Low-Energy Dynamics of Supersymmetric Solitons

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ABSTRACT

In bosonic field theories the low-energy scattering of solitons that saturate Bogomol’nyi-type bounds can be approximated as geodesic motion on the moduli space of static solutions. In this paper we consider the analogous issue within the context of supersymmetric field theories. We focus our study on a class of $N = 2$ non-linear sigma models in $d = 2 + 1$ based on an arbitrary Kähler target manifold and their associated soliton or “lump” solutions. Using a collective coordinate expansion, we construct an effective action which, upon quantisation, describes the low-energy dynamics of the lumps. The effective action is an $N = 2$ supersymmetric quantum mechanics action with the target manifold being the moduli space of static charge $N$ lump solutions of the sigma model. The Hilbert space of states of the effective theory consists of anti-holomorphic forms on the moduli space. The normalisable elements of the dolbeault cohomology classes $H^{(0,p)}$ of the moduli space correspond to zero energy bound states and we argue that such states correspond to bound states in the full quantum field theory of the sigma model.

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1. Introduction

The study of the low-energy scattering of soliton-like objects in bosonic field theories was initiated by the work of Manton on monopoles [1]. It was argued that the scattering of slowly moving BPS monopoles could be approximated as geodesic motion on the moduli space of static solutions. Atiyah and Hitchin then continued the programme by explicitly constructing the metric on the moduli space of two monopoles and then employing it to construct geodesics (see [2] and references therein). One of the most striking conclusions is that in a head on collision two monopoles scatter at 90°.

These techniques have since been applied to a variety of other models, including Abelian-Higgs vortices in two dimensions [3], “lumps” of non-linear sigma models [4] and extremal black holes [5]. A common feature of all these models is that the static multi-soliton solutions saturate a Bogomol’nyi-type bound. This implies that there is no net force between the solitons, an essential ingredient in the geodesic approximation. The presence of the Bogomol’nyi bounds is a result of the fact that the models have supersymmetric extensions. Topological charges appear in the supersymmetry algebra from which one can deduce the Bogomol’nyi bound.

Given this deep connection with supersymmetry, it is natural to consider the scattering of solitons in the supersymmetric theory itself. Since the fermionic equations of motion of the supersymmetric theory are trivially satisfied by setting all the fermions to zero, the static soliton solutions of the bosonic theory continue to be solutions in the supersymmetric theory. However, the inclusion of the fermions dramatically changes the quantum theory. Now there will be fermionic zero modes in addition to the bosonic zero modes in the fluctuations about the classical soliton solution. After quantisation, the fermionic zero modes imply that there is a multiplet of soliton states degenerate in energy (assuming that the supersymmetry is not dynamically broken).

This illustrates a crucial difference in principle between the scattering of solitons in the bosonic and supersymmetric theories. In the bosonic theory the geodesic
approximation is developed by viewing the time evolution of the classical solitons as that of a fictitious particle moving in an infinite dimensional configuration space, the space of time independent, finite energy field configurations. That is, the analysis is purely classical (although the quantum case can, of course, also be considered [6,7,8]). In the supersymmetric case the rôle of the fermionic zero modes is only manifest after quantisation. To discuss the scattering of solitons in this case, it is thus necessary to construct an effective action which, upon quantisation, describes the low energy dynamics of the system. In this paper we construct such an action for a particular class of models by employing a collective co-ordinate expansion.

In the bosonic case, the collective co-ordinate expansion is essentially an equivalent way to develop the geodesic approximation. The fluctuations about the soliton solution contain zero modes that determine the low-energy dynamics. For each zero mode one must introduce a collective co-ordinate. The collective co-ordinates are the arbitrary parameters or moduli that a general static soliton solution depends on i.e. they can be considered as providing co-ordinates on the moduli space of static solutions. More physically, they can be interpreted as corresponding to the positions and “charges” of the solitons. By allowing the collective co-ordinates to depend on time and ignoring the contributions of the non-zero modes one constructs an ansatz for the low-energy fields. After substituting this ansatz into the action one obtains an effective action describing the low-energy dynamics of the system. The effective action is that of a free particle propagating on the moduli space of static solutions, with the metric being naturally induced from the kinetic energy functional of the parent field theory. The solutions to the equations of motion of the effective action are simply the geodesics on the moduli space.

If the exact soliton solutions are explicitly known, this procedure can be carried out directly. This is the approach employed in the study of the scattering of the lumps of bosonic sigma models [4]. For most models, however, less is known about the soliton solutions. Nevertheless, in some special cases, such as the scattering of two solitons (the case of most interest), the geometry of the moduli space can still be derived using indirect methods relying on identifying the various symmetries of
the model. This approach has successfully been used in the monopole system [2].

In the supersymmetric case we can use a blend of techniques. It is a generic phenomenon that the solutions saturating the Bogomol’nyi bound, i.e. the static soliton solutions, only break half of the supersymmetry. We will show how the unbroken supersymmetry pairs the bosonic zero modes with the fermionic zero modes. Using this we can formally perform the collective co-ordinate expansion for the fermions and bosons. After identifying the geometric structures of the moduli space we can then utilise any of the results of the bosonic case that have been obtained either directly or indirectly. We will see that this naturally leads to an effective action which is a kind of supersymmetric quantum mechanics on the moduli space. We note that a similar method was employed in [9] to show that the low-energy scattering of axions with 5-branes in superstring theory is related to the Donaldson polynomial. This connection between geometry and low-energy supersymmetric dynamics was one of the inspirations for this work.

We note that the procedure outlined above for deriving the low energy effective action involves neglecting all of the modes apart from the zero modes. Upon quantisation this is equivalent to assuming that the low energy dynamics may be described by a wave function of field configurations in which all but the zero modes are in the ground state. One may wonder about the validity of this approximation. In particular this approximation seems most vulnerable for the situations where the field theory has a continuous spectrum (e.g. monopoles, sigma model lumps). In these cases one may ask whether or not solitons with fermionic zero modes excited are stable against the emmission of soft “photinos”. If supersymmetry is not dynamically broken then the spectrum of soliton states will be exactly degenerate which implies the stability of the states. In some cases, including the models considered in this paper, the issue of supersymmetry breaking can be decided by calculating the Witten index Tr(−1)F or one of its variants [10].

Our approach is to assume that at low enough energy all radiation processes are significantly supressed and that the effective action just depending on the zero
modes is a good approximation. The range of validity of this approximation will depend on the detailed dynamics of the model considered. Of course for models such as the Abelian Higgs model for which there is a mass gap we expect the effective action to be a particularly robust approximation to the true dynamics.

An important class of states of the effective action are the normalisable wave functions with lowest energy. These correspond to bound states of the full quantum field theory as there are no other states with lower energy into which they could decay. We will see that the normalisable states of the effective action with zero energy are given by normalisable harmonic forms on the moduli space.

There is a variety of interesting supersymmetric models to consider. It would be extremely interesting to investigate the Montonen and Olive conjecture [11] concerning the self-duality of $N = 4$ super Yang-Mills under the interchange of magnetic and electric charges, along with the monopole and elementary particle sectors of the spectrum and the interchange of strong and weak coupling. Presumably some insight into this conjecture could be attained by studying the low-energy scattering of monopoles. Another interesting case to consider would be supersymmetric vortices. It is known that in a head on collision bosonic vortices scatter at $90^0$ at least in the abelian-Higgs model [3]. It has been further shown that this provides a plausibility argument as to why cosmic strings intercommute [12], a crucial property in cosmic string based cosmology. It is natural to wonder how the presence of fermionic zero modes could alter these phenomena.

