THE EQUIVARIANT COHOMOLOGY OF
HAMILTONIAN $G$-SPACES FROM RESIDUAL $S^1$
ACTIONS

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Abstract. We show that for a Hamiltonian action of a compact torus $G$ on a compact, connected symplectic manifold $M$, the $G$-equivariant cohomology is determined by the residual $S^1$ action on the submanifolds of $M$ fixed by codimension-1 tori. This theorem allows us to compute the equivariant cohomology of certain manifolds, which have pieces that are four-dimensional or smaller. We give several examples of the computations that this allows.

1. Introduction

It has long been a “folk theorem” that, for a Hamiltonian torus action on a symplectic manifold, the associated equivariant cohomology is determined by $S^1$ actions on certain submanifolds. Recently, Tolman and Weitsman [TW] used equivariant Morse theory to prove that the cohomology is determined by that of the one-skeleton, the subspace given by the closure of all points whose orbit under the torus action is one-dimensional. Here we use a powerful result of Chang and Skjelbred [CS] to give a short proof of a slightly stronger statement.

Let $M$ be a compact, connected symplectic manifold, and let $G$ be a compact torus acting on $M$ in a Hamiltonian fashion. Let $M^G$ denote the fixed point set of the action. The inclusion $r : M^G \hookrightarrow M$ induces a map

$$r^* : H^*_G(M) \rightarrow H^*_G(M^G).$$

F. Kirwan [K] proved that $r^*$ is an injection. We find a simple description of the image of $r^*$ in terms of the $S^1$-equivariant cohomology of submanifolds of $M$ fixed by any codimension-1 subtorus of $G$.

We first fix some notation. Let $H \subset G$ be a codimension-1 torus and let $M^H$ denote its fixed point set. Let $X_H$ denote a connected component of $M^H$. Let

$$r^*_{X_H} : H^*_G(X_H) \rightarrow H^*_G(X_H^G).$$
be the map induced by the inclusion of the fixed points $X^G_H$ into $X_H$ and

$$i^*_{X_H} : H^*_G(M^G) \to H^*_G(X^G_H)$$

be the map induced by the inclusion of $X^G_H = X_H \cap M^G$ into $M^G$. We can reduce the computation of $G$-equivariant cohomology of $M$ to the computation of $S^1$-equivariant cohomology of submanifolds of $M$ as follows.

**Theorem 1.** Let $M$ be a compact, connected symplectic manifold with a Hamiltonian action of a torus $G$. Let $r^* : H^*_G(M) \to H^*_G(M^G)$ be the map induced by the inclusion of the fixed point set. A class $f \in H^*_G(M^G)$ is in the image of $r^*$ if and only if

$$i^*_{X_H}(f) \in r^*_{X_H}(H^*_G(X_H)).$$

for all codimension-1 subtori $H \subset G$ and connected components $X_H$ of $M^H$.

In particular, consider the case in which $G$ acts with isolated fixed points, and dim $X_H \leq 2$ for all $X_H$. Theorem [1] gives an explicit description of $r^*$, which was proven in significant generality in [GKM]. See, for example, [GS], Chapter 11. First we find the cohomology of each component $X_H$. If dim $X_H = 2$, then $X_H$ is diffeomorphic to $S^2$ with a Hamiltonian $S^1 \cong G/H$ action with fixed points denoted $\{N, S\}$. Suppose first that $G \cong S^1$ and $H = \{0\}$. In that case,

$$r^*_{X_H} : H^*_S(S^2) \hookrightarrow H^*_S(\{N, S\})$$

is the inclusion induced by $\{N, S\} \subset S^2$. By the Atiyah-Bott Berline-Vergne localization theorem [AB], [BV], any element $f \in H^*_S(\{N, S\})$ in the image of $r^*_{X_H}$ must satisfy the property that

$$(1) \quad f_N - f_S \in x \cdot \mathbb{C}[x]$$

where $f_N$ and $f_S$ are the restrictions of $f$ to the points $N$ and $S$, respectively. We have identified the equivariant cohomology of a point $H^*_S(pt)$ with $\mathbb{C}[x]$.

