Hadron resonance gas and mean-field nuclear matter for baryon number fluctuations

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We give an estimate for the skewness and the kurtosis of the baryon number distribution in two representative models; i.e., models of a hadron resonance gas and relativistic mean-field nuclear matter. We emphasize formal similarity between these two descriptions. The hadron resonance gas leads to a deviation from the Skellam distribution if quantum statistical correlation is taken into account at high baryon density, but this effect is not strong enough to explain fluctuation data seen in the beam-energy scan at RHIC/STAR. In the calculation of mean-field nuclear matter the density correlation with the vector \( \omega \)-field rather than the effective mass with the scalar \( \sigma \)-field renders the kurtosis suppressed at higher baryon density so as to account for the experimentally observed behavior of the kurtosis. We finally discuss the difference between the baryon number and the proton number fluctuations from correlation effects in isospin space. Our numerical results suggest that such effects are only minor even in the case of complete randomization of isospin.

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I. INTRODUCTION

The phase diagram of matter described by quantum chromodynamics (QCD) in terms of quarks and gluons, i.e., the QCD phase diagram has not been unveiled yet in spite of tremendous theoretical and experimental efforts [1, 2]. The severest obstacle lies in the notorious sign problem which prevents the first-principle lattice QCD simulation from working reliably at finite baryon density [3]. There are so many theoretical speculations on the QCD phase structures but it is next to impossible to constrain them enough to pin the right one down or to eliminate unphysical ones. Even if there were a way to evade the sign problem, it would still be a highly non-trivial question whether the lattice simulation can correctly identify the genuine ground state [4]. Taking the continuum limit and overcoming the lattice artifact should be crucial to resolve intricate structures such as the critical point [5] (see also Ref. [6] for a heuristic argument) and the crystalline state with inhomogeneous chiral condensates [7, 8] (see Ref. [9] for an argument parallel to Ref. [6] and also Ref. [10] for a comprehensive review).

It is thus our hope that the experimental data should be able to constrain diverse candidates of the QCD phase diagram, so that we can identify the correct answer. Now that we have established the formation of the quark-gluon plasma at high energy where colored degrees of freedom are active in thermodynamics, some of future heavy-ion collision programs are directed toward higher baryon density with lower collision energies. Such a project to explore the QCD phase diagram by tuning the collision energy is often called the beam-energy scan (BES) and the STAR Collaboration at Relativistic Heavy Ion Collider (RHIC) already published the first BES (i.e., BES-I) results [11]. The primary mission of the BES was to discover the so-called QCD critical point by looking at fluctuations of conserved quantities such as the baryon number and the strangeness [5, 12, 13].

So far, there is no appreciable indication that signals for the critical behavior, and nevertheless, the BES has turned out to be extremely intriguing for QCD physics, for our understanding of finite-density QCD is severely limited and any hint would be useful. With accumulation of abundant experimental data, it might be even feasible to find a way for drastic simplification leading to pragmatic modeling. We have already witnessed such simplification in RHIC at high temperature \( T \) and low baryon chemical potential \( \mu_B \); the statistical thermal fit [14–16] and the hadron resonance gas (HRG) model (see Ref. [17] and references therein and also Ref. [18] for a recent study) stunningly reproduce the experimental yields of particles and the lattice-QCD numerics for thermodynamics. Nobody had believed in the reality of such an oversimplified description of non-interacting hadrons before the good agreement to data was confirmed. Although the theoretical foundation needs more investigations, this a bit expedient but profitable tool for data analysis is as effective for analyzing experimental data taken by the ALICE Collaboration at Large Hadron Collider (LHC) (see Ref. [19] and references therein).

We cannot, of course, trust the HRG model over the entire QCD phase diagram away from the chemical freeze-out line. It is obvious that the HRG should break down in the region of nuclear matter at low-\( T \) and high-\( \mu_B \). Nuclear physics at \( T = 0 \) has revealed that a first-order phase transition of liquid-gas (or liquid-vacuum at \( T = 0 \)) should take place at \( \mu_B = M_N - B \) with \( M_N \) and \( B \) being the nucleon mass and the binding energy \( B \simeq 16 \text{ MeV} \) [20]. Some years ago an interesting possibility was demonstrated [21]: the chemical freeze-out condition at low-\( T \) and high-\( \mu_B \) could be rather sensitive to nuclear matter properties. The present work aims to pursue the idea along the same line to show the agreement for not only the chemical freeze-out condition but also the fluctuations.

