Rigidity dimension - a homological dimension measuring resolutions of algebras by algebras of finite global dimension

Hongxing Chen*, Ming Fang, Otto Kerner, Steffen Koenig and Kunio Yamagata

Abstract

A new homological dimension is introduced to measure the quality of resolutions of ‘singular’ finite dimensional algebras (of infinite global dimension) by ‘regular’ ones (of finite global dimension). Upper bounds are established in terms of extensions and of Hochschild cohomology, and finiteness in general is derived from homological conjectures. Then invariance under stable equivalences is shown to hold, with some exceptions when there are nodes in case of additive equivalences, and without exceptions in case of triangulated equivalences. Stable equivalences of Morita type and derived equivalences, both between self-injective algebras, are shown to preserve rigidity dimension as well.

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1 Introduction

A finite dimensional algebra $E$ may be called ‘regular’ if it has finite global dimension, that is, cohomology $\text{Ext}^n_{E}(X,Y)$ between $E$-modules vanishes from a certain degree $N_0$ on. Given a ‘singular’ algebra $A$, of infinite global dimension, ‘resolving’ it by $E$ means, in the weakest sense, that there exists

*Corresponding author
an idempotent \( e = e^2 \in E \) such that \( A = eEe \). By a construction due to Auslander, this is always possible. When \( A \) is self-injective this implies that \( eE \) is a generator-cogenerator in \( E\mod \); in general one may require that. There always is a double centraliser property between \( E \) and \( A = eEe \) on the module \( eE \). Algebras \( E \) ‘resolving’ \( A \) then are the endomorphism rings of generator-cogenerators over \( A \). Among the algebras \( E \) one has to single out those with finite global dimension. In this sense, a classical or quantised Schur algebra is a resolution of the group algebra of a symmetric group or a Hecke algebra, respectively. In Rouquier’s terminology [31], the Schur algebra is a ‘quasi-hereditary’ cover, where quasi-hereditary refers to the additional structure of its module category being a highest weight category.

An appropriate measure of the ‘quality’ of the resolution is the dominant dimension of \( E \), as has been demonstrated in [12, 14, 10, 11] in the case of Schur algebras and more generally of gendo-symmetric algebras. The choice of dominant dimension to measure how close \( E \) and \( A \) are related (the closer, the larger the dominant dimension of \( E \) is) is compatible with Rouquier’s setup, and it reflects the role dominant dimension is playing in Morita-Tachikawa correspondence, Auslander’s correspondence, Auslander’s representation dimension and Iyama’s higher representation dimension [24]. There also are close relations to Iyama’s maximal orthogonal modules [23] and to Iyama and Wemyss’ cluster tilting theory for commutative algebras [25].

The aim of this article is to introduce a new homological dimension, which provides information on the ‘best possible’ resolution of a fixed algebra \( A \):

\[
\text{rigdim}(A) := \sup \left\{ \text{domdim} \End_A(M) \mid M \text{ is a generator-cogenerator in } A\mod, \text{ and } \text{gldim} \End_A(M) < \infty \right\}.
\]

By Theorem [2.3] (Müller), when \( M \) is a generator-cogenerator in \( A\mod \), the dominant dimension \( \text{domdim} \End_A(M) \) of its endomorphism ring is determined by the rigidity degree of \( M \) (Definition [2.2]), which measures vanishing of self-extensions of \( M \). Therefore, the new dimension is called \textbf{rigidity dimension} of \( A \). Note that non-semisimple algebras of finite global dimension always have finite dominant dimension, so the supremum is taken over a set of natural numbers, unless \( A \) is semisimple.

When \( A \) has infinite global dimension, \( \text{rigdim}(A) \) measures how close \( A \) can come to an endomorphism ring \( E = \End_A(M) \) of a generator-cogenerator \( M \), with \( E \) having finite global dimension. Semisimple algebras obviously have infinite rigidity dimension. When \( A \) is not self-injective, \( \text{rigdim}(A) \) still is defined, but its meaning is less clear. We will see that always \( \text{rigdim}(A) \geq 2 \).

A combination of global and dominant dimension also occurs in the definitions of Auslander’s representation dimension, which is always finite, and Iyama’s higher representation dimension, which is often infinite. It turns out that rigidity dimension controls finiteness of higher representation dimension: \( \text{repdim}_n(A) \) is finite if and only if \( \text{rigdim}(A) \geq n + 1 \). Thus, rigidity dimension can be viewed as a companion of higher representation dimension.

We will address two basic questions about this new dimension: \textit{finiteness} and \textit{invariance under equivalences}.

Three approaches are developed to establish finiteness. The first approach provides an upper bound for \( \text{rigdim}(A) \) in terms of the smallest degree \( n \geq 1 \), if existent, for which \( \text{Ext}^n_A(D(A), A) \) does not vanish (Theorem [3.1]). Here \( D \) is the usual \( k \)-duality over the ground field \( k \). This can be applied to maximal orthogonal modules and to gendo-symmetric algebras, including Schur algebras of algebraic groups and blocks of the Bernstein-Gelfand-Gelfand category \( O \) of semisimple complex Lie algebras. For the latter algebras, this method implies that further ‘resolving’ a gendo-symmetric algebra \( E \) of finite global dimension, for instance a Schur algebra \( S(r, r) \), which resolves a symmetric algebra \( A \) like the group algebra \( k\Sigma_r \) of a symmetric group, cannot produce a better resolution of \( A \), with respect to \( \text{rigdim} \).

The second approach provides an upper bound in terms of the smallest positive degree of a non-nilpotent homogenous generator of Hochschild cohomology (Theorem [3.6]). This implies finiteness of rigidity dimension for all non-semisimple group algebras (Theorem [3.7]). Symonds’ proof of Benson’s
regularity conjecture then implies that the order of the group is an explicit, but weak, upper bound for \( \mathrm{rigdim}(kG) \).

The third approach derives finiteness of rigidity dimension for all non-semisimple algebras from homological conjectures (Theorem 3.5): Assuming Tachikawa’s first conjecture yields finiteness for algebras that are not self-injective, as an application of the first approach. Finiteness in general is shown to follow from Yamagata’s conjecture. These conjectures are in general open; for finite-dimensional algebras, no counterexamples are known. Tachikawa’s first conjecture is part of a reformulation of Nakayama’s conjecture.

Morita equivalences do preserve rigidity dimension, which is shown also to be invariant under stable equivalences provided the algebras involved do not have nodes (Theorem 4.5). In the presence of nodes (which does not happen frequently), rigidity dimension can change, as we show by examples. Algebras without nodes always have minimal rigidity dimension in their stable equivalence class (Theorem 4.5). Non-invariance disappears when the equivalences preserve the triangulated structure which stable categories of self-injective algebras are known to have (Theorem 4.6).

Stable equivalences of Morita type, and thus also derived equivalences, between self-injective algebras do leave rigidity dimension invariant. More precisely, stable equivalences of adjoint type are shown always to preserve rigidity dimension, for all finite dimensional algebras (Theorem 5.2).

The organisation of this article is as follows:

Section 2 starts by recalling some known material and then gives the definition of rigidity dimension, some basic properties, and connections with higher representation dimension, maximal orthogonal modules and weakly Calabi-Yau properties. Section 3 provides three approaches to proving finiteness of rigidity dimension; first, an upper bound in terms of vanishing of extensions between injective and projective modules, then an upper bound in terms of degrees of generators of reduced Hochschild cohomology, and finally a general proof of finiteness, for non-semisimple algebras, based on assuming validity of homological conjectures. In Section 4, rigidity dimension is shown to be invariant under stable equivalences between algebras without nodes. More generally, algebras without nodes are shown to have minimal rigidity dimension in their stable equivalence class. The inequalities may be proper, as we illustrate by examples. Stronger results are shown for self-injective algebras; in particular, invariance of rigidity dimension holds when equivalences are required to preserve the triangulated structure. In Section 5, stable equivalences of adjoint type are shown to preserve rigidity dimension; this implies invariance of rigidity dimension under stable equivalences of Morita type between self-injective algebras, and hence also under derived equivalences between self-injective algebras.

In the subsequent article [7], rigidity dimensions of classes of examples will be determined.

**Notation.** Throughout this paper, \( k \) is an arbitrary but fixed field. Unless stated otherwise, all algebras are finite-dimensional associative \( k \)-algebras with unit, and all modules are finite-dimensional left modules. The set of positive integers is denoted by \( \mathbb{N} \); the set of non-negative integers is denoted by \( \mathbb{N}_0 \).

Let \( A \) be an algebra. By \( A\text{-mod} \), we denote the category of all left \( A \)-modules. The syzygy and cosyzygy operators of \( A\text{-mod} \) are denoted by \( \Omega_A \) and \( \Omega_A^{\text{op}} \), respectively. Let \( A^{\text{op}} \) be the opposite algebra of \( A \). Then \( D := \text{Hom}_k(-, k) \) is a duality between \( A\text{-mod} \) and \( A^{\text{op}}\text{-mod} \).

Let \( \mathcal{X} \) be a class of \( A \)-modules. By \( \text{add}(\mathcal{X}) \), we denote the smallest full subcategory of \( A\text{-mod} \) which contains \( \mathcal{X} \) and is closed under finite direct sums and direct summands. When \( \mathcal{X} \) consists of only one object \( X \), we write \( \text{add}(X) \) for \( \text{add}(\mathcal{X}) \). In particular, \( \text{add}(_{A}A) \) is exactly the category of projective \( A \)-modules and also denoted by \( A\text{-proj} \). Let \( \mathcal{P}_A \) and \( \mathcal{I}_A \) stand for the set of isomorphism classes of indecomposable projective and injective \( A \)-modules, respectively.

Let \( n \) be a natural number. We denote by \( \mathcal{X}^{\perp_n} \) (respectively, \( _{\perp_n}\mathcal{X} \)) the full subcategory of \( A\text{-mod} \) consisting of modules \( Y \) such that \( \text{Ext}^i_A(X, Y) = 0 \) (respectively, \( \text{Ext}^i_A(Y, X) = 0 \)) for \( 1 \leq i \leq n \) and \( X \in \mathcal{X} \). Similarly, we write \( X^{\perp_n} \) for \( \mathcal{X}^{\perp_n} \) whenever \( \mathcal{X} = \{ X \} \).

The module \( M \) is called **basic** if every indecomposable direct summand of \( M \) is multiplicity-free. Moreover, the head, radical and socle of \( M \) are denoted by \( \text{hd}(M) \), \( \text{rad}(M) \) and \( \text{soc}(M) \), respectively.
The composition of two morphisms \( f : X \to Y \) and \( g : Y \to Z \) in \( A\text{-mod} \) is denoted by \( fg : X \to Z \). In this sense, \( \text{Hom}_A(X,Y) \) is an \( \text{End}_A(X)\text{-End}_A(Y) \)-bimodule.

2 Rigidity dimension

The main object of study in this article, rigidity dimension, will be introduced in the second subsection. Before, we recall the definitions of global and dominant dimension and define the rigidity degree of a module. The third subsection then provides basic properties and examples. This includes the connection with Iyama’s higher representation dimension, the relation with maximal orthogonal modules, which can be used to provide lower bounds for rigidity dimension, and finally a result on rigidity dimension of weakly Calabi-Yau self-injective algebras, which implies that preprojective algebras of Dynkin type have rigidity dimension exactly three.

2.1 Global and dominant dimension, and rigidity

An \( \text{A-module} \) \( M \) is called a generator if \( A \in \text{add}(M) \), a cogenerator if \( D(A_A) \in \text{add}(M) \), a generator-cogenerator if it is both a generator and a cogenerator.

Dominant dimension was introduced by Nakayama and later systematically study by Tachikawa, Morita, M"uller, Yamagata and many others, see [31, 38, 40]. See also [10, 12, 13, 14, 26, 41] for some recent developments partly motivating the present article.

