SEMIGROUPS OF COMPOSITION OPERATORS AND INTEGRAL OPERATORS IN BMOA-TYPE SPACES

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Abstract. The aim of this article is to study semigroups of composition operators $T_t = f \circ \phi_t$ on the BMOA-type spaces $BMOA_p$, and on their "little oh" analogues $VMOA_p$. The spaces $BMOA_p$ were introduced by R. Zhao as part of the large family of $F(p, q, s)$ spaces, and are the Möbius invariant subspaces of the Dirichlet spaces $D_p^{p−1}$. We study the maximal subspace $[\phi_t, BMOA_p]$ of strong continuity, providing a sufficient condition on the infinitesimal generator of $\{\phi_t\}$, under which $[\phi_t, BMOA_p] = VMOA_p$, and a related necessary condition in the case where the Denjoy-Wolff point of the semigroup is in $D$. Further, we characterize those semigroups, for which $[\phi_t, BMOA_p] = VMOA_p$, in terms of the resolvent operator of the infinitesimal generator of $(T_t|_{VMOA_p})$. In addition we provide a connection between the maximal subspace of strong continuity and the Volterra-type operators $T_g$. We characterize the symbols $g$ for which $T_g: BMOA \to BMOA_1$ is bounded or compact, thus extending a related result to the case $p = 1$. We also prove that for $1 < p < 2$ compactness of $T_g$ on $BMOA_p$ is equivalent to weak compactness.

1. Introduction

Let $\mathbb{D}$ be the unit disc of the complex plane and $\partial \mathbb{D}$ its boundary. We will denote by $H(\mathbb{D})$ the space of all analytic functions on the disc and by $H^\infty$ the subspace of bounded analytic functions. A family $\{\phi_t\}_{t \geq 0}$ of analytic self-maps of the disc is a semigroup of functions if:

1. $\phi_0(z) = z$
2. $\phi_t \circ \phi_s = \phi_{t+s}$, for $t, s \geq 0$
3. $\phi_t \to \phi_0$, uniformly on compact subsets of $\mathbb{D}$, as $t \to 0$.

The identity maps $\phi_t(z) = z$, $t \geq 0$, form the trivial semigroup. In any other case we say $\{\phi_t\}_{t \geq 0}$ is nontrivial. Each family $\{\phi_t\}_{t \geq 0}$ induces a semigroup of composition operators on $H(\mathbb{D})$,

$T_t(f) = f \circ \phi_t$, $t \geq 0$.

If $X$ is a Banach space of analytic functions on $\mathbb{D}$ on which the composition operators $T_t: X \to X$ are bounded, there arises the question whether $(T_t)$ is strongly continuous, that is if

$\lim_{t \to 0^+} \|T_t(f) − f\|_X = 0$, $f \in X$.

This question of strong continuity was first studied by E. Berkson and H. Porta [7] on Hardy spaces. They proved that each $\{\phi_t\}$ induces a strongly

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continuous composition semigroup \((T_t)\) on each Hardy space \(H^p\), \(1 \leq p < \infty\). Several other authors studied the question on classical spaces of analytic functions, in particular on Bergman spaces, the Dirichlet space, the space \(BMOA\) and the Bloch space among others.

This article studies strong continuity of composition semigroups in the family of spaces \(BMOA_p\), \(1 \leq p < \infty\). When \(p \geq 2\) it is immediate that every semigroup \(\{\phi_t\}\) induces a semigroup \((T_t)\) of bounded composition operators on \(BMOA_p\). We show that the same is true in the non-obvious case \(1 \leq p < 2\) and also for \(VMOA_p\). We then study the maximal subspace \([\phi_t, BMOA_p]\) of strong continuity on these spaces. We show that \([\phi_t, BMOA_p] \subseteq BMOA_p\) for nontrivial \(\{\phi_t\}\) and provide conditions under which \([\phi_t, BMOA_p] = VMOA_p\).

We also consider Volterra-type operators \(T_g\) on the above spaces and characterize the symbols \(g\) for which \(T_g : BMOA \rightarrow BMOA_1\) is bounded or compact, extending a result from [38] to the case \(p = 1\). The key point for proving this extension is the use of the Garsia norm for functions in \(BMOA\). Finally we prove that for \(1 < p < 2\) compactness of \(T_g\) on \(BMOA_p\) is equivalent to its weak compactness, and find a connection between the maximal subspace of strong continuity and the mapping properties of \(T_\gamma\) for a specific symbol \(\gamma\) determined by the inducing semigroup.

2. Background and Main Results

2.1. The spaces \(BMOA_p\). Let \(0 < p < \infty\). A function \(f \in H(\mathbb{D})\) belongs to the space \(BMOA_p\) if

\[
(2.1) \quad \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-2} (1 - |\varphi_a(z)|^2) \, dm(z) < \infty,
\]

where \(\varphi_a(z) = \frac{a - z}{1 - \overline{a}z}\), \(a \in \mathbb{D}\), are the Möbius automorphisms of the disc and \(dm(z) = r \, dr \, d\theta / \pi\) is the normalized Lebesgue area measure of \(\mathbb{D}\). The subspace \(VMOA_p\) contains those \(f\) for which

\[
(2.2) \quad \lim_{|a| \to 1} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-2} (1 - |\varphi_a(z)|^2) \, dm(z) = 0.
\]

The spaces \(BMOA_p\) are part of the large family of spaces \(F(p, q, s)\) introduced by R. Zhao in [39]. An \(f \in H(\mathbb{D})\) is in \(F(p, q, s)\) if

\[
\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{q} (1 - |\varphi_a(z)|^2)^s \, dm(z) < \infty,
\]

and \(f \in F_0(p, q, s)\) if the above integral tends to 0 as \(|a| \to 1\). The parameters are \(p > 0\), \(q > -2\), \(s > 0\), and the spaces are nontrivial when \(q + s > -1\). These spaces were further studied by J. Rättyä in [26].

Clearly

\(BMOA_p \equiv F(p, p - 2, 1)\) and \(VMOA_p \equiv F_0(p, p - 2, 1)\),

and the value \(p = 2\) corresponds to the classical spaces \(BMOA\) and \(VMOA\).

We recount some basic properties of \(BMOA_p\), most of which can be found in [39]. For \(p \geq 1\), \(BMOA_p\) equipped with the norm defined by

\[
(2.3) \quad \|f\|_{BMOA_p}^p = |f(0)|^p + \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-2} (1 - |\varphi_a(z)|^2) \, dm(z)
\]
is a Banach space, with \( VMOA_p \) being the closure of the polynomials in \( BMOA_p \). The identity

\[
1 - |\varphi_a(z)|^2 = |\varphi_a'(z)|(1 - |z|^2), \quad a, z \in \mathbb{D},
\]

and a change of variables \( z = \varphi_a(w) \) gives

\[
(2.4) \quad \|f\|_{BMOA_p} = |f(0)|^p + \sup_{a \in \mathbb{D}} \|f \circ \varphi_a - f(a)\|_{D_{p-1}}^p.
\]

where

\[
\|f\|_{D_{p-1}}^p = |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p(1 - |z|^2)^{p-1}dm(z)
\]

is a norm defining the Dirichlet spaces \( D_{p-1} = \{ f \in H(\mathbb{D}) : \|f\|_{D_{p-1}} < \infty \} \). Thus (2.4) says that \( f \in BMOA_p \) if and only if the set of hyperbolic translates

\[
\{ f \circ \varphi_a - f(a) : a \in \mathbb{D} \}
\]

is bounded in \( D_{p-1} \), i.e. \( BMOA_p \) is the Möbius invariant version of \( D_{p-1} \). For more information on \( D_{p-1} \) see [14, 33].

It is to be noticed at this point that the spaces \( D_{p-1} \) and \( BMOA_p \) are part of the larger families of spaces \( D_p \) and their Möbius invariant subspaces \( M(D_p) \) respectively, which were studied by M. D. Contreras, S. Díaz-Madrigal and D. Vukotić in [11]. Specifically, given a positive Borel measure \( \mu \) on \( \mathbb{D} \) consider the Dirichlet-type spaces

\[
D_p = \{ f \in H(\mathbb{D}) : \int_{\mathbb{D}} |f'(z)|^p \ d\mu(z) < \infty \}
\]

and their Möbius invariant versions

\[
M(D_p) = \{ f \in D_p : \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |(f \circ \varphi_a)'(z)|^p \ d\mu(z) < \infty \},
\]

along with the little-oh spaces \( M_0(D_p) \) containing those \( f \) for which

\[
\lim_{|a| \to 1} \int_{\mathbb{D}} |(f \circ \varphi_a)'(z)|^p \ d\mu(z) = 0.
\]

The choice \( d\mu(z) = (1 - |z|^2)^{p-1}dm(z) \) gives \( D_{p-1} = D_p \), and then

\[
BMOA_p = M(D_p) \text{ and } VMOA_p = M_0(D_p).
\]

Even though there is no inclusion relation between \( D_{p-1} \) and \( D_{q-1} \) for \( p \neq q \), their Möbius invariant subspaces \( BMOA_p \) form a chain that increases in size with \( p \), [39]. If \( 0 < p < q < \infty \) then

\[
BMOA_p \subsetneq BMOA_q \subsetneq B,
\]

where \( B \) is the Bloch space of \( f \in H(\mathbb{D}) \) with \( \sup_{z \in \mathbb{D}} |1 - |z|^2| |f'(z)| < \infty \). An analogous containment relation is valid for \( VMOA_p \). For the sake of completeness we prove the following lemma which can also be proved using the closed graph theorem.

