Gravitational effects on the Heisenberg Uncertainty Principle: a geometric approach

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The Heisenberg Uncertainty Principle (HUP) limits the accuracy in the simultaneous measurements of the position and momentum variables of any quantum system. This is known to be true in the context of non-relativistic quantum mechanics. Based on a semiclassical geometric approach, here we propose an effective generalization of this principle, which is well-suited to be extended to general relativity scenarios as well. We apply our formalism to Schwarzschild and de Sitter spacetime, showing that the ensuing uncertainty relations can be mapped into well-known deformations of the HUP. We also infer the form of the perturbed metric that mimics the emergence of a discrete spacetime structure at Planck scale, consistently with the predictions of the Generalized Uncertainty Principle. Finally, we discuss our results in connection with other approaches recently appeared in the literature.

Keywords: Uncertainty Principle(s), Schwarzschild spacetime, de Sitter spacetime, Planck scale

I. INTRODUCTION

The Heisenberg Uncertainty Principle (HUP) plays a pivotal rôle within the framework of non-relativistic Quantum Mechanics (QM) and can be regarded as one of the cornerstones of quantum theory. However, in high energy/large distance regimes, it is expected to acquire corrections in order to accommodate effects that escape the domain of QM. For instance, several models of quantum gravity, such as String Theory [1], Loop Quantum Gravity [2] and Doubly-Special Relativity [3] predict the emergence of a minimal length at Planck scale, which is at odds with the standard Heisenberg’s formula. This feature is taken into account by generalizing the HUP to the so-called Generalized Uncertainty Principle (GUP), which naturally embeds the UV length cutoff into a minimal position uncertainty. On the other hand, in cosmological scenarios it has been argued that the HUP should be modified by introducing corrections proportional to the cosmological constant, which reflects the existence of a maximum length of the order of the cosmological horizon [4]. Such kind of modification is referred to as Extended Uncertainty Principle (EUP). Deformations that combine both the Generalized and Extended Uncertainty Principles (GEUP) are also being considered as models for a more complete picture of the quantum spacetime [5].

While providing an effective description of QM in extreme regimes, the GUP and HUP break down Lorentz covariance by introducing a universal length scale. This poses the problem of how to extend these models to the relativistic theory. Attempts to tackle this issue have been carried out in Refs. [6–8] via the introduction of the most general covariant form of the quadratic GUP algebra in Minkowski spacetime. However, challenging problems remain open, such as the derivation of the uncertainty relation (UR) for a generic (curved) background or, by reversing the perspective, the determination of the metric capable of mimicking a given deformed UR. In this regard, we mention that gravitationally induced UR’s in the context of both General Relativity (GR) and extended theories of gravity have been recently investigated in Ref. [9] by assuming the wave function of the quantum system to be confined to a geodesic ball on a given space-like hypersurface whose radius is a measure of the position uncertainty. Furthermore, a metric-dependent uncertainty relation based on a proper redefinition of the translation operator has been exhibited in Ref. [10].

Starting from the above premises, in this work we propose a semiclassical geometric approach to compute a metric-dependent generalization of the HUP. Specifically, we first extend the canonical UR to Minkowski spacetime through a suitable redefinition of the scalar product. Then, we apply our considerations to curved manifolds endowed with a time-like Killing vector. As particular examples, we deal with Schwarzschild, perturbed weak-field Schwarzschild space...
and de Sitter metrics, showing how the ensuing uncertainty relations can be mapped into generalizations of the HUP which are well-established in the literature. Letting ourselves be guided by the phenomenological predictions of GUP, we also provide some hints towards understanding how Minkowski metric should be perturbed at quantum gravity scale in order to reproduce the emergent discrete structure of spacetime.

The remainder of the work is organized as follows: in Sec. II we set the stage for the generalization of the Heisenberg Uncertainty Principle. The ensuing relation is applied to some specific metrics in Sec. III. Conclusions and outlook are summarized in Sec. IV. Throughout all the paper, we set the (reduced) Planck’s constant $\hbar$, the Newton’s gravitational constant $G$, the speed of light in vacuum $c$ and Boltzmann’s constant $k_B$ equal to unity. Furthermore, we use the metric with the conventional spacelike signature $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$.

