Delta_0 = B_J \) and \( \varepsilon_j = 2e_j \), \( (\varepsilon_j \pm \varepsilon_k)^* = e_j \pm e_k \) that \( \mathcal{T} \) is the subgroup of \( V \) generated by the elements
\[ 2(e_j \pm e_k), j \neq k \quad \text{and} \quad 2e_j, j \in J. \]

(ii) For \( C_J^{(2)} \) we have \( \Delta_1 = D_J \) and \( \Delta_0 = C_J \), which leads to the generators
\[ \pm e_j \pm e_k, j \neq k \quad \text{and} \quad 2e_j, j \in J. \]

(iii) For \( BC_J^{(2)} \) we obtain from \( \Delta_1 = BC_J \) and \( \Delta_0 = B_J \) the generators \( \pm e_j, \pm e_j \pm e_k \) for \( j \neq k \). \( \square \)
Note that
\[ \hat{W}d = Nd = \left\{ \left( \frac{||x||^2}{2}, x, 1 \right) : x \in T \right\} \]
leads to
\[ (\hat{W}\lambda)(d) = \left\{ \lambda_c \frac{||x||^2}{2} + \lambda(x) + \lambda_d : x \in T \right\}. \tag{7} \]
This formula shows immediately that if \( \lambda_c > 0 \) and
\[ ||\lambda||_2^2 := \sum_{j \in J} |\lambda_j|^2 < \infty, \]
which is in particular the case if \( \text{supp}(\lambda) \) is finite, then \( (\hat{W}\lambda)(d) \) is bounded from below. Since \( T \) is not a vector space, we cannot expect that the condition that \( (\hat{W}\lambda)(d) \) is bounded from below implies that \( ||\lambda||_2 < \infty \), and Theorem 3.10 does indeed show that this is not the case. It only implies that \( \lambda \) is bounded. The following example illustrates the situation further.

**Example 3.13.** We provide an example of an element \( \lambda \in \hat{V}^* \) for which \( (\hat{W}\lambda)(d) \) is bounded from below, but contains no minimum.

We consider the root system \( A_\mathbb{N} (J = \mathbb{N}) \) and \( \lambda = (1, \lambda, 0) \in \hat{V}^* \) defined by
\[ \lambda : \mathbb{N} \to \mathbb{R}, \quad \lambda_{2k} = 0 \quad \text{and} \quad \lambda_{2k-1} = 1 + \frac{1}{k^2} \quad \text{for} \quad k \in \mathbb{N}. \]
On \( T = \{ x \in \mathbb{Z}^J : \sum_n x_n = 0 \} \) we then consider the function
\[ f : T \to \mathbb{R}, \quad f(x) := \frac{1}{2} ||x||^2 \lambda_c + \lambda(x) = \frac{1}{2} ||x||^2 + \sum_{n=1}^{\infty} x_n \lambda_n := \frac{1}{2} ||x||^2 + \sum_{k=1}^{\infty} x_{2k-1} \left( 1 + \frac{1}{k^2} \right). \]
We claim that \( f \) is bounded from below but that it does not have a minimal value.

If \( x_{2k-1} \leq -3 \) for some \( k \), then we consider the element \( \bar{x} := x + e_{2k-1} - e_{2k} \), where \( \bar{k} \) is such that \( x_{\bar{k}} = 0 \). Then
\[ f(x) - f(\bar{x}) = \frac{1}{2} (x_{2k-1}^2 - (x_{2k-1} + 1)^2 - 1) - \left( 1 + \frac{1}{k^2} \right) = -x_{2k-1} - 2 - \frac{1}{k^2} \geq 0. \]
To show that \( f(T) \) is bounded from below, it therefore suffices to consider \( f(x) \) for elements \( x \in T \) satisfying \( x_{2k-1} \geq -2 \) for every \( k \in \mathbb{N} \). This leads to
\[ f(x) = \frac{1}{2} ||x||^2 + \sum_{k=1}^{\infty} x_{2k-1} \left( 1 + \frac{1}{k^2} \right) \geq \frac{1}{2} ||x||^2 + \sum_{k=1}^{\infty} x_{2k-1} - 2 \sum_{k=1}^{\infty} \frac{1}{k^2}, \]
and since \( \frac{1}{2} ||x||^2 + \sum_{k=1}^{\infty} x_{2k-1} \geq 0 \) for every \( x \in T \) by Theorem 3.10(A\(^{(1)}\)), we see that \( f \) is bounded from below.

If \( f(x_0) = \min f(T) \), then
\[ f(x_0 + x) = \frac{1}{2} ||x + x_0||^2 + \langle \lambda, x + x_0 \rangle = \frac{1}{2} ||x||^2 + \langle \lambda + x_0, x \rangle + f(x_0) \]
implies that the function \( \overline{\lambda} := \lambda + x_0 \) defines a \( d \)-minimal functional \( \mu = (1, \overline{\lambda}, 0) \), so that
\[ \sup \overline{\lambda} - \inf \overline{\lambda} \leq \mu_c = 1. \]
Since \( x_0 \) has finite support, \( \sup \overline{\lambda} > 1 \), so that \( \inf \mu \leq 0 \) leads to a contradiction. Therefore \( \hat{W}\lambda \) contains no \( d \)-minimal element.
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