Dirichlet principal eigenvalue comparison theorems in geometry with torsion

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Abstract: We describe min-max formulas for the principal eigenvalue of a V-drift Laplacian defined by a vector field V on a geodesic ball of a Riemannian manifold N. Then we derive comparison results for the principal eigenvalue with the one of a spherically symmetric model space endowed with a radial vector field, under pointwise comparison of the corresponding radial sectional and Ricci curvatures, and of the radial component of the vector fields. These results generalize the known case \( V = 0 \).

1 Introduction

Given a vector field \( V \) on a \( m \)-dimensional Riemannian manifold \((N, g)\), the V-drift Laplacian \( \Delta_V u = \Delta u - g(V, \nabla u) \) can be introduced in the context of Riemannian geometry with torsion. If \( \nabla^V \) is a metric connection with vectorial torsion defined by \( V \), \( \Delta_V u \) is the trace of the covariant derivative of \( du \). If \( V = 0 \), \( \Delta_0 u \) is the usual Laplacian for the Levi-Civita connection. The purpose of this work is twofold. First, to prove the existence of a principal

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eigenvalue of the operator $-\Delta V$, $\lambda^*_V$, for any vector field $V$ on a regular domain $\bar{M}$ of $N$, under the Dirichlet boundary condition. Second, to establish a variational principle for $\lambda^*_V$, and use it to obtain comparison results when $\bar{M}$ is a geodesic ball. In Lemma 4, we show that there is a weight function $f$, such that $\Delta V$ is self-adjoint with respect to the $L^2$ space with measure weighted by $e^{-f}$ if and only if $V = \nabla f$. In this case, $\Delta V$ is the Bakry-Émery $f$-Laplacian $\Delta_f$. If $\bar{M} = M \cup \partial M$ is a compact domain of $N$ with smooth boundary $\partial M$, the spectrum for the eigenvalue problem of $-\Delta_f$ with Dirichlet boundary condition is a discrete sequence of positive real values converging to infinity. Furthermore, each eigenvalue has a variational characterization of Rayleigh type, as shown in [19]. On the other hand, any vector field, $V$, which is not a gradient gives rise to an operator $\Delta V$ for which there is no canonically associated Hilbert space on which this operator is self-adjoint. As a consequence, standard arguments used to establish a variational principle for an eigenvalue may not be applied.

We obtain existence of a principal eigenvalue $\lambda^*_V$ using Krein-Rutman theory for compact operators on $C^{1,\alpha}_0(\bar{M})$. This eigenvalue is a distinguished one, simple, with a positive eigenfunction $\omega V$, only vanishing on $\partial M$. The eigenvalue $\lambda^*_V$ is positive by a maximum principle argument. In Proposition 6 we show that $-\Delta V$ and its formal adjoint operator $-\Delta^*_V$ have the same set of eigenvalues, which form a discrete set that may only accumulate at infinity. Furthermore, they have the same principal eigenvalue. As a consequence, the weak maximum principle also holds for $\Delta^*_V$.

In Theorem 9, using principal eigenfunctions, we give a simple proof of Barta’s type inequalities (20)-(21) for $\lambda^*_V$ on a regular domain $\bar{M}$. This is a well known inequality for the case $V = 0$ ([6], III.2., Lemma 1), and it is a useful tool for estimating principal eigenvalues. It consists of a min-max formula for the ratio $-\Delta_V u / u$, taken over all functions $u$ on a positive cone of $H^1_0(M)$.

In Theorem 12, we describe a Rayleigh type variational principle for $\lambda^*_V$ on a regular coordinate chart $\bar{M}$. This variational principle was initially due to Holland [18] for a certain type of second-order linear elliptic equations on domains of Euclidean space. Later, it was reformulated by Godoy, Gossez and Paczka in [11], using suitable weighted Sobolev spaces. This provided an alternative proof of the formula. Instead of Holland’s method which uses ergodic measures to obtain a positive solution, $G_V$, of a related degenerate elliptic second order differential equation, it consists of applying Krein-Rutman theory for compact, positive and irreducible operators on weighted $L^2$ spaces. It thereby requires less regularity conditions on the domain and coefficients of the operator. We follow this second approach, taking weighted Sobolev spaces on $\bar{M}$ weighted by the square of the intrinsic distance function to $\partial M$, $d_{\partial M}(p) = \inf_{x \in \partial M} d(p, x)$, for $p \in \bar{M}$, defined in (22)-(23). In Lemma 11(3) we obtain Sobolev embedding theorems in case $\bar{M}$ is a global chart domain, generalizing the known
Euclidean case. The variational principle is given by

$$\lambda_V = \inf_{\{u \in D_0 : \|u\|_{L^2} = 1\}} \left( \mathcal{L}(u, u) - \inf_{v \in H^1_0(M)} Q_u(v) \right),$$

where

$$\left\{ \begin{array}{ll}
\mathcal{L}(u, u) &= \int_M (|\nabla u|^2 + u g(V, \nabla u)) dM, \\
Q_u(v) &= \int_M u^2 (|\nabla v|^2 - g(V, \nabla v)) dM.
\end{array} \right.$$ 

Here, the infimum is taken over all $L^2$-unit functions of the class

$$D_0 := \left\{ u \in H^1(M) : \frac{u(p)}{d_M(p)} \in [C_1, C_2] \text{ for some constants } C_i > 0 \right\}. \quad (1)$$

This infimum is achieved at a function $u_V \in D_0$, $L^2$-normalized, and given by the product $u_V = \omega_V \cdot \sqrt{G_V}$, where $G_V$ is a bounded, positive, weak solution of the degenerate elliptic differential equation, $\text{div}^0(\omega_V^2(\nabla G + GV)) = 0$, that is, a solution of the integral equation (25). Furthermore, it is unique in the weighted Sobolev space (23), up to a multiplicative constant. In Proposition [13], we show that, when $V$ is the gradient of a function $f$, it turns out that $G_V = e^{-f}$, and this variational principle reduces to the Rayleigh variational principle for the first eigenvalue $\lambda_f$, given in equation (10).

These formulas allow us to obtain comparison results for the principal eigenvalue on geodesic balls, under pointwise comparison of the radial curvatures and the radial component of $V$, with the ones of model spaces. On $N$, the radial direction from a point $p_0$ is defined by $\partial_t(p) = \nabla r(p)$, where $t = r(p) = d(p, p_0)$ is the intrinsic distance of $p$ to $p_0$. The exponential map of $N$ defines the spherical geodesic parametrization of a closed geodesic ball $\bar{M} = \bar{B}_{r_0}(p_0)$. Namely, $p = \exp_{p_0}(t\xi) = : \Theta(t, \xi)$, with $\xi$ in the unit sphere of $T_{p_0}N$, and $0 \leq t \leq r_0$. The radial component of $V$ is given by its projection onto the radial direction, $h_1(t, \xi) = g(V(p), \partial_t(p))$. Our model spaces are geodesic balls $\bar{M}^\rho$ of spherically symmetric spaces, $N^\rho = [0, l] \times_\rho S^{m-1}$, endowed with a radial vector field, $V^\rho = h(t)\partial_t$. The warping function $\rho$ is chosen based on pointwise comparison of the radial curvatures with the ones of $N$. The function $h(t)$ is chosen based on pointwise comparison with $h_1(t, \xi)$ of $V$. Comparison theorems for the first eigenvalue of $-\Delta_0$ on a geodesic ball were obtained by Cheng in [7], using space forms as model spaces. These theorems were generalized by Freitas, Mao and the second author in [9], taking as model spaces the larger class of spherically symmetric spaces.

Next, we state our two main theorems on a closed geodesic ball $\bar{M} = \bar{B}_{r_0}(p_0)$, endowed with a vector field $V$. We are assuming $r_0 < \min\{\text{inj}(p_0), l\}$, where $\text{inj}(p_0)$ is the injectivity radius of $p_0 \in N$. The ball $M^\rho$ in the model space is centered at the origin and has radius $r_0$. The radial sectional curvature of $N^\rho$ is given by $-\rho'(t)/\rho(t)$. This curvature is a constant $\kappa$ in the case of space forms. Namely, the spheres when $\rho(t) = (\sqrt{\kappa})^{-1} \sin \sqrt{\kappa}t$, for $\kappa > 0,$
the Euclidean space when \( \rho(t) = t \), for \( \kappa = 0 \), and the hyperbolic spaces when \( \rho(t) = (\sqrt{-\kappa})^{-1} \sinh \sqrt{-\kappa} t \), for \( \kappa < 0 \). The radial vector field on the model space, \( V^\rho \), depends on \( t \) only, with initial condition \( h(0) = 0 \). The principal eigenvalue is the first eigenvalue \( \lambda_{p,H} \) of the Bakry-Émery \( H \)-Laplacian \( \Delta^\rho_H \) on \( \tilde{M}^\rho \), with Dirichlet boundary condition, where \( H'(t) = h(t) \). In Section 4, we describe properties of the corresponding principal eigenfunction \( \omega_{p,H} \).

We also describe the whole spectrum of \(-\Delta_{p,H}\), relating to a family of one-dimensional eigenvalue problems and the spectrum of the \((m - 1)\)-sphere.

**Theorem 1.** We assume the radial sectional curvatures of \( M \), \( K(\partial_t, X) \), and the radial component of \( V \) satisfy at each point \( p = \Theta(t, \xi) \),

\[
K(\partial_t, X) \leq -\frac{\rho''(t)}{\rho(t)},
\]

\[
h_1(t, \xi) \leq h(t),
\]

for all \( 0 \leq t \leq r_0 \), and unit vectors \( \xi \in T_{p_0}M \) and \( X \in T_p M \) orthogonal to \( \partial_t(p) \). Then, we have, \( \lambda_V \geq \lambda_{p,H} \). Furthermore, equality of the eigenvalues holds if and only if \( M \) is isometric to \( M^\rho \) and \( h_1(t, \xi) = h(t) \), for all \( (t, \xi) \). In this case, the principal eigenfunctions are the same, that is, \( \omega_V = \omega_{p,H} \).

If \( V = 0 \) and \( H = 0 \), this is Theorem 4.4 of [9]. Applying Theorem 1 to vector fields \( V \) on a geodesic ball of a model space \( N^\rho \), we conclude that the principal eigenvalue \( \lambda_V \) is just \( \lambda_{p,H} \), if the radial component of \( V \) depends on \( t \) only. Therefore, in this case, the principal eigenvalue does not depend on the non-radial component of \( V \), and the principal eigenfunction \( \omega_V \) of the \( V \)-drift Laplacian is the radial first eigenfunction \( \omega_{p,H} \) for the \( H \)-Laplacian.

**Theorem 2.** We are given radial vector fields, \( V(p) = h_1(t, \xi) \partial_t \) on \( \tilde{M} \), and \( V^\rho = h(t) \partial_t \) on \( \tilde{M}^\rho \). We assume that \( h(t) \geq 0 \), and \( h_1(0, \xi) = h(0) = 0 \) holds for all unit vectors \( \xi \in T_{p_0}M \). We also assume that the radial Ricci curvatures of \( M \) and \( V \) satisfy the following inequalities

\[
\text{Ricci}(\partial_t, \partial_t) \geq -(m - 1)\frac{\rho''(t)}{\rho(t)},
\]

\[
\text{div}^0(V)(t, \xi) - \frac{1}{2}|V|^2(t, \xi) \geq \text{div}^0(V^\rho)(t) - \frac{1}{2}|V^\rho|^2(t),
\]

for all \( t, \xi \), with \( t \leq r_0 \). Then \( \lambda_V \leq \lambda_{p,H} \), and equality of the eigenvalues holds if and only if \( M \) is isometric to \( M^\rho \) and equality holds in (3), for all \( 0 \leq t \leq r_0 \). In this case, the principal eigenfunctions are related by the formula, \( \omega_V(t, \xi) = \omega_{p,H}(t)e^{-\int_{H(t)}^{H_1(t, \xi)} \frac{H''(t)}{2} dt} \), where \( \frac{dH(t)}{dt}(t, \xi) = h_1(t, \xi) \), and \( H'(t) = h(t) \). If, additionally, \( \rho(t), h(t), \) and \( h_1(t, \xi) \) are analytic on \( t \in [0, r_0] \), then \( h_1 = h \) and \( \omega_V = \omega_{p,H} \).
The above theorem, in case \( V = V^p = 0 \) coincides with Theorem 3.6 of [9]. The assumption \( r_0 < \text{inj}(p_0) \) can be dropped if the min-max formula in Theorem 12 is valid on domains with less boundary regularity. Inequality (5) at \( t = 0 \) means \( h'_1(0, \xi) \geq h'(0) \), for all \( \xi \).

In [14, 15, 16], comparison results are obtained on an open domain \( \Omega \) of Euclidean space, where the infimum and the supremum of \( \lambda^*_V \) are searched among all vector fields \( V \) with \( \|V\| \leq \tau \), for a fixed constant \( \tau \geq 0 \). The model space is the Euclidean disk with volume \( |\Omega| \), endowed with the bounded radial vector field \( \tau x/|x| \), not defined at \( x = 0 \). In [17], \( \tau \) is allowed to be a radial function \( \tau(|x|) \), and a suitable symmetric rearrangement of the drift-Laplacian on the disk is taken. The results are obtained under comparison of \( L^\infty \) or \( L^2 \) norms of the vector field. Presently, we allow our model spaces to be geodesic disks of any spherically symmetric space, endowed with any smooth radial vector field \( V(r(x)) \), vanishing at the origin. Our method is based on comparing pointwise the radial part of the vector fields and the radial curvatures. Radial curvature comparison conditions, as stated in the above theorems, can be translated into comparison conditions between volumes of geodesic balls of \( N \) and \( N^p \) of radius \( t \leq r_0 \). Namely, (2) and (4) correspond to nondecreasing and nonincreasing ratio volume elements \( \theta(t, \xi) \), defined in (33), respectively (see [9]).

