Horizon Branes and Chiral Strings

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String Theory is known to be one possible model to unify all the known forces of the Nature by a universal concept, that of strings. All fundamental particles, including gravitons with a given energy, are supposed to be oscillating states of tensive open or closed strings. The string concept introduces some non-locality in the gravitational interactions, making the self-interacting Feynman diagrams finite order by order. The discovery of D-branes has created a new situation in string theory, as those objects are non-perturbative solutions dual to solitonic charged objects describing black p-branes, solutions of the supergravity equations. Since then many occurrences of dualities have been found in String Theory where all five different type of vacua are related by M-theory. On the other hand, a well defined quantum gravity theory should statistically count the number of quantum states giving rise to the Bekenstein-Hawking entropy of any black hole horizon. A partial success has been achieved in the particular case when the near horizon geometry is that of Anti-de Sitter, giving rise to the general AdS/CFT holographic principle.

In this thesis I present a new type of brane - H-brane - where the role of time in String Theory is considered as a primary concept in the search for still unknown black hole physics. We start to study the physics of charged open strings immersed in a critical electric field. Using the lightcone gauge, we find that open strings are naturally described by a worldsheet with null boundaries that define the H-brane. Using the basic tools of boundary conformal field theory, we describe the H-branes both in the open and closed string channels and examine how they fit naturally in the known D-brane moduli space. In particular we compute their Ishibashi states and quantize the system using the first order formalism. We find that the geometry associated to target null coordinates is non-commutative. This is a possible way to solve the information loss paradox.

We conjecture that any quantum horizon - black hole, cosmological, etc. - is phenomenologically described in time-dependent String Theory by a chiral and non-normalized squeezed state.
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Chapter 1

Introduction

The discovery of $D_p$-branes by Polchinski provided a non-perturbative string theory description of the supergravity black $p$-brane solutions carrying RR charges. Since then many dualities in String Theory have been found where non-perturbative effects become an accessible problem after a strong-weak coupling duality. String Theory has taught us that $D$-branes provide a useful tool to obtain a microscopic (dual) picture of the Bekenstein-Hawking entropy of an extremal charged black hole. Near the horizon, such extremal black holes are solutions of supergravity on $AdS$ which, by the holographic principle, are dual to the $\mathcal{N} = 4$ Yang-Mills theory at the fixed point. One counter example where the use of duality does not permit us to solve the problem is the relative motion of a $D$-brane which is dual to a free open string, with endpoints carrying an electric charge $e$, immersed in a constant electric field. The unknown behavior of the system when the electric field approaches the critical limit $E \to E_{\text{crit}} = (2\pi\alpha')^{-1}$ is dual to the unknown behavior of a D-brane moving with velocity close to the speed of light $V \to V_{\text{crit}} = 1$. What is missing in the known $D$-brane moduli space is some kind of an infinitely boosted brane solution.

In [38] the existence of a new type of brane in the time dependent string theory, namely the nullbranes, was postulated. The nullbranes were introduced in the literature to provide a supplementary tool to study unknown properties of black holes for more general situations than that of the extremal charged regime. In the closed string channel, our nullbranes are described by chiral and non-normalized squeezed Ishibashi states, the properties that are believed to be phenomenologically associated
to quantum event horizons - black holes, cosmological (de-Sitter), etc. For this reason our nullbrane was renamed as a **Horizon-brane**, or H-brane for short \(^1\).

This thesis is organized as follows. In Sect. 2.1 we start to review some mathematical techniques of quantization of \(d\)-dimensional systems, as applied to \(d = 2\) field theory at a renormalization group fixed point which in Sect. 2.2 and Sect. 2.3 we see to provide the worldsheet description of a closed or open string respectively. In Sect. 2.4 we focus on the case of boundary conformal field theory (BCFT), useful for the treatment of \(D\)-branes in flat backgrounds. Our aim is to study the \(D\)-brane behavior under an infinite boost. In Sect. 2.5 we start to calculate the correlation functions for a free string perturbed by an external electromagnetic field and in Sect. 2.6 we review the general mathematical treatment of boundary perturbations by self-dual fields, where the Dissipative Hofstadter Model is one particular example.

In Sect. 3.1 we describe how boosted \(D\)-branes in the Minkowskian spacetime are related by \(T\)-duality to the charged open strings immersed in a constant electric field and explain why there is an upper limit for the electric field at which the theory breaks down.

From the results in the previous sections on the two-point correlation functions, we consider two distinct cases of charged open strings, one with a magnetic background (Sect. 3.2) and the other one with an electric background (Sect. 3.3). We show how the magnetic case leads to non-commutativity of space coordinates and gives some hints about the spacetime geometry in the electric case. Sect. 3.4 analyzes the tensionless strings for which all points travel at the speed of light. Such strings are described by the Schild action whose quantization leads to non-commutativity of the spacetime coordinates of string endpoints.

In Sect. 4.1 we give the technical definition of an \(H\)-brane in terms of boundary conditions and see how they fit naturally in a Bosonic String Theory in the lightcone gauge. We proceed to their quantization in Sect. 4.2 using the first order formalism, where there is no room for guesswork. In Sect. 4.3 we give some hints on the supersymmetric extension to our \(H\)-branes. Sect. 4.4 shows that there is enough structure in the null directions so that we may neglect one chiral sector of the closed string

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\(^1\)Let us note that nevertheless there is some information about null structure in the letter \(H\) too: Russian \(H = \) Latin \(N\).
without introducing singularities into the system. In Sect. 4.5 we show that chiral closed strings are coupled naturally to $H$-branes and calculate the Ishibashi states by the first order formalism.

In Sect. 5 we give some hints on our motivations to describe quantum black hole horizons by H-branes.

Conclusions give a brief summary of what was achieved in the thesis and discuss open questions to be addressed in the future. More technical calculations referred to in the main text have been collected in Appendices.
Chapter 2

D-branes and Conformal Field Theory

2.1 The first-order formalism

To find the symplectic structure of the phase space of a physical system and proceed with the Dirac quantization, we use the first order formalism where there is no room for guesswork [27]. The basic object is a $d$-form $\alpha$ on a bundle over the $d$-dimensional spacetime $\Sigma$ through which the first-order action is expressed. In our case it is $d = 2$ worldsheet with the string target coordinates considered as fields $\Phi$ depending on two parameters $\tau$ and $\sigma$. The original worldsheet action is of the form

$$S = \int_{\Sigma} L(x^a, \phi^\mu, \partial_a \phi^\mu)$$

(2.1.1)

where $\mu$ run over the target coordinates $0, ..., D - 1$ and $a$ over the two-dimensional worldsheet space $\Sigma$ that might or might not have a boundary $\partial \Sigma$ depending on whether the string is open or closed respectively. In general, such an action leads to variational equations with second order derivatives. To obtain the first-order formalism, one defines

$$\xi^\mu_a = \partial_a \phi^\mu, \quad \Pi^a_\mu = \frac{\partial L}{\partial \xi^\mu_a}$$

(2.1.2)

and introduces the 2-form

$$\alpha = \mathcal{L}dx^0 \wedge dx^1 + \Pi^0_\mu (d\phi^\mu - \xi^\mu_0 dx^0) \wedge dx^1 + \Pi^1_\mu dx^0 \wedge (d\phi^\mu - \xi^\mu_1 dx^1)$$

(2.1.3)
on the space with coordinates \((x^a, \phi^\mu, \xi_a^\mu)\) which forms a bundle \(P\) over \(\Sigma\). The fields \(\Phi = (\phi^\mu, \xi_a^\mu)\) are geometrically interpreted as sections of \(P\). The first-order action is given by

\[
S' = \int_\Sigma \Phi^* \alpha
\]

and it coincides with the original one if \(\xi_a^\mu = \partial_\alpha \phi^\mu\). The variational equations \(\delta S'(\Phi) = 0\) take the geometric form

\[
\Phi^* (i(\delta \Phi) d\alpha) = 0 \quad \text{on} \quad \Sigma
\]
\[
\Phi^* (i(\delta \Phi) \alpha) = 0 \quad \text{along} \quad \partial \Sigma
\]

for any vector field \(\delta \Phi\) tangent to \(P\) and respecting the boundary condition when present (such vector fields describe infinitesimal variations of \(\Phi\)). The bulk equations reduce to the standard Euler-Lagrange equations plus the conditions \(\xi_a^\mu = \partial_\alpha \phi^\mu\). The boundary equations may additionally restrict the behavior of the solutions on the boundary. The space of classical solutions carries a closed 2-form

\[
\Omega(\delta_1 \Phi, \delta_2 \Phi) = \int_{\Sigma_t} \Phi^* (i(\delta_2 \Phi) i(\delta_1 \Phi) d\alpha) - \int_{\partial \Sigma_t} \Phi^* (i(\delta_2 \Phi) i(\delta_1 \Phi) \alpha)
\]

where \(\delta_i \Phi\) are vectors tangent to the space of the classical solutions and \(\Sigma_t\) is a slice of the worldsheet, usually the line of constant time where the Cauchy data may be defined. If the form \(\Omega\) is non-degenerate, then it provides the space of classical solutions with a symplectic structure. Otherwise one should identify the solutions that are connected by one-parameter families tangent to the degeneration directions. \(\Omega\) descends then to a symplectic form on the quotient space \(\tilde{P}\) that forms the phase space of the theory. The symplectic structure leads to the Hamiltonian vector fields \(X_\mathcal{F}\) corresponding to functions \(\mathcal{F}\) of the phase space, such that \(d\mathcal{F} = i_{X_\mathcal{F}} \Omega\) and the Poisson bracket \(\{\mathcal{F}, \mathcal{F}'\} = X_\mathcal{F}(\mathcal{F}')\).

### 2.2 Closed strings and Conformal Field Theory

There is a well known relation between closed strings in curved spacetimes \(M\) and the two-dimensional sigma model defined on closed 2d worldsheets \(\Sigma\) with the spacetime \(M\) as the target. Let \(X^\mu\) denote the coordinates of \(M\) and \(G_{\mu \nu}\) and \(R\)
the metric tensor and the Ricci scalar of $M$, respectively. We also turn on the Kalb-Ramond $B_{\mu\nu}$ field tensor. The closed string coupling constant $g_c$ is given by the dilaton field as $g_c \propto e^\Phi$. The closed string immersed in a curved spacetime is described by specifying the dependence of the coordinates $X^\mu$ on the worldsheet point. The string tension $T$ is given by the Regge slope as $T = 1/(2\pi\alpha')$. From the non-linear sigma model with the action

$$S_\sigma = \frac{1}{4\pi\alpha'} \int d\tau d\sigma \gamma^{1/2} \left[ (\gamma^{ab} G_{\mu\nu}(X) + i\epsilon^{ab} B_{\mu\nu}(X)) \partial_a X^\mu \partial_b X^\nu + \alpha' R \Phi(X) \right]$$

one calculates the trace of the energy-momentum tensor and from that the renormalization group $\beta$-functions describing the scale dependence of $G_{\mu\nu}$, $B_{\mu\nu}$ and $\Phi$. The string equations of motion are defined by imposing the fixed point condition on the sigma model ($\beta = 0$), as required by the Weyl invariance, i.e. closed strings are described by 2d Conformal Field Theory (CFT) on the worldsheet. Performing perturbation theory by expanding the fields around flat solutions, we may explore the low-energy regime for which the radius of curvature is much larger then the characteristic string length scale $\alpha'^{1/2}$. In the target space, the equations of motion are described by the low-energy action of an effective field theory, as we have ignored the internal structure of the string.

On the other hand, we can associate order by order perturbative terms in the sigma model to vertex operators in the target space. The scattering of $n$ particles with each of them carrying momentum $k_i$ is given by the $S$-matrix with entries

$$S_{j_1,...,j_n}(k_1,...,k_n) = \sum \int \frac{DX}{Vol_{diff} \times Weyl} \frac{D\gamma}{\gamma} e^{-S_\sigma - \lambda \chi} \prod_{i=1}^n \int V_i(k_i; \tau, \sigma) \gamma^{1/2}(\tau, \sigma) d\tau d\sigma$$

where $V_i$ is the vertex operator inserted at position $(\tau, \sigma)$ on the worldsheet and the sum is over all compact 2d topologies. They are characterized by the Euler number $\chi$ and the measure is normalized by the volume of the diffeomorphism and Weyl symmetry groups for each compact worldsheet. $S_\sigma$ is the action of the sigma model for a flat spacetime. We obtain the vertex operators by, for example, expanding about the flat spacetime, $G_{\mu\nu} = \eta_{\mu\nu} + \chi_{\mu\nu}(X)$, in (2.2.7), substituting into the Polyakov path-integral and comparing order by order with (2.2.8). The result is that a curved spacetime is in fact a coherent background of gravitons whose vertex operators are of
the form

$$V_G \propto -4\pi g_c \partial_a X^\mu \partial_b X^\nu e^{ik \cdot X} \gamma^{ab} X_{\mu\nu}$$  \hspace{1cm} (2.2.9)$$

The CFT worldsheet description of string theory comes as one of the fundamental ideas of string theory to treat spacetime as a derived concept rather than as part of the input data \[46\].

To import the known mathematical techniques of CFT from statistical physics, let us restrict ourselves to a closed superstring model in a Minkowski spacetime with vanishing Kalb-Ramond and dilaton fields with the action

$$S = \frac{1}{4\pi\alpha'} \int d\tau d\sigma \gamma^{ab} \partial_a X^\mu \partial_b X^\mu + \frac{i}{2\pi} \int d\tau d\sigma (\Psi^\mu \partial_- \Psi_{\mu} + \bar{\Psi}^\mu \partial_+ \bar{\Psi}_{\mu})$$  \hspace{1cm} (2.2.10)$$

with $\partial_\pm = (\partial_\tau \pm \partial_\sigma)/2$ and $\Psi^\mu$ are 2d Majorana fields. The general solution of the equations of motion is given by splitting the fields into left- and right-chiral movers

$$X^\mu(\tau, \sigma) = X^\mu(x^+) + \tilde{X}^\mu(x^-) \hspace{1cm} \Psi^\mu(\tau, \sigma) = \begin{pmatrix} \Psi^\mu(x^-) \\ \bar{\Psi}^\mu(x^+) \end{pmatrix}$$  \hspace{1cm} (2.2.11)$$

with $x^\pm = \tau \pm \sigma$. The action must be invariant under periodicity $\sigma \sim \sigma + 2\pi$ which allows us to impose the conditions

$$X^\mu(\tau, \sigma + 2\pi) = X^\mu(\tau, \sigma)$$
$$\Psi^\mu(\tau, \sigma + 2\pi) = \exp(2\pi i\nu) \Psi^\mu(\tau, \sigma)$$
$$\bar{\Psi}^\mu(\tau, \sigma + 2\pi) = \exp(-2\pi i\tilde{\nu}) \bar{\Psi}(\tau, \sigma)$$  \hspace{1cm} (2.2.12)$$

where $\nu$ and $\tilde{\nu}$ can take the values 0 or $1/2$, depending on whether we choose the Ramond (R) or the Neveu-Schwarz (NS) sector, respectively. Note that as we have left- and right-chiral sectors, there are four $(\nu, \tilde{\nu})$ possible types of closed superstrings.

The field Fourier expansions respecting the above periodicity conditions are

$$X^\mu(\tau, \sigma) = \hat{q}^\mu + \hat{\alpha}_0^\mu x^- + \hat{\alpha}_0^\mu x^+ + i\alpha'^{1/2} \frac{\hat{\alpha}_n^\mu}{n} e^{-inx^-} + \frac{\hat{\alpha}_n^\mu}{n} e^{-inx^+}$$
$$\Psi^\mu(\tau, \sigma) = \sum_{r \in \mathbb{Z} + \nu} \psi^\mu_r e^{-irx^-}, \hspace{1cm} \bar{\Psi}^\mu(\tau, \sigma) = \sum_{r \in \mathbb{Z} + \tilde{\nu}} \bar{\psi}^\mu_r e^{-irx^+}$$  \hspace{1cm} (2.2.13)$$

with mode-oscillator elements whose Poisson brackets may be canonically quantized.
From the energy momentum tensor

\[ T_{++} = \frac{1}{\alpha'} \partial_+ X^\mu \partial_+ X_\mu + \frac{i}{2} \Psi^\mu \partial_+ \Psi_\mu \]
\[ T_{--} = \frac{1}{\alpha'} \partial_- \bar{X}^\mu \partial_- \bar{X}_\mu + \frac{i}{2} \bar{\Psi}^\mu \partial_- \bar{\Psi}_\mu \] (2.2.14)

and the supercurrents

\[ J_+ = \Psi^\mu \partial_+ X_\mu , \quad J_- = \bar{\Psi}^\mu \partial_- \bar{X}_\mu \] (2.2.15)

we define their Fourier components by the following expressions

\[ L_m = \frac{1}{2\pi} \int_0^{2\pi} d\sigma e^{imx^-} T_{--} , \quad \bar{L}_m = \frac{1}{2\pi} \int_0^{2\pi} d\sigma e^{imx^+} T_{++} \]
\[ G_r = \frac{1}{2\pi} \int_0^{2\pi} d\sigma e^{irx^-} J_- , \quad \bar{G}_r = \frac{1}{2\pi} \int_0^{2\pi} d\sigma e^{irx^+} J_+ \] (2.2.16)

for \( n \neq 0 \) and \( r \in \mathbb{Z} + (\nu, \bar{\nu}) \) as appropriate.

The canonical commutation relations for the holomorphic bosonic coordinates give the relations on the mode oscillators

\[ [\alpha^\mu_m, \alpha^\nu_n] = m\delta_{m+n,0} \eta^{\mu\nu} \quad [\hat{x}^\mu, \hat{p}^\nu] = i\eta^{\mu\nu} . \] (2.2.17)

For the fermionic fields we obtain the canonical anti-commutation relations

\[ \{ \psi^\mu_r, \psi^\nu_s \} = \eta^{\mu\nu} \delta_{r+s,0} \] (2.2.18)

with similar expressions on the anti-holomorphic fields. The Laurent modes for the energy-momentum tensor and for the supercurrent are given respectively by the expressions

\[ L_m = \frac{1}{2} \sum_{n \in \mathbb{Z}} : \alpha^\mu_{m-n} \alpha^\mu_m : + \frac{1}{4} \sum_{r \in \mathbb{Z}+\nu} (2r - m) : \psi^\mu_{m-r} \psi^\mu_r : + a\delta_{m,0} \]
\[ G_r = \sum_{n \in \mathbb{Z}} \alpha^\mu_n \psi^\nu_{m-r-n} \] (2.2.19)

where \( : : \) denotes creation-annihilation normal ordering and \( a \) the normal ordering constant with the value depending on whether we are in the R-sector \((a = 1/2)\) or the NS-sector \((a = 0)\). The super Virasoro algebra follows

\[ [L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12} m(m^2 - m) \delta_{m+n,0} \]
\[ \{ G_r, G_s \} = 2L_{r+s} + \frac{c}{12} (4r^2 - 1) \delta_{r+s,0} \]
\[ [L_m, G_r] = \frac{m - 2r}{2} G_{m+r} \] (2.2.20)
with similar expressions for the antiholomorphic sector. The conditions that a physical state associated to a closed string must obey are

\[ L_m^{(C)}|\Psi_{\text{phys}}\rangle = 0 \quad m > 0 \tag{2.2.21} \]
\[ L_0^{(C)}|\Psi_{\text{phys}}\rangle = (1/2 - a)|\Psi_{\text{phys}}\rangle \]
\[ G_r^{(C)}|\Psi_{\text{phys}}\rangle = 0 \quad r \geq 0 \]

with the same conditions involving the anti-holomorphic sector, e.g. \( \bar{L}_m^{(C)}|\Psi_{\text{phys}}\rangle = 0 \quad \forall m > 0. \)

### 2.3 Open strings and Boundary CFT

We have seen in the last section how closed worldsheet diagrams are associated to closed string vacuum amplitudes. We shall now consider the diagrams with worldsheets with boundary associated to open string. The simplest case is obtained by restricting to the strip diagram with \(-\infty < \tau < \infty\) and \(0 < \sigma < \pi\). By the map \((z, \bar{z}) = (e^{i\sigma - \tau}, e^{-i\sigma - \tau})\) the strip is mapped to the upper half complex plane \(\mathcal{H}_0^+ = \{z | \Im(z) \geq 0\}\) with boundaries \(\sigma = 0, \pi\) mapped to the real line \(z = \bar{z}\). The holomorphic and anti-holomorphic sectors are reflected at the boundary, with oscillator modes related by a gluing map \(\Omega\)

\[ \alpha_n^\mu + \Omega \bar{\alpha}_n^\mu = 0 \]
\[ \psi_r^{\mu} + \Omega \bar{\psi}_r^{\mu} = 0 \tag{2.3.22} \]

at \(z = \bar{z}\). Note that a creation oscillator mode is coupled to an annihilation mode. The condition for no energy flow crossing the boundary is given by

\[ T(z) = \bar{T}(\bar{z}) \quad J(z) = \bar{J}(\bar{z}) \tag{2.3.23} \]

and it should be universal, i.e., it must not depend on the gluing map\(^1\). Even if far away from the boundary the theory behaves as a usual CFT, the above super-Virasoro fields defined only at the upper half-plane, are not sufficient to give two commuting

\(^1\text{We will see later that in fact we can relax such condition after we choose different type of worldsheet boundaries in time-dependent string theory}\)
super-Virasoro algebras. Nevertheless we can construct one chiral algebra after we define the open super Virasoro operators by the method of images

\[
L_n^O := \frac{1}{2\pi i} \int z^{n+1} T(z) dz - \frac{1}{2\pi i} \int \bar{z}^{n+1} \bar{T}(\bar{z}) d\bar{z}
\]
\[
G_r^O := \frac{1}{2\pi i} \int z^{n+1/2} J(z) dz - \frac{1}{2\pi i} \int \bar{z}^{n+1/2} \bar{J}(\bar{z}) d\bar{z}
\]

where the integrals are over a semi-circle in \( \mathcal{H}_0^+ \) and the label \( (O) \) specifies that we are in the open channel. This means that the coupled holomorphic and anti-holomorphic sectors of the boundary CFT defined in the upper half-plane are replaced by a holomorphic sector of a chiral CFT defined on the whole plane. The physical states must obey the conditions

\[
L^{(O)}_m |\Psi_{\text{phys}}\rangle = 0 \quad m > 0
\]
\[
L^{(O)}_0 |\Psi_{\text{phys}}\rangle = (1/2 - a)|\Psi_{\text{phys}}\rangle
\]
\[
G^{(O)}_r |\Psi_{\text{phys}}\rangle = 0 \quad r \geq 0
\]

In the target space, BCFT’s are associated to the scattering of open string whose endpoints must obey some boundary condition consistent with the no-flow of energy requirement. The boundaries are then mapped to a defect in spacetime. Open string perturbative analysis will give the formulation of D-branes, a local perturbation on the background geometry.

