A family of ternary decagonal tilings

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Abstract. A new family of decagonal quasiperiodic tilings are constructed by the use of
generalized point substitution processes, which is a new substitution formalism developed by the
author [N. Fujita, Acta Cryst. A 65, 342 (2009)]. These tilings are composed of three prototiles:
an acute rhombus, a regular pentagon and a barrel shaped hexagon. In the perpendicular space,
these tilings have windows with fractal boundaries, and the windows are analytically derived as
the fixed sets of the conjugate maps associated with the relevant substitution rules. It is shown
that the family contains an infinite number of local isomorphism classes which can be grouped
into several symmetry classes (e.g., $C_{10}$, $D_5$, etc.). The member tilings are transformed into
one another through collective simpleton flips, which are associated with the reorganization in
the window boundaries.

1. Introduction
The standard projection formalism [1, 2, 3, 4] for constructing quasiperiodic point sets had
evolved into the so called dualization method, which can be used to construct extensive
quasiperiodic tilings; a complete account of the dualization method can be found in [5] and
references cited therein. The generated tilings however are restricted to those whose windows
are given as convex polytopes. Recall that some quasiperiodic tilings are known to have
windows which have fractal boundaries, take for instance the dodecagonal square-triangle tilings
[6, 7]. Such tilings, including many unknowns too, are unlikely to be constructed by way of
projection because of the following difficulties. (i) Handling directly the broad range of fractal
windows, which also fulfil the non-trivial constraints for tilings with a moderately small number
of prototiles, is generally rather difficult. (ii) Even if the window is known, it is practically
difficult to avoid minute numerical errors associated with the fine structures of the window
when projecting the higher-dimensional structure down to the physical space.

These difficulties however are not relevant if instead a substitution (or an inflation) formalism
is used for construction. Recall that various substitution rules for constructing quasiperiodic
tilings with fractal windows can be found in the literature [8, 9, 10, 11, 12]. We describe a
substitution rule for a tiling by the following two steps: (S1) expansive similarity transformation
of the tiling and (S2) division of the expanded tiles into tiles of the original size. For a
deterministic tilings, it is commonly assumed that the division of every expanded tile is
determined uniquely by the local surroundings of that tile. Moreover, division rules for
neighbouring expanded tiles must not compete with each other because tiles are not allowed to
overlap. To search for a substitution rule of a new tiling, trial and error has been commonplace.
This uneconomic situation has led me to propose a more systematic way for composing
substitution rules for generating quasiperiodic tilings. The new method is called **generalized point substitution processes (GPSPs)** [13], in which Step S2 is divided into two sub-steps: (S2a) decoration of the expanded tiles with points which would be the candidate positions of the vertices of the next tiling and (S2b) choose an appropriate subset of the candidate positions so that the chosen points can be connected by unit edges to generate the next tiling. These sub-steps operate in a local manner so that they conform to the local uniqueness of the division Step S2.

In the subsequent two sections, the basic information about the decagonal Bravais module and the GPSP scheme is summarized. Then in Sec.4 the scheme is extensively applied to construct the new family of ternary decagonal quasiperiodic tilings. Note that Similar tilings have already been discussed in [13]. The aim is to somewhat extend the method used there so that it can handle a broader class of tilings. It turns out that the present method can generate tilings whose point groups are $D_5$, $C_5$, $C_2$, $D_1$ and $C_1$, in addition to $C_{10}$ which has been the only point group for the ternary tilings discussed in [13]. In the final section, we introduce a randomization procedure into the GPSP scheme, where we obtain further variety of tilings with partial disorder. The effects of simpleton flips in the tilings are also considered.

2. Decagonal Bravais module

The five crystallographic axes for decagonal systems can be represented by the five unit vectors, $e_j = (\cos (j\theta), \sin (j\theta))$ with $j = 0, 1, ..., 4$ and $\theta = 2\pi/5$. The first four members are taken as the basis set for indexing the lattice points in the system. The basis set generates a $Z$-module of rank four called the decagonal Bravais module [14], denoted by the symbol $\Lambda_{10}$. $\Lambda_{10}$ has the point symmetry of the dihedral group $D_{10}$, while it has an important property of scaling invariance $\tau \Lambda_{10} = \Lambda_{10}$, where $\tau = (1 + \sqrt{5})/2$ is the golden mean.