A simple application of the min-max formulas leads to some comparison results between \( \lambda_0 \) and \( \lambda^*_V \) in Proposition 15. In Corollary 16 we conclude that, if \( \text{div}^0(V) \leq 0 \), then \( \lambda_0 \leq \lambda^*_V \). In the particular case \( V = \nabla f \), we get the following conclusion for a variation on the first eigenvalue \( \lambda_f \).

**Proposition 3.** If \( f \in C^\infty(\bar{M}) \) has constant \( 0 \)-Laplacian, \( \Delta_0 f = 2c_0 \), then \( \frac{d}{d\epsilon}\big|_{\epsilon=0}\lambda_{\epsilon f} \) exists and it is equal to \( -c_0 \).

We may question geometric properties of eigenvalues of \( -\Delta_V \), real or complex; not only the principal eigenvalue. Another natural development will be the study, in the Riemannian context, of the variation of the principal eigenvalue for domain variations under variational constraints. An extension of the variational principle for \( \lambda^*_V \) to any regular Riemannian domain could be obtained by extending to such domains the embedding results on weighted Sobolev spaces given in Lemma 11(3), and main result of [22].

## 2 The \( V \)-Laplacian

We consider \( \bar{M} = M \cup \partial M \) a smooth, compact domain with boundary, which is contained in a smooth \( m \)-dimensional Riemannian manifold \((N, g)\). Denote by \( \nabla^0 \) its Levi Civita connection. We will use the subscript or superscript 0 on geometric objects that are defined with respect to \( \nabla^0 \). Given a smooth vector field \( V \) on \( \bar{M} \), we define a new connection by

\[
\nabla^V_X Y = \nabla^0_X Y + \frac{1}{m-1}(g(X,Y)V - g(V,Y)X).
\]

(6)
This is a metric connection, i.e. \( 0 = \nabla_Z g(X,Y) = Z \cdot (g(X,Y)) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y) \).

The torsion is given by

\[
T(X,Y) = \frac{1}{m-1} (g(V,X)Y - g(V,Y)X),
\]

and it is one of the two distinguished types out of the three torsion types for metric connections, namely the vectorial torsion as named by Cartan [5]. For each function \( f \) and it is one of the two distinguished types out of the three torsion types for metric connections, namely the vectorial torsion as named by Cartan [5]. For each function \( u : M \to \mathbb{R} \) of class \( C^2 \), the Laplacian of \( u \) with respect to the affine connection \( \nabla^V \) is given by

\[
\Delta_V u := \text{tr}(\nabla^V du) = \Delta_0 u - g(V, \nabla u),
\]  

(7)

where \( \nabla u \) is the \( g \)-gradient of \( u \). This is the so-called Laplacian with drift the vector field \( V \). In case \( V \) is the gradient of a function \( f \), this is the Bakry-Émery \( f \)-Laplacian,

\[
\Delta_f u = \Delta_0 u - g(\nabla f, \nabla u) = e^f \text{div}^0(e^{-f} \nabla u).
\]

The \( f \)-Laplacian is self-adjoint for the \( e^{-f} \)-weighted \( L^2 \) space, \( L^2_{e^{-f}}(M) \), that is

\[
\int_M v \Delta_f u e^{-f} dM = \int_M u \Delta_f v e^{-f} dM, \quad \forall u, v \in C_c^\infty(M),
\]

where \( C_c^k(M) \) is the space of functions of class \( C^k \) \((0 \leq k \leq +\infty)\) with compact support in the interior of \( M \). Equivalently, \( \Delta_f \) is \( L^2 \)-self-adjoint for the conformally equivalent metric \( \hat{g} = e^{-\frac{2}{m}f} g \). Note that \( \hat{\Delta}u = e^{\frac{2}{m}f} \Delta_V u \), where \( V = -\frac{2(m-1)}{m} \nabla f \) is of gradient type.

**Lemma 4.** The \( V \)-Laplacian is \( L^2_{e^{-f}} \)-self-adjoint for some density function \( e^{-f} \) if and only if \( V = \nabla f \).

**Proof.** Let \( u, v \in C_c^2(M) \). Applying Stokes’s theorem to \( \text{div}^0(e^{-f}(u \nabla v - v \nabla u)) \) we get,

\[
\int_M (u \Delta_V v - v \Delta_V u) e^{-f} dM = \int_M g(\nabla f - V, u \nabla v - v \nabla u) e^{-f} dM.
\]  

Assume \( \Delta_V \) is \( L^2_{e^{-f}} \)-self-adjoint. Hence [2] \( = 0 \) holds. Take any \( u \in C_c^2(M) \). Let \( v \in C_c^2(M) \) with \( v = 1 \) on a neighbourhood of the support of \( u \). The above equality implies \( \int_M g(\nabla f - V, \nabla u) dM = 0 \). Thus, for any \( u, v \in C_c^2(M) \), we have \( \int_M g(\nabla f - V, \nabla (uv)) dM = 0 \). From \( \nabla(uv) = u \nabla v + v \nabla u \) and [2] \( = 0 \) we obtain \( \int_M 2uv g(\nabla f - V, \nabla u) dM = 0 \). Since \( v \) is arbitrary, then \( g(\nabla f - V, \nabla u) = 0 \) for all \( p \in M \), and \( V = \nabla f \), necessarily. \( \square \)

The formal adjoint of \( \Delta_V : C_c^2(M) \to C_c^0(M) \) is the operator \( \Delta^*_V : C_c^2(M) \to C_c^0(M) \), given by

\[
\Delta^*_V v = \Delta_0 v + g(V, \nabla v) + \text{div}^0(V) v.
\]  

(9)
It can be extended as an operator defined on $H^1_0(M)$ (see notations in Section 3), which satisfies for all $u, v \in C^2_c(M)$
\[
\int_M v \Delta u dM = \int_M u \Delta v dM.
\]

If $V = \nabla f$, for some function $f \in C^2(\overline{M})$, it is known that the Dirichlet eigenvalue problem $\Delta f u + \lambda u = 0$, $u = 0$ on $\partial M$, consists of a discrete sequence $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \ldots \to +\infty$, [19]. Furthermore, assuming each eigenvalue is repeated the number of times equal to its multiplicity, we may take $\{\phi_1, \phi_2, \ldots\}$ a complete orthonormal basis of $L^2_{\overline{M}}(M)$, composed of the corresponding eigenfunctions. The first eigenvalue $\lambda_f := \lambda_1$ is positive, of multiplicity one, and satisfies a Rayleigh variational principle:
\[
\lambda_f = \inf_{u \in H^1_0(M)} \frac{\int_M \nabla u^2 e^{-f} \, dM}{\int_M u^2 e^{-f} \, dM} = \inf_{u \in C^2_c(M)} \frac{\int_M -u \Delta f u e^{-f} \, dM}{\int_M u^2 e^{-f} \, dM}.
\]

The infimum is achieved at $u$ if and only if $u$ is the $\lambda_f$-eigenfunction $\omega_f$. All eigenvalues satisfy a similar variational principle (cf. [19]).

3 The principal eigenvalue

As in the previous section, we are assuming $\overline{M} = M \cup \partial M$ is a smooth compact domain with boundary contained in a complete Riemannian manifold $N$. We consider the Sobolev space
\[
H^1(M) = \{u \in L^2(M) : \exists \nabla u \in L^2(TM)\},
\]
endowed with the $H^1$-norm $\|u\|_1^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2$, where $\|u\|_{L^2}^2 = \int_M u^2 \, dM$ and $\|\nabla u\|_{L^2}^2 = \int_M \|\nabla u\|_{L^2}^2 \, dM$. Here, $\nabla u \in L^2(TM)$ means the weak gradient of $u$ (cf. [1], I.5, Definition 4). The subspace $H^1_0(M)$ is the $H^1$-closure of $C^\infty_c(M)$, and $C^\infty_c(M)$ is $L^2$-dense on $L^2(M)$. We recall that $H^1_0(M) \cap C(\overline{M}) \subset C_0(\overline{M})$, where $C(\overline{M})$ is the space of continuous functions on $\overline{M}$ and $C_0(\overline{M})$ its subspace of functions that vanish on $\partial M$. Conversely, $C^1(M) \cap C_0(M) \subset H^1_0(M)$ (2, 3.50 and Remark). A more recent result (cf. [22]), guarantees that if $M$ is diffeomorphic to a Euclidean domain with boundary of class $C^1$, then $H^1(M) \cap C_0(M) \subset H^1_0(M)$.

Let $H^1_0(M)'$ be the dual space of $H^1_0(M)$ of the bounded linear functionals on $H^1_0(M)$ with supremum norm $\|F\| := \sup_{\|u\|_1 = 1} F(u)$, $\forall F \in H^1_0(M)'$. By the Riesz representation theorem, $J : H^1_0(M) \to H^1_0(M)'$, $J(u)(v) := (u, v)_1$, is a continuous isomorphic surjective isometry with continuous inverse $J^{-1}$. The linear operator $I : L^2(M) \to H^1_0(M)'$, $I f(v) := F_f(v) := \int_M f v \, dM$, defines a continuous embedding of $L^2(M)$ into $H^1_0(M)'$ with $\|I f\| \leq \|f\|_{L^2}$. The usual embedding $H^1(M) \subset L^2(M)$ is compact for $m \geq 2$ (2). Kondrakov
Theorem 2.34), inducing a compact operator $I : H^1_0(M) \rightarrow H^1_0(M)'$. We have $\|F_f\| = \|u_f\|_1 \leq \|f\|_{L^2}$, where $u_f = J^{-1}(F_f)$, and $(f, v)_{L^2} = F_f(v) = (u_f, v)_1$ for $v \in H^1_0(M)$.

For each constant $\epsilon$, we consider the $H^1$-continuous bilinear functionals $\mathcal{L}_\epsilon$, $\mathcal{L}_\epsilon^* : H^1_0(M) \times H^1_0(M) \rightarrow \mathbb{R}$,

$$\mathcal{L}_\epsilon(u, v) = \int_M g(\nabla u, \nabla v) dM + \int_M g(V, \nabla u)v dM + \epsilon \int_M uv dM,$$

$$\mathcal{L}_\epsilon^*(u, v) = \int_M g(\nabla u, \nabla v) dM - \int_M g(V, \nabla u)v dM + \int_M (\epsilon - \text{div}^0(V))uv dM.$$

The norm of a multilinear operator that is considered is the supremum norm, $\|\mathcal{L}_\epsilon\| = \sup_{\|u\|_1 = \|v\|_1 = 1} |\mathcal{L}_\epsilon(u, v)|$. We have $\mathcal{L}_\epsilon^*(u, v) = \mathcal{L}_\epsilon(v, u)$, and continuous operators $L_\epsilon : H^1_0(M) \rightarrow H^1_0(M)'$, and $\hat{L}_\epsilon : H^1_0(M) \rightarrow H^1_0(M)$ (similarly $\mathcal{L}_\epsilon^*$ and $\hat{L}_\epsilon^*$), defined by $\mathcal{L}_\epsilon(u, v) = L_\epsilon u(v) = (\hat{L}_\epsilon u, v)_1$. We can naturally extend the operators $\mathcal{L}_\epsilon(u, v)$ and $\mathcal{L}_\epsilon^*(u, v)$, for $u \in H^1(M)$ and $v \in H^1_0(M)$.

Given continuous functions $f \in C(M)$ and $\phi \in C(\partial M)$, a strong solution is a function $u \in C^2(M) \cap C(\bar{M})$ that satisfies $-\Delta u + \epsilon u = f$ at every $p \in M$ and $u = \phi$ at any $p \in \partial M$. Given $F \in H^1_0(M)'$, a weak solution of $-\Delta u + \epsilon u = F$, $u = 0$ on $\partial M$, is an element $u \in H^1_0(M)$ that satisfies $\mathcal{L}_\epsilon(u, v) = F(v)$, for all $v \in H^1_0(M)$. We have $\hat{L}_\epsilon u = \hat{L}u + \epsilon J^{-1}Iu$, $L_\epsilon u = Lu + \epsilon Iu$, or in simplified notation, $L_\epsilon u = Lu + \epsilon u = -\Delta u + \epsilon u$.

Using local coordinate charts on $M$, we apply theorems of Chapter 8 of [10], Section 3.6 of [2], or Section 6.3 of [8], to determine regularity of solutions on open domains $M$. If $u \in H^1_0(M)$ is a weak solution of $-\Delta u + \epsilon u = f \in L^2(M) \subset H^1_0(M)'$, then $u \in H^2(M)$. Furthermore, under additional conditions, we have the following conclusions. If $f \in H^k(M)$, then $u \in H^{k+2}(M)$ and $\|u\|_{k+2} \leq C(\|u\|_{L^2} + \|f\|_k)$. If $f \in C^{k,\alpha}(\bar{M})$ (resp. $C^\infty(M)$), then $u \in C^{k+2,\alpha}(M)$ (resp. $C^\infty(M)$). For regularity up to the boundary we have the Sobolev theorem on any smooth Riemannian manifolds with boundary (Theorem 2.30). Namely, if $u \in H^k(M)$ and $2k \geq n + 2s$, where $s \geq 0$, then $u \in C^{2s}(\bar{M})$. In this case, since we are assuming $u \in H^1_0(M)$, $u = 0$ on $\partial M$. Furthermore, if $s \geq 2$, then $u$ satisfies $-\Delta u + \epsilon u \in L^2(M)$ if $-\Delta u + \epsilon u = 0$ are in $C^\infty(\bar{M})$ and vanish on $\partial M$. The same conclusions hold for $\Delta^*_\epsilon u = \epsilon$. If $\epsilon = 0$ we omit the number in $L_\epsilon$. There is uniqueness of weak solutions $u \in H^1_0(M)$ of $\Delta u - \epsilon u = f$, for all $\epsilon \geq 0$. The same holds for $\Delta^*_\epsilon u = \epsilon$ if $\text{div}^0(V) \leq \epsilon$. They are consequences of uniqueness of generalized Dirichlet problems (cf. [10] Corollary 8.2 and Theorem 8.3, whose arguments are valid in any regular Riemannian compact domain). Uniqueness results can be derived from maximum principles, and existence results from Fredholm theory for compact operators on Hilbert spaces. Both theories are used to prove the existence of a principal eigenvalue. We recall some maximum principles that we need (cf. [2], Theorem 3.74).
Theorem 5. Assume $M$ compact with boundary, and let $\nu$ be the unit outer normal to $\partial M$. Let $u \in C^2(M)$ such that $\Delta_V u \geq 0$.

