DRAZIN INVERTIBILITY OF LINEAR OPERATORS ON QUATERNIONIC BANACH SPACES

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Abstract. Let $A$ be a right linear operator on a two-sided quaternionic Banach space $X$. The paper studies the Drazin inverse for right linear operators on a quaternionic Banach space. It is shown that if $A$ is Drazin invertible then the Drazin inverse of $A$ is given by $f(A)$ where $f$ is 0 in an axially symmetric neighborhood of 0 and $f(q) = q^{-1}$ in an axially symmetric neighborhood of the nonzero spherical spectrum of $A$. Some results analogous to the ones concerning the Drazin inverse of operators on complex Banach spaces are proved in the quaternionic context.

1. Introduction and preliminaires

We denote by $H$ the algebra of quaternions, introduced by Hamilton in 1843. An element $q$ of $H$ is of the form

$$q = a + bi + cj + dk; a, b, c, d \in \mathbb{R}$$

where $i, j$ and $k$ are imaginary units. By definition, they satisfy

$$i^2 = j^2 = k^2 = ij = -1.$$ 

Given $q = a + bi + cj + dk$, then

- the conjugate quaternion of $q$ is $\bar{q} = a - bi - cj - dk$;
- the norm of $q$ is $|q| = \sqrt{q\bar{q}} = \sqrt{a^2 + b^2 + c^2 + d^2}$;
- the real and the imaginary parts of $q$ are respectively $\Re(q) := \frac{1}{2}(q + \bar{q}) = a$ and $\Im(q) := \frac{1}{2}(q - \bar{q}) = bi + cj + dk$.

The unit sphere of imaginary quaternions is given by

$$S = \{q \in H : q^2 = -1\}.$$

Let $p$ and $q$ be two quaternions. $p$ and $q$ are said to be conjugated, if there is $s \in H \setminus \{0\}$ such that $p = sqs^{-1}$. The set of all quaternions conjugated with $q$, is equal to the 2-sphere

$$[q] = \{\Re(q) + |\Im(q)|j : j \in S\} = \Re(q) + |\Im(q)|S.$$

For every $j \in S$, denote by $C_j$ the real subalgebra of $H$ generated by $j$; that is,

$$C_j := \{u + vj : u, v \in \mathbb{R}\}.$$
We say that \( U \subseteq \mathbb{H} \) is axially symmetric if \([q] \subset U\) for every \( q \in U\).

For a thorough treatment of the algebra of quaternions \( \mathbb{H} \), the reader is referred, for instance, to [4].

**Definition 1.1** ([1, Definition 2.1.2], Slice hyperholomorphic functions). Let \( U \subseteq \mathbb{H} \) be an axially symmetric open set and let \( \mathcal{U} = \{(u, v) \in \mathbb{R}^2 : u + vS \subset U\} \).

A function \( f : U \to \mathbb{H} \) is called a left slice function if there exist two functions \( f_0, f_1 : \mathcal{U} \to \mathbb{H} \) such that:

\[
f(q) = f_0(u, v) + j f_1(u, v) \quad \text{for every } q = u + vj \in U
\]

and if \( f_0, f_1 \) satisfy the compatibility conditions

\[
f_0(u, -v) = f_0(u, v), \quad f_1(u, -v) = -f_1(u, v).
\]

(1.1)

If in addition \( f_0 \) and \( f_1 \) satisfy the Cauchy-Riemann equations

\[
\frac{\partial}{\partial u} f_0(u, v) - \frac{\partial}{\partial v} f_1(u, v) = 0, \\
\frac{\partial}{\partial v} f_0(u, v) + \frac{\partial}{\partial u} f_1(u, v) = 0,
\]

(1.2)

then \( f \) is called left slice hyperholomorphic. We denote the set of all left slice functions on \( U \) by \( \mathcal{SF}_L(U) \) and the set of all left slice hyperholomorphic functions on \( U \) by \( \mathcal{SH}_L(U) \).

A function \( f : U \to \mathbb{H} \) is called a right slice function if there exist two functions \( f_0, f_1 : \mathcal{U} \to \mathbb{H} \) such that:

\[
f(q) = f_0(u, v) + f_1(u, v)j \quad \text{for every } q = u + vj \in U
\]

and if \( f_0, f_1 \) satisfy the compatibility conditions (1.1). If in addition \( f_0 \) and \( f_1 \) satisfy the Cauchy-Riemann equations (1.2), then \( f \) is called right slice hyperholomorphic. We denote the set of all right slice functions on \( U \) by \( \mathcal{SF}_R(U) \) and the set of all right slice hyperholomorphic functions on \( U \) by \( \mathcal{SH}_R(U) \).

If \( f \) is a left (or right) slice function such that \( f_0 \) and \( f_1 \) are real-valued, then \( f \) is called intrinsic. The set of all intrinsic slice functions on \( U \) will be denoted by \( \mathcal{FN}(U) \) and the set of all intrinsic slice hyperholomorphic functions on \( U \) will be denoted by \( \mathcal{N}(U) \).

**Lemma 1.2** ([1, Lemma 2.1.6], Splitting lemma). Let \( U \subseteq \mathbb{H} \) be an axially symmetric open set and let \( i, j \in S \) with \( ij = -ji \). If \( f \in \mathcal{SH}_L(U) \), then the restriction \( f_j = f|_{U \cap \mathbb{C}_j} \) satisfies

\[
\frac{1}{2} \left( \frac{\partial}{\partial u} f_j(z) + j \frac{\partial}{\partial v} f_j(z) \right) = 0,
\]

for all \( z = u + vj \in U \cap \mathbb{C}_j \). Hence

\[
f_j(z) = F_0(z) + F_1(z)i
\]
with holomorphic functions \( F_0, F_1 : U \cap \mathbb{C}_j \to \mathbb{C}_j \).

If \( f \in \mathcal{SH}_R(U) \), then the restriction \( f_j = f|_{U \cap \mathbb{C}_j} \) satisfies
\[
\frac{1}{2} \left( \frac{\partial}{\partial u} f_j(z) + \frac{\partial}{\partial v} f_j(z) \right) = 0,
\]
for all \( z = u + v j \in U \cap \mathbb{C}_j \). Hence
\[
f_j(z) = F_0(z) + iF_1(z)
\]
with holomorphic functions \( F_0, F_1 : U \cap \mathbb{C}_j \to \mathbb{C}_j \).

**Theorem 1.3** ([1, Theorem 2.1.21], Cauchy’s integral theorem). Let \( U \subseteq \mathbb{H} \) be open, \( j \in \mathbb{S} \) and \( D_j \subset U \cap \mathbb{C}_j \) be a bounded open subset of the complex plane \( \mathbb{C}_j \) with \( \partial D_j \subset U \cap \mathbb{C}_j \) such that \( \partial D_j \) is a finite union of piecewise continuously differentiable Jordan curves. Then for all \( f \in \mathcal{SH}_L(U) \) and all \( g \in \mathcal{SH}_R(U) \)
\[
\int_{\partial D_j} g(s) ds_j f(s) = 0,
\]
where \( ds_j = ds(-j) \).

