Basic Twist Quantization of $osp(1|2)$ and $\kappa$–Deformation of $D = 1$ Superconformal Mechanics

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Abstract

The twisting function describing a nonstandard (super-Jordanian) quantum deformation of $osp(1|2)$ is given in explicite closed form. The quantum coproducts and universal $R$-matrix are presented. The non-uniqueness of the twisting function as well as two real forms of the deformed $osp(1|2)$ superalgebras are considered. One real quantum $osp(1|2)$ superalgebra is interpreted as describing the $\kappa$-deformation of $D = 1$, $N = 1$ superconformal algebra, which can be applied as a symmetry algebra of $N = 1$ superconformal mechanics.
1 Introduction

It is well-known that for the Lie algebra $\mathfrak{sl}(2; \mathbb{R}) \cong \mathfrak{o}(2, 1)$ do exist only two inequivalent deformations, generated by the classical $r$-matrices with the following antisymmetric parts $(a \wedge b \equiv a \otimes b - b \otimes a)$:

i) Standard deformation $[1, 2, 3, 4]$ with

$$ r_{DJ} = \gamma e_+ \wedge e_-, $$

where

$$ [h, e_\pm] = \pm e_\pm, \quad [e_+, e_-] = 2h $$

defines the Cartan-Chevalley basis of $\mathfrak{sl}(2; \mathbb{R}) \cong \mathfrak{o}(2, 1)$, and $\gamma$ is a deformation parameter. Classical $r$–matrix (1) determines the term linear in deformation parameter in the coproduct of Drinfeld-Jimbo quantum algebra $U_q(\mathfrak{sl}(2))$ where $\mathfrak{sl}(2) = \mathfrak{sl}(2; \mathbb{R})$ or $\mathfrak{sl}(2; \mathbb{C})$. The classical $r$-matrix (1) satisfies the modified YB equation (MYBE).

ii) Nonstandard (Jordanian) deformation $[5, 6, 7]$ with

$$ r_J = \xi h \wedge e_+. $$

The classical $r$-matrix (3) satisfies the classical YB equation (CYBE), and the corresponding quantization of $U(\mathfrak{sl}(2))$ can be obtain by using Drinfeld twist technique $[8, 9]$ with the twisting two-tensor $[6]$

$$ F_J = \exp(\xi h \otimes E_+) $$

where

$$ E_+ = \frac{1}{\xi} \ln(1 + \xi e_+) = e_+ + \mathcal{O}(\xi) $$

describes deformed the generator $e_+$ with proper no-deformation limit $\xi \to \infty$. The quantum Hopf algebra $U_\xi(\mathfrak{sl}(2))$ has the classical non-deformed algebra sector but the deformed twisted coproduct and antipode

$$ \Delta_\xi(a) = F_J \Delta^{(0)}(a) F_J^{-1}, \quad S_\xi(a) = uS(a)u^{-1} \quad (a \in U(\mathfrak{sl}(2))), $$

where $\Delta^{(0)}(a) = a \otimes 1 + 1 \otimes a$ for any $a \in \mathfrak{sl}(2)$, and $u = \sum_i f^{(1)}_i S(f^{(2)}_i)$ provided that the twist factor (4a) is written $F_J = \sum_i f^{(1)}_i \otimes f^{(2)}_i$. The nonstandard quantum Hopf algebra $U_\xi(\mathfrak{sl}(2))$ can be also described in a nonclassical algebra basis $[7]$ with deformed $\mathfrak{sl}(2)$ Lie-algebraic relations.

The algebra $[2]$ with the following reality conditions

$$ e^\dagger_\pm = -e_\pm, \quad h^\dagger = -h $$

can be considered as $D = 1$ conformal algebra $\mathfrak{sl}(2; R) \cong \mathfrak{o}(2, 1)$

$$ [\mathcal{D}, \mathcal{H}] = i \mathcal{H}, \quad [\mathcal{D}, \mathcal{K}] = -i \mathcal{K}, \quad [\mathcal{H}, \mathcal{K}] = -2i \mathcal{D}, $$

