A Bond Option Pricing Formula in the Extended CIR Model, with an Application to Stochastic Volatility

Zheng Liu∗, Qidi Peng† and Henry Schellhorn‡

October 22, 2014

Abstract

We provide a complete representation of the interest rate in the extended CIR model. Since it was proved in Maghsoodi (1996) that the representation of the CIR process as a sum of squares of independent Ornstein-Uhlenbeck processes is possible only when the dimension of the interest rate process is integer. We use a slightly different representation, valid when the dimension is not integer. Our representation uses an infinite sum of squares of basic processes. Each basic process can be described as an Ornstein-Uhlenbeck process with jumps at fixed times. In this case, the price of a bond option resembles the Black-Scholes formula, where the normal distribution is replaced by a new distribution, which generalizes the non-central chi-square distribution. For practical purposes, the bond price is however better calculated by inverting a Laplace transform. We generalize the result to a model where volatility is stochastic.

1 Introduction

The Cox-Ingersoll-Ross term structure model, in short the CIR model, was introduced in Cox et al. (1985a), (1985b). In this model, the spot interest rate \( r(t) \) is assumed to follow a squared Bessel process:

\[
\begin{align*}
    dr(t) &= (-b(t)r(t) + \theta(t)) dt + \sigma(t) \sqrt{r(t)} dB(t) \\
    r(0) &= r_0 > 0,
\end{align*}
\]

where \( B \) is a standard Brownian motion. In the original specification of Cox et al. (1985a), (1985b), the speed of mean reversion \( b \geq 0 \), the volatility \( \sigma > 0 \) and the parameter \( \theta > 0 \) are assumed constant. In this paper, we consider the case where all these parameters are continuous functions of time. This is sometimes referred to as the extended CIR model.

Several features of the CIR model are particularly attractive. Firstly, it can be justified by general equilibrium considerations, see Cox et al. (1985a). Secondly, the interest rate is always positive and stationary. Cox et al. found that its distribution follows a non-central chi-square distribution. Finally, there is a closed form formula for the bond price. For practitioners however, the main shortcoming of the constant parameters version of the model is that it cannot reproduce the original term structure of interest rates. This fact was highlighted by several authors (Hull (1990), Keller-Ressel and Steiner (2008), Yang (2006) and all the references therein): yield curves can be only normal, inverse, or humped. The extended CIR model, however, has enough parameters to be fitted to the original yield curve.

Maghsoodi (1996), Jamshidian (1995), and Rogers (1995) propose a representation of the extended CIR model as a sum of squares of Ornstein-Uhlenbeck processes when the dimension \( d \equiv 4\theta(t)/\sigma^2(t) \) is constant and integer. As a consequence, the interest rate follows the generalized chi-square distribution. Maghsoodi (1996), and Shirakawa (2002) also propose a representation of the interest rate as a time-changed lognormal process. However, as the latter author states, “it is difficult to derive the probability distribution of the squared Bessel processes with time-varying dimensions.

∗Morgan Stanley, New York city, E-mail: Zheng.Liu@cgu.edu
†Institute of Mathematical Sciences, Claremont Graduate University, E-mail: Qidi.Peng@cgu.edu
‡Corresponding author, Institute of Mathematical Sciences, Claremont Graduate university, E-mail: Henry.Schellhorn@cgu.edu
explicitly”. Brigo and Mercurio (2006) state that no solution to that problem has been found. We view this as the principal contribution of our paper. While the characteristic function of the interest rate has a remarkably simple structure, there are many different ways to invert it to characterize the probability distribution. One of them is by moment approximation. In the constant parameters case, the bond price is determined by solving a Riccati equation. There is in general no closed form formula for the bond price, and authors generally claim that their solution is in closed form "up to solution of an ODE”. We note that Mannoli et al. (2008) also study the extended CIR model, and provide a closed form solution for the bond price as an infinite series. We then generalize Maghsoodi’s result (1996) for the price of a bond option to the case where the dimension $d(t)$ is any positive continuous function.

There are alternate approaches to extend the CIR model. Brigo and Mercurio (2001) find that a deterministic shift of the CIR model is analytically tractable. The obvious drawback of making the parameters of the CIR model functions of time is the problem of overparameterization: the parameters will not be robust in a change of regime. Several authors consider instead a CIR model of interest rates with constant parameters and stochastic volatility. For instance, Longstaff and Schwartz (1992), and Duffie and Kan (1996), consider a generalized two-factor CIR model, where one factor is the interest rate, and the other one is its volatility. The volatility in that model is a variation of the volatility in the popular Heston (1993) model. Cotton, Fouque, Papanicolaou and Sircar (2004) calculate an asymptotic expansion of the bond price in such a model (with constant parameters) when the speed of mean reversion is fast. Fouque and Lorig (2011) generalize this model to a model with a volatility of volatility.

Finally, we note that several authors have generalized the CIR model in a different way using multiple factors. We refer the reader to the references contained in Chen, Filipovic, and Poor (2004) and Gourieroux and Monfort (2011).

We then incorporate stochastic volatility $v(t)$ in our extended CIR model. If $b$ and $\theta$ are deterministic, and $v(t)$ is independent of $r(t)$ the distribution of the interest rate, conditional on volatility, is the same as in the deterministic volatility case. As an illustration of our approach, we give a semi-closed formula of the price of a bond option when the price of the underlying bond at time $t$ can be expressed a function as a function of $r(t), v(t)$ and time $t$. We also consider a model where volatility is correlated with the interest rate.

In the same way that it is not too difficult to generalize the one-factor CIR model to multiple factors (see Duffie and Kan (1996)), we believe it is not difficult to generalize our results on the extended CIR model to multiple factors. However, we leave this for future research.

The structure of this paper is as follows. In Section 2, we consider the case where volatility is deterministic, namely the extended CIR model per se. In Section 3, we provide an analytic formula (up to solution of ODEs) for a bond option with deterministic volatility. In Section 4, we extend our results to stochastic volatility.

2 The Extended CIR Model

Definition 2.1 We let $T > 0$ be the maturity of the bond underlying the bond option.

Definition 2.2 We define the number of degrees of freedom (also called dimension in Maghsoodi)

$$d(t) = \frac{4\theta(t)}{\sigma^2(t)}.$$  

Since $\theta$ is assumed to be strictly positive, and $\sigma$ non-zero, $d(t)$ is also strictly positive.

2.1 An Informal View of Our Representation

2.1.1 The Rate as a Sum of Squared Gaussian Processes: an Incomplete, but Useful Initial Concept

Our goal is to generalize the representation of the rate $r(t)$ as a sum with a constant number of terms of squares of Ornstein-Uhlenbeck processes into a sum of time-varying numbers of such terms. Heuristically, it is easier to represent this sum as an integral. Before introducing the rigorous definitions, we explain informally the method. We introduce
2.1.3 The Rate as a Double Limit

It turns out that for some unknown functional "plug". In particular, if we take the representation:

\[ r(t) = \int_0^{d(t)} x(t, u)^2 \, du. \]

Then informally, taking derivatives results in:

\[ dr(t) = \int_0^{d(t)} \frac{\partial(x(t, u)^2)}{\partial t} \, du + x(t, d(t))^2 \, d'(t). \]

We see that:

\[ dr(t) = (-b(t)r(t) + \frac{\sigma^2(t)d(t)}{4}) \, dt + \sigma(t)\sqrt{r(t)} \, dB(t) + \int_{u=0}^{d(t)} x(t, u)^2n \, "plug" \, du + x(t, d(t))^2 \, d'(t). \]

If the last two terms cancel out, by Definition 2.1, we obtain the right equation for \( r \). Intuitively, the last two terms will cancel out if the "plug" term includes a Dirac delta function:

\[ "plug" = -\delta(u - d^{-1}(t))d'(t), \]

provided \( d \) is invertible.

2.1.2 Application of the Infinite Divisibility of the Chi-Squared Distribution

We want to represent \( \int_0^{d(t)} x(t, u)^2 \, du \) in (2.1.1) as a sum. To this end we discretize the \( x(t, u) \) process in the \( u \) direction and replace \( \int_0^{d(t)} x(t, u)^2 \, du \) by a sum of squares of basic processes \( \{\hat{x}_i(t)\} \). Since the "number" of terms \( d(t) \) is real-valued, an idea is to approximate \( d(t) \) by the ratio \( \frac{\lfloor nd(t) \rfloor}{n} \). We then construct a sum of \( \lfloor nd(t) \rfloor \) terms, which we call 2. Z\(_{\text{low}}^{(n)}(t)\):

\[ Z^{(n)}_{\text{low}}(t) = \sum_{i=1}^{\lfloor nd(t) \rfloor} \hat{x}_i(t)^2. \]

We should then "divide" \( Z^{(n)}_{\text{low}}(t) \) by \( n \), and then make \( n \) go to infinity. The reason why we put quotes around "divide" is that the variance of \( Z^{(n)}_{\text{low}}(t)/n \) tends to zero, and thus \( Z^{(n)}_{\text{low}}(t)/n \) is not the right representation for the rate. Rather, if \( Z^{(n)}_{\text{low}}(t) \) had a non-central chi-squared distribution, we could represent \( Z^{(n)}_{\text{low}} \) as a sum of i.i.d. processes \( \{r^{(n)}_{j,\text{low}}(t)\}_j \)

\[ Z^{(n)}_{\text{low}}(t) = \sum_{j=1}^{n} r^{(n)}_{j,\text{low}}(t). \]

2.1.3 The Rate as a Double Limit

It turns out that \( Z^{(n)}_{\text{low}}(t) \) does not have the scaled non-central chi-squared distribution. However, it is possible to define a process \( r^{(n,M)}_{j,\text{mid}}(t) \) so that, conditionally on the information available at time \( \frac{(m-1)}{M}T \), the random variable \( r^{(n,M)}_{j,\text{mid}}(\frac{m}{M}T) \) has approximately the scaled non-central chi-squared distribution. Since the latter is computable, we can compute the characteristic function of \( r^{(n,M)}_{j,\text{mid}}(\frac{m}{M}T) \) conditionally on \( r^{(n,M)}_{j,\text{mid}}(\frac{(m-1)}{M}T) \). We then iterate this backward induction by calculating this characteristic function conditionally on \( r^{(n,M)}_{j,\text{mid}}(\frac{(m-2)}{M}T) \), and so far until time zero. We will then take a double limit of this characteristic function, by making both \( n \) and \( M \) go to infinity.

\(^1\)Rather than starting with a countably infinite collection of processes \( x(., u) \) we will start with a countable finite collection of processes in the formal proof \( \{\tilde{x}(., \cdot)\} \). See below.

\(^2\)The subscript \( \text{low} \) indicates that we truncate the integral into a lower value.
2.2 Definitions

Let \((\Omega, \mathcal{F}_t, \mathbb{P})\) be a probability space where the filtration \(\mathcal{F}_t\) is generated by a countably infinite collection \(\{W_i\}\) of independent Brownian motions. The probability measure \(\mathbb{P}\) is the risk-neutral measure.

Observation

Most subsequent definitions will not be needed to understand part (i) of Theorem 2.4, and the reader may skip this tedious part, which will be necessary only in the stochastic volatility section.

We start by defining

\[
\begin{align*}
     d^{(n)}(t) &= nd(t); \\
     M_{d}^{(n)} &= \max_{t \in [0,T]} d^{(n)}(t); \\
     t_{m}^{M} &= \frac{m}{M} T \quad 1 \leq m \leq M.
\end{align*}
\]  

(2.1)

Definition (2.1) will be a standing definition, in the sense that, for each \(m, M\), (2.1) will hold throughout the paper.

Assumption 1

We suppose that, over the interval \([0, T]\), \(d(t)\) satisfies Dirichlet’s condition, i.e., it is differentiable, the number \(Q\) of minima and maxima of \(d(t)\) is finite, and that the set of critical points of \(d\) (i.e. the points where \(d'(t) = 0\)) has measure zero. As mentioned above, \(d(t)\) is strictly positive. Also, we assume that \(\theta(t), b(t)\) and \(\sigma(t)\) are positive real-valued continuous functions on \([0, T]\). This ensures that (1.1) has a pathwise unique strong solution (see Magshoofi (1996)).

2.2.1 Definition of \(\tilde{x}^{(n)}_i(t)\)

We have to consider separately each decreasing and increasing branch of \(d^{(n)}(t)\). We now have 3 subcases. Accordingly, the processes \(\tilde{x}^{(n)}_i(t)\) can at any time \(t\) either:

- not jump;
- jump up by a half;
- jump down by a half.

