Path Integral Method for Step Option Pricing

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Path integral method in quantum mechanics offers a new thinking for barrier option pricing. For double step barrier options, option changing process is analogous to a particle moving in a finite symmetric square potential well. Using energy level approximate formulas, the analytical expression of option price could be acquired. Numerical results show that the up-and-out call double step barrier option price decreases with the increasing of discounting time and exercise price, but increases with the increasing of underlying price for \( S_0 < B \). For a fixed underlying price, exercise price or discounting time, the option price decreases with the increasing of the height of the barrier \( V \).

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I. INTRODUCTION

In 1973, Black and Scholes derived the analytical expression for fixed-volatility option price by solving stochastic differential equations [1]. Financial mathematics applied in derivative pricing has made great progress from then on [2, 3]. Recently, more complex options have emerged in financial engineering, which collectively called exotic options. An exotic option is the one attached some conditions to an ordinary option, set some barrier for example: when the underlying price reaches this barrier, the option contract will be activated, which is called knock-in option; when the underlying price reaches this barrier, the option contract is invalid, which is call knock-out option. Snyder had discussed down-and-out option in 1969 [4]. Baaquie et al discussed this kind of option by path integral method, and derived the corresponding analytical expression [5]. Similar to one-dimensional infinite square potential well in quantum mechanics, they derived the analytical expression for double-knock-out option price [6], which is in accordance with the result derived by mathematical method [7]. In addition, path integral method has been applied to the research of interest rate derivative pricing [8, 9].

In this paper, we will discuss a kind of barrier option. Taking up-and-out option for example, when the underlying price touches and passes the barrier, the option contract is not invalid immediately, but the option is knocked out gradually, such an option is called step barrier up-and-out option [10]. We focus on the relation between step option pricing and finite symmetric square well. The boundaries of the well are regarded as two barriers. When a particle moving ahead and going through the boundary, the wave function begins to decay exponentially, which is similar to an option knocked out over time.

Our work is organized as follows. In Section 2, we review Black-Scholes Model in path integral method, and derive the expression for European option. In Section 3, comparing double step barrier option to finite symmetric square well, we derive the corresponding option pricing kernel. In Section 4, we show the numerical results for option price as a function of underlying price, exercise price and discounting time, respectively. We summarize our main results in Section 5. Appendix A gives the energy level approximation analytical expression.
II. PATH INTEGRAL METHOD FOR BLACK-SCHOLES MODEL

According to Ref [6], starting from Black-Scholes pricing formula, the price of European option can be derived by path integral method. The Black-Scholes formula is

\[
\frac{\partial C}{\partial t} + rS \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = 0
\]  

(1)

where \( C \) is European option price, \( S \) is the underlying asset price, \( \sigma \) is the fixed volatility, and \( r \) is the interest rate. Let

\( S = e^x, \ (-\infty < x < +\infty) \)

(2)

and (1) can be denoted as

\[
\frac{\partial C}{\partial t} = \left[ -\frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \left( \frac{1}{2} \sigma^2 - r \right) \right] C
\]

(3)

let

\[
H_{BS} = -\frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \left( \frac{1}{2} \sigma^2 - r \right) \frac{\partial}{\partial x} + r
\]

(4)

the Black-Scholes equation is written as

\[
\frac{\partial C}{\partial t} = H_{BS} C
\]

(5)

Comparing (5) to Schrödinger equation, we have

\[
\sigma^2 \sim \frac{1}{m^2}, \ C \sim \psi(x)
\]

(6)

where \( m \) is the particle mass, and \( \psi(x) \) is the wave function. The Black-Scholes Hamiltonian (4) in momentum representation can be denoted as

\[
H_{BS} = \frac{1}{2} \sigma^2 p^2 + i \left( \frac{1}{2} \sigma^2 - r \right) p + r
\]

