Distribution-Agnostic Model-Agnostic Meta-Learning

Liam Collins∗, Aryan Mokhtari∗, Sanjay Shakkottai∗

February 13, 2020

Abstract

The Model-Agnostic Meta-Learning (MAML) algorithm [Finn et al., 2017] has been celebrated for its efficiency and generality, as it has demonstrated success in quickly learning the parameters of an arbitrary learning model. However, MAML implicitly assumes that the tasks come from a particular distribution, and optimizes the expected (or sample average) loss over tasks drawn from this distribution. Here, we amend this limitation of MAML by reformulating the objective function as a min-max problem, where the maximization is over the set of possible distributions over tasks. Our proposed algorithm is the first distribution-agnostic and model-agnostic meta-learning method, and we show that it converges to an ϵ-accurate point at the rate of $O\left(\frac{1}{\epsilon^2}\right)$ in the convex setting and to an $(\epsilon, \delta)$-stationary point at the rate of $O\left(\max\{\frac{1}{\epsilon^5}, \frac{1}{\delta^5}\}\right)$ in nonconvex settings. We also provide numerical experiments that demonstrate the worst-case superiority of our algorithm in comparison to MAML.

1 Introduction

Broadly speaking, the extent to which a model learns from data is dependent upon the model’s properties, including its architecture, initial setting, and update rule. Meta-learning has gained popularity as a paradigm to learn optimal model properties and thereby improve model performance with more experience, i.e., to learn how to learn [Thrun and Pratt, 2012, Bengio et al., 1990, Hochreiter et al., 2001]. One especially popular meta-learning problem proposed by [Finn et al., 2017] is to learn an initialization for a gradient-based update, such that upon seeing a new task, the model performs well on that task after performing one step of gradient descent from the global initialization. This problem is also known as gradient-based Model-Agnostic Meta-Learning (MAML), and is formally defined as

$$\min_{w \in W} \left\{ \mathbb{E}_{i \sim P} \left[ f_i(w) \right] := \mathbb{E}_{i \sim P} \left[ f_i(w - \alpha \nabla f_i(w)) \right] \right\},$$

(1)

where $W \subset \mathbb{R}^d$ is the feasible set, $P$ is the probability distribution over tasks indexed by $i$ and $f_i$ is the stochastic loss function associated with the $i$-th task. In particular, each $f_i : W \rightarrow \mathbb{R}$ is such that $f_i(w) := \mathbb{E}_{\theta \sim Q_i} \left[ f_i(w, \theta) \right]$, where $Q_i$ is the distribution over the data from the sample space $\Omega_i$ associated with the $i$-th task, $\theta$ is a sample drawn from $Q_i$, and $f_i(w, \theta)$ is the approximation of $f_i(w)$ at the data point $\theta$. Note that since often the probability distribution $P$ is unknown and only $m$ realizations from $P$ are accessible, we settle for solving the sample average approximation problem, i.e.,

$$\min_{w \in W} \frac{1}{m} \sum_{i=1}^{m} f_i(w) := \frac{1}{m} \sum_{i=1}^{m} f_i(w - \alpha \nabla f_i(w)).$$

(2)

However, the formulation (2) yields meta-learning algorithms that are flawed in three key areas:

∗Department of Electrical and Computer Engineering, The University of Texas at Austin, Austin, TX, USA. (Email: liamc@utexas.edu, mokhtari@austin.utexas.edu, sanjay.shakkottai@utexas.edu).
1. **Worst case performance.** Since the objective function of (2) is the average loss over all the drawn tasks, models trained to optimize the objective prioritize performance on tasks that appear the most often from $P$ while devaluing performance on infrequent tasks, leading to potentially arbitrarily poor performance in the worst case. Yet in many situations, arbitrarily poor worst-case performance is unacceptable.

2. **Fairness.** Following the same line of reasoning, algorithms that solve (2) are biased towards tasks that appear more frequently from $P$ than others. This may result in high variance in model performance across tasks, which is unfair to those tasks at the low end of the spectrum.

3. **Reliance on Task Similarity.** State-of-the-art algorithms to solve (2) are forms of stochastic gradient descent that compute their next iterates as functions of the stochastic gradients of only a relatively small number of tasks sampled from $P$ on each iteration. Therefore, if the gradients of commonly occurring tasks are far from each other, these methods may not converge. See for example the proof of convergence of the MAML algorithm in [Fallah et al. 2019], which relies on the assumption of bounded variance of the gradients of the tasks drawn from $P$ (Assumption 4.5).

All three of these deficiencies caused by the formulation (2) originate from the underlying assumption that the tasks come from a particular distribution $P$, signaling the need for a meta-learning problem formulation that does not make this assumption, just as the MAML formulation has removed the assumption that the objective is to learn a particular type of model. This begs the question:

*Can we develop meta-learning methods that are not only model-agnostic, but also distribution-agnostic?*

**Contributions.** In this paper, we answer the above question in the affirmative by developing a novel min-max meta-learning formulation (Section 3) and introducing a method (Section 4) which efficiently solves it in three distinct cases (Section 5). In particular, a list of our detailed contributions follows.

- We propose a min-max variant of the MAML formulation in (2) which corrects the three deficiencies of the original MAML problem discussed above by making no assumptions about the tasks being generated from some distribution, yielding algorithms that perform well in the worst case and fairly treat tasks that may be arbitrarily dissimilar from each other.

- In the convex setting, we show that our proposed methods converge to a point that is $\epsilon$-close to an optimal solution of the proposed min-max MAML problem after $O(1/\epsilon^2)$ stochastic function evaluations, matching the optimal rate studied in Nemirovski et al. [2009].

- We also consider two nonconvex settings - one where the problem is unconstrained, i.e., $W = \mathbb{R}^d$, and another where the problem is constrained over a convex and compact set. In the unconstrained case, we show that our method requires at most $O(\max\{1/\epsilon^{2/\beta}, 1/\beta^{1/\min(2\beta, 1-2\beta)}\})$ iterations to find an $(\epsilon, \delta)$-stationary point for any $\beta \in (0, 1/2)$, where the norm of the gradient with respect to $w$ of our objective is at most $\epsilon$. In the constrained case, we show that the complexity to achieve the same bound on the norm of the projected gradient is of $O(\max\{1/\epsilon^{(2+2\beta)/\beta}, 1/\beta^{(1+\beta)/\min(\beta, 1-\beta)}\})$ for any $\beta \in (0, 1)$. For both cases, an appropriate setting of $\beta$ yields an $O(\max\{1/\epsilon^5, 1/\delta^5\})$ convergence rate.

- Additionally, we provide experimental results that demonstrate the improved worst-case performance across tasks compared to state-of-the-art methods.

### 2 Related Work

Recently many works have studied Meta-learning techniques in various contexts, including few-shot learning [Vinyals et al. 2016, Ravi and Larochelle 2016, Snell et al. 2017], reinforcement learning [Duan et al. 2016, Wang et al. 2016] and online learning [Finn et al. 2019, Khodak et al. 2019b,a]. Of interest in this paper are gradient-based meta-learning methods [Andrychowicz et al. 2016, Finn et al. 2017, Zhou et al. 2019] which have an inner-outer loop structure that executes a learning
algorithm in the inner loop and use the results to update the algorithm parameters in the outer loop with some function of the gradient of the meta-learning objective.

Our work contributes a worst-case optimal version of the seminal gradient-based meta-learning technique, MAML [Finn et al., 2017]. MAML has achieved success in regression, few-shot image classification, and reinforcement learning [Finn et al., 2017] and has inspired numerous related algorithms studied experimentally [Al-Shedivat et al., 2017, Li et al., 2017, Nichol and Schulman, 2018].

The convergence properties of MAML and two of its first-order variants were analyzed in Fallah et al. [2019], and other works have established regret bounds in terms of expected loss over tasks for the online version of MAML [Finn et al., 2019, Zhuang et al., 2019, Khodak et al., 2019]. However, to the best of our knowledge, no other work has proposed a version of MAML that optimizes the worst case performance over tasks.

Robustness in meta-learning has been studied recently by Zügner and Günnemann [2019] and Yin et al. [2018]. However, the notion of robustness in both of these papers is with respect to perturbations in the samples, not with respect to the distribution of tasks - both works attempt to minimize some expected loss over a particular distribution of tasks. Meanwhile, meta-learning is closely related to hyper-parameter optimization (HPO) and bilevel programming (BLP), and both meta-learning and HPO have jointly been formulated in terms of BLP [Franceschi et al., 2018]. Still, we are unaware of any analogous worst-task performance analysis in either the HPO or BLP literature.

There also exist many works outside of meta-learning that have considered min-max optimization problems of the stochastic finite-sum form we will discuss. In the context of distributionally robust optimization, Shalev-Shwartz and Wexler [2016] and Sinha et al. [2017] argued that minimizing the maximal loss of a model over a set of possible distributions can provide better generalization performance than minimizing the average loss. When each function being summed is convex, Nemirovski et al. [2009] showed that stochastic mirror descent-ascent algorithm achieves the asymptotically optimal $O(\epsilon^{-2})$ rate of convergence to an $\epsilon$-accurate solution. The literature has treated the case when the outer minimization problem is nonconvex far less thoroughly. Rafique et al. [2018] proposed a stochastic inexact proximal point method that attains $O(\epsilon^{-4})$ convergence in terms of the outer minimization problem when that problem is nonsmooth and weakly convex, while Qian et al. [2019] showed $O(\epsilon^{-4})$ convergence for a smooth problem. Also, Chen et al. [2017] and Jin et al. [2019] analyzed first order methods that improve on this convergence rate but rely on an oracle to solve the inner maximization. In contrast, neither the smoothness nor oracle assumptions apply to our setting.

3 Problem Formulation

In this section, we reformulate the MAML objective in (1) to yield a both model-agnostic and distribution-agnostic problem that amends the deficiencies of MAML as discussed in the introduction. In particular, our problem formulation tries to find the initial point that performs best after one step of gradient descent for the worst-case task. We later show that to solve this problem using iterative methods we never need to assume any conditions about the similarity of the tasks and in general they can be arbitrarily chosen.

Consider the following min-max problem

$$
\min_{w \in W} \max_{i \in [m]} \{f_i(w - \alpha \nabla f_i(w))\}
$$

(3)

which finds the initial point that minimizes the objective function after one step of gradient over all possible loss functions (i.e., over the loss corresponding to all possible tasks). We can think of (3) as a distribution-agnostic formulation of MAML.

The min-max problem in (3) is equivalent to the problem of finding the $w^*$ that minimizes the worst-case performance over all possible distributions of tasks, since the worst-case distributions will occur at the extreme points of the probability simplex. We write this relaxed problem as

$$
\min_{w \in W} \max_{\mu \in \Delta} \sum_{i=1}^{m} p_i f_i(w - \alpha \nabla f_i(w)),
$$

(4)

where $p_i$ is the probability associated with task $i$, the vector $\mu \in \mathbb{R}^m$ is the concatenation of probabilities $p_1, \ldots, p_m$, and $\Delta$ is the standard simplex in $\mathbb{R}^m$, i.e., $\Delta = \{\mu \in \mathbb{R}_+^m | \sum_{i=1}^{m} p_i = 1\}$. 

3
By optimizing worst-case performance, the formulation in [1] also encourages a fair solution \( w^* \). Instead of being allowed to disregard performance on some tasks, any algorithm that solves [4] must try to perform reasonably well on all of them. Indeed, as Duch et al. [2016] observed, the min-max formulation implicitly regularizes the variance of the losses \( f_i(w - \alpha \nabla f_i(w)) \), meaning that the training procedure tries to produce a model that performs similarly on all tasks. Such a model aligns with the objective of “good intent fairness” proposed in Mohri et al. [2019].

In addition to replacing the expectation over a particular distribution with a maximization over all possible distributions of tasks, we also reorder the expectations implicit in the MAML objective [1]. Note that \( \nabla f_i(w) \) is an expectation over many data points, thus it is intractable to compute exactly, so instead we must approximate it using a finite number of samples - perhaps as few as 1 or 5 for few-shot learning - at test time. Thus, it makes sense to optimize the expectation over the drawn samples of the function value after taking one step of stochastic gradient descent - in other words, to move the expectation over the finite sample gradient approximation outside of the function evaluation - as opposed to optimizing the function value after the true gradient step.

Combining both our modifications, the problem that we aim to solve is as follows:

\[
\min_{w \in W} \max_{p \in \Delta} \left\{ \phi(w, p) := \sum_{i=1}^{m} p_i E_{\theta \sim Q_i} \left[ f_i(w - \alpha \nabla \hat{f}_i(w, \theta)) \right] \right\}
\]  

(5)

Note that this objective corresponds to the case that in the test phase we only run one step of stochastic gradient descent, and the expectation is taken over \( \theta \). Indeed, one can also consider the case where we run multiple steps of stochastic gradient descent, but to simplify the expressions for the proposed method and convergence analysis we only focus on the single step case, without loss of generality.

4 Algorithm

Our proposed algorithm to solve [5] is a version of stochastic projected gradient descent-ascent, and is inspired by the Euclidean version of the Saddle Point Mirror SA algorithm proposed by Nemirovski et al. [2009]. Before stating our method, let us first mention that the gradients of the function \( \phi(w, p) \) defined in [5] with respect to \( w \) and \( p \), which we denote by \( g_w(w, p) \) and \( g_p(w, p) \), respectively, are given by

\[
g_w(w, p) = \sum_{i=1}^{m} p_i E_{\theta \sim Q_i} \left[ (I - \alpha \nabla^2 \hat{f}_i(w, \theta)) \nabla f_i(w - \alpha \nabla \hat{f}_i(w, \theta)) \right]
\]

(6)

\[
g_p(w, p) = \left[ E_{\theta \sim Q_i} \left[ f_i(w - \alpha \nabla \hat{f}_i(w, \theta)) \right] \right]_{1 \leq i \leq m}
\]

(7)

The notation \( [a_i]_{1 \leq i \leq m} \) corresponds to a vector of size \( m \) with elements \( a_1, \ldots, a_m \). Note that \( g_w(w, p) \) is the weighted average of the gradients of each of the task losses weighted by \( p \), whereas each component of \( g_p(w, p) \) is the loss for a distinct task. Each of the above expressions involves two expectations: one over the samples \( \theta \), as is written, and another in the definitions of \( \nabla f_i \) and \( \nabla \hat{f}_i \), as these functions are the expected values of their sample evaluations over the data corresponding to the \( i \)-th task. Since the number of data points corresponding to a task may be large or even infinite, exactly computing the expectations may be intractable. Likewise, we cannot hope to repetitively estimate the gradients corresponding to every task because the number of tasks \( m \) may be large. Thus, we must estimate \( g_w \) and \( g_p \) by sampling both tasks and data.

