Intuitionistic Fuzzy Small Submodules and Their Properties

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ABSTRACT
The concept of an intuitionistic fuzzy set, which is a generalisation of fuzzy set, was introduced by K. T. Atanassov in 1986. In this paper using of intuitionistic fuzzy small submodules we get some results about this kind of fuzzy submodules. First we give some preliminary properties of intuitionistic fuzzy submodules. Then we attempt to investigate various properties of intuitionistic fuzzy small submodules. A necessary and sufficient condition for intuitionistic fuzzy small submodules is established. We investigate the nature of intuitionistic fuzzy small submodules under direct sum. Also we study on relation between intuitionistic fuzzy small submodules and level subsets of them, and get some interesting results in this sense.

1. Introduction

Fuzzy sets as a method of representing uncertainty in real physical world, is introduced by L. A. Zadeh [1] at first. Since then many authors have been applied this concept to introduce some classes of fuzzy sets.

In 1971 A. Rosenfeld [2] applied the notion of fuzzy sets to algebra and introduced fuzzy subgroups of a group. C. V. Negoita and D. A. Ralescu [3] applied this concept to modules and defined fuzzy submodules of a module. Then some classes of fuzzy submodules have been proceeded by some authors, such as finitely generated fuzzy submodules [4], primary fuzzy submodules [5], fuzzy essential submodules [6] and fuzzy small submodules [7].

As a generalisation of fuzzy sets, the concept of fuzzy intuitionistic sets was introduced by K. T. Atanassov in [8]. Using this idea, B. Davvas [9] established the intuitionistic fuzzification of the concept of submodules of a module. Intuitionistic fuzzy sets are applied in many area of applied mathematics such as graph theory [10], Rough set theory [11], biology [12], medicine [13] and ring theory [14]. Refs. [15–21] are some other researches about intuitioniditic fuzzy sets.

In this paper after some essential preliminaries of intuitionistic fuzzy sets and submodules, we discuss on the class of intuitionistic fuzzy small submodules. We investigate various characteristics of intuitionistic fuzzy small submodules. Necessary and sufficient conditions for intuitionistic fuzzy small submodules are established. We also attempt to investigate well-known properties on intuitionistic small submodules. The nature of intuitionistic fuzzy submodules under fuzzy direct sums is investigated in this paper.

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This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.
Throughout article $R$ means a commutative ring with unity and $M$ denotes a unitary left $R$-module. $\lor$ and $\land$ denote respectively the maximum and minimum in the unit interval $[0,1]$.

Let $X$ be a set. A map $\mu : X \rightarrow [0,1]$ is called a fuzzy subset of $X$. The complement of $\mu$, denoted by $\mu^c$, is a fuzzy subset of $X$ defined by $\mu^c(x) = 1 - \mu(x)$ for every $x \in X$.

2. Intuitionistic Fuzzy Sets and Intuitionistic Fuzzy Submodules

In this section we give some preliminary definitions and results about intuitionistic fuzzy sets which will be used later.

**Definition 2.1 ([8]):** An intuitionistic fuzzy set (briefly an IFS) $A$ of a non-void set $X$ is an object having the form $A = \{(x, \mu_A(x), \nu_A(x)); x \in X\}$, where the maps $\mu_A : X \rightarrow [0,1]$ and $\nu_A : X \rightarrow [0,1]$, are fuzzy subsets of $X$, denote respectively the degree of membership namely $\mu_A(x)$ and the degree of non-membership namely $\nu_A(x)$ for each element $x \in X$, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$.

For the sake of simplicity, we denote an IFS, $A = \{(x, \mu_A(x), \nu_A(x)); x \in X\}$ of the set $X$ by $A = (\mu_A, \nu_A)$ or briefly $A$, and the set of all IFS of $X$ by $IFS(X)$.

If $X$ is a non-empty set and $A = (\mu_A, \nu_A)$, $B = (\mu_B, \nu_B)$ are two IFS of $X$, then [8]

1. $A \subseteq B$, if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_B(x) \leq \nu_A(x)$, for all $x \in X$;
2. $A = B$ if and only if $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$, for all $x \in X$;
3. $A^c = (\nu_A, \mu_A)$;
4. $A \cap B = \{(x, \mu_A(x) \land \mu_B(x), \nu_A(x) \lor \nu_B(x)); x \in X\}$;
5. $A \cup B = \{(x, \mu_A(x) \lor \mu_B(x), \nu_A(x) \land \nu_B(x)); x \in X\}$.

[18] Let $\{A_i = (\mu_{A_i}, \nu_{A_i})\}_{i \in I}$ be a family of IFS of $X$. Then

\[
\bigcap_{i \in I} A_i = (\mu_{\bigcap_{i \in I} A_i}, \nu_{\bigcap_{i \in I} A_i}) = \{(x, \bigwedge_{i \in I} \mu_{A_i}(x), \bigvee_{i \in I} \nu_{A_i}(x)); x \in X\}
\]

and

\[
\bigcup_{i \in I} A_i = (\mu_{\bigcup_{i \in I} A_i}, \nu_{\bigcup_{i \in I} A_i}) = \{(x, \bigvee_{i \in I} \mu_{A_i}(x), \bigwedge_{i \in I} \nu_{A_i}(x)); x \in X\}
\]

**Definition 2.2 ([17]):** Let $G$ be a group. An IFS $A = (\mu_A, \nu_A)$ of $G$ is called an intuitionistic fuzzy subgroup of $G$ if the following conditions hold for all $x, y \in G$

1. $\mu_A(xy) \geq \mu_A(x) \land \mu_A(y)$;
2. $\nu_A(xy) \leq \nu_A(x) \lor \nu_A(y)$;
3. $\mu_A(x^{-1}) \geq \mu_A(x)$ (consequently $\mu_A(x^{-1}) = \mu_A(x)$);
4. $\nu_A(x^{-1}) \geq \nu_A(x)$ (consequently $\nu_A(x^{-1}) = \nu_A(x)$).