**Lemma 1.**

1. For each \( 0 < q < \infty \) there is a constant \( C = C_q \) such that

\[
\|f\|_B \leq C\|f\|_{BMOA_q}, \text{ for all } f \in BMOA_q.
\]

2. For each pair \( (p, q) \) with \( 0 < p < q < \infty \) there is a constant \( C = C_{p,q} \) such that

\[
\|f\|_{BMOA_q} \leq C\|f\|_{BMOA_p}, \text{ for all } f \in BMOA_p.
\]
Proof. (1) For \( f \in BMOA_q \) and \( a \in \mathbb{D} \) let

\[
I(f, q, a) = \int_\mathbb{D} \frac{|f'(z)|^q(1 - |z|^2)^{q-2}(1 - |\varphi_a(z)|^2)}{(1 - |z|^2)^2} dm(z).
\]

According to [39, Lemma 2.9] there is a constant \( K_q \) such that

\[
|f'(a)|^q(1 - |a|^2)^{q-2} \leq K_q I(f, q, a),
\]

for all \( a \in \mathbb{D} \). Thus we have

\[
\left( |f(0)| + |f'(a)| \right)^q \leq 2^q \left( |f(0)|^q + |f'(a)|^q \right) \leq 2^q \left( |f(0)|^q + K_q I(f, q, a) \right) \leq 2^q \max\{1, K_q\} \left( |f(0)|^q + I(f, q, a) \right) \leq K_q' \|f\|_{BMOA_q}^q,
\]

and taking the sup on \( a \) of the quantity in left hand side, we have the desired inequality with \( C_q = (K_q')^{1/q} \).

(2) Let \( f \in BMOA_p \) then for \( I(f, q, a) \) as defined above we have

\[
I(f, q, a) = \int_\mathbb{D} \left( (1 - |z|^2)|f'(z)| \right)^p \left( (1 - |z|^2)|f'(z)| \right)^q \frac{(1 - |\varphi_a(z)|^2)}{(1 - |z|^2)^2} dm(z)
\]

\[
\leq \sup_{z \in \mathbb{D}} \left( |f'(z)|^p \right)^q \int_\mathbb{D} \left( |f'(z)|^p \right)^q \frac{(1 - |\varphi_a(z)|^2)}{(1 - |z|^2)^2} dm(z)
\]

\[
\leq \|f\|_{BMOA_p}^q \int_\mathbb{D} \left( |f'(z)|^p \right)^q \frac{(1 - |\varphi_a(z)|^2)}{(1 - |z|^2)^2} dm(z)
\]

\[
\leq \|f\|_{BMOA_p}^q \|f\|_{BMOA_p}^p \|f\|_{BMOA_p}^p \quad \text{by using (i)}
\]

\[
= C_q^n \|f\|_{BMOA_p}^q.
\]

Thus

\[
\|f\|_{BMOA_q}^q = \left( |f(0)|^q + \sup_{a \in \mathbb{D}} I(f, q, a) \right)^{p/q}
\]

\[
\leq 2^{p/q} \left( |f(0)|^p + \sup_{a \in \mathbb{D}} I(f, q, a)^{p/q} \right)
\]

\[
\leq 2^{p/q} \left( \|f\|_{BMOA_p}^p + \left( C_q^n \|f\|_{BMOA_p}^p \right)^{p/q} \right)
\]

\[
= 2^{p/q} (1 + C_\varphi (\frac{p+q}{q})) \|f\|_{BMOA_p}^p,
\]

and the conclusion follows with \( C_{p,q} = (2^{p/q}(1 + C_\varphi (\frac{p+q}{q})))^{1/q} \).

Since \( H^\infty \subset BMOA = BMOA_2 \) it follows that \( H^\infty \subset BMOA_p \) for \( p \geq 2 \), but this containment is no longer true for \( 1 \leq p < 2 \). In fact it is shown in [20, Proposition A (7)] and [26, Theorem 2.3.4] that the disc algebra \( A \) is not contained in \( BMOA_p \) for \( p < 2 \).

Yet another description of the spaces \( BMOA_p \) is in terms of Carleson measures. Recall that a positive measure \( \mu \) on \( \mathbb{D} \) is a Carleson measure if

\[
\|\mu\|_{CM} = \sup_{I \subset \mathbb{D}} \frac{\mu(S(I))}{|I|} < \infty,
\]
where \( I \subset \partial \mathbb{D} \) is an arc with length \(|I|\), and

\[
S(I) = \{ re^{it} : e^{it} \in I, \text{ and } 1 - |I| < r < 1 \}
\]

is the Carleson square for \( I \). Further, \( \mu \) is a vanishing Carleson measure if

\[
\lim_{|I| \to 0} \frac{\mu(S(I))}{|I|} = 0.
\]

The defining property (2.5) is equivalent to that the Hardy space \( H^2 \) is boundedly embedded in \( L^2(\mathbb{D}, \mu) \), i.e. there is a constant \( C_\mu \) such that

\[
(2.6) \quad \int_{\mathbb{D}} |f(z)|^2 d\mu(z) \leq C_\mu \|f\|^2_{H^2}
\]

for all \( f \in H^2 \). Vanishing Carleson measures are those for which the embedding \( i : H^2 \to L^2(\mathbb{D}, \mu) \) is a compact operator. It is shown in [39] that \( f \in BMOA_p \) if and only if

\[
\sup_{I \subset \partial \mathbb{D}} \frac{1}{|I|} \int_{S(I)} |f'(z)|^p (1 - |z|^2)^{p-1} dm(z) < \infty,
\]

equivalently the measure \( d\mu(z) = |f'(z)|^p (1 - |z|^2)^{p-1} dm(z) \) is a Carleson measure on \( \mathbb{D} \). \( VMOA_p \) then consists of those \( f \) for which \( d\mu \) is vanishing Carleson. A consequence is that

\[
|f(0)|^p + \sup_{I \subset \partial \mathbb{D}} \frac{\mu(S(I))}{|I|} \simeq \|f\|^p_{BMOA_p}.
\]

Using the above description, Z. Wu [35] (see also [33]) gave a characterization of pointwise multipliers of \( D^p_{p-1} \), i.e. of functions in the space

\[
\mathcal{M}(D^p_{p-1}) = \{ g \in H(\mathbb{D}) : gf \in D^p_{p-1} \text{ for each } f \in D^p_{p-1} \}.
\]

This characterization for \( 0 < p \leq 2 \) can be restated as follows

\[
g \in \mathcal{M}(D^p_{p-1}) \iff g \in H^\infty \cap BMOA_p.
\]

Another important class of spaces contained in family \( F(p,q,s) \) are the \( Q_s \) spaces, obtained as \( Q_s = F(2,0,s) \) and \( Q_{s,0} = F_0(2,0,s) \). They were introduced by Aulaskari, Xiao and Zhao in [6], see also [4]. For \( s = 1 \) we have \( Q_1 = BMOA \), while for all \( s > 1 \), \( Q_s = B \). For \( 0 < s < 1 \) these spaces are distinct and they increase in size with \( s \). In [5], Aulaskari and Tovar proved that

\[
\bigcup_{s \in (0,1)} Q_s \subset \bigcap_{0 < p \leq 2} BMOA_p,
\]

which indicates that \( BMOA_p \) are considerably larger spaces than \( Q_s \). A good reference for \( Q_s \) spaces is Xiao’s monographs [36] and [37].

We note that in the literature, the spaces \( BMOA_p \) are also referred to as Besov-type spaces, see for example [5] or [21].

2.2. Semigroups of composition operators. If \( \{\phi_t\}_{t \geq 0} \) is a semigroup then each \( \phi_t \) is univalent and the limit

\[
G(z) = \lim_{t \to 0^+} \frac{\phi_t(z) - z}{t}
\]
exists uniformly on compact subsets of \( D \). The analytic function \( G(z) \) is the \textit{infinitesimal generator} of \( \{\phi_t\} \) and uniquely determines the semigroup. Moreover \( G \) satisfies
\[
(2.7) \quad G(\phi_t(z)) = \frac{\partial \phi_t(z)}{\partial t} = G(z) \frac{\partial \phi_t(z)}{\partial z}, \quad z \in D, \ t \geq 0.
\]
The infinitesimal generator \( G \) can be uniquely represented in terms of the Denjoy-Wolff point \( b \) of the semigroup (see [7]) as
\[
(2.8) \quad G(z) = (\overline{bz} - 1)(z - b)P(z), \quad z \in D,
\]
where \( b \in \overline{D} \) and \( P \in H(D) \) with \( \Re(P(z)) \geq 0 \) for all \( z \in D \).

If \( X \) is a Banach space consisting of analytic functions on \( D \) we denote by \([\phi_t, X]\) the maximal closed subspace of \( X \) on which \( \{\phi_t\} \) generates a strongly continuous composition semigroup \((T_t)\), that is,
\[
[\phi_t, X] := \{f \in X : \lim_{t \to 0^+} \|f \circ \phi_t - f\|_X = 0\}.
\]
It is shown in [9] that if \( X \) contains the constants and \( \sup_{0 < t < 1} \|T_t\| < \infty \), then
\[
(2.9) \quad [\phi_t, X] = \{f \in X : Gf' \in X\}
\]
with \( G \) the generator of \( \{\phi_t\} \). It is well known that for every semigroup \( \{\phi_t\} \),
\[
[\phi_t, X] = X
\]
when \( X \) is any of the following spaces: the Hardy spaces \( H^p \), \( 1 \leq p < \infty \) [7], the Bergman spaces \( A^p_a \), \( 1 \leq p < \infty \), \( a > -1 \) [29], the Dirichlet space \( D \) [30], the space \( \text{VMOA} \) and the little Bloch space \( \mathcal{B}_0 \) [9], and \( Q_s \) [34]. Note that in each of the above spaces, polynomials are dense in \( X \), a property that plays a key role in proving strong continuity.

This is no longer true when \( X = H^\infty \) or \( X = B \). For these spaces and for nontrivial \( \{\phi_t\} \) we have,
\[
[\phi_t, H^\infty] \subsetneq H^\infty \quad \text{and} \quad [\phi_t, B] \subsetneq B,
\]
see for example [9], [28]. Extending this result, A. Anderson, M. Jovovic and W. Smith proved in [3] that whenever \( X \) is a space such that
\[
H^\infty \subseteq X \subseteq B
\]
then \([\phi_t, X] \subsetneq X\). In particular for \( X = \text{BMOA}_p, p \geq 2\),
\[
[\phi_t, \text{BMOA}_p] \subsetneq \text{BMOA}_p
\]
for all nontrivial \( \{\phi_t\} \), since for such \( p \) we have \( H^\infty \subseteq \text{BMOA}_p \subseteq B \). However if \( 1 \leq p < 2 \) then \( H^\infty \not\subseteq \text{BMOA}_p \), so the result of Anderson, Jovovic and Smith does not apply for this range of \( p \) and the question of strong continuity needs a different treatment.

