Classical and Quantum Transport in One-Dimensional Periodically Kicked Systems

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This paper is a brief review of classical and quantum transport phenomena, as well as related spectral properties, exhibited by one-dimensional periodically kicked systems. Two representative and fundamentally different classes of systems will be considered, those satisfying the classical Kolmogorov-Arnol’d-Moser scenario and those which not. The experimental realization of some of these systems using atom-optics methods will be mentioned.

Key words: classical Hamiltonian chaos, quantum chaos, kicked systems, diffusion, ratchet transport, Anderson localization, quantum resonance, quantum antiresonance, quantum diffusion, quasienergy spectra, atom optics.

I. INTRODUCTION

During the last four decades, there has been much interest in the problem of “Quantum Chaos”, i.e., understanding the properties and dynamics of quantum systems whose classical Hamiltonian counterparts are non-integrable and exhibit chaos. In developing this understanding, there was a natural attempt to focus, at least at a first stage, on model systems that are as simple as possible but still feature some typical behavior of more complex and/or “realistic” nonintegrable systems. The minimal number of degrees of freedom required by a nonlinear Hamiltonian system to be nonintegrable is “1.5”, namely one dimension (1D) with time dependence. A simple time dependence is a periodic one and a very simple periodic dependence is that of a periodic delta function. This corresponds to 1D periodically “kicked” systems. The most attractive feature of these systems, and also of kicked systems in higher dimensions, is that both their classical Poincaré map and quantum map in one time period can be written explicitly in closed form. This fact already makes it possible to derive several exact results and/or to make rigorous statements about the properties of classical and quantum kicked systems.

The many theoretical investigations of these relatively simple systems, some of which eventually became paradigmatic models in the field of Quantum Chaos, have led to the discovery of an unexpected rich variety of fascinating classical and quantum transport phenomena. These systems have also proved to be quite realistic. In fact, during the last two decades, several of the quantum phenomena have been experimentally realized using atom-optics methods with cold atoms or Bose-Einstein condensates. In most cases, the experimental results agreed well with the theoretical predictions. Some phenomena were first discovered experimentally and later explained theoretically.

This paper is a brief review of classical and quantum transport phenomena, as well as related spectral properties, exhibited by representative classes of 1D periodically kicked systems. This review will include significant contributions made by Paul Brumer and co-workers to the field, see Secs. III and IVD. The paper is organized as follows. In Secs. II and III, we consider kicked systems satisfying the classical Kolmogorov-Arnol’d-Moser (KAM) scenario. Generalized versions of paradigmatic models in this class of systems, the kicked rotor and the kicked particle, will be treated in some detail in Sec. II; variants of these models, the modulated kicked rotors, will be the subject of Sec. III. In Sec. IV, we consider the fundamentally different class of “non-KAM” kicked systems, represented by generalized kicked charged particles in a uniform magnetic field or, equivalently, generalized kicked harmonic oscillators; a small sub-class of these systems is exactly equivalent to generalized versions of the paradigmatic “kicked Harper models”. Conclusions are presented in Sec. V.

II. KAM SYSTEMS: KICKED ROTOR AND KICKED PARTICLE

A. Classical Kicked Rotor, KAM Scenario, Chaotic Normal and Anomalous Diffusion

The classical periodically kicked rotor (KR), or Chirikov-Taylor system, is defined, in its generalized version and in dimensionless scaled variables, by the Hamiltonian:

$$H = \frac{L^2}{2} + kV(\theta) \sum_{t=-\infty}^{\infty} \delta(t’ - t),$$

(1)

where $L$ is angular momentum, $\theta$ is angle, $k$ is a nonintegrability parameter, $V(\theta)$ is a general $2\pi$-periodic potential, $t’$ is the usual (continuous) time, and $t$ is a “discrete” time taking all integer values. The classical map for $\Pi$, from time $t’ = t - 0$ to time $t’ = t + 1 - 0$, is:

$$L_{t+1} = L_t + kf(\theta_s), \quad \theta_{t+1} = \theta_t + L_{t+1} \mod(2\pi),$$

