Adaptive Temporal Difference Learning with Linear Function Approximation

Tao Sun, Han Shen, Tianyi Chen, and Dongsheng Li

Abstract—This paper revisits the temporal difference (TD) learning algorithm for the policy evaluation tasks in reinforcement learning. Typically, the performance of TD(0) and TD(λ) is very sensitive to the choice of stepsizes. Oftentimes, TD(0) suffers from slow convergence. Motivated by the tight link between the TD(0) learning algorithm and the stochastic gradient methods, we develop a provably convergent adaptive projected variant of the TD(0) learning algorithm with linear function approximation that we term AdaTD(0). In contrast to the TD(0), AdaTD(0) is robust or less sensitive to the choice of stepsizes. Analytically, we establish that to reach an ε-accuracy, the number of iterations needed is $O(\epsilon^{-2} \ln^2 1/\epsilon)$ in the general case, where $\rho$ represents the speed of the underlying Markov chain converges to the stationary distribution. This implies that the iteration complexity of AdaTD(0) is no worse than that of TD(0) in the worst case. When the stochastic semi-gradients are sparse, we provide theoretical acceleration of AdaTD(0). Going beyond TD(0), we develop an adaptive variant of TD(λ), which is referred to as AdaTD(λ). Empirically, we evaluate the performance of AdaTD(0) and AdaTD(λ) on several standard reinforcement learning tasks, which demonstrate the effectiveness of our new approaches.

Index Terms—Temporal Difference, Linear Function Approximation, Adaptive Step Size, MDP, Finite-Time Convergence

1 INTRODUCTION

Reinforcement learning (RL) involves a sequential decision-making procedure, where an agent takes (possibly randomized) actions in a stochastic environment over a sequence of time steps, and aims to maximize the long-term cumulative rewards received from the interacting environment. Owing to its generality, RL has been widely studied in many areas, such as control theory, game theory, operations research, multi-agent systems [1]. Temporal Difference (TD) learning is one of the most commonly used algorithms for policy evaluation in RL [2]. TD learning provides an iterative procedure to estimate the value function with respect to a given policy based on samples from a Markov chain. The classical TD algorithm adopts a tabular representation for the value function, which stores value estimates on a per state basis. In large-scale settings, the tabular-based TD learning algorithm can become intractable due to the increased number of states, and thus function approximation techniques are often combined with TD for better scalability and efficiency [3], [4].

The idea of TD learning with function approximation is essentially to parameterize the value function with a linear or nonlinear combination of fixed basis functions induced by the states that are termed feature vectors, and estimates the combination parameters in the same spirit of the tabular TD learning. Similar to all other parametric stochastic optimization algorithms, however, the performance of the TD learning algorithm with function approximation is very sensitive to the choice of stepsizes. Oftentimes, it suffers from slow convergence [5]. Ad-hoc adaptive modification of TD with function approximation has often been used empirically, but their convergence behavior and rate have not been fully understood. When implementing TD-learning, many practitioners use the adaptive optimizer but without theoretical guarantees. This paper is devoted to the development of a provably convergent adaptive algorithm to accelerate the TD(0) and TD(λ) algorithms. The key difficulty here is that the update used in the original TD does not follow the (stochastic) gradient direction of any objective function in an optimization problem, which prevents the use of the popular gradient-based optimization machinery. And the Markovian sampling protocol naturally involved in the TD update further complicates the analysis of adaptive and accelerated optimization algorithms.

1.1 Related works

We first briefly review related works in both the areas of TD learning and adaptive stochastic gradient.

Temporal difference learning. The great empirical success of TD [2] motivated active studies on the theoretical foundation of TD. The first convergence analysis of TD was given by [6] using stochastic approximation techniques. In [4], the characterization of limit points in TD with linear function approximation has been studied, giving new intuition about the dynamics of TD learning. The ODE-based method (e.g., [2]) has dramatically inspired the subsequent development of research on asymptotic convergence of TD. Early convergence results of TD learning were mostly asymptotic, e.g., [3], because the TD update does not follow
the (stochastic) gradient direction of any fixed objective function. Non-asymptotic analysis for the gradient TD — a variant of the original TD has been first studied by [9], in which the authors reformulate the original problem as new primal-dual saddle point optimization. The finite-time analysis of TD with linear function approximation under i.i.d observation has been studied in [10]. In particular, it is assumed that observations in each iteration of TD are independently drawn from the steady-state distribution. In a concurrent line of research, TD has been considered in the view of the stochastic linear system, whose improved results are given by [11]. Even without any fixed objective function to optimize, the proofs of [10], [11] still follow a Lyapunov analysis like the SGD due to the i.i.d sampling assumption and quadratic functions structure. Nevertheless, a more realistic assumption for the data sampling in TD is the Markov rather than the i.i.d process. The finite-time convergence analysis under Markov sampling is first presented in [12], whose results are based on the controls of the gradient basis [Lemma 9, [12]] and a coupling [Lemma 11, [12]]. The finite-time analysis for stochastic linear system under the Markov sampling is established by [13], [14], which is a general formulate of TD and applies potentially to other problems. The finite-time analysis of multi-agent TD is proved by [15]. However, all the aforementioned work leverages the original TD update. In [16], the authors proposed an improvement of Q-learning, which can be used to TD, but with only asymptotic analysis being provided. An adaptive variant of two time-scale stochastic approximation was introduced by [17], that can also be applied to TD. In [18], the authors introduce the momentum techniques for reinforcement learning and an extension on DQN. In [19], the authors propose an adaptive scaling mechanism for TRPO and show that it is the “RL version” of traditional trust-region methods from convex analysis.

Adaptive stochastic gradient descent. In machine learning areas different but related to RL, adaptive stochastic gradient descent methods have been actively studied. The first adaptive gradient (AdaGrad) is proposed by [20], [21], and the algorithm demonstrated impressive numerical results when the gradients are sparse. While the original AdaGrad has a performance guarantee only in the convex case, the nonconvex AdaGrad has been studied by [22]. Besides the convex results, sharp analysis for nonconvex AdaGrad has also been investigated in [23]. Variants of AdaGrad have been developed in [24], [25], which use alternative updating schemes (the exponential moving average schemes) rather than the average of the square of the past iterate. The momentum technique applied to the adaptive stochastic algorithms gives birth to Adam and Nadam [26], [27]. However, in [28], the authors demonstrate that Adam may diverge under certain circumstances, and provide a new convergent Adam algorithm called AMSGrad. Another method given by [29] is the use of decreasing factors for moving the average of the square of the past iterates. In [30], the convergence for generic Adam-type algorithms has been studied, which contains various adaptive methods.

1.2 Comparison with existing analysis

Our analysis considers the Markov sampling setting and thus differs [10], [11]. It is worth mentioning that the stepsizes chosen in this paper are different from that of [17]: in [17], the “adaptive” means that the learning rate is reduced by multiplying a preset factor when transient error is dominated by the steady-state error; while in our paper, the “adaptive” inherits the notion from Ada training method, i.e., using the learning rate associated with the past gradients. Our analysis cannot directly follow the techniques given by [12], [13], [14], [16] since the learning rate in our algorithm is statistically dependent on past information, which breaks previous Lyapunov analysis.

Thus, this paper uses a delayed expectation technique rather than bounding a coupling [Lemma 11, [12]]. Furthermore, our analysis needs to deal with several terms related to adaptive style learning rates, which are quite complicated but absent in [12]. Since the adaptive learning rate consists of past iterates instead of being preset, these terms do not enjoy simple explicit bounds. To this end, in this paper, we develop novel techniques (Lemmas 5 and 7). On the other hand, our analysis is also different from the adaptive SGD, because TD update is not the stochastic gradient of any objective function; it uses biased samples generated from the Markov chain.

1.3 Our contributions

Complementary to existing theoretical RL efforts, we propose the first provably convergent adaptive projected variant of the TD learning algorithm with linear function approximation that has finite-time convergence guarantees. For completeness of our analytical results, we investigate both the TD(0) algorithm as well as the TD(λ) algorithm. In a nutshell, our contributions are summarized in threefold:

1. We develop the adaptive variants of the TD(0) and TD(λ) algorithms with linear function approximation. The new algorithms AdaTD(0) and AdaTD(λ) are simple to use.
2. We establish the finite-time convergence guarantees of AdaTD(0) and AdaTD(λ), and they are not worse than those of TD and TD(λ) algorithms in the worst case.
3. We test our AdaTD(0) and AdaTD(λ) on several standard RL benchmarks and show how these compare favorably to existing alternatives like TD(0), TD(λ), etc.

2 Preliminaries

This section introduces the notation, assumptions about the underlying MDP, and the setting of TD learning with linear function approximation.

Notation: The coordinate j of a vector x is denoted by xj and xT is transpose of x. We use E[·] to denote the expectation with respect to the underlying probability space, and 1 1 for ℓ2 norm. Given a constant R > 0 and y ∈ Rd, ProjR(y) denotes the projection of y to the ball {x ∈ Rd | ||x|| ≤ R}. For a matrix A ∈ R d×d, ProjA(y) denotes the projection to space {Ax | x ∈ Rd}. We denote the sub-algebra as σk := σ(θ0, θ1, . . . , θk), where θk is the kth iterate. We use Ω(b) and Θ(b) to hide the logarithmic factor of b.
denotes the action space, $\mathcal{P}$ represents the transition matrix, $\mathcal{R}$ is the reward function, and $0 < \gamma < 1$ is the discount factor. In this case, let $\mathcal{P}(s'|s)$ denote the transition probability from state $s$ to state $s'$. The corresponding transition reward is $\mathcal{R}(s, s')$. We consider the finite-state case, i.e., $S$ consists of $|S|$ elements, and a stochastic policy $\mu : S \rightarrow A$ that specifies an action given the current state $s$. We use the following two assumptions on the stationary distribution and the reward.

**Assumption 1.** The transition rewards are uniformly bounded, i.e.,

$$|\mathcal{R}(s, s')| \leq B, \forall s, s' \in S.$$ 

**Assumption 2.** For any two states $s, s' \in S$, it holds that

$$\pi(s') = \lim_{t \to \infty} \mathcal{P}(s_t = s'|s_0 = s) > 0.$$ 

There exist $\bar{\kappa} > 0$ and $0 \leq \rho < 1$ such that

$$\sup_{s \in S} \left\{ \sum_{s' \in S} |\mathcal{P}(s_{t+1} = s' \mid s_t = s) - \pi(s')| \right\} \leq \bar{\kappa} \rho^t.$$ 

Assumptions 1 and 2 are standard in MDP. For irreducible and aperiodic Markov chains, Assumption 2 can always hold [31]. The constant $\rho$ represents the Markov chain’s speed accessing the stationary distribution $\pi$. When the number of states is finite, the Markovian transition kernel is a matrix $\mathcal{P}$, and $\rho$ is identical to the second largest eigenvalue of $\mathcal{P}$. An important notion in the Markov chain is the mixing time, which measures the time that a Markov chain needs for its current state distribution roughly matches the stationary one $\pi$. Given an $\epsilon > 0$, the mixing time is defined as

$$\tau(\epsilon) := \min_{t \geq 1} \left\{ \bar{\kappa} \rho^t \leq \epsilon \right\}.$$ 

With Assumption 2, we can see $\tau(\epsilon) = O(\ln 1/\epsilon / \ln 1/\rho)$. That means if $\rho$ is small, the mixing time is short.

