SOLITON-LIKE SOLUTIONS OF THE GRAD-SHAFRANOV EQUATION

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(Dated: March 20, 2022)

Abstract

A new class of soliton-like solutions is derived for the Grad-Shafranov (GS) equations. A mathematical analogy between the GS equation for MHD equilibria and the cubic Schrödinger (CS) equation for non-linear wave propagation forms the basis to derive the new class of solutions. The soliton-like solutions are considered for their possible relevance to astrophysics and solar physics problems. We discuss how a soliton-like solution can be generated by a repetitive process of magnetic arcade stretching and plasmoid formation induced by the differential rotation of the solar photosphere or of an accretion disk.

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We present a new class of soliton-like solutions of the Grad-Shafranov equation. The Grad-Shafranov (GS) equation governs the equilibrium conditions of a magnetic field embedded in a plasma.

Countless applications rely on the GS equation to determine the equilibrium steady state condition [1]. The equilibrium field in magnetic fusion devices is computed with the GS equation. In the present letter, the attention focuses more on problems where the GS equation can be of importance for space and astrophysical systems.

In the solar corona, field lines emerging from the photosphere form arcades whose evolution is crucial in solar flares and coronal mass ejections [2]. Similarly, magnetic fields tied to an accretion disk rotating around a massive object (such as a star or a black hole) can form magnetic arcades [3]. In both cases, photospheric motions or the differential rotation of the accretion disk cause a shear of the footpoints of the magnetic arcade. The shear motion, in turn, injects toroidal magnetic flux and causes the expansion of the arcade away from the photosphere or from the accretion disk.

In this scenario, the role of the GS equation has been traditionally tied to the understanding of the expansion of the magnetic arcade. A series of papers have investigated the expansion phase using self-similar solutions of the GS equation, proving that the field expands away from a differentially rotating surface at a characteristic angle of 60 degrees [4]. However, the applicability of the GS equation to this scenario is intrinsically limited by two factors. First, the GS neglects inertia effects that can be important in determining the expansion of the arcade. Secondly, the presence of dissipation processes (such as anomalous resistivity or kinetic effects) can lead to reconnection processes that alter the topology of the expanding arcade. Several papers have investigated this process proving that reconnection can lead to the formation and ejection of a plasmoid, i.e. a blob of plasma encircled by field lines at least partially detached from the original footpoints. In the solar case such plasmoid can be interpreted as a coronal mass ejection.

Recently it has been pointed out that the process of arcade expansion and plasmoid ejection can repeat itself in time at the same location [5]. There is observational evidence to support this claim. In the solar case, it has been observed that flares can repeat at the same location on a cycle of several hours to few days [5]. In the case of accretion disks, the repetitive ejections of plasma "bullets" has been observed for SS433 [6] and jets emitted from accretion disks often present a knotty structure that suggests the presence of multiple
islands. In the present letter, we discuss a new model of such repetitive series of plasmoids. The model interprets such observational occurrences in the mathematical framework of soliton theory. A jet of plasma composed by a series of plasmoids is represented mathematically as a soliton being propagated from the source (the photosphere in the solar case and the accretion disk for the astrophysical case).

Below, we show that a mathematical analogy exists between the GS equation and the cubic Schrödinger (CS) equation typical of the soliton theory. Such partial analogy is used to find a class of exact solutions of the GS equation that have the same mathematical structure of the soliton solutions of the CS equation. Finally we discuss the likelihood of such solutions existing in practice.

**Grad-Shafranov Equation:** Magnetic equilibria are computed using the GS equation. The GS equation is derived straightforwardly from the momentum equation assuming equilibrium conditions and neglecting inertia:

\[ \mathbf{J} \times \mathbf{B} = \nabla p \] (1)

Assuming a Cartesian geometry with axis \((x, y, z)\), we assume all quantities to depend only on \((x, y)\) and assume the third dimension to be a symmetry direction. In most of the applications, axisymmetric coordinates would be a more appropriate choice, but not within the scope of the present work. The use of the axisymmetric coordinates would obscure the mathematical equivalence that we want to prove between the GS equation and the CS equation. The extension of the derivations below to axisymmetric coordinates will be the topic of a future paper.

