Degree-degree distribution in a power law random intersection graph with clustering

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Abstract

The bivariate distribution of degrees of adjacent vertices, degree-degree distribution, is an important network characteristic defining the statistical dependencies between degrees of adjacent vertices. We show the asymptotic degree-degree distribution of a sparse inhomogeneous random intersection graph and discuss its relation to the clustering and power law properties of the graph.

key words: degree-degree distribution, power law, clustering coefficient, random intersection graph, affiliation network.

1 Introduction

Correlations between degrees of adjacent vertices influence many network properties including the component structure, epidemic spreading, random walk performance, network robustness, etc., see [2], [8], [12], [18], [19] and references therein. The correlations are defined by the degree-degree distribution, i.e., the bivariate distribution of degrees of endpoints of a randomly chosen edge. In this paper we present an analytic study of the degree-degree distribution in a mathematically tractable random graph model of an affiliation network possessing tunable power law degree distribution and clustering coefficient. Our study is motivated by the interest in tracing the relation between the degree-degree distribution and clustering properties in a power law network.

Affiliation network and random intersection graph. An affiliation network defines adjacency relations between actors by using an auxiliary set of attributes. Let \( V \) denote the set of actors (nodes of the network) and \( W \) denote the auxiliary set of attributes. Every actor \( v \in V \) is prescribed a collection \( S_v \subset W \) of attributes and two actors \( u, v \in V \) are declared adjacent in the network if they share some common attributes. For example one may interpret elements of \( W \) as weights and declare two actors adjacent whenever the total weight of shared attributes is above some threshold value. Here we consider the simplest case, where \( u, v \in V \) are called adjacent whenever they share at least one common attribute, i.e., \( S_u \cap S_v \neq \emptyset \). Two popular examples of real affiliation networks are the film actor network, where two actors are declared adjacent if they have played in the same movie, and the collaboration network, where two scientists are declared adjacent if they have coauthored a publication.
A plausible model of a large affiliation network is obtained by prescribing the collections of attributes to actors at random. In order to model the heterogeneity of human activity, every actor \( v_j \in V \) is prescribed a random weight \( y_j \) reflecting their activity. Similarly, a random weight \( x_i \) is prescribed to each attribute \( w_i \in W \) to model its attractiveness. Now an attribute \( w_i \in W \) is included in the collection \( S_{ij} \) at random and with probability proportional to the attractiveness \( x_j \) and activity \( y_j \) (cf., \cite{4}, \cite{17}). In this way we obtain a random graph on the vertex set \( V \) sometimes called the inhomogeneous random intersection graph, see \cite{5} and references therein. Before giving a detailed definition of this random graph model we mention a recent publication \cite{16}, which argues convincingly that in some social networks the ‘heavy-tailed degree distribution is causally determined by similarly skewed distribution of human activity’. The empirical evidence reported in \cite{16} suggests that the inhomogeneous random intersection graph can be considered as a realistic model of a power-law affiliation network.

**Rigorous model.** Let \( X_1, \ldots, X_m, Y_1, \ldots, Y_n \) be independent non-negative random variables such that each \( X_i \) has the probability distribution \( P_1 \) and each \( Y_j \) has the probability distribution \( P_2 \). Given realized values \( X = \{X_i\}_{i=1}^m \) and \( Y = \{Y_j\}_{j=1}^n \) we define the random bipartite graph \( H_{X,Y} \) with the bipartition \( V \cup W \), where \( V = \{v_1, \ldots, v_n\} \) and \( W = \{w_1, \ldots, w_m\} \). Every pair \( \{w_i, v_j\} \) is linked in \( H_{X,Y} \) with probability

\[
p_{ij} = \min\{1, \lambda_{ij}\}, \quad \text{where} \quad \lambda_{ij} = \frac{X_i Y_j}{\sqrt{nm}}
\]

independently of the other pairs \( \{w, v\} \in W \times V \). For large \( n \) and \( m \), we typically have \( \lambda_{ij} < 1 \) so that the probability \( p_{ij} \) is proportional to the “activity” \( Y_j \) of \( v_j \) and the “attractiveness” \( X_i \) of \( w_i \). The inhomogeneous random intersection graph \( G = G(P_1, P_2, n, m) \) defines the adjacency relation on the vertex set \( V \): vertices \( u, v \in V \) are declared adjacent (denoted \( u \sim v \)) whenever \( u \) and \( v \) have a common neighbor in \( H_{X,Y} \). We call this neighbor a witness of the adjacency relation \( u \sim v \).

The random graph \( G \) has several features that make it a convenient theoretical model of a real complex network. Firstly, the statistical dependence of neighboring adjacency relations in \( G \) mimics that of real affiliation networks. In particular, \( G \) admits a tunable clustering coefficient: For \( m/n \to \beta \in (0, +\infty) \) as \( m, n \to +\infty \), we have, see \cite{7},

\[
P(v_1 \sim v_2 | v_1 \sim v_3, v_2 \sim v_3) = \frac{\kappa}{\kappa + \sqrt{\beta}} + o(1).
\]

Here \( \kappa := b_1 b_2^{-1} a_3 a_2^{-2} \) and \( a_i = \mathbf{E} X_i^3, b_j = \mathbf{E} Y_j^2 \). Secondly, an important feature of the model is its ability to produce a rich class of (asymptotic) degree distributions including power laws. Let \( d(v) \) denote the degree of a vertex \( v \in V \) in \( G \). We note that, by the symmetry, random variables \( d(v_1), \ldots, d(v_n) \) have the same probability distribution. The following result of \cite{3} describes the asymptotic distribution of \( d(v_1) \) as \( n, m \to +\infty \).

**Theorem 1.** Let \( m, n \to \infty \).

(i) Assume that \( m/n \to \beta \) for some \( \beta \in (0, +\infty) \). Suppose that \( \mathbf{E} X_1^2 < \infty \) and \( \mathbf{E} Y_1 < \infty \). Then \( d(v_1) \) converges in distribution to the random variable

\[
d_* = \sum_{j=1}^{\Lambda_1} \tau_j,
\]

where \( \tau_1, \tau_2, \ldots \) are independent and identically distributed random variables independent of the random variable \( \Lambda_1 \). They are distributed as follows. For \( r = 0, 1, 2, \ldots \), we have

\[
P(\tau_1 = r) = \frac{r + 1}{\mathbf{E} \Lambda_0} P(\Lambda_0 = r + 1) \quad \text{and} \quad P(\Lambda_i = r) = \mathbf{E} e^{-\lambda_i} \frac{\lambda_i^r}{r!}, \quad i = 0, 1.
\]
Here \( \lambda_0 = X_1 b_1 \beta^{-1/2} \) and \( \lambda_1 = Y_1 a_1 \beta^{1/2} \).

(ii) Assume that \( m/n \to +\infty \). Suppose that \( E X_1^2 < \infty \) and \( E Y_1 < \infty \). Then \( d(v_1) \) converges in distribution to a random variable \( \Lambda_3 \) having the probability distribution

\[
P(\Lambda_3 = r) = E e^{-\lambda_3} \frac{\lambda_3^r}{r!}, \quad r = 0, 1, \ldots
\]

Here \( \lambda_3 = Y_1 a_2 b_1 \).

(iii) Assume that \( m/n \to 0 \). Suppose that \( E X_1 < \infty \). Then \( P(d(v_1) = 0) = 1 - o(1) \).

We briefly explain the origin of \( \Lambda_i \), \( i = 0, 1, 3 \). The random variables \( \Lambda_0 \) and \( \Lambda_1 \) are limits (in distribution) of the degrees in the bipartite graph \( H_{X,Y} \) of \( w_1 \) and \( v_1 \) respectively. That is, the number of attributes linked to \( v_1 \) converges in distribution to \( \Lambda_1 \) and the number of vertices linked to \( w_1 \) converges in distribution to \( \Lambda_0 \). Furthermore, the size-biased random variable \( \tau_j \) counts the neighbors in \( G \) of \( v_1 \) witnessed by an attribute \( w_j \), given the event that \( w_j \) is linked to \( v_1 \). As for \( \Lambda_3 \) it refers to the case where \( m/n \to +\infty \). Here the number of attributes linked to \( v_1 \) grows to infinity (at the rate \( \Theta(\sqrt{m/n}) \)), while the number of vertices linked to any given attribute vanishes (an attribute produces a single link with a small probability of order \( \Theta(\sqrt{m/n}) \) or has no link at all). Thus among many attributes linked to \( v_1 \) only a few contribute to the degree \( d(v_1) \) by witnessing a single neighbor each. The number of neighbors converges in distribution to a mixed Poisson random variable \( \Lambda_3 \).

Using the fact that a Poisson random variable is highly concentrated around its mean one can show that for a power law distribution \( P(\lambda_i > r) \approx c_i r^{-\kappa_i} \), with some \( c_i, \kappa_i > 0 \), we have \( P(\Lambda_i > r) \approx c_i r^{-\kappa_i} \), for \( i = 0, 1, 3 \), see Lemma 4. Here and below for real sequences \( \{t_r\}_{r \geq 1} \) and \( \{s_r\}_{r \geq 1} \) we denote \( t_r \approx s_r \) whenever \( t_r/s_r \to 1 \) as \( r \to +\infty \). Furthermore, the tail \( P(d^* > r) \) of a randomly stopped sum \( d_* \) is as heavy as the heavier one of \( Y_1 \) and \( \tau_1 \), see, e.g., [1]. Hence, choosing a power law weights \( X \) and \( Y \) we obtain a power law asymptotic degree distributions, namely, the distributions of \( d^* \) and \( \Lambda_3 \).

In what follows we will focus on local probabilities. Given \( c > 0 \) and \( \kappa > 1 \), we say that a non-negative random variable \( Z \) has the power law property \( P_{c,\kappa} \) (denoted \( Z \in P_{c,\kappa} \)) if either \( Z \) is integer valued and satisfies \( P(Z = r) \approx cr^{-\kappa} \), or \( Z \) is absolutely continuous with density \( f_Z \) satisfying \( f_Z(t) = (c + o(1))t^{-\kappa} \) as \( t \to +\infty \).

**Remark 1.** Let \( c, x > 0 \). Let \( r \to +\infty \).

(i) Let \( a > 0 \) and \( \kappa > 3 \). Assume that \( E e^{a Y_1} < \infty \) and \( X_1 \in P_{c,\kappa} \). Then \( P(d_* = r) \approx cb_1^{x-1} b^{3(\kappa-x)/2} r^{1-\kappa} \).

(ii) Let \( \kappa > 2 \). Assume that \( Y_1 \in P_{c,\kappa} \) and \( P(X_1 = x) = 1 \). Then \( P(d_* = r) \approx c(x^2 b_1)^{x-1} r^{-\kappa} \).

To show (i) we exploit the power law properties of the local probabilities of randomly stopped sums, like \( d_* \), in the case where the summands are heavy tailed and the number of summands has a light tail (see, e.g., Theorem 4.30 of [9]). Unfortunately we are not aware of rigorous results establishing power law properties of the local probabilities of randomly stopped sums, like \( d_* \), in the case where the number of summands is heavy tailed.

**Degree-degree distribution.** We are interested in the bivariate distribution of degrees of adjacent vertices. Denote \( d_1 = d(v_1) \), \( d_2 = d(v_2) \) and let

\[
p(k_1, k_2) = P(d_1 = k_1 + 1, d_2 = k_2 + 1 | v_1 \sim v_2), \quad k_1, k_2 = 0, 1, \ldots
\]

denote the probabilities defining the conditional bivariate distribution of the ordered pair \( (d_1, d_2) \), given the event that vertices \( v_1 \) and \( v_2 \) are adjacent. Let \((u^*, v^*)\) be an ordered pair of distinct
In Theorem 2 below we show a first order asymptotics of $Y$. Formulating the theorem we introduce some notation. We remark that $d$ vertices chosen uniformly at random from $V$ invariant under permutations of its vertices, we have

$$p(k_1, k_2) = p(k_2, k_1) = \mathbf{P}(d(u^*) = k_1 + 1, d(v^*) = k_2 + 1 | u^* \sim v^*).$$

In Theorem 2 below we show a first order asymptotics of $p(k_1, k_2)$ as $n, m \to +\infty$. Before formulating the theorem we introduce some notation. We remark that $d$, defined by (2) depends on $Y_1$. By conditioning on the event $\{Y_1 = y\}$ we obtain another random variable, denoted $d_y^*$, which has the compound Poisson distribution

$$\mathbf{P}(d_y^* = r) = \mathbf{P}(d_y = r | Y_1 = y) = \mathbf{P}\left(\sum_{j=1}^{N} \tau_j = r\right), \quad r = 0, 1, \ldots.$$

Here $N = N_y$ denotes a Poisson random variable which is independent of the iid sequence $\{\tau_j\}_{j \geq 1}$ and has mean $\mathbf{E}N_y = ya_1\beta^{1/2}$, $y \geq 0$. Given integers $k_1, k_2, r \geq 0$, denote

$$q_r = \mathbf{E}(Y_1 \mathbf{P}(d_y^* = r | Y_1)) = \mathbf{E}(Y_1 \mathbf{P}(d_y = r | Y_1)),$$

$$p_{\beta}(k_1, k_2) = \frac{\beta}{b_1 a_2} \sum_{r=0}^{k_1 k_2} (r + 1)(r + 2)\mathbf{P}(A_0 = r + 2)q_{k_1-r}q_{k_2-r},$$

$$\tilde{p}(r) = r\mathbf{P}(A_3 = r) (\mathbf{E}A_3)^{-1}, \quad p_{\infty}(k_1, k_2) = \tilde{p}(k_1 + 1)\tilde{p}(k_2 + 1).$$

Our main result is the following theorem.

