ON THE LEMNISCATE COMPONENTS CONTAINING NO CRITICAL POINTS OF A POLYNOMIAL EXCEPT FOR ITS ZEROS

Let $P$ be a complex polynomial of degree $n$ and let $E$ be a connected component of the set $\{z : |P(z)| \leq 1 \}$ containing no critical points of $P$ different from its zeros. We prove the inequality $|(z-a)P'(z)/P(z)| \leq n$ for all $z \in E \setminus \{a\}$, where $a$ is the zero of the polynomial $P$ lying in $E$. Equality is attained for $P(z) = cz^n$ and any $z, c \neq 0$.

Bibliography: 4 titles.

Introduction

Let $R$ be a rational function of degree $n$ represented as $R = P/Q$, where $P$ and $Q$ are polynomials of degrees $n$ and not exceeding $n$, respectively, which have no zeros in common. Assume that $R(0) = R'(0) \neq 0$. Sheil-Small [1, 10.3.2] posed a question on finding a neighborhood of the point $w = 0$ in which one could distinguish a one-valued branch, $f(w)$, of the inverse function $z = R^{-1}(w)$, $f(0)=0$, satisfying the inequality

$$\text{Re} \frac{wf'(w)}{f(w)} \geq \frac{1}{n}$$

The inequality obtained would be of interest in connection with the shape of the level curves $|R(z)| = \text{const}$ in the domain where the function $R$ is univalent. In the present note, a closely related problem for polynomials $P(Q \equiv 1)$ is considered. More precisely, the following result is proved.

**Theorem.** Let $P$ be a polynomial of degree not exceeding $n$ and let $E$ be a connected component of the lemniscate $|P(z)| \leq 1$ containing no critical points of the polynomial $P$ different from its zeros. Then, for any point $z \in E \setminus \{a\}$,

$$\left| \frac{(z-a)P'(z)}{P(z)} \right| \leq n,$$

where $a$ is the zero of the polynomial $P$ belonging to the component $E$. Equality in (1) is attained for any point $z$ in the case where $P(z) = cz^n, c \neq 0$.

If the component $E$ contains no critical points, then from (1) it follows that for the corresponding branch $f$ of the function inverse to the polynomial $P$, the inequality

$$\left| \frac{wf'(w)}{f(w) - a} \right| \geq \frac{1}{n}.$$
holds in the disk $|\omega| < 1$. This result is weaker than the Sheil-Small statement. However, contrary to [1], critical points in $E$ are allowed. The result obtained has the following geometric interpretation. Assume, for simplicity, that $a = 0$ and let $\log (\cdot)$ denote the one-valued branch of the logarithm mapping the plane slit along the real positive semiaxis onto a strip of width $2\pi$. For any level curve $c(\tau)(|P(z)| = \tau < 1)$, the "curve" $\gamma(\tau) = \log c(\tau)$ connects the opposite sides of the strip mentioned; consequently, its length is no less than $2\pi$. On the other hand, the image $\gamma(\tau)$ under the map $\log P(\exp(\cdot))$ covers a vertical interval of length $2\pi$ no more than $n$ times. Therefore, on the curve $\gamma(\tau)$, there is a point $\zeta$ at which the distortion coefficient satisfies the condition.

$$|\log P(\exp \zeta)|' \leq n.$$ 

This means that inequality (1) holds at a certain point $z = \exp \zeta$ of the level curve $c(\tau)$. The theorem of the present paper claims that this inequality holds at any point of the curve $c(\tau)$. The assumption that the set $E$ contains no critical points different from the polynomial zero is essential. For instance, for the polynomial $P(z) = z^3/2 - 3z^2/4$, the lemniscate $|P(z)| \leq 1$ contains both critical points $z = 0$ and $z = 1$, whence it is connected. The point $z = 2$ belongs to this lemniscate, but

$$\frac{2P'(2)}{P(2)} = 6 > 3.$$

**Corollary.** If, under the assumption of the theorem, the inequality $P(z) > 0$ holds at point $z \in E$, then the polar derivative with respect to the point $a$, $D_a P$, satisfies the bound

$$\text{Re} D_a P(z) = \text{Re} [nP(z) - (z - a)P'(z)] \geq 0, \quad \text{Im}.$$ 

The theorem will be proved in Sec. 2. Ideologically, it comes back to the proof of Hayman’s conjecture on coverings of vertical under a conformal mapping of the disk [2].

