EIGENVALUES OF PERIODIC DIFFERENCE OPERATORS ON LATTICE OCTANT

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Abstract. Consider a difference operator $H$ with periodic coefficients on the octant of the lattice. We show that for any integer $N$ and any bounded interval $I$, there exists an operator $H$ having $N$ eigenvalues, counted with multiplicity on this interval, and does not exist other spectra on the interval. Also right and to the left of it are spectra and the corresponding subspaces have an infinite dimension. Moreover, we prove similar results for other domains and any dimension. The proof is based on the inverse spectral theory for periodic Jacobi operators.

1. Introduction and main results

We consider a operator $H = H_1 + H_2 + V$ acting on domain $D$ and $V$ is the multiplication operator on $\ell^2(D)$:

$$ (Vf)_z = V(z)f_z, \quad f = f_z, \quad z = (x,y) \in D = \mathbb{Z}_+^{d_1} \times \mathbb{Z}_+^{d_2} \subset \mathbb{Z}^d, $$ (1.1)

$$ d_1 + d_2 = d \geq 2, \quad d_1, d_2 \geq 0. $$

Here $H_1$ is the difference operator on the octant $\mathbb{Z}_+^{d_1}$ with the Dirichlet boundary conditions on the boundary $\partial \mathbb{Z}_+^{d_1}$ (i.e., $g = 0$ on $\mathbb{Z}_+^{d_1} \setminus \mathbb{Z}_+^{d_1}$ in (1.2)) and $H_2$ is the difference operator on $\mathbb{Z}_+^{d_2}$ defined by

$$ (H_1 g)_x = \sum_{i=1}^{d_1} (a_{x-c_i} g_{x+c_i} + a_i g_{x+c_i}), \quad x \in \mathbb{Z}_+^{d_1}, \quad g = (g_x) \in \ell^2(\mathbb{Z}_+^{d_1}), $$

$$ (H_2 u)_y = \sum_{i=d_1+1}^{d} (a_{y-c_i} u_{y+c_i} + a_i u_{y+c_i}), \quad y \in \mathbb{Z}_+^{d_2}, \quad u = (u_y) \in \ell^2(\mathbb{Z}_+^{d_2}). $$ (1.2)

Here $\varsigma_1 = (1,0,0,..), ..., \varsigma_d = (0,0,0,..,1)$ is the standard basis in $\mathbb{Z}^d$ and $\mathbb{Z}_+ = \{1,2,3,...\}$. We assume that the potential $V$ and the coefficients $a_i$ are real octant periodic, i.e., they have decompositions (1.3).

In order to define octant periodic functions we introduce a sequence $\omega = (\omega_j)^m_1$, where $\omega_j = +$ or $\omega_j = -$ and the set of all such sequences we denote by $\Omega_m$. For any $\omega \in \Omega_m$ we define the octants $Z_{\omega} \subset \mathbb{Z}^m$ by

$$ Z_{\omega} = Z_{\omega_1} \times Z_{\omega_2} \times ... \times Z_{\omega_m}, \quad \omega = (\omega_j)^m_1 \in \Omega_m, \quad Z_- = \mathbb{Z} \setminus \mathbb{Z}_+ = \{..., -3, -2, -1, 0\}. $$

In particular, if $d = 2$ and $\omega = (+,+)$, then we have the positive octant $Z_{\omega} = \mathbb{Z}_+^2$. Note that two axial lines $(x_1 = 1, x_2 = 1)$ divide space $\mathbb{Z}^2$ into four quadrants, each with a coordinate signs from $(-,-)$ to $(+,+)$.

Date: April 30, 2019.

Key words and phrases. eigenvalues, discrete Schrödinger operator, lattice.
A function $F(z), z = (x, y) \in \mathbb{Z}_+^{d_1} \times \mathbb{Z}^{d_2}$ is called octant periodic if it has the decomposition

$$F(x, y) = \sum_{\omega \in \Omega_{d_2}} F_{\omega}(x, y) \chi_{\omega}(y),$$

(1.3)

where $\chi_{\omega}$ is the characteristic function of the octant $\mathbb{Z}_{\omega}$ and the function $F_{\omega}(z), z = (x, y)$ is periodic in $\mathbb{Z}^d$ and satisfies

$$F_{\omega}(z + p_\omega \varsigma_i) = F_{\omega}(z), \quad \text{for all } (z, i) \in \mathbb{Z}^d \times \mathbb{N}_d$$

(1.4)

for some constants $p_\omega = p_\omega(\omega) > 0$, where $\mathbb{N}_d = \{1, 2, ..., d\}$.

For each $\omega \in \Omega_d$ we define difference operators $H_{\omega}$ with periodic functions $a_{\omega,i}^\omega, V_{\omega}$ on $\mathbb{Z}^d$ by

$$H_{\omega} f = \sum_{i=1}^{d} \left( (a_{\omega,i}^\omega f_{z+i} + a_{-\omega,i}^\omega f_{z-i}) \right) + V_{\omega}, \quad f = (f_z) \in \ell^2(\mathbb{Z}^d).$$

(1.5)

It is well known that the spectrum of each operator $H_{\omega}$ is absolutely continuous and is an union of a finite number of bounded intervals. In the next theorem we show the existence of eigenvalues of $H$ with some octant periodic functions $a_{\omega,i}^\omega$ and $V_{\omega}$.

**Theorem 1.1.** i) Let an operator $H$ be given by (1.1), (1.2) with octant periodic coefficients. Then

$$\bigcup_{\omega \in \Omega_{d_2}} \sigma(H_{\omega}) \subseteq \sigma_{\text{ess}}(H).$$

(1.6)

ii) Let $I \subset \mathbb{R}$ be a finite open interval. Then for any integer $N \geq 0$ there exists an operator $H$ given by (1.1), (1.2) and having $N$ eigenvalues, counted with multiplicity on this interval, and does not exist other spectra on the interval. Also right and to the left of it are spectra and the corresponding subspaces have an infinite dimension.

**Remark.** 1) The result of i) is standard and its proof is based on the Floquet theory.

2) We do not know any information about absolutely continuous spectrum of $H$. We only show that the operator $H$ can have any number $N \geq 1$ of eigenvalues for specific coefficients.

3) In the case of the continuous Schrödinger operator with octant periodic potentials on $\mathbb{R}^d$ the top of the spectra is a isolated simple eigenvalue for specific potentials [34]. In the discrete case we have no any information about it.

1.1. **Historical review.** Firstly we discuss the continuous case. The one dimensional model of octant periodic potentials on $\mathbb{R}^1$ is considered by Korotyaev [22], [21]. The corresponding multidimensional model of octant periodic potentials is considered recently by Korotyaev and Moller [34]. Hempel and Kohlmann [7],[8] discuss different types of dislocation problem in solid state physics.

Secondly, we discuss the discrete case. Local defects are considered by different authors. For the discrete Schrödinger operators most of the results were obtained for uniformly decaying potentials for the $\mathbb{Z}$ case, see, for example, [35]. There are results about spectral properties of discrete Schrödinger operators on the lattice $\mathbb{Z}^d$, the simplest example of periodic graphs. Schrödinger operators with decaying potentials on the lattice $\mathbb{Z}^d$ are considered by Boutet de Monvel-Sahbani [4], Hundertmark-Simon [10], Isozaki-Korotyaev [15], Isozaki-Morioka [17], Korotyaev-Moller [28], Nakamura [38], Parra and Richard [40], Rosenblum-Solomjak [42], Shaban-Vainberg [43] and see references therein. Gieseker-Knörrer-Trubowitz [11] consider Schrödinger operators with periodic potentials on the lattice $\mathbb{Z}^2$. Korotyaev-Kutsenko [25]
study the spectra of the discrete Schrödinger operators on graphene nano-ribbons in external electric fields. The inverse spectral theory for the discrete Schrödinger operators with finitely supported potentials on some graphs were discussed by Ando [1], Ando-Isozaki-Morioka [2, 3], Isozaki-Korotyaev [15]. Scattering on periodic metric graphs was considered by Korotyaev-Saburova [29]. Laplacians on periodic graphs with non-compact perturbations and the stability of their essential spectrum were considered in [9], [44]. Korotyaev-Saburova [30], [31] considered Schrödinger operators with periodic potentials on periodic discrete graphs with by so-called guides, which are periodic in some directions and finitely supported in others. They described some properties of so-called guided spectrum. Note that line defects on the lattice were considered in [5], [9], [35], [36], [39]. Hempel, Kohlmann, Stautz and Voigt [9] discussed nano-tubes with a dislocation.

We shortly describe the plan of the paper. In Section 2 we present the main properties of the periodic Jacobi operator on the half lattice $\mathbb{Z}_+$. In Section 3 we discuss half-solid models on the lattice $\mathbb{Z}$. Section 4 is a collection of needed facts about difference operators on $D$, when the variables are separated. In Section 5 we prove main theorems.