Definition 2.1 The dominant dimension of \( \text{A}\), denoted by \( \text{domdim} \quad \text{A} \), is the largest \( t \in \mathbb{N}_0 \), or \( \infty \), such that in a minimal injective resolution

\[
0 \to AA \to I^0 \to I^1 \to \cdots \to I^{t-1} \to I^t \to \cdots
\]

all \( I^i \) are projective for \( 0 \leq i < t \).

Note that \( \text{domdim} \quad \text{A} = \text{domdim} \quad \text{A}^{op} \) (see [31 Theorem 4]). If \( \text{domdim} \quad \text{A} \geq 1 \), then the injective envelope of \( AA \) is faithful and projective. If \( \text{domdim} \quad \text{A} \geq 2 \), then any faithful projective-injective \( \text{A-module} \) \( P \) has the double centralizer property, that is, with \( \Lambda = \text{End}_A(P) \) there is an equality \( A \cong \text{End}_{\Lambda^{op}}(P)^{op} \). In this case, also \( A \cong \text{End}_A(D(P)) \), and \( D(P) \) is a generator-cogenerator in \( \Lambda\text{-mod} \). In general, for calculating dominant dimensions of endomorphism algebras, the following definition is useful.

Definition 2.2 Let \( M \) be an \( \text{A-module} \). The rigidity degree of \( M \), denoted by \( \text{rd}(\text{A}M) \), is the maximal \( n \in \mathbb{N}_0 \), or \( \infty \), such that \( \text{Ext}^i_A(M,M) \) vanishes for all \( 1 \leq i \leq n \).

In other words, \( \text{rd}(\text{A}M) \geq n \) if and only if \( \text{Ext}^i_A(M,M) = 0 \) for all \( 1 \leq i \leq n \).

In this article, rigidity degrees of modules, measuring the vanishing of self-extensions, are of particular interest, due to their connections with both dominant dimensions and Hochschild cohomology rings. The connection with dominant dimension is provided by a result due to M"uller:

Theorem 2.3 (M"uller [31, Lemma 3]) Let \( A \) be an algebra and \( M \) a generator-cogenerator in \( \text{A-mod} \). Then \( \text{domdim} \quad \text{End}_A(M) = \text{rd}(\text{A}M) + 2 \).

2.2 Definition of rigidity dimension

Both Rouquier’s theory of quasi-hereditary covers [34] and the theory of non-commutative crepant resolutions due to Van den Bergh, Iyama and Wemyss, see for instance [24], ‘resolve’ algebras of infinite global dimension by algebras of finite global dimension. The quality of such a resolution can be measured by the dominant dimension of the resolving algebra or by the rigidity degree of the generator-cogenerator over the algebra being resolved. The homological dimension to be defined now, aims at measuring the quality of the best possible such resolution.
Definition 2.4 The rigidity dimension of an algebra $A$ is defined to be

$$\text{rigdim}(A) = \sup \left\{ \text{domdim} \text{End}_A(M) \mid M \text{ is a generator-cogenerator in } A\text{-mod} \text{ and } \text{gldim} \text{End}_A(M) < \infty \right\}.$$ 

The construction in the proof of Iyama’s finiteness theorem on representation dimension, [22, Lemma 2.2], ensures the existence of a generator-cogenerator $M$ in $A$-mod with $\text{End}_A(M)$ being quasi-hereditary and hence $\text{gldim} \text{End}_A(M) < \infty$. Thus, the supremum is taken over a non-empty set.

The rigidity dimension of a semisimple algebra is always $\infty$, as the dominant dimension of semisimple algebras is $\infty$. In Section 3, we will give criteria to check finiteness of rigidity dimension for many non-semisimple algebras, and we will prove finiteness for all non-semisimple algebras after assuming Tachikawa’s conjectures and Yamagata’s conjecture.

Morita-Tachikawa correspondence includes the statement that endomorphism rings of generator-cogenerators have dominant dimension at least two. Thus, values of rigdim always are at least two.

Corollary 2.5 For any algebra $A$, $\text{rigdim} A \geq 2$.

This also follows from a reformulation of the definition, using Müller’s Theorem [22].

$$\text{rigdim}(A) = \sup \left\{ \text{rd}(A M) \mid M \text{ is a generator-cogenerator in } A\text{-mod} \text{ and } \text{gldim} \text{End}_A(M) < \infty \right\} + 2.$$

2.3 Basic properties, examples and connections

Proposition 2.6 Let $A$ and $B$ be algebras. Then

1. $\text{rigdim}(A) = \text{rigdim}(A^{\text{op}})$ and $\text{rigdim}(A \times B) = \min\{\text{rigdim}(A), \text{rigdim}(B)\}$.

2. If $A$ and $B$ are Morita equivalent, then $\text{rigdim}(A) = \text{rigdim}(B)$.

3. If $k$ is perfect, then $\text{rigdim}(A \otimes_k B) \geq \min\{\text{rigdim}(A), \text{rigdim}(B)\}$.

Proof. (1) and (2) are consequences of well-known facts: Both global dimension and dominant dimension are invariant under taking opposite algebras or passing to Morita equivalent algebras; the dominant dimension (respectively, global dimension) of the product of two algebras is the minimum (respectively, the maximum) of their dominant dimensions (respectively, global dimensions).

(3) Let $X$ and $Y$ be generator-cogenerators in $A$-mod and $B$-mod, respectively. Then $X \otimes_k Y$ is a generator-cogenerator in $(A \otimes_k B)$-mod. Now $\text{End}_{A \otimes_k B}(X \otimes_k Y) \cong \text{End}_A(X) \otimes_k \text{End}_B(Y)$ as $k$-algebras, and $\text{domdim}(\text{End}_A(X) \otimes_k \text{End}_B(Y)) = \min\{\text{domdim} \text{End}_A(X), \text{domdim} \text{End}_B(Y)\}$ by [31, Lemma 6], and $\text{gldim}(\text{End}_A(X) \otimes_k \text{End}_B(Y)) = \text{gldim} \text{End}_A(X) + \text{gldim} \text{End}_B(Y)$ whenever $k$ is a perfect field. Therefore, $\text{rigdim}(A \otimes_k B) \geq \min\{\text{rigdim}(A), \text{rigdim}(B)\}$. \hfill \Box

2.3.1 Relation with higher representation dimension

The following result exhibits rigidity dimension as a counterpart of the higher representation dimension introduced by Iyama [24, Definition 5.4]. Recall that for a natural number $n$, the $n$-th representation dimension $\text{repdim}_n(A)$ of an algebra $A$ is defined to be

$$\text{repdim}_n(A) = \inf \left\{ \text{gldim} \text{End}_A(M) \mid M \text{ is a generator-cogenerator in } A\text{-mod} \text{ and } \text{domdim} \text{End}_A(M) \geq n + 1 \right\}.$$ 

Auslander’s classical representation dimension is $\text{repdim}_1$. As Iyama has shown [22], $\text{repdim}_1$ is always finite. For $n \geq 2$, infinite values do occur.

Proposition 2.7 Let $A$ be an algebra and $n$ a positive integer. Then $\text{repdim}_n(A) < \infty$ if and only if $\text{rigdim}(A) \geq n + 1$. In particular, let $M$ be a generator-cogenerator in $A$-mod with $\text{rd}(M) \geq \text{rigdim}(A) - 1$. Then $\text{gldim} \text{End}_A(M) = \infty$ unless $\text{End}_A(M)$ is semisimple.
Proof. If \( \text{repdim}_n(A) < \infty \), then \( \text{rigdim}(A) \geq n + 1 \) by definition. Conversely, if \( \text{rigdim}(A) \geq n + 1 \), then there exists a generator-cogenerator \( M \) in \( A\mod \) such that \( \text{domdim} \text{End}_A(M) \geq n + 1 \) and \( \text{gldim} \text{End}_A(M) < \infty \). Hence, \( \text{repdim}_n(A) < \infty \). \( \Box \)

When \( n = 1 \), the statement \( \text{rigdim}(A) \geq 2 \) for all \( A \) in Corollary 2.8, is a reformulation of Iyama's finiteness result, which has been used in the definition of rigidity dimension.

**Example 2.8** Let \( A \) be a finite dimensional non-simple self-injective local \( k \)-algebra with \( \text{rad}^3(A) = 0 \). Then \( \text{rigdim}(A) = 2 \), since by [19, Theorem 3.4] every non-projective \( A\)-module has non-trivial self-extensions.

### 2.3.2 Relation with maximal orthogonal modules, and a lower bound for rigidity dimension

Recall that Iyama [23] has defined a module \( M \) to be maximal \( n \)-orthogonal (for \( n \geq 1 \)) if it satisfies \( M^{\perp_n} = \perp_n M = \text{add}(M) \). Such a module \( M \) automatically is a generator-cogenerator. Endomorphism rings of maximal \( n \)-orthogonal modules always have finite global dimension.

**Proposition 2.9** Let \( A \) be a non-semisimple algebra and \( n \) a natural number. If there exists a maximal \( n \)-orthogonal \( A\)-module, then \( \text{rigdim}(A) \geq n + 2 \).

**Proof.** Let \( _AM \) be a maximal \( n \)-orthogonal \( A\)-module. By [21, Theorem 0.2], \( M \) is a generator-cogenerator with \( \text{rd}(M) = n \) and \( \text{gldim End}_A(M) < \infty \). Therefore, \( \text{rigdim}(A) \geq n + 2 \). \( \Box \)

**Remark.** Using Theorem 3.1(2) below, if furthermore either \( 1 \leq \text{injdim}(A_A) \leq n + 1 \) or \( 1 \leq \text{injdim}(A_A^\perp) \leq n + 1 \), then \( \text{rigdim}(A) = n + 2 \). Indeed, under this assumption \( \text{rigdim}(A) \leq n + 2 \) by Theorem 3.1(2), and therefore \( \text{rigdim}(A) = n + 2 \).

### 2.3.3 Weakly Calabi-Yau self-injective algebras

Let \( A \) be a self-injective algebra. The stable module category \( A\mod -\) of \( A \) is a \( k \)-linear Hom-finite triangulated category, and its shift functor \( \Sigma \) is the cosyzygy functor \( \text{Ω}^{-1}_A \) [18 Section 2.6]. Recall that \( A\mod -\) is said to be weakly \( n \)-Calabi-Yau for a natural number \( n \) if there are natural \( k \)-linear isomorphisms

\[
\text{Hom}_A(Y, \Sigma^n(X)) \cong D \text{Hom}_A(X, Y)
\]

for any \( X, Y \in A\mod -\). Since \( A\mod -\) has a Serre duality \( \Omega_A \nu_A [9, \text{Prop. 1.2}] \), \( A \) is weakly \( n \)-Calabi-Yau if and only if \( \Omega_A^n \) and \( \Omega_A \nu_A \) are naturally isomorphic as auto-equivalences of \( A\mod -\). Equivalently, \( \Omega_A^{n+1} \nu_A \) is naturally isomorphic to the identity functor of \( A\mod -\). If \( A \) is symmetric, then it is weakly \( n \)-Calabi-Yau if and only if \( \Omega_A^{n+1} \) is naturally isomorphic to the identity functor of \( A\mod -\).

**Proposition 2.10** Let \( A \) be a non-semisimple self-injective algebra. If \( A\mod -\) is weakly \( n \)-Calabi-Yau with \( n \geq 1 \), then \( \text{rigdim}(A) \leq n + 1 \).

