A matrix product solution for a nonequilibrium steady state of an XX chain

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Abstract

A one-dimensional XX spin chain of finite length coupled to reservoirs at both ends is solved exactly in terms of a matrix product state ansatz. An explicit representation of matrices of fixed dimension 4 independent of the chain length is found. Expectations of all observables are evaluated, showing that all connected correlations, apart from the nearest neighbor $z \rightarrow z$, are zero.

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1. Introduction

Simplification leading to understanding of essential features of physical systems is one of the leading principles in theoretical physics. Finding simple solutions to seemingly complicated models is one of the ways to approach this goal. In quantum physics the complexity of a system grows exponentially with the number of particles. Even if an exact solution is possible, representing it in a compact way is nontrivial for many-body systems. One approach to represent a given state is to express its expansion coefficients in a suitable basis in terms of the product of matrices, the so-called matrix product states [1, 2], used before in a wholly different context of a 2D classical lattice model [3]. If a quantum state is only weakly correlated, the resulting matrices can be small. In condensed matter such an ansatz has been used to exactly describe ground states of many low-dimensional spin systems, among the first one for instance the AKLT chain [4] or the ladder system [5]. In addition to ground states, description in terms of the products of matrices is also successfully used in algorithms for the simulation of quantum systems [6]. Crucial for the efficiency of such simulation is the necessary matrix dimension $D$. Unfortunately, generic coherent quantum evolution will cause $D$ to grow exponentially with the simulation time [7], rendering simulation inefficient. However, in some special cases of the Heisenberg evolution of certain operators in integrable systems, such as the transverse Ising chain, the dimension $D$ is small and does not grow with time [7, 8]. Efficient matrix product description with the time-independent $D$ is possible also for certain open versions of the same integrable models [9].
Besides quantum systems, matrix product states are also widely used in classical stochastic models. There a matrix product formulation is used to study nonequilibrium stochastic lattice models, in particular, their nonequilibrium stationary state (NESS). Since its first use for the exact solution of a one-dimensional asymmetric exclusion process, a matrix product formulation has been utilized in a plethora of models, for a review see. Expectations in the NESS can sometimes be calculated with the help of algebra only, sometimes though an explicit representation of matrices is required. It is not known in general when a finite-dimensional matrix representation is possible.

In this work, we provide an explicit finite-dimensional matrix product solution for a stationary state of quantum system in a nonequilibrium situation. This extends the applicability of matrix product states in quantum domain from ground states and time evolutions to the NESS. The model we consider is a one-dimensional XX spin chain coupled to reservoirs at chain ends. Using the Jordan–Wigner transformation, one can reformulate the system in terms of spinless fermions. In the fermionic language, the system is composed of free fermions. Hamiltonian part is therefore trivially integrable. In fact, for our choice of bath Lindblad operators, the whole superoperator is quadratic in fermionic operators and can therefore be diagonalized in the operator space. An explicit solution for the first two orders in the driving can also be obtained as a particular case of an analytic solution for the XX chain with dephasing. A slightly different model resulting in the same solution, in which the XX chain repeatedly interacts with independent spins of the bath, has been explicitly solved in [15]. For studies of nonequilibrium states in a doubly infinite chain, see [16–18]. Our primary goal here is therefore not to solve the system but instead to provide a compact form for the exact NESS in terms of a matrix product ansatz with the matrices of low dimension that is independent of the chain length. This in turn enables one to explicitly evaluate the expectation value in the NESS of an arbitrary observable.

The Hamiltonian of the XX spin chain is given by

$$H = \sum_{j=1}^{n-1} \left( \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y \right),$$

with standard Pauli matrices; lower indices running from $j = 1, \ldots, n$ denote a site position. Dynamics of the spin chain coupled to the environment will be described in an effective way using the Lindblad master equation

$$\frac{d}{dt} \rho = i[\rho, H] + \mathcal{L}_{\text{bath}}^L(\rho) + \mathcal{L}_{\text{bath}}^R(\rho) = \mathcal{L}(\rho).$$

To induce a nonequilibrium situation, we couple the chain at the first and the last site to a ‘bath’, modeled here by the linear operator $\mathcal{L}_{\text{bath}}$ which is expressed in terms of the Lindblad operators $L_{1,2}^L$ acting on the first site, and $L_{1,2}^R$ acting on the last site,

$$\mathcal{L}_{\text{bath}}(\rho) = \sum_k \left( [L_k^{LR}, \rho L_k^{LR}] + [L_k^{L}, \rho L_k^{R}] \right).$$