**Definition 1.4** ([1, Definition 2.1.23]). We define the left slice hyperholomorphic Cauchy kernel as:
\[
S_L^{-1}(s, q) := -(q^2 - 2\text{Re}(s)q + |s|^2)^{-1}(q - \bar{s}); \quad q \notin [s],
\]
and the right slice hyperholomorphic Cauchy kernel as:
\[
S_R^{-1}(s, q) := -(q - \bar{s})(q^2 - 2\text{Re}(s)q + |s|^2)^{-1}; \quad q \notin [s].
\]

**Lemma 1.5** ([1, Lemma 2.1.27]). Let \( q, s \in \mathbb{H} \) with \( s \notin [q] \).
The left slice hyperholomorphic Cauchy kernel \( S_L^{-1}(s, q) \) is left slice hyperholomorphic in \( q \) and right slice hyperholomorphic in \( s \).
The right slice hyperholomorphic Cauchy kernel \( S_R^{-1}(s, q) \) is left slice hyperholomorphic in \( s \) and right slice hyperholomorphic in \( q \).

**Definition 1.6** ([1, Definition 2.1.30], Slice Cauchy domain). An axially symmetric open set \( U \subset \mathbb{H} \) is called a slice Cauchy domain if \( U \cap \mathbb{C}_j \) is a Cauchy domain in \( \mathbb{C}_j \) for every \( j \in \mathbb{S} \). More precisely, \( U \) is a slice Cauchy domain if for every \( j \in \mathbb{S} \) the boundary \( \partial(U \cap \mathbb{C}_j) \) of \( U \cap \mathbb{C}_j \) is the union a finite number of nonintersecting piecewise continuously differentiable Jordan curves in \( \mathbb{C}_j \).

**Theorem 1.7** ([1, Theorem 2.1.32], Cauchy’s formulas). Let \( U \subset \mathbb{H} \) be a bounded slice Cauchy domain, let \( j \in \mathbb{S} \), and set \( ds_j = ds(-j) \). If \( f \) is a left slice hyperholomorphic function on a set that contains \( \overline{U} \), then
\[
f(q) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} S_L^{-1}(s, q) ds_j f(s), \text{ for every } q \in U.
\]
If \( f \) is a right slice hyperholomorphic function on a set that contains \( \overline{U} \), then
\[
f(q) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} f(s) ds_j S_R^{-1}(s, q), \text{ for every } q \in U.
\]
These integrals depend neither on \( U \) nor on the imaginary unit \( j \in \mathbb{S} \).
Definition 1.8 ([1, Definition 2.3.1]). Let \((X, +)\) be an abelian group.

- \(X\) is a right quaternionic vector space denoted by \(X_R\) if it is endowed with a right quaternionic multiplication \((X, \mathbb{H}) \to X, (u, q) \mapsto uq\) such that for all \(u, v \in X\) and all \(p, q \in \mathbb{H}\),
  \[u(p + q) = up + uq, (u + v)q = uq + vq, (up)q = u(pq)\text{ and } u1 = u.\]

- \(X\) is a left quaternionic vector space denoted by \(X_L\) if it is endowed with a left quaternionic multiplication \((\mathbb{H}, X) \to X, (q, u) \mapsto qu\) such that for all \(u, v \in X\) and all \(p, q \in \mathbb{H}\),
  \[(p + q)u = pu + qu, q(u + v) = qu + vq, q(pu) = (qp)u\text{ and } 1u = u.\]

- \(X\) is a two-sided quaternionic vector space if it is endowed with a left and a right quaternionic multiplication such that \(X\) is both a left and a right quaternionic vector space and such that \(ru = ur\) for all \(r \in \mathbb{R}\), and \((pu)q = p(Qu)\) for all \(p, q \in \mathbb{H}\) and all \(u \in X\).

Definition 1.9. Let \(X_R\) be a right quaternionic vector space. A function \(\| \cdot \| : X_R \to [0; +\infty)\) is called a norm on \(X_R\), if it satisfies

(i) \(\|u\| = 0\) if and only if \(u = 0\);
(ii) \(\|uq\| = \|u\||q|\) for all \(u \in X_R\) and all \(q \in \mathbb{H}\);
(iii) \(\|u + v\| \leq \|u\| + \|v\|\) for all \(u, v \in X_R\).

If \(X_R\) is complete with respect to the metric induced by \(\| \cdot \|\), we call \(X_R\) a right quaternionic Banach space.

Let \(X_L\) be a left quaternionic vector space. A function \(\| \cdot \| : X_L \to [0; +\infty)\) is called a norm on \(X_L\), if it satisfies (i), (iii) and

(ii') \(\|qu\| = |q|\|u\|\) for all \(u \in X_L\) and \(q \in \mathbb{H}\).

If \(X_L\) is complete with respect to the metric induced by \(\| \cdot \|\), we call \(X_L\) a left quaternionic Banach space.

Finally, a two-sided quaternionic vector space \(X\) is called a two-sided quaternionic Banach space if it is endowed with a norm \(\| \cdot \|\) such that it is both a left and a right quaternionic Banach space.

Remark 1.10. If \(X\) is a two-sided quaternionic Banach space, then \(\|qu\| = \|uq\| = |q|\|u\|\) for all \(u \in X\) and all \(q \in \mathbb{H}\).

Definition 1.11 ([1, Definition 2.3.9], Slice hyperholomorphic vector-valued functions). Let \(U \subseteq \mathbb{H}\) be an axially symmetric open set and let \(\mathcal{U} = \{(u, v) \in \mathbb{R}^2 : u + v\mathfrak{S} \subset U\}\). A function \(f : U \to X_L\) with values in a left quaternionic Banach space \(X_L\) is called a left slice function if it is of the form:

\[f(q) = f_0(u, v) + jf_1(u, v)\text{ for every } q = u + vj \in U\]

with two functions \(f_0, f_1 : \mathcal{U} \to X_L\) that satisfy the compatibility conditions (1.1).

If in addition \(f_0\) and \(f_1\) satisfy the Cauchy-Riemann equations (1.2), then \(f\) is called left slice hyperholomorphic.
A function \( f : U \to X_R \) with values in a right quaternionic Banach space \( X_R \) is called a right slice function if it is of the form:

\[
f(q) = f_0(u,v) + f_1(u,v)j \quad \text{for every } q = u + vj \in U
\]

with two functions \( f_0, f_1 : U \to X_R \) that satisfy the compatibility conditions (1.1). If in addition \( f_0 \) and \( f_1 \) satisfy the Cauchy-Riemann equations (1.2), then \( f \) is called right slice hyperholomorphic.