**Definition 2.3 ([9]):** Let $M$ be an $R$-module and $A = (\mu_A, \nu_A)$ an IFS of $M$. Then $A$ is called an intuitionistic fuzzy submodule of $M$ if $A$ satisfies the following

1. $\mu_A(0) = 1, \nu_A(0) = 0$
2. $\mu_A(x+y) \geq \mu_A(x) \land \mu_A(y)$, for all $x, y \in M$
   \[\nu_A(x+y) \leq \nu_A(x) \lor \nu_A(y), \text{ for all } x, y \in M\]
3. $\mu_A(rx) \geq \mu_A(x)$, for all $x \in M$ and $r \in R$
   \[\nu_A(rx) \leq \nu_A(x), \text{ for all } x \in M \text{ and } r \in R\]
If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy submodule of an $R$-module $M$, we write $A$ is an IFM of $M$ and denote by $A \leq_{IF} M$. In this case we say $A$ is an intuitionistic fuzzy module too.

**Definition 2.4 ([8]):** Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two IF$S$'s of $M$. Then the IF$S$, $A + B$ of $M$ is $A + B = \{(x, \mu_{A+B}(x), \nu_{A+B}(x)) : x \in M\}$ defined as

$$\mu_{A+B}(x) = \bigvee \{\mu_A(y) \land \mu_B(z) : y = y + z; \ y, z \in M\}$$

$$\nu_{A+B}(x) = \bigwedge \{\nu_A(y) \lor \nu_B(z) : y = y + z; \ y, z \in M\}$$

For an IF$S$, $A = (\mu_A, \nu_A)$ of $M$ and for any $r \in R$, define the IF$S$, $rA = (\mu_{rA}, \nu_{rA})$ such that for every $x \in M$

$$\mu_{rA}(x) = \bigvee \{\mu_A(y) : y \in rM\} \quad \text{and} \quad \nu_{rA}(x) = \bigwedge \{\nu_A(y) : y \in rM\}$$

So the IF$S$, $-A = (\mu_{-A}, \nu_{-A})$ will be defined as $\mu_{-A}(x) = \mu_A(-x)$ and $\nu_{-A}(x) = \nu_A(-x)$ for every $x \in M$.

**Proposition 2.5:** Let $A, B$ be two IF$S$'s of an $R$-module $M$. Then $A + B$ and $rA$ for every $r \in R$, are IF$S$'s of $M$.

**Proof:** It is straightforward and follows from definitions.

**Definition 2.6:** Let $M$ be an $R$-module, $N \subseteq M$ and $\alpha \in [0, 1]$. Define the IF$S$ $\alpha_N = (\mu_{\alpha_N}, \nu_{\alpha_N})$ of $M$ as follows

$$\mu_{\alpha_N}(x) = \begin{cases} \alpha & x \in N \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \nu_{\alpha_N}(x) = \begin{cases} 1 - \alpha & x \in N \\ 1 & \text{otherwise} \end{cases}$$

for all $x \in M$.

If $\alpha = 1$, then $\mu_{\alpha_N} = \chi_N$ and $\nu_{\alpha_N} = \chi_N^c$, where $\chi_N$ denotes the characteristic function of $N$. In this case we write $\alpha_N = \chi_N^{IF} = (\chi_N, \chi_N^c)$.

**Proposition 2.7:** Let $M$ be an $R$-module and $N \subseteq M$. Then $N \leq M$ if and only if $\chi_N^{IF} \leq_{IF} M$.

**Proof:** Suppose that $N$ is a submodule of $M$. Then $0 \in N$ and hence $\chi_N(0) = 1$ and $\chi_N^c(0) = 0$.

Now let $x, y \in M$. If $x, y \in N$, then $x + y \in N$, so $1 = \chi_N(x + y) \geq \chi_N(x) \land \chi_N(y)$ and $0 = \chi_N^c(x + y) \leq \chi_N^c(x) \lor \chi_N^c(y)$.

If $x \not\in N$, then $\chi_N(x + y) \geq \chi_N(x) \land \chi_N(y) = 0$ and $\chi_N^c(x + y) \leq \chi_N(x) \lor \chi_N(y) = 1$. Similar to this case we get if $y \not\in N$.

Now let $x \in M$ and $r \in R$. If $x \in N$, then $rx \in N$ and so we have $1 = \chi_N(rx) \geq \chi_N(x)$ and $0 = \chi_N^c(rx) \leq \chi_N^c(x)$.

If $x \in N$, then $0 = \chi_N(x) \leq \chi_N(rx)$ and also $1 = \chi_N^c(x) \geq \chi_N^c(rx)$.