We take the simplest Lindblad operators of the form

$$L_1^L = \sqrt{\Gamma_L \left( 1 - \frac{\mu}{2} + \bar{\mu} \right)} \sigma_1^+, \quad L_2^L = \sqrt{\Gamma_L \left( 1 + \frac{\mu}{2} - \bar{\mu} \right)} \sigma_1^-,$$

on the first site, while on the nth site we have

$$L_1^R = \sqrt{\Gamma_R \left( 1 + \frac{\mu}{2} + \bar{\mu} \right)} \sigma_n^+, \quad L_2^R = \sqrt{\Gamma_R \left( 1 - \frac{\mu}{2} - \bar{\mu} \right)} \sigma_n^-,$$

$$\sigma_j^\pm = (\sigma_j^x \pm i \sigma_j^y)/2.\text{ Such bath operators induce an imbalance in magnetization, causing a flow of magnetization from one to the other end. For bath operators only, the stationary state,}$$
for which $L_{\text{bath}}^{\text{bath}}(\rho) = 0$ holds, is diagonal in the eigenbasis of $\sigma_i^z$, with the expectation value $\langle \sigma_i^z \rangle = \text{tr}(\rho \sigma_i^z) = \bar{\mu} - \frac{\mu^2}{2}$, and correspondingly for $L_R^{\text{bath}}$. Therefore, the parameter $\bar{\mu}$ plays the role of average magnetization, $\mu$ is its difference between the right and left ends, while $\Gamma_{L,R}$ are the coupling strengths. The matrix representation of the superoperators $L_{L,R}^{\text{bath}}$ can be found in the appendix. We are interested in a stationary solution of the whole master equation, $\mathcal{L}(\rho) = 0$, which we call a nonequilibrium stationary state, NESS for short, and denote simply by $\rho$.

2. Matrix product ansatz

We shall write the NESS $\rho$ with the matrix product operator (MPO) ansatz,

$$\rho = \frac{1}{2^n} \sum_{\alpha_1, \alpha_2, \ldots, \alpha_n} \langle 1 | A_1^{(\alpha_1)} A_2^{(\alpha_2)} \cdots A_n^{(\alpha_n)} | 1 \rangle \sigma_1^{\alpha_1} \sigma_2^{\alpha_2} \cdots \sigma_n^{\alpha_n}. \quad (6)$$

The indices $\alpha_i$ run over the labels of Pauli matrices forming an operator basis, $\alpha_i \in \{x, y, z, \mathbb{I}\}$, with the convention $\sigma_i^\mathbb{I} = \mathbb{I}_j$, the matrices $A_j^{(\alpha)}$ are of dimension $D \times D$, while $|1\rangle$ is a $D$-dimensional unit vector. We arbitrarily choose its components to be $\delta_{j,1}$.

The goal is to write the NESS in terms of as small matrices as possible. Our method of solution shall be the following. In [14], an exact solution for an XX model with dephasing has been provided of which our current XX chain is a special limit (the limit of zero dephasing). It has been observed that in a model without dephasing there are no long-range correlations, that is, all connected correlations are zero. This leads us to think that one could perhaps construct an exact MPO solution just by observing nontrivial one- and two-point observables; in our case these are magnetization, current and $z$-$z$ correlations. Therefore, based on the solution from [14], we are first going to construct an MPO that accounts only for few-point observables.

In [14] it has been found that (for $\Gamma_L = \Gamma_R = 1$, $\bar{\mu} = 0$) the solution is (up to normalization)

$$\rho \approx \mathbb{I} + \frac{\mu}{4} (-\sigma_i^z + \sigma_n^z) - \frac{\mu}{4} \sum_{j=1}^{n-1} (\sigma_j^z \sigma_{j+1}^z - \sigma_j^x \sigma_{j+1}^x) - \frac{\mu^2}{16} (\sigma_i^z \sigma_n^z + \sum_{j=1}^{n-1} \sigma_j^z \sigma_{j+1}^z) + \mathcal{O}(\mu^3). \quad (7)$$

In the above expression, we write only linear terms in $\mu$ and the $z$-$z$ term of order $\mu^2$. All other are inessential for the following discussion. How can we write such a state in terms of matrices? As an easy overture, we start with a simpler operator $\rho$ obtained by keeping in the NESS (7) only terms with $\sigma_j^x$, i.e. dropping the current term in equation (7). One can easily convince oneself that the matrices $A_j^{(x)} = a_j |1\rangle \langle 1| + |2\rangle \langle 2| - \frac{\mu^2}{16} |1\rangle \langle 1| + |1\rangle \langle 2| - |2\rangle \langle 1| + |2\rangle \langle 2|$ and $A_j^{(y)} = |1\rangle \langle 1|$ where $a_1 = -\frac{\mu}{2}$, $a_2 = \frac{\mu}{2}$, $a_{32} = a_{n-1} = a_{\text{bath}} = 0$, while trivially $A_j^{(y,3)} = 0$ gives the wanted state. Therefore, for this simple operator MPO ansatz with dimension $D = 2$ would suffice. To describe also the current though, we have to allow for at least two additional basis states in the matrices $A_j$. We found that the matrices of size $D = 4$ are sufficient. To correctly describe all terms of orders $\mu$ and $\mu^2$ (also those of order $\mu^3$ that are not explicitly