Let \(\mathcal{J}_{\text{down}}^{(n)}(i)\) for \(i \geq 1\) be the set of solutions of:

\[
\frac{dd^{(n)}(t)}{dt} = i \quad \text{such that} \quad \frac{dd^{(n)}(t)}{dt} > 0, \quad t \in [0, T].
\]

Let \(\mathcal{J}_{\text{up}}^{(n)}(i)\) for \(i \geq 1\) be the set of solutions of:

\[
\frac{dd^{(n)}(t)}{dt} = i \quad \text{such that} \quad \frac{dd^{(n)}(t)}{dt} < 0, \quad t \in [0, T].
\]

We do not consider the case \(\frac{dd^{(n)}(t)}{dt}|_t = 0\) and \(d^{(n)}(t) = i\) \(^3\) Denote by \(N_{\text{up}}^{(n)}(i) \ (N_{\text{down}}^{(n)}(i))\) the cardinality of the set \(\mathcal{J}_{\text{up}}^{(n)}(i) \ (\mathcal{J}_{\text{down}}^{(n)}(i))\), and designate the elements of these sets followingly:

\[
\begin{align*}
    \mathcal{J}_{\text{up}}^{(n)}(i) &= \{J_{k,\text{up}}^{(n)}(i) | 0 \leq k \leq N_{\text{up}}^{(n)}(i)\}, \\
    \mathcal{J}_{\text{down}}^{(n)}(i) &= \{J_{p,\text{down}}^{(n)}(i) | 0 \leq p \leq N_{\text{down}}^{(n)}(i)\}.
\end{align*}
\]

\(^3\)It is fairly easy to note that the cases where \(t\) is such that \(\frac{dd^{(n)}(t)}{dt}|_t = 0\) and \(d^{(n)}(t) = i\) play no role in the proof (provided that the set of points where \(d\) is flat has measure zero). From now on, we continue the development without taking care of that case, in order to save space.
where for convenience we set \( J_{0,up}^{(n)}(i) = J_{0,down}^{(n)}(i) = -\infty \). Finally, we call the sets of all jumps together with the terminal time:

\[
\mathcal{J}^{(n)} = \bigcup_{i=1}^{N_{up}^{(n)}(i)} \mathcal{J}_{up}^{(n)}(i) \bigcup_{i=1}^{N_{down}^{(n)}(i)} \mathcal{J}_{down}^{(n)}(i) \bigcup \{T\}
\]

We call \( Z_{k,up}^{(n)}(i) (Z_{p,down}^{(n)}(i)) \) the first minimizer (maximizer) of \( d^{(n)} \) after \( J_{k,up}^{(n)}(i) (J_{p,down}^{(n)}(i)) \). We have then (again, barring the case \( \frac{dd^{(n)}(t)}{dt} = 0 \) and \( d^{(n)}(t) = 1 \)):

\[
J_{k,up}^{(n)}(i) < Z_{k,up}^{(n)}(i) < J_{p(k),down}^{(n)}(i) < Z_{p(k),down}^{(n)}(i) < J_{k+1,up}^{(n)}(i),
\]

where:

\[
k(p) = \begin{cases} p & \text{if } d^{(n)} \text{ is decreasing at } t = 0 \\ p + 1 & \text{if } d^{(n)} \text{ is increasing at } t = 0, \end{cases}
\]

\[
p(k) = \begin{cases} p & \text{if } d^{(n)} \text{ is decreasing at } t = 0 \\ k - 1 & \text{if } d^{(n)} \text{ is increasing at } t = 0. \end{cases}
\]

For definiteness, we set \( Z_{0,up}^{(n)}(i) = Z_{0,down}^{(n)}(i) = 0 \).

We now define recursively \( \tilde{x}^{(n)}(t) \) for all times \( t \). In the course of the definition, we also define what we mean by “times before the first jump”, “times between up and down jump” and “times between down and up jump” for a particular process \( \tilde{x}^{(n)}(t) \). We start with

\[
\tilde{x}^{(n)}(0) = \frac{r(0)}{\sqrt{d^{(n)}(0)}}.
\]

In the subcase when \( t < \min \{ J_{1,down}^{(n)}(i), J_{1,up}^{(n)}(i) \} \) (times before the first jump), we have:

\[
\tilde{x}^{(n)}(t) = \tilde{x}^{(n)}(0) \exp\left(- \int_0^t \frac{b(u)}{2} du + \int_0^t \exp(- \int_u^t \frac{b(s)}{2} ds) \frac{\sigma(u)}{2} dW_i(u) \right).
\]

In the case where \( J_{p-1,down}^{(n)}(i) \leq J_{k,up}^{(n)}(i) \leq t < J_{p,down}^{(n)}(i) \leq J_{k+1,up}^{(n)}(i) \) (times between up and down jump), we define:

\[
\tilde{x}^{(n)}(t) = \tilde{x}^{(n)}(J_{k,up}^{(n)}(i))(1 + \frac{1}{2} \frac{dd^{(n)}}{dt} |_{t = J_{k,up}^{(n)}(i)}) \exp\left(- \int_{J_{k,up}^{(n)}(i)}^t \frac{b(u)}{2} du + \int_u^t \exp(- \int_u^t \frac{b(s)}{2} ds) \frac{\sigma(u)}{2} dW_i(u) \right).
\]

In the case where \( J_{k-1,up}^{(n)}(i) \leq J_{p,down}^{(n)}(i) \leq t < J_{k,up}^{(n)}(i) \leq J_{p+1,down}^{(n)}(i) \) (times between down and up jump), we define:

\[
\tilde{x}^{(n)}(t) = \tilde{x}^{(n)}(J_{p,down}^{(n)}(i))(1 + \frac{1}{2} \frac{dd^{(n)}}{dt} |_{t = J_{p,down}^{(n)}(i)}) \exp\left(- \int_{J_{p,down}^{(n)}(i)}^t \frac{b(u)}{2} du + \int_u^t \exp(- \int_u^t \frac{b(s)}{2} ds) \frac{\sigma(u)}{2} dW_i(u) \right)
\]

We summarize this into the following definitions.

\[
\frac{d\tilde{x}^{(n)}(t)}{dt} = \frac{-b(t)}{2} + \frac{g^{(n)}(t,i)}{2}\tilde{x}^{(n)}(t) dt + \frac{\sigma(t)}{2} dW_i(t),
\]

with

\[
g^{(n)}(t,i) = \left( \sum_{k=1}^{N_{up}^{(n)}(i,t)} \delta\left(t - \frac{1}{2} J_{k,up}^{(n)}(i)\right) + \sum_{p=1}^{N_{down}^{(n)}(i,t)} \delta\left(t - J_{p,down}^{(n)}(i)\right) \right) \left. \frac{dd^{(n)}}{dt}\right|_t.
\]
Fact 1
There is no jump of $\tilde{x}_i^{(n)}(t)$ in the interval $[J_{p, \text{down}}^{(n)}(i) \leq t < J_{k(p), \text{up}}^{(n)}(i)]$. Likewise, there is no jump of $\tilde{x}_i^{(n)}(t)$ in the interval $[J_{k, \text{up}}^{(n)}(i) \leq t < J_{p(k), \text{down}}^{(n)}(i)]$.

We further define:

$$W^{(n)}(t, a, \omega) = \sum_{i=1}^{\infty} \{i - 1 < d^{(n)}(a) \leq i\} W_i(t, \omega),$$

$$x^{(n)}(t, u) = \tilde{x}_i^{(n)}(t)1\{i - 1 < u \leq i\},$$

where $1\{i - 1 < d^{(n)}(a) \leq i\}$ denotes an indicator function. We also introduce the following notations:

$$Z_{\text{mid}}^{(n,0,M)}(0) = \int_{u=0}^{d^{(n)}(t)} x^{(n)}(0, u)^2 \, du,$$

$$d\tilde{x}_i^{(n,0,M)}(t) = \frac{d^{(n)}(t)}{2}( - b(t) + g^{(n)}(t, i)) \tilde{x}_i^{(n,0,M)}(t) \, dt + \frac{\sigma(t)}{2} W_i(t) \text{ for } 0 \leq t < t_m^M,$$

and for each $1 \leq m < M$:

$$\tilde{x}_i^{(n,m,M)}(t_m^M) = \frac{Z_{\text{mid}}^{(n,m,M)}(t_m^M)}{d^{(n)}(t_m^M)} ,$$

$$d\tilde{x}_i^{(n,m,M)}(t) = \frac{1}{2}( - b(t) + g^{(n)}(t, i)) \tilde{x}_i^{(n,m,M)}(t) \, dt + \frac{\sigma(t)}{2} W_i(t) \text{ for } t_m^M \leq t < t_{m+1}^M,$$

$$x_{\text{mid}}^{(n,m,M)}(t, u) = \tilde{x}_i^{(n,m,M)}(t)1\{i - 1 < u \leq i\},$$

in which we define $Z_{\text{mid}}^{(n,m,M)}(t)$ for all $1 \leq m < M$ as:

$$Z_{\text{mid}}^{(n,m,M)}(t) = \int_{u=0}^{d^{(n)}(t)} (x^{(n,m,M)}(t, u))^2 \, du \text{ for } t_m^M \leq t < t_{m+1}^M.$$

We denote by (for the $k$ branches) lower integration bounds:

$$L_{\text{up}}(m, M, k, i, n) = \min(t_m^M, J_{k, \text{up}}^{(n)}(i))$$

and upper integration bounds:

$$U_{\text{up}}(m, k, i, n) = \min(J_{\text{next}}^{(n)}(J_{k, \text{up}}^{(n)}(i)), t_{m+1}^M),$$

where $J_{\text{next}}^{(n)}$ is defined as in [5,3].

Similarly we can define $L_{\text{down}}(m, M, p, i, n)$ and $L_{\text{down}}(m, M, k, i, n)$ for the $p$ branches. We define a remainder by

$$R^{(n,m,M)}(t) = \sum_{i=1}^{M^{(n)}_{\text{down}}(i)} \sum_{k=1}^{U_{\text{up}}(m, M, k, i, n)} \int_{L_{\text{up}}(m, M, k, i, n)} \left( (\tilde{x}_i^{(n,m,M)}(s)) \frac{dd^{(n)}}{ds} - (\tilde{x}_i^{(n,m,M)}(J_{k, \text{up}}^{(n)}(i))) \frac{dd^{(n)}}{ds} \right) ds + \int_{L_{\text{down}}(m, M, p, i, n)} \left( (\tilde{x}_i^{(n,m,M)}(s)) \frac{dd^{(n)}}{ds} - (\tilde{x}_i^{(n,m,M)}(J_{p, \text{down}}^{(n)}(i))) \frac{dd^{(n)}}{ds} \right) ds.$$
We now define \( r_{j,mid}^{(n,m,M)}(t) \) by, for \( t_M^m \leq t < t_{M+1}^m \):

\[
r_{j,mid}^{(n,m,M)}(t_M^m) = \frac{Z_{mid}^{(n,m,M)}(t_M^m)}{n},
\]

and as a consequence,

\[
r_{j,mid}^{(n,m,M)}(t) - r_{j,mid}^{(n,m,M)}(t_{M+1}^m)
= -\int_{s=t_{M+1}^m}^{t} (b(s)r_{j,mid}^{(n,m,M)}(s) + \frac{\sigma^2(s)d(n,s)}{4}) ds + \int_{s=t_{M+1}^m}^{t} \sigma(s)\sqrt{r_{j,mid}^{(n,m,M)}(s)} dB_{j,mid}^{(n,m,M)}(s)
+ R_{j,mid}^{(n,m,M)}(t) .
\]

As usual \( \{B_{j,mid}^{(n,m,M)}(s)\}_j \) are independent Brownian motions. We can thus define the process:

\[
r_{j,mid}^{(n,M)}(t) = \sum_{m=0}^{M} r_{j,mid}^{(n,m,M)}(t) 1\{t_M^m \leq t < t_{M+1}^m\} .
\] (2.2)

### 2.2.2 The Scaled Non-central (SNC) Chi-squared Distribution

For any \( \lambda > 0, \lambda_2 \geq 0 \) and \( c > 0 \), and for any \( x \in \mathbb{R} \), we define:

\[
g_{\lambda_1,\lambda_2,c}(x) = \frac{1}{c^2} \sum_{i=0}^{\infty} \frac{e^{-\lambda_2/2}(\lambda_2/2)^i}{i!} f_{\lambda_1+2i}(\frac{x}{c^2}),
\]

where

\[
f_{\lambda}(x) = \begin{cases} \frac{x^{\lambda-1}e^{-x/2}}{2^{\lambda/2}\Gamma(\lambda/2)} & \text{if } x \geq 0 \\ 0 & \text{otherwise}. \end{cases}
\]

It is straightforward to verify that \( g_{\lambda_1,\lambda_2,c} \) is a probability density. We say that a random variable \( X \sim \chi^2(\lambda_1, \lambda_2, c) \) if the density of \( X \) is \( g_{\lambda_1,\lambda_2,c} \). In words, \( X \) is a scaled non-central chi-square with real-valued degrees of freedom \( \lambda_1 \). When \( c = 1 \) and \( \lambda_1 \) is integer, we obtain the standard non-central chi-square distribution. When \( \lambda_1 \) is integer the random variable \( X \) is the sum of squares of independent normal random variables \( X_i \) with mean \( \mu_i \) and variance \( \sigma^2 \):

\[
X = \sum_{i=1}^{\lambda_1} X_i^2
\]

where the parameter

\[
\lambda_2 = \sum_{i=1}^{\lambda_1} \frac{\mu_i^2}{\sigma^2}
\]

As the usual chi-square distribution, its generalization to real-valued degrees of freedom is infinitely divisible.

**Theorem 2.3** For any \( X \sim \chi^2(\lambda_1, \lambda_2, c) \) there exists \( n \) independent and identically distributed random variables \( X_1, \ldots, X_n \) such that:

\[
X \overset{d}{=} \sum_{k=1}^{n} X_k.
\]

Moreover:

\[
X_1 \sim \chi^2(\frac{\lambda_1}{n}, \frac{\lambda_2}{n}, \frac{c}{n}).
\]
2.3 Main Result

Theorem 2.4 Suppose Assumption 1 holds. Then

(i) Assume that \( d(\cdot) \) is differentiable over \([0,T]\), bounded away from zero, and has a finite number of optima. For each \( t \in [0,T] \) the characteristic function of \( r(t) \) is: for \( \omega \in \mathbb{R} \),

\[
E[\exp(i\omega r(t))] = \exp \left( \frac{i\omega r(0) e^{-\int_0^t b(u) du}}{1 - 2i\omega \Sigma(0,t)} + \int_0^t \exp\left(-\int_s^t b(u) du\right) \theta(s) \frac{ds}{1 - 2i\omega \Sigma(s,t)} \right), \tag{2.3}
\]

where

\[
\Sigma(s,t) := \frac{1}{4} \int_s^t \exp\left(-\int_v^t b(u) du\right) \sigma^2(v) dv.
\]

(ii) For any \( t \in [0,T] \), the following convergence holds in distribution:

\[
\lim_{n \to \infty} r_{1,mid}^{(n,n^{4Q})}(t) = r(t),
\]

where the definition of \( r_{1,mid}^{(n,n^{4Q})}(t) \) is given in (2.2).

