Calderón–Zygmund Operators in the Bessel Setting
for All Possible Type Indices

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Abstract We show that many harmonic analysis operators in the Bessel setting, including maximal operators, Littlewood–Paley–Stein type square functions, multipliers of Laplace or Laplace–Stieltjes transform type and Riesz transforms are, or can be viewed as, Calderón–Zygmund operators for all possible values of type parameter $\lambda$ in this context. This extends results existing in the literature, but being justified only for a restricted range of $\lambda$.

Keywords Bessel operator, Bessel semigroup, maximal operator, square function, multiplier, Riesz transform, Calderón–Zygmund operator

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1 Introduction and Preliminaries

Let $n \geq 1$ and $\lambda \in (-1/2, \infty)^n$. We consider the Bessel differential operator

$$\Delta_\lambda = -\Delta - \sum_{i=1}^{n} \frac{2\lambda_i}{x_i} \partial_{x_i},$$

where $\Delta$ stands for the Euclidean Laplacian in $\mathbb{R}_+^n = (0, \infty)^n$. The operator $\Delta_\lambda$ is symmetric and nonnegative in $C_c^\infty(\mathbb{R}_+^n) \subset L^2(\mathbb{R}_+^n, d\mu_\lambda)$, where $d\mu_\lambda$ is the doubling measure given by

$$d\mu_\lambda(x) = \prod_{i=1}^{n} x_i^{2\lambda_i} dx_i, \quad x \in \mathbb{R}_+^n.$$ 

It is well known that $\Delta_\lambda$ has a self-adjoint extension, here still denoted by $\Delta_\lambda$, whose spectral decomposition is given via the Hankel transform, see [7] for details.

The semigroup \{$W_t^\lambda\}_{t>0}$ generated by $-\Delta_\lambda$ has the integral representation

$$W_t^\lambda f(x) = \int_{\mathbb{R}_+^n} W_t^\lambda(x,y) f(y)d\mu_\lambda(y), \quad x \in \mathbb{R}_+^n, \; t > 0,$$

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where the Bessel heat kernel is given by
\[
W_t^\lambda(x, y) = \frac{1}{(2t)^n} \exp \left( -\frac{1}{4t} (|x|^2 + |y|^2) \right) \prod_{i=1}^n (x_i y_i)^{-\lambda_i + 1/2} I_{\lambda_i - 1/2} \left( \frac{x_i y_i}{2t} \right); \tag{1.1}
\]
here \( x, y \in \mathbb{R}^n_+ \), \( t > 0 \), and \( I_\nu \) denotes the modified Bessel function of the first kind and order \( \nu \), cf. [23, p. 395]. Note that \( \{W_t^\lambda\} \) is a symmetric diffusion semigroup in the sense of Stein’s monograph (see [19, p. 65]).

In this setting, the \( n \)-dimensional Hankel transform \( h_\lambda \) plays the same role as the Fourier transform in the Euclidean context. It is given by
\[
h_\lambda f(x) = \int_{\mathbb{R}^n_+} \varphi_\lambda^h(y) f(y) \, d\mu_\lambda(y), \quad x \in \mathbb{R}^n_+,
\]
with the kernel
\[
\varphi_\lambda^h(y) = \prod_{i=1}^n (x_i y_i)^{-\lambda_i + 1/2} J_{\lambda_i - 1/2}(x_i y_i), \quad x, y \in \mathbb{R}^n_+,
\]
where \( J_\nu \) stands for the Bessel function of the first kind and order \( \nu > -1 \).

We investigate the following multi-dimensional Bessel operators defined initially either in \( L^2(d\mu_\lambda) \) in the cases of (1)–(4), or in \( C^\lambda \) (the space of smooth \( L^2(d\mu_\lambda) \)-functions whose Hankel transform \( h_\lambda \) is also smooth and compactly supported) in the case of Riesz transforms (5) (see [7, Section 4.4]).

1. The maximal operator
\[
W^\lambda_s f = \|W^\lambda_s f\|_{L^\infty(dt)}.
\]

2. Littlewood–Paley–Stein type mixed square functions
\[
g^\lambda_{m,k}(f)(x) = \|\partial^m_i \partial^k_t W_t^\lambda f(x)\|_{L^2(t^{|m|+2k-1}dt)},
\]
where \( m \in \mathbb{N}^n \), \( k \in \mathbb{N} \), \( |m| + k > 0 \).

3. Multipliers of Laplace transform type
\[
T^\lambda_M f = h_\lambda(Mh_\lambda f),
\]
where \( M(z) = |z|^2 \int_0^\infty e^{-t|z|^2} \psi(t) \, dt \) with \( \psi \in L^\infty(dt) \).

4. Multipliers of Laplace–Stieltjes transform type
\[
T^\lambda_M f = h_\lambda(Mh_\lambda f),
\]
where \( M(z) = \int_{(0, \infty)} e^{-t|z|^2} \, d\nu(t) \) with \( \nu \) being a complex Borel measure on \((0, \infty)\).

5. Riesz transforms of order \( m \)
\[
R^\lambda_m f(x) = \partial^m_x h_\lambda(|\cdot|^{-|m|}h_\lambda f)(x),
\]
where \( m \in \mathbb{N}^n \) and \( |m| > 0 \).

In [7], Betancor et al. showed that the above (vector-valued) operators, excluding (4), are Calderón–Zygmund in the sense of the space of homogeneous type \( (\mathbb{R}^n_+, d\mu_\lambda, |\cdot|) \), but under the restriction \( \lambda \in [0, \infty)^n \). The objective of this paper is to extend that result to the full range of \( \lambda \in (-1/2, \infty)^n \), see Theorem 2.1 below. Typically, the main technical difficulty connected with the Calderón–Zygmund approach is to show the relevant kernel estimates. Here we follow