Quantum operators will be distinguished from classical variables by the usual symbol $\hat{}$. However, the same notation for both quantities will be employed where no ambiguity arises.

### II. GENERALIZATION OF HEISENBERG UNCERTAINTY PRINCIPLE

Let us consider a particle of position vector $x \equiv (x, y, z)$ and denote by $E, p \equiv (p_x, p_y, p_z)$ its energy and three-momentum. According to the standard HUP applied, for example, along the $x$-direction, it is well-known that

$$\Delta x \Delta p_x \simeq \frac{1}{2},$$

(2)

where $\Delta x$ and $\Delta p_x$ are the position and momentum uncertainties of the particle state, respectively.

By assuming that the uncertainties along the three axes do not affect each other, the above relation can be equally written for the remaining spatial dimensions, yielding

$$\Delta y \Delta p_y \simeq \frac{1}{2}, \quad \Delta z \Delta p_z \simeq \frac{1}{2}.$$  

(3)

Likewise, one can introduce an equivalent time-energy uncertainty relation in the form

$$\Delta t \Delta E \simeq \frac{1}{2}.$$  

(4)

Following one of the earliest proposed versions of this relation, here we identify $\Delta E$ and $\Delta t$ with the uncertainty on the energy measurement and the duration of such a measuring process, respectively. However, this way of interpreting Eq. (4) is not unique and several reformulations have appeared in the literature over the years. For a more detailed review, one can refer to [11].

We can now recast the UR’s in Eqs. (2) and (3) in a more compact form by introducing the notation

$$\Delta x \equiv (\Delta x, \Delta y, \Delta z),$$

(5)

$$\Delta p \equiv (\Delta p_x, \Delta p_y, \Delta p_z).$$

(6)

This leads to

$$\Delta x \cdot \Delta p = \Delta x \Delta p_x + \Delta y \Delta p_y + \Delta z \Delta p_z \simeq \frac{3}{2},$$

(7)

Here we have used the symbol “·” to denote the scalar product between three-vectors. Notice that a similar multi-dimensional generalization of HUP has been discussed in Ref. [12].

Equation (7) provides the starting point of our analysis. Indeed, one can naturally extend it to the four-dimensional Minkowski spacetime by implementing the following replacement

$$\Delta x \cdot \Delta p \rightarrow |\Delta x^\mu \langle \hat{\eta} \rangle_{\mu\nu} \Delta p^\nu|,$$  

(8)

where $\langle \hat{\eta} \rangle_{\mu\nu}$ is defined as the tensor whose generic $(\alpha, \beta)$-element $(\alpha, \beta = \{0, 1, 2, 3\})$ is the $(\alpha, \beta)$-element of $\eta_{\mu\nu}$ evaluated on the expectation value in the quantum state of the system, i.e.

$$\langle \hat{\eta}(\hat{x}) \rangle_{\alpha\beta} \equiv \eta_{\alpha\beta}(\hat{x}).$$

(9)

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1 Notice that in our quantum picture the presence of the expectation value in Eq. (8) follows from the fact that the metric $\hat{\eta}_{\mu\nu}$ must be regarded as being composed by operators (rather than classical variables) acting on the state of the quantum system.
The presence of the absolute value in the definition (8) follows from the fact that the metric tensor contains in general negative terms. Clearly, for the specific case of Minkowski metric, we simply have \( \langle \hat{\eta} \rangle_{\mu\nu} = \eta_{\mu\nu} \). Furthermore, we have defined
\[
\Delta x^\mu = (\Delta t, \Delta x, \Delta y, \Delta z),
\]
\[
\Delta p^\nu = (\Delta E, \Delta p_x, \Delta p_y, \Delta p_z).
\]
In the above setting, by plugging Eqs. (9)-(11) into (8), we obtain
\[
|\Delta x^\mu \eta_{\mu\nu} \Delta p^\nu| = \left| -\Delta t \Delta E + \Delta x \Delta p_x + \Delta y \Delta p_y + \Delta z \Delta p_z \right| \approx 1.
\]
It is worth noting that the one-dimensional UR’s (2)-(4) can be recovered by properly restricting the dimension of the metric tensor and assuming uncertainties along different directions to be independent from each other. For instance, with reference to the \( \alpha \)-th-component, \( \eta_{\mu\nu} \) in Eq. (8) must be replaced by its \((\alpha,\alpha)\)-element. This straightforwardly gives
\[
|\Delta x_\alpha \eta_{\alpha\alpha} \Delta p_\alpha| = \Delta x_\alpha \Delta p_\alpha \approx \frac{1}{2}, \quad \alpha = \{0, 1, 2, 3\}.
\]
We stress that the above relation must not be intended as summed over \( \alpha \). Furthermore, the r.h.s. has been fixed by properly taking into account the dimensional restriction. Alternatively, one can obtain the usual UR’s in Minkowski spacetime by using the vierbein formalism, i.e. by projecting the commutator and the metric tensor on the tangent space.