To do so, we first sample a subset \( C \) of \( C \) tasks uniformly and independently from the set of all tasks. For each sampled task \( T_i \in C \), we independently sample a batch \( D_i \) of \( D \) pairs \( \{(\theta_{ij}^m, \theta_{ij}^{out})\}_{j=1}^{D} \) from the data distribution \( Q_i \times Q_i \) corresponding to \( T_i \). To estimate \( g_w \), for each pair \( (\theta_{ij}^m, \theta_{ij}^{out}) \in D_i \), we compute a stochastic estimate of the gradient of the objective \( \phi(w, p) \) with respect to \( w \) at the \( i \)-th task, using \( \theta_{ij}^m \) to estimate the Hessian and the inner gradient, and \( \theta_{ij}^{out} \) to estimate the outer gradient. We finally average these values across the \( C \) sampled tasks and \( D \) sampled data pairs and multiply by \( m \) to obtain our unbiased stochastic gradient \( \hat{g}_w(w, p) \), i.e.,

\[
\hat{g}_w(w, p) = \frac{m}{C} \sum_{i \in C} \sum_{j=1}^{D} \sum_{i=1}^{D} p_i (I - \alpha \nabla^2 \hat{f}_i(w, \theta_{ij}^m)) \nabla \hat{f}_i(w - \alpha \nabla \hat{f}_i(w, \theta_{ij}^m), \theta_{ij}^{out})
\]

(8)
The averaging is performed outside of the function evaluations, and the same data is used to estimate the Hessian and gradient to ensure \( \hat{g}_w(w, p) \) is an unbiased estimate of \( g_w \).

To estimate \( g_p \), we follow a similar procedure: for each \((\theta^n_{ij}, \theta^{out}_{ij}) \in D_i\), we compute a stochastic estimate of the gradient of the objective \( \phi(w, p) \) with respect to \( p \) at the \( i \)-th task, using \( \hat{\theta}^n_{ij} \) to estimate the gradient of \( f_i \) and \( \hat{\theta}^{out}_{ij} \) to estimate the function \( f_i \). We then average these values across the data sampled for each task, and set the \( i \)-th element of \( \hat{g}_p(w, p) \) to be the average corresponding to the \( i \)-th task, as written in Equation (9). Note that the computation of each \( \hat{f}_i(w, \theta^n_{ij}) \) can be reused in computing \( \hat{g}_p(w, p) \) after evaluating it to compute \( \hat{g}_w(w, p) \), thus the total number of stochastic function, gradient and Hessian evaluations required to estimate the gradients per iteration is \( 4CD \).

\[
\hat{g}_p(w, p) = \left[ \frac{m}{CD} \sum_{i=1}^m \sum_{j=1}^D \hat{f}_i(w - \alpha \nabla \hat{f}_i(w, \theta^n_{ij}, \theta^{out}_{ij})) \right]_{1 \leq i \leq m}.
\] (9)

Now that we have a procedure to compute the stochastic gradients, we are ready to discuss our algorithm to solve (3). The algorithm initializes \( p^1 = [1/m]_{1 \leq i \leq m} \) and \( w^1 \in W \), then iteratively executes alternating projected stochastic gradient descent-ascent steps. From now on, we denote the stochastic gradients in (8) and (9) on the \( t \)-th iteration as

\[
\hat{g}_w^t := \hat{g}_w(w^t, p^t), \quad \text{and} \quad \hat{g}_p^t := \hat{g}_w(w^t, p^t).
\]

From iterations \( t = 1 \) to \( T - 1 \), the algorithm updates \( w^{t+1} \) and \( p^{t+1} \) on the \( t \)-th iteration as

\[
w^{t+1} = \Pi_W(w^t - \eta^n_w \hat{g}_w(w^t, p^t))
\] (10)

\[
p^{t+1} = \Pi_\Delta(p^t + \eta^p_p \hat{g}_p(w^t, p^t))
\] (11)

where \( \eta^n_w, \eta^p_p \in \mathbb{R} \) are step sizes and \( \Pi_W(u) = \arg\min_{w \in W} \|u - w\|_2 \) and \( \Pi_\Delta(q) = \arg\min_{p \in \Delta} \|p - q\|_2 \). These projections are convex programs and can be solved efficiently using standard convex minimization techniques. After \( T \) iterations, the algorithm terminates in one of two ways, depending on the convexity of the minimization problem:

- **Case T1 (Convex Case):** If each \( \hat{F}_i \) is convex, the algorithm outputs

\[
w^*_T := \frac{1}{T} \sum_{t=1}^T w^t \quad \text{and} \quad p^*_T := \frac{1}{T} \sum_{t=1}^T p^t
\] (12)

- **Case T2 (Nonconvex Case):** If any function \( \hat{F}_i \) is nonconvex, the algorithm samples \( \tau \) uniformly at random from \( \{1, \ldots, T\} \) and outputs \( w^*_T := w^\tau \) and \( p^*_T := p^\tau \).

The full procedure is outlined in Algorithm 1.

**Implementation.** In order to implement Algorithm 1, we execute a routine with the inner-outer loop structure of the MAML algorithm, proposed in [Finn et al., 2017], and other gradient-based meta-learning methods. In the inner loop, the algorithm executes individual stochastic gradient
We undertake these changes so that the stochastic gradients are unbiased estimates of the updated Algorithm 1. To approximate optimal solutions to (5), and the motivation for this objective was given in the introduction. Like MAML, Algorithm 1 differs from MAML in that the data used to approximate the Hessian is the same as the data used to compute the inner loop parameter update. Furthermore, the averaging of the stochastic gradients in Algorithm 1 is over single-sample, full meta-gradient computations, whereas in MAML, each stochastic component of the meta-gradient is averaged before combining. We undertake these changes so that the stochastic gradients are unbiased estimates of the updated objective (5), and the motivation for this objective was given in the introduction. Like MAML, Algorithm 1 can be modified to only entail computing first-order derivatives in the interest of computational efficiency. The analogous min-max algorithms to FO-MAML [Finn et al., 2017], Reptile [Nichol and Schulman, 2018], and Hessian-Free MAML [Fallah et al., 2019] can be developed to approximate optimal solutions to (5) without requiring Hessian computation.

### Algorithm 2: DA-MAML: Implementation

**Input:** \( m \) functions \( f_i \), number of iterations \( T \), inner step size \( \alpha \), method step sizes \( \{\eta_i^p\}_t \), \( \{\eta_i^m\}_t \), convex set \( W \).

Initialize \( p^1 = \{1/m\}_{1 \leq i \leq m} \) and \( w^1 \in W \) arbitrarily.

for \( t = 1 \) to \( T - 1 \) do

Sample a batch \( C \) of \( C \) tasks independently from the uniform distribution over tasks.

for \( T_i \in C \) do

Sample a batch \( D_i \) of \( D \) pairs of samples \( \{(\theta_i^{in}, \theta_i^{out})\}_{j=1}^{D} \) independently from \( Q_i \times Q_i \).

for \( j = 1 \) to \( D \) do

Set \( w_{1j}^t \leftarrow w^t - \alpha \nabla \tilde{f}_i(w^t, \theta_i^{in}) \)
end for
end for

Denoting \( \Lambda_{ij} := I - \alpha \nabla^2 \tilde{f}_i(w, \theta_i^{in}) \), update

\[
w_{t+1}^i \leftarrow \Pi_W(w^t - \frac{n_{im}^m}{CD} \sum_{i:T_i \in C} \sum_{j=1}^{D} \Lambda_{ij} \nabla \tilde{f}_i(w_{1j}^t, \theta_i^{out}))
\]

\[
p_{t+1}^i \leftarrow \Pi_{\Delta}(p^t + \frac{n_{im}^m}{CD} \sum_{j=1}^{D} \tilde{f}_i(w_{1j}^t, \theta_i^{out}))_{1 \leq i \leq m}
\]

end for

**Output:** See Cases T1 and T2.

Descent using each sample in \( D_i \) for each of the set of sampled tasks, computing the stochastic gradient update

\[
w_{1j}^t \leftarrow w^t - \alpha \nabla \tilde{f}_i(w^t, \theta_i^{in})
\]

for the \( j \)-th sample for the \( i \)-th task. In the outer loop, i.e. the meta-learning phase, the algorithm updates \( w_{t+1}^i \) by executing projected stochastic gradient descent on the loss of the model corresponding to the sampled tasks after taking the local stochastic gradient descent steps, where the losses with respect to the tasks are weighed by \( p^t \). Updating \( p_{t+1}^i \) during the meta-learning phase also amounts to taking the stochastic gradient of the local losses, this time with respect to \( p^t \), and executing projected stochastic gradient ascent. In order to facilitate the meta-learning phase, the algorithm computes the second-order term \( \Lambda_{ij} \leftarrow I - \alpha \nabla^2 \tilde{f}_i(w, \theta_i^{in}) \) for each sample \( \theta_i^{in} \) and each task \( T_i \) passed over in the inner loop. Note that the total number of stochastic function, gradient and Hessian evaluations required per iteration is still \( 4CD \). The full procedure is outlined in Algorithm 2.

### 4.1 Comparison to standard MAML

MAML attempts to solve (1) by executing one step of SGD for each of the tasks sampled in the inner loop, then updating the initial weights \( w \) using SGD on the net loss, or test error, for all of the inner loop SGD steps [Finn et al., 2017]. Algorithm 1 shares the structure of MAML, but instead of computing the meta-stochastic gradient as the average of the stochastic gradients corresponding to the sampled tasks, Algorithm 1 weighs more heavily the stochastic gradients of tasks with higher loss.

Algorithm 1 also differs from MAML in that the data used to approximate the Hessian is the same as the data used to compute the inner loop parameter update. Furthermore, the averaging of the stochastic gradients in Algorithm 1 is over single-sample, full meta-gradient computations, whereas in MAML, each stochastic component of the meta-gradient is averaged before combining. We undertake these changes so that the stochastic gradients are unbiased estimates of the updated objective (5), and the motivation for this objective was given in the introduction. Like MAML, Algorithm 1 can be modified to only entail computing first-order derivatives in the interest of computational efficiency. The analogous min-max algorithms to FO-MAML [Finn et al., 2017], Reptile [Nichol and Schulman, 2018], and Hessian-Free MAML [Fallah et al., 2019] can be developed to approximate optimal solutions to (5) without requiring Hessian computation.
5 Theoretical Results

For our convergence results we will need unbiased stochastic gradients with bounded second moments. These properties are standard assumptions in the stochastic optimization literature, see, e.g., [Nemirovski et al. 2009, Radé et al. 2018, Qian et al. 2019]. However, it is not obvious that they are satisfied for the stochastic gradients of our meta-learning objective $\phi(w, p)$ given similar assumptions about the functions $f_i$ due to the nested stochastic gradients involved in $\phi(w, p)$, so we establish them in the following lemmas. We provide all proofs in the supplementary material.

**Assumption 1 (Unbiased samples).** Recall that $Q_i$ is the distribution over the data associated with the $i$-th task; $\theta$ is a sample drawn from $Q_i$, and $\hat{f}_i(w, \theta)$ is the approximation of $f_i(w)$ at the data point $\theta$. Then, $\forall w \in W, i \in [m]$: 

$$E_{\theta \sim Q_i} \left[ \hat{f}_i(w, \theta) - f_i(w) \right] = 0, \quad (14)$$

$$E_{\theta \sim Q_i} \left[ \nabla f_i(w, \theta) - \nabla f_i(w) \right] = 0, \quad (15)$$

$$E_{\theta \sim Q_i} \left[ \nabla^2 \hat{f}_i(w, \theta) - \nabla^2 f_i(w) \right] = 0. \quad (16)$$

The conditions in Assumption 1 are required to ensure the estimates of functions, gradients, and Hessians are unbiased.

**Lemma 1.** Suppose Assumption 1 holds, then the stochastic gradients $\hat{g}_w^i$ and $\hat{g}_p^i$ defined in $\eqref{eq:gradient}$ and $\eqref{eq:gradient_p}$, respectively, are unbiased estimates of $g_w^i$ and $g_p^i$, i.e., for all $t \geq 1$

$$E[\hat{g}_w^i - g_w^i] = E[\hat{g}_p^i - g_p^i] = 0. \quad (17)$$

**Assumption 2.** $f_i$ and $\hat{f}_i$, $\forall i \in [m]$, have these properties:

1. $f_i$ has bounded function values: $\exists B \in \mathbb{R}$ s.t. $\forall w \in W$ we have $|f_i(w)| \leq B$.
2. $f_i$ is L-Lipschitz continuous: $\exists L \in \mathbb{R}$ s.t. $\forall u, v \in W$, we have $|f_i(u) - f_i(v)| \leq L \|u - v\|_2$.
3. $\hat{f}_i$ is smooth: Each $\hat{f}_i$ is differentiable, and $\exists M \in \mathbb{R}$ such that $\|\nabla \hat{f}_i(u, \theta) - \nabla \hat{f}_i(v, \theta)\|_2 \leq M \|u - v\|$, $\forall u, v \in W, \theta \in \Omega_i$. Hence, each $f_i$ is also $M$-smooth.
4. Bounded variance: $\exists \sigma, \sigma_R, \sigma_H \in \mathbb{R}$ s.t. $\forall w \in W$:

$$E_{\theta \sim Q_i} \left[ \left| \hat{f}_i(w, \theta) - f_i(w) \right|^2 \right] \leq \sigma^2 \quad (18)$$

$$E_{\theta \sim Q_i} \left[ \| \nabla \hat{f}_i(w, \theta) - \nabla f_i(w) \|^2 \right] \leq \sigma_R^2 \quad (19)$$

$$E_{\theta \sim Q_i} \left[ \| \nabla^2 \hat{f}_i(w, \theta) - \nabla^2 f_i(w) \|^2 \right] \leq \sigma_H^2. \quad (20)$$

5. Hessian second-order Lipschitz continuity in expectation: $\exists H \in \mathbb{R}$ s.t. $E_{\theta \sim Q_i}[\| \nabla^2 f_i(u, \theta) - \nabla^2 f_i(v, \theta)\|_2^2] \leq H^2 \|u - v\|_2^2, \forall u, v \in W, i \in [m]$.

