EXTRA-FINE SHEAVES AND INTERACTION DECOMPOSITIONS

DANIEL BENNEQUIN, OLIVIER PELTRE, GRÉGOIRE SERGEANT-PERTHUIS, AND JUAN PABLO VIGNEAUX

ABSTRACT. We introduce an original notion of extra-fine sheaf on a topological space, for which Čech cohomology in strictly positive dimension vanishes. We provide a characterization of such sheaves when the topological space is a partially ordered set (poset) equipped with the Alexandrov topology. Then we further specialize our results to some sheaves of vector spaces and injective maps, where extra-fineness is (essentially) equivalent to the decomposition of the sheaf into a direct sum of subfunctors, known as interaction decomposition, and can be expressed by a sum-intersection condition. We use these results to compute the dimension of the space of global sections when the presheaves are freely generated over a functor of sets, generalizing classical counting formulae for the number of solutions of the linearized marginal problem (Kellerer and Matúš). We finish with a comparison theorem between the Čech cohomology associated to a covering and the topos cohomology of the poset with coefficients in the presheaf, which is also the cohomology of a cosimplicial local system over the nerve of the poset. For that, we give a detailed treatment of cosimplicial local systems on simplicial sets. The appendixes present presheaves, sheaves and Čech cohomology, and their application to the marginal problem.

CONTENTS

1. Introduction 1
2. Fine, extra-fine, super-local and acyclic 5
3. Alexandrov topologies and sheaves 9
4. Interaction decomposition 12
5. Factorization of free sheaves 18
6. Nerves of categories and nerves of coverings 23
Appendix A. Topology and sheaves 37
Appendix B. Čech cohomology 39
Appendix C. Finite probability functors 40
References 44

1. Introduction

This article develops cohomological tools to study collections of data associated to hypergraphs, or to more general partially ordered sets (posets). The kind of data we will consider is organized in families of sets indexed by the elements of the poset, forming covariant and contravariant functors with respect to the partial ordering,
which are called respectively copresheaves and presheaves over the poset. Such functors have been applied to several problems at the crossroad of data analysis, information theory, coding theory, logic, computation, and bayesian learning. We will mention below some of these problems and develop several applications of the cohomological approach.

In this work, we see a partially ordered set (poset) $\mathcal{A}$ as a small category such that:

1. There is at most one morphism between two objects;
2. If $a \to b$ and $b \to a$, then $a = b$.

An hypergraph is a particular case of poset, whose objects are some finite subsets of an index set $I$, and there exist a morphism $S \to S'$ whenever $S' \subseteq S$. An abstract simplicial complex is an hypergraph $K$ that satisfies an additional property: if $S$ belongs to $K$, then every subset of $S$ belongs to $K$ too.

A presheaf on a category $\mathcal{A}$ is a contravariant functor $F$ from $\mathcal{A}$ to the category of sets $\mathcal{S}$, in other terms it is a covariant functor on the opposite category $F: \mathcal{A}^{\text{op}} \to \mathcal{S}$. A copresheaf is just a covariant functor $F: \mathcal{A} \to \mathcal{S}$. The presheaves of classical sheaf theory on topological spaces [15] are obtained when $\mathcal{A}$ is the category of open sets of some topological space, which is an example of poset.

Abstract simplicial complexes play a prominent role in persistent homology [8,14], a technique to extract topological features that is a cornerstone of applied algebraic topology. The basic idea is to replace a sequence of data points in a metric space by an abstract simplicial complex induced by a proximity parameter (e.g. the Čech complex or the Vietoris-Rips complex). Then homological tools (spectral sequences) are applied to an increasing family of complexes for defining invariant quantities of the data.

Curry’s dissertation [10] showed that persistent homology can be extended in several directions involving sheaves on posets of parameters.

Curry [10] also gave a systematic treatment of sheaves defined on another kind of complexes, the cellular complexes (giving cellular sheaves and cosheaves), which he traces back to Zeeman’s Ph.D. thesis [47]. A spectral theory of such sheaves was later developed by Hansen and Ghrist [18]. Those works list several situations that can be modeled by cellular (co)sheaves, which include network coding, sensor networks, distributed consensus, flocking, synchronization and opinion dynamics, among other things.

Along similar lines, a series of works by Robinson and collaborators [29, 31, 30] argued that sheaves are a canonical model for the integration of information provided by interconnected sensors. In those works, the vertices of an abstract simplicial complex represent heterogeneous data sources and the abstract simplexes some sort of interaction between these sources. It is claimed that sheaves constitute a canonical data structure if one requires sufficient generality to represent all sensors of interest and the ability to summarize information faithfully. A similar approach is taken by Mansourbeigi in his doctoral dissertation [23].

Independently, Abramsky and his collaborators (see e.g. [2, 1]) have used sheaves and cosheaves on simplicial complexes to study contextuality. In this situation, the
vertices represent observables, the simplices represent joint measurements (measurement contexts) and the maximal faces of the complex are called maximal contexts. The functor associates to each context a set of possible outcomes or a set of probabilities on those outcomes. Contextuality refers to the fact that it can happen that some sections of the probability functor (i.e. coherent collections of “local” probabilities) are not compatible with a globally defined probability law. In this article, we refer to this problem as the probabilistic marginal problem. There are also linearized versions of this problem, as well as “possibilistic” versions.

In all these examples, homology and cohomology is used to determine the “shape” of the simplicial complex or the relevant geometrical invariants of the associated sheaves.

Simplicial complexes are particularly convenient because they have a geometric realization as CW-complexes, so they can be studied using standard tools in algebraic topology e.g. standard homology and cohomology theories. Hypergraphs were introduced in combinatorics, not in geometry, hence their geometrical study is less straightforward. There have been several proposals to define (co)homological invariants of hypergraphs. A recent paper by Bressan, Li, Ren and Wu [7] defines the embedded homology of an hypergraph \( H \), which equals the homology of the smallest abstract simplicial complex that contains \( H \). A specific cohomology of \( k \)-regular hypergraphs (i.e. containing only subsets of cardinality \( k \)) was introduced by Chung and Graham [9] motivated by some problems in combinatorics.

The present article develops an alternative approach, based on sheaf theory and simplicial methods. We equip the poset \( A \) with the lower or upper Alexandroff topology (see Section 3), obtaining the topological space \( X_A \) or \( X^{op} \), respectively. There is a bijection between covariant (resp. contravariant) set-valued functors on \( A \) and sheaves on \( X_A \) (resp. \( X^{op} \)) i.e.

\[
[A, S] \cong \text{Sh}(X_A), \quad [A^{op}, S] \cong \text{Sh}(X^{op}).
\]

In other words, we can see a (co)presheaf on \( A \) as a usual sheaf on a topological space, where Čech cohomology can be used. This cohomology is convenient from a computational viewpoint and well adapted to study the global sections of the sheaf.

The article presents and studies in detail several equivalent definitions of this cohomology, from simplicial methods involving nerves of categories, and from topos theory (i.e. derived algebra and geometry), all presenting a particular interest for some specific problem.

Here, we are particularly interested in the following setting, which is adapted to a wide variety of problems, as mentioned above. One introduces an hypergraph \( A \) with vertex set \( I \). The elements of \( I \) represent elementary observables or sources, and the elements \( \alpha \) of \( A \) represent interactions or joint measurements. To take into account the internal degrees of freedom of each object of \( A \), one introduces a covariant set-valued functor \( E : A \to S \) of possible outcomes, associating to each object \( \alpha \) of \( A \) a set \( E_\alpha \), and to each arrow \( \alpha \to \beta \) a surjective map \( E_\alpha \to E_\beta \). The local probabilities on each \( E_\alpha \) or the functions over each \( E_\alpha \) give rise to other important functors, that can be covariant or contravariant. For instance, the vector spaces \( \{ V_\alpha \}_{\alpha \in A} \) of numerical functions on the sets \( E_\alpha \), and the inclusion \( j_{\alpha \beta} : V_\beta \to V_\alpha \) form a contravariant functor (an example of an injective presheaf, see Section 4).
In particular, the study of the special case of real-valued functions of the probability laws on finite sets \( E_\alpha \) over a simplicial complex \( \mathcal{A} \) gives a natural interpretation in terms of topos theory and cohomology [3] of the information quantities defined by Shannon and Kullback, or by Von Neumann in the quantum case, cf. [6, 11]. These results were later extended to presheaves of functions of statistical frequencies, and to gaussian laws in Euclidean space [12]. The cohomologies which were used here are not of the type of Čech, they are based on the action of variables on probabilities by conditioning, expressed as non-trivial modules in the topos of presheaves over \( \mathcal{A} \). A conjecture is that computing cohomology in degrees higher than one will give entirely new information quantities.

Furthermore, there exist several notions of morphisms from a module \((\mathcal{A}; V)\) to a module \((\mathcal{B}; W)\). A natural hope is the existence of convenient categories of sheaves on hypergraphs that would be suitable to construct a new kind of geometric topology and homotopy theory in this setting. A similar approach was taken by Friedman in the case of graphs in order to prove Hanna Neumann’s conjecture [13]: the category of directed graphs over a given directed graph \( G \) can be faithfully embedded in \( \text{Sh}(G) \), but such embedding is not full and the kernels of some new morphisms (\( \rho \)-kernels) play a fundamental role in the proof (which essentially reduces the problem to the vanishing of the homology of those kernels).

We also expect that sheaf-theoretic constructions on hypergraphs will give a better understanding of certain algorithms in Statistics or Machine learning. In this direction, Olivier Peltre (cf. [27] and his thesis [28]) has developed a cohomological understanding of the Belief Propagation Algorithm (in the generalized version of [46]); the algorithm appears as a non-linear dispersion flow. Higher dimensional analogs are promising tools. Grégoire Sergeant-Perthuis (cf. [34, 35, 36, 37] and his forthcoming thesis) focussed on defining the thermodynamical limit in the category of Markov Kernels, extending several constructions of statistics and statistical physics such as the decomposition into interaction subspaces, first introduced for factor spaces [20, 40], the space of Hamiltonians, infinite-volume Gibbs state, and the renormalisation group.

In both these works, a same result appears: the vanishing of sheaf cohomology (in the toposic form, or in Čech form respectively) in degree larger than one (i.e. acyclicity, without contractility) for the case of an injective presheaf \( V \) over \( \mathcal{A} \), under a certain condition relating the intersections and the sums of the subspaces given by the faces: the condition \([G]\) in Section 3. The main goal of this article is to enunciate and prove this result, and to place it in a topological context.

The condition \([G]\) is satisfied for hypergraphs and freely generated modules when the hypergraph is closed by intersection. As a corollary we get a proof by homology of the Marginal Theorem of [19], showing the existence of perhaps non-positive measures on the joint measurement represent by the index set \( I \), with prescribed compatible marginals along the arrows of \( \mathcal{A} \), and computing their dimensions with the Möbius function of \( A \), cf. also [24]. Remark that [13] showed that the necessary and sufficient condition for extension by positive measures of every collection of compatible marginals is the regularity of the complex \( \mathcal{A} \), which is a convenient notion of contractility in this context (cf. [28]).

Section 2 defines an original notion of extra-fine presheaf over a topological space \( X \), that is reminiscent of the classical notion of fine sheaf. Then we prove, as it was
the case for fine presheaves on paracompact spaces, that extra-fine implies acyclic for the Čech cohomology (Theorem 1) on any topological space. In Section 3 we characterize extra-fine sheaves on the Alexandrov spaces $X_\mathcal{A}$ or $X^\mathcal{A}$ by the property of interaction decomposition.

In Section 4 we consider presheaves $V$ of $\mathbb{K}$ vector spaces (over any field $\mathbb{K}$) and injective maps. For such presheaves, we give an alternative characterization of extra-fineness through the sum-intersection condition; this is the main result of the article (Theorem 4.4). We do that without any finiteness condition on the vector spaces, and only weak finiteness conditions for the poset. Then we study duality, proving the acyclicity theorem for the weak dual cosheaves.

Section 5 contains the definition of free presheaves generated by a covariant set-valued functor $E$ over a commutative field $\mathbb{K}$ (the usual case in data analysis over hypergraphs). We establish the condition $G$ for the injective presheaf $V$ of functions from $E$ to $\mathbb{K}$, when the poset $\mathcal{A}$ has conditional coproducts (meaning stable by non-empty intersections in the case of hypergraphs), then its acyclicity (Theorem 5.7). The acyclicity is deduced for the sheaf generated by $E$ over $\mathbb{K}$ on $X_\mathcal{A}$. Then we compute the cohomology on hypergraphs (Theorem 5.10): it is the sum of the ordinary cohomology of $\mathcal{A}$ in all degrees, and of the cohomology of degree zero of a restricted sheaf of functions (where the sum of coordinates is zero). We also prove a version of the marginal theorem (surjection in Čech cohomology, Theorem 5.15), which seems to be new in this generality. We deduce an index theorem for the Euler characteristic of the marginal sheaves (Theorem 5.16).

Finally, Section 6 comes back to a general topological space $X$ and presheaves of abelian groups, to provide the homotopy equivalence of the Čech cochain complex of an open covering of a presheaf with the cochain complex of the nerve of the category generated by the covering; this is done for a general notion of cosimplicial coefficients (Theorem 6.31). This answers a natural question in our framework, but the proof is surprisingly cumbersome, which is reminiscent of the known fact that there exists an homotopy equivalence between a finite simplicial complex and its barycentric subdivision but non-canonically.

In all the above sections we take care of morphisms between presheaves and naturality behaviors, or functoriality.

Three appendices are added at the end, where we summarize the main objects and constructions involved in the article: sheaves, Čech cohomology, and Möbius functions, among other things.

2. Fine, extra-fine, super-local and acyclic

In this section, we consider presheaves of abelian groups over a topological space $X$. See Appendix A for some basic topological definitions and notations. We use Čech cohomology as presented in any standard reference, e.g. [39], but all relevant definitions can also be found in Sections 6.1 and 6.2 under the formalism of cosimplicial local systems.

**Definition 2.1 (Fine presheaf, cf. [39 Sec. 6.8]).** A presheaf $F$ of abelian groups over a topological space $X$ is said to be fine if for every open covering $\mathcal{U}$ of $X$, there exists a family $\{e_V\}_{V \in \mathcal{U}}$ of endomorphisms of $F$ (i.e. natural transformations $e_V : F \to F$, whose components $e_V(W)$ we denote by $e_V|_W$), such that:
(i) For all \( V \in \mathcal{U} \) and every open set \( W \), one has \( e_{V|\mathcal{W}}(W \setminus V) = 0 \).

(ii) For every open \( W \) and every \( x \in F(W) \), there exists only a finite number of elements \( V \in \mathcal{U} \) such that \( e_{V|\mathcal{W}}(x) \neq 0 \), and we have \( x = \sum_{V \in \mathcal{U}} e_{V|\mathcal{W}}(x) \).

Under the second condition, the family \( \{e_V\}_{V \in \mathcal{U}} \) is named a partition of unity (or partition of identity) adapted to \( \mathcal{U} \). If the two conditions are satisfied we say that the partition of unity is supported by \( \mathcal{U} \).

Fine presheaves are part of the classical literature on sheaf theory, see also [15 Sec. 3.7] and [16 p. 42], although the classical definitions require \( \mathcal{U} \) to be locally finite. Positive dimensional cohomology of a paracompact topological space with coefficients in a fine presheaf vanishes [39 Thm. 6.8.4], and this fact has important implications in the comparison of Alexander and Čech cohomology. We propose here a specialization of this notion that plays a fundamental role in our investigations.

**Definition 2.2 (Extra-fine presheaf).** A presheaf \( F \) of abelian groups over the topological space \( X \) is said to be extra-fine if for every open covering \( \mathcal{V} \) of \( X \), there exists a finer open covering \( \mathcal{U} \) and a partition of unity \( \{e_V\}_{V \in \mathcal{U}} \) adapted to \( \mathcal{U} \) (i.e. \( 2.1\)(ii) is satisfied), such that

(i) for all \( V \in \mathcal{U} \) and \( W \in \mathcal{U} \), \( e_{V|\mathcal{W}} \neq 0 \) implies \( W \subseteq V \);

(ii) for all \( V, W \in \mathcal{U} \) such that \( V \neq W \), \( e_V \circ e_W = e_W \circ e_V = 0 \).

**Lemma 2.3.** If a partition of unity satisfies condition \( (iii) \) then for all \( V \in \mathcal{U} \) the equality \( e_V \circ e_V = e_V \) holds.

**Proof.** For any \( s_V \in F(V) \), we have a finite decomposition \( s_V = \sum_W e_W(s_V) \), then \( e_V(s_V) = \sum_W e_V \circ e_W(s_V) \). Therefore \( e_V(s_V) = e_V \circ e_V(s_V) \), because all the other elements in the sum are zero. \( \square \)

Thus a partition of unity \( \{e_V\}_{V \in \mathcal{U}} \) that satisfies \( (iii) \) is a family of projections, decomposing the presheaf \( F \) is a direct sum; we refer to this as a local orthogonal decomposition of the functor. If \( (i') \) is also satisfied, we speak of a super-local orthogonal decomposition.

The condition \( (i') \) for a partition of unity is named super-locality; it is certainly exceptional for usual topologies, but useful for the particular topologies we are interested in in this text.

When a presheaf admits a partition of unity satisfying \( (i') \) in addition to \( (ii) \) but perhaps not \( (iii) \), we say that the presheaf \( F \) is fine and super-local.

**Proposition 2.4.** If \( F \) is a presheaf of abelian groups on \( X \), then it is fine if, for any open covering \( \mathcal{V} \) of \( X \), there exists a finer covering \( \mathcal{U} \), and a partition of unity \( \{e_V\}_{V \in \mathcal{U}} \) of endomorphisms of \( F \) supported by \( \mathcal{U} \) (i.e. satisfying \( (i') \) and \( (ii) \)). Moreover, every open covering \( \mathcal{V} \) admits a partition of unity of \( F \) which is orthogonal as soon as it is true for the finer covering \( \mathcal{U} \).

**Proof.** Suppose given a partition of unity \( \{e_V\}_{U \in \mathcal{U}} \) (resp. an orthogonal decomposition) and an arbitrary open cover \( \mathcal{V} \) of \( X \) coarser than \( \mathcal{U} \); one can build a partition of unity \( \{e_V\}_{V \in \mathcal{V}} \) subordinated to \( \mathcal{V} \) (resp. an orthogonal decomposition) as follows.

---

1Here \( (W \setminus V) \) denotes postcomposition \( e_{V|\mathcal{W}} \circ F(\iota) \) with the map \( F(\iota) : F(W \setminus V) \to F(W) \) induced by the inclusion \( \iota : W \setminus V \hookrightarrow W \).
For each \( U \in \mathcal{U} \), we choose an element \( V(U) \) of \( \mathcal{V} \) such that \( U \subseteq V(U) \). For each \( V \in \mathcal{V} \), let \( A_V \) be the set of \( U \in \mathcal{U} \) such that \( V(U) = V \). The subsets \( \{ A_V \}_{V \in \mathcal{V}} \) are two by two disjoint and their union is \( \mathcal{U} \). Define \( e_V = \sum_{U \in A_V} e_U \).

Let us show that the resulting \( \{ e_V \}_{V \in \mathcal{V}} \) form a partition of the unity (Definition 2.1). Let \( W \) be an open set in \( X \), and \( v \in F(W) \). Let \( A(v) \) be the set of \( U \in \mathcal{U} \) such that \( e_{U|W}(v) \neq 0 \); by hypothesis, this set is finite, and we have \( v = \sum_{U \in A(v)} e_{U|W}(v) \). Since the sets \( \{ A_V \}_{V \in \mathcal{V}} \) partition \( \mathcal{U} \), the set

\[
\mathcal{V}(v) = \{ V \in \mathcal{V} \mid A_V \cap A(v) \neq \emptyset \}
\]

is also finite. Now, if an element \( V \in \mathcal{V} \) is such that \( e_{V|W}(v) \neq 0 \), the corresponding set \( A_V \) is non-empty, then \( V = V(U) \) for some \( U \in \mathcal{U} \), and \( A_V \) contains at least one \( U \) such that \( e_{U|W}(v) \neq 0 \), thus \( V \) belongs to \( \mathcal{V}(v) \). This shows that the axiom (ii) of 2.2 is satisfied. Moreover, if \( V \in \mathcal{V} \) and \( W \) is open, consider the open set \( W' = W \setminus (W \cap V) \). For any \( U \subseteq V \), we have \( W' \subseteq W' \cap V \). But for every \( U \in \mathcal{U} \), we have \( e_{U|W|W'}(v) = 0 \), then for each \( U \in A_V \), by naturality of \( e_U \), we have

\[
e_{U|W|W'} = 0.
\]

This proves the condition (i).

If the decomposition \( e_U : U \in \mathcal{U} \) is orthogonal, for two different elements \( V, W \) of \( \mathcal{V} \), the sets \( A_V \) and \( A_W \) are disjoint, then the above definition of \( e_V \) and \( e_W \) shows that \( e_V \circ e_W = 0 \).

The proposition above does not extend to the super-locality; this property cannot be transferred to coarser coverings. From this result, we see that a fine presheaf can be defined analogously to an extra-fine presheaf, by the existence of a finer covering which supports a partition of unity. But extra-fine presheaves cannot be defined on the model of fine presheaves i.e. by the existence of an adapted super-local partition of identity for every open covering.

Remark 2.5 (Lack of functoriality). Let \( f : X \to Y \) be a continuous map and \( \mathcal{F} \) a fine presheaf of abelian groups over \( X \), the presheaf \( \mathcal{G} = f_* \mathcal{F} \) on \( Y \) is fine [39, Thm. 6.8.3], but it can happen that \( \mathcal{F} \) is extra-fine on \( X \) and that \( \mathcal{G} = f_* \mathcal{F} \) is not extra-fine. The problematic property is super-locality. For the inverse image of a presheaf \( \mathcal{G} \) over \( Y \), both fine and extra-fine fail to be transmitted from \( \mathcal{G} \) to \( f^{-1} \mathcal{G} \).

We shall see that positive dimensional Čech cohomology of a super-local presheaf vanishes. To fix some notations, we summarize here the construction of Čech cohomology; more details can be found in Sections 6.1 and 6.2.

