PERSISTENCE OF BANACH LATTICES UNDER NONLINEAR ORDER ISOMORPHISMS

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Abstract. Ordered vector spaces $E$ and $F$ are said to be order isomorphic if there is a (not necessarily linear) bijection $T : E \to F$ such that $x \geq y$ if and only if $Tx \geq Ty$ for all $x, y \in E$. We investigate some situations under which an order isomorphism between two Banach lattices implies the persistence of some linear lattice structure. For instance, it is shown that if a Banach lattice $E$ is order isomorphic to $C(K)$ for some compact Hausdorff space $K$, then $E$ is (linearly) isomorphic to $C(K)$ as a Banach lattice. Similar results hold for Banach lattices order isomorphic to $c_0$, and for Banach lattices that contain a closed sublattice order isomorphic to $c_0$.

Two ordered vector spaces $E$ and $F$ are said to be order isomorphic if there is a (not necessarily linear) bijection $T : E \to F$ so that $x \geq y$ if and only if $Tx \geq Ty$ for all $x, y \in E$. In this case, we call $T$ an order isomorphism. When $E$ and $F$ are Banach lattices, there is the well studied notion of (vector) lattice isomorphism: $E$ and $F$ are lattice isomorphic if there is a linear bijection $T : E \to F$ such that $T|x| = |Tx|$ for all $x \in E$. This is equivalent to the existence of a linear order isomorphism from $E$ onto $F$. It is well known that a lattice isomorphism $T$ between Banach lattices must also be an isomorphism between the underlying Banach spaces; that is, both $T$ and $T^{-1}$ must be bounded. It is easy to see that, in general, two Banach lattices that are order isomorphic need not be lattice isomorphic. Indeed, for any measure space $(\Omega, \Sigma, \mu)$ and any $1 < p < \infty$, the map $f \mapsto |f|^p \text{sgn } f$ is an order isomorphism from $L^p(\Omega, \Sigma, \mu)$ onto $L^1(\Omega, \Sigma, \mu)$. However, $L^p(\Omega, \Sigma, \mu)$ and $L^1(\Omega, \Sigma, \mu)$ are not lattice isomorphic unless they are finite dimensional. In contrast to the situation for $L^p$ spaces, it is shown in this paper that some vector lattice properties pertaining to $AM$-(or abstract $M$-) spaces persist under order isomorphisms. For the definition of $AM$-spaces, as well as for general background with regard to the theory of Banach lattices, we refer the reader to [5, 7]. By the well known Kakutani’s representation theorem, a Banach lattice is an $AM$-space if and only if it is isometrically lattice isomorphic to a closed sublattice of $C(K)$ for some

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compact Hausdorff space $K$; see, e.g., [4] Theorem 1.b.6]. Our first result is quite simple. If $u$ is a positive element in a Banach lattice $E$, let $E_u$ be the closed ideal in $E$ generated by $u$,

$$E_u = \{ x \in E : |x| \leq nu \text{ for some } n \in \mathbb{N} \}.$$ 

$u$ is an order unit of $E$ if $E_u = E$. It is a standard fact that if $E$ has an order unit, then $E$ is lattice isomorphic to $C(K)$ for some compact Hausdorff space $K$; see [7] Proposition II.7.2 and Corollary 1 to Theorem II.7.4].

**Theorem 1.** Let $E$ be a Banach lattice. If $E$ is order isomorphic to $C(K)$ for some compact Hausdorff space $K$, then $E$ is lattice isomorphic to $C(K)$.

**Proof.** Let $T : C(K) \to E$ be an order isomorphism. We may assume that $T0 = 0$. For any $n \in \mathbb{N}$, we use the same symbol $n$ denote the constant function on $K$ with value $n$. Then $C(K)_+ = \cup_n [0, n]$. Hence $E_+ = \cup_n [0, x_n]$, where $x_n = Tn$. By the Baire Category Theorem, there exists $n_0$ such that $[0, x_{n_0}]$ contains nonempty interior. Thus $E_+$ has an interior point $u$. By [5] Corollary 1.2.14], $u$ is an order unit of $E$. It follows that $E$ is lattice isomorphic to $C(L)$ for some compact Hausdorff space $L$. In this case, $C(K)$ and $C(L)$ are nonlinearly order isomorphic. By [1] Proposition 3, $K$ and $L$ are homeomorphic. Thus $C(K)$ and $C(L)$ are lattice isomorphic. Since $E$ is lattice isomorphic to $C(L)$, the proof is complete. \[ \square \]