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1 Two quadratic terms not written in equation (7) are $\frac{\mu^2}{4} (\sigma_i^z + \sigma_n^z) \sum_{j=1}^{n-1} (\sigma_j^z \sigma_{j+1}^z - \sigma_j^x \sigma_{j+1}^x) + \frac{\mu^2}{16} \sum_{j=1}^{n-1} (\sigma_j^z \sigma_{j+1}^z - \sigma_j^x \sigma_{j+1}^x)(\sigma_j^x + \sigma_{j+1}^x)$ and $\frac{\mu^2}{16} \sum_{j=1}^{n-1} (\sigma_j^x \sigma_{j+1}^x - \sigma_j^z \sigma_{j+1}^z)(\sigma_j^x \sigma_{j+1}^x - \sigma_j^z \sigma_{j+1}^z)$.
written out in equation (7)) the following set of matrices gives the correct operator:

\[
\begin{align*}
A^{(c)}_i &= \begin{pmatrix}
 a_i & -t^2 & 0 & 0 \\
 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
\end{pmatrix}, & \quad A^{(d)}_i &= \begin{pmatrix}
 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & t \\
 0 & 0 & t & 0 \\
\end{pmatrix}, \\
A^{(s)}_i &= (-P, -P, P, -P, \ldots), & \quad A^{(y)}_i &= (-R, R, R, -R, -R, \ldots),
\end{align*}
\]

\[P = \begin{pmatrix}
 0 & 0 & 0 & t \\
 0 & 0 & 0 & 0 \\
 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
\end{pmatrix}, \quad R = \begin{pmatrix}
 0 & 0 & t & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 1 & 0 & 0 & 0 \\
\end{pmatrix},
\]

where we use the notation \(A^{(k,y)}_i = (A^{(k,y)}_1, A^{(k,y)}_2, \ldots)\). The matrices \(A^{(x)}_i\) and \(A^{(y)}_i\) are periodic with period 4, \(A^{(x)}_{i+4} = A^{(x)}_i\), and can be concisely written as \(A^{(x)}_i = (\cos \frac{\pi}{2} j - \sin \frac{\pi}{2} j) P\), while \(A^{(y)}_i = -(\cos \frac{\pi}{2} j + \sin \frac{\pi}{2} j) R\). We shall show that the MPO with the above matrices (8) and appropriately chosen three parameters \(t, a_1, a_2\) and \(a_{\text{bulk}} = a_{2,\ldots,n-1}\) is an exact NESS solution for the Lindblad equation (2) with arbitrary \(\Gamma_{L,R}, \mu, \bar{\mu}\). To show this we are going to use an algebraic approach similar to the one used in the solutions of classical stochastic processes.

Let us denote by \(L^{(H)}_{i,i+1}\) the superoperator from the commutator part of the master equation corresponding to the nearest-neighbor 2-spin term in the Hamiltonian, \(\sigma^x_i \sigma^x_{i+1} + \sigma^y_i \sigma^y_{i+1}\) in our case. Writing four MPO matrices in a vector of matrices as \(A_i = (A^{(x)}_i, A^{(y)}_i, A^{(z)}_i, A^{(d)}_i)\), we can form all 16 different products of two matrices at consecutive sites through \(A_i \otimes A_{i+1}\) (a \(4 \times 4\) matrix, each element being a product of two \(D \times D\) matrices). Depending on the action of \(L^{(H)}_{i,i+1}\) on the products of operators, i.e. on \(L^{(H)}_{i,i+1}(A_i \otimes A_{i+1})\), see also the appendix, solving for the NESS is relatively simple in two cases: (i) if the NESS is separable or has only two-particle entanglement, like for instance in valence bond states, then we can have \(L^{(H)}_{i,i+1}(A_i \otimes A_{i+1}) = 0\); (ii) other relatively simple situation is when \(L^{(H)}_{i,i+1}(A_i \otimes A_{i+1})\) results in a divergence-like term, that is,

\[L^{(H)}_{i,i+1}(A_i \otimes A_{i+1}) = A_i \otimes M_{i+1} - M_i \otimes A_{i+1},\]

with some matrices \(M_i = (M^{(x)}_i, M^{(y)}_i, M^{(z)}_i, M^{(d)}_i)\). Note that this ansatz is a trivial inhomogeneous extension of a standard procedure used in classical nonequilibrium systems [20]. The reason to allow for spatially dependent matrices is that we want to find the MPO solution with the smallest \(D\). We find that the representation with \(D = 4\) is possible irrespective of the chain length \(n\). Any inhomogeneous solution can be written as a site-independent one but with larger matrices. One possibility is to make a block site-independent matrix \(\tilde{A}\) of size \(D \cdot n\) out of site-dependent matrices \(A_i\) as

\[
\tilde{A} = \begin{pmatrix}
 0 & A_1 & 0 & \cdots & 0 \\
 0 & 0 & A_2 & \cdots & 0 \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & 0 & \cdots & A_{n-1} \\
 A_n & 0 & 0 & \cdots & 0 \\
\end{pmatrix},
\]

\[\tilde{A} = \begin{pmatrix}
 0 & A_1 & 0 & \cdots & 0 \\
 0 & 0 & A_2 & \cdots & 0 \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & 0 & \cdots & A_{n-1} \\
 A_n & 0 & 0 & \cdots & 0 \\
\end{pmatrix}.
\]