**Theorem 1.12** ([1, Theorem 2.3.19], Vector-valued Cauchy formulas). Let \( U \subset \mathbb{H} \) be a bounded slice Cauchy domain, let \( j \in \mathbb{S} \), and set \( ds_j = ds(-j) \). If \( f \) is a left slice hyperholomorphic function with values in a left quaternionic Banach space \( X_L \) that is defined on a set that contains \( \overline{U} \), then

\[
f(q) = \frac{1}{2\pi} \int_{\partial(U \cap C_j)} S_L^{-1}(s,q)ds_jf(s), \quad \text{for every } q \in U.
\]

If \( f \) is a right slice hyperholomorphic function with values in a right quaternionic Banach space \( X_R \) that is defined on a set that contains \( \overline{U} \), then

\[
f(q) = \frac{1}{2\pi} \int_{\partial(U \cap C_j)} f(s)ds_jS_R^{-1}(s,q), \quad \text{for every } q \in U.
\]

These integrals depend neither on \( U \) nor on the imaginary unit \( j \in \mathbb{S} \).

**Definition 1.13.** Let \( X \) be a two-sided quaternionic Banach space. A right (resp. left) linear operator on \( X \) is a map \( T : X \to X \) such that:

\[
T(u+p+v) = (Tu)p+Tv \quad \text{(resp. } T(pu+v) = p(Tu)+Tv) \quad \text{for all } u, v \in X \text{ and all } p \in \mathbb{H}.
\]

A right or left linear operator \( T \) on \( X \) is called bounded if

\[
\|T\| := \sup\{\|Tu\| : u \in X, \|u\| = 1\} < \infty.
\]

The set of all right (resp. left) linear bounded operators on \( X \) is denoted by \( B_R(X) \) (resp. \( B_L(X) \)). \( B_R(X) \) (resp. \( B_L(X) \)) is viewed as a two-sided quaternionic vector space equipped with the metric \( B_R(X) \times B_R(X) \ni (A,B) \mapsto \|A-B\| \) (resp. \( B_L(X) \times B_L(X) \ni (A,B) \mapsto \|A-B\| \)).

In a two-sided quaternionic Banach space \( X \), we can define a left and a right quaternionic multiplication on \( B_R(X) \) (resp. \( B_L(X) \)) by

\[
(Tq)u = T(qu) \quad \text{(resp. } Tq(u) = q(Tu)) \quad \text{for all } q \in \mathbb{H}, u \in X \text{ and all } T \in B_R(X)
\]

resp. \( (Tq)u = T(u)q \) and \( (qT)(u) = T(uq) \) for all \( q \in \mathbb{H}, u \in X \text{ and all } T \in B_L(X) \).

The spectral theory over quaternionic Hilbert spaces has been developed in [4] and [6].

In the remainder of this paper, \( X \) will be a two-sided quaternionic Banach space. We will consider just right linear operators on \( X \). The theory we develop here also applies in the case of left linear operators with obvious modifications.
Let $T \in B_R(X)$. We want to define the notion of spectrum for right linear operators on $X$ such that this notion generalize the known results on the spectrum in the complex case (for instance, the compactness of the spectrum, the spectrum of self-adjoint operators is real). Note that if $q \in \mathbb{H}$, $T - Iq = T - qI$, where $I$ is the identity operator on $X$. Take $X = \mathbb{H} \oplus \mathbb{H}$ equipped with the standard scalar product:

$$\langle \begin{bmatrix} p \\ q \end{bmatrix} ; \begin{bmatrix} p' \\ q' \end{bmatrix} \rangle = \bar{p}p' + \bar{q}q' \text{ for all } p, q, p', q' \in \mathbb{H}.$$ 

Then $T := \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$ is self-adjoint. Let $u = \begin{bmatrix} 1 \\ -k \end{bmatrix}$, then $(T - jI)u = 0$, hence $j$ is an eigenvalue of $T$, thus the spectrum of $T$ is not real, and so the operators $T - Iq$ and $T - qI$ should not be used to define the spectrum of $T$. F. Colombo et al. [2] extended the definitions of the spectrum and resolvent in quaternionic Banach spaces as follows.

**Definition 1.14.** Let $T \in B_R(X)$. For $q \in \mathbb{H}$, we set

$$Q_q(T) := T^2 - 2\text{Re}(q)T + |q|^2I.$$ 

Where $I$ is the identity operator on $X$. We define the S-resolvent set $\rho_S(T)$ of $T$ as:

$$\rho_S(T) := \{ q \in \mathbb{H} : Q_q(T) \text{ is invertible in } B_R(X) \},$$

and we define the S-spectrum $\sigma_S(T)$ of $T$ as:

$$\sigma_S(T) := \mathbb{H} \setminus \rho_S(T).$$

**Proposition 1.15 ([1, Proposition 3.1.8]).** Let $T \in B_R(X)$. The sets $\sigma_S(T)$ and $\rho_S(T)$ are axially symmetric.

**Theorem 1.16 ([1, Theorem 3.1.13], Compactness of the S-spectrum).** Let $T \in B_R(X)$. The S-spectrum $\sigma_S(T)$ of $T$ is a nonempty compact set contained in the closed ball \( \{ q \in \mathbb{H} : |q| \leq \|T\| \} \).

Let $T \in B_R(X)$. Then the S-spectral radius of $T$ is defined to be the nonnegative real number

$$r_S(T) := \sup \{|q| : q \in \sigma_S(T)\}.$$

**Theorem 1.17 ([1, Theorem 4.2.3]).** For $T \in B_R(X)$, we have

$$r_S(T) = \lim_{n \to +\infty} \|T^n\|^\frac{1}{n}.$$

**Theorem 1.18.** Let $T \in B_R(X)$ and $q \in \mathbb{H}$ with $r_S(T) < |q|$. Then

$$(T^2 - 2\text{Re}(q)T + |q|^2I)^{-1} = \sum_{n=0}^{+\infty} T^n \sum_{k=0}^{n} \bar{q}^{-k-1}q^{-n+k-1},$$

where this series converges in the operator norm.
Proof. Let

\[ a_n := \sum_{k=0}^{n} q^{-k-1} q^{-n+k-1}, \]
then

\[ |a_n| \leq (n+1)|q|^{-n-2}. \]
Hence

\[ \|T^n \sum_{k=0}^{n} q^{-k-1} q^{-n+k-1}\| \leq \|T^n\|(n+1)|q|^{-n-2}. \]
We have

\[ \lim_{n \to \infty} \left( \|T^n\|(n+1)|q|^{-n-2} \right)^{\frac{1}{n}} = \frac{r_{S}(T)}{|q|}. \]
Thus the series \( \sum_{n=0}^{\infty} T^n \sum_{k=0}^{n} q^{-k-1} q^{-n+k-1} \) converges in the operator norm. The rest follows from the proof of [1, Theorem 3.1.5]. \( \square \)

**Definition 1.19** ([1, Definition 3.2.5], S-functional calculus). Let \( T \in \mathcal{B}_{R}(X) \). Let \( U \supset \sigma_{S}(T) \) be a bounded slice Cauchy domain, let \( j \in \mathbb{S} \), and set \( ds_{j} = ds(-j) \). For every \( f \in \mathcal{SH}_{L}(U) \), we define

\[ f(T) := \frac{1}{2\pi} \int_{\partial(U \cap C_{j})} S_{L}^{-1}(s, T) ds_{j} f(s). \]
For every \( f \in \mathcal{SH}_{R}(U) \), we define

\[ f(T) := \frac{1}{2\pi} \int_{\partial(U \cap C_{j})} f(s) ds_{j} S_{R}^{-1}(s, T). \]
Let \( K \subseteq \mathbb{H} \). In the following, we mean by \( \mathcal{SH}_{R}(K) \) (resp. \( \mathcal{SH}_{L}(K), \mathcal{N}(K) \)), the set of all right (resp. left, intrinsic) slice hyperholomorphic functions on an open axially symmetric set \( U \) that contains \( K \).