§1. Auxiliary constructions and assertions

Let $P$ be a polynomial of degree $n$ and let $E$ be a connected component of the lemniscate $|P(z)| \leq 1$ that contains no critical points of the polynomial $P$ other than its
zeros (i.e., no points $\zeta$ such that $P'(\zeta) = 0$ and $P(\zeta) \neq 0$). Let $a$ be the zero of $P$ lying in $E$ and let $z_0$ be a point of the component $E$ such that $P(z_0) > 0$.

By $\mathcal{R}$ denote the Riemann surface of the function $\mathcal{P}^{-1}$ inverse to the polynomial $P$. In what follows, we consider the function $\mathcal{P}^{-1}$ as a one-valued function given on the surface $\mathcal{R}$. Let $P : \overline{\mathbb{C}} \rightarrow \mathcal{R}$ be the function inverse to $\mathcal{P}^{-1}$ in this sense. The projection of a point $W \in \mathcal{R}$ is defined as the point $P(\mathcal{P}^{-1}(W)) \in \overline{\mathbb{C}}$.

Assume that the ray \{w : \Im w = 0, 0 < \Re w < \infty\} contains no critical values of the polynomial $P$ (i.e., no points $P(\zeta)$ such that $P'(\zeta) = 0$ for a certain $\zeta$). By $L$ denote the ray on the surface $\mathcal{R}$ or, more exactly, the Jordan curve univalently lying over the above ray of the sphere $\overline{\mathbb{C}}$ and connecting the points $P(a)$ and $P(\infty)$. Let $T = \{t_k\}_{k=0}^m$, $0 = t_0 < P(z_0) = t_1 < \ldots < t_{m-1} < t_m = \infty$, be a partition of the interval $0 \leq t \leq \infty$ containing all those values of $t$ in $1 < t < \infty$ at which the circle $\gamma(t) := \{w : |w| = t\}$ contains at least one critical value $P(\zeta)$ with $\zeta \in E$.

Finally, by $C(t)$ denote the closed Jordan curve on $\mathcal{R}$ intersecting the ray $L$ and lying over the circle $\gamma(t)$ whose orientation corresponds to the positive orientation on the projection $\gamma(t)$, $0 < t < \infty$, $t \notin T$; $c(t)$ is the image of the curve $C(t)$ under the mapping $\mathcal{P}^{-1}$.

**Lemma 1.** The argument increment

$$\Delta_{c(t)} \arg P(z)$$

is a nondecreasing function of $t$ on the set $\{t : 0 < t < \infty, t \notin T\}$.

**Proof.** Let $0 < t' < t'' < \infty$, $t', t'' \notin T$. The points $P(a)$ and $P(\infty)$ are located on different sides of the curves $C(t')$ and $C(t'')$ on the surface $\mathcal{R}$. Therefore, the nonintersecting Jordan curves $c(t')$ and $c(t'')$ separate the point $a$ from $\infty$ on the sphere $\overline{\mathbb{C}}$. Consequently, one of them lies in the interior of the other. Furthermore, as we move along the ray $L$ from the point $P(a)$ to the point $P(\infty)$, we first meet the curve $C(t')$ and then the curve $C(t'')$. This means that the curve $c(t')$ lies in the interior of the curve $c(t'')$. Therefore, the number $N_{t'}$ of the zeros of the polynomial $P$ lying inside $c(t')$ does not exceed the number $N_{t''}$ of the zeros lying inside $c(t'')$ (with account for their multiplicities). It remains to apply the argument principle:

$$2\pi N_t = \Delta_{c(t)} \arg P(z), \quad t = t', t''.$$ 

The lemma is proved.
The points \( \mathcal{P}(a) \) and \( \mathcal{P}(\infty) \) lie on different sides of the curve \( \mathcal{C}(t) \) for any \( t, 0 < t < \infty, t \notin T \). This implies that for every \( k = 0, \ldots, m - 1 \), the doubly-connected domain

