First and Second Order Phase Transitions in Maxwell–Chern-Simons Theory Coupled to Fermions *

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Abstract

In the Maxwell–Chern-Simons theory coupled to \( N_f \) flavors of 4-component fermions (or even number of 2-component fermions) we construct the gauge-covariant effective potential written in terms of two order parameters which are able to probe the breakdown of chiral symmetry and parity. In the absence of the bare Chern-Simons term, we show that the chiral symmetry is spontaneously broken for fermion flavors \( N_f \) below a certain finite critical number \( N_f^c \), while the parity is not broken spontaneously. This chiral phase transition is of the second order. In the presence of the bare Chern-Simons term, on the other hand, the chiral phase transition associated with the spontaneous breaking of chiral symmetry is shown to continue to exist, although the parity is explicitly broken. However it is shown that the existence of the bare Chern-Simons term changes the order of the chiral transition into the first order, no matter how small the bare Chern-Simons coefficient may be. This gauge-invariant result is consistent with that recently obtained by the Schwinger-Dyson equation in the non-local gauge.

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1 Introduction

The Chern-Simons term \cite{1, 2, 3} generates a gauge-invariant mass for the gauge field, combined with the ordinary Maxwell term. Formally, in the Chern-Simons theory which does not include the kinetic term for the gauge field, the gauge sector of the theory is strictly renormalizable, although the Maxwell theory is superrenormalizable in 2+1 space-time dimensions. The Chern-Simons term affects the long-distance behavior of the theory and the perturbative expansion of Chern-Simons theory contains logarithmic divergences \cite{4, 5, 6}. However, the topological nature of the theory should allow only trivial, finite renormalization \cite{4}.

Actually it is verified that this happens in the various renormalization scheme. The massive matter fields may have a finite one-loop radiative correction to the statistical parameter $\theta$, the coefficient of the Chern-Simons term. It is generally known as the Coleman-Hill nonrenormalization theorem \cite{7} that the topological mass term at one-loop order is the only one radiative correction and there are no radiative corrections from higher-order loops to the topological mass term. Therefore the coefficient of the topological mass term at one-loop order is exact to all higher orders. The requirement of this nonrenormalization theorem is the gauge invariance and analyticity of the one-loop $n$-point functions.

Indeed, from the explicit consideration of the Feynman graph, the following results are obtained. When there is no topological mass at the tree level, no topological mass terms are induced radiatively to the two-loop order in either Abelian or non-Abelian gauge theories with massive fermions in 2+1 space-time dimensions \cite{8}. Moreover, if there is a nonzero tree-level topological mass, the radiative correction to the topological mass is shown to vanish in Abelian gauge theory coupled to the massive fermion \cite{9}. However it should be remarked that the existence of massless charged particles or spontaneous gauge symmetry breaking violates the initial assumptions of the Coleman-Hill theorem and higher loop renormalizations of the Chern-Simons term appear \cite{10, 11, 12}.

The nonrenormalization theorem in the weak coupling expansion can be extended to the large flavor ($N_f$) case, i.e., to the $1/N_f$ expansion: there is no radiative corrections to the Chern-Simons coefficient $\theta$ in the $1/N$ expansion \cite{13}. This argument do not apply to the case in which the fermions remain massless due to absence of the analyticity. In the Abelian case, the nonrenormalization theorem to all orders is also shown by using the effective potential \cite{14}. It will be interesting to extend these analyses to the non-Abelian case \cite{14} such as QCD3 \cite{15, 16}.

In this paper we adopt the $1/N_f$ expansion as a systematic truncation scheme. The $1/N_f$ expansion is quite interesting from the viewpoint of the non-perturbative study, since a class of theories with a four-fermion interaction is shown to be renormalizable in the framework of $1/N_f$ expansion \cite{18} in spite of its nonrenormalizability in the weak coupling expansion. The Chern-Simons term may affect the high-energy behavior of the gauged four-fermion model \cite{13, 19} and the pattern of dynamical generation for the fermion mass, although the four-fermion interaction is not investigated in this paper.
In this paper we regard the dimensional coupling constant $e$, the electric charge, as a genuine dimensional parameter that sets the natural scale of the theory. This viewpoint has been taken by Appelquist et al. \[20]\ and several authors. The dynamically generated fermion will have a mass $m_d$ which is proportional to this scale: $m_d \sim e^2$. The fermion mass is generated when $1/N_f > 1/N_c^2$ where $N_f$ is called the critical number of fermion flavors. Another way of viewing the dimensional parameter is to take it as the ultraviolet (UV) cutoff of the theory where $\Lambda$ is the UV cutoff and $g = e^2/\Lambda$ is a dimensionless coupling \[21\]. In this case, the only dimensional parameter in the model is the UV cutoff. In this standpoint, if the dynamical mass is generated, it will be independent of the UV cutoff $\Lambda$. This leads to the non-perturbative renormalization for $g$, i.e., the coupling is running, i.e., $\beta(g) := \Lambda \frac{dg}{d\Lambda} \neq 0$. Therefore, the first viewpoint is not compatible with the second one and two viewpoints are in many ways different from each other.

The Maxwell-Chern-Simons theory \[1, 2, 3, 4, 5, 6\] with $N_f$ flavors of Dirac fermions in (2+1)-dimensions is defined by the following euclidean Lagrangean density:

$$
\mathcal{L} = \frac{\beta}{4} F_{\mu\nu}^2 + \frac{i\theta}{2} \epsilon_{\mu\nu\rho} A_\mu \partial_\nu A_\rho + \mathcal{L}_{GF} + \bar{\Psi}^\alpha (i\gamma_\mu \partial_\mu - m_e - m_{\sigma T}) \Psi^\alpha + e \bar{\Psi}^\alpha \gamma_\mu \Psi^\alpha A_\mu,
$$

where the index $\alpha$ denotes the fermion flavors ($\alpha = 1, ..., N_f$) and $\mathcal{L}_{GF}$ is the gauge-fixing term specified in the next section. In the above Lagrangean density we have introduced an interpolating parameter $\beta$ between the Maxwell ($\beta = 1, \theta = 0$) and the Chern-Simons ($\beta = 0, \theta \neq 0$) theories. Our analysis based on the effective potential in this paper can be applied equally to the case of $\beta = 0$ as well as $\beta \neq 0$. The (2+1)-dimensional QED (QED3) with a Chern-Simons term is a special case of this model. For the Chern-Simons theory, the term $\frac{\beta}{4} F_{\mu\nu}^2$ can be interpreted as a regularization which is removed finally, i.e. $\beta \to 0$ in this case.

We start from the 4-component formulation for fermions in (2+1)-dimensions \[24\]. Each 4-component fermion $\Psi$ is decomposed into two 2-component fermions $\psi_1, \psi_2$ as

$$
\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},
$$

Then the Dirac conjugate $\bar{\Psi} := \Psi^\dagger \gamma_0$ is written as

$$
\bar{\Psi} = (\bar{\psi}_1, -\bar{\psi}_2)
$$

by using $\bar{\psi}_a := \psi_a^\dagger \sigma_3 (a = 1, 2)$, see Appendix A. Accordingly the gamma matrices satisfying the Clifford algebra $\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}$ are written by $4 \times 4$ matrices. In this reducible representation, there exist 4 $4 \times 4$ matrices anticommuting with all

---

1. We use $\Psi$ without the index $\alpha$ to denote each 4-component fermion.

2. In what follows we use the notation $A := B$ in the sense that $A$ is defined by $B$, while $A \equiv B$ implies that $A$ is identically equal to $B$.
the gamma matrices, say $\gamma_3, \gamma_5$, in sharp contrast to the irreducible representation corresponding to 2-component fermions \[^{20}\]. Then we can define the chiral symmetry similarly as in the (3+1)-dimensional case:

$$\Psi \rightarrow \exp(i\vartheta \gamma_3)\Psi, \quad \bar{\Psi} \rightarrow \bar{\Psi} \exp(i\vartheta \gamma_5),$$

(1.4)

under which $\bar{\Psi} i\gamma_{\mu} \partial_{\mu} \Psi$ is invariant. Moreover, we can define the parity transformation:

$$\begin{align*}
\Psi(x_0, \vec{x}) & \rightarrow P\Psi(x_0, \vec{x}_P), \\
\bar{\Psi}(x_0, \vec{x}) & \rightarrow \bar{\Psi}(x_0, \vec{x}_P)P, \\
A_{\mu}(x_0, \vec{x}) & \rightarrow (-1)^{\delta^{\mu}_{1}} A_{\mu}(x_0, \vec{x}_P),
\end{align*}$$

(1.5)

where $\vec{x}_P = (-x_1, x_2)$ for the spacial coordinate $\vec{x} = (x_1, x_2)$ and $P = -i\gamma_5\gamma_1$. The parity transformation \(^{1.5\)} is equivalent to $\psi_1(x_0, \vec{x}) \rightarrow \sigma_1\psi_2(x_0, \vec{x}_P)$ and $\psi_2(x_0, \vec{x}) \rightarrow \sigma_1\psi_1(x_0, \vec{x}_P)$, for 2-component fermions. The transformation property of each term in the Lagrangian under two transformations is summarized as follows.

| Transformation property | chiral | parity |
|-------------------------|--------|--------|
| $\Psi \Psi$             | odd    | even   |
| $\Psi \tau \Psi$        | even   | odd    |
| $\epsilon_{\mu\nu\rho}A_{\mu} \partial_{\nu} A_{\rho}$ | even   | odd    |

The order parameter for chiral symmetry breaking is defined by

$$\langle \bar{\Psi} \Psi \rangle \equiv \langle \bar{\psi}_1 \psi_1 \rangle - \langle \bar{\psi}_2 \psi_2 \rangle,$$

(1.6)

while the order parameter for parity breaking is defined by

$$\langle \bar{\Psi} \tau \Psi \rangle \equiv \langle \bar{\psi}_1 \psi_1 \rangle + \langle \bar{\psi}_2 \psi_2 \rangle.$$

(1.7)

According to analytical and numerical studies \(^{22}\) of the Schwinger-Dyson equation for QED3 with a Chern-Simons term ($\beta = 1$, $\theta \neq 0$, $m_e = m_o = 0$), we can write the schematic phase diagram for the Maxwell-Chern-Simons theory with $N_f$ flavors of 4-component fermions, see Fig. 1. First note that, when $\theta \neq 0$, the parity is explicitly broken, $\langle \bar{\Psi} \tau \Psi \rangle \neq 0$. The solid line in Fig. 1 denotes the critical line of chiral phase transition below which the chiral symmetry is spontaneously broken, $\langle \bar{\Psi} \Psi \rangle \neq 0$. In Fig. 1 we have chosen typical five points, assuming that the critical number of flavors $N_f^c$ is nonzero and finite: $0 < N_f^c < \infty$. The points A and E in Fig. 1 are in the chiral symmetric phase; the point A is in the parity even phase, while E is in the parity odd phase. The point D is located just on the critical line. The points B and C are in the chiral-symmetry-breaking phase; the point B is in the parity even phase, while C is in the parity odd phase. For the order of phase transition, it is well known that the point $(N_f, \theta) = (N_f^c, 0)$ exhibits the
continuous second order transition through the analyses of the Schwinger-Dyson equation \[23\].

By using the Schwinger-Dyson equation, the critical number of flavors \(N_f\) has been evaluated: \(N_f^c = \frac{24}{3\pi^2} \cong 3.2\) first in the Landau gauge by Appelquist, Nash and Wijewardhana \[23\], and \(N_f^c = \frac{128}{3\pi^2} \cong 4.3\) in the heuristic treatment of the non-local gauge by Nakatani \[24\] and \(N_f^c \cong 3 \sim 4\) by Nash \[23\]. These results are confirmed by the systematic investigation of the non-local gauge by Kondo et al. \[21\] and by Kondo and Maris \[22\]. On the other hand, infinity of \(N_f\), \(N_f = \infty\), was claimed by \[27, 28\] and subsequent works \[29, 30\]. Different approximations to the vertex function and the existence of infrared cutoff may lead to different results, see \[31\] and \[32\]. Such a controversy exists \[33, 34\] also in the approach of the Cornwall-Jackiw-Tomboulis (CJT) effective potential \[35\]. The Monte Carlo simulation of non-compact lattice QED3 shows \(N_f = 3.5 \pm 0.5\) according to Dagotto, Kogut and Kocic \[36\]. The result are controversial up to now. This shows that it is extremely difficult to find a consistent truncation scheme in the Schwinger-Dyson approach which is able to deduce the gauge-invariant result. For more details on the Schwinger-Dyson equation of QED3, see for example a review \[37\].

In the absence of the bare Chern-Simons term, the fermion mass is dynamically generated for \(N_f < N_f^c\) in such a way that the parity is preserved, i.e., \(\langle \bar{\Psi} \Psi \rangle \neq 0\) and \(\langle \bar{\Psi} \tau \Psi \rangle = 0\). This is rephrased in terms of the fermion dynamical mass \(m_1, m_2\) for two 2-component fermions, \(\psi_1, \psi_2\), respectively. Here the chiral symmetry breaking, \(\langle \bar{\Psi} \Psi \rangle \neq 0\), implies \(m_1 \neq m_2\) and no parity breaking, \(\langle \bar{\Psi} \tau \Psi \rangle = 0\), implies more specific pattern for the fermion dynamical mass:

\[
\begin{array}{cccccccc}
  m, m, \ldots, m, & -m, -m, \ldots, -m \\
  \hline
  N_f & N_f
\end{array}
\]

This result is consistent with various analyses so far \[38, 39, 40, 41, 42, 26\] as well as the general consideration by Vafa and Witten \[43\].

A new and remarkable feature discovered in \[22\] is that the order of chiral phase transition turns into the first order when \(\theta \neq 0\). Namely the chiral phase transition is of the first order on the whole critical line except for one point \((N_f, \theta) = (N_f^c, 0)\) in the phase diagram, which is in sharp contrast to the previous analysis \[14\]. The meaning of the first order transition is explained from the behavior of dynamically generated fermion mass as shown in Fig. 2. In Fig. 2, fermion masses \(m_1, m_2\) are shown as different functions of \(\theta\) when \(N_f < N_f^c\). At the point B, \(\theta = 0\) and fermion masses for two 2-component fermions are generated in such a way that the parity is preserved: \(m_1 = -m_2 > 0\) (we choose a convention such that \(m_1 > 0\) without loss of generality) and the chiral symmetry is spontaneously broken, \(\langle \bar{\Psi} \Psi \rangle \neq 0\). When \(\theta \neq 0\), we see this pattern for mass generation is destroyed so that \(|m_1| \neq |m_2|\) \((m_1 > 0, m_2 < 0)\). As \(\theta\) is increased while \(N_f\) is kept fixed, each \(m_a (a = 1, 2)\) increases monotonically. However, at \(\theta = \theta_c\), \(m_2(\theta)\) jumps discontinuously towards \(m_1(\theta)\). For \(\theta > \theta_c\), it is realized that \(m_1(\theta) = m_2(\theta)\), which implies restoration of the chiral symmetry: \(\langle \bar{\Psi} \Psi \rangle = 0\). In this sense the chiral phase transition is of the first order when \(\theta \neq 0\).
However this result is available only in a specific gauge. In the Schwinger-Dyson equation approach \cite{22}, the non-local gauge was adopted in order to avoid the vertex correction. A possible way to improve the gauge-dependence in the Schwinger-Dyson equation approach is to improve the vertex so as to be consistent with the Ward-Takahashi identity. However this way is not unique.

