PATHWISE INTEGRALS AND ITÔ–TANAKA FORMULA FOR GAUSSIAN PROCESSES

TOMMI SOTTINEN AND LAURI VIITASAARI

Abstract. We prove the Itô–Tanaka formula and the existence of pathwise stochastic integrals for a wide class of Gaussian processes. Motivated by financial applications, we define the stochastic integrals as forward-type pathwise integrals introduced by Föllmer and as pathwise generalized Lebesgue–Stieltjes integrals introduced by Zähle. As an application, we illustrate the importance of Itô–Tanaka formula for pricing and hedging of financial derivatives.

1. Introduction

Let $Y$ be a continuous Gaussian stochastic process on the time interval $[0,T]$ and let $f$ be a linear combination of convex functions. We are interested in which generality and for what kind of integrals and notion of local time $L_T^a$ we can obtain the Itô–Tanaka formula

$$f(Y_T) = f(Y_0) + \int_0^T f'(Y_u) \, dY_u + \frac{1}{2} \int_{-\infty}^{\infty} L_T^a(Y) \, f''(da).$$

Since we do not assume that $Y$ is a semimartingale, standard stochastic integration theory cannot be applied here and we have to determine in which sense the stochastic integral in (1.1) exists.

Motivated by financial applications we consider pathwise integrals that are generalizations of the financially natural Riemann–Stieltjes integral. These generalizations go back to Young [16]. In particular, we consider the pathwise forward-type Riemann–Stieltjes integral introduced by Föllmer [9] and the pathwise generalized Lebesgue–Stieltjes integrals introduce by Zähle [17] and further studied e.g. by Nualart and Răşcanu [11].

Let us note that for Gaussian processes one can also consider Skorokhod integrals and e.g. in the particular case of fractional Brownian motion the Itô–Tanaka formula (1.1) is established in [8]. However, the Skorokhod integrals do not admit an economical interpretation in any obvious way; see [15] for details.

Our work is related to [1, 2, 3, 11], where only the case of fractional Brownian motion was studied. We extend the results to a more general class of Gaussian processes. Bertoin [7] also established the Itô–Tanaka formula (1.1) in the Föllmer sense for a very general class of Dirichlet process. In that

2010 Mathematics Subject Classification. 60G15; 60H05; 91G20.

Key words and phrases. Föllmer integral; Gaussian processes; generalized Lebesgue–Stieltjes integral; Itô–Tanaka formula; mathematical finance; pathwise stochastic integral.

L. Viitasaari was financed by the Finnish Doctoral Programme in Stochastics and Statistics.
sense our work is merely a special case of [7]. However, in [7] it was a priori assumed that the local time \( L_T(Y) \) exists as the Lebesgue density of the occupation measure in the \( L^2 \) sense. This actually also implies the existence of the associated Föllmer integral. We do not assume the existence of the density or the integral a priori. Also, the generalized Lebesgue–Stieltjes integrals were not studied in [7].

The rest of the paper is organized as follows. In section 2 we introduce the generalized Lebesgue–Stieltjes integrals and the Föllmer integrals. The main section 3 begins with introducing our assumptions with discussion and examples. Then we prove several fundamental lemmas after which we state and prove our main results on the existence of the generalized Lebesgue–Stieltjes integrals, the Föllmer integrals and mixed Föllmer–generalized Lebesgue–Stieltjes integrals. Then we prove the Itô–Tanaka formula. In Section 4 we discuss the importance of the Itô–Tanaka formula for the hedging and pricing of financial derivatives. Finally, a technical lemma on the level-crossing probabilities of Gaussian processes is given in the appendix.

2. Pathwise Integrals

We recall two notions of pathwise stochastic integrals: the generalized Lebesgue–Stieltjes integral and the Föllmer integral.

**Generalized Lebesgue–Stieltjes Integral.** The generalized Lebesgue–Stieltjes integral is based on fractional integration and fractional Besov spaces. For details on fractional integration we refer to [13] and for fractional Besov spaces we refer to [11].

We first recall the definitions for fractional Besov norms and Lebesgue–Liouville fractional integrals and derivatives.

**Definition 2.1.** Fix \( 0 < \beta < 1 \).

(i) The fractional Besov space \( W^\beta_1 = W^\beta_1([0,T]) \) is the space of real-valued measurable functions \( f : [0,T] \to \mathbb{R} \) such that

\[
\|f\|_{1,\beta} = \sup_{0 \leq s < t \leq T} \left( \frac{|f(t) - f(s)|}{(t-s)^\beta} + \int_s^t \frac{|f(u) - f(s)|}{(u-s)^{1+\beta}} \, du \right) < \infty.
\]

(ii) The fractional Besov space \( W^\beta_2 = W^\beta_2([0,T]) \) is the space of real-valued measurable functions \( f : [0,T] \to \mathbb{R} \) such that

\[
\|f\|_{2,\beta} = \int_0^T \frac{|f(s)|}{s^\beta} \, ds + \int_0^T \int_0^s \frac{|f(u) - f(s)|}{(u-s)^{1+\beta}} \, duds < \infty.
\]

**Remark 2.1.** Let \( C^\alpha = C^\alpha([0,T]) \) denote the space of Hölder continuous functions of order \( \alpha \) on \([0,T]\) and let \( 0 < \epsilon < \beta \wedge (1-\beta) \). Then

\[ C^{\beta+\epsilon} \subset W^\beta_1 \subset C^{\beta-\epsilon} \quad \text{and} \quad C^{\beta+\epsilon} \subset W^\beta_2. \]
Lemma 2.1. Let

\[ \mathcal{I}_0 \]

\[ \mathcal{I}_1 \]

where \( \Gamma \) is the Gamma-function. The Riemann–Liouville fractional derivatives \( D_{0+}^\beta \) and \( D_{1-}^\beta \) are the left-inverses of the corresponding integrals \( I_{0+}^\beta \) and \( I_{1-}^\beta \). They can be also define via the Weyl representation as

\[ (D_{0+}^\beta f)(s) = \frac{1}{\Gamma(1-\beta)} \left( f(s) s^\beta + \beta \int_0^s \frac{f(u) - f(u)}{(s-u)^{\beta+1}} \, du \right), \]

\[ (D_{1-}^\beta f)(s) = \frac{(-1)^{-\beta}}{\Gamma(1-\beta)} \left( f(s) (t-s)^\beta + \beta \int_s^t f(u) - f(u) \frac{(u-s)^{\beta+1}}{\beta+1} \, du \right) \]

if \( f \in I_{0+}^\beta(L^1) \) or \( f \in I_{1-}^\beta(L^1) \), respectively.

Denote \( g_{1-}(s) = g(s) - g(t-) \).

The generalized Lebesgue–Stieltjes integral is defined in terms of fractional derivative operators according to the next proposition.

Proposition 2.1. [11] Let \( 0 < \beta < 1 \) and let \( f \in W_2^\beta \) and \( g \in W_1^{1-\beta} \). Then for any \( t \in (0, T] \) the generalized Lebesgue–Stieltjes integral exists as the following Lebesgue integral

\[ \int_0^t f(s) \, dg(s) = \int_0^t (D_{0+}^\beta f)(s)(D_{1-}^{1-\beta} g_{1-})(s) \, ds \]

and is independent of \( \beta \).

Remark 2.2. It is shown in [17] that if \( f \in C^\gamma \) and \( g \in C^\eta \) with \( \gamma + \eta > 1 \), then the generalized Lebesgue–Stieltjes integral \( \int_0^t f(s) \, dg(s) \) exists and coincides with the classical Riemann–Stieltjes integral, i.e., as a limit of Riemann–Stieltjes sums. This is natural, since in this case one can also define the integrals as Young integrals [16].

We will also need the following estimate in order to prove our main theorems.

Theorem 2.1. [11] Let \( f \in W_2^\beta \) and \( g \in W_1^{1-\beta} \). Then we have the estimate

\[ \left| \int_0^t f(s) \, dg(s) \right| \leq \sup_{0 \leq s < t \leq T} \left| D_{1-}^{1-\beta} g_{1-}(s) \right| \| f \|_{2, \beta}. \]

To conclude the section we also recall Garsia–Rademich–Rumsey inequality (see [11] and [10]).

