Quantum fluctuations of lightcone
in 4-dimensional spacetime with parallel plane boundaries

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Abstract

Quantum fluctuations of lightcone are examined in a 4-dimensional spacetime with two parallel planes. Both the Dirichlet and the Neumann boundary conditions are considered. In all the cases we have studied, quantum lightcone fluctuations are greater where the Neumann boundary conditions are imposed, suggesting that quantum lightcone fluctuations depend not only on the geometry and topology of the spacetime as has been argued elsewhere but also on boundary conditions. Our results also show that quantum lightcone fluctuations are larger here than that in the case of a single plane. Therefore, the confinement of gravitons in a smaller region by the presence of a second plane reinforces the quantum fluctuations and this can be understood as a consequence of the uncertainty principle.

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I. INTRODUCTION

Quantization of gravity is more difficult than quantization of the other fundamental interactions. Although a lot of efforts have been made, a consistent theory of quantum gravity is still elusive. If, however, the basic quantum principles we are already familiar with apply as well to a quantum theory of gravity, we can make some predictions about expected quantum effects, even in the absence of a fundamental underlying theory. One such effect is the fluctuations of lightcone induced by quantum fluctuations of the space-time metric to be expected in any theory of quantum gravity. The study of quantum lightcone fluctuations has attracted much attention recently. A model of lightcone fluctuations on a flat background has been developed [1], where the fluctuations are produced by gravitons propagating on the background. This model has been further developed [2,3] and applied to study the lightcone fluctuations in spacetimes with a compactified spatial section and with a single plane boundary [3] as well as in spacetimes with extra dimensions [4,5]. It has also been applied to a microscopic recoil model for lightcone fluctuations in a quantum gravity framework inspired by string theory [6]. Because the lightcone fluctuates, the propagation time of a classical light pulse over distance $r$ is no longer precisely $r$, but undergoes fluctuations around a mean value of $r$. The quantum lightcone fluctuations can be thought of as giving rise to stochastic fluctuations in the speed of light, which may produce an observable time delay or advance in the arrival times of pulses from distant astrophysical sources, or the broadening of spectral lines. They also lead to an intrinsic uncertainty in measuring positions of objects and thus constitute a noise from quantum gravity in modern gravity-wave interferometers. The possible observable distortion of the cosmic microwave background (CMB) radiation spectrum due to the lightcone fluctuations has been recently discussed [7,8]. These quantum gravity effects as well as others predicted by various other approaches [9–13] open up the possibility of testing theories of quantum gravity in the near future or even at the present experiments. This might suggest that an era of quantum-gravity phenomenology may be just around the corner [14].

In this paper, our aim is to extend the analysis performed in Ref. [3] on lightcone fluctuations in a spacetime with a single plane boundary to the case where there are two parallel plane boundaries, and examine what influence of the presence of the second plane will have on the lightcone fluctuations. Both the Dirichlet and the Neumann boundary conditions will be considered. Note that in Ref. [3] only the Neumann boundary conditions were studied for the single plane. The framework we are going to use is that developed in [1,3,4]. In Sect. II, we will give a brief review of the formalism and then study in detail the lightcone fluctuations for both longitudinal and transverse light propagations in both cases of the Dirichlet and the Neumann boundary conditions. Our results will be summarized in Sect. III.

II. LIGHTCONE FLUCTUATIONS IN SPACETIME WITH TWO PARALLEL PLANE BOUNDARIES

To begin, let us consider a flat background spacetime with a linearized perturbation $h_{\mu\nu}$ propagating upon it, so the spacetime metric may be written as

$$ds^2 = (\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu = dt^2 - dx^2 + h_{\mu\nu} dx^\mu dx^\nu,$$  

(1)
where the indices $\mu, \nu$ run through 0, 1, 2, 3. Let $\sigma(x, x')$ be one half of the squared geodesic distance between a pair of spacetime points $x$ and $x'$, and $\sigma_0(x, x')$ be the corresponding quantity in the flat background. In the presence of a linearized metric perturbation $h_{\mu\nu}$, we may expand

$$\sigma = \sigma_0 + \sigma_1 + O(h_{\mu\nu}^2) .$$

(2)

Here $\sigma_1$ is first order in $h_{\mu\nu}$. If we quantize $h_{\mu\nu}$, then quantum gravitational vacuum fluctuations will lead to fluctuations in the geodesic separation, and therefore induce lightcone fluctuations. In particular, we have $\langle \sigma_1^2 \rangle \neq 0$, since $\sigma_1$ becomes a quantum operator when the metric perturbations are quantized. The quantum lightcone fluctuations give rise to stochastic fluctuations in the speed of light, which may produce an observable time delay or advance $\Delta t$ in the arrival times of pulses. Note that this model uses a linearized approach to quantum gravity which is expected to be a limit of a more exact theory. In the absence of a full theory, this seems to be the most conservative way to compute quantum gravity effects.

Let us consider the propagation of light pulses between a source and a detector separated by a distance $r$ on a flat background with quantized linear perturbations. It has been shown that the root-mean-squared fluctuation in the propagation time is given by [4]

$$\Delta t = \frac{\sqrt{\langle \sigma_1^2 \rangle_R}}{r} ,$$

(3)

where $\langle \sigma_1^2 \rangle_R$ is a renormalized expectation value. In order to find $\Delta t$ we need to calculate the quantum expectation value of the mean squared fluctuation $\langle \sigma_1^2 \rangle$ of the geodesic interval function, which is given by [1,3]

$$\langle \sigma_1^2 \rangle_R = \frac{1}{8}(\Delta r)^2 \int_{r_0}^{r_1} dr \int_{r_0}^{r_1} dr' n^{\mu} n'^{\nu} n^{\rho} n'^{\sigma} G_{\mu\nu\rho\sigma}^R(x, x') ,$$

(4)

where $dr = |dx|$, $\Delta r = r_1 - r_0$, $n^{\mu} = dx^{\mu}/dr$, and $G_{\mu\nu\rho\sigma}^R(x, x')$ is the suitably renormalized graviton Hadamard function. A natural starting point for quantum calculations may be the graviton two point function of Minkowski spacetime. If, however, it is not renormalized, the integral diverges. The usual response to this problem is to require that the renormalized two point function vanish in Minkowski spacetime. If this is the correct solution, then $\Delta t = 0$, and there are no lightcone fluctuations in Minkowski spacetime. It should be pointed out that many current theories of quantum gravity do not, however, use 4-dimensional Minkowski spacetime as the starting point. For example, quantum gravity derived in the context of string theory [6] and canonical quantum gravity in the loop approximation [11] all predict lightcone fluctuations. From now on, we will assume that a 4-dimensional spacetime configuration in which there are two two-dimensional parallel planes located at $z = 0$ and $z = L$ respectively. We will examine the quantum lightcone fluctuations in this scenario for both the Dirichlet and the Neumann boundary conditions.

A. Dirichlet boundary conditions

First we will consider the case where gravitons satisfy the Dirichlet boundary conditions. Assuming that $G_{ijkl}^{(1)}(t, x, y, z; t', x', y', z')$ is the Hadamard function for the Minkowski vacuum
state, the renormalized Hadamard function for the case of two parallel planes can be found, by using the method of images, as

\[
G_{ijkl}^R(t, x, y, z; t', x', y', z') = \sum_{n=-\infty}^{+\infty} G_{ijkl}^{(1)}(t, x, y, z; t', x', y', z' + 2nL) - \sum_{n=-\infty}^{+\infty} G_{ijkl}^{(1)}(t, x, z; t', x', y', -z' + 2nL). \tag{5}
\]

Here the prime denotes that the \(n = 0\) term which is the Hadamard function of Minkowski spacetime is omitted in the summation. Since the formalism is gauge invariant [3], we will choose to work in the TT gauge.

1. Parallel propagation case

Now let us consider a light ray traveling parallel to the planes, along the \(x\)-direction from point \((t, a, 0, z)\) to point \((t', b, 0, z)\). Define \(\rho = x - x'\), and \(\Delta z = z - z'\) and keep in mind that the integration in Eq. (4) is to be carried out along the null ray on which \(\rho = t - t'\), we obtain the relevant graviton two-point function in the Minkowski vacuum state in the TT gauge [3]

\[
G_{xxxx}^{(1)}(t, x, 0, z; t', x', 0, z') = \frac{2}{\pi^2} \left[ -\frac{\rho^2 \Delta z^4}{8(\rho^2 + \Delta z^2)^4} - \frac{\rho^6}{3(\rho^2 + \Delta z^2)^4} + \frac{47\rho^4 \Delta z^2}{12(\rho^2 + \Delta z^2)^4} \right.
\]

\[\left. - \frac{1}{(\rho^2 + \Delta z^2)^{9/2}} \left( -\frac{1}{32}\rho \Delta z^6 - \frac{3}{8}\rho^3 \Delta z^4 + \frac{3}{4}\rho^5 \Delta z^2 \right) \cdot \ln \left( \frac{\sqrt{\rho^2 + \Delta z^2} + \rho}{\sqrt{\rho^2 + \Delta z^2} - \rho} \right)^2 \right], \tag{6}\]

Let \(r = b - a\), we have

\[
\int_a^b dx \int_a^b dx' G_{xxxx}^{(1)}(t, x, 0, z; t', x', 0, z') = 2 \int_0^r d\rho (r - \rho) G_{xxxx}(\rho, \Delta z)
\]

\[= \frac{2}{\pi^2} \left[ -\frac{2r^4 + \Delta z^2 r^2}{4(r^2 + \Delta z^2)^2} + \frac{8r^5 + 8r^3 \Delta z^2 + 3r^2 \Delta z^4}{24(r^2 + \Delta z^2)^{5/2}} \ln \frac{\sqrt{r^2 + \Delta z^2} + r}{\sqrt{r^2 + \Delta z^2} - r} \right]
\]

\[\equiv f(\Delta z, r). \tag{7}\]

Plugging the above result into Eq. (5) and Eq. (4) and letting \(z = z'\) yields

\[
\langle \sigma_1^2 \rangle_R = \frac{r^2}{8} \left[ -f(2z, r) + \sum_{n=1}^{+\infty} (2f(2nL, r) - f(2z - 2nL, r)) \right] - \frac{r^2}{8} \left[ -f(2z, r) - f(2z - 2L, r) + g \right], \tag{8}\]

1By Minkowski, we mean flat spacetime without any boundaries
where we have defined
\[ g = \sum_{n=1}^{+\infty} [2f(2nL, r) - f(2z + 2nL, r)] - \sum_{n=2}^{+\infty} f(2z - 2nL, r) . \] (9)

Since the first term in Eq.(8) diverges when \( z \to 0 \), and the second term blows up as \( z \) approaches \( L \), the mean squared geodesic interval fluctuation is ill-behaved on the boundaries. This might be a result of our assumption that the boundaries are rigid and at fixed positions. A similar example is the divergence of energy density of a quantized field as the boundary is approached, where it has been shown if one treats the boundaries as quantum objects with a nonzero position uncertainty, the singularity in energy density is removed [15]. However it remains to be checked whether the divergence here can also be eliminated when the quantum fluctuations of the boundaries are taken into account.

Finding a closed-form result for the summations in function \( g \) seems to be a very difficult task. We will not make the attempt in this paper, instead we will try to analyze two special cases, i.e., \( \epsilon = r/L \gg 1 \) and \( \epsilon = r/L \ll 1 \). When the distance travelled by the light ray is much larger than the separation of the two parallel planes such that \( \epsilon \gg 1 \), or \( r \gg L \), the summation in Eq. (8) can be approximated by integration, for example,
\[ \sum_{n=1}^{+\infty} f(2nL, r) \approx \frac{\epsilon}{\pi^2} \int_{1/\epsilon}^{\infty} dx \left[ - \frac{2 + 4x^2}{2(1 + 4x^2)^2} + \frac{8 + 32x^2 + 48x^4}{12(1 + 4x^2)^{5/2}} \ln \frac{\sqrt{1 + 4x^2} + 1}{\sqrt{1 + 4x^2} - 1} \right] \equiv \frac{\epsilon}{\pi^2} h(1/\epsilon) . \] (10)

It then follows that
\[ g \approx \frac{\epsilon}{\pi^2} \left[ 2h(1/\epsilon) - h(1/\epsilon + z/r) - h(2/\epsilon - z/r) \right] . \] (11)

Performing the integration by parts and series expanding the result to order \( o(1/\epsilon) \), we obtain
\[ g \approx \frac{4}{3\pi^2} \left[ \ln \epsilon - \left( 1 + \frac{z}{L} \right) \ln \left( 1 + \frac{z}{L} \right) - \left( 2 - \frac{z}{L} \right) \ln \left( 2 - \frac{z}{L} \right) \right] . \] (12)

Since \( z \leq L \), we must have \( \lambda \gg 1 \) when \( \epsilon \gg 1 \). Therefore, similarly, one has
\[ f(2z, r) \approx \frac{4}{3\pi^2} \ln \frac{r}{z} \] (13)
\[ f(2z - 2L, r) \approx \frac{4}{3\pi^2} \ln \frac{r}{L - z} . \] (14)

Hence, in the case of \( r \gg L \)
\[ \langle \sigma_1^2 \rangle_R \approx -\frac{r^2}{6\pi^2} \ln \frac{rL}{z(L - z)} , \] (15)
and the mean deviation from the classical propagation time is

$$\Delta t \approx \sqrt{\frac{1}{6\pi^2} \ln \frac{rL}{z(L - z)}}. \quad (16)$$

A few comments are now in order. This result reveals that the mean deviation due to lightcone fluctuations grows as $r$, the travel distance, increases, and it is symmetric under $z \leftrightarrow L - z$ as it should be. At the same time, one can see that the deviation increases as $L$ decreases. This indicates that the presence of a second plane reinforces the lightcone fluctuations. In fact, if we expand Eq.(16) in the limit of large $L$, the result is

$$\Delta t \approx \sqrt{\frac{1}{6\pi^2} \ln \frac{r}{z} + \frac{z}{L} + \frac{1}{2} \frac{z^2}{L^2}}, \quad (17)$$

which gives us a idea what the contributions are due to the presence of the second brane in this case. Finally, let us note that the mean deviation has a minimum at $z = L/2$.

If $\epsilon \ll 1$, i.e. $L \gg r$, we can series expand $g$ and $f(2z - 2L, r)$ with $r/L$ to get

$$g + f(2z - 2L, r) \approx \frac{2}{\pi^2} \sum_{n=1}^{\infty} \left[ \frac{r^6}{90n^6L^6} - \frac{r^6}{180(nL + z)^6} - \frac{r^6}{180(nL - z)^6} \right]. \quad (18)$$

The summation can be calculated and the result can be expanded as a power series of $z/L$ to yield

$$g + f(2z - 2L, r) \approx -\frac{\pi^6}{10 \times 45^2} \left( \frac{r}{L} \right)^6 \left( \frac{z}{L} \right)^2. \quad (19)$$

Meanwhile $f(2z, r)$ is given by Eq.(13) if $r \gg z$, and $f(2z, r) \approx 0$ if $r \ll z$. Consequently, one has for the mean deviation in the case of $L \gg r$ and $r \gg z$

$$\langle \sigma_1^2 \rangle_R \approx -\frac{r^2 \ln(r/z)}{6\pi^2} - \frac{r^2 \pi^6}{80 \times 45^2} \left( \frac{r}{L} \right)^6 \left( \frac{z}{L} \right)^2 \quad (20)$$

and

$$\Delta t = \frac{\sqrt{\langle \sigma_1^2 \rangle_R}}{r} \approx \sqrt{\frac{\ln(r/z)}{6\pi^2} + \frac{\pi^6}{80 \times 45^2} \left( \frac{r}{L} \right)^6 \left( \frac{z}{L} \right)^2}. \quad (21)$$

And in the case of $L \gg r$ and $r \ll z$, the mean deviation is given instead by

$$\Delta t \approx \frac{\pi^3}{180\sqrt{5}} \left( \frac{r}{L} \right)^3 \left( \frac{z}{L} \right). \quad (22)$$

These results also show enhancing lightcone fluctuations as the travel distance increases or the separation of the plane decreases.
Let us turn our attention to the case where the light ray propagates perpendicularly to the plane, from point \((t, 0, 0, a)\) to \((t', 0, 0, b)\). Rewrite Eq. (5) as

\[
G_{zzzz}^R(t, z; t', z') = -G_{zzzz}^{(1)}(t, z; t', -z') + \sum_{n=-\infty}^{+\infty} G_{zzzz}^{(1)}(t, z; t', z' + 2nL) \\
- G_{zzzz}^{(1)}(t, z; t', -z' + 2nL) .
\]

(23)

Here the prime on the summation indicates that the \(n = 0\) term is excluded and the notation \((t, 0, 0, z) \equiv (t, z)\) has been adopted. Then we have [3]

\[
G_{zzzz}^{(1)}(t, z; t', z') = -\frac{2}{\pi^2} \frac{\Delta t^2}{(\Delta z)^2} + \frac{\Delta t^3}{4(\Delta z)^3} \ln \left( \frac{\Delta z - \Delta t}{\Delta z + \Delta t} \right) \\
- \frac{2}{3(2a + r)^3} - \frac{\Delta t}{4(\Delta z)^2} \ln \left( \frac{\Delta z - \Delta t}{\Delta z + \Delta t} \right) ,
\]

(24)

where \(\Delta t = t - t'\) and \(\Delta z = z - z'\). Taking into account that for a null geodesic \(\Delta t = \Delta z\) and as before letting \(r = b - a\), we find, after performing the integrations

\[
\int_{a}^{b} dz \int_{a}^{b} dz' G_{zzzz}^{(1)}(t, z; t', -z') \\
= \frac{r}{6\pi^2(2a + r)^3} \left\{ (4ar + 2r^2) - (6a^2 + 6ar + r^2) \ln \frac{a^2}{(a + r)^2} \right\} ,
\]

(25)

\[
\sum_{n=-\infty}^{+\infty} \int_{a}^{b} dz \int_{a}^{b} dz' G_{zzzz}^{(1)}(t, z; t', z' + 2nL) \\
= \frac{2}{3\pi^2} \sum_{n=1}^{+\infty} \frac{(r + nL)^3}{(r + 2nL)^3} \ln \left( 1 + \frac{r}{nL} \right)^2 + \frac{(r - nL)^3}{(r - 2nL)^3} \ln \left( 1 - \frac{r}{nL} \right)^2 \\
+ \frac{4r^2(r^2 - 2n^2L^2)}{(r^2 - 4n^2L^2)^2} ,
\]

(26)

and

\[
\sum_{n=-\infty}^{+\infty} \int_{a}^{b} dz \int_{a}^{b} dz' G_{zzzz}^{(1)}(t, z; t', -z' + 2nL) \\
= \frac{1}{3\pi^2} \sum_{n=1}^{+\infty} \frac{r(6a^2 + 6n^2L^2 - 6nLr + r^2 + 6a(r - 2nL))}{2(2a - 2nL + r)^3} \ln \left( \frac{a - nL}{a - nL + r} \right)^2 \\
+ \frac{r^2}{(2a - 2nL + r)^2} ,
\]

(27)

Since here \(r\) is always less than \(L\), we will only consider the case where \(L \gg r\). Again Eq.(26) and Eq.(27) can be approximately evaluated, by expanding the summands as a power series of \(r/L\), keeping only the leading term and doing the summation afterwards,
\[ \sum_{n=-\infty}^{+\infty} \int_{a}^{b} dz \int_{a}^{b} dz' G_{zzzz}^{(1)}(t, z; t', z' + 2nL) \approx \sum_{n=1}^{\infty} \frac{2r^2}{3\pi^2n^2L^2} = \frac{r^2}{9L^2}, \quad (28) \]

\[ \sum_{n=-\infty}^{+\infty} \int_{a}^{b} dz \int_{a}^{b} dz' G_{zzzz}^{(1)}(t, z; t', -z' + 2nL) \]

\[ \approx \sum_{n=-\infty}^{+\infty} \frac{r^2}{3\pi^2(a-nL)^2} = \frac{1}{3} \frac{r^2}{L^2} \csc^2(\pi a/L) - \frac{r^2}{3\pi^2a^2}. \quad (29) \]

Here we have used the following two formulas in calculating the summation [16]

\[ \sum_{n=0}^{\infty} \frac{1}{(n^2 - a^2)^2} = \frac{1}{2a^4} + \frac{\pi}{4a^3} \cot(\pi a) + \frac{\pi^2}{4a^2} \csc^2(\pi a), \quad (30) \]

and

\[ \sum_{n=0}^{\infty} \frac{n^2}{(n^2 - a^2)^2} = -\frac{\pi}{4a} \cot(\pi a) + \frac{\pi^2}{4} \csc^2(\pi a). \quad (31) \]

As for Eq. (25), we need to consider two special cases. One is when \( r \ll a \), where one finds

\[ \int_{a}^{b} dz \int_{a}^{b} dz' G_{zzzz}^{(1)}(t, z; t', -z') \approx \frac{r^2}{3\pi^2a^2}, \quad (32) \]

and the other is when \( r \gg a \) and there one has

\[ \int_{a}^{b} dz \int_{a}^{b} dz' G_{zzzz}^{(1)}(t, z; t', -z') \approx \frac{1}{3\pi^2} [1 + \ln(r/a)]. \quad (33) \]

To summarize, when \( L \gg r \) and \( r \gg a \), we have that

\[ \langle \sigma_{11}^{2} \rangle_R = \frac{r^2}{8} \int_{a}^{b} dz \int_{a}^{b} dz' G_{zzzz}^{R}(t, z; t', z') \]

\[ \approx -\frac{1}{24\pi^2} [1 + \ln(r/a)] + \frac{r^2}{72L^2} + \frac{r^2}{24\pi^2a^2} - \frac{1}{24} \frac{r^2}{L^2} \csc^2(\pi a/L) \]

\[ \approx -\frac{1}{24\pi^2} [1 + \ln(r/a)] - \frac{\pi^2}{360} \left( \frac{r}{L} \right)^2 \left( \frac{a}{L} \right)^2. \quad (34) \]

Therefore, the mean deviation due to lightcone fluctuations is

\[ \Delta t = \sqrt{|\langle \sigma_{11}^{2} \rangle_R / r|} \approx \sqrt{\frac{1}{24\pi^2} [1 + \ln(r/a)] + \frac{\pi^2}{360} \left( \frac{r}{L} \right)^2 \left( \frac{a}{L} \right)^2}. \quad (35) \]

The last term in square-root represents the contribution due to the presence of a second plane. However, when \( L \gg r \) and \( r \ll a \), Eq. (34) becomes
\left\langle \sigma_1^2 \right\rangle_R = \frac{r^2}{8} \int_a^b \int_a^b dz \int_a^b dz' G_{zzz}^{R}(t, z; t', z') \\
\approx -\frac{1}{24} \frac{r^2}{L^2} \csc^2(\pi a / L) + \frac{1}{72} \frac{r^2}{L^2}.

(36)

Consequently, the mean deviation reads

$$
\Delta t = \frac{\sqrt{\left\langle \sigma_1^2 \right\rangle_R}}{r} \approx \sqrt{\frac{1}{24} \frac{r^2}{L^2} \csc^2(\pi a / L) - \frac{1}{72} \frac{r^2}{L^2}}.
$$

(37)

If we further assume that \( L \gg a \), then the above result becomes

$$
\Delta t \approx \sqrt{\frac{r^2}{24\pi^2 a^2} + \frac{\pi^2}{360} \left( \frac{r}{L} \right)^2 \left( \frac{a}{L} \right)^2}.
$$

(38)

This reveals a linear growth over the travel distance, \( r \).

A few comments on the results obtained above are in order here. Let us recall that \( \left\langle \sigma_1^2 \right\rangle_R \) was assumed to be positive when the relation between \( \Delta t \) and \( \left\langle \sigma_1^2 \right\rangle_R \) was first derived [1,3]. Thus, if \( \left\langle \sigma_1^2 \right\rangle_R \) is negative, then \( \Delta t \) would be imaginary. To solve this problem, an alternative derivation was proposed based upon averaging \( \sigma^4 \) over a given quantum state of gravitons and applying Wick’s theorem [4]. This derivation yielded Eq. (3) in which the absolute value of \( \left\langle \sigma_1^2 \right\rangle_R \) is used. All the calculations performed so far on the mean deviation in propagation time, \( \Delta t \), are based on this new formula. Although Eq. (3) can formally deal with situations where \( \left\langle \sigma_1^2 \right\rangle_R \) is negative and gives a real thus physically meaningful \( \Delta t \), it does not fully resolve the issue physically. It just circumvents the sign problem and tells us little about the physical significance of the sign in the geodesic distance fluctuation. For all the cases we have discussed above in the case of the Dirichlet boundary conditions, the mean squared fluctuations in the geodesic distance are all negative, while, as we will see in the next subsection, they are all positive in the corresponding cases in the case of the Neumann boundary conditions. Now let us note that to build a reflecting wall for gravitons one would need negative mass. Therefore imposing the Dirichlet boundary conditions for gravitons in the present context seems unphysical, as it has the same effect as having sources of negative mass. This might suggest that the negative sign for the geodesic distance fluctuation might be a result of the imposition of the unphysical boundary conditions. However, it is worth pointing out that much more work needs to be done to fully understand the meaning of the sign in the geodesic distance fluctuation and its relation with boundary conditions, geometry and topology of spacetime as well as other physical conditions. Finally, recall that our renormalization scheme is to subtract the mean squared fluctuation of the geodesic interval function in the Minkowski spacetime. Thus our result may also indicate that the Dirichlet boundary condition reduce the lightcone fluctuations that may already present in the Minkowski spacetime, while the Neumann boundary conditions enhance them.

### B. Neumann boundary conditions

After having considered the case of the Dirichlet boundary conditions, here we turn our attention to the case in which gravitons satisfy the Neumann boundary conditions. The renor-
malized Hadamard function obeying the boundary conditions can also be found, by using the method of images, as

\[
G^R_{ijkl}(t, x, y, z; t', x', y', z') = \sum_{n=-\infty}^{+\infty} G^{(1)}_{ijkl}(t, x, y, z; t', x', y', z' + 2nL) + \sum_{n=-\infty}^{+\infty} G^{(1)}_{ijkl}(t, x, y, z; t', x', y', -z' + 2nL).
\]  (39)

Note the sign change of the second term in the above expression as compared to that in Eq. (5).

1. Parallel propagation case

First let us examine the case in which a light ray propagates parallel to the planes from point \((t, a, 0, z)\) to point \((t', b, 0, z)\). Using the same notation and methods as before, we find that the mean squared geodesic interval fluctuation \(\langle \sigma^2_1 \rangle_R\) can be expressed as

\[
\langle \sigma^2_1 \rangle_R = \frac{r^2}{8} [f(2z, r) + f(2z - 2L, r) + \sum_{n=1}^{+\infty} (2f(2nL, r) + f(2z + 2nL, r)) + \sum_{n=2}^{+\infty} f(2z - 2nL, r)]. \]  (40)

In the limit of \(r \gg L\), the summation part in the above expression can be calculated, in the same way as that in the proceeding subsection, to get

\[
\sum_{n=1}^{+\infty} (2f(2nL, r) + f(2z + 2nL, r)) + \sum_{n=2}^{+\infty} f(2z - 2nL, r)
\approx \frac{4}{3\pi^2} \left[ 3A^2 \epsilon - 5 \ln \epsilon + \left(1 + \frac{z}{L}\right) \ln \left(1 + \frac{z}{L}\right) + \left(2 - \frac{z}{L}\right) \ln \left(2 - \frac{z}{L}\right) \right], \]  (41)

where

\[
A^2 = \frac{1}{4} \int_0^\infty dx \frac{\ln(2x + \sqrt{1 + 4x^2})}{x\sqrt{1 + 4x^2}} \approx 0.6168. \]  (42)

Using Eq. (13), Eq. (14) and Eq. (41), we find that

\[
\langle \sigma^2_1 \rangle_R \approx \frac{r^2}{8} \left[ \frac{4}{3\pi^2} \left( 3A^2 \frac{r}{L} - 3 \ln \frac{r}{L} + \ln \frac{L^2}{z(L - z)} \right) \right] \approx \frac{A^2r^2}{2\pi^2 L}. \]  (43)

Here in the last step we have assumed that \(z \neq 0\) and \(z \neq L\). Thus

\[
\Delta t \approx \frac{A}{\sqrt{2\pi}} \sqrt{\frac{r}{L}}. \]  (44)
A comparison of Eq. (16) and the above result shows that while the growth of the mean deviation due to the lightcone fluctuations over the travel distance, \( r \), is that of a square root of the logarithm of \( r \) in the Dirichlet boundary condition case, it is a square root of \( r \) here. Therefore, as \( r \) the travel distance increases, the lightcone fluctuations grow much faster in the present case than that in the corresponding case with the Dirichlet boundary conditions. The square root behavior of growth seems to suggest a fluctuation of random walk nature. Note also that this behavior is the same as that in the case where one spatial dimension is periodically compactified, i.e., the space topology is that of a cylinder (see Eq. (78) of Ref. [3]), since the constant \( A \) equals \( c_1/2 \) given by Eq. (75) in Ref. [3].

We now turn to the case in which \( L \gg r \) and \( r \gg z \). Here the summation part can be approximately calculated by series expanding the summand first. The result is

\[
\sum_{n=1}^{+\infty} [2f(2nL, r) + f(2z - 2nL, r) + f(2z + 2nL, r)] \approx \frac{2\pi^4}{45^2 \times 21} \frac{r^6}{L^6}.
\]

(45)

Therefore, if in addition, \( r \gg z \), we have

\[
\langle \sigma_1^2 \rangle_R \approx \frac{r^2}{8} \left[ \frac{4}{3\pi^2} \ln \frac{r}{z} + \frac{2\pi^4}{45^2 \times 21} \frac{r^6}{L^6} \right].
\]

(46)

Thus

\[
\Delta t \approx \sqrt{\frac{\ln(r/z)}{6\pi^2}} + \frac{2\pi^4}{45^2 \times 21} \left( \frac{r}{L} \right)^6.
\]

(47)

The leading term here is the same as that in the case of the Dirichlet boundary conditions (refer to Eq. (21)). But the higher order corrections are suppressed by a square factor of \( z/L \) in the Dirichlet boundary conditions as compared to that of the Neumann ones. So again, the lightcone fluctuations here are greater. If we let \( L \to \infty \), then the result of a single plane is recovered as expected (see Eq. (91) in Ref. [3]). However, if \( r \ll z \), then we have

\[
\langle \sigma_1^2 \rangle_R \approx \frac{r^2}{8} \left[ \frac{2\pi^4}{45^2 \times 21} \frac{r^6}{L^6} \right].
\]

(48)

As a result

\[
\Delta t \approx \frac{\pi^2}{90\sqrt{21}} \left( \frac{r}{L} \right)^3.
\]

(49)

If we compare Eq. (22) with the above result, we can see that the mean deviation in the present case is larger than that of the Dirichlet boundary conditions since \( L \gg z \).

2. Perpendicular propagation case

Here we study the case in which a light ray travels along \( z \) axis from point \((t, 0, 0, a)\) to \((t', 0, 0, b)\). Making use of Eq. (28), Eq. (29), Eq. (32), Eq. (33), and Eq. (39), we arrive at the following results.
If \( L \gg r \) and \( r \gg a \), then
\[
\langle \sigma^2 \rangle_R \approx \frac{r^2}{8} \left[ \frac{1}{3\pi^2} [1 + \ln(r/a)] + \frac{2r^2}{9L^2} + \frac{\pi^2}{45} \left( \frac{r}{L} \right)^2 \left( \frac{a}{L} \right)^2 \right],
\] (50)

and
\[
\Delta t = \sqrt{\frac{|\langle \sigma^2 \rangle_R|}{r}} \approx \sqrt{\frac{1}{24\pi^2} [1 + \ln(r/a)] + \frac{1}{36} \left( \frac{r}{L} \right)^2}.
\] (51)

The \( L \)-independent terms in the above expression agree with the result of a single plane (see Eq. (85) in Ref. [3]), while the \( L \)-dependent term gives the contribution due to the presence of the second plane. The above result reveals that the lightcone fluctuations are reinforced with the introduction of a second plane. Recall Eq. (35), one can see that the \( L \)-dependent term here is of lower order than that in the case of the Dirichlet boundary conditions.

If, however, \( r \ll a \), then
\[
\langle \sigma^2 \rangle_R \approx \frac{r^2}{8} \left[ \frac{1}{3\pi^2} [1 + \ln(r/a)] + \frac{2r^2}{9L^2} + \frac{\pi^2}{45} \left( \frac{r}{L} \right)^2 \csc^2 \left( \frac{\pi a}{L} \right) \right],
\] (52)

Thus
\[
\Delta t = \sqrt{\frac{|\langle \sigma^2 \rangle_R|}{r}} \approx \sqrt{\frac{r^2}{72L^2} + \frac{1}{24L^2} \csc^2 \left( \frac{\pi a}{L} \right)}.
\] (53)

If we further assume that \( L \gg a \), then the above result becomes
\[
\Delta t \approx \sqrt{\frac{1}{24\pi^2} \left( \frac{r}{a} \right)^2 + \frac{1}{36} \left( \frac{r}{L} \right)^2}.
\] (54)

This shows that the lightcone fluctuations grow linearly with \( r \) in this case, and the result reduces that of a single plane when \( L \) approaches infinity, i.e., one plane is infinitely away from the other. Once again, the mean deviation is larger here than in the case of the Dirichlet boundary conditions.

### III. CONCLUSION

We have examined the quantum lightcone fluctuations in a spacetime with two parallel planes based upon a framework proposed in [1] and subsequently further developed in [2–4]. Lightcone fluctuations due to gravitons satisfying the Dirichlet and the Neumann boundary conditions are all considered. For each boundary condition, the mean deviation from the classical propagation time is calculated both for light rays travelling parallel to the planes and for those propagating perpendicularly to the planes. For all cases, we find that lightcone fluctuations are larger when the Neumann boundary conditions are imposed than that when the
Dirichlet boundary conditions are enforced. This reveals that quantum lightcone fluctuations depend not only on the geometry and topology of the spacetime as has been argued [3] but also on the boundary conditions.

In the case of longitudinal (parallel) light propagation, when the travel distance is large compared to the separation of the planes, the fluctuation increases with the squared root of the distance travelled for the case of the Neumann boundary conditions, indicating a fluctuation of a random walk nature. It is worth pointing out that this behavior is basically the same as that in the corresponding case in a spacetime where one spatial section is periodically compactified [3]. However, if the boundary condition is that of the Dirichlet, then the growth becomes the square root of the logarithm of the distance travelled, which is much slower than the square root growth over the travel distance in the Neumann boundary condition case.

The lightcone fluctuations discussed in this paper are frequency-independent. However, it is interesting to note that frequency-dependent even helicity-dependent lightcone fluctuations are expected in other theories of quantum gravity [6,11]. It should be pointed out that the growth of lightcone fluctuation here is in general a rather complicated function of the light travel distance and it can be linear, square root, etc., depending on the relative size of the travel distance and the characteristic scales of the configuration. Therefore contingent on the physical scales involved, the effects of lightcone fluctuations obtained here can either be larger or smaller than those effects discussed in [6] using the same theoretic formalism together with a string theory motivated microscopic model.

Finally, for all the cases we have investigated for both longitudinal and transverse light propagations in both cases of the Dirichlet and the Neumann boundary conditions, our results show that lightcone fluctuations are greater in the spacetime with two parallel planes than that in the spacetime with just a single plane. Thus the presence of a second plane reinforces the quantum lightcone fluctuations. This phenomenon may be considered as due to enhanced quantum fluctuations stemming from confining gravitons in a smaller region. It can be understood as a consequence of the uncertainty principle and an analogy of what happens to any quantum particle confined in a smaller region of space.

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