**Theorem 2.** Let $m, n \to \infty$. Suppose that $\mathbf{E}X_2^2 < \infty$ and $\mathbf{E}Y_1 < \infty$.

(i) Assume that $m/n \to \beta$ for some $\beta \in (0, +\infty)$. Then for every $k_1, k_2 \geq 0$ we have

$$p(k_1, k_2) = p_{\beta}(k_1, k_2) + o(1), \quad (7)$$

(ii) Assume that $m/n \to +\infty$. Then for every $k_1, k_2 \geq 0$ we have

$$p(k_1, k_2) = p_{\infty}(k_1, k_2) + o(1). \quad (8)$$

We note that the moment conditions $\mathbf{E}X_2^2 < \infty$ and $\mathbf{E}Y_1 < \infty$ of Theorem 2 are the minimal ones as the numbers $a_2 = \mathbf{E}X_2^2$ and $b_1 = \mathbf{E}Y_1$ enter (implicitly) both formulas (7) and (8).

**Remark 2.** In the case where $m/n \to +\infty$, the size biased probability distribution $\{\tilde{p}(r)\}_{r \geq 1}$ is the limiting distribution of $d(v_1)$ conditioned on the event $v_1 \sim v_2$, i.e.,

$$\mathbf{P}(d(v_1) = r | v_1 \sim v_2) = \tilde{p}(r) + o(1), \quad r = 1, 2, \ldots \quad (9)$$

Let us examine how the clustering property (presence of non-vanishing clustering coefficient) affects the structure of the asymptotic degree-degree distribution. Firstly we consider the case $m/n \to +\infty$, where the clustering coefficient vanishes (cf. (7)). In this case one may expect that the statistical dependence between neighboring adjacency relations fades away as $m, n \to +\infty$ and that the degrees of adjacent vertices are asymptotically independent. This is indeed the case as (8) and (9) imply that

$$p(k_1, k_2) = \mathbf{P}(d(v_1) = k_1 + 1 | v_1 \sim v_2)\mathbf{P}(d(v_2) = k_2 + 1 | v_1 \sim v_2) + o(1).$$

For $m/n \to \beta \in (0, +\infty)$ the random graph $G$ admits a non-vanishing clustering coefficient. Now, the statistical dependence of neighboring adjacency relations persists as $n, m \to +\infty$, and
Theorem 3. Let \( a, \beta, c, x > 0 \).

(i) Assume that \( E X_1^2 < \infty \) and \( Y_1 \in \mathcal{P}_{c, \kappa} \), for some \( \kappa > 2 \). Then for \( k_1, k_2 \to +\infty \) we have

\[
p_{\infty}(k_1, k_2) = (1 + o(1))c_2 a_2^{2x-3} b_1^{2x-6} (k_1 k_2)^{1-\kappa}.
\]

(ii) Assume that \( E e^{a Y_1} < \infty \) and \( X_1 \in \mathcal{P}_{c, \kappa} \), for some \( \kappa > 3 \). Let \( k_1, k_2 \to +\infty \) so that \( k_1 \leq k_2 \). Suppose that either \( k_2 - k_1 \to +\infty \) or \( k_2 - k_1 = k \), for an arbitrary, but fixed integer \( k \geq 0 \). Then

\[
p_{\beta}(k_1, k_2) = (1 + o(1)) \frac{\beta}{b_1 a_2} \times \begin{cases} b_1 c_1^* c_2^* k_1^{2-\kappa} (k_2 - k_1)^{1-\kappa} & \text{if } k_2 - k_1 \to +\infty, \\ c_3^* k_1^{2-\kappa} & \text{if } k_2 - k_1 = k. \end{cases}
\]

Here \( c_1^* = c(b_1 \beta^{-1/2})^{x-1}, c_2^* = c b_1^{x-2} b_2^{3-\kappa/2}, \) and \( c_3^* = \sum_{i \geq 0} q_i q_{k+i} \). Furthermore, we have \( q_i \approx c_1^* i^{x-1} \). We recall that \( q_i \) is defined in \( [6] \).

(iii) Assume that \( P(X_1 = x) = 1 \) and \( Y_1 \in \mathcal{P}_{c, \kappa} \), for some \( \kappa > 2 \). For \( k_1, k_2 \to +\infty \) we have

\[
p_{\beta}(k_1, k_2) = (1 + o(1)) e_2^{2x-3} b_1^{2x-6} (k_1 k_2)^{1-\kappa}.
\]

We remark, that example (iii) refers to the clustering regime \((m/n \to \beta \in (0, +\infty))\), where neighboring adjacency relations are statistically dependent. Hence \( p_{\beta}(., .) \) is not a product of marginal probabilities. An interesting fact is that for \( k_1, k_2 \to +\infty \) the tail of \( p_{\beta}(k_1, k_2) \) is asymptotically a product of independent marginals. Here we observe a situation, where heavy tailed weights of actors \( Y_j \) define the power law tails of \( p_{\beta}(., .) \) and outperform the light weights of attributes \( X_i \).

Our final remark is about the case where \( m/n \to 0 \). By Theorem 1 in this case the edges of a sparse inhomogeneous random intersection graph span a subgraph on \( o(n) \) randomly selected vertices leaving the remaining \((1 - o(1))n\) vertices isolated. Consequently, the subgraph is relatively dense and we do not expect stochastically bounded degrees of endpoints of adjacent vertices.

Related work. The influence of degree-degree correlations on the network properties have been studied by many authors, see, e.g., \([8], [12], [18], [19]\) and references therein. The asymptotic degree-degree distribution in a preferential attachment random graph with tunable power law degree distribution was shown in \([11]\). Our model and approach are much different. To our best knowledge the present paper is the first attempt to trace the relation between the degree-degree distribution and the clustering property in a power law network. Connections between Newman’s assortativity coefficient and the clustering coefficient in related random graph models have been discussed in \([6]\).

The present paper complements, revises and extends the results of \([3]\), presented at the 12th Workshop on Algorithms and Models for the Web Graph, WAW 2015. In particular, the factor \( b_1 \) is included in \([11]\). It was missing in the respective formula (7) of \([3]\).

2 Proof

Here we prove Theorems 2, 3 and Remarks 1, 2. Before the proof we collect some notation. We will assume throughout the proof that \( m = m(n) \to +\infty \) as \( n \to +\infty \). The expressions...
$o(\cdot)$, $O(\cdot)$ will always refer to the case where $n \to +\infty$. We use the notation $o_P(\cdot)$ and $O_P(\cdot)$ consistently with [13]. Given two real sequences $\{a_n\}_{n \geq 1}$ and $\{b_n\}_{n \geq 1}$ we write $a_n \sim b_n$ to denote the fact that $(a_n - b_n)mn = o(1)$. By $I_A$ we denote the indicator function of an event $A$ and $\overline{I}_A = 1 - I_A = I_{A^c}$ denotes the indicator of the complement event $A$.

In the proof we often use the following facts: For a random variable $Z$ and a sequence of events $\{\mathcal{K}_n\}_{n \geq 1}$ defined on the same probability space we have that $E|Z| < \infty$ and $\lim n P(\mathcal{K}_n) = 0$ imply $E(\overline{I}_{\mathcal{K}_n} Z) = o(1)$; for a sequence of random variables $\{Z_n\}_{n \geq 1}$ the conditions $\exists C > 0$ such that $|Z_n| \leq C$ almost surely $\forall n$, and $Z_n = o_P(1)$ imply $E|Z_n| = o(1)$.

We denote by $P^*$ and $E^*$ (respectively $\overline{P}$ and $\overline{E}$) the conditional probability and expectation given $X_1, Y_1, Y_2$ (respectively $X, Y$). For random variables $\xi, \zeta$ defined on the same probability space as $X, Y$ we denote by $d_{TV}(\xi, \zeta)$ (respectively $\tilde{d}_{TV}(\xi, \zeta)$) the total variation distance between the conditional distributions of $\xi$ and $\zeta$ given $X_1, Y_1, Y_2$ (respectively $X, Y$). Given a sequence of random variables $\{Z_n\}_{n \geq 1}$, defined on the same probability space as $Y_1, Y_2, X_1$, the notation $Z_n = o_P(1)$ means that, given any realized values $Y_1, Y_2, X_1$ and any $\varepsilon > 0$, we have $P^*(Z_n > \varepsilon) = o(1)$. We note that for $\{Z_n\}_{n \geq 1}$ independent of $Y_1, Y_2, X_1$ we have $Z_n = o_P(1) \iff Z_n = o_P(1)$.

The degree of a vertex $v_i \in V$ is denoted $d_i = d(v_i)$. The number of common neighbors of $v_i, v_j \in V$ is denoted $d_{ij}$. For a vertex $v \in V$ and attribute $w \in W$ we denote by $\{w \rightarrow v\}$ the event that $v$ and $w$ are linked in $H$. Introduce the events

$$A_i = \{w_i \rightarrow v_1, w_i \rightarrow v_2\}, \quad 1 \leq i \leq m.$$ 

We write for short

$$I_{ij} = I_{(w_i, v_j)}, \quad I_{ij} = 1 - I_{ij}, \quad I_{ij} = I_{(\lambda_{ij} \leq 1)}, \quad I_{ij} = 1 - I_{ij}.$$ 

Let $L = (L_0, L_1, L_2)$ denote the random vector with marginal random variables

$$L_0 = u_1, \quad L_1 = \sum_{2 \leq i \leq m} I_{1i} u_i, \quad L_2 = \sum_{2 \leq i \leq m} I_{2i} u_i, \quad u_i = \sum_{3 \leq j \leq n} I_{ij}, \quad 1 \leq i \leq m.$$ 

Let $\Lambda_i$, $0 \leq i \leq 4$ denote random variables having mixed Poisson distributions

$$P(\Lambda_i = r) = E(e^{-\lambda_i} \lambda_i^r/r!), \quad r = 0, 1, 2, \ldots, \quad (13)$$

where

$$\lambda_0 = X_1 b_1 \beta^{-1/2}, \quad \lambda_1 = Y_1 a_1 \beta^{1/2}, \quad \lambda_2 = Y_2 a_1 \beta^{1/2}, \quad \lambda_3 = Y_1 a_2 b_1, \quad \lambda_4 = Y_2 a_2 b_1$$

are random variables. We assume that conditionally, given $Y_1, Y_2, X_1$, the random variables $\Lambda_i$, $0 \leq i \leq 4$ are independent. Define random variables $d_{1}^* = \sum_{i=1}^{\Lambda_1} \tau_i$ and $d_{2}^* = \sum_{i=1}^{\Lambda_2} \tau_i$. Here $\tau_i$, $\tau_i^*$, $i \geq 1$ are independent and identically distributed random variables, which are independent of $Y_1, Y_2, X_1$ and have distribution [3]. Define the events

$$U_{k_1, k_2} = \{d_1 = k_1 + 1, d_2 = k_2 + 1\}, \quad U_{r_1, r_2}^* = \{L = (r, r_1, r_2)\},$$

$$U_{r_1, r_2} = \{\Lambda_0 = r, d_{1}^* = r_1, d_{2}^* = r_2\}, \quad U_{r_1, r_2} = \{\Lambda_4 = r_1, \Lambda_3 = r_2\}.$$ 

Define random variables $\hat{b}_k = m^{-1} \sum_{2 \leq i \leq m} X_i^k$ and $\hat{b}_k = n^{-1} \sum_{3 \leq j \leq n} Y_i^k$.

**Proof of Theorem 3** Proof of (i). The intuition behind formula [7] is that with a high probability the adjacency relation $v_1 \sim v_2$ as well as all common neighbors of $v_1$ and $v_2$ are witnessed, by
the same common attribute (all attributes having equal chances). Furthermore, conditionally on the event that this attribute is \( v_1 \), and given \( Y_1, Y_2, X_1 \), we have that the random variables \( d_{12}, \Delta_1 - d_{12} =: \Delta_1' \), \( d_{2} - d_{12} =: \Delta_2' \) are asymptotically independent. We note that \( \Delta_1' \) and \( \Delta_2' \) count individual (not shared) neighbors of \( v_1 \) and \( v_2 \). The individual neighbors of \( v_1 \) (of \( v_2 \)) are attracted by all attributes linked to \( v_1 \) (to \( v_2 \)), but \( w_1 \), while the common neighbors are attracted by the attribute \( w_1 \). The (conditional given \( Y_1, Y_2, X_1 \) asymptotic independence of \( d_{12}, \Delta_1', \Delta_2' \) comes from the fact that these random variables are mainly related to each other via average characteristics \( \hat{a}_1, \hat{b}_1 \) which are asymptotically constant, by the law of large numbers. Now, using Theorem \( 1 \) we identify limiting distributions of \( \Delta_1', \Delta_2' \). Finally, the limiting distribution for the number of vertices from \( \{v_1, \ldots, v_n\} \) attracted by \( w_1 \) is that of \( \Lambda_0 \).