We do not know if a Banach lattice that is order isomorphic to an AM-space must be lattice isomorphic to an AM-space. In this direction, there is a useful characterization of AM-spaces due to Cartwright and Lotz; see [2] and [5] Theorem 2.1.12]. A subset $A$ in an ordered vector space $E$ is order bounded if there are $u, v \in E$ such that $u \leq x \leq v$ for all $x \in A$. A sequence $(x_n)$ in a vector lattice is disjoint if $|x_m| \wedge |x_n| = 0$ whenever $m \neq n$.

**Theorem 2.** (Cartwright and Lotz) A Banach lattice $E$ is lattice isomorphic to an AM-space if and only if every disjoint norm null sequence in $E$ is order bounded in $E''$.

With the help of this theorem, we offer a partial solution to the problem raised above. A subspace $F$ of a Banach lattice $E$ is an (order) ideal if $y \in F$ for all $y \in E$ such that $|y| \leq |x|$ for some $x \in F$. By [5] Proposition 2.1.9], every closed ideal in $C(K)$ has the form

$$I = \{ f \in C(K) : f = 0 \text{ on } K_0 \}$$

for some closed subset $K_0$ of $K$.

**Proposition 3.** Let $E$ be a Banach lattice. If $E$ is order isomorphic to a closed ideal of some space $C(K)$, where $C(K)$ is separable, then $E$ is lattice isomorphic to an AM-space.

**Proof.** Let $T : E \to I$ be an order isomorphism, where $I$ is a closed ideal in $C(K)$, with $C(K)$ separable. We may assume that $T0 = 0$. Since $C(K)$ is separable, $K$ is metrizable. Let $d$ be a metric on $K$ generating the given topology. There is a closed set $K_0$ in $K$ so that $I$ consists of all functions
in $C(K)$ that vanish on $K_0$. By Theorem 2 it suffices to show that every disjoint norm null sequence in $E$ is order bounded in $E$. Let $(x_n)$ be a disjoint null sequence in $E$. Define $f_n = T|x_n|$ for all $n$. Then $(f_n)$ is a disjoint nonnegative sequence in $I$. If $(f_n)$ is not norm bounded, there is a subsequence $(f_{n_k})$ such that $\|x_{n_k}\| \leq 1/2^k$ and $\|f_{n_k}\| > k$ for all $k$. The sum $x = \sum |x_{n_k}|$ converges in $E$. Clearly $Tx \geq f_{n_k} \geq 0$ for all $k$. This implies that $\|Tx\| > k$ for all $k$, which is absurd. Therefore, there exists $c_0$ such that $c_0 > \|f_n\|$ for all $n$.

**Claim.** Let $c_k = \sup\{d(t, K_0) : d(t, K_0) \leq 1/k, n \in \mathbb{N}\}$. Then $(c_k)$ is a nonincreasing null sequence.

Clearly $(c_k)$ is a nonincreasing sequence. If $(c_k)$ is not a null sequence, there exists $\varepsilon > 0$ such that $c_k > \varepsilon$ for all $k$. By uniform continuity of $f_n$, for each $n$, $\lim_k \sup\{d(t, K_0) : d(t, K_0) \leq 1/k\} = 0$. Thus, there exist $n_1 < n_2 < \cdots$ and $(t_i) \in K$, $d(t_i, K_0) \to 0$, such that $f_{n_i}(t_i) > \varepsilon$ for all $i$. By taking a further subsequence if necessary, we may also assume that $\|x_{n_i}\| \leq 1/2^i$ for all $i$. Now $x = \sum |x_{n_i}|$ converges in $E$ and $Tx \geq f_{n_i}$ for all $i$. Then $Tx(t_i) \geq f_{n_i}(t_i) > \varepsilon$ for all $i$. Since $d(t_i, K_0) \to 0$, this contradicts the fact that $Tx \in I$.