Let \( \mathcal{U} \) be an open covering of a topological space \( X \), and for each \( n \in \mathbb{N} \), let \( K_n(\mathcal{U}) \) denote the set of sequences of length \( n+1 \), \( u = (U_0, ..., U_n) \), of elements of \( \mathcal{U} \) such that the intersection \( U_n = U_0 \cap ... \cap U_n \) is non-empty. For \( n \in \mathbb{N} \), a Čech cochain of \( \mathcal{F} \) of degree \( n \) with respect to \( \mathcal{U} \) is a element \( \{ c(u) \}_{u \in K_n(\mathcal{U})} \) of \( \prod_{u \in K_n(\mathcal{U})} F(U_u) \). The set of \( n \)-cochains is denoted \( C^n(\mathcal{U}; F) \); it is an abelian group.

A coboundary operator \( \delta : C^n(\mathcal{U}; F) \to C^{n+1}(\mathcal{U}; F) \) is then introduced, as a linear map such that

\[
(\delta c)(U_0, ..., U_{n+1}) = \sum_{i=0}^{i=n+1} (-1)^i c(U_0, ..., \widehat{U_i}, ..., U_{n+1})|U_0 \cap ... \cap U_{n+1},
\]

where \( \widehat{U_i} \) means that \( U_i \) is omitted. When we want to be more precise we write \( \delta = \delta^{n+1} \) at degree \( n \).
It is well known that $\delta \circ \delta = 0$, which allows one to define the Čech cohomology of $F$ over $U$ in degree $n$ as the quotient abelian group $H^n(U; F) = \ker(\delta_{n+1}^U)/\text{im}(\delta_n^U)$. As explained in Appendix [13], the set of open coverings of $X$ with the relation of refinement is a directed set. And the Čech cohomology of $F$ over $X$ is defined as

$$\forall n \in \mathbb{N}, \quad H^n(X; F) = \lim_{\rightarrow} H^n(U; F).$$

From the definition of $\delta^1_0$, it is clear that the group $H^0(U; F)$ can be identified with the group of global sections of $F$ over $X$, for any open covering $U$. Hence $H^0(X; F)$ coincides with every $H^0(U; F)$ and also corresponds to global sections.

A presheaf is called acyclic if its cohomology is zero for every degree $n \geq 1$.

**Theorem 2.6.** A presheaf $F$ of abelian groups which is fine and super-local is acyclic. More precisely, for every open covering $V$, and every integer $n \geq 1$, there exists an open covering $U$ finer than $V$ such that the cohomology group $H^n(U; F)$ is zero.

**Proof.** We adapt a more elaborate argument given by Spanier in the case of para-compact spaces [13] Thm. 6.8.4.

Given $V$, we take for $U$ the covering which satisfies (U) the condition of superlocality.

Consider a cochain $\psi$ for $U$ and $F$ of degree $q \geq 1$, which is a cocycle i.e. $\delta \psi = 0$. Then for every collection $U_0, U_1, ..., U_q, U_{q+1}$ of elements of $U$, we have

$$\psi(U_1, ..., U_{q+1})|_{U_0 \cap ... \cap U_{q+1}} = \sum_{k=1}^{q+1} (-1)^{k+1} \psi(U_0, ..., \widehat{U_k}, ..., U_{q+1})|_{U_0 \cap ... \cap U_{q+1}}.$$

Set $U_0 = U$. We deduce from (2.4) that when $U$ contains $U_1 \cap ... \cap U_{q+1}$,

$$e_U \psi(U_1, ..., U_{q+1})|_{U_1 \cap ... \cap U_{q+1}} = \sum_{k=1}^{q+1} (-1)^{k+1} e_U \psi(U_1, ..., \widehat{U_k}, ..., U_{q+1})|_{U_1 \cap ... \cap U_{q+1}},$$

and when $U$ does not contain $U_1 \cap ... \cap U_{q+1}$, the super-locality implies that

$$e_U \psi(U_1, ..., U_{q+1}) = 0.$$

For any $U \in U$, we define a $(q-1)$-cochain $\phi_U$ for $F$ and the covering $U$ as follows: given $V_0, ..., V_{q-1} \in U$, if $V_0 \cap ... \cap V_{q-1} \subseteq U$ then

$$\phi_U(V_0, ..., V_{q-1}) = e_V(\psi(U, V_0, ..., V_{q-1})|_{V_0 \cap ... \cap V_{q-1}}),$$

and if $V_0 \cap ... \cap V_{q-1} \not\subseteq U$ then

$$\phi_U(V_0, ..., V_{q-1}) = 0.$$

By definition of the coboundary operator, in both cases we have

$$\delta \phi_U(U_1, ..., U_{q+1}) = \sum_{k=1}^{q+1} (-1)^{k+1} \phi_U(U_1, ..., \widehat{U_k}, ..., U_{q+1})|_{U_1 \cap ... \cap U_{q+1}};$$
which gives, when $U$ contains $U_1 \cap \ldots \cap U_{q+1}$,

\begin{equation}
(\delta \varphi_V)(U_1, \ldots, U_{q+1}) = \sum_{k=1}^{q+1} (-1)^{k+1} e_V(\psi(U_1, U_1, \ldots, \mathring{U}_k, \ldots, U_{q+1})|U_1 \cap \ldots \cap U_{q+1}),
\end{equation}

and, when $U$ does not contain $U_1 \cap \ldots \cap U_{q+1}$, gives $(\delta \varphi_V)(U_1, \ldots, U_{q+1}) = 0$.

Consequently, in any case we get

\begin{equation}
\delta \varphi_U(U_1, \ldots, U_{q+1}) = e_V(\psi(U_1, \ldots, U_{q+1})).
\end{equation}

Then we define $\phi$ by summing over the open sets $U$ in $\mathcal{U}$, and using \[^3\] we obtain $\delta \phi = \psi$, which proves the theorem. \[\square\]

3. **ALEXANDROV TOPOLOGIES AND SHEAVES**

3.1. **Basic definitions.** A partially ordered set (poset) is set with a binary relation $\leq$ that is reflexive, antisymmetric and transitive. Equivalently, it is a small category $\mathcal{A}$ such that:

1. For any pair of objects $\alpha$, $\beta$, there is at most one morphism from $\alpha$ to $\beta$, and
2. If there is a morphism from $\alpha$ to $\beta$ and a morphism from $\beta$ to $\alpha$, then $\alpha = \beta$.

Starting with a partially ordered set $\text{Ob}\, \mathcal{A}$, there exists an arrow $\alpha \to \beta$ if and only if $\beta \leq \alpha$. This convention is chosen to agree with other studies on categories of random variables, such that arrows are in the sense of fine to coarse, cf. \[^41\]. A covariant functor between two posets is simply a monotone map. We write $\alpha \in \mathcal{A}$ instead of $\alpha \in \text{Ob}\, \mathcal{A}$ if there is no risk of ambiguity.

The categorical coproduct between two objects $\alpha$ and $\alpha'$ of $\mathcal{A}$ is an object $\beta$ such that $\alpha \to \beta$ and $\alpha' \to \beta$, that additionally satisfies the following property: for any $\omega \in \mathcal{A}$, if $\alpha \to \omega$ and $\alpha' \to \omega$, then $\beta \to \omega$. Such $\beta$ is denoted $\alpha \vee \alpha'$ and called coproduct (or sup) of $\alpha$ and $\alpha'$; it is unique. We shall not suppose that our categories have all finite coproducts, but sometimes we impose the following conditional existence of coproducts: for any $\alpha, \alpha' \in \mathcal{A}$, if there exists $\omega \in \mathcal{A}$ such that $\alpha \to \omega$ and $\alpha' \to \omega$, then $\alpha \vee \alpha'$ exists.

The dual notion is the product $\alpha \land \alpha'$ of $\alpha$ and $\alpha'$, called meet. In \[^41\], Vigneaux introduced posets subject to conditional existence of meets under the name of conditional meet semilattices; they are the fundamental ingredient to introduce information cohomology.

**Example 3.1.** Let $\mathcal{K}$ be an abstract simplicial complex, i.e., a family of subsets of a given set $I$ such that if $\alpha \in \mathcal{K}$, then every subset of $\alpha$ is also in $\mathcal{K}$. In this structure all coproducts exist, $\alpha \vee \beta = \alpha \land \beta$, but meets only exist conditionally.

P. S. Alexandrov introduced a natural topology on the set of objects of a poset $\mathcal{A}$, given by a basis of open sets $U_\alpha = \{\beta \mid \alpha \to \beta\}$, indexed $\alpha \in \mathcal{A}$\[^2\]. We will name this topology the lower Alexandrov topology (A-topology) of $\mathcal{A}$, and denote $X_\mathcal{A}$ the topological space obtained in this way.

---

\[^2\] To justify the definition, one must verify that an intersection $U_\alpha \cap U_{\alpha'}$ is a union of sets $U_\beta$, $\beta \in B$; but if $\alpha \to \beta$ and $\alpha' \to \beta$, we have $U_\beta \subseteq U_\alpha \cap U_{\alpha'}$, then $U_\alpha \cap U_{\alpha'} = \bigcup_{\beta \in U_\alpha \cap U_{\alpha'}} U_\beta$. The same argument shows that the intersection of every family of open sets is an open set.
Dually, the upper sets \( U^\beta = \{ \alpha \mid \alpha \to \beta \} \), indexed by objects \( \beta \in \mathcal{A} \), form the basis of a topology that we call upper \( \mathcal{A} \)-topology of \( \mathcal{A} \). The corresponding topological space is denoted \( X^\mathcal{A} \). Clearly, it is the lower \( \mathcal{A} \)-topology of the opposite category \( \mathcal{A}^{op} \).

Remark that if \( \alpha \to \beta \) then \( U_\alpha \supseteq U_\beta \) and \( U_\alpha \subseteq U^\beta \). Also, whenever \( \mathcal{A} \) possesses conditional coproducts and \( U_\alpha \cap U_{\alpha'} \) is non-empty, one has \( U_\alpha \cap U_{\alpha'} = U_{\alpha \vee \alpha'} \); the element in \( U_\alpha \cap U_{\alpha'} \) is a common upper bound of \( \alpha \) and \( \alpha' \).

A general reference for Alexandrov spaces, finite topological spaces, and their relations to simplicial complexes is [5]. For instance, the reader can find there the following result.

**Lemma 3.2** ([5 Prop. 1.2.1]). Let \( \mathcal{A}, \mathcal{B} \) be posets. A map \( f : \text{Ob} \mathcal{A} \to \text{Ob} \mathcal{B} \) is order preserving (equivalently, defines a covariant functor from \( \mathcal{A} \) to \( \mathcal{B} \)) if and only if \( f \) is continuous for the lower (or upper) \( \mathcal{A} \)-topology.

The functors on a poset \( \mathcal{A} \) can be seen as classical sheaves on the associated topological space.

**Proposition 3.3** (cf. [10 Thm. 4.2.10]). Every covariant functor \( F \) from \( \mathcal{A} \) to the category of sets, can be extended to a sheaf on \( X_\mathcal{A} \), and this extension is unique.

**Proof.** Let \( F \) be a covariant functor on \( \mathcal{A} \). Suppose that \( F \) extends to a sheaf on \( X_\mathcal{A} \). For any open set \( U = \bigcup_{\alpha \in U} U_\alpha \), we must have \( F(U) = \lim_{\alpha \in U} F(\alpha) \), which is the set of sections \((s_\alpha)_{\alpha \in U}\), with \( s_\alpha \in F(\alpha) \), such that for any pair \( \alpha, \alpha' \) in \( U \) and any element \( \beta \) in \( U_\alpha \cap U_{\alpha'} \), the images of \( s_\alpha \) and \( s_{\alpha'} \) in \( F(\beta) \) coincide (“coherent collection”). This proves the uniqueness of the extension. In any case, this formula defines a presheaf \( F \) on \( X_\mathcal{A} \) i.e. for the lower \( \mathcal{A} \)-topology.

Let us verify that \( F \) is a sheaf. First, let \( U \) be a covering of an open \( U \), and \( s, s' \) two elements of \( F(U) \) such that \( s|_V = s'|_V \) for all \( V \in \mathcal{U} \); in this case, for each \( \alpha \in U \), the components \( s_\alpha \) and \( s'_\alpha \) (in \( F(\alpha) \)) of \( s \) and \( s' \) are necessarily the same, so \( s = s' \). Concerning the second axiom of a sheaf, suppose that a collection \( s V \) is defined for \( V \in \mathcal{U} \), and that \( s V|_{V \cap W} = s W|_{V \cap W} \) whenever \( V, W \in \mathcal{U} \) have nonempty intersection, then by restriction to the \( U_\alpha \) for \( \alpha \in U \) we get a coherent section over \( U \). This proves the existence of the extension. \( \square \)

Neither the (conditional) existence of coproducts or products nor any finiteness hypothesis are used in the previous proof. A similar proposition holds for the upper topology, but in this case the sheaves are in correspondence with contravariant functors on \( \mathcal{A} \) (i.e. presheaves on \( \mathcal{A} \)).

**Remark 3.4** (Functoriality). In the case of posets and their associated Alexandrov topologies, the direct images and inverse images of sheaves (or presheaves) are easy to handle.

Let \( f : \mathcal{A} \to \mathcal{B} \) be a morphism of posets, i.e. an increasing map; \( f \) is continuous for the lower and the upper topologies.

If \( G \) is a sheaf of sets on \( \mathcal{B} \) for the lower \( \mathcal{A} \)-topology, its inverse image is defined at the level of germs of sections by the formula \((f^* G)(\alpha) = G(f(\alpha))\), which gives the stack in \( \alpha \).

If \( F \) is a sheaf of sets on \( \mathcal{A} \) for the lower \( \mathcal{A} \)-topology; its direct image is defined by \((f_! F)(\beta) = (f_! F)(U_\beta) = (f^{-1} F)(U_\beta) = F(f^{-1}(U_\beta))\), where the open set \( f^{-1}(U_\beta) \) is the set of elements \( \alpha \in \mathcal{A} \) such that \( f(\alpha) \subseteq \beta \).
If \( \beta \) does not meet the image of \( f \), this is the empty set. For sheaves of abelian groups, and \( \beta \) non-intersecting \( f(\mathcal{A}) \), we have \( f_* F(\beta) = 0_{\beta} \). For instance, if \( \mathcal{A} \) is a sub-poset of \( \mathcal{B} \), and \( J \) the injection: \( J_* F \) coincides with \( F \) on \( \mathcal{A} \) and is zero in its complement.

Analog results hold true for the upper topology and for the contravariant functors on \( \mathcal{A} \) and \( \mathcal{B} \).

### 3.2. Extra-fine presheaves on posets.

Consider a poset \( \mathcal{A} \), and the induced topological space \( X_\mathcal{A} \) whose underlying set is \( \text{Ob}\mathcal{A} \), equipped with the lower \( \mathcal{A} \)-topology. Let us denote by \( U_\mathcal{A} \) the covering of \( X_\mathcal{A} \) by the open sets \( U_\alpha, \alpha \in \mathcal{A} \).

By definition of the lower \( \mathcal{A} \)-topology, \( U_\mathcal{A} \) refines any other open covering. So by taking the injective limit on the category of coverings pre-ordered by refinement (cf. Appendix B), Theorem 1 implies that if \( F \) is an extra-fine sheaf on \( X_\mathcal{A} \), for any \( n \geq 1 \) we have \( H^n(X; F) = 0 \).

Due to the maximality of the open covering \( U_\mathcal{A} \), the existence of a super-local orthogonal decomposition for \( F \) subordinated to \( U_\mathcal{A} \) implies that \( F \) is extra-fine. Proposition 2.4 tells that also the notions of fine sheaf and of orthogonality can be tested on \( U_\mathcal{A} \).

However, in general, \( U_\mathcal{A} \) is not the only finest open covering of \( \mathcal{A} \). For instance, one can take all the intersections (or finite intersections) of the elements of \( U_\mathcal{A} \); when \( \mathcal{A} \) is not stable by arbitrary coproducts (resp. finite coproducts), the resulting covering \( U^+_\mathcal{A} \) is strictly larger than \( U_\mathcal{A} \), but \( U_\mathcal{A} \) is also a refinement of \( U^+_\mathcal{A} \). In such case, the relation of refinement only defines a pre-order.

Therefore it can happen that a sheaf \( F \) is extra-fine, but that \( F \) is not super-local for \( U_\mathcal{A} \).

In the applications, the covering \( U_\mathcal{A} \) is super-local for the sheaf \( F \); in this case we say that \( F \) over \( X_\mathcal{A} \) is **canonically extra-fine**. This property implies that \( F \) is extra-fine, because every open covering is less fine than \( U_\mathcal{A} \), and extra-fine implies canonically extra-fine when \( \mathcal{A} \) is stable by any non-empty coproducts.

If \( F \) is canonically extra-fine, we can describe completely the group \( H^0(X; F) = 0 \); if \( F \) is canonically extra-fine, there is a super-local orthogonal decomposition \( \{ e_\alpha \}_{\alpha \in \mathcal{A}} \) associated to the covering \( U_\mathcal{A} \). In this situation, the covering \( U'_\mathcal{A} \) of the axiom (i) can be replaced by \( U_\mathcal{A} \) itself, which is finer. From the axioms, the images \( S_\alpha = \text{im } e_\alpha \) define sub-sheaves of \( F \) such that \( F = \bigoplus_{\alpha \in \mathcal{A}} S_\alpha \). Moreover, for any open set \( U \) of \( X_\mathcal{A} \), \( e_\alpha(U) \) is the projection on \( S_\alpha(U) \) parallel to \( \bigoplus_{\beta: \beta \neq \alpha} S_\beta(U) \).

We will see the relation with the interaction decomposition in the next section.

This can be summarized in the following result.

**Proposition 3.5.** Let \( F \) be a canonically extra-fine sheaf over \( X_\mathcal{A} \), where \( X_\mathcal{A} \) denotes the topological space defined by a poset \( \mathcal{A} \) equipped with its lower \( \mathcal{A} \)-topology. Then,

\[
H^0(X_\mathcal{A}; F) = \bigoplus_{\alpha \in \mathcal{A}} H^0(U_\mathcal{A}; S_\alpha).
\]
exists a natural number $d_\alpha$ respect to $j$ the case for the poset associated to a CW complex (cf. [45]).

A more convenient condition for us will be the hypothesis of locally finite dimension. For any $\alpha$ in $A$, there exists a minimal object $\omega$ such that $\alpha \to \omega$, and there exists a natural number $d$, such that for every minimal object $\omega$ under $\alpha$ the height of $\alpha$ over $\omega$ is smaller than $d$. The smallest such $d$ is called the dimension of $\alpha$.

If $A$ is of locally finite dimension, then it is also of locally finite relative dimension. This is easily verified: consider $\alpha \to \beta$ and $\beta \to \omega$ where $\omega$ is terminal, if the height of $\alpha$ over $\beta$ were infinite, it should be the same for the height of $\alpha$ over $\omega$, then for the dimension of $\alpha$.

Note that the conditions of local finiteness and locally finite relative length are self-dual, i.e. they hold for $A$ if and only they hold for $A^{op}$. This is not the case for closure finiteness or locally finite dimension. The posets $A$ and $A^{op}$ are both closure finite if and only if they are finite. The posets $A$ and $A^{op}$ are both of locally finite dimension if and only if there exists a number $d$ such that any sequence $\alpha \to \beta$ of length bigger than $d+1$ has a repetition; in this case we say that $A$ has finite dimension, or finite depth.

The most elegant finiteness condition is Lower Well Foundedness: there exists no infinite chain without repetition (cf. [45]).

In the case of finite posets and sheaves of finitely generated abelian groups, we can assert that the cohomology is finitely generated.

4. Interaction decomposition

4.1. Condition $G$ and the equivalence theorem. Let $A$ be an arbitrary poset, and let $V$ be a contravariant functor on $A$, valued in the category of vector spaces over a commutative field $\mathbb{K}$. We suppose that for each $\rho : \alpha \to \beta$ in $A$, the map $j_{\alpha,\beta} = V(\rho) : V(\beta) \to V(\alpha)$ is injective. We call $V$ an injective presheaf.
To get a sheaf on a topological space from $V$, we must consider the upper $A$-topology and not the lower one, because $U^\beta \supseteq U^\alpha$ whenever $\alpha \rightarrow \beta$. In what follows we denote by $X^A$ the set $\text{Ob} \ A$ equipped with the upper $A$-topology.

We write $V_{\alpha \beta}$ instead of $j_{\alpha \beta}(V_\beta)$. For a partition of unity associated to $V$, if it exists, $e_{\alpha|\beta}$ is an endomorphism of $V_\beta$.

**Definition 4.1.** An *interaction decomposition* of an injective presheaf $V$ is a family of vector sub-spaces $S_\gamma$ of $V_\gamma$, indexed by $\gamma \in A$, such that

\[(4.1) \quad \forall \alpha \in A, \quad V_\alpha = \bigoplus_{\beta \subseteq \alpha} j_{\alpha \beta} S_\beta.\]

Let us introduce, for every $\gamma \in A$, the vector space

\[(4.2) \quad \forall \alpha \in A, \quad S^\gamma(\alpha) = \begin{cases} S_{\alpha \gamma} = j_{\alpha \gamma} S_\gamma & \text{if } \alpha \rightarrow \gamma, \\ 0 & \text{if } \alpha \not\rightarrow \gamma, \end{cases}\]

this defines a presheaf $S^\gamma$ on $A$. The interaction decomposition corresponds to a decomposition of the presheaf $V$:

\[(4.3) \quad V = \bigoplus_{\gamma \in A} S^\gamma.\]

**Remark 4.2.** The name *interaction decomposition* comes from Statistical Physics, where the spaces $\{V_\alpha\}_{\alpha \in A}$ are spaces of functions depending on local variables over a lattice. An important old example corresponds to Wick’s theorem, used in renormalization theory and Wiener analysis: a particular case is the decomposition of functions in sum of Bernoulli polynomials or Hermite polynomials, cf. Sinai’s *Theory of Phase Transition: rigorous results* [38]. The notion of interaction decomposition also plays a fundamental role in other domains of Probability and Statistics, cf. [20].

For injective presheaves, the concepts of canonical extra-fine and interaction decomposition are equivalent, as shown by the following lemma in combination with the construction at the beginning of Section 3.2.