**Theorem 1.20** ([1, Theorem 4.1.3], Product rule). Let \( T \in \mathcal{B}_{R}(X) \), \( f \in \mathcal{N}(\sigma_{S}(T)) \) and \( g \in \mathcal{SH}_{L}(\sigma_{S}(T)) \) or \( g \in \mathcal{SH}_{R}(\sigma_{S}(T)) \). Then

\[ (fg)(T) = f(T)g(T). \]

**Theorem 1.21** ([1, Theorem 4.2.1], The spectral mapping theorem). Let \( T \in \mathcal{B}_{R}(X) \) and \( f \in \mathcal{N}(\sigma_{S}(T)) \). Then

\[ \sigma_{S}(f(T)) = \sigma(f_{S}(T)) := \{ f(q) : q \in \sigma_{S}(T) \}. \]

**Theorem 1.22** ([1, Theorem 4.2.4], Composition rule). Let \( T \in \mathcal{B}_{R}(X) \) and \( f \in \mathcal{N}(\sigma_{S}(T)) \). If \( g \in \mathcal{SH}_{L}(\sigma_{S}(f(T))) \), then \( g \circ f \in \mathcal{SH}_{L}(\sigma_{S}(T)) \), and if \( g \in \mathcal{SH}_{R}(\sigma_{S}(T))) \), then \( g \circ f \in \mathcal{SH}_{R}(\sigma_{S}(T)) \). In both cases,

\[ g(f(T)) = (g \circ f)(T). \]

A bounded right projection \( P \in \mathcal{B}_{R}(X) \) (or simply projection when no confusion can arise) is such that \( P^2 = P \). If \( P \) is a projection, then so is \( I - P \), and their null spaces and ranges are related as follows:

\[ \mathcal{R}(P) = \mathcal{N}(I - P) \text{ and } \mathcal{N}(P) = \mathcal{R}(I - P). \]
The range and the kernel form a pair of algebraic complements,
\[ \mathcal{R}(P) + \mathcal{N}(P) = X \text{ and } \mathcal{R}(P) \cap \mathcal{N}(P) = \{0\}. \]

2. Generalized inverse

In this section, we study the generalized invertibility of right linear operators on quaternionic Banach spaces.

**Definition 2.1.** An operator \( B \in \mathcal{B}_R(X) \) is called a generalized inverse of \( A \in \mathcal{B}_R(X) \) if \(ABA = A \) and \( BAB = B \).

**Remark 2.2.** Let \( A, B \in \mathcal{B}_R(X) \).

1) If \( B \) is a generalized inverse of \( A \), then \( AB \) and \( BA \) are projections. Indeed \( (AB)^2 = (ABA)B = AB \), \( (BA)^2 = B(ABA) = BA \).

2) If \( ABA = A \), then \( T := BAB \) is a generalized inverse of \( A \).

3) If \( A \) is left (resp. right) invertible, then it is generalized invertible.

**Theorem 2.3.** An operator \( A \in \mathcal{B}_R(X) \) has a generalized inverse if and only if both the range \( \mathcal{R}(A) \) and the null space \( \mathcal{N}(A) \) are closed complemented subspaces of \( X \).

**Proof.** Let \( Y \) and \( Z \) be two closed complemented subspaces of \( X \). If \( X = \mathcal{R}(A) \oplus Y = \mathcal{N}(A) \oplus Z \), then the operator defined by \( B(az + y) = z \) with \( z \in Z \) is right linear on \( X \). It is easy to check that \( B \) is a generalized inverse of \( A \). Conversely, if \( A \) has a generalized inverse \( B \in \mathcal{B}_R(X) \), then by Remark 2.2, \( \mathcal{R}(AB) = \mathcal{R}(A) \) and \( \mathcal{N}(BA) = \mathcal{N}(A) \). Since \( AB \) and \( BA \) are bounded right projections, both the range \( \mathcal{R}(AB) \) and the null space \( \mathcal{N}(BA) \) are closed complemented subspaces of \( X \), then so is \( \mathcal{R}(A) \) and \( \mathcal{N}(A) \).

The generalized inverse is not unique in general, the following theorem describes all generalized inverses of \( A \in \mathcal{B}_R(X) \).

**Theorem 2.4.** Suppose \( B \in \mathcal{B}_R(X) \) is a generalized inverse of \( A \in \mathcal{B}_R(X) \). Then the set of all generalized inverses of \( A \) consists of all operators of the form:

\[ T = PBQ, \]

where \( Q \) is a projection onto \( \mathcal{R}(A) \) and \( P \) is a projection whose kernel coincides with \( \mathcal{N}(A) \).

**Proof.** Let \( B \) and \( T \) be two generalized inverses of \( A \), then \( ABA = ATA = A \) and \( TAT = T \), hence \( T(ABA)T = (TA)B(AT) = T \). By Remark 2.2, \( TA \) and \( AT \) are projections and such that \( \mathcal{N}(TA) = \mathcal{N}(A) \) and \( \mathcal{R}(A) = \mathcal{R}(AT) \).

Conversely, if \( B, P \) and \( Q \) are as in Theorem 2.4, then \( QA = A = AP \), hence \( A(PBQ)A = A \) and \( (PBQ)A(PBQ) = PBQ \).
Lemma 2.5. Let $A \in B_{\mathcal{R}}(X)$. Then $A$ has a generalized inverse $B$ such that $AB = BA$ if and only if $X$ can be written as $X = \mathcal{R}(A) \oplus \mathcal{N}(A)$. In such a case, $B$ is unique.

Proof. By Remark 2.2, $X = \mathcal{R}(A) \oplus \mathcal{N}(B)$ and since $AB = BA$, $\mathcal{N}(B) = \mathcal{N}(AB) = \mathcal{N}(BA) = \mathcal{N}(A)$, then $X = \mathcal{R}(A) \oplus \mathcal{N}(A)$.