By the Claim, there exists a continuous function $g$ on $[0, \infty)$ such that $g(0) = 0$, $g(s) \geq c_k$ if $\frac{1}{k+1} \leq s < \frac{1}{k}$, where we take $1/0 = \infty$. Define $f : K \to \mathbb{R}$ by $f(t) = g(d(t, K_0))$. Then $f \in C(K)$ and $f = 0$ on $K_0$. Hence $f \in I$. For any $n$, if $d(t, K_0) = 0$, then $t \in K_0$ and hence $f_n(t) = 0 \leq f(t)$. On the other hand, if $\frac{1}{k+1} \leq d(t, K_0) < \frac{1}{k}$, then $f_n(t) \leq c_k \leq f(t)$. Thus $f \geq f_n$ for all $n \in \mathbb{N}$. Then $T^{-1}f \geq |x_n|$ for all $n \in \mathbb{N}$. Therefore, $(x_n)$ is order bounded in $E$, as desired.

Now we can show that the Banach lattice $c_0$ is stable under nonlinear order isomorphisms.

**Theorem 4.** Let $E$ be a Banach lattice. The following are equivalent.

(a) $E$ is lattice isomorphic to $c_0$.

(b) $E$ is order isomorphic to $c_0$.

(c) $E$ is order isomorphic to an infinite dimensional closed sublattice of $c_0$.

**Proof.** The implications (a) $\implies$ (b) $\implies$ (c) are immediate. By [6, Corollary 5.3], every infinite dimensional closed sublattice of $c_0$ is lattice isomorphic to $c_0$. The implication (c) $\implies$ (b) follows. Now assume that $E$ is order isomorphic to $c_0$. Let $T : c_0 \to E$ be an order isomorphism such that $T0 = 0$. Denote by $(e_n)$ the unit vector basis of $c_0$ and let $x_n = Te_n$ for each $n$. If $m \neq n$, $0 = T(e_m \wedge e_n) = x_m \wedge x_n$. That is, $(x_n)$ is a disjoint positive sequence in $E$. Also, since $[0, e_n]$ is a totally ordered set, so is $[0, x_n]$. It follows that $[0, x_n] = \{cx_n : 0 \leq c \leq 1\}$.

**Claim.** For each $n \in \mathbb{N}$ and any $a \geq 0$, there exists $b \geq 0$ such that $T(ax_n) = bx_n$. 

Let $a \geq 0$ be given and define $b = \sup \{c \geq 0 : cx_n \leq T(\alpha e_n) \}$. Obviously, the set on the right contains 0 and hence is nonempty. Also $cx_n \leq T(\alpha e_n)$ implies that $|c||x_n| \leq ||T(\alpha e_n)||$. Since $x_n \neq 0$, it follows that $b < \infty$. There exist $c_k \geq 0$ such that $c_k x_n \leq T(\alpha e_n)$ and $c_k \to b$. Since $E_+$ is a closed set, $b x_n \leq T(\alpha e_n)$. Let $x = T(\alpha e_n) - b x_n$. Then $x \geq 0$. Thus $T^{-1} x = \sum a_m e_m = \sqrt{a_m e_m} \geq 0$ in $c_0$. If $m \neq n$,

$$0 = T(\alpha e_n \wedge e_m) = T(\alpha e_n) \wedge x_m \geq x \wedge x_m \geq 0.$$ 

Thus $x \wedge x_m = 0$ if $m \neq n$. On the other hand, since $x \wedge x_n \in [0, x_n]$, there exists $0 \leq c \leq 1$ such that $x \wedge x_n = cx_n$. Then

$$T(\alpha e_n) - b x_n = x \geq x \wedge x_n = cx_n$$

and hence $T(\alpha e_n) \geq (b + c)x_n$. By definition of $b, c = 0$. Hence $x \wedge x_n = 0$. Therefore, $T^{-1} x \wedge e_i = T^{-1} (x \wedge x_i) = 0$ for all $i$. Clearly, this means that $T^{-1} x = 0$ and hence $x = 0$. So we have shown that $T(\alpha e_n) = b x_n$, as desired. This completes the proof of the Claim.

Let $x$ be any positive element in $E$. Then $T^{-1} x = \sqrt{\alpha n} e_n$ for some nonnegative sequence $(\alpha_n) \in c_0$. Thus $x = \sqrt{T(\alpha_n e_n)}$. By the Claim, $x = \sqrt{b_n x_n}$ for some nonnegative scalars $b_n$. If $\|b_n x_n = \sqrt{b_n} x_n$, where $b_n, b_n' \geq 0$ and both suprema exist, then using the distributivity of the lattice operations, it is easy to see that $b_n = b_n'$ for all $n$.

Now we show that for any $x = \sqrt{b_n x_n}$ as described above, $\lim \|b_n x_n\| = 0$. Otherwise, there exists $\varepsilon > 0$ and an infinite subset $I$ of $\mathbb{N}$ so that $\|b_n x_n\| \geq \varepsilon$ for all $n \in I$. For each $k \in \mathbb{N}$, $T^{-1}(k b_n x_n) = \sqrt{\sum_m a_k, m e_m}$. If $i \neq n$,

$$0 = T^{-1}(k b_n x_n) \wedge x_i = \sqrt{a_k, m e_m} \wedge e_i = (a_k, i) \wedge 1 e_i.$$ 

Thus $a_k, i = 0$ if $i \neq n$. Hence $T^{-1}(k b_n x_n) = a_k, n e_n$. Then $T^{-1}(k x) = \sqrt{a_k, n e_n}$. In particular, $\lim a_k, n = 0$ for all $k$. Choose $n_1 < n_2 < \cdots$ in $I$ so that $\lim_k a_k, n_k = 0$. We have $z = \sqrt{a_k, n_k e_n} \in c_0$ and $z \geq T^{-1}(k b_n x_n)$ for all $k$. Hence $Tz \geq k b_n x_n$ for all $k$. But then $\|Tz\| \geq \|k b_n x_n\| \to \infty$, which is impossible. This proves that $\lim \|b_n x_n\| = 0$.

To recap, we have shown that if $x \in E_+$, then $x$ has a unique representation $x = \sqrt{b_n x_n}$, where $b_n$ are nonnegative scalars so that $\lim \|b_n x_n\| = 0$. Note that $c_0$ is a closed ideal in the space $C(\mathbb{N}_*)$, where $\mathbb{N}_*$ is the 1-point compactification of $\mathbb{N}$, and that $C(\mathbb{N}_*)$ is separable. By Proposition 3, $E$ is lattice isomorphic to an $AM$-space. Consider the linear map $S : c_0 \to E$ given by $S(b_n) = \sum b_n x_n / \|x_n\|$. Note that if $(b_n) \in c_0$, then $\sum b_n x_n / \|x_n\|$ converges in $E$ since $E$ is an $AM$-space. Since $(x_n)$ is a disjoint sequence, $S$ is an injection. If $x \in E_+$, then $x = \sqrt{b_n x_n}$, where $b_n$ are nonnegative scalars so that $\lim \|b_n x_n\| = 0$. Thus $S(b_n \|x_n\|) = \sum b_n x_n = \sqrt{b_n x_n} = x$. Hence the range of $S$ contains $E_+$. It follows that $S$ is onto. It is clear that $S(b_n) \geq 0$ if $(b_n) \geq 0$. Since $S$ is a bijection as well, $S$ is an order isomorphism. Hence it is a linear order isomorphism and thus a lattice isomorphism. □
In view of Theorems 1 and 3 and the example of $L^p$ spaces mentioned in the introduction, it seems reasonable to ask the following question.

**Problem.** Suppose that $E$ is a Banach lattice so that any Banach lattice that is order isomorphic to $E$ is (linearly) lattice isomorphic to $E$. Must $E$ be an $AM$-space?

We can offer a partial solution to the problem. An element $e \geq 0$ in a Banach lattice is an *atom* if the ideal generated by $e$ is one dimensional. A Banach lattice is *atomic* if there is a maximal orthogonal set consisting of atoms. Let $E$ be an atomic Banach lattice and let $(e_\gamma)_{\gamma \in \Gamma}$ be a maximal orthogonal set consisting of normalized atoms. Any element $x \in E$ has a unique representation

$$x = \bigvee_{\gamma \in \Gamma_1} a_\gamma e_\gamma - \bigvee_{\gamma \in \Gamma_2} a_\gamma e_\gamma,$$