**Lemma 4.3.** If $\{S_\gamma\}_{\gamma \in A}$ defines an interaction decomposition of the injective presheaf $V$, the family of endomorphisms $\{e_\beta\}_{\beta \in A}$ such that $e_{\beta|\alpha} : V(\alpha) \rightarrow V(\alpha)$ is the projection onto $S_{\alpha \beta}$ parallel to $\bigoplus_{\beta' : \beta \neq \beta'} S_{\alpha \beta'}$ forms a super-local orthogonal decomposition.

**Proof.** By the maximality of the covering $U^A$ of $X^A$ by the $\{U^\alpha\}_{\alpha \in A}$, it is sufficient to verify the axioms (i'), (ii) and (iii) for this covering $U^A$. For the condition (i'), if $U^\alpha \not\subseteq U^\beta$, then $\alpha \rightarrow \beta$, which in turn implies that $S_{\alpha \beta} = 0$ so $e_{\beta|\alpha} = 0$. For (ii) let us consider $x \in V_\alpha$; to have $e_{\gamma|\alpha}(x) \neq 0$, we must have $\gamma \rightarrow \alpha$, but the definition of interaction decomposition tells that $x$ belongs to the direct sum of the spaces $S_{\alpha \beta}$, thus only a finite number of the $e_{\gamma|\alpha}(x)$ are different from zero. For (iii) consider $\beta \neq \beta'$ lower than $\alpha \in A$, then by definition of the projector $e_{\beta|\alpha}$ the space $S_{\alpha|\beta'}$ belongs to its kernel. \qed
Condition G:

\[(G) \quad \forall \alpha, \beta \in \mathcal{A} \text{ such that } \alpha \to \beta, \quad V_{\alpha \beta} \cap \left( \sum_{\gamma: \alpha \to \gamma, \gamma \neq \beta} V_{\alpha \gamma} \right) \subset \sum_{\gamma: \alpha \to \gamma, \beta \not\rightarrow \gamma} V_{\alpha \gamma}, \]

where \( \beta \not\rightarrow \gamma \) means that \( \beta \to \gamma \) and \( \beta \neq \gamma \).

**Theorem 4.4.** Let \( V \) be an injective presheaf on a poset \( \mathcal{A} \).

1. If the condition \( \mathcal{G} \) is satisfied and \( \mathcal{A} \) is of locally finite dimension (in the lower direction), the sheaf defined by \( V \) on \( X^\mathcal{A} \) is canonically extra-fine.
2. If the sheaf induced by \( V \) on \( X^\mathcal{A} \) is canonically extra-fine, the condition \( \mathcal{G} \) is satisfied.

**Proof.** In view of the preceding decomposition results, we establish the first claim showing that there exists an interaction decomposition associated to the presheaf \( V \).

For each \( \alpha \in \mathcal{A} \), we define the *boundary sum* \( V'_\alpha = \sum_{\beta: \alpha \to \beta, \gamma \neq \beta} V_{\alpha \beta} \), and we choose any supplementary space \( S_\alpha \) of it. Hence it remains to prove that \( V'_\alpha \) is the direct sum \( \bigoplus_{\beta: \alpha \to \beta, \gamma \neq \beta} S_{\alpha \beta} \). We prove this by recurrence in the dimension of \( \alpha \).

First, if \( \alpha \) has dimension zero, then it is maximal, which means that \( \alpha \to \beta \) implies \( \alpha = \beta \). So \( V'_\alpha = 0 \) and the claim is then trivially true.

Let us suppose now that the recurrence hypothesis holds true in dimension smaller or equal than \( r - 1 \), for some \( r \geq 1 \), and consider \( \alpha \) of dimension \( r \).

Let \( \mathcal{B} \) be the set of maximal cells \( \beta \) such that \( \alpha \to \beta \) and \( \alpha \neq \beta \). And for \( \beta \in \mathcal{B} \), consider \( x \in V_{\alpha \beta} \), and suppose it also belongs to the algebraic sum \( \sum_{\gamma: \alpha \to \gamma, \beta \neq \gamma, \gamma \neq \alpha} V_{\alpha \gamma} \). As \( \beta \) is maximal, we have

\[
(4.4) \quad x \in \sum_{\gamma: \alpha \to \gamma, \beta \neq \gamma, \gamma \neq \alpha} V_{\alpha \gamma}.
\]

Then, applying the condition \( \mathcal{G} \) to \( x \), we deduce that \( x \) belongs to the sum of \( V_{\alpha \gamma} \) over the \( \gamma \in \mathcal{A} \) such that \( \alpha \not\rightarrow \gamma \) and \( \beta \not\rightarrow \gamma \), which coincides with \( V'_{\alpha \beta} = \sum_{\gamma: \alpha \to \gamma, \beta \neq \gamma} V_{\alpha \gamma} \), consequently

\[
(4.5) \quad V'_\alpha = S_{\alpha \beta} \oplus (V'_{\alpha \beta} + \sum_{\beta' \neq \beta, \beta' \in \mathcal{B}} V_{\alpha \beta'}). \]

For \( \beta' \in \mathcal{B}, \beta' \neq \beta \), consider \( x' \in V_{\alpha \beta'} \), and suppose it also belongs to the algebraic sum \( \sum_{\beta'' \in \mathcal{B}, \beta'' \neq \beta, \beta'' \neq \beta'} V_{\alpha \beta''} \). As \( \beta' \) is maximal, we have

\[
(4.6) \quad x \in \sum_{\gamma: \alpha \to \gamma, \beta \neq \gamma} V_{\alpha \gamma}.
\]

Then, applying the condition \( \mathcal{G} \) to \( x' \), we deduce that \( x' \) belongs to the sum of \( V_{\alpha \gamma} \) over the \( \gamma \in \mathcal{A} \) such that \( \alpha \not\rightarrow \gamma \) and \( \beta' \not\rightarrow \gamma \), which is the space \( V'_{\alpha \beta'} = \sum_{\gamma: \alpha \to \gamma, \beta \neq \gamma} V_{\alpha \gamma} \), consequently

\[
(4.7) \quad V'_{\alpha} = S_{\alpha \beta} \oplus S_{\alpha \beta'} \oplus (V'_{\alpha \beta'} + V'_{\alpha \beta''} + \sum_{\beta'' \in \mathcal{B} \setminus \{\beta, \beta'\}} V_{\alpha \beta''}).
\]
By (possibly transfinite) induction, we get
\[(4.8) \quad V'_\alpha = \bigoplus_{\beta \in B} S_{\alpha\beta} \oplus \sum_{\beta \in B} V'_\beta.\]

Then we conclude by applying the recurrence hypothesis to the spaces $V'_\beta$, and the transitivity, $j_{\alpha\gamma} = j_{\alpha\beta} \circ j_{\beta\gamma}$.

We prove now the second claim. As we saw above, if the sheaf $V$ on $X^A$ is extra-fine then there exists a super-local partition of unity subordinate to the covering $U^A$, say $\{e_\alpha\}_{\alpha \in A}$ where $e_\alpha = e_{U^\alpha}$. Setting $S_\gamma = \operatorname{im} e_\gamma$, one has $F = \bigoplus_{\gamma \in A} S_\gamma$ and the formulae (4.1) hold with $S_\gamma = S^\gamma(U^\gamma)$, so we have an interaction decomposition.

Let us fix $\alpha \in A$, look at $\alpha \to \beta$, and consider a vector $x$ in $S_{\alpha\beta}$. Suppose that this vector is equal to a finite sum $y_1 + \ldots + y_m$ of elements of $V_{\alpha\beta_1}, \ldots, V_{\alpha\beta_m}$ respectively, with $\alpha \to \beta_i$ but $\beta \not\to \beta_i$. Applying to each of these vectors $y_i$ the projector $e_{\beta_i\alpha} = e_{U^\beta\beta_1(U^\alpha)}$, we find zero, in reason of super-locality; however $e_{\beta_i\alpha}(x) = x$ by definition of $S_{\alpha\beta}$, thus $x = 0$. Now consider any vector $z$ in $V_{\alpha\beta}$; by interaction decomposition, $z$ is (in a unique way) the sum of a vector $x \in S_{\alpha\beta}$, and a vector $y$ in the space $V'_{\alpha\beta}$, equal to the sum of the $S_{\alpha\gamma}$ for $\beta \not\to \gamma$, which is included in the sum of all the $V_{\alpha\gamma}$ for $\beta \not\to \gamma$. Then the condition [G] is proved. \qed

In this generality, the theorem above appears for the first time in [36], by G. Sergeant-Perthus. It holds true if we replace the property of local finite dimension by the noetherian property of well-foundedness.

We also refer to condition [G] as sum-intersection property. Before the work of G. Sergeant-Perthus, a particular case of this property appeared in the book of Lauritzen (see Proposition B.5 in the Appendix B of [20]), as a corollary of the interaction decomposition. Lauritzen considers a finite poset $A$ and a presheaf of finite dimensional vector spaces $\{V_a\}_{a \in A}$ that admits an interaction decomposition; the property is stated for two open subsets $U, V$ of the lower space $X_A$ (named generating classes, the topology was not mentioned) in the following form:
\[(4.9) \quad \sum_{c \in R(U \cap V)} V_c = \left( \sum_{a \in R(U)} V_a \right) \cap \left( \sum_{b \in R(V)} V_b \right),\]

where $R(U)$ denotes the set of all elements $a$ of $A$ such that $U_a$ is included in $U$.

In [36] it is assumed that all the $V(\alpha)$, $\alpha \in A$, belong to a fixed vector space $V$, this is not a restriction, because it is always possible to inject all of them in the colimit of $V$, seen as a diagram over $A^{op}$. A traditional name for this special colimit is direct limit, denoted $\lim_{\rightarrow A^{op}} V$ or simply $\lim_{\rightarrow} V$. It is the quotient of the direct sum $\bigoplus_{a \in A} V_a$ subject to the relations $j_{\alpha\beta}(x_\beta) = x_\beta$, for each arrow $\alpha \to \beta$. Every space $V_\alpha$ goes naturally into $\lim_{\rightarrow} V$; we denote by $j^\alpha$ this map. The space $\lim_{\rightarrow} V$ is universal in the sense that if there exist a vector space $W$ and homomorphisms $f_\alpha : V_\alpha \to W; \alpha \in A$, such that $f_\beta = f_\alpha \circ j_{\alpha\beta}$ every time it has a meaning, there exists a unique homomorphism $f : \lim_{\rightarrow} V \to W$ such that $\forall \alpha \in A, f \circ j^\alpha = f_\alpha$.

From this description, it is clear that $\lim_{\rightarrow} V$ corresponds to set $H^0(A, V)$ of global section of $V$ over $A$, which is in turn isomorphic to $H^0(X^A, V)$. 

EXTRA-FINE SHEAVES AND INTERACTION DECOMPOSITIONS 15
If $V$ is an injective presheaf, then every $j^\alpha$ is injective. Moreover, if $V$ has an interaction decomposition $S$, we have

$$
\lim_{\to} V = \bigoplus_{\alpha \in \mathcal{A}} j^\alpha S_{\alpha}.
$$

4.2. Duality. We suppose that the sheaf $V$ on $X^A$ is extra-fine. For each $\alpha \in \mathcal{A}$, let $V_{\alpha}^*$ be the (algebraic) dual vector space of the vector space $V_{\alpha} = V(\alpha)$; the transpose maps $j_{\alpha\beta}; \beta \subseteq \alpha$ define a covariant functor on $\mathcal{A}$, then a presheaf for the lower topology on $\mathcal{A}$. But the transposed maps $e_{\alpha}; \alpha \in \mathcal{A}$ give a decomposition of this presheaf into the product of the presheaves $S_{\alpha}^*; \alpha \in \mathcal{A}$, not into a direct sum. In general $V^*$ is not extra-fine, the condition of super-locality fails for the lower topology. Only the condition $[\text{iii}]$ is satisfied. Therefore we need to follow another way to dualize Theorem 4.4.

This can be done adding a further hypothesis. Let us suppose that there exists a covariant functor $F$ on $\mathcal{A}$ (equivalently, a topological sheaf on $X_\mathcal{A}$), with surjective arrows $\pi^{\alpha\beta}$ for $\alpha \to \beta$, such that $V_{\alpha} = F_{\alpha}^*$ and $j_{\alpha\beta} = \pi^{\beta\alpha}$ for all pairs $\alpha, \beta$ with $\alpha \to \beta$. Then for every $\alpha \in \mathcal{A}$, the space $F_{\alpha}$ embeds naturally in $V_{\alpha}^*$, in such a manner that, for every pair $\alpha, \beta$ with $\alpha \to \beta$, $j_{\alpha\beta}$ induces the map $\pi^{\beta\alpha}$. Let us denote by $e_{\alpha}^*$ the restriction of $e_{\alpha}$ to $F$. Given the following lemma, this gives a family of orthogonal projectors from $F$ to the dual copresheaf $V^*$ (by the same argument given in the proof of Lemma 4.3). We do not ask that $e_{\alpha}$ preserves $F$.

**Lemma 4.5.** $\text{id}_F = \sum_{\alpha \in \mathcal{A}} e_{\alpha}^*$, in the sense of finite sum when applied to a given vector $g \in F_{\beta}$ for any $\beta \in \mathcal{A}$.

**Proof.** Consider $\gamma \in \mathcal{A}$ and a basis $\{f_j \mid j \in J\}$ of $F_\gamma$ as a vector space over $\mathbb{K}$, the space $V_\gamma = F_\gamma^*$ is isomorphic to the product $\mathbb{K}^J$, in such a manner that the duality is given by the natural evaluation. The space $F_\gamma$ itself is isomorphic to the space of scalar functions on $J$ which are zero outside a finite subset.

For $j \in J$, note $x_j = f_j^*$ the element of $V_\gamma$ corresponding to $f_j$ (in the dual basis). The set $A_j$ of elements $\beta \in \mathcal{A}$ such that $e_{\beta}(x_j) \neq 0$ is finite, and we have

$$
(4.11) \quad x_j = \sum_{\beta \in A_j} e_{\beta|\gamma}(x_j).
$$

Choose $g \in F_\gamma$. For any $\alpha \in \mathcal{A}$, we have

$$
(4.12) \quad \langle x_j, e_{\alpha|\gamma}(g) \rangle = \langle e_{\alpha|\gamma}(x_j), g \rangle,
$$

where the bracket denotes the form of incidence from $V_\gamma^* \times V_\gamma$ to $\mathbb{K}$. Then

$$
(4.13) \quad \langle x_j, g \rangle = \sum_{\beta \in A_j} \langle e_{\beta|\gamma}(x_j), g \rangle = \langle x_j, \sum_{\beta \in A_j} e_{\beta|\gamma}^*(g) \rangle.
$$

If $\alpha$ does not belong to $A_j$, we have

$$
(4.14) \quad 0 = \langle e_{\beta|\gamma}(x_j), g \rangle = \langle x_j, e_{\alpha|\gamma}^*(g) \rangle,
$$

i.e. $x_j$ vanishes at $e_{\alpha|\gamma}^*(g)$. Therefore, for every $j \in J$ and $g \in F$,

$$
(4.15) \quad \langle x_j, g \rangle = \langle x_j, \sum_{\beta \in \mathcal{A}} e_{\beta|\gamma}^*(g) \rangle.
$$
Let us denote by $B_g$ a finite set of indexes $k \in J$ such that
\begin{equation}
(4.16) \quad g = \sum_{k \in B_g} g^k f_k.
\end{equation}
Equivalently, if $j$ does not belong to $B_g$, we have
\begin{equation}
(4.17) \quad 0 = \langle x_j, g \rangle = \langle x_j, \sum_{\beta \in A} e_{\beta|\gamma}^* (g) \rangle,
\end{equation}
and if $j \in B_g$,
\begin{equation}
(4.18) \quad g^j = \langle x_j, g \rangle = \langle x_j, \sum_{\beta \in A} e_{\beta|\gamma}^* (g) \rangle.
\end{equation}
Now, consider any element $x \in V^\gamma = F^\gamma_*$, it is identified with the numerical function that assigns $x(j) \in K$ to $j \in J$. Then, using the above equations (4.17) and (4.18), we get
\begin{align*}
\langle x, g \rangle &= \langle x, \sum_{k \in B_g} g^k f_k \rangle \\
&= \sum_{k \in B_g} g^k x(k) = \sum_{k \in B_g} \langle x(k) x_k, \sum_{\beta \in A} e_{\beta|\gamma}^* (g) \rangle \\
&= \langle x, \sum_{\beta \in A} e_{\beta|\gamma}^* (g) \rangle,
\end{align*}
which implies the desired result. 

Note that the axiom (1') is not verified for $F$ and the family of projectors $e_{\alpha|\beta}^*$.

The images of $e_{\alpha|\beta}^*$ for $\beta$ describing $A$ define a sub-sheaf of $V^*$, that we denote $T_\alpha$. And we denote by $T_\alpha$ its stack at $\alpha \in A$.

**Corollary 4.6.** The presheaf $F$ (for the lower $\mathcal{A}$-topology) defined above is acyclic and $H^0(X^\gamma, F) \cong \bigoplus_{\alpha \in A} H^0(X^\gamma, T^\alpha) = \bigoplus_{\alpha \in A} T_\alpha$.

**Proof.** It is sufficient to prove the acyclicity for the covering by the $\{U_\alpha\}_{\alpha \in A}$.

For any $\gamma \in A$, the presheaf $T^\gamma$ is zero outside $U^\gamma$ and is $(j_\alpha, S_\gamma)^*$ for $\alpha \in U^\gamma$.

The Čech cohomology of a direct sum of sheaves is the direct sum of their cohomology; this follows by projection of the cochain and naturality of the coboundary operator. Therefore the corollary 1 results from the following lemma. 

**Lemma 4.7.** Let $T$ be a presheaf on $A$, equipped with the lower $\mathcal{A}$-topology, that is supported on a set $U^\gamma$ for $\gamma \in A$. If for every $\alpha, \beta \in U^\gamma$ such that $\alpha \to \beta$ the morphism $\pi^{\beta\alpha}$ is an isomorphism, then $T$ is acyclic and $H^0(X^\gamma, T) = T^\gamma$.

**Proof.** Again, it is sufficient to prove the acyclicity for the covering by the $\{U_\alpha\}_{\alpha \in A}$. Every space $T_\alpha$ is zero except if $\alpha \to \gamma$, then we can consider that every cochain takes its value in $T_\gamma$, whatever being its degree. Considering a cochain $c$ of degree $n$, if it is a cocycle, for any family $\alpha_1, ..., \alpha_{n+1}$ in $A$, we have, in $T_\gamma$:
\begin{equation}
(4.19) \quad c(\alpha_1, ..., \alpha_{n+1}) = \sum_{k=1}^{n+1} (-1)^{k+1} c(\gamma, \alpha_1, ..., \widehat{\alpha_k}, ..., \alpha_{n+1}),
\end{equation}
which tells that \( c \) is equal to \( \delta \phi \), where \( \phi \) is the \((n-1)\)-cochain defined by
\[
(4.20) \quad \forall \beta_1, \ldots, \beta_n \in \mathcal{A}, \quad \phi(\beta_1, \ldots, \beta_n) = c(\gamma, \beta_1, \ldots, \beta_n).
\]
This establishes the lemma.

In the following section we will need a variant of this lemma, concerning the relative cohomology. Suppose that \( \mathcal{A} \) is a sub-poset of \( \mathcal{B} \), and that we have a presheaf \( T \) on \( X_\mathcal{B} \) (i.e. for the lower \( A \)-topology \( \mathcal{B} \)), which is supported on a set \( U^\gamma \) for \( \gamma \in \mathcal{B} \), such that every morphism \( \pi^\beta \) with \( \alpha \to \beta \to \gamma \) is an isomorphism. Then we consider the sheaf \( S \) over \( \mathcal{A} \), obtained by restriction.

**Lemma 4.8.** Under the above hypotheses, \( \forall n \geq 1, H^n(\mathcal{B}, \mathcal{A}; T, S) = 0 \).

See Appendix [II] for the definition of relative cohomology.

**Proof.** It is sufficient to prove the result for the cohomology of the covering by the open sets \( \{ U_\beta \}_{\beta \in \mathcal{B}} \), and their traces on \( \mathcal{A} \). By definition, a relative cochain \( c \in C^n(\mathcal{B}, \mathcal{A}; T, S) \) takes the value 0 on every family of \( n+1 \) elements of \( \mathcal{A} \). If it is a cocycle, for any family \( \alpha, \alpha_1, \ldots, \alpha_{n+1} \) of \( n+2 \) elements in \( \mathcal{B} \), we have, in \( T_\gamma \):
\[
(4.21) \quad c(\alpha, \alpha_1, \ldots, \alpha_{n+1}) = \sum_{k=1}^{n+1} (-1)^{k+1} c(\alpha, \alpha_1, \ldots, \alpha_k, \ldots, \alpha_{n+1}),
\]
which tells that \( c \) is equal to \( \delta \phi \), where \( \phi \) is the \((n-1)\)-cochain defined by
\[
(4.22) \quad \forall \beta_1, \ldots, \beta_n \in \mathcal{B}, \quad \phi(\beta_1, \ldots, \beta_n) = c(\alpha, \beta_1, \ldots, \beta_n).
\]
If \( U^\gamma \) has empty intersection with \( \mathcal{A} \), taking \( \alpha = \gamma \), we have \( \phi \in C^{n-1}(\mathcal{B}, \mathcal{A}; T, S) \), and \( c = \delta \phi \). And if \( \alpha \) belongs to \( \mathcal{A} \cap U^\gamma \), the cochain \( \phi \) belongs to \( C^{n-1}(\mathcal{B}, \mathcal{A}; T, S) \), and \( c = \delta \phi \). This establishes the lemma.

5. **Factorization of free sheaves**

5.1. **Free presheaves and intersection properties.** In many applications to Statistical Physics or Bayesian Learning, the presheaves that appear are free modules, generated by subsets of a fixed set.