Conversely, since $X = \mathcal{R}(A) \oplus \mathcal{N}(A)$, the restriction $T := A/\mathcal{R}(A)$ is invertible, then $B = T^{-1} \oplus 0$ is a generalized inverse of $A$ which commutes with $A$. If $B$ and $C$ are two generalized inverses of $A$ that commute with $A$, then $B = AB^2 = CA^2B^2 = CAB = C^2A^2B = C^2A = C$. □

Theorem 2.6. Suppose $A \in B_{\mathcal{R}}(X)$ with generalized inverse $B$ such that $AB = BA$. Then

$$\sigma_S(B) \setminus \{0\} = \{q^{-1} : q \in \sigma_S(A) \setminus \{0\}\}.$$  

Proof. By Lemma 2.5, $X = \mathcal{R}(A) \oplus \mathcal{N}(A)$, then $A = T \oplus 0$ on $\mathcal{R}(A) \oplus \mathcal{N}(A)$ and $B = T^{-1} \oplus 0$. We have $Q_q(B) = Q_q(T^{-1}) \oplus Q_q(0)$, for all $q \in \mathbb{H}$. Then we have $\sigma_S(B) = \sigma_S(T^{-1}) \cup \sigma_S(0)$, since $Q_q(0) = |q|^2I$ is always invertible where $I$ is the identity operator on $\mathcal{N}(A)$,

$$\sigma_S(B) \setminus \{0\} = \sigma_S(T^{-1}) \setminus \{0\}.$$  

The function $f : \mathbb{H} \setminus \{0\} \ni q \mapsto q^{-1}$ is intrinsic slice hyperholomorphic (because $q^{-1} = \frac{q}{|q|^2}$), then by Theorem 1.21, $\sigma_S(T^{-1}) = \sigma_S(f(T)) = \{q^{-1} : q \in \sigma_S(T)\}$. Thus

$$\sigma_S(B) \setminus \{0\} = \{q^{-1} : q \in \sigma_S(A) \setminus \{0\}\}.$$  

□

3. Drazin inverse

In this section, we study the Drazin invertibility of right linear operators acting on a quaternionic Banach space.

Definition 3.1. Let $A \in B_{\mathcal{R}}(X)$. An element $B \in B_{\mathcal{R}}(X)$ is a Drazin inverse of $A$, written $B = A^d$, if

$$AB = BA, \ AB^2 = B, \ A^{k+1}B = A^k,$$  

for some nonnegative integer $k$. The least nonnegative integer $k$ for which these equations hold is the Drazin index $i(A)$ of $A$.

Definition 3.2. An element $A$ of $B_{\mathcal{R}}(X)$ is called quasinilpotent if $\sigma_S(A) = \{0\}$. The set of all quasinilpotent elements in $B_{\mathcal{R}}(X)$ will be denoted by $QN(B_{\mathcal{R}}(X))$.

Proposition 3.3. An element $A$ of $B_{\mathcal{R}}(X)$ is quasinilpotent if and only if, for every $T$ commuting with $A$, we have $I - TA$ is invertible.

Proof. Let $A \in B_{\mathcal{R}}(X)$, assume that for every $T \in B_{\mathcal{R}}(X)$ commuting with $A$, we have $I - TA$ is invertible. Let $T = \frac{-1}{|q|}A + \frac{2\text{Re}(q)}{|q|^2}I$ with $q \in \mathbb{H} \setminus \{0\}$, clearly $T$ commutes with $A$ and $I - TA = \frac{1}{|q|}[A^2 - 2\text{Re}(q)A + |q|^2I]$ is invertible, hence $\sigma_S(A) = \{0\}$. 


Conversely, if $\sigma_S(A) = \{0\}$. Let $T \in \mathcal{B}_R(X)$ commutes with $A$, then by Theorem 1.17, $r_S(TA) \leq r_S(T)r_S(A) = 0$ and hence $\sigma_S(TA) = \{0\}$. Then by Theorem 1.21, $\sigma_S(I - TA) = \{1\}$ and hence $I - TA$ is invertible.

An operator $A \in \mathcal{B}_R(X)$ is said to be nilpotent if there exists $k \in \mathbb{N}$ such that $A^k = 0$. The least nonnegative integer $k$ for which $A^k = 0$ is called the nilpotency index of $A$ and the set of all nilpotent elements in $\mathcal{B}_R(X)$ is denoted by $N(\mathcal{B}_R(X))$.

**Lemma 3.4.** In $\mathcal{B}_R(X)$, (3.1) is equivalent to

$$AB = BA, AB^2 = B, A - A^2B \in N(\mathcal{B}_R(X)).$$

The Drazin index $i(A)$ is equal to the nilpotency index of $A - A^2B$.

**Proof.** If $AB = BA$ and $AB^2 = B$, then $I - AB$ is a projection. Hence the equivalence and the last statement are given by this equalities $(A - A^2B)^k = A^k(I - AB)k = A^k(I - AB) = A^k - A^{k+1}B$. □

Koliha [5, Definition 2.3] generalized the notion of Drazin invertibility in a complex Banach algebra. According to this definition one can generalize the notion of Drazin invertibility in $\mathcal{B}_R(X)$.

**Definition 3.5.** Let $A \in \mathcal{B}_R(X)$. An element $B \in \mathcal{B}_R(X)$ is a generalized Drazin inverse of $A$, written $B = A^D$, if

$$AB = BA, AB^2 = B, A - A^2B \in QN(\mathcal{B}_R(X)).$$

**Lemma 3.6.** In $\mathcal{B}_R(X)$, an element $A$ has a Drazin (resp. generalized Drazin) inverse if and only if there is a projection $P$ commuting with $A$ such that

$$AP \in N(\mathcal{B}_R(X)) (\text{resp. } AP \in QN(\mathcal{B}_R(X))) \text{ and } A + P \text{ is invertible.}$$

A Drazin (resp. generalized Drazin) inverse of $A$ is given by

$$A^d = (A + P)^{-1}(I - P) (\text{resp. } A^D = (A + P)^{-1}(I - P)).$$

**Proof.** Suppose that there is a projection $P$ commuting with $A$ and satisfying (3.4). Set $B = (A + P)^{-1}(I - P)$, then $AB = BA$, $AB^2 = B$ and $A - A^2B = AP \in N(\mathcal{B}_R(X))$ (resp. $A - A^2B = AP \in QN(\mathcal{B}_R(X))$). Conversely, suppose that $B$ satisfies (3.2) (resp. (3.3)) and set $P = I - AB$. Since $AB^2 = B$, $P$ is a projection commuting with $A$ and $AP = A - A^2B$, then $AP \in N(\mathcal{B}_R(X))$ (resp. $AP \in QN(\mathcal{B}_R(X))$). Furthermore $(A + P)(B + P) = (B + P)(A + P) = I + AP$ and $I + AP$ is invertible because $\sigma_S(I + AP) = \{1\}$, then $A + P$ is invertible in $\mathcal{B}_R(X)$. □

