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Gap Reduced Minimum Error Robust Simultaneous Estimation For Unstable Nano Air Vehicle

Jinraj V. Pushpangathan, Harikumar Kandath, Member, IEEE, Suresh Sundaram, Senior Member, IEEE, and Narasimhan Sundararajan, Life Fellow, IEEE

Abstract—This paper proposes a novel Gap Reduced Minimum Error Robust Simultaneous (GRMERS) estimator for resource-constrained Nano Aerial Vehicle (NAV) that enables a single estimator to provide simultaneous and robust estimation for a given N unstable and uncertain NAV plant models. The estimated full state feedback enables a stable flight for NAV. The GRMERS estimator is implemented utilizing a Minimum Error Robust Simultaneous (MERS) estimator and Gap Reducing (GR) compensators. The MERS estimator provides robust simultaneous estimation with minimal largest worst-case estimation error even in the presence of a bounded energy exogenous disturbance signal. The GR compensators reduce the gap between the graphs of N linear plant models to decrease the estimation error generated by the MERS estimator. A sufficient condition for the existence of a simultaneous estimator is established using LMIs and robust estimation theory. Further, MERS estimator and GR compensator design are formulated as non-convex tractable optimization problems and are solved using the population-based genetic algorithms. The performance of the GRMERS estimator consisting of MERS estimator and GR compensators from the population-based genetic algorithms is validated through simulation studies. The study results indicate that a single GRMERS estimator can produce state estimates with reduced errors for all flight conditions. The results indicate that the single GRMERS estimator is robust than the individually designed $H_{\infty}$ filters.

Index Terms—Linear matrix inequality, Nano air vehicle, Robust simultaneous estimator, €-gap metric

I. INTRODUCTION

Recent trends in Micro Air Vehicles (MAVs) point to the development of a new class of small air vehicles called Nano Air Vehicles (NAVs) that execute specific missions undetected with a high degree of agility. NAVs can be broadly classified into three categories, viz., fixed-wing NAVs, rotary wing NAVs, and flapping-wing NAVs. They are widely used for intelligence operations, battlefield surveillance, reconnaissance, and disaster assessment missions. These small vehicles have severe dimensional and weight constraints as their overall dimensions and weights need to be lower than 75 mm and 20 g, respectively [1]. Figure 1(a) shows a typical fixed-wing NAV that weighs 19.4 g and has an overall dimension of 75 mm [2].

In general, the plant models of these NAVs are multi-input-multi-output (MIMO), unstable, uncertain, adversely coupled, and have a different number of unstable poles [3], [5]. In this paper, for convenience, we use the terms plant and estimator to represent the plant model and estimator model, respectively. More details about these plant characteristics are given in [1], [2]. Due to these plant characteristics, the NAV’s require complex feedback controllers to accomplish a mission. For using the existing closed-form solutions of the full state feedback for designing a feasible controller for a NAV requires the measurements or estimates of all the system’s state variables. However, due to the weight and dimensional constraints, the autopilot hardware of a NAV like the one shown in Fig. 1(b) [2] has severe resource constraints, such unavailability of lightweight sensors to measure every state variable (like translational velocities, angle-of-attack, airspeed, side slip angle). Moreover, these autopilots have both limited computational and memory powers. Hence, one needs to design a computationally simple full state estimator from the available measurements. Extended Kalman Filter (EKF) is the de-facto standard for UAV estimation schemes. However, EKF is not suitable for a NAV, as its autopilot hardware has limited computational and memory resources, specifically for Jacobian computing. A gain-scheduled EKF reduces the computational cost of calculating the estimator gain [6], [7], but it still requires the measurement of scheduling variables like airspeed etc. Further, the significant model uncertainties in NAVs can induce notable errors in the state estimates. These difficulties clearly point out to the need for an estimation algorithm that is computationally simpler and robust to model uncertainties. This algorithm should also cater to both stable/unstable plant models and should not require a computationally expensive gain-scheduling approach. To put the problem in a sharp focus, a NAV requires a single computationally less intensive and robust estimator to estimate the states of a finite set of MIMO LTI uncertain unstable plants. Herein, such an estimator is

![Figure 1: 75 mm wingspan NAV and autopilot hardware](image-url)
referred to as a Robust Simultaneous (RS) estimator. The overall problem definition of simultaneous estimation is given below.

**Simultaneous Estimation problem:** Let us consider a finite set, $\mathcal{P} = \{P_i(s) \in \mathcal{RL}_\infty \mid i \in \{1, \ldots, 1, \ldots, N\}$, containing stabilizable and detectable MIMO LTI plant models in a transfer function matrix form (i.e., $P_i(s) \in \mathcal{RL}_\infty$). Here, $\mathcal{RL}_\infty$ symbolizes the space of proper, real-rational functions of $s \in \mathbb{C}$ which has no poles in the imaginary axis of $s$-plane.

The state-space form of $P_i(s)$ is given by

$$
P_i(s): \begin{cases}
\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t) \\
y_i(t) = C_i x_i(t) + v(t) \\
z_i(t) = C_z x_i(t)
\end{cases}
$$

where $x_i(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}^m$, $y_i(t) \in \mathbb{R}^r$, and $z_i(t) \in \mathbb{R}^{q \leq n}$ represent the state vector, the control input vector, the measurement vector, and the vector that contains those states that need to be estimated, respectively. Besides this, $v(t) \in \mathbb{R}^n$ is a bounded energy measurement noise vector. Note that the noise considered in this article is zero-mean Gaussian white. Further, $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m}$, and $C_i \in \mathbb{R}^{r \times n}$ are the system, input, and output matrices, respectively. Also, $C_z \in \mathbb{R}^{q \times n}$ is a constant matrix for all the plants. Let $P_i(s) \in \mathcal{RH}_\infty$ be a suitable estimator for all the plant models belonging to $\mathcal{P}$. Here, $\mathcal{RH}_\infty$ denotes the space of proper, real-rational functions of $s \in \mathbb{C}$ which are analytic in $\mathbb{C}_s$. Let the state-space form of $P_i(s)$ be formed using the state-space matrices of $P_i(s) \in \mathcal{P}$. Following this, one of the state-space realizations of $P_i(s)$ is given by

$$
P_i(s): \begin{cases}
\dot{\hat{x}}_i(t) = A_i \hat{x}_i(t) + B_i u_i(t) + L_i (y_i(t) - C_i \hat{x}_i(t)) \\
\hat{z}_i(t) = C_z \hat{x}_i(t)
\end{cases}
$$

where $\hat{x}_i(t) \in \mathbb{R}^n$ is the state vector of the estimator, $\hat{z}_i(t) \in \mathbb{R}^q$ is the output vector of the estimator which provides the estimate of $z_i(t)$ of $P_i(s) \in \mathcal{P}$ for all $i \in \{1, \ldots, 1, \ldots, N\}$, and $L_i \in \mathbb{R}^{r \times r}$ is the estimator gain. In this article, $u(t)$ and $y(t)$ are the inputs and the estimator from the plant. For example, $u(t) = u_i(t)$ and $y(t) = y_i(t)$ when $P_i(s)$ becomes the estimator of $P_i(s)$. Also, for all $i \in \{1, \ldots, 1, \ldots, N\}$, $e_{z_i}(t) = z_i(t) - \hat{z}_i(t) \in \mathbb{R}^q$ be the estimation error vector when $P_i(s)$ estimates $z_i(s)$ of $P_i(s) \in \mathcal{P}$, respectively. Let $d(t)$ be a bounded energy exogenous signal vector that adversely affect the estimation error dynamics of the estimator by increasing the estimation errors (e.g. $v(t)$ given in (1)).

Now, the simultaneous estimation problem is about finding an estimator $P_i(s)$ for $\mathcal{P}$ such that the following two conditions hold $\forall i \in \{1, \ldots, 1, \ldots, N\}$.

**Condition I:** when $d(t) = 0$, all the estimation error vectors should asymptotically converge to zero, i.e.,

$$
\lim_{t \to \infty} e_{z_i}(t) = 0 \quad (3)
$$

**Condition II:** when $d(t) \neq 0$, the root mean square (RMS) gain from $d(t)$ should be bounded, i.e.,

$$
\sup_{d(t) \in L_2[0, \infty)} \| e_{z_i}(t) \|_2 < \gamma \quad (4)
$$

where $\gamma$ is a constant that satisfies $1 > \gamma > 0$ and $\sup$ symbolizes supremum. In this paper, only **Condition II** is considered as the measurements of NAV are always affected by $v(t)$. Before formulating the problem of simultaneous estimation for a NAV, we provide first a brief review of the existing simultaneous estimation research work in the literature below.

The problem of simultaneous observation was first studied in [8]. In this paper, coprime factorization technique was utilized to solve the simultaneous observation problem. For the finite set that contains at least one stable plant, the necessary and sufficient conditions for the existence of simultaneous observations were obtained. Using the proposed method, a simultaneous observer for two plants was designed. In [9], a stable inverse approach was employed to synthesize a simultaneous observer for a given set of plants. The restrictions on these plants were that they should not have any right half plane zeros besides satisfying the condition, $(\hat{m} + \hat{r}) > \hat{m}N$. For plants that have the same number of inputs and outputs without common eigenvalues, the necessary and sufficient conditions for the existence of a simultaneous functional observer were presented in [10]. In [11], algebraic geometry tools were presented to characterize the simultaneous observability of a set of linear single-input single-output plants and also to design a simultaneous state observer for the same. The methods presented in [8]-[11] are not suitable for synthesizing a simultaneous estimator for a NAV due to the following reasons: All the plants of the NAV may be unstable and also can have zeros on the right half of the $s$-plane [12]. Also, the number of outputs of the NAV is more than the number of inputs violating the condition mentioned in [10]. Furthermore, in the case of a NAV, satisfying the condition, $(\hat{m} + \hat{r}) > \hat{m}N$ mentioned in [9] is not possible. For example, a NAV with three inputs and five outputs, a simultaneous estimator can not be synthesized only for two plants. The measured outputs of the plants of a NAV are also affected by noise. Apart from this, the plants of the NAV are subjected to higher model uncertainties. In [8]-[11], no method is explicitly proposed to provide the desired performance (to achieve the **condition II**) by the estimator when the plants are subjected to measurement noise and higher model uncertainties. To overcome the above-mentioned limitations, one needs to develop a new method to synthesis a simultaneous estimator for a NAV.

