Article

Certain Applications of Generalized Kummer’s Summation Formulas for \(2F_1\)

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Abstract: We present generalizations of three classical summation formulas \(2F_1\) due to Kummer, which are able to be derived from six known summation formulas of those types. As certain simple particular cases of the summation formulas provided here, we give a number of interesting formulas for double-finite series involving quotients of Gamma functions. We also consider several other applications of these formulas. Certain symmetries occur often in mathematical formulae and identities, both explicitly and implicitly. As an example, as mentioned in Remark 1, evident symmetries are naturally implicated in the treatment of generalized hypergeometric series.

Keywords: Gamma function; Beta function; Psi function; generalized hypergeometric function

MSC: 33B20; 33C20; 33B15; 33C05; 44A10

1. Introduction and Preliminaries

The generalized hypergeometric series \(pF_q\) \((p, q \in \mathbb{N}_0)\) is defined by:

\[
pF_q\left[\begin{array}{c}
\alpha_1, \ldots, \alpha_p \\
\beta_1, \ldots, \beta_q
\end{array}; z\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} = pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z)
\]

where:

\[
\omega = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j \quad (\beta_j \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}, j = 1, \ldots, q),
\]

and \((\lambda)_\nu\) indicates the Pochhammer symbol, which is defined (for \(\lambda, \nu \in \mathbb{C}\)), in terms of the well-known Gamma function \(\Gamma\), by:

\[
(\lambda)_\nu := \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} = \begin{cases} 
1 & (\nu = 0; \lambda \in \mathbb{C} \setminus \{0\}) \\
\lambda(\lambda + 1) \cdots (\lambda + n - 1) & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}),
\end{cases}
\]

it being accepted formally that \((0)_0 := 1\) (see, e.g., [1,2]). Here and elsewhere, let \(\mathbb{C}, \mathbb{R}, \mathbb{Z}\), and \(\mathbb{N}\) be the sets of complex numbers, real numbers, integers, and positive integers, respectively, and let:

\[
\mathbb{N}_0 := \mathbb{N} \cup \{0\}, \quad \mathbb{Z}_{\leq \ell} := \{j \in \mathbb{Z} : j \leq \ell\}, \quad \text{and} \quad \mathbb{Z}_{\geq \ell} := \{j \in \mathbb{Z} : j \geq \ell\},
\]
for some integer $\ell$.

Since Gauss discovered the following acclaimed summation theorem (see, e.g., [2] (p. 64, Equation (7))):

$$
_{2}F_{1}(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}
$$

(3)

$$
(\Re(c-a-b) > 0; \ c \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}),
$$

triggered by the great reputation and far-reaching practicality of the hypergeometric function $_{2}F_{1}$ and the generalized hypergeometric functions $_{p}F_{q}$, a remarkably large number of summation formulas for $_{2}F_{1}$ and $_{p}F_{q}$ with particular arguments and specified parameters have been offered.

In this paper, we consider only three summation formulas for $_{2}F_{1}$ due to Kummer [3] (p. 134, Entries 1, 2, and 3) (see, e.g., [4] (Equations (1.3), (1.4), and (1.5)); see also [5–8]) (for the first contributor of the following three summation formulas, we refer to [4] (p. 853)):

$$
_{2}F_{1}\left[\frac{a}{2}(a+b+1); \frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{a+b+1}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)};
$$

(4)

$$
_{2}F_{1}\left[\frac{a}{1+a-b}; \frac{1-a}{1-a}\right] = \frac{\Gamma\left(1+\frac{a}{2}\right)\Gamma\left(1+a-b\right)}{\Gamma(1+a)\Gamma\left(1+\frac{a}{2}-b\right)};
$$

(5)

$$
_{2}F_{1}\left[\frac{a-1-a}{b}; \frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}b\right)\Gamma\left(\frac{1}{2}b+1\right)}{\Gamma\left(\frac{1}{2}b+\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b-\frac{1}{4}a+\frac{1}{2}\right)}.
$$

(6)

The following generalizations of (4)–(6) are recalled (see, e.g., [9,10]):

$$
_{2}F_{1}\left[\frac{a}{1+a-b+m}; \frac{1-a-b-m}{1-a-b}\right] = \frac{2^{m-2b}\Gamma\left(b-m\right)\Gamma\left(1+a-b+m\right)}{\Gamma\left(b\right)\Gamma\left(1+a-2b+m\right)}
$$

$$
\times \sum_{r=0}^{m} (-1)^{r} \binom{m}{r} \frac{\Gamma\left(\frac{a+r+m+1}{2}-b\right)}{\Gamma\left(\frac{a+r-m+1}{2}\right)} (m \in \mathbb{N}_{0})
$$

(7)

and:

$$
_{2}F_{1}\left[\frac{a}{1+a-b-m}; \frac{1-a-b-m}{1-a-b}\right] = \frac{2^{-m-2b}\Gamma\left(1+a-b-m\right)}{\Gamma\left(a-2b-m+1\right)}
$$

$$
\times \sum_{r=0}^{m} \binom{m}{r} \frac{\Gamma\left(\frac{a+r+m+1}{2}-b\right)}{\Gamma\left(\frac{a+r-m+1}{2}\right)} (m \in \mathbb{N}_{0});
$$

(8)

$$
_{2}F_{1}\left[\frac{a}{2}(a+b+m+1); \frac{1}{2}\right] = \frac{2^{b-1}\Gamma\left(\frac{a+b+m+1}{2}\right)\Gamma\left(\frac{a-b-m+1}{2}\right)}{\Gamma\left(b\right)\Gamma\left(\frac{a-b+m+1}{2}\right)}
$$

$$
\times \sum_{r=0}^{m} (-1)^{r} \binom{m}{r} \frac{\Gamma\left(\frac{b+r}{2}\right)}{\Gamma\left(\frac{a+r-m+1}{2}\right)} (m \in \mathbb{N}_{0})
$$

(9)
which was established by Fox [15], who used the Mellin–Barnes-type contour integral. Indeed, the substitution $a$ into $a - m$ in (10) leads to the identity (10), while the replacement $a$ by $a + m$ in (10) yields the identity (13).

The identity (10) was recorded in [11,12] (Entry 7.3.7-2, p. 491) (see also [13,14]). Qureshi and Baboo [12] pointed out that Formula (10) is equivalent to the following summation formula:

$$2F_1 \left[ \frac{1}{2} (a + b + 1); \frac{1}{2} \right] = \frac{2^{b-1} \Gamma \left( \frac{a+b+1}{2} \right)}{\Gamma (b)} \sum_{r=0}^{m} \frac{(m)_r \Gamma \left( \frac{b+r}{2} \right)}{\Gamma \left( \frac{a-r+1}{2} \right)} (m \in \mathbb{N}_0),$$

which was established by Fox [15], who used the Mellin–Barnes-type contour integral. Indeed, the substitution $a$ into $a - m$ in (13) leads to the identity (10), while the replacement $a$ by $a + m$ in (10) yields the identity (13).

