\[ N = 4 \] Instanton Calculus in \( \Omega \) and R-R Backgrounds

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Abstract

We study the instanton calculus for \( \mathcal{N} = 4 \) super Yang-Mills theory in ten-dimensional \( \Omega \)-background with the R-symmetry Wilson line gauge field. From the ADHM construction of instantons in the background, we obtain the deformed instanton effective action. For a certain case we get the effective action of \( \mathcal{N} = 2^* \) theory in the \( \Omega \)-background. We also study the low-energy effective D\((-1)\)-brane action for the D3/D\((-1)\)-brane system in the R-R 3-form field strength backgrounds and find that the action agrees with the instanton effective action in the \( \Omega \)-background.
1 Introduction

The Ω-background deformation [1] is used to perform the integrals over the moduli spaces in various supersymmetric gauge theories via the localization technique. In particular, Nekrasov computed the instanton partition function for \( \mathcal{N} = 2 \) supersymmetric Yang-Mills theory from the Ω-deformation [2], which is defined by the dimensional reduction from the six-dimensional Ω-background with the R-symmetry Wilson line gauge field [3, 4].

It has been known that the Ω-background deformation can be interpreted as a certain \( \mathcal{N} = 2 \) supergravity background. For the (anti-)self-dual Ω-background, the instanton partition function corresponds to the partition function of topological string theory extracted from the scattering amplitudes including the self-dual graviphoton vertex operators in type II superstring theory [5, 6]. The topological partition functions has been studied in the non-(anti-)self-dual case [7, 8, 9]. Recently, it has been pointed out in [10, 11] that the partition functions for the general (non-(anti-)self-dual) Ω-background correspond to the scattering amplitudes including the anti-self-dual graviphoton and the self-dual gauge field associated with the matter vector multiplets.
Such closed string backgrounds also change the microscopic description of instantons realized by D-branes. It has been shown in [12] that the low-energy effective action of the D(-1)-branes for the D3/D(-1)-brane system at the fixed point of $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ in the self-dual R-R 3-form field strength background coincides with the $\mathcal{N} = 2$ instanton effective action in the self-dual $\Omega$-background. In [13], we have extended this result to the general $\Omega$-background, where the corresponding closed string backgrounds are the $(S,A)$- and the $(A,S)$-types of the R-R 3-form field strengths. The $(S,A)$-type R-R 3-form field strengths have the same tensor structures as the $\Omega$-backgrounds, which give the mass for the bosonic and the fermionic instanton moduli. On the other hand, the $(A,S)$-type field strengths correspond to the Wilson line gauge fields, which give the masses for the fermionic moduli. This correspondence is similar to the relation between the system in the uniform magnetic fields and the one in the rotating frame.

In this work we will investigate further generalization of the $\Omega$-background and its string theory description in order to understand this correspondence in more general setup. This is also useful for studying non-perturbative and stringy aspects of the supersymmetric gauge theories constructed by various D-branes. The $\Omega$-background deformation of D-brane systems in various dimensions has been studied in [14, 15, 16, 17, 18, 19]. In particular, we have proposed the ten-dimensional $\Omega$-background for the self-dual case and studied the deformed $\mathcal{N} = 4$ super Yang-Mills theories [20]. We have studied the ADHM construction of instantons in the background and showed that the deformed instanton effective action agrees with the D(-1)-brane effective action for the D3/D(-1) system in the self-dual R-R 3-form background of the $(S,A)$-type. We also found that the $(A,S)$-type R-R 3-form background gives the holomorphic mass deformation to the fermionic moduli in the instanton effective action.

In this paper we study the $\mathcal{N} = 4$ action in the general non-(anti-)self-dual $\Omega$-background with the R-symmetry Wilson line gauge field and the deformed instanton effective action. We also investigate the correspondence between the ten-dimensional $\Omega$-background and the R-R 3-form backgrounds. We will study the ADHM construction of instantons in the $\Omega$-deformed $\mathcal{N} = 4$ super Yang-Mills theory. We find that this $\Omega$-deformation includes the mass deformation ($\mathcal{N} = 2^*$ deformation) as an example, which is useful for the calculation of the $\mathcal{N} = 4$ instanton partition function [21, 22]. We will
calculate the D(−1)-brane effective action for the D3/D(−1) system in the R-R 3-form backgrounds and find that the deformed action agrees with the instanton effective action in the ten-dimensional Ω-background.

This paper is organized as follows: in Section 2, we introduce the ten-dimensional Ω-background and the R-symmetry Wilson line gauge field. We will discuss the ADHM construction of instantons in the deformed $\mathcal{N}=4$ theory and calculate the instanton effective action. In Section 3 we will study the D(−1)-brane effective action for the D3/D(−1) system in the R-R 3-form backgrounds. Section 4 devotes the conclusions and discussion. In Appendix A, we summarize our notations of the sigma matrices in four and six dimensions. In Appendix B, we describe detailed calculations of the disk amplitudes in the R-R 3-form backgrounds.

2 $\mathcal{N}=4$ instanton effective action in Ω-background

2.1 $\mathcal{N}=4$ super Yang-Mills theory in Ω-background

The $\mathcal{N}=4$ super Yang-Mills theory with the gauge group $U(N)$ contains a gauge field $A_m$ ($m = 1, 2, 3, 4$), Weyl fermions $\Lambda^\alpha_A$, $\bar{\Lambda}_{\dot{\alpha}}^A$ ($\alpha, \dot{\alpha} = 1, 2$, $A = 1, 2, 3, 4$) and six real scalar fields $\varphi_a$ ($a = 1, \cdots, 6$). The fields belong to the adjoint representation of the gauge group. Since we are interested in the instanton calculus, we study the theory defined in the Euclidean spacetime. The left and the right spinor indices of the Lorentz group $SO(4) \cong SU(2)_L \times SU(2)_R$ are denoted by $\alpha$ and $\dot{\alpha}$ respectively. They are raised and lowered by the anti-symmetric $\epsilon$-symbols $\epsilon_{\alpha\beta}$ and $\epsilon_{\dot{\alpha}\dot{\beta}}$ with $\epsilon^{12} = -\epsilon_{12} = 1$. The index $A$ labels the (anti-)fundamental representation of the R-symmetry group $SU(4)_I$. The Lagrangian is

$$\mathcal{L}_0 = \frac{1}{\kappa} \text{Tr} \left[ \frac{1}{4} F^{mn} F_{mn} + \frac{i\theta g^2}{32\pi^2} F^{mn} \tilde{F}_{mn} + \Lambda^A \sigma^m D_m \bar{\Lambda}_A + \frac{1}{2} (D_m \varphi_a)^2 \right. \right.$$

$$\left. \left. \quad - \frac{g}{2} (\Sigma_a)^{AB} \bar{\Lambda}_A [\varphi_a, \bar{\Lambda}_B] - \frac{g}{2} (\Sigma_a)^{AB} \Lambda^A [\varphi_a, \Lambda^B] - \frac{g^2}{4} [\varphi_a, \varphi_b]^2 \right] \right),$$

(2.1)

where $g$ is the gauge coupling constant and $\theta$ is the theta angle parameter. $D_m \ast = \partial_m \ast + ig [A_m, \ast]$ is the gauge covariant derivative, $F_{mn} = \partial_m A_n - \partial_n A_m + ig [A_m, A_n]$ is the gauge field strength and $\tilde{F}_{mn} = \frac{1}{2} \epsilon_{mnpq} F^{pq}$ is its dual. We normalize the generators
of the gauge group $U(N)$ as $\text{Tr}(T^u T^v) = \kappa \delta^{uv}$. $\sigma^m$ and $\bar{\sigma}^m$ are the sigma matrices in four dimensions, while $(\Sigma_A)^{AB}$ and $(\bar{\Sigma}_A)^{AB}$ are the sigma matrices in six dimensions. We summarize their properties in Appendix A.

The four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory is obtained by the dimensional reduction of the $\mathcal{N} = 1$ super Yang-Mills theory in ten dimensions [23]. In this work, we consider the $\mathcal{N} = 1$ super Yang-Mills theory in the ten-dimensional $\Omega$-background and its reduction. We introduce the ten-dimensional coordinates $x^M = (x^m, x^{a+4}) (M = 1, \cdots, 10)$ and the metric $g_{MN}$ defined by

$$ds^2 = g_{MN} dx^M dx^N = \left( dx^{a+4} \right)^2 + \left( dx^m + \Omega^m_a dx^{a+4} \right)^2,$$

(2.2)

with $\Omega^m_a \equiv \Omega^{mn}_a x_n$ where $\Omega_{mna}$ are six constant anti-symmetric matrices $\Omega^m_a = -\Omega^m_a$. The Lagrangian of the $\mathcal{N} = 1$ super Yang-Mills theory in this background is

$$\mathcal{L}_{10D} = \frac{1}{\kappa} \sqrt{- \det g_{MN}} \text{Tr} \left[ - \frac{1}{4} g^M_P g^N_Q F_{MPN} F_{PQ} - \frac{i}{2} \bar{\Psi} e^M_M \Gamma^M D_M \Psi \right],$$

(2.3)

where $F_{MN} = \partial_M A_N - \partial_N A_M + ig [A_M, A_N]$ is the field strength of the gauge field $A_M = (A_m, \varphi_a)$ and $\Psi$ is the ten-dimensional Majorana-Weyl spinor. $\Gamma^M$ is the gamma matrices in ten dimensions and $e^M_M$ is the vielbein. The vector indices in the local Lorentz frame is denoted by $M, N, \cdots$. The covariant derivative in the curved spacetime for spinors is given by

$$D_M = D_M - \frac{1}{2} \omega_{M,MN} \Gamma^{MN},$$

(2.4)

where $D_M \equiv \partial_M * + ig [A_M, *]$ is the gauge covariant derivative, $\Gamma^{MN} = \frac{1}{4} [\Gamma^M, \Gamma^N]$ is the Lorentz generator in ten dimensions and $\omega_{M,MN}$ is the spin connection. The non-zero components of the spin connection are

$$\omega_{a,mn} = -\Omega_{mna}, \quad \omega_{m,ab} = \frac{1}{2} x^n O_{nmab}, \quad \omega_{a,mb} = \frac{1}{2} x^n O_{mnab}, \quad \omega_{a,bc} = -\frac{1}{2} x^m O_{mnbc} \Omega^m_a,$$