These assumptions on the task loss functions allow us to derive properties of the meta-learning task loss functions, which we write as follows, for all $w \in W$ and $i \in [m]$:

$$\hat{F}_i(w) := E_{\theta \sim Q_i}[f_i(w - \alpha \nabla \hat{f}_i(w, \theta))] \quad (21)$$

**Lemma 2.** Suppose Assumptions 1 and 2 hold. Then for all $i \in [m]$, $\hat{F}_i(w)$ defined in $\eqref{eq:meta_loss}$ is $\hat{L}$-Lipschitz, where

$$\hat{L} := L(1 + \alpha M + \alpha \sigma_H) \quad (22)$$

Next, we show that the variance of gradient estimations and the expected squared norm of stochastic gradients are uniformly bounded. We will use these results later to characterize the convergence properties of our proposed method.

**Lemma 3.** Suppose Assumptions 1 and 3 hold and the stochastic gradients $\hat{g}_w^i$ and $\hat{g}_p^i$ are computed as in $\eqref{eq:gradient}$ and $\eqref{eq:gradient_p}$. Then defining $\hat{\delta}_w^i := g_w^i - \hat{g}_w^i$, and $\hat{\delta}_p^i := g_p^i - \hat{g}_p^i$ for all $t \geq 1$,

$$E\left[ ||\hat{\delta}_w^i||^2 \right] \leq \sigma_w^2, \quad E\left[ ||\hat{\delta}_p^i||^2 \right] \leq \sigma_p^2 \quad (23)$$

where $\sigma_w^2 := \frac{m - 1}{C} \hat{L}^2 + \frac{m^2}{CD} (1 + \alpha M^2) (L^2 + \sigma_R^2)$ and $\sigma_p^2 := \frac{m^2 \alpha^2}{CD}$. Moreover, $E[||\hat{g}_w^i||_2^2] \leq \hat{G}_w^2$, and $E[||\hat{g}_p^i||_2^2] \leq \hat{G}_p^2$, where $\hat{G}_w^2 := \hat{L}^2 + \sigma_w^2$ and $\hat{G}_p^2 := mB^2 + \sigma_p^2$. 

7
5.1 Convex setting

We first consider the case when each of the functions \( \tilde{F}_i(w) \) are convex, which implies \( \phi(w,p) \) is convex in \( w \). In the following lemma we show that \( f_i(w) \) being strongly convex implies \( \tilde{F}_i(w) \) is also strongly convex.

Lemma 4. Suppose that if \( \alpha < 1/M \) and in addition to satisfying Assumption 2, each \( f_i \) is also \( \mu \)-strongly convex. Then each \( \tilde{F}_i \) is \( \mu(1 - \alpha M)^2 \) - strongly convex.

Note that the convexity of \( f_i \) does not imply the convexity of \( \tilde{F}_i \) (consider for example the counterexample in one dimension \( f_i(w) = \frac{1}{2} \) for \( w \in \mathbb{R}_+ \)). We also assume that the set \( W \) is contained in a ball of radius \( R_W \). The optimal rate of convergence for solving convex-concave stochastic min-max problems is \( O(1/\epsilon^2) \), where convergence is measured in terms of the expected number of stochastic gradient computations required to achieve a duality gap of \( \epsilon \) [Nemirovski et al., 2009]. If \( \phi^* \) is the min-max optimal value of \( \phi \), then the duality gap of the pair \((w,\hat{p})\) is defined as

\[
\max_{p \in \Delta} \phi(w,p) - \min_{w \in W} \phi(w,\hat{p})
\]

(24)

It is known that if the stochastic gradients of \( \phi(w,p) \) are unbiased and have bounded second moments, then alternating stochastic projected gradient descent-ascent achieves the optimal \( O(1/\epsilon^2) \) convergence rate when \( \phi(w,p) \) is a finite sum of convex functions of \( w \) weighted linearly by the components of \( p \), with \( p \) in the simplex and \( W \) contained in a ball of finite radius [Nemirovski et al., 2009]. We have established that all the conditions required for the existing analysis to apply to our convex setting, so we can apply the standard analysis to show the optimal convergence rate of \( O(1/\epsilon^2) \) of Algorithm 1. We adapt the result from Mohri et al. [2019], which in turn is a simplified version of Algorithm 1 and Theorem 1 from Juditsky et al. [2011].

Theorem 1. (Adapted from Mohri et al. [2019]) Consider problem 6 when each \( \tilde{F}_i \) is convex and Assumptions 2 and 3 hold. Suppose also that there exists a ball of radius \( R_W \) that contains \( W \). Let \( w_T^\tau \) and \( p_T^\tau \) be the output of Algorithm 1 run for \( T \) iterations with termination case T1. Then if we choose the step sizes as \( \eta_w = (2R_W)/(G_w\sqrt{T}) \) and \( \eta_p = 2/(G_p\sqrt{T}) \), the following bound holds:

\[
\mathbb{E} \left[ \max_{p \in \Delta} \phi(w_T^\tau,p) - \min_{w \in W} \phi(w,p_T^\tau) \right] \leq \frac{3R_W \hat{G}_w + 3\hat{G}_p}{\sqrt{T}}
\]

Thus, Algorithm 1 requires \( O(1/\epsilon^2) \) iterations to reach a solution with an expected duality gap of at most \( \epsilon \). Since Algorithm 1 computes a constant number of stochastic function, gradient and Hessian evaluations per iteration, we have that its convergence rate is the optimal \( O(1/\epsilon^2) \) stochastic oracle calls to reach an \( \epsilon \)-accurate solution.

5.2 Nonconvex setting

Next, we study the case that the functions \( f_i \) in 5 may not be convex and as a result the function \( \phi(w,p) \) be nonconvex in \( w \). In this case, the inner maximization remains concave with a linear objective and convex constraints. Thus, we must evaluate the pair \((w_T,p_T)\) returned by our algorithm using distinct methods with respect to \( w \) and \( p \). With respect to \( p \), we still care about the closeness of \( \phi(w_T,p_T) \) to global maximum of \( \phi(w,\cdot) \), as solving concave programs is generally tractable. Meanwhile, we care about the closeness of \( w \) to a stationary point of \( \phi(\cdot,p) \), as \( \phi \) is nonconvex in \( w \) and finding a global minimum of a nonconvex function is generally intractable. To capture this, we say that \((w',p')\) is an \((\epsilon,\delta)\)-stationary point of \( \phi \) if

\[
\|\nabla_w \phi(w',p')\|_2 \leq \epsilon \quad \text{and} \quad \phi(w',p') \geq \max_{p \in \Delta} \phi(w',p) - \delta,
\]

where \( \epsilon,\delta > 0 \), assuming that \( W = \mathbb{R}^d \). In the case that \( W \) does not contain all of \( \mathbb{R}^d \), we consider the projected gradient at the point in question, which we will discuss later. In either case we will leverage smoothness. Unfortunately, the function of \( w \) that we aim to minimize, \( \max_{p \in \Delta} \phi(w,p) \), is nonsmooth because of the maximization. However, we show that each \( \tilde{F}_i \) is smooth under the previous assumptions on \( f_i \).

Lemma 5. Suppose Assumptions 2 and 3 hold. Then for all \( i \in [m] \), \( \tilde{F}_i \) is \( \tilde{M} \)-smooth, where

\[
\tilde{M} := M (1 + \alpha M)^2 + \alpha LH
\]

We are now ready to present our results for the nonconvex setting, including both the case when \( W \) is equal to \( \mathbb{R}^d \) and when \( W \) is a compact, convex set, starting with the former.
5.2.1 Unconstrained case ($W = \mathbb{R}^d$)

Proposition 1. Suppose Assumptions 1-4 hold and $W = \mathbb{R}^d$. Let $\eta_w$ and $\eta_p$ be constant over all $t$, denoted by $\eta_w$ and $\eta_p$, respectively, where $\eta_w < (2/M)$. Let $(w_T^*, p_T^*)$ be the solution returned by Algorithm 1 after $T$ iterations. Then,

$$\mathbb{E}[\|\nabla w(\tilde{w}_T^*, p_T^*)\|^2_2] \leq \frac{2(\phi(w^1, p^1) + B)}{T(2\eta_w - \eta_w^2)M} + \frac{4\eta_p \sqrt{2} B G_p}{(2\eta_w - \eta_w^2)M} + \eta_w \tilde{M} \sigma_w^2 (2 - \eta_w M).$$

$$\mathbb{E}[\phi(\tilde{w}_T^*, p_T^*)] \geq \max_{p \in \Delta} \{\mathbb{E}[\phi(\tilde{w}_T^*, p)]\} - \frac{1}{\eta_p T} - \frac{\eta_p \tilde{G}_p^2}{2}.$$

Theorem 2. In the setting of Proposition 1 if the step sizes are set as $\eta_w = T^{-\beta}$, and $\eta_p = 2^{-1/2} \tilde{G}_p^{-1} T^{-2\beta}$ for $\beta \in (0, \frac{1}{2})$, and $T^\beta > \tilde{M}/2$, then

$$\mathbb{E}[\|\nabla w(\tilde{w}_T^*, p_T^*)\|^2_2] \leq \frac{\phi(w^1, p^1) + B + \sqrt{2m} B + 2\tilde{M} \sigma_w^2}{T^{\beta} - M/2}$$

$$\mathbb{E}[\phi(\tilde{w}_T^*, p_T^*)] \geq \max_{p \in \Delta} \{\mathbb{E}[\phi(\tilde{w}_T^*, p)]\} - \frac{\sqrt{2} \tilde{G}_p}{T^{\text{min}(2\beta, 1-2\beta)}}.$$
(w^T_T, p^T_T) be the solution returned by Algorithm 1 after T iterations. Then we have

\[ \mathbb{E}[\|\nabla_w \phi(w^T_T, p^T_T)\|^2] \leq \frac{2(\phi(w^1, p^1) + B)}{T(2\eta_w - \eta_w^2 M)} + \frac{4\eta_p \sqrt{m} B \tilde{G}_p}{(2\eta_w - \eta_w^2 M)} + \frac{\sigma_w^2}{2 - \eta_w M}. \]

\[ \mathbb{E}[\phi(w^T_T, p^T_T)] \geq \max_{p \in \Delta} \left\{ \mathbb{E}[\phi(w^T_T, p)] \right\} - \frac{1}{\eta_p T} \frac{\eta_p \tilde{G}_p^2}{2}. \]

The only significant difference between this bound and the bound derived in Proposition 1 is that the term with \( \sigma_w^2 \) is not multiplied by the step size \( \eta_w \), thus appears to asymptotically behave as a constant. Therefore, in order to show that the right hand side in the above bound converges, we must treat \( \sigma_w^2 \) as a function of the number of stochastic gradients computed during each iteration. Recall that \( \sigma_w^2 \) is an upper bound on the variance of \( \tilde{g}_w \), and note from Lemma 3 that we can write it as \( \sigma_w^2 = \tilde{\sigma}_w^2 / C \), where \( C \) is the number of sampled tasks used for each stochastic gradient computation, and each sampled task involves a constant number of single-sample function, gradient and Hessian evaluations if \( D \) is fixed. In the following theorem, we choose \( C \) as a function of \( T \) to show convergence.

**Theorem 3.** Suppose that the setting of Proposition 1 holds. If the step sizes are chosen as \( \eta_w = 1/2M \) and \( \eta_p = \tilde{G}_p / \sqrt{2T} \) for all \( t = 1, ..., T \) and the task batch size is chosen as \( C = T^\beta \) for \( \beta \in (0, 1) \), then

\[ \mathbb{E}[\|\tilde{\gamma}_T\|^2] \leq \frac{8M(\Phi(w^1, p^1) + B)}{3T} + \frac{8M B \sqrt{m} + 4\tilde{\sigma}_w^2}{3T^\beta}, \]

\[ \mathbb{E}[\phi(w^T_T, p^T_T)] \geq \max_{p \in \Delta} \left\{ \mathbb{E}[\phi(w^T_T, p)] \right\} - \frac{\tilde{G}_p}{\sqrt{2T^{\min(\beta, 1-\beta)}}}. \]

Hence, if \( N \) represents the number of single-sample function, gradient, and Hessian evaluations \( (N = CT = T^{1+\beta} \text{ thus } T = N^{1/(1+\beta)}) \), Algorithm 1 converges to an \((\epsilon, \delta)\)-stationary point after \( \mathcal{O}(\max\{1/(\epsilon^{(2+2\beta)/\beta}), 1/\delta^{(1+\beta)/\min(\beta, 1-\beta)}\}) \) single-sample function, gradient, and Hessian evaluations. We can tune \( \beta \) to favor convergence with respect to \( w \) or \( p \). By setting \( \beta = \frac{3}{4} \) we treat convergence with respect to both equally, leading to a complexity of \( \mathcal{O}(\max\{1/\epsilon^{5}, 1/\delta^{5}\}) \).

**6 Experimental Results**

In this section we compare the empirical performance of the DA-MAML algorithm with the MAML algorithm on image classification problems. We are particularly interested in the extent to which each algorithm can be considered fair. To define fairness we adopt the definition provided in Mohri et al. [2019], which considers an algorithm fair if it tries to minimize the maximum loss over a set of objectives. This means that the algorithm does not allow poor performance on some objectives in order to boost performance on others, such as those which may contribute heavily to minimizing the average loss, even if the training data itself is biased. Our experiments test whether DA-MAML and/or MAML yield a fair solution under exactly this type of scenario.