Now, the most direct way to extend the relation (12) to a generic spacetime of diagonal metric \( g_{\mu\nu} \) endowed with a time-like Killing vector reads\(^2\)
\[
\langle \hat{\eta} \rangle_{\mu\nu} \rightarrow \langle \hat{g} \rangle_{\mu\nu} \quad \Rightarrow \quad |\Delta x^\mu \langle \hat{g} \rangle_{\mu\nu} \Delta p^\nu| \approx 1,
\]
where the last equality follows from the fact that Eq. (12) must be recovered when going back to Minkowski spacetime\(^3\).

We notice that a similar generalization of Heisenberg relation to curved spacetime has been proposed in [13] at level of the canonical commutator between position and momentum operators.

Before applying the above formalism to some specific examples, we point out that the connection between extensions of the uncertainty relation and spacetime metrics has also been investigated in Ref. [10]. In that case a translation operator acting in a space with a diagonal metric is introduced to describe the motion of a particle in a quantum system. As a result, it is shown that the momentum operator and the uncertainty relation acquire a metric-dependent structure. Furthermore, for any metric expanded up to the second order, such a formalism naturally leads to an Extended Uncertainty Principle with a minimum momentum dispersion. The present analysis goes in the direction of [14], where a duality between theories yielding generalized uncertainty principles and quantum mechanics on nontrivial momentum space is found.

## III. APPLICATIONS

For the sake of concreteness, in this Section we consider some applications of the formula (14). We derive the corrections to the Heisenberg principle arising in Schwarzschild, perturbed weak-field Schwarzschild and de Sitter backgrounds. We also infer the form of the perturbed metric which is best suited to the description of spacetime fluctuations at Planck scale according to the predictions of GUP.

### A. Schwarzschild spacetime

As a first example, let us consider the spacetime geometry around a Schwarzschild black hole of mass \( M \) and Schwarzschild radius \( r_s = 2M \). In spherical coordinates \( (t, r, \theta, \varphi) \), this is described by the well-known metric tensor
\[
g_{\mu\nu} = \text{diag}(-e^{2\alpha(r)}, e^{-2\alpha(r)}, r^2, r^2 \sin^2 \theta),
\]
where the generalization of our formalism to non-diagonal metrics, as well as to metric that do not admit a time-like Killing vector is not immediate and will be discussed elsewhere.

More generally, the r.h.s. of Eq. (14) may depend on some invariants of the metric, in such a way that, for \( g_{\mu\nu} \rightarrow \eta_{\mu\nu} \), Eq. (14) is recovered in its current form. In the absence of a definitive way to fix these extra-terms, in what follows we stick to the simplest generalization (14), leaving a more in-depth study for future work.
where
\[ e^{2\alpha(r)} = 1 - \frac{r_s}{r}, \quad r > r_s. \]  
(16)

By introducing the analogues of Eqs. (10) and (11) in spherical coordinates for an observer in Schwarzschild metric, we obtain from Eq. (14)
\[ |\Delta x^\mu (\hat{y})_{\mu\nu} \Delta p^\nu| = |e^{2\alpha(\hat{r})} \Delta t \Delta E + e^{-2\alpha(\hat{r})} \Delta r \Delta p_r + (\hat{\theta})^2 \Delta \theta \Delta p_\theta + (\hat{\phi} \sin \hat{\theta})^2 \Delta \phi \Delta p_\phi| \approx 1, \]  
(17)

where we have denoted by \( p_r, p_\theta \) and \( p_\phi \) the radial and angular components of the particle momentum, respectively.