where $\mathcal{H} = 1/2 (e_+ + e_-)$, $\mathcal{D} = 1/2 (e_+ - e_-)$, and $\mathcal{K} = i (e_+ e_- - e_- e_+)$. The algebra $[2]$ is the nonstandard conformal algebra $\mathfrak{o}(2, 1) \setminus \{0\}$.
where $H$ describes the time translation generator (Hamiltonian), $D$ is a scale generator, and $K$ is a conformal acceleration generator. We can identify (2) and (7) if
\[
    e_+ = iH, \quad e_- = -iK, \quad h = -iD.
\]
One gets
\[
    r_{D,J} = \gamma H \wedge K, \quad r_J = \xi D \wedge H.
\]
Further we shall choose classical $r$-matrix purely imaginary under the involution $^{1}\mathcal{O}$, which implies after the assumption $(a \otimes b)^\dagger = a^\dagger \otimes b^\dagger$ that $\gamma$ and $\xi$ are purely imaginary. We see that under the physical scaling transformations
\[
    H' = \lambda H, \quad K' = \lambda^{-1} K, \quad D' = D, \quad (10)
\]
the parameter $\gamma$ is dimensionless, and $\xi$ has the inverse mass dimension. If we put $\xi = \frac{i}{2\kappa}$ we see that the Jordanian deformation of $sl(2)$ describes the $\kappa$-deformation of $D = 1$ conformal algebra with $\kappa$ describing the fundamental mass deformation parameter and classical limit given by $\kappa \to \infty (\xi \to 0)$.

The aim of this paper is to provide explicite formulae for the super-Jordanian twist quantization of the superalgebra $osp(1|2; \mathbb{R})$ which is supersymmetric extension of $sl(2; \mathbb{R}) \simeq sp(2; \mathbb{R}) \simeq o(2, 1)$ \cite{12, 13}. The classical $r$-matrices \cite{11} and \cite{14} are supersymmetrically extended as follows
\[
    r^{\text{susy}}_{DJ} = \gamma(e_+ \wedge e_- + 2v_+ \wedge v_-), \quad (11a)
\]
\[
    r^{\text{susy}}_J = \xi (h \wedge e_+ - v_+ \wedge v_+), \quad (11b)
\]
where for odd generators $a \wedge b = a \otimes b + b \otimes a$, and the standard relations of the $osp(1|2)$ Cartan–Chevalley basis looks as follows
\[
    [h, v_\pm] = \pm \frac{1}{2} v_\pm, \quad \{v_+, v_-\} = -\frac{1}{2} h, \quad (12a)
\]
\[
    e_\pm = \pm 4 (v_\pm)^2. \quad (12b)
\]
One sees that $e_\pm$ play the role of composite ”double root” generators, extending $osp(1|2)$ Cartan–Chevalley basis \cite{12a}.

One can consider two different reality conditions which represent two possible supersymmetric extensions of the antilinear antiinvolution defining \cite{6} – we shall call them Hermitean and graded or super-Hermitean:

(i) The Hermitean reality conditions defined as follows
\[
    (v_\pm)^\dagger = i v_\pm \quad (13a)
\]
\[
    (v_\pm)^\dagger = i v_\pm \quad (13b)
\]
\[
    \text{In such a case we obtain a form of quantum group, which was also called an imaginary form (see e.g. \cite{10}).} \]
provided that

\[(ab)^\dagger = b^\dagger a^\dagger \quad \text{and} \quad (a \otimes b)^\dagger = (-1)^{\text{deg}a \text{deg}b} a^\dagger \otimes b^\dagger\]  \hspace{1cm} (13b)

for any homogeneous elements \(a, b \in U(osp(1|2))\).

\((ii)\) The super-Hermitean reality conditions we define

\[(v_\pm)^\dagger = v_\pm\]  \hspace{1cm} (13c)

provided that

\[(ab)^\dagger = (-1)^{\text{deg}a \text{deg}b} b^\dagger a^\dagger \quad \text{and} \quad (a \otimes b)^\dagger = a^\dagger \otimes b^\dagger\]  \hspace{1cm} (13d)

for any homogeneous elements \(a, b \in U(osp(1|2))\).

The reality conditions (13a) and (13c) for odd generators will provide new real \(osp(1|2)\) \(\star\)-Hopf superalgebras generated by the classical \(r\)-matrix (11b).