(a) (weak maximum principle) If $u \in C(\bar{M})$ and $u_{\mid \partial M} \leq 0$, then $u \leq 0$.

(b) (Hopf maximum principle) If $u$ achieves a nonnegative maximum $T \geq 0$ at $p \in M$, then $u$ is constant.

(c) (boundary condition) Assume $u \in C(\bar{M})$ and $u \leq 0$. If $u$ is not constant, and $u(p_1) = 0$ at $p_1 \in \partial M$, then $\frac{\partial u}{\partial \nu}(p_1) > 0$, provided this derivative exists.

The same holds for $\Delta_V - \epsilon$ for any constant $\epsilon \geq 0$, and for $\Delta_V^\star - \epsilon$ if $\epsilon \geq \text{div}^0(V)$.

We will see that the weak maximum principle holds for $\Delta_V^\star$ in Proposition 6.

Next we recall how the Fredholm alternative theorem describes the spectrum of $-\Delta_V$. If $V$ is not of gradient type, the set of eigenvalues $\Lambda(-\Delta_V)$ may have complex numbers, with complex eigenfunctions. Moreover, a non self-adjoint $V$-Laplacian for any $L^2$-inner product does not have an $L^2$-diagonalization process that splits $L^2(M)$ into eigenspaces. A long computation can show that, in general, $\Delta_V$ is not a normal operator unless $V$ is a parallel vector field.

From the Rayleigh principle for $V = 0$, we can take $\epsilon$ sufficiently large such that coerciveness of $L_\epsilon$ is satisfied, that is, $L_\epsilon(u,u) \geq \beta \|u\|_1^2$, for a positive constant $\beta$. Hence, the bounded linear operators $L_\epsilon$ and $L_\epsilon^\star$ are isomorphisms with bounded inverses. The continuous operator $T_\epsilon = L_\epsilon^{-1} \circ I : L^2(M) \to H_0^1(M)$ satisfies $\mathcal{L}_\epsilon(T_\epsilon f, v) = (f, v)_{L^2}$, and it is a compact operator as an operator on $L^2(M)$ (as well as on $H_0^1(M)$). Similarly, we define a compact operator $H_\epsilon$ from $L_\epsilon^\star$. Then we have, for any $u, v \in C_0^\infty(M),$

$$(u,v)_{L^2} = Iu(v) = L_\epsilon(T_\epsilon u,v) = L_\epsilon^\star(v, T_\epsilon u) = L_\epsilon^\star(H_\epsilon H_\epsilon^{-1}v, T_\epsilon u) = (H_\epsilon^{-1}v, T_\epsilon u)_{L^2} = (T_\epsilon^* H_\epsilon^{-1}v, u)_{L^2},$$

where $T_\epsilon^*$ is the adjoint operator of $T_\epsilon$ for the $L^2$-norm. Thus, $H_\epsilon$ is just $T_\epsilon^*$. Consequently, $H_\epsilon$ and $T_\epsilon$ have the same spectrum ($\mu = 0$ included). Moreover, $u$ is an eigenfunction of $T_\epsilon$ for a nonzero eigenvalue $\mu$, i.e. $T_\epsilon u = \mu u$, if and only if $u$ is an eigenfunction of $-\Delta_V$ for the eigenvalue $\lambda = \mu^{-1} - \epsilon$. From the Fredholm theory applied to the compact operator $T_\epsilon$ and its adjoint on $L^2(M)$ (as in [8]) (we can also use $H_0^1(M)$ as in [10]), the set of eigenvalues of $T_\epsilon$ is the same of its adjoint. It is either a finite set or a sequence converging to zero, and the dimension of each eigenspace is finite. Among these eigenvalues there is a distinguished one, the principal eigenvalue, that can be described using Krein-Rutman theory. This theory only requires $\partial M$ to be of class $C^{2,\alpha}$, and the metric $g$ and the vector field $V$ of class $C^{1,\alpha}$ on $\bar{M}$. One considers $T_\epsilon$ as a compact operator on $C_0^{1,\alpha}(\bar{M})$, $T_\epsilon : C_0^{1,\alpha}(\bar{M}) \to C_0^{1,\alpha}(\bar{M})$. In this case $K = \{v \in C_0^{1,\alpha}(\bar{M}) : v \geq 0\}$ is a solid cone, whose interior is given by $K^\circ = \{v \in K : v > 0 \text{ on } M, \partial v/\partial \nu < 0\}$. This cone $K^\circ$ is not empty.
since $\partial M$ is of class $C^1$, as we can see from Lemma 11. We can build a function $v \in K^o$ gluing a constant function $v_0$ with $d_{\partial M}$, using a partition of unity. If $v \in K \setminus \{0\}$, applying the maximum principle with respect to $\Delta_V - \epsilon$, $u = T_\epsilon v \in K$, and by the Hopf maximum principle we get $u \in K^o$. Thus, $T_\epsilon$ is strongly positive with respect to $K$. Then, the Krein-Rutman theory states that the spectral radius $r(T_\epsilon)$ of $T_\epsilon$ is a simple eigenvalue of $T_\epsilon$, with eigenfunction $v_\epsilon \in K$. Hence, $\omega_V := r(T_\epsilon)v_\epsilon = T_\epsilon v_\epsilon \in K^o$ is a principal eigenfunction of $-\Delta_V$ for the principal eigenvalue $\lambda_V = r(T_\epsilon)^{-1} - \epsilon$, that is,

$$\Delta_V \omega_V + \lambda_V \omega_V = 0.$$ 

This means the pair $(\lambda_V, \omega_V)$ satisfies the following conditions (1)-(4). Moreover, if we choose $\epsilon$ sufficiently large so that the maximum principle also holds for $\Delta_V^* - \epsilon$, then we have a similar construction of a pair $(\lambda_V^*, \omega_V^*)$ satisfying the same conditions:

1. $\lambda_V$ is a simple eigenvalue, $\omega_V > 0$ on $M$, $\omega_V = 0$ on $\partial M$.
2. $\lambda_V < Re(\lambda)$, $\forall \lambda \neq \lambda_V$ (complex) eigenvalue of $\Delta_V$.
3. No other eigenvalue has a positive eigenfunction on $M$.
4. $\partial \omega_V / \partial \nu < 0$, that is, $\omega_V \in K^o$.

It is known that (cf. [4], and [20] for any compact Riemannian domain $\bar{M}$ with smooth boundary):

5. $\lambda_V > 0$ if and only if the weak maximum principle holds.

This is the case of $\Delta_V$ as we stated in Theorem 5. On the other hand, we have equality of the spectral radius $r(T_\epsilon^*) = r(T_\epsilon)$ and we may conclude that $\lambda_V^* = \lambda_V$, and so $\lambda_V^*$ is also positive. Note that $\omega_V, \omega_V^* \in C_0^{1,0}(\bar{M}) \subset H_0^1(M)$, and so they are in $C^\infty(M) \cap C_0(\bar{M})$ as well. Furthermore, applying the Fredholm alternative theorem to $T_\epsilon$ ([10], Theorem 5.11), we obtain, in the following proposition, a description of the eigenvalues of $\Delta_V$ and $\Delta_V^*$, as operators on $L^2(M)$.

**Proposition 6.** The Laplacians $-\Delta_V$ and $-\Delta_V^*$ have the same set $\Lambda$ of eigenvalues, a discrete set that can be either a finite set or a sequence $|\lambda_k| \to +\infty$. The corresponding eigenspaces are finite dimensional subspaces of $L^2(M)$. The principal eigenfunctions $\omega_V$ and $\omega_V^*$ lie in $C_0^{1,0}(\bar{M}) \subset H_0^1(M)$, and are smooth on $M$, and vanish on $\partial M$. The weak maximum principle also holds for $\Delta_V^*$, independently of $\text{div}^0(V) \leq 0$ holding or not.

The common positive principal eigenvalue of $-\Delta_V$ and of $-\Delta_V^*$ will be denoted by $\lambda_V^*$. 


4 Model spaces

A spherically symmetric space is a warped product space, \( N^ρ = [0, l] \times ρ S^{m-1} \), endowed with the warped metric,

\[
g_ρ = dt^2 + ρ^2(t)dσ^2,
\]

where \( ρ \in C^∞([0, l]) \) satisfies \( ρ > 0 \) on \((0, l)\), \( ρ(0) = ρ''(0) = 0 \), and \( ρ'(0) = 1 \). Here, \( dσ^2 \) denotes the usual metric on the unit \((m-1)\)-sphere. The origin of \( N^ρ \) is the point \( p_ρ \) defined by identifying all pairs \((0, ξ)\), where \( ξ \in S^{m-1} \). The metric is smooth away from the origin, and smooth at \( p_ρ \) if we assume all even derivatives of \( ρ \) vanish at \( t = 0 \). The distance function to \( p_ρ \) is given by \( r(t, ξ) = t \). Hence, \( t\partial_t = \frac{1}{2}\nabla r^2 \) is a smooth vector field. We consider closed geodesic balls \( M^ρ := \bar{B}_{r_0}(p_ρ) \), centered at \( p_ρ \) and of radius \( r_0 < l \). The radial sectional curvature, and the radial Ricci curvature of \( N^ρ \) are defined at each point \( p = (t, ξ) \), (cf. [9]) by

\[
K^ρ(\partial_t, W) = -\frac{ρ''(t)}{ρ(t)}, \quad \text{Ricci}^ρ(\partial_t, \partial_t) = -(m-1)\frac{ρ''(t)}{ρ(t)},
\]

respectively, where \( W \in T_ξS^{m-1} \) has unit \( g_ρ \)-norm.

A function \( F(t, ξ) \) is said radial if it only depends on \( t \), that is, \( F(t, ξ) = F(t) \). We will consider vector fields in the radial direction, depending on \( t \) only, that is, \( V^ρ = h(t)\partial_t = \nabla H \), where \( H \in C^∞([0, r_0]) \), and \( h(t) = H'(t) \). We are always assuming that \( h(t) \) is of the form \( h(t) = \tilde{h}(t) \) for \( t \) near \( 0 \), and for some smooth function \( \tilde{h} \). Hence, \( h(0) = 0 \) and \( V^ρ = \tilde{h}(t)(t\partial_t) \) is smooth on \([0, r_0]\). This vector field is of gradient type, and so it defines a Bakry-Émery model space \((N^ρ, g_ρ, e^{-H}dV)\). We will denote the \( V^ρ \)-Laplacian \( Δ_{V^ρ} \), by the \( H \)-Laplacian \( Δ_ρ^H \). Fixing a \( g_ρ \)-orthonormal basis \( \partial_t, e_i \), with \( e_i \in T_ξS^{m-1} \), for \( 1 ≤ i ≤ m-1 \), we have

\[
\nabla_\partial_t V^ρ = h'(t)\partial_t, \quad \nabla_{e_i} V^ρ = ρ'(t) \frac{h(t)}{ρ(t)} e_i, \\
div^0(V^ρ) = h'(t) + (m-1)\frac{h(t)}{ρ(t)} ρ'(t).
\]

Thus, \( \lim_{t \to 0^+} \text{div}^0(V^ρ)(t) = mh'(0) \). Consider the function

\[
p(t) = ρ^m(t)e^{-H(t)}, \quad \text{where} \quad H(t) = h'(t).
\]

The Laplacian for the Levi-Civita connection has the following expression (cf. [9]),

\[
Δ_0^ρ u = \frac{d^2u}{dt^2} + (m-1)\frac{ρ'}{ρ} \frac{du}{dt} + \frac{1}{ρ^2} Δ_{S^{m-1}} u.
\]

Hence, the \( H \)-Laplacian of a function \( u(t, ξ) \) is given by,

\[
Δ_ρ^H u = \frac{d^2u}{dt^2} + \frac{p'}{p} \frac{du}{dt} + \frac{1}{ρ^2} Δ_{S^{m-1}} u.
\]

In the following proposition we describe the properties of the principal eigenvalue \( ω_{ρ,H} \) of the \( H \)-Laplacian on \( M^ρ \), with Dirichlet boundary conditions:
Proposition 7. Let $V^\rho = h(t)\partial_t$, with $h(0) = 0$. Then, $\omega_{\rho,H}$ is radial, and for each $t \in (0, r_0)$ it satisfies

$$\omega''_{\rho,H}(t) + \frac{p'(t)}{p(t)}\omega'_{\rho,H}(t) + \lambda_{\rho,H}\omega_{\rho,H}(t) = 0.$$  \hfill (13)

Furthermore, $\omega_{\rho,H}(t) > 0$ and $\omega'_{\rho,H}(t) < 0$ on $(0, r_0)$, $\omega_{\rho,H}(r_0) = 0 = \omega'_{\rho,H}(0)$, and $\omega'_{\rho,H}(r_0) < 0$.