Lemma 2.1. Let \( p \geq 1 \) and \( \alpha > \frac{1}{p} \). Then there exists a constant \( C = C(\alpha, p) > 0 \) such that for any continuous function \( f \) on \([0, T] \), and for all \( 0 \leq s, t \leq T \) we have

\[ |f(t) - f(s)|^p \leq CT^{\alpha p - 1}|t - s|^\alpha p - 1 \int_0^T \int_0^T \frac{|f(x) - f(y)|^p}{|x - y|^{\alpha p + 1}} \, dx \, dy. \]
Föllmer integral. We recall the definition of a forward-type Riemann–Stieltjes integral due to Föllmer [9] (see [14] for English translation) and discuss its connection to the generalized Lebesgue–Stieltjes integral of Proposition 2.1.

Definition 2.3. Let \((\pi_n)_{n=1}^\infty\) be a sequence of partitions \(\pi_n = \{0 = t_0^n < \ldots < t_{k(n)}^n = T\}\) such that \(|\pi_n| = \max_{j=1,\ldots,k(n)} |t_j^n - t_{j-1}^n| \to 0\) as \(n \to \infty\). Let \(X\) be a continuous process. The Föllmer integral along the sequence \((\pi_n)_{n=1}^\infty\) of \(Y\) with respect to \(X\) is defined as

\[
\int_0^t Y_u \, dX_u = \lim_{n \to \infty} \sum_{t_j^n \in \pi_n \cap (0,t]} Y_{t_j^n} (X_{t_j^n} - X_{t_{j-1}^n}),
\]

if the limit exists almost surely.

Remark 2.3. (i) The Föllmer integral is a forward-type Riemann–Stieltjes integral. Thus, if the Riemann–Stieltjes integral exists, so does the Föllmer integral, but not necessarily vice versa. Also, it is clear that the Föllmer integral is a pathwise generalization of the Itô integral, if one takes the sequence of partitions \((\pi_n)_{n=1}^\infty\) to be refining.

(ii) The Föllmer integral is the natural notion of integration in mathematical finance. Indeed, the budget constraint of a self-financing trading strategy translates in the limit as a Föllmer integral. See [4, 5] for further discussion.

In general, it is very difficult to prove the existence of Föllmer integral. In the case of so-called quadratic variation processes the existence is guaranteed by the Itô–Föllmer formula of Lemma 2.2 below, which shows that the Föllmer integral behaves like the Itô integral in the case of integrators with quadratic variation.

Definition 2.4. Let \((\pi_n)_{n=1}^\infty\) be a sequence of partitions \(\pi_n = \{0 = t_0^n < \ldots < t_{k(n)}^n = T\}\) such that \(|\pi_n| = \max_{j=1,\ldots,k(n)} |t_j^n - t_{j-1}^n| \to 0\) as \(n \to \infty\). Let \(X\) be a continuous process. Then \(X\) is a quadratic variation process along the sequence \((\pi_n)_{n=1}^\infty\) if the limit

\[
\langle X \rangle_t = \lim_{n \to \infty} \sum_{t_j^n \in \pi_n \cap (0,t]} (X_{t_j^n} - X_{t_{j-1}^n})^2
\]

exists almost surely.

Remark 2.4. (i) For standard Brownian motion \(W\) we have \(d\langle W \rangle_t = dt\) if the sequence \((\pi_n)\) is refining. This follows from the Borel-Cantelli lemma.

(ii) For continuous martingales \(M\) their bracket \(\langle M \rangle\) is also their quadratic variation (in the pathwise, Föllmer, sense) for suitably chosen sequences \((\pi_n)\).

(iii) If \(Z\) is a continuous process with zero quadratic variation along \((\pi_n)\) and \(X\) is a continuous quadratic variation process along \((\pi_n)\) then \(d\langle X + Z \rangle_t = d\langle X \rangle_t\). This follows from the Cauchy-Schwartz inequality.
(iv) If $X$ is a quadratic variation process along $(\pi_n)$ and $f \in C^1$ then $Y = f \circ X$ is also a quadratic variation process along $(\pi_n)$. Indeed,
$$d\langle Y \rangle_t = f'(X_t) \, d\langle X \rangle_t.$$  

**Lemma 2.2.** Let $X$ be a continuous quadratic variation process and let $f \in C^{1,2}([0, T] \times \mathbb{R})$. Let $0 \leq s < t \leq T$. Then
$$f(t, X_t) = f(s, X_s) + \int_s^t \frac{\partial f}{\partial t}(u, X_u) \, du + \int_s^t \frac{\partial f}{\partial x}(u, X_u) \, dB_u$$
$$+ \frac{1}{2} \int_s^t \frac{\partial^2 f}{\partial x^2}(u, X_u) \, d\langle X \rangle_u.$$  

In particular, the Föllmer integral exists and has a continuous modification.

### 3. Main Results

#### Notations, Definitions and Auxiliary Results.

**Definition 3.1.** Let $X$ be a centered Gaussian process. We denote by $R(t, s)$, $W(t, s)$, and $V(t)$ its covariance, incremental variance and variance, i.e.
$$R(t, s) = \mathbb{E}[X_t X_s],$$
$$W(t, s) = \mathbb{E}[(X_t - X_s)^2],$$
$$V(t) = \mathbb{E}[X_t^2].$$

We denote by $w^*(t)$ the "worst case" incremental variance
$$w^*(t) = \sup_{0 \leq s \leq t} W(t + s, s).$$

We begin with the following technical Lemma.

**Lemma 3.1.** Let $R$ be a covariance of a centered process with $R(s, t) > 0$. Let $0 \leq s \leq t \leq T$.

(i) If $R(s, s) \leq R(s, t)$, then
$$1 - \frac{R(s, s)}{R(t, s)} \leq \frac{\sqrt{W(t, s)}}{\sqrt{V(s)}},$$

(ii) if $R(s, s) > R(s, t)$, then
$$\frac{R(s, s)}{R(t, s)} - 1 \leq \frac{\sqrt{W(t, s)} \, R(s, s)}{\sqrt{V(s)} \, R(t, s)}.$$  

**Proof.** For the claim (i), note that we have always $R(t, s)^2 \leq R(t, t)R(s, s)$. Hence
$$\frac{R(t, s)^2}{R(s, s)} + R(s, s) - 2R(t, s) \leq R(t, t) + R(s, s) - 2R(t, s).$$

Clearly we have
$$\frac{R(t, s)^2}{R(s, s)} + R(s, s) - 2R(t, s) = \left( \frac{R(t, s)}{\sqrt{R(s, s)}} - \sqrt{R(s, s)} \right)^2.$$
and 
\[ R(t, t) + R(s, s) - 2R(t, s) = W(t, s). \]
Hence by taking square root on both sides of (3.1) we obtain 
\[ \frac{R(t, s)}{\sqrt{R(s, s)}} - \sqrt{R(s, s)} \leq \sqrt{W(t, s)}. \]

It remains to note that 
\[ 1 - \frac{R(s, s)}{R(t, s)} \leq \frac{R(t, s)}{R(s, s)} - 1. \]

For the claim (ii), arguing as in the the proof of the claim (i) above, we obtain that 
\[ 1 - \frac{R(s, s)}{R(t, s)} \leq \sqrt{W(t, s)} \sqrt{V(s)}. \]
Multiplying by \( \frac{R(s, s)}{R(t, s)} \) gives the result. \( \square \)

The key lemma to our analysis is the following estimate for the probability that the process \( X \) crosses a fixed level: \( \mathbb{P}(X_s < a < X_t) \). Depending on the values of \( t, s \) and \( a \) one can obtain different estimates and detailed bounds that are given in Lemma A.1 in the appendix. For our purposes we need the following universal estimate which holds for every value \( s, t \) and \( a \).

**Lemma 3.2.** Let \( X \) be a centered Gaussian process with strictly positive and bounded covariance function \( R, 0 < s < t \leq T \) and \( a \in \mathbb{R} \). Then there exists a universal constant \( C \) such that
\[
\mathbb{P}(X_s < a < X_t) \leq C \sqrt{W(t, s)} \sqrt{V(s)} \left[ 1 + \frac{R(s, s)}{R(t, s)} + \frac{|a| e^{-\frac{2}{\sigma^2}}}{\sqrt{V(s)}} \max \left( 1, \frac{R(s, s)}{R(t, s)} \right) \right],
\]
where
\[ V^* = \sup_{s \leq T} V(s). \]

**Proof.** The claim follows directly by applying Lemma 3.1 and the inequality \( \sigma^2 \leq W(t, s) \) on the terms in Lemma A.1. \( \square \)

Recall that a process \( X = (X_t)_{t \in [0, T]} \) is **Hölder continuous of order** \( \alpha \) if there exists almost surely finite random variable \( C_T \) such that
\[ |X_t - X_s| \leq C_T |t - s|^\alpha \]
almost surely for all \( s, t \in [0, T] \).