2. Periodic Jacobi operators on the half-lattice

2.1. Periodic Jacobi operators. Let $N_\pm = \{\pm 1, \pm 2, \pm 3, \ldots\}$. Recall that $N_+ = \mathbb{Z}_+ = \{1, 2, 3, \ldots\}$ and $\mathbb{Z}_- = \mathbb{Z} \setminus \mathbb{Z}_+$. We consider the $p$-periodic Jacobi operator $J_\pm$ on $\ell^2(N_\pm)$ given by

\[
(J_{\pm f})_x = a_x f_{x-1} + a_x f_{x+1} + b_x f_x, \quad f_0 = 0, \quad x \in N_\pm,
\]

and in particular,

\[
\begin{align*}
(J_- f)_1 &= a_{-1} f_{-1} + a_{-2} f_{-2} + b_{-1} f_{-1}, \\
(J_- f)_{-1} &= a_{-2} f_{-2} + b_{-1} f_{-1}, \quad f_1 = 0,
\end{align*}
\]

where $a_x > 0, b_x \in \mathbb{R}, x \in \mathbb{Z}$ are $p$ periodic sequences and the product $\prod_{j=1}^{p} a_j = 1$. It is well known that the spectrum of $J_+$ has absolutely continuous part $\sigma_{ac}(J_+) = \sigma_{ac}(J_-)$ (the union of the bands $\sigma_0, \sigma_n, n \in N_{p-1}$ separated by gaps $\gamma_n$) plus at most one eigenvalue of $J_-$ or $J_+$ in each non-empty gap $\gamma_x, n \in N_{p-1}$. The bands $\sigma_n$ and gaps $\gamma_n$ are given by

\[
\sigma_0 = [\lambda_1^+, \lambda_1^-], \quad \sigma_n = [\lambda_n^+, \lambda_{n+1}^-], \quad \gamma_n = (\lambda_n^+, \lambda_n^-), \quad n \in N_{p-1},
\]

\[
\lambda_1^- < \lambda_1^+ < \ldots < \lambda_{p-1}^- < \lambda_p^+ < \lambda_p^-,
\]

(see Fig. 1) and recall that $N_p = \{1, 2, \ldots, p\}$. The bands satisfy (see e.g., [37], [33])

\[
\sum_{n=0}^{p-1} |\sigma_n| \leq 4.
\]

If a gap $\gamma_n$ is degenerate, i.e. $|\gamma_n| = 0$, then the corresponding segments $\sigma_n, \sigma_{n+1}$ merge. We introduce fundamental solutions $\varphi = (\varphi_x(\lambda))_{x \in \mathbb{Z}}$ and $\vartheta = (\vartheta_x(\lambda))_{x \in \mathbb{Z}}$ of the equation

\[
a_{x-1} f_{x-1} + a_x f_{x+1} + b_x f_x = \lambda f_x, \quad (\lambda, x) \in \mathbb{C} \times \mathbb{Z},
\]

with initial conditions $\varphi_0 = \vartheta_1 = 0$ and $\varphi_1 = \vartheta_0 = 1$. Recall that the zeros of $\varphi_x(\lambda)$ are real, simple and strictly interlace those of $\varphi_{x+1}(\lambda)$. Moreover, the zeros of $\vartheta_x(\lambda)$ are real, simple and strictly interlace those of $\varphi_x(\lambda)$. Define the Lyapunov function $\mathcal{F}$ by

\[
\mathcal{F}(\lambda) = \frac{1}{2} (\varphi_{p+1}(\lambda) + \vartheta_p(\lambda)), \quad \lambda \in \mathbb{C}.
\]
We recall the well known asymptotics as \( \lambda \to \infty \):

\[
\vartheta_x(\lambda) = -\frac{a_0\lambda^{x-2}(1 + O(\frac{1}{\lambda}))}{a_{1..a_{x-1}}}, \quad \varphi_x(\lambda) = \frac{\lambda^{x-1}(1 + O(\frac{1}{\lambda}))}{a_{1..a_{x-1}}},
\]

The functions \( \mathfrak{F}, \varphi_x \) and \( \vartheta_x, x \geq 1 \) are polynomials of \((\lambda, a, b) \in \mathbb{C}^{2p+1}\). We have the following identities

\[
\sigma_{ac}(J_{\pm}) = \{ \lambda \in \mathbb{R} : |\mathfrak{F}(\lambda)| \leq 1 \} \quad \text{and} \quad (-1)^{p-n} \mathfrak{F}(\lambda_n^+) = 1, \quad n = 0, 1, 2, \ldots, p.
\]

For any sequences \( u = (u_x)_{x=\infty}^{\infty}, f = (f_x)_{x=\infty}^{\infty} \) we define the Wronskian

\[
\{f, u\}_x = a_x(f_x u_{x+1} - u_x f_{x+1}), \quad x \in \mathbb{Z}.
\]

If \( f, u \) are some solutions of (2.4), then \( \{f, u\}_x \) does not depend on \( x \). In particular, we have

\[
\vartheta_p \varphi_{p+1} - \varphi_p \vartheta_{p+1} = 1,
\]

since \( \{\vartheta, \varphi\}_p = a_p(\vartheta_p \varphi_{p+1} - \varphi_p \vartheta_{p+1}) = \{\vartheta, \varphi\}_0 = a_0 \). Thus we obtain

\[
\mathfrak{F}_a^2 - \mathfrak{F}_b^2 + 1 = -\varphi_p \vartheta_{p+1}.
\]

We define the Jacobi operator \( J_D \) on \( \mathbb{N}_{p-1} \) with the Dirichlet boundary conditions by

\[
(J_D f)_x = a_{x-1} f_{x-1} + a_x f_{x+1} + b_x f_x, \quad x \in \mathbb{N}_{p-1}, \quad f_0 = f_p = 0.
\]

Denote its corresponding Dirichlet eigenvalues by \( \mu_n, n \in \mathbb{N}_{p-1} \). It is well known that the eigenvalues \( \mu_n \) are simple and are zeros of the polynomial \( \varphi_p(\lambda) \) and satisfy

\[
\mu_n \in [\lambda_n^-, \lambda_n^+] \quad \text{for all } n \in \mathbb{N}_{p-1}.
\]

### 2.2. Riemann surface.

For the operator \( J_{\pm} \) we introduce the two-sheeted Riemann surface \( \Lambda \) obtained by joining the upper and lower rims of two copies of the cut plane \( \mathbb{C} \setminus \sigma_{ac}(J_{\pm}) \) in the usual (crosswise) way. We denote the \( n \)-th gap on the first physical sheet \( \Lambda_1 \) by \( \gamma_n^1 \) and the same gap but on the second nonphysical sheet \( \Lambda_2 \) by \( \gamma_n^2 \), and set a circle gap \( \gamma_n^c \) by

\[
\gamma_n^c = \overline{\gamma_n^1} \cup \overline{\gamma_n^2} \quad \text{for all } n \in \mathbb{N}_{p-1},
\]

see Fig. 2. Note that \( \Lambda \) is the two-sheeted Riemann surface for \( \sqrt{1 - \mathfrak{F}^2(\lambda)} \). The polynomial \( \mathfrak{F}(\lambda) \) is real on the real line. We use the standard definition of the root: \( \sqrt{1} = 1 \) and fix the branch of the function \( f(\lambda) = \sqrt{1 - \mathfrak{F}^2(\lambda)} \) on \( \mathbb{C} \) by demanding

\[
f(\lambda) = \sqrt{1 - \mathfrak{F}^2(\lambda)} < 0, \quad \text{for } \lambda \in (\lambda_{p-1}^+, \lambda_p^-).
\]
2.3. Bloch functions. Define the cut spectral domain $L$ and the cut quasimomentum domain $K_p$ by

$$L = \mathbb{C} \setminus \cup_{n=1}^{p-1} \gamma_n, \quad K_p = \left\{ -\pi p < \text{Re} \ k < 0 \right\} \setminus \cup_{n=1}^{p-1} \Gamma_n,$$

$$\Gamma_n = (\pi n + i h_n, \pi n - i h_n), \quad \text{ch} \ h_n = \max_{\lambda \in \gamma_n} |\mathfrak{F}(\lambda)| = (-1)^{n-p} \mathfrak{F}(\alpha_n),$$

where $h_n \geq 0$ is defined by the equation $\cosh h_n = |\mathfrak{F}(\alpha_n)|$, where $\alpha_n$ is a zero of $\mathfrak{F}'(\lambda)$ in the close gap $[\lambda_n^-, \lambda_n^+]$. For each Jacobi operator $J_\pm$ there exist a unique conformal mapping $k : L \to K_p$ such that $\mathfrak{F}(\lambda) = \cos k(\lambda)$, $\lambda \in L$ and following identities and asymptotics hold true:

$$k(\mathbb{C}) = K_p, \quad (\mathbb{C} \pm) = K_p \cap \mathbb{C} \pm, \quad (k(\gamma_n \pm \text{i}0) = \Gamma_n \cap \mathbb{C} \pm, \quad k(\lambda) \to \pm i \infty \quad \text{as} \ \text{Im} \ \lambda \to \pm \infty,$$

see [11]. The quasimomentum $k(\lambda)$ satisfies $\sigma_{ac}(J_\pm) = \{ \lambda \in \mathbb{R}; \ \text{Im} k(\lambda) = 0 \}$. We define the Bloch functions functions $\psi^\pm_x$ and the Weyl-Titchmarsh function $m_\pm$ by

$$\psi^\pm_x(\lambda) = \vartheta_x(\lambda) + m_\pm(\lambda) \varphi_x(\lambda), \quad m_\pm(\lambda) = \frac{\mathfrak{F}_o(\lambda) \pm i \sin k(\lambda)}{\varphi_p(\lambda)}, \quad \lambda \in \Lambda_1,$$

where $\sin k(\lambda) = \sqrt{1 - \mathfrak{F}^2(\lambda)}$, $\lambda \in \Lambda_1$ and satisfies (2.12). Due to the properties of $\sin k(\lambda) = \sqrt{1 - \mathfrak{F}^2(\lambda)}$, $\lambda \in \Lambda_1$ the functions $\psi^\pm_x$ and $m_\pm$ have analytic extensions from the first sheet $\Lambda_1$ onto the whole two-sheeted Riemann surface $\Lambda$. Let $k = u(\lambda) + iv(\lambda)$.