This paper considers the on-policy setting, where both the target and behavior policies are $\mu$. For a given policy $\mu$, since the actions or the distribution of actions will be uniquely determined, we thus eliminate the dependence on the action in the rest of the paper. We denote the expected reward at a given state $s$ by

$$\mathcal{R}(s) := \sum_{s' \in S} \mathcal{P}(s' \mid s) \mathcal{R}(s, s').$$ 

The value function $V_\mu : S \rightarrow \mathbb{R}$ associated with a policy $\mu$ is the expected cumulative discounted reward from a given state $s$, that is

$$V_\mu(s) = \mathbb{E} \left[ \sum_{t=0}^{\infty} \gamma^t \mathcal{R}(s_t) \mid s_0 = s \right],$$ 

where the expectation is taken over the trajectory of states generated under $\mathcal{P}$ and $\mu$. The restriction on discount $0 < \gamma < 1$ can guarantee the boundedness of $V_\mu(s)$. The Markovian property of MDP yields the well-known Bellman equation

$$T_\mu V_\mu = V_\mu,$$ 

where the operator $T_\mu$ on $V$ is defined as

$$(T_\mu V)(s) := \mathcal{R}(s) + \gamma \sum_{s' \in S} \mathcal{P}(s' \mid s) V(s'), \ s \in S.$$ 

Solving the (linear) Bellman equation allows us to find the value function $V_\mu$ induced by a given policy $\mu$. However, in practice, $|S|$ is usually very large and computationally intractable. An alternative method is to leverage the linear [11] or non-linear function approximations (e.g., kernels and neural networks [32]). We focus on the linear case here, that is

$$V_\mu(s) \approx V_\theta(s) := \phi(s)^\top \theta,$$ 

where $\phi(s) \in \mathbb{R}^d$ is the feature vector for state $s$, and $\theta \in \mathbb{R}^d$ is a parameter vector. To reduce difficulty caused by the dimension, $d$ is set smaller than $|S|$. With the linear function approximator, the vector $V_\theta \in \mathbb{R}^{|S|}$ becomes

$$V_\theta = \Phi \theta,$$ 

where the feature matrix is defined as

$$\Phi := [\phi(s_1), \phi(s_2), \ldots, \phi(s_{|S|})]^\top \in \mathbb{R}^{|S| \times d}$$ 

with $s_n$ being the $n$th state.

**Assumption 3.** For any state $s \in S$, we assume the feature vector is uniformly bounded such that $\|\phi(s)\| \leq 1$, and the feature matrix $\Phi$ is full column-rank.

It is not hard to guarantee Assumption 3 since the feature map $\phi$ is chosen by the users and $|S| > d$. With Assumptions 2 and 3, we can see that the matrix $\Phi^\top \text{Diag}(\gamma) \Phi$ is positive define, and we denote its minimal eigenvalue as follows

$$\omega := \lambda_{\min}(\Phi^\top \text{Diag}(\gamma) \Phi) > 0.$$ 

With the linear approximation of value function, the task then is tantamount to finding $\theta \in \mathbb{R}^d$ that obeys the Bellman equation given by

$$\Phi \theta = T_\mu \Phi \theta.$$ 

However, $\theta$ that satisfies such an equation may not exist if $V_\mu \not\in \{\Phi \theta \mid \theta \in \mathbb{R}^d\}$. Instead, there always exists a unique solution $\theta^*$ for the projected Bellman equation [4], given by

$$\Phi \theta^* = \text{Proj}_\Phi(T_\mu \Phi \theta^*),$$ 

where $\text{Proj}_\Phi$ is the projection onto the span of $\Phi$’s columns.

### 3 Adaptive Temporal Difference Learning

#### 3.1 TD with linear function approximation

TD(0) algorithm starts with an initial parameter $\theta^0$. At iteration $k$, after sampling states $s_k, s_{k+1}$, and reward $r(s_{k+1}, s_k)$ from a Markov chain, we can compute the TD (temporal difference) error which is also called the Bellman error:

$$d_k := r(s_{k+1}, s_k) + \gamma V_\theta(s_{k+1}) - V_\theta(s_k).$$ 

The TD error is subsequently used to compute the stochastic semi-gradient:

$$g(\theta^k; s_k, s_{k+1}) := d_k \nabla V_{\theta^k}(s_k) = d_k \phi(s_k).$$ 

The traditional TD(0) with linear function approximation performs SGD-like update as

$$\theta^{k+1} = \theta^k + \eta g(\theta^k; s_k, s_{k+1}).$$
The update TD(0) makes sense because the direction \( g(\theta; s_k, s_{k+1}) \) is a good one since it is asymptotically close to the direction whose limit point is \( \theta^* \). Specifically, it has been established that \[ \lim_{k \to \infty} E[ g(\theta; s_k, s_{k+1}) ] = g(\theta), \] where \( g(\theta) \) is defined as
\[ g(\theta) := \Phi^T \text{Diag}(\pi) (\hat{T}_\pi \Phi \theta - \Phi \theta). \]
We term \( g(\theta) \) as the limiting update direction, ensuring that \( g(\theta^*) = 0 \). Note that while \( g(\theta; s_k, s_{k+1}) \) is an unbiased estimate under the stationary \( \pi \), it is not for a finite \( k \) due to the Markovian property of \( s_k \). Therefore, the TD(0) update is asymptotically akin to the stochastic approximation
\[ \theta^{k+1} = \theta^k + \eta G(\theta^k; s_k, s_{k+1}), \] where \( s, s' \) are independently drawn from the stationary distribution \( \pi \).

Nevertheless, an important property of the limiting direction \( g(\theta) \), found by [14], is that: for any \( \theta \in \mathbb{R}^d \), we have
\[ \langle \theta^* - \theta, g(\theta) \rangle \geq (1 - \gamma) \omega \| \theta^* - \theta \|^2. \] (8)
An important observation follows from this inequality readily: only one \( \theta^* \) satisfies \( g(\theta^*) = 0 \). We can show this by contradiction. If there exists another \( \bar{\theta} \) such that \( g(\bar{\theta}) = 0 \), we have \( 0 = \langle \theta^* - \bar{\theta}, g(\bar{\theta}) \rangle \geq (1 - \gamma) \omega \| \theta^* - \bar{\theta} \|^2 \), which again means \( \theta^* = \bar{\theta} \).

To ensure the boundedness of \( \theta^k \) and simplify the convergence analysis, projection is used in [7] (see e.g., [12]). In [12], it has been shown that if \( R \geq 2B/\sqrt{(1 - \gamma)^2} \) (\( B \) has appeared in Assumption 1), the projected TD(0) does not exclude all the limit points of the TD(0) (such a fact still holds for our proposed algorithms). The finite-time convergence of projected TD(0) is analyzed by [12]. The projection step is removed in [13], with almost the same results being proved. But in [13], the authors just studied the constant stepsize, while [12] shows more cases, including diminishing stepsize cases. In this paper, we consider a more complicated scheme that cannot be analyzed using techniques in [13]. Thus, the projection is still needed.

### 3.2 Adaptive TD development

Motivated by the recent success of adaptive SGD methods such as [20], [24], [28], this paper aims to develop an adaptive version of TD with linear function approximation that we term AdaTD(0) (adaptive TD). Unlike TD(0) method, in which step size is often a constant, AdaTD(0) seeks to scale the step size depending on the norm of stochastic gradient. Additionally, instead of using a one-step gradient, AdaTD(0) utilizes momentum, which gives the algorithm a memory of history information.

As presented in the last section, \( \bar{g}(\theta; s_k, s_{k+1}) \) is a stochastic estimate of \( g(\theta) \). Based on this observation, we develop the adaptive scheme for TD(0). Different from projected TD(0), we use the update direction \( m^k \), which is the exponentially weighted average of stochastic gradients. It is further scaled by \( v^k \), moving the average of the squared norm of stochastic semi-gradients. Intuitively, when the gradient is large, the algorithm will take smaller steps. To prevent a too large scaling, we use a positive hyper-parameter \( \delta \) for numerical stability. The AdaTD(0) update is given by [7]. The key difference between AdaTD(0) and the TD(0) method is that AdaTD(0) utilizes the history information in the update of both first moment estimate \( m^k \) and second moment estimate \( v^k \). Unlike projected TD(0) whose asymptotically expected TD update, \( g(\theta^*) \), is a good direction as is evident from [8], the update direction \( m^k \) in AdaTD(0) is an exponentially weighted version of \( g(\theta^*; s_k, s_{k+1}) \), which makes our analysis more challenging.

Although the variance term in AdaTD(0) is used as a sum form, it can be rewritten as an exponentially weighted moving average. If we denote \( \hat{v}^k := \sum_{i=1}^k \| G_i \|^2 = v^k \), the last two steps of AdaTD(0) can be reformulated as
\[ \theta^k = (1 - \frac{1}{k}) \hat{v}^{k-1} + \frac{1}{k} \theta^k, \]
\[ \theta^{k+1} = \text{Proj}_{B}(\theta^k + \eta m^k / (\hat{v}^k + \delta)^{1/2}), \] Note that in this form, weights are \( \{1/k\}_{k \geq 1} \) and step-sizes are \( \{\eta/k\}_{k \geq 1} \), which obey the sufficient conditions to guarantee the convergence of Adam-type algorithms with exponentially weighted average variance term [29], [30].

#### 3.3 Finite-time analysis of projected AdaTD(0)

Because the main results depend on constants related to the bounds, we present them in Lemma 1.

**Lemma 1.** The following bounds hold for \( (\theta^k)_{k \geq 0} \) generated by AdaTD(0)
\[ ||\theta^k - \theta^*|| \leq 2R, \quad ||g^k|| \leq G, \quad ||m^k|| \leq G \] (11)
where we define \( g^k := \bar{g}(\theta^k; s_k, s_{k+1}) \), and \( G := 2R + B \).

Lemma 1 follows readily. The bounds presented in Lemma 1 are critical for the subsequent analysis.