(2)

where the subscripts indicate the times above at which $(L, \theta)$ are evaluated and $f(\theta) = -dV/d\theta$ is the force function. For $V(\theta) = \cos(\theta)$ or $f(\theta) = \sin(\theta)$, Eqs. (2) give the famous “standard map”, a prototypical Hamiltonian map exhibiting the celebrated Kolmogorov-Arnol’d-Moser (KAM) scenario. As $k$ is gradually increased from 0, local chaotic layers develop around “stochastic resonances” which extend “horizontally” from $\theta = 0$ to...
KAM tori break into “cantori” where $k > k_0$ larger than $k$ extending from $L = -\infty$ to $L = \infty$. Then, unbounded chaotic diffusion takes place in the $L$ direction, characterized by the diffusion coefficient
\begin{equation}
D = \lim_{t \to \infty} \frac{\langle (L_t - L_0)^2 \rangle}{2t},
\end{equation}
where $\langle \cdot \rangle$ denotes average over an ensemble $\{(L_0, \theta_0)\}$ of initial conditions in the chaotic region. For $k$ sufficiently larger than $k_0$, the standard map features generalized periodic orbits, the so-called “accelerator modes” satisfying
\begin{equation}
L_m = L_0 + 2\pi w, \quad \theta_m = \theta_0 \mod(2\pi),
\end{equation}
where $m$ is the minimal period and $w \neq 0$ is some integer. Clearly, the orbit (4) performs ballistic motion in the $L$ direction with mean acceleration $2\pi w/m$. If this orbit is stable, it generates a chain of $m$ islands embedded in the chaotic sea and all the points within these islands perform ballistic motion with the same mean acceleration:
\begin{equation}
\langle (L_{mt} - L_0)^2 \rangle \approx 4\pi^2 w^2 t^2,
\end{equation}
where the average is over an initial ensemble within the islands. A chaotic orbit will usually stick to the boundaries of the island chain for a long time, performing ballistic motion during this time, it will then return to the chaotic sea where it will diffuse for some time, and it will eventually stick again to the island-chain boundaries.

The net result of these transitions over a very long time is an anomalous chaotic diffusion:
\begin{equation}
\langle (L_{mt} - L_0)^2 \rangle \propto t^\mu, \quad 1 < \mu < 2,
\end{equation}
where the average is again over an initial ensemble in the chaotic region and $\mu$ is the anomalous-diffusion exponent. The latter ranges between the values of 1 [normal diffusion (3)] and 2 [ballistic motion (5)]. It corresponds therefore to a “superdiffusion”, which appears to be the most well-established kind of Hamiltonian anomalous diffusion (there exists, however, a rare case of Hamiltonian subdiffusion, $\mu < 1$). It must be remarked that the existence of accelerator modes (4) and the associated chaotic superdiffusion (5) is a consequence of an important feature of the map (2), i.e., its translational invariance in the $L$ direction (with period $2\pi$). Other implications of this invariance are considered below.

### B. Quantum Kicked Rotor and Dynamical Localization

The quantum map for (1), analogous to the classical one (2), is given by the evolution operator in one time period, from $t' = t - 0$ to $t' = t + 1 - 0$:
\begin{equation}
\hat{U} = \exp[-i\hat{L}^2/(2\hbar)] \exp[-ikV(\theta)/\hbar],
\end{equation}
where $\hbar$ is a dimensionless scaled Planck constant and $\hat{L}$ is the angular-momentum operator $\hat{L} = -i\hbar \partial/\partial \theta$ with eigenvalues $n\hbar$, $n$ integer. It is natural to compare the time evolution of the expectation value $\langle \hat{L}_z^2/2 \rangle_t = \langle \phi^{(t)}|\hat{L}_z^2/2|\phi^{(t)} \rangle$, with the classical diffusive evolution expected from (3) in the global chaotic regime ($k \gg k_0$). First numerical experiments indicated that for sufficiently small $\hbar$ (semiclassical regime) $\langle \hat{L}_z^2/2 \rangle_t$ mimics the classical chaotic diffusion up to some break-time $t = t_B$ but for $t > t_B$ $\langle \hat{L}_z^2/2 \rangle_s$ stabilizes, becoming bounded in time.