where $\Gamma_1$ and $\Gamma_2$ are disjoint subsets of $\Gamma$ and $0 < a_\gamma \in \mathbb{R}$ for all $\gamma \in \Gamma_1 \cup \Gamma_2$. See, e.g., [4, Exercise II.7]. For $1 < p < \infty$, the $p$-convexification of $E$, denoted by $E(p)$, as defined on p.53 in [4], may be presented as follows. $E(p)$ is the set of all real sequences $(a_\gamma)_{\gamma \in \Gamma}$ such that $\bigvee |a_\gamma|^p e_\gamma \in E$, endowed with the norm $\|\|(a_\gamma)\|| = \|\bigvee |a_\gamma|^p e_\gamma\|^{1/p}$. $E(p)$ is a Banach lattice (in the pointwise order). For each $\gamma \in \Gamma$, let $u_\gamma = (a_\xi)_{\xi \in \Gamma}$ with $a_\xi = 1$ if $\xi = \gamma$ and $a_\xi = 0$ otherwise. Then $(u_\gamma)_{\gamma \in \Gamma}$ is a maximal orthogonal set in $E(p)$ consisting of normalized atoms. The map $T : E(p) \to E$,

$$T(a_\gamma) = \bigvee_{\{\gamma : a_\gamma \geq 0\}} |a_\gamma|^p e_\gamma - \bigvee_{\{\gamma : a_\gamma < 0\}} |a_\gamma|^p e_\gamma$$

is a nonlinear order isomorphism. The norm on a Banach lattice $X$ is said to be *weakly Fatou* [5, Definition 2.4.18] if there is a constant $K < \infty$ so that if $0 \leq x_\tau \uparrow x$, then $\|x\| \leq K \sup_\tau \|x_\tau\|$.

**Theorem 5.** Let $E$ be an atomic Banach space and let $(e_\gamma)_{\gamma \in \Gamma}$ be a maximal orthogonal set consisting of normalized atoms. Suppose that any Banach lattice $F$ that is (nonlinearly) order isomorphic to $E$ is (linearly) lattice isomorphic to $E$. Then the closed sublattice generated by $(e_\gamma)_{\gamma \in \Gamma}$ in $E$ is lattice isomorphic to $c_0(\Gamma)$. Furthermore, if the norm on $E$ is weakly Fatou, then $E$ is lattice isomorphic to a closed sublattice of $\ell^\infty(\Gamma)$.

**Proof.** Let $F$ be the 2-convexification of $E$ and let $(u_\gamma)$ be the maximal orthogonal set of normalized atoms in $F$ as described above. Since $F$ is order isomorphic to $E$, it is lattice isomorphic to $E$ by the assumption. Let $T : E \to F$ be a lattice isomorphism. For each $\gamma$, $Te_\gamma$ is a nonzero positive element in $F$ and $[0, Te_\gamma] = T[0, e_\gamma]$ lies within a 1-dimensional subspace. Hence there exist $\pi(\gamma) \in \Gamma$ and $c_\gamma > 0$ such that $Te_\gamma = c_\gamma u_{\pi(\gamma)}$. Since $T$ is a lattice isomorphism, $\pi : \Gamma \to \Gamma$ is a permutation on $\Gamma$ and
0 < \inf c_\gamma \leq \sup c_\gamma < \infty. For any finite subset \( I \) of \( \Gamma \), we have

\[
\frac{1}{\sup c_\gamma} \| \bigvee_{\gamma \in I} e_\gamma \| \leq \| \bigvee_{\gamma \in I} \frac{1}{c_\gamma} e_\gamma \| = \| T^{-1} \bigvee_{\gamma \in I} u_{\pi(\gamma)} \|
\]

\[
\leq \| T^{-1} \| \cdot \| \bigvee_{\gamma \in I} u_{\pi(\gamma)} \| = \| T^{-1} \| \| \bigvee_{\gamma \in I} e_{\pi(\gamma)} \|^{1/2}.
\]

Let \( C = \sup c_\gamma \| T^{-1} \| \). For any \( m \in \mathbb{N} \), let

\[
\mu_m = \sup \{ \| \bigvee_{n \in I} e_n \| : \#I = m \}.
\]

Clearly, \( \mu_m < \infty \). Let \( I \) be such that \( \#I = m \) and \( \| \bigvee_{n \in I} e_n \| \geq \mu_m / 2 \). Then

\[
\mu_m \geq \| \bigvee_{\gamma \in \pi(I)} e_\gamma \| \geq \frac{1}{C^2} \| \bigvee_{\gamma \in I} e_\gamma \|^2 \geq \frac{\mu_m^2}{4C^2}.
\]

Therefore, \( \mu_m \leq 4C^2 \).