The convergence analysis of AdaTD(0) is more challenging than that of both TD and adaptive SGD. Compared with the analysis of adaptive SGD methods in e.g., [20], [24], [28], even under the i.i.d. sampling, the stochastic direction \( \bar{g}(\theta^*; s_k, s_{k+1}) \) used in [7] fails to be the stochastic gradient of any objective function, aside from the fact that samples are drawn from a Markov chain. Compared with TD(0), the actual update of \( \theta^* \) in AdaTD(0) involves the history information of both the first and the second moments of \( g(\theta^*; s_k, s_{k+1}) \), which makes the analysis of TD in e.g., [12], [13] intractable.
Algorithm 2 Projected Adaptive TD(0) Learning

1: Parameters: $\eta, \beta, \gamma, R$.
2: Initialization: $g^0 = 0$, $m^0 = 0$, $v^0 = 0$
3: for $k = 1, 2, \ldots$ do
4: sample a state transition $s_k \rightarrow s_{k+1}$ from $\mu$
5: calculate $\nabla f(s_k; s_{k+1})$ in (5)
6: update the parameter $\theta^k$ as
   \[ m^k = \beta m^{k-1} + (1 - \beta) \nabla f(s_k; s_{k+1}), \]
   \[ v^k = v^{k-1} + \beta \| \nabla f(s_k; s_{k+1}) \|^2, \]
   \[ \theta^{k+1} = \text{Proj}_R(\theta^k + \eta m^k / \sqrt{v^k + \delta}). \]
7: end for

**Theorem 1.** Suppose $(\theta^k)_{k \geq 0}$ are generated by AdaTD(0) with $R \geq 2B / \sqrt{\omega(1 - \gamma)}$ under the Markovian observation. Given $\eta > 0, \delta > 0, 0 \leq \beta < 1$, we have
   \[
   \min_{1 \leq k \leq K} \mathbb{E}(\| \theta^k - \theta^* \|^2) \leq \left[ C_1 \log \left( \frac{\delta + KG^2}{\delta} \right) \right] / \sqrt{K} + C_2 / \sqrt{K},
   \]
   where $C_1$ and $C_2$ are given as
   \[
   C_1 := \frac{16 \log K / \ln \delta}{(1 - \gamma) \omega^2} + \frac{2\eta \beta G}{(1 - \beta)(1 - \gamma) \omega} + \frac{\eta G}{(1 - \gamma) \omega} + \frac{4RG^2}{(1 - \gamma) \omega^2} = O\left( \left( \log K / \ln \frac{1}{\rho} \right)^2 \right),
   \]
   \[
   C_2 := \frac{4RG^2}{(1 - \gamma) \omega^2} + \frac{4RG^2}{(1 - \gamma) \omega^2} + 4R \left( B + 2G + \frac{R}{\omega} \right) \leq O\left( \frac{1}{\sqrt{\delta}} \right).
   \]

With Theorem 1, to achieve $\epsilon$-accuracy for $\min_{1 \leq k \leq K} \mathbb{E}(\| \theta^k - \theta^* \|^2)$, we need
   \[
   C_2 / \sqrt{K} \leq \frac{\epsilon}{2}, \quad C_1 \log \left( \frac{\delta + KG^2}{\delta} \right) / \sqrt{K} \leq \frac{\epsilon}{2}. \tag{12}
   \]

Since $C_2 = O(1)$, with the first inequality of (12), $K = O\left( \frac{\epsilon}{\delta} \right)$. Further, due to $C_1 = O\left( \log K / \ln \frac{1}{\rho} \right)^2 = O\left( \left( \frac{1}{\sqrt{\delta}} \ln \frac{1}{\delta} \right)^2 \right)$, with the second inequality of (12), we have $C_1 \log K / \sqrt{K} = O\left( C_2 / \sqrt{K} \right) = O(\epsilon)$. Therefore, we obtain a solution whose square distance to $\theta^*$ is $\epsilon$, the iteration needed is
   \[
   \hat{O}\left( C_2 / \epsilon^2 \right) = \hat{O}\left( \ln^4 \frac{1}{\epsilon^2} / \epsilon^2 \ln^4 \frac{1}{\rho} \right). \tag{13}
   \]

When $\rho$ is much smaller than 1, the term $\ln \frac{1}{\rho} / \ln \frac{1}{\rho}$ keeps at a relatively small level. Recall that the state-of-the-art convergence result of TD(0) given in (12) is $O\left( \ln^2 \frac{1}{\epsilon^2} / (\epsilon^2 \ln^2 \frac{1}{\rho}) \right)$, the rate of AdaTD(0) is close to TD(0) in the general case. We do not present a faster speed for technical reasons. In fact, this is also the case for the adaptive SGD [23], [24].

Specifically, although numerical results demonstrate the advantage of adaptive methods, the worst-case convergence rate of adaptive methods is still similar to that of the stochastic gradient descent method. To reach a desired error as $\min_{1 \leq k \leq K} \mathbb{E}(\| \nabla f(x^k) \|^2) \leq \epsilon$ in adaptive SGD, where $x^k$ is the $k$th iterate and $\epsilon > 0$, the iteration number $K$ needs to be set as $O\left( \frac{1}{\epsilon^2} \right)$, which is identical to the SGD.

**Algorithm 3** Projected Adaptive TD($\lambda$) Learning

1: parameters: $\eta, \beta, \gamma, R$.
2: initialization: $g^0 = 0$, $x^0 = 0$, $m^0 = 0$, $v^0 = 0$
3: for $k = 1, 2, \ldots$ do
4: sample a state transition $s_k \rightarrow s_{k+1}$ from $\mu$
5: update $x^k = (\gamma) x^{k-1} + \sigma(s_k)$
6: update $g^k(\theta^k; s_k, s_{k+1}) := r(s_{k+1}; s_k) x^k$
7: update $\theta^k = \text{Proj}_R(\theta^k + \eta g^k / \sqrt{v^k + \delta}).$
8: end for

**Sketch of the proofs:** Now, we present the sketch of the proofs for the main result. Because AdaTD(0) does not have any objective function to optimize, we consider sequence of the $(\| \theta^k - \theta^* \|^2)_{k \geq 0}$.

Using the update [9], we have
   \[
   \| \theta^k - \theta^{k+1} \|^2 - \| \theta^k - \theta^* \|^2 \\
   \leq 2\eta(m_k^k, \theta^k - \theta^*) + \frac{\eta^2}{2} m_k^k || \| m_k^k ||^2 \\
   \leq 2\eta(\theta^k - \theta^*, m_k^k)/(v_k^k + \delta)^{1/2} + \frac{\eta^2}{2} m_k^k || \| m_k^k ||^2 / (v_k^k + \delta) \\
   \leq 2\gamma^k - \theta^*, m_k^k)/(v_k^k + \delta)^{1/2} - 1/(v_k^k + \delta)^{1/2},
   \]

Here, we split the term $\langle m_k^k, \theta^k - \theta^* \rangle / (v_k^k + \delta)^{1/2}$ as $I_1^k + I_3^k$ because both $m_k^k$ and $v_k^k$ statistically depend on $g^k(\theta^k; s_k, s_{k+1})$, which makes it hard to calculate the expectation. Subtracting $-I_1^k$ to both sides of (13), we get
   \[
   -I_1^k \leq \| \theta^* - \theta^k \|^2 - \| \theta^* - \theta^{k+1} \|^2 + I_2^k + I_3^k. \tag{15}
   \]

With division (15), the analysis can be divided into three parts: A) determine the lower bound of $-E[I_1^k]$ (i.e., the upper bound of $E[I_1^k]$); B) prove the summability of $(E[I_2^k])_{k \geq 1}$, i.e., $\sum_{k=1}^{+\infty} I_2^k < +\infty$; C) prove the summability of $(E[I_3^k])_{k \geq 1}$.

A) Calculating the upper bound of $E[I_1^k]$ is the most difficult part, which consists of two substeps. Since $m_k^k$ is a convex combination of $m_k^{k-1}$ and $g^k(\theta^k; s_k, s_{k+1})$. We recursively bound $E[I_1^k]$ by analyzing $E[\| \theta^k - \theta^* - g^k(\theta^k; s_k, s_{k+1})/2\|]$. Assume that the Markov chain has reached its stationary distribution, in which, $E[\| \theta^k - \theta^* - g^k(\theta^k; s_k, s_{k+1})/2\|] = g$. But the stationary distribution is not assumed in our setting. Therefore, we need to bound the difference between $E[\| \theta^k - \theta^* - g^k(\theta^k; s_k, s_{k+1})/2\|]$. Unlike i.i.d setting, directly bounding them is difficult in the Markovian setting since it will lead to some uncontrollable error in the final convergence result. To solve this technical issue, we consider $E[\| \theta^k - \theta^* - g^k(\theta^k; s_k, s_{k+1})/2\|]$. This is because although $s_k$ is biased, $\mathbb{P}(s_k|s_k-T = s)$ is very close to $\pi(s)$ when $T$ is large. Using this technique, we prove the following Lemma.

**Lemma 2.** Assume $(\theta^k)_{k \geq 0}$ are generated by AdaTD(0).

Given an integer $K_0 \in \mathbb{Z}^+$, we have
   \[
   \left\| \mathbb{E}\left[ g^k(\theta^k - K_0; s_k, s_{k+1}) \right] - g^k(\theta^k - K_0) \right\| \leq K_0 \rho^{K_0}, \tag{16}
   \]
where the constant is defined as $κ := \bar{κ}(B + Rγ + R)$. In the second substep, because we have got the biased bound caused by the Markovian stochastic process, it is possible to bound the difference $E[(θ^k − θ∗, g(θ^k; s_k, s_{k+1}))/((v^{k-1} + δ)^{1/2})]$ between $E[(θ^k − θ∗, g(θ^k))/((v^{k-1} + δ)^{1/2})]$ with the following lemma.

**Lemma 3.** Given $K_0 \in \mathbb{Z}^+$, we have
\[
E[(θ^k − θ∗, g(θ^k; s_k, s_{k+1}))/((v^{k-1} + δ)^{1/2})] \\
\leq \frac{1}{2} E[(θ^k − θ∗, g(θ^k))/((v^{k-1} + δ)^{1/2})] + \frac{RκρK_0}{\sqrt{δ}} \\
+ \frac{8K_0}{δ^{1/2}(1 − γ)} \sum_{k = K_0}^∞ E[∥m_k - k∥^2/((v^{k-1} + δ)^{1/2})]. (17)
\]

**B** The second part is to bound $\sum_k E[∥m_k - k∥^2/((v^{k-1} + δ)^{1/2})]$. Because $v^k$ depends on $∥g∥^2 \geq 0 (g(θ^k; s_k, s_{k+1}))$, we expand $∥m_k - k∥^2$ by $∥g∥^2 \geq 0 (g(θ^k; s_k, s_{k+1}))$, in Lemma 3 in the Appendix. We then apply a provable result (Lemma 3 in the Appendix) to the right side of (18) and get (19).