**Definition 3.2.** A centered continuous Gaussian process \( X = (X_t)_{t \in [0, T]} \) with covariance \( R \) belongs to the class \( X^\alpha \) if
- (i) \( R(s, t) > 0 \) for every \( s, t > 0 \),
- (ii) the "worst case" incremental variance satisfies, at \( t = 0 \),
\[ w^*(t) = C t^{2\alpha} + o(t^{2\alpha}), \]
where \( C > 0 \) and \( 0 < \alpha < 1 \).
Definition 3.2 of the class $\mathcal{X}^\alpha$ is rather technical. However, the next remarks and examples should convince the reader that the assumptions are relatively natural and that the class $\mathcal{X}^\alpha$ quite large.

Remark 3.1. (i) The first condition on strictly positive autocorrelation $R(t, s) > 0$ is rather restrictive. However, it seems that most Gaussian models do indeed satisfy it. Also, if one uses the generalized Lebesgue–Stieltjes approach, one has to assume $R(s, t) > 0$. Indeed, otherwise our integrands would not belong to the required fractional Besov spaces.

(ii) The second condition on the “worst case” incremental variance is the most important assumption: it implies that $X$ has a version which is Hölder continuous of order $r$ for any $r < \alpha$ on $[0, T]$. Indeed, now

$$\mathbb{E}[(X_t - X_s)^p] \leq C_p^p|t - s|^{p\alpha}$$

for every $p \geq 1$. Hence the Hölder continuity follows directly from the Kolmogorov continuity theorem.

(iii) The third condition implies that the process is not too smooth. Indeed, if the variance $V(s)$ behaves like $s^\gamma$ for some $\gamma \geq 2$ near zero, we obtain that the process is differentiable in the mean square sense. As a consequence we could apply standard integration techniques for such cases.

(iv) Finally, the fourth assumption is quite mild as for it we simply need that when $s$ and $t$ are both close to each other and at the same time near to zero, the variance $V(s)$ is not “too far” from the covariance $R(s, t)$. The fourth condition is also connected to the notion of local non-determinism introduced by Berman [6] in connection to the existence of local time as an occupation density.

Example 3.1. Let $0 \leq s \leq t \leq T$.

(i) For processes with stationary increments

$$R(t, s) = \frac{1}{2}[V(t) + V(s) - V(t - s)],$$

$$W(t, s) = V(t - s),$$

$$w^*(t) = V(t).$$

Consequently, a process with stationary increments belongs to $\mathcal{X}^\alpha$ if and only if $V(t) > 0$ for all $t > 0$ and $V(t) = Ct^{2\alpha} + o(t^{2\alpha})$ at $t = 0$.

In particular, the fractional Brownian motion with index $\alpha \in (0, 1)$ belongs to the space $\mathcal{X}^\alpha$. 
(ii) For stationary processes

\[ R(t, s) = r(t - s), \]
\[ W(t, s) = 2[r(0) - r(t - s)], \]
\[ V(t) = r(0), \]
\[ w^*(t) = 2[r(0) - r(t)]. \]

Consequently, a stationary process belongs to the class \( X^\alpha \) if and only if \( r(t) > 0 \) for all \( t \) and at \( t = 0 \) we have

\[ r(0) - r(t) = C t^{2\alpha} + o(t^{2\alpha}). \]

In particular, the fractional Ornstein–Uhlenbeck process \( e^\alpha \) with index \( \alpha \in (0, 1) \) belongs to the space \( X^\alpha \).

**Existence of Generalized Lebesgue–Stieltjes Integral.** We consider existence of pathwise integrals of the type

\[ \int_0^T f'-(X_u) \, dY_u, \tag{3.2} \]

where \( X \) and \( Y \) are Gaussian processes and \( f'-(X) \) is left derivative of a convex function. We begin by showing that the integral can be understood in generalised Lebesgue–Stieltjes sense, and then we prove that it can also be understood as a forward type integral in the sense of Föllmer. The reason for considering a mixed type integral with two processes \( X \) and \( Y \) is that we want to later consider mixed processes \( Y = M + X \), where \( M \) is a Gaussian martingale with Lipschitz-variance and \( X \) is a Gaussian process from the class \( X^\alpha \).

**Theorem 3.1.** Let \( X \in X^\alpha \). If \( \alpha > \beta \), then for every convex function \( f \) we have \( f'-(X) \in W^\beta \) almost surely.

**Proof.** Let us first note that it is sufficient to consider convex functions of the form \( f(x) = (x - a)^+ \), \( a \in \mathbb{R} \). Indeed, assume for a moment that the result is valid for these particular convex functions and let \( f \) be a general convex function for which the measure \( f'' \) has compact support. Then

\[ |f'-(X_t) - f'-(X_s)| \leq \int_{\text{supp}(f'')} 1_{X_t < a < X_s} + 1_{X_s < a < X_t} f''(da). \tag{3.3} \]

Now, by applying Tonelli’s theorem, and the result for the functions \( (x-a)^+ \), \( a \in \mathbb{R} \), we obtain the result for the convex functions \( f \) with \( \text{supp}(f'') \) compact. Finally, the general case follows by approximating a convex function \( f \) with a sequence \( f_n \) of convex functions for which \( f'' \) has compact support (see [2] or [3] for details).

Consider then the functions \( f(x) = (x - a)^+ \), \( a \in \mathbb{R} \). Now,

\[ |f'-(X_t) - f'-(X_s)| = 1_{X_t < a < X_s} + 1_{X_s < a < X_t}. \]

For the first term in the fractional Besov norm we obtain

\[ \int_0^T \frac{1_{X_t > a}}{t^\beta} \, dt \leq \int_0^T \frac{1}{t^\beta} \, dt < \infty. \]
It follows that we only have to prove that
\[ I = \int_0^T \int_0^t \frac{1_{X_s < a < X_t} + 1_{X_t < a < X_s}}{(t-s)^{\beta+1}} \, ds \, dt < \infty. \]

Now, the iterated integral above can be split as
\[ I = \int_0^T \int_0^t \frac{1_{X_s < a < X_t} + 1_{X_t < a < X_s}}{(t-s)^{\beta+1}} \, ds \, dt \]
\[ + \int_0^T \int_{t-\delta}^{t-\delta} \frac{1_{X_s < a < X_t} + 1_{X_t < a < X_s}}{(t-s)^{\beta+1}} \, ds \, dt \]
\[ + \int_0^T \int_{t-\delta}^{t-\delta} \frac{1_{X_s < a < X_t} + 1_{X_t < a < X_s}}{(t-s)^{\beta+1}} \, ds \, dt \]
\[ + \int_0^T \int_0^t \frac{1_{X_s < a < X_t} + 1_{X_t < a < X_s}}{(t-s)^{\beta+1}} \, ds \, dt. \]

In the integrals \( I_3 \) and \( I_4 \) we can bound indicators with one and they are still finite. Hence, it is sufficient to consider the integrals \( I_1 \) and \( I_2 \). First, we note that by taking expectation and applying Tonelli’s theorem we have the result if
\[ (3.4) \quad \int_{2\delta}^T \int_{t-\delta}^t \frac{\mathbb{P}(X_t < a < X_s) + \mathbb{P}(X_t > a > X_s)}{(t-s)^{\beta+1}} \, ds \, dt < \infty, \]
and
\[ (3.5) \quad \int_0^{2\delta} \int_{t-\delta}^t \frac{\mathbb{P}(X_t < a < X_s) + \mathbb{P}(X_t > a > X_s)}{(t-s)^{\beta+1}} \, ds \, dt < \infty. \]

We begin with the term \((3.4)\). By symmetry we can only analyze the term \( \mathbb{P}(X_t > a > X_s) \). We have
\[ W(t, s) \leq w^*(t-s). \]