**Definition 3.7.** Let $A \in \mathcal{B}_R(X)$. For $s \in \rho_S(T)$, we define the left $S$-resolvent operator as

$$S_L^{-1}(s, A) = -Q_s(T)^{-1}(T - sI).$$

**Theorem 3.8** ([1, Theorem 4.1.5]). Let $A \in \mathcal{B}_R(X)$ and assume that $\sigma_S(A) = \sigma_1 \cup \sigma_2$ with

$$\text{dist}(\sigma_1, \sigma_2) > 0.$$
We choose an open axially symmetric set \( O \) with \( \sigma_1 \subset O \) and \( \overline{O} \cap \sigma_2 = \emptyset \), and define a function \( \chi_{\sigma_1} \) on \( \mathbb{H} \) by \( \chi_{\sigma_1}(s) = 1 \) for \( s \in O \) and \( \chi_{\sigma_1}(s) = 0 \) for \( s \notin O \). Then \( \chi_{\sigma_1} \in \mathcal{N}(\sigma_S(A)) \), and for an arbitrary imaginary unit \( j \) in \( \mathbb{S} \) and an arbitrary bounded slice Cauchy domain \( U \subset \mathbb{H} \) such that \( \overline{U} \subset O \), we have

\[
P_{\sigma_1} := \chi_{\sigma_1}(A) = \frac{1}{2\pi} \int_{\partial(U \cap C_j)} S_L^{-1}(s, A)ds_j
\]

is a continuous projection that commutes with \( A \). Hence \( P_{\sigma_1}(X) \) is a right linear subspace of \( X \) that is invariant under \( A \).

**Remark 3.9.** Let \( q \in \mathbb{H} \). If \( \sigma_1 = \{q\} \), we say that the projection \( P_{\sigma_1} \) is the Riesz’s projection of \( A \) corresponding to \( q \).

Denote by acc \( U \) (resp. iso \( U \)) the set of all accumulation (resp. isolated) points of a set \( U \subset \mathbb{H} \).

**Theorem 3.10.** Let \( A \in \mathcal{B}_R(X) \). Then \( 0 \notin \text{acc} \ \sigma_S(A) \) if and only if there is a projection \( P \in \mathcal{B}_R(X) \) commuting with \( A \) such that

\[
AP \in QN(\mathcal{B}_R(X)) \quad \text{and} \quad A + P \text{ is invertible in } \mathcal{B}_R(X).
\]

Moreover, \( 0 \in \text{iso} \ \sigma_S(A) \) if and only if \( P \neq 0 \), in which a case \( P \) is the Riesz’s projection of \( A \) corresponding to \( q = 0 \).

**Proof.** Clearly, \( 0 \notin \sigma_S(A) \) if and only if \( 3.6 \) holds with \( P = 0 \).

Assume that \( 0 \in \text{iso} \ \sigma_S(A) \). Let \( P \) be the spectral projection of \( A \) corresponding to \( q = 0 \), then \( P \neq 0 \), commutes with \( A \) and \( AP = \text{id}(A)\chi_{\{0\}}(A) = (\text{id}\chi_{\{0\}})(A) \) where \( \text{id} : \mathbb{H} \to \mathbb{H}, q \mapsto q \). Hence \( \sigma_S(AP) = \text{id}\chi_{\{0\}}(\sigma_S(A)) = \{0\} \), thus \( AP \in QN(\mathcal{B}_R(X)) \). Similarly \( A + P = \text{id}(A) + \chi_{\{0\}}(A) = (\text{id} + \chi_{\{0\}})(A) \), then \( 0 \notin \sigma_S(A + P) = (\text{id} + \chi_{\{0\}})\sigma_S(A) \), hence \( A + P \) is invertible.

Conversely, assume that there is a nonzero projection \( P \) commuting with \( A \) such that \( 3.6 \) holds. For any \( q \in \mathbb{H} \), we have

\[
A^2 - 2\text{Re}(q)A + |q|^2I
= P((AP)^2 - 2\text{Re}(q)AP + |q|^2I) + (I - P)((A + P)^2 - 2\text{Re}(q)(A + P) + |q|^2I).
\]

There is \( r > 0 \) such that if \( |q| < r \) then \( (A + P)^2 - 2\text{Re}(q)(A + P) + |q|^2I \) is invertible. Since \( AP \in QN(\mathcal{B}_R(X)) \), \( (AP)^2 - 2\text{Re}(q)AP + |q|^2I \) is invertible of all \( q \neq 0 \). Hence for all \( 0 < |q| < r \), it is easy to check that

\[
(A^2 - 2\text{Re}(q)A + |q|^2I)^{-1} = P((AP)^2 - 2\text{Re}(q)AP + |q|^2I)^{-1} + (I - P)((A + P)^2 - 2\text{Re}(q)(A + P) + |q|^2I)^{-1}.
\]

That is,

\[
Q_q(A)^{-1} = PQ_q(AP)^{-1} + (I - P)Q_q(A + P)^{-1}.
\]

Since \( S_L^{-1}(q, A) = -Q_q(A)^{-1}(A - \bar{q}I) \), it is easy to see that

\[
S_L^{-1}(q, A) = PS_L^{-1}(q, AP) + (I - P)S_L^{-1}(q, A + P).
\]
Since $P \neq 0$, $0 \in \text{iso}_S(A)$. Indeed, if $A$ is invertible, then $A^{-1}AP = P$, so that $r_S(A^{-1}AP) \leq r_S(A^{-1}) r_S(AP) = 0$ and $r_S(P) = 1$. To show that $P$ is the Riesz's projection of $A$ corresponding to $q = 0$. Let $j$ and $U$ as in Theorem 3.12, then
\[
\chi(\{0\})(A) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} S_{L}^{-1}(s, A) ds. 
\]
If we take $U$ such that $U \subset \{q \in \mathbb{H} : |q| < \frac{1}{2}\}$, then by (3.7)
\[
\chi(\{0\})(A) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} S_{L}^{-1}(s, A) ds_j = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} PS_{L}^{-1}(s, AP) + (I - P)S_{L}^{-1}(s, A + P) ds_j = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} PS_{L}^{-1}(s, AP) ds + \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} (I - P)S_{L}^{-1}(s, A + P) ds_j = \frac{P}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} S_{L}^{-1}(s, AP) ds_j + (I - P) \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} S_{L}^{-1}(s, A + P) ds_j. 
\]
Since $S_{L}^{-1}(-, A + P)$ is right slice hyperholomorphic function on $U$ (see, [1, Lemma 3.11]),
\[
\int_{\partial(U \cap \mathbb{C}_j)} S_{L}^{-1}(s, A + P) ds_j = 0. 
\]
On the other hand,
\[
\frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} S_{L}^{-1}(s, AP) ds_j = I, 
\]
because $\sigma_S(AP) = \{0\} \subset U$. Hence $\chi(\{0\})(A) = P$. This completes the proof. \hfill \Box

**Corollary 3.11.** The Drazin (resp. generalized Drazin) inverse of an operator $A \in \mathcal{B}_R(X)$ is uniquely determined.