We briefly outline the proof. In the first step we show that

\[
p(k_1, k_2) = \frac{\mathbb{P}(U_{k_1, k_2} \cap \{v_1 \sim v_2\})}{\mathbb{P}(v_1 \sim v_2)} = \frac{\mathbb{P}(U_{k_1, k_2} \cap (\cup_i A_i))}{\mathbb{P}(\cup_i A_i)} = \frac{nm}{a_2 b_1^2} \mathbb{P}(U_{k_1, k_2} \cap A_1) + o(1). \tag{14}
\]

Then using the total probability formula we split

\[
\mathbb{P}(U_{k_1, k_2} \cap A_1) = \sum_{r=0}^{k_1 \wedge k_2} \mathbb{P}(U_{k_1, k_2} \cap \{d_{12} = r\} \cap A_1). \tag{15}
\]

In the second step we show that for every \( r = 0, 1, \ldots, k_1 \wedge k_2 \)

\[
\mathbb{P}(U_{k_1, k_2} \cap \{d_{12} = r\} \cap A_1) \approx \mathbb{P}(U_{r, k_1 - r, k_2 - r} \cap A_1). \tag{16}
\]

In the final step we show that for \( r, r_1, r_2 = 0, 1, 2 \ldots \)

\[
\mathbb{P}(U_{r, r_1, r_2} \cap A_1) \approx \mathbb{E}(\lambda_{11} \lambda_{12} \mathbb{P}^*(U_{r, r_1, r_2}^*)). \tag{17}
\]

Now the simple identity

\[
\frac{nm}{a_2 b_1^2} \mathbb{E}(\lambda_{11} \lambda_{12} \mathbb{P}^*(U_{r, r_1, r_2}^*)) = \frac{\beta}{b_1}(r + 1)(r + 2)\mathbb{P}(\Lambda_0 = r + 2) q_{r_1} q_{r_2}
\]

completes the proof of \( 7 \). Finally, we remark that in order to prove the result under the minimal moment conditions \( \mathbb{E}Y_1 < \infty \) and \( \mathbb{E}X_1^2 < \infty \), we invoke, when necessary, a truncation argument, which makes our presentation some more involved.

**Step 1.** Here we prove \( \mathbb{P}(A_i) \leq \mathbb{E} \lambda_{11} \lambda_{12} = a_2 b_1^2 (nm)^{-1} \), relations \( 18 \) imply \( 14 \). We only prove the second relation. The proof of the first one is much the same. Fix a small \( 0 < \delta < 1 \) and introduce truncation events

\[
\mathcal{H}_i = \{Y_i < \delta \sqrt{m}\}, \quad i = 1, 2.
\]

We have

\[
\mathbb{P}(\{v_1 \sim v_2\}) = \mathbb{P}(\{v_1 \sim v_2\} \cap \mathcal{H}_1 \cap \mathcal{H}_2) + \mathbb{P}(\{v_1 \sim v_2\} \cap \mathcal{H}_1 \cap \bar{\mathcal{H}}_2) + \mathbb{P}(\{v_1 \sim v_2\} \cap \bar{\mathcal{H}}_1 \cap \mathcal{H}_2) + \mathbb{P}(\{v_1 \sim v_2\} \cap \bar{\mathcal{H}}_1 \cap \bar{\mathcal{H}}_2) =: p'_1 + p'_2 + p'_3 + p'_4. \tag{19}
\]
Now we evaluate $p'_i$ and construct upper bounds for $p'_i$, $i = 2, 3, 4$. Using the independence of $Y_1, Y_2$ and Markov’s inequality we obtain

$$p'_4 \leq P(\mathcal{H}_1 \cap \mathcal{H}_2) \leq \delta^{-2} n^{-1} (E(Y_1 || H_1))^2. \tag{20}$$

Next, using the identity $\{v_1 \sim v_2\} = \cup_{1 \leq i \leq m} A_i$ and inequality

$$P(A_1 \cap \mathcal{H}_1) = E(P^*(A_1 || H_1)) \leq E(\lambda_{11, \lambda_{21}} H_1)$$

we obtain

$$p'_2 \leq \sum_{1 \leq i \leq m} P(A_i \cap \mathcal{H}_1) = mP(A_1 \cap \mathcal{H}_1) \leq \frac{a_2 b_1}{n} EY_1 || H_1. \tag{21}$$

Clearly, (21) extends to $p'_3$. An upper bound on $p'_1$ is obtained in a similar way,

$$p'_1 \leq P(\{v_1 \sim v_2\}) \leq \sum_{1 \leq i \leq m} P(A_i) \leq \sum_{1 \leq i \leq m} E\lambda_{11, \lambda_{2i}} = n^{-1} a_2 b_1^2. \tag{22}$$

To get a lower bound we invoke inclusion-exclusion. We have

$$p'_1 \geq \sum_{1 \leq i \leq m} P(A_i \cap \mathcal{H}_1 \cap \mathcal{H}_2) - \sum_{1 \leq i < j \leq m} P(A_i \cap A_j \cap \mathcal{H}_1 \cap \mathcal{H}_2) =: m p' - \binom{m}{2} p'',$

where $p' = P(A_1 \cap \mathcal{H}_1 \cap \mathcal{H}_2)$ and $p'' = P(A_1 \cap A_2 \cap \mathcal{H}_1 \cap \mathcal{H}_2)$. In order to evaluate $p' = E(P^*(A_1 || H_1 \cap H_2))$ we invoke the inequalities

$$(1 - \mathbb{1}_{11} - \mathbb{1}_{12}) \lambda_{11, \lambda_{12}} \leq \mathbb{1}_{11} \mathbb{1}_{12} \lambda_{11, \lambda_{12}} = \mathbb{1}_{11} \mathbb{1}_{12} P^*(A_1) \leq P^*(A_1) \leq \lambda_{11, \lambda_{12}}. \tag{24}$$

We obtain

$$p' \geq E(P^*(A_1 || \mathcal{H}_1 \cap \mathcal{H}_2) \mathbb{1}_{11} \mathbb{1}_{12}) \geq (nm)^{-1}(a_2 b_1^2 - R), \tag{25}$$

where $R = E(Y_1 Y_2 X_1^2 (\mathbb{1}_{11} + \mathbb{1}_{12} + \mathbb{1}_{\mathcal{H}_1 \cap \mathcal{H}_2}))$. The next upper bound for $p''$ is simple

$$p'' = E(P^*(A_1) P^*(A_2 || \mathcal{H}_1 \cap \mathcal{H}_2)) \leq E(\lambda_{11, \lambda_{12}} \lambda_{21} \lambda_{22} || \mathcal{H}_1 \cap \mathcal{H}_2) \leq \delta^2 n^{-1} a_2 b_1^2. \tag{26}$$

Collecting (25), (26) in (23) we obtain a lower bound for $p'_1$. Combining this lower bound with (22) we obtain

$$n^{-1}(a_2 b_1^2 - R - \delta^2 a_2 b_1^2) \leq p'_1 \leq n^{-1} a_2 b_1^2. \tag{27}$$

Finally, we choose $\delta = \delta_n$ converging to 0 slowly enough so that $\delta \sqrt{n} \to +\infty$ and $\delta^{-1} E(Y_1 || H_1) \to 0$. We obtain from (20), (21), (27) that $p'_2, p'_3, p'_4 = o(n^{-1})$ and (27) implies that $p'_1 = n^{-1} a_2 b_1^2 + o(n^{-1})$. Hence (18) follows from (19).

**Step 2.** Here we prove (16). Let us note that event $A_1$ implies that $L_0$ counts common neighbors of $v_1$ and $v_2$ witnessed by $w_1$. In the case where $L_0 \neq d_{12}$ there should be a common neighbor of $v_1, v_2$ witnessed by some attribute $v_i$ other than $v_1$ or witnessed by two different attributes, say $w_{i_1}, w_{i_2} \in W \setminus \{w_1\}$. We introduce related events

$$B_{i0} = \{I_{i_1} I_{i_2} = 1 \text{ for some } 2 \leq i \leq m\},$$

$$B_{i1} = \{I_{i_1} I_{i_1}, i_2 \leq \bar{I}_{i_2} I_{i_2} = 1 \text{ for some } 3 \leq j \leq n \text{ and } 2 \leq i_1 \neq i_2 \leq m\}$$

and observe that on the event $A_1 \cap (B_{i0} \cup B_{i1})$ we have $L_0 = d_{12}$. Next, assuming that events $A_1$ and $L_0 = d_{12}$ hold we consider $L_k$ for $k = 1, 2$. The sum $L_k$ counts pairs $w_i \to v_j$, for $i \geq 1$, 

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j ≥ 3, such that v_j is a neighbor of v_k witnessed by w_i. Generally, we have L_k ≥ d_k − 1 − d_{12}. This inequality is strict if some common neighbor of v_1, v_2 witnessed by w_i is also witnessed as a neighbor of v_k by some other attribute w_{i'}, i ≥ 2. The inequality is also strict in the case where some neighbor of v_k is witnessed by two or more distinct attributes. If we rule out both of these possibilities, we have the equality L_k = d_k − 1 − d_{12}. We introduce the corresponding undesired events

\[ B_{k0} = \{ \| i_k \|_{i_j} = 1 \text{ for some } 3 \leq j \leq n \text{ and } 2 \leq i \leq m \}, \]

\[ B_{k1} = \{ \| i_{1k} \|_{\| i_j \|_{i_j} = 1 \text{ for some } 3 \leq j \leq n \text{ and } 2 \leq i_1 < i_2 \leq m \}. \]

Finally, we conclude that if A_1 holds and at least one of the following three relations fails

\[ L_0 = d_{12}, \quad L_1 + L_0 = d_1 - 1, \quad L_2 + L_0 = d_2 - 1 \]

then at least one of the events B_l, B_l, 0 ≤ l ≤ 2, 0 ≤ j ≤ 1, occurs. Hence, we have

\[ |P(U_{k_1, k_2} \cap \{ d_{12} = r \} \cap A_1) - P(U_{r, k_1 - r, k_2 - r} \cap A_1) | \leq \sum_{0 \leq l \leq 2} \sum_{0 \leq j \leq 1} P(A_1 \cap B_{lj}). \]

Next we prove that the quantity on the right is o((nm)^{-1}). For this purpose we estimate

\[ P(A_1 \cap B_{lj}) = E(P^*(A_1)P^*(B_{lj})) \leq (mn)^{-1}E(Y_1Y_2X_i^2P^*(B_{lj})) \]

and show that for any realized values Y_1, Y_2, X_1 we have

\[ P^*(B_{lj}) = o(1), \quad \text{for } 0 \leq l \leq 2, \quad 0 \leq j \leq 1. \] (29)

Since \( P^*(\cdot) \leq 1 \) and \( EY_1Y_2X_i^2 < \infty \), Lebesgue’s dominated convergence theorem then implies the desired bound \( E(Y_1Y_2X_i^2P^*(B_{lj})) = o(1) \).

Let us show (29). For this purpose we write events B_{lj} in the form B_{lj} = \{ S_{lj} ≥ 1 \}, where

\[ S_{00} = \sum_{2 \leq i_1 \leq m} \| i_{1} \|_{i_2}, \quad S_{01} = \sum_{3 \leq j \leq n} \sum_{2 \leq i_2 \leq m} \| i_{1} \|_{i_{1j}} \| i_{2} \|_{i_{2j}}, \]

\[ S_{k0} = \sum_{3 \leq j \leq n} \sum_{2 \leq i_2 \leq m} \| i_{k} \|_{i_1} \| i_{1j}, \quad S_{k1} = \sum_{3 \leq j \leq n} \sum_{2 \leq i_2 \leq m} \| i_{k} \|_{i_{1j}} \| i_{2} \|_{i_{2j}}, \] (30)

and apply the inequality

\[ P^*(S_{lj} ≥ 1) ≤ \varepsilon + P^*(\tilde{E}S_{lj} ≥ \varepsilon), \quad \forall \varepsilon > 0. \] (31)

Let us briefly explain (31). We apply the trivial inequality \( Z ≤ \varepsilon + Z1_{\{Z > \varepsilon\}} \) to the random variable \( Z = \tilde{P}(S_{lj} ≥ 1) \) and obtain

\[ \tilde{P}(S_{lj} ≥ 1) ≤ \varepsilon + \| P(S_{lj} ≥ 1) > \varepsilon \| ≤ \varepsilon + \| E\mathbb{S}_{lj} > \varepsilon \|. \] (32)

The last inequality follows by Markov’s inequality, \( \tilde{P}(S_{lj} ≥ 1) ≤ E\mathbb{S}_{lj} \). Taking \( E^*\)-expected values in (32) we arrive to (31) since \( P^*(S_{lj} ≥ 1) = E^*(\tilde{P}(S_{lj} ≥ 1)) \).
To show that the latter bound holds true we estimate
\[ \mathbb{E}S_{00} \leq \sum_{2 \leq i \leq m} \lambda_i \lambda_{2i} = Y_1 Y_2 n^{-1} \hat{a}_2 = o_P(1), \]
\[ \mathbb{E}S_{01} \leq \sum_{3 \leq j \leq n} \sum_{2 \leq i \leq m} \lambda_{i1} \lambda_{i2j} \lambda_{i2j} \leq Y_1 Y_2 (\hat{a}_2)^2 \hat{b}_2 n^{-1} = o_P(1), \]
\[ \mathbb{E}S_{k0} \leq \sum_{3 \leq j \leq n} \sum_{2 \leq i \leq m} \lambda_{i1} \lambda_{i1j} \lambda_{i2j} \leq Y_1 X_1 \beta n^{-1/2} \hat{a}_2 \hat{b}_2 n^{-1} = o_P(1), \]
\[ \mathbb{E}S_{k0} \leq \sum_{3 \leq j \leq n} \sum_{2 \leq i \leq m} \lambda_{i1} \lambda_{i2j} \lambda_{i2j} \leq Y_2^2 (\hat{a}_2)^2 \hat{b}_2 n^{-1} = o_P(1). \]
Here we used the fact that \( \mathbb{E}X_1^2 < \infty \) implies \( \hat{a}_2 = O_P(1) \) and \( \mathbb{E}Y_1 < \infty \) implies \( \hat{b}_2 n^{-1} = o_P(1) \).