The two-dimensional axis vectors in $E^2$ are lifted into the four-dimensional Euclidean hyperspace $E_4$ as $e_j = (e_j, e_j^\perp)$, where the complementary components are defined by $e_j^\perp := e_{2j} (mod 5)$ where $j = 0, 1, ..., 4$. The first four members of $\{e_j\}$ are the generators of the decagonal lattice $L_{10}$. The orthogonal projection of $L_{10}$ onto $E^2$ gives nothing but the decagonal Bravais module $\Lambda_{10}$ [15]. The two-dimensional perpendicular space $E^\perp$ is defined as the orthogonal complement to $E^2$ in $E_4$; that is, $E_4 = E^2 \oplus E^\perp$. Then $L_{10}$ can be projected onto $E^\perp$ as well, generating the conjugate module $\Lambda^\perp_{10}$. Since the orthogonal projections from $L_{10}$ to both $\Lambda_{10}$ and $\Lambda^\perp_{10}$ are bijections, one can introduce a natural bijection $\pi$ between the modules; $\Lambda^\perp_{10} = \pi \Lambda_{10}$.

3. Generalized point substitution processes

We make a restricted use of the term **planar tilings** for disjoint coverings of the plane by copies of a finite number of polygonal prototiles while fulfilling edge-to-edge condition. From such a tiling, a point set can always be defined by taking the set of vertices. In contrast, however, it is not always possible to define a tiling from a point set. Let us confine ourselves to the case when points from the set can be connected with uncrossed unit edges given by the vectors $e_j (j = 0, 1, ..., 4)$ and these edges define a tesselation of the plane by a finite kinds of polygons. Then the point set can be called **unit connective**. It is further assumed that the vertex set $\Sigma_T$ of a tiling $T$ is a subset of the Bravais module $\Lambda_{10}$

The window $W_T \subset E^2$ is so defined that $\Sigma_T$ is the orthogonal projection of $(E^2 + W_T) \cap L_{10}$ onto $E^2$. \(^1\) Then the image of $\Sigma_T$ in $E^2$, denoted as $\Sigma_T^\perp := \pi \Sigma_T$, is a dense subset of the window $W_T$, which on the other hand should be included in the closure of $\Sigma_T^\perp$, i.e. $\Sigma_T \subset W_T \subset \overline{\Sigma_T^\perp}$.

A GPSP [13] is described as a three-step process for composing a substitution rule: (G1) **expansive similarity transformation of the original tiling by the ratio $\sigma = \tau^n (\tau : the \ Pisot \ unit, or \ the \ golden \ mean \ for \ the \ decagonal \ case)$$,$ (G2) replication of the basic motif $S$ on every vertex

\(^1\) Here the + symbol implies that $A + B \equiv \{a + b | a \in A, b \in B\}$. 

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and (G3) elimination of excess points, so that the remaining points are unit connective. Steps G1 and G2 together constitute a point inflation rule, which has been introduced for constructing a quasiperiodic point set [16], whereas Step G3 is an crucial step for constructing quasiperiodic tilings [13].

4. RPH tilings

RPH tilings have three prototiles, R (a rhombus with acute angles of 36°), P (a regular pentagon) and H (a barrel shaped hexagon). Every edge corresponds to one of the five unit vectors \( \{e_j\} \) defined above. In [13], two GPSPs have been identified as substitution rules for RPH tilings. These GPSPs are the mirror images of one another and are connected through simpleton flips involving three adjacent tiles, R, P and H. The two GPSPs can be applied in a mixed and arbitrary order so that an infinite number of tilings with the point group \( C_{10} \) can be obtained. The conjugate maps for these GPSPs are defined in the perpendicular space, and the windows with fractal boundaries for the relevant RPH tilings are derived analytically as the fixed sets of these conjugate maps.

Let us provide some information about the two GPSPs for RPH tilings. The scaling ratio for the expansion step G1 is taken as the square of the golden mean \( \tau \). The basic motif \( S \) for Step G2 is a centred decagon, with the representative indices being [0000] (centre) and [1100] (the decagonal shell). The first two steps, G1 and G2, would generate the candidate positions for the vertices of the next RPH tiling. These positions are superfluous for the vertices of an RPH tiling, so that an excess part should be eliminated at Step G3. To be specific, either of the two points that lie on every acute angle of expanded rhombi at the unit distance from the apex is to be eliminated. There are four ways to eliminate such points from an expanded rhombus, as shown in Fig.1; one can assign an arrow to each of the acute angles according to the choice of a point that is eliminated.

Since the two elimination rules labelled l and r (signifying the left- and right-handed chiralities, respectively) in Fig.1 are invariant under the two-fold rotation of the rhombus, either of these rules can be applied uniquely to all the expanded rhombi in a single iteration. It has been argued that the relevant two chiral GPSPs could be applied in an arbitrary order, resulting in a family of RPH tilings with the point group \( C_{10} \) [13]. For the left-handed GPSP, the arrows assigned to all the acute angles are symmetrically equivalent. Let us represent this situation as the wheel diagram shown in the upper left part of Fig.3.