Proof. In this proof we will denote $S^{m-1}$ by $S$. Let $0 < \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_l \leq \ldots \to +\infty$ be the set of eigenvalues of $\Delta_H^0$ on $M^\rho$ with Dirichlet boundary conditions. Let $u(t, \xi) \in C_0^\infty(M^\rho)$ be an eigenfunction for one of the eigenvalues $\lambda = \lambda_l$. As in [6], p. 40-43 (for $H = 0$), we will decompose $u(t, \xi)$ in a sum of products of an eigenfunction of a one-dimensional eigenvalue problem, with a homogeneous harmonic polynomial. Let $\nu_k = k(k + m - 1)$, $k = 0, 1, \ldots$, be the eigenvalues of the Laplacian $-\Delta_S$ on the $(m - 1)$-sphere for the closed eigenvalue problem. There is a complete orthonormal system of eigenfunctions $G_{k,\alpha}$ of $-\Delta_S$, defining a basis of $L^2(S^{m-1})$. The index $k$ corresponds to the eigenvalue $\nu_k$, and $\alpha$ runs from 1 to $N_k$, the multiplicity of $\nu_k$. If we fix $t$, then

$$u(t, \xi) = \sum_k \sum_\alpha a_{k,\alpha}(t)G_{k,\alpha}(\xi),$$  \hfill (14)

for some constants $a_{k,\alpha}(t) = (u(t, \cdot), G_{k,\alpha})_{L^2(S)}$. We have

$$\|u^2(t, \cdot)\|_{L^2(S)}^2 = \sum_{k,\alpha} a_{k,\alpha}^2(t), \quad a_{k,\alpha}^{(s)}(t) = \left(\frac{d^su}{dt^s}(t, \cdot), G_{k,\alpha}\right)_{L^2(S)}$$

Since $\nu_k G_{k,\alpha} = -\Delta_S G_{k,\alpha}$, using (12) and $\Delta_H^0 u = -\lambda u$, we have

$$\nu_k a_{k,\alpha}(t) = -(u(t, \cdot), \Delta_S G_{k,\alpha})_{L^2(S)} = -(\Delta_S u(t, \cdot), G_{k,\alpha})_{L^2(S)}$$

$$= \rho^2 \left( -\lambda u + \sum_{l,\beta} a''_{l,\beta}(t)G_{l,\beta} + \sum_{l,\beta} \frac{p'(t)}{p(t)} a'_{l,\beta}(t)G_{l,\beta}, G_{k,\alpha}\right)_{L^2(S)}$$

$$= \rho^2 \left[ -\lambda a_{k,\alpha}(t) - \frac{1}{p(t)} (pa'_{k,\alpha})'(t) \right] .$$

Hence, for each $k$, if $a_{k,\alpha} \neq 0$ for some $\alpha$, then $\lambda = \lambda_l$ is a solution of the one-dimensional eigenvalue problem

$$(pa')'(t) + (\lambda - \rho^{-2}\nu_k)p(t)a(t) = 0,$$  \hfill (15)

or equivalently, a solution of

$$a''(t) + \left( (m - 1)\frac{p(t)}{\rho(t)} - h(t) \right) a'(t) + (\lambda - \nu_k \rho^{-2})a(t) = 0.$$  \hfill (16)
From the Dirichlet boundary conditions on \( u \), and smoothness of \( u \) at \( t = 0 \), we must impose the following boundary conditions on \( a(t) \),

\[
    a(r_0) = 0 = a'(0). \tag{17}
\]

Therefore, we conclude that each eigenvalue \( \lambda_l \) of the \( H \)-Laplacian arises as a solution of at least one of the eigenvalue problems (16), with boundary condition (17), with respect to some \( k \). Moreover, each \( a_{k,\alpha} \) lies in the eigenspace \( E_{k,\lambda} \) of the eigenvalue problem (16), when \( k \) is fixed.

Reciprocally, let us we fix an eigenfunction on the \((m-1)\)-sphere, \( G_k(\xi) \), with eigenvalue \( \nu_k \), and an eigenfunction \( a_k(t) \) of the eigenvalue problem (16)–(17), with respect to \( \nu_k \) and with eigenvalue \( \lambda \). Set \( u(t, \xi) := a_k(t)G_k(\xi) \). It satisfies \( u = 0 \) on \( \partial M^\rho \), and for any \( 0 < t < r_0 \) and \( \xi \in S \), we have

\[
    \Delta_H^\rho u(t, \xi) + \lambda u(t, \xi) = -\rho^{-2} a_k(t) \nu_k G_k(\xi) + \frac{a_k(t)}{\rho^2(t)} \nu_k G_k(\xi) = 0.
\]

Thus, \( u(t, \xi) \) is an eigenfunction of \( -\Delta_H^\rho \) with eigenvalue \( \lambda \). In particular, for each \( k \), the set of eigenvalues of the one-dimensional eigenvalue problem (16)–(17) is a subset \( \{\lambda_{k,1}, \ldots, \lambda_{k,2}, \ldots\} \) of \( \{\lambda_1, \lambda_2, \ldots\} \).

Now we consider \( k \) fixed, and two solutions of (16), \( a_{k,i}, a_{k,j} \), with eigenvalues \( \lambda_{k,i} \) and \( \lambda_{k,j} \), respectively. Then

\[
    \int_0^{r_0} (pa'_{k,i})'a_{k,j}dt = -\int_0^{r_0} (pa'_{k,j})a_{k,i}dt. \tag{18}
\]

On the other hand,

\[
    \int_0^{r_0} (pa'_{k,i})'a_{k,j}dt = -\lambda_{k,i} \int_0^{r_0} pa_{k,i}a_{k,j}dt + \nu_k \int \rho^{-2} pa_{k,i}a_{k,j}dt.
\]

Hence, \( (\lambda_{k,j} - \lambda_{k,i}) \int_0^{r_0} pa_{k,i}a_{k,j}dt = 0 \). That is, if \( i \neq j \), \( a_{k,i} \) and \( a_{k,j} \) are \( L^2_{p(t)} \) orthogonal on \([0, r_0] \). For \( i = j \) we have,

\[
    \int_0^{r_0} p(a'_{k,i})^2dt = \lambda_{k,i} \int_0^{r_0} p(a_{k,i})^2dt - \nu_k \int_0^{r_0} \rho^{-2} p(a_{k,i})^2dt.
\]

We also note that, if \( k = 0 \), (15) gives

\[
    p(t)a'_{0,i}(t) = -\lambda_{0,i} \int_0^t p(\tau)a_{0,i}(\tau)d\tau. \tag{18}
\]

Similarly, if \( a_k(t) \) and \( a_s(t) \) are solutions of (16) for the same eigenvalue \( \lambda \), associated with \( \nu_k \) and \( \nu_s \), respectively, then \( (\nu_k - \nu_s) \int_0^{r_0} p\rho^{-2}a_ka_sdt = 0 \). Hence, for \( k \neq s \), \( a_k \) and \( a_s \) are \( L^2_{pp^{-2}} \)-orthogonal on \([0, r_0] \). Now, the volume element of \((N^\rho, g_\rho)\) is given by \( \rho^{m-1}dt \land dS \),
where $dS$ is the volume element of $S^{m-1}$. Thus, the norm of the eigenfunction $u(t, \xi)$ on the $e^{-H}$-weighted $L^2$-space on $M^\rho$ is given by

$$
\|u\|_{L^2_e H (M^\rho)}^2 = \int_0^{r_0} \rho^{m-1}(t)e^{-H(t)} \left( \int_S u^2(t, \xi) dS \right) dt \\
= \sum_{k, \alpha} \int_0^{r_0} p(t) a_{k, \alpha}^2(t) dt = \sum_{k, \alpha} \|a_{k, \alpha}\|_{L^2_p(t)}^2.
$$

The arguments given in [6], pp. 41, are valid concerning the eigenvalue problem (16), since $p(t) = \rho(t)^{m-1}e^{-H(t)}$ is qualitatively the same as $H = 0$ (see also [24], p. 209). Thus, for each $k = 0, 1, \ldots$, the solutions of the eigenvalue problem (16), with boundary condition (17), consists of an increasing sequence $\lambda_{k,i}$, converging to infinity when $i \to +\infty$. Each $\lambda_{k,i}$ is simple, and the corresponding eigenfunction $a_{k,i}$ ($L^2_{p(t)}$-normalized) has $i - 1$ zeros on $(0, r_0)$. Now, $u$ in (14) is a $-\Delta^\rho_H$-eigenfunction for at least one of the eigenvalues $\lambda_{k,i} = \lambda_l$. Then, for each $\alpha$, $a_{k,i}(t) = c_{k,i,\alpha}a_{k,i}(t)$, for some constants $c_{k,i,\alpha}$. Consequently, we have the following representation of a $\lambda_l$-eigenfunction $u$ as a $L^2_{e^{-H}}(M^\rho)$-convergent series

$$
u(t, \xi) = \sum_{\{k,i: \lambda_{k,i} = \lambda_l\}} a_{k,i}(t)G_{k,i}(\xi), \quad \text{where} \quad G_{k,i}(\xi) = \sum_{\alpha} c_{k,i,\alpha}G_{k,\alpha}(\xi).
$$

Moreover, $\sum_\alpha \|a_{k,\alpha}\|_{L^2_p(t)}^2 = \sum_\alpha c_{k,i,\alpha}^2 = \|G_{k,i}\|_{L^2(S)}^2$, and so

$$
\|u\|_{L^2_e H (M^\rho)}^2 = \sum_{\{k,i: \lambda_{k,i} = \lambda_l\}} \|G_{k,i}\|_{L^2(S)}^2 = \sum_{\{k,i: \lambda_{k,i} = \lambda_l, \alpha\}} c_{k,i,\alpha}^2.
$$

Note that the only eigenfunction $G_{k,\alpha}$ that does not change of sign in $S$ is the constant function $G_{0,1} = 1$. Now, the principal eigenvalue of $-\Delta^\rho_H$, is the lowest eigenvalue $\lambda_{p,H}$, and the first eigenfunction, $\omega_{p,H}$, is positive on $M^\rho$, only vanishing along the boundary. Hence, $\omega_{p,H}(t, \cdot)$ corresponds to the lowest eigenvalue of (16) with $k = 0$. Consequently, $\lambda_{p,H} = \lambda_{0,1}$, and $\omega_{p,H}(t, \xi) = a_{1,0}(t)$, up to a multiplicative positive constant. It is radial, positive for $t \in [0, r_0)$, vanishes at $t = r_0$, and satisfies (16), and thus, (13). From (18), and since $a_{0,1}(s) > 0$, we conclude that the sign of $a'_{0,1}(t)$ is the same of $-\lambda_1$, and $a'_{0,1}(r_0) < 0$. This completes the proof. \hfill \Box

In the above proof, we also have obtained the following conclusions.

**Proposition 8.** (1) For each $k$ fixed, the one-dimensional eigenvalue problem (16) with boundary condition (17), consists of an increasing sequence of simple eigenvalues, $0 < \lambda_{k,1} < \lambda_{k,2} < \ldots \to +\infty$. Furthermore, eigenfunctions $a_{k,i}$ and $a_{k,j}$, with respect to different eigenvalues $\lambda_{k,i}$ and $\lambda_{k,j}$, are $L^2_{p(t)}$-orthogonal. This means $(a_{k,i}, a_{k,j})_{L^2_p(0, r_0)} := \int_0^{r_0} a_{k,i} a_{k,j} p dt = 0,$
for all \( i \neq j \). Moreover, if \( k = 0 \), \( \| a_{0,i} \| _{L_{p}^{2}(0,r_{0})}^{2} = \lambda_{0,i} \| a_{0,i} \| _{L_{p}^{2}(0,r_{0})}^{2} \).

(2) The discrete set of eigenvalues \( \lambda_{i} \to +\infty \) of \( -\Delta_{H}^{0} \) on the ball of radius \( r_{0} \), \( M^{0} \), with Dirichlet boundary condition, consists of the set \( \{ \lambda_{k,i}, k = 0,1,2,\ldots, i = 1,2,\ldots \} \). Furthermore, each function \( F \in L_{0}^{2}(M^{0}) \) can be expressed as an \( L_{0}^{2}(M^{0}) \)-convergent sum of \( -\Delta_{H}^{0} \)-eigenfunctions \( u_{i}(t,\xi) \), and so, as a convergent sum whose terms consists of products of an eigenfunction of (16) with an eigenfunction of \( -\Delta_{\mathbb{S}^{m-1}} \).

5 Min-max formulas for the principal eigenvalue

We first extend to the Riemannian case a min-max formula obtained by Protter-Winberger [21] in 1966 (see also [4]), for open regular domains of \( \mathbb{R}^{n} \). We assume that \( \bar{M} \) is a compact regular domain of a complete Riemannian manifold \((N,g)\). We consider the following cone of \( H_{0}^{1}(M) \),

\[
D_{0}^{+} = \{ u \in C^{2}(M) \cap C(\bar{M}) \cap H_{0}^{1}(M) : u > 0 \text{ on } M, u_{\partial M} = 0 \}. \tag{19}
\]

**Theorem 9** (Min-Max formula). The following min-max formula holds for the principal eigenvalue of \( \Delta_{V} \),

\[
\lambda_{V}^{*} = \sup_{u \in D_{0}^{+}} \inf_{p \in M} \frac{\Delta_{V}u(p)}{u(p)} = \inf_{u \in D_{0}^{+}} \sup_{p \in M} \frac{\Delta_{V}u(p)}{u(p)}.
\]

We have the same formula with respect to \( \Delta_{V}^{*} \).