### 2.3.1 The Cardy program

It is possible to relate the open string boundary conditions to closed strings by considering a one-loop diagram obtained by the periodic identification of the worldsheet time \( \tau \sim \tau + 2\pi T \). By worldsheet duality, we may swap the roles of time and space and the diagram becomes a closed string tree-level diagram with 2\( \pi \)-periodic space coordinate and boundaries at \( t_\alpha = 0 \) and \( t_\beta = \pi/T \). This is the first step of the Cardy program, that allows us to relate by worldsheet duality the partition function on the annulus with \( \alpha \)- and \( \beta \)-boundary conditions to the closed string propagating on the cylinder from an initial boundary state to a final one:

\[
Z_{\alpha\beta}(q) := Tr_{\mathcal{H}_0} q^{H^{(O)}} = \langle \alpha | (q^{1/2})^{H^{(C)}} | \beta \rangle
\]
where the trace is taken over the open channel space of states with boundary conditions \( \alpha \) and \( \beta \), associated to the boundary states \( |\alpha\rangle \) and \( |\beta\rangle \) in the closed channel space of states. The hamiltonians are given by the zero-mode Virasoro operators
\[
H^{(O)} = L_0^{(O)} - c/24 \quad H^{(C)} = L_0^{(C)} + \bar{L}_0^{(C)} - c/12
\]  
(2.3.27)
of the open channel and closed channel respectively. Finally, the parameters \( q = e^{2\pi i T} \) and \( \tilde{q} = e^{-2\pi i/T} \) are related by euclidian worldsheet duality \( iT \to (iT)^{-1} \).

The closed channel space of states decomposes into a sum of products of left- and right-chiral sectors
\[
\mathcal{H} = \bigoplus_{\alpha} \mathcal{H}_{\alpha} \otimes \tilde{\mathcal{H}}_{\alpha}
\]  
(2.3.28)
where each chiral space \( \mathcal{H}_{\alpha} \) carries an irreducible representation of the chiral algebra: the left-moving (super-)Virasoro algebra or its extension and similarly for the right-movers (we restrict ourselves to diagonal CFT’s). There is a single vacuum sector \( \mathcal{H}_{\epsilon} \otimes \tilde{\mathcal{H}}_{\epsilon} \) with the vacuum state
\[
|\Psi_{\text{vacuum}}\rangle := |0\rangle|\tilde{0}\rangle.
\]  
(2.3.29)
The operator-state correspondence associates to a primary state \( \alpha \) in each chiral sector a primary field \( \phi_{\alpha} \). The fields obey the fusion algebra
\[
\phi_{\alpha} \star \phi_{\beta} = \sum_i N^{\gamma}_{\alpha\beta} \phi_{\gamma}
\]  
(2.3.30)
where \( N^{\gamma}_{\alpha\beta} \) are the entries of the fusion matrices that can also be seen as the number of copies of the representation labeled by \( \gamma \) occurring in the open string spectrum
\[
Z_{\alpha\beta} = \sum_i N^{\gamma}_{\alpha\beta} \chi_{\gamma}(q)
\]  
(2.3.31)
with \( \chi_{\alpha}(q) \) the character of the representation of the chiral algebra in \( \mathcal{H}_{\alpha} \)
\[
\chi_{\alpha}(q) = \text{Tr}_{\mathcal{H}_{\alpha}} q^{L_0^{(O)} - \frac{c}{24}}
\]  
(2.3.32)
Under worldsheet duality \( q \to \tilde{q} \), they transform under the modular \( S \)-matrix
\[
\chi_{\alpha}(\tilde{q}) = \sum_{\beta} S_{\alpha\beta} \chi_{\beta}(q)
\]  
(2.3.33)
with
\[ SS^* = 1, \quad S = S^t, \quad S^2 = C \] (2.3.34)
where \( C \) is the charge conjugation matrix. The second step of Cardy was to express the boundary states as linear combinations of the so called Ishibashi states
\[ |\alpha\rangle = \sum_{\beta} S_{\alpha\beta} \sqrt{S_{0\beta}} |D_{\beta}\rangle \] (2.3.35)
in the closed channel space of states. The Ishibashi states were defined in \[32\] has the unique solution in \( \mathcal{H} \) (up to a normalization factor) of the equation
\[ (J_{a}^{n} + \bar{J}_{-a}^{-n})|D_{\beta}\rangle = 0 \] (2.3.36)
which impose that \( |D_{\beta}\rangle \) does not break the current algebra symmetry. The solution is given by the (formal) expression
\[ |D_{\beta}\rangle = \sum_{\{\beta_{n}\}} |\phi_{n}^{\beta}\rangle \otimes |\bar{\phi}_{n}^{\beta}\rangle \] (2.3.37)
where \( \{\phi_{n}^{\beta}\} \) is a complete orthonormal basis of \( \mathcal{H}_{\beta} \). They obey the condition
\[ (L_{m}^{(C)} - \bar{L}_{-m}^{(C)})|D_{\beta}\rangle = 0 \quad \forall m \] (2.3.38)
The relation (2.3.26) is proven by the use of Verlinde formula
\[ N^{\gamma}_{\alpha\beta} = \sum_{i} S_{\alpha i} S_{\beta i} S_{\gamma i} \] (2.3.39)

It is also interesting to write the behaviour of the bulk fields \( \phi_{\alpha} \) of conformal weight \( h \) when we approach the boundary \( z = \bar{z} \) corresponding to the boundary state \( \alpha \). It is given by the bulk-boundary OPE
\[ \phi_{\alpha}(z)\phi_{\alpha}(\bar{z}) = \sum_{i} (z - \bar{z})^{h_{i} - 2h} \phi_{i}^{\alpha A_{\alpha}}(x) + ... \] (2.3.40)
where \( x = (z + \bar{z})/2 \) and \( \phi_{i}^{\beta \alpha}(x) \) denotes boundary operators of conformal weight \( h_{i} \) changing the boundary condition \( \beta \) to \( \gamma \) (label A takes care of the multiplicity).
2.4 Boundary state formalism

The open string theory is described by the same sigma model as in the closed string case but now with the integration restricted to a worldsheet with boundary that, in the Lorentz picture, is parametrized by the coordinates \((\tau, \sigma)\) with metric \(\gamma_{ab}\) of signature \((-\, +)\). The simplest open worldsheet is the timelike strip with \(0 \leq \sigma \leq \pi\). The conformal boundary conditions along the timelike lines \(\sigma = 0, \pi\) are a combination of Neumann and Dirichlet conditions that in the target space are interpreted as free open strings and strings stretched between D-branes, respectively.

If we want to explore time-dependent string theory within the CFT on the worldsheet, we may try to explore other types of boundaries - spacelike or null - where for each of them we impose Neumann or Dirichlet conditions on the target fields. This simple consideration comes from the fact that in special relativity, we may treat the time and space coordinates similarly, so that all spacetime boundaries are at equal footing in our considerations.

So from now on we will use two labels for the lorentzian BCFT to specify the type of the boundary conditions. The Neumann condition with respect to a timelike boundary will be called an \((N; t)\)-boundary condition, the Dirichlet condition with respect to a null boundary will be called \((D; n)\) and so on.

2.4.1 D-branes and the non-chiral coherent Ishibashi states

In this section we describe the boundary states in the closed string sector corresponding to open strings whose endpoints obey a Neumann or a Dirichlet condition. They will be linear combinations of Ishibashi states that fulfil the boundary conditions that we impose on the open string endpoints. We first restrict ourselves to the case of the bosonic string described by the following worldsheet action with timelike boundaries

\[
S_{\text{open}} = \frac{1}{4\pi \alpha'} \int_0^\pi d\sigma \int_{-\infty}^{\infty} d\tau \partial_a X^\mu \partial^a X_\mu \tag{2.4.41}
\]

where we have considered the diagonal worldsheet metric \(ds^2 = -d\tau^2 + d\sigma^2\) [63]. At timelike boundaries we can either have a Neumann boundary condition \((N; t)\)

\[
\partial_\sigma X = 0 \quad (\sigma = 0, \pi) \tag{2.4.42}
\]
or a Dirichlet boundary condition \((D; t)\)
\[
\delta X = 0 \Leftrightarrow \partial_{\tau} X = 0 \quad (\sigma = 0, \pi)
\] (2.4.43)
as these make the boundary contributions to the equation \(\delta S_{\text{open}} = 0\) vanish. If in the action we restrict the time integral to the interval \([0, T]\) and the field configurations to the periodic ones \(X(\sigma, 0) = X(\sigma, T)\) then we may apply the worldsheet duality, where space and time are interchanged. Under such interchange the Neumann boundary condition and simple rescaling, \((N; t)\) is replaced by the \((N; s)\)-condition
\[
\partial_{\tau} X = 0 \quad \text{at } \tau = 0, \pi/T
\] (2.4.44)
and the Dirichlet boundary condition \((D; t)\) by the \((D; s)\)-condition
\[
\partial_{\sigma} X = 0 \quad \text{at } \tau = 0, \pi/T
\] (2.4.45)
in the closed string sector.

The solutions of the above equations are nicely related by noting that the introduction of boundaries allow us to relate the \(U(1)\)-chiral currents \(J(x^+) = \partial_{+} X_L(x^+)\) and \(J(x^-) = \partial_{-} X_R(x^-)\) of the bulk theory by a gluing map \(\Omega\)
\[
J(x^-) = \Omega J(x^+)
\] (2.4.46)
on the boundary. Considering the Fourier expansion of the chiral bosonic fields
\[
X_L(x^-) = \frac{\bar{x}}{2} + \alpha_0 x^- + \frac{1}{2} \sum_{n \neq 0} \frac{\alpha_n}{n} e^{-inx^-}
\]
\[
X_R(x^+) = \frac{\bar{x}}{2} + \tilde{\alpha}_0 x^+ + \frac{1}{2} \sum_{n \neq 0} \frac{\tilde{\alpha}_n}{n} e^{-inx^+}.
\] (2.4.47)
and substituting them in the definition of the \(U(1)\) currents, we find that \(\Omega = 1\) for Neumann condition and \(\Omega = -1\) for the Dirichlet boundary condition at \(\sigma = 0, \pi\) and the other way around for \(\tau = 0, \pi/T\), with the coupling of the chiral modes
\[
\alpha_n + \Omega \tilde{\alpha}_{-n} = 0 \quad \forall n \neq 0
\] (2.4.48)
and the field solution \(X(\tau, \sigma) = X(x^+) + X(x^-)\). These conditions were used in [16] to define the boundary state associated with them.

Consider the example of a \(D_p\)-brane defined by imposing Neumann conditions on its \(p\) longitudinal coordinates and Dirichlet conditions at the \(D-p\) transverse coordinates.
We define the boundary state as a sum of Ishibashi states in the closed string sector satisfying

\[
\partial_\tau X^\alpha \mid_{\tau=0} \mid B \rangle_X = 0 \quad \alpha = 0, ..., p \\
X^i \mid_{\tau=0} \mid B \rangle_X = y^i \quad i = p + 1, ..., D - 1
\]

that in terms of the closed string oscillators is translated on

\[
(\alpha_n^\mu + S_{\nu}^\mu \tilde{\alpha}_{-n}^\nu) \mid B \rangle_X = 0 \quad \forall n \neq 0 \\
\tilde{p}^\alpha \mid B \rangle_X = 0 \\
(q^i - y^i) \mid B \rangle_X = 0
\]

where we have introduced the matrix

\[
S^{\mu\nu} = (\eta^{\alpha\beta}, -\delta^{ij})
\]

The eigenstates of (2.4.50) are obtained, up to a normalization factor, by a Bogoliubov transformation

\[
\mid B \rangle_X = N_p \delta^{25-p} (q^i - y^i) \exp \left( -\sum_{n=1}^{\infty} \frac{1}{n} \alpha_{-n} \cdot S \cdot \alpha_{-n} \right) \mid 0 \rangle \mid \bar{0} \rangle \mid p = 0 \rangle
\]

To those states one can add the Majorana field contributions as well as the bosonic and fermionic Faddev-Popov ghosts

\[
\mid B \rangle_\psi = \exp \left( \pm i \sum_{r>0} \psi_{\mu,r} \tilde{\psi}_{\mu,-r} \right) \mid 0 \rangle \\
\mid B \rangle_{gh} = \exp \left( \sum_{m=1}^{\infty} [c_{-m} \tilde{b}_{-m} + \bar{c}_{-m} \tilde{b}_{-m}] \pm i \sum_{r>0} [\gamma_{-r} \tilde{\beta}_{-r} - \bar{\gamma}_{-r} \bar{\beta}_{-r}] \right) \frac{1}{2} \langle c_0 + \bar{c}_0 \rangle \mid \downarrow \rangle
\]

where \( \mid \downarrow \rangle \) is annihilated by all positive frequency ghost and superghost oscillators and the anti-ghost zero-modes. The above states are the Bogoliubov solutions of the boundary conditions

\[
\psi_{\mu,r} = \pm i \tilde{\psi}_{\mu,-r} \\
c_m = -\bar{c}_{-m} \quad b_m = \tilde{b}_{-m} \\
\gamma_r = \mp i \bar{\gamma}_{-r} \quad \beta_r = \mp i \bar{\beta}_{-r}
\]
The full boundary state \( |B\rangle = |B\rangle_X |B\rangle_\psi |B\rangle_{gh} \) is a sum of **coherent** and **nonchiral** Ishibashi states. The imposition of no flux-energy crossing non-null boundaries is translated to the condition of conformal invariance

\[
(L_n^{(C)} - \bar{L}_n^{(C)})|B\rangle = 0
\]  

(2.4.55)

that comes from \( \alpha_n + \Omega \bar{\alpha}_{-n} \) with \( \Omega^2 = 1 \). This was interpreted in [16] as a diffeomorphism invariance condition under the group \( Diff(S^1) \) of diffeomorphisms of the circle. Explicitly, we define the operator \( S_n = L_n - \bar{L}_{-n} \) that can be seen to obey the algebra

\[
[S_n, S_m] = (n - m)S_{n+m}
\]

(2.4.56)

coming from the Virasoro algebra after we impose \( c = \bar{c} \).

**Summarizing:** Boundary states must obey two conditions, a no-flux of energy crossing the physical boundary and a conformal \( Diff(S^1) \) invariance that in the present case coincide.

### 2.4.2 Deformed Ishibashi states and the one-dimensional path integral approach

One would like to calculate the boundary states for a \( D_p \)-brane immersed in a background gauge field \( A_\mu(X) \) with an interaction given in the closed string channel at \( \tau = 0 \) by

\[
S_A = \frac{1}{4\pi} \int_0^{2\pi} d\sigma \left( A_\mu(X) \partial_\sigma X^\mu - \frac{i}{2} F_{\mu\nu}(X) \theta^\mu \theta^\nu \right)
\]

(2.4.57)

in the euclidian picture, where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) and \( \theta^\mu = \psi^\mu \pm i\bar{\psi}^\mu \) for R sector or NS sector. The bosonic boundary conditions in the closed string sector become a mixing of Neumann and Dirichlet conditions

\[
\partial_\tau X_\alpha + F_{\alpha\beta} \partial_\sigma X^\beta = 0
\]

(2.4.58)

at \( \tau = 0, \pi/T \). How does this affect the previous boundary states?

To answer this question we consider the example of a simple harmonic oscillator. Let us search for the oscillator state satisfying for a given value of the momentum \( p \)
the eigenvalue condition

\[(a - a^\dagger - ip)|p\rangle = 0\]  \tag{2.4.59}

where \(a\) and \(a^\dagger\) are respectively the annihilation and creation operators. We want as well that the solution obey the completeness relation

\[\int_{-\infty}^{\infty} dp |p\rangle\langle p| = 1\]  \tag{2.4.60}

After some algebra we find the eigenstates

\[|p\rangle = (2\pi)^{1/4} e^{-p^2/4} e^{\pm (a^\dagger)^2/2 + ipa^\dagger} |0\rangle\]  \tag{2.4.61}

Note that after integration over all possible momentum values, we find a non-normalized squeezed state

\[|\Psi\rangle = \int dp |p\rangle = (8\pi)^{1/4} e^{-(a^\dagger)^2/2} |0\rangle\]  \tag{2.4.62}

as we have chosen from the beginning a state perfectly localized in momentum space, see (2.4.59). On the other hand, the insertion of a perturbative action \(S_{\text{pert}}(p)\) at a given time serves to create the perturbed oscillator state \(|\Psi_{\text{pert}}\rangle\) with

\[|\Psi_{\text{pert}}\rangle = \int dp e^{-S_{\text{pert}}(p)} |p\rangle\]  \tag{2.4.63}

In the string case, the boundary perturbation at a given time will change the free boundary states by a multiplication of a Wilson line factor

\[\text{Tr} \ P \ \text{exp}(-S_A)\]  \tag{2.4.64}

in the Polyakov path integral where \(P\) denotes path (integral) ordering operator. We thus obtain the perturbed state as

\[|A\rangle = \int D\bar{p} Dp D\Theta D\bar{\Theta} \text{Tr} \ P \ \text{exp}(-S_A)|p, \bar{p}\rangle |\Theta, \bar{\Theta}; \pm\rangle\]  \tag{2.4.65}

where \(|p, \bar{p}\rangle\) are bosonic eigenstates associated to the free theory of two commuting set of bosonic coordinates as we will soon see. The same holds for the Majorana contribution \(|\Theta, \bar{\Theta}; \pm\rangle\) associated to two anticommuting sets of fermionic fields.