With the geometry chosen above, a solution of eq. (1) can be found in the form:

\[ \mathbf{B} = \hat{z} \times \nabla \Psi + B_z \hat{z} \] (2)

where \(\hat{z}\) is the unit vector in the ignorable direction \(z\). From the magnetic field given in eq. (2), the current follows immediately:

\[ \mathbf{J} = \nabla \times \mathbf{B} = \nabla B_z \times \hat{z} + \nabla^2 \Psi \hat{z} \] (3)

Substituting the expression for the current and the field, the balance eq. (1) assumes a new form:

\[ (\nabla B_z \times \hat{z}) \cdot \nabla \Psi \hat{z} = \nabla p + \nabla^2 \Psi \nabla \Psi + B_z \nabla B_z \] (4)
From this rather involved equation, the GS equation follows, noting that the left hand side is directed along the $z$ direction and the right hand side is directed on the $(x, y)$ plane. It follows that the two sides of the equations must both be zero for the complete equation to be satisfied:

\[
\begin{align*}
(\nabla \Psi \times \nabla B_z) \cdot \hat{z} &= 0 \\
\nabla p + \nabla^2 \Psi \nabla \Psi + B_z \nabla B_z &= 0
\end{align*}
\]  

\[ (5) \]

where the left hand side of eq. (4) has been manipulated into a more convenient form.

The two equations must be both satisfied to obtain a complete equilibrium. Note that the first equation simply states that the gradients of $B_z$ and of $\Psi$ must be parallel. This requirement is typically enforced by requiring that $B_z$ is itself a function of $\Psi$ (the same being also true for $p$):

\[
\begin{align*}
p &= g(\Psi), \quad B_z = f(\Psi) \\
\nabla^2 \Psi &= -g' - ff'
\end{align*}
\]

\[ (6) \]

where the prime represent derivative with respect to $\Psi$. Equation (6) is the classic form of the GS equation.

Typically it is convenient to work on the classic form (6) of the GS equation obtained above, but we will show that it has a limitation: it does not allow to consider the extension of $\Psi$ in the complex plane. Often in mathematical physics, it can be advantageous to extend an equation in the complex plane to obtain its solutions, returning to the real axis after the solution is obtained. To obtain the class of soliton-like solutions derived below, that is indeed the procedure to follow. To that end, the original form (5) of the GS equations is to be preferred, as the independent variables $(x, y)$ remain real even when the dependent variable $\Psi$ is extended in the complex plane. In the classic form of the GS equation, the extension of $\Psi$ to the complex plane would entice the consideration of complex functions of complex variable. The use of the original form (5) requires only the consideration of complex functions of real variables.

**Analogies for the Grad-Shafranov Equation:** When studying the GS equation, one is confronted with two possible analogies. One is well known and has been often used in previous works: the analogy with the Helmholtz problem of mathematical physics. The other is new and will be presented here for the first time: the analogy with the CS equation.
Eq. (6) when considered in the traditional way as an equation for a real function of a real variable \( \Psi(x, y) \) is an application of the Helmholtz problem:

\[
\triangle \Upsilon = \lambda \Upsilon \quad (7)
\]

for the eigenvalue \( \lambda \) and the eigenfunction \( \Upsilon \).

In studies of the GS equation applied to space and astrophysics systems, the analogy with the Helmholtz problem \( \ref{eq:helmholtz} \) has been very fruitful in studying arcade expansion \cite{4}. However, all solutions provided by the analogy with the Helmholtz problem do not include the presence of reconnection. As noted above, reconnection is believed to be present as suggested by simulation and supported by observation. To arrive to a model that incorporates the effects of reconnection and the presence of plasmoids, we need to abandon the analogy with the Helmholtz equation and propose a new one: the analogy with the CS equation.

The inductive process that leads from the Helmholtz problem to the CS equation is typical of the discipline of nonlinear propagation in optical waveguides. Wave propagation in optical waveguides is governed by a Helmholtz equation for the electromagnetic field \cite{10}:

\[
\triangle \Psi = -k^2 n^2 \Psi \quad (8)
\]