**Proof.** We take \( \omega_{V}, \omega_{V}^{*} \in D_{0}^{+} \cap C^{\infty}(\bar{M}) \), principal eigenfunctions of \( -\Delta_{V} \) and \( -\Delta_{V}^{*} \), respectively. Then,

\[
\lambda_{V}^{*} = -\frac{\Delta_{V}\omega_{V}}{\omega_{V}} = \inf_{M} \frac{-\Delta_{V}\omega_{V}}{\omega_{V}} \leq \sup_{u \in D_{0}^{+}} \inf_{M} \frac{-\Delta_{V}u}{u}.
\]

On the other hand, for any \( u \in D_{0}^{+} \),

\[
\inf_{M} \left( -\frac{\Delta_{V}u}{u} \right) \int_{M} u \omega_{V}^{*} dM \leq \int_{M} \frac{-\Delta_{V}u}{u} u \omega_{V}^{*} dM = -\int_{M} (\Delta_{V}u) \omega_{V}^{*} dM = -\int_{M} u (\Delta_{V}^{*} \omega_{V}) dM = \lambda_{V}^{*} \int_{M} u \omega_{V}^{*} dM.
\]

Thus \( \inf_{M} \left( -\frac{\Delta_{V}u}{u} \right) \leq \lambda_{V}^{*} \). Similarly, we have \( \sup_{M} \left( -\frac{\Delta_{V}u}{u} \right) \geq \lambda_{V}^{*} \).

The above theorem is just a Barta’s type result (for \( V = 0 \) see [3], or [4], III.1, Lemma 1).
Corollary 10 (Generalized Barta’s type inequality). For any $u \in D_0^+$,

$$\inf_M \left( -\frac{\Delta V u}{u} \right) \leq \lambda_* \leq \sup_M \left( -\frac{\Delta V u}{u} \right)$$

(20)

$$\inf_M \left( -\frac{\Delta^* V u}{u} \right) \leq \lambda_* \leq \sup_M \left( -\frac{\Delta^* V u}{u} \right).$$

(21)

Equalities hold in (20) (in (21), respectively) if and only if $u = \omega_V$ ($u = \omega^*_V$, respectively).

The min-max formula we will describe next is due to Holland [18] on Euclidean domains. This formula was reformulated by Godoy, Gossez and Paczka [11], using weighted Sobolev spaces. We give a sketch of the proof, valid at least for the case $\bar{M}$ a coordinate chart, which formally follows the same steps as in [11]. We also provide some formulas that we will need.

The weighted Sobolev spaces with weight the square of the distance function to $\partial M$, $d_{\partial M}(p) = \inf_{x \in \partial M} d(p, x)$, for $p \in \bar{M}$, are defined by

$$L^2_{\partial}(M) = \left\{ u : M \to \mathbb{R} \text{ measurable : } \int_M d_{\partial M}^2 u^2 dM < +\infty \right\},$$

(22)

$$H^1_{\partial}(M) = \left\{ u \in H^1_{loc}(M) : \int_M d_{\partial M}^2 (u^2 + |\nabla u|^2) dM < +\infty \right\},$$

(23)

with the weighted Sobolev norms,

$$\|u\|_{L^2_{\partial}} = \int_M d_{\partial M}^2 u^2 dM, \quad \|u\|_{H^1_{\partial}} = \int_M d_{\partial M}^2 (u^2 + |\nabla u|^2) dM,$$

respectively. If $\bar{M}$ is a smooth Euclidean domain, it is shown in [11] that $H^1_{\partial}(M)$ is continuously embedded into $L^2(M)$ and compactly embedded into $L^2_{\partial}(M)$. In Lemma [11](3), we show this is also true when $\bar{M}$ is a smooth compact Riemannian domain that is diffeomorphic to an Euclidean domain. This is clear when $M$ is a geodesic ball $B_{r_0}(p_0)$ of $N$ with $r_0 < \text{inj}(p)$. The exponential map of $N$ defines a diffeomorphism $\exp_{p_0} : \bar{D}_{r_0} \to \bar{B}_{r_0}(p_0)$ from the Euclidean closed $m$-ball $\bar{D}_{r_0}$ of radius $r_0$. For each $t < r_0$, and $\xi \in S^{m-1}$ the distance functions to the boundaries are related by $d_{\partial M}(\exp_{p_0}(t\xi)) = r_0 - t = d_{\partial D_{r_0}}(t\xi)$. We also show in the following lemma that the principal eigenfunction $\omega_V$ lies in $D_0$. Let $\mathcal{O}$ be a small tubular neighbourhood of $\partial M$ in $N$ such that normal minimizing geodesics starting from $\partial M$ are unique.

Lemma 11. Assume $\partial M$ is a smooth hypersurface of $N$ and $\nu$ is its unit outer normal with respect to $\bar{M}$. We have the following:

(1) The distance function, $d_{\partial M} : \bar{M} \to [0, +\infty)$, lies in $C^\infty(\bar{M} \cap \mathcal{O}) \cap C_0(\bar{M})$ and satisfies $\frac{d_{\partial M}}{d\nu}(p) = -1$, for all $p \in \partial M$;
(2) \( \omega_V \in \mathcal{D}_\theta \cap \mathcal{D}_o^1 \cap H^1_0(M) \);

(3) If \( \Phi : \tilde{D} \to \tilde{M} \) is a diffeomorphism from an Euclidean domain \( \tilde{D} \) onto \( \tilde{M} \), then we can find some constants \( c_i > 0 \) such that,

\[
c_1 d_{euc}(x, \partial D) \leq d_\partial M(\Phi(x)) \leq c_2 d_{euc}(x, \partial D),
\]

holds for all \( \Phi(x) \in \mathcal{O} \cap \tilde{M} \). Furthermore, \( H^1_0(M) \) is continuously embedded into \( L^2(M) \) and compactly embedded into \( L^2_0(M) \). Moreover, \( \mathcal{D}_\theta \subset H^1_0(M) \).

**Proof.** (1) and (2). Locally, \( \partial M \) is the hyperplane of \( \mathbb{R}^m \), \( \{0\} \times \mathbb{R}^{m-1} \), \( \tilde{M} \) is the half space \( \{x_1, x_2, \ldots, x_m\} \) with \( x_1 \leq 0 \), and \( d_\partial M(x_1, \ldots, x_m) = -x_1 \). Hence, locally, \( d_\partial M \) has smooth extensions on a neighbourhood of each point \( p_1 \in \partial M \) in \( N \). Therefore, \( d_\partial M \in C^\infty(\tilde{M} \cap \mathcal{O}) \). Now, normal geodesics starting from \( \partial M \) are of the form \( \gamma(t) = \exp_{p_1}( \tau t \nu(p_1) ) \), for \( t \in [0, \epsilon) \), with \( p_1 \in \partial M \). The sign corresponds to a geodesic lying in \( \tilde{M} \cap \mathcal{O} \). We only consider these geodesics. Then we have, \( \nabla d_\partial M(\gamma(t)) = \gamma'(t) \). This can be shown using Fermi coordinates on \( \mathcal{O} \) ([12], Lemma 2.7, Lemma 2.8 (2.25)). Therefore, \( \nabla d_\partial M(p_1) = \lim_{t \to 0^+} \nabla d_\partial M(\gamma(t)) = -\nu(p_1) \). It follows that \( \frac{d_{\partial M}(p_1)}{\partial M}(p_1) = -1 \). Consequently, \( \lim_{p \to p_1} \omega_V(p)/d_\partial M(p) = -\left( \partial \omega_V/\partial \nu \right)(p_1) \). This equality and the fact that \( \omega_V \in K^0 \) implies \( \omega_V \in \mathcal{D}_\theta \).

(3). Coordinates charts on \( \partial D \) are transported by \( \Phi \) into coordinate charts on \( \partial M \). Hence, we may build simultaneously a Fermi coordinate system on a tubular neighbourhood \( \mathcal{O}_{euc} \) of \( \partial D \) and another on \( \mathcal{O} \), such that, for each \( x \in \partial D \) and \( t_1 \in [0, \epsilon) \), \( d(-t_1 \nu_{euc}(x), \partial D) = t_1 = d_\partial M(\exp_{\Phi(x)}(-t_1 \nu(\Phi(x))) \), ([12], Chapter 2, Section 2.1., (2.4)). Now, the first statement follows naturally. Obviously, the metric \( g \) is equivalent to the one induced by the Euclidean one via \( \Phi \). These two facts imply that the weighted Sobolev norms defined for functions \( u \) on \( M \) and for functions \( \tilde{u} = u \circ \Phi \) on \( D \) are equivalent. This implies the second last statement of (3) is true for \( M \), knowing it is true for \( D \) ([11], Lemma 4.1). Finally, we have \( \mathcal{D}_\theta \subset H^1(M) \cap C_0(M) \). In case \( \tilde{M} \) is a global chart the latter set is contained in \( H^1_0(M) \), [22].

For each \( u \in \mathcal{D}_\theta \) we consider the continuous functional \( Q_u : H^1_0(M) \to \mathbb{R} \), given by

\[
Q_u(v) := \int_M u^2 (|\nabla v|^2 - g(V, \nabla v)) dM.
\]

Now we may present the Holland-Godoy-Gossez-Paczka formula ([18] [11]).

**Theorem 12** (Min-max integral formula). If \( \tilde{M} \) is a regular compact domain of a coordinate chart, then

\[
\lambda^*_V = \inf_{\{u \in \mathcal{D}_\theta : \|u\|_{L^2} = 1\}} \left( \mathcal{L}(u, u) - \inf_{v \in H^1_0(M)} Q_u(v) \right).
\]
Equality is achieved at $u_V = \omega_V \sqrt{G_V}$ (normalized). Here, $G_V \in H^1_0(M)$ is the unique solution, up to a multiplicative constant, of the integral equation

$$\int_M g(\nabla G + GV, \nabla \phi) \omega_V^2 dM = 0, \quad \forall \phi \in H^1_0(M),$$

(25)

satisfying $0 < c_1 \leq G_V \leq c_2$, for some positive constants $c_i$. In particular $u_V \in D_0 \cap H^1_0(M)$ and

$$\lambda^*_V = \inf_{\{u \in D_0 \cap H^1_0(M): \|u\|_{L^2} = 1\}} \left( \mathcal{L}(u, u) - \inf_{v \in H^1_0(M)} Q_u(v) \right).$$

The proof is based on two existence results, (A) and (B), and makes use of the completing of a square algebraic inequality (C), described below:

(A) Given $u \in D_0$, there exists $w_u \in H^1_0(M)$ such that $Q_u(w_u) = \inf_{v \in H^1_0(M)} Q_u(v)$. It is unique on $H^1_0(M)$ up to an additive constant (a.e.). Computing the Euler Lagrange equation we see that,

$$Q_u(w_u) = - \int_M |\nabla w_u|^2 u^2 dM = - \frac{1}{2} \int_M g(V, \nabla w_u) u^2 dM \leq 0.$$  

(26)

In case $V = 0$ we must have $w_u = 0$ up to a constant (a.e.), and the min-max formula is the usual Rayleigh formula for the first eigenvalue of the 0-Laplacian.

(B) Given $u \in D_0$, there exists $G_u \in H^1_0(M)$ such that

$$B(G_u, \phi) := \int_M g(\nabla G_u + G_u V, \nabla \phi) u^2 dM = 0, \quad \forall \phi \in H^1_0(M).$$

(27)

It is unique up to a multiplicative positive constant, satisfying $0 < c_1 \leq G \leq c_2$ for some positive constants $c_i$.

(C) For all $X, Z \in T_p M$,

$$- \|X\|^2 - g(V, X) \leq g(X, Z) + \frac{1}{4} |V + Z|^2.$$  

(28)

Equality holds if and only if $Z = -2X - V$.

The proof of (A) on the existence of a unique minimum $w_u$ relies on the strict convexity of the integrand $F(P, v, p) = u^2(|P|^2 - g(V(p), P))$ in the variable $P$, and on the coerciveness of $Q_u$ on the subspace $H^1_{0, B}(M) = \{u \in H^1_0(M) : \int_B u = 0\}$. Here, $B$ is some fixed small ball in the interior of $M$. Coerciveness of $Q_u$ can be shown using the compactness of the embedding $H^1_0(M)$ into $L^2_0(M)$, given in Lemma [11]. A minimum $w_u$ is a critical point of $Q_u$. It is a (weak) solution of the degenerate elliptic second order differential equation $\text{div}^0(u^2(2\nabla w - V)) = 0$, for $w \in H^1_0(M)$. A critical point $w_u$ satisfies $Q_u(w_u) = - \int_M u^2 |\nabla w_u|^2 \leq 0$. The proof of the second existence result (B) is more complex because one
has to find a positive solution. A solution is a weak solution of the degenerate elliptic second
order differential equation \( \text{div}^0(u^2(\nabla G + VG)) = 0 \), that is, it satisfies \( B(G, \phi) = 0 \) for all 
\( \phi \in H^1_0(M) \). In \cite{18}, Holland proved the existence of \( G \geq 0 \) by finding an ergodic measure with
probability density \( G \). The alternative proof in \cite{11} consists of taking \( \epsilon \) sufficiently
large such that \( B_\epsilon(G, \phi) = B(G, \phi) + \epsilon \int_M G \phi u^2 dM \) is coercive for the \( H^1_0 \)-norm. Its inverse
operator \( T_{d,\epsilon} : L^2_0(M) \to L^2_0(M) \) can be shown to be compact, and satisfies the following
property: if \( 0 \neq f \geq 0 \), then \( \text{ess inf}_B T_{d,\epsilon} f > 0 \), for any open domain \( B \) with compact closure
in the interior of \( M \). This implies that the compact operator \( T_{d,\epsilon} \) is a positive and irreducible
operator on \( L^2 \) spaces, in the sense of Schwartz \cite{23}, which forces the spectral radius of \( T_{d,\epsilon} \)
to be positive. This is a sufficient condition for the existence of a principal eigenvalue,
and a principal eigenfunction \( G_u \geq 0 \). Performing a Moser type iteration technique from
\( B(G_u, \phi) = 0 \), leads to the conclusion that \( G_u \) is uniformly bounded for all weighted \( L^p \)
norms. Consequently, \( G_u \) is bounded from above by a positive constant \( c_2 \). Coerciveness
of a related modified operator implies \( G_u \geq c_1 > 0 \). The divergence of the vector field
\( U = -u^2 \nabla \log \omega_V \) is given by

\[
-\text{div}^0(U) = \frac{2u}{\omega_V} \nabla u - \frac{u^2}{\omega_V^2} \nabla \omega_V, \nabla \omega_V - u^2 \lambda^*_V + \frac{u^2}{\omega_V} g(V, \nabla \omega_V).
\]