To calculate the bosonic eigenstates we consider the closed string as a combination of left- and right-oscillators created at \(\tau = 0\)

\[X^\mu(\sigma, 0) = q^\mu + (\alpha')^{1/2} \sum_{m \neq 0} |m|^{-1/2} \left[ a^\mu_m e^{-im\sigma} + \bar{a}^\mu_m e^{i\sigma} \right]\]  \tag{2.4.66}
for which we impose the Dirichlet condition

$$\partial_\sigma X^\mu = 0 \quad (\tau = 0) \quad (2.4.67)$$

so that the left/right oscillator coupling is given by $a_\mu + \tilde{a}_\mu = 0$, for all $m \neq 0$. In this way, the target coordinate with a Dirichlet boundary condition is given by combinations of the following two sets

$$i \mathcal{P}_m = a_\mu - \tilde{a}_\mu^\dagger$$
$$-i \tilde{\mathcal{P}}_m = a_\mu^\dagger - \tilde{a}_\mu$$

for $m > 0$ where we have denoted $a_\mu^\dagger = a_{-\mu}$. These two combinations form a complete commuting set of bosonic oscillators. As in the one-dimensional oscillator case, we interpret these equations as eigenvalue conditions that should define certain eigenstates $|p, \bar{p}\rangle$. After we impose the completeness condition on them

$$\int \mathcal{D}p \mathcal{D}\bar{p} \langle p, \bar{p}| \langle p, \bar{p}| = 1 \quad (2.4.69)$$

we find the solution

$$|p, \bar{p}| = \exp \left[ -\frac{1}{2} \mathcal{P}|p| + (a^\dagger|\tilde{a}^\dagger) + i(a^\dagger|p) + i(p|\tilde{a}^\dagger) \right]|0\rangle|\bar{0}\rangle \quad (2.4.70)$$

where we use the notation

$$(\bar{p}|p) = \sum_{\mu=0}^{D-1} \sum_{m=1}^{\infty} \mathcal{P}_m \mathcal{P}_m^\mu$$

with $D = 26$ for the bosonic case or $D = 10$ for the superstring case. Similar considerations hold if we had considered Neumann boundary conditions [16].

The same analysis holds for the Majorana contribution by writing the set of two anticommuting eigenstates

$$(\tilde{\Theta}_r^\mu - \psi_\mu^\dagger \mp i \bar{\psi}_\mu)|\Theta, \tilde{\Theta}; \pm\rangle = 0$$
$$(\Theta_r^\mu - \psi_\mu^\dagger \pm i \bar{\psi}_\mu)|\Theta, \tilde{\Theta}; \pm\rangle = 0 \quad (2.4.72)$$

for $r > 0$ and with solution

$$|\Theta, \tilde{\Theta}; \pm\rangle = \exp \left[ -\frac{1}{2}(\Theta|\Theta) \pm i(\psi^\dagger|\tilde{\psi}^\dagger) + (\psi^\dagger|\Theta) \mp i(\tilde{\Theta}|\tilde{\psi}^\dagger) \right]|0; \pm\rangle \quad (2.4.73)$$

that obeys the completeness relation.
Insert these states corresponding to Dirichlet and Neumann boundary conditions in the path integral (2.4.65) and perturb the system by a constant external field. In this case, the perturbed boundary term reduces to a quadratic form

\[
S_A = \frac{i}{8\pi} F_{\mu\nu} \int_0^{2\pi} d\sigma \left( X^\mu \partial_\sigma X^\nu - i\theta^\mu \theta^\nu \right)
\]

so that the functional integrals in (2.4.65) become gaussian. In what follows, we are interested on the case of vanishing gauge fields in the directions transverse to the \(D_p\) - brane. After a gaussian integration, the bosonic and fermionic contributions to the perturbed boundary state \(|A\rangle\) are [16]

\[
|B_X\rangle = \sqrt{-\det(\eta + F)} \delta^{D-p-1}(q - y) \exp \left( -\sum_{n=1}^\infty \frac{1}{n} \alpha^\mu_n M_{\mu\nu} \tilde{\alpha}_n^\nu \right) |0\rangle
\]

\[
|B_\psi\rangle = |B_R\rangle = \frac{i}{\sqrt{-\det(\eta + F)}} \exp \left( i \sum_{n=1}^\infty \psi^\mu_{-n} M_{\mu\nu} \tilde{\psi}_{-n}^\nu \right) |0_R\rangle
\]

\[
|B_{\psi}\rangle = |B_{NS}\rangle = -i \exp \left( i \sum_{n=1/2}^\infty \psi^\mu_{-n} M_{\mu\nu} \tilde{\psi}_{-n}^\nu \right) |0_{NS}\rangle
\]

where

\[
M_{\mu\nu} = \left( \frac{1 - F}{1 + F} \right)_{\alpha\beta}, -\delta_{ij}
\]

and we have replaced the \(a_n\) oscillator modes by the \(\alpha_n\), compare (2.4.66) with (2.2.14).

The above states satisfy the boundary conditions for the bosonic contribution

\[
\hat{p}^\beta |B_X\rangle = 0 \quad , \quad q^i |B_X\rangle = y^i |B_X\rangle \quad , \quad (\alpha^i_n - \tilde{\alpha}_{-n}^i) |B_X\rangle = 0
\]

\[
\left[ (1 + F)_{\alpha\beta} \alpha^\beta_n + (1 - F)_{\alpha\beta} \tilde{\alpha}_{-n}^\beta \right] |B_X\rangle = 0
\]

and the fermionic contribution

\[
(\psi^i_n \pm i \tilde{\psi}_{-n}^i) |B_\psi\rangle = 0
\]

\[
\left[ (1 + F)_{\alpha\beta} \psi^\beta_n \mp i(1 - F)_{\alpha\beta} \tilde{\psi}_{-n}^\beta \right] |B_\psi\rangle = 0
\]

Here we have labelled the indices of transverse coordinates to the \(D_p\) - brane by Latin letters and the longitudinal coordinates by Greek letters.

It is interesting to note in (2.4.76) that there is a relative boost of left and right-movers as the external gauge field approaches to a critical limit \(F \to 1\). Here there
is a complete decoupling of left- and right-chiral sector, as we can see in the above boundary conditions along the transverse directions to the brane

$$F \to 1 \quad \Rightarrow \quad \Omega \to 0$$  \hspace{1cm} (2.4.79)

Moreover the normalization factors of the above coherent and non-chiral perturbed boundary states become singular at this critical limit. We will later show that the alternative boundary states associated to the decoupling limit are **chiral** and **squeezed** states, solutions of Dirichlet conditions with respect to worldsheet null boundaries.

### 2.5 Calculation of the $D$-brane tension

The normalization factor $N_p$ in the front of the boundary state associated to the $D_p$-brane is still unknown. To calculate it we consider a Cardy type program in the target space of string theory. For the sake of simplicity we restrict ourselves to the case of bosonic strings and starting to neglect the ghosts contribution. First compute the one-loop free energy of an open string stretching between the two $D_p$-branes, given by the Coleman-Weinberg formula [45]

$$F = \int_0^\infty \frac{dT}{2T} \text{Tr}_{n,k} \left[ e^{-2\pi T (L_0^{(O)}-1)} \right]$$

$$= \int_0^\infty \frac{dT}{2T} \text{Tr}_{n,k} \left[ e^{-2\pi T \alpha'(k^2+M^2)} \right]$$  \hspace{1cm} (2.5.80)

where $T$ parametrizes the periodic time variable $T \sim T + 2\pi$ and $L_0^{(O)}$ is the Virasoro operator on the open string sector related to the mass spectrum

$$M^2 = \frac{1}{\alpha'} \left( \sum_{n=1}^\infty \alpha_{-n} \cdot \alpha_n - 1 \right) + \frac{|y|^2}{4\pi^2\alpha'}$$  \hspace{1cm} (2.5.81)

and $|y|$ is the value of the distances between the two $D_p$-branes. The trace $\text{Tr}_{n,k}$ is to be read as a integration over the momentum along the brane and a trace over the oscillator modes. After substitution of (2.5.81) into (2.5.80) we have

$$F = 2V_{p+1} \int_0^\infty \frac{dT}{2T} (8\pi^2 \alpha'T)^{-\frac{p+1}{2}} e^{-\frac{|y|^2 T}{2\pi \alpha'}} f_{-26}^{-1}(e^{-\pi T})$$  \hspace{1cm} (2.5.82)

with the front factor of two coming from the freedom of changing the oriented string endpoints, and

$$f_1^{-1}(e^{-\pi T}) := \prod_{n=1}^\infty \left( \frac{1}{1 - e^{-2\pi T \alpha_n}} \right) = \text{Tr}_n \prod_{n=1}^\infty e^{-2\pi T \alpha_n \alpha_n}$$  \hspace{1cm} (2.5.83)
without summation on the index $\mu$. Under the worldsheet modular transformation $T \rightarrow \tau = 1/T$, the function $f_1$ has the property

$$f_1(e^{-\pi T}) = \sqrt{T} f_1(e^{-\pi \tau}) . \quad (2.5.84)$$

The result should be compared to the free-energy $F = \langle B_X | D | B_X \rangle$ of a closed string created at one brane and propagating freely until annihilated by the second brane. Under the change of variable $z = e^{-i\pi \tau}$, the cylinder amplitude may be calculated using the disk operator

$$D_a = \frac{\alpha'}{4\pi} \int |z| \leq 1 \left| \frac{d^2z}{z^2} \right| L_0-a \bar{L}_0-a , \quad (2.5.85)$$

which is the closed string propagator written in terms of the Hamiltonian given by the closed string zero-mode Virasoro operators $H^{(c)} = L_0^{(c)} + \bar{L}_0^{(c)}$. The constant $a$ comes from normal ordering and in the case at hand we set $a = 1$. We reproduce here the bosonic boundary state for the bosonic string

$$|B_X\rangle = N_p \delta^{25-p}(q^i - y^i) \prod_{n=1}^\infty e^{-\frac{1}{2} n^{\mu\nu} S_{\mu\nu} n |0\rangle |\bar{0}\rangle |p = 0\rangle \quad (2.5.86)$$

and the Virasoro operators

$$L_0^{(c)} = \frac{\alpha'}{4} \hat{p}^2 + \sum_{n=1}^\infty \alpha_{-n} \cdot \alpha_n , \quad \bar{L}_0^{(c)} = \frac{\alpha'}{4} \hat{\bar{p}}^2 + \sum_{n=1}^\infty \bar{\alpha}_{-n} \cdot \bar{\alpha}_n . \quad (2.5.87)$$

By substitution in the amplitude, we first split the zero mode from the non-zero oscillator mode terms. The non-zero modes give the contribution

$$f_1^{-26}(|z|) = \prod_{n=1}^\infty \left( \frac{1}{1 - |z|^{2n}} \right)^{26} . \quad (2.5.88)$$

The zero mode contribution will be the important one to obtain the value of the normalization factor, so we will be examine it carefully

$$A_0 = \langle p = 0 | \delta^{25-p}(q^i - y^i) |z|^{2\hat{p}^2} \delta^{25-p}(q^i - y^i) |p = 0\rangle \quad . \quad (2.5.89)$$

Writing $A_0$ as a Fourier transformation we get

$$A_0 = \int \int \frac{d^{25-p}k}{(2\pi)^{25-p}} \frac{d^{25-p}k'}{(2\pi)^{25-p}} \langle p = 0 | e^{ik\hat{q}|z|^{2\hat{p}^2}} e^{ik'(q-y)} |p = 0\rangle \quad (2.5.90)$$

where $k$ is the one-loop transverse momentum. Using the identities

$$e^{ik\hat{q}|p = 0\rangle} = e^{ik\hat{q}|p_\perp = 0\rangle |p_\parallel = 0\rangle} \quad = |p_\perp = k\rangle |p_\parallel = 0\rangle$$

$$\hat{p}^2 |p_\perp = k\rangle |p_\parallel = 0\rangle = k^2 |p_\perp = k\rangle |p_\parallel = 0\rangle$$

$$\langle p = k | p = k' \rangle \quad = 2\pi \delta(k - k')$$

$$V_d := (2\pi)^d \delta^d(0) \quad (2.5.91)$$
and changing the variable $\tau$, we end up with the gaussian integral

$$A_0 = V_{p+1} \int \frac{d^{25-p} \epsilon}{(2\pi)^{25-p}} e^{-\frac{\tau}{2} \alpha' k^2 + ik \cdot y}$$

$$= V_{p+1} e^{-y^2/(2\pi \tau \alpha')} (2\pi^2 \tau \alpha')^{(25-p)/2}$$

(2.5.92)

The total contribution

$$\langle B_X | D_1 | B_X \rangle = (N_p)^2 V_{p+1} \frac{\pi \alpha'}{2} \int_0^\infty d\tau (2\pi^2 \tau \alpha')^{-(25-p)/2} e^{-y^2/(2\pi \tau \alpha')} f_{1}^{26} (e^{-\pi \tau})$$

should be compared to the one-loop open string calculation by performing the world-sheet duality $\tau = 1/T$ and using the modular property of $f_1$, see (2.5.84). The ghost contribution corrects the power by replacing 26 with 24. After this remark, we are able to find the normalization factor in the boundary state of the $D_p$ brane

$$N_p = \sqrt{\frac{\pi}{2}} T_p, \quad T_p = \left(\frac{2\pi \sqrt{\alpha'}}{24}\right)^{11-p} .$$

(2.5.93)

As shown in [11, 23] we can read from the boundary state the meaning of certain coupling constants on the low-energy effective field theory by taking projections

$$C_{\psi} \sim \langle \Psi | B_X \rangle$$

(2.5.94)

where $|\Psi\rangle$ denotes closed string state associated to the massless field in question. In particular

$$A^{\mu \nu} := \langle 0; k | \alpha'^{\mu} \tilde{\alpha}^{\nu} | B_X \rangle = -\frac{T_p}{2} V_{p+1} S^{\mu \nu}$$

(2.5.95)

and by saturating it with the symmetric graviton entering in the vertex operator (2.2.9)

$$A_{grav} = A^{\mu \nu} \epsilon_{\mu \nu} = -T_p V_{p+1} \eta^{\alpha \beta} \chi_{\alpha \beta} .$$

(2.5.96)

This is interpreted as the value of the tension (energy per unit brane volume) due to exchange of gravitons with the brane.

### 2.6 The Dissipative Hofstadter Model and Open Strings

It is known that open strings immersed in a constant electromagnetic background may be described by one dimensional field theory of dissipative quantum mechanics at
the critical point. A particular case is the dissipative Hofstadter model that we shall treat here. Consider a particle trajectory described by the position vector $\vec{X}$ immersed in a thermal bath of an infinite number of harmonic oscillators with frequency $\omega_\alpha$ coupled linearly to $\vec{X}$ with a given coupling constant $C_\alpha$. This can be described macroscopically as a dissipative force of the form $-\eta \vec{X}$ where $\eta$ is the coefficient of friction. Such macroscopic friction may be rendered microscopically by the relation

$$\eta \omega = \sum_\alpha \frac{C^2_\alpha}{\omega_\alpha} \delta(\omega - \omega_\alpha)$$

(2.6.97)

where $\omega$ is the frequency of $\vec{X}$. This is known as the Caldeira-Leggett model, a particular Dissipative Quantum Mechanics model (DQM for short). The DQM models convert quantum mechanics into a one-dimensional statistical mechanics problem. Such models becomes interesting when applied to the quantum mechanics of an electron moving in two dimensions subject to a uniform magnetic field and a periodic potential, usually called the Hofstadter model. The introduction of the Caldeira-Leggett thermal bath to the Hofstadter model will define the Dissipative Hofstadter Model (DHM).

The Lagrangian of such system is given by

$$S_{DHM} = \int_0^T dt \left[ \frac{1}{2} M \dot{\vec{X}}^2 + V(\vec{X}) + iA_j(\vec{X})\dot{X}^j \right] + \frac{\eta}{4\pi} \int_0^T dt \int_{-\infty}^\infty dt' \frac{(\vec{X}(t) - \vec{X}(t'))^2}{(t-t')^2}$$

where $V$ is a scalar potential and $A$ is the vector potential of the magnetic field.

In the euclidian path integral

$$Z(\eta, B, V, 0) = \int DX(t) e^{-S_{DHM}}$$

we may add a linear source term

$$S_F = \int \vec{F}(t) \cdot \vec{X}(t) dt$$

(2.6.99)

and define the generating functional

$$Z(\eta, B, V, F) = \int DX(t) e^{-S_{DHM} - S_F}$$

(2.6.100)

that allows to compute the correlation functions. In particular, the two-point correlation function is

$$\langle X^\mu(t_1)X^\nu(t_2) \rangle = \frac{1}{Z(\eta, B, V, 0)} \frac{\delta^2 Z(\eta, B, V, F)}{\delta F_\mu(t_1)\delta F_\nu(t_2)} \bigg|_{F=0}$$

(2.6.101)
The kinematic term is quadratic and therefore irrelevant in the renormalization group so we set $M = 0$. Consider the case of the charged particle immersed in a magnetic field given by the linear gauge

$$(A_x, A_y) = \frac{1}{2}(By, -Bx)$$

(2.6.102)

and take the periodic potential

$$V(x, y) = V_0 \cos(2\pi x/a) + V_0 \cos(2\pi y/a).$$

(2.6.103)

In this case the two-point correlation function is

$$\langle X_i(t)X_j(t') \rangle = -\frac{\alpha}{\alpha^2 + \beta^2} \log(t - t')^2 \delta_{ij} - i\frac{\pi \beta}{\alpha^2 + \beta^2} \text{sign}(t - t')\epsilon_{ij}$$

(2.6.104)

for $\alpha = \eta a^2/2\pi\hbar$ and $\beta = eBa^2/2\pi\hbar c$ where $\hbar$ is the Planck constant and $c$ is the speed of light.

The first term on the right hand side measures the delocalization of the particle wave function, called mobility in the condensed matter literature. The logarithmic growth is a transition between two extreme limits defined by the long-time behaviour: bound by a constant or growing without limit. The coefficient at the front of the logarithm is the value of the critical mobility.

The second term can be interpreted as a Hall effect, it measures the response of the system in the transverse directions to the applied magnetic field.

### 2.7 Boundary Deformation Theory

Given a closed string background described by a bulk CFT one may ask what are the possible D-brane configurations associated to boundary CFT’s. It turns out that one has multiple choices parametrized by marginal boundary fields, forming the D-brane moduli space for a given bulk background. There may be more symmetries on BCFT than those generated by the Virasoro algebra. The most general symmetries are generated by certain currents forming a $\mathcal{W}$ algebra, where the Virasoro is a subalgebra of $\mathcal{W}$. By the usual method of images we may construct the general chiral algebra in the open channel by computing the chiral currents

$$W_n^{(O)} := \frac{1}{2\pi i} \int z^{n+h_W-1}W(z)dz - \frac{1}{2\pi i} \int \bar{z}^{n+h_W-1}\Omega(W)(\bar{z})d\bar{z}$$

(2.7.105)
where the integrals are performed along a semi-circle in the upper half plane. Here $h_W$ is the conformal height of the operator $W$ and $\Omega$ denotes the automorphism relating left-movers to the right-movers.

Marginal boundary fields are introduced in the free system as boundary perturbations that will change the boundary conditions given by the gluing map $\Omega$ of the current algebra, but without affecting the local bulk properties of the system. An example of that is the Dissipative Quantum Mechanics that we have described in the previous section.

Depending on the value of the conformal weight of the boundary field, we may distinguish three general behaviours of the renormalization group flow under the boundary perturbation. For $h > 1$ the perturbative field is said to be irrelevant as the RG-flow leads to the same original BCFT. In the marginal case ($h = 1$) we do not introduce any length scale in the theory so that under RG-flow we most probably end up in the same fixed point of the theory or in another point of the fixed point manifold.

The most interesting case arises when $h < 1$ where the perturbations are relevant. In such case, it is difficult to follow the RG-flow, which will end up in some unknown fixed point of the theory. We stress here that it is assumed that the perturbations do not affect the local bulk properties.

A more systematic treatment was done in [47] where the authors consider two types of marginal deformations - chiral and non-chiral. In chiral deformations branes (BCFT) are related to each other by continuous symmetries on the target space. In the second case, non-chiral deformations can for example push the brane to some singularity on the target space.