**Proof.** If $A$ is invertible, then it has a unique Drazin inverse which coincides with its inverse $A^{-1}$. Assume that $A$ is not invertible and let $B$ and $C$ be two Drazin (resp. generalized Drazin) inverses of $A$, then by the proof of Lemma 3.6, $I - BA$ and $I - AC$ are two projections commuting with $A$ and satisfying (3.6). Then by Theorem 3.10, $I - BA = I - AC$, and so $BA = AC$, thus $B = B^2A = BAC = AC^2 = C$. Hence the Drazin (resp. generalized Drazin) inverse of $A$ is unique. \hfill \Box

**Theorem 3.12.** Let $A \in \mathcal{B}_R(X)$. If $0 \in \text{iso}_S(A)$, then
\[ A^D = f(A), \]
where $f \in \mathcal{N}(\sigma_S(A))$ is such that $f$ is 0 in an axially symmetric neighborhood of 0 and $f(q) = q^{-1}$ in an axially symmetric neighborhood of $\sigma_S(A) \setminus \{0\}$, and
\[ \sigma_S(A^D) \setminus \{0\} = \{q^{-1} : q \in \sigma_S(A) \setminus \{0\}\}. \]

**Proof.** Let $O_1$ be an axially symmetric open neighborhood of 0 and $O_2$ be an axially symmetric open neighborhood of $\sigma_S(A) \setminus \{0\}$ with $\overline{O_1} \cap \overline{O_2} = \emptyset$. Define $f$ by $f(q) = 0$ if $q \in O_1$ and $f(q) = q^{-1}$ if $q \in O_2$, clearly $f \in \mathcal{N}(\sigma_S(A))$. By
Theorem 1.20 and Theorem 1.21, it is easy to see that (3.3) holds for $A$ and $f(A)$. By Theorem 1.21, it follows that $\sigma_s(A^D) \setminus \{0\} = \sigma_s(f(A)) \setminus \{0\} = \{f(q) : q \in \sigma_s(A) \setminus \{0\}\}. \hfill \square$

**Theorem 3.13.** Let $A \in \mathcal{B}_R(X)$. The following conditions are equivalent:

(i) $A$ is generalized Drazin invertible;
(ii) $0 \notin \text{acc } \sigma_S(A)$;
(iii) $A = A_1 \oplus A_2$, where $A_1$ is invertible on some closed subspace $X_1$ of $X$ and $A_2$ is quasinilpotent on some complemented subspace $X_1$ of $X$.

**Proof.** (i) $\Leftrightarrow$ (ii) Already proved in Lemma 3.6 and Theorem 3.10.
(ii) $\Rightarrow$ (iii) By Lemma 3.6 there exists a projection $P = I - AA^D$ such that $AP$ is quasinilpotent and $AP = PA$, then $\mathcal{R}(P)$ and $\mathcal{N}(P)$ are invariant under $A$, that is $AR(P) \subset \mathcal{R}(P)$ and $AN(P) \subset \mathcal{N}(P)$. Let $u \in \mathcal{N}(P)$, then $u = AA^Du$, thus the restriction of $A$ to the kernel of $P$ is injective and surjective, and so invertible. If we write $A = A_1 \oplus A_2$ on $X = \mathcal{N}(P) \oplus \mathcal{R}(P)$, then $A_2 \in \mathcal{B}_R(X_1)$ is quasinilpotent and $A_1 \in \mathcal{B}_R(X_2)$ is invertible.
(iii) $\Rightarrow$ (i) It is easy to check that $A^D = A_1^{-1} \oplus 0$. \hfill \square

**Corollary 3.14.** Let $A \in \mathcal{B}_R(X)$. The following conditions are equivalent:

(i) $A$ is Drazin invertible;
(ii) $0 \notin \sigma(A)$;
(iii) $A = A_1 \oplus A_2$, where $A_1$ is invertible on some closed subspace $X_1$ of $X$, $A_2$ is nilpotent on some complemented subspace $X_1$ of $X$ and the nilpotency index of $A_2$ is the Drazin index of $A$.

**Proof.** Assume that $A$ is Drazin invertible, then by Theorem 3.13 (iii), $A = A_1 \oplus A_2$ and $A^k = A_1^{-1} \oplus 0$. Hence, by (3.1), $A^{k+1}A^D = A^k$, then $A_2^k \oplus 0 = A_1^k \oplus A_2^k$, thus $A_2^k = 0$, so that the nilpotency index of $A_2$ is less than the Drazin index of $A$.
Conversely, let $B = A_1^{-1} \oplus 0$, where $A_1$ is invertible and $A_2$ is nilpotent, then (3.1) holds for $A$, $B$ and the nilpotency index of $A_2$. Hence $A$ is Drazin invertible and the Drazin index of $A$ is less than the nilpotency index of $A_2$. \hfill \square

**Theorem 3.15.** Suppose that $A \in \mathcal{B}_R(X)$ has the generalized Drazin inverse $A^D$.

Then

(i) $(A^k)^D = (A^D)^k$ for all $k \in \mathbb{N}$;
(ii) $(A^D)^D = A^2A^D$;
(iii) $(A^D)^D = A^D D$;
(iv) $A^D (A^D)^D = AA^D$.

**Proof.** Let $f \in \mathcal{N}(\sigma_S(A))$ such that $f$ is 0 in an axially symmetric neighborhood of 0 and $f(q) = q^{-1}$ in an axially symmetric neighborhood of $\sigma_S(A) \setminus \{0\}$. By Theorem 3.12, $A^D = f(A)$. Let $k \in \mathbb{N}$ and $g_k \in \mathcal{N}(\mathbb{H})$ such that $g_k(q) = g^k$ if $q \in \mathbb{H}$. Clearly $f \circ g_k = g_k \circ f$ for all $k \in \mathbb{N}$, $f \circ f = g_2 \circ f$, $f \circ f \circ f = f$ and $f(f \circ f) = g_1f$. The above assertions are easily verified by using the previous equalities, Theorem 1.20 and Theorem 1.22. \hfill \square
Theorem 3.16. Let $A, B \in \mathcal{B}_R(X)$ be commuting elements such that $A^d$ and $B^d$ exist. Then $(AB)^d$ exists and 

$$(AB)^d = A^d B^d.$$ 

Proof. By [3, Theorem 1], $A, B, A^d$ and $B^d$ commute mutually, then the result follows by Definition 3.1.

Given a right linear operator $T$ on $X$. $T$ is said to have finite ascent if there is an integer $k$ such that $\mathcal{N}(T^k) = \mathcal{N}(T^{k+1})$, the smallest such positive integer $k$ is called the ascent of $T$ and denoted by $a(T)$. If there is no such integer we set $a(T) := \infty$. Analogously, $T$ is said to have finite descent if there is an integer $k$ such that $T^{k+1}(X) = T^k(X)$, the smallest such positive integer $k$ is called the descent of $T$ and denoted by $d(T)$. If there is no such integer we set $d(T) := \infty$.