On the other hand using the fact that the closure of the polynomials in \( \text{BMOA}_p \) is \( \text{VMOA}_p \) we will show that each \( \{\phi_t\} \) induces a strongly continuous composition semigroup on \( \text{VMOA}_p \), so we have
\[
\text{VMOA}_p \subseteq [\phi_t, \text{BMOA}_p]
\]
for \( p \geq 1 \). This containment can be proper as the following example shows. Let
\[
f(z) = \log(1 - z) \in \text{BMOA}_p \setminus \text{VMOA}_p
\]
and
\[ \phi_t(z) = e^{-t}z + 1 - e^{-t}, \quad t \geq 0. \]
Then \( \lim_{t \to 0} \| f \circ \phi_t - f \|_{BMOA_p} = 0 \), thus \( f \in [\phi_t, BMOA_p] \) so, in this case,
\[ VMOA_p \subsetneq [\phi_t, BMOA_p]. \]
Thus for a general semigroup \( \{\phi_t\} \) the question arises to relate function-theoretic properties of \( \{\phi_t\} \) to the size of the maximal subspace of strong continuity of \( (T_t) \).

2.3. Main results. In section 3 we study the boundedness of composition operators
\[ C_\psi(f)(z) = f(\psi(z)), \]
on \( BMOA_p \) and \( VMOA_p \), where \( \psi \) is an analytic self-map of the disc. It is well known that \( C_\psi \) is bounded on \( BMOA \) \([10]\) and on \( B \) for all \( \psi \), and it is not difficult to check that this remains true for \( BMOA_p \) for \( p > 2 \). When \( 1 \leq p < 2 \) we prove that if \( \psi \) is univalent then \( C_\psi \) is bounded on \( BMOA_p \) and \( VMOA_p \), and this is sufficient for our work with semigroups.

In section 4, we study semigroups of composition operators on \( BMOA_p \) and \( VMOA_p \), \( p \in [1, 2) \). It is easy to see that for the dilations \( \phi_t(z) = e^{-t}z \) or the rotations \( \phi_t = e^{it}z \), we have a proper inclusion
\[ [\phi_t, BMOA_p] \subsetneq BMOA_p. \]
Indeed taking any function \( g \in BMOA_p \setminus VMOA \), if we assume that \( g \in [\phi_t, BMOA] \) then \( g \in [\phi_t, BMOA] \), since
\[ \lim_{t \to 0} \| g \circ \phi_t - g \|_{BMOA} \leq C \lim_{t \to 0} \| g \circ \phi_t - g \|_{BMOA_p}, \quad 1 \leq p < 2. \]
Then, by Sarason’s theorem \([27]\), \( g \in VMOA \) which is a contradiction. Examples of such functions are \( g(z) = \log(1 - z) \), and the class of inner functions
\[ g(z) = \exp \left( \frac{\gamma z + w}{z - w} \right), \]
where \( 0 < \gamma < \infty \) and \( w \in \partial \mathbb{D} \) (see \([20\) Theorem 1.1]). More generally using an argument similar to that in \([3\) we will show that for any nontrivial \( \{\phi_t\} \) we always have
\[ [\phi_t, BMOA_p] \subsetneq BMOA_p, \quad 1 \leq p < 2. \]

On the other hand we will show that if the infinitesimal generator \( G \) of \( \{\phi_t\} \) satisfies the condition
\[ \lim_{|I| \to 0} \left( \frac{\log 2}{|I|} \right)^p \int_{S(I)} \frac{(1 - |z|^2)^{p-1}}{|G(z)|^p} \, dm(z) = 0, \]
then \( VMOA_p = [\phi_t, BMOA_p] \), and will observe that this condition is satisfied for a large family of semigroups. We also prove a (different) necessary condition for this equality, for semigroups with Denjoy-Wolff point inside \( \mathbb{D} \).

We finish the section by observing that a characterization of the equality \( [\phi_t, BMOA_p] = VMOA_p \) may be given in terms of the resolvent operator \( R(\lambda, \Gamma) = (\lambda - \Gamma)^{-1} \) of the infinitesimal generator \( \Gamma \) of the semigroup \( (T_t) \) acting on \( VMOA_p \) when \( 1 < p \leq 2 \), and that this is equivalent to the weak compactness of \( R(\lambda, \Gamma) \) on \( VMOA_p \).
In section 5, we study the compactness and weak compactness of the Volterra-type operators

\[ T_g = \int_0^z f(\zeta)g'(\zeta) \, d\zeta \]
on \(BMOA_p\) and \(VMOA_p\). We give a condition on \(g\) that characterizes when \(T_g : BMOA \to BMOA_1\) is bounded or compact thus extending [28, Theorem 14 and Corollary 16 (3)] to the case \(p = 1\). To prove this extension, we consider the Garsia norm ([13]) for functions in \(BMOA\). In addition we give equivalent conditions for \(T_g\) to be compact or weakly compact when it acts between the spaces \(BMOA_p\) and \(VMOA_p\) for \(1 < p < 2\), and relate this to a Carleson type condition on \(g\).

Finally, we show a connection between the mapping properties of specific Volterra-type operators with the size of the maximal subspace of strong continuity \([\phi_t, BMOA_p]\), extending a related result on BMOA from [9, Corollary 2].

We will use the notation \(C, C', C_1\ldots\) to denote constants, the values of which may change from one step to the next. The notation \(X \simeq Y\) means that the two quantities \(X, Y\) are comparable.

3. Composition Operators

In this section we discuss boundedness of composition operators induced by univalent symbols on \(BMOA_p\) and \(VMOA_p\).

**Theorem 1.** Let \(1 \leq p < 2\), and \(\psi : \mathbb{D} \to \mathbb{D}\) analytic and univalent. Then:

(1) the composition operator \(C_\psi : BMOA_p \to BMOA_p\) is bounded, and

\[
\|C_\psi\|_{BMOA_p \to BMOA_p} \leq C \left(1 + \log \frac{1 + |\psi(0)|}{1 - |\psi(0)|}\right),
\]

where \(C\) is a constant depending only on \(p\).

(2) \(C_\psi : VMOA_p \to VMOA_p\) is also bounded.

**Proof.** (1) For \(f \in BMOA_p\) and \(a \in \mathbb{D}\) we write

\[
I(f, p, a) = \int_\mathbb{D} |(f \circ \varphi_a)'(z)|^p (1 - |z|^2)^{p-1} \, dm(z),
\]

then we have

\[
I(f \circ \psi, p, a) = \int_\mathbb{D} |(f \circ \psi \circ \varphi_a)'(z)|^p (1 - |z|^2)^{p-1} \, dm(z)
\]

\[
= \int_\mathbb{D} |(f \circ \varphi_{\psi(a)} \circ \varphi_{\psi(a)} \circ \varphi_a)'(z)|^p (1 - |z|^2)^{p-1} \, dm(z)
\]

\[
= \int_\mathbb{D} |(f \circ \varphi_{\psi(a)})(\sigma_a(z))|^p |\sigma'_a(z)|^p (1 - |z|^2)^{p-1} \, dm(z).
\]

where the function \(\sigma_a = \varphi_{\psi(a)} \circ \psi \circ \varphi_a\).
is a self-map of $\mathbb{D}$ and $\sigma_a(0) = 0$. Applying Hölder’s inequality with exponents $2/p$ and $2/(2 - p)$ we get

$$I(f \circ \psi, p, a) \leq \left( \int_{\mathbb{D}} |(f \circ \varphi_{\psi(a)})'(\sigma_a(z))|^p |\sigma'_a(z)|^2 (1 - |z|^2)^{p-1} dm(z) \right)^{\frac{2}{2-p}} \cdot \left( \int_{\mathbb{D}} |(f \circ \varphi_{\psi(a)})'(\sigma_a(z))|^p (1 - |z|^2)^{p-1} dm(z) \right)^{\frac{2-p}{2}} = I_1^p \cdot I_2^{\frac{2-p}{2}}.$$  

For the first integral $I_1$ we use the inequality $1 - |z|^2 \leq 1 - |\sigma_a(z)|^2$, which is a consequence of Schwarz’s Lemma on $\sigma_a$, and make the change of variables $w = \sigma_a(z)$ to obtain

$$I_1 \leq \int_{\sigma_a(\mathbb{D})} |(f \circ \varphi_{\psi(a)})'(w)|^p (1 - |w|^2)^{p-1} dm(w) \leq \int_{\mathbb{D}} |(f \circ \varphi_{\psi(a)})'(w)|^p (1 - |w|^2)^{p-1} dm(w) = I(f, p, \psi(a)).$$

The second integral $I_2$ can be viewed as a weighted area integral of the composite function $g \circ \sigma_a$, where $g = (f \circ \varphi_{\psi(a)})'$, against the weight $(1 - |z|^2)^{p-1}$. Using known estimates for composition operators on weighted Bergman spaces (see for example [29, Lemma 1]) we obtain

$$I_2 = \int_{\mathbb{D}} |g(\sigma_a(z))|^p (1 - |z|^2)^{p-1} dm(z) \leq \left( \frac{\|\sigma_a\|_{\infty} + |\sigma_a(0)|}{\|\sigma_a\|_{\infty} - |\sigma_a(0)|} \right)^{p+1} \int_{\mathbb{D}} |g(z)|^p (1 - |z|^2)^{p-1} dm(z) = \int_{\mathbb{D}} |(f \circ \varphi_{\psi(a)})'(z)|^p (1 - |z|^2)^{p-1} dm(z) \quad (\text{since } \sigma_a(0) = 0) = I(f, p, \psi(a)).$$

Putting these together we obtain

$$I(f \circ \psi, p, a) \leq I(f, p, \psi(a))$$

for each $a \in \mathbb{D}$, so

$$\sup_{a \in \mathbb{D}} I(f \circ \psi, p, a) \leq \sup_{a \in \mathbb{D}} I(f, p, \psi(a)) \leq \sup_{b \in \mathbb{D}} I(f, p, b) \leq \|f\|_{BMOA_p}^p.$$  

Next using the growth estimate

$$|f(z)| \leq C_1 (1 + \log \frac{1 + |z|}{1 - |z|}) \|f\|_{BMOA_p}$$  

for functions in $BMOA_p$ which is implicitly proved in [39] we obtain

$$\|f \circ \psi\|_{BMOA_p} = |f(\psi(0))|^p + \sup_{a \in \mathbb{D}} I(f \circ \psi, p, a) \leq C_2 \left( 1 + \log \frac{1 + |\psi(0)|}{1 - |\psi(0)|} \right)^p \|f\|_{BMOA_p}^p,$$

from which (3.1) follows.
(2) We are going to use the fact that polynomials are dense in $VMOA_p$, and also the fact that \( \lim_{r \to 1} \| f - f_r \|_{BMOA_p} = 0 \) for each $f \in VMOA_p$, where $f_r(z) = f(rz)$, $0 < r < 1$, are the dilations of $f$. These two properties are in fact equivalent and each of them characterizes membership of functions in $VMOA_p$, see [11, Theorem 3] or [17, Proposition 2.3].