The quantization with \(r_{susy}^{D,J}\) describes Drinfeld-Jimbo type of the quantum superalgebra, \(U_q(osp(1|2))\), studied firstly in [15, 16]. The quantization with \(r_{susy}^{D,J}\) is a subject of the present paper. One can show that (11b) satisfies graded CYBE and can be quantized by superextension of the Drinfeld twisting procedure. Further introducing the \(D = 1\) conformal supercharges \(Q, S\) which transform under scaling (10) as follows

\[Q' = \lambda^{1/2} Q, \quad S' = \lambda^{-1/2} S,\]  \hspace{1cm} (14)

we easily see that the classical \(r\)-matrix (11b) describes via identification \(\xi = \frac{1}{2\kappa^2}\) the \(\kappa\)-deformation of \(D = 1, N = 1\) superconformal algebra.

We would like to point out that in this paper we complete the discussion of the super-Jordanian deformation of \(osp(1|2)\) presented in [14, 19]. In [19] the proposed ansatz for the super-Jordanian twisting two-tensor was not properly chosen what subsequently did not allow to complete the twist quantization procedure generated by the classical \(r\)-matrix (11b), and in particular it was not possible to write down all coproduct formulae. It should be admitted however, that several results on the super-Jordanian deformation of \(osp(1|2)\) which did not require the complete knowledge of the twisting two-tensor were presented in [14, 19].

The plan of our paper is the following:

In Sect. 2 we present an explicite formula for the super-Jordanian twisting two-tensor, calculate coproducts for all generators and present the universal \(R\)-matrix. We shall also discuss in Sec. 2 the real forms of quantum \(osp(1|2)\) superalgebras as well as non-uniqueness of the twisting procedure. In Sect. 3 we interpret the super-Jordanian deformation of \(osp(1|2; \mathbb{R})\) generated by the classical \(r\)-matrix (11b) as a \(\kappa\)-deformation of the \(D = 1\) superconformal algebra. Final Sect. 4 contains an outlook.

\section{Super-Jordanian Twist Quantization of \(osp(1|2)\)}

Firstly we shall outline basic elements of Drinfeld's theory of twisting quantization of Hopf algebras [9]. A Hopf algebra \(A := A(m, \Delta, \epsilon, S)\) with a multiplication \(m : A \otimes A \rightarrow A\), a
coproduct $\Delta : A \to A \otimes A$, a counit $\epsilon : A \to \mathbb{C}$, and an antipode $S : A \to A$ due to twisting procedure can be transformed with a help of an invertible element $F \in A \otimes A$, $F = \sum_i f_i^{(1)} \otimes f_i^{(2)}$, into a twisted Hopf algebra $A_\xi := A_\xi(m, \Delta_\xi, \epsilon, S_\xi)$ which has the same multiplication $m$ and the counit mapping $\epsilon$ but the twisted coproduct and antipode

$$
\Delta_\xi(a) = F \Delta(a) F^{-1}, \quad S_\xi(a) = u S(a) u^{-1}, \quad u = \sum_i f_i^{(1)} S(f_i^{(2)}) \quad (a \in A).
$$

(15)

The twisting element (twisting two-tensor) $F$ satisfies the cocycle equation

$$
F^{12}(\Delta \otimes \mathrm{id})(F) = F^{23}(\mathrm{id} \otimes \Delta)(F),
$$

(16)

and the "unital" normalization condition

$$
(\epsilon \otimes \mathrm{id})(F) = (\mathrm{id} \otimes \epsilon)(F) = 1.
$$

(17)

The Hopf algebra $A$ is called quasitriangular if it has an additional invertible element (universal $R$-matrix) $R$ [2, 9] which relates the coproduct $\Delta$ with its opposite coproduct $\Delta^{op}$ by the similarity transformation

$$
\Delta^{op}(a) = R \Delta(a) R^{-1} \quad (a \in A),
$$

(18)

with $R$ satisfying the quasitriangularity conditions

$$
(\Delta \otimes \mathrm{id})(R) = R^{13} R^{23}, \quad (\mathrm{id} \otimes \Delta)(R) = R^{13} R^{12}.
$$

(19)

The twisted ("quantized") Hopf algebra $A_\xi$ is also quasitriangular with the universal $R$-matrix $R_\xi$ defined as follows

$$
R_\xi = F^{21} R F^{-1},
$$

(20)

where $F^{21} = \sum_i f_i^{(2)} \otimes f_i^{(1)}$ provided $F = \sum_i f_i^{(1)} \otimes f_i^{(2)}$. For the nondeformed, classical case $A = U(g)$, where $g$ is a simple Lie algebra, the universal $R$-matrix is trivial, i.e. $R = 1$.