In this paper we study the low energy dynamics of the lumps of $N = 2$ non-linear sigma models in $d=2 + 1$ based on an arbitrary Kähler target manifold. Such sigma models have been studied for a variety of reasons. In $d = 2 + 1$ the non-linear sigma model is not a renormalisable quantum field theory so it should be viewed as describing some low-energy effective behaviour in planar physics with an in built cutoff. Our main motivation for studying these models is that they do not have any of the additional complications that the gauge invariance introduces in the models mentioned above. We will return to a study of those models in the
The Witten index of the $CP^N$ sigma models is given by $Tr(-1)^F = N + 1$ [10]. Thus supersymmetry is not dynamically broken for these models and this ensures the stability of the lump states with fermionic zero modes excited, despite the fact that the system has a continuous spectrum. The non-renormalisability of the model is not relevant for our discussions as we are only interested in the very low-energy dynamics. There is one feature of sigma model solitons that is not shared by other models: the number of normalisable zero modes is smaller than the dimension of the moduli space. We will discuss this point further in the text and see that it brings in some complications in the analysis of the quantum theory. Despite this latter point, we believe that this model is a good laboratory to both investigate most conceptual issues and to develop some techniques for discussing the low-energy dynamics of supersymmetric solitons in general.

The classical scattering of lumps in the bosonic $CP^1$ and $CP^N$ sigma-models has been studied by various authors [4,14-16]. The geometry of the moduli space for an arbitrary Kähler target was elucidated by Ruback [17] and his results will be important in the following. After first reviewing the bosonic case and introducing some notation in section 2, we will then discuss the supersymmetric sigma models in section 3. We discuss how the soliton solutions break half of the supersymmetry and show how the unbroken supersymmetry pairs the bosonic and fermionic zero modes. The collective co-ordinate expansion will then lead to an effective supersymmetric quantum mechanics action describing the low-energy dynamics of the solitons. The quantisation of the effective action is discussed in section 4. We show that the Hilbert space of states is given by anti-holomorphic forms on the moduli space of static solutions. We also show that the zero energy bound states are given by normalisable elements of the cohomology classes $H^{(0,p)}$ of the moduli space and we argue that they correspond to bound lump states in the full quantum field theory of the sigma model. In an appendix we discuss how the operator ordering in the quantum mechanics of the effective action is related to the operator ordering in the vacuum sector of the parent sigma model. Section 5 contains some conclusions.
2. Bosonic Lumps

Consider a bosonic non-linear sigma model with an arbitrary Kähler target manifold $M$, described by the action

$$S = -\frac{1}{2} \int d^3 x g_{ij}(\phi(x)) \partial_m \phi^i \partial_n \phi^j \eta^{mn}$$  \hspace{1cm} (2.1)

where $\phi^i$ and $g_{ij}$, $i, j = 1...2n$ are co-ordinates and the metric on $M$, respectively.

To obtain the static soliton or lump solutions to the equations of motion, we first rewrite the action in the form

$$S = \int dt (K - V)$$  \hspace{1cm} (2.2)

where the kinetic energy functional is given by

$$K = \frac{1}{2} \int d^2 x g_{ij} \dot{\phi}^i \dot{\phi}^j$$  \hspace{1cm} (2.3)

and the potential energy functional is

$$V = \frac{1}{2} \int d^2 x g_{ij} \partial_\mu \phi^i \partial_\mu \phi^j$$  \hspace{1cm} (2.4)

where $\mu = 1, 2$. The total conserved energy is $E = K + V$.

The configuration space of the system, $C$, is the set of all finite energy maps from $R^2$ to $M$. The action can then be viewed as describing the motion of a fictitious particle moving on the infinite dimensional space $C$ under the influence of the potential (2.4). In this spirit, the kinetic energy functional naturally defines a metric on $C$. Specifically, if we let $\chi^i(x)$, $\rho^i(x)$ be two tangent vectors above the point $\phi^i(x) \in C$, then the metric $\tilde{g}$ is given by

$$\tilde{g}(\chi, \rho) \equiv \int d^2 x g_{ij}(\phi(x)) \chi^i(x) \rho^j(x)$$  \hspace{1cm} (2.5)

Similarly, the complex structure $J$ on the Kähler target induces a natural complex
structure $\tilde{J}$ on $\mathcal{C}$ via

$$
\left[ \tilde{J} \dot{\chi} \right]^i (x) \equiv J^i_j(\phi(x)) \dot{\chi}^j(x) . \tag{2.6}
$$

The geometry of $\mathcal{C}$ was described in detail in [17] and, assuming that $\mathcal{C}$ is at least an incomplete manifold (as we also will), it was shown that $(\mathcal{C}, \tilde{g}, \tilde{J})$ is in fact Kähler.

Static solutions to the equations of motion of (2.1) are obtained by minimising the static energy functional $V$. As is well known, (2.4) can be recast in the form

$$
V = \frac{1}{4} \int d^2 x D^\pm_\mu \phi^i D^\pm_\nu \phi^j g_{ij} \mp T , \tag{2.7}
$$

where

$$
D^\pm_\mu \phi^i = \partial_\mu \phi^i \pm J^i_j \epsilon_{\mu\nu} \partial_\nu \phi^j \tag{2.8}
$$

and

$$
T = \int_{\mathbb{R}^2} \phi^*(\omega) = \int d^2 x g_{ik} J^k_j \partial_\mu \phi^i \partial_\nu \phi^j \epsilon_{\mu\nu} \tag{2.9}
$$

is the integral of the pull back of the Kähler form on $M$. Since the Kähler form defines a non-trivial element of the second cohomology class $H^2(M,R)$, $T$ is a topological invariant. (For $M = CP^N$, it is, up to normalisation, an integer labelling the 2-d instanton number; the instantons in d=2 become the static lumps in d=2+1). The configuration space $\mathcal{C}$ is partitioned into different topological sectors labelled by the topological charge $T$.

From (2.7) we can deduce the Bogomol’nyi bound on the static energy functional

$$
E \geq |T| . \tag{2.10}
$$

Within each topological class of maps labelled by the topological charge $T$, the energy is minimised when the Bogomol’nyi bound (2.10) is saturated. Thus the
static soliton solutions to the equations of motion are obtained by solving the first order differential equations $D^\pm_i \phi^i = 0$. By switching to complex co-ordinates it is easily seen that this corresponds to (anti-)holomorphic maps: $\partial \bar{z} \phi^\alpha = 0$ ($\partial z \phi^\alpha = 0$). In the following we will restrict our considerations to the case of holomorphic maps; the analysis can be trivially extended to the case of anti-holomorphic maps.