A set \( I \) is given (non-necessarily finite) and the poset \( \mathcal{A} \) is a sub-poset (i.e. a subcategory) of the poset \( (\mathcal{P}(I), \rightarrow) \) of finite subsets of \( I \), ordered in such a way that \( A \to B \) iff \( B \subseteq A \). The poset \( \mathcal{A} \) is automatically of locally finite dimension. The pair \( (\mathcal{A}, I) \) is named an hyperset. We consider a covariant functor (a.k.a. copresheaf) of sets \( E \) on \( \mathcal{A} \), such that, for every \( \alpha \in \mathcal{A} \), the set \( E_\alpha = E(\alpha) \) can be identified with the cartesian product \( \prod_{i \in \alpha} E_i \) by surjective maps \( \pi^\alpha = E(\alpha \to i) \). By naturality, all the maps \( \pi^\beta : E_\alpha \to E_\beta \) are surjective. If the empty set \( \emptyset \) belongs to \( \mathcal{A} \), the set \( E_\emptyset \) is a singleton \( \ast = \{ \emptyset \} \). In this case, for every element \( \alpha \in \mathcal{A} \), there exists a unique map \( \pi^{\emptyset \alpha} : E_\alpha \to E_\emptyset \).

Note that \( E \) is a sheaf of sets for the lower \( A \)-topology on \( \mathcal{A} \), and for every arrow \( \alpha \to \beta \), the map \( \pi^{\beta \alpha} \) is the restriction of sections from the open set \( U_\alpha \) to the open set \( U_\beta \).

A commutative field \( K \) of any characteristic is given. For every \( \alpha \in \mathcal{A} \), we define \( V_\alpha \) as the space of all functions from \( E_\alpha \) to \( K \). We say that \( V \) is the free presheaf generated by \( E \). If \( \emptyset \in \mathcal{A} \), the space \( V_\emptyset \) is canonically isomorphic to \( K \). If \( \alpha \to \beta \), i.e. \( \beta \subseteq \alpha \), we get a natural application \( j_{\alpha \beta} : V_\beta \to V_\alpha \), which is linear and injective. As before, \( V_{\alpha \beta} \) designates the image of \( j_{\alpha \beta} \) in \( V_\alpha \). Using the projection \( \pi^{\beta \alpha} \) we
can identify $V_{\alpha \beta}$ with the space of numerical functions of $x_\alpha$ that depend only on the variables $x_\beta$, these functions are named the cylindrical functions with respect to $\pi^{\beta\alpha}$.

**Definition 5.1** (Reduced functor). The sub-functor of constants $K_A$ maps each $\alpha \in A$ to the one dimensional vector subspace $K_\alpha$ of constant functions, embedded in $V_\alpha$. The reduced functor (or reduced free presheaf) $V_\alpha; \alpha \in A$ is made of the quotient vector spaces $V_\alpha/K_{\alpha}$.

If $\emptyset \in A$, for every $\alpha \in A$, we have $K_\alpha = V_{\alpha \emptyset}$.

**Definition 5.2** (Intersection property). The hypergraph $(A, I)$ satisfies the strong (resp. weak) intersection property, if, for every pair $(\alpha, \alpha')$ in $A$ (resp. every pair having non-empty intersection in $P(I)$), the intersection $\alpha \cap \alpha'$ belongs to $A$.

**Remark 5.3.** If $A$ satisfies the strong intersection property, all the coproducts $\alpha \vee \alpha'$ exist; if $A$ satisfies the weak intersection property, the coproducts exist conditionally, i.e. $\alpha \vee \alpha'$ exists as soon as $\alpha$ and $\alpha'$ have a common majorant (under the relation $\rightarrow$).

If $A$ has non-intersecting elements, the strong intersection property implies that the empty set $\emptyset$ belongs to $A$, then $A$ possesses a unique final element, that is $\emptyset$. If $A$ satisfies the weak intersection property, it possesses conditional coproducts (here intersections) in the categorical sense of Section 3.

**Proposition 5.4.** If $A$ has the strong intersection property, the condition $\mathbb{G}$ is satisfied by the free presheaf $V$.

**Proof.** Consider $\alpha \rightarrow \beta$ in $A$ (i.e. $\beta \subseteq \alpha$), and a vector $v$ in $V_{\alpha \beta}$ that satisfies

$$v = \sum_{\gamma: \alpha \rightarrow \gamma, \gamma \neq \alpha, \gamma \neq \beta} v_\gamma,$$

for some $v_\gamma \in V_{\alpha \gamma}$.

The above decomposition tells that for every $x_\beta \in E_{\beta}$, and for any collection of elements $\{y_j\}_{j \in \alpha \setminus \beta}$, where $y_j \in E_j$, we have

$$v(x_\beta, y_{\alpha \setminus \beta}) = \sum_{\gamma: \alpha \rightarrow \gamma, \gamma \neq \alpha, \gamma \neq \beta} v_\gamma(x_\gamma);$$

where on the right, the components of $x_\gamma$ are $x_i$ with $i \in \beta \cap \gamma$ and $y_j$ with $j \in (\alpha \setminus \beta) \cap \gamma$.

For each index $k \in \alpha \setminus \beta$, we choose a fixed $y_k^0$, and replace everywhere in the formula the variable $x_k$ by this value. The formula continues to hold true. In the expression $v_\gamma(x_\gamma)$, the variables $x_i$ that do not belong to $\beta \cap \gamma$, are constants $y_k^0; k \in \alpha \setminus \beta$. Moreover the intersection of $\beta$ and $\gamma$ is a strict subset of $\beta$, because $\beta$ is assumed to be not included in $\gamma$.

This gives

$$v(x_\beta, y_{\alpha \setminus \beta}^0) = \sum_{\gamma: \alpha \rightarrow \gamma, \gamma \neq \alpha, \beta \rightarrow \beta \cap \gamma, \beta \neq \beta \cap \gamma} v_{\beta \cap \gamma}^0(x_{\beta \cap \gamma}).$$
And, for all possible $\omega \in A$, $\omega \subset \beta$, $\beta \neq \omega$, if we bring together the $\gamma$ such that $\alpha \rightarrow \gamma, \gamma \neq \alpha, \beta \cap \gamma = \omega$, this gives

\begin{equation}
(5.4) \quad v(x_\beta, y_\alpha | \beta) = \sum_{\omega: \alpha \rightarrow \omega, \omega \neq \alpha, \beta \rightarrow \omega, \beta \neq \omega} w_\omega(x_\omega).
\end{equation}

Which is the expected result.

\begin{remark}
Without the strong intersection property the results is false. Take for instance, $I = \{i, j\}, A = \{i; j; \alpha = (i, j)\}$, a non-zero constant function belongs to $V_\alpha$, but cannot belong to the image of a strict subset of $\{j\}$.
\end{remark}

**Proposition 5.6.** If $A$ has the weak intersection property, the condition \(G\) is satisfied by the reduced functor $V$.

**Proof.** Repeat the proof of Proposition 5.4 but distinguish the cases where $\beta \cap \gamma$ is empty or not. When it is empty the respective function $v_\beta$ of $(x_\gamma, y_\gamma')$ belongs to the constants.

Now remind that, by construction, the poset $A$ is of locally finite dimension (it is even locally finite), then the following proposition results directly from the prop. 5.4 (resp. 5.6) and the Theorem 4.4.

**Theorem 5.7.** If $A$ has the strong (resp. weak) intersection property, the sheaf $V$ (resp. \(\nabla\)) is extra-fine for the upper $A$-topology.

Remind that under the strong (resp. weak) condition of intersections, extra-fine is equivalent to canonically extra-fine.

Theorem 5.7 generalizes a theorem of existence of an interaction decomposition for factor spaces that, under different forms, has been known for long time in probability theory, but only for finite posets and finite dimensional vector spaces, cf. \cite{19, 24, 40, 20}.

As in the preceding section, denote by $V'_\alpha$ the sum of the $V_{\alpha \beta}$ over $\beta \subsetneq \alpha$, (resp. $\nabla'_\alpha$ the sum of the $\nabla_{\alpha \beta}$ over $\beta \subsetneq \alpha$) and take a supplementary subspace $S_\alpha$ of $V'_\alpha$ in $V_\alpha$ (resp. $\overline{S}_\alpha$ of $\nabla'_\alpha$ in $\nabla_\alpha$). The interaction decomposition gives

\begin{equation}
(5.5) \quad \forall \alpha \in A, \quad V_\alpha = \bigoplus_{\beta \subseteq \alpha} S_\beta,
\end{equation}

resp.

\begin{equation}
(5.6) \quad \forall \alpha \in A, \quad \nabla_\alpha = \bigoplus_{\beta \subseteq \alpha} \overline{S}_\beta.
\end{equation}

5.2. **Duality:** Free copresheaves. Note $F_\alpha = K(E_\alpha)$ the space of functions with finite supports, which can be seen as the vector spaces freely generated by the set $E_\alpha$ over the field $K$. Its dual space is $V_\alpha = K^{E_\alpha}$ and the transpose of the natural map $\pi^{\beta \alpha} : F_\alpha \rightarrow F_\beta$ is $j_{\alpha \beta}$. The vector spaces $F_\alpha$ and the maps $\pi^{\beta \alpha}$ define a covariant functor (i.e. a copresheaf) over $A$ (resp. a sheaf on $X_A$) named the free copresheaf (resp. the free sheaf) generated by $E$.

We can apply Corollary 6.32 in the preceding section to get the following result.
Proposition 5.8. When $\mathcal{A}$ satisfies the strong intersection property, $F$ is acyclic and $H^0(X_{\mathcal{A}}, F) \cong \bigoplus_{\alpha \in \mathcal{A}} S^*_\alpha$.

Respectively, denote by $F_\alpha$ the subspace of $F_\alpha = \mathbb{K}^{(E_\alpha)}$ defined by annihilating the sum of the coordinates in the canonical basis. Its dual space is $\overline{V}_\alpha = \mathbb{K}^{E_\alpha}/\mathbb{K}_\alpha$. The transpose of the natural map $\pi^{\alpha}: F_\alpha \to F_\beta$ is again $j_{\alpha\beta}$. This forms a sheaf over $X_{\mathcal{A}}$, named the restricted free sheaf generated by $E$. As before, we obtain the following.

Proposition 5.9. When $\mathcal{A}$ satisfies the weak intersection property, $\overline{F}$ is acyclic and $H^0(X_{\mathcal{A}}, \overline{F}) \cong \bigoplus_{\alpha \in \mathcal{A}} \overline{F}_\alpha$.

In the case of finite sets $\{E_i\}_{i \in I}$ and $I$ finite, this result was established in H.G. Kellerer [19]. See also [24] and Appendix C below.

Theorem 5.10. If the hypergraph $(\mathcal{A}, I)$ satisfies the weak intersection hypothesis, for any covariant functor of sets $E$ on the category $\mathcal{A}$, the Čech cohomology $H^*(X_{\mathcal{A}}; F)$ of the induced free sheaf $F$ is naturally isomorphic to the sum of $H^*(X_{\mathcal{A}}; \overline{F})$ which is concentrated in degree zero, and of the full Čech cohomology (with trivial coefficients $\mathbb{K}$) of the topological space $X_{\mathcal{A}}$ (i.e. the poset $\mathcal{A}$ equipped with the lower Alexandrov topology).

Proof. The sheaf $F$ over $\mathcal{A}$ is decomposed into the sum of the sheaf $\overline{F}$ and the constant sheaf $\mathbb{K}_A$; this induces a decomposition in direct sum of the cochain complexes. One of them gives gives $H^*(X_{\mathcal{A}}; \overline{F})$, which is concentrated in degree zero as just said by the preceding proposition, whereas the other one gives the standard Čech cohomology of $\mathcal{A}$.

When $\mathcal{A}$ is the poset of a finite simplicial complex, it satisfies the weak intersection property, and the standard Čech cohomology (with constant coefficients) on $X_{\mathcal{A}}$ is isomorphic to the singular or simplicial cohomology with coefficients in $\mathbb{K}$.

5.3. Relative cohomology. In addition to $\mathcal{A} \subseteq \mathcal{P}_I(I)$, consider another poset $\mathcal{B}$ satisfying the same kind of hypotheses, with respect to a set $J$, i.e. $\mathcal{B} \subseteq \mathcal{P}_J(J)$.

Definition 5.11. A strict morphism from $(\mathcal{A}, I)$ to $(\mathcal{B}, J)$ is the pair $(f, f_I)$ of a functor (i.e. an increasing map) $f: \mathcal{A} \to \mathcal{B}$, and a map $f_I: I \to J$, such that $\forall i \in I$ and all $\alpha \in \mathcal{A}$ such that $i \in \alpha$, one has $f_I(i) \in f(\alpha) \subseteq J$. For simplicity, we will denote $f_I = f$.

As before, let $E$ be the sheaf of sets over $\mathcal{A}$ given by products of the sets $\{E_i\}_{i \in I}$ i.e. such that $\alpha \mapsto E_\alpha \cong \prod_{i \in \alpha} E_i$; we call the $E_i$ basic sets. Consider a strict morphism $f: \mathcal{A} \to \mathcal{B}$. For every $j \in J$, let us define $E'_j$ as the product of the $E_i$ for $i \in I$ such that $f(i) = j$.

Proposition 5.12. The direct image $f_* E$ over $\mathcal{B}$ is given by the products of the basic sets $\{E'_j\}_{j \in J}$.

Proof. For $\beta \in \mathcal{B}$, the set $f_* E(U_\beta)$ (in the lower $A$-topology) is the subset of the product of the $E_{\alpha}$ over $\alpha \in f^{-1}(\beta)$ formed by the families $s_{\alpha}: f(\alpha) \subseteq \beta$, that are compatible on the intersections $U_{\alpha} \cap U_{\alpha'}$. Each set $E_\alpha$ is the product of the sets $E_i; i \in \alpha$, and the compatibility condition tells that for any pair $\alpha, \alpha'$ with $f(\alpha) \subseteq \beta$ and $f(\alpha') \subseteq \beta$, the restriction of $s_{\alpha}$ and $s_{\alpha'}$ to their common terminal points coincide. This implies that $f(E(f^{-1}(U_\beta))$ is the product of the $E_i; i \in I$ such that $f(i) \in \beta$, then it is the product of the $E'_j$ for $j \in \beta$. \qed
In particular, $E'_i$ coincides with the set $(f_*E)_j$ which corresponds to the direct image of sheaves.

**Definition 5.13.** A simplicial morphism from $A$ to $B$ is a strict morphism $f : A \to B$, such that $\forall \alpha \in A$, the restriction of $f_1$ to the set $\alpha \in \mathcal{P}_f(I)$ is surjective onto the set $f(\alpha) \in \mathcal{P}_f(J)$.

**Proposition 5.14.** Let $f : A \to B$ be injective and simplicial, and let $F'$ be a sheaf on $B$, given by products of the basic sets $E'_j ; j \in J$. The inverse image $f^*F'$ over $A$ is given by the products of the basic sets $(E_i = E'_{f(i)})_{i \in I}$.

**Proof.** For $\alpha \in A$, by definition of $f^{-1}F'$ (which coincide with $f^*F'$ in the case of posets), $(f^*E')_\alpha = E'_{f(\alpha)}$ is the product of the sets $E'_j ; j \in f(\alpha)$, and this product coincide with the product of the sets $E'_{f(i)}$ for $i \in \alpha$ because $f$ is simplicial and injective.

In the following result, we consider the restricted subsheaves $\overline{F}$ and $\overline{F'}$, and we assume that both $A$ and $B$ verify the weak intersection property.

**Theorem 5.15.** Let $J : A \to B$ be an inclusion of posets, strict and simplicial. If $F'$ is a restricted free copresheaf over $B$, then the inverse image $\overline{F} = J^*\overline{F'}$ over $A$ is restricted, and we have a natural surjection from $\overline{F'}$ to $J_*\overline{F}$, and the induced natural map in cohomology $J^* : H^0(B; \overline{F'}) \to H^0(A; \overline{F})$ is surjective.

**Proof.** Using the sheaves $\overline{F}$ and $\overline{F'}$, we get the following exact sequence:

\[0 \to H^0(B, A; \overline{F'}) \to H^0(B; \overline{F'}) \to H^0(A; \overline{F}) \to H^1(B, A; \overline{F'}; \overline{F}) \to 0.\]  \hfill (5.7)

Then the theorem is equivalent to the vanishing of $H^1(B, A; \overline{F'}; \overline{F}) = 0$. To prove the latter, we proceed as in the proof of Corollary 6.32: we decompose $\overline{F'}$ over $B$ and then $\overline{F}$ accordingly over $A$ in direct sums of sheaves $T^0; \beta \in B$ and $S^\alpha; \alpha \in A$ respectively, which satisfy the hypotheses of Lemma 11.3. Then we conclude by applying Lemma 11.8 and the natural isomorphism in cohomology between $A$ (resp. $B$) and $\overline{A}$ (resp. $\overline{B}$) for the restricted sheaves.

In the context of finite probabilities, reducing corresponds to the tangent equation of the probability restriction of sum 1, and Theorem 5.15 is equivalent to a result of H.G. Kellerer [19].

**5.4. Marginal theorem.** Given a set $I$, let $\{E_i\}_{i \in I}$ be a collection of sets. For any subset $\beta$ of $I$, define $E_\beta = \prod_{i \in \beta} E_i$. The (discrete) probabilistic marginal problem is the following: given a subposet $A$ of $\mathcal{P}_f(I)$ and a family of probability laws $\{P_\alpha\}_{\alpha \in A}$ over the respective sets $\{E_\alpha\}_{\alpha \in A}$, which satisfy the compatibility conditions over all the intersections $U_\alpha \cap U_\beta$, for $\alpha, \beta \in A$, determine if there exists a probability $Q : E = \prod_{i \in I} E_i \to [0, 1]$ such that for every $\alpha \in A$,

\[\forall \alpha \in A, \forall x_\alpha \in E_\alpha, \sum_{x_{i \setminus \alpha} \in E_{i \setminus \alpha}} Q(x_\alpha, x_{i \setminus \alpha}) = P_\alpha(x_\alpha),\]  \hfill (5.8)

The linearized marginal problem asks for a function $Q : E \to \mathbb{R}$ that is not necessarily positive.
In the last appendix below, based on the preceding sections, we prove the following index formula, which generalizes the result of Kellerer [19] and Matúš [24].

**Theorem 5.16.** If the poset \( \mathcal{A} \) is finite and satisfies the weak intersection property, and if the \( \{ E_i \}_{i \in I} \) are finite sets, then

\[
\chi(\mathcal{A}; V) = \sum_{k=0}^{\infty} (-1)^k \dim_k H^k(\mathcal{U}_\mathcal{A}; V) = \sum_{\alpha, \beta \in \mathcal{A}} \mu_{\alpha, \beta} N_\beta;
\]

where \( \mu_{\alpha, \beta} \) is the Möbius function of \( \mathcal{A} \), and, for each \( \alpha \in \mathcal{A} \), \( N_\alpha \) denotes the cardinality of \( E_\alpha \).

We will also prove that

\[
\chi(\mathcal{A}; V) = \dim_k H^0(\mathcal{U}_\mathcal{A}; V) + \chi(\mathcal{A});
\]

where \( \chi(\mathcal{A}) \) denotes the Euler characteristic of \( \mathcal{A} \), in every possible sense: as a metric subspace of the simplex \( \mathcal{P}(I) \), as the lower or upper Hausdorff topological space in Čech cohomology, or as an abstract poset; this is also the Euler characteristic of the nerve of the category \( \mathcal{A} \).

6. Nerves of categories and nerves of coverings

Any contravariant functor \( G \) of abelian groups on a poset \( \mathcal{A} \) produces a sheaf, also denoted by \( G \), on the topological space \( \mathcal{X}_\mathcal{A} \), whose underlying set is \( \text{Ob} \mathcal{A} \), equipped with the upper \( \mathcal{A} \)-topology. This is equivalent with the dual statement for covariant functors on \( \mathcal{A}^{\text{op}} \). But \( G \) is also an abelian object in the topos \( \text{PSh}(\mathcal{A}) \), cf. [4], [26].

And in the context of topos theory, it is customary to study another cohomology, that is the graded derived functor \( H^\bullet(\mathcal{A}, -) \) of

\[
\Gamma_\mathcal{A}(-) = \text{Hom}_{\text{Ab}(\mathcal{A})}(\mathbb{Z}, -) \cong \text{Hom}_{\text{PSh}(\mathcal{A})}(\ast, -);
\]

cf. [44]. In the following lines, we give a more explicit and topological definition of this functor, according to [1], [26].

The nerve of a small category \( \mathcal{C} \) is the simplicial set whose \( n \)-simplices are sequences \( c_0 \to \cdots \to c_n \) of composable arrows in \( \mathcal{C} \), and whose face operators are

\[
d_i(c_0 \xrightarrow{f_0} \cdots \xrightarrow{f_n} c_n) = \begin{cases} c_0 \to \cdots \to c_{i-1} & \text{if } i = 0 \\ c_0 \to \cdots \to c_{i-1} \xrightarrow{f_i} c_{i+1} \to \cdots \to c_n & \text{if } 0 < i < n \\ c_0 \to \cdots \to c_{n-1} & \text{if } i = n \end{cases}
\]

For background and details, see Section 6.1 below. This permits to define a canonical cochain complex \( (C^n(\mathcal{A}, G), d) \) whose cohomology is precisely \( H^\bullet(\mathcal{A}, G) \), cf. [26] Prop. 6.1. This complex comes from a canonical projective resolution of the constant presheaf \( \mathbb{Z} \) [1 Ex. V.2.3.6].

The \( n \)-cochains are

\[
C^n(\mathcal{A}, G) = \prod_{a_n \to \cdots \to a_0 \text{ in } \mathcal{A}} G(a_n) = \prod_{a_0 \to \cdots \to a_n \text{ in } \mathcal{A}^{\text{op}}} G(a_n)
\]
and the coboundary \( \delta : C^{n-1}(A, G) \to C^n(A, G) \) is given by

\[
(\delta g)_{a_0 \to \cdots \to a_n} = \sum_{i=0}^{n-1} (-1)^i g_{d_i(a_0 \to \cdots \to a_n)} + (-1)^n G(\varphi_n) g_{d_n(a_0 \to \cdots \to a_n)},
\]

where \( \varphi_n \) is the \( A \)-morphism from \( a_n \) to \( a_{n-1} \) in the sequence \( a_n \to \cdots \to a_0 \).

