LEFT ORDERABLE SURGERIES FOR GENUS ONE TWO-BRIDGE KNOTS

ANH T. TRAN

Abstract. For a knot $K$ in $S^3$, a rational number $r$ is called a left orderable slope for $K$ if the 3-manifold obtained from $S^3$ by $r$-surgery along $K$ has left orderable fundamental group. In this note we consider the genus one two-bridge knots $C(2m, -2n)$ in the Conway notation, where $m, n$ are integers $\geq 1$. We show that any rational number $r \in (-\infty, 0)$ is a left orderable slope for $C(2m, -2n)$. Combining this with results in [HTe1, Tr], we conclude that any rational number $r \in (-\infty, \max\{4m, 4n\})$ is a left orderable slope for $C(2m, -2n)$, where $m, n$ are integers $\geq 1$.

1. Introduction

The motivation of this note is the L-space conjecture of Boyer, Gordon and Watson [BGW] which states that an irreducible rational homology 3-sphere is an L-space if and only if its fundamental group is not left orderable. Here a rational homology 3-sphere $Y$ is an L-space if its Heegaard Floer homology $\widehat{HF}(Y)$ has rank equal to the order of $H_1(Y; \mathbb{Z})$, and a non-trivial group $G$ is left orderable if it admits a total ordering $<$ such that $g < h$ implies $fg < fh$ for all elements $f, g, h$ in $G$. A knot $K$ in $S^3$ is called an L-space knot if it admits a positive Dehn surgery yielding an L-space. It is known that non-torus alternating knots are not L-spaces, see [OS]. In view of the L-space conjecture, this would imply that any non-trivial Dehn surgery along a non-torus alternating knot produces a 3-manifold with left orderable fundamental group.

For a knot $K$ in $S^3$, a rational number $r$ is called a left orderable slope for $K$ if the 3-manifold obtained from $S^3$ by $r$-surgery along $K$ has left orderable fundamental group. As mentioned above, one would expect that any rational number is a left orderable slope for a non-torus alternating knot and, in particular, for a non-torus two-bridge knot. It is known that any rational number $r \in (-4, 4)$ is a left orderable slope for the figure eight knot, and any rational number $r \in [0, 4]$ is a left orderable slope for the hyperbolic twist knot 5$_2$, see [BGW] and [HTe2] respectively. For non-torus genus one two-bridge knots $C(2m, \pm 2n)$ in the Conway notation, where $m, n$ are integers $\geq 1$, it was shown in [HTe1, Tr] that any rational number $r \in (-4n, 4m)$ is a left orderable slope for $C(2m, 2n)$ and any rational number $r \in [0, \max\{4m, 4n\})$ is a left orderable slope for $C(2m, -2n)$. Note that $C(2, -2)$ is the trefoil knot, which is the (2, 3)-torus knot.

2000 Mathematics Subject Classification. Primary 57M27, Secondary 57M25.

Key words and phrases. Dehn surgery, left orderable, L-space, Riley polynomial, two-bridge knot.
In this note, we extend the range of left orderable slopes for $C(2m, -2n)$.

**Theorem 1.** Let $m$ and $n$ be integers $\geq 1$. Then any rational number $r \in (-\infty, 0)$ is a left orderable slope for $C(2m, -2n)$.

Combining this with results in [HTe1, Tr], we conclude that any rational number $r \in (-\infty, \max\{4m, 4n\})$ is a left orderable slope for $C(2m, -2n)$, where $m, n$ are integers $\geq 1$.

As in [BGW, HTe1, HTe2, Tr, CD], the proof of Theorem 1 is based on the existence of certain non-abelian representations of the knot group of $C(2m, -2n)$ into $SL_2(\mathbb{R})$.

This note is organized as follows. In Section 1, we study certain real roots of the Riley polynomial of $C(2m, -2n)$, whose zero locus describes all non-abelian representations of the knot group of $C(2m, -2n)$ into $SL_2(\mathbb{R})$. In Section 2, we prove Theorem 1.