\footnote{For an explicit small-dimensional MPO construction of some simple operators, see [21].}
If equation (9) holds, terms from consecutive $L^{(i)}_{j,i+1}$ will pairwise cancel, except at the boundaries. To ensure stationarity we have to enforce an additional condition at the boundaries. For our case of bath acting on a single boundary spin we get two equations:

\[
\begin{align*}
&[\mathcal{L}^{\text{bath}}_1(A_1) - M_1] = 0, \\
&[\mathcal{L}^{\text{bath}}_R(A_n) + M_n] = 0.
\end{align*}
\]  

(11)

If one manages to find such $A_i$ and the associated $M_i$ that equations (9) and (11) are satisfied, one has found the NESS solution of the master equation (2). Condition (9) can in our case of the Pauli basis of the operator space be written as a set of $4^2$ matrix equations of size $D \times D$. As we have only $2 \times 4$ matrices $A_i$ and $M_i$ of size $D \times D$, with $8D^2$ unknown parameters, it is not guaranteed that the solution exists.

We are now going to show that for the XX model it actually does exist. We are going to find an explicit representation of the matrices $M_i$, showing that they, together with $A_i$ (8), satisfy equations (9) and (11). Similarly as for $A^{(x,y)}_i$, the matrices $M^{(x,y)}_i$ also have the periodicity 4. Their explicit $(D = 4)$-dimensional representation in the bulk, that is, for the sites $i = 2, \ldots, n-1$, is

\[
M^{(4)}_i = -2a_{\text{bulk}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad M^{(2)}_i = -2 \begin{pmatrix} 2t & 0 & 0 & 0 \\ 0 & 2t & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},
\]

\[
M^{(x)} = (S, S, -S, -S, \ldots), \quad M^{(y)} = (T, -T, -T, T, T, \ldots), \quad S = 2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad T = 2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & t & 0 & 0 \end{pmatrix}.
\]

(12)

One can convince oneself by direct calculation that the above $M_i$ together with $A_i$ (8) satisfy condition (9), written out in full in the appendix, equation (A.4). To also fulfill the two boundary conditions (11), $M_1$ and $M_n$ must have additional matrix elements, and the parameters $t, a_1, a_n$ and $a_{\text{bulk}}$ must take specific values. We have $M^{(x)}_1 = S + 2(a_{\text{bulk}} - a_1)\langle 1 | 4 \rangle$, $M^{(y)}_1 = T + 2(a_{\text{bulk}} - a_1)\langle 1 | 3 \rangle$ and $M^{(x)}_n = M^{(y)}_n = 2t(a_{\text{bulk}} - a_n)\langle 3 | 1 \rangle$. On the right end we have $M^{(x)}_n = M^{(y)}_n = \frac{a_{\text{bulk}} - a_n}{2} (S + 2(a_{\text{bulk}} - a_n)\langle 1 | 2 \rangle \langle 3 | 4 \rangle)$, where $\text{sgn}_x$ and $\text{sgn}_y$ are the signs in front of last $S$ or $T$ in equation (12) and depend on the site $n$. In addition, for the boundary terms to zero the values of the parameters must be

\[
\begin{align*}
t &= \frac{\Gamma_L \Gamma_R}{(1 + \Gamma_L \Gamma_R)(\Gamma_L + \Gamma_R)}, \\
a_1 &= \frac{\mu}{2} \frac{(\Gamma_L - \Gamma_R) + \Gamma_L \Gamma_R}{(1 + \Gamma_L \Gamma_R)(\Gamma_L + \Gamma_R)} \\
a_2, \ldots, a_{n-1} &= a_{\text{bulk}} = \frac{\mu}{2} \frac{(\Gamma_L - \Gamma_R)(1 - \Gamma_L \Gamma_R)}{(1 + \Gamma_L \Gamma_R)(\Gamma_L + \Gamma_R)} \\
a_n &= \frac{\mu}{2} \frac{(\Gamma_L - \Gamma_R) - \Gamma_L \Gamma_R}{(1 + \Gamma_L \Gamma_R)(\Gamma_L + \Gamma_R)}.
\end{align*}
\]  

(13)

Note that we have $a_1 - a_{\text{bulk}} = -t\Gamma_L, a_n - a_{\text{bulk}} = t\Gamma_R$ and $a_n - a_1 = t(\Gamma_L + \Gamma_R)$. Parameters (13) together with matrices (8) form an exact MPO solution of the NESS for the XX chain.