The identity (9) was recalled in [12] (Equation (13)). Exchanging the role of $a$ and $b$ in (9) and replacing $a$ by $a - m$ in the resulting identity, we obtain a companion of the Fox summation Formula (13) (cf. [12] (Equation (14))):

$$2F_1 \left[ \frac{1}{2} (a + b + 1); \frac{1}{2} \right] = \frac{2^{b-1} \Gamma \left( \frac{a+b+1}{2} \right)}{\Gamma (b)} \sum_{r=0}^{m} \frac{(m)_r \Gamma \left( \frac{b+r}{2} \right)}{\Gamma \left( \frac{a-r+1}{2} \right)} (m \in \mathbb{N}_0).$$

Qureshi and Baboo [12] (Equation (16)) established the following summation formula:

$$2F_1 \left[ \frac{1}{2} (a + b - m); \frac{1}{2} \right] = \frac{2^{b-1} \Gamma \left( \frac{a+b-m}{2} \right)}{\Gamma (a)} \sum_{r=0}^{m} \frac{(m)_r \Gamma \left( \frac{b+r}{2} \right)}{\Gamma \left( \frac{a-r+1}{2} \right)} \left\{ \frac{\Gamma \left( \frac{a+r+1}{2} \right)}{\Gamma \left( \frac{b+r-m}{2} \right) + \Gamma \left( \frac{b+r+1}{2} \right)} (m \in \mathbb{N}_0).$$
We present a companion form of (15):
\[
\begin{align*}
2F_1 & \left[ \frac{a, b; 1}{2} \right](a + b + m; \frac{1}{2}) = 2^{a-1} \Gamma \left( \frac{a+b+m}{2} \right) \Gamma(a) \left( \frac{b-a-m}{2} \right)^m \\
& \times \sum_{r=0}^{m} (-1)^r \binom{m}{r} \left( \frac{\Gamma \left( \frac{a+r}{2} \right)}{\Gamma \left( \frac{b+r-m}{2} \right)} + \frac{\Gamma \left( \frac{a+r+1}{2} \right)}{\Gamma \left( \frac{b+r-m+1}{2} \right)} \right) (m \in \mathbb{N}_0).
\end{align*}
\]  
(16)

Equation (16) is equivalent to the identity [12] (Equation (16)) due to Qureshi and Baboo, who corrected the formula [16] (p. 582, Entry 8.1.1-130).

By using the Pfaff–Kummer transformation for \(2F_1\) (see, e.g., [2] (p. 67, Equation (19))):
\[
2F_1[a, b; c; z] = (1-z)^{-a} 2F_1[a, c-b; c; \frac{z}{1-z}]
\]  
(17)

\[\Leftrightarrow
2F_1 \left[ \frac{a, b; \frac{z}{z-1}}{2} \right] = (1-z)^a 2F_1[a, c-b; c; z]
\]  
(18)

we obtain:
\[
2F_1 \left[ \frac{a, b; 1}{2} \right] = 2^a 2F_1[a, c-b; c; -1]
\]  
(19)

\[\Leftrightarrow
2F_1[a, b; c; -1] = 2^{-a} 2F_1[a, c-b; c; \frac{1}{2}] .
\]  
(20)

The relation (19) or (20) reveals that the summation formulas for \(2F_1(1/2)\) yield those for \(2F_1(-1)\) and vice versa.

If we set \(m = 0, 1, 2, 3, 4, 5\) in (7) and (8), (9) and (10), and (11) and (12), we can obtain the summation formulas for \(2F_1\) in Lavoie et al. [17]. Further, the formulas in (7) and (8) when \(m = 0, 1, \ldots, 9\) were given in [18].

The particular case \(m = 0\) in (7) or (8), (9) or (10), or (11) or (12) recovers the classical Kummer’s summation theorems (4)–(6).

In this paper, we aim to establish summation formulas for:
\[
\begin{align*}
2F_1 & \left[ \frac{a + p, b + q; 1}{2} \right] \left( a + b + r \right; \frac{1}{2}) ; \\
2F_1 & \left[ \frac{a + p, b + q; 1}{a - b + r; -1} ; \\
2F_1 & \left[ \frac{a + p, b + q - a; 1}{b + r; \frac{1}{2}}  
\end{align*}
\]

where \(p, q, r \in \mathbb{N}_0\). We also consider several applications for those summation formulas presented here.

For this, the Beta function and Euler’s integral representation for \(2F_1\) are recalled (see, e.g., respectively, [2] (Section 1.1 and p. 65, Equation (10)):
\[
B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \ dt \quad (\min\{\Re(\alpha), \Re(\beta)\} > 0)
\]  
(21)

\[
\Gamma(\alpha) \Gamma(\beta) \quad (a, \beta \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0})
\]
and:
\[
2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b} \, dt
\] (22)

**Remark 1.** Certain symmetries ubiquitously appear explicitly and implicitly in the mathematical formulas and identities. Here, for example, in the definition of the generalized hypergeometric series \( pF_q (p, q \in \mathbb{N}_0) \) in (1), any permutation of the numerator parameters \( \alpha_1, \ldots, \alpha_p \) and the denominator parameters \( \beta_1, \ldots, \beta_q \) is easily found to give the same identity as:

\[
pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) = pF_q(\alpha_p, \ldots, \alpha_1; \beta_q, \ldots, \beta_1; z).
\]

For another example, in (21), obviously, \( B(\alpha, \beta) = B(\beta, \alpha) \).

2. Summation Formulas

We establish generalizations of the summation Formula (4) in the following theorem.

**Theorem 1.** Let \( p, q, r \in \mathbb{N}_0 \). Then, the following summation formulas hold.

\[
2F_1 \left[ \frac{a + p, b + q; 1}{2(a + b - r), \frac{1}{2}} \right] = \frac{2^{a+p-1} \Gamma \left( \frac{a+b-r}{2} \right)}{\Gamma(a+p)} \times \sum_{k=0}^{p+q+r+1} \binom{r}{k} \sum_{j=0}^{p+q+1} \binom{p+q+1}{j} \frac{\Gamma \left( \frac{a+p+k+j}{2} \right)}{\Gamma \left( \frac{b-p-r+k+j}{2} \right)}
\] (23)

\[
2F_1 \left[ \frac{a + p, b + q; 1}{2(a + b + r), \frac{1}{2}} \right] = \frac{2^{a+p-1} \Gamma \left( \frac{a+b+r}{2} \right) \Gamma \left( \frac{b-a-r-2p}{2} \right)}{\Gamma(a+p) \Gamma \left( \frac{b-a+r+2p}{2} \right)} \times \sum_{k=0}^{r} (-1)^k \binom{r}{k} \sum_{j=0}^{p+q+1} \binom{p+q+1}{j} \frac{\Gamma \left( \frac{a+p+k+j}{2} \right)}{\Gamma \left( \frac{b-p-r+k+j}{2} \right)}
\] (24)

\[
2F_1 \left[ \frac{a - p, b + q; 1}{2(a + b - r), \frac{1}{2}} \right] = \frac{2^{a-p-1} \Gamma \left( \frac{b-a-r}{2} \right)}{\Gamma(a-p) \Gamma \left( \frac{b-a+r+2p}{2} \right)} \times \sum_{k=0}^{p} (-1)^k \binom{p}{k} \sum_{j=0}^{q+r+1} \binom{q+r+1}{j} \frac{\Gamma \left( \frac{a-p+k+j}{2} \right)}{\Gamma \left( \frac{b-p-r+k+j}{2} \right)}
\] (25)

