AN EXTENSION OF HYBRID METHOD WITHOUT EXTRAPOLATION STEP TO EQUILIBRIUM PROBLEMS

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Abstract. In this paper, we introduce a new hybrid algorithm for solving equilibrium problems. The algorithm combines the generalized gradient-like projection method and the hybrid (outer approximation) method. In this algorithm, only one optimization program is solved at each iteration without any extra-step dealing with the feasible set like as in the hybrid extragradient method and the hybrid Armijo linesearch method. A specially constructed half-space in the hybrid method is the reason for the absence of an optimization program in the proposed algorithm. The strongly convergent theorem is established and several numerical experiments are implemented to illustrate the convergence of the algorithm and compare it with others.

1. Introduction. The equilibrium problem (EP) [2] which was considered as the Ky Fan inequality [9] is very general in the sense that it includes, as special cases, many mathematical models such as: variational inequalities, fixed point problems, optimization problems, Nash equilibrium problems, complementarity problems, see [2, 8, 25] and the references therein. Many methods have been proposed for solving EPs, for instance [2, 7, 13, 14, 15, 16, 23, 25, 29]. The most solution approximations to EPs are often based on the resolvent of equilibrium bifunction (see, for instance [7]) in which a strongly monotone regularization equilibrium problem (REP) is solved at each iterative step. It is also called the proximal point method (PPM). This method was first introduced by Martinet [22] for variational inequalities, and then it was extended by Rockafellar [31] to the problem of finding a zero point of a monotone operator. In 2000, Konnov [18] further extended PPM to Ky Fan inequalities for monotone or weakly monotone bifunctions.

A special case of EP is the variational inequality problem (VIP). The simplest method for VIPs is the gradient projection method in which only one projection on the feasible set is computed. However, in order to obtain the convergence, the method requires the restrictive assumption that operators are strongly (or inverse strongly) monotone. To overcome this, Korpelevich [19] introduced the extragradient method (double projection method) where two metric projections onto the feasible set are implemented at each iteration. The convergence of the extragradient method was proved under the weaker assumption that operators are only monotone
The extragradient method has received a lot of attention by several authors and it has been modified in various ways, see [5, 6, 12, 21] and the references therein. For instance, the authors in [5] replaced the second projection onto the feasible set in the extragradient method by one onto a half-space and proposed the subgradient extragradient method for VIPs in Hilbert spaces.

The Korpelevich’s extragradient method has been naturally extended to EPs for monotone (or pseudomonotone) and Lipschitz-type continuous bifunctions and widely studied both theoretically and algorithmically [15, 27, 28, 29, 32, 33]. In the extended extragradient methods to EPs, we need to solve two strongly convex optimization programs on a closed convex constrained set (see, Algorithms 3 and 4 in Section 2). They are generalizations of two projections in Korpelevich’s extragradient method. The advantage of the extragradient method is that two optimization programs are solved at each iteration which seems to be numerically easier than the non-linear inequality (or REP) in PPM.

In this paper, motivated by the hybrid method without the extrapolation step [21] for variational inequalities, the extragradient method [29] and the hybrid method, we have proposed a new hybrid algorithm for solving EPs. In this algorithm, by constructing a specially cutting - halfspace in the hybrid method, we only need to solve a strongly convex optimization program onto the feasible set at each iteration. The absence of an optimization program in the proposed algorithm (compare with the extragradient method) can be considered as an improvement in each computational step of the results in [15, 20, 27, 28, 32, 33].

The remainder of the paper is organized as follows: Section 2 introduces a new algorithm and some related works. In Section 3, we collect some definitions and preliminary results used in the paper. Section 4 deals with proving the convergence of the proposed algorithm. Some applications of the algorithm to Gâteaux differentiable EPs and multivalued variational inequalities are presented in Section 5. Finally, in Section 6 we provide some numerical examples to illustrate the convergence of the proposed algorithm and compare it with others.

2. Algorithm and related works. Let $H$ be a real Hilbert space, $C$ be a non-empty closed convex subset of $H$ and $f : C \times C \to \mathbb{R}$ be a bifunction with $f(x, x) = 0$ for all $x \in C$. The equilibrium problem (EP) for the bifunction $f$ on $C$ is to find $x^* \in C$ such that
\[ f(x^*, y) \geq 0, \quad \forall y \in C. \tag{1} \]

The solution set of EP (1) is denoted by $EP(f, C)$. In this paper, we introduce the following hybrid algorithm for solving EP (1).

**Algorithm 1** (An extended hybrid algorithm without extrapolation step to EPs).
\[
\begin{aligned}
y_{n+1} &= \arg\min_{y \in C} \{\lambda f(y_n, y) + \frac{1}{2}||x_n - y||^2\}, \\
x_{n+1} &= P_{C_n \cap Q_n}(x_0),
\end{aligned}
\]

where $x_0 \in H$, $y_0 \in C$, $\lambda$ is a suitable parameter and $C_n, Q_n$ are two specially constructed half-spaces (see Algorithm 1 in Section 4 below).
In the special case, \( f(x, y) = \langle A(x), y - x \rangle \) where \( A : C \to H \) is a nonlinear operator then EP becomes the following variational inequality problem (VIP): Find \( x^* \in C \) such that
\[
\langle A(x^*), y - x^* \rangle \geq 0, \quad \forall y \in C.
\]
Then, the proposed algorithm (Algorithm 1) becomes the following hybrid algorithm without extrapolation step which was introduced in [21] for VIPs.

**Algorithm 2** (The hybrid algorithm without extrapolation step for VIPs).
\[
\begin{align*}
y_{n+1} &= PC (x_n - \lambda A(y_n)), \\
x_{n+1} &= PC \cap Q_n (x_0).
\end{align*}
\]

In 2008, Quoc et al. [29] extended the Korpelevich’s extragradient method [19] to EPs in Euclidean spaces in which two optimization programs are solved at each iteration. Recently, Nguyen et al. [28] also have done in that direction and proposed the general extragradient method which consists of solving three optimization programs on the feasible set. In Euclidean spaces, the convergence of the sequences generated by the extragradient methods \([28, 29]\) was proved under the assumptions of pseudomonotonicity and Lipschitz-type continuity of equilibrium bifunctions. The problem which arises in infinite dimensional Hilbert spaces is how to design an algorithm which provides the strong convergence. In 2012, Vuong et al. [33] used the extragradient method in [29] and the hybrid (outer approximation) method to obtain the following strongly convergent hybrid algorithm.

**Algorithm 3.**
\[
\begin{align*}
y_n &= \arg\min_{y \in C} \{ \lambda f(x_n, y) + \frac{1}{2} ||x_n - y||^2 \}, \\
z_n &= \arg\min_{y \in C} \{ \lambda f(y_n, y) + \frac{1}{2} ||x_n - y||^2 \}, \\
C_n &= \{ z \in C : ||z_n - z||^2 \leq ||x_n - z||^2 \}, \\
Q_n &= \{ z \in C : \langle x_0 - x_n, z - x_n \rangle \leq 0 \}, \\
x_{n+1} &= PC \cap Q_n (x_0),
\end{align*}
\]
where \( x_0 \in C \) and
\[
0 < \lambda < \min \left\{ \frac{1}{2c_1}, \frac{1}{2c_2} \right\}
\]
with \( c_1, c_2 \) being two Lipschitz-type constants of the bifunction \( f \) (see, Definition 3.1(iii) below). In 2013, another hybrid algorithm [27, Algorithm 1] was also proposed in this direction as

**Algorithm 4.**
\[
\begin{align*}
y_n &= \arg\min_{y \in C} \{ \lambda f(x_n, y) + \frac{1}{2} ||x_n - y||^2 \}, \\
z_n &= \arg\min_{y \in C} \{ \lambda f(y_n, y) + \frac{1}{2} ||x_n - y||^2 \}, \\
C_{n+1} &= \{ z \in C_n : ||z_n - z||^2 \leq ||x_n - z||^2 \}, \\
x_{n+1} &= PC_{n+1} (x_0),
\end{align*}
\]
where \( x_0 \in C, C_0 = C \) and \( \lambda \) satisfies condition (3). The authors in [27, 33] proved that the sequences \( \{x_n\} \) generated by Algorithms 3 and 4 converge strongly to \( P_{EP(f,C)} (x_0) \) under the hypotheses of the pseudomonotonicity and Lipschitz-type continuity of \( f \). Also in [27], in order to avoid the condition of the Lipschitz-type continuity of the bifunction \( f \), the authors replaced the second optimization problem.
in the extragradient method by the Armijo linesearch technique and obtained the following hybrid algorithm.