In general, the holomorphic maps will depend on a continuous set of parameters, or moduli. These moduli can be interpreted as the positions and “charges” of the static lumps. The moduli space $M_N \subset \mathcal{C}$ is defined as the set of all holomorphic maps within a given topological class $T = N$. Clearly the dimension of the moduli space is determined by the number of moduli. The existence of exact static charge $N$-lump configurations relies on the Bogomol’nyi bound being saturated. This can be interpreted as saying that there is no net force between the (static) lumps. This is entirely analogous to the case of BPS monopoles. Thus, following the work by Manton [1], Ward suggested in [4] that the scattering of slowly moving lumps could be approximated by assuming that the evolution is adiabatic in the space of static solutions. That is, the time evolution is determined by geodesic motion in the moduli space $M_N$.

The low energy scattering problem is thus reduced to determining the geometry of $M_N$ and calculating its geodesics. We first introduce some notation. We denote the most general N-lump solution by $\phi^i_o(x, X)$ where the moduli $X^a$, $a = 1, \ldots, 2k = \text{dim}(M_N)$, can be considered as co-ordinates on $M_N$. The maps $\phi^i_o(x, X)$ can be considered as providing alternative co-ordinates on $M_N$. By letting the moduli $X$ depend on a parameter $t$, which we interpret as time, we obtain the tangent vectors on $M_N$ by differentiation:

$$\dot{\phi}^i_o(x, X(t))|_{t=0} = \frac{\partial \phi^i_o}{\partial X^a} \bigg|_{t=0} \dot{X}^a \equiv \delta^i_a \phi^i \dot{X}^a \ . \quad (2.11)$$

Geometrically, $\delta^i_a \phi^i$ is the matrix corresponding to a change of co-ordinates. Physically, for each $a$ they are simply the zero modes in the fluctuations about the
solution \( \phi^i_o(x) \equiv \phi^i_o(x, X(0)) \). This can be seen by noting that, by definition, 
\( V(\phi^i_o(x), X(t)) = V(\phi^i_o(x)) \).

A metric \( \mathcal{G} \) on \( M_N \) is naturally induced by the restriction of the metric (2.5) on \( \mathcal{C} \):
\[
\mathcal{G}(\dot{\phi}_o, \dot{\phi}_o) \equiv \mathcal{G}_{ab}(\phi_o) \dot{X}^a \dot{X}^b ,
\]
where
\[
\mathcal{G}_{ab} = \int d^2 x g_{ij}(\phi_o(x)) \delta^a_i \delta^b_j
\]
is the representation in the co-ordinates \( \{X^a\} \). We note that the physical assumption, due to finiteness of energy, of only considering zero modes with finite norm (i.e. \( \mathcal{G}_{ab} \) finite) is equivalent to only considering tangent vectors with finite length. Thus, although it might seem that an arbitrary holomorphic perturbation of a holomorphic map within a given topological class would be a zero mode, only a finite number of them will have finite norm. A feature peculiar to sigma model solitons is that the number of normalisable zero modes is less than the dimension of \( M_N \) [4]. Thus the time evolution of slowly moving lumps will be geodesic motion on some submanifold \( \tilde{M}_N \subset M_N \), defined by fixing all the collective co-ordinates corresponding to zero modes with infinite norm to be constant.

It was shown in [17] that if \( \dot{\phi}_o \) is a tangent vector to \( M_N \) then so is \( [\tilde{J} \dot{\phi}_o] \). Thus the restriction of the complex structure (2.6) on \( \mathcal{C} \) provides a natural complex structure on the moduli space. In the co-ordinates \( \{X^a\} \) it has the following form:
\[
\mathcal{J}^a_b = \mathcal{G}^{ac} \int d^2 x \mathcal{J}^i_j g_{ik} \delta^c_i \delta^b_k \delta^a_j
\]
The property \( \mathcal{J}^2 = -1 \) is most easily verified using the co-ordinates \( \phi^i_o(x, X) \). By a generalisation of the argument in [17], it is clear that (2.14) provides a complex structure on \( \tilde{M}_N \) by restricting the zero modes to have finite norm. Thus \( (\tilde{M}_N, \mathcal{G}, \mathcal{J}) \) is a Kähler manifold.
We conclude this section with a brief discussion of the symmetry groups of the various spaces we have been discussing. From (2.5) it is clear that the isometry group of \((C, \tilde{g})\) is given by the product of the isometry group of \(R^2\) with the isometry group of the target manifold \(M\). Considering (2.6) we conclude that the subgroup given by the product of the isometry group of \(R^2\) with the group of holomorphic isometries on \(M\) are holomorphic isometries of \((C, \tilde{g}, \tilde{J})\). The holomorphic isometries of \((C, \tilde{g}, \tilde{J})\) are holomorphic isometries of \((M_N, G, J)\) since they both act covariantly on the static equations of motion and they preserve the topological charge (2.9). The combination of these isometries that do not shift the collective co-ordinates held fixed in defining \(\tilde{M}_N\) provide the isometries of this space.

3. Supersymmetric Lumps

The sigma model (2.1) has a supersymmetric extension given by

\[
S = -\frac{1}{2} \int d^3x \left\{ g_{ij} \partial_m \phi^i \partial_n \phi^j \eta^{mn} + i \bar{\psi}^i \slashed{D} \psi^j g_{ij} + \frac{1}{6} R_{ijkl} \bar{\psi}^i \psi^k \bar{\psi}^j \psi^l \right\}
\]

(3.1)

where \(\psi^i\) is a set of two component Majorana SL(2,R) spinors that transform as a vector on the target manifold \(M\). The covariant derivative in (3.1) is defined using the pullback of the connection on \(M\):

\[
D_m \psi^i = \partial_m \psi^i + \Gamma^i_{jk} \partial_m \phi^j \psi^k.
\]

(3.2)

Our conventions for the gamma matrices are as follows: \((\gamma^1)_{\alpha \beta} = (\sigma^1)_{\alpha \beta}\), \((\gamma^2)_{\alpha \beta} = (\sigma^3)_{\alpha \beta}\), \((\gamma^0)_{\alpha \beta} = C_{\alpha \beta} = (i \sigma^2)_{\alpha \beta}\), where \(C\) is the charge conjugation matrix. The Dirac adjoint is thus given by \(\bar{\psi}^i = \psi^i \gamma^0 = \psi^i T C\).

The action is invariant under the supersymmetry transformation

\[
\delta \phi^i = i \epsilon^1 \psi^i
\]

\[
\delta \psi^i = \bar{\phi}^j \epsilon^1 - i \Gamma^i_{jk} \bar{\psi}^j \epsilon^1 \psi^k
\]

(3.3)

where \(\epsilon^1\) is a constant anticommuting Majorana spinor. Additional supersymmetries require a reduction of the holonomy of \(M\) [18,19]. In the case at hand, \(M\) is
Kähler and hence there is an additional supersymmetry given by

\[ \delta \phi^i = i\epsilon^2 J^i_j \psi^j \]
\[ \delta \psi^i = -J^i_j \bar{\phi}^j \epsilon^2 - i\Gamma_{i j k} J^k_l \epsilon^2 \psi^l \psi^j \]  

where \( \epsilon^2 \) is a second constant anticommuting Majorana spinor.

The fermionic equations of motion of (3.1) are trivially satisfied by setting all of the fermions to zero. Thus the static soliton solutions of the bosonic theory continue to be solitons in the supersymmetric theory. In the following we will call the solution \( \partial \bar{z} \phi^i = \psi^i = 0 \) the supersymmetric solution, where \( \alpha = 1, \ldots, n \) are holomorphic co-ordinates on the target.

