Supersymmetric Black Holes with a Single Axial Symmetry in Five Dimensions

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Abstract: We present a classification of asymptotically flat, supersymmetric black hole and soliton solutions of five-dimensional minimal supergravity that admit a single axial symmetry which ‘commutes’ with the supersymmetry. This includes the first examples of five-dimensional black hole solutions with exactly one axial Killing field that are smooth on and outside the horizon. The solutions have similar properties to the previously studied class with biaxial symmetry, in particular, they have a Gibbons–Hawking base and the harmonic functions must be of multi-centred type with the centres corresponding to the connected components of the horizon or fixed points of the axial symmetry. We find a large moduli space of black hole and soliton spacetimes with non-contractible 2-cycles and the horizon topologies are $S^3$, $S^1 \times S^2$ and lens spaces $L(p, 1)$.

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1. Introduction

The classification of higher dimensional stationary black hole solutions of general relativity remains a major open problem [1]. The black hole uniqueness theorems of four dimensions do not generalise to higher dimensions, even for asymptotically flat vacuum spacetimes. This was first demonstrated by the discovery of an explicit counterexample known as the black ring [2], which is an asymptotically flat, five-dimensional, vacuum black hole solution with horizon topology $S^1 \times S^2$. Furthermore, for a range of asymptotic charges, there exist two different black ring solutions, as well as a spherical Myers-Perry black hole solution.

A number of general results are known which constrain the topology and symmetry of higher-dimensional black hole spacetimes [3]. These are particularly restrictive for asymptotically flat five-dimensional stationary spacetimes, which will be the focus of this paper. Topological censorship guarantees that the domain of outer communication (DOC) is simply connected [4]. The horizon topology theorem states that cross-sections of the horizon must be $S^3$, $S^1 \times S^2$, $S^3/\Gamma$ where $\Gamma$ is a discrete group (or connected sums thereof) [5]. The rigidity theorem guarantees (under the assumption of analyticity) that rotating black holes must have an axial $U(1)$ symmetry that commutes with the stationary symmetry (for the non-extremal case see [6,7], and for generic extremal black holes see [8]). Motivated by this, further constraints on the topology of the horizon and the DOC have been derived for stationary black holes with a $U(1)$ axial symmetry [9]. Furthermore, classification theorems have been proven for stationary black holes with a $U(1)^2$ biaxial symmetry [10–12].

Black hole non-uniqueness is also present in five-dimensional (minimal) supergravity theory, even for asymptotically flat, supersymmetric black hole solutions. First, the BMPV solution was found [13], which is a charged, rotating black hole with $S^3$ horizon topology and equal angular momenta in the two orthogonal 2-planes. A uniqueness theorem for the BMPV black hole has been proven for locally $S^3$ horizons under the assumption that the stationary Killing field (the existence of which is necessary by supersymmetry) is timelike outside the black hole [14]. Later, supersymmetric black rings were constructed [15], moreover, it was found that concentric black ring solutions can possess the same asymptotic charges as the BMPV black hole [16].

More recently, new classes of supersymmetric black holes with non-trivial topology have been found in this theory. Black holes with lens space horizon topology were first found for $L(2, 1)$ topology [17] and then generalised to $L(p, 1)$ topology [18–20]. Black holes with $S^3$ horizons with non-trivial spacetime topology have also been constructed [19–21]. These solutions have a DOC with non-trivial topology due to the presence of non-contractible 2-cycles and are similar to the ‘bubbling’ microstate geometries [22]. These new types of black hole evade the aforementioned uniqueness theorem for the BMPV solution because they possess ergosurfaces on which the stationary Killing field is null. Interestingly, some of the black holes in bubbling spacetimes can have the same...
asymptotic charges as a BMPV black hole, and rather surprisingly, there even exist black holes whose horizon area exceeds that of the corresponding BMPV black hole with the same conserved charges [20,23]. This result is in conflict with the microscopic derivation of the BMPV black hole entropy in string theory [13,24], a contradiction which remains to be resolved. This highlights the importance of determining of the full moduli space of five-dimensional supersymmetric black holes.

The general local form of supersymmetric solutions in minimal supergravity has been known for some time [25]. It is determined using Killing spinor bilinears which define a scalar function, a causal Killing field, and three 2-forms on spacetime. When the Killing field is timelike, the metric takes the form of a timelike fibration over a hyper-Kähler base space, and the 2-forms are the complex structures of the base space. If there exists an axial symmetry of the base space that preserves the complex structures (i.e. the $U(1)$ action is triholomorphic), the base metric is a Gibbons-Hawking space [26], and if the full solution is also invariant under this axial symmetry it is determined by four harmonic functions on $\mathbb{R}^3$ [25]. It turns out the known supersymmetric black hole solutions discussed above all belong to this class. However, despite this local solution being known for nearly 20 years, no general global analysis of supersymmetric solutions with a Gibbons-Hawking base has been performed. One of the purposes of this paper is to perform such an analysis to determine all asymptotically flat black hole and soliton spacetimes in this class.

In fact, a number of classification theorems for supersymmetric solutions of minimal supergravity are already known. The near-horizon geometries of supersymmetric black holes in this theory were completely determined [14], and it was found that the horizon geometries are a (squashed) three-sphere $S^3$, lens space $L(p, q)$, and $S^1 \times S^2$ (the $T^3$ geometry is excluded by [5]). It turns out that in all cases the near-horizon geometry must have a $U(1) \times U(1)$ biaxial symmetry group in addition to the stationary symmetry. In fact, all known regular black hole solutions in five dimensions possess such a biaxial symmetry. Under the assumption of biaxial symmetry, a classification of asymptotically flat, supersymmetric black hole and soliton solutions in minimal supergravity has been achieved [19]. These solutions all have a Gibbons-Hawking base and the associated harmonic functions on $\mathbb{R}^3$ have collinear simple poles which correspond to either horizon components or fixed points of the triholomorphic axial Killing field. This reveals a large moduli space of black holes with $S^3$, $S^1 \times S^2$ and lens space $L(p, q)$ horizons and non-contractible 2-cycles in the DOC, which contains all the above examples. In particular, this rules out lens space horizons $L(p, q)$ for $q \neq 1$, at least in this symmetry class.

Reall conjectured that higher dimensional rotating black holes with exactly one axial symmetry should exist [14]. Evidence supporting this has been obtained from approximate solutions [28] and the analysis of linearised perturbations of odd dimensional Myers-Perry black holes for $D \geq 9$ [29]. This conjecture was motivated by the rigidity theorem, which only applies to black holes that are rotating in the sense that the stationary Killing field is not null on the horizon. Even though supersymmetric black holes are non-rotating in this sense (the stationary Killing field is null on the horizon), one may also expect supersymmetric black holes with a single axial symmetry to exist. There have been a number of constructions of such solutions in the literature [31–34], however, these have all resulted in solutions for which the metric or matter fields are not smooth at the horizon [35,36]. In this work we will show that one can easily construct

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1 A global analysis of a subclass of supersymmetric solutions with a Gibbons-Hawking base which reduce to four-dimensional euclidean Einstein–Maxwell solutions was performed in [27].

2 It is possible to construct near-horizon geometries with a single axial symmetry in $D \geq 6$ [30].
examples of five-dimensional supersymmetric black hole solutions with a single axial symmetry, that are smooth on and outside the horizon, by working within the class of supersymmetric solutions with a Gibbons-Hawking base.

The main goal of this paper is to obtain a classification of asymptotically flat, supersymmetric black hole and soliton solutions to five-dimensional minimal supergravity, that possess a single axial symmetry and are smooth on and outside a horizon (if there is one). This generalises the classification derived under the stronger assumption of a biaxial symmetry [19]. Our main assumption is that the axial symmetry ‘commutes’ with the supersymmetry in the sense that it preserves the Killing spinor. It then easily follows that the $U(1)$ action is triholomorphic and commutes with the stationary Killing field. Our main result can be summarised in the following theorem (the full statement is given in Theorem 3).

**Theorem 1.** Consider an asymptotically flat, supersymmetric black hole or soliton solution of $D = 5$ minimal supergravity, with an axial symmetry that preserves the Killing spinor. In addition, assume that the domain of outer communication is globally hyperbolic, on which the span of Killing fields is timelike. Then, the solution must have Gibbons-Hawking base and the associated harmonic functions are of multi-centred type, where the poles correspond to connected components of the horizon or fixed points of the axial symmetry, and the parameters must satisfy a complicated set of algebraic equations and inequalities. Furthermore, the cross-section of each horizon component must have $S^3$, $S^1 \times S^2$ or lens space $L(p, 1)$ topology.

This theorem is completely analogous to the case with biaxial symmetry [19]. However, the method of proof is rather different since it requires an analysis of the possible three-dimensional orbit spaces. These have been analysed in detail in [9]. We find that supersymmetry strongly constrains the orbit space and that it can be identified with the $\mathbb{R}^3$ base of the Gibbons-Hawking base. In contrast, in the biaxially symmetric case the orbit space is a two-dimensional manifold with boundaries and corners, which can be identified with a half-plane where the boundary is divided into rods (this is encoded by the rod structure). In that case it was also found that supersymmetry constrains the possible orbit spaces (that is, the rod structures are constrained). It is interesting that supersymmetry leads to such constraints on the spacetime topology.

It turns out that the constraints on the parameters in Theorem 1 are exactly the same as for the biaxisymmetric case, but there is no requirement for the centres to be collinear on $\mathbb{R}^3$. As we will show, this means that generically these spacetimes have $\mathbb{R} \times U(1)$ symmetry. The existence of these solutions depends on whether a complicated set of constraint equations and inequalities on the parameters can be simultaneously satisfied. Unfortunately, in general we do not have analytic control over these constraints. However, for three-centred solutions, we present numerical evidence that solutions do exist in the case of non-collinear centres, which correspond to black holes with exactly one axial Killing field. This extends the systematic study of three-centred solutions with biaxial symmetry (collinear centres) [20]. To our knowledge, this is the first explicit construction of higher-dimensional black holes with $\mathbb{R} \times U(1)$ symmetry (that are smooth on and outside the horizon), which confirms Reall’s conjecture for supersymmetric black holes.

The outline of this paper and of the proof of Theorem 1 is as follows. In Sects. 2.1–2.2 we review the local form of a supersymmetric solution, state our assumptions, and show that the general solution must have a Gibbons-Hawking base. In Sect. 2.3 we derive the constraints imposed by asymptotic flatness. In section 2.4 we combine the classification of near-horizon geometries [14] with our assumptions, to prove that the
horizon corresponds to a single point in the $\mathbb{R}^3$ cartesian coordinates of the Gibbons-Hawking base, and that the associated harmonic functions have at most simple poles at these points. In Sect. 2.5 we show that the $\mathbb{R}^3$ cartesian coordinates provide a global chart on the orbit space, and the associated harmonic functions have at most simple poles at fixed points of the $U(1)$ Killing field. This implies that the general form of such solutions is of multi-centred type with simple poles in the harmonic functions, see Theorem 2. In Sect. 3 we perform a general regularity analysis of multi-centred solutions, in particular we derive necessary and sufficient conditions for the solution to be smooth at a horizon and at a fixed point. In Sect. 4 we prove our main classification result which is stated in Theorem 3, give the asymptotic charges and show the symmetries of these spacetimes are generically $\mathbb{R} \times U(1)$. We conclude the paper with a discussion of the results in Sect. 5. A number of technical details are relegated to several Appendices. This includes a derivation of the general cohomogeneity-1 hyper-Kähler metric with triholomorphic euclidean $E(2)$ symmetry in Appendix E.

2. Supersymmetric Solutions with Axial Symmetry

2.1. Supersymmetric solutions and global assumptions. We will consider supersymmetric solutions to $D = 5$ ungauged minimal supergravity. The bosonic field content of this theory consists of a spacetime metric $g$ and a Maxwell field $F$, defined on a 5-dimensional spacetime manifold $\mathcal{M}$. The action is given by that of Einstein–Maxwell theory coupled to a Chern-Simons term for the Maxwell field. A solution is supersymmetric if it admits a supercovariantly constant spinor $\epsilon$ (Killing spinor). This condition is highly restrictive; Gauntlett et al. in [25] derived the general local form of all supersymmetric solutions using Killing spinor bilinears. Let us now briefly summarise some of their results.

We will work in the conventions of [14], so in particular the metric signature is ‘mostly plus’. From Killing spinor bilinears, one can construct a function $f$, a vector field $V$, and three 2-forms $X^{(i)}$, $i = 1, 2, 3$. These satisfy certain algebraic identities, in particular,

$$g(V, V) = -f^2, \quad (1)$$

$$\iota_V X^{(i)} = 0, \quad (2)$$

$$X^{(i)}_{\mu \gamma} X^{(j)}_{\nu} = \delta_{ij} (f^2 g_{\mu \nu} + V_\mu V_\nu) - f \epsilon_{ijk} X^{(k)}_{\mu \nu}, \quad (3)$$

where $\epsilon_{ijk}$ is the alternating symbol with $\epsilon_{123} = 1^3$. Importantly, (1) shows that $V$ is causal everywhere. Furthermore, it was also shown that $V$ is non-vanishing on any region where the Killing spinor is non-vanishing. These quantities must also satisfy certain differential identities, in particular, $V$ is a Killing vector field on $(\mathcal{M}, g)$, the 2-forms $X^{(i)}$ are closed and

$$\iota_V F = -\frac{\sqrt{3}}{2} d f. \quad (4)$$

This last relation implies that $\mathcal{L}_V F = 0$, that is, the Maxwell field is preserved by the Killing vector field $V$. 

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3 Greek indices $\mu, \nu, \ldots$ denote spacetime indices and are raised and lowered with the spacetime metric $g_{\mu \nu}$. 
On regions where $V$ is timelike, i.e. $f \neq 0$, the metric can be written as
\[
g = -f^2(dt + \omega)^2 + f^{-1}h, \tag{5}\]
where $V = \partial_t$, and $h$ is a Riemannian metric on a four-dimensional base space $B$ orthogonal to the orbits of $V$. Note that the base metric $h$ can be invariantly defined on regions where $V$ is timelike by
\[
h_{\mu\nu} = f(g_{\mu\nu} + V_\mu V_\nu f^2), \tag{6}\]
whereas the 1-form $\omega$ may be defined by $\iota_V \omega = 0$ and $d\omega = -d(f^{-2}V)$ (which fixes it up to a gradient) and hence can be regarded as a 1-form on $B$. The constraints from supersymmetry imply that the base space $(B, h)$ is hyper-Kähler with complex structures given by the 2-forms $X^{(i)}$. In particular, (2) implies that $X^{(i)}$ can be viewed as 2-forms on $B$ and (3) implies that they obey the quaternion algebra on $(B, h)^4$
\[
X^{(i)}_{ac} X^{(j)c}{}_{b} = -\delta_{ij}h_{ab} + \epsilon_{ijk}X^{(k)}_{ab}. \tag{7}\]
Furthermore, it can be shown that $X^{(i)}$ are parallel with respect to the Levi-Civita connection of $h$ and are anti-self dual with respect to the orientation $\eta$ on $B$ defined by the spacetime orientation $f(dt + \omega) \wedge \eta$. The Maxwell field can be written as
\[
F = -\frac{\sqrt{3}}{2}d\left(\frac{V}{f}\right) - \frac{1}{\sqrt{3}}G^+, \tag{8}\]
where $G^+$ is the self-dual part of $f d\omega$ with respect to the base space metric $h$.

We now turn to our global assumptions.

**Assumption 1.** $(\mathcal{M}, g, F)$ is a solution of $D = 5$ minimal supergravity such that:

(i) The solution is supersymmetric in the sense that it admits a globally defined Killing spinor $\epsilon$,

(ii) The supersymmetric Killing field $V$ is complete,

(iii) The domain of outer communication (DOC), denoted by $\mathcal{M}$, is globally hyperbolic,

(iv) $\mathcal{M}$ is asymptotically flat, that is, it has an end diffeomorphic to $\mathbb{R} \times (\mathbb{R}^4 \setminus B^4)$ where $B^4$ is a 4-ball, such that on this end,

(a) The metric $g = -du^0 du^0 + \delta_{IJ} du^I du^J + O(R^{-\tau})$ and some decay rate $\tau > 0$, where $u^0, (u^I)_{I=1}^4$ are the pull-back of the cartesian coordinates on $\mathbb{R} \times \mathbb{R}^4$, $R := \sqrt{u^I u^J \delta_{IJ}}$, and in these coordinates $\partial_\mu g_{\nu\rho} = O(R^{-\tau - 1}),$

(b) The supersymmetric Killing field in these coordinates is $V = \partial/\partial u^0$, so we also refer to it as the stationary Killing field,

(v) Each connected component of the event horizon $\mathcal{H}$ has a smooth cross-section $H_i$, that is, a 3-dimensional spacelike submanifold transverse to the orbits of $V$, which is compact,

(vi) There exists a Cauchy surface $\Sigma$ that is the union of a compact set, an asymptotically flat end as in (iv), and a finite number of asymptotically cylindrical ends diffeomorphic to $\mathbb{R} \times H_i$ each corresponding to a connected component of the horizon,

\footnote{Latin indices $a, b, \ldots$ denote base space indices and are raised and lowered with the base metric $h_{ab}$.}
(vii) The metric $g$ and the Maxwell field $F$ are smooth ($C^\infty$) on $\mathcal{M}$ and at the horizon (if there is one).

Remarks. 1. Assumption 1 (i) implies that the spinor bilinears $f$, $V$ and $X^{(i)}$ introduced above are globally defined on spacetime.
2. Under these assumptions it follows that $\mathcal{M}$ simply connected, by the topological censorship theorem [4].
3. From global hyperbolicity and completeness of $V$ it follows that each integral curve of $V$ intersects a spacelike Cauchy surface $\Sigma$ exactly once. The flow-out from $\Sigma$ along $V$ is injective otherwise there would be closed causal curves in $\mathcal{M}$, contradicting global hyperbolicity. Using the Flow-out Theorem (see e.g. [37]) one can see that $\mathcal{M}$ is diffeomorphic to $\mathbb{R} \times \Sigma$, and that the orbit space $\mathcal{M}/\mathbb{R}V$ is a manifold homeomorphic to $\Sigma$. Furthermore, the base space $B$ can be identified with the open subset of the orbit space $\mathcal{M}/\mathbb{R}V$ corresponding to timelike orbits of $V$ and the base metric $h$ is the corresponding orbit space metric [38].
4. The assumption that $V$ is timelike in the asymptotic region Assumption 1 (iv) implies that the metric can be written as timelike fibration over a hyper-Kähler base space (5), at least in the asymptotic region. We emphasise that we do not assume that $V$ is strictly timelike everywhere in $\mathcal{M}$ and therefore the hyper-Kähler structure is not globally defined. Notably, the Killing field $V$ must be tangent to the event horizon and therefore tangent to the null generators of the horizon.
5. The horizon corresponds to asymptotically cylindrical ends due to the well-known fact the horizon of a supersymmetric black hole must be extremal (see discussion around equation (43) for the argument in our context). By Assumption 1 (vi) we can compactify these ends of $\Sigma$ by adding boundaries diffeomorphic to $H_i$. Thus, we can also view $\Sigma$ as an asymptotically flat 4-manifold with boundaries $H_i$ corresponding to each connected component of the horizon.

2.2. Including axial symmetry. Let us now turn to the general analysis of a supersymmetric solution admitting a compatible axial symmetry. In particular, we will make the following assumptions.

Assumption 2. The supersymmetric background $(\mathcal{M}, g, F, \epsilon)$ admits a globally defined spacelike Killing field $W$ such that:

(i) Its flow has periodic orbits, that is, it is an ‘axial’ Killing field in the sense it generates a $U(1)$ isometry
(ii) It preserves the Maxwell field, $L_W F = 0$
(iii) It preserves the Killing spinor $L_W \epsilon = 0$, that is, commutes with the remaining supersymmetry
(iv) At each point of $\mathcal{M}$ there exists a linear combination of $V$ and $W$ which is timelike.

Assumption 2 (iii) is a supersymmetric generalisation of the usual requirement that the axial Killing field commutes with the stationary Killing field. This is revealed by the following lemma, which also greatly restricts the possible base space geometries.

Lemma 1. Under Assumption 2 (iii) the following hold:

(a) The Killing spinor bilinears $(f, V, X^{(i)})$ are preserved by the axial Killing field $W$; in particular, the supersymmetric and axial Killing fields commute, $[V, W] = 0$,
(b) Wherever $V$ is timelike, the data on the base $(f, h, X^{(i)})$ and $\omega$ (in an appropriate gauge) are preserved by the axial Killing field $W$; in particular, $W$ defines a triholomorphic Killing field of the hyper-Kähler structure $(B, h, X^{(i)})$ which must therefore be of Gibbons-Hawking form.

Proof. By Assumption 2 (iii) $W$ preserves the Killing spinor, so by the Leibniz rule, it also preserves Killing spinor bilinears, i.e. $W(f) = 0$, $[W, V] = 0$, $\mathcal{L}_W X^{(i)} = 0$.

Next, the base metric $h$ is invariantly defined when $f \neq 0$ by (6), and as $g$, $f$ and $V$ are preserved by both Killing fields, so is $h$. Now, $\omega$ is only defined up to a gauge transformation $\omega \rightarrow \omega + d\lambda$ generated by $t \rightarrow t - \lambda$, where $\lambda$ is a function on $B$. We may partially fix this gauge by requiring $\mathcal{L}_W t = 0$, so that $\omega$ is also invariant under $W$. In this gauge it is manifest that $W$ can be regarded as a vector field on the base. Since $X^{(i)}$ are the complex structures of $B$, and they are preserved by the Killing field $W$ (i.e. it is triholomorphic), the metric $h$ can always be written in Gibbons-Hawking form [26].

Remarks. 1. The following converse of Lemma 1 is also true: a Killing field that is triholomorphic on the base and commutes with the stationary Killing field preserves the Killing spinor. This is shown in Appendix A.

2. Lemma 1 shows that Assumption 2 (ii) is redundant, because, when $f \neq 0$, $F$ can be expressed in terms of $U(1)$-invariant quantities as in (8). We will show that the region $f \neq 0$ is dense in $\mathcal{M}$ (see Corollary 4), therefore, using continuity of $\mathcal{L}_W F$, $F$ is preserved by $W$ on and outside the horizon.

We have established that on regions where $f \neq 0$, the base metric $h$ has Gibbons-Hawking form. Recall the local form of this is

$$h = \frac{1}{H}(d\psi + \chi)^2 + H dx^i dx^i,$$

where $x^i, i = 1, 2, 3$ are cartesian coordinates on $\mathbb{R}^3$, $H$ and $\chi$ is a harmonic function and a 1-form on $\mathbb{R}^3$, respectively, satisfying

$$\ast_3 d\chi = dH,$$

where $\ast_3$ denotes the Hodge star operator on $\mathbb{R}^3$ with respect to the euclidian metric. In these coordinates, the complex structures are

$$X^{(i)} = (d\psi + \chi) \wedge dx^i - \frac{1}{2} H \epsilon_{ijk} dx^j \wedge dx^k,$$

and the triholomorphic Killing field is $W = \partial_\psi$.

Remarkably, it has been shown [25] that if the triholomorphic Killing field $W$ of the base is a Killing field of the five-dimensional metric, as is the case for us, then the

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5 Evidently, $W$ being well-defined on the base is not gauge-dependent. To be more precise, we can define $\tilde{W} := \pi_\ast W$ on $B$, where $\pi : \mathcal{M} \rightarrow B$ is the quotient map by $V$. This is a projection, in a coordinate basis $(W^i, W^a) \mapsto (W^a)$, and it is well-defined since $\mathcal{L}_V W = 0$. Then $\mathcal{L}_{\tilde{W}} h = 0$ on the spacetime implies $\mathcal{L}_{\tilde{W}} h = 0$ on $B$, and similarly for other $U(1)$-invariant tensors well-defined on the base. In the following, we will not distinguish between $W$ and $\tilde{W}$, except for Appendix A.
general solution is completely determined by four harmonic functions, $H, K, L, M$ on $\mathbb{R}^3$ as follows. Let $\omega_\psi$ be a function and $\hat{\omega}$ and $\xi$ be 1-forms on $\mathbb{R}^3$ satisfying
\[
\omega_\psi = \frac{K^3}{H^2} + \frac{3}{2} \frac{KL}{H} + M, \tag{12}
\]
\[
\star_3 d\hat{\omega} = H dM - M dH + \frac{3}{2} (K dL - L dK), \tag{13}
\]
\[
\star_3 d\xi = -dK. \tag{14}
\]
Then $f$ and $\omega$ can be written as
\[
f = \frac{H}{K^2 + HL}, \tag{15}
\]
\[
\omega = \omega_\psi (d\psi + \chi) + \hat{\omega}, \tag{16}
\]
while the Maxwell field takes the form
\[
F = dA = \frac{\sqrt{3}}{2} d \left( f (dT + \omega) - \frac{K}{H} (d\psi + \chi) - \xi \right). \tag{17}
\]
We emphasise that at this stage, the local form of the solution is now fully determined, up to four harmonic functions on $\mathbb{R}^3$.

We will now introduce several spacetime invariants that are useful for our global analysis, following [19]. Invariance of the Maxwell field under the Killing fields $V, W$ allows us to introduce an electric and magnetic potential $\Phi, \Psi$ satisfying
\[
\sqrt{3} \frac{d\Phi}{2} := \iota_V F, \tag{18}
\]
\[
\sqrt{3} \frac{d\Psi}{2} := \iota_W F, \tag{19}
\]
which are globally defined functions (up to an additive constant) on the DOC since it is simply connected. In fact, by (4) we must have $\Phi = -f + \text{const}$ so the electric potential for supersymmetric solutions does not give an independent invariant. These potentials are preserved by the Killing fields. Indeed, using (18–19), the electric and magnetic potentials satisfy $\mathcal{L}_V \Phi \propto \iota_V i_V F = 0, \mathcal{L}_W \Psi \propto \iota_W i_W F = 0, \mathcal{L}_V \Psi = -\mathcal{L}_W \Phi = i_W df = 0$ where in the final step we used the relation between $\Phi$ and $f$.

It turns out that a key spacetime invariant is given by the determinant of the inner product matrix of Killing fields:
\[
N := \frac{\det}{g(V, V) g(W, W)} = \frac{f}{H} = \frac{1}{K^2 + HL}, \tag{20}
\]
where the last two equalities are valid for solutions with a Gibbons-Hawking base as above. By Lemma 1, $N$ is preserved by both Killing fields, because it is defined in terms of $V, W$ and $g$. The significance of $N$ is that Assumption 2 (iv) implies that $N > 0$ everywhere on $\mathcal{M}$ \ $\mathcal{F}$ where
\[
\mathcal{F} := \{ p \in \mathcal{M} \mid W_p = 0 \} \tag{21}
\]
denotes the set of fixed points of $W$ in the DOC (see proof of Lemma 3 below).

Let us now recall a useful result proven in [19, Lemma 1].
Lemma 2. A supersymmetric solution \((\mathcal{M}, g, F)\) with a Gibbons-Hawking base is smooth on the region \(N > 0\) if and only if the associated harmonic functions \(H, K, L, M\) are smooth and \(K^2 + HL > 0\).

In particular, if \(N > 0\), the harmonic functions can be expressed in terms of spacetime invariants:

\[
\begin{align*}
H &= \frac{f}{N}, \\
L &= \frac{fg(W, W) + 2g(V, W)\Psi - f\Psi^2}{N}, \\
K &= \frac{f\Psi - g(V, W)}{N}, \\
M &= \frac{g(W, W)g(V, W) - 3f\Psi g(W, W) - 3\Psi^2 g(V, W) + f\Psi^3}{2N}.
\end{align*}
\]  

(22)

This shows that the harmonic functions \(K, L, M\) are only defined up to a gauge transformation \(\Psi \to \Psi + c\) where \(c\) is a constant. This allows us to deduce the following important result.

Lemma 3. The harmonic functions \(H, K, M, L\) are well-defined and smooth on \(\mathcal{M}\) \(\setminus \mathcal{F}\), and they are preserved by the two Killing fields \(V, W\).

Proof. Assumption 2 (iv) implies that if \(f(p) = 0\) at some \(p \in \mathcal{M}\) then \(W \cdot V \neq 0\) at \(p\), otherwise there would not exist a timelike linear combination of the Killing fields at \(p\). It follows that the zeros of the invariant \(N = f^2|W|^2 + (V \cdot W)^2\) in \(\mathcal{M}\) coincide with the zeros of \(W\). Hence, for any \(p \in \mathcal{M}\) such that \(W_p \neq 0\) we have \(N > 0\). Therefore, by Lemma 2, the associated harmonic functions are well-defined and smooth at every point in \(\mathcal{M}\) that is not a fixed point of \(W\). The harmonic functions can be expressed in terms of invariants as (22), and since these invariants are preserved by both Killing fields, so are the harmonic functions. \(\Box\)

Remark. From Assumption 2 (iv) it follows that \(f \neq 0\) at a fixed point of \(W\). Therefore, wherever \(f = 0\) in \(\mathcal{M}\), we must have \(N > 0\), so the associated harmonic functions are well-defined, even though the base is not. Therefore, from (22) it can be seen that the zeros of \(f\) and \(H\) must coincide in \(\mathcal{M}\). These so-called ‘evanescent ergosurfaces’ have been analysed in great detail in [39]. These are smooth, timelike hypersurfaces outside the horizon on which the stationary Killing field becomes null. This way black hole solutions can evade the uniqueness theorem of [14], which assumed that the stationary Killing field is strictly timelike outside the horizon.

From (11), it is immediate that the cartesian coordinates satisfy

\[
dx^i = \iota_W X^{(i)}.
\]  

(23)

The right-hand side \(\iota_W X^{(i)}\) is a globally defined 1-form on spacetime, and closed everywhere as a consequence of \(\mathcal{L}_W X^{(i)} = 0\), and the fact that \(X^{(i)}\) are closed [25]. Since \(\mathcal{M}\) is simply connected, equation (23) allows us to introduce globally defined functions \(x^i\) on \(\mathcal{M}\) (up to an additive constant) that coincide with the local cartesian coordinates of the Gibbons-Hawking base. From (2) it follows that the functions \(x^i\) are preserved by both \(V\) and \(W\). In the following sections, we will determine the behaviour of \(x^i\) in the asymptotically flat region and near the horizon.
2.3. Asymptotic flatness. In this section we use asymptotic flatness as stated in our definition Assumption 1 (iv) to deduce the behaviour of the base metric $h$, the complex 2-forms $X^{(i)}$, and the $\mathbb{R}^3$ cartesian coordinates $x^i$ of the Gibbons-Hawking base, near spatial infinity.

First observe by Assumption 1 (iv)b we may identify $u^I, I = 1, 2, 3, 4$ as coordinates on the base space $B$. Asymptotic flatness Assumption 1 (iv) then implies that in the asymptotic end,

$$f = 1 + O(R^{-\tau}) \, ,$$
$$\omega = O(R^{-\tau}) du^I \, ,$$
$$h = (\delta_{IJ} + O(R^{-\tau})) du^I du^J \, ,$$

so, in particular, $(B, h)$ has an asymptotically euclidean end diffeomorphic to $\mathbb{R}^4 \setminus B^4$. The next result constrains the behaviour of the hyper-Kähler structure near infinity.

**Lemma 4.** On the asymptotically flat end the complex structures of $(B, h)$ can be written in cartesian coordinates as

$$X^{(i)} = \Omega^{(i)}_+ + O(R^{-\tau}) \, ,$$

where $\Omega^{(i)}$ are a standard basis of anti-self-dual 2-forms on $\mathbb{R}^4$,

$$\Omega^{(1)}_+ = du^1 \wedge du^4 + du^3 \wedge du^2 \, ,$$
$$\Omega^{(2)}_+ = du^1 \wedge du^3 + du^2 \wedge du^4 \, ,$$
$$\Omega^{(3)}_+ = du^1 \wedge du^2 + du^3 \wedge du^4 \, .$$

**Proof.** The 2-forms $X^{(i)}$ satisfy the quaternion algebra (7), which in particular implies that $X^{(i)}_{ab} X^{(i)ab} = -4$ (no sum over $i$). Hence, from (26) we immediately deduce that in the cartesian coordinates on the asymptotic end $X^{(i)}_{IJ} = O(1)$. Next, we use the fact that $X^{(i)}$ are parallel with respect to the Levi-Civita connection $\nabla$ defined by $h$. By Assumption 1 (iv)a the derivatives of the metric in cartesian coordinates are $\partial_I h_{JK} = O(R^{-\tau-1})$ and therefore the covariant derivative $\nabla_I X^{(i)}_{JK} = \partial_I X^{(i)}_{JK} + O(R^{-\tau-1})$, so we deduce that $X^{(i)}_{IJ} = \tilde{X}^{(i)}_{IJ} + O(R^{-\tau})$ where $\tilde{X}^{(i)}_{IJ}$ are constants.

Finally, we use that $X^{(i)}$ is ASD with respect to the base metric $h$. To this end, let us decompose $\tilde{X}^{(i)} = \tilde{X}^{(i)}_+ + \tilde{X}^{(i)}_-$ where $\tilde{X}^{(i)}_\pm$ are the SD/ASD parts with respect to the euclidean metric on the asymptotic end, that is, $\star_h \tilde{X}^{(i)}_\pm = \pm \tilde{X}^{(i)}_\pm$. Then

$$\star_h X^{(i)} = \star_h \tilde{X}^{(i)} + O(R^{-\tau}) = \star_h \tilde{X}^{(i)}_+ + O(R^{-\tau}) = \tilde{X}^{(i)}_+ - \tilde{X}^{(i)}_- + O(R^{-\tau}) \, ,$$

where in the first equality we have used (26) to write $\star_h$ in terms of $\star_\delta$ and $O(R^{-\tau})$ terms and $X^{(i)}_{IJ} = O(1)$. Hence, $\star_h X^{(i)} = -X^{(i)}$ implies that $\tilde{X}^{(i)}_+ = O(R^{-\tau})$ and therefore since they are constants $\tilde{X}^{(i)}_+ = 0$. We have therefore shown that

$$X^{(i)} = \tilde{X}^{(i)}_- + O(R^{-\tau}) \, ,$$

where $\tilde{X}^{(i)}_-$ are constant components of ASD 2-forms on $(\mathbb{R}^4, \delta)$. Thus, the quaternion algebra (7) implies that $\tilde{X}^{(i)}_-$ obey the quaternion algebra with respect to the euclidean metric $\delta$. Therefore, by performing a constant $SO(3)$ rotation on $X^{(i)}$, we may always set $\tilde{X}^{(i)}_+ = \Omega^{(i)}_+$, where $\Omega^{(i)}$ are a basis of ASD 2-forms on $\mathbb{R}^4$ given in the lemma. \qed
We have shown that asymptotic flatness implies that the base space is asymptotically euclidean and the hyper-Kähler structure is asymptotically that of euclidean space. However, we also know that the hyper-Kähler structure is of Gibbons-Hawking form. Combining these facts we deduce the following.

**Lemma 5.** The triholomorphic Killing field can be written as

\[ W = \frac{1}{2} (J_{12} + J_{34}) + O(R^{1-\tau}) , \]  

where \( J_{IJ} := u^I \partial_I - u_I \partial^J \). The corresponding \( \mathbb{R}^3 \)-cartesian coordinates \( x^i \) defined by (23) are

\[ x^1 = \frac{1}{2} (u^1 u^3 + u^2 u^4) + O(R^{2-\tau}) , \quad x^2 = \frac{1}{2} (u^2 u^3 - u^1 u^4) + O(R^{2-\tau}) , \quad x^3 = \frac{1}{4} \left( (u^1)^2 + (u^2)^2 - (u^3)^2 - (u^4)^2 \right) + O(R^{2-\tau}) . \]  

(32)

In particular, \( r = \sqrt{x^i x^i} = R^2/4 + O(R^{2-\tau}) \) and so \( r \to \infty \) in the asymptotically flat end.

**Proof.** The axial Killing field \( W \) can be written as an \( \mathbb{R} \)-linear combination of rotational Killing fields of \( \mathbb{R}^4 \) up to \( O(R^{-\tau}) \) corrections [40]. Therefore, without loss of generality (by rotating \( u^I \) coordinates if necessary) we can write

\[ W = \alpha (J_{12} + J_{34}) + \beta (J_{12} - J_{34}) + O(R^{1-\tau}) , \]  

where \( \alpha, \beta \) constants. One can check that \( J_{12} + J_{34} \) preserves \( \Omega_1^{(i)} \) while \( J_{12} - J_{34} \) rotates \( \Omega_1^{(i)} \) and \( \Omega_2^{(i)} \) into the one another. Hence, from (27), \( W \) is triholomorphic if and only if \( \beta = 0 \), and requiring \( 4\pi \)-periodicity of orbits of \( W \) fixes \( \alpha = 1/2 \).

For the cartesian coordinates of the Gibbons-Hawking metric we have

\[ dx^i = \iota_W X^{(i)} = \frac{1}{2} \iota_{J_{12}} \Omega_1^{(i)} + \frac{1}{2} \iota_{J_{34}} \Omega_2^{(i)} + O(R^{1-\tau}) , \]  

(34)

and a straightforward computation yields (32) upon integration. \( \square \)

From this we deduce a number of important corollaries.

**Corollary 1.** The asymptotic end of \( (B, h) \) is diffeomorphic to an \( S^1 \)-fibration over \( \mathbb{R}^3 \setminus B^3 \) where \( S^1 \) is the orbits of \( W \) and \( B^3 \) is a 3-ball in \( \mathbb{R}^3 \).

**Corollary 2.** On the asymptotic end the harmonic function associated to the Gibbons-Hawking base takes the form

\[ H = \frac{1}{r} + \sum_{l \geq 1, m} h_{lm} r^{-l-1} Y_l^m , \]  

(35)

where \( Y_l^m \) are the spherical harmonics and \( h_{lm} \) are constants.

**Proof.** For the second corollary observe that the harmonic function of the Gibbons-Hawking metric (9) can be written invariantly on the base as \( H = h(W, W)^{-1} \) where recall in these coordinates \( W = \partial \psi \). On the other hand, from Lemma 5 we can compute the norm of \( W \) and find \( h(W, W) = \frac{1}{4} R^2 (1 + O(R^{-\tau})) = r (1 + O(r^{-\tau/2})) \), which establishes the claimed leading term. The form of the subleading terms follows from the fact \( H \) is harmonic on \( \mathbb{R}^3 \setminus B^3 \). \( \square \)
Remarks. 1. The above proof also shows that we can fix the decay rate to \( \tau = 2 \).

2. Our Assumption 1 (iv) implies that the invariants \( g(V, W) = -f^2\ell W \omega = O(R^{1-\tau}) \) and \( g(W, W) = f^{-1}h(W, W) - f^2(\ell W \omega)^2 = \frac{3}{4}R^2(1 + O(R^{-\tau})) \) and so in particular \( N = \frac{1}{2}R^2(1 + O(R^{-\tau})) \). Furthermore, it also implies the Maxwell field \( (8) \)

\[
F_{\mu\nu} = O(R^{-\tau-1}) \quad \text{in cartesian coordinates. From (19) and setting } \tau = 2, \text{ it follows that the magnetic potential is } \Psi = \Psi_0 + O(R^{-1}) \text{ where } \Psi_0 \text{ is a constant. Therefore, from (22), we may deduce that the other harmonic functions behave as}
\]

\[
L = 1 + O(r^{-1}) , \quad K = O(r^{-1}) , \quad M = -\frac{3}{2} \Psi_0 + O(r^{-1}) , \quad (36)
\]

where the form of the subleading terms is fixed by harmonicity.

The leading term in \( H = 1/r + O(r^{-2}) \) gives euclidean space. To see this, first we integrate for the 1-form \( \chi \) using (10) which gives, up to a gauge transformation,

\[
\chi = (\tilde{\chi}_0 + \cos \theta) d\phi + O(r^{-1}) . \quad (37)
\]

where \((r, \theta, \phi)\) are spherical polar coordinates on \( \mathbb{R}^3 \) and \( \tilde{\chi}_0 \) is an integration constant. Under a coordinate change \((\psi, \phi) \rightarrow (\psi + c\phi, \phi)\), the constant \( \tilde{\chi}_0 \rightarrow \tilde{\chi}_0 - c \) so we can fix it to any value that we like. It turns out that a convenient choice, which we will make, is to fix \( \tilde{\chi}_0 \) to be an odd integer. In particular, \( \tilde{\chi}_0 = \pm 1 \) removes the Dirac string singularity in \( \chi \) on the lower (upper) half \( z \)-axis on \( \mathbb{R}^3 \).

To determine the identification lattice of the angular directions let us define new coordinates

\[
\tilde{\psi} = \psi + \tilde{\chi}_0 \phi , \quad \tilde{\phi} = \phi , \quad (38)
\]

and \( r = \frac{1}{4}R^2 \). Then the Gibbons-Hawking metric to leading order in \( R \) becomes

\[
h_0 = dR^2 + \frac{1}{4}R^2 ((d\tilde{\psi} + \cos \theta d\tilde{\phi})^2 + d\theta^2 + \sin^2 \theta d\tilde{\phi}^2) . \quad (39)
\]

This is isometric to \( \mathbb{R}^4 \) in spherical coordinates where the radial coordinate of \( \mathbb{R}^4 \) is given by \( R \) and \((\theta, \tilde{\phi}, \tilde{\psi})\) are Euler-angles of \( S^3 \) with their identification lattice generated by

\[
P: (\tilde{\psi}, \tilde{\phi}) \sim (\tilde{\psi} + 4\pi, \tilde{\phi}) , \quad R: (\tilde{\psi}, \tilde{\phi}) \sim (\tilde{\psi} + 2\pi, \tilde{\phi} + 2\pi) . \quad (40)
\]

In the original coordinates, with \( \tilde{\chi}_0 \) an odd integer, this identification is equivalent to

\[
P: (\psi, \phi) \sim (\psi + 4\pi, \phi) , \quad R: (\psi, \phi) \sim (\psi, \phi + 2\pi) . \quad (41)
\]

that is, the angles are independently periodic so \( \phi \) can be thought of as the standard azimuthal angle on \( \mathbb{R}^3 \). In these coordinates it is thus manifest that the asymptotic end is diffeomorphic to an \( S^1 \)-fibration over \( \mathbb{R}^3 \setminus B^3 \) where \( \psi \) is a coordinate on \( S^1 \) and \( x^i \) are coordinates on \( \mathbb{R}^3 \).
2.4. Near-horizon geometry. By Lemma 3 the harmonic functions are smooth at generic points of \( M \), so the only potentially singular behaviour occurs at the horizon and fixed points of \( W \). In this section we will investigate the constraints imposed by a smooth black hole horizon on a supersymmetric solution with axial symmetry satisfying our above assumptions. In particular, we will deduce that connected components of the horizon correspond to points in the \( \mathbb{R}^3 \) coordinates of the Gibbons-Hawking base and that the harmonic functions have at most simple poles at these points. We will heavily rely on the known classification of near-horizon geometries [14] which uses Gaussian null coordinates adapted to the horizon. In particular, our strategy will be to impose that the spacetime near the horizon admits an axial Killing field \( W \) that preserves the 2-forms \( \Omega^\lambda \), and then deduce the cartesian coordinates \( x^i \) of the Gibbons-Hawking base in terms of Gaussian null coordinates using (23).

The supersymmetric Killing field \( V \) must be tangent to the event horizon \( \mathcal{H} \) and therefore (1) implies it must be null on the horizon and thus tangent to its null generators. Furthermore, (1) implies the horizon is extremal, that is, \( d(g(V,V)) = 0 \) at the horizon. Thus, \( \mathcal{H} \) is an extremal Killing horizon of \( V \). Next, by our Assumption 1 (v) each component of the horizon \( \mathcal{H} \) has a smooth cross-section \( H \) transverse to the orbits of \( V \). Let \( (y^A) \) be coordinates on \( H \). Then, following [14], in a neighbourhood of the horizon \( \mathcal{H} \) we may introduce Gaussian null coordinates \( (v, \lambda, y^A) \) adapted so \( V = \partial_v \) where \( U = \partial_\lambda \) is tangent to affine null geodesics transverse to the horizon that are normalised so \( g(V,U) = 1 \) and synchronised so the horizon is at \( \lambda = 0 \) (note \( U \) is past-directed and \( \lambda > 0 \) is outside the horizon). It can be shown that in such a neighbourhood of the horizon the metric takes the form

\[
g = -\lambda^2 \Delta^2 dv^2 + 2dv d\lambda + 2\lambda h_A dv dy^A + \gamma_{AB} dy^A dy^B ,
\]

where the metric components are smooth functions of \( \lambda, y^A \) for sufficiently small \( \lambda \). The quantities \( \Delta, h_A, \gamma_{AB} \) define components of a function, 1-form and Riemannian metric on the 3d surfaces of constant \( (v, \lambda) \) which include the horizon cross-sections \( H \) \( (v = \text{constant}, \lambda = 0) \). We will denote any quantity evaluated at \( \lambda = 0 \) by \( \Delta := \Delta|_{\lambda=0} \), \( h_A := h_A|_{\lambda=0} \), \( \gamma_{AB} := \gamma_{AB}|_{\lambda=0} \) etc., so in particular \( (H, \tilde{y}) \) is a 3d Riemannian manifold.

As is typical for spacetimes containing extremal horizons, the horizon corresponds to an asymptotically cylindrical end of the space orthogonal to the orbits of \( V \). We recall the argument for this in the present context. The orbit space metric is \( \hat{g}_{\mu\nu} := g_{\mu\nu} - \frac{V\mu V\nu}{g(V,V)} \) in Gaussian null coordinates is

\[
\hat{g} = \frac{(d\lambda + \lambda h_A dy^A)^2}{\Delta^2 \lambda^2} + \gamma_{AB} dy^A dy^B .
\]

When we approach the horizon on a geodesic \( y^A = \text{const} \), the proper distance \( \int^0 d\lambda / (\lambda \Delta) \) diverges at least logarithmically, so it takes infinite proper distance to reach the horizon in the orbit space. Thus, the horizon corresponds to an asymptotic end diffeomorphic to \( \mathbb{R} \times H \).

Now we consider the axial Killing field \( W \) near the horizon. It must be tangent to the event horizon \( \mathcal{H} \) and therefore in Gaussian null coordinates \( W^\lambda = 0 \) at \( \lambda = 0 \). Furthermore, \( [W, V] = 0 \) implies all components of \( W \) are \( v \)-independent. Then, by evaluating \( \mathcal{L} \hat{W} = 0 \) on the horizon it follows that \( \mathcal{L} \hat{y} = 0 \) where \( \hat{W} := \hat{W}^A \partial_{y^A} \), that is, \( \hat{W} \) is a Killing field of \( (H, \hat{y}) \). Therefore, by smoothness of \( W \) at the horizon, we may write \( W \) in Gaussian null coordinates as

\[
W = \hat{W}^A \partial_{y^A} + \lambda \hat{\omega}_A \partial_\lambda + O(\lambda) \partial_{y^A} + W^v \partial_v ,
\]

where the metric components are smooth functions of \( \lambda, y^A \) for sufficiently small \( \lambda \). The quantities \( \Delta, h_A, \gamma_{AB} \) define components of a function, 1-form and Riemannian metric on the 3d surfaces of constant \( (v, \lambda) \) which include the horizon cross-sections \( H \) \( (v = \text{constant}, \lambda = 0) \). We will denote any quantity evaluated at \( \lambda = 0 \) by \( \Delta := \Delta|_{\lambda=0} \), \( h_A := h_A|_{\lambda=0} \), \( \gamma_{AB} := \gamma_{AB}|_{\lambda=0} \) etc., so in particular \( (H, \tilde{y}) \) is a 3d Riemannian manifold.

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\[
\hat{g} = \frac{(d\lambda + \lambda h_A dy^A)^2}{\Delta^2 \lambda^2} + \gamma_{AB} dy^A dy^B .
\]

When we approach the horizon on a geodesic \( y^A = \text{const} \), the proper distance \( \int^0 d\lambda / (\lambda \Delta) \) diverges at least logarithmically, so it takes infinite proper distance to reach the horizon in the orbit space. Thus, the horizon corresponds to an asymptotic end diffeomorphic to \( \mathbb{R} \times H \).

Now we consider the axial Killing field \( W \) near the horizon. It must be tangent to the event horizon \( \mathcal{H} \) and therefore in Gaussian null coordinates \( W^\lambda = 0 \) at \( \lambda = 0 \). Furthermore, \( [W, V] = 0 \) implies all components of \( W \) are \( v \)-independent. Then, by evaluating \( \mathcal{L} \hat{W} = 0 \) on the horizon it follows that \( \mathcal{L} \hat{y} = 0 \) where \( \hat{W} := \hat{W}^A \partial_{y^A} \), that is, \( \hat{W} \) is a Killing field of \( (H, \hat{y}) \). Therefore, by smoothness of \( W \) at the horizon, we may write \( W \) in Gaussian null coordinates as

\[
W = \hat{W}^A \partial_{y^A} + \lambda \hat{\omega}_A \partial_\lambda + O(\lambda) \partial_{y^A} + W^v \partial_v ,
\]

where the metric components are smooth functions of \( \lambda, y^A \) for sufficiently small \( \lambda \). The quantities \( \Delta, h_A, \gamma_{AB} \) define components of a function, 1-form and Riemannian metric on the 3d surfaces of constant \( (v, \lambda) \) which include the horizon cross-sections \( H \) \( (v = \text{constant}, \lambda = 0) \). We will denote any quantity evaluated at \( \lambda = 0 \) by \( \Delta := \Delta|_{\lambda=0} \), \( h_A := h_A|_{\lambda=0} \), \( \gamma_{AB} := \gamma_{AB}|_{\lambda=0} \) etc., so in particular \( (H, \tilde{y}) \) is a 3d Riemannian manifold.

As is typical for spacetimes containing extremal horizons, the horizon corresponds to an asymptotically cylindrical end of the space orthogonal to the orbits of \( V \). We recall the argument for this in the present context. The orbit space metric is \( \hat{g}_{\mu\nu} := g_{\mu\nu} - \frac{V\mu V\nu}{g(V,V)} \) in Gaussian null coordinates is

\[
\hat{g} = \frac{(d\lambda + \lambda h_A dy^A)^2}{\Delta^2 \lambda^2} + \gamma_{AB} dy^A dy^B .
\]

When we approach the horizon on a geodesic \( y^A = \text{const} \), the proper distance \( \int^0 d\lambda / (\lambda \Delta) \) diverges at least logarithmically, so it takes infinite proper distance to reach the horizon in the orbit space. Thus, the horizon corresponds to an asymptotic end diffeomorphic to \( \mathbb{R} \times H \).

Now we consider the axial Killing field \( W \) near the horizon. It must be tangent to the event horizon \( \mathcal{H} \) and therefore in Gaussian null coordinates \( W^\lambda = 0 \) at \( \lambda = 0 \). Furthermore, \( [W, V] = 0 \) implies all components of \( W \) are \( v \)-independent. Then, by evaluating \( \mathcal{L} \hat{W} = 0 \) on the horizon it follows that \( \mathcal{L} \hat{y} = 0 \) where \( \hat{W} := \hat{W}^A \partial_{y^A} \), that is, \( \hat{W} \) is a Killing field of \( (H, \hat{y}) \). Therefore, by smoothness of \( W \) at the horizon, we may write \( W \) in Gaussian null coordinates as

\[
W = \hat{W}^A \partial_{y^A} + \lambda \hat{\omega}_A \partial_\lambda + O(\lambda) \partial_{y^A} + W^v \partial_v ,
\]
where $\tilde{W}_i$ is a smooth function at the horizon. In fact, the $v$-component of $W$ does not feature in the subsequent calculations, and therefore we do not need its detailed form here. Note that $W$ cannot vanish identically on the horizon, because if it did, it then follows $W$ vanishes everywhere. This is because, if $W = 0$ on $\mathcal{H}$ then all tangential derivatives of $W$ also vanish on $\mathcal{H}$, hence by Killing’s equation all first derivatives of $W$ must vanish on $\mathcal{H}$, which implies that $W$ vanishes everywhere.

We now prove one of the main results of this section which relates the Gibbons-Hawking coordinates to Gaussian null coordinates.

**Lemma 6.** The cartesian coordinates $x^i$ defined by (23) are constant on a connected component of the horizon. Furthermore, the euclidean distance from a connected component of the horizon $x^i = a_i$, is $r := |x - a| = c\lambda + \mathcal{O}(\lambda^2)$ for some positive constant $c$.

**Proof.** In the following we will use the notation and results of [14]. In the neighbourhood of $\mathcal{H}$ the hyper-Kähler 2-forms can be written in Gaussian null coordinates as

$$X^{(i)} = d\lambda \wedge Z^{(i)} + \lambda (h \wedge Z^{(i)} - \Delta \ast_3 Z^{(i)}) ,$$

where $h = h_A dy^A$, $\ast_3$ is the Hodge-dual with respect to $\gamma = \gamma_{AB} dy^A dy^B$, and $Z^{(i)} = Z^{(i)}_A dy^A$ satisfy

$$\ast_3 Z^{(i)} = \frac{1}{2} \epsilon_{ijk} Z^{(j)} \wedge Z^{(k)}, \quad Z^{(i)} \cdot Z^{(j)} = \delta_{ij} ,$$

where the inner product $\cdot$ is defined by $\gamma$. Since $V$ is Killing and preserves $X^{(i)}$, the $Z^{(i)}$ are all preserved by its flow. Furthermore, in [14] it was shown that compactness of $H$ (Assumption 1 (v)) implies that on each connected component of the horizon $\Delta$ is constant, $\hat{h}^A$ is a Killing vector of $(H, \hat{\gamma})$, $\hat{h}^2 := \hat{h}^A \hat{h}_A$ is constant, $L_h \hat{Z}^{(i)} = 0, 6$ and in the neighbourhood of $\mathcal{H}$

$$dh = \Delta \ast_3 h + \mathcal{O}(\lambda) dy^A + \mathcal{O}(1) d\lambda , \quad d \ast_3 h = \mathcal{O}(\lambda) dy^A + \mathcal{O}(1) d\lambda .$$

Further analysis depends on whether $\Delta$ and $h$ vanish on the horizon. There are three cases to consider which we examine in detail below.

**Case 1a:** $\hat{\Delta} \neq 0$ and $\hat{h} \neq 0$. One can define the following functions and 1-forms on (a possibly smaller) neighbourhood of $\mathcal{H}$,

$$\hat{x}^{(i)} := \frac{1}{\sqrt{\hat{h}^2}} h \cdot Z^{(i)} , \quad \sigma_L^{(i)} := \frac{\Delta^2 + h^2}{\Delta} Z^{(i)} + \frac{1}{\Delta} d\left( h \cdot Z^{(i)} \right) ,$$

which satisfy

$$d\sigma_L^{(i)} = -\frac{1}{2} \epsilon_{ijk} \sigma_L^{(j)} \wedge \sigma_L^{(k)} + \mathcal{O}(\lambda) dy^A \wedge dy^B + \mathcal{O}(1) d\lambda \wedge dy^A$$

and $\hat{x}^{(i)} \hat{x}^{(i)} = 1$. Now define vector fields $\xi_L^i$ by $\langle \sigma_L^{(i)}, \xi_L^j \rangle = \delta_{ij}$ and $\xi_L^i (v) = 0$ and $\xi_L^i (\lambda) = 0$, so in particular $\hat{\xi}_L^i$ are the dual vectors to $\hat{\sigma}_L^{(i)} := (\sigma_L^{(i)})^A dy^A$ (note by our definitions $\sigma_L^{(i)}$ has a $\lambda$-component but $\xi_L^i$ does not). The near-horizon analysis [14] shows

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6 Throughout this section indices $A, B, \ldots$ of $Z^{(i)}$, $h$ etc are raised and lowered with $\gamma_{AB}$ (and its inverse).

7 Compared to [14], here we do not restrict these equations to the horizon.
that $\xi^i_L$ are Killing vector fields of $(H, \hat{\gamma})$ that commute with $\hat{h}$. It follows that in the generic case ($\hat{h} \neq 0$), the geometry of $H$ is locally isometric to that of a squashed three-sphere $S^3$ and the Killing fields of $H$ are exactly $\hat{h}$ and $\xi^i_L$, which generate a $U(1) \times SU(2)$ isometry. Therefore, we deduce that the axial Killing field $W$ restricted to the horizon $\hat{W}$ must be an $\mathbb{R}$-linear combination of $\hat{h}, \xi^i_L$. Hence, from (44) we can write $W$ in some neighbourhood of the horizon as

$$W = W_0 h + W_i \xi^i_L + \lambda W_\lambda(y) \partial_\lambda + O(\lambda) \partial_y^A + O(\lambda^2) \partial_\lambda,$$

(50)

where $W_0, W_i$ are constants, $W_\lambda$ is a function of $y^A$, and we have adjusted the subleading terms as necessary.

We are now in a position to compute $\iota_W X^i$ near the horizon and hence use (23) to determine the Gibbons-Hawking coordinates $x^i$ in terms of Gaussian null coordinates. For this it is useful to use the following identities:

$$d\hat{x}^i = \epsilon^{ijk} \hat{x}^j \sigma^k_L + O(\lambda) dy^A + O(1) d\lambda$$

(51)

and

$$h = \frac{\Delta \sqrt{h^2}}{\Delta^2 + h^2} \hat{x}^i \sigma^i_L + O(\lambda) dy^A + O(1) d\lambda.$$

(52)

Then, using definitions (48), the expression for the complex structures (45), and the form of $W$ in (50), we find after a tedious calculation that

$$\iota_W X^i = -d \left( W_0 \sqrt{h^2} \hat{x}^i \right)$$

$$+ W_j \left( -\frac{\Delta}{\Delta^2 + h^2} \delta_{ij} d\lambda + \frac{\sqrt{h^2}}{\Delta^2 + h^2} \epsilon_{ipj} \hat{x}^j \sigma^k_L \right)$$

$$+ \lambda W_\lambda \left( \frac{\Delta}{\Delta^2 + h^2} \sigma^i_L - \frac{\sqrt{h^2}}{\Delta^2 + h^2} \epsilon_{ijk} \sigma^k_L \right) + O(\lambda^2) dy^A + O(\lambda) d\lambda.$$

(53)

Thus, taking the exterior derivative of (53) we get

$$d\iota_W X^i = \left[ \delta_{iq} \left( \frac{\sqrt{h^2}}{\Delta^2 + h^2} W_j \hat{x}^j - \frac{\Delta}{\Delta^2 + h^2} W_\lambda \right)$$

$$+ \epsilon_{iqj} \left( -\frac{\Delta}{\Delta^2 + h^2} W_j - \frac{\sqrt{h^2}}{\Delta^2 + h^2} W_\lambda \hat{x}^j \right)$$

$$- \frac{\sqrt{h^2}}{\Delta^2 + h^2} W_q \hat{x}^i + O(\lambda) \right] \left( \sigma^i_L \wedge d\lambda \right) + O(\lambda) dy^A \wedge dy^B.$$

(54)

Now, recall that triholomorphicity of $W$ implies $d\iota_W X^i = 0$. The $\sigma^i_L \wedge d\lambda$ are linearly independent on the horizon, and by continuity, also on some neighbourhood of the horizon, so the coefficient of these terms must vanish for all $i, q$. Contracting the coefficient
of these terms with $\delta_{iq}$ and $\frac{1}{2} \epsilon_{ijm}$ and requiring them to vanish at $\lambda = 0$, then yields the following linear system of equations for $W_i$,

$$A_{mj} W_j := \left( \frac{2h^2}{3 \Delta} \hat{x}^{(j)} \hat{x}^{(m)} + \Delta \delta_{jm} + \frac{1}{2} \sqrt{h^2} \epsilon_{ijm} \hat{x}^{(i)} \right)_{\lambda = 0} W_j = 0 ,$$

with $W_\lambda = \frac{2}{3} (\sqrt{h^2} \hat{x}^{(j)})_{\lambda = 0} W_j$. Since the determinant of the matrix

$$\det A = \frac{(2 \hat{h}^2 + 3 \hat{\Delta}^2)(\hat{h}^2 + 4 \hat{\Delta}^2)}{12 \hat{\Delta}} \neq 0 ,$$

it follows that $W_i = 0$, which also implies that $W_\lambda = 0$.

Therefore, we have shown that $W = W_0 h + O(\lambda) \partial_A + O(\lambda^2) \partial_\lambda$ for some constant $W_0 \neq 0$. Thus substituting back into (53) the definition for the cartesian coordinates (23) yields

$$dx^i = -d \left( W_0 \sqrt{h^2} \lambda \hat{x}^{(i)} \right) + O(\lambda^2) dy^A + O(\lambda) d\lambda .$$

At $\lambda = 0$, $dx^i \propto d\lambda$, therefore $x^i$ are constant on the horizon, and the cross-section $H$ of each connected component of the horizon corresponds to a single point in $\mathbb{R}^3$. Integrating (57) and taking $a^i$ to correspond to a connected component of the horizon yields

$$x^i - a^i = -W_0 \sqrt{h^2} \lambda \hat{x}^{(i)} + O(\lambda^2) ,$$

and the euclidean distance on $\mathbb{R}^3$ is

$$r = \sqrt{(x^i - a^i)(x^i - a^i)} = |W_0| \sqrt{h^2} \lambda + O(\lambda^2) .$$

**Case 1b:** $\hat{h} = 0$. In this case one must have $\hat{\Delta} \neq 0$ and $h = O(\lambda) dy^A$. We again define $\sigma_L^{(i)}$ as in (48) which satisfies (49), and dual vectors $\xi_L^{(i)}$ that satisfy $\xi_L^{(i)}(v) = \xi_L^{(i)}(\lambda) = 0$. Thus, in this case we can write

$$Z^{(i)} = \frac{\sigma_L^{(i)}}{\Delta} + O(\lambda) dy^A + O(1) d\lambda .$$

Then, using (45), we find

$$X^{(i)} = d \left( \frac{\lambda \sigma_L^{(i)}}{\Delta} \right) + O(\lambda^2) dy^A \wedge dy^B + O(\lambda) d\lambda \wedge dy^A .$$

The horizon geometry $\gamma = \sigma_L^{(i)} \sigma_L^{(i)} / \hat{\Delta}^2$ where we again define $\sigma_L^{(i)} := (\sigma_L^{(i)})_A dy^A$. It follows that the horizon geometry is isometric to a round $S^3$ or a lens space. The dual vectors $\xi_L^{(i)}$ are now all Killing vector fields of $(H, \gamma)$, and thus we can write the axial Killing field on the horizon as $\hat{W} = W_i \xi_L^{(i)} + \hat{W}_R$ where $W_i \in \mathbb{R}$ and $\hat{W}_R$ is a left-invariant vector field so $\mathcal{L}_{\hat{W}_R} \sigma_L^{(i)} = 0$. Therefore, by (44), in a neighbourhood of the horizon we can write

$$W = \hat{W}_R + W_i \xi_L^{(i)} + \lambda W_\lambda (y) \partial_\lambda + O(\lambda) \partial_A + O(\lambda^2) \partial_\lambda .$$
Then a short computation using the above gives

\[ \iota_W X^{(i)} = -d\left(\frac{W_R^i}{\Delta}\right) + \frac{\lambda}{\Delta} W_{\lambda} \sigma_L^{(i)} - W_j \left(\frac{\delta_{ij} d\lambda}{\Delta} + \frac{\lambda}{\Delta} \epsilon_{ijk} \sigma^{(k)}\right) + O(\lambda^2) dy^A + O(\lambda) d\lambda , \]

(63)

where \( W_R^i := \iota_{\hat{\sigma}_L} \hat{\sigma}^{(i)}_L \). It follows that

\[ dt_W X^{(i)} = \sigma_L^{(k)} \wedge d\lambda \left[ \frac{W_J \epsilon_{ijk}}{\Delta} - \frac{\delta_{ik} W_{\lambda}}{\Delta} + O(\lambda) \right] + O(\lambda) dy^A \wedge dy^B , \]

(64)

and therefore we deduce that triholomorphy of \( W \) implies \( W_i = W_{\lambda} = 0 \). Therefore, from (23) we again find that \( d\chi^i \propto d\lambda \) at the horizon, so each connected component of the horizon is a point say \( a^i \in \mathbb{R}^3 \). By integrating (63) we find

\[ x^i - a^i = -\frac{W_R^i}{\Delta} \lambda + O(\lambda^2) , \quad r = \left| \frac{W_0}{\Delta} \right| \lambda + O(\lambda^2) , \]

(65)

where \( W_0^2 := \frac{J}{\Delta} W_R^i W_R^i \) is a constant (since \( dW_R^i = \epsilon_{ijk} W_R^j \hat{\sigma}_L^{(k)} \)), which must be non-zero to avoid \( W \) vanishing identically on the horizon.

**Case 2:** \( \Delta = 0 \). In this case \( Z^{(i)} \) can be written in terms of coordinates \( z^i \) as [14]

\[ Z^{(i)} = K (z^i) dz^i + O(\lambda) dz^i , \quad h = d \log K + O(\lambda) dz^i , \]

(66)

with \( K = L/\sqrt{z^i z^i} \) for some constant \( L \). Let us introduce standard spherical polar coordinates \( (z^i) \to (R, \theta, \phi) \), and define \( \psi := \log R \). The near-horizon data is then

\[ \hat{\gamma} = L^2 (d\psi^2 + d\theta^2 + \sin^2 \theta d\phi^2) , \quad \hat{h} = -d\psi , \]

(67)

and

\[ X^{(i)} = L d\left(\lambda (\hat{\chi}^{(i)} d\psi + d\hat{\chi}^{(i)})\right) + O(\lambda^2) dy^A \wedge dy^B + O(\lambda) d\lambda \wedge dy^A , \]

(68)

where \( \hat{\chi}^{(i)}(\theta, \phi) := z^i / R \). The horizon cross-section \( H \) is locally isometric to \( S^1 \times S^2 \), and its independent Killing fields are \( \partial_\psi \) and the standard Killing fields of \( S^2 \). Therefore, without loss of generality we can always adapt the coordinates on \( S^2 \) so that \( \hat{W} = W_\psi \partial_\psi + W_\phi \partial_\phi \), where \( W_\psi, W_\phi \) are constants, and hence write \( W \) in the neighbourhood of the horizon (44) as

\[ W = W_\psi \partial_\psi + W_\phi \partial_\phi + \lambda W_\lambda (\theta, \phi, \psi) \partial_\lambda + O(\lambda) \partial_y^A + O(\lambda^2) \partial_\lambda , \]

(69)

which yields

\[ \iota_W X^{(i)} = -d\left(LW_\psi \lambda \hat{\chi}^{(i)} + LW_\phi (\lambda d\psi - d\lambda) \partial_\phi \hat{\chi}^{(i)} + \right. \]

\[ \left. + LW_\lambda (\hat{x}^{(i)} d\psi + d\hat{x}^{(i)}) + O(\lambda^2) dy^A + O(\lambda) d\lambda \right) , \]

(70)

It follows that

\[ dt_W X^{(i)} = L \left(W_\phi \partial_\phi \hat{x}^i + W_\lambda \hat{x}^i + O(\lambda)\right) d\lambda \wedge d\psi + \ldots , \]

(71)
where \ldots represent terms not proportional to \( d\lambda \wedge d\psi \). Therefore, we deduce that \( t W X^{(i)} \) is closed (to leading order) if and only if \( W_\phi = W_\lambda = 0 \) (to see this note \( \dot{x}^i \partial_\phi \dot{x}^i = 0 \)). As before, in order for \( W \) not to vanish identically, \( W_\psi \) must be non-zero. From (23) we also find that \( x^i \) are constant on the horizon, and (70) can be integrated to get

\[
x^i - a^i = -LW_\psi \lambda \dot{x}^{(i)} + O(\lambda^2) , \quad r = |LW_\psi| \lambda + O(\lambda^2) .
\]  

(72)

\[ \square \]

Remark. In Gaussian null coordinates (42) one can check \((L W g)_{\lambda \lambda} = 2 \partial_\lambda W^v\) so that \( W^v = W^v(y) \). If one assumes that \( W \) is tangent to the cross-section \( H \) of the horizon it therefore follows that \( W^v = 0 \). Then, \((L W g)_{\lambda A} = 0 \) implies \( \partial_\lambda W^A = 0 \) and in turn \((L W g)_{\nu \lambda} = 0 \) implies \( \partial_\lambda W^\nu = 0 \). By assumption, \( W \) is tangent to the horizon and hence \( W^\nu |_{\lambda = 0} = 0 \) as in (44), so we deduce that \( W^\nu = 0 \). This shows that in Gaussian null coordinates, any Killing vector that is tangent to the cross-section takes the form \( W = W^A(y) \partial_{\gamma A} \) in the whole neighbourhood of the horizon (not just on the horizon), that is, \( W \) is tangent to constant \((v, \lambda)\) surfaces even for \( \lambda \neq 0 \). Therefore, if one assumes that \( W \) is tangent to \( H \) (which we have not) the proof of Lemma 6 simplifies somewhat.

The proof of Lemma 6 also reveals the following important property of the axial Killing field.

Corollary 3. The axial Killing field \( W \) has no zeroes on the horizon.

We are now in a position to determine the precise singular behaviour of the harmonic functions near a horizon.

Lemma 7. The associated harmonic functions in a neighbourhood of a connected component of the horizon \( x^i = a^i \) are of the form

\[
H = \frac{h}{|x - a|} + \tilde{H} , \quad K = \frac{k}{|x - a|} + \tilde{K} , \quad L = \frac{l}{|x - a|} + \tilde{L} , \quad M = \frac{m}{|x - a|} + \tilde{M} ,
\]  

(73)

where \( h, k, l, m \) are (possibly zero) constants and \( \tilde{H}, \tilde{K}, \tilde{L}, \tilde{M} \) are harmonic functions regular at \( a^i \).

Proof. In Gaussian null coordinates the invariants \( g(V, V) = -\lambda^2 \Delta^2 \), \( g(V, W) = \lambda \dot{W} \cdot \dot{h} + O(\lambda^2) \) and hence (20) gives

\[
N = \lambda^2 (N_0 + O(\lambda)) , \quad N_0 := |\dot{W}|^2 \Delta^2 + (\dot{W} \cdot \dot{h})^2 ,
\]  

(74)

where we have used (44), together with the fact that \( \dot{W}_\lambda = O(\lambda) \) which follows from the proof of Lemma 6. Crucially, since \( W \) has no zeroes on the horizon (Corollary 3) the function \( N_0 \) is strictly positive on the horizon for all types of near-horizon geometry (see three cases in proof of Lemma 6). Therefore, the harmonic functions (22) near the horizon take the form

\[
H = \frac{\Delta}{N_0 \lambda} + O(1) , \quad L = \frac{\Delta |W|^2 + 2(W \cdot h) \Psi - \Delta \Psi^2}{N_0 \lambda} + O(1) ,
\]  

(75)

\[
K = \frac{\Delta \Psi - W \cdot \dot{h}}{N_0 \lambda} + O(1) ,
\]

\[
M = \frac{|W|^2 (W \cdot \dot{h}) - 3\Delta \Psi |W|^2 - 3\Psi^2 (W \cdot \dot{h}) + \Delta \Psi^3}{2N_0 \lambda} + O(1) .
\]

Therefore, by Lemma 6 we deduce that \( |x - a| H = O(1) \) as \( x^i \to a^i \) and similarly for the other harmonic functions. The claim now follows by standard harmonic function theory. \[ \square \]
2.5. Orbit space and general form of harmonic functions. The orbit space is defined by

$$\hat{\Sigma} := \mathfrak{M}/[\mathbb{R} \times U(1)] \cong \Sigma/U(1),$$

(76)

where the $\mathbb{R} \times U(1)$ action is defined by the flow of the Killing fields $V$, $W$ and the second equality follows from Remark 3. By Corollary 1 we deduce that the orbit space $\hat{\Sigma}$ has an end diffeomorphic to $\mathbb{R}^3 \setminus B^3$ on which the Gibbons-Hawking cartesian coordinates $x^i$ are a global chart. We now turn to a detailed study of the orbit space.

An extensive analysis of the structure of such orbit spaces has been performed in [9]. In general, $\hat{\Sigma}$ is a simply connected topological space with a boundary $\partial \hat{\Sigma} = \hat{H} \cup \bigcup_{i=1}^3 S^1_i \cup S^2_{\infty}$, where $\hat{H}$ is the orbit space of the event horizon, $S^2_{\infty}$ denotes the asymptotic boundary and $S^2_i$ correspond to fixed points of the $U(1)$ action (i.e. zeroes of $W$ corresponding to ‘bolts’ [41]). The interior of the orbit space is the union of three kinds of points $\hat{\Sigma} = \hat{L} \cup \hat{E} \cup \hat{F}$, corresponding to regular orbits (trivial isotropy), exceptional orbits (discrete isotropy) and fixed points ($U(1)$ isotropy), respectively. $\hat{L}$ is open in $\hat{\Sigma}$ and has a structure of a manifold, internal fixed points of $\hat{\Sigma}$ are isolated, and $\hat{E}$ are smooth arcs ending on either fixed points or $\hat{H}$ (they cannot form closed loops [42]).

**Lemma 8.** The interior of the orbit space $\hat{\Sigma} = \hat{L} \cup \hat{F}$ where $\hat{L}$ corresponds to regular orbits and $\hat{F}$ to a finite number of isolated fixed points, i.e., the $U(1)$-action has no exceptional orbits. Furthermore, its boundary $\partial \hat{\Sigma} = \hat{H} \cup S^2_{\infty}$, i.e. there are no fixed points corresponding to bolts.

**Proof.** First recall that Assumption 2 (iv) implies $f \neq 0$ at the zeros of $W$, so the fixed points correspond to internal points of the base manifold $B$. Let us define $\hat{W}$ to be the metric dual of $W$ with respect to $h$. As a consequence of triholomorphicity, $d\hat{W}$ is self-dual in $B$ (see e.g. [26]), hence each of the fixed points of $W$ corresponds to a ‘nut’ of type $(\pm 1, \pm 1)$ and ‘bolts’ are not possible [41]. As a result, fixed points of $W$ must be isolated in $B$ and hence correspond to internal fixed points in $\hat{\Sigma}$. $\hat{W}$ has no zeros in a neighbourhood of horizon components (Corollary 3) or at spatial infinity (Lemma 5), hence by Assumption 1 (vi) $\hat{F}$ is contained in a compact set. $\hat{F}$ is also closed in $\hat{\Sigma}$, thus it must be finite. Furthermore, since fixed points correspond to a ‘nut’ of type $(\pm 1, \pm 1)$, there cannot be any arc of exceptional orbits ending on them (see [9]). Thus, if there are arcs of exceptional orbits, those arcs must end on $\hat{H}$. Assume for contradiction that there is an arc of exceptional orbit ending on a horizon component $\hat{H}_i$. $\hat{H}_i$ cannot have $S^2 \times S^1$ topology, because in that case all points of $\hat{H}_i$ have the same isotropy group (see Case 3 in Section 2.4 and [14]). It follows that $\hat{\Delta} \neq 0$ for $\hat{H}_i$, therefore there must be exceptional orbits with $f > 0$ in their neighbourhood. In the base of such a neighbourhood we can use Gibbons-Hawking coordinates $(\psi, x^i)$, which excludes the possibility of an exceptional orbit since on such a chart the period of $\psi$ is fixed. (A more detailed technical argument is given in Appendix B).

The spacetime invariants $f, \Psi, N, x^i$ that we have constructed are preserved by the Killing fields $V$, $W$ and therefore descend to functions on the orbit space.\textsuperscript{8} It follows by Lemma 8 that fixed points in Gibbons-Hawking coordinates $x^i$ correspond to points in $\mathbb{R}^3$. We shall now prove that $x^i$ can be used as global coordinates on $\hat{L}$, so that in

\textsuperscript{8} By abuse of notation, we denote such invariant functions on $\mathfrak{M}$ and their corresponding functions on the orbit space by the same letter.
particular $\hat{L}$ is diffeomorphic to $\mathbb{R}^3$ with a finite set of points removed corresponding to the image of fixed points in $\hat{F}$ and horizon components $\hat{H}_i$ (recall a horizon component in Gibbons-Hawking coordinates also corresponds to a point in $\mathbb{R}^3$ by Lemma 6).

**Lemma 9.** The function $x : \hat{L} \to \mathbb{R}^3 \setminus x(\hat{H} \cup \hat{F})$ is a diffeomorphism.

**Proof.** Let us start by showing that $x$ is a local diffeomorphism on $\hat{L}$. From Lemma 1 it follows that $N$ is preserved by $V$, $W$, so it descends to the orbit space, furthermore $N > 0$ on $\hat{L}$. Recall, that $\hat{L}$ is a manifold. The algebraic relations (3) and definition (23) and (20) imply that on the spacetime

$$g^{-1}(dx^i, dx^j) = W^\mu W^\nu X^{(i)}_{\mu \rho} X^{(j)\rho} = W^\mu W^\nu \delta_{ij} (f^2 g_{\mu \nu} + V_\mu V_\nu) = N \delta_{ij},$$

(77)

hence $dx^i$ are linearly independent in $T^*_{q}M$ for any $q \in M$ where $N(q) > 0$, which is the case in $\langle \mathcal{M} \rangle$ on regular orbits. Since $\iota_W dx^i = \iota_V dx^i = 0$, $dx^i$ are also linearly independent in $T^*_p \hat{L}$ for all $p \in \hat{L}$. Therefore, $x$ is a local diffeomorphism on $\hat{L}$. For it to be a diffeomorphism onto $\mathbb{R}^3 \setminus x(\hat{H} \cup \hat{F})$, we need to show surjectivity and global injectivity.

First, we shall prove surjectivity. Let us define the dual vectors $e_j$ on $\hat{L}$, i.e. $dx^i(e_j) = \delta^j_i$. Let $a := x(p) \in \mathbb{R}^3$ for some $p \in \hat{L}$, and $y \in \mathbb{R}^3 \setminus x(\hat{F} \cup \hat{H})$ an arbitrary point. Consider a path in $\mathbb{R}^3 \setminus x(\hat{F} \cup \hat{H})$ from $a$ to $y$ that is a union of line segments, such that straight continuation of any line stay in $\mathbb{R}^3 \setminus x(\hat{F} \cup \hat{H})$ (i.e. there is no fixed point or horizon mapped to the continuation of the segments). This can be done using two segments due to the fact that $|\hat{F}|$ and the number of horizon components are finite (Assumption 1 (vi) and Lemma 8). For the segment ending on $a$ let the vector tangent to it be $u'e_i$ with $\{e_i\}$ being the standard basis of $\mathbb{R}^3$. Then consider the maximal integral curve $\gamma$ of $U = u'e_i$ starting at $p$ in $\hat{L}$. Assume for contradiction that it is incomplete, which means that $x^i(\gamma(t)) = a^i + t u^i$ is bounded. However, due to the Escape Lemma (see e.g. [37]), the image of $\gamma$ cannot be contained in any compact subset of $\hat{L}$, therefore it must approach the asymptotically flat end (recall by construction $\gamma$ does not approach a horizon or fixed point and our Assumption 1 (vi)). Thus, $x^i$ is bounded along $\gamma$ as we approach the asymptotically flat end, which by Lemma 5 is a contradiction. Therefore, $U = u'e_i$ must be a complete vector field. One can similarly show that the vector field $V = v'e_i$, where $v'e_i$ is tangent to the line in $\mathbb{R}^3$ that ends at $y$, is complete. This shows that starting at $p \in \hat{L}$, we can follow the integral curves of $U$ and $V$ to reach a point $q \in \hat{L}$ such that $x(q) = y$. But $y$ is arbitrary in $\mathbb{R}^3 \setminus x(\hat{F} \cup \hat{H})$ and hence $x$ is surjective.

We next show that $x$ is injective. For contradiction, let us assume that $p \neq q \in \hat{L}$ and $x(p) = x(q) := x_0$. As above, let us choose a straight line through $x_0$ in $\mathbb{R}^3 \setminus x(\hat{F} \cup \hat{H})$ with tangent vector $U = u'e_i$, and let $\gamma_p(t)$ and $\gamma_q(t)$ denote the two integral curves of $U$ in $\hat{L}$ starting at $\gamma_p(0) = p$ and $\gamma_q(0) = q$. The two curves are disjoint by the uniqueness of integral curves. The straight line in $\mathbb{R}^3$ does not go through any fixed points or horizon components, hence by using the argument of the previous paragraph $\gamma_p$ and $\gamma_q$ are complete. We claim that $\gamma_p, \gamma_q$ must enter the asymptotically flat end of $\hat{L}$. For contradiction, suppose the contrary, so that these curves are contained in a compact set $K \subset \hat{L}$. Then by continuity of $x$ the image $x(K)$ is a compact subset of $\mathbb{R}^3$. On the other hand, by completeness of the curves $x(\gamma_p) = x(\gamma_q) = \{x_0 + ut, t \geq 0\}$ is unbounded,$^9$

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$^9$ For compact notation we will denote the vector of functions $x^i$ by $x$. 
so cannot be contained in a compact subset of $\mathbb{R}^3$. Therefore, we have a contradiction, so $\gamma_p, \gamma_q$ must enter the asymptotically flat end of $\hat{L}$ as $t \to \infty$. This means that for any large enough $|x|$, there exist two distinct points with the same $x$ value. This violates asymptotic flatness, since by Corollary 1 the $x$ are global coordinates on the asymptotically flat end of $\mathbb{R}^3$ (thus injective). Therefore, we have obtained a contradiction, and we deduce that $x$ is globally injective and hence a diffeomorphism.

**Remark.** $\hat{\Sigma} \cup \{\hat{H}_i\}$ is in bijection with $\mathbb{R}^3$ (here we are adding each horizon component as a single point). This follows from continuity of $x$ on $\mathcal{M}$ and injectivity on $\hat{L}$.

**Corollary 4.** The set $\{p \in \langle \mathcal{M} \rangle : f(p) \neq 0\}$ is dense in $\langle \mathcal{M} \rangle$.

**Proof.** Since $H$ is harmonic on $\mathbb{R}^3$, it is also real-analytic in $x^i$ (on its domain). It follows that if $H = 0$ on some open set of $\mathbb{R}^3$, it is zero everywhere, and so by (22) $f$ vanishes identically, which cannot happen (e.g. by asymptotic flatness). Therefore, the set $\{x \in \mathbb{R}^3 : f(x) \neq 0\}$ is dense in $\mathbb{R}^3$. By Lemma 9, the $x^i$ are global coordinates on the orbit space $\hat{\Sigma}$, so $\{p \in \hat{\Sigma} : f(p) \neq 0\}$ is also dense in $\hat{\Sigma}$, and since the quotient map is open, $\{p \in \langle \mathcal{M} \rangle : f(p) \neq 0\}$ is also dense in $\langle \mathcal{M} \rangle$, as claimed. \qed

We now determine the behaviour of the harmonic functions $H, K, L, M$ at a fixed point.

**Lemma 10.** Let $p \in \hat{\Sigma}$ be a fixed point of $W$ as above. $H, K, L, M$ have (at most) simple poles at $x(p) \in \mathbb{R}^3$.

**Proof.** By the remark below Lemma 3 $f$ is non-zero on some neighbourhood of $p$ in $\hat{\Sigma}$ and $N > 0$ on this neighbourhood except at $p$. Therefore, from (22) we see that $H$ is also non-zero on some neighbourhood of $p$ in $\hat{\Sigma}$. By Lemma 9, $x : \hat{\Sigma} \to \mathbb{R}^3$ is surjective to some neighbourhood of $x(p)$ and therefore $\hat{H}$ is a harmonic function non-vanishing on a neighbourhood of $x(p)$ in $\mathbb{R}^3$. Using Böcher’s theorem for harmonic functions on $\mathbb{R}^3$ (see e.g. [43]), we see that $H$ has a simple pole at $x(p)$ on $\mathbb{R}^3$. By (22), it follows that all other harmonic functions $K, L, M$ have (at most) simple poles at $x(p)$. \qed

We can now put together the results we have obtained so far to completely fix the functional form of the harmonic functions for any solution satisfying our assumptions.

**Theorem 2.** Any solution $(\mathcal{M}, g, F)$ of $D = 5$ minimal supergravity satisfying assumption 1 and 2 must have a Gibbons-Hawking base (wherever $f \neq 0$) and the associated harmonic functions $H, K, L, M$ are of ‘multi-centred’ form

$$H = \sum_{i=1}^{N} \frac{h_i}{r_i}, \quad K = \sum_{i=1}^{N} \frac{k_i}{r_i}, \quad L = 1 + \sum_{i=1}^{N} \frac{l_i}{r_i}, \quad M = \sum_{i=1}^{N} \left( -\frac{3}{2}k_i + \frac{m_i}{r_i} \right),$$

(78)

where $r_i := |x - a_i|$ and the centres $a_i \in \mathbb{R}^3$ are the coordinates of fixed points of $W$ or connected horizon components, and $h_i, k_i, l_i, m_i$ are constants satisfying

$$\sum_{i=1}^{N} h_i = 1.$$

(79)
Proof. Lemmas 3, 7, 9, 10 imply that the harmonic functions can be written as

\[ H = \sum_{i=1}^{N} \frac{h_i}{r_i} + \tilde{H} \]  

(80)

for some constants \( h_i \), where \( \tilde{H} \) is a harmonic function that is regular everywhere in \( \mathbb{R}^3 \).

The finiteness of the number of centres follows from Assumption 1 (vi) and Lemma 8. On the other hand, asymptotic flatness implies Corollary 2, which implies (79) and that \( \tilde{H} \to 0 \) in the asymptotically flat end. Therefore, \( \tilde{H} \) is a bounded everywhere regular harmonic function on \( \mathbb{R}^3 \) and hence must be a constant, and this constant vanishes using (35) again. Thus, \( H \) takes the claimed form. An identical argument works for the other harmonic functions \( K, L, M \) using (36) giving the claimed form for \( L, K \). For \( M \) this shows that \( M = m + O(r^{-1}) \) where \( m \) is some constant. Then (12) implies \( \omega_{\psi} = m + \frac{3}{2} \sum_{i=1}^{N} k_i + O(r^{-1}) \) and since asymptotic flatness implies \( \omega_{\psi} \to 0 \) at infinity we deduce

\[ m = -\frac{3}{2} \sum_{i=1}^{N} k_i , \]  

(81)

as required. \( \square \)

A consequence of Theorem 2 is that a solution in this class is determined by choosing \( N \) points on \( \mathbb{R}^3 \) corresponding to the simple poles of the harmonic functions, and assigning weights to each of the poles. However, it is not guaranteed that all such solutions correspond to a solution that is smooth in the DOC and at the horizon. In the next section we will determine the necessary and sufficient criteria for this.

3. Smoothness of Multi-centred Solutions

In this section we will determine the conditions required for smoothness of the solution in Theorem 2 at the horizon and the fixed points. In each case the strategy is the same: we locally expand the harmonic functions in terms of spherical harmonics around a centre.

3.1. Regularity and topology of the horizon. As we showed in Lemma 6, a connected component of the horizon corresponds to a simple pole in \( \mathbb{R}^3 \) of the harmonic functions associated to the Gibbons-Hawking base space. Without loss of generality we can take a horizon component at the origin of \( \mathbb{R}^3 \), so the harmonic functions take the form

\[ H = \frac{h_{-1}}{r} + h_0 + \tilde{H} , \]  

(82)

where \( h_{-1}, h_0 \) are constant, \( \tilde{H} \) is a harmonic function which is smooth (in fact analytic) and vanishes at \( r = 0 \), and similarly for \( K, L, M \). It will sometimes be useful to expand \( \tilde{H} = \sum_{l \geq 1, |m| \leq l} h_{lm} r^l Y_l^m \) where \( Y_l^m (\theta, \phi) \) are the spherical harmonics on \( S^2 \) and \( h_{lm} \)

\footnote{While \( \sum_{i=1}^{N} k_i \) and \( m \) are not invariant under a constant shift in the magnetic potential \( \Psi \to \Psi + c \), equation (81) is invariant.}
are constants. It then follows from (10) and (14) that the 1-forms take the form, up to a gradient,

\[ \chi = (h_{-1} \cos \theta + \chi_0) d\phi + \tilde{\chi}, \quad (83) \]
\[ \xi = (-k_{-1} \cos \theta + \xi_0) d\phi + \tilde{\xi}, \quad (84) \]

where we have used the identity \( \ast_3 d(\cos \theta d\phi) = d(r^{-1}) \), \( \chi_0, \xi_0 \) are constants and \( \tilde{\chi}, \tilde{\xi} \) are 1-forms that satisfy \( \ast_3 d\tilde{\chi} = d\tilde{H} \) and \( \ast_3 d\tilde{\xi} = d\tilde{K} \). Therefore, in particular, \( \tilde{\chi}, \tilde{\xi} \) must be smooth 1-forms on \( \mathbb{R}^3 \). Upon expanding the harmonic functions in spherical harmonics we find that, up to a gradient,

\[ \tilde{\chi} = \sum_{l \geq 1, |m| \leq l} \frac{h_{lm} r^{l+1}}{l + 1} \ast_2 dY_l^m, \quad (85) \]

where \( \ast_2 \) is the Hodge star operator for the metric \( d\Omega^2 \) on the unit \( S^2 \), and similarly for \( \tilde{\xi} \).

In order to determine the other 1-form \( \hat{\omega} \) we need to solve (13), which is a bit more complicated. We can decompose this, up to a gradient, as

\[ \hat{\omega} = (\omega_0 + \omega_{-1} \cos \theta) d\phi + \hat{\omega}, \quad \hat{\omega} := \hat{\omega}_{\text{sing}} + \hat{\omega}_{\text{reg}}, \quad (86) \]

where \( \omega_0, \omega_{-1} \) are constants and \( \hat{\omega}_{\text{sing}}, \hat{\omega}_{\text{reg}} \) are 1-forms defined by

\[ \omega_{-1} := h_0 m_{-1} - m_0 h_{-1} + \frac{3}{2} (k_0 l_{-1} - l_0 k_{-1}), \quad (87) \]
\[ \ast_3 d\hat{\omega}_{\text{sing}} = \frac{1}{r} dF - F d\left( \frac{1}{r} \right), \quad F := h_{-1} \tilde{M} - m_{-1} \tilde{H} + \frac{3}{2} (k_{-1} \tilde{L} - l_{-1} \tilde{K}), \quad (88) \]
\[ \ast_3 d\hat{\omega}_{\text{reg}} = d\left( h_0 \tilde{M} - m_0 \tilde{H} + \frac{3}{2} (k_0 \tilde{L} - l_0 \tilde{K}) \right) + \tilde{H} d\tilde{M} - \tilde{M} d\tilde{H} + \frac{3}{2} (\tilde{K} d\tilde{L} - \tilde{L} d\tilde{K}). \quad (89) \]

In particular, \( \hat{\omega}_{\text{reg}} \) is determined by the regular parts of the harmonic functions and therefore must be a smooth 1-form on \( \mathbb{R}^3 \). On the other hand, \( \hat{\omega}_{\text{sing}} \) receives contributions from the singular parts of the harmonic functions and thus requires a little more care. In fact, by expanding the harmonic functions in spherical harmonics, \( F = \sum_{l \geq 1, |m| \leq l} f_{lm} r^l Y_l^m \) for constants \( f_{lm} \), one can derive the explicit expression (again up to a gradient),

\[ \hat{\omega}_{\text{sing}} = \sum_{l \geq 1, |m| \leq l} \frac{f_{lm} r^l}{l} \ast_2 dY_l^m. \quad (90) \]

In particular, \( \hat{\omega}_{\text{sing}} \) is a smooth 1-form on \( S^2 \) for each fixed value of \( r \), since the spherical harmonics \( Y_l^m \) are smooth on \( S^2 \), and vanishes at \( r = 0 \).

We now have all the ingredients to construct the spacetime metric and gauge field near the horizon. In fact, since the first two orders in the \( r \)-expansions of the harmonic functions are \( \phi \)-independent, the analysis is essentially identical to that in the case of solutions with biaxial symmetry [19].

Using (12), (15) and (20) it follows that near the horizon the invariants take the form

\[ N^{-1} = \frac{\alpha_0}{r^2} + \frac{\alpha_1}{r} + O(1), \quad f = \frac{h_{-1}}{\alpha_0} r^2 + \frac{h_0 \alpha_0 - h_{-1} \alpha_1}{\alpha_0^2} r^2 + O(r^3), \quad (91) \]
\[ g_{\psi \psi} = \beta_0 + r \beta_1 + O(r^2), \quad g_{\tau \psi} = r (\gamma_0 + r \gamma_1) + O(r^3), \quad (92) \]
where \( \alpha_i, \beta_i, \gamma_i \) are constants and the error terms are analytic in \( r \) and smooth on \( S^2 \) (since they depend on the spherical harmonics). Since \( N > 0 \) in the DOC away from fixed points and \( W = \partial_\psi \) is spacelike it follows that \( \alpha_0 > 0 \) and \( \beta_0 > 0 \) respectively. In fact, using the explicit expressions for these constants it turns out that these inequalities are equivalent to the single condition \( \alpha_0^2 \beta_0 > 0 \) which reads [19]

\[
-h_{-1}^2 m_{-1}^2 - 3h_{-1}k_{-1}l_{-1}m_{-1} + h_{-1}l_{-1}^3 - 2k_{-1}^3 m_{-1} + \frac{3}{4} k_{-1}^2 l_{-1}^2 > 0 .
\]

(93)

In fact this is not only necessary, but also sufficient for the existence of a smooth horizon away from the axes \( \theta = 0, \pi \). This is revealed by performing the coordinate change \((t, \psi, r, \theta, \phi) \rightarrow (v, \psi', r, \theta, \phi')\) defined by

\[
dr = dv + \left( \frac{A_0}{r^2} + \frac{A_1}{r} \right) dr , \quad d\psi = d\psi' + \frac{B_0}{r} dr - \chi_0 d\phi' , \quad d\phi = d\phi' ,
\]

with

\[
A_0^2 = \beta_0 \alpha_0^2 , \quad B_0 = - \frac{A_0 \gamma_0}{\beta_0} , \quad A_1 = \frac{\alpha_0 \beta_0}{2A_0} \left( B_0^2 \beta_1 + 2B_0 A_0 \gamma_1 + \alpha_1 - \frac{2h_{-1}(h_0 \alpha_0 - h_{-1} \alpha_1)}{\alpha_0^3} A_0^2 \right) .
\]

(95)

We emphasise that the single condition (93) (which is equivalent to \( \alpha_0 > 0, \beta_0 > 0 \)) is sufficient for this coordinate change to exist. This coordinate change is the same as in the case with biaxial symmetry [19]. The metric in the new chart can be written as

\[
g = - f^2 (dv + \dot{\omega}')^2 + 2g_{tr} (dv + \dot{\omega}') (d\psi' + h_{-1} \cos \theta d\phi' + \tilde{\chi}') + 2g_{\psi \phi} (dv + \dot{\omega}') dr + g_{rr} dr^2 + 2g_{\psi \phi} (d\psi' + h_{-1} \cos \theta d\phi' + \tilde{\chi}') dr + g_{\psi \phi} (d\psi' + h_{-1} \cos \theta d\phi' + \tilde{\chi}')^2 + r^2 N^{-1} (d\theta^2 + \sin^2 \theta d\phi'^2) ,
\]

where \( \tilde{\chi}', \dot{\omega}' \) denote the 1-forms \( \tilde{\chi}, \dot{\omega} \) with \( \phi \) replaced by \( \phi' \) and

\[
g_{\psi \phi} = - f^2 \left( \frac{A_0}{r^2} + \frac{A_1}{r} + \frac{\omega \psi B_0}{r} \right) , \quad g_{\psi \phi} = \frac{NB_0}{f^2 r} + g_{tr} \left( \frac{A_0}{r^2} + \frac{A_1}{r} + \frac{\omega \psi B_0}{r} \right) ,
\]

(97)

\[
g_{rr} = \frac{1}{N} + \frac{NB_0^2}{f^2 r^2} - f^2 \left( \frac{A_0}{r^2} + \frac{A_1}{r} + \frac{\omega \psi B_0}{r} \right)^2 .
\]

(98)

Using the expansion of the invariants (91), (92) and the form of the coordinate change (95), it follows that

\[
g_{\psi \phi} = \frac{1}{\sqrt{B_0}} + \mathcal{O}(r) , \quad g_{rr} = \mathcal{O}(1) , \quad g_{\psi \phi} = \mathcal{O}(1) ,
\]

(99)

where the error terms are analytic in \( r \) and smooth on \( S^2 \). Therefore, we deduce from (91), (92), (85), (86), (90), that the spacetime metric (96) and its inverse are analytic in \( r \) at \( r = 0 \) and can be analytically extended to \( r \leq 0 \). The hypersurface \( r = 0 \) is a Killing horizon of \( V = \partial_v \) and the metric induced on the cross-section of the horizon \( v = \text{const} \), \( r = 0 \) is

\[
\beta_0 (d\psi' + h_{-1} \cos \theta d\phi')^2 + \alpha_0 (d\theta^2 + \sin^2 \theta d\phi'^2) .
\]

(100)
Furthermore, it can be shown that the Maxwell field is also analytic at $r = 0$ and the near-horizon limit of the solution takes the same form as in the biaxisymmetric case.

We will now turn to analysing regularity at the axes $\theta = 0, \pi$ including where these intersect the horizon at $r = 0$. By inspecting the horizon metric (100) it is clear the vector fields that vanish at the axes are

$$K^\pm = \partial \phi^\pm \mp h_{-1} \partial \psi^\pm,$$

in particular, $K_+ = 0$ at $\theta = 0$ and $K_- = 0$ at $\theta = \pi$. Therefore, smoothness of the spacetime metric at the axis $\theta = 0, \pi$ requires

$$g_{\mu \rho} K^\rho_\pm = 0 \quad \text{at } \theta = 0, \pi,$$

respectively. (102)

For $\mu = v$ this condition is equivalent to

$$0 = f^2 (\tilde{\omega}_\phi + \omega_\psi (\chi_\phi - \chi_0 \mp h_{-1})) = f^2 (\pm \omega_{-1} + \omega_0) \quad \text{at } \theta = 0, \pi,$$

where we have used (94) and the second equality follows from (83), (86) and the fact that $\tilde{\chi}_\phi = \tilde{\omega}_\phi = 0$ at $\theta = 0, \pi$ for any $r$ (this is because $\partial_\phi = 0$ at $\theta = 0, \pi$ and $\tilde{\chi}_\phi = \iota_{\partial_\phi} \tilde{\chi}$ where $\tilde{\chi}$ is a smooth 1-form on $S^2$ and similarly for $\tilde{\omega}_\phi$). Therefore, since $\omega_{-1}, \omega_0$ are constants the condition $g_{v \rho} K^\rho_\pm = 0 \quad \text{at } \theta = 0, \pi$ is equivalent to

$$\omega_{-1} = \omega_0 = 0.$$

(104)

It can be similarly shown that (104) are sufficient for (102) to hold for all other components $\mu$. Therefore, (102) is equivalent to (104).

To verify smoothness at the axes we also need to check that all higher order terms in the expansion around $\theta = 0$ and $\theta = \pi$ are suitably smooth. We will return to this point below. First it is convenient to perform a global analysis of the horizon geometry in order to deduce the possible horizon topologies. In fact, we will now show that asymptotic flatness imposes global constraints that restrict the horizon topology as in the biaxisymmetric case.

**Lemma 11.** For a multi-centred solution as given in Theorem 2 the topology of cross-sections of each connected component of the event horizon is $S^3$, $S^2 \times S^1$ or a lens space $L(p, 1)$.

**Proof.** The analysis splits into two cases depending on if $h_{-1}$ vanishes. First suppose $h_{-1} \neq 0$. It is convenient to define coordinates adapted to the vectors that vanish on the axes, that is, $K_\pm = \partial \phi^\pm$ where

$$\phi^\pm := \frac{1}{2} \left( \phi' \mp \frac{\psi'}{h_{-1}} \right).$$

(105)

By asymptotic flatness the coordinates $\psi, \phi$ satisfy (41), which is equivalent to $(\theta, \tilde{\phi}, \tilde{\psi})$, defined by (38), being Euler angles on the $S^3$ at spatial infinity. Using the coordinate change (94) this is equivalent to the identifications on the $\phi^\pm$ coordinates

$$P : (\phi^+, \phi^-) \sim \left( \phi^+ - 2\pi \frac{1}{h_{-1}}, \phi^- + 2\pi \frac{1}{h_{-1}} \right),$$

$$R : (\phi^+, \phi^-) \sim \left( \phi^+ + 2\pi \frac{h_{-1} - \chi_0}{2h_{-1}}, \phi^- + 2\pi \frac{h_{-1} + \chi_0}{2h_{-1}} \right).$$

(106)
Recall that the identification lattice of \( L(p, q) \) is generated by

\[
Q : (\phi^+, \phi^-) \sim \left( \phi^+ + 2\pi \frac{1}{p}, \phi^- + 2\pi \frac{q}{p} \right), \\
S : (\phi^+, \phi^-) \sim (\phi^+, \phi^- + 2\pi). \tag{107}
\]

The requirement that (100) extends to a smooth metric on a compact manifold means that the lattice generated by \( \{P, R\} \) must be the same as the one generated by \( \{Q, S\} \). It can be shown that this condition is equivalent to \( h_{-1} = \pm p, \chi_0 \equiv h_{-1} \mod 2 \) and \( q \equiv -1 \mod p \). In particular, notice that \( h_{-1} \) and \( \chi_0 \) are required to be integers with the same parity. Therefore, the allowed topologies are \( L(\pm h_{-1}, -1) \cong L(|h_{-1}|, 1) \) or \( S^3 \) for \( h_{-1} = \pm 1 \).

In the case \( h_{-1} = 0 \) the horizon geometry (100) extends to a smooth metric on a compact manifold if and only if \( \psi' \) and \( \phi' \) are independently periodic and the periodicity of \( \phi' \) is \( 2\pi \). The horizon topology in this case is \( S^1 \times S^2 \). Again, by asymptotic flatness \( \psi \) and \( \phi \) are independently periodic with periodicities \( 4\pi, 2\pi \), respectively (41), which in terms of the coordinates (94) is equivalent to

\[
P : (\psi', \phi') \sim (\psi' + 4\pi, \phi'), \\
R : (\psi', \phi') \sim (\psi' + 2\pi \chi_0, \phi' + 2\pi). \tag{108}
\]

Thus, in order for \( \psi' \) and \( \phi' \) to be independently periodic, \( \chi_0 \) must be an even integer. \( \square \)

Now we have the global geometry of the horizon, we can calculate its area, which yields

\[
A_H = 16\pi^2 \sqrt{-h_{-1}^2 m_{-1}^2 - 3h_{-1}k_{-1}l_{-1}m_{-1} + h_{-1}l_{-1}^3 - 2k_{-1}m_{-1} + \frac{3}{4} l_{-1}^2}. \tag{109}
\]

The quantity inside the square-root is always positive due to (93).

We now return to verifying smoothness at the axes \( \theta = 0, \pi \). For definiteness, we focus on the \( \theta = 0 \) axis, although the argument for \( \theta = \pi \) is identical. First consider the case \( h_{-1} \neq 0 \) and introduce coordinates \( \phi^\pm \) adapted to the vectors \( K_\pm \) that vanish on the axes as in (105). In particular, \( K_+ = \partial_\phi^+ \) vanishes at \( \theta = 0 \) and from the periodicities (107) it is easy to see that \( \phi^+ \) must be \( 2\pi \)-periodic for fixed \( \phi^- \). Now, inverting (105) we have \( \phi' = \phi^+ + \phi^- \) and \( \psi' = h_{-1}(\phi^- - \phi^+) \) which allows us to easily write the full metric (96) in terms of \( \phi^\pm \). In particular, the explicit dependence of the metric components on \( \phi^\pm \) comes from the dependence on \( \phi' \) of the higher order terms in \( r \), which in turn arises from the \( \phi \)-dependence of the spherical harmonics

\[
Y_{l}^{m}(\theta, \phi) = c_{lm} P_{l}^{m}(\cos \theta) e^{im\phi'} = c_{lm} P_{l}^{m}(\cos \theta) e^{im(\phi^+ + \phi^-)} = Y_{l}^{m}(\theta, \phi^+) e^{im\phi^-}. \tag{110}
\]

Therefore, since \( Y_{l}^{m} \) are smooth on \( S^2 \) they will be smooth at the pole \( \theta = 0 \) of the sphere parameterised by \( (\theta, \phi^+) \) (recall \( \phi^+ \) is \( 2\pi \)-periodic for fixed \( \phi^- \)). Furthermore, the 1-form

\[
d\psi' + h_{-1} \cos \theta d\phi' = h_{-1} \left( (1 + \cos \theta) d\phi^- + (-1 + \cos \theta) d\phi^+ \right) \tag{111}
\]

is manifestly smooth at the axis \( \theta = 0 \). Hence, from the \( r \)-expansion of the functions (91), (92) and the 1-forms (85), (86), (90) together with (104), it now follows that the full metric...
is smooth at the axis $\theta = 0$, at least on a neighbourhood of the horizon $r = 0$ (the domain of convergence of the expansion of the harmonic functions into spherical harmonics). This establishes that the metric is smooth at the axis including up to where it intersects the horizon if and only if (104) holds. It can be easily seen that the Maxwell field is also smooth at the axes including at the intersection with the horizon (this essentially follows from the fact that the only new type of term is from $d\xi = k - 1 \sin \theta d\theta \wedge d\phi + d\tilde{\xi}$ using (84)).

Finally, let us consider the case $h - 1 = 0$. This is simpler since as observed above $\psi', \phi'$ are independently periodic and correspond to the angle coordinates on the $S^1$ and $S^2$ factors of the horizon respectively. Therefore, the metric (96) is smooth on the $S^2$ parameterised by $(\theta, \phi')$ in a neighbourhood of $r = 0$ if and only if (104), since the dependence of the higher order terms in $r$ on $(\theta, \phi')$ is through the spherical harmonics.

To summarise, we have shown that the solution is smooth at a horizon, including where it intersects the axis, if and only if the coefficients in the expansions (82), (83) etc, satisfy (93), (104),

$$h - 1 \in \mathbb{Z}, \quad \chi_0 + h - 1 \in 2\mathbb{Z},$$

and the horizon topology is $S^1 \times S^2$ if $h - 1 = 0$, $S^3$ if $h - 1 = \pm 1$ and $L(|h - 1|, 1)$ otherwise.

3.2. Smoothness at the fixed points. In this section we will derive the necessary and sufficient criteria for the solution to be smooth around a fixed point of the axial Killing field $W$. As shown in Lemma 10 a fixed point corresponds to a simple pole of the harmonic functions $H, K, L, M$, so without loss of generality, we will take this to be at the origin. We then expand these functions in the same way that we did for a horizon in equation (82). It follows that the 1-forms $\chi, \xi, \omega$ can also be written in the same form as for a horizon, namely these are given by equations (83–90).

As argued earlier, at a fixed point we must have $f \neq 0$ and therefore the base metric is well-defined at least on a neighbourhood of such points. It follows that the base metric $h$, the 1-form $\omega$ and the function $f$ must be smooth at the fixed points. Let us first consider the base metric near a fixed point at $r = 0$. It is convenient to introduce coordinates

$$r = \frac{1}{4} R^2, \quad \psi' = \psi + \chi_0 \phi', \quad \phi' = h - 1 \phi,$$

so that the base metric takes the form

$$h = G \left( dR^2 + \frac{1}{4} R^2 \left[ d\theta^2 + \sin^2 \theta \frac{d\phi'^2}{h_{-1}^2} + \frac{1}{G^2} (d\psi' + \cos \theta d\phi' + \tilde{\chi})^2 \right] \right),$$

where we have defined $G := r H = h_{-1} + O(R^2)$ and $\tilde{\chi} = O(R^4)$ is defined by the regular part $\tilde{H}$ of the harmonic function which can be written as (85). In order to avoid a curvature singularity of the base metric as $R \to 0$ we must require

$$h_{-1} = \pm 1,$$

in which case (114) approaches a locally flat metric on $\pm \mathbb{R}^4$. Furthermore, to avoid any conical singularities at $R = 0$ the angles $(\theta, \phi', \psi')$ must be Euler angles on $S^3$ with identifications as in (40), in which case the base metric approaches the flat metric on $\pm \mathbb{R}^4$ (note $G = 1$ and $\tilde{\chi} = 0$ corresponds to the flat metric on $\mathbb{R}^4$). On the other hand, by
asymptotic flatness, \((\psi, \phi)\) must obey identifications \((41)\), which implies that \((\psi', \phi')\) have the correct identification lattice \((40)\) if and only if \(\chi_0\) is an odd integer.

To verify the base metric is smooth at the fixed point requires control of the higher order terms as \(r \to 0\). In particular, we must check that the metric is smooth at the origin of \(\mathbb{R}^4\). For this, it is easiest to use cartesian coordinates \((u_I)_{I=1,2,3,4}\) on \(\mathbb{R}^4\), which are introduced as follows. We first define the double-polar coordinates \(11\)

\[
X_+ = R \cos \frac{\theta}{2}, \quad X_- = R \sin \frac{\theta}{2}, \quad \phi^\pm = \frac{1}{2}(\psi' \pm \phi'),
\]

in terms of which the leading order metric is

\[
h \sim \pm \left( dX_+^2 + X_+^2(d\phi^+)^2 + dX_-^2 + X_-^2(d\phi^-)^2 \right).
\]

Hence, cartesian coordinates on \(\mathbb{R}^4\) are given by

\[
u_1 = X_+ \cos \phi^+, \quad \nu_2 = X_+ \sin \phi^+, \quad \nu_3 = X_- \cos \phi^-, \quad \nu_4 = X_- \sin \phi^-.
\]

It is helpful for the analysis to note that the following \(\mathbb{R}^3\)-functions are smooth on \(\mathbb{R}^4\):

\[
r = \frac{1}{4} (u_1^2 + u_2^2 + u_3^2 + u_4^2), \quad r \cos \theta = \frac{1}{4} (u_1^2 + u_2^2 - u_3^2 - u_4^2),
\]

\[
r \sin \theta \exp(i\phi) = \frac{1}{2} (u_1 + ih_{-1}u_2)(u_3 - ih_{-1}u_4).
\]

Furthermore, noting that the \(\mathbb{R}^3\) cartesian coordinates satisfy \(x^1 + i x^2 = r \sin \theta e^{i\phi}\) and \(x^3 = r \cos \theta\), it immediately follows that \(x^i\) are smooth functions on \(\mathbb{R}^4\), and more generally any smooth function \(f(x)\) on \(\mathbb{R}^3\) is also a smooth function on \(\mathbb{R}^4\). In particular, any regular harmonic function on \(\mathbb{R}^3\) is automatically smooth on \(\mathbb{R}^4\). Therefore, the function \(G = h_{-1} + r(h_0 + \tilde{H})\) defined above is smooth on \(\mathbb{R}^4\). It is also helpful to note that the 1-form

\[
r(d\psi' + \cos \theta d\phi') = \frac{1}{2} (X_+^2 d\phi^+ + X_-^2 d\phi^-)
\]

is smooth on \(\mathbb{R}^4\) (convert to cartesian coordinates \(u_I\)), and the 1-form \(\tilde{\chi}\) which is defined by \(\star_3 d\tilde{\chi} = d\tilde{H}\) where \(\tilde{H}\) is the regular part of the harmonic function in \((82)\) must also be smooth on \(\mathbb{R}^4\) (up to a gauge transformation). Now, we can write the base metric \((114)\) as

\[
h = G du_I du_I - \frac{(h_{-1} + G)(h_0 + \tilde{H})}{G} r^2 (d\psi' + \cos \theta d\phi')^2 + \frac{2}{G} r(d\psi' + \cos \theta d\phi') \tilde{\chi} + \frac{r}{G} \tilde{\chi}^2,
\]

where we used \((82)\) and the definition \(G = rH\). It is now manifest that the base metric is smooth at the origin of \(\mathbb{R}^4\), since all functions and 1-forms that we have written it in terms of are smooth by the above comments.

We now turn to smoothness of the function \(f\) on the base. Using \((82)\), we have the useful identity

\[
(rH)^{-1} = h_{-1} - rH_0 + r^2 G_1
\]

\(\phi^\pm\) in this section are defined differently to those in Sect. 3.1.
where \( H_0 := h_0 + r \tilde{H} \) and \( G_1 := \tilde{H}_0^2 / (h_{-1} + rH_0) \) is a smooth function on \( \mathbb{R}^4 \). Then, using (15) and (122), it is easy to see that

\[
f^{-1} = \frac{l_{-1} + k_{-1}^2 h_{-1}}{r} + l_0 - h_0 k_{-1}^2 + 2h_{-1}k_{-1}k_0 + \mathcal{O}(r) ,
\]

where the error terms are smooth on \( \mathbb{R}^4 \). Therefore, since we must have \( f \neq 0 \) at a fixed point, smoothness requires

\[
l_{-1} = -h_{-1}k_{-1}^2 ,
\]

\[
h_{-1}(l_0 - h_0 k_{-1}^2 + 2h_{-1}k_{-1}k_0) > 0 ,
\]

where the sign of the inequality is chosen to ensure the spacetime metric has the correct signature at the fixed point. We deduce that these are the necessary and sufficient conditions for \( f \) to be smooth at the fixed point.

Let us now look at the 1-form \( \omega \) which decomposes as (16). The invariant \( g(V, W) = -f^2 \omega \psi \), together with the fact that \( f \neq 0 \) at the fixed point, implies that \( \omega \psi \) must be a smooth function that vanishes at the fixed point. By expanding (12) near \( r = 0 \) and using (124) one finds

\[
\omega \psi = m_{-1} - \frac{1}{2}k_{-1}^3 + \mathcal{O}(1) ,
\]

where the error terms are smooth. Thus, in particular we must require

\[
m_{-1} = \frac{1}{2}k_{-1}^3 .
\]

Then, using (122) together with (124), (127), we can write (12) as

\[
\omega \psi = h_{-1}(-\omega_{-1} + F) + r \tilde{G}_1 ,
\]

where \( \omega_{-1} \) is the constant defined in (87), \( F \) is the harmonic function defined in (88), and \( \tilde{G}_1 \) is a smooth function on \( \mathbb{R}^4 \). Therefore, since \( F \) vanishes at \( r = 0 \), the vanishing of \( \omega \psi \) at \( r = 0 \) occurs iff

\[
\omega_{-1} = 0 .
\]

Thus, together with the form for \( \hat{\omega} \) given in (86), we can write

\[
\omega = h_{-1} F(d\psi' + \cos \theta d\phi') + \hat{\omega}_{\text{sing}} + \omega_0 h_{-1} d\phi' + \tilde{G}_1 r (d\psi' + \cos \theta d\phi') + \omega_\psi \tilde{\chi} + \hat{\omega}_{\text{reg}} ,
\]

where the fact that the last three terms are smooth immediately follows from our above analysis. Therefore, smoothness of \( \omega \) reduces to that of \( \alpha \). From (90) and the fact that \( F \) is harmonic it automatically follows that both terms of \( \alpha \) are smooth at \( r > 0 \), but smoothness at \( r = 0 \) remains to be checked. To this end, a short computation reveals that

\[
\star_8 \delta \alpha = -\frac{h_{-1}}{r} \star_3 dF + r(d\psi' + \cos \theta d\phi') \wedge (h_{-1} F \star_3 d\psi' + d\hat{\omega}_{\text{sing}})
\]

\[
= -\frac{h_{-1}}{r} \star_3 dF + (d\psi' + \cos \theta d\phi') \wedge dF ,
\]

where
where $\star_\delta$ denotes the Hodge dual with respect to the flat metric $\delta = r (d\psi' + \cos \theta d\phi')^2 + r^{-1} dx^i dx^i$ on $\mathbb{R}^4$ with orientation $\epsilon_{\psi'123} = 1$, and the second line is obtained using (88). Then, using the fact that $F$ is harmonic, we deduce

$$d \star_\delta d\alpha = 0. \quad (132)$$

Therefore, $\alpha$ must be a smooth 1-form on $\mathbb{R}^4$ (up to a gauge transformation). In Appendix C we show that $\alpha$ is a smooth 1-form on $\mathbb{R}^4$ by an explicit calculation. We deduce that $\omega$ is smooth on $\mathbb{R}^4$ if and only if the constant $\omega_0 = 0$.

We finish this section with the analysis of the Maxwell field, which takes the form

$$F = \frac{\sqrt{3}}{2} d \left[ f (dt + \omega) - \frac{K}{H} (d\psi' + \cos \theta d\phi') + h_{-1} k_{-1} \cos \theta d\phi' - \frac{K}{H} \tilde{\chi} + \tilde{\xi} \right], \quad (133)$$

where we have used (17) and (84). The 1-form $\tilde{\xi}$ is smooth on $\mathbb{R}^4$ by the same argument for $\tilde{\chi}$, and by the above analysis we also know that $f (dt + \omega)$ and $(K/H) \tilde{\chi}$ are smooth. Using (122), the middle two terms in the gauge field can be rewritten as

$$- \frac{K}{H} (d\psi' + \cos \theta d\phi') + h_{-1} k_{-1} \cos \theta d\phi' = -k_{-1} h_{-1} d\psi'$$

$$+ \left( k_{-1} (H_0 - r G_1) - (k_0 + \tilde{K})(r H)^{-1} \right) r (d\psi' + \cos \theta d\phi'), \quad (134)$$

where the terms on the second line are manifestly smooth. Thus, the only non-smooth term is pure gauge, so the Maxwell field is indeed smooth on $\mathbb{R}^4$. This concludes our analysis at a fixed point of $W$.

To summarise, we have shown that the solution at a fixed point $r = 0$ is smooth if and only if the parameters satisfy (115), (124), (125), (127), (129), $\omega_0 = 0$ and

$$\chi_0 \in 2\mathbb{Z} + 1. \quad (135)$$

The spacetime in a neighbourhood of a such point is then diffeomorphic to $\mathbb{R}^5$.

4. General Black Hole and Soliton Solutions

4.1. Classification theorem. We are now ready to deduce the main result of this paper which provides a classification of black hole and solitons spacetimes that satisfy our assumptions.

**Theorem 3.** Any asymptotically flat, supersymmetric black hole or soliton solution $(\mathcal{M}, g, F)$ of $D = 5$ minimal supergravity, with an axial symmetry, satisfying assumption 1 and 2 must have a Gibbons-Hawking base (wherever $f \neq 0$) with 'multi-centred' harmonic functions

$$H = \sum_{i=1}^{N} \frac{h_i}{r_i}, \quad K = \sum_{i=1}^{N} \frac{k_i}{r_i}, \quad L = 1 + \sum_{i=1}^{N} \frac{l_i}{r_i}, \quad M = m + \sum_{i=1}^{N} \frac{m_i}{r_i}, \quad (136)$$
where \( r_i := |x - a_i| \), and \( a_i = (x_i, y_i, z_i) \in \mathbb{R}^3 \) correspond to fixed points of the axial Killing field or the connected components of the horizon, and the 1-forms can be written as

\[
\chi = \sum_{i=1}^{N} \left( \chi_i^0 + \frac{h_i(z - z_i)}{r_i} \right) d\phi_i , \\
\xi = -\sum_{i=1}^{N} \frac{k_i(z - z_i)}{r_i} d\phi_i , \tag{137}
\]

\[
\hat{\omega} = \sum_{i, j=1 \atop i \neq j}^{N} \left( h_i m_j + \frac{3}{2} k_i l_j \right) \beta_{ij} , \tag{138}
\]

where

\[
d\phi_i := \frac{(x - x_i)dy - (y - y_i)dx}{(x - x_i)^2 + (y - y_i)^2} , \tag{139}
\]

\[
\beta_{ij} := \frac{(x - a_i) \cdot (a_i - a_j)}{|a_i - a_j| r_i} - \frac{(x - a_j) \cdot (a_i - a_j)}{|a_i - a_j| r_j} - \frac{(x - a_i) \cdot (x - a_j)}{r_i r_j} + 1 \tag{140}
\]

The parameters \( h_i, k_i, l_i, m_i \) must satisfy

\[
\sum_{i=1}^{N} h_i = 1 , \\
m = -\frac{3}{2} \sum_{i=1}^{N} k_i , \tag{141}
\]

and for each \( i = 1, \ldots, N \),

\[
h_i m + \frac{3}{2} k_i + \sum_{j=1 \atop j \neq i}^{N} \frac{h_i m_j - m_i h_j + \frac{3}{2} (k_i l_j - k_j l_i)}{|a_i - a_j|} = 0 . \tag{142}
\]

Moreover, if \( a_i \) is a fixed point \( \chi_i^0 \in 2\mathbb{Z} + 1 \),

\[
h_i = \pm 1 , \\
l_i = -h_i k_i^2 , \\
m_i = \frac{1}{2} k_i^3 , \tag{143}
\]
\[ h_i + \sum_{j=1 \atop j \neq i}^{N} \frac{2k_i k_j - h_i (h_j k_i^2 - l_j)}{|a_i - a_j|} > 0, \]  
\begin{equation}
(144)
\end{equation}

whereas if \( a_i \) is a horizon component \( h_i \in \mathbb{Z}, \chi_0^i + h_i \in 2\mathbb{Z} \) and

\[ -h_i^2 m_i^2 - 3h_i k_i l_i m_i + h_i l_i^3 - 2k_i^2 m_i + \frac{3}{4}k_i^2 l_i^2 > 0, \]
\begin{equation}
(145)
\end{equation}

and the horizon topology is \( S^1 \times S^2 \) if \( h_i = 0 \), \( S^3 \) if \( h_i = \pm 1 \) and a lens space \( L(h_i, 1) \) otherwise. Finally, the harmonic functions must satisfy

\[ K^2 + H L > 0 \]
\begin{equation}
(146)
\end{equation}

for all \( x \in \mathbb{R}^3 \setminus \{a_1, \ldots, a_N\} \).

**Proof.** Theorem 2 shows that the solution must have a Gibbons-Hawking base with harmonic functions of multi-centred form (136), and that asymptotic flatness implies the parameter constraints (141). The 1-forms are determined as follows. First note that \( \beta_i := (z - z_i) d\phi_i / r_i \) obeys \( \star_3 d\beta_i = d(1 / r_i) \). Therefore, solving (10), (14) immediately gives that \( \chi, \xi \) can be written in the claimed form, where \( \chi_0^i \) are integration constants introduced for convenience. To determine \( \hat{\omega} \) it is convenient to follow [22] (see also [18, 27]) and define the 1-forms (140) which are a solution to

\[ \star_3 d\beta_{ij} = \frac{1}{r_i} d \left( \frac{1}{r_j} \right) - \frac{1}{r_j} d \left( \frac{1}{r_i} \right) + \frac{1}{r_{ij}} d \left( \frac{1}{r_i} - \frac{1}{r_j} \right), \]
\begin{equation}
(147)
\end{equation}

where \( r_{ij} := |a_i - a_j| \). Note that, in contrast to \( \beta_i \), the 1-forms \( \beta_{ij} \) are free of Dirac string singularities, indeed \( \beta_{ij} \) are smooth 1-forms on \( \mathbb{R}^3 \) except at the corresponding centres \( a_i, a_j \). Then, we can solve (13) and write

\[ \hat{\omega} = \sum_{i, j=1 \atop i \neq j}^{N} \left( h_i m_j + \frac{3}{2} k_i l_j \right) \beta_{ij} + \sum_{i=1}^{N} \omega_{-1}^i \beta_i, \]
\begin{equation}
(148)
\end{equation}

where we have defined the constants

\[ \omega_{-1}^i := -h_i m - \frac{3}{2} k_i = \sum_{j=1 \atop j \neq i}^{N} \frac{h_i m_j - m_i h_j + \frac{3}{2}(k_i l_j - k_j l_i)}{r_{ij}}. \]
\begin{equation}
(149)
\end{equation}

The functional form of the solution is now fully fixed. In particular, notice that asymptotically as \( r \to \infty \),

\[ \chi = \left( \sum_{i=1}^{N} \chi_0^i + \cos \theta \right) d\phi + \mathcal{O}(r^{-1}), \]
\begin{equation}
(150)
\end{equation}

so upon comparison to (37) we deduce \( \tilde{\chi}_0 = \sum_{i=1}^{N} \chi_0^i \) where (38) give the coordinates that are manifestly asymptotically flat. Furthermore,

\[ \hat{\omega} = \sum_{i=1}^{N} \omega_{-1}^i \cos \theta d\phi + \mathcal{O}(r^{-1}) = \mathcal{O}(r^{-1}), \]
\begin{equation}
(151)
\end{equation}
where the second equality uses that $\sum_{i=1}^{N} \omega_{i-1} = 0$ which follows from (141). Thus, the solution is asymptotically flat.

Next, by expanding the harmonic functions around a centre $x = a_i$ they take the form (82) where the coefficients are given by

\begin{align}
    h_{-1} &= h_i , \quad h_0 = \sum_{j=1}^{N} \frac{h_j}{r_{ij}} , \quad k_{-1} = k_i , \quad k_0 = \sum_{j=1}^{N} \frac{k_j}{r_{ij}} , \\
    l_{-1} &= l_i , \quad l_0 = 1 + \sum_{j=1}^{N} \frac{h_j}{r_{ij}} , \quad m_{-1} = m_i , \quad m_0 = m + \sum_{j=1}^{N} \frac{m_j}{r_{ij}} .
\end{align}

(152)

(153)

Necessary and sufficient conditions for smoothness at the event horizon and fixed points were determined in Sects. 3.1 and 3.2 in terms of these coefficients. For smoothness at fixed points, we must require (115), (124), (125), (127) and (135), which upon use of the above coefficients give (143), (144) and that $\chi_0^i$ must be an odd integer. For smoothness at the horizon, we require (93), (112), which upon use of the above coefficients gives (145), $h_i \in \mathbb{Z}$ and $\chi_0^i + h_i$ must be an even integer. The horizon topology is determined in Lemma 11. Notice in particular that $h_i$ and $\chi_0^i$ have the same parity at each centre. Now, recall that we chose a gauge where $\tilde{\chi}_0$ is an odd integer, so we must check that this is compatible with the smoothness constraints at all centres. Indeed, we have $\tilde{\chi}_0 = \sum_{i=1}^{N} \chi_i = \sum_{i=1}^{N} h_i = 1 \mod 2$, where in the second equality we used that $\chi_i$ and $h_i$ are of the same parity and in the final equality (141). Therefore, $\tilde{\chi}_0$ is automatically odd which means that the smoothness constraints for $\chi$ at all centres can be satisfied simultaneously.

The remaining conditions for smoothness at both fixed points and horizons are $\omega_{-1} = 0$ and $\omega_0 = 0$, which are defined by (86) and (87). Upon using the above coefficients $\omega_{-1} = 0$ is equivalent to the constraint (142) for each $i = 1, \ldots, N$, which in fact is the same as $\omega_{i-1} = 0$ (see definition (149)) so that the 1-form $\omega$ now has the claimed form (138). Furthermore, from the explicit form of $\beta_{ij}$ one can check that $\omega_0 = 0$ for each centre (this is due to the aforementioned fact that $\beta_{ij}$ has no string singularities). Therefore, all necessary and sufficient conditions for smoothness at fixed points and horizons are now satisfied.

Finally, (146) is required by Lemma 2 and is equivalent to smoothness of the solution in the DOC away from the fixed points.

Remarks. 1. It is not known if the conditions (141)-(146) in the above theorem are also sufficient for the solution to have a globally hyperbolic DOC, although it is clear that other assumptions in (1) and (2) are indeed satisfied. In particular, it is not known if they are even sufficient for stable causality $g^{tt} < 0$ (which is a consequence of global hyperbolicity). In [44] the authors argue that for solitons (146) implies stable causality and support this with numerical evidence. In Appendix D we present a numerical analysis of three-centred solutions and find that those configurations which satisfy (141)-(146) together with positive mass $M > 0$ (given by (155)) are indeed stably causal. This adds to previous evidence for biaxially symmetric solutions that stable causality is a consequence of these conditions [17,19–21].

2. The explicit form of the 1-forms (137) possess $N$ Dirac string singularities parallel to the $z$-axis and is a gauge choice. While these cannot be removed simultaneously, by a local coordinate transformation each string can be rotated into any direction. The spacetime can be covered by a family of charts in which the strings are arranged to be
between every other consecutive centre (for some arbitrary ordering of the centres), similarly to the multi-centred gravitational instantons [45].

4.2. Conserved charges and fluxes. Conserved charges associated to Killing fields can be determined by Komar integrals. Therefore the mass $M$ and angular momentum $J_\psi$ can be computed using Komar integrals with respect to the Killing vectors $\partial_t$, $\partial_\psi$ respectively. However, as we will show in the next section, generically $\partial_\phi$ is not a Killing vector. In order to compute the angular momentum $J_\phi$ associated to this, one can compare the asymptotic form of the metric to that of a localised source with known charges [1]. In order for the metric to be in the right form asymptotically one must align the $z$-axis in $\mathbb{R}^3$ with the direction of $D$ defined as [46]

$$D = \sum_{i=1}^{N} \left( k_i - h_i \sum_{j=1}^{N} k_j \right) a_i. \quad (154)$$

Note that this does not depend on the choice of origin in $\mathbb{R}^3$ due to (141). One then finds that the conserved charges of the solution in Theorem 3 are

$$M = 3\pi \left( \left( \sum_{i=1}^{N} k_i \right)^2 + \sum_{i=1}^{N} l_i \right), \quad Q = 2\sqrt{3}\pi \left( \left( \sum_{i=1}^{N} k_i \right)^2 + \sum_{i=1}^{n} l_i \right),$$

$$J_\psi = 2\pi \left( \left( \sum_{i=1}^{N} k_i \right)^3 + \frac{3}{2} \sum_{i=1}^{N} k_i l_j + \sum_{i=1}^{n} m_i \right), \quad J_\phi = 3\pi |D|. \quad (155)$$

As expected, the solutions saturate the BPS bound with $M = \sqrt{3}Q/2$. Notice these take the same form as in the biaxially symmetric case [19].

Each curve in $\mathbb{R}^3$ that connects two centres lifts to a non-contractible 2-cycle in spacetime ending on fixed points of $W$ or a horizon component. These 2-cycles either smoothly cap off at the fixed points as the length of orbits of $W$ goes to zero, or end on the horizon. They have topology of $S^2$ (between fixed points), a disc (between a fixed point and a horizon), or a tube (between horizon components). The flux through a 2-cycle $C_{ij}$ between centres $a_i$ and $a_j$ is defined as

$$\Pi_{ij} := \frac{1}{4\pi} \int_{C_{ij}} F = \lim_{r_i \to 0} A_\psi - \lim_{r_j \to 0} A_\psi, \quad (156)$$

where we used Stokes’ theorem. Using the explicit form of the Maxwell field (17), and taking the limit yields

$$\lim_{r_i \to 0} A_\psi = \begin{cases} \frac{-\sqrt{3} k_i}{2 h_i}, & \text{if } i \text{ corresponds to a fixed point,} \\ \frac{\sqrt{3} h_i m_i + \frac{1}{2} k_i l_i}{k_i^2 + h_i l_i}, & \text{if } i \text{ corresponds to a horizon.} \end{cases} \quad (157)$$
4.3. Symmetries. Theorem 3 shows that there is no requirement that forces the centres $a_i \in \mathbb{R}^3$ to be collinear on $\mathbb{R}^3$. Naturally, one would expect that a solution with centres in generic positions (although still satisfying the constraints of the theorem) has less symmetry than those with collinear centres. In this section we will show that this is indeed the case. We will only consider Killing fields of $(\mathcal{M}, g)$ that commute with the supersymmetric Killing field $V$, that is, symmetries which are also symmetries of the base space (wherever it is defined).

Due to asymptotic flatness any Killing field of $(\mathcal{M}, g)$ must approach a linear combination of those of Minkowski at asymptotic infinity [40], and in Lemma 5 we showed that $W$ generates an isoclinic\(^\text{12}\) rotation at infinity, i.e. $W = \frac{1}{2}(J_{12} + J_{34}) + O(R^{-1})$, where $J_{ij}$ were defined in Lemma 5. Therefore, excluding boosts (as they do not commute with $V$) and $V$ itself, the algebra of the Killing fields of $(\mathcal{M}, g)$ must be isomorphic to some subalgebra of the Lie algebra of the four-dimensional euclidean group $E(4)$ that contains $J_{12} + J_{34}$. Such algebras must contain one of the following [47]:

(i) A Killing field that commutes with $W$, e.g. a subalgebra $\langle J_{12}, J_{34} \rangle$,
(ii) An $SU(2)$ subalgebra $\langle J_{12} + J_{34}, J_{13} - J_{24}, J_{14} + J_{23} \rangle$,
(iii) An $E(2)$ subalgebra $\langle J_{12} + J_{34}, P_1, P_2 \rangle$,

where the translations $P_I = \partial_I$ in the coordinates of Lemma 5. We will now consider each case in turn.

For case (i) let us denote the additional Killing field by $\xi$, which by assumption commutes with $V$ and $W$. The orbit space $\Sigma$ inherits a metric from the spacetime $q_{\mu\nu} := g_{\mu\nu} - G^{AB}g_{A\mu}g_{B\nu}$, where $G^{AB}$ is the inverse of matrix $G_{AB}$ in (20), and indices $\{A,B\}$ correspond to $\{t,\psi\}$. We find that this orbit space metric is

$$q = \frac{1}{N}dx^i dx^i,$$  \hfill (158)

which is non-singular on the region $N > 0$ (i.e. in the DOC away from fixed points). Now, since $\xi$ preserves both the spacetime metric and Killing fields $V$ and $W$, it follows that $\mathcal{L}_\xi q = 0$ and $\mathcal{L}_\xi N = 0$, i.e. the orbit space has a Killing field that preserves $N$. From the explicit form of the orbit space metric (158) it follows that $\xi$ is a Killing field of the euclidean metric on $\mathbb{R}^3$. From (22) it follows that the harmonic functions $H, K, L, M$ are invariant under $\xi$. Now, 1-parameter subgroups of the isometry group of euclidean space $\mathbb{R}^3$ are either closed (rotation) or unbounded (translation with possibly a rotation). However, since $H$ is invariant under this subgroup and $H \rightarrow 0$ at infinity by asymptotic flatness, it must be that $\xi$ generates a rotation. Thus, $\xi$ is an axial Killing field of $\mathbb{R}^3$ and hence the centres $a_i$ must be collinear. In this case the full spacetime solution has $\mathbb{R} \times U(1)^2$ symmetry and corresponds to the biaxially symmetric case [19]. Therefore, if the centres are not collinear the abelian isometry group of the solution cannot be larger than $\mathbb{R} \times U(1)$.

In case (ii) the $U(1)$ isometry generated by $W$ is a subgroup of the $SU(2)$ isometry. The complex structures of the hyper-Kähler base space must carry a real three-dimensional representation of any subgroup of its isometry group, which for $SU(2)$ must be either the trivial or adjoint representation. The latter is excluded by the triholomorphicity of $W$, therefore the whole $SU(2)$ symmetry is necessarily triholomorphic. Furthermore, from the asymptotic form of the Killing fields it follows that around spatial infinity the $SU(2)$ action has three-dimensional orbits. Hyper-Kähler manifolds with

\(^{12}\) That is, $W$ generates a rotation around two orthogonal axes with the same angle.
cohomogeneity-1 triholomorphic $SU(2)$ symmetry belong to the BGPP class of solutions [48]. In Appendix E we show that the only multi-centred asymptotically euclidean BGPP solution is the trivial flat metric on $\mathbb{R}^4$. Therefore, case (ii) cannot happen.

In case (iii) two of the generators corresponding to translations commute, therefore there exists a linear combination $K$ of them which is triholomorphic [26]. Since $W$ has closed orbits it must correspond to the rotation in $E(2)$ and hence $W, K$ are linearly independent triholomorphic Killing fields. It follows that $[W,K]$ is also triholomorphic and by the algebra of $E(2)$ it must be a third independent Killing field. Therefore, the whole $E(2)$ symmetry must be triholomorphic. Similarly to case (ii), the $E(2)$ action has generically three-dimensional orbits at spatial infinity. In Appendix E we derive the most general hyper-Kähler metric with cohomogeneity-1 triholomorphic $E(2)$ symmetry and show that the only multi-centred Gibbons-Hawking metrics in this class is the trivial flat metric on $\mathbb{R}^4$. Therefore, case (iii) also cannot happen.

Therefore, we have shown that our general solution in Theorem 3 generically possesses an abelian isometry group $\mathbb{R} \times U(1)$ and not larger. Furthermore, this is enhanced to $\mathbb{R} \times U(1)^2$ precisely if the centres are collinear in $\mathbb{R}^3$. To our knowledge the construction outlined in this paper provides the first examples of smooth, asymptotically flat black hole solutions in higher dimensions with exactly a single axial symmetry (on top of a stationary symmetry), confirming the conjecture of Reall at least for supersymmetric black holes [14]. In Appendix D we present a numerical study of three-centred solutions and find that the parameter constraints (141–145) can be easily satisfied even for non-collinear centres. Furthermore, (146) can be proven to hold for the case of a black lens $L(3,1)$. For the other horizon topologies we found a large set of parameters that numerically satisfy the parameter constraints and for which (146) holds. Therefore, we expect that there is a vast moduli space of asymptotically flat supersymmetric black holes with exactly a single axial symmetry.

5. Discussion

In this work we have presented a classification of asymptotically flat, supersymmetric black holes in five-dimensional minimal supergravity. Our main assumption is that there is an axial symmetry that ‘commutes’ with the remaining supersymmetry, i.e. it preserves the Killing spinor. It is an interesting question whether there are black hole solutions if this assumption is relaxed. A natural possibility would be to only require that the axial symmetry commutes with the supersymmetric Killing field, but that the Killing spinor is not preserved by its flow. Then, the complex structures on the base space (spinor bilinears) are no longer preserved by the axial symmetry, i.e. the axial Killing field is not triholomorphic. In this case the hyper-Kähler base is not a Gibbons-Hawking metric, but instead can be obtained by solving the $SU(\infty)$ Toda equation [49].

There exist constructions of supersymmetric solutions without a triholomorphic axial symmetry [50], however no smooth black hole solution of this class is known.

There have been a number of previous attempts to construct supersymmetric black holes with exactly a single axial symmetry, however in all these cases the spacetime metric itself or the matter fields are not smooth at the horizon [31–36]. It is worth noting that even the Majumdar-Papapetrou static multi-black hole spacetimes do not have smooth horizons in five (or higher) dimensions [51–54]. In contrast to our construction, a common feature of all these solutions is that they break the $U(1)$ rotational symmetry that corresponds to the triholomorphic Killing field $\partial_\psi$ in our setting. Generally, it is expected that breaking a rotational symmetry leads to non-smooth horizons, essentially
because coordinate changes of the type (94) necessarily introduce logarithmically divergent terms in the corresponding angular coordinate (ψ in this case), which would cause infinite oscillation in the metric and or matter fields near the horizon [35,36]. Our construction avoids this problem because we break the symmetry in a direction which is not rotational13 near the horizon (i.e. the angular coordinate φ does not diverge at the horizon by (94)). Furthermore, the classification of near-horizon geometries [14] shows that the rotational symmetry near the horizon is always triholomorphic, which suggests that regular black hole solutions without a triholomorphic rotational symmetry may not exist. However, to properly check this, one would need to carefully analyse the near-horizon geometry in coordinates corresponding to the aforementioned Toda system.

For the class of solutions studied in this work, it is not obvious how large the moduli space of black holes is. Theorem 3 imposes equations and inequalities not only on the parameters of the harmonic functions, but also on a combination of harmonic functions (146) which must be satisfied everywhere on \( \mathbb{R}^3 \). It would significantly advance our understanding of the moduli space of black holes if an equivalent condition were known on the parameters of harmonic functions. The positivity of the ADM mass has been conjectured to be sufficient (together with the other parameter constraints listed in Theorem 3) [20], however our numerical results show that this is unfortunately not the case (see Appendix D), even for the biaxially symmetric case. This prevents us from having a totally explicit description of the black hole moduli space in this theory.

Finally, as mentioned in the introduction, the rigidity theorem does not apply to supersymmetric black holes since the stationary Killing field is null on the horizon. It is therefore possible that there are supersymmetric black holes with no axial symmetry at all. Obtaining a complete classification of supersymmetric black holes in this theory would require analysis of this case too. This would for example include the Majumdar-Papapetrou multi-black holes, which have been shown to be the most general asymptotically flat, static, supersymmetric black holes in this theory (although the horizon has low regularity) [55]. However, the analysis of general supersymmetric solutions with no axial symmetry likely would require new techniques. We leave this as an interesting problem for the future.

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Declarations

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13 By rotational here we mean that the near-horizon geometry has non-zero angular momentum associated to that direction, i.e. in our coordinates \( J_\psi \neq 0 \), while \( J_\phi = 0 \).
Data Availability  The datasets generated or analysed during the current study are available from the corresponding author on reasonable request.

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A. Killing Spinor is Preserved by Triholomorphic Symmetry

In this section we show that the Killing spinor is preserved by a triholomorphic Killing field that commutes with the supersymmetric Killing field. We will work on regions where \( f \neq 0 \), which are dense in \( \mathcal{M} \) (Corollary 4), and by continuity of the Lie-derivative we argue that it must be preserved on the whole spacetime.

Let us choose a pseudo-orthonormal co-frame \( e^\mu \), where \( \mu = 0, i, \) and \( i = 1, 2, 3, 4, \)
\[
e^0 := -\frac{\nabla^b}{f} = f(dt + \omega) , \quad e^i := \tilde{e}^\tilde{i} ,
\]
where \( \tilde{e}^\tilde{i} \), with \( \tilde{i} = i \), are an orthonormal co-frame of the base metric \( h \),\(^{14}\) with spin connection 1-forms \( \tilde{\omega}^\tilde{i}{}^\tilde{j} \) defined by \( d\tilde{e}^\tilde{j} = -\tilde{\omega}^\tilde{i}{}^\tilde{j} \wedge \tilde{e}^\tilde{i} \). Now, by Lemma 1 the axial Killing field \( W \) preserves the base data \( (f, \omega, h) \), so we are free to choose our co-frame such that \( \mathcal{L}_W \tilde{e}^\tilde{i} = \mathcal{L}_W e^i = 0 \), and \( \mathcal{L}_W e^0 = 0 \) is automatic. The spin connection 1-forms \( \omega^\mu_\nu \) of the spacetime are given by\(^{15}\)
\[
\omega^0_i = f_i e^0 + \frac{1}{2} G_{ij} e^j , \quad \omega^j_i = \tilde{\omega}^i{}^j + \frac{1}{2} G_{ij} e^0 + \frac{1}{2} (f_i e^j - f_j e^i) ,
\]
where \( f_i e^j := f^{-1} df \) and \( \frac{1}{2} G_{ij} e^i \wedge e^j := f \omega \).

Let \( \{ E_\mu \} \) be the dual frame of vector fields to \( \{ e^\mu \} \) and expand the Killing field \( W \) in this frame,
\[
W = (\iota_W e^0) E_0 + (\iota_W e^i) E_i .
\]

The corresponding metric dual with respect to \( g \) can be written as
\[
W^b = -(\iota_W e^0) e^0 + (\iota_W e^i) e^i = -(\iota_W e^0) e^0 + \frac{\tilde{W}^b}{f} ,
\]
where \( \tilde{W}^b \) is the metric dual of \( W \) with respect to the base metric \( h \).

By definition [56], the Lie-derivative of the Killing spinor by \( W \) is
\[
\mathcal{L}_W \epsilon := \nabla_W \epsilon - \frac{1}{4} dW^b \cdot \epsilon
\]
\[
= W(\epsilon) - \frac{1}{4} (\iota_W \omega_{\mu\nu}) \gamma^\mu \gamma^\nu \epsilon - \frac{1}{8} (dW^b)_{\mu\nu} \gamma^\mu \gamma^\nu \epsilon ,
\]
where the second line follows from the expression for the spinorial Levi-Civita connection \( \nabla_X \epsilon = X(\epsilon) - \frac{1}{4} (\iota_W \omega_{\mu\nu}) \gamma^\mu \gamma^\nu \epsilon \), and for the Clifford algebra we use conventions
\(^{14}\) In this section we distinguish quantities on the base space with a tilde.
\(^{15}\) The spin connection should not be confused with the 1-form \( \omega \) defined on the base.
\[ \gamma^\mu \gamma^v + \gamma^v \gamma^\mu = -2g^{\mu v}. \] Using (160) and \( 0 = \mathcal{L}_\omega e^\mu = \iota_\omega de^\mu + d\iota_\omega e^\mu, \) a calculation yields

\[ dW^b = -2(\iota_\omega \omega_{0i})e^0 \wedge e^i - (\iota_\omega e^j) f_i e^i \wedge e^j - \frac{1}{2} (\iota_\omega e^0) G_{ij} e^i \wedge e^j + \frac{1}{f} d\tilde{W}^b. \] (165)

Substituting into (164), and using again (160) for \( \omega_{ij}, \) we get

\[ \mathcal{L}_\omega e = \tilde{D}_{\tilde{W}} e - \frac{1}{4f} d\tilde{W}^b \cdot e, \] (166)

where \( \tilde{D} \) is the Levi-Civita connection of the base metric \( h, \) and \( \tilde{W} = \pi_\ast W \) the projection of \( W \) to the base (see the proof of Lemma 1). We can expand \( d\tilde{W}^b \) on the base tetrad as

\[ d\tilde{W}^b = (d\tilde{W}^b)_{ij} \tilde{e}^i \wedge \tilde{e}^j = f (d\tilde{W}^b)_{ij} e^i \wedge e^j. \] (167)

This factor of \( f \) cancels the one in the second term of (166) and we get that the Lie-derivative of the Killing spinor on the spacetime is the same as on the base, and

\[ \mathcal{L}_\omega e = \tilde{D}_{\tilde{W}} e - \frac{1}{8} (d\tilde{W}^b)_{ij} \gamma^i \gamma^j \epsilon = \tilde{\mathcal{L}}_{\tilde{W}} e. \] (168)

The Killing spinor \( \epsilon \) on the r.h.s can be interpreted as a spinor on the base and takes the form [25]

\[ \epsilon = \sqrt{f} \eta, \] (169)

where \( \tilde{D} \eta = 0. \) It follows (since \( f \) is also preserved by \( W \)) that the first term of (168) vanishes. Now we use the fact that \( \tilde{W} \) is triholomorphic to deduce that \( d\tilde{W}^b \) is self-dual [26], and since \( \gamma^{[i} \gamma^{j]} \epsilon \) is anti-self-dual [25] on the base, the second term of (168) also vanishes. Thus, the Killing spinor is preserved by a triholomorphic Killing field that also commutes with the supersymmetric Killing field, as claimed.

**B. On the Non-existence of Exceptional Orbits**

Here we give a more detailed argument against the existence of exceptional orbits in the DOC. As discussed in the proof of Lemma 8, such an exceptional orbit can only end on a horizon component with spherical topology, which has a neighbourhood where \( f > 0. \) It follows that there exists an open region \( \tilde{U}_0 \subset \Sigma \) with a Gibbons-Hawking coordinate chart such that \( U_0 \cap E \neq \emptyset. \) By making \( U_0 \) smaller if necessary, we may assume without loss of generality that all points in \( U_0 \cap E \) have the same isotropy group \( \mathbb{Z}_n \) for some integer \( n > 1, \) i.e. the parameter \( \psi \) of orbits of \( W \) is \( 4\pi/n \)-periodic (assuming regular orbits have \( 4\pi \)) periodicity).

Let \( \Omega_0 \) be the \( \psi = 0 \) hypersurface in \( U_0. \) Each orbit of \( W \) intersects \( \Omega_0 \) at most once, since \( x^i \) are constants on orbits of \( W, \) and \( x \) is injective on \( \Omega_0). \) It follows from the periodicities of the orbits that the flowout from \( \Omega_0 \) along \( W \) is injective for flow parameters \((-2\pi/n, 2\pi/n), \) so \( U_0 \) can be chosen such that it is diffeomorphic to \((-2\pi/n, 2\pi/n) \times \Omega_0 \) by the Flowout Theorem (see e.g. [37]).

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16 Since we are in an orthonormal frame the gamma matrices will have the exact same form as the spatial gamma matrices of the spacetime, i.e. \( \gamma^i = \gamma^i. \)
C. Smoothness of \( \omega \) at Fixed Points

In this section we show explicitly that the 1-form \( \alpha \) defined in (130) is smooth on \( \mathbb{R}^4 \) around a fixed point \( r = 0 \). The harmonic function \( F \) defined in (88) can locally be expanded as

\[
F = \sum_{k=1}^{\infty} \sum_{-k \leq m \leq k} r^k f_{km} P_k^m (\cos \theta) e^{im\phi},
\]

where \( P_k^m(x) \) are the associated Legendre functions and \( f_{km} \) some complex coefficients. For reference, the associated Legendre functions for \( m \geq 0 \) can be written as

\[
P_k^m(x) = (-)^m \frac{(k + m)!}{(k - m)!} P_k^{-m}(x) = (-)^m (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_k(x),
\]

where \( P_k(x) \) are the Legendre polynomials. Therefore, for any \( |m| \leq k \) they are a product of \( (\sin \theta)^{|m|} \) and a polynomial of \( x := \cos \theta \) of order \( k - |m| \).

Integrating (88), or using (90), yields\(^{17}\)

\[
\hat{\omega}_{\text{sing}} = \sum_{k=1}^{\infty} \sum_{-k \leq m \leq k} \frac{r^k}{k} f_{km} \left[ (\sin \theta)^2 \dot{P}_k^m e^{im\phi} d\phi + \frac{im}{\sin \theta} P_k^m e^{im\phi} d\theta \right].
\]

The 1-form \( \alpha \) defined by (130) in this gauge is given by

\[
\alpha = \sum_{k=1}^{\infty} \sum_{-k \leq m \leq k} r^k f_{km} e^{im\phi} \left[ h_{-1} P_k^m (d\psi' + \cos \theta d\phi') + \frac{(\sin \theta)^2}{k h_{-1}} \dot{P}_k^m d\phi' + \frac{im}{k \sin \theta} P_k^m d\theta \right].
\]

The smoothness of axisymmetric terms \( (m = 0) \) has been checked in [19], but for completeness we also include it here. The axisymmetric terms for a given \( k \), using (116) and basic properties for Legendre polynomials, can be written as

\[
h_{-1} f_{k0} r^k \left[ P_k (d\psi' + \cos \theta d\phi') + \frac{(\sin \theta)^2}{k} \dot{P}_k d\phi' \right] = h_{-1} f_{k0} r^k \left[ (P_k + P_{k-1}) d\phi^* + (P_k - P_{k-1}) d\phi^- \right],
\]

\(^{17}\) Here we are omitting the argument of associated Legendre functions which we always take to be \( x = \cos \theta \), and dot denotes the derivative with respect to \( x \).
where we have used the identity \((1 \pm x)P_k \pm k^{-1}(1 - x^2)\tilde{P}_k = P_k \pm P_{k-1}\). From the recursion formula for the Legendre polynomials it follows that \(r^k(P_k \pm P_{k-1})d\phi^\pm = r(1 \pm \cos \theta)\tilde{G}_\pm d\phi^\pm = \frac{1}{2}\tilde{G}_\pm X_\pm^2 d\phi^\pm\) for some smooth function \(\tilde{G}\) of \(X_\pm^2\) and therefore these terms are indeed smooth on \(\mathbb{R}^4\) [19].

We now consider the non-axisymmetric terms in \(\alpha\) for a given \(k\) and \(m \neq 0\). We focus on terms with \(m > 0\) \((P_l^m\) and \(P_l^{-m}\) only differ by a constant factor, so the analysis is essentially identical for \(m < 0\), which can be written using (115) as

\[
h_{-1} r^k \left( \frac{(\sin \theta)^2}{k} \tilde{P}_k^m + (1 + \cos \theta) P_k^m \right) e^{im\phi} d\phi^+ \\
+ h_{-1} r^k \left( -\frac{(\sin \theta)^2}{k} \tilde{P}_k^m + (1 - \cos \theta) P_k^m \right) e^{im\phi} d\phi^- \\
+ r^k \frac{im}{k \sin \theta} P_k^m e^{im\phi} d\theta. \tag{175}
\]

For the last line we use the recursion formula

\[
\frac{1}{\sqrt{1 - x^2}} P_k^m = \frac{-1}{2m} \left[ P_{k-1}^{m+1} + (k + m - 1)(k + m) P_{k-1}^{m-1} \right]. \tag{176}
\]

The term containing \(P_{k-1}^{m+1}\) can be written as

\[
r^k P_{k-1}^{m+1} e^{im\phi} d\theta \sim (r \sin \theta e^{i\phi})^m \left[ (r \cos \theta)^{k-m-2} + \ldots \right] \left[ X_+^2 d(X_+^2) - X_-^2 d(X_+^2) \right], \tag{177}
\]

where \(\ldots\) represents lower order terms in \(\cos \theta\). Since this term is explicitly smooth, we will omit it in the further analysis.

In the first two lines of (175) we can use the identity

\[
(1 - x^2) \tilde{P}_k^m + kx P_k^m = (k + m) P_{k-1}^m, \tag{178}
\]

and after omitting the smooth factor \(\frac{1}{k} (r \exp(i\phi) \sin \theta)^{m-1}\), we get

\[
h_{-1} r^{k-m+1} \sin \theta \left( (k + m) \tilde{P}_{k-1}^m + k \tilde{P}_k^m \right) e^{i\phi} d\phi^+ \\
+ h_{-1} r^{k-m+1} \sin \theta \left( -(k + m) \tilde{P}_{k-1}^m + k \tilde{P}_k^m \right) e^{i\phi} d\phi^- \\
- \frac{i}{2} r^{k-m+1} (k + m - 1)(k + m) \tilde{P}_{k-1}^{m-1} e^{i\phi} d\theta. \tag{179}
\]

where we defined \(\tilde{P}_k^m := \frac{P_k^m}{(\sin \theta)^m} = (-)^m \frac{d^m}{d\theta^m} P_k\). Let us now look at the real part of (179) (the analysis of imaginary part gives the same result), which can be written as

\[
\frac{1}{2} h_{-1} r^{k-m} (u_1 u_3 + u_2 u_4) [(k + m) \tilde{P}_{k-1}^m + k \tilde{P}_k^m] \frac{u_1 du_2 - u_2 du_1}{X_+^2} \\
+ \frac{1}{2} h_{-1} r^{k-m} (u_1 u_3 + u_2 u_4) [-(k + m) \tilde{P}_{k-1}^m + k \tilde{P}_k^m] \frac{u_3 du_4 - u_4 du_3}{X_-^2} \\
+ \frac{1}{4} h_{-1} r^{k-m} (u_2 u_3 - u_4 u_1) (k + m - 1)(k + m) \tilde{P}_{k-1}^{m-1} \left( \frac{u_1 du_1 + u_2 du_2}{X_+^2} + \frac{u_3 du_3 + u_4 du_4}{X_-^2} \right). \tag{180}
\]

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By standard properties of the Legendre polynomials one can show that we can write the following polynomials as

\[ k \tilde{P}_k^m \pm (k + m) \tilde{P}_{k-1}^m + \frac{1}{2} (k + m - 1)(k + m) \tilde{P}_{k-1}^{m-1} = (1 \pm x) Q_{km}^{\pm} \]  

(181)

with some polynomials \( Q_{km}^{\pm}(x) \) of order \( k - m - 1 \). Using this property with the upper (lower) sign for the \( du_1 \) and \( du_2 \) (\( du_3 \) and \( du_4 \)) terms, one can check that (180) is smooth. In detail, we can rearrange the \( du_1 \) terms as

\[
\frac{h_{-1} r^{k-m}}{2} \frac{X^2}{X^2_k} \left\{ \left( u_1^2 u_4 - u_1 u_2 u_3 \right) \left[ k \tilde{P}_k^m + (k + m) \tilde{P}_{k-1}^m + \frac{1}{2} (k + m - 1)(k + m) \tilde{P}_{k-1}^{m-1} \right] \right.

- \left[ k \tilde{P}_k^m + (k + m) \tilde{P}_{k-1}^m \right] \left( u_1^2 + u_2^2 \right) u_4 \right\} du_1 =

= \frac{h_{-1}}{2} \left\{ \left( \frac{1}{2} u_1^2 u_4 - u_1 u_2 u_3 \right) r^{k-m-1} Q^+_km - r^{k-m} \left( k \tilde{P}_k^m + (k + m) \tilde{P}_{k-1}^m \right) u_4 \right\} du_1 ,

\]

where we used (181) and \( r(1 + x) = r(1 + \cos \theta) = \frac{1}{2} X^2_k \). Since \( Q^+_km(x) \) is of order \( k - m - 1 \), the last line is explicitly smooth. The argument for the \( du_{A=2,3,4} \) terms is identical. This proves our claim that \( \alpha \) is smooth on \( \mathbb{R}^4 \).

**D. Three-Centred Solutions**

The simplest examples with a single axial symmetry are three-centred solutions. We will focus on single black hole solutions, where without loss of generality, we will choose coordinates such that the origin corresponds to the black hole. We can use the redundancy \( \Psi \rightarrow \Psi + c \) in the harmonic functions so that they have the following form:

\[
H = \frac{h_0}{r} + \frac{h_1}{r_1} + \frac{h_2}{r_2} , \quad K = \frac{k_1}{r_1} + \frac{k_2}{r_2} ,

L = 1 + \frac{l_0}{r} - \frac{h_1 k_1^2}{r_1} - \frac{h_2 k_2^2}{r_2} , \quad M = -\frac{3}{2}(k_1 + k_2) + \frac{m_0}{r} + \frac{k_1^3}{2r_1} + \frac{k_2^3}{2r_2} .
\]

(183)

The remaining constraints of Theorem 3 on the parameters are

\[
h_1^2 = h_2^2 = 1 , \quad h_0 + h_1 + h_2 = 1 , \quad h_0^3 - h_0^2 m_0 > 0 , \quad h_i \left( h_0 k_i^2 - l_0 \right) - h_1 h_2 \left( k_1 h_2 - k_2 h_1 \right)^2 > 0 , \quad \text{for } i = 1, 2 .
\]

(184) \hspace{1cm} (185) \hspace{1cm} (186) \hspace{1cm} (187)

There are three different choices for the parameters \( h_i \):

(i) \( h_0 = 1 \), \( h_1 = 1 \), \( h_2 = -1 \),
(ii) \( h_0 = -1 \), \( h_1 = h_2 = 1 \),
(iii) \( h_0 = 3 \), \( h_1 = h_2 = -1 \).
Table 1. Examples of parameters satisfying the constraint equations and inequalities (184-190). For all cases $x_1 = y_1 = y_2 = 0$, and $l_0, m_0$ are determined by (185)

| $h_0$ | $h_1$ | $h_2$ | $k_1$ | $k_2$ | $z_1$ | $z_2$ | $x_2$ |
|-------|-------|-------|-------|-------|-------|-------|-------|
| 1     | 1     | -1    | -3.467| -7.256| 0.2422| -5.083| 1.606 |
| 1     | 1     | -1    | 1.112 | -5.219| 1.479 | 0.3712| 1.810 |
| 1     | 1     | -1    | -1.009| 9.371 | 8.756 | 9.130 | -9.022|
| -1    | 1     | 1     | -8.155| -9.288| 1.562 | 7.167 | -5.651|
| -1    | 1     | 1     | -3.061| -5.430| 0.02954| -2.200| 7.453 |
| -1    | 1     | 1     | 6.501 | 7.217 | 2.931 | 0.6217| -6.930|
| 3     | -1    | -1    | 5.049 | 7.799 | 0.2238| -8.775| 3.644 |
| 3     | -1    | -1    | -9.477| -8.117| 7.736 | -0.3053| 0.4992|
| 3     | -1    | -1    | 7.571 | 5.287 | 9.393 | 0.3150| -1.048|

Cases (i) and (ii) correspond to a horizon topology $S^3$, while (iii) is a black lens $L(3, 1)$. There are two further conditions that have to be satisfied everywhere on $\mathbb{R}^3$ for smoothness and stable causality:

\[ N^{-1} \equiv K^2 + HL > 0 , \]  \hspace{1cm} (188)

\[ g^{tt} < 0 . \]  \hspace{1cm} (189)

In [20] it has been conjectured that in the $U(1)^2$-symmetric case (184-187) together with the positivity of the total mass implies that (188-189) are automatically satisfied. Hence, for the analysis presented here, we add the requirement of positivity of mass $M > 0$, which in terms of the parameters is

\[ l_0 - h_1 k_1^2 - h_2 k_2^2 + (k_1 + k_2)^2 > 0 . \]  \hspace{1cm} (190)

For the $U(1)^2$-symmetric black lens this conjecture was analytically proven for (188), while for the other two cases numerical evidence was presented. As mentioned in the remarks below Theorem 3, it has been argued that for soliton solutions (189) follows from (188) [44].

For the numerical checks, sets of parameters are constructed as follows. Without loss of generality, we fix coordinates for the centres $a_i = (x_i, y_i, z_i)$ such that $x_1 = y_1 = y_2 = 0$. For each case (i)-(iii), we first choose the parameters $k_1, k_2, z_1, z_2, x_2$ randomly and then determine $l_0$ and $m_0$ as a solution of the linear system of equations (185)\(^{18}\). Then those sets of parameters that fail to satisfy (186-187) or (190) are discarded. Finally, (188–189) are checked numerically at random points. For concreteness, some examples satisfying all conditions (184–190) are shown in Table 1. We will now consider the three cases in turn.

For case (iii), the black lens, the proof that (188) is automatically satisfied carries over to the case with non-collinear centres as follows. Using (187) for $N^{-1}$ yields

\[ N^{-1} > (k_1 - k_2)^2 \left( \frac{|a_1|}{|a_1 - a_2|rr_1} + \frac{|a_2|}{|a_1 - a_2|rr_2} - \frac{1}{rr_1rr_2} \right) \]

\[ + \frac{3l_0}{r^2} + \left( \frac{3}{r} + \frac{|a_1|}{rr_1} + \frac{|a_2|}{rr_2} - \frac{1}{r_1} - \frac{1}{r_2} \right) . \]  \hspace{1cm} (191)

\(^{18}\) The linear equations are not invertible for some non-generic values of $k_1, k_2$, so these cases have been treated separately.
The first term is non-negative by Ptolemy’s inequality. The second term is positive due to (186), and the last term is non-negative by the triangle inequality. The stable causality condition (189) was checked numerically. In the numerical checks we have looked at $10^4$ random points in a radius $R$ of the origin such that $R > 3 \max(|a_1|, |a_2|)$, chosen either uniformly, or centred around the origin, closer to the singular points. We found that (189) was satisfied by all $10^4$ set of parameters that satisfy (184–187) and (190). For case (i) and (ii), we can only check (188–189) numerically on parameters that satisfy (184)–(187) and (190). The method of the numerical checks is identical to the one described in case (iii). For each case we have found a large parameter space for which all the constraints are satisfied. We have found that for case (ii) equations (184)–(187) and (190) are sufficient to guarantee (188–189) for the tested $10^4$ set of parameters. In contrast, rather surprisingly, for case (i) there are even collinear (therefore $U(1)^2$-symmetric) configurations that satisfy (184)–(187) and (190) but violate (188–189). This disproves the conjecture of [20]. When the centres are collinear, these configurations all have $k_2/k_1 \sim O(1)$, and even in that parameter range they appear with small probability when the parameters are chosen uniformly, which explains why they have not been found previously. None of these violating configurations have equal momenta, which is the relevant case when one would like to compare these black holes to the BMPV solution. Also, no configuration have been found which satisfy (188) but violate (189), inline with the conjecture of [44] (although that conjecture is for solitons).

We have also looked at soliton solutions, where instead of (186), we require $l_0 = 0$, $m_0 = 0$, and

$$h_0 \left(1 - \frac{h_1k_1^2}{|a_1|} - \frac{h_2k_2^2}{|a_2|}\right) > 0 \, . \tag{192}$$

For these configurations, we have found that for all tested configurations satisfying (184)–(187) and (190), (188–189) are also satisfied.

### E. Hyper-Kähler Metrics with Triholomorphic $SU(2)$ or $E(2)$ Isometry

In this section we determine the possible multi-centred Gibbons-Hawking solutions with triholomorphic cohomogeneity-1 $SU(2)$ or $E(2)$ symmetry. For the latter case we derive the general cohomogeneity-1 hyper-Kähler metric with triholomorphic $E(2)$ symmetry.

#### E.1. $SU(2)$

Hyper-Kähler manifolds with triholomorphic cohomogeneity-1 $SU(2)$ symmetry belong to the BGPP class [48]. In Gibbons-Hawking form adapted to a triholomorphic subgroup $U(1) \subset SU(2)$ the associated harmonic function can be written as

$$H(x) = \left(\prod_{i=1}^{3}(\beta(x) - \beta_i)\right)^{-1/2} \left(\sum_{i=1}^{3} \frac{x_i^2}{(\beta(x) - \beta_i)^2}\right)^{-1} \, , \tag{193}$$

where $\beta(x)$ is an algebraic root of

$$\sum_{i=1}^{3} \frac{x_i^2}{\beta - \beta_i} = C \, , \tag{194}$$

19 When the centres are not co-planar, a lower estimate can be used by looking at a co-planar arrangement, keeping $r_1$, $r_2$, $r$, $|a_1|$ and $|a_2|$ the same, and increasing $|a_1 - a_2|$. 

with \( \beta_1, C \) constants \([57,58]\). First note that if \( \beta_1 = \beta_2 = \beta_3 \) then \( H = 1/(\sqrt{C}r) \), and if we take \( C = 1 \) the solution reduces to flat euclidean space. We will therefore exclude this case.

A simple pole in \( H \) can occur at \( x_0 = (x_0, y_0, z_0) \) only if \( \beta(x_0) = \beta_i \) for some \( i \).\(^{20}\) For definiteness, assume that \( H \) is singular at \( \beta(x_0) = \beta_1 \), so the analysis splits into the cases:

(i) \( \beta_1 = \beta_2 \neq \beta_3 \),

(ii) \( \beta_i \) are distinct.

Now consider case (i) for which \( H \) is necessarily axisymmetric. From (193) we can see that \( H \) is singular at \( x_0 \) if \( (x^2 + y^2)/(\beta - \beta_1) \to 0 \) as \( x \to x_0 \). Then (194) implies that \( z_0^2 = C(\beta_1 - \beta_3) \geq 0 \). If \( C(\beta_1 - \beta_3) > 0 \), there are isolated singularities at \( x_0 = y_0 = 0, z_0 = \pm \sqrt{C(\beta_1 - \beta_3)} \), and the corresponding harmonic function takes the form

\[
H(x) = \frac{1}{2\sqrt{C}} \left( \frac{1}{|x - a|} \pm \frac{1}{|x + a|} \right),
\]

with \( a = (0, 0, \sqrt{C(\beta_1 - \beta_3)}) \). This family includes the Eguchi-Hanson metric (with + sign and \( C = 1/4 \)). These configurations, however, do not correspond to asymptotically euclidean metrics. If \( C = 0 \), the harmonic function is given by \( H = \text{const} \tilde{z}^2 \), so this solution does not have a simple pole.

Finally, let us look at case (ii). From (193) it can be shown that if \( H \) has a simple pole at a given \( x_0 \) then

\[
\lim_{x \to x_0} \frac{x^2}{(\beta(x) - \beta_1)^{3/2}} = 0.
\]

Using (196) in (194) implies that \( (x_0, y_0, z_0) \) is on the curve defined by

\[
\frac{y^2}{\beta_1 - \beta_2} + \frac{z^2}{\beta_1 - \beta_3} = C, \quad x = 0.
\]

Let \( S \) be the connected component of (197) containing \( x_0 \). We now show that \( H \) is singular on \( S \). Recall that having a simple pole requires \( \lim_{x \to x_0} \beta = \beta_1 \). Taking this limit in the direction along the curve (197) and using the fact that the other root of (194) on (197) is separated from \( \beta_1 \), it follows that \( \beta = \beta_1 \) on \( S \). Let \( (0, \tilde{y}, \tilde{z}) \in S \), and let us look at the limit

\[
\lim_{x \to 0}(\beta(x, \tilde{y}, \tilde{z}) - \beta_1)^{3/2} = \lim_{x \to 0} \frac{C - \tilde{y}^2}{\beta - \beta_1} \frac{\tilde{z}^2}{\beta - \beta_3} = \lim_{x \to 0} 2\left( \frac{\tilde{y}^2}{(\beta - \beta_2)^2} + \frac{\tilde{z}^2}{(\beta - \beta_3)^2} \right) \sqrt{\beta - \beta_1} = 0,
\]

where for the second line we used L’Hôpital’s rule. Thus, by (193) \( H \) diverges on \( S \), hence the singularity of \( H \) at \( x_0 \) cannot be isolated.

\(^{20}\) If \( C \neq 0 \) then \( H \) is clearly non-singular at all points \( \beta(x) \neq \beta_1 \). If \( C = 0 \) and \( \beta(x) \neq \beta_1 \) then \( H \) can only be singular at the origin in which case \( \beta \) has a direction dependent limit, so it cannot be a simple pole of \( H \).
E.2. $E(2)$. We will first derive the general form for cohomogeneity-1 hyper-Kähler metrics with triholomorphic $E(2)$ symmetry. Without loss of generality we can write this as

$$h = (\det h(\rho))d\rho^2 + h_{ij}(\rho)\sigma^i_R\sigma^j_R,$$

where $\sigma^i_R$, $i, j = 1, 2, 3$, are left-invariant 1-forms on $E(2)$, satisfying

$$d\sigma^i_R = \frac{1}{2}c^i_{jk}\sigma^j_R \wedge \sigma^k_R,$$

with $c^i_{jk}$ being the structure constants of the Lie algebra of $E(2)$. The Killing fields of (199) are given by the right-invariant vector fields $L_i$ which satisfy $[L_i, L_j] = c^k_{ij}L_k$.

Let us define $\iota^m := \frac{1}{2}\epsilon^{ijk}c^m_{jk}$. For $E(2)$, $\iota^{ij}$ is a symmetric, positive semi-definite matrix with one zero eigenvalue (VII.0 in the Bianchi classification).

By a change of basis $\sigma^i_{\prime R} = L^i_{\prime j}\sigma^j_R$, where $L \in GL(3, \mathbb{R})$, one can simultaneously diagonalise $h$ and $t$ at a given $\rho = \rho_0$ as follows. Under such change $h^\prime = (L^{-1})^T h L^{-1}$ and $t^\prime = (\det L)^{-1} L t L^T$. Since $h$ is symmetric and positive definite, we can change basis such that $h^\prime = 1$. As $t^\prime$ is symmetric, we can diagonalise it by an orthogonal matrix so that $t^{\prime\prime} = \text{diag}(0, t_2, t_3)$ and $h^{\prime\prime} = 1$. Finally, we can rescale each direction such that $t^{\prime\prime\prime} = \text{diag}(0, 1, 1)$ and $h^{\prime\prime\prime}$ is diagonal.

We now drop primes and assume that $h_{ij}(\rho_0)$ is diagonal and $t = \text{diag}(0, 1, 1)$. The latter are equivalent to the structure constants $c^3_{12} = 1 = c^2_{31}$ with the rest vanishing.

Now, following [59, 60], the Einstein condition for the metric (199) then gives at $\rho_0$ (omitting primes)

$$\dot{h}_{13} = \dot{h}_{12} = 0, \quad (h^{22} - h^{33})\dot{h}_{23} = 0,$$

where dot means derivative with respect to $\rho$. Generically, if no other symmetry is assumed, this implies that the first derivative of the off-diagonal metric components vanish.

If there is additional symmetry and $h^{22} = h^{33}$, then $h$ is automatically diagonal [60]. By further differentiation of the Einstein condition, using real-analyticity\footnote{$h_{ij}$ and $K_{ij} := \dot{h}_{ij}$ satisfy a system of first order ODEs (the Einstein equations) which are analytic in $(h_{ij}, K_{ij})$ if $\det h \neq 0$, so by standard results, there is a unique solution analytic at $\rho = \rho_0$ for given ‘initial’ values at $\rho = \rho_0$.} in $\rho$, one can see that the metric remains diagonal for all $\rho$.

After diagonalisation, the metric can be put into the form

$$h = \omega_1 \omega_2 \omega_3 d\rho^2 + \omega_2 \omega_3 \omega_1 (\sigma_R^1)^2 + \omega_1 \omega_2 \omega_3 (\sigma_R^2)^2 + \omega_1 \omega_2 \omega_3 (\sigma_R^3)^2,$$

for some functions $\omega_i(\rho)$. We can take a basis of anti-self-dual (ASD) 2-forms

$$\Omega^1 = \omega_2 \omega_3 d\rho \wedge \sigma_R^1 - \omega_1 \sigma_R^2 \wedge \sigma_R^3,$$

$$\Omega^2 = \omega_1 \omega_3 d\rho \wedge \sigma_R^2 - \omega_2 \sigma_R^1 \wedge \sigma_R^3,$$

$$\Omega^3 = \omega_1 \omega_2 d\rho \wedge \sigma_R^3 - \omega_3 \sigma_R^1 \wedge \sigma_R^2,$$

where we choose the orientation $d\rho \wedge \sigma_R^1 \wedge \sigma_R^2 \wedge \sigma_R^3$. The requirement that $\Omega^i$ are closed is equivalent to the systems of ODE:

$$\dot{\omega}_1 = 0, \quad \dot{\omega}_2 = -\omega_1 \omega_3, \quad \dot{\omega}_3 = -\omega_1 \omega_2.$$
In particular, $\omega_1$ is a constant which must be non-vanishing, so without loss of generality we can assume $\omega_1 > 0$ (this can be arranged using the discrete symmetry $\omega_1 \to -\omega_1, \omega_2 \to -\omega_2$ of the above system). We can now easily integrate for $\omega_2, \omega_3$, which gives us two qualitatively different classes of solutions. The first is $\omega_2 = Ae^{\pm \omega_1 \rho}$, $\omega_3 = \mp Ae^{\mp \omega_1 \rho}$, and it is easily checked that this gives euclidean space. The second class of solutions can be written as

$$
\omega_2 = -A \sinh(\omega_1(\rho - \rho_0)) , \quad \omega_3 = A \cosh(\omega_1(\rho - \rho_0)) ,
$$

(205)

with $A$ and $\rho_0$ arbitrary constants and without loss of generality we can set $\rho_0 = 0$. Let us focus on this second non-trivial solution.

Now, by defining a new coordinate $\hat{\rho} := -\omega_1 \rho$ and by rescaling $\hat{\sigma}_R^{2,3} = \sqrt{\omega_1} \sigma_R^{2,3}$, which does not change the structure constants, we obtain the metric (dropping hats)

$$
h = K^2 \sinh \rho \cosh \rho (d\rho^2 + (\sigma_1^1)^2) + \coth \rho (\sigma_2^1)^2 + \tanh \rho (\sigma_3^1)^2 ,
$$

(206)

where $K^2 := A^2/\omega_1$. The ASD 2-forms are

$$
\Omega^1 = -K^2 \sinh \rho \cosh \rho d\rho \wedge \sigma_1^1 - \sigma_2^2 \wedge \sigma_3^3 , \\
\Omega^2 = -K (\cosh \rho d\rho \wedge \sigma_2^1 + \sinh \rho \sigma_3^1 \wedge \sigma_1^1) , \\
\Omega^3 = -K (\sinh \rho d\rho \wedge \sigma_3^1 + \cosh \rho \sigma_1^1 \wedge \sigma_2^2) ,
$$

(207)

where note that the orientation is now $-d\rho \wedge \sigma_1^1 \wedge \sigma_2^2 \wedge \sigma_3^3$. It is useful to note that we can always choose local coordinates $(Z, x, y)$ on $E(2)$ so

$$
\sigma_1^1 = dZ , \quad \sigma_2^2 = -\sin Z dx + \cos Z dy , \quad \sigma_3^3 = \cos Z dx + \sin Z dy .
$$

(208)

In these coordinates the Killing fields that generate the $E(2)$ algebra are

$$
L_1 = \partial_Z + x \partial_y - y \partial_x , \quad L_2 = \partial_y , \quad L_3 = \partial_x .
$$

(209)

This completes the derivation of the general form of the metric which is given by (206). This is the analogue of the BGPP metric for $E(2)$-symmetry.

As $\rho \to \infty$ (206) approaches the metric

$$
h_0 = dR^2 + R^2 dZ^2 + dx^2 + dy^2 ,
$$

(210)

with $R := K^2/2 \exp(\rho)$. Even though (210) is (locally) isometric to the flat euclidean metric on $\mathbb{R}^4$, it is only valid at $R \to \infty$, thus (206) is not asymptotically euclidean (one can also check that $R_{abcd} R^{abcd} \to 0$ iff $R \to \infty$). This excludes (206) as a possible base for an asymptotically flat supersymmetric solution.

Even though (206) does not have the appropriate asymptotic behaviour, it is interesting to write it in Gibbons-Hawking form with respect to the Killing field $W' = \frac{1}{2} L_1$ (the analogue of an isoclinic Killing field $W$ for asymptotically euclidean metrics). The coordinates $(\psi, x^i)$ adapted to $W'$ are defined by $dx^i = \iota_W \Omega^i$ and $W' = \partial_\psi$. We find that they take a simpler form in polar coordinates $(x, y) \to (r, \theta)$, and a computation using the above ASD 2-forms reveals that

$$
\chi^1 = K^2 \sinh^2 \rho - r^2 , \quad \chi^2 = \frac{K}{2} r \sin \rho \cos \phi ,
$$

(211)

$$
\chi^3 = \frac{K}{2} r \cos \rho \sin \phi , \quad \psi = Z + \theta ,
$$

(212)
where $\phi := Z - \theta$. The associated harmonic function is given by $H = 1/h(W', W')$, and we find

$$H = \frac{16 \sinh(2\rho)}{K^2(\cosh(4\rho) - 1) + 4r^2(\cosh(2\rho) + \cos(2\phi))}. \quad (213)$$

This diverges on a parabola in $\mathbb{R}^3$ parametrised by $r$:

$$x^1 = -\frac{r^2}{4}, \quad x^2 = 0, \quad x^3 = \pm \frac{K}{2}r, \quad (214)$$

and therefore its singularity is not isolated.

References

1. Emparan, R., Reall, H.S.: Black Holes in Higher Dimensions. Living Rev. Rel. 11, 6 (2008). https://doi.org/10.12942/lrr-2008-6. arXiv:0801.3471 [hep-th]
2. Emparan, R., Reall, H.S.: A Rotating black ring solution in five-dimensions. Phys. Rev. Lett. 88, 101101 (2002). https://doi.org/10.1103/PhysRevLett.88.101101. arXiv:hep-th/0110260
3. Hollands, S., Ishibashi, A.: Black hole uniqueness theorems in higher dimensional spacetimes. Class. Quant. Grav. 29, 163001 (2012). https://doi.org/10.1088/0264-9381/29/16/163001. arXiv:1206.1164 [gr-qc]
4. Friedman, J.L., Schleich, K., Witt, D.M.: Topological censorship. Phys. Rev. Lett. 71, 1486–1489 (1993). arXiv:gr-qc/9305017. https://doi.org/10.1103/PhysRevLett.71.1486. [Erratum: Phys.Rev.Lett. 75, 1872 (1995)]
5. Galloway, G.J., Schoen, R.: A Generalization of Hawking’s black hole topology theorem to higher dimensions. Commun. Math. Phys. 266, 571–576 (2006). https://doi.org/10.1007/s00220-006-0019-z. arXiv:gr-qc/0509107
6. Hollands, S., Ishibashi, A., Wald, R.M.: A Higher dimensional stationary rotating black hole must be axisymmetric. Commun. Math. Phys. 271, 699–722 (2007). https://doi.org/10.1007/s00220-007-0216-4. arXiv:gr-qc/0605106
7. Moncrief, V., Isenberg, J.: Symmetries of higher dimensional black holes. Class. Quant. Grav. 25, 195015 (2008). https://doi.org/10.1088/0264-9381/25/19/195015. arXiv:0805.1451 [gr-qc]
8. Hollands, S., Ishibashi, A.: On the ‘stationary implies axisymmetric’ theorem for extremal black holes in higher dimensions. Commun. Math. Phys. 291, 403–441 (2009). https://doi.org/10.1007/s00220-009-0841-1. arXiv:0809.2659 [gr-qc]
9. Hollands, S., Holland, J., Ishibashi, A.: Further restrictions on the topology of stationary black holes in five dimensions. Ann. Henri Poincare 12, 279–301 (2011). https://doi.org/10.1007/s00023-011-0079-2. arXiv:1002.0490 [gr-qc]
10. Hollands, S., Yazadjiev, S.: Uniqueness theorem for 5-dimensional black holes with two axial Killing fields. Commun. Math. Phys. 283, 749–768 (2008). https://doi.org/10.1007/s00220-008-0516-3. arXiv:0707.2775 [gr-qc]
11. Hollands, S., Yazadjiev, S.: A Uniqueness theorem for 5-dimensional Einstein–Maxwell black holes. Class. Quant. Grav. 25, 095010 (2008). https://doi.org/10.1088/0264-9381/25/9/095010. arXiv:0711.1722 [gr-qc]
12. Hollands, S., Yazadjiev, S.: A Uniqueness theorem for stationary Kaluza–Klein black holes. Commun. Math. Phys. 302, 631–674 (2011). https://doi.org/10.1007/s00220-010-1176-7. arXiv:0812.3036 [gr-qc]
13. Brekenridge, J.C., Myers, R.C., Peet, A.W., Vafa, C.: D-branes and spinning black holes. Phys. Lett. B 391, 93–98 (1997). https://doi.org/10.1016/S0370-2693(96)01460-8. arXiv:hep-th/9602065
14. Reall, H.S.: Higher dimensional black holes and supersymmetry. Phys. Rev. D 68, 024024 (2003). https://doi.org/10.1103/PhysRevD.70.089902. [Erratum: Phys. Rev. D 70, 089902 (2004)]. arXiv:hep-th/0211290
15. Elvang, H., Emparan, R., Mateos, D., Reall, H.S.: A Supersymmetric black ring. Phys. Rev. Lett. 93, 211302 (2004). https://doi.org/10.1103/PhysRevLett.93.211302. arXiv:hep-th/0407065
16. Gauntlett, J.P., Gutowski, J.B.: Concentric black rings. Phys. Rev. D 71, 025013 (2005). https://doi.org/10.1103/PhysRevD.71.025013. arXiv:hep-th/0408010
17. Kunduri, H.K., Lucietti, J.: Supersymmetric black holes with lens-space topology. Phys. Rev. Lett. 113(21), 211101 (2014). https://doi.org/10.1103/PhysRevLett.113.211101. arXiv:1408.6083 [hep-th]
18. Tomizawa, S., Nozawa, M.: Supersymmetric black lenses in five dimensions. Phys. Rev. D 94(4), 044037 (2016). https://doi.org/10.1103/PhysRevD.94.044037. arXiv:1606.06643 [hep-th]
46. Berglund, P., Gimon, E.G., Levi, T.S.: Supergravity microstates for BPS black holes and black rings. JHEP **06**, 007 (2006). https://doi.org/10.1088/1126-6708/2006/06/007. arXiv:hep-th/0505167

47. Fushchich, V.I., Barannik, A.F., Barannik, L.F.: Continuous subgroups of a generalized Euclidean group. Ukr. Math. J. **38**(1), 58–63 (1986). https://doi.org/10.1007/BF01056758

48. Belinsky, V.A., Gibbons, G.W., Page, D.N., Pope, C.N.: Asymptotically Euclidean Bianchi IX metrics in quantum gravity. Phys. Lett. B **76**, 433–435 (1978). https://doi.org/10.1016/0370-2693(78)90899-7

49. Bakas, I., Sfetsos, K.: Toda fields of SO(3) hyperKahler metrics and free field realizations. Int. J. Mod. Phys. A **12**, 2585–2612 (1997). https://doi.org/10.1142/S0217751X97001456. arXiv:hep-th/9604003

50. Bena, I., Bobev, N., Warner, N.P.: Bubbles on manifolds with a U(1) isometry. JHEP **08**, 004 (2007). https://doi.org/10.1088/1126-6708/2007/08/004. arXiv:0705.3641 [hep-th]

51. Welch, D.L.: On the smoothness of the horizons of multi-black hole solutions. Phys. Rev. D **52**, 985–991 (1995). https://doi.org/10.1103/PhysRevD.52.985. arXiv:hep-th/9502146

52. Candlish, G.N., Reall, H.S.: On the smoothness of static multi-black hole solutions of higher-dimensional Einstein-Maxwell theory. Class. Quant. Grav. **24**, 6025–6040 (2007). https://doi.org/10.1088/0264-9381/24/23/022. arXiv:0707.4420 [gr-qc]

53. Gowdigere, C.N., Kumar, A., Raj, H., Srivastava, Y.K.: On the smoothness of multi center coplanar black hole and membrane horizons. Gen. Rel. Grav. **51**(11), 146 (2019). https://doi.org/10.1007/s10714-019-2634-y. arXiv:1401.5189 [hep-th]

54. Gowdigere, C.N.: On the smoothness of horizons in the most generic multi center black hole and membrane solutions (2014) arXiv:1407.5338 [hep-th]

55. Lucietti, J.: All higher-dimensional Majumdar–Papapetrou Black Holes. Ann. Henri Poincare **22**(7), 2437–2450 (2021). https://doi.org/10.1007/s00023-021-01037-0. arXiv:2009.05358 [gr-qc]

56. Fatibene, L., Ferraris, M., Francaviglia, M., Godina, M.: A Geometric definition of Lie derivative for spinor fields. In: 6th International Conference on Differential Geometry and Applications (1996)

57. Gibbons, G.W.: Gravitational instantons, confocal quadrics and separability of the Schrodinger and Hamilton–Jacobi equations. Class. Quant. Grav. **20**, 4401–4408 (2003). https://doi.org/10.1088/0264-9381/20/20/305. arXiv:math/0303191

58. Dunajski, M.: Harmonic functions, central quadrics, and twistor theory. Class. Quant. Grav. **20**, 3427–3440 (2003). https://doi.org/10.1088/0264-9381/20/15/311. arXiv:math/0303181

59. Tod, K.: Cohomogeneity-one metrics with self-dual weyl tensor. In: Huggett, S. (ed.) Twistor Theory, pp. 171–184. Routledge, New York (1995). https://doi.org/10.1201/9780203734889-13

60. Dammerman, B.: Diagonalizing cohomogeneity-one Einstein metrics. J. Geom. Phys. **59**(9), 1271–1284 (2009). https://doi.org/10.1016/j.geomphys.2009.06.010

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