\(\theta\) SCHEME WITH TWO DIMENSIONAL WAVELET-LIKE INCREMENTAL UNKNOWNS FOR A CLASS OF POROUS MEDIUM DIFFUSION-TYPE EQUATIONS

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Abstract. In this article, a \(\theta\) scheme based on wavelet-like incremental unknowns (WIU) is presented for a class of porous medium diffusion-type equations. Through some important norm inequalities, we prove the stability of \(\theta\) scheme. Compared to the classical scheme, the stability conditions are improved. Numerical results show that the \(\theta\) scheme based on the WIU decomposition is efficient.

1. Introduction. Incremental unknowns (IU) method which arises from the dynamical systems theory of Navier Stokes equations (see [9]) is a very powerful tool for the computation and analysis of fluid flows. The primary motivation of the IUs theory is to approximate inertial manifolds when finite differences are used for the spatial discretization (see [5]). However, Chen and Temam proved that IU allows to define hierarchical preconditioners for discrete operators arising from elliptic-like problems (see [1], [2]). Many studies on IU have been carried out (see [8], [10]–[15]). Wavelet-like incremental unknowns (WIU) deserve special attention because they enjoy the \(L^2\) orthogonality property between different levels of unknowns. This makes the method with multilevel wavelet-like incremental unknowns particularly appropriate for the approximation of evolution equation. Chen and Temam applied the multilevel WIU to a Reaction-Diffusion equation (see [3]), they presented several numerical schemes using one dimensional WIU. The fully discretized explicit and semi-explicit schemes for the reaction-diffusion equation are presented and analyzed, the stability conditions are improved with the corresponding algorithms. The author in [13] established two semi-implicit schemes with multilevel WIU methods for the same equation, the stability conditions of the two schemes become better. [8] proposed a new type of WIU method for the two-dimensional reaction-diffusion equations with a polynomial nonlinear term and analysed the stability of Euler explicit and semi-implicit schemes.

The purpose of this paper is to apply WIU to two dimensional reaction-diffusion equations with more general nonlinear term and present a \(\theta\) Scheme which includes explicit scheme, implicit scheme and Grank-Nicolson scheme. We will prove some important inequalities, then we prove that under suitable conditions, the \(\theta\) scheme

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is stable. Compared with the classical theme discretized without WIU, the time step can be larger. Numerical result shows that the θ scheme save more CPU time than the scheme without WIU.

The paper is organized as follows. In Section 2 we present the porous medium diffusion-type equation and its finite difference discretization. Then in Section 3 we recall the definition of the two dimensional WIU and the multilevel discretization in space. New θ schemes based on multilevel WIU are established in Section 4. In Section 5, firstly we write the variational form of the approximate scheme, then we develop the stability criteria of the schemes. Finally, numerical results show the efficiency of the new methods.

2. Dynamic Equation and Discretization. Let Ω be an open-bounded subset of $\mathbb{R}^n$. We consider the model

$$
P : \begin{cases}
\frac{\partial u}{\partial t} - \nu \Delta u + g(t, x, u) = 0, & \text{in } (0, T) \times \Omega, \\
u u(x, t) = 0, & \text{on } (0, T) \times \partial \Omega, \\
u u(x, 0) = u_0, & \text{in } \Omega.
\end{cases}
$$

(1) under the following assumptions:

(H1) For each $\xi \in \mathbb{R}$, the map $(t, x) \mapsto g(t, x, \xi)$ is measurable and almost everywhere in $\Omega \times \mathbb{R}$ $\xi \mapsto g(t, x, \xi)$ is continuous and differentiable.

(H2) We assume that there exist $c_1 > 0, c_2 > 0$ and $c_3 > 0$ such that

$$
\begin{align*}
\text{sign}(\xi) \cdot g(t, x, \xi) &\geq c_1 |\xi|^{q-1} - c_2, & q > 2, \\
\limsup_{t \to 0} |g(t, x, \xi)| &\leq c_3 (|\xi|^{q-1} + 1), \\
|g(x, \xi)| &\leq a(|\xi|),
\end{align*}
$$

(2)

where $a : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is an increasing function and the sign function is defined as follows:

$$
\tau \mapsto \text{sign}(\tau) = \begin{cases} 
1, & \text{if } \tau > 0, \\
0, & \text{if } \tau = 0, \\
-1, & \text{if } \tau < 0.
\end{cases}
$$

(H3) There exists $c_7 > 0$ such that for almost every $(t, x) \in \mathbb{R}^+ \times \Omega : \xi \mapsto g(t, x, \xi) + c_7 \xi$ is an increasing function.

(H4) $u_0 \in L^2(\Omega)$.

Using the above assumptions, we have the following existence and uniqueness result (see [4]):

**Theorem 2.1.** Under the assumptions (H1) to (H4), there exists a unique solution $u \in L^q(0, T; L^q(\Omega)) \cap L^2(0, T; H_0^1(\Omega)) \cap L^\infty(\eta, T; L^\infty(\Omega)), \forall \eta > 0$, satisfies P.

Now, we consider spatial discretization by finite difference with mesh size $h_d = 1/(2^d N + 1)$, where $N \in \mathbb{N}$. For the sake of simplicity, here let $\Omega = [0, 1]^2$. We have

$$
\frac{\partial U_{h_d}}{\partial t} + \nu A_{h_d} U_{h_d} + g(t, X, U_{h_d}) = 0,
$$

(3)

where $U_{h_d} \in \mathbb{R}^{(2^d N)^2}$ is the vector that consists of approximate values of $u$ at the grid points and $X = (x_1, x_2, ..., x_{(2^d N)^2})^T$, $A_{h_d}$ is a matrix of order $(2^d N)^2$ which
has the form
\[ A_d = \frac{1}{h_d^2} \begin{pmatrix} C & -I \\ -I & \ddots & \ddots \\ & \ddots & \ddots & -I \\ -I & C \end{pmatrix}, \quad C = \begin{pmatrix} 4 & -1 \\ -1 & \ddots & \ddots \\ & \ddots & \ddots & -1 \\ -1 & 4 \end{pmatrix}. \]

Here, \( I \) is the identity matrix, \( C \) and \( I \) are both of order \( 2^d N \).
For simplicity, we write \( A_d = A_{h_d}, U_d = U_{h_d}, g(t, X, U_d) = g(U_d) \).

3. **Multilevel Wavelet-like Incremental Unknowns.** We introduce the wavelet-like incremental unknowns equation (3) by recalling the definition of WIU in [3].
We separate evenly the unknowns into four different parts according to the grid, see Fig.1.

![Fig.1 Coarse grid points(×) and fine grid points(◦), d=1, N=4.](image)

Here \( u_{2i,2j}^d \) is the value of \( u \) corresponding to coarse grid \( M_1 \), \( u_{2i-1,2j}^d \) \( u_{2i,2j-1}^d \) and \( u_{2i-1,2j-1}^d \) are the values of \( u \) corresponding to complementary grids \( A_1, A_2 \) and \( A_3 \).
For \( i = 1, \cdots, 2^d-1, j = 1, \cdots, 2^d-1 \), the first separation of new variables is defined by
\[
\begin{align*}
\begin{cases}
y_{2i,2j}^d = \frac{1}{4} (u_{2i,2j}^d + u_{2i-1,2j}^d + u_{2i,2j-1}^d + u_{2i-1,2j-1}^d), \\
z_{2i-1,2j}^d = u_{2i-1,2j}^d - y_{2i,2j}^d, \\
z_{2i,2j-1}^d = u_{2i,2j-1}^d - y_{2i-1,2j}^d, \\
z_{2i-1,2j-1}^d = u_{2i-1,2j-1}^d - y_{2i-1,2j}^d.
\end{cases}
\end{align*}
\]

(4)

Inversely, we have
\[
\begin{align*}
\begin{cases}
u_{2i,2j}^d &= y_{2i,2j}^d - z_{2i-1,2j}^d - z_{2i,2j-1}^d - z_{2i-1,2j-1}^d, \\
u_{2i-1,2j}^d &= y_{2i-1,2j}^d + z_{2i,2j-1}^d, \\
u_{2i,2j-1}^d &= y_{2i,2j-1}^d + z_{2i-1,2j}^d, \\
u_{2i-1,2j-1}^d &= y_{2i-1,2j-1}^d + z_{2i-1,2j-1}^d.
\end{cases}
\end{align*}
\]

(5)
We reorder $U_d$ into $\hat{U}_d$ by letting
\[
\hat{U}_d = \left( U_0^d, U_1^d, U_2^d, U_3^d \right)^T,
\]
where $U_0^d, U_1^d, U_2^d, U_3^d$ are column vectors ordered in lexical order (from left to right, from down to up) separately by four different types of grid points $u_{2i,2j}, u_{2i,2j-1}, u_{2i-1,2j}, u_{2i-1,2j-1}$. Denoting
\[
\hat{U}_d = \left( Y^d, Z_1^d, Z_2^d, Z_3^d \right)^T,
\]
while $Y^d, Z_1^d, Z_2^d, Z_3^d$ are column vectors ordered in lexical order separately by four different types of grid points $y_{2i,2j}, z_{2i,2j-1}, z_{2i-1,2j}, z_{2i-1,2j-1}$. Obviously we have
\[
U_d = P_d \hat{U}_d, \quad \hat{U}_d = S_d U_d,
\]
where $P_d$ is a permutation matrix of order $(2^d N)^2$ which changes $\hat{U}_d$ to $U_d$ and $P_d$ has the following form
\[
P_d = \left[ E_d \otimes E_d, \ V_d \otimes E_d, \ E_d \otimes V_d, \ V_d \otimes V_d \right],
\]
That is
\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & 0 \\
E_d & 0 & \cdots & 0 & 0 \\
0 & E_d & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & E_d \\
0 & 0 & \cdots & 0 & \vdots \
\end{pmatrix}
\begin{pmatrix}
E_d & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & \vdots 
\end{pmatrix},
\]
with
\[
E_d = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & 0 \\
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 \\
\end{pmatrix},
V_d = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\end{pmatrix}.
\]
Here denote the zero matrix of the order $2^d N \times 2^d - 1 \times N$. $S_d$ is the transfer matrix of order $(2^d N)^2$ which transfers $\hat{U}_d$ to $U_d$. $S_d$ has the following form
\[
S_d = \begin{pmatrix}
I_{d-1} & \cdots & -I_{d-1} & \cdots & -I_{d-1} \\
I_{d-1} & \cdots & I_{d-1} & \cdots & 0 \\
I_{d-1} & \cdots & 0 & \cdots & I_{d-1} \\
I_{d-1} & \cdots & 0 & \cdots & 0 \\
\end{pmatrix},
\]
where the $I_{d-1}$ is the identity matrix of order $(2^{d-1} N)^2$. Substituting (8) into the finite difference equation (3) and multiplying the equation by $(P_d S_d)^T$, we find
\[
(P_d S_d)^T \frac{\partial P_d S_d \hat{U}_d}{\partial t} + \nu (P_d S_d)^T A_d P_d S_d \hat{U}_d + (P_d S_d)^T g(P_d S_d \hat{U}_d) = 0.
\]
(9)
Noting that $P_d^T P_d = I_d$, $P_d^T$ and $g$ can commute, we obtain
\[
S_d^T S_d \frac{\partial \hat{U}_d}{\partial t} + \nu (P_d S_d)^T A_d P_d S_d \hat{U}_d + S_d^T g(S_d \hat{U}_d) = 0,
\]
(10)
which is the 2-level WIU scheme. The next level of WIU on \( Y^d \) can be introduced by repeating the same procedure. We now separate \( Y^d \) into four parts and denote that

\[
\tilde{Y}^d = \begin{pmatrix}
Y^d-1 \\
Z_1^{d-1} \\
Z_2^{d-1} \\
Z_3^{d-1}
\end{pmatrix}
\]

where \( Y^d-1, Z_1^{d-1}, Z_2^{d-1}, Z_3^{d-1} \) are column vectors ordered in lexical order separately by \( Y^d_{4i,4j}, Z^d_{4i,4j-2}, Z^d_{4i-2,4j}, Z^d_{4i-2,4j-2} \) four different type of grid points. Similar to (5), for \( i = 1, \cdots, 2^{d-2}N, j = 1, \cdots, 2^{d-2}N, \) we define

\[
\begin{align*}
&y^d_{4i,4j} = y^d_{4i,4j} - z^d_{4i-2,4j} - z^d_{4i,4j-2} - z^d_{4i-2,4j-2}, \\
&y^d_{4i,4j-2} = y^d_{4i,4j} + z^d_{4i,4j-2}, \\
&y^d_{4i-2,4j} = y^d_{4i-2,4j} + z^d_{4i-2,4j}, \\
&y^d_{4i-2,4j-2} = y^d_{4i-2,4j} + z^d_{4i-2,4j-2}.
\end{align*}
\]

Therefore, we can obtain the equality

\[
Y^d = P_{d-1}S_{d-1}\tilde{Y}^d,
\]

with

\[
P_{d-1} = [E_{d-1} \otimes E_{d-1} V_{d-1} \otimes E_{d-1} E_{d-1} \otimes V_{d-1} V_{d-1} \otimes V_{d-1}]
\]

and

\[
S_{d-1} = \begin{pmatrix}
I_{d-2} & -I_{d-2} & -I_{d-2} & -I_{d-2} \\
I_{d-2} & I_{d-2} & 0 & 0 \\
I_{d-2} & 0 & I_{d-2} & 0 \\
I_{d-2} & 0 & 0 & I_{d-2}
\end{pmatrix}.
\]

Here, \( I_{d-2} \) denote the identity matrix of order \((2^{d-2}N)^2\). \( P_{d-1}, S_{d-1} \) have the similar structures as \( P_d, S_d \) respectively but they are both matrices of order \((2^{d-1}N)^2\).

Let

\[
\tilde{U}_{d-1} = \begin{pmatrix}
Y^d-1 \\
Z_1^{d-1} \\
Z_2^{d-1} \\
Z_3^{d-1}
\end{pmatrix}, \quad \tilde{P}_{d-1} = \begin{pmatrix}
P_{d-1} & 0 & 0 & 0 \\
0 & I_{d-1} & 0 & 0 \\
0 & 0 & I_{d-1} & 0 \\
0 & 0 & 0 & I_{d-1}
\end{pmatrix},
\]

and

\[
\tilde{S}_{d-1} = \begin{pmatrix}
S_{d-1} & 0 & 0 & 0 \\
0 & I_{d-1} & 0 & 0 \\
0 & 0 & I_{d-1} & 0 \\
0 & 0 & 0 & I_{d-1}
\end{pmatrix}.
\]

Noting that \( \tilde{P}_{d-1}, \tilde{S}_{d-1} \) are matrices of order \((2^dN)^2\), we can find

\[
\tilde{U}_d = \tilde{P}_{d-1}\tilde{S}_{d-1}\tilde{U}_{d-1}.
\]

Substituting (13) into (10) and multiplying the equation by \((\tilde{P}_{d-1}\tilde{S}_{d-1})^T\), we can see that

\[
S^T S \frac{\partial \tilde{U}_{d-1}}{\partial t} + \nu S^T A_d S \tilde{U}_{d-1} + S^T g(S\tilde{U}_{d-1}) = 0.
\]
Here, \( S = P_d S_d \tilde{P}_{d-1} \tilde{S}_{d-1} \). Generally, for \( l = d - 1, d - 2, \ldots, 1 \), we can introduce the next level of WIU on \( Y^{l+1} \) by repeating the same method. Let

\[
\tilde{Y}^{l+1} = \begin{pmatrix} Y^{l} \\ Z_1^{l} \\ Z_2^{l} \\ Z_3^{l} \end{pmatrix},
\]

(15)

We can see that

\[
Y^{l+1} = P_l S_l \tilde{Y}^{l+1},
\]

(16)

\( P_l, S_l \) are the matrices of order \((2^l N)^2\). Now, we can include (16) with \( l = d \) by let \( Y^{d+1} = U_d \), \( \tilde{Y}^{d+1} = \tilde{U}_d \). Setting

\[
\tilde{U}_l = \begin{pmatrix} Y^{l} \\ Z_1^l \\ Z_2^l \\ \vdots \\ Z_d^l \end{pmatrix} \quad \text{with} \quad Z^l = \begin{pmatrix} Z_1^l \\ Z_2^l \\ \vdots \\ Z_d^l \end{pmatrix}, \quad Z_d^l \in \mathbb{R}^{(2^l - 1) N^2},
\]

(17)

then we can write

\[
\tilde{U}_{l+1} = \tilde{P}_l \tilde{S}_l \tilde{U}_l,
\]

(18)

where

\[
\tilde{P}_l = \begin{pmatrix} P_l & 0 \\ 0 & I_k \end{pmatrix}, \quad \tilde{S}_l = \begin{pmatrix} S_l & 0 \\ 0 & I_k \end{pmatrix},
\]

with \( k = (2^d - N)^2 - (2^l - N)^2 \). Substituting (18) into (10) with \( l = d - 1, d - 2, \ldots, 1 \) and using the same method, we can obtain the \( d + 1 \)-level WIUs in terms of \( Y \) and \( Z \),

\[
S^T S \frac{\partial \tilde{U}_1}{\partial t} + \nu S^T A_d S \tilde{U}_1 + S^T g(S \tilde{U}_1) = 0.
\]

(19)

Here \( S = \tilde{P}_d \tilde{S}_d \tilde{P}_{d-1} \tilde{S}_{d-1} \ldots \tilde{P}_1 \tilde{S}_1, Y = Y^1 \in \mathbb{R}^{N^2}, Z = (Z^1, Z^2, \ldots, Z^d)^T \) with \( Z^l \in \mathbb{R}^{(2^l - 1) N^2} \).

4. Approximate Scheme. We now propose \( \theta \) scheme based on WIU introduced in section 3. Firstly, we find a method to compute the nonlinear term \( S_d^T g(S_d \tilde{U}_d) \). The equation (19) with \( S = \tilde{P}_d \tilde{S}_d \) has the form

\[
S_d^T S_d \frac{\partial \tilde{U}_d}{\partial t} + v(P_d S_d)^T A_d P_d S_d \tilde{U}_d + S_d^T g(S_d \tilde{U}_d) = 0.
\]

According to some computations, we get

\[
S_d^T g(S_d \tilde{U}_d) = \begin{pmatrix} 4g(Y^d) + O(|Z_1^d|^2) + O(|Z_2^d|^2) + O(|Z_3^d|^2) \\ O(|Z_1^d| + |Z_2^d| + |Z_3^d|) \\ O(|Z_1^d| + |Z_2^d| + |Z_3^d|) \\ O(|Z_1^d| + |Z_2^d| + |Z_3^d|) \end{pmatrix}.
\]

(20)
In fact, using the definition of \( S_d \), the left hand of equation (20) equals to

\[
\begin{pmatrix}
g(Y^d - Z_1^d - Z_2^d - Z_3^d) + g(Y^d + Z_1^d) + g(Y^d + Z_2^d) + g(Y^d + Z_3^d) \\
g(Y^d + Z_1^d) - g(Y^d - Z_1^d - Z_2^d - Z_3^d) \\
g(Y^d + Z_2^d) - g(Y^d - Z_1^d - Z_2^d - Z_3^d) \\
g(Y^d + Z_3^d) - g(Y^d - Z_1^d - Z_2^d - Z_3^d)
\end{pmatrix}.
\]

(21)

Using Tylor expansion at \( Y^d \) and neglecting the terms \( O(|Z_d|^2) \) and \( O(|Z_d|) \), we obtain

\[
S_d^T S_d \frac{\partial \hat{U}_d}{\partial t} + \nu S_d^T P_d A_d P_d S_d \hat{U}_d + 4 \left( \begin{array}{c} g(Y^d) \\ 0 \end{array} \right) = 0. \tag{22}
\]

For two level wave-like incremental unknowns, the form of the equation (19) with \( S = \tilde{P}_d \tilde{S}_d \tilde{P}_d^{-1} \tilde{S}_d^{-1} \) is

\[
S^T S \frac{\partial \hat{U}_d}{\partial t} + \nu S^T A_d S \hat{U}_d + \nu S^T g(S \hat{U}_d) = 0.
\]

Using the same approximation as (20), the approximate equation above becomes

\[
S^T S \frac{\partial \hat{U}_d}{\partial t} + \nu S^T A_d S \hat{U}_d + 4^2 \left( \begin{array}{c} g(Y^{d-1}) \\ 0 \end{array} \right) = 0. \tag{23}
\]

In fact,

\[
S^T g(S \hat{U}_d) = \tilde{S}_{d-1}^T \tilde{P}_{d-1}^T \tilde{P}_{d} \tilde{S}_{d} \tilde{P}_{d}^T \tilde{P}_{d} \tilde{S}_{d} \tilde{P}_{d} \hat{U}_d (\hat{P}_d \tilde{S}_d \hat{P}_d^{-1} \tilde{S}_d^{-1} \hat{U}_d - 1)
\]

(24)

Using the above approximation (20) and the definition of \( \hat{S}_{d-1}, \hat{P}_{d-1} \), we have

\[
S^T g(S \hat{U}_d) = 4 \left( \begin{array}{cc} S_{d-1}^T & 0 \\ 0 & I \end{array} \right) \left( \begin{array}{cc} P_{d-1}^T & 0 \\ 0 & I \end{array} \right) \left( \begin{array}{c} g(Y^d) \\ 0 \end{array} \right)
\]

\[
= 4 \left( \begin{array}{cc} S_{d-1}^T & 0 \\ 0 & I \end{array} \right) \left( \begin{array}{cc} P_{d-1}^T & 0 \\ 0 & I \end{array} \right) \left( \begin{array}{c} g(P_{d-1} S_{d-1} Y^d) \\ 0 \end{array} \right)
\]

\[
= 4 \left( \begin{array}{cc} S_{d-1}^T & 0 \\ 0 & I \end{array} \right) \left( \begin{array}{c} 0 \\ 0 \end{array} \right)
\]

Finally, the \((d + 1)\)-level incremental unknowns equation (19) is approximated by

\[
S^T S \frac{\partial}{\partial t} \left( \begin{array}{c} Y \\ Z \end{array} \right) + \nu S^T A_d S U_1 + 4^d \left( \begin{array}{c} g(Y) \\ 0 \end{array} \right) = 0. \tag{25}
\]

Now as for time discretization, we propose \( \theta \) scheme as follows

\[
\frac{S^T S}{\tau} \left( \begin{array}{c} Y^{n+1} - Y^n \\ Z^{n+1} - Z^n \end{array} \right) + \nu A^* \left[ (1 - \theta) \left( \begin{array}{c} Y^n \\ Z^n \end{array} \right) + \theta \left( \begin{array}{c} Y^{n+1} \\ Z^{n+1} \end{array} \right) \right] + 4^d \left( \begin{array}{c} g(Y^n) \\ 0 \end{array} \right) = 0. \tag{26}
\]

Here, \( A^* = S^T A_d S, \tau \) is the time step and \( \theta \in [0, 1] \) is a parameter.
5. Stability Analysis of $\theta$ scheme. In this section, firstly, we write the equivalent variational formulation of scheme (25), then under suitable conditions, we present the stability analysis of $\theta$ scheme (26). Let $V_{h_d}$ (or simply $V_d$) be the function space spanned by the basis functions $w_{h_d,M_{ij}}(x)$, Here $M_{ij} = (ih_d, jh_d), i = 1, 2, \cdots, 2^d N, j = 1, 2, \cdots, 2^d N,$ and $w_{h_d,M_{ij}}(x)$ satisfies
\[
 w_{h_d,M_{ij}} = \begin{cases} 
 1, & x \in K_{ij}, \\
 0, & \text{otherwise} 
\end{cases}
\]
where $K_{ij} = [ih_d, (i+1)h_d) \times [jh_d, (j+1)h_d)$, $u_d(x)$ be a step function in $V_d$, and
\[
u u_d(x) = \sum_{i=1}^{2^d N} \sum_{j=1}^{2^d N} u_d(M_{ij}) w_{h_d,M_{ij}}(x), \; x \in \Omega.
\]
We introduce two finite difference operators $\nabla_{1,h_d}$ and $\nabla_{2,h_d}$:
\[
\nabla_{1,h_d} \Phi(x) = \frac{1}{h_d} (\Phi(x + h_de_1) - \Phi(x)),
\]
\[
\nabla_{2,h_d} \Phi(x) = \frac{1}{h_d} (\Phi(x + h_de_2) - \Phi(x)),
\]
where $e_1 = (1,0), \; e_2 = (0,1)$. We define the following discrete scalar product
\[
((u_d,v_d))_{h_d} = (\nabla_{1,h_d} u_d, \nabla_{1,h_d} v_d) + (\nabla_{2,h_d} u_d, \nabla_{2,h_d} v_d),
\]
where $(\cdot, \cdot)$ is the scalar product in $L^2(\Omega)$. Let $\|\cdot\|_{h_d} = (\cdot, \cdot)_{h_d}^{-\frac{1}{2}}$ and observe that $\|\cdot\|_{h_d}$ and $|\cdot|$ are Hilbert norms on $V_d$.

With the help of step functions, we can write the finite difference discretization scheme (3) in a variational form
\[
\frac{\partial u_d}{\partial t} + \nu ((u_d, \tilde{u}))_{h_d} + (g(u_d), \tilde{u}) = 0, \; \forall \tilde{u} \in V_d. \tag{27}
\]
Scheme (3) can be recovered by choosing $\tilde{u} = w_{h_d,M_{ij}}(x)$. We now separate space $V_d$ into two spaces $\mathcal{Y}^d$ and $\mathcal{Z}^d$ according to the definition of wavelet-like incremental unknowns. Let $\mathcal{Y}^d$ as the space spanned by the basis functions $\Psi_{2h_d,M_{2i,2j}}(x), i = 1, 2, \cdots, 2^{d-1} N, \; j = 1, 2, \cdots, 2^{d-1} N,$ and
\[
\Psi_{2h_d,M_{2i,2j}}(x) = \begin{cases} 
 1, & x \in [(2i-1)h_d, (2i+1)h_d) \times [(2j-1)h_d, (2j+1)h_d), \\
 0, & \text{otherwise} 
\end{cases}
\]
Thus for every $y^d \in \mathcal{Y}^d$,
\[
y^d(x) = \sum_{i=1}^{2^{d-1} N} \sum_{j=1}^{2^{d-1} N} y^d(M_{2i,2j}) \Psi_{2h_d,M_{2i,2j}}(x), \; x \in \Omega.
\]
Let $\mathcal{Z}^d$ as the space spanned by the basis function $\chi_{h_d,A}(x)$, there are three kinds of points of $A$. For $i, j = 1, 2, \cdots, 2^{d-1} N,$
- $A_1 = (2ih_d - h_d, 2jh_d)$,
- $A_2 = (2ih_d + h_d, 2jh_d)$,
- $A_3 = (2ih_d, 2jh_d)$.
\]
\[
\chi_{h_d,A_1}(x) = \begin{cases} 
 1, & x \in [2ih_d - h_d, 2ih_d) \times [2jh_d, 2jh_d + h_d), \\
 -1, & x \in [2ih_d, 2ih_d + h_d) \times [2jh_d, 2jh_d + h_d), \\
 0, & \text{otherwise} 
\end{cases}
\]
\[
\chi_{h_d,A_2}(x) = \begin{cases} 
 1, & x \in [2ih_d - h_d, 2ih_d) \times [2jh_d, 2jh_d + h_d), \\
 -1, & x \in [2ih_d, 2ih_d + h_d) \times [2jh_d, 2jh_d + h_d), \\
 0, & \text{otherwise} 
\end{cases}
\]
\[
\chi_{h_d,A_3}(x) = \begin{cases} 
 1, & x \in [2ih_d - h_d, 2ih_d) \times [2jh_d, 2jh_d + h_d), \\
 -1, & x \in [2ih_d, 2ih_d + h_d) \times [2jh_d, 2jh_d + h_d), \\
 0, & \text{otherwise} 
\end{cases}
\]
Before presenting the stability theory, let us introduce some useful lemmas.

\[ \chi_{h,d,A_2}(x) = \begin{cases} 
1, & x \in [2ih_d, 2ih_d + h_d) \times [2jh_d - h_d, 2jh_d), \\
-1, & x \in [2ih_d, 2ih_d + h_d) \times [2jh_d, 2jh_d + h_d), \\
0, & \text{otherwise.}
\end{cases} \]

\[ \chi_{h,d,A_3}(x) = \begin{cases} 
1, & x \in [2ih_d - h_d, 2ih_d) \times [2jh_d - h_d, 2jh_d), \\
-1, & x \in [2ih_d, 2ih_d + h_d) \times [2jh_d, 2jh_d + h_d), \\
0, & \text{otherwise.}
\end{cases} \]

Therefore, for all \( z^d \in \mathbb{Z}^d \), \( x \in \Omega \), we have

\[ z^d(x) = \sum_{i=1}^{2^{d-1}N} \sum_{j=1}^{2^{d-1}N} (z^d(A_1)\chi_{h,d,A_1}(x) + z^d(A_2)\chi_{h,d,A_2}(x) + z^d(A_3)\chi_{h,d,A_3}(x)), \tag{28} \]

From the definition of \( \Psi_{2h,d,M}(x) \) and \( \chi_{h,d,A}(x) \), for all \( z^d \in \mathbb{Z}^d \), \( y^d \in \mathcal{Y}^d \), we can obtain three conclusions:

1. \( \int_{\Omega} z^d \, dx = 0 \),
2. \( \int_{\Omega} z^d y^d \, dx = 0 \),
3. The decomposition of \( V_d \) makes (5) hold.

Thus the space \( V_d \) can be decomposed as

\[ V_d = \mathcal{Y}^d \oplus \mathbb{Z}^d. \]

Obviously, for all \( u_d \in V_d \), we have

\[ u_d = y^d + z^d, \quad y^d \in \mathcal{Y}^d, \quad z^d \in \mathbb{Z}^d. \tag{29} \]

With above decomposition, we can see that the variational form

\[ \left( \frac{\partial y^d}{\partial t}, \tilde{y} \right) + \nu ((y^d + z^d, \tilde{y}))_{h,d} + (g(y^d), \tilde{y}) = 0, \quad \forall \tilde{y} \in \mathcal{Y}^d, \tag{30} \]

\[ \left( \frac{\partial z^d}{\partial t}, \tilde{z} \right) + \nu ((y^d + z^d, \tilde{z}))_{h,d} = 0, \quad \forall \tilde{z} \in \mathbb{Z}^d. \tag{31} \]

is identical to (22). Multilevel incremental unknowns can be recovered in a similar fashion, we decompose \( \mathcal{Y}^l \), \( l = d, ..., 1 \) into

\[ \mathcal{Y}^{l+1} = \mathcal{Y}^l \oplus \mathbb{Z}^l. \]

Remember that \( Y^{d+1} \in \mathcal{Y}^{d+1} = U_d \). Therefore for any function \( u_d \in V_d \), we can write it as

\[ u_d = y + z, \]

where \( y = y^1 \in \mathcal{Y} = \mathcal{Y}^1 \) and \( z \in \mathbb{Z} = \mathbb{Z}^1 \oplus \mathbb{Z}^2 \oplus \cdots \oplus \mathbb{Z}^d \). Using the above decomposition, we can prove that the following variational form is identical to (25).

\[ \begin{cases} 
(\frac{\partial y}{\partial t}, \tilde{y}) + \nu((y + z, \tilde{y}))_{h,d} + (g(y), \tilde{y}) = 0, \quad \forall \tilde{y} \in \mathcal{Y}, \\
(\frac{\partial z}{\partial t}, \tilde{z}) + \nu((y + z, \tilde{z}))_{h,d} = 0, \quad \forall \tilde{z} \in \mathbb{Z}. \tag{32} \end{cases} \]

Before presenting the stability theory, let us introduce some useful lemmas.
Lemma 5.1. For every function $u_h \in V_d$, we have

$$2|u_h| \leq \|u_h\|_{h_d} \leq \frac{1}{S_1(h_d)}|u_h|$$

where $S_1(h_d) = \frac{h_d}{\sqrt{2}}$.

Proof. Firstly, we prove the right hand side of (33). Due to the Dirichlet boundary condition, we agree $u_{\alpha \beta} = 0$ if $\alpha$ or $\beta = 0$ or $2N$. For every function $u_h \in V_h$, we have

$$\|u_h\|_{h_d}^2 = \sum_{\alpha=0}^{2^d N} \sum_{\beta=0}^{2^d N} (u_{\alpha+1,\beta} - u_{\alpha,\beta})^2 + \sum_{\alpha=1}^{2^d N} \sum_{\beta=0}^{2^d N} (u_{\alpha,\beta+1} - u_{\alpha,\beta})^2$$

$$\leq 2 \sum_{\alpha=0}^{2^d N} \sum_{\beta=0}^{2^d N} (u_{\alpha+1,\beta}^2 + u_{\alpha,\beta}^2) + 2 \sum_{\alpha=1}^{2^d N} \sum_{\beta=0}^{2^d N} (u_{\alpha,\beta+1}^2 + u_{\alpha,\beta}^2)$$

$$= 8 \sum_{\alpha=0}^{2^d N} \sum_{\beta=0}^{2^d N} u_{\alpha,\beta}^2$$

and

$$|u_h|^2 = h_d^2 \sum_{\alpha=0}^{2^d N} \sum_{\beta=0}^{2^d N} u_{\alpha,\beta}^2.$$

It is easy to see that

$$\|u_h\|_{h_d} \leq \frac{1}{S_1(h_d)}|u_h|,$$

with

$$S_1(h_d) = \frac{h_d}{\sqrt{2}}.$$

Secondly, we prove the left hand side of (33). Thanks to Cauchy’s inequality, we have

$$|u_h(i, j)|^2 = \sum_{k=0}^{i-1} \sum_{k=0}^{j-1} \left| \nabla_{1,h_d} u(k, j) h_d \right|^2 \leq \sum_{k=0}^{i-1} h_d \sum_{k=0}^{j-1} \left| \nabla_{1,h_d} u(k, j) \right|^2$$

$$= ih_d^2 \sum_{k=0}^{2^d N} \left| \nabla_{1,h_d} u(k, j) \right|^2.$$

$$|u_h(i, j)|^2 = \sum_{k=0}^{i-1} \sum_{k=0}^{j-1} \left| \nabla_{2,h_d} u(i, k) h_d \right|^2 \leq \sum_{k=0}^{i-1} h_d \sum_{k=0}^{j-1} \left| \nabla_{2,h_d} u(i, k) \right|^2$$

$$= jh_d^2 \sum_{k=0}^{2^d N} \left| \nabla_{2,h_d} u(i, k) \right|^2.$$

Then we obtain

$$|u_h(i, j)|^2 \leq \frac{1}{2} \left[ i h_d^2 \sum_{k=0}^{2^d N} \left| \nabla_{1,h_d} u(k, j) \right|^2 + j h_d^2 \sum_{k=0}^{2^d N} \left| \nabla_{2,h_d} u(i, k) \right|^2 \right].$$

(34)
Using the definition of $|\cdot|$ and (34), we have
\[
|u_h|^2 = h_d^2 \sum_{i=0}^{2^d N} \sum_{j=0}^{2^d N} |u(i, j)|^2 \\
\leq \frac{1}{2} \sum_{i=0}^{2^d N} \sum_{j=0}^{2^d N} h_d^2 |u(i, j)|^2 + j h_d^2 \sum_{k=0}^{2^d N} |\nabla_{1,h_d} u(k, j)|^2 \\
\leq \frac{1}{2} \sum_{i=0}^{2^d N} \sum_{j=0}^{2^d N} |u(i, j)|^2 h_d^2 \\
+ \sum_{j=0}^{2^d N} j h_d^2 \sum_{i=0}^{2^d N} |\nabla_{2,h_d} u(i, k)|^2 h_d^2 \\
\leq \frac{1}{2} \frac{2^d N(2^d N + 1)}{2} h_d^2 \sum_{j=0}^{2^d N} |\nabla_{1,h_d} u(k, j)|^2 h_d^2 \\
+ \frac{2^d N(2^d N + 1)}{2} h_d^2 \sum_{i=0}^{2^d N} |\nabla_{2,h_d} u(i, k)|^2 h_d^2 \\
\leq \frac{1}{4} \|u\|^2_{h_d}.
\]