The supersymmetric solutions break half of the supersymmetry. By this we mean that of the four dimensional space spanned by the supersymmetry parameters, the solution is left invariant by a two dimensional subspace. To see this we use holomorphic co-ordinates and introduce the following hermitean projection operator in the spinor space:

\[ \Gamma = \frac{1}{4} \gamma^z \gamma^\bar{z}, \]  

satisfying

\[ \Gamma^2 = \Gamma, \quad \Gamma + \Gamma^* = 1, \quad \Gamma \Gamma^* = 0 \]  

where \( \gamma^z \equiv \gamma^1 + i\gamma^2 \).

After redefining the supersymmetry parameters via

\[ \chi \equiv \epsilon^1 + i\epsilon^2 \quad \Gamma \chi \equiv \rho \quad \Gamma^* \chi \equiv \sigma \]  

the \( N = 2 \) supersymmetry transformations (3.3), (3.4) acting on a static bosonic
holomorphic map $\phi_\sigma^\alpha$ are given by

$$\delta_\sigma \phi^\alpha = i \tilde{\sigma}^* \psi^\alpha$$

$$\delta_\sigma \psi^\alpha = -i \Gamma^\alpha_{\beta \gamma} |_{\phi, \tilde{\sigma}} \psi^\beta \psi^\gamma$$

and

$$\delta_\rho \phi^\alpha = 0$$

$$\delta_\rho \psi^\alpha = \gamma^z \partial_z \phi_\sigma^\alpha \rho^* - i \Gamma^\alpha_{\beta \gamma} |_{\phi, \rho} \psi^\beta \psi^\gamma$$

where we have used the fact that for a Kähler manifold the Christoffel symbol is pure in its holomorphic indices. When the fermions are set to zero, $\delta_\sigma$ is the unbroken supersymmetry leaving the supersymmetric solution invariant and $\delta_\rho$ is the broken supersymmetry.

This partial breaking of supersymmetry is a generic feature of supersymmetric field theories admitting topologically non-trivial solutions. It was first noticed by Witten and Olive [20] and is best understood by showing that the algebra of supersymmetry charges are modified by topological charges (see [21] for a model independent discussion of this). For the N=2 supersymmetric sigma model it was shown in [17] that the supersymmetry algebra is given by

$$\left\{ Q^I_\alpha, Q^J_\beta \right\} = \delta^{IJ} P^m (\gamma_m)_{\alpha \beta} + T \epsilon^{IJ} C_{\alpha \beta}$$

where $T$ is the topological charge (2.9). Furthermore, from (3.10) we can deduce the Bogomol’nyi bound (2.10) and that the bound is saturated iff the solution breaks half of the supersymmetry.

We now turn to a discussion of the zero modes in the fluctuations about the supersymmetric solution. The bosonic zero modes are exactly the same as for the bosonic sigma model since after setting the fermions to zero in the equations of motion of the supersymmetric model one obtains the same equations of motion as in the bosonic model. The fermionic zero modes are normalisable c-number solutions to the Dirac equation in the presence of the soliton background. A metric on the
space of fermion zero modes is induced by the fermion kinetic term in the action (3.1):

\[ G'_{\alpha'b'} = \int d^2x g_{ij}(\phi_o(x))(\psi^i_{a'})^T \psi^j_{b'} \]  

(3.11)

where we have denoted the fermionic zero modes by \( \psi^i_{a'} \). Restricting our considerations to normalisable zero modes is equivalent to demanding that \( G'_{\alpha'b'} \) has finite entries.

Two normalisable fermionic zero modes are immediately obtained from the broken supersymmetry. The fermionic part of the supersymmetry transformation of the supersymmetric solution is given by

\[ \psi^\alpha = \gamma^z \partial_z \phi^\alpha \rho^* \]  

(3.12)

and it is straightforward to verify that it satisfies \( D \psi^\alpha = 0 \). Clearly these zero modes satisfy \( \Gamma \psi^\alpha = \psi^\alpha \). These modes can be interpreted as the Goldstone modes of the broken supersymmetry.

Following an argument by Zumino [22] in the context of instantons, we will now show that the unbroken supersymmetry pairs all of the bosonic and fermionic zero modes. This pairing will be crucial to the derivation of the effective action. After imposing the following restrictions

\[ \partial_{\bar{z}} \phi^\alpha = 0 \quad \psi^\alpha = \Gamma \psi^\alpha \]  

(3.13)

the equations of motion for time independent configurations take the form

\[ \partial_{\bar{z}} \phi^\alpha = \partial_{\bar{z}} \psi^\alpha = 0 \]  

(3.14)

Returning to the supersymmetric solution \( \partial_{\bar{z}} \phi^\alpha = \psi^\alpha = 0 \), this seems to indicate that any holomorphic bosonic or fermionic perturbations are zero modes. In particular, it is surprising that the Dirac equation does not seem to depend on the
background. We discussed in detail in the last section that the normalisable bosonic zero modes are restricted by requiring that the metric on the moduli space (2.13) has finite entries. The fermionic zero modes are similarly restricted by demanding that the c-number holomorphic perturbations be normalisable (finite entries in (3.11)). We note that this also resolves the apparent paradox of the background independence of the Dirac equation in (3.14).

The linearly realised supersymmetry now reads

\[ \delta_\sigma \phi^\alpha = i \bar{\sigma}^* \psi^\alpha \quad \delta_\sigma \psi^\alpha = 0 \quad . \quad \tag{3.15} \]

If we let \( \psi^\alpha \) be a normalisable fermionic zero mode and we let \( \sigma \) also be a c-number spinor, it is clear that (3.15) generates a normalisable bosonic zero mode. This would seem to imply that for each fermionic zero mode there are two bosonic zero modes. However, because the supersymmetry algebra (3.10) has an \( SO(2) \) automorphism group, these bosonic zero modes are not independent. Using this we can invert (3.15) to supersymmetrically pair the zero modes via

\[ \psi_p^\alpha = \delta_p^\alpha \psi^\alpha \epsilon \quad \tag{3.16} \]

where \( \epsilon \) is a c-number spinor satisfying \( \Gamma \epsilon = \epsilon, \epsilon^\dagger \epsilon = 1 \) and here and in the following \( p, q, r \) are holomorphic co-ordinates on \( \tilde{M}_N \). Thus for each normalisable bosonic zero mode there is in fact one normalisable fermionic zero mode. We further note that (3.16) implies that the metrics (2.13) and (3.11) are equal, consistent with the unbroken supersymmetry.

At this stage we have shown that the fermionic zero modes satisfying \( \Gamma \psi^\alpha = \psi^\alpha \) are paired with the bosonic zero modes. In principle there could other fermionic zero modes and one would need a kind of index argument to determine whether or not they are present. Although we do not have a general proof that there are not additional fermionic zero modes, we do not expect them. In all supersymmetric models that we know of where there is a corresponding index theorem, a simple
counting argument shows that all of the bosonic and fermionic zero modes are paired by the unbroken supersymmetry.