Let $R$ be the graded ring $H^*_S(\{N, S\})$ subject to the above restriction. A quick dimension check shows that as modules over $H^*_S(pt)$, $H^*_S(S^2) = R$. However, the module structure forces the rings to be equal, so that condition (1) is the only condition for $f \in im(r^*_{X_H})$.

Suppose now that $G \cong T^n$.

**Proposition 1.1.** Suppose that $S^2$ is a Hamiltonian $G$-space for $G \cong T^n$. Let $H$ be a codimension 1 subtorus which acts trivially. Then a
function \( f = (f_N, f_S) \in S(\mathfrak{g}^*) \oplus S(\mathfrak{h}^*) \) is in the image of \( r^*: H^*_G(S^2) \to H^*_G(\{N, S\}) \) if and only if
\[
f_N - f_S \in \ker(\pi_H),
\]
where \( \pi_H : S(\mathfrak{g}^*) \to S(\mathfrak{h}^*) \) is induced by the projection \( \mathfrak{g}^* \to \mathfrak{h}^* \).

**Proof.** Choose a complement \( L \) to \( H \) in \( G \) and write \( S(\mathfrak{t}^*) \cong \mathbb{C}[x] \). We note that \( H^*_G(S^2) = H^*_L(S^2) \otimes H^*_H(pt) \) and \( H^*_G(\{N, S\}) = H^*_L(\{N, S\}) \otimes H^*_H(pt) \) because \( H \) acts trivially on \( S^2 \). Furthermore, the map
\[
r^*: H^*_L(S^2) \otimes S(\mathfrak{h}^*) \to H^*_L(\{N, S\}) \otimes S(\mathfrak{h}^*),
\]
is the identity of the second component, where \( S(\mathfrak{h}^*) \) is the \( H \)-equivariant cohomology of a point. Thus, by \( \text{(1)} \), \( f \in H^*_G(\{N, S\}) \) is in the image of \( r^*H^*_G(S^2) \to H^*_G(\{N, S\}) \) if and only if \( f_N - f_S \in (x) \cdot \mathbb{C}[x] \otimes S(\mathfrak{h}^*) \).
But this is precisely the kernel of \( \pi_H \).

Using this description of \( H^*_G(X_H) \) in the case that \( \dim(X_H) \leq 2 \), we have the following corollary, a theorem of Goresky, Kottwitz and MacPherson \([GKM]\).

**Corollary 1.2.** \([GKM]\) Let \( M \) be a compact, symplectic manifold with a Hamiltonian action of a compact torus \( G \). Assume that \( M^G \) consists of isolated fixed points \( \{p_1, \ldots, p_d\} \) and that each component \( X_H \) of \( M^H \) has dimension 0 or 2 for \( H \subset G \) a codimension-1 torus. Let \( f_i \) be the restriction of a class \( f \in H^*_G(M) \) to the fixed point \( p_i \). Let \( \pi_H : \mathfrak{g}^* \to \mathfrak{h}^* \) be the projection induced by the inclusion \( \mathfrak{h} \hookrightarrow \mathfrak{g} \). Then the map
\[
r^*: H^*_G(M) \longrightarrow H^*_G(M^G) = \bigoplus_{p \in M^G} H^*_G(pt)
\]
has image \( (f_1, \ldots, f_d) \) such that
\[
\pi_H(f_i) = \pi_H(f_j)
\]
whenever \( \{p_i, p_j\} = X_H \cap M^G \).

This theorem can be stated in terms of graphs (cf. \([GZ]\)). The natural generalization of graphs are hypergraphs. These are explored by the authors in \([GH]\).

We now discuss the more general case, which extends the above corollary. We allow the fixed point sets of codimension 1 subtori of \( G \) to have dimension 0, 2, or 4. The equivariant cohomology of \( M \) in this case can be computed as follows.

