The regularity criterion for 3D Navier-Stokes Equations involving one velocity gradient component

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Abstract

In this article, we establish sufficient conditions for the regularity of solutions of Navier-Stokes equations based on one of the nine entries of the gradient tensor. We improve the recently results of C.S. Cao, E.S. Titi (Arch. Rational Mech.Anal. 202 (2011) 919-932) and Y. Zhou, M. Pokorný (Nonlinearity 23, 1097-1107 (2010)).

Keywords: 3D Navier-Stokes equations; Leray-Hopf weak solution; Regularity criterion

1. Introduction

We consider sufficient conditions for the regularity of weak solutions of the Cauchy problem for the Navier-Stokes equations

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p &= 0, \quad \text{in } \mathbb{R}^3 \times (0, T), \\
\nabla \cdot u &= 0, \quad \text{in } \mathbb{R}^3 \times (0, T), \\
u(x, 0) &= u_0, \quad \text{in } \mathbb{R}^3,
\end{aligned}
\]  

(1.1)

where \( u = (u_1, u_2, u_3) : \mathbb{R}^3 \times (0, T) \to \mathbb{R}^3 \) is the velocity field, \( p : \mathbb{R}^3 \times (0, T) \to \mathbb{R}^3 \) is a scalar pressure, and \( u_0 \) is the initial velocity field, \( \nu > 0 \) is the viscosity. We set \( \nabla_h = (\partial_{x_1}, \partial_{x_2}) \) as the horizontal gradient operator and \( \Delta_h = \partial_{x_1}^2 + \partial_{x_2}^2 \) as the horizontal Laplacian, and \( \Delta \) and \( \nabla \) are the usual Laplacian and the gradient operators, respectively. Here we use the classical notations

\[
(u \cdot \nabla)v = \sum_{i=1}^{3} u_i \partial_{x_i} v_k, \quad (k = 1, 2, 3), \quad \nabla \cdot u = \sum_{i=1}^{3} \partial_{x_i} u_i,
\]

and for sake of simplicity, we denote \( \partial_{x_i} \) by \( \partial_i \).

We set

\[
\mathcal{V} = \{ \phi : \text{the 3D vector valued } C_0^\infty \text{ functions and } \nabla \cdot \phi = 0 \},
\]

which will form the space of test functions. Let \( H \) and \( V \) be the closure spaces of \( \mathcal{V} \) in \( L^2 \) under \( L^2 \)-topology, and in \( H^1 \) under \( H^1 \)-topology, respectively.

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For \( u_0 \in H \), the existence of weak solutions of (1.1) was established by Leray \([1]\) and Hopf in \([2]\), that is, \( u \) satisfies the following properties:

(i) \( u \in C_w([0, T]; H) \cap L^2(0, T; V) \), and \( \partial_t u \in L^1(0, T; V') \), where \( V' \) is the dual space of \( V \);

(ii) \( u \) verifies (1.1) in the sense of distribution, i.e., for every test function \( \phi \in C^\infty([0, T]; V) \), and for almost every \( t, t_0 \in (0, T) \), we have

\[
\int_{\mathbb{R}^3} u(x, t) \cdot \phi(x) dx - \int_{\mathbb{R}^3} u(x, t_0) \cdot \phi(x) dx = \int_{t_0}^t \int_{\mathbb{R}^3} [u(x, t) \cdot (\phi_t(x, t) + \nu \Delta \phi(x, t))] dx ds \\
+ \int_{t_0}^t \int_{\mathbb{R}^3} [(u(x, t) \cdot \nabla) \phi(x, t) \cdot u(x, t)] dx ds;
\]

(iii) The energy inequality, i.e.,

\[
\|u(\cdot, t)\|^2_{L^2} + 2\nu \int_{t_0}^t \|\nabla u(\cdot, s)\|^2_{L^2} ds \leq \|u_0\|^2_{L^2},
\]

for every \( t \) and almost every \( t_0 \).

It is well known, if \( u_0 \in V \), a weak solution of (1.1) on \((0, T)\) becomes strong if it satisfies

\[
u \in C([0, T); V) \cap L^2(0, T; H^2) \text{ and } \partial_t u \in L^2(0, T; H).
\]

We know the strong solution is regular (say, classical) and unique (see, for example, \([3]\), \([4]\)).

For the 2D case, just as the authors said in \([5]\), the Navier-Stokes equations (1.1) have unique weak and strong solutions which exist globally in time. However, the global regularity of solutions for the 3D Navier-Stokes equations is a major and challenging problem, the weak solutions are known to exist globally in time, but the uniqueness, regularity, and continuous dependence on initial data for weak solutions are still open problems. Furthermore, strong solutions in the 3D case are known to exist for a short interval of time whose length depends on the initial data. Moreover, this strong solution is known to be unique and to depend continuously on the initial data.

There are many interesting sufficient conditions which guarantee that a given weak solution is smooth (see, for example, \([6]-[11]\)), and the first result is usually referred as Prodi-Serrin conditions (see \([12]\) and \([13]\)), which states that if a weak solution \( u \) is in the class of

\[
u \in L^t(0, T; L^s(\mathbb{R}^3)), \quad \frac{2}{t} + \frac{3}{s} = 1, \quad s \in [3, \infty],
\]

then the weak solution becomes regular.

A better result was showed by Neustupa, Novotny, and Penel (see \([14]\)). More precisely, the solution is regular if

\[
u_3 \in L^t(0, T; L^s(\mathbb{R}^3)), \quad \frac{2}{t} + \frac{3}{s} \leq \frac{1}{2}, \quad s \in (6, \infty].
\]
This result was improved in [15] to
\[ u_3 \in L^t(0, T; L^s(\mathbb{R}^3)), \quad \frac{2}{t} + \frac{3}{s} \leq \frac{5}{8}, \quad s \in \left(\frac{24}{5}, \infty\right]. \]

C.S. Cao, E.S. Titi in [5] considered the regularity of solutions to the 3D Navier-Stokes equations subject to periodic boundary conditions or in the whole space and obtained better results
\[ u_3 \in L^t(0, T; L^s(\mathbb{R}^3)), \quad \frac{2}{t} + \frac{3}{s} \leq \frac{2(s + 1)}{3s}, \quad s > \frac{7}{2}, \]
or
\[ u_3 \in L^\infty(0, T; L^s(\mathbb{R}^3)), \quad \text{with} \quad s > \frac{7}{2}. \]

Furthermore, this work was improved by Y. Zhou, M. Pokorný in [16], the authors considered the following additional assumptions to get the regularity of solution of 3D Navier-Stokes equations
\[ u_3 \in L^t(0, T; L^s(\mathbb{R}^3)), \quad \frac{2}{t} + \frac{3}{s} \leq \frac{3}{4} + \frac{1}{2s}, \quad s > \frac{10}{3}. \]

The full regularity of weak solutions can also be proved under alternative assumptions on the gradient of the velocity \( \nabla u \). Specifically (see [14]), if
\[ \nabla u \in L^t(0, T; L^s(\mathbb{R}^3)), \quad \frac{2}{t} + \frac{3}{s} = 2, \quad s \in \left[\frac{3}{2}, \infty\right]. \]

A comparable result for the gradient of one velocity component was improved in [18] to
\[ \nabla u_3 \in L^t(0, T; L^s(\mathbb{R}^3)), \quad \frac{2}{t} + \frac{3}{s} \leq \frac{3}{2}, \quad s \in [2, \infty]. \]

There are many similar results, we refer to the references [19]. In [16], the authors also studied the regularity of the solutions of the Navier-Stokes equations under the assumption on \( \partial_3 u_3 \), namely,
\[ \partial_3 u_3 \in L^\beta(0, T; L^\alpha(\mathbb{R}^3)), \quad \frac{3}{\alpha} + \frac{2}{\beta} < \frac{4}{5}, \quad \alpha \in \left(\frac{15}{4}, \infty\right]. \tag{1.2} \]