For $f \in VMOA_p$, we need to show that $f \circ \psi \in VMOA_p$. Equivalently we will show that
\[
\lim_{r \to 1} \| (f \circ \psi)_r - f \circ \psi \|_{BMOA_p} = 0.
\]
Note that $(f \circ \psi)_r = f \circ \psi_r$ and $\psi_r(0) = \psi(0)$ for each $r$. For a polynomial $P$ we have (dropping the subscript of the norm)
\[
\| (f \circ \psi)_r - f \circ \psi \| = \| f \circ \psi_r - f \circ \psi \|
\leq \| f \circ \psi_r - P \circ \psi_r \| + \| P \circ \psi_r - P \circ \psi \| + \| P \circ \psi - f \circ \psi \|
= \| (f - P) \circ \psi_r \| + \| P \circ \psi_r - P \circ \psi \| + \| (f - P) \circ \psi \|
\leq 2C \left( 1 + \log \frac{1 + |\psi(0)|}{1 - |\psi(0)|} \right) \| f - P \| + \| P \circ \psi_r - P \circ \psi \|.
\]
Now since $\psi$ is univalent and bounded an easy argument through the integrability of $|\psi'(z)|^2$ over $\mathbb{D}$ gives that $\psi \in VMOA_p$, and subsequently that $\psi(z)^n \in VMOA_p$ for each positive integer $n$. Therefore $P \circ \psi \in VMOA_p$ for every polynomial $P$. Given $\varepsilon > 0$ we can chose a polynomial $P$ such that $2C(1 + \log \frac{1 + |\psi(0)|}{1 - |\psi(0)|})\| f - P \| < \varepsilon/2$. Then we can find $r < 1$ such that $\| P \circ \psi_r - P \circ \psi \| < \varepsilon/2$, and the conclusion follows. \(\square\)

4. Semigroups of Composition Operators on $BMOA_p$

First we show that all semigroups $\{\phi_t\}$ induce strongly continuous composition semigroups $(T_t)$ on $VMOA_p$.

**Theorem 2.** Let $\{\phi_t\}$ be a semigroup of functions and $p \geq 1$. Then the induced composition semigroup $(T_t)$ is strongly continuous on $VMOA_p$.

**Proof.** We need to show that if $f \in VMOA_p$ then,
\[
\lim_{t \to 0^+} \| T_t(f) - f \| = 0,
\]
(dropping again the subscript of the $BMOA_p$-norm). For a polynomial $P$ we have
\[
\| T_t(f) - f \| \leq \| T_t(f) - T_t(P) \| + \| T_t(P) - P \| + \| P - f \|
\leq \| T_t \| \| f - P \| + \| T_t(P) - P \| + \| P - f \|
\leq (1 + \| T_t \|) \| P - f \| + \| T_t(P) - P \|.
\]
Since the polynomials are dense in $VMOA_p$ and $\sup_{t \leq 1} \| T_t \| < \infty$, it suffices to prove the claim for $f = P$, a polynomial. Now the set of polynomials is contained in the classical Dirichlet space $\mathcal{D}$ consisting of those those $f$ for which
\[
\| f \|^2_{\mathcal{D}} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \, dm(z) < \infty.
\]
An application of Hölder’s inequality shows that \( \mathcal{D} \subset VMOA_p \) and there is a constant \( C = C_p \), such that
\[
\|f\| \leq C \|f\|_{\mathcal{D}}, \quad \forall f \in \mathcal{D}.
\]
In particular for a polynomial \( P \) we have \( P \circ \phi_t - P \in \mathcal{D} \) since \( \phi_t \) is univalent, and
\[
\|T_t(P) - P\| \leq C \|T_t(P) - P\|_{\mathcal{D}}, \quad t > 0.
\]
But composition semigroups are strongly continuous on \( \mathcal{D} \) \[30\], so the last inequality implies
\[
\lim_{t \to 0^+} \|T_t(P) - P\| = 0
\]
for each polynomial \( P \) and the proof is complete. \( \square \)

**Theorem 3.** Suppose \( \{\phi_t\} \) is a nontrivial semigroup and \( 1 \leq p < \infty \). Then \( [\phi_t, BMOA_p] \not\subset BMOA_p \).

**Proof.** The result is obtained by observing that the proof of \[3\, \text{Theorem 3.1}\], whose conclusion is used to prove the analogous result \[3\, \text{Theorem 1.1}\], in fact applies here. More precisely, in the proof of \[3\, \text{Theorem 3.1}\] the authors use appropriate test functions \( f \in H^\infty \) which satisfy
\[
\liminf_{t \to 0} \|f \circ \phi_t - f\|_B \geq \delta > 0,
\]
for each nontrivial \( \{\phi_t\} \). Those test functions \( f \) are infinite interpolating Blaschke products whose zeros lie on a radius \( \{r \gamma_0 : 0 < r < 1\} \) for some \( \gamma_0 \in \partial \mathbb{D} \). On the other hand from \[15, \text{Theorem 3.12}\] (see also \[12\]), we know that for such Blaschke products \( B(z) \),
\[
B(z) \in \bigcap_{s \in (0,1)} Q_s.
\]
Since \( \bigcup_{0 < s < 1} Q_s \subset BMOA_p \) for all \( p > 0 \), these test functions can be used in our context, and the proof can be completed by following the lines of the proof in \[3\]. \( \square \)

**Remark 1.** From the above it follows that the hypothesis \( H^\infty \subset X \subset B \) in the theorem of Anderson, Jovovic and Smith \[3\, \text{Theorem 1.1}\] can be replaced by
\[
\bigcap_{s \in (0,1)} Q_s \subset X \subset B
\]
or by some similar condition which will imply that interpolating Blaschke products belong to \( X \).

**Theorem 4.** Let \( 1 \leq p < 2 \) and let \( \{\phi_t\} \) be a semigroup with infinitesimal generator \( G \). If
\[
(4.1) \quad \lim_{|I| \to 0} \left( \frac{\log 2}{|I|} \right)^p \int_{S(I)} \frac{(1 - |z|^2)^{p-1}}{|G(z)|^p} \, dm(z) = 0,
\]
then \( VMOA_p = [\phi_t, BMOA_p] \).
Proof. Using the description (2.9) of the maximal subspace of strong continuity, it suffices to show that if \( h \in BMOA_p \) is such that \( Gh' \in BMOA_p \) then \( h \in VMOA_p \). Set \( f = Gh' \) for such an \( h \) and select \( r \) sufficiently close to 1 so that \( G \) has no zeros in the ring \( D_r = \{ z \in \mathbb{D} : r < |z| < 1 \} \). Then \( 1/G \) is analytic on \( D_r \). Let \( I \subset \partial \mathbb{D} \) be an arc sufficiently small so that \( S(I) \subset D_r \), write \( z_I = (1 - |I|)\xi \) where \( \xi \in \partial \mathbb{D} \) is the center of \( I \), and set

\[
d\mu(z) = \frac{(1 - |z|^2)^{p-1}}{|G(z)|^p} \, dm(z).
\]

Then \( h' = f/G \) and we have

\[
\frac{1}{|I|} \int_{S(I)} |h'(z)|^p (1 - |z|^2)^{p-1} \, dm(z) = \frac{1}{|I|} \int_{S(I)} |f(z)|^p \, d\mu(z)
\]

\[
\leq 2^p \frac{1}{|I|} \int_{S(I)} |f(z) - f(z_I)|^p \, d\mu(z) + 2^p \frac{|f(z_I)|^p}{|I|} \int_{S(I)} d\mu(z)
\]

\[
\leq 2^p \frac{1}{|I|} \int_{S(I)} \left| f(z) - f(z_I) \right|^p \left| 1 - \bar{z}_I z \right|^p \, d\mu(z) + 2^p C_p \left( \log \frac{2}{|I|} \right)^p \, \mu(S(I))
\]

where we have used the inequality \((x + y)^p \leq 2^p(x^p + y^p)\) and the growth estimate \(|g(z)| \leq C \log \left( \frac{2}{|I|} \right)\) for functions \( g \in BMOA_p \). By hypothesis (1.1),

\[
B_I = \frac{\left( \log \frac{2}{|I|} \right)^p}{|I|} \mu(S(I)) \to 0, \quad \text{as } |I| \to 0.
\]

We will show that the same is true for \( A_I \). Using the estimate \(|1 - \bar{z}_I z| \simeq |I|\) for \( z \in S(I) \), and applying Hölder’s inequality we have

\[
A_I \leq C |I|^{p-1} \int_{S(I)} \left| \frac{f(z) - f(z_I)}{1 - \bar{z}_I z} \right|^p \, d\mu(z)
\]

\[
\leq C |I|^{p-1} \left( \int_{S(I)} \left| \frac{f(z) - f(z_I)}{1 - \bar{z}_I z} \right|^2 \, d\mu(z) \right)^{\frac{p}{2}} \left( \int_{S(I)} \, d\mu(z) \right)^{\frac{2-p}{2}}
\]

\[
= C |I|^{p-1} \mu(S(I))^{\frac{2-p}{2}} \left( \int_{\mathbb{D}} \left| \frac{f(z) - f(z_I)}{1 - \bar{z}_I z} \right|^2 \, d\mu(z) \right)^{\frac{p}{2}}.
\]