2.4. Eigenvalues and resonances. It is well known (see e.g. [12]) that, for each finitely supported $f \in \ell^2(\mathbb{N}_\pm)$, $f \neq 0$, the function $g(\lambda) = ((J_\pm - \lambda)^{-1} f, f)$ has a meromorphic extension from the physical sheet $\Lambda_1$ into the whole Riemann surface $\Lambda$. It is well known that the function $g(\lambda)$ has only tree following kinds of singularity on $\Lambda$:

- $g$ has a pole at some $\lambda_o \in \bigcup_{n=1}^{p-1} \gamma^{(1)}_n \subset \Lambda_1$ for some $f$ and $\lambda_o$ is an eigenvalue of $J_\pm$.

- $g$ has a pole at some $\lambda_o \in \bigcup_{n=1}^{p-1} \gamma^{(2)}_n \subset \Lambda_2$ for some $f$ and $\lambda_o$ is called a resonance of $J_\pm$.

- The function $g(\lambda_o + z^2)$ has a pole at $z = 0$ for some $\lambda_o \in \{ \lambda^+_n, \lambda^-_n \}, n = \mathbb{N}_{p-1}$ and $\lambda_o$ is called a virtual state of $J_\pm$.

We call $\lambda_o$ a state if $\lambda_o$ is an eigenvalue or a resonance or a virtual state. It is well known that if some gap $\gamma_n \neq \emptyset$, $n \in \mathbb{N}_{p-1}$, see e.g., [12], then the operator $J_\pm$ has exactly one state $\mu^+_n$ on each ”circle” gap $\gamma_n$ and there are no others. The projection of $\mu^+_n \in \gamma_n$ onto the complex plane coincides with the eigenvalue $\mu_n$ of the operator $J_D$ with the Dirichlet boundary conditions (2.10). There are no other states of $J_\pm$. If there are exactly $n_o \geq 1$ non-degenerate finite gaps
in the spectrum of $\sigma_{ac}(J_{\pm})$, then the operator $J_{\pm}$ has exactly $n_{\ast}$ states; the closed gaps $\gamma_{n} = \emptyset$ and the semi-infinite gaps $(-\infty, \lambda_{0}^{+})$ and $(\lambda_{p}^{+}, \infty)$ do not contribute any states. In particular, if $\gamma_{n} = \emptyset$ for all $n \in \mathbb{N}_{p-1}$, then all $\sigma_{n} = 1, b_{n} = 0$ (see e.g., [23]) and thus $J_{\pm}$ has no states. A more detailed description of the states of $J_{\pm}$ is given below.

Lemma 2.1. Let $\lambda \in \Lambda_{1} \setminus \{\mu_{j}, j \in \mathbb{N}_{p-1}\}$. Then

$$m_{\pm}(\lambda) = \frac{e^{\pm ik(\lambda)} - \varphi_{p}(\lambda)}{\varphi_{p}(\lambda)} = \frac{\varphi_{p+1}(\lambda) - e^{\mp ik(\lambda)}}{\varphi_{p}(\lambda)},$$

$$m_{+}m_{-} = -\frac{\varphi_{p+1}}{\varphi_{p}},$$

and

$$\psi_{0}^{\pm}(\lambda) = 1, \quad \psi_{p}^{\pm}(\lambda) = e^{\pm ik(\lambda)},$$

$$\psi_{p+1}(\lambda) = e^{\pm ik(\lambda)}m_{\pm}(\lambda).$$

Remark. This relation (2.17) considered at zeros of $\varphi_{p}(\lambda)$ shows that if $\mu_{n}^{\pm} \in \gamma_{n}^{c} \neq \emptyset$, then $m_{\pm}$ has simple pole at $\mu_{n}^{\pm}$ and the function $m_{\pm}$ is regular at $\mu_{n}^{\pm}$.

Proof. Let for shortness $\vartheta_{x} = \vartheta_{x}(\cdot)$, $\varphi_{x} = \varphi_{x}(\cdot)$,... We consider the case $+$, the proof for the case $-$ is similar. Using the definitions $\tilde{\gamma}_{n}^{c} = \tilde{\gamma}_{n}^{c} = \cos k$ and $\tilde{\gamma}_{0} = \frac{\varphi_{p+1} - \varphi_{p}}{2}$ we obtain

$$e^{ik} - \varphi_{p} = \tilde{\gamma}_{n}^{c} + i \sin k - \varphi_{p} = \tilde{\gamma}_{0} + i \sin k,$$

$$\varphi_{p+1} - e^{-ik} = \varphi_{p+1} - \tilde{\gamma}_{n}^{c} + i \sin k = \tilde{\gamma}_{0} + i \sin k,$$

which yields (2.16). From (2.9), (2.8) we obtain

$$\varphi_{p}^{2}m_{+}m_{-} = \tilde{\gamma}_{0}^{2} - (\tilde{\gamma}_{0}^{2} - 1) = 1 + \varphi_{p} \varphi_{p+1} = -\varphi_{p} \varphi_{p+1},$$

which yields (2.17). We show (2.18). From (2.3) at $n = 0$ and $n = p$ and (2.16) have

$$\psi_{0}^{+} = 1, \quad \psi_{p}^{+} = \vartheta_{p} + e^{ik} - \varphi_{p} = e^{ik}.$$ (2.19)

From (2.16), (2.8) we have

$$\psi_{p+1}^{+} = \varphi_{p+1} + \frac{e^{ik} - \varphi_{p}}{\varphi_{p}} \varphi_{p+1} = \frac{\varphi_{p+1} e^{ik} - 1}{\varphi_{p}} = e^{ik} m_{+},$$

which yields (2.18). □

We recall the results from [12]

Lemma 2.2. Let a finite gap $\gamma_{n} \neq \emptyset$ for some $n \in \mathbb{N}_{p-1}$. Then

i) the operator $J_{\pm}$ has exactly one state $\mu_{n}^{\pm}$ on $\gamma_{n}^{c} = \gamma_{n}^{c} \cap \gamma_{n}^{o}$ and its projection on $\mathbb{C}$ coincides with the Dirichlet eigenvalue $\mu_{n}^{\pm}$.

ii) the state $\mu_{j}^{+} \in \gamma_{n}^{c}$ iff the state $\mu_{j}^{+} \in \gamma_{n}^{o}$. Moreover, if $\mu_{n}^{\pm} \in \{\lambda_{n}^{+}, \lambda_{n}^{-}\}$ is a virtual state, then $\mu_{n}^{+} = \mu_{n}^{-}$.

iii) Let $\mu_{j}^{+} \in \gamma_{n}^{c}$ be an eigenvalue of $J_{+}$. Then $\varphi(\mu_{j}^{+}) \in \ell^{2}(\mathbb{Z}_{+})$.

iv) Let $\lambda_{o} \in \gamma_{n}^{c} \neq \emptyset$ be a state of $J_{+}$ for some $n \in \mathbb{N}_{p-1}$. Then

$$\begin{align*}
\text{if } |\varphi_{p+1}(\lambda_{o})| < 1 & \Rightarrow \lambda_{o} \in \gamma_{n}^{c}, \\
\text{if } |\varphi_{p+1}(\lambda_{o})| > 1 & \Rightarrow \lambda_{o} \in \gamma_{n}^{c}, \\
\text{if } |\varphi_{p+1}(\lambda_{o})| = 1 & \Rightarrow \lambda_{o} \in \{\lambda_{n}^{+}, \lambda_{n}^{-}\}
\end{align*}$$ (2.20)
2.5. **Inverse problem.** We need the following results from the inverse spectral theory for the operator \( J_x \) on the half-line, in the form convenient for us. Let \( v_{1x} = \log a_x \in R, v_{2x} = b_x \). We can take \((a_x, b_x)\) as a vector \( v \) in the form:

\[
v = (v_{1x}, v_{2x})^p \in \mathcal{H}^2, \quad \mathcal{H} := \left\{ b \in \mathbb{R}^p : \sum_{i=1}^{p} b_x = 0 \right\}. \tag{2.21}
\]