Our goal is to construct the twisting two-tensor $F_{sJ} := F$ for $U(osp(1|2))$, such that the universal $R$-matrix

$$
R_{sJ} = F_{sJ}^{21} F_{sJ}^{-1}
$$

has the form

$$
R_{sJ} = 1 - r_{sJ} + \mathcal{O}(\xi^2),
$$

(21)

where $r_{sJ}$ is the classical $r$-matrix [11b] linear in deformation parameter $\xi$. The Taylor-series expansion of $F_{sJ}$ with respect to the parameter $\xi$ looks as follows

$$
F_{sJ} = 1 + \xi (h \otimes e_+ - v_+ \otimes v_+) + \mathcal{O}(\xi^2)
$$

(22)

and it is consistent with the relations (21), (22). One can show that (see also [19]) the twisting two-tensor $F_{sJ}$ describing the quantization of the classical $r$-matrix [11b] can be factorized as follows:

$$
F_{sJ} = F_s F_J,
$$

(23)
where $F_J$ is the Jordanian twisting two-tensor (4a) for $sl(2)$ depending on $h$ and $\sigma$ (i.e. $h$ and $e_\pm$) and $F_s$, the supersymmetric part, depends only on odd generator $v_\pm$ (remember $e_\pm = 4v_\pm^2$). The Jordanian twisting two-tensor $F_J$ as well as $F_{sj}$ should satisfy the cocycle and "unital" conditions (16), (17).

Substituting (24) into (16) one obtains the following twisted cocycle condition for $F_s$

$$F_s^{12}(\Delta \otimes 1)(F_s) = F_s^{23}(1 \otimes \Delta)(F_s),$$

where

$$\Delta_j(a) = F_{J}^{(0)}(a)F_{J}^{-1} \quad (a \in U(sl(2))).$$

We mention that because $sl(2) \subset osp(1|2)$ the classical $r$-matrix (3) for $sl(2)$ satisfying CYBE can be considered also as the classical $r$-matrix for $osp(1|2)$. The coproducts (26) have been calculated in [18] and are given by the formulae:

\begin{align*}
\Delta_j(e^\pm) &= e^\pm \otimes e^\pm, \\
\Delta_j(e_+) &= \frac{1}{\xi}(e^{2\sigma} \otimes e^{2\sigma} - 1) = e_+ \otimes e^{2\sigma} + 1 \otimes e_+, \\
\Delta_j(h) &= h \otimes e^{-2\sigma} + 1 \otimes h, \\
\Delta_j(v_+) &= v_+ \otimes e^\sigma + 1 \otimes v_+, \\
\Delta_j(v_-) &= v_- \otimes e^{-\sigma} + 1 \otimes v_- + \xi h \otimes v_+ e^{-2\sigma},
\end{align*}

where $\sigma = \frac{\xi}{2}E_+ = \frac{1}{2}\ln(1 + \xi e_+)$. The coproducts $\Delta_j(e_-)$ can be calculated from (12b) and (27e). We also recall the antipodes given by the formulae [18]

\begin{align*}
S_j(e^\pm) &= e^\mp, \\
S_j(e_+) &= -e_+ e^{-2\sigma}, \\
S_j(h) &= -h e^{2\sigma}, \\
S_j(v_+) &= -e^{-\sigma} v_+, \\
S_j(v_-) &= -v_- e^{\sigma} + \xi h v_+ e^{2\sigma}.
\end{align*}

It should be added that the coproduct $\Delta_j$ is real under both involutions (13a, c), i.e.