**Step 3.** Here we prove (17). Denote
\[ \Delta_1 = \mathbb{P}^*(U^*_{r_1, r_2}) - \mathbb{P}^*(U^{**}_{r_1, r_2}), \quad \Delta_2 = \mathbb{P}^*(A_1) - \lambda_{11} \lambda_{12}. \]

From identities
\[ \mathbb{P}(U^*_{r_1, r_2} \cap A_1) = \mathbb{E}(\mathbb{P}^*(U^*_{r_1, r_2}) \mathbb{P}^*(A_1)), \quad \mathbb{P}(U^{**}_{r_1, r_2} \cap A_1) = \mathbb{E}(\mathbb{P}^*(U^{**}_{r_1, r_2}) \mathbb{P}^*(A_1)) \]
we obtain
\[ \mathbb{P}(U^*_{r_1, r_2} \cap A_1) - \lambda_{11} \lambda_{12} \mathbb{P}(U^{**}_{r_1, r_2}) = \mathbb{E}(\Delta_1 \mathbb{P}^*(A_1)) + \mathbb{E}(\Delta_2 \mathbb{P}^*(U^{**}_{r_1, r_2})). \]  \(33\)
We shall show in Lemma 2 below that for any realized values \( Y_1, Y_2, X_1 \) we have \( \Delta_1 = o(1) \). This, together with the inequality \( \mathbb{P}^*(A_1) \leq \lambda_{11} \lambda_{12} \) implies
\[ \mathbb{E}(\Delta_1 \mathbb{P}^*(A_1)) \leq (nm)^{-1} \mathbb{E}(\Delta_1 Y_1 Y_2 X_1^2) = o((nm)^{-1}), \]  \(34\)
by Lebesgue’s dominated convergence theorem. Furthermore, (24) implies
\[ \mathbb{E}(\Delta_2 \mathbb{P}^*(U^{**}_{r_1, r_2})) \leq \mathbb{E} |\Delta_2| \leq (nm)^{-1} \mathbb{E} Y_1 Y_2 X_1^2 (I_{11} + I_{12}) = o((nm)^{-1}). \]  \(35\)
Collecting (34) and (35) in (33) we obtain (17).

**Proof of (ii).** The proof is similar to that of (i). It makes use the observation that the typical adjacency relation is witnessed by a single attribute. One difference is that now the size of the collection of attributes, prescribed to the typical vertex, tends to infinity as \( m/n \to +\infty \) while the number of vertices sharing any given attribute tends to zero. As a consequence we obtain that \( d_{12} = o_P(1) \).

The first several steps of the proof are the same as that of (7). Namely, relations (14), (15), (16) hold true as the argument of their proof remains valid for \( m/n \to +\infty \). Further steps of the proof are a bit different. We show that
\[ \mathbb{P}(U^*_{r,k_1-k_2-r} \cap A_1) \simeq 0, \quad \text{for} \quad r = 1, 2, \ldots, k_1 \wedge k_2, \]  \(36\)
and
\[ \mathbb{P}(U^*_{k_1,k_2} \cap A_1) \simeq \mathbb{P}([L_1 = k_1, L_2 = k_2] \cap A_1). \]  \(37\)
Finally, we show that
\[ \mathbb{P}([L_1 = k_1, L_2 = k_2] \cap A_1) \simeq \mathbb{E}(\lambda_{11} \lambda_{12} \mathbb{P}^*(U^{**}_{k_1,k_2})). \]  \(38\)
Now the simple identity
\[ a_2^{-1}b_1^{-2}nme(\lambda_1\lambda_2)P^*(U_{k_1,k_2}^{*}) = \bar{p}(k_1+1)\bar{p}(k_2+1) \]
completes the proof of (8). It remains to prove (36), (37), (38).

Let us prove (38). The proof is similar to that of (17). We write
\[ P\{L_1 = k_1, L_2 = k_2, \Lambda \} - P\{U_{k_1,k_2}^{*} \cap A_1 \} \leq R', \quad r = 1, 2, \ldots, \]
(39)
and the bound \( R' = o((nm)^{-1}) \). To prove the latter bound we write
\[ R' = E(P^*(L_0 \geq 1)P^*(A_1)) \leq (nm)^{-1}E(Y_1Y_2X_{i}^{*}P^*(L_0 \geq 1)) = o((nm)^{-1}). \]
(41)
Here we used inequalities \( P^*(A_1) \leq \lambda_1\lambda_2 \leq (nm)^{-1}Y_1Y_2X_{i}^{*} \) and the bound \( E(Y_1Y_2X_{i}^{*}P^*(L_0 \geq 1)) = o(1) \), which follows by Lebesgue’s dominated convergence theorem, since
\[ P^*(L_0 \geq 1) \leq E^*L_0 \leq \sum_{3 \leq j \leq n} \lambda_{1j} = b_1X_1((n-2)/n)β_n/n^{1/2} = o(1). \]

Let us prove (38). The proof is similar to that of (17). We write
\[ P\{L_1 = k_1, L_2 = k_2, \Lambda \} - \lambda_1\lambda_2P(U_{k_1,k_2}^{*}) = E(\Delta_3P^*(A_1)) + E(\Delta_2P^*(U_{k_1,k_2}^{*})), \]
(42)
where
\[ \Delta_2 = P^*(A_1) - \lambda_1\lambda_2, \quad \Delta_3 = P^*(L_1 = k_1, L_2 = k_2) - P^*(U_{k_1,k_2}^{*}) \]
and show that both summands in the right of (42) are \( o((nm)^{-1}) \), cf. proof of (17) above. In fact, the only thing that remains to show is that for any realized values \( Y_1, Y_2, X_1 \) we have \( \Delta_3 = o(1) \). This is shown in Lemma 3 below.

\[ \square \]

Proof of Remark 2 Let us prove (i). Lemma 4 implies
\[ P(\tau_1 = r) \approx c_1^*r^{-\gamma}, \quad \text{where} \quad c_1^* = c(b_1β^{-1/2})^{r^{-1}}. \]
(43)
Consequently,
\[ P(\tau_1 = r) \approx ca_1^{-1}(b_1β^{-1/2})^{r^{-2}r^{1-\gamma}}. \]
(44)
From the latter relation we conclude that the sequence of probabilities \( \{P(\tau_1 = r)\}_{r \geq 0} \) is long-tailed and sub-exponential, that is, it satisfies conditions of Theorem 4.30 of [9]. This theorem implies \( P(\sum_{1\leq t \leq \Lambda_1} \tau_1 = r) \approx P(\tau_1 = r)E\Lambda_1 \), thus completing the proof.

Let us prove (ii). We observe that \( \tau_1 \) has Poisson distribution with mean \( \lambda_0 = xb_1β^{-1/2} \). Hence, given \( \Lambda_1 \), the random variable \( d_1 \) has Poisson distribution with mean \( \lambda_0\Lambda_1 \). Now statement (iii) of Lemma 4 implies \( P(\Lambda_1 = r) \approx c_er^{-\gamma}, \) where \( c_e = c(a_1\beta^{1/2})^{-1} \). Next, we apply statement (iii) of Lemma 4 once again and obtain \( P(d_1 = r) \approx c_e\lambda_0^{r-1}r^{-\gamma} \).

\[ \square \]

Proof of Remark 3 The proof is standard. We present it here for reader’s convenience. Let \((v_1', v_2')\) denote an ordered pair of distinct vertices drawn uniformly at random and let \( P' \) denote the conditional probability given all the random variables considered, but \((v_1', v_2')\). We have
\[ P(d(v_1) = r | v_1 \sim v_2) = \frac{P(d(v_1) = r, v_1 \sim v_2)}{P(v_1 \sim v_2)} = \frac{P(d(v'_1) = r, v'_1 \sim v'_2)}{P(v_1 \sim v_2)}. \]
(45)
The denominator is evaluated in \([18]\). We have
\[
P(\{v_1 \sim v_2\}) = n^{-1}a_2b_1^2 + o(n^{-1}) = n^{-1}\mathbf{E}\Lambda_3 + o(n^{-1}).
\]
In the last step we used the simple identities \(\mathbf{E}\Lambda_3 = \mathbf{E}\Lambda_3 = a_2b_1^2\). In order to evaluate the numerator we combine identities
\[
P'(d(v_1') = r, v_1' \sim v_2') = P'(v_1' \sim v_2')P(d(v_1') = r) = \frac{r}{n-1}P'(d(v_1') = r)
\]
\[
P(d(v_1') = r, v_1' \sim v_2') = \mathbf{E}(P'(d(v_1') = r, v_1' \sim v_2')).
\]
We obtain \(P(d(v_1') = r, v_1' \sim v_2') = (r/(n-1))P(d(v_1) = r)\). Hence, by \([43]\),
\[
P(d(v_1) = r|v_1 \sim v_2) = rP(d(v_1) = r)(\mathbf{E}\Lambda_3)^{-1} + o(1).
\]
Now, the statement (ii) of Theorem \([1]\) completes the proof of \([9]\).

**Proof of Theorem** \([3]\). (i) follows from the relation \(P(\Lambda_3 = r) \approx c(a_2b_1)^{r-1}r^{-\kappa}\), see Lemma \([4]\).

In the proof of (ii) and (iii) we assume that \(k_1 \leq k_2\). We recall that \(q_i\) is defined in \([6]\) and introduce the notation
\[
S_A = \sum_{r \in A}(r + 1)(r + 2)P(\Lambda_0 = r + 2)q_{k_1-r}q_{k_2-r}, \quad A \subset [0, k_1].
\]

Let us prove (ii). We observe that \(\mathbf{E}(e^{aY_1}) < \infty\) implies that \(\mathbf{E}Y_1 e^{a'\Lambda_1} < \infty\) for some \(a' > 0\).

Using this observation and the fact that the sequence of probabilities \(\{P(\tau_1 = r)\}_{r \geq 0}\) is long-tailed and sub-exponential (see \([44]\) and \([9]\)) we show that
\[
\mathbf{E}(Y_1P(d(v_1') = r|Y_1)) \approx (\mathbf{E}(Y_1\Lambda_1))P(\tau_1 = r).
\]

The proof of \([46]\) is much the same as that of Theorem 4.30 in \([9]\). Next, we invoke in \([46]\) the identity \(\mathbf{E}(Y_1\Lambda_1) = \mathbf{E}(Y_1\Lambda_1) = a_1b_2\beta^{3/2}\) and \([44]\), and obtain
\[
\mathbf{E}(Y_1P(d(v_1') = r|Y_1)) \approx c_2^r r^{-\kappa}, \quad \text{where} \quad c_2^* = cb_1^{r-2}b_2\beta^{3-\kappa}/2.
\]

Hence we have \(q_r \approx c_2^* r^{-\kappa}\) and \(P(\Lambda_0 = r) \approx c_1^* r^{-\kappa}\), see \([43]\).