**Proof.** Since \( A \) is self-injective and \( n \geq 1 \), we have \( \text{Ext}^n_A(X, X) \cong \text{Hom}_A(X, \Omega_A^{-n}(X)) \) for any \( A\)-module \( X \). Recall that the shift functor \( \Sigma \) of \( A\mod -\) is given by the cosyzygy functor \( \Omega_A^{-1} \). This yields \( \text{Hom}_A(X, \Omega_A^{-n}(X)) = \text{Hom}_A(X, \Sigma^n(X)) \). Since \( A\mod -\) is weakly \( n \)-Calabi-Yau, it follows that \( \text{Hom}_A(X, \Sigma^n(X)) \cong D \text{Hom}_A(X, X) \). Thus \( \text{Ext}^n_A(X, X) \cong D \text{Hom}_A(X, X) \). This implies that if \( X \) is non-projective, then \( \text{Ext}^n_A(X, X) \) does not vanish. In this case, \( \text{rd}(A_A) \leq n - 1 \). Now, it is clear that \( \text{rigdim}(A) \leq n + 1 \). \( \Box \)

**Corollary 2.11** Let \( A \) be a preprojective algebra of Dynkin type over an algebraically closed field. Then \( \text{rigdim}(A) = 3 \).

**Proof.** It is known that \( A\mod -\) is weakly 2-Calabi-Yau. Proposition 2.10 implies \( \text{rigdim}(A) \leq 3 \). Further, by [16, Theorem 2.2 and Corollary 2.3] there exists a maximal 1-orthogonal \( A\)-module. It follows from Proposition 2.9 that \( \text{rigdim}(A) \geq 3 \). Thus \( \text{rigdim}(A) = 3 \). \( \Box \)
3 Finiteness

When defining a homological dimension, a basic question is: On which algebras does it take finite values? Semisimple algebras have infinite rigidity dimension, for trivial reasons, which distinguish them from all other algebras. In the first two subsections we provide two methods to prove finiteness. The first one is using extension groups between injective and projective modules; this works for algebras of finite global dimension and for gendo-symmetric algebras. The second one is using Hochschild cohomology; this works for group algebras of finite groups. The third subsection then derives finiteness in general from (still unproven) homological conjectures due to Tachikawa and Yamagata.

3.1 Finiteness I: Relation with the extension groups $\text{Ext}^*(D(A), A)$

As $D(A) \oplus A$ appears as a direct summand of every generator-cogenerator $M$ (up to multiplicities), the groups $\text{Ext}^*_A(D(A), A)$ naturally occur in the computation of the Yoneda algebra $\text{Ext}^*_A(M, M)$, and therefore in the computation of the rigidity dimension of $A$.

**Theorem 3.1** Let $A$ be a non-selfinjective $k$-algebra. Then

1. $\text{rigdim}(A) \leq \sup\{n \in \mathbb{N}_0 \mid \text{Ext}^j_A(D(A), A) = 0 \text{ for } 1 \leq j \leq n\} + 2$. Equality holds if the endomorphism algebra of $A \oplus D(A)$ has finite global dimension.
2. $\text{rigdim}(A) \leq \text{injdim}(A) + 1 \leq \text{gldim}(A) + 1$.

**Proof.** (1) Let $d := \sup\{n \in \mathbb{N}_0 \mid \text{Ext}^j_A(D(A), A) = 0 \text{ for } 1 \leq j \leq n\}$. Then, $d = \text{rd}(A \oplus D(A))$ since $\text{Ext}^j_A(D(A), A) \cong \text{Ext}^j_A(A \oplus D(A), A \oplus D(A))$ for any $i \geq 1$. Now let $M$ be a generator-cogenerator in $A$-mod. As $A \oplus D(A) \in \text{add}(M)$, it follows that $\text{rd}(A \oplus D(A)) = \text{rd}(A \oplus D(A))$ and therefore $\text{rigdim}(A) \leq d + 2$. If furthermore $\text{gldim}(A \oplus D(A)) < \infty$, then $\text{rigdim}(A) = \text{rd}(A \oplus D(A)) + 2 = d + 2$.

2. (2) Let $m := \text{injdim}(A)$. Then $m \geq 1$ since otherwise, $A$ is self-injective. When $m$ is infinite, there is nothing to show. Suppose $m$ is finite. Then $\text{Ext}^j_A(D(A), A) \neq 0$ and (1) implies $\text{rigdim}(A) \leq m - 1 + m + 1$. □

**Example 3.2** Let $A$ be a non-semisimple hereditary algebra. Then $\text{rigdim}(A) = 2$ since for any generator-cogenerator $M$ in $A$-mod, we have $D(A) \oplus A \in \text{add}(M)$ and $\text{Ext}^1_A(D(A), A) \neq 0$.

**Corollary 3.3** If $A$ is a gendo-symmetric algebra, i.e., the endomorphism algebra of a generator over some symmetric algebra, then $\text{rigdim}(A) \leq \text{domdim}(A)$.

**Proof.** By [13, Proposition 3.3], the dominant dimension of $A$ is at least two, and equals

$$\sup\{n \in \mathbb{N}_0 \mid \text{Ext}^i_A(D(A), A) = 0, 1 \leq i \leq n\} + 2.$$ 

This implies $\text{rigdim}(A) \leq \text{domdim}(A)$ by Theorem 3.1(1). □

**Example 3.4** In particular, a gendo-symmetric algebra $A$ with $\text{domdim}(A) = 2$ satisfies $\text{rigdim}(A) = 2$. Examples of such algebras are the non-simple blocks of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ of semisimple complex Lie algebras. The global dimension of these algebras always is an even number, which can be arbitrarily large.

3.2 Finiteness II: Relation with Hochschild cohomology

Let $A$ be a $k$-algebra and $A^\text{op}$ its opposite algebra. The Hochschild cohomology ring $HH^*(A)$ is the Yoneda extension algebra $\text{Ext}^*_A(A, A)$ where $A^\text{co} = A \otimes_k A^\text{op}$, the enveloping algebra of $A$. So, $HH^*(A)$ is the direct sum of $HH^i(A) := \text{Ext}^i_A(A, A)$ for $i \in \mathbb{N}_0$. This is an $\mathbb{N}_0$-graded $k$-algebra. In general, it is not commutative, but graded commutative. Moreover, it may be infinite-dimensional as a vector space over $k$. However, if $A^\text{co}$ has finite global dimension, then $HH^*(A)$ is finite-dimensional. The
reduced Hochschild cohomology ring $\text{HH}^i(A)$ is the quotient $\text{HH}^i(A)/\mathcal{N}$ where $\mathcal{N}$ is the ideal of $\text{HH}^i(A)$ consisting of nilpotent elements. Although the problem whether the (reduced) Hochschild cohomology ring is finitely generated has been widely studied, very little seems to be known about the degrees of the homogeneous generators. We will show that rigidity dimension of $A$ is closely related to the minimal degree of non-nilpotent homogeneous generators of positive degree.

We will use the following result, which is essentially combining [5] Theorems 2.13 and 5.9] in our situation. Note that a main tool in [5] is the grade defined by Auslander and Bridger, which is closely related to dominant dimension.

**Theorem 3.5 (Buchweitz)** Let $M$ be a generator-cogenerator in $A$-mod and let $E = \text{End}_A(M)$. Then there is an $\mathbb{N}_0$-graded algebra homomorphism $\text{HH}^i(E) \to \text{HH}^i(A)$ such that $\text{HH}^i(E) \to \text{HH}^i(A)$ is an isomorphism for each $0 \leq i \leq \text{rd}(A,M)$.

The connection with rigidity dimension is as follows:

**Theorem 3.6** Let $A$ be an algebra over a perfect field $k$. Suppose that $\text{HH}^i(A)$ is not concentrated in degree zero. Then $\text{rigdim}(A)$ is finite.

More precisely, let $\delta(A) := \inf\{i \geq 1 \; | \; \text{HH}^i(A) \neq 0\}$. Then $\text{rigdim}(A) \leq \delta(A) + 1$. If $k$ has characteristic different from 2 and $\text{rigdim}(A)$ is even, then $\text{rigdim}(A) \leq \delta(A)$.

**Proof.** Let $M$ be a generator-cogenerator in $A$-mod and let $E = \text{End}_A(M)$. By Theorem 3.5, there is a graded algebra homomorphism $\varphi : \text{HH}^i(E) \to \text{HH}^i(A)$ such that $\varphi_i : \text{HH}^i(E) \to \text{HH}^i(A)$ is an isomorphism for $0 \leq i \leq \text{rd}(A,M) = \text{domdim}E - 2$, where the final equality follows from Müller's Theorem 2.3.

If $\text{gldim}E < \infty$, then $\text{HH}^i(E)$ is a finite dimensional $k$-algebra, and therefore all homogeneous elements in $\text{HH}^i(E)$ of positive degrees are nilpotent. Using $\varphi$ and $\varphi_m$, all homogeneous elements in $\text{HH}^i(A)$ of degree $m$ with $1 \leq m \leq \text{domdim}E - 2$ are seen to be nilpotent. Hence, $\text{domdim}E - 1 \leq \delta(A)$ since $\delta(A)$ detects the minimal degree of homogeneous generators of positive degrees in $\text{HH}^i(A)$. Thus $\text{rigdim}(A) \leq \delta(A) + 1$.

If $k$ has characteristic different from 2, then homogeneous elements in $\text{HH}^i(A)$ of odd degrees are nilpotent, and therefore $\delta(A)$ must be an even number or $\infty$. When $\text{rigdim}(A)$ is an even number, then $\text{rigdim}(A) \leq \delta(A)$ as claimed.

**Remark.** (1) Theorem 3.6 provides an upper bound for the rigidity dimension of $A$ as long as some information on its Hochschild cohomology ring is known. Conversely, a lower bound for rigidity dimension is a lower bound for the degree of homogeneous generators in the reduced Hochschild cohomology ring. Examples in [7] will show that the bound in Theorem 3.6 is optimal.

(2) The reduced Hochschild cohomology ring $\text{HH}_r(A)$ is not finitely generated in general. The first example has been a seven dimensional $k$-algebra $A$ found by Xu [39]. If $k$ has characteristic two, then $\delta(A) = 1$ [39], see also [36] Theorem 4.5, and therefore $\text{rigdim}(A) = 2$.

(3) Even when $A$ has infinite global dimension, the reduced Hochschild cohomology ring $\text{HH}_r(A)$ can be concentrated in degree zero, see [36] [15] for examples.

Theorem 3.6 can be applied to group algebras, for which we obtain finiteness in all non-semisimple cases.

**Theorem 3.7** Let $G$ be a finite group and $k$ a perfect field of characteristic $p \geq 0$.

Then $\text{rigdim}(kG) < \infty$ if and only if $p$ divides the order $|G|$ of $G$ (if and only if $kG$ is not semisimple). In this case, $\text{rigdim}(kG) \leq |G|$.

**Proof.** By [27] Proposition 4.5, there is an injective graded $k$-algebra homomorphism

$$\theta : \text{HH}^i(G,k) \to \text{HH}^i(kG)$$
where $H^*(G, k) = \text{Ext}_{kG}^*(k, k)$ is the cohomology ring of the group algebra $kG$. Let $\overline{H}^*(G, k)$ be the cohomology ring $H^*(G, k)$ modulo nilpotent elements. Then $\theta$ induces an injective morphism $\overline{H}^*(G, k) \to \text{HH}^*(kG)$.