The hypothesis (4.1) implies that for \( |I| \) sufficiently small \( \mu(S(I)) \leq |I| \left( \log \frac{r}{|I|} \right)^{-p} \), and that \( \mu \) is a Carleson measure, i.e. (2.6) holds. Thus we have
\[ A_I \leq C|I|^{p-1} \left( |I| \left( \log \frac{2}{|I|} \right)^{-p} \right) \left( \int_{T} \left| \frac{f(z) - f(z_I)}{1 - \bar{z}_I z} \right|^2 d\mu(z) \right)^{\frac{p}{2}} \]
\[ \leq CC_p^{p/2} \frac{|I|^{\frac{p}{2}}}{(\log \frac{2}{|I|})^{\frac{p}{2} - p}} \left( \int_{T} \left| \frac{f(\zeta) - f(z_I)}{1 - \bar{z}_I \zeta} \right|^2 |d\zeta| \right)^{\frac{p}{2}} \]
\[ \simeq \frac{1}{(\log \frac{2}{|I|})^{\frac{p}{2} - p}} \left( (1 - |z_I|) \int_{T} \left| \frac{f(\zeta) - f(z_I)}{1 - \bar{z}_I \zeta} \right|^2 |d\zeta| \right)^{\frac{p}{2}} \]
\[ \leq \frac{1}{(\log \frac{2}{|I|})^{\frac{p}{2} - p}} \left( \sup_{a \in T} (1 - |a|^2) \int_{T} \left| \frac{f(\zeta) - f(a)}{1 - \bar{a} \zeta} \right|^2 |d\zeta| \right)^{\frac{p}{2}}. \]

Now the quantity inside the parenthesis is the square of the Garsia norm of \( f \in BMOA_p \subset BMOA \). Since the Garsia norm is comparable to the BMOA norm \([13]\), i.e.

\[ \| f \|^2_{BMOA} \simeq \sup_{a \in T} \int_{T} \left| f(\zeta) - f(a) \right|^2 \frac{1 - |a|^2}{1 - \bar{a} \zeta} |d\zeta|, \]

we get

\[ A_I \leq \frac{1}{(\log \frac{2}{|I|})^{\frac{p}{2} - p}} \| f \|^p_{BMOA} \to 0 \quad \text{as} \ |I| \to 0. \]

It follows that

\[ \lim_{|I| \to 0} \frac{1}{|I|} \int_{S(I)} |h'(z)|^p(1 - |z|^2)^{p-1} \, dm(z) = 0 \]

which means that \( h \) is in \( VMOA_p \) and the proof is complete. \( \square \)

The next Corollary says that there are plenty of semigroups for which \([\phi_t, BMOA_p] = VMOA_p\).

**Corollary 4.1.** Let \( 1 \leq p < 2 \) and \( a \in (0, 1) \). If

\[ \frac{(1 - |z|)^a}{G(z)} = O(1), \quad |z| \to 1, \]

then \([\phi_t, BMOA_p] = VMOA_p\).

**Proof.** The hypothesis \([4.3]\) implies

\[ \frac{(1 - |z|^2)^{p-1}}{|G(z)|^p} \leq C(1 - |z|)^{(p(1-a)-1)}, \quad |z| \to 1, \]

therefore

\[ \int_{S(I)} \frac{(1 - |z|^2)^{p-1}}{|G(z)|^p} \, dm(z) \leq C \int_{S(I)} (1 - |z|)^{(p(1-a)-1)} \, dm(z) \]
\[ \leq C|I| \int_{1-|I|}^1 (1 - r)^{(p(1-a)-1)} \, dr 
\]
\[ = C|I|^{p(1-a)+1} \frac{1}{p(1-a)}. \]
Now since
\[
\left( \log \frac{2}{|I|} \right)^p \frac{|I|^{p(1-a)}}{p(1-a)} \rightarrow 0 \quad \text{as } |I| \rightarrow 0,
\]
the condition (4.1) is satisfied and the conclusion follows. \qed

Next we prove a necessary condition for \([\phi_t, \text{BMOA}_p] = \text{VMOA}_p\) for semi-
groups with Denjoy-Wolff point in the open disc.

**Theorem 5.** Let \(\{\phi_t\}_{t \geq 0}\) be a semigroup with infinitesimal generator \(G\) and
Denjoy-Wolff point \(b \in \mathbb{D}\). If \(\text{VMOA}_p = [\phi_t, \text{BMOA}_p]\), then
\[
(4.4) \quad \lim_{|I| \rightarrow 0} \frac{1}{|I|} \int_{S(I)} \frac{(1-|z|^2)^{p-1}}{|G(z)|^p} \, dm(z) = 0.
\]

**Proof.** We assume, without loss of generality, that \(b = 0\). Then, from (2.8)
\(G(z) = -zP(z), \quad \text{Re}(P) \geq 0\).

Next, we consider the function
\[
g(z) = \int_0^z \frac{u}{G(u)} \, du = -\int_0^z \frac{1}{P(u)} \, du,
\]
Since \(\text{Re}(\frac{1}{P}) \geq 0\) the function \(g\) is univalent [24, Proposition 1.10], and from
the growth estimate \(|1/P(z)| = O(\frac{1}{|z|^2})\), \(z \in \mathbb{D}\), for functions of non-negative
real part it follows that \((1-|z|)|g'(z)|\) is bounded on \(\mathbb{D}\), so \(g \in B\). But univalent
functions of the Bloch space are the same as those in \(\text{BMOA}_p\) [21, p. 134],
\(g \in \text{BMOA}_p\).

In addition \(G(z)g'(z) = z \in \text{BMOA}_p\). This means that \(g \in [\phi_t, \text{BMOA}_p]\)
and by hypothesis \(g \in \text{VMOA}_p\). That is
\[
\lim_{|I| \rightarrow 0} \frac{1}{|I|} \int_{S(I)} \frac{|z|^p}{|G(z)|^p} (1-|z|^2)^{p-1} \, dm(z) = 0
\]
and (4.4) follows since \(|z| \simeq 1\) for \(z \in S(I)\) when \(I\) is small. \qed

Note that there is a gap between the conditions (4.1) and (4.4), and finding
a characterization of the equality \(\text{VMOA}_p = [\phi_t, \text{BMOA}_p]\) in terms of the
generator \(G\) seems to be difficult. Such a characterization however can be
given, following [9], in terms of the resolvent operator
\[
\mathcal{R}(\lambda, \Gamma) = (\lambda - \Gamma)^{-1}
\]
of the infinitesimal generator \(\Gamma(f) = Gf'\) of the semigroup \((T_t)\) acting on
\(\text{VMOA}_p\) when \(1 < p < 2\). For those values of \(p\) it follows from the work of
K.M. Perfekt [22] that we have the duality
\[
(4.5) \quad \text{VMOA}^{**}_p = \text{BMOA}_p.
\]
Indeed Perfekt shows that if \(X\) is a reflexive and separable Banach space, then
its Möbius invariant subspace
\[
M(X) = \{ f \in X : \sup_{a \in \mathbb{D}} \| f \circ \phi_a - f \|_X < \infty \}
\]
and the little-o version
\[
M_0(X) = \{ f \in X : \lim_{|a| \rightarrow 1} \| f \circ \phi_a - f \|_X = 0 \}
\]
have the duality property
\[ M_0(X)^{**} = M(X). \]

More details on the results from [22] will be given in Section 5. If \( X = D^p_{p-1}, 1 < p < 2 \), then \( M(X) = BMOA_p \) and \( M_0(X) = VMOA_p \) and \( D^p_{p-1} \) is reflexive, as it can be seen by adjusting Luecking’s result on the dual spaces of weighted Bergman spaces [18, Theorem 2.1]. Thus we get the duality (4.5).

A characterization of the semigroups for which \( [\phi_t, BMOA] = VMOA \) was given in [9] in terms of properties of the resolvent operator. Following the same steps the result is seen to hold for \( BMOA_p \) for all \( 1 < p < 2 \) as the following proposition describes.

**Proposition 1.** Let \( \{\phi_t\} \) be a semigroup and let \( \Gamma \) be the infinitesimal generator of the composition semigroup \((T_t)\) acting on \( VMOA_p, 1 < p < 2 \). Let \( \rho(\Gamma) = \{\lambda \in \mathbb{C} : R(\lambda, \Gamma) = (\lambda - \Gamma)^{-1} : VMOA_p \to VMOA_p \text{ is bounded}\} \), the resolvent set. Then for \( \lambda \in \rho(\Gamma) \) the following statements are equivalent:

1. \( [\phi_t, BMOA] = VMOA \).
2. \( R(\lambda, \Gamma)^{**}(BMOA_p) \subset VMOA_p. \)
3. \( R(\lambda, \Gamma) \) is weakly compact on \( VMOA_p. \)

As mentioned above the proof follows the lines of [9, Theorem 4] and we omit the details.

5. **Volterra-type Operators on \( BMOA_p \)**

In this last section we study Volterra-type operators \( T_g \) and point out a connection between the maximal subspace \( [\phi_t, BMOA_p] \) and some properties of these operators. The operators \( T_g \) are defined by

\[ T_g(f)(z) = \int_0^z f(\zeta)g'(\zeta) \, d\zeta \quad f \in H(\mathbb{D}), \]

where \( g \in H(\mathbb{D}) \) is the inducing symbol. They were first considered by Ch. Pommerenke [25]. He proved that \( T_g \) is bounded on the Hardy space \( H^2 \) if and only if \( g \in BMOA \). This characterization of boundedness was extended to all Hardy spaces \( H^p \) by A. Aleman and A. Siskakis [1] and the operator was subsequently studied on several spaces of analytic functions by many authors.

On the spaces of our concern, \( T_g \) were studied by J. Pau and R. Zhao [19]. They considered \( T_g \) on the more general family of the spaces \( F(p, q, s) \) and in particular they proved for \( BMOA_p = F(p, p - 2, 1) \) and for \( 1 \leq p < 2 \) that \( T_g : BMOA_p \to BMOA_p \) is bounded if and only if \( g \) satisfies

\[ \sup_{I \subset \partial \mathbb{D}} \left( \frac{\log \frac{2}{|I|}}{|I|} \right)^p \int_{S(I)} |g'(z)|^p (1 - |z|^2)^{p-1} \, dm(z) < \infty \]

and is compact if and only if the corresponding little-oh condition is valid, i.e. the quantity inside the parenthesis goes to 0 as \( |I| \to 0. \)

These results were extended by C. Yuan and C. Tong who proved that for \( 1 < p < 2 \), the operator \( T_g : BMOA \to BMOA_p \) is bounded if and only if the same above condition (5.1) holds for \( g \), and is compact if and only if the corresponding little-oh condition is valid [38, Theorem 14 & Corollary 16, (3)]

In the next theorem we show that for the missing case \( p = 1 \) in the results of [38], an analogous characterization of boundedness and compactness holds.
Theorem 6. Let \( g \) be an analytic function on \( \mathbb{D} \). Then:

1. The operator \( T_g : \text{BMOA} \rightarrow \text{BMOA}_1 \) is bounded if and only if

\[
\sup_{I \subset \partial \mathbb{D}} \left( \frac{\log \frac{2}{|I|}}{|I|} \int_{S(I)} |g'(z)| \, dm(z) \right) < \infty.
\]

2. The following are equivalent:
   (i) \( T_g : \text{BMOA} \rightarrow \text{BMOA}_1 \) is compact.
   (ii) The function \( g \) satisfies,

\[
\lim_{|I| \to 0} \left( \frac{\log \frac{2}{|I|}}{|I|} \int_{S(I)} |g'(z)| \, dm(z) \right) = 0.
\]

(iii) \( T_g(\text{BMOA}) \subset \text{VMOA}_1 \).

Proof. (1) Suppose (5.2) holds and let \( f \in \text{BMOA} \). Let \( z_I = (1 - |I|) \xi \), where \( \xi \in \partial \mathbb{D} \) is the center of \( I \). Then

\[
\|T_g(f)\|_{\text{BMOA}_1} \simeq \sup_{I \subset \partial \mathbb{D}} \frac{1}{|I|} \int_{S(I)} |f(z)g'(z)| \, dm(z).
\]

Setting \( d\mu(z) = d\mu_g(z) := |g'(z)| \, dm(z) \),

(this notation will be used throughout the proof) we have

\[
\frac{1}{|I|} \int_{S(I)} |f(z)g'(z)| \, dm(z) = \frac{1}{|I|} \int_{S(I)} |f(z)| \, d\mu(z)
\leq \frac{1}{|I|} \int_{S(I)} |f(z) - f(z_I)| \, d\mu(z) + |f(z_I)| |I| \int_{S(I)} |g'(z)| \, dm(z)
\leq \frac{1}{|I|} \int_{S(I)} |f(z) - f(z_I)| \, d\mu(z) + C \left( \frac{\log \frac{2}{|I|}}{|I|} \int_{S(I)} |g'(z)| \, dm(z) \right) \|f\|_{\text{BMOA}}
\leq A_I(f) + C' \|f\|_{\text{BMOA}},
\]

where we have used the growth estimate for \( \text{BMOA} \) functions and the hypothesis (5.2). Using the estimate \( |1 - \bar{z}_I z| \simeq |I| \) for \( z \in S(I) \), the quantity \( A_I(f) \) is

\[
A_I(f) = \frac{1}{|I|} \int_{S(I)} \left| \frac{f(z) - f(z_I)}{1 - \bar{z}_I z} \right| |1 - \bar{z}_I z| \, d\mu(z)
\leq C \int_{S(I)} \left| \frac{f(z) - f(z_I)}{1 - \bar{z}_I z} \right| \, d\mu(z)
\]

We can then continue with the last integral by repeating the steps in the proof of Theorem 4. In specific, apply a Cauchy-Schwarz inequality, use [5.2] and its implication that \( d\mu(z) \) is a Carleson measure, and change to polar coordinates to obtain

\[
\int_{S(I)} \left| \frac{f(z) - f(z_I)}{1 - \bar{z}_I z} \right| \, d\mu(z) \leq \frac{C}{(\log \frac{2}{|I|})^{1/2}} \|f\|_G
\]

where \( \|f\|_G \) is the Garsia norm of \( f \), which is equivalent to \( \|f\|_{\text{BMOA}} \). It follows then from (5.1) that \( \|T_g(f)\|_{\text{BMOA}_1} \leq C \|f\|_{\text{BMOA}} \) for some constant \( C \) and for all \( f \in \text{BMOA} \), i.e. \( T_g : \text{BMOA} \rightarrow \text{BMOA}_1 \) is bounded.
For the converse, assume that \( T_g : BMOA \to BMOA_1 \) is bounded and use the test functions
\[
    f_a(z) = \log \frac{1}{1 - az}, \quad a \in \mathbb{D},
\]
which form a bounded set in \( BMOA \). If \( a = z_I \) then \( |f_a(z)| \simeq \log \frac{2}{|z|} \) for \( z \in S(I) \) and so
\[
    \log \frac{2}{|z|} \int_{S(I)} |g'(z)| \, dm(z) \leq C \frac{1}{|I|} \int_{S(I)} |f_a(z)||g'(z)| \, dm(z)
\]
\[
    \leq C' \|T_g(f_a)\|_{BMOA_1}
\]
\[
    \leq C' \|T_g\| \|f_a\|_{BMOA}
\]
\[
    \leq C'' \|T_g\|
\]
so \((5.2)\) holds.

(2) We first show that (i) and (ii) are equivalent. First assume (ii) holds, then we will show that \( T_g : BMOA \to BMOA_1 \) is compact. Using a well known characterization of compactness of operators on function spaces, see for example [32, Lemma 3.7], it suffices to show that if \( (f_n) \) is a bounded sequence in \( BMOA \) such that \( f_n \to 0 \) uniformly on each compact subset of \( \mathbb{D} \) then \( \|T_g(f_n)\|_{BMOA_1} \to 0 \). Equivalently we want to show that
\[
    \sup_{I} \frac{1}{|I|} \int_{S(I)} |f_n(z)||g'(z)| \, dm(z) \to 0 \quad \text{as} \ n \to \infty.
\]
To show this we write
\[
    \frac{1}{|I|} \int_{S(I)} |f_n(z)||g'(z)| \, dm(z) \leq \frac{1}{|I|} \int_{S(I)} |f_n(z) - f_n(z_I)||g'(z)| \, dm(z)
\]
\[
    + |f_n(z_I)| \frac{1}{|I|} \int_{S(I)} |g'(z)| \, dm(z)
\]
\[
    = A_n(I) + B_n(I),
\]
where \( z_I \) is as before.

We first treat \( B_n(I) \). Using the assumptions on \( (f_n) \) we have
\[
    |f_n(z_I)| \leq C \|f_n\|_{BMOA} \log \frac{2}{1 - |z_I|} \leq C' \log \frac{2}{|I|},
\]
for all \( I \) and \( n \), so that
\[
    B_n(I) \leq C' \log \frac{2}{|I|} \int_{S(I)} |g'(z)| \, dm(z)
\]
for all \( n \). Now given \( \varepsilon > 0 \), by the hypothesis \((5.3)\) and the above inequality, there is a \( \delta > 0 \) such that if \( |I| < \delta \) then \( B_n(I) < \varepsilon/2 \) for all \( n \), thus, \( \sup_{|I|<\delta} B_n(I) \leq \varepsilon/2 \) for all \( n \).

On the other hand if \( I \) is an arc with \( |I| \geq \delta \) then \( |z_I| = 1 - |I| \leq 1 - \delta \), so \( z_I \) is inside the closed disc \( \{ |z| \leq 1 - \delta \} \). This and the assumption of the convergence of \( (f_n) \) to 0, uniformly on compact sets, imply that there is an \( N_0 \) such that for \( n \geq N_0 \)
\[
    |f_n(z_I)| \frac{1}{|I|} \int_{S(I)} |g'(z)| \, dm(z) \leq \left( \sup_{|z| \leq 1-\delta} |f_n(z_I)||g| \right) \|f_n\|_{BMOA_1} \leq \varepsilon/2,
\]
thus $$\sup_{|I| \geq \delta} B_n(I) \leq \varepsilon / 2$$, for $$n \geq N_0$$. It follows then that

$$\sup_I B_n(I) = \max \{ \sup_{|I| < \delta} B_n(I), \sup_{|I| \geq \delta} B_n(I) \} \leq \sup_{|I| \leq \delta} B_n(I) + \sup_{|I| \geq \delta} B_n(I) \leq \varepsilon,$$

for $$n \geq N_0$$, therefore

$$\lim_{n \to \infty} \sup_I B_n(I) = \lim_{n \to \infty} \left( \sup_I |f_n(z_I)| \frac{1}{|I|} \int_{S(I)} |g'(z)| \, dm(z) \right) = 0. \quad (5.6)$$

Now we consider $$A_n(I)$$. Using the estimate $$|1 - \bar{z}_I z| \simeq |I|$$ for $$z \in S(I)$$ and recalling that $$d\mu(z) = |g'(z)| \, dm(z)$$ we have

$$A_n(I) \simeq \int_{S(I)} \frac{|f_n(z) - f_n(z_I)|}{|1 - \bar{z}_I z|} \, d\mu(z). \quad (5.7)$$

The hypothesis (5.3) implies

$$\mu(S(I)) \leq C \frac{|I|}{\log \frac{2}{|I|}},$$

and applying the Cauchy-Schwartz inequality we obtain

$$A_n(I)^2 \simeq \left( \int_{S(I)} \frac{|f_n(z) - f_n(z_I)|}{|1 - \bar{z}_I z|} \, d\mu(z) \right)^2 \leq \mu(S(I)) \int_{S(I)} \left| \frac{f_n(z) - f_n(z_I)}{1 - \bar{z}_I z} \right|^2 \, d\mu(z) \leq C \frac{|I|}{\log \frac{2}{|I|}} \int_{D_r} \left| \frac{f_n(z) - f_n(z_I)}{1 - \bar{z}_I z} \right|^2 \, d\mu(z) \simeq \frac{|I|}{\log \frac{2}{|I|}} \int_{D_r \setminus D_{r'}} \left| \frac{f_n(z) - f_n(z_I)}{1 - \bar{z}_I z} \right|^2 \, d\mu(z) + \frac{|I|}{\log \frac{2}{|I|}} \int_{D_{r'}} \left| \frac{f_n(z) - f_n(z_I)}{1 - \bar{z}_I z} \right|^2 \, d\mu(z),$$

where $$D_r = \{ z \in \mathbb{D} : |z| \leq r \}$$ and $$r \in (0, 1)$$ will be chosen later. We consider each term of this sum separately.