Very recently, C.S. Cao and E.S. Titi considered the more general case in [20], in which authors provided sufficient conditions, in terms of only one of the nine components of the gradient of velocity field (i.e., the velocity Jacobian matrix) that guarantee the global regularity of the 3D Navier-Stokes equations. The authors divided into cases to discuss the regularity of the weak solution, namely, given the condition
\[ \frac{\partial u_j}{\partial x_k} \in L^\beta(0, T; L^\alpha(\mathbb{R}^3)), \quad \text{when} \quad j \neq k \]
and where \( \alpha > 3, 1 \leq \beta < \infty, \) and
\[ \frac{3}{\alpha} + \frac{2}{\beta} \leq \frac{\alpha + 3}{2\alpha}, \tag{1.3} \]
or

\[
\frac{\partial u_j}{\partial x_j} \in L^\beta(0, T; L^\alpha(\mathbb{R}^3)),
\]

and where \( \alpha > 2, 1 \leq \beta < \infty \), and \( \frac{3}{\alpha} + \frac{2}{\beta} \leq \frac{3(\alpha + 2)}{4\alpha} \). \hspace{1cm} (1.4)

Moreover, Z.J. Zhang studied the Cauchy problem for the 3D Navier-Stokes equations, and proved some scaling-invariant regularity criteria involving only one velocity component in [21]. The author proved that the weak solution \( u \) to \((1.1)\) with datum \( u_0 \in V \) is regular, if

\[
u_3 \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \partial_3u_3 \in L^\beta(0, T; L^\alpha(\mathbb{R}^3)), \]

with \( 1 \leq p, q, \beta, \alpha \leq \infty \), \( 0 \leq \lambda, \gamma < \infty \) satisfying

\[
\begin{cases}
\frac{2}{p} + \frac{3}{q} = \lambda, & \frac{3}{\beta} + \frac{3}{\alpha} = \gamma, \\
(1 - \frac{1}{\alpha})q = \frac{1/\beta + 3/8}{3/8 - 1/p} = \frac{9/4 - \gamma}{\lambda - 3/4} > 1, \\
\beta < \infty \text{ or } p < \infty.
\end{cases}
\] \hspace{1cm} (1.6)

Motivated by [20] and [21], in this article, we consider the alternative assumptions on one velocity gradient component, and we improve the results of [20]. The key point of our approach is that we start with the estimate of the norm \( \|u_3\|_q \), where \( q \) satisfies \( q \geq 2 \), and then construct some new estimates. We also improve the result of [16]. From our argument, one can know which cause the difference of the both results in [16] and [20].

Our main results can be stated in the following:

**Theorem 1.1.** Let \( u_0 \in V \), and assume \( u \) is a Leray-Hopf weak solution to the 3D Navier-Stokes equations \((1.1)\). Suppose for any \( j, k \) with \( 1 \leq j, k \leq 3 \), we have

\[
\partial_j u_k \in L^\infty(0, T; L^3(\mathbb{R}^3)).
\] \hspace{1cm} (1.7)

Then \( u \) is regular.

**Theorem 1.2.** Let \( u_0 \) and \( u \) be as in Theorem [1.1]. For any \( j, k \) with \( 1 \leq j, k \leq 3 \).

(i) For \( j \neq k \), suppose that \( u \) satisfies

\[
\int_0^T \|\partial_j u_k\|_q^2 d\tau \leq M, \text{ for some } M > 0,
\] \hspace{1cm} (1.8)

with

\[
\frac{3}{2\alpha} + \frac{2}{\beta} \leq f(\alpha), \quad \alpha \in (3, \infty) \text{ and } 1 \leq \beta < \infty,
\] \hspace{1cm} (1.9)

where

\[
f(\alpha) = \frac{\sqrt{103\alpha^2 - 12\alpha + 9} - 9\alpha}{2\alpha};
\] \hspace{1cm} (1.10)
(ii) For \( j = k \), suppose
\[
\int_0^T \| \partial_k u_k \|^\beta \alpha \, d\tau \leq M, \text{ for some } M > 0,
\]
with
\[
\frac{3}{2\alpha} + \frac{2}{\beta} \leq g(\alpha), \quad \frac{9}{5} < \alpha < \infty \text{ and } 1 \leq \beta < \infty,
\]
where
\[
g(\alpha) = \frac{\sqrt{289\alpha^2 - 264\alpha + 144} - 7}{8\alpha}.
\]

Then \( u \) is regular.

**Remark 1.3.** Theorem 1.1 gives us an endpoint version of regularity criterion, which is a complement of [20]. From the proof of it, one can know this result in fact should have been included in [20]. Compared with the results of [20], it is easy to check that Theorem 1.2 (i) is an improvement of (1.3) (see Figure 1). For Theorem 1.2 (ii), the allowed region of \((\alpha, \beta)\) in our result is much larger than those of [16] and [20] (see Figure 2).

For the convenience, we recall the following version of the three-dimensional Sobolev and Ladyzhenskaya inequalities in the whole space \( \mathbb{R}^3 \) (see, for example, [22] - [24]). There exists a positive constant \( C \) such that
\[
\| u \|_r \leq C \| u \|_{3/2} \| \partial_1 u \|_{2/r} \| \partial_2 u \|_{2/r} \| \partial_3 u \|_{2/r},
\]
for every \( u \in H^1(\mathbb{R}^3) \) and every \( r \in [2, 6] \), where \( C \) is a constant depending only on \( r \). Taking \( \nabla \text{div} \) on both sides of (1.1) for smooth \((u, p)\), one can obtain
\[
-\Delta (\nabla p) = \sum_{i,j} \partial_i \partial_j (\nabla (u_i u_j)),
\]
therefore, the Calderon-Zygmund inequality in \( \mathbb{R}^3 \) (see [25])
\[
\| \nabla p \|_q \leq C \| \nabla u \|_{3/2} \| u \|_q, 1 < q < \infty,
\]
holds, where \( C \) is a positive constant depending only on \( q \). And there is another estimate for pressure
\[
\| p \|_q \leq C \| u \|_{3/2}^2, \quad 1 < q < \infty.
\]

Let
\[
1 \leq q_i < \infty, \quad i = 1, ..., n.
\]
Then, for all \( u \in C_0^\infty(\mathbb{R}^n) \) the following Troisi inequality holds (see [23]):
\[
\| u \|_s \leq C \prod_{i=1}^n \| D_i u \|_{q_i}^{1/n},
\]
where \( q_i, n \) and \( s \) satisfy
\[
\sum_{i=1}^n q_i^{-1} > 1, \quad \text{and } s = \frac{n}{\sum_{i=1}^n q_i^{-1} - 1}.
\]
2. *A Priori* Estimates

In this section, under the assumptions of Theorem [1.1 1.2] we will prove some *a priori* estimates, which are needed in the proof of our results. First of all, we note that the Leray-Hopf weak solutions have the energy inequality (see, for example, [3], [4], [22] for detail)

\[ \| u(\cdot, t) \|_{L^2}^2 + 2\nu \int_0^t \| \nabla u(\cdot, s) \|_{L^2}^2 ds \leq K_1, \quad (2.1) \]

for all \( 0 < t < T \), where \( K_1 = \| u_0 \|_{L^2}^2 \).

Then, an estimate of \( \| u_3 \|_q \) can be read in the following lemma:

**Lemma 2.1.** Assume that

\[ 3 \leq \alpha < \infty, \quad 1 < \sigma \leq \frac{9}{8}, \quad \text{and} \quad q \geq 2, \quad (2.2) \]

where \( q, \alpha, \) and \( \sigma \) satisfy

\[ \frac{1}{\sigma} + \frac{q - 2}{q} + \frac{1}{3\alpha} = 1, \quad (2.3) \]

then we have the following estimates

\[ \frac{1}{2} \frac{d}{dt} \| u_3 \|_q^q + C(q)\nu \| \nabla u_3 \|_{L^2}^{\frac{2}{3}-s} \| \partial_1 u_3 \|_{L^\infty}^{1/3}, \quad \text{with} \quad s = \frac{3 - 2\sigma}{\sigma}. \quad (2.4) \]

**Proof.** We use \( |u_3|^{q-2} u_3, q \geq 2, \) as test function in the equation (1.1) for \( u_3 \). By using of Gagliardo-Nirenberg, Hölder’s inequalities, (1.14) and (1.16), we have

\[
\frac{1}{q} \frac{d}{dt} \| u_3 \|_q^q + C(q)\nu \| \nabla u_3 \|_{L^2}^{\frac{2}{3}-s} \| \partial_1 u_3 \|_{L^\infty}^{1/3} \leq \int_{\mathbb{R}^3} \partial_3 p|u_3|^{q-2} u_3 dx \\
\leq \| \partial_3 p \|_{L^\infty} \| u_3 \|_{L^q}^{q-2} \| u_3 \|_{L^{3\alpha}} \\
\leq C \| \nabla u \|_{L^\infty} T\| u_3 \|_{L^q}^{q-2} \| u_3 \|_{L^{3\alpha}} \| \partial_1 u_3 \|_{L^\infty}^{1/3} \| \partial_2 u_3 \|_{L^2}^{1/3} \| \partial_3 u_3 \|_{L^2}^{1/3} \]  

\[
\leq C \| \nabla u \|_2 \| u \|_{L^\infty}^{2-\sigma} \| u_3 \|_{L^{3\alpha}}^{q-2} \| \partial_1 u_3 \|_{L^\infty}^{1/3} \| \nabla u \|_{L^2}^{2/3} \]  

\[
\leq C \| u \|_2 \| \nabla u \|_2^{\frac{8}{3}-s} \| u_3 \|_{L^{3\alpha}}^{q-2} \| \partial_1 u_3 \|_{L^\infty}^{1/3}. \]