In this paper, we propose a novel Gap Reduced Minimum Error Robust Simultaneous (GRMERS) estimator to handle unstable plants with model uncertainties and measurement noises. The GRMERS estimator incorporates the solution of two problems: a Minimum Error Robust Simultaneous (MERS) estimation problem and a Gap Reduced (GR) compensator problem. The MERS estimation problem finds a single estimator referred to as the MERS estimator that accomplishes robust state estimation with minimal (largest) worst-case estimation error for $N$ number of unstable uncertain MIMO linear plants of the NAV. The estimation error of the MERS estimator is further reduced by decreasing the gap (see Definition 13 for definition) between the graphs (see Definition 11) of the plants in $\mathcal{P}$ by cascading these plants with suitable pre/post compensators. These compensators are called the GR compensators, and
the corresponding synthesis problem is termed as the GR compensators problem. The GR compensators are defined by first-order differential equations which can be solved by using the limited computational capabilities of the NAV’s autopilot hardware. Using the robust estimation theory, the MERS estimator and GR compensator designs problems are devised as non-convex optimization problems following the robust estimation theory, formulated in terms of Linear Matrix Inequalities (LMI) and the properties of \( \nu \)-gap metric, respectively. The major highlights of the proposed GRMERS estimator in this paper are:

1) This approach can handle the robust simultaneous (RS) estimation of more than three \( (N > 3) \) minimum/minimum phase unstable plants even having common eigenvalues.

2) To our best knowledge, it has been shown here (for the first time) that cascading the plants with compensators (GR compensators) that reduce the gap between the graphs of the plants can reduce the root mean square value of estimation errors of a simultaneous estimator.

The effectiveness of both the MERSE and GR algorithms are demonstrated by generation of the MERS estimator and the GR compensators to synthesise the GRMERS estimator for four unstable plants of the NAV mentioned in [2]. The stability, nominal, and robust performances of the GRMERS estimator are 1) validated through numerical simulations with Gaussian measurement noise and 2) compared with the performances of MERS estimator and \( H_{\infty} \) filter. For this purpose, individual \( H_{\infty} \) filters are designed separately for each plant. The nominal performance analysis indicates that the best performance is given by individual \( H_{\infty} \) filters, followed by the GRMERS estimator. As compared to the MERS estimator, the GRMERS estimator yields up to 55\% reduction in estimation error, which substantiates the effectiveness of providing GR compensators.

The robust performance analysis, however, shows that the GRMERS estimator has a lower estimation error of up to 43 \% compared to the \( H_{\infty} \) filters.

The paper is organized as follows. The dynamics of a fixed-wing NAV is adversely coupled, and unstable as seen from dynamics of the 75 mm wingspan fixed-wing NAV mentioned in [2]. Generally, in a fixed-wing aircraft, the actuator’s bandwidth would be much higher than that of the plant, whereas it is not true in the case of a NAV. Hence, in the flight controller and estimator design of a NAV, one has to explicitly include the dynamics of the actuator along with the plant dynamics. Besides, the dynamics of a single propeller fixed-wing NAV has significant cross-coupling effects. Based on these considerations, a suitable linear model for a single propeller fixed-wing NAV with both the coupling effects and actuator dynamics is the linear coupled model [2] given by

\[
x_i(t) = A_i x_i(t) + B_i u_i(t)
\]

where \( x_i(t) \in \mathbb{R}^{n_i} \), \( u_i(t) \in \mathbb{R}^{m_i} \), \( A_i \in \mathbb{R}^{(n_i \times n_i)} = \begin{bmatrix} A_{L_{a_i}} & A_{L_{b_i}} \\ A_{L_{a_i}} & A_{L_{b_i}} \end{bmatrix} \), and \( B_i \in \mathbb{R}^{(n_i \times m_i)} = \begin{bmatrix} B_{L_{a_i}} & B_{L_{b_i}} \\ B_{L_{a_i}} & B_{L_{b_i}} \end{bmatrix} \) are the state vector, the control input vector, the system matrix, and the control matrix, respectively. Here, \( A_{L_{a_i}} \in \mathbb{R}^{6 \times 6} \), \( A_{L_{b_i}} \in \mathbb{R}^{6 \times 5} \), \( B_{L_{a_i}} \in \mathbb{R}^{6 \times 2} \), and \( B_{L_{b_i}} \in \mathbb{R}^{5 \times 1} \) represent the system and control matrices of the longitudinal and lateral state-space models, respectively. Also, \( A_{L_{a_i}}^{11} \in \mathbb{R}^{6 \times 5} \), \( A_{L_{b_i}}^{11} \in \mathbb{R}^{6 \times 4} \), \( B_{L_{a_i}}^{11} \in \mathbb{R}^{6 \times 4} \), and \( B_{L_{b_i}}^{11} \in \mathbb{R}^{5 \times 2} \) are the longitudinal coupling block of \( A_i \), lateral coupling block of \( A_i \), longitudinal coupling block of \( B_i \), and lateral coupling block of \( B_i \), respectively. Furthermore, \( x_i(t) \) and \( u_i(t) \) in (5) are defined as

\[
x_i(t) = \begin{bmatrix} u & w & q & \theta & \delta_e & \delta_r & v & p & r & \phi \end{bmatrix}^T(t)
\]

\[
u_i(t) = \begin{bmatrix} \delta_{eu} & \delta_{ru} & \delta_{ru} \end{bmatrix}^T(t)
\]

where \( \begin{bmatrix} u(t) & v(t) & w(t) \end{bmatrix}^T \) is the body-fixed linearized translational velocities in m/s, \( \begin{bmatrix} p(t) & q(t) & r(t) \end{bmatrix}^T \) is the body-fixed linearized rotational velocities in rad/s, and \( \begin{bmatrix} \theta(t) & \phi(t) \end{bmatrix}^T \) is the body-fixed linearized Euler’s angles in rad. Also, \( \delta_{eu}(t), \delta_{ru}(t), \delta_{ru}(t) \) are the linearized elevator deflection (in rad), rudder deflection (in rad), and propeller speed (in rpm-revolution per second), respectively. In (7), \( \delta_{eu}(t) \) (rad), \( \delta_{ru}(t) \) (in rad), and \( \delta_{ru}(t) \) (in rpm) represent the inputs to the elevator actuator, the input to the rudder actuator, and the input to the electric motor that drives the propeller, respectively.

The linear dynamics of a fixed-wing NAV is adversely coupled, uncertain, and unstable as seen from dynamics of the 75 mm wingspan fixed-wing NAV mentioned in [2]. Hence, the NAVs similar to the 75 mm wingspan NAV require flight controllers to handle all these complexities and accomplish the desired mission. This controller can use either full state feedback or output feedback strategy. Generally, the full state feedback strategy is preferred as various closed-form solutions are available when compared with the output feedback strategy. The development of a well-proven full state feedback flight controller requires the measurement of all the state variables. If all the state variables can not be measured, then estimates of unmeasured states are required. Among all the state variables of the NAV, the measured state variables are \( q(t), \theta(t), p(t), r(t), \) and \( \phi(t) \) can be directly used for control. Hence, the
measurement vector of the NAV, $y_i(t) \in \mathbb{R}^{\tilde{r} \times 5}$, is given by

$$y_i(t) = [q \theta p r \phi]^T(t) \quad (8)$$

Following this, the measurement equation of the NAV is given by

$$y_i(t) = Cx_i(t) + v(t) \quad (9)$$

where $v(t) \in \mathbb{R}^5$ and $C \in \mathbb{R}^{5 \times 11}$. Due to the absence of lightweight sensors for measurement, $u(t), v(t), u(t), \delta_i, \delta_T$, and $\delta_r$ need to be estimated. Thus, the estimation vector for a NAV, $z_i(t) \in \mathbb{R}^{\hat{r} \times 6}$, is given by

$$z_i(t) = [u w \delta_c \delta_T v \delta_r]^T(t) \quad (10)$$

Then, the equation of $z_i(t)$ is given by

$$z_i(t) = Cz_i(t) \quad (11)$$

where $C_z \in \mathbb{R}^{6 \times 11}$. Also, note that, in the case of a NAV, $n_\pi = t + \hat{q}$. Consequently, the state-space model of the NAV used for designing the estimator is given by (1) with $B_1 = B, C_1 = C, n_\pi = 11, m_\pi = 3, \hat{r} = 5, \hat{q} = 6$. Let $\hat{P}_i(t) \in \mathcal{P}$ represents an estimator. The estimator, $\hat{P}_i(s)$, is formed using the state-space matrices of $P_i(s) \in \mathcal{P}$ and a suitable estimator gain, $L_t \in \mathbb{R}^{11 \times 5}$. Following this, the state-space model of $\hat{P}_i(s)$ is given by (2) with $\hat{x}_i(t) \in \mathbb{R}^{11 \times 1}$ and $\hat{z}_i(t) \in \mathbb{R}^{5 \times 1}$. Here, $\hat{z}_i(t)$ provides the estimation of $z_i(t)$, $\forall i \in \{1, \ldots, N\}$. When we consider $\hat{P}_i(s)$ as the common estimator of $P_i(s) \in \mathcal{P} \forall i \in \{1, \ldots, N\}$, then the state estimation error vectors and the estimation error vectors are denoted by $e_{x_i}(t) = x_i(t) - \hat{x}_i(t)$ and $e_{z_i}(t) = z_i(t) - \hat{z}_i(t)$ for all $i \in \{1, \ldots, l, \ldots, N\}$, respectively. Similarly, for all $i \in \{1, \ldots, l, \ldots, N\}$, $e_{y_i}(t) = y_i(t) - Cx_i(t)$. Now, consider the case where $\hat{P}_i(s)$ is employed to estimate $z_i(t)$, $\forall i \in \{1, \ldots, N\}$ of $P_i(s) \in \mathcal{P} \forall i \in \{1, \ldots, N\}$, respectively. Then, one can obtain $N$ number of estimator error models that are given by

$$\dot{e}_{x_i}(t) = A_i e_{x_i}(t) + \Delta A_{il} x_i(t) + B_c u_c(t) \quad (12)$$

$$e_{x_i}(t) = C e_{x_i}(t) \quad$$

$$\dot{e}_{y_i}(t) = C e_{x_i}(t) + v(t) \quad (13)$$

where $\Delta A_{il}$ is the difference between the system matrices, $A_l$ and $A_i$ of $P_l(s)$ and $P_i(s)$, respectively, $x_i(t)$ is the state vector of $P_i(s)$, $B_c = I \in \mathbb{R}^{11 \times 1}$ is the input matrix of error dynamics, and $u_c(t) = -L_t e_{y_i}(t)$. Applying this $u_c(t)$ in

$$\hat{e}_{x_i}(t) = (A_l - B_c C L_t) e_{x_i}(t) + \Delta A_{il} x_i(t) - B_c L_t v(t) \quad (13)$$

Equation (13) suggests that, along with $v(t)$, $x_i(t)$ also becomes an exogenous signal vector that adversely affect the state estimation error dynamics. This is because of the difference between the system matrices of $P_l(s)$ and $P_i(s)$, respectively. Hence, when $\hat{P}_i(s)$ estimates $z_i(t)$ of $P_i(s) \in \mathcal{P}$, then $d \in \mathbb{R}^{n \times r \times 1}.$

Note that $x_i(t)$ needs to be a bounded energy exogenous signal. For that either $P_i(s)$ or its closed-loop plant must to be stable. Further, when $\hat{P}_i(s)$ becomes the common estimator of all the plants belonging to $\mathcal{P}$, then there exist $N$ number of closed-loop transfer function matrices, $e(s)_{P_i, \hat{P}_i} \in \mathcal{R} H_{\infty}$, from $d(t) \in \mathbb{R}^{n \times r \times 1}$ to $e_i(t)$, for all $i \in \{1, \ldots, l, \ldots, N\}$. Now, consider $N$ number of estimators, $\hat{P}_1(s), \ldots, \hat{P}_i(s), \ldots, \hat{P}_N(s)$, and define the plant as unstable coupled uncertain plants of MERS estimator is given in the next subsection.

B. Minimum Error Robust Simultaneous Estimation Problem

To describe the MERS estimation problem, consider $\mathcal{P} \subset \mathcal{R} L_{\infty}$ containing $N$ number of stabilizable and detectable LTI MIMO unstable and coupled uncertain plants of the NAV given in (2). The state-space form of any plant belonging to $\mathcal{P}$ is given by (1) with $B_1 = B, C_1 = C, n_\pi = 11, m_\pi = 3, \hat{r} = 5, \hat{q} = 6$. Let $\hat{P}_i(s) \in \mathcal{R} H_{\infty}$ represents an estimator. The estimator, $\hat{P}_i(s)$, is formed using the state-space matrices of $P_i(s) \in \mathcal{P}$ and a suitable estimator gain, $L_t \in \mathbb{R}^{11 \times 5}$. Following this, the state-space model of $\hat{P}_i(s)$ is given by (2) with $\hat{x}_i(t) \in \mathbb{R}^{11 \times 1}$ and $\hat{z}_i(t) \in \mathbb{R}^{5 \times 1}$. Here, $\hat{z}_i(t)$ provides the estimation of $z_i(t)$, $\forall i \in \{1, \ldots, N\}$. When we consider $\hat{P}_i(s)$ as the common estimator of $P_i(s) \in \mathcal{P} \forall i \in \{1, \ldots, N\}$, then the state estimation error vectors and the estimation error vectors are denoted by $e_{x_i}(t) = x_i(t) - \hat{x}_i(t)$ and $e_{z_i}(t) = z_i(t) - \hat{z}_i(t)$ for all $i \in \{1, \ldots, l, \ldots, N\}$, respectively. Similarly, for all $i \in \{1, \ldots, l, \ldots, N\}$, $e_{y_i}(t) = y_i(t) - Cx_i(t).$ Now, consider the case where $\hat{P}_i(s)$ is employed to estimate $x_i(t) \forall i \in \{1, \ldots, N\}$ of $P_i(s) \in \mathcal{P} \forall i \in \{1, \ldots, N\}$, respectively. Then, one can obtain $N$ number of estimator error models that are given by

$$\dot{e}_{x_i}(t) = A_i e_{x_i}(t) + \Delta A_{il} x_i(t) + B_c u_c(t) \quad (12)$$

$$e_{x_i}(t) = C e_{x_i}(t) \quad$$

$$\dot{e}_{y_i}(t) = C e_{x_i}(t) + v(t) \quad (13)$$

where $\Delta A_{il}$ is the difference between the system matrices, $A_l$ and $A_i$ of $P_l(s)$ and $P_i(s)$, respectively, $x_i(t)$ is the state vector of $P_i(s)$, $B_c = I \in \mathbb{R}^{11 \times 1}$ is the input matrix of error dynamics, and $u_c(t) = -L_t e_{y_i}(t)$. Applying this $u_c(t)$ in

$$\dot{e}_{x_i}(t) = (A_l - B_c C L_t) e_{x_i}(t) + \Delta A_{il} x_i(t) - B_c L_t v(t) \quad (13)$$

Now, if we consider $\hat{P}_i(s) \in \mathcal{P}$ as the simultaneous estimator of all the plants belonging to $\mathcal{P}$, then $\|e(s)_{P_i, \hat{P}_i}\|_\infty \forall i \in \{1, \ldots, l, \ldots, N\}$ are the worst-case gains from $d(s)$ associated with $\hat{P}_i(s)$. Also, the largest worst-case gain from $d(s)$ associated with $\hat{P}_i(s)$ is $\max \{\|e(s)_{P_i, \hat{P}_i}\|_\infty \forall i \in \{1, \ldots, l, \ldots, N\}\}$. The largest worst-case gain from $d(s)$ associated with an estimator belongs to $\mathcal{P}$ occurs while estimating the desired state variables of a plant referred to as the worst plant from the perspective of the simultaneous estimation process. This worst plant is indicated through the subscript $k$ along with a superscript that shows its association with the corresponding estimator. Following this, the largest worst-case gain from $d(s)$ associated with $\hat{P}_i(s) \in \mathcal{P}$.

$$\|e(s)_{P_i, \hat{P}_i}\|_\infty = \max\{\|e(s)_{P_i, \hat{P}_i}\|_\infty \forall i \in \{1, \ldots, l, \ldots, N\}\} \quad (15)$$

From (4) and (15), the sufficient conditions for considering
\( \hat{P}_j(s) \in \mathcal{P} \) as a simultaneous estimator of all the plants belonging to \( \mathcal{P} \) with respect to Condition II are \( e(s)_{\hat{P}_j} < \gamma \forall i \in \{1, \ldots, l, \ldots, N\} \). An equivalent single condition that satisfy the above conditions can be obtained using (16), and is given by

\[
\| e(s)_{\hat{P}_j} \|_\infty < \gamma
\]  

(17)

Now, the MERS estimation problem can be defined precisely as: Given \( \mathcal{P} \), find \( \hat{P}_j(s) \in \mathcal{P} \) along with \( L_j \) such that 1) \( \hat{P}_j(s) \) simultaneously estimates \( z_i(t) \) of \( P_i(s) \in \mathcal{P} \) \( \forall i \in \{1, \ldots, j, \ldots, N\} \) even when these plants have model uncertainties and 2) the condition given by

\[
\| e(s)_{\hat{P}_j} \|_\infty = \min \left\{ \| e(s)_{\hat{P}_j} \|_\infty \mid l \in \{1, \ldots, N\} \right\} < \gamma
\]

(18)

is satisfied.

The solution to this problem is a RS estimator referred to as the MERS estimator whose \( \| e(s)_{\hat{P}_j} \|_\infty \) is the smallest among \( \{\| e(s)_{\hat{P}_j} \|_\infty \mid l \in \{1, \ldots, j, \ldots, N\} \} \) with \( \| e(s)_{\hat{P}_j} \|_\infty < \gamma \). This suggests that the largest worst-case estimation error of MERS estimator is the minimal among the largest worst-case estimation errors of simultaneous estimators. In this paper, we consider only parametric uncertainties in the form of bounded perturbations in the system matrices. The performance of the MERS estimator can be further improved by appending suitable compensators to the plant dynamics. This is discussed in the next section.

