 Cliques and the Spectral Radius

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Abstract

We prove a number of relations between the number of cliques of a graph $G$ and the largest eigenvalue $\mu (G)$ of its adjacency matrix. In particular, writing $k_s (G)$ for the number of $s$-cliques of $G$, we show that, for all $r \geq 2$,

$$\mu^{r+1} (G) \leq (r+1) k_{r+1} (G) + \sum_{s=2}^{r} (s-1) k_s (G) \mu^{r+1-s} (G),$$

and, if $G$ is of order $n$, then

$$k_{r+1} (G) \geq \left( \frac{\mu (G)}{n} - 1 + \frac{1}{r} \right) \frac{r (r-1)}{r+1} \left( \frac{n}{r} \right)^{r+1}.$$  

Keywords: number of cliques, clique number, spectral radius, stability

1 Introduction

Our graph-theoretic notation is standard (e.g., see [1]); in particular, we write $G (n)$ for a graph of order $n$. Given a graph $G$, a $k$-walk is a sequence of vertices $v_1, \ldots, v_k$ of $G$ such that $v_{i-1}$ is adjacent to $v_i$ for all $i = 2, \ldots, k$. We write $w_k (G)$ for the number of $k$-walks in $G$ and $k_r (G)$ for the number of its $r$-cliques. We order the eigenvalues of the adjacency matrix of a graph $G = G (n)$ as $\mu (G) = \mu_1 (G) \geq \ldots \geq \mu_n (G)$.

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Let \( \omega = \omega(G) \) be the clique number of \( G \). Wilf [12] proved that
\[
\mu(G) \leq \frac{\omega - 1}{\omega} v(G) = \frac{\omega - 1}{\omega} w_1(G),
\]
and Nikiforov [9] extended this, showing that the inequality
\[
\mu^s(G) \leq \frac{\omega - 1}{\omega} w_s(G)
\]
holds for every \( s \geq 1 \). Note that for \( s = 2 \) inequality (1) implies a concise form of Turán’s theorem. Indeed, if \( G \) has \( n \) vertices and \( m \) edges, then \( \mu(G) \geq 2m/n \), and so,
\[
\left( \frac{2m}{n} \right)^2 \leq \mu^2(G) \leq \frac{\omega - 1}{\omega} w_2(G) = \frac{\omega - 1}{\omega} 2m.
\]
This shows that
\[
m \leq \frac{\omega - 1}{2\omega} n^2,
\]
which is best possible whenever \( \omega \) divides \( n \). If we combine (1) with other lower bounds on \( \mu(G) \), e.g., with
\[
\mu^2(G) \geq \frac{1}{n} \sum_{u \in V(G)} d^2(u),
\]
we obtain generalizations of (2).

Moreover, inequality (1) follows from a result of Motzkin and Straus [7] following in turn from (2) (see [10]). The implications
\[
(1) \implies (2) \implies \text{MS} \implies (1)
\]
justify regarding inequality (1) as a spectral form of Turán’s theorem, well suited for nontrivial generalizations. For example, the following conjecture seems to be quite subtle.

**Conjecture 1** Let \( G \) be a \( K_{r+1} \)-free graph with \( m \) edges. Then
\[
\mu_1^2(G) + \mu_2^2(G) \leq \frac{r - 1}{r} 2m.
\]
If true, this conjecture is best possible whenever \( r \) divides \( n \). Indeed, for \( r \mid n \), \( n = qr \), the Turán graph \( T_r(n) \) (i.e., the complete \( r \)-partite graph \( K_r(q) \) with \( q \) vertices in each class) has \( r(r - 1)q^2/2 \) edges, and there are three eigenvalues: \((r - 1)q\), with multiplicity 1, \(-q\), with multiplicity \( r - 1 \), and 0, with multiplicity \( r(q - 1) \), so that \( \mu_1(G) = (r - 1)q \) and \( \mu_2(G) = 0 \).
The aim of this note is to prove further relations between $\mu(G)$ and the number of cliques in $G$. In [8] it is proved that

$$\mu^\omega (G) \leq \sum_{s=2}^{\omega} (s - 1) k_s (G) \mu^{\omega-s} (G) \quad (3)$$

with equality holding if and only if $G$ is a complete $\omega$-partite graph with possibly some isolated vertices. It turns out that this inequality is one of a whole sequence of similar inequalities.