The purpose of this paper is to show the existence of the first order phase transition in the presence of the bare Chern-Simons term as well as the second order transition in the absence of the bare Chern-Simons term in the gauge-invariant way. For this, we construct the gauge-covariant (gauge-parameter independent) effective potential in terms of two order parameters: \( \phi = \langle \bar{\Psi} \Psi \rangle \) and \( \chi = \langle \bar{\Psi} \tau \Psi \rangle \), up to the leading order of \( 1/N_f \) expansion. To construct the effective potential, we adopt the inversion method \cite{45, 46}.

This paper is organized as follows. In section 2, we show how the gauge-covariant result is obtained for the gauge-invariant order parameters through the evaluation of the effective potential in the scheme of \( 1/N_f \) expansion. In section 3, after introducing two external sources which trigger the spontaneous breaking of the chiral symmetry and parity, we show how explicitly the order parameter is related to the vacuum polarization of the photon propagator. In section 4, we observe the dependence of the order parameter on two external sources in detail. Sections 2 to 4 are preparations for performing the inversion. Now, in section 5, we state a general strategy of obtaining the non-trivial solutions through the inversion method in the presence of two external sources. In section 6, as a special case we study the system with a single order parameter and in a slightly simplified situation we show that the chiral symmetry is spontaneously broken in the three-dimensional gauge theory, e.g., QED3 by using the inversion method. In section 7, we give the argument on the behavior of the effective potential near the stationary point in the absence of the Chern-Simons term. The stationary point corresponds to the non-trivial solution (order parameter) corresponding to the spontaneous breakdown of the relevant symmetry. In section 8, we study the effect of the Chern-Simons term on the chiral phase transition by extending the effective potential given in section 7. The final section is devoted to conclusion and discussion. In Appendix A, we give a formulation of 4-component fermion in (2+1)-dimensions. In Appendix B, we derive the gauge covariant formula in the \( 1/N_f \) expansion given in section 2 for the gauge-invariant order parameter. In Appendix C, the vacuum polarization tensor in three-dimensional gauge theory is obtained together with its power-series expansion in the external source.

2 Order parameter in the \( 1/N_f \) expansion

In this paper we take the covariant gauge-fixing:

\[
\mathcal{L}_{GF} = \frac{1}{2\xi}(\partial_{\mu}A_\mu)^2. \tag{2.1}
\]
From the Lagrangian (1.1) with the gauge-fixing term (2.1), it is easy to see that the inverse of the bare photon propagator reads

\[
D^{(0)}_{\mu\nu}^{-1}(k) = \beta k^2 \left( \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) + \frac{1}{\xi} k_\mu k_\nu + \theta \epsilon_{\mu\nu\rho} k_\rho,
\]

and hence the bare photon propagator is given by

\[
D^{(0)}_{\mu\nu}(k) = \beta k^2 \left( \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) + \frac{1}{\xi} k_\mu k_\nu + \theta \epsilon_{\mu\nu\rho} k_\rho, \tag{2.3}
\]

where we have used \(\epsilon_{012} = 1\) and \(\epsilon_{\mu\nu\lambda} \epsilon_{\rho\sigma\lambda} = \delta_{\mu\rho} \delta_{\nu\sigma} - \delta_{\mu\sigma} \delta_{\nu\rho}\).

The general form of the vacuum polarization tensor in three-dimensional gauge theory is written as

\[
\Pi_{\mu\nu}(k) = \left[ \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right] \Pi_T(k) + \epsilon_{\mu\nu\rho} \frac{k_\rho}{\sqrt{k^2}} \Pi_O(k). \tag{2.4}
\]

The full photon propagator \(D_{\mu\nu}(k)\) is related to the vacuum polarization tensor \(\Pi_{\mu\nu}(k)\) and the bare photon propagator \(D^{(0)}_{\mu\nu}(k)\) through the Schwinger-Dyson equation:

\[
D_{\mu\nu}^{-1}(k) = D^{(0)}_{\mu\nu}^{-1}(k) - \Pi_{\mu\nu}(k). \tag{2.5}
\]

Therefore the full photon propagator is written in the form

\[
D_{\mu\nu}(k) = \left[ \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right] D_T(k) + \frac{k_\mu k_\nu}{k^2} D_L(k) + \epsilon_{\mu\nu\rho} \frac{k_\rho}{\sqrt{k^2}} D_O(k), \tag{2.6}
\]

with

\[
D_T(k) = \frac{\beta k^2 - \Pi_T(k)}{[\beta k^2 - \Pi_T(k)]^2 + [\theta \sqrt{k^2} - \Pi_O(k)]^2},
\]

\[
D_O(k) = \frac{\theta \sqrt{k^2} - \Pi_O(k)}{[\beta k^2 - \Pi_T(k)]^2 + [\theta \sqrt{k^2} - \Pi_O(k)]^2},
\]

\[
D_L(k) = \frac{\xi}{k^2}. \tag{2.7}
\]

In order to calculate the order parameter, we introduce the source term

\[
\mathcal{L}_J = J_\epsilon \bar{\Psi}^\alpha(x) \Psi^\alpha(x) + J_o \bar{\Psi}^\alpha(x) \tau \Psi^\alpha(x). \tag{2.8}
\]

Then the partition function in the presence of the source is defined by

\[
Z[J_\epsilon, J_o] = \int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \mathcal{D}A_\mu \exp \left[ - \int d^3x (\mathcal{L} + \mathcal{L}_J) \right], \tag{2.9}
\]

and the generating functional is given by

\[
W[J_\epsilon, J_o] = -\ln Z[J_\epsilon, J_o]. \tag{2.10}
\]
The order parameter is obtained by differentiating the generating functional with respect to the corresponding source. For the translation-invariant order parameter, we obtain

\[ \langle \bar{\Psi}^\alpha \Psi^\alpha \rangle = \frac{1}{\Omega} \frac{\delta}{\delta J_\alpha} W[J_\alpha, J_\alpha] \bigg|_{J_\alpha = J_\alpha = 0}, \]

\[ \langle \bar{\Psi}^\alpha \tau^\alpha \rangle = \frac{1}{\Omega} \frac{\delta}{\delta J_\alpha} W[J_\alpha, J_\alpha] \bigg|_{J_\alpha = J_\alpha = 0}, \] (2.11)

where \( \Omega = \int d^3 x \) is the space-time volume. The generating functional is given by the sum of all the bubble diagrams which is depicted in Fig. 3 up to the leading order of \( 1/N_f \) expansion. Separating the bare (interaction free) part with the superscript \( (0) \), the order parameter which we calculate can be written in the form (see Appendix B):

\[ \langle \bar{\Psi} \Psi \rangle := \frac{\langle \bar{\Psi}^\alpha \Psi^\alpha \rangle}{N_f} = \langle \bar{\Psi} \Psi \rangle^{(0)} + \frac{1}{2} N_f^{-1} \int \frac{d^3 k}{(2\pi)^3} D^{(1)}_{\mu\nu}(k) \frac{\partial}{\partial J_\mu} \Pi^{(1)}_{\mu\nu}(k), \]

\[ \langle \bar{\Psi} \tau \rangle := \frac{\langle \bar{\Psi}^\alpha \tau^\alpha \rangle}{N_f} = \langle \bar{\Psi} \tau \rangle^{(0)} + \frac{1}{2} N_f^{-1} \int \frac{d^3 k}{(2\pi)^3} D^{(1)}_{\mu\nu}(k) \frac{\partial}{\partial J_\mu} \Pi^{(1)}_{\mu\nu}(k). \] (2.12)

Here \( \Pi^{(m)}_{\mu\nu} \) denotes the vacuum polarization in which all the radiative corrections up to \( m \)-fermion loops are included, and \( D^{(n)}_{\mu\nu} \) denotes the photon propagator which is constructed from Eq. (2.5) by using \( \Pi^{(n)}_{\mu\nu} \). We use the convention: \( \Pi^{(0)}_{\mu\nu}(k) \equiv 0 \equiv \Pi^{(0)}_O(k) \) and \( D^{(0)}_{\mu\nu} \) is the bare photon propagator.

To carry out the calculation in the \( 1/N_f \) scheme, therefore, we need to calculate

\[ \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} D^{(n)}_{\mu\nu}(k) \frac{\partial}{\partial J} \Pi^{(m)}_{\mu\nu}(k), \]

\[ = \int \frac{d^3 k}{(2\pi)^3} \left[ D^{(n)}_{\mu\nu}(k) \frac{\partial}{\partial J} \Pi^{(m)}_{\mu\nu}(k) + D^{(n)}_O(k) \frac{\partial}{\partial J} \Pi^{(m)}_O(k) \right], \]

\[ = \int \frac{d^3 k}{(2\pi)^3} \frac{[\beta k^2 - \Pi^{(m)}_{\mu\nu}(k)] \frac{\partial}{\partial J} \Pi^{(m)}_{\mu\nu}(k) + [\theta \sqrt{k^2} - \Pi^{(m)}_O(k)] \frac{\partial}{\partial J} \Pi^{(m)}_O(k)}{[\beta k^2 - \Pi^{(n)}_{\mu\nu}(k)]^2 + [\theta \sqrt{k^2} - \Pi^{(n)}_O(k)]^2}, \] (2.13)

where we have used \( \epsilon_{\mu\nu\alpha} \epsilon_{\mu\nu\beta} = 2\delta_{\alpha\beta} \).

Up to the leading order of \( 1/N_f \) expansion, we have only to consider the case: \( n = 1, m = 1 \), which is shown in Appendix B. This is a novel observation in this paper. This should be compared with the usual perturbation theory with respect to the coupling constant \( e^2 \). In this case, the order of the graph is counted with respect to the coupling, \( e^2 \). In the lowest order of the perturbation series with respect to the coupling constant, we need to choose \( n = 0, m = 1 \) as is well known in the four-dimensional case \( D = 4 \). It should be remarked that the expression (2.13) is free from the gauge-fixing parameter \( \xi \). Therefore, if we calculate the gauge-invariant order parameter according to this expression, we will get the gauge-covariant (i.e. gauge-parameter independent) result.
This should be compared with the Schwinger-Dyson equation approach in which a specific gauge has to be chosen even in calculating the gauge-invariant quantity. The Schwinger-Dyson equation approach does not in general guarantee that the gauge-invariant quantity calculated in one gauge coincides with that calculated in other gauges.

Even in the approach of the effective potential, a special gauge is usually taken such as the Landau gauge, see [12]. Note that if we are allowed to change the order of calculating the graph or equivalently performing the integration over the internal momenta, we get (see Fig. 3(b))

\[
\int \frac{d^3 k}{(2\pi)^3} D^{(n)}_{\mu \nu}(k) \Pi^{(m)}_{\mu \nu}(k) = \int \frac{d^3 p}{(2\pi)^3} S^{(0)}(p) \Sigma^{(1)}(p).
\]  

(2.14)

In this case, even if we start from the second expression, we should obtain the same result as long as it is finite. However, it is divergent and hence we need to regularize it. The naive cutoff procedure usually taken in the calculation of the effective potential may break the gauge invariance in the sense that this reordering is not allowed and hence the result may be not gauge-covariant. Therefore, if we start from the gauge-noninvariant quantity \( \Sigma(p) \), it is quite difficult to recover the gauge covariance after the calculation.

3 Order parameter and vacuum polarization

We introduce the projection operators

\[
\chi_{\pm} := \frac{1}{2} (1 + \tau), \quad \tau = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},
\]  

(3.1)

which satisfy the properties: \( \chi_{\pm}^2 = \chi_{\pm}, \chi_+ \chi_- = 0 \) and \( \chi_+ + \chi_- = 1 \). Then the source term (2.8) is rewritten as

\[
\mathcal{L}_J = J_+ \bar{\Psi}^\alpha(x) \Psi^\alpha(x) + J_- \bar{\Psi}^\alpha(x) \Psi^\alpha(x),
\]  

(3.2)

where we have defined

\[
J_\pm := J_e \pm J_o,
\]  

(3.3)

or equivalently

\[
J_e = \frac{1}{2} (J_+ + J_-), \quad J_o = \frac{1}{2} (J_+ - J_-).
\]  

(3.4)

3.1 free part

The bare fermion propagator \( S^{(0)}(p) \) is also decomposed as\(^3\)

\[
S^{(0)}(p) = \frac{1}{p + J_e + J_o \tau} = S_+^{(0)}(p) \chi_+ + S_-^{(0)}(p) \chi_-,
\]  

(3.5)

\(^3\)Note that the signature of \( J_e, J_o \) is opposite to that of the bare fermion mass \( m_e, m_o \).
where
\[ S_{\pm}^{(0)}(p) := \frac{1}{p + J_{\pm}}. \] (3.6)

Then the free parts of two order parameters are decomposed as
\[
\langle \bar{\Psi} \Psi \rangle^{(0)} = \langle \bar{\Psi} \Psi \rangle^{(0)}_{+} + \langle \bar{\Psi} \Psi \rangle^{(0)}_{-}, \\
\langle \bar{\Psi} \tau \Psi \rangle^{(0)} = \langle \bar{\Psi} \Psi \rangle^{(0)}_{+} - \langle \bar{\Psi} \Psi \rangle^{(0)}_{-},
\] (3.7)
with
\[
\langle \bar{\Psi} \Psi \rangle^{(0)}_{\pm} = \int \frac{d^3p}{(2\pi)^3} \text{tr}[S_{\pm}^{(0)}(p) \chi_{\pm}].
\] (3.8)

By introducing the UV cutoff \( \Lambda_f \), we have
\[
\langle \bar{\Psi} \Psi \rangle^{(0)}_{\pm} = \int_0^{\Lambda_f} dp \frac{p^2 J_{\pm}}{p^2 + J_{\pm}^2}.
\] (3.9)

Here it should be remarked that the quantity \( \langle \bar{\Psi} \Psi \rangle^{(0)}_{\pm} \) defined by eq. (3.9) has an ambiguity arising from the lower bound of the integration, no infrared cutoff, unless we specify the signature of \( J_{\pm} \). In order to define this quantity unambiguously, we introduce the small and positive infrared cutoff \( \epsilon > 0 \) and then take the limit \( \epsilon \downarrow 0 \). Thus we obtain
\[
\pi^2 \langle \bar{\Psi} \Psi \rangle^{(0)}_{\pm} = \lim_{\epsilon \downarrow 0} \int_{\epsilon}^{\Lambda_f} dp \frac{p^2 J_{\pm}}{p^2 + J_{\pm}^2}
\]
\[
= \lim_{\epsilon \downarrow 0} \left[ J_{\pm}(\Lambda_f - \epsilon) + J_{\pm}^2 \left( \arctan \frac{J_{\pm}}{\Lambda_f} - \arctan \frac{J_{\pm}}{\epsilon} \right) \right]
\]
\[
= J_{\pm} \Lambda_f + J_{\pm}^2 \left( \arctan \frac{J_{\pm}}{\Lambda_f} - \frac{\pi}{2} \text{sgn}(J_{\pm}) \right),
\] (3.10)
where \( \text{sgn}(J_{\pm}) \) denotes the signature of \( J_{\pm} \). Therefore we have
\[
\frac{\langle \bar{\Psi} \Psi \rangle^{(0)}_{\pm}}{\Lambda_f^2} = \frac{1}{\pi^2} \left[ \frac{J_{\pm}}{\Lambda_f} - \text{sgn}(J_{\pm}) \frac{\pi J_{\pm}^2}{2 \Lambda_f^2} + \frac{J_{\pm}^3}{3 \Lambda_f^3} + O(J_{\pm}^3) \right].
\] (3.11)

In what follows, we assume that \( J_{\pm} \) is always positive, \( J_{\pm} > 0 \), without loss of generality. Then, depending on the signature of \( J_{\pm} \), there are two possibilities: \( J_e > J_o \) for \( J_{\pm} > 0 \) and \( J_e < J_o \) for \( J_{\pm} < 0 \). Here it should be remarked that the signature of \( J_e \) and \( J_o \) are not specified.