**Algorithm 5.**

\[
\begin{aligned}
&y_n = \arg\min_{y \in C}\{\lambda f(x_n, y) + \frac{1}{2\lambda}||x_n - y||^2\}, \\
m_n = \text{the smallest integer number such that} \\
&w_n = (1 - \eta^{m_n})x_n + \eta^{m_n}y_n, \\
f(w_n, x_n) - f(w_n, y_n) \geq \frac{\alpha}{2\lambda}||x_n - y_n||^2; \\
z_n = P_C(x_n - \sigma_ng_n), \\
C_{n+1} = \{z \in C : ||z_n - z||^2 \leq ||x_n - z||^2\}, \\
x_{n+1} = P_{C_{n+1}}(x_0), \quad n \geq 1,
\end{aligned}
\]

where \(x_0 \in H\), \(C_1 = C\), \(x_1 = PC_{\alpha}(x_0)\), \(\lambda \in (0, 1)\), \(\alpha \in (0, 2)\), \(\eta \in (0, 1)\), \(g_n \in \partial f_2(w_n, x_n)\) and \(\sigma_n = f(w_n, x_n)||g_n||^2\) if \(y_n \neq x_n\) and \(\sigma_n = 0\) otherwise. According to Algorithm 5, we still have to solve an optimization program on \(C\) for \(z_n\), find an optimization direction for \(w_n\), and compute a projection onto \(C\) for \(z_n\) at each step. We emphasize that the projection \(PC_{\alpha \cap Q_n}(x_0)\) in Algorithm 3 and the projection \(PC_{n+1}(x_0)\) in Algorithms 4 and 5 still deal with the constrained set \(C\) while the sets \(C_n\) and \(Q_n\) in Algorithm 1 are two half-spaces, and so \(x_{n+1}\) can be expressed by an explicit formula (see, for instance [7]).

3. **Preliminaries.** In this section, we recall some definitions and results for further use. Let \(C\) be a nonempty closed convex subset of a real Hilbert space \(H\). We begin with some concepts of the monotonicity of a bifunction (see [2, 25] for more details).

**Definition 3.1.** A bifunction \(f : C \times C \to \mathbb{R}\) is said to be

(i) monotone on \(C\) if

\[f(x, y) + f(y, x) \leq 0, \quad \forall x, y \in C;\]

(ii) pseudomonotone on \(C\) if

\[f(x, y) \geq 0 \implies f(y, x) \leq 0, \quad \forall x, y \in C;\]

(iii) Lipschitz-type continuous on \(C\) if there exist two positive constants \(c_1, c_2\) such that

\[f(x, y) + f(y, z) \geq f(x, z) - c_1||x - y||^2 - c_2||y - z||^2, \quad \forall x, y, z \in C.\]

From the definitions above, it is clear that a monotone bifunction is pseudomonotone, i.e., (i) \(\implies\) (ii). For solving EP (1), we assume that the bifunction \(f\) satisfies the following conditions:

(A1) \(f\) is pseudomonotone on \(C\) and \(f(x, x) = 0\) for all \(x \in C\);

(A2) \(f\) is Lipschitz-type continuous on \(C\) with two constants \(c_1, c_2\);

(A3) \(\limsup_{n \to \infty} f(x_n, y) \leq f(\bar{x}, y)\) for each sequence \(\{x_n\} \subset C\) converging weakly to \(\bar{x}\) and every fixed \(y \in C\);

(A4) \(f(x, .)\) is convex and subdifferentiable on \(C\) for every fixed \(x \in C\).

It is easy to show that under assumptions (A1) – (A4), the solution set \(EP(f, C)\) of EP (1) is closed and convex (see, for instance [29]). In this paper, we assume that the solution set \(EP(f, C)\) is nonempty.

The metric projection \(PC : H \to C\) is defined by \(PCx = \arg\min_{y \in C}\{||y - x|| : y \in C\}\). Since \(C\) is nonempty, closed and convex, \(PCx\) exists and is unique. It is also known that \(PC\) has the following characteristic properties, see [10] for more details.
Lemma 3.2. Let $P_C : H \to C$ be the metric projection from $H$ onto $C$. Then

(i) $P_C$ is firmly nonexpansive, i.e.,

$$\langle P_C x - P_C y, x - y \rangle \geq \| P_C x - P_C y \|^2, \quad \forall x, y \in H.$$ 

(ii) For all $x \in C, y \in H$,

$$\| x - P_C y \|^2 + \| P_C y - y \|^2 \leq \| x - y \|^2. \quad (4)$$

(iii) $z = P_C x$ if and only if

$$\langle x - z, z - y \rangle \geq 0, \quad \forall y \in C. \quad (5)$$

Let $g : C \to \mathbb{R}$ be a function. The subdifferential of $g$ at $x$ is defined by

$$\partial g(x) = \{ w \in H : g(y) - g(x) \geq \langle w, y - x \rangle, \quad \forall y \in C \}.$$ 

We recall that the normal cone of $C$ at $x \in C$ is defined by

$$N_C(x) = \{ w \in H : \langle w, y - x \rangle \leq 0, \quad \forall y \in C \}.$$ 

Definition 3.3 (Weakly lower semicontinuity). A function $\varphi : H \to \mathbb{R}$ is called weakly lower semicontinuous at $x \in H$ if for any sequence $\{ x_n \}$ in $H$ converging weakly to $x$ then

$$\varphi(x) \leq \liminf_{n \to \infty} \varphi(x_n).$$ 

It is well-known that the functional $\varphi(x) := \| x \|^2$ is convex and weakly lower semicontinuous. Any Hilbert space has the Kadec-Klee property (see, for instance [11]), i.e., if $\{ x_n \}$ is a sequence in $H$ such that $x_n \rightharpoonup x$ and $\| x_n \| \to \| x \|$ then $x_n \to x$ as $n \to \infty$.

Finally, we have the following technical lemma.

Lemma 3.4. [21] Let $\{ \alpha_n \}, \{ \beta_n \}, \{ \gamma_n \}$ be nonnegative real sequences, $a, b \in \mathbb{R}$ and for all $n \geq 0$, the following inequality holds

$$\alpha_n \leq \beta_n + b \gamma_n - a \gamma_{n+1}.$$ 

If $\sum_{n=0}^{\infty} \beta_n < +\infty$ and $a > b \geq 0$ then $\lim_{n \to \infty} \alpha_n = 0$.

4. Convergence analysis. In this section, we present the detailed algorithm and prove the convergence of it.

Algorithm 1. (An extended hybrid algorithm without extrapolation step)

Initialization. Choose $x_0 = x_1 \in H$, $y_0 \in C$ and set $C_0 = Q_0 = H$. The parameters $\lambda$ and $k$ satisfy the following conditions:

$$0 < \lambda < \frac{1}{2(c_1 + c_2)}, \quad k > \frac{1}{1 - 2\lambda(c_1 + c_2)}.$$ 

Compute $y_1$ by

$$y_1 = \arg\min_{y \in C} \{ \lambda f(y_0, y) + \frac{1}{2} \| x_0 - y \|^2 \}.$$ 

Step 1. Solve a strongly convex program

$$y_{n+1} = \arg\min_{y \in C} \{ \lambda f(y_n, y) + \frac{1}{2} \| x_n - y \|^2 \}, \quad n \geq 1.$$
If \( y_{n+1} = y_n = x_n \) then stop.

**Step 2.** Compute \( x_{n+1} = P_{C_n \cap Q_n}(x_0) \), where
\[
C_n = \{ z \in H : ||y_{n+1} - z||^2 \leq ||x_n - z||^2 + \epsilon_n \},
\]
\[
Q_n = \{ z \in H : \langle x_0 - x_n, z - x_n \rangle \leq 0 \},
\]
and \( \epsilon_n = k\|x_n - x_{n-1}\|^2 + 2\lambda c_1\|y_n - y_{n-1}\|^2 - (1 - \frac{1}{k} - 2\lambda c_2)||y_{n+1} - y_n||^2 \). Set \( n := n + 1 \) and go back **Step 1**.

We have the following result which gives us a stopping criterion of Algorithm 1.