By analogy with Eqs. (13), let us see how the uncertainty principle (17) appears for each couple of variables separately. In particular, we focus on the uncertainty relation between \( r \) and \( p_r \), which takes the form\(^4\)
\[ \Delta r \Delta p_r \approx \frac{1}{2} \left( 1 - \frac{r_s}{\hat{r}} \right), \]  
(18)

Quite unexpectedly, from this relation it follows that the closer the particle gets to the Schwarzschild horizon, the more classical its behavior becomes, since the r.h.s. approaches zero (see Fig. 1). A possible interpretation for this effect is provided below in connection with the classicalization (i.e. the recovery of a classical behavior) of gravitational interaction in high energy regimes predicted by some alternative theories of gravity.

One can now recognize in Eq. (18) a GUP-like deformation of the uncertainty principle. In fact, as suggested by gedanken experiments involving the formation of either gravitational instabilities in high energy scatterings of strings\(^{[15–18]}\) or micro black holes\(^{[19]}\), it is expected that the standard Heisenberg relation gets non-trivially modified at Planck scale. Specifically, by considering a micro black hole of gravitational radius \( r_s = r_s(E) = 2E \) proportional to the (centre-of-mass) scattering energy \( E \), the HUP should take the form\(^{[19]}\)
\[ \Delta r \approx \frac{1}{2E} + \beta r_s(E). \]  
(19)

We note that this kind of modification was also proposed in Ref.\(^{[20]}\). Furthermore, it can be recast in the most common form of quadratic Generalized Uncertainty Principle
\[ \Delta r \Delta p_r \approx \frac{1}{2} \left( 1 + \beta \ell_p^2 \Delta p_r^2 \right), \]  
(20)

where \( \ell_p \) is the Planck length. In principle, the deformation parameter \( \beta \) is not fixed by the theory, although it is generally assumed to be of order unity in some models of string theory\(^{[15–18]}\). We also mention that many studies have recently attempted to constrain \( \beta \) by means of different quantum mechanical or field theoretical approaches\(^{[21–34]}\) (see Ref.\(^{[35]}\) for a recent review). Concerning the sign of \( \beta \), it is easy to show that Eq. (20) leads to a minimal position uncertainty of the order of Planck length for \( \beta > 0 \), while no threshold is predicted for \( \beta < 0 \)\(^{[36, 37]}\).

Now, as displayed in Eq. (18), in the gravitational field of a Schwarzschild black hole, the limit on the simultaneous measurement of the position and momentum of a quantum system is lowered with respect to Heisenberg bound. From Eqs. (19) and (20), it is evident that this feature is characteristic of GUP models with \( \beta < 0 \). Although most of studies and gedanken experiments on the GUP seem to support the \( \beta > 0 \) scenario, the possibility of having negative \( \beta \) has not yet been fully ruled out. For instance, in Ref.\(^{[38]}\) it has been shown that the GUP with \( \beta < 0 \) would lead to a description of the Universe with an underlying crystal lattice-like structure. Further arguments have been proposed in Refs.\(^{[34]}\) and\(^{[39]}\), where the \( \beta < 0 \) framework has proved to be consistent with the corpuscular picture of gravity\(^{[40–42]}\) and the phenomenologically observed Chandrasekhar limit for white dwarfs, respectively. Notice also that our result is in line with Ref.\(^{[43]}\), where it has been argued that a spacetime metric is able to reproduce the GUP-deformed Hawking temperature, provided that the parameter \( \beta \) is assumed to be negative. Remarkably, in all these contexts it has been highlighted that, if \( \beta < 0 \), there exists a maximum \( \Delta p \) such that \( \Delta r \Delta p_r \approx 0 \) as one approaches Planck scale. Contrary to common belief, it thus follows that physics in high energy regime would look classical again, similarly as in 't Hooft’s approach to deterministic quantum mechanics\(^{[44, 45]}\) and in deformed special relativity\(^{[46]}\). From this perspective, the result (18) is easily explained: since the higher the energy scale, the more classical the behavior of gravity and vice-versa, effects of quantum fluctuations are expected to become increasingly