We employ the MNIST [LeCun and Cortes 2010] and Fashion MNIST (FMNIST) [Xiao et al. 2017] datasets in order to make this evaluation. Both these datasets contain 70,000 28-by-28 grayscale images split evenly into 10 classes. In MNIST these classes are the handwritten digits, and in FMNIST they are more complex fashion products. Following Finn et al. [2017], we consider tasks as N-way image classification problems, and we evaluate meta-learning performance by observing how well the model classifies N images from N distinct classes after learning from K labelled training examples from each class, where K is small. An algorithm’s success on this K-shot, N-way classification problem is an appropriate measure of its meta-learning capability because K being small exposes how quickly the algorithm learns from training data. Furthermore, in all the distributions of tasks that we consider, the probabilities of drawing the tasks composed of the same classes but any permutation of labelings are equivalent, so all models are expected to classify with \( 1/N \)-accuracy on randomly drawn tasks without access to any training examples.

Our first experiment involves meta-learning eight two-way classification tasks with a two-layer, \( \text{tanh} \)-activated neural network. Two of the tasks are to classify pullovers and coats from FMNIST (where in each task, each class has the opposite label), and the remaining six MINIST tasks are similarly to classify images belonging to three distinct pairs of classes. To simulate a biased training
environment, we enforce that MAML will see fewer training samples of FMNIST tasks than MNIST tasks by setting the ambient distribution of the tasks such that the probability of drawing an FMNIST task is only 0.1, and fixing the number of task samples per iteration and the total number of iterations at 3 and 3000, respectively. One would predict that in this situation a combination of factors likely biases MAML against trying to perform well on the FMNIST tasks: the fact that the MAML objective is to optimize the expected loss over tasks drawn from the ambient distribution and the FMNIST tasks have a low probability of being drawn from this distribution, the greater computational resources that solving the FMNIST tasks presumably requires, the possibility that the optimal solution to solve FMNIST tasks is far from the optimal solution to solve MNIST tasks, and the small number of FMNIST training examples MAML can learn from. In contrast, DA-MAML optimizes over worst case instead of expected performance, and samples tasks uniformly, so we expect that DA-MAML still tries to perform well on the FMNIST tasks.

Our results support this hypothesis, showing that DA-MAML yields a more fair solution. We report the testing accuracy vs. number of iterations for both MAML and DA-MAML in Figure 1. For testing, we take the current model and perform 500 episodes of 5-shot, 2-way classification for each task and consider the average classification accuracy to be the test accuracy corresponding to the task. DA-MAML clearly outperforms MAML in terms of the worst performing task (which are FMNIST tasks in this case).

A Proof of Lemma 1

Proof. Recall that \( \hat{g}_w(w, p) \) is computed as follows:

\[
\hat{g}_w(w, p) = \frac{m}{\mathcal{C}} \sum_{i : \mathcal{T}_i \in \mathcal{C}} \frac{1}{D} \sum_{j=1}^D p_i (I - \alpha \nabla^2 \hat{f}_i(w, \theta_{ij}^{\text{in}})) \nabla \hat{f}_i(w - \alpha \nabla \hat{f}_i(w, \theta_{ij}^{\text{in}}, \theta_{ij}^{\text{out}}))
\]

Denote \( c_i \) as the number of times the task \( \mathcal{T}_i \) appears in the batch \( \mathcal{C} \), and \((\theta_{ij,k}^{\text{in}}, \theta_{ij,k}^{\text{out}})\) as the \( j \)-th data pair sampled from the \( k \)-th instance of \( \mathcal{T}_i \). Then we can write \( \hat{g}_w(w, p) \) as

\[
\hat{g}_w(w, p) = \frac{m}{\mathcal{C}} \sum_{i=1}^m \frac{c_i}{D} \sum_{k=1}^D p_i (I - \alpha \nabla^2 \hat{f}_i(w, \theta_{ij,k}^{\text{in}})) \nabla \hat{f}_i(w - \alpha \nabla \hat{f}_i(w, \theta_{ij,k}^{\text{in}}, \theta_{ij,k}^{\text{out}}))
\]

(27)
Taking the expectation over the random variables \( \{c_i\} \) and \( \{\theta_{i,j,k}^{\text{in}}, \theta_{i,j,k}^{\text{out}}\} \) and using the linearity of expectation,

\[
\mathbb{E}[\hat{g}_w(w, p)] = \mathbb{E} \left[ \sum_{i=1}^{m} \mathbb{E}_{c_i} \left[ \sum_{k=1}^{m} \frac{1}{D} \sum_{j=1}^{D} p_i (I - \alpha \nabla^2 \hat{f}_i(w, \theta_{i,j,k}^{\text{in}})) \nabla \hat{f}_i(w - \alpha \nabla \hat{f}_i(w, \theta_{i,j,k}^{\text{in}}), \theta_{i,j,k}^{\text{out}}) \right] c_i \right] \\
= \sum_{i=1}^{m} \frac{1}{C} \mathbb{E}_{c_i} \left[ \sum_{k=1}^{m} \frac{1}{D} \sum_{j=1}^{D} p_i (I - \alpha \nabla^2 \hat{f}_i(w, \theta_{i,j,k}^{\text{in}})) \nabla \hat{f}_i(w - \alpha \nabla \hat{f}_i(w, \theta_{i,j,k}^{\text{in}}), \theta_{i,j,k}^{\text{out}}) \right] c_i \\
= \sum_{i=1}^{m} \frac{1}{C} \mathbb{E}_{c_i} \left[ \sum_{k=1}^{m} \frac{1}{D} \sum_{j=1}^{D} p_i \mathbb{E}_{g_{i,j,k}^{\text{in}}, \theta_{i,j,k}^{\text{out}}} \left[ (I - \alpha \nabla^2 \hat{f}_i(w, \theta_{i,j,k}^{\text{in}})) \nabla \hat{f}_i(w - \alpha \nabla \hat{f}_i(w, \theta_{i,j,k}^{\text{in}}), \theta_{i,j,k}^{\text{out}}) \right] \right] c_i \\
= \sum_{i=1}^{m} \frac{1}{C} \mathbb{E}_{c_i} \left[ c_i \sum_{k=1}^{m} \frac{1}{D} \sum_{j=1}^{D} p_i \nabla_w \hat{F}_i(w) \right] \\
= \sum_{i=1}^{m} p_i \nabla_w \hat{F}_i(w) \\
= g_w(w, p)
\]

To see that \( \hat{g}_p(w, p) \) is unbiased, note that if we again use \( c_i \) to denote the number of times the task \( T_i \) is chosen for each computation of \( \hat{g}_p(w, p) \), we can write \( \hat{g}_p(w, p) \) as:

\[
\hat{g}_p(w, p) = \left[ \frac{m}{C} \sum_{k=1}^{m} \sum_{j=1}^{D} \hat{f}_i(w - \alpha \nabla \hat{f}_i(w, \theta_{i,j,k}^{\text{in}}), \theta_{i,j,k}^{\text{out}}) \right]_{1 \leq i \leq m}
\]

Observe that the mean of \( C_i \) is \( C/m \), thus similarly applying the Law of Iterated Expectations, the linearity of expectation, the independence of the samples, and the unbiasedness of the single-sample function evaluations yields that the expected value of the \( i \)-th element of \( \hat{g}_p(w, p) \) is \( \hat{F}_i(w) \), for all \( i \in [m] \), as expected.

\[ \Box \]

### B Proof of Lemma 2

**Proof.** To begin, note that

\[
\nabla \hat{F}_i(w) = \mathbb{E}_{\theta \sim Q_i} \left[ (I - \alpha \nabla^2 \hat{f}_i(w, \theta)) \nabla \hat{f}_i(w - \alpha \nabla \hat{f}_i(w, \theta)) \right]
\]

Thus using Jensen’s Inequality followed by the Cauchy-Schwarz Inequality twice (recalling that the Cauchy-Schwarz Inequality implies \( \mathbb{E}[XY] \leq \sqrt{\mathbb{E}[X^2]\mathbb{E}[Y^2]} \) for any two random variables \( X \) and...
Thus it only remains to compute \( \text{Var}(\hat{F}_i(w)) \), we have

\[
\|\nabla \hat{F}_i(w)\| = \|E_{\theta \sim Q_i} [(I - \alpha \nabla^2 \hat{f}_i(w, \theta)) \nabla f_i(w - \alpha \nabla \hat{f}_i(w, \theta))]\|
\leq E_{\theta \sim Q_i} \| (I - \alpha \nabla^2 \hat{f}_i(w, \theta)) \nabla f_i(w - \alpha \nabla \hat{f}_i(w, \theta)) \|
\leq E_{\theta \sim Q_i} \| (I - \alpha \nabla^2 \hat{f}_i(w, \theta)) \| \| \nabla f_i(w - \alpha \nabla \hat{f}_i(w, \theta)) \|
\leq \sqrt{E_{\theta \sim Q_i} \| I - \alpha \nabla^2 \hat{f}_i(w, \theta) \|^2} \| \nabla f_i(w - \alpha \nabla \hat{f}_i(w, \theta)) \|^2
\leq \sqrt{L^2 E_{\theta \sim Q_i} \| I - \alpha \nabla^2 \hat{f}_i(w, \theta) \|^2}
\]  

(29)

where (29) follows from the \( L \)-Lipschitzness of \( f_i \). Considering the term inside the square root, we have

\[
E_{\theta \sim Q_i} \| I - \alpha \nabla^2 \hat{f}_i(w, \theta) \|^2 = E_{\theta \sim Q_i} \left\| (\alpha \nabla^2 f_i(w) - \alpha \nabla^2 \hat{f}_i(w, \theta)) + I - \alpha \nabla^2 f_i(w) \right\|^2
\leq E_{\theta \sim Q_i} \left\| (\alpha \nabla^2 f_i(w) - \alpha \nabla^2 \hat{f}_i(w, \theta)) \right\|^2 + \| I - \alpha \nabla^2 f_i(w) \|^2
\leq E_{\theta \sim Q_i} \left\| (\alpha \nabla^2 f_i(w) - \alpha \nabla^2 \hat{f}_i(w, \theta)) \right\|^2 + (1 + \alpha M)^2
\]

(30)

(31)

where (30) follows from the triangle inequality, (31) follows from the \( M \)-smoothness of \( f_i \), and (32) follows from the definition of \( \sigma_H^2 \), noting that

\[
E_{\theta \sim Q_i} \| \alpha \nabla^2 f_i(w) - \alpha \nabla^2 \hat{f}_i(w, \theta) \|^2 \leq \| \alpha \nabla^2 f_i(w) - \alpha \nabla^2 \hat{f}_i(w, \theta) \|^2 \leq \sigma_H^2
\]

(33)

Combining (29) and (32) gives us that \( \hat{F}_i \) is \( \hat{L} := L(1 + \alpha M + \alpha \sigma_H) \)-Lipschitz for all \( i \in [m] \).

C Proof of Lemma 3

Proof. We denote the true gradient of \( \phi(w, p) \) with respect to \( w \) corresponding to the \( i \)-th task as

\[
\rho_i := p_i \nabla_w \hat{F}_i(w_i) = p_i E_{\theta} [(I - \alpha \nabla^2 \hat{f}_i(w, \theta)) \nabla f_i(w - \alpha \nabla \hat{f}_i(w, \theta)]
\]

(34)

Following (27), we denote the stochastic gradient corresponding to the \( i \)-th task as

\[
\hat{\rho}_i := \frac{m}{C} \sum_{j=1}^{c_i} \frac{1}{D} \sum_{k=1}^{D} p_j (I - \alpha \nabla^2 \hat{f}_i(w, \theta_{j,k}) \nabla f_i(w - \alpha \nabla \hat{f}_i(w, \theta_{j,k}))
\]

(35)

where \( c_i \) is the number of times the \( i \)-th task is chosen during the computation of \( \hat{g}_w(w, p) \). From Lemma 1 and the linearity of expectation, we have that \( E[\hat{\rho}_i] = \rho_i \), so using the independence of the sampled tasks and data, we have

\[
E[\|\delta_w\|^2] = \text{Var}(\hat{g}_w(w, p)) = E[\|\sum_{i=1}^{m} \hat{\rho}_i - \rho_i \|^2] = \sum_{i=1}^{m} E[\|\hat{\rho}_i - \rho_i \|^2] = \sum_{i=1}^{m} \text{Var}(\hat{\rho}_i)
\]

(36)

Thus it only remains to compute \( \text{Var}(\hat{\rho}_i) \). Note that each \( c_i \) is a binomial random variable with parameters \( (C, 1/m) \). By the Law of Total Variance,

\[
\text{Var}(\hat{\rho}_i) = \text{Var}(E[\hat{\rho}_i | c_i]) + E[\text{Var}(\hat{\rho}_i | c_i)]
\]

(37)

Conditioned on \( c_i \), \( \hat{\rho}_i \) has mean \( \frac{m}{c_i} \rho_i \), thus

\[
\text{Var}(E[\hat{\rho}_i | c_i]) = \text{Var}(\frac{m}{c_i} \rho_i) = \frac{m-1}{c_i} \| \rho_i \|^2 \leq \frac{m-1}{c_i} \bar{L}^2 \rho_i^2
\]