It turns out that the window for an RPH tiling generated by the two GPSPs has a fractal boundary with the point symmetry \( C_{10} \). The window can be analytically represented as the limit figure obtained by successively applying the conjugate maps associated with the GPSPs. A conjugate map operates in \( \mathbb{E}_\perp \) and comprises the following three steps: (D1) contractive similarity transformation by the ratio \( \tilde{\sigma} \), which is the algebraic conjugate to \( \sigma \) and \( |\tilde{\sigma}| = 1/|\sigma| < 1 \), (D2) replication of the contracted figure onto every point of the conjugate motif \( \tilde{S} \) and (D3) subtraction of an excess part of the resulting figure corresponding to the elimination step G3 in the physical space. Note that, for decagonal RPH tilings, the algebraic conjugate to \( \sigma = \tau^2 \) is given by \( \tilde{\sigma} = 1/\tau^2 \) for Step D1. It has been shown [13] that Step D3 for RPH tilings can be described as a carving process of a small portion near the boundary of the figure obtained after Step D2. Note that ten strips are indicated by the broken lines in the window diagrams in Fig.3. Corresponding to the arrows for the ten acute angles in the wheel diagram, arrows can be associated with the ten strips. Then the aft-end of each strip (tail of the arrow) is so carved that it would coincide with the fore-end of the strip through the \( \tau \)-translation.

In Fig.2, a 1/20 sector of the boundary of the window is shown for first several iterations of the conjugate map for the left-handed GPSP. At each iteration, each line segment is replaced with an array of three segments, which are \( 1/\tau^2 \) times shorter than the original one. As the
conjugate map is repeated infinitely, the boundary would be a generalized von Koch curve whose fractal dimension is simply calculated as $\text{dim}(\partial W_T) = \frac{\ln 3}{\ln(\tau^2)} \approx 1.1415$.

It is also tempting to allow different elimination rules (i.e., arrow designations) for each expanded rhombus. Let us confine ourselves to the case when the choice of an elimination rule for each expanded rhombus only depends on the orientation of the rhombus. The choice for all the orientations can be specified by an wheel diagram. There are $2^{10} (= 1024)$ distinct arrow

![Diagram](image1)

**Figure 1.** Points can be eliminated in four distinct ways from an expanded rhombus. The eliminated points are represented as open circles. The choice of a side for each acute angle can be represented by an arrow.

![Diagram](image2)

**Figure 2.** A $1/20$ sector of the boundary of the window is constructed in a recursive manner. The corresponding window (Fig.3, upper left) is for the case when only the $l$-type elimination rule is applied.

![Diagram](image3)

**Figure 3.** Four different ways to assign arrows to acute angles depending on their directions are represented by the wheel diagrams. Their point group symmetries are as indicated. The windows for the RPH tilings generated by successively applying the relevant GPSPs are depicted.
designations for wheel diagrams. In Fig.3, four wheel diagrams as well as the corresponding windows are shown. Different shapes of the windows imply that the relevant tilings belong to distinct local isomorphism classes. The global point symmetry of a tiling (which is the same as that of the window) can be inferred from the point symmetry of the relevant wheel diagram.

It is readily seen that a tiling with a mirror symmetry could be generated only if the elimination rule \( m \) or \( m' \) (see Fig.1) would take place for at least one orientation of expanded rhombi. Since these elimination rules do not allow a two-fold axis, a two-fold rotational symmetry is incompatible with a mirror symmetry. Therefore, a global mirror symmetry exists only when a two-fold rotational symmetry is absent; the only allowed point groups are \( D_5 \) and \( D_1 \). In addition, \( C_2 \) and \( C_1 \) may also be the point group of a wheel diagram with arrows. Moreover, if different GPSPs are applied in a mixed and arbitrary order, the resulting point group would be a common subgroup to the point groups of the relevant GPSPs; \( C_5 \) is allowed in this way. Obviously, most of the 1024 possible wheel diagrams have the point group \( C_1 \). In Fig.4, an RPH tiling generated by four-cycles of GPSPs is demonstrated along with the corresponding window.

In [13], statistics of the local tile arrangements in RPH tilings have been analysed by two substitution matrices associated with the GPSPs of the types \( l \) and \( r \). It has been shown that the relative frequencies of the three prototiles are invariably given by \( n(R) : n(P) : n(H) = 1 : 2 : 1/\tau \). An alternative way to prove the statistics is to take the advantage of the four-dimensional description of the structures.