**Lemma 4.1.** If Algorithm 1 finishes at some iteration step \( n < \infty \), then \( x_n \in EP(f, C) \).

**Proof.** Assume that \( y_{n+1} = y_n = x_n \). From the definition of \( y_{n+1} \),
\[
x_n = \arg \min \left\{ \lambda_n f(x_n, y) + \frac{1}{2}\|x_n - y\|^2 : y \in C \right\}.
\]
Thus, from [23, Proposition 2.1], one has \( x_n \in EP(f, C) \). The proof of Lemma 4.1 is complete.

We need the lemma below which is an infinite version of Theorem 27.4 in [31] and is similarly proved by using Moreau-Rockafellar Theorem to find the subdifferential of a sum of a convex function \( g \) and the indicator function \( \delta_C \) to \( C \) in a real Hilbert space.

**Lemma 4.2.** Let \( C \) be a nonempty convex subset of a real Hilbert space \( H \) and \( g : C \to \mathbb{R} \) be a convex, subdifferentiable and lower semicontinuous function on \( C \). Then, \( x^* \) is a solution to the following convex optimization problem
\[
\min \{ g(x) : x \in C \}
\]
if and only if \( 0 \in \partial g(x^*) + N_C(x^*) \), where \( \partial g(.) \) denotes the subdifferential of \( g \) and \( N_C(x^*) \) is the normal cone of \( C \) at \( x^* \).

Based on Lemma 4.2 and Lemma 6 in [21], we obtain the following central lemma which is used to prove the convergence of Algorithm 1.

**Lemma 4.3.** Assume that \( x^* \in EP(f, C) \). Let \( \{x_n\}, \{y_n\} \) be the sequences generated by Algorithm 1. Then, there holds the relation
\[
||y_{n+1} - x^*||^2 \leq ||x_n - x^*||^2 + \epsilon_n,
\]
where \( \epsilon_n \) is defined by **Step 2** of Algorithm 1.

**Proof.** From the definition of \( y_{n+1} \) and Lemma 4.2,
\[
0 \in \partial_2 \left( \lambda f(y_n, y) + \frac{1}{2}\|x_n - y\|^2 \right)(y_{n+1}) + N_C(y_{n+1}).
\]
Thus, there exist \( w \in \partial_2 f(y_n, y_{n+1}) := \partial f(y_n, \cdot)(y_{n+1}) \) and \( \bar{w} \in N_C(y_{n+1}) \) such that
\[
\lambda w + y_{n+1} - x_n + \bar{w} = 0.
\]
Hence,
\[
\langle y_{n+1} - x_n, y - y_{n+1} \rangle = \lambda \langle w, y_{n+1} - y \rangle + \langle \bar{w}, y_{n+1} - y \rangle, \forall y \in C.
\]
This together with the definition of \( N_C \) implies that
\[
\langle y_{n+1} - x_n, y - y_{n+1} \rangle \geq \lambda \langle w, y_{n+1} - y \rangle, \forall y \in C.
\]
By \( w \in \partial_2 f(y_n, y_{n+1}) \),
\[
f(y_n, y) - f(y_n, y_{n+1}) \geq \langle w, y - y_{n+1} \rangle, \quad \forall y \in C.
\]
From the last two inequalities, we obtain
\[
\langle y_{n+1} - x_n, y - y_{n+1} \rangle \geq \lambda (f(y_n, y_{n+1}) - f(y_n, y)), \quad \forall y \in C. \tag{6}
\]
Similarly, by replacing \( n + 1 \) by \( n \), we also have
\[
\langle y_n - x_{n-1}, y - y_n \rangle \geq \lambda (f(y_{n-1}, y) - f(y_{n-1}, y)), \quad \forall y \in C. \tag{7}
\]
Substituting \( y = y_{n+1} \) into (7) and a straightforward computation yield
\[
\lambda (f(y_{n-1}, y_{n+1}) - f(y_{n-1}, y_n)) \geq \langle y_n - x_{n-1}, y_n - y_{n+1} \rangle. \tag{8}
\]
Substituting \( y = x^* \) into (6) we also obtain
\[
\langle y_{n+1} - x_n, x^* - y_{n+1} \rangle \geq \lambda (f(y_n, y_{n+1}) - f(y_n, x^*)). \tag{9}
\]
Since \( x^* \in EP(f, C) \) and \( y_n \in C \), \( f(x^*, y_n) \geq 0 \). Thus, from the pseudomonotonicity of \( f \) one has \( f(y_n, x^*) \leq 0 \). This together with (9) implies that
\[
\langle y_{n+1} - x_n, x^* - y_{n+1} \rangle \geq \lambda f(y_n, y_{n+1}). \tag{10}
\]
By the Lipschitz-type continuity of \( f \),
\[
f(y_{n-1}, y_n) + f(y_n, y_{n+1}) \geq f(y_{n-1}, y_{n+1}) - c_1 \|y_{n-1} - y_n\|^2 - c_2 \|y_n - y_{n+1}\|^2.
\]
Thus,
\[
f(y_n, y_{n+1}) \geq f(y_{n-1}, y_{n+1}) - f(y_{n-1}, y_n) - c_1 \|y_{n-1} - y_n\|^2 - c_2 \|y_n - y_{n+1}\|^2. \tag{11}
\]
Relations (10) and (11) lead to
\[
\langle y_{n+1} - x_n, x^* - y_{n+1} \rangle \geq \lambda \{ f(y_{n-1}, y_{n+1}) - f(y_{n-1}, y_n) \}
\]
\[
- \lambda c_1 \|y_{n-1} - y_n\|^2 - \lambda c_2 \|y_n - y_{n+1}\|^2.
\]
This together with relation (8) implies that
\[
\langle y_{n+1} - x_n, x^* - y_{n+1} \rangle \geq \langle y_n - x_{n-1}, y_n - y_{n+1} \rangle - \lambda c_1 \|y_{n-1} - y_n\|^2
\]
\[
- \lambda c_2 \|y_n - y_{n+1}\|^2.
\]
Thus,
\[
2 \langle y_{n+1} - x_n, x^* - y_{n+1} \rangle - 2 \langle y_n - x_{n-1}, y_n - y_{n+1} \rangle \geq -2 \lambda c_1 \|y_{n-1} - y_n\|^2
\]
\[
-2 \lambda c_2 \|y_n - y_{n+1}\|^2. \tag{12}
\]
We have the following fact:
\[
2 \langle y_{n+1} - x_n, x^* - y_{n+1} \rangle = \|x_n - x^*\|^2 - \|y_{n+1} - x^*\|^2 - \|x_n - y_{n+1}\|^2
\]
\[
= \|x_n - x^*\|^2 - \|y_{n+1} - x^*\|^2 - \|x_n - x_{n-1}\|^2 - 2 \langle x_n - x_{n-1}, x_{n-1} - y_{n+1} \rangle
\]
\[
- \|x_{n-1} - y_{n+1}\|^2
\]
\[
= \|x_n - x^*\|^2 - \|y_{n+1} - x^*\|^2 - \|x_n - x_{n-1}\|^2 - 2 \langle x_n - x_{n-1}, x_{n-1} - y_{n+1} \rangle
\]
\[
- \|x_{n-1} - y_{n+1}\|^2 - 2 \langle x_{n-1} - y_n, y_n - y_{n+1} \rangle - \|y_n - y_{n+1}\|^2. \tag{13}
\]
By the triangle, Cauchy-Schwarz and Cauchy inequalities,
\[
-2 \langle x_n - x_{n-1}, x_{n-1} - y_{n+1} \rangle \leq 2 \|x_n - x_{n-1}\| \|x_{n-1} - y_{n+1}\|
\]
\[
\leq 2 \|x_n - x_{n-1}\| \|x_{n-1} - y_n\| + 2 \|x_n - x_{n-1}\| \|y_n - y_{n+1}\|
\]
\[
\leq \|x_n - x_{n-1}\|^2 + \|x_{n-1} - y_n\|^2 + 2 \|x_n - x_{n-1}\|^2 + \frac{1}{k} \|y_n - y_{n+1}\|^2.
\]
This together with (13) implies that
\[2 \langle y_{n+1} - x_n, x^* - y_{n+1} \rangle \leq ||x_n - x^*||^2 - ||y_{n+1} - x^*||^2 + k||x_n - x_{n-1}||^2 \]
\[+ 2 \langle y_n - x_{n-1}, y_n - y_{n+1} \rangle + \left( \frac{1}{k} - 1 \right) ||y_n - y_{n+1}||^2.\]