**Theorem 2.** Suppose that \( M \) is a compact, connected symplectic manifold with an effective Hamiltonian \( G \) action. Suppose further that the \( G \) action has only isolated fixed points \( M^G = \{p_1, \ldots, p_d\} \) and that \( \dim X_H \leq 4 \) for all \( H \subset G \) of codimension 1 and \( X_H \) a connected
component of $M^H$. As before, let $f_i \in H_G^*(pt)$ denote the restriction of $f \in H_G^*(M)$ to the fixed point $p_i$. The image of the injection $r^* : H_G^*(M) \to H_G^*(M^G)$ is the subalgebra of functions $(f_1, \ldots, f_d) \in \bigoplus_{i=1}^d S(g^*)$ which satisfy
$$\begin{cases} \pi_H(f_{i_1}) = \pi_H(f_{i_k}) & \text{if } \{p_{i_1}, \ldots, p_{i_k}\} = X_H^G \\ \sum_{j=1}^d \frac{f_j}{\alpha_1^{ij} \alpha_2^{ij}} & \in S(g^*) & \text{if } \{p_{i_1}, \ldots, p_{i_k}\} = X_H^G \text{ and } \dim X_H = 4 \end{cases}$$
for all $H \subset G$ codimension-1 tori, where $\alpha_1^{ij}$ and $\alpha_2^{ij}$ are the (linearly dependent) weights of the $G$ action on $T_{p_{i_j}} X_H$.

The results we present here are not generally true outside the symplectic setting. We rely heavily on the fact that the restriction map to the equivariant cohomology of the fixed point set is an injection. We also make use of the Morse theory associated to a Morse function obtained from the moment map for the Hamiltonian action.

In Section 2 we prove Theorem 1. In Section 3 we prove Theorem 2. Lastly, in Section 4 we give several examples of the computations allowed by this theorem.

2. Reduction to the study of circle actions

The proof of Theorem 1 is based on the following result of Chang and Skjelbred [CS]. The description given here is due to Brion and Vergne [BrV].

**Lemma 2.1.** [CS] The image of $r^* : H_G^*(M) \to H_G^*(M^G)$ is the set
$$\bigcap_H r^*_M(H_G^*(M^H)),$$
where the intersection in $H_G^*(M^G)$ is taken over all codimension-one subtori $H$ of $G$ and $r^*_M$ is the inclusion of $M^G$ into $M^H$.

**Remark 2.2.** In fact, the only nontrivial contributions to this intersection are those codimension-one subtori $H$ which appear as isotropy groups of elements of $M$. Since $M$ is compact, there are only finitely many such isotropy groups.

We will now restate and prove the theorem reducing the computation of $H_G^*(M)$ to computations of $H_G^*(X)$ for various submanifolds $X \subseteq M$.

**Theorem 1.** Let $r^* : H_G^*(M) \to H_G^*(M^G)$ be the map induced by the inclusion of the fixed point set. Let $i^*_H : H_G^*(M^G) \to H_G^*(X_H^G)$ and
r^*_X: H^*_G(X_H) \to H^*_G(X^G_H) be the maps defined in Section 1. A class \( f \in H^*_G(M^G) \) is in the image of \( r^* \) if and only if
\[
i^*_X(f) \in r^*_X(H^*_G(X_H)).
\]
for all codimension-1 subtori \( H \subset G \) and connected components \( X_H \) of \( M^H \).

**Proof.** By Lemma 2.1, \( f \in \text{im}(r^*) \) if and only if \( f \) is in the intersection of \( r^*_M(X_H) \) over all codimension-one subtori \( H \), where \( r^*_M : H^*_G(M^H) \to H^*_G(M^G) \). Equivalently,
\[
f \in \bigcap_H r^*_M(\bigoplus_{X_H} H^*_G(X_H)),
\]
where the direct sum is taken over all connected components \( X_H \) of \( M^H \). Let \( k_{X_H} : H^*_G(X_H) \to H^*_G(M^H) \) be the map which extends any class on \( X_H \) to 0 on other components of \( M^H \). Let \( k_{X^G_H} : H^*_G(X^G_H) \to H^*_G(M^G) \) be the same map on the fixed point sets. Then
\[
r^*_M(\bigoplus_{X_H} H^*_G(X_H)) = \bigoplus_{X_H} r^*_M \circ k_{X_H}(H^*_G(X_H)).
\]
As \( k_{X^G_H} \circ r^*_X = r^*_M \circ k_{X_H} \), we have that \( f \) is in \( \text{im}(r^*) \) if and only if
\[
f \in \bigoplus_{X_H} k_{X^G_H} \circ r^*_X(H^*_G(X_H)),
\]
for all \( H \). Now note that \( i^*_X \circ k_{X^G_H} = id \). Because the \( X_H \) are disjoint, we can now apply \( i^*_X \) to (2) to get
\[
i^*_X(f) \in r^*_X(H^*_G(X_H)),
\]
for every \( H \) and \( X_H \). However, since \( \bigoplus_{X_H} i^*_X \) is an injection, we can apply \( \bigoplus_{X_H} k_{X^G_H} \) to (3) to get (2). Thus, (3) and (2) are equivalent. This completes the proof. \( \square \)

This provides another proof for a result of Tolman and Weitsman [TW].