(2.5)

where $O_{mnab} \equiv \Omega_m^p \Omega_pnb - \Omega_m^p \Omega_pna$ is the commutator of the $\Omega$-matrices. The condition $O_{mnab} = 0$ implies that the associated $U(1)^6$ vector fields $\Omega^m_a \partial_m$ commute with each other.
After the dimensional reduction of the Lagrangian \([2.3]\) to four dimensions, the Wick rotation and adding the theta term, we obtain the following deformed Lagrangian:

\[
\mathcal{L}_\Omega = \frac{1}{\kappa} \text{Tr} \left[ \frac{1}{4} F^{mn} F_{mn} + \frac{ig}{32\pi^2} F^{mn} \tilde{F}_{mn} + \Lambda^A \sigma^m D_m \tilde{\Lambda}_A + \frac{1}{2} (D_m \varphi_a - g F_{mn} \Omega^n_a)^2 
- \frac{g^2}{2} \langle \Omega_a \rangle^{AB} \tilde{\Lambda}_A \varphi_a - \frac{g^2}{2} \langle \Omega_a \rangle^{AB} \Lambda^A [\varphi_a, \Lambda^B] 
- \frac{g^2}{4} (\varphi_a, \varphi_b) + i \Omega^a \Omega^b D_m \varphi_b - i \Omega^a \Omega^b D_m \varphi_a - ig F_{mn} \Omega_a \Omega^n_b \right]^2 
- \frac{ig}{2} \Omega^a \langle \Omega_a \rangle^{AB} \Lambda^A D_m \tilde{\Lambda}_B + (\sum_a) \Lambda^A \Lambda^A \Omega^B 
+ \frac{ig}{4} \Omega_{mn} (\langle \Omega_a \rangle^{AB} \tilde{\Lambda}_A \sigma^m \Lambda_B + (\sum_a) \Lambda^A \sigma^m \Lambda^A \Omega^B \right] + \mathcal{L}_O, \tag{2.6}
\]

where

\[
\mathcal{L}_O = \frac{1}{\kappa} \text{Tr} \left[ - \frac{ig^3}{8} \left( \langle \Omega_a \rangle^{bc} \right)^A \Lambda^A \Lambda^B + (\sum_a) \Lambda^A \Omega^B \right] x^p \Omega^a_m O_{mnbc} 
+ \frac{ig^3}{16} \left( \langle \Omega_a \rangle^{AB} (\sigma^m \sigma^n)_{\alpha}^B \Lambda^A \Lambda^B + (\sum_a) \Lambda^A (\sigma^m \sigma^n)_{\alpha}^A \tilde{\Lambda}_B \right] x^p \Omega^b_m O_{mnab} 
- \frac{ig^4}{8} \left( \langle \Omega_a \rangle^{AB} \sigma^m \tilde{\Lambda}_A \Lambda_B + (\sum_a) \Lambda^A \sigma^m \tilde{\Lambda}_B \right] x^p \Omega^c_m \Omega^n_c O_{mnab} \right]. \tag{2.7}
\]

The terms in \(\mathcal{L}_O\) arise from the part of the spin connection \([2.5]\) and are proportional to the commutator \(O_{mnab}\).

The deformed theory has no supersymmetry in general since the \(\Omega\)-background breaks the Poincaré symmetry. However, a part of the supersymmetry can be recovered by choosing the parameters of the background. For example, when the \(\Omega\)-matrices are self-dual and satisfy the commuting conditions \(O_{mnab} = 0\), the deformed theory reduces to the one obtained in \([20]\). We can show that it preserves a half of \(\mathcal{N} = 4\) supersymmetry, which is given by

\[
\delta A_m = - \xi_A \sigma_m \Lambda^A, \\
\delta \Lambda^A = i (\sum_a)^{AB} \sigma^m \xi_B (D_m \varphi_a - g F_{mn} \Omega^n_a), \\
\delta \tilde{\Lambda}_A = \tilde{\sigma}^{mn} \xi_A F_{mn} + ig (\tilde{\sigma}_{ab})^A B \xi_B (\bar{\varphi}_a, \varphi_b) + i \Omega^a_m D_m \varphi_b - i \Omega^b_m D_m \varphi_a - ig \Omega_m^a \Omega^n_b F_{mn}, \\
\delta \varphi_a = - i \xi_A (\sum_a)^{AB} \tilde{\Lambda}_B - g \xi_A \sigma_m \Lambda^A \Omega_m^a. \tag{2.8}
\]

In the case of the non-(anti-)self-dual \(\Omega\)-matrices, the theory has supersymmetry by introducing the R-symmetry Wilson line gauge field and by choosing the deformation parameters, as in the case of the \(\mathcal{N} = 2\) theory \([4, 13]\). The R-symmetry Wilson
line is introduced by gauging the subgroup $SO(6)$ of the ten-dimensional Lorentz group with a constant gauge field $(A_a)^A{}_B$, which takes values in the adjoint representation of $SU(4)_I \sim SO(6)$. In the ten-dimensional theory, the Wilson line modifies the gauge covariant derivative along the $x^{a+4}$ directions. For the spinor fields the gauge covariant derivative is changed as

$$D_{a+4} \to D_{a+4} + iA_a.$$ (2.9)

The gauge covariant derivative for the gauge field is also modified as

$$D_{a+4} \to D_{a+4} + iA^\text{vec}_a,$$ (2.10)

where $A^\text{vec}_a$ is the Wilson line gauge field in the vector representation of the $SO(6)$, which is equivalent to the anti-symmetric representation of $SU(4)_I$. In the viewpoint of the four-dimensional theory, the Wilson line shifts the commutator containing the scalar fields $\varphi_a$.

For the spinor fields, it changes the following terms in the Lagrangian (2.6) as

$$[\varphi_a, \Lambda^A] \to [\varphi_a, \Lambda^A] + (A_a)^A{}_B \Lambda^B,$$

$$[\varphi_a, \bar{\Lambda}_A] \to [\varphi_a, \bar{\Lambda}_A] - \bar{\Lambda}_B (A_a)^B{}_A.$$ (2.11)

In contrast to the $\mathcal{N} = 2$ case, the commutator of the scalar fields $[\varphi_a, \varphi_b]$ is also changed due to (2.10). Its shift is given by

$$[\varphi_a, \varphi_b] \to [\varphi_a, \varphi_b] - \frac{1}{2} \left( (\Sigma_a \bar{\Sigma}_c)^A{}_B \varphi_c(A_a)^B{}_A - (\Sigma_a \bar{\Sigma}_c)^A{}_B \varphi_c(A_b)^B{}_A \right).$$ (2.12)

Finally, we obtain the Lagrangian in the $\Omega$-background with the R-symmetry Wilson line as

$$\mathcal{L}_{(\Omega,A)} = \frac{1}{\kappa} \text{Tr} \left[ \frac{1}{4} F^{mn}F_{mn} + \frac{i\theta g^2}{32\pi^2} F^{mn}F^m_n + \Lambda^A \Lambda^m D_m \Lambda_A \right.$$

$$- \frac{g}{2} (\Sigma_a)^A{}_B \Lambda_A [\varphi_a, \bar{\Lambda}_B] - \frac{g}{2} (\Sigma_a)^A{}_B \Lambda^A [\varphi_a, \Lambda^B]$$

$$- \frac{g^2}{4} \left( [\varphi_a, \varphi_b] + i\Omega_a^m D_m \varphi_b - i\Omega_b^m D_m \varphi_a - i\Omega_{mn} F_{mn} \Omega^m_a \Omega^n_b \right.$$

$$- \frac{1}{2} \left( (\Sigma_b \bar{\Sigma}_c)^A{}_B \varphi_c(A_a)^B{}_A - (\Sigma_a \bar{\Sigma}_c)^A{}_B \varphi_c(A_b)^B{}_A \right) \left. \right)^2$$

$$- \frac{i}{2} \Omega_a^m (\Sigma_a)^A{}_B \Lambda_A D_m \bar{\Lambda}_B + (\Sigma_a)^A{}_B \Lambda^A D_m \Lambda^B)$$
\[ + \frac{ig}{4} \Omega_{mna} \left( (\Sigma_a)^{AB} \bar{\Lambda}_A \bar{\sigma}^{mn} \bar{\Lambda}_B + (\bar{\Sigma}_a)^{AB} \Lambda^A \sigma^{mn} \Lambda^B \right) \]
\[ + \frac{g}{2} (\Sigma_a)^{AB} \bar{\Lambda}_A \bar{\Lambda}_D (A_a)^D_B - \frac{g}{2} (\bar{\Sigma}_a)^{AB} \Lambda^A (A_a)_B^D \Lambda^D \right] + \mathcal{L}_O. \] (2.13)

Note that one can recover the \( \Omega \)-deformed \( \mathcal{N} = 2 \) super Yang-Mills theory \[4, 13\] from the Lagrangian (2.13) by the \( \mathbb{Z}_2 \) orbifold projection \[12, 13\]. To see this, we decompose the \( \mathcal{N} = 4 \) vector multiplet \((A_m, \Lambda^A, \bar{\Lambda}_A, \varphi_a)\) into the \( \mathcal{N} = 2 \) vector multiplet and the \( \mathcal{N} = 2 \) adjoint hypermultiplet. We consider the subgroup \( SU(2)_I \times SU(2)_{I'} \) of \( SU(4)_I \) such that the \( SU(4)_I \) index \( A = 1, 2 \) corresponds to that of \( SU(2)_I \) and \( A = 3, 4 \) to that of \( SU(2)_{I'} \). We label the indices of the fundamental representations of \( SU(2)_I \) and \( SU(2)_{I'} \) as \( A' \) and \( \hat{A} \) respectively. We define \( \varphi^{AB}(= -\varphi^{BA}) \) and \( \bar{\varphi}_{AB}(= -\bar{\varphi}_{BA}) \) by
\[ \varphi^{AB} = \frac{i}{\sqrt{2}} (\Sigma_a)^{AB} \varphi_a, \]
\[ \bar{\varphi}_{AB} = -\frac{i}{\sqrt{2}} (\bar{\Sigma}_a)^{AB} \varphi_a, \] (2.14)
and decompose them as
\[ \varphi^{AB} = \begin{pmatrix} \varphi^{A'B'} \\ \varphi^{\hat{A}'B'} \end{pmatrix}, \quad \bar{\varphi}_{AB} = \begin{pmatrix} \bar{\varphi}_{A'B'} & \varphi^{A'B} \\ \bar{\varphi}_{\hat{A}'B'} & -\varphi^{\hat{A}'B} \end{pmatrix}. \] (2.15)

We note that \( \varphi^{A'B} \) and \( \bar{\varphi}_{AB'} \) are related by
\[ \varphi^{A'B} = \epsilon^{A'C'} \epsilon^{B'D} \bar{\varphi}_{D'C'}, \] (2.16)
which is shown by using (A.5). Under this decomposition, the \( \mathcal{N} = 4 \) vector multiplet is divided into the \( \mathcal{N} = 2 \) vector multiplet \((A_m, \Lambda^{A'}, \bar{\Lambda}_{A'}, \varphi, \bar{\varphi})\) and the \( \mathcal{N} = 2 \) adjoint hypermultiplet \((\Lambda^{\hat{A}}, \bar{\Lambda}_{\hat{A}}, \varphi^{\hat{A}'B'})\). We also define \( \Omega_{mn}, \Omega_{mn}, \Omega^{AB}_{mn} \) and \((A)^{C'D}, (A)^{C'D}_D, (A^{A'B})^{C'D}_D\) from \( \Omega_{mna} \) and \((A_a)^{C'D}_D\) respectively. Now we consider the \( \mathbb{Z}_2 \) subgroup of \( SU(2)_{I'} \), which changes the signs of the fields and parameters with odd \( \mathbb{Z}_2 \) charges. Under the \( \mathbb{Z}_2 \) projection, the hypermultiplet, \( \Omega^{AB}_{mn} \), the off-diagonal blocks \((A)^{A'B}_B, (A)^{\hat{A}'B}_B, (\bar{A})^{A'B}, (\bar{A})^{\hat{A}'B}\) of \( A, \bar{A} \) and the diagonal blocks \((A^{A'B})^{C'D}_D, (A^{A'B})^{C'D}_D\) of \( A^{A'B} \) are projected out. After the projection and imposing the commuting condition for the \( \Omega \)-matrices as
\[ \Omega_{mn} \bar{\Omega}_{np} - \bar{\Omega}_{mn} \Omega_{np} = 0, \] (2.17)
we get the Lagrangian of the \( \Omega \)-deformed \( \mathcal{N} = 2 \) super Yang-Mills theory obtained in \[4\].