\(^4\) Notice that, if we considered the generalized UR (14) in the most common inequality form, then Eq. (18) itself can be cast as an inequality by using Jensen’s relation \( f(\hat{r}) \leq \langle f(\hat{r}) \rangle \) for the convex function \( f(r) = e^{-2\alpha(r)} \).
negligible the closer we get to the black hole horizon. In turns, this results into a trivialization of the uncertainty relation at Schwarzschild radius.

Now, by exploiting the above formalism, one might reverse the problem and look for the metric that exactly fits the GUP (20). We expect that the most suitable candidate for this rôle is a solution of the would-be quantum reformulation of GR, which must include among its distinctive features the emergence of a discrete structure of the spacetime at Planck scale. In the absence of a consistent theory of quantum gravity, we then resort to an effective semiclassical description. In particular, by following the approach of Ref. [47, 48], we regard the spacetime as a classical background consistent with the GUP (20) can be described as a perturbed Minkowski metric, with a correction where we have reabsorbed the numerical factor in the definition of \( \beta \).

By retracing the same steps as in Eq. (18), we are led to

\[
\langle \hat{g} \rangle_{11} \Delta r \Delta p_r \approx \left( 1 + \langle \hat{h} \rangle_{11} \right) \Delta r \Delta p_r \approx \frac{1}{2},
\]

which implies

\[
\Delta r \Delta p_r \approx \frac{1}{2} \left( 1 + \langle \hat{h} \rangle_{11} \right)^{-1} \approx \frac{1}{2} \left( 1 - \langle \hat{h} \rangle_{11} \right),
\]

to the leading order in the perturbation. By requiring consistency between Eqs. (20) and (23), we can infer the relation

\[
\langle \hat{h} \rangle_{11} \approx -\beta \ell_p^2 \Delta r^2 = -\beta \ell_p^2 \left( \langle \hat{p}_r \rangle - \langle \hat{p}_r \rangle^2 \right).
\]

Clearly, to get positive spacetime fluctuations as discussed in [49], we have to set \( \beta < 0 \).

Now, to the leading order in the deformation parameter \( \beta \), we can safely use the standard HUP and approximate \( \Delta p_r \approx 1/(2\Delta r) \), thus yielding

\[
\langle \hat{h} \rangle_{11} \approx -\beta \left( \frac{\ell_p}{\Delta r} \right)^2,
\]

where we have reabsorbed the numerical factor in the definition of \( \beta \). Therefore, based on our approach, the semiclassical background consistent with the GUP (20) can be described as a perturbed Minkowski metric, with a correction given by Eq. (25). For small perturbations, we can still require that \( g_{00} \approx -g^{-1}_{11} \approx -1 + \langle \hat{h} \rangle_{11} = -1 - \beta \ell_p^2 / \Delta r^2 \).

As expected, deviations from Minkowski background become relevant only at Planck scale, beyond which the form (21) for the perturbed metric ceases to be valid due to the singularity \( \Delta r = \sqrt{\beta} \ell_p \). We also emphasize that an alternative possibility to take into account spacetime discreteness has been proposed in the context of conformally-quantized gravity by considering fluctuations of the conformal factor only and quantizing them [50]. For an almost comprehensive review of various approaches to the description of spacetime fluctuations, see Ref. [51].