$$\Delta_j(a^*) = (\Delta_j(a))^*$$

as well as antipode $S_j$ satisfies the consistency condition (see e.g. [17], § 1.7)

$$S_j((S_j(a^*))^*) = a$$

$(a \in U(osp(1|2)); \ast = \dagger, \check{\dagger})$ as well as $\epsilon(a^*) = \overline{\epsilon(a)}$ is trivially valid. We see therefore that the formulae (27a - e) define two real quantum $osp(1|2)$ superalgebras subject to a choice of reality condition (13a) or (13c).
The "super" part $F_s$ of the twist element (24) can be given as solution of Eq. (25) by the following explicit formula

$$F_s = 1 - 4 \xi \frac{v_+}{e^\sigma + 1} \otimes \frac{v_+}{e^\sigma + 1}, \quad (31)$$

and obviously

$$(\epsilon_j \otimes 1)(F_s) = (1 \otimes \epsilon_j)(F_s) = 1, \quad (32)$$

where $\epsilon_j(\sigma) = \epsilon_j(v_\pm) = 0$, $\epsilon_j(1) = 1$. Further one can rewrite (31) as follows

$$F_s = 1 - \xi \frac{v_+ e^{-\frac{1}{2} \sigma}}{\cosh \frac{1}{2} \sigma} \otimes \frac{v_+ e^{-\frac{1}{2} \sigma}}{\cosh \frac{1}{2} \sigma}. \quad (33)$$

It can be also shown that the element

$$F_s^{-1} = \frac{\cosh \frac{1}{2} \sigma \otimes \cosh \frac{1}{2} \sigma + \xi v_+ e^{-\frac{1}{2} \sigma} \otimes v_+ e^{-\frac{1}{2} \sigma}}{\cosh \frac{1}{2} \Delta_j(\sigma)} \quad (34)$$

is inverse to $F_s$, i.e. $F_s^{-1} F_s = F_s F_s^{-1} = 1$, where $\Delta_j(\sigma) = \sigma \otimes 1 + 1 \otimes \sigma$.

Using the formula $\Delta_{S_J}(a) = \tilde{F}_s \Delta_j(a) F_s^{-1}$, which follows from (24), and (27a-e), (31), (34) one can calculate explicitly the super-Jordanian twisted coproducts for $osp(1|2)$. One gets after quite involved calculations that

$$\Delta_{S_J}(h) = h \otimes e^{-2\sigma} + 1 \otimes h + \xi v_+ e^{-\sigma} \otimes v_+ e^{-2\sigma} - \frac{1}{4} (e^{-\sigma} - 1) \otimes (e^{-\sigma} - 1) e^{-\sigma}, \quad (35a)$$

$$\Delta_{S_J}(v_+) = v_+ \otimes 1 + e^\sigma \otimes v_+, \quad (35b)$$

$$\Delta_{S_J}(v_-) = v_- \otimes e^{-\sigma} + 1 \otimes v_- + \frac{\xi}{4} \left\{ \left( \{h, e^\sigma\} \otimes v_+ e^{-2\sigma} - \{h, v_+\} \otimes (e^\sigma - 1) e^{-2\sigma} + 2 v_+ \otimes h - \left\{ h, \frac{v_+ e^\sigma}{e^\sigma + 1} \right\} \otimes (e^\sigma - 1) e^{-\sigma} + (e^\sigma - 1) \otimes \left\{ h, \frac{v_+}{e^\sigma + 1} \right\} \right\} \right. + \frac{1}{4} \left( \frac{v_+ e^\sigma}{e^\sigma + 1} \otimes (e^\sigma - 1) e^\sigma + (e^\sigma - 1) e^\sigma \otimes \frac{v_+}{e^\sigma + 1} - v_+ (e^\sigma - 1) \otimes (e^\sigma - 1) \right) \left( e^{-\sigma} \otimes e^{-2\sigma} \right) \left. \right) \quad (35c)$$

and $\Delta_{S_J}(e_-) = -4(\Delta_{S_J}(v_-))^2$, where we use the denotation $\left\{ a, b \right\} := ab + ba$. It can be added that the coproduct relations (27a-b) remain still valid.

The formula (35b) and an analog of (35a) (see (43a)) were given by Kulish [10]; due to the explicit knowledge of the super-Jordanian twisting two-tensor we calculated also the coproduct (35c).

The formulae for the antipode $S_{S_J}$ look as follows:

$$S_{S_J}(h) = -h e^{2\sigma} + \frac{1}{2} (e^\sigma - 1), \quad (36a)$$

$$S_{S_J}(v_+) = -e^{-\sigma} v_+, \quad (36b)$$

$$S_{S_J}(v_-) = -v_- e^{\sigma} + \xi h v_+ e^{\sigma} - \frac{1}{2} \xi v_+ e^{\sigma} \frac{e^\sigma}{e^\sigma + 1}. \quad (36c)$$
Supplementing with $\epsilon_{sj}(v_{\pm}) = \epsilon_{sj}(h) = 0$ we obtain the complete set of formulae describing the super-Jordanian deformation $U_\xi(osp(1|2))$ as non-cocommutative Hopf algebra.