If $p$ does not divide $|G|$, then $kG$ is semisimple and hence $\text{rigdim}(kG) = \infty$. If $p$ divides $|G|$, then by Chouinard’s theorem in group cohomology, see [4, Lemma 5.2.3 and Theorem 5.2.4], $k$ is not projective and

$$\gamma(kG) := \inf\{i \geq 1 \mid \overline{H}^i(G, k) \neq 0\} < \infty.$$  

The embedding $\overline{H}^*(G, k) \to \text{HH}^*(kG)$ gives $\gamma(kG) \geq \delta(kG)$, the minimal degree of homogeneous generators of $\text{HH}^*(kG)$ of positive degrees. Using Theorem 3.6, it follows that

$$\text{rigdim}(kG) \leq \delta(kG) + 1 \leq \gamma(kG) + 1 < \infty.$$  

The explicit upper bound follows from a major result by Symonds, who proved Benson’s regularity conjecture. According to [37, Proposition 0.3], which is a consequence of the main result in [37], for a finite group $G$ with more than one element, group cohomology $H^*(G, k)$ has a set of homogeneous generators of degree at most $|G| - 1$. As a consequence, the reduced group cohomology ring has a set of generators in degrees smaller than $|G|$, and thus

$$\gamma(kG) := \inf\{i \geq 1 \mid \overline{H}^i(G, k) \neq 0\} \leq |G| - 1.$$  

Remark. (1) Symonds’ result [37] provides an upper bound for the degrees of homogeneous generators of $H^*(G, k)$, Theorem 3.7 shows that rigidity dimension of the group algebra $kG$ may be used to provide a lower bound for the degrees of homogeneous generators of the Hochschild cohomology ring modulo nilpotents.

(2) In [7], the rigidity dimensions of defect one blocks of group algebras will be determined.

3.3 Finiteness III: Using homological conjectures

Is it to be expected that $\text{rigdim}$ is always finite, except for semisimple algebras? Some evidence is provided here by deriving this statement from some homological conjectures.

(TC1) **Tachikawa’s First Conjecture** [38, p. 115] Let $A$ be a finite dimensional $k$-algebra. Suppose $\text{Ext}_A^i(D(A), A) = 0$ for all $i \geq 1$. Then $A$ is self-injective.

(TC2) **Tachikawa’s Second Conjecture** [38, p. 116] Let $A$ be a finite dimensional self-injective $k$-algebra and $M$ a finitely generated $A$-module. Suppose $\text{Ext}_A^i(M, M) = 0$ for all $i \geq 1$. Then $M$ is a projective $A$-module.

(YC) **Yamagata’s Conjecture** [35, p. 876] There exists a function $\varphi : \mathbb{N} \to \mathbb{N}$ such that for any finite dimensional $k$-algebra $A$ with finite dominant dimension, $\text{domdim} A \leq \varphi(n)$ where $n$ is the number of isomorphism classes of simple $A$-modules.

(TC1) and (TC2) together are equivalent to Nakayama’s conjecture.

**Theorem 3.8** Let $A$ be a finite dimensional non-semisimple $k$-algebra.

1. If $A$ is not self-injective, then (TC1) implies $\text{rigdim}(A) < \infty$.

2. If (YC) holds, then $\text{rigdim}(A) < \infty$.  


Proof. (1) If $A$ is not self-injective, then by (TC1), $\text{Ext}^n_A(D(A), A) \neq 0$ for some natural number $n \geq 1$. Therefore, by Theorem 3.3, $\text{rigdim}(A) \leq n + 2$.

(2) Let $M$ be a generator-cogenerator in $A\text{-mod}$. Up to multiplicities of direct summands, suppose $AM = A \oplus D(A) \oplus \bigoplus_{i=1}^m M_i$, where $m \geq 1$, and $M_i$ is either zero or indecomposable, non-projective and non-injective.

Claim. $\text{rd}(M) = \min \{ \text{rd}(A \oplus D(A) \oplus M_i \oplus M_j) \mid 1 \leq i, j \leq m \}$.

Proof of claim. For $1 \leq i, j \leq m$, set $M_{i,j} := A \oplus D(A) \oplus M_i \oplus M_j$. Since $M_{i,j} \in \text{add}(M)$, \text{rd}(M) \leq \text{rd}(M_{i,j})$. If \text{rd}(M) is infinite, there is nothing to show. Suppose \text{rd}(M) = s < \infty. Then $\text{Ext}^t_A(M, M) = 0$ for $1 \leq t \leq s$ and $\text{Ext}^{s+1}_A(M, M) \neq 0$. Since $\text{add}(M) = \text{add}((\bigoplus_{1 \leq i \leq m}(A \oplus D A \oplus M_i)))$, there is a pair $(u, v)$ of integers with $1 \leq u, v \leq m$ such that $\text{Ext}^{u,v}_A(A \oplus D A \oplus M_u, A \oplus D A \oplus M_v) \neq 0$. This implies $\text{rd}(M_{u,v}) \leq s$, and thus $\text{rd}(M_{u,v}) = s$.

Suppose $E := \text{End}_A(M)$ has finite global dimension. By assumption, $A$ is not semisimple, hence $E$ is not semisimple either. It follows that $\text{domdim} E \leq \text{gldim} E < \infty$. Moreover, by Theorem 2, $\text{domdim} E = \text{rd}(M) + 2$ and $\text{domdim} \text{End}_A(M_{u,v}) = \text{rd}(M_{u,v}) + 2$. Then the equality $\text{rd}(M_{u,v}) = \text{rd}(M)$ implies that $\text{domdim} \text{End}_A(M_{u,v}) = \text{domdim} E < \infty$. Denote by $d$ the number of isomorphism classes of indecomposable objects in $\text{add}(A \oplus D(A))$. Then $M_{u,v}$ has at least $d$ and at most $d + 2$ non-isomorphic indecomposable direct summands. Applying $(YC)$ to $\text{End}_A(M_{u,v})$ yields the inequality

$$\text{domdim} \text{End}_A(M_{u,v}) \leq \max \{ \varphi(d), \varphi(d + 1), \varphi(d + 2) \}.$$

Consequently, $\text{domdim} E$ has a uniform upper bound, which only depends on the function $\varphi$ and $d$. Thus $\text{rigdim}(A) < \infty$. \hfill $\Box$

4 Stable equivalences and invariance I

In this and the next section, we discuss the invariance of rigidity dimension under stable or derived equivalences. The first main result in this section shows that algebras without nodes have minimal rigidity dimension in their stable equivalence class. In particular, two stably equivalent algebras without nodes have the same rigidity dimension. The second main result provides stronger information for stable equivalences between self-injective algebras. In particular, stable equivalences preserving the triangulated structure are shown to preserve rigidity dimension.

4.1 Notation and definitions, and some correspondences

Let $A$ be an algebra. By $A\text{-mod}$ we denote the stable module category of $A$. It has the same objects as $A\text{-mod}$, but the morphism set between two $A$-modules $X$ and $Y$ is given by

$$\text{Hom}(X, Y) := \text{Hom}_A(X, Y)/\mathcal{P}(X, Y)$$

where $\mathcal{P}(X, Y)$ consists of homomorphisms factoring through projective $A$-modules.

We will use functor categories to translate from stable equivalences to equivalences of abelian categories. The category of finitely presented functors from $(A\text{-mod})^{\text{op}}$ to the category $\mathcal{A}$ of abelian groups is denoted by $(A\text{-mod})^{\text{op}}$-$\mathcal{A}$. Recall that a functor $F : (A\text{-mod})^{\text{op}} \to \mathcal{A}$ is finitely presented if there is a morphism $f : X \to Y$ in $A\text{-mod}$ inducing an exact sequence of functors $\text{Hom}_A(-, X) \to \text{Hom}_A(-, Y) \to F \to 0$. Note that $(A\text{-mod})^{\text{op}}$-$\mathcal{A}$ is an abelian category with enough projective objects and injective objects. For each $X \in A\text{-mod}$, $\text{Hom}_A(-, X)$ is a projective object and $\text{Ext}^1_A(-, X)$ is an injective object; all projective (respectively, injective) objects are of these forms (for more details, see [1], [2]).

Let $A\text{-mod}_{\mathcal{A}}$ and $A\text{-mod}_{\mathcal{P}}$ be the full subcategories of $A\text{-mod}$ consisting of all modules without projective and injective direct summands, respectively. Then there is a one-to-one correspondence between indecomposable projective (respectively, injective) objects in $(A\text{-mod})^{\text{op}}$-$\mathcal{A}$ and indecomposable modules in $A\text{-mod}_{\mathcal{A}}$ (respectively, $A\text{-mod}_{\mathcal{P}}$).
Definition 4.1 Two algebras $A$ and $B$ are stably equivalent if $A$-$\text{mod}$ and $B$-$\text{mod}$ are equivalent as additive categories; equivalently, $(A$-$\text{mod})$-$\text{mod}$ and $(B$-$\text{mod})$-$\text{mod}$ are equivalent as abelian categories.

The main complication when studying invariance of homological dimensions under stable equivalences, comes from a particular class of modules called nodes:

Definition 4.2 An indecomposable $A$-module $S$ is called a node if it is neither projective nor injective, and there is an almost split sequence $0 \to S \to P \to T \to 0$ with $P$ a projective $A$-module.

A node $S$ must be simple, see, for example, [3] V.Theorem 3.3. A node does not occur as a composition factor of $\text{rad}(A)/\text{soc}(A)$.

Two stably equivalent algebras without nodes and without semi-simple summands share many homological invariants, such as stable Grothendieck groups, global dimensions and dominant dimensions (see [30]). Such stable equivalences do preserve projective dimensions. However, if one algebra has a node, simple examples of stable equivalences already show that projective and global dimensions are not preserved.

Self-injective algebras with nodes are quite well-known (see [3]): When a self-injective connected algebra $A$ has a node $S$, there exists an indecomposable projective (and injective) $A$-module whose radical is $S$. Moreover, the Loewy length of $A$ must be two and $A$ must be a Nakayama algebra. The non-projective indecomposable $A$-modules are all simple, and each simple $A$-module is a node satisfying $D\text{Tr}(S) \cong \Omega_A(S)$ and $\Omega^m_A(S) \cong S$ for some positive integer $m$. This justifies the following notation:

Definition 4.3 Let $A$ be a self-injective algebra with nodes. Then $\rho(A)$ denotes the smallest positive integer $m$ such that there exists a node $\sim A S$ and an isomorphism $S \cong \Omega_A^m(S)$.

We will use various correspondences between objects in two equivalent stable categories and in related categories.

Let $F: A$-$\text{mod} \to B$-$\text{mod}$ be an equivalence of additive categories. It induces quasi-inverse equivalences of abelian categories:

$$\alpha: (A$-$\text{mod})$-$\text{mod} \cong (B$-$\text{mod})$-$\text{mod} \text{ and } \beta: (B$-$\text{mod})$-$\text{mod} \cong (A$-$\text{mod})$-$\text{mod}.$

Moreover, there are one-to-one correspondences

$$\tilde{\alpha}: A$-$\text{mod}, \alpha': A$-$\text{mod}, \tilde{\beta}: B$-$\text{mod}, \beta': B$-$\text{mod} \leftrightarrow B$-$\text{mod}, \alpha': A$-$\text{mod}, \tilde{\beta}: B$-$\text{mod}, \beta': B$-$\text{mod}$$

such that

$$\alpha(\text{Hom}_A(\_ ,X)) \cong \text{Hom}_B(\_ ,\tilde{\alpha}(X)) \text{ and } \alpha(\text{Ext}^1_A(\_ ,Y)) \cong \text{Ext}^1_B(\_ ,\alpha'(Y)),$$

$$\beta(\text{Hom}_B(\_ ,M)) \cong \text{Hom}_A(\_ ,\tilde{\beta}(M)) \text{ and } \beta(\text{Ext}^1_B(\_ ,N)) \cong \text{Ext}^1_A(\_ ,\beta'(N))$$

where $X \in A$-$\text{mod}, Y \in A$-$\text{mod}, M \in B$-$\text{mod},$ and $N \in B$-$\text{mod}$. When applied to $A$-modules, $F$ and $\tilde{\alpha}$ coincide.

Formally, we set $\tilde{\alpha}(P) = 0$ and $\alpha'(I) = 0$ when $\sim A P$ is projective and $\sim A I$ is injective. Readers familiar with the Auslander Reiten translation $D\text{Tr}$, may observe that the correspondences are related by $D\text{Tr}\tilde{\alpha} \cong \alpha'D\text{Tr}$.