(38)
by the $\tilde{L}$-Lipschitzness of $\tilde{F}_i$, and

\[
\text{Var}(\hat{p}_i | c_i) = E_{\tilde{g}_{ij,k}^{\text{in}}, \theta_{ij,k}^{\text{out}}} \left[ \frac{m}{C} c_i \hat{p}_i - \frac{m}{C} \sum_{j=1}^{c_i} \frac{1}{D} \sum_{k=1}^{D} p_i (I - \alpha \nabla^2 \tilde{f}_i (w, \theta_{ij,k}^{\text{in}})) \nabla \tilde{f}_i (w - \alpha \nabla \tilde{f}_i (w, \theta_{ij,k}^{\text{in}}), \theta_{ij,k}^{\text{out}}) \right]_2^2 | c_i
\]

\[
= E_{\tilde{g}_{ij,k}^{\text{in}}, \theta_{ij,k}^{\text{out}}} \left[ \frac{m}{C} \sum_{j=1}^{c_i} \left( p_i - \frac{1}{D} \sum_{k=1}^{D} p_i (I - \alpha \nabla^2 \tilde{f}_i (w, \theta_{ij,k}^{\text{in}})) \nabla \tilde{f}_i (w - \alpha \nabla \tilde{f}_i (w, \theta_{ij,k}^{\text{in}}), \theta_{ij,k}^{\text{out}}) \right) \right]_2^2 | c_i
\]

\[
= \frac{m^2}{CD} \sum_{j=1}^{c_i} \sum_{k=1}^{D} E_{\tilde{g}_{ij,k}^{\text{in}}, \theta_{ij,k}^{\text{out}}} \left[ \left| p_i - \frac{1}{D} \sum_{k=1}^{D} p_i (I - \alpha \nabla^2 \tilde{f}_i (w, \theta_{ij,k}^{\text{in}})) \nabla \tilde{f}_i (w - \alpha \nabla \tilde{f}_i (w, \theta_{ij,k}^{\text{in}}), \theta_{ij,k}^{\text{out}}) \right|_2^2 \right]
\]

where (39) follows from the independence of the sampled data across task instances, and (40) follows from the independence of each of the samples within each task instance. Thus we are left with a sum of finite-sample stochastic gradient variances. For any $i, j, k$, we can bound this variance by first computing its second moment of the single-sample stochastic gradient:

\[
E_{\tilde{g}_{ij,k}^{\text{in}}, \theta_{ij,k}^{\text{out}}} \left[ \left| p_i (I - \alpha \nabla^2 \tilde{f}_i (w, \theta_{ij,k}^{\text{in}})) \nabla \tilde{f}_i (w - \alpha \nabla \tilde{f}_i (w, \theta_{ij,k}^{\text{in}}), \theta_{ij,k}^{\text{out}}) \right|_2^2 \right]
\]

\[
\leq p_i^2 \mathbb{E}_{\tilde{g}_{ij,k}^{\text{in}}, \theta_{ij,k}^{\text{out}}} \left[ \left| I - \alpha \nabla^2 \tilde{f}_i (w, \theta_{ij,k}^{\text{in}}) \right|_2^2 \right] \left| \nabla \tilde{f}_i (w - \alpha \nabla \tilde{f}_i (w, \theta_{ij,k}^{\text{in}}, \theta_{ij,k}^{\text{out}}) \right|_2^2 \right]
\]

\[
\leq p_i^2 (1 + \alpha M)^2 \mathbb{E}_{\tilde{g}_{ij,k}^{\text{in}}, \theta_{ij,k}^{\text{out}}} \left[ \left| \nabla \tilde{f}_i (w - \alpha \nabla \tilde{f}_i (w, \theta_{ij,k}^{\text{in}}), \theta_{ij,k}^{\text{out}}) \right|_2^2 \right]
\]

\[
= p_i^2 (1 + \alpha M)^2 \mathbb{E}_{\tilde{g}_{ij,k}^{\text{in}}, \theta_{ij,k}^{\text{out}}} \left[ \left| \nabla \tilde{f}_i (w - \alpha \nabla \tilde{f}_i (w, \theta_{ij,k}^{\text{in}}), \theta_{ij,k}^{\text{out}}) - \nabla f_i (w - \alpha \nabla f_i (w, \theta_{ij,k}^{\text{in}})) \right|_2^2 \right]
\]

\[
\leq p_i^2 (1 + \alpha M)^2 (L^2 + \sigma_R^2)
\]

where (41) follows from the Cauchy-Schwarz Inequality, (42) follows from the $M$-smoothness of $f_i(\cdot, \theta)$ for all $\theta$, (44) follows from the fact that $\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$ for any random variable $X$, and (45) follows from the $L$-Lipschitz continuity of $f_i$ and the bounded single-sample variance of $\nabla f_i$. Plugging this result back into (40), we have

\[
\text{Var}(\hat{p}_i | c_i) \leq \frac{m^2 c_i}{C^2 D} p_i^2 (1 + \alpha M)^2 (L^2 + \sigma_R^2)
\]

thus taking the expectation over $c_i$ yields

\[
\mathbb{E}[\text{Var}(\hat{p}_i | c_i)] \leq \frac{m}{C} p_i^2 (1 + \alpha M)^2 (L^2 + \sigma_R^2)
\]

Using (37), we have

\[
\text{Var}(\hat{p}_i) \leq \left( \frac{m - 1}{C} \tilde{L}^2 + \frac{m^2}{CD} (1 + \alpha M)^2 (L^2 + \sigma_R^2) \right) p_i^2
\]

Summing over $i$, by (36) we have

\[
\mathbb{E}[\|\delta_w\|_2^2] \leq \text{Var}(\tilde{g}_w (w, p)) \leq \left( \frac{m - 1}{C} \tilde{L}^2 + \frac{m^2}{CD} (1 + \alpha M)^2 (L^2 + \sigma_R^2) \right) \sum_{i=1}^{m} p_i^2
\]

\[
\leq \frac{m - 1}{C} \tilde{L}^2 + \frac{m^2}{CD} (1 + \alpha M)^2 (L^2 + \sigma_R^2)
\]

where (46) holds because the probability simplex has diameter 1.
Next we bound $\mathbb{E}[\| \delta_p \|^2_2]$. Note that

\[
\mathbb{E}[\| \delta_p \|^2_2] = \sum_{i=1}^{m} \mathbb{E}[\| \frac{m}{C} \sum_{j=1}^{c_i} \frac{1}{D} \sum_{k=1}^{D} \hat{f}_i(w - \alpha \nabla \hat{f}_i(w, \theta_{i,k}^{in}, \theta_{i,k}^{out}) - \hat{F}_i(w)\|_2^2)\\
= \frac{m^2}{C^2} \sum_{i=1}^{m} c_i \mathbb{E}[\| \frac{1}{D} \sum_{j=1}^{c_i} \sum_{k=1}^{D} \hat{f}_i(w - \alpha \nabla \hat{f}_i(w, \theta_{i,k}^{in}, \theta_{i,k}^{out}) - \hat{F}_i(w)\|_2^2)\\
= \frac{m^2}{C^2} \sum_{i=1}^{m} c_i \mathbb{E}[\| \sum_{j=1}^{c_i} \sum_{k=1}^{D} \hat{f}_i(w - \alpha \nabla \hat{f}_i(w, \theta_{i,k}^{in}, \theta_{i,k}^{out}) - \hat{F}_i(w)\|_2^2)\\
(47)
\]

where (47) follows from the Law of Iterated Expectation and (48) follows from the independence of the samples. We can bound the inner expectation as follows:

\[
\mathbb{E}[\| \sum_{j=1}^{c_i} \sum_{k=1}^{D} \hat{f}_i(w - \alpha \nabla \hat{f}_i(w, \theta_{i,k}^{in}, \theta_{i,k}^{out}) - \hat{F}_i(w)\|_2^2)\\
= \mathbb{E}[\| \sum_{j=1}^{c_i} \sum_{k=1}^{D} \hat{f}_i(w - \alpha \nabla \hat{f}_i(w, \theta_{i,k}^{in}, \theta_{i,k}^{out}) - \hat{F}_i(w)\|_2^2)\\
\leq \mathbb{E}[\| \sum_{j=1}^{c_i} \sum_{k=1}^{D} \hat{f}_i(w - \alpha \nabla \hat{f}_i(w, \theta_{i,k}^{in}, \theta_{i,k}^{out}) - \hat{F}_i(w)\|_2^2)\\
\leq \sigma^2\\
(48)
\]

where (48) follows from Jensen’s Inequality and (49) follows from the assumption of bounded variance of single-sample function evaluations. Combining (50) with (48) yields

\[
\mathbb{E}[\| \delta_p \|^2_2] \leq \frac{m^2}{C^2} \sum_{i=1}^{m} c_i \mathbb{E}[\| \sum_{j=1}^{c_i} \sum_{k=1}^{D} \sigma_k^2)\\
= \frac{m^2}{C^2} \sum_{i=1}^{m} c_i \frac{\sigma^2}{D}\\
= \frac{m^2 \sigma^2}{CD}\\
(51)
\]

Now that we have bounds on $\mathbb{E}[\| \delta_w \|^2_2]$ and $\mathbb{E}[\| \delta_p \|^2_2]$, it is straightforward to bound the second moments of the stochastic gradients with respect to $w$ and $p$. Denoting the upper bound on $\mathbb{E}[\| \delta_w \|^2_2]$ in (46) as $\sigma_w^2$, we have that since $\tilde{g}_w$ is unbiased,

\[
\mathbb{E}[\| \tilde{g}_w \|^2_2] = \| g_w \|^2_2 + \mathbb{E}[\| \tilde{g}_w - g_w \|^2_2)\\
\leq \| \sum_{i=1}^{m} p_i \nabla_w \hat{F}_i(w)\|_2^2 + \sigma_w^2\\
= \sum_{i=1}^{m} p_i \sum_{j=1}^{m} p_j (\nabla_w \hat{F}_i(w)\|_2^2 + \sigma_w^2\\
\leq \sum_{j=1}^{m} p_j \| p \|_1 \max_{j \in [m]} \| \sum_{i=1}^{m} (\nabla_w \hat{F}_i(w)\|_2^2 + \sigma_w^2\\
\leq \| p \|_1 \max_{i \in [m]} \| \sum_{j \in [m]} (\nabla_w \hat{F}_i(w)\|_2^2 + \sigma_w^2\\
\leq \| p \|_1 \max_{i \in [m]} \| \tilde{F}_i(w)\|_2^2 + \sigma_w^2\\
(52)
\]

where (52) and (53) follow from Hölder’s Inequality, and (54) follows from the facts that $\| p \|_1 = 1$ since $p \in \Delta$ and each $\tilde{F}_i$ is $L$-Lipschitz. To show the bound on $G_p$, we denote the upper bound on $\mathbb{E}[\| \delta_p \|^2_2]$ in (51) as $\sigma_p^2$, and similarly use the unbiasedness of $\tilde{g}_p$ to obtain

\[
\mathbb{E}[\| \tilde{g}_p \|^2_2] = \| g_p \|^2_2 + \mathbb{E}[\| \tilde{g}_p - g_p \|^2_2)\\
\leq \sum_{i=1}^{m} \tilde{F}_i(w)^2 + \sigma_p^2\\
\leq mB^2 + \sigma_p^2\\
(55)
\]

where (55) follows from the boundedness of each $f_i$. □
D Proof of Lemma 4

Proof. We show the strong convexity of \( \tilde{F}_i \) when \( \alpha < 1/M \) and \( f_i \) is \( \mu \)-strongly convex in addition to satisfying Assumptions 1 and 2. We have

\[
\| \nabla \tilde{F}_i(u) - \nabla \tilde{F}_i(v) \| = \| \mathbb{E}_{q_i \sim Q_i} [(I - \alpha \nabla^2 \tilde{f}_i(u, \theta)) (\nabla f_i(u - \alpha \nabla \tilde{f}_i(u, \theta)) - \nabla f_i(v - \alpha \nabla \tilde{f}_i(v, \theta))] + (I - \alpha \nabla^2 \tilde{f}_i(v, \theta)) \nabla f_i(v - \alpha \nabla \tilde{f}_i(v, \theta))] \| \geq \| \mathbb{E}_{q_i \sim Q_i} [(I - \alpha \nabla^2 \tilde{f}_i(u, \theta)) (\nabla f_i(u - \alpha \nabla \tilde{f}_i(u, \theta)) - \nabla f_i(v - \alpha \nabla \tilde{f}_i(v, \theta))] \| - \| \mathbb{E}_{q_i \sim Q_i} [(I - \alpha \nabla^2 \tilde{f}_i(v, \theta)) \nabla f_i(v - \alpha \nabla \tilde{f}_i(v, \theta))] \| \geq \| \mathbb{E}_{q_i \sim Q_i} [(I - \alpha \nabla^2 \tilde{f}_i(u, \theta)) (\nabla f_i(u - \alpha \nabla \tilde{f}_i(u, \theta)) - \nabla f_i(v - \alpha \nabla \tilde{f}_i(v, \theta))] \| - \| \mathbb{E}_{q_i \sim Q_i} [(I - \alpha \nabla^2 \tilde{f}_i(v, \theta)) \nabla f_i(v - \alpha \nabla \tilde{f}_i(v, \theta))] \| \quad (56)
\]

To lower bound the first term, we use the \( M \)-smoothness of \( \tilde{f}_i \), which implies that the minimum eigenvalue of \( I - \alpha \nabla^2 \tilde{f}_i(u) \) is at least \( 1 - \alpha M \) for all \( u \in W \). Thus,

\[
\| \mathbb{E}_{q_i \sim Q_i} [(I - \alpha \nabla^2 \tilde{f}_i(u, \theta)) (\nabla f_i(u - \alpha \nabla \tilde{f}_i(u, \theta)) - \nabla f_i(v - \alpha \nabla \tilde{f}_i(v, \theta))] \| \geq (1 - \alpha M) \| \mathbb{E}_{q_i \sim Q_i} [(\nabla f_i(u - \alpha \nabla \tilde{f}_i(u, \theta)) - \nabla f_i(v - \alpha \nabla \tilde{f}_i(v, \theta))] \| \quad (57)
\]

By the \( \mu \)-strong convexity of \( f_i \) and the triangle inequality, we have

\[
\| \mathbb{E}_{q_i \sim Q_i} [\nabla f_i(u - \alpha \nabla \tilde{f}_i(u, \theta)) - \nabla f_i(v - \alpha \nabla \tilde{f}_i(v, \theta))] \| \geq \mu \| \mathbb{E}_{q_i \sim Q_i} [u - \alpha \nabla \tilde{f}_i(u, \theta) - (v - \alpha \nabla \tilde{f}_i(v, \theta))] \| \geq \mu \| u - v \| - \alpha \| \mathbb{E}_{q_i \sim Q_i} [\nabla \tilde{f}_i(v, \theta) - \alpha \nabla \tilde{f}_i(u, \theta)] \| \geq \mu \| u - v \| - \alpha \| u - \alpha M \| u - v \| \quad (60)
\]