Now, since \( X \) is continuous on \([0,T]\), \( V^* < \infty \). Hence, by assumption (iii) of Definition 3.2 and Lemma 3.2 it is sufficient to consider integral of form
\[ \int_{2\delta}^T \int_{t-\delta}^t \frac{\sqrt{W(t, s)}}{(t-s)^{\beta+1}} \, ds \, dt. \]

Let now \( \delta \) be small enough such that, by assumption (ii) of Definition 3.2 we have
\[ w^*(t) \leq C t^{2\alpha} \]
for some constant \( C \). Since \( \alpha > \beta \) we obtain that \((3.4)\) holds almost surely. Consider next term in \((3.5)\). Note first that by applying Lemma 3.2 together with (iii) and (iv) of Definition 3.2 we obtain
\[ \int_0^{2\delta} \int_{t-\delta}^{t-\delta} \frac{(t-s)^\alpha}{s(t-s)^{\beta+1}} \, ds \, dt \leq C \int_0^{2\delta} t^{-1} t^{\alpha-\beta} \, dt < \infty. \]

Hence it is sufficient to study term of form
\[ e^{-\frac{2}{\pi \nu}} \frac{|a| \sqrt{W(t,s)}}{V(s)}. \]
The case $a = 0$ is trivial. Let now $a \neq 0$ and introduce time points $t_0 = 2\delta$,

$$t_k = 2\delta - 2\delta \sum_{j=1}^{k} \left( \frac{1}{2} \right)^j, \quad k \geq 1.$$ 

Now the integral can be split as

$$\int_0^{2\delta} \int_{t_k}^t \ldots dsdt = \sum_{k=0}^{\infty} \int_{t_{k+1}}^{t_k} \int_{t_k}^t \ldots dsdt.$$ 

Note next that assumptions (i) and (iii) of Definition 3.2 implies that either $\inf_{s \in [0,T]} V(s) > 0$ or else $X_0$ is a constant. In the first case the proof is trivial. On the latter case, since $X$ is centered, we have $X_0 = 0$ and in this case

$$\sup_{t_{k+1} \leq s \leq t_k} V(s) \leq C t_k^{2\alpha}.$$ 

Hence by applying Lemma 3.2 we obtain

$$\sum_{k=0}^{\infty} \int_{t_k}^{t_{k+1}} \int_{t_k}^t \frac{\mathbb{P}(X_t > a > X_s)}{(t-s)^{\beta+1}} dsdt \leq C |d| \sum_{k=0}^{\infty} \int_{t_k}^{t_{k+1}} \int_{t_k}^t \frac{e^{-\frac{a^2}{2Ck}} (t-s)^{\alpha}}{s^2(t-s)^{\beta+1}} dsdt$$

$$\leq C \sum_{k=0}^{\infty} \int_{t_k}^{t_{k+1}} e^{-\frac{a^2}{2Ck}} t^{-2} t^{\alpha-\beta} dt$$

$$\leq C \sum_{k=0}^{\infty} e^{-\frac{a^2}{2Ck}} t_k^{\alpha-\beta-1}$$

$$< \infty.$$ 

Hence the result is valid for convex function $f(x) = (x-a)^+$. Consider next more general convex function for which $f''$ has compact support. Using (3.3) we have

$$\int_{\text{supp}(f'')} 1_{X_t < a < X_s} + 1_{X_s < a < X_t} f''(da)$$

$$= \int_{\text{supp}(f'') : a \in [0,1]} 1_{X_t < a < X_s} + 1_{X_s < a < X_t} f''(da)$$

$$+ \int_{\text{supp}(f'') : a \in [-1,0]} 1_{X_t < a < X_s} + 1_{X_s < a < X_t} f''(da)$$

$$+ \int_{\text{supp}(f'') : |a| > 1} 1_{X_t < a < X_s} + 1_{X_s < a < X_t} f''(da)$$

$$=: J_1 + J_2 + J_3.$$ 

For $J_3$ we have lower bound for $a$ and since $\int_{\text{supp}(f'') : |a| > 1} f''(da) < C$ the result follows immediately from the result for functions $(x-a)^+$. Consider next the term $J_1$ and take a sequence of functions $f_n \in C^2$ such that

$$\int_{\mathbb{R}} g(x) f_n''(x) dx \to \int_{\mathbb{R}} g(x) f''(dx)$$

$\cdots$
for every function $g \in C^1$ with compact support (see [2]). Then noting that $3\alpha > \beta + 1$ we get
\[ \sum_{k=0}^{\infty} \int_{0}^{1} ae^{-\frac{a^2}{2C_k^2} t_k^{\alpha-\beta-1} f''(da)} = \sum_{k=0}^{\infty} \limsup_{n} \int_{0}^{1} ae^{-\frac{a^2}{2C_k^2} t_k^{\alpha-\beta-1} f''(a)da} \leq \sum_{k=0}^{\infty} \limsup_{n} \max_{0 \leq a \leq 1} f''(a) \int_{0}^{1} ae^{-\frac{a^2}{2C_k^2} t_k^{\alpha-\beta-1} da} \leq C \limsup_{n} \max_{0 \leq a \leq 1} f''(a) \sum_{k=0}^{\infty} t_k^{3\alpha-\beta-1} \leq C \limsup_{n} \max_{0 \leq a \leq 1} f''(a) \leq C \limsup_{n} \max_{0 \leq a \leq 1} f''(a) \leq \infty \]
since for every fixed $a$ we have $f''(a) \to f''(a)$ and we consider maximum on a compact interval (see [2] for details). The term $J_2$ can be treated similarly and hence we are done. □

**Corollary 3.1.** Let $f$ be a linear combination of convex functions, $Y \in W^{1-\beta}_{1}$ and $X \in X^{\alpha}$. If $\alpha > \beta$, then the integral
\[ \int_{0}^{T} f'(X_u) \, dY_u \]
exists almost surely in the sense of generalized Lebesgue-Stieltjes integral.

**Corollary 3.2.** Let $f$ be a linear combination of convex functions, $Y \in W^{1-\beta}_{1}$ and $X \in X^{\alpha}$. Put $S = g(X)$ for some strictly monotone function $g \in C^{1}$. If $\alpha > \beta$, then the integral
\[ \int_{0}^{T} f'_-(S_u) g'(X_u) \, dY_u \]
exists almost surely in the sense of generalized Lebesgue-Stieltjes integral.

**Proof.** Without loss of generality we can assume that $g$ is strictly increasing. Now the inverse $g^{-1}$ exists and we have
\[ 1_{S_t > a > S_s} = 1_{X_t > g^{-1}(a) > X_s}. \]
Hence, by following the proof in [2] we obtain the result from our main theorem if
\[ \int_{0}^{T} \int_{0}^{t} |f'_-(S_t)||g'(X_t) - g'(X_s)| \frac{ds}{(t-s)^{\beta+1}} \, dt < \infty. \]
We obtain by Taylor’s theorem that
\[ |g'(X_t) - g'(X_s)| = |g'(\xi)||X_t - X_s| \]
for some random point $\xi$ between $X_s$ and $X_t$. It remains to note that
\[ \mathbb{E}|X_t - X_s| \leq \sqrt{\mathbb{E}|X_t - X_s|^2} = \sqrt{W(t,s)}, \]
and the claim follows. \[\square\]

**Remark 3.2.** For financial applications natural candidate is $g(x) = e^x$.

Next our aim is to define integrals over the random interval $[0, \tau]$, where $\tau$ is almost surely finite random variable instead of deterministic time.

**Theorem 3.2.** Let $X_t \in \mathcal{X}_\alpha$ and let $\tau \leq T$ be a random time. If $\alpha > \beta$, then for every convex function $f$ we have $f'(X)1_{\tau \leq \tau} \in W_2^{\beta}$ almost surely.

**Proof.** For the first term in the fractional Besov norm we can simply make upper bound
\[ 1_{\tau \leq \tau} \leq 1 \]
and we can proceed as in the proof of our main theorem 3.1.