It is easy to see that the formulae (35a)–(35c) do not satisfy neither the reality conditions (13a) nor (13c), i.e. $(\Delta_{sj}(x))^\ast \neq \Delta_{sj}(x^\ast)$ for some element $x \in U_\xi(osp(1|2))$, where $\ast = \dagger$ or $\ddagger$. However, if we require that the super-Jordanian coproduct $\tilde{\Delta}_{sj} := \tilde{F}_s \Delta_{sj} \tilde{F}_s^{-1}$ satisfy the $\ast$-reality condition it is necessary and sufficient to assume the following unitarity condition for the twisting two-tensor $\tilde{F}_s$:

$$\tilde{F}_s^\ast = \tilde{F}_s^{-1}. \quad (37)$$

The condition (37) can be achieved because the twisting element (24) is not unique -- without modifying corresponding $r$-matrix (11b) it can be multiplied by suitable multiplicative factor $\Phi \in U(osp(1|2)) \otimes U(osp(1|2))$

$$\Phi = \frac{f(e^\sigma) \otimes f(e^\sigma)}{f(e^\sigma \otimes e^\sigma)}. \quad (38)$$

provided that $f(1) = 1$. It can be shown that modified twisting element

$$\tilde{F}_s = \Phi F_s, \quad (39)$$

satisfies the relations (16) and (17). Different choices of $f$ in the multiplicative factor $\Phi$ describe nonuniqueness (up to Hopf automorphism [9]) of the twist quantization of the classical $r$-matrix (11b). In particular, we can choose the element $f$ such that the twisting two-tensor will satisfy the reality condition (37). Indeed, choosing in (39) $f(\sigma) = \sqrt{\frac{1}{2}(e^\sigma + 1)}$ one obtains the formula for the super-Jordanian two-tensor

$$\tilde{F}_s = \Phi \left(1 - 4\xi \frac{v_+}{e^\sigma + 1} \otimes \frac{v_+}{e^\sigma + 1}\right) \quad (40)$$

with

$$\Phi = \sqrt{\frac{(e^\sigma + 1) \otimes (e^\sigma + 1)}{2(e^\sigma \otimes e^\sigma + 1)}}, \quad (41)$$

which satisfies with respect to the Hermitean (13a) and super-Hermitean conjugation (13c) the following unitarity condition

$$\tilde{F}_s^\ast = \tilde{F}_s^{-1} \quad \text{for } \ast = \dagger \text{ or } \ddagger, \quad (42)$$

provided that the parameter $\xi$ is purely imaginary, $\xi^\ast := \bar{\xi} = -\xi$. Such choice will modify the coproduct, $\tilde{\Delta}_{sj} = \Phi \Delta_{sj} \Phi^{-1}$, and we obtain

$$\tilde{\Delta}_{sj}(h) = h \otimes e^{-2\sigma} + 1 \otimes h + \xi v_+ e^{-\sigma} \otimes v_+ e^{-2\sigma}, \quad (43a)$$

$$\tilde{\Delta}_{sj}(v_{\pm}) = v_{\pm} \otimes 1 + e^\sigma \otimes v_{\pm}, \quad (43b)$$
\[ \tilde{s}_{SJ}(v_-) = v_- \otimes e^{-\sigma} + 1 \otimes v_- + \xi \left\{ \left\{ h, e^\sigma \right\} \otimes v_+ e^{-2\sigma} - \left\{ h, v_+ \right\} \otimes (e^\sigma - 1)e^{-2\sigma} + 2v_+ \otimes h - \left\{ h, \frac{v_+ e^\sigma}{e^\sigma + 1} \right\} \otimes (e^\sigma - 1)e^{-\sigma} + (e^\sigma - 1)\otimes \left\{ h, \frac{v_+ e^\sigma}{e^\sigma + 1} \right\} \right\}, \] (43c)