Now we are ready to prove \([11]\). Let \(\varepsilon = \ln(k_1 \wedge (k_2 - k_1))\) for \(k_2 - k_1 \rightarrow +\infty\), and \(\varepsilon = \ln k_1\) otherwise. Split \(S_{[0,k_1]} = S_{A_1} + S_{A_2} + S_{A_3}\), where
\[
A_1 = [0, k_1/2], \quad A_2 = (k_1/2, k_1 - \varepsilon], \quad A_3 = (k_1 - \varepsilon, k_1].
\]

In the remaining part of the proof we shall show that \(S_{A_1}, S_{A_2}\) are negligibly small compared to \(S_{A_3}\) and determine the first order asymptotics of \(S_{A_3}\) as \(k_1, k_2 \rightarrow +\infty\). We have for some \(\bar{c} > 0\) (independent of \(k_1, k_2\))
\[
S_{A_1} \leq \bar{c} \sum_{i \in A_1} \frac{1}{(k_1 - i)^{1-\kappa}} \frac{1}{k_1} \frac{1}{(k_2 - i)^{1-\kappa}} = O\left(k_1^{4-2\kappa}k_2^{1-\kappa}(1 + \Delta)\right).
\]

Here \(\Delta = \ln n\) for \(\kappa = 3\) and \(\Delta = 0\) otherwise. Furthermore, for \(k_2 - k_1\) bounded we have
\[
S_{A_2} \leq \frac{\bar{c}}{k_1^{1-\kappa}} \sum_{i \in A_2} \frac{1}{(k_1 - i)^{2-\kappa}} = O\left(\varepsilon^{3-2\kappa}k_1^{2-\kappa}\right) = o(k_1^{2-\kappa}).
\]

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For \( k_2 - k_1 \to +\infty \) we have
\[
S_{A_2} \leq \frac{\bar{c}}{k_1^{x-2}} \sum_{i \in A_2} \left( \frac{1}{(k_1 - i)^{\kappa - 1}} \right) \frac{1}{(k_2 - i)^{\kappa - 1}} \leq \frac{\bar{c}}{k_1^{x-2}(k_2 - k_1)^{\kappa - 1}} \sum_{i \in A_2} \frac{1}{(k_1 - i)^{\kappa - 1}}.
\]
Since \( \sum_{i \in A_2} \frac{1}{(k_1 - i)^{\kappa - 1}} = O(\varepsilon^{2-\kappa}) = o(1) \) we obtain \( S_{A_2} = o(k_1^{2-\kappa}(k_2 - k_1)^{1-\kappa}) \).
Finally, using the approximation \((i + 1)(i + 2)P(A_0 = i + 2) \approx c_1^2 k_1^{2-\kappa}\) uniformly in \( i \in A_3 \) we obtain for \( k_2 - k_1 \to +\infty \)
\[
S_{A_3} = \frac{c_1^4 (1 + o(1))}{k_1^{x-2}} \sum_{i \in A_3} q_{k_1^i} - q_{k_2^i} \approx \frac{c_1^4}{k_1^{x-2}} \sum_{i \in A_3} q_{k_1^i} - q_{k_2^i} \approx \frac{c_1^4 c_3 b_1}{k_1^{x-2}(k_2 - k_1)^{\kappa - 1}}.
\]
Here we used \( \sum_{r \geq 0} q_r = b_1 \). Similarly, in the case where \( k_2 - k_1 = k \) for some fixed \( k \) we have
\[
S_{A_3} = \frac{c_1^4 (1 + o(1))}{k_1^{x-2}} \sum_{i \in A_3} q_{k_1^i} - q_{k_2^i} \approx \frac{c_1^4 c_3 k}{k_1^{x-2}}, \quad \text{where} \quad c_3 k = \sum_{i \geq 0} q_i q_{k+i}.
\]
Let us prove (iii). We observe that \( A_0 \) has the Poisson distribution with (non-random) mean \( \lambda_0 \). Using the identity \((r + 1)(r + 2)P(A_0 = r + 2) = \lambda_0^2 P(A_0 = r) \) we write
\[
S_{[0,k_1]} = \lambda_0^2 \sum_{r=0}^{k_1} P(A_0 = r) q_{k_1^r} q_{k_2^r} = \lambda_0^2 E(q_{k_1 - \lambda_0} q_{k_2 - \lambda_0}) = \lambda_0^2 (J_1 + J_2).
\]
\[
J_1 = E(q_{k_1 - \lambda_0} q_{k_2 - \lambda_0}) I_{\{\lambda_0 < \sqrt{k_1}\}}, \quad J_2 = E(q_{k_1 - \lambda_0} q_{k_2 - \lambda_0}) I_{\{\sqrt{k_1} \leq \lambda_0 \leq k_1\}}.
\]
Next, combining the fast decay of Poisson tail probability \( P(A_0 > t) \) as \( t \to +\infty \) with the relation, which is shown below,
\[
q_r \approx c_0 r^{1-\kappa}, \quad c_0 = c(x^2 b_1)^{1-2}; \quad (47)
\]
we estimate \( J_1 = (1 + o(1))\lambda_0^2 c_0^2 (k_1 k_2)^{1-\kappa} \) and \( J_2 = o((k_1 k_2)^{1-\kappa}) \). We obtain that \( S_{[0,1]} = (1 + o(1))\lambda_0^2 J_1 \). Now the identity \( p_{\beta}(k_1, k_2) = \beta b_1^{-4} x^{-2} S_{[0,1]} \) completes the proof of (12).

Let us prove (47). Since \( \tau_1 \) has Poisson distribution with mean \( \lambda_0 = x b_1 \beta^{-1/2} \), we obtain
\[
P(d_{Y_1}^* = k| Y_1 = y) = E\left(P(d_{Y_1}^* = k| A_1, Y_1 = y)| Y_1 = y\right) = E\left(e^{-\lambda_0 A_1} \frac{(\lambda_0 A_1)^k}{k!} | Y_1 = y\right) = \sum_{i \geq 0} e^{-\lambda_0 b_1} \frac{(\lambda_0 i)^k}{k!} e^{-B y} \frac{(B y)^i}{i!}.
\]
Here we denote \( B = x \beta^{1/2} \). After we write the product \( y^i \) in the form
\[
\sum_{i \geq 0} e^{-\lambda_0 i} \frac{(\lambda_0 i)^k}{k!} e^{-B y} \frac{(B y)^i}{(i + 1)!} \frac{i + 1}{B} = E\left(e^{-\lambda_0(A_1 - 1)} \frac{(\lambda_0 (A_1 - 1))^k}{k!} \frac{A_1}{B} \right)| Y_1 = y \)
\]
we obtain the following expression for the expectation \( q_k = E(Y_1 P(d_{Y_1}^* = k| Y_1)) \)
\[
q_k = \frac{P(A_1 \geq 1)}{B} (I_{1,k} + I_{2,k}), \quad I_{1,k} = E\left(e^{-\lambda_0 Z} \frac{(\lambda_0 Z)^k}{k!}\right), \quad I_{2,k} = E\left(Z e^{-\lambda_0 Z} \frac{(\lambda_0 Z)^k}{k!}\right). \quad (48)
\]
Here $Z$ denotes a random variable with the distribution

$$P(Z = r) = P(A_1 = r + 1)/P(A_1 \geq 1), \quad r = 0, 1, \ldots.$$  

We note that

$$P(Z = r) \approx c'r^{-\kappa}, \quad \text{where} \quad c' = cB^{\kappa-1}/P(A_1 \geq 1). \quad (49)$$

Indeed, (49) follows from the relation $P(A_1 = r) \approx cB^{\kappa-1}r^{-\kappa}$, which is a simply consequence of the property $P_{c,\kappa}$ of the distribution of $Y_1$, see Lemma 3. Next, we show that

$$I_{1,k} \approx c'\lambda_0^{\kappa-1}k^{-\kappa}, \quad I_{2,k} \approx c'\lambda_0^{\kappa-2}k^{1-\kappa}. \quad (50)$$

The first relation follows from (49), by Lemma 4. The second relation follows from the first one via the simple identity $I_{2,k} = (k + 1)\lambda_0^{-1}I_{1,k+1}$. Finally invoking (50) in (48) we obtain (47). \(\square\)

### 3 Appendix A

Here we prove the bounds $\Delta_1 = o(1)$ and $\Delta_3 = o(1)$, see (33) and (42) above. In the proof we apply and further extend the approach of [4]. An important tool used below is the following inequality referred to as Le Cam’s lemma, see e.g., [20].

**Lemma 1.** Let $S = \sum_{i=1}^{n} I_i$ be the sum of independent random indicators with probabilities $P(I_i = 1) = p_i$. Let $\Lambda$ be Poisson random variable with mean $p_1 + \cdots + p_n$. The total variation distance between the distributions $P_S$ of $P_\Lambda$ of $S$ and $\Lambda$

$$\sup_{A \subseteq \{0,1,2,\ldots\}} |P(S \in A) - P(\Lambda \in A)| = \frac{1}{2} \sum_{k \geq 0} |P(S = k) - P(\Lambda = k)| \leq \sum_{i} p_i^2. \quad (51)$$

**Lemma 2.** Assume that conditions of part (i) of Theorem 2 are satisfied. Then for any realized values $Y_1, Y_2, X_1$ and any integers $r, r_1, r_2 \geq 0$ we have

$$P^*(U^{n*}_{r,r_1,r_2}) - P^*(U^{n*}_{r_1,r_2}) = o(1). \quad (52)$$

**Proof of Lemma 2.** Denote

$$L^{(0)} = (L_0^{(0)}, L_1^{(0)}, L_2^{(0)}) := (L_0, L_1, L_2), \quad L^{(4)} = (L_0^{(4)}, L_1^{(4)}, L_2^{(4)}) := (d_{Y_1}^*, d_{Y_2}^*).$$

In the proof we construct random vectors $L^{(h)} = (L_0^{(h)}, L_1^{(h)}, L_2^{(h)}), \quad h = 1, 2, 3$, defined on the same probability space as $Y_1, Y_2, X_1$ such that for any realized values $Y_1, Y_2, X_1$ we have

$$d_{TV}(L^{(h)}, L^{(h+1)}) = o(1), \quad h = 0, 1, \quad (53)$$

$$P^*(L^{(h)}) = (r, r_1, r_2)) - P^*(L^{(h+1)}) = o(1), \quad h = 2, 3, \quad (54)$$

for any integers $r, r_1, r_2 \geq 0$. We note that (53), (54) imply (52). Let us define random vectors $L^{(h)}, \quad h = 1, 2, 3$. Given $X, Y$, let $(\eta_{ki})_{i=2}^m, \quad k = 1, 2$ and $(\xi_{ki})_{i=1}^m, \quad h = 2, 3$, be sequences of Poisson random variables which are (conditionally, given $X, Y$) independent within each sequence and have mean values

$$E\eta_{ki} = \lambda_{ki}, \quad E\xi_{2i} = \sum_{3 \leq j \leq m} \lambda_{ij}, \quad E\xi_{3i} = \frac{b_1}{\sqrt{\beta}} X_i.$$
In addition, we assume that (conditionally, given $X, Y$) the sequences $\{\eta_{ki}\}_{i=2}^{m}, \ k = 1, 2,$ are independent and they are independent of the sequences $\{\xi_{hi}\}_{i=1}^{m}, \ h = 2, 3, 4.$ Denote for $k = 1, 2$ and $h = 2, 3$ 

\[ L^{(1)}_0 = u_1, \quad L^{(h)}_0 = \xi_{h1}, \quad L^{(1)}_k = \sum_{2 \leq i \leq m} \eta_{ki} u_i, \quad L^{(h)}_k = \sum_{2 \leq i \leq m} \eta_{hi} \xi_{hi}. \]

Let us prove (53) for $h = 0.$ In the proof we use the following simple inequalities. Let $\tilde{d}_TV(\zeta, \theta)$ denote the total variation distance between the conditional distributions of random variables/vectors $\zeta$ and $\theta$ given $X, Y, u_1, \ldots, u_m.$ Then we have

\[ \tilde{d}_TV(\zeta, \theta) \leq \mathbf{E} \tilde{d}_TV(\zeta, \theta), \quad \tilde{d}_TV(\zeta, \theta) \leq \mathbf{E} * \tilde{d}_TV(\zeta, \theta) \leq \mathbf{E} * \tilde{d}_TV(\zeta, \theta). \] (55)

Introduce random variables for $k = 1, 2$ and $t = 1, \ldots, m$

\[ L^{[t]}_k = \sum_{2 \leq i \leq t} \mathbb{I}_{ik} u_i + \sum_{t + 1 \leq i \leq m} \eta_{ki} u_i. \]

Note that $L^{[1]}_k = L^{(1)}_k$ and $L^{[m]}_k = L^{(0)}_k.$ We have, by the triangle inequality,

\[ \tilde{d}_TV(L^{[1]}_k, L^{[m]}_k) \leq \sum_{2 \leq t \leq m} \tilde{d}_TV(L^{[t-1]}_k, L^{[t]}_k) \leq \sum_{t=2}^{m} \lambda^2_{tk}. \]

Here we estimated each summand

\[ \tilde{d}_TV(L^{[t-1]}_k, L^{[t]}_k) \leq \tilde{d}_TV(\mathbb{I}_{ik}, \eta_{kt}) \leq \lambda^2_{tk}. \] (56)

In the first inequality of (56) we use the fact that $L^{[t-1]}_k$ and $L^{[t]}_k$ only differ in the $t$-th summand. The second inequality is trivial for $\lambda_{tk} \geq 1$ and it follows by Le Cam’s inequality, see Lemma 1, for $\lambda_{tk} < 1.$

Next, we use the fact that given $X, Y, u_1, \ldots, u_m$ the random vectors $\mathbf{L}^{(0)}$ and $\mathbf{L}^{(1)}$ have conditionally independent marginals. In particular, we can apply the triangle inequality

\[ \tilde{d}_TV(\mathbf{L}^{(0)}, \mathbf{L}^{(1)}) \leq \sum_{0 \leq k \leq 2} \tilde{d}_TV(L^{(0)}_k, L^{(1)}_k) \leq \sum_{2 \leq t \leq m} (\lambda^2_{t1} + \lambda^2_{t2}). \]

In the last step we use $\tilde{d}_TV(L^{(0)}_0, L^{(1)}_0) = 0.$ Finally, (55) implies

\[ \tilde{d}_TV(\mathbf{L}^{(0)}, \mathbf{L}^{(1)}) \leq \mathbf{E} * \sum_{2 \leq t \leq m} (\lambda^2_{t1} + \lambda^2_{t2}) = \frac{2 m - 2}{m} a_2 (Y_1^2 + Y_2^2) = o(1). \]

Let us prove (53) for $h = 1.$ We note that conditionally, given $X, Y,$ the random variable $L^{(1)}_0$ is independent of $(L^{(1)}_1, L^{(1)}_2),$ and $L^{(0)}_0$ is independent of $(L^{(2)}_1, L^{(2)}_2).$ Hence, we have, by triangle inequality,

\[ \tilde{d}_TV(\mathbf{L}^{(1)}, \mathbf{L}^{(2)}) \leq \tilde{d}_TV(L^{(1)}_0, L^{(0)}_0) + \tilde{d}_TV((L^{(1)}_1, L^{(1)}_2), (L^{(2)}_1, L^{(2)}_2)) =: Z_1 + Z_2. \]

We shall show that $\mathbf{E}^* Z_i = o(1), i = 1, 2.$ These bounds combined with the second inequality of (55) imply $\tilde{d}_TV(\mathbf{L}^{(1)}, \mathbf{L}^{(2)}) = o(1).$
We firstly estimate $Z_1$. We have
\[ Z_1 = \tilde{d}_{TV}(u_1, \xi_{21}) \leq \sum_{3 \leq j \leq n} \lambda_{1j}^{-1} X_j \tilde{b}_2 n^{-1}. \] (57)
Indeed, for max, $\lambda_{1j} < 1$ this inequality follows by Lemma 1. Otherwise the inequality is trivial, since the total variation distance is always less than or equal to 1. Next, we use the fact that $EY_1 < \infty$ implies $\tilde{b}_2 n^{-1} = o_P(1)$. We obtain $Z_1 = o_P(1)$. This bound together with the inequality $Z_1 \leq 1$ implies $E^* Z_1 = o(1)$.