Let \( G \) be the closed sublattice of \( E \) generated by \( (e_\gamma)_{\gamma \in \Gamma} \). Since \( (e_\gamma)_{\gamma \in \Gamma} \) is a disjoint set, \( G \) is the same as the closed subspace generated by \( (e_\gamma)_{\gamma \in \Gamma} \). Clearly, \( \sum a_\gamma e_\gamma \in G \) implies that \( (a_\gamma) \in c_0(\Gamma) \). Conversely, suppose that \( (a_\gamma) \in c_0(\Gamma) \). For any \( \varepsilon > 0 \), there exists a finite subset \( I \) of \( \Gamma \) such that \( |a_\gamma| \leq \varepsilon \) for all \( \gamma \notin I \). If \( J \) is a finite subset of \( \Gamma \) disjoint from \( I \), then

\[
\| \sum_{\gamma \in J} a_\gamma e_\gamma \| \leq \max_{\gamma \in J} |a_\gamma| \sum_{\gamma \in J} e_\gamma \leq \varepsilon \cdot 4C^2.
\]

Thus, \( \sum a_\gamma e_\gamma \) converges in \( G \) if \( (a_\gamma) \in c_0(\Gamma) \). It is now clear that the map \( S : c_0(\Gamma) \to G \) defined by \( S(a_\gamma) = \sum a_\gamma e_\gamma \) is a lattice isomorphism.

Finally, suppose that the norm on \( E \) is weakly Fatou with constant \( K \). By the discussion preceding the theorem, each \( x \in E \) has a unique representation \([\Gamma]\). Clearly, for \( \gamma \in \Gamma_1 \cup \Gamma_2 \), \( |a_\gamma| = \| a_\gamma e_\gamma \| \leq \| x \| \). Define

\[
x(\gamma) = \begin{cases} a_\gamma & \text{if } \gamma \in \Gamma_1, \\ -a_\gamma & \text{if } \gamma \in \Gamma_2, \\ 0 & \text{otherwise.}\end{cases}
\]

Then \( (x(\gamma)) \in \ell^\infty(\Gamma) \) and \( \|(x(\gamma))\|_{\infty} \leq \|x\| \). On the other hand \( \bigvee_{\gamma \in I} |x(\gamma)| e_\gamma \uparrow |x| \), where \( I \) runs through the directed set of all finite subsets of \( \Gamma \). By assumption,

\[
\|x\| = \|x\| \leq K \sup_I \| \bigvee_{\gamma \in I} |x(\gamma)| e_\gamma \|
\]

\[
= K \sup_I \| S(|x(\gamma)|) \| \leq K \| S \|(x(\gamma))\|_{\infty}.
\]

It is now clear that the map \( R : E \to \ell^\infty(\Gamma) \) given by \( Rx = (x(\gamma)) \) is a lattice isomorphism from \( E \) onto a closed sublattice of \( \ell^\infty(\Gamma) \). \( \square \)
Our final result shows that containment of a closed sublattice isomorphic to \( c_0 \) is also a stable property under order isomorphisms. This holds in fact in the category of quasi-Banach lattices. Let \( E \) be a real or complex vector space. A quasi-norm on \( E \) is a functional \( \| \cdot \| \) on \( E \) such that

(a) \( \| x \| > 0 \) if \( x \neq 0 \),
(b) \( \| ax \| = |a| \| x \| \) for any scalar \( a \) and any \( x \in E \),
(c) There is a constant \( C < \infty \) such that \( \| x + y \| \leq C(\| x \| \lor \| y \|) \) for all \( x, y \in E \).