The construction of the effective action describing the low energy dynamics now proceeds by a collective co-ordinate expansion. For each normalisable zero mode, we introduce a collective co-ordinate. For the bosonic zero modes this amounts to allowing the moduli associated with finite norm zero modes to depend on time. For the fermionic zero modes we use (3.16) to introduce the collective co-ordinates in a way that preserves the unbroken supersymmetry. Specifically, we are led to the following low-energy ansatz for the time varying fields:

\[ \phi^{\alpha}(t, z) = \phi^{\alpha}_0(z, X^P(t)) + \ldots \]
\[ \psi^{\alpha}(t, z) = \delta_p \phi^{\alpha} e^P(t) + \ldots \]

(3.17)

where \( e^P \) is now a Grassmann odd spinor satisfying \( \Gamma e^P = e^P \) and the neglected terms correspond to non-zero modes. This expansion corresponds to a change of variables from \( \phi \) and \( \psi \) to their infinite mode expansions and the low-energy approximation is imposed by neglecting all but the zero modes. After substituting the ansatz (3.17) into the action (3.1) and exploiting the fact that the moduli space \( \tilde{M}_N \) is Kähler, we obtain the following effective action describing the low-energy dynamics:

\[ S_{\text{eff}} = \int dt G_{pq} \left\{ \dot{X}^p \dot{\bar{X}}^q + i \lambda^p D_t \bar{\lambda}^q \right\} \]

(3.18)

where \( e^p = \frac{1}{\sqrt{2}} \begin{pmatrix} \lambda^p \\ -i \lambda^p \end{pmatrix} \) and the covariant derivative is defined using the pullback of the Christoffel connection \( \gamma \) on the moduli space:

\[ D_t \lambda^\bar{p} = \dot{\lambda}^\bar{p} + \gamma^\bar{p} \bar{q} \dot{X}^\bar{q} \lambda^\bar{r} \]

(3.19)

The action can be recast into a slightly more familiar form using real co-ordinates:

\[ S_{\text{eff}} = \frac{1}{2} \int dt G_{ab} \left\{ \dot{X}^a \dot{X}^b + i \lambda^a D_t \lambda^b \right\} \]

(3.20)

This action is a variation of the usual supersymmetric quantum mechanics action. Usually, the fermions are two component Majorana spinors that transform as real
vectors on the target. In the present case the $\lambda^a$ are a set of one component Grassmann odd objects that transform as real vectors on the target. This fact and the ansatz (3.17) resolve the apparent contradiction discussed in the conclusion of [17].

The unbroken supersymmetry of the underlying sigma model provides an N=2 supersymmetry of (3.18) i.e. there exist two one component anti-commuting supercharges $\star$. To show this we define the unbroken supersymmetry parameter by

$$\sigma = \frac{1}{\sqrt{2}} \left( \begin{array}{c} i\kappa \\ -\kappa \end{array} \right)$$

satisfying $\Gamma^*\sigma = \sigma$, and using (3.3),(3.4),(3.7) and (3.17) we deduce that

$$\delta X^p = i\kappa \lambda^p$$
$$\delta \lambda^p = -\dot{X}^p \kappa^*$$  \hspace{1cm} (3.21)

The supersymmetry charges can be derived using Noether’s theorem and we find

$$Q = g_{p\bar{q}} \dot{X}^{\bar{q}} \lambda^p$$
$$Q^* = g_{p\bar{q}} \dot{X}^{\bar{q}} \lambda^p$$  \hspace{1cm} (3.22)

Thus the effective action describing the low energy dynamics of a set of N supersymmetric solitons is given by an N=2 supersymmetric quantum mechanics with the moduli space of N static solutions as a target manifold.

It will be useful in the following to know how the angular momentum operator of the sigma model depends on the zero modes. This could be derived by first calculating the angular momentum operator of the sigma model and then substituting (3.17). A more direct way is the following. We first define the two supersymmetry charges of the sigma model via

$$\delta \phi^i = (i\bar{\epsilon}^1 Q^1 + i\bar{\epsilon}^2 Q^2)\phi^i$$  \hspace{1cm} (3.23)

Using (3.3) and (3.4) a short calculation shows that up to a factor the supersymmetry charge $Q$ in (3.22) corresponds to the following linear combination of the

\footnote{On a general manifold the action (3.20) is sometimes called an $N = \frac{1}{2}$ supersymmetric quantum mechanics. In the present context where we have two $N = \frac{1}{2}$ supersymmetries the nomenclature is clearly clumsy.}
components of the sigma model supersymmetry charges defined in (3.23):

\[ Q = \frac{1}{2} \left[ Q_1^1 + Q_2^2 + i(Q_2^1 - Q_1^2) \right] . \]  

(3.24)

In [23] it was shown that the d=2+1 superPoincare algebra implies

\[ [J, Q] = -\frac{1}{2}Q \quad [J, Q^\ast] = \frac{1}{2} \]  

(3.25)

where \( J \equiv M_{12} \) is the angular momentum generator in d=2+1. After carrying out a canonical analysis of (3.18) we can construct the zero mode contribution to \( J \) by demanding that it satisfies (3.25) and commutes with the effective Hamiltonian. This is done in the next section.

Before concluding this section, we note that the consistency condition for the existence of the N=2 supersymmetric quantum mechanics is that the moduli space of static solutions be Kähler, as indeed it is. It is tempting to suggest that this logic can be reversed; that the Kähler nature of the moduli space can be deduced from the presence of the unbroken supersymmetry. This connection could be a general feature of field theories with Bogomol’nyi bounds, as they all have unbroken supersymmetry and the moduli spaces of static solutions are Kähler and hyperKähler in the situations when we expect N=2 and N=4 supersymmetric quantum mechanics, respectively. It is possible that this connection is just a formal one and not a substitute for hard analysis.
4. Quantisation of the Effective Action

To further pursue the analysis of the low energy dynamics of the supersymmetric lumps, one needs to quantize the effective action (3.18). We will see that additional information from the parent field theory is still required to resolve some ambiguities in the quantisation procedure. As we mentioned in the introduction, the quantum mechanics of the effective action describes the dynamics of the full field theory assuming that all but the zero modes are in the ground state.

To simplify the discussion we will discuss the quantisation in the context of the $CP^1$ sigma model. The moduli space $M_N$ for this target consists of all rational functions of degree $N$. The most general one lump solution is given by

$$\phi_o = \alpha + \beta (z + \gamma)^{-1}$$  \hspace{1cm} (4.1)

where the moduli $\alpha, \beta, \gamma$ are arbitrary complex numbers. Only the modular parameter $\gamma$ is associated to a zero mode of finite norm, so only $\gamma$ is allowed to be time dependent in (3.17). Fixing $\alpha$ and $\beta$, the reduced moduli space $\tilde{M}_N$ for one lump is two dimensional with a flat metric and $\gamma$ can be interpreted as the location of the soliton in the plane. For this case $M_N$ is simply the orbit space of the isometries induced by translations in $R^2$.

