Relation between Scattering and Production Amplitude
—Case of Intermediate $\sigma$-Particle in $\pi\pi$-System—

Muneyuki Ishida*, Shin Ishida† and Taku Ishida‡

* Department of Physics, University of Tokyo, Tokyo 113, † Atomic Energy Research Institute, College of Science and Technology, Nihon University, Tokyo 101 and ‡ KEK, Oho, Tsukuba, Ibaraki 305, JAPAN

Abstract. The relation between scattering and production amplitudes are investigated, using a simple field theoretical model, from the general viewpoint of unitarity and the applicability of final state interaction(FSI-) theorem. The IA-method and VMW-method, which are applied to our phenomenological analyses [2,3] suggesting the $\sigma$-existence, are obtained as the physical state representations of scattering and production amplitudes, respectively. Moreover, the VMW-method is shown to be an effective method to obtain the resonance properties from general production processes, while the conventional analyses based on the “universality” of $\pi\pi$-scattering amplitude are powerless for this purpose.

(General problem) In our phenomenological analyses suggesting the $\sigma$-existence the $\pi\pi$-scattering amplitude $T$ and the production amplitude $F$ are parametrized by IA-method and VMW-method, respectively. In treating $T$ and $F$ there are two general problems to be taken into account: The $T$ must satisfy the unitarity, 

$$T = T^\dagger = 2i \rho \rho^\dagger,$$

and the $F$ must have the same phase [7] as $T$: $T \propto e^{i\delta} \rightarrow F \propto e^{i\delta}$, in case that the initial states have no strong phases. Moreover, on the basis of the “Universality” [4,5] the more restrictive relation than FSI-theorem between $F$ and $T$ is required: $F = \alpha(s)T$ with a slowly varying real function $\alpha(s)$ of $s$.

We re-examine the relation between $F$ and $T$ concretely, by using a simple model [8,9]: The pion $\pi$ and the resonant particles such as $\sigma(600)$ or $f_0(980)$ are introduced equally as bare states, denoted as $|\bar{\alpha}\rangle = |\bar{\pi}\rangle$, $|\bar{\sigma}\rangle$, $|\bar{f}\rangle$, which are stable particles with zero widths. By taking into account the residual strong interaction between these color-singlet states (and a production channel “$P$”),

$$L_{\text{scatt}}^{\text{int}} = \sum \bar{g}_\alpha \bar{\alpha} \pi \pi + \bar{g}_2 \pi (\pi)^4 \quad (L_{\text{prod}}^{\text{int}} = \sum \bar{\xi}_\alpha \bar{\alpha} "P" + \bar{\xi}_2 \pi \pi "P") ,$$

(1)

the bare states change into the physical states acquiring finite widths. In the following we consider only the repetition of the $\pi\pi$-loop effects.
(3-different ways of description of scattering amplitudes) There are three different ways of description of scattering amplitudes, corresponding to the three sorts of basic states for describing the resonant particles: the bare states \(|\hat{\alpha}\rangle\), the \(K\)-matrix states \(|\hat{\alpha}\rangle\), and the physical states \(|\alpha\rangle\) with a definite mass and lifetime.

First we consider the two \((\sigma, f)\) resonance-dominative case. The \(T\) is represented in terms of the \(\pi\pi\)-coupling constants \(\bar{g}_\alpha\) and the propagator matrix \(\tilde{\Delta}\) as

\[
T^{\text{Res}} = \bar{g}_\alpha \tilde{\Delta}^{-1}_{\alpha\beta} \bar{g}_\beta; \quad \tilde{\Delta}^{-1} = (\tilde{M}^2 - s - i\tilde{G})_{\alpha\beta}; \quad \tilde{G}_{\alpha\beta} = \bar{g}_\alpha \rho \bar{g}_\beta.
\]  

(2)

This \(T^{\text{Res}}\) is easily shown to satisfy the unitarity.