C. Gap Reducing Compensator Problem

The Gap Reducing (GR) compensator problem is about finding those compensators that modify the input-output characteristics of all the plants belonging to \( \mathcal{P} \) for reducing further the estimation errors arising from the differences between the system matrices of the MERS estimator and \( P_j(s) \forall i \in \{1, \ldots, j, \ldots, N\} \) (\( \Delta A_{ij} \)). The GR compensator problem can be stated as: Assume that there exists a MERS estimator, \( \hat{P}_j(s) \in \mathcal{P} \), for the plants belonging to \( \mathcal{P} \). Now, consider the plant, \( P_j(s) \in \mathcal{P} \) and assume \( \mathcal{P} \) as an uncertainty model set with \( P_j(s) \) as the nominal plant. Besides this, assume also that the plants belonging to \( \mathcal{P} \) are the perturbed plants of \( P_j(s) \). Let \( N_j(s) \in \mathcal{R} \mathcal{H}_\infty \) and \( M_j(s) \in \mathcal{R} \mathcal{H}_\infty \) with \( \text{det}(M_j(s)) \neq 0 \) are the normalized right coprime factors of \( P_j(s) \). Subsequently, \( P_j(s) \) is given by

\[
P_j(s) = N_j(s)M^{-1}_j(s)
\]

(19)

Let \( \Delta N_{p_j}(s) \in \mathcal{R} \mathcal{H}_\infty \) and \( \Delta M_{p_j}(s) \in \mathcal{R} \mathcal{H}_\infty \) are the right coprime factor perturbations of \( N_j(s) \) and \( M_j(s) \) with

\[
\| [\Delta N_{p_j}(s) \Delta M_{p_j}(s)]^T \|_\infty \leq \epsilon_{p_j}, \quad \text{respectively.}
\]

Here, \( \epsilon_{p_j} \) is the least upper bound on the right coprime factor perturbations. Then, \( P_i(s) \forall i \in \{1, \ldots, N\} \setminus j \) are defined as

\[
P_i(s) = (N_j(s) + [\Delta N_{p_j}(s)])(M_j(s) + [\Delta M_{p_j}(s)])^{-1}
\]

(20)

\( \forall i \in \{1, \ldots, N\} \setminus j \)

Now, consider the scenario where \( v(t) = 0 \) and \( \hat{P}_j(s) \) is employed to estimate \( z_i(t) \) of \( P_j(s) \). To realize this scenario,

the state vector of the plant needs to be bounded. For that, a feedback controller is employed as shown in Fig. 2. Now, assume \( P_j(s) \) is perturbed to form \( P_j(s) \in \mathcal{P} \) when \( \hat{P}_j(s) \) estimates \( z_i(t) \) of \( P_j(s) \). In that case, \( P_j(s) \) receives \( u_i(t) = u_j(t) + \Delta u_i(t) \) and \( y_i(t) = y_j(t) + \Delta y_i(t) \) instead of \( u_j(t) \) and \( y_j(t) \), respectively as shown in Fig. 2. Here, \( \Delta u_i(t) \) and \( \Delta y_i(t) \) are the perturbations in the inputs and outputs of \( P_j(s) \), respectively arising from \( \Delta A_{ij} \). So when we develop the estimation error dynamics offline, the plant considered is \( P_j(s) \). But in reality, the inputs to \( \hat{P}_j(s) \) will be \( u_i(s) \) and \( y_i(s) \) of the perturbed plant of \( P_j(s) \). Following this, the estimation error dynamics is given by

\[
e_{x_{ij}}(t) = (A_j - B_jL_jC)e_{x_{ij}}(t) - B\Delta u_i(t) - L_j\Delta y_i(t)
\]

\[
e_{z_{ij}}(t) = C_ze_{x_{ij}}(t)
\]

(21)

Let the eigenvalues of \( (A_j - B_jL_jC) \) belong to \( \mathbb{C} \). Then, (21) suggests that \( e_{x_{ij}}(t) \) converges to zero when \( u_i(t) = u_j(t) \) and \( y_i(t) = y_j(t) \). Following this, \( e_{x_{ij}}(t) \) will be closer to zero if we make \( u_i(t) \) and \( y_i(t) \) closer to \( u_j(t) \) and \( y_j(t) \), respectively.

This indicates that when \( e_{x_{ij}}(t), e_{y_{ij}}(t) \in \{1, \ldots, N\} \setminus \{j\} \) become closer to \( u_j(t) \), then the estimation error due to \( \Delta A_{ij} \) at \( t \in \{1, \ldots, N\} \setminus \{j\} \) becomes close to zero.

To state the GR compensator problem, consider the following definition.

**Definition II.1. Graph, \( \mathcal{G}, \) of an operator:** Let \( P_j(s) : \mathcal{H} \to \mathcal{H} \) be any linear operator in Hilbert space \( \mathcal{H} \) defined on the domain, \( D(P_j(s)) \subseteq \mathcal{H} \). Then, the **graph** of \( P_j(s) \), \( \mathcal{G}(P_j) \) is defined as

\[
\mathcal{G}(P_j) = \{ (u_j(s), y_j(s)) \in \mathcal{H} \times \mathcal{H} : u_j(s) \in D(P_j(s)), y_j(s) = P_j(s)u_j(s) \in \mathcal{H} \}
\]

(22)

This definition suggests that \( \mathcal{G}(P_j) \) is the set of all pairs of \( (u_j(s), y_j(s)) \) with \( u_j(s) \in D(P_j(s)) \). Also, this definition indicates that making \( \mathcal{G}(P_j) \forall i \in \{1, \ldots, N\} \setminus \{j\} \) closer to \( \mathcal{G}(P_j) \) increases the closeness between \( (u_i(t), y_i(t)) \forall i \in \{1, \ldots, N\} \setminus \{j\} \) and \( (u_j(t), y_j(t)) \). Casading the plants with pre and post compensators, \( W_{in}(s) \in \mathcal{R} \mathcal{H}_\infty \) and \( W_{out}(s) \in \mathcal{R} \mathcal{H}_\infty \) respectively, we modify the input-output characteristics and thereby the **graphs** of the plants. Now, the GR compensator problem can be stated as: Find \( W_{in}(s) \in \mathcal{R} \mathcal{H}_\infty \) and \( W_{out}(s) \in \mathcal{R} \mathcal{H}_\infty \) such that

1. \( \max\{\text{gap}(\mathcal{G}(P_j(s)), \mathcal{G}(\hat{P}_j(s))) \mid i \in \{1, \ldots, j, \ldots, N\} \} \) is lower than \( \max\{\text{gap}(\mathcal{G}(P_j(s)), \mathcal{G}(P_i(s))) \mid i \in \{1, \ldots, j, \ldots, N\} \} \)
2. \( \max\{\text{gap}(\mathcal{G}(P_j(s)), \mathcal{G}(\hat{P}_j(s))) \mid i \in \{1, \ldots, j, \ldots, N\} \} \) is closer to zero.

where \( \hat{P}_j(s) = W_{out}(s)P_j(s)W_{in}(s) \) and \( \hat{P}_j(s) = W_{out}(s)P_j(s)W_{in}(s) \).

In the next section, we present the synthesis of GRMERS estimator for a typical unstable and highly coupled plants of a NAV.
III. SYNTHESIS OF GAP REDUCED MINIMUM ERROR
ROBUST SIMULTANEOUS ESTIMATOR

The Gap Reduced Minimum Error Robust Simultaneous
(GRMERS) estimator comprises of a MERS estimator and
the GR compensators. Here, the MERS estimator performs
robust simultaneous estimation with minimal largest worst-
error is analyzed first. In this section, we first explain the procedure for synthesising
the matrices of plants and estimator as indicated by (21). In this
section, we first explain the procedure for synthesising
the MERS estimator model. Before carrying out this procedure,
as preliminaries, the effect of $v(t)$ and $x_i(t)$ on the estimation
error is analyzed first.

A. Preliminaries: Analysis of the Effect of Measurement Noise
and State Vector on Estimation Error Dynamics

Here, the effects of $v(t)$ and $x_i(t)$ on the estimation error
are briefly analyzed. For that, consider $\hat{P}_i(s) \in \mathcal{P}$ as the
simultaneous estimator for estimating $z_i(t)$ of $P_i(s) \in \mathcal{P}$
for all $i \in \{1, \ldots, l, \ldots, N\}$. We now define $e(s)_{P_i\hat{P}_i} \forall i \in \{1, \ldots, l, \ldots, N\}$ using (12) and $u_e(t) = -L_i e_{x_i}(t)$ as

$$
e(s)_{P_i\hat{P}_i} = -C_z(sI - (A_l - B_L L_C))^{-1} A_{il}$$

$$- C_z(sI - (A_l - B_L L_C))^{-1} B_L e_i; \quad (23)$$

In (23), $-C_z(sI - (A_l - B_L L_C))^{-1} A_{il}$ is the transfer
function matrix from $x_i(t)$ to $e_{z_i}(s)$ and $-C_z(sI - (A_l - B_L L_C))^{-1} B_L e_i$ is the transfer function matrix from $v(t)$ to $e_{z_i}(s)$. Now, when $\hat{P}_i(s)$ estimates $z_i(t)$ of $P_i(s)$, then $A_{il}$ is a null matrix and $x_i(t) = 0$. Following this, if $d(t) = v(t) = 0$ and all the eigenvalues of $[A_l - B_L L_C]$ belong to $\mathbb{C}_-$, then
$
\lim_{t \to \infty} e_{z_i}(t) = 0.
$

This indicates that when any estimator, say $\tilde{P}_i(s) \in \tilde{P}$, estimates $z_i(t)$ of $P_i(s) \in \tilde{P}$, then
$\lim_{t \to \infty} e_{z_i}(t) = 0$ if $v(t)=0$ and $\hat{P}_i(s) \in \mathcal{R}H_{\infty}$. However, when $\hat{P}_i(s)$ estimates $z_i(t)$ of $P_i(s)$ with $v \neq 0$ and $x_i(t) = 0$, then $e_{z_i}(s) = -C_z(sI - (A_l - B_L L_C))^{-1} B_L e_i v(t)$. Subsequently, in time domain,$
e_{z_i}(t) = \int_0^t -C_z u([A_l - B_L L_C](t-\tau)) B_L e_i v(\tau) d\tau.
$

This integral will never be zero when $v(t) \neq 0$, the estimation errors do not converge to zero during the simultaneous estimation process. The same
phenomenon will be there even when $x_i(t) \neq 0$. Equation (23) indicates that $-C_z(sI - (A_l - B_L L_C))^{-1} B_L e_i$ are same for all $i \in \{1, \ldots, l, \ldots, N\}$. This indicates that the effect of $v(t)$ on the estimation errors remains identical
when an estimator belonging to $\hat{P}$ performs simultaneous
estimation. However, (23) shows that $-C_z(sI - (A_l - B_L L_C))^{-1} B_L e_i$ are different when $A_{il} \neq 0$. This proposes that the effect of $x_i(t)$ on the estimation errors may be
dissimilar when an estimator belongs to $\hat{P}$ executes simultaneous
estimation. Applying $\mathcal{H}_{\infty}$ norm on both sides of (23)
and then using triangle and Cauchy-Schwarz inequalities, the
$|e(s)_{P_i\hat{P}_i}|_{\infty} \forall i \in \{1, \ldots, l, \ldots, N\}$ can be written as

$$|e(s)_{P_i\hat{P}_i}|_{\infty} \leq ||-C_z(sI - (A_l - B_L L_C))^{-1}||\|A_{il}\|_{\infty} + ||-C_z(sI - (A_l - B_L L_C))^{-1} B_L e_i||_{\infty}; \quad (24)$$

Equation (24) indicates that the estimation error due to $x_i(t)$
increases with the increase of $||A_{il}\|_{\infty}$. Next, the develop-
ment of a sufficient condition for the existence of a RS estimator based on above arguments is presented.