If we add a homogeneous magnetic field in the $z$-direction to our Hamiltonian, that is, the term of the form $B \sum_{j=1}^{2} \sigma_j^z$, the model can again be solved exactly. In fact, the single-site superoperator due to the magnetic field acts as $\mathcal{L}^{(B)}(\sigma^z_j) = 2B \sigma^z_j$, $\mathcal{L}^{(B)}(\sigma_j^x) = -2B \sigma^x_j$, while
\( \mathcal{L}^{(B)}_i (\sigma_i^{z,1}) = 0 \). Due to the symmetry of the NESS without the field (see also the explicit form of all nonzero terms given in [14]) and the minus sign in the action of \( \mathcal{L}^{(B)}_i \), we have \( \sum_i \mathcal{L}^{(B)}_i (\rho) = 0 \). This means that the solution presented (8) is also the exact NESS in the presence of an arbitrary homogeneous field of strength \( B \).

3. Expectations of observables

With an explicit representation of the matrix product solution at hand we can evaluate expectations of various operators in the NESS. Let us first evaluate the expectation of a series of \( \sigma_i^z \) operators, such as \( \langle \sigma_i^z \sigma_j^z \cdots \rangle \). One or two \( \sigma^z \)'s are simple to evaluate by direct calculation; we obtain \( \langle \sigma_i^z \rangle = a_i \), and \( \langle \sigma_i^z \sigma_j^z \rangle = a_i a_j - t^2 \delta_{i+l,j} \). The connected correlation function of \( n \) operators, \( \langle O_1 O_2 \cdots O_n \rangle_c \), is obtained from an ordinary expectation value by subtracting the product of all connected correlations where each involves less than \( n \) operators. For instance, \( \langle O_1 O_2 \rangle_c = \langle O_1 \rangle \langle O_2 \rangle - \langle O_1 O_2 \rangle \), \( \langle O_1 O_2 O_3 \rangle_c = \langle O_1 O_2 O_3 \rangle - \langle O_1 O_2 \rangle \langle O_3 \rangle - \langle O_1 O_3 \rangle \langle O_2 \rangle - \langle O_2 O_3 \rangle \langle O_1 \rangle \), and so on. The connected correlation function of two \( \sigma^z \)'s is therefore nonzero only on neighboring sites, \( \langle \sigma_i^z \sigma_j^z \rangle_c = -t^2 \delta_{i+l,j} \). We are now going to show by induction that all higher order connected correlation functions involving more than two \( \sigma^z \)'s are identically zero. Assume that the statement holds for the products of up to \( n \) \( \sigma^z \)'s. Denoting by \( Z = \sigma_i^z \sigma_j^z \cdots \sigma_k^z = Z \sigma_k^z \) a product of \( n \) not necessarily neighboring \( \sigma_i^z \)'s, we would like to show that the connected correlation \( \langle Z \sigma_k^z \rangle_c \) is zero for \( n \geq 2 \). Assuming that the connected correlations of more than two \( \sigma^z \)'s are zero, we can write

\[
\langle Z \sigma_i^z \rangle_c = \langle Z \sigma_i^z \rangle - \left( \sum_c \langle Z \rangle_c \right) \langle \sigma_i^z \rangle_c - \left( \sum_c \langle Z \rangle_c \right) \langle \sigma_i^z \rangle_c + \langle \sigma_i^z \rangle_c.
\]

where we denoted by \( \sum_c \langle Z \rangle_c \) a sum of all products of connected correlations, each involving \( \leq n \) operators \( \sigma^z \). For instance, \( \sum_c \langle \sigma_i^z \sigma_j^z \rangle_c = \langle \sigma_i^z \sigma_j^z \rangle_c + \langle \sigma_i^z \rangle_c \langle \sigma_j^z \rangle_c \). We used the fact that to have a nonzero connected correlation, \( \sigma_j^z \) must be paired with at most one other \( \sigma_i^z \) and that it must be its neighbor (non-nearest-neighbor connected correlations are zero). Using our explicit MPO representation (8) of matrices for \( A^{(y)} \) and \( A^{(k)} \), we will now show that the right-hand side of (14) is zero. First, observe that the matrix corresponding to the product of matrices occurring in the operator \( Z \) has an upper-left \( 2 \times 2 \) block equal to

\[
Z = \begin{pmatrix}
\langle Z \rangle & -t^2 \langle Z \rangle \\
0 & 0
\end{pmatrix}.
\]

(15)

The upper-left element is \( \langle Z \rangle \) by definition, while the upper right follows through a simple multiplication of the matrix corresponding to \( Z \) by \( A^{(k)}_{i+1} \). With the explicit form of \( Z \), we have

\[
\langle Z \sigma_i^z \rangle_c = \langle Z \rangle \langle \sigma_i^z \rangle_c - t^2 \langle Z \rangle \delta_{i+1,l}.
\]

(16)

where the Kronecker delta takes into account the multiplication of the matrix for \( Z \) by \( A^{(k)}_{i+1} \) in the case of non-neighboring sites \( k \) and \( l \). Plugging this into equation (14) we see that the \( (n+1) \)-point connected correlation of \( \sigma^z \) is indeed zero. This completes the proof.