As in the complex case, we have the following result.

Theorem 3.17. Suppose that $T$ is a right linear operator on $X$ and let $k \in \mathbb{N}$. Then $a(T) = d(T) \leq k$ if and only if, we have the decomposition

$$X = \mathcal{R}(T^k) \oplus \mathcal{N}(T^k).$$

Proof. Let $k = a(T) = d(T)$, then $\mathcal{R}(T^k) \cap \mathcal{N}(T^k) = \{0\}$. Indeed, let $u \in \mathcal{R}(T^k) \cap \mathcal{N}(T^k)$, then there is a vector $v \in X$ such that $u = T^kv$, hence $T^ku = T^{2k}v = 0$. Since $\mathcal{N}(T^{2k}) = \mathcal{N}(T^k)$, $u = T^kv = 0$. On the other hand, we have $\mathcal{R}(T^k) \cap \mathcal{N}(T^k) = X$. Indeed, let $u \in X$, since $\mathcal{R}(T^k) = \mathcal{R}(T^{2k})$, there is a vector $v \in X$ such that $T^ku = T^{2k}v$, hence $u = T^kv + u - T^kv$. Thus $X = \mathcal{R}(T^k) \oplus \mathcal{N}(T^k)$. Conversely, let $u \in \mathcal{N}(T^{k+1})$. Since $\mathcal{R}(T^k) \cap \mathcal{N}(T^k) = \{0\}$, $T^ku = 0$. Hence $\mathcal{N}(T^{k+1}) = \mathcal{N}(T^k)$. On the other hand, let $u \in \mathcal{R}(T^k)$, then there is a vector $v \in X$ such that $u = T^kv$. Since $v \in \mathcal{R}(T^k) \oplus \mathcal{N}(T^k)$, $u = T^kv \in \mathcal{R}(T^{k+1})$. Hence $\mathcal{R}(T^k) = \mathcal{R}(T^{k+1})$. Let now $p = a(T)$ and $q = d(T)$, we can suppose that $p > 0$ and $q > 0$, assume that $p \leq q$, let $u \in \mathcal{R}(T^p)$, then there is a vector $v \in X$ such that $u = T^pv$. Since $v \in \mathcal{R}(T^q) \oplus \mathcal{N}(T^p)$, $u = T^pv \in \mathcal{R}(T^{p+q}) \subseteq \mathcal{R}(T^{p+1})$. Thus $p = q \leq k$. Assume that $q \leq p$, let $u \in \mathcal{N}(T^{q+1})$, then $T^{q}u = 0$, that is $T^q u \in \mathcal{R}(T^q) \cap \mathcal{N}(T)$. Since $X = \mathcal{R}(T^q) \oplus \mathcal{N}(T^q)$, $T^q u = 0$, and so $u \in \mathcal{N}(T^q)$. Thus $p = q \leq k$. Hence $a(T) = d(T) \leq k$.

Theorem 3.18. An operator $A$ in $\mathcal{B}_R(X)$ has a Drazin inverse if and only if it has finite ascent and descent. In such a case, the Drazin index of $A$ is equal to the common value of $a(A)$ and $d(A)$.

Proof. By Corollary 3.14, $A$ is Drazin invertible if and only if $A = A_1 \oplus A_2$ with $A_1$ is invertible and $A_2$ is nilpotent. Let $k$ be the nilpotency index of $A_2$, then $k$ is the least integer such that $X = \mathcal{R}(T^k) \oplus \mathcal{N}(T^k)$, hence $a(A) = d(A) = k$. By Corollary 3.14, again, $k = i(A)$, thus $a(A) = d(A) = i(A)$.

Definition 3.19. A two-sided quaternionic Banach algebra is a two-sided quaternionic Banach space $\mathcal{A}$ that is endowed with a product $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such that:

(i) The product is associative and distributive over the sum in $\mathcal{A}$;
(ii) one has \((qx)y = q(xy)\) and \(x(yq) = (xy)q\) for all \(x, y \in \mathcal{A}\) and all \(q \in \mathbb{H}\);

(iii) one has \(\|xy\| \leq \|x\|\|y\|\) for all \(x, y \in \mathcal{A}\).

If in addition there exists \(e \in \mathcal{A}\) such that \(exe = x\) for all \(x \in \mathcal{A}\), then \(\mathcal{A}\) is called a two-sided quaternionic Banach algebra with unit.

One can prove that \(\mathcal{B}_R(X)\) and \(\mathcal{B}_L(X)\) are two-sided quaternionic Banach algebras with unit.

**Definition 3.20.** Let \(\mathcal{A}\) be a two-sided quaternionic Banach algebra and \(a \in \mathcal{A}\). An element \(b \in \mathcal{A}\) is a Drazin inverse of \(a\), written \(b = a^d\), if

\[
ab = ba, \quad ab^2 = b, \quad a^{k+1}b = a^k,
\]

for some nonnegative integer \(k\). The least nonnegative integer \(k\) for which these equations hold is the Drazin index \(i(a)\) of \(a\).

Let \(\mathcal{A}\) be a two-sided quaternionic Banach algebra and \(a \in \mathcal{A}\). For any \(a \in \mathcal{A}\) we define the left multiplication of \(a\) by \(L_a(b) = ab\), for all \(b \in \mathcal{A}\). Then \(L_a \in \mathcal{B}_R(\mathcal{A})\), we have \(\|L_a\| = \|a\|\).

**Theorem 3.21.** Let \(\mathcal{A}\) be a two-sided quaternionic Banach algebra and \(a \in \mathcal{A}\) with unit. Then \(a\) is Drazin invertible if and only if \(L_a\) is Drazin invertible. In such a case, \(L_a^d = L_a\) and \(i(L_a) = i(a)\).

*Proof.* Let \(a \in \mathcal{A}\) such that \(a\) is Drazin invertible. For every \(b \in \mathcal{A}\), we have \(L_a L_b = L_{ab}\), hence it is easy to check that \(L_a^d = L_a\) and then \(i(L_a) \leq i(a)\).

Conversely, assume that \(L_a\) is Drazin invertible and let \(b = L_a(c)\). Since \(L_a^{k+1}L_a^d = L_a^k\), \(a^k b = a^k\). Hence \(L_a^{k+1}L_b = L_a^k = L_a L_a^{k+1}\), then by [3, Theorem 4] and its proof, \(L_a^d = L_a^k L_a^{k+1} = L_a^{k+1} L_c = L_a L_c L_a\), \(L_a^2 = L_c, L_a^{k+1} L_c = L_a^k\), hence \(ac = ca, ac^2 = c, a^{k+1}c = a^k\). Thus \(a\) is Drazin invertible and then \(i(a) \leq i(L_a)\).

\[\square\]

**References**

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