For the index in above inequality, we know \( 1 < \sigma \leq \frac{9}{8} < \frac{3}{2}, \)

\[ 2 < \frac{2\sigma}{2 - \sigma} < 6, \]

and \( s \) satisfies

\[ \frac{2 - \sigma}{2\sigma} = \frac{s}{2} + \frac{1 - s}{6}, \text{ namely, } s = \frac{3 - 2\sigma}{\sigma}, \text{ by (2.2)}. \]
In view of (2.1), (2.5) implies that

\[
\frac{1}{2} \frac{d}{dt} \|u_3\|_q^2 \leq C \|\nabla u\|_2^{\frac{8}{3} - s} \|\partial_1 u_3\|_\alpha^{1/3}.
\]

The proof is thus completed. \(\square\)

Next, we estimate \(\|\nabla h u\|_2\):

**Lemma 2.2.** Assume that \(\alpha\) and \(q\) satisfy the conditions in Lemma 2.1. Set

\[
r = \frac{(q + 1)\alpha - q}{\alpha},
\]

then we have the following estimates

\[
\|\nabla h u\|_2^2 + \nu \int_0^t \|\nabla h \nabla u\|_2^2 d\tau \leq \|\nabla h u(0)\|_2^2
\]

\[
+ C \int_0^t \|u_3\|_q^{\frac{2(r-1)}{r-2}} \|\partial_1 u_3\|_\alpha^{\frac{r}{r-1}} \|\nabla u\|_2^{\frac{r-2}{r-1}} \|
\]

\[
dt \|\Delta u\|_2^{\frac{r-2}{r-1}} \|
\]

(2.7)

and

\[
\|\nabla h u\|_2^2 + \nu \int_0^t \|\nabla h \nabla u\|_2^2 d\tau
\]

\[
\leq \|\nabla h u(0)\|_2^2 + C \int_0^t \|u_3\|_q^{\frac{2(r-1)}{r-2}} \|\partial_3 u_3\|_\alpha^{\frac{2}{r-2}} \|\nabla u\|_2^2 d\tau.
\]

**Proof.** Taking the inner product of the equation (1.1) with \(-\Delta h u\) in \(L^2\), applying Hölder’s inequality several times, we obtain

\[
\frac{1}{2} \frac{d}{dt} \|\nabla h u\|_2^2 + \nu \|\nabla h \nabla u\|_2^2
\]

\[
= \int_{\mathbb{R}^3} [(u \cdot \nabla) u] \Delta h u dx
\]

\[
\leq C \int_{\mathbb{R}^3} |u_3| |\nabla u| |\nabla h \nabla u| dx \quad \text{(see [20])}
\]

\[
\leq C \int_{\mathbb{R}^2} \max_{x_1} |u_3| (\int_{\mathbb{R}} |\nabla u|^2 dx_1)^{\frac{1}{2}} (\int_{\mathbb{R}} |\nabla h \nabla u|^2 dx_1)^{\frac{1}{2}} dx_2 dx_3
\]

\[
\leq C \int_{\mathbb{R}^2} (\max_{x_1} |u_3|)^r dx_2 dx_3 \left[ \int_{\mathbb{R}} (\int_{\mathbb{R}} |\nabla u|^2 dx_1)^{\frac{r}{r-2}} dx_2 dx_3 \right]^\frac{r-2}{2r}
\]

\[
\times \left[ \int_{\mathbb{R}^2} |\nabla h \nabla u|^2 dx_1 dx_2 dx_3 \right]^{\frac{1}{2}}
\]

(2.9)

\[
\leq C \int_{\mathbb{R}^3} |u_3|^{r-1} \partial_1 u_3 dx_2 dx_3 \left[ \int_{\mathbb{R}} (\int_{\mathbb{R}} |\nabla u|^2 dx_1)^{\frac{r}{r-2}} dx_2 dx_3 \right]^\frac{r-2}{2r}
\]

\[
\times \left[ \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}} |\nabla u| \right|^{\frac{r-2}{2}} dx_2 dx_3 \left| \int_{\mathbb{R}} \right|^{\frac{r-2}{2}} dx_1 \right]^{\frac{1}{2}}
\]

\[
\leq C \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}} |\nabla u| \right|^{\frac{r-2}{2}} dx_2 dx_3 \left| \int_{\mathbb{R}} \right|^{\frac{r-2}{2}} dx_1 \right]^{\frac{1}{2}}
\]

\[
\leq C \|u_3\|_q^{\frac{2}{r-1}} \|\partial_1 u_3\|_2^{\frac{1}{r-1}} \|\nabla u\|_2^{\frac{r-2}{r}} \|\partial_2 u\|_2 \|\partial_3 u\|_2 \|\nabla h \nabla u\|_2.
\]
By (2.6), and the fact \( \alpha \geq 3, q \geq 2 \), we have
\[
\frac{q}{q - r + 1} = \alpha, \quad \text{and} \quad \frac{7}{3} \leq r < q + 1.
\]  
(2.10)

To prove (2.7), applying Young’s inequality to (2.9), we obtain
\[
\frac{1}{2} \frac{d}{dt} \| \nabla_h u \|_2^2 + \nu \| \nabla_h \nabla u \|_2^2 \\
\leq C \| u_3 \|_q^2 \| \partial_1 u_3 \|_{\alpha-1}^2 \| \nabla u \|_2^{2r-1} \| \Delta u \|_2^{-\frac{r}{2}} + \nu \| \nabla_h \nabla u \|_2^2.
\]  
(2.10)

Absorbing the last term in right hand side and integrating the above inequality, using Hölder’s inequality, we have
\[
\| \nabla_h u \|_2^2 + \nu \int_0^t \| \nabla_h \nabla u \|_2^2 d\tau \\
\leq \| \nabla_h u(0) \|_2^2 + C \int_0^t \| u_3 \|_q^2 \| \partial_1 u_3 \|_{\alpha-1}^2 \| \nabla u \|_2^{2r-1} \| \Delta u \|_2^{-\frac{r}{2}} d\tau \\
\leq \| \nabla_h u(0) \|_2^2 + C \int_0^t \| u_3 \|_q^{2r-2} \| \partial_1 u_3 \|_{\alpha-2}^2 \| \nabla u \|_2^2 d\tau \times \left[ \| \Delta u \|_2^2 d\tau \right]^{-\frac{1}{2}}.
\]

To prove (2.8), firstly, we note that we can get a similar inequality to (2.9) as follows
\[
\frac{1}{2} \frac{d}{dt} \| \nabla_h u \|_2^2 + \nu \| \nabla_h \nabla u \|_2^2 \\
\leq C \| u_3 \|_q^{2r-2} \| \partial_3 u_3 \|_{\alpha-2}^2 \| \nabla u \|_2^2 \| \partial_2 \nabla u \|_2^2 \| \nabla_h \nabla u \|_2^2.
\]  
(2.11)

Applying Young’s inequality to (2.11), we have
\[
\frac{1}{2} \frac{d}{dt} \| \nabla_h u \|_2^2 + \nu \| \nabla_h \nabla u \|_2^2 \\
\leq C \| u_3 \|_q^{2r-2} \| \partial_3 u_3 \|_{\alpha-2}^2 \| \nabla u \|_2^2 + \frac{\nu}{2} \| \nabla_h \nabla u \|_2^2.
\]  
(2.12)

As above, absorbing the last term in right hand side of (2.12) and integrating the above inequality, using Hölder’s inequality, we obtain
\[
\| \nabla_h u \|_2^2 + \nu \int_0^t \| \nabla_h \nabla u \|_2^2 d\tau \leq \| \nabla_h u(0) \|_2^2 + C \int_0^t \| u_3 \|_q^{2r-2} \| \partial_3 u_3 \|_{\alpha-2}^2 \| \nabla u \|_2^2 d\tau
\]

The proof of Lemma 2.2 is completed.