Consider then the second term in the fractional Besov norm:
\[ \int_0^T \int_0^t \frac{|f'(X_t)1_{\tau \leq \tau} - f'(X_s)1_{\tau \leq \tau}|}{(t-s)^{\beta+1}} \, ds \, dt. \]

Now either $s \leq t \leq \tau$ in which case we can proceed as for deterministic time or we have $t > \tau$ and $s < \tau$. In this case we get
\begin{align*}
\int_0^\tau \int_s^\tau \frac{|f'(X_s)|}{(t-s)^{\beta+1}} \, ds \, dt & \leq C(\beta) \sup_{s \in [0,T]} |f'(X_s)| \int_0^\tau (u - \tau)^{-\beta} \, du \\
& \leq C(\beta) \sup_{s \in [0,T]} |f'(X_s)| T^{1-\beta}.
\end{align*}

This completes the proof. \[\square\]

**Corollary 3.3.** Let $f$ be a linear combination of convex functions, $Y \in W_1^{1-\beta} \cdot X \in \mathcal{X}_\alpha$, and let $\tau \leq T$ be a bounded random time. If $\alpha > \beta$, then the integral
\[ \int_0^\tau f'(X_u) \, dY_u \]
exists almost surely in the sense of generalized Lebesgue–Stieltjes integral.

**Remark 3.3.** Let $g$ be strictly monotone, and set $S = g(X)$. Then similar results hold for integrals
\[ \int_0^\tau f'_-(S_u)g'(X_u) \, dY_u. \]

**Existence of Föllmer Integral.** As noted e.g. by Zähle [17] we can sometimes approximate the generalized Lebesgue–Stieltjes integral with Riemann–Stieltjes sums. This is the topic of Theorem 3.3 below. The proof follows exactly the same arguments as for the particular case for fractional Brownian motion in [2]. Hence we only give the idea of the proof and details are left to the reader.
Theorem 3.3. Let $f$ be a linear combination of convex functions, $Y \in W^{1-\beta}$ and $X \in \mathcal{X}^\alpha$. If $\alpha > \beta$, then

$$
\sum_{j=1}^{k(n)} f'_-(X_u^n)(Y_{u_{j+1}^n} - Y_{u_j^n}) \to \int_0^T f'_-(X_u) dY_u
$$

almost surely for any partition $\pi_n = \{0 = u_0^n < \ldots < u_{k(n)}^n = T\}$ such that $|\pi_n| \to 0$.

Proof. Again we can assume that the measure $f''$ has compact support. Now,

$$
\int_0^T f'_-(X_u) dY_u - \sum_{j=1}^{k(n)} f'_-(X_{u_j^n})(Y_{u_{j+1}^n} - Y_{u_j^n}) = \int_0^T h_n(X_u) dY_u,
$$

where

$$
h_n(u) = \sum_{j=1}^{k(n)} \left(f'_-(X_{u_j^n}) - f'_-(X_u)\right) 1_{u_j^n < u \leq u_{j+1}^n}.
$$

To conclude the proof we have to show that $\|h_n\|_{\beta,2} \to 0$ almost surely. Following arguments in [2] we obtain an integrable upper bound in both terms of the fractional Besov norm, and hence the result follows by using the dominated convergence theorem. More precisely, we have

$$
|h_n(t)| \leq 2 \sup_{0 \leq u \leq T} |f'_-(X_u)|,
$$

and

$$
|h_n(t) - h_n(s)| \leq C \int 1_{X_s < a < X_t} f''(da)
$$

on set $\{X_s < a < X_t\}$ and

$$
|h_n(t) - h_n(s)| \leq C \int 1_{X_t < a < X_s} f''(da)
$$

on set $\{X_t < a < X_s\}$. Hence we have integrable upper bound in all the cases and hence the result follows by dominated convergence theorem together with the fact that $|h_n(t)| \to 0$ almost surely. \qed

Remark 3.4. Let $g$ be strictly monotone and set $S = g(X)$. Let $\tau \leq T$ be a random time. Then similar result holds for integrals of the form

$$
\int_0^\tau f'_-(S_u) g'(X_u) dY_u.
$$
Existence of Mixed Integrals. The particular reason why we considered integrals of form (3.2) with arbitrary process $Y$ instead of $X$ is that now we can apply our result for processes of type $Y = M + X$

where $M$ is a centered Gaussian martingale with Lipschitz continuous variance function $(M)$. These kind of mixed processes are especially interesting in mathematical finance, see [4, 5]

Note that Lipschitz continuity of the variance on $M$ implies that

$$E[w_M(t)] \leq Ct$$

Theorem 3.4. Let $M$ be centered Gaussian martingale with Lipschitz variance and let $X$ be a $\alpha$-Hölder continuous Gaussian process for some $\alpha > 1/2$. Denote $Y = M + X$. Let $f$ be a linear combination of convex functions and let $\tau \leq T$ be a stopping time. If $Y$ satisfies assumptions (i), (iii) and (iv) of Definition 3.2, then the integral

$$\int_0^\tau f'_-(Y_u) dY_u = \int_0^\tau f'_-(Y_u) dM_u + \int_0^\tau f'_-(Y_u) dX_u$$

exists as a Föllmer integral.

Remark 3.5. Note that we do not assume that the processes $M$ and $X$ are independent.

Proof. Consider first the first integral on the right-hand side of (3.6). Since $M$ is a martingale this integral exists as Itô integral. By using suitable subdivision sequence $(\pi_n)_{n=0}^\infty$ it exists pathwise as a Föllmer integral.

Consider then the second integral on the right-hand side of (3.6). Since $M$ has Lipschitz variance we have that $Y \in \mathcal{X}^{1/2}$. Consequently, $f'_-(Y)1_{\tau \leq \tau} \in W^\beta_2$ for all $\beta \in (1 - \alpha, 1/2)$. Therefore, the generalized Lebesgue–Stieltjes integral $\int_0^\tau f'_-(Y_u) dX_u$ exists.

Finally, all the integrals in (3.6) can also be understood as Föllmer integrals due to Theorem 3.3.

Itô–Tanaka Formula. We begin with the following Itô formula for smooth functions. The proof is based on Taylor expansion and is the same as in the semimartingale case, or indeed, in the classical case.

Proposition 3.1. Let $X \in \mathcal{X}^\alpha$ with $\alpha > \frac{1}{2}$ and let $f \in C^2$. Then

$$f(X_T) = f(X_0) + \int_0^T f'(X_u) dX_u$$

The Itô formula of Proposition 3.1 can be extended to convex functions. Indeed, the arguments presented in [2] imply that:

Theorem 3.5. Let $X \in \mathcal{X}^\alpha$ with $\alpha > \frac{1}{2}$ and let $f$ be a linear combination of convex functions. Then

$$f(X_T) = f(X_0) + \int_0^T f'_+(X_u) dX_u.$$
Corollary 3.4. Let $g$ be a strictly monotone smooth function and set $S = g(X)$ for some $X \in X^\alpha$ with $\alpha > \frac{1}{2}$. Let $f$ be a linear combination of convex functions and let $\tau \leq T$ be a random time. Then
\[
f(S_\tau) = f(S_0) + \int_0^\tau f'_-(S_u) \, dS_u.
\]

Let us now consider the non-smooth case. Föllmer [9] showed that for any process $Y$ with finite quadratic variation $\langle Y \rangle$ and $f \in C^2$ we have
\[
f(Y_T) = f(Y_0) + \int_0^T f'(Y_u) \, dY_u + \frac{1}{2} \int_0^T f''(Y_u) \, d\langle Y \rangle_u,
\]
which also implies the existence of the Föllmer integral $\int_0^T f'(Y_u) \, dY_u$. We will extend this result to convex functions $f$. We will, however, not consider general quadratic variation processes. Instead, we consider processes that are of the form $Y = M + X$ as in Theorem 3.4.

The non-trivial quadratic variation gives rise to local time:

Definition 3.3. Let $Y$ be a continuous process with quadratic variation $\langle Y \rangle$. Its local time $L^a_T(Y)$ is the process defined by the occupation time formula
\[
\int_{-\infty}^{\infty} g(a) L^a_T(Y) \, da = \int_0^T g(Y_u) \, d\langle Y \rangle_u
\]
almost surely for every bounded, Borel measurable function $g$.

Theorem 3.6. Let $f$ and $Y = M + X$ satisfy the conditions of Theorem 3.4. Then there exists a local time process $L^a_T(Y)$ such that
\[
f(Y_T) = f(Y_0) + \int_0^T f'_-(Y_u) \, dY_u + \frac{1}{2} \int_{-\infty}^{\infty} L^a_T(Y) \, f''(da).
\]

Proof. Note first that Theorem 3.4 implies the existence of the integrals as mixed Föllmer and generalized Lebesgue–Stieltjes integral. Let us consider these integrals for a while.