In the following we start from the \("K\)-matrix" states, which are able to be identified with the bare states \(|\hat{\alpha}\rangle(\equiv |\hat{\alpha}\rangle)\), and suppose \((\bar{M}^2 - s)_{\alpha\beta} = (\bar{m}^2_\alpha - s)\delta_{\alpha\beta}\), without loss of essential points.\(^1\) The \(T^{\text{Res}}\) can be expressed in the form representing concretely the repetition of the \(\pi\pi\)-loop, as

\[
T^{\text{Res}} = K^{\text{Res}}/(1 - i\rho K^{\text{Res}}); \quad K^{\text{Res}} = \bar{g}_\sigma (\bar{m}^2_\sigma - s)^{-1} \bar{g}_\sigma + \bar{g}_f (\bar{m}^2_f - s)^{-1} \bar{g}_f.
\]  

(3)

This is the same form as the conventional \(K\)-matrix in potential theory. From the viewpoint of the present field-theoretical model, this \("K\)-matrix" has a physical meaning as the propagators of bare particles with infinitesimal imaginary widths, \(\bar{m}^2_\alpha \rightarrow \bar{m}^2_\alpha - i\epsilon\), while the original \(K\)-matrix in potential theory is purely real and has no direct meaning.

(Relation between \(T\) and \(F\)) The production amplitude \(F\) is obtained, by the substitution, \(\bar{g}^2 \rightarrow \bar{g} \xi\) in the numerator \(K^{\text{Res}}\) in Eq.(3) (\(\xi\) being the production coupling-constant), as

\[
F^{\text{Res}} = P^{\text{Res}}/(1 - i\rho K^{\text{Res}}); \quad P^{\text{Res}} = \bar{\xi}_\sigma (\bar{m}^2_\sigma - s)^{-1} \bar{g}_\sigma + \bar{\xi}_f (\bar{m}^2_f - s)^{-1} \bar{g}_f.
\]  

(4)

The FSI-theorem is automatically satisfied since both \(K^{\text{Res}}\) and \(P^{\text{Res}}\) can be treated as real and the phases of \(T^{\text{Res}}\) and \(F^{\text{Res}}\) come from the common factor \((1 - i\rho K^{\text{Res}})^{-1}\).

By diagonalizing \(\tilde{G}\) in Eq.(2) further by a complex orthogonal matrix \(u_{\alpha\alpha}\) we obtain the \(T^{\text{Res}}\) in the physical state representation given by

\[
T^{\text{Res}} = F_{\alpha\beta} \Delta^{-1}_{\alpha\beta} F_{\beta\gamma} = F_{\sigma} (\lambda_\sigma - s)^{-1} F_{\sigma} + F_{f} (\lambda_f - s)^{-1} F_{f};
\]  

(5)

\(^1\) The bare states are related to the \("K\)-matrix" states through the orthogonal transformation, which does not change the reality of coupling constant. The real part of the mass correction generally do not have sharp \(s\)-dependence. Thus, the coupling constant in the \("K\)-matrix" representation remains almost \(s\)-independent except for the threshold region.

\(^2\) The criticism on our present work, raised by M.R.Pennington [6], that a spurious zero of \(T\) transmits to \(F\) unphysically, is due to his misunderstanding the relation between Eqs.(3) and (4). The positions of zero in \(T\) and \(F\), which are determined through \(K^{\text{Res}} = 0\) and \(P^{\text{Res}} = 0\), are \(s = s^0_{\alpha} = (\bar{g}^2_\alpha \bar{m}^2_\alpha + \bar{g}_f \bar{m}^2_\alpha)/(\bar{g}^2_\alpha + \bar{g}^2_f)\) and \(s = s^0_{\beta} = (\bar{g}_\sigma \bar{m}^2_\sigma + \bar{g}_f \bar{m}^2_\sigma)/(\bar{g}_\sigma \bar{m}^2_\sigma + \bar{g}_f \bar{m}^2_f)\), respectively. The \(s^0_{\alpha}\) are dependent on the production couplings, \(\xi_\sigma\) and \(\xi_f\), in the respective processes, and generally different from \(s^T_0\) except for the special case \(\xi_\sigma/\bar{g}_\sigma = \xi_f/\bar{g}_f\).
where the $\lambda_\alpha$ is the physical squared mass of the $\alpha$-state, and the $F_\alpha$ is the coupling constant in physical state representation, which is generally complex. By using the real physical coupling $g_\alpha$ defined by $g_\alpha^2 \equiv -\text{Im} \lambda_\alpha/\rho$, the $\mathcal{T}^{\text{Res}}$ is rewritten into the following form:

$$\mathcal{T}^{\text{Res}} = \frac{g_\sigma^2}{\lambda_\sigma - s} + \frac{g_f^2}{\lambda_f - s} + 2i\rho \frac{g_\sigma^2}{\lambda_\sigma - s} \frac{g_f^2}{\lambda_f - s}, \quad (6)$$

where the $\lambda_\alpha$ and $g_\alpha$ are represented by $\bar{m}_\alpha$, $\bar{g}_\alpha$, and the $\pi\pi$-state density $\rho(=\sqrt{1-4m_\pi^2/s/16\pi})$, and accordingly are also almost $s$-independent except for the threshold region. Thus Eq.(6) is just the form of scattering amplitude, applied in IA-method.

Similarly the $F^{\text{Res}}$ in the physical state representation is given by

$$F^{\text{Res}} = r_\sigma e^{i\theta_\sigma} \frac{\lambda_\sigma - s}{\lambda_\sigma - s} + r_f e^{i\theta_f} \frac{\lambda_f - s}{\lambda_f - s}, \quad (7)$$

where the $r_\sigma e^{i\theta_\sigma} \equiv \Sigma_\sigma F_\sigma$ and $r_f e^{i\theta_f} \equiv \Sigma_f F_f$, being the complex physical production coupling defined by $\Sigma_\alpha \equiv \bar{g}_\beta u_{\beta\alpha}$. The $r_\sigma$, $r_f$, $\theta_\sigma$ and $\theta_f$ are given by

$$r_\alpha e^{i\theta_\alpha} = [\bar{r}_\alpha(m_\beta^2 - \lambda_\alpha) + \bar{r}_\beta(m_\alpha^2 - \lambda_\alpha)]/[(\lambda_\beta - \lambda_\alpha); \quad \bar{r}_\alpha \equiv \bar{g}_\alpha \xi_\alpha, \quad (8)$$

where $(\alpha, \beta) = (\sigma, f)$ or $(f, \sigma)$. As can be seen by Eq.(8) the $r_\alpha$ and $\theta_\alpha$ are almost $s$-independent except for the threshold region. Thus, the Eq.(7) is just the same formula applied in VMW-method.

In the VMW-method essentially the three new parameters, $r_\sigma$, $r_f$ and the relative phase $\theta(\equiv \theta_\sigma - \theta_f)$, independent of the scattering process, characterize the relevant production processes. Presently they are represented by the two production coupling constants, $\xi_\sigma$ and $\xi_f$ (or equivalently $\bar{r}_\sigma$ and $\bar{r}_f$). Thus, among the three parameters in VMW-method there is one constraint due to the FSI-theorem.

**Background effect** Next we consider the effect of the non-resonant background phase $\delta_{BG}$. Applying a general prescription in the IA-method [2], the amplitudes are obtained in a similar manner to Eq.(4), as

$$\mathcal{T} = \frac{\mathcal{K}^{\text{Res}} + \mathcal{K}^{BG}}{(1 - i\rho\mathcal{K}^{\text{Res}})(1 - i\rho\mathcal{K}^{BG})} \rightarrow \mathcal{F} = \frac{\mathcal{P}^{\text{Res}} + \mathcal{P}^{BG}}{(1 - i\rho\mathcal{K}^{\text{Res}})(1 - i\rho\mathcal{K}^{BG})}, \quad (9)$$

where $\mathcal{K}^{\text{Res}}$ and $\mathcal{K}^{BG}$ ( $\mathcal{P}^{\text{Res}}$ and $\mathcal{P}^{BG}$ ) is, respectively, the resonant and background $\mathcal{K}$-matrix in scattering (production) process. The $\mathcal{K}^{BG}(\mathcal{P}^{BG})$ is equal to the background coupling $\bar{g}_{2\pi}(s)(\xi_{2\pi}(s))$. This $\mathcal{F}$ automatically satisfies the FSI-theorem. The $\mathcal{F}$ is rewritten into the same form as Eq.(7) of VMW-method, except for