Remark 6.1. This complex and its analog for a covariant homology were rediscovered by O. Peltre in the context of his doctoral work [28], which gives a homological interpretation of the generalized Belief Propagation algorithm [46], which is applied in statistical physics, bayesian learning and decoding processes. One of the initial motivations behind the present article was to understand better the connections of it with Čech cohomology and sheaf cohomology.

In this section, we want to compare this cohomology with the topological Čech cohomology of the sheaf \( G \) on \( X \). That we have studied in the previous sections. In fact we will prove that they are naturally isomorphic.

When \( G \) is the constant sheaf \( \mathbb{Z} \), \( H^\bullet(C(N(A), \mathbb{Z}), d) \) corresponds to the simplicial cohomology of \( |N(A)| \), the geometric realization of the nerve \( N(A) \) (see Remark 6.7); it is known to be naturally isomorphic to the singular cohomology of \( |N(A)| \) (cf. [11]). In turn, the Čech cohomology \( \check{H}^\bullet(X^A, \mathbb{Z}) \) is isomorphic to the singular cohomology of \( X^A \). Hence the isomorphism between \( H^\bullet(C(N(A), \mathbb{Z}), d) \cong \check{H}^\bullet(X^A, \mathbb{Z}) \) is implied by the homotopy equivalence between \( |N(A)| \) and \( X^A \), see May [25]. Thus we are looking here for an extension of this result in the context of sheaves.

For that purpose, we introduce a general framework of cosimplicial local systems on simplicial sets. We will remind below the definition of simplicial sets and simplicial objects in a category. The nerve \( K_\bullet(U) \) of a covering \( U \) introduced in Section 2 and the nerve \( N(C) \) of a category \( C \) are examples of simplicial sets. Cosimplicial local systems are functorial assignments of local data to the simplexes and morphisms of a simplicial set. It appears that both Čech cohomology and the cohomology introduced by (6.3)-(6.4) become particular cases of this general construction and can be compared in this framework.

Remark 6.2. It is not excluded that spectral sequences, as defined in Segal [33], can be used for establishing the comparison, but we have not seen how this can be done directly.

6.1. Simplicial sets and nerves of coverings. Simplicial sets can be traced back to Eilenberg and Zilber [12]—under the name “complete semi-simplicial sets”. They became ubiquitous in algebraic topology, due to the works of Segal, Grothendieck, Kan, Quillen, May and many others. The subject was treated in great detail by May in [25]; also [44, Ch. 8] is a good introduction.

Let \( \Delta \) be the category whose objects are the finite ordered sets \( [n] = \{0 < 1 < \cdots < n\} \), for each \( n \in \mathbb{N} \), and whose morphisms are nondecreasing monotone functions. Given any category \( C \), a simplicial object \( S \) in \( C \) is a contravariant functor from \( \Delta \) to \( C \) i.e. \( S : \Delta^{op} \to C \). When \( C \) is the category of sets, \( S \) is called a simplicial set. One can define analogously simplicial groups, modules, etc.

Although \( \Delta \) has many morphisms, which seem complicated at first sight, they can be conveniently expressed in terms of certain morphisms known as face and
degeneracy maps. For each \( n \in \mathbb{N} \) and \( i \in [n] \), the face map \( d_i^n : [n] \to [n + 1] \) is given by
\[
d_i^n(j) = \begin{cases} j & \text{if } j < i, \\ j + 1 & \text{if } j \geq i. \end{cases}
\]
Similarly, for each \( i \in [n + 1] \), the degeneracy map \( s_i^{n+1} : [n + 1] \to [n] \) is
\[
s_i^{n+1}(j) = \begin{cases} j & \text{if } j \leq i, \\ j - 1 & \text{if } j > i. \end{cases}
\]

Normally the super-index is dropped.

Remark that \( \phi \). For each degeneracy maps commutes.

Def. I.5.4], which is the presheaf represented by \[
d(6.9)
\]
Similarly, for each \( i \) equals Hom([\( n \)]) of finite sequences of ele-

\( \{X\) such that \( \varphi(j) = \varphi(j + 1) \). Then
\[
\varphi = d_{i_1} \cdots d_{i_s} s_{j_1} \cdots s_{j_t}.
\]
Remark that \( m - t + s = n \). This factorization is unique [25, Sec. I.2].

The face and degeneracy maps satisfy some relations:
\[
\forall n \in \mathbb{N}, 0 \leq j < k \leq n, \quad s_j^{n+1} \circ d_k^n = d_{k-1}^n \circ s_j^{n+1},
\]
\[
\forall n \in \mathbb{N}, 0 \leq j \leq n + 1, \quad s_j^{n+1} \circ d_j^n = s_{j+1}^{n+1} \circ d_j^n = \text{id}_{n+1},
\]
\[
\forall n \in \mathbb{N}, n + 1 \geq j > k + 1 \geq 1, \quad s_j^{n+1} \circ d_k^n = d_k^n \circ s_j^{n+1}.
\]

A (simplicial) morphism from a simplicial set \( S \) to a simplicial set \( S' \) is a natural transformation of functors: a collection of maps \( \{f_n : S([n]) \to S'([n])\}_{n \in \mathbb{N}} \) such that, for each morphism \( \varphi : [m] \to [n] \) in \( \Delta \), the diagram
\[
\begin{array}{ccc}
S([n]) & \xrightarrow{S\varphi} & S'([m]) \\
\downarrow f_n & & \downarrow f_m \\
S'([n]) & \xrightarrow{S\varphi} & S'([m])
\end{array}
\]
commutes.

Example 6.3 (Simplex). A basic example of simplicial set is the \( k \)-simplex \( \Delta^k \) [25 Def. I.5.4], which is the presheaf represented by \([k]\). This means that \( \Delta^k = \Delta^k([n]) \) equals Hom([\( n \)], [\( k \)]), and the map \( \Delta^k \varphi : \text{Hom}([m], [k]) \to \text{Hom}([n], [k]) \) induced by \( \varphi : [m] \to [n] \) is given by precomposition with \( \varphi \).

Example 6.4 (Nerve of a covering). Let \( X \) be a topological space and \( \mathcal{U} \) an open covering of \( X \). The nerve of the covering \( \mathcal{U} \) is the set \( K(\mathcal{U}) \) of finite sequences of elements of \( \mathcal{U} \) having a non-empty intersection. It has a natural structure of simplicial set: \( K_n = K([n]) \) is the set of sequences of length \( n + 1 \), denoted \((U_0, \ldots, U_n)\), and for any nondecreasing function \( \varphi_{m,n} \) from \( m \) to \( n \), there is a map \( \varphi_{m,n}^* : K_n \to K_m \) given by
\[
\varphi_{m,n}^*(V_0, \ldots, V_n) = (V_{\varphi(0)}, \ldots, V_{\varphi(m)}).
\]
In other terms, \( K_n(\mathcal{U}) \) is the set of maps \( u : [n] \to \mathcal{U} \) such that the intersection of the images are non-empty, and if \( \varphi : [m] \to [n] \) is a morphism and \( u \in K_n(\mathcal{U}) \), then \( \varphi_{m,n}^*(u) = u \circ \varphi \).

Hence the map \( s_i^k = K(s_i^{n+1}) \) is given by
\[
s_i^k(U_0, \ldots, U_n) = (U_0, \ldots, U_i-1, U_i, U_{i+1}, \ldots, U_n);
\]
is also called degeneracy map, whereas \( d_i^k = K(d_i^{n+1}) \) is given by
\[
d_i^k(U_0, \ldots, U_{n+1}) = (U_0, \ldots, U_{i-1}, U_i, U_{i+1}, \ldots, U_{n+1}),
\]
simplicial maps.

Definition 6.8. Simplicial homotopy is an equivalence relation, compatible with composition of simplicial maps.

Example 6.5 (Nerve of a category). To any small category $\mathcal{C}$ is naturally associated a simplicial set $N(\mathcal{C})$, named its nerve: the elements of $N_n(\mathcal{C})$ are the covariant functors from the poset $[n]$ to $\mathcal{C}$, and the morphisms are obtained by right composition.

Concretely an element of degree $n$ is a sequence

$$a = \alpha_0 \to \alpha_1 \to \ldots \to \alpha_n.$$  

The action $s^*_i$ of $s_i$ is the repetition of the object $\alpha_i$ via the insertion of an identity $\text{id}_{\alpha_i}$; the action $d^*_i$ of $d_i$ is the deletion of $\alpha_i$ via the composition of $\alpha_{i-1} \to \alpha_i$ and $\alpha_i \to \alpha_{i+1}$.

More generally if

$$b = \beta_0 \to \beta_1 \to \ldots \to \beta_m$$

belongs to $N_m(\mathcal{C})$, and $\varphi : n \to m$ is non-decreasing, then

$$\varphi^*(b) = \beta_{\varphi(0)} \to \beta_{\varphi(1)} \to \ldots \to \beta_{\varphi(n)}.$$  

Example 6.6 (Barycentric subdivision of the nerve of a covering). Consider the category $\mathcal{C}(\mathcal{U})$ which has for objects the non-empty intersections of the elements of $\mathcal{U}$, and for morphisms the inclusions, then the nerve $N(\mathcal{C}(\mathcal{U}))$ is the barycentric subdivision of the simplicial set $K(\mathcal{U})$. This was remarked by Segal [33], interpreting [12].

Remark 6.7 (Geometric realization). It is reassuring to know that any simplicial set gives rise to a CW-complex, even if this is not directly used in the present text. The geometric realization $|K|$ of the simplicial set $K$ is a topological space obtained as the quotient of the disjoint union of the products $K_n \times \Delta(n)$, where $K_n = K([n])$ and $\Delta(n) \subseteq \mathbb{R}^{n+1}$ is the geometric standard simplex, by the equivalence relation that identifies $(x, \varphi_*(y))$ and $(\varphi^*(x), y)$ for every nondecreasing map $\varphi : [m] \to [n]$, every $x \in K_n$ and every $y \in \Delta(m)$; here $f^*$ is $K(f)$ and $f_*$ is the unique linear map from $\Delta(n)$ to $\Delta(m)$ that maps the canonical vector $e_i$ to $e_{f(i)}$. For every $n \in \mathbb{N}$, $K_n$ is equipped with the discrete topology and $\Delta(n)$ with its usual compact topology, the topology on the union over $n \in \mathbb{N}$ is the weak topology, i.e. a subset is closed if and only if its intersection with each closed simplex is closed, and the realization is equipped with the quotient topology. In particular, even it is not evident at first sight, the realization of the simplicial set $\Delta^k$ is the standard simplex $\Delta(k)$. See [25, Ch. III].

The cartesian product of two simplicial sets $K$ and $L$ is taken as it must be for functors to $\mathcal{E}$, that is term by term: $(K \times L)([n]) = K([n]) \times L([n])$ at the level of objects, and similarly for the maps.

Definition 6.8. Let $f : K \to L$ and $g : K \to L$ be two simplicial maps, a simplicial homotopy from $f$ to $g$ is a simplicial map $h : K \times \Delta^1 \to L$, such that $f = h \circ (\text{id}_K \times d_0)$ and $g = h \circ (\text{id}_K \times d_1)$.

Simplicial homotopy is an equivalence relation, compatible with composition of simplicial maps.
Example 6.9 (Homotopy induced by a projection of coverings). A covering $\mathcal{U}$ is called a refinement of another covering $\mathcal{U}'$ when every set of $\mathcal{U}$ is contained in some set of $\mathcal{U}'$. In that case, there exists a map $\lambda : \mathcal{U} \to \mathcal{U}'$, called projection, such that for every $U \in \mathcal{U}$ one has $U \subseteq \lambda(U)$. It is also said that $\mathcal{U}$ is finer than $\mathcal{U}'$ \footnote{This is a footnote.}. A projection map $\lambda : \mathcal{U} \to \mathcal{U}'$ induces a simplicial morphism $\lambda_*$ from the simplicial set $K(\mathcal{U})$ to the simplicial set $K(\mathcal{U}')$:

(6.18) $\lambda_*(u) = \lambda \circ u$;

Proposition 6.10. If $\mathcal{U}$ is a refinement of $\mathcal{U}'$, two projections $\lambda, \mu$ from $\mathcal{U}$ to $\mathcal{U}'$ induce homotopic simplicial maps $\lambda_*, \mu_*$ from $K(\mathcal{U})$ to $K(\mathcal{U}')$.

Proof. Let $u = (U_0, ..., U_n)$ be an element of $K_n(\mathcal{U})$, and $\varphi_i = (0, ..., 0, 1, ..., 1) \in \Delta^1_n$ with the first 1 at place $i$ between 0 and $n + 1$, we put

(6.19) $h(u, \varphi_i) = (\lambda(U_0), ..., \lambda(U_{i-1}), \mu(U_i), ..., \mu(U_n))$.

$\square$

6.2. Cosimplicial local systems and their cohomology. We present here a general definition of cohomology for cosimplicial local systems on simplicial sets.

Definition 6.11. A cosimplicial local system of sets $F$ over the simplicial set $K$ is a family $F_\alpha$ indexed by the elements $\alpha$ of $K$, and, for any morphism $\varphi : [m] \to [n]$ and any $v \in K_n$, a given application $F(\varphi, v) : F_\alpha \to F_\beta$, where $u = \varphi^*_{m,n}(v)$, such that $F(\psi, w) \circ F(\varphi, v) = F(\varphi \circ \psi, w)$, for $\varphi : [m] \to [n]$, $\psi : [n] \to [p]$, $w \in K_p$, $v = \psi^*_{\alpha,\beta}(w)$, $u = \varphi^*_{m,n}(v)$.

Remark 6.12. A definition of simplicial local systems appeared in the work of Halperin \footnote{Halperin's work.}. In his case the arrows are in the reverse direction, i.e. for $\varphi : [m] \to [n]$, $v \in K_n$, a map $\varphi^*_v : F_v \to F_{\varphi^*(v)}$.

Example 6.13 (Čech system). Take a presheaf $F$ over the topological space $X$ and consider an open covering $\mathcal{U}$ of $X$. Then for $u \in K(\mathcal{U})$, define $F_u = F(U_u)$, and for $\varphi : [m] \to [n]$, $v \in K_n$, take for $F(\varphi, v)$ the restriction from $F(U_{\varphi^*(v)})$ to $F(U_v)$. This defines a cosimplicial system over $K(\mathcal{U})$.

Example 6.14 (Upper and lower systems associated to a functor). Let $F$ be a contravariant functor from $\mathcal{C}$ to the category of sets $\mathcal{E}$. We can define a cosimplicial local system $F^*$ over the nerve $N(\mathcal{C})$ (see Example 6.5 for the definition and the notation), named the upper system, by taking $F^*(a) = F(\alpha_n)$, and for a morphism $\psi : n \to p$, an element $b = \beta_0 \to ... \to \beta_p$ in $N_p(\mathcal{C})$, denoting by $\alpha_n$ the last element of $a = \psi^*(b)$, we have $\alpha_n = \beta_\psi(n)$, and this comes with a canonical arrow $f$ going to $\beta_p$, then we take $\psi^*_b = f^*$, going from $F_b = F(\beta_p)$ to $F_a = F(\alpha_n)$.

In the dual manner, if $F$ is covariant, we can define the lower cosimplicial system $F_*$, by taking $F_*(a) = F(\alpha_0)$. Taking again the element $b$ and the morphism $\psi$, we use now the fact that the first element of $a = \psi^*(b)$ is $\alpha_0 = \beta_\psi(0)$, which comes with a canonical arrow $g$ in $\mathcal{C}$ from $b_0$ to it, and we can take $\varphi^*_b = g_\ast$ from $F_*(b)$ to $F_*(a)$.

Replacing $\mathcal{C}$ by the opposite (or dual) category $\mathcal{C}^{op}$, we exchange contravariant functors with covariant ones, and lower systems with upper ones.

Remark 6.15. Introduce the category $\mathcal{S}(K)$, having for objects the elements of $K$, and for arrows between two elements $v \in K_n$ and $u \in K_m$ the elements $\varphi$ of $\Delta(m, n)$...
such that $\varphi^*(v) = u$. Then a cosimplicial local system $F$ over $K$ is a contravariant functor (i.e. a presheaf) from $\mathcal{S}(K)$ to the category of sets.

**Definition 6.16.** Let $F$ be a cosimplicial local system over the simplicial set $K$; for each $n \in \mathbb{N}$, a *simplicial cochain* of $F$ of degree $n$ is an element $(c_u)_{u \in K_n}$ of the product $C^n(K; F) = \prod_{u \in K_n} F_u$.

When $F$ is a local system of abelian groups, $C^n(K; F)$ has a natural structure of abelian group. In what follows, we stay in this abelian context.

The *coboundary operator* $\delta : C^n(K; F) \to C^{n+1}(K; F)$ is defined by

$$
(\delta c)(v) = \sum_{i=0}^{n+1} (-1)^i F(d_i, v)(c(d_i^* (v))),
$$

for any element $v$ of $C^{n+1}(K; F)$. In the expression, $d_i^*$ is $K(d_i)$; remark that the sum takes place in $F_v$. When we want to be more precise, we write $\delta = \delta_{n+1}$ at degree $n$. The operator $\delta$ is also named the differential of the cochain complex $C^n(K; F), n \in \mathbb{N}$.

**Proposition 6.17.** For all $n \in \mathbb{N}$, the equality $\delta_{n+1} \circ \delta_n = 0$ holds. In short, $\delta \circ \delta = 0$.

**Proof.** The expression of $\delta \circ \delta(v)(w)$ is the sum of elementary terms of the form $(-1)^k F(d_j^* \circ d_i^*, w)(c(d_i^* \circ d_j^* w))$, with $i \neq j$ and $k = i + j$ if $j < i$, $k = i + j + 1$ if $j > i$.

It is easy to verify that the maps $d$ satisfy the relation $d_j d_i = d_i d_{i+1}$ if $i < j$ [44, Ex. 8.1.1]. It follows that the terms in the sum cancel two by two. \qed

A sequence $\{C^n\}_{n \in \mathbb{N}}$ of abelian groups with an operator $\delta$ of degree +1 and square zero, is named a differential complex, or a cochain complex.

**Definition 6.18.** (Cohomology of a cosimplicial local system). The cohomology group in degree $n \in \mathbb{N}$ of the local system $F$ over the simplicial set $K$ is the quotient abelian group

$$
H^n(K; F) = \ker(\delta_{n+1})/\text{im}(\delta_n).
$$

By convention $\delta_{-1} = 0$.

Equivalently, the cohomology $H^\bullet(K; F)$ of $F$ over $K$, seen as graded vector space, is the cohomology of the complex of simplicial cochains $(C^\bullet(K, F), \delta)$.

As usual, the elements of $\ker \delta_n$ are called $n$-cocycles, and those in the image of $\delta_{n-1}$ are the $n$-coboundaries.

For example, a 0-cochain is a collection $(c_u)_{u \in K_0}$, and it a 0-cocycle if for any $v \in K_2$,

$$
0 = F(d_0, v)(c_{d_0^* v}) - F(d_1, v)(c_{d_1^* v}).
$$

In the particular case of an open covering $\mathcal{U}$ of a topological space $X$, and a presheaf of abelian groups $F$ on $X$, the group $H^0(\mathcal{U}; F)$ (for the associated local system on the nerve of the covering $\mathcal{U}$) coincide with the set of global sections of $F$ on $X$, i.e. the families $(c_U) \in \prod_{U \in \mathcal{U}} F(U)$ of local sections of $F$ whose restriction coincide on the non-empty intersections $U \cap V$, $U, V \in \mathcal{U}$.

---

[4]This is a short argument with big consequences. When did it appear for the first time? Who came up with it? Euler, Poincaré, Noether, Lefschetz, Alexander?
Remark 6.19. Let $G$ be a contravariant functor over a poset $A$. The simplicial cochain complex $(C^n(N(A^{op}), G^*), \delta)$ associated to the upper local system of $G : A^{op} \to E$, in the sense of the preceding definitions, is precisely the cochain complex introduced by $[13, 14]$. One could also compute this cohomology from the complex $(C^n(N(A), G^*), \delta)$: the cochains are the same, and the differential only differs by a sign when $n$ is odd.

Similarly, if $G$ is covariant over $A$, then one can see it as a presheaf on $A^{op}$; its sheaf cohomology can be computed as the cohomology of $(C^n(N(A), G^*), \delta)$ or $(C^n(N(A^{op}), G^*), \delta)$.

As mentioned before, these complexes were rediscovered by O. Peltre $[27]$ for understanding geometrically the generalized belief propagation algorithm of $[46]$.

Given two cochain complexes of abelian groups $C^*$ and $D^*$, whose differential operators have degree $+1$ (i.e. ascending complexes) a cochain map (or cochain morphism) is a collection \( \{ f^n : C^n \to D^n \}_{n \in \mathbb{Z}} \) of morphism of groups that commute with the differentials. In other words, it is a morphism of graded abelian groups of degree zero that commute with the differentials.

A chain map between two cochain complexes sends coboundaries to coboundaries and cocycles to cocycles, thus it induces a map at the level of cohomology.

**Example 6.20.** Let $U$ be a refinement of $U'$, and $\lambda : U \to U'$ an adapted projection. The simplicial morphism $\lambda_*$ of the simplicial set $K(U)$ to the simplicial set $K(U')$ in the last subsection induces, for each integer $n$, a map $\lambda^n$ from $C^n(U'; F) \to C^n(U; F)$ defined by $\lambda^n(c') = c' \circ \lambda_*$. More concretely,

\[
(\lambda^n c')(U_0, ..., U_n) = c'(\lambda(U_0), ..., \lambda(U_n)).
\]

This map commutes with the Čech differential, then it induces a map in cohomology

\[
\lambda^n : H^n(U'; F) \to H^n(U; F).
\]

Given two cochain maps $f^*, g^*$, a cochain homotopy from $f^*$ to $g^*$ is a morphism of graded groups $h$ from $C^*$ to $D^*$ of degree $-1$ such that

\[
d \circ h + h \circ d = g - f.
\]

This defines an equivalence relation on cochain maps (of degree zero) which is compatible with the composition of maps.

**Proposition 6.21 ([H1 Thm. 4.4]).** If two cochain morphisms are homotopic, they give the same application in cohomology.