(7)

where \( p = -i \frac{\partial}{\partial x} \). The pricing kernel is

\[
\langle x | e^{-\tau H_{BS}} | x' \rangle = \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \langle x | e^{-\tau H_{BS}} | p \rangle \langle p | x' \rangle
\]

\[
e^{-\tau} \int_{-\infty}^{+\infty} \frac{dp}{2\pi} e^{\frac{1}{2} \tau \sigma^2 \left( \frac{p - x' + x_0}{\tau \sigma^2} \right)^2 - \frac{(x' - x_0)^2}{2 \tau \sigma^2}}
\]

\[
e^{-\frac{1}{2} \tau \sigma^2 (x' - x_0)^2}
\]

(8)
where the completeness relation has been used, and

\[ x_0 = x + \tau \left( r - \frac{\sigma^2}{2} \right) \]  \hspace{1cm} (9)

The European call option price can be denoted as

\[
C(x, \tau) = e^{-r\tau} \int_{-\infty}^{+\infty} \frac{dx'}{\sqrt{2\pi} \tau \sigma^2} \left( e^{x'} - K \right) e^{-\frac{1}{2\tau\sigma^2}(x'-x_0)^2} \]

\[
= e^{-r\tau} \int_{\ln K - x_0}^{+\infty} \frac{dx'}{\sqrt{2\pi} \tau \sigma^2} \left( e^{x'+x_0} - K \right) e^{-\frac{1}{2\tau\sigma^2}x'^2} \]

\[
= SN(d_+) - e^{-r\tau} KN(d_-) \hspace{1cm} (10)
\]

where

\[ N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}z^2} dz, \quad d_\pm = \frac{\ln S_K + (r \pm \frac{\sigma^2}{2}) \tau}{\sigma \sqrt{\tau}} \]  \hspace{1cm} (11)

III. DOUBLE STEP OPTION PRICING

The price changing of a double step option could be analogous to a particle moving in a symmetric square potential well, and the potential can be denoted as

\[ V(x) = \begin{cases} 
  r, & |x| \leq B, \\
  V_0, & |x| > B.
\end{cases} \hspace{1cm} (12) \]

for \(|x| > B\), the wave function decays with the increasing of \(|x|\), which is similar to an option touches a barrier and knocked out gradually. The Hamiltonian for a double step option is

\[ H_{DSO} = \begin{cases} 
  -\frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \left( \frac{1}{2} - r \right) \frac{\partial}{\partial x} + r, & |x| \leq B, \\
  -\frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \left( \frac{1}{2} - V_0 \right) \frac{\partial}{\partial x} + V_0, & |x| > B.
\end{cases} \hspace{1cm} (13) \]

which is a non-Hermitian Hamiltonian. Considering the following transformation

\[ H_{DSO} = e^{\alpha x} H_{eff} e^{-\alpha x} = \begin{cases} 
  e^{\alpha x} \left[ -\frac{\sigma^2}{2} \left( \frac{\partial^2}{\partial x^2} - \gamma_1 \right) \right] e^{-\alpha x}, & |x| \leq B, \\
  e^{\alpha x} \left[ -\frac{\sigma^2}{2} \left( \frac{\partial^2}{\partial x^2} - \gamma_2 \right) \right] e^{-\alpha x}, & |x| > B.
\end{cases} \hspace{1cm} (14) \]
where

\[ \alpha = \frac{1}{\sigma^2} \left( \frac{\sigma^2}{2} - V_0 \right) \]

\[ \gamma_1 = \frac{\left( \frac{\sigma^2}{2} + r \right)^2}{\sigma^4} \]

\[ \gamma_2 = \frac{\left( \frac{\sigma^2}{2} + V_0 \right)^2}{\sigma^4} \]

(15)

and \( H_{\text{eff}} \) is a Hermitian Hamiltonian which is considered as the symmetric square potential well Hamiltonian.