B. Minimum Error Robust Simultaneous Estimator

In this subsection, we describe the development of the MERS estimator model and the MERSE algorithm. At first, the
sufficient condition for the existence of a robust simultaneous estimator is derived.

1) Sufficient Condition for the Existence of a Robust Simultaneous Estimator: We now state the sufficient condition for
the existence of a RS estimator through the following theorem.

\textit{Theorem III.1.} Given $\mathcal{P}$, $\bar{P}$, and $\gamma$. Consider $\tilde{P}_i(s) \in \mathcal{P}$ as the simultaneous estimator of $P_i(s) \in \mathcal{P} \forall i \in \{1, \ldots, k, \ldots, l, \ldots, N\}$. Let $P_k(s) \in \mathcal{P}$ satisfies the condition given by

$$||A_{ik}\|_{\infty} = \max\{||A_{ik}\|_{\infty} \} \forall i \in \{1, \ldots, k, \ldots, l, \ldots, N\} \quad (25)$$

Then, $P_k(s) = P_k(s)$ and the sufficient condition for the existence of $\tilde{P}_i(s)$ as the RS estimator of all the plants of $\mathcal{P}$ is given by

$$|e(s)_{P_i\tilde{P}_i}|_{\infty} < \gamma \quad (26)$$

\textit{Proof.} Using (24), let $|e(s)_{P_i\tilde{P}_i}|_{\infty} \forall i \in \{1, \ldots, l, \ldots, N\}\} \{k\}$ are expressed as
These perturbed plants arise due to the perturbations in the condition for the existence of $P_k(s)$ as the MERS estimator is given in (18). Hence, to solve MERS estimation problem, we have to solve (18).

2) Method to Solve the Minimum Error Robust Simultaneous Estimation Problem: We now present the method that solves the MERS estimation problem. The MERS estimator synthesis is about finding $\hat{P}_j(s)$ with a static gain $L_j$, such that the condition given in (18) is satisfied. Equation (18) indicates that the solution of it follows by solving (32). For this, the inequalities given in (32) is formulated in terms of linear matrix inequalities (LMIs) using bounded real lemma [14]. If there exists $Q_l > 0$ and $Y_l \in \mathbb{R}^{n \times d}$, then from the bounded real lemma, the LMIs corresponding to (32) for $0<\gamma<1$ are given by

$$\begin{bmatrix}
Q_l A_l^T - Y_l C - C^T Y_l^T & \dot{Y}_l \\
* & -\gamma I
\end{bmatrix} < 0$$

where $Q_l = Q_l^{-1} Y_l \forall l \in \{1, \ldots, j, \ldots, N\}$ and the post compensator, $\hat{P}_j(s)$ satisfies (25), Since, RHS of (29) is negative, we can rewrite (29) as

$$\|e(s)\|_{\infty} - \|\hat{e}(s)\|_{\infty} \leq -\Lambda_i$$

where $\Lambda_i$ is a positive constant. Note that $\|e(s)\|_{\infty} > 0 \forall i \in \{1, \ldots, l, \ldots, N\} \setminus \{k\}$ and $\|\hat{e}(s)\|_{\infty} > 0$. Consequently, the condition given in (30) ensures the condition given by

$$\|e(s)\|_{\infty} \forall i \in \{1, \ldots, l, \ldots, N\} \setminus \{k\} < \|\hat{e}(s)\|_{\infty}$$

Equation (31) implies $P_{i\rightarrow}^k(s) = P_k(s)$ and $\|e_{i\rightarrow}^k(s)\|_{\infty} = \|e_{i\rightarrow}^k(s)\|_{\infty}$ when $P_k(s)$ satisfies (25). Then, from (17), the sufficient condition for the existence of $P_i(s)$ as the simultaneous estimator of $N$ number of plants belonging to $P$ is (26).

Now consider the case of RS estimation. For that, let $P_i(s) \notin P \forall i \in \{1, \ldots, k, \ldots, l, \ldots, N\}$ are the perturbed plants of $P_i(s) \in P \forall i \in \{1, \ldots, k, \ldots, l, \ldots, N\}$, respectively. These perturbed plants arise due to the perturbations in the system matrix of $P_i(s) \in P \forall i \in \{1, \ldots, k, \ldots, l, \ldots, N\}$. Following this, the system matrix of $P_i(s)$ be $A_i = A_i + \Delta A_i$ with $\|\Delta A_i\|_{\infty} < \|\Delta A_{kl}\|_{\infty}$. Here, $\Delta A_i$ is the bounded perturbation of $A_i$. Because of $\|\Delta A_i\|_{\infty} \leq \|\Delta A_{kl}\|_{\infty}$ it obvious that $\|e(s)\|_{\infty} < \|\hat{e}(s)\|_{\infty} \forall i \in \{1, \ldots, k, \ldots, l, \ldots, N\}$. So, if $\|e(s)\|_{\infty} < \gamma$ then $\|e(s)\|_{\infty} < \gamma \forall i \in \{1, \ldots, k, \ldots, l, \ldots, N\}$. Hence, concerning $P_i(s)$, $\|e(s)\|_{\infty} < \gamma$ is the sufficient condition for the existence of $P_i(s)$ as the RS estimator of all the plants of $P$. This establishes the proof of the theorem.

Following Theorem [III.1] the sufficient condition for the existence of the estimators, $P_l(s) \in P \forall l \in \{1, \ldots, i, \ldots, N\}$, as a RS estimator is given by

$$\|e(s)\|_{\infty} < \gamma \forall l \in \{1, \ldots, i, \ldots, N\}$$

Furthermore, Theorem [III.1] suggests that the sufficient condition for the existence of $P_j(s)$ as the MERS estimator is given in (18). Hence, to solve MERS estimation problem, we have to solve (18).
Then, the output of $\widetilde{W}_{eo}(s)$ from $v(t), \tilde{v}(t)$, is defined as $\tilde{v}(t) = L^{-1}[W_{eo}(s)v(s)](t)$. Here, $L^{-1}$ denotes the inverse Laplace transform. Considering this, the state-space form of $P_i(s)$ is given by

$$
\begin{align*}
\dot{\tilde{x}}_i(t) &= A_{\tilde{x}}\tilde{x}_i(t) + B\tilde{u}_i(t) \\
\tilde{z}_i(t) &= C_{\tilde{x}}\tilde{x}_i(t) + \tilde{v}(t)
\end{align*}
$$

where $\tilde{x}_i(t) = [x_{eo\,i}, x_{eo\,i}, x_{eo\,i}]^T \in \mathbb{R}^{22}$ is the state vector. Here, $x_{eo\,i} \in \mathbb{R}^3$ and $x_{eo\,i} \in \mathbb{R}^3$ are the state vectors of post and pre compensators, respectively. Besides, $A_{\tilde{x}} \in \mathbb{R}^{22 \times 22}, B \in \mathbb{R}^{22 \times 3}$, and $C_{\tilde{x}} \in \mathbb{R}^{5 \times 22}$ are the system matrix, the control input matrix, and the measurement matrix, respectively. Likewise, $C_{\tilde{x}} = [0_{3,5} | C_{\tilde{x}} | 0_{5,3}]$ is a constant matrix. Additionally, $\tilde{z}_i(t)$ is the vector to be estimated. The characteristic of $\tilde{x}_i(t)$ and $C_{\tilde{x}}$ suggests $\tilde{z}_i(t) = z_i(t)$. Now, consider a finite set, $\mathcal{P} = \{P_i(s) \in \mathcal{RL} \mid P_i(s) = \tilde{W}_{eo}(s)P_i(s)\tilde{W}_{eo}(s), P_i(s) \in \mathcal{P}, i \in \{1, \ldots, n\}\}$. The state-space form of any plant belongs to $\mathcal{P}$ is given by (34). Now, consider $N$ number of estimators, $\hat{P}_1(s), \ldots, \hat{P}_i(s), \ldots, \hat{P}_N(s)$, each formed using state-space matrices of $P_i(s) \in \mathcal{P}, \ldots, P_i(s) \in \mathcal{P}, \ldots, P_i(s) \in \mathcal{P}$, respectively. Let the finite set, $\mathcal{P}$, contains all these $N$ estimators. The state-space forms of these estimators are given as