**Proof.** If $c$ is a cocycle of $C$, and if $h$ is an homotopy from $f$ to $g$, then $dh(c) = g(c) - f(c)$, thus $g(c)$ and $f(c)$ have the same classes in cohomology. \hfill \Box

**Definition 6.22** (Lift of simplicial map to a local system). Let $F$ (resp. $G$) be a cosimplicial local system over the simplicial set $K$ (resp. $L$), and $f : K \to L$ a simplicial map, a lift $\tilde{f}$ of $f$ from $G$ to $F$ is family of maps $\tilde{f}_u : G_f(u) \to F_u$, for each $u \in K$, such that, for any morphism $\varphi : m \to n$, any $v \in K_n$, $u = \varphi^* v \in K_m$,

\[
F(\varphi, v) \circ \tilde{f}_u = \tilde{f}_v \circ G(\varphi, f(v)).
\]

An example is given by a morphism $(f, \varphi)$ between a presheaf $\mathcal{F}$ over $X$ and a presheaf $\mathcal{G}$ over $Y$, when we consider two open coverings $U$ and $V$ of $X$ and $Y$ respectively, such that $U$ is finer than the open covering $f^{-1}(V)$. In this case, we choose a projection $\lambda$ from $U$ to $V$ i.e. $\forall U \in \mathcal{U}, U \subseteq f^{-1} \lambda(U)$. As we have
seen, this defines a simplicial map from $K(U)$ to $K(V)$, that we write $f_\lambda$; then, for 
$u = (U_0, \ldots, U_n)$, the group $G(f_\lambda(u))$ can be identified with $G(\bigcap_{i=0}^n(\lambda(U_i))$, and for 
every element $g$ in this group, we pose 

\begin{equation}
(6.26) \quad \tilde{f}_\lambda(u)(g) = \varphi(g) \in F(\bigcap_{i=0}^n(U_i));
\end{equation}

because $\varphi$ can be seen as a morphism of $f^{-1}G$ to $\mathcal{F}$.

**Definition 6.23.** Two pairs $(f, \tilde{f}), (g, \tilde{g})$ of morphisms and lifts are *simplicially homotopic* if there exists a simplicial homotopy $h : K \times \Delta^1 \to L$ from $f$ to $g$, and 
a family of maps $h_{u,s}, u \in K, s \in \Delta^1$ from $G_{h(u,s)}$ to $F_u$, such that $\tilde{f} = \tilde{j}_0 \circ h$ and 
$\tilde{g} = \tilde{j}_1 \circ \tilde{h}$, where $j_0 = \text{id}_K \times s_0$ and $j_1 = \text{id}_K \times s_1$.

As a consequence of Proposition [6,10] two choices of projections in the construction 
of the map of local systems associated to a morphism of presheaves gives two homotopic morphisms in the simplicial sense.

Suppose given two local systems $F, G$ over $K, L$ respectively, and a lift $\tilde{f}$ of $f : K \to L$. The following formula defines a natural morphism $\tilde{f}^*$ from $C^*(L; G)$ to $C^*(K; F)$: 

\begin{equation}
(6.27) \quad \tilde{f}^*(c_L)(u) = \tilde{f}_u(c_L(f(u)).
\end{equation}

**Lemma 6.24.** $\tilde{f}^*$ commutes with the differentials.

**Definition 6.25.** Let $G$ be a cosimplicial local system over a simplicial set $L$, and 
$f$ a simplicial map from $K$ to $L$, the family $F_u = G_{f(u)}$, for $u \in K$, with the maps 
$F(\varphi, v) = G(\varphi, f(v))$ is a cosimplicial local system over $K$, named the pull-back of $G$, and denoted by $f^*(G)$.

**Example 6.26.** Start with a cosimplicial local system $F$ over a simplicial set $K$; 
then, over the product $K \times \Delta^1$, we define a local system $\pi^*F$ by taking, for $u \in K_n$ 
and $s \in \Delta^1$, $\pi^*F(u, s) = F(u)$. Then consider the two injections $j_0 = \text{Id}_K \times s_0$ and 
$j_1 = \text{Id}_K \times s_1$ from $K = K \times \Delta^0$ to $K \times \Delta^1$; the two pull-back $j_0^*\pi^*F$ and 
j_1^*\pi^*F coincide with $F$. Thus we have evident lifts $\tilde{j}_0$ and $\tilde{j}_1$ from $F$ to $\pi^*F$. They 
are homotopic in the simplicial sense, the map $h$ from $K \times \Delta^1$ to itself being the identity, 
and the lift being the natural identification.

From $C^*(K \times \Delta^1; \pi^*F)$ to $C^*(K; F)$, the two chain maps $\tilde{j}_0^*$ and $\tilde{j}_1^*$ are given 
by the following formulas, for $c$ be a $n$-cochain of $K \times \Delta^1$ with value in $\pi^*F$, and 
u an element of $K_n$

\begin{equation}
(6.28) \quad \tilde{j}_0^* c(u) = c(u, (0, \ldots, 0)),
\end{equation}

\begin{equation}
(6.29) \quad \tilde{j}_1^* c(u) = c(u, (1, \ldots, 1)).
\end{equation}

**Lemma 6.27.** $\tilde{j}_0^*$ and $\tilde{j}_1^*$ are homotopic as chain maps.

**Proof.** Let $C$ be a $(n+1)$-cochain of $K \times \Delta^1$ with value in $\pi^*F$, and $u$ an element 
of $K_n$, we pose 

\begin{equation}
(6.30) \quad H(C)(u) = \sum_{j=0}^{n+1} (-1)^j F(s_j, u)(C(s_j^*(u), 1_j^{n+1})).
\end{equation}
where, for \( n \in \mathbb{N} \) and \( j \in [n+2] \), \( 1_j^{n+1} \) denotes the element \((0, \ldots, 0, 1, \ldots, 1)\) of \( \Delta_1^{n+1} \), where the first 1 is at the place \( j \). That gives \( 1_{n+2}^{n+1} = (0, \ldots, 0) = s_0(0) \) and \( 1_0^{n+1} = (1, \ldots, 1) = s_1(0) \).

This defines an endomorphism \( H \) of degree -1 of \( C^\bullet(K; F) \). Now we compute, for \( c \in C^n(K \times \Delta^1; \pi^*F) \):

\[
H(\delta c)(u) = \sum_{j=0}^{n+1} \sum_{k=0}^{n+1} (-1)^{j+k} F(s_j, u) \circ F(d_k, s_j^*u) c(d_k^*s_j^*u, d_k^*1_j^{n+1}),
\]

and

\[
\delta(H(c))(u) = \sum_{j=0}^{n} \sum_{k=0}^{n} (-1)^{j+k} F(d_k, u) \circ F(s_j, d_k^*u) c(s_j^*d_k^*u, 1_j^{n+1}).
\]

Let us add \( H(\delta c)(u) \) and \( \delta(H(c))(u) \), in virtue of the relations (6.8), most of the terms annihilate. The only ones that survive correspond to the terms \((s_jd_j)^*\) and \((s_jd_{j-1})^*\). Note that \( d_j^*1_j^{n+1} = 1_j^n \) and that \( d_{j-1}^*1_j^{n+1} = 1_j^{n-1} \), then, due to the signs, they annihilate two by two, except the extreme terms, for \( j = 0 \) and \( j = n + 1 \), giving

\[
\delta(H(c))(u) + H(\delta c)(u) = c(u, 1_0^n) - c(u, 1_{n+1}^n)
\]

\[
= c(u, (1, \ldots, 1) - c(u, (0, \ldots, 0)).
\]

Then \( H \) is a chain homotopy operator from \( \tilde{j}_0^* \) to \( \tilde{j}_1^* \), as we desired. \( \square \)

**Proposition 6.28.** Let \( f, g \) be two simplicial maps from the simplicial set \( K \) to the simplicial set \( L \), let \( F, G \) be cosimplicial systems over \( K \) and \( L \) respectively, and \( \tilde{f}, \tilde{g} \) two lifts over \( K \); suppose that the pairs \((f, \tilde{f})\), \((g, \tilde{g})\) are simplicially homotopic, then the induced maps of cochains complexes are homotopic.

**Proof.** The map \( \tilde{h} \) is a pullback of the simplicial map \( h \) to the local systems \( \pi^*F \) on \( K \times \Delta^1 \) and \( G \) on \( L \). Thus we get a chain-map

\[
\tilde{h}^* : C^\bullet(L; G) \to C^\bullet(K \times \Delta^1; \pi^*F).
\]

On the other side, we have two natural cochain maps, for \( k = 0, 1 \),

\[
\tilde{j}_k^* : C^\bullet(K \times \Delta^1; \pi^*F) \to C^\bullet(K; F).
\]

Applying the Lemma 6.24, there exists an homotopy from \( \tilde{j}_0^* \) to \( \tilde{j}_1^* \):

\[
H : C^\bullet(K; F) \to C^{\bullet-1}(K; F);
\]

therefore, applying the Lemma 6.24

\[
\tilde{g}^* - \tilde{f}^* = \tilde{h}^* \circ (\tilde{j}_1^* - \tilde{j}_0^*)
= \tilde{h}^* \circ (d \circ H + H \circ d) = d \circ (\tilde{h}^* \circ H) + (\tilde{h}^* \circ H).
\]

\( \square \)

**Corollary 6.29.** The induced morphisms in cohomology are the same.

**Theorem 6.30** (cf. [13] Ch. IX)). If \( \mathcal{U} \) is a refinement of \( \mathcal{U}' \), two projections \( \lambda, \mu \) from \( \mathcal{U} \) to \( \mathcal{U}' \) give the same application in cohomology.
Proof: The simplicial maps $\lambda_\ast$ and $\mu_\ast$ from $K(\mathcal{U})$ to $K(\mathcal{U}')$ are homotopic in the simplicial sense, then the maps $\lambda^\ast$ and $\mu^\ast$ are homotopic in the sense of maps of differential complexes, cf. last section). \hfill \Box

6.3. **Comparison theorems.** Given a covering $\mathcal{U}$ of a topological space $X$, let $\mathcal{A}(\mathcal{U})$ denote the poset whose objects are finite intersections of elements of $\mathcal{U}$, ordered by inclusion (thus the morphisms go from intersections to partial intersections), and $N(\mathcal{U}) = N(\mathcal{A}(\mathcal{U}))$ denotes the nerve of the category $\mathcal{A}(\mathcal{U})$, cf. Example 6.6.

The objects of $\mathcal{A}(\mathcal{U})$ make an open covering of $X$ which is finer than $\mathcal{U}$. By choosing for each non-empty finite intersection of elements of $\mathcal{U}$ one of these elements, we obtain a map from $\mathcal{A}(\mathcal{U})$ to $\mathcal{U}$, that we denote $\pi$; it is a projection in the sense of Eilenberg-Steenrod, see Example 6.9. In what follows we will always assume that for $U \in \mathcal{U}$, $\pi(U) = U$. The map $\pi$ induces a simplicial map $\pi_\ast$ from the simplicial set $N(\mathcal{U})$ to the simplicial set $K(\mathcal{U})$ (which is the usual nerve of the covering $\mathcal{U}$ in the sense of Example 6.4); it maps the sequence $V_0 \to \cdots \to V_n$ of elements of $\mathcal{A}(\mathcal{U})$ to the sequence $(\pi(V_0), \ldots, \pi(V_n))$ of elements of $\mathcal{U}$.

Given a presheaf $F$ of abelian groups over $X$, we have defined the cosimplicial local system of Čech $F_\lambda$ over $K(\mathcal{U})$ (cf. Example 6.13). To define a local system over $N(\mathcal{U})$, we restrict $F$ to a presheaf on $\mathcal{A}(\mathcal{U})$ and we take the lower cosimplicial local system $F_\ast$ over $N(\mathcal{U})$, as in Example 6.14. Given an element $v = (V_0 \to \cdots \to V_n)$ of $N_n(\mathcal{U})$, remark that $V_0 = \bigcap_{i=0}^n V_i \subseteq \bigcap_{i=0}^n \pi(V_i)$, hence there is a well-defined restriction map from $F_\lambda(\pi_*v)$ to $F_\ast(v)$. This defines a lift $\tilde{\pi}$ of $\pi_\ast$ from $F_\lambda$ to $F_\ast$ in the sense of Definition 6.22, hence a morphism

\[
(6.38) \quad \pi^\ast : C^\bullet(K(\mathcal{U}); F_\lambda) \to C^\bullet(N(\mathcal{U}); F_\ast)
\]

**Theorem 6.31.** The map $\pi^\ast$ is an homotopy equivalence between $C^\bullet(K(\mathcal{U}))$ and $C^\bullet(N(\mathcal{U}))$.

Before presenting the proof, let us see how this implies the isomorphism between topos cohomology and Čech cohomology in the case of abelian presheaves on a poset, provided it is a conditional meet semilattice i.e. that products exists conditionally. Let $G$ be a contravariant functor of abelian groups on a poset $\mathcal{A}$, and let $\tilde{G}$ be the induced sheaf on the upper $A$-space $X^A$. We have seen that the topos cohomology of $G \in \text{PSh}(\mathcal{A})$ is isomorphic to the cohomology of the cochain complex $(C^\bullet(N(\mathcal{A}), G_*), \delta)$, whereas the Čech cohomology of $\tilde{G}$ is the cohomology of the complex $(C^\bullet(K(\mathcal{U}^A), G^\circ), \delta)$.

The space $X^A$ has a finest canonical open covering $\mathcal{U}^A$ made by the upper sets $U^\alpha; \alpha \in \mathcal{A}$. An inclusion $U^\alpha \subseteq U^\beta$ corresponds to an arrow $\alpha \to \beta$, then the natural inclusion $\mathcal{A} \hookrightarrow \mathcal{A}(\mathcal{U}^A)$ is a covariant functor, and induces an injective simplicial covariant functor $\iota : N(\mathcal{A}) \hookrightarrow N(\mathcal{U}^A)$.

We have a diagram of simplicial sets

\[
(6.39) \quad N(\mathcal{A}) \xrightarrow{\iota} N(\mathcal{U}^A) \xrightarrow{\pi} K(\mathcal{U}^A)
\]

where the last arrow is induced by any projection $\pi : \mathcal{A}(\mathcal{U}^A) \to \mathcal{U}^A$ that is the identity on $\mathcal{U}^A$. Let us denote by $j$ the simplicial map $\pi_* \circ \iota$. Given $a = (\alpha_0 \to \cdots \to \alpha_n) \in N(\mathcal{A})$, we have

\[
(6.40) \quad G(\alpha_0) = G_\ast(a) = G^\circ(j(a)) = \tilde{G}(\cap_{i=0}^n U^{\alpha_i}) = \tilde{G}(U^{\alpha_0}),
\]
then there is a lift of \( j \) from \( G' \) to \( G_* \) (cf. Definition 6.22) given by identities. Thus we deduce a morphism of chain complexes

\[
(6.41) \quad j^* : C^\bullet(K(A); G') \to C^\bullet(N(A); G_*)
\]
that induces a morphism in cohomology.

**Corollary 6.32.** If products exist conditionally in \( A \), the chain map \( j^* \) is a chain equivalence up to homotopy, thus induces an isomorphism in cohomology.

**Proof.** Under the hypothesis, one has \( \mathcal{A}(\mathcal{U}A) = \mathcal{U}A \), since every intersection \( U^{\alpha_0} \cap \cdots \cap U^{\alpha_n} \) equals \( U^{\alpha_0 \land \cdots \land \alpha_n} \). Hence \( N(A) \cong N(\mathcal{U}A) \). The map \( \pi_* \) is induced by \( \pi = \text{id}_{\mathcal{U}A} \). The claim then follows from Theorem 6.31. \( \square \)

The equivalence above is natural in the category of posets with presheaves up to homotopy.

We close this section with the proof of Theorem 6.31 inspired by a classical argument of Eilenberg and Steenrod: starting with a simplicial complex \( K \), they associated to it the poset \( A(K) \), whose elements are the faces (simplices) of \( K \); the nerve \( N \) of this poset is naturally isomorphic to the barycentric subdivision of \( K \) (cf. [33]). In [12, pp. 177-178], the authors proved that there exists an homotopy equivalence between \( N \) and \( K \). The following proof is an adaptation of their argument to this more general setting.

**Proof of Theorem 6.31**

1) first we construct by recurrence over \( n \) a linear application \( Sd^n \) from \( C^n(N(\mathcal{U})) \) to \( C^\bullet(K(\mathcal{U})) \), having the two following properties:

(i) (locality) for any \( c \in C^n(N(\mathcal{U})) \) and any collection \( u = (U_0,...,U_n) \), the value \( (Sd^n c)(u) \) in \( F(U_u) \) depends only of the values of \( c \) on the descendent of the open sets \( U_i \); \( i = 0,...,n \), i.e. the values \( c(v) \in F(V_n) \) for the sequences \( v = (V_0,...,V_n) \), where each \( V_i \); \( i = 0,...,n \) is included in a \( U_j \); \( j = 0,...,n \);

(ii) (morphism of cochain complex) \( d \circ Sd^n = Sd^\circ d \).

For \( n = 0 \), and \( U_0 \in \mathcal{U} = K_0(\mathcal{U}) \), we take \( Sd^0(c)(U_0) = c(U_0) \), this is allowed because \( U_0 \) is also an element of \( N_0(\mathcal{U}) \). The condition (i) is evidently satisfied, and (ii) is empty in this degree.

For \( n = 1 \), \( c \in C^1(N(\mathcal{U})) \) and \( u = (U_0, U_1) \), we pose \( Sd^1 c(U_0, U_1) = c(U_0, U_u) - c(U_1, U_u) \), where \( U_u = U_0 \cap U_1 \). This is local, and for \( c_0 \in C^0(N(\mathcal{U})) \):

\[
(6.42) \quad Sd^1(dc_0)(U_0, U_1) = (c_0(U_0) - c_0(U_u)) - (c_0(U_1) - c_0(U_u)) \nonumber
= dc_0(U_0, U_1) \nonumber = d \circ Sd^1(c)(U_0, U_1).
\]

Then take \( n \geq 2 \), and suppose that a map \( Sd^q \) is constructed for every \( q \leq n - 1 \), satisfying (i) and (ii). Take a cochain \( c \in C^n(N(\mathcal{U})) \), and consider an element \( u = (U_0,...,U_n) \) of \( K_n(\mathcal{U}) \); remind we note \( U_u \) the intersection (necessary non-empty) of the \( U_i \); \( i = 0,...,n \). We define an element \( c_u \in C^{n-1}(N(\mathcal{U})) \) by taking on every decreasing sequence \( v = (V_0,...,V_{n-1}) \),

\[
(6.43) \quad c_u(v) = c(V_0,...,V_{n-1}, U_u),
\]

if \( V_{n-1} \) contains \( U_u \) and taking \( c_u(V_0,...,V_{n-1}) = 0 \) in the opposite case.

Then we define

\[
(6.44) \quad Sd^n(c)(u) = \sum_{i=0}^{n} (-1)^{n-i} Sd^{n-i}(c_u)(U_0,...,\hat{U}_i,...,U_n)|U_u.
\]
The locality (i) follows from the recurrence hypothesis: the definition of \( c_u \) depends only of \( U_u \) which is a descendent of \( u \), and this is the same for the restriction to \( U_u \), moreover the value of \( Sd^{n-1}(c_u) \) on \( (U_0, \ldots, \widehat{U}_i, \ldots, U_n) \) depends only of the values of \( c_u \) on the sequences of descendent of \( (U_0, \ldots, \widehat{U}_i, \ldots, U_n) \).

For (ii), we have to compute \( Sd^n(de)(u) \) for a cochain \( c \in C^{n-1}(N(U); F) \). For a decreasing sequence \( V_0, \ldots, V_{n-1} \), then, by writing \( V_n = U_u \), we have

\[
(dc)_u(V_0, \ldots, V_{n-1}) = dc(V_0, \ldots, V_{n-1}, U_u)
\]

\[
= \sum_{j=0}^{n} (-1)^j c(V_0, \ldots, \widehat{V}_j, \ldots, V_n)|U_u
\]

\[
= d(c_u)(V_0, \ldots, V_{n-1})|U_u + (-1)^n c(V_0, \ldots, V_{n-1})|U_u,
\]

where \( c_u \) is also defined by \( c_u(V_0, \ldots, V_{n-2}) = c(V_0, \ldots, V_{n-2}, U_u) \) if \( V_{n-2} \) contains \( U_u \) and \( c_u(V_0, \ldots, V_{n-2}) = 0 \) in the opposite case. Which gives for reference, when \( c \) belongs to \( C^{n-1}(N(U); F) \):

\[
(6.45) \quad d(c_u) = (dc)_u + (-1)^{n-1} c.
\]

It follows from the recurrence hypothesis that

\[
Sd^n(de)(u) = \sum_{i=0}^{n} (-1)^{n-i} Sd^{n-1}((dc)_u)(U_0, \ldots, \widehat{U}_i, \ldots, U_n)|U_u
\]

\[
= \sum_{i=0}^{n} (-1)^{n-i} (Sd^{n-1}d(c_u))(U_0, \ldots, \widehat{U}_i, \ldots, U_n)|U_u
\]

\[
+ \sum_{i=0}^{n} (-1)^i (Sd^{n-1}c)(U_0, \ldots, \widehat{U}_i, \ldots, U_n)|U_u
\]

\[
= \sum_{i=0}^{n} (-1)^{n-i} (d \circ Sd^{n-1}(c_u))(U_0, \ldots, \widehat{U}_i, \ldots, U_n)|U_u
\]

\[
+ \sum_{i=0}^{n} (-1)^i (Sd^{n-1}c)(U_0, \ldots, \widehat{U}_i, \ldots, U_n)|U_u
\]

\[
= (-1)^n (d \circ d) Sd^n(c_u)(u) + d(Sd^{n-1}(c))(u)
\]

\[
= d \circ Sd^{n-1}(c)(u).
\]

Therefore \( Sd^n \) verifies (ii).

2) let us prove that the composition \( Sd^* \circ \pi^* \) is homotopic to the identity of \( C^*(K(U); F) \).