**Definition 2.3.** Let \( N \subset M \) be the set of points whose orbits under the \( G \) action are 1-dimensional. The one-skeleton of \( M \) is the closure \( \overline{N} \).

Tolman and Weitsman show that the image of \( r^* : H^*_G(M) \to H^*_G(M^G) \) is equal to the image of the cohomology of the one-skeleton.

**Theorem 2.4.** [TW] Let \( M \) be a compact symplectic manifold with a Hamiltonian torus action by \( G \). Let \( M^G \) be the fixed point set, and \( \overline{N} \) be the one-skeleton. Let \( r : M^G \hookrightarrow M \) be the inclusion of the fixed point
set to $M$ and $j : M^G \rightarrow \overline{N}$ be the inclusion to $\overline{N}$. The induced maps $r^* : H^*_G(M) \rightarrow H^*_G(M^G)$ and $j^* : H^*_G(\overline{N}) \rightarrow H^*_G(M^G)$ on equivariant cohomology have the same image.

Proof. Because $G$ acts effectively, $N$ consists of points fixed by some codimension-1 torus $H \subset G$ but not by all of $G$, i.e.

$$N = \bigcup_H M^H \setminus M^G$$

where the union is taken over all codimension-1 tori $H \subset G$. As noted above, this is a finite union over all codimension-1 $H$ which appear as isotropy subgroups of points in $M$. Then $\overline{N} = \bigcup_H M^H$, and the inclusion $\gamma_H : M^H \hookrightarrow M$ factors through the inclusion $\gamma : \overline{N} \hookrightarrow M$ for each codimension-1 torus $H$ in $G$. It follows that the induced maps in cohomology also factor. Furthermore, there is an inclusion

$$H^*_G(\overline{N}) \hookrightarrow \bigoplus_{i=1}^k H^*_G(M^{H_i}),$$

where $H_i, i = 1, \ldots, k$ are the codimension-1 tori which appear as isotropy subgroups of $G$. Theorem 1 implies that the map $r^* : H^*_G(M) \rightarrow H^*_G(M^G)$ factors through the map

$$\bigoplus_{i=1}^k r^*_M : \bigoplus_{i=1}^k H^*_G(M^{H_i}) \longrightarrow H^*_G(M^G)$$

But then $r^*$ must factor through $j^* : H^*_G(\overline{N}) \rightarrow H^*_G(M^G)$.

Now suppose that $M^G$ consists of isolated fixed points. Then

$$H^*_G(M^G) = \bigoplus_{p \in M^G} S(g^*)$$

and any $f \in H^*_G(M^G)$ is a map $f : M^G \rightarrow S(g^*)$. Furthermore, as $X_H$ and $X_H^G$ have trivial $H$ actions, we can rewrite Theorem 1 in the following way.

**Theorem 2.5.** Under the above hypotheses, the image of $r^*$ is the set of $f : M^G \rightarrow S(g^*)$ such that

$$i^*_X(f) : X^G_H \rightarrow S(g^*)$$

is in the image of

$$r^*_{X_H} : H^*_G(X_H) \rightarrow H^*_G(X^G_H) = \bigoplus_{p \in X^G_H} S(g^*),$$

where $r^*_{X_H}$ is restriction for each fixed point.
3. AN EXTENSION OF A THEOREM OF GKM

We use Theorem 2.5 to compute the equivariant cohomology of $M$ in the case in which $\dim X_H \leq 4$ for all codimension-1 tori $H \subset G$ and connected components $X_H$ of the fixed point set $M^H$. In Section 1 we considered the case that $\dim X_H \leq 2$.