One might have an impression that the HRG is a sort of opposite to nuclear matter and one should abandon
the HRG immediately to switch to the nuclear physics terrain. This intuition is not totally correct, however, and we know that the independent quasi-particle picture makes good sense inside of nuclei and nuclear matter. Hence, on the formal level, the HRG-like model with “renormalized” parameters may have a chance to work continuously from low-\(\mu_B\) to high-\(\mu_B\). Indeed, the relativistic mean-field (RMF) model of nuclear matter is designed in this spirit. The simplest setup of the RMF is the \(\sigma\)-\(\omega\) model [22] as was adopted in Ref. [21]. This model deals with nucleons as relativistic quasi-particles moving in the scalar mean-field \(\sigma\) and the vector mean-field \(\omega\). We note that we can safely neglect \(\pi\) fluctuations as long as we concern the baryon number at small \(T\). If needed, we can extend our present analysis so as to include \(\pi\) fluctuations, for example, with the renormalization group improvement [23].

**II. SUMMARY OF CENTRAL RESULTS**

We summarize our central results in advance to going into technical details. In this way we here make it clear what we address in this paper. As a preparation to discuss the numerical results, we should elucidate physical observables of our interest. We follow the standard convention as used in Ref. [18] for thermal fluctuations which are derived from the derivatives of the pressure with respect to the relevant chemical potentials. For the baryon number fluctuation, thus, we calculate the following dimensionless quantities:

\[
\chi^{(n)}_B \equiv \frac{\partial^n p}{\partial (\mu_B/T)^n T^4} ,
\]

from which we can construct the mean value (i.e., the particle number): \(M \equiv VT^3 \chi_{B}^{(1)}\). For an arbitrary distribution we can define the Gaussian width \(\sigma^2\) together with the non-Gaussian fluctuations such as the skewness \(S\) and the kurtosis \(\kappa\) as [13, 18]:

\[
\frac{\sigma^2}{M} \equiv \frac{\chi_B^{(2)}}{\chi_B^{(1)}}, \quad S \equiv \frac{\chi_B^{(3)}}{\chi_B^{(2)}}, \quad \kappa \equiv \frac{\chi_B^{(4)}}{\chi_B^{(2)}}.
\]

Therefore, once a model provides us with the pressure \(p\) as a function of \(\mu_B\), we can give an estimate for these fluctuations under an assumption of the dominance of thermal fluctuations.

Furthermore, to make a contact with the collision experiment, it is necessary to relate the collision energy \(E_{\text{NN}}\) and \(T\) and \(\mu_B\). Fortunately, such parametrization of \(T(\sqrt{s_{\text{NN}}})\) and \(\mu_B(\sqrt{s_{\text{NN}}})\) has been well established along the chemical freeze-out line [14] that reads:

\[
T(\mu_B) = a - b \mu_B^2 - c \mu_B^4 ,
\]

\[
\mu_B(\sqrt{s_{\text{NN}}}) = \frac{d}{1 + e \sqrt{s_{\text{NN}}}} ,
\]

where parameters are chosen as \(a = 0.166\) GeV, \(b = 0.139\) GeV\(^{-1}\), \(c = 0.053\) GeV\(^{-3}\), \(d = 1.308\) GeV, and \(e = 0.273\) GeV\(^{-1}\) to reproduce experimentally observed particle yields. Charge and strangeness chemical potentials, \(\mu_Q\) and \(\mu_S\), are also parametrized in a similar manner. In our present analysis, we numerically checked that the inclusion of \(\mu_Q\) and \(\mu_S\) hardly changes the fluctuation results, and so we neglect them for clarity of presentation.

Figures 1 and 2 show our results for \(S\) and \(\kappa\) estimated in the HRG (green dotted line) and in the RMF (blue dashed line) on top of the BES/STAR data (red dots). Now let us briefly discuss particular two of the non-trivial features noticeable in these figures.

