Three-Dimensional Analyses of Functionally Graded Multi-Layered Systems

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Abstract

Functionally Graded Materials (FGMs) are composites with continuously varying volume fractions of their constituent homogeneous phases. When the characteristic length scale of such variations is much larger than the size of the phases, the model of the inhomogeneous solids is able to describe the macroscopic behaviour of these materials by means of homogenization techniques. By considering materials in which this assumption is possible, some elastic solutions for graded inhomogeneous materials have been obtained in the literature in order to tailor the material parameters for specific requirements. The main purpose of the paper is devoted to analyse the effects of FGM elements in sandwich structures, where the mismatch between the properties of the core and the face-sheets can produce interface damage especially when the structures are subjected to extreme conditions. Two different cases concerning the bending of a thick circular sandwich plate are investigated in the framework of the linear elasticity theory: (a) sandwich plates with homogeneous face sheets and FGM-core and (b) sandwich plates with FGM face sheets and homogeneous core. The related elastic solutions are obtained in the framework of the elasticity theory and explicit solutions are discussed by considering only the first contribution of the series in order; in such a way, we have a solution in closed-form and no convergence effects on the solutions. Three different aspects are studied in detail in terms of the geometric and material parameter: the semi-inverse solution method adopted with the consequence on boundary conditions, the non-linear behaviour of the radial displacement and, the shear and circumferential stress behaviour in the thickness of the plate. A comparative study between sandwich with FGM elements and conventional multi-layered systems is performed in order to highlight the effects of the inhomogeneity.

Keywords: sandwich plate, functionally graded materials, elasticity, plates
1. Introduction

In multi-layered systems the mismatch between the properties of the single layers can produce interface damages; in order to reduce these effects, one of the recent alternatives, adopts of functionally graded materials (FGMs) for the entire multi-layered structure or for single elements [1,2,3]. In this paper, the interest is devoted to sandwich plates to reduce interface delamination effects by using FGM core or face-sheets. Different approaches are presented in the literature to describe FGM structures and much of the work has been carried out by using refined plate theories or/and numerical methods [4,5,6].

In the framework of elasticity theory, some analytical solutions have recently been obtained by Kashtalyan and Meshykova [7,8] for squared sandwich panels with graded elements (core or face-sheets). This paper presents some analytical solutions for circular sandwich plates with FGM elements subjected to axisymmetric loading conditions. The Young modulus in the graded elements is assumed exponentially dependent on the transverse direction while the Poisson ratio is uniform. The circular geometry requires to solving inherent mathematical difficulties concerning the solution method and the boundary conditions. A particular representation form, that reduces the field equations in terms of potential functions for any heterogeneous or homogeneous layer, is adopted [9]. Then, a semi-inverse solution method is used to solve the partial differential equation systems in terms of Bessel’s functions and, as consequence of the solution method, particular boundary conditions on the lateral surface of the plate are required.

Numerical investigations allow us to highlight the effects of graded elements in sandwich plates in comparison with conventional homogeneous sandwich structures. Further, the solutions may give rise a benchmark to investigate the agreement with numerical or structural refined theory results.

2. Problem Statement

The geometry of the sandwich plates is described in Fig.1. We chose a cylindrical coordinate system with the middle plane of the sandwich of radius \( b \) and total thickness \( h \); the face-sheet and core thicknesses are denoted with \( h_f \) and \( 2h_c \), respectively.

![Fig. 1. Circular sandwich plate geometry and loading conditions](image-url)

This study investigates the bending of the sandwich plate by considering transversal axisymmetric loading conditions and partial boundary conditions on the lateral surface; this, to better control the consequence of the semi-inverse solution method adopted. In particular, we consider free traction on the bottom of the sandwich and transversal load on the upper surface in the form:
\[
\sigma_{zz}^{(1)}(r, h_z + h_f) = -p(r) \quad \tau_{rz}^{(1)}(r, -(h_z + h_f)) = 0 \\
\sigma_{zz}^{(4)}(r, -(h_z + h_f)) = 0 \quad \tau_{rz}^{(4)}(r, -(h_z + h_f)) = 0
\] (1)

where the index \(i=1,4\) denotes quantities referred to the upper (1) and bottom face-sheet (4) while \(i=2,3\) quantities referred to the core as in the Fig.1. On the lateral surface, only the following condition on the transversal displacement \(u_z\) is assumed:

\[ u_z(b, z) = 0 \] (2)