\[
2F_1 \left[ \frac{a + p, b - q; 1}{2(a + b - r), \frac{1}{2}} \right] = \frac{2^{a+p-1} \Gamma \left( \frac{b-a-r-2q}{2} \right)}{\Gamma(a+p) \Gamma \left( \frac{b-a+r-2p}{2} \right)} \times \sum_{k=0}^{q} (-1)^k \binom{q}{k} \sum_{j=0}^{p+r+1} \binom{p+r+1}{j} \frac{\Gamma \left( \frac{a+p+k+j}{2} \right)}{\Gamma \left( \frac{b-p-2q-r+k+j}{2} \right)}
\] (26)
\[2F_1 \left[ \frac{a-p, b+q; 1}{2(a+b+r)} \frac{1}{2} \right] = \frac{2^{a-p-1} \Gamma \left( \frac{b-a+r}{2} \right) \Gamma \left( \frac{b-a-r}{2} \right)}{\Gamma(a-p) \Gamma \left( \frac{b-a+r+2p}{2} \right)} \]  
\[\times \sum_{k=0}^{\nu+r} (-1)^k \binom{p+r}{k} \binom{q+1}{j} \frac{\Gamma \left( \frac{a-p+k}{2} \right)}{\Gamma \left( \frac{b-p-r-2q+k}{2} \right)} \]  
\[\left( \Re(a-p) > 0, \Re(b-a-r) > 0; \right) \]  
\[\left( \Re(a+p) > 0, \Re(b-a-r-2p-2q) > 0; \right) \]  
\[\begin{aligned} 2F_1 \left[ \frac{a-p, b+q; 1}{2(a+b+r)} \frac{1}{2} \right] &= \frac{2^{a-p-1} \Gamma \left( \frac{b+a+r}{2} \right) \Gamma \left( \frac{b-a-r-2q}{2} \right)}{\Gamma(a-p) \Gamma \left( \frac{b-a+r+2p}{2} \right)} \\
&\times \sum_{k=0}^{\nu+q+r} (-1)^k \binom{p+q+r}{k} \frac{\Gamma \left( \frac{a-p+k}{2} \right)}{\Gamma \left( \frac{b-p-r-2q+k}{2} \right)} + \frac{\Gamma \left( \frac{a-p+k+1}{2} \right)}{\Gamma \left( \frac{b-p-r-2q+k+1}{2} \right)} \right) \]  
\[\left( \Re(a-p) > 0, \Re(b-a-r-2q) > 0; \right) \]  
\[\begin{aligned} 2F_1 \left[ \frac{a-p, b-q; 1}{2(a+b-r)} \frac{1}{2} \right] &= \frac{2^{a-p-1} \Gamma \left( \frac{b+a-r}{2} \right) \Gamma \left( \frac{b-a-r-2q}{2} \right)}{\Gamma(a-p) \Gamma \left( \frac{b-a+r+2p}{2} \right)} \\
&\times \sum_{k=0}^{\nu+q} (-1)^k \binom{p+q}{k} \sum_{j=0}^{r} \binom{r+1}{j} \frac{\Gamma \left( \frac{a-p+k+j}{2} \right)}{\Gamma \left( \frac{b-p-r-2q-j+k}{2} \right)} \]  
\[\left( \Re(a-p) > 0, \Re(b-p-2q-r) > 0; \right) \]  

Here, by the principle of analytic continuation, the condition given in each formula can be greatly extended, as noted in Remark 2.

**Proof.** Let \( \mathcal{L}_2 \) be the left member of (24). We first use (19) and then (22) to find:

\[\mathcal{L}_2 = \frac{2^{a+p} \Gamma \left( \frac{a+b+r}{2} \right)}{\Gamma(a+p) \Gamma \left( \frac{b-a+r-2p}{2} \right)} \mathcal{J}_2, \]  

where:

\[\mathcal{J}_2 = \int_0^1 t^{a+p-1} (1-t)^{\frac{b-a-r-2q}{2}-1} (1+t)^{\frac{b-a-r}{2}+q} \, dt. \]  

We have:

\[\mathcal{J}_2 = \int_0^1 t^{a+p-1} (1-t^2)^{\frac{b-a-r-2p}{2}-1} (1+t)^{p+q+1} \, dt \]
\[= \sum_{k=0}^{r} (-1)^k \binom{r}{k} \sum_{j=0}^{p+q+1} \binom{p+q+1}{j} \int_0^1 t^{a+p+k+j-1} (1-t^2)^{\frac{b-a-r-2p}{2}-1} \, dt \]
\[= \frac{1}{2} \sum_{k=0}^{r} (-1)^k \binom{r}{k} \sum_{j=0}^{p+q+1} \binom{p+q+1}{j} \int_0^1 u^{\frac{a+p+k+j}{2}-1} (1-u)^{\frac{b-a-r-2q}{2}-1} \, du. \]
Now, we are ready to apply (21) to evaluate $F_2$. Now, this final result for $F_2$ is used in (31) to prove Formula (24).

Similarly, the other formulas can be proven. The details are omitted. $\square$

Similar to the proof of Theorem 1, in particular, using (22), we present generalizations of the summation Formula (5) in the following theorem.

**Theorem 2.** Let $p, q, r \in \mathbb{N}_0$. Then, the following summation formulas hold.

\[
\begin{align*}
\sum_{k=0}^{p+q+r} (-1)^k \binom{p+q+r}{k} \sum_{j=0}^{p+1} \binom{p+1}{j} \frac{\Gamma\left(\frac{a+p+k+j}{2}\right)}{\Gamma\left(\frac{a-2b-p-2q+2k+j}{2}\right)} \\
\sum_{k=0}^{q+r} (-1)^k \binom{q+r}{k} \sum_{j=0}^{p+1} \binom{p+1}{j} \frac{\Gamma\left(\frac{a+p+k+j}{2}\right)}{\Gamma\left(\frac{a-2b-p-2q+2k+j}{2}\right)} \\
\sum_{k=0}^{p+q} (-1)^k \binom{p+q}{k} \sum_{j=0}^{r+1} \binom{r+1}{j} \frac{\Gamma\left(\frac{a+p+k+j}{2}\right)}{\Gamma\left(\frac{a-2b-p-2q+2k+j}{2}\right)} \\
\sum_{k=0}^{p+q+r} (-1)^k \binom{p+q+r}{k} \sum_{j=0}^{p+q+r+1} \binom{p+q+r+1}{j} \frac{\Gamma\left(\frac{a+p+k+j}{2}\right)}{\Gamma\left(\frac{a-2b-p-2q+2k+j}{2}\right)} \\
\sum_{k=0}^{p+q} (-1)^k \binom{p+q}{k} \sum_{j=0}^{p+q+1} \binom{p+q+1}{j} \frac{\Gamma\left(\frac{a+p+k+j}{2}\right)}{\Gamma\left(\frac{a-2b-p-2q+2k+j}{2}\right)} \\
\sum_{k=0}^{p+q+r} (-1)^k \binom{p+q+r}{k} \sum_{j=0}^{p+q+r+1} \binom{p+q+r+1}{j} \frac{\Gamma\left(\frac{a+p+k+j}{2}\right)}{\Gamma\left(\frac{a-2b-p-2q+2k+j}{2}\right)}
\end{align*}
\]
Let $p$ greatly extended, as noted in Remark 2. We derive generalizations of the summation Formula (6) in the following theorem. Here, by the principle of analytic continuation, the condition given in each formula can be greatly extended, as noted in Remark 2.

We derive generalizations of the summation Formula (6) in the following theorem.