Thus, the inequality (33) holds. \(\square\)

**Lemma 5.2.** Let $K_{ij}^1 = [ih_1, (i+1)h_1) \times [jh_1, (j+1)h_1)$, $M_{ij}^1 = (ih_1, jh_1)$, $\forall y \in \mathcal{Y}$ is a step function, we have the following equality
\[
\int_{K_{ij}^1} (y(x + h_1e_1) - y(x))^2 dx = 2^d \int_{K_{ij}^1} (y((i+1)h_1, jh_1) - y(ih_1, jh_1))^2 dx \\
\int_{K_{ij}^1} (y(x + h_2e_2) - y(x))^2 dx = 2^d \int_{K_{ij}^1} (y(ih_1, (j+1)h_1) - y(ih_1, jh_1))^2 dx
\]

\[\begin{array}{cccccccc}
\Omega_1 & \Omega_2 & \Omega_3 & \Omega_4 & \Omega_5 & \Omega_6 & \Omega_7 & \Omega_8 \\
\Omega_9 & \Omega_{10} & \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} & \Omega_{15} & \Omega_{16}
\end{array}\]

Fig. 2 Coarse grid points(×), finer grid points(○) and the finest grid points (⋄)

**Proof.** For simplicity, we only prove (35) when $d = 2$. From Fig. 2, we can see that if $d = 2$, the square $K_{ij}^1$ is divided into 16 small squares, we order them with
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\[ \Omega, \ i = 1, 2, \cdots, 16. \text{ Since } y(x) \text{ is a constant in } K_{ij}^1, \text{ we can find} \]

\[ \int_{\Omega_i} (y(x + h_1 e_1) - y(x))^2 \, dx = 0, \ i \neq 4, 8, 12, 16 \]

\[ \int_{\Omega_i} (y(x + h_2 e_2) - y(x))^2 \, dx = 0, \ i = 5, 6, \cdots, 16. \]

Thus

\[ \int_{K_{ij}^1} (y(x + h_1 e_1) - y(x))^2 \, dx \]

\[ = \sum_{i=4, 8, 12, 16} \int_{\Omega_i} (y(x + h_1 e_1) - y(x))^2 \, dx \]

\[ = \sum_{i=4, 8, 12, 16} \int_{\Omega_i} (y((i + 1)h_1, jh_1) - y(ih_1, jh_1))^2 \, dx \]

\[ = 4 \int_{\Omega} (y((i + 1)h_1, jh_1) - y(ih_1, jh_1))^2 \, dx \]

\[ \int_{K_{ij}^1} (y(x + h_2 e_2) - y(x))^2 \, dx \]

\[ = \sum_{i=1}^4 \int_{\Omega_i} (y(x + h_2 e_2) - y(x))^2 \, dx \]

\[ = \sum_{i=1}^4 \int_{\Omega_i} (y(ih_1, (j + 1)h_1) - y(ih_1, jh_1))^2 \, dx \]

\[ = 4 \int_{\Omega} (y(ih_1, (j + 1)h_1) - y(ih_1, jh_1))^2 \, dx \]

Here, \( \Omega \) denotes any small square in \( K_{ij}^1 \). The proof is completed. \( \square \)

**Lemma 5.3.** For every function \( y \in \mathcal{V} \),

\[ S_2(h_1) |y|_{\infty} \leq |y|, \quad S_2(h_1) = h_1, \]

\[ \overline{S_1}(h_1) \|y\|_{h_1} \leq |y|, \quad \overline{S_1}(h_1) = \frac{h_1}{\sqrt{8 \cdot 2^d}}. \]

Here \( |y|_{\infty} \) is the maximum norm of \( y \).

**Proof.** Let \( K_{ij}^1 = [ih_1, (i + 1)h_1] \times [jh_1, (j + 1)h_1], M_{ij}^1 = (ih_1, jh_1) \). Using the expression of \( y \), we have

\[ |y|^2 = \int_{\Omega} (\sum_{i,j=1}^N y(M_{ij}^1) \Psi_{h_1, M_{ij}^1})^2 \, dx \]

\[ = \sum_{i,j=1}^N \int_{K_{ij}^1} (\sum_{i,j=1}^N y(M_{ij}^1) \Psi_{h_1, M_{ij}^1})^2 \, dx \]

\[ = h_1^2 \sum_{i,j=1}^N y^2(M_{ij}^1) \geq h_1^2 |y|_{\infty}^2. \]
Then we prove (37). Using the definition of $\|\cdot\|_{h_d}$, $|\cdot|$, Lemma 5.2 and $(a - b)^2 \leq 2(a^2 + b^2)$, it suffices to prove that

$$\|\| y \|\|_{h_d}^2 = \int_{\Omega} \frac{y(x + h_d e_1) - y(x)}{h_d}^2 dx + \int_{\Omega} \frac{y(x + h_d e_2) - y(x)}{h_d}^2 dx$$

$$= \frac{1}{h_d^2} \sum_{i,j=1}^{N} \int_{K_{ij}} (y(x + h_d e_1) - y(x))^2 dx + \frac{1}{h_d^2} \sum_{i,j=1}^{N} \int_{K_{ij}} (y(x + h_d e_2) - y(x))^2 dx$$

$$= \frac{1}{h_d^2} \sum_{i,j=1}^{N} 2^d \int_{K_{ij}} (y((i + 1)h_1,jh_1) - y(ih_1,jh_1))^2 dx$$

$$+ \frac{1}{h_d^2} \sum_{i,j=1}^{N} 2^d \int_{K_{ij}} (y(ih_1,(j + 1)h_1) - y(ih_1,jh_1))^2 dx$$

$$\leq 2^d \sum_{i=1}^{N} \sum_{j=1}^{N} 2(y^2((i + 1)h_1,jh_1) + y^2(ih_1,jh_1))$$

$$+ 2^d \sum_{i,j=1}^{N} 2(y^2(ih_1,(j + 1)h_1) + y^2(ih_1,jh_1))$$

$$= 2^d \sum_{i,j=1}^{N} 8 y^2(ih_1,jh_1)$$

and

$$|y|^2 = h_d^2 \sum_{i=1}^{N} \sum_{j=1}^{N} y^2(ih_1,jh_1).$$

By simple computations, we prove (37) holds. 

\[\square\]

**Lemma 5.4.** For given time step $\tau = \frac{T}{K}$, if $c\tau < 1$, then for $\forall n \leq K$, we have

$$\left(\frac{1 + c\tau}{1 - c\tau}\right)^n < c^*, \quad c^* = e^{2cT}.$$

Now, we have the following theorem for $\theta$ scheme.

**Theorem 5.5.** Assuming that $\tau \leq \min\{K_0, \frac{1}{4\nu}\}$ for some fixed $K_0 > 0$ and $c_4, c_5, c_6$ are positive constants. Let

$$M_1(\theta) = \frac{2^d(q-2)}{2c_3^2(2-\theta)D(\theta)^{q-2}}$$

$$M_2(\theta) = \frac{1}{8\nu} \cdot \frac{(1 - 2\theta)^2}{(1 - \theta)^2 + 2 - 3\theta}$$

$$M_3(\theta) = \frac{1}{16\nu} \cdot \frac{(2\theta - 1)^2}{1 - \theta}.$$
where

\[
D(\theta) = \begin{cases} 
|u^0_{h_d}|^2 + \frac{c_4 + K_0c_5}{2\nu} |\Omega|, & \theta = 0, \\
|u^0_{h_d}|^2 + \frac{2c_4 + (2 - \theta)K_0c_5}{\nu(1 - \theta)(1 - 2\theta)} |\Omega|, & 0 < \theta < 0.5, \\
c^*(|y^0|^2 + |z^0|^2) + \frac{c^* - 1}{2c_6} (2c_4 + 1.5K_0c_5)\Omega, & \theta = 0.5, \\
|u^0_{h_d}|^2 + \frac{2c_4 + (2 - \theta)K_0c_5}{\nu\theta(2\theta - 1)} |\Omega|, & 0.5 < \theta < 1, \\
|u^0_{h_d}|^2 + \frac{2c_4 + K_0c_5}{8\nu} |\Omega|, & \theta = 1.
\end{cases}
\]

We define \(r(q) = \max\{\frac{x}{h_d^2}, \frac{1}{h_d^2}, \frac{x}{h_d^2}\}\), then the following estimate for any \(n \geq 0\),

\[
|u_n^0|^2 = |y^n|^2 + |z^n|^2 \leq D(\theta)
\]

holds, if

\[
\begin{cases} 
\frac{r(q)}{\min\{M_1(\theta), M_2(\theta)\}}, & \theta = 0, \\
\frac{r(q)}{\min\{M_1(\theta), M_2(\theta), \frac{1 - \theta}{8\theta}\}}, & 0 < \theta < 0.5, \\
\frac{r(q)}{\min\{M_1(\theta), \frac{2d_c}{32\nu^2}\}}, & \theta = 0.5, \\
\frac{r(q)}{\min\{M_1(\theta), M_2(\theta), \frac{\theta}{8(1 - \theta)}\}}, & 0.5 < \theta < 1, \\
\frac{r(q)}{\min\{M_1(\theta)\}}, & \theta = 1.
\end{cases}
\]