**Lemma 4.** Let $(m_k)_{k \geq 0}$ be defined in (26), we have
\[
\sum_{k = 1}^K m_k \leq \sum_{j = 1}^{K-1} E[∥g∥^2/((v^j + δ)^{1/2})]. (18)
\]

Further, with the boundedness of $∥g∥ \geq 0$, we then get
\[
\sum_{k = 1}^K m_k \leq \ln((δ + (K - 1)G^2)/δ). (19)
\]

**C** While the third part is the easiest and obvious. The boundedness of the points indicates the uniform bound of $∥θ^k − θ∗∥$. The monotonicity of $v^k$ then yields the summable bound.

Summing (15) from $k = 1$ to $K$, with the proved bounds in these three parts, we then can bound $\sum_{k=1}^K (θ^k − θ∗, g(θ^k))/((v^k + δ)^{1/2})$. Once with the descent property $(θ^k − θ∗, g(θ^k)) \geq (1 − γ)ω∥θ^k − θ∗∥^2$, we can derive the main convergence result.

The acceleration of adaptive SGD is proved under extra assumptions like the sparsity of the stochastic gradients. In the following, we present the acceleration result of AdaTD(0) also with an extra assumption.

**Theorem 2.** Suppose the conditions of Theorem 1 hold and $v^k \leq cK^0$, (20)

where $c > 0$ is a universal constant and $0 < ϵ < 1$. To achieve $ε$-accuracy for $\min_{1 \leq k \leq K} E[∥θ^k − θ∗∥^2]$, the needed iteration is $O \left( \frac{1}{(ln \frac{1}{1 − ϵ})^4} \right)$. When $ε \ll ln \frac{1}{1 − ϵ}$ and $0 < ϵ < 1$, the result in Theorem 2 significantly improves the speed of TD. When $ν = 1$, the complexity is the same as [13].

We explain a little about assumption (20). Note that the boundedness of the sequence $(θ^k)_{k \geq 0}$ together with Assumptions 1 and 3 directly yields $v^k \leq O(k)$ (i.e., $ν = 1$). In fact, assumption (20) is very standard for analyzing adaptive stochastic optimization [33, 20, 28, 34, 30, 35]. As far as we know, there is no very good explanation of the superiority of the adaptive SGD rather than assumption (20); it is widely used in previous works because many training tasks enjoy sparse stochastic gradients.

In our AdaTD algorithm, we have
\[
g^k = r(s_{k+1}, s_k) + γVθ(s_{k+1}) - Vθ(s_k)\phi(s_k),
\]
which keeps the sparsity of $\phi(s_k)$ because $r(s_{k+1}, s_k) + γVθ(s_{k+1}) - Vθ(s_k) \in \mathbb{R}$. Then if the features are sparse, the stochastic semi-gradients are also sparse. In RL, a class of methods, called as state-aggregation based approaches [36, 37, 38, 39, 40, 41], usually uses sparse features to approximate Bellman function equation. The main idea of state-aggregation is dividing the whole state space into a few mutually disjoint clusters (i.e., $S = \bigcup_i^d S_i$ and $S_i \bigcap S_j = \emptyset$ if $i \neq j$) and regards each cluster as a meta-state. Compared with the vanilla MDP with linear approximation, the state-aggregation based one employs a structured feature matrix $\tilde{Φ} := [\phi(s_1), \phi(s_2), \ldots, \phi(s_{|S|})]^T \in \mathbb{R}^{|S| \times d}$, where $\phi(s_i)[j] = 1$ if $s_i \in S_j$ and $\phi(s_i)[j] = 0$ if $s_i \notin S_j$. We can see that $\phi(s_i)$ has only one non-zero element, which is very sparse. And then, the stochastic semi-gradients in the AdaTD used to the state-aggregation MDP are very sparse.

State aggregation is a degenerated form of linear representations. More general is the tile coding, which still retains the sparsity structure [42].

4 EXTENSION TO PROJECTED ADAPTIVE TD(λ)

This section contains the adaptive TD(λ) algorithm and its finite-time convergence analysis.

4.1 Algorithm development

Using existing analysis of TD(λ) [11, 4], if $V_{μ}$ solves the Bellman equation (1), it also solves $\nabla \mu = V_{μ} \in \mathbb{Z}^+$, where $\nabla \mu$ denotes the $m$-step of $\nabla \mu$. In this case, we can also represent $V_{μ}$ as
\[
V_{μ}(s) = E[\sum_{t=0}^n γ^t R(s_t) + γ^{m+1}V_{μ}(s_{m+1}) | s_0 = s].
\]
Given $λ \in [0, 1]$ and $V_{μ}(s) = (1 − λ) \sum_{m=0}^∞ λ^m V_{μ}(s)$, $∀ s \in S$, the $λ$-averaged Bellman operator is given by
\[
(\nabla \mu)^λ \in S = (1 − λ) \sum_{m=0}^∞ λ^m \sum_{t=0}^m γ^t R(s_t) + γ^{m+1}V_{μ}(s_{m+1}) | s_0 = s \right].
\]
Comparing operator $\nabla \mu$ and the $λ$-averaged Bellman operator, it is clear that $\nabla \mu = \nabla \mu$.

By defining
\[
g^λ(θ) = Φ^T \text{Diag}(π)(\nabla \mu^λ \Phi θ - Φ θ),
\]
the stochastic update in TD(λ) can be presented as (10). Similar to TD(0), it has been established in [4] and [43] that
\[
\lim_{k \to ∞} E[g^λ(θ; s_k, s_{k+1}, z^k)] = g^λ(θ).
\]
Like the limiting update direction \( \mathbf{g}(\theta) \) in TD(0), a critical property of the update direction in TD(\( \lambda \)) is given by
\[
(\theta^* - \theta, \mathbf{g}(\theta)) \geq (1 - \alpha)\omega(\theta^* - \theta^*^2),
\]
where \( \alpha := \gamma(1 - \lambda)/(1 - \gamma \lambda) \) for any \( \theta \in \mathbb{R}^d \). By denoting \( z^{\infty} := \sum_{s=0}^{\infty} (\gamma^*)^s \mathbf{s} \), where \( (\mathbf{s}_1, \mathbf{s}_2, \ldots) \) is the stationary sequence. Then, it also holds
\[
\mathbb{E}[\mathbf{g}(\theta; \mathbf{s}_k, \mathbf{s}_{k+1}, z^{\infty})] = \mathbf{g}(\theta).
\]
We present AdaTD(\( \lambda \)) in Algorithm 3 Adadelta(\( \lambda \)) and AdaTD(0) differ in a different update protocol. We directly have the following bounds for AdaTD(\( \lambda \))
\[
\|\theta^k - \theta^*\| \leq 2R, \quad \|g^k\| \leq \bar{G}, \quad \|m^k\| \leq \bar{G}
\]
where \( \bar{G} := 2R + B \).

4.2 Finite-time analysis of projected AdaTD(\( \lambda \))

The analysis of TD(\( \lambda \)) is more complicated than TD due to the existence of \( z^k \). To this end, we need to bound the sequence \((\mathbb{E}\mathbf{z}^k)\) in Lemma 8. Similar to the analysis of AdaTD(0) in Sec. 3, we need to consider the “delayed” expectation. For a fixed \( k \), we consider the following error decomposition
\[
\mathbb{E}[\mathbf{g}(\theta^{k-K_0}; \mathbf{s}_k, \mathbf{s}_{k+1}, z^{\infty})] = \mathbb{E}[\mathbf{g}(\theta^{k-K_0}; \mathbf{s}_k, \mathbf{s}_{k+1}, z^{\infty})] + \mathbb{E}[\mathbf{g}(\theta^{k-K_0}; \mathbf{s}_k, \mathbf{s}_{k+1}, z^{\infty})] - \mathbb{E}[\mathbf{g}(\theta^{k-K_0}; \mathbf{s}_k, \mathbf{s}_{k+1}, z^{\infty})].
\]
Therefore, our problem becomes bounding the difference between \( \mathbb{E}[\mathbf{g}(\theta^{k-K_0}; \mathbf{s}_k, \mathbf{s}_{k+1}, z^{\infty})] \) and \( \mathbf{g}(\theta) \), where the proof is similar to Lemma 2. We can also establish the Lemma 10. We present the convergence of AdaTD(\( \lambda \)) as follows.

**Theorem 3.** Suppose \((\theta^k)_{k \geq 0}\) are generated by AdaTD(\( \lambda \)) with
\[
R \geq 2B/(\sqrt{\omega}(1 - \gamma)\sqrt{1 - \gamma(1 - \lambda)/(1 - \gamma \lambda)}),
\]
under the Markovian observation. Given \( \eta > 0, \delta > 0, 0 \leq \beta < 1, 0 \leq \lambda \leq 1 \), we have
\[
\min_{1 \leq k \leq K} \mathbb{E}[\|\theta^k - \theta^*\|^2] \leq \left( C^2 \ln \left( \frac{\delta + KG^2}{\delta} \right) / \sqrt{K} \right)
\]
\[
+ C^2_\lambda \sqrt{K},
\]
where \( C^2_\lambda \) and \( C_\lambda^2 \) are given as
\[
C^2_1 := \frac{16(\ln K)/K + 1/2G^2}{\ln \left( \frac{1}{\eta} (1 - \gamma)(1 - \alpha)\omega^2 \right) + 2\eta\beta G}
\]
\[
+ \eta G (1 - \alpha)\omega + 4RG^2 \left( \frac{1}{\ln \left( \frac{1}{\eta} (1 - \gamma)(1 - \alpha)\omega^2 \right) + 2\eta\beta G} \right),
\]
\[
C_2^2 := \frac{4RG^2 \eta G (1 - \alpha)\omega + 4RG^2 \eta}{\eta G (1 - \alpha)\omega + 4RG^2 \eta}
\]
\[
+ \frac{2RG^2 \sum_{i=1}^\infty \zeta_k i}{\sqrt{\delta}(1 - \gamma)(1 - \alpha)\omega} + \frac{2RG^2 \sum_{i=1}^\infty \zeta_k (B + \hat{R} + \tilde{R}) G}{\sqrt{\delta}(1 - \gamma)(1 - \alpha)\omega}
\]
\[
= \frac{O\left( \frac{1}{\sqrt{\delta}(1 - \beta)} + \frac{1}{\sqrt{\delta}(1 - \gamma)(1 - \alpha)\omega} \right)}{\max\{\gamma \lambda, \rho\}},
\]
and \( \zeta_k := \sum_{i=1}^k (\gamma \lambda)^{k-i} \rho^i \).

If \( \lambda = 0 \), it then holds \( \zeta_k = 0 \). It is also easy to see \( \zeta_k \leq k(\max\{\gamma \lambda, \rho\})^k \).

5 NUMERICAL SIMULATIONS

To validate the analysis and show the effectiveness of our algorithms, we tested AdaTD(0) and AdaTD(\( \lambda \)) on several commonly used RL tasks.