The single point expectations \( \langle \sigma_i^z \rangle \) and \( \langle \sigma_i^z \rangle_c \) are zero. Among two-point expectations on neighboring sites, besides \( s-s \), the only nonzero terms are two from the spin current, that is, \( \langle \sigma_i^z \sigma_j^z \rangle = -t \langle \sigma_i^z \sigma_{i+1}^z \rangle = -t \). Therefore, the expectation value of the spin current operator \( j_k = 2 \langle \sigma_k^z \sigma_{k+1}^z \rangle - \langle \sigma_k^z \sigma_k^z \rangle \) is \( \langle j_k \rangle = -4t \). If we have a product of non-overlapping current operators at different sites, \( \langle j_i j_{i+1} \cdots \rangle \), (with \( k > i+1 \)), the expectation value is simple. Observe that \( A^{(y)}_{i+1} A^{(k)}_{i+1} = -t \langle | 1 \rangle \langle 1 | + \langle 3 | \rangle \langle 3 | \rangle \rangle \). Because of the form of \( A^{(k)} \), if we
have a product of only $A_j^{(k)}$s and $p$ terms $A_i^{(s)}A_{i+1}^{(y)}$ at various sites $i$, their expectation value is simply equal to $t^p$. This means that the connected correlation function of the non-overlapping operators $j_i$ is zero, $\langle j_i j_i \cdots j_i \rangle_c = 0$, apart from $\langle j_i \rangle_c = -4t$ if two current operators overlap, i.e. $j_i j_{i+1}$, with the Hermitian $(j_i j_{i+1} + j_{i+1} j_i)/2 = -4(\sigma^x_i \sigma^x_{i+1} + \sigma^y_i \sigma^y_{i+1})$, the corresponding product of matrices is $A_i^{(s)}A_j^{(k)}A_{i+1}^{(y)} = -(i^2|1\rangle\langle1| + i|3\rangle\langle4|)$. From this form, similarly as before, we can see that for the products of $j_i$ the term $|3\rangle\langle4|$ is not important, resulting again in all connected correlations being zero. Similar argument holds for the products of more than two overlapping current operators.

Finally, let us discuss the expectations of the products of $\sigma^x_j$ and $j_k$, the only remaining nonzero terms in the NESS. For overlapping sites, $j_j \sigma_j^x + \sigma_j^x j_k = 0$ (the same for $j_{i+1}$), and we have to consider only non-overlapping operators. The two-point expectation is $\langle j_i j_j \rangle = -4t a_j$, connected correlation being therefore zero. Again, taking into account that if we consider only the products of $A_i^{(s)}$, $A_{i+1}^{(y)}$, and $A_{i+1}^{(s)}$, the term $|3\rangle\langle4|$ in $A_i^{(s)}A_{i+1}^{(y)}$ is irrelevant because there are no $|3\rangle\langle1|, |4\rangle\langle1|$ or their Hermitian conjugates in any of the three matrices. The expectations are therefore the trivial products of scalar quantities in front of the $|1\rangle\langle1|$ terms, resulting in all connected correlations being zero. If we have a product of neighboring currents, like for instance in $(j_i j_{i+1} + j_{i+1} j_i)/2$, similar argument holds. Incidentally, we also see that the expectation of the energy current $j_i^c = 2(\sigma^x_{j_{i+1}} - \sigma^x_{j_i} - \sigma^x_{j_{i-1}})$, such that $\i(t^2|j_i^c\rangle\langle j_i| + t^2|j_i^c\rangle\langle j_i| + t^2|\sigma^x_j\rangle\langle j| + t^2|j_{i-1}\rangle\langle j_{i+1}| + t^2|j_{i+1}\rangle\langle j_{i-1}|) = 0$. In the NESS with our choice of baths, no energy current flows.