First by Taylor expansion we have the result for $f \in C^2$. The general case follows with same arguments as for semimartingale case (see [12], pp. 221–224). Indeed, let $f$ be a convex function for which the measure $f''$ has compact support and define
\[
f_n(x) = n \int_{-\infty}^{0} f(x + y) j(ny) \, dy,
\]
where $j(y)$ is a positive $C^\infty$-function with compact support in $(-\infty, 0]$ such that
\[
\int_{-\infty}^{0} j(y) \, dy = 1.
\]
Now $f_n$ is well-defined and smooth for every $n$. Moreover, $f_n$ converges to $f$ pointwise and $(f_n)'_-$ increases to $f'_-$. Now,
\[
f_n(Y_T) = f_n(Y_0) + \int_0^T (f_n)'_-(Y_u) \, dY_u + \frac{1}{2} \int_0^T f''_n(Y_u) \, d\langle M \rangle_u.
\]
Moreover, by examining the proof of Theorem 3.5 we can conclude that
\[ \int_0^T (f_n)'_-(Y_u) dX_u \rightarrow \int_0^T f'_-(Y_u) dX_u, \]
and for the Itô integral we get by standard arguments that
\[ \int_0^T (f_n)'_-(Y_u) dM_u \rightarrow \int_0^T f'_-(Y_u) dM_u. \]
Consequently, we obtain that for every convex function \( f \) there exists a process \( A_f \) such that
\[ f(Y_T) = f(Y_0) + \int_0^T f'_-(Y_u) dY_u + \frac{1}{2} A_f^T. \]
Applying this result to the convex functions \((x-a)^+\) and \((x-a)^-\) we obtain
\[ (Y_T - a)^+ = (Y_0 - a)^+ + \int_0^T 1_{Y_u > a} dY_u + \frac{1}{2} A_+^T \]
and
\[ (Y_T - a)^- = (Y_0 - a)^- - \int_0^T 1_{Y_u < a} dY_u + \frac{1}{2} A_-^T. \]
Subtract second equation from the first to get that
\[ A_+^T = A_-^T \]
almost surely, and we define the local time process \( L_\alpha^T(Y) \) as the process
\[ |Y_T - a| = |Y_0 - a| + \int_0^T |Y_u - a| dY_u + L_\alpha^T(Y). \]
In order to have (3.8) for convex function \( f \) for which \( f'' \) has compact support we use representations
\[ f(x) = \alpha x + \beta + \int_{-\infty}^{\infty} |x - a| f''(da) \]
and
\[ f'_-(x) = \alpha + \int_{-\infty}^{\infty} \text{sgn}(x - a) f''(da). \]
By using the representation formula for the convex function \(|Y_T - a|\) we obtain
\[ f(Y_T) = f(Y_0) + \alpha (Y_T - Y_0) \]
\[ + \int_{-\infty}^{\infty} \frac{1}{2} \left( \int_0^T \text{sgn}(Y_u - a) dY_u + L_\alpha^T(Y) \right) f''(da). \]
Hence, for convex functions \( f \) for which \( f'' \) has compact support, the process \( A_f \) is given by
\[ A_f^T = \int_{-\infty}^{\infty} L_\alpha^T(Y) f''(da) + \int_{-\infty}^{\infty} \int_0^T \text{sgn}(Y_u - a) dY_u f''(da) \]
\[ - \int_0^T f'_-(Y_u) dY_u + \alpha (Y_T - Y_0). \]
It remains to apply Fubini’s Theorem to stochastic integrals. For the martingale part we use classical Stochastic Fubini’s Theorem (see [12]) and for pathwise integral we can use classical Fubini’s Theorem for $\sigma$-finite measures. Now, we obtain the occupation time formula by the same argument as in the semimartingale case.

**Corollary 3.5.** Let $M$ and $X$ be processes such that assumptions of Theorem 3.4 are satisfied. Then the local time $L^a_T(Y)$ of the process $Y = X + M$ is almost surely continuous in $t$ and in $a$ and we have the local representation

$$L^a_T(Y) = \lim_{\epsilon \to 0^+} \frac{1}{2\epsilon} \int_0^T 1_{a-\epsilon < Y_u < a + \epsilon} d\langle Y \rangle_u$$

almost surely.

**Proof.** Obviously $(Y_t - a)^+$ is continuous in $t$ and $a$. Hence it is enough to show that the stochastic integral

$$\int_0^t 1_{Y_u > a} dY_u$$

is continuous in $t$ and $a$. For this it is enough to show that

$$\|1_{X_u > a} 1_{u \leq t}\|_{2,\beta}$$

is continuous in $t$ and $a$. But continuity in $t$ is evident and the continuity in $a$ follows from Lebesgue dominated convergence theorem.

Finally, the local representation follows from continuity of the local time in $a$ and the occupation time formula [3.7].

**Remark 3.6.**

(i) In the case of standard Brownian motion $W$ with $g(x) = 1_A$ the occupation time formula reads that

$$\int_A L^a_T(W) \, d\alpha = \int_0^T 1_{(W_u \in A)} \, d\alpha.$$

So the local time is exactly the density with respect to Lebesgue measure for the occupation measure. In the Itô–Tanaka formula the the local time for $Y$ is the density of the occupation measure with respect to the “clock” $d\langle Y \rangle$.

(ii) Berman [6] showed that a Gaussian process $X$ admits local time as occupation density with respect to the “Lebesgue clock” $du$ if and only if its incremental variance $W$ satisfies

$$\int_0^T \int_0^T \frac{1}{\sqrt{W(t,s)}} \, ds \, dt < \infty.$$

Consequently, every stationary or stationary increment process $X \in \mathcal{X}^\alpha$ admits a local time defined as the density of the occupation measure with respect to the “Lebesgue clock”. If $\alpha > 1/2$, then $L^a_T(X) = 0$.

(iii) In Corollary 3.5 we have

$$d(Y)_u = d\langle M \rangle_u = (\langle M' \rangle_u)_u.$$

(iv) Again with similar arguments we can obtain the Itô–Tanaka formula for transformation $S = g(Y)$ with obvious changes.
4. Implications to Option-Pricing

We explain the implications of Itô–Tanaka formula for pricing and hedging of European options in the spirit on Sondermann [14, Ch. 6]. For the use of non-semimartingales and pathwise Föllmer integration in mathematical finance we refer to [5].

Let $S$ be the discounted stock-price process given by the dynamics
\[
\frac{dS_t}{S_t} = \mu(t)dt + dY_t,
\]
where $Y$ is a centered continuous quadratic variation process. Then, by [5],
\[
S_t = S_0 \exp \left\{ \int_0^t \mu(u) \, du + Y_t - \frac{1}{2} \langle Y \rangle_t \right\}.
\]

Suppose we want to replicate the European call-option $(S_T - K)^+$. Suppose $Y \in \mathcal{X}^\alpha$ for some $\alpha > 1/2$. Then $\langle Y \rangle = 0$, and the Itô–Tanaka formula takes the form
\[
(S_T - K)^+ = (S_0 - K)^+ + \int_0^T \mathbf{1}_{S_u \geq K} \, dS_u.
\]
This means that one can replicate the European call-option $(S_T - K)^+$ with a fair price $(S_0 - K)^+$ by dynamically buying at time $t$ one share of the stock if the option goes from out-of-the-money into in-the-money and selling the stock if the option goes from out-of-the-money into in-the-money. But this is silly because of at least two reasons:

(i) Take $K > S_0$. Then out-of-the-money options are worthless. This is obviously nonsense!

(ii) Take $K = S_0$. Then one could make profit without risk by selling cheap and buying expensive. This is against any reasonable business sense!

Thus, we see that in order to avoid this buy–sell paradox the option-pricing model must include a quadratic variation.

Suppose then that $Y = M + X$ such that the assumption of Theorem 3.6 are satisfied. Then the Itô–Tanaka formula takes the form
\[
(S_T - K)^+ = (S_0 - K)^+ + \int_0^T \mathbf{1}_{S_u \geq K} \, dS_u + \frac{1}{2} L^K_T(S).
\]
This solves the buy–sell paradox. Indeed, now the buy–sell strategy $\mathbf{1}_{S_u \geq K}$ is no longer a self-financing hedging strategy for the call-option $(S_T - K)^+$.