The effective action (3.18) is thus an $N=2$ supersymmetric quantum mechanics based on $R^2$. We can rewrite it in hamiltonian form as

$$S_{eff} = \int dt \left\{ P_a \dot{X}^a + \frac{im}{2} \lambda^a \dot{\lambda}^b \delta_{ab} - \frac{P^2}{2m} \right\}$$ \hspace{1cm} (4.2)

where $a, b = 1, 2$ and the mass of the lumps $m$ has been defined via $G_{ab} = m \delta_{ab}$. In complex co-ordinates the supersymmetry charges (3.22) are given by

$$Q = \lambda P \quad Q^* = \lambda^* P^*$$ \hspace{1cm} (4.3)

The non-vanishing canonical quantum commutation relations can be read off di-
rectly from (4.2), viz

\[ [X^a, P_b] = i\delta^a_b \quad \{\lambda, \lambda^*\} = m^{-1} \]  

(4.4)

where here and in the following we set \( \hbar = 1 \). Defining a state \( |0\rangle \) satisfying \( \lambda |0\rangle = 0 \), the Hilbert space consists of two types of states: \( |0\rangle f(X) \) and \( \lambda^* |0\rangle g(X) \). The momentum operator is realised on these states in the usual way: \( P_a = -i\frac{\partial}{\partial X^a} \). The Hamiltonian is constructed using the supersymmetry algebra:

\[ H = \{Q, Q^*\} = \frac{p^2}{2m} \]  

(4.5)

The operator given by

\[ S = -\frac{1}{2}\lambda\lambda^* \]  

(4.6)

clearly commutes with the Hamiltonian (4.5) and satisfies (3.25). Thus we can interpret it as the semiclassical spin operator in the one-lump sector. Noting that

\[ [S, \lambda] = -\frac{1}{2}\lambda \quad [S, \lambda^*] = \frac{1}{2}\lambda^* \]  

(4.7)

we deduce that the spin of the two types of states differ by a half and that the Hilbert space consists of a d=2+1 supermultiplet. An alternative way to arrive at this conclusion would be to use the arguments developed in [23] to show that the N=2 superparticle in d=2+1 is a more accurate description of the dynamics of a single supersymmetric lump since it posesses all of the appropriate symmetries: broken symmetries are non-linearly realised and unbroken symmetries are linearly realised. In fact (4.2) is a gauged fixed version of this superparticle action. The quantisation of superparticle actions naturally leads to supermultiplets. A more sophisticated analysis is required to determine the exact spin content of the spectrum. It is natural to expect that there is one boson and one fermion state and that an anyonic supermultiplet could be arrived at if one included a Hopf term in the sigma-model action.
We now turn to a discussion of the quantisation in the multi-lump sectors. Because of the translation invariance of the underlying filed theory, the moduli space \( \tilde{M}_N \) will factorise into a flat piece and a non-trivial piece \( \tilde{M}^0_N \) [4]. The flat piece corresponds to the centre of mass motion, the orbit space of the translations, and its quantisation is the same as for the one lump case leading to a \( d=2+1 \) supermultiplet. The quantum states of (3.18) are then obtained as a tensor product of these states and those coming from the quantisation of the non-trivial piece. Thus the non-trivial aspect of the dynamics concerns quantising on the non-trivial part of the moduli space so we now turn to a discussion of this.

We want to quantise the action (3.18) on an arbitrary Kähler manifold \( \tilde{M}^0_N \). The quantisation of the action (3.18) on an arbitrary compact manifold has been considered in [24,25]. It was shown that a natural way of quantising the system leads to a Hilbert space of states consisting of spinors on the target space (assuming the target admits a spin structure). In the present case, where the target space is Kähler we will show that an equally natural quantisation procedure leads to a Hilbert space of states consisting of the anti-holomorphic forms on the target. The difference between the two quantisations is due to an operator ordering ambiguity in the transition from classical to quantum mechanics. The operator ordering can be determined by demanding that the operator ordering of the zero modes is consistent with the operator ordering of the parent field theory in the vacuum sector. In the appendix we show that an operator ordering of the sigma-model consistent with the ordering adopted in [10] to calculate the Witten index of the theory, leads to an ordering producing the Hilbert space of anti-holomorphic forms on the target.

We now show how this quantisation procedure works. The quantisation of (3.18) is facilitated by introducing tangent space indices. Since a Kähler manifold

\[ * \text{If the moduli space were a Calabi-Yau manifold then these two quantisations would be equivalent.} \]
has $U(N)$ holonomy, we can introduce a basis of one forms $\{e_p^A, e_{\bar{p}}^{\bar{A}}\}$ satisfying

$$G_{p\bar{q}} = e_p^A e_{\bar{q}}^{\bar{B}} \delta_{AB} \quad (4.8)$$

with $(e_p^A)^* = e_{\bar{p}}^{\bar{A}}$. Furthermore the nonvanishing components of the spin connection can be chosen to take the form

$$\omega_p^A \, _B = e_q^A (E_q^B p + \gamma_q \, _{pr} E^r_B) \quad \text{and c.c.}$$
$$\omega_{\bar{p}}^{\bar{A}} \, _B = e_{\bar{q}}^{\bar{A}} E_{\bar{q}}^B \bar{p} \quad \text{and c.c.} \quad (4.9)$$

where we have introduced $E_p^B$ satisfying $e_p^A E_p^B = \delta_A^B$, $e_p^A E_{\bar{q}}^{\bar{A}} = \delta_p^{\bar{q}}$ and “c.c” denotes the complex conjugates. Using these definitions we can rewrite the action (3.18) in the form

$$S = \int dt \left\{ G_{p\bar{q}} \dot{X}^p \dot{X}^{\bar{q}} + i \lambda^A D_t \lambda^{\bar{B}} \delta_{A\bar{B}} \right\} \quad (4.10)$$

where the covariant derivative is now defined using the pullback of the spin connection from the target to the worldline.

The canonical momenta are given by

$$P_p = G_{p\bar{q}} \dot{X}^{\bar{q}} + i \lambda^A \lambda^{\bar{B}} \omega_{pAB} \quad \text{and c.c.}$$
$$\mathcal{P}_A = \frac{\delta L}{\delta \lambda^A} = 0$$
$$\mathcal{P}_{\bar{B}} = \frac{\delta L}{\delta \lambda^{\bar{B}}} = -i \lambda^A \delta_{A\bar{B}} \quad . \quad (4.11)$$

The definition of the momenta conjugate to the fermionic variables necessarily contain second class constraints. These can be eliminated by replacing the following non-zero graded Poisson brackets

$$\{X^p, P_q\}_\text{pb} = \delta^p_q \quad \{\lambda^p, \mathcal{P}_q\}_\text{pb} = -\delta^p_q \quad \text{and c.c.} \quad (4.12)$$

by Dirac brackets. To canonically quantise one then replaces the (graded) Dirac brackets by (graded) commutators via $\{ \, , \}_D \rightarrow -i[ \, , ]$. A short calculation
shows that the non-zero commutators are given by

\[ [X^p, P_q] = i\delta^p_q \text{ and } \text{c.c.} \]

\[ \{\lambda^A, \lambda^B\} = \delta^{AB} \] .

The holomorphic and anti-holomorphic components of the fermions are thus raising and lowering operators. Defining a state \(|0\rangle\) satisfying \(\lambda^A |0\rangle = 0\) the Hilbert space consists of the following states:

\[ |f\rangle = \lambda^{\tilde{A}_1} \ldots \lambda^{\tilde{A}_p} |0\rangle \frac{1}{p!} f_{\tilde{A}_1 \ldots \tilde{A}_p} (X) \]

\[ = \lambda^{\tilde{p}_1} \ldots \lambda^{\tilde{p}_p} |0\rangle \frac{1}{p!} f_{\tilde{p}_1 \ldots \tilde{p}_p} (X) \] .