First, let $X$ be a compact, connected symplectic four-manifold with an effective Hamiltonian $G = S^1$ action with isolated fixed points. Then the equivariant cohomology can be computed as follows.

**Proposition 3.1.** Let $X$ be a compact, connected symplectic 4-manifold with an effective Hamiltonian $S^1$ action with isolated fixed points $X_{S^1} = \{p_1, \ldots, p_d\}$. The map $r^*: H_{S^1}^*(X) \to H_{S^1}^*(X_{S^1})$ induced by inclusion is an injection with image

\[(4) \{ (f_1, \ldots, f_d) \in \bigoplus_{i=1}^d S(\mathfrak{s}^*) | f_i - f_j \in x \cdot \mathbb{C}[x], \sum_{i=1}^d \frac{f_i}{\alpha_1^i \alpha_2^i} \in S(\mathfrak{s}^*) \},\]

where $\alpha_1^i$ and $\alpha_2^i$ are the (linearly dependent) weights of the $S = S^1$ isotropy action on $T_{p_i} X$.

**Proof.** The map $r^*$ is injective because $X$ is equivariantly formal. We know that the $f_i$ must satisfy the first condition because the functions constant on all the vertices are the only equivariant classes in degree 0, as $\dim H^0_{S^1}(X) = 1$. The second condition is necessary as a direct result of the Atiyah-Bott Berline-Vergne (ABBV) localization theorem ([AB], [BV]). Notice that this condition gives us one relation in degree 2 cohomology. A dimension count shows us that these conditions are sufficient. As an $S(\mathfrak{s}^*)$-module, $H^*_{S^1}(X) \cong H^*(X) \otimes H^*_{S^1}(pt)$. Thus, the equivariant Poincaré polynomial is

$$P^S_t(X) = (1 + (d - 2)t^2 + t^4) \cdot (1 + t^2 + t^4 + \ldots) = 1 + (d - 1)t^2 + dt^4 + \cdots + dt^{2n} + \ldots.$$ 

As $H^*_{S^1}(X)$ is generated in degree 2, the $d - 1$ degree 2 classes given by the $(f_1, \ldots, f_d)$ subject to the ABBV condition generate the entire cohomology ring. Thus we have found all the conditions. \qed

We now prove a slightly more general proposition.

**Proposition 3.2.** Let $X$ be a compact, connected symplectic 4-manifold with a Hamiltonian $G$ action with isolated fixed points $X^G = \{p_1, \ldots, p_d\}$. Suppose further that there is a codimension-1 subtorus $H$ which acts trivially. The map $r^*: H^*_G(X) \to H^*_G(X^G)$ induced by inclusion is an
injection with image

\{(f_1, \ldots, f_d) \in \bigoplus_{i=1}^d S(g^*) \mid f_i - f_j \in \ker(\pi_H), \sum_{i=1}^d \frac{f_i}{\alpha_1^i \alpha_2^i} \in S(g^*)\},

where \(\pi_H\) is the map \(S(g^*) \to S(h^*)\), and \(\alpha_1^i\) and \(\alpha_2^i\) are the (linearly dependent) weights of the \(G\) isotropy action on \(T_{p_i}X\).

Proof. As in the case where \(X \cong S^2\),

\[H_G^*(X) = H_G^*(X) \otimes S(h^*).\]

Furthermore,

\[H_G^*(X^G) = H_{G/H}^*(X^G) \otimes S(h^*).\]

Again, choose a complement \(L\) to \(H\), and write \(S(t^*) \cong \mathbb{C}[x]\). Then \(H_{G/H}^*(X^G)\) can be identified with \(\bigoplus_{p \in X \cap H} \mathbb{C}[x]\). By Proposition 3.1, we have \(f \in H_{G/H}^*(X^G)\) is in the image of \(r^* : H_{G/H}^*(X) \to H_{G/H}^*(X^G)\) if and only if the component of \(f\) in \(H_{G/H}^*(X^G) \cong \bigoplus_{p \in X \cap H} \mathbb{C}[x]\) satisfies the conditions (4) of Proposition 3.1. But then \(f\) must satisfy the conditions (3).