Thus,
\[2 \langle y_{n+1} - x_n, x^* - y_{n+1} \rangle - 2 \langle y_n - x_{n-1}, y_n - y_{n+1} \rangle \leq ||x_n - x^*||^2 - ||y_{n+1} - x^*||^2 \]
\[- ||y_n - x^*||^2 + k||x_n - x_{n-1}||^2 + \left( \frac{1}{k} - 1 \right) ||y_n - y_{n+1}||^2.\]

Combining (12) and (14) we obtain
\[-2\lambda_1 ||y_n - y_n||^2 - 2\lambda_2 ||y_n - y_{n+1}||^2 \leq ||x_n - x^*||^2 - ||y_{n+1} - x^*||^2 \]
\[+ k||x_n - x_{n-1}||^2 + \left( \frac{1}{k} - 1 \right) ||y_n - y_{n+1}||^2.\]

Thus, from the definition of \(\epsilon_n\) we obtain
\[||y_{n+1} - x^*||^2 \leq ||x_n - x^*||^2 + k||x_n - x_{n-1}||^2 + 2\lambda_1 ||y_{n-1} - y_n||^2 \]
\[- \left( 1 - \frac{1}{k} - 2\lambda_2 \right) ||y_n - y_{n+1}||^2.\]

\[= ||x_n - x^*||^2 + \epsilon_n.\]

Lemma 4.3 is proved.

Lemma 4.4. Let \(\{x_n\}, \{y_n\}\) be the sequences generated by Algorithm 1. Then, there hold the following relations

(i) \(EP(f, C) \subset C_n \cap Q_n\) for all \(n \geq 0\).

(ii) The sequences \(\{x_n\}\) and \(\{y_n\}\) are bounded and
\[\lim_{n \to \infty} ||x_{n+1} - x_n|| = \lim_{n \to \infty} ||y_{n+1} - y_n|| = \lim_{n \to \infty} ||y_{n+1} - y_n|| = \lim_{n \to \infty} ||x_n - x_{n-1}|| = 0.\]

Proof. (i). From the definitions of \(C_n\) and \(Q_n\), we see that they are the half-spaces. Thus, \(C_n\) and \(Q_n\) are closed and convex for all \(n \geq 0\). Lemma 4.3 and the definition of \(C_n\) ensure that \(EP(f, C) \subset C_n\) for all \(n \geq 0\). It is clear that \(EP(f, C) \subset C_0 \cap Q_0\). Assume that \(EP(f, C) \subset C_n \cap Q_n\) for some \(n \geq 0\). From \(x_{n+1} = P_{C_n \cap Q_n}(x_n)\) and Lemma 3.2(iii) we see that \(\langle z - x_{n+1}, x_0 - x_{n+1} \rangle \leq 0\) for all \(z \in C_n \cap Q_n\). This is also true for all \(z \in EP(f, C)\) because \(EP(f, C) \subset C_n \cap Q_n\). From the definition of \(Q_{n+1}\), \(EP(f, C) \subset Q_{n+1}\). By the induction, \(EP(f, C) \subset C_n \cap Q_n\) for all \(n \geq 0\). Since \(EP(f, C)\) is nonempty, so \(C_n \cap Q_n\) is. Thus, \(P_{C_n \cap Q_n}(x_0)\) is well-defined.

(ii). From the definition of \(Q_n\) and Lemma 3.2(iii), \(x_n = P_{Q_n}(x_0)\). Thus, from Lemma 3.2(ii) we have
\[||z - x_n||^2 \leq ||z - x_0||^2 - ||x_n - x_0||^2, \forall z \in Q_n.\]  \hspace{1cm} (15)

Substituting \(z = x^\dagger := P_{EP(f,C)}(x_0) \in Q_n\) into (15), one has
\[||x^\dagger - x_0||^2 - ||x_n - x_0||^2 \geq ||x^\dagger - x_n||^2 \geq 0.\]  \hspace{1cm} (16)

Thus, the sequence \(\{|x_n - x_0|\}\), therefore \(\{x_n\}\), are bounded. Substituting \(z = x_{n+1} \in Q_n\) into (15), one also has
\[0 \leq ||x_{n+1} - x_n||^2 \leq ||x_{n+1} - x_0||^2 - ||x_n - x_0||^2.\]  \hspace{1cm} (17)
This implies that \( \{ ||x_n - x_0|| \} \) is non-decreasing. Hence, there exists the limit of \( \{ ||x_n - x_0|| \} \). By (17),
\[
\sum_{n=1}^{K} ||x_{n+1} - x_n||^2 \leq ||x_{K+1} - x_0||^2 - ||x_1 - x_0||^2, \quad \forall K \geq 1.
\]
Passing to the limit in the last inequality as \( K \to \infty \), we obtain
\[
\sum_{n=1}^{\infty} ||x_{n+1} - x_n||^2 < +\infty.
\]
Thus,
\[
\lim_{n \to \infty} ||x_{n+1} - x_n|| = 0.
\]
From the definition of \( C_n \) and \( x_{n+1} \in C_n \),
\[
||y_{n+1} - x_{n+1}||^2 \leq ||x_n - x_{n+1}||^2 + \epsilon_n.
\]
Set \( \alpha_n = ||y_{n+1} - x_{n+1}||^2, \beta_n = ||x_n - x_{n+1}||^2 + k||x_n - x_{n-1}||^2, \gamma_n = ||y_n - y_{n-1}||^2 \),
\( b = 2\lambda c_1 \), and \( a = 1 - \frac{1}{2} - 2\lambda c_2 \). From the definition of \( \epsilon_n \), \( \epsilon_n = k||x_n - x_{n-1}||^2 + b\gamma_n - a\gamma_{n+1} \). Thus, from (20),
\[
\alpha_n \leq \beta_n + b\gamma_n - a\gamma_{n+1}.
\]
From the hypotheses of \( \lambda \), \( k \) and (18), we see that \( a \geq b \geq 0 \) and \( \sum_{n=1}^{\infty} \beta_n < +\infty \).
Lemma 3.4 and (21) imply that \( \alpha_n \to 0 \), or
\[
\lim_{n \to \infty} ||y_{n+1} - x_{n+1}|| = 0.
\]
This together with relation (19) and the inequality \( ||y_{n+1} - y_n|| \leq ||y_{n+1} - x_{n+1}|| + ||x_{n+1} - x_n|| + ||x_n - y_n|| \) implies that
\[
\lim_{n \to \infty} ||y_{n+1} - y_n|| = 0.
\]
Similarly, from (19), (22) and the triangle inequality, we also obtain \( ||x_n - y_{n+1}|| \to 0 \). Finally, the sequence \( \{ y_n \} \) is bounded because of the boundedness of \( \{ x_n \} \). Lemma 4.4 is proved.

Thanks to Lemma 4.1, we see that if Algorithm 1 terminates at some iterate \( n \) then a solution of EP can be found. Otherwise, if Algorithm 1 does not terminate then we have the following main result.

**Theorem 4.5.** Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \). Assume that the bifunction \( f \) satisfies all conditions (A1) – (A4). In addition the solution set \( EP(f,C) \) is nonempty. Then, the sequences \( \{ x_n \}, \{ y_n \} \) generated by Algorithm 1 converge strongly to \( P_{EP(f,C)}(x_0) \).