$$
\begin{align*}
\begin{cases}
\dot{\tilde{x}}_i(t) &= A_{\tilde{x}}\tilde{x}_i(t) + B\tilde{u}_i(t) + \tilde{L}_i(t) (\tilde{y}(t) - C_{\tilde{x}}\tilde{x}_i(t)) \\
\tilde{z}_i(t) &= C_{\tilde{x}}\tilde{x}_i(t); \forall l \in \{1, \ldots, i, \ldots, j, \ldots, N\}
\end{cases}
\end{align*}
$$

where $\tilde{u}_i(t) \in \{\tilde{u}_i(t), \ldots, \tilde{u}_i(t), \ldots, \tilde{u}_i(t)\}$ and $\tilde{y}_i(t) \in \{\tilde{y}_i(t), \ldots, \tilde{y}_i(t), \ldots, \tilde{y}_i(t)\}$. Here, $\tilde{u}_i(t)$ and $\tilde{y}_i(t)$ are the control input and measurement vectors of $P_i(s) \in \mathcal{P}$, respectively. Also, $\tilde{x}_i(t) \in \mathbb{R}^{22}$ and $\tilde{u}_i(t) \in \mathbb{R}^3$ is the state vector of $P_i(s)$. Additionally, $\tilde{z}_i(t) \in \mathbb{R}^3$ is the estimate of $z_i(t)$. Using Theorem III.1 the sufficient condition for the existence of estimators, $P_i(s) \in \mathcal{P} \forall l \in \{1, \ldots, i, \ldots, j, \ldots, N\}$, as a RS estimator is given by

$$
\|e(s)\|_\infty < \gamma; \forall l \in \{1, \ldots, i, \ldots, j, \ldots, N\}
$$

We solve (35) by solving the equivalent LMIs to obtain the estimator gains, $\tilde{L}_l \forall l \in \{1, \ldots, i, \ldots, j, \ldots, N\}$, mentioned in (37). If there exists $\tilde{Q}_l > 0 \in \mathbb{S}^n$ and $\tilde{Y}_l \in \mathbb{R}^{n \times d}$, then from the bounded real lemma [14], the LMIs corresponding to (36) for a $0 < \gamma \leq 1$ are given by

$$
\begin{align*}
\begin{bmatrix}
\tilde{Q}_l A_{\tilde{x}} + A_{\tilde{x}}^T \tilde{Q}_l - \tilde{Y}_l C - C^T \tilde{Y}_l^T & \tilde{Q}_l B_t - \tilde{Y}_l D_t & C_{\tilde{x}}^T \\
-\gamma I & 0 & 0 \\
* & \gamma & -\gamma I
\end{bmatrix} < 0
\end{align*}
$$

where $\tilde{B}_t = [\Delta \bar{A}_{kl} 0]$ and $\bar{D} = [0 I] \in \mathbb{R}^{5 \times 5}$. Here, $\Delta \bar{A}_{kl}$ is the difference between the system matrices of $P_i(s)$ and $P_i(s)$. Now, for a given $\gamma$, solve (37) for all $\tilde{Q}_l$ and $\tilde{Y}_l$. Thereafter, recover all the estimator gain using $\tilde{L}_l = \tilde{Q}_l^{-1}\tilde{Y}_l \forall l \in \{1, \ldots, i, \ldots, j, \ldots, N\}$. Theorem III.1 suggests that the estimators belong to $\mathcal{P}$.
Algorithm 1 Pseudocode of MERSE algorithm

1: Initialize: Genetic algorithm
2: Input: $P$, maximum number of generations, and $\gamma$
3: if number of generation of GA solver $\leq$ maximum value then
4:   GA obtain feasible values of search variables
5:   Compute: $\hat{W}_{ei}(s)$ and $\hat{W}_{eo}(s)$ using (15) and (16) of [12].
6:   Compute: $\hat{L}_i \forall i \in \{1, \ldots, j, \ldots, N\}$ by solving N LMI$s$ given in (37).
7:   Compute: $J$ for fitness evaluation
8:   Fitness value of GA $=$ $J$
9:   go to 3
10: else
11:   if Fitness value $< 1$ then
12:      Find $\hat{P}_j(s)$ and $\hat{L}_j$
13:      Output: feasible $\hat{W}_{ei}(s)$ and $\hat{W}_{eo}(s)$, $\hat{P}_j(s)$, $\hat{L}_j$
14:      Exit
15:   else
16:      Output: no feasible solution to (40)
17:      Exit
18: end if
19: end if

Figure 3: MERS estimator for the NAV (from Algorithm 1) shown inside the orange box

C. Gap Reducing Compensators

The development of the GRC algorithm is presented in this section. The gap reducing compensators obtained through the GRC algorithm are augmented with the plants in $P$ in the implementation of GRMERS estimator to further reduce the estimation error obtained from the MERS estimator. The method developed in this section depends only on the plants in $P$ and is independent of the MERS compensators $W_{ei}(s)$, $W_{eo}(s)$ and the gain $L_j$. From the output $P_j(s)$ of the MERSE algorithm, the maximum value of $\nu$-gap metric of the associated plant $P_j(s) \in P$ with the other plants in $P$ is reduced further by adding suitable pre and post compensators to the plants in $P$. From the definition of $G(P_j(s))$, making $G(P_j(s)) \forall i \in \{1, \ldots, N\} \setminus \{j\}$ closer to $G(P_j)$ increases closeness between $(u_i, y_i) \forall i \in \{1, \ldots, N\} \setminus \{j\}$ and $(u_j, y_j)$. To make $G(P_j(s)) \forall i \in \{1, \ldots, N\} \setminus \{j\}$ closer to $G(P_j(s))$, the gap between $G(P_i(s)) \forall i \in \{1, \ldots, N\} \setminus \{j\}$ and $G(P_j(s))$ needs to be reduced. For developing an algorithm that minimizes the gap between $G(P_i(s)) \forall i \in \{1, \ldots, N\} \setminus \{j\}$ and $G(P_j(s))$, it is necessary to compute the gap between the graphs. Let $\delta_1(P_1(j\omega), P_2(j\omega)) \in [0, 1]$ be the $\nu$-gap metric (see [12] for details) between two plants $P_1(s)$ and $P_2(s)$. Then, the gap between two graphs is given by [13]

$$\text{gap}(G(P_1), G(P_2)) = \delta_1(P_1(j\omega), P_2(j\omega))$$

Using (41), gap between two graphs is computed. Let $\epsilon_{P_j}$ denotes the maximum $\nu$–gap metric of $P_j(s)$. Then, $\epsilon_{P_j} = \max \{ \text{gap}(P_j(j\omega), P_i(j\omega)) \mid P_i(s), P_i(s) \in P \forall i \in \{1, 2, \ldots, N\} \}$. Now, using (41), $\epsilon_{P_j}$ is rewritten as

$$\epsilon_{P_j} = \max \{ \delta_1(P_j(j\omega), P_i(j\omega)) \mid P_j(s), P_i(s) \in P \forall i \in \{1, 2, \ldots, N\} \}$$

Equation (42) suggests that to make $G(P_j) \forall i \in \{1, \ldots, N\} \setminus \{j\}$ closer to $G(P_j)$, we need to reduce $\epsilon_{P_j}$ and bring it closer towards zero. The maximum gap of $P_j(s)$ is improved by cascading these models with suitable pre and post compensators, $W_{in}(s) \in RH_{\infty}$ and $W_{ot}(s) \in RH_{\infty}$, respectively [12]. Simultaneously, if required, these compensators can be employed to improve the frequency characteristics of the plants in $P$. The basic structure of $W_{in}(s)$ and $W_{ot}(s)$ are the same as that of $\hat{W}_{in}(s)$ and $\hat{W}_{ot}(s)$, respectively. However, $W_{ot}(s)$ needs to be strictly proper. Let $\kappa = \{ \hat{P}_i(s) \in RH_{\infty} \mid \hat{P}_i(s) = W_{ot}(s)P_i(s)W_{in}(s), P_i(s) \in P, W_{ot}(s) \in RH_{\infty}, W_{in}(s) \in RH_{\infty}, \forall i \in \{1, 2, \ldots, N\} \}$. Now, $\epsilon_{P_j}$ is defined as

$$\epsilon_{P_j} = \max \{ \delta_1(\hat{P}_j(j\omega), \hat{P}_i(j\omega)) \mid \hat{P}_j(s), \hat{P}_i(s) \in \kappa, \forall i \in \{1, 2, \ldots, N\} \}$$

Then, from the GR compensators problem statement, the feasible $W_{in}(s)$ and $W_{ot}(s)$ are those that achieve the following.

1. $\epsilon_{P_j} < \epsilon_{P_j}$ and bring $\epsilon_{P_j}$ closer to zero.
2. $W_{in}(s)$ and $W_{ot}(s)$ induce desired frequency characteristics on all the plants belonging to $\kappa$.

As there does not exist any closed-form solution for the feasible $W_{in}(s)$ and $W_{ot}(s)$, an optimization problem is formulated and is given by

$$\min_{Q} J_1 = \epsilon_{P_j}$$

subject to

1. Bound constraints on the coefficients of pre and post compensators
2. No pole-zero cancellation between compensators and the plants of $P$

In (44), $Q$ represents the set that contains the coefficients of $W_{in}(s)$ and $W_{ot}(s)$. The bound constraints on the coefficients of compensators provide desired frequency characteristics to the plants of $P$. These constraints prevent the
minimization of $J_1$ with any $W_{\text{in}}(s)$ and $W_{\text{ot}}(s)$ that degrade the frequency characteristics of all the augmented plants. Note that the pre and post compensators are physically present in the closed-loop and therefore, these compensators need to be appended to the hardware. The performance index of (44) is non-convex and non-smooth. Hence, the optimization problem given in (44) is solved using an iterative algorithm referred to as the GRC algorithm that has a population-based genetic algorithm (GA) solver. In that solver, GA employs the same steps as in MERSE algorithm.