One is that the HRG model may have a richer structure than the Skellam distribution. Actually it was clearly stated in Ref. [18] that the Skellam predictions come from the Boltzmann approximation. If the baryon density gets
large, therefore, we naturally expect modifications on the distribution. More specifically, as seen in Figs. 1 and 2, the kurtosis is not necessarily the unity at small $\sqrt{s_{NN}}$. This effect is not such substantial, but it would be interesting to reveal how the quantum correlation would affect the distribution in a wider region away from the chemical freeze-out line.

The other is that $\kappa\sigma^2$ in the RMF is suppressed at smaller $\sqrt{s_{NN}}$ thus larger $\mu_B$. In fact the RMF-estimated $\kappa\sigma^2$ happens to approach the experimental data. It is, of course, the interaction effect that modifies $S\sigma$ and $\kappa\sigma^2$ and we should further identify which of the mean-fields, $\sigma$ and $\omega$, is more responsible for this. Our study, as we will explain later, brings us a conclusion that the renormalization of $\mu_B$ caused by $\omega$ suppresses $\kappa\sigma^2$, while the in-medium mass coupled with $\sigma$ does the opposite. We comment that, in view of Figs. 1 and 2, the fluctuations grow up again when $\sqrt{s_{NN}}$ reaches below $\sim 4$ GeV. This low-$\sqrt{s_{NN}}$ enhancement of the fluctuations is simply because of the criticality when the chemical freeze-out line hits the liquid-gas critical point of nuclear matter [25] that is located at $T \approx 21$ MeV and $\mu_B \approx 906$ MeV in our RMF setup.

### III. Hadron Resonance Gas

In the estimate with non-interacting hadrons (in which the canonical factor $\gamma$ is not included) we make use of the standard expression of the free grand canonical partition function. That is, the pressure from baryons (fermions) is prescribed as

$$p_{\text{free}}(m_N, \mu_B) = \sum_i^N 2T \int \frac{d^3p}{(2\pi)^3} \left\{ \ln[1 + e^{-(\varepsilon_p - \mu_B)/T}] + \ln[1 + e^{-(\varepsilon_p + \mu_B)/T}] \right\}. \tag{5}$$

Here $N$ is 2 for nucleons corresponding to the isospin degeneracy and the pressure depends on the nucleon mass $m_N$ through the energy dispersion relation: $\varepsilon_p \equiv \sqrt{p^2 + m_N^2}$. We can then take the derivatives of the above expression, which results in

$$\chi^{(n)}_B = \frac{4}{T^3} \int \frac{d^3p}{(2\pi)^3} X^{(n)}(p), \tag{6}$$

where 4 appears from the spin and the isospin degeneracy (for $N = 2$) and the integrands read:

$$X^{(1)} = n_p - \bar{n}_p,$$

$$X^{(2)} = n_p(1 - n_p) + \bar{n}_p(1 - \bar{n}_p),$$

$$X^{(3)} = n_p(1 - n_p)(1 - 2n_p) - \bar{n}_p(1 - \bar{n}_p)(1 - 2\bar{n}_p), \tag{7}$$

$$X^{(4)} = (1 - 6n_p + 6n_p^2)n_p(1 - n_p) + (1 - 6\bar{n}_p + 6\bar{n}_p^2)\bar{n}_p(1 - \bar{n}_p)$$

with $n_p \equiv [e^{(\varepsilon_p - \mu_B)/T} + 1]^{-1}$ and $\bar{n}_p \equiv [e^{(\varepsilon_p + \mu_B)/T} + 1]^{-1}$ being the Fermi-Dirac distribution functions for nucleons and anti-nucleons. We can continue taking the derivatives for even larger $n$ if needed.

In the Boltzmann approximation that is valid when $n_p$ and $\bar{n}_p$ are both dilute, we can neglect the quantum statistical factors of non-linear $n_p$ and $\bar{n}_p$ terms. Then, we can approximate Eq. (7) as $X^{(2)} \approx X^{(4)} \approx (e^{\mu_B/T} + e^{-\mu_B/T})e^{-\varepsilon_p/T}$ and $X^{(3)} \approx (e^{\mu_B/T} - e^{-\mu_B/T})e^{-\varepsilon_p/T}$. In this particular limit we can readily derive:

$$S\sigma = \tanh(\mu_B/T), \quad \kappa\sigma^2 = 1, \tag{8}$$

which are nothing but the Skellam expectations. We can easily generalize the above derivation of Eq. (8) to a superposition of arbitrary $N$ with different masses to find that Eq. (8) still holds after all. This is because $e^{\mu_B/T} \pm e^{-\mu_B/T}$ is always factored out and the remaining integrand is common for $X^{(2)}$, $X^{(3)}$, and $X^{(4)}$. 