Integrability of \( \text{div}^0(U) \) on \( M \) follows from the properties of \( \omega_V/d_{\partial M} \) and \( u/d_{\partial M} \) near \( \partial M \).
On the other hand, \( U \) continuously extends to zero on \( \partial M \). Considering for each \( \epsilon > 0 \),
\( M_\epsilon = \{ p \in M : d_{\partial M}(p) \geq \epsilon \} \), and \( \nu_\epsilon \) the outward unit of its boundary, we have

\[
\int_M \text{div}^0(U) dM = \lim_{\epsilon \to 0} \int_{M_\epsilon} \text{div}^0(U) dM = \lim_{\epsilon \to 0} \int_{\partial M_\epsilon} g(U, \nu_\epsilon) dM = 0.
\]

This fact, and following \cite{11}, taking the \( u^2 dM \) integration of the algebraic inequality \cite{28} in
(C) with, \( X = -\nabla \log \omega_V, Z = -V + 2 \nabla (\log(u + w_u)), \) gives the inequality \( \lambda^*_V \int_M u^2 dM \leq
\mathcal{L}(u, u) - Q_u(w_u) \). Equality holds at

\[
u = u_V := \omega_V \sqrt{G_V}, \tag{29}
\]

where \( G_V \) is the solution \( G_u \) given in (B) with respect to \( u = \omega_V \), with

\[
w_{u_V} = -(\log G_V)/2. \tag{30}
\]

This solution \( u_V \) lies in \( H^1_0(M) \). To see this we first recall that \( \omega_V \in \mathcal{D}_0 \). Now, \( G_V \) is a weak
solution of an elliptic operator of second order with smooth coefficients on any subdomain \( \Omega \)
with smooth compact closure in the interior of \( M \). Moreover, \( G_V \in H^1(\Omega) \) and it is bounded.
Hence, \( G_V \in C^\infty(\Omega) \) (cf. \cite{2}, Theorem 3.55). In particular, \( G_V \in C(M) \). From

\[
|\nabla(\omega_V \sqrt{G_V})|^2 \leq 2|\nabla \omega_V|^2|G_V| + \frac{1}{2} \frac{|\omega_V|^2}{d_{\partial M}^2} \frac{|\nabla G_V|^2}{G_V^2} d_{\partial M},
\]
we conclude that \( u_V \in H^1(M) \), and so \( u_V \in \mathcal{D}_0 \cap C_0(\bar{M}) \). Consequently, \( u_V \in H^1_0(M) \) (cf. [22]).

In the particular case of the Bakry-Émery Laplacian, straightforward computations prove the following.

**Proposition 13.** If \( V = \nabla f \), then for any \( u \in \mathcal{D}_0 \), \( w_u = f/2 \) and \( G_u = e^{-f} \). Moreover

\[
\lambda^*_V = \inf_{u \in \mathcal{D}_0} \frac{\mathcal{L}(\tilde{u}, \tilde{u}) - \inf_{v} Q_{\tilde{u}}(v)}{\int_M \tilde{u}^2 dM} = \inf_{u \in \mathcal{D}_0} \frac{\int_M |\nabla \tilde{u} + \tilde{u}V f|^2 dM}{\int_M \tilde{u}^2 dM}.
\]

The infimum is achieved at \( \tilde{u} = \omega e^{-f/2} \in \mathcal{D}_0 \cap H^1_0(M) \). Writing \( u = \tilde{u} e^{f/2} \), and recalling that \( C^2_c(M) \subset \mathcal{D}_0 \) is dense in \( H^1_0(M) \), we obtain the Rayleigh variational characterization for the first eigenvalue of the \( f \)-Laplacian given in (10). That is, \( \lambda^*_V = \lambda_f \), and \( u = \omega e^{-f} \in \mathcal{D}_0 \cap H^1_0(M) \) is the \( \lambda_f \)-eigenfunction for \(-\Delta_f\).

**Corollary 14.** If \( V = 0 \), the min-max formula in Theorem 12 reduces to the Rayleigh variational characterization of the first eigenvalue of \(-\Delta_0\).

### 6 Comparison results

Let \( \bar{M} \) be a smooth compact domain endowed with a smooth vector field \( V \) for which the min-max formulas of Section 5 hold. This is the case when \( \bar{M} \) is the domain of a smooth coordinate chart. Our first proposition is a straightforward application of the min-max integral formula for the principal eigenvalues \( \lambda^*_V \) and \( \lambda^*_0 \), the second one for \( V = 0 \). Let \( \omega_V \) and \( \omega_0 \) be the respective \( L^2 \)-unit principal eigenfunctions, and \( w_{\omega_0} \in H^1_0(M) \) the function that realizes \( \inf_{v} Q_{\omega_0}(v) \) in the integral inequality (24).

**Proposition 15.** The following inequalities hold:

\[
\lambda^*_V + \frac{1}{2} \int_M (\text{div}^0(V) - 2|\nabla w_{\omega_0}|^2)\omega_0^2 dM \leq \lambda^*_0 \leq \lambda^*_V + \frac{1}{2} \int_M \text{div}^0(V)\omega_V^2 dM.
\]

Furthermore, equality holds for the right hand side inequality if and only if \( \omega_V = \omega_0 \). In this case, \( \lambda^*_V - \lambda^*_0 = g(V, \nabla \log \omega_0) \) on \( M \), \( g(V, \nu) = 0 \) on \( \partial M \), and \( \int_M \text{div}^0(V) dM = 0 \), where \( \nu \) is the unit outer normal of \( \partial M \). Equality holds for the left hand side inequality if and only if \( \omega_V = \alpha \omega_0 e^{u_{\omega_0}} \), where \( \alpha \) is a normalizing constant.
Proof. Using the Rayleigh characterization of \( \lambda_0^* \) applied to \( \omega_V \), and Stokes’s theorem,

\[
\lambda_0^* \leq \int_M \| \nabla \omega_V \|^2 dM
= \lambda_V^\star \int_M \omega_V^2 dM - \frac{1}{2} \int_M g(V, \nabla \omega_V^2) dM = \lambda_V^\star + \frac{1}{2} \int_M \text{div}^0(V) \omega_V^2 dM.
\]

Equality holds if and only if \( \omega_V = \omega_0 \). In this case,

\[
\lambda_V^\star \omega_V = -\Delta_V \omega_V = -\Delta \omega_0 = -\Delta_0 \omega_0 + g(V, \nabla \omega_0) = \lambda_0^\star \omega_0 + g(V, \nabla \omega_0).
\]

Thus, \( (\lambda_V^\star - \lambda_0^\star) \omega_0 = g(V, \nabla \omega_0) \). Consequently, along \( \partial M \), \( g(V, \nabla \omega_0) = \frac{\partial \omega}{\partial \nu} g(V, \nu) \) must vanish. This is possible only if \( g(V, \nu) = 0 \). Now, applying the min-max integral formula for \( \lambda_V^\star \) with respect to \( \omega_0 \) we get the left hand side inequality. Equality holds if and only if \( \omega_V = \omega_0 \) or \( \partial M \). By \((30)\), \( \sqrt{G} = e^{-w_{\omega V}} = e^{-w_{\omega 0}} \).

As a consequence of the previous proposition and its proof, and of Proposition \ref{thm:13}, we have the following two corollaries:

**Corollary 16.** If \( \text{div}^0(V) \leq 0 \), then \( \lambda_0^* \leq \lambda_V^\star \). Equality of the eigenvalues holds if and only if \( \text{div}^0(V) = 0 \) and \( \omega_V = \omega_0 \). In this case \( V \perp \nabla \omega_0 \) on \( M \) and \( V \perp \nu \) along \( \partial M \).

**Corollary 17.** Let us suppose \( V = \nabla f \), for some \( f \in C^\infty(M) \).

1. **Assume** \( \Delta_0 f \leq 0 \). Then \( \lambda_0^* \leq \lambda_f \), and equality holds if and only if \( f \) is a harmonic function and \( \omega_0 = \omega_f \). In this case, \( \nabla f \perp \nabla \omega_0 \) pointwise on \( M \).
2. **Assume** for some constant \( \epsilon > 0 \), \( \Delta_0 f \geq \epsilon |\nabla f|^2 \) holds. Then \( \lambda_0^* \geq \lambda_{\epsilon f} \), and equality holds if and only if \( \Delta_0 f = \frac{\epsilon}{2} |\nabla f|^2 \).

Proposition \ref{thm:15} with \( V = \epsilon \nabla f \), and \( \epsilon > 0 \) a constant, give us

\[
\frac{\epsilon}{2} \int_M (\Delta_0 f - \frac{\epsilon}{2} |\nabla f|^2) \omega_0^2 dM \leq \lambda_0^* - \lambda_{\epsilon f} \leq \frac{\epsilon}{2} \int_M \Delta_0 f \omega_{\epsilon f}^2 dM.
\]

Similar reasoning for \( \epsilon < 0 \) leads to the following consequence for the Bakry-Émery first eigenvalue.

**Corollary 18.** If for some sequence \( \epsilon_i \to 0 \), the limit \( \lambda_f = \lim_{\epsilon_i \to 0} \frac{1}{\epsilon_i} (\lambda_{\epsilon_i f} - \lambda_0^*) \) exists, then \( -\frac{1}{2} (\sup_M \Delta_0 f) \leq \lambda_f \leq -\frac{1}{2} (\inf_M \Delta_0 f) \). Consequently, if \( f \in C^\infty(M) \) is a harmonic function on \( M \), or more generally, \( \Delta_0 f = 2c_0 \) a constant, then \( \frac{d}{d\epsilon}|_{\epsilon = 0} \lambda_{\epsilon f} \) exists and it is equal to \(-c_0\).
Next we will define suitable models spaces, based on pointwise estimates of the radial curvatures and of the radial component of $V$. These model spaces will establish estimates for the principal eigenvalue of a geodesic ball of $N$ by comparing it with the corresponding ones of the model spaces.

The exponential map of $N$ from a given point $p_0$ is a smooth diffeomorphism $\exp_{p_0} : \mathcal{D}_{p_0} \to N \setminus C(p_0)$ from the star-shaped open set $\mathcal{D}_{p_0} = \{ t\xi : 0 \leq t \leq d_\xi, \xi \in S^{m-1}_{p_0} \subset T_{p_0} N \}$, onto the open dense set $N \setminus C(p_0)$ of $N$, where $C(p_0)$ is the cut locus at $p_0$ and $d_\xi$ is the largest $t$ for which $\gamma_\xi(s) = \exp_{p_0}(s\xi)$ is a minimizing geodesic for all $0 < s \leq t$. This diffeomorphism defines on $N \setminus C(p_0)$ the geodesic coordinate chart, $\tilde{\Theta}(t, \xi) = \exp_{p_0}(t\xi)$. In these coordinates the metric $g$ can be expressed as

$$g(\tilde{\Theta}(t, \xi)) = dt^2 + |A(t, \xi)d\xi|^2, \quad \forall t\xi \in \mathcal{D}_{p_0}.$$ 

Here, $A(t, \xi) : \xi^\perp \to \xi^\perp$ is the linear operator given by $A(t, \xi) = \gamma_t^{-1}\nabla\eta$, where $\nabla\eta(t) = d(\exp_{p_0}(t\xi))(\eta)$ is the Jacobi field along the geodesic $\gamma_\xi(t)$, with initial conditions, $\nabla\eta(0) = 0$, $\nabla\partial_t \nabla\eta(0) = \eta$, and $\gamma_t : T_{p_0} M \to T_{\gamma_\xi(t)}$ is the parallel transport along $\gamma_\xi$. It satisfies the Jacobi equation $A'' + RA = 0$, with $A(0, \xi) = 0$, $A'(0, \xi) = Id$, where $R(t) : \xi^\perp \to \xi^\perp$ is the self-adjoint operator, $R(t)\eta = (\tau_t)^{-1}R(\gamma_\xi(t), \tau_t\eta)\gamma_\xi(t)$. The trace of $R(t)$ is the radial Ricci tensor, $\text{Ricci}(\gamma_\xi(0))(\gamma_\xi'(t), \gamma_\xi'(t))$. We define a non-negative smooth function $J$ on $\mathcal{D}_{p_0}$, such that

$$J^{m-1} = \det A.$$ 

Let $dS$ be the volume element of $S^{m-1}$, and $r(p) = d(p, p_0)$ the intrinsic distance of $p$ to $p_0$ in $N$. The square $r^2(p)$ is smooth on $N \setminus C(p_0)$ (cf. [12], Section 3.2), and the gradient $\nabla r$ is a unit vector field. For $t > 0$, it satisfies the equality, $\nabla r(p) = \gamma_\xi'(t)$, for $p = \gamma_\xi(t)$, and defines the radial direction $\partial_t(p) = \nabla r(p)$ at each $p \neq p_0$. In these geodesic coordinates $(t, \xi)$, $dV_M = J^{m-1}(t, \xi) dt dS$ expresses the volume element of $M$. The function $J$ satisfies the following equations and inequalities (cf.[9])

$$\Delta_0 r = \partial_t \ln(J^{m-1}), \quad \partial_t \Delta_0 r + \|\text{Hess} \ r\|^2 = -\text{Ricci}(\partial_t, \partial_t),$$

$$\begin{cases} (m - 1)J''(t, \xi) + \text{Ricci}(\gamma_\xi'(t), \gamma_\xi'(t))J(t, \xi) \leq 0, \\ J(0, \xi) = 0, \quad J'(0, \xi) = 1. \end{cases}$$