To understand the latter phenomenon, it is necessary to investigate the nature of the eigenvalues and eigenstates of $\hat{U}$. Since $\hat{U}$ is a unitary operator, its eigenvalues must lie on the unit circle in the complex plane and can be written as $\exp(-i\omega t)$, where the real quantity $\omega$ is the so-called “quasienergy” ($\omega$) and ranges in the interval $0 \leq \omega < 2\pi$. The QE eigenvalue problem for $\hat{U}$ is then:
\begin{equation}
\hat{U} \Psi_{\omega} = \exp(-i\omega t) \Psi_{\omega},
\end{equation}
where $\{|\Psi_{\omega}\rangle\}$ are the QE eigenstates. It was shown that Eq. (15), written in the angular-momentum representation $\langle n|\Psi_{\omega}\rangle$, is exactly equivalent to the eigenvalue problem of a tight-binding chain of “sites” $n$, with an on-site potential which is pseudorandom for generic, irrational values of $\hbar/(2\pi)$. By assuming that such a pseudorandom disorder is effectively similar to a truly random one which causes Anderson localization, one would conclude that all the eigenstates $\langle n|\Psi_{\omega}\rangle$ are exponentially localized in angular-momentum space and the QE spectrum $\omega$ is then discrete. These conclusions were extensively verified numerically. A discrete QE spectrum implies quasiperiodic and bounded quantum motion, as observed in the first numerical experiments mentioned above. This bounded quantum motion is known as “dynamical localization”. It was experimentally observed using atom-optics methods, i.e., ultracold atoms “kicked” by an optical standing wave.

An important consequence of the exponential localization of QE eigenstates is that the distribution of the QE level spacings is Poisson. This is due to the fact that most of the exponentially localized eigenstates have far-separated centers on the infinite angular-momentum space, $-\infty < n\hbar < \infty$, and therefore do not overlap, i.e., they are uncorrelated.

It was also found later that in a semiclassical regime of small $\hbar$ the exponential localization length $\xi$ of the QE eigenstates is roughly proportional to the diffusion coefficient (3). This relation between $L$ and $D$ is a significant quantum signature of classical chaos and it was experimentally verified using again atom-optics methods. These experiments also showed that in parameter regimes where classical accelerator modes exist and chaotic superdiffusion occurs (see Sec. IIA) the quantum angular-momentum distributions acquire “shoulders”, i.e., their...
initial decay is slower than the exponential one in the case of normal chaotic diffusion.

C. Quantum Kicked Particle and the Quasimomentum

In the atom-optics experiments mentioned in Sec. IIB, the quantum KR is actually realized as a kicked-particle system, since atoms move on lines and not on circles like rotors. However, the quantum kicked particle can be exactly and simply related to quantum KR as follows. The one-period evolution operator for the quantum kicked particle is given by Eq. (7) with \( L \) replaced by \( \hat{p} \) (linear-momentum operator) and with \( \theta \) replaced by \( \hat{x} \) (position operator):

\[
\hat{U} = \exp \left( -i \frac{p^2}{2\hbar} \right) \exp \left[ -i k V(\hat{x}) / \hbar \right]. \tag{9}
\]

The QE eigenvalue problem for \( \Psi_\omega(x) \) in the \( x \) representation is:

\[
\hat{U} \Psi_\omega(x) = \exp(-i\omega) \Psi_\omega(x). \tag{10}
\]

The \( 2\pi \)-periodicity of \( \Psi_\omega(x) \) in \( \hat{x} \) implies that \( \Psi_\omega(x) \) can be chosen to have the Bloch form:

\[
\Psi_\omega(x) = \exp(i\beta x) \psi_{\beta,\omega}(x), \tag{11}
\]

where \( \beta \) is the quasimomentum \((0 \leq \beta < 1)\), whose meaning is explained below, and \( \psi_{\beta,\omega}(x) \) is \( 2\pi \)-periodic in \( x \). After inserting (11) into Eq. (10), one easily finds that \( \psi_{\beta,\omega}(x) \) is an eigenstate of

\[
\hat{U}_\beta = \exp[ -i(\hat{p} + \beta \hbar)^2 / (2\hbar)] \exp[-i k V(\hat{x}) / \hbar] \tag{12}
\]

with eigenvalue \( \exp(-i\omega) \). Due to the \( 2\pi \)-periodicity of \( \psi_{\beta,\omega}(x) \), one can interpret \( x \) as an angle \( \theta \) and \( \hat{p} \) in Eq. (12) as an angular-momentum operator \( \hat{L} \) with eigenvalues \( n\hbar \). Then, \( \hat{U}_\beta \) is the evolution operator of a “\( \beta \)-KR” and \( \beta \) is conserved during the evolution. To illustrate this conservation and the physical meaning of \( \beