Generic thin-shell gravastars

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Abstract. We construct generic spherically symmetric thin-shell gravastars by using the cut-and-paste procedure. We take considerable effort to make the analysis as general and unified as practicable; investigating both the internal physics of the transition layer and its interaction with “external forces” arising due to interactions between the transition layer and the bulk spacetime. Furthermore, we discuss both the dynamic and static situations. In particular, we consider “bounded excursion” dynamical configurations, and probe the stability of static configurations. For gravastars there is always a particularly compelling configuration in which the surface energy density is zero, while surface tension is nonzero.

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4 Summary and Discussion
1. Introduction

Gravastars (gravitational-vacuum stars) are hypothetical objects mooted as alternatives to standard Schwarzschild black holes [1, 2]. Typically the interior is some simple nonsingular spacetime geometry such as de Sitter space, the exterior is some close approximation to the Schwarzschild geometry, and there is some complicated transition layer near the location where the event horizon would otherwise have been expected to form. Thus, in the traditional gravastar picture, the transition layer replaces both the de Sitter and the Schwarzschild horizons, and consequently the gravastar model has no singularity at the origin and no event horizon, as its surface is located at a radius slightly greater than the Schwarzschild radius. In this model, the quantum vacuum undergoes a phase transition at or near the location where the event horizon is expected to form. Considerable attention has been devoted to these objects, and to closely related “dark stars”, “quasi black holes”, “monsters”, “black stars”, and the like [3, 4, 5, 6]. Related models, analyzed in a different context, have also been considered by Dymnikova [7]. Some models use a continuous distribution of stress-energy, which on rather general grounds must be anisotropic in the transition layer [8, 9]. Other models idealize the transition layer to being a thin shell [10] and apply a version of the Sen–Lanczos–Israel junction condition formalism [11]. In fact, the latter approaches have been extensively analysed in the literature, and applied to a wide variety of scenarios [12, 13]. Several criteria related to potential observability have been explored [14].

The key point of the present paper is to develop an extremely general and robust framework that can quickly be adapted to wide classes of generic thin-shell gravastars. We shall consider standard general relativity, with gravastars that are spherically symmetric, with the transition layer confined to a thin shell. The bulk spacetimes (interior and exterior) on either side of the transition layer will be spherically symmetric and static but otherwise arbitrary. (So the formalism is simultaneously capable of dealing with gravastars embedded in Schwarzschild, Reissner–Nordström, Kottler, or de Sitter spacetimes, or even “stringy” black hole spacetimes. Similarly the gravastar interior will be kept as general as possible for as long as possible.) The thin shell (transition layer) will be permitted to move freely in the bulk spacetimes, permitting a fully dynamic analysis. This will then allow us to perform a general stability analysis, where gravastar stability is related to the properties of the matter residing in the thin-shell transition layer.
Many of the purely technical aspects of the analysis are very similar in spirit to that encountered in a companion paper analyzing thin-shell traversable wormholes [15] — mathematically there are a few strategic sign flips — but physically the current framework is significantly different. Consequently we shall very rapidly find our analysis diverging from the traversable wormhole case. Further afield, we expect that the mathematical formalism developed herein will also prove useful when considering spacetime “voids” (manifolds with boundary) [16].

This paper is organized in the following manner: In Section 2 we outline in great detail the general formalism of generic dynamic spherically symmetric thin shells, and provide a novel approach to the linearized stability analysis around a static solution. In Section 3 we provide specific examples and consider a stability analysis by applying the generic linearized stability formalism outlined in section 2. Finally, in Section 4 we shall draw some general conclusions. Throughout this work, we adopt the sign conventions of Misner–Thorne–Wheeler [17], with $c = G = 1$.

2. General formalism

To set the stage, consider two distinct spacetime manifolds, an exterior $\mathcal{M}_+$, and an interior $\mathcal{M}_-$, that are eventually to be joined together across some surface layer $\Sigma$. Let the two bulk spacetimes have metrics given by $g^\pm_{\mu\nu}(x^\pm_\mu)$ and $g^-_{\mu\nu}(x^-_\mu)$, in terms of independently defined coordinate systems $x^\pm_\mu$ and $x^-_\mu$. In particular, consider two generic static spherically symmetric spacetimes given by the following line elements:

$$ds^2 = -e^{2\Phi}(r) \left[ 1 - \frac{b_\pm(r)}{r_\pm} \right] dr^2_\pm + \left[ 1 - \frac{b_\pm(r)}{r_\pm} \right]^{-1} r^2_\pm d\Omega^2_\pm. \quad (1)$$

We take $+$ to refer to the exterior geometry and $-$ to refer to the interior geometry. For simplicity we assume that the exterior geometry is asymptotically flat and define

$$R_- = \max \{ r : b_+(r) = r \}. \quad (2)$$

(More generally, if the exterior geometry is not asymptotically flat, the maximum should be taken over whatever black hole horizons are present, but excluding the cosmological horizons.) Similarly we assume that the interior geometry is regular at the origin ($r = 0$) and define

$$R_+ = \min \{ r : b_-(r) = r \}. \quad (3)$$

Note the (at first glance) counter-intuitive placement of the $\pm$ on the quantities $R_\pm$: The conventions are chosen so that $R_+$ is the furthest outwards one can extend the
interior geometry before hitting a horizon, whereas $R_-$ is the furthest inwards one can extend the exterior geometry before hitting a horizon.

Since the whole point of a gravastar model is to avoid horizon formation we certainly desire $R_- < R_+$. In particular, if the thin-shell transition layer $\Sigma$ is located at $r = a(\tau)$, then to avoid horizon formation we demand

$$R_- < a(\tau) < R_+. \quad (4)$$

The key issue of central interest in this article is the dynamics of this surface layer.

2.1. Bulk Einstein equations

Using the Einstein field equation, $G_{\mu\nu} = 8\pi T_{\mu\nu}$ (with $c = G = 1$), the (orthonormal) stress-energy tensor components in the bulk are given by

$$\rho(r) = \frac{1}{8\pi r^2} b', \quad (5)$$

$$p_r(r) = -\frac{1}{8\pi r^2} [2\Phi'(b - r) + b'], \quad (6)$$

$$p_t(r) = -\frac{1}{16\pi r^2} [(-b + 3rb' - 2r)\Phi' + 2r(b - r)(\Phi')^2 + 2r(b - r)\Phi'' + b''r], \quad (7)$$

where the prime denotes a derivative with respect to the radial coordinate. Here $\rho(r)$ is the energy density, $p_r(r)$ is the radial pressure, and $p_t(r)$ is the lateral pressure measured in the orthogonal direction to the radial direction. The $\pm$ subscripts were (temporarily) dropped so as not to overload the notation. Note that in obtaining the individual field equations (5)-(7), instead of using an orthonormal basis in the Einstein field equation one could simply, (because of the diagonal form of the metric in the current situation), consider the mixed tensor components, i.e., $G_{\mu\nu} = 8\pi T_{\mu\nu}$.

2.2. Null energy condition

Consider the null energy condition (NEC): $T_{\mu\nu} k^\mu k^\nu \geq 0$, where $T_{\mu\nu}$ is the stress-energy tensor and $k^\mu$ any null vector. Then along the radial direction, with $k^{\hat{\mu}} = (1, \pm 1, 0, 0)$ in the orthonormal frame where $T_{\hat{\mu}\hat{\nu}} = \text{diag}[\rho(r), p_r(r), p_t(r), p_t(r)]$, we have the particularly simple condition

$$T_{\hat{\mu}\hat{\nu}} k^{\hat{\mu}} k^{\hat{\nu}} = \rho(r) + p_r(r) = \frac{(r - b)\Phi'}{4\pi r^2} \geq 0. \quad (8)$$

By hypothesis $r > b(r)$ in both the interior and exterior regions of the gravastar, so the radial NEC reduces to $\Phi'(r) > 0$. The NEC in the transverse direction, $\rho + p_r \geq 0$,
does not have any direct simple interpretation in terms of the metric components. In most gravastar models the NEC is taken to be satisfied, though the status of the NEC as fundamental physics is quite dubious \[18\].

2.3. Transition layer

The interior and exterior manifolds are bounded by isometric hypersurfaces $\Sigma_+$ and $\Sigma_-$, with induced metrics $g^+_{ij}$ and $g^-_{ij}$. By assumption $g^+_{ij}(\xi) = g^-_{ij}(\xi)$, with natural hypersurface coordinates $\xi^i = (\tau, \theta, \phi)$. A single manifold $\mathcal{M}$ is obtained by gluing together $\mathcal{M}_+$ and $\mathcal{M}_-$ at their boundaries. So $\mathcal{M} = \mathcal{M}_+ \cup \mathcal{M}_-$, with the natural identification of the boundaries $\Sigma = \Sigma_+ = \Sigma_-$. The intrinsic metric on $\Sigma$ is

$$ds^2_{\Sigma} = -d\tau^2 + a(\tau)^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$

(9)

The position of the junction surface is given by $x^\mu(\tau, \theta, \phi) = (t(\tau), a(\tau), \theta, \phi)$, and the respective 4-velocities (as measured in the static coordinate systems on the two sides of the junction) are

$$U^\mu_\pm = \left( e^{-\Phi_{\pm}(a)} \sqrt{1 - \frac{b_{\pm}(a)}{a}} \hat{a}^2, \hat{a}, 0, 0 \right).$$

(10)

The overdot denotes a derivative with respect to $\tau$, the proper time of an observer comoving with the junction surface. The Israel formalism requires that the normals point from $\mathcal{M}_-$ to $\mathcal{M}_+$ \[11\]. The unit normals to the junction surface are

$$n^\mu_\pm = \left( e^{-\Phi_{\pm}(a)} \hat{a}, \sqrt{1 - \frac{b_{\pm}(a)}{a}} + \hat{a}^2, 0, 0 \right).$$

(11)

In view of the spherical symmetry these results can easily be deduced from the contractions $U^\mu U_\mu = -1$, $U^\mu n_\mu = 0$, and $n^\mu n_\mu = +1$. The extrinsic curvature, or the second fundamental form, is defined as $K_{ij} = n_{\mu\nu} e^\mu_{(i)} e^\nu_{(j)}$. Differentiating $n_{\mu} e^\mu_{(i)} = 0$ with respect to $\xi^j$, we have

$$n_{\mu} \frac{\partial^2 x^\mu}{\partial \xi^i \partial \xi^j} = -n_{\mu\nu} \frac{\partial x^\mu}{\partial \xi^i} \frac{\partial x^\nu}{\partial \xi^j},$$

(12)

so that general the extrinsic curvature is given by

$$K^\pm_{ij} = -n_{\mu} \left( \frac{\partial^2 x^\mu}{\partial \xi^i \partial \xi^j} + \Gamma^\mu_{\alpha\beta} \frac{\partial x^\alpha}{\partial \xi^i} \frac{\partial x^\beta}{\partial \xi^j} \right).$$

(13)
For a thin shell $K_{ij}$ is not continuous across $\Sigma$. For notational convenience, the discontinuity in the second fundamental form is defined as $\kappa_{ij} = K_{ij}^+ - K_{ij}^-$. The non-trivial components of the extrinsic curvature can easily be computed to be

$$K_{\theta\theta}^\pm = \frac{n_r}{a},$$  \hfill (16)

where the prime now denotes a derivative with respect to the coordinate $a$.

- Note that $K_{\theta\theta}^\pm$ is independent of the quantities $\Phi_{\pm}$. This is most easily verified by noting that in terms of the normal distance $\ell$ to the shell $\Sigma$ the extrinsic curvature can be written as $K_{ij} = \frac{1}{2} \partial_i g_{ij} = \frac{1}{2} n^\mu \partial_\mu g_{ij} = \frac{1}{2} n^r \partial_r g_{ij}$, where the last step relies on the fact that the bulk spacetimes are static. Then since $g_{\theta\theta} = r^2$, differentiating and setting $r \rightarrow a$ we have $K_{\theta\theta} = a n_r$. Thus

$$K_{\theta\theta}^\pm = \frac{n_r}{a},$$  \hfill (16)

which is a particularly simple formula in terms of the radial component of the normal vector, and which easily lets us verify (14).

- For $K_{\tau\tau}$ there is an argument (easily extendable to the present context) in reference [19] (see especially pages 181–183) to the effect that

$$K_{\tau\tau}^\pm = g_{\pm} = \text{magnitude of the physical 4-acceleration of the transition layer}.$$  \hfill (17)

This gives a clear physical interpretation to $K_{\tau\tau}^\pm$ and rapidly allows one to verify (15).

- There is also an important differential relationship between these extrinsic curvature components

$$\frac{d}{d\tau} \left\{ a e^{\Phi_{\pm}} K_{\theta\theta}^\pm \right\} = e^{\Phi_{\pm}} K_{\tau\tau}^\pm \dot{a}.$$  \hfill (18)

The most direct way to verify this is to simply differentiate, using (14) and (15) above. Geometrically, the existence of these relations between the extrinsic curvature components is ultimately due to the fact that the bulk spacetimes have been chosen to be static. By noting that

$$\frac{d}{da} \left( \frac{1}{2} \dot{a}^2 \right) = \left( \frac{d}{da} \dot{a} \right) \ddot{a} = \dddot{a},$$  \hfill (19)
we can also write this differential relation as
\[
\frac{d}{da} \left( a \ e^{\Phi \pm} \ K_{\theta}^{\pm} \right) = e^{\Phi \pm} \ K_{\tau}^{\pm}.
\] (20)

2.4. Lanczos equations: Surface stress-energy

The Lanczos equations follow from the Einstein equations applied to the hypersurface joining the bulk spacetimes, and are given by
\[
S_{ij} = -\frac{1}{8\pi} \left( \kappa_{ij} - \delta_{ij} \kappa_k^k \right).
\] (21)
Here \( S_{ij} \) is the surface stress-energy tensor on \( \Sigma \). In particular, because of spherical symmetry considerable simplifications occur, namely \( \kappa_{ij} = \text{diag} (\kappa_\tau \tau, \kappa_\theta \theta, \kappa_\theta \theta) \). The surface stress-energy tensor may be written in terms of the surface energy density, \( \sigma \), and the surface pressure, \( \mathcal{P} \), as \( S_{ij} = \text{diag} (-\sigma, \mathcal{P}, \mathcal{P}) \). The Lanczos equations then reduce to
\[
\sigma = -\frac{\kappa_\theta \theta}{4\pi}, \quad \mathcal{P} = \frac{\kappa_\tau \tau + \kappa_\theta \theta}{8\pi}, \quad \sigma + 2\mathcal{P} = \frac{\kappa_\tau \tau}{4\pi}.
\] (22)

From equations (14)–(15), we see that:
\[
\sigma = -\frac{1}{4\pi a} \left[ \sqrt{1 - \frac{b_+(a)}{a} + \dot{a}^2} - \sqrt{1 - \frac{b_-(a)}{a} + \dot{a}^2} \right],
\] (23)
\[
\mathcal{P} = \frac{1}{8\pi a} \left[ \sqrt{1 + \dot{a}^2 + a\ddot{a} - \frac{b_+(a) + a\dot{b}_+(a)}{2a}} + \sqrt{1 - \frac{b_+(a)}{a} + \dot{a}^2} a\Phi_+(a) 
- \frac{1 + \dot{a}^2 + a\ddot{a} - \frac{b_-(a) + a\dot{b}_-(a)}{2a}}{\sqrt{1 - \frac{b_-(a)}{a} + \dot{a}^2}} - \sqrt{1 - \frac{b_-(a)}{a} + \dot{a}^2} a\Phi_-(a) \right],
\] (24)
and finally
\[
\sigma + 2\mathcal{P} = \frac{[g]}{4\pi} = \frac{1}{4\pi} \left[ \frac{\ddot{a} + \frac{b_+(a) - a\dot{b}_+(a)}{2a^2}}{\sqrt{1 - \frac{b_+(a)}{a} + \dot{a}^2}} + \sqrt{1 - \frac{b_+(a)}{a} + \dot{a}^2} \Phi_+(a) 
- \frac{\ddot{a} + \frac{b_-(a) - a\dot{b}_-(a)}{2a^2}}{\sqrt{1 - \frac{b_-(a)}{a} + \dot{a}^2}} - \sqrt{1 - \frac{b_-(a)}{a} + \dot{a}^2} \Phi_-(a) \right].
\] (25)

Note that \( \sigma + 2\mathcal{P} \) has a particularly simple physical interpretation in terms of \([g]\), the discontinuity in 4-acceleration. (This is ultimately related to the fact that the
quantity $\sigma + 2\mathcal{P}$ for a thin shell has properties remarkably similar to those of the quantity $\rho + 3p$ for a bulk spacetime.) Furthermore the surface energy density $\sigma$ is independent of the quantities $\Phi_{\pm}$. The surface mass of the thin shell is given by $m_s = 4\pi a^2 \sigma$.

Independent of the state of motion of the thin shell, we have $\sigma(a) > 0$ whenever $b_+(a) > b_-(a)$, and $\sigma(a) < 0$ whenever $b_+(a) < b_-(a)$. The situation where $\sigma = 0$ corresponds to $b_-(a) = b_+(a)$, and is precisely the case where all the discontinuities are concentrated in $K_\tau^\tau$ while $K_\theta^\theta$ is continuous. This phenomenon, the vanishing of $\sigma$ at certain specific shell radii given by $b_-(a) = b_+(a)$, is generic to gravastars but (because of a few key sign flips) cannot occur for the thin-shell traversable wormholes considered in [15, 19, 20]. This is perhaps the most obvious of many properties differentiating gravastars from the thin-shell traversable wormhole case, although wormhole geometries surrounded by thin shells, similar to the cases explored in this work, have also been analyzed in the literature [21, 22].

2.5. Static gravastars

Assume, for the sake of discussion, a static solution at some $a_0 \in (R_-, R_+)$. Then

$$\sigma(a_0) = -\frac{1}{4\pi a_0} \left[ \sqrt{1 - \frac{b_+(a_0)}{a_0}} - \sqrt{1 - \frac{b_-(a_0)}{a_0}} \right],$$

(26)

$$\mathcal{P}(a_0) = \frac{1}{8\pi a_0} \left[ \frac{1 - b_+(a_0) + a_0 b_+''(a_0)}{2a_0} \frac{1}{\sqrt{1 - \frac{b_+(a_0)}{a_0}}} + \sqrt{1 - \frac{b_+(a_0)}{a_0}} a_0 \Phi_+'(a_0) \right. 
\left. - \frac{1 - b_-(a_0) + a_0 b_-''(a_0)}{2a_0} \frac{1}{\sqrt{1 - \frac{b_-(a_0)}{a_0}}} - \sqrt{1 - \frac{b_-(a_0)}{a_0}} a_0 \Phi_-'(a_0) \right],$$

(27)

and finally

$$\sigma(a_0) + 2\mathcal{P}(a_0) = \frac{[g_0]}{4\pi} = \frac{1}{4\pi} \left[ \frac{b_+(a) - a b_+''(a)}{2a^2} \frac{1}{\sqrt{1 - \frac{b_+(a)}{a}}} + \sqrt{1 - \frac{b_+(a)}{a}} \Phi_+'(a) 
\right. 
\left. - \frac{b_-(a) - a b_-''(a)}{2a^2} \frac{1}{\sqrt{1 - \frac{b_-(a)}{a}}} - \sqrt{1 - \frac{b_-(a)}{a}} \Phi_-'(a) \right].$$

(28)
(See also equation (32) of [23].) Now taking \( a_0 \to R_- \), we have

\[
\sigma(R_-) = + \frac{1}{4\pi R_-} \sqrt{1 - \frac{b_-(R_-)}{R_-}} > 0.
\]

However taking \( a_0 \to R_+ \) we have

\[
\sigma(R_+) = - \frac{1}{4\pi R_+} \sqrt{1 - \frac{b_+(R_+)}{R_+}} < 0.
\]

That is:

\[
\sigma(R_\pm) = \mp \frac{1}{4\pi R_\pm} \sqrt{1 - \frac{b_\pm(R_\pm)}{R_\pm}}.
\]

Applying the mean value theorem, for gravastars there will always be some \( R_0 \in (R_-,R_+) \), possibly many such \( R_0 \), such that \( \sigma(R_0) = 0 \). (This phenomenon cannot occur for the thin-shell traversable wormholes considered in [15, 19, 20], because of key sign flips — for thin-shell traversable wormholes we always have \( \sigma < 0 \).) The \( R_0 \) such that \( \sigma(R_0) = 0 \) is clearly a special place for gravastars. Explicitly this occurs when

\[
b_+(R_0) = b_-(R_0),
\]

and in fact for

\[
R_- < b_+(R_0) = b_-(R_0) < R_+.
\]

At this special point the discontinuities are concentrated in \( K_{\tau}^\tau \) while \( K_{\theta}^\theta \) is continuous. That is

\[
\mathcal{P}(R_0) = \frac{1}{16\pi R_0} \left[ \frac{b'_-(R_0) - b'_+(R_0)}{\sqrt{1 - \frac{b_+(R_0)}{R_0}}} + 2 \frac{1 - \frac{b_+(R_0)}{R_0}}{R_0} R_0 [\Phi'_+(R_0) - \Phi'_-(R_0)] \right],
\]

so that

\[
\mathcal{P}(R_0) = \frac{[K_\tau^\tau]}{8\pi} = \frac{[g_0]}{8\pi}.
\]

Even though \( \sigma = 0 \), one needs \( \mathcal{P} \neq 0 \) because of the non-zero 4-acceleration \( g_0 \).

If one also demands that the surface pressure at \( R_0 \) also be zero, \( \mathcal{P}(R_0) = 0 \), one must impose the additional condition

\[
b'_+ - b'_- = 2 (R_0 - b_\pm) (\Phi'_+ - \Phi'_-),
\]
which is equivalent to
\[ (e^{2\Phi} [1 - b_+/a_0])'\big|_{R_0} = (e^{2\Phi} [1 - b_-/a_0])'\big|_{R_0}. \]  
(37)

If \( \Phi'_\pm(R_0) \), then not only does \( b_+(R_0) = b_-(R_0) \) but also \( b'_+(R_0) = b'_-(R_0) \) — these two conditions give a zero pressure and zero density shell. More specifically, in terms of standard nomenclature, if the surface stress-energy terms are zero, the junction is denoted as a boundary surface; if surface stress terms are present, the junction is called a thin shell.

### 2.6. Surface stress estimates

It is interesting to obtain some estimates of the surface stresses. For this purpose, consider for simplicity an exterior geometry that is Schwarzschild-de Sitter spacetime, so that
\[ b_+(r) = 2M + \frac{\Lambda}{3} r^3, \quad \Phi_+(r) = 0. \]  
(38)

An important quantity that will be play a fundamental role throughout this paper is the surface mass of the thin shell, which is given by \( m_s = 4\pi a^2 \sigma \). For the exterior Schwarzschild-de Sitter spacetime, and considering an arbitrary interior geometry, the surface mass of the static thin shell is given by
\[ m_s(a_0) = a_0 \left( \sqrt{1 - \frac{b_-(a_0)}{a_0}} - \sqrt{1 - \frac{2M}{a_0} - \frac{\Lambda}{3} a_0^2} \right). \]  
(39)

Note that one may interpret \( M \) as the total mass of the system, as measured in an asymptotic region of the spacetime. Solving for \( M \), we have
\[ M = \frac{b_-(a_0)}{2} + m_s(a_0) \left( \sqrt{1 - \frac{b_-(a_0)}{a_0}} - \frac{m_s(a_0)}{2a_0} \right) - \frac{\Lambda}{6} a_0^3. \]  
(40)

For the Schwarzschild–de Sitter spacetime \( \Lambda > 0 \). For the range \( 0 < 9\Lambda M^2 < 1 \), the factor \( g_{rr}^{-1} = -g_{tt} = (1 - 2M/r - \Lambda r^2/3) \) possesses two positive real roots, \( r_b \) and \( r_c \), corresponding to the black hole and the cosmological event horizons:
\[ r_b = 2\Lambda^{-1/2} \cos(\alpha/3), \]  
(41)
\[ r_c = 2\Lambda^{-1/2} \cos(\alpha/3 + 4\pi/3). \]  
(42)

Here \( \cos \alpha \equiv -3M\Lambda^{1/2} \), with \( \pi < \alpha < 3\pi/2 \). In this domain we have \( 2M < r_b < 3M \) and \( r_c > 3M \).
Defining suitable dimensionless parameters, equations (26)–(27) take, for the Schwarzschild-de Sitter solution, the form

\[
\mu(a_0) = x \left( \sqrt{1 - x \bar{b}(x)} - \sqrt{1 - x - \frac{4\beta}{27x^2}} \right),
\]

\[
\Pi(a_0) = x \left( \frac{1 - \frac{x}{2} - \frac{8\beta}{27x^2}}{\sqrt{1 - x - \frac{4\beta}{27x^2}}} - \frac{1 - \frac{x}{2} \left[ x\bar{b}(x) + b' \right]}{\sqrt{1 - x\bar{b}(x)}} - \zeta \sqrt{1 - x\bar{b}(x)} \right).
\]

Here we define: \(x = 2M/a\), \(\beta = 9\Lambda M^2\), \(\bar{b}(x) = b(a)/(2M)\), \(\mu = 8\pi M\sigma\), and set \(\Pi = 16\pi M\mathcal{P}\). In the analysis that follows we shall assume that \(M\) is positive, \(M > 0\).

As a specific application, consider the standard gravastar picture, which consists of a Schwarzschild exterior geometry and an interior de Sitter spacetime. Thus the surface stresses, equations (43)–(44), are obtained by setting \(\Lambda = 0\), (i.e., \(\beta = 0\)), while \(b_-(a_0) = a_0^3/R^2\) and \(\Phi_-(a_0) = 0\). Here we have defined \(R^2 = 3/\Lambda_-\), for simplicity. For this case the total mass of the system \(M\) is given by

\[
M = \frac{a_0^2}{2R^2} + m_s(a_0) \left( \sqrt{1 - \frac{a_0^2}{R^2}} - \frac{m_s}{2a_0} \right).
\]

Using the dimensionless parameters, \(x = 2M/a_0\) and \(y = 2M/R\), while \(\mu(a_0) = 8\pi M\sigma(a_0)\) and \(\Pi(a_0) = 16\pi M\mathcal{P}(a_0)\), equations (43)–(44), take the form

\[
\mu(a_0) = -x \left( \sqrt{1 - x} - \sqrt{1 - \frac{y^2}{x^2}} \right),
\]

\[
\Pi(a_0) = x \left( \frac{1 - \frac{x}{2} - \frac{2\beta}{27x^2}}{\sqrt{1 - x}} - \frac{1 - \frac{x}{2} \left[ x\bar{b}(x) + b' \right]}{\sqrt{1 - x\bar{b}(x)}} \right).
\]

The qualitative behaviour is depicted in figure 1. The left plot represents the surface energy density, \(\sigma\). Note that \(\sigma\) is positive in the range \(2M < a_0 < (2MR^2)^{1/3}\), and negative for \((2MR^2)^{1/3} < a_0 < R\). The right plot represents the surface pressure, \(\mathcal{P}\). The latter diverges as \(a_0\) approaches \(2M\) (\(R\)) from the right (left). A detailed stability analysis of the thin shell for this specific configuration will be presented in section 3.1.
Figure 1. Schwarzschild–de Sitter gravastar: These plots depict the qualitative behaviour of the surface stresses of a gravastar with interior de Sitter and exterior Schwarzschild spacetimes. We have considered the dimensionless parameters, $x = 2M/a_0$ and $y = 2M/R$, and $\mu(a_0) = 8\pi M\sigma(a_0)$ and $\Pi(a_0) = 16\pi MP(a_0)$. The left plot represents the surface energy density, $\sigma$. Note that $\sigma$ is positive in the range $2M < a_0 < (2MR^2)^{1/3}$, and negative in the range $(2MR^2)^{1/3} < a_0 < R$. The right plot represents the surface pressure, $P$. The latter surface pressure diverges as $a_0$ approaches $2M$ ($R$) from the right (left). See the text for more details.

2.7. Conservation identity

The first contracted Gauss–Codazzi equation$\S$ is
\[ G_{\mu\nu} n^\mu n^\nu = \frac{1}{2} \left( K^2 - K_{ij}K^{ij} - 3R \right). \tag{48} \]

The second contracted Gauss–Codazzi equation$\S$ is
\[ G_{\mu\nu} e^\mu_{(i)} n^\nu = K^2_{ij} - K_{ij}. \tag{49} \]

Together with the Lanczos equations this provides the conservation identity
\[ S^i_{j|i} = \left[ T_{\mu\nu} e^\mu_{(j)} n^\nu \right]_+, \tag{50} \]

$\S$ Sometimes called simply the Gauss equation, or in general relativity more often referred to as the “Hamiltonian constraint”.

$\S$ Sometimes called simply the Codazzi or the Codazzi–Mainardi equation, or in general relativity more often referred to as the “ADM constraint” or “momentum constraint”.
where the convention \([X]_+^\dagger \equiv X^+|_{\Sigma} - X^-|_{\Sigma}\) is used. When interpreting this conservation identity, consider first the momentum flux defined by

\[
\left[T_{\mu\nu} \varepsilon_{(\tau)} n^\nu\right]^+ = \left[T_{\mu\nu} U^\mu n^\nu\right]^+ = \left(T_{\hat{t}\hat{t}} + T_{\hat{r}\hat{r}}\right) \frac{\dot{a} \sqrt{1 - \frac{b(a)}{a} + \dot{a}^2}}{1 - \frac{b(a)}{a}} , 
\]

(51)

where \(T_{\hat{t}\hat{t}}\) and \(T_{\hat{r}\hat{r}}\) are the bulk stress-energy tensor components given in an orthonormal basis. This flux term corresponds to the net discontinuity in the (bulk) momentum flux \(F_\mu = T_{\mu\nu} U^\nu\) which impinges on the shell. Applying the (bulk) Einstein equations we see

\[
\left[T_{\mu\nu} \varepsilon_{(\tau)} n^\nu\right]^+ = \frac{\dot{a}}{4\pi a} \left[\Phi'_{\tau}(a) \sqrt{1 - \frac{b_{\tau}(a)}{a} + \dot{a}^2} - \Phi'_{\tau}(a) \sqrt{1 - \frac{b_{\tau}(a)}{a} + \dot{a}^2}\right] , 
\]

(52)

It is useful to define the quantity

\[
\Xi = \frac{1}{4\pi a} \left[\Phi'_{\tau}(a) \sqrt{1 - \frac{b_{\tau}(a)}{a} + \dot{a}^2} - \Phi'_{\tau}(a) \sqrt{1 - \frac{b_{\tau}(a)}{a} + \dot{a}^2}\right] , 
\]

(53)

and to let \(A = 4\pi a^2\) be the surface area of the thin shell. Then in the general case, the conservation identity provides the following relationship

\[
\frac{d\sigma}{d\tau} + (\sigma + P) \frac{1}{A} \frac{dA}{d\tau} = \Xi \dot{a} , 
\]

(54)

or equivalently

\[
\frac{d(\sigma A)}{d\tau} + P \frac{dA}{d\tau} = \Xi A \dot{a} . 
\]

(55)

The first term represents the variation of the internal energy of the shell, the second term is the work done by the shell’s internal force, and the third term represents the work done by the external forces. Once could also brute force verify this equation by explicitly differentiating (23) using (24) and the relations (18). If we assume that the equations of motion can be integrated to determine the surface energy density as a function of radius \(a\), that is, assuming the existence of a suitable function \(\sigma(a)\), then the conservation equation can be written as

\[
\sigma' = -\frac{2}{a} (\sigma + P) + \Xi , 
\]

(56)

where \(\sigma' = d\sigma/da\). We shall carefully analyze the integrability conditions for \(\sigma(a)\) in the next sub-section. For now, note that the flux term (external force term) \(\Xi\) is
automatically zero whenever $\Phi_{\pm} = 0$; this is actually a quite common occurrence, for instance in either Schwarzschild or Reissner–Nordström geometries, or more generally whenever $\rho + p_r = 0$, so it is very easy for one to be mislead by those special cases. In particular, in situations of vanishing flux $\Xi = 0$ one obtains the so-called “transparency condition”, $[G_{\mu\nu}, U^\mu n^\nu]^+ = 0$, see [24]. The conservation identity, equation (50), then reduces to the simple relationship $\dot{\sigma} = -2(\sigma + P)\dot{a}/a$. But in general the “transparency condition” does not hold, and one needs the full version of the conservation equation as given in equation (55).

2.8. Integrability of the surface energy density

When does it make sense to assert the existence of a function $\sigma(a)$? Let us start with the situation in the absence of external forces (we will rapidly generalize this) where the conservation equation,

$$\dot{\sigma} = -2(\sigma + P)\dot{a}/a,$$

(57)

can easily be rearranged to

$$\frac{\dot{\sigma}}{\sigma + P} = -2\frac{\dot{a}}{a}.$$  

(58)

Assuming a barotropic equation of state $P(\sigma)$ for the matter in the gravastar transition layer, this can be integrated to yield

$$\int_{\sigma_0}^{\sigma} \frac{d\tilde{\sigma}}{\tilde{\sigma} + P(\tilde{\sigma})} = -2 \int_{a_0}^{a} \frac{d\tilde{a}}{\tilde{a}} = -2 \ln(a/a_0).$$  

(59)

This implies that $a$ can be given as some function $a(\sigma)$ of $\sigma$, and by the inverse function theorem implies over suitable domains the existence of a function $\sigma(a)$. Now this barotropic equation of state is a rather strong assumption, albeit one that is very often implicitly made when dealing with thin-shell gravastar (or thin-shell wormholes, or other thin-shell objects). As a first generalization, consider what happens if the surface pressure generalized is to be of the form $P(a, \sigma)$, which is not barotropic. Then the conservation equation can be rearranged to be

$$\sigma' = -2\frac{[\sigma + P(a, \sigma)]}{a}.$$  

(60)

This is a first-order (albeit nonlinear and non-autonomous) ordinary differential equation, which at least locally will have solutions $\sigma(a)$. There is no particular
reason to be concerned about the question of global solutions to this ODE, since in applications one is most typically dealing with linearization around a static solution.

If we now switch on external forces, one way of guaranteeing integrability would be to demand that the external forces are of the form $\Xi(a, \sigma)$, since then the conservation equation would read

$$\sigma' = -\frac{2[\sigma + P(a, \sigma)]}{a} + \Xi(a, \sigma),$$

which is again a first-order albeit nonlinear and non-autonomous ordinary differential equation. But how general is this $\Xi = \Xi(a, \sigma)$ assumption? There are at least two nontrivial situations where this definitely holds:

- If $\Phi_+(a) = \Phi_-(a) = \Phi(a)$, then $\Xi = -\Phi'(a) \sigma$, which is explicitly of the required form.
- If $b_+(a) = b_-(a) = b(a)$, *but the $\Phi_\pm$ are unequal*, then $\sigma \equiv 0$ regardless of the location and state of motion of the transition layer. Furthermore $\Xi \equiv 2P/a$. (This would make for a somewhat unusual gravastar.)
- If both $b_+(a) = b_-(a) = b(a)$ and $\Phi_+(a) = \Phi_-(a) = \Phi(a)$, then the situation is vacuous. There is then no discontinuity in extrinsic curvatures and the thin shell carries no stress-energy; so this in fact corresponds to a “continuum” gravastar with $b(r)/r < 1$ for all $r \in (0, \infty)$.

But in general we will need a more complicated set of assumptions to assure integrability, and the consequent existence of a function $\sigma(a)$. A model that is always *sufficient* (not necessary) to guarantee integrability is to view the exotic material in the transition layer as a two-fluid system, characterized by $\sigma_\pm$ and $P_\pm$, with two (possibly independent) equations of state $P_\pm(\sigma_\pm)$. Specifically, take

$$\sigma_\pm = -\frac{1}{4\pi} (K_\pm)^\theta_\theta, \quad P_\pm = \frac{1}{8\pi} \left\{ (K_\pm)^\tau_\tau + (K_\pm)^\theta_\theta \right\}. \quad (62)$$

In view of the differential identities

$$\frac{d}{dt} \{ a e^{\Phi_\pm} K_\theta^\pm \} = e^{\Phi_\pm} K_\tau^\pm \dot{a}, \quad (64)$$

each of these two fluids is independently subject to

$$\frac{d}{dt} \{ e^{\Phi_\pm} \sigma_\pm \} = -\frac{2e^{\Phi_\pm}}{a} \{ \sigma_\pm + P_\pm \} \dot{a}, \quad (65)$$
which is equivalent to
\[ \{ e^{\Phi_\pm} \sigma_\pm \}' = -\frac{2e^{\Phi_\pm}}{a} \{ \sigma_\pm + \mathcal{P}_\pm \}. \] (66)

With two equations of state \( \mathcal{P}_\pm(\sigma_\pm) \) these are two nonlinear first-order ordinary differential equations for \( \sigma_\pm \). These equations are integrable, implicitly defining functions \( \sigma_\pm(a) \), at least locally. Once this is done we define
\[ \sigma(a) = \sigma_+(a) - \sigma_-(a), \] (67)
and
\[ m_s(a) = 4\pi \sigma(a) a^2. \] (68)

While the argument is more complicated than one might have expected, the end result is easy to interpret: We can simply choose \( \sigma(a) \), or equivalently \( m_s(a) \), as an arbitrarily specifiable function that encodes the (otherwise unknown) physics of the specific form of matter residing on the gravastar transition layer.

2.9. Equation of motion

To qualitatively analyze the stability of the gravastar, assuming integrability of the surface energy density, (that is, the existence of a function \( \sigma(a) \)), it is useful to rearrange equation (23) into the form
\[ \frac{1}{2} \frac{d^2 a}{dt^2} + V(a) = 0, \] (69)
where the potential \( V(a) \) is given by
\[ V(a) = \frac{1}{2} \left\{ 1 - \frac{\bar{b}(a)}{a} - \left[ \frac{m_s(a)}{2a} \right]^2 - \left[ \frac{\Delta(a)}{m_s(a)} \right]^2 \right\}. \] (70)

Here \( m_s(a) = 4\pi a^2 \sigma(a) \) is the mass of the thin shell. The quantities \( \bar{b}(a) \) and \( \Delta(a) \) are defined, for simplicity, as
\[ \bar{b}(a) = \frac{b_+(a) + b_-(a)}{2}, \] (71)
\[ \Delta(a) = \frac{b_+(a) - b_-(a)}{2}, \] (72)
† This equation only valid for \( \sigma \neq 0 \) due to a divide-by-zero problem. The \( \sigma = 0 \) case is a special one worth separate consideration:
\[ V(a) = \frac{1}{2} \left\{ 1 - \frac{\bar{b}(a)}{a} - \left( \frac{8\pi \mathcal{P}}{\Phi'_+ - \Phi'_-} \right)^2 \right\}. \]
respectively. This gives the potential $V(a)$ as a function of the surface mass $m_s(a)$. By differentiating with respect to $a$, (using (19)), we see that the equation of motion implies
\[ \ddot{a} = -V'(a). \] (73)

It is sometimes useful to reverse the logic flow and determine the surface mass as a function of the potential. Following the techniques used in [10, 15], suitably modified for the present context, a brief calculation yields
\[ m_s(a) = -a \left[ \sqrt{1 - \frac{b_+(a)}{a} - 2V(a)} - \sqrt{1 - \frac{b_-(a)}{a} - 2V(a)} \right], \] (74)

with the negative root now being necessary for compatibility with the Lanczos equations. Note the logic here — assuming integrability of the surface energy density, if we want a specific $V(a)$ this tells us how much surface mass we need to put on the transition layer (as a function of $a$), which is implicitly making demands on the equation of state of the matter residing on the transition layer. In a completely analogous manner, the assumption of integrability of $\sigma(a)$ implies that after imposing the equation of motion for the shell one has
\[ \sigma(a) = -\frac{1}{4\pi a} \left[ \sqrt{1 - \frac{b_+(a)}{a} - 2V(a)} - \sqrt{1 - \frac{b_-(a)}{a} - 2V(a)} \right], \] (75)

while
\[ \mathcal{P} = \frac{1}{8\pi a} \left[ \frac{1 - 2V(a) - aV'(a) - \frac{b_+(a) + ab'_+(a)}{2a}}{\sqrt{1 - \frac{b_+(a)}{a} - 2V(a)}} + \sqrt{1 - \frac{b_-(a)}{a} - 2V(a)} a\Phi'_+(a) 
- \frac{1 - 2V(a) - aV'(a) - \frac{b_-(a) + ab'_-(a)}{2a}}{\sqrt{1 - \frac{b_-(a)}{a} - 2V(a)}} - \sqrt{1 - \frac{b_-(a)}{a} - 2V(a)} a\Phi'_-(a) \right], \] (76)

and
\[ \Xi(a) = \frac{1}{4\pi a} \left[ \Phi'_+(a) \sqrt{1 - \frac{b_+(a)}{a} - 2V(a)} - \Phi'_-(a) \sqrt{1 - \frac{b_-(a)}{a} - 2V(a)} \right]. \] (77)

The three quantities \{\sigma(a), \mathcal{P}(a), \Xi(a)\} (or equivalently \{m_s(a), \mathcal{P}(a), \Xi(a)\}) are related by the differential conservation law, so at most two of them are functionally independent.
2.10. Linearized equation of motion

Consider a linearization around an assumed static solution (at \( a_0 \)) to the equation of motion \( \frac{1}{2} \dot{a}^2 + V(a) = 0 \), and so also a solution of \( \ddot{a} = -V'(a) \). Generally a Taylor expansion of \( V(a) \) around \( a_0 \) to second order yields

\[
V(a) = V(a_0) + V'(a_0)(a - a_0) + \frac{1}{2} V''(a_0)(a - a_0)^2 + O[(a - a_0)^3]. \tag{78}
\]

But since we are expanding around a static solution \( \ddot{a}_0 = \dddot{a}_0 = 0 \), we automatically have \( V(a_0) = V'(a_0) = 0 \), so it is sufficient to consider

\[
V(a) = \frac{1}{2} V''(a_0)(a - a_0)^2 + O[(a - a_0)^3]. \tag{79}
\]

The assumed static solution at \( a_0 \) is stable if and only if \( V(a) \) has a local minimum at \( a_0 \), which requires \( V''(a_0) > 0 \). This will be our primary criterion for gravastar stability, though it will be useful to rephrase it in terms of more basic quantities.

For instance, it is extremely useful to express \( m'_s(a) \) and \( m''_s(a) \) by the following expressions:

\[
m'_s(a) = \frac{m_s(a)}{a} + \frac{a}{2} \left\{ \frac{(b_+ (a)/a)' + 2V'(a)}{\sqrt{1 - b_+(a)/a - 2V(a)}} - \frac{(b_- (a)/a)' + 2V'(a)}{\sqrt{1 - b_-(a)/a - 2V(a)}} \right\}, \tag{80}
\]

and

\[
m''_s(a) = \frac{a}{4} \left\{ \frac{[(b_+ (a)/a)' + 2V'(a)]^2}{[1 - b_+(a)/a - 2V(a)]^{3/2}} - \frac{[(b_- (a)/a)' + 2V'(a)]^2}{[1 - b_-(a)/a - 2V(a)]^{3/2}} \right\} + \frac{a}{2} \left\{ \frac{(b_+ (a)/a)'' + 2V''(a)}{\sqrt{1 - b_+(a)/a - 2V(a)}} - \frac{(b_- (a)/a)'' + 2V''(a)}{\sqrt{1 - b_-(a)/a - 2V(a)}} \right\}. \tag{81}
\]

Doing so allows us to easily study linearized stability, and to develop a simple inequality on \( m''_s(a_0) \) by using the constraint \( V''(a_0) > 0 \). Similar formulae hold for \( \sigma'(a) \), \( \sigma''(a) \), for \( \mathcal{P}'(a) \), \( \mathcal{P}''(a) \), and for \( \Xi'(a) \), \( \Xi''(a) \). In view of the redundancies coming from the relations \( m_s(a) = 4\pi \sigma(a)a^2 \) and the differential conservation law, the only interesting quantities are \( \Xi'(a) \), \( \Xi''(a) \).

It is similarly useful to consider

\[
4\pi \Xi(a) a = \left[ \Phi'_+(a)\sqrt{1 - \frac{b_+(a)}{a} - 2V(a)} - \Phi'_-(a)\sqrt{1 - \frac{b_-(a)}{a} - 2V(a)} \right]. \tag{82}
\]
for which an easy computation yields:

$$\frac{4\pi}{3} \Xi(a) a'' = + \left\{ \Phi''_+(a) \sqrt{1 - b(a)/a - 2V(a)} - \Phi''_-(a) \sqrt{1 - b_-(a)/a - 2V(a)} \right\}$$

$$- \frac{1}{2} \left\{ \Phi'_+(a) \frac{(b(a)/a)'+ 2V'(a)}{\sqrt{1 - b(a)/a - 2V(a)}} - \Phi'_-(a) \frac{(b_-(a)/a)'+ 2V'(a)}{\sqrt{1 - b_-(a)/a - 2V(a)}} \right\},$$

and

$$\frac{4\pi}{3} \Xi(a) a''' = \left\{ \Phi'''_+(a) \sqrt{1 - b(a)/a - 2V(a)} - \Phi'''_-(a) \sqrt{1 - b_-(a)/a - 2V(a)} \right\}$$

$$- \left\{ \Phi''_+(a) \frac{(b(a)/a)' + 2V'(a)}{\sqrt{1 - b(a)/a - 2V(a)}} - \Phi''_-(a) \frac{(b_-(a)/a)' + 2V'(a)}{\sqrt{1 - b_-(a)/a - 2V(a)}} \right\}$$

$$- \frac{1}{4} \left\{ \Phi'_+(a) \frac{[(b(a)/a)'+ 2V'(a)]^2}{[1 - b(a)/a - 2V(a)]^{3/2}} - \Phi'_-(a) \frac{[(b_-(a)/a)'+ 2V'(a)]^2}{[1 - b_-(a)/a - 2V(a)]^{3/2}} \right\}$$

$$- \frac{1}{2} \left\{ \Phi'_+(a) \frac{(b(a)/a)'' + 2V''(a)}{\sqrt{1 - b(a)/a - 2V(a)}} - \Phi'_-(a) \frac{(b_-(a)/a)'' + 2V''(a)}{\sqrt{1 - b_-(a)/a - 2V(a)}} \right\}.$$  

(83)

We shall now evaluate these quantities at the assumed stable solution $a_0$.

2.11. The master equations

In view of the above, to have a stable static solution at $a_0$ we must have:

$$m_s(a_0) = -a_0 \left\{ \sqrt{1 - \frac{b_+(a_0)}{a_0}} - \sqrt{1 - \frac{b_-(a_0)}{a_0}} \right\},$$

(85)

while

$$m_s'(a_0) = \frac{m_s(a_0)}{2a_0} - \frac{1}{2} \left\{ \frac{1 - b_+(a_0)}{\sqrt{1 - b_+(a_0)/a_0}} - \frac{1 - b_-(a_0)}{\sqrt{1 - b_-(a_0)/a_0}} \right\}.$$  

(86)

The inequality one derives for $m_s''(a_0)$ is now trickier since the relevant expression contains two competing terms of opposite sign. Provided $b_+(a_0) \geq b_-(a_0)$, which is equivalent to demanding $\sigma(a_0) \geq 0$, one derives

$$m_s''(a_0) \geq + \frac{1}{4a_0^3} \left\{ \frac{[b_+(a_0) - a_0b_+(a_0)]^2}{[1 - b_+(a_0)/a_0]^{3/2}} - \frac{[b_-(a_0) - a_0b_-(a_0)]^2}{[1 - b_-(a_0)/a_0]^{3/2}} \right\}$$

$$+ \frac{1}{2} \left\{ \frac{b_+(a_0)}{\sqrt{1 - b_+(a_0)/a_0}} - \frac{b_-(a_0)}{\sqrt{1 - b_-(a_0)/a_0}} \right\}.$$  

(87)
However if \( b_+ (a_0) \leq b_- (a_0) \) the direction of the inequality is reversed. This last formula in particular translates the stability condition \( V''(a_0) \geq 0 \) into a rather explicit and not too complicated inequality on \( m''_2 (a_0) \), one that can in particular cases be explicitly checked with a minimum of effort.

In the absence of external forces this inequality is the only stability constraint one requires. However, once one has external forces (\( \Xi \neq 0 \) which requires \( \Phi \pm \neq 0 \)), there is additional information:

\[
[4 \pi \Xi (a) a]'' |_{a_0} = \pm \left( \Phi''_+(a) \sqrt{1 - b_+(a)/a} - \Phi''_- (a) \sqrt{1 - b_-(a)/a} \right) |_{a_0}
- \frac{1}{2} \left\{ \Phi'_+(a) \frac{(b_+(a)/a)'}{\sqrt{1 - b_+(a)/a}} - \Phi'_-(a) \frac{(b_-(a)/a)'}{\sqrt{1 - b_-(a)/a}} \right\} |_{a_0}.
\]

Provided \( \Phi'_+(a_0)/\sqrt{1 - b_+(a_0)/a_0} \geq \Phi'_-(a_0)/\sqrt{1 - b_- (a_0)/a_0} \), we have

\[
[4 \pi \Xi (a) a]''' |_{a_0} \leq \left\{ \Phi'''_+(a) \sqrt{1 - b_+(a)/a} - \Phi'''_- (a) \sqrt{1 - b_-(a)/a} \right\} |_{a_0}
- \frac{1}{2} \left\{ \Phi''_+(a) \frac{(b_+(a)/a)''}{\sqrt{1 - b_+(a)/a}} - \Phi''_-(a) \frac{(b_-(a)/a)''}{\sqrt{1 - b_-(a)/a}} \right\} |_{a_0}
- \frac{1}{4} \left\{ \Phi'_+(a) \frac{[(b_+(a)/a)']^2}{[1 - b_+(a)/a]^{3/2}} - \Phi'_-(a) \frac{[(b_-(a)/a)']^2}{[1 - b_-(a)/a]^{3/2}} \right\} |_{a_0}
- \frac{1}{2} \left\{ \Phi'_+(a) \frac{(b_+(a)/a)'''}{\sqrt{1 - b_+(a)/a}} - \Phi'_-(a) \frac{(b_-(a)/a)'''}{\sqrt{1 - b_-(a)/a}} \right\} |_{a_0}.
\]

If \( \Phi'_+(a_0)/\sqrt{1 - b_+(a_0)/a_0} \leq \Phi'_-(a_0)/\sqrt{1 - b_- (a_0)/a_0} \) then the direction of the inequality is reversed. Note that these last two equations are entirely vacuous in the absence of external forces, which is why they have not appeared in the literature until now.

3. Specific gravastar models

In discussing specific gravastar models one now “merely” needs to apply the general formalism described above. Up to this stage we have kept the formalism as general as possible with a view to future applications, but we shall now focus on some more specific situations.
3.1. Schwarzschild exterior, de Sitter interior

The traditional gravastar, first considered in [1] and [2], has a Schwarzschild exterior (with \( b_+ = 2M \) and \( \Phi_+ = 0 \)) and a de Sitter interior (with \( b_- = r^3/R^2 \) and \( \Phi_- = 0 \)), but with a complicated transition layer. Thin-shell Schwarzschild-de Sitter gravastars were first explicitly discussed in [10]. The parameters are chosen such that the transition layer is located at some \( 2M < a < R \). (So the transition layer is situated outside the region where the Schwarzschild event horizon would normally form, and inside the region where the de Sitter cosmological horizon would form). One normally is rather noncommittal regarding the physics of the transition region, however, for the present case, the surface stresses of the thin shell are given by

\[
\sigma = -\frac{1}{4\pi a} \left[ \sqrt{1 - \frac{2M}{a}} + \dot{a}^2 - \sqrt{1 - \frac{a^2}{R^2}} + \dot{a}^2 \right], \tag{90}
\]

\[
\mathcal{P} = \frac{1}{8\pi a} \left[ \frac{1 + \dot{a}^2 + a\ddot{a} - \frac{M}{a}}{\sqrt{1 - \frac{2M}{a}} + \dot{a}^2} - \frac{1 + \dot{a}^2 + a\ddot{a} - 2\frac{a^2}{R^2}}{\sqrt{1 - \frac{a^2}{R^2}} + \dot{a}^2} \right]. \tag{91}
\]

The external forces vanish (\( \Xi = 0 \)), as \( \Phi_\pm = 0 \), and \( \sigma > 0 \) for \( a(\tau) < (2MR^2)^{1/3} \), as can be readily verified from equation (90). To have a stable static solution at \( a_0 \) we must have:

\[
\sigma(a_0) = -\frac{1}{4\pi a_0} \left[ \sqrt{1 - \frac{2M}{a_0}} - \sqrt{1 - \frac{a_0^2}{R^2}} \right], \tag{92}
\]

\[
\mathcal{P}(a_0) = \frac{1}{8\pi a_0} \left[ \frac{1 - \frac{M}{a_0}}{\sqrt{1 - \frac{2M}{a_0}}} - \frac{1 - 2\frac{a_0^2}{R^2}}{\sqrt{1 - \frac{a_0^2}{R^2}}} \right], \tag{93}
\]

with \( 2M < a_0 < R \), a situation which has already been extensively analysed in Section 2.6. As pointed out in the general discussion, \( \sigma \) takes finite values with different signs in the endpoints of the range between \( 2M \) and \( R \), being positive for \( 2M < a_0 < (2MR^2)^{1/3} \) and negative for \( (2MR^2)^{1/3} < a_0 < R \). The surface pressure \( \mathcal{P} \) tends to \(+\infty\) when \( a_0 \) approaches \( 2M \) (\( R \)) from the right (left). It can be seen that \( \mathcal{P} \) never vanishes in this interval, and that its derivative with respect to \( a_0 \) tends to \(-\infty\) \((+\infty)\) when \( a_0 \) goes to \( 2M \) (\( R \)) from the right (left). That is, \( \mathcal{P}(a_0) \) evolves from infinitely large values, to a minimum non-vanishing value, before then again going to infinity.
Now consider the mass

\[ m_s(a_0) = -a_0 \left\{ \sqrt{1 - \frac{2M}{a_0}} - \sqrt{1 - \frac{a_0^2}{R^2}} \right\}, \]  

(94)

while

\[ m'_s(a_0) = - \left\{ \frac{1 - M/a_0}{\sqrt{1 - 2M/a_0}} - \frac{1 - 2a_0^2/R^2}{\sqrt{1 - a_0^2/R^2}} \right\}. \]  

(95)

The inequality one derives for \( m''_s(a_0) \) is now trickier since the relevant expression contains two competing terms of opposite sign. However, considering that the physical solution should have \( \sigma > 0 \) (equivalent to \( a_0 < (2MR^2)^{1/3} \)), one finds that stability requires

\[ m''_s(a_0) \geq - \frac{1}{4a_0^2} \left\{ \frac{[2M]^2}{[1 - 2M/a_0]^{3/2}} - \frac{[2a_0^3/R^2]^2}{[1 - a_0^2/R^2]^{3/2}} \right\} - \frac{3a_0/R^2}{\sqrt{1 - a_0^2/R^2}}. \]  

(96)

We can recast this as

\[ a_0 m''_s(a_0) \geq \frac{(M/a_0)^2}{[1 - 2M/a_0]^{3/2}} - \frac{(a_0/R)^2 [3 - 2 (a_0/R)^2]}{[1 - (a_0/R)^2]^{3/2}}. \]  

(97)

In figure 2 we show the surface which is produced when this inequality saturates, where the stability region are represented above this surface. It is interesting to note that the possible positions of a static thin shell were studied in reference [10]. As those solutions, \( a_0 \), were obtained by considering a particular equation of state for the matter on the shell, and \( \sigma \) and \( \mathcal{P} \) are independent of \( V'' \), then the solutions would be the same for the zero potential as for the linearized potential. The only difference between the two models is that whereas if \( V = 0 \) then the solution would be on the surface depicted in figure 2, if we consider the linearized potential then the solutions should be in the region above the surface in order to assure stability of the static solution (which is equivalent to demanding \( V''(a_0) > 0 \)).

### 3.2. Bounded excursion gravastars

On the other hand, once one obtains a static solution at \( a_0 \) by requiring a specific equation of state (for example, stiff matter on the shell \( \sigma = \mathcal{P} \)), and verifies the stability by checking that the second derivative of the mass of the thin shell is in the stability region, it is easy to obtain dynamic “stable” solutions of the ‘bounded excursion’ type by deforming the linearized potential. Thus, considering
Figure 2. Schwarzschild-de Sitter gravastar: The function $a_0 m''_s$ in the case that $V = 0$. The surface is only defined for values of $a_0 < R$ and $2M < a_0$, as expected. Models producing a function $a_0 m''_s$ above this surface would be in the stability region.

$$V(a) = \frac{\gamma^2}{2} (a - a_0)^2 - \frac{\epsilon^2}{2},$$

with $\epsilon$ sufficiently small, one can obtain the equation of motion of the shell; this is

$$a(\tau) = a_0 + \frac{\epsilon}{\gamma} \sin \left[ \gamma (\tau - \tau_0) \right].$$

(98)

Therefore, the shell expands from a minimum size with $a_1 = a_0 - \epsilon/\gamma$ to a maximum size corresponding to $a_2 = a_0 + \epsilon/\gamma$. It then contracts to $a_1$, starting a new cycle of evolution after reaching this value. Now, it can be clearly understood what we mean with $\epsilon$ sufficiently small, because this behavior makes sense for a stable gravastar only if $a_1 > 2M$ and $a_2 < R$, which implies $\epsilon < \gamma (a_0 - 2M)$ and $\epsilon < \gamma (R - a_0)$, respectively. In figure 3 we show the behavior of the potential, which is related with $\dot{a}$ through the equation of motion, as a function of $a$. This potential vanishes at $a_1$ and $a_2$, where $\dot{a} = 0$ but $\ddot{a} = \pm \epsilon \gamma$, respectively. This acceleration imposes that the shell rolls down the potential when it has reached both its minimum and maximum sizes. The shell continues evolving when its radius takes the value $a_0$ because at this point $\dot{a} \neq 0$.

Note that since

$$\frac{d\tau}{dt_{\pm}} = \frac{1 - b_{\pm}(a)/a}{\sqrt{1 - b_{\pm}(a)/a + \dot{a}^2}},$$

(99)
one would have, in general, a different equation of motion of the shell in terms of the time coordinate of the exterior geometry and the time coordinate of the interior geometry, that is $a(t_+) \neq a(t_-)$. Nevertheless, as $\tau$ can be considered as a parameter in both regions of the space, the shell radius would always be bounded by $a_1$ and $a_2$. Thus, if $a_1$ and $a_2$ can be reached at a finite $t_\pm$, then the shell would be vibrating, although with a different kind of vibration as seen using $t_+$ or $t_-$. (See, for instance, [25].)

Finally, we should comment on some features of the material on the shell. Although we are deforming a solution corresponding to a particular kind of material on the shell, that is with a given equation of state parameter relating $\sigma$ and $P$, the corresponding dynamic solution would not have a constant equation of state parameter, at least in the general case, because the surface stresses, $\sigma(a)$ and $P(a)$, have a different dependence on the trajectory (see equations (90)). Therefore, the shell would be filled by material changing its behavior during the evolution of the shell. Moreover, even if the original static solution corresponds to a material on the
shell fulfilling the energy conditions, its dynamic generalization could easily violate those conditions at some stage of the evolution of the shell. Thus, one should carefully study that this is not the case in each particular model. In figure 4 we have depicted the equation of state parameter \( w = \frac{P}{\sigma} \) as a function of time, and the energy density on the shell as a function of the pressure for a particular model obtained by deforming a stable static solution with dust matter on the shell. It can be seen that the pressure on the dynamic shell never vanishes, although the solution is obtained by deforming one with \( P(a_0) = 0 \).

![Figure 4](image)

**Figure 4.** Bounded excursion: We consider the deformation \( \epsilon = 0.01 \) of a stable static solution with dust matter on the shell, \( M = 1 \), \( a_0 = 4.178821374980832 \), \( R = 22.3607 \) and \( \gamma = 1 \). In the left figure we show the evolution of \( w \) in terms of \( \tau \). It decreases from its maximum value to its minimum value from \( \tau = \tau(a = a_1) \) to \( \tau = \tau(a = a_2) \) and then increases to its maximum value for the next cycle. Evolution of the energy density in terms of the pressure is depicted in the right figure. Consideration of more than one cycle would lead to the same graphic.

### 3.3. Close to critical: Non-extremal

Another common feature of gravastars is that the transition layer is typically taken to be “close” to horizon formation. That is \( b_\pm(a)/a \approx 1 \). In fact, it is useful to take a linear approximation

\[
b_\pm(a)/a = 1 \mp \gamma_\pm (a - R_\mp) + \mathcal{O}([a - R_\pm]^2),
\]

(100)

where \( R_- \) is where the “black hole” horizon would have formed in the exterior spacetime, \( R_+ \) is where the “cosmological” horizon would have formed in the interior spacetime. We take the transition layer to be at a position \( a \) such that \( R_- < a < R_+ \),
and $\gamma_\pm > 0$ because we are (for now) avoiding “extremal” geometries. Dismissing second order terms and considering vanishing external forces ($\Phi_\pm(r) = 0$), we have
\[
\sigma \simeq -\frac{1}{4\pi a} \left[ \sqrt{\gamma_+ (a - R_-)} + \dot{a}^2 - \sqrt{\gamma_- (R_+ - a)} + \dot{a}^2 \right],
\]
(101)
\[
P \simeq \frac{1}{8\pi a} \left[ \frac{\dot{a}^2 + a\ddot{a} + \gamma_+ (3a/2 - R_-)}{\sqrt{\gamma_+ (a - R_-)} + \dot{a}^2} - \frac{\dot{a}^2 + a\ddot{a} + \gamma_- (R_+ - 3a/2)}{\sqrt{\gamma_- (R_+ - a)} + \dot{a}^2} \right].
\]
(102)

Therefore, $\sigma(a) > 0$ for $a(\tau) < (\gamma_+ R_- + \gamma_- R_+)/\gamma_+(\gamma_+ + \gamma_-)$.

This “close to the horizon” approximation would be valid only if the trajectory of the transition layer, which can be obtained from the equation of motion (or equivalently considering some equation of state), is always in the region $R_- \lesssim a \lesssim R_+$. In order to analyze the accuracy of this approximation, we consider a stable static solution, implying
\[
\sigma(a_0) \simeq -\frac{1}{4\pi a_0} \left[ \sqrt{\gamma_+ (a_0 - R_-)} - \sqrt{\gamma_- (R_+ - a_0)} \right],
\]
(103)
\[
P(a_0) \simeq \frac{1}{8\pi a_0} \left[ \frac{\gamma_+ (3a_0/2 - R_-)}{\sqrt{\gamma_+(a_0 - R_-)}} - \frac{\gamma_- (R_+ - 3a_0/2)}{\sqrt{\gamma_- (R_+ - a_0)}} \right].
\]
(104)

Consider stiff matter on the shell, $\sigma(a_0) = P(a_0)$. Therefore, we have
\[
\frac{\gamma_+^{1/2} (3R_+ - 7a_0/2)}{\sqrt{R_+ - a_0}} \simeq \frac{\gamma_-^{1/2} (7a_0/2 - 3R_-)}{\sqrt{a_0 - R_-}}.
\]
(105)

Squaring both sides, and defining the dimensionless quantities $\alpha = a_0/R_-$, $\beta = R_-/R_+$, $\Gamma_+ = \gamma_+ R_-$, and $\Gamma_- = \gamma_- R_-$, with $\alpha > 1$ and $0 < \beta < 1$, one has
\[
\frac{49}{4} (\Gamma_+ + \Gamma_-) \alpha^3 - \frac{49}{4} \left( \Gamma_- + \frac{\Gamma_+}{\beta} \right) + 21 \left( \frac{\Gamma_-}{\beta} + \Gamma_+ \right) \alpha^2 + \frac{21}{\beta} (\Gamma_+ + \Gamma_-) + 9 \left( \frac{\Gamma_-}{\beta^2} + \Gamma_+ \right) \alpha - \frac{9}{\beta} \left( \frac{\Gamma_-}{\beta} + \Gamma_+ \right) \simeq 0.
\]
(106)

Thus, by considering the expansion of the background geometries close to where the horizon would be formed, we have reduced the problem of finding static and stable solutions (which usually involves some highly nontrivial equation) to solving a cubic, which can be done analytically.

Nevertheless, some comments are in order. In the first place, as the RHS of equation (105) is always positive, the solutions of equation (106) would correspond to solutions of our problem only if $R_+ > 7a_0/6$, i.e. $\beta < 6/7$. In the second place, one can consider that the approximation would not be accurate enough if
(a_0 - R_-)/R_- < 1 and (R_+ - a_0)/a_0 < 1 are not satisfied; thus, the solutions would be reliable only if \( \alpha < 2 \) and \( 1/4 < \beta \). In summary, we should consider \( 1/4 < \beta < 6/7 \) to solve equation (106), and give physical meaning only to solutions with \( 1 < \alpha < 2 \), if any. In fact, one can see that the results that can be obtained by using this approximation are compatible with those of a stiff matter gravastar with a Schwarzschild exterior and a de Sitter interior when \( 1 < \alpha < 2 \) (see equation (60) of reference [10] for the equation that must be solved in that case).

On the other hand, we can study the stability of the static solutions. In this case the inequality (87), for \( \sigma > 0 \), leads to

\[
 m''_s(a_0) \geq -\frac{\gamma_+^2(3a_0/4 - R_-)}{[\gamma_+(a_0 - R_-)]^{3/2}} - \frac{\gamma_-^2(R_+ - 3a_0/4)}{[\gamma_-(R_+ - a_0)]^{3/2}}. \tag{107}
\]

It is more useful to consider \( m''_s(a_0)/\gamma_+ \), which can be written in terms of the dimensionless quantities previously introduced, and is given by

\[
 \frac{m''_s(a_0)}{\gamma_+} \geq -\frac{3\alpha/4 - 1}{(\alpha - 1)^{3/2}} - \sqrt{\frac{\Gamma_-}{\Gamma_+}} \frac{1/\beta - 3\alpha/4}{(1/\beta - \alpha)^{3/2}}. \tag{108}
\]

In figure 5 we have drawn this function for the case that the inequality saturates for two particular values of \( a_0 \). Thus the region above the surface corresponds to the stability region.

![Figure 5](image_url)

**Figure 5.** Close to critical (non-extremal): The left and right figures correspond to \( a_0 = 1.1R_- \) and \( a_0 = 1.5R_- \), respectively. It can be noticed that the function is not defined for values of \( R_+ < a_0 \), since we must consider \( R_- < a_0 < R_+ \).
3.4. Close to critical: Extremal

We now consider a gravastar with a transition layer “close” to where the horizon would be expected to form, when both bulk geometries are “extremal”. This implies that the coefficient of the first order term in the expansion \((100)\) vanishes. Then, assuming that the coefficient of the second order term is not vanishing, one has

\[
\frac{b_\pm(a)}{a} = 1 - \gamma_\pm^2 (a - R_\mp)^2 + \mathcal{O}([a - R_\pm]^3),
\]

with \(R_- < a < R_+\), and \(\gamma_\pm > 0\). Note particularly that we now have a different definition for \(\gamma_\pm\).

Again considering the situation \(\Phi_\pm(r) = 0\), we can express the quantities related to the thin-shell as

\[
\sigma \approx -\frac{1}{4\pi a} \left[ \sqrt{\gamma_\pm^2 (a - R_-)^2 + \dot{a}^2} \right],
\]

(110)

\[
\mathcal{P} \approx \frac{1}{8\pi a} \left[ \frac{\dot{\gamma}_\pm^2 (a - R_-)^2 + \dot{a}^2}{\sqrt{\gamma_\pm^2 (a - R_-)^2 + \dot{a}^2}} - \frac{\dot{\gamma}_\pm^2 (a - R_-)^2 + \dot{a}^2}{\sqrt{\gamma_-^2 (R_- - a)^2 + \dot{a}^2}} \right].
\]

(111)

Thus, we recover that \(a(\tau) < \frac{(\gamma_+ R_- + \gamma_- R_+) / (\gamma_+ + \gamma_-)}{\gamma_\pm} \) for \(\sigma(a) > 0\), although \(\gamma_\pm\) would be different than in the former case analyzed. Considering a static solution, one obtains

\[
\sigma (a_0) \approx -\frac{1}{4\pi a_0} [\gamma_+ (a_0 - R_-) - \gamma_- (R_+ - a_0)],
\]

(112)

\[
\mathcal{P} (a_0) \approx \frac{1}{8\pi a_0} \left[ \frac{\gamma_- (2a_0^2 + R_-^2 - 3R_- a_0)}{a_0 - R_-} - \frac{\gamma_- (2a_0^2 + R_+^2 - 3R_+ a_0)}{R_+ - a_0} \right],
\]

(113)

where we have taken into account \(R_- < a < R_+\), when simplifying. Following a similar procedure to that considered in the non-extremal case, one can see that a stiff matter shell must fulfil the equation

\[
\frac{\gamma_- (4a_0^2 + 3R_+^2 - 7R_- a_0)}{R_+ - a_0} \approx \frac{\gamma_+ (4a_0^2 + 3R_-^2 - 7R_- a_0)}{a_0 - R_-},
\]

(114)

which, taking the same definition of \(\Gamma_\pm, \beta\) and \(\alpha\) introduced in the former case, leads to

\[
4 (\Gamma_+ + \Gamma_-) \alpha^3 - 4 \left( \Gamma_- + \frac{\Gamma_+}{\beta} \right) + 7 \left( \frac{\Gamma_-}{\beta} + \Gamma_+ \right) \alpha^2 + \left[ \frac{7}{\beta} (\Gamma_+ + \Gamma_-) + 3 \left( \frac{\Gamma_-}{\beta^2} + \Gamma_+ \right) \right] \alpha - \frac{3}{\beta} \left( \frac{\Gamma_-}{\beta} + \Gamma_+ \right) \approx 0.
\]

(115)
This is again a cubic equation. However, whereas for non-extremal geometries we have obtained equation (106) by squaring equation (105), in this case that step was not necessary. Thus, all solutions of equation (115) are also solutions of equation (114). Therefore, noticing the values of $\beta$ and $\alpha$ for which the approximation would be sufficiently accurate, we should solve equation (115) for $1/4 < \beta < 1$ and then consider only the solutions with $1 < \alpha < 2$.

A special characteristic of these static solutions can be seen when studying their stability. Considering $\sigma > 0$, the inequality (87) can be simplified to obtain

$$\frac{m_s''}{\gamma_+} \geq -2 \left( 1 + \frac{\gamma_-}{\gamma_+} \right),$$

which is independent of $a_0$. As the position $a_0$ can be obtained by considering a particular equation of state for the material on the shell, this implies that the stability of the solution is independent of that equation of state. In figure 6 we show the behavior of this function.

![Figure 6](image)

*Figure 6.* Close to critical (extremal): We show the function $m_s''/\gamma_+$ in the case that $V = 0$. The stability region would be above this curve, independent of the value of $a_0$.

Finally, some comments about the case with one non-extremal and one extremal background geometries are in order. In this case, for a stiff matter gravastar, one would obtain an equation with the RHS of equation (105) and the LHS of

\[\int_0^\infty \frac{dF}{d\gamma_+} \leq \int_0^\infty \frac{dF}{d\gamma_-}.\]

\[\int_0^\infty \frac{dF}{dP} \leq 0\]

\[\sigma (not \ on \ P), \ if \ \sigma \neq 0.\]
equation (114), or vice versa. Thus, after squaring, the analogue of equation (106) or (115) would be a quintic. Therefore, the consideration of the close-to-horizon approximation in this situation would generally not significantly simplify the problem of obtaining static solutions, because we are only reducing the polynomial equation by one degree (see also reference [10]).

3.5. Charged dilatonic exterior, de Sitter interior

An interesting solution to consider is that of an interior de Sitter spacetime, (where the metric functions are given by $b_-(r) = r^3/R^2$ and $\Phi_-(r) = 0$), while the exterior spacetime is given by the dilaton black hole solution, which corresponds to an electric monopole. The latter de Sitter-charged dilatonic gravastar is considered here for the first time. In Schwarzschild coordinates, this exterior solution is described by [26, 27, 28],

$$ds^2_+ = - \left(1 - \frac{2M}{\beta + \sqrt{r^2 + \beta^2}}\right) dt^2 + \left(1 - \frac{2M}{\beta + \sqrt{r^2 + \beta^2}}\right)^{-1} \frac{r^2}{r^2 + \beta^2} dr^2$$

$$+ r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2).$$

(117)

Here we have dropped the subscripts + for notational convenience. The Lagrangian that describes this combined gravitational-electromagnetism-dilaton system is given by [26, 27, 28]

$$\mathcal{L} = \sqrt{-g} \left\{ -R/8\pi + 2(\nabla \psi)^2 + e^{-2\psi} F^2 / 4\pi \right\}.$$  

(118)

(Note that the first charged dilatonic solutions, including black hole solutions, were considered by Bronnikov et al. [29]). The non-zero component of the electromagnetic tensor is given by $F_{t\hat{r}} = Q/r^2$, and the dilaton field is given by $e^{2\psi} = 1 - Q^2/M(\beta + \sqrt{r^2 + \beta^2})$. The parameter $\beta$ is defined by $\beta \equiv Q^2/2M$.

In terms of the formalism developed in this paper, the metric functions of the exterior spacetime are

$$b_+(r) = r \left[ 1 - \left(1 + \frac{\beta^2}{r^2}\right) \left(1 - \frac{2M}{\beta + \sqrt{r^2 + \beta^2}}\right) \right],$$

(119)

$$\Phi_+(r) = -\frac{1}{2} \ln \left(1 + \frac{\beta^2}{r^2}\right).$$

(120)
An event horizon exists at \( r_b = 2M \sqrt{1 - \beta/M} \). Note that \( \Phi_+ (r) \) is given by

\[
\Phi'_+ (r) = \frac{\beta^2}{a(\beta^2 + a^2)},
\]

which is positive (and so satisfies the NEC) throughout the spacetime. Thus, taking into account that \( \Phi_- = 0 \), we verify that the stability condition imposed by the presence of the flux term is governed by inequality (89).

The thin shell is placed in the region \( r_b < a < R \), so that \( b_+(a) \geq b_-(a) \) (and consequently \( \sigma > 0 \)) is satisfied. Consequently the stability regions are governed by inequality (87). The expressions for inequalities (87) and (89) are extremely lengthy, so that rather than write them down explicitly, we will analyse the qualitative behaviour of the stability regions depicted in figures 7–9.

Consider as a first example the case for \( \beta/M = 1/2 \), so that the stability regions, governed by the inequalities (87) and (89), are depicted in figure 7. The left plot describes the stability regions, governed by (87), which lie above the surface. The right plot describes the stability regions depicted below the surface, provided by inequality (89). As a second case, consider the value \( \beta/M = 3/4 \), depicted in figure 8. We verify that the qualitative results are similar to the specific case of \( \beta/M = 1/2 \), considered above. That is, the stability condition dictated by both the inequalities (87) show that the stability regions decrease significantly as one approaches the black hole event horizon. The final stability regions are depicted in figure 9 and are shown in between the shaded regions. Note that the respective stability regions increase for increasing values of \( \beta/M \).

4. Summary and Discussion

In this work, we have developed an extremely general and robust framework leading to the linearized stability analysis of dynamical spherically symmetric thin-shell gravastars, a framework that can quickly be adapted to wide classes of generic thin-shells. We have built on a companion paper, which analyzed thin-shell traversable wormholes [15]. We emphasize that mathematically there are a few strategic sign flips, which physically implies significant changes in the analysis. An important difference is the possibility of a vanishing surface energy density at certain specific shell radii, for the generic thin-shell gravastars considered in this work. Due to the key sign flips, ultimately arising from the definition of the normals on the junction
interface, the surface energy density is always negative for the thin-shell traversable wormholes considered in [15]. We have also explored static gravastar configurations to some extent, and considered the generic qualitative behavior of the surface stresses. Relative to the conservation law of the surface stresses, we have analysed in great detail the most general case, widely ignored in the literature, namely, the presence of a flux term, corresponding to the net discontinuity in the (bulk) momentum flux...
Figure 9. Dilaton–de Sitter gravastar: The final stability regions for a thin shell dilaton gravastar for $\beta/M = 0.5$ and $\beta/M = 0.75$ depicted in the left and right plots, respectively. The stability regions are represented above the green surface, which is given by inequality (87); and below the blue surface, given by inequality (89). Thus the final stability regions are given in between both surfaces. Note that the stability regions increase for increasing values of $\beta/M$.

which impinges on the shell. Physically, the latter flux term can be interpreted as the work done by external forces on the thin shell.

In the context of the linearized stability analysis we have reversed the logic flow typically considered in the literature and introduced a novel approach. More specifically, we have considered the surface mass as a function of the potential, so that specifying the latter tells us how much surface mass we need to put on the transition layer. This procedure implicitly makes demands on the equation of state of the matter residing on the transition layer and demonstrates in full generality that the stability of the gravastar is equivalent to choosing suitable properties for the material residing on the thin shell. We have applied the latter stability formalism to a number of specific cases, namely, to the traditional gravastar picture, where the transition layer separates an interior de Sitter space and an exterior Schwarzschild geometry; bounded excursion; close-to-horizon models, in both the extremal and non-extremal regimes; and finally, to specific example of an interior de Sitter spacetime matched to a charged dilatonic exterior. This latter case is particularly interesting, as it involves the flux term.

In conclusion, by considering the matching of two generic static spherically symmetric spacetimes using the cut-and-paste procedure, we have analyzed the
stability of thin-shell gravastars. The analysis provides a general and unified framework for simultaneously addressing a large number of gravastar models scattered throughout the literature. As such we hope it will serve to bring some cohesion and focus to what is otherwise a rather disorganized and disparate collection of results.

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