Here we have \((v_{1x})^p \in \mathcal{H}^2\), since \( \prod_{1} a_x = 1 \). Using symmetrization, we construct a gap length mapping \( \psi : \mathcal{H}^2 \to \mathbb{R}^{2p-2} \) by:

\[
v \to \psi(v) = (\psi_x(v))^{p-1}, \quad \psi_x = (\psi_{1x}, \psi_{2x}) \in \mathbb{R}^2,
\]

and the components have the form:

\[
\psi_{1n} = \frac{\lambda^+_n + \lambda^-_n}{2} - \mu_n, \quad \psi_{2n} = \left| \frac{g_n}{4} - \psi^2_{1n} \right|^{\frac{1}{2}} \eta_n, \quad \begin{cases} 
\eta_n = 1 & \text{if } \mu_n^+ \in \gamma^+_n, \\
\eta_n = -1 & \text{if } \mu_n^+ \in \gamma^-_n, \\
\eta_n = 0 & \text{if } \mu_n^+ \in \{\lambda^+_n, \lambda^-_n\}
\end{cases} \tag{2.22}
\]

Due to (2.20) we have \( \eta_n = \text{sign} \log |\varphi_{p+1}(\mu_n)| \) and note that \( \varphi_{p+1}(\mu_n) \neq 0 \). In order to construct the vector \( \psi \) we need: the gap length \( |\gamma_n| \), \( \psi_{1n} \) and the sign \( \eta_n \) for all \( n \in \mathbb{N}_{p-1} \). We formulate the result about the mapping \( \psi \), which is similar to results from [23] and it is some analogous of the gap-length mapping for periodic Schrödinger operators on \( \mathbb{R}^+ \) from [24].

**Theorem 2.3.** The mapping \( \psi : \mathcal{H}^2 \to \mathbb{R}^{2p-2} \) given by (2.22) is a real analytic isomorphism between \( \mathcal{H}^2 \) and \( \mathbb{R}^{2p-2} \).

**Proof.** In [23] we consider the mapping, where \( \mu_n \) are the zeroes of \( \vartheta_{p+1}(\lambda) \), i.e., we use the Neumann eigenvalues. In the present paper we discuss the case, where \( \mu_n \) are the zeroes of \( \varphi_p(\lambda) \), i.e., we use the Dirichlet eigenvalues. We omit the proof of theorem for the Dirichlet eigenvalues, since it is very similar to the case of the Neumann eigenvalues in [23].

3. **ONE DIMENSIONAL HALF-SOLID**

We discuss a half-solid model in \( \mathbb{Z} \). In this case we consider the Jacobi operator \( T_{\tau}, \tau \in \mathbb{R} \) on \( \ell^2(\mathbb{Z}) \) given by

\[
(T_{\tau}f)_x = a_{x-1}f_{x-1} + a_xf_{x+1} + b_xf_x, \quad n \in \mathbb{Z}, \tag{3.1}
\]

where \( \tau > 1 \) is large enough and the coefficients \( a_x, b_x \) satisfy

\[
\begin{cases}
  a_x, b_x, x \geq 1, \text{ are } p \text{ - periodic, } \\
  \prod_{1} a_j = 1, \\
  a_x = 1, b_x = \tau, x \leq 0
\end{cases} \tag{3.2}
\]
By the physical point of view, $b_x, x \geq 1$ is a crystal potential and the real constant $\tau$ is the vacuum potential. Define two operators $J_+$ on $\ell^2(\mathbb{Z}_+)$ and $J_-$ on $\ell^2(\mathbb{Z}_-)$ by

\begin{align}
(J_+ f)_x &= a_{x-1} f_{x-1} + a_x f_{x+1} + b_x f_x, \quad x \geq 1, \quad f_0 = 0, \\
(J_- f)_x &= f_{x-1} + f_{x+1} + \tau f_x, \quad x \leq 0, \quad f_1 = 0.
\end{align}

(3.3)

Let $P_{\pm}$ be the projector from $\ell^2(\mathbb{Z})$ onto $\ell^2(\mathbb{Z}_{\pm})$. We rewrite the operator $T_\tau$ in the form

\begin{equation}
T_\tau P_{\pm} = J_{\pm} P_{\pm},
\end{equation}

where

\begin{align}
(T_\tau f)_1 &= a_0 f_0 + (J_+ P_+ f)_1 \\
(T_\tau f)_{-1} &= f_0 + (J_- P_- f)_{-1} \\
(T_\tau f)_0 &= f_1 + a_0 f_1 + \tau f_0
\end{align}

(3.4)

In fact, we discuss the case of one-dimensional octant periodic potentials in the specific form given by (3.2). In order to describe the spectrum of $T_\tau$ we use some properties of the operator $J_+$ on the half-line $\mathbb{Z}_+$ from Section 2. We recall needed results about operators $T_\tau$. We have the following simple results about the spectrum of $\sigma(T_\tau)$ given by

\begin{equation}
\sigma(T_\tau) = \sigma_{ac}(T_\tau) \cup \sigma_d(T_\tau), \quad \sigma_{ac}(T_\tau) = \sigma_{ac}(J_+) \cup \sigma_{ac}(J_-), \quad \sigma_{ac}(J_+) = [\tau - 2, \tau + 2].
\end{equation}

Recall that we assume that the parameter $\tau > 1$ is large enough. In this case we have

\begin{align}
\sigma_{ac}(T_\tau) &= \cup_{n=0}^{p} \sigma_{n}(T_\tau), \quad \sigma_{n}(T_\tau) = \sigma_{n}(J_+), n = 0, 1, \ldots, p - 1, \quad \sigma_{p}(T_\tau) = [\tau - 2, \tau + 2].
\end{align}

(3.6)

Thus, all possible gaps in the spectrum $\sigma_{ac}(T_\tau)$ are given by

\begin{equation}
\gamma_{n}(T_\tau) = \gamma_{n}(J_+), \quad n \in \mathbb{N}_{p-1}, \quad \gamma_{p}(T_\tau) = (\lambda_{p+1}, \tau - 2).
\end{equation}

(3.7)

We begin to describe eigenvalues of $T_\tau$. For the operator $T_\tau$ we introduce the Jacobi equation

\begin{equation}
a_{x-1} f_{x-1} + a_x f_{x+1} + b_x f_x = \lambda f_x, \quad x \in \mathbb{Z}.
\end{equation}

(3.8)

For the operator $J_+$ we define the Weyl function $\psi_{\tau}^{\pm}$ by

\begin{equation}
\psi_{\tau}^{\pm} = \vartheta_{\tau} + m_{\tau} \varphi_{\tau}, \quad x \geq 1,
\end{equation}

where $\vartheta_{\tau}, \varphi_{\tau}$ are solutions of the equation (3.8) under the conditions $\varphi_0 = \vartheta_1 = 0$ and $\varphi_1 = \vartheta_0 = 1$. Note that $\psi_{\tau}^{\pm}$ depends on $a_p, a_x, b_x, x \geq 1$ only.

For the operator $J_-$ we define the Weyl function $\psi_{\tau}^{-}, x \leq 0$. The equation (3.8) has the form

\begin{equation}
\psi_{\tau}^{-} = \lambda - \tau + \frac{1}{z}, \quad x \leq 0,
\end{equation}

(3.10)

where $z \in \mathbb{D}_1 = \{ \lambda \in \mathbb{C} : |\lambda| < 1 \}$ is defined by $\lambda - \tau = z + \frac{1}{z}$. Thus we have

\begin{equation}
\psi_{\tau}^{-} = z^{-x}, \quad x \leq 1,
\end{equation}

\begin{equation}
z = z(\lambda) = t - \sqrt{t^2 - 1} \in \mathbb{D}_1, \quad t = \frac{\lambda - \tau}{2},
\end{equation}

(3.11)

\begin{equation}
z(\lambda) = \frac{1}{2t} + O(1),
\end{equation}

as $t \to \infty$.