**Theorem 3.** Let $p, q, r \in \mathbb{N}_0$. Then, the following summation formulas hold.

$$
2F_1\left[\begin{array}{c}
-a-p, b-q; \\
a-b+r;
\end{array}; -1\right] = \frac{\Gamma(a-b+r)\Gamma(-b)}{2\Gamma(a-p)\Gamma(-b+p+r)}
\times \sum_{k=0}^{p+r} (-1)^k \left(\begin{array}{c}
p+r \\
k
\end{array}\right) \sum_{j=0}^{q+1} \left(\begin{array}{c}
q+1 \\
j
\end{array}\right) \frac{\Gamma\left(\frac{a-p+k+j}{2}\right)}{\Gamma\left(\frac{a-2b-p+k+j}{2}\right)}
$$

(38)

$$
2F_1\left[\begin{array}{c}
a-p, b-q; \\
a-b-r;
\end{array}; -1\right] = \frac{\Gamma(a-b-r)\Gamma(-b-r)}{2\Gamma(a-p)\Gamma(-b+p-r)}
\times \sum_{k=0}^{p} (-1)^k \left(\begin{array}{c}
p \\
k
\end{array}\right) \sum_{j=0}^{q+r+1} \left(\begin{array}{c}
q+r+1 \\
j
\end{array}\right) \frac{\Gamma\left(\frac{a-p+k+j}{2}\right)}{\Gamma\left(\frac{a-2b-p-2k+j}{2}\right)}
$$

(39)

Here, by the principle of analytic continuation, the condition given in each formula can be greatly extended, as noted in Remark 2.
can be derived directly from Equations (7)–(12). Take an example in Theorem 1: cases of certain identities in Theorems 1–3. In this regard, those identities in Theorems 1–3 seem to and Equation (47) can hold for \( b \neq a \).

The convergence conditions given in each identity in Theorems 1–3 are derived from the Remark 2.

Here, by the principle of analytic continuation, the condition given in each formula can be greatly extended, as noted in Remark 2.

**Proof.** From (17) and (22), we find:

\[
2F_1 \left[ a - p, -q - a; \frac{1}{2} \right] = \frac{2^{a-p-1} \Gamma(b-r)}{\Gamma(a+b+q-r)} \times \sum_{j=0}^{p+q+1} \binom{p+q+1}{j} \frac{\Gamma(1+\frac{a+b+q+r+j}{2})}{\Gamma\left(\frac{b-a-q+r+j}{2}\right)} \quad (44)
\]

\[
\text{for } \Re(a+p+q) < 0, \Re(a+b+q-r) > 0;
\]

\[
2F_1 \left[ a - p, q - a; \frac{1}{2} \right] = \frac{2^{a-p-1} \Gamma(b+r)}{\Gamma(-a+q)} \frac{\Gamma(a+b+q-r)}{\Gamma(a+b+q-r)} \times \sum_{k=0}^{q} (-1)^k \binom{q}{k} \sum_{j=0}^{p+1} \binom{p+1}{j} \frac{\Gamma(a+b+q+r+k+j)}{\Gamma\left(\frac{b-a-q+r+k+j}{2}\right)} \quad (45)
\]

\[
\text{for } \Re(a) < 0, \Re(a+b+q-r) > 0;
\]

\[
2F_1 \left[ a + p, -q - a; \frac{1}{2} \right] = \frac{2^{a+p-1} \Gamma(b-r)}{\Gamma(-a-q)} \frac{\Gamma(a+b+q-r)}{\Gamma(a+b+q-r)} \times \sum_{k=0}^{p} (-1)^k \binom{p}{k} \sum_{j=0}^{q+1} \binom{q+1}{j} \frac{\Gamma(a+b+q+r+k+j)}{\Gamma\left(\frac{b-a-q+r+k+j}{2}\right)} \quad (46)
\]

\[
\text{for } \Re(a+p+q) < 0, \Re(a+b+q+r) > 0;
\]

\[
2F_1 \left[ a - p, -q - a; \frac{1}{2} \right] = \frac{2^{a-p-1} \Gamma(b+r)}{\Gamma(a+b+q-r)} \times \sum_{j=0}^{p+q+1} \binom{p+q+1}{j} \frac{\Gamma(a+b+q+r+j)}{\Gamma\left(\frac{b-a-q+r+j}{2}\right)} \quad (47)
\]

\[
\text{for } \Re(a+q) < 0, \Re(a+b+q+r) > 0.
\]

With the aid of (48), similar to the proof of Theorem 1, we can establish each identity here. We omit the details. □

**Remark 2.** The convergence conditions given in each identity in Theorems 1–3 are derived from the process of their proofs. Yet, by the principle of analytic continuation, the convergence region of each formula can be greatly extended. For example, Equation (23) can be true for \( a+b+2 \in \mathbb{C} \setminus \mathbb{Z} \leq 0 \), and Equation (47) can hold for \( b+r, \frac{a+b+q+r}{2} \in \mathbb{C} \setminus \mathbb{Z} \leq 0 \).

**Remark 3.** It is easy to see that all of the summation formulas, except (3), in Section 1 are particular cases of certain identities in Theorems 1–3. In this regard, those identities in Theorems 1–3 seem to be generalizations of those formulas in Equations (7)–(12). Indeed, the formulas in Theorems 1–3 can be derived directly from Equations (7)–(12). Take an example in Theorem 1:

\[
2F_1 \left[ a + p, b + q; \frac{1}{2} \right] = 2F_1 \left[ \frac{a'}{2} \right] \text{ for } \frac{a'}{2} + b' - p - q \pm \frac{1}{2}.
\]
where \( a' = a + p \) and \( b' = b + q \). Then, applying (9) or (10), depending on whether \(-p - q \pm r\) is positive or negative, to the right member of (49) yields the formulas in Theorem 1. Similarly, (7) or (8) gives the formulas in Theorem 2, and (11) or (12) produces the formulas in Theorem 3.

3. Applications

We consider several applications in the following subsections.

3.1. Summation Formulas for the Kampé de Fériet Function

The enormous popularity and broad usefulness of the hypergeometric function \(_2F_1\) and the generalized hypergeometric functions \(_pF_q\) (\( p, q \in \mathbb{N}_0 \)) of one variable have inspired and stimulated a large number of researchers to introduce and investigate hypergeometric functions of two or more variables (see, e.g., [1,5,11,16,19,20]). A serious, significant, and systematic study of the hypergeometric functions of two variables was initiated by Appell [21], who offered the so-called Appell functions \( F_1, F_2, F_3, \) and \( F_4 \) which are generalizations of the Gauss hypergeometric function (see, e.g., [20,22–24] (pp. 22–23)). The confluent forms of the Appell functions were studied by Humbert [25]. A complete list of these functions can be seen in the standard literature (see, e.g., [26]). Later, the four Appell functions and their confluent forms were further generalized by Kampé de Fériet [27], who introduced more general hypergeometric functions of two variables. The notation defined and introduced by Kampé de Fériet for his double-hypergeometric functions of superior order was subsequently abbreviated by Burchnall and Chaundy [28,29]. We recall here and introduced the so-called Appell functions due to their double-hypergeometric functions of superior order was subsequently abbreviated by Burchnall and Chaundy [28,29]. We recall here the definition of a more general double-hypergeometric function (than the one defined by Kampé de Fériet) in a slightly modified notation given by Srivastava and Panda [30] (p. 423, Equation (26)). The convenient generalization of the Kampé de Fériet function (KdF function) is defined as follows:

\[
P^{H;A,B}_{G;C,D}
\begin{bmatrix}
    (h_H) : (a_A); (b_B); (c_C); (d_D); x, y \\
    (g_G) : \\
\end{bmatrix}
= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(h_H)_{m+n} (a_A)_m (b_B)_n}{(g_G)_{m+n} (c_C)_m (d_D)_n} \frac{x^m y^n}{m! n!},
\]

where \((h_H)\) denotes the sequence of parameters \((h_1, h_2, \ldots, h_H)\) and \((h_H)_n\) is defined by the following product of Pochhammer symbols:

\[
(h_H)_n := (h_1)_n (h_2)_n \cdots (h_H)_n \quad (n \in \mathbb{N}_0),
\]

where the product when \(n = 0\) is understood as unity. For more details about the function (50), including its convergence and further generalization, we refer, for example, to [19,20,31].