**Theorem 1** For every graph $G$ and $r \geq 2$,

$$\mu^{r+1} (G) \leq (r + 1) k_{r+1} (G) + \sum_{s=2}^{r} (s - 1) k_s (G) \mu^{r+1-s} (G).$$

Observe that, with $r = \omega + 1$, Theorem 1 implies (3). Theorem 1 also implies a lower bound on the number of cliques of any given order, as stated below.

**Theorem 2** For every graph $G = G(n)$ and $r \geq 2$,

$$k_{r+1} (G) \geq \left( \frac{\mu (G)}{n} - 1 + \frac{1}{r} \right) r (r - 1) \left( \frac{n}{r} \right)^{r+1}.$$

We also prove the following extension of an earlier result of ours [2].

**Theorem 3** Let $1 \leq s \leq r < \omega (G)$ and $\alpha \geq 0$. If $G = G(n)$ and

$$(s + 1) k_{s+1} (G) \geq n^{s+1} \prod_{t=1}^{s} \left( \frac{r - t}{rt} + \alpha \right), \quad (4)$$

then

$$k_{r+1} (G) \geq \alpha \frac{r^2}{r + 1} \left( \frac{n}{r} \right)^{r+1}. \quad (5)$$

Note that Theorems 3 and 2 hold for all values of the parameters satisfying the conditions there; in particular, $\alpha$ may depend on $n$.

Our final theorem is the following stability result.
Theorem 4 For all \( r \geq 2 \) and \( 0 \leq \alpha \leq 2^{-10}r^{-6} \), if \( G = G(n) \) is a \( K_{r+1} \)-free graph with
\[
\mu(G) \geq \left( 1 - \frac{1}{r} - \alpha \right)n, \tag{6}
\]
then \( G \) contains an induced \( r \)-partite graph \( G_0 \) of order \( v(G_0) > (1 - 3\alpha^{1/3})n \) and minimum degree
\[
\delta(G_0) > \left( 1 - \frac{1}{r} - 6\alpha^{1/3} \right)n.
\]

2 Proofs

2.1 Proof of Theorem \[1\]

For a vertex \( u \in V(G) \), write \( w_l(u) \) for the number of \( l \)-walks starting with \( u \) and \( k_r(u) \) for the number of \( r \)-cliques containing \( u \). Clearly, it is enough to prove the assertion for \( 2 \leq r < \omega(G) \), since the case \( r \geq \omega(G) \) follows easily from \( [3] \).

It is shown in \[8\] that for all \( 2 \leq s \leq \omega(G) \) and \( l \geq 2 \),
\[
\sum_{u \in V(G)} \left( k_s(u) w_{l+1}(u) - k_{s+1}(u) w_l(u) \right) \leq (s - 1) k_s(G) w_l(G), \tag{7}
\]

Summing these inequalities for \( s = 2, \ldots, r \), we obtain
\[
\sum_{u \in V(G)} \left( k_2(u) w_{l+r-1}(u) - k_{r+1}(u) w_l(u) \right) \leq \sum_{s=2}^{r} (s - 1) k_s(G) w_{l+r-s}(G),
\]

and so, after rearranging,
\[
 w_{l+r}(G) - \sum_{s=2}^{r} (s - 1) k_s(G) w_{l+r-s}(G) \leq \sum_{u \in V(G)} k_{r+1}(u) w_l(u). \]

Noting that \( w_l(u) \leq w_{l-1}(G) \), this implies that
\[
\sum_{u \in V(G)} k_{r+1}(u) w_l(u) \leq w_{l-1}(G) \sum_{u \in V(G)} k_{r+1}(u) = (r + 1) k_{r+1}(G) w_{l-1}(G),
\]

and so,
\[
\frac{w_{l+r}(G)}{w_{l-1}(G)} - \sum_{s=2}^{r} (s - 1) k_s(G) \frac{w_{l+r-s}(G)}{w_{l-1}(G)} \leq (r + 1) k_{r+1}(G).
\]
Given $n$, there are non-negative constants $c_1, \ldots, c_n$ such that for $G = G(n)$ we have

$$w_l(G) = c_1 \mu_1^{l-1}(G) + \cdots + c_n \mu_n^{l-1}(G),$$

(See, e.g., [3], p. 44.) Since $\omega > 2$, our graph $G$ is not bipartite and so $|\mu_n(G)| < \mu_1(G)$. Therefore, for every fixed $q$, we have