\[
\mathcal{G}_k = \bigcup_{t_k < t < t_{k+1}} \mathcal{C}(t)
\]

also separates the points \( \mathcal{P}(a) \) and \( \mathcal{P}(\infty) \). At the same time, the curve \( \mathcal{P}(H), H = \{ z : z_0 + (a - z_0)\tau, 1 \leq \tau \leq \infty \} \), connects these points. Therefore, for every \( k = 0, \ldots, m - 1 \), there is at least one Jordane arc \( \mathcal{H}_k \), on the curve \( \mathcal{P}(H) \) that lies in the domain \( \mathcal{G}_k \) and connects its boundary components. Thus, in the above notation, the following assertion holds.

**Lemma 2.** For any \( k = 0, \ldots, m - 1 \) the domain \( \mathcal{G}_k \setminus \mathcal{H}_k \) is simply connected.

Below, we will need the notion of condenser capacity (e.g., see [3]). For sufficiently small positive \( r \) and \( \rho \) on the sphere \( \overline{\mathbb{C}}_z \), consider the condensers

\[
C(r) = (H, \{ z : |z - z_0| \leq r \})
\]

and

\[
C(r, \rho) = (H \cup \{ z : |z - a| \leq \rho \} \cup \{ z : |z| \geq 1/\rho \} \cup \bigcup_{P'(\zeta) = 0} \{ z : |z - \zeta| \leq \rho \}, \{ z : |z - 1| \leq r \}).
\]

**Lemma 3.** For a fixed \( r, 0 < r < |a - z_0| \), the condensers capacities satisfy the relation

\[
\lim_{\rho \to 0} \text{cap} \ C(r, \rho) = \text{cap} \ C(r).
\]

**Proof.** We make use of the continuity of the capacity and of the fact that the latter is invariant under addition of a finite number of points to points to the condenser’s plates:

\[
\lim_{\rho \to 0} \text{cap} \ C(r, \rho) = \text{cap} \left( H \cup \bigcup_{P'(\zeta) = 0} \{ \zeta \}, \{ z : |z - z_0| \leq r \} \right) = \text{cap} \ C(r)
\]

(see Propositions 1.4 and 1.6 in [3]). This proves the lemma.

Below, we introduce new notation and give some comments.

\( \zeta = f_k(W) \) is the one-valued branch of the function \( \zeta = \log(W/P(z_0)) \) that maps the domain \( \mathcal{G}_k \setminus \mathcal{H}_k \) conformally and univalently into the "strip" \( \Pi_k := \{ \zeta : \xi_k < \Re \zeta < \)
ξ_{k+1}, k = 0, \ldots, m - 1. Here, \( \xi_k = \log(t_k/P(z_0)) \), \( k = 0, 1, \ldots, m \). The choice of such a branch is feasible in view of Lemma 2. For \( k = 1 \) and \( k = m \), \( \Pi_k \) is a half-plane.

\( u(z) \) is the potential function of the condenser \( C(r, \rho) \), i.e., the resl-valued function continuous on \( \overline{C}_z \), vanishing on the first plate of the condenser \( C(r, \rho) \), equal to unity on its second plate, and harmonic in the complement to these plates:

\[
v_k(\zeta) = \begin{cases} 
  u(D^{-1}(f_k^{-1}(\zeta))), & \zeta \in f_k(G_k \setminus \mathcal{H}_k), \\
  0, & \zeta \in \Pi_k \setminus f_k(G_k \setminus \mathcal{H}_k),
\end{cases} \quad k = 0, \ldots, m - 1
\]

On \( \partial \Pi_k \) the function \( v_k \) is defined by continuity. The function obtained in this way is also denoted by \( v_k \). As is not difficult to see, the function \( v_k \) satisfies the Lipschitz condition in the strip \( \Pi_k \), \( k = 0, \ldots, m - 1 \), whereas the function \( v_j \) is equal to unity on the set \( f_j(D(\{z : |z - z_0| \leq r\}) \cap \mathcal{G}_j) \), \( j = 0, 1 \).