The theory is perturbed by the boundary operator

$$I_{\lambda\Psi} = P \exp\{\lambda S_\Psi\} = P \exp\{\lambda \int_{-\infty}^{\infty} \Psi(x) \frac{dx}{2\pi}\}$$ (2.7.106)

where $P$ denotes path ordering in the Polyakov path integral and we are using the complex plane where the boundary is located at the real line. For simplicity we impose that the boundary field $\Psi(x)$ should be mutual- (and self-) local with respect to another boundary field $\Phi(x')$, that is

$$\Psi(x_1)\Phi(x_2) = \Phi(x_2)\Psi(x_1) \quad (x_1 < x_2) \quad .$$ (2.7.107)

An example of a chiral marginal perturbation is the one given by the $U(1)$ current
for which the gluing map is deformed as

\[ W(z) = \Omega \circ \gamma_J(\bar{W})(\bar{z}) \quad (z = \bar{z}) \]  

(2.7.108)

with

\[ \gamma_J(W) = \exp(-i\lambda J_0)W\exp(i\lambda J_0) \]  

(2.7.109)

an inner automorphism of the chiral \( W \) algebra. In particular, the inner automorphism acts trivial on the Virasoro field leading to the condition \( T = \bar{T} \) interpreted as no flow of energy across the timelike boundaries \( \sigma = 0, \pi \). This last condition continues to be valid even if we deform the theory by self-local non-chiral marginal boundary fields.

For the other currents the gluing map is deformed by general local marginal boundary fields as

\[ W(z)e^{\lambda\int \frac{dx}{2\pi} \psi(x)} = e^{\lambda\int \frac{dx}{2\pi} \psi(x)} \left[ e^{\lambda \psi \bar{W}} \right](\bar{z}) \]  

(2.7.110)

with

\[ \left[ e^{\lambda \psi \bar{W}} \right](\bar{z}) := \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \int_{\gamma_1} \frac{dx_1}{2\pi} \cdots \int_{\gamma_n} \frac{dx_n}{2\pi} \bar{W}(\bar{z}_\delta)\psi(x_1)\cdots\psi(x_n) \]  

(2.7.111)

where we have used \( \bar{z}_\delta = z - 2i\delta \) for \( \delta \) a positive infinitesimal real parameter and the integrations are taken over straight lines \( \gamma_i \) at the upper half plane, getting closer and closer to the real line as \( i \) runs from 1 to \( n \).

An example of non-chiral self-local deformation was developed in [14] where to a free field we add a boundary perturbation given by a periodic complex potential with strength \( |\lambda| \). The system is described by the Lagrangian

\[ L = -\frac{1}{8\pi} \int d\tau d\sigma \left( (\partial_\tau X)^2 - (\partial_\sigma X)^2 \right) - \frac{1}{2\sqrt{2}\pi} \int d\tau \left( \lambda e^{iX(0)/\sqrt{2}} - \bar{\lambda} e^{-iX(0)/\sqrt{2}} \right) \]

where \( X \) is the target coordinate that parametrizes an open string. By imposing a Dirichlet boundary condition at \( \sigma = \pi \), the boundary condition at the first endpoint \( \sigma = 0 \) becomes dynamical

\[ -\partial_\sigma X + i\lambda e^{iX/\sqrt{2}} - i\bar{\lambda} e^{-iX/\sqrt{2}} = 0 \]  

(2.7.112)

and by varying \( |\lambda| \) from zero to infinity, we are able to interpolate between Neumann to Dirichlet boundary conditions. More systematically, consider the case when \( \lambda \) is
real. The perturbation is generated by two boundary fields

\[ \psi^1(x) = \left( e^{i\sqrt{2}X(x)} + e^{-i\sqrt{2}X(x)}/\sqrt{2} \right) \]
\[ \psi^2(x) = \left( e^{i\sqrt{2}X(x)} - e^{-i\sqrt{2}X(x)}/\sqrt{2} \right), \] (2.7.113)

that are local with respect to themselves. It can be seen that the gluing condition in the energy momentum tensor is unchanged \( T(z) = \tilde{T}(\bar{z}) \) but on the currents \((J, \tilde{J}) = (\partial X, \bar{\partial} X)\), the above boundary fields will deform the initial gluing map \( J(z) = \tilde{J}(\bar{x}) \) as follows; under the perturbation by \( \psi^1 \) we have

\[ J(z) = \sin(\sqrt{2}\lambda)\psi^2(x) + \cos(\sqrt{2}\lambda)\tilde{J}(\bar{z}) \] (2.7.114)

and by \( \psi^2 \) perturbation

\[ J(z) = -\sin(\sqrt{2}\lambda)\psi^1(x) + \cos(\sqrt{2}\lambda)\tilde{J}(\bar{z}) \] (2.7.115)

for \( z = \bar{z} = x \). In particular, for \( \lambda = (2n + 1)\frac{\pi}{\sqrt{2}} \), the original Neumann conditions for \( J \) turn into Dirichlet conditions.
Chapter 3

Electrically charged open string and boosted $D$-branes

It is known that there is pair production of particles in a Rindler spacetime describing a constantly accelerated particle detector in a Minkowskian background. The particle production comes from the existence of an external force to keep the motion of the detector, creating a thermal bath around it of a given temperature proportional to the acceleration [62]. There is a critical limit associated to the relativistic velocity of the particle. A second example of pair production is the case of a Minkowskian spacetime with an electric field. Nevertheless, there is no critical behaviour as we raise the electric field strength.

The existence of an upper limit in the presence of a constant electric field was found in the context of string theory. Consider a bosonic open string whose endpoints carry an electric charge $e$. The system is described by the following action

$$S = -\frac{1}{4\pi\alpha'} \int d\sigma d\tau \partial_{\mu} X^\mu \partial^\mu X_{\mu} + \frac{e}{2} \int d\tau F_{\mu\nu} X^\nu \partial_{\tau} X^\mu$$

where the string tension $T$ is related to the Regge slope by $T = (2\pi\alpha')^{-1}$. Classically, above the critical electric field $E_{\text{crit}} = (2\pi\alpha'e)^{-1}$, the tension can not hold the string together [12].

A more refined argument was presented in [4], where they make a Schwinger’s type of calculation on the pair production of open unoriented bosonic strings in the presence of a constant electric field ($F_{01} = E$). They concluded that the rate of production diverges when the electric field approaches to the critical value $E_{\text{crit}}$. For
future reference we shall describe the main results on the quantization of the system [25]. Since we are in the case of a pure constant electric field, only the light-cone coordinates involving the direction of the field are affected. The boundary conditions become a mixture of Dirichlet and Neumann conditions

$$\partial_\sigma X^\pm = 2\pi \alpha' eE \partial_\tau X^\pm$$

(3.0.2)

at $\sigma = 0, \pi$. In this way, the light-cone coordinates have the mode expansion

$$X^\pm(\sigma, \tau) = \hat{x}^\pm + ia_0^\pm \phi_0^\pm + i(\alpha')^{1/2} \sum_{n=1}^{\infty} (a_n^\pm \phi_n^\pm(\epsilon) - h.c)$$

(3.0.3)

where

$$\phi_n^\pm(\epsilon) = (n \pm i\epsilon)^{-1/2} e^{-i(n \pm i\epsilon)\tau} \cos [(n \pm i\epsilon)\sigma \mp i\pi\epsilon/2]$$

(3.0.4)

are a complete set of normalized mode functions that satisfy the wave equation and the above boundary conditions. They depend on the parameter $\epsilon = 2 \arctanh(2\pi \alpha' eE)/\pi$.

The inner product is defined by the following expression

$$\int_0^{\pi} \frac{d\sigma}{\pi} \phi_n^\pm(\tau, \sigma) \left[i \frac{\partial}{\partial \tau} + 2\pi \alpha' eE \left[\delta(\sigma) - \delta(\pi - \sigma)\right]\right] \phi_m^\pm(\tau, \sigma) = \delta_{mn} \text{sign}(n \pm \epsilon)$$

where $\phi \frac{\partial}{\partial \tau} \psi = \phi \partial_\tau \psi - \psi \partial_\tau \phi$. After a gauge transformation, the momentum density is

$$\pi P_\pm = \partial_\tau X_\mp + i\pi \alpha' eE X_\mp(\delta(\sigma) - \delta(\pi - \sigma))$$

(3.0.5)

Considering the usual canonical commutation relations

$$[X_\mu(\tau, \sigma), X_\nu(\tau, \sigma')] = 0$$

$$[P_\mu(\tau, \sigma), P_\nu(\tau, \sigma')] = 0$$

$$[X_\mu(\tau, \sigma), P_\nu(\tau, \sigma')] = i\delta_{\mu\nu}\delta(\sigma - \sigma')$$

(3.0.6)

we find for $\mu, \nu = -, +$ the (gauge invariant) relations

$$[a_n^-, a_m^{+\dagger}] = \delta_{nm}$$

$$[a_n^+, a_m^{-\dagger}] = \delta_{nm}$$

(3.0.7)

$$[\hat{x}^+ , \hat{x}^- ] = \frac{1}{4\alpha' eE}$$
Contrary to what we may think, \( \hat{x}^\pm \) are not identified with the center-of-mass coordinate since the functions \( \phi_n \) are not periodic in \( \sigma \) and so they do not vanish by taking their mean-value integral.

The open string Virasoro zero-mode restricted to the lightcone coordinates given by the energy-momentum tensor is

\[
L^0_{\text{(0,} \ell))} = -\sum_{n=1}^{\infty} (n - i\epsilon) (a_n^+)^* a_n^- - \sum_{n=0}^{\infty} (n + i\epsilon) (a_n^-)^* a_n^+ + a(\epsilon) \tag{3.0.8}
\]

with normal order constant \( a(\epsilon) = i\epsilon(1 - i\epsilon)/2 \). By a similar computation of the one-loop free energy \( F \) as in Sect. 2.5, considering now four contributions from the unoriented open and closed strings (the annulus, Möbius strip, torus and Klein bottle), the rate of pair production per unit volume given by the imaginary part of \( F \) is

\[
\omega := \Im(F) = \frac{1}{(2\pi)^{25}} \frac{\alpha'eE}{\epsilon} \sum_{i,k} (-1)^{k+1} \left( \frac{|\epsilon|}{k} \right)^{13} e^{-\pi k(M_i^2 + \epsilon^2)/|\epsilon|} \tag{3.0.9}
\]

where the sum is taken over the physical states and the internal momentum. We see that the rate production diverges as \( \epsilon \to \infty \) corresponding to the upper bound on the electric field \( E_{\text{crit}} = (2\pi\alpha'e)^{-1} \).

The dual model can be roughly seen by noting that the dynamics of the system is governed by the Born-Infeld action [15]. Under the change \( 2\pi\alpha'eE \leftrightarrow V \), this becomes the action for a relativistic point particle,

\[
L_{\text{BI}} \propto \sqrt{1 - (2\pi\alpha'E)^2} \leftrightarrow L_{\text{particle}} \propto \sqrt{1 - V^2}. \tag{3.0.10}
\]

The existence of a critical electric field is dual to the existence of a critical relativistic velocity. This will become more explicit in the next section.

### 3.1 The T-duality \( E \leftrightarrow V \) on the D-brane action

The bosonic \( D_{25} \)-brane action with \( A_\mu \) gauge fields on its world volume is given by

\[
S = \frac{1}{4\pi\alpha'} \int d\tau d\sigma d^a X^\mu \partial_a X_\mu + i\epsilon \sum_{\mu=0}^{25} \int d\tau A^\mu(X^0, ..., X^p) \partial_\tau X_\mu \tag{3.1.11}
\]

where the gauge fields are restricted to depend only on the set of \((X^0, ..., X^p)\) coordinates and we have neglected the Kalb-Ramond \( B_{\mu\nu} \) field contribution. Here all the string target space coordinates obey the Neumann boundary condition. Take the
gauge $A^M = F^{M\alpha}X_\alpha$ for $\alpha = 0, 1, \ldots, p$ and $M = p + 1, \ldots, 25$. Considering only the disk-interactions, the low energy effective Lagrangian is given by the Born-Infeld Lagrangian

$$L_{BI} \sim \sqrt{-\det(\eta_{\mu\nu} + 2\pi\alpha'eF_{\mu\nu})} \quad (3.1.12)$$

where $\eta_{\mu\nu}$ is the metric on the brane. Performing a $T$-duality along the coordinates $X^M$, we obtain a $D_p$-brane describing some trajectory in the $D = 26$ Minkowski spacetime given by $Y^M = 2\pi\alpha'eA^M$. Under such transformation, the above Born-Infeld action (3.1.12) changes to the Nambu-Goto action of a relativistic boosted $D_p$-brane

$$L_{D_p} \sim \sqrt{-\det(\eta_{\alpha\beta} + \partial_\alpha Y^M \partial_\beta Y_M)} \quad (3.1.13)$$

Here $Y^M$ is written in the gauge where the $p + 1$ longitudinal coordinates parametrize the worldsheet of the brane.

On the other hand, the sigma model describing the dynamics of a bosonic $D_p$-brane is given by

$$S = \frac{1}{4\pi\alpha'} \int d\tau d\sigma \partial^\alpha X^\mu \partial_\alpha X_\mu + \frac{1}{2\pi\alpha'} \sum_{M=p+1}^{25} \int d\tau Y^M(X^0, \ldots, X^p) \partial_\sigma X_M \quad (3.1.14)$$

where $Y^M$ describes the trajectory of the $D_p$-brane and boundaries are taken to be the timelike surfaces $\sigma = 0, \pi$. From the actions (3.1.11) and (3.1.14) we may compute the two point correlation functions. The Neumman coordinates transform under T-duality to Dirichlet coordinates. Both are related by

$$<\partial_\sigma X^i(\tau) \partial_\sigma X^j(\tau') >_{\text{Dirichlet}} = -<\partial_\tau X^i(\tau) \partial_\tau X^j(\tau') >_{\text{Neumann}} = \frac{2\alpha'\delta^{ij}}{(\tau - \tau')^2} \quad (3.1.15)$$

The dynamics of the brane is visualized by choosing a reference frame defined by a second $D_p$-brane in which the first brane is moving at velocity $V$. Open strings stretched between the two branes obey Neumann boundary conditions along $X^{1, \ldots, p}$ coordinates and Dirichlet conditions on $X^{p+1, \ldots, 24}$ and

$$X^{25} = \partial_\sigma X^0 = 0 \quad \text{at } \sigma = 0$$

$$X^{25} - vX^0 = \partial_\sigma (X^0 - vX^{25}) = 0 \quad \text{at } \sigma = \pi \quad (3.1.16)$$
The solutions of these constraints are technically similar to closed string in a twisted sector of an orbifold with imaginary twist angle given by the velocity \( e := \text{arctanh}(V) \)

\[
X^0 \pm X^{25} := X^\pm = i\sqrt{\alpha'} \sqrt{\frac{1 \pm V}{1 \mp V}} \times \sum_{n=-\infty}^{\infty} \left[ \frac{a_n}{\sqrt{n+\epsilon}} e^{-i(n+\epsilon)(\tau \mp \sigma)} + \frac{\tilde{a}_n}{\sqrt{n-\epsilon}} e^{-i(n-\epsilon)(\tau \pm \sigma)} \right]
\]

(3.1.17)

where left-mode oscilators obey the canonical commutation relations

\[
[a_n, a_m] = \delta_{n+m,0}
\]

(3.1.18)

with the same condition on the right-movers. The existence of a relativistic limit in the electric field is now interpreted as an upper bound on the relativistic velocity, as was nicely explained in [5].

One may also look for the states associated to moving D-branes [11][26]. As we have seen in Sect.2.4, the boundary state associated to a static \( D_p \)-brane at position \( y_i \) is given by

\[
|B_X\rangle = (2\pi \sqrt{\alpha'})^{25-p} \delta^{25-p} (q^i - y^i) \exp \left( -\sum_{n=1}^{\infty} a^-_n S_{\mu\nu} a^\mu_n \right) |0\rangle \langle 0| p = 0
\]

(3.1.19)

where

\[
S_{\mu\nu} = (\eta_{\alpha\beta} , -\delta_{ij})
\]

(3.1.20)

and we have chosen indices \( \alpha, \beta (i,j) \) to label the components longitudinal (transverse) to the \( D_p \)-brane. The state associated to the boosted \( D_p \)-brane is characterized by

\[
|B, y, V\rangle = \exp^{ij} J_{ij} |B, y(V)\rangle
\]

\[
y^i(V) = y^i + V^i(V) y^j (\cosh |V| - 1)/|V|^2
\]

(3.1.21)

where \( J^{\mu\nu} \) is the boost operator

\[
J^{\mu\nu} = q^\mu p^\nu - q^\nu p^\mu - i \sum_{n>0} (a^-_n a^\nu_n - a^-_n a^\mu_n + \tilde{a}^-_n \tilde{a}^\nu_n - \tilde{a}^-_n \tilde{a}^\mu_n)
\]

(3.1.22)

Considering the boost in a single transverse direction label as the 25—direction, the boosted state becomes

\[
|B, y, V\rangle = (2\pi \sqrt{\alpha'})^{25-p} \sqrt{1 - V^2} \delta(q^1 - q^0 V - y^1) \prod_{i \neq 1} \delta(q^i - y^i) \times \exp \left( -\sum_{n=1}^{\infty} a^-_n S_{\mu\nu} a^\mu_n \right) \exp \left( -(a^-_0 a^1_n) M(V)^2 \left( \frac{\tilde{a}^-_0}{a^1_n} \right) \right) |0\rangle \langle 0| p = 0
\]
with
\[
M(V) = \begin{pmatrix}
\cosh V & -\sinh V \\
-\sinh V & \cosh V
\end{pmatrix}
\]
where the notation \( S_{\mu\nu}|_{\mu\neq 0,25} \) means that we neglect the directions \( \mu = 0,25 \) of the target coordinates, longitudinal to the motion. Note that in the above state there is a boost of left-movers relative to the right-movers, similar to what we have found for charged open strings immersed in a constant electric field [15], see end of Sect.2.4.2.

We see from the above remarks that a priori the boost limit \( V \to 1 \) is non-smooth as expected to happen for any massive (tensive) object. This does not mean that in the moduli space there is no brane that may be interpreted as an infinitely boosted \( D \)-brane. So far, the dynamics of D-branes tells us that there is a relative boost of the left-movers with respect to the right-movers. We thus expect that a nullbrane should be described by a purely single chiral sector, left or right depending on its orientation in the spacetime. By the \( V \leftrightarrow E \) duality, a formulation of a nullbrane will presumably give a picture of open strings with charged endpoints at a critical electric field.