The eigenfunction can be denoted as

\[ \psi(x) = \begin{cases} 
A_1 e^{k_2 x}, & x \leq -B, \\
A_2 \sin(k_1 x + \delta), & -B \leq x \leq B, \\
A_1 e^{-k_2 x}, & x > B.
\end{cases} \]

(16)

where \( k_1 \) and \( k_2 \) are particle momenta for different regions, \( A_1 \) and \( A_2 \) are undetermined coefficients. Considering the continuity for both wave function and it’s derivative at \( x = B \), we have

- **Odd Parity Case**

\[ \psi(x) = \begin{cases} 
-\sqrt{\frac{1}{B} \sqrt{\frac{\eta}{\eta + 1}}} e^{\eta} \sin \xi e^{k_2 x}, & x \leq -B, \\
\sqrt{\frac{1}{B} \sqrt{\frac{\eta}{\eta + 1}}} \sin(k_1 x), & -B \leq x \leq B, \\
\sqrt{\frac{1}{B} \sqrt{\frac{\eta}{\eta + 1}}} e^{\eta} \sin \xi e^{-k_2 x}, & x > B.
\end{cases} \]

(17)

- **Even Parity Case**

\[ \psi(x) = \begin{cases} 
\sqrt{\frac{1}{B} \sqrt{\frac{\eta}{\eta + 1}}} e^{\eta} \cos \xi e^{k_2 x}, & x \leq -B, \\
\sqrt{\frac{1}{B} \sqrt{\frac{\eta}{\eta + 1}}} \cos(k_1 x), & -B \leq x \leq B, \\
\sqrt{\frac{1}{B} \sqrt{\frac{\eta}{\eta + 1}}} e^{\eta} \cos \xi e^{-k_2 x}, & x > B.
\end{cases} \]

(18)
where \( \xi = k_1 B \), \( \eta = k_2 B \). Considering the option price changing in the range of \( |x| \leq B \), using (14), the pricing kernel is

\[
\langle x | e^{-\tau_1 H_{DSO}} | x' \rangle = e^{\alpha(x-x')} \langle x | e^{-\tau_1 H_{eff}} | x' \rangle
\]

\[
= e^{\alpha(x-x')} \sum_n \langle x | e^{-\tau_1 H_{eff}} | n \rangle \langle n | x' \rangle
\]

\[
= e^{-\tau_1} e^{\alpha(x-x')} \sum_n e^{-\tau_1 E_n} \psi_n(x) \psi_n(x')
\]

where \( \tau_1 \) is the option moving time in \( |x| \leq B \), \( |n \rangle \) is the energy eigenstate, and \( \psi_n(x) \) is the energy eigenstate in coordinate representation. In general, there is no analytical solution for energy eigenvalue. In Ref [11], the author gives the following energy equation

\[
k_{1n} = \frac{n\pi}{2} - \arcsin \frac{k_{1n}}{\beta}
\]

where \( \beta = B \sqrt{2(V_0 - r)/\sigma} \), and \( k_{1n} \) is the discrete momentum values for \( k_1 \).

For low energy approximation, \( n \ll n_{\text{max}} \), only first-order approximation for (20) would be considered. The momentum for particle moving in \( |x| \leq B \) is

\[
k_{1n} \approx \frac{\beta n\pi}{2(\beta + 1)} B
\]

where

\[
n_{\text{max}} = \left[ \frac{2\beta}{\pi} \right]
\]

is the maximum number of energy levels in \( |x| \leq B \). The bracket represents the minimal integer not less than \( 2\beta/\pi \). The relative error of \( k_{1n} \) is

\[
\delta \kappa_n = \left| \frac{\Delta \kappa_n}{\kappa_n} \right| \approx \frac{(n\pi)^2}{24\beta(\beta + 1)^2}
\]

For high energy level situation, the momentum approximation is

\[
k_{1n} \approx \frac{1}{B} \sqrt{\frac{n - 1}{2\pi} + \sqrt{2 - \frac{n - 1}{\beta}}} \pi
\]
and the relative error of $k_{1n}$ is

$$
\delta k_n = \left| \frac{\Delta k_n}{k_n} \right| \approx \frac{[2 - (n - 1)\pi/\beta]^{3/2}}{12(n - 1)\pi}
$$