$$\lim_{l \to \infty} \frac{w_{l+q}(G)}{w_{l-1}(G)} = \mu^{q+1}(G),$$

and the assertion follows. \hfill \Box

### 2.2 Proof of Theorem 3

Moon and Moser [6] stated the following result (for a proof see [4] or [5], Problem 11.8): if $G = G(n)$ and $k_s(G) > 0$, then

$$\frac{(s + 1) k_{s+1}(G)}{sk_s(G)} - \frac{n}{s} \geq \frac{sk_s(G)}{(s - 1) k_{s-1}(G)} - \frac{n}{s - 1}. $$

Equivalently, for $1 \leq s < t < \omega(G)$, we have

$$\frac{(t + 1) k_{t+1}(G)}{tk_t(G)} - \frac{n}{t} \geq \frac{(s + 1) k_{s+1}(G)}{sk_s(G)} - \frac{n}{s}. \quad (8)$$

Let $s \in [1, r]$ be the smallest integer for which (11) holds. This implies either $s = 1$ or

$$sk_s(G) < n^s \prod_{t=1}^{s-1} \left( r - \frac{t}{rt} + \alpha \right) \quad (9)$$

for some $s \in [2, r]$. Suppose first that $s = 1$. (This case is considered in [2], but for the sake of completeness we present it here.) We have

$$\frac{2k_2(G)}{k_1(G)} - n \geq \left( \frac{r - 1}{r} + \alpha \right) n - n = \alpha n - \frac{n}{r};$$

and so, for all $t = 1, \ldots, r$, inequality (8) implies that

$$\frac{(t + 1) k_{t+1}(G)}{tk_t(G)} \geq \alpha n + \frac{n}{t} - \frac{n}{r}. $$

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Multiplying these inequalities for \( t = 1, \ldots, r \), we obtain that
\[
(r + 1) k_{r+1}(G) \geq n^{r+1} \prod_{t=1}^{r} \left( \frac{r-t}{rt} + \alpha \right) \geq \alpha r^2 \left( \frac{n}{r} \right)^{r+1} \prod_{t=1}^{r-1} \frac{r-t}{t} = \alpha r^2 \left( \frac{n}{r} \right)^{r+1},
\]
proving the result in this case.

Assume now that (9) holds for some \( s \in [2, r] \). Then we have
\[
\frac{(s+1) k_{s+1}(G)}{sk_s(G)} > \left( \frac{r-s}{rs} + \alpha \right) n.
\]
and so, for every \( t = s, \ldots, r \),
\[
\frac{(t+1) k_{t+1}(G)}{tk_t(G)} > \frac{n}{s} - \frac{n}{s} + \frac{r-s}{rs} n + \alpha n = \left( \frac{r-t}{rt} + \alpha \right) n.
\]
Multiplying these inequalities for \( t = s + 1, \ldots, r \), we obtain
\[
\frac{(r+1) k_{r+1}(G)}{(s+1) k_{s+1}(G)} > n^{r-s} \prod_{t=s+1}^{r} \left( \frac{r-t}{rt} + \alpha \right).
\]
Appealing to (10), this implies that
\[
(r + 1) k_{r+1}(G) > n^{r+1} \prod_{t=1}^{r} \left( \frac{r-t}{rt} + \alpha \right) = \alpha n^{r+1} \prod_{t=1}^{r-1} \left( \frac{r-t}{rt} + \alpha \right) \geq \alpha r^2 \left( \frac{n}{r} \right)^{r+1},
\]
as required. \( \square \)

### 2.3 Proof of Theorem [2]

Set
\[
\alpha = \frac{\mu}{n} - 1 + \frac{1}{r-1}.
\]
Clearly we may assume that \( \alpha > 0 \), since otherwise the assertion is trivial. Suppose that
\[
sk_s(G) > n^s \prod_{t=1}^{s-1} \left( \frac{r-t}{rt} + \alpha \right)
\]
for some \( s \in [2, r] \). Then, by Theorem [3]
\[
(r + 1) k_{r+1}(G) > \alpha \frac{r^2}{r+1} \left( \frac{n}{r} \right)^{r+1} \geq \alpha \frac{r (r-1)}{r+1} \left( \frac{n}{r} \right)^{r+1},
\]
completing the proof. Thus we may and shall assume that (10) fails for every \( s \in [r - 1] \).