Therefore $\chi_N^{IF}$ is an IF$M$ of $M$.

For converse suppose that $\chi_N^{IF}$ is an IF$M$ of $M$. So $\chi_N(0) = 1$ and hence $0 \in N$. Now let $x, y \in N$ and $r \in R$, then $\chi_N(rx + y) \geq \chi_N(rx) \land \chi_N(y) \geq \chi_N(x) \land \chi_N(y) = 1$. So $rx + y \in N$.

That is $N$ is a submodule of $M$. 

■
Example 2.8: 1. Let $M = \mathbb{Z}$ over $\mathbb{Z}$. By above proposition $\chi^{IF}_{A^c}$ is an intuitionistic fuzzy submodule of $M$ for every $n \in \mathbb{Z}$.

2. Let $M = \mathbb{Z}_{12}$ over $\mathbb{Z}$ and $N = \mathbb{Z}_{12}$, $K = \mathbb{Z}_{12}$. Then $\chi^{IF}_N$ and $\chi^{IF}_K$ are IFM's of $M$, by above proposition.

Let $A = (\mu_A, \nu_A)$ be an IF $\mu$ of $M$. Define

$$\mu^*_A = \{x \in M | \mu_A(x) > 0\} \quad \text{and} \quad \nu^*_A = \{x \in M | \nu_A(x) < 1\}.$$ 

Also $\mu_*A = \{x \in M | \mu_A(x) = \mu_A(0)\}$ and $\nu_*A = \{x \in M | \nu_A(x) = \nu_A(0)\}$.

In general for for every $t \in M$ define level subsets

$$(\mu)_t = \{x \in M | \mu_A(x) > t\} \quad \text{and} \quad (\nu)_t = \{x \in M | \nu_A(x) < 1 - t\}.$$ 

If $A$ is an IFM of $M$, then it is not difficult to see that $\mu_*A = M$ if and only if $\mu_A = \chi_M$ and $\nu_*A = M$ if and only if $\nu_A = \chi_M$.

If $A = (\mu_A, \nu_A) \subseteq B = (\mu_B, \nu_B)$ are two IFM's of an $R$-module $M$, then obviously $\mu^*_A \subseteq \mu^*_B$, $\nu^*_A \subseteq \nu^*_B$, $\mu_*A \subseteq \mu_*B$ and $\nu_*A \subseteq \nu_*B$.

For an IFM $A = (\mu_A, \nu_A)$ of $M$ define

$$A^* = \mu^*_A \cap \nu^*_A \quad \text{and} \quad A_* = \mu_*A \cap \nu_*A$$

It is easy to see that

1. If $A$ is an IFM of $M$ then $\mu^*_A = A^* \subseteq \nu^*_A$ and $\mu_*A = A_*$.  
2. If $A \leq IF M$ then $\mu_*A = A_* \subseteq \nu_*A$, and also $A_* = M$ if and only if $A = \chi^IF_M$.  
3. $A_* = 0$ if and only if $A = \chi^IF_M$.  
4. If $A \leq IF M$, then $\chi^IF_{A^*} \subseteq A$ and $A_* \subseteq A_*$.  
5. If $\chi^IF_{A^*} = A$, then $M = A^*$. $\chi^IF_{A_*} = A$ if and only if $M = A_*$.  
6. If $A \subseteq B$ are two IFM's of $M$, then $A^* \subseteq B^*$ and $A_* \subseteq B_*$.  

Proposition 2.9: Let $M$ be an $R$-module and $A = (\mu_A, \nu_A)$ an IFM of $M$. Then

1. If $A \leq IF M$, then $\mu^*_A \leq M$ and $\nu^*_A \leq M$.  
2. If $A \leq IF M$, then $\mu_*A \leq M$ and $\nu_*A \leq M$.  
3. If $A \leq IF M$, then $A^* \leq M$ and $A_* \leq M$.  

Proof: 1 and 2 follow from definitions and 3 follows from 1 and 2. 

Let $A \subseteq B$ be two IFM's of the module $M$. Define the quotient intuitionistic fuzzy module $\frac{B}{A}$ as an IFM of the module $\frac{M}{A^*}$ by $\frac{B}{A} = (\mu_{\frac{B}{A}}, \nu_{\frac{B}{A}})$ such that for every $x \in B^*$

$$(\mu_{\frac{B}{A}})([x]) = \bigvee \{\mu_{\frac{B}{A}}(z) | z \in [x]\} \quad \text{and} \quad (\nu_{\frac{B}{A}})([x]) = \bigwedge \{\nu_{\frac{B}{A}}(z) | z \in [x]\}$$

where $[x] = x + A^* \in \frac{M}{A^*}$. 