When some considerably generalized special functions such as (50) are introduced, it has been an intriguing and usual research subject to study certain reducibilities of the functions. In this vein, the KdF function has attracted many mathematicians to investigate its reducibility and transformation formulas. In fact, there are numerous reduction formulas and transformation formulas of the KdF function in the literature (see, e.g., [24,31–35]). In the above-cited references, most of the reduction formulas were linked to the cases \(H + A = 3\) and \(G + C = 2\). In 2010, by using Euler’s transformation formula for \(_2F_1\), Cvijović and Miller [34] established a reduction formula for the case \(H + A = 2\) and \(G + C = 1\) (see also [32,36,37]). Inspired by the work [34], in the recent past, Liu and Wang [38] employed Euler’s first and second transformation formulas for \(_2F_1\) and some classical summation theorems for \(_pF_q\) to afford a number of very amusing reduction formulas and then derived summation formulas for the KdF function. Here, we present one special case of Equation (2.11) of [38]:

\[
_1F_1_{1;2;3}^{1;1;1} \begin{bmatrix}
a + 2b + q + r; \quad a + p, b + r; \\
    a - p, b + r; \\
\end{bmatrix}
\begin{array}{c}
a : \\
a + b + r; \\
\end{array}
= (1 - x)^{-2b - q + r} _2F_1[a - p, -b - q; a + b + r; x],
\]

\[
(51)
\]
which, upon setting $x = -1$ and using (38), gives the following summand formula:

$$\begin{align*}
F_{1:1}^{1:2;1} \left[ \begin{array}{c}
a + 2b + q + r : p; \\
\quad a - p, b + r; \\
\quad a + b + r; \\
\quad -1, -1
\end{array} \right] \\
= \frac{2^{-2b-p-q+r-1} \Gamma(b) \Gamma(a + b + r)}{\Gamma(a - p) \Gamma(b + p + r)} \\
\times \sum_{k=0}^{p+r} (-1)^k \binom{p + r}{k} \sum_{j=0}^{q+1} \left( \binom{q + 1}{j} \frac{\Gamma(a + 2b - p + k + j)}{2} \right)
\end{align*}$$

(52)

We can give a relationship between the derivative of $pF_q$ and the KdF function. For simplicity, we deal with $2F_1$ to offer two identities in the subsequent lemma.

**Lemma 1.** The following formulas hold true.

$$\frac{\partial}{\partial a} 2F_1[a, b; c; z] = \frac{bz}{c} F_{2:2;1}^{2:2;1} \left[ \begin{array}{c}a + 1, b + 1 : a, 1; \\
\quad c + 1, 2 : a + 1; \\
\quad - ; \ z, z
\end{array} \right]$$

(53)

and:

$$\frac{\partial}{\partial c} 2F_1[a, b; c; z] = -\frac{abz}{c^2} F_{2:1;1}^{2:2;1} \left[ \begin{array}{c}a + 1, b + 1 : c, 1; \\
\quad c + 1, 2 : c + 1; \\
\quad - ; \ z, z
\end{array} \right]$$

(54)

$(a \in C \setminus z \leq -1, c \in C \setminus z \leq 0; |z| < 1)$.

**Proof.** We prove only (53). Let $\psi(a)$ be the left member of (53). Differentiating term-by-term the $2F_1$ with respect to the parameter $a$, we find:

$$\begin{align*}
\psi(a) &= \frac{\partial}{\partial a} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n \\
&= \sum_{n=1}^{\infty} \frac{(b)_n}{(c)_n (1)_n} z^n \frac{d}{da} (a)_n \\
&= \frac{bz}{c} \sum_{n=0}^{\infty} \frac{(b + 1)_n}{(c + 1)_n (2)_n} z^n \frac{d}{da} (a)_{n+1},
\end{align*}$$

(55)

whose validity can be verified. Using the Psi function $\psi(z) = \frac{d}{dz} \log \Gamma(z)$ (see, e.g., [2] (Section 1.3)), we find:

$$\frac{d}{da} (a)_{n+1} = (a)_{n+1} \left\{ \psi(a + n + 1) - \psi(a) \right\}$$

$$= (a)_{n+1} \sum_{k=0}^{n} \frac{1}{a + k}$$

(56)

Inserting (56) into (55) and manipulating the double series, we obtain:

$$\psi(a) = \frac{bz}{c} \sum_{n,k=0}^{\infty} \frac{(a + 1)_{n+k} (b + 1)_{n+k} (a)_k (1)_n}{(c + 1)_{n+k} (2)_{n+k} (a + 1)_k} \frac{z^{n+k}}{n! k!},$$

which, in view of (50), leads to the right member of (53).

Similarly, we can prove (54). The details are omitted. □
We chose to apply (54) to only (6) to give:

\[
F^{2:2:1}_{2:1:0} \left[ \begin{array}{ccc} a + 1, 2 - a & b, 1; & 1; 1 \\ b + 1, 2 & b + 1; & 2, 2 \end{array} \right] = b^2 \frac{\Gamma \left( \frac{1}{2} b \right) \Gamma \left( \frac{1}{2} b + \frac{1}{2} \right)}{a(a - 1) \Gamma \left( \frac{1}{2} b + \frac{1}{2} a \right) \Gamma \left( \frac{1}{2} b - \frac{1}{2} a + \frac{1}{2} \right)}
\times \left\{ \psi \left( \frac{1}{2} b \right) + \psi \left( \frac{1}{2} b + 1 \right) - \psi \left( \frac{1}{2} b + \frac{1}{2} a \right) - \psi \left( \frac{1}{2} b - \frac{1}{2} a + 1 \right) \right\},
\]

where the involved parameters are suitably restricted so that this identity is meaningful.

### 3.2. Deducible Summation Formulas

Making use of Kummer’s summation theorems ((4)–(6)) for, respectively, the left members of the identities in Theorems 1–3, we obtain a family of intriguing summation formulas, which are presented in Corollaries 1–3. Here, in order to simplify the identities, we try to reduce the number of parameters involved and use the following fundamental relation for the Gamma function \( \Gamma(z + 1) = z \Gamma(z) \) and Legendre’s duplication formula for the Gamma function:

\[
\Gamma \left( \frac{1}{2} \right) \Gamma (z) = 2^{z - 1} \Gamma \left( \frac{z}{2} \right) \Gamma \left( \frac{z + 1}{2} \right).
\]

In the following, we assume that \( (0)_0 := 1 \).

**Corollary 1.** Let \( p, q \in \mathbb{N}_0 \). Then, the following summation formulas hold.

\[
\sum_{k=0}^{p} (-1)^k \binom{p + q}{k} \sum_{j=0}^{p+q+1} \binom{p + q + 1}{j} \Gamma \left( \frac{a + k + j}{2} \right) \Gamma \left( \frac{b - 2p - q - 1 + k + j}{2} \right)
= \frac{\Gamma \left( \frac{a}{2} \right) \Gamma \left( \frac{b - a + q + 1}{2} \right)}{\Gamma \left( \frac{b + q + 1}{2} \right) \Gamma \left( b - a - 2p - q - 1 \right)}
\tag{57}
\]

\[
\left( \frac{a}{2} \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}; \frac{b - a + q + 1}{2} \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}; \frac{a + k + j}{2} \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0} (0 \leq k, j \leq p + q + 1) \right);
\]

\[
\sum_{k=0}^{p} (-1)^k \binom{p}{k} \sum_{j=0}^{p} \binom{p}{j} \Gamma \left( \frac{a - p + k + j}{2} \right) \Gamma \left( \frac{b - 2p - 1 + k + j}{2} \right)
= \frac{\Gamma \left( \frac{a - p}{2} \right) \Gamma \left( \frac{b + a + p + 1}{2} \right) \Gamma \left( \frac{b - a + p + 1}{2} \right)}{\Gamma \left( \frac{b + 1}{2} \right) \Gamma \left( \frac{b + a - p + 1}{2} \right) \Gamma \left( \frac{b - a - p + 1}{2} \right)}
\tag{58}
\]

\[
\left( \frac{a - p}{2} \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}; \frac{b + a + p + 1}{2} \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}; \frac{a - p + k + j}{2} \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0} (0 \leq k, j \leq p) \right);
\]
Corollary 2. Let $p, q \in \mathbb{N}_0$. Then, the following summation formulas hold.

\[
\sum_{k=0}^{q} (-1)^k \binom{q}{k} \sum_{j=0}^{q} \binom{q}{j} \frac{\Gamma\left(\frac{a+k+j}{2}\right)}{\Gamma\left(\frac{b-2q+1+k+j}{2}\right)}
= \frac{\Gamma\left(\frac{a}{2}\right) \Gamma\left(\frac{b+q+1}{2}\right)}{\Gamma\left(\frac{b-a+q+1}{2}\right)}
\]

\[\left(\frac{a}{2} \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}; \quad \frac{b+a+1}{2} \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}; \quad \frac{a+k+j}{2} \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0} \ (0 \leq k, j \leq q)\right);\]

\[
\sum_{k=0}^{q+1} (-1)^k \binom{q+1}{k} \sum_{j=0}^{q+1} \binom{q+1}{j} \frac{\Gamma\left(\frac{a+k+j}{2}\right)}{\Gamma\left(\frac{b-a-1+k+j}{2}\right)}
= \frac{\Gamma\left(\frac{a}{2}\right) \Gamma\left(\frac{b-a+q+1}{2}\right)}{\Gamma\left(\frac{b-a+q-1}{2}\right)}
\]

\[\left(\frac{a}{2} \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}; \quad \frac{b-a+q+1}{2} \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}; \quad \frac{a+k+j}{2} \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0} \ (0 \leq k, j \leq q+1)\right);\]

\[
\sum_{k=0}^{p+1} (-1)^k \binom{p+1}{k} \sum_{j=0}^{p+1} \binom{p+1}{j} \frac{\Gamma\left(\frac{a+k+j}{2}\right)}{\Gamma\left(\frac{b-2(p-1)+1+k+j}{2}\right)}
= \frac{\Gamma\left(\frac{a}{2}\right) \Gamma\left(\frac{b-a+2p-1}{2}\right)}{\Gamma\left(\frac{b-a+q+1}{2}\right)}
\]

\[\left(\frac{a}{2} \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}; \quad \frac{b-a+1}{2} \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}; \quad \frac{a+k+j}{2} \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0} \ (0 \leq k, j \leq p+1)\right);\]

\[
\sum_{k=0}^{p+q} (-1)^k \binom{p+q}{k} \sum_{j=0}^{p+q} \binom{p+q}{j} \frac{\Gamma\left(\frac{a+p+k+j}{2}\right)}{\Gamma\left(\frac{b-2p-2q+1+k+j}{2}\right)}
= \frac{\Gamma\left(\frac{a}{2}\right) \Gamma\left(\frac{b-a+p+1}{2}\right)}{\Gamma\left(\frac{b-a+p-2q+1}{2}\right)}
\]

\[\left(\frac{a-p}{2} \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}; \quad \frac{b-a+p+1}{2} \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}; \quad \frac{a-p+k+j}{2} \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0} \ (0 \leq k, j \leq p+q)\right).\]
\[
\sum_{k=0}^{q} (-1)^k \binom{q}{k} \sum_{j=0}^{q} \binom{q}{j} \frac{\Gamma\left(\frac{a+k+j}{2}\right)}{\Gamma\left(a-\frac{2(b-4q+2+k+j)}{2}\right)} = \frac{\Gamma(-b-q+1) \Gamma\left(\frac{q}{2}\right)}{\Gamma(-b-2q+1) \Gamma\left(a-\frac{2b-2q+2}{2}\right)}
\]

\[b \in \mathbb{C} \setminus \mathbb{Z}_{\geq q+1}; \frac{a}{2} \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}; \]

\[a + k + j \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0} \quad (0 \leq k, j \leq q);\]

\[
\sum_{k=0}^{p+q} (-1)^k \binom{p+q}{k} \sum_{j=0}^{p+q} \binom{p+q}{j} \frac{\Gamma\left(\frac{a+k+j}{2}\right)}{\Gamma\left(a-\frac{2b-2p-2q+2+k+j}{2}\right)} = \frac{\Gamma(-b+1) \Gamma\left(\frac{a+p}{2}\right)}{\Gamma(-b-q+1) \Gamma\left(a-\frac{2b+p+2q+2}{2}\right)}
\]

\[b \in \mathbb{C} \setminus \mathbb{N}; \frac{a}{2} \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}; \]

\[a + k + j \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0} \quad (0 \leq k, j \leq p + q);\]

\[
\sum_{k=0}^{q+1} (-1)^k \binom{q+1}{k} \sum_{j=0}^{q+1} \binom{q+1}{j} \frac{\Gamma\left(\frac{a+k+j}{2}\right)}{\Gamma\left(a-\frac{2b+k+j+2}{2}\right)} = \frac{\Gamma(-b+q+1) \Gamma\left(\frac{a+p}{2}\right)}{\Gamma(-b) \Gamma\left(a-\frac{2b+2q+2}{2}\right)}
\]

\[b \in \mathbb{C} \setminus \mathbb{Z}_{\geq q+1}; \frac{a+p}{2} \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}; \]

\[a + p + k + j \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0} \quad (0 \leq k, j \leq p + q + 1);\]

\[
\sum_{k=0}^{p} (-1)^k \binom{p}{k} \sum_{j=0}^{p} \binom{p}{j} \frac{\Gamma\left(\frac{a+k+j}{2}\right)}{\Gamma\left(a-\frac{2b-2p+2+k+j}{2}\right)} = \frac{\Gamma(-b+1) \Gamma\left(\frac{q}{2}\right)}{\Gamma(-b-p+1) \Gamma\left(a-\frac{2b+2}{2}\right)}
\]

\[b \in \mathbb{C} \setminus \mathbb{N}; \frac{a}{2} \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}; \]

\[a + k + j \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0} \quad (0 \leq k, j \leq p).\]
Corollary 3. Let $p$, $q$, $r \in \mathbb{N}_0$. Then, the following summation formulas hold.