2. REAL ROOTS OF THE RILEY POLYNOMIAL

For a knot $K$ in $S^3$, let $G(K)$ denote the knot group of $K$ which is the fundamental group of the complement of an open tubular neighborhood of $K$.

Let $K_{m,n}$ be the genus one two-bridge knot $C(2m, -2n)$ in the Conway notation, where $m, n$ are integers $\geq 1$. Note that $K_{m,n}$ is the mirror image of the double twist knot $J(2m, 2n)$ in [HS]. By [HS], the knot group of $K_{m,n}$ has a presentation $G(K_{m,n}) = \langle a, b \mid aw^n = w^n b \rangle$, where $a, b$ are meridians and $w = (ab^{-1})^m(a^{-1}b)^m$.

Suppose $\rho : G(K_{m,n}) \to SL_2(\mathbb{C})$ is a nonabelian representation. Up to conjugation, we may assume that

$$\rho(a) = \begin{bmatrix} M & 1 \\ 0 & M^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} M & 0 \\ 2 - y & M^{-1} \end{bmatrix}$$

where $(M, y) \in \mathbb{C}^2$ satisfies the matrix equation $\rho(aw^n) = \rho(w^n b)$. It is known that this matrix equation is equivalent to a single polynomial equation $R_{K_{m,n}}(x, y) = 0$, where $x = \text{tr} \rho(a) = M + M^{-1}$ and $R_{K_{m,n}}(x, y)$ is called the Riley polynomial of $K_{m,n}$, see [Ri]. The polynomial $R_{K_{m,n}}(x, y)$ is described via the Chebychev polynomials as follows.

Let $\{S_j(v)\}_{j \in \mathbb{Z}}$ be the Chebychev polynomials in one variable $v$ defined by $S_0(v) = 1, S_1(v) = v$ and $S_j(v) = vS_{j-1}(v) - S_{j-2}(v)$ for all integers $j$. Note that $S_j(\pm 2) = (\pm 1)^j(j+1)$ and $S_j(v) = \frac{v^{j+1} - (j+1)v^{j+1}}{s - s^{-1}}$ if $v = s + s^{-1} \neq \pm 2$. Using these identities one can prove that
The equation we have

\[ t = \text{tr} \rho(w) = 2 + (y + 2 - x^2)(y - 2)S_{m-1}^2(y), \]

\[ z = 1 + (y + 2 - x^2)S_{m-1}(y)(S_m(y) - S_{m-1}(y)). \]

See e.g. [12, Pa]. The following describes certain real roots of \( R_{K_{m,n}}(x, y) \).

**Lemma 2.1.** For every real number \( x \in \left[ \sqrt{4 - \frac{1}{mn}}, 2 \right] \), there exists a unique real number \( y = y(x) \in [2, \infty) \) such that \( R_{K_{m,n}}(x, y) = 0 \). Moreover, \( y = y(x) \) is a continuous function in \( x \in \left[ \sqrt{4 - \frac{1}{mn}}, 2 \right] \).

**Proof.** Fix a real number \( x \in \left[ \sqrt{4 - \frac{1}{mn}}, 2 \right] \). Consider \( y \in [2, \infty) \). Since \( y \geq 2 \geq x^2 - 2 \), we have \( t \geq 2 \) and \( z \geq 1 \). This implies that \( zS_{n-1}(t) - S_{n-2}(t) \geq S_{n-1}(t) - S_{n-2}(t) > 0 \). The equation \( R_{K_{m,n}}(x, y) = 0 \) is then equivalent to

\[ \begin{align*}
(S_n(t) - zS_{n-1}(t)) & \left( zS_{n-1}(t) - S_{n-2}(t) \right) = 0.
\end{align*} \]

Let \( P(y) \) denote the LHS of equation (2.1). Since \( S_n(t) + S_{n-2}(t) = tS_{n-1}(t) \) and \( S_n(t)S_{n-2}(t) = S_{n-1}^2(t) - 1 \), we have

\[ P(y) = (z^2 - tz + 1)S_{n-1}^2(t) - 1. \]

By a direct calculation, using \( S_m^2(y) + S_{m-1}^2(y) = 1 + yS_m(y)S_{m-1}(y) \), we have

\[ z^2 - tz + 1 = (z - 1)^2 - (t - 2)z \]

\[ = (y + 2 - x^2)S_{m-1}^2(y)[(y + 2 - x^2)(S_m(y) - S_{m-1}(y) \]

\[ - (y - 2)(1 + (y + 2 - x^2)S_{m-1}(y)(S_m(y) - S_{m-1}(y)))] \]

\[ = (y + 2 - x^2)S_{m-1}^2(y)[4 - x^2 + (y + 2 - x^2)(y - 2)S_{m-1}^2(y)] \]

\[ = (y + 2 - x^2)S_{m-1}^2(y)(t + 2 - x^2). \]

Hence \( P(y) = (y + 2 - x^2)S_{m-1}^2(y)(t + 2 - x^2)S_{n-1}^2(t) - 1. \)

For a fixed real number \( x \in \left[ \sqrt{4 - \frac{1}{mn}}, 2 \right] \), it is easy to see that \( P(y) \) is a strictly increasing function on \( y \in [2, \infty) \). Moreover, \( \lim_{y \to \infty} P(y) = \infty \) and

\[ \lim_{y \to 2^+} P(y) = (4 - x^2)^2m^2n^2 - 1 \leq 0. \]

Hence there exists a unique real number \( y = y(x) \in [2, \infty) \) such that \( P(y) = 0 \). Moreover, by the inverse function theorem \( y = y(x) \) is a continuous function in \( x \in \left[ \sqrt{4 - \frac{1}{mn}}, 2 \right] \).

This completes the proof of Lemma 2.1. \( \square \)
3. Proof of Theorem 1

Recall that $K_{m,n}$ is the genus one two-bridge knot $C(2m, -2n)$ in the Conway notation, where $m, n$ are integers $\geq 1$. Let $M_{m,n}^r$ denote the 3-manifold obtained from $S^3$ by $r$-surgery along $K_{m,n}$. Using Lemma 2.1, we first prove the following.

**Proposition 3.1.** For each rational number $r \in (-\infty, 0)$, there exists a representation of the fundamental group of $M_{m,n}^r$ into $SL_2(\mathbb{R})$.

**Proof.** Let $\theta_0 = \arccos \frac{1}{4mn}$. Note that $0 < \theta_0 \leq \frac{\pi}{6}$. For $\theta \in (0, \theta_0)$ we let $M = e^{i\theta}$. By Lemma 2.1, there exists a unique real number $y = y(\theta) \in (2, \infty)$ such that $R_{K_{m,n}}(x, y) = 0$, where $x = M + M^{-1} = 2 \cos \theta$. Then there exists a non-abelian representation $\rho : G(K_{m,n}) \to SL_2(\mathbb{C})$ such that

$$
\rho(a) = \begin{bmatrix} M & 1 \\ 0 & M^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} M & 0 \\ 2 - y & M^{-1} \end{bmatrix}.
$$

We now determine the image of a canonical longitude under $\rho$. Recall that $K_{m,n}$ is the mirror image of the double twist knot $J(2m, 2n)$ in [HS], and hence the canonical longitude corresponding to the meridian $\mu = a$ is $\lambda = (w^m \overline{w}^n)^{-1}$, where $\overline{w}$ is the word obtained by writing $w$ in the reversed order. Moreover, we have $\rho(\lambda) = \begin{bmatrix} L & * \\ 0 & \bar{L}^{-1} \end{bmatrix}$, where

$$
L = \frac{M^{-1}(S_m(y) - S_{m-1}(y)) - M(S_m(y) - S_{m-1}(y))}{M(S_m(y) - S_{m-1}(y)) - M^{-1}(S_m(y) - S_{m-1}(y))}.
$$

See e.g. [HS, Pe, Tr]. Let $\alpha = S_m(y) - S_{m-1}(y)$ and $\beta = S_{m-1}(y) - S_{m-2}(y)$. Then

$$
L = -\frac{M^{-1}\alpha - M\beta}{M\alpha - M^{-1}\beta}.
$$

Note that $\alpha > \beta > 0$, since $y > 2$.