For the integral in the first term, putting $$d\mu_r = d\mu|_{\mathbb{D} \setminus D_r}$$ and applying (2.6) in the first inequality, we have

$$\int_{D\setminus D_r} \left| \frac{f_n(z) - f_n(z_I)}{1 - \bar{z}_I z} \right|^2 \, d\mu(z) = \int_D \left| \frac{f_n(z) - f_n(z_I)}{1 - \bar{z}_I z} \right|^2 \, d\mu_r(z) \leq C \| \mu_r \|_{CM} \int_{D_{2\delta}} \left| \frac{f_n(z) - f_n(z_I)}{1 - \bar{z}_I \zeta} \right|^2 \, |d\zeta| \leq C' \| \mu_r \|_{CM} \frac{1 - |z_I|}{|I|} \int_{D_{2\delta}} \left| \frac{f_n(z) - f_n(z_I)}{1 - \bar{z}_I \zeta} \right|^2 \, |d\zeta| \leq C'' \| \mu_r \|_{CM} \left(\sup_{a \in \mathbb{D}} (1 - |a|^2) \int_{D_{2\delta}} \left| \frac{f_n(\zeta) - f_n(a)}{1 - \bar{a} \zeta} \right|^2 \, |d\zeta|\right) \leq C''' \| \mu_r \|_{CM} \| f_n \|_{BMOA}^2 \leq C'''' \| \mu_r \|_{CM}.$$
for all $I$. At this point, since $\mu$ is a vanishing Carleson measure, we can use Lemma 15 which says that $\lim_{r \to 1^-} \|\mu_r\|_{CM} = 0$. Thus given $\varepsilon > 0$ we can find an $r$ such that

$$\sup_{I} \frac{|I|}{\log \frac{2}{|I|}} \int_{D \setminus D_r} \left| \frac{f_n(z) - f_n(z_l)}{1 - \bar{z}_lz} \right|^2 d\mu(z) < \varepsilon, \quad \text{for all } n. \tag{5.8}$$

Now we consider the second term, with $r$ fixed as above. For $|z| \leq r$ we have $|1 - \bar{z}_lz|^2 > (1 - r)^2$ and $\sup_{|z| \leq r} |g'(z)| < \infty$, so there are constants such that

$$\int_{D_r} \left| \frac{f_n(z) - f_n(z_l)}{1 - \bar{z}_lz} \right|^2 d\mu(z) \leq C_r \int_{D_r} |f_n(z)|^2 d\mu(z) + C_r |f_n(z_l)|^2 \int_{D_r} d\mu(z) \leq C'_r \sup_{|z| \leq r} |f_n(z)|^2 + C'_r |f_n(z_l)|^2,$$

and we then have

$$\frac{|I|}{\log \frac{2}{|I|}} \int_{D_r} \left| \frac{f_n(z) - f_n(z_l)}{1 - \bar{z}_lz} \right|^2 d\mu(z) \leq C''_r \sup_{|z| \leq r} |f_n(z)|^2 + C'_r \frac{|I|}{\log \frac{2}{|I|}} |f_n(z_l)|^2 \leq C''_r \sup_{|z| \leq r} |f_n(z)|^2 + C'_r \frac{|I|}{\log \frac{2}{|I|}} |f_n(z_l)|^2.$$

By the hypothesis on $(f_n)$, the first term above tends to 0 as $n \to \infty$. In addition for each small positive $\delta$ we have,

$$\sup_{|I| < \delta} \frac{|I|}{\log \frac{2}{|I|}} |f_n(z)|^2 \leq \sup_{|I| < \delta} \frac{|I|}{\log \frac{2}{|I|}} |f_n(z_l)|^2 + \sup_{|I| \geq \delta} \frac{|I|}{\log \frac{2}{|I|}} |f_n(z_l)|^2 \leq C \sup_{|I| < \delta} \frac{|I|}{\log \frac{2}{|I|}} \left( \log \frac{2}{|I|} \right)^2 + \sup_{|I| \geq \delta} \frac{|I|}{\log \frac{2}{|I|}} |f_n(z_l)|^2 = C \sup_{|I| < \delta} \frac{|I|}{\log \frac{2}{|I|}} \log \frac{2}{|I|} + \sup_{|I| \geq \delta} \frac{|I|}{\log \frac{2}{|I|}} |f_n(z_l)|^2.$$

Now for the given $\varepsilon$ we can choose $\delta$ such that if $|I| < \delta$, then $C |I| \log \frac{2}{|I|} < \varepsilon$, so that

$$\sup_{|I| < \delta} \frac{|I|}{\log \frac{2}{|I|}} |f_n(z_l)|^2 < \varepsilon,$$

for all $n$. At the same time if $|I| \geq \delta$ then $|z_l| \leq 1 - \delta$ and by the hypothesis on $(f_n)$ and the fact that $\frac{|I|}{\log \frac{2}{|I|}}$ remains bounded, we can find $N_1$ such that

$$\sup_{|I| \geq \delta} \frac{|I|}{\log \frac{2}{|I|}} |f_n(z_l)|^2 < \varepsilon$$

for $n \geq N_1$. It follows from the last two inequalities that

$$\lim_{n \to \infty} \sup_{I} \frac{|I|}{\log \frac{2}{|I|}} |f_n(z_l)|^2 = 0.$$

Putting the above together gives

$$\lim_{n \to \infty} \left( \sup_{I} \frac{|I|}{\log \frac{2}{|I|}} \int_{D_r} \left| \frac{f_n(z) - f_n(z_l)}{1 - \bar{z}_lz} \right|^2 d\mu(z) \right) = 0. \tag{5.9}$$
From this and \((5.8)\) we obtain \(\lim_{n \to \infty} A_n(I) = 0\), and finally in combination with \((5.6)\) the desired conclusion \((5.5)\) is obtained.

To show the converse assume \(T_g : BMOA \to BMOA_1\) is compact. Then, taking into account that there is a constant \(C\) such that for each \(f \in BMOA_1\), \(\|f\|_{BMOA} \leq C\|f\|_{BMOA_1}\), we can easily verify that the restriction
\[
T_g|_{BMOA_1} : BMOA_1 \to BMOA_1
\]
is a compact operator. It then follows from [19, Theorem 5.2 (ii)], applied for the space \(BMOA_1 = F(1,-1,1)\), that
\[
\lim_{|l| \to \infty} \log \frac{2}{1 - |l|^2} \int_D |g'(z)|(1 - |\phi_a(z)|^2) dm(z) = 0
\]
and this condition on \(g\) is well known to be equivalent to \((5.3)\).

Next we show that conditions (ii) and (iii) are equivalent. Suppose (ii) holds and \(f \in BMOA\), we need to show that \(T_g(f) \in VMOA\) or equivalently that
\[
\lim_{|l| \to 0} \frac{1}{|l|} \int_{S(I)} |f(z)||g'(z)| dm(z) = 0.
\]
To do this we can follow the steps in the proof of Theorem 4 for the value \(p = 1\), setting \(d\mu(z) = |g'(z)| dm(z)\). As in that proof the calculations lead to an inequality involving the Garsia norm of \(f\) and finally to the conclusion that the above limit is 0, so condition \((5.3)\) implies that \(T_g(BMOA) \subset VMOA_1\).

To show the converse we use the test functions
\[
f_I(z) = \log \frac{1}{1 - \bar{z}_I z} \in BMOA,
\]
thus \(T_g(f_I) \in VMOA_1\). Using the standard estimate \(|f_I(z)| \simeq \log \frac{2}{|l|}\) for \(z \in S(I)\) we have
\[
\frac{\log \frac{2}{|l|}}{|l|} \int_{S(I)} |g'(z)| dm(z) \leq C \int_{S(I)} |f_I(z)||g'(z)| dm(z) = C \int_{S(I)} |T_g(f_I)'(z)| dm(z)
\]
which gives the desired conclusion. \(\square\)

To continue the study of compactness and weak compactness of Volterra operators on \(BMOA_p\) we will need some results of K.M. Perfekt from [22, 23]. In these articles there is a general construction of pairs of spaces \((M_0, M)\) obtained by little oh and big oh conditions respectively. More precisely let \(X, Y\) be two Banach spaces, where \(X\) is reflexive and separable, and let \(\mathcal{L}\) be a family of bounded linear operators \(L : X \to Y\), equipped with a \(\sigma\)-compact, locally compact Hausdorff topology \(\tau\), such that for every \(x \in X\) the map \(T_x : \mathcal{L} \to Y\), \(T_x(L) = L(x)\) is continuous. A pair of spaces is then defined
\[
M = M(X, \mathcal{L}) = \{x \in X : \sup_{L \in \mathcal{L}} \|L(x)\|_Y < \infty\},
\]
\[
M_0 = M_0(X, \mathcal{L}) = \{x \in M : \lim_{L \to \infty} \|L(x)\|_Y = 0\}
\]
where \(L \to \infty\) in the definition of \(M_0\) means that \(L\) eventually escapes all compact subsets of the topological space \((\mathcal{L}, \tau)\). Under appropriate assumptions on \(\mathcal{L}\) the quantity \(\|x\|_M = \sup_{L \in \mathcal{L}} \|L(x)\|_Y\) is a norm on \(M\) and, by
replacing $X$ by the closure of $M$ in $X$ and repeating the construction, it may be assumed that $M$ is dense in $X$. It is clear that $M_0$ is a closed subspace of $M$. This construction puts in a very general frame classical pairs such as the mean oscillation pair $(VMOA, BMOA)$, the Bloch pair $(B_0, B)$, other Möbius invariant pairs, Lipschitz-Hölder spaces, etc. Using functional analysis methods it is then proved under mild additional assumptions that the duality $M_0^{**} = M$ holds in this general context and also the following theorem on weak compactness.

**Theorem A.** ([23, Theorem 3.2]). Let $Z$ a Banach space. A bounded linear operator $T : M_0(X, \mathcal{L}) \to Z$ is weakly compact if and only if for each $\varepsilon > 0$, there is a $C > 0$ such that

$$\|T(x)\|_Z \leq C\|x\|_X + \varepsilon\|x\|_{M}, \quad \text{for each } x \in M_0(X, \mathcal{L})$$

In addition as a corollary, the same above characterization holds for weak compactness of operators $T : M(X, \mathcal{L}) \to Z$ provided that they are weak$^\ast$-weak continuous with respect to the duality $M_0^{**} = M$.