First we consider the case (I) in which \( J_{\pm} > 0 \), i.e. \( J_e > J_o \). Thus we have
\[
\frac{\langle \bar{\Psi} \Psi \rangle^{(0)}}{\Lambda_f^2} = \frac{1}{\pi^2} \left[ \frac{J_+ + J_-}{\Lambda_f} - \frac{\pi J_+^2 + J_-^2}{2 \Lambda_f^2} + O(J_{\pm}^3) \right]
\]
\[
= \frac{2}{\pi^2} \left[ \frac{J_e}{\Lambda_f} - \frac{\pi J_e^2 + J_o^2}{2 \Lambda_f^2} + O(J_{\pm}^3) \right],
\]
\[
\frac{\langle \bar{\Psi} \tau \Psi \rangle^{(0)}}{\Lambda_f^2} = \frac{1}{\pi^2} \left[ \frac{J_+ - J_-}{\Lambda_f} - \frac{\pi J_+^2 - J_-^2}{2 \Lambda_f^2} + O(J_{\pm}^3) \right]
\]
\[
= \frac{2}{\pi^2} \left[ \frac{J_o}{\Lambda_f} - \frac{\pi 2J_eJ_o}{2 \Lambda_f^2} + O(J_{\pm}^3) \right].
\] (3.12)
In the second case (II) where $J_+ > 0, J_- < 0$, i.e. $J_o > J_e$. we obtain

$$\frac{\langle \bar{\Psi} \Psi \rangle^{(0)}}{\Lambda_f^2} = \frac{1}{\pi^2} \left[ \frac{J_+ + J_-}{\Lambda_f} - \frac{\pi J_o^2}{2} \Lambda_f^3 + \mathcal{O}(J^3) \right]$$

$$= \frac{2}{\pi^2} \left[ \frac{J_e}{\Lambda_f} - \frac{\pi J_e J_o}{2} \Lambda_f^3 + \mathcal{O}(J^3) \right],$$

$$\frac{\langle \bar{\Psi}_T \Psi \rangle^{(0)}}{\Lambda_f^2} = \frac{1}{\pi^2} \left[ \frac{J_+ - J_-}{\Lambda_f} - \frac{\pi J_e^2}{2} \Lambda_f^3 + \mathcal{O}(J^3) \right]$$

$$= \frac{2}{\pi^2} \left[ \frac{J_o}{\Lambda_f} - \frac{\pi J_e^2}{2} \Lambda_f^3 + \mathcal{O}(J^3) \right].$$

(3.13)

### 3.2 vacuum polarization

By using the decomposition (3.5) for $S^{(0)}(p)$, the vacuum polarization tensor at one-loop is also decomposed as

$$\Pi_{\mu\nu}^{(1)}(k) \equiv -N_f e^2 \int \frac{d^3p}{(2\pi)^3} \text{tr}[\gamma_\mu S^{(0)}(p)\gamma_\nu S^{(0)}(p + k)]$$

$$= N_f [\Pi_{\mu\nu}^{(1)}(k; J_+) + \Pi_{\mu\nu}^{(1)}(k; J_-)],$$

(3.14)

where we have defined

$$\Pi_{\mu\nu}^{(1)}(k; J_\pm) \equiv -e^2 \int \frac{d^3p}{(2\pi)^3} \text{tr}[\gamma_\mu S_\pm^{(0)}(p)\gamma_\nu S_\pm^{(0)}(p + k)\chi_{\pm}].$$

(3.15)

For the one-loop vacuum polarization tensor with the tensor structure:

$$\Pi_{\mu\nu}^{(1)}(k) = \left[ \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right] \Pi_T^{(1)}(k) + \epsilon_{\mu\nu\rho} \frac{k_\rho}{\sqrt{k^2}} \Pi_O^{(1)}(k),$$

(3.16)

we find

$$\Pi_T^{(1)}(k) = N_f [\Pi_T^{(1)}(k; J_+) + \Pi_T^{(1)}(k; J_-)],$$

$$\Pi_O^{(1)}(k) = N_f [\Pi_O^{(1)}(k; J_+) - \Pi_O^{(1)}(k; J_-)],$$

(3.17)

since the tensor $\Pi_{\mu\nu}^{(1)}(k; J_\pm)$ should have the same tensor structure as $\Pi_{\mu\nu}^{(1)}(k)$. Note that there exists a minus sign in $\Pi_O^{(1)}(k)$. Actual calculation shows (see Appendix C) that

$$\Pi_T^{(1)}(k; J) = -\frac{e^2}{8\pi} \left[ 2|J| + \frac{k^2 - 4J^2}{k} \arctan \frac{k}{2|J|} \right],$$

(3.18)

and

$$\Pi_O^{(1)}(k; J) = \frac{e^2}{2\pi} J \arctan \frac{k}{2|J|}.$$

(3.19)
It should be remarked that $\Pi_T^{(1)}(k; J)$ is an even function of $J$ and $\Pi_O^{(1)}(k; J)$ is odd in $J$. Therefore, if $J_e = 0$, then $J_\pm = \pm J_o$ and

$$\Pi_T^{(1)}(k) = 2N_f\Pi_T^{(1)}(k; J_o), \quad \Pi_O^{(1)}(k) = 2N_f\Pi_O^{(1)}(k; J_o). \quad (3.20)$$

On the other hand, if $J_o = 0$, then $J_\pm = J_e$ and

$$\Pi_T^{(1)}(k) = 2N_f\Pi_T^{(1)}(k; J_e), \quad \Pi_O^{(1)}(k) = 0. \quad (3.21)$$

The usual criterion for generation of the induced Chern-Simons term is given by $\Pi_O^{(1)}(k \to 0) \neq 0$. The induced Chern-Simons term is generated when $J_+ > 0, J_- < 0 (J_e < J_o)$, since

$$\Pi_O^{(1)}(k \to 0) = \frac{N_f e^2}{4\pi} [\text{sgn}(J_+) - \text{sgn}(J_-)]$$

$$= \frac{N_f e^2}{4\pi} [\text{sgn}(J_e + J_o) - \text{sgn}(J_e - J_o)]$$

$$= \begin{cases} 0 & (J_e > J_o; J_+, J_- > 0) \\ \frac{N_f e^2}{2\pi} & (J_e < J_o; J_+ > 0, J_- < 0) \end{cases}. \quad (3.22)$$

4 Order parameter as a function of source

We introduce new mass parameters (or sources) $J_1, J_2$ corresponding to two 2-component fermions, $\psi_1, \psi_2$:

$$J_1 = J_+, \quad J_2 = -J_- \quad (4.1)$$

so that the source term (2.8) is rewritten as

$$L_J = J_1 \bar{\psi}_1(x)\psi_1(x) + J_2 \bar{\psi}_2(x)\psi_2(x), \quad (4.2)$$

where

$$J_e = \frac{1}{2}(J_1 - J_2), \quad J_o = \frac{1}{2}(J_1 + J_2), \quad (4.3)$$

or

$$J_1 = J_e + J_o, \quad J_2 = -J_e + J_o. \quad (4.4)$$

Similarly in the preceding section, the free parts of the order parameter are decomposed as

$$\langle \bar{\Psi}\Psi \rangle^{(0)} = \langle \bar{\psi}_1\psi_1 \rangle^{(0)} - \langle \bar{\psi}_2\psi_2 \rangle^{(0)},$$

$$\langle \bar{\Psi}_T\Psi \rangle^{(0)} = \langle \bar{\psi}_1\psi_1 \rangle^{(0)} + \langle \bar{\psi}_2\psi_2 \rangle^{(0)}, \quad (4.5)$$

where we have defined:

$$\langle \bar{\psi}_a\psi_a \rangle^{(0)} = \int_0^\Lambda dp \frac{p^2 J_a}{\pi^2 p^2 + J_a^2} \quad (a = 1, 2). \quad (4.6)$$
We apply the above decomposition also to the interaction part. By taking into account eq. (2.12) and
\[ \frac{\partial}{\partial J_e} = \frac{\partial}{\partial J_1} - \frac{\partial}{\partial J_2}, \quad \frac{\partial}{\partial J_o} = \frac{\partial}{\partial J_1} + \frac{\partial}{\partial J_2}, \]
the two order parameters are rewritten in the form:
\[ \langle \bar{\Psi} \Psi \rangle = \langle \bar{\Psi}_1 \Psi_1 \rangle - \langle \bar{\Psi}_2 \Psi_2 \rangle, \]
\[ \langle \bar{\Psi} \tau \Psi \rangle = \langle \bar{\Psi}_1 \Psi_1 \rangle + \langle \bar{\Psi}_2 \Psi_2 \rangle, \]
where
\[ \langle \bar{\Psi}_a \Psi_a \rangle = \langle \bar{\Psi}_a \Psi_a \rangle^{(0)} + \frac{N_f}{2} \int \frac{d^3k}{(2\pi)^3} D_{\mu\nu}(k) \frac{\partial}{\partial J_a} \Pi_{\mu\nu}^{(1)}(k), \]

\[ \Pi_{\mu\nu}^{(1)}(k) = \alpha k \left[ 1 + 2 \sum_a \frac{J_a^2}{k^2} - \frac{32}{3\pi} \sum_a \frac{\sigma_a J_a^3}{k^3} \right] + \mathcal{O}(J_a^4), \]
\[ \Pi_{\mu\nu}^{(1)}(k) = 2\alpha \left[ \sum_a J_a - \frac{4}{\pi} \sum_a \frac{\sigma_a J_a^2}{k} \right] + \mathcal{O}(J_a^4), \]

\[ \frac{\partial}{\partial J_a} \Pi_{\mu\nu}^{(1)}(k) = 4\alpha \left[ \frac{J_a}{k} - \sigma_a \frac{8}{\pi} \frac{J_a}{k^2} \right] + \mathcal{O}(J_a^4), \]
\[ \frac{\partial}{\partial J_a} \Pi_{\mu\nu}^{(1)}(k) = 2\alpha \left[ 1 - \sigma_a \frac{8}{\pi} \frac{J_a}{k} \right] + \mathcal{O}(J_a^3), \]

where a parameter \( \alpha \) is defined by
\[ \alpha := \frac{N_f e^2}{8}, \]
which has the dimension of mass in three-dimensions.
4.1 case (I)

First we consider the case of $J_1 > 0, J_2 < 0$ ($J_+ > 0, J_- > 0$). The numerator of the integrand in eq. (4.9) has the following dependence on the source $J_a$:

$$
[\beta k^2 - \Pi_T^{(1)}(k)] \frac{\partial \Pi_T^{(1)}(k)}{\partial J_a} + [\theta \sqrt{k^2} - \Pi_O^{(1)}(k)] \frac{\partial \Pi_O^{(1)}(k)}{\partial J_a}
$$

\[
= P_0 + P_1^a J_a + P_2 \sum_b J_b + P_3^a J_a^2 + P_4 \sum_b \sigma_b J_b^2 + P_5^a J_a \sum_b J_b + \mathcal{O}(J_a^3)
\]

\[
(4.14)
\]

where

$$
P_0(k) = 2\theta \alpha k,
$$

$$
P_1^a(k) = -\sigma_a \frac{16}{\pi} \theta \alpha + 4\alpha(\beta k + \alpha),
$$

$$
P_2(k) = -4\alpha^2 < 0,
$$

$$
P_3^a(k) = -\sigma_a \frac{32\alpha \beta k + \alpha}{\pi k} = \sigma_a P_3,
$$

$$
P_4(k) = \frac{16}{\pi} \alpha^2 \frac{k}{k} > 0,
$$

$$
P_5^a(k) = \sigma_a \frac{32\alpha^2}{\pi k} = \sigma_a P_5.
$$

(4.15)

On the other hand, the denominator of the integrand in eq. (4.9) has

$$
\{[\beta k^2 - \Pi_T^{(1)}(k)]^2 + [\theta \sqrt{k^2} - \Pi_O^{(1)}(k)]^2\}^{-1}
$$

\[
= [F(k)]^{-1} \left[1 + Q_0(k) \sum_a J_a
\right.

\[
+ Q_1(k) \sum_a \sigma_a J_a^2 + Q_2(k) \sum_a J_a^2 + Q_3(k) \left(\sum_a J_a\right)^2 + \mathcal{O}(J_a^3),
\]

\[
(4.16)
\]

where

$$
Q_0(k) = \frac{4\theta \alpha k}{F(k)},
$$

$$
Q_1(k) = -4\alpha \frac{\theta}{F(k)},
$$

$$
Q_2(k) = 4\alpha \frac{\beta k + \alpha}{F(k)},
$$

$$
Q_3(k) = -4\alpha^2 \frac{[\beta k^2 + \alpha k]^2 + (\theta k)^2 - 4\theta^2 k^2}{F(k)^2},
$$

(4.17)

and

$$
F(k) := (\beta k^2 + \alpha k)^2 + (\theta k)^2.
$$

(4.18)
Substituting eq. (4.14) and eq. (4.16) into eq. (4.9), we obtain for \( a = 1, 2 \)

\[
\langle \bar{\psi}_a \psi_a \rangle = \langle \bar{\psi}_a \psi_a \rangle^{(0)} + N_f^{-1} \int \frac{d^3k}{(2\pi)^3} F(k)^{-1} \left[ P_0 + P_1^a J_a + A \sum_b J_b \right. \\
+ P_3^a J_a^2 + B \sum_b \sigma_b J_b^2 + C^a J_a b \sum_b J_b + D \sum_b J_b^2 + E(\sum_b J_b)^2 \bigg] \\
+ \mathcal{O} \left( J^3 \right),
\]

where

\[
A(k) = P_0(k) Q_0(k) + P_2(k), \\
B(k) = P_0(k) Q_1(k) + P_4(k), \\
C^a(k) = P_1^a(k) Q_0(k) + P_5^a(k), \\
D(k) = P_0(k) Q_2(k), \\
E(k) = P_0(k) Q_3(k) + P_2 Q_0(k).
\]

Thus we obtain the order parameter in the form:

\[
\varphi_1 := \langle \bar{\psi}_1 \psi_1 \rangle / N_f = \varphi_0 + u_1 J_1 + v J_2 + s_1 (J_1)^2 + t_1 (J_2)^2 + r_1 J_1 J_2 + \mathcal{O} \left( J^3 \right), \\
\varphi_2 := \langle \bar{\psi}_2 \psi_2 \rangle / N_f = \varphi_0 + v J_1 + u_2 J_2 + t_2 (J_1)^2 + s_2 (J_2)^2 + r_2 J_1 J_2 + \mathcal{O} \left( J^4 \right),
\]

where

\[
\begin{align*}
\varphi_0 &= \frac{1}{N_f} \int \frac{d^3k}{(2\pi)^3} P_0(k) / F(k), \\
\varphi_1 &= \frac{1}{N_f} \int \frac{d^3k}{(2\pi)^3} \left( P_1^a(k) + A(k) \right) / F(k), \\
\varphi_2 &= \frac{1}{N_f} \int \frac{d^3k}{(2\pi)^3} \left( P_3^a(k) + \sigma_a B(k) + C^a(k) + D(k) + E(k) \right) / F(k), \\
\varphi_3 &= \frac{1}{N_f} \int \frac{d^3k}{(2\pi)^3} \left( -\sigma_a B(k) + D(k) + E(k) \right) / F(k), \\
\varphi_4 &= \frac{1}{N_f} \int \frac{d^3k}{(2\pi)^3} \left( C^a(k) + 2E(k) \right) / F(k),
\end{align*}
\]

and

\[
\varphi_0 = \frac{1}{N_f} \int \frac{d^3k}{(2\pi)^3} P_0(k) / F(k).
\]