Remarks:

1. The representation (5.5), which together with (5.6) and (5.7), states that:

\[
E[\exp(i\omega r(t))] = \exp \left( \frac{i\omega r(0) e^{-\int_0^t b(u) du}}{1 - 2i\omega \Sigma(0,t)} + \int_0^t \exp\left(-\int_s^t b(u) du\right) \theta(s) \frac{ds}{1 - 2i\omega \Sigma(s,t)} \right).
\]

Therefore Equation (2.3) results from the following integration by parts

\[
\frac{1}{2} \int_0^t d'(s) \left( 1 - 2i\omega \int_s^t e^{-\int_v^t b(u) du} \sigma^2(v)/4 dv \right) ds
= -\frac{d(0)}{2} \log \left( 1 - 2i\omega \int_s^t e^{-\int_v^t b(u) du} \sigma^2(v)/4 dv \right) + \frac{1}{2} \int_0^t \frac{d(s)}{1 - 2i\omega \int_s^t e^{-\int_v^t b(u) du} \sigma^2(v)/4 dv} ds.
\]

2. When \( b, \sigma, \) and \( d \) are constants over \([0,T]\), i.e., \( d'(s) = 0 \), we see from (2.4) that \( r(t) \) has the SNC chi-squared distribution with \( d(0) \) degrees of freedom. Our characteristic function thus properly generalizes a well-known result.

3. Casual observation of Lemma (5.12) seems to highlight a much simpler proof: namely, approximate the time-dependent parameters by step functions, i.e., construct an Euler approximation of (1.1), calculate its characteristic function like in Lemma (5.12), and make the time-step go to zero. Unfortunately, we could not find any proof of strong convergence of the Euler approximation for the CIR process: in our proof we had to resort to the extra power coming from our representation (2.3), and to weak convergence. Even if there was a classical proof of strong convergence of the Euler approximation with deterministic volatility, the extension to stochastic volatility would be difficult, unlike what we obtain in the next section.

4. We have not found in the literature any distribution with characteristic function equal to (2.3). This seems to be a new, or at least independently rediscovered distribution. We discuss in the stochastic volatility section a method to approximate this distribution by its moments.
5. It can be verified that (2.3) solves the equivalent of the Fokker-Planck equation for characteristic functions. We emphasize again that the main strength of our method of proof, compared to solving the Fokker-Planck equation directly, is that it can be extended to stochastic volatility.

The Fokker-Planck equation for the density $f(r, t)$ of $r(t)$ given in (1.1) is:

\[
\frac{\partial f(r,t)}{\partial t} = -\frac{\partial}{\partial r}[(b(t)r + \theta(t))f(r,t)] + \frac{1}{2} \frac{\partial^2}{\partial r^2} [\sigma^2(t)r f(r,t)]
\]

\[
= b(t)f(r,t) + (\sigma^2(t) + b(t)r - \theta(t)) \frac{\partial f(r,t)}{\partial r} + \frac{\sigma^2(t)}{2} r \frac{\partial^2 f(r,t)}{\partial r^2}.
\]

Let the Fourier transform with respect to $x$ be defined as $\hat{f}(x, t) = \frac{1}{\sqrt{2\pi}} \int e^{-ix \tau} f(r,t) \, d\tau$, then the Fourier transform with respect to $x$ of the Fokker-Planck equation is given as

\[
\frac{\partial \hat{f}(x,t)}{\partial t} = -b(t)x \frac{\partial \hat{f}(x,t)}{\partial x} - \theta(t)ix \hat{f}(x,t) - \frac{\sigma^2(t)}{2} x^2 \frac{\partial^2 \hat{f}(x,t)}{\partial x^2}.
\]

Writing $\Phi(x,t) = E[e^{ixr(t)}]$, then:

\[
\Phi(x,t) = \exp\left( ix \left( \frac{r(0)e^{-\int_0^t b(u) \, du}}{1 - 2ix \int_0^t e^{-\int_0^s b(u) \, du} \sigma^2(v) / 4 \, dv} \right) + \int_0^t \frac{\theta(s)e^{-\int_s^t b(u) \, du}}{1 - 2ix \int_s^t e^{-\int_s^u b(u) \, du} \sigma^2(v) / 4 \, dv} \, ds \right).
\]

Hence the following relation holds:

\[
\Phi(-x,t) = \sqrt{2\pi} \hat{f}(x,t).
\]

We first compute the left hand side (LHS) of (2.4). Observe that

\[
\sqrt{2\pi} \frac{\partial \hat{f}(x,t)}{\partial t} = \Phi(-x,t) \left( -ix \frac{\partial}{\partial t} \left( \frac{r(0)e^{-\int_0^t b(u) \, du}}{1 + 2ix \int_0^t e^{-\int_0^s b(u) \, du} \sigma^2(v) / 4 \, dv} \right) + \int_0^t \frac{\theta(s)e^{-\int_s^t b(u) \, du}}{1 + 2ix \int_s^t e^{-\int_s^u b(u) \, du} \sigma^2(v) / 4 \, dv} \, ds \right).
\]

We calculate

\[
\frac{\partial}{\partial t} \left( \frac{r(0)e^{-\int_0^t b(u) \, du}}{1 + 2ix \int_0^t e^{-\int_0^s b(u) \, du} \sigma^2(v) / 4 \, dv} \right) + \int_0^t \frac{\theta(s)e^{-\int_s^t b(u) \, du}}{1 + 2ix \int_s^t e^{-\int_s^u b(u) \, du} \sigma^2(v) / 4 \, dv} \, ds
\]

\[
= \frac{r(0)e^{-\int_0^t b(u) \, du}(-b(t) - ix\sigma^2(t)/2)}{(1 + 2ix \int_0^t e^{-\int_0^s b(u) \, du} \sigma^2(v) / 4 \, dv)^2} + \theta(t) + \int_0^t \frac{\theta(s)e^{-\int_s^t b(u) \, du}(-b(t) - ix\sigma^2(t)/2)}{1 + 2ix \int_s^t e^{-\int_s^u b(u) \, du} \sigma^2(v) / 4 \, dv} \, ds
\]

\[
= \theta(t) - \left( b(t) + \frac{ix\sigma^2(t)}{2} \right) \left( \frac{r(0)e^{-\int_0^t b(u) \, du}}{(1 + 2ix \int_0^t e^{-\int_0^s b(u) \, du} \sigma^2(v) / 4 \, dv)^2} \right) + \int_0^t \frac{\theta(s)e^{-\int_s^t b(u) \, du}}{1 + 2ix \int_s^t e^{-\int_s^u b(u) \, du} \sigma^2(v) / 4 \, dv} \, ds.
\]

Secondly we compute the following item for the right hand side (RHS) of (2.4):

\[
\sqrt{2\pi} \frac{\partial \hat{f}(x,t)}{\partial x} = \Phi(-x,t) \frac{\partial}{\partial x} \left( \frac{-ixr(0)e^{-\int_0^t b(u) \, du}}{1 + 2ix \int_0^t e^{-\int_0^s b(u) \, du} \sigma^2(v) / 4 \, dv} + \int_0^t \frac{-ix\theta(s)e^{-\int_s^t b(u) \, du}}{1 + 2ix \int_s^t e^{-\int_s^u b(u) \, du} \sigma^2(v) / 4 \, dv} \, ds \right).
\]

On one hand,
On the other hand,

\[
\frac{\partial}{\partial x} \left( \frac{-ixr(0)e^{-\int_0^t b(u) \, du}}{1 + 2ix \int_0^t e^{-\int_s^t b(u) \, du} \sigma^2(v) / 4 \, dv} \right) = -\frac{ir(0)e^{-\int_0^t b(u) \, du}}{(1 + 2ix \int_0^t e^{-\int_s^t b(u) \, du} \sigma^2(v) / 4 \, dv)^2}.
\]

Multiplying both sides of Equation (2.4) by $\sqrt{2\pi/\Phi(-x, t)}$ we get:

\[
LHS = -ix\theta(t) + ix \left( b(t) + ix \frac{\sigma^2(t)}{2} \right) \left( \frac{r(0)e^{-\int_0^t b(u) \, du}}{1 + 2ix \int_0^t e^{-\int_s^t b(u) \, du} \sigma^2(v) / 4 \, dv} + \int_0^t \frac{\theta(s)e^{-\int_s^t b(u) \, du}}{(1 + 2ix \int_0^t e^{-\int_s^t b(u) \, du} \sigma^2(v) / 4 \, dv)^2} \, ds \right),
\]

and

\[
RHS = -ix\theta(t) - \left( \frac{i\sigma^2(t)}{2} \sigma^2 + b(t)\sigma^2 \right) \left( -\frac{ir(0)e^{-\int_0^t b(u) \, du}}{1 + 2ix \int_0^t e^{-\int_s^t b(u) \, du} \sigma^2(v) / 4 \, dv} - \int_0^t \frac{i\theta(s)e^{-\int_s^t b(u) \, du}}{(1 + 2ix \int_0^t e^{-\int_s^t b(u) \, du} \sigma^2(v) / 4 \, dv)^2} \, ds \right).
\]

It is easy to see LHS and RHS are algebraically equal. Thus the Fokker-Planck equation (2.4) has been verified.

### 3 Option Pricing

We call $P(t, T)$ the price at time $t$ of a zero-coupon bond with maturity $T$, i.e.:

\[
P(t, T) = E\left[ \exp\left( -\int_t^T r(s) \, ds \right) | \mathcal{F}_t \right].
\]

A standard result (see, e.g. Shreve (2004)) is that:

\[
P(t, T) = \exp \left( -r(t)C(t, T) + A(t, T) \right),
\]

where $C(t, T)$ satisfies the Riccati equation:

\[
C_t(t, T) - b(t)C(t, T) - \frac{1}{2} \sigma^2(t)C(t, T)^2 + 1 = 0 \quad \text{for } t \geq T
\]

\[
C(T, T) = 0
\]

and:

\[
A(t, T) = -\int_t^T \theta(u)C(u, T) \, du.
\]

There is a well-known analytical solution in case the parameters $b, \sigma$ and $\theta$ are constant.
3.1 Call Option Price

By moving to the forward measure, we can separate discounting and evaluation of the expected value of the payoff. We can thus apply Theorem 2.4.

Let now $0 \leq t \leq T$. The price $C(0)$ at initial time 0 of a European option (expiring at time $t$ and with exercise price $K$) on the $T$ maturity bond is given by:

$$C(0) = E\left[ \exp\left(-\int_0^t r(u) \, du\right) \max(P(t, T) - K, 0) \right].$$

The $t$-forward probability measure is the probability measure $P^t$ under which $P(\cdot, t)$ is a martingale. In this measure the process $\{B^t(s)\}_s$ is Brownian motion where:

$$dB^t(s) = dB(t) - \sigma_P(s, t) \, dt,$$

where the bond volatility $\sigma_P(s, t)$ is given by:

$$\sigma_P(s, t) = \begin{cases} -\sigma(s)\sqrt{r(s)C(s, t)} & \text{if } 0 \leq s \leq t \\ 0 & \text{else.} \end{cases}$$

Thus:

$$dr(s) = \left( (-b(s) + \sigma^2(s)C(s, t))r(s) + \theta(s) \right) ds + \sigma(s)\sqrt{r(s)} dB^t(s).$$

We define:

$$b^t(s) = -b(s) + \sigma^2(s)C(s, t) \quad \Sigma^t(s, t) = \frac{1}{4} \int_0^t \exp\left(-\int_s^t b^t(u) \, du\right) \sigma^2(s) \, ds$$

as well as the Laplace transform of the density of $r(t)$ in the $t$-forward measure:

$$\hat{F}_{r(t)}(p) = \exp\left(-p\left(\frac{r(0)e^{-\int_0^t b^t(u) \, du}}{1 + 2p\Sigma^t(0, t)} + \int_0^t \exp\left(-\int_0^t \int_s^t b^t(u) \, du\right) \theta(s) \right) ds \right).$$

**Theorem 3.1** Let $C(s, T)$ and $A(s, T)$ solve (3.1) and (3.2). The price of a call option is given by Laplace inversion:

$$C(0) = \frac{P(0, t)}{2\pi i} \lim_{b \to \infty} \int_{a-i \infty}^{a+i \infty} e^{p \hat{C}_t(p; t)} \hat{F}_{r(t)}(p) \, dp, \quad (3.3)$$

where,

$$\hat{C}_t(p) = \frac{Kp - C(t, T) e^{A(t, T)} - (p + C(t, T)) K^{p+1}}{C(t, T)p - p^2}. \quad (3.4)$$

**Proof** Using the change of measure developed by Jamshidian (1989) and Geman (1989):

$$C(0) = P(0, t) E^{pt}\left[ \max(\exp(-r(t)C(t, T) + A(t, T)) - K, 0) \right].$$
The Laplace transform of $\max(P(t,T) - K, 0)$ is (see, e.g., Lewis (2000)):

$$\int_0^\infty e^{-pt} \max \left( e^{-C(t,T)r + A(t,T)} - K, 0 \right) dr = \hat{C}(p),$$

which is equal to the right hand side of Equation (3.4). It is well-known (see, e.g., Lewis (2000) again) that the price of an option is given by the discounted inverse Laplace transform of the product of the transform of the payoff and the transform of the distribution. Therefore we obtain (3.3). □

4 Stochastic Volatility

We now assume that both $\sigma$ and $\theta$ are continuous functions of infinite variation, but that the dimension $d(t)$ satisfies assumption 1. This assumption is not unrealistic, in the sense that only the moments of $v$ intervene in the calculations of the moments of $r(t)$. Given a time-series of $r$, it would actually be difficult to invalidate the hypothesis that $\sigma$ is of infinite variation. Thus, given the limited practical use of generalizing our results to the case where $d$ can be of infinite variation, we leave this task as an open conjecture for future research. There is an interesting parallel between the deterministic and the stochastic volatility case. In the deterministic case it was deemed natural since Magshoodi (1996) to take the simplifying assumption that $d$ be integer-valued. In the stochastic case, we deem natural to consider the case where $d$ if of finite variation.