(25)

In Table I, the numerical results for (23) and (25) are shown. For $B = 5.7$, $V = 0.2$, $\sigma = 0.3$ and $r = 0.05$, we have $\beta = 10.41$ and $n_{\text{max}} = 6$. It is shown that, with the increasing of $n$, the error of (23) increases while the error of (25) decreases. For $n \leq 4$, the error of (23) is smaller, and for $n \geq 5$, the error of (25) is smaller.

We will use different energy formulas for different $n$ to give a more accurate pricing kernel.

Allowing for (17)-(18) and (21), the pricing kernel (19) can be written as

$$
\langle x | e^{-\tau_1 H_{DOS}} | x' \rangle = \frac{1}{2} e^{\alpha(x-x')} e^{-\gamma_1 \tau_1} \times
$$

$$
\sum_{m=1}^{m_{1}} e^{-\frac{1}{2} \tau_1^2 \sigma^2 k_{1,n,odd,low}^2 k_{2,n,odd,low}^2} \frac{k_{2,n,odd,low}}{k_{2,n,odd,low} B + 1} \left[ \cos(k_{1,n,odd,low}(x + x')) + \cos(k_{1,n,odd,low}(x - x')) \right] \\
+ \sum_{m=m_{1}+1}^{m_{\text{max,odd}}} e^{-\frac{1}{2} \tau_1^2 \sigma^2 k_{1,n,odd,high}^2 k_{2,n,odd,high}^2} \frac{k_{2,n,odd,high}}{k_{2,n,odd,high} B + 1} \left[ \cos(k_{1,n,odd,high}(x + x')) + \cos(k_{1,n,odd,high}(x - x')) \right] \\
+ \sum_{m=1}^{m_{2}} e^{-\frac{1}{2} \tau_1^2 \sigma^2 k_{1,n,even,low}^2 k_{2,n,even,low}^2} \frac{k_{2,n,even,low}}{k_{2,n,even,low} B + 1} \left[ \sin(k_{1,n,even,low}(x + x')) - \sin(k_{1,n,even,low}(x - x')) \right] \\
+ \sum_{m=m_{2}+1}^{m_{\text{max,even}}} e^{-\frac{1}{2} \tau_1^2 \sigma^2 k_{1,n,even,high}^2 k_{2,n,even,high}^2} \frac{k_{2,n,even,high}}{k_{2,n,even,high} B + 1} \left[ \sin(k_{1,n,even,high}(x + x')) + \sin(k_{1,n,even,high}(x - x')) \right]
$$

(26)
where \( m_1 \) and \( m_2 \) are boundaries of low and high energy levels for odd and even parities, respectively, and

\[
\begin{align*}
k_{1n,\text{odd,low}} &= \frac{\beta(2m-1)\pi}{2(\beta + 1)B} \\
k_{1n,\text{odd,high}} &= \frac{1}{B} \left[ (m-1)\pi + \sqrt{2 - \frac{2(m-1)\pi}{\beta}} \right] \\
k_{1n,\text{even,low}} &= \frac{2\beta m \pi}{2(\beta + 1)B} \\
k_{1n,\text{even,high}} &= \frac{1}{B} \left[ \frac{(2m-1)\pi}{2} + \sqrt{2 - \frac{(2m-1)\pi}{\beta}} \right]
\end{align*}
\]

are momenta for low energy, odd parity; high energy, odd parity; low energy even parity; and high energy, even parity, respectively. Allowing for the relation between \( k_{1n} \) and \( k_{2n} \)

\[
k_{1n}^2 + k_{2n}^2 = \frac{2B^2}{\sigma^2} (V_0 - r)
\]