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The $\Omega$-matrices satisfying (2.17) are expressed by the $\epsilon$-parameters $\epsilon_1, \epsilon_2$ as

$$
\Omega_{mn} = \frac{1}{2\sqrt{2}} \begin{pmatrix}
0 & i\epsilon_1 & 0 & 0 \\
-i\epsilon_1 & 0 & 0 & 0 \\
0 & 0 & 0 & -i\epsilon_2 \\
0 & 0 & i\epsilon_2 & 0
\end{pmatrix}, \quad \bar{\Omega}_{mn} = \frac{1}{2\sqrt{2}} \begin{pmatrix}
0 & -i\bar{\epsilon}_1 & 0 & 0 \\
\bar{\epsilon}_1 & 0 & 0 & 0 \\
0 & 0 & 0 & i\bar{\epsilon}_2 \\
0 & 0 & i\bar{\epsilon}_2 & 0
\end{pmatrix}.
(2.18)
$$

The $\Omega$-deformed $\mathcal{N} = 2$ super Yang-Mills theory has one fermionic charge when we choose the R-symmetry Wilson line gauge field as $[3, 4, 13]$

$$
A_{A' B'} = -\frac{1}{2} \Omega_{mn} (\bar{\sigma}^{mn})^{A' B'}, \quad \bar{A}_{A' B'} = -\frac{1}{2} \bar{\Omega}_{mn} (\bar{\sigma}^{mn})^{A' B'},
(2.19)
$$

and $(A^{A'B'})_{C D'} = (A^{A'B'})_{C D'} = 0$. We note that the other components of the Wilson line gauge field do not appear in the Lagrangian because they couple only with the hypermultiplet.

One can also obtain the mass deformed $\mathcal{N} = 4$ theory or the $\mathcal{N} = 2^*$ theory in the $\Omega$-background when the $\mathbb{Z}_2$-projected $\Omega$-background satisfies (2.17) and the R-symmetry Wilson line gauge fields take the following form:

$$
A^{A'B'} = \begin{pmatrix}
A_{A' B'} & 0 \\
0 & M^{\tilde{A} B}
\end{pmatrix}, \quad \bar{A}^{A'B'} = \begin{pmatrix}
\bar{A}_{A' B'} & 0 \\
0 & \bar{M}^{\tilde{A} B}
\end{pmatrix}, \quad (A^{C'D'})_{A B} = 0.
(2.20)
$$

Here $A^{A'B'}$ and $\bar{A}^{A'B'}$ are the $\mathcal{N} = 2$ R-symmetry Wilson lines (2.19). The mass deformation parameters of the $\mathcal{N} = 2^*$ theory are given by $M^{\tilde{A} B}$ and $\bar{M}^{\tilde{A} B}$. Taking them to be proportional to $\tau^3$ by the $SU(2)_{I'}$ transformation as

$$
M^{\tilde{A} B} = \begin{pmatrix}
m & 0 \\
0 & -m
\end{pmatrix}, \quad \bar{M}^{\tilde{A} B} = \begin{pmatrix}
\bar{m} & 0 \\
0 & -\bar{m}
\end{pmatrix},
(2.21)
$$

the mass of the hypermultiplet is $\sqrt{m\bar{m}}$. The action of the $\Omega$-deformed $\mathcal{N} = 2^*$ theory is invariant under the following supersymmetry transformations:

$$
\delta A_m = \xi \sigma_m A B \Lambda^{aA'}, \\
\delta \Lambda_{A} = -\sqrt{2} \xi A' B' \sigma_{AB}^m (D_m \phi - g F_{mn} \Omega^n), \\
\delta \Lambda_{\hat{A}} = -\sqrt{2} \xi \sigma_{AB}^m D_m \phi^{AB'},
$$

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\[ \delta \tilde{\Lambda}^\hat{A} = -\tilde{\xi}(\bar{\sigma}^{\mu n})^\hat{A}F_{\mu n} + ig\bar{\xi}\delta^\hat{A}([\varphi, \bar{\varphi}] + i\Omega^m D_m \bar{\varphi} - i\bar{\Omega}^m D_m \varphi - igF_{\mu n}\Omega^n) \\
+ ig\bar{\xi}\delta^{\hat{B}}_B[\varphi^{\hat{B}'\hat{C}}, \bar{\varphi}^{\hat{C}A}], \]
\[ \delta \tilde{\Lambda}^{\hat{A}} = -2ig\bar{\xi}\tilde{\epsilon}^{\hat{A}\hat{B}}([\varphi, \bar{\varphi}]_{\hat{A}\hat{B}} + i\Omega^m D_m \bar{\varphi}_{\hat{A}\hat{B}} - A^{\hat{A}'\hat{B}'}\bar{\varphi}_{\hat{A}\hat{B}} - M^{\hat{A}'}A^{\hat{B}'}\bar{\varphi}_{\hat{B}'\hat{B}}), \]
\[ \delta \varphi = g\bar{\xi}\sigma^{\hat{m}A'}\Lambda^{\hat{m}A'}\Omega_m, \]
\[ \delta \bar{\varphi} = -\sqrt{2}\bar{\xi}\tilde{\epsilon}^{\hat{A}\hat{B}}\tilde{\Lambda}_{\hat{A}\hat{B}} + g\bar{\xi}\sigma^{\hat{m}A'}\Lambda^{\hat{m}A'}\tilde{\Omega}_m, \]
\[ \delta \bar{\varphi}^{\hat{A}'\hat{B}} = \sqrt{2}\bar{\xi}\delta^{\hat{A}}A\tilde{\bar{\Lambda}}_{\hat{A}\hat{B}}. \] (2.22)

Here \( \Omega_m = \Omega_{mn}x^n \) and \( \bar{\Omega}_m = \bar{\Omega}_{mn}x^n \). The parameter \( \tilde{\xi} \) is obtained by taking the diagonal part of the \( N = 2 \) supersymmetry transformation parameters \( \tilde{\xi}^{\hat{A}} \) as \( \tilde{\xi} = \delta^{\hat{A}'}_A \tilde{\xi}^{\hat{A}} \) [24]. The transformations (2.22) can be regarded as the \( \Omega \)-deformation of the topologically twisted supersymmetry.

### 2.2 Instanton calculus in deformed \( N = 4 \) theory

We now study the instanton calculus in the deformed \( N = 4 \) \( U(N) \) super Yang-Mills theory. We are interested in the integration over the zero modes around the instanton solutions in the path integral. We first examine the solutions to the equations of motion around the instantons. In the Coulomb branch of the theory, the adjoint scalar fields \( \varphi_a \) have vacuum expectation values (VEVs) \( \langle \varphi_a \rangle = \delta^0_a \). We take the VEVs to be diagonal such that they commute with each other. It is very difficult to solve the equations exactly even in the case where all the deformation parameters vanish. Instead we expand the fields in the coupling constant \( g \). Its approximation is valid when the VEVs are large [25]. The leading term of the gauge field satisfies the (anti-)self-dual equation. Here we consider the case of the self-dual instanton solution. The \( g \)-expansions of the fields are given by

\[
\begin{align*}
A_m &= g^{-1}A^{(0)}_m + g^1A^{(1)}_m + \cdots, \\
\varphi_a &= g^0\varphi^{(0)}_a + g^2\varphi^{(1)}_a + \cdots, \\
\Lambda^A &= g^{-1/2}\Lambda^{(0)A} + g^{3/2}\Lambda^{(1)A} + \cdots, \\
\tilde{\Lambda}_A &= g^{1/2}\tilde{\Lambda}^{(0)}_A + g^{5/2}\tilde{\Lambda}^{(1)}_A + \cdots. \tag{2.23}
\end{align*}
\]

Here \( A^{(0)}_m \) satisfies the self-dual equation:

\[
F^{(0)}_{mn} = \tilde{F}^{(0)}_{mn}, \tag{2.24}
\]
where \( F_{mn}^{(0)} \) is the field strength of \( A_m^{(0)} \). From the Lagrangian (2.13) the equations of motion for the other fields at the leading order in \( g \) are obtained as

\[
\bar{\sigma}^{m\dot{\alpha}} \nabla_m \Lambda^{(0)A}_{\dot{\alpha}} = 0, \tag{2.25}
\]

\[
\nabla^2 \varphi_a^{(0)} + F_{mn}^{(0)} \Omega^{mn}_a + (\bar{\Sigma}_a)_{AB} \Lambda^{(0)\alpha A} \Lambda^{(0)B}_{\alpha} = 0, \tag{2.26}
\]

\[
\sigma_{a\dot{\alpha}} m \nabla_m \Lambda^{(0)\dot{\alpha}} A - (\bar{\Sigma}_a)_{AB} [\varphi_a^{(0)} , \Lambda^{(0)B}_{\dot{\alpha}}] - i \Omega_a^m (\bar{\Sigma}_a)_{AB} \nabla_m \Lambda^{(0)B}_{\alpha} + \frac{i}{2} \Omega_{mna} (\bar{\Sigma}_a)_{AB} (\sigma^{mn})_{\alpha}^{\beta} \Lambda^{(0)B}_{\beta} - (\bar{\Sigma}_a)_{AB} (\Lambda_a)^B C \Lambda^{(0)C}_{\alpha} = 0, \tag{2.27}
\]

where \( \nabla_m* = \partial_m* + i [A_m^{(0)},*] \) is the gauge covariant derivative in the instanton background (2.24). The parameters \( \Omega_{mna} \) and \( \Lambda_a \) are of order \( g^0 \). The equations (2.24) and (2.25) are not deformed, while only the self-dual part of \( \Omega_{mna} \) contributes in the equation (2.26) since it is contracted with the self-dual \( F_{mn}^{(0)} \). Then the solutions to (2.24)–(2.26) are the same as those in the self-dual \( \Omega \)-background and without the R-symmetry Wilson line, which have been obtained in [20]. The solution to (2.27) does not contribute to the instanton effective action as we will see later.