Acting on these states the bosonic momenta are realised in the usual way: \( P_p = -i \partial_{X^p} \) and \( P_{\bar{p}} = -i \partial_{\bar{X}^p} \). Thus the Hilbert space can be identified with the space of anti-holomorphic forms on the moduli space.

The natural inner product is given by the hermitean inner product of the differential forms. If \(|f\rangle\) and \(|g\rangle\) are two states corresponding to p-forms the inner product is given by

\[ <f|g> = \frac{1}{p!} \int dX d\bar{X} \sqrt{G} f^{\tilde{p}_1 \ldots \tilde{p}_p} g_{\tilde{p}_1 \ldots \tilde{p}_p} . \]

The inner product of two states corresponding to different rank forms is zero and the inner product of arbitrary states is obtained by linearity. We note here that using this inner product implies that \( P_p^\dagger = P_{\bar{p}} - i\gamma^q \frac{\partial}{\partial \bar{X}^q} \).

The supersymmetry charges are given by

\[ Q = \lambda^p \pi_p \]

\[ Q^* = Q^\dagger = \lambda^{\bar{p}} \pi_{\bar{p}} \] \( \) ,

where we have defined

\[ \pi_p = P_p + i\lambda^{\bar{B}} \lambda^A \omega_{pAB} \]

\[ \pi_{\bar{p}} = P_{\bar{p}} + i\lambda^{\bar{B}} \lambda^A \omega_{\bar{p}AB} \] .

In defining the quantum supersymmetry charges we have made a definite choice of
operator ordering. We argue in the appendix that this is consistent with a natural operator ordering choice in the full field theory.

To see how the supercharges act on the states, we first present the following useful commutation relations:

\[
\{ \lambda^p, \lambda^{\bar{q}} \} = G^{p\bar{q}} \\
[\pi_p, \lambda^q] = i\gamma^r_{pr} \lambda^r \quad \text{and} \quad \text{c.c.} \\
[\pi_p, \pi_{\bar{q}}] = -R_{\bar{p}pq} \lambda^p \lambda^q
\] (4.18)

Using these it is straightforward to show that acting on the states (4.14), \(\pi_p = -i\nabla_p\) and \(\pi_{\bar{p}} = -i\nabla_{\bar{p}}\). A short calculation then shows that the supersymmetry charges act on the states as follows

\[
Q^* |f\rangle = \lambda^{\bar{p}1} \ldots \lambda^{\bar{p}_{p+1}} |0\rangle \frac{-i}{p!} \nabla_{[\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_{p+1}]}(X) \\
Q |f\rangle = \lambda^{\bar{p}1} \ldots \lambda^{\bar{p}_{p-1}} |0\rangle \frac{-i}{(p-1)!} \nabla_{\bar{q}_{\bar{p}_1}, \ldots, \bar{p}_{p-1}}(X)
\] (4.19)

Thus we identify the supersymmetry charges as the anti-holomorphic exterior derivative and its adjoint:

\[
Q^* = -i\bar{\partial} \\
Q = i\bar{\partial}^\dagger
\] (4.20)

where the adjoint is defined with respect to the inner product (4.15) (and thus is strictly only defined on normalisable states).

The quantum Hamiltonian is calculated using the supersymmetry algebra:

\[
H = \{ Q, Q^* \} = \bar{\partial}^\dagger \bar{\partial} + \partial \bar{\partial}^\dagger = G^{p\bar{q}} \pi_p \pi_{\bar{q}} + R_{\bar{p}pq} \lambda^{\bar{q}} \lambda^p
\] (4.21)

On a Kähler manifold \(\bar{\partial}^\dagger \bar{\partial} + \partial \bar{\partial}^\dagger = \frac{1}{2}(dd^\dagger + d^\dagger d)\) is simply half the Laplacian acting on differential forms. The quantum problem can now be tackled in the usual
manner. We look for energy eigenstates of the Hamiltonian (4.21), interpreting the normalisable wave functions as bound states and the non-normalisable ones as scattering states.

We first discuss the bound states. As we have noted a quantum state of the effective action corresponds to a state of the full sigma model field theory assuming that all but the zero modes are in their ground states. Hence, the bound states of the effective theory with lowest energy must correspond to bound states in the full theory as there are no states with lower energy into which they could decay. The other bound states could correspond to bound states of the full theory or possibly resonances, determining which seems a difficult problem.

Thus the normalisable states of lowest energy are an important class of states. Because the effective action is supersymmetric, the energy of the states are all greater than zero. If a state has zero energy then it must be annihilated by all the supersymmetry charges. In the present case this is equivalent to the state corresponding to a normalisable harmonic \((0,p)\) form on the moduli space \(\tilde{M}_N^0\). That is, a non-trivial element \(\bar{\partial}\)-cohomology class of the moduli space \(\tilde{M}_N^0\) with the proviso that the state is normalisable. This proviso is necessary since the space is non-compact.

We note that the spin of these states could be ascertained by constructing the angular momentum operator. The rotation invariance of the parent sigma model implies that the metric on \(\tilde{M}_N^0\) is \(SO(2)\) invariant. Thus there is a corresponding angular momentum operator of the effective theory that commutes with the Hamiltonian. Since the bound states have zero energy and hence zero orbital angular momentum, the angular momentum of the states is just the spin of the states.

The next stage in the analysis of the low-energy dynamics of the supersymmetric lumps would be to investigate the scattering states. This problem seems to require a rather detailed knowledge of the geometry of the moduli space. We note that one of the difficulties in pursuing the quantum mechanics further is that it will depend on which moduli space \(\tilde{M}_N\) one is considering. Depending on how one fixes
the moduli that are not allowed to vary in time one will obtain different geometries and possibly topologies and hence different quantum systems. It would be wise to investigate the quantum scattering in the purely bosonic case first, which has not yet been attempted.

5. Conclusions

In this paper we have initiated the investigation of the low-energy dynamics of solitons of supersymmetric field theories by presenting some detailed calculations within the context of a class of $N=2$ non-linear sigma models. We developed a number of techniques that can be applied to many other models. We conclude by summarising the important points. Generically, the solitons break half of the supersymmetry and the bosonic and fermionic zero modes form a multiplet with respect to the unbroken supersymmetry. This can be used to carry out a supersymmetric collective co-ordinate expansion and the low-energy effective action will be that of supersymmetric quantum mechanics based on the moduli space of static bosonic soliton solutions. The operator ordering ambiguities in the quantum theory can be reduced by demanding that the operator ordering in the soliton sectors be the same as that in the vacuum sector. It seems likely that the Hilbert space of states will be isomorphic to some kind of differential forms on the moduli space. Bound states of zero energy will then correspond to normalisable harmonic forms on the moduli space and correspond to bound states in the spectrum of the parent field theory. Detailed calculations of the scattering theory of the effective action is an issue that we hope to report on in the future.

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APPENDIX

In general there are operator ordering ambiguities in the quantisation of any classical system. Demanding that observables be self adjoint operators and that the Hilbert space furnish a representation of any symmetry groups present constrains the operator ordering. In the semi-classical quantisation of theories with solitons a further constraint is that one must ensure that the operator ordering in the vacuum sector is the same as that in the soliton sectors [26].