We compared our algorithms with other policy evaluation methods using the runtime mean squared Bellman error (RMSBE). In each test, the policy is the same for all the algorithms when the value parameter is updated separately. In the first two tasks, the value function is approximated using linear functions. In the last two tasks, the value function is parameterized by a neural network. In the linear tasks, for different values of \( \lambda \), we compared AdaTD(\( \lambda \)) algorithm, the TD(\( \lambda \)), and ALRR algorithm in [17]. For fair comparison, we changed the update step in the original ALRR algorithm to a single time scale TD(\( \lambda \)) update. In the non-linear tasks, we

1. For convenience, we follow the convention \( 0^0 = 0 \).
extended our algorithm to non-linear cases and compared it with TD(λ). Since ALRR was not designed for the neural network-parameterized cases, we did not include it in the non-linear TD tests. In all tests, the curves are generated by taking the average of 10 Monte-Carlo runs.

5.1 Experiment Settings

**Mountain Car.** Algorithms were tested when λ = 0, 0.3, 0.5 and 0.7. For all methods, we set max episode = 300, batch size (horizon) = 16. In vanilla TD method, η = 0.7. In ALRR-TD(λ), η₀ = 1.0, σ = 0.001, ξ = 1.2. In AdaTD(λ), η₀ = 3.0, δ = 0.01, β = 0.3.

**Acrobot.** In all three methods, max episode = 1000 and batch size = 48. In vanilla TD(λ), η = 0.05 when λ = 0, otherwise η = 0.04. In ALRR-TD(λ), η₀ = 0.001, σ = 0.001, ξ = 1.2. In AdaTD(λ), η₀ = 9, δ = 1. When λ = 0 and 0.3, β = 0.5, otherwise β = 0.9.

**CartPole.** We used a neural network to approximate the value function. The neural network has two hidden layers each with 128 neurons and ReLU activation. For both methods, we set max episode = 500 and batch size = 32. In TD(λ), η = 0.02. In AdaTD(λ), η₀ = 1.5, δ = 0.01, β = 0.2.

**Navigation.** We used a neural network to approximate the value function. The neural network has two hidden layers each with 64 neurons and ReLU activation. For both methods, we set max episode = 200 and batch size = 20. In TD(λ), η = 0.2. In AdaTD(λ), η₀ = 0.7, δ = 0.01, β = 0.2.
5.2 Numerical Results

In the test of Mountain Car, the performance of all three methods is close, while AdaTD(λ) still has a small advantage over other two when λ is small. In the Acrobot task, the initial step size is relatively large for AdaTD(λ), but AdaTD(λ) is able to adjust the large initial step size and guarantee afterwards convergence. Note there is a major fluctuation in average loss around episode 30. TD(λ) has constant step size, and thus it is more vulnerable to the fluctuation than AdaTD(λ). As a result, our algorithm demonstrates better overall convergence speed over TD(λ).

In Figures 4 and 5, we tested our algorithm with neural network parameterization. In these tests, the step size of TD(λ) cannot be large due to stability issues. As a result, TD(λ) is outperformed by AdaTD(λ) where large step size is allowed. In fact, when λ is large, a small step size of η = 0.02 still cannot guarantee the stability of TD(λ). It can be observed in Figure 4 that when λ gets larger, i.e. the gradient magnitude is larger, original TD(λ) becomes less stable. In comparison, AdaTD(λ) has exhibited robustness to the choice of λ and the large initial step size in this test.

We conduct two experiments in Figures 3 and 6 where we deliberately skip the step size tuning process and applies large step size to all methods. We show that our algorithm is more robust to large step size and therefore is easier to tune in practice. The hyper-parameters of the two tests are the same as before except for step sizes. In Acrobot, step size is set to 1. In Cartpole, step size is set to 0.8.

In Figure 7 we have also provided another test to evaluate the performance of AdaTD under sparse state features. The MDP has a discrete state space of size 50, and a discrete action space of size 4. The transition matrix and reward table are randomly generated with each element in (0, 1). To ensure the sparsity of gradients, we construct sparse state features as described in Section 3.3. In this test, we select η = 0.45 for TD, and η₀ = 0.5, δ = 1.0, β = 0.5 for AdaTD.

6 PROOFS

6.1 Proofs for AdaTD(0)

This part contains the proofs of the main results and leaves the proofs of the technical lemmas in the appendix.

6.1.1 Technical Lemmas

Lemma 5 (50, 22). For 0 ≤ αₖ ≤ a and δ > 0, we have

\[ \sum_{t=1}^{K_0} \frac{\alpha_t}{\delta + \sum_{i=1}^{t} \alpha_i} \leq \ln(\delta + K_0 a) - \ln \delta. \]

Lemma 6 (H). For any \( \theta \in \mathbb{R}^d \), it follows

\[ \langle \theta^* - \theta, g(\theta) \rangle \geq (1 - \gamma) \omega \| \theta^* - \theta \|^2. \]
In the proofs, we use three shorthand notation for simplifications. Those three notation are all related to the iteration \( k \). Assume \((m^k)_{k \geq 0}, (\theta^k)_{k \geq 0}, (v^k)_{k \geq 0}\) are all generated by AdaTD(0). We denote

\[
\begin{align*}
\hat{\mathbf{m}}_k &:= \mathbb{E} \left[ \|m^k\|^2 / (v^k + \delta) \right], \\
\Delta_k &:= \mathbb{E} \left[ (\theta^k - \theta^*, \mathbf{m}^k) / (v^k + \delta)^{1/2} \right], \\
\phi_k &:= \mathbb{E} \left[ (\theta^* - \theta^k, g(\theta^k)) / (v^k + \delta) \right], \\
\mathcal{R}_k &:= \frac{8K_0(1-\gamma)}{\delta^2(1-\gamma)\omega} \sum_{h=K_0}^{K} \hat{\mathbf{m}}_{k-h} + \eta \beta \hat{\mathbf{m}}_k + 2RG(1-\beta)^{1/2} \ln (K_0) \sqrt{\delta} \sum_{h=K_0}^{K} \mathcal{R}_{k-h} + 2R(1-\beta) \omega K_0 \kappa \rho K_0 \sqrt{\delta}.
\end{align*}
\]

The above notations will be used in the remaining lemmas.

**Lemma 7.** Let \((\Delta_k)_{k \geq 0}\) and \((\mathcal{R}_k)_{k \geq 0}\) be defined as \((26)\), the following result holds for AdaTD(0)

\[
\Delta_k + \frac{1-(1-\beta)}{2} \phi_k \leq \beta \Delta_{k-1} + \mathcal{R}_k.
\]

On the other hand, we can bound \(\Delta_k\) as \(|\Delta_k| \leq \eta \frac{2RG}{\delta^{1/2}}\).

6.1.2 Proof of Theorem

Given \( K, K_0 \in \mathbb{Z}^+ \) and \( K \geq K_0 - 1 \), Lemma 7 tells

\[
\Delta_k + \frac{1-(1-\beta)}{2} \phi_k \leq \beta \Delta_{k-1} + \mathcal{R}_k,
\]

where \(K_0 + 1 \leq k \leq K\). Summing together, we then get

\[
\begin{align*}
\sum_{k=K_0+1}^{K} \phi_k &\leq \beta \sum_{k=K_0+1}^{K} \Delta_k + \sum_{k=K_0+1}^{K} \mathcal{R}_k \\
&\leq (\beta - 1) \sum_{k=K_0}^{K-1} \Delta_k + \sum_{k=K_0+1}^{K} \mathcal{R}_k \\
&\leq (\beta - 1) \sum_{k=K_0}^{K-1} \Delta_k + \sum_{k=K_0+1}^{K} \mathcal{R}_k + \frac{2RG}{\delta^{1/2}}.
\end{align*}
\]

With direct calculations, we get

\[
\begin{align*}
\|\theta^* - \theta^{k+1}\|^2 &\leq \|\theta^* - \theta^k - \eta m^k / \sqrt{v^k + \delta}\|^2 \\
&\leq \|\theta^* - \theta^k\|^2 + \frac{2\eta \|m^k\|^2}{(v^k + \delta)^{1/2}} + \frac{\eta^2 \|m^k\|^2}{v^k + \delta}.
\end{align*}
\]

Taking total condition expectation gives us

\[
\mathbb{E}\|\theta^* - \theta^{k+1}\|^2 \leq \mathbb{E}\|\theta^* - \theta^k\|^2 + 2\eta \Delta_k + \eta^2 \hat{\mathbf{m}}_k.
\]

That is also

\[
\sum_{k=K_0}^{K-1} \Delta_k \leq \frac{\mathbb{E}\|\theta^* - \theta^k\|^2}{2\eta} + \frac{\eta}{2} \sum_{k=K_0}^{K-1} \hat{\mathbf{m}}_k.
\]

Combining \((28)\), we are then led to

\[
\begin{align*}
\sum_{k=K_0+1}^{K} \phi_k &\leq 2 \sum_{k=K_0}^{K-1} (-\Delta_k) + \frac{2}{1-\beta} \sum_{k=K_0+1}^{K} \mathcal{R}_k + \frac{4RG}{\delta^{1/2}} (1-\beta) \\
&\leq \mathbb{E}\|\theta^* - \theta^k\|^2 + \eta \sum_{k=K_0}^{K-1} \hat{\mathbf{m}}_k \\
&\quad + \sum_{k=K_0+1}^{K} \mathcal{R}_k + \frac{4RG}{\delta^{1/2}} (1-\beta) \\
&\leq \frac{4R^2}{\eta^2} + \eta \sum_{k=K_0+1}^{K} \hat{\mathbf{m}}_k + \frac{16K_0^2}{\delta^{1/2} (1-\gamma)\omega} + \frac{2\eta \beta}{1-\beta} \sum_{k=K_0+1}^{K} \hat{\mathbf{m}}_k \\
&\quad + \frac{4RG}{\delta} \sum_{k=K_0+1}^{K} \|g^k\|^2 + \frac{4RG \rho K_0}{\sqrt{\delta}} (K - K_0).\tag{29}
\end{align*}
\]