The equation (2.24) for the instantons is solved by the ADHM construction [26]. The solution with the instanton number \( k \) is parametrized by the position moduli \( a'_m \) and the size moduli \( w_{\dot{\alpha}}, \bar{w}^{\dot{\alpha}} \). Here \( a'_m \) are the \( k \times k \) Hermitian matrices and \( w_{\dot{\alpha}}, \bar{w}^{\dot{\alpha}} \) are the \( N \times k \) and \( k \times N \) complex matrices respectively, which are Hermitian conjugate to each other. The variables \( a'_m, w_{\dot{\alpha}} \) and \( \bar{w}^{\dot{\alpha}} \) are called the bosonic ADHM moduli and satisfy the bosonic ADHM constraints

\[
(\tau^c)^{\dot{\alpha}}_{\beta}(\bar{w}^{\dot{\beta}} w_{\dot{\alpha}} + a_{\dot{\alpha}a}^{(c)} a'_m) = 0, \quad c = 1, 2, 3, \tag{2.28}
\]

where \( a_{\dot{\alpha}a}^{(c)} \) are defined by \( a_{\dot{\alpha}a}^{(c)} = \sigma_{a\dot{\alpha}}^m a'_m \). The instanton moduli space is described by the variables satisfying (2.28), divided by the \( U(k) \) action

\[
a'_m \to u^{-1} a'_m u, \quad w_{\dot{\alpha}} \to w_{\dot{\alpha}} u, \quad \bar{w}^{\dot{\alpha}} \to u^{-1} \bar{w}^{\dot{\alpha}}, \quad u \in U(k). \tag{2.29}
\]

We can also solve the fermionic zero mode equation (2.25) by the ADHM construction [27]. The fermionic zero modes are parametrized by the Grassmann-odd matrices \( M_{\alpha}^A, \mu^A \) and \( \bar{\mu}^A \) called the fermionic ADHM moduli. They are the superpartners of \( a'_m, w_{\dot{\alpha}} \) and \( \bar{w}^{\dot{\alpha}} \) respectively and then have the same size of matrices with their partners. The
fermionic ADHM moduli satisfy the fermionic ADHM constraints
\[ \bar{\mu}^A w_{\dot{\alpha}} + \bar{w}_{\dot{\alpha}} \mu^A + [\mathcal{M}^{\alpha A}, a'_{\alpha \dot{\alpha}}] = 0. \]  
(2.30)

The \( U(k) \) group also acts on the fermionic moduli as
\[ \mathcal{M}_\alpha^A \rightarrow u^{-1} \mathcal{M}_\alpha^A u, \quad \mu^A \rightarrow \mu^A, \quad \bar{\mu}^A \rightarrow u^{-1} \bar{\mu}^A. \]  
(2.31)

The solution to the equation (2.26) for the scalar fields does not have zero modes and is expressed in terms of the ADHM moduli, the scalar VEVs and the deformation parameter \( \Omega_{mna} \) [20].

By substituting the expansions (2.23) into (2.13), the spacetime action \( S = \int d^4 x L(\Omega, A) \) is expanded as
\[ S = \left( \frac{8\pi^2}{g^2} - i\theta \right) k + g^0 S^{(0)} + \mathcal{O}(g^2), \]  
(2.32)

where \( S^{(0)} \) is given by
\[ S^{(0)} = \frac{1}{\kappa} \int d^4 x \text{Tr} \left[ \frac{1}{2} \left( \nabla_m \phi_a^{(0)} - F_{mn} \Omega^n_a \right)^2 - \frac{1}{2} (\bar{\sigma}_a)_{AB} \Lambda^{(0)\alpha A} \left[ \varphi_a, \Lambda^{(0)B}_\alpha \right] ight. 
\[ - \frac{i}{2} \Omega^m_a (\bar{\sigma}_a)_{AB} \Lambda^{(0)\alpha A} \nabla_m \Lambda^{(0)B}_\alpha + \frac{i}{4} \Omega_{mna} (\bar{\sigma}_a)_{AB} \Lambda^{(0)\alpha A} (\sigma^{mn})_\alpha^\beta \Lambda^{(0)B}_\beta 
\]  
\[ - \frac{1}{4} (\bar{\sigma}_a)_{AB} \Lambda^{(0)\alpha A} (\mathcal{A}_a)_B^C \Lambda^{(0)C}_\alpha \right]. \]  
(2.33)

As we mentioned before, \( S^{(0)} \) does not depend on \( \bar{\Lambda}_A^{(0)} \). We evaluate the integral in (2.33) by substituting the solutions to (2.24)–(2.26) and we express \( S^{(0)} \) as the function of the ADHM moduli. In the case of the self-dual \( \Omega_{mna} \) and the vanishing R-symmetry Wilson line, (2.33) is reduced to the one in [20]. In the general case we have the contributions from the anti-self-dual part of \( \Omega_{mna} \) and the Wilson line, which can be computed in a similar way as in the case of the \( \mathcal{N} = 2 \) theory [13]. Then \( S^{(0)} \) is evaluated as
\[ S^{(0)} = \frac{2\pi^2}{\kappa} \text{tr}_k \left[ -\left( \frac{1}{4} (\bar{\sigma}_a)_{AB} \left( \bar{\mu}^A \mu^B + \mathcal{M}^{\alpha A} \mathcal{M}_\alpha^B \right) + \bar{w}_{\dot{\alpha}} \phi^0_a w_{\dot{\alpha}} - i \Omega^{+mna}_{\alpha \dot{\alpha}} [a'_m, a'_{n\dot{\alpha}}] \right) \right. 
\[ \times L^{-1} \left( \frac{1}{4} (\bar{\sigma}_a)_{CD} \left( \bar{\mu}^C \mu^D + \mathcal{M}^{\beta C} \mathcal{M}_\beta^D \right) + \bar{w}_{\dot{\beta}} \phi^0_a w_{\dot{\beta}} - i \Omega^{+pqa}_{\beta \dot{\beta}} [a'_p, a'_{q\dot{\beta}}] \right) 
\]  
\[ + \frac{i}{8} (\bar{\sigma}_a)_{AB} \Omega^{mna} (\sigma^{mn})_\alpha^\beta \mathcal{M}^{\alpha A} \mathcal{M}_\beta^B + \frac{1}{4} \Omega^{-mna} \Omega_{mna} \bar{w}_{\dot{\alpha}} w_{\dot{\alpha}} \right]. \]
\[ + \frac{1}{2} (\Sigma^a)_{A'B'} \phi_a^0 \phi_a^0 - \bar{w}_a \phi_a^0 \phi_a^0 \bar{w}_a + \imath \Omega_{mna} (\sigma_{m\alpha}^{\beta} \phi_a^0 \phi_a^0 \bar{w}_a + \imath \Omega_{mna} (\sigma_{m\alpha}^{\beta} \phi_a^0 \phi_a^0 \bar{w}_a
\]
\[ + \Omega_{mna}^{\alpha} \omega_{\alpha}^{\beta} \phi_a^0 \phi_a^0 \bar{w}_a - m_{AB} \left( 2 \phi_a^0 \phi_a^0 + \mathcal{M}^{\alpha A} \mathcal{M}^{\beta B} \right), \tag{2.34} \]

where \( \text{tr}_k \) denotes the trace for the \( k \times k \) matrix. We use the same normalization factor \( \kappa \) for the \( U(k) \) generators. \( \Omega_{mna}^{\pm} = \frac{1}{2} (\Omega_{mna} \pm \bar{\Omega}_{mna}) \) is the (anti-)self-dual part of \( \Omega_{mna} \).

The operator \( \mathcal{L} \) acting on the \( k \times k \) matrix is defined by
\[ \mathcal{L} = \frac{1}{2} \{ \bar{w}_a w_a, * \} + [a^m, [a'_m, *]]. \tag{2.35} \]

The parameters \( m_{AB} \), which are symmetric in the indices \( A \) and \( B \), are expressed by the Wilson line gauge field \( (A_a)^{A'B'} \) as
\[ m_{AB} = \frac{1}{8} ((\Sigma_a)_{AC} (A_a)^{C'B'} + (\Sigma_a)_{BC} (A_a)^{C'A}). \tag{2.36} \]

In \[2.34\] the ADHM moduli obey the ADHM constraints. We introduce the auxiliary variables \( D^\tilde{c} \) and \( \bar{\psi}_A^{\tilde{c}} \) as the Lagrange multipliers for the constraints \[2.28\] and \[2.30\] respectively such that we can regard the ADHM moduli as the independent variables in the path integral. We also introduce the auxiliary variables \( \chi_a \) in the path integral such that the \( \mathcal{L}^{-1} \)-terms in \[2.34\] become the Gaussian form. Then \( S^{(0)} \) can be rewritten as
\[ S^{(0)}_{\text{eff}} = \frac{2\pi^2}{\kappa} \text{tr}_k \left[ -[\chi_a, a'_m] - i \Omega_{mna} a^m \right] \left([\chi_a, a^m] - i \Omega_{\alpha p} a'_p \right) \]
\[ - \frac{1}{4} \mathcal{M}^{\alpha A} \left( (\Sigma_a)_{AB} \left([\chi_a, \mathcal{M}^{\beta B}_{\alpha}] - i \Omega_{mna} (\sigma_{mn})^{\alpha\beta} \mathcal{M}^{\beta B}_{\alpha} \right) + 4m_{AB} \mathcal{M}^{\beta B}_{\alpha} \right) \]
\[ + \left( \chi_a \bar{w}_a - \bar{w}_a \phi_a^0 - i \Omega_{mna} (\sigma_{mn})^{\alpha\beta} \bar{w}_a \right) \left( w_\bar{a} \chi_a - \phi_a^0 w_\bar{a} - i \Omega_{\beta p} (\bar{\sigma}_{pq})^{\gamma\delta} \bar{w}_\gamma \right) \]
\[ + \frac{1}{2} \mu^A \left( (\Sigma_a)_{AB} \left( \mu^A \chi_a - \phi_a^0 \mu^B \right) - 4m_{AB} \mu^B \right) \]
\[ - i \bar{\psi}_A^{\tilde{c}} (\mu^A \bar{w}_\bar{a} + \bar{w}_\bar{a} \mu^A + [\mathcal{M}^{\alpha A}_{\alpha}, a'^{\alpha A}_{\alpha}] ) + i D^\tilde{c} (\tilde{\tau}_a) \bar{w}_a \bar{w}_a + \bar{a}^{\tilde{c}} a'^{\alpha A}_{\alpha} \right) - i \zeta^\tilde{c} D^\tilde{c}, \tag{2.37} \]

which is called the instanton effective action. Here we have introduced the Fayet-Iliopoulos (FI) parameters \( \zeta^\tilde{c} \) to resolve the small instanton singularity, which modifies the bosonic ADHM constraints \[2.28\] as
\[ (\tilde{\tau}_a) \bar{w}_a + \bar{a}^{\tilde{c}} a'^{\alpha A}_{\alpha} = \zeta^\tilde{c}. \tag{2.38} \]
The parameters $\zeta^\hat{c}$ are also interpreted as the noncommutativity parameters in space-time \[28\]. In the instanton effective action (2.37), the $\Omega$-background parameters $\Omega_{mna}$ give the mass terms for the bosonic moduli $(a'_m, w_\alpha, \bar{w}^\dot{\alpha})$ and for the fermionic moduli $\mathcal{M}_{\alpha}^A$, while the parameters $m_{AB}$ give the mass terms for the fermionic moduli $(\mathcal{M}_{\alpha}^A, \mu^A, \bar{\mu}^A)$.

For the case with the self-dual $\Omega$-background and without the Wilson line the effective action (2.37) is reduced to the one obtained in \[20\], where the action is invariant under the deformed supersymmetry. We can also reduce the effective action (2.37) to that in $N=2$ case by the $\mathbb{Z}_2$ orbifold projection which acts on the parameters with the $SU(2)_F$ indices. Here the moduli variables and the deformation parameters having odd $\mathbb{Z}_2$ charges are projected out. To describe the $N=2$ theory, we decompose the fermionic moduli $\mathcal{M}_{\alpha}^A$, $\mu^A$, the auxiliary variables $\chi_a$, $\bar{\psi}_{\dot{\alpha}}^\dot{A}$ and the scalar VEVs $\phi_0^A$ to $(\mathcal{M}_{\alpha}^A, \mu^A, \bar{\mu}^A)$, $\chi$, $\bar{\chi}$, $\chi^A \bar{\chi}$, $(\bar{\psi}_{\dot{\alpha}}^\dot{A}, \bar{\psi}_{\dot{\alpha}}^\dot{A})$ and $(\phi_0^A, \bar{\phi}_0^A, \phi_0^A)$ respectively. The deformation parameters $\Omega_{mna}$ and $(A_a)_{AB}$ are decomposed as in the previous subsection. Then the mass parameter $m_{AB}$ is decomposed to $(m_{A'B'}, m_{A'A}, m_{\hat{A}\hat{B}})$. Here $m_{A'B'}$ is related to the $N=2$ Wilson line gauge field $A^A_{B'}$ as $m_{A'B'} = -2\sqrt{2} i \epsilon_{A'C'} \bar{A}_{C'B'}$. Under the orbifold projection the moduli $(\mathcal{M}_{\alpha}^A, \mu^A, \bar{\mu}^A, \bar{\psi}_{\dot{\alpha}}^\dot{A})$, the scalar VEVs $\phi_0^A$ and the parameters $(\Omega^A_{mn}, m_{A'A})$ are projected out. The parameter $m_{\hat{A}\hat{B}}$ survives in the projection but does not contribute to the effective action since it only couples to the fermionic moduli $\mathcal{M}_{\alpha}^A$ and $\mu^A$. Then (2.37) is reduced to the deformed $N=2$ effective action, which has one deformed supersymmetry under the conditions (2.17) and (2.19) \[2\] [13].

In the case without the $\Omega$-background and with the non-zero Wilson line parameters $M_{\hat{A}\hat{B}}$, we find that the instanton effective action (2.37) becomes that of the undeformed $N=2^*$ theory \[21\]. The $N=2$ mass parameters $M_{\hat{A}\hat{B}}$ are related with $m_{AB}$ via $M_{\hat{A}\hat{B}} = -2\sqrt{2} i \epsilon^{A'C'} m_{C'B}$. We note that the effective action (2.37) does not depend on $M_{\hat{A}\hat{B}}$, since the terms in the spacetime action depending on $M_{\hat{A}\hat{B}}$ are of higher order in the coupling constant expansion (2.32). We have to set $\phi_0^A = 0$ since it breaks the $N=2$ supersymmetry due to the mass terms for the scalar fields in the hypermultiplet \[21\].

In the $\Omega$-background and the Wilson lines satisfying $\Omega_{mn} = 0$, (2.17), (2.19) and (2.20), we obtain the instanton effective action of the deformed $N=2^*$ theory. The
action (2.37) is invariant under the following supersymmetry transformations:

\[
\begin{align*}
\delta a'_{a\dot{a}} &= \bar{\xi} \epsilon_{\dot{a}A'} M'^A_{\dot{a}}, \\
\delta M'^A_{\dot{a}} &= \sqrt{2} \xi \epsilon^{A\dot{a}} (2i [a'_{a\dot{a}}, \chi] - \Omega^+_{mn}(\sigma^{mn})_{\dot{a}a} a'_{\dot{a}a} + \Omega^-_{mn}(\bar{\sigma}^{mn})_{\dot{a}a} a'_{\dot{a}a}), \\
\delta w_{a\dot{a}} &= \bar{\xi} \epsilon_{\dot{a}A'} \mu'^{A}, \\
\delta \mu'^{A} &= \sqrt{2} \xi \epsilon^{A\dot{a}} (2i (w_{a\dot{a}} \chi - \phi^0 w_{\dot{a}}) + \Omega^-_{mn}(\bar{\sigma}^{mn})_{\dot{a}a} w_{\dot{a}}), \\
\delta \bar{w}_{\dot{a}a} &= \bar{\xi} \delta^{\dot{a}}_A \bar{\mu}^{A'}, \\
\delta \bar{\mu}^{A'} &= -\sqrt{2} \xi \delta^{A'}_{\dot{a}} (2i (\chi \bar{w}_{\dot{a}} - \bar{w}_{\dot{a}} \phi^0) + \Omega^+_{mn}(\sigma^{mn})_{\dot{a}a} \bar{w}_{\dot{a}}), \\
\delta \chi &= 0, \\
\delta \chi^\dot{a} A &= \sqrt{2} \xi \bar{\psi}_{A'} \psi^{\dot{a}A'}, \\
\delta \bar{\psi}^{\dot{a}A'} &= \bar{\xi} (-i \delta^{\dot{a}}_A [\chi, \chi] + (\tau^\dot{c})^{\dot{a}A'} D^\dot{c} + i \bar{\delta}^{\dot{a}}_{A'} [\chi B^A, \bar{\chi}_A])], \\
\delta \chi^\dot{a} A &= \sqrt{2} \xi \bar{\psi}_{A'} \psi^{\dot{a}A'}, \\
\delta \bar{\psi}^{\dot{a}A'} &= \bar{\xi} (-i \delta^{\dot{a}}_A [\chi, \chi] - \Omega^+_{mn}(\sigma^{mn})_{\dot{a}a} \chi^\dot{a} A - 2i M^A B \chi^\dot{a} B), \\
\delta D^\dot{c} &= \frac{i}{\sqrt{2}} \bar{\xi} (\tau^\dot{c})^{\dot{a}A'} [\bar{\psi}^{\dot{a}A'}, \chi] - \sqrt{2} i \bar{\xi} (\tau^\dot{c})^{\dot{a}A'} [\bar{\psi}^{\dot{a}A'}, \bar{\chi}_A] + \frac{1}{\sqrt{2}} \bar{\xi} \bar{\delta}^{\dot{c}}_{\bar{\delta}} \Omega^+_{mn} \bar{\eta}_{mn} (\tau^\dot{c})^{\dot{a}A'} \bar{\psi}^{\dot{a}A'}, \\
\delta M'^A_{\dot{a}} &= -2\sqrt{2} i \bar{\xi} [a'_{a\dot{a}}, \chi^\dot{a} A], \\
\delta \mu'^{A} &= -2\sqrt{2} i \bar{\xi} w_{a\dot{a}} \chi^\dot{a} A, \\
\delta \bar{\mu}^{A'} &= -2\sqrt{2} i \bar{\xi} \bar{\mu}^{A'} \bar{\chi}_{B\dot{a}} \bar{w}_{\dot{a}}, \\
\end{align*}
\]

where the anti-self-dual 't Hooft \( \eta \)-symbol \( \bar{\eta}^\dot{c}_{mn} \) is defined in \((A.3)\).

We have constructed the \( \mathcal{N} = 4 \) instanton effective action in the \( \Omega \)-background from the ADHM method. In the next section, we will study the relation between the deformed instanton effective action and the \( D(-1) \)-brane effective action for the \( D3/D(-1) \)-brane system deformed in the R-R 3-form backgrounds.

### 3 D\((-1)\)-brane effective action in R-R backgrounds

#### 3.1 Disk amplitudes in closed string backgrounds

The \( \mathcal{N} = 4 \) super Yang-Mills theory with gauge group \( U(N) \) is realized as the low-energy effective theory of \( N \) coincident D3-branes. The zero modes of open strings with both the end points on the D3-branes correspond to the \( \mathcal{N} = 4 \) vector multiplet. One can introduce instantons with the topological number \( k \) as \( k \) D\((-1)\)-branes embedded into the D3-brane world-volume \([29]\). The zero modes of open strings at least one of whose end points is on the D\((-1)\)-branes correspond to the ADHM moduli \( a'_m, w_{a\dot{a}}, \bar{w}_{\dot{a}a}, M'^A_{\dot{a}}, \mu^A, \bar{\mu}^{A} \) and the auxiliary variables \( \chi_a, D^\dot{c}, \bar{\psi}^{\dot{a}}_A \) \([30]\).
The instanton effective action of the $N = 4$ super Yang-Mills theory is also obtained as the low-energy effective action of the $D(-1)$-branes for the $D3/D(-1)$-brane system [12]. The action is derived from the zero-slope limit $\alpha' \to 0$ of the open string disk amplitudes with the boundary lying on the $D(-1)$-branes, where $\alpha'$ is the Regge slope parameter. The amplitudes include the vertex operators associated with the zero modes. We summarize our conventions and notations for the calculations of the amplitudes in Appendix B. The vertex operators for the open string zero modes are found in Table 1 in Appendix B. We keep the D3-brane (Yang-Mills) coupling constant $g = (2\pi)^{\frac{1}{2}} g_s^3$ finite [31]. Then the zero-slope limit corresponds to the limit $g_0 \to \infty$, where $g_0 = (2\pi)^{-\frac{3}{4}} \alpha'^{-1} g_s$ is the gauge coupling constant for the $D(-1)$-branes and $g_s$ is the string coupling constant. Some of the ADHM moduli in the vertex operators must be rescaled by $g_0$ to reproduce the field theory calculations [32].

The $D(-1)$-brane action $S_{D(-1)}^0$ which reproduces the amplitudes in the zero-slope limit [32] is given by

$$S_{D(-1)}^0 = \frac{2\pi^2}{\kappa'} \text{tr}_{\kappa} \left[ Y_{ma} Y_{ma} - X_{\dot{a}a} X^{\dot{a}a} + 2Y_{ma} [\chi_a, \alpha'_m] + iD^\dot{\alpha}(\tau^\dot{\alpha})_\beta \left( \bar{\psi}_\dot{\alpha} w_{\dot{a}} + \bar{a}^{\beta\alpha} a'_{\alpha a} \right) 
- X_{\dot{a}a} (\chi_a \bar{w}_{\dot{a}a} - \bar{w}_{\dot{a}a} \phi^0_a) - (w_{\dot{a}a} \chi_a - \phi^0_a w_{\dot{a}a}) X^{\dot{a}a} 
+ \frac{1}{2} (\Sigma^a)_{AB} \bar{\mu}^A B(\chi_a \bar{w}_{\dot{a}a} A + \phi^0_{a \mu} B) - \frac{1}{2} (\Sigma^a)_{AB} M^{\alpha A} M'_{B} \chi_a 
- i\bar{\psi}_\dot{\alpha} A (\bar{\mu}^A w_{\dot{a}a} + \bar{w}_{\dot{a}a} \mu^A + [M^{\alpha A}, \alpha'_a]) \right].$$

(3.1)

Here we have introduced the auxiliary fields $Y_{ma}, X_{\dot{a}a}, X^{\dot{a}a}$ to disentangle the higher point interactions in the low-energy effective action [32]. After eliminating these auxiliary fields by using their equations of motion, the action $S_{D(-1)}^0$ reduces to the one (2.37) where all the deformation parameters are set to be zero.

We now study the deformation of the $D(-1)$-brane action in the R-R backgrounds. The R-R field strengths $F_M, F_{MNP}, F_{MNPQR}$ can be combined into the bi-spinor form $\bar{F}^{\dot{A}B}$, where $\dot{A}, \dot{B}$ are 16 component spinor indices in ten dimensions. Since the D3-branes break the $SO(10)$ Lorentz symmetry down to $SO(4) \times SO(6)$, the R-R backgrounds are decomposed into

$$\bar{F}^{\dot{A}B} = (F^{\alpha \beta AB}, F^{\alpha A B}, F^{\alpha B A}, F^{\dot{\alpha} \dot{\beta} AB}).$$

(3.2)
We consider the deformations of the effective action by the constant R-R backgrounds with the (S,A)-type $F^{(\alpha\beta)[AB]}$, $F_{(\dot{\alpha}\dot{\beta})[AB]}$ and the (A,S)-type $F^{[\alpha\beta](AB)}$, $F_{[\dot{\alpha}\dot{\beta}](AB)}$ [33]. Here the round parentheses $(\cdot\cdot)$ denote symmetrization of the indices and the square bracket $[\cdot\cdot]$ stands for anti-symmetrization. The (S,A)-type backgrounds have the index structure $F_{mna}$ while the (A,S)-type backgrounds have the index structure $F_{abc}$. In [20], we have calculated the D(−1)-brane effective action in the presence of the constant self-dual R-R background of the (S,A)-type and have found that it agrees with the $\mathcal{N} = 4$ instanton effective action in the self-dual $\Omega$-background without the R-symmetry Wilson line by the identifications of $\Omega^{\pm}_{mna}$ with $F^{(\alpha\beta)[AB]}$. We have also observed that the (A,S)-type background $F_{[\dot{\alpha}\dot{\beta}](AB)}$ gives the holomorphic mass terms for the fermionic ADHM moduli in the corresponding effective action. In order to generalize this result to the non-(anti-)self-dual $\Omega$-background with the Wilson line, we will consider both of the self-dual and the anti-self-dual parts of the (S,A)- and the (A,S)-type backgrounds. These backgrounds have the same tensor structures as those of the deformation parameters and give rise to the bilinear couplings with the massless fields in the D3/D(−1) brane system. We also introduce the constant NS-NS B-field background. The low-energy effective theories of D-branes in the presence of the constant NS-NS B-field can be described by gauge theories in the noncommutative spacetime [34, 35]. When one considers instantons in this background, the NS-NS B-field corresponds to the FI parameters in (2.37).

We study the effects of the closed string backgrounds in the D(−1)-brane effective theory by calculating the string disk amplitudes that contain the open and the closed string vertex operators. The vertex operators for the closed string backgrounds are summarized in Table 2 in Appendix B. As we discussed in [20], we consider the zero-slope limit with finite $(2\pi\alpha')^{\frac{1}{2}}F$. Here $F$ represents the component of the (S,A)- and the (A,S)-type backgrounds.

In order to cancel the overall factor $1/g_0^2$ of the disk amplitudes (3.6) in the zero-slope limit, the vertex operators for the following combinations of the fields need to be inserted in the disk amplitudes together with that for $F$:

$$\mu\bar{\mu}, \ Y\alpha', \ M'M', \ w\bar{X}, \ \bar{w}X, \ X\bar{X}, \ w\bar{w}, \ YY, \ a'a'. \quad (3.3)$$

By dimensional analysis and the conservation law of the charges associated with the spin
operators and the twist fields, we find that the irreducible amplitudes that contain one vertex operator for the (S,A)- or the (A,S)-type backgrounds include only the vertex operators associated with the first five combinations in (3.3).

In the following, we calculate the non-zero amplitudes that contain the closed string backgrounds.

**Amplitudes with (S,A)-type backgrounds** We first consider the amplitudes that contain the self-dual part of the (S,A)-type backgrounds [20]. These have been evaluated as

\[
\langle \langle \nu_2(0) \nu_1(0) \rangle \langle \langle V_{\mathcal{M}^\prime}^{(-\frac{1}{2})} V_{\mathcal{F}^\prime} \rangle \rangle \rangle = \frac{2\pi^2}{\kappa} \text{tr}_k \left[ -\pi \nu_2(0) \nu_1(0) \right],
\]

\[
\langle \langle \nu_2(0) \nu_1(0) \rangle \langle \langle V_{\mathcal{M}^\prime}^{(-\frac{1}{2})} V_{\mathcal{F}^\prime} \rangle \rangle \rangle = \frac{2\pi^2}{\kappa} \text{tr}_k \left[ -\pi \nu_2(0) \nu_1(0) \right].
\]
**Amplitudes with NS-NS B-field background**  We now calculate the disk amplitudes in the constant NS-NS B-field background. We first consider the amplitudes which do not contain any vertex operators for the R-R backgrounds but contain one vertex operator for the NS-NS B-field background and that for the open string zero modes. By dimensional analysis, the following amplitudes should be considered:

$$
\langle \langle V^{(0)}(0) \partial V^{(-1,-1)}(0,0) B \rangle \rangle,
\langle \langle V^{(0)}(0) \chi V^{(0)}(0) \partial V^{(-1,-1)}(0,0) B \rangle \rangle,
\langle \langle V^{(0)}(0) \phi V^{(0)}(0) \partial V^{(-1,-1)}(0,0) B \rangle \rangle,
\langle \langle V^{(0)}(0) \chi V^{(0)}(0) \phi V^{(-1,-1)}(0,0) B \rangle \rangle.
$$

The second, the third and the fourth ones vanish because the amplitudes are proportional to the factor $B_{mn} B^{mn}$. The first amplitude was evaluated in [36]. The result is

$$
\langle \langle V^{(0)}(0) B^{-1,1} \rangle \rangle = \frac{2\pi^2}{\kappa} \text{tr}_k[iD_c \zeta^c],
$$

where we have defined $\zeta^c \equiv \bar{\eta}^c_{mn} B^{mn}$.

Next we consider the case where both the NS-NS B-field and the R-R 3-form backgrounds are turned on. Once these backgrounds are introduced simultaneously, there is a possibility of non-zero amplitudes containing both $B$ and $F$. By dimensional analysis, we find that the vertex operators for the open string zero modes in the amplitudes must not have $g_0$ dependence and the sum of their powers of $\alpha'$ must be 1/2. The only possible candidates for such vertex operators are $V^{(-1)}_\chi$ and $V^{(-1)}_\phi$. Therefore we examine the following amplitudes:

$$
\langle \langle V^{(-1)}_\chi V^{(0,0)}_B V^{(-1,1)} F \rangle \rangle,
\langle \langle V^{(-1)}_\phi V^{(0,0)}_B V^{(-1,1)} F \rangle \rangle,
$$

where $V^{(-1,1)}_F$ is a vertex operator for the (S,A)- or the (A,S)-type backgrounds. Calculating these amplitudes are cumbersome since they contain the five-point world-sheet correlators. Instead we evaluate the amplitudes from the Wess-Zumino term of the D$(-1)$-brane action in the NS-NS B-field and the R-R backgrounds [37]. We find that the corresponding interaction term vanishes for the constant NS-NS B-field and the (S,A)- or the (A,S)-type backgrounds. Then the amplitudes (3.10) are zero. The amplitudes that contain more than one R-R vertex operator are reducible or of higher order in $\alpha'$ and vanish in the zero-slope limit.

---

1 The terms that contain the VEVs $\phi^0_\alpha$ can not be derived from the Wess-Zumino term in the effective action in [37]. However, the second amplitude in (3.10) vanishes when the first amplitude is zero. This is because the structure of the vertex operator for $\chi_\alpha$ and $\phi^0_\alpha$ is the same.
3.2 Deformed D(−1)-brane effective action

We now consider the deformed D(−1)-brane effective action for the D3/D(−1)-brane system. The amplitudes (3.4)–(3.9) are reproduced by the following effective action:

\[ S'_{D(-1)} = S^0_{D(-1)} + S_{\text{closed}}, \tag{3.11} \]

where

\[ S_{\text{closed}} = \frac{2\pi^2}{\kappa} \text{tr}_k \left[ 2Y_{ma}a'_nC_{mna} - \frac{1}{8}(\Sigma^a)_{AB}\mathcal{M}^A_{\alpha}\mathcal{M}^B_{\beta}C_{mna}\epsilon^\gamma(\sigma_{mn})_{\gamma} \right. \]
\[ - \frac{1}{2} \chi_{\hat{\alpha}a}(\bar{\sigma}_{mn})_{\hat{\alpha}a}w_{\hat{\alpha}a}C_{mna} - \frac{1}{2} w_{\hat{\alpha}a}(\bar{\sigma}_{mn})_{\hat{\alpha}a}X_{\hat{\alpha}a}C_{mna} \]
\[ + \left. \left( -2\bar{\mu}A\mu - \mathcal{M}^A_{\alpha}\mathcal{M}^A_{\beta} \right) \tilde{m}_{AB} - iD\bar{\xi}^\beta \right]. \tag{3.12} \]

Here we have defined the deformation parameters \( C_{mna} \) from the backgrounds \( \mathcal{F}^{(\alpha\beta)|AB} \), \( \mathcal{F}^{(\hat{\alpha}\hat{\beta})|AB} \) and \( \tilde{m}_{AB} \) from \( \mathcal{F}^{(\hat{\alpha}\hat{\beta})|AB} \) as

\[ C_{mna} = -2\pi(2\pi\alpha')^2 \left[ \mathcal{F}^{(\alpha\beta)|AB}\epsilon_{\beta\gamma}(\sigma_{mn})_{\gamma}^{\alpha}\mathcal{F}^{(\hat{\alpha}\hat{\beta})|AB}\epsilon_{\hat{\beta}\hat{\gamma}}(\bar{\sigma}_{mn})_{\hat{\beta}}^{\hat{\alpha}}\mathcal{F}^{(\hat{\alpha}\hat{\beta})|AB} \right], \tag{3.13} \]
\[ \tilde{m}_{AB} = \pi i(2\pi\alpha')^2 \mathcal{F}^{(\hat{\alpha}\hat{\beta})|AB}\epsilon_{\hat{\alpha}\hat{\beta}}. \tag{3.14} \]