where \( n \) is the refractive index of the media, \( k \) the wavenumber and \( \Psi \) is a complex function, interpreted physically as a polarization component of the electromagnetic field. In non-homogeneous media \( n \) is a function of space and in non-linear optical materials it is a function of the electric field amplitude: \( n = n(|\Psi|) \). In the most common case of the Kerr effect \cite{10}, the dependence of the refractive index upon the amplitude of the field is quadratic: \( n = n_0 + n_2 |\Psi|^2 \). Following the classic textbook derivation \cite{10}, we assume a 2D system (in analogy with the system under consideration for the GS equation) and we rewrite the unknown function using a leading harmonic dependence of the field along the propagating direction (that we assume to be \( y \) in the present case):

\[
\Psi(x, y) = \varphi(x, y)e^{-jkny} \quad (9)
\]

where \( j \) is the imaginary unit. Absolute generality is still provided by retaining the most general dependence of the factor \( \varphi(x, y) \). For the new unknown function \( \varphi \), the complex Helmholtz equation \( \ref{eq:helmholtz} \) yields:

\[
\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} - 2jk\varphi_0 \frac{\partial \varphi}{\partial y} + k^2 (n^2 - n_0^2) \varphi = 0 \quad (10)
\]
Traditionally, a further approximation is made, assuming \((n^2 - n_0^2) \approx 2n_0(n - n_0)\) and requiring the applicability of the paraxial (or Fresnel) approximation that prescribes \(|\frac{\partial^2 \varphi}{\partial y^2}| \ll 2kn_0|\frac{\partial \varphi}{\partial y}|\). The final equation typically used to study light propagation in nonlinear media becomes:

\[
\frac{\partial \varphi}{\partial y} = -j \frac{1}{2kn_0} \frac{\partial^2 \varphi}{\partial x^2} - jkn_2|\varphi|^2 \varphi \tag{11}
\]

none others than the cubic Schrödinger (CS) equation \([8]\). A well known theoretical result is that the CS equation above admits a soliton solution of the form \([8, 10]\):

\[
\varphi = \varphi_p \text{sech} \left( \frac{x}{y_0} \right) \exp \left( \frac{-jy}{2n_0y_0^2} \right) \tag{12}
\]

where \(y_0\) is a free parameter and \(|\varphi_p|^2 = 2/n_0n_2y_0^2\).

The formal analogy of the original Helmholtz equation in the complex plane used to study light propagation with the GS equation allows to derive a new class of soliton-like solution similar to eq. (refsolitonCS).

**Soliton solutions of the Grad-Shafranov Equation:** To use the analogy with the CS equation outlined above, we consider the GS equation in its original form \([5]\) and assume the following choice of the free functions \(B_z\) and \(p\):

\[
B_z \nabla B_z = \alpha_0^2 \Psi \nabla \Psi
\]

\[
\nabla p = \alpha_0^2 |\Psi|^2 \Psi \nabla \Psi \tag{13}
\]

The GS equation \([5]\) is extended in the complex plane and complex functions \(\Psi\) of real variables \((x, y)\) are sought as solutions. The choice above \([13]\) is a possible and legitimate choice of the free functions in the GS equation. Although at first sight it might appear arbitrary, the choice selected above has a simple physical meaning: it ensures that \(\nabla B_z^2/\nabla p\) is independent of the \(y\) coordinate along which the soliton-like solution propagates. With this choice the first of the GS eq. \([5]\) is automatically satisfied and the second becomes identical to eq.\([8]\) with a non-linear refractive index of the form prescribed by the Kerr effect:

\[
\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = -\alpha_0^2(1 + |\Psi|^2)\varphi \tag{14}
\]

A soliton solution can be readily obtained in the form:

\[
\Psi(x, y) = \Psi_p \text{sech}(x/L)e^{-j(\alpha_0+1/2n_0y_0^2)y} \tag{15}
\]
FIG. 1: Contour plot of the real part of the flux function, corresponding to the section of the magnetic surfaces on the \((x, y)\) plane.

with \(L = (1/y_0^2 + 1/4\alpha_0^2 y_0^4)^{-1/2}\) and \(\Psi_p = \sqrt{2/\alpha_0 L}\).

Note that the soliton solution (15) is exact and no paraxial approximation has been made in the present case, since the exact solution can be found even for the exact GS equation and the physical significance of the paraxial approximation is peculiar to optical waveguide propagation, and of no meaning in the case of the GS equation.