It is easy to see that $|L| = \sqrt{LL} = 1$, where $\bar{L}$ denotes the complex conjugate of $L$. Moreover, by a direct calculation, we have

$$
\begin{align*}
\Re(L) &= \frac{(2\alpha\beta - (\alpha^2 + \beta^2) \cos 2\theta)/|M\alpha - M^{-1}\beta|^2}, \\
\Im(L) &= \frac{(\alpha^2 - \beta^2) \sin 2\theta/|M\alpha - M^{-1}\beta|^2}.
\end{align*}
$$

Since $\alpha > \beta > 0$ and $0 < \theta < \frac{\pi}{6}$, we have $\Im(L) > 0$. Let

$$
\varphi = \varphi(\theta) := \arccos \frac{(2\alpha\beta - (\alpha^2 + \beta^2) \cos 2\theta)/|e^{i\theta} \alpha - e^{-i\theta} \beta|^2}.
$$

Note that $\varphi \in (0, \pi)$ and $L = e^{i\varphi}$.

The function $f(\theta) := -\frac{\varphi}{\theta}$ is a continuous function in $\theta \in (0, \theta_0)$. As $\theta \to 0^+$ we have $M \to 1$ and $L \to -1$, so $\varphi \to \pi$. As $\theta \to \theta_0^-$ we have $y \to 2$ and $\alpha, \beta \to 1$, so $L \to 1$ and $\varphi \to 0$. This implies that

$$
\lim_{\theta \to 0^+} -\frac{\varphi}{\theta} = -\infty \quad \text{and} \quad \lim_{\theta \to \theta_0^-} -\frac{\varphi}{\theta} = 0.
$$
Hence the image of the function $f(\theta)$ contains the interval $(-\infty, 0)$.

Suppose $r = \frac{p}{q}$ is a rational number such that $r \in (-\infty, 0)$. Since $r$ belongs to the image of $f$, we have $r = -\frac{\theta}{\pi}$ for some $\theta \in (0, \theta_0)$. Then $M^p L^q = e^{i(p\theta + q\varphi)} = 1$, and hence $\rho(\mu^p \lambda^q) = 1$. This means that the non-abelian representation $\rho : G(K_{m,n}) \to SL_2(\mathbb{C})$ extends to a representation $\rho : \pi_1(M^r_{m,n}) \to SL_2(\mathbb{C})$.

Finally, since $2 - y < 0$, a result in [Kh, page 786] implies that $\rho$ can be conjugated to an $SL_2(\mathbb{R})$-representation. This completes the proof of Proposition 3.1.

We now finish the proof of Theorem 1. Suppose $r$ is a rational number such that $r \in (-\infty, 0)$. By Proposition 3.1 there exists a representation of $\rho : \pi_1(M^r_{m,n}) \to SL_2(\mathbb{R})$. Since $K_{m,n}$ is a knot of genus one, by [HTe1, Lemma 7.1], the representation $\rho$ lifts to a representation $\rho : \pi_1(M^r_{m,n}) \to \tilde{SL}_2(\mathbb{R})$, where $\tilde{SL}_2(\mathbb{R})$ is the universal covering group of $SL_2(\mathbb{R})$. Note that $M^r_{m,n}$ is an irreducible 3-manifold (by [HTh]) and $\tilde{SL}_2(\mathbb{R})$ is a left-orderable group (by [Be]). Hence [BRW, Theorem 1.1] implies that $\pi_1(M^r_{m,n})$ is a left-orderable group.

Acknowledgements

The author was partially supported by a grant from the Simons Foundation (#354595).

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Department of Mathematical Sciences, The University of Texas at Dallas, 800 West Campbell Road FO 35, Richardson, TX 75080-3021, USA

E-mail address: att140830@utdallas.edu