we have

\[
\begin{align*}
k_{2n,\text{odd,low}} &= \sqrt{\frac{2}{\sigma^2} (V_0 - r) - k_{1n,\text{odd,low}}^2} \\
k_{2n,\text{odd,high}} &= \sqrt{\frac{2}{\sigma^2} (V_0 - r) - k_{1n,\text{odd,high}}^2} \\
k_{2n,\text{even,low}} &= \sqrt{\frac{2}{\sigma^2} (V_0 - r) - k_{1n,\text{even,low}}^2} \\
k_{2n,\text{even,high}} &= \sqrt{\frac{2}{\sigma^2} (V_0 - r) - k_{1n,\text{even,high}}^2}
\end{align*}
\]
Similar to (26), the pricing kernel for $|x| > B$ can be denoted as

$$\langle x' | e^{-\tau_2 H_{DSO}} | x'' \rangle = e^{\alpha(x' - x'')} e^{-\gamma_2 \tau_2} \times$$

\[
\left[ \sum_{m=1}^{m_2'} e^{-\frac{1}{2} \tau_2 \sigma^2 k_{2n,even,high}^2} \frac{k_{2n,even,low}^2}{k_{2n,even,high}^2} \frac{k_{2n,even,low} B + 1}{k_{2n,even,high} B + 1} e^{2Bk_{2n,even,low}} \cos^2 k_{1n,even,high} e^{-k_{2n,even,low}(x' + x'')} \right] + \left[ \sum_{m=m_1' + 1}^{m_{max,odd}} e^{-\frac{1}{2} \tau_2 \sigma^2 k_{2n,odd,low}^2} \frac{k_{2n,odd,low}^2}{k_{2n,odd,high}^2} \frac{k_{2n,odd,low} B + 1}{k_{2n,odd,high} B + 1} e^{2Bk_{2n,odd,low}} \cos^2 k_{1n,odd,high} e^{-k_{2n,odd,low}(x' + x'')} \right] + \left[ \sum_{m=m_{max,odd}}^{m_1} e^{-\frac{1}{2} \tau_2 \sigma^2 k_{2n,odd,high}^2} \frac{k_{2n,odd,low}^2}{k_{2n,odd,high}^2} \frac{k_{2n,odd,low} B + 1}{k_{2n,odd,high} B + 1} e^{2Bk_{2n,odd,low}} \cos^2 k_{1n,odd,high} e^{-k_{2n,odd,low}(x' + x'')} \right] + \left[ \sum_{m=m_2 + 1}^{m_{max,even}} e^{-\frac{1}{2} \tau_2 \sigma^2 k_{2n,even,high}^2} \frac{k_{2n,even,low}^2}{k_{2n,even,high}^2} \frac{k_{2n,even,low} B + 1}{k_{2n,even,high} B + 1} e^{2Bk_{2n,even,low}} \cos^2 k_{1n,even,high} e^{-k_{2n,even,low}(x' + x'')} \right]
\]

(30)

Considering a simple case, the initial underlying price is in the range of $S_0 < B$, and the final underlying is in the range of $S > B$. The option price can be denoted as

$$C(x, \tau_1, \tau_2) = \int_{\ln K}^{+\infty} dx'' \langle x'' | e^{-\tau_1 H_{DSO}} | B \rangle \langle B | e^{-\tau_2 H_{DSO}} | x'' \rangle (e^{x''} - K)$$

(31)

where $K$ is the exercise price, and we have used the boundary condition $x' = B$. For an up-and-out call option, the lower limit of integral is $\ln K$. In the following section, we will discuss the numerical results for (31).

### IV. NUMERICAL RESULTS

Considering a simple case, the initial underlying price is in the well, and the final price is in the range of $|x| \geq B$.

In Fig. 1, we show the double step up-and-call option price as a function of underlying price. It is shown that with the increasing of discounting potential $V$, the option price decreases for the same underlying price. For a fixed $V$, the option price increases with the underlying price increasing.
The double step up-and-call option price as a function of discounting time is shown in Fig. 2. For a fixed $V$, with the increasing of discounting time, the option price decreases. On the other hand, for a fixed discounting time, the option price decreases with the increasing of $V$.