**Proof.** From Lemma 4.4, the sequence \( \{ x_n \} \) is bounded. Assume that \( p \) is any weak cluster point of \( \{ x_n \} \). Without loss of generality, we can write \( x_n \to p \) as \( n \to \infty \). Thus, \( y_n \to p \) because \( ||x_n - y_n|| \to 0 \). Since \( C \) is closed and convex in the Hilbert space \( H \), \( C \) is weakly closed. Thus \( p \in C \) because of \( \{ y_n \} \subset C \). Now, we show that \( p \in EP(f,C) \). From (6), we get
\[
\lambda f(y_n, y) \geq \lambda f(y_n, y_{n+1}) + \langle x_n - y_{n+1}, y - y_{n+1} \rangle, \quad \forall y \in C.
\]
From relations (8) and (11),
\[
\lambda f(y_n, y_{n+1}) \geq \langle y_n - x_{n-1}, y_{n+1} - y_{n} \rangle - \lambda c_1 ||y_{n+1} - y_{n}||^2 - \lambda c_2 ||y_n - y_{n+1}||^2.
\]
Combining (23) and (24), we obtain
\[
\lambda f(y_n, y) \geq \left( y_n - x_{n-1}, y_n - y_{n+1} \right) - \lambda c_1 \left\| y_{n-1} - y_n \right\|^2 - \lambda c_2 \left\| y_n - y_{n+1} \right\|^2
\]
\[+ \left( x_n - y_{n+1}, y - y_{n+1} \right). \]

Passing to the limit in the last inequality as \( n \to \infty \) and using Lemma 4.4(ii), the boundedness of \( \{y_n\} \), \( \lambda > 0 \) and (A3), we obtain
\[
f(p, y) \geq \lim_{n \to \infty} \sup_{y_n, y} f(y_n, y) \geq 0, \quad \forall y \in C.
\]

Thus, \( p \in EP(f, C) \). Finally, from inequality (16), we get
\[
||x_n - x_0|| \leq ||x^\dagger - x_0||,
\]
where \( x^\dagger = P_{EP(f,C)}(x_0) \). By the weakly lower semicontinuity of the norm ||.,|| and \( x_n \to p \), we have
\[
||p - x_0|| \leq \lim_{n \to \infty} \inf ||x_n - x_0|| \leq \lim_{n \to \infty} \sup ||x_n - x_0|| \leq ||x^\dagger - x_0||.
\]

By the definition of \( x^\dagger \), \( p = x^\dagger \) and \( \lim_{n \to \infty} ||x_n - x_0|| = ||x^\dagger - x_0|| \). By \( x_n - x_0 \to x^\dagger - x_0 \) and the Kadec-Klee property of the Hilbert space \( H \), we obtain \( x_n - x_0 \to x^\dagger - x_0 \). Thus, \( x_n \to x^\dagger = P_{EP(f,C)}x_0 \) as \( n \to \infty \). From Lemma 4.4(ii), we also see that \( \{y_n\} \) converges strongly to \( P_{EP(f,C)}x_0 \). Theorem 4.5 is proved. \( \Box \)

5. Applications. In this section, we introduce several applications of Algorithm 1 to Gâteaux differentiable EPs and multivalued variational inequalities.

5.1. Gâteaux differentiable equilibrium problems. We consider EPs for Gâteaux differentiable bifunctions. We denote \( \nabla_2 f(x, y) \) by the Gâteaux derivative of the function \( f(x,.) \) at \( y \). For solving EP (1), we assume that the bifunction \( f \) satisfies the following conditions:

(B1). \( f \) is monotone on \( C \) and \( f(x, x) = 0 \) for all \( x \in C \);
(B2). \( f(x,.) \) is convex and Gâteaux differentiable on \( C \);
(B3). There exists a constant \( L > 0 \) such that
\[
||\nabla_2 f(x, x) - \nabla_2 f(y, y)|| \leq L||x - y|| \quad \forall x, y \in C;
\]
(B4). \( \lim_{t \to 0^+} \sup f(x + t(z - x), y) \leq f(x, y) \) for all \( x, y \in C \).

Remark 1. If EP (1) is reduced to VIP (2) for the operator \( A : C \to H \) then the condition (B3) is equivalent to the Lipschitzianity of \( A \) with the constant \( L > 0 \).

We need the following result.

Lemma 5.1. [20, Lemma 2] Suppose that conditions (B1), (B2), (B4) hold. Then,

(i) The operator \( A(x) = \nabla_2 f(x, x) \) is monotone on \( C \).
(ii) \( EP(f, C) = VI(A, C) \).

Thanks to Lemma 5.1, instead of EP (1) we solve VIP (2) for the operator \( A(x) = \nabla_2 f(x, x) \) onto \( C \). It is emphasized that (B2) and (B3) are slightly strong conditions. However, in this case, we can use the existing methods for VIPs to solve
we obtain the following hybrid algorithm for solving EP (1) we first use the subgradient extragradient method [6, Algorithm 3.6] to obtain
\[ \begin{align*}
    y_n &= P_C(x_n - \lambda \nabla f(x_n)), \\
    z_n &= \alpha_n x_n + (1 - \alpha_n) P_{T_n}(x_n - \lambda \nabla f(y_n)), \\
    C_n &= \{ z \in H : \| z_n - z \| \leq \| x_n - z \| \}, \\
    Q_n &= \{ z \in H : \langle x_n - x_n, z - x_n \rangle \leq 0 \}, \\
    x_{n+1} &= P_{C_n \cap Q_n}(x_n),
\end{align*} \tag{25}
\]
where \( T_n = \{ z \in H : \langle x_n - \lambda \nabla f(x_n), z - y_n \rangle \leq 0 \} \). If conditions \((B1) - (B4)\) hold for all \( x, y \in H \) then \( \{ x_n \} \) generated by \((25)\) converges strongly to \( P_{EP(f,C)}(x_0) \).

In this subsection, using Algorithm 2 which is a special case of Algorithm 1 for the operator \( A(x) = \nabla f(x, x) \), we come to the following result.

**Theorem 5.2.** Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \). Assume that the bifunction \( f \) satisfies all conditions \((B1) - (B4)\) such that \( EP(f,C) \) is nonempty. Let \( \{ x_n \} \) be the sequence generated by the following manner:
\[ \begin{align*}
    y_n &= P_C(x_n - \lambda \nabla f(y_n)), \\
    C_n &= \{ z \in H : \| y_{n+1} - z \| \leq \| x_{n+1} - z \| + \epsilon_n \}, \\
    Q_n &= \{ z \in H : \langle x_n - x_n, z - x_n \rangle \leq 0 \}, \\
    x_{n+1} &= P_{C_n \cap Q_n}(x_n), \quad n \geq 1,
\end{align*} \]
where \( \epsilon_n, \lambda, k \) are defined as in Algorithm 1 with \( c_1 = c_2 = L/2 \). Then, the sequence \( \{ x_n \} \) converges strongly to \( P_{EP(f,C)}(x_0) \).

### 5.2. Multivalued variational inequalities

In this subsection, we consider the following multivalued variational inequality problem (MVIP)
\[ \begin{align*}
    \text{Find } x^* \in C \text{ and } v^* \in A(x^*) \\
    \text{such that } \langle v^*, y - x^* \rangle \geq 0, \quad \forall y \in C,
\end{align*} \tag{26}
\]
where \( A : C \to 2^H \) is a multivalued compact operator. For each pair \( x, y \in C \), we put
\[ f(x, y) = \sup_{u \in A(x)} \langle u, y - x \rangle. \tag{27} \]
It is easy to show that \( x^* \) is a solution of MVIP \((26)\) if and only if \( x^* \) is a solution of EP for the bifunction \( f \) defined by \((27)\). We recall the following definitions.

**Definition 5.3.** A multivalued operator \( A : C \to 2^H \) is said to be:
\( \text{(i) monotone on } C \) if
\[ \langle u - v, x - y \rangle \geq 0 \quad \forall x, y \in C, \quad \forall u \in A(x), \quad \forall v \in A(y); \]
\( \text{(ii) pseudomonotone on } C \) if
\[ \langle u, x - y \rangle \geq 0 \implies \langle v, y - x \rangle \leq 0 \quad \forall x, y \in C, \quad \forall u \in A(x), \quad \forall v \in A(y); \]
\( \text{(iii) } L \text{- Lipschitz continuous if there exists a positive constant } L \text{ such that} \)
\[ \sup_{u \in A(x)} \inf_{v \in A(y)} \| u - v \| \leq L \| x - y \| \quad \forall x, y \in C. \]
Remark 2. If we denote $h(C_1, C_2)$ by the Hausdorff distance between two sets $C_1$ and $C_2$ then definition (iii) means that

$$h(A(x), A(y)) \leq L\|x - y\| \quad \forall x, y \in C.$$  

We can easily check that if $A$ is pseudomonotone and $L$-Lipschitz continuous then $f$ is also pseudomonotone and Lipschitz-type continuous with two constants $c_1 = c_2 = L/2$. Note that, when $A$ is singlevalued then Algorithm 1 becomes the hybrid algorithm without the extrapolation step for variational inequalities [21]. When $A$ is multivalued then Algorithm 1 can be applied for the bifunction $f$ defined by (27). A disadvantage of performing Algorithm 1 in this case is that it is not easy to choose an approximation of the bifunction $f(x, y)$. In fact, if $A$ is a monotone and $L$-Lipschitz continuous multi-valued operator, by repeating the proof of Theorem 1 in [21] with $A(y_n)$ being replaced by $u_n$ and using Definition 5.3(ii) and (iii), we can obtain the strong convergence of the following algorithm

$$
\begin{align*}
&x_0 = x_1 \in H, \ y_0 \in C, \\
y_{n+1} = P_C(x_n - \lambda u_n), \ u_n \in A(y_n), \\
C_n = \{z \in H : \|y_{n+1} - z\|^2 \leq \|x_n - z\|^2 + \epsilon_n\}, \\
Q_n = \{z \in H : \langle x_0 - x_n, z - x_n \rangle \leq 0\}, \\
x_{n+1} = P_{C_n \cap Q_n}(x_0),
\end{align*}
$$  

where $\epsilon_n, \lambda, k$ are defined as in Algorithm 1 with $c_1 = c_2 = L/2$.