Another way to write the NESS is to rewrite it in terms of an exponential function as $\rho = \exp(-\hat{H})$. Doing the calculation we observe that the operator $\hat{H}$ contains only few nonzero terms. These are $\sigma^x_j$ at all $n$ sites (with different prefactors on different sites), spin current $j_k = 2h_k^{(2)}$ at all $n - 1$ sites (the same prefactor on all sites) as well as $h_j^{(2k+1)}$, with $k = 1, \ldots$, and $b_j^{(2k)}$ with $k = 2, \ldots$, where $h_j^{(k)} = \sigma^x_{j+1} \cdots \sigma^y_{j+k-1} \cdots \sigma^z_{j+k-2} \sigma^y_{j+k-1}$ and $b_j^{(k)} = \sigma^x_{j+1} \cdots \sigma^y_{j+k-2} \sigma^z_{j+k-1} - \sigma^y_{j+k-1} \cdots \sigma^z_{j+k-2} \sigma^y_{j+k-1}$. This shows that the NESS in the open XX chain cannot be exactly written as a quasi-equilibrium generalized Gibbs state $\rho \sim \exp(\sum_j \kappa_j Q_j)$ with $\kappa_j$ the locally varying fields and $Q_j$ the conserved quantities of the corresponding Hamiltonian system [22]. Namely, conserved quantities in integrable systems can depend on small changes in the system, for instance, on boundary conditions. For a 1D XX chain with periodic boundary conditions, two infinite sequences of conserved quantities exist: one set is $Q_{2k} = \sum_j h_j^{(2k)}$ and $Q_{2k+1} = \sum_j b_j^{(2k+1)}$, another $Q_{2k} = \sum_j b_j^{(2k)}$ and $Q_{2k+1} = \sum_j h_j^{(2k+1)}$. If we change boundary conditions to open, only half of the conserved quantities survive, that is, for open boundary conditions, the conserved quantities are only $Q_{2k} = Q_{2k} + q_{2k}$ and $Q_{2k+1} = Q_{2k+1} + q_{2k+1}$, where $q_j$ are the additional boundary terms [23]. First few conserved quantities for the open XX chain are $Q_2 = \sum_j h_j^{(2)}$, $Q_3 = \sum_j h_j^{(3)} + \sigma^x_j + \sigma^y_j$, $Q_4 = \sum_j h_j^{(4)} - h_j^{(2)}$, $Q_5 = \sum_j h_j^{(5)} + \sigma^x_i h_i^{(1)} + \sigma^x_j h_j^{(1)} - h_j^{(2)}$.

We can see that $\hat{H}$ cannot be written as a sum of the conserved quantities $Q$ of an open chain nor that of a periodic one. A change in the boundary condition can therefore globally influence the constants of motion of the Hamiltonian system. This sensitivity also translates to open systems described by the master equation: a change in the boundary operators of the bath can have a large influence on the NESS. In short, for open integrable systems there is an ambiguity over which ‘conserved’ quantities $Q_j$ one should use in the generalized Gibbs ensemble, or, in other words, the $Q_j$ depends on the bath. Such non-universality, where the functional form of the NESS does not depend only on the Hamiltonian but also on reservoirs, is probably a generic situation for open versions of integrable systems.
To summarize, we have shown that all connected correlations, apart from neighboring \( \langle \sigma_z^i \sigma_z^{i+1} \rangle_c \), are identically zero. This should not come as a surprise. In fact, working in the fermionic language the system is quadratic; therefore, all expectations of the products of fermionic operators can be evaluated in terms of two-point expectations using Wick’s theorem.

4. Conclusion

We have found an exact solution of an open XX chain in terms of the matrix product ansatz. An explicit \((D = 4)\)-dimensional representation of matrices is found, enabling us to evaluate arbitrary expectations. All connected correlation functions, apart from the correlation function of magnetization at nearest-neighbor sites, are identically zero. The results presented extend the applicability of matrix product states to quantum nonequilibrium systems. Furthermore, the method of the solution is an algebraic one borrowed from the field of classical stochastic processes where it has been used very successfully. It is hoped that this will lead to new exactly solvable nonequilibrium quantum systems. One such instance is an open XX chain with dephasing, a solvable diffusive model \([14]\) where, based on our experience, a compact matrix product solution is also possible. Unfortunately though, in an even more interesting XXZ model, the algebra seems to be more complicated.

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Appendix

A.1. Representation of bath superoperators

Using a basis of Pauli matrices, and tensor products thereof, a single-site superoperator for the bath is

\[
\mathcal{L}^{\text{bath}}_L = \Gamma_L \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -4 & -2(\mu - 2\bar{\mu}) \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]  

(A.1)

The superoperator \(\mathcal{L}^{\text{bath}}_R\) for the right bath is obtained by replacing \(\Gamma_L\) with \(\Gamma_R\) and \(\mu\) with \(-\mu\). Basis elements are ordered as \((\sigma^x, \sigma^y, \sigma^z, \mathbb{1})\). Matrix representation gives us the operation of \(\mathcal{L}^{\text{bath}}_L\) on the expansion coefficients of operators in the Pauli basis, e.g., writing

\[
\rho = \sum \mathcal{c}_\alpha \sigma^\alpha,
\]

we have

\[
\mathcal{L} \rho = \mathcal{L} \rho' = \sum \mathcal{c}'_\alpha \sigma^\alpha,
\]

with \(\mathcal{c}_\alpha = \sum_\beta \mathcal{L}_{\alpha,\beta} c_\beta\). Alternatively, the same matrices can be thought of as transforming matrices in the MPO ansatz, e.g.,

\[
[\mathcal{L}^{\text{bath}}_L (A_1)]_i = \sum_{j=1}^d [\mathcal{L}^{\text{bath}}_L]_{i,j} [A_1]_j.
\]

A.2. Representation of \(\mathcal{L}^{(H)}_{i,i+1}\)