The materials of the sandwich elements (face-sheets or core) are isotropic and the following two different cases are investigated: (a) sandwich with graded core and homogeneous face-sheets and (b) sandwich with graded face-sheets and homogeneous core. The elastic properties of the graded elements are considered exponentially dependent by \(z\). For the generic FGM element denoted by the index \((i)\), the Young modulus assumes the following form:

\[ E(i)(z) = E_0 e^{az} \] (3)

where \(E_0\) and \(a\) are constants detailed in subsections 2.1 and 2.2. Furthermore, the face-sheets and the core are assumed perfectly bonded together:

\[
\begin{align*}
\sigma_{zz}^{(1)}(r, h_z) &= \sigma_{zz}^{(2)}(r, h_z) & \tau_{rz}^{(1)}(r, h_z) &= \tau_{rz}^{(2)}(r, h_z) \\
u_z^{(1)}(r, h_z) &= u_z^{(2)}(r, h_z) & u_z^{(1)}(r, h_z) &= u_z^{(2)}(r, h_z) \\
\sigma_{zz}^{(3)}(r, -h_z) &= \sigma_{zz}^{(4)}(r, -h_z) & \tau_{rz}^{(3)}(r, -h_z) &= \tau_{rz}^{(4)}(r, -h_z) \\
u_z^{(3)}(r, -h_z) &= u_z^{(4)}(r, -h_z) & u_z^{(3)}(r, -h_z) &= u_z^{(4)}(r, -h_z)
\end{align*}
\] (4)

In the following sub-sections the specific variation laws in the thickness of the plate are detailed for the two cases analyzed.

2.1. Sandwich with graded core

The case of sandwich plate with graded core is analyzed by assuming the elastic properties in the thickness of the plate as:

\[
E(z) = \begin{cases} 
E_z(z) = \bar{E} & \text{if } h_z \leq z \leq h_z + h_f \\
E_z(z) = E_0 e^{kz} & \text{if } 0 \leq z \leq h_z \\
E_z(z) = E_0 e^{-kz} & \text{if } -h_z \leq z \leq 0 \\
E_z(z) = \bar{E} & \text{if } -(h_z + h_f) \leq z \leq -h_z
\end{cases}
\] (5)

where the elastic property in the bottom and upper face-sheets is \(\bar{E} = E_0 e^{2kh_c}\) and \(E_0\) is the value of the elastic property in the middle of the graded core. The inhomogeneity parameter is assumed positive to describe a continuous increase of the elastic property from the middle of the plate to the face-sheet interface. The Poisson ratios are assumed constant and equal in all elements of the sandwich plate.
2.2. Sandwich with graded face-sheets

The case of sandwich plate with graded face-sheets is analyzed by assuming a symmetric behavior with respect to the middle plane of the plate in this way:

\[
E(z) = \begin{cases} 
E_0 e^{k_z z} & h_c \leq z \leq h_f + h_c \\
E_0 & 0 \leq z \leq h_f \\
E_0 e^{k_z z} & -h_c \leq z \leq 0 \\
E_0 e^{-k_z z} & (h_f + h_c) \leq z \leq -h_c 
\end{cases}
\]  
(6)

where

\[
k_i = \frac{h_c}{h_f} \ln \left( \frac{E_i'}{E_0} \right), \quad k_2 = \frac{1}{2h_f} \ln \left( \frac{E_1'}{E_0} \right), \quad E' = E_0 e^{k_z (h_f + h_c)}
\]

Doing so, the constant \( E' \) results as the elastic modulus on the free surface of the plate, assumed decreasing from the free surface in the thickness of the plate face-sheets.

3. Solution method

To obtain the explicit solution in the framework of the linear elasticity, the transversal and radial displacement fields are written in terms of potential functions \( L^{(i)}(r, z) \) in the form [9]:

\[
u_e^{(i)}(r, z) = -\frac{1 + \nu}{E_i(z)} \frac{\partial}{\partial r} \left( \nu \nabla^2 L^{(i)}(r, z) - (1 - \nu) \frac{\partial^2}{\partial z^2} L^{(i)}(r, z) \right),
\]

\[
u_r^{(i)}(r, z) = \frac{2(1 + \nu)}{E_i(z)} \frac{\partial}{\partial z} \nabla^2 L^{(i)}(r, z) + (1 + \nu) \frac{\partial}{\partial z} \left[ \frac{1}{E_i(z)} \left( \nu \nabla^2 L^{(i)}(r, z) - (1 - \nu) \frac{\partial^2}{\partial z^2} L^{(i)}(r, z) \right) \right]
\]

The potential functions \( L^{(i)}(r, z) \), called Plevako’s functions, must satisfy the following uncoupled partial differential equation for any layer of the sandwich plate:

\[
\nabla^2 \left( \frac{1}{E_i(z)} \nabla^2 L^{(i)}(r, z) \right) - \frac{1}{1 - \nu} \nabla^2 L^{(i)}(r, z) \frac{d^2}{dz^2} \frac{1}{E_i(z)} = 0
\]

(10)

By assuming the distribution laws in the thickness of the plate for Young’s modulus (5) or (6), we get four uncoupled partially differential equations (PDEs) for the case of graded core (sub-section 2.1), while we get three uncoupled PDEs for the graded face-sheets (sub-section 2.3).