At last, we estimate \( \| \nabla u \|_2 \) :
Lemma 2.3. Let $\alpha$, $q$ and $r$ satisfy Lemma 2.1 and Lemma 2.2, then we have the following estimates:

(i),

$$
\|\nabla u\|^2_2 + \nu \int_0^t \|\Delta u\|^2_2 d\tau \\
\leq C \int_0^t \|u_3\|^{\frac{8(r-1)}{q-2}}_q \|\partial_1 u_3\|^{\frac{8}{\alpha}} \|\nabla u\|^2_2 d\tau + \|\nabla u(0)\|^2_2 
$$

(2.13)

$$
+ C \int_0^t \|u_3\|^{\frac{2(r-1)}{q-2}}_q \|\partial_1 u_3\|^{\frac{2}{\alpha}} \|\nabla u\|^2_2 d\tau + C
$$

if $3r - 7 > 0$.

(ii),

$$
\|\nabla u\|^2_2 + \frac{5\nu}{4} \int_0^t \|\Delta u\|^2_2 d\tau \\
\leq \|\nabla u(0)\|^2_2 + C \int_0^t \|u_3\|^6_q \|\partial_1 u_3\|^6_\alpha \|\nabla u\|^2_2 d\tau + C
$$

(2.14)

$$
+ C \int_0^t \|u_3\|^{\frac{8(r-1)}{q-2}}_q \|\partial_3 u_3\|^{\frac{8}{3(r-2)}} \|\nabla u\|^2_2 d\tau + C
$$

if $3r - 7 = 0$. Moreover, we have

$$
\|\nabla u\|^2_2 + \frac{\nu}{2} \int_0^t \|\Delta u\|^2_2 d\tau \\
\leq \|\nabla u(0)\|^2_2 + C \int_0^t \|u_3\|^{\frac{2(r-1)}{q-2}}_q \|\partial_3 u_3\|^{\frac{2}{\alpha}} \|\nabla u\|^2_2 d\tau
$$

(2.15)

$$
+ C \int_0^t \|u_3\|^{\frac{8(r-1)}{q-2}}_q \|\partial_3 u_3\|^{\frac{8}{3(r-2)}} \|\nabla u\|^2_2 d\tau + C
$$

Proof. Taking the inner product of the equation (1.1) with $-\Delta u$ in $L^2$, we obtain

$$
\frac{1}{2} \frac{d}{dt} \|\nabla u\|^2_2 + \nu \|\Delta u\|^2_2 \\
= \sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} u_i \partial_i u_j \partial_k u_j dx
$$

$$
= \sum_{i,j,k=1}^2 \int_{\mathbb{R}^3} u_i \partial_i u_j \partial_k u_j dx + \sum_{i=1}^3 \sum_{j,k=1}^2 \int_{\mathbb{R}^3} u_i \partial_i u_j \partial_3 u_j dx
$$

$$
+ \sum_{j,k=1}^2 \sum_{j,k=1}^3 \int_{\mathbb{R}^3} u_3 \partial_3 u_j \partial_k u_j dx + \sum_{i,k=1}^3 \int_{\mathbb{R}^3} u_i \partial_i u_3 \partial_k u_3 dx
$$

$$
= I_1(t) + I_2(t) + I_3(t) + I_4(t).
$$

The calculation is similar to Lemma 2.2 in [26], for the convenience of readers, we show it below. By integrating by parts several times and using the incompressibility condition, we get

$$
I_1(t) = \frac{1}{2} \sum_{i,j,k=1}^2 \int_{\mathbb{R}^3} \partial_i u_i \partial_k u_j \partial_k u_j dx - \sum_{i,j,k=1}^2 \int_{\mathbb{R}^3} \partial_k u_i \partial_i u_j \partial_k u_j dx = I_1^1(t) + I_1^2(t).
$$
The terms $I^1_1(t), I^1_2(t), I_3(t)$ and $I_4(t)$ read as

$$I^1_1(t) = -\frac{1}{2} \sum_{j,k=1}^{2} \int_{\mathbb{R}^3} \partial_3 u_3 \partial_k u_j \partial_k u_j dx,$$

$$I^2_1(t) = -\sum_{i,j,k=1}^{2} \int_{\mathbb{R}^3} \partial_k u_i \partial_i u_j \partial_k u_j dx$$

$$= \int_{\mathbb{R}^3} \partial_2 u_1 \partial_1 u_2 \partial_2 u_2 dx + \int_{\mathbb{R}^3} \partial_1 u_2 \partial_2 u_1 \partial_1 u_1 dx + \int_{\mathbb{R}^3} \partial_1 u_1 \partial_1 u_2 \partial_2 u_2 dx$$

$$+ \int_{\mathbb{R}^3} \partial_2 u_1 \partial_2 u_1 \partial_1 u_1 dx + \int_{\mathbb{R}^3} \partial_1 u_1 \partial_2 u_2 \partial_1 u_2 dx$$

$$+ \int_{\mathbb{R}^3} \partial_2 u_2 \partial_1 u_1 \partial_2 u_2 dx + \int_{\mathbb{R}^3} \partial_2 u_2 \partial_2 u_1 \partial_1 u_2 dx$$

$$= -\int_{\mathbb{R}^3} (\partial_2 u_1 \partial_1 u_2 \partial_3 u_3 + \partial_3 u_3 \partial_1 u_2 \partial_2 u_2 + \partial_2 u_1 \partial_3 u_3 \partial_2 u_1) dx$$

$$- \int_{\mathbb{R}^3} (\partial_1 u_1 \partial_1 u_1 \partial_3 u_3 + \partial_3 u_3 \partial_2 u_2 \partial_2 u_2 - \partial_1 u_1 \partial_3 u_3 \partial_2 u_2) dx,$$

$$I_3(t) = \sum_{j=1}^{2} \sum_{k=1}^{2} \int_{\mathbb{R}^3} u_3 \partial_3 u_3 \partial_3 u_j \partial_3 u_j dx$$

$$= -\sum_{j=1}^{2} \sum_{k=1}^{2} \int_{\mathbb{R}^3} \partial_3 u_3 \partial_3 u_j \partial_3 u_j dx - \sum_{j=1}^{2} \sum_{k=1}^{2} \int_{\mathbb{R}^3} u_3 \partial_3 u_j \partial_3 u_j dx$$

$$= -\sum_{j=1}^{2} \sum_{k=1}^{2} \int_{\mathbb{R}^3} \partial_3 u_3 \partial_3 u_j \partial_3 u_j dx + \frac{1}{2} \sum_{j=1}^{2} \sum_{k=1}^{2} \int_{\mathbb{R}^3} \partial_3 u_3 \partial_3 u_j \partial_3 u_j dx,$$

$$I_4(t) = \sum_{i,k=1}^{3} \int_{\mathbb{R}^3} u_i \partial_i u_3 \partial_3 u_3 dx$$

$$= -\sum_{i,k=1}^{3} \int_{\mathbb{R}^3} \partial_3 u_3 \partial_i u_3 \partial_3 u_3 dx - \sum_{i,k=1}^{3} \int_{\mathbb{R}^3} u_i \partial_3 u_3 \partial_3 u_3 dx$$

$$= -\sum_{i,k=1}^{3} \int_{\mathbb{R}^3} \partial_3 u_3 \partial_i u_3 \partial_3 u_3 dx.$$
Thus, above inequalities imply that
\[
\frac{1}{2} \frac{d}{dt} \|\nabla u\|^2 + \nu \|\Delta u\|^2 \leq C \int_{\mathbb{R}^3} |u_3| \|\nabla u\| \|\Delta u\| dx + C \int_{\mathbb{R}^3} |\nabla_h u| \|\nabla u\|^2 dx.
\]
\[
= J_1(t) + J_2(t).
\]  
(2.16)

Similar to (2.9), by Young’s and Hölder’s inequalities, we have
\[
J_1(t) \leq C \|u_3\|^{\frac{r-1}{2}} \|\partial_t u_3\|^{\frac{1}{2}} \|\nabla u\|^{\frac{r-2}{2}} \|\Delta u\|^{\frac{r+2}{2}},
\]
\[
\leq C \|u_3\|^{\frac{2(r-1)}{q-2}} \|\partial_t u_3\|^{\frac{2}{\alpha-2}} \|\nabla u\|^2 + \frac{\nu}{4} \|\Delta u\|^2,
\]  
(2.17)

and
\[
J_2(t) \leq C \|\nabla_h u\| \|\nabla u\|^2 \leq C \|\nabla_h u\| \|\nabla u\|^2 \|\Delta u\|^\frac{1}{2}.
\]  
(2.18)