The formulae for the antipode \( \tilde{S}_{SJ} \) look as follows:

\[ \tilde{s}_{SJ}(h) = -he^{2\sigma} + \frac{1}{4}(e^{2\sigma} - 1), \] (44a)

\[ \tilde{s}_{SJ}(v_+) = -e^{-\sigma}v_+, \] (44b)

\[ \tilde{s}_{SJ}(v_-) = -v_- e^\sigma + \xi hv_+ e^\sigma - \frac{\xi}{4} v_+ e^\sigma. \] (44c)

It is easy to see that the formulae (43a–c) satisfy the reality condition \( (\tilde{s}_{SJ}(a))^* = \tilde{s}_{SJ}(a^*) \) for \( \star = \dagger, \check{\dagger} \) and any \( a \in \text{osp}(1|2) \) and the antipodes (44a–c) satisfy respectively the condition (30).

Subsequently, we can state that the relations (43a–c) and (44a–c) describe two real quantum \( \text{osp}(1|2) \) Hopf superalgebras.

The universal \( R \)-matrix has the form

\[ \tilde{R}_{SJ} = \tilde{F}_S^{21}R_j \tilde{F}_S^{-1}, \] (45)

where \( \tilde{F}_S^{-1} = \tilde{F}_S^2(\xi) = \tilde{F}_S(-\xi) \) and

\[ R_j = F_j^{21}F_j^{-1} = e^{2\sigma \otimes h} e^{-2h \otimes \sigma}. \] (46)

If \( \xi \) is purely imaginary one can show that the universal \( R \)-matrices \( R_j \) and \( \tilde{R}_{SJ} \) are unitary or superunitary, what depends on the choice of reality condition and is inherited from analogous properties of \( F_j \) and \( \tilde{F}_S \).

The twist deformations described by the formulae (43a–c) and (43c) locates whole deformation in coalgebra sector. In order to distribute “more evenly” the deformation in algebraic and coalgebraic sector one should introduce the suitable deformation map from classical to deformed \( \text{osp}(1|2) \) quantum superalgebra basis. It should be added that recently an interesting deformed basis in the algebraic sector of \( U_\xi(\text{osp}(1|2)) \) was proposed [20]. Unfortunately, these authors did not find neither the explicit formula for two-tensor \( F_S \) nor for the universal \( R \)-matrix.

### 3 Quantum \( U_\xi(\text{osp}(1|2)) \) and Deformed \( D = 1 \) Superconformal Mechanics

The \( D = 1 \) conformal algebra (7) can be extended to the \( D = 1 \) simple superconformal algebra by adding two Hermitean (real) odd supercharges \( Q \) and \( S \), describing the "supersymmetric
roots” of the momenta \( P \) and conformal momenta \( K \) (see e.g. \[21, 22, 23, 24\]). In order to have a standard description of supercharges in Hilbert space we shall use the reality condition (13a) with the antilinear antiinvolutive mapping \( \dagger \) satisfying properties (13b).

From the reality condition (13a) follow the following definitions of real supercharges

\[
Q = \sqrt{-i} v_+ , \quad S = \sqrt{-i} v_-
\]

and from (12a-b) one gets

\[
\{Q, Q\} = \frac{1}{2} \mathcal{H} , \quad \{S, S\} = \frac{1}{2} \mathcal{K} , \quad \{S, Q\} = \frac{1}{2} \mathcal{D} .
\]

The \( sp(2; \mathbb{R}) \simeq o(2, 1) \) covariance relations for the supercharges \( Q, S \) look as follows

\[
[\mathcal{H}, Q] = 0 , \quad [\mathcal{H}, S] = -i Q ,
\]

\[
[K, Q] = i S , \quad [K, S] = 0 ,
\]

\[
[D, Q] = \frac{i}{2} Q , \quad [D, S] = -\frac{i}{2} S .
\]

The relations (7) and (48a)–(48b) after taking into consideration the relation (8) and (47) can be identified with \( osp(1, 2) \) algebra.