We now consider two cases. The easiest case to consider is when the volatility is independent on the interest rate. We then build a tractable model where volatility is correlated with the interest rate.

4.1 Independent Volatility

We consider the model

$$dr(t) = (-b(t)r(t) + \theta(t)) dt + \sqrt{v(t)r(t)} dB(t),$$  \hspace{1cm} (4.1)

$$r(0) = r_0 > 0;$$

$$dv(t) = \mu(v(t), t) dt + \xi(v(t), t) dB^v(t),$$

$$v(0) = v_0 > 0,$$  \hspace{1cm} (4.2)

where the coefficients $\mu$ and $\xi$ are such that $0 < v(t) < K$ almost surely. We assume

$$dB(t) dB^v(t) = 0.$$

Under Assumption 1, all the arguments in the proof of Theorem 2.4 hold, and, by conditioning on the path of volatility, we obtain the characteristic function of the rate:

$$E[\exp(i\omega r(t))] = E \left[ \exp \left( i\omega \left( r(0)e^{-\int_0^t b(u) du} \right) + \int_0^t \exp(-\int_0^t b(u) du)\theta(s) ds \right) \right].$$

(4.3)

When volatility is stochastic, it is often possible to write the price of a bond at time $t$ as a function $f$ of both the rate and volatility:

$$P(t, T, \omega) = f(r(t, \omega), v(t, \omega), t; T)$$

In this case, to calculate the price $C(0)$ of a bond option, one is more interested in the conditional Laplace transform function of the density of the rate, given volatility $v(t)$ at time $t$, which we call $\tilde{F}_r(t|v(t))$. One can then develop the latter in a Taylor series:

$$\tilde{F}_r(t|v(t))(p) = E \left[ \exp \left( -p \left( r(0)e^{-\int_0^t b'(u) du} \right) + \int_0^t \exp(-\int_0^t b'(u) du)\theta(s) ds \right) \right] |v(t)|$$

$$= \sum_{k=0}^{\infty} \frac{(-p)^k}{k!} \frac{d^k \tilde{F}_r(t|v(t))}{dp^k} \bigg|_{p=0}.$$
For instance the first term is:

\[
\frac{dF_t(v(t))}{dp} \bigg|_{p=0} = -E \left[ \left( r_0 e^{-\int_0^t b(u)\,du} + \int_0^t \exp\left(-\int_s^t b(u)\,du\right)\theta(s) \right) v(t) \right].
\]  

(4.4)

It is well-known that the derivative of the Laplace transform of the density is equal to the negative of the first moment. By Itô’s lemma, we can verify that the (conditional) first moment of the rate is equal to the negative of the right hand-side of (4.4), which provides a pleasant confirmation of our result. Since \( \theta(s) = d(s)\sigma^2(s)/4 \), it is enough to compute of \( E[\sigma^2(s)|v(t)] \) in order to compute explicitly (4.4). We can then easily reconstruct the conditional distribution of \( r(t) \) given \( v(t) \) by inverse Laplace transformation, provided convergence conditions are met.

4.2 A Tractable Model with Correlated Volatility

We consider the following model of volatility, which is a special case of (4.2).

\[
dw(t) = (-b(t)v(t) + \theta_w(t))\,dt + \sqrt{v(t)}\,dB(t),
\]

\[
r(0) = r_0 > 0;
\]

\[
dv(t) = (-b(t)v(t) + \theta_v(t))\,dt + \xi v(t)\,dB^\nu(t),
\]

\[
v(0) = v_0 > 0;
\]

\[
r(t) = w(t) + v(t).
\]  

(4.5)

We let

\[
d_w(t) = \frac{4\theta_w(t)}{v^2(t)},
\]

\[
d_v(t) = \frac{\theta_v(t)}{4v^2(t)}
\]

and suppose that both of them satisfy assumption (1). Even if \( B \) and \( B^\nu \) are uncorrelated, \( r(t) \) is correlated with \( v(t) \). It is then clear that

\[
E[\exp(i\omega r(t))|v(t)] = E \left[ \exp \left( i\omega \left( \frac{w(0)e^{-\int_0^t b(u)\,du}}{1 - 2i\omega\Sigma(0,t)} + \int_0^t \exp \left(-\int_s^t b(u)\,du\right)\theta(s) \right) ds \right) \right] |v(t)|
\]

\[
\times \exp(i\omega v(t)).
\]

The advantage of this model is that the conditional characteristic function of the rate is explicit. Is this model a stochastic CIR model? We would like to leave this exciting (and, as far as we know, unsolved) question as an open problem to the community.

**Open problem:**

Is there a Brownian motion \( B_\nu(t) \) such that the distribution of \( r(t) \) defined in (4.5) agrees with the distribution at time \( t \) of the solution of the following SDE:

\[
dx(t) = -b(t)x(t)\,dt + (\theta_w(t) + \theta_v(t))\,dt + \sqrt{v(t)}x(t)\,dB_\nu(t) \?
\]  

(4.6)

**Observation**

One avenue of proof is to reuse our framework. Indeed, let:

\[
d\tilde{x}^{(n)}_{i,w}(t) = -\frac{b(t)}{2} x^{(n)}_{i,w}(t)\,dt + \frac{\sqrt{v(t)}}{2} \,dW_{i,w}(t),
\]

\[
d\tilde{x}^{(n)}_{i,v}(t) = -\frac{b(t)}{2} x^{(n)}_{i,v}(t)\,dt + \frac{\sqrt{v(t)}}{2} \,dW_{i,v}(t),
\]

\[
E[\exp(i\omega r(t))|v(t)] = E \left[ \exp \left( i\omega \left( \frac{w(0)e^{-\int_0^t b(u)\,du}}{1 - 2i\omega\Sigma(0,t)} + \int_0^t \exp \left(-\int_s^t b(u)\,du\right)\theta(s) \right) \right) \right] |v(t)|
\]

\[
\times \exp(i\omega v(t)).
\]

The advantage of this model is that the conditional characteristic function of the rate is explicit. Is this model a stochastic CIR model? We would like to leave this exciting (and, as far as we know, unsolved) question as an open problem to the community.

**Open problem:**

Is there a Brownian motion \( B_\nu(t) \) such that the distribution of \( r(t) \) defined in (4.5) agrees with the distribution at time \( t \) of the solution of the following SDE:

\[
dx(t) = -b(t)x(t)\,dt + (\theta_w(t) + \theta_v(t))\,dt + \sqrt{v(t)}x(t)\,dB_\nu(t) \?
\]  

(4.6)

**Observation**

One avenue of proof is to reuse our framework. Indeed, let:

\[
d\tilde{x}^{(n)}_{i,w}(t) = -\frac{b(t)}{2} x^{(n)}_{i,w}(t)\,dt + \frac{\sqrt{v(t)}}{2} \,dW_{i,w}(t),
\]

\[
d\tilde{x}^{(n)}_{i,v}(t) = -\frac{b(t)}{2} x^{(n)}_{i,v}(t)\,dt + \frac{\sqrt{v(t)}}{2} \,dW_{i,v}(t),
\]
where \( W_{i,w} \) and \( W_{i,v} \) are independent Brownian motions, and \( g_{w}^{(n)}(t, i) \) and \( g_{v}^{(n)}(t, i) \) are defined in a manner similar with \( \Phi \). Then, (conditionally on \( v \)), independent copies \( w_{j,low}^{(n,M)} \) and \( v_{j,low}^{(n,M)} \) exist that satisfy the representation:

\[
\sum_{j=1}^{n} w_{j,low}^{(n,M)}(t_{m}) = \sum_{i=1}^{[nd_{w}(t)]} x_{i,w}^{2}(t_{m})
\]

\[
\sum_{j=1}^{n} v_{j,low}^{(n,M)}(t_{m}) = \sum_{i=1}^{[nd_{v}(t)]} x_{i,v}^{2}(t_{m})
\]

It then remains to prove that, for some appropriately defined processes \( u_{1,mid}^{(n,M)} \) and \( v_{1,mid}^{(n,M)} \):

\[
r^{(n,M)} = u_{1,mid}^{(n,M)} + v_{1,mid}^{(n,M)}
\]

converges somehow to a process \( r^{(\infty, \infty)} \) that satisfies \( 4.6 \), and for which \( r^{(\infty, \infty)}(t) \) agrees with \( r(t) \) in distribution.

**Acknowledgements**

We would like to thank the participants at the 2014 Claremont Symposium on Interest rates, as well as the participants of the 2014 Bachelier conference. We would also thank Professor John Angus for the valuable communication with him. We received very helpful advice from two anonymous referees. All errors are ours.

### 5 Appendix

#### 5.1 Proof of Theorem \( 2.3 \)

The characteristic function of \( Y \sim \chi^{2}(\lambda_{1}) \) when \( \lambda_{1} > 0 \) is real-valued is:

\[
\Phi_{Y}(\omega) = (1 - 2i\omega)^{-\lambda_{1}/2}.
\]

(5.1)

Let \( X \sim \chi^{2}(\lambda_{1}, \lambda_{2}, 1) \). It follows from (5.1) that:

\[
\Phi_{X}(\omega) = \int_{\mathbb{R}} e^{i\omega x} g_{\lambda_{1},\lambda_{2}}(x) \, dx = \sum_{k=0}^{+\infty} \frac{e^{-\lambda_{2}/2}(\lambda_{2}/2)^{k}}{k!} \int_{0}^{+\infty} e^{i\omega x} f_{\lambda_{1}+2k}(x) \, dx
\]

\[
= \sum_{k=0}^{+\infty} \frac{e^{-\lambda_{2}/2}(\lambda_{2}/2)^{k}}{k!} (1 - 2i\omega)^{-\lambda_{1}/2-k} = (1 - 2i\omega)^{-\lambda_{1}/2} \sum_{k=0}^{+\infty} \frac{e^{-\lambda_{2}/2}(\lambda_{2}/2)(1 - 2i\omega)^{-1}}{k!}
\]

\[
= \exp\left(\frac{i\omega}{1 - 2i\omega}\right)^{\lambda_{1}/2}.
\]

(5.2)

On the other hand, since \( X_{1}, \ldots, X_{n} \) are i.i.d,

\[
\Phi_{\sum_{k=1}^{n} X_{k}}(t) = (\Phi_{X_{1}}(t))^{n}.
\]

By the facts that \( X = \sum_{k=1}^{n} X_{k} \), we obtain

\[
\Phi_{X_{1}}(\omega) = (\Phi_{X}(\omega))^{1/n} = \frac{\exp\left(\frac{i(\lambda_{2}/n)\omega}{1 - 2i\omega}\right)}{(1 - 2it)^{(\lambda_{1}/n)/2}}, \text{ for } k = 0, \ldots, n - 1.
\]

The fact that \( \Phi_{X_{1}}(0) = 1 \) guarantees

\[
\Phi_{X_{1}}(\omega) = \frac{\exp\left(\frac{i(\lambda_{2}/n)\omega}{1 - 2i\omega}\right)}{(1 - 2i\omega)^{\lambda_{1}/2}}
\]
Again, it results from (5.2) and the above equation that:
\[ X_1 \sim \chi^2\left(\frac{\lambda_1}{n}, \frac{\lambda_2}{n}, 1\right). \]

The case where \( c \neq 1 \) follows trivially. □

### 5.2 Proof of Theorem 2.4

The proofs of the lemmas necessary to this proof are in a second section of Appendix.

For convenience, we define a function \( J_{next}^{(n)}(t) \) which associates to a point in time \( t \) the next jump after \( t \):
\[
J_{next}^{(n)}(t) = \min\{J \in J^{(n)} | J > t\}. \tag{5.3}
\]

The stochastic Fubini theorem (Ikeda and Watanabe 1981, Williams 1979 p. 44, Heath Jarrow Morton 1992, Lemma 0.1) states the following.

**Lemma 5.1** [Heath-Jarrow-Morton (1992)]

Let \( \{\Phi(t, a, \omega) : (t, a) \in [0, \tau] \times [0, \tau]\} \) be a family of real random variables such that:

(i) \((t, \omega, a) \in ([0, \tau] \times [0, \tau]) \rightarrow \Phi(t, a, \omega) \) is \( L \times B[0, \tau] \) measurable where \( L \) is the predictable \( \sigma \)-field;

(ii) \( \int_0^t \Phi^2(s, a, \omega) \, ds < \infty \) a.e. for all \( t \in [0, \tau] \) and \( \omega \in \Omega \);

(iii) \( \int_0^t \left( \int_0^\tau \Phi(s, a, \omega) \, da \right)^2 \, ds < \infty \) a.e. for all \( t \in [0, \tau] \) and \( \omega \in \Omega \).

If \( t \rightarrow \int_0^\tau \int_0^\tau \Phi(s, a, \omega) \, dW_s(\omega) \, da \) is continuous a.e. for all \( t \in [0, \tau] \) and \( \omega \in \Omega \), then:
\[
\int_0^t \int_0^\tau \Phi(s, a, \omega) \, dW_s(\omega) = \int_0^\tau \int_0^\tau \Phi(s, a, \omega) \, dW_s(\omega) \, da,
\] a.e. for all \( t \in [0, \tau] \) and \( \omega \in \Omega \).

**Definition 5.2** We define:

\[
\mu^{(n)}(t, u) = \sum_{i=1}^{\infty} (-b(s) + g(t, i))x_i^{(n)}(t)^2 \frac{\sigma^2(t)}{4} 1\{i-1 < u \leq i\};
\]
\[
\Phi^{(n)}(t, u) = \sum_{i=1}^{\infty} \sigma(t)x_i^{(n)}(t, u);
\]
\[
y^{(n)}(t, u) = x_i^{(n)}(t, u)^2;
\]
\[
Z^{(n)}_{mid}(t) = \int_{u=0}^\tau y^{(n)}(t, u) \, du.
\]

The SDE for \( y^{(n)}(t, u) \) spells then:
\[
dy^{(n)}(t, u) = \mu^{(n)}(t, u) \, dt + \Phi^{(n)}(t, u) \, d\tilde{W}^{(n)}(t, u).
\]

Using the convention that the differential operator \( d_s \) applies to the first parameter of \( W^{(n)}(s, a) \), this definition results in:
\[
\int_0^\tau \int_0^\tau \Phi^{(n)}(s, a) \partial_s W^{(n)}(s, a) \, da = \sum_{i=1}^{\infty} \int_0^\tau 1\{i-1 < d^{(n)}(a) \leq i\} \int_0^\tau \sigma(s)x_i^{(n)}(s) \, dW_i(s) \, da.
\]

We have then the obvious lemma.
Lemma 5.3 For a.e. \( t \in [0, \tau] \), the following equation holds almost surely:

\[
\int_{s=0}^{t} \int_{a=0}^{\tau} \Phi^{(n)}(s, a) \, da \, dB(s, a) = \int_{a=0}^{\tau} \int_{s=0}^{t} \Phi^{(n)}(s, a) \, da \, W(s, a) \, da.
\]

Definition 5.4 We define the remainder:

\[
R^{(n)} = \sum_{i=1}^{M^{(n)}_{\uparrow}} \sum_{k=1}^{N_{\uparrow}(i)} J^{(n)}_{k, \uparrow}(i) \int_{s=J^{(n)}_{k, \uparrow}(i)}^{t} \left( \tilde{x}^{(n)}_{i}(s) \right)^2 \frac{dd^{(n)}}{ds} \mid_{s} - \frac{d^{(n)}}{ds} \left( \tilde{x}^{(n)}_{i}(J^{(n)}_{k, \uparrow}(i)) \right) \, ds + \sum_{p=1}^{N_{\downarrow}(i)} J^{(n)}_{k, \downarrow}(i) \int_{s=J^{(n)}_{k, \downarrow}(i)}^{t} \left( \tilde{x}^{(n)}_{i}(s) \right)^2 \frac{dd^{(n)}}{ds} \mid_{s} - \frac{d^{(n)}}{ds} \left( \tilde{x}^{(n)}_{i}(J^{(n)}_{k, \downarrow}(i)) \right) \, ds.
\]

Lemma 5.5 There exists a collection of Brownian motions \( B^{(n)}_{Z, \text{mid}} \) such that:

\[
Z^{(n)}_{\text{mid}}(t) - Z^{(n)}_{\text{mid}}(0) = \int_{s=0}^{t} -b(s)Z^{(n)}_{\text{mid}}(s) - \frac{\sigma^{2}(s)d^{(n)}(s)}{4} \, ds + \int_{s=0}^{t} \sigma(s)\sqrt{Z^{(n)}_{\text{mid}}(s)} \, dB^{(n)}_{Z, \text{mid}}(s) + R^{(n)}(t).
\]

Fix a time \( t^{M}_{m} \). We cannot estimate the remainder \( R^{(n)}(t) \) properly since each term \( \tilde{x}^{(n)}_{i} \) may have jumped over \([0, t^{M}_{m}]\), which makes them of different orders of magnitude. To this effect, we refine our representation by resetting it at each time \( t^{M}_{m} \). This allows us to better control the remainder.

Lemma 5.6 There exists a collection of Brownian motions \( B^{(n,m,M)}_{Z, \text{mid}} \) such that, for \( t^{M}_{m} \leq t \leq t^{M}_{m+1} \):

\[
Z^{(n,m,M)}_{\text{mid}}(t) - Z^{(n,m,M)}_{\text{mid}}(t^{M}_{m}) = \int_{s=t^{M}_{m}}^{t} -b(s)Z^{(n,m,M)}_{\text{mid}}(s) - \frac{\sigma^{2}(s)d^{(n)}(s)}{4} \, ds + \int_{s=t^{M}_{m}}^{t} \sigma(s)\sqrt{Z^{(n,m,M)}_{\text{mid}}(s)} \, dB^{(n,m,M)}_{Z, \text{mid}}(s) + R^{(n,m,M)}(t).
\]

We notice that the proof of Lemma 5.5 can be carried over exactly to Lemma 5.6 so we omit it.

Lemma 5.7 Let \( M(n) = n^{4Q} \). The remainder \( R^{(n,m,M(n))}(t)/n \) converges to zero in probability when \( n \to \infty \).

We define

\[
\tilde{Z}^{(n,M)}_{\text{mid}}(t) = \sum_{m=0}^{M} Z^{(n,m,M)}_{\text{mid}}(t)1\{t^{M}_{m} \leq t < t^{M}_{m+1}\}
\]

By the fact that \( Z^{(n,m-1,M)}_{\text{mid}}(t^{M}_{m}) = Z^{(n,m,M)}_{\text{mid}}(t^{M}_{m}) \), the trajectory \( \tilde{Z}^{(n,M)}_{\text{mid}}(\cdot, \omega) \) is continuous.

Lemma 5.8 The trajectory \( r_{j, \text{mid}}^{(n,m,M)}(\cdot, \omega) \) is continuous almost surely. Also, the following relations hold:

\[
r_{j, \text{mid}}^{(n,m-1,M)}(t^{M}_{m}) = \frac{r_{j, \text{mid}}^{(n,m,M)}(t^{M}_{m})}{n},
\]

\[
\tilde{Z}^{(n,M)}_{\text{mid}}(t^{M}_{m+1}) = \sum_{j=1}^{n} r_{j, \text{mid}}^{(n,m,M)}(t^{M}_{m+1}).
\]
Lemma 5.9 Take $M(n) = n^{4Q}$. The process $r_{j, mid}^{(n, M(n))}$ converges to $r$ in distribution when $n$ goes to $\infty$.

Definition 5.10 We define:

$$
d_{low}(s, u) = \left[ \min_{s \leq t < u} d^{(n)}(t) \right];
$$

$$
Z_{\chi^2}^{(n, m, M)}(t) = \sum_{i=0}^{d_{low}(t, \sigma^2(v)/4 \mathrm{d}v)} (\tilde{X}_i(t))^{2} i_{m}^{M} \leq t < i_{m+1}^{M};
$$

$$
\tilde{Z}_{\chi^2}^{(n, M)}(t) = \sum_{m=1}^{M} Z_{\chi^2}^{(n, m, M)}(t) 1\{i_{m}^{M} \leq t < i_{m+1}^{M}\}.
$$

We also say that a $\mathcal{F}_t$ - measurable random variable $X$ satisfies

$$
X = o(\Delta; t)
$$

if

$$
E[X(t)] = o(\Delta).
$$

Each term in the sum in the RHS of (5.4) has (conditionally on $\mathcal{F}_{t_M}$) the SNC chi-squared distribution with one degree of freedom, and the same scale factor $c^{(m, n, M)}$. Thus $Z_{\chi^2}^{(n)}(t_{m}^{M})$ has (conditionally on $\mathcal{F}_{t_{M}}$) the SNC chi-squared distribution with $d_{low}(t_{m}^{M}, t_{m+1}^{M})$ degrees of freedom, and scale factor $c^{(m, n, M)}$. By the infinite divisibility of the SNC chi-squared distribution (see Theorem 2.3), we can define $j$ conditionally independent copies $r_{j, low}^{(n, m, M)}$ such that

$$
\sum_{j=1}^{n} r_{j, low}^{(n, m, M)}(t) = Z_{\chi^2}^{(n, m, M)}(t) (5.4)
$$

Lemma 5.11 There exists continuous functions $\tilde{b}$ and $\tilde{\sigma}$ such that:

$$
E[\exp(i \omega \tilde{Z}_{\chi^2}^{(n, M)}(t)/n)] = \exp\left(\frac{i \omega \tilde{Z}^{(n)}(0)}{1 - 2i \omega \int_{0}^{t} \sigma^{2}(u) \mathrm{d}u} \right) - \frac{1}{2} \int_{0}^{t} d'(s) \log \left(1 - 2i \omega \int_{s}^{t} e^{-\int_{s}^{t} b(u) \mathrm{d}u} \sigma^{2}(u) \mathrm{d}u \right) ds) + o(n^{4Q}/M).
$$

Lemma 5.12 Take $M(n) = n^{4Q}$. There exists continuous functions $\tilde{b}$ and $\tilde{\sigma}$ such that

$$
\lim_{n \to \infty} E[\exp(i \omega r_{mid}^{(n, M(n))}(t))] = \exp\left(\frac{i \omega \tilde{r}(0)}{1 - 2i \omega \int_{0}^{t} e^{-\int_{0}^{t} b(u) \mathrm{d}u} \tilde{\sigma}^{2}(u) \mathrm{d}u} \right) - \frac{1}{2} \int_{0}^{t} d'(s) \log \left(1 - 2i \omega \int_{s}^{t} e^{-\int_{s}^{t} b(u) \mathrm{d}u} \tilde{\sigma}^{2}(u) \mathrm{d}u \right) ds) (5.5)
$$

Taking the sequence $M(n) = n^{4Q}$, Lemma 5.9 and Lemma 5.12 show that $r(t)$ has characteristic function given by (5.5). Formula (2.3) obtains by integration by parts. Since both forms of the characteristic functions (before and after integration by parts) are interesting, we show this calculation in the main text. We calculate the first two moments of $r(t)$ by ways of the characteristic function (5.5) and match them with the analytical results obtained from applying Itô’s lemma to (1.1). Using (2.3), it is easy to see that the only valid choice (for all $t$) of a function $\tilde{b}$ and $\tilde{\sigma}$ is:

$$
\tilde{b} = b (5.6)
$$

$$
\tilde{\sigma} = \sigma (5.7)
$$

Note that it is also possible to see directly from our definitions that (5.6), (5.7) hold, but it is more cumbersome.
5.3 Proofs of All Lemmas

Proof of Lemma 5.5

The function \( t \to \int_0^t \Phi^{(n)}(s, a) \, dW_s(a) \, da \) is continuous a.e. and a.s. even at a discontinuity point. However, that the function:

\[
\frac{d^{(n)}(t)}{t} \quad t \to \int_0^t \int \mu^{(n)}(s, a, \omega) \, ds \, da
\]

is discontinuous at \( t \). The latter case is however covered by the regular Fubini theorem (see e.g. Hunter and Nachtergaele 2001, p.350). Applying both Fubini theorems we have:

\[
Z^{(n)}_{mid}(t) - Z^{(n)}_{mid}(0) = (A) + (B) + (C),
\]

where:

\[
(A) = \int \int \mu^{(n)}(s, u) \, du \, ds;
\]

\[
(B) = \int \int \Phi^{(n)}(s, u) \, du \, \partial_s W(s, u);
\]

\[
(C) = \int \int \Phi^{(n)}(s, u) \, du \, ds + \int \int \Theta^{(n)}(s, u) \, du \, \partial_s W(s, u)
\]

\[
= \int \int \mu^{(n)}(s, d^{(n)}(a)) \frac{d\Phi^{(n)}(s, u)}{da} \bigg|_a \, da \, ds + \int \int \Phi^{(n)}(s, d^{(n)}(a)) \frac{d\mu^{(n)}(s, u)}{da} \bigg|_a \, da \, \partial_s W(s, d(a))
\]

\[
= \int \int \mu^{(n)}(s, d^{(n)}(a)) \frac{d\mu^{(n)}(s, u)}{da} \bigg|_a \, da \, ds + \int \int \Phi^{(n)}(s, d^{(n)}(a)) \partial_s W(s, d(a)) \frac{d\mu^{(n)}(s, u)}{da} \bigg|_a \, da
\]

\[
= \int \int (\Phi^{(n)}(s, u))^2 \frac{d\mu^{(n)}(s, u)}{da} \bigg|_a \, da \, ds
\]

\[
= \sum_{i=1}^{M_d} \sum_{k=1}^{N_{next}(d^{(n)}_{k,up}(s))} \int (\hat{\Phi}^{(n)}_i(s))^2 \frac{d\mu^{(n)}(s, u)}{da} \bigg|_a \, ds + \sum_{p=1}^{N_{down}(i)} \int (\hat{\Phi}^{(n)}_i(s))^2 \frac{d\mu^{(n)}(s, u)}{da} \bigg|_a \, ds.
\]

Calculation of (A)

We split \((A)\) in two parts:

\[
(A) = (A1) + (A2)
\]

\[\text{The function } t \to \int_0^t \Phi^{(n)}(s, a) \, dW_s(a) \text{ is not continuous at a discontinuity } t = d^{-1}(i/n), \text{ but the discontinuity is "integrated" when this function is integrated w.r.t. } a.\]
where

\[ (A1) = \int_{s=0}^{t} \int_{u=0}^{d^n(s)} (-b(s)(\tilde{x}_i^{(n)}(s))^2 - \frac{\sigma^2(s)}{4}) \, du \, ds \]
\[ = \int_{s=0}^{t} (-b(s)Z_{\text{mid}}^{(n,M)}(s) - \frac{\sigma(s)^2 d^n(s)}{4}) \, ds. \]

The calculation of \( (A2) \) is more complicated. It is easier to isolate the up branches and the down branches of \( d^n \):

\[ (A2) = \sum_{i=1}^{M_d^{(n)}} N_{\text{up}}^{(n)}(i) (A2_{k,\text{up},i,t}) + \sum_{p=1}^{N_{\text{down}}^{(n)}(i)} (A2_{p,\text{down},i,t}) \]

where we define:

\[ (A2_{k,\text{up},i,t}) = - \int_{s=J_{k,\text{up}}^{(n)}(i)}^{d^n(s)} \int_{u=0}^{J_{k,\text{up}}^{(n)}(i)} \delta(s - J_{k,\text{up}}^{(n)}(i)) (\tilde{x}_i^{(n)}(s))^2 1\{i - 1 < u \leq i\} \, du \, ds; \]
\[ (A2_{p,\text{down},i,t}) = - \int_{s=J_{p,\text{down}}^{(n)}(i)}^{d^n(s)} \int_{u=0}^{J_{p,\text{down}}^{(n)}(i)} \delta(t - J_{p,\text{down}}^{(n)}(i)) (\tilde{x}_i^{(n)}(s))^2 \frac{dd^n}{ds} |_{s=1} 1\{i - 1 < u \leq i\} \, du \, ds. \]

**Calculation of (B)**

By Levy’s theorem there exists a collection of Brownian motions \( B^{(n)}(s) \) such that:

\[ (B) = \int_{s=0}^{t} \int_{u=0}^{d^n(s)} \sigma(s)\tilde{x}_i^{(n)}(s, \omega) \sum_{i=1}^{\infty} 1\{i - 1 < u \leq i\} \, du \, dW_i(s) \]
\[ = \int_{s=0}^{t} \sum_{i=1}^{[d^n(s)]} \sigma(s)\tilde{x}_i^{(n)}(s) \, dW_i(s) + (d^n(s) - [d^n(s)])\sigma(s)\tilde{x}^{(n)}_{[d^n(s)]+1}(s) \, dW_{[d^n(s)]+1}(s) \]
\[ = \int_{s=0}^{t} \sigma(s)\sqrt{Z_{\text{mid}}^{(n)}} \sum_{i=1}^{[d^n(s)]} \tilde{x}_i^{(n)}(s) \, dW_i(s) + (d^n(s) - [d^n(s)])\tilde{x}^{(n)}_{[d^n(s)]+1}(s) \, dW_{[d^n(s)]+1}(s) \]
\[ = \int_{s=0}^{t} \sigma(s)\sqrt{Z_{\text{mid}}^{(n)}} \, dB^{(n)}(s). \]
Finally Lemma 5.5 follows from 5.8 and regrouping $(A) = (A1) + (A2), (B)$ and (C).

**Proof of Lemma 5.7**

We observe that:

$$J_{next}^{(n)}(J_{k,up}^{(n)}(i)) - J_{k,up}^{(n)}(i) = o\left(\frac{1}{n}\right)$$

$$J_{next}^{(n)}(J_{p,down}^{(n)}(i)) - J_{p,down}^{(n)}(i) = o\left(\frac{1}{n}\right).$$

Clearly $\frac{dd^{(n)}}{ds}|_s = o(n)$. There is no jump of $\tilde{x}_{i,mid}^{(n,m,M)}$ in the interval $[J_{k,up}^{(n)}(i) \leq s < J_{p,down}^{(n)}(i)]$ if $s \in [\frac{t_{m}, t_{m+1}}{M}, \frac{t_{m+1}}{M}].$

A fortiori there is no jump of $\tilde{x}_{i,mid}^{(n,m,M)}$ in the interval $[J_{k,up}^{(n)}(i) \leq s < J_{k,up}^{(n)}(i)]$ if $s \in [\frac{t_{m}, t_{m+1}}{M}, \frac{t_{m+1}}{M}].$ Thus $E[(\tilde{x}_{i,mid}^{(n,m,M)}(s)) - (\tilde{x}_{i,mid}^{(n,m,M)}(J_{k,up}^{(n)}(i)))]^2$ is of order $1/M$ on $s \in [L_{up}^{(m,M,k,i,n)}, U_{up}^{(m,M,k,i,n)}]$, and:

$$E\left[ \int_{L_{up}^{(m,M,k,i,n)}}^{U_{up}^{(m,M,k,i,n)}} \left| (\tilde{x}_{i,mid}^{(n,m,M)}(s)) - (\tilde{x}_{i,mid}^{(n,m,M)}(J_{k,up}^{(n)}(i))) \right|^2 ds \right] = o\left(\frac{1}{M^2}\right).$$

Thus:

$$E\left[ \int_{L_{up}^{(m,M,k,i,n)}}^{U_{up}^{(m,M,k,i,n)}} \left( (\tilde{x}_{i,mid}^{(n,m,M)}(s)) - (\tilde{x}_{i,mid}^{(n,m,M)}(J_{k,up}^{(n)}(i))) \right)^2 \left| s - (\tilde{x}_{i,mid}^{(n,m,M)}(J_{k,up}^{(n)}(i))) \right| \frac{dd^{(n)}}{ds} \frac{dd^{(n)}}{ds} J_{k,up}^{(n)}(i) \right] ds = o\left(\frac{n^2}{M^2}\right).$$

Since $Var[(\tilde{x}_{i}^{(n,m,M)}(s)) - (\tilde{x}_{i}^{(n,m,M)}(J_{k,up}^{(n)}(i)))^2]$ is also of order $1/M$ on $s \in [L_{up}^{(m,M,k,i,n)}, U_{up}^{(m,M,k,i,n)}]$,:

$$Var\left[ \int_{L_{up}^{(m,M,k,i,n)}}^{U_{up}^{(m,M,k,i,n)}} \left( (\tilde{x}_{i,mid}^{(n,m,M)}(s)) - (\tilde{x}_{i,mid}^{(n,m,M)}(J_{k,up}^{(n)}(i))) \right)^2 \frac{dd^{(n)}}{ds} \frac{dd^{(n)}}{ds} J_{k,up}^{(n)}(i) \right] ds = o\left(\frac{n^2}{M^2}\right).$$

The double sum $\sum_{i=1}^{M} \sum_{k=1}^{N_{up}(i)} + \sum_{k=1}^{N_{down}(i)}$ contributes to a number of jumps equal to $card\{ J^{(n)} \} - 1$, which is of order $n$. Thus:

$$E\left[ \frac{\hat{R}^{(n,m,M)}(t)}{n} \right] = o\left(\frac{n^2}{M^2}\right)$$

The variance of $\frac{\hat{R}^{(n,m,M)}(t)}{n}$ is a bit more difficult to calculate, since we focus only on the up branches, but the

---

5We focus only on the up branches, but the computation for the down branches is similar.
computation for the down branches is similar):

\[
\begin{align*}
&\text{Var}\left[ \sum_{k=1}^{N_{up}^{(n)}(i)} \int_{L_{up}(m,M,k,i,n)} (\hat{x}_{i,mid}^{(n,m,M)}(s))^{2} \frac{dd^{(n)}}{ds}|s - (\hat{x}_{i,mid}^{(n)}(j_{k,up}^{(n)})^{(n)}(i))) \frac{dd^{(n)}}{ds}|n_{ext}(j_{k,up}^{(n)}) \right] ds = \\
&\text{Var}\left[ \sum_{k=1}^{N_{up}^{(n)}(i)} \sum_{k'=1}^{N_{up}^{(n)}(i)} \int_{L_{up}(m,M,k,i,n)} (\hat{x}_{i,mid}^{(n,m,M)}(s))^{2} \frac{dd^{(n)}}{ds}|s - (\hat{x}_{i,mid}^{(n)}(j_{k,up}^{(n)}))^{(n)}(i))) \frac{dd^{(n)}}{ds}|n_{ext}(j_{k,up}^{(n)}) \right] ds + \\
&\text{Cov}\left[ \sum_{k=1}^{N_{up}^{(n)}(i)} \sum_{k'=1}^{N_{up}^{(n)}(i)} \int_{L_{up}(m,M,k,i,n)} (\hat{x}_{i,mid}^{(n,m,M)}(s))^{2} \frac{dd^{(n)}}{ds}|s - (\hat{x}_{i,mid}^{(n)}(j_{k,up}^{(n)}))^{(n)}(i))) \frac{dd^{(n)}}{ds}|n_{ext}(j_{k,up}^{(n)}) \right] ds,
\end{align*}
\]

(5.12)

However the number \(N_{up}^{(n)}(i)\) of terms of \(i\) is of order 1, thus the number of terms in (5.12) is still of order 1, and:

\[
\begin{align*}
\text{Var}\left[ \frac{R^{(n,m,M)}(t)}{n} \right] &= \mathcal{O}\left( \frac{n^{2}}{M^{2}} \right) \\
\end{align*}
\]

(5.13)

Then by Markov’s inequality: for all \(\epsilon > 0\) arbitrarily small,

\[
P\left( \left| \frac{R^{(n,m,M)}(t)}{M} \right| \leq \frac{R^{(n,m,M)}(t)}{M} \right) = \frac{R^{(n,m,M)}(t)}{M} \xrightarrow{n \to +\infty} 0.
\]

As a result, \(\frac{R^{(n,m,M)}(t)}{n}\) tends to 0 in probability when \(M \to \infty\).

**Proof of Lemma 5.8** Fix any \(\epsilon, \delta > 0\). The proof is by induction on \(n\). The case \(n = 1\) is trivial. Suppose that it is true for \(n - 1\).

Let \(A_{\epsilon,\delta}\) be the event such that \(\{|r^{(n,m,M)}_{n,mid}(t_{m+1}) - r^{(n,m,M)}_{n,mid}(t_{m+1} - \epsilon)| > \delta\}\). Set \(B^{(n)}_{j,mid} = B^{(n-1)}_{j,mid}\) for \(1 \leq j \leq n - 1\). Then:

\[
P\left( \left| \sum_{j=1}^{n-1} t^{(n,M)}_{j,mid}(t_{m+1}) - r^{(n,M)}_{j,mid}(t_{m+1} - \epsilon) \right| > \delta \right) = 0.
\]

Suppose \(P(A_{\epsilon,\delta}) > 0\). We have then

\[
P\left( \left| \sum_{j=1}^{n} t^{(n,M)}_{j,mid}(t_{m+1}) - r^{(n,M)}_{j,mid}(t_{m+1} - \epsilon) \right| \right) > \delta \right) > 0.
\]

By Levy’s theorem, we arrive at the contradiction:

\[
P\left( \left| \sum_{j=1}^{n} Z^{(n,mid)}_{m+1} - Z^{(n,mid)}_{m+1} - \epsilon) \right| > \delta \right) > 0.
\]

**Proof of Lemma 5.9**

Define

\[
\hat{R}^{(n,M)}(t) = \sum_{m=1}^{M} R^{(n,m,M)}(t)1\{t_{m} \leq t < t_{m+1}\}
\]
Step (i): tightness of \( r_{1,mid}^{(n,M)} \)

Let
\[
h_1^{(n,M)}(t) = \int_{s=0}^{t} -b(s)r_{1,mid}^{(n,M)}(s) - \frac{\sigma^2(s)d(s)}{4} \, ds + \int_{s=0}^{t} \sigma(s)\sqrt{r_{1,mid}^{(n,M)}(s)} \, dB_{1,mid}^{(n,M)}(s). \tag{5.14}
\]

Thus:
\[
r_{1,mid}^{(n,M)}(t) = h_1^{(n,M)}(t) + \frac{\tilde{R}^{(n,M)}(t)}{n}.
\]

By Lemma 5.8, the process \( r_{1,mid}^{(n,M)} \) is continuous. Let the modulus of continuity be:
\[
w(r_{1,mid}^{(n,M)}, \delta) = \sup_{|s-t|<\delta, 0 \leq s,t \leq T} |r_{1,mid}^{(n,M)}(t) - r_{1,mid}^{(n,M)}(s)|.
\]

We bound its expected value:
\[
E[w(r_{1,mid}^{(n,M)}, \delta)] \leq (A) + (B) + (C),
\]
where:
\[
(A) = E[\sup_{|s-t|<\delta, 0 \leq s,t \leq T} \int_{u=s}^{t} -b(u)r_{1,mid}^{(n,M)}(u) - \frac{\sigma^2(u)d(u)}{4} \, du] < K_1 \delta
\]
for some constant \( K_1 > 0 \). With the assumption that \( \sigma \) is bounded:
\[
(B) = E[\sup_{|s-t|<\delta} \int_{u=s}^{t} \sigma(u)\sqrt{r_{1,mid}^{(n,M)}(u)} \, dB_{1,mid}^{(n,M)}(u)]
\leq E[\sup_{s \in [0,T]} |\sigma(s)\sqrt{r_{1,mid}^{(n,M)}(s)}| \sup_{|s-t|<\delta} |B_{1,mid}^{(n,M)}(t) - B_{1,mid}^{(n,M)}(s)|]
\leq (E[\sup_{s \in [0,T]} |\sigma(s)\sqrt{r_{1,mid}^{(n,M)}(s)}|^2])^{1/2} E[\sup_{|s-t|<\delta} |B_{1,mid}^{(n,M)}(t) - B_{1,mid}^{(n,M)}(s)|^2]^{1/2} \text{ (Hölder inequality)}
\leq K_2 \delta^{1/2},
\]
where \( K_2 > 0 \) is a constant. The last line follows because the expected value of the maximum of Brownian motion \( B \) is equal to :
\[
E[\sup_{s<t} |B(s)|] = \sqrt{\frac{2t}{\pi}}.
\]