It is easy to check that

\[
 r_1 = 2t_2, \quad r_2 = 2t_1,
\]

since \( P_5 = 2P_4 \) and \( P_1^a Q_0 = 2P_0 (Q_2 + \sigma_a Q_1) \).
The parity even and odd order parameters are written as
\[
\varphi_e := \langle \bar{\Psi}^a \Psi^a \rangle / N_f = \varphi_1 + \varphi_2 = \varphi_1 - \varphi_2,
\]
\[
\varphi_o := \langle \bar{\Psi}^o \Psi^o \rangle / N_f = \varphi_1 - \varphi_2 = \varphi_1 + \varphi_2.
\]
They obey
\[
\varphi_e = (u_1 + u_2 - 2v)J_e + (u_1 - u_2)J_o + [s_1 - s_2 + 3(t_1 - t_2)]J_e^2 + [s_1 - s_2 - (t_1 - t_2)]J_o^2 + 2[s_1 + s_2 - (t_1 + t_2)]J_eJ_o + O(J^3),
\]
\[
\varphi_o = 2\varphi + (u_1 - u_2)J_e + (u_1 + u_2 + 2v)J_o + [s_1 + s_2 - (t_1 + t_2)]J_e^2 + [s_1 + s_2 + 3(t_1 + t_2)]J_o^2 + 2[s_1 - s_2 - (t_1 - t_2)]J_eJ_o + O(J^3).
\]

In order to study the small \(\theta\) case, we first observe that
\[
A(k) = P_2(k) + O(\theta^2) < 0,
\]
\[
B(k) = P_4(k) + O(\theta^2) > 0,
\]
\[
C^a(k) = P_5^a(k) + \frac{16\alpha^2 k(\beta k + \alpha)}{F_0(k)} \theta + O(\theta^2),
\]
\[
D(k) = \frac{8\alpha^2 k(\beta k + \alpha)}{F_0(k)} \theta + O(\theta^2),
\]
\[
E(k) = -\frac{24\alpha^3 k}{F_0(k)} \theta + O(\theta^2),
\]
where
\[
F_0(k) := (\beta k^2 + \alpha k)^2.
\]
Note that when \(\theta = 0\), \(P_0 = 0\), \(Q_0 = 0\), \(Q_1 = 0\) and \(P_1^1 = P_1^2\). Therefore, the functions \(Q_a(a = 1, 2, 3)\) do not contribute to the order parameter.

Thus the coefficient is calculated up to \(O(1/N_f)\) and \(O(\theta)\):
\[
\theta_a = \frac{1}{N_f} \int \frac{d^3k}{(2\pi)^3} \frac{4\alpha\beta k}{F_0(k)} \left( \varphi_1 + \varphi_2 \right)_a,
\]
\[
u = \frac{1}{N_f} \int \frac{d^3k}{(2\pi)^3} \frac{4\alpha^2}{F_0(k)} \left( \varphi_1 - \varphi_2 \right)_a,
\]
\[
s_a = \frac{1}{N_f} \int \frac{d^3k}{(2\pi)^3} \frac{\sigma_a [P_3 + P_4(k) + P_5(k)]}{F_0(k)} + \frac{\theta}{N_f} \int \frac{d^3k}{(2\pi)^3} \frac{24\alpha^2 \beta k}{F_0(k)^2} + O(\theta^2/N_f^2),
\]
\(a = 1, 2, 3\).
\[
\begin{align*}
t_a &= -\frac{1}{N_f} \int \frac{d^3k}{(2\pi)^3} \frac{\sigma_a P_4(k)}{F_0(k)} \\
&\quad + \frac{\theta}{N_f} \int \frac{d^3k}{(2\pi)^3} \frac{8\alpha^2 k(\beta k - 2\alpha)}{F_0(k)^2} + O\left(\frac{\theta^2}{N_f^2}\right),
\end{align*}
\]
and
\[
\varphi_\theta = \frac{\theta}{N_f} \int \frac{d^3k}{(2\pi)^3} \frac{2\alpha k}{F_0(k)} + O\left(\frac{\theta^3}{N_f^2}\right).
\]

It is easy to see that
\[
\begin{align*}
& u_1 - u_2 = O\left(\frac{\theta}{N_f}\right), \quad u_1 + u_2 = (u_1 + u_2)_{\theta=0} + O\left(\frac{\theta^2}{N_f^2}\right), \\
& s_1 + s_2 = O\left(\frac{\theta}{N_f}\right), \quad s_1 - s_2 = (s_1 - s_2)_{\theta=0} + O\left(\frac{\theta^2}{N_f^2}\right), \\
& t_1 + t_2 = O\left(\frac{\theta}{N_f}\right), \quad t_1 - t_2 = (t_1 - t_2)_{\theta=0} + O\left(\frac{\theta^2}{N_f^2}\right).
\end{align*}
\]

Hence, for example, we obtain \(u_1 + u_2 \pm 2v = 2(u_{\theta=0} \pm v_{\theta=0}) + O\left(\frac{\theta^2}{N_f^2}\right)\).

Finally, we consider the limit \(\theta \to 0\), i.e., the Chern-Simons term is absent. In this limit, we find \(s_1 + s_2 \to 0\) and \(t_1 + t_2 \to 0\), and \(u_1, u_2\) approach to the same limit: \(u_1, u_2 \to u\). Thus, in the limit \(\theta \to 0\), two order parameters obey
\[
\begin{align*}
\varphi_e &= 2(u - v)J_e + [s_1 - s_2 + 3(t_1 - t_2)]J_e^2 + [s_1 - s_2 - (t_1 - t_2)]J_o^2 + O(J^3), \\
\varphi_o &= 2(u + v)J_o + 2[s_1 - s_2 - (t_1 - t_2)]J_e J_o + O(J^3),
\end{align*}
\]
where
\[
\begin{align*}
u &= \int_0^{\Lambda_f} \frac{dp}{\pi^2} + \frac{1}{N_f} \int \frac{d^3k}{(2\pi)^3} \frac{4\alpha \beta k}{F_0(k)} + O\left(\frac{1}{N_f^2}\right), \\
v &= -\frac{1}{N_f} \int \frac{d^3k}{(2\pi)^3} \frac{4\alpha^2}{F_0(k)} + O\left(\frac{1}{N_f^2}\right), \\
s_a &= \frac{1}{N_f} \int \frac{d^3k}{(2\pi)^3} \frac{\sigma_a}{F_0(k)} \left(\frac{16\alpha}{\pi}\right) \frac{\alpha^2}{k} + O\left(\frac{1}{N_f^2}\right), \\
t_a &= -\frac{1}{N_f} \int \frac{d^3k}{(2\pi)^3} \frac{\sigma_a}{F_0(k)} \left(\frac{16\alpha}{\pi}\right) \frac{\alpha^2}{k} + O\left(\frac{1}{N_f^2}\right).
\end{align*}
\]

For example, if we introduce another cutoff \(\Lambda_p\), we get
\[
2(u - v) = \frac{2\Lambda_f}{\pi^2} \left[1 + \frac{K_1}{N_f}\right],
\]
where
\[
\begin{align*}
K_1 &= \left. \frac{\pi^2}{\Lambda_f} \int \frac{d^3k}{(2\pi)^3} \frac{4\alpha (\beta k + \alpha)}{F_0(k)^2} \right|_{\Lambda_p} \\
&= \frac{2 \Lambda_p}{\Lambda_f} \left(\frac{\beta \Lambda_p}{\alpha}\right)^{-1} \ln \left(1 + \frac{\beta \Lambda_p}{\alpha}\right).
\end{align*}
\]
4.2 case (II)

Next we consider the case of $J_1 > 0, J_2 > 0 \ (J_+ > 0, J_- < 0)$.

In this case we take $\sigma_a = 1$ for $a = 1, 2$ in Eq. (4.11) and Eq. (4.12). Then we obtain

$$\varphi_1 = \varphi_\theta + u J_1 + v J_2 + s (J_1)^2 + t (J_2)^2 + 2 t J_1 J_2 + \mathcal{O} (J^3),$$

$$\varphi_2 = \varphi_\theta + v J_1 + u J_2 + t (J_1)^2 + s (J_2)^2 + 2 t J_1 J_2 + \mathcal{O} (J^3),$$

where the coefficients $u, v, s, t$ are given by Eq. (4.29) with $\sigma_a \equiv 1$. Hence the order parameter is written in the form:

$$\varphi_e = 2(u - v) J_e + 4(s - t) J_e J_o + \mathcal{O}(J^3),$$

$$\varphi_o = 2 \varphi_\theta + 2(u + v) J_o + 2(s + t) J_e^2 + 2(s + 3 t) J_o^2 + \mathcal{O}(J^3).$$

When $\theta = 0$, for example, we get

$$2(u + v) = \frac{2 \Lambda f}{\pi^2} \left[ 1 + \frac{L_1}{N_f} \right],$$

where

$$L_1 = \frac{\pi^2}{\Lambda f} \int_{-\Lambda_f}^{\Lambda_f} d^3 k \frac{4 \alpha (\beta k - \alpha)}{(2\pi)^3 (\beta k^2 + \alpha k)^2}$$

$$= \frac{2 \Lambda_p}{\Lambda f} \left( \frac{\beta \Lambda_p}{\alpha} \right)^{-1} \left[ \ln \left( 1 + \frac{\beta \Lambda_p}{\alpha} \right) - 2 \frac{\beta \Lambda_p}{\alpha} \left( 1 + \frac{\beta \Lambda_p}{\alpha} \right) \right].$$

5 Inversion method

In this section we restrict our consideration to the case in which the bare Chern-Simons term is absent. The effect of the bare Chern-Simons term will be investigated in section 8. In this paper we use the inversion method to obtain the effective potential and find the non-trivial solution as its stationary point. The inversion method was proposed by Fukuda [45] and was applied to gauge theories: for example, QCD4 [15], the strong coupling QED4 [46, 47], the gauged Yukawa model [48, 49] and the Thirring model (reformulated as a gauge theory) [50].

5.1 general strategy

Let $g$ denote an expansion parameter in a certain scheme of the expansion. For example, $g$ may be the coupling constant $e^2$ in the usual perturbation theory or the expansion parameter $1/N_f$ in the $1/N_f$ expansion. In order to calculate the order parameter $\langle \bar{\Psi} \Psi \rangle$ and/or $\langle \bar{\Psi} \tau \Psi \rangle$ we introduce the source term
We define the dimensionless order parameter $\phi$ and $\chi$ by

\[
\phi := \frac{\pi^2}{2\Lambda_f^2} \frac{\langle \bar{\Psi}^\alpha \Psi^\alpha \rangle}{N_f},
\]
\[
\chi := \frac{\pi^2}{2\Lambda_f^2} \frac{\langle \bar{\Psi}^\alpha \tau \Psi^\alpha \rangle}{N_f},
\]

(5.1)

with an appropriate factor which simplifies the following calculations. In a certain expansion scheme, suppose that the order parameter is calculated in the power series of $g$:

\[
\phi = h_e[J_e, J_o] = \sum_{n=0}^{\infty} g^n \phi_n(J_e, J_o),
\]
\[
\chi = h_o[J_e, J_o] = \sum_{n=0}^{\infty} g^n \chi_n(J_e, J_o).
\]

(5.2)

Then we invert these equations with respect to the source:

\[
J_e = f_e[\phi, \chi] = \sum_{m=0}^{\infty} g^m f_m^e(\phi, \chi),
\]
\[
J_o = f_o[\phi, \chi] = \sum_{m=0}^{\infty} g^m f_m^o(\phi, \chi).
\]

(5.3)

We call this procedure the inversion. Here the coefficient functions $f_m^e(\phi, \chi)$ and $f_m^o(\phi, \chi)$ are determined from the requirement that the following consistency condition should hold:

\[
\phi = h_e[f_e[\phi, \chi], f_o[\phi, \chi]],
\]
\[
\chi = h_o[f_e[\phi, \chi], f_o[\phi, \chi]],
\]

(5.4)

and are written by using $\phi_n(J_e, J_o)$ and $\chi_n(J_e, J_o)$. In the inverted series eq. (5.3), we look for the non-trivial solution for $\phi$ and $\chi$ in the limit, $J_e \to 0$ and $J_o \to 0$, besides the trivial solution: $\phi = 0$ and $\chi = 0$. This is the inversion method. Thus we can get the non-perturbative solution $\phi \neq 0$ or $\chi \neq 0$, if it exists at all, by the inversion method [45].

5.2 case (I)

In the case of $J_1 > 0, J_2 < 0 (J_e > J_o)$, eq. (1.32) shows that, up to the leading order of $g = 1/N_f$, $\phi$ and $\chi$ are written as functions of two sources $J_e, J_o$:

\[
\phi := K \bar{J}_e - (P \bar{J}_e^2 + Q \bar{J}_o^2) + O(J^3),
\]
\[
\chi := L \bar{J}_o - R(2 \bar{J}_e \bar{J}_o) + O(J^3),
\]

(5.5)

4 Here we have adopted the UV cutoff $\Lambda_f$ to make the order parameter dimensionless. However this is not a unique choice. We can take other quantities which play the same role, e.g., $\alpha := N_f e^2$. The result on the existence of the phase transition is unchanged irrespective of what quantity we may choose to define the dimensionless order parameter, as already pointed out in [40].
where we have defined the dimensionless source:
\[
\tilde{J}_{e,o} = \frac{J_{e,o}}{\Lambda_f}.
\] (5.6)

Here we find that the cross term \(J_e J_o\) does not appear in \(\phi\) and that \(J_e^2, J_o^2\) terms do not appear in \(\chi\). In what follows \(\mathcal{O}(J^3)\) implies \(\mathcal{O}(J_e^3), \mathcal{O}(J_o^3), \mathcal{O}(J_e^2 J_o)\) or \(\mathcal{O}(J_o^2)\). The coefficients \(K, L, P, Q, R\) are written as power series of \(g\), e.g.,
\[
K = \sum_{n=0}^{\infty} g^n K_n.
\]
In this paper we pay attention to the first two terms \(K_0, K_1\) up to \(\mathcal{O}(g)\) and similarly for other coefficients, \(L, P, Q, R\). Our definition of \(\phi, \chi\) leads to
\[
K_0 = L_0 = 1, \quad P_0 = Q_0 = R_0 = \frac{\pi}{2},
\] (5.7)
as already shown in section 3.

Now we invert the above set of equations with respect to two sources, \(J_e\) and \(J_o\). We assume the inverted form as
\[
\begin{align*}
\tilde{J}_e &= \tau_e \phi + A \phi^2 + B \chi^2 + C \phi \chi + \mathcal{O}(\varphi^3), \\
\tilde{J}_o &= \tau_o \chi + D \phi^2 + E \chi^2 + F \phi \chi + \mathcal{O}(\varphi^3),
\end{align*}
\] (5.8)

where \(\mathcal{O}(\varphi^3)\) denotes \(\mathcal{O}(\phi^3), \mathcal{O}(\phi^2 \chi), \mathcal{O}(\phi \chi^2)\) or \(\mathcal{O}(\chi^3)\).