When working with generators, the following correspondences will be used:

$$\Phi: A$-$\text{mod} \to B$-$\text{mod} \text{ and } \Psi: B$-$\text{mod} \to A$-$\text{mod} \leftrightarrow B$-$\text{mod},$$

$$U \mapsto \tilde{\alpha}(U) \oplus \bigoplus_{Q \in \mathcal{P}_B} Q, \text{ and } V \mapsto \tilde{\beta}(V) \oplus \bigoplus_{P \in \mathcal{P}_A} P.$$

Moreover, if $X$ is indecomposable, not projective, not injective and not a node, then $\tilde{\alpha}(X) \cong \alpha'(X)$ (see [2] Lemma 3.4).

A particular set of nodes will turn out to play a crucial role (compare [17] Section 3):
Definition 4.4 Let \( n_F(A) \) be the set of isomorphism classes of indecomposable, neither projective nor injective \( A \)-modules \( U \) such that \( \tilde{\alpha}(U) \not\cong \alpha'(U) \).

By definition, \( n_F(A) \) depends on the given equivalence \( F \). In this situation, \( n_{F^{-1}}(B) \) denotes the set of isomorphism classes of indecomposable, neither projective nor injective \( B \)-modules \( V \) such that \( \beta(V) \not\cong \beta'(V) \).

The elements of \( n_F(A) \) are nodes; in general, \( n_F(A) \) is a proper subset of the set of isomorphism classes of nodes. It can be empty although there are nodes.

4.2 Invariance under general stable equivalences and under triangulated stable equivalences

We state the main result and briefly outline its proof, which will occupy the rest of this section. We also provide examples showing that the assumptions are necessary and algebras with nodes really behave differently.

**Theorem 4.5** (I) Let \( A \) and \( B \) be stably equivalent algebras. Suppose \( A \) has no nodes. Then \( \text{rigdim}(A) \leq \text{rigdim}(B) \). In particular, if neither \( A \) nor \( B \) has nodes, then \( \text{rigdim}(A) = \text{rigdim}(B) \).

(II) More precisely, let \( A \) and \( B \) be stably equivalent by an equivalence \( F \). If \( n_F(A) \) is empty, then \( \text{rigdim}(A) \leq \text{rigdim}(B) \). In particular, if both \( n_F(A) \) and \( n_{F^{-1}}(B) \) are empty, then \( \text{rigdim}(A) = \text{rigdim}(B) \).

Statement (II) implies (I), since \( n_F(A) \) is contained in the set of nodes. Statement (I) may, however, be easier to apply, since it uses a property of the algebra, not of the given equivalence. By (I), the rigidity dimension of an algebra without nodes is minimal in its stable equivalence class.

The stable module category of a self-injective algebra carries additional structure; it is a triangulated category (see [18, 3]). A stable equivalence between self-injective algebras may be a triangulated equivalence or just an additive equivalence. It turns out that triangulated equivalences do respect rigidity dimension even when there are nodes. Also, in the additive case, more can be said when the algebras are self-injective.

**Theorem 4.6** Let \( A \) and \( B \) be stably equivalent self-injective algebras. Then:

(I) \( \text{rigdim}(A) = \text{rigdim}(B) \) in each of the following three cases:

1. \( A \) has no nodes.
2. \( A \) and \( B \) are symmetric algebras.
3. \( A \text{-mod} \) and \( B \text{-mod} \) are equivalent as triangulated categories.

(II) If both \( A \) and \( B \) have nodes, then

\[ |\text{rigdim}(A) - \text{rigdim}(B)| \leq |\rho(A) - \rho(B)|. \]

In particular, \( \text{rigdim}(A) < \infty \) if and only if \( \text{rigdim}(B) < \infty \).

In the proof of the two theorems, we will have to control what happens to generator-cogenerators and their endomorphism algebras, under stable equivalences. In every step, nodes will cause problems. Therefore, in the first part of the proof, we will have to use various correspondences between objects in the stable categories and also in the functor categories. These correspondences are compatible with Guo’s results [17], which thus can be used to compare global dimensions of endomorphism rings. The core of the proof then is to compare also dominant dimensions, which needs another technically involved subsection. Finally, everything can be put together to derive Theorem 4.5. To prove Theorem 4.6 we will in addition use the description of self-injective algebras with nodes in [3].

In general, that is in the presence of nodes, stable equivalences do not preserve rigidity dimensions, as the following examples illustrate.
Example 4.7 (a) Let $B$ be the $k$-algebra given by the quiver $1 \xrightarrow{\alpha} 2$ with relations $\{\alpha \beta, \beta \alpha\}$. Then $B$ is self-injective with radical square zero. $B$ is stably equivalent to $A := T_2(k) \times T_2(k)$, where $T_2(k) := \begin{pmatrix} k & 0 \\ k & k \end{pmatrix}$. The algebra $B$ has two nodes, but $A$ has no nodes. The rigidity dimensions of $B$ and $A$ can be calculated as follows.

Since $T_2(k)$ is hereditary, rigdim$(T_2(k)) = 2$ by Theorem 3.1. Together with Proposition 2.6, rigdim$(A) = \text{rigdim}(T_2(k)) = 2$. To calculate rigdim$(B)$, let $S_1$ and $S_2$ denote the simple $B$-modules corresponding to the vertices 1 and 2, respectively. Note that $B$ has only four basic generator-cogenerators (up to isomorphism): $B$, $B \oplus S_1$, $B \oplus S_2$ and $B \oplus S_1 \oplus S_2$. Moreover the endomorphism algebras, except for $B$ itself, have finite global dimension; domdim$\text{End}_B(B \oplus S_1) = 3 = \text{domdim} \text{End}_B(B \oplus S_2)$, but domdim$\text{End}_B(B \oplus S_1 \oplus S_2) = 2$. Thus rigdim$(B) = 3 > \text{rigdim}(A)$, illustrating that the inequality in Theorem 4.3 cannot be improved. In this example, the set $\mathcal{GCN}_{F-1}(B)$, to be introduced in the next subsection, consists only of the Auslander generator $B \oplus S_1 \oplus S_2$ since $B$ has two nodes $S_1$ and $S_2$.

(b) Now, we give an example showing that two stably equivalent self-injective algebras may have different rigidity dimensions. Let $B$ be as in (a) and let $C := k[x]/(x^2) \times k[x]/(x^2)$. This is a self-injective algebra. To check that $B$ and $C$ are stably equivalent it is enough to count indecomposable objects in $B$-mod and in $C$-mod. But $C$ has nodes and rigdim$(C) = 2 \neq \text{rigdim}(B)$. As the stable category of a Frobenius category is triangulated, both $B$-mod and $C$-mod are triangulated categories. However, the shift functor of $B$-mod permutes simple modules, while the shift functor of $C$-mod is the identity. This means that $B$-mod and $C$-mod are equivalent as additive categories, but not as triangulated categories.

4.3 Further preparations for the proofs

Throughout this subsection, $A$ and $B$ are stably equivalent algebras, possibly with nodes. We continue setting up correspondences between objects in the two stable categories and in related categories. Let $F : A$-mod $\rightarrow$ $B$-mod be an equivalence of additive categories.

Denote by $\mathcal{P}_A$ and $\mathcal{I}_A$ the set of isomorphism classes of indecomposable projective and injective $A$-modules, respectively. Further, set

$$
\triangle_A := n_F(A) \cup (\mathcal{P}_A \setminus \mathcal{I}_A) \quad \text{and} \quad \nabla_A := n_F(A) \cup (\mathcal{I}_A \setminus \mathcal{P}_A).
$$

Let $\triangle_A'$ be the class of indecomposable, non-injective $A$-modules which do not belong to $\triangle_A$. Then each module $Y \in A$-mod$_F$ admits a unique decomposition (up to isomorphism)

$$
Y \cong Y_\triangle \oplus Y'
$$

with $Y_\triangle \in \text{add}(\triangle_A)$ and $Y' \in \text{add}(\triangle_A')$. The module $Y_\triangle$ is called the $\triangle_A$-component of $Y$.

In the following, we denote by $\mathcal{GCN}_F(A)$ the class of basic $A$-modules $X$ which are generator-cogenerators with $n_F(A) \subseteq \text{add}(X)$. In particular, if $A$ has no nodes, then $\mathcal{GCN}_F(A)$ is exactly the class of basic generator-cogenerators for $A$-mod. In the same situation, we similarly use the notation $\mathcal{GCN}_{F-1}(B)$.

Here are some results from [17] for later use.

Lemma 4.8 (compare [17, Lemmas 3.1 and 3.2])

1. There are one-to-one correspondences

$$
\alpha : \nabla_A \leftrightarrow \nabla_B : \beta, \quad \alpha' : \triangle_A \leftrightarrow \triangle_B : \beta' \quad \text{and} \quad \alpha' : \triangle_A' \leftrightarrow \triangle_B' : \beta'.
$$

2. The correspondences $\Phi$ and $\Psi$ restrict to one-to-one correspondences between $\mathcal{GCN}_F(A)$ and $\mathcal{GCN}_{F-1}(B)$. Moreover, if $X \in \mathcal{GCN}_F(A)$, then $\Phi(X) \cong \alpha'(X) \oplus \bigoplus_{I \in \mathcal{I}_B} I$. 

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Proof. (1) By [17, Lemma 3.1], we only need to show that \(X \in \mathfrak{n}_F(A)\) implies \(\alpha'(X) \in \Delta_B\).

Let \(X \in \mathfrak{n}_F(A)\). Then \(X\) is neither projective nor injective. Consequently, \(\alpha'(X)\) is not injective. Recall that \(\beta'(V) \cong \beta'(V)\) for any \(V \in \Delta^c_B\). Suppose \(\alpha'(X) \in \Delta^c_B\). Then \(\alpha'(X)\) is not projective and \(\beta'(\alpha'(X)) \cong X\). It follows that \(\alpha'(X) \cong \tilde{\alpha}\beta'(\alpha'(X)) \cong \tilde{\alpha}(X)\). This is a contradiction since \(X \in \mathfrak{n}_F(A)\). Thus \(\alpha'(X) \in \Delta_B\).

Note that \(\tilde{\alpha}(U) \cong \alpha'(U)\) for any \(U \in \Delta^c_A\). Now, (2) follows from (1). \(\square\)

Generator-cogenerators in \(\mathcal{GC}N_F(A)\) are better to control under stable equivalences. In particular, global dimensions of their endomorphism algebras are preserved under stable equivalences:

**Lemma 4.9 ([17, Lemma 3.5])** If \(X \in \mathcal{GC}N_F(A)\), then

\[
\text{gldim} \text{End}_A(X) = \text{gldim} \text{End}_B(\Phi(X)).
\]

### 4.4 Stable equivalences and dominant dimension of endomorphism rings

To compare rigidity dimension under stable equivalences, we need an analogue of Lemma 4.9 for dominant dimensions; this is the main part of the proof of Theorem 4.5.

**Proposition 4.10** Let \(A\) and \(B\) be stably equivalent by an equivalence \(F\). If \(X \in \mathcal{GC}N(A)\), then

\[
\text{domdim} \text{End}_A(X) = \text{domdim} \text{End}_B(\Phi(X)).
\]

The proof of Proposition 4.10 will need three lemmas. The first one extends the first part of [17, Lemma 3.3].

**Lemma 4.11** Assume that \(AZ\) is indecomposable and non-projective. Let

\[
0 \longrightarrow X \oplus X' \longrightarrow Y \oplus P \overset{g}{\longrightarrow} Z \longrightarrow 0
\]

be an exact sequence of \(A\)-modules without a split exact sequence as a direct summand, such that \(X \in \text{add}(\Delta^c_A), X' \in \text{add}(\Delta_A), Y \in A\)-mod\(\neq\) and \(P \in \text{add}_A\). Then there is an exact sequence of \(B\)-modules

\[
0 \longrightarrow \tilde{\alpha}(X) \oplus \alpha'(X') \longrightarrow \tilde{\alpha}(Y) \oplus Q \overset{g'}{\longrightarrow} \tilde{\alpha}(Z) \longrightarrow 0
\]

without a split direct summand such that \(g' = F(g)\) in \(B\)-mod with \(Q \in \text{add}(B)\).