Finally
\[
(C) = E\left[\sup_{|s-t|<\delta, 0 \leq s,t \leq T} \left| \frac{\tilde{R}^{(n,M)}(t)}{n} - \frac{\tilde{R}^{(n,M)}(s)}{n} \right|\right].
\]

Since \( r_{1,mid}^{(n,M)} \) is continuous at \( t_{k}^{M} \), let \( t_{k-1}^{M} \leq s < t_{k}^{M} < \ldots < t_{K}^{M} < t < t_{K+1}^{M} \). Then
\[
\left| \frac{\tilde{R}^{(n,M)}(t)}{n} - \frac{\tilde{R}^{(n,M)}(s)}{n} \right| < \frac{\tilde{R}^{(n,M)}(t_{k}^{M})}{n} - \frac{\tilde{R}^{(n,M)}(t_{k}^{M})}{n} + \sum_{i=k}^{K} \frac{\tilde{R}^{(n,M)}(t_{i+1}^{M})}{n} - \frac{\tilde{R}^{(n,M)}(t_{i}^{M})}{n} + \frac{\tilde{R}^{(n,M)}(t_{K}^{M})}{n} - \frac{\tilde{R}^{(n,M)}(t_{K}^{M})}{n}
\leq 2\delta \max_{k-1 \leq i \leq K+1} \left| \frac{\tilde{R}^{(n,M)}(t_{i}^{M})}{n} \right|.
\]

Therefore
Thus, by (5.11):

\[
(C) = E\left[ \sup_{|s-t| \leq \delta} |\frac{\tilde{R}^{(n,M)}(t)}{n} - \frac{\tilde{R}^{(n,M)}(s)}{n}| \right] = O\left(\frac{n^2 \delta}{M^2}\right) \tag{5.15}
\]

Grouping the results, and passing to the sequence \(M(n) = n^{4Q}\), we have:

\[
E[w(r_{1,\text{mid}}^{(n,M(n))}, \delta)] \leq K_1 \delta + K_2 \delta^{1/2}
\]

We first now demean the sequence \(w(r_{1,\text{mid}}^{(n,M(n))}, \delta)\). Clearly \(E[w(r_{1,\text{mid}}^{(n,M(n))}, \delta)] > 0\). Then for any \(\varepsilon > 0\), using Markov’s inequality:

\[
P(w(r_{1,\text{mid}}^{(n,M(n))}, \delta) \geq \varepsilon) \leq \frac{E[w(r_{1,\text{mid}}^{(n,M(n))}, \delta)]}{\varepsilon}
\]

Clearly, for some constant \(K_4 > 0\):

\[
E[w(r_{1,\text{mid}}^{(n,M(n))}, \delta)] \leq \sup_{|s-t| < \delta} (E[(r_{1,\text{mid}}^{(n,M(n))}(t) - r_{1,\text{mid}}^{(n,M(n))}(s))^2])^{1/2} \leq K_5 \delta^{1/2}.
\]

Thus:

\[
P(w(r_{1,\text{mid}}^{(n,M(n))}, \delta) \geq \varepsilon) \leq \frac{K_5 \delta^{1/2}}{\varepsilon}
\]

We now invoke Theorem 8.2 in Billingsley (1968). Tightness occurs if, for each positive \(\varepsilon\) and \(\eta\) there must exist of \(\delta\) such that, for all \(n \geq n_0\)

\[
P(w(r_{1,\text{mid}}^{(n,M(n))}, \delta) \geq \varepsilon) \leq \eta.
\tag{5.16}
\]

As a result we can select \(\delta\) such that:

\[
\frac{K_5 \delta^{1/2}}{\varepsilon} \leq \eta.
\]

We thus select

\[
\delta = \min(\eta \varepsilon^2, K_5, 1).
\]

and, for all \(n \geq n_0 = \frac{n_0}{\delta}\), (5.16) occurs.

**Step (ii):** convergence of \(r_{1,\text{mid}}^{(n,M(n))}\) to a solution of (1.1). By Lemma 5.8, for any \(\varepsilon > 0\) and any \(t\):

\[
\lim_{n \to \infty} P(r_{1,\text{mid}}^{(n,M(n))}(t) - h_1^{(n)}(t) > \varepsilon) = 0
\]

Thus,

\[
r_{1,\text{mid}}^{(n,M(n))}(t) - h_1^{(n)}(t) \overset{d}{\to} 0
\]

Because of the Markov property, with \(t_1 < t_2\) :

\[
(r_{1,\text{mid}}^{(n,M(n))}(t_1) - h_1^{(n)}(t_1), r_{1,\text{mid}}^{(n,M(n))}(t_2) - r_{1,\text{mid}}^{(n,M(n))}(t_1) - h_1^{(n)}(t_2) - h_1^{(n)}(t_1)) \overset{d}{\to} (0,0).
\]

By Corollary 1 to Theorem 5.1 in Billingsley (1968):

\[
(r_{1,\text{mid}}^{(n,M(n))}(t_1) - h_1^{(n)}(t_1), r_{1,\text{mid}}^{(n,M(n))}(t_2) - h_1^{(n)}(t_2)) \overset{d}{\to} (0,0).
\]
Repeating this argument, we can prove equality of the finite dimensional distributions of \( r_{1,mid}^{(n,M(n))} - h_1^{(n)} \) to zero. In other terms:

\[
\lim_{n \to \infty} P(r_{1,mid}^{(n,M(n))} = h_1^{(n)}) = 1
\]

We can extract a subsequence \( \{n_k(i)\} \) so that \( k \geq k(i) \) implies that \( P(|r_{1,mid}^{(n_k,M(n_k))} - h_1^{(n_k)}| \geq i^{-1}) = 2^{-i} \). By the first Borel-Cantelli lemma there is a probability 1 that \( |r_{1,mid}^{(n_k,M(n_k))} - h_1^{(n_k)}| \leq i^{-1} \) for all but finitely many \( i \). Therefore:

\[
\lim_{k \to +\infty} r_{1,mid}^{(n_k,M(n_k))} = \lim_{k \to +\infty} h_1^{(n_k)} \quad \text{a.s. } P
\]

In other terms:

\[
P(r_{1,mid}^{(\infty,\infty)} = r_{1,mid}^{(\infty,\infty)}(0) - \int_{s=0}^t b(s) r_{1,mid}^{(\infty,\infty)}(s) - \frac{\sigma^2 d^{(n)}(s)}{4n} \, ds + \int_{s=0}^t \sigma(s) \sqrt{r_{1,mid}^{(\infty,\infty)}(s)} \, dB_{1,mid}(s) = 1.
\]

Thus \( (r_{1,mid}^{(\infty,\infty)}, P_{1,mid}^{(\infty)}) \) is a weak solution to the stochastic differential Equation 1.1. Maghsoodi (1996) proved that under our conditions to 1.1 there is pathwise uniqueness.

**Proof of Lemma 5.11**

By differentiability of \( d^{(n)} \), for any \( t_m^M \leq t < t_{m+1}^M \):

\[
d^{(n)}(t) - d_{low}(t_m^M, t_{m+1}^M) = o\left(\frac{n}{M}\right).
\]

By definition of \( Z_{x}^{(n,m,M)} \) and \( \tilde{Z}_{mid}^{(n,M)} \)

\[
E[Z_{x}^{(n,m,M)}(t_{m+1}^M) - Z_{x}^{(n,m,M)}(t_{m}^M) | F_{t_m}^{M}] = o\left(\frac{n^2 Q}{M^2} \cdot t_m^M\right)
\]

(5.17)

\[
Var[Z_{x}^{(n,m,M)}(t_{m+1}^M) - Z_{x}^{(n,m,M)}(t_{m}^M) | F_{t_m}^{M}] = o\left(\frac{n^3 Q}{M^2} \cdot t_m^M\right).
\]

(5.18)

By differentiability of the characteristic function, the following Taylor series holds:

\[
E[\exp(i\omega Z_{mid}^{(n,M)}(t)) | F_{t_m}^{M}] = E[\exp(i\omega (Z_{x}^{(n,m,M)}(t)) | F_{t_m}^{M}] + i\omega E[Z_{mid}^{(n,M)}(t) - Z_{x}^{(n,m,M)} | F_{t_m}^{M}] - \frac{\omega^2}{2} E[(Z_{mid}^{(n,M)}(t) - Z_{x}^{(n,m,M)})^2 | F_{t_m}^{M}] + ...
\]

Let \( X \) be a \( F_{t_m}^{M} \)-measurable random variable with bounded mean. Suppose that

\[
X \geq E[\exp(i\omega Z_{mid}^{(n,M)}(t)) | F_{t_m}^{M}] - E[\exp(i\omega Z_{x}^{(n,m,M)}(t)) | F_{t_m}^{M}] \geq X\left(\frac{K_1}{M^2} - \frac{K_2}{M^2}\right),
\]

then:

\[
|\omega E[Z_{mid}^{(n,M)}(t) - Z_{x}^{(n,m,M)} | F_{t_m}^{M}] - \frac{\omega^3}{3!} E[(Z_{mid}^{(n,M)}(t) - Z_{x}^{(n,m,M)})^2 | F_{t_m}^{M}] + ...| \geq X \frac{K_1}{M^2}
\]

(5.19)

\[
| - \frac{\omega^2}{2} E[(Z_{mid}^{(n,M)}(t) - Z_{x}^{(n,m,M)})^2 | F_{t_m}^{M}] + \frac{\omega^4}{4!} E[(Z_{mid}^{(n,M)}(t) - Z_{x}^{(n,m,M)})^3 | F_{t_m}^{M}] + ...| \geq X \frac{K_2}{M^2}
\]

(5.20)

The left hand side of (5.19) and (5.20) can be majorized by the terms below, and we obtain

\[
X \frac{K_1}{M^2} \leq \exp(\omega E[Z_{mid}^{(n,M)}(t) - Z_{x}^{(n,m,M)} | F_{t_m}^{M})] - 1 \leq X \frac{K_3}{M^2},
\]

\[
X \frac{K_2}{M^2} \leq \exp(\omega E[(Z_{mid}^{(n,M)}(t) - Z_{x}^{(n,m,M)})^2 | F_{t_m}^{M}] - 1 \leq X \frac{K_4}{M^2}
\]

24
which contradicts (5.17) and (5.18). Thus

\[ E[\exp(i\omega Z_{\text{mid}}^{(n,M)}(t))|F_{t_m}] - E[\exp(i\omega(Z_{\chi^2}^{(n,m,M)}(t))|F_{t_m}]] = o\left(\frac{n^{4Q}}{M^2}; t_m^M\right). \]  

(5.21)

From now on within this proof, we suppress the subscript \( n \) from our variables. For the moment we fix \( M \), and take it out of the subscripts of our variables. We let \( \Delta = 1/M \), and suppose that \( K\Delta = t \). We find it more convenient to shorten further the notation by:

\[ Z_{\text{mid}}^{(m,M)}(t_m+1) \rightarrow Z_{\text{mid}}((m+1)\Delta) \]
\[ d_{\text{low}}(m\Delta, (m+1)\Delta) \rightarrow \tilde{d}(m+1)\Delta) \]
\[ E[X|Z_{\text{mid}}^{(n,M)}(t_m)] \rightarrow E_{(m\Delta)}[X] \]
\[ c^{(n,m,M)} \rightarrow c(m\Delta). \]

We denote the conditional non-centrality parameter and scale factor of \( Z_{\chi^2}((m+1)\Delta) \) given \( Z_{\text{mid}}^{(n,M)}(t_m) \) by:

\[ \lambda(m\Delta) = c(m\Delta)Z_{\text{mid}}^{(n,M)}(m\Delta)e^{-\hat{b}(m\Delta)\Delta} \]  

(5.22)
\[ c(m\Delta) = \frac{(1-e^{-\hat{b}(m\Delta)\Delta})\tilde{\sigma}^2(m\Delta)}{4\hat{b}(m\Delta)}. \]  

(5.23)

where, so far \( \hat{b} \) and \( \tilde{\sigma} \) are unknown functions. By the characteristic function of non-central chi-square distribution, the assumption that \( M(n) = n^{4Q} \) and (5.21), for \( k = 0, \ldots, K-1 \)

\[ E_{k\Delta}[\exp(i\omega Z_{\text{mid}}((k+1)\Delta))] = \frac{\exp\left(i\omega\frac{Z_{\text{mid}}(k\Delta)e^{-\hat{b}(k\Delta)\Delta}}{1-2i\omega c(k\Delta)}\right)}{(1-2i\omega c(k\Delta))^{d((k+1)\Delta)/2}} + o(\Delta; k\Delta) \]  

(5.24)

As a way to find the general iteration formula, we study\(^6\) the case where \( k = 2 \). Defining:

\[ \omega_2 = \frac{\omega e^{-\hat{b}(2\Delta)\Delta}}{1-2i\omega c(2\Delta)}, \]

we calculate

\[ E_{2\Delta}[\exp(i\omega_2 Z_{\text{mid}}(2\Delta))] = \frac{\exp\left(i\omega_2 Z_{\text{mid}}(\Delta)e^{-\hat{b}(\Delta)\Delta}\frac{1}{1-2i\omega_2 c(\Delta)}\right)}{(1-2i\omega_2 c(\Delta))^{d(2\Delta)/2}} + o(\Delta; \Delta). \]