![FIG. 3. Skewness of the baryon number estimated in the HRG by the (red) fine mesh. The (blue) sparse mesh represents the Skellam expectation: $\tan(\mu_B/T)$.

![FIG. 4. Kurtosis of the baryon number estimated in the HRG by the (red) fine mesh. The (blue) sparse mesh represents the Skellam expectation that is the unity.](image_url)
Baryon Density $[\text{fm}^{-3}]$

\begin{align*}
\nu & = 0.05 \\
\nu & = 0.1 \\
\nu & = 0.2
\end{align*}

Baryon Chemical Potential $[\text{MeV}]$

\begin{align*}
\mu_B & = 8 \text{GeV}
\end{align*}

FIG. 5. HRG-estimated baryon density (including not only nucleons but all baryonic resonances of the particle data contained in the THERMUS2.3 package) as a function of $T$ and $\mu_B$. The nucleon contribution is nearly a half of shown results. The vertical lines correspond to $\sqrt{\sigma_{NN}} \simeq 8 \text{ GeV}$. The chemical freeze-out line is drawn according to Eqs. (3) and (4).

Let us then quantify the breakdown of the Boltzmann approximation explicitly by scanning the 3D landscape of $S\sigma$ and $\kappa\sigma^2$ for various $T$ and $\mu_B$. We show our results of the HRG using the particle data contained in the THERMUS2.3 package (by red fine mesh) as well as the Skellam predictions (by blue sparse mesh) in Figs. 3 and 4. It is clear from the figures that the quantum correlation certainly suppresses both $S\sigma$ and $\kappa\sigma^2$ in the high-density region where $n_p$ is not really dilute. Although this suppression effect is noticeable along the chemical freeze-out line as in Figs. 1 and 2, it is not sufficiently strong for reproducing the trend of the experimental data. In short, the quantum correlation is weak, as correctly speculated in Ref. [18], because the baryon density never gets large enough on the chemical freeze-out line.

To have a feeling about how the baryon density behaves on the chemical freeze-out line, we shall make a plot of the integrated baryon density in the standard unit of $\text{fm}^{-3}$ in Fig. 5. The vertical lines correspond to the collision energy $\sqrt{\sigma_{NN}}$ with spacing by 1 GeV. The lowest collision energy in Fig. 5 starts with $\sqrt{\sigma_{NN}} = 2 \text{ GeV}$, and so the maximum of the baryon density is located at $\sqrt{\sigma_{NN}} \sim 8 \text{ GeV}$. It is interesting that this maximum position precisely coincides with the triple-point-like region as speculated in Ref. [24]. This coincidence is not accidental; in Ref. [24] the triple-point-like region was recognized based on the horn structure in $K^+/\pi^+$ that is sensitive to the strangeness chemical potential; $\mu_S$. If the bulk system maintains zero strangeness, it is not hard to confirm that $\mu_S$ is almost proportional to $\mu_B$ within the effective model framework [26]. In this way, naturally, $K^+/\pi^+, A/\pi^+, \Xi/\pi^-$, etc have a peak structure at $\sqrt{\sigma_{NN}} \simeq 8 \text{ GeV}$ with which the baryon density is maximized.

As a final related remark we point out that the effect of the strangeness and the charge conservation is only of a few percent order in $S\sigma$ and $\kappa\sigma^2$ along the chemical freeze-out line. We have checked this numerically by adopting $\mu_Q$ and $\mu_S$ parametrized along the chemical freeze-out line [18]. We then observed that $S\sigma$ and $\kappa\sigma^2$ in Figs. 1 and 2 are pushed down by a few percent at most as compared to the $\mu_Q = \mu_S = 0$ case. This check justifies our later discussions without $\mu_Q$ and $\mu_S$ taken into account.

IV. SIMILARITY BETWEEN HRG AND RMF

It is nuclear matter (that is a self-bound system of infinite nucleons) that lies in the opposite limit to the non-interacting matter described by the HRG model. Nevertheless, on the technical level in theory, the formulation of nuclear matter, namely the RMF, is not so far from the HRG model or they actually share similarity to some extent.