**Proof.** By the proof of the first part of [17, Lemma 3.3], the following two statements hold.

1. There is an exact sequence of \(B\)-modules

\[
0 \longrightarrow N \oplus N' \longrightarrow \tilde{\alpha}(Y) \oplus Q \overset{g'}{\longrightarrow} \tilde{\alpha}(Z) \longrightarrow 0
\]

without a split direct summand such that \(g' = \alpha(g)\) in \(B\)-mod, and that \(N \in \text{add}(\Delta^c_B), N' \in \text{add}(\Delta_B)\) and \(Q \in \text{add}(B)\).

2. \(\alpha(\text{Ext}^1_A(-, X \oplus X')) \cong \text{Ext}^1_B(-, N \oplus N')\) in \((B\text{-mod})\)-mod.

Since \(X \oplus X' \in A\)-mod\(\neq\), there is an isomorphism \(\alpha(\text{Ext}^1_A(-, X \oplus X')) \cong \text{Ext}^1_A(-, \alpha'(X) \oplus \alpha'(X'))\). Note that \(\alpha'(X) \cong \tilde{\alpha}(X)\) since \(X \in \text{add}(\Delta^c_A)\). Thus

\[
\text{Ext}^1_B(-, N \oplus N') \cong \alpha(\text{Ext}^1_A(-, X \oplus X')) \cong \text{Ext}^1_B(-, \tilde{\alpha}(X) \oplus \alpha'(X')).
\]

This implies \(\tilde{\alpha}(X) \oplus \alpha'(X') \cong N \oplus N'\) in \(B\)-mod\(\neq\). Since \(\tilde{\alpha}(X) \in \text{add}(\Delta^c_B)\) and \(\alpha'(X') \in \Delta_B\) by Lemma 4.8(1), there are isomorphisms \(\tilde{\alpha}(X) \cong N\) and \(\alpha'(X') \cong N'\). By (1), Lemma 4.11 follows. \(\square\)

The second lemma establishes a connection between different syzygy modules under stable equivalences.
Lemma 4.12 Let $X \in A\text{-mod}$ and $n$ a positive integer. Then
\[
\tilde{\alpha}(\Omega^n_A(Y)) \oplus \bigoplus_{j=1}^{n} \Omega^{n-1-j}_B(\alpha'(\Omega^j_A(X)_\triangle)) \cong \Omega^n_B(\tilde{\alpha}(X)) \oplus \bigoplus_{j=1}^{n} \Omega^{n-1-j}(\tilde{\alpha}(\Omega^j_A(X)_\triangle)),
\]
where $\Omega^j_A(X)_\triangle$ stands for the $\triangle_A$-component of the $A$-module $\Omega^j_A(X)$.

Proof. Lemma 4.12 will be shown by induction on $n$. If $n = 1$, then it suffices to check
\[
(\ast) \quad \tilde{\alpha}(\Omega_A(X)) \oplus \alpha'(\Omega_A(X)_\triangle) \cong \Omega_B(\tilde{\alpha}(X)) \oplus \tilde{\alpha}(\Omega_A(X)_\triangle).
\]
If $X$ is projective, then both sides are zero, and there is nothing to prove. It is enough to show $(\ast)$ when $X$ is indecomposable and non-projective. Let
\[
0 \to \Omega_A(X) \to P \to X \to 0
\]
be an exact sequence of $A$-modules such that $P$ is a projective cover of $X$. Then this sequence contains no split direct summand and $Y := \Omega_A(X) \in A\text{-mod}_f$. So $Y$ has a decomposition as $Y \cong Y_\triangle \oplus Z$, where $Y_\triangle \in \text{add}(\triangle_A)$ and $Z \in \text{add}(\triangle_A^c)$. Applying Lemma 4.11 to the sequence
\[
0 \to Z \oplus Y_\triangle \to P \to X \to 0
\]
yields the following exact sequence of $B$-modules
\[
0 \to \tilde{\alpha}(Z) \oplus \alpha'(Y_\triangle) \to Q \to \tilde{\alpha}(X) \to 0
\]
without split direct summands, such that $Q \in \text{add}(B)$. Thus $Q$ is a projective cover of $\tilde{\alpha}(X)$ and further, $\Omega_B(\tilde{\alpha}(X)) \cong \tilde{\alpha}(Z) \oplus \alpha'(Y_\triangle)$. Hence
\[
\tilde{\alpha}(Y) \oplus \alpha'(Y_\triangle) \cong \tilde{\alpha}(Y_\triangle) \oplus \tilde{\alpha}(Z) \oplus \alpha'(Y_\triangle) \cong \Omega_B(\tilde{\alpha}(X)) \oplus \tilde{\alpha}(Y_\triangle).
\]
This shows the isomorphism $(\ast)$.

Let $n \geq 2$. Suppose that for any $A$-module $U$, there is an isomorphism of $B$-modules
\[
(\ast\ast) \quad \tilde{\alpha}(\Omega^{n-1}_A(U)) \oplus \bigoplus_{j=1}^{n-1} \Omega^{n-1-j}(\alpha'(\Omega^j_A(U)_\triangle)) \cong \Omega^{n-1}_B(\tilde{\alpha}(U)) \oplus \bigoplus_{j=1}^{n-1} \Omega^{n-1-j}(\tilde{\alpha}(\Omega^j_A(U)_\triangle)).
\]
Choosing $U = Y = \Omega_A(X)$ gives
\[
\tilde{\alpha}(\Omega^n_A(X)) \oplus \bigoplus_{j=1}^{n-1} \Omega^{n-1-j}(\alpha'(\Omega^j_A(X)_\triangle)) \cong \Omega^{n-1}_B(\tilde{\alpha}(\Omega_A(X))) \oplus \bigoplus_{j=1}^{n-1} \Omega^{n-1-j}(\tilde{\alpha}(\Omega^j_A(X)_\triangle)).
\]
Thus
\[
\tilde{\alpha}(\Omega^n_A(X)) \oplus \bigoplus_{j=1}^{n} \Omega^{n-j}(\alpha'(\Omega^j_A(X)_\triangle))
\]
\[
\cong \tilde{\alpha}(\Omega^n_A(X)) \oplus \bigoplus_{j=1}^{n-1} \Omega^{n-1-j}(\alpha'(\Omega^j_A(X)_\triangle)) \oplus \Omega^{n-1}_B(\alpha'(\Omega_A(X)_\triangle))
\]
\[
\cong \Omega^{n-1}_B(\tilde{\alpha}(\Omega_A(X)) \oplus \alpha'(\Omega_A(X)_\triangle)) \oplus \bigoplus_{j=1}^{n-1} \Omega^{n-1-j}(\tilde{\alpha}(\Omega^j_A(X)_\triangle)) \quad (\text{by } (\ast\ast))
\]
\[
\cong \Omega^{n-1}_B(\tilde{\alpha}(X)) \oplus \bigoplus_{j=1}^{n-1} \Omega^{n-(j+1)}(\tilde{\alpha}(\Omega^{j+1}_A(X)_\triangle)) \quad (\text{by } (\ast))
\]
\[
\cong \Omega^{n}_B(\tilde{\alpha}(X)) \oplus \bigoplus_{j=1}^{n} \Omega^{n-j}(\tilde{\alpha}(\Omega^j_A(X)_\triangle)).
\]
The third lemma identifies extension groups of modules preserved under stable equivalences.

**Lemma 4.13** Let $X \in A$-mod, $Y \in \mathcal{GCN}_F(A)$ and $n$ a positive integer.

1. Then $\mathrm{Ext}^1_A(X, Y) \cong \mathrm{Ext}^1_B(\Phi(X), \Phi(Y))$.

2. If $\mathrm{Ext}^i_B(N, \Phi(Y)) = 0$ for each $N \in \nabla_B$ and $1 \leq i \leq n$, then $\mathrm{Ext}^{n+1}_A(X, Y) \cong \mathrm{Ext}^{n+1}_B(\Phi(X), \Phi(Y))$.

**Proof.** Set $C := (A$-mod$)$-mod and $D := (B$-mod$)$-mod. Then $\alpha : C \rightarrow D$ is an equivalence with a quasi-inverse $\beta$. By Yoneda’s Lemma,

$$\mathrm{Ext}^1_A(X, Y) \cong \mathrm{Hom}_C(\mathrm{Hom}_A(-, X), \mathrm{Ext}^1_A(-, Y)) \cong \mathrm{Hom}_D(\alpha(\mathrm{Hom}_A(-, X)), \alpha(\mathrm{Ext}^1_A(-, Y))).$$

Since $\alpha(\mathrm{Hom}_A(-, X)) \cong \mathrm{Hom}_B(-, \tilde{\alpha}(X))$ and $\alpha(\mathrm{Ext}^1_A(-, Y)) \cong \mathrm{Ext}^1_B(-, \alpha'(Y))$, it follows that

$$\mathrm{Ext}^1_A(X, Y) \cong \mathrm{Ext}^1_B(\tilde{\alpha}(X), \alpha'(Y)).$$

Moreover, by Lemma 4.8(2), $\Phi(Y) = \alpha'(Y) \oplus \bigoplus_{I \in \mathcal{J}_B} I$, since $Y \in \mathcal{GCN}_F(A)$. Also, $\Phi(X) = \tilde{\alpha}(X) \oplus \bigoplus_{Q \in \mathcal{P}_B} Q$, by definition. Consequently, $\mathrm{Ext}^1_A(X, Y) \cong \mathrm{Ext}^1_B(\Phi(X), \Phi(Y))$ Thus (1) holds

The proof of (1) also implies

$$\mathrm{Ext}^{n+1}_A(X, Y) \cong \mathrm{Ext}^1_A(\Omega^n_B(X), Y) \cong \mathrm{Ext}^1_B(\tilde{\alpha}(\Omega^n_A(X)), \alpha'(Y)) \cong \mathrm{Ext}^1_B(\tilde{\alpha}(\Omega^n_A(X)), \Phi(Y)).$$

By Lemma 4.12 $\tilde{\alpha}(\Omega^n_A(X)) \oplus L \cong \Omega^n_B(\tilde{\alpha}(X)) \oplus R$ in $B$-mod, where

$$L := \bigoplus_{j=1}^n \Omega_B^{n-j}(\tilde{\alpha}(\Omega^j_A(X)_\Delta)) \quad \text{and} \quad R := \bigoplus_{j=1}^n \Omega_B^{n-j}(\tilde{\alpha}(\Omega^j_A(X)_\Delta)).$$

Since $\Omega^j_A(X)_\Delta \in \text{add}(\Delta_A)$ and $\Delta_A = \mathcal{N}_F \cup (\mathcal{J}_A \setminus \mathcal{A})$, it follows from Lemma 4.8(1) that

$$\alpha'(\Omega^j_A(X)_\Delta) \in \text{add}(\Delta_B) \quad \text{and} \quad \tilde{\alpha}(\Omega^j_A(X)_\Delta) \in \text{add}(\tilde{\alpha}(\mathcal{N}_F) \cup (\mathcal{J}_A \setminus \mathcal{A})).$$

where $\nabla_B := \mathcal{N}_F \cup (\mathcal{J}_A \setminus \mathcal{A})$. So, if $\mathrm{Ext}^i_B(N, \Phi(Y)) = 0$ for each $N \in \nabla_B$ and $1 \leq i \leq n$, then $\mathrm{Ext}^1_B(L, \Phi(Y)) = 0 = \mathrm{Ext}^1_B(R, \Phi(Y))$, and thus

$$\mathrm{Ext}^1_B(\tilde{\alpha}(\Omega^n_A(X)), \Phi(Y)) \cong \mathrm{Ext}^1_B(\Omega^n_B(\tilde{\alpha}(X)), \Phi(Y)) \cong \mathrm{Ext}^{n+1}_B(\Phi(X), \Phi(Y)).$$