Next we upper bound the second term in (57). We have

\[
\mathbb{E}_{q_i \sim Q_i} \left[ \| (\alpha \nabla^2 \tilde{f}_i(v, \theta) - \alpha \nabla^2 \tilde{f}_i(u, \theta)) \nabla f_i(v - \alpha \nabla \tilde{f}_i(v, \theta)) \| \right] \leq \mathbb{E}_{q_i \sim Q_i} \left[ \| (\alpha \nabla^2 \tilde{f}_i(v, \theta) - \alpha \nabla^2 \tilde{f}_i(u, \theta)) \nabla f_i(v - \alpha \nabla \tilde{f}_i(v, \theta)) \| \right] \leq \mathbb{E}_{q_i \sim Q_i} \left[ \| (\alpha \nabla^2 \tilde{f}_i(v, \theta) - \alpha \nabla^2 \tilde{f}_i(u, \theta)) \| \right] \leq \sqrt{\mathbb{E}_{q_i \sim Q_i} \left[ \| (\alpha \nabla^2 \tilde{f}_i(v, \theta) - \alpha \nabla^2 \tilde{f}_i(u, \theta)) \| ^2 \right]} \mathbb{E}_{q_i \sim Q_i} \left[ \| \nabla f_i(v - \alpha \nabla \tilde{f}_i(v, \theta)) \| ^2 \right] \leq \alpha L \mathbb{E}_{q_i \sim Q_i} \left[ \| (\alpha \nabla^2 \tilde{f}_i(v, \theta) - \alpha \nabla^2 \tilde{f}_i(u, \theta)) \| ^2 \right] \leq \alpha LH \| u - v \| \quad (63)
\]

where (61) follows from Jensen’s Inequality, (62) and (63) follow from the Cauchy-Schwarz Inequality, (64) follows from the \( L \)-Lipschitzness of \( f_i \) and (65) follows from Assumption 2. Combining (57), (59), and (60) and (65) yields that \( \tilde{F}_i \) is \( \tilde{\mu} := (\mu(1 - \alpha M)^2 - \alpha LH) \)-strongly convex under the given conditions.

E Proof of Theorem 1

Proof. We adapt the arguments from [Mohri et al., 2019] to our nested gradients case. First observe that since each \( \tilde{F}_i(w) \) is convex, \( \phi(w, p) \) is convex in \( w \) and linear, thus concave, in \( p \). Therefore we can write:

\[
\max_{p \in D} \phi(w^T, p) - \min_{w \in W} \phi(w, p^T) = \max_{p \in D} \left\{ \phi(w^T, p) - \min_{w \in W} \phi(w, p^T) \right\} = \max_{p \in D, w \in W} \left\{ \phi(w^T, p) - \phi(w, p^T) \right\} \leq \frac{1}{T} \max_{p \in D, w \in W} \left\{ \sum_{t=1}^{T} \phi(w^T, p) - \phi(w, p^T) \right\} \quad (66)
\]
where (66) follows from the convexity of $\phi$ in $w$ and the concavity of $\phi$ in $p$. Again using the convexity of $\phi$ in $w$ along with the linearity of $\phi$ in $p$, we have that for any $t \geq 1$,

$$
\phi(w^t, p) - \phi(w, p^t) = \phi(w^t, p) - \phi(w^t, p^t) + \phi(w^t, p^t) - \phi(w, p^t)
$$

$$
\leq \langle (p - p^t), \nabla_p \phi(w^t, p^t) \rangle + \langle (w^t - w), \nabla_w \phi(w^t, p^t) \rangle
$$

$$
= \langle (p - p^t), \hat{g}^t_p \rangle + \langle (w^t - w), \hat{g}^t_w \rangle + \langle (p - p^t), (\nabla_p \phi(w^t, p^t) - \hat{g}^t_p) \rangle + \langle (w^t - w), (\nabla_w \phi(w^t, p^t) - \hat{g}^t_w) \rangle
$$

Thus by rearranging terms and the subadditivity of max,

$$
\max_{p \in \Delta, w \in W} \left\{ \sum_{t=1}^T \phi(w^t, p) - \phi(w, p^t) \right\}
$$

$$
\leq \max_{p \in \Delta, w \in W} \left\{ \sum_{t=1}^T \langle (p - p^t), \hat{g}^t_p \rangle + \langle (w^t - w), \hat{g}^t_w \rangle \right\}
$$

$$
+ \max_{p \in \Delta, w \in W} \left\{ \sum_{t=1}^T \langle p, (\nabla_p \phi(w^t, p^t) - \hat{g}^t_p) \rangle + \langle w, (\hat{g}^t_w - \nabla_w \phi(w^t, p^t)) \rangle \right\}
$$

$$
- \left( \sum_{t=1}^T \langle p^t, (\nabla_p \phi(w^t, p^t) - \hat{g}^t_p) \rangle - \langle w^t, (\nabla_w \phi(w^t, p^t) - \hat{g}^t_w) \rangle \right)
$$

(67)

We bound the expectation of each of the above terms separately, starting with the first one. Note that since $2ab = a^2 + b^2 - (a-b)^2$, for any $w \in W$, using a constant step size $\eta_w$ we have

$$
\sum_{t=1}^T \langle (w^t - w), \hat{g}^t_w \rangle = \frac{1}{2} \sum_{t=1}^T \frac{1}{\eta_w} \|w^t - w\|^2_2 + \eta_w \|\hat{g}^t_w\|^2_2 - \frac{1}{\eta_w} \|w^t - \eta_w \hat{g}^t_w - w\|^2_2
$$

$$
\leq \frac{1}{2\eta_w} \sum_{t=1}^T \|w^t - w\|^2_2 + (\eta_w)^2 \|\hat{g}^t_w\|^2_2 - \|w^{t+1} - w\|^2_2
$$

(68)

$$
= \frac{1}{2\eta_w} (\|w^1 - w\|^2_2 - \|w^{T+1} - w\|^2_2) + \frac{\eta_w}{2} \sum_{t=1}^T \|\hat{g}^t_w\|^2_2
$$

(69)

$$
\leq \frac{1}{2\eta_w} \|w^1 - w\|^2_2 + \frac{\eta_w}{2} \sum_{t=1}^T \|\hat{g}^t_w\|^2_2
$$

(70)

where (68) follows from the projection property and (69) is the result of the telescoping sum. Since (70) holds for all $w \in W$ and its right hand side does not depend on $w$, we can take the maximum over $w \in W$ on the left hand side, and the expectation of both sides with respect to the stochastic gradients, to obtain

$$
\mathbb{E} \left[ \max_{w \in W} \sum_{t=1}^T \langle (w^t - w), \hat{g}^t_w \rangle \right] \leq \frac{2R^2_W}{\eta_w} + \frac{\eta_w T \hat{G}^2_w}{2}
$$

(71)

where we have used Lemma 3 for the bounds on the stochastic gradients. Using analogous arguments and noting that the radius of $\Delta$ is 1, we can show that

$$
\mathbb{E} \left[ \max_{p \in \Delta} \sum_{t=1}^T \langle (p - p^t), \hat{g}^t_p \rangle \right] \leq \frac{2}{\eta_p} + \frac{\eta_p T \hat{G}^2_p}{2}
$$

(72)
Next, for the second term in (67), we can use the Cauchy-Schwarz Inequality and again the fact that max_{p\in\Delta} \|p\|_2 = 1 to write
\[
\max_{p\in\Delta} \sum_{t=1}^{T} \langle p, \nabla_p \phi(w_t, p') - \hat{g}_p^t \rangle = \max_{p\in\Delta} \left( \sum_{t=1}^{T} \nabla_p \phi(w_t, p') - \hat{g}_p^t \right) \\
\leq \| \sum_{t=1}^{T} \nabla_p \phi(w_t, p') - \hat{g}_p^t \|_2
\] (73)

Recall from Definition 1 that \( E[\|\nabla_p \phi(w_t, p') - \hat{g}_p^t\|_2^2] \leq \sigma_p^2 \) for all \( t \geq 1 \). Note that because the batch selections are independent, the \( \nabla_p \phi(w_t, p') - \hat{g}_p^t \) terms are uncorrelated random variables with mean 0 and variance at most \( \sigma_p^2 \), which means that \( \sum_{t=1}^{T} \nabla_p \phi(w_t, p') - \hat{g}_p^t \) is a random variable with mean 0 and variance at most \( T\sigma_p^2 \). Since
\[
E[\| \sum_{t=1}^{T} \nabla_p \phi(w_t, p') - \hat{g}_p^t \|_2^2] \leq \text{Var} \left( \sum_{t=1}^{T} \nabla_p \phi(w_t, p') - \hat{g}_p^t \right) \leq T\sigma_p^2
\] (74)
we have that \( E[\| \sum_{t=1}^{T} \nabla_p \phi(w_t, p') - \hat{g}_p^t \|_2] \leq \sqrt{T}\sigma_p \). Using this relation after taking the expectation of both sides of (73) yields
\[
E \left[ \max_{p\in\Delta} \sum_{t=1}^{T} \langle p, \nabla_p \phi(w_t, p') - \hat{g}_p^t \rangle \right] \leq \sqrt{T}\sigma_p
\] (75)

Using similar arguments, with this time using \( R_w \) to bound \( \max_{w\in W} \|w\|_2 \) after the analogous Cauchy-Schwarz step as in (73) we have
\[
E \left[ \max_{w\in W} \sum_{t=1}^{T} \langle w, \nabla_w \phi(w_t, p') - \hat{g}_w^t \rangle \right] \leq R_w \sqrt{T}\sigma_w
\] (76)

For the third and final term in (67), note that by the Law of Iterated Expectations and the unbiasedness of the stochastic gradients, we have that for any \( t \geq 1 \),
\[
E[(p', (\nabla_w \phi(w_t, p') - \hat{g}_w^t)) - (w', (\nabla_w \phi(w_t, p') - \hat{g}_w^t))] \\
= E[E[(p', (\nabla_w \phi(w_t, p') - \hat{g}_w^t)) - (w', (\nabla_w \phi(w_t, p') - \hat{g}_w^t))]|w', p']] \\
= 0
\]
Recalling (66) and (67), by combining the bounds on each of the terms and dividing by \( T \), we obtain
\[
E \left[ \max_{p\in\Delta} \phi(w_{T}^*, p) - \min_{w\in W} \phi(w, p_{T}^*) \right] \leq \frac{2R_w^2}{\eta_w T} + \frac{\eta_w \bar{G}_w^2}{2} + \frac{2}{\eta_p T} + \frac{\eta_p \bar{G}_p^2}{2} + \frac{R_w \sigma_w}{\sqrt{T}} + \frac{\sigma_p}{\sqrt{T}}
\] (77)
We minimize the above bound by setting the step sizes as
\[
\eta_w = \frac{2R_w}{\bar{G}_w \sqrt{T}}, \quad \eta_p = \frac{2}{\bar{G}_p \sqrt{T}}
\] (78)

to complete the proof, noting that \( \sigma_w \leq \bar{G}_w \) and \( \sigma_p \leq \bar{G}_p \). \( \square \)

F Proof of Lemma 5

Proof. We show the smoothness of each \( \bar{F}_i \) by upper bounding the norm of the difference of its gradients. Using (66) and the triangle inequality,
\[
\| \nabla \bar{F}_i(u) - \nabla \bar{F}_i(v) \| \leq ||E_{\theta \sim Q} \left[ (I - \alpha \nabla^2 \bar{f}_i(u, \theta))(\nabla f_i(u - \alpha \nabla \bar{f}_i(u, \theta)) - (\nabla f_i(v - \alpha \nabla \bar{f}_i(v, \theta))) \right] \| + ||E_{\theta \sim Q} \left[ (I - \alpha \nabla^2 \bar{f}_i(u, \theta)) - (I - \alpha \nabla^2 \bar{f}_i(v, \theta)) \right] \| \nabla \bar{f}_i(v - \alpha \nabla \bar{f}_i(v, \theta)) \| \| \| (79)
\]
where (79) follows from the triangle inequality. We consider the two terms in the right hand side of the second equation separately. Denoting the first term as \( \Xi \), we use Jensen’s Inequality then the Cauchy-Schwarz Inequality twice, as in (29) and (82), to obtain

\[
\Xi \leq \sqrt{\mathbb{E}_{\theta \sim Q_i} \left[ \| I - \alpha \nabla^2 \hat{f}_i(u, \theta) \|_2^2 \right]} \mathbb{E}_{\theta \sim Q_i} \left[ \| \nabla \hat{f}_i(u - \alpha \nabla \hat{f}_i(u, \theta)) - \nabla \hat{f}_i(v - \alpha \nabla \hat{f}_i(v, \theta)) \|_2^2 \right]
\]

(80)

\[
\leq (1 + \alpha M) \sqrt{\mathbb{E}_{\theta \sim Q_i} \left[ \| \nabla \hat{f}_i(u - \alpha \nabla \hat{f}_i(u, \theta)) - \nabla \hat{f}_i(v - \alpha \nabla \hat{f}_i(v, \theta)) \|_2^2 \right]}
\]

(81)

where to obtain (81) we have used the \( M \)-smoothness of \( \hat{f}_i \). Considering the term remaining inside the square root, we have

\[
\mathbb{E}_{\theta \sim Q_i} \left[ \| \nabla \hat{f}_i(u - \alpha \nabla \hat{f}_i(u, \theta)) - \nabla \hat{f}_i(v - \alpha \nabla \hat{f}_i(v, \theta)) \|_2^2 \right]
\]

(82)

\[
\leq M^2 \mathbb{E}_{\theta \sim Q_i} \left[ \| u - \alpha \nabla \hat{f}_i(u, \theta) - (v - \alpha \nabla \hat{f}_i(v, \theta)) \|_2^2 \right]
\]