Integrating (2.16) and combing (2.1), (2.7), (2.17) and (2.18), we obtain
\[
\|\nabla u\|^2 + \frac{7\nu}{4} \int_0^t \|\Delta u\|^2 d\tau \\
\leq \|\nabla u(0)\|^2 + C \int_0^t \|u_3\|^{\frac{2(r-1)}{q-2}} \|\partial_t u_3\|^{\frac{2}{\alpha-2}} \|\nabla u\|^2 d\tau \\
+ (\sup_{0 \leq s \leq t} \|\nabla_h u\|)(\int_0^t \|\nabla u\|^2 d\tau)^\frac{1}{2} \\
\times (\int_0^t \|\nabla_h \nabla u\|^2 d\tau)^\frac{1}{2} (\int_0^t \|\Delta u\|^2 d\tau)^\frac{1}{2} \\
\leq \|\nabla u(0)\|^2 + C \int_0^t \|u_3\|^{\frac{2(r-1)}{q-2}} \|\partial_t u_3\|^{\frac{2}{\alpha-2}} \|\nabla u\|^2 d\tau \\
+ C \int_0^t \|u_3\|^{\frac{2(r-1)}{q-2}} \|\partial_t u_3\|^{\frac{2}{\alpha-2}} \|\nabla u\|^2 d\tau \\
\times [\int_0^t \|\Delta u\|^2 d\tau]^\frac{1}{2} + \frac{3\nu}{4} \int_0^t \|\Delta u\|^2 d\tau \\
+ \|\nabla u(0)\|^2 (\int_0^t \|\Delta u\|^2 d\tau)^\frac{1}{4}.
\]  
(2.19)

If \(3r - 7 > 0\), we have
\[
\frac{1}{r-1} + \frac{1}{4} < 1,
\]
then, by Hölder’s and Young’s inequalities, from (2.19) we get
\[
\|\nabla u\|^2 + \frac{7\nu}{4} \int_0^t \|\Delta u\|^2 d\tau \\
\leq \|\nabla u(0)\|^2 + C \int_0^t \|u_3\|^{\frac{2(r-1)}{q-2}} \|\partial_t u_3\|^{\frac{2}{\alpha-2}} \|\nabla u\|^2 d\tau \\
+ C \int_0^t \|u_3\|^{\frac{2(r-1)}{q-2}} \|\partial_t u_3\|^{\frac{2}{\alpha-2}} \|\nabla u\|^2 d\tau \\
\times [\int_0^t \|\Delta u\|^2 d\tau]^\frac{1}{2} + \frac{3\nu}{4} \int_0^t \|\Delta u\|^2 d\tau \\
\leq C \int_0^t \|u_3\|^{\frac{8(r-1)}{3r-7}} \|\partial_t u_3\|^{\frac{8}{3r-7}} \|\nabla u\|^2 d\tau \times [\int_0^t \|\Delta u\|^2 d\tau]^\frac{1}{2} \\
+ \|\nabla u(0)\|^2 + C \int_0^t \|u_3\|^{\frac{2(r-1)}{q-2}} \|\partial_t u_3\|^{\frac{2}{\alpha-2}} \|\nabla u\|^2 d\tau + \frac{3\nu}{4} \int_0^t \|\Delta u\|^2 d\tau.
\]
Absorbing the last term and applying (2.1), it follows that
\[
\| \nabla u \|^2 + \nu \int_0^t \| \Delta u \|^2 d\tau \leq C \int_0^t \| u_3 \|_{q/2}^{8(\epsilon - 1)} \| \partial_1 u_3 \|_{\alpha}^{\frac{8}{\alpha - 2}} \| \nabla u \|^2 d\tau + \| \nabla u(0) \|^2 \\
+ C \int_0^t \| u_3 \|_{q/2}^{2(\epsilon - 1)} \| \partial_1 u_3 \|_{\alpha}^{\frac{2}{\alpha - 2}} \| \nabla u \|^2 d\tau + C,
\]
therefore we prove (2.13). If \( 3r - 7 = 0 \), we have
\[
\frac{1}{r - 1} + \frac{1}{4} = 1,
\]
then by Young’s inequality and (2.19) we have
\[
\| \nabla u \|^2 + \frac{5\nu}{4} \int_0^t \| \Delta u \|^2 d\tau \\
\leq \| \nabla u(0) \|^2 + C \int_0^t \| u_3 \|_{q/2} \| \partial_1 u_3 \|_{\alpha}^{\frac{6}{\alpha - 2}} \| \nabla u \|^2 d\tau + C \\
+ C \int_0^t \| u_3 \|_{q/2} \| \partial_1 u_3 \|_{\alpha}^{\frac{6}{\alpha - 2}} \| \nabla u \|^2 d\tau \times \int_0^t \| \Delta u \|^2 d\tau,
\]
which shows that the proof of (ii) is completed.

Finally, we prove (2.15) in a similar way, integrating (2.16) and using (2.1), (2.8), (2.17) and (2.18), we obtain
\[
\| \nabla u \|^2 + \frac{\nu}{2} \int_0^t \| \Delta u \|^2 d\tau \\
\leq \| \nabla u(0) \|^2 + C \int_0^t \| u_3 \|_{q/2}^{2(\epsilon - 1)} \| \partial_3 u_3 \|_{\alpha}^{\frac{2}{\alpha - 2}} \| \nabla u \|^2 d\tau \\
+ \left( \sup_{0 \leq s \leq t} \| \nabla_h u \|_2 \right) \left( \int_0^t \| \nabla u \|^2 d\tau \right)^{\frac{1}{2}} \\
\times \left( \int_0^t \| \nabla u \|^2 d\tau \right)^{\frac{1}{2}} \left( \int_0^t \| \Delta u \|^2 d\tau \right)^{\frac{1}{2}} \\
\leq \| \nabla u(0) \|^2 + C \int_0^t \| u_3 \|_{q/2}^{2(\epsilon - 1)} \| \partial_3 u_3 \|_{\alpha}^{\frac{2}{\alpha - 2}} \| \nabla u \|^2 d\tau \\
+ C \int_0^t \| u_3 \|_{q/2}^{2(\epsilon - 1)} \| \partial_3 u_3 \|_{\alpha}^{\frac{2}{\alpha - 2}} \| \nabla u \|^2 d\tau \times \left[ \int_0^t \| \Delta u \|^2 d\tau \right]^{\frac{1}{2}} \\
\times C \int_0^t \| u_3 \|_{q/2}^{2(\epsilon - 1)} \| \partial_3 u_3 \|_{\alpha}^{\frac{2}{\alpha - 2}} \| \nabla u \|^2 d\tau + C.
\]
By using Hölder’s and Young’s inequalities, we immediately have
\[
\| \nabla u \|^2 + \frac{\nu}{2} \int_0^t \| \Delta u \|^2 d\tau \leq \| \nabla u(0) \|^2 + C \int_0^t \| u_3 \|_{q/2}^{2(\epsilon - 1)} \| \partial_3 u_3 \|_{\alpha}^{\frac{2}{\alpha - 2}} \| \nabla u \|^2 d\tau \\
+ C \int_0^t \| u_3 \|_{q/2}^{\frac{8(\epsilon - 1)}{3(\epsilon - 2)}} \| \partial_3 u_3 \|_{\alpha}^{\frac{8}{\alpha - 2}} \| \nabla u \|^2 d\tau + C.
\]
Therefore, we complete the proof of Lemma 2.3. \( \square \)
3. Proof of Main Results

In this section, we prove our main results.

Proof of Theorem 1.1 The framework of the proof is standard, we refer to [5].
Without loss of generality, in the proof, we will assume that \( j = 1, k = 3 \), the other cases can be discussed in the same way (for details see Remark 3.1 below).

It is well known that there exists a unique strong solution \( u \) for a short time interval if \( u_0 \in V \). In addition, this strong solution \( u \in C([0, T^*); V) \cap L^2(0, T^*; H^2(\mathbb{R}^3)) \) is the only weak solution with the initial datum \( u_0 \), where \( (0, T^*) \) is the maximal interval of existence of the unique strong solution. If \( T^* \geq T \), then there is nothing to prove. If, on the other hand, \( T^* < T \), then our strategy is to show that the \( H^1 \) norm of this strong solution is bounded uniformly in time over the interval \( (0, T^*) \), provided condition (1.7) is valid. As a result the interval \( (0, T^*) \) can not be a maximal interval of existence, and consequently \( T^* \geq T \), which concludes our proof.