3) Moreover, the $\mathcal{F}$ has the overall phase factor $e^{i\delta_{BG}}$ which plays a role only in the angular analysis through the scalar-tensor interfering term. This factor has a dull $s$-dependence and its effect may be regarded as being included in the phase parameters, the $\theta_\alpha$, of VMW-method.
the $s$-dependence of production couplings $r_\sigma(s), \theta_\sigma(s), r_f(s), \theta_f(s)$, which are obtained by substituting to the $\bar{r}_\alpha(\bar{\alpha} = \bar{\sigma}, \bar{f})$ in Eq.(8) the $\bar{r}_\alpha(s)$:

$$\bar{r}_\sigma(s) = \bar{r}_\sigma \cos \delta_{BG} + f_{BG}(s)(\bar{m}_\sigma^2 - s), \quad \bar{r}_f(s) = \bar{r}_f \cos \delta_{BG},$$

(10)

where the $f_{BG}(s)$ is defined by $f_{BG}(s)e^{i\bar{\theta}_{str}} \equiv F_{BG} = P_{BG}/(1 - i\rho K_{BG})$. In the case where the production coupling $\xi_{2\pi}$ is so small as $\bar{r}_\sigma \gg f_{BG} \bar{m}_\sigma^2$, the $\bar{r}_\sigma(s)$ and $\bar{r}_f(s)$ have dull $s$-dependence, and are approximated with constants, $\bar{r}_\sigma(\bar{m}_\sigma^2)$ and $\bar{r}_f(\bar{m}_f^2)$, respectively, and the VMW-method with constrained phase parameters is reproduced effectively in this case with a non-resonant $\delta_{BG}$.

(Applicability of FSI-theorem) Here it should be noted that the FSI-theorem is only applicable to the case of the initial state having no strong phase $\bar{\theta}_{str}$. However, the initial $\bar{\theta}_{str}$ exists generally in all processes under the effect of strong interactions. For example, the $pp$-central collision is largely affected by the $\Delta$-production. The $J/\Psi \rightarrow \omega\pi\pi$-decay process is also affected by the $b_1$-resonance effect. In these cases we have few knowledge on the initial phases, and we are forced to treat the parameters in VMW-method as being effectively free. The analyses presented by K.Takamatsu [3] were done from this standpoint.

(Physical meaning of “Universality”) In the “Universality” argument the masses and widths of resonances are determined only from the $\pi\pi$-scattering. In actual analyses the $\alpha(s)$ is arbitrarily chosen as, $\alpha(s) = \sum_{n=0}^{\infty} \alpha_n s^n$, and the analyses of respective production processes become nothing but the determination of the $\alpha_n$, which has no direct physical meaning.

On the other hand in the VMW-method, the difference between the spectra of $F$ and $T$ is explained intuitively by taking the relations such as $\bar{\xi}_s/\bar{g}_\sigma \gg \bar{\xi}_{2\pi}/\bar{g}_{2\pi}$, that is, the background effects are comparatively weaker in the production processes than in the scattering process. Thus in this case the large low-energy peak structure in $|F|^2$ shows directly the $\sigma$-existence. In this situation the properties of $\sigma$ can be obtained more exactly in the production processes than in the scattering process, and the VMW-method is effective for this purpose. (See, more detail in the contribution [1].)

REFERENCES

1. S. Ishida et al., this conference; Prog. Theor. Phys. 95 (1996) 745; 98 (1997) 1005, 621.
2. T. Ishida et al., this conference. Doctor thesis, Univ. of Tokyo 1996, KEK Report 97-8(1997).
3. K. Takamatsu et al., this conference.
4. M.R.Pennington, Proc. of Int.Conf. Hadron’95, p.3, Manchester, July 1995, World Scientific.
5. K.L.Au, D.Morgan and M.R.Pennington, Phys. Rev. D35, 1633 (1987). D.Morgan and M.R.Pennington, Phys. Rev. D48, 1185 (1993).
6. M.R.Pennington, hep-ph/9710456.
7. K. M. Watson, Phys. Rev. 95, 228 (1954).
8. I. J. L. Aitchson, Nucl. Phys. A189, 417 (1972).
9. S. U. Chung et al., Ann. Physik. 4, 404 (1995).

4) The effect of this unknown strong phase [9] is able to be introduced in the VMW-method by substitution of $\bar{r}_\alpha \rightarrow \bar{r}_\alpha e^{i\bar{\theta}_{str}}$. 

4)