The superoperator \(\mathcal{L}^{(H)}_{i,i+1}\) transforms operators according to their commutator with the Hamiltonian. It is therefore fully specified by its operation on a 16-dimensional basis of 2-site operators. We choose the products of Pauli matrices as a basis. As an example, for the XX chain we have, for instance,

\[
[\mathcal{L}^{(H)}_{i,i+1} (\mathbb{1} \sigma^i_{i+1}) = -2 \sigma^y_i \sigma^z_{i+1}.
\]

Acting on the MPO ansatz, this
for instance gives $\mathcal{L}_{i+1}^{(d)}(\cdots A_{i+1}^{(d)} \cdot A_{i+1}^{(x)} \cdots) = \{\cdots (-2A_{i+1}^{(y)} A_{i+1}^{(x)}) \cdots) \sigma_{i+1}^y \sigma_{i+1}^z$. In the transformed state, the term that comes in the product of matrices at the position of $A_{i+1}^{(y)} A_{i+1}^{(x)}$ is therefore equal to $(-2A_{i+1}^{(y)} A_{i+1}^{(x)})$. Because we are interested in how $\mathcal{L}_{i+1}^{(d)}$ transforms the MPO, it is handy to represent its action in terms of a $4 \times 4$ matrix whose element $[\mathcal{L}_{i+1}^{(d)}]_{\alpha, \beta}$ gives us the term in front of $\sigma_{i+1}^{\alpha} \sigma_{i+1}^{\beta}$. For the XX chain we have

$$\mathcal{L}_{i+1}^{(d)} = 2 \begin{pmatrix} 0 & B & A_{i+1}^{(y)} A_{i+1}^{(x)} & A_{i+1}^{(z)} A_{i+1}^{(x)} \\ -B & 0 & -A_{i+1}^{(y)} A_{i+1}^{(x)} & -A_{i+1}^{(z)} A_{i+1}^{(x)} \\ A_{i+1}^{(y)} A_{i+1}^{(x)} & -A_{i+1}^{(y)} A_{i+1}^{(x)} & 0 & C \\ -A_{i+1}^{(z)} A_{i+1}^{(x)} & -A_{i+1}^{(z)} A_{i+1}^{(x)} & C & 0 \end{pmatrix},$$  

(A.2)

where $B = A_{i+1}^{(y)} A_{i+1}^{(x)} - A_{i+1}^{(z)} A_{i+1}^{(y)}$ and $C = A_{i+1}^{(y)} A_{i+1}^{(y)} - A_{i+1}^{(z)} A_{i+1}^{(z)}$, explicitly giving the transformation

$$\mathcal{L}_{i+1}^{(d)}(\rho) = \sum_{\alpha_1, \alpha_2} \cdots [(\mathcal{L}_{i+1}^{(d)})_{\alpha_1, \alpha_2}] \cdots) \sigma_{i+1}^{\alpha_2} \sigma_{i+1}^{\alpha_1}.$$

(A.3)

Written explicitly, the algebra (9) of the matrices $A_j$ and $M_j$ in the bulk that is induced by $\mathcal{L}_{i+1}^{(d)}$ is for the XX model,

$$2(A_{i+1}^{(y)} A_{i+1}^{(x)} - A_{i+1}^{(z)} A_{i+1}^{(x)}) = A_{i+1}^{(x)} M_{i+1}^{(x)} - M_{i+1}^{(x)} A_{i+1}^{(x)}$$

$$-2(A_{i+1}^{(y)} A_{i+1}^{(x)} - A_{i+1}^{(z)} A_{i+1}^{(y)}) = A_{i+1}^{(y)} M_{i+1}^{(y)} - M_{i+1}^{(y)} A_{i+1}^{(y)}$$

$$2(A_{i+1}^{(y)} A_{i+1}^{(x)} - A_{i+1}^{(z)} A_{i+1}^{(y)}) = A_{i+1}^{(x)} M_{i+1}^{(y)} - M_{i+1}^{(x)} A_{i+1}^{(y)}$$

$$-2(A_{i+1}^{(y)} A_{i+1}^{(x)} - A_{i+1}^{(z)} A_{i+1}^{(y)}) = A_{i+1}^{(x)} M_{i+1}^{(y)} - M_{i+1}^{(x)} A_{i+1}^{(y)}$$

$$2A_{i+1}^{(y)} A_{i+1}^{(x)} = A_{i+1}^{(x)} M_{i+1}^{(y)} - M_{i+1}^{(x)} A_{i+1}^{(y)}$$

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$$-2A_{i+1}^{(y)} A_{i+1}^{(x)} = A_{i+1}^{(x)} M_{i+1}^{(y)} - M_{i+1}^{(x)} A_{i+1}^{(y)}$$

The representation given in equations (8) and (12) satisfies this algebra for the arbitrary values of four parameters $a_1, a_2, a_{\text{bulk}}$ and $t$.

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