\( v_k^*(\zeta) \) is the result of Steiner symmetrization of the function \( v_k(\zeta) \), \( \zeta \in \overline{\Pi}_k \), with respect to the real axis (see [4]). Every function \( v_k^*(\zeta) \) is a Lipschitz function in \( \overline{\Pi}_k \) and vanishes on the set \( \{\zeta \in \overline{\Pi}_k : |\text{Im} \ \zeta| \geq \pi n\} \), \( k = 0, \ldots, m - 1 \). Lemma 1 implies the following inequalities:

\[
v_{k-1}^*(\xi_k + i\eta) \leq v_k^*(\xi_k + i\eta), \quad -\infty < \eta < \infty, \quad k = 2, \ldots, m - 1.
\]

\( \zeta = F(z) \) is the function that maps the unit disk \( |z| < 1 \), conformally and univalently, onto the strip \( |\text{Im} \ \zeta| < \pi n \) in such a way that \( F(0) = 0, \ F'(0) > 0 \).

\( \tilde{r} \) is the upper bound for all \( r \) for which the set \( F(\{z : |z| < r\}) \cap \{\zeta : \text{Re} (-1)^j \zeta < 0\} \)
belongs to the result of Steiner symmetrization with respect to the real axis of the set \( f_j(D(\{z : |z - z_0| \leq r\}) \cap \mathcal{G}_j) \) for \( j = 0 \) and \( j = 1 \).

\( v(\zeta) \) is the potential function of the condenser \( \tilde{C}(\tilde{r}) = (\overline{C}_\zeta \setminus \{\zeta : |\text{Im} \ \zeta| < \pi n\}, F(\{z : |z| \leq \tilde{r}\})) \). It is readily seen that

\[
\frac{\partial v}{\partial \xi} = 0 \quad \text{on the line} \quad \text{Re} \ \zeta = 0,
\]

\[
\frac{\partial v}{\partial \xi} \leq 0 \quad \text{on every line} \quad \text{Re} \ \zeta = \xi > 0.
\]

The level curves of the potential function \( v \) coincide with the level curves of the function \( F \) (i.e., with the curves \( |F^{-1}(\zeta)| = \text{const} \)).
Given a sufficiently smooth function $\lambda$ on an open set $\Omega \subset \mathbb{C}$, denote

$$I(\lambda, \Omega) = \int_\Omega |\nabla \lambda|^2 d\sigma.$$ 

**Lemma 4.** The following inequality holds:

$$\sum_{k=0}^{m-1} I(v_k^*, \Pi_k) \geq I(v, \mathbb{C}).$$

**Proof.** Set $G_k = \{ \zeta \in \Pi_k : |\text{Im} \zeta| < \pi n \}$, $k = 0, 1, \ldots, m - 1$, and $l_k = \{ \zeta : \text{Re} \zeta = \xi_k, |\text{Im} \zeta| < \pi n \}$, $k = 2, \ldots, m - 1$. For every $k$, $0 \leq k \leq m - 1$, we have

$$I(v_k^*, \Pi_k) = I(v_k^*, G_k) = I(v_k^* - v + v, G_k) = I(v_k^* - v, G_k) + I(v, G_k) +$$

$$+ 2 \iint_{G_k} \left[ \frac{\partial(v_k^* - v)}{\partial \xi} \frac{\partial v}{\partial \xi} + \frac{\partial(v_k^* - v)}{\partial \eta} \frac{\partial v}{\partial \eta} \right] d\xi d\eta \geq I(v, G_k) - 2 \int_{\partial G_k} (v_k^* - v) \frac{\partial v}{\partial n} ds,$$

where $\partial/\partial n$ means differentiation along the inward normal to the boundary of the domain $G_k$ (angle points are excluded). With account for relations (2) and (3), we derive

$$\sum_{k=0}^{m-1} I(v_k^*, \Pi_k) \geq \sum_{k=0}^{m-1} I(v, G_k) - 2 \sum_{k=1}^{m-1} \int_{\partial G_k} (v_k^* - v) \frac{\partial v}{\partial n} ds =$$

$$= \sum_{k=0}^{m-1} I(v, G_k) - 2 \sum_{k=2}^{m-1} \int_{l_k} \left[ (v_{k-1}^* - v) \left( -\frac{\partial v}{\partial \xi} \right) + (v_k^* - v) \frac{\partial v}{\partial \xi} \right] ds =$$

$$= \sum_{k=0}^{m-1} I(v, G_k) + 2 \sum_{k=2}^{m-1} \int_{l_k} (v_{k-1}^* - v_k^*) \frac{\partial v}{\partial \xi} ds \geq I(v, \mathbb{C})$$

This completes the proof.