### 3.2 Space/space noncommutative field theory

To find the propagator for an open string immersed in a constant electromagnetic field in a target space with constant metric \( g_{\mu\nu} \), we have to solve the following constraints for the Green’s function
\[
D^{\mu\nu}(z,z') = \langle X^\mu(z)X^\nu(z') \rangle \\
g_{\mu\nu}\partial\bar{\partial}D^{\mu\nu}(z,z') = -2\pi\alpha'\delta(z-z') \\
\left[ g_{\mu\nu}(\partial - \bar{\partial}) + 2\pi\alpha'F_{\mu\rho}(\partial + \bar{\partial}) \right] D^{\mu\nu}(z,z') \big|_{z=\bar{z}=0} = 0
\]
where \( \partial = \partial/\partial z, \bar{\partial} = \partial/\partial \bar{z} \) and \( \Im z \geq 0 \). The solution is
\[
D^{\mu\nu}(z,z') = -\frac{1}{2}\alpha' g_{\mu\nu} \ln |z - z'|^2 + \frac{1}{2}\alpha' \left( \frac{g + 2\pi\alpha'F}{g - 2\pi\alpha'} \right)_{\mu\nu} \ln(z - \bar{z}') \\
+ \frac{1}{2}\alpha' \left( \frac{g - 2\pi\alpha'F}{g + 2\pi\alpha'F} \right)_{\mu\nu} \ln(\bar{z} - z')
\]
(3.2.23)
The propagator at the boundary \( \sigma = \sigma' = 0 \) in the limit when \( \tau \to \tau' \) is
\[
D_{\mu\nu}(\tau \to \tau') = -\alpha' \left[ g + \frac{1}{2} \left( \frac{g - 2\pi\alpha'F}{g + 2\pi\alpha'F} \right) + \frac{1}{2} \left( \frac{g + 2\pi\alpha'F}{g - 2\pi\alpha'F} \right) \right]_{\mu\nu} \ln \Lambda
\]
\[ = -2\alpha' \left[ \frac{g}{(g - 2\pi\alpha'F)(g + 2\pi\alpha'F)} \right]_{\mu\nu} \ln \Lambda \quad (3.2.25) \]

where \( \Lambda \) is a short distance cutoff. We may now find the 1-loop contribution to the perturbed system [25]

\[ S_1 = \frac{i}{2\pi} \int_{z = \bar{z}} ds \Gamma_\mu \partial_s X^\mu \]
\[ = -\frac{i}{4\pi\alpha'} \int ds \partial_\nu F_{\mu\lambda} \partial_\tau X^\mu D^{\nu\lambda}(\tau, \tau') |_{\tau \to \tau'} \quad (3.2.26) \]

and calculate the beta function, from which we get the equations of motion by imposing the fixed point condition

\[ \beta_\mu = \lambda \frac{\partial}{\partial \lambda} \Gamma_\mu \]
\[ = \partial^\nu F^\lambda_\mu \left[ \frac{g}{(g - 2\pi\alpha'F)(g + 2\pi\alpha'F)} \right]_{\lambda\nu} = 0 \quad . \quad (3.2.27) \]

They are also given by the effective Langrangian

\[ L_{\text{eff}} \sim \sqrt{-\det(g_{\mu\nu} + 2\pi\alpha' F_{\mu\nu})} \quad . \quad (3.2.28) \]

A more general expression of the propagator at the boundary is [53]

\[ \langle X^\mu(\tau)X^\nu(\tau') \rangle = -\alpha' G^{\mu\nu} \log(\tau - \tau')^2 + \frac{i}{2} \Theta_{\mu\nu} \text{sign}(\tau - \tau') \quad (3.2.29) \]

that should be compared with the two-point correlation function of the dissipative Hofstadter model, see end of Sect.2.6. Here the mobility and the Hall term are given by

\[ G_{\mu\nu} = \left( \frac{1}{g + 2\pi\alpha' B} \right)^{\mu\nu} \],
\[ \Theta^{\mu\nu} = -(2\pi\alpha')^2 \left( \frac{1}{g + 2\pi\alpha' B} \right)^{\mu\nu} \quad . \quad (3.2.30) \]

The term \( G^{\mu\nu} \) is known in the string literature as the inverse effective metric seen by the open strings, that depends on the closed string metric \( g_{\mu\nu} \). The Hall effect term is interpreted in condensed matter physics as the quantity that gives the commutation relations of operators from the short distance behaviour of time ordered product (the famous BJL relation, for a recent discussion see for example [46] and references therein)

\[ [X^\mu(\tau), X^\nu(\tau')] = T \left( X^\mu(\tau)X^\nu(\tau^-) - X^\mu(\tau)X^\nu(\tau^+) \right) = i\Theta^{\mu\nu} \quad (3.2.31) \]
that is interpreted as a spacetime with a noncommutative geometry. From the boundary correlation functions, we see that the spacelike coordinates of the open string ends become noncommutative in the allowed limit of infinite magnetic field $B_{ij} \to \infty$. This limit is the same as if we kept the magnetic field constant and take otherwise the field theory limit $\alpha' \to 0$, for which the string oscillators disappear.

### 3.3 Space/time noncommutativity: Field Theory vs String Theory

We might think that the space/space noncommutativity in the presence of a magnetic field could be extended to its space/time counterpart by turning on a constant electric field, say along the $X$-direction

$$\Delta T \Delta X \sim \theta$$

in the field theory limit. This is however not the case since there is a critical value of the electric field such that above it String Theory has no physical meaning. Thus it is not possible to take the field theory limit by considering the limit $E \to +\infty$. Nevertheless, it is possible to find a certain limit for which space/time noncomutativity arises in a new noncritical String Theory where open strings decouple from the closed strings, in particular from the gravitational sector of the theory [51]. Here, the open strings are confined to the branes, that is to say, they cannot interact with each other in the usual way to transform into a closed string leaving the brane and propagating freely in the bulk. In the new theory such a process would cost an infinite amount of energy. Charged open strings behave as a dipole oriented by the electric field lines. At the critical electric field limit, an infinite amount of energy would be necessary to push both ends of the string to meet. Such a phenomenon is seen by a given worldsheet observer for which the boundary conditions are consistent with the modified light cone gauge

$$X^+ = \tau + \tilde{E}\sigma$$

given in terms of the normalized electric field $\tilde{E} = 2\pi\alpha' E$.

Note that for vanishing electric field and non-zero magnetic field, the gauge $X^+ = \tau$ is perfectly consistent with the familiar boundary conditions on timelike boundaries. If
we want to keep in mind that physics should be gauge invariant, we note that there is something peculiar in the consistency of boundary conditions with respect to timelike boundaries when we turn on the electric field. In the next Chapter we shall take the general expression for the Polyakov action that depends on the worldsheet metric, and we shall study the consistency of boundary conditions in the light-cone gauge $X^+ = \tau$. We will see that the critical electric field is related to a Penrose limit in the worldsheet metric where boundaries become lightlike.

Nevertheless, it is not possible to find a field theory limit in the presence of the electric field. In fact, space/time noncommutative field theory is acausal and non-unitary, so we are glad not to have such a limit from string theory.

To see this, we compute the $S$-matrix describing the scattering of two particles in 1+1 dimensions. Let us call $iM$ the non-trivial components of it

$$
\langle k_1, k_2 | S | p_1, p_2 \rangle = (2\pi)^2 \delta^2 (k_1 + k_2 - p_1 - p_2) (1 + iM) .
$$

Consider the tree-level amplitude of $\phi^4$-theory with coupling constant $g$. At the center of mass frame, where the incoming and outgoing particles have the same absolute value of spatial momenta but with opposite directions, the non-trivial contribution to the $S$-matrix is

$$
iM \sim ig .
$$

In the presence of space/time noncommutativity, the usual field multiplication is replaced by the $\star$-product with Moyal phase depending on the energy modes

$$
[t, x] = i\theta \\
\phi_1 \star \phi_2(x, t) = e^{i\theta \left( \partial_x \partial_t^\dagger - \partial_t \partial_x^\dagger \right)} \phi(w) \phi(z) \big|_{w=z=(x,t)} .
$$

We now have

$$
iM \sim ig(\cos(4p^2\theta) + 2)
$$

where $p$ is the center of mass frame momenta of the incoming (and outgoing) particles. The presence of non-vanishing Moyal phase $\theta$ will produce an outgoing wave-function that is split into three parts, two of them concentrated at $x \sim \pm p_0 \theta$ and the third one at $x = 0$. The first two are interpreted as advanced and retarded waves, that is
to say, they are created before and after the collision takes place. Since the Moyal phase depends on the energy of the particles, the bigger the energy, the bigger the time shifts. The problem comes from the advanced wave - if we want to preserve Lorentz invariance, it violates causality. The scattering particles resemble rigid rods that expand as their momentum increases, see [52].

Fortunately, those pathologies are not present for space/time noncommutative string theory. Here the string oscillator modes play an essential role since they do not allow the creation of the pathological advanced wave packet. To see that, we start to compute the disk Veneziano amplitude without the presence of an electric field of four massless open strings. Using the Mandelstam variables

\[
s = 2p_1p_2 , \quad t = 2p_1p_4 , \quad u = 2p_1p_3
\]

the disk amplitude splits into three terms

\[
A_{D_2}(p_1, p_2, p_3, p_4) \sim g_s \delta(\sum_i p_i) \left[ I(s, t) + I(t, u) + I(u, s) \right]
\]

where \( g_s \) is the open string coupling and the functions \( I(x, y) \) are expressed in terms of the gamma functions. Consider now the case when an electric field is present. The new disk amplitude is obtained from the above one by replacing the metric \( \eta_{\mu\nu} \) by the effective open string metric \( G_{\mu\nu} \), \( g_s \) by \( G_s \) and by multiplying the terms by the Moyal phases. After proper modifications, the first term on (3.3.39) is now given by

\[
I^\theta(s, t) \sim G_s \left( K_{st} \exp^{2\pi i \tilde{E}s\ell_s^2} + K_{st} \exp^{-2\pi i \tilde{E}s\ell_s^2} \right) \gamma(-2s\ell_s^2)\gamma(2s\ell_s^2)
\]

where we have expressed the kinematic terms involving the momenta \( p_i \) by factors \( K \) that in the noncommutative case are proportional to \( s^2 \). The second (advanced) phase is the one responsible for acausality in the field theory case. In the string case, the dependence on the gamma functions is crucial. If we expand their product into power series, the acausal behavior is cancelled and we are left with

\[
I^\theta(s, t) \sim G_s s \sum_{n>0 \text{ odd}} a_1 \exp^{2\pi i(n+\tilde{E})s\ell_s^2} + a_2 \exp^{2\pi i(n-\tilde{E})s\ell_s^2}
\]

for some constants \( a_1 \) and \( a_2 \) independent of \( s \). Let us remember that the new non-critical string theory was obtained by scaling the metric

\[
g^{-1} \sim 1 - \tilde{E}^2 \quad \ell_s^2 = \alpha'/g
\]
where the noncommutative parameter in this limit is finite

$$\theta \sim 2\pi \alpha'. \quad (3.3.43)$$

By comparing the above string amplitude to the field theory case, we note that the string oscillators modify the noncommutative parameter by

$$\theta_n = 2\pi(n \pm \tilde{E})\ell_s^2, \quad n > 0, \text{ odd}. \quad (3.3.44)$$

Nevertheless, the string grows with energy, contrary to the intuitive Lorentz contraction. But this phenomenon is well known and explained in [58] as a consequence that at high energy, we are able to see more string-oscillating modes than at low energy and consequently, the string seems to grow with the energy. This is an important point of the black hole complementarity paradox.

We may interpret the phenomena as the occurrence of a phase transition as we reach the critical limit for the electric field. It is possible that the phase transition gives a new unstable vacuum where the usual phase space is replaced by a noncommutative spacetime, i.e., the phase transition is a space/phase-space transmutation that might be associated to a new vacuum state. This raises the question of looking for a model where space/time noncommutativity arises without the presence of an electric field. In the next section we will see that the Schild action is one possible model [61].

### 3.4 Tensionless strings and the Schild action

It is well known that the Polyakov action is classically equivalent to the Nambu-Goto action $S_{NG}$, which has the simple $n$-extension

$$S_n = -\int d\tau d\sigma \left[ \frac{1}{e^n} \left( -\frac{1}{2\lambda^2}(\varepsilon^{ab}\partial_a X^\mu \partial_b X^\nu)^2 \right)^{n/2} + n - 1 \right]$$

$$S_1 = S_{NG}. \quad (3.4.45)$$

The case $n = 2$ was first proposed by Schild [48]. His action describes tensionless null strings travelling at the speed of light, including their endpoints for the case of open strings. We emphasize here that from the classical equations of motion for a free Nambu-Goto string, we see that the endpoints travel at the speed of light but the Nambu-Goto string should not be confused with the null Schild string, where all the points travel at the speed of light and the string is tensionless.
The quantization of the null string is given in the Appendix A where we show that the endpoint target coordinates describe a non-commutative spacetime geometry.

On the other hand, the Polyakov action seems to be the unique two dimensional action which is tractable in the path-integral approach. Nevertheless, attempts are made for the case of the Schild action whose path-integral is of the form

\[ Z = \int D\gamma D\chi D\chi e^{-S_{\text{schild}}}. \]  

(3.4.46)

Here we have included the fermionic counterpart. The path-integral can be approximated by a quantum matrix IKKT-model [33]

\[ Z = \sum_{n=0}^{\infty} \int dA d\psi e^{-S_{\text{IKKT}}}. \]  

(3.4.47)

with

\[ S_{\text{IKKT}} = \alpha \left( -\frac{1}{4} \text{Tr}[X_\mu, X_\nu]^2 - \frac{1}{2} \text{Tr}(\bar{\psi}\gamma^\mu [X_\mu, \psi]) \right) + \beta \text{Tr}1 \]  

(3.4.48)

where \( X_\mu \) and \( \psi \) are bosonic and fermionic \( N \times N \) hermitian matrices respectively. In the large \( N \) limit they correspond to the classical spacetime coordinates and the fermionic counterpart fields. The Schild action coincides with the IKKT action in the large \( N \) limit after we replace the commutator and the trace by the Poisson bracket and the integration

\[ -i[\cdot, \cdot] \Rightarrow \{\cdot, \cdot\}_{PB} \]

\[ \text{Tr} \Rightarrow \int d^2 \sigma \sqrt{\gamma} \ . \]  

(3.4.49)

Moreover, it follows that the Schild action is conformal invariant only if we impose the constraint on the Poisson bracket

\[ -\frac{1}{2}(\{X_\mu, X_\nu\})^2 = \lambda^2 \]  

(3.4.50)

which is interpreted as a spacetime noncommutativity condition. After Dirac quantization, the spacetime coordinates are described by hermitian matrices and their physics by the IKKT-model.
Chapter 4

Flat $H$-branes and time-dependent Conformal Field Theory

We have seen in the last Chapter that charged open strings in the presence of an electric field are related by T-duality to boosted $D$-branes, both bounded by a certain critical value. In the electric case, because of the pair-production, we require to spend an infinite amount of energy to reach the critical value on the electric field. In the $T$-dual picture this phenomenon is interpreted as an infinite amount of energy necessary to infinitely boost a tensive D-brane. Nevertheless, in the worldsheet sense, the behaviour of the theory exactly at the critical value seems to present no pathology - it would describe some sort of a nullbrane.

The picture we have in mind is that in the critical limit there is a space/phase-space transmutation, that is to say, the usual momenta-coordinate commutation relations are replaced by the canonical commutation relations of a noncommutative spacetime vacuum. At this phase transition, infinitely boosted $D$-branes are replaced in the moduli space by $H$-branes, described by chiral and squeezed Ishibashi states

$$D - brane \rightarrow H - brane \quad . \quad (4.0.1)$$

Note that to describe a nullbrane in terms of boundary conditions, we need to have a Neumann condition for, say, $X^+$ and a Dirichlet condition for $X^-$, where $X^\pm$ are target lightcone coordinates. Is this possible?

To answer this let us recall how one gets boundary conditions for the bosonic open strings (for more details see [63]). The starting point is the variation of the Polyakov
action that gives equations of motion from the bulk terms plus boundary conditions from the boundary term

\[ \delta S_P = -\frac{1}{4\pi\alpha'} \int_{\Sigma} d\tau d\sigma \delta X^\mu \partial_a \left( (-\gamma)^{1/2} \gamma^{ab} \partial_b X_\mu \right) - \frac{1}{4\pi\alpha'} \int_{\partial\Sigma} ds \delta X^\mu t^a \partial_a X_\mu \] (4.0.2)

where \( t^a \) is a unit tangent vector to the boundary. Take the worldsheet metric in the conformal gauge and the familiar timelike worldsheet boundaries. The boundary conditions on the null and transverse target coordinates are

\[
\begin{align*}
\delta X^+ \partial_\sigma X^- &= 0 \\
\delta X^- \partial_\sigma X^+ &= 0
\end{align*}
\]

where \( i = 2, 3, ..., 25 \) labels the transverse coordinates. From the first line of (4.0.3), we see that if we choose the Dirichlet condition for say \( X^+ \), then \( \partial_\sigma X^+ \neq 0 \). Then from the second line we are forced to choose also a Dirichlet condition for \( X^- \), since \( X^+ \) do not obey the Neumann condition. By the same argument, if we choose the Neumann condition for \( X^- \), then we will end also with the Neumann condition for \( X^+ \). On the transverse coordinates there are no such restrictions since boundary conditions do not have a cross product as in \(+, -\) case. We can choose for each transverse coordinates, either Dirichlet or Neumann conditions. It seems that a \( D \)-brane with a single null direction is impossible.

Is this really so?

### 4.1 Deformation of worldsheet boundaries

We start to analyze time-dependent conformal field theory considering boundary perturbations as a source of a dynamical deformation of the usual timelike worldsheet boundaries \( \sigma = \text{const} \) [38]. Consider the string worldsheet \( \Sigma \) parametrized by the coordinates \((\tau, \sigma)\) of signature \((-+, +)\), with \( \tau \) the dynamical variable and boundaries defined at \( \sigma = 0, \ell \). We take the non-linear sigma model describing charged open string immersed in a constant (normalized) electric field \( 0 \leq \tilde{E} \leq 1 \) along the \((T, X)\)
spacetime plane \(^1\)

\[ S_{E}^{\text{open}} = -\frac{1}{4\pi\alpha'} \int_{\Sigma} d\tau d\sigma (-\gamma)^{1/2} \gamma^a_\tau \partial_{a} X^\mu \partial_{\mu} X_\tau - \frac{\bar{E}}{4\pi\alpha'} \int_{\partial \Sigma} ds X^+ t^a \partial_a X^- \]  

(4.1.4)

where \( t^a \) is a unit vector tangent to the boundary and we have defined \( X^\pm := (T \pm X)/\sqrt{2} \). Now we dynamically construct the string worldsheet metric as in [63] by working in the lightcone gauge \( X^+ = \tau \) with two more conditions on the worldsheet metric

\[ \partial_\sigma \gamma_{\sigma\tau} = 0 \quad \text{det}(\gamma_{ab}) = -1 \quad . \]  

(4.1.5)

Note that the lightcone gauge is different from (3.3.33) and this is the cause for the dynamical deformation of worldsheet boundaries from the familiar timelike to lightlike one.

From the variational principle for the above action, the boundary condition for \( X^+ = \tau \) target coordinate gives the following restriction on the worldsheet metric [38]

\[ ds^2 = \gamma_{\tau\tau} d\tau^2 + \gamma_{\sigma\sigma} d\sigma^2 - 2\bar{E} d\tau d\sigma \]  

(4.1.6)

with

\[ \gamma_{\sigma\sigma} \gamma_{\tau\tau} = \bar{E}^2 - 1 \]  

(4.1.7)

where

\[ \gamma_{\sigma\sigma} = \gamma_{\sigma\sigma}(\tau) \quad . \]  

(4.1.8)

The complete set of boundary conditions are

\[ \partial_{\sigma} X^\pm = 0 \]  

\[ 2\bar{E} \partial_{\tau} X^- = f(\bar{E}) \partial_{\sigma} X^- \]  

\[ \bar{E} \partial_{\tau} X^i = f(\bar{E}) \partial_{\sigma} X^i \]  

(4.1.9)

where \( f(\bar{E}) = -\gamma_{\tau\tau}(\tau) \) and we impose that \( f(0) = 1 \) and \( f(1) = 0 \), consistently with \( (4.1.7) \). In general we have a mixing of Dirichlet and Neumann conditions as expected.

\(^1\)In [38] we have conjectured the existence of a boundary term that was added to the Polyakov action with a \( \beta \) parameter taking values between 0 and 1. We have later identified the physics of such conjectured model with the present one where \( \beta = \bar{E} \).
Nevertheless, for \( \tilde{E} \rightarrow \tilde{E}_{\text{crit}} = 1 \) there is a clear separation of boundary conditions on the target lightcone coordinates

\[
\partial_\sigma X^+ = 0 \ , \quad \partial_\tau X^- = 0 \ , \quad \partial_\tau X^i = 0
\] (4.1.10)

at \( \sigma = 0, \ell \). This is the technical starting point to define a nullbrane!