For the operator $T_\tau$ we introduce the Weyl-type functions $\Psi_{\tau}^{\pm}(\lambda)$, which are solutions of the equation (3.8) and satisfy

\begin{equation}
(\Psi_{\tau}^{\pm}(\lambda))_{x \in \mathbb{Z}_{\pm}} \in \ell^2(\mathbb{Z}_{\pm}), \quad \forall \lambda \in \mathcal{L} := \mathbb{C} \setminus \sigma_{ac}(T_\tau).
\end{equation}
For $\lambda \in \mathcal{L}$ they have the forms
\[
\Psi_{x}^{+}(\lambda) = \psi_{x}^{+}(\lambda) = \vartheta_{x}(\lambda) + m_{+} \varphi_{x}(\lambda), \quad n \geq 1;
\]
\[
\Psi_{x}^{-}(\lambda) = z^{-n}, \quad n \leq 1.
\] (3.12)

These functions $\Psi_{x}^{\pm}(\lambda)$ are analytic in the cut domain $\mathcal{L}$ and are continuous up to the boundary. We compute $\Psi_{0}^{+}$. From (3.8) and (3.12), we get
\[
\Psi_{0}^{+} + a_{1} \psi_{1}^{+} + (b_{1} - \lambda) \psi_{1}^{+} = 0 \Leftrightarrow \Psi_{0}^{+} - a_{p} = 0 \Leftrightarrow \Psi_{0}^{+} = a_{p}.
\] (3.13)

Thus due to (3.12) - (3.13) and $a_{0} = 1$ we obtain
\[
w = \{\Psi^{-}, \Psi^{+}\}_{0} = a_{0}(\psi_{1}^{-} \psi_{1}^{+} - \Psi_{0}^{+} \Psi_{1}^{-}) = m_{+} - \frac{a_{p}}{z}.
\] (3.14)

The function $w(\lambda)$ is analytic on the domain $\mathcal{L}$ and has finite number of zeros, which are simple and coincide with eigenvalues of the operator $T_{\tau}$. In Lemma 3.1 we show that in each open gap $\gamma_{j}(T_{\tau}) \neq 0, j \in \mathbb{N}_{p-1}$ there is at most one eigenvalue $\mu_{j}(\tau) \sim \mu_{j}$ at large $\tau$. We discuss the eigenvalues of $T_{\tau}$ in the gaps $\gamma_{n}(T_{\tau}), n \geq 0$, and determine how these eigenvalues depend on $\tau$ large enough.

**Lemma 3.1.** Let the operator $J_{+}$ on $l^{2}(\mathbb{Z}_{+})$ defined by (3.3) have an open gap $I = (\lambda^{-}, \lambda^{+})$ in the continuous spectrum and an eigenvalue $\mu \in (\lambda^{-}, \lambda^{+})$ for some $p$-periodic $a, b$. Then for any constant $\tau$ large enough the operator $T_{\tau}$ defined by (3.1), (3.2) has exactly one eigenvalue $\mu_{\tau}$ in the gap $I$ such that
\[
\mu_{\tau} = \mu + \frac{c(\mu)}{\tau} + \frac{O(1)}{\tau^{2}} \quad \text{as} \quad \tau \to \infty,
\] (3.15)

where $c(\mu) = \frac{2 \varphi_{p}(\mu)}{a_{p} \varphi_{p}(\mu)} \neq 0$. Moreover, if $J_{+}$ has a resonance on the interval $I^{2} = (\lambda^{-}, \lambda^{+})$ on the second sheet of the operator $J_{+}$, then for any constant $\tau$ large enough the operator $T_{\tau}$ defined by (3.1), (3.2) has not any eigenvalue in the gap $I$.

**Proof.** Using (3.14), (3.11) we rewrite the Wronskian $w(\lambda)$ in the gap $\gamma_{n} \subset \mathcal{L}$ in the form
\[
w(\lambda) = m^{+}(\lambda) - \frac{a_{p}}{z} = \frac{\varphi(\lambda) - b(\lambda)}{\varphi(1, \lambda)} + a_{p} z_{1}(\lambda),
\] (3.16)
\[
z_{1}(\lambda) = t + \sqrt{t^{2} - 1}, \quad t = \frac{\lambda - \tau}{2}, \quad \lambda \in \gamma_{n} \subset \mathcal{L},
\]
since $zz_{1} = 1$, and
\[
z_{1}(\lambda) = t + \sqrt{t^{2} - 1} = \lambda - \tau - \frac{O(1)}{\tau}, \quad \text{as} \quad \tau \to \infty, \lambda \in \mathcal{L},
\] (3.17)
\[(-1)^{n}b(\mu) = \sqrt{\varphi^{2}(\lambda) - 1} > 0, \quad \text{if} \quad \lambda \in \gamma_{n} \subset \mathcal{L}.
\]

The eigenvalues of $T_{\tau}$ are zeros of the Wronskian $w(\lambda)$, given by (3.10), on the domain $\mathcal{L}$. Consider the two functions $m_{+}(\lambda)$ and $z_{1}(\lambda)$ on the gap $(\lambda^{-}, \lambda^{+})$, where $\tau \to \infty$. The point $\mu \in I$ is an eigenvalue of the operator $J_{+}$. Then due to (2.9) we have $\varphi_{p}(\mu) = b_{2}(\mu) \neq 0$ and $\varphi_{p}(\mu) = -b(\mu) \neq 0$ since the functions $m_{+}(\lambda)$ has the pole at $\mu \in \gamma_{n}^{1}$. Then the function $m_{+}(\lambda)$ is a meromorphic in the disk $\{\lambda \in \Lambda_{1} : |\lambda - \mu| < \varepsilon\}$ around $\mu \in \gamma_{n}^{1}$ and has the following asymptotics
\[
\frac{m_{+}(\lambda)}{a_{p}} = \frac{c(\mu)}{\lambda - \mu} + O(1) \quad \text{as} \quad \lambda \to \mu, \quad c(\mu) = \frac{2 \varphi_{p}(\mu)}{a_{p} \varphi_{p}(\mu)}.
\] (3.18)
We have also \( z_1(\lambda) = \lambda - \tau + O(\frac{1}{\tau}) \) as \( \tau \to +\infty \) locally uniformly in \( \lambda \in \mathbb{C} \). Thus the equation \( m_+(\lambda) = z_1(\lambda) \) has a unique solution \( \mu_+ \to \mu \) as \( \tau \to \infty \) given by (3.15), since 
\[
\frac{1}{\mu_+} = \tau - \lambda + O(1).
\]

Let \( J_+ \) have a resonance on the interval \( I^2 = (\lambda^-, \lambda^+)^2 \) on the second sheet of the operator \( J_+ \). Then due to Lemma 2.2 the function \( m_+ \) is analytic the interval \( I = (\lambda^-, \lambda^+) \) on the first sheet of the operator \( J_+ \) and the function \( m_+ \) is uniformly bounded on \([\lambda^-, \lambda^+]\). Then due to the simple asymptotics (3.11), the Wronskian \( w = m_+ - \frac{a}{\tau^2} \) has not any zero on \( I = (\lambda^-, \lambda^+) \) for any constant \( \tau \) large enough. Thus for any constant \( \tau \) large enough the operator \( T_\tau \) defined by (3.1), (3.2) has not any eigenvalue in the gap \( \gamma \).

Now we prove the main result of this section. Recall that \( \mathbb{N}_m = \{1, 2, \ldots, m\} \).

**Lemma 3.2.** i) Let integer \( p \geq 2 \) and let \( \gamma > 0 \). Then there exist \( p \)-periodic sequences \( a_n, b_n \) such that all \( p - 1 \) gaps in the spectrum of the operator \( J_+ \) on \( \ell^2(\mathbb{Z}_+) \) are open and satisfy
\[
|\gamma j | = \gamma, \quad \forall j \in \mathbb{N}_{p-1}.
\]
In addition, for any points \( \lambda_j \in \gamma_j^c \), \( j \in \mathbb{N}_{p-1} \), exist unique \( p \)-periodic sequences \( a_n, b_n \) such that each \( \lambda_j = \mu_j^+ \), is a state of the operator \( J_+ \).

ii) Let in addition the operator \( T_\tau \) be given by (3.1), (3.2) and let \( \tau \) be large enough. If \( \mu_j^+ \in \gamma_j^c \) is an eigenvalue of the operator \( J_+ \), then the operator \( T_\tau \) has a unique eigenvalue \( \mu_j(\tau) \) on the gap \( \gamma_j^c \) such that for some constant \( c(\mu) \neq 0 \):
\[
\mu_j(\tau) - \mu_j^+ = \frac{c(\mu_j)}{\tau} + O(\frac{1}{\tau^2}) \quad \text{as} \quad \tau \to \infty.
\]

If \( \mu_j^+ \in \gamma_j^c \) is a resonance of the operator \( J_+ \) for some \( j \in \mathbb{N}_{p-1} \), then the operator \( T_\tau \) has not eigenvalues on the gap \( \gamma_j^c \).

**Proof** of i) follows from Theorem 2.3. The proof of ii) follows from i) and Lemma 3.1. ❑

### 4. Difference operators on the lattice

**4.1. Specific periodic Jacobi operators on the half-line.** Consider the Jacobi operator \( J_+ \) on \( \ell^2(\mathbb{Z}_+) \) given by (2.1). Recall that the spectrum of \( J_+ \) consists of an absolutely continuous part (which is a union of non-degenerate spectral bands \( \sigma_n = [\lambda_n^-, \lambda_n^+] \), \( n = 0, 1, \ldots, p - 1 \) plus at most one eigenvalue in each open gap \( \gamma_n = (\lambda_n^-, \lambda_n^+) \), \( n \in \mathbb{N}_{p-1} \).