Remark 4.1. (i) The local time $\frac{1}{2} L^K_T(S)$ in the hedging formula (4.1) can be interpreted as transaction costs in the following way noted by Sondermann [14]: Assume that one tries to apply the buy–sell strategy $\mathbf{1}_{S_u \geq K}$ to hedge the European call-option $(S_T - K)^+$, i.e., buy the stock at price $K$ when up-crossing the barrier $K$, sell it again when down-crossing the barrier. But you cannot sell it at the same price. You need a so-called cutout. You can place only limit orders of the form: buy at $K$, sell at $K - \epsilon$ for some $\epsilon > 0$. The smaller you choose $\epsilon$, the more cutouts you will face, and in the limit the sum of these cutouts is just equal to the transaction costs.
(ii) The true hedging strategy for the European call-option in the models $Y = M + X$ satisfying the assumptions of Theorem 3.6 can be calculated just like in the martingale case $Y = M$ by using the Black–Scholes BPDE. See [5] for details.

**Appendix A. Level-Crossing Lemma**

The key lemma in our analysis is the following estimate for the probability that a Gaussian process $X$ crosses a fixed level. Actually, in [3] the authors proved the lemma in the particular case of fractional Brownian motion. We extend the result here for more general Gaussian process. We consider only probability $P(X_s < a < X_t)$ and a case $\sup_{s \leq T} V(s) \leq 1$. However, by considering processes $Y = -X$ and $Y = X/\sqrt{V}$ we obtain same bound for $P(X_s > a > X_t)$ and for the general case $\sup_{s \leq T} V(s) < \infty$. Also, note that continuous Gaussian processes on compact time intervals satisfy $V^* < \infty$.

**Lemma A.1.** Let $X$ be a Gaussian process with positive covariance function $R$. Denote

$$
\sigma^2 = \frac{R(s,s)R(t,t) - R(t,s)^2}{R(s,s)},
$$

and fix $0 < s < t \leq T$ and $a \in \mathbb{R}$. Assume that the variance function satisfies

$$
V^*: = \sup_{s \leq T} V(s) \leq 1.
$$

(i) If

$$
\frac{R(s,s)}{R(t,s)}(a - 1) < a,
$$

then there exists a constant $C$, independent of $s$, $t$, $T$ and $a$, such that

$$
P(X_s < a < X_t) \leq I_1 + I_2 + I_3 + I_4,
$$

where

$$
I_1 \leq C \min[\sqrt{V(s)}\sigma, \sigma^2] e^{-\frac{a^2}{2}},
$$

$$
I_2 \leq Ce^{-\frac{\min[\sigma^2, (a-1)^2]}{2\sigma^2}} \left(1_{|a|>2} + \left(a - \frac{R(s,s)}{R(t,s)}(a - 1)\right)1_{|a|\leq 2}\right),
$$

$$
I_3 \leq C\frac{R(s,s)}{R(t,s)} \sigma e^{-\frac{\min[\sigma^2, (a-1)^2]}{2\sigma^2}},
$$

$$
I_4 \leq e^{-\frac{\sigma^2}{2V(s)}} \left[ a \left(1 - \frac{R(s,s)}{R(t,s)}\right) \right];
$$

(ii) If

$$
\frac{R(s,s)}{R(t,s)}(a - 1) \geq a,
$$

then there exists a constant $C$, independent of $s$, $t$, $T$ and $a$, such that

$$
P(X_s < a < X_t) \leq C \min[\sqrt{V(s)}\sigma, \sigma^2] e^{-\frac{a^2}{2}}.
$$

In the proof we use the following well-known estimate.
Lemma A.2. Let $Z$ be a standard normal random variable and fix $a > 0$. Then

$$\mathbb{P}(Z > a) \leq \frac{1}{\sqrt{2\pi a}} e^{-\frac{a^2}{2}}.$$ 

Proof of Lemma A.1. We make use of decomposition

$$X_t = \frac{R(t, s)}{R(s, s)} X_s + \sigma Y,$$

where $Y$ is $N(0, 1)$ random variable independent of $X_s$ and

$$\sigma^2 = \frac{R(t, t)R(s, s)}{R(t, s)^2} - \frac{R(t, s)^2}{R(s, s)}.$$

Assume that

$$\frac{R(s, s)}{R(t, s)} (a - 1) < a.$$

Then we obtain

$$\mathbb{P}(X_s < a < X_t)$$

$$= \int_{-\infty}^{a} \mathbb{P} \left( Y \geq \frac{a - \frac{R(t, s)}{R(s, s)} x}{\sigma} \right) \frac{1}{\sqrt{2\pi \sqrt{V(s)}}} e^{-\frac{x^2}{2V(s)} \frac{1}{\sqrt{V(s)}}} dx$$

$$= \int_{-\infty}^{\frac{R(s, s)}{R(t, s)} (a-1)} \mathbb{P} \left( Y \geq \frac{a - \frac{R(t, s)}{R(s, s)} x}{\sigma} \right) \frac{1}{\sqrt{2\pi \sqrt{V(s)}}} e^{-\frac{x^2}{2V(s)} \frac{1}{\sqrt{V(s)}}} dx$$

$$+ \int_{\frac{R(s, s)}{R(t, s)} (a-1)}^{a} \mathbb{P} \left( Y \geq \frac{a - \frac{R(t, s)}{R(s, s)} x}{\sigma} \right) \frac{1}{\sqrt{2\pi \sqrt{V(s)}}} e^{-\frac{x^2}{2V(s)} \frac{1}{\sqrt{V(s)}}} dx$$

$$= I_1 + A_1.$$ 

Moreover, if

$$\frac{R(s, s)}{R(t, s)} (a - 1) \geq a,$$

then

$$\mathbb{P}(X_s < a < X_t) \leq I_1.$$ 

Note that here $I_1$ corresponds the one given in the Lemma and $A_1$ contains $I_2$, $I_3$ and $I_4$. We shall use similar technique for the rest of the proof.

We begin with $I_1$. By Lemma A.2 we have

$$\mathbb{P} \left( Y \geq \frac{a - \frac{R(t, s)}{R(s, s)} x}{\sigma} \right) \leq \frac{1}{\sqrt{2\pi A(x)}} e^{-\frac{A(x)^2}{2}},$$
where $A(x) = \frac{R(s, s) - R(t, s)}{\sigma}$. Hence

$$I_1 \leq \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi A(x)}} e^{-\frac{A(x)^2}{2}} \frac{1}{\sqrt{2\pi \sqrt{V(s)}}} e^{-\frac{x^2}{2V(s)}} dx$$

$$\leq \frac{\sigma}{\sqrt{V(s)}} e^{-\frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{A(x)^2}{2}} \frac{x^2}{2V(s)} + \frac{a^2}{2} dx}$$

$$= R(s, s) \frac{\sigma}{\sqrt{V(s)}} e^{-\frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2\sigma^2} x^2} \left[ \frac{R(s, s)}{R(t, s)} (a - y) \right]^2 + \frac{a^2}{2} dy}$$

We proceed to study the integral. Now,

$$y^2 - \frac{(R(s, s) (a - y))^2}{2V(s)} + \frac{a^2}{2}$$

$$= -\frac{1}{2\sigma^2} \left[ \left( y - \frac{R(s, s)}{R(t, s)} \right)^2 + a^2 \left( \frac{R(s, s)}{R(t, s)} \theta^2 - \theta^2 - \frac{R(s, s)^2}{R(t, s)^4 \theta^4} \right) \right]$$

where

$$\frac{1}{\sigma^2} = 1 + \frac{R(s, s)}{R(t, s)^2}.$$ 

Now

$$\frac{1}{\sigma^2} \geq 1$$

and since $V^* \leq 1$, we also have

$$\left( \frac{R(s, s)}{R(t, s)} \theta^2 - \theta^2 - \frac{R(s, s)^2}{R(t, s)^4 \theta^4} \right) \geq 0.$$ 