Proof. Firstly, We write the \(\theta\) scheme (26) in its variational form

\[
\begin{align*}
(y^{n+1} - y^n, \hat{y}) + \tau\nu(1 - \theta)((y^n + z^n, \hat{y}))_{h_d} + \tau\nu\theta((y^{n+1} + z^{n+1}, \hat{y}))_{h_d} + \tau\nu(g(y^n), \hat{y}) &= 0, \\
(z^{n+1} - z^n, \hat{z}) + \tau\nu(1 - \theta)((y^n + z^n, \hat{z}))_{h_d} + \tau\nu\theta((y^{n+1} + z^{n+1}, \hat{z}))_{h_d} + \tau\nu(g(y^n), y^n) &= 0,
\end{align*}
\]

Let \(\hat{y} = 2(1 - \theta)y^n + 2\theta y^{n+1}\) and \(\hat{z} = 2(1 - \theta)z^n + 2\theta z^{n+1}\), we have

\[
\begin{align*}
2(1 - \theta)(y^{n+1} - y^n, y^n) + 2\theta(y^{n+1} - y^n, y^{n+1}) + 2\tau\nu(1 - \theta)^2((y^n + z^n, y^n))_{h_d} + 2\tau\nu\theta(1 - \theta)((y^n + z^n, y^{n+1}))_{h_d} + 2\tau\nu\theta(1 - \theta)((y^{n+1} + z^{n+1}, y^{n+1}))_{h_d} + 2\tau\nu\theta^2((y^{n+1} + z^{n+1}, y^{n+1}))_{h_d} + 2\tau\nu(g(y^n), y^{n+1}) + \tau(1 - \theta)(g(y^n), y^n) &= 0,
\end{align*}
\]

and

\[
\begin{align*}
2(1 - \theta)(z^{n+1} - z^n, z^n) + 2\theta(z^{n+1} - z^n, z^{n+1}) + 2\tau\nu(1 - \theta)^2((y^n + z^n, z^n))_{h_d} + 2\tau\nu\theta(1 - \theta)((y^n + z^n, z^{n+1}))_{h_d} + 2\tau\nu\theta(1 - \theta)((y^{n+1} + z^{n+1}, z^{n+1}))_{h_d} + 2\tau\nu\theta^2((y^{n+1} + z^{n+1}, z^{n+1}))_{h_d} &= 0.
\end{align*}
\]

Adding (40) and (41) with

\[
2(a - b, b) = |a|^2 - |b|^2 - |a - b|^2, 2(a - b, a) = |a|^2 - |b|^2 + |a - b|^2,
\]

...
we have
\[ |y^{n+1}|^2 - |y^n|^2 + |z^{n+1}|^2 - |z^n|^2 + (2\theta - 1)|y^{n+1} - y^n|^2 \]
\[ + (2\theta - 1)|z^{n+1} - z^n|^2 + 2\nu(1 - \theta)^2\|y^n + z^n\|_{h_d}^2 + 2\nu\theta^2\|y^{n+1} + z^{n+1}\|_{h_d}^2 \]
\[ + 4\nu\theta(1 - \theta)(|y^n + z^n, y^{n+1} + z^{n+1}|)_{h_d} + 2\tau(1 - \theta)(g(y^n), y^n) \]
\[ + 2\tau(g(y^n), y^n) + 2\tau(g(y^n), y^{n+1} - y^n) = 0. \]

For simplicity, we denote
\[ \Gamma_0 = |y^{n+1}|^2 + |z^{n+1}|^2 - |y^n|^2 - |z^n|^2, \]
\[ \Gamma_1 = \tau\nu(1 - \theta)(2\theta - 1), \]
\[ \Gamma_2 = \tau\nu\theta(2\theta - 1). \]

Since
\[ 2(|y^n + z^n, y^{n+1} + z^{n+1}|)_{h_d} \leq 2\|y^n + z^n\|_{h_d}\|y^{n+1} + z^{n+1}\|_{h_d} \]
\[ \leq \|y^n + z^n\|_{h_d}^2 + \|y^{n+1} + z^{n+1}\|_{h_d}^2, \]
and
\[ (g(y^n), y^{n+1} - y^n) \leq |g(y^n)| \cdot |y^{n+1} - y^n|, \]
we get
\[
\begin{align*}
\Gamma_0 + & (2\theta - 1) \left( |y^{n+1} - y^n|^2 + |z^{n+1} - z^n|^2 \right) - 2\Gamma_1\|y^n + z^n\|_{h_d}^2 \quad (42) \\
+ & 2\Gamma_2\|y^{n+1} + z^{n+1}\|_{h_d}^2 - 2\tau\nu\|g(y^n)\| |y^{n+1} - y^n| + 2\tau(g(y^n), y^n) \leq 0.
\end{align*}
\]

• Firstly, we consider the case with \(0 \leq \theta < 1/2\). Since
\[ \tau|g(y^n)| |y^{n+1} - y^n| \leq \tau^2|g(y^n)|^2 + \frac{1}{4}|y^{n+1} - y^n|^2, \]
From (42),
\[
\begin{align*}
\Gamma_0 + & \frac{1}{2}(3\theta - 2)|y^{n+1} - y^n|^2 + (2\theta - 1)|z^{n+1} - z^n|^2 - 2\Gamma_1\|y^n + z^n\|_{h_d}^2 \quad (43) \\
+ & 2\Gamma_2\|y^{n+1} + z^{n+1}\|_{h_d}^2 - 2\theta\tau^2|g(y^n)|^2 + 2\tau(g(y^n), y^n) \leq 0.
\end{align*}
\]
Let \(\hat{y} = y^{n+1} - y^n\) in the first equality of (39), By using the Cauchy-Schwarz inequality and Lemma 5.3, we obtain
\[
|y^{n+1} - y^n|^2 \leq \tau\nu(1 - \theta)|y^n + z^n|_{h_d}|y^{n+1} - y^n|_{h_d} \]
\[ + \tau|g(y^n)| \cdot |y^{n+1} - y^n| + \tau\nu\|g(y^n)\| |y^{n+1} + z^{n+1}|_{h_d} |y^{n+1} - y^n|_{h_d}, \]
\[ \leq \frac{\tau\nu(1 - \theta)}{S_1(h_1)} \|y^n + z^n\|_{h_d}|y^{n+1} - y^n| + \tau|g(y^n)| \cdot |y^{n+1} - y^n| \]
\[ + \frac{\tau\nu\theta}{S_1(h_1)}|y^{n+1} + z^{n+1}|_{h_d} |y^{n+1} - y^n|, \]
\[ \leq \frac{\tau^2\nu^2\Gamma_3}{S_1(h_1)^2} + \frac{1}{2}|y^{n+1} - y^n|^2 + \tau^2|g(y^n)|^2, \]
That is
\[ |y^{n+1} - y^n|^2 \leq \frac{2\tau^2\nu^2\Gamma_3}{S_1(h_1)^2} + 2\tau^2|g(y^n)|^2 \quad (44) \]
where $\Gamma_3 = (1 - \theta)\|y^n + z^n\|^2_{h_d} + \theta\|y^{n+1} + z^{n+1}\|^2_{h_d}$. Replace $\overline{S}_1(h_1)$ with $h_1/\sqrt{8 \cdot 2^d}$, we have

$$|y^{n+1} - y^n|^2 \leq \frac{16\tau^2\nu^2}{2^d h_d^2} \Gamma_3 + 2\tau^2|g(y^n)|^2. \quad (45)$$

Let $\tilde{z} = z^{n+1} - z^n$ in the second equality of (39), we can get the following inequality by using the Cauchy-Schwarz inequality and Lemma 5.1

$$\|z^{n+1} - z^n\|^2 \leq \frac{\tau\nu(1 - \theta)}{S_1(h_d)} \|y^n + z^n\|_h\|z^{n+1} - z^n\| + \frac{\tau\nu\theta}{S_1(h_d)} \|y^{n+1} + z^{n+1}\|_h\|z^{n+1} - z^n\|,
\leq \frac{\tau^2\nu^2\Gamma_3}{2S_1(h_d)^2} + \frac{1}{2}\|z^{n+1} - z^n\|^2.$$

Resolving the above inequality, we have

$$|z^{n+1} - z^n|^2 \leq \frac{\tau^2\nu^2\Gamma_3}{S_1(h_d)^2} = \frac{8\tau^2\nu^2\Gamma_3}{h_d^2}.$$

Thus, we have from (43),

$$\Gamma_0 + (-2\Gamma_1 + \frac{8\tau\nu\Gamma_1}{h_d^2} + \frac{8\tau^2\nu^2(1 - \theta)(3\theta - 2)}{2^d h_d^2})\|y^n + z^n\|^2_{h_d}
\leq \frac{8\tau^2\nu^2\Gamma_3}{2S_1(h_d)^2} + \tau^2\|g(y^n)|^2 + 2\tau|g(y^n), y^n| \leq 0.$$