Now we begin to construct a specific Jacobi operator \( J_+ \). Here we use results about the gap-lengths mapping from Lemma 3.2 i). Due to these results about the gap-lengths mapping, we take the coefficients \( a_n, b_n \) such that all \( p - 1 \) gaps \( \gamma_1, \ldots, \gamma_{p-1} \) are open in the spectrum of \( J_+ \) and satisfy
\[
\lambda_n^0 = 0, \quad \gamma = |\gamma_1| = |\gamma_2| = |\gamma_3| = \cdots |\gamma_{p-1}|.
\]
Let \( S_n = \sum_{j=0}^{n-1} |\sigma_j| \). Thus (4.1) and the estimate (2.3) give
\[
\lambda_n^+ = \gamma(n - 1) + S_n, \quad \lambda_n^- = \gamma n + S_n, \quad |S_n| \leq 4.
\]
Due to Lemma 3.2 i) in each gap \( \gamma_n, n \in \mathbb{N}_{p-1} \) of \( J_+ \) we choose exactly one eigenvalue \( \mu_n^+ \) by
\[
\mu_n^+ = \gamma e_n \in \gamma_n^1, \quad e_n = n - 1 + e_1, \quad e_1 = \frac{1}{4d}.
\]
It is convenient to define the **normalized** operator \( \mathcal{J}_\gamma = \frac{1}{\gamma} J_+ \). Then the spectrum of \( \mathcal{J}_\gamma \) consists of union of bands \( s_n = \frac{1}{\gamma} \sigma_n \) part plus exactly one eigenvalue \( e_n = \frac{\kappa_n}{\gamma} \) in each open gap \( g_n = \frac{1}{\gamma} \gamma_n \). Thus due to (4.1), (4.2) we have

\[
s_0 = \frac{\sigma_0}{\gamma}, \quad s_n = \frac{\sigma_n}{\gamma} = \left[ n + \frac{\mathcal{S}_n}{\gamma}, n + \frac{\mathcal{S}_{n+1}}{\gamma} \right], \quad g_n = \frac{\gamma_n}{\gamma},
\]

(4.4)

for all \( n \in \mathbb{N}_{p-1} \), where \( \mathcal{S}_n \) is defined in (4.2). Thus each spectral band \( s_n \) is very small and is very close to the point \( n \) and satisfies

\[
|s_n| \leq \frac{4}{\gamma}, \quad |s_n| \leq \frac{4}{\gamma}, \quad |g_n| = 1,
\]

(4.5)

In each open gap \( g_n, n \in \mathbb{N}_{p-1} \), there exists exactly one eigenvalue \( e_n \) of \( J_\gamma \) such that

\[
e_n = \frac{\kappa_n}{\gamma}.
\]

(4.6)

4.2. **Difference operators on** \( \mathbb{Z}^2_+ \). We consider the difference periodic operator \( H_0 = J_1 + J_2 \) on the corner \( \mathbb{Z}^2_+ \) acting on the functions \( f_x, x = (x_1, x_2) \in \mathbb{Z}^2_+ \). Here \( J_1, J_2 = 1, 2 \) is the \( p \) periodic Jacobi operator on the half-lattice \( \mathbb{Z}^+_+ \) and given by

\[
(J_f)_m = a_{m-1}f_{m-1} + a_m f_{m+1} + b_m f_m, \quad f_0 = 0, \quad m = x_j \in \mathbb{Z}^+_+ = \{1, 2, ...\}.
\]

We assume that the Jacobi operators \( J_1 \) and \( J_2 \) satisfy (4.1)- (4.3) with large gaps in the spectrum. For a large constant \( \gamma \) we define a new **normalized** operator \( \mathcal{H}_\gamma = \frac{\mathcal{H}_0}{\gamma} = J_\gamma \). We take the operator \( \mathcal{H}_\gamma \), when the variables are separated. We show that \( \mathcal{H}_\gamma \) has bands which are very small and their positions are very close to the integer \( n \). The union of group of bands close to the integer \( n \) forms a cluster. Between the two neighbor clusters there exists a big gap. On this gap there exist eigenvalues. We describe these clusters and eigenvalues.

- We define the basic bands \( S_{i,j}^0 \) of the operator \( \mathcal{H}_\gamma \) and their clusters \( K_n^0 \) by

\[
S_{i,j}^0 = s_i + s_j, \quad i, j = 0, 1, ..., p - 1, \quad K_n^0 = \bigcup_{i+j=n} S_{i,j}^0, \quad n = 0, 1, ..., 2p - 2,
\]

(4.7)

where we define \( A + B \) for sets \( A, B \) by \( A + B = \{ z = x + y : (x, y) \in A \times B \} \). In particular, we have

\[
K_0^0 = S_{0,0}^0, \quad K_1^0 = S_{1,0}^0, \quad K_2^0 = S_{2,0}^0 \cup S_{1,1}^0, \ldots,
\]

(4.8)

If \( \gamma \) is large enough, then due to (4.4), (4.5) we estimate the position of bands \( S_{i,j}^0 \), their lengths \( |S_{i,j}^0| \) and their cluster \( K_n^0 \) by

\[
\text{dist}\{S_{i,j}^0, i + j\} \leq \frac{2}{\gamma}, \quad \text{dist}\{K_n^0, n\} \leq \frac{2}{\gamma}.
\]

(4.9)

- A surface band is created by an eigenvalue \( e_j \) and a band \( s_i \) of Jacobi operators. We define the surface bands \( S_{i,j}^1 \) and their clusters \( K_n^1 \) of the operator \( \mathcal{H}_\gamma \) by

\[
S_{i,j}^1 = e_i + s_j, \quad i \in \mathbb{N}_{p-1}, \quad j = 0, 1, 2, 3, ..., \quad K_n^1 = \bigcup_{i+j=n+1} S_{i,j}^1, \quad n = 0, 1, ..., p.
\]

(4.10)

In particular, we have

\[
K_0^1 = S_{1,0}^1, \quad K_1 = S_{2,0}^1 \cup S_{1,1}^1, \quad K_2^1 = S_{3,0}^1 \cup S_{1,2}^1 \cup S_{2,1}^1, \ldots.
\]

(4.11)
The position of the guided bands $S_{i,j}^1$ and the cluster $K_n^1$ is given by

$$S_{i,j}^1 \sim (i + j - 1) + e_1 = n + e_1, \quad K_n^1 \sim n + e_1. \quad (4.12)$$

- The operator $\mathcal{H}_\gamma$ has eigenvalues $K_n^\sigma, n \geq 0$ with multiplicity $n+1$ given by

$$K_n^\sigma = E_{i,j} := e_i + e_j = n + 2e_1, \quad e_1 = \frac{1}{4d}, \quad i + j = n + 2, n \geq 0 \quad (4.13)$$

for all $i, j = 1, 2, ..., p$. In particular, we have

$$K_0^e = E_{1,1}, \quad K_1^e = E_{1,2} = E_{2,1}, \quad K_3^e = E_{1,3} = E_{2,2} = E_{3,1}, .... \quad (4.14)$$

- Thus we can describe $\sigma_{ac}(\mathcal{H}_\gamma)$ and $\sigma_{disc}(\mathcal{H}_\gamma)$ by

$$\sigma_{ac}(\mathcal{H}_\gamma) = \bigcup_{n \geq 0} (K_n^0 \cup K_n^1), \quad \sigma_{disc}(\mathcal{H}_\gamma) = \bigcup_{n \geq 0} K_n^e, \quad (4.15)$$

where

$$K_n^0 \sim n, \quad K_n^1 \sim n + e_1, \quad K_n^e = n + 2e_1. \quad (4.16)$$

We have two types of band clusters $K_n^0$ and $K_n^1$. These clusters are separated by gaps. Now combining all estimates (4.7)-(4.16) we deduce that there exists an interval $I_n$ such that

$$I_n = [K_n^e - r, K_n^e + r], \quad \sigma_{ac}(\mathcal{H}_\gamma) \cap I_n = \emptyset, \quad \text{where } r = \frac{e_1}{2} = \frac{1}{8d}. \quad (4.17)$$

for some $\gamma > 0$ large enough. Thus the spectral interval $\mathcal{J}_{n,\gamma} = \gamma I_n$ satisfies

$$\mathcal{J}_{n,\gamma} = \gamma I_n = [\gamma (K_n^e - r), \gamma (K_n^e + r)], \quad \text{dist}(\mathcal{J}_{n,\gamma}, \sigma_{ac}(H_0)) \geq 3. \quad (4.18)$$

Then interval $\mathcal{J}_{n,\gamma} \cap \sigma_{ac}(H_0) = \emptyset$ and the operator $H_0$ has the eigenvalue $\gamma K_n^e \in \mathcal{J}_{n,\gamma}$ of multiplicity $n + 1$. Moreover, the interval $\mathcal{J}_{n,\gamma}$ does not contain other spectrum and to the right and to the left of it there is an essential spectrum. In fact we have proved Theorem 1.1 ii) for the case $H_0$.