**§2. Proof of the theorem**

It is sufficient to prove inequality (1) at an arbitrary point $z_0 \in E \setminus \{a\}$ such that $P(z_0) > 0$ under the assumption that the ray $\{ w : \text{Im} w = 0, 0 < \text{Re} w < \infty \}$ contains no critical points of the polynomial $P$. We use the notation introduced in Sec. 1.
chain of relations below stems from the conformal invariance of the Dirichlet integral, from the Pólya and Szegő theorem on function symmetrization (see [4]), and from Lemma 4:

\[
\operatorname{cap} C(r, \rho) = I(u, C) \geq \sum_{k=0}^{m-1} I(v_k, \Pi_k) \geq \sum_{k=0}^{m-1} I(v_k^*, \Pi_k) \geq I(v, \mathbb{C}) = \operatorname{cap} \tilde{C}(\tilde{r}).
\]

In view of Lemma 3, we ultimately obtain

\[
\operatorname{cap} C(r) \geq \operatorname{cap} \tilde{C}(\tilde{r}). \quad (4)
\]

In order to compute the asymptotics of the condenser capacity as \( r \to 0 \), we use known formulas (e.g., see [3, (1.6) and (1.8)]), in which \( r(B, a) \) stands for the inner radius of the domain \( B \) with respect to a point \( a \in B \). As a result, we obtain

\[
\operatorname{cap} C(r) = -\frac{2\pi}{\log r} - \frac{1}{2\pi} (\log r(C_z \setminus H, z_0)) \left( \frac{2\pi}{\log r} \right)^2 + o \left( \left( \frac{1}{\log r} \right)^2 \right) =
\]

\[
= -\frac{2\pi}{\log r} - 2\pi (\log[4|a - z_0|]) \left( \frac{1}{\log r} \right)^2 + o \left( \left( \frac{1}{\log r} \right)^2 \right), \quad r \to 0.
\]

Further, the second plate of the condenser \( \tilde{C}(\tilde{r}) \) is an "almost disk" of radius \( (r|P'(z_0)|/P(z_0))(1 + o(1)) \) as \( r \to 0 \). Consequently,

\[
\operatorname{cap} \tilde{C}(\tilde{r}) = -\frac{2\pi}{\log(r|P'(z_0)|/P(z_0))} - 2\pi (\log r(\{\zeta : \text{Im } \zeta < \pi n\}, 0)) \left( \frac{1}{\log(r|P'(z_0)|/P(z_0))} \right)^2 +
\]

\[
+ o \left( \left( \frac{1}{\log r} \right)^2 \right) = -\frac{2\pi}{\log(r|P'(z_0)|/P(z_0))} -
\]

\[
-2\pi (\log(4n)) \left( \frac{1}{\log(r|P'(z_0)|/P(z_0))} \right)^2 + o \left( \left( \frac{1}{\log r} \right)^2 \right) =
\]

\[
= -\frac{2\pi}{\log r} \left( 1 - \frac{\log |P'(z_0)/P(z_0)|}{\log r} + o \left( \frac{1}{\log r} \right) \right) -
\]

\[
-2\pi (\log(4n)) \left( \frac{1}{\log r} \right)^2 + o \left( \left( \frac{1}{\log r} \right)^2 \right) = -\frac{2\pi}{\log r} -
\]

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\[-2\pi (\log |4nP(z_0)/P'(z_0)|) \left( \frac{1}{\log r} \right)^2 + o \left( \left( \frac{1}{\log r} \right)^2 \right), \quad r \to 0.\]

Substituting the asymptotics obtained into inequality (4), we arrive at the inequality

\[|a - z_0| \leq |nP(z_0)/P'(z_0)|.\]

Under the assumptions considered, the latter relation coincides with (1) \((z = z_0)\). The case of equality is verified straightforwardly.

The theorem is proved.

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