6. Numerical examples. In this section, we consider some numerical examples in Euclidean space $\mathbb{R}^m$. The purpose of these experiments is to illustrate the convergence of Algorithm 1 and compare the efficiency of it with Algorithms 3, 4 and 5. The advantage of the proposed algorithm is a computation over each iteration. Of course, there are many mathematical models for EPs in infinite dimensional Hilbert spaces, see for instance [2], and the norm convergence of algorithms is more necessary than the weak convergence. The ability of the implementation of these algorithms has been discussed in Sections 1 and 2. Firstly, we present briefly the problems of constructing the sets $C_n$, $Q_n$ and computing the projections $P_{C_n+1}(x_0)$ and $P_{C_n \cap Q_n}(x_0)$ for Algorithms 3, 4 and 5 in the first four examples when $C$ is given as a polyhedron. Since $C$ is a polyhedron convex set, it can be reformulated by $C = \{x \in \mathbb{R}^m : A x \leq b\}$, where $A \in \mathbb{R}^{l \times m}$ is a matrix, $b \in \mathbb{R}^l$ is a vector. For Algorithms 4 and 5, by setting $A_0 = A$, $b_0 = b$ then $C_0 = C = \{x \in \mathbb{R}^m : A_0 x \leq b_0\}$ is a polyhedron. We claim that $C_n$ is also a polyhedron defined by $C_n = \{x \in \mathbb{R}^m : A_n x \leq b_n\}$ for all $n \geq 0$, where $A_n \in \mathbb{R}^{(l+n) \times m}$ and $b \in \mathbb{R}^{l+n}$. Indeed, assume that $C_n = \{x \in \mathbb{R}^m : A_n x \leq b_n\}$ for some $n \geq 0$, by the definition of $C_{n+1}$ in Algorithms 4 and 5, we see that $C_{n+1} = C_n \cap H_n$, where

$$H_n = \{x \in \mathbb{R}^m : \|z_n - x\|^2 \leq \|x_n - x\|^2\} = \{x \in \mathbb{R}^m : 2 \langle x - z_n, x \rangle \leq \|x_n\|^2 - \|z_n\|^2\}.$$  

By setting

$$A_{n+1} = \begin{pmatrix} A_n \\ 2(x_n - z_n)^T \end{pmatrix} \in \mathbb{R}^{(l+n+1) \times m}, \ b_{n+1} = \begin{pmatrix} b_n \\ \|x_n\|^2 - \|z_n\|^2 \end{pmatrix} \in \mathbb{R}^{l+n+1}.$$  

Then $C_{n+1} = \{x \in \mathbb{R}^m : A_{n+1} x \leq b_{n+1}\}$. By the induction, we can obtain the desired conclusion. The sets $C_n, Q_n$ in Algorithm 3 are also polyhedral convex sets and similarly constructed by adding one linear inequality constraint to the ones of $C$ per each step. Projections $P_{C_{n+1}}(x_0)$ and $P_{C_n \cap Q_n}(x_0)$ are equivalently rewritten to
convex quadratic optimization programs onto polyhedral convex sets, respectively. While the sets $C_n$, $Q_n$ in Algorithm 1 are two half-spaces, and so we use the explicit formula in [7] to compute $x_{n+1} = P_{C_n \cap Q_n}(x_0)$. All convex quadratic optimization programs in the algorithms can be easily solved by the MALAB Optimization Toolbox. In the final example, when $C$ is given as a generalized convex feasible set [34], then optimization programs and projections $P_{C_n \cap Q_n}(x_0)$ and $P_{C_{n+1}}(x_0)$ in Algorithms 3, 4, 5 seem to be more complex. The problem of solving these subproblems is presented in Example 5.

Next, we present five numerical examples and compute them on a PC Desktop Intel(R) Core(TM) i5-3210M CPU @ 2.50GHz 2.50 GHz, RAM 2.00 GB. For a given tolerance $TOL$, we compare number of iterations (Iter.) and execution time (Time - in second) for above mentioned algorithms with choosing different starting points and stepsizes.

**Example 1.** We consider the bifunction $f : C \times C \to \mathbb{R}$ in $\mathbb{R}^2$ [24, Example 3] as $f(x, y) = (x_1 + x_2 - 1)(y_1 - x_1) + (x_1 + x_2 - 1)(y_2 - x_2)$ and the feasible set $C = [0, 1] \times [0, 1]$. It is easy to show that $f$ is monotone (so pseudomonotone) and Lipschitz-type continuous with $c_1 = c_2 = 1$. The solution set of EP for $f$ on $C$ is $EP(f, C) = \{x \in C : x_1 + x_2 - 1 = 0\}$. In this example, for a starting point $x_0$ then the sequence $\{x_n\}$ generated by Algorithms 1, 3, 4 and 5 converges to $x^\dagger = P_{EP(f, C)}(x_0)$ which is easily known because $EP(f, C)$ is explicit. The termination criterion in all algorithms is $||x_n - x^\dagger|| \leq TOL = 10^{-6}$. The parameters are chosen as follows $\lambda = 0.2$ for all the algorithms, $k = 6$ for Algorithm 1 and $\eta = 0.5$, $\alpha = 1.5$ for Algorithm 5. In Algorithm 1, we choose $x_1 = x_0$, $y_0 = (0, 0)^T$. The results are shown in Table 1.

| $x_0$ | Iter. | Time |
|-------|-------|------|
|       | Alg. 1 | Alg. 3 | Alg. 4 | Alg. 5 | Alg. 1 | Alg. 3 | Alg. 4 | Alg. 5 |
| (2,5) | 241    | 111   | 221   | 122    | 1.928  | 2.331  | 1.768  | 4.636  |
| (5,5) | 192    | 108   | 215   | 122    | 2.880  | 2.700  | 2.580  | 3.294  |
| (4.4,5) | 194   | 108   | 215   | 122    | 2.910  | 2.700  | 2.795  | 3.294  |
| (-0.75,0) | 196 | 108   | 215   | 122    | 3.724  | 3.132  | 2.795  | 3.172  |

**Example 2.** We consider the pseudomonotone bifunction $f$ which comes from the Nash-Cournot equilibrium model in [29, 32]. It is defined by

$$f(x, y) = \langle Px + Qy + q, y - x \rangle,$$

where $q \in \mathbb{R}^5$, $P$, $Q \in \mathbb{R}^{5 \times 5}$ are two matrices of order 5 such that $Q$ is symmetric, positive semidefinite and $Q - P$ is negative semidefinite. The feasible set $C$ is a polyhedral convex set defined by

$$C = \left\{ x \in \mathbb{R}^5 : \sum_{i=1}^{5} x_i \geq -1, \ -5 \leq x_i \leq 5, \ i = 1, \ldots, 5 \right\}.$$
This example is tested with \( q = (1, -2, -1, 2)^T \),

\[
P = \begin{pmatrix}
3.1 & 2 & 0 & 0 & 0 \\
2 & 3.6 & 0 & 0 & 0 \\
0 & 0 & 3.5 & 2 & 0 \\
0 & 0 & 2 & 3.5 & 0 \\
0 & 0 & 0 & 0 & 3
\end{pmatrix},
Q = \begin{pmatrix}
1.6 & 1 & 0 & 0 & 0 \\
1 & 1.6 & 0 & 0 & 0 \\
0 & 0 & 1.5 & 1 & 0 \\
0 & 0 & 1 & 1.5 & 0 \\
0 & 0 & 0 & 0 & 2
\end{pmatrix}.
\]

Two Lipschitz-type constants are \( c_1 = c_2 = \|P - Q\|/2 = 1.5 \). The parameters are chosen as \( \lambda = \frac{1}{2c_1} \) for Algorithms 1, 3, 4 and \( \lambda = 0.5, \eta = 0.5, \alpha = 1.5 \) for Algorithm 5, \( k = 6 \) for Algorithm 1. We also choose \( x_1 = x_0 \) and \( y_0 = (0, 0, 0, 0, 0)^T \).