This shows (2). □

**Proof of Proposition 4.10**

If $\text{domdim} \mathrm{End}_B(\Phi(X)) = n + 2$ for some $n \in \mathbb{N}$, then $\mathrm{Ext}^i_B(\Phi(X), \Phi(X)) = 0$ for $1 \leq i \leq n$ and $\mathrm{Ext}^{n+1}_B(\Phi(X), \Phi(X)) \neq 0$ due to Theorem 2.3. By Lemma 4.8(2), $\Phi(X) \in \mathcal{GCN}_{F^{-1}}(B)$. Thus, $\nabla_B \subset \text{add}(\Phi(X))$. Therefore, Lemma 4.13 implies $\mathrm{Ext}^j_B(X, X) \cong \mathrm{Ext}^j_B(\Phi(X), \Phi(X))$ for $1 \leq j \leq n + 1$. Then $\text{domdim} \mathrm{End}_A(X) = n + 2$, again by Theorem 2.3. Similarly, if $\text{domdim} \mathrm{End}_B(\Phi(X)) = \infty$, then $\text{domdim} \mathrm{End}_A(X) = \infty$. □

**Remark.** When neither $A$ nor $B$ has nodes, both Lemma 4.12 and Lemma 4.13 can be simplified. In Lemma 4.12 the isomorphism becomes $\tilde{\alpha}(\Omega^n_A(X)) \oplus Q \cong \Omega^n_B(\tilde{\alpha}(X))$, where $Q$ is a projective $B$-module without injective direct summands. This implies a stronger form of Lemma 4.12. If $X \in A$-mod and $Y \in \mathcal{GCN}_F(A)$, then $\mathrm{Ext}^1_A(X, Y) \cong \mathrm{Ext}^1_B(\Phi(X), \Phi(Y))$ for any $n \geq 1$. Note that both isomorphisms can be obtained from [2] Theorem 3.6 and [30] Section 1, Corollary, and Proposition 2.2]. Thus, under the stronger assumption that $A$ and $B$ are two stably equivalent algebras without nodes and without semi-simple blocks, Proposition 4.10 can also be derived from the results in [30].
4.5 Completion of proofs, and an application to representation dimension

Proof of Theorem 4.5

Let $X \in \mathcal{GCN}_F(A)$. Then $\Phi(X) \in \mathcal{GCN}_{F-1}(B)$. By Lemma 4.9, both $\text{End}_A(X)$ and $\text{End}_B(\Phi(X))$ have the same global dimension. Moreover, by Proposition 4.10, they have the same dominant dimension. When $n_F(A)$ is the empty set, then $\mathcal{GCN}_F(A)$ is exactly the class of basic generator-cogenerators in $A$-mod. It follows that $\text{rigdim}(A) \leq \text{rigdim}(B)$. If $n_{F-1}(B)$ is also empty, then $\text{rigdim}(B) \leq \text{rigdim}(A)$, and thus $\text{rigdim}(A) = \text{rigdim}(B)$. □

To prepare for the proof of Theorem 4.6, we describe the rigidity dimensions of Nakayama self-injective algebras with radical square zero.

Proposition 4.14 Suppose that $A$ is an indecomposable non-simple Nakayama self-injective algebra with radical square zero. Let $e$ be the number of isomorphism classes of simple $A$-modules. Then $\text{rigdim}(A) = e + 1$.

Proof. Since the radical square of $A$ is zero, every indecomposable, non-projective $A$-module is simple. So, for any generator $M$ in $A$-mod, if $\text{gldim} \text{End}_A(M) < \infty$, then $M$ contains at least one simple module, say $S$, as a direct summand. In particular, by Theorem 2.3

$$\text{domdim} \text{End}_A(M) \leq \text{domdim} \text{End}_A(A \oplus S)$$

Note that $\Omega^*_{A}(S) \cong S$ and $\{\Omega^*_{A}(S) \mid 1 \leq i \leq e\}$ is the complete set of isomorphism classes of indecomposable, non-projective $A$-modules. The following equalities

$$S^\perp = \text{add}(A \oplus S) = \perp^{e-1} S.$$

can be verified by writing down projective and injective resolutions of $S$. This implies that $A \oplus S$ is a maximal $(e-1)$-orthogonal $A$-module. By [24, Theorem 0.2], $\text{domdim} \text{End}_A(A \oplus S) = e + 1 = \text{gldim} \text{End}_A(A \oplus S)$. Thus $\text{rigdim}(A) = \text{domdim} \text{End}_A(A \oplus S) = e + 1$ as claimed. □

Proof of Theorem 4.6

Since $A$ and $B$ are self-injective, it follows from [3, X.1: Proposition 1.6] that $\tilde{\alpha}$ and $\tilde{\beta}$ restrict to one-to-one correspondences between the set of isomorphism classes of nodes of $A$ and the set of nodes of $B$. If $A$ has no nodes, then so does $B$. Thus (1) holds by Theorem 4.5.

Suppose that both $A$ and $B$ have nodes and no semisimple blocks. We first show (II), and then use (II) to show (2) and (3).

Let $A = A_1 \times A_2$ and $B = B_1 \times B_2$ be decompositions of algebras, such that $A_2$ and $B_2$ are the products of all blocks of $A$ and $B$ without nodes, respectively. In other words, all nodes of $A$ and $B$ only belong to $A_1$-mod and $B_1$-mod, respectively. Then, all indecomposable non-projective $A_1$-modules (and similarly, $B_1$-modules) are nodes. This is a consequence of the following facts:

As already mentioned, by [3, X.1: Proposition 1.8], if $\Lambda$ is an indecomposable self-injective algebra with nodes, then it is a Nakayama algebra with radical square zero. In this case, each simple $\Lambda$-module $N$ is a node, $\text{DT}(N) \cong \Omega_A(N)$ and $\Omega^{\rho_A}(N) \cong N$. Moreover, $\{\Omega_i^A(N) \mid 1 \leq i \leq \rho(\Lambda)\}$ is the complete set of isomorphism classes of indecomposable objects in $\Lambda$-mod. By Proposition 4.11, $\text{rigdim}(\Lambda) = \rho(\Lambda) + 1$.

Consequently, both $A_1$ and $B_1$ are products of indecomposable Nakayama algebras with radical square zero. Moreover, $\tilde{\alpha}$ and $\tilde{\beta}$ induce a stable equivalence between $A_1$ and $B_1$ and a stable equivalence between $A_2$ and $B_2$. By Proposition 2.7, $\text{rigdim}(A) = \min\{\rho(A_1) + 1, \text{rigdim}(A_2)\}$ and $\text{rigdim}(B) = \min\{\rho(B_1) + 1, \text{rigdim}(B_2)\}$. Since $A_2$ and $B_2$ have no nodes, $\rho(A_1) = \rho(A)$ and $\rho(B_1) = \rho(B)$. By (1), $\text{rigdim}(A_2) = \text{rigdim}(B_2)$. Now, it is easy to check the inequality $|\text{rigdim}(A) - \text{rigdim}(B)| \leq |\rho(A) - \rho(B)|$. This shows (II).

In particular, (II) implies that if $\rho(A) = \rho(B)$, then $\text{rigdim}(A) = \text{rigdim}(B)$. A sufficient condition to guarantee $\rho(A) = \rho(B)$ is that $\tilde{\alpha}(\Omega_A(S)) \cong \Omega_B(\tilde{\alpha}(S))$ in $B$-mod for any node $S$ of $A$. In this case, both $S$ and $\tilde{\alpha}(S)$ are $\Omega$-periodic of the same period.
When $A$ and $B$ are symmetric algebras, it follows from [3] X.1: Proposition 1.12 that the correspondence $\tilde{\alpha}$ between objects in $A$-$\text{mod}_\mathcal{J}$ and $B$-$\text{mod}_\mathcal{J}$ commutes with the syzygy functor $\Omega$. This shows (2). Recall that the shift functor of the triangulated category $A$-$\text{mod}$ is the syzygy functor $\Omega_A^{-1}$. So, in (3), $\tilde{\alpha}$ commutes with $\Omega^{-1}$, and thus also with $\Omega$. This shows (3). $\Box$

Remark. In Theorem 4.6 (2) and (3), both $n_F(A)$ and $n_{F-1}(B)$ are empty. This follows from (*) in the proof of Lemma 4.12, or alternatively since the functors $D\text{Tr}$ and $\Omega$ coincide when applied to nodes, and since the correspondences $\tilde{\alpha}$ and $\tilde{\beta}$ commute with $\Omega$ as is shown in the proof of Theorem 4.6.

Finally, we explain how our results can be used to compare higher representation dimensions of stably equivalent algebras. Recall that classical representation dimension $\text{repdim}_1$ is preserved under arbitrary stable equivalences of algebras (see [17]). However, the following result suggests that in the case of higher representation dimension, the situation is similar to rigidity dimension and depending on the presence of nodes.

Corollary 4.15 Let $A$ and $B$ be stably equivalent by an equivalence $F$, and let $n \in \mathbb{N}$. Suppose that $n_F(A)$ is empty and $n+1 \leq \text{rigdim}(A)$. Then $\text{repdim}_n(B) \leq \text{repdim}_n(A) < \infty$. If additionally $n_{F-1}(B)$ is empty, too, then $\text{repdim}_n(A) = \text{repdim}_n(B)$.

Proof. Since $n+1 \leq \text{rigdim}(A)$, Proposition 2.7 implies $\text{repdim}_n(A) < \infty$. The inequality $\text{rigdim}(A) \leq \text{rigdim}(B)$ follows from Theorem 3.5. This implies $n+1 \leq \text{rigdim}(B)$, and thus $\text{repdim}_n(B) < \infty$. By assumption, the set $n_F(A)$ is empty and $\mathcal{GC}_F(A)$ is exactly the class of basic generator-cogenerators in $A$-$\text{mod}$. Now, Corollary 4.15 follows from Proposition 4.10 and Lemma 4.9. $\Box$

5 Stable equivalences and invariance II

In this section, we show invariance of rigidity dimension under stable equivalences of adjoint type, which implies invariance under stable equivalences of Morita type under very mild assumptions, and thus also invariance under certain derived equivalences.

5.1 Definitions and main result

Definition 5.1 Two algebras $A$ and $B$ are stably equivalent of Morita type if there exist an $(A,B)$-bimodule $M$ and a $(B,A)$-bimodule $N$ such that

(i) $M$ and $N$ are both projective as one sided modules,

(ii) $M \otimes_B N \cong A \oplus P$ as $A$-$A$-bimodules for some projective $A$-$A$-bimodule $P$,

(iii) $N \otimes_A M \cong B \oplus Q$ as $B$-$B$-bimodules for some projective $B$-$B$-bimodule $Q$.

Further, if $(M \otimes_B - , N \otimes_A - )$ and $(N \otimes_A - , M \otimes_B - )$ are adjoint pairs of functors, then $A$ and $B$ are stably equivalent of adjoint type.

If $A$ and $B$ are stably equivalent of Morita type, then $(M \otimes_B - , N \otimes_A - )$ induce a stable equivalence between $A$ and $B$.

Theorem 5.2 (a) Let $A$ and $B$ be stably equivalent of adjoint type. Then $\text{rigdim}(A) = \text{rigdim}(B)$.

(b) Let $A$ and $B$ be stably equivalent of Morita type. Then $\text{rigdim}(A) = \text{rigdim}(B)$ in each of the following three cases:

1. $A$ and $B$ have no simple blocks.
2. $A$ and $B$ are algebras over a perfect field $k$.
3. $A$ and $B$ are self-injective algebras.
Remark. We don’t know whether Theorem 5.2 holds for general stable equivalences of Morita type. Our proof of Theorem 5.2 (b) depends on the relevant functors forming an adjoint pair, and thus part (a) being applicable. For an arbitrary stable equivalence of Morita type, it is not clear if tensor functors induced by two bimodules preserve injective modules.