The simplest RMF is known as the $\sigma-\omega$ model defined by the partition function:

$$
p = 2 \cdot 2T \int \frac{d^3p}{(2\pi)^3} \left\{ \ln \left[ 1 + e^{-(\varepsilon_p - \mu_p^*)/T} \right] + \ln \left[ 1 + e^{-(\varepsilon_p + \mu_p^*)/T} \right] \right\} - \frac{m^2_{\sigma} \sigma^2}{2} + \frac{m^2_{\omega} \omega^2}{2},
$$

where the quasi-particle dispersion relation is $\varepsilon_p = \sqrt{p^2 + m^2_{\sigma}}$. Here, quantities with asterisk are “in-medium” or “renormalized” ones which contain a shift by the mean-field as

$$
m^*_{N} \equiv m_{N} - \sigma g_{\sigma}, \quad \mu^*_{B} \equiv \mu_{B} - g_{\omega} \omega.
$$

These mean-fields of $\sigma$ and $\omega$, or equivalently, $m^*_{N}$ and $\mu^*_{B}$ are determined with the stationary conditions: $\partial\Omega/\partial\sigma = \partial\Omega/\partial\omega = 0$, which lead to the gap equations. By choosing the model parameters appropriately [27]; i.e., $m_{N} = 939 \text{ MeV}$, $m_{\pi} = 550 \text{ MeV}$, $m_{\omega} = 783 \text{ MeV}$, $g_{\sigma} = 10.3$, $g_{\omega} = 12.7$, we can reproduce the saturation properties of symmetric nuclear matter with the saturation density given by 0.17 nucleons/$\text{fm}^3$ and the binding energy per nucleon given by 16.3 MeV. We note that this simplest $\sigma-\omega$ model fails in reproducing the empirical value of the compressibility of symmetric nuclear matter [28]. It is possible to overcome this problem by extending the model with self-coupling potential of the mean-fields. For the fluctuations of our present interest, however, such improvement of the model makes only minor modifications on the final results [29].

From Eq. (9) it is obvious that the RMF estimate should reduce to nothing but the HRG estimate if we freeze the implicit dependence on $\mu_B$ through the solutions of $\sigma$ and $\omega$, or equivalently, $m^*_{N}$ and $\mu^*_{B}$. Then, an immediate question that comes to our mind is which of $\sigma$ and $\omega$ should be more responsible for the suppression seen in Fig. 2. One may well consider that the in-medium effective mass can bring about the leading effect.
of the interactions, which is indeed the case whenever the Hartree approximation works. In the present problem, as we will see in the next section, the situation is rather involved. Because we take the $\mu_B$ derivatives to compute the baryon fluctuations, it turns out to be $\mu_B^*$ and thus $\omega$ that play the essential role for forming a peculiar shape of $\kappa\sigma^2$ in Fig. 2.

V. WHAT CAUSES THE SUPPRESSION?

As we mentioned previously, if we fix $m_N^*$ and $\mu_B^*$ at the vacuum values; i.e., $m_N$ and $\mu_B$, and then take the $\mu_B$ derivatives, the results for $\Sigma\sigma$ and $\kappa\sigma^2$ are identical to what is referred to by the HRG in Figs. 1 and 2, which we have numerically checked. They are not exactly the same because the genuine HRG results have contributions also from higher baryonic resonances.

If we include the in-medium mass effect only, the $\mu_B$ derivative hits the implicit dependence in $n_p$ and $\bar{n}_p$ and, for example, the first derivative reads:

$$\frac{\partial n_p}{\partial (\mu_B/T)} = (1 - \epsilon_p')n_p(1 - n_p) \simeq (1 - \epsilon_p')n_p$$

in the Boltzmann approximation. Here $\epsilon_p'$ represents $\partial \varepsilon_p/\partial \mu_B$. We find a similar expression for $\bar{n}_p$ with an overall minus sign and with $-\epsilon_p'$ changed to $+\epsilon_p$.

At high density we can neglect the anti-particle contribution from $\bar{n}_p$, and moreover, $\epsilon_p'$ is negative because the effective mass $m_N^*$ generally decreases with increasing density. This means that $\partial n_p/\partial (\mu_B/T)$ is greater than $n_p$ by an enhancement factor $1 - \epsilon_p' > 1$. In the approximation to neglect higher derivatives in terms of $\mu_B$, therefore, $\Sigma\sigma$ and $\kappa\sigma^2$ should get larger, respectively, by $(1 - \epsilon_p')^3$ and $(1 - \epsilon_p')^4$.