We now discuss the more general case, which extends the result due to \([GKM]\) (Corollary 1.2). Suppose that \(M\) is a compact, connected symplectic manifold with an effective Hamiltonian \(G\) action. Suppose further that this \(G\) action has only isolated fixed points \(M^G = \{p_1, \ldots, p_d\}\) and that \(\dim X_H \leq 4\) for all \(H \subset G\) and \(X_H\) a connected component of \(M^H\), as above. As before, let \(f_i \in H_G^*(pt)\) denote the restriction of \(f \in H_G^*(M)\) to the fixed point \(p_i\). The equivariant cohomology of \(M\) can be computed as follows.

**Theorem 2.** Under the above hypotheses, the image of the injection \(r^* : H_G^*(M) \to H_G^*(M^G)\) is the subalgebra of functions \((f_1, \ldots, f_d) \in \bigoplus_{i=1}^d S(g^*)\) which satisfy

\[\left\{ \begin{array}{ll}
\pi_H(f_i) = \pi_H(f_k) & \text{if } \{p_i, \ldots, p_k\} = X_H^G \\
\sum_{j=1}^d \frac{f_j}{\alpha_1^i \alpha_2^i} \in S(g^*) & \text{if } \{p_i, \ldots, p_j\} = X_H^G \text{ and } \dim X_H = 4
\end{array} \right.\]

for all \(H \subset G\) codimension-1 tori, where \(\alpha_1^i\) and \(\alpha_2^i\) are the (linearly dependent) weights of the \(G\) action on \(T_{p_i}X_H\).

Proof. By Theorem 2.1, \(\text{im}(r^*)\) consists of \((f_1, \ldots, f_d)\) which have certain properties restricted to each \(X_H\). Proposition 3.2 lists these restrictions for each \(X_H\) of dimension 2. The conditions for \(X_H\) of dimension 2 are discussed in Section 4. A quick check shows that these are exactly the conditions listed above. \(\square\)
4. Examples

Here we demonstrate the use of Theorem 2 in computing equivariant cohomology. In the first example, we compute the $S^1$-equivariant cohomology of $\mathbb{C}P^2$ with a Hamiltonian circle action. In the second, we calculate the $T^2$-equivariant cohomology of $\mathbb{C}P^3$. Finally, we find the $S^1$-equivariant cohomology of a 4-dimensional manifold obtained by symplectic reduction.

Example 1. Consider $\mathbb{C}P^2$ with homogeneous coordinates $[z_0 : z_1 : z_2]$. Let $T = S^1$ act on $\mathbb{C}P^2$ by

$$e^{i\theta} \cdot [z_0 : z_1 : z_2] = [e^{-i\theta}z_0 : e^{i\theta}z_1 : z_2].$$

This action has three fixed points: $[1 : 0 : 0]$, $[0 : 1 : 0]$ and $[0 : 0 : 1]$. The weights at these fixed points are

| Fixed point | Weights |
|-------------|---------|
| $p_1 = [1 : 0 : 0]$ | $x, 2x,$ |
| $p_2 = [0 : 1 : 0]$ | $-x, x,$ |
| $p_3 = [0 : 0 : 1]$ | $-2x, -x,$ |

where we have identified $t^*$ with degree one polynomials in $\mathbb{C}[x]$. As cohomology elements, these are assigned degree two. The image of the moment map for this action is show in the figure below.

We use the cohomology computed above to compute the $T^2$-equivariant cohomology of $\mathbb{C}P^3$.

Example 2. The second example we consider is a $T^2$ action on $\mathbb{C}P^3$. Consider $\mathbb{C}P^3$ with homogeneous coordinates $[z_0 : z_1 : z_2 : z_3]$. Let $T^2$ act on $\mathbb{C}P^3$ by

$$(e^{i\theta_1}, e^{i\theta_2}) \cdot [z_0 : z_1 : z_2 : z_3] = [e^{-i\theta_1}z_0 : e^{i\theta_1}z_1 : e^{i\theta_2}z_2 : e^{i\theta_2}z_3].$$

This action has four fixed points, $[1 : 0 : 0 : 0]$, $[0 : 1 : 0 : 0]$, $[0 : 0 : 1 : 0]$ and $[0 : 0 : 0 : 1]$. The image of the moment map for this action is show in the figure below.
Figure 1. This shows the image of the moment map for $T^2$ acting on $M = \mathbb{C}P^3$, as described above.