### B. Perturbed weak-field Schwarzschild spacetime

Let us now consider the effects of a nonminimally coupled (NMC) model of gravity on a perturbed Minkowski metric. For the description of this model, we basically follow Ref. [52]. The metric we use is the one associated with an asymptotically flat spacetime around a spherical object of mass \( M \), radius \( r_s \) and static radial mass density \( \rho(r) \). In spherical coordinates and in the weak-field limit, it is given by the following perturbation of Minkowski metric

\[
g_{\mu\nu} = \text{diag} \left( -[1 + 2\Psi(r)], 1 + 2\Phi(r), r^2, r^2 \sin^2 \theta \right),
\]

where \( \Psi \) and \( \Phi \) are the perturbing functions such that \( |\Psi(r)| \ll 1 \) and \( |\Phi(r)| \ll 1 \). Following Ref. [52] and solving the linearized field equations, it is possible to show that, outside the spherical body \( r > r_s \), these functions are given by

\[
\Psi(r) = -\frac{M}{r} \left[ 1 + \left( \frac{1}{3} - 4\xi \right) A(m, r_s) e^{-mr} \right],
\]

where we have reabsorbed the numerical factor in the definition of \( \beta \). Therefore, based on our approach, the semiclassical background consistent with the GUP (20) can be described as a perturbed Minkowski metric, with a correction given by Eq. (25). For small perturbations, we can still require that \( g_{00} \approx -g^{-1}_{11} \approx -1 + \langle \hat{h} \rangle_{11} = -1 - \beta \ell_p^2 / \Delta r^2 \).

As expected, deviations from Minkowski background become relevant only at Planck scale, beyond which the form (21) for the perturbed metric ceases to be valid due to the singularity \( \Delta r = \sqrt{\beta} \ell_p \). We also emphasize that an alternative possibility to take into account spacetime discreteness has been proposed in the context of conformally-quantized gravity by considering fluctuations of the conformal factor only and quantizing them [50]. For an almost comprehensive review of various approaches to the description of spacetime fluctuations, see Ref. [51].
FIG. 1: Uncertainty relation for Minkowski (red solid line), Schwarzschild (blue dotted line) and perturbed weak-field Schwarzschild (green dashed line) metrics. For each metric, the allowed region is that above the corresponding curve. For the perturbed weak-field Schwarzschild plot, we have set the sampling values $M = 1$, $\xi = 10^{-2}$, $m = 10^{-4}$ as in Ref. [52]. Notice that, far enough from the source, both the blue-dotted and green-dashed lines converge to the standard Heisenberg limit, as expected.

\[ \Phi(r) = \frac{M}{r} \left[ 1 - \left( \frac{1}{3} - 4\xi \right) A(m, r_s) e^{-mr(1 + mr)} \right], \quad (28) \]

where $m$ is a characteristic mass scale, $\xi$ a dimensionless parameter specific of the NMC model and $A(m, r_s)$ a form factor which can be found by integrating the field equations of NMC gravity. Note that, in the GR limit, $\xi = 0$ and $m \to \infty$, so that the exponential term in both the perturbing functions $\Psi$ and $\Phi$ vanishes and we recover the weak-field approximation of Schwarzschild metric, as expected.

As shown in Ref. [52], the expression of $A(m, r_s)$ strongly depends on the mass density $\rho(r)$. However, by considering a uniform density profile and assuming not to be too far from the gravity source, we can approximate $A(m, r_s) \approx 1$. In this setting, by resorting to Eq. (14), the modified uncertainty relation between $r$ and $p_r$ in the weak-field limit becomes

\[ \Delta r \Delta p_r \simeq \frac{1}{2} (1 + 2\Phi(\hat{r}))^{-1}. \quad (29) \]

From the above equation, we obtain

\[ \Delta r \Delta p_r \simeq \frac{1}{2} \left( 1 - 2\Phi(\hat{r}) \right) \]

\[ = \frac{1}{2} \left\{ 1 - 2 \frac{M}{\hat{r}} \left[ 1 - \left( \frac{1}{3} - 4\xi \right) e^{-m(\hat{r})(1 + m(\hat{r}))} \right] \right\}, \quad (30) \]

where in the first step we have expanded around $|\Phi(r)| \ll 1$. As a result, one can see that the quantum gravitational threshold on the simultaneous measurement of $r$ and $p_r$ is slightly increased with respect to the bound in Eq. (18) (see Fig. 1). However, the latter is recovered for $\xi = 0$ and $m \to \infty$, as discussed above. We also point out that both bounds in Eqs. (18) and (30) converge to the standard Heisenberg limit far enough from the source.