With Lemma 8 the bound is further bounded by

\[
\begin{align*}
&\left( \eta + \frac{16K_0^2}{\delta^{1/2} (1-\gamma)\omega} + \frac{2\eta \beta}{1-\beta} \right) \sum_{k=K_0}^{K} \hat{\mathbf{m}}_k \\
&\quad + 4RG \delta \sum_{k=K_0}^{K} \|g^k\|^2 + \frac{4RG \rho K_0}{\sqrt{\delta}} (K - K_0) \\
&\leq \left( \frac{16K_0^2}{\delta^{1/2} (1-\gamma)\omega} + \frac{2\eta \beta}{1-\beta} + \eta \right) \ln \frac{\delta + (K-1)G^2}{\delta} \\
&\quad + \frac{4RG}{\delta} \ln \frac{\delta + K^2G}{\delta} + \frac{4RG \rho K_0}{\sqrt{\delta}} (K - K_0).\tag{30}
\end{align*}
\]

where we used the inequality \( \sum_{k=1}^{K} \|g^k\|^2 \leq \ln (\frac{K+G^2}{\delta}) \) (Lemma 5). We set \( K_0 = \gamma \ln K / \ln \frac{1}{\beta} \). It is easy to see that \( K \geq \frac{\gamma}{2} (K_0 + 1) \) as \( K \) is large. On the other hand, with Lemma 9 we can get

\[
\begin{align*}
\sum_{k=K_0}^{K} \phi_k &\geq \frac{K}{\|G^2 + \delta\|} \sum_{k=K_0}^{K} \|\theta^* - \theta^k\|^2 \\
&\geq \left( \sqrt{(K-1)G^2 + \delta} - \sqrt{(K_0-1)G^2 + \delta} \right) \|G^2\| \\
&\quad \times (1-\gamma) \omega \min_{K_0 \leq k \leq K} \mathbb{E}\|\theta^* - \theta^k\|^2 \\
&\geq (1-\gamma) \omega \sqrt{K} / G \cdot \min_{K_0 \leq k \leq K} \mathbb{E}\|\theta^* - \theta^k\|^2,\tag{31}
\end{align*}
\]

where we used \( K \geq \frac{\gamma}{2} (K_0 + 1) \) to get

\[
2(\sqrt{(K-1)G^2 + \delta} - \sqrt{(K_0-1)G^2 + \delta}) \geq \sqrt{KG}.
\]
In this case, we then derive
\[(1 - \gamma)\omega \min_{1 \leq k \leq K} \mathbb{E}[\|\theta^* - \theta_k\|^2] \leq (1 - \gamma)\omega \min_{K_0 \leq k \leq K} \mathbb{E}[\|\theta^* - \theta_k\|^2] \leq 4R\kappa \rho K G / \sqrt{\delta} \cdot \sqrt{K} \left( \theta_0 + \frac{2G\eta}{(1 - \beta)} + \eta G \right) \cdot \frac{4R^2G}{\delta} \cdot \frac{1}{\sqrt{K}} \left( \frac{1}{\sqrt{\delta}} + \frac{1}{\sqrt{\delta(1 - \beta)}} \right). \]

By setting \( K_0 := \ln K / \ln^{1/\rho} \eta \) and defining the constants involved in the theorem as given in Theorem [1] we then proved the results.

6.1.3 Proof of Theorem 2

The proof lies on a re-estimate of (31). Under condition (20), we can derive
\[ \sum_{k = K_0}^{K} \phi_k \geq \sum_{k = K_0}^{K} \frac{1}{|c(k - 1)^{1/\rho} + \delta|^{1/\rho}} \min_{K_0 \leq k \leq K} \mathbb{E}[\|\theta^* - \theta_k\|^2] \geq K^{1 - \frac{1}{\rho}} \left( 1 - \gamma \right) / 2 \left( 1 - \nu / 2 \right) / \sqrt{C} \left( K_0 \leq k \leq K, \mathbb{E}[\|\theta^* - \theta_k\|^2] \right), \] (32)
as \( K \geq 2^{1 - \frac{1}{\rho}} K_0 \). Noticing that (30) still holds, which means
\[ \min_{1 \leq k \leq K} \mathbb{E}[\|\theta^* - \theta_k\|^2] = O \left( K_0 / K^{1 - \frac{1}{\rho}} + \rho K_0 K^{1 - \frac{2}{\rho}} \right). \]

To achieve the \( O(\epsilon) \)-accuracy for \( \min_{1 \leq k \leq K} \{ \mathbb{E}[\|\theta^* - \theta_k\|^2] \} \), we need
\[ \rho K_0 K^{1 - \frac{2}{\rho}} \leq \epsilon, \quad K_0 / K^{1 - \frac{1}{\rho}} \leq \epsilon. \]

We then have \( K_0 = O(\ln^{1/\rho} K^{1 - \frac{1}{\rho}}) \) and \( K = O \left( \ln^{1/\rho} K^{1 - \frac{1}{\rho}} \left( \ln^{1/\rho} K^{1 - \frac{1}{\rho}} \right)^{1/2} \right) \).

6.2 Proofs for AdaTD(\lambda)

The proof of AdaTD(\lambda) is similar to AdaTD(0) but with involved technical lemmas being modified. The complete proof is given in the appendix.

7 CONCLUSIONS

We developed an improved variant of the celebrated TD learning algorithm in this paper. Motivated by the close link between TD(0) and stochastic gradient-based methods, we developed adaptive TD(0) and TD(\lambda) algorithms. The finite-time convergence analysis of the novel adaptive TD(0) and TD(\lambda) algorithms has been established under the Markovian observation model. While the worst-case convergence rates of Adaptive TD(0) and TD(\lambda) are similar to those of the original TD(0) and TD(\lambda), our numerical tests on several standard benchmark tasks demonstrate the effectiveness of our adaptive approaches. Future work includes variance-reduced and decentralized AdaTD approaches.
8 Proofs of Technical Lemmas for ADATD(0)

8.1 Proof of Lemma 2

Given a fixed integer $K_0$, with Assumptions 1 and 2 by using a shorthand notation

$$\Xi := \left( R(s, s') + \gamma \phi(s')^T \theta^{k-K_0} - \phi(s)^T \theta^{k-K_0} \right) \phi(s),$$

we have

$$\mathbb{E}(g(\theta^{k-K_0}, s_k, s_{k+1}) | \sigma^{k-K_0}) = \sum_{s,s' \in S} \pi(s) \cdot \mathcal{P}(s'|s) \cdot \Xi$$

$$+ \sum_{s,s' \in S} \left( \mathcal{P}(s_k | s_{k-K_0} = s) - \pi(s) \right) \cdot \mathcal{P}(s'|s) \cdot \Xi. \quad (33)$$

Noticing that the following expectation

$$\sum_{s,s' \in S} \pi(s) \cdot \mathcal{P}(s'|s) \cdot \Xi = g(\theta^{k-K_0}).$$

With Assumption 2, the second term in right side of (33) can then be bounded by $S\kappa(B + R\gamma + R)p^{K_0}$. By setting $\kappa := S\kappa(B + R\gamma + R)$ and using $\mathbb{E}(\mathbb{E}(\cdot | \sigma)) = \mathbb{E}(\cdot)$, we then proved the result.

8.2 Proof of Lemma 3

With the fact

$$g^k = g(\theta^k) + g^{k-K_0} - \mathbb{E}(g(\theta^{k-K_0}, s_k, s_{k+1}) | \sigma^{k-K_0}) - g(\theta),$$

it then follows

$$\mathbb{E}\left[ (\theta^k - \theta^*; g^k)/(\nu^{k-1} + \delta)^{\frac{1}{2}} \right]$$

$$= \mathbb{E}\left[ (\theta^k - \theta^*; g(\theta^k))/(\nu^{k-1} + \delta)^{\frac{3}{2}} \right] + \text{I} + \text{II} + \text{III}. \quad (34)$$

where

$$\text{I} := \mathbb{E}(\langle \theta^k - \theta^*; g(\theta^{k-K_0}, s_k, s_{k+1}) \rangle),$$

$$\text{II} := \mathbb{E}(\langle \theta^k - \theta^*; [g(\theta^{k-K_0}, s_k, s_{k+1}) - g(\theta^{k-K_0})] \rangle),$$

$$\text{III} := \mathbb{E}(\langle \theta^k - \theta^*; [g(\theta^k) - g(\theta^*)] \rangle).$$

Now, we bound I, II and III. Note that I and III enjoy the same upper bound as

$$\text{I}(\text{III}) \leq 2\mathbb{E}\left( \|\theta^k - \theta^*\| \cdot \|\theta^k - \theta^{k-K_0}\|/(\nu^{k-1} + \delta)^{\frac{1}{2}} \right).$$

With Lemma 2 we have II $\leq \frac{2R\kappa p^{K_0}}{\sqrt{\delta}}$. Thus, we get

$$\text{I} + \text{II} + \text{III} \leq 4 \sum_{h=1}^{K_0} \mathbb{E}\left[ \|\theta^k - \theta^*\| \cdot \|\theta^{k+1-h} - \theta^{k-h}\|/(\nu^{k-1} + \delta)^{\frac{1}{2}} \right] + \frac{2R\kappa p^{K_0}}{\sqrt{\delta}}$$

$$\leq 4 \sum_{h=1}^{K_0} \mathbb{E}\left[ \|\theta^k - \theta^*\| \cdot \|m^{k-h}\|/(\nu^{k-1} + \delta)^{\frac{1}{2}} \right] + \frac{2R\kappa p^{K_0}}{\sqrt{\delta}}. \quad (35)$$
On the other hand, with the Cauchy-Schwarz inequality, we have
\[
\sum_{k=1}^{K_0} \frac{1}{\delta^{1/4}} \sum_{h=1}^{K_0} \left( \frac{\|\theta^k - \theta^*\| \cdot \|m^{k-h}\|}{(v^{k-1} + \delta)^{1/2}} \right) \geq \frac{1}{\delta^{1/4} (1 - \gamma) \omega} \frac{\|\theta^k - \theta^*\|^2}{(v^{k-1} + \delta)^{1/2}} + \frac{2K_0 \sum_{h=1}^{K_0} \|m^{k-h}\|^2}{\delta^{1/4} (1 - \gamma) \omega} \geq \left( \frac{1}{(1 - \gamma) \omega} + \frac{2K_0}{\delta^{1/4}} \right) \frac{\|\theta^k - \theta^*\|^2}{\delta^{1/4} (1 - \gamma) \omega} \geq \left( \frac{1}{(1 - \gamma) \omega} + \frac{2K_0}{\delta^{1/4}} \right) \frac{\|\theta^k - \theta^*\|^2}{\delta^{1/4} (1 - \gamma) \omega} .
\]
Combining (34), (35) and (36), we then get the result.

8.3 Proof of Lemma 4
For simplicity, we define \( g^k := \mathbf{r}^k; s_k, s_k+1 \). Recall \( m^k = (1 - \beta) \sum_{j=1}^{k-1} \beta^{k-1-j} g^j \). We have
\[
\|m^k\|^2/(v^k + \delta) \leq \left( 1 - \beta \right) \frac{1}{2} \sum_{j=1}^{k-1} \beta^{k-1-j} \|g^j\|^2/(v^k + \delta) \leq \left( 1 - \beta \right) \frac{1}{2} \sum_{j=1}^{k-1} \beta^{k-1-j} \|g^j\|^2/(v^k + \delta) = \left( 1 - \beta \right) \frac{1}{2} \sum_{j=1}^{k-1} \beta^{k-1-j} \|g^j\|^2/(v^k + \delta) \leq \left( 1 - \beta \right) \frac{1}{2} \sum_{j=1}^{k-1} \beta^{k-1-j} \|g^j\|^2/(v^k + \delta) \leq \left( 1 - \beta \right) \frac{1}{2} \sum_{j=1}^{k-1} \beta^{k-1-j} \|g^j\|^2/(v^k + \delta) .
\]
where \( a) \) uses the fact \( (\sum_{j=1}^{k-1} a_j b_j)^2 \leq \sum_{j=1}^{k-1} a_j^2 \sum_{j=1}^{k-1} b_j \) with \( a_j = \beta^{k-1-j} \) and \( b_j = \beta^{k-1-j} \|g^j\|^2/(v^k + \delta)^{1/2} \). Combining the inequalities above, we then get the result. By applying Lemma 5, we then get the second bound.