In order to prove the \( H^1 \) norm of the strong solution \( u \) is bounded on interval \( (0, T^*) \), combing with the energy equality (2.1), it is sufficient to prove

\[
\|\nabla u\|_2^2 + \nu \int_0^t \|\Delta u\|_2^2 d\tau \leq C, \quad \forall \ t \in (0, T^*), \tag{3.1}
\]

where the constant \( C \) depends on \( T, K_1 \) and \( M \).

Firstly, by energy inequality (2.1), we have

\[
\|u_3\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \leq C, \tag{3.2}
\]

where \( C \) depends only on \( K_1 \). Then we show (3.1) is true on a small interval \( (0, t_1) \) with some \( 0 < t_1 < T^* \), because the constant \( C \) in (3.1) depends only on \( K_1 \) and \( M \), we give the same process to treat \( t_1 \) as the start point. After finite steps, we get (3.1) holds true on the whole interval \( (0, T^*) \).

Now, by using of Lemma 2.1, Lemma 2.2 and Lemma 2.3 with \( \alpha = 3, \sigma = \frac{9}{8} \) and \( q = 2 \), then form (2.4) and (2.6), we have

\[
r = \frac{7}{3}, \quad s = \frac{2}{3}. \tag{3.3}
\]

Applying (3.2) and (3.3), (2.14) becomes

\[
\|\nabla u\|_2^2 + \frac{5\nu}{4} \int_0^t \|\Delta u\|_2^2 d\tau \\
\leq C \int_0^t \|u_3\|_3^2 \|\partial_1 u_3\|_3^2 \|\nabla u\|_2^2 d\tau + \|\nabla u(0)\|_2^2 + C \\
+ C \int_0^t \|u_3\|_3^2 \|\partial_1 u_3\|_3^2 \|\nabla u\|_2^2 d\tau \frac{1}{2} \times \int_0^t \|\Delta u\|_2^2 d\tau. \tag{3.4}
\]

In view of (1.7), we can choose \( 0 < t_1 < T^* \) small enough such that

\[
C \|u_3\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}^2 \|\partial_1 u_3\|_{L^\infty(0,T;L^3(\mathbb{R}^3))} \int_0^{t_1} \|\nabla u\|_2^2 d\tau \frac{1}{2} \leq \frac{\nu}{4}. \]
Then (3.4) becomes
\[ \sup_{0 \leq t \leq t_1} \| \nabla u \|_2^2 + \nu \int_0^{t_1} \| \Delta u \|_2^2 d\tau \leq \| \nabla u(0) \|_2^2 + C. \tag{3.5} \]

From (3.5), we have
\[ \| \nabla h(u(t_1)) \|_2 < \| \nabla u(0) \|_2 + C, \quad \| \nabla u(t_1) \|_2 < \| \nabla u(0) \|_2 + C, \]
so we can repeat the above argument with initial value at \( t_1 \) to obtain the similar estimate (2.7) in Lemma 2.2, and we have, for \( t_1 < t < T^* \),
\[ \| \nabla h(u(t_1)) \|_2^2 + \nu \int_{t_1}^t \| \Delta u \|_2^2 d\tau \leq \| \nabla u(t_1) \|_2^2 \times \| \nabla u(t_1) \|_2^2 \]
\[ + \| \nabla h(u(t_1)) \|_2^2. \]

From above inequality, we obtain a similar estimate as (3.4), for \( t_1 < t < T^* \),
\[ \| \nabla u \|_2^2 + \frac{5\nu}{4} \int_{t_1}^t \| \Delta u \|_2^2 d\tau \]
\[ \leq C \int_{t_1}^t \| u_3 \|_{L^3} \| \partial_1 u_3 \|_{L^6} \| \nabla u \|_2^2 d\tau + \| \nabla u(t_1) \|_2^2 + C \]
\[ + C \int_{t_1}^t \| u_3 \|_{L^3} \| \partial_1 u_3 \|_{L^6} \| \nabla u \|_2^2 d\tau \times \int_{t_1}^t \| \Delta u \|_2^2 d\tau. \]

There exists a number \( t_2 \) such that
\[ C \| u_3 \|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \| \partial_1 u_3 \|_{L^\infty(0,T;L^3(\mathbb{R}^3))} \int_{t_1}^{t_2} \| \nabla u \|_2^2 d\tau \frac{\nu}{4}, \]
and we have
\[ \sup_{t_1 \leq t \leq t_2} \| \nabla u \|_2^2 + \nu \int_{t_1}^{t_2} \| \Delta u \|_2^2 d\tau \leq \| \nabla u(t_1) \|_2^2 + C \leq \| \nabla u(0) \|_2^2 + C. \]

Then we can repeat the above process from \( t_2 \), if \( t_2 < T^* \). Actually, since \( \partial_1 u_3 \in L^\infty(0,T;L^3(\mathbb{R}^3)) \), and the coefficients involving \( \| \partial_1 u_3 \|_{L^\infty(0,T;L^3(\mathbb{R}^3))} \), depend only on \( K_1 \) and \( M \), after finite steps of the process of the bootstrap iteration, we can get an estimate on the whole time interval
\[ \| \nabla u \|_2^2 + \nu \int_0^t \| \Delta u \|_2^2 d\tau \leq \| \nabla u(0) \|_2^2 + C, \]
for all \( t \in (0,T^*) \). Therefore, the \( H^1 \) norm of the strong solution \( u \) is bounded on the maximal interval of existence \( (0,T^*) \). This completes the proof of Theorem 1.1.
Remark 3.1. In above proof, we note that if we give the additional assumptions on $\partial_3 u_3$, namely, we choose $j = 3, k = 3$, then the inequality \((2.5)\) may be replaced by
\[
\frac{1}{q} \frac{d}{dt} \|u_3\|^q_q + C(q) \nu \|\nabla |u_3|^2\|_2^2 = \|u_3\|^\frac{q}{2} \|\nabla u_3\|^\frac{q-2}{2} \|\partial_3 u_3\|^{1/3}_q,
\]
and the inequality \((2.9)\) may be replaced by
\[
\frac{1}{2} \frac{d}{dt} \|\nabla h u\|_2^2 + \nu \|\nabla h \nabla u\|_2^2 \leq C \|\partial_3 u_3\|^2 \|\nabla u\|_2^2 \|\Delta u\|_2^2 + \frac{\nu}{2} \|\nabla h \nabla u\|_2^2,
\]
and \((2.7)\) becomes
\[
\|\nabla h u\|_2^2 + \nu \int_0^t \|\nabla h \nabla u\|_2^2 d\tau \leq C \int_0^t \|u_3\|^\frac{q}{2} \|\partial_3 u_3\|^\frac{q-2}{2} \|\nabla u\|_2^2 d\tau \|\Delta u\|_2^2 \|\Delta u\|_2^2 \times \left[ \int_0^t \|\Delta u\|_2^2 d\tau \right]^{\frac{1}{r-1}} + C \|\nabla h u(0)\|^2.
\]

The other cases can be done in a similar way. Since if we want to provide the conditions on $\partial_j u_2$, $j = 1, 2, 3$, in Lemma 2.1 we will use $|u_2|^{q-2} u_2$, as test function in the equation for $u_2$, and we can get the similar results to that in Lemma 2.2 and Lemma 2.3. Therefore, this method is suitable for every one of the nine entries of the gradient tensor.

Proof of Theorem 1.2 We take different strategy to prove Theorem 1.2 (i) and (ii) in turn. The framework of our proof of Theorem 1.2 is also standard. As to the second part of this theorem, we give another inequality on $u_3$, and then we prove \((3.1)\).

• $j \neq k$

To prove Theorem 1.2 (i), we take the same strategy as that in Theorem 1.1 and \((0, T^*)\) is the maximal interval of existence of the strong solution. Next, we show \((3.1)\) is true under the condition of \((1.8)-(1.9)\). Similar to the proof of Theorem 1.1 we take $j = 1, k = 3$, and prove the boundedness of $u_3$ in $L^\infty(0, T; L^q)$ with some $q$ at first, then apply Lemma 2.3 and Gronwall’s inequality to get \((3.1)\).