In passing, we mention that a similar connection between the deformed Schwarzschild metric and modified uncertainty relations has been investigated in Ref. [53] within the framework of non-commutative geometry [54–56]. In that case, the authors relate the deformation parameter of non-commutative geometry to the $\beta$ coefficient appearing in the GUP via the computation of the modified Hawking temperature. Even in that context a negative value for $\beta$ is obtained, thus suggesting a granular structure of spacetime at the Planck scale.
C. de Sitter spacetime

As a further application of the above formalism, let us now consider the case of the four-dimensional de Sitter spacetime. It is well-known that the main use of de Sitter space in General Relativity is to describe a simple mathematical model of the Universe consistent with its observed accelerating expansion. More specifically, de Sitter metric represents the maximally symmetric solution of Einstein’s field equations in vacuum with a positive cosmological constant, corresponding to a positive vacuum energy density and negative pressure.

In static coordinates \((t, r, \theta, \varphi)\), de Sitter metric takes the form

\[
g_{\mu\nu} = \text{diag}(-e^{2\beta(r)}, e^{-2\beta(r)}, r^2, r^2 \sin^2 \theta),
\]

where

\[
e^{2\beta(r)} = 1 - \frac{r^2}{l_H^2}, \quad r < l_H.
\]

Here \(l_H\) is the de Sitter radius, which can be expressed in terms of the cosmological constant \(\Lambda\) as \(l_H^2 = 1/\Lambda\), with \(\Lambda > 0\).

The computation of the uncertainty relations for the metric (31) closely follows the one carried out for Schwarzschild spacetime. Indeed, with the particular choice of coordinates we have adopted, the metric tensors (31) and (15) can be mapped into each other by making the substitution \(e^{2\omega(r)} \rightarrow e^{2\beta(r)}\). In this way, we obtain

\[
|\Delta e^\mu (\hat{g})_{\mu\nu} \Delta p^\nu| = | - e^{2\beta(r)} \Delta r \Delta E + e^{-2\beta(r)} \Delta r \Delta p_r + \langle \hat{r} \rangle^2 \Delta \theta \Delta p_\theta + \langle \hat{r} \sin \hat{\theta} \rangle^2 \Delta \varphi \Delta p_\varphi| \simeq 1.
\]

Again, we focus on the uncertainty relation between \(r\) and \(p_r\), which can be now rearranged as

\[
\Delta r \Delta p_r \simeq \frac{1}{2} \left( 1 - \frac{\langle \hat{r} \rangle^2}{l_H^2} \right).
\]

It is interesting to observe that this equation is comparable with the well-known deformation of HUP arising at cosmic scale, here rewritten as in [57]

\[
\Delta r \Delta p_r \simeq \frac{1}{2} \left[ 1 + \frac{\gamma}{2} \left( \frac{\Delta r}{l_H} \right)^2 \right],
\]

provided that one assumes \(\langle \hat{r} \rangle \simeq \Delta r\), which is quite reasonable at cosmological-scale distances.

The generalization (35) is usually referred to as Extended Uncertainty Principle. In this case the rôle of the deformation parameter is played by the \(\gamma\) coefficient. Of course, the matching between Eqs. (34) and (35) is met for \(\gamma < 0\). This finds confirmation in the analysis of Refs. [4] and more recently of [57], where the value \(\gamma = -1/\pi^2 < 0\) has been obtained by requiring consistency between deformations of HUP at cosmic scales in de Sitter space and predictions by Modified Newtonian dynamics (MoND) theories. In this regard, we emphasize that, while for \(\gamma > 0\) Eq. (35) implies the existence of a minimal momentum \(p_{min} \sim \sqrt{\gamma}/l_H\), the \(\gamma < 0\) framework is characterized only by a maximum length given of course by the radius \(l_H\) of the cosmological horizon [4]. Clearly, for \(\gamma = 0\) and/or \(\Delta r/l_H \ll 1\), the standard HUP is recovered.

Phenomenological implications of the EUP have been considered in a variety of contexts, ranging from black hole physics [58–62], to the thermodynamics of the FRW universe [59] and Unruh effect [62].