Definition 2.10: Let $M, N$ be two $R$-modules and $f : M \rightarrow N$ an $R$-homomorphism. Let $A = (\mu_A, \nu_A) \leq IF M$ and $B = (\mu_B, \nu_B) \leq IF N$. Then $f(A) = (\mu_{f(A)}, \nu_{f(A)})$ and $f^{-1}(B) = (\mu_{f^{-1}(B)}, \nu_{f^{-1}(B)})$ are IFM's of $N$ and $M$ respectively, such that for all $y \in N$

\[
(\mu_{f(A)})(y) = \begin{cases} 
\bigvee \{ \mu_A(x) | y = f(x) \} & y \in \text{Im}(f) \\
0 & y \notin \text{Im}(f)
\end{cases}
\]

and

\[
(\nu_{f(A)})(y) = \begin{cases} 
\bigwedge \{ \nu_A(x) | y = f(x) \} & y \in \text{Im}(f) \\
1 & y \notin \text{Im}(f)
\end{cases}
\]

and for every $x \in M$

\[
(\mu_{f^{-1}(B)})(x) = \mu_B(f(x)) \quad \text{and} \quad (\nu_{f^{-1}(B)})(x) = \nu_B(f(x))
\]

3. Intuitionistic Fuzzy Small Submodules

S. Rahman and H. K. Saikia introduced and studied the class of fuzzy small submodules as a class of fuzzy submodules [7]. In this section we will introduce intuitionistic fuzzy small submodules. We investigate various properties of this class of intuitionistic fuzzy submodules. First recall the small submodules and give some properties of these submodules.

Definition 3.1 ([22]): Let $M$ be a module and $K \leq M$ is called a small submodule of $M$ (denoted by $K \ll M$) if, $K + L \neq M$ for every proper submodule $L$ of $M$.

In any module $M$, trivially zero is a small submodule of $M$ called trivial small submodule. Also $M$ is not small in $M$.

Proposition 3.2: Let $M$ be a module and $K \leq N \leq M$ and $L \leq M$. Then

1. $L + K \ll M$ if and only if $L \ll M$ and $K \ll M$.
2. $N \ll M$ if and only if $K \ll M$ and $N/K \ll M/K$.
3. If $K \ll N$, then $K \ll M$.
4. If $M = M_1 \oplus M_2$ and $K_i \leq M_i$ for $i = 1, 2$; then $K_1 \oplus K_2 \ll M_1 \oplus M_2$ if and only if $K_1 \ll M_1$ and $K_2 \ll M_2$.

Proof: See Proposition 5.17, Lemma 5.18 and Proposition 5.20 of [22].

Definition 3.3: Let $M$ be an $R$-module. An IFM, $A = (\mu_A, \nu_A)$ is called an intuitionistic fuzzy small submodule of $M$ (denoted by $A \ll IF M$) if, $A + B \neq \chi_{IF} M$ for every IFM, $B \neq \chi_{IF} M$. Equivalently if $A + B = \chi_{IF} M$, then $B = \chi_{IF} M$.

Proposition 3.4: Let $M$ be a module and $N \leq M$. Then $N \ll M$ if and only if $\chi_{IF} N \ll IF M$.

Proof: Let $A = \chi_{IF} N$. Suppose that $N \leq M$ and $B = (\mu_B, \nu_B)$ is an IFM of $M$ and $A + B = \chi_{IF} M$. So we have

(i) $\mu_{A+B} = \chi_M$ and (ii) $\nu_{A+B} = \chi_M^c$.  

Let $x \in M$, then by (i) we have $1 = (\mu_{A+B})(x) = \sqrt{\{\mu_A(y) \land \mu_B(z) \mid y + z = x\}}$ and so there exist $y_0, z_0 \in M$ such that $x = y_0 + z_0$ and $\mu_A(y_0) \land \mu_B(z_0) = 1$. Hence $\mu_A(y_0) = \mu_B(z_0) = 1$, implies $y_0 \in N$ and $z_0 \in \mu_{A+B}$. Therefore $N + \mu_{A+B} = M$. Since $N \ll M$, so $\mu_{A+B} = M$ and hence $\mu = \chi_M$.

In the other hand by (ii) we have $0 = (\nu_{A+B})(x) = \bigwedge\{\nu_A(y) \lor \nu_B(z) \mid y + z = x\}$. So there exist $y_0, z_0 \in M$ such that $x = y_0 + z_0$ and $\nu_A(y_0) = \nu_B(z_0) = 0$. Hence $\nu_A(y_0) = \nu_B(z_0) = 0$, implies $y_0 \in N$ and $z_0 \in \nu_{A+B}$. Therefore $N + \nu_{A+B} = M$. Since $N \ll M$, we get $\nu_{A+B} = M$ and hence $\nu_B = \chi^C_M$.

Thus we obtain $B = \chi^F_M$; i.e. $A = \chi^F_N \ll M$.

For the converse assume $A = \chi^F_N \ll M$. Suppose that $N + K = M$, where $K$ is a proper submodule of $M$. Note that $\chi^F_K \neq \chi^F_M$ because $K$ is a proper submodule of $M$. Let $x \in M$, so there exist $n \in N$ and $k \in K$ such that $x = n + k$. Let $B = \chi^F_K$. Then

$$(\mu_{A+B})(x) = \sqrt{\{\chi_N(y) \land \chi_K(z) \mid y + z = x\}} \geq \chi_N(n) \land \chi_K(k) = 1$$

Therefore $\mu_{A+B} = \chi_M$.

Moreover

$$(\nu_{A+B})(x) = \bigwedge\{\chi_N(y) \lor \chi_K(z) \mid y + z = x\} \leq \chi_N(n) \lor \chi_K(k) = 0$$

Therefore $\nu_{A+B} = \chi^C_M$.

Since $x$ is arbitrary, so we conclude $A + B = \chi^F_M$ that is a contradiction by $A \ll M$ and $B \neq \chi^F_M$. Hence $N \ll M$.