In Fig. 3, we show option price as a function of exercise price $K$. For a fixed $K$, option price decreases with the increasing of $v$; for a fixed $V$, the option price decreases with the increasing of $K$. 

FIG. 1. Plot of double step call-and-out option price against underlying price for different discounting potentials. Parameters: $B = 5.7$, $K = 200$, $r = 0.05$, $\sigma = 0.3$, $\tau_1 = \tau_2 = 0.5$. 

FIG. 3. Plot of double step call-and-out option price against exercise price $K$ for different discounting potentials.
FIG. 2. Plot of double step call-and-out option price against discounting time for different discounting potentials. Parameters: $x = 4.605$, $B = 5.7$, $K = 200$, $r = 0.05$, $\sigma = 0.3$.

V. CONCLUSION

Path-integral is an effective method linking option price changing to a particle moving in potential wells. Here we have studied a simple case: the initial underlying price $S_0 = e^x < e^B$, and the final underlying price $S > e^B$, which could be analogous to a particle moving in a symmetric square potential well. We have presented option prices changing with the initial underlying prices, discounting times, and exercise prices, respectively. More complicated cases could be discussed by adding more pricing kernels, and the pricing of other barrier options could be studied by defining appropriate potentials $V$. 
FIG. 3. Plot of double step call-and-out option price against discounting time for different discounting potentials. Parameters: $x = 4.605$, $B = 5.7$, $K = 200$, $r = 0.05$, $\sigma = 0.3$.

Appendix A: Energy level approximate formulas for symmetric square well

Consider the following one-dimension symmetric square potential well

$$V(x) = \begin{cases} 
0, & |x| \leq a, \\
V_0, & |x| > a.
\end{cases} \tag{A1}$$
where $a$ is the width of the well. The corresponding Schrödinger equation is

\[
\begin{cases}
- \frac{\hbar^2}{2m} \frac{d^2 \phi_0}{dx^2} = E \phi_0, & |x| \leq a, \\
- \frac{\hbar^2}{2m} \frac{d^2 \phi_1}{dx^2} + V_0 \phi_1 = E \phi_1, & |x| > a.
\end{cases}
\tag{A2}
\]

Define

\[
\xi = \frac{x}{a}
\tag{A3}
\]

(A2) can be written as

\[
\begin{cases}
\frac{d^2 \phi_0}{d\xi^2} + \epsilon \phi_0 = 0, & |\xi| \leq 1, \\
\frac{d^2 \phi_1}{d\xi^2} - (u - E) \phi_1 = 0, & |\xi| > 1.
\end{cases}
\tag{A4}
\]

where

\[
\epsilon = \frac{2ma^2E}{\hbar^2}, \quad u = \frac{2ma^2U_0}{\hbar^2}
\tag{A5}
\]

- **Odd Parity Case**

Considering boundary conditions, the solution of (A4) for odd parity is

\[
\psi(x) = \begin{cases}
-B e^{-\sqrt{\beta^2 - \kappa^2} \xi}, & \xi < -1, \\
A \sin \kappa \xi, & |\xi| \leq 1, \\
B e^{-\sqrt{\beta^2 - \kappa^2} \xi}, & \xi > 1.
\end{cases}
\tag{A6}
\]

where

\[
\kappa = \sqrt{\epsilon}, \quad \beta = \sqrt{u}
\tag{A7}
\]

Using the continuous condition at $\xi = 1$, we have the following energy level equation

\[
\kappa \cot \kappa = -\sqrt{\beta^2 - \kappa^2}
\tag{A8}
\]

define an angle parameter

\[
\theta = \arcsin \frac{\kappa}{\beta}, \quad \theta \in \left(0, \frac{\pi}{2}\right)
\tag{A9}
\]
which gives

\[ \kappa = \ell \pi - \theta, \quad \ell \in \mathbb{Z}^+ \]  \hspace{1cm} (A10)