\[
\sum_{k=0}^{p+1} (-1)^k \binom{p+1}{k} \sum_{j=0}^{p+1} \binom{p+1}{j} \frac{\Gamma\left(\frac{b+a-p+1+k+j}{2}\right)}{\Gamma\left(\frac{b-a-p+1+k+j}{2}\right)} = \frac{\Gamma\left(\frac{b+a-p-1}{2}\right) \Gamma(1+p-a)}{\Gamma(-a) \Gamma\left(\frac{b-a+p+1}{2}\right)}
\]

(70)

\[
\sum_{k=0}^{p} (-1)^k \binom{p}{k} \sum_{j=0}^{p} \binom{p}{j} \frac{\Gamma\left(\frac{b+a-1+k+j}{2}\right)}{\Gamma\left(\frac{b-a+1+k+j}{2}\right)} = \frac{\Gamma\left(\frac{b+a-1}{2}\right) \Gamma(1-a)}{\Gamma(-a-p+1) \Gamma\left(\frac{b-a+1}{2}\right)}
\]

(71)

\[
\sum_{k=0}^{p+1} (-1)^k \binom{p+1}{k} \sum_{j=0}^{p+1} \binom{p+1}{j} \frac{\Gamma\left(\frac{b+a-p+1+k+j}{2}\right)}{\Gamma\left(\frac{b-a-p+1+k+j}{2}\right)}
\]

(72)
3.3. Transformation Formulas for \( _2F_1 \)

We can use some transformation formulas for \( _2F_1 \) to give numerous summation formulas \( _2F_1(z) \) with modified parameters and different arguments \( z \) from \( z = -1, \frac{1}{2} \) of those identities in Theorems 1–3. We demonstrate this by recalling a transformation formula (see [26] (p. 66, Equation (33))):

\[
{ _2F_1(a, b; 2b; z) = \left( 1 - \frac{1}{2}z \right)^{-a} _2F_1 \left[ \frac{1}{2}a, \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}z; \frac{z}{2 - z} \right]}. \tag{74}
\]

The particular case \( z = -1 \) of (74) gives:

\[
{ _2F_1 \left[ \frac{1}{2}a, \frac{1}{2}a + \frac{1}{2}b + 1; \frac{1}{9} \right] = \left( \frac{3}{2} \right)^a _2F_1[a, b; 2b; -1]}. \tag{75}
\]

We can apply (75) to the identities in Theorem 2 to give some corresponding summation formulas for \( _2F_1(1/9) \) (see, e.g., [11] (p. 495, Equations (26)–(28))). For example, from (32), we have:

\[
{ _2F_1 \left[ \frac{3}{2}b + \frac{p}{2} + q - r \frac{3}{2}, \frac{3}{2}b + q - r \frac{1}{2}; b + q + \frac{1}{2} \right] = \left( \frac{3}{2} \right)^{3b + p + 2q - r} \frac{\Gamma(2b + 2q) \Gamma(-b - p - q)}{2 \Gamma(3b + p + 2q - r) \Gamma(r - p - b)} \times \sum_{k=0}^{q+r} (-1)^k \binom{q + r}{k} \sum_{j=0}^{p+1} \binom{p + 1}{j} \frac{\Gamma \left( \frac{3b + p + 2q - r + k + j}{2} \right)}{\Gamma \left( \frac{b - p - r + k + j}{2} \right)}}, \tag{76}
\]

\[
\left( p, q, r \in \mathbb{N}_0; 2b + 2q + 1 \left( 3b + p + 2q - r - 1 \right) \in \mathbb{C} \setminus \mathbb{Z}_0; \Re(b) < \frac{1}{2}(1 + r - p - q) \right).
\]

3.4. From \( _{p+1}F_p \) to \( _{p+2}F_{p+1} \)

Recall the elementary relation (see, e.g., [39] (p. 216))

\[
{ _{p+1}F_{q+1} \left[ \begin{array}{c} 1 + \alpha_1, \ldots, 1 + \alpha_p, 1 \vspace{0.03in} \\ 1 + \beta_1, \ldots, 1 + \beta_q, 2 \end{array} \vspace{0.03in} ; z \right] = \frac{\beta_1 \cdots \beta_q}{\alpha_1 \cdots \alpha_p} \left\{ _pF_q \left[ \begin{array}{c} \alpha_1, \ldots, \alpha_p \vspace{0.03in} \\ \beta_1, \ldots, \beta_q \vspace{0.03in} ; z \end{array} \vspace{0.03in} ; -1 \right] \right\}. \tag{77}
\]
We can apply the identities in Theorems 1–3 in (77) to give certain summation formulas for \(3F_2\). For example, from (32), we find:

\[
3F_2 \left[ \frac{1 + a + p, 1 + b + q, 1;}{1 + a - b + r, 2;} : -1 \right] = \frac{1 + a - b + r}{(1 + a + p)(1 + b + q)} \times \left\{ 1 - \frac{\Gamma(a - b + r) \Gamma(-b - p - q)}{2 \Gamma(a + p) \Gamma(r - p - b)} \times \sum_{k=0}^{q+r} (-1)^k \left( \frac{q + r}{k} \right) \sum_{j=0}^{p+1} \left( \frac{p + 1}{j} \right) \frac{\Gamma\left(\frac{a + p + k + j}{2}\right)}{\Gamma\left(\frac{a - 2b - p - 2q + k + j}{2}\right)} \right\} 
\]

\[ (78) \]

\[
\left( p, q, r \in \mathbb{N}_0; a - b + r + 1 \in \mathbb{C} \setminus \mathbb{Z}_0^+; \Re(b) < \frac{1}{2}(r - p - q) \right).
\]

Exton [40] (Equations (14) and (15)) investigated a certain reducibility of the KdF function to present two impressive identities showing relations between \(2F_1\) and \(3F_2, 4F_3\), for example,

\[
4F_3[a + b - 1, -a, 2 - b - 2a, b + 2a - 1; b, b + a, 1 - b - 2a; z] = (1 - z)^a 2F_1[a + b - 1, -2a; b; z]. \quad (79)
\]

Demonstrating by applying (40) to (79), we obtain:

\[
4F_3 \left[ a + p, \frac{1}{2}a + \frac{1}{2}q, -\frac{3}{2}a + \frac{1}{2}q - p + 1, \frac{3}{2}a - \frac{1}{2}q + p; \frac{1}{2} \right] = \left[ \frac{\Gamma\left(\frac{1}{2}q + \frac{1}{2}a + p + r + 1\right) \Gamma\left(\frac{1}{2}q - \frac{1}{2}a - p\right)}{\Gamma\left(\frac{1}{2}q - \frac{1}{2}a\right) \Gamma(a + p - q + r - 1)} \times \sum_{k=0}^{p+q} (-1)^k \left( \frac{p + q}{k} \right) \frac{\Gamma\left(\frac{a + p - q + r + k + 1}{2}\right)}{\Gamma\left(\frac{-p + r + k + 1}{2}\right)} + \frac{\Gamma\left(\frac{a + p - q + r + k + 2}{2}\right)}{\Gamma\left(\frac{-p + r + k + 2}{2}\right)} \right] 
\]

\[ (80) \]

\[
\left( p, q, r \in \mathbb{N}_0; a + p + 1, \frac{1}{2}(q + 2p + 2a), \frac{1}{2}(q - 2p - 3a) \in \mathbb{C} \setminus \mathbb{Z}_0^+ \right).
\]