8.4 Proof of Lemma 7
With direct computations, we have
\[
\Delta_k = E \left( \langle \theta^k - \theta^*, m^k \rangle/(v^{k-1} + \delta)^{1/2} \right) + E \left( \langle \theta^k - \theta^*, m^k \rangle/(v^k + \delta)^{1/2} \right) \leq \left( \frac{\|\theta^k - \theta^*\| \cdot \|m^k\|}{(v^{k-1} + \delta)^{1/2}} \right) \frac{\|g^k\|^2}{\|g^k\|^2} \leq \left( \frac{\|\theta^k - \theta^*\| \cdot \|m^k\|}{(v^{k-1} + \delta)^{1/2}} \right) \frac{\|g^k\|^2}{\|g^k\|^2} \leq \frac{2RG}{\delta} \frac{\|g^k\|^2}{v^k + \delta} .
\]
Now, we bound I and II. The Cauchy's inequality then gives us
\[
\leq \frac{2RG}{\delta} \frac{\|g^k\|^2}{v^k + \delta} .
\]
With scheme of the algorithm, we use a shorthand notation \( \Lambda := E(\langle \theta^k - \theta^*, \mathbf{g}^k \rangle/(v^{k-1} + \delta)^{1/2} | \sigma^k ) \) and then get
\[
I = E \left( (\theta^k - \theta^*, \mathbf{g}^k)/(v^{k-1} + \delta)^{1/2} \right) = \frac{\beta - 1}{\Lambda} + \beta \|\theta^k - \theta^*\| \cdot \|m^{k-1}\|/(v^{k-1} + \delta)^{1/2} \leq \frac{\beta - 1}{\Lambda} + \beta \|\theta^k - \theta^*\| \cdot \|m^{k-1}\|/(v^{k-1} + \delta)^{1/2} \leq \frac{\beta - 1}{\Lambda} + \beta \|\theta^k - \theta^*\| \cdot \|m^{k-1}\|/(v^{k-1} + \delta)^{1/2} \leq \frac{\beta - 1}{\Lambda} + \beta \|\theta^k - \theta^*\| \cdot \|m^{k-1}\|/(v^{k-1} + \delta)^{1/2} \leq \frac{\beta - 1}{\Lambda} + \beta \|\theta^k - \theta^*\| \cdot \|m^{k-1}\|/(v^{k-1} + \delta)^{1/2} .
\]
where \( a \) uses the Cauchy's inequality, \( b \) depends on the scheme of AdaTD(0). Combination of the inequalities I, II and Lemma 3 gives the final result.

The bound for \( \Delta_k \) a direct result from the bound of \( m^k \) and \( g^k \).

9 PROOFS FOR ADAJD(\( \lambda \))
9.1 Technical Lemmas
Although \( (m^k)_{k \geq 0} \) and \( (v^k)_{k \geq 0} \) are generated as the same as AdaTD(0), the difference scheme of updating \( (g^k)_{k \geq 0} \) makes them different. Thus, we use different notation here. Like previous proofs, we denote three items for AdaTD(\( \lambda \)) as follows
\[
\begin{align*}
\hat{m}_k &:= E \left( \|m^k\|^2/(v^k + \delta) \right) , \\
\hat{\Delta}_k &:= E \left( (\theta^k - \theta^*, m^k)/(v^k + \delta)^{1/2} \right) , \\
\hat{\phi}_k &:= E \left( (\theta^k - \theta^*, \mathbf{g}^k)/(v^{k-1} + \delta)^{1/2} \right) , \\
\hat{\gamma}_k &:= E \left( \sum_{i=K+1}^{K_0} \frac{\|\theta^i - \theta^*\|}{\sqrt{(1 - \gamma) \omega}} \sum_{j=K+1}^{K_0} \frac{\|\theta^j - \theta^*\|}{\sqrt{(1 - \gamma) \omega}} \right) .
\end{align*}
\]
Direct computing gives
\[
\sum_{k=1}^{\infty} \hat{\gamma}_k \leq \frac{\max \{ \gamma_\lambda, \rho \}}{(1 - \max \{ \gamma_\lambda, \rho \})} .
\]
\[
\text{Lemma 8. Assume } (\mathbb{E} z^k)_{k \geq 0} \text{ is generated by AdaTD}(\lambda) . \text{ It then holds that}
\]
\[
\|\mathbb{E} z^k - \mathbb{E} z^\infty\| \leq \frac{k^c}{1 - \gamma_\lambda} \text{ and } \|z^\infty\| \leq \frac{1}{1 - \gamma_\lambda} .
\]
\[
\text{Lemma 9. Assume } (m^k)_{k \geq 0} \text{ and } (v^k)_{k \geq 0} \text{ are given by (37). We have}
\]
\[
\sum_{k=1}^{K} \hat{m}_k \leq \sum_{j=1}^{K} \mathbb{E} \|g^j\|^2/(v^j + \delta) .
\]
Further, with the boundedness of \((\|g^k\|)_{k \geq 0}\), we then get
\[
\sum_{k=1}^{K} \hat{m}_k \leq \ln(\frac{\delta + (K-1)\hat{G}^2}{\delta}).
\]

**Lemma 10.** Assume \((\theta^k)_{k \geq 0}\) is generated by AdaTD(\(\lambda\)). Given integer \(K_0 \in \mathbb{Z}^+\), we then get
\[
\left\| \mathcal{E} \left[ \mathbf{g}^\prime \left( \theta^k, K_0 ; s_k, s_{k+1}, z^k \right) \right] - g^\lambda \left( \theta^k, K_0 \right) \right\| \leq \frac{\kappa}{1 - \gamma \lambda} \rho \hat{K}_0 + \frac{\hat{G}}{1 - \gamma \lambda} \zeta_k.
\]

**Lemma 11.** Assume \((\theta^k)_{k \geq 0}\) is generated by AdaTD(\(\lambda\)). Given \(K_0 \in \mathbb{Z}^+\), we have
\[
\mathbb{E} \left[ \theta^k - \theta^\ast, g^\lambda (\theta^k) \right] \leq \frac{1}{\beta} \phi_k + \frac{8K_0}{\delta^{1/2}(1-\gamma \omega)(1-\gamma)} + \frac{2R}{\delta} \left( \frac{\kappa}{1 - \gamma \lambda} \rho \hat{K}_0 + \frac{\hat{G}}{1 - \gamma \lambda} \zeta_k \right).
\]

**Lemma 12.** Assume \(\hat{m}_k, \hat{\Delta}_k\) are denoted by \((\gamma, \delta)\), we then have the following result
\[
\hat{\Delta}_k + \frac{(1 - \beta)}{2} \phi_k \leq \beta \hat{\Delta}_{k-1} + \hat{R}_k.
\]

On the other hand, we have another bound for \(\hat{\Delta}_k\) as
\[
|\hat{\Delta}_k| \leq \eta \frac{R \hat{G}^2}{\delta^2}.
\]

### 9.2 Proof of Theorem 3

Similar to \((29)\), we derive
\[
\sum_{k=K_0+1}^{K} \frac{(\theta^k - \theta^\ast, g^\lambda (\theta^k))}{(v^k+1)^{\frac{1}{2}}} \leq \frac{\mathbb{E} \|\theta^\ast - \theta^{K_0}\|^2}{\eta} + \eta \sum_{k=K_0+1}^{K} \hat{m}_k + \frac{2}{1 - \beta} \sum_{k=K_0+1}^{K} \hat{R}_k + \frac{4R \hat{G}^2}{\delta (1 - \beta)}.
\]

**Lemma 5** gives us
\[
\eta \sum_{k=K_0+1}^{K} \hat{m}_k + \frac{2}{1 - \beta} \sum_{k=K_0+1}^{K} \hat{R}_k \leq \eta \sum_{k=K_0+1}^{K} \hat{m}_k + \left( \frac{16K_0^2}{(1 - \gamma \omega)^2} + \frac{2\eta \beta}{1 - \beta} \right) \sum_{k=K_0+1}^{K} \hat{m}_k + \frac{4R \hat{G}^2}{\delta} \sum_{k=K_0+1}^{K} \frac{\|g^k\|^2}{v^k + \delta} + \frac{4R \hat{K} \rho \hat{K}_0 (K - K_0)}{(1 - \gamma \lambda) \sqrt{\delta}} + \frac{2R \hat{G}}{\sqrt{(1 - \gamma \lambda)}} \zeta_k.
\]

Further with Lemma 3, the right hand is bounded by
\[
\left( \frac{16K_0^2}{(1 - \gamma \omega)^2} + \frac{2\eta \beta}{1 - \beta} + \eta \right) \ln(\delta + (K-1)\hat{G}^2) + \frac{4R \hat{G}^2}{\delta} \ln(\delta + K \hat{G}^2) + \frac{4R \hat{K} \rho \hat{K}_0 (K - K_0)}{\sqrt{(1 - \gamma \lambda)}} + \frac{2R \hat{G} \sum_{k=1}^{\infty} \zeta_k}{\sqrt{(1 - \gamma \lambda)}}
\]

The rest of the proof is almost identical to the one of Theorem 1. Letting \(K_0 = \lceil \ln K / \ln \frac{1 - \rho}{\rho} \rceil\) and
\[
C_1^\lambda := \frac{16K_0^2 \hat{G}_1}{(1 - \gamma \omega)^2} + \frac{2\eta \beta \hat{G}}{(1 - \beta)} (1 - \gamma \omega) + \frac{4R \hat{G}^2}{(1 - \alpha) \omega} = O \left( \frac{\ln K / \ln \frac{1 - \rho}{\rho}}{\sqrt{(1 - \gamma \lambda)}} \right),
\]
\[
C_2^\lambda := \frac{4R \hat{G}^2}{\eta (1 - \gamma \omega) \omega} + \frac{4R \hat{K} \rho \hat{K}_0 (K - K_0)}{(1 - \gamma \lambda) \sqrt{\delta}} + \frac{4R \hat{G} (B + \hat{R} + \hat{G})}{(1 - \alpha) \sqrt{\delta} \omega} - O \left( \frac{1}{\sqrt{(1 - \gamma \lambda)}} \right).
\]

we then proved the results.