In contrast to this behavior of $m_N^*$, the effect of the renormalized chemical potential $\mu_B^*$ yields a suppression factor by $\partial \mu_B^*/\partial \mu_B = 1 - g_\omega(\partial \omega/\partial \mu_B)$ where $\omega$ is proportional to the baryon density, so that we can conclude that $\partial \omega/\partial \mu_B > 0$. The above-mentioned arguments have been carefully confirmed in our numerical calculations.

Let us see the numerical check from a different view point. We change the strength of the vector coupling $g_\omega$ by hand to find that $\kappa\sigma^2$ is certainly modified in a way consistent with the above qualitative arguments, as is transparent in Fig. 6: the entire curve goes down for larger $g_\omega$. We should note, however, that we cannot infer $g_\omega$ from a fit of the model results to the experimental data. This is because we simply vary $g_\omega$ not adjusting other parameters to reproduce the saturation properties of nuclear matter. In this sense, thus, our results in Fig. 6 should not be regarded as anything beyond a test purpose.

VI. EFFECTS OF ISOSPIN CORRELATIONS

So far, we have discussed a quantitative comparison assuming that the experimentally measurable quantities of the proton number fluctuations are somehow to be identified as the baryon number fluctuations. One may have wondered if it really works or not. In fact such identification requires a non-trivial assumption about independence between neutrons and protons as is the case in the HRG calculation. We can readily understand this by expanding higher powers of $N_B = N_p + N_n$ where $N_p$ and $N_n$ are, respectively, the (net) proton number and the (net) neutron number. For the simplest example, the quadratic fluctuation consists of

$$\chi^{(2)}_B = \frac{1}{V T^3} \left( \langle N_B^2 \rangle - \langle N_B \rangle^2 \right) = \chi^{(2)}_p + \chi^{(2)}_n + 2\chi^{(2)}_{pn},$$

where

$$\chi^{(2)}_{pn} \equiv \frac{1}{V T^3} \left( \langle N_p N_n \rangle - \langle N_p \rangle \langle N_n \rangle \right).$$

If the proton and the neutron behave independently from their isospin partners, there is no connected contribution in the correlation function of $N_p$ and $N_n$; i.e., $\langle N_p N_n \rangle = \langle N_p \rangle \langle N_n \rangle$ and the last term involving $\chi^{(2)}_{pn}$ in Eq. (12) vanishes. As long as we do not consider isospin symmetry violation, the neutron fluctuation should be just identical with the proton fluctuation, so that we can conclude $\chi^{(2)}_B = 2\chi^{(2)}_p$ immediately from Eq. (12). We can continue similar arguments to deduce that $\chi^{(n)}_B = 2\chi^{(n)}_p$ in general. Therefore, obviously, this factor 2 is canceled out in the dimensionless ratios and $\Sigma\sigma$ and $\kappa\sigma^2$ of protons take the same value as those of baryons (nucleons).

This argument is valid as long as we consider a free gas of baryons only. It is known, however, that off-diagonal components of the susceptibility such as $\chi^{(2)}_{ud}$ are non-vanishing as observed in the lattice-QCD simulation [30] as well as in the model studies [31, 32]. We
do not go into technical details here but simply note that non-zero $\chi_{ud}$ is induced by different behavior of the Polyakov loop and the anti-Polyakov loop in a finite-density environment described by the Polyakov-loop extended Nambu–Jona-Lasinio model \cite{33, 34}. Physically speaking, different flavors communicate to each other through confining gluons to form pions. It is important to mention that $\chi_{ud}$ itself is finite also in the HRG calculation, which is attributed to pions rather than baryons. Then, a non-zero $\chi_{spin}^{(2)}$ of baryons should be induced by $\chi_{ud} \neq 0$ after all. Since we cannot avoid relying on another assumption to give a concrete estimate of induced $\chi_{spin}^{(2)}$, we shall postpone numerical analyses along this line into another publication.

Recently a more dynamical origin of isospin correlations has been discussed in Ref. \cite{35}. That is, residual interactions after the chemical freeze-out can change the isospin, it is a natural anticipation to presume that each in isospin space.

