Systolic volume and complexity of 3-manifolds

Lizhi Chen

School of Mathematics and Statistics
Lanzhou University

2018 Workshop on Algebraic and Geometric Topology
Southwest Jiaotong University
July 28 – 31, 2018
1 Background and motivation
2 Main problem
3 Proof of the main theorem
1. Loewner’s Inequality

Theorem (Loewner, around 1949)

For any Riemannian metric \( G \) defined on a torus \( \mathbb{T}^2 \), the length \( \ell \) of a shortest noncontractible geodesic loop satisfies

\[
\ell^2 \leq \frac{2}{\sqrt{3}} \text{Area}_G(\mathbb{T}^2),
\]

where the equality holds for a flat hexagonal metric.

- Inequality on a manifold similar to (1) is called systolic inequality.
2. Systole and systolic volume

Let $M$ be a closed $n$-dimensional manifold endowed with the Riemannian metric $\mathcal{G}$, denoted $(M, \mathcal{G})$.

**Definition**

- The **homotopy 1-systole** of $(M, \mathcal{G})$, denoted $\text{Sys}_{\pi_1}(M, \mathcal{G})$, is defined to be the shortest length of a noncontractible geodesic loop in $M$.

- The **systolic volume** of $M$, denoted $\text{SR}(M)$, is defined to be

$$\inf_{\mathcal{G}} \frac{\text{Vol}_{\mathcal{G}}(M)}{\text{Sys}_{\pi_1}(M, \mathcal{G})^n},$$

where the infimum is taken over all Riemannian metrics $\mathcal{G}$ on $M$. 
A closed $n$-dimensional manifold $M$ is aspherical if all higher homotopy groups $\pi_i(M)$ vanish, with $i \geq 2$.

Examples:

1. An $n$-dimensional torus is aspherical, with $n \geq 2$.
2. A closed hyperbolic $n$-manifold ($n \geq 2$) is aspherical.
3. Gromov’s systolic inequality

**Theorem (Gromov 1983)**

For any Riemannian metric $G$ defined on a closed $n$-dimensional aspherical manifold $M$,

$$\text{Sys } \pi_1(M, G)^n \leq C_n \text{Vol}_G(M),$$

where the constant $C_n$ is only depending on the manifold dimension.
Kuratowski embedding and isoperimetric inequality

Let \((M, G)\) be a closed Riemannian \(n\)-manifold. The Riemannian metric \(G\) on \(M\) induces a metric \(\text{dist}(\cdot, \cdot)\). Let \(L^\infty(M)\) be the set of all bounded Borel function defined on \(M\). Define the supremum norm \(\| \cdot \|_\infty\) on \(L^\infty(M)\). Then \((L^\infty(M), \| \cdot \|_\infty)\) is a Banach space.

The metric space \((M, \text{dist}(\cdot, \cdot))\) is embedded into \(L^\infty(M, \| \cdot \|)\) by

\[
K : x \rightarrow K(x)
\]

with \(K(x)(y) = \text{dist}(x, y)\).
Lemma

\[ \text{dist}(x, y) = \| K(x) - K(y) \|_\infty. \]

Hence, we know that the Kuratowski embedding is isometric.
Filling Volume

**Definition**

- For a singular $n$-dimensional cycle $z$ in a Riemannian manifold $M$, the filling volume of $z$, denoted $\text{Fill Vol}(z)$, is defined to be

  $$\inf_{c} \text{Vol}_G(c),$$

  where the infimum is taken over all $(n + 1)$-dimensional cycles $c$ for which $\partial c = z$.

- The filling volume of a manifold $V$ with a given metric is defined as

  $$\text{Fill Vol}(V) = \text{Fill Vol}(V \subset L^\infty(V)).$$
Isoperimetric Inequality

Theorem (Gromov 1983)

The filling volume of every $n$-dimensional cycle $z$ in an arbitrary space $L^\infty$ satisfies

$$\text{Fill Vol}(z) \leq C_n [\text{Vol}(z)]^{\frac{n+1}{n}},$$

for some universal constant in the interval $0 < C_n < n^n \sqrt{(n+1)!}$. In particular, all closed connected Riemannian manifold $V$ have

$$\text{Fill Vol}(M) \leq C_n [\text{Vol}(M)]^{\frac{n+1}{n}}.$$
4. Systolic volume of aspherical manifolds

**Proposition (Babenko 1991)**

*For a closed orientable aspherical manifold \( M \), the systolic volume \( SR(M) \) is a homotopy invariant.*
(1) Estimates of systolic volume

- In dimension 2, for a closed aspherical surface $\Sigma$,
  \[ \text{SR}(\Sigma) \geq \frac{2}{\pi}. \]
  In particular, we know that $\text{SR}(\mathbb{T}^2) = \frac{\sqrt{3}}{2}$, $\text{SR}(\mathbb{RP}^2) = \frac{2}{\pi}$, $\text{SR}(\mathbb{RP}^2 \# \mathbb{RP}^2) = \frac{2\sqrt{2}}{\pi}$.