For that purpose we construct a sequence of homomorphisms,

\[
(6.46) \quad D_{K}^{n+1} : C^{n+1}(K(U); F) \to C^n(K(U); F),
\]

for \( n \geq 0 \), by recurrence over the integer \( n \), such that

\[
(6.47) \quad Id - Sd^n \circ \pi^n = d \circ D_{K}^{n} + D_{K}^{n+1} \circ d.
\]

For \( n = 0 \), and \( c \in C^1(K(U); F) \), we simply take \( D_{K}^{1}(c)(U) = 0 \). This works because, if \( c \) is a 0-cochain of \( K(U) \), \( F \),

\[
(6.48) \quad (Sd^0 \circ \pi^0 c)(U_0) = c(U_0).
\]
For $n = 1$, and $c \in C^2(K(\mathcal{U}); F)$, take

\begin{equation}
D^2_K c(U_0, U_1) = c(U_0, U_1, \pi(U_0 \cap U_1)|U_0 \cap U_1).
\end{equation}

This gives for $c' \in C^2(K(\mathcal{U}); F)$:

\begin{equation}
D^2_K (dc')(U_0, U_1) = c'(U_1, \pi(U_u))|U_u - c'(U_0, \pi(U_u))|U_u + c'(U_0, U_1);
\end{equation}

where as usual we have denoted $U_0 \cap U_1$ by the symbol $U_u$. On the other side,

\begin{equation}
(Sd^1 \circ \pi^1 c')(U_0, U_1) = -(\pi^1 c')_u(U_1)|U_u + (\pi^1 c')_u(U_0)|U_u
= -(\pi^1 c')(U_1, U_u)|U_u + (\pi^1 c')(U_0, U_u)|U_u
= -c'(U_1, \pi(U_u))|U_u + c'(U_0, \pi(U_u))|U_u.
\end{equation}

Then $Id - Sd^1 \circ \pi^1 = D^2_K \circ d + d \circ D^1_K$, as we expected.

More generally, for any $n$, consider consider a $n$-cochain $c$ of $K(\mathcal{U})$ with respect to the local system $F$. For a sequence $u' = (U'_0, ..., U'_{n-1})$ in $\mathcal{U}$, let us define

\begin{equation}
\pi^u(U'_0, ..., U'_{n-1}) = c(U'_0, ..., U'_{n-1}, \pi(U_u))
\end{equation}

if $U'_u \supseteq \pi(U_u)$, and $c^u_n(U'_0, ..., U'_{n-1}) = 0$ if not.

Then consider a decreasing sequence $\nu = (V_0, ..., V_{n-1})$ in $\mathcal{A}_\mathcal{U}$. If $U_u \subseteq U_{\nu(u)} = \bigcap_{i=0}^{n-1} \pi(V_i)$,

\begin{equation}
(\pi^u c)(V_0, ..., V_{n-1}) = \pi^u c(V_0, ..., V_{n-1}, U_u)
= c(\pi(V_0), ..., \pi(V_{n-1}), \pi(U_u))
= c^u_n(\pi(V_0), ..., \pi(V_{n-1}))
= \pi^{n-1}(c^u_n)(V_0, ..., V_{n-1}).
\end{equation}

If $U_u \notin U_{\nu(u)}$, we have $(\pi^u c_u)(\nu) = 0 = (\pi^{n-1}(c^u_n))(\nu)$. Therefore, in all cases

\begin{equation}
(\pi^u c_u)(\nu) = \pi^{n-1}(c^u_n)(\nu).
\end{equation}

Now assume that $D^{n+1}_K$ is defined for $q \leq n$, satisfying the homotopy relation for $Id - Sd^q \circ \pi^q$, and consider a $n$-cochain $c$ of $K(\mathcal{U})$ with respect to the local system $F$; for every sequence $u = (U_0, ..., U_n)$ in $\mathcal{U}$, the chosen definition of $Sd^n$ gives

\begin{equation}
(Sd^n \circ \pi^n c)(U_0, ..., U_n) = \sum_{i=0}^{n} (-1)^{n-i}(Sd^{n-1}(\pi^n c_u))(U_0, ..., \widehat{U_i}, ..., U_n)|U_u.
\end{equation}

Thus, applying (6.52) we get

\begin{equation}
(Sd^n \circ \pi^n c)(U_0, ..., U_n) = \sum_{i=0}^{n} (-1)^{n-i}Sd^{n-1} \circ \pi^{n-1}(c^u_n)(U_0, ..., \widehat{U_i}, ..., U_n)|U_u.
\end{equation}

By applying the hypothesis of recurrence, we get

\begin{equation}
(Sd^n \circ \pi^n c)(U_0, ..., U_n) = (-1)^n \sum_{i=0}^{n} (-1)^ic^u_n(U_0, ..., \widehat{U_i}, ..., U_n)|U_u
+ \sum_{i=0}^{n} (-1)^{n+1-i}D^i_K \circ d(c^u_n)(U_0, ..., \widehat{U_i}, ..., U_n)|U_u
+ \sum_{i=0}^{n} (-1)^{n+1-i}D^i_K \circ d(c^u_n(U_0, ..., \widehat{U_i}, ..., U_n)|U_u.
\end{equation}
The last sum is zero due to $d \circ d = 0$, and the first sum is $(-1)^n d(c^n_u)$, therefore
\begin{align*}
(6.6) \quad (Sd^n \circ \pi^n(c))(U_0, ..., U_n) &= (-1)^n d(c^n_u)(U_0, ..., U_n) \\
&\quad + \sum_{i=0}^{n} (-1)^{n+1-i} D_K^n \circ d(c^n_u)(U_0, ..., \widehat{U_i}, ..., U_n)|U_u
\end{align*}

As we obtained a formula for $d(c_u^n)$, we obtain a formula for $d(c_u^n)|U_u$. In fact, writing $U_{n+1} = \pi(U_u)$,
\[(dc)_u^n(U_0, ..., U_n)|U_u = dc(U_0, ..., U_n, \pi(U_u))|U_u \]
\[= \sum_{j=0}^{n+1} (-1)^j c(U_0, ..., \widehat{U}_j, ..., U_n)|U_u \]
\[= d(c_u^n)(U_0, ..., U_n)|U_u + (-1)^{n+1} c(U_0, ..., U_n)|U_u;\]

Then, replacing $d(c_u^n)|U_u$ by $(dc)_u^n + (-1)^n c$ in the formula (6.55), we get
\begin{align*}
(6.57) \quad (Sd^n \circ \pi^n(c))(U_0, ..., U_n) &= c(U_0, ..., U_n) + (-1)^n (dc)_u^n(U_0, ..., U_n) \\
&\quad + (-1)^{n+1} d \circ D_K^n \circ (dc)_u^n(U_0, ..., \widehat{U}_i, ..., U_n)|U_u - d \circ D_K^n(c)(U_0, ..., U_n).
\end{align*}

Assuming that we have defined $D_K^n$ on $C^n(K(U); \mathcal{F})$, we define $D_{K}^{n+1}$ on $C^{n+1}(K(U); \mathcal{F})$ by the following formula:
\begin{align*}
(6.58) \quad D_{K}^{n+1}(c') &= (-1)^{n+1}(c')_u^n + (-1)^{n+1} dD_K^n(c')_u^n.
\end{align*}

This gives the awaited result.

3) To finish the proof of the theorem, we have to demonstrate that the composition $\pi^* \circ Sd^*$ is homotopic to the identity of $C^\bullet(N(U); \mathcal{F})$. For that, we construct a sequence of homomorphisms,
\begin{align*}
(6.59) \quad D_{N}^{n+1}: C^{n+1}(N(U); \mathcal{F}) &\to C^n(N(U); \mathcal{F}),
\end{align*}

by recurrence over the integer $n \geq 0$, such that
\begin{align*}
(6.60) \quad Id - \pi^* \circ Sd^n = d \circ D_{N}^n + D_{N}^{n+1} \circ d.
\end{align*}

For $n = 0$, and $c \in C^1(N(U); \mathcal{F})$, we define $D_{N}^1(c)(V_0) = c(\pi(V_0), V_0)$. Remember that if $c$ is a zero cochain for $N(U)$, $Sd^0 c(U_0) = c(U_0)$. Then
\begin{align*}
(6.61) \quad \pi^0 Sd^0 c(V_0) &= c(\pi(V_0)) = c(V_0) + (c(\pi(V_0)) - c(V_0)) = c(V_0) - D_{N}^1(c)(V_0);
\end{align*}

which gives $c - \pi^0 Sd^0 c = D_{N}^1(c)$ as desired.

Now assume the recurrence hypothesis, that there exist operators $D_{K}^{q+1}$ for $q \leq n$, satisfying the homotopy relation for $Id - \pi^q \circ Sd^q$, and consider a $n$-cochain $c$ of $N(U)$ with respect to the local system $\mathcal{F}$; for every decreasing sequence $v = (V_0, ..., V_n)$ in $\mathcal{A}_U$, we have
\begin{align*}
(\pi^n \circ Sd^n(c))(v) &= (Sd^n)(\pi(V_0), ..., \pi(V_n))|V_n \\
&= \sum_{i=0}^{n} (-1)^{n-i} Sd^{n-1}(\pi(V_0), ..., \pi(V_i), ..., \pi(V_n))|V_n \\
&\quad + \sum_{i=0}^{n} (-1)^{n-i} \pi^{n-1}(Sd^{n-1}(c)(\pi(v)))(V_0, ..., \widehat{V}_i, ..., V_n)|V_n;
\end{align*}
which gives by applying the hypothesis of recurrence:

\[
(\pi^n \circ Sd^n(c))(V_0, ..., V_n) = \sum_{i=0}^{n} (-1)^{n-i} c_{\pi(v)}(V_0, ..., \hat{V}_i, ..., V_n)|V_n
\]

\[
+ \sum_{i=0}^{n} (-1)^{n+1-i} D_N^{-1}(d_{\pi(v)})(V_0, ..., \hat{V}_i, ..., V_n)|V_n
\]

\[
+ \sum_{i=0}^{n} (-1)^{n+1-i} d \circ D_N^{-1}(c_{\pi(v)})(V_0, ..., \hat{V}_i, ..., V_n)|V_n.
\]

The last sum is zero due to \(d \circ d = 0\), the first one is equal to \((-1)^n d(c_{\pi(v)})(v)\), and the second one to \((-1)^{n+1} d(D_N^{-1}(d_{\pi(v)})(v))\), that is

\[
(\pi^n \circ Sd^n(c))(v) = (-1)^n d(c_{\pi(v)}(v)) + (-1)^{n+1} d(D_N^{-1}(d_{\pi(v)})(v)).
\]

But the relation \((6.45)\) tells that

\[
d(c_{\pi(v)})(v) = (dc)_{\pi(v)}(v) + (-1)^c(v).
\]

Thus by substituting, we get

\[
(\pi^n \circ Sd^n(c))(v) = c(v) + (-1)^n (dc)_{\pi(v)}(v)
\]

\[
- d(D_N^n(c)(v)) - (-1)^n d(D_N^n(d(c_{\pi(v)}))(v));
\]

which gives the expected result,

\[
c(v) - (\pi^n \circ Sd^n(c))(v) = d(D_N^n(c)(v)) + D_N^{n+1}(dc)(v)_{\pi(v)}(v);
\]

if we define, for any \(c' \in C^{n+1}(N(U); F)\) and any \(v\) in \(N_n(U)\):

\[
D_N^{n+1}(c')(v) = (-1)^{n+1} c'_{\pi(v)}(v) + (-1)^n d D_N^n(c'_{\pi(v)})(v).
\]

This ends the proof. \(\square\)

The constructions made in the proof show that the homotopy equivalence is natural in the category of open covering of topological spaces and morphisms of local systems.

**Appendix A. Topology and sheaves**

Remind that a **topological space** is a set \(X\), equipped with a subset \(T\) of the set of parts \(\mathcal{P}(X)\)—named its **topology**—that is supposed to contain \(X\) and the empty set \(\emptyset\), and to be closed under union and finite intersection. A map \(f : X \to Y\) between topological spaces is said **continuous** if the inverse image of an open set is an open set. A topology \(T\) is said **finer** than a topology \(T'\) if the identity is continuous from \(X_T\) to \(X_{T'}\). It is equivalent to ask that \(T' \subseteq T\) as elements of \(\mathcal{P}(\mathcal{P}(X))\).

An **open covering** of an open set \(V \in T\) is a subset \(U \subseteq T\) such that \(V = \bigcup_{U \in U} U\).

A topology \(T\) can be seen as a category, whose objects are the open sets of \(X\) (i.e. the elements of \(T\)); whenever \(U \subseteq V\), there is one \(U \to V\). The resulting category \(T\) is a poset (see Section 3).

A **presheaf** \(\mathcal{F}\) over a topological space \(X\) is a contravariant functor from \(T\) to the category of sets \(\mathcal{E}\), i.e. a family of sets \(\{F(U) = F_U\}_{U \in T}\), and maps \(\{\pi_{U} V\}_{(U \to V) \in T}\) such that \(\pi_{UU} = Id_{F(U)}\) and \(\pi_{WV} \circ \pi_{UV} = \pi_{WU}\) when \(W \subseteq V \subseteq U\). Frequently
we will note \( \pi_{V}(s) = s|V \), as a restriction. Sometimes, the elements \( s \) of \( F_{U} \) are named sections of \( F \) over \( U \).

A sheaf is a presheaf which satisfies the two following axioms:

1. For every \( V \in \mathcal{T} \) and every open covering \( U \subseteq \mathcal{T} \) of \( V \), if \( s, t \) are two elements of \( F_{V} \) such that for any \( U \in \mathcal{U} \) we have \( s|U = t|U \), then \( t = s \).
2. For every \( V \in \mathcal{T} \) and every open covering \( U \subseteq \mathcal{T} \) of \( V \), if a family \( (s_{U})_{U \in \mathcal{U}} \in \prod_{U \in \mathcal{U}} F_{U} \) is such that for all \( U, U' \in \mathcal{U} \), \( s_{U}((U \cap U') = s_{U'}((U \cap U') \), then there exists \( s \in F_{V} \) such that for all \( U \in \mathcal{U} \), \( s|U = s_{U} \).

The notion of presheaf extends to any category \( \mathcal{C} \) in place of \( \mathcal{E} \); just take a contravariant functor from \( T \) to \( \mathcal{C} \). However the definition of sheaf requires a priori that \( \mathcal{C} \) is a sub-category of \( \mathcal{E} \).

One of the main theorems in sheaf theory is the existence of a canonical sheaf \( \mathcal{F}^{\sim} \) associated to a presheaf \( \mathcal{F} \) on \((X, \mathcal{T})\), built as follows \[22\text{, Sec. II.5}\]. One says \( s \in F_{U} \) and \( t \in F_{V} \) have the same germ at \( x \) if there exists \( W \subseteq U \cap V \) such that \( s|W = t|W \). Having the same germ at \( x \) is an equivalence relation and one denotes \( \text{germ}_{x} s \) the corresponding equivalence class. More precisely, one can describe the set of all germs as a colimit \( \lim_{\gamma \in \mathcal{U}} F_{U} \) over all open neighborhoods of \( U \); the resulting set \( \mathcal{F}_{x} \) is called the stalk of \( \mathcal{F} \) at \( x \). Set \( \Lambda F = \prod_{x \in X} \mathcal{F}_{x} \), and introduce the obvious projection \( p : \Lambda F \to X \). Any \( s \in F_{U} \) determines a map \( \dot{s} : U \to \Lambda F \), \( x \mapsto (x, \text{germ}_{x} s) \), which is a section of \( p \). The set \( \Lambda F \) is topologized introducing \( \{ \dot{s}(U) \mid U \in \mathcal{T}, s \in F_{U} \} \) as a basis of open sets. Then \( \mathcal{F}^{\sim} \) is defined as the sheaf of (continuous) sections of \( \Lambda F \) over the opens of \( X \). This means that an element of \( \mathcal{F}^{\sim}(U) \) is a family \( (s_{x}) \in \prod_{x \in U} \mathcal{F}_{x} \) which is locally a germ of \( \mathcal{F} \); for all \( y \in U \), there exist \( V \in \mathcal{T} \) and \( t \in F_{V} \) such that \( y \in V \subseteq U \) and for all \( x \in V \), \( \text{germ}_{x} t = s_{x} \). The map \( s \mapsto \dot{s} \) defines a natural transformation \( F \to \mathcal{F}^{\sim} \), which is an isomorphism when \( F \) is a sheaf.

We consider now the functoriality of sheaves. Let \( f : X \to Y \) be a continuous map; it induces a functor \( f^{-1} : \mathcal{T}_{Y} \to \mathcal{T}_{X} \) between the topologies (seen as categories) of \( Y \) and \( X \), respectively.

1. If \( \mathcal{F} \) is a presheaf over \( X \), the direct image \( f_{*} \mathcal{F} \) is defined on \( Y \) by the formula: \( f_{*} \mathcal{F}(V) = \mathcal{F}(f^{-1}(V)) \). If \( \mathcal{F} \) is a sheaf, this is also the case for \( f_{*} \mathcal{F} \)[22]. (In fact, \( f_{*} \mathcal{F} \) is also the pullback \((f^{-1})^{*} F \) of \( F \) under the functor \( f^{-1} \) according to \[3\text{, Sec. I.5}\].)
2. If \( \mathcal{G} \) is a presheaf over \( Y \), the inverse image \( f^{-1} \mathcal{G} \) is defined on \( X \) by the formula: \( f^{-1} \mathcal{G}(U) = \lim_{V \supseteq f(U)} \mathcal{G}(V) \), where the limit is taken over the directed family of opens subsets \( V \) of \( Y \) which contain \( f(U) \). Even if \( \mathcal{F} \) is a sheaf, in general \( f^{-1} \mathcal{G} \) is not a sheaf. We make use of the sheafification, and define the pullback of \( \mathcal{G} \) by \( f^{*} \mathcal{G} = (f^{-1} \mathcal{G})^{\sim} \).

The functors \( f_{*}, f^{-1} \) between the corresponding categories of presheaves are adjoint i.e. for any presheaves \( \mathcal{F} \) on \( X \) and \( \mathcal{G} \) on \( Y \), there exist natural bijections

\[
\text{Hom}_{X}(f^{-1} \mathcal{G}, \mathcal{F}) \cong \text{Hom}_{Y}(\mathcal{G}, f_{*} \mathcal{F}).
\]

Similarly, \( f^{*} \) is left adjoint to \( f_{*} \) in the categories of sheaves.

**Definition A.1.** A map of presheaves (resp. sheaves) from \((X, \mathcal{F})\) to \((Y, \mathcal{G})\) is a pair \((f, \varphi)\), where \( f : X \to Y \) is continuous, and \( \varphi \) is a morphism from \( \mathcal{G} \) to \( f_{*} \mathcal{F} \), or equivalently a morphism \( \varphi^{*} \) from \( f^{-1} \mathcal{G} \) (resp. \( f^{*} \mathcal{G} \)) to \( \mathcal{F} \).
Appendix B. Čech cohomology

We summarize some facts concerning Čech cohomology.

B.1. Limit over coverings. A preorder is a set \( P \) with a binary relation that is transitive and reflexive. Equivalently, is a small category \( \mathcal{P} \) where there exists at most one arrow between two objects. The preorder \( \mathcal{P} \) is called directed if for any objects \( a \) and \( b \) of \( \mathcal{P} \), there exists an object \( c \) such that \( a \to c \) and \( b \to c \).

As we saw in Section 6.2, a covering \( U \) is called a refinement of another covering \( U' \) when every set of \( U \) is contained in some set of \( U' \). In that case, there exists a map \( \lambda : U \to U' \), called projection, such that for every \( U \in \mathcal{U} \) one has \( U \subseteq \lambda(U) \). It is also said that \( U \) is finer than \( U' \).

This notion of refinement does not give in general a partial ordering among coverings, but only a pre-order. So it is unlike the notion of finer topology, which corresponds to the natural partial ordering by inclusion of subsets. This can be illustrated with two coverings of \( \mathbb{R} \), such that \( U = \{n, \infty| \ n \text{ even}\} \) and \( U' = \{n, \infty| \ n \text{ odd}\} \).

Lemma B.1. The category of open coverings of \( X \), such that \( U \to U' \) if \( U' \) refines \( U \), is a directed set.

Proof. If \( U \) and \( V \) are open coverings of \( X \), the set of non-empty intersections \( U \cap V \), for \( U \in \mathcal{U} \) and \( V \in \mathcal{V} \) is a refinement of both \( U \) and \( V \). \( \square \)

Given a directed set \( \mathcal{P} \), a directed system of sets (associated to \( \mathcal{P} \)) is a covariant functor from \( \mathcal{P} \) to a category \( \mathcal{C} \), i.e. a family of objects \( E_a \) for \( a \in \mathcal{P} \), and a family \( f_{ab} \) of morphisms \( E_a \to E_b \), associated to ordered pairs \( a \leq b \), such that for all \( a \), \( f_{aa} = 1_{E_a} \), and for all \( a, b, c \), \( a \leq b \leq c \Rightarrow f_{ac} = f_{bc} \circ f_{ab} \).

By definition a direct limit of such a direct system in the category \( \mathcal{C} \) is an object \( E \) with a set of morphisms \( E_a \to E, a \in \mathcal{P} \), such that for any \( a \leq b \) \( f_{ab} \circ f_{ab} = \varphi_a \), which is initial, i.e. for any object \( Y \) and set \( \psi_a : E_a \to Y \) verifying the same rule there exist a morphism \( h : E \to Y \) making all evident diagrams commutative. If such a limit exists it is unique up to unique isomorphism, and denoted \( \varinjlim E_a \).

When \( \mathcal{C} \) is the category of sets \( \mathcal{E} \), the direct limit always exists, it is the quotient of the union of the disjoint sets \( \tilde{E}_a = E_a \times \{a\} \) by the equivalence relation \( e_a \approx e_b \) if there exist \( c \in \mathcal{C} \), with \( a \preceq c, b \preceq c \) and \( f_{ac}(e_a) = f_{bc}(e_b) \), i.e. asymptotic equality. If the category of \( \mathcal{C} \) is the subcategory of \( \mathcal{E} \) made by abelian groups and their morphisms, the direct limit is an abelian group.

Definition B.2. For all \( n \in \mathbb{N} \), \( H^n(X; F) = \varinjlim H^n(U; F) \), the direct limit being associated to the directed set of open coverings of \( X \).