From Theorem 1 we have

\[
(r + 1) k_{r+1} (G) \geq \mu^{r+1} (G) - \sum_{s=2}^{r} (s - 1) k_s (G) \mu^{r+1-s} (G).
\]  

(11)

Substituting the bounds on \( k_s (G) \) into (11), and setting \( \mu = \mu (G) / n \), we obtain

\[
\frac{(r + 1) k_{r+1} (G)}{n^{r+1}} \geq \mu^{r+1} - \sum_{s=2}^{r} \frac{r - 1 - s}{s} \prod_{t=1}^{s-1} \left( \frac{r - t}{rt} + \alpha \right)
\]

\[
\geq \mu^{r+1} - \mu^{r+1-2} \frac{1}{2} \left( \frac{r - 1}{r} + \alpha \right) + \sum_{s=3}^{r} \frac{s - 1}{s} \mu^{r+1-s} \prod_{t=1}^{s-1} \left( \frac{r - t}{rt} + \alpha \right)
\]

\[
\geq \mu^{r+1-2} \left( \mu^2 - \frac{1}{2} \left( \frac{r - 1}{r} + \alpha \right) \right) + \sum_{s=3}^{r} \frac{s - 1}{s} \mu^{r+1-s} \prod_{t=1}^{s-1} \left( \frac{r - t}{rt} + \alpha \right)
\]

\[
\geq \mu^{r+1-2} \left( \frac{r - 1}{r} + \alpha \right) \left( \frac{r - 2}{2r} + \alpha \right) + \sum_{s=3}^{r} \frac{s - 1}{s} \mu^{r+1-s} \prod_{t=1}^{s-1} \left( \frac{r - t}{rt} + \alpha \right).
\]

By induction on \( k \) we prove that, for all \( k = 2, \ldots, r \),

\[
\frac{(r + 1) k_{r+1} (G)}{n^{r+1}} \geq \mu^{r+1-k} \prod_{t=1}^{k} \left( \frac{r - t}{rt} + \alpha \right) - \sum_{s=k+1}^{r} \frac{s - 1}{s} \mu^{r+1-s} \prod_{t=1}^{s-1} \left( \frac{r - t}{rt} + \alpha \right)
\]

and hence,

\[
\frac{(r + 1) k_{r+1} (G)}{n^{r+1}} \geq \mu \prod_{t=1}^{r} \left( \frac{r - t}{rt} + \alpha \right) \geq \alpha \frac{r - 1}{r} \prod_{t=1}^{r-1} \frac{r - t}{rt} = \alpha \frac{r - 1}{r^r}.
\]

It follows that

\[
k_{r+1} (G) \geq \alpha \frac{r (r - 1)}{r + 1} \left( \frac{n}{r} \right)^{r+1},
\]

as required. \( \square \)

### 2.4 Proof of Theorem 4

Inequality (11) for \( s = 2 \) together with (6) implies that

\[
2 \frac{r - 1}{r} e (G) \geq \mu^2 (G) \geq \left( \frac{r - 1}{r} - \alpha \right)^2 n^2 \geq \left( \left( \frac{r - 1}{r} \right)^2 - 2 \alpha \frac{r - 1}{r} \right) n^2.
\]
and so,
\[ e(G) \geq \left( \frac{r - 1}{2r} - 2\alpha \right) n^2. \]

To complete our proof, let us recall the following stability theorem proved by Nikiforov and Rousseau in [11]. Let \( r \geq 2 \) and \( 0 < \beta \leq 2^{-9}r^{-6} \), and let \( G = G(n) \) be a \( K_{r+1} \)-free graph satisfying
\[ e(G) \geq \left( \frac{r - 1}{2r} - \beta \right) n^2. \]

Then \( G \) contains an induced \( r \)-partite graph \( G_0 \) of order \( v(G_0) > (1 - 2\alpha^{1/3}) n \) and with minimum degree
\[ \delta(G_0) \geq \left( 1 - \frac{1}{r} - 4\beta^{1/3} \right) n. \]

Setting \( \beta = 2\alpha \), in view of \( 4 \cdot 2^{1/3} < 6 \), the required inequalities follow. \( \square \)

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