From the condition for the inverted series eq. (5.8) to be consistent with the above equation eq. (5.5), the coefficient is determined. Hence the inverted series in the case (I) is obtained as
\[
\begin{align*}
\tilde{J}_e &= \tau_e \phi + A \phi^2 + B \chi^2 + \mathcal{O}(\varphi^3), \\
\tilde{J}_o &= \tau_o \chi + F \phi \chi + \mathcal{O}(\varphi^3),
\end{align*}
\] (5.9)

where
\[
\begin{align*}
\tau_e &= K^{-1}, \\
\tau_o &= L^{-1}, \\
A &= P \tau_e^2 K^{-1}, \\
B &= Q \tau_o^2 K^{-1} = Q K^{-1} L^{-2}, \\
F &= 2 R \tau_e \tau_o L^{-1} = 2 R K^{-1} L^{-2},
\end{align*}
\] (5.10)
together with \(C = D = E = 0\). Therefore, up to the leading order of \(g\), we obtain
\[
\begin{align*}
\tau_e &= 1 - K_1 g + \mathcal{O}(g^2), \\
\tau_o &= 1 - L_1 g + \mathcal{O}(g^2), \\
A &= P_0 + (P_1 - 3 K_1 P_0) g + \mathcal{O}(g^2), \\
B &= Q_0 + (Q_1 - K_1 Q_0 - 2 L_1 Q_0) g + \mathcal{O}(g^2), \\
F &= 2 R_0 + 2 (R_1 - K_1 R_0 - 2 L_1 R_0) g + \mathcal{O}(g^2).
\end{align*}
\] (5.11)
5.3 case (II)

Next we consider the case where \( J_1, J_2 > 0 \) \((J_e < J_o)\). In this case eq. (4.37) leads to

\[
\phi := K \tilde{J}_e - U(2 \tilde{J}_e \tilde{J}_o) + \mathcal{O}(J^3),
\]

\[
\chi := L \tilde{J}_o - (S \tilde{J}_e^2 + T \tilde{J}_o^2) + \mathcal{O}(J^3). \tag{5.12}
\]

Using the consistency condition, we obtain the inverted equation:

\[
\tilde{J}_e = \tau_e \phi + C \phi \chi + \mathcal{O}(\varphi^3),
\]

\[
\tilde{J}_o = \tau_o \chi + D \phi^2 + E \chi^2 + \mathcal{O}(\varphi^3), \tag{5.13}
\]

where

\[
\tau_e = K^{-1},
\]

\[
\tau_o = L^{-1},
\]

\[
C = 2U \tau_e \tau_o K^{-1},
\]

\[
D = S \tau_e^2 L^{-1},
\]

\[
E = T \tau_o^2 L^{-1}, \tag{5.14}
\]

and \( A = B = F \). Since

\[
K_0 = L_0 = 1, \quad S_0 = T_0 = U_0 = \frac{\pi^2}{2}, \tag{5.15}
\]

we obtain

\[
C_0 = 2D_0 = 2E_0. \tag{5.16}
\]

6 Chiral symmetry breaking in pure QED3

In this section we consider the pure QED3 without the bare Chern-Simons term. The induced Chern-Simons term \( \Pi^{(1)}_O(k) \) is also neglected in this section as well as the bare Chern-Simons term. The Chern-Simons terms are included in the subsequent sections based on the result of section 4. This section is an independent one which illustrates the inversion method in a slightly simplified setting.

6.1 single order parameter in QED3

As a special case of the inversion scheme presented in the previous section, we introduce the single source term \( \mathcal{L}_J = J_e \bar{\Psi}(x) \Psi(x) \) and consider a single order parameter \( \phi \) which has the power-series expansion with respect to the expansion parameter \( g \) \((g = e^2 or 1/N_f)\):

\[
\phi = h[J_e] = \sum_{n=0}^{\infty} g^n \phi_n(J_e). \tag{6.1}
\]
After inversion, we will get
\[ J_e = f[\phi] = \sum_{m=0}^{\infty} g^m f_m(\phi). \] (6.2)

The coefficient function \( f_m(\phi) \) is related to \( \phi_n(J_e) \) from the consistency condition:
\[ \phi = h[f[\phi]]. \] (6.3)

In QED3 without the Chern-Simons term \((\beta = 1, \theta = 0)\), it is shown in the following that \( \phi \) obeys
\[ \phi := K\tilde{J}_e - P\tilde{J}_e^2 + \mathcal{O}(\tilde{J}_e^3). \] (6.4)

The inverted equation is obtained as
\[ \tilde{J}_e = \tau \phi + A\phi^2 + \mathcal{O}(\phi^3), \] (6.5)
where the consistency condition leads to
\[ \tau = K^{-1}, \quad A = P\tau^2 K^{-1}. \] (6.6)

Hence, to the leading order of \( g \),
\[ \tau = 1 - K_1 g + \mathcal{O}(g^2), \quad A = P_0 + (P_1 - 3K_1 P_0) g + \mathcal{O}(g^2). \] (6.7)

Even in the limit \( J_e \to 0 \), \( \phi \) in eq. (6.5) has a non-trivial solution \( \phi = -\tau A^{-1} > 0 \) when \( \tau < 0 \) (for \( A > 0 \)). Moreover, the effective potential is obtained as
\[ V(\phi) = \frac{\tau}{2} \phi^2 + \frac{A}{3} \phi^3, \] (6.8)
from the relation
\[ \frac{\delta V(\phi)}{\delta \phi} = \tilde{J}_e. \] (6.9)

Hence the non-trivial solution \( \phi > 0 \) is energetically more favorable than the trivial one \( \phi = 0 \) when \( \tau < 0 \) (as long as \( A > 0 \)). Therefore the chiral symmetry is spontaneously broken for \( g > g_c = K_1^{-1} \) where \( g_c \) is identified with the critical point.

### 6.2 quenched QED3

In this subsection we consider the chiral symmetry breaking in the quenched QED3. Here the term "quenched" implies that all the fermion loop correction to the photon propagator are neglected. This situation is realized by taking the
limit $N_f \to 0$ in an appropriate way. Therefore we estimate the critical point in the lowest order of the coupling constant $e$ instead of $1/N_f$. As shown in section 2, we need to calculate

$$
\frac{1}{2} \int \frac{d^3k}{(2\pi)^3} D^{(0)}_{\mu\nu}(k) \frac{\partial}{\partial J_e} \Pi^{(1)}_{\mu\nu}(k)
= \frac{1}{4\pi^2} \int_0^{\Lambda_p} k^2 dk D^{(0)}_{\mu\nu}(k) \frac{\partial}{\partial J_e} \Pi^{(1)}_{\mu\nu}(k)
= \frac{1}{4\pi^2} \int_0^{\Lambda_p} dk \frac{2}{\beta} \frac{\partial}{\partial J_e} \left[ \Pi^{(1)}_{\mu\nu}(k) \right].
$$

(6.10)

To the lowest order $\mathcal{O}(e^2)$, we obtain

$$
\frac{1}{2} \int \frac{d^3k}{(2\pi)^3} D^{(0)}_{\mu\nu}(k) \frac{\partial}{\partial J_e} \Pi^{(1)}_{\mu\nu}(k)
= \frac{4\alpha}{\pi^2 \beta} \int_\mu^\Lambda_p dk \left( \frac{J_e}{k} - \frac{8 J_e^2}{\pi k^2} \right) + \mathcal{O}(J^4),
$$

(6.11)

where we have temporarily introduced the infrared (IR) cutoff $\mu$ besides the ultra-violet (UV) cutoff $\Lambda_p$. For a while, we put $\beta = 1$. In the quenched limit, all the loop corrections are neglected and hence the photon propagator reduces to the bare one and the order parameter $\phi$ is defined by taking the limit $N_f \to 0$. Then we have

$$
\phi := \lim_{N_f \to 0} \frac{\pi^2}{2} \frac{\langle \bar{\Psi}^\alpha \Psi^\alpha \rangle}{N_f \Lambda_f^2} = \tilde{J}_e \left( 1 + \frac{e^2}{8 \Lambda_f} \ln \frac{\Lambda_p}{\mu} \right) - P \tilde{J}_e^2 + \mathcal{O}(J^4).
$$

(6.12)

After inversion up to $\mathcal{O}(e^2)$, we get

$$
\tilde{J}_e := \frac{J_e}{\Lambda_f} = \phi \left( 1 - \frac{e^2}{8 \Lambda_f} \ln \frac{\Lambda_p}{\mu} \right) + A \phi^2 + \mathcal{O}(\phi^4).
$$

(6.13)

Even when $J_e = 0$, there may exist a non-trivial solution $\phi \neq 0$ besides the trivial one, $\phi = 0$:

$$
\phi = A^{-1} \left( \frac{e^2}{8 \Lambda_f} \ln \frac{\Lambda_p}{\mu} - 1 \right).
$$

(6.14)

Actually, defining the dimensionless inverse coupling constant $\beta_e := \Lambda_f/e^2$, we see there is a non-trivial solution when $\beta_e < \beta_e^c$ for a critical value:

$$
\beta_e^c := \frac{1}{8} \ln \frac{\Lambda_p}{\mu}.
$$

(6.15)

This result however shows that $\beta_e^c \uparrow \infty$ as $\mu \downarrow 0$. Naively this may be interpreted as follows. In the quenched QED3, the chiral symmetry is spontaneously broken

\footnote{Note that we obtain

$$
P_1 = \frac{\pi^2 4\alpha}{2 \pi^2} \int_\mu^{\Lambda_p} \frac{dk}{\beta k^2},
$$

which diverges as $\mu \downarrow 0$.}
for arbitrary values of the coupling constant $e^2 \neq 0$. This agrees with the results of the Schwinger-Dyson equation approach [51] and the Monte Carlo simulation of the lattice non-compact QED3 [36].

### 6.3 unquenched QED3 in the $1/N_f$ expansion

Next we consider the problem of chiral symmetry breaking in unquenched QED3 based on the scheme of the $1/N_f$ expansion. To the leading order in $1/N_f$ expansion, we wish to calculate

$$
\frac{1}{2} \int \frac{d^3k}{(2\pi)^3} D^{(1)}_{\mu\nu}(k) \frac{\partial}{\partial J_e} \Pi^{(1)}_{\mu\nu}(k)
= \frac{1}{2\pi^2} \int_0^{\Lambda_f} k^2 dk D^{(1)}_{\mu\nu}(k) \frac{\partial}{\partial J_e} \Pi^{(1)}_{\mu\nu}(k)
= \frac{1}{2\pi^2} \int_0^{\Lambda_f} k^2 dk \frac{1}{[\beta k^2 - \Pi^{(1)}_T(k)]} \frac{\partial \Pi^{(1)}_T(k)}{\partial J_e}. 
$$

(6.16)

Up to $O(J_e^2)$, this reduces to

$$
\frac{1}{2} \int \frac{d^3k}{(2\pi)^3} D^{(1)}_{\mu\nu}(k) \frac{\partial}{\partial J_e} \Pi^{(1)}_{\mu\nu}(k)
= \frac{1}{2\pi^2} \int_0^{\Lambda_f} k^2 dk \frac{1}{[\beta k^2 + \alpha k]} 8\alpha \left( \frac{J_e}{k} - \frac{8J^2_e}{\pi^2 k^2} \right) + O(J_e^3). 
$$

(6.17)

Therefore we obtain

$$
\phi := \frac{\pi^2}{2} \frac{\langle \bar{\Psi}^\alpha \Psi^\alpha \rangle}{N_f \Lambda_f^2} = \tilde{J}_e \left( 1 + \frac{K_1}{N_f} \right) - P \tilde{J}_e^2 + O(J_e^3). 
$$

(6.18)

After inversion, we get

$$
\tilde{J}_e = \left( 1 - \frac{N_f^c}{N_f} \right) \phi + A\phi^2 + O(\phi^3), 
$$

(6.19)

with

$$
N_f^c = K_1 = \frac{\pi^2}{2\Lambda_f} \frac{4\alpha}{\pi^2} \int_0^{\Lambda_f} dk \frac{1}{\beta k + \alpha} = \frac{2\alpha}{\beta \Lambda_f} \ln(1 + \frac{\beta \Lambda_f}{\alpha}). 
$$

(6.20)

In the leading order of the $1/N_f$ expansion, we have shown the existence and finiteness of the critical number of flavors $N_f^c$, below which the chiral symmetry is

\[ \text{Note that we obtain} \]
\[ P_1 = \frac{\pi^2}{2} \frac{4\alpha}{\pi^2} \frac{8}{\pi} \int_0^{\Lambda_f} dk \frac{1}{k(\beta k + \alpha)}. \]

\[ \text{Naively this diverges as } \mu \downarrow 0. \text{ The infrared divergence in } P_1 \text{ cannot be removed in this method. This term may induce the fluctuation-induced first order transition as discussed in the final section.} \]
spontaneously broken and above which the chiral symmetry restores. This result should be compared with the previous analysis in the quenched case. The reason why we can obtain a finite critical number is that the \(1/N_f\) expansion alters the infrared behavior of the gauge boson propagator from \(1/k^2\) to \(1/(\alpha k)\) due to the existence of \(\Pi_T^{(1)}(k)\) in the denominator. This eliminates the infrared divergence encountered in the previous subsection.

Moreover, even in the Chern-Simons limit \(\beta \to 0\), i.e., without the kinetic term for the gauge boson, the finite critical number of flavors is obtained: \(N^c_f(\beta = 0) = 2\). When \(\beta \neq 0\), the naive "continuum" (infinite cutoff) limit \(\Lambda/\alpha \to \infty\) leads to \(N^f_\beta \to 0\), i.e., no breaking of the chiral symmetry. However the first viewpoint explained in the introduction that the dimensional quantity \(\alpha\) gives a natural scale of the theory implies that the relevant physics comes essentially from the region \(k \sim \alpha\) and the region \(k \gg \alpha\) does not give the essential contribution. Hence we should normalize the physical quantity by \(\alpha\) (instead of \(\Lambda_f\)) and set the upper bound of integration as \(\Lambda_\rho \sim \alpha\). This standpoint gives a finite and nonzero value for \(N^f_\beta\).

7 Effective potential and stability

7.1 chiral-symmetry violation (case (I))

We define the dimensionless effective potential \(\tilde{\Gamma}[\phi, \chi]\) by

\[
\tilde{\Gamma}[\phi, \chi] = \frac{\pi^2}{2N_f} \frac{\Gamma[\phi, \chi]}{\Lambda^3},
\]

in conformity with the definition of the dimensionless order parameter (5.1). We consider the effective potential

\[
\tilde{\Gamma}[\phi, \chi] = \frac{1}{2} \tau_e \phi^2 + \frac{1}{2} \tau_o \chi^2 + \frac{1}{3} A \phi^3 + \frac{1}{2} F \phi \chi^2 + \mathcal{O}(\phi^4),
\]

which satisfies the following relation associated with the case (I):

\[
\frac{\delta \tilde{\Gamma}[\phi, \chi]}{\delta \phi} \equiv \tilde{J}_e = \tau_e \phi + A \phi^2 + \frac{1}{2} F \phi \chi^2 + \mathcal{O}(\phi^3),
\]

\[
\frac{\delta \tilde{\Gamma}[\phi, \chi]}{\delta \chi} \equiv \tilde{J}_o = \tau_o \chi + F \phi \chi + \mathcal{O}(\phi^3).
\]

It is easy to see that, even when \(\tilde{J}_e = \tilde{J}_o = 0\), a pair of eq. (7.3) has non-trivial solutions besides the trivial one \(\phi = \chi = 0\). First of all, we observe that a pair of solutions such that \(\chi \neq 0\) and \(\phi = 0\) does not exist, which is consistent with the assumption \(J_e > J_o\) in case (I).