Hence,

\[ E_{3\Delta}[\exp(i\omega Z_{\text{mid}}(3\Delta))] = \frac{\exp\left(i\omega Z_{\text{mid}}(\Delta)e^{-\hat{b}(\Delta)\Delta}\frac{1}{1-2i\omega c(\Delta)}\right)}{(1-2i\omega c(\Delta))^{d(2\Delta)/2}(1-2i\omega c(2\Delta))^{d(3\Delta)/2}} + o(\Delta^2; \Delta) \]  

\[ = \frac{\exp\left(i\omega Z_{\text{mid}}(\Delta)e^{-\hat{b}(\Delta)\Delta}\frac{1}{1-2i\omega c(\Delta)}\right)}{(1-2i\omega c(\Delta) + e^{-\hat{b}(2\Delta)\Delta}c(\Delta))^{d(2\Delta)/2}(1-2i\omega(c(2\Delta))^{d(3\Delta)-d(2\Delta)/2}} + o(\Delta; \Delta). \]

\(^6\)We thank an anonymous referee for this judicious construction.
Observe that, for a general case,

$$E[\exp (i\omega Z_{mid}(K\Delta))] = \frac{\exp \left( i\frac{\omega Z_{mid}(0) e^{-\sum_{k=0}^{K-1} i(k\Delta)_{c}}}{1 - 2i\omega \sum_{k=0}^{K-1} e^{-\Delta \sum_{l=k+1}^{K} b(l\Delta)_{c}(k\Delta)}} \right)}{(1 - 2i\omega \sum_{k=0}^{K-1} e^{-\Delta \sum_{l=k+1}^{K} b(l\Delta)_{c}(k\Delta)})^{d(\Delta)/2} A_K} + o(\Delta),$$

where the denominator $A_K$ is given by:

$$A_K = \prod_{j=0}^{K-1} \left( 1 - 2i\omega \sum_{k=j}^{K} e^{-\Delta \sum_{l=k+1}^{K} b(l\Delta)_{c}(k\Delta)} \right)^{(d((j+1)\Delta) - d(j\Delta)) / 2}.$$

In order to show the limit of the above sequence, we take the logarithm and get:

$$\log A_K = \frac{1}{2} \sum_{j=0}^{K-1} (d((j+1)\Delta) - d(j\Delta)) \log \left( 1 - 2i\omega \sum_{k=j}^{K-1} e^{-\Delta \sum_{l=k+1}^{K} b(l\Delta)_{c}(k\Delta)} \right).$$

We now return to placing subscripts $(M)$ to our expressions. Let

$$\log A_{K/M}^{(M)} = \frac{1}{2} \sum_{j=0}^{K-1} h^{(M)}(\frac{j}{M}) g^{(M)}(\frac{j}{M})$$

where:

$$h^{(M)}(\frac{j}{M}) = \tilde{d}(\frac{j+1}{M}) - \tilde{d}(\frac{j}{M})$$

$$= d_{low}(\frac{j}{M}, \frac{j+1}{M}) - d_{low}(\frac{j-1}{M}, \frac{j}{M})$$

$$g^{(M)}(\frac{j}{M}) = \log \left( 1 - 2i\omega \sum_{k=j}^{K-1} e^{-\Delta \sum_{l=k+1}^{K} b(l\Delta)_{c}(\frac{k}{M})} \right).$$

Let $\mathcal{R}(M) = \{ j | h(\frac{j}{M}) \neq d'(\frac{j}{M}) + o(\frac{1}{M}) \}$. Since there is a finite number of minima of $d''$, there is a sequence $M_K$ (take for instance a dyadic sequence) such that for all $K > K_0$ the set $\mathcal{R}(M_K)$ is finite and constant. We can thus split the calculations into

$$\log A^{(M_K)}_K = \frac{1}{2} \sum_{j=0}^{K-1} h^{(M_K)}(\frac{j}{M_K}) g^{(M_K)}(\frac{j}{M_K})$$

$$= \sum_{j \in \{0,...,K\} \cap \mathcal{R}(M_K)} h^{(M_K)}(\frac{j}{M_K}) g^{(M_K)}(\frac{j}{M_K}) + \sum_{j \in \mathcal{R}(M_K)} h^{(M_K)}(\frac{j}{M_K}) g^{(M_K)}(\frac{j}{M_K}).$$

The sum in (5.26) has a finite number of terms, while the sum in (5.25) has a number of terms that tends to infinity when $K \to \infty$. 

26
Also, observe that, by using the mean value theorem,

\[ c(k\Delta) = \frac{(1 - e^{-\hat{b}(k\Delta)\Delta})\hat{\sigma}^2(k\Delta)}{4\hat{b}(k\Delta)} = \frac{\hat{\sigma}^2(k\Delta)\Delta}{4} + o(\Delta). \]

Thus we can write:

\[ \lim_{K \to \infty} \log A^{(M_K)}_K = \frac{1}{2} \int_0^T d'(s)g(s) \, ds, \]

with

\[ g(s) = \log(1 - 2i\omega \int_s^t e^{-\int_u^s \hat{b}(u) \, du} \hat{\sigma}^2(v) \, dv)/4 \, dv). \]

Equivalently:

\[ \lim_{K \to \infty} A^{(M_K)}_K = \exp\left(\frac{1}{2} \int_0^t d'(s) \log(1 - 2i\omega \int_s^t e^{-\int_u^s \hat{b}(u) \, du} \hat{\sigma}^2(v) \, dv) \, ds\right). \]

We also observe that, by Riemann sum,

\[ \frac{i\omega Z_{m,M}(0)e^{-\left(\sum_{k=0}^{K-1} \frac{1}{2}\hat{b}(k\Delta)\Delta\right)}}{1-2i\omega \sum_{k=0}^{K-1} e^{-\Delta \sum_{l=k+1}^{K-1} \hat{b}(l\Delta)\Delta)}} \]

\[ \frac{\exp(i\omega Z_{m,M}(0)e^{-\left(\sum_{k=0}^{K-1} \frac{1}{2}\hat{b}(k\Delta)\Delta\right)}}{1-2i\omega \sum_{k=0}^{K-1} e^{-\Delta \sum_{l=k+1}^{K-1} \hat{b}(l\Delta)\Delta)}} \to \exp(i\omega \int_0^t e^{-\int_u^s \hat{b}(u) \, du} \hat{\sigma}^2(v) \, dv) \, dv) \, ds(0)/2. \]

Then Lemma 5.11 follows.

**Proof of Lemma 5.12**

The proof is analogous to the proof of Lemma 5.11 and we show only the first inductive step. By (5.4),

\[ E[\exp(i\omega Z_{m,M}(t_{m+1})|F_{t_m}^M] \]

\[ = E[\exp(i\omega \sum_{j=1}^n h_{j,mid}(t_{m+1})^M)|F_{t_m}^M] \]

\[ = \exp(i\omega \sum_{j=1}^n h_{j,mid}(t_{m+1})^M) E[\exp(i\omega \sum_{j=1}^n h_{j,mid}(t_{m+1})^M + R^{(n,m,M)}(t_{m+1}^M))|F_{t_m}^M] \]

\[ = \exp(i\omega Z_{m,M}(t_{m+1}) E[\exp(i\omega \sum_{j=1}^n h_{j,mid}(t_{m+1})^M + R^{(n,m,M)}(t_{m+1}^M))|F_{t_m}^M]. \]

We use 5.11, 5.13 as well as the arguments presented at the beginning of Lemma 5.11 to obtain

\[ E[\exp(i\omega Z_{m,M}(t_{m+1})|F_{t_m}^M] \]

\[ = \exp(i\omega Z_{m,M}(t_{m+1}) E[\exp(i\omega \sum_{j=1}^n h_{j,mid}(t_{m+1})^M)|F_{t_m}^M] + o\left(\frac{n^4Q^2}{M^2}; t_{m+1}^M \right) \]

\[ = E \left[ \exp \left( i\omega \sum_{j=1}^n \left( R_{j,mid}(t_{m+1})^M + h_{j,mid}(t_{m+1})^M \right) \right) |F_{t_m}^M \right] + o\left(\frac{n^4Q^2}{M^2}; t_{m+1}^M \right). \]

However each copy of \( R_{j,mid}(t_{m+1})^M + h_{j,mid}(t_{m+1})^M \) is conditionally independent. Thus

\[ E[\exp(i\omega Z_{m,M}(t_{m+1})|F_{t_m}^M] = \left( E[\exp(i\omega \left( R_{1,mid}(t_m^M) + h_{1,mid}(t_{m+1})^M \right)) |F_{t_m}^M] \right)^n + o\left(\frac{n^2}{M^2}; t_{m+1}^M \right). \]
Using the same method as before:

\[ E[\exp \left( i\omega \left( r_{1,\text{mid}}^{(n,M)}(t_m^M) + h_{1,\text{mid}}^{(n,M)}(t_{m+1}^M) \right) \right) | F_{t_m^M}] - E[\exp(i\omega r_{1,\text{mid}}^{(n,M)}(t_m^M)) | F_{t_m^M}] = o\left( \frac{n^{2Q}}{M^2}; t_m^M \right). \]

Thus:

\[ E[\exp(i\omega \hat{Z}_{\text{mid}}^{(n,M)}(t_{m+1}^M)) | F_{t_m^M}] = \left( E[\exp(i\omega r_{1,\text{mid}}^{(n,M)}(t_{m+1}^M)) | F_{t_m^M}] \right)^n + o\left( \frac{n^{4Q}}{M^2}; t_m^M \right). \]

Then Lemma 5.12 holds by using the infinite divisibility of SNC chi-squared distribution and Lemma 5.11. □

References

[1] Billingsley, P. (1968). *Convergence of Probability Measures*. Wiley, New York.

[2] Brigo, D., and F. Mercurio (2001). A Deterministic-shift Extension of Analytically-tractable and Time-homogeneous Short-rate Models. *Finance and Stochastics*, Vol 5. 369-387.

[3] Brigo, D., and F. Mercurio (2006). *Interest Rate Models: Theory and Practice - with Smile, Inflation, and Credit*. Springer.

[4] Chen, L., D. Filipovic, and H. Vincent Poor (2004). Quadratic Term Structure Models for Risk-free and Defaultable Rates. *Mathematical Finance*, Vol. 14, No 4, 515-536.

[5] Cotton, P., J.P. Fouque, G. Papanicolaou, and R. Sircar (2004). Stochastic Volatility Corrections for Interest Rate Derivatives. *Mathematical Finance*, vol. 14, No 2, 173-200.

[6] Cox, J.C., J.E. Ingersoll, and S.A. Ross (1985a). An Intertemporal General Equilibrium Model of Asset prices. *Econometrica*, vol. 53, no 2, 363-384.

[7] Cox, J.C., J.E. Ingersoll, and S.A. Ross (1985b). A Theory of the Term Structure of Interest Rates. *Econometrica*, 53, 385-487.

[8] Duffie, D., and R. Kan (1996). A Yield Factor Model of Interest Rates. *Mathematical Finance*, 6, 379-406.

[9] Fouque, J.-P., and M. Lorig (2011). A Fast Mean-Reverting Correction to Heston’s Stochastic Volatility Model. *SIAM Journal of Financial Mathematics*, Vol. 2, p. 221-254.

[10] Geman, H. (1989). L’Importance de la Probabilité Forward Neutre dans une Approche Stochastique des Taux d’Intérêt, ESSEC Working Paper (Univ. Paris Panthéon Sorbonne PhD Dissert)

[11] Gourieroux, C., and A. Monfort (2011). Bilinear Term Structure Models. *Mathematical Finance*, vol. 21, no. 1, 1-19.

[12] Heston, S. (1993). A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options. *The Review of Financial Studies* vol. 6, no 2, 327-343.

[13] Hull, J.C., and A. White (1990). Pricing Interest Rate derivative Securities. *The Review of Financial Studies*, vol. 3.

[14] Hunter, J. and B. Nachtergaele (2001). *Applied Analysis*. World Scientific.

[15] Jamshidian, F. (1989). An Exact Bond Option Pricing Formula. *The Journal of Finance* 44: 205-209.

[16] Jamshidian, F., (1993b). A Simple Class of Square-Root Models. Fuji International Finance Working Paper.
[17] Karatzas, I., and Shreve, S.E. (1991). *Brownian Motion and Stochastic Calculus*. Springer-Verlag.

[18] Keller-Ressel, M., and T. Steiner (2008). Yield Curve Shapes and the Asymptotic Short Rate Distribution in Affine One-factor Models. *Finance and Stochastics*, vol. 2, Issue 2,149-172.

[19] Lewis, A. (2000). Option Valuation under Stochastic Volatility. Finance Press.

[20] Longstaff, F., and E. Schwartz (1992). Interest Rate Volatility and the Term Structure: a Two-Factor General Equilibrium Model. *Journal of Finance*, vol. 47, no 4, 1259-1282.

[21] Maghsoodi, Y. (1996). Solutions of the Extended CIR Term Structure and Bond Option Valuation. *Mathematical Finance*, vol. 6, no 1, 89-109.

[22] Mannolini, A., C. Mari, and R. Reno (2008). Pricing Caps and Floors with the Extended CIR Model. *International Journal of Finance and Economics*, vol. 13, 386-400.

[23] Revuz, D., and M. Yor (1991). Continuous Martingales and Brownian Motion. Springer.

[24] Rogers, L.C.G. (1995). Which Model for the Term Structure of Interest Rates Should One Use? In *Mathematical Finance*, ed. M. Davis, D. Suffie, W. Fleming, and S. Shreve. New York: Springer-Verlag, 93-116.

[25] Shirakawa, H. (2002) Squared Bessel Processes and their Applications to the Square Root Interest Rate Model. *Asia-Pacific Financial Markets* 9, 169-190.

[26] Shreve, S. (2004). *Stochastic Calculus for Finance*, vol. 2. Springer.

[27] Yang, H. (2006). Calibration of the Extended CIR Model. *SIAM Journal of Applied Mathematics*, vol. 6, no 2, 721-735.