We will argue at a formal level at least that the operator ordering chosen in section 4 is consistent with a natural operator ordering in the canonical quantisation of the full supersymmetric sigma model (3.1). The operator ordering of the sigma model that we will use agrees with that chosen by Witten in [10] to calculate the Witten index of the model.

Using the components of the spinor field $\psi^i_\alpha(x)$ we first define

$$
\psi^i = \frac{1}{\sqrt{2}}(\psi^i_1 + i\psi^i_2) \quad \psi^{*i} = \frac{1}{\sqrt{2}}(\psi^i_1 - i\psi^i_2)
$$

(A.1)

We next introduce an orthonormal basis of one forms on the target manifold of the sigma model \{$e^P_j(\phi(x))\}$ and a spin connection defined by

$$
\omega^P_j(\phi(x)) = e^P_j(\partial_i E^j_Q + \Gamma^i_{jk} E^k_Q)
$$

(A.2)

where we have also introduced the inverse matrix $E^i_P(\phi(x))$. In terms of these variables the supersymmetric sigma model action (3.1) can be written in the form

$$
S = -\frac{1}{2} \int d^3x \left\{ g_{ij} \partial_m \phi^i \partial_n \phi^j \eta^{mn} + i\psi^P \slashed{D} \psi^{*Q} \delta_{PQ} + \frac{1}{2} R_{PQRS} \psi^{*P} \psi^{*Q} \psi^R \psi^S \right\}
$$

(A.3)

where we have defined $\psi^P = e^P_i \psi^i$.

Considering (A.3) as a supersymmetric quantum mechanics based on the infinite dimensional target $\mathcal{C}$, the canonical quantisation is now similar to the quantisation of the supersymmetric quantum mechanics discussed in section 4. Without
presenting all the details, the non-vanishing equal time commutation relations are given by

\[ [\phi^i(x), P_j(y)] = i\delta^i_j \delta(x - y) \]
\[ \{\psi^P(x), \psi^{* Q}(y)\} = \delta^{PQ} \delta(x - y) \] (A.4)

where \( P_i(x) \) is the momentum conjugate to \( \phi^i(x) \). Defining a state \( |0\rangle \) satisfying

\[ \psi^i(x) |0\rangle = 0 \] (A.5)

the Hilbert space consists of states of the form

\[ \int dx^{i_1} \ldots dx^{i_n} dF[\phi]_1 \ldots \phi_n(x^{i_1} \ldots x^{i_n}) \psi^{i_1}(x^{i_1}) \ldots \psi^{i_n}(x^{i_n}) |0\rangle \] (A.6)

which we can formally identify with the differential forms on \( C \). The momentum operator is realised on these states as a functional derivative: \( P_i(x) = -i \frac{\delta}{\delta \phi^i(x)} \).

The components of the supersymmetry charges are given by

\[ \tilde{Q} = \int d^2 x \psi^i \Pi_i \quad \tilde{Q}^* = \int d^2 x \psi^{* i} \Pi_i \]
\[ S = \int d^2 x J^i_\phi(\phi(x)) \psi^i \Pi_j \quad S^* = \int d^2 x J^{* i}_\phi(\phi(x)) \psi^{* i} \Pi_j \] (A.7)

where we have defined

\[ \Pi_i(x) = P_i(x) + i\omega_i P Q \psi^{* Q} \psi^P \] (A.8)

In [10] the quantisation of this model was considered in the zero momentum sector (fields independent of \( x \)) in order to calculate the Witten index of the model. It was shown that with a particular choice of operator ordering the quantisation led to a Hilbert space of states isomorphic to the differential forms on the target manifold of the sigma model and that the supersymmetry charges acted as the exterior derivative and its adjoint. It can be checked that the operator ordering we have chosen is consistent with that chosen in [10].
Using the supersymmetry algebra to calculate the Hamiltonian we find

\[ 2H = \{ \tilde{Q}, \tilde{Q}^* \} = \int d^2 x \left( \frac{1}{\sqrt{g}} \Pi_i \sqrt{g} g^{ij} \Pi_j + R_{iklj} \psi^i \psi^j \psi^k \psi^l \right) \]  

(A.9)

where we have used heavily the Kähler property of \( C \). We now want to truncate the theory to the zero-mode sector. We view the collective co-ordinate expansion (3.17) as change of co-ordinates on \( C \) from the field \( \phi^i(x) \) to its infinite mode expansion. Ignoring all but the collective co-ordinates will give us the desired operator ordering in the lump sectors. We first consider the kinetic term in (A.9). Formally it is the Laplacian acting on the differential forms on \( C \). Since this is independent of the choice of co-ordinates on \( C \) it truncates to the Laplacian acting on the differential forms on \( \tilde{M}_N \). Thus after substituting (3.17) into the curvature term in (A.9) we obtain the following truncated Hamiltonian

\[ H = \frac{1}{2} \left[ \pi^a \sqrt{G} G^{ab} \pi_b - R_{pqrs} \lambda^p \lambda^q \lambda^r \lambda^s \right] \]

(A.10)

Comparing with (4.21) we verify that the operator orderings agree.

REFERENCES

1. N. Manton, Phys. Lett. **110B** (1982) 54.
2. M. Atiyah and N. Hitchin, The Geometry and Dynamics of Magnetic Monopoles, Princeton University Press, 1988.
3. P. Ruback, Nucl. Phys. **B296** (1988) 669.
4. R.S. Ward, Phys. Lett. **158B** (1985) 424.
5. G.W. Gibbons and P. Ruback, Phys. Rev. Lett. **57** (1986) 1492.
6. G.W. Gibbons and N.S. Manton, Nucl. Phys. **B274** (1986) 183.
7. B. Schroers, preprint DAMTP-91-05
8. T.M. Samols, preprint DAMTP-91-13
9. J. Harvey and A. Strominger, preprint EFI-91-30.
10. E. Witten, Nucl. Phys. **B202** (1982) 253.
11. C. Montonen and D. Olive, Phys. Lett. **72B** (1977) 213.
12. E.P.S. Shellard, Cosmic Strings: the Current Status, 1988.
13. J.P. Gauntlett and J. Harvey, in preparation.
14. A.M. Din and W.J. Zakzrewski, Nucl. Phys. **B253** (1985) 77.
15. I. Stokoe and W.J. Zakzrewski, Z. Phys. **C34** (1987) 491.
16. R. Leese, Nucl. Phys. **B344** (1990) 33.
17. P. Ruback, Commun. Math. Phys. **116** (1988) 645.
18. B. Zumino, Phys. Lett. **87B** (1979) 203.
19. L. Alvarez-Gaumé and D.Z. Freedman, Commun. Math. Phys. **80** (1981) 443.
20. E. Witten and D. Olive, Phys. Lett. **78B** (1978) 97.
21. J.A. de Azcarraga, J.P. Gauntlett, J.M. Izquierdo and P.K. Townsend, Phys. Rev. Lett. **63** (1989) 2443.
22. B. Zumino, Phys. Lett. **69B** (1977) 369.
23. J.P. Gauntlett and C. Yastremiz, Class. and Quantum Grav. **7** (1990) 2089.
24. L. Alvarez-Gaume, Commun. Math. Phys. **90** (1983) 161.
25. D. Friedan and P. Windey, Nucl. Phys. **B235** (1984) 395.
26. R. Rajaraman, Solitons and Instantons, North-Holland, 1987