Since 0 is a small submodule in any module $M$, so $\chi^F_0 \ll M$ by above proposition.

**Example 3.5:**

1. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}$. For every $0 \neq n \in \mathbb{Z}$ the submodule $n\mathbb{Z}$ is not small in $M$. Hence $\chi^F_{n\mathbb{Z}} \ll M$ for every $0 \neq n \in \mathbb{Z}$, by above proposition.
2. Consider the submodules $K = \{0, \bar{2}, \bar{4}, 6, 8, 10\}$ and $L = \{0, 6\}$ of $M = \mathbb{Z}_{12} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8}, \bar{9}, \bar{10}, \bar{11}\}$ as $\mathbb{Z}$-module. It is easy to check that $K \ll M$ and $L \ll M$. So $\chi^F_K \ll M$ and $\chi^F_L \ll M$, by above proposition.

Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two IFM’s of the module $M$ such that $A \subseteq B$. We say $A$ is a small intuitionistic fuzzy submodule of $B$ (denoted by $A \ll_{IF} B$) if, $A \ll_{IF} B$; that is, for ever IFM, $C$ of $M$ satisfying $C|_{B^*} \neq \chi^F_{B^*}$, we have $A|_{B^*} + C|_{B^*} \neq \chi^F_{B^*}$, where $A|_{B^*} = (\mu_A|_{B^*}, \nu_A|_{B^*})$ such that $\mu_A|_{B^*}$ and $\nu_A|_{B^*}$ are the restriction mapping of $\mu_A$ and $\nu_A$ on $B^*$ respectively.

**Definition 3.6:** Let $M, N$ be any two modules over a ring $R$. An epimorphism $f : M \longrightarrow N$ is called an intuitionistic fuzzy small epimorphism if $f^{-1}(\chi^F_M) \ll_{IF} M$ so that $f^{-1}(\chi^F_M) = \chi^F_{\text{Ker}(f)}$ clearly.

**Proposition 3.7:** Let $M$ be a module and $A = (\mu_A, \nu_A) \leq_{IF} M$. Then $A \ll_{IF} M$ if and only if $A_0 \ll M$.

**Proof:** Suppose that $A \ll_{IF} M$. Let $A_0 + K = M$ for some submodule $K$ of $M$. We claim $A + B = \chi^F_M$, where $B = \chi^F_K$. To see this let $x \in M$. Then there exist $a \in A_0$ and $k \in K$ such that
\( x = a + k \). Now

\[
(\mu_{A+B})(x) = \bigvee \{\mu_A(y) \wedge \chi_K(z) \mid y + z = x\} \geq \mu_A(a) \wedge \chi_K(k) = 1
\]

and

\[
(\nu_{A+B})(x) = \bigwedge \{\nu_A(y) \vee \chi_K(z) \mid y + z = x\} \leq \nu_A(a) \vee \chi_K(k) = 0
\]

Hence \( \mu_{A+B} = \chi_M \) and \( \nu_{A+B} = \chi_M^c \); that is, \( A + B = \chi_M^{IF} \), implies \( \chi_M^{IF} = B = \chi_M^{IF} \), as \( A \ll_{IF} M \). Hence \( K = M \).

For the converse assume \( A_\ast \ll_{IF} M \) and \( A + B = M \) for some \( B = (\mu_B, \nu_B) \leq_{IF} M \). We claim \( A_\ast + B_\ast = M \): let \( x \in M \). Then \( 1 = (\mu_{A+B})(x) = \bigvee \{\mu_A(y) \wedge \chi_K(z) \mid y + z = x\} \) and hence there exist \( y_0, z_0 \in M \) such that \( x = y_0 + z_0 \) and \( \mu_A(y_0) \wedge \chi_K(z_0) = 1 \), implies \( \mu_A(y_0) = \mu_B(z_0) = 1 \). So \( y_0 \in A_\ast \) and \( z_0 \in B_\ast \) and consequently \( A_\ast + B_\ast = M \). Since \( A_\ast \ll_{IF} M \), so \( M = B_\ast \) and hence \( \chi_M^{IF} = (\chi_M, \chi_M^c) = (\mu_B, \nu_B) = B \); that is \( A \ll_{IF} M \).

**Example 3.8:** 1. Let \( M = \mathbb{Z}_6 \) and \( S = \{\bar{0}, \bar{2}, \bar{4}\} \). Define the IFM, \( A = (\mu_A, \nu_A) \) of \( \mathbb{Z}_6 \) by

\[
\mu_A = \begin{cases} 1 & x \in S \\ \frac{1}{2} & \text{otherwise} \end{cases} \quad \text{and} \quad \nu_A = \begin{cases} 0 & x \in S \\ \frac{1}{3} & \text{otherwise} \end{cases}
\]

Then \( A_\ast = S \) is not a small submodule of \( M \) (\( S + \{\bar{0}, \bar{3}\} = M \)), so by above proposition \( A \ll_{IF} M \).

2. Let \( M = \mathbb{Z}_8 \) and \( S = \{0, \bar{2}, \bar{4}, \bar{6}\} \) which is a small submodule of \( M \). Define the IFM, \( A = (\mu_A, \nu_A) \) of \( M \) by

\[
\mu_A = \begin{cases} 1 & x \in S \\ \alpha & \text{otherwise} \end{cases} \quad \text{and} \quad \nu_A = \begin{cases} 0 & x \in S \\ \beta & \text{otherwise} \end{cases}
\]

where \( 0 < \alpha + \beta < 1 \). Then \( A_\ast = S \) is small in \( M \), so \( A \ll_{IF} M \) by above proposition.