There are two types of non-trivial solutions. One type of non-trivial solutions is given by

\[
\chi^{(1)} = 0, \quad \phi^{(1)} = -\tau_e A^{-1} > 0,
\]

25
where \( \tau_e < 0 \) and \( A > 0 \). At this solution, the effective potential takes the value:

\[
\tilde{\Gamma}[\phi^{(1)}, \chi^{(1)}] = \frac{1}{6} \tau_e^3 A^{-2} < 0, \tag{7.5}
\]

which is lower than the trivial one: \( \tilde{\Gamma}[\phi = 0, \chi = 0] = 0 \). This solution implies the spontaneous breaking of chiral symmetry and no spontaneous breaking of parity.

Another non-trivial solution is given by

\[
\phi^{(2)} = -\tau_o F^{-1}, \quad \chi^{(2)} = \sqrt{2} \tau_o \left(F \tau_e - A \tau_o\right) F^{-1}. \tag{7.6}
\]

The solution \( \phi^{(2)} \) is positive for \( \tau_o < 0 \) and \( \chi^{(2)} \) is real and positive if \( \tau_o < 0 \) and \( F \tau_e < A \tau_o \), as long as \( A, F > 0 \). This solution implies that the chiral symmetry and the parity are simultaneously broken. It gives the effective potential

\[
\tilde{\Gamma}[\phi^{(2)}, \chi^{(2)}] = \frac{1}{6} \tau_o^3 F^{-3} (3F \tau_e - 2A \tau_o) < \frac{1}{6} \tau_e^3 A F^{-3} < 0. \tag{7.7}
\]

Then this solution also gives lower effective potential than \( \tilde{\Gamma}[0, 0] = 0 \).

Which solution gives the absolute minimum? If the second solution is the absolute minimum, the chiral symmetry and the parity is spontaneously broken simultaneously. However we can show that the first solution is energetically favorable:

\[
\tilde{\Gamma}[\phi^{(1)}, \chi^{(1)}] < \tilde{\Gamma}[\phi^{(2)}, \chi^{(2)}] < 0, \tag{7.8}
\]

since

\[
\tilde{\Gamma}[\phi^{(1)}, \chi^{(1)}] - \tilde{\Gamma}[\phi^{(2)}, \chi^{(2)}] = [2A \tau_o]^3 + |F \tau_e|^3 - 3 |A \tau_o|^2 |F \tau_e|] /[6 A^2 F^3] < 0, \tag{7.9}
\]

in the allowed region of parameters, \( A, F, \tau_e, \tau_o \).

Now we consider the stability of the solution around the respective stationary point: \( \phi = \phi^{(a)}, \chi = \chi^{(a)} \) at which \( \frac{\partial \Gamma}{\partial \phi} = 0 \) and \( \frac{\partial \Gamma}{\partial \chi} = 0 \). The effective potential can be expanded around the stationary point as

\[
\Gamma[\phi, \chi] = \Gamma[\phi^{(a)}, \chi^{(a)}] + \frac{1}{2} (\delta \phi \delta \chi) H_{\Gamma}[\phi^{(a)}, \chi^{(a)}] \left( \frac{\delta \phi}{\delta \chi} \right) + \mathcal{O}((\delta \phi)^3), \tag{7.10}
\]

where \( \delta \phi \) denotes deviation \( \delta \phi \) or \( \delta \chi \) from the respective stationary point: \( \delta \phi := \phi - \phi^{(a)}, \delta \chi := \chi - \chi^{(a)} \) and \( H_{\Gamma}[\phi, \chi] \) is the Hessian matrix defined by

\[
H_{\Gamma}[\phi, \chi] := \left( \begin{array}{cc}
\frac{\partial^2 \Gamma}{\partial \phi^2} & \frac{\partial^2 \Gamma}{\partial \phi \partial \chi} \\
\frac{\partial^2 \Gamma}{\partial \chi \partial \phi} & \frac{\partial^2 \Gamma}{\partial \chi^2}
\end{array} \right). \tag{7.11}
\]

For the above effective potential \( \tilde{\Gamma} \), the Hessian reads

\[
H_{\Gamma}[\phi, \chi] = \begin{pmatrix}
\tau_e + 2A \phi & F \chi \\
F \chi & \tau_o + F \phi
\end{pmatrix}. \tag{7.12}
\]
Around the first solution,
\[ H_{\Gamma}[\phi^{(1)}, \chi^{(1)}] = \begin{pmatrix} -\tau_e & 0 \\ 0 & (A\tau_o - F\tau_e)/A \end{pmatrix}. \] (7.13)

Note that \(-\tau_e > 0\) and \((A\tau_o - F\tau_e)/A > 0\). Hence \(\text{tr} H_{\Gamma} > 0\) and the Hessian is positive definite, i.e. \(\det H_{\Gamma} > 0\). Then the first solution \(\phi^{(1)}, \chi^{(1)}\) gives the local minimum.

Around the second solution,
\[ H_{\Gamma}[\phi^{(2)}, \chi^{(2)}] = \begin{pmatrix} (F\tau_e - 2A\tau_o)/F & F\chi^{(2)} \\ F\chi^{(2)} & 0 \end{pmatrix}. \] (7.14)

The Hessian is negative definite, since \(\det H_{\Gamma} = -(F\chi^{(2)})^2 < 0\). Hence the second solution \(\phi^{(2)}, \chi^{(2)}\) corresponds to the saddle point.

Thus the first solution gives the absolute minimum. Therefore we can conclude that there are no stable (absolute) minima at which both \(\phi\) and \(\chi\) are simultaneously non-zero. This conclusion agrees with the result of Semenoff and Wijewardhana [42]. Moreover we have shown up to \(O(1/N_f)\) that the chiral symmetry is broken spontaneously for \(N_f < N^c_f = K_1\), while the spontaneous breaking of parity does not occur. In the broken phase the non-trivial order parameter behaves near the critical point as
\[ \phi^{(1)} = -\tau_e A^{-1} = (K_1/N_f - 1)P_0^{-1} + O(1/N_f^2), \] (7.15)
where the value \(K_1\) is given by eq. (4.35). Note that \(K_1\) is always positive and monotonically decreasing in \(\beta\lambda/\alpha\) and has a maximum value 2 at the Chern-Simons limit: \(\beta\lambda/\alpha = 0\). Hence \(N^c_f\) has a finite upper bound 2. For example, when \(\beta\lambda/\alpha = 1\), \(K_1 = 2 \ln(2) \approx 1.4\).

### 7.2 parity violation (case (II))

Next we consider the case where \(J_1 > 0, J_2 > 0 \ (J_e < J_o)\). This case corresponds to the effective potential:
\[ \hat{\Gamma}[\phi, \chi] = \frac{1}{2}\tau_e \phi^2 + \frac{1}{2}\tau_o \chi^2 + \frac{1}{3}(E\chi^3 + \frac{3}{2}C\phi^2) + O(\phi^4), \] (7.16)
since this gives the following relation for the case (II):
\[ \frac{\delta \hat{\Gamma}[\phi, \chi]}{\delta \phi} \equiv \hat{J}_e = \tau_e \phi + C\phi \chi + O(\phi^3), \]
\[ \frac{\delta \hat{\Gamma}[\phi, \chi]}{\delta \chi} \equiv \hat{J}_o = \tau_o \chi + \frac{C}{2} \phi^2 + E\chi^2 + O(\phi^3). \] (7.17)

One non-trivial solution is given by
\[ \phi^{(1)} = 0, \ \chi^{(1)} = -\tau_o E^{-1} > 0, \] (7.18)
where \( \tau_o < 0 \) and \( E > 0 \). This solution gives the effective potential

\[
\Gamma[\phi^{(1)}, \chi^{(1)}] = \frac{1}{6} \tau_o^3 E^{-2} < 0, \tag{7.19}
\]

which is lower than \( \Gamma[\phi = 0, \chi = 0] = 0 \). This solution implies the spontaneous breaking of parity and no spontaneous breaking of chiral symmetry.

Another non-trivial solution is given by

\[
\chi^{(2)} = -\tau_e C^{-1}, \quad \phi^{(2)} = \sqrt{\frac{2\tau_e(C\tau_o - E\tau_e)}{C^3}}. \tag{7.20}
\]

The solution \( \chi^{(2)} \) is positive for \( \tau_e < 0 \) and \( \chi^{(2)} \) is real and positive if \( \tau_e < 0, C\tau_o < E\tau_e \), as long as \( C, E > 0 \). This solution gives the effective potential

\[
\Gamma[\phi^{(2)}, \chi^{(2)}] = \frac{1}{6} \tau^2 C^{-3}(3C\tau_o - 2E\tau_e) < \frac{1}{6} \tau^3 C^{-3} E < 0, \tag{7.21}
\]

which is lower than \( \Gamma[0, 0] = 0 \).

We can show that the first solution is energetically favorable:

\[
\bar{\Gamma}[\phi^{(1)}, \chi^{(1)}] < \bar{\Gamma}[\phi^{(2)}, \chi^{(2)}] < 0, \tag{7.22}
\]

in the same way as the previous case (I).

For the above effective potential (7.16), the Hessian reads

\[
H_{\Gamma}[\phi, \chi] = \begin{pmatrix}
\tau_e + C\chi & C\phi \\
C\phi & 2E
\end{pmatrix}. \tag{7.23}
\]

Around the second solution,

\[
H_{\Gamma}[\phi^{(2)}, \chi^{(2)}] = \begin{pmatrix}
0 & C\phi^{(2)} \\
C\phi^{(2)} & 2E
\end{pmatrix}. \tag{7.24}
\]

The Hessian is negative definite, since \( \det H_{\Gamma} = -(C\phi^{(2)})^2 < 0 \). Hence the second solution \( \phi^{(2)}, \chi^{(2)} \) corresponds to the saddle point.

Around the first solution,

\[
H_{\Gamma}[\phi^{(1)}, \chi^{(1)}] = \begin{pmatrix}
(E\tau_e - C\tau_o)/E & 0 \\
0 & 2E
\end{pmatrix}. \tag{7.25}
\]

Note that \( E > 0, (E\tau_e - C\tau_o)/E > 0 \). Hence \( \text{tr} H_{\Gamma} > 0 \) and the Hessian is positive definite, i.e. \( \det H_{\Gamma} > 0 \). Then the first solution \( \phi^{(1)}, \chi^{(1)} \) gives the absolute minimum. Therefore we can conclude that there are no stable (absolute) minima at which both \( \phi \) and \( \chi \) are simultaneously non-zero.

Hence we are tempted to conclude that up to \( \mathcal{O}(1/N_f) \) the parity symmetry may be broken spontaneously for \( N_f < N_c f = L_1 \), while the spontaneous breaking of chiral symmetry does not occur. If so, the parity-odd order parameter should behave as

\[
\chi^{(1)} = -\tau_o E^{-1} = (L_1/N_f - 1) S_0^{-1} + \mathcal{O}(1/N^2_f) > 0. \tag{7.26}
\]
However note that the value $L_1$ given by eq. (1.39) has a maximum value 0.1208 (at $x := \beta \Lambda / \alpha \cong 13.3$) which is extremely small. Above which $L_1$ decreases monotonically and goes to zero at $x \to \infty$, while $L_1$ becomes negative below $x = 3.92$ and reaches to $L_1 = -2$ at the Chern-Simons limit: $x = 0$. This seems to show that the spontaneous breaking of parity does not occur actually. The absence of spontaneous breakdown of parity in (2+1)-dimensional gauge theories is consistent with results of other works \[38, 39, 40, 41, 26\] and general consideration by Vafa and Witten \[43\].

8 First order transition

8.1 effective potential in the presence of bare Chern-Simons term

In this section we study the effect of the bare Chern-Simons term. In what follows we restrict our consideration to small

Vafa and Witten \[43\].

with results of other works \[38, 39, 40, 41, 26\] and general consideration by Vafa and Witten \[43\].
from the result of section 5. As in section 7, the effective potential is obtained by integrating eq. (8.3):

\[ \tilde{\Gamma}[\phi,\chi] = C_\theta \chi + E_\theta \phi \chi + \frac{1}{2} \tau_e \phi^2 + \frac{1}{2} \tau_o \chi^2 + \frac{1}{3} A \phi^3 + \frac{1}{2} F \phi \chi^2 + \mathcal{O}(\theta \phi^3, \varphi^4), \] (8.5)

so that the effective potential satisfies

\[ \frac{\delta \tilde{\Gamma}}{\delta \phi} = \tilde{J}_e, \quad \frac{\delta \tilde{\Gamma}}{\delta \chi} = \tilde{J}_o, \] (8.6)

where we have used \( E_\theta = D_\theta \).

Let \( \phi^*, \chi^* \) denote the stationary point of the effective potential \( \tilde{\Gamma}[\phi,\chi] \), i.e.,

\[ \frac{\delta \tilde{\Gamma}}{\delta \phi} \bigg|_{\phi=\phi^*,\chi=\chi^*} = 0, \quad \frac{\delta \tilde{\Gamma}}{\delta \chi} \bigg|_{\phi=\phi^*,\chi=\chi^*} = 0. \] (8.7)

In order to specify the location of the stationary point, we put \( r := \chi^*/\phi^* \). From the first equation of eq. (8.3), we get

\[ \phi^* = -\frac{\tau_e + E_\theta r}{A + Br^2}. \] (8.8)

On the other hand, the second equation of eq. (8.3) leads to

\[ \phi^* = -\frac{E_\theta + \tau_o r \pm \sqrt{(E_\theta + \tau_o r)^2 - 4FC_\theta r}}{2Fr}. \] (8.9)

Equating two values for \( \phi^* \) in the above two equations, we get the algebraic equation for \( r \) which is 4th order in \( r \). Since we restrict our attention to small \( \theta \), \( r \) is obtained up to \( \mathcal{O}(\theta^2) \) as

\[ r_\theta = A \frac{AC_\theta - E_\theta \tau_e}{A\tau_e \tau_o - F\tau_e} + \mathcal{O}(\theta^2), \] (8.10)

since \( C_\theta, E_\theta \sim \mathcal{O}(\theta) \).

At the stationary point we define an order parameter \( \varphi \) by

\[ \phi^* = \frac{1}{(1 + r_\theta^2)^{1/2}} \varphi, \quad \chi^* = \frac{r_\theta \varphi}{(1 + r_\theta^2)^{1/2}}. \] (8.11)

In the case of \( \theta = 0 \), \( r_\theta = 0 \) and hence \( (\phi^*, \chi^*) = (\varphi, 0) \) which is consistent with the result of section 7.1. In the plane \( \chi^* = r_\theta \phi^* \) including the stationary point \( (\phi^*, \chi^*) \), the effective potential behaves as (see Fig. 4)

\[ \Gamma[\varphi] = \frac{r_\theta}{(1 + r_\theta^2)^{1/2}} C_\theta \varphi + \frac{1}{2} \left( \frac{1}{1 + r_\theta^2} \right) (\tau_e + r_\theta^2 \tau_o + 2Er_\theta) \varphi^2 \]
\[ + \frac{1}{3} \left( \frac{1}{1 + r_\theta^2} \right)^{3/2} (A + \frac{3}{2} r_\theta^2 F) \varphi^3 + \mathcal{O}(\theta \varphi^3, \varphi^4), \] (8.12)

This is a generalization of eq. (7.2) to the case of \( \theta \neq 0 \).
8.2 1st order transition

In the presence of the bare Chern-Simons term, we are forced to investigate the behavior of the effective potential of the form:

\[ \Gamma[\varphi] = A_1(N_f, \theta)\varphi + \frac{1}{2} A_2(N_f, \theta)\varphi^2 + \frac{1}{3} A_3(N_f, \theta)\varphi^3 + \mathcal{O}(\varphi^4), \]  

(8.13)

where all the coefficient \(A_i(i = 1, 2, \ldots)\) are functions of the bare parameters: \(N_f\) and \(\theta\).