We secondly estimate $Z_2$. Introduce random variables for $k = 1, 2$ and $t = 1, \ldots, m$
\[ L_k^{(t)} = \sum_{2 \leq i \leq t} \eta_{ki} u_i + \sum_{t+1 \leq i \leq m} \eta_{ki} \xi_{2i}. \]
Note that $L_k^{(1)} = L_k^{(2)}$ and $L_k^{(m)} = L_k^{(1)}$. We apply triangle inequality
\[ \tilde{d}_{TV}\left((L_1^{(m)}, L_2^{(m)}), (L_1^{(1)}, L_2^{(1)})\right) \leq \sum_{t=2}^{m} \tilde{d}_{TV}\left((L_1^{(t-1)}, L_2^{(t-1)}), (L_1^{(t)}, L_2^{(t)})\right) \leq \tilde{d}_{TV}\left((u_i \eta_{1i}, u_i \eta_{2i}, (\xi_{2i} \eta_{1i}, \xi_{2i} \eta_{2i})\right) \leq (\lambda_{t1} + \lambda_{t2}) \min\{1, \beta_n^{-1} X_t^2 \tilde{b}_2 n^{-1}\}. \] (58)

In (59) we use the fact that $L_k^{(t-1)}$ and $L_k^{(t)}$ only differ in the $t$-th summand. In (60) we first estimate the total variation distance between conditional distributions of $(u_i \eta_{1i}, u_i \eta_{2i})$ and $(\xi_{2i} \eta_{1i}, \xi_{2i} \eta_{2i})$ given $\eta_{1i}, \eta_{2i}, X, Y$ from above by $\tilde{d}_{TV}(u_i, \xi_{2i})$ and then take the expected value $E$. In (61) we estimate $E(\eta_{1i}, \eta_{2i}) \neq (0, 0) \leq \lambda_{t1} + \lambda_{t2}$. This bound is trivial for $\lambda_{t1} + \lambda_{t2} \geq 1$. Otherwise it follows from the inequalities
\[ \tilde{d}_{TV}(u_i, \xi_{2i}) \leq E\left|\eta_{1i} \right| + E\left|\eta_{2i} \right| = e^0 - e^{-\lambda_{t1}} + e^0 - e^{-\lambda_{t2}} \leq \lambda_{t1} + \lambda_{t2}. \]

In (61) we invoke the inequality $\tilde{d}_{TV}(u_i, \xi_{2i}) \leq \min\{1, \beta_n^{-1} X_t^2 \tilde{b}_2 n^{-1}\}$, cf. (57) above.

Next, using the identities $L_k^{(1)} = L_k^{(2)}$ and $L_k^{(m)} = L_k^{(1)}$ we obtain from (58), (61) that
\[ Z_2 \leq \sum_{t=2}^{m} (\lambda_{t1} + \lambda_{t2}) \min\{1, \beta_n^{-1} X_t^2 \tilde{b}_2 n^{-1}\} = (Y_1 + Y_2) \beta_n^{1/2} \sum_{t=2}^{m} X_t \{1, X_t^2 \tilde{b}_2 \beta_n^{-1}\}. \] (62)

Finally, we have,
\[ E^* Z_2 = \beta^{1/2}(Y_1 + Y_2) E^* X_2 \{1, X_2^2 \tilde{b}_2 \beta_n^{-1} n^{-1}\} = \beta^{1/2}(Y_1 + Y_2) E X_2 \{1, X_2^2 \tilde{b}_2 \beta_n^{-1} n^{-1}\} = o(1), \]
since $\tilde{b}_2$ is independent of $X_2$ and $E X_2 < \infty$, and $\tilde{b}_2 n^{-1} = o_P(1)$.

Let us prove (51) for $h = 2$. Given $X, Y$, let $\{\xi_i\}_{i=1}^{m}, \{\Delta_i\}_{i=1}^{m}, \{\Delta_i'\}_{i=2}^{m}$ be independent Poisson random variables, which are independent of $\{\eta_i\}_{i=2}^{m}, \{\eta_3\}_{i=2}^{m}$, and have mean values
\[ \tilde{E}\xi_i = b X_i, \quad \tilde{E}\Delta_i' = \delta' X_i, \quad \tilde{E}\Delta_i'' = \delta'' X_i, \quad 1 \leq i \leq m. \] (63)
Here $\delta' = \tilde{b}_1 \beta_n^{-1/2} - b$ and $\delta'' = b_1 \beta_n^{-1/2} - b$, and $b = \min\{\tilde{b}_1 \beta_n^{-1/2}, b_1 \beta_n^{-1/2}\}$. Define the random vectors $\tilde{L}^{(2)}$ and $\tilde{L}^{(3)}$ in the same way as $L^{(2)}$ and $L^{(3)}$ above, but with $\xi_{2i}$ and $\xi_{3i}$ replaced by...
\[ \tilde{\xi}_2 = \xi_i + \Delta'_i \] and \[ \tilde{\xi}_3 = \xi_i + \Delta''_i \] respectively. We note that given \( Y_1, Y_2, X_1 \) the random vector \( \tilde{L}^{(2)} \) has the same conditional distribution as \( L^{(2)} \), and \( \tilde{L}^{(3)} \) has the same conditional distribution as \( L^{(3)} \). Hence in order to prove (54) for \( h = 2 \) it suffices to show that for any realized values \( Y_1, Y_2, X_1 \) we have
\[
P^*\left( |L_k^{(2)} - \tilde{L}_k^{(3)}| \geq 1 \right) = o(1), \quad k = 0, 1, 2.
\] (64)
Here \( \tilde{L}_k^{(i)} \) denotes the \( k \)-th coordinate of the vector \( \tilde{L}^{(i)} \). We derive (64) from the bounds \( E|L_k^{(2)} - \tilde{L}_k^{(3)}| = o_P(1) \) using (31). To prove these bounds we calculate
\[
E|L_0^{(2)} - \tilde{L}_0^{(3)}| = (\delta' + \delta'')X_1,
E|L_k^{(2)} - \tilde{L}_k^{(3)}| = \sum_{2 \leq i \leq m} \tilde{\nu}_{ki}(\delta' + \delta'')X_i = \beta_n^{1/2}Y_k(\delta' + \delta''), \quad k = 1, 2
\]
and use the fact that \( \delta' + \delta' = |\hat{b}_1\beta_n^{-1/2} - \beta_1^{-1/2}| = o_P(1) \). Indeed, we have \( \beta_n \to \beta \) and, by the law of large numbers, \( \hat{b}_1 - \beta_1 = o_P(1) \).

Let us prove (53) for \( h = 3 \). We note that \( L_0^{(3)} \) has the same distribution as \( \Lambda_0 \). Furthermore, given \( Y_1, Y_2, X_1 \), the random variable \( L_0^{(3)} \) is conditionally independent of \( (L_1^{(4)}, L_2^{(4)}) \). Hence, it suffices to show that \( \mathbf{P}^*((L_1^{(4)}, L_2^{(4)}) = (r_1, r_2)) - \mathbf{P}^*((L_1^{(4)}, L_2^{(4)}) = (r_1, r_2)) = o(1) \) for any integers \( r_1, r_2 \geq 0 \). For this purpose we prove the convergence of the conditional characteristic functions
\[
\mathbf{E}^* e^{iL_1^{(3)} + Is_1^{(3)}} \to \mathbf{E}^* e^{iL_1^{(4)} + Is_1^{(4)}}, \quad \forall s, t \in (-\infty, +\infty).
\] (65)
Here \( i \) denotes the imaginary unit.

In the proof of (65) we exploit the compound Poisson structure of the distributions of \( L_k^{(h)} \), \( h = 2, 3, k = 1, 2 \). Using the fact that, given \( Y_1, Y_2, X_1 \), the marginals of \( (L_1^{(4)}, L_2^{(4)}) \) are conditionally independent and have compound Poisson distributions, we write the characteristics function in the form
\[
\mathbf{E}^* e^{iL_1^{(3)} + Is_1^{(3)}} = (\mathbf{E}^* e^{iL_1^{(4)}}) \cdot (\mathbf{E}^* e^{iL_2^{(4)}}) = e^{(f_r(t) - 1)\lambda_1} \cdot e^{(f_r(s) - 1)\lambda_2} =: f_1(t) \cdot f_2(s).
\] (66)
Here \( f_r(t) = \mathbf{E} e^{it\tau} \) denotes the characteristic function of \( \tau \). Similarly, using the fact that given \( X, Y, \xi_{32}, \ldots, \xi_{3m} \) the marginals of \( (L_1^{(3)}, L_2^{(3)}) \) are conditionally independent and have compound Poisson distributions, we write the conditional characteristics function in the form
\[
\tilde{\mathbf{E}}^* e^{iL_1^{(3)} + Is_1^{(3)}} = (\tilde{\mathbf{E}}^* e^{iL_1^{(3)}}) \cdot (\tilde{\mathbf{E}}^* e^{iL_2^{(3)}}) = e^{(f_r(t) - 1)\tilde{\lambda}_1} \cdot e^{(f_r(s) - 1)\tilde{\lambda}_2} =: g_1(t) \cdot g_2(s).
\] (67)
Here \( \tilde{\mathbf{E}} \) denotes the conditional expectation given \( X, Y, \xi_{32}, \ldots, \xi_{3m} \). Furthermore, we denote \( \tilde{\lambda}_k = \sum_{2 \leq j \leq m} \lambda_{jk} \) and
\[
\tilde{f}_r(t) = \sum_{r \geq 0} e^{itr} \tilde{p}_r, \quad \text{where} \quad \tilde{p}_r = \tilde{\lambda}_1^{-1} \sum_{2 \leq j \leq m} \lambda_{j1} \mathbb{1}_{\{\xi_{3j} = r\}}.
\]
We observe that \( \tilde{\lambda}_k = Y_k \tilde{a}_1 \sqrt{\beta_r} \) and each ratio \( \lambda_{jk}/\tilde{\lambda}_k = X_j/\tilde{a}_1 \) does not depend on \( k = 1, 2 \).

Finally, using (66) (67) we write
\[
\mathbf{E}^* e^{iL_1^{(3)} + Is_1^{(3)}} - \mathbf{E}^* e^{iL_1^{(4)} + Is_1^{(4)}} = \mathbf{E}^* (g_1(t) - f_1(t)) g_2(s) + \mathbf{E}^* (g_2(s) - f_2(s)) f_1(t)
\]
and invoke the bounds \( \mathbf{E}^* (g_1(t) - f_1(t)) g_2(s) = o(1) \) and \( \mathbf{E}^* (g_2(s) - f_2(s)) f_1(t) = o(1) \), which are obtained in the same way as relation (22) in [4]. We note that the proof of (22) in [4] uses the moment conditions \( \mathbf{E} X_1^2 < \infty, \mathbf{E} Y_1 < \infty \) that are assumed to hold in the statement of our Theorem 2.
Lemma 3. Assume that conditions of part (ii) of Theorem 2 are satisfied. Then for any realized values $Y_1, Y_2, X_1$ and any integers $k_1, k_2 \geq 0$ we have

$$P^*(L_1 = k_1, L_2 = k_2) - P^*(U^*_{k_1, k_2}) = o(1).$$  \hspace{5cm} \text{(68)}$$

Proof of Lemma 3. Before the proof we introduce some notation. Let $\varepsilon > 0$. For $2 \leq i \leq m$ denote

$$L_i' = L_i' (\varepsilon) = \mathbb{L}(X_i, b_i - b_1 < \varepsilon), \quad \gamma_i = X_i b_i^{-1/2} b_1 L_i' \quad \text{and} \quad \theta_k = \sum_{2 \leq i \leq m} \lambda_{ik} \gamma_i, \quad k = 1, 2.$$ 

Given $X, Y$, let $\bar{\xi}_2, \ldots, \bar{\xi}_m$ be independent Bernoulli random variables with success probabilities

$$P(\bar{\xi}_i = 1) = 1 - P(\bar{\xi}_i = 0) = \gamma_i,$$

and let $\tilde{\xi}_{hi}, 2 \leq i \leq m, h = 1, 2$, be independent Poisson random variables with mean values

$$\mathbb{E}[\tilde{\xi}_{hi}] = \sum_{3 \leq j \leq n} \lambda_{ij}, \quad \mathbb{E}[\tilde{\xi}_{2i}] = \frac{b_1}{\sqrt{\beta_n}} X_i.$$ 

We assume that given $X, Y$ the sequences $\{\bar{\xi}_i\}_{i=2}^m$, $\{\tilde{\xi}_i\}_{i=2}^m$, and $\{\tilde{\xi}_{hi}\}_{i=2}^m, h = 1, 2$ are independent.