Thus,

$$\int_{1}^{\infty} \frac{1}{2\pi} e^{-\frac{y^2}{2\sigma^2} - \frac{(R(s, s) (a - y))^2}{2V(s)} + \frac{a^2}{2}} dy$$

$$\leq \int_{1}^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2\sigma^2} (y - \frac{R(s, s)}{R(t, s)} \theta^2)^2} dy$$

$$\leq \frac{1}{\sqrt{2\pi}}.$$ 

Hence, we obtain for $I_1$ that

$$I_1 \leq C \frac{R(s, s)}{R(t, s)} \frac{\sigma}{\sqrt{V(s)}} e^{-\frac{a^2}{2\sigma^2}}.$$ 

Now we have

$$\sigma^2 = \frac{\sigma^2 R(t, s)^4}{\sigma^2 + R(s, s)^4}$$

and hence for $I_1$ there exists a constant $C$ such that

$$I_1 \leq C \min \sqrt{V(s) \sigma, \sigma^2} e^{-\frac{a^2}{2\sigma^2}}.$$
For the term $A_1$ we have

$$A_1 = \int_{\frac{R(s,s)}{R(t,s)}(a-1)}^{a} \int_{A(x)}^{\infty} \frac{1}{2\pi} e^{-\frac{y^2}{2}} dy \frac{1}{\sqrt{2\pi(s)}} e^{-\frac{R(s)^2}{2V(s)}} dx$$

$$= \int_{\frac{R(s,s)}{R(t,s)}(a-1)}^{a} \int_{\frac{R(s,s)}{R(t,s)}(a-1)}^{\frac{1}{2}} \ldots dy dx + \int_{\frac{R(s,s)}{R(t,s)}(a-1)}^{a} \int_{\frac{1}{2}}^{\infty} \ldots dy dx$$

$$= A_2 + I_2.$$

Consider then $I_2$. Applying Lemma [A.2] we obtain

$$I_2 \leq C \frac{\sigma}{\sqrt{V(s)}} e^{-\frac{\sigma^2}{2V(s)}} \int_{\frac{R(s,s)}{R(t,s)}(a-1)}^{a} e^{-\frac{R(s)^2}{2V(s)}} dx.$$ 

Note that $\sigma^2 \geq 0$. Therefore,

$$\frac{R(s,s)}{R(t,s)^2} \geq \frac{1}{R(t,t)} \geq \frac{1}{V^*}.$$ 

Now if $|a| > 2$, we can apply Lemma [A.2] to obtain

$$\int_{\frac{R(s,s)}{R(t,s)}(a-1)}^{a} e^{-\frac{R(s)^2}{2V(s)}} dx \leq C e^{-\frac{\min[a^2/(a-1)^2]}{V^*}}.$$ 

As a consequence, we obtain the required upper bound for $I_2$. Now, if $|a| \leq 2$ we obtain

$$\int_{\frac{R(s,s)}{R(t,s)}(a-1)}^{a} e^{-\frac{R(s)^2}{2V(s)}} dx \leq C e^{-\frac{\min[a^2/(a-1)^2]}{V^*}} \left[ a - \frac{R(s,s)}{R(t,s)}(a-1) \right].$$

To conclude we study the term $A_2$. If we have

$$\frac{R(s,s)}{R(t,s)}a < a,$$

then by applying the Tonelli theorem we obtain

$$A_2 = \int_{\frac{R(s,s)}{R(t,s)}(a-1)}^{a} \frac{1}{2\pi} \int_{A(x)}^{\infty} e^{-\frac{y^2}{2}} dy \frac{1}{\sqrt{2\pi(s)}} e^{-\frac{R(s)^2}{2V(s)}} dx$$

$$= \int_{\frac{1}{2}}^{a} \left[ 1 - \frac{R(s,s)}{R(t,s)}(a-1) \right] \frac{R(s,s)}{R(t,s)} \ldots dx dy$$

$$= \int_{\frac{1}{2}}^{a} \left[ 1 - \frac{R(s,s)}{R(t,s)} \right] \frac{R(s,s)}{R(t,s)} \ldots dx dy + \int_{\frac{1}{2}}^{1} \frac{R(s,s)}{R(t,s)} \ldots dx dy$$

$$= I_3 + I_4.$$ 

Moreover, if

$$\frac{R(s,s)}{R(t,s)}a \geq a,$$

then

$$A_2 \leq I_3.$$
Now, for $I_3$ we have
\[
\int 1_{(1-R(t,s)) \sigma} \int \frac{R(s,s)}{R(t,s)} \sqrt{V(s)} e^{-\frac{y^2}{2V(s)}} e^{-\frac{2}{\sqrt{V(t)}}} dy dx \leq C \sigma \sqrt{V(s)} e^{-a^2/2V(s)} |a| e^{-\frac{2}{\sqrt{V(t)}}}.
\]
Hence, we have the required upper bound for $I_3$. To conclude, note that for $I_4$ we have
\[
I_4 = \int 1_{(1-R(t,s)) \sigma} \int a^2 \frac{R(s,s)}{R(t,s)} \sqrt{V(s)} e^{-\frac{y^2}{2V(s)}} e^{-\frac{2}{\sqrt{V(t)}}} dy dx \leq C \frac{1}{\sqrt{V(s)}} e^{-\frac{2}{\sqrt{V(s)}}} \sigma \left| 1 - \frac{R(s,s)}{R(t,s)} \right|.
\]
This finishes the proof of Lemma A.1. □

REFERENCES

[1] E. Azmoodeh. Riemann–Stieltjes Integrals with Respect to Fractional Brownian Motion and Applications. PhD Thesis. Helsinki University of Technology Institute of Mathematics Research Reports A590, 2010.

[2] E. Azmoodeh, Y. Mishura, and E. Valkeila. On hedging european options in geometric fractional brownian motion market model. Statistics & Decisions, 27:129–143, 2010.

[3] E. Azmoodeh and L. Viitasaari. Rate of convergence for discretization of integrals with respect to fractional Brownian motion. Journal of Theoretical Probability, Doi 10.1007/s10959-013-0495-y, 2013.

[4] C. Bender, T. Sottinen, and E. Valkeila. Arbitrage with fractional brownian motion? Theory of Stochastic Processes, 13(29):23–27, 2007.

[5] C. Bender, T. Sottinen, and E. Valkeila. Pricing by hedging and no-arbitrage beyond semimartingales. Finance and Stochastics, 12:441–468, 2008.

[6] S.M. Berman. Local nondeterminism and local times of gaussian processes. Indiana Univ. Math. J., 23:69–94, 1973.

[7] J. Bertoin. Temps locaux et intégration stochastique pour les processus de dirichlet. Séminaire de probabilités, 21:191–205, 1987.

[8] L. Coutin, D. Nualart, and C. Tudor. Tanaka formula for the fractional brownian motion. Stochastic Process. Appl., 94(2):301–315, 2001.

[9] H. Föllmer. Calcul d’îto sans probabilités. Séminaire de probabilités, 15:143–150, 1981.

[10] A. Garsia, E. Rademich, and H. Rumsey. A real variable lemma and the continuity of paths of some gaussian processes. Indiana Univ. Math. Journal, 20:565–578, 1970/71.

[11] D. Nualart and A. Râşcanu. Differential equations driven by fractional brownian motion. Collect. Math., 53:55–81, 2002.

[12] D. Revuz and M. Yor. Continuous Martingales and Brownian Motion. Springer, 1999.

[13] S.G. Samko, A.A. Kilbas, and O.I. Marichev. Fractional integrals and derivatives, Theory and applications. Gordon and Breach Science Publishers, Yvendon, 1993.

[14] D. Sondermann. Introduction to Stochastic Calculus for Finance: A New Didactic Approach. Springer, 2006.

[15] T. Sottinen and E. Valkeila. On arbitrage and replication in the fractional black-scholes pricing model. Statistics and Decisions, 21:93–107, 2003.

[16] L. C. Young. An inequality of the hölder type, connected with stieltjes integration. Acta Math., 67:251–282, 1936.

[17] M. Zähle. Integration with respect to fractal functions and stochastic calculus. part i. Probab. Theory Relat. Fields, 111:333–372, 1998.
DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF VAASA, P.O. BOX 700, FIN-65101 VAASA, FINLAND

DEPARTMENT OF MATHEMATICS AND SYSTEM ANALYSIS, AALTO UNIVERSITY SCHOOL OF SCIENCE, HELSINKI, P.O. BOX 11100, FIN-00076 AALTO, FINLAND