Due to assumption H2, we can prove that exist constants $c_4 > 0, c_5 > 0$ such that the function $g$ above satisfies:

$$ug(u) \geq \frac{1}{2}c_1|u|^q - c_4, \quad (46)$$
$$g(u)^2 \leq 2c_3^2|u|^{2q-2} + c_5.$$

Using Lemma 5.3, we have

$$(g(y^n), y^n) = \int_\Omega g(y^n)y^n \, dx \geq \frac{1}{2}c_1 \int_\Omega |y^n|^q \, dx - c_4|\Omega|, \quad (47)$$

and

$$\tau^2|g(y^n)|^2 \leq 2\tau^2c_3^2 \int_\Omega |y^n|^q \, dx + \tau^2c_5|\Omega| 
\leq 2\tau^2c_3^2 \int_\Omega |y^n|^{q-2} \, dx + \tau^2c_5|\Omega| 
\leq \frac{2\tau^2c_3^2}{2^{(q-2)h_d^{d-2}}} \int_\Omega |y^n|^{q-2} \, dx + \tau^2c_5|\Omega|. \quad (48)$$

According to (47), (48) and $\tau < M_2(\theta)h_d^2$ in the first condition of (38), we have

$$\Gamma_0 - \Gamma_1\|y^n + z^n\|^2_{h_d} + 3\Gamma_2\|y^{n+1} + z^{n+1}\|^2_{h_d} + \Gamma_5(\theta) \int_\Omega |y^n|^q \, dx \leq \Gamma_4(\theta).$$

with

$$\Gamma_4(\theta) = (2\tau c_4 + (2 - \theta)\tau^2 c_5)|\Omega|.$$
we obtain that

\[ \Gamma_5(\theta) = \tau c_1 - (2 - \theta) \frac{2\tau^2 c_3^2}{[2^d h_4]^{q-2}} |y^n|^q. \]

Using Lemma 5.1, the above inequality becomes

\[ \Gamma_0 - 4\Gamma_1 |y^n + z|^2 + \frac{24}{h_4^2} \Gamma_2 |y^{n+1} + z^{n+1}|^2 + \Gamma_5(\theta) \int_\Omega |y^n|^q \, dx \leq \Gamma_4(\theta). \] (49)

(1) \( \theta = 0 \). Obviously, \( \Gamma_1 < 0, \Gamma_2 = 0 \).

From the inequality (49), We get

\[ \Gamma_0 - 4\Gamma_1 |y^n + z|^2 + \Gamma_5(0) \int_\Omega |y^n|^q \, dx \leq \Gamma_4(0). \] (50)

By simple computations, from the first inequality of the condition (38), we get

\[ |u_{h_4}^n|^2 = (1 + 4\Gamma_1)^{n+1} (|y^0|^2 + |z|^2) - \frac{\Gamma_4(0)}{4\Gamma_1} \leq D(0). \]

(2) \( 0 < \theta < 1/2 \). we prove the estimate by strong math induction:

- For \( k = 0 \) is obvious since \( |y^0|^2 + |z|^2 \leq D(\theta) \).
- Assuming the conclusion is correct up to \( k = n \).
- For \( k = n + 1 \): Using the second condition of (38) \( \frac{1}{h_4^2} \leq \frac{1 - \theta}{8\eta} \) and assumption

\[ |y^n|^2 + |z|^2 \leq D(\theta), \]

we have

\[ |y^{n+1}|^2 + |z^{n+1}|^2 - |y^n|^2 - |z|^2 - 4\Gamma_1 |y^n + z|^2 + 3\Gamma_1 |y^{n+1} + z^{n+1}|^2 \leq \Gamma_4(\theta). \]

That is

\[ |y^{n+1}|^2 + |z^{n+1}|^2 \leq \left( \frac{1 + 4\Gamma_1}{1 + 3\Gamma_1} \right)^{n+1} (|y^0|^2 + |z|^2) + \frac{\Gamma_4(\theta)}{1 + 3\Gamma_1} \sum_{i=0}^{n} \left( \frac{1 + 4\Gamma_1}{1 + 3\Gamma_1} \right)^i \]
\[ \leq \left( \frac{1 + 4\Gamma_1}{1 + 3\Gamma_1} \right)^{n+1} (|y^0|^2 + |z|^2) - \frac{\Gamma_4(\theta)}{\Gamma_1} \]
\[ \leq |y^0|^2 + |z|^2 + \frac{2c_4 + (2 - \theta) K_0 c_5}{\nu (1 - \theta) (1 - 2\theta)} |\Omega|. \]

Therefore, we obtain the estimate

\[ |u_{h_4}^n|^2 = |y^n|^2 + |z|^2 \leq D(\theta). \]

- Secondly we consider \( \theta = 0.5 \).

Noting that when \( \theta = 0.5, \Gamma_1 = 0, \Gamma_2 = 0 \). From (42), we get

\[ \Gamma_0 - 2\tau |g(y^n)| |y^{n+1} - y^n| + 2\tau (g(y^n), y^n) \leq 0. \]

since

\[ 2\tau |g(y^n)| |y^{n+1} - y^n| \leq \tau^2 |g(y^n)|^2 + |y^{n+1} - y^n|^2, \]

and

\[ |y^{n+1} - y^n|^2 \leq \frac{16\tau^2 \nu^2}{2^d h_4^2} \Gamma_3 + 2\tau^2 |g(y^n)|^2. \]

we obtain that

\[ \Gamma_0 - \frac{4\tau^2 \nu^2}{2^d h_4^2} \left( \|y^n + z\|_{h_4}^2 + \|y^{n+1} + z^{n+1}\|_{h_4}^2 \right) \geq \frac{3}{2} \tau^2 |g(y^n)|^2 + 2\tau (g(y^n), y^n) \leq 0. \] (51)
Due to (47), (48) and Lemma 5.1, we obtain
\[
\Gamma_0 - \frac{32\tau^2\nu^2}{2^d h_d^4} \left[ |y^n + z^n|^2 + |y^{n+1} + z^{n+1}|^2 \right] + \Gamma_5(0.5) \leq \Gamma_4(0.5).
\]  
(52)

Using the third condition of (38), we get
\[
\Gamma_0 - \frac{32\tau^2\nu^2}{2^d h_d^4} \left[ |y^n + z^n|^2 + |y^{n+1} + z^{n+1}|^2 \right] \leq \Gamma_4(0.5).
\]  
(53)

That is
\[
|y^{n+1}|^2 + |z^{n+1}|^2 \leq \frac{1 + \frac{32\tau^2\nu^2}{2^d h_d^4} (|y^n|^2 + |z^n|^2)}{1 - \frac{32\tau^2\nu^2}{2^d h_d^4}} \Gamma_4(0.5).
\]  
(54)

Then using the third condition of (38), we get
\[
|y^{n+1}|^2 + |z^{n+1}|^2 \leq \frac{1 + c_6 \tau}{1 - c_6 \tau} (|y^n|^2 + |z^n|^2) + \frac{\Gamma_4(0.5)}{1 - c_6 \tau}.
\]  
(55)

Using Lemma 5.4, we get
\[
|y^{n+1}|^2 + |z^{n+1}|^2 \leq c^* (|y^0|^2 + |z^0|^2) + \tilde{c}
\]  
(56)

where \(\tilde{c} = \frac{c^* - 1}{2c_6} (2c_4 + 1.5 K_0 c_5) \omega\).

- Thirdly we consider the other case with \(\frac{1}{2} < \theta \leq 1\).

Since \((2\theta - 1)|z^{n+1} - z^n|^2 > 0\) and
\[
2\tau |g(y^n)| \left| y^{n+1} - y^n \right| \leq \tau^2 |g(y^n)|^2 + |y^{n+1} - y^n|^2,
\]
From (42), we get the following inequality:
\[
\Gamma_0 - 2\Gamma_1 \|y^n + z^n\|_{h_d}^2 + 2\Gamma_2 \|y^{n+1} + z^{n+1}\|_{h_d}^2
\]
\[-(1 - \theta) |y^{n+1} - y^n|^2 - \theta \tau^2 |g(y^n)|^2 + 2\tau (g(y^n), y^n) \leq 0.
\]  
(57)

Here we neglect the term \((2\theta - 1)|z^{n+1} - z^n|^2\). Due to (45), we obtain
\[
\Gamma_0 - 2\Gamma_1 \|y^n + z^n\|_{h_d}^2 + 2\Gamma_2 \|y^{n+1} + z^{n+1}\|_{h_d}^2
\]
\[-(1 - \theta) \frac{16\tau^2\nu^2}{2^d h_d^4} (1 - \theta) \|y^n + z^n\|_{h_d}^2 + \frac{16\tau^2\nu^2}{2^d h_d^4} \theta \|y^{n+1} + z^{n+1}\|_{h_d}^2 \]
\[-(2 - \theta) \tau^2 |g(y^n)|^2 + 2\tau (g(y^n), y^n) \leq 0.
\]  
(58)

By some computations, (58) becomes
\[
\Gamma_0 - 2\Gamma_1 \|y^n + z^n\|_{h_d}^2 + 2\Gamma_2 \|y^{n+1} + z^{n+1}\|_{h_d}^2
\]
\[-(2 - \theta) \tau^2 |g(y^n)|^2 + 2\tau (g(y^n), y^n) \leq 0.
\]  
(59)

Because of \(\tau \leq M_3(\theta) h_d^2\), we have \(\frac{8\tau \nu (1 - \theta)}{2^d h_d^4 (2\theta - 1)} < \frac{1}{2}\), (59) becomes
\[
\Gamma_0 - 3\Gamma_1 \|y^n + z^n\|_{h_d}^2 + \Gamma_2 \|y^{n+1} + z^{n+1}\|_{h_d}^2 - (2 - \theta) \tau^2 |g(y^n)|^2 + 2\tau (g(y^n), y^n) \leq 0.
\]
(60)

Due to Lemma 5.1, we have
\[
\Gamma_0 - 2\Gamma_1 \|y^n + z^n\|_{h_d}^2 + 4\Gamma_2 \|y^{n+1} + z^{n+1}\|_{h_d}^2 + \Gamma_5(\theta) \int_{\Omega} |y^n|^q dx \leq \Gamma_4(\theta),
\]  
(60)

(2) \(\frac{1}{2} < \theta < 1\). we prove the estimate by strong math induction:

- \(k = 0\) is obvious since \(|y^0|^2 + |z^0|^2 \leq D(\theta)\).