For $\alpha \in (3, \infty)$,
we choose the parameters in the following form

\[
\begin{align*}
\frac{1}{\sigma_1} &= \frac{(7 - \frac{2}{\alpha}) + \sqrt{\frac{9}{\alpha^2} - \frac{12}{\alpha} + 103}}{18}, \\
q_1 &= \frac{6\alpha\sigma_1}{3\alpha + \sigma_1}, \\
s_1 &= \frac{\sigma_1}{3 - 2\sigma_1}, \\
r_1 &= \frac{6(\alpha - 1)\sigma_1}{3\alpha + \sigma_1} + 1.
\end{align*}
\] (3.6)

Then, the above parameters satisfy (2.3) and (2.4), namely,

\[
\frac{2}{q_1} = \frac{1}{3\alpha} + \frac{1}{\sigma_1} \quad \text{and} \quad \frac{2}{3s_1 - 2} = \frac{2\sigma_1}{9 - 8\sigma_1}.
\]

We choose

\[
\beta = \frac{2\sigma_1}{9 - 8\sigma_1},
\] (3.7)

then we have

\[
\frac{1}{\sigma_1} = \frac{2}{9}(\frac{1}{\beta} + 4),
\]

and (3.6), we get \(\frac{2}{3\alpha} + \frac{2}{\beta} = f(\alpha)\). We denote

\[
V_1(t) = \int_0^t \|\partial_1 u_3\|_{\alpha}^\beta \|\nabla u\|_2^2 d\tau = \int_0^t \|\partial_1 u_3\|_{\alpha}^{\frac{2\sigma_1}{9 - 8\sigma_1}} \|\nabla u\|_2^2 d\tau.
\] (3.8)

By the following fact

\[
\frac{18\sigma - 5\sigma^2}{3\sigma^2 + 14\sigma - 18} - \frac{\sigma}{\sigma - 1} = \frac{(-8\sigma + 9)\sigma^2}{(3\sigma^2 + 14\sigma - 18)(\sigma - 1)} > 0
\]

(when \(\sqrt{103 - 7}/3 = \sigma_2 < \sigma < 9/8\),

where \(\sigma_2 = \sqrt{103 - 7}/3\) is the positive solution of \(3\sigma^2 + 14\sigma - 18 = 0\). Therefore, for every \(\alpha\), we have (in fact, from (3.6) we get \(\alpha = \frac{18\sigma_1 - 5\sigma_1^2}{3(3\sigma_1^2 + 14\sigma_1 - 18)}\))

\[
\alpha > \frac{\sigma_1}{3(\sigma_1 - 1)}.
\] (3.10)

Applying (3.6) and (3.10), we get

\[
\frac{2}{q_1} < \frac{\sigma_1 - 1}{\sigma_1} + \frac{1}{\sigma_1} = 1 \implies q_1 > 2,
\] (3.11)

thus (2.2) is satisfied with \(\alpha, \sigma_1\) and \(q_1\), and for \(s_1\), we have

\[
\frac{2}{3} < s_1 < 1.
\]
Integrating (2.1) with \( q = q_1, \sigma = \sigma_1 \) and \( s = s_1 \), in view of H"older’s inequality and (2.1), we get

\[
\| u_3 \|_{q_1}^2 \leq \| u_3(0) \|_{q_1}^2 + C \int_0^t \| \nabla u \|_{2}^{2-q_1} \| \partial_1 u_3 \|_{\alpha}^{1/3} d\tau
\]

\[
\leq \| u_3(0) \|_{q_1}^2 + C \left[ \int_0^t \| \nabla u \|_{2}^{2-q_1} d\tau \right]^{\frac{3\sigma_1}{6}} \left[ \int_0^t \| \partial_1 u_3 \|_{\alpha}^{\frac{2}{\sigma_1}} d\tau \right]^{\frac{3(\sigma_1 - 2)}{6}}
\]

\[
= \| u_3(0) \|_{q_1}^2 + C \left[ \int_0^t \| \partial_1 u_3 \|_{\alpha}^{\frac{2}{\sigma_1}} d\tau \right]^{\frac{9-8\sigma_1}{6\sigma_1}}.
\]  \( (3.12) \)

From (3.6), we have \( \alpha = \frac{18\sigma_1 - 5\sigma^2}{3(3\sigma_1 + 14\sigma_1 - 18)} \), it is not difficult to see that \( k(\sigma) = \frac{18\sigma - 5\sigma^2}{3(3\sigma + 14\sigma - 18)} \)

is a decreasing function with respect to the variable \( \sigma \), and

\[
\lim_{\sigma \to \frac{9}{17}} \frac{18\sigma - 5\sigma^2}{3(3\sigma^2 + 14\sigma - 18)} = 3, \quad \text{and} \quad \lim_{\sigma \to \frac{9}{17}} \frac{18\sigma - 5\sigma^2}{3(3\sigma^2 + 14\sigma - 18)} = \infty. \]  \( (3.13) \)

By (3.13), we have that \( \sigma_1 \) satisfies \( \sigma_2 < \sigma_1 < \frac{9}{17} \). Together with \( 3 < \alpha < \infty \), we obtain \( 2 < q_1 < \frac{9}{17} \). From \( u_0 \in V \), by Sobolev embedding we have \( \| u_3(0) \|_{q_1} < C \) for some \( C > 0 \). By virtue of

\[
\frac{4(\sigma + 3\alpha)}{3(3\sigma - 2)\alpha - 11\sigma} - \frac{2\sigma}{9 - 8\sigma} = \frac{2[-5\sigma^2 + 18\sigma + 3(-3\sigma^2 - 14\sigma + 18)\alpha]}{(3(3\sigma - 2)\alpha - 11\sigma)(9 - 8\sigma)},
\]  \( (3.14) \)

we have

\[
\frac{4(\sigma + 3\alpha)}{3(3\sigma_1 - 2)\alpha - 11\sigma_1} = \frac{2\sigma_1}{9 - 8\sigma_1},
\]  \( (3.15) \)

where \( 3(3\sigma_1 - 2)\alpha - 11\sigma_1 > 0 \) (see (3.20) and (3.21) below). Therefore, applying (3.12) and (3.15), we obtain

\[
\| u_3 \|_{q_1}^2 \leq \| u_3(0) \|_{q_1}^2 + C \left[ \int_0^t \| \partial_1 u_3 \|_{\alpha}^{\frac{2}{\sigma_1}} d\tau \right]^{\frac{9-8\sigma_1}{6\sigma_1}}.
\]  \( (3.16) \)

From the condition (1.9), we have

\[
\| u_3 \|_{L^\infty(0,T;L^{q_1} (\mathbb{R}^3))} \leq C,
\]  \( (3.17) \)

where \( C \) depend on \( T, K_1, M \). The selected \( r_1 \) in (3.6) satisfies

\[
r_1 = \frac{(q_1 + 1)\alpha - q_1}{\alpha},
\]  \( (3.18) \)

and from (2.3), (2.4) and (2.6), we have

\[
\frac{8}{3r_1 - 7} = \frac{4(\sigma_1 + 3\alpha)}{3(3\sigma_1 - 2)\alpha - 11\sigma_1}.
\]  \( (3.19) \)

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Because
\[
\frac{\sigma}{\sigma - 1} - \frac{11\sigma}{3\sigma - 2} = \frac{(-8\sigma + 9)\sigma}{(\sigma - 1)(3\sigma - 2)} > 0, \quad \forall 1 < \sigma < 9/8,
\]  
(3.20)
and take into account of (3.10), (3.20) implies that
\[
\alpha > \frac{11\sigma_1}{3(3\sigma_1 - 2)}, \text{ namely } 3(3\sigma_1 - 2)\alpha - 11\sigma_1 > 0,
\]  
(3.21)
and then \(\frac{8}{3r_1 - 7} > 0\) in (3.19). From (3.18), the fact that \(3 < \alpha < \infty\) and \(q_1 > 2\), one has
\[
r_1 > \frac{7}{3}.
\]  
(3.22)

By Lemma 2.3, we have
\[
\|\nabla u\|_2^2 + \nu \int_0^t \|\Delta u\|_2^2 d\tau 
\leq C \int_0^t \|u_3\|_{\frac{8(r_1-1)}{3r_1-7}} \|\partial_1 u_3\|_{\frac{8}{3r_1-7}} \|\nabla u\|_2^2 d\tau + \|\nabla u(0)\|_2^2 
\]  
(3.23)
\[
+ C \int_0^t \|u_3\|_{\frac{2(r_1-1)}{r_1-2}} \|\partial_1 u_3\|_{\frac{2}{r_1-2}} \|\nabla u\|_2^2 d\tau + C.
\]

Applying (3.23) and (3.17), and the fact \(\frac{8}{3r_1 - 7} > \frac{2}{r_1 - 2}\) for all \(r\) satisfying (2.10), we have
\[
\|\nabla u\|_2^2 + \nu \int_0^t \|\Delta u\|_2^2 d\tau \leq CV_1(t) + \|\nabla u(0)\|_2^2 + C.
\]

The result for \(\alpha > 3\) follows from Gronwall’s inequality, and this end the proof of (i).