Search Variables: The search variables of the GA solver are the coefficients of $W_{\text{ot}}(s)$ and $W_{\text{in}}(s)$. The feasible values of these search variables are those which satisfy all the constraints of the problem given in (44).

Fitness functions: The fitness function of the GA solver is the performance index, $J_1$, of the optimization problem given in (44).

Termination Conditions: The iteration terminates when the number of generations of GA solver exceeds the set maximum value.

The pseudocode for the iterative algorithm is given in Algorithm 2. The RS estimation using GRMERS estimator is depicted in Fig. 4 This figure consists of the MERS estimator (shown inside the orange box) obtained from Algorithm 1 and the gap reducing compensator (shown inside the blue box) obtained from Algorithm 2. In Fig. 4, the blocks, $W_{\text{in}}(s) \triangleq \begin{bmatrix} A_{\text{in}} & B_{\text{in}} \\ C_{\text{c}} & D_{\text{c}} \end{bmatrix}$ and $W_{\text{ot}}(s) \triangleq \begin{bmatrix} A_{\text{ot}} & B_{\text{ot}} \\ C_{\text{c}} & D_{\text{c}} \end{bmatrix}$ denote the state-space representation of $W_{\text{in}}(s)$ and $W_{\text{ot}}(s)$ respectively. The plants of $\mathcal{P}$ are unstable and hence they are stabilized using a full state feedback controller, $K$ that uses all the states of $W_{\text{ot}}(s)P_i(s)W_{\text{in}}(s)$ for feedback. These states are obtained by augmenting states of $W_{\text{in}}(s)$ and $W_{\text{ot}}(s)$ and the estimated states of $P_i(s)$, $\hat{x}_i(t)$ as shown in Fig. 4. Here, $\hat{x}_i(t)$ is obtained by augmenting $y_i(t)$ with $z_i(t)$. Note that if we employ MERS estimator only, then the full state vector for feedback is $\hat{z}_i(t)$.

![Figure 4: GRMERS estimator for the NAV with the MERS estimator (from Algorithm 1) and the GR compensators (from Algorithm 2) shown inside the orange and blue boxes, respectively](image)

Algorithm 2 Pseudocode of GRC algorithm

1: Initialize: Genetic algorithm
2: Input: $\mathcal{P}$, $P_j(s)$ (from Algorithm 1), maximum value of generations, and $cP_j$.
3: if number of generation of GA solver $\leq$ maximum value then
4: GA obtain feasible values of search variables
5: Compute: $W_{\text{in}}(s)$ and $W_{\text{ot}}(s)$
6: Compute: $J_1$ for fitness evaluation
7: Fitness value of GA $= J_1$
8: go to 3
9: else
10: if Fitness value $< cP_j$ then
11: Output: feasible compensators $W_{\text{in}}$ and $W_{\text{ot}}$.
12: else
13: Output: no feasible compensators
14: Exit
15: end if
16: end if

IV. SYNTHESIS OF THE GRMERS ESTIMATOR FOR THE CANDIDATE NAV

In this section, the synthesis of a GRMERS estimator for the candidate NAV described in Section II-III of [2] is presented. To this end, a GRMERS estimator is designed for four unstable MIMO plants ($\mathcal{P} = \{P_1(s), P_2(s), P_3(s), P_4(s)\}$) of this NAV by designing a suitable MERS estimator and the GR compensators. The plant, $P_1(s)$ is associated with the steady turn and climb flight condition at $V_a$ (air speed) of 9 m/s, climb rate ($h$) of 1 m/s, turn radius ($R$) of 30 m. Similarly, $P_2(s)$, $P_3(s)$ and $P_3(4)$ are associated with the flight condition at ($V_a$ = 10 m/s, $h$ = 1 m/s, $R$ = 30 m), ($V_a = 10$ m/s, $h = 0.5$ m/s, $R = 30$ m) and ($V_a = 10$ m/s, $h = 0$ m/s, $R = 30$ m) respectively. The state-space matrices of all the four plants including actuator dynamics are given in the supporting material.

A. Synthesis of MERS estimator

For synthesizing the MERS estimator, the MERSE algorithm was run with the $\gamma = 1$ and the maximum number of generations set to 200. This algorithm provides $P_j(s) = P_4(s)$ with $\|e_{P_4P_4}(s)\|_\infty = 0.65$. The corresponding compensators $W_{\text{ei}}(s)$ and $W_{\text{eo}}(s)$ are given by

$$W_{\text{ei}}(s) = \text{diag} \begin{bmatrix} 47.3s + 13.85 & 1.243s + 0.6176 & 0.05965s + 1.118 \\ s + 8.196 & s + 2.576 & s + 0.7056 \end{bmatrix}$$

$$W_{\text{eo}}(s) = \text{diag} \begin{bmatrix} 323.6s + 517.2 & 2447s + 3710 & 228.1s + 163 \\ s + 0.932 & s + 0.4487 & 1453s + 803.5 \\ 867.4s + 564.4 & s + 0.6791 & s + 0.8048 \end{bmatrix}$$
Now, the MERS estimator is $\hat{P}_4(s)$ with estimator gain $L_4 \in \mathbb{R}^{19 \times 5}$. This gain is given in the supporting material and the state-space form of $\hat{P}_4(s)$ is given by

$$\dot{\hat{x}}_4 = A_4\tilde{x}_4 + B_4\tilde{u}_i + L_4(y_i - \tilde{C}_x\dot{x}_4)$$

(47)

The dynamics of $\hat{P}_4(s)$ is asymptotically stable as eigenvalues of $\hat{P}_4(s)$ belongs to $\mathcal{C}_-$ as shown in Fig. 3 of the supporting material.

B. Synthesis of GR compensators

The maximum $v-$gap metric of the plant associated with the MERS estimator, $P_4(s)$, is $\epsilon_{P_4} = 0.2915$. The GRC algorithm was run to find the pre and post compensators that reduce the maximum $v-$gap metric associated with the $P_4(s)$. These compensators are given by

$$W_{in}(s) = diag \left[ \frac{6723s + 6409}{s + 1}, \frac{0.00026s + 0.06368}{s + 2.333}, \frac{0.00278s + 0.9258}{s + 2.322} \right]$$

(48)

$$W_{out}(s) = diag \left[ \frac{0.0002834}{s + 0.3444}, \frac{0.0002312}{s + 0.7408}, \frac{0.0003528}{s + 0.7941}, \frac{0.00005}{s + 2.917} \right]$$

(49)

Also, the obtained value of $J^*_1$ is 0.29793 and is considerably lower than $\epsilon_{P_4}(s) = 0.2915$. This suggests that the closeness between $\hat{P}_4(s)$ and other plants in $\kappa$ is increased.

C. Stability, performance, and robustness of GRMERS estimator

In this subsection, a study has been conducted to evaluate the stability, performance, and robustness of the GRMERS estimator synthesised for the plants belonging to $\mathcal{P}$. For this, the GRMERS estimator implementation follows the architecture shown in Fig. 2 with $j = 4$. To study the effectiveness of the GR compensators, we utilize the MERS estimator to estimate the desired states of all the nominal plants of $\mathcal{P}$. The MERS estimator implementation for this purpose follows the architecture shown in Fig. 2 with $j = 4$. It is necessary to compare the nominal and robust performance of the GRMERS estimator while estimating all the plants belonging to $\mathcal{P}$ with a benchmark estimator. If the GRMERS estimator is designed to estimate a single plant alone, then the GR compensators are not required to improve the performance. In that case, the GRMERS estimator can resemble a $H_\infty$ filter. Following this, a single $H_\infty$ filter is designed for each nominal plant belonging to $\mathcal{P}$ (refer to the supporting material for more details). The nominal and robust performances of these individual filters while estimating the corresponding nominal and perturbed plants with measurement noise are then evaluated and compared with the performance of the GRMERS estimator while estimating all the plants belonging to $\mathcal{P}$. The measurement noise considered in the simulations is that of the rate-gyro which is used for measuring $p$, $q$, and $r$. For simulating the effect of this noise in MATLAB, the noise with a zero mean, RMS (root mean squared) value of 0.06 $^\circ$/$s$, and a spectral density of 0.005 $^\circ$/$s\sqrt{\text{Hz}}$ [3] is added to $p$, $q$, and $r$ (rate-gyro output). In all the simulations, the plants are excited with a doublet thrust input. Since all the four plants are unstable, a static full state feedback controller was implemented first to make them stable. The controller, $K_4 \in \mathbb{R}^{3 \times 11}$ associated with both the MERS estimator and $H_\infty$ filter are the same. However, the controller, $K_4 \in \mathbb{R}^{3 \times 19}$ used along with the GRMERS estimator has different dimension because of the presence of the GR compensators. The details of the controllers are given in the supporting material.