The weights at these fixed points are

| Fixed point $p_i$ | Weights                      |
|-------------------|-------------------------------|
| $p_1 = [1 : 0 : 0 : 0]$ | $x, 2x, x + y$               |
| $p_2 = [0 : 1 : 0 : 0]$ | $-x, x, y$                    |
| $p_3 = [0 : 0 : 1 : 0]$ | $-2x, -x, y - x$              |
| $p_4 = [0 : 0 : 0 : 1]$ | $-x - y, -y, x - y$           |

Theorem 2 tells us that the image of the equivariant cohomology $H^*_{T^2}(\mathbb{C}P^3)$ in $H^*_{T^2}(\{p_1, p_2, p_3, p_4\}) \cong \bigoplus_{i=1}^4 \mathbb{C}[x, y]$ is the ring of functions $(f_1, f_2, f_3, f_4)$ such that

$$f_i - f_j \in (x) \cdot \mathbb{C}[x, y] \quad \text{for every } i, j \in \{1, 2, 3\},$$

$$\frac{f_1}{2x^2} - \frac{f_2}{x^2} + \frac{f_3}{2x^2} \in \mathbb{C}[x, y],$$

$$f_1 - f_4 \in (y + x) \cdot \mathbb{C}[x, y],$$

$$f_2 - f_4 \in (y) \cdot \mathbb{C}[x, y],$$

$$f_3 - f_4 \in (y - x) \cdot \mathbb{C}[x, y].$$

Figure 2. This shows an equivariant class of $H^*_{T^2}(\mathbb{C}P^3)$, shown as an element of the equivariant cohomology of the fixed points.
Example 3. Let $O_{\lambda}$ be the coadjoint orbit of $SU(3)$ through the generic point $\lambda \in t^*$, the dual of the Lie algebra $t$ of the maximal 2-torus $T$ in $SU(3)$. Recall that $T$ acts on $O_{\lambda}$ in a Hamiltonian fashion, and (one choice of) the moment map

$$\Phi_T : O_{\lambda} \rightarrow t^*$$

takes each matrix to its diagonal entries. Equivalently, $\Phi_T$ is the composition of the inclusion of $O_{\lambda}$ into $\mathfrak{su}(3)^*$ and projection of $\mathfrak{su}(3)^*$ onto $t^*$.

We compute the equivariant cohomology of $M = O_{\lambda}/H$, the symplectic reduction of $O_{\lambda}$ by a circle $H$ chosen such that the reduced space is a manifold. Let $H \subset T$ be any copy of $S^1$ which fixes a two-sphere in $O_{\lambda}$. Then the moment map $\Phi_H : O_{\lambda} \rightarrow \mathfrak{h}^*$ for the $H$ action is the map $\Phi_T$ followed by the projection $\pi_H : t^* \rightarrow \mathfrak{h}^*$ induced by the inclusion $\mathfrak{h} \hookrightarrow t$. The symplectic reduction at $\mu$ by $H$ is by definition

$$M = O_{\lambda}/H := \Phi^{-1}_H(\mu)/H,$$

where $\mu$ is a regular value for $\Phi_H$. Note that there is a residual $T/H \cong S^1$ action on $M$. We use Theorem 1 to calculate the corresponding equivariant cohomology of $M$.

One can easily see that there are four fixed points of this action, which we denote by $p_i$ for $i = 1, \ldots, 4$. For each $p_i$, $\Phi_T^{-1}(p_i)$ lies on a two-sphere in $O_{\lambda}$, denoted $S^2_i$, which is fixed by a a subgroup $H_i \cong S^1$ of $T$. Note that $H_i$ is complementary to $H$ in $T$.

The weights of the $T/H$ action on the tangent space $T_{p_i}M$ are determined by the $T$ action on $S^2_i$. Let $n_i$ and $s_i$ be fixed points of the $T$ action on $S^2_i$. Note that the condition that $\mu$ be a regular value of $\Phi_H$ ensures that $\Phi_T^{-1}(p_i) \neq n_i, s_i$. Furthermore, by assumption the
set $\Phi^{-1}_T(p_i)$ is point-wise fixed by $H_i$. Thus in the reduction, the $T/H$ action on $T_p M$ is isomorphic to the $H_i$ action on this space.