Figure 1 shows the contour lines of the real part of the flux function \(\Psi\), physically such contour lines correspond to the section of the magnetic surfaces on the \((x, y)\) plane. The field lines belong on the flux surfaces but have a third component away from the plane and given by \(B_z\):

\[ B_z = \alpha_0 \Psi \]  \hspace{1cm} (16)

The magnetic field topology presents a series of magnetic islands piled one above the other. Physically, this solution can be considered the end result of a repetitive series of processes of arcade stretching and plasmoid ejections that compose a jet of plasma and magnetic field emitted from a differentially rotating surface (such as the photosphere or an accretion disk as described above).

It is interesting to observe that the primary balance of force in the present case is between the magnetic tension of the field lines and the plasma pressure. Using the solution (15), the initial choice (13) for the free functions in the GS equation yield an equation for the pressure perturbation:

\[ \Delta \delta p = \nabla \cdot \left( \alpha_0^2 |\Psi|^2 \psi \nabla \psi \right) \]  \hspace{1cm} (17)
FIG. 2: Dependence on $x$ of the pressure (solid) and of the flux function (dashed), $\alpha_0 = 1$, $y_0 = 1$.

In astrophysical systems, it is reasonable to assume a background pressure that is being perturbed by the magnetic structure [3]. Eq. (17) can be readily solved observing that the $y$ dependence of $\delta p$ is harmonic:

$$\delta p = \chi(x)e^{-2J/2(\alpha_0 + 1/2\alpha_0 y_0^2)}y$$

(18)

Substituting into eq. (17), a simple ordinary equation is obtained for the $x$-dependent factor $\chi$ of the pressure perturbation:

$$\frac{d^2 \chi}{dx^2} - 4 \left( \alpha_0 + \frac{1}{2\alpha_0 y_0^2} \right)^2 \chi = - \left( \frac{\text{sech}^2(x/L)}{4\alpha_0^4 y_0^3 L^2} \right)^2 \cdot$$

(19)

$$\left( 4 \left( \alpha_0 y_0^2 + \frac{1}{L^2} \right) + \left( 4\alpha_0 y_0^2 - \frac{1}{L^2} \right) \cosh(2x/L) \right)$$

that is solved numerically.

Figure 2 shows the $x$ dependence of the pressure $\chi$. The $y$ dependence is harmonic and is given by eq. (18).

Existence in Nature of soliton-like solutions: The solution derived above is an exact and legitimate solution of the GS equation. It has been derived in the complex plane and to obtain a physically viable solution one has simply to take either the real or the imaginary part.

Once we have shown the mathematical existence of soliton-like solutions, we need to consider the possibility that in practice they can be identified as realistic descriptions of processes observed in nature or generated in laboratory. Here we propose that soliton-like solutions can explain the observed structure of astrophysics jets. Astrophysics jets are observed to be emitted from rotating accretion disks. The dynamo effect inside the accretion disks generate a magnetic field that is then emitted with the flow of the jet [3]. Observations
show that often jets present a so called "knotty" structure where the jet is composed by a number of islands (knots). This feature suggests the possibility to model the magnetic structure of jets using multiple islands solutions of the Grad-Shafranov equation. While the solution-like solutions are by no means the only solutions presenting multiple islands (e.g. for another example), they present the right qualitative behaviour observed in astrophysics jets. Two issues need to be considered to propose credibly the applicability of soliton-like solutions.

The first question is whether a soliton-like solution is stable. Previous studies have considered the stability of equilibria composed by periodic magnetic islands, proposing the existence of the coalescence instability. For sufficiently large aspect ratio islands, i.e. for islands stretched along the axis of the jet (the $y$ axis of the solution above) the instability is greatly reduced allowing the soliton-like solution to remain stable for long times (that observationally turn into long distances, recalling that the jets are moving at relativistic speeds). One can indeed propose that the eventual onset of this instability can explain why the jets after remaining collimated for kiloparsec distances eventually destabilize in large magnetic clouds.

The second question is whether a soliton-like solution can be created by a disk. To answer this question one should do a simulation of the accretion disk and of the jets emitted from them. A complete treatment is extremely challenging and beyond the scope of the present work (if at all possible). Yet limited studies that focus on just a part of the problem are possible and have been performed. The creation of repetitive islands similar to the islands featured by the soliton-like solution have indeed been observed in simulations of a problem closely related to jet formation: the study of plasma emitted from the solar corona. And indeed the ejection of multiple magnetic islands has been observed for the jet from SS433.

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