4.3. Difference operators on $\mathbb{Z}_+^3$. We consider difference operators $J = J_1 + J_2 + J_3$ on the corner $\mathbb{Z}_+^3$ and acting on the functions $f_x, x = (x_1, x_2, x_3) \in \mathbb{Z}_+^3$. Here $J_j, j = 1, 2, 3$ is the $p$ periodic Jacobi operator on the half-line $\mathbb{Z}_+$ and given by

$$(J_j f)_m = a_{m-1} f_{m-1} + a_m f_{m+1} + b_m f_m, \quad f_0 = 0, \quad m = x_j \in \mathbb{Z}_+. \quad (4.19)$$

We assume that the Jacobi operators $J_j$ satisfy (4.11)-(4.3) with large gaps in the spectrum. For large constant $\gamma$ we define a new normalized operator by

$$J_\gamma = J = \frac{J_1 + J_2 + J_3}{\gamma} = J_{1,\gamma} + J_{2,\gamma} + J_{3,\gamma}, \quad J_{j,\gamma} = \frac{J_j}{\gamma}. \quad (4.20)$$

- We define basic bands $S_{i,j}^0$ of the operator $J_\gamma$ and their clusters $K_n^0, n = 0, 1, ...$ by

$$S_{i,j}^0 = s_i + s_j + s_k, \quad i, j = 0, 1, 2, ..., \quad k \in \mathbb{N}_{p-1}, \quad K_n^0 = \bigcup_{i+j+k=n} S_{i,j,k}^0, \quad (4.19)$$

and in particular,

$$K_0^0 = S_{0,0,0}^0 = s_0 + s_0 + s_0, \quad K_1^0 = S_{0,0,1}^0, \quad K_2^0 = S_{0,0,2}^0 \cup S_{0,1,1}^0, .... \quad (4.20)$$

Recall that we define $A + B$ for sets $A, B$ by $A + B = \{z = x + y : (x, y) \in A \times B\}$. Similar to 2-dim case we deduce that

$$S_{i,j,k}^0 \sim i + j + k, \quad K_n^0 \sim n, \quad \forall \quad n = 1, 2, ..., N. \quad (4.20)$$

In 3-dimensional case we have two types of the surface bands $S_{i,j,k}^1$ and $S_{i,j,k}^2$. 
• **The first type of surface bands.** We define the surface bands $S^1_{i,j,k}$ of the operator $J_\gamma$ and their clusters $K^1_n$, $n = 1, 2, ..., k$ by

$$S^1_{i,j,k} = s_i + s_j + e_k, \quad K^1_n = \bigcup_{i+j+k=n+1} S^1_{i,j,k}, \quad i, j = 0, 1, 2, ..., k \in \mathbb{N}_{p-1}. \quad (4.21)$$

The position of surface bands $S^1_{i,j,k}$ and their clusters $K^1_n$ are given by

$$S^1_{i,j,k} \sim i + j + k - 1 + e_1 = n - 1 + e_1, \quad K^1_n \sim n - 1 + e_1. \quad (4.22)$$

These clusters are separated by gaps. Thus we have

$$K^1_1 = S^1_{0,0,1}, \quad K^1_2 = S^1_{0,0,2} \cup S^1_{0,1,1}, \quad K^1_3 = S^1_{0,0,3} \cup S^1_{0,1,2} \cup S^1_{1,1,1}, \quad \ldots. \quad (4.23)$$

• **The second type of surface (guided) bands.** We define the surface (guided) bands $S^2_{i,j,k}$ of the operator $J_\gamma$ and their clusters $K^2_n$, $n = 0, 1, \ldots$ by

$$S^2_{i,j,k} = e_i + e_j + s_k, \quad K^2_n = \bigcup_{i+j+k=n+2} S^2_{i,j,k}, \quad i, j \in \mathbb{N}_{p-1}, \quad k = 0, 1, 2, \ldots \quad (4.24)$$

The positions of the surface bands $S^2_{i,j,k}$ and the cluster $K^2_n$ are given by

$$S^2_{i,j,k} \sim i + j - 2 + 2e_1 + k = n + 2e_1, \quad K^2_n \sim n + 2e_1. \quad (4.25)$$

These clusters are separated by gaps. Thus we have

$$K^2_0 = S^2_{1,1,0}, \quad K^2_1 = S^2_{1,1,1} \cup S^2_{2,1,0}, \quad K^2_2 = S^2_{2,1,0} \cup S^2_{2,1,1} \cup S^2_{1,1,2}, \ldots. \quad (4.26)$$

• **Eigenvalues.** Due to (4.6) the operator $J_\gamma$ has eigenvalues $K^e_n$ given by

$$K^e_n = e_i + e_j + e_k = i + j + k - 3 + 3e_1 = n + 3e_1, \quad i, j, k \in \mathbb{N}_{p-1}, \quad i + j + k = n + 3, \quad (4.27)$$

$n = 0, 1, \ldots$ The sets $\sigma_{ac}(J_\gamma)$ and $\sigma_{disc}(J_\gamma)$ are given by

$$\sigma_{ac}(J_\gamma) = \bigcup_{n \geq 0} (K^e_n \cup K^e_n \cup K^e_n), \quad \sigma_{disc}(J_\gamma) = \bigcup_{n \geq 1} K^e_n. \quad (4.28)$$

Later on we repeat the consideration for the case $d = 2$.

### 4.4 **Specific 1dim half-solid potentials.** Consider the Jacobi operator as a half-solid model in $\mathbb{Z}$. In this case we consider the Jacobi operator $T_\tau$ on $l^2(\mathbb{Z})$ given by

$$(T_\tau f)_x = a_{x-1}f_{x-1} + a_xf_{x+1} + b_xf_x, \quad x \in \mathbb{Z}. \quad (4.29)$$

Let $\tau$ be large enough and the coefficients $a_x, b_x$ satisfy

$$\begin{cases} a_0, a_x, b_x, \text{ are } p - \text{ periodic} & x \geq 1, \\ a_x = 1, b_x = \tau = b_0, & x \leq -1 \end{cases} \quad (4.30)$$

Let an integer $p \geq 2$ be large enough. Due to Lemma 3.2 for large $\gamma > 1$ we obtain that there exists $p$-periodic $a_x, b_x, x \geq 1$ sequences such that (4.1) holds true. Thus by (3.5)-(3.7), all gaps $\gamma_j, j \in \mathbb{N}_{p-1}$ in the spectrum of the operators $J_\gamma$ and $T_\tau$ are open. Moreover, there exists an eigenvalue $\mu_j(\tau)$ in each this gap $\gamma_j$ and they satisfy

$$\mu_j(\tau) \in \gamma_j \quad (4.31)$$

Here the bands $\sigma_0, \sigma_1, \ldots, \sigma_{p-1}$ are separated by gaps $\gamma_j, j \in \mathbb{N}_{p-1}$ and the bands $\sigma_{p-1}$ and $\bar{\sigma}$ are separated by a gap $\bar{\gamma}_p = (\lambda_p^+, \tau - 2)$ and each eigenvalue $\mu_n(\tau)$ satisfies (3.20).
Define a new **normalized** operator $T_{\tau, \gamma} = \frac{1}{\gamma} T_{\tau}$. From the properties of $T_{\tau}$ we deduce that the spectrum of $T_{\tau, \gamma}$ consists of an absolutely continuous part

$$\sigma_{ac}(T_{\tau, \gamma}) = \bigcup_{n=0}^{p} s_n, \quad s_n = \frac{\sigma_n}{\gamma}, \quad n \in \mathbb{N}_{p-1}, \quad s_p = \frac{\bar{\sigma}}{\gamma},$$

(4.33)

plus at most one eigenvalue in each non-empty finite gap $g_n, n \in \mathbb{N}_p$, given by

$$g_n = \frac{\gamma_n}{\gamma}, \quad n \in \mathbb{N}_{p-1},$$

and they satisfy (4.4)-(4.6). In each gap $g_n, n \in \mathbb{N}_{p-1}$, there exists exactly one eigenvalue $\tilde{e}_n$ given by

$$\tilde{e}_n = \frac{\tilde{\mu}_n}{\gamma} = e_n + \varepsilon_n, \quad e_n = n - 1 + \frac{1}{4d}, \quad |\varepsilon_n| \leq \frac{1}{\gamma}, \quad n \in \mathbb{N}_{p-1},$$

(4.34)

since $\tau > 1$ is large enough. Thus roughly speaking the spectrum of the operators $T_{\tau, \gamma}$ on $\ell^2(\mathbb{Z})$ and $J_\tau$ (on $\ell^2(\mathbb{Z}^+)$) is the same on the interval $[0, \lambda_p^+]$. They have the same bands $\sigma_0, \sigma_j, j \in \mathbb{N}_{p-1}$ and the same gaps $\gamma_j, j \in \mathbb{N}_{p-1}$. Moreover, their eigenvalues $\tilde{e}_j$ and $e_j$ in each gap $\gamma_j$ are very close, since we take $\tau$ large enough.