= \( M^2 \mathbb{E}_{\theta \sim Q_i} \left[ \| u - v \|_2^2 + 2\alpha (u - v)^T (\nabla \hat{f}_i(u, \theta) - \nabla \hat{f}_i(v, \theta)) + \alpha^2 \| \nabla \hat{f}_i(u, \theta) - \nabla \hat{f}_i(v, \theta) \|_2^2 \right] \)

\[
= M^2 \left( \| u - v \|_2^2 + 2\alpha \| u - v \| M + \alpha^2 M^2 \right)
\]

(83)

\[
= M^2 (1 + \alpha M)^2 \| u - v \|_2
\]

(84)

where (83) follows from the \( M \)-smoothness of \( \hat{f}_i \) and the Cauchy Schwarz Inequality. Thus we have

\[
\Xi \leq M (1 + \alpha M)^2 \| u - v \|
\]

(85)

Note that we have already upper bounded the second term in (79) in the previous lemma (see Equation (65)). Thus we have that the smoothness parameter of \( \hat{F}_i \) is

\[
\hat{M}_i := M (1 + \alpha M)^2 + \alpha LH
\]

(86)

\[ \Box \]

### G Proof of Proposition 1

**Proof.** Note that

\[
\mathbb{E}[\| \nabla w \phi(w^t, p^t) \|_2^2] = \mathbb{E}_t \left[ \mathbb{E}_{\tau} \left[ \| \nabla w \phi(w^\tau, p^\tau) \|_2^2 \right] \right]
\]

(87)

\[
= \mathbb{E}_t \left[ \frac{1}{T} \sum_{t=1}^{T} \| \nabla w \phi(w^t, p^t) \|_2^2 \right]
\]

(88)

\[
= \frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \left[ \| g^t \|_2^2 \right]
\]

(89)

where the un-subscripted expectation in the right hand sides of (87) and (88) is over the stochastic gradients which determine the sequence \( \{(w^t, p^t)\}_t \). Thus to show the bound on \( \mathbb{E}[\| \nabla w \phi(w^t, p^t) \|_2^2] \) in Proposition 1 we bound the right hand side of (89). To do so we borrow ideas from the proof of Theorem 1 in [Qian et al., 2019]. First recall that by Lemma 5, \( \hat{F}_i \) is \( M \)-smooth for each

\[
\hat{F}_i \leq \hat{F}_i(u) + \nabla \hat{F}_i(v)^T (u - v) + \frac{\hat{M}}{2} \| u - v \|^2
\]

(90)

Condition on the history up to iteration \( t \), denoted by \( \mathcal{F}^t \), the above equation implies

\[
\mathbb{E} \left[ \sum_{i=1}^{m} p_i^{t} \hat{F}_i(w^{t+1}) | \mathcal{F}^t \right]
\]

\[
\leq \mathbb{E} \left[ \sum_{i=1}^{m} p_i^{t} \hat{F}_i(w^t) + \left( \nabla w \sum_{i=1}^{m} p_i^{t} \hat{F}_i(w^t) \right)^T (w^{t+1} - w^t) + \frac{\hat{M}}{2} \| w^{t+1} - w^t \|^2 | \mathcal{F}^t \right]
\]

(91)
where the expectation is conditioned on the prior stochastic gradient computations up to iteration $t$. Note that $\nabla_w \sum_{i=1}^m p_i F_i(w^t) = g_w^t$ and $w^{t+1} - w^t = -\eta_w \hat{g}_w^t$. Thus, we have

$$E \left[ \sum_{i=1}^m p_i^t \hat{F}_i(w^{t+1}) | F^t \right] = E \left[ \sum_{i=1}^m p_i^t \hat{F}_i(w^t) - \eta_w (g_w^t)^T \hat{g}_w^t + \frac{M^2}{2} \eta^2_w | g_w^t \|^2 | F^t \right]$$

$$= \sum_{i=1}^m p_i^t \hat{F}_i(w^t) - \eta_w \| g_w^t \|^2 + \frac{M^2}{2} \eta^2_w (\| g_w^t \|^2 + E [\| \hat{g}_w^t - g_w^t \|^2 | F^t])$$

(93) follows because $\hat{g}_w^t$ is an unbiased estimate of $g_w^t$. Using Lemma 3, we have

$$E \left[ \sum_{i=1}^m p_i^t \hat{F}_i(w^{t+1}) | F^t \right] \leq \sum_{i=1}^m p_i^t \hat{F}_i(w^t) - \left( \eta_w - \frac{\eta^2_w M}{2} \right) \| g_w^t \|^2 + \frac{\eta^2_w M \sigma^2_w}{2}$$

(94)

Rearranging the terms, we obtain

$$\left( \eta_w - \frac{\eta^2_w M}{2} \right) \| g_w^t \|^2 \leq E \left[ \sum_{i=1}^m p_i^t \hat{F}_i(w^t) - \sum_{i=1}^m p_i^t \hat{F}_i(w^{t+1}) | F^t \right] + \frac{\eta^2_w M \sigma^2_w}{2}$$

(95)

$$= \sum_{i=1}^m p_i^t \hat{F}_i(w^t) - \sum_{i=1}^m p_i^t \hat{F}_i(w^{t+1}) | F^t$$

(96)

$$+ E \left[ \sum_{i=1}^m p_i^t \hat{F}_i(w^{t+1}) - \sum_{i=1}^m p_i^t \hat{F}_i(w^{t+1}) | F^t \right] + \frac{\eta^2_w M \sigma^2_w}{2}$$

(97)

We bound the second expectation in the above equation:

$$E \left[ \sum_{i=1}^m p_i^t \hat{F}_i(w^{t+1}) - \sum_{i=1}^m p_i^t \hat{F}_i(w^{t+1}) | F^t \right] = E \left[ \sum_{i=1}^m (p_i^{t+1} - p_i^t) \hat{F}_i(w^{t+1}) | F^t \right]$$

$$\leq E \left[ \| p^{t+1} - p^t \|_2 \sum_{i=1}^m \left( \hat{F}_i(w^{t+1}) \right)^{1/2} | F^t \right]$$

(98)

$$\leq \sqrt{mB} E \left[ \| p^{t+1} - p^t \|_2 | F^t \right]$$

(99)

$$\leq 2 \sqrt{mB} E \left[ \| \eta \hat{g}_w^t \|_2 | F^t \right]$$

(100)

$$= 2 \eta \sqrt{mB} \hat{G}_p$$

(101)

where (98) follows from the Cauchy-Schwarz Inequality, (99) follows by the bound on $f_i$ for all $i$, and (100) follows from the update rule for $p$ combined with the projection property (since $p^t \in \Delta$, $\| p^t - (p^t + \eta \hat{g}_w^t) \| \geq \| p^t - (1 - \Delta) \|_2$). Using this result, summing (97) from $t = 1$ to $T$, and taking the expectation over all the stochastic gradients of both sides and using the Law of Iterated Expectations to remove the conditioning on $F^t$, we obtain

$$\left( \eta_w - \frac{\eta^2_w M}{2} \right) \sum_{t=1}^T E [\| g_w^t \|^2] \leq E \left[ \sum_{i=1}^m p_i^1 \hat{F}_i(w^1) \right] - E \left[ \sum_{i=1}^m p_i^t \hat{F}_i(w^{t+1}) \right] + 2 T \eta \sqrt{mB} \hat{G}_p + \frac{T \eta^2_w M \sigma^2_w}{2}$$

$$\leq \phi(w^1, p^1) + B + 2 T \eta \sqrt{mB} \hat{G}_p + \frac{T \eta^2_w M \sigma^2_w}{2}$$

Next, dividing both sides by $T \left( \eta_w - \frac{\eta^2_w M}{2} \right)$ we have

$$\frac{1}{T} \sum_{t=1}^T E [\| g_w^t \|^2] \leq \frac{\phi(w^1, p^1) + B}{T \left( \eta_w - \frac{\eta^2_w M}{2} \right)} + \frac{2 \eta \sqrt{mB} \hat{G}_p}{T \left( \eta_w - \frac{\eta^2_w M}{2} \right)} + \frac{\eta_w M \sigma^2_w}{2}$$

(102)

which by (89) is the desired bound on $E [\| \hat{g}_w^t \|^2]$. Next we show the bound on the optimality of $p^*_w$. As before, we start by evaluating the
expectation over \( \tau \):

\[
\mathbb{E} [\phi(w_T^\tau, p_T^\tau)] = \mathbb{E} [\mathbb{E}_\tau [\phi(w_T^\tau, p_T^\tau)]]
\]

\[= \mathbb{E} \left[ \frac{1}{T} \sum_{t=1}^T \phi(w^t, p^t) \right] \tag{103} \]

\[= \frac{1}{T} \sum_{t=1}^T \mathbb{E} [\phi(w_T^\tau, p_T^\tau)] \tag{104} \]

Next, since \( \phi(w, p) \) is linear in \( p \), we have that for any \( p \in \Delta \) and any \( t \in \{1, ..., T\} \),

\[
\mathbb{E} [\phi(w^t, p) - \phi(w^t, p^t)|\mathcal{F}^t] = \mathbb{E} [(p - p^t)g_p^t|\mathcal{F}^t]
\]

\[= \mathbb{E} [(p - p^t)\hat{g}_p^t|\mathcal{F}^t] + \mathbb{E} [(p - p^t)\hat{g}_p^t|\mathcal{F}^t] \] \tag{106}

\[= \mathbb{E} [(p - p^t)g_p^t|\mathcal{F}^t] \tag{107} \]

where (107) follows because \( \hat{g}_p^t \) is an unbiased estimate of \( g_p^t \). Using (107) and the identity

\[2ab = a^2 + b^2 - (a - b)^2 \] with \( a = p - p^t \) and \( b = \eta_p \hat{g}_p^t \) yields

\[
\mathbb{E} [\phi(w^t, p) - \phi(w^t, p^t)|\mathcal{F}^t] = \mathbb{E} \left[ \frac{1}{2\eta_p} \left( \|p - p^t\|_2^2 + (\eta_p)\|\hat{g}_p^t\|_2^2 - \|p - (p^t + \eta_p \hat{g}_p^t)\|_2^2 \right) \right] |\mathcal{F}^t | \tag{108}
\]

\[\leq \frac{1}{2\eta_p} \left( \|p - p^t\|_2^2 + (\eta_p)\|\hat{g}_p^t\|_2^2 - \|p - p^t+1\|_2^2 \right) |\mathcal{F}^t | \tag{109}
\]

\[\leq \frac{1}{2\eta_p} \left( \|p - (p^t + \eta_p \hat{G}_p^2)\|_2^2 - \|p - p^t+1\|_2^2 \right) |\mathcal{F}^t | \tag{110}
\]

where (107) follows from the projection property and (110) follows from the definition of \( \hat{G}_p \). Summing from \( t = 1 \) to \( T \) and taking the expectation over all the stochastic gradients of both sides and using the Law of Iterated Expectations to remove the conditioning on \( \mathcal{F}^t \), we obtain

\[
\sum_{t=1}^T \mathbb{E} [\phi(w^t, p) - \phi(w^t, p^t)] \leq \sum_{t=1}^T \frac{1}{2\eta_p} \mathbb{E} [\|p - p^t\|_2^2] - \frac{1}{2\eta_p} \mathbb{E} [\|p - p^t+1\|_2^2] + \frac{\eta_p}{2} \hat{G}_p^2 \tag{112}
\]

\[= \frac{1}{2\eta_p} \mathbb{E} [\|p - p^t\|_2^2] + \frac{\eta_p}{2} \hat{G}_p^2 \tag{113}\]

\[\leq \frac{1}{\eta_p} + \frac{\eta_p T \hat{G}_p^2}{2} \tag{114}
\]

where (112) follows from the telescoping sum and (114) follows from the fact that \( p, p^t \in \Delta \) and \( \Delta \) is contained in an \( \ell_2 \) ball of radius 1. Dividing both sides of (114) by \( T \) and rearranging terms

\[
\frac{1}{T} \sum_{t=1}^T \mathbb{E} [\phi(w^t, p^t)] \geq \mathbb{E} [\phi(w_T^\tau, p)] - \left( \frac{1}{\eta_p} + \frac{\eta_p \hat{G}_p^2}{2} \right)
\]

Finally, since (114) holds for all \( p \in \Delta \), we maximize the right hand side over \( p \in \Delta \), yielding

\[
\frac{1}{T} \sum_{t=1}^T \mathbb{E} [\phi(w^t, p^t)] \geq \max_{p \in \Delta} \mathbb{E} [\phi(w_T^\tau, p)] - \left( \frac{1}{\eta_p} + \frac{\eta_p \hat{G}_p^2}{2} \right)
\]

From (105), the left hand side above is equal to \( \mathbb{E} [\phi(w_T^\tau, p_T^\tau)] \), thus completing the proof. \( \square \)