- In dimension 3, for a closed aspherical manifold $M$, Nakamura proved
  \[ \text{SR}(M) \geq \frac{1}{6}. \]
(2) Systolic volume and other topological invariants

**Theorem (Gromov 1996)**

For a closed hyperbolic surface $\Sigma_g$ with genus $g$,

$$\text{SR}(\Sigma_g) \geq C \frac{g}{\log^2 g},$$

where $C$ is a fixed constant.
Denote by $\|M\|$ the simplicial volume of a closed $n$-manifold $M$.

**Theorem (Gromov 1983)**

For a closed and aspherical $n$-dimensional manifold $M$,

$$
\text{SR}(M) \geq C \frac{\|M\|}{\log^n (1 + \|M\|)},
$$

where $C$ is a fixed constant only depending on $n$. 
Let \( M \) be a closed aspherical \( n \)-manifold \((n \geq 2)\). Denote by \( \#_k(M) \) the connected sum of \( k \) copies of \( M \).

**Theorem (Sabourau 2006)**

For a closed \( n \)-dimensional aspherical manifold \( M \),

\[
SR(\#_kM) \geq C \frac{k}{\exp \left( C' \sqrt{\log k} \right)},
\]

where \( C \) and \( C' \) are two constants only depending on the manifold dimension \( n \).
1. Complexity of 3-manifolds

Let $M$ be a closed irreducible 3-manifold distinct from $S^3$, $\mathbb{RP}^3$, $L(3, 1)$.

**Definition**

The complexity of $M$, denoted $c(M)$, is defined to be the minimum number of tetrahedra in a triangulation of $M$.

**Proposition**

For two closed irreducible 3-manifolds $M_1$ and $M_2$,

$$c(M_1 \# M_2) = c(M_1) + c(M_2).$$
2. Main problem

**Problem**

For a closed irreducible aspherical 3-manifold, is the complexity $c(M)$ a lower bound to the systolic volume $SR(M)$?

**Theorem**

For a closed aspherical 3-manifold $M$,

$$SR(M) \geq C_1 \frac{c(M)}{\exp \left( C_2 \sqrt{\log c(M)} \right)},$$

where $C_1$ and $C_2$ are two fixed positive constants.
Proof of the theorem

1.

Assume that $\mathcal{T}$ is a smooth triangulation of $M$, with $V$ the corresponding polyhedron.

There exists a sequence of piecewise smooth Riemannian metrics $G_k$ on $V$, such that

$$\lim_{k \to \infty} \frac{\text{Vol}_{G_k}(V)}{\text{Sys} \pi_1(V, G_k)^3} = \text{SR}(M).$$

Denote $(V, G_k)$ by $V_k$. 
2.

- Choose a finite and sufficiently dense set $V_0 \subset V$, and denote by $L_0^\infty$ the Banach space $L^\infty(V_0)$.

- For a small positive number $\delta$, in terms of $V_k$, we construct a cubic complex contained in $L_0^\infty$, denoted by $I_\delta(V_k)$.

There is an embedding $J : V_k \to I_\delta(V_k)$. Denote by $V'_k$ the image $J(V_k)$. 
Proposition

When \( k \) is large enough, for each point \( v \in V'_k \) and \( 0 < R < \frac{1}{2} \text{Sys } \pi_1(V'_k) \), we have

\[
\text{Vol}_{G_k}(B_v(R)) \geq A \cdot R^3,
\]

where \( A \) is a fixed constant.
3.

- Let $\{B_{v_i}(R_i)\}$ be an appropriate maximal system of disjoint open balls in $V'_k$, for $v_i \in V'_k$, and $0 < R_i \leq \frac{1}{2} \text{Sys } \pi_1(V'_k)$. Then $\{B_{v_i}(2R_i)\}$ is an open cover of $V'_k$.

- Construct a nerve $P$ associated with the open cover $\{B_{v_i}(2R_i)\}$, which is a polyhedron homotopic to $V'_k$.

- By counting the number of pairs of intersections, we have

$$N_k \leq \text{SR}(M) \exp \left( C \sqrt{\text{SR}(M)} \right),$$

where $N_k$ is the number of $k$-simplices of $P$, and $C$ is a fixed constant.
Thank you!