Next, we introduce random vectors $\bar{L}^{(h)} = (\bar{L}_1^{(h)}, \bar{L}_2^{(h)}), 0 \leq h \leq 6$. Denote

$$\bar{L}^{(0)} = (\bar{L}_1^{(0)}, \bar{L}_2^{(0)}) := (L_1, L_2), \quad \bar{L}^{(6)} = (\bar{L}_1^{(6)}, \bar{L}_2^{(6)}) := (\Lambda_3, \Lambda_4).$$

For $k = 1, 2$ and $h = 1, 2$ denote

$$\bar{L}_k^{(h)} = \sum_{2 \leq i \leq m} \mathbb{I}_{ik} \bar{\xi}_{hi}, \quad \bar{L}_k^{(3)} = \sum_{2 \leq i \leq m} \mathbb{I}_{ik} L_i', \quad \bar{L}_k^{(4)} = \sum_{2 \leq i \leq m} L_i' \bar{\xi}_{2i}.$$ 

Furthermore, given $X, Y$, let $\bar{L}_1^{(5)}$ and $\bar{L}_2^{(5)}$ be independent Poisson random variables with mean values $\mathbb{E}[\bar{L}_k^{(5)}] = \theta_k$.

We shall show below that for any integers $r_1, r_2 \geq 0$ we have

$$d_{TV}(L^{(0)}, L^{(1)}) = o(1),$$  \hspace{5cm} \text{(69)}$$

$$P^*(\bar{L}^{(1)} = (r_1, r_2)) - P^*(\bar{L}^{(2)} = (r_1, r_2)) = o(1),$$  \hspace{5cm} \text{(70)}$$

$$E^*[L_k^{(2)} - L_k^{(3)}] = o(1), \quad k = 1, 2,$$  \hspace{5cm} \text{(71)}$$

$$d_{TV}(L^{(3)}, L^{(4)}) \leq \varepsilon (Y_1 + Y_2) a_2 b_1,$$  \hspace{5cm} \text{(72)}$$

$$|P^*(\bar{L}^{(4)} = (r_1, r_2)) - P^*(\bar{L}^{(6)} = (r_1, r_2))| \leq \varepsilon (Y_1 + Y_2) a_2 b_1 + o(1).$$  \hspace{5cm} \text{(73)}$$

Since $\varepsilon > 0$ can be chosen arbitrarily small, relations (69), (70), (71), (72), and (73) imply (68).

Let us prove (69). Introduce random variables

$$\bar{L}_k^{(t)} = \sum_{2 \leq i \leq m} \mathbb{I}_{ik} u_i + \sum_{t+1 \leq i \leq m} \mathbb{I}_{ik} \bar{\xi}_{ii}, \quad t = 1, \ldots, m, \quad k = 1, 2.$$ 

Note that $\bar{L}_k^{(1)} = \bar{L}_k$ and $\bar{L}_k^{(m)} = \bar{L}_k^{(0)}$. We apply triangle inequality

$$d_{TV}((\bar{L}_1^{(m)}, \bar{L}_2^{(m)}), (\bar{L}_1^{(1)}, \bar{L}_2^{(1)})) \leq \sum_{t=2}^{m} d_{TV}((\bar{L}_1^{(t-1)}, \bar{L}_2^{(t-1)}), (\bar{L}_1^{(t)}, \bar{L}_2^{(t)}))$$  \hspace{5cm} \text{(74)}$$
and estimate each summand as follows

\[
\bar{d}_{TV}(L^{(t-1)}_1, L^{(t-1)}_2, (L^{(t)}_1, L^{(t)}_2)) \leq \bar{d}_{TV}((I_{1t}u_t, I_{2t}u_t), (I_{1t}\xi_{1t}, I_{2t}\xi_{1t})) \leq \bar{P}((I_{1t}, I_{2t}) \neq (0, 0)) \bar{d}_{TV}(u_t, \xi_{1t}) \leq (\lambda_1 + \lambda_2)\beta^{-1/2}X_1(\tilde{b}_2n^{-1})^{1/2}. \tag{75}
\]

In (75) we use the fact that \(\tilde{L}^{(t-1)}_k\) and \(\tilde{L}^{(t)}_k\) only differ in the \(t\)-th summand. In (76) we first estimate the total variation distance between conditional distributions of \((I_{1t}u_t, I_{2t}u_t)\) and \((I_{1t}\xi_{1t}, I_{2t}\xi_{1t})\) given \(I_{1t}, I_{2t}, X, Y\) from above by \(\bar{P}((\bar{a}_1, \bar{b}_2) \neq (0, 0)) \bar{d}_{TV}(u_t, \xi_{1t})\) and then take the expected value \(\bar{E}\). In (77) we estimate \(\bar{P}((I_{1t}, I_{2t}) \neq (0, 0)) \leq \lambda_1 + \lambda_2\) and invoke the inequality

\[
\bar{d}_{TV}(u_t, \xi_{1t}) \leq \min\{1, \beta^{-1}X_1^2\tilde{b}_2n^{-1}\} \leq (\beta^{-1}X_1^2\tilde{b}_2n^{-1})^{1/2},
\]

cf. (77) above.

Next, using the identities \(\tilde{L}^{(1)}_k = \tilde{L}^{(0)}_k\) and \(\tilde{L}^{(m)}_k = \tilde{L}^{(0)}_k\) we obtain from (74), (77) that

\[
\bar{d}_{TV}(L^{(0)}, L^{(1)}) \leq \sum_{i=2}^{m} (\lambda_1 + \lambda_2)\beta^{-1/2}X_1(\tilde{b}_2n^{-1})^{1/2} = (Y_1 + Y_2)\bar{a}_2(\tilde{b}_2/n)^{1/2}. \tag{78}
\]

Finally, since \(EX_1^2 < \infty\) and \(EY_1 < \infty\) imply \(\bar{a}_2 = O_P(1)\) and \(\tilde{b}_2/n = o_P(1)\), we conclude that \(\bar{d}_{TV}(L^{(0)}, L^{(1)}) = o_P(1)\). Hence,

\[
\bar{d}_{TV}(L^{(0)}, L^{(1)}) = \bar{E}^{*}d_{TV}(L^{(0)}, L^{(1)}) = \bar{E}^{*}\min\{1, \bar{d}_{TV}(L^{(0)}, L^{(1)})\} = o(1).
\]

Let us prove (70). We proceed as in the proof of (54) for \(h = 2\) above. Denote \(b = \min\{\bar{b}_1, b_1\}\) and \(\delta' = \bar{b}_1 - b, \delta'' = b_1 - b\). Given \(X, Y\), let \(\xi_i, \Delta_i', \Delta_i'', 2 \leq i \leq m\), be independent Poisson random variables, which are independent of \(\bar{b}_2, \ldots, \bar{b}_m\), and having mean values

\[
\bar{E}\xi_i = \beta^{-1/2}X_i\bar{b}, \quad \bar{E}\Delta_i' = \beta^{-1/2}X_i\delta', \quad \bar{E}\Delta_i'' = \beta^{-1/2}X_i\delta''.
\]

Denote

\[
\hat{L}^{(1)}_k = \sum_{2 \leq i \leq m} \bar{I}_{ik}(\xi_i + \Delta_i'), \quad \hat{L}^{(2)}_k = \sum_{2 \leq i \leq m} \bar{I}_{ik}(\xi_i + \Delta_i''), \quad k = 1, 2.
\]

We note that \(\hat{L}^{(h)} := (\hat{L}^{(1)}_1, \hat{L}^{(1)}_2)\) has the same distribution as \(L^{(h)}\), \(h = 1, 2\). Hence it suffices to show (70) for \(L^{(1)}_1, L^{(1)}_2\). Proceeding as in the proof of (54) for \(h = 2\) above, we reduce the problem to showing that

\[
\bar{E}|\hat{L}^{(1)}_k - \hat{L}^{(2)}_k| = \sum_{2 \leq i \leq m} \lambda_i\beta^{-1}(\delta' + \delta'') = Y_k\bar{a}_2|\bar{b}_1 - b_1| = o(1), \quad k = 1, 2.
\]

Let us prove (71). We have, for \(k = 1, 2\),

\[
\bar{E}|L^{(2)}_k - L^{(3)}_k| = \sum_{2 \leq i \leq m} (1 - \bar{I}_i') (\bar{E}\xi_{2i}) (\bar{E}\xi_{2i}) \leq Y_k\bar{b}_1m^{-1} \sum_{2 \leq i \leq m} (1 - \bar{I}_i') X_i^2.
\]

Taking \(\bar{E}^{*}\) expected value we obtain

\[
\bar{E}^{*}|L^{(2)}_k - L^{(3)}_k| = \bar{E}^{*}(\bar{E}|L^{(2)}_k - L^{(3)}_k|) \leq Y_k\bar{b}_1\bar{E}X_2^2\bar{I}_{\{X_2\bar{b}_1^{-1/2}b_1 \geq \varepsilon\}} = o(1).
\]
We note that the expectation on right tends to zero since $\beta_n \to +\infty$.

Let us prove (72). We proceed as in the proof of (69) for $h = 0$ above. We have

$$d_{TV}(L^{(3)}, L^{(4)}) \leq \sum_{2 \leq i \leq m} I_i^r \tilde{P}(I_{i1}, I_{i2}) \neq (0, 0) d_{TV}(\tilde{\xi}_{i1}, \tilde{\xi}_{i2}).$$

Next, we estimate $I_i^r d_{TV}(\tilde{\xi}_{i1}, \tilde{\xi}_{i2}) \leq \gamma_i^2$, by Le Cam’s inequality (51), and invoke the inequality $P(I_{i1}, I_{i2}) \neq (0, 0) \leq \lambda_{i1} + \lambda_{i2}$. We obtain

$$d_{TV}(L^{(3)}, L^{(4)}) \leq \sum_{2 \leq i \leq m} \lambda_i(r_{i1} + \lambda_{i2}) \gamma_i^2 \leq \varepsilon \sum_{2 \leq i \leq m} \lambda_i(r_{i1} + \lambda_{i2}) \gamma_i \leq \varepsilon (Y_1 + Y_2) b_1 a_2.$$

Here we used inequality $\gamma_i^2 \leq \varepsilon \gamma_i$. It follows now that

$$d_{TV}(L^{(3)}, L^{(4)}) \leq E^* \tilde{d}_{TV}(L^{(3)}, L^{(4)}) \leq \varepsilon (Y_1 + Y_2) b_1 a_2.$$

Let us prove (73). For $A \subset W$ denote $\tilde{S}(A) = \sum_{w_i \in A} \bar{I}_i$. For $k = 1, 2$ we write $\bar{L}^{(4)}_k = \tilde{S}(A_k)$, where $A_k$ denote the set attributes from $W \setminus \{w_1\}$ that are prescribed to $v_k$, i.e., $A_k = \{w_i \in W \setminus \{w_1\} : w_i \rightarrow v_k\}$. Given $r_1, r_2 \geq 0$ introduce events $D_k = \{\tilde{S}(A_k) = r_k\}$, $k = 1, 2$, and $D = \{|A_1 \cap A_2| \geq 1\}$.