Let us denote by \( v \) the density of the vertices of an RPH tiling. Then it is given by the formula: \( v = w/\Omega_4 \), where \( w \) is the area of the window and \( \Omega_4 \) the four-dimensional volume of the primitive unit cell of the decagonal lattice \( L_{10} \). For the family of RPH tilings, \( w = 2\sqrt{5}a \) and \( \Omega_4 = 5\sqrt{5}/4 \), leading to \( v = (8/5)a \), where \( a = \sin(\pi/5) = \tau^{-1/2}5^{1/4}/2 \) is the area of an R tile. The point density is related to the densities of R, P and H tiles in the following way:

\[
n(R) + \frac{3}{2}n(P) + 2n(H) = \frac{8}{5}a. \tag{1}
\]

Recall also that an RPH tiling is a projection of a ‘puckered’ net which is embedded in \( L_{10} \) along

**Figure 4.** Part (a) shows a square patch of the RPH tiling generated by successively applying the four-cycle of GPSPs that are represented in Part (b). Part (c) depicts the window for the tiling. There is no point group symmetry except for the trivial one.
the physical space. The puckered net can be projected onto a lattice plane spanned by two of the five axis vectors $\mathbf{e}_i$ and $\mathbf{e}_j$ ($i \neq j$). A specific shape (R, P or H) and orientation of tiles in the physical space corresponds to a specific contribution to the area of the projected image. The total area of the projected image is proportional to that of the tiling in $E^\parallel$. There are ten lattice planes onto which the tiling can be mapped, and accordingly there are ten equations for the densities of the relevant tiles. By appropriately adding the left and right hand sides of these equations, one derives the following two equations for the density of the three prototiles:

$$n(R) + \frac{1}{2}n(P) + n(H) = \frac{4}{5}a,$$

$$\frac{3}{2}n(P) + 2n(H) = \frac{4}{5}\tau a.$$  

From Eqs.(1-3), it turns out that the densities for the three prototiles are

$$n(R) = \frac{4}{5\tau^2}a, \quad n(P) = \frac{8}{5\tau^2}a, \quad n(H) = \frac{4}{5\tau^3}a.$$  

The right hand sides of Eqs.(1-3) may be modified if there exists a linear phason strain, which however is assumed to be absent in the present report.

5. Random RPH tilings

The 1024 GPSPs as described above can be applied in an arbitrary order, resulting in an unlimited number of RPH tilings or, more precisely, local isomorphism classes; these tilings still maintain perfect quasiperiodicity. Let us now lift the constraint on the uniqueness of the elimination rule for all the expanded rhombi in the same orientation. If a new GPSP, in which the elimination rule for each expanded rhombus is chosen at random, is used at each iteration, a member of a stochastic ensemble of RPH tilings will be obtained. It will still fulfil the edge-to-edge condition. The number of possible patterns within a patch of diameter $R$ will diverge exponentially as $R$ is increased. Such a randomized GPSP can be thought of as a realization of the concept of random substitution rules proposed by Gummelt [17], who gave the example of randomized Penrose/Tübingen-triangle tilings.

In every GPSP described so far, irrespective of it being randomized or not, the elimination step G3 takes a subset of a larger point set, which is generated through Steps G1 and G2. It follows that by repeating the latter two steps only, a unique super-set can be defined from which the vertices of any RPH tiling originate. The super-set is the set of vertices of the para-Penrose tiling [16], whose window is a moth-eaten version of the regular decagon; see Fig.4-no.1 in [16]. Then for any of the randomized RPH tilings described above, points close to the boundary of this window is partially removed. The partial occupancies in certain triangular regions close to the boundary of the window are the same as the deterministic case as given in the end of the caption of Fig.8 in [13]; the only difference lies in that in the present case points are removed in a stochastic manner and that there is no clear boundary between occupied and unoccupied parts. In all cases the main part of the window is fully occupied except the partially occupied boundary regions, hence the quasiperiodicity is almost maintained. The relatively small amount of randomness will cause weak diffuse components superposed in the diffraction spectra.

The elimination rules for expanded rhombus as shown in Fig.1 can be mutually connected through simpleton flips involving three tiles, R, P and H. Every flip is caused by a hop of a vertex by a distance of $1/\tau$ in the parallel direction to the relevant arrow. This corresponds to a transport of a point in the perpendicular space by a distance of $\tau$ along the associated strip; see Fig.3. Moreover, all the RPH tilings we have introduced so far are mutually connected through collective simpleton flips of the same kind, because such flips can be repeated indefinitely so that the boundary regions of the window can be reorganized.
We have also performed a preliminary Monte Carlo study in which simpleton flips are excited stochastically when there is the combination of three tiles R, P and H. It has turned out that the resulting tilings are genuinely random, where the quasiperiodic order in the initial tiling has been completely dissolved while only the orientational order has been maintained.

Finally, it may be helpful to redirect the reader to some physical counterparts. The same kind of simpleton flips as discussed above has been observed in situ in a $d$-Al-Cu-Co quasicrystal at a temperature of 1123 K [18]. Furthermore, the centres of atomic clusters in a high-temperature phase of $d$-Al-Ni-Co quasicrystal can be described rather nicely by the vertices of an RPH tiling [19, 20]. Therefore, the family of RPH tilings may serve as templates for physical models of decagonal quasicrystals.

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