On the other hand, one can repeat the same calculation as above by considering anti-de Sitter spacetime as background metric. As opposed to de-Sitter, this metric describes a Universe with a negative cosmological constant corresponding to a slowed down expansion. In this case, it is straightforward to see that the opposite condition \(\gamma > 0\) is obtained.

Let us finally observe that the results obtained for the Schwarzschild and de Sitter spacetime can be merged by looking at the structure of the uncertainty principle in de Sitter-Schwarzschild metric. This metric describes a spherically symmetric solution with a positive cosmological constant, providing a spacetime background with both event and cosmological horizons. In this framework, the metric tensor can be written as

\[
g_{\mu\nu} = \text{diag}(-e^{2\omega(r)}, e^{-2\omega(r)}, r^2, r^2 \sin^2 \theta),
\]

with

\[
e^{2\omega(r)} = 1 - \frac{r_s}{r^2} + \frac{r_s^2}{l_H^2}, \quad r_s < r < l_H.
\]

Following the same considerations as above, one can see that, far enough from both the Schwarzschild and cosmological horizon radii, the ensuing HUP resembles the so-called Generalized Extended Uncertainty Principle (GEUP) [5, 20], which combines together the effects of the GUP and EUP models.
IV. CONCLUSIONS AND OUTLOOK

The generalizations of the Heisenberg Uncertainty Principle deduced in the last decades from quantum gravity and cosmological models have been mimicked starting from well-known metric solutions of General Relativity and beyond. This result has been achieved by extending the HUP on the basis of a semiclassical geometric approach. The cases of Schwarzschild and de Sitter spacetime have been studied in detail, showing that they lead to deformations of the uncertainty relation which are formally similar to the Generalized and Extended Uncertainty Principles, respectively. Furthermore, by requiring consistency with the most common form of GUP predicted by the string theory, we have inferred the would-be perturbed Minkowski metric that accounts for the emergence of a discrete spacetime at Planck scale. In this framework, we have argued that the characteristic deformation parameter must be negative. This is in line with the result commonly found in the literature, i.e. that a negative $\beta$ typically arises in non-trivial space-time having an emergent reticular nature [36, 37]. A similar argument has also been exhibited in Ref. [38] by showing that the GUP with $\beta < 0$ would be consistent with a description of the Universe with an underlying crystal lattice-like structure.

Let us emphasize that semiclassical attempts to establish a connection between geometric properties of spacetime and uncertainty relations are not completely novel in the literature. For instance, a derivation of GUP from Quantum Geometry has been proposed in Ref. [13] in Caianiello’s theory of maximal acceleration. In that case, gravitational effects are directly implemented through a generalization of the canonical commutator between position and momentum operators, in such a way that the quadratic GUP is recovered in the framework of Quantum Geometry theory. A careful investigation of the relation between our extension and that proposed in [13] deserves more attention.

On the other hand, in Ref. [9] the influence of space-time geometry on the uncertainty relation has been investigated by assuming the quantum wave function of the system to be confined to a geodesic ball of radius proportional to the position uncertainty and defining a hermitian momentum operator that complies with the canonical commutation relations in the non-relativistic limit of the $3 + 1$ formalism. Computations have been developed for some metrics arising in the context of both General Relativity and extended theories of gravity, showing that there might be a direct link between deformations of the Heisenberg Uncertainty Principle and the curvature in energy-momentum space. In light of this result, it would be interesting to see how such a formalism interfaces with our geometric generalization of HUP. Another challenging perspective would be the investigation of our formalism in the context of Banados-Teitelboim-Zanelli black holes in light of the outcome of Ref. [63].

Besides the above issues, some other aspects remain to be addressed. Indeed, the full understanding of how to deal with deformed uncertainty relations in the relativistic framework provides an essential element to explore the phenomenological implications of a minimal/maximal length scale in Quantum Field Theory, where a systematic treatment of the problem is still missing. In this sense, the present analysis should be intended as a further step toward this goal. Work along this and other directions is presently under active investigation and will be elaborated elsewhere.

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