### 9.3 Proof of Proposition 1

Based on Theorem 3, in the general case, we directly get the complexity of AdaTD(\(\lambda\)). Beginning from \((42)\), like the proof of Theorem 2, the improved complexity can be obtained with condition \((20)\).

### 10 PROOFS OF TECHNICAL LEMMAS FOR ADA-TD(\(\lambda\))

#### 10.1 Proof of Lemma 8

With the scheme of updating \(\mathbf{z}^k\), it follows \(\mathbb{E} \mathbf{z}^k = \sum_{i=1}^{k} (\gamma \lambda)^{k-1} \mathbb{E} \phi(s_i)\). Assume \(\pi^0\) is the initial probability of \(s_0\), then \(\mathbb{E} \phi(s_i) = \Phi^T \mathcal{P}^i \pi^0\) which yields
\[
\mathbb{E} \mathbf{z}^k = \sum_{i=1}^{k} (\gamma \lambda)^{k-1} \Phi^T \mathcal{P}^i \pi^0.
\]

From \([?]\), there exists orthogonal matrix \(U\) such that
\[
\mathcal{P} = U^T \text{Diag}(1, \lambda_2, \ldots, \lambda_S) U
\]

with \(\lambda_2 \geq \ldots \geq \lambda_S \geq 0\) and \(\lambda_2 \leq \rho\). Without loss of generation, we assume \(\gamma \lambda > \rho\) and then derive
\[
\mathbb{E} \mathbf{z}^k = \Phi^T U^T \Gamma U \pi^0,
\]

where
\[
\Gamma := \text{Diag}(\sum_{i=1}^{k} (\gamma \lambda)^{k-1}, \ldots, \sum_{i=1}^{k} (\gamma \lambda)^{k-1} \lambda_2^{i-1}, \ldots, \sum_{i=1}^{k} (\gamma \lambda)^{k-1} \lambda_S^{i-1}).
\]

We also see \(\mathbb{E} \mathbf{z}^\infty = \Phi^T U^T \text{Diag}(\frac{1}{1 - \gamma \lambda}, 0, \ldots, 0) U \pi^0\). And hence, it follows
\[
\mathbb{E} \mathbf{z}^k - \mathbb{E} \mathbf{z}^\infty = \Phi^T U^T \hat{\Gamma} U \pi^0,
\]

where
\[
\hat{\Gamma} := \text{Diag}(\frac{-(\gamma \lambda)^{k-1}}{1 - \gamma \lambda}, \sum_{i=1}^{k} (\gamma \lambda)^{k-1} \lambda_2^{i-1}, \ldots, \sum_{i=1}^{k} (\gamma \lambda)^{k-1} \lambda_S^{i-1}).
\]

For \(i \in \{2, 3, \ldots, S\}\), \(\sum_{i=1}^{k} (\gamma \lambda)^{k-1} \lambda_i^{j} \leq \sum_{i=1}^{k} (\gamma \lambda)^{k-1} \rho^j\). It is easy to see
\[
\zeta_k := \sum_{i=1}^{k} (\gamma \lambda)^{k-1} \rho^i \leq \sum_{i=1}^{k} (\max\{\gamma \lambda, \rho\})^{k-1} (\max\{\gamma \lambda, \rho\})^i = k (\max\{\gamma \lambda, \rho\})^k.
\]
10.2 Proof of Lemma 9

This proof is identical to the one of Lemma 4 and will not be reproduced.

10.3 Proof of Lemma 10

With the boundedness, we are led to

\[
\left\| \mathbb{E}\left[ g^\lambda(\theta^{k-K_0}; s_k, s_{k+1}, z^\infty) \right] - \mathbb{E}\left[ g^\lambda(\theta^{k-K_0}; s_k, s_{k+1}, z^\infty) \right] \right\| \leq \frac{G}{1 - \gamma \lambda} k \zeta_k.
\]

By using a shorthand notation

\[\Upsilon := \mathcal{R}(s, s') + \gamma \phi(s')^T \theta^{k-K_0} - \phi(s)^T \theta^{k-K_0},\]

with direct computation, we have

\[
\mathbb{E}\left[ g^\lambda(\theta^{k-K_0}; s_k, s_{k+1}, z^\infty) \mid \sigma(\theta^{k-K_0}, z^\infty) \right] = \sum_{s, s' \in S} \pi(s) \mathcal{P}(s' | s) \cdot \Upsilon \cdot z^\infty + \sum_{s, s' \in S} \left( \mathcal{P}(s_k | s_{k-K_0} = s) - \pi(s) \right) \mathcal{P}(s' | s) \cdot \Upsilon \cdot z^\infty.
\]

Noticing that

\[
\sum_{s, s' \in S} \pi(s) \mathcal{P}(s' | s) \cdot \Upsilon \cdot z^\infty = \mathbb{E}\left[ g^\lambda(\theta^{k-K_0}; s_k, s_{k+1}, z^\infty) \mid \sigma(\theta^{k-K_0}, z^\infty) \right].
\]

Direct calculation gives us

\[
\left\| \sum_{s, s' \in S} \left( \mathcal{P}(s_k | s_{k-K_0} = s) - \pi(s) \right) \mathcal{P}(s' | s) \cdot \Upsilon \cdot z^\infty \right\| \leq \frac{k \rho_{K_0}}{1 - \gamma \lambda}.
\]

10.4 Proof of Lemma 11

Noting

\[
\mathbb{E}\left[ g^\lambda(\theta; \tilde{s}_k, \tilde{s}_{k+1}, z^\infty) \right] = g^\lambda(\theta),
\]

where \((\tilde{s}_1, \tilde{s}_2, \ldots)\) is the stationary sequence. Then for any \(\theta^1, \theta^2 \in \mathbb{R}^d\),

\[
\left\| g^\lambda(\theta^1) - g^\lambda(\theta^2) \right\| = \left\| \mathbb{E}\left[ g^\lambda(\theta^1; \tilde{s}_k, \tilde{s}_{k+1}, z^\infty) \right] - \mathbb{E}\left[ g^\lambda(\theta^2; \tilde{s}_k, \tilde{s}_{k+1}, z^\infty) \right] \right\| \leq \frac{2\left\| \theta^1 - \theta^2 \right\|}{1 - \gamma \lambda}.
\]

With the fact \(g^k = g^\lambda(\theta^k) + g^k - \mathbb{E}\left[ g^\lambda(\theta^{k-K_0}; s_k, s_{k+1}, z^k) + g^\lambda(\theta^{k-K_0}; s_k, s_{k+1}, z^k) - g^\lambda(\theta^{k-K_0}) + g^\lambda(\theta^{k-K_0}) - g^\lambda(\theta^k) \right] \), we can have

\[
\mathbb{E}\left[ (\theta^k - \theta^*, g^k) / (v^{k-1} + \delta) \right]^2 = \mathbb{E}\left[ (\theta^k - \theta^*, g^\lambda(\theta^k)) / (v^{k-1} + \delta) \right]^2
\]

\[
+ \mathbb{E}\left[ (\theta^k - \theta^*, g^k - \mathbb{E}\left[ g^\lambda(\theta^{k-K_0}; s_k, s_{k+1}, z^k) \right]) / (v^{k-1} + \delta) \right]^2
\]

\[
+ \mathbb{E}\left[ (\theta^k - \theta^*, g^\lambda(\theta^{k-K_0}; s_k, s_{k+1}, z^k) - g^\lambda(\theta^{k-K_0})) / (v^{k-1} + \delta) \right]^2.
\]

We bound I, II and III in the following. We can see I and III have the same bound

\[
I(III) \leq \frac{2}{1 - \gamma \lambda} \mathbb{E}\left[ \left\| \theta^k - \theta^* \right\| \left\| \theta^k - \theta^{k-K_0} \right\| \right] / (v^{k-1} + \delta)^2
\]

and with Lemma 10

\[
II \leq \frac{2R}{\sqrt{\delta}} \left( \frac{k}{1 - \gamma \lambda} \rho_{K_0} + \frac{G}{1 - \gamma \lambda} \zeta_k \right).
\]

Hence, we have

\[
I + II + III
\]

\[
\leq \frac{4}{1 - \gamma \lambda} \sum_{h=K_0}^{1} \mathbb{E}\left[ \left\| \theta^k - \theta^* \right\| \left\| \theta^{k-1-h} - \theta^{k-h} \right\| \right] / (v^{k-1} + \delta)^2
\]

\[
+ \frac{2R}{\sqrt{\delta}} \left( \frac{k}{1 - \gamma \lambda} \rho_{K_0} + \frac{G}{1 - \gamma \lambda} \zeta_k \right)
\]

\[
\leq \frac{4}{1 - \gamma \lambda} \sum_{h=K_0}^{1} \mathbb{E}\left[ \left\| \theta^k - \theta^* \right\| \left\| m^{k-h} \right\| \right] / (v^{k-1} + \delta)^2
\]

\[
+ \frac{2R}{\sqrt{\delta}} \left( \frac{k}{1 - \gamma \lambda} \rho_{K_0} + \frac{G}{1 - \gamma \lambda} \zeta_k \right).
\]

On the other hand, with the Cauchy-Schwarz inequality, we derive

\[
\sum_{h=K_0}^{1} \mathbb{E}\left( \left\| \theta^k - \theta^* \right\| \left\| m^{k-h} \right\| \right) / (v^{k-1} + \delta)^2 \left| \sigma^k \right|
\]

\[
\leq 1 / \delta^3 / 4 \sum_{h=K_0}^{1} \left( \left\| \theta^k - \theta^* \right\| \left\| m^{k-h} \right\| \right) / (v^{k-1} + \delta)^2
\]

\[
\leq \frac{1}{2\delta^3 / 4} \sum_{h=K_0}^{1} \left( \delta^3 / 4 (1 - \alpha) \omega(1 - \gamma \lambda) \left\| \theta^k - \theta^* \right\|^2 \right) / (v^{k-1} + \delta)^2
\]

\[
+ \frac{4K_0}{\delta^3 / 4 (1 - \alpha) \omega(1 - \gamma \lambda) (v^{k-1} + \delta)}.
\]
The right hand of (46) is further bounded by
\[
\frac{(1 - \alpha)\omega(1 - \gamma\lambda)}{8} \|\theta^k - \theta^*\|^2 / (v^{k-1} + \delta)^{1/2}
\]
\[
+ \frac{2K_0}{\delta^{1/2}(1 - \alpha)\omega(1 - \gamma\lambda)} \sum_{h=1}^{K_0} \hat{m}_{k-h}
\]
\[
\leq \frac{(1 - \gamma\lambda)}{8} \tilde{\phi}_k + \frac{2K_0}{\delta^{1/2}(1 - \alpha)\omega(1 - \gamma\lambda)} \sum_{h=1}^{K_0} \hat{m}_{k-h}. \tag{47}
\]

Combining (44) and (47), we then get the result.

10.5 Proof of Lemma 12
The proof is identical to Lemma 7 and will not be repeated.