Note that $r\Delta_0 r = \frac{1}{2}(\Delta_0 r^2 - 1)$ is smooth, and applying L’Hôpital’s rule,

$$\lim_{t \to 0^+} r \Delta_0 r(t, \xi) = \lim_{t \to 0^+} (m - 1) \frac{t}{J(t, \xi)} J'(t, \xi) = (m - 1). \quad (31)$$

We are assuming $r_0 < \text{inj}(p_0)$, so that $\tilde{M} \subset N \setminus C(p_0)$, and $d_\xi \geq \text{inj}(p_0), \forall \xi$. The restriction of the geodesic coordinates, $\tilde{\Theta} : [0, r_0] \times S^{m-1} \to \tilde{M}$, defines the spherical geodesic coordinate.
of $M$ centered at $p_0$. It satisfies $\frac{d\hat{\rho}}{dt} = \nabla r$. We have the following identities holding for any function $\phi \in C^1(M)$,

\[
\frac{d\phi}{dt} = \phi'(t, \xi) = \frac{d(\phi \circ \hat{\Theta})}{dt} = g(\nabla \phi, \nabla r) = \delta_t \phi.
\]

For any radial function $F$ on $M$, i.e $F(p) = T(r(p))$, where $T : [0, r_0] \to \mathbb{R}$ is of class $C^2$, satisfying $T'(0) = 0$, we have (cf. [4])

\[
\nabla F(p) = T'(r(p))\nabla r = T' \delta_t \\
\Delta_0 F(p) = T''(r(p)) + (\delta_r \log J^{m-1})T'(r(p)) = T'' + \Delta_0 T'.
\]  

(32)

We decompose the vector field $V$ as $V = V_{rad} + V_s$, where $V_{rad} = h_1(t, \xi)\nabla r$, with $h_1(t, \xi) = g(V, \nabla r)$ the radial component of $V$, and $V_s$ the $g$-orthogonal complement of $V_{rad}$. We say $V$ is a radial vector field if $V = V_{rad}$. It is smooth if $h(0, \xi) = 0$ for any $\xi$, to be more precise, if $h(t, \xi) = \tilde{h}(t, \xi)$ for some smooth function $\tilde{h}$. If $V$ is not radial, $V_{rad}$ is not assumed to be smooth.

As in [9], we consider a model space $N^p = [0, l] \times \rho S^{m-1}$, and a geodesic ball $M^p$ centered at the origin $p_0$ with radius $r_0$. We are assuming $r_0 < \min\{l, \text{inj}(p_0)\}$, and take a radial vector field $V^p = h(t)\partial_t = \nabla H$, where $H'(t) = h(t)$ and $h(0) = 0$. On the model space, $\mathcal{A}(t, \xi) = \rho(t)I_{d} J = \rho$, and $d((t, \xi), \partial M^p) = r_0 - t$, for $t \leq r_0$. The properties of the positive principal eigenfunction $\omega_{\rho, H}(r)$ on $M^p$ are described in Proposition [7].

The ratio of the volume elements of $M$ and $M^p$ is a fundamental tool to derive comparison results based on relations between radial curvatures:

\[
\theta(t, \xi) = \frac{dM(p)}{dM^p(p)} = \left[\frac{J(t, \xi)}{\rho(t)}\right]^{m-1}, \quad \theta(0, \xi) = 1,
\]  

(33)

where $p = \hat{\Theta}(t, \xi)$. Comparison on radial curvatures corresponds to nondecreasing or nonincreasing $\theta(t, \xi)$ on $[0, r_0)$, and consequent inequality on the volumes of the geodesic balls of radius $t < r_0$ (see [7], generalized Bishop’s comparison Theorems 4.2 and 3.3). We start by recalling the comparison result of [9] for the first eigenvalue of $\Delta_0$ on a geodesic ball with radial sectional curvatures bounded from above by those of the model space.

**Theorem 19.** Assume the radial sectional curvatures of $\tilde{M} = B_{r_0}(p_0)$ are bounded from above by the ones of the model space $(\tilde{M}^p, H = 0)$, that is, $K(\partial_t, X) \leq -\frac{\rho''(t)}{\rho'(t)}$, for all unit $X \in T_p M$ orthogonal to $\partial_t(p)$. Then $\lambda_0^* \geq \lambda_{\rho, H=0}$, and equality holds if and only if $M$ is isometric to $M^p$.

This is an extension of Cheng’s comparison result reduced to the case $-\frac{\rho''(t)}{\rho'(t)} = \text{constant}$, the case of a space form [7]. We now extend this result to the $V$-Laplacian, obtaining Theorem [1].
Theorem 20. On $\tilde{M} = \tilde{B}_{r_0}(p_0)$ it is given a vector field $V$, where $V_{rad} = h_1(t, \xi)\partial_t$. Assume the radial sectional curvatures are bounded from above by the one of the model space $(\tilde{M}^\rho, V^\rho)$, that is, $K(\partial_t, X) \leq -\frac{\rho''(t)}{\rho(t)}$. Additionally, assume $h_1(t, \xi) \leq h(t)$. Then $\lambda_V^* \geq \lambda_{p, H}$. If equality holds on the eigenvalues, then $M$ is isometric to $M^\rho$ and $h_1(t, \xi) = h(t)$. In that case $\omega_V = \omega_{p, H}$.

Proof. By the curvature conditions and the generalized Rauch-Bishop’s comparison theorem ([9], Theorem 4.2) $\theta'(t, \xi) \geq 0$, where $\theta$ is defined in (33). Equivalently,

$$\Delta_0^r = (m-1)(\log J)' \geq \Delta_0^\rho = (m-1)(\log \rho)'$$

with equality if and only if $A = C = \rho(t)Id$, that is $M$ is isometric to $M^\rho$. On the spherical geodesic coordinates of $M$, we define $\tilde{\omega}(t, \xi) := \omega_{p, H}(t)$ extending the principal eigenfunction of the model space for the $\Delta_0^\rho$-Laplacian to a radial function on $M$. Recall that $\omega_{p, H}'(t) < 0$ on $(0, r_0]$ and $\omega_{p, H}(r_0) = \omega_{p, H}'(0) = 0$. It is clear that $\tilde{\omega} \in D_0^+$ on $M$. Then by (32) and Proposition 7

$$-\Delta_V \tilde{\omega} \over \tilde{\omega} = -\Delta_0 \tilde{\omega} - g(V, \nabla \tilde{\omega})$$

$$= -\frac{1}{\omega_{p, H}(t)} \left( \omega_{p, H}''(t) + \left( (\log J^m(t))' - h_1(t, \xi) \right) \omega_{p, H}'(t) \right)$$

$$\geq -\frac{1}{\omega_{p, H}(t)} \left( \omega_{p, H}''(t) + \left( (m-1) \frac{\rho'(t)}{\rho(t)} - h(t) \right) \omega_{p, H}'(t) \right)$$

$$= \lambda_{p, H}.$$  

From the generalized Barta’s inequality in Corollary 10 for the $V$-Laplacian,

$$\lambda_V^* \geq \inf_M -\Delta_V \tilde{\omega} \over \tilde{\omega} \geq \lambda_{p, H}.$$

Equality holds if and only if $M$ is isometric to $M^\rho$, $h_1 = h$ and $\omega_V = \omega_{p, H}$. 

The radial sectional curvatures do not depend on the vector field $V$. The following corollary is an immediate consequence of the proof of the above theorem.

Corollary 21. On a geodesic ball $M^\rho$ of a model space $N^\rho$, if the radial part of a vector field $V$ satisfies $h_1(t, \xi) = h(t) = H'(t)$ for a smooth function $h \in C^\infty([0, r_0])$, with $h(0) = 0$, then $\lambda_V^* = \lambda_{p, H}$ and $\omega_V(t, \xi) = \omega_{p, H}(t)$.

Proof. As in the previous proof, we get equality of (34) with (35), where the last is constant. Thus, we have $-\Delta_V \tilde{\omega} = \lambda_{p, H} \tilde{\omega}$, and $\tilde{\omega} > 0$ on $M = M^\rho$, $\tilde{\omega} = 0$ on $\partial M$, that is, $\tilde{\omega}$ is a principal eigenvalue on $M$ for the $V$-Laplacian. Consequently, $\omega_V(t, \xi) = \tilde{\omega}(t, \xi) = \omega_{p, H}(t)$ and $\lambda_V^* = \lambda_{p, H}$. 

Now we recall the comparison theorem in [9] for $\Delta_0$ with radial Ricci curvature bounded from below.

**Theorem 22.** Assume the radial Ricci curvature of a geodesic ball $M$ of radius $r_0$ is bounded from below by the one of the model space $M^\rho$, i.e. $\text{Ricci}(\partial_t, \partial_t) \geq -(m - 1)\frac{\rho''(t)}{\rho(t)}$. Then $\lambda^*_V \leq \lambda_{\rho,H=0}$, and equality holds if and only if $M$ is isometric to $M^\rho$.

Next we extend the above results to the $V$-Laplacian when $V$ is a radial vector field, but not necessarily a gradient one. That is, $V = h_1(t, \xi)\partial_t$, where $h_1(t, \xi)$ may depend on $\xi$.

**Theorem 23.** Let $h(t) = H'(t)$, where $h \in C^\infty([0, r_0])$, $h(t) \geq 0$ and $h(0) = 0$. Assume $V(t, \xi) = h_1(t, \xi)\partial_t$ for a function $h_1 \in C^\infty(M)$, satisfying $h_1(0, \xi) = 0$, $\forall \xi$, and the following inequalities take place at each $(t, \xi)$,

$$\text{Ricci}(\partial_t, \partial_t) \geq -(m - 1)\frac{\rho''(t)}{\rho(t)},$$

$$h'_1 - \frac{h_1^2}{2} + h_1 \Delta_0 r \geq h' - \frac{h^2}{2} + h \Delta^\rho_r.$$  \hspace{1cm} (36)

Then $\lambda^*_V \leq \lambda_{\rho,H}$, and equality holds if and only if $M$ is isometric to $M^\rho$ and equality holds in (36). In the latter case, $\omega_V(t, \xi) = \omega_{\rho,H}(t)e^{-H(t)+H_1(t,\xi)/2}$, where $H'_1(t, \xi) = h_1(t, \xi)$. Additionally, if $\rho(t)$, $h(t)$ and $h_1(t, \xi)$ are analytic functions on $t \in [0, r_0]$, then $h_1(t, \xi) = h(t)$ and $\omega_V = \omega_{\rho,H}$.

The second inequality of (36) is just (5) in Theorem 2. We need the following lemmas to prove the theorem:

**Lemma 24.** If $V(t, \xi) = h_1(t, \xi)\partial_t$, then $\text{div}^0(V) = h'_1(t, \xi) + h_1(t, \xi)\Delta^0 r$. In this case, for any $u \in H^1_0(M)$ the solution $w_u$ is described as follows:

$$w_u(t, \xi) = \frac{1}{2} \int_0^t h_1(\tau, \xi)d\tau, =: \frac{1}{2}H_1(t, \xi),$$

$$Q_u(w_u) = \inf_v Q_u(v) = -\frac{1}{4} \int_M u^2 h_1^2 dM.$$ \hspace{1cm} (37) \hspace{1cm} (38)

**Proof.** We prove the two equations. For any $v \in H^1_0(M)$,

$$Q_u(v) = \int_M u^2 \left(|\nabla v|^2 - h_1(t, \xi)\frac{dv}{dt}\right)dM \geq \int_M u^2 \left(\frac{dv}{dt}\right)^2 - h_1(t, \xi)\frac{dv}{dt}\right)dM.$$ 

Hence,

$$\inf_v Q_u(v) \geq \inf_v \int_M u^2 \left(\frac{dv}{dt}\right)^2 - h_1(t, \xi)\frac{dv}{dt}\right)dM.$$ \hspace{1cm} (39)
The infimum on the right hand side of (39) is achieved when \( \frac{dh}{dt}(t, \xi) = h_1(t, \xi)/2 \), that is, \( v(t, \xi) = \frac{1}{2} \int_0^t h_1(\tau, \xi)d\tau + c(\xi) \), giving an equality in (39). We may choose \( c(\xi) = 0 \), defined a.e. on \( M \). In this case it must be \( w_u \). \( \square \)

Remark 1. In the above Lemma, if \( h_1(t, \xi) \geq 0 \), then \( \text{div}^0(V) \leq 0 \) is only possible if \( h_1 \equiv 0 \). Indeed, if we assume \( \text{div}^0(V)(t, \xi) \leq 0 \) is possible for a fixed \( \xi \) and all \( t \in [0, t_2] \), then

\[
(\log h_1)' \leq (\log(J^{1-m}))'
\]

on that interval. Integration of this inequality on \( t_1 < t_2 \), gives

\[
h_1(t_2, \xi)/h_1(t_1, \xi) \leq J^{m-1}(t_1, \xi)/J^{m-1}(t_2, \xi),
\]

that is, \( (h_1(t, \xi)J^{m-1}(t, \xi))' \leq 0 \) on \([0, t_2]\). Since \( h_1(t, \xi)J^{m-1}(t, \xi) \) vanishes at \( t = 0 \), \( h_1(t, \xi)J^{m-1}(t, \xi) \leq 0 \). By assumption \( h_1 \geq 0 \), hence we must have \( h_1 \equiv 0 \). Thus, we have the following conclusion, where (b) is proved using a similar reasoning.