• \(j = k\)

For Theorem 1.2 (ii), without loss of generality, we assume \(j = 3, k = 3\). For every
\[
\alpha \in \left(\frac{9}{5}, \infty\right), \quad \text{(3.24)}
\]
we set
\[
\left\{
\begin{array}{l}
\frac{1}{\mu} = 1 - \frac{12}{\alpha} + \sqrt{\frac{144}{\alpha^2} - \frac{264}{\alpha} + 289} \\
q_2 = \frac{2\alpha\mu}{\alpha + \mu}, \\
r = \frac{2\mu\alpha - \mu + \alpha}{\alpha + \mu}.
\end{array}
\right.
\]  
(3.25)

From (3.25), we have
\[
\alpha = \frac{12\mu + 5\mu^2}{6\mu^2 + \mu - 12}.
\]  
(3.26)
and note that \(h(\mu) := \frac{12\mu + 5\mu^2}{6\mu^2 + \mu - 12}\) is a decreasing function of \(\mu\), and
\[
\lim_{\mu \to 3^-} \frac{12\mu + 5\mu^2}{6\mu^2 + \mu - 12} = \frac{9}{5} \quad \text{and} \quad \lim_{\mu \to \frac{4}{3}^+} \frac{12\mu + 5\mu^2}{6\mu^2 + \mu - 12} = \infty.
\]  
(3.27)
By (3.24) and (3.27), we have
\[
\frac{4}{3} < \mu < 3, \quad (3.28)
\]
and (3.25) follows
\[
\frac{1}{\mu} + \frac{1}{\alpha} + \frac{q_2 - 2}{q_2} = 1. \quad (3.29)
\]
On the other hand,
\[
\frac{12\mu + 5\mu^2}{6\mu^2 + \mu - 12} - \frac{\mu}{\mu - 1} = \frac{\mu^2(6 - \mu)}{(6\mu^2 + \mu - 12)(\mu - 1)} > 0 \text{ with } \frac{4}{3} < \mu < 3.
\]
Combing (3.26) and above inequality, we have \(\alpha > \frac{\mu}{\mu - 1}\), and hence (3.25) implies \(q_2 > 2\). We choose
\[
\beta = \frac{2\mu}{3 - \mu}, \quad (3.30)
\]
then we have
\[
\frac{1}{\mu} = \frac{2}{3} (\frac{1}{\beta} + \frac{1}{2}),
\]
by (3.25) and (3.30), we have \(\frac{3}{2\alpha} + \frac{2}{\beta} = g(\alpha)\). We denote
\[
V_2(t) = \int_0^t \|\partial_3 u_3\|_\alpha^2 \|\nabla u\|_2^2 d\tau = \int_0^t \|\partial_3 u_3\|_\alpha^{2\mu} \|\nabla u\|_2^2 d\tau. \quad (3.31)
\]
Next, we give another estimates on \(u_3\). We use \(|u_3|^{q_2} u_3\) as test function in the equation (1.1) for \(u_3\). By using of Gagliardo-Nirenberg and Hölder’s inequalities, and applying the inequality (1.15), we have
\[
\frac{1}{q} \frac{d}{dt} \|u_3\|_{q_2}^{q_2} + C(q_2)\nu \|\nabla|u_3|^{q_2/2}\|_{2}^{2} = - \int_{\mathbb{R}^3} \partial_3 p |u_3|^{q_2-2} u_3 dx \\
\leq C \int_{\mathbb{R}^3} |p| |u_3|^{q_2-2} |\partial_3 u_3| dx \\
\leq C \|p\|_{\mu} \|u_3\|_{q_2}^{q_2-2} \|\partial_3 u_3\|_{\alpha} \quad (by (3.29)) \\
\leq C \|u\|_{2\mu}^{2} \|u_3\|_{q_2}^{q_2-2} \|\partial_3 u_3\|_{\alpha} \quad (by (1.15)) \\
\leq C \|u\|_{2\mu}^{\frac{3-\mu}{2}} \|\nabla u\|_{2}^{\frac{3(\mu-1)}{2\mu}} \|u_3\|_{q_2}^{q_2-2} \|\partial_3 u_3\|_{\alpha}.
\]

The above inequality immediately implies that
\[
\frac{1}{2} \frac{d}{dt} \|u_3\|_{q_2}^{2} \leq C \|u\|_{2\mu}^{\frac{3-\mu}{2}} \|\nabla u\|_{2}^{\frac{3(\mu-1)}{2\mu}} \|\partial_3 u_3\|_{\alpha}. \quad (3.33)
\]
In view of (3.28), we have \(\frac{3(\mu-1)}{\mu} < 2\), applying Young’s inequality, we have
\[
\frac{1}{2} \frac{d}{dt} \|u_3\|_{q_2}^{2} \leq C \|\nabla u\|_{2}^{2} + \|u\|_{2}^{2} \|\partial_3 u_3\|_{\alpha}^{2\mu}. \quad (3.34)
\]
Integrating (3.34) on time, and by energy inequality (2.1), we obtain

$$
\|u_3\|_{q_2}^2 \leq \|u_3(0)\|_{q_2}^2 + C \int_0^t \|\partial_3 u_3\|_{2-\frac{2}{\alpha}} d\tau.
$$

(3.35)

By the condition (1.12), (3.25) and (3.28), we have $q_2 < 6$. Note that $\|u_3(0)\|_{q_2} < C$ for some $C > 0$, we get

$$
u u_3 \in L^\infty(0, T; L^{q_2}(\mathbb{R}^3)).
$$

(3.36)

Keeping in mind that we have another estimates in Lemma 2.3

$$
\|\nabla u\|_2^2 + \frac{\nu}{2} \int_0^t \|\Delta u\|_2^2 d\tau
\leq \|\nabla u(0)\|_2^2 + C \int_0^t \|u_3\|_{\frac{2(r-1)}{q_2}} \|\partial_3 u_3\|_{\frac{2}{\alpha}} \|\nabla u\|_2^2 d\tau
$$

(3.37)

By (3.25), we have $r = \frac{(q_2+1)\alpha-q_2}{\alpha}$, and $r > 2$, therefore

$$
\frac{8}{3(r-2)} = \frac{8(\mu + \alpha)}{3(2\mu\alpha - 3\mu - \alpha)} > 0.
$$

Moreover, from (3.26), we have (note that $2\mu\alpha - 3\mu - \alpha > 0$)

$$
\frac{8(\mu + \alpha)}{3(2\mu\alpha - 3\mu - \alpha)} - \frac{2\mu}{3 - \mu} = \frac{2[12\mu + 5\mu^2 + (-6\mu^2 - \mu + 12)\alpha]}{3(2\mu\alpha - 3\mu - \alpha)(3 - \mu)} = 0.
$$

Combing (3.36) and (3.37), and the fact $\frac{2}{r-2} < \frac{8}{3(r-2)}$, we have

$$
\|\nabla u\|_2^2 + \nu \int_0^t \|\Delta u\|_2^2 d\tau \leq CV_2(t) + \|\nabla u(0)\|_2^2 + C,
$$

and end the proof for $\alpha \in \left(\frac{9}{5}, \infty\right)$ by using of Gronwall’s inequality.

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The line ",(1)" is the result of C.S. Cao, E.S. Titi in [20], which signifies (1.3). The line ",(2)" is our result, which means (1.9).

Figure 1: Case of \( j \neq k \)

The line "(1)" is the result of C.S. Cao, E.S. Titi in [20], which signifies (1.3). The line "(2)" is our result, which means (1.9).

The line "(1)" signifies (1.4), which is also considered by C.S. Cao, E.S. Titi in [20]. The line "(3)" is our result, which means (1.12). The result of Y. Zhou, M. Pokorný in [16] is showed by line "(2)".

Figure 2: Case of \( j = k \)