In this special case of complete randomization of isospin, it is a natural anticipation to presume that each in isospin space. Still, such a mixing between the life time of matter in the heavy-ion collision is of or-
tation, we do not have to think of weak processes because $n$ into $p$ and $p$ ↔ $\pi^0$ through an intermediate state of $\Delta^+ (1232)$ and $\Delta^0 (1232)$. It should be a quite complicated procedure to establish any reliable evaluation for these contributions to $\chi_{spin}^{(2)}$, but we can drastically simplify the theoretical calculation in the limit of complete mixing or randomization, that is the limit opposite to complete independence in isospin space.

In this special case of complete randomization of isospin, it is a natural anticipation to presume that each (anti-) nucleon is either a (anti-) proton or a (anti-) neutron with equal probability. Therefore, the distribution of $N_p$ is the binomial one with the mean value given by $N_B/2$ \cite{35}, where $N_p$ and $N_B$ are not the net quantities but the absolute proton number and the absolute baryon (nucleon) number. That is, $N_p = N_p - N_{\bar{p}}$, $N_B = N_B - N_{\bar{B}}$, etc. Thus, for a given $N_B$ and $N_{\bar{B}}$ (for which the average is denoted by $\langle \cdots \rangle_B$), we expect:

$$\langle N_p \rangle_B = \frac{1}{2} N_B ,$$

$$\langle (N_p - \langle N_p \rangle_B)^2 \rangle_B = \frac{1}{4} N_B ,$$

$$\langle (N_p - \langle N_p \rangle_B)^3 \rangle_B = 0 ,$$

$$\langle (N_p - \langle N_p \rangle_B)^4 \rangle_B = \frac{1}{16} N_B (3N_B - 2) ,$$

and so on according to the binomial distribution. We note that Eqs. (14)-(17) are $T$ independent unlike the thermal distribution.

We are now ready to express the proton number fluctuations in terms of baryons. For $n$-th order fluctuation we have:

$$\chi_p^{(n)} = \frac{1}{VT^3} \langle \langle (N_p - N_p - \langle N_B - N_{\bar{B}} \rangle_B / 2)^n \rangle_B \rangle_B ,$$

(18)

where $\langle \cdots \rangle$ represents an average over the distribution of $N_B$ and $N_{\bar{B}}$.

Using these relations we can easily prove, for example, the following of the quadratic ($n = 2$) fluctuation:

$$\chi_p^{(2)} = \frac{1}{4} \chi_B^{(2)} + \frac{1}{4VT^3} \langle 2N_B + N_{\bar{B}} \rangle_B ,$$

(19)

where we used independence of the baryon and the anti-baryon distributions. It should be noted that Eq. (19) exactly coincides with the formula derived in Ref. \cite{35}.

Let us see how large the second term could be, and for this purpose, we make use of an expression for the free baryon gas. Then, we numerically confirm that this second term is very close to the first term at good precision; i.e., $\langle 2N_B + N_{\bar{B}} \rangle_B \approx \chi_B^{(2)}$ within 1% level at large $\sqrt{s_{NN}}$ and at most 5% level at smaller $\sqrt{s_{NN}}$ of a few GeV. We can then approximate $\chi_p^{(2)}$ as $\chi_p^{(2)} \approx (1/2) \chi_B^{(2)}$. This means that both Eq. (19) and the previous relation in the HRG model eventually lead to the same answer; $\chi_p^{(2)} = (1/2) \chi_B^{(2)}$ after all, though they superficially look quite different from each other.

We next proceed to the $n = 3$ case. Then, after some calculations, we can arrive at:

$$\chi_p^{(3)} = \frac{1}{8} \chi_B^{(3)} + 3 \frac{8}{3} \chi_B^{(2)} - \chi_B^{(2)} ,$$

(20)

where we defined $\chi_B^{(2)}$ and $\chi_B^{(4)}$ as:

$$\chi_B^{(2)} = \frac{1}{VT^3} \langle \langle N_B^2 \rangle - \langle N_B \rangle^2 \rangle_B ,$$

(21)

$$\chi_B^{(4)} = \frac{1}{VT^3} \langle \langle N_B^4 \rangle - \langle N_B \rangle^4 \rangle_B .$$

(22)

So, the ordinary quadratic fluctuation is given as $\chi_B^{(2)} = \chi_p^{(2)} + \chi_B^{(2)}$. Our result above is again equivalent to the formula listed in Ref. \cite{35}. It is also easy to check that this latter term in Eq. (20) gives the same answer as $\chi_B^{(3)}$ within a few % as long as the baryon distribution is thermal. Therefore, $\chi_p^{(3)} \approx (1/2) \chi_B^{(3)}$ follows.