In this example, the exact solution is not known. Thus, the stopping criterion is used in the algorithms as \( \|y_n - x_n\| \leq TOL = 10^{-4} \). Table 2 shows the numerical results for choosing randomly different starting points as \( x_1^0 = (1, 3, 1, 1, 2)^T, x_2^0 = (-1, 1, 2, 0, 0)^T, x_3^0 = (1, 0, 1, 0, 2)^T \).

| \( x_0 \) | Iter. | \( \text{Alg. 1} \) | \( \text{Alg. 3} \) | \( \text{Alg. 4} \) | \( \text{Alg. 5} \) | \( \text{Alg. 1} \) | \( \text{Alg. 3} \) | \( \text{Alg. 4} \) | \( \text{Alg. 5} \) |
|---|---|---|---|---|---|---|---|---|---|
| \( x_1^0 \) | 1918 | 1022 | 225 | 812 | 15.344 | 10.220 | 7.875 | 334.544 |
| \( x_2^0 \) | 1470 | 524 | 173 | 924 | 10.290 | 8.384 | 5.709 | 370.524 |
| \( x_3^0 \) | 1421 | 980 | 296 | 1043 | 12.789 | 32.34 | 10.360 | 617.456 |

Note that the range of stepsize \( \lambda \) in Algorithm 1 is smaller than the ones in Algorithms 3 and 4. Next, we perform numerical experiments for the best step-size in these algorithms. We chose \( \lambda = \min \{ \frac{1}{2c_1} - \tau, \frac{1}{2c_2} - \tau \} \) for Algorithms 3 and 4, where \( \tau = 10^{-p}, p = 2, 4, 6 \). The parameter \( k \) in Algorithm 1 is chosen as \( k = \frac{1}{8(c_1+c_2)} > \frac{1}{2A(c_1+c_2)} \) and the starting points are \( x_0 = x_1 = (1, 1, 1, 1, 1)^T, y_0 = (0, 0, 0, 0, 0)^T \). The stopping criterion is \( \|y_n - x_n\| \leq 10^{-4} \). The numerical results are showed in Table 3. All the sequences generated by the algorithms converge to the same approximation solution

\[
x^* \approx (-0.7226033378, 0.8009657032, 0.6861451896, -0.810040850, 0.2004196232).
\]

| \( \tau \) | Iter. | \( \text{Alg. 1} \) | \( \text{Alg. 3} \) | \( \text{Alg. 4} \) | \( \text{Alg. 1} \) | \( \text{Alg. 3} \) | \( \text{Alg. 4} \) |
|---|---|---|---|---|---|---|---|
| \( 10^{-2} \) | 1423 | 572 | 351 | 25.614 | 20.020 | 15.093 |
| \( 10^{-4} \) | 1341 | 655 | 373 | 22.797 | 20.960 | 16.039 |
| \( 10^{-6} \) | 1093 | 635 | 367 | 18.581 | 17.145 | 15.414 |

**Table 3.** Results for different given parameter \( \lambda \) in **Example 2**.

**Example 3.** We consider the bifunction \( f \) is defined as (28) in \( \mathbb{R}^m \) with \( m = 10, 15, 20, 50 \) where all entries of the vector \( q \in \mathbb{R}^m \) are randomly generated in
$[-m, m]$ and two matrices $P, Q \in \mathbb{R}^{m \times m}$ are also generated randomly\(^1\) such that $Q$ is symmetric, positive semidefinite and $Q - P$ is negative semidefinite. The feasible set $C$ is a polyhedral convex set defined by

$$C = \{ x \in \mathbb{R}^m : Ax \leq b \},$$

where $A$ is a $l \times m$ matrix ($l = 50$) with its entries being generated randomly in $[-m, m]$ and the vector $b \in \mathbb{R}^l$ is generated randomly with the elements in $[1, m]$ (note that $C \neq \emptyset$ because of $0 \in C$). The bifunction $f$ satisfies the conditions of Theorem 4.5 with the constants $c_1 = c_2 = \|P - Q\|/2$. We chose $\lambda = 1/4.5c_1$ for Algorithms 1, 3, 4 and $\lambda = 0.5, \eta = 0.5, \alpha = 1.5$ for Algorithm 5, $k = 10$ for Algorithm 1. The starting point $x_0 = x_1 = (1, 1, \ldots, 1)^T \in \mathbb{R}^m$ and $y_0 = 0 \in \mathbb{R}^m$. We use the stopping criterion as $\|y_n - x_n\| \leq 10^{-3}$. The numerical results are showed in Table 4. When $m = 50$, the convergence of all algorithms is slightly slow (Slow).

**Table 4.** Numerical results for Example 3.

| $m$ | Iter. | Alg. 1 | Alg. 3 | Alg. 4 | Alg. 5 | Alg. 1 | Alg. 3 | Alg. 4 | Alg. 5 |
|-----|-------|--------|--------|--------|--------|--------|--------|--------|--------|
| 10  | 521   | 212    | 183    | 312    | 17.193 | 13.356 | 17.934 | 37.440 |
| 15  | 674   | 406    | 366    | 467    | 31.004 | 25.254 | 32.074 | 30.897 |
| 20  | 912   | 388    | 348    | Slow   | 127.680| 111.356| 98.136 |
| 50  | Slow  | -      | -      | -      | -      | -      | -      | -      |

**Example 4.** We consider the nonsmooth equilibrium bifunction $f : C \times C \to \mathbb{R}$ in $\mathbb{R}^6$ [30] defined by

$$f(x, y) = ((A + B)x + By + a, y - x) + c(y) - c(x),$$

where $C$ is defined like as in [30] by $C = \{ x \in \mathbb{R}^6 : x_{ib} \leq x \leq x_{ub} \}$ with $x_{ib} = (0, 0, 0, 0, 0)^T$ and $x_{ub} = (80, 80, 50, 55, 30, 40)^T$. The values of $A, B, a, c(x)$ [30] are defined by

$$A = 2 \sum_{i=1}^{3} q^i(q^i)^T, \quad B = 2 \sum_{i=1}^{3} q^i(q^i)^T, \quad a = -378.4 \sum_{i=1}^{3} q^i, \quad c(x) = 6 \sum_{j=1}^{6} c_j(x_j),$$

where $q^i = (q_i^1, \ldots, q_i^6)^T, \ i = 1, 2, 3$ with

$$q_j^i = \begin{cases} 1 & \text{if } j \in I_i, \\ 0 & \text{otherwise}, \end{cases}$$

and $I_1 = \{1\}, \ I_2 = \{2, 3\}, \ I_3 = \{4, 5, 6\}; \ q^i = (q_1^i, \ldots, q_6^i)$ with $q_j^i = 1 - q_j, \ (j = 1, \ldots, 6); \ c_j(x_j) = \max \{\hat{c}_j(x_j), \tilde{c}(x_j)\}$ with

$$\hat{c}_j(x_j) = \frac{\hat{a}_j}{2} x_j^2 + \hat{\beta}_j x_j + \hat{\gamma}_j,$$

$$\tilde{c}_j(x_j) = \tilde{a}_j x_j + \frac{\tilde{\beta}_j}{\tilde{\beta}_j + 1} \tilde{\gamma}_j^{-1/\tilde{\beta}_j} (x_j)^{\tilde{\beta}_j + 1}/\tilde{\beta}_j,$$

and $\hat{\alpha}_j, \hat{\beta}_j, \hat{\gamma}_j, \tilde{\alpha}_j, \tilde{\beta}_j, \tilde{\gamma}_j, (j = 1, 2, \ldots, 6)$ are given in Table 5. Note that $c(x)$ is a

\(^1\)Two matrices $P, Q$ are randomly generated as follows: we randomly choose $\lambda_{1k} \in [0, m], \ \lambda_{2k} \in [-m, 0], \ k = 1, \ldots, m$. Set $Q, T$ as two diagonal matrices with eigenvalues $\{\lambda_{1k}\}_{k=1}^m$ and $\{\lambda_{2k}\}_{k=1}^m$, respectively. Then, we make a positive semidefinite matrix $Q$ and a negative semidefinite matrix $T$ by using $Q$ and $T$ with two random orthogonal matrices, respectively. Finally, set $P = Q - T$. 