**H Proof of Proposition 2**

**Proof.** Our proof makes analogous initial arguments to the proof of Theorem 2 in [Ghadimi et al., 2016], and cites two results on the properties of the prox operation from the same paper. Let
\( \bar{g}^t := P_W(w^t, p^t, g^t_w, \eta^t_w) \) for all \( t \geq 1 \). By the \( \bar{M} \)-smoothness of \( \bar{F}_i \) for each \( i \), we have equation \( \text{[90]} \), and thus for any \( t \in \{1, \ldots, T\} \),

\[
\sum_{i=1}^{m} p_i^t \bar{F}_i(w^{t+1}) \leq \sum_{i=1}^{m} p_i^t \bar{F}_i(w^t) + \left( \nabla_w \sum_{i=1}^{m} p_i^t \bar{F}_i(w^t) \right)^T \left( w^{t+1} - w^t \right) + \frac{\bar{M}}{2} \| w^{t+1} - w^t \|^2_2
\]

\[
= \sum_{i=1}^{m} p_i^t \bar{F}_i(w^t) - \eta^t_w \left( \nabla_w \sum_{i=1}^{m} p_i^t \bar{F}_i(w^t) \right)^T \bar{g}^t + \frac{\bar{M}}{2} (\eta^t_w)^2 \| \bar{g}^t \|^2_2
\]

\[
= \sum_{i=1}^{m} p_i^t \bar{F}_i(w^t) - \eta^t_w (\bar{g}^t) \left( \bar{g}^t \right)^T \bar{g}^t + \frac{\bar{M}}{2} (\eta^t_w)^2 \| \bar{g}^t \|^2_2 + \eta^t_w (\delta^t_w)^T \bar{g}^t
\]

where in the identity we have used the definitions of \( \bar{g}^t \) and \( \delta^t_w \). Next, using Lemma 1 in \( \text{Ghadimi et al. 2016} \) with \( x = w^t, \gamma = \eta^t_w \), and \( g = \bar{g}^t \), we obtain

\[
\sum_{i=1}^{m} p_i^t \bar{F}_i(w^{t+1}) \leq \sum_{i=1}^{m} p_i^t \bar{F}_i(w^t) - [\eta^t_w \| \bar{g}^t \|^2_2 + h(w^{t+1}) - h(w^t)] + \frac{\bar{M}}{2} (\eta^t_w)^2 \| \bar{g}^t \|^2_2 + \eta^t_w (\delta^t_w)^T \bar{g}^t
\]

\[
= \sum_{i=1}^{m} p_i^t \bar{F}_i(w^t) - [\eta^t_w \| \bar{g}^t \|^2_2 + h(w^{t+1}) - h(w^t)] + \frac{\bar{M}}{2} (\eta^t_w)^2 \| \bar{g}^t \|^2_2 + \eta^t_w (\delta^t_w)^T \bar{g}^t
\]

where \( g^t := P_W(w^t, p^t, g^t_w, \eta^t_w) \) is the projection of the true gradient with respect to \( w \). Thus after rearranging terms,

\[
\Phi(w^{t+1}, p^t) \leq \Phi(w^t, p^t) - \left( \eta^t_w - \frac{\bar{M}}{2} (\eta^t_w)^2 \right) \| \bar{g}^t \|^2_2 + \eta^t_w (\delta^t_w)^T \bar{g}^t + \eta^t_w \| \delta^t_w \|^2
\]

\[
\leq \Phi(w^t, p^t) - \left( \eta^t_w - \frac{\bar{M}}{2} (\eta^t_w)^2 \right) \| \bar{g}^t \|^2_2 + \eta^t_w (\delta^t_w)^T \bar{g}^t + \eta^t_w \| \delta^t_w \|^2
\]

where the last inequality follows from Proposition 1 in \( \text{Ghadimi et al. 2016} \) with \( x = w^t, \gamma = \eta^t_w, g_1 = \bar{g}^t \), and \( g_2 = g^t \). Rearranging terms, we have

\[
\left( \eta^t_w - \frac{\bar{M}}{2} (\eta^t_w)^2 \right) \| \bar{g}^t \|^2_2 \leq \Phi(w^t, p^t) - \Phi(w^{t+1}, p^t) + \eta^t_w (\delta^t_w)^T \bar{g}^t + \eta^t_w \| \delta^t_w \|^2
\]

\[
= (\Phi(w^t, p^t) - \Phi(w_{t+1}^{t+1}, p^t+1)) + (\Phi(w^{t+1}, p^{t+1}) - \Phi(w^{t+1}, p^t)) + \eta^t_w (\delta^t_w)^T \bar{g}^t + \eta^t_w \| \delta^t_w \|^2
\]

\[
= (\Phi(w^t, p^t) - \Phi(w^{t+1}, p^{t+1})) + (\phi(w^{t+1}, p^{t+1}) - \phi(w^{t+1}, p^t)) + \eta^t_w (\delta^t_w)^T \bar{g}^t + \eta^t_w \| \delta^t_w \|^2
\]

Taking the expectation with respect to the stochastic gradients conditioned on the history up to time \( t \) of each side, we have

\[
\left( \eta^t_w - \frac{\bar{M}}{2} (\eta^t_w)^2 \right) E[\| \bar{g}^t \|^2_2 | F^t]
\]

\[
\leq E \left[ (\Phi(w^t, p^t) - \Phi(w^{t+1}, p^{t+1})) | F^t \right] + E \left[ (\phi(w^{t+1}, p^{t+1}) - \phi(w^{t+1}, p^t)) | F^t \right]
\]

\[
+ \eta^t_w E \left[ (\delta^t_w)^T \bar{g}^t | F^t \right] + \eta^t_w E \left[ \| \delta^t_w \|^2 | F^t \right]
\]

\[
= E \left[ (\Phi(w^t, p^t) - \Phi(w^{t+1}, p^{t+1})) | F^t \right] + E \left[ \sum_{i=1}^{m} (p_i^{t+1} - p_i^t) \bar{F}_i(w^{t+1}) | F^t \right]
\]

\[
+ \eta^t_w E \left[ (\delta^t_w)^T \bar{g}^t | F^t \right] + \eta^t_w E \left[ \| \delta^t_w \|^2 | F^t \right]
\]

(115)
Note that we can use the Hölder Inequality to bound the second expectation in (115). In doing so we obtain

\[
\left(\eta_w - \frac{\bar{M}}{2}(\eta_w)^2\right) \sum_{t=1}^{T} \mathbb{E}[\|g^t\|^2_2] \leq \Phi(w^1, p^1) - \mathbb{E}[\Phi(w^{T+1}, p^{T+1})] + 2T\eta_pB\sqrt{m}G_p + T\eta_w\sigma_w^2 \leq \Phi(w^1, p^1) + B + 2T\eta_pB\sqrt{m}G_p + T\eta_w\sigma_w^2
\]

Next we divide both sides by \(T\left(\eta_w - \frac{\bar{M}}{2}(\eta_w)^2\right)\) to yield

\[
\left(\eta_w - \frac{\bar{M}}{2}(\eta_w)^2\right) \sum_{t=1}^{T} \mathbb{E}[\|g^t\|^2_2] \leq \frac{2(\phi(w^1, p^1) + B)}{T(2\eta_w - \eta_w^2M)} + \frac{4\eta_p\sqrt{m}BG_p}{(2\eta_w - \eta_w^2M)} + \frac{\sigma_w^2}{(2 - \eta_wM)}
\]

Using (89), we have that the left hand side of the above equation is equal to \(\mathbb{E}[\|\tilde{g}_T^t\|^2]\), thus we have completed the proof of the convergence result in \(w\).

For the convergence with respect to \(p\), note that the update rule for \(p^{t+1}\) is identical to the update rule analyzed in Proposition 4 and the output procedure is the same for both algorithms. Furthermore, since the convergence analysis of \(p\) does not depend on the update rule for \(w\), the analysis with respect to \(p\) in the proof of Proposition 4 still applies here, thus we have the same bound.

\[
\square
\]

**References**

Maruan Al-Shedivat, Trapit Bansal, Yuri Burda, Ilya Sutskever, Igor Mordatch, and Pieter Abbeel. Continuous adaptation via meta-learning in nonstationary and competitive environments. arXiv preprint arXiv:1710.03641, 2017.

Marcin Andrychowicz, Misha Denil, Sergio Gomez, Matthew W Hoffman, David Pfau, Tom Schaul, Brendan Shillingford, and Nando De Freitas. Learning to learn by gradient descent by gradient descent. In Advances in neural information processing systems, pages 3981–3989, 2016.

Antreas Antoniou, Harrison Edwards, and Amos Storkey. How to train your maml. arXiv preprint arXiv:1810.09502, 2018.

Yoshua Bengio, Samy Bengio, and Jocelyn Cloutier. Learning a synaptic learning rule. Université de Montréal, Département d’informatique et de recherche . . . , 1990.
Robert S Chen, Brendan Lucier, Yaron Singer, and Vasilis Syrgkanis. Robust optimization for non-convex objectives. In Advances in Neural Information Processing Systems, pages 4705–4714, 2017.

Yan Duan, John Schulman, Xi Chen, Peter L Bartlett, Ilya Sutskever, and Pieter Abbeel. RL2: Fast reinforcement learning via slow reinforcement learning. arXiv preprint arXiv:1611.02779, 2016.

John Duchi, Peter Glynn, and Hongseok Namkoong. Statistics of robust optimization: A generalized empirical likelihood approach. arXiv preprint arXiv:1610.03425, 2016.

Alireza Fallah, Aryan Mokhtari, and Asuman Ozdaglar. On the convergence theory of gradient-based model-agnostic meta-learning algorithms. arXiv preprint arXiv:1908.10400, 2019.

Chelsea Finn, Pieter Abbeel, and Sergey Levine. Model-agnostic meta-learning for fast adaptation of deep networks. In Proceedings of the 34th International Conference on Machine Learning-Volume 70, pages 1126–1135. JMLR. org, 2017.

Chelsea Finn, Aravind Rajeswaran, Sham Kakade, and Sergey Levine. Online meta-learning. arXiv preprint arXiv:1902.08438, 2019.

Luca Franceschi, Paolo Frasconi, Saverio Salzo, Riccardo Grazzi, and Massimilano Pontil. Bilevel programming for hyperparameter optimization and meta-learning. arXiv preprint arXiv:1806.04910, 2018.

Saeed Ghadimi, Guanghui Lan, and Hongchao Zhang. Mini-batch stochastic approximation methods for nonconvex stochastic composite optimization. Mathematical Programming, 155(1-2):267–305, 2016.

Sepp Hochreiter, A Steven Younger, and Peter R Conwell. Learning to learn using gradient descent. In International Conference on Artificial Neural Networks, pages 87–94. Springer, 2001.

Chi Jin, Praneeth Netrapalli, and Michael I Jordan. Minmax optimization: Stable limit points of gradient descent ascent are locally optimal. arXiv preprint arXiv:1902.00618, 2019.

Anatoli Juditsky, Arkadi Nemirovski, and Claire Tauvel. Solving variational inequalities with stochastic mirror-prox algorithm. Stochastic Systems, 1(1):17–58, 2011.

Mikhail Khodak, Maria-Florina Balcan, and Ameet Talwalkar. Provable guarantees for gradient-based meta-learning. arXiv preprint arXiv:1902.10644, 2019a.

Mikhail Khodak, Maria-Florina F Balcan, and Ameet S Talwalkar. Adaptive gradient-based meta-learning methods. In Advances in Neural Information Processing Systems, pages 5915–5926, 2019b.

Yann LeCun and Corinna Cortes. MNIST handwritten digit database. 2010. URL http://yann.lecun.com/exdb/mnist/.

Zhengu Li, Fengwei Zhou, Fei Chen, and Hang Li. Meta-sgd: Learning to learn quickly for few-shot learning. arXiv preprint arXiv:1707.09835, 2017.

Mehryar Mohri, Gary Sivek, and Ananda Theertha Suresh. Agnostic federated learning. arXiv preprint arXiv:1902.00146, 2019.

Arkadi Nemirovski, Anatoli Juditsky, Guanghui Lan, and Alexander Shapiro. Robust stochastic approximation approach to stochastic programming. SIAM Journal on optimization, 19(4): 1574–1609, 2009.

Alex Nichol and John Schulman. Reptile: a scalable metalearning algorithm. arXiv preprint arXiv:1803.02999, 2018.

Qi Qian, Shenghuo Zhu, Jiasheng Tang, Rong Jin, Baigui Sun, and Hao Li. Robust optimization over multiple domains. In Proceedings of the AAAI Conference on Artificial Intelligence, volume 33, pages 4739–4746, 2019.
Hassan Rafique, Mingrui Liu, Qihang Lin, and Tianbao Yang. Non-convex min-max optimization: Provable algorithms and applications in machine learning. *arXiv preprint arXiv:1810.02060*, 2018.

Sachin Ravi and Hugo Larochelle. Optimization as a model for few-shot learning. 2016.

Shai Shalev-Shwartz and Yonatan Wexler. Minimizing the maximal loss: How and why. In ICML, pages 793–801, 2016.

Aman Sinha, Hongseok Namkoong, and John Duchi. Certifying some distributional robustness with principled adversarial training. *arXiv preprint arXiv:1710.10571*, 2017.

Jake Snell, Kevin Swersky, and Richard Zemel. Prototypical networks for few-shot learning. In *Advances in neural information processing systems*, pages 4077–4087, 2017.

Sebastian Thrun and Lorien Pratt. *Learning to learn*. Springer Science & Business Media, 2012.

Oriol Vinyals, Charles Blundell, Timothy Lillicrap, Koray Kavukcuoglu, and Daan Wierstra. Matching networks for one shot learning. In D. D. Lee, M. Sugiyama, U. V. Luxburg, I. Guyon, and R. Garnett, editors, *Advances in Neural Information Processing Systems 29*, pages 3630–3638. 2016.

Jane X Wang, Zeb Kurth-Nelson, Dhruva Tirumala, Hubert Soyer, Joel Z Leibo, Remi Munos, Charles Blundell, Dharshan Kumaran, and Matt Botvinick. Learning to reinforcement learn. *arXiv preprint arXiv:1611.05763*, 2016.

Han Xiao, Kashif Rasul, and Roland Vollgraf. Fashion-mnist: a novel image dataset for benchmarking machine learning algorithms. *arXiv preprint arXiv:1708.07747*, 2017.

Chengxiang Yin, Jian Tang, Zhiyuan Xu, and Yanzhi Wang. Adversarial meta-learning. *arXiv preprint arXiv:1806.03316*, 2018.

Pan Zhou, Xiaotong Yuan, Huan Xu, Shuicheng Yan, and Jiashi Feng. Efficient meta learning via minibatch proximal update. In *Advances in Neural Information Processing Systems*, pages 1532–1542, 2019.

Zhenxun Zhuang, Yunlong Wang, Kezi Yu, and Songtao Lu. Online meta-learning on non-convex setting. *arXiv preprint arXiv:1910.10196*, 2019.

Daniel Zügner and Stephan Günnemann. Adversarial attacks on graph neural networks via meta learning. *arXiv preprint arXiv:1902.08412*, 2019.