A quasi-norm on \( E \) generates a Hausdorff linear topology where the sets \( \{ x : \| x \| < 1/n \} \) form a neighborhood basis at 0. If this topology is completely metrizable, then we say that the quasi-norm is complete and that \( E \) is a quasi-Banach space. A quasi-Banach lattice is a real vector lattice equipped with a complete quasi-norm \( \| \cdot \| \) such that \( |x| \leq |y| \) implies \( \| x \| \leq \| y \| \) for all \( x, y \in E \). Refer to [3] for more information regarding quasi-Banach spaces and quasi-Banach lattices. Given a quasi-norm \( \| \cdot \| \) with associated constant \( C \), it is evident that

\[
\| \sum_{k=1}^{n} x_k \| \leq \max_{1 \leq k \leq n-1} C^k \| x_k \| \lor C^{n-1} \| x_n \|.
\]

It follows that if \( (x_k) \) is a sequence in a quasi-Banach space with \( \lim \| x_k \| = 0 \), then there is a subsequence \( (x_{k_i}) \) such that \( \sum x_{k_i} \) converges. It is easy to see that the positive cone \( \{ x : x \geq 0 \} \) is a closed set in a quasi-Banach lattice; equivalently, the limit of any positive sequence is positive. Consider the following statements.

**Theorem 6.** Let \( E \) and \( F \) be order isomorphic quasi-Banach lattices. If \( E \) contains a closed sublattice (nonlinearly) order isomorphic to \( c_0 \), then \( F \) contains a closed sublattice linearly lattice and topologically isomorphic to \( c_0 \).

**Theorem 7.** Let \( E \) and \( F \) be order isomorphic quasi-Banach lattices. If \( E \) contains a closed sublattice linearly order isomorphic to \( c_0 \), then \( F \) contains a closed sublattice linearly lattice and topologically isomorphic to \( c_0 \).

Evidently Theorem 6 is stronger than Theorem 7. But, in fact, the two results are equivalent. Indeed, assume that Theorem 7 holds. If \( G \) is a quasi-Banach lattice order isomorphic to \( c_0 \), then, taking \( E \) to be \( c_0 \) and \( F \) to be \( G \) in Theorem 7 one concludes that \( G \) contains a closed sublattice linearly order isomorphic to \( c_0 \). Thus any \( E \) that satisfies the hypothesis of Theorem 6 also fulfills the condition of Theorem 7.

We now proceed to prove Theorem 7 (and hence also Theorem 6). First observe that in order to produce a closed sublattice of \( F \) linearly order and topologically isomorphic to \( c_0 \), it suffices to obtain a disjoint sequence \( (y_i) \) in \( F \) such that \( \inf \| y_i \| > 0 \) and \( \sup_j \| \sum_{i=1}^{j} y_i \| < \infty \).
Lemma 8. Let \( G \) be a quasi-Banach lattice and let \( S : c_0 \to G \) be a (linear) lattice isomorphism. Denote by \((e_k)\) the unit vector basis of \( c_0 \). Then \( \inf \|Se_k\| > 0 \).

**Proof.** Otherwise, by the observation preceding Theorem \([6]\) there is a subsequence \((e_{k_i})\) such that \( x = \sum Se_{k_i} \) converges in \( G \). Since the positive cone of \( G \) is closed, \( x \geq \sum_{i=1}^m Se_{k_i} = S(\sum_{i=1}^m e_{k_i}) \) for all \( m \). Then \( S^{-1}x \geq \sum_{i=1}^m e_{k_i} \) for all \( m \), which is clearly absurd. \( \square \)

Lemma 9. Let \( E \) and \( F \) be quasi-Banach lattices and let \( T : E \to F \) be an order isomorphism such that \( T(0) = 0 \). If \((x_k)\) is a disjoint sequence in \( E_+ \) with \( \inf \|x_k\| > 0 \), then there exists \( N \in \mathbb{N} \) such that \( \lim \sup \|T(Nx_k)\| > 0 \).

**Proof.** Otherwise, there is a subsequence \((x_{k_i})\) such that \( \lim \|T(i\eta x_{k_i})\| = 0 \). By using a further subsequence if necessary, we may assume that \( y = \sum T(i\eta x_{k_i}) \) converges in \( F \). Since \( T \) is an order isomorphism and \( T(0) = 0 \), \( T(i\eta x_{k_i}) \geq 0 \) for all \( i \). Thus \( y \geq T(i\eta x_{k_i}) \geq 0 \) for all \( i \). Hence \( T^{-1}y \geq i\eta x_{k_i} \geq 0 \) for all \( i \). Therefore, \( \|T^{-1}y\| \geq \inf \|x_k\| \) for all \( i \), which is impossible. \( \square \)