1) Stability and nominal performance analysis: The nominal performance of the estimators is obtained through the estimation of the nominal plants, $P_{i}(s) \in \mathcal{P} \forall i \in \{1, \ldots, 4\}$. The estimated vectors of these plants are: $z_i(t) = [u(t), v(t), w(t), \delta_c(t), \delta_T(t), \delta_r(t)]^T \forall i \in \{1, \ldots, 4\}$. The boundedness of the estimated vector and the estimation errors indicate that the GRMERS and MERS estimator are stable (refer to the supporting material for the plots of the estimated vector and the estimation errors). In this paper, the Normalized-Root-Mean-Squared Error (NRMSE), $x^{NE}$ is used as a quantitative measure for the estimator’s performance in estimating any scalar variable, $x(t)$. The NRMSE, $x^{NE}$, of $x(t)$ is defined as

$$x^{NE} = \frac{\sqrt{\sum_{t=1}^{T}(x^i - \hat{x}^i)^2}}{max(x) - min(x)}$$

(50)

where $T$ is the number of observation, $x^i$ is the $i$-th observation of $x$, $\hat{x}^i$ is the $i$-th estimation of $x$, $max(x)$ is the maximum value of $x$, and $min(x)$ is the minimum value of $x$. Now, the normalized error vector, $z^{i}_r$, of $z_i(t)$ is defined as

$$z^{i}_r = [u^{NE}_i, v^{NE}_i, w^{NE}_i, \delta^{NE}_c, \delta^{NE}_T, \delta^{NE}_r]$$

(51)

where $u^{NE}_i, v^{NE}_i, w^{NE}_i, \delta^{NE}_c, \delta^{NE}_T, \delta^{NE}_r$ are the NRMSE of $u(t), v(t), w(t), \delta_c(t), \delta_T(t)$, and $\delta_r(t)$ of $i$th plant, respectively. The nominal and robust performances of the robust simultaneous estimator are acceptable if $||z^{i}_r||_2 \forall i \in \{1, \ldots, 4\}$ are closer to zero. The $||z^{i}_r||_2$ of the estimators are given in Table 1. The values shown in this table suggest that the performance of the individual $H_\infty$ filters is the best followed by the GRMERS estimator. Moreover, the values of $||z^{i}_r||_2$ associated with the GRMERS estimator is smaller than the values of the MERS estimator as indicated by Table 1. Following this, the reduction in the estimation error caused by the GR compensators with reference to the MERS estimator expressed as the percentage when the GRMERS estimator estimates $P_{1}(s), P_{2}(s)$, and $P_{3}(s)$ are 41.13 $\%$, 55.8344 $\%$, and 47.5410 $\%$, respectively. This suggests that a GRMERS estimator, formed by integrating GR compensators and a MERS estimator, outperforms a sole MERS estimator. Note that the estimation error reduction by reducing the gap between the plants using GR compensators for $P_4(s)$ is not required as the design of GRMERS and MERS estimators are based on same plant, $P_4(s)$.

2) Robust performance analysis: Here, the robust performance of the GRMERS estimator is presented. For this
Table I: Nominal performance comparison between GRMERS, MERS and individual $H_\infty$ filters

| Estimator | $P_1(s)$ | $P_2(s)$ | $P_3(s)$ | $P_4(s)$ |
|-----------|----------|----------|----------|----------|
| GRMERS    | $\|z_1^o\|_2$ | $\|z_2^o\|_2$ | $\|z_3^o\|_2$ | $\|z_4^o\|_2$ |
| MERS      | 3.85e-2  | 3.52e-2  | 1.92e-2  | 4.9e-3   |
| $H_\infty$ filter | 3.80e-4 | 1.40e-4 | 1.14e-4 | 3.82e-4 |

purposes, $P_1(s)$, $P_2(s)$, $P_3(s)$, and $P_4(s)$ are perturbed into $P_1(s)$, $P_2(s)$, $P_3(s)$, and $P_4(s)$, respectively, by inducing $8.5\%$, $13\%$, $10\%$, and $5\%$ parametric uncertainties into the system matrices of $P_1(s)$, $P_2(s)$, $P_3(s)$, and $P_4(s)$ such that $\|\Delta A_k\|_\infty < \|\Delta A_k\|_\infty \forall i \in \{1, \ldots, 4\}$. The state-space matrices of the perturbed plants are given the supporting material. The robust performance of the GRMERS estimator is compared with the individual $H_\infty$ filters designed around each plant belonging to $P$. A simulation setup similar to the one explained earlier is used to study the robustness of the GRMERS and the individual $H_\infty$ filters for each plant. Table II shows the robust estimation performances of all the estimators and filters considered in this paper. This table indicates that the robust estimation performance of GRMERS estimator is better than the individual $H_\infty$ filters as the $\|z_i^o\|_2$ of the GRMERS estimator is lower than the $H_\infty$ filters. The GRMERS estimator’s estimation errors are $17.86\%$, $40.35\%$, $41.13\%$, and $43.00\%$ smaller than $H_\infty$ filters while estimating $P_1(s)$, $P_2(s)$, $P_3(s)$, and $P_4(s)$, respectively.

Table II: Robust performance comparison between GRMERS, and individual $H_\infty$ filters

| Estimator | $P_1(s)$ | $P_2(s)$ | $P_3(s)$ | $P_4(s)$ |
|-----------|----------|----------|----------|----------|
| GRMERS    | 5.52e-2  | 5.38e-2  | 4.39e-2  | 2.73e-2  |
| $H_\infty$ filter | 6.76e-2 | 9.02e-2 | 8.10e-2 | 4.80e-2 |

V. CONCLUSION

In this paper, a novel robust simultaneous estimator referred to as the GRMERS estimator has been developed to estimate the states of a finite set of unstable MIMO plants of a NAV. This GRMERS estimator comprises of a MERS estimator and GR compensators where the former provides robust simultaneous estimation with minimal largest worst-case estimation error and the latter reduces this estimation error further by decreasing the gap between the graphs of $N$ linear plants. For a given set of stable/unstable plants, a sufficient condition for the existence of a MERS estimator has been presented using LMIs and robust estimation theory. Two separate non-convex tractable optimization problems, one for the solution of the sufficient conditions and the other to obtain the GR compensators, are formulated in terms of LMIs using robust estimation theory and $\nu$-gap metric, respectively. The solutions for these optimization problems are obtained using two GA-based iterative algorithms. The tractability of these algorithms is successfully demonstrated by the generation of a feasible MERS estimator and GR compensators for four unstable plants of a typical fixed-wing NAV. The simulation results highlight that the GRMERS estimator is easily implementable in a typical NAV, and its performance is within the acceptable limit. Further, the 2-norm of normalized error of the GRMERS estimator is lower than that of the MERS estimator, which indicates that the GR compensators are effective in reducing the simultaneous estimation errors. The nominal and robust performance of the GRMERS estimator is compared with the individually designed $H_\infty$ filters. The performance of GRMERS and individual $H_\infty$ filters are evaluated for both nominal and perturbed plants, and the results indicate that the GRMERS estimator is robust under perturbation than $H_\infty$ filters and GRMERS estimator provides satisfactory performance for all nominal plants. The novel GRMERS estimator is ideal for implementation in computational-resource constrained systems.

REFERENCES

[1] J. V. Pushpangathan, K. Harikumar, S. Sundaram, and N. Sundararajan, “Robust Simultaneously Stabilizing Decoupling Output Feedback Controllers for Unstable Adversely Coupled Nano Air Vehicles,” IEEE Trans. Syst., Man, Cybern., Syst., early access, Sept. 13, 2020, DOI: 10.1109/TSMC.2020.3012507.
[2] J. V. Pushpangathan, M. S. Bhat, and K. Harikumar, “Effects of Gyroscopic Coupling and Countertorque in a Fixed-Wing Nano Air Vehicle,” J. Aircraft, vol. 55, no.1, pp. 239-250, Jan.-Feb. 2018.
[3] J. V. Pushpangathan, “Design and Development of 75 mm Fixed-Wing Nano Air Vehicle,” Ph.D. dissertation, Dept. Aero. Eng., Indian Institute of Science, Bangalore, India, 2018.
[4] A. Mouy, A. Rossi, and H. E. Taha, “Coupled Unsteady Aero-Flight Dynamics of Hovering Insects/Flapping Micro Air Vehicles,” J. Aircraft, vol. 54, no.5, pp. 1738-1749, Sept.-Oct. 2017.
[5] S. M. Nogar, A. Serrani, A. Gogulapatii, J. J. McNamara, M. W. Oppenheimer, and D. B. Doman, “Design and Evaluation of a Model-Based Controller for Flapping-Wing Micro Air Vehicles,” J. Guid. Control Dyn., vol. 41, no.12, pp. 2513-2528, Dec. 2018.
[6] M. D. Pham, K. S. Low, S. T. Goh, and S. Chen, “Gain-Scheduled Extended Kalman Filter for Nano Satellite Attitude Determination System,” IEEE Trans. Aeros. Electron. Syst., vol. 51, no.2, pp. 1017-1028, April 2015.
[7] D. P. Horkheimer, “Gain Scheduling of an Extended Kalman Filter for Use in an Attitude/Heading Estimation System,” M.S. thesis dissertation, University of Minnesota, Minneapolis, 2012.
[8] Y. X. Yao, M. Darouch, and J. Schaefers, “Simultaneous Observation of Linear Systems,” IEEE Trans. Autom. Control, vol. 40, no. 4, pp. 696-699, April 1995.
[9] R. Kovacevic, Y. X. Yao, and Y. M. Zhang, “Observer Parameterization for Simultaneous Observation,” IEEE Trans. Autom. Control, vol. 41, no. 2, pp. 255-259, Feb. 1996.
[10] J. A. Moreno, “Simultaneous Observation of Linear Systems: A State-Space Interpretation,” IEEE Trans. Autom. Control, vol. 50, no. 7, pp. 255-259, July 2005.
[11] L. Menini, C. Possieri, and A. Tornambe, “Algebraic Approaches for the Design of Simultaneous Observers for Linear Systems,” IET Control Theory and Applications, vol. 14, no. 1, pp. 52-62, Jan. 2020.
[12] J. V. Pushpangathan, M. S. Bhat, and H. Kandath, “$\nu$-Gap Metric–Based Simultaneous Frequency-Shaping Stabilization for Unstable Multi-Input Multi-Output Plants,” J. Guid., Control Dyn., vol. 41, no. 12, pp. 2687-2694, Dec. 2018.
[13] G. Viannicome, “Frequency Domain Uncertainty and the Graph Topology,” IEEE Trans. Autom. Control, vol. 38, no. 9, pp. 1371-1383, Sep. 1993.
[14] P. Gahinet, “Explicit controller formulas for LMI-based $H_\infty$ synthesis,” Automatica, vol. 32, no. 7, pp. 1007-1014, 1996.