B.2. Functoriality. Suppose given a map of presheaves \( (f, \varphi) : (X, \mathcal{F}) \to (Y, \mathcal{G}) \), and two open coverings \( \mathcal{U}, \mathcal{V} \) of \( X \) and \( Y \) respectively, such that \( \mathcal{U} \) is a refinement of \( f^{-1}(\mathcal{V}) \).

We can choose a projection map \( \lambda \) from \( \mathcal{U} \) to \( \mathcal{V} \), i.e. \( \forall U \in \mathcal{U}, U \subseteq f^{-1}(\lambda(V)) \).

From Proposition 6.10, two such maps are homotopic in the simplicial sense. This induces a natural application of chain complexes:

\[ (f, \varphi, \lambda)^* : C^*(\mathcal{V}; \mathcal{G}) \to C^*(\mathcal{U}; \mathcal{F}), \]

which commutes with the coboundary operators.

Consider the particular case of an inclusion \( J : X \to Y \). A covering \( \mathcal{V} \) of \( Y \) induce a covering \( \mathcal{U} \) of \( X \), made of the (non-empty) intersections \( V \cap X \) for \( V \in \mathcal{V} \);
there is an evident projection $\lambda$ from $\mathcal{U}$ to $\mathcal{V}$.

**Hypothesis:** the map $\varphi$ is surjective, i.e. for any open set $V$ in $Y$ the map $\varphi|_V : \mathcal{G}(V) \to \mathcal{F}(V \cap X)$ is surjective.

In particular this happens if $\mathcal{G} = J_* (\mathcal{F})$ over $Y$.

If $c \in C^n(\mathcal{U}; \mathcal{F})$, there exists $\tilde{c} \in C^n(\mathcal{V}; \mathcal{G})$ such that, for any family $V_0, \ldots, V_n$ of elements of $\mathcal{V}$, we have

$$\varphi(\tilde{c}(V_0, \ldots, V_n)) = c(V_0 \cap X, \ldots, V_n \cap X) \in \mathcal{F}(\bigcap_{i=0}^n (V_i \cap X) = \varphi(\bigcap_{i=0}^n (V_i)).$$

This gives $\tilde{f}_\lambda^* (\tilde{c}) = c$, then the map $\tilde{f}_\lambda^* = (f, \varphi, \lambda)^*$ is surjective.

Let us denote by $D_\alpha$ the dimension of $S_\alpha$, for $\alpha \in \mathcal{A}$. We have $N_\alpha = \prod_{i \in \alpha} N_i$.

If we suppose that $\mathcal{A}$ satisfies the strong intersection property, the sheaf $V$ has an interaction decomposition:

$$\forall \alpha \in \mathcal{A}, \quad V_\alpha = \bigoplus_{\beta \subseteq \alpha} S_\beta,$$

Let us denote by $D_\alpha$ the dimension of $S_\alpha$, for $\alpha \in \mathcal{A}$. We have $N_\alpha = \sum_{\alpha \to \beta} D_\beta$. Then the Möbius inversion formula gives

$$\forall \alpha \in \mathcal{A}, \quad D_\alpha = \sum_{\alpha \to \beta} \mu_{\alpha, \beta} N_\beta;$$

where the integral numbers $\mu_{\alpha, \beta}$ are the Möbius coefficients of $\mathcal{A}$.

Let us remind what are these coefficients [32], Black 2015. For any locally finite poset $\mathcal{A}$, they are defined by the two following equations:

$$\forall \alpha, \gamma, \quad \delta_{\alpha = \gamma} = \sum_{\beta: \alpha \to \beta \to \gamma} \mu_{\alpha, \beta} = \sum_{\alpha \to \beta \to \gamma} \mu_{\beta, \gamma}. $$

$$\forall \alpha, \beta, \quad \alpha \to \beta \Rightarrow \mu_{\alpha, \beta} = 0.$$

This gives a function from $\mathcal{A} \times \mathcal{A}$ to $\mathbb{Z}$, which is named the Möbius function of the poset. The Möbius function of $\mathcal{A}^{op}$ is given by $\mu^*_{\beta, \alpha} = \mu_{\alpha, \beta}$. 

**Appendix C. Finite Probability Functors**

This is a continuation of Section 5 on free sheaves. Our aim is to give a proof of the Theorem 5.16.

We introduce now the hypothesis that the $\{E_i\}_{i \in I}$ are finite sets, of respective cardinality $N_i$.

If $N_\alpha$ denotes the cardinality of $E_\alpha$, we have $N_\alpha = \prod_{i \in \alpha} N_i$.
For example, if $\mathcal{A}$ is the full set of parts of a finite set $I$, including the empty set or not, we have, for $\beta \subseteq \alpha$:

\[(C.5) \quad \mu_{\alpha,\beta} = (-1)^{|\alpha|-|\beta|},\]

where $|\alpha|$ denotes the cardinality of $\alpha$, for any $\alpha \in \mathcal{A}$. This formula is called the inclusion-exclusion principle. When $\beta = \emptyset$, the above formula holds true if we pose $|\emptyset| = -1$.

If $\alpha \supseteq \omega$ are two elements of $\mathcal{A}$, and if $\mathcal{A}(\alpha,\omega)$ is the sub-poset of $\mathcal{A}$ made by the elements $\beta$ such that $\alpha \rightarrow \beta \rightarrow \omega$, the restriction of the M"obius function of $\mathcal{A}$ to $\mathcal{A}(\alpha,\omega)$ coincides with the M"obius function of $\mathcal{A}(\alpha,\omega)$.

The formula (C.5) extends to the poset associated to any simplicial complex. This follows from the preceding assertion, because in the case of a manifold, for every pair of elements $\alpha,\omega$ of $\mathcal{A}$ such that $\omega \subseteq \alpha$, the elements $\beta$ between $\alpha$ and $\omega$ are the same in $\mathcal{A}$ or in the simplex defined by $\alpha$.

If $\mathcal{A}$ verifies the strong intersection property, for each $\alpha \in \mathcal{A}$, the dimension of $H^0(\mathcal{U}_\mathcal{A}; S_\alpha)$ is $D_\alpha$, then Theorem 5.7 and Proposition 5.8 imply:

**Proposition C.1.** If the poset $\mathcal{A}$ is finite and satisfies the strong intersection property, and if the $\{E_i\}_{i \in I}$ are finite sets,

\[(C.6) \quad \dim_k H^0(\mathcal{U}_\mathcal{A}; V) = \sum_{\alpha,\beta \in \mathcal{A}} \mu_{\alpha,\beta} N_\beta.\]

In particular for the full simplex $\Delta(n-1) = \mathcal{P}(J)$, if $J$ has cardinality $n$, and if $N_i = N$ for any vertex, the dimension of $H^0(\mathcal{A}; V)$ is $N^n$.

**Proof.** Since we include the empty set, with $V_\emptyset = S_\emptyset$ of dimension 1, we get:

\[
\sum_{\alpha,\beta \in \mathcal{A}} \mu_{\alpha,\beta} N_\beta = \sum_{k=0}^{n} C_n^k \sum_{l=0}^{k} C_l^k (-1)^{k-l} N^l \\
= \sum_{k=0}^{n} (-1)^k C_n^k (1-N)^k = \sum_{k=0}^{n} C_n^k (N-1)^k \\
= (N-1+1)^n = N^n.
\]

**Remark C.2.** In this case, if we remove the empty set, and compute the expression we get the same result

\[
\sum_{\alpha,\beta \in \mathcal{A}} \mu_{\alpha,\beta} N_\beta = \sum_{k=1}^{n} C_n^k \sum_{l=1}^{k} C_l^k (-1)^{k-l} N^l = \sum_{k=1}^{n} (-1)^k C_n^k ((1-N)^k - 1) \\
= \sum_{k=1}^{n} C_n^k (N-1)^k - \sum_{k=1}^{n} C_n^k (-1)^k \\
= ((N-1+1)^n - 1) - ((1-1)^n - 1) = N^n.
\]

Let us now delete the maximal face $\alpha = I$, then the poset $\mathcal{A}$ becomes the boundary $\partial \Delta(n-1)$ of the $(n-1)$-simplex. If we include the empty set in $\mathcal{A}$, and
compute the dimension of $H^0$; we obtain

$$\sum_{\alpha, \beta \in A^+} \mu_{\alpha \beta} N_\beta = \sum_{k=0}^{n-1} C_n^k \sum_{l=0}^k (-1)^{k-l} N^l$$

$$= \sum_{k=0}^{n-1} (-1)^k C_n^k (1 - N)^k = \sum_{k=0}^{n-1} C_n^k (N - 1)^k$$

$$= (N - 1 + 1)^n - (N - 1)^n$$

$$= N^n - (N - 1)^n.$$

**Remark C.3.** Now the expression $\sum_{\alpha, \beta \in A} \mu_{\alpha \beta} N_\beta$ is not the same if we exclude $\emptyset$, because in this case, we have

$$\sum_{\alpha, \beta \in A} \mu_{\alpha \beta} N_\beta = \sum_{k=1}^{n-1} C_n^k \sum_{l=1}^k (-1)^{k-l} N^l$$

$$= \sum_{k=1}^{n-1} (-1)^k C_n^k ((1 - N)^k - 1) = \sum_{k=1}^{n-1} C_n^k (N - 1)^k - \sum_{k=1}^{n-1} C_n^k (-1)^k$$

$$= ((N - 1 + 1)^n - 1 - (N - 1)^n) - ((1 - 1)^n - 1 - (-1)^n)$$

$$= N^n - (N - 1)^n + (-1)^n.$$

We will see just below why there is a difference for the boundary $\partial \Delta(n - 1)$ and not for the simplex $\Delta(n - 1)!$.

If $A$ satisfies the weak intersection property, then

(C.7) \[ \forall \alpha \in A, \quad \overline{V}_\alpha = \bigoplus_{\beta \leq \alpha} S_\beta. \]

**Proposition C.4.** If the poset $A$ is finite and satisfies the weak intersection property, and if the \( \{E_i\}_{i \in I} \) are finite sets,

(C.8) \[ \dim K^0(U_A; \overline{V}) = \sum_{\alpha, \beta \in A} \mu_{\alpha \beta} (N_\beta - 1). \]

**Proof.** We apply Proposition 5.9 as we applied Proposition 6.8 to prove Proposition C.1. \qed

In disguise, this result is known under the name of the Marginal Theorem of H.G. Kellerer [19] (see also F. Matúš [24]).

**Definition C.5.** Let $A$ be a finite poset, the *Euler characteristic of $A$* is defined by

(C.9) \[ \chi(A) = \sum_{\alpha, \beta \in A} \mu_{\alpha \beta}. \]

In fact, the Euler characteristic was defined by Rota [22], when $A$ contains a maximal element $I$ and a minimal element $\emptyset$, by the formula

(C.10) \[ E(A) = 1 + \mu_{I, \emptyset}. \]

But take any finite poset $A$, and add formally to $A$ a maximal element 1 and a minimal element 0, obtaining a poset $A^+$. Then, for any $\alpha \in A$,

(C.11) \[ 0 = \mu(\alpha, 0) + \sum_{\beta \in A, \beta \leq \alpha} \mu(\alpha, \beta). \]
EXTRA-FINE SHEAVES AND INTERACTION DECOMPOSITIONS 43

and

\[(C.12) \quad 0 = \mu(1,0) + \mu(0,0) + \sum_{\alpha \in \mathcal{A}} \mu(\alpha,0).\]

Consequently

\[(C.13) \quad \chi(\mathcal{A}) = \mu(1,0) + \mu(0,0) = E(\mathcal{A}^+).\]

Therefore the two definitions accord. See also the categorical extension of these ideas by Tom Leinster [21].

The Hall formula (cf. [32]), tells that

\[(C.14) \quad E(\mathcal{A}) = r_0 - r_1 + r_2 - \ldots;\]

where each \(r_k\) is the number of non degenerate chains of length \(k\) in \(\mathcal{A}\). This number is the Euler characteristic of the nerve \(N(\mathcal{A})\) of the category \(\mathcal{A}\), therefore \(\chi(\mathcal{A})\) coincides with the Euler characteristics of \(N(\mathcal{A})\). But we have seen in Section 6 that the Čech cohomology of the (lower) Hausdorff space \(\mathcal{A}\) with coefficients in \(\mathbb{Z}\), is isomorphic to the simplicial cohomology of \(N(\mathcal{A})\). Then \(\chi(\mathcal{A})\) also coincides with the Euler-Čech characteristic of the (lower) Hausdorff space \(\mathcal{A}\). By duality of the Möbius function, this is also true for the upper topology.

From the inclusion-exclusion formula, it is easy to show that for the poset of a simplicial complex, the number \(\chi(\mathcal{A})\) is the alternate sum of the numbers of faces of each dimension:

\[(C.15) \quad \chi(\mathcal{A}) = a_0 - a_1 + \ldots\]

as in the original definition by Euler.

Now consider \(\mathcal{A}\) (finite) as a topological subspace \(\mathcal{A}_t\) of the simplex \(P(I)\); its closure \(\overline{\mathcal{A}}\) is a simplicial complex. Moreover, if \(\mathcal{A}\) satisfies the weak intersection property, the inclusion of \(\mathcal{A}_t\) in \(\overline{\mathcal{A}}\) is an equivalence of homotopy; therefore, in this case, \(\chi(\mathcal{A})\) is also the usual Euler characteristic of the metric space \(\mathcal{A}\).

Consequently, Proposition [C.4] can be rephrased by the following formula

\[(C.16) \quad \dim \ker H^0(\mathcal{U}_A; \mathcal{V}) + \chi(\mathcal{A}) = \sum_{\alpha, \beta \in \mathcal{A}} \mu_{\alpha \beta} N_{\beta}.\]

Applying Theorem 5.7 we get the following result:

**Theorem 5.16.** If the poset \(\mathcal{A}\) is finite and satisfies the weak intersection property, and if the \(\{E_i\}_{i \in I}\) are finite sets, then

\[(C.17) \quad \chi(\mathcal{A}; V) = \sum_{k=0}^{\infty} (-1)^k \dim \ker H^k(\mathcal{U}_A; \mathcal{V}) = \sum_{\alpha, \beta \in \mathcal{A}} \mu_{\alpha \beta} N_{\beta}.\]

Remark that we also have

\[(C.18) \quad \chi(\mathcal{A}; V) = \dim \ker H^0(\mathcal{U}_A; \mathcal{V}) + \chi(\mathcal{A}).\]

In the example of \(\Delta(n-1)\), we have \(\chi(\mathcal{A}) = 1\), and when \(\mathcal{A} = \partial \Delta(n-1)\) we have \(\chi(\mathcal{A}) = 1 + (-1)^n\), therefore, with all the \(N_i\) equals to \(N\), this explains the results obtained in the previous remarks.

The standard marginal problem: when compatible measures of sum 1 over the poset \(\partial \Delta(n-1)\) come from a global measure, corresponds to Proposition [C.4] then the measure always exists but it depends on \((N-1)^n\) degrees of freedom. Moreover, in general none of these measures is positive.
References

1. Samson Abramsky, Rui Soares Barbosa, Kohei Kishida, Raymond Lal, and Shane Mansfield, Contextuality, Cohomology and Paradox, 24th EACSL Annual Conference on Computer Science Logic (CSL 2015) (Dagstuhl, Germany) (Stephan Kreutzer, ed.), Leibniz International Proceedings in Informatics (LIPIcs), vol. 41, Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2015, pp. 211–228.
2. Samson Abramsky and Adam Brandenburger, The sheaf-theoretic structure of non-locality and contextuality, New Journal of Physics 13 (2011), no. 11, 113036.
3. Michael Artin, Alexander Grothendieck, and Jean-Louis Verdier, Théorie des topos et cohomologie étale des schémas: Séminaire de géométrie algébrique du bois-marie 1963/64 - sga 4. Tome 1, Lecture notes in mathematics, Springer-Verlag, 1972.
4. Théorie des topos et cohomologie étale des schémas: Séminaire de géométrie algébrique du bois-marie 1963/64 - sga 4. Tome 2, Lecture notes in mathematics, Springer-Verlag, 1972.
5. Jonathan A. Barmak, Algebraic topology of finite topological spaces and applications, Lecture Notes in Mathematics, Springer Berlin Heidelberg, 2011.
6. Pierre Baudot and Daniel Bennequin, The homological nature of entropy, Entropy 17 (2015), no. 5, 3253–3318.
7. Stephane Bressan, Jingyan Li, Shiquan Ren, and Jie Wu, The embedded homology of hypergraphs and applications, Asian Journal of Mathematics 23 (2019), no. 3, 479–500.
8. Erik Carlsson, Gunnar Carlsson, and Vin De Silva, An algebraic topological method for feature identification, International Journal of Computational Geometry & Applications 16 (2006), no. 04, 291–314.
9. FRK Chung and RL Graham, Cohomological aspects of hypergraphs, Transactions of the American Mathematical Society 334 (1992), no. 1, 365–388.
10. Justin Curry, Sheaves, cosheaves and applications, Ph.D. thesis, The University of Pennsylvania, 2013, arXiv:1303.3255.
11. Samuel Eilenberg and Normal Steenrod, Foundations of algebraic topology, Princeton University Press, 1952.
12. Samuel Eilenberg and Joseph A Zilber, Semi-simplicial complexes and singular homology, Annals of Mathematics (1950), 499–513.
13. Joel Friedman, Sheaves on graphs, their homological invariants, and a proof of the hanna neumann conjecture, Memoirs of the American Mathematical Society, American Mathematical Society, 2014.
14. Robert Ghrist, Barcodes: the persistent topology of data, Bulletin of the American Mathematical Society 45 (2008), no. 1, 61–75.
15. Roger Godement, Topologie algébrique et théorie des faisceaux, Actualités scientifiques et industrielles, no. v. 1, Hermann, 1964.
16. Phillip Griffiths and Joseph Harris, Principles of algebraic geometry, Wiley Classics Library, John Wiley & Sons, 1978.
17. Stephen Halperin, Lectures on minimal models, vol. 9, Gauthier-Villars, 1983.
18. Jakob Hansen and Robert Ghrist, Toward a spectral theory of cellular sheaves, arXiv preprint arXiv:1808.01513 (2018).
19. Hans G Kellerer, Masstheoretische marginalprobleme, Mathematische Annalen 153 (1964), no. 3, 168–198.
20. Steffen L Lauritzen, Graphical models, Oxford Science Publications, 1996.
21. Tom Leinster, The Euler characteristic of a category, Documenta Mathematica 13 (2008), 21–49.
22. Saunders Mac Lane and Ieke Moerdijk, Sheaves in geometry and logic: A first introduction to topos theory, Mathematical Sciences Research Institute Publications, Springer-Verlag, New York, USA, 1992.
23. Seyed M-H Mansourbeigi, Sheaf theory as a foundation for heterogeneous data fusion, Ph.D. thesis, Utah State University, 2010.
24. František Matuš, Discrete marginal problem for complex measures, Kybernetika 24 (1988), no. 1, 36–46.
25. J. Peter May, Simplicial objects in algebraic topology, Chicago Lectures in Mathematics, University of Chicago Press, 1992.
26. Ieke Moerdijk, *Classifying spaces and classifying topoi*, Lecture Notes in Mathematics, Springer Berlin Heidelberg, 2006.
27. Olivier Peltre, *A homological approach to belief propagation and bethe approximations*, International Conference on Geometric Science of Information, Springer, 2019, pp. 218–227.
28. Olivier Peltre, *Homology of Message-Passing Algorithms*, [http://opeltre.github.io](http://opeltre.github.io), 2020, Ph.D. thesis (preprint).
29. Michael Robinson, *Understanding networks and their behaviors using sheaf theory*, 2013 IEEE Global Conference on Signal and Information Processing, IEEE, 2013, pp. 911–914.
30. Michael Robinson, *Sheaves are the canonical data structure for sensor integration*, Information Fusion 36 (2017), 208–224.
31. Michael Robinson, Cliff Joslyn, Emilie Hogan, and Chris Capraro, *Conglomeration of heterogeneous content using local topology*, Tech. report, American University, Mar, 2015.
32. Gian-Carlo Rota, *On the foundations of combinatorial theory i. Theory of m"obius functions*, Probability theory and related fields 2 (1964), no. 4, 340–368.
33. Graeme Segal, *Classifying spaces and spectral sequences*, Publications Mathématiques de l’IHÉS 34 (1968), 105–112.
34. Grégoire Sergeant-Perthuis, *Around the intersection property and the interaction decomposition*, 2019.
35. Grégoire Sergeant-Perthuis, *Bayesian/graphoid intersection property for factorisation models*, arXiv preprint arXiv:1903.06026 (2019).
36. Grégoire Sergeant-Perthuis, *Intersection property and interaction decomposition*, arXiv preprint arXiv:1904.09017 (2019).
37. Ya G Sinai, *Theory of phase transitions: rigorous results*, vol. 108, Elsevier, 2014.
38. Edwin H. Spanier, *Algebraic topology*, McGraw-Hill, 1966.
39. Terry P. Speed, *A note on nearest-neighbour gibbs and markov probabilities*, Sankhyā: The Indian Journal of Statistics, Series A (1979), 184–197.
40. Juan Pablo Vigneaux, *Generalized information structures and their cohomology*, arXiv e-prints (2017), arXiv:1709.07807 [cs.IT].
41. Juan Pablo Vigneaux, *Topology of statistical systems: A cohomological approach to information theory*, Ph.D. thesis, Université de Paris, 2019.
42. Nikolai N. Vorob’ev, *Consistent families of measures and their extensions*, Theory of Probability & Its Applications 7 (1962), no. 2, 147–163.
43. Charles A. Weibel, *An introduction to homological algebra*, Cambridge University Press, Printed in USA, 1994.
44. John HC Whitehead, *Combinatorial homotopy. I*, Bulletin of the American Mathematical Society 55 (1949), no. 3, 213–245.
45. Jonathan S Yedidia, William T Freeman, and Yair Weiss, *Constructing free-energy approximations and generalized belief propagation algorithms*, IEEE Transactions on information theory 51 (2005), no. 7, 2282–2312.
46. Erik Christopher Zeeman, *Dihomology: I. relations between homology theories*, Proceedings of the London Mathematical Society 3 (1962), no. 1, 609–638.