From the non-linear sigma model action we define its Lagrangian by considering \( \tau \) as the dynamical variable. The Hamiltonian in lightcone gauge is \[38\]

\[
H_{\text{open}}^{\tilde{E}} = \frac{\ell}{4\pi\alpha'} \int_0^\ell d\sigma \left[ 2\pi\alpha' \Pi^i \Pi_i + \frac{1}{2\pi\alpha'} \partial_\sigma X^i \partial_\sigma X^i + 2\tilde{E} \Pi^i \partial_\sigma X^i \right] - \frac{\tilde{E}}{2\pi\alpha'} \int_0^\ell d\sigma \partial_\tau Y^- - \frac{f(\tilde{E})}{4\pi\alpha'} \left[ X^+ \partial_\sigma Y^- |_{\sigma=0} + X^+ \partial_\sigma Y^- |_{\sigma=\ell} \right] (4.1.11)
\]

where we have split \( X^-(\tau, \sigma) = Y^-(\tau, \sigma) + \bar{X}^-(\tau) \) with

\[
\bar{X}^-(\tau) = \frac{1}{\ell} \int_0^\ell d\sigma X^-(\tau, \sigma) .
\] (4.1.12)

The momentum conjugate to \( \bar{X}^- \) is

\[
p_- = -p^+ = -\frac{\ell}{2\pi\alpha'} \gamma_{\sigma\sigma}(\tau)
\] (4.1.13)

and for \( X^i \) is

\[
\Pi^i = \frac{p^+}{\ell} \partial_\tau X^i - \frac{\tilde{E}}{2\pi\alpha'} \partial_\sigma X^i .
\] (4.1.14)

The equations of motion are

\[
\partial_\tau p^+ = 0
\]

\[
-\frac{1}{c} \partial_\tau^2 X^i - 2\tilde{E} \partial_\tau \partial_\sigma X^i + c(1 - \tilde{E}^2) \partial_\sigma^2 X^i = 0
\] (4.1.15)

for \( c = \frac{\ell}{2\pi\alpha' p^+} \) where \( p^+ \) is a conserved quantity. The equation of motion (4.1.15) becomes the wave equation

\[
\gamma^{ab} \partial_a \partial_b X = -\gamma_{\sigma\sigma} \partial_\tau^2 X + 2\gamma_{\tau\sigma} \partial_\tau \partial_\sigma X - \gamma_{\tau\tau} \partial_\sigma^2 X = 0
\] (4.1.16)

if we take

\[
\gamma_{\sigma\sigma} = 1/c \quad \gamma_{\tau\tau} = -c(1 - \tilde{E}^2) ,
\] (4.1.17)
see (4.1.6). Moreover, it’s convenient to take the combination $\ell/p^\perp$ such that we have $c\sqrt{1 - \tilde{E}^2} = 1$ and $f(\tilde{E}) = -\gamma_{\tau\tau} = \sqrt{1 - \tilde{E}^2}$, so that in particular the critical limit takes the form

$$c \to \infty \quad \text{when} \quad \tilde{E} \to 1 \quad \text{such that} \quad c(1 - \tilde{E}^2) \to 0 \quad . \quad (4.1.18)$$

We see that by turning on a boundary perturbation given by the $U(1)$-currents $(\partial_\tau X^-, \partial_\tau X^+)$, we have dynamically deformed the familiar timelike boundaries $\sigma = \text{const}$ to a new form that can be read from the perturbed worldsheet metric in Lorentz signature

$$ds^2 = -\sqrt{1 - \tilde{E}^2}d\tau^2 - 2\tilde{E}d\tau d\sigma + \sqrt{1 - \tilde{E}^2}d\sigma^2 \quad (4.1.19)$$

where $\tau$ is the dynamical variable.

At the critical limit, the equations of motion are

$$\gamma^{ab}\partial_a\partial_b X = \partial_\tau \partial_\sigma X = 0 \quad (4.1.20)$$

with general solution given as usual by a linear combination of left- and right-chiral movers

$$X = X_L(\tau) + X_R(\sigma) \quad (4.1.21)$$

with the worldsheet coordinates $\tau$ and $\sigma$ parametrizing the left- and right-chiral movers. So they are nullcoordinates, consistent with (4.1.6) where we have at the critical limit $ds^2 = d\tau d\sigma$ and the worldsheet boundaries $\sigma = \text{const}$ become null. By a careful analysis, it is interesting to note that the Hamiltonian takes the unusual form [38]

$$H_{1}^{\text{open}} = -\frac{1}{2\pi \alpha'} \int_0^\ell d\sigma \partial_\sigma Y^- \quad (4.1.22)$$

with transverse conjugate momenta

$$\Pi^i = -\frac{1}{2\pi \alpha'} \partial_\sigma X^i \quad . \quad (4.1.23)$$

Note that all transverse oscillators are frozen in time.

Similarly, at the critical limit we may add the fermionic contribution to the system given by the action

$$S_{1}^{\text{open}}(\psi) = \frac{1}{8\pi} \int_0^\ell d\sigma \int_{-\infty}^\infty d\tau \left( \psi^\mu_L \partial_\tau \psi^\mu_L + \psi^\mu_R \partial_\sigma \psi^\mu_R \right) \quad (4.1.24)$$
where $\psi_L, \psi_R$ are the left- and right-Majorana spinors. The momenta conjugate to the fermionic fields are

$$
\Pi^\mu_L = \frac{\delta L}{\delta (\partial_\tau \psi^\mu_L)} = \frac{1}{8\pi} \psi^\mu_L
$$

and

$$
\Pi^\mu_R = \frac{\delta L}{\delta (\partial_\tau \psi^\mu_R)} = 0
$$

(4.1.25)

and the fermionic contribution to the Hamiltonian is

$$
H^{open}_1(\psi) = -\frac{1}{8\pi} \int d\tau d\sigma \psi^\mu_R \partial_\sigma \psi^\mu_R.
$$

(4.1.26)

### 4.2 Perturbative analyses of $H$-branes in the open string channel

Let us define the variables $\xi^\pm = \delta_\tau X^\pm, \xi^\pm_\sigma = \delta_\sigma X^\pm$ where $X^\pm$ are the null target coordinates. The perturbative action at the critical electric field $\tilde{E} = 1$ is

$$
-\frac{1}{2\pi\alpha'} \int_\Sigma \mathcal{L} = -\frac{1}{2\pi\alpha'} \int_\Sigma d\tau d\sigma \xi^-_\tau \xi^+_\sigma
$$

(4.2.27)

with $\Sigma = \{(\tau, \sigma) \mid d\mathcal{S}^2 = 2d\tau d\sigma\}$ with boundary $\delta\Sigma = \{(\tau, \sigma) \mid \sigma = 0, \pi\}$. We use the first order formalism to quantize the system. The one-form is

$$
\alpha = -\xi^-_\tau \xi^+_\sigma d\tau \wedge d\sigma + \xi^+_\sigma dX^- \wedge d\sigma + \xi^-_\tau d\tau \wedge dX^+.
$$

(4.2.28)

The equations of motion are

$$
X^*(i_{\delta X}d\alpha) = 0 \iff \partial_\sigma \partial_\tau X^\pm = 0
$$

(4.2.29)

with boundary conditions

$$
X^*(i_{\delta X}+\alpha) = 0 \iff \xi^- = 0 \quad \text{at} \quad \partial\Sigma.
$$

(4.2.30)

The space of solutions are given in terms of left- and right-sectors

$$
X^+(\tau, \sigma) = X^+_R(\sigma) + X^+_L(\tau) + X^+_0
$$

$$
X^-(\tau, \sigma) = X^+_R(\sigma) + X^-_0
$$

(4.2.31)

with

$$
\Omega = \int_0^\pi \partial_\sigma (\delta X^+) \wedge \delta X^- R d\sigma
$$

(4.2.32)
the symplectic two-form of the phase-space. The degeneracy of this form is given by the left-sector of the $X^+$ target variable plus its zero mode. They are gauged away from the space of solutions

$$X_R^+(\sigma) + X_L^+(\tau) + X_0^+ \longrightarrow X_R^+(\sigma) \quad (4.2.33)$$

so that they become non-dynamical, i.e. all string oscillators are frozen in time.

We proceed with the quantization of the system by looking at the Poisson brackets. Given a certain function of the phase space $F$ with variation

$$\delta F = \int_0^\pi f^-(\sigma) \delta X^+ d\sigma + \int_0^\pi f^+(\sigma) \delta X^- d\sigma \quad (4.2.34)$$

we define the associated Hamiltonian vector field

$$\mathcal{X}_F = \int_0^\pi g^-(\sigma) \frac{\delta}{\delta X^-} d\sigma + \int_0^\pi g^+(\sigma) \frac{\delta}{\delta X^+} d\sigma \quad (4.2.35)$$

using the phase space symplectic structure $\delta F = i_{\mathcal{X}_F} \Omega$ giving the relations

$$f^+(\sigma) = \partial_\sigma g^+(\sigma)$$
$$f^-(\sigma) = \partial_\sigma g^-(\sigma) \quad , \quad g^-(0) = g^-(\pi) = 0 \quad (4.2.36)$$

so that in particular

$$\int_0^\pi f^-(\sigma) d\sigma = 0 \quad . \quad (4.2.37)$$

From the Poisson bracket $\{\mathcal{F}, \mathcal{F}'\}_{PB} = \mathcal{X}_F(\mathcal{F}')$ we find the relations

$$\{X_R^-(\sigma), X^+(\sigma', \tau)\}_{PB} = \Theta(\sigma - \sigma')$$
$$\{X^+(\sigma, \tau), X_R^-(\sigma')\}_{PB} = \Theta(\sigma - \sigma') \quad (4.2.38)$$

with $\Theta(\sigma)$ the Heaviside function. Note that both relations are compatible with each other because the phase space symplectic structure has a degeneracy so that the $X^+$ coordinate is only defined up to a $\tau$-dependent function.

We can see that the complete set of mode oscillators satisfying (4.2.36) is

$$X_m^- = \sqrt{2/\pi} \int_0^\pi \sin(m\sigma) X_R^-(\sigma) d\sigma$$
$$X_m^+ = \sqrt{2/\pi} \int_0^\pi \cos(m\sigma) X_R^+(\sigma) d\sigma \quad (4.2.39)$$
for \( m > 0 \). They are respectively a set of Dirichlet and Neumann modes where the Neumann zero mode is gauged away. The canonical commutation relations are

\[
\{X_m^-, X_n^+\} = \delta_{m,n}. \tag{4.2.40}
\]

They should be interpreted as non-commutation relations among the null target coordinates. Since these oscillator modes behave as an \((x, p)\) set, we can define the creation and annihilation set as \((x + ip, x - ip)\) by

\[
\frac{1}{n}\alpha_n = \frac{1}{\sqrt{2}}(X_n^- + iX_n^+), \quad \frac{1}{n}\alpha_n^* = \frac{1}{n}\alpha_{-n} = \frac{1}{\sqrt{2}}(X_n^- - iX_n^+) \tag{4.2.41}
\]

for \( n > 0 \) and

\[
[\alpha_n, \alpha_{-n}] = in \tag{4.2.42}
\]

for which we define the Virasoro zero mode

\[
L_0 = \sum_{n \neq 0} \alpha_{-n}\alpha_n. \tag{4.2.43}
\]

### 4.3 Majorana spinors, ghost, superghost and the null boundaries

We now generalize the previous analysis of null worldsheet boundaries to the case of superstrings, where the fermionic contribution is given by (4.1.24) and we continue to restrict ourselves to lightcone target fields. Since the new type of boundaries will not modify the familiar equations of motion, we continue to expand the Majorana spinors as

\[
\psi_L^\pm(\tau) = \sum_{r \in \mathbb{Z}+\nu} \psi_{r}^\pm e^{-ir\tau},
\]

\[
\psi_R^\pm(\sigma) = \sum_{r \in \mathbb{Z}+\tilde{\nu}} \tilde{\psi}_{r}^\pm e^{-ir\sigma} \tag{4.3.44}
\]

where \( \nu \) and \( \tilde{\nu} \) can take the values 0 (Ramond sector), or 1/2 (Neveu-Schwarz sector). From the boundary contribution of the variation principle we have

\[
\psi_R^- \delta\psi_R^+ = \psi_R^+ \delta\psi_R^- = 0 \tag{4.3.45}
\]

at \( \sigma = 0, \pi \). It is sufficient to consider for example

\[
\psi_R^-(\tau, 0) = \psi_R^-(\tau, \pi) = 0 \tag{4.3.46}
\]
without any boundary condition on $\Psi_R^\dagger$. Since $\psi^-_R = \psi^-_R(\sigma)$ we have
\[ \tilde{\psi}^-_r = 0 \quad \forall r \in \mathbb{Z} + \tilde{\nu} . \tag{4.3.47} \]

Nevertheless, we could have imposed the same boundary conditions on fermionic field $\psi_R^+$
\[ \tilde{\psi}^+_r = 0 \quad \forall r \in \mathbb{Z} + \tilde{\nu} . \tag{4.3.48} \]

Note that if we had taken the null boundaries at $\tau = 0, \pi$, we would have the previous conditions on opposite chiral $\psi^\pm_L$ fields
\[ \psi^-_r = 0 \quad \text{or} \quad \psi^+_r = 0 \quad \forall r \in \mathbb{Z} + \nu . \tag{4.3.49} \]

Moreover, we would like to have at the null boundaries $\sigma = 0, \pi$ the BRST invariance given by the condition on the BRST charges
\[ Q + \tilde{Q} = 0 \tag{4.3.50} \]
where
\[ Q = \sum_n : (L_n^{(\alpha,\psi)} c_n + G_n^{(\alpha,\psi)} \gamma) : \]
\[ - \frac{1}{2} \sum_{m,n} (m-n) : c_{-m} c_{-n} b_{m+n} : \]
\[ + \sum_{m,n} (\frac{3}{2} n + m) : c_{-n} \beta_{-m} \gamma_{m+n} : \]
\[ - \sum_{m,n} : \gamma_{-m} \gamma_{-n} b_{m+n} : - ac_0 \tag{4.3.51} \]

with a similar expression for the right chiral fields. As we have imposed the boundary conditions on the bosonic and Majorana fields
\[ \tilde{\alpha}^+_n = \tilde{\psi}^+_n = 0 \quad \forall n \tag{4.3.52} \]
we see (so far we are neglecting transverse coordinates to the spacetime lightcone) $\tilde{L}_m^{(\alpha,\psi)} = \tilde{G}_m^{(\alpha,\psi)} = 0$ so that for condition (4.3.50) to hold, we need to impose the boundary conditions on the ghosts and superghosts
\[ c_n = \gamma_n = 0 \]
\[ \tilde{b}_n = \tilde{\beta}_n = 0 \quad \forall n \tag{4.3.53} \]
at $x^+ = 0, \pi$. Taking a variation on the Faddeev-Popov action

$$S_{FP} = \frac{1}{\pi} \int d\tau \int_0^\pi d\sigma (c^- \partial_\sigma b_{--} + c^+ \partial_\tau b_{++})$$

(4.3.54)

the null boundary conditions

$$c^- \delta b_{--} = 0$$

(4.3.55)

are in fact satisfied by considering only $c^-$ and we are free to set $b_{++} = 0$ as we have in (4.3.53).

### 4.4 Chiral closed strings and null boundaries

Our main qualitative result is easy to understand. The usual mixing boundary conditions for charged open strings immersed in a constant electric field

$$\sqrt{1 - \tilde{E}^2} \partial_\sigma X - \tilde{E} \partial_\tau X = 0 \quad \sigma = 0, \pi$$

(4.4.56)

are in the critical limit $\tilde{E}_{\text{crit}} = 1$ rewritten as a single boundary condition

$$\partial_\tau X = 0 \quad \sigma = 0, \pi$$

(4.4.57)

so that we have interpolated the Neumann condition for $\tilde{E} = 0$ to a Dirichlet condition for $\tilde{E} = 1$. On the other hand, the induction of null boundaries on the worldsheet allows us to interpret this condition as a Dirichlet condition with respect to the null boundary, i.e., a $(D; n)$-condition in time-dependent string theory.

It is easy to visualize that the no-flux of energy condition through the null boundary is $\tilde{T}(\tau) = 0$, so that the boundary states must satisfy the condition

$$\bar{L}_n^{(C; \ell)} |B\rangle = 0 \ .$$

(4.4.58)

The restriction that we had on the central charge $c = \bar{c}$ by requiring $\text{Diff}(S^1)$ modular invariance of the boundary states, is now replaced by the condition $\bar{c} = 0$. Such a condition has appeared in the past few years in the study of LCFT, see [34] for a review. The second consequence of null boundaries is that clearly there is no coupling between left- and right-chiral oscillator movers. In the target space, the defect caused by the boundary is interpreted as some kind of infinitely boosted D-brane; an H-brane.
We now show that it is possible to have purely Ishibashi chiral states along the
target null direction without introducing singularities in quantities that describe the
physical system, as for example, the (general) tree-level closed string amplitudes. Ne-
glecting the zero-modes, note that the closed string left-modes on the null direction
obey the Heisenberg algebra
\[
\begin{align*}
\left[ \alpha_m^+, \alpha_n^- \right] &= m \delta_{mn} \\
\left[ \alpha_m^-, \alpha_n^- \right] &= -m \delta_{mn}
\end{align*}
\] (4.4.59)
for \( m, n \geq 1 \). The chiral lightcone Heisenberg algebra is of the same form as if we had
considered left- and right-chiral Heisenberg algebras for a given transverse coordinate
\[
\begin{align*}
\left[ \alpha_m, \alpha_{-n} \right] &= m \delta_{mn} \\
\left[ \tilde{\alpha}_{-m}, \tilde{\alpha}_n \right] &= -m \delta_{mn}
\end{align*}
\] (4.4.60)
for \( m, n \geq 1 \). As both algebras are indistinguishable, we can make the next identifica-
tions by the one-to-one maps
\[
\begin{align*}
\alpha_m \leftrightarrow \alpha_m^+ \\
\alpha_{-m} \leftrightarrow \tilde{\alpha}_{-m} \\
\tilde{\alpha}_m \leftrightarrow \alpha_m^- \\
\tilde{\alpha}_{-m} \leftrightarrow \alpha_{-m}^+
\end{align*}
\] (4.4.61)
for \( m > 0 \). We see that the familiar non-chiral and coherent Ishibashi state of a
D-brane
\[
|D\rangle = \exp \left( \sum_{m \geq 1} \frac{1}{m} \alpha_{-m} \tilde{\alpha}_{-m} \right) |0\rangle
\] (4.4.62)
is mapped to the chiral and coherent Ishibashi state along a null direction
\[
|N\rangle = \exp \left( \sum_{m \geq 1} \frac{1}{m} \alpha_{-m}^+ \alpha_{-m} \right) |0\rangle
\] (4.4.63)
as well as
\[
\langle D | = \langle 0 | \exp \left( \sum_{m \geq 1} \frac{1}{m} \alpha_m \tilde{\alpha}_m \right)
\] (4.4.64)
to
\[
\langle N | = \langle 0 | \exp \left( \sum_{m \geq 1} \frac{1}{m} \alpha_m^+ \alpha_m^- \right) .
\] (4.4.65)
We proceed to the analysis of the Majorana spinors where for \( m, n > 0 \) the anticommutation relations for a single coordinate

\[
\{ \psi_r, \psi_{-t} \} = -\{ \bar{\psi}_{-r}, \bar{\psi}_t \} = \delta_{rt}
\]

obeys the same algebra as for Majorana spinors in lightcone coordinates

\[
\{ \psi^+_r, \psi^-_{-t} \} = -\{ \psi^-_{-r}, \psi^+_t \} = \delta_{rt}
\]

for \( r, t > 0 \). We thus extend the previous map to

\[
\begin{align*}
\psi_r &\leftrightarrow \psi^+_r \\
\psi_{-r} &\leftrightarrow \psi^-_{-r} \\
\bar{\psi}_r &\leftrightarrow \psi^-_{-r} \\
\bar{\psi}_{-r} &\leftrightarrow \psi^+_r
\end{align*}
\]

for \( r > 0 \). The fermionic D-Ishibashi state

\[
|D\rangle_\psi = \exp \left( \pm i \sum_{r > 0} \psi_{-r} \bar{\psi}_{-r} \right) |0\rangle
\]

is mapped to a fermionic N-Ishibashi state

\[
|N\rangle_\psi = \exp \left( \pm i \sum_{r > 0} \psi^-_{-r} \psi^+_r \right) |0\rangle
\]

with analogous relations for \( \langle D|_\psi \) and \( \langle N|_\psi \). Moreover, we also see that the non-chiral closed string Hamiltonian restricted to a single coordinate

\[
H^{(C)} = L^{(C)}_0 + \bar{L}^{(C)}_0 = \\
= \sum_{n \geq 1} (\alpha^-_n \alpha^+_n + \bar{\alpha}_n \bar{\alpha}_n) + \sum_{r \in \mathbb{Z}^+ - \nu} r \psi_{-r} \psi_r + \sum_{r \in \mathbb{Z}^+ - \bar{\nu}} r \bar{\psi}_{-r} \bar{\psi}_r
\]

is mapped to the chiral contribution of a closed string Hamiltonian restricted to the lightcone coordinates

\[
L^{(C; \ell)}_0 = \sum_{n \geq 1} (\alpha^-_n \alpha^+_n + \alpha^+_n \alpha^-_n) + \sum_{r \in \mathbb{Z}^+ - \nu} r (\psi^-_{-r} \psi^+_r + \psi^+_r \psi^-_{-r})
\]

where we have imposed \( \nu = \bar{\nu} \).