First we consider the situation in which the Chern-Simons term is absent: \(A_1 = 0\) and \(A_3 > 0\). For positive \(A_2 > 0\) (point A), the effective potential is a convex function of \(\varphi\) in the region \(\varphi \geq 0\) and has an absolute minimum at \(\varphi = 0\), see Fig. 5. For negative \(A_2 < 0\) (point B), it has a minimum at \(\varphi = \varphi_m := -A_2/A_3 > 0\). This region of parameters corresponds to the chiral symmetry-breaking phase. As the parameter \(A_2\) passes through \(A_2 = 0\) from the positive side to negative one, the absolute minimum at \(\varphi = 0\) moves continuously to the absolute minimum at \(\varphi = \varphi_m \neq 0\). Therefore this theory exhibits a second order phase transition at \(A_2(N_f = N_f^c, 0) = 0\). This determines \(N_f^c\), the critical number of flavors, below which \((N_f < N_f^c)\) the chiral symmetry is spontaneously broken.

Now we can show that the non-vanishing \(A_1\) changes the type of the phase transition drastically. In this case, there occurs a first order phase transition at a certain value of \(A_1\). In order to show this, we start from a point B in the broken-symmetry phase of the phase diagram \((N_f, \theta)\) such that \(N_f < N_f^c\) and \(\theta = 0\). At this point, \(A_1(N_f, \theta = 0) = 0\) and \(A_2(N_f, \theta = 0) < 0\). As the Chern-Simons coefficient \(\theta\) grows from zero, the shape of the effective potential \(\Gamma[\varphi]\) changes as shown in Fig. 5. For small \(\theta\) (point C), \(A_1 > 0\) is also small and \(\Gamma[\varphi]\) has an absolute minimum at \(\varphi = \varphi_m(N_f, \theta)\) away from \(\varphi = 0\). As \(A_1\) is increased, the value \(\Gamma[\varphi_m]\) at \(\varphi = \varphi_m\) increases monotonically and approaches to zero at \(A_1 = A_1^c\) (point D). Therefore there is a certain value of \(A_1\) such that \(\Gamma[\varphi_c] = 0\) at \(A_1 = A_1^c\) and above which \((A_1 > A_1^c)\) the absolute minimum moves to the origin discontinuously (point E). This is nothing but the first order phase transition point. At the critical point, the effective potential takes the form:

\[ \Gamma[\varphi] = \frac{1}{3} A_3(N_f, \theta)\varphi(\varphi - \varphi_c)^2 + \mathcal{O}(\varphi^4). \]  

(8.14)

Therefore there is a critical value \(\theta_c(> 0)\) of Chern-Simons coefficient at which the discontinuous first order phase transition occurs in the region \(N_f < N_f^c\). By comparing eq. (8.14) with eq. (8.13), the critical value of \(\theta_c = \theta_c(N_f)\) as a function of \(N_f\) is determined: \(\theta_c\) satisfies the following equation:

\[ A_1(N_f, \theta) = \frac{3}{16} \frac{A_2^2(N_f, \theta)}{A_3(N_f, \theta)} (\ll 1), \]  

(8.15)

and \(\varphi_c\) is given by

\[ \varphi_c = -\frac{3}{4} \frac{A_2(N_f, \theta_c)}{A_3(N_f, \theta_c)} = \sqrt{\frac{3A_1(N_f, \theta_c)}{A_3(N_f, \theta_c)}} (> 0). \]  

(8.16)
Here note that \( r_\theta \) is proportional to the Chern-Simons coefficient and hence \( \theta = 0 \) implies \( A_1 = 0 \). Therefore the first order phase transition in Maxwell-Chern-Simons theory with fermions occurs only when there is a bare Chern-Simons term, \( \theta \neq 0 \).

Therefore, comparing eq. (8.12) with eq. (8.13), we find that

\[
\begin{align*}
A_1 &= k_0 \frac{\theta^2}{N_f} + \mathcal{O}(\theta^4), \\
A_2 &= k_1 \left( 1 - \frac{N_f^c}{N_f} \right) + \tilde{k}_1 \frac{\theta^2}{N_f^2} + \mathcal{O}(\theta^4), \\
A_3 &= k_2 \frac{1}{N_f} + \mathcal{O}(\theta^2),
\end{align*}
\]

and hence the critical line for the first order chiral phase transition is given by

\[
\theta \sim \left( 1 - \frac{N_f^c}{N_f} \right),
\]

in the neighborhood of the critical point \((N_f, \theta) = (N_f^c, 0)\). This result should be compared with that of Schwinger-Dyson equation approach \[22\]:

\[
\theta \sim \exp \left[ -\frac{\pi}{\sqrt{N_f^c/N_f - 1}} \right].
\]

The result eq. (8.18) is reasonable, since it is obtained up to the leading order of \( 1/N_f \) expansion. To the lowest order of \( 1/N_f \) expansion, we can not recover the essential singularity behavior even if it is correct, see \[49\].

9 Conclusion and discussion

In the leading order of \( 1/N_f \) expansion, we have constructed the gauge-covariant (gauge-parameter-independent) effective potential in terms of two order parameters, \( \langle \bar{\Psi} \Psi \rangle \) and \( \langle \bar{\Psi} \tau \Psi \rangle \), in the Maxwell and/or Chern-Simons theory coupled to fermionic matter. From the viewpoint of the stability of the solution around the stationary point of the effective potential, we have shown that the spontaneous breaking of both the chiral-symmetry and the parity can not occur simultaneously. This result in the absence of the bare Chern-Simons term agrees with that obtained only in the Landau gauge by Semenoff and Wijewardhana \[42\]. In \( (2+1) \)-dimensional gauge theories without the bare Chern-Simons term (\( \theta = 0 \)), there may exist a finite number of critical flavors \( N_f^c \) such that for \( N_f < N_f^c \) the chiral symmetry is spontaneously broken. On the other hand, the spontaneous breakdown of parity does not occur. This conclusion agrees with the previous analyses \[38, 39, 40, 41, 42, 26\] and general consideration \[43\]. We wish to emphasize that all the results obtained in this paper are gauge-parameter independent.
Moreover, in sharp contrast to the Schwinger-Dyson equation approach of QED3 up to now (see e.g. [22]), we have included the effect coming from the vacuum polarization to the photon propagator exactly, up to the leading order of $1/N_f$ expansion. In the Schwinger-Dyson equation approach for QED3 so far, the vacuum polarization effect to the photon propagator has been included through the expression $\Pi_T^{(1)}(k) = \alpha k$ which is obtained in the massless fermion limit of $\Pi_T^{(1)}(k)$. In order to include the vacuum polarization effect from the massive fermion in the Schwinger-Dyson equation approach, we must solve the Schwinger-Dyson equation for the photon propagator simultaneously with the Schwinger-Dyson equation for the fermion propagator. In the four-dimensional case such a calculation has been performed in the strong-coupling phase of QED4 [52]. Therefore the result obtained in this paper based on the inversion method goes beyond the Schwinger-Dyson equation approach for QED3 done so far in a sense just mentioned. However our analysis shows that the inclusion of the massive fermion does not change the critical number of flavors, if any.

The critical value $N_f^c$ obtained here seems rather small compared with the result of the Schwinger-Dyson equation $N_f^c = 3 \sim 4$ and the Monte Carlo simulation, $N_f^c \cong 3.5 \pm 0.5$. This will be due to the fact that the above calculation is restricted to the leading order of $1/N_f$ expansion, although the calculation is systematic. This is also the case for QED4, see [49]. To estimate more accurately the critical flavor, we need to perform the higher order calculation.

The order of the chiral phase transition is of the second order in the absence of the Chern-Simons term. This is consistent with the result of [53]. However, in the presence of the Chern-Simons term, we have shown that the chiral phase transition turns into the first order transition, as discovered firstly in the Schwinger-Dyson equation [22].

Now we come to the stage of discussing the meaning of the "continuum" limit (of removing the cutoffs) in the non-perturbative sense [54, 55]. The point $(N_f^c, 0)$ is the only one point which exhibits the continuous second order chiral transition. Therefore, in order to obtain the continuum theory with a finite fermion mass, the bare parameters must be adjusted so as to approach the critical point $(N_f^c, 0)$.

It should be remarked that the bare Chern-Simons term plays completely different role from the induced Chern-Simons term. It is the bare Chern-Simons term which causes the first order transition, while the induced Chern-Simons term does not leads to the first order transition, even if the Chern-Simons term may be induced by radiative corrections at one-loop. The Chern-Simons coefficient $\theta$ is not subject to renormalization according to the Coleman-Hill theorem [7] and the beta function of the renormalization group is identically zero: $\beta(\theta) \equiv 0$. The continuum theory obtained from the massive phase in this continuum limit may have no Chern-Simons term in the renormalized sense.

In this paper we have neglected the possibility of the fluctuation-induced first order transition. If this happens in this theory, the point $(N_f^c, 0)$ will be no longer of the second order transition point. To clarify this problem, we need to perform more careful analysis on the infrared behavior of the theory. However the absence
of such a first order transition has been reported recently in [53].

Finally we remark the possible effect coming from higher orders in $1/N_f$ expansion [56]. Our analysis of the effective potential up to the leading order of $1/N_f$ expansion shows

$$\frac{\langle \bar{\Psi} \Psi \rangle}{\alpha^2} \sim \left( 1 - \frac{N_c}{N_f} \right), \quad (9.1)$$

in QED3 without the Chern-Simons term. In the Schwinger-Dyson equation approach the chiral order parameter $\langle \bar{\Psi} \Psi \rangle$ as well as the dynamically generated fermion mass $m_d$ exhibits the essential singularity at the critical point $N_f = N_c^c$:

$$\frac{\langle \bar{\Psi} \Psi \rangle}{\alpha^2} \sim \left( \frac{m_d}{\alpha} \right)^{3/2} \sim \exp \left[ -\frac{3\pi}{\sqrt{N_c^c/N_f - 1}} \right]. \quad (9.2)$$

In the four-dimensional case, we have shown [49] how the result obtained in the Schwinger-Dyson equation approach for QED4 can be reproduced from the higher order calculation in the inversion method. For this, it is helpful to enlarge the model so as to include the four-fermion interaction. We can expect that this scenario also holds in the three-dimensional case, see e.g., [41]. However, even if this conjecture is correct, we will need the infinite order result of $1/N_f$ expansion to recover the essential singularity exactly. A similar feature can be seen in the expression of the critical line between eq. (8.18) and eq. (8.19). The result $N_f^c = 2$ up to the leading order for the Chern-Simons theory ($\beta = 0$) with $N_f$ flavors of 4-component fermions is not so large that the reliability of $1/N_f$ expansion may be not very definite. To see the improvement of the expansion by higher orders, we need to calculate the next-to-leading order. This is the subject of the subsequent work.

One more comment is on the choice of order parameter. In the analysis of this paper, we did not consider the order parameter

$$\omega := \frac{1}{\Omega} \int d^3 x \epsilon^{\mu \nu \rho} A_\mu(x) \partial_\nu A_\rho(x). \quad (9.3)$$

This order parameter $\omega$ as well as $\chi$ is able to signal the spontaneous breakdown of parity after taking the limit $\theta \to 0$. In this case we must obtain the effective potential as a function of three order parameters $\phi$, $\chi$ and $\omega$. This is possible in principle, but is somewhat tedious. To discuss the connection of our result with the anyonic superconductivity, we must include the chemical potential into this theory. This will be done in the forthcoming paper. The first order phase transition induced by the topological term may have other important implications, see [57].

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A Clifford algebra and its representation

In the Minkowski space-time, the gamma matrices $\gamma^\mu_M$ satisfy the Clifford algebra:

$$\gamma^\mu_M \gamma^\nu_M + \gamma^\nu_M \gamma^\mu_M = 2g^{\mu\nu}, \quad (A.1)$$

which is satisfied for example by

$$\gamma^0_M = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \quad \gamma^1_M = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix}, \quad \gamma^2_M = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}, \quad (A.2)$$

with Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (A.3)$$

In the euclidean space, we define the gamma matrices satisfying

$$\gamma^\mu_E \gamma^\nu_E + \gamma^\nu_E \gamma^\mu_E = -2\delta^{\mu\nu}, \quad (A.4)$$

by $\gamma^0_E := -i\gamma^0_M, \gamma^1_E := \gamma^1_M, \gamma^2_E := \gamma^2_M$, i.e.,

$$\gamma^0_E = \begin{pmatrix} -i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{pmatrix}, \quad \gamma^1_E = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix}, \quad \gamma^2_E = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}. \quad (A.5)$$

Note that

$$(\gamma^\mu_E)^2 = -\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \quad (\mu = 0, 1, 2) \quad (A.6)$$

and

$$\gamma^0_E \gamma^1_E \gamma^2_E = \begin{pmatrix} i\sigma_3 \sigma_1 \sigma_2 & 0 \\ 0 & -i\sigma_3 \sigma_1 \sigma_2 \end{pmatrix} = -\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \quad (A.7)$$

Then we define

$$\gamma^3_E := \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad (A.8)$$

$$\gamma^5_E := \gamma^0_E \gamma^1_E \gamma^2_E \gamma^3_E = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \quad (A.9)$$

and

$$\tau = \gamma^3_E \gamma^5_E = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \quad (A.10)$$

For the projection operator:

$$\chi_+ = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad \chi_- = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}. \quad (A.11)$$
we obtain the following trace formulae:

\[
\begin{align*}
\text{tr}[\chi_{\pm}] &= 2, \\
\text{tr}[\gamma_E^\mu \gamma_E^\nu \chi_{\pm}] &= -2\delta_{\mu\nu}, \\
\text{tr}[\gamma_E^\mu \gamma_E^\nu \gamma_E^\rho \chi_{\pm}] &= \mp 2\epsilon_{\mu\rho\nu}, \\
\text{tr}[\gamma_E^\mu \gamma_E^\nu \gamma_E^\rho \gamma_E^\sigma \chi_{\pm}] &= 2(\delta_{\mu\nu}\delta_{\alpha\beta} - \delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha}).
\end{align*}
\]

(A.12)

and

\[
\begin{align*}
\text{tr}[\tau] &= 0, \\
\text{tr}[\gamma_E^\mu \gamma_E^\nu \tau] &= \mp 2\delta_{\mu\nu}, \\
\text{tr}[\tau \chi_{\pm}] &= \pm 2, \\
\text{tr}[\gamma_E^\mu \gamma_E^\nu \chi_{\pm}] &= -2\delta_{\mu\nu}, \\
\text{tr}[\gamma_E^\mu \gamma_E^\nu \gamma_E^\rho \chi_{\pm}] &= \mp 2\epsilon_{\mu\rho\nu}, \\
\text{tr}[\gamma_E^\mu \gamma_E^\nu \gamma_E^\rho \gamma_E^\sigma \chi_{\pm}] &= 2(\delta_{\mu\nu}\delta_{\alpha\beta} - \delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha}).
\end{align*}
\]