**Corollary 3.9:** Let \( A, B \) be two IFM’s of an R-module \( M \) such that \( A \subseteq B \). Then \( A \ll_{IF} B \) if and only if \( A_\ast \ll_{IF} B_\ast \).

**Proof:** Suppose that \( A \ll_{IF} B \); i.e. \( A \ll_{IF} B_\ast \). Then by Proposition 3.7, \( A_\ast \ll_{IF} B_\ast \).

For the converse assume \( A_\ast \ll_{IF} B_\ast \). Again by Proposition 3.7, \( A \ll_{IF} B_\ast \); i.e. \( A \ll_{IF} B \).

**Example 3.10:** Consider \( M = \frac{\mathbb{Z}}{24\mathbb{Z}} \) and \( N = \frac{12\mathbb{Z}}{24\mathbb{Z}} \) as \( \mathbb{Z} \)-modules. Let \( A = (\mu_A, \nu_A) \) and \( B = (\mu_B, \nu_B) \) be two IFM’s of \( M \) such that for every \( x \in M \):

\[
\mu_A(x) = \begin{cases} 1 & x \in N \\ \frac{3}{4} & \text{otherwise} \end{cases} \quad \text{and} \quad \nu_A(x) = \begin{cases} 0 & x \in N \\ \frac{1}{5} & \text{otherwise} \end{cases}
\]

and

\[
\mu_B(x) = \begin{cases} 1 & x = 0 \\ \frac{3}{4} & x \in N \{0\} \\ \frac{1}{2} & \text{otherwise} \end{cases}
\]
If $A$, Proposition 3.11:

Let $M$ be a module, $N \leq M$ and $A = (\mu_A, \nu_A)$ an IFM of $M$ such that $A \subseteq \chi^\text{IF}_M$. If $A|_N \ll_{IF} N$, then $A \ll_{IF} M$.

Proof: Suppose that $B = (\mu_B, \nu_B)$ is an IFM of $M$ such that $A + B = \chi^\text{IF}_M$. We claim $A|_N + C = \chi^\text{IF}_N$, where $C = B|_N \cap \chi^\text{IF}_N$. To see this, let $x \in N$. Then

$$\left(\mu_{A|_N + C}\right)(x) = \bigvee \{\mu_{A|_N}(y) \land \mu_{B|_N \cap \chi^\text{IF}_N}(z) \mid y, z \in N; y + z = x\}$$

and also

$$\left(\nu_{A|_N + C}\right)(x) = \bigwedge \{\nu_A(y) \lor \nu_B(z) \lor \chi^\text{IF}_N(z) \mid y, z \in N; y + z = x\}$$

Hence we conclude $A|_N + (B|_N \cap \chi^\text{IF}_N) = \chi^\text{IF}_N$.

Now since $A|_N \ll_{IF} \chi^\text{IF}_N$, so $B|_N \cap \chi^\text{IF}_N = \chi^\text{IF}_N$, implies $\chi^\text{IF}_N \subseteq B|_N$. Hence $A \subseteq \chi^\text{IF}_N \subseteq B|_N$ and so $\chi^\text{IF}_M = A + B \subseteq B \subseteq \chi^\text{IF}_M$, implies $B = \chi^\text{IF}_M$ as desired.

Corollary 3.12: Let $M$ be a module and $A, B$ be two IFM's of $M$ such that $A \subseteq B$. If $A \ll_{IF} B$, then $A \ll_{IF} M$.

Proof: $A \ll_{IF} B$ means $A \ll_{IF} B^*$. So by 3.11, $A \ll_{IF} M$.

In the next lemma we prove the modularity law in the lattice of intuitionistic fuzzy submodules of a module. Next we use this lemma to get some results about the intuitionistic fuzzy small submodules.

Lemma 3.13: Let $M$ be a module and $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B)$ and $C = (\mu_C, \nu_C)$ be IFM's of $M$. Then $A \cap (B + C) \supseteq (A \cap B) + (A \cap C)$. Moreover if $B \subseteq A$, then $A \cap (B + C) = B + (A \cap C)$.

Proof: The first statement is clear.

To see the second statement, suppose that $B \subseteq A$. Then for every $x \in M$ we have

$$\left(\mu_{B + (A \cap C)}\right)(x) = \bigvee \{\mu_B(y) \land (\mu_A(z) \land \mu_C(z)) \mid y + z = x; (x - y = z)\}$$

$$\geq \bigvee \{\mu_B(y) \land (\mu_A(x) \land \mu_C(y)) \land \mu_C(z) \mid y + z = x\} = (\text{since } B \subseteq A)$$

$$\bigvee \{\mu_B(y) \land \mu_A(x) \land \mu_C(z) \mid y + z = x\}$$
Lemma 3.16: Let $M$ be a module and $A$, $B$ be two IFM’s of $M$ such that $A \subseteq B$ and $c \subseteq A$. Then $A$ is called an IF direct sum of $B$ and $C$ if,

$$A = B + C \quad \text{and} \quad B \cap C = \chi_{0}^{IF}$$

In this case we write $A = B \oplus_{IF} C$ and say $B, C$ are IF direct summands of $A$.