From a methodical exploitation of the simplest relations existing between contiguous series (see, e.g., [5,41] (p. 80)), 39 and 25 contiguous extensions of the classical Dixon’s (see, e.g., [2] (p. 351, Equation (4))) and Watson’s summation theorems for \(3F_2\) were presented in terms of single formulas (see, respectively, [42,43]). On the other hand, Lavoie et al. [17] used the different extensions of Dixon’s theorem [42] to give 39 contiguous extensions of the classical Whipple’s summation theorem for \(3F_2\) in terms of a single formula (see, e.g., [17] (Equation (3))). Indeed, Lavoie et al. [17] (Equation (2)) presented the following transformation:

\[
3F_2 \left[ a, 1 - a + i + j, c; \frac{1}{e}, 1 + 2c - e + i; \frac{1}{1} \right] = \frac{\Gamma(e) \Gamma(c - i)}{\Gamma(e - a) \Gamma(a + e - j)} 3F_2 \left[ a + 2c - e - j, 1 + c - e - i, a; \frac{1}{a + c - j, 1 + 2c - e + i; \frac{1}{1}} \right]. \quad (81)
\]

The left member of (81) when \((i, j) = (0, 0)\) is in the form of Whipple’s classical theorem, which can be obtained by using Dixon’s theorem in the right member of (81) when \((i, j) = (0, 0)\). Additional contiguous extensions of Dixon’s summation theorem for
where the constant \( \gamma \) where (13), (15) and (16), which are expressed in terms of Gamma functions.

Due to the identities in Theorem 2, by employing the same method as in [1,18,42], further contiguous extensions of Dixon’s summation theorem for a \( _3F_2 \) as many as one wishes are sure to be established.

Recall the following Laplace transform (see, e.g., [46] (p. 219, Equation (6)):

\[
\mathcal{L}\left\{ t^{\lambda-1} \frac{\alpha}{\beta} z \right\} = \frac{\Gamma(\lambda)}{s^\lambda} \cdot \frac{\alpha}{\beta} z
\]

where \( n \in \mathbb{N}_0 \) and \( (c_p) \) stands for the array of parameters \( a_1, a_2, \ldots, a_p \). We assume that the parameters are such that the right-hand side is not singular and that when \( p+1 \leq q \), \(|z| < \infty\), and when \( p = q \), \(|z| < 1\).

It is easy to see that the applications of (82) to some results in Theorems 1–3 can yield corresponding formulas for finite sums of \( _3F_2 \).

3.5. Laplace Transforms and Inverse Laplace Transforms

Let the function \( f(t) \) be piecewise continuous on the closed interval \( 0 \leq t \leq T \) for every finite \( T > 0 \). Furthermore, let:

\[
f(t) = O(e^{\alpha t}) \quad (t \to \infty)
\]

for some \( \alpha \in \mathbb{R} \). The classical Laplace transform of \( f(t) \) is defined by:

\[
\mathcal{L}\{f(t) : s\} = \int_0^\infty e^{-st} f(t) \, dt = F(s) \quad (\Re(s) > \alpha).
\]

Assuming \( f(t) \) to be continuous for each \( t \geq 0 \) and to satisfy (83), the function \( f(t) \) is retrieved by means of the inverse Laplace transform:

\[
\mathcal{L}^{-1}\{F(s) : t\} = \frac{1}{2\pi i} \text{P.V.} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} F(s) \, ds = f(t),
\]

where the constant \( \gamma \) is any real number so large that the singularities of \( F \) all lie to the left of the vertical line \( \Re(s) = \gamma \) and \( \gamma > \alpha \). For an extensive treatment of such details regarding Laplace transforms, see [44]; for an application to residue calculus, see also [45] (Section 66).

Recall the following Laplace transform (see, e.g., [46] (p. 219, Equation (6))):

\[
\mathcal{L}\left\{ t^{\lambda-1} \frac{\alpha}{\beta} z \right\} = \frac{\Gamma(\lambda)}{s^\lambda} \cdot \frac{\alpha}{\beta} z
\]

Its inverse Laplace transform is given by:

\[
\mathcal{L}^{-1}\left\{ s^{-\lambda} \frac{\alpha}{\beta} \frac{z}{s} \right\} = \frac{\Gamma(\lambda)}{\Gamma(\lambda)} \frac{\alpha}{\beta} \frac{z}{t}
\]

Qureshi and Baboo [12] presented some Laplace transforms of the form in (86), whose arguments are specialized to be adjusted for general summation formulas for \( _2F_1(1/2) \) ((9), (13), (15) and (16)), which are expressed in terms of Gamma functions.
Here, as in [12], with the help of the summation formulas in Theorems 1–3, we can give many Laplace transforms of a similar type. We chose to apply only (23) in (86) to obtain:

\[
\mathcal{L}\left\{\frac{b+q}{s}\left(\frac{b+q}{s}\right)^{a+p-1}e^{st}\mathbf{1}_{\alpha}F_{\beta}\left[\begin{array}{c}
\frac{b+q}{s}+\frac{a+1}{2}dr\end{array};\frac{a+1}{2}\right] ; s \right\} = \frac{1}{2} \left(\frac{2}{s-\xi}\right)^{a+p} \Gamma\left(\frac{a+b-r}{2}\right) \times \sum_{k=0}^{p+a+r+1} \binom{p+q+r+1}{k} \Gamma\left(\frac{a+p+k}{2}\right) \Gamma\left(\frac{b-p-r+k}{2}\right).
\]

(88)

\[
(p, q, r \in \mathbb{N}_0; \Re(a+p) > 0, \Re(s-\xi) > 0, \frac{a+b-r}{2} \in \mathbb{C} \setminus \mathbb{Z}_0^-).
\]

4. Concluding Remarks and Posing of Problems

A remarkably large number of summation formulas for \(pF_q\) \((p, q \in \mathbb{N}_0)\) have been presented, and some of them have been applied in diverse ways.

In this paper, we established certain generalized summation formulas for \(2F_1\) with the arguments \(-1\) and \(\frac{1}{2}\) and specified arguments that have been found to generalize Kummer’s three classical summation theorems, (4)–(6), as well as all of those extensions of Kummer’s summation formulas. We also exhibited several examples in order to show the diverse applicability of the general summation formulas provided in Theorems 1–3. In connection with such demonstrations of applications in Section 3, we pose the following problems:

- Using the general summation formulas in Theorems 1–3, establish certain further identities similar to those in each of the subsections in Section 3: summation formulas for Kampé de Fériet function; deducible summation formulas; transformation formulas for \(2F_1\); from \(p+1F_p\) to \(p+2F_{p+1}\); Laplace transforms and inverse Laplace transforms;
- Provide other possible applications of the general summation formulas in Theorems 1–3.

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