Now we can make a guess that probably $\chi_p^{(4)} \approx (1/2) \chi_B^{(4)}$ and let us explicitly make it sure. In the same way we can write $\chi_p^{(4)}$ down as:

$$\chi_p^{(4)} = \frac{1}{16} \chi_B^{(4)} + 3 \frac{8}{3} \chi_B^{(3)} - \chi_B^{(2)}$$

+ 3 \frac{16 \chi_B^{(2)}}{16} - \frac{1}{8VT^3} \langle 2N_B + N_{\bar{B}} \rangle_B .$$

(23)

This is indeed close to $(1/2) \chi_B^{(4)}$ but shows a deviation as $\sqrt{s_{NN}}$ gets smaller. We present our numerical results in Fig. 7. It is clear from Fig. 7 that $\chi_p^{(4)} \approx (1/2) \chi_B^{(4)}$ is the case as long as $\sqrt{s_{NN}}$ is sufficiently large, while it increases by about 10% at smaller $\sqrt{s_{NN}}$. Our conclusion is that, contrary to what is claimed in Ref. \cite{35}, the isospin correlation does not help us with explaining...
a suppression tendency in the kurtosis at smaller $\sqrt{s_{NN}}$; the effect is in a wrong direction. In any case, the 10% correction is just too minor to account for almost 50% suppression in the experimental data as seen in Fig. 2.

Here we make a remark that we can easily give a general proof of $\chi_p^{(n)} \approx (1/2)\chi_B^{(n)}$ if we can make the Boltzmann approximation for the baryon distribution. Therefore, in this sense, the 10% deviation seen in Fig. 7 can be attributed to the violation of the Boltzmann approximation that is quantified by the deviation from the unity in Fig. 2, which is also of the 10% level. The bottom line of our analysis is that we can safely neglect the difference between the baryon number and the proton number fluctuations.

VII. SUMMARY

We investigated the baryon number fluctuations using the hadron resonance gas model and the mean-field model of nuclear matter. We found that the mean-field description yields fairly good results which look quite consistent with the skewness and the kurtosis measured in the beam-energy scan.

Because the mean-field approximation is based on the quasi-particle treatment, in fact, it is not much different from the hadron resonance gas model except for the interaction effects incorporated in terms of the scalar and the vector mean-fields. We numerically checked that the kurtosis is suppressed at smaller collision energy (i.e., higher baryon density) due to the vector mean-field that is directly coupled to the baryon density.

Finally, in the present study, we discussed the effects of isospin correlations and reached a conclusion that such effects are only minor such that we can ignore them in the first approximation. Even in the case of strong residual interactions that realize complete randomization in isospin space, we found that the deviation from the HRG prediction is at most 10% at the smallest collision energy of a few GeV. Therefore, for a semi-quantitative estimate, we can simply identify the proton number fluctuations as (a half of) the baryon number fluctuations.

In this paper we only mentioned another possibility of flavor mixing through the off-diagonal susceptibility: $\chi_{ud}$. This non-zero $\chi_{ud}$ arises from the pion dynamics, and so it is quite non-trivial how we can relate $\chi_{ud}$ to the correlations purely among the proton number $N_p$ and the neutron number $N_n$. We are now making progress in this direction in order to refine relationship between $\chi_p^{(n)}$ and $\chi_B^{(n)}$.

Although the $\sigma$-$\omega$ model is one of the simplest methods to capture the essential features of nuclear matter, it would be more desirable to develop quantitative investigations by means of more systematic approaches such as the Chiral Perturbation Theory. It would be definitely worth attempting the fully quantitative comparisons for $Sr$ and $K\sigma^2$ within the framework of the Chiral Perturbation Theory and also more established Bruckner-type calculations. This is one of our future problems and the results shall be reported in follow-ups hopefully soon.

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