After integrating out the auxiliary fields \( Y_{ma}, X_{\hat{\alpha}a}, \bar{X}_{\hat{\alpha}a} \), we finally obtain the following effective action

\[ S_{D(-1)} = \frac{2\pi^2}{\kappa} \text{tr}_k \left[ - (\mathcal{X}_{a} + d'_{m})^2 \right. \]
\[ + \left( w_{a}\mathcal{X}_{a} - \phi_{a}^{0}w_{a} + \frac{1}{2} C^{mna}(\bar{\sigma}_{mn})_{\hat{\alpha}}w_{\hat{\alpha}} \right)(\mathcal{X}_{a} w_{a} - \bar{w}_{\hat{\alpha}} \phi_{a}^{0} + \frac{1}{2} C^{ppq}(\bar{\sigma}_{pq})_{\hat{\alpha}}w_{\hat{\alpha}}) \]
\[ + \frac{1}{2} (\Sigma^{a})_{AB}\bar{\mathcal{M}}^{A}_{\alpha}\left( -\chi_{a}\mu^{B} + \phi^{0}_{a}\mu^{B} \right) - \frac{1}{8} (\Sigma^{a})_{AB}\epsilon^{\alpha\gamma}(\sigma_{mn})^{\beta} C^{mna}\mathcal{M}^{A}_{\alpha}\mathcal{M}^{B}_{\beta} \]
\[ - \frac{1}{2} (\Sigma^{a})_{AB}\mathcal{M}^{\alpha A}_{\alpha}\mathcal{M}^{A}_{\beta} - (2\bar{\mu}A\mu^{B} + \mathcal{M}^{A}_{\alpha}\mathcal{M}^{B}_{\alpha}) \tilde{m}_{AB} \]
\[ - i\bar{\xi}_{\hat{\alpha}}^{\beta}(\bar{\mu}^{A}w_{\hat{\alpha}} + \bar{w}_{\hat{\alpha}}\mu^{A} + [\mathcal{M}^{\alpha A}, a'_{aA}]) \]
\[ + iD^{\hat{\alpha}}(\tau^{\beta}\bar{\xi}^{\hat{\beta}} \bar{\xi}^{\beta} + \bar{w}_{\hat{\alpha}} w_{\hat{\alpha}} + a'^{\alpha}_{a}_{aa} a'_{a}_{aa} - \frac{1}{2}(\tau^{\beta}\bar{\xi}^{\hat{\beta}})_{\hat{\alpha}}^{\hat{\alpha}} \right]. \tag{3.15} \]

In this effective action, the \((S,A)\)-type R-R 3-form backgrounds \( C^{mna} \) give the mass terms for the bosonic moduli \((a'_{m}, w_{a}, \bar{w}_{\hat{\alpha}})\) and for the fermionic moduli \( \mathcal{M}^{A}_{\alpha} \) while the \((A,S)\)-type backgrounds \( \tilde{m}_{AB} \) give the mass terms for the fermionic moduli \( (\mathcal{M}^{A}_{\alpha}, \mu^{A}, \bar{\mu}^{A}) \). In
fact, if we identify the deformation parameters $C_{mna}$ and $\tilde{m}_{AB}$ with the $\Omega$-background and the mass parameters by $C_{mna} = -i\Omega^{mna}$, $\tilde{m}_{AB} = m_{AB}$, the low-energy effective action of the D$(-1)$-branes in the constant NS-NS B-field, the (S,A)- and the (A,S)-type backgrounds coincides with the instanton effective action $S_{\text{eff}}^{(0)}$ in the $\Omega$-background with the R-symmetry Wilson line.

4 Conclusions and Discussion

In this paper we have studied the $\mathcal{N} = 4$ super Yang-Mills theory deformed in the ten-dimensional $\Omega$-background with the R-symmetry Wilson line gauge field. We have obtained the deformed spacetime action in the general $\Omega$-background. For the self-dual $\Omega$-background without the Wilson line we have shown that the spacetime action is invariant under the anti-chiral supersymmetry. For the general $\Omega$-background one expects that a part of the $\mathcal{N} = 4$ supersymmetry is preserved by choosing the Wilson line gauge field appropriately. In particular, we have shown that the action becomes the $\mathcal{N} = 2^*$ deformed one for the specific background, where the hypermultiplet mass is introduced by the Wilson line.

We have constructed the supersymmetry transformations for the $\Omega$-deformed $\mathcal{N} = 2^*$ theory explicitly. In the undeformed case, these transformations lead to the nilpotent fermionic charge where the action is written as the exact form with respect to the charge $^{[39]}$. In the deformed case, we can show that the deformed $\mathcal{N} = 2^*$ action is written in the exact form with respect to the charge defined by (2.22). The fermionic charge is obtained by the topological twist of $\mathcal{N} = 4$ supersymmetry, where one identifies the $SU(2)_R$ Lorentz group with the $SU(2)$ subgroup of the $SU(4)_I$ R-symmetry group. One can consider three types of the different twists $^{[40,41]}$. Among them, the half twist and the Vafa-Witten twist $^{[42]}$ are particularly interesting. Since the ten-dimensional $\Omega$-background contains many deformation parameters, it is interesting to explore the deformed supersymmetry of the theory and to construct the $\Omega$-deformed topologically twisted theories.

We have also studied the ADHM construction of instantons in the $\Omega$-background.

$^2$A similar bi-spinor coupling has been studied for the matrix model in the pp-wave background $^{[38]}$. 

20
Using the solutions to the equations of motion for the fields at the leading order in the coupling constant, which are the same as the ones in the self-dual Ω-background, we got the deformed instanton effective action in the Ω-background. We have calculated the low-energy effective action of the D(−1)-branes for the D3/D(−1) system in the R-R 3-form field strength backgrounds of the (S,A)- and the (A,S)-types and found that it agrees with the instanton effective action in the Ω-background. As in the case of the spacetime action, it is interesting to examine the deformed supersymmetry when the theory has nilpotent fermionic charges.

The string theory calculation of the instanton effective action is rather straightforward compared to the ADHM method, where we need to solve the deformed equations of motion for the fields in the instanton background. We can apply the R-R 3-form field strength deformations to various D-brane systems. However the existence of the deformed supersymmetry transformations is not obvious in this approach. There is also a problem of the back reaction terms in the deformed D3-brane action, which is necessary to reproduce the deformed instanton effective action \[20\].

We note that the effective action \[3.15\] is different from the one discussed in \[22\] where the deformed action depends only on the ε-parameters holomorphically. They also differ even in the case where all the deformation parameters vanish. The action in \[22\] contains the quartic terms of \(a'_m, \chi_a\) and the quadratic term of \(D^2\) which are absent in our action. This difference comes from the fact that when we consider the zero-slope limit of the amplitudes, we have rescaled some of the ADHM moduli by \(g_0\) in the vertex operators as found in Table 1. On the other hand, in \[22\], the authors studied the D(−1)-brane effective action without use of this rescaling. However both the effective actions provide the same instanton partition function \[4\] (see \[48\] for the Ω-deformed topological string amplitudes related to the \(\mathcal{N} = 2^*\) partition function). These problems will be discussed elsewhere.

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A Sigma matrices in four and six dimensions

In this appendix, we present our conventions of the sigma matrices in four and six Euclidean dimensions. The sigma matrices $\sigma^m_{\alpha \dot{\alpha}}$ and $\bar{\sigma}^{m \dot{\alpha} \alpha}$ in four dimensions are defined by

$$
\sigma^m = \begin{pmatrix} i\tau_1, i\tau_2, i\tau_3, 1_2 \end{pmatrix}, \quad \bar{\sigma}^m = \begin{pmatrix} -i\tau_1, -i\tau_2, -i\tau_3, 1_2 \end{pmatrix},
$$

(A.1)

where $\tau_c$ ($c = 1, 2, 3$) are the Pauli matrices and $1_2$ denotes the $2 \times 2$ identity matrix. We define the Lorentz generators $\sigma^{mn}$ and $\bar{\sigma}^{mn}$ by

$$
\sigma^{mn} = \frac{1}{4}(\sigma^m \bar{\sigma}^n - \sigma^n \bar{\sigma}^m), \quad \bar{\sigma}^{mn} = \frac{1}{4}(\bar{\sigma}^m \sigma^n - \bar{\sigma}^n \sigma^m).
$$

(A.2)

They are related with the Pauli matrices by using the 't Hooft $\eta$-symbol:

$$
\sigma_{mn} = \frac{i}{2} \eta_{mn} \tau^c, \quad \bar{\sigma}_{mn} = \frac{i}{2} \bar{\eta}^{\bar{c}} \tau_{\bar{c}}.
$$

(A.3)

The sigma matrices $(\Sigma_a)^{AB}$ and $(\bar{\Sigma}_a)_{AB}$ in six dimensions are defined by

$$
\Sigma_a = \begin{pmatrix} \eta^3, -i\bar{\eta}^3, \eta^2, -i\bar{\eta}^2, \eta^1, i\bar{\eta}^1 \end{pmatrix}, \quad \bar{\Sigma}_a = \begin{pmatrix} -\eta^3, -i\bar{\eta}^3, -\eta^2, -i\bar{\eta}^2, -\eta^1, -i\bar{\eta}^1 \end{pmatrix}.
$$

(A.4)

They are related by using the rank-4 $\epsilon$-symbol $\epsilon^{ABCD}$ with $\epsilon^{1234} = 1$ as

$$
(\Sigma_a)^{AB} = -\frac{1}{2} \epsilon^{ABCD} (\bar{\Sigma}_a)_{CD}.
$$

(A.5)

The Lorentz generators $\Sigma_{ab}$ and $\bar{\Sigma}_{ab}$ in six dimensions are defined by

$$
\Sigma_{ab} = \frac{1}{4}(\Sigma_a \Sigma_b - \Sigma_b \Sigma_a), \quad \bar{\Sigma}_{ab} = \frac{1}{4}(\bar{\Sigma}_a \bar{\Sigma}_b - \bar{\Sigma}_b \bar{\Sigma}_a).
$$

(A.6)
B Calculations of string disk amplitudes

In this appendix, we calculate the string amplitudes (3.5)–(3.7) in Section 3. We first summarize our notations and conventions of the world-sheet fields in type IIB superstring theory in ten-dimensional flat Euclidean spacetime. The world-sheet coordinates are denoted by \( z, \bar{z} \). The fields \( X^M(z, \bar{z}), \psi^M(z), \bar{\psi}^M(\bar{z}), (M = 1, \cdots, 10) \) are the bosonic and the fermionic string coordinates. The left moving fields satisfy the free field OPEs

\[
X^M(z)X^N(w) \sim \delta^{MN} \ln(z - w), \quad \psi^M(z)\psi^N(w) \sim \delta^{MN}(z - w)^{-1}.
\]  

(B.1)

The right moving fields satisfy the similar OPEs. In the presence of parallel D3-branes, the \( SO(10) \) Lorentz symmetry is broken down to \( SO(4) \times SO(6) \) and the string coordinates are decomposed as \( X^M = (X^m, X^{a+4}), \psi^M = (\psi^m, \psi^{a+4}), (m = 1, \cdots, 4, a = 1, \cdots, 6) \), where \( X^m \) spans the world-volume of the D3-branes and \( X^{a+4} \) represents the transverse directions to the D3-branes.

The spin fields are defined by

\[
S^\lambda = e^{\lambda \phi^i} c_\lambda, \quad \phi \cdot e = \phi^i e_i \quad (i = 1, \cdots, 5),
\]

\[
\lambda = \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5) \equiv \lambda_i e_i, \quad \lambda_i = \pm \frac{1}{2},
\]  

(B.2)

where \( e_i \) are the orthonormal basis in five dimensions, \( c_\lambda \) is the cocycle factor and \( \phi^i \) are the free bosons obtained by the bosonization of \( \psi^M \) [13]. The OPE of these free bosons is given by

\[
\phi^i(z)\phi^j(w) \sim \delta^{ij} \ln(z - w).
\]  

(B.3)

The weight vector \( \lambda \) specifies the ten-dimensional 32 spinor components. After the GSO projection, the spin fields corresponding to the weight vectors \( \lambda \) that contain odd number of minus components survive. The ten-dimensional spin fields in the presence of the D3-branes are decomposed as

\[
S^\lambda \to (S_\alpha S_A, S^{\dot{\alpha}} S^{A}),
\]  

(B.4)

where \( \alpha, \dot{\alpha} = 1, 2 \) are the \( SO(4) \) indices and \( A = 1, 2, 3, 4 \) are the \( SO(6) \) spinor \( SU(4) \) (anti-)fundamental indices. The explicit relation between the four-dimensional spinor indices and the spin states is found in [13].
We also introduce the free boson field $\phi$ which is obtained by the bosonization of the superconformal ghost field and specifies the picture number of vertex operators $\{44\}$. The OPE is given by

$$\phi(z)\phi(w) \sim -\ln(z - w).$$

We consider disk amplitudes containing closed and open string vertex operators. A disk is realized as the upper half-plane parameterized by $z, \overline{z}$ and its boundary is the real axis parametrized by $y$. We employ the doubling trick where the right moving fields of the closed string vertex operators are located on the lower-half plane and the left and the right moving fields are identified on the boundary.