If whenever $A \cap B = \chi_{0}^{IF}$ for two IFM’s of any module $M$, then we denote $A \oplus_{IF} B$ instead of $A + B$.

Proposition 3.15: Let $M$ be a module and $A, B$ two IFM’s of $M$ such that $A \subseteq B$. Then $A \ll_{IF} B$ if and only if $A_{*} \ll_{IF} B_{*}$.

Proof: Suppose that $A \ll_{IF} B$ and $K$ is a proper submodule of $B_{*}$. So $\chi_{K}^{IF} \neq \chi_{B_{*}}^{IF}$. Since $A \ll_{IF} B$, we have $A + \chi_{K}^{IF} \neq \chi_{B_{*}}^{IF}$. Hence there exists $x_{0} \in B_{*}$ such that $x_{0} \notin A_{*} + K$. Then $A_{*} + K \neq B_{*}$, implies $A_{*} \ll_{IF} B_{*}$.

For the converse suppose that $A_{*} \ll_{IF} B_{*}$. Then $A_{*} \ll_{IF} B_{*}$ by Proposition 3.2 (3). Hence $A \ll_{IF} B$ by Corollary 3.9.

Lemma 3.16: Let $M$ be a module and $A, B$ be two IFM’s of $M$ such that $\chi_{M}^{IF} = A \oplus_{IF} B$. Then $M = A^{*} \oplus B^{*} = A_{*} \oplus B_{*}$.

Proof: Since $A_{*} \subseteq A^{*}$ for every IFM of $M$, it suffices to show that $M = A_{*} + B_{*}$ and $A^{*} \cap B^{*} = 0$.

Let $x \in M$, then

$$1 = \chi_{M}(x) = \mu_{A+B}(x) = \bigvee \{\mu_{A}(y) \wedge \mu_{B}(z) | y + z = x\}$$

So there exist $y_{0}, z_{0} \in M$ such that $x = y_{0} + z_{0}$ and $\mu_{A}(y_{0}) \wedge \mu_{B}(z_{0}) = 1$ and hence $\mu_{A}(y_{0}) = \mu_{B}(z_{0}) = 1$, implies $\nu_{A}(y_{0}) = \nu_{B}(z_{0}) = 0$. Therefore $y_{0} \in A_{*}$ and $z_{0} \in B_{*}$, i.e. $M = A_{*} + B_{*}$.
Now let $x \in A^* \cap B^*$, then $\mu_A(x) > 0$ and $\mu_B(x) > 0$. By hypothesis

\[ 0 < \mu_A(x) \wedge \mu_B(x) = \chi_0(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases} \]

This implies $x = 0$; i.e. $A^* \cap B^* = 0$.

**Proposition 3.17:** Let $A, B$ be two IFM's of any module $M$ such that $A \subseteq B$ and $B$ is an IF direct summand of $M$. Then $A \ll_IF M$ if and only if $A \ll_IF B$.

**Proof:** Write $\chi^M = B \oplus_IF C$. Suppose that $A \ll_IF M$. By Lemma 3.16, $M = B^* \oplus A^*$. By Proposition 3.7, $A \ll M$ and so $A \ll B^*$ by [23, 2.2(6)]. Now use Corollary 3.9 to get $A \ll_IF B$.

The converse follows from Corollary 3.12.

Let $M = N \oplus K$ be modules, then it is easy to see that $\chi^M = \chi_N \oplus \chi_K$. Now we state the following example.

**Example 3.18:** Let $M = \frac{\mathbb{Z}}{12\mathbb{Z}}$ and $M_1 = \frac{\mathbb{Z}}{12\mathbb{Z}}, M_2 = \frac{4\mathbb{Z}}{12\mathbb{Z}}$. Put $A = \chi^M, B = \chi^M_1$, and $C = \chi^M_2$. It is easy to see that $N = \frac{6\mathbb{Z}}{12\mathbb{Z}} \ll M$, so $\chi^N \ll_IF M$ by Proposition 3.4. We have $M = M_1 \oplus M_2$ and hence $\chi^M = A \oplus IF B$. Since $\chi^N \subseteq B$, so $\chi^N \ll_IF B$ by Proposition 3.17.

**Lemma 3.19:** Let $A, B$ be two IFM's of the module $M$. Then

1. $(A + B)^* = A^* + B^*$.
2. $(A + B)_* = A_* + B_*$.

**Proof:** 1. Let $x \in (A + B)^*$, then

\[ 0 < \mu_{A+B}(x) = \sqrt{\{\mu_A(y) \wedge \mu_B(z) \mid y + z = x; y, z \in M\}} \]

So there exist $y_0, z_0 \in M$ such that $x = y_0 + z_0$ and $\mu_A(y_0) \wedge \mu_B(z_0) > 0$, implies $\mu_A(y_0) > 0$ and $\mu_B(z_0) > 0$ and so $y_0 \in A^*, z_0 \in B^*$. Hence $x = y_0 + z_0 \in A^* + B^*$. Therefore $(A + B)^* \subseteq A^* + B^*$.

In the other hand let $x \in A^* + B^*$. Then there exist $y_0 \in A^*$ and $z_0 \in B^*$ such that $x = y_0 + z_0$. Now

\[ \mu_{A+B}(x) = \sqrt{\{\mu_A(y) \wedge \mu_B(z) \mid y + z = x; y, z \in M\}} \geq \mu_A(y_0) \wedge \mu_B(z_0) > 0 \]

Hence $x \in \mu_{(A+B)}^* = (A + B)^*$. Therefore $A^* + B^* \subseteq (A + B)^*$.

The proof of (2) is similar to the proof of (1).

The next corollary immediately follows from Lemma 3.16 and Lemma 3.19.

**Corollary 3.20:** Let $A, B$ be two IFM's of the module $M$ such that $A \cap B = \chi^M_0$. Then

1. $(A \oplus_IF B)^* = A^* \oplus B^*$.
2. $(A \oplus_IF B)_* = A_* \oplus B_*$. 
**Proposition 3.21:** Let $M, N$ be two $R$-modules and $f : M \rightarrow N$ an $R$-homomorphism. If $A = (\mu_A, \nu_A)$ is a small IFM of $M$, then $f(A)$ is a small IFM of $N$.

**Proof:** Suppose that $B$ is an IFM of $N$ such that $f(A) + B = \chi_N$. First we show that $A + f^{-1}(B) = \chi_M$. To see this, let $x \in N$, so

$$1 = \mu_{f(A) + B}(x) = \bigvee \{\mu_{f(A)}(y) \land \mu_B(z) \mid x = y + z; y, z \in N\}$$

$$= \bigvee \{\{\mu_A(m) \mid y = f(m); m \in M\} \land \mu_B(z) \mid f(m) + z = x\}$$

So there exist $y_0, z_0 \in N$ such that $x = y_0 + z_0$ and $\bigvee\{\mu_A(m) \mid y_0 = f(m); m \in M\} = \mu_B(z_0) = 1$. Especially there exists $m_0 \in M$ such that $\mu_A(m_0) = \mu_B(z_0) = 1$ where $x = f(m_0) + z_0$.

Now for $t \in M$ we have

$$\mu_{(A + f^{-1}(B))}(t) = \bigvee \{\mu_A(r) \land \mu_{f^{-1}(B)}(s) \mid r + s = t; r, s \in M\}$$

$$\bigvee \{\mu_A(r) \land \mu_B(f(s)) \mid f(r) + f(s) = f(t)\}$$

Since $f(t) \in N$, there exist $m_0 \in M$ and $z_0 \in N$ such that $f(x) = f(m_0) + z_0$ and $\mu_A(m_0) = \mu_B(z_0) = 1$. Since $z_0 = f(t - m_0) \in \text{im}(f)$, so

$$\bigvee \{\mu_A(r) \land \mu_B(f(s)) \mid r + s = t\} \geq \mu_A(m_0) \land \mu_B(z_0) = 1$$

Thus we have $\mu_{(A + f^{-1}(B))} = \chi_M$.

Similarly we can see $\nu_{(A + f^{-1}(B))} = \chi_M$. Thus $A + f^{-1}(B) = \chi_M$. Since $A \ll_M M$, we conclude $f^{-1}(B) = \chi_M$. In the other hand $A \subseteq \chi_M$, so $f(A) \subseteq B$ implies $B = \chi_M$, as desired. $lacksquare$

**Corollary 3.22:** Let $A \subseteq B$ be two IFM’s of any module $M$. If $B \ll_M M$, then $A \ll_M M$.

**Proof:** Follows immediately from Proposition 3.21. $lacksquare$

**Lemma 3.23:** Let $A, B$ be two IFM’s of the module $M$. Then $A \ll_M M$ and $B \ll_M M$ if and only if $A + B \ll_M M$.

**Proof:** Suppose that $A \ll_M M$ and $B \ll_M M$. Then by Proposition 3.15, $A_* \ll_M M$ and $B_* \ll_M M$. By Proposition 3.2 (1) and Lemma 3.19 we conclude $(A + B)_* \ll M$. Now again by Proposition 3.15, $A + B \ll_M M$.

The converse follows from Corollary 3.22. $lacksquare$

**Proposition 3.24:** Let $A, B$ be two IFM’s of $M$ such that $A \oplus_M B = \chi_M$. Moreover let $C, D$ be two IFM’s of $M$ such that $C \subseteq A$ and $D \subseteq B$. Then $C \oplus_M D \ll_M A \oplus_M B$ if and only if $C \ll_M A$ and $D \ll_M B$.

**Proof:** Assume $C \oplus_M D \ll_M A \oplus_M B$. Then $C = \chi_M A, D = \chi_M B, C \ll_M A$ and $D \ll_M B$, as desired. Similarly it can be seen that $A \ll_M B$.

For the converse suppose that $C \ll_M A$ and $D \ll_M B$. By Corollary 3.22, $C \ll_M A \oplus_M B$ and $D \ll_M A \oplus_M B$. By the Proposition 3.23, $C \ll_M A \oplus_M B$ and $D \ll_M A \oplus_M B$. By Proposition 3.23, $C \ll_M A \oplus_M B$. $lacksquare$
4. Conclusion

The study of properties of intuitionistic fuzzy submodules of a module is a meaningful research topic in the theory of intuitionistic fuzzy sets. In this paper we focus our research on a special property of this kind of submodules namely intuitionistic fuzzy small submodules and study on some classical properties of them in details.

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