To obtain the pair of spaces $(VMOA_p, BMOA_p)$, we apply the above construction with $X = Y = D^p_{p-1}, 1 < p < 2$, and $\mathcal{L} = \{L_a : a \in \mathbb{D}\}$, $L_a(f) = f \circ \varphi_a - f(a)$, with $\varphi_a$ the Möbius automorphisms of $\mathbb{D}$. We can then apply the above theorem as in [23, Example 4] to obtain information about the Volterra operators $T_g$ acting on the spaces, thus generalizing results from [31], [9], [16], concerning compactness and weak compactness of $T_g$ on the pair $(VMOA, BMOA)$.

**Theorem 7.** Let $1 < p < 2$ and $g$ analytic on $\mathbb{D}$ such that the Volterra operator $T_g : BMOA_p \to BMOA_p$ is bounded. Then the following are equivalent:

(i) $T_g : VMOA_p \to VMOA_p$ is weakly compact

(ii) $T_g(BMOA_p) \subseteq VMOA_p$

(iii) $T_g : BMOA_p \to BMOA_p$ is weakly compact

(iv) The function $g$ satisfies

$$\lim_{|I| \to 0} \left(\frac{\log 2/|I|}{|I|}\right)^p \int_{S(I)} |g'(z)|^p (1 - |z|^2)^{p-1} \, dm(z) = 0.$$

**Proof.** Starting with the duality $(VMOA_p)^{**} = BMOA_p$ it is easy to verify that $(T_g(VMOA_p))^{**} = T_g$. Hence, the equivalence of (i), (ii) and (iii) is immediate due to Gantmacher’s Theorem [2, Theorem 5.23]. We will prove the equivalence between (iii), (iv).

Suppose (iii) holds. Consider a sequence of arcs $\{I_n\}$ such that $|I_n| \to 0$, and let $w_n = (1 - |I_n|)\zeta_n$ where $\zeta_n \in \partial \mathbb{D}$ is the center of the arc $I_n$. Without loss of generality we may assume that $\lim_{n \to \infty} w_n = \zeta \in \partial \mathbb{D}$. Consider the functions

$$f_n(z) = \log \frac{1}{1 - \bar{w}_nz}, \quad f_0(z) = \log \frac{1}{1 - \zeta z}, \quad h_n(z) = \log \frac{1 - \bar{\zeta}z}{1 - \bar{w}_nz},$$

and define $g_n(z)$ and $g_0(z)$ accordingly. We have

$$g_n(z) = \frac{1}{g'(z)} f_n(z) + f_0(z) - f_0(z) + h_n(z),$$

and

$$g_0(z) = \frac{1}{g'(z)} f_0(z) + f_0(z) - f_0(z) + h_n(z).$$

Since $T_g(BMOA_p) \subseteq VMOA_p$, we have

$$\int_{\partial \mathbb{D}} |g'(z)|^p (1 - |z|^2)^{p-1} \, dm(z) = 0.$$

Theorem A then implies that

$$\lim_{|I| \to 0} \frac{1}{|I|} \int_{S(I)} |g'(z)|^p (1 - |z|^2)^{p-1} \, dm(z) = 0.$$
and notice that \( \|h_n\|_{BMO_A} \leq C \log \frac{1}{1-z} \|BMO_A \), and also that
\begin{equation}
\lim_{|I_n| \to 0} \|h_n\|_{D_{p-1}^p} = 0.
\end{equation}
For \( z \in S(I_n) \) we have that \( |f_n(z)| \lesssim \log \frac{2}{|I_n|} \), hence
\begin{align*}
\left( \log \frac{2}{|I_n|} \right)^p \int_{S(I_n)} |g'(z)|^p (1 - |z|^2)^{p-1} \, dm(z) & \leq C \frac{1}{|I_n|} \int_{S(I_n)} |f_n(z)|^p |g'(z)|^p (1 - |z|^2)^{p-1} \, dm(z) \\
& \leq C \frac{1}{|I_n|} \int_{S(I_n)} |f_0(z)|^p |g'(z)|^p (1 - |z|^2)^{p-1} \, dm(z) \\
& \quad + C \frac{1}{|I_n|} \int_{S(I_n)} |f_n(z) - f_0(z)|^p |g'(z)|^p (1 - |z|^2)^{p-1} \, dm(z) \\
& \leq C \frac{1}{|I_n|} \int_{S(I_n)} |(T_g(f_0)(z))|^p (1 - |z|^2)^{p-1} \, dm(z) + C \|T_0(h_n)\|_{BMO_A}^p \\
& = A_n + B_n.
\end{align*}

Since \( f_0 \in BMO_A \) from the equivalence of (ii) and (iii) we have that \( T_g(f_0) \in VMO_A \) thus \( \lim_{n \to \infty} A_n = 0 \). In addition by the hypothesis and the theorem mentioned above \([23] \) Theorem 3.2, for \( \varepsilon > 0 \) there is a constant \( C \) such that
\begin{align*}
\|T_g(h_n)\|_{BMO_A} & \leq C \|h_n\|_{D_{p-1}^p} + \varepsilon \|h_n\|_{BMO_A} \\
& \leq C \|h_n\|_{D_{p-1}^p} + \varepsilon C' \log \frac{1}{1-z} \|BMO_A\).
\end{align*}
In combination with (5.10) it follows then that \( \lim_{n \to \infty} B_n = 0 \) and the desired condition (iv) follows.

Conversely if (iv) holds then from \([19] \) Theorem 5.2(ii) \( T_g \) is compact on \( BMO_A \) and (iii) follows. \( \square \)

We collect in a corollary the information about Volterra operators on \( BMO_A \).

**Corollary 5.1.** Let \( 1 < p < 2 \) and let \( g \) be analytic on \( \mathbb{D} \), such that the operator \( T_g : BMO_A \to BMO_A \) is bounded. Then the following are equivalent:
(i) \( T_g : BMOA \to BMOA \) is compact
(ii) \( T_g : BMOA \to BMOA \) is compact
(iii) \( T_g : BMOA \to BMOA \) is weakly compact
(iv) \( T_g : VMOA \to VMOA \) is weakly compact
(v) \( T_g(BMOA) \subseteq VMOA \)
(vi) \( T_g(BMOA) \subseteq VMOA \)
(vii) The function $g$ satisfies
\[
\lim_{|I| \to 0} \left( \log \frac{2}{|I|} \right)^p \int_{S(I)} |g'(z)|^p (1 - |z|^2)^{p-1} \, dm(z) = 0.
\]

5.1. An application to composition semigroups. We will apply the above results on $T_g$, with a specific choice of the symbol $g$, to characterize the maximal subspace of strong continuity $[\phi_t, BMOA_p]$ when the inducing semigroup of functions $\{\phi_t\}$ has its Denjoy-Wolff point inside the disc, thus extending results in [3, Corollary 2].

Given a semigroup $\{\phi_t\}$ with Denjoy-Wolff point $b$ and infinitesimal generator $G$, define the associated $g$-symbol $\gamma(z)$ as follows,

1. If $b \in \mathbb{D}$ let $\gamma(z) = \int_b^z \frac{\xi - b}{G(\xi)} \, d\zeta$,
2. If $b \in \partial \mathbb{D}$ let $\gamma(z) = \int_0^z \frac{\xi - b}{G(\xi)} \, d\zeta$,

and consider the Volterra operators $T_\gamma$. Notice that if $b \in \partial \mathbb{D}$, then $\gamma$ coincides with the Koenigs function $h$ associated to $\{\phi_t\}$ [28, page 234]. The function $h$ has the geometric property that for every $w \in h(\mathbb{D})$, the half-line $\{w + t : t \geq 0\}$ is contained in $h(\mathbb{D})$. Thus the range of $h$ contains an infinite strip, and an immediate consequence is that $h$ does not belong to the little Bloch space $B_0$. This implies that $T_\gamma$ cannot be bounded on $BMOA_p$. For if we assume that $T_\gamma$ is bounded then condition (5.1) holds for $g = \gamma$. In particular, since $(\log \frac{2}{|I|})^p \to \infty$ as $|I| \to 0$ we must have
\[
\lim_{|I| \to 0} \frac{1}{|I|} \int_{S(I)} |\gamma'(z)|^p (1 - |z|^2)^{p-1} \, dm(z) = 0,
\]
thus $h = \gamma \in VMOA_p \subset B_0$ and this is a contradiction.

**Corollary 5.2.** Let $1 < p < 2$ and $\{\phi_t\}$ a semigroup with Denjoy-Wolff point $b \in \mathbb{D}$ and associated $g$-symbol $\gamma(z)$, such that $T_\gamma$ is bounded on $BMOA_p$. Then $[\phi_t, BMOA_p] = VMOA_p$ if and only if $T_\gamma$ is (weakly) compact on $BMOA_p$.

**Proof.** Suppose $T_\gamma$ is (weakly) compact. From Corollary (5.1), condition (vii) holds with $\frac{\xi - b}{G} = \gamma'$, and consequently Theorem [4] gives the desired conclusion.

Conversely, suppose that $[\phi_t, BMOA_p] = VMOA_p$. Note that $f'/f' = Gf'/(z - b)$ and that, since $G(b) = 0$, we have $Gf' \in BMOA_p$ if and only if $Gf'/(z - b) \in BMOA_p$. By the assumption and from (2.9), we have that
\[
VMOA_p = \overline{\{f \in BMOA_p : Gf' \in BMOA_p\}}.
\]

Now if $f \in BMOA_p$ is such that $Gf' \in BMOA_p$, this is equivalent to $f'/\gamma' = m \in BMOA_p$ and further to
\[
f(z) = \int_0^z m(\zeta) \gamma'(\zeta) \, d\zeta + c, \quad c \text{ a constant},
\]
for some $m(z) \in BMOA_p$. Thus $f \in BMOA_p \cap (T_\gamma(BMOA_p) \oplus \mathbb{C})$. But $T_\gamma$ is bounded on $BMOA_p$ so this intersection is just the image $T_\gamma(BMOA_p)$ plus the constants. Since $VMOA_p$ is closed, it follows that $T_\gamma(BMOA_p) \subset VMOA_p$ and Corollary [5.1] then says that $T_\gamma$ is (weakly) compact on $BMOA_p$. 

\[\square\]
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