The algebra (7) can be realized in two-dimensional phase space, generated by one pair of phase space variables \((x, p)\) satisfying Heisenberg relations \([x, p] = i\) (we put \( \hbar = 1 \)). One can assume (see e.g. [25]):

\[
\mathcal{H} = \frac{p^2}{2m} , \quad \mathcal{D} = \frac{1}{4} (px + xp) , \quad \mathcal{K} = \frac{1}{2} mx^2 .
\]

Adding one real fermionic variable satisfying the anticommutation relation

\[
\{\psi, \psi\} = 1 ,
\]

one can realize the superalgebra (48a-b) if we supplement the generators (49) by the following odd ones

\[
Q = \frac{p \cdot \psi}{\sqrt{4m}} , \quad S = \sqrt{\frac{m}{4}} x \cdot \psi.
\]

The realization (49)–(51) can be generalized to \( N \)-dimensional superconformal mechanics in curved target superspace with a suitable torsion [26, 27], with the generators described by 2N bosonic phase space variables \((x_i, p_i)\) \((i = 1, \ldots N)\) and \( N \) real vectorial fermionic variables \( \psi_i \), satisfying \( N \)-dimensional Clifford algebra.

We would like to point out that quantum \( \kappa \)-deformed \( osp(1|2) \) superalgebra, obtained via twist quantization technique, can be applied to the deformation of supersymmetric conformal mechanics. If we use the classical \( osp(1|2) \) basis, satisfying the relations (18a1), (18b1), one can
use coproduct of energy operator $\mathcal{H}$ in order to describe the two–particle interactions $^2$. Using the formulæ (12b), (47), (43b) and the property of graded tensor product one gets (recall that $i\mathcal{H} = e_+ = 4v_+^2 = 4iQ^2$ and we put $\xi = \frac{1}{2\kappa}$)

\[ \Delta(\mathcal{H}) = 4(\Delta(Q))^2 = \mathcal{H} \otimes 1 + (1 + \frac{1}{2\kappa}\mathcal{H}) \otimes \mathcal{H}, \tag{52} \]

i.e. we obtain the energy of two–particle system described by the formula

\[ \mathcal{E}_{1+2} = \mathcal{E}_1 + \mathcal{E}_2 + \frac{1}{2\kappa}\mathcal{E}_1 \cdot \mathcal{E}_2. \tag{53} \]

We see that the deformation parameter $\kappa$ describes a geometric mass parameter and its inverse $\frac{1}{\kappa}$ can be interpreted as coupling constant in superconformal two–particle dynamics. Surprisingly, the formulæ (52), (53) are the same as for the well-known bosonic Jordanian deformation of $sl(2) \simeq o(2, 1)$, given by the twist (4a); the supersymmetric corrections depending on the supercharges (17) will be however presented in coproducts of $\mathcal{D}$ and $\mathcal{K}$.

In this section we applied the quantum deformation of $osp(1|2, \mathbb{R})$ with the reality condition (13a). It is interesting to find an application of quantum deformation of $osp(1|2, \mathbb{R})$ with the reality condition (13a) which, as we expect, can be employed in the superspace formulation of dynamical models.

### 4 Outlook

The super–Jordanian twist and super–Jordanian deformation of $osp(1|2)$ should play a basic role in the description of twist quantizations of superalgebras. The role of $osp(1|2)$ in the theory of superalgebras is analogous to the role played by $sl(2)$ in the theory of Lie groups and Lie algebras. In particular any superextension of the twist quantization techniques of classical Lie algebras (see e.g. [29, 30]) requires the twist element (4a) as its basic building block. We expect that the results of this paper can help substantially for the construction of extended twist elements for arbitrary orthosymplectic algebra $osp(N|2M)$, which for $M = 1$ and arbitrary $N$ should describe $N$–extended supersymmetric conformal mechanics, and for $M = 2$ has physical interpretation as describing four–dimensional anti–de–Sitter supersymmetries.

Also it should be recalled here that the twist element (4) has been used for deformations of quantum infinite–dimensional algebras, e.g. the Yangians [31, 32]. Having an explicite form of super–Jordanian twist [24] it is a matter of standard calculation to obtain the twisted form of $osp(2|1)$ super–Yangian$^3$. Other possible application of our deformation of $osp(1|2)$ are integrable vertex models (see e.g. [34]) and $osp(1|2)$ Gaudin lattice models [35, 36]. Also the knowledge of super–Jordanian twist (14) permits to calculate the Clebsch-Gordon coefficients of $\kappa$–deformed $osp(1|2)$ by using the technique of projection operators proposed in [13].

$^2$For $D = 4$ relativistic case an attempt in this direction leading to analogous formula was proposed in [28].

$^3$The $osp(1|2)$ super–Yangian has been described recently by one of the authors [33].
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