(A.13)

**B Generating functional**

For the bosonic field, \(A\), the functional integration is performed as

\[
\int D A \exp \left[ -\frac{1}{2} A D^{-1} A \right] = (\text{Det} D^{-1})^{-1/2} = \exp \left[ \frac{1}{2} \ln \text{Det} D \right].
\]

(B.1)

Since the full boson propagator \(D\) satisfies the Schwinger-Dyson equation:

\[
D = [D^{(0)}^{-1} - \Pi]^{-1} = [1 - D^{(0)}\Pi]^{-1} D^{(0)},
\]

(B.2)

we have

\[
\ln \text{Det}[D] = \text{Tr} \ln[D] \\
= \text{Tr} \ln[D^{(0)}] - \text{Tr} \ln[1 - D^{(0)}\Pi] \\
= \text{Tr} \ln[D^{(0)}] + \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr}[(D^{(0)}\Pi)^n].
\]

(B.3)

Then the derivative takes the form:

\[
\frac{\partial}{\partial J} \ln \text{Det}[D] = \frac{\partial}{\partial J} \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr}[(D^{(0)}\Pi)^n] \\
= \sum_{n=0}^{\infty} \text{Tr} \left[ (D^{(0)}\Pi)^n D^{(0)} \frac{\partial}{\partial J} \Pi^{(1)} \right] \\
= \text{Tr} \left[ (1 - D^{(0)}\Pi)^{-1} D^{(0)} \frac{\partial}{\partial J} \Pi \right] \\
= \text{Tr} \left[ D \frac{\partial}{\partial J} \Pi \right].
\]

(B.4)
Especially, for the gauge field, we obtain
\[
\frac{\partial}{\partial J} \ln \det[D] = \int \frac{d^Dk}{(2\pi)^D} D^{(1)}_{\mu\nu}(k) \frac{\partial}{\partial J} \Pi^{(1)}_{\mu\nu}(k). \tag{B.5}
\]

On the other hand, for the fermionic field, \(\bar{\psi}, \psi\), we have
\[
\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp[\bar{\psi}S^{-1}\psi] = \det S^{-1} = \exp[-\ln \det S]. \tag{B.6}
\]

Since the full fermion propagator \(S\) satisfies the Schwinger-Dyson equation:
\[
S = \left[S^{(0)} - \Sigma\right]^{-1} = \left[1 - S^{(0)} \Sigma\right]^{-1} S^{(0)}, \tag{B.7}
\]
we obtain in the similar way to the bosonic case
\[
\ln \det[S] = \text{Tr} \ln[S] = \text{Tr} \ln[S^{(0)}] - \text{Tr} \ln[1 - S^{(0)} \Sigma] = \text{Tr} \ln[S^{(0)}] + \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr}[(S^{(0)} \Sigma)^n], \tag{B.8}
\]
and
\[
\frac{\partial}{\partial J} \ln \det[S] = \frac{\partial}{\partial J} \text{Tr} \ln[S^{(0)}] + \text{Tr} \left[S \frac{\partial}{\partial J} \Sigma\right]. \tag{B.9}
\]

In the scheme of \(1/N_f\) expansion, we find
\[
\Pi^{(1)}_{\mu\nu} \sim e^2 N_f = \alpha, \\
\Sigma^{(1)} \sim e^2 = \frac{\alpha}{N_f}. \tag{B.10}
\]

Therefore, in the leading order of \(1/N_f\) expansion, we have only to calculate
\[
- \frac{\partial}{\partial J} \text{Tr} \ln[S^{(0)}] + \frac{1}{2} \int \frac{d^Dk}{(2\pi)^D} D^{(1)}_{\mu\nu}(k) \frac{\partial}{\partial J} \Pi^{(1)}_{\mu\nu}(k). \tag{B.11}
\]

\section{C Vacuum polarization}

\subsection{C.1 Vacuum polarization tensor}

The vacuum polarization tensor at one-loop is obtained from
\[
\Pi^{(1)}_{\mu\nu}(k; -m) = -e^2 \int \frac{d^Dp}{(2\pi)^D} \text{Tr} \left[ \gamma_\mu \frac{1}{\not{p} + k - m} \gamma_\nu \frac{1}{\not{p} - m} \chi^\pm \right] = -e^2 \int \frac{d^Dp}{(2\pi)^D} \frac{\text{Tr}[\gamma_\mu(\not{p} + k + m)\gamma_\nu(\not{p} + m)\chi^\pm]}{((p + k)^2 + m^2)(p^2 + m^2)}. \tag{C.1}
\]
Hence the vacuum polarization tensor is decomposed into two parts:
\[-e^2 \int \frac{d^3 p}{(2\pi)^3} \text{tr}[\gamma_\mu (\not{p} + \not{k}) \gamma_\nu \not{p} \chi_{\pm}] + m^2 \text{tr}[\gamma_\mu \gamma_\nu \chi_{\pm}],\]
(C.2)
and
\[-e^2 m \int \frac{d^3 p}{(2\pi)^3} \text{tr}[\gamma_\mu (\not{p} + \not{k}) \gamma_\nu \not{p} \chi_{\pm}] + m^2 \text{tr}[\gamma_\mu \gamma_\nu \not{p} \chi_{\pm}].\]
(C.3)
The parity even part is calculated as
\[-e^2 \int \frac{d^3 p}{(2\pi)^3} \text{tr}[\gamma_\mu (\not{p} + \not{k}) \gamma_\nu \not{p} \chi_{\pm}] + m^2 \text{tr}[\gamma_\mu \gamma_\nu \not{p} \chi_{\pm}]
= -k^2 \left( \frac{\delta_{\mu\nu} - k_\mu k_\nu}{k^2} \right) \frac{e^2}{2\pi} \int_0^1 dx \frac{x(1-x)}{[m^2 + x(1-x)k^2]^{1/2}}
= - \left( \frac{\delta_{\mu\nu} - k_\mu k_\nu}{k^2} \right) \frac{e^2}{2\pi} \left[ \frac{1}{2} \sqrt{m^2 + \frac{k^2 - 4m^2}{4k}} \arctan \frac{k}{2\sqrt{m^2}} \right].\]
(C.4)
On the other hand, the odd part reads
\[-e^2 m \int \frac{d^3 p}{(2\pi)^3} \text{tr}[\gamma_\mu (\not{p} + \not{k}) \gamma_\nu \not{p} \chi_{\pm}] + m^2 \text{tr}[\gamma_\mu \gamma_\nu \not{p} \chi_{\pm}]
= +e^2 m \int \frac{d^3 p}{(2\pi)^3} \mp 2\epsilon_{\mu\nu\rho} k_\rho \frac{1}{[p^2 + 2xk \cdot p + xk^2 + m^2]^2}
= +e^2 m \frac{\epsilon_{\mu\nu\rho}}{4\pi} \epsilon_{\mu\nu\rho} k_\rho \int_0^1 dx \frac{1}{[m^2 + x(1-x)k^2]^{1/2}}
= +e^2 m \frac{\epsilon_{\mu\nu\rho}}{4\pi} \epsilon_{\mu\nu\rho} k^2 k \arctan \frac{k}{2\sqrt{m^2}},\]
(C.5)
where we have used \(\text{tr}[\gamma_\mu \gamma_\nu \gamma_\rho \chi_{\pm}] = \mp 2\epsilon_{\mu\nu\rho}\) and the Feynman parameter formula
\[
\frac{1}{AB} = \int_0^1 dx \frac{1}{(xA + (1-x)B)^2}.
\]
For the 2-component fermion, we have
\[
\Pi_T^{(1)}(k; J) = -\frac{e^2}{8\pi} \left[ 2|J| + \frac{k^2 - 4J^2}{k} \arctan \frac{k}{2|J|} \right],\]
(C.6)
\[
\Pi_O^{(1)}(k; J) = \frac{e^2}{2\pi} J \arctan \frac{k}{2|J|},\]
(C.7)
and the derivative with respect to the source \(J\) is given by
\[
\frac{\partial}{\partial J} \Pi_T^{(1)}(k; J) = -\frac{e^2}{4\pi} \left[ \text{sgn}(J) \frac{8J^2}{k^2 + 4J^2} - \frac{4J}{k} \arctan \frac{k}{2|J|} \right],\]
(C.8)
\[
\frac{\partial}{\partial J} \Pi_O^{(1)}(k; J) = \frac{e^2}{4\pi} \left[ -\frac{4|J|}{k^2 + 4J^2} + \frac{2}{k} \arctan \frac{k}{2|J|} \right],\]
(C.9)
where \(k := \sqrt{k^2}\).
C.2 power-series expansion (I)

Using the mathematical identities:

\[
\arcsin \frac{x}{\sqrt{1 + x^2}} = \arctan x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} x^{2n-1} = \frac{\pi}{2} - \arctan(x^{-1}),
\]

we obtain the following expansions.

For \(|\frac{2J}{k}| < 1\), we get

\[
\Pi_T^{(1)}(k; J) = \frac{e^2}{16} k \left[ -1 + 4 \frac{J^2}{k^2} - \text{sgn}(J) \frac{8}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{4n^2-1} \left( \frac{2J}{k} \right)^{2n+1} \right]
= \frac{e^2}{16} k \left[ -1 + 4 \frac{J^2}{k^2} - 64 \left| J \right|^3 \frac{1}{3\pi} k^{-3} + \mathcal{O} \left( \left( \frac{J}{k} \right)^4 \right) \right],
\]

\[
\Pi_O^{(1)}(k; J) = \frac{e^2}{4} J \left[ 1 - \text{sgn}(J) \frac{2}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{2n-1} \left( \frac{2J}{k} \right)^{2n-1} \right]
= \frac{e^2}{4} J \left[ 1 - 4 \frac{|J|}{\pi k} + \mathcal{O} \left( \left( \frac{J}{k} \right)^3 \right) \right],
\]

and

\[
\frac{\partial}{\partial J} \Pi_T^{(1)}(k; J) = \frac{e^2}{2} \left[ \frac{1}{k} - \text{sgn}(J) \frac{2}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{2n-1} \left( \frac{2J}{k} \right)^{2n-1} \right]
= \frac{e^2}{2} \left[ \frac{1}{2k} - \text{sgn}(J) \frac{4}{\pi} \frac{J^2}{k^2} + \mathcal{O} \left( \left( \frac{J}{k} \right)^4 \right) \right],
\]

\[
\frac{\partial}{\partial J} \Pi_O^{(1)}(k; J) = \frac{e^2}{4} \left[ 1 - \text{sgn}(J) \frac{4}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{2n-1} \left( \frac{2J}{k} \right)^{2n-1} \right]
= \frac{e^2}{4} \left[ 1 - 8 \frac{|J|}{\pi k} + 64 \left| J \right|^3 \frac{1}{3\pi} k^{-3} + \mathcal{O} \left( \left( \frac{J}{k} \right)^4 \right) \right].
\]

C.3 power-series expansion (II)

For \(|\frac{k}{2J}| < 1\), we obtain

\[
\Pi_T^{(1)}(k; J) = \text{sgn}(J) \frac{e^2}{2\pi} k \sum_{n=1}^{\infty} (-1)^{n} \frac{n}{4n^2-1} \left( \frac{k}{2J} \right)^{2n-1}
= \text{sgn}(J) \frac{e^2}{4\pi} k \left[ -\frac{1}{3} \left( \frac{k}{J} \right)^3 + \mathcal{O} \left( \left( \frac{k}{J} \right)^4 \right) \right],
\]

\[
\Pi_O^{(1)}(k; J) = \text{sgn}(J) \frac{e^2}{2\pi} J \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{2n-1} \left( \frac{k}{2J} \right)^{2n-1}
= \text{sgn}(J) \frac{e^2}{4\pi} k \left[ 1 + \mathcal{O} \left( \left( \frac{k}{J} \right)^2 \right) \right],
\]

39
\[
\frac{\partial}{\partial J} \Pi_T^{(1)}(k; J) = sgn(J) \frac{e^2}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{2n+1} \left( \frac{k}{2J} \right)^{2n}
\]
\[
= sgn(J) \frac{e^2}{4\pi} \left[ \frac{1}{3} \left( \frac{k}{J} \right)^4 + \mathcal{O}\left( \left( \frac{k}{J} \right)^5 \right) \right], \quad (C.17)
\]

\[
\frac{\partial}{\partial J} \Pi_O^{(1)}(k; J) = sgn(J) \frac{e^2}{\pi} \sum_{n=2}^{\infty} (-1)^{n} \frac{n-1}{2n-1} \left( \frac{k}{2J} \right)^{2n-1}
\]
\[
= sgn(J) \frac{e^2}{4\pi} \left[ \frac{1}{6} \left( \frac{k}{J} \right)^3 + \mathcal{O}\left( \left( \frac{k}{J} \right)^4 \right) \right]. \quad (C.18)
\]
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Figure Captions

Figure 1: Schematic phase diagram \((N_f, \theta)\). The solid line denotes the phase transition line below which the chiral symmetry is spontaneously broken.

| Typical points in the phase diagram |
|-----------------------------------|
| A \( \theta = 0 \) \( N_f > N_f^c \) \( \langle \bar{\Psi} \Psi \rangle = 0 \) \( \langle \bar{\Psi} \tau \Psi \rangle = 0 \) chiral, parity symm. |
| B \( \theta = 0 \) \( N_f < N_f^c \) \( \langle \bar{\Psi} \Psi \rangle \neq 0 \) \( \langle \bar{\Psi} \tau \Psi \rangle = 0 \) chiral B, parity symm. |
| C \( 0 < \theta < \theta_c \) \( N_f < N_f^c \) \( \langle \bar{\Psi} \Psi \rangle \neq 0 \) \( \langle \bar{\Psi} \tau \Psi \rangle \neq 0 \) chiral B, parity B |
| D \( \theta = \theta_c \) \( N_f < N_f^c \) \( \langle \bar{\Psi} \Psi \rangle \neq 0 \) \( \langle \bar{\Psi} \tau \Psi \rangle \neq 0 \) 1st order transition |
| E \( \theta > \theta_c \) \( N_f < N_f^c \) \( \langle \bar{\Psi} \Psi \rangle = 0 \) \( \langle \bar{\Psi} \tau \Psi \rangle \neq 0 \) chiral symm., parity B |

Figure 2: Dynamical fermion masses as functions of \( \theta \) for a pair of 2-component fermions.

Figure 3: Feynman diagrams needed to evaluate the effective potential up to the leading order of \( 1/N_f \) expansion. The solid line corresponds the fermion propagator and the broken line to the gauge boson propagator.

Figure 4: The effective potential in terms of two order parameters \( \phi = \varphi_e \) and \( \chi = \varphi_o \) in the chiral-symmetry breaking phase. When \( \theta = 0 \) (the point \( B \)), the effective potential has a minimum at \( (\varphi_e^*, 0) \). When \( \theta \neq 0 \), the location of the stationary point moves to \( (\varphi_e^*, \varphi_o^*) \) where \( \varphi_e^* \neq 0 \) and \( \varphi_o^* \neq 0 \).

Figure 5: The shape of the effective potential corresponding to the respective point, \( A, B, C, D, E \), in the phase diagram of Figure 1. The phase transition from \( A \) to \( B \) corresponds to the second order one, while the phase transition at \( D \) is the first order one.
FIGURE 1
FIGURE 2
FIGURE 3

(a)  

(b)