### 4.5. Model difference operators on $\mathbb{Z}^2$

We consider difference operators $H_0 = T_{\tau, 1} + T_{\tau, 2}$ on the lattice $\mathbb{Z}^2$, where $T_{\tau, 1}, j = 1, 2$ is the Jacobi operator on the lattice $\mathbb{Z}$, discussed in Subsection 4.4. The spectrum of $T_{\tau, j}$ and $J_{\tau, j}$ are similar on the interval $[\lambda_0^+, \lambda_p^+]$. Then the spectrum of the sum $T_{\tau, 1} + T_{\tau, 2}$ is similar to the spectrum of $J_1 + J_2$ on the interval $[\lambda_0^+, 2\lambda_p^+]$. The proof repeats the case $J_1 + J_2$. Moreover, using similar arguments we prove Theorem 1.1 for the operator $T = T_{\tau, 1} + T_{\tau, 2}$. The proof for the case $\mathbb{Z}^d, d \geq 3$ is similar.

### 4.6. Model difference operators on $\mathbb{Z}_+ \times \mathbb{Z}$

Consider the operator $H_0 = J_1 + T_{\tau, 2}$ on the half-lattice $\mathbb{Z}_+ \times \mathbb{Z}$, where the operator $J_1$ acts on the half-line and depends on one variable $x_1 \in \mathbb{Z}_+$; the operator $T_{\tau, 2}$ (depending on one variable $x_2 \in \mathbb{Z}$) acts on $\mathbb{Z}$ and given by (4.29), (4.30) and the constant $\tau \geq 1$ is large enough. The spectrum of $T_{\tau, 2}$ and $J_1$ are similar on the interval $[\lambda_0^+, \lambda_p^+]$ for $p, \tau$ large enough. The proof repeats the case $J_1 + J_2$. Moreover, using similar arguments we prove Theorem 1.1 for the operator $H_0 = J_1 + T_{\tau, 2}$.

### 5. Proof of main Theorems

**Proof Theorem 1.1** i) We consider an operator $H_+ = -\Delta + V$ on $\mathbb{R}^2_+$, where the potential $V \in \ell^\infty(\mathbb{Z}^2_+)$ is octant periodic, the proof for other cases is similar. Without loss of generality, we assume that $V$ is $(p\mathbb{Z})^2$-periodic for some $p > 1$. Let $H = -\Delta + V$ on $\mathbb{Z}^2$. Define functions $g_n \in \ell^\infty(\mathbb{Z}^2_+)$ and $G_n \in \ell^2(\mathbb{Z}^2_+), n \geq 1$ by:

$$g_n|_{w_n} = 1, \quad g_n|_{\mathbb{Z}_+ \setminus w_n} = 0, \quad w_n = [4^n, 4^n + n + 1], \quad w_n \cap w_{n+1} = \emptyset,$$

$$G_n(x) = g_n(x_1)g_n(x_2), \quad x = (x_1, x_2) \in \mathbb{Z}^2, \quad \text{supp} G_n \subset \mathbb{Z}^2_+.$$  

(5.1)

Let $T = \mathbb{Z}^2/(p\mathbb{Z})^2$. For any $\lambda \in \sigma(H)$ there exists a function $\psi_\infty = e^{i(k,x)}u(x, k)$, which satisfies

$$(-\Delta + V(x))\psi_\infty(k) = \lambda \psi_\infty(k), \quad \forall x \in \mathbb{Z}^2,$$

$$u(\cdot, k) \in \ell^2(T), \quad \sum_{x \in T} |u(x, k)|^2 = 1,$$

(5.2)
see [11] for some $k \in \mathbb{R}^2$. For this fix $k \in \mathbb{R}^2$ we define the sequence $f_n(x) = \frac{1}{c_n}G_n(x)\psi_x$, where $c_n > 0$ is given by

$$c_n^2 = \sum_{x \in \mathbb{Z}^2} |G_n(x)\psi_x(k)|^2.$$  

The function $u(x, k)$ is $(p\mathbb{Z})^2$ periodic, then due to (5.2) we obtain

$$c_n^2 = \sum_{x \in \mathbb{Z}^2} |G_n(x)\psi_x(k)|^2 = n^2 + O(n) \quad \text{(5.3)}$$

as $n \to \infty$. Thus the sequence $f_n$ satisfies

1) $\|f_n(\cdot, k)\| = 1$ and $\Delta f_n \in \ell^2(\mathbb{Z}^2_+)$, for all $n \in \mathbb{N}$,
2) $f_n \perp f_m$ for all $n \neq m$, and $f_n \to 0$ weakly as $n \to \infty$.

Thus $\lambda \in \sigma_{\text{ess}}(H_+)$, which yields (1.7), since standard arguments imply

$$\| (H_+ - \lambda)f_n \| = \| (H - \lambda)f_n \| \to 0 \quad \text{as} \quad n \to \infty.$$  

We prove ii) for the case $D = \mathbb{Z}^2_+$ and $N \geq 1$, the proof of other cases is similar. Consider the operator $H_0 = J_1 + J_2$, where $J_1$ and $J_2$ is the described in Subsection 4.2 and $J_1, J_2$ are the Jacobi operator on $\mathbb{Z}_+$. We assume that they have the properties in (1.1)-(1.5) for some $p$-periodic sequences $(a_n, b_n) \in \mathbb{R}_+ \times \mathbb{R}$. Due to (4.18) for each $n$ the operator $H_0$ has the eigenvalue $E = \gamma(n + 2\epsilon_1)$ of multiplicity $n + 1$ and the the interval $I_{n, \gamma}$ such that

$$I_{n, \gamma} = \gamma I_n = [E - \gamma r, E + \gamma r], \quad I_{\gamma, r} \cap \sigma_{\text{ac}}(H_0) = \emptyset, \quad \text{where} \quad r = \frac{\epsilon_1}{2}.$$  

Moreover, the interval $I_{n, \gamma}$ does not contain other spectrum and to the right and to the left of it there is an essential spectrum. In fact we have proved ii) for the case $H_0$.

We consider an operator $H_\varepsilon = H_0 + \varepsilon W$ on $\ell^2(\mathbb{R}^2_+)$ and $W$ is the multiplication operator. Here $H_\varepsilon$ is the difference operator on the quadrant $\mathbb{Z}^2_+$ with the Dirichlet boundary conditions on the boundary $\partial\mathbb{Z}^2_+$ with octant periodic coefficients. We assume that the perturbation $W$ satisfies

$$W = \sum_{i=1}^{2}\left(\tilde{a}^i U_i + U_{-\tilde{a}}^i\right) + \tilde{V}, \quad \text{(5.4)}$$

where $(U_i f)_x = f_{x+e_i}$ and $(U_{-\tilde{a}}^i f)_x = f_{x-e_i}$ for $f = (f_x) \in \ell^2(\mathbb{Z}^2_+)$ and $i = 1, 2$. We also assume that $\tilde{a}^i > 0$ and $\tilde{V}$ are the octant periodic functions on $\mathbb{Z}^2_+$. Thus we obtain

$$\|W\| \leq 5. \quad \text{(5.5)}$$

We define contours $c = \{ \lambda \in \mathbb{C} : |\lambda - E| = 2\}$. Due to (4.17) the operator $H_0$ has an eigenvalue $E = n + 2\epsilon_1 \in I_{n, \gamma}$ of multiplicity $n + 1$ inside the contours $c$. Using the simple identities we deduce that the resolvents $R_0(\zeta) = (H_0 - \zeta)^{-1}$ and $R_\varepsilon(\zeta) = (H_\varepsilon - \zeta)^{-1}$ satisfy

$$\|R_\varepsilon(\zeta)\| \leq 2, \quad \|R_0(\zeta)\| \leq 1,$$

$$\|R_\varepsilon(\zeta) - R_0(\zeta)\| \leq 5\varepsilon\|R_0(\zeta)\|\|R_\varepsilon(\zeta)\| \leq 10\varepsilon \quad \text{(5.6)}$$

for all $\zeta \in c$ since $\dist\{\sigma(H_\varepsilon), E \pm 1\} \geq 1 - 5\varepsilon \geq \frac{1}{2}$. Then we obtain

$$P(\varepsilon) = -\frac{1}{2\pi i} \int_{\varepsilon} R_\varepsilon(\zeta) d\zeta,$$

$$R_\varepsilon(\zeta) = R_0(\zeta) - R_0(\zeta)\varepsilon W R_\varepsilon(\zeta), \quad \text{(5.7)}$$
which yields \( \| P(\varepsilon) - P_n(0) \| < 1 \). Thus the projectors \( P(\varepsilon) \) and \( P(0) \) have the same dimension \( n + 1 \) for all \( \varepsilon > 0 \) small enough. We use similar arguments in order to show that to the right and to the left of the interval \([E - 1, E + 1]\) there is spectra and its corresponding subspaces have infinite dimension. ■

Acknowledgments. This work was supported by the RSF grant No. 18-11-00032. We thank Natalia Saburova for Fig. 2.

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