In the Cardy program we have seen by worldsheet duality that \( D \)-branes are naturally coupled to non-chiral closed strings, see (2.3.26), and that the closed string amplitude is finite. Nevertheless, we have the straightforward relation

\[
\langle D_\beta | \bar{q}^{\frac{1}{2}} (L^{(C)}_0 + \bar{L}^{(C)}_0) | D_\alpha \rangle = \langle N_\beta | \bar{q}^{\frac{1}{2}} L^{(C; \ell)}_0 | N_\alpha \rangle
\]
since the algebras on both hand sides are the same by the previous identifications.

In this simple example we see that there is enough structure on the lightcone mode oscillators so that we may disregard one of the two possible chiral sectors after imposing null boundaries, without introducing singularities in the physical system. Despite the fact that the chiral coherent states might seem good candidates to describe our $H$-branes, we will later see that the latter are instead described by squeezed chiral states. See [50] for a review of squeezed states and their applications to optics.

### 4.5 $H$-branes and chiral squeezed Ishibashi states

For the calculation of the Ishibashi state corresponding to the $(D; n)$-boundary condition, we use in particular the one-dimensional path integral approach of [16] for the boundary state formalism. We consider a one-loop open string diagram by making a periodic identification on the variable $\sigma \sim \sigma + 2\pi T$. At the endpoints we may impose the conditions

\begin{align*}
(D; n) : & \quad \partial_\tau X = 0 \quad \text{open} \\
(N; n) : & \quad \partial_\sigma X = 0 \quad \text{string}
\end{align*}

at $\sigma = 0, \pi$. By interchange $\tau \leftrightarrow \sigma$, the one-loop diagram is now a tree-level closed string channel. After we rescale the periodicity of $\sigma$ to $\sigma \sim \sigma + 2\pi$ we have respectively

\begin{align*}
(D; n) : & \quad \partial_\sigma X = 0 \quad \text{closed} \\
(N; n) : & \quad \partial_\tau X = 0 \quad \text{string}
\end{align*}

at $\tau = 0, \pi/T$. Note that the interchange $\tau \leftrightarrow \sigma$ comes naturally associated to the interchange of left-movers with right-movers

$$\tau \leftrightarrow \sigma \quad \iff \quad \alpha_n \leftrightarrow \tilde{\alpha}_n \, .$$

We start to examine the case of closed string $(D; n)$-boundary condition for the coordinate $X = X^-$. In general we might think that the $N$-brane emits a closed string of the form $X^-(\tau, \sigma) = X^-(\tau) + X^-(\sigma)$ a general solution of the equations of motion $\partial_\tau \partial_\sigma X^\pm = 0$. By imposing the closed $(D; n)$-boundary condition we see in fact that
the $H$-brane has created a **chiral closed string** $X^- = X^-(\tau)$ that will be annihilated by the second $H$-brane

\[
X^-(\tau) = \hat{q}^- + \frac{\alpha^-_0}{2} \tau + \frac{i}{2} \sum_{n \neq 0} \frac{\alpha^-_n e^{-i n \tau}}{n} = \hat{q}^- + \frac{\alpha^-_0}{2} \tau + \frac{1}{2} \sum_{n > 0} \frac{X^-_n}{\sqrt{n}}
\]

(4.5.77)

where

\[
X^-_n = a^-_n e^{-i n \tau} + a^+_n e^{i n \tau}
\]

(4.5.78)

for $\tau = 0, \pi/T$. We have used the redefinitions $a^-_n = \frac{i}{\sqrt{n}} \alpha^-_n$ and $a^+_n = \frac{i}{\sqrt{n}} \alpha^-_{-n}$ with $n > 0$ that for the closed string channel obey the usual canonical commutation relations

\[
[a^+_m, a^-_n] = \delta_{m+n,0}
\]

(4.5.79)

for $m \neq 0$. Given the single set (4.5.78) of bosonic coordinates, we interpret it as an eigenvalue condition for a given eigenstate

\[
(a^-_n e^{-i n \tau} + a^+_n e^{i n \tau} - X^-_n)|X^-\rangle = 0 \quad \forall n > 0
\]

(4.5.80)

A possible solution is

\[
|X^-\rangle = \prod_{n=1}^{\infty} |X^-_n\rangle
\]

\[
|X^-_n\rangle = (2\pi)^{-1/4} e^{\left[-\frac{i}{2}(X^-_n)^2 - a^+_n a^-_n e^{i n \tau} + X^-_n a^+_n e^{-i n \tau} + X^-_n a^-_n e^{i n \tau}\right]} |0\rangle
\]

\[
= (2\pi)^{-1/4} e^{-\frac{i}{2}(X^-_n - e^{i n \tau} (a^+_n + a^-_n))} e^{\frac{i}{2} e^{2i n \tau} (a^+_n)^2 + (a^-_n)^2} |0\rangle
\]

(4.5.81)

where we have used

\[
a^-_n e^{a^+_n f} |0\rangle = f e^{a^+_n f} |0\rangle
\]

(4.5.82)

for $f$ a commuting operator with respect to $a^-_n$ and $a^+_n$. The $(D; n)$-Ishibashi state is

\[
|(D; n)\rangle = \prod_{n=1}^{\infty} \int \mathcal{D}X^-_n |X^-_n\rangle
\]

\[
= \prod_{n=1}^{\infty} \exp \left(\frac{e^{2n i \tau}}{2} \left[(a^+_n)^2 + (a^-_n)^2\right]\right) |0\rangle
\]

(4.5.83)

where we have performed a gaussian integration. The closed string is created at $\tau = 0$ and annihilated at $\tau = \pi/T$. The $\tau$ dependence can be factored out by integration

\[
\prod_{n=1}^{\infty} \int_{-\infty}^{0} d\sigma \exp \left(\frac{e^{2n i \tau}}{2} \left[(a^+_n)^2 + (a^-_n)^2\right]\right) |0\rangle
\]

(4.5.84)
where in the Appendix B we show that it gives

$$\prod_{n=1}^{\infty} \int_{-\infty}^{0} d\tau e^{i\tau L_0^{(C;\ell)}} e^{\frac{i}{2}(\langle a_n^+\rangle^2 + \langle a_n^-\rangle^2)} |0\rangle$$

$$\int_{-\infty}^{0} d\tau D_{ch} |H\rangle$$

and we see that the $H$-brane, described by the chiral and squeezed Ishibashi state

$$|H\rangle = \prod_{n=1}^{\infty} e^{\frac{i}{2}(\langle a_n^+\rangle^2 + \langle a_n^-\rangle^2)} |0\rangle$$


couples naturally to the chiral closed string disk propagator

$$D_{ch} = e^{i\tau L_0^{(C;\ell)}}$$

with

$$L_0^{(C;\ell)} = \sum_{n=1}^{\infty} n(a_n^+)^\dagger a_n^- + \sum_{n=1}^{\infty} n(a_n^-)^\dagger a_n^+ .$$

In the Appendix C we show that the disk amplitude neglecting the zero-modes is

$$\langle H|D_{ch}|H\rangle = \prod_{n=1}^{\infty} \frac{1}{1 - z_n}$$

$$z_n = e^{2i\sigma} .$$

Similar analysis follow for the case of closed $(N; n)$-boundary condition on the $X^+$ coordinate, where we only have to interchange left- with right-movers

$$|\langle (N; n)\rangle = \prod_{n=1}^{\infty} \exp \left(\frac{e^{2i\sigma}}{2}[(\tilde{a}_n^+)^2 + (\tilde{a}_n^-)^2]\right) |0\rangle$$

for $\sigma \sim \sigma + 2\pi$. The vacuum is defined in the euclidean picture from the conditions that $T(z)|0\rangle$ and $\tilde{T}(\bar{z})|0\rangle$ are well-defined as $z, \bar{z} \to 0$ which implies

$$L_n|0\rangle = 0 , \quad \tilde{L}_n|0\rangle = 0$$

for all $n \geq -1$, and in particular is invariant under global conformal transformations with the $SL(2, \mathbb{R})_L \otimes SL(2, \mathbb{R})_R$ group generated by $L_{-1}$, $L_0$ and $L_1$ and their anti-holomorphic counterparts. We will suppose that in the closed string channel, this definition is still valid in the Lorentz picture. Nevertheless a proper definition of the vacuum state in time-dependent conformal field theory is an open problem.
We note that

\[
\langle (D; n)|\tilde{L}^{(C,\ell)}_m|(D; n) \rangle = 0 \\
\langle (N; n)|L^{(C,\ell)}_m|(N; n) \rangle = 0 \quad \forall m \in \mathbb{N}
\]

(4.5.92)
since left- and right-chiral movers commute between each other. If we want to interpret these conditions as \( Diff(S^1) \) invariance, we must have \( c = 0 \) and \( \tilde{c} = 0 \) respectively.

For the fermionic contribution to the \( H \)-Ishibashi states, we add to the above states their Majorana components. In the closed string channel boundaries are set at \( \tau = 0, \pi/T \) and that gives conditions on the left-sector, see (4.3.49). In the case at hand we have to flip left- from right-chiral mode sector when we move from the open string sector to the closed sector, see (4.5.76). In all we have the same conditions as in the open sector, \( \Psi^{-}_R = 0 \). The non-vanishing counterpart is

\[
\Psi^{-}_L = \sum_{r \in \mathbb{Z} + \nu} \psi^{-}_r e^{-ir\tau} = \sum_{r \geq \nu} \theta^{-}_r
\]

(4.5.93)

where

\[
\theta^{-}_r = \psi^{-}_r e^{-ir\tau} + \psi^{-\dagger}_r e^{ir\tau}
\]

(4.5.94)

for \( \tau = 0, \pi/T \) and we have set \( \psi^{-\dagger}_r = \psi^{-}_r \) for \( r \geq \nu \). Again we set this condition as an eigenvalue for a given eigenstate \( |\Theta^{-}\rangle \)

\[
(\theta^{-}_r - \psi^{-}_r e^{-ir\tau} - \psi^{-\dagger}_r e^{ir\tau}) |\Theta^{-}\rangle = 0
\]

(4.5.95)

for \( r > 0 \) and with solution

\[
|\Theta^{-}\rangle = \prod_{r>0} \exp \left( -\frac{1}{2}(\theta^{-}_r)^2 + \psi^{-\dagger}_r \theta^{-}_r e^{ir\tau} + \psi^{-\dagger}_r \theta^{-\dagger}_r e^{ir\tau} - \psi^{\dagger}_r \psi^{-\dagger}_r e^{2ir\sigma} \right) |0\rangle.
\]

(4.5.96)

Finally

\[
|N^{\psi}_\sigma\rangle = \int \mathcal{D}\Theta^{-}\mid\Theta^{-}\rangle = \prod_{r>0} \exp \left( \frac{1}{2} e^{2ir\tau} \psi^{-\dagger}_r \psi^{-\dagger}_r \right)
\]

(4.5.97)

for \( \sigma = 0, \pi/T \).
Chapter 5

Black hole quantum horizons

It was shown in the early seventies by Christodoulou [20], Penrose and Floyd [44] and Hawking [29] that, in a classical theory, the area of a black hole horizon cannot decrease. In his seminal paper [8], Bekenstein used this fact to postulate an identification between the horizon area and the black hole entropy. Later, Bardeen, Carter and Hawking found four laws of black hole mechanics analogous to the four laws of thermodynamics [7]. The pioneer postulate of Bekenstein was finally realized to be correct when Hawking found, using quantum perturbations near the horizon, that a black hole radiates energy and behaves like a hot object with a given temperature that is identified with the horizon surface gravity [30]. We conclude that the four mechanical laws of black holes are in fact the four laws of thermodynamics. In particular we identify the area of the event horizon of any black hole with the Bekenstein-Hawking entropy (in Planck units) up to factor

\[ S_{BH} = \frac{1}{4} A. \]  

As in any thermodynamical system, the entropy must have some microscopic origin. Here the black hole entropy should be statistically given by a consistent quantum gravity theory, where the Ashtekar approach is one candidate. Recently the case of the Schwarzschild black hole has been treated within this approach [2].

In [9] Bekenstein has done another important step to understand the black hole physics. There he suggested that the horizon area \( A \) (and consequently the mass) must be quantized in Planck units. This suggests that a quantum horizon behaves as a phase space formed by independent patches of equal Planck size areas, bearing
in mind that each patch can locate one degree of freedom as in a usual phase space patchwork. This is now called the Holographic Principle [31] [59].

What’s wrong with this picture?

Fig. 1

The idea of a discrete spectrum proposed by Bekenstein was later discussed independently by Mukhanov [43] and Kogan [36]. Arguments used in [43] were based on entropy and further developments are contained in [10]. In recent years the discrete spectrum of quantum black holes was examined in numerous papers, see for example
[35] and references therein. The black hole area quantization has been extensively discussed recently in Ashtekar’s approach to quantum gravity [3]. However it is not clear whether the Ashtekar program is consistent with the Holographic Principle.

String theory is another candidate for a quantum gravity theory. In paper [36] (see also [37]) a stringy approach to the black hole area quantization was suggested. It was based on the consistency of a \textit{chiral closed string} moving in a Euclidean black hole background. The fact that it leads to the same discrete spectrum as Bekenstein’s was an indication that chiral sectors should be relevant to the statistical counting of black hole entropy.

In [55] Strominger and Vafa have for the first time counted in a controllable way the correct number of microscopic states that gives rise to the Bekenstein-Hawking entropy of an extremal charged black hole, with a AdS geometry near the horizon, using D-brane technology. See [39] for a review and subsequent important developments. The counting of states is done in a dual theory of supergravity on AdS spacetime in the UV limit. That is the IR limit of a gauge theory living on the D-branes. Such duality has given rise to the AdS/CFT holographic principle [42].

Nevertheless, in the UV limit the black hole has become smaller than the string scale, the spacetime geometry becomes quite fuzzy and there is no concept of an horizon at all. Thus it is difficult to understand how one can explain the universality of the Bekenstein-Hawking entropy law [57]. Such a dual description is also problematic if we want to discuss the unitarity of the black hole evolution, see Fig. 1 for the embedding of the an extremal black hole geometry in a AdS spacetime [57]. Here curved rows represent the black hole time isometries and the straight lines the time isometries of the gauge theory that lives on the D-brane. From the AdS/CFT dual picture, straight lines describe a unitary evolution of conformal field theory so by duality the black hole evolution should be unitary. However, it is hard to understand why this should be so: any initial configuration evolves into the black hole and we progressively loose information that was outside the black hole. Once inside it, the information cannot flow outside the black hole and continue to propagate freely to infinity. Clearly the evolution is not unitary! Nevertheless, we stress here that the statistical count was performed by time independent String Theory and the information loss paradox is a time-dependent phenomena.
To summarize, in black hole physics we need a quantum model where the number of microstates gives rise in a controllable way to the Bekenstein-Hawking entropy and to an area quantization with the given discrete spectrum proposed by Bekenstein, consistent with the holographic principle. It is interesting to note that when we try to get a stringy description of a quantum horizon we either have to deal with open strings attached to $D$-branes [55] or with a chiral sector of closed strings [36]. The only common feature is that both have one Virasoro algebra. As we have seen, $H$-branes are naturally coupled to chiral closed strings. They are also associated with a non-commutativity along the light-cone coordinates, proposed by Susskind in [60] to solve the information loss paradox. At last, from their time-dependent geometric shape, it is conceivable that $H$-branes can provide a phenomenological picture (instead of a dual picture) of the black hole horizon quantization. In this way it was conjectured in [38] that in a stringy picture, quantum horizons are described by chiral and non-normalized squeezed states.

In the following subsections, we review the BTZ black hole geometry and give some hints as to how $H$-branes fit into it. We are, however, far from giving a clear picture on the BTZ black hole horizon quantization using $H$-branes.

### 5.1 The BTZ black hole

The BTZ black hole is a solution of the Einstein equations in 3-dimensions with a negative cosmological constant $\Lambda = -1/\ell^2$ [6]. The global geometry of this solution may be described by certain identifications on the $AdS_3$ spacetime. The solution depends on the ADM mass $M$ and angular momentum defined at spatial infinity $J$ that enter in the metric in the following way

$$ds^2 = -(N^\perp)^2 dt^2 + f^{-2} dr^2 + r^2(d\phi + N^\phi dt)^2$$

with lapse and shift functions

$$N^\perp = f = \left(-M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2}\right)^{1/2}, \quad N^\phi = -\frac{J}{2r^2}$$

$$\phi \sim \phi + 2\pi.$$  \hspace{1cm} (5.1.3)

There is a mass gap between the energy of the BTZ-solution with $M = J = 0$ and the $AdS_3$ vacuum energy. The last is described by the BTZ-metric with $J = 0$ and
\( M = -\frac{1}{8G} \). Naked singularities appear in the interpolations of those two solutions. Nevertheless, the existence of a tachyon does not mean instability of the \( AdS_3 \) vacuum. The \( AdS \) vacuum is in fact perfectly stable. When \( M > 0 \) and \( |J| \leq M\ell \), the solution describes a black hole with inner \( r_- \) and outer \( r_+ \) horizon with

\[
 r^2_\pm = \frac{M\ell^2}{2} \left( 1 \pm \sqrt{1 - (J/M\ell)^2} \right) \quad (5.1.4)
\]
i.e.,

\[
 M = \frac{r^2_+ + r^2_-}{\ell^2}, \quad J = \frac{2r_-r_+}{\ell}. \quad (5.1.5)
\]

The Bekenstein-Hawking entropy is given by one-quarter of the horizon area (in Planck units)

\[
 S = \frac{A}{4G} = \frac{2\pi r^+}{4G} \quad (5.1.6)
\]
where \( G \) is the 3-d Newton constant. The Hawking temperature is

\[
 T = \frac{r^+_2 - r^-_2}{r_+\ell^2} \quad (5.1.7)
\]

In the euclidian picture, the \( AdS \) geometry is a hyperbolic space whose geometry is given by the standart Poincare-metric

\[
 ds^2 = \frac{\ell^2}{z^2}(dx^2 + dy^2 + dz^2), \quad z > 0 \quad (5.1.8)
\]
where the cartesian coordinates may be expressed by the spherical coordinates

\[
 (x, y, z) = (R \cos \theta \cos \chi, R \sin \theta \cos \chi, R \sin \chi) \quad . \quad (5.1.9)
\]

On the other hand, the outer region of the BTZ black hole is given by certain identifications on the above hyperbolic space. Explicitly, we first do a coordinate transformation

\[
 x = \left( \frac{r^2 - r^2_+}{r^2 - r^2_-} \right)^{1/2} \cosh \left( \frac{r_+}{\ell^2} - \frac{r_-}{\ell} \phi \right) \exp \left( \frac{r_+}{\ell} \phi - \frac{r_-}{\ell^2} t \right)
\]
\[
 y = \left( \frac{r^2 - r^2_+}{r^2 - r^2_-} \right)^{1/2} \sinh \left( \frac{r_+}{\ell^2} - \frac{r_-}{\ell} \phi \right) \exp \left( \frac{r_+}{\ell} \phi - \frac{r_-}{\ell^2} t \right)
\]
\[
 z = \left( \frac{r^2 - r^2_+}{r^2 - r^2_-} \right)^{1/2} \exp \left( \frac{r_+}{\ell} \phi - \frac{r_-}{\ell^2} t \right) \quad (5.1.10)
\]
followed by the discrete identifications in the spherical coordinates

\[
 (R, \theta, \chi) \sim (Re^{2\pi r_+/\ell}, \theta + \frac{2\pi r_-}{\ell}, \chi) \quad (5.1.11)
\]
In particular, in the euclidian picture we see that for the non-extremal regime \( r_+ \neq r_- \), the event horizon shrinks to a circle \( (x = y = 0) \) and at the extremal regime, it shrinks to a point

\[
r = r_{\text{horizon}} \to (x, y, z) = (0, 0, 0), \quad (J = M\ell) \quad (5.1.12)
\]

where, as expected, the Hawking temperature vanishes.