We first show that

$$P^*(\bar{L}^{(4)}_1 = r_1, \bar{L}^{(4)}_2 = r_2) = E^*(\tilde{P}(\bar{L}^{(4)}_1 = r_1) \bar{P}(\bar{L}^{(4)}_2 = r_2)) + o(1).$$

It is convenient to write (79) in the form

$$P^*(D_1 \cap D_2) = E^*(\tilde{P}(D_1) \bar{P}(D_2)) + o(1).$$

Now, we observe that given $X, Y$, and $A_1, A_2$, satisfying $A_1 \cap A_2 = \emptyset$, the random variables $\tilde{S}(A_1)$ and $\tilde{S}(A_2)$ are independent. Hence for $A_1 \cap A_2 = \emptyset$ we have

$$\tilde{P}(D_1 \cap D_2 | A_1, A_2) = \tilde{P}(D_1 | A_1) \bar{P}(D_2 | A_2).$$

This identity implies

$$\tilde{P}(D_1 \cap D_2 \cap D) = E(\tilde{P}(D_1 \cap D_2 | A_1, A_2) \bar{I}_D) = E(\tilde{P}(D_1 | A_1) \bar{P}(D_2 | A_2) \bar{I}_D).$$

Next, combining inequalities

$$0 \leq \tilde{P}(D_1 \cap D_2) - \tilde{P}(D_1 \cap D_2 \cap D) \leq \tilde{P}(D),$$

$$0 \leq E(\tilde{P}(D_1 | A_1) \bar{P}(D_2 | A_2)) - E(\tilde{P}(D_1 | A_1) \bar{P}(D_2 | A_2) \bar{I}_D) \leq \tilde{P}(D)$$

with (80) and using the identity (which holds, since given $X, Y$, the random sets $A_1$ and $A_2$ are independent)

$$E(\tilde{P}(D_1 | A_1) \bar{P}(D_2 | A_2)) = E(\tilde{P}(D_1 | A_1)) E(\bar{P}(D_2 | A_2)) = \tilde{P}(D_1) \bar{P}(D_2)$$

we obtain that

$$|P^*(D_1 \cap D_2) - E^*(\tilde{P}(D_1) \bar{P}(D_2))| \leq E^*(\tilde{P}(D)).$$

It remains to show that $E^* \tilde{P}(D) = o(1)$. To this aim we apply Markov’s inequality

$$\tilde{P}(D) \leq \sum_{2 \leq i \leq m} E \bar{I}_{i1} I_{i2} \leq \sum_{2 \leq i \leq m} E \lambda_{i1} \lambda_{i2} = n^{-1} Y_1 Y_2 a_2.$$

(81)
and obtain \( \mathbf{E}^*(\tilde{P}(\mathcal{D})) \leq n^{-1}Y_1Y_2a_2 = o(1) \) thus completing the proof of (79).

We secondly show that
\[
\mathbf{E}^* |\tilde{P}(\bar{L}_1^{(4)} = r_1)\tilde{P}(\bar{L}_2^{(4)} = r_2) - \tilde{P}(\bar{L}_1^{(5)} = r_1)\tilde{P}(\bar{L}_2^{(5)} = r_2)| \leq \varepsilon (Y_1 + Y_2)b_1a_2. \tag{82}
\]

Denote, for short, the integrand in (82) by \(|q_1q_2 - h_1h_2|\). Since \(0 \leq q_1, q_2, h_1, h_2 \leq 1\), we have \(|q_1q_2 - h_1h_2| \leq |q_1 - h_1| + |q_2 - h_2|\). We apply Le Cam’s inequality (51) and obtain
\[
|q_k - h_k| \leq \sum_{2 \leq i \leq m} \sum_{2 \leq i \leq m} \|_{\lambda_{ik}\lambda_i}^2 \leq \varepsilon \sum_{2 \leq i \leq m} \|_{\lambda_{ik}\lambda_i} \leq \varepsilon Y_1b_2a_2, \quad k = 1, 2.
\]

Here we estimate \( \gamma_i^2 \leq \varepsilon \gamma_i \). Clearly, inequality \( \mathbf{E}^*|q_k - h_k| \leq \varepsilon Y_1b_1a_2 \) imply (82).

Finally, we show that
\[
\mathbf{E}^*(\tilde{P}(\bar{L}_1^{(5)} = r_1)\tilde{P}(\bar{L}_2^{(5)} = r_2)) \to \mathbf{P}^*(\Lambda = r_1)\mathbf{P}^*(\Lambda = r_2). \tag{83}
\]

Here we will use the fact that the almost sure convergence \( \hat{a}_2 \to a_2 \) implies the convergence in probability
\[
(\theta_1, \theta_2) \xrightarrow{P} (\lambda_3, \lambda_4). \tag{84}
\]

Since for \( s, t \geq 0 \) the function \((s, t) \to e^{-t-s}e^{r_1!r_2!/(r_1!r_2!)}\) is continuous and bounded, (84) implies the convergence of expected values
\[
\mathbf{E}^* e^{-\theta_1 e^{-\theta_2 \theta_1 \theta_2 r_1 r_2}} \to \mathbf{E}^* e^{-\lambda_3 e^{-\lambda_4 \lambda_3 \lambda_4 r_1 r_2}} = \frac{e^{-\lambda_3 e^{-\lambda_4 \lambda_3 \lambda_4 r_1 r_2}}}{r_1!r_2!}. \tag{85}
\]

We observe that the quantities on the left (right) sides of (83) and (85) are the same. Hence (85) implies (83).

It remains to prove (84). Denote \( \hat{\lambda}_{k+2} = Y_kb_1\hat{a}_2, k = 1, 2 \). We have for any \( \delta > 0 \)
\[
\mathbf{P}^*(|\theta_k - \hat{\lambda}_{k+2}| \geq \delta) \leq \delta^{-1} \mathbf{E}^*|\theta_k - \hat{\lambda}_{k+2}| \leq \delta^{-1} Y_kb_1\mathbf{E}X^2_2(1 - \|\|) = o(1). \tag{86}
\]

In the last step we used \( \mathbf{E}X^2_2(1 - \|\|) = o(1) \). Now, (84) follows from (86) and the bounds \( \hat{\lambda}_{k+2} - \lambda_{k+2} = o_P(1) \), \( k = 1, 2 \), which are simple consequences of the fact that \( \mathbf{E}X^2_2 < \infty \) implies \( \hat{a}_2 - a_2 = o_P(1) \).

4 Appendix B

**Lemma 4.** Let \( c, \kappa, h > 0 \). Let \( Z, \Lambda_Z \) be non-negative random variables such that \( \mathbf{P}(\Lambda_Z = r) = \mathbf{E}(e^{-Z}Z^r/r!) \), \( r = 0, 1, \ldots \).

(i) The relation \( \mathbf{P}(Z > t) = (c + o(1))t^{-\kappa} \) as \( t \to +\infty \) implies
\[
\mathbf{P}(\Lambda_Z > t) = (c + o(1))t^{-\kappa} \quad \text{as} \quad t \to +\infty. \tag{87}
\]

(ii) If \( Z \in \mathcal{P}_{c,\kappa} \) then \( \mathbf{P}(\Lambda_Z = r) \approx cr^{-\kappa} \).

(iii) If \( hZ \) is integer valued and satisfies \( \mathbf{P}(hZ = r) \approx c(h/r)^\kappa \) then \( \mathbf{P}(\Lambda_Z = r) \approx chr^{-\kappa} \).

**Proof of Lemma 4.** Let us prove (i). We first collect auxiliary inequalities. For a Poisson random variable \( \Lambda \) with mean \( z > 0 \) and \( 0 < s < z < t \) we have, see [15],
\[
\mathbf{P}(\Lambda \geq t) \leq e^{-z}(ez/t)^t, \quad \mathbf{P}(\Lambda \leq s) \leq e^{-z}(ez/s)^s. \tag{88}
\]
For $0 < x < 1$ and $y > 1$ we have

$$\ln(1-x) \leq -x - 0.5x^2, \quad \ln(1+x) \leq x - 0.25x^2, \quad \ln(1+y) \leq y \ln 2. \quad (89)$$

In order to prove (87) we split the probability $P(A > t) = P_1 + P_2 + P_3$, where

$$P_1 = P(A > t, Z < t_1), \quad P_2 = P(A > t, Z \in [t_1, t_2]), \quad P_3 = P(A > t, Z > t_2),$$

$$t_1 = t(1-\varepsilon), \quad t_2 = t(1+\varepsilon), \quad \varepsilon = t^{-1/3},$$

and show that $P_2 = (c + o(1))t^{-\kappa}$ and $P_k = o(t^{-\kappa})$, $k = 1, 2$.

Next, we estimate $P_1$. Given $z < t_1$ we denote $\bar{z} = 1 - z/t$. Using (88), we obtain

$$P(A > t | Z = z) \leq e^{-z(ez/t)^t} = e^{t^{1-1/3} \ln(1-z)} \leq e^{t-z-t\ln(1-z)} \leq e^{-0.5z^2 \leq e^{-0.5te^2}}.$$

Hence, $P_1 = E[P(A > t | Z) 1_{[Z < t_1]} \leq e^{-0.5z^2} \leq o(t^{-\kappa})$. In order to evaluate $P_3$ we observe that $P(Z > t_2) = (c + o(1))t^{-\kappa}$ and write

$$P_3 = P(Z > t_2) - P(A \leq t, Z > t_2).$$

Then we estimate

$$P(A \leq t, Z > t_2) = E[P(A > t | Z) 1_{[Z > t_1]} = o(t^{-\kappa})$$

using the inequality

$$P(A \leq t | Z = z) \leq \max\{e^{-0.25z^2}, e^{-t\ln(e/2)}\}, \quad \forall z > t_2. \quad (90)$$

It remains to prove (90). We have, see (88),

$$P(A \leq t | Z = z) \leq e^{-z(ez/t)^t} = e^{t^{1-1/3} \ln(1+y)}.$$

Here $y = zt^{-1} - 1$. For $0 < y < 1$ the quantity on the right is less than $e^{-0.25z^2t}$, by the second inequality of (89). For $y \geq 1$, by the third inequality of (89), the quantity on the right is less than $e^{-(z-t)(1-\ln 2)} \leq e^{-t\ln(e/2)}$, since $z - t = ty \geq t$.

Let us prove (ii). We only consider the case of integer valued $Z$. For an absolute continuous $Z$ proof is the same. Denote $\varepsilon_r = r^{-1/2} \ln r$. We split

$$P(A = r) = E(e^{-Z Z^r/r!}) = J_1 + J_2 + J_3, \quad J_k = E(1_{(Z \in Q_k)} e^{-Z Z^r/r!}),$$

$$Q_1 = [0, r(1 - \varepsilon_r)), \quad Q_2 = [r(1 - \varepsilon_r), r(1+ \varepsilon_r)], \quad Q_3 = (r(1+ \varepsilon_r), +\infty),$$

and show that

$$J_1 = o(r^{-\kappa}), \quad J_3 = o(r^{-\kappa}), \quad J_2 = (c + o(1))r^{-\kappa} \quad as \quad r \rightarrow +\infty. \quad (91)$$

In the proof we use the following bounds related to the integral representation of Euler’s Gamma function. Given interval $Q \subset [0, +\infty)$, denote

$$I_Q = \int_Q e^{-x x^r/r!} dx, \quad S_Q = \sum_{j \in Q} e^{-j j^r/r!}.$$
For large $r$ we have
\[ I_{Q_1} \leq r^{-0.3\ln r}, \quad I_{Q_3} \leq r^{-0.3\ln r}, \quad 1 - 2r^{-0.3\ln r} \leq I_{Q_2} \leq 1. \] (92)

Relations (91) follow from (92) and the approximation $S_{Q_k} = (1 + o(1))I_{Q_k}$, $k = 1, 2, 3$,
\[ J_k = \sum_{j \in Q_k} e^{-jr}j^r r! P(Z = r) \leq S_{Q_k} = (1 + o(1))I_{Q_k}, \quad k = 1, 3, \]
\[ J_2 = \sum_{j \in Q_2} e^{-jr}j^r r! P(Z = r) = \frac{c + o(1)}{r^{3\alpha}} S_{Q_2} = \frac{c + o(1)}{r^{3\alpha}} (1 + o(1))I_{Q_2}. \]

For reader’s convenience we provide a proof of (92). We note that the third relation of (92) follows from the first two and the well known fact that $I_{(0, +\infty)} = 1$. In the proof, for sufficiently small $\varepsilon > 0$, we apply the inequalities, which follow by Taylor’s expansion,
\[ e^{\varepsilon}(1 - \varepsilon) \leq 1 - \varepsilon^2/3, \quad e^{-\varepsilon}(1 + \varepsilon) \leq 1 - \varepsilon^2/3. \]

Since the function $x \rightarrow e^{-x}x^r$ increases on $Q_1$, we have $I_{Q_1} \leq ze^{-z}z^r/r!$, where $z = r(1 - \varepsilon_r)$ is the right end-point of $Q_1$. Using Stirling’s formula we obtain for large $r$
\[ ze^{-z}z^r/r! = (1 + o(1)) e^{r\varepsilon_r} (1 - \varepsilon_r)^r + 1 (r/2\pi)^{1/2} \leq r^{1/2}(1 - \varepsilon_r^2/3) \leq r^{-0.3\ln r}. \]

In the last step we used $1 - \varepsilon \leq e^{-\varepsilon}$. Furthermore, for $y = r(1 + \varepsilon_r)$ we have decreases on $Q_3$ we have
\[ I_{Q_3} \leq \frac{e^{-y}y^r}{r!} \int_y^{+\infty} e^{-(x-y)} \left(1 + \frac{x-y}{y}\right)^r dx. \] (93)

Using Stirling’s formula we obtain
\[ e^{-y}y^r/r! = (1 + o(1)) (e^{-x_r} (1 + \varepsilon_r))^r (2\pi r)^{-1/2} \leq r^{-1/2}(1 - \varepsilon_r^2/3)^r \leq r^{-0.5 - 0.3\ln r}. \]

Combining this inequality with the following upper bound on the integral of (93)
\[ \int_0^{+\infty} e^{-t}(1 + t/y)^r dt \leq \int_0^{+\infty} e^{-t} e^{(rt/y)} dt = 1 + \varepsilon_r^{-1} \leq r^{1/2} \]
we obtain (92). Here we used the inequality $(1 + t/y)^r \leq e^{(rt/y)}$.

We omit the proof of (iii) since it is similar to that of (ii).

\[ \square \]

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