Lemma 25. Let \( V = h_1(t, \xi)\partial_t \) be a radial vector field with \( h_1(t_1, \xi) = 0 \). The following statements hold:

(a) If \( h_1(t, \xi) \geq 0 \) on \([t_1, t_2]\), and \( \text{div}^0(V) \leq 0 \) then \( h_1 \equiv 0 \) on \([t_1, t_2]\).

(b) If \( h_1(t, \xi) \geq 0 \) on \([t_1, t_2]\), then \( \text{div}^0(V) \geq 0 \) if and only if \( h_1J^{m-1} \) is a nondecreasing on \([t_1, t_2]\).

Lemma 26. Given functions, \( u \in C^1_0(\bar{M}) \), and \( \phi \in C^1(\bar{M}) \) satisfying \( \phi(p_0) = 0 \), where \( \bar{M} = B_{r_0}(p_0) \), we have

\[
\int_M \phi \frac{du}{dt} dM = -\int_M u(\frac{d\phi}{dt} + \phi \Delta_0 r) dM.
\]

Proof. The vector field \( W = u\phi \nabla r \) is continuous on \( \bar{M} \), of class \( C^1 \) on \( M \setminus \{p_0\} \), and vanishes at \( \partial M \). On the other hand,

\[
\text{div}^0(W) = g(\nabla(u\phi), \nabla r) + u\phi \text{div}^0(\nabla r) = \phi \frac{du}{dt} + u \frac{d\phi}{dt} + u\phi \Delta_0 r.
\]

By L’Hôpital’s rule, for each \( \xi \), and using (31)

\[
\lim_{t \to 0^+} \phi \Delta_0 r(t, \xi) = \lim_{t \to 0^+} \frac{\phi(t, \xi)}{t^m} r(t, \xi) \Delta_0 r(t, \xi) = (m - 1) \frac{d\phi}{dt}(0, \xi).
\]

Thus, \( \text{div}^0(W) \in L^1(M) \). Applying Stokes’s theorem on \( M_\epsilon := M \setminus B_\epsilon(p_0) \), for all \( \epsilon > 0 \) small, and using the fact that \( \partial M_\epsilon = \partial M \cup \partial B_\epsilon(p_0) \) we get,

\[
\int_M \text{div}^0(W) dM = \lim_{\epsilon \to 0} \int_{M_\epsilon} \text{div}^0(W) dM = \int_{\partial M_\epsilon} g(W, \nu) dS - \lim_{\epsilon \to 0} \int_{\partial B_\epsilon(p_0)} g(W, \nu') dS.
\]
Here, \( \nu^\epsilon \) is the outward unit of \( \partial B_\epsilon(p_0) \) and \( dS \) is the volume element of hypersurfaces. The area \( |\partial B_\epsilon(p_0)| \) converges to zero when \( \epsilon \to 0 \). Hence,

\[
\left| \int_{\partial B_\epsilon(p_0)} g(W, \nu^\epsilon) dS \right| \leq |\partial B_\epsilon(p_0)| \sup_M |W| \to 0, \quad \text{when } \epsilon \to 0.
\]

Since \( W \) vanish on \( \partial M \), \( \int_M \text{div}^0(W) dM = 0 \), which proves the lemma.

**Proof of Theorem 23** Let \( \omega_{\rho,H} \) be a positive principal eigenvector for the \( H \)-drift Laplacian on the geodesic ball \( M^\rho \) of radius \( r_0 \) and center at the origin. We take \( \tilde{w}(t) = \omega_{\rho,H}(t) e^{-H(t)/2} \).

Using the spherical geodesic coordinates, we extend it as a function on \( \mathbb{R} \) with \( \tilde{w} \).

Thus,

\[
\lambda_\nu + \inf_v Q_{\tilde{w}}(v) \leq \int_M (\| \nabla \tilde{w} \|^2 + g(V, \nabla \tilde{w}) \tilde{w}) dM.
\]

We express the integrals using the spherical geodesic coordinate \( \hat{\Theta} \), giving

\[
\int_M g(V, \nabla \tilde{w}) \tilde{w} dM = \int_{\xi \in S^{m-1}} \left[ \int_0^{r_0} g(V, \partial_t) \frac{d\tilde{w}}{dt}(t)\rho(t)^{m-1} \theta(t, \xi) dt \right] dS,
\]

and

\[
\int_M \| \nabla \tilde{w} \|^2 dM = \int_{S^{m-1}} \left[ \int_0^{r_0} (\frac{d\tilde{w}}{dt})^2 \rho(t)^{m-1} \theta(t, \xi) dt \right] dS =
\]

\[
= \int_{S^{m-1}} \left\{ \frac{w}{\rho(t)} \frac{d\tilde{w}}{dt}(t)\rho(t)^{m-1} \theta(t, \xi) \right\} \left. \right|_0^{r_0} - \int_0^{r_0} \frac{\tilde{w}}{\rho(t)} \frac{d\tilde{w}}{dt} \rho(t)^{m-1} \theta(t, \xi) dt \right\} dS \]

\[
= \int_{S^{m-1}} \left\{ \frac{w}{\rho(t)} \frac{d\tilde{w}}{dt}(t)\rho(t)^{m-1} \theta(t, \xi) \right\} \left. \right|_0^{r_0}
\]

\[
- \int_0^{r_0} \tilde{w} \cdot \left( \frac{d^2 \tilde{w}}{dt^2} + (m-1) \frac{\rho'}{\rho} \right) \left( \frac{\rho'}{\rho} \right) \theta(t, \xi) \right\} dS.
\]

From equation (13) we have,

\[
\tilde{w}'' + (m-1) \frac{\rho'(t)}{\rho(t)} \tilde{w}' + (B(t) + \lambda_{\rho,H}) \tilde{w}(t) = 0,
\]

where

\[
B(t) := \frac{1}{2} h'(t) - \frac{1}{4} (h(t))^2 + \frac{(m-1) \rho'(t)}{\rho(t)} h(t).
\]
Furthermore, Proposition 7 and the assumption \( h \geq 0 \) imply that for \( t > 0 \),

\[
\tilde{w}'(t) = e^{-\frac{h(t)}{2}}(\omega_{\rho,H}'(t) - \frac{h(t)}{2} \omega_{\rho,H}(t)) < 0,
\]

and \( \tilde{w}'(0) = \tilde{w}(r_0) = 0 \). Hence, the term [10] vanishes. Under the curvature conditions, we apply the generalized Bishop’s comparison theorem I (see [9], Theorem 3.3) to get

\[
(J/\rho)'(t, \xi) \leq 0,
\]

with equality if and only if \( A = \rho(t)Id \) and \( M \) isometric to \( M^\rho \). Consequently,

\[
\Delta_0r = (m - 1)(\log J)' \leq \Delta_0^r = (m - 1)(\log \rho)'.
\]

From [11] and [12], and using the fact that \( \tilde{w}' \leq 0 \), we arrive at

\[
\lambda^*_V + \inf_v Q_{\tilde{w}}(v) \leq \lambda_{\rho,H} + \int_M \tilde{w}^2 dM.
\]

In the last integral the function \( B(t) \) is considered extended as a radial function on \( M \) via \( \tilde{\Theta} \). Applying Lemma 24, we have \( \inf_v Q_{\tilde{w}}(v) = -\frac{1}{4} \int_M \tilde{w}^2 \sqrt{\gamma_1} dM \). Applying Lemma 26 on the last term of [14], with \( \phi = \frac{h_1}{2} \) and \( u = \tilde{w}^2 \), we obtain the inequality

\[
\lambda^*_V \leq \lambda_{\rho,H} + \int_M \tilde{w}^2 \left( \frac{h_1^2}{4} + \frac{h_1'}{2} - \frac{h_2}{4} + \Delta_0^r \frac{h}{2} - \frac{h_1'}{2} - \Delta_0^r \frac{h_1}{2} \right) dM.
\]

The assumption [36] implies \( \lambda^*_V \leq \lambda_{\rho,H} \). Equality holds if and only if \( J/\rho \equiv 1 \), \( M \) is isometric to \( M^\rho \), equality holds in [36], and the min-max formula is achieved at \( \tilde{w} \). In this case \( u_V = \omega_V \sqrt{\gamma_1} = \tilde{w} = \omega_{\rho,H}(t)e^{-\frac{h(t)}{2}} = u_H \), where \( u_H = u_{\gamma_1} \). As we have seen in [30], \( w_{uv} = -\frac{1}{2} \log \gamma_1 \) and by Lemma 24 \( w_{uv} = \frac{1}{2} H_1 \). Thus \( \sqrt{\gamma_1} = e^{-\frac{h(t)}{2}} \), which proves the relation between \( \omega_V \) and \( \omega_{\rho,H} \) stated in the theorem.

Now we assume \( \tilde{M} \) is isometric to \( \tilde{M}^\rho \), and for each \( \xi, h_1(t, \xi), h(t), \) and \( \rho(t) \) are analytic functions on \([0, r_0]\). Next we show that under these conditions, and the initial condition \( h_1(0, \xi) = h(0) = 0 \), equality in [36] implies \( h_1(t, \xi) = h(t), \forall t, \xi \), and consequently \( \omega_V = \omega_{\rho,H} \). For each fixed \( \xi \in S^{n-1} \), we define a Riccati equation on \( h_1(t, \xi) \),

\[
h_1' = q_0 + q_1 h_1 + q_2 h_1^2,
\]

(45)
with initial condition $h_1(0, \xi) = 0$, where the coefficients are given by

$$
q_0 = h' - \frac{h^2}{2} + h \Delta_0^2 r, \quad q_1 = -\Delta_0^2 r, \quad q_2 = \frac{1}{2}.
$$

(46)

From $\Delta_0^2 r = (m - 1) \frac{d}{dt} \log \rho$, we conclude that $q_1$ has a simple pole at $t = 0$. Since $h(0) = 0$, the coefficient $q_0$ is analytic at $t = 0$ with $q_0(0) = mh'(0)$. Equality in (36) is equivalent to $h_1$ satisfying (45) with (46). The function $h$ trivially solves (45) with the same initial condition $h(0) = 0$. We use the Frobenius method to show there is uniqueness of solutions of equation (45) with the same initial condition at the regular singular point $t = 0$. We set

$$
h_1 = -\frac{1}{q_2} u' = -\frac{2}{u},
$$

(47)

assuming $u(0) = 1$ without loss of generality, and equation (45) turns into

$$
\begin{aligned}
&\left\{ \begin{array}{l}
u'' - q_1 u' + \frac{q_2}{2} u = 0, \\
u(0) = 1, \quad u'(0) = 0
\end{array} \right.
\end{aligned}
$$

(48)

The indicial equation is given by $I(\alpha) := \alpha^2 + (P(0) - 1)\alpha + Q(0) = 0$, where $P(0) = \lim_{t \to 0} -tq_1(t) = m - 1$ and $Q(0) = \lim_{t \to 0} t^2q_0 = 0$. Hence $I(\alpha) = \alpha(\alpha + m - 2) = 0$. The roots are $\alpha_1 = 0$, and $\alpha_2 = 2 - m$. If $m = 2$ we have a double root $\alpha_1 = 0$, and so a unique analytic solution $u(t)$ exists, uniquely determined by its value at $t = 0$ (cf. the detailed exposition in [13]). Now, $H_1(t, \xi) = \log(u^{-2}(t)) + C'$. We fix $u_0$ the solution corresponding to $h$, with $u_0(0) = 1$, and $u_0'(0) = 0$. Any other solution $u_1(t)$ must be $u_0$. This corresponds to $h_1(t, \xi) = h(t)$. If $m \geq 3$, then $\alpha_1 - \alpha_2 = m - 2$. One of the solutions is given by $u_1$ as in case $m = 2$. The other type of solution is of the form $u_2(t) = c \log t \ u_1(t) + t^{2-m} \sigma(t)$, with $c$ a constant and $\sigma(t)$ an analytic function satisfying $\sigma(0) \neq 0$, giving a corresponding solution $h_2(t, \xi)$ unbounded at $t = 0$, thus it cannot satisfy the initial condition. This completes the uniqueness proof.

**Remark 2.** The radial function $\tilde{w}$ in the previous proof is in $C^\infty({\bar{M}})$ and vanishes on the boundary $\partial M$. We could apply Lemma [26] to $\phi = \frac{d\tilde{w}}{dt}$ and $u = \tilde{w}$ to get

$$
\int_M \|\nabla \tilde{w}\|^2 dM = \int_M \left(\frac{d\tilde{w}}{dt}\right)^2 dM = -\int_M \tilde{w} \left(\frac{d^2 \tilde{w}}{dt^2} + \frac{d\tilde{w}}{dt} \Delta_0^2 r\right) dM.
$$

This is just the same expression as in the proof using the spherical geodesic coordinates. We choose to explicitly use the coordinate chart to see that, if $r_0$ is not smaller than $\text{inj}(p_0)$, we still can get (10) \leq 0 as in [9], by using $d_\xi$ instead $r_0$. If the min-max formula is valid on domains $\hat{M}$ with less regular $\partial M$, we can obtain the same conclusion in Theorem [23] for geodesic balls with radius exceeding the injectivity radius.
Remark 3. The study of the spectrum of the Laplacian with respect to a metric connection $\nabla$ is only interesting if $B(X,Y) = \nabla_X Y - \nabla^0_X Y$ has a nonzero symmetric part, as it is the present case. For instance, connections with skew torsion (see definition in [1]) have the same Laplacian as the Levi-Civita connection one.

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