To obtain the solutions of these PDEs, the functions \( L^{(i)}(r, z) \) are written as a zero-order Bessel function expansion in the form [10]:

\[
L^{(i)}(r, z) = \sum_{j=1}^{\tilde{g}} L^{(i)}_j(z) J_0(\varphi_r r), \quad \varphi_r = z_j^{(i)} / h_f, \quad L^{(i)}_j(z) = \frac{2}{b^2 J_1^2(\varphi_r b)} \int_0^{\varphi_r} L^{(i)}(\rho, z) J_0(\varphi_r \rho) \rho d \rho
\]

(11)
where \( z_j^{(0)} \) are the zero of the \( J_0 \)-Bessel function. We observe that this solution technique carries out specific data on the boundary conditions of the lateral surface of the plate, as shown in details in [11]. Indeed, the boundary condition on the lateral surface \( (r=b) \) requires only transversal displacement zero for all \( z \) (see (2)) that corresponds to the usual condition in clamped or simply supported plate in structural theories (for \( z=0 \)); this condition is satisfied by considering (11) and (9). The condition on zero rotation, for clamped plate, or zero moment, for simply supported plate, are not automatically verified by expansions (11). This fact is due to the solution method in which we assume a priori the form (11) that gives rise to a radial displacement field in \( r=b \) in the thickness of the plate or, in equivalent form, to a radial resultant moment to satisfy exactly the elastic solution obtained. In other word, we obtain a solution for a plate with assigned conditions on radial displacement or stress in the thickness of the plate in \( r=b \). To obtain exact solution, for example for clamped plate, it is possible to solve the problem by using superposition method as shown in detail in [12].

Finally, substituting the expansions (11) in the PDEs (10), we obtain the elastic solutions in terms of the Bessel expansions and a number of unknown constants (16 for the graded core case and 12 for graded face-sheets case). By using boundary and continuity conditions, the explicit values of the constants are determined; details on the solutions may be found in [13] and [14].

4. Numerical results

In this section, we highlight the effects of graded elements in sandwich plates in comparison with conventional sandwich structures with homogeneous layers, by considering numerical examples for the two different cases studied.

In both case a suitable loading condition is introduced to obtain closed-form solutions; to this end, the load condition equivalent to the first term of the Bessel function is considered on the top of the plate as shown in Fig.1. Furthermore, we consider a thin plate \( (h/b=0.1) \), a moderately thick plate \( (h/b=0.3) \) and a thick plate \( (h/b=0.6) \) to investigate the inhomogeneity effects for different modulus ratio. The face-sheet thickness \( t_f \) is assumed 1/10 of the plate thickness. An investigation with a corresponding conventional sandwich is performed.

4.1. Sandwich with graded core

The behavior of radial displacement is strongly nonlinear already for thin plate \( (h/b=0.1) \) if the moduli ratio is severe (1000); in the case of thick plate the radial displacement assumes nonlinear behavior already for the moduli ratio equal to 10.

Concerning the shear stress in the thickness of the sandwich plate, we observe that the core graded properties permit us to reduce the pick of shear stress in the face-sheets that occurs in conventional thick sandwiches; in this way, the stress is absorbed by the core with linear behavior in thin plate and in thick plate with a non-linear one.

The radial and hoop stresses assume in all case uniform behavior in the core thickness with value comparable with those of homogeneous core. No change of sign in the thickness of the homogeneous face-sheets, presents only for moderately thick and thick plates of conventional sandwich, compares in the case of graded core. In the Fig.2 the radial displacement (a) and shear stress (b) in the thickness of a moderately thick plate are shown for different ratios of elastic properties of the graded core \( (E_f/E_0=1,10,100) \) [13].

4.2. Sandwich with graded face-sheets

The sensitivity analysis for sandwich with graded properties of face-sheets is performed. Assuming severe moduli ratio (1000) the radial displacement remains linear for thin plate, on the contrary with respect the case of section 4.1, and becomes nonlinear with the increase of the plate thickness (for moderately and thick plates) in similar way to the homogeneous case.

The shear stress presents regular behavior for thin and moderately thick plate; only for thick plate the shear increases in the face-sheet with respect to the core of the plate, near the free surface of the plate. The graded layers
overlap the pick of shear stress in the face-sheets that occurs already for moderately thick plate in sandwich with homogeneous layers.

In similar way, the radial and hoop stress for thin plate increases with respect to the corresponding homogeneous face-sheets with elastic properties equal to the maximum value in graded material but the sign not change in the face-sheet thickness in both case. On the contrary, for moderately and thick plate, in conventional moderately and thick sandwich we observe a change of sign in the face-sheet thickness that the graded property permit us to eliminate with a regular behavior that increases and assume constant value in the core.

![Fig. 2. Radial displacement and shear stress in sandwich plate of subsection 4.1](image)

5. Conclusions

By solving the equilibrium equations of linear elasticity theory for homogeneous and heterogeneous material, this study investigates the bending response of sandwich plates with functionally graded elements. This analysis
reveals the sensitivity of the displacement and stress to the geometric and material parameters. Such knowledge may be useful in controlling results obtained by using approximate solution methods.

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