5.2 The \( AdS_3/CFT_2 \) holographic principle

Recently a correspondence between supergravity on \( AdS_D \) and the Yang-Mills \( \mathcal{N} = 4 \) Theory at fixed point in one lower dimension was introduced [42]. However, in a different context, gravity in \( AdS_3 \) with cosmological constant \( \Lambda = -1/\ell^2 \) was shown to be described at infinity by a CFT with central charge \( c = \frac{3\ell}{2G} \) [22]. This can be realized by considering the following boundary conditions of an asymptotically \( AdS_3 \) metric

\[
\begin{align*}
 g_{tt} &= -\frac{r^2}{\ell^2} + O(1) \quad g_{t\phi} = O(1) \quad g_{tr} = O\left(\frac{1}{r^3}\right) \\
g_{rr} &= \frac{\ell^2}{r^2} + O\left(\frac{1}{r^4}\right) \quad g_{r\phi} = O\left(\frac{1}{r^3}\right) \quad g_{\phi\phi} = r^2 + O(1) \quad . \quad (5.2.13)
\end{align*}
\]

The diffeomorphism group that preserves these boundary conditions can be seen to be generated by the following vector fields \( \zeta^a(r, t, \phi) \)

\[
\begin{align*}
 \zeta^t &= \ell(T^+ + T^-) + \frac{\ell^3}{2r^2}(\partial_+^2 T^+ + \partial_-^2 T^-) + O\left(\frac{1}{r^4}\right) \\
 \zeta^\phi &= T^+ - T^- - \frac{\ell^2}{2r^2}(\partial_+^2 T^+ - \partial_-^2 T^-) + O\left(\frac{1}{r^4}\right) \\
 \zeta^r &= -r(\partial_+ T^+ + \partial_- T^-) + O\left(\frac{1}{r}\right) \quad (5.2.14)
\end{align*}
\]

where

\[
2\partial_\pm := \ell \frac{\partial}{\partial t} \pm \frac{\partial}{\partial \phi} \quad . \quad (5.2.15)
\]

They are written in term of two independent diffeomorphisms \( T^\pm(r, t, \phi) = T^\pm(\frac{t}{\ell} \pm \phi) \). By expanding them in a Fourier series with \( L_n (\tilde{L}_n) \) generating the diffeomorphism \( T^\pm = e^{in(\frac{t}{\ell} \pm \phi)} \), it was shown that the asymptotic isometries are given by two commuting Virasoro algebras of central charge \( c = \frac{3\ell}{2G} \). They represent a CFT that lives in the \( (t, \phi) \) cylinder at spatial infinity.
5.3 Counting the microstates of a BTZ black hole horizon

Strominger approach [56]: From the above $AdS_3/CFT_2$ duality, we define the ground state of the CFT, that lives in the cylinder at spatial infinity, by imposing the conditions on the zero-mode Virasoro operators $L_0 = \tilde{L}_0 = 0$. We would like that by duality, the CFT vacuum state corresponds to the $J = M = 0$ BTZ-solution. Furthermore, we may postulate the following holographic identifications between the Virasoro operators and the physical parameters that describe the spinning BTZ black hole

$$M = \frac{1}{\ell}(L_0 + \tilde{L}_0)$$

$$J = L_0 - \tilde{L}_0$$

(5.3.16)

In the semiclassical limit $\ell \gg G$ one may use the Cardy formula to count the asymptotic number of states in a CFT of a given central charge $c$

$$S = 2\pi \sqrt{\frac{cL_0}{6} + 2\pi \sqrt{\frac{cL_0}{6}}}$$

(5.3.17)

where in our case $c = \frac{3\ell}{2G}$. By a short algebra we see that the result is in exact agreement with the Bekenstein-Hawking entropy. This argument should be valid in any consistent quantum theory of gravity, as is the case for String Theory.

However the boundary microstates live on the cylinder at spatial infinity and it makes no direct connection to the number of states that we expected to find on the horizon, see [18] for a discussion of this point.

Carlip approach: A second way to count the microstates that gives rise to the Bekenstein-Hawking entropy may be found in [19] where the author considers the horizon as a marginal trapped surface. It is known that 3-d gravity with local $AdS$ geometry may be treated in the bulk as a Chern-Simons theory. Treating the horizon as a boundary, the would-be-pure gauge degrees of freedom of the Chern-Simons theory become physical degrees of freedom of a WZW model at the 2-dimensional boundary surface with target group manifold $SL(2, R)_{-k} \otimes SL(2, R)_k$. Here the level is related to the cosmological constant by $k = \frac{\sqrt{2\ell}}{8G}$. Note that the CFT living at the horizon is different from the one that lives on the cylinder at spatial infinity. In particular, in
the classical limit the present CFT has central charge \( c = 6 \) whereas for the previous one \( c \gg 1 \).

It is known that in the WZW model, the Virasoro mode operators are formulated in terms of the chiral U(1)-current algebra \( J \) and \( \tilde{J} \) by the Sugawara construction. In particular \( L_0 \) and \( \tilde{L}_0 \) are

\[
L_0 = -\frac{2}{2k+1} \sum_{m=-\infty}^{\infty} : J_{-n}^a J_n^b : \eta_{ab} \\
\tilde{L}_0 = \frac{2}{2k-1} \sum_{m=-\infty}^{\infty} : \tilde{J}_{-n}^a \tilde{J}_n^b : \eta_{ab}
\]

(5.3.18)

with the currents obeying the algebra

\[
\left[ J_m^a, J_n^b \right] = i f_{c}^{ab} J_{m+n}^c - k m \eta^{ab} \delta_{m+n,0} \\
\left[ \tilde{J}_m^a, \tilde{J}_n^b \right] = i \tilde{f}_{c}^{ab} \tilde{J}_{m+n}^c + k m \eta^{ab} \delta_{m+n,0} 
\]

(5.3.19)

Here we are using \((- , +, +)\) spacetime signature. The vacuum state \(| \Omega \rangle\) is annihilated by the current algebra generators \( J_m \) and \( \tilde{J}_m \) for \( m > 0 \) and any physical state is given in terms of raising operators - \( J_{-n} \) and \( \tilde{J}_{-n} \) with \( n > 0 \) - acting in the vacuum state

\[
| \phi, N, \tilde{N} \rangle \simeq | (n_1, a_1), (n_2, a_2), \ldots) \rangle (\tilde{n}_1, \tilde{a}_1), (\tilde{n}_2, \tilde{a}_2), \ldots) \rangle = (J_{-n_1})^{a_1} (J_{-n_2})^{a_2} \ldots (\tilde{J}_{-\tilde{n}_1})^{\tilde{a}_1} (\tilde{J}_{-\tilde{n}_2})^{\tilde{a}_2} \ldots | \Omega \rangle 
\]

(5.3.20)

Here \( N = \sum a_i n_i \) and \( \tilde{N} = \sum \tilde{a}_i \tilde{n}_i \) are the number operators of the left and right moving oscillators respectively. The Hamiltonian describing the evolution of these states is, at the classical limit (large \( k \)), given by

\[
H = L_0 + \tilde{L}_0 = N + \tilde{N} - \frac{2}{2k+1} (J_0)^2 + \frac{2}{2k-1} (\tilde{J}_0)^2 
\]

(5.3.21)

The conjecture in [18] was to consider the horizon to have only left-moving excited states, i.e., \( \tilde{N} = 0 \). Note that such chiral-state was postulated in [36] to describe a chiral closed string used to quantize the energy levels of the horizon.

Further we impose the physical condition \( H| \phi, N, 0 \rangle = 0 \) where we have neglected the constant from normal ordering as it will not affect the final semi-classical result. Such a condition imposes a certain value for the number operator \( N \). Moreover from the boundary condition that was used to define the horizon as a trapped surface, the main contributions to the number of states came from the identifications

\[
J_0 | \Omega \rangle = -k \frac{r^+}{\sqrt{2\ell}} (2k + 1)| \Omega \rangle
\]
\[ \tilde{J}_0 |\Omega\rangle = -k \frac{r^+}{\sqrt{2\ell}} (2k - 1) |\Omega\rangle . \]  (5.3.22)

It is interesting to see that the momentum operator \( P = L_0 - \tilde{L}_0 \) is given by

\[ P = N - 4k^3 \frac{(r^+)^2}{\ell^2} . \]  (5.3.23)

Using the Cardy formula to count the number of states of the closed “chiral excited” string (remember that \( \tilde{N} = 0 \))

\[ S = 2\pi \sqrt{\frac{cN}{6}} \]  (5.3.24)

with \( c = 6 \), it is easy to see that the condition \( L_0 + \tilde{L}_0 = 0 \) translates now to \( N = \left( \frac{r^+}{4G} \right)^2 \); when inserted in (5.3.24) this gives the exact Bekenstein-Hawking entropy of the BTZ black hole. Finally, the momentum operator takes the value

\[ P = -\frac{2k^2(r^+)^2}{\ell^2} (2k - 1) . \]  (5.3.25)

### 5.4 The black hole complementarity principle

The previous calculation by Strominger is consistent with the one performed by Callan and Maldacena for the 5-dimensional extremal supersymmetric black hole, using the so called \( D_1/D_5 \) system, a gas of open strings stretching between a number \( Q_5 \) of \( D_5 \)-branes and \( Q_1 \) \( D_1 \)-branes, all wrapped on a five-torus and for which we give the system a Kaluza-Klein momentum \( N \) in one of the directions. In this picture, the entropy is given by

\[ Z = \prod_{n=1}^{\infty} \left( \frac{1 + q^n}{1 - q^n} \right)^{4Q_1Q_5} = \sum d(N)q^N \]  (5.4.26)

where the integers \( d(N) \) represent the degeneracy of the state with Kaluza-Klein momentum number \( N \)

\[ d(N) \to \int dq \frac{Z(q)}{q^{N+1}} \]  (5.4.27)

that can be estimated by the saddle point method. For \( N \to \infty \), keeping fixed the product \( Q_1Q_5 \) fixed, this gives the entropy

\[ S = \log d(N) \sim 2\pi \sqrt{NQ_1Q_5} \]  (5.4.28)
and that agrees with the classical black hole entropy. The problem is that a similar model does not work for black holes whose Schwarzschild radius exceeds the compactification scale \[ [41] \]. For these so called fat black holes, we take the limit where \( Q_1, Q_5 \) and \( N \) tend to infinity in fixed proportion

\[
S = \log d(N) \rightarrow N \log N
\]  

which does not agree with the black hole entropy. The picture of the black hole complementarity comes from the D-brane picture in a straightforward way, by replacing the gas of \( Q_1Q_5 \) species by a single string and the level number \( N \) by \( N' = NQ_1Q_5 \). The entropy of the fat black hole is the same as the entropy carried by a single long string with central charge \( c = 6 \) (as in the Carlip calculation) and a string tension \( T \sim \frac{1}{g\alpha'}Q_5 \).

The question we would like to address is to know exactly what are the properties of this long string. We suggest that such long string is open with the endpoints fixed at an \( H \)-brane. The states associated to the Bekenstein-Hawking entropy should be of open strings stretching between stacks of \( H \)-branes. In the BTZ black hole, the state associated to this stack of \( H \)-branes should be chiral (\( \tilde{N} = 0 \)) with the conditions

\[
H^h|\phi, N, 0\rangle = \left(\frac{r^+}{4G}\right)^2
\]

\[
P^h|\phi, N, 0\rangle = -\left(\frac{r^+}{4G}\right)^2(2k - 1)
\]  

with

\[
H^h = \sum_{n>0} (\alpha^+_n\alpha^-_n + \alpha^-_n\alpha^+_n)
\]

\[
P^h = \sum_{n>0} (\alpha^+_n\alpha^-_n - \alpha^-_n\alpha^+_n)
\]  

with
being the chiral hamiltonian and chiral momentum associated to null coordinates.

**Summarizing:** We conjecture that quantum horizons - black hole, cosmological, etc - are described in String Theory by chiral and non-normalized squeezed states.
Chapter 6

Conclusion

In this work we note the absence in the known $D$-brane moduli space of Bosonic String Theory of an infinitelly boosted brane - a nullbrane. By $T$-duality, the nullbrane should also describe free open charged strings immersed in a critical constant electric field, where the pair-production of strings diverges. In both situations, from the worldsheet point of view the left- and right-chiral mode oscillators decouple so that the theory becomes chiral and moreover the string endpoints are lightlike separate; the open string is naturally described by a worldsheet sigma model with null boundaries.\footnote{For an early discussion of boundary conditions of nullbranes in the context of conjugacy classes in group manifolds, see [54] (I thank V. Schomerus for calling attention to this reference).}

By working in the lightcone gauge, we show how the familiar worldsheet timelike boundaries are dynamically deformed to null boundaries at the critical limit. Null worldsheet boundaries set the correct boundary conditions that one would like to have on a nullbrane in Minkowskian spacetime - a Neumann condition for say $X^+$ and a Dirichlet condition for $X^-$, where $X^\pm$ are the target lightcone coordinates. We then carry the quantization of the system in the open string channel using the first-order formalism and find a space/time noncommutative geometry. From the analysis at the closed string channel, we calculate the $H$-brane Ishibashi states using the first-order formalism. Contrary to the case of $D$-branes that are described by non-chiral coherent Ishibashi states, $H$-brane Ishibashi states seem to be chiral and squeezed - $H$-branes are naturally coupled to chiral closed strings. Nevertheless, the result should be checked by carrying out a Cardy program in a time-dependent BCFT, but that is outside the scope of this M.Sc. thesis.
Let us note that $H$-branes where suggested by us from an initial attempt to find an alternative/complementary model in String Theory to the well established $AdS/CFT$ correspondence in order to describe quantum horizons. Because of their properties, we have conjectured that a stack of coinciding $H$-branes may \textit{phenomenologically} describe any quantum horizon - black hole or cosmological - so that the states associated to quantum horizons should be in this way non-normalized, chiral and squeezed.

We think that studying the concept of time in String Theory would answer open problems posed by quantum gravity, such as the information loss paradox. Nevertheless it is necessary to have at hand tools to describe String Theory in a mathematically consistent way, in particular for the study of branes - see [28] and references therein.
Chapter 7

Appendixes

Appendix A

Here we show that nullstrings described by the Schild action induce a spacetime noncommutativity on their endpoints. First note that the Schild action is already at first order

\[ S_{\text{Schild}} = \int \epsilon_{\mu\nu} dX^\mu \wedge dX^\nu \]  

(A.1)

interpreted as a tensionless open string immersed in a constant electromagnetic field \( F_{\mu\nu} \propto \epsilon_{\mu\nu} \). We see that the equations of motion are given by \( dF = 0 \) without involving the target coordinates and the boundary conditions are

\[ \epsilon_{\mu\nu} \delta X^\mu dX^\nu = 0 \]  

(A.2)

The space of solutions is the set of coordinates with some constant value at the string endpoints

\[ X^\mu(\tau, 0) = x^\mu_0 \quad X^\mu(\tau, \pi) = x^\mu_1 \]  

(A.3)

The symplectic structure of the phase-space is given by

\[ \Omega = \epsilon_{\mu\nu} \delta x^\mu_0 \wedge \delta x^\nu_0 - \epsilon_{\mu\nu} \delta x^\mu_1 \wedge \delta x^\nu_1 \]  

(A.4)

Here \( \delta x^\mu_s (s = 0, 1) \) is a vector tangent to the space of solutions where the variation is given by a change of the constant \( x^\mu_s \) from solution to solution. It is clear that it gives a noncommutative geometry at the nullstring endpoints.
Appendix B

In this appendix we calculate the chiral closed string disk propagator. First relabel
the lightcone operators
\[
\begin{align*}
    a_n^+ &\rightarrow a_n & (a_n^+)^\dagger &\rightarrow \tilde{a}_n^+ \\
    a_n^- &\rightarrow \tilde{a}_n & (a_n^-)^\dagger &\rightarrow a_n^+ 
\end{align*}
\]  
(B.1)

Consider
\[
Z = h n a_n^\dagger a_n 
\]
and
\[
Y = a_n^\dagger a_n^- 
\]

we have
\[
[Z, Y] = 2hnY 
\]  
(B.3)

so that by the Hausdorff formula we find
\[
e^Z e^Y = \exp(e^{2hnY})e^Z 
\]  
(B.4)

and since
\[
Z|0\rangle = 0 
\]  
(B.5)

we evaluate the integrand of (4.5.84)
\[
\prod_{n=1}^{\infty} \exp \left[ \frac{e^{2inx^+}}{2} \left( (\tilde{a}_n^\dagger)^2 + (a_n^\dagger)^2 \right) \right] |0\rangle 
\]
\[
= e^{ix^+(\tilde{L}_0^C + \tilde{L}_0^{\bar{C}})} \prod_{n=1}^{\infty} e^{\frac{1}{2}(\tilde{a}_n^\dagger)^2 + (a_n^\dagger)^2}} |0\rangle 
\]  
(B.6)

where
\[
L_0^{(C)} = \sum_{n=1}^{\infty} na_n^\dagger a_n 
\]
\[
\tilde{L}_0^{(C)} = \sum_{n=1}^{\infty} n\tilde{a}_n^\dagger \tilde{a}_n 
\]

we evaluate the integrand of (4.5.84)
\[
\prod_{n=1}^{\infty} \exp \left[ \frac{e^{2inx^+}}{2} \left( (\tilde{a}_n^\dagger)^2 + (a_n^\dagger)^2 \right) \right] |0\rangle 
\]
\[
= e^{ix^+(\tilde{L}_0^C + \tilde{L}_0^{\bar{C}})} \prod_{n=1}^{\infty} e^{\frac{1}{2}(\tilde{a}_n^\dagger)^2 + (a_n^\dagger)^2}} |0\rangle 
\]  
(B.6)

where
\[
L_0^{(C)} = \sum_{n=1}^{\infty} na_n^\dagger a_n 
\]
\[
\tilde{L}_0^{(C)} = \sum_{n=1}^{\infty} n\tilde{a}_n^\dagger \tilde{a}_n 
\]

After relabel again to the lightcone oscillator modes
\[
L_0^{(C)} + \tilde{L}_0^{(C)} \rightarrow L_0^{(C,\ell)} 
\]  
(B.8)

it is easy to see that (4.5.84) gives the desire result for the chiral closed string disk propagator.
Appendix C

In this appendix we calculate the chiral closed string amplitude between two H-branes. Consider

\[ X = g(a_k)^2 \quad Y = (a_k^\dagger)^2 \]  

(C.1)

and

\[ \langle 0 | (a_k)^{2n}(a_k^\dagger)^{2n} | 0 \rangle = \langle 0 | (a_k)^{2n-1}(a_k^\dagger)^{2n-1} | 0 \rangle + \langle 0 | (a_k)^{2n-1}a_k^\dagger a_k(a_k^\dagger)^{2n-1} | 0 \rangle \\
= 2n \langle 0 | (a_k)^{2n-1}(a_k^\dagger)^{2n-1} | 0 \rangle \\
= (2n)! \]  

(C.2)

so that

\[ \langle 0 | X^n e^Y | 0 \rangle = \frac{1}{n!} \langle 0 | X^n Y^n | 0 \rangle = \frac{g^n}{n!} (2n)! \]  

(C.3)

and finally

\[ \langle 0 | e^X e^Y | 0 \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \langle 0 | X^n e^Y | 0 \rangle \\
= \sum_{n=0}^{\infty} g^n \frac{(2n)!}{n!n!} \\
= \sum_{n=0}^{\infty} C_n^{2n} g^n \\
= \frac{1}{\sqrt{1 - 4g}}. \]  

(C.4)

The disk amplitude presented in the text follows easily. We have to relabel the previous oscillators as the lightcone oscillators.
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