- Assuming the conclusion is correct up to \(k = n\).
\begin{itemize}
\item For \(k = n+1\): from (60), using the fourth inequality \(\frac{\tau}{h_d^2} \leq \frac{\theta}{8(1-\theta)}\) in condition (38), and the assumption \(|y^n|^2 + |z^n|^2 \leq D(\theta)|, we have

\[|y^{n+1}|^2 + |z^{n+1}|^2 - |y^n|^2 - |z^n|^2 - 3\Gamma_2|y^n + z^n|^2 + 4\Gamma_2|y^{n+1} + z^{n+1}|^2 \leq \Gamma_4(\theta).\]

That is

\[|y^{n+1}|^2 + |z^{n+1}|^2 \leq \frac{1 + 3\Gamma_2}{1 + 4\Gamma_2}(|y^n|^2 + |z^n|^2) + \frac{\Gamma_4(\theta)}{1 + 4\Gamma_2},\]

\[\leq |y^0|^2 + |z^0|^2 + \frac{2c_4 + (2-\theta)K_0c_5}{\nu(2\theta-1)}|\Omega|.\]

Therefore, we obtain the estimate

\[|u^n_{h_d}|^2 = |y^n|^2 + |z^n|^2 \leq D(\theta).\]

(3) \(\theta = 1\). Obviously, \(\Gamma_1 = 0, \Gamma_2 = \tau \nu\).

Inequality (60) becomes

\[\Gamma_0 + 8\tau \nu|y^{n+1} + z^{n+1}|^2 + \Gamma_5(1)\int_\Omega |y^n|^q dx \leq \Gamma_4(1),\]

(61)

Using the condition the last inequality of (38), we get

\[|y^{n+1}|^2 + |z^{n+1}|^2 \leq \frac{1}{1 + 8\tau \nu}(|y^n|^2 + |z^n|^2) + \frac{1}{1 + 8\tau \nu}\Gamma_4(1).\]

By simple computations, we can prove that

\[|u^n_{h_d}|^2 = |y^0|^2 + |z^0|^2 + \frac{2c_4 + K_0c_5}{8\nu}|\Omega| = D(1).\]

\[\square\]

**Remark 1.** When \(\theta = 0\), the scheme (26) is an explicit scheme, From Theorem 5.5, the stability condition of the explicit scheme is

\[\frac{\tau}{h_d^2} \leq \frac{1}{8\nu} \cdot \frac{2^d}{2 + 2^d}\]

and

\[\frac{\tau}{h_d^{q-2}} \leq \frac{2^{d(q-2)}}{4c_3^2 \cdot D(0)^{q-2}}\]

Using the same method to the classical explicit approximation scheme of (27) (i.e two levels in time, one level in space), the stability condition is

\[\frac{\tau}{h_d^2} \leq \frac{1}{40\nu}\]

and

\[\frac{\tau}{h_d^{q-2}} \leq \frac{1}{4c_3^2 \cdot D(0)^{q-2}}\]

we can see that when level \(d > 1\) and the nonlinear effect is strong, the stability condition of the scheme (26) based on WIU is better, the time step can be taken about \(2^{d(q-2)}\) larger than the step size if we deal with \(u_{h_d}\) directly.

When \(\theta = 1\), the scheme (26) is an implicit scheme. Compared with the classical scheme, the time step in implicit scheme can be also taken \(2^{d(q-2)}\) larger than the normal step size.
6. **Numerical illustration.** we consider the following equation:

\[
\begin{align*}
\frac{\partial u}{\partial t} - \nu \Delta u + u &= 0, \\
u(x, y, t) &= 0, \\
(u(x, y, 0) = \sin(\pi x)\sin(\pi y), (x, y) &\in \Omega, \ t = 0.
\end{align*}
\]

with \( \Omega = [0, 1]^2 \), \( \nu = 1/(\pi^2) \), \( T=1 \). It can be check that the exact solution of this equation is \( u = \sin(\pi x)\sin(\pi y)e^{-3t} \). We can verify the above equation satisfies the assumptions (H1)-(H4). We solve the problem with explicit scheme \((\theta = 0)\), Crank-Nicolson scheme \((\theta = 0.5)\) and implicit scheme \((\theta = 1)\) respectively with different \( \tau \), \( d \) and \( N \), let \( M_1 \) represent WIU method, \( M_2 \) represent classical finite difference method without WIU. \( \|\text{error}\| \) denote the 2-norm of the difference between the exact solution and approximate solution, the comparison of the numerical results is shown in Tables 1-3. From those results, we can see that the precision of two methods are nearly the same, but the scheme using the WIU method can increase the speed of solving equation, save more time than the scheme without using the WIU.

| Parameters | Methods | \( M_1 \) | \( M_2 \) |
|------------|----------|----------|----------|
|            | CPU | \|error\| | CPU | \|error\| |
| \( \tau = 0.002, d = 1, N = 15 \) | 1.1719 | 1.2e-4 | 1.6094 | 4e-4 |
| \( \tau = 0.001, d = 1, N = 18 \) | 6.4844 | 1e-5 | 8.2500 | 2e-4 |
| \( \tau = 0.001, d = 1, N = 20 \) | 7.9688 | 4e-5 | 11.4063 | 2e-4 |
| \( \tau = 0.001, d = 2, N = 10 \) | 8.0313 | 6e-4 | 10.6250 | 2e-4 |
| \( \tau = 0.0005, d = 2, N = 15 \) | 73.1250 | 3e-4 | 99.9844 | 1e-4 |

Table 2. Comparison of CPU time and error with different \( d \) and \( N \) when \( \theta = 0.5 \)

| Parameters | Methods | \( M_1 \) | \( M_2 \) |
|------------|----------|----------|----------|
|            | CPU | \|error\| | CPU | \|error\| |
| \( \tau = 0.005, d = 1, N = 15 \) | 1.0938 | 5e-5 | 3.0156 | 6e-4 |
| \( \tau = 0.005, d = 1, N = 18 \) | 1.6094 | 1.5e-4 | 6.7031 | 4e-4 |
| \( \tau = 0.005, d = 1, N = 20 \) | 1.9531 | 4e-4 | 8.9650 | 4e-4 |
| \( \tau = 0.005, d = 2, N = 10 \) | 2.5608 | 4e-4 | 9.3750 | 6e-4 |
| \( \tau = 0.005, d = 2, N = 12 \) | 4.9688 | 3e-4 | 23.8964 | 5e-4 |

7. **Concluding remarks.** In this paper, for a class of porous medium diffusion-type equations, we present a \( \theta \) scheme based on wave-like incremental unknowns(WIU). We prove the stability of the \( \theta \) scheme and show that the stability condition is better when compared with the classical scheme. Since WIU enjoys the \( L^2 \) orthogonality property between different levels of unknowns, thus it deserves special attention. We can apply WIU to different kinds of equations and construct new schemes.
Table 3. Comparison of CPU time and error with different $d$ and $N$ when $\theta = 1$

| Parameters | Methods | $M_1$ | $M_2$ |
|------------|---------|-------|-------|
| $\tau = 0.005, d = 1, N = 15$ | CPU | $1.1875$ | $3.1875$ |
| | $\|error\|$ | $8e-4$ | $6e-4$ |
| $\tau = 0.005, d = 1, N = 18$ | CPU | $1.719$ | $6.3594$ |
| | $\|error\|$ | $8e-4$ | $6e-4$ |
| $\tau = 0.005, d = 1, N = 20$ | CPU | $2.1875$ | $9.3750$ |
| | $\|error\|$ | $6e-4$ | $6e-4$ |
| $\tau = 0.005, d = 2, N = 10$ | CPU | $3.4375$ | $9.5196$ |
| | $\|error\|$ | $1.2e-4$ | $6e-4$ |
| $\tau = 0.005, d = 2, N = 12$ | CPU | $5.2675$ | $21.6094$ |
| | $\|error\|$ | $4e-4$ | $4e-4$ |

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