Denote the weights of the $T$ action on $T_p O_\lambda$ by $\pm \alpha_1, \pm \alpha_2$, and $\pm \alpha_3 = \pm (\alpha_1 + \alpha_2)$, where the signs depend on $i$. The weights of the $H$ action on the reduction $M$ are determined by projecting the $\alpha_i$ to $h^\ast$.

At $p_1$, the weight $\alpha_3$ projects to $0$ and the other two weights both project to the generator $x$ of $S(h_1^\ast) \cong \mathbb{C}[x]$. Similarly, at $p_2$ the weights are $x$ and $-x$, at $p_3$ they are $x$ and $-x$ and at $p_4$ they are both $-x$. The image of the moment map $\Phi_H : M \to h^\ast$, with weights, is shown in Figure 4.

\begin{figure}[h]
\centering
\begin{tabular}{c|c|c|c|c}
Points & $p_1$ & $p_2$ & $p_3$ & $p_4$ \\
Weights & $\frac{x}{2}$ & $-\frac{x}{2}$ & $\frac{x}{2}$ & $-\frac{x}{2}$
\end{tabular}
\caption{The image of the moment map for the $T/H$ action on $M = O_\lambda // H$, with the weights for the isotropy action on the tangent space of the fixed points.}
\end{figure}

Finally, this tells us that the equivariant cohomology of $M$ is

$$H^*_{T^1}(M) \cong \{ f : V \to \mathbb{C}[x] \ | \ f_i - f_j \in x \cdot \mathbb{C}[x], \frac{f_1}{x^2} - \frac{f_2}{x^2} - \frac{f_3}{x^2} + \frac{f_4}{x^2} \in \mathbb{C}[x] \}.$$ 

Notice that this computation leads us to the $T/S^1$-equivariant cohomology of $M \cong O_\lambda // S^1$ for a coadjoint orbit of $SU(n)$, as the submanifolds that appear are identical to those shown the above $SU(3)$ case. ♦

References

[AB] M. Atiyah and R. Bott. The moment map and equivariant cohomology. Topology , 23 (1984), 1–28.

[BV] N. Berline and M. Vergne. Classes caractéristiques équivariantes. Formules de localisation en cohomologie équivariante. C.R. Acad. Sci. Paris Sér. I Math., 295 (1982), 539–541.

[B] A. Borel. Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts. Ann. of Math. (2), 57 (1953), 115–207.

[BrV] M. Brion and M. Vergne, On the localization theorem in equivariant cohomology, in M. Brion, Equivariant cohomology and equivariant intersection theory, Chapter 2. Representation Theories and Algebraic Geometry, A. Broer et G. Sabidussi, eds., Nato ASI Series 514, Kluwer 1998, 1–37.

[CS] T. Chang and T. Skjelbred, The topological Schur lemma and related results, Ann. of Math. 100 (1974), 307–321.
[GKM] M. Goresky, R. Kottwitz, and R. MacPherson, *Equivariant cohomology, Koszul duality, and the localization theorem*, Invent. math. 131 (1998), 25–83.

[GH] R. Goldin and T. Holm, *Hypergraphs and equivariant cohomology*, in preparation.

[GS] V. Guillemin and S. Sternberg, *Supersymmetry and Equivariant deRham Cohomology*, Springer Verlag, Berlin 1999.

[GZ] V. Guillemin and C. Zara, *Equivariant de Rham theory and graphs* Asian J. Math. 3 (1999), no. 1, 49–76.

[K] F. Kirwan, *Cohomology of Quotients in Symplectic and Algebraic Geometry*, Princeton University Press, Princeton, NJ, 1984.

[TW] S. Tolman and J. Weitsman, *The cohomology rings of Hamiltonian T-Spaces*, Proc. of the Northern CA Symplectic Geometry Seminar, Y. Eliashberg et al. eds. AMS Translations Series 2, vol 196: Advances in Mathematical Sciences (formerly Advances in Soviet Mathematics) 45, (1999) 251–258.

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