nonsmooth and convex function, $B$ is a symmetric positive semidefinite matrix and the matrix $A$ is not positive semidefinite. Thus, $f(x,.)$ is convex and nonsmooth on $C$. Moreover, $f$ is not monotone. Thus, we can not apply directly Algorithm 1 to solve EP in this case. However, from Lemma 7 in [30], we can reformulated equivalently to a monotone equilibrium problem with the following bifunction

$$f_1(x,y) = \langle (A_1 + B_1)x + B_1y + a, y - x \rangle + c(y) - c(x),$$

where $A_1 = A + \frac{3}{2}B$ and $B_1 = \frac{1}{2}B$. Since $c(x)$ is continuous, $f$ satisfies the conditions (A1), (A3), (A4). By arguing similarly to Lemma 6.2 in [29], we can conclude that $f$ is Lipschitz-type continuous with $c_1 = c_2 = ||A_1||/2 = ||A+3B||/4$. Thus, $f$ satisfies the hypothesis (A2). Contrary to convex quadratic optimization problems in Examples 2 and 3, in this example we need to solve the following convex optimization problem

$$\min \left\{ \frac{1}{2}x^TH_nx + h_n^T x + \lambda c(x) : x_{lb} \leq x \leq x_{ub} \right\},$$

where $H_n = \lambda B + I$ and $h_n = \lambda \left[ (A + \frac{3}{2}B)y_n + a \right]$. We use the Ipopt package (https://projects.coin-or.org/Ipopt) for solving Problem (29). We choose $\lambda = \frac{1}{5\alpha_1}$ for Algorithms 1, 3, 4 and $\lambda = 0.5$, $\eta = 0.5$, $\alpha = 1.5$ for Algorithm 5, $k = 3$ for Algorithm 1. The starting point $y_0$ is $y_0 = 0 \in \mathbb{R}^6$ and four points $x_0$ are $x_{10} = (0,0,0,0,0,0)$, $x_2 = (1,1,1,1,1,1)$, $x_3 = (1,0,1,0,-1,0)$, $x_4 = (-3,4,2,7,-10,8)$. The stopping criterion is $||y_n - x_n|| \leq 10^{-4}$. The numerical results are showed in Table 6.

### Table 6. Results for given starting points in **Example 4**.

| $x_0$ | Alg. 1 | Alg. 3 | Alg. 4 | Alg. 5 | Alg. 1 | Alg. 3 | Alg. 4 | Alg. 5 |
|-------|--------|--------|--------|--------|--------|--------|--------|--------|
| $x_{10}$ | 1965 | 934 | 557 | 805 | 47.160 | 36.426 | 23.394 | 79.695 |
| $x_2$ | 2001 | 1023 | 573 | 928 | 58.029 | 31.713 | 26.358 | 144.768 |
| $x_3$ | 1877 | 1279 | 567 | 1011 | 46.925 | 48.602 | 24.948 | 105.144 |
| $x_4$ | 1574 | 1089 | 598 | 873 | 39.350 | 34.848 | 24.518 | 139.680 |

Finally, we consider an example for $C$ being a generalized convex feasible set introduced in [34].

**Example 5.** The bifunction $f$ in $\mathbb{R}^5$ is defined by (28) and $P, Q, q$ are chosen like as in **Example 2**. Let $K, C_1, \ldots, C_l$ be nonempty closed convex subsets of
\( \mathbb{R}^5 \) such that \( K \cap \bigcap_{j=1}^l C_j = \emptyset \) and at least one \( K, C_j, j = 1, \ldots, l \) (\( l = 100 \)) is bounded. For each point \( x \in \mathbb{R}^5 \), we set \( \Phi(x) = \frac{1}{l} \sum_{j=1}^l w_j d^2(x, C_j) \), where \( \{w_j\}_{j=1}^l \subset (0, 1) \), \( \sum_{j=1}^l w_j = 1 \) and \( d(x, C_j) = \inf \{||x - y|| : y \in C_j\} \). We consider the feasible set \( C \) which is called the generalized convex feasible set [34, Definition 4.1] as follows

\[
C = \left\{ x \in K : \Phi(x) = \min_{y \in K} \Phi(y) \right\}. \quad (30)
\]

Note that \( C \) is a nonempty closed convex subset of \( K \), see [34, Proposition 4.2 and Remark 4.3]. In this experiment, we chose \( w_j = \frac{1}{l} \) and \( K = \{x \in \mathbb{R}^5 : ||x - a|| \leq 4.5\} \), \( C_1 = \{x \in \mathbb{R}^5 : ||x|| \leq 1\} \), \( C_j = \{x \in \mathbb{R}^5 : \langle c_j x, y \rangle \leq b_j\} \), \( j = 2, \ldots, l \), where \( a = (10, 0, 0, 0, 0)^T \) and the vectors \( c_j, b_j \) are generated randomly with their entries in \([-2, 2]\). We need to solve the following program at each step

\[
\min \left\{ \frac{1}{2} x^T H x + h^T x : x \in C \right\}, \quad (31)
\]

where \( H = 2\lambda Q + I \) and \( h \) is a vector in \( \mathbb{R}^5 \). The first problem here is how to solve problem (31) when the feasible set \( C \) is formulated in implicit form (30). Now, we set

\[
T = P_K \left( \sum_{j=1}^l w_j P_{C_j} \right),
\]

then \( C = Fix(T) \), see [34, Proposition 4.2b] for \( \beta = 1 \). Thus, problem (31) becomes an optimization problem over the fixed point set of the nonexpansive mapping \( T \). We use the hybrid steepest descent method (HSDM) in [17, 34] to obtain the solution of problem (31) with the tolerance \( TOL = 10^{-4} \). The second problem is how to find the next iterate \( x_{n+1} \) in Algorithm 3. We rewrite \( x_{n+1} \) in this algorithm as \( x_{n+1} = P_{H_{1n} \cap H_{2n} \cap C}(x_0) \) where \( H_{1n}, H_{2n} \) are two spaces \( H_{1n} = \{z \in \mathbb{R}^5 : ||z - z_n||^2 \leq ||z - x_n||^2\} \), \( H_{2n} = \{z \in \mathbb{R}^5 : \langle x_0 - x_n, z - x_n \rangle \leq 0\} \). We combine the Haugazeau’s method [1, Corollary 29.8] and the HSDM in [17, 34] to obtain \( x_{n+1} \) with the tolerance \( TOL \). In this example, it seems to be not easy to find the next iterate \( x_{n+1} \) in Algorithms 4 and 5. Note that the projection \( x_{n+1} = P_{C_n \cap Q_n}(x_0) \) in Algorithm 1 is explicit. We perform numerical tests for Algorithms 1 and 3 with the control parameters \( \lambda, k \) and three starting points \( x_0 \) and \( y_0 \) as in the first experiment of Example 2. The results are reported in Table 7. From this table, we see that the whole time for performing Algorithm 3 is significantly larger than the one for Algorithm 1. Of course, this comes from the problems of solving additionally an optimization program and finding the next iterate \( x_{n+1} \) per each iteration.

**Table 7.** Results for given starting points in Example 5.

| \( x_0 \) | Iter. | Time |
|---|---|---|
| \( x_0^0 \) | 2365 | 2059.915 |
| \( x_0^1 \) | 1565 | 3428.915 |
| \( x_0^2 \) | 2417 | 1437 |
| \( x_0^3 \) | 2877 | 1729 |
| \( x_0^4 \) | 1729 | 2557.653 |
| \( x_0^5 \) | 3577.301 |
The study of examples here is preliminary and it is clear that EP depends on the structures of the feasible set $C$ and of the bifunction $f$. The advantage of the proposed algorithm is a computation per each iteration. The convergence and the efficiency of the algorithm have been illustrated by the numerical results in Tables 1 - 4 and 6, 7 on five test problems.

7. **Concluding remarks.** The paper proposes a novel algorithm for solving EPs for a class of pseudomonotone and Lipschitz-type continuous bifunctions. By constructing specially cutting halfspaces, we have designed the algorithm with a more simple and elegant structure and without any extra-step dealing with the feasible set. The strong convergence of the algorithm is proved. It is also emphasized that we still have to solve exactly an optimization problem in each step. This, in general, is a disadvantage of the algorithm (also, of the extragradient methods and the Armijo linesearch methods) when equilibrium bifunctions and feasible sets have complex structures. The paper also helps us in the design and analysis of more practical algorithms to be seen.

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