The open string vertex operators corresponding to the ADHM moduli, the auxiliary variables, the auxiliary fields $Y_{\alpha a}, X_{\alpha a}, \overline{X}_{\alpha a}$ and the VEVs of the scalar fields are summarized in Table 1. The zero modes of the D$(-1)$/D$(-1)$ open strings are in the adjoint representation of $U(k)$ while those associated with the D3/D$(-1)$ strings are in the bifundamental representations of $U(k)$ and $U(N)$. The $U(N)$ adjoint scalar VEVs are in the D3/D3 sector and are taken to be diagonal. The powers of $\alpha'$ in the vertex operators are determined such that the zero modes have the canonical dimensions. In order to reproduce the undeformed instanton effective action in the zero-slope limit, some of the zero modes should be rescaled by $g_0$ $\{32\}$. The fields $\Delta$ and $\overline{\Delta}$ are the twist fields which interchange the D3 and the D$(-1)$ boundaries $\{45, 46, 47\}$. The twist fields appear as a pair of $\Delta$ and $\overline{\Delta}$ in the non-zero amplitudes.

The vertex operators for the NS-NS B-field, the (S,A)- and the (A,S)-type backgrounds are given in Table 2. As mentioned in Section 3, we consider the scaling such that $(2\pi\alpha')^{\frac{1}{2}}F$ is finite in the zero-slope limit. Here $F$ is the component of the (S,A)- and the (A,S)-type backgrounds.

The disk amplitude which contains $n_o$ open string vertex operators $V^{(q_i)}_{\Psi_i}(y_i)$ and $n_c$ closed string vertex operators $V^{(r_i, \tilde{r}_i)}_{C_i}$ is given by

$$\langle \langle V^{(q_1)}_{\Psi_1}(y_1) \cdots V^{(r_1, \tilde{r}_1)}_{C_1} \cdots \rangle \rangle = C_0 \int \prod_{i=1}^{n_o} dy_i \prod_{j=1}^{n_c} dz_j d\bar{z}_j \langle V^{(q_1)}_{\Psi_1}(y_1) \cdots V^{(r_1, \tilde{r}_1)}_{C_1}(z_1, \bar{z}_1) \cdots \rangle.$$  

(B.6)
| Brane sectors | Vertex operators | Zero modes |
|---------------|------------------|------------|
| D(-1)/D(-1)  | $V_a^{(-1)}(y) = \pi(2\pi\alpha')^\frac{\tau}{2} g_0 \alpha_m^a \psi^m e^{-\phi}(y)/\sqrt{2}$ | ADHM moduli |
|              | $V_A^{(-\frac{1}{2})}(y) = \pi(2\pi\alpha')^\frac{\tau}{2} g_0 M_{\alpha A}^a S_A^\alpha e^{-\frac{1}{2}\phi}(y)$ | Auxiliary variables |
|              | $V_\chi^{(-1)}(y) = (2\pi\alpha')^{\frac{\tau}{2}} \chi_a \psi^{a+4} e^{-\phi}(y)/\sqrt{2}$ | Auxiliary fields |
|              | $V_{\phi}^{(-\frac{1}{2})}(y) = 2(2\pi\alpha')^\frac{\tau}{2} \dot{\psi}_a^A S_A^\alpha e^{-\frac{1}{2}\phi}(y)$ | Scalar VEVs |
|              | $V_D^{(0)}(y) = 2(2\pi\alpha') D \bar{\psi}_{\mu m} \psi^m \psi^\mu(y)$ | |
| D3/D(-1)     | $V_w^{(-1)}(y) = \pi(2\pi\alpha')^\frac{\tau}{2} g_0 w_a \Delta S_A^\alpha e^{-\phi}(y)/2$ | ADHM moduli |
|              | $V_{\tilde{w}}^{(-1)}(y) = \pi(2\pi\alpha')^\frac{\tau}{2} g_0 \tilde{w}_a \Delta S_A^\alpha e^{-\phi}(y)/2$ | |
|              | $V_\mu^{(-\frac{1}{2})}(y) = (2\pi\alpha')^\frac{\tau}{2} g_0 \mu A \Delta S_A e^{-\frac{1}{2}\phi}(y)$ | |
|              | $V_{\tilde{\mu}}^{(-\frac{1}{2})}(y) = (2\pi\alpha')^\frac{\tau}{2} g_0 \tilde{\mu} A \Delta S_A e^{-\frac{1}{2}\phi}(y)$ | |
|              | $V_X^{(0)} = 2\sqrt{2\pi}(2\pi\alpha') g_0 X_{\alpha A} \Delta S_A^\alpha \psi^{a+4}(y)$ | |
|              | $V_{\tilde{X}}^{(0)} = 2\sqrt{2\pi}(2\pi\alpha') g_0 \tilde{X}_{\alpha A} \Delta S_A^\alpha \psi^{a+4}(y)$ | |

Table 1: Vertex operators for the open string zero modes in the D-brane sectors. We denote the vertex operator for a field $\Psi$ in the $q$-picture by $V_q^{(\Psi)}$. We omit the normal ordering symbol.

Here $C_0$ is the disk normalization factor which is given by

$$C_0 = \frac{1}{2\pi^2 \alpha'^2 \kappa g_6^2}. \quad (B.7)$$

The factor $dV_{CKG}$ is the $SL(2, \mathbb{R})$-invariant volume element to fix three positions $x_1$, $x_2$ and $x_3$ among $y_i$, $z_j$ and $\tilde{z}_j$’s. This is given by

$$dV_{CKG} = \frac{dx_1 dx_2 dx_3}{(x_1 - x_2)(x_2 - x_3)(x_3 - x_1)}. \quad (B.8)$$

We fix one position of an open string vertex operator to $y_1 \to \infty$ and the other positions of a closed string vertex operator to $z = i$, $\tilde{z} = -i$. Note that in the disk amplitude $C_0$, the sum of the picture numbers must be $-2$.

In the following, we calculate the amplitudes $[3.5]-[3.7]$. Using the formula $[B.6]$, the
amplitude (3.3) is evaluated as

\[
\langle\langle V_Y^{(0)} V_{a'}^{(-1)} V_{F(-)}^{(-\frac{1}{2},-\frac{1}{2})}\rangle\rangle = \frac{1}{2\pi^2\alpha'^2} \frac{1}{\kappa g_0^2} (2\pi\alpha')^2 g_0^2 \left(\frac{4\pi^2}{\sqrt{2}}\right) \text{tr}_k \left[ Y_m a'_n (2\pi\alpha')^{\frac{3}{2}} F_{(\dot{a}\dot{b})[AB]} \right]
\times \int_{-\infty}^{y_1} dy_2 \left( y_1 - z \right) \left( y_1 - \bar{z} \right) (z - \bar{z})
\times \left( e^{-\phi(y_2)} e^{-\frac{1}{2}\phi(z)} e^{-\frac{1}{2}\phi(\bar{z})} \right)
\times \left( \psi^m \psi^{n+4} (y_1) \psi^a (y_2) S^\alpha (z) S^A (z) S^\beta (\bar{z}) S^B (\bar{z}) \right).
\] (B.9)

The correlators involving the world-sheet free fields are evaluated by the formulas in [43]. After fixing three positions of the vertices and performing the integration of \( y_2 \), we obtain

\[
\langle\langle V_Y^{(0)} V_{a'}^{(-1)} V_{F(-)}^{(-\frac{1}{2},-\frac{1}{2})}\rangle\rangle = -\frac{2\pi^2}{\kappa} \text{tr}_k \left[ 2Y_m a'_n C^{(-)mna} \right],
\] (B.10)

where the cocycle phase factor becomes 1 and \( C^{(-)mna} \) is the anti-self-dual part of the deformation parameter (3.13).

The amplitude (3.6) is given by

\[
\langle\langle V_X^{(0)} V_{\dot{a}}^{(-1)} V_{F(-)}^{(-\frac{1}{2},-\frac{1}{2})}\rangle\rangle = \frac{1}{2\pi^2\alpha'^2} \frac{1}{\kappa g_0^2} (2\sqrt{2}\pi) \frac{\pi}{2} \text{tr}_k \left[ X_{\dot{a}\dot{a}} \tilde{w}_{\dot{b}} (2\pi\alpha')^{\frac{3}{2}} F_{(\dot{a}\dot{b})[AB]} \right]
\times \int_{-\infty}^{y_1} dy_2 \left( y_1 - z \right) \left( y_1 - \bar{z} \right) (z - \bar{z}) \times \left( e^{-\phi(y_2)} e^{-\frac{1}{2}\phi(z)} e^{-\frac{1}{2}\phi(\bar{z})} \right)
\times \left( \Delta(y_1) \Delta(y_2) \right) \left( S^\alpha (y_1) S^\beta (y_2) S^i (z) S^j (\bar{z}) \right) \left( \psi^{a+4} (y_1) S^A (z) S^B (\bar{z}) \right).
\] (B.11)
Again, the correlators which contain the ghost and the spin fields are evaluated using the formulas in [43]. The twist field correlator is evaluated as [45, 46, 47]

$$\langle \Delta(y_1) \bar{\Delta}(y_2) \rangle = (y_1 - y_2)^{-\frac{1}{2}}. \quad (B.12)$$

Then the result is

$$\langle \langle V^{(0)}_X V^{(-1)}_w \bar{w} V^{(-1)}_{-\frac{1}{2},-\frac{1}{2}} F^{-1} \rangle \rangle = \frac{2\pi^2}{\kappa} \text{tr}_k \left[ \frac{1}{2} X_{\dot{a}\dot{a}} (\bar{\sigma}_{mn})^{\dot{a} \dot{b}} \bar{w}^\dot{b} C^{mna} \right], \quad (B.13)$$

where the cocycle phase factor $-i$ has been included.

The amplitude (3.7) is evaluated in the same manner. The result is

$$\langle \langle V^{(-1)}_w V^{(0)}_X \bar{w} V^{(-1)}_{-\frac{1}{2},-\frac{1}{2}} F^{-1} \rangle \rangle = \frac{2\pi^2}{\kappa} \text{tr}_k \left[ \frac{1}{2} w_{\dot{a}} (\bar{\sigma}_{mn})_{\dot{a} \dot{b}} \bar{X}^{\dot{b}a} C^{mna} \right]. \quad (B.14)$$

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