**Proof of Theorem \([4]\)**. Let \( G \) be a closed sublattice of \( E \) and let \( S : c_0 \to G \) and \( T : E \to F \) be order isomorphisms, where \( S \) is linear and, without loss of generality, \( T(0) = 0 \). Denote the unit vector basis of \( c_0 \) by \((e_k)\). By Lemma \( 8 \) \( \inf \|Se_k\| > 0 \). Let \( x_k = Se_k \). Since \((x_k)\) is a disjoint sequence in \( E_+ \), by Lemma \( 9 \) there exists \( N \in \mathbb{N} \) and an infinite subset \( I \) of \( \mathbb{N} \) so that \( \inf_{k \in I} \|T(Nx_k)\| > 0 \).

Assume that there exists \( \eta > 0 \) such that \( \inf_{k \in I} \|T(\eta x_k)\| = 0 \). There is an increasing sequence \((k_i)\) in \( I \) such that \( y = \sum T(\eta x_{k_i}) \) converges in \( F \). Then \( y \geq T(\eta x_{k_i}) \) and hence \( T^{-1}y \geq \eta x_{k_i} \) for all \( i \). Thus \( x = (N/\eta)T^{-1}y \geq Nx_{k_i} \) and so \( Tx \geq T(Nx_{k_i}) \) for all \( i \). Let \( y_i = T(Nx_{k_i}) \) for all \( i \). Then \((y_i)\) is a disjoint sequence in \( F \) such that \( \inf \|y_i\| > 0 \). Furthermore, \( 0 \leq \sum_{i=1}^j y_i = \sqrt{\sum_{i=1}^j y_i} \leq Tx \) for all \( j \). Hence \( \sum_{i=1}^j y_i \leq \|Tx\| \) for all \( j \). By the remark preceding Lemma \( 8 \) \( F \) has a closed sublattice linearly order and topologically isomorphic to \( c_0 \).

Finally, suppose that \( \inf_{k \in I} \|T(\eta x_k)\| > 0 \) for all \( \eta > 0 \). Let \((k_i)\) be an increasing sequence in \( I \). We claim that there exists \( \varepsilon > 0 \) such that \( \sup_j \|T(\varepsilon \sum_{i=1}^j x_{k_i})\| < \infty \). Otherwise, there is an increasing sequence \((j_m)\) such that \( \|T(2^{-m} \sum_{i=1}^{j_m} x_{k_i})\| > m \) for all \( m \). The element

\[
u = \sum_{m=1}^{\infty} 2^{-m} \sum_{i=1}^{j_m} e_{k_i}
\]

belongs to \( c_0 \) and majorizes \( 2^{-m} \sum_{i=1}^{j_m} e_{k_i} \) for each \( m \). Since \( S \) is linear and order preserving, \( x = Su \geq 2^{-m} \sum_{i=1}^{j_m} x_{k_i} \) and thus \( Tx \geq T(2^{-m} \sum_{i=1}^{j_m} x_{k_i}) \geq 0 \) for all \( m \). But then \( \|Tx\| > m \) for all \( m \), reaching a contradiction. Hence the claim is verified. Let \( y_i = T(\varepsilon x_{k_i}) \). Then \((y_i)\) is a disjoint sequence and
\[ \inf \|y_i\| > 0. \] For any \( j \),
\[ 0 \leq \sum_{i=1}^{j} y_i = \bigvee_{i=1}^{j} y_i = \bigvee_{i=1}^{j} T(\varepsilon x_{k_i}) = T(\bigvee_{i=1}^{j} \varepsilon x_{k_i}) = T(\sum_{i=1}^{j} \varepsilon x_{k_i}). \]
Therefore,
\[ \sup_{j} \| \sum_{i=1}^{j} y_i \| \leq \sup_{j} \| T(\sum_{i=1}^{j} \varepsilon x_{k_i}) \| < \infty. \]
Again, by the remark preceding Lemma \( \S \) we conclude that \( F \) has a closed sublattice linearly order and topologically isomorphic to \( c_0 \). \( \square \)

**Remark.** If \( E \) is a Banach lattice, then \( E \) does not contain a closed sublattice lattice isomorphic to \( c_0 \) if and only if \( E \) is weakly sequentially complete. Thus Theorem \( \S \) shows that the topological property of weak sequential completeness is preserved under nonlinear order isomorphisms between Banach lattices.

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