5.2 Proof of the main result

In the proof of Theorem 5.2 (b), the following result will be used, which is likely to be known to experts. We thank Yuming Liu for pointing out the present proof.

Lemma 5.3 Let $A = A_1 \times A_2$ and $B = B_1 \times B_2$, where $A_1$ and $B_1$ are separable, and $A_2$ and $B_2$ have no separable blocks. If $A$ and $B$ are stably equivalent of Morita type, then so are $A_2$ and $B_2$.

Proof. It suffices to verify the following claim:

If $\Lambda$ is a non-zero algebra and $S$ is a separable algebra, then $\Lambda$ and $\Lambda \times S$ are stably equivalent of Morita type.

For checking this, let $\Gamma := \Lambda \times S$, $M := \Lambda \times (\Lambda \otimes_k S)$ and $N := \Lambda \times (S \otimes_k \Lambda)$. Then $M$ can be endowed with a $\Lambda$-$\Gamma$-bimodule structure: For $\lambda, \lambda_1, \lambda_2 \in \Lambda$ and $s, s' \in S$,

$$\lambda(\lambda_1 \otimes s) = (\lambda \lambda_1, \lambda \lambda_2 \otimes s) \quad \text{and} \quad \lambda_1, \lambda_2 \otimes s)(\lambda', s') = (\lambda_1 \lambda', \lambda_2 \otimes ss').$$

Similarly, $N$ can be endowed with a $\Gamma$-$\Lambda$-bimodule structure. Moreover, $M \otimes_\Gamma N \cong \Lambda \oplus (\Lambda \otimes \Gamma \otimes \Lambda)$ as $\Lambda$-$\Lambda$-bimodules and $N \otimes_\Lambda M \cong \Lambda \oplus \Lambda \otimes \Gamma \otimes \Lambda \oplus \Gamma \otimes \Lambda \otimes \Gamma \otimes \Lambda$ as $\Gamma$-$\Gamma$-bimodules. Then $\Lambda \otimes \Gamma \Gamma \otimes \Lambda$ is a projective $\Lambda$-$\Lambda$-bimodule. As

$$\Gamma \otimes_k \Gamma \cong (\Lambda \otimes k \Lambda^\text{op}) \times (\Lambda \otimes k S^\text{op}) \times (S \otimes k \Lambda^\text{op}) \times (S \otimes k S^\text{op}),$$

$S \otimes k S$, $\Lambda \otimes k S$ and $S \otimes k \Lambda$ are projective $\Gamma$-$\Gamma$-bimodules. The $S$-$S$-bimodule $S$ is projective, since $S$ is a separable algebra. Furthermore, the multiplication map $S \otimes_k S \to S$ is a homomorphism of $S$-$S$-bimodules, and $S$ is a direct summand of $S \otimes k S$ as bimodules. Since $S \otimes k S \otimes k S \cong (S \otimes k S)^n$ with $n := \dim(k) \geq 1$, it follows that $S$ is a direct summand of the projective bimodule $S \otimes k \Lambda \otimes k S$. Consequently, there is a projective $\Gamma$-$\Gamma$-bimodule $Q$ such that $N \otimes_\Lambda M \cong (\Lambda \oplus S) \oplus Q \cong \Gamma \oplus Q$ as $\Gamma$-$\Gamma$-bimodules. So $\Lambda$ and $\Gamma$ are stably equivalent of Morita type.

Proof of Theorem 5.2

Let $A M_B$ and $B N_A$ be bimodules defining a stable equivalence of Morita type (not necessarily of adjoint type) between $A$ and $B$. Let $A X$ be a generator in $A$-mod. We claim that $N \otimes_\Lambda X$ is a generator in $B$-mod and $\text{gldim} \text{End}_A(X) = \text{gldim} \text{End}_B(N \otimes_\Lambda X)$. Indeed, since $N \otimes_\Lambda M \cong B \oplus Q$ as $B$-bimodules for some projective $B$-bimodule $Q$, it follows that $B B \in \text{add}(N \otimes_\Lambda M)$. Then $A M$ being projective implies $B \in \text{add}(B N)$. In other words, $B N$ is a projective generator, and thus $A M_B$ is a generator. By [29, Theorem 1.1], $\text{End}_A(X)$ and $\text{End}_B(N \otimes_\Lambda X)$ are stably equivalent of Morita type. As global dimensions are preserved by stable equivalences of Morita type, there is an equality $\text{gldim} \text{End}_A(X) = \text{gldim} \text{End}_B(N \otimes_\Lambda X)$.

(a) Now, suppose that the pair $(M, N)$ defines a stable equivalence of adjoint type. In other words, the pairs $(M \otimes B, N \otimes A)$ and $(N \otimes A, M \otimes B)$ are adjoint pairs of functors. Further, assume $X$ to be a cogenerator in $A$-mod. We claim that $N \otimes A X$ is a cogenerator in $B$-mod and $\text{domdim} \text{End}_A(X) = \text{domdim} \text{End}_B(N \otimes_\Lambda X)$. Since $A M \otimes B \cong B B$ is exact with a right adjoint $N \otimes A \otimes B$ is a cogenerator in $B$-mod, and $N \otimes A (A \oplus B)$ is a generator-cogenerator in $B$-mod. This implies that $N \otimes A X$ is a generator-cogenerator in $B$-mod.

Since $A$ and $B$ are stably equivalent of adjoint type, it follows from [29, Theorem 1.3] that $\text{End}_A(X)$ and $\text{End}_B(N \otimes_\Lambda X)$ are stably equivalent of adjoint type, too. As such stable equivalences preserve dominant dimension by [29, Lemma 4.2(2)], there is an equality $\text{domdim} \text{End}_A(X) = \text{domdim} \text{End}_B(N \otimes_\Lambda X)$. 

By the definition of rigidity dimension, $\text{rigdim}(A) \leq \text{rigdim}(B)$. Swapping the roles of $A$ and $B$ yields $\text{rigdim}(B) \leq \text{rigdim}(A)$. Thus $\text{rigdim}(A) = \text{rigdim}(B)$.

(b) Under some mild assumptions, stable equivalences of Morita type are of adjoint type. Using this, the claims in (b) can be derived from (a) as follows:

(1) If neither $A$ nor $B$ is a simple module, then each stable equivalence of Morita type between $A$ and $B$ is of adjoint type by [8, Lemma 4.1] and [28, Lemma 4.8]. Thus $\text{rigdim}(A) = \text{rigdim}(B)$ by (a).

(2) Let $A = A_1 \times A_2$ and $B = B_1 \times B_2$ such that $A_1$ and $B_1$ are semi-simple and that $A_2$ and $B_2$ have no semi-simple blocks. Since $k$ is perfect, the class of finite-dimensional semisimple $k$-algebras coincides with that of finite-dimensional separable $k$-algebras. So both $A_1$ and $B_1$ are separable. By Lemma 5.3 both $A_2$ and $B_2$ are stably equivalent of Morita type. It follows from (1) that $\text{rigdim}(A_2) = \text{rigdim}(B_2)$. Since $\text{rigdim}(A_1) = \text{rigdim}(B_1) = \infty$, Proposition 2.6 implies $\text{rigdim}(A) = \text{rigdim}(B)$.

(3) By [28, Lemma 4.8], if two self-injective algebras without separable blocks, are stably equivalent of Morita type, then there is a stable equivalence of adjoint type between them. Separable algebras are semi-simple, hence have infinite rigidity dimension. Now, (3) follows from Lemma 5.3 and (a) together with Proposition 2.6. □

5.3 Applications to derived equivalences

Any derived equivalence between self-injective algebras induces a stable equivalence of Morita type, see [32, Corollary 2.2]. The following result is a consequence of Theorem 5.2(b)(3). Alternatively, it can be derived from Theorem 4.6(a)(3), since any derived equivalence between self-injective algebras does induce a triangle equivalence between their stable module categories.

Corollary 5.4 Let $A$ and $B$ be self-injective algebras. Suppose $A$ and $B$ are derived equivalent. Then $\text{rigdim}(A) = \text{rigdim}(B)$.

This can be extended to algebras that are not necessarily self-injective, by restricting the class of derived equivalences to certain derived equivalences which induce stable equivalences of Morita type. These are the almost $\nu$-stable derived equivalences introduced in [20]. Every derived equivalence between two self-injective Artin algebras induces an almost $\nu$-stable derived equivalence (see [20, Proposition 3.8]). In general, there are still many examples of almost $\nu$-stable derived equivalences, for example, between Artin algebras constructed in some way from self-injective algebras.

Lemma 5.5 ([21, Corollary 1.2]) Let $A$ be a self-injective algebra and $X$ an $A$-module. Then $\text{End}_A(A \oplus X)$ and $\text{End}_A(A \oplus \Omega_A(X))$ are almost $\nu$-stable derived equivalent.

Almost $\nu$-stable derived equivalences in many respects have better properties than general derived equivalences, for instance in the following way.

Lemma 5.6 ([20, Theorem 1.1]) If $A$ and $B$ are almost $\nu$-stable derived equivalent, then they are stably equivalent of Morita type. In this case, both $A$ and $B$ have the same global dimension and dominant dimension.

The following result extends Corollary 5.4 to non-selfinjective algebras.

Proposition 5.7 If $A$ and $B$ are almost $\nu$-stable derived equivalent, then $\text{rigdim}(A) = \text{rigdim}(B)$.

Proof. Let $A = A_1 \times A_2$ and $B = B_1 \times B_2$ such that $A_1$ and $B_1$ are semi-simple and that $A_2$ and $B_2$ have no semi-simple blocks. Since derived equivalences preserve semi-simplicity of blocks, both $A_i$ and $B_i$ are derived equivalent for $i = 1,2$. Moreover, $A_2$ and $B_2$ are almost $\nu$-stable derived equivalent. By Lemma 5.6 they are stably equivalent of Morita type. It follows from Theorem 5.2(b)(1) that $\text{rigdim}(A_2) = \text{rigdim}(B_2)$. Since a semi-simple algebra has infinite rigidity dimension, $\text{rigdim}(A) = \text{rigdim}(B)$ by Proposition 2.6. □
Corollary 5.8 Let $A$ be a self-injective algebra and $X$ an $A$-module. Then \( \text{rigdim} \left( \text{End}_A(A \oplus X) \right) = \text{rigdim} \left( \text{End}_A(A \oplus \Omega_A(X)) \right) \).

Proof. By Lemma 5.5, $\text{End}_A(A \oplus X)$ and $\text{End}_A(A \oplus \Omega_A(X))$ are almost $\nu$-stable derived equivalent. Now, Corollary 5.8 follows from Proposition 5.7. \qed

Finally, we show that some assumptions are needed on derived equivalences to preserve rigidity dimension.

Example 5.9 In general, rigidity dimensions are not preserved under derived equivalences. The following is a counterexample.

Let $A$ be the path algebra over $k$ given by the quiver $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$, and let $B$ be the quotient algebra of $A$ modulo the ideal generated by $\alpha \beta$. Then $A$ and $B$ are derived equivalent via a tilting module of projective dimension one. Since $\text{gldim}(A) = 1$ and $\text{gldim}(B) = 2$, it follows from Theorem 3.1 that $\text{rigid}(A) = 2$ and $\text{rigid}(B) \leq 3$. Note that both the global dimension and the dominant dimension of $\text{End}_B(B \oplus D(B))$ are equal to 3. This implies $\text{rigid}(B) \geq 3$, and so $\text{rigid}(B) = 3$. Thus $\text{rigid}(A) \neq \text{rigid}(B)$.

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