• Odd Parity Case

Even parity solution for (A4) is

\[ \psi(x) = \begin{cases} A \cos \kappa \xi, & |\xi| \leq 1, \\ B e^{-\sqrt{\beta^2 - \kappa^2} \xi}, & |\xi| > 1. \end{cases} \]  \hspace{1cm} (A11)

and the energy level equation is

\[ -\kappa \tan \kappa = -\sqrt{\beta^2 - \kappa^2} \]  \hspace{1cm} (A12)

which gives

\[ \kappa = \ell \pi + \frac{\pi}{2} - \theta, \quad \ell \in \mathbb{N} \]  \hspace{1cm} (A13)

Combining (A10) and (A13), we have

\[ \kappa_n = \frac{n\pi}{2} - \theta_n, \quad n \in \mathbb{Z}^+ \]  \hspace{1cm} (A14)

where \( n \) even for odd parity solution, and \( n \) odd for even parity solution. Take (A9) and (A14) into account, the range of \( \kappa_n \) is

\[ \kappa_n \in \left( \frac{n\pi}{2} - \frac{\pi}{2}, \frac{n\pi}{2} \right) \]  \hspace{1cm} (A15)

According to (A9), \( \kappa = \beta \sin \theta < \beta \), and the range of \( \beta \) is

\[ \beta \in \left( \frac{n_{\text{max}}\pi}{2} - \frac{\pi}{2}, \frac{n_{\text{max}}\pi}{2} \right) \]  \hspace{1cm} (A16)

where

\[ n_{\text{max}} = \left\lceil \frac{2\beta}{\pi} \right\rceil \]  \hspace{1cm} (A17)

is the number for the highest energy level. For low energy case, considering only the first order approximation, (A14) can be simplified into

\[ \kappa_n = \frac{n\pi}{2} - \arcsin \frac{\kappa_n}{\beta} \approx \frac{n\pi}{2} - \frac{\kappa_n}{\beta} \]  \hspace{1cm} (A18)
and the energy formula for low energy level is

\[ \kappa_n = \frac{\beta n \pi}{2(\beta + 1)}, \quad n \ll n_{\text{max}} \]  

(A19)

the error for \( \kappa_n \) is

\[ \Delta \kappa_n \approx \frac{1}{6} \left( \frac{\kappa_n}{\beta} \right)^3 = -\frac{(n\pi)^3}{48(\beta + 1)^3} \]  

(A20)

where we have ignored the third and higher order approximations. The relative error for \( \kappa_n \) is

\[ \delta \kappa_n = \left| \frac{\Delta \kappa_n}{\kappa_n} \right| \approx \frac{(n\pi)^2}{24\beta(\beta + 1)^2} \]  

(A21)

For high energy case, \( \kappa_n/\beta \approx 1 \), and

\[ x = 1 - \frac{\kappa_n}{\beta} \ll 1 \]  

(A22)

considering only the first order approximation of the following Taylor expansion

\[ \arcsin(1-x) \approx \frac{\pi}{2} - \sqrt{2x} - \frac{(\sqrt{2x})^3}{24} - ... \]  

(A23)

we have

\[ \kappa_n \approx \frac{n\pi}{2} - \frac{\pi}{2} + \sqrt{2 \left( 1 - \frac{\kappa_n}{\beta} \right)} \]  

(A24)

the error and relative error are

\[ \Delta \kappa_n \approx \frac{1}{24} \left( 2 - \frac{2\kappa_n}{\beta} \right)^{3/2} \approx \frac{1}{24} \left[ 2 - \frac{(n - 1)\pi}{\beta} \right]^{3/2} \]  

\[ \delta \kappa_n = \left| \frac{\Delta \kappa_n}{\kappa_n} \right| \approx \frac{2 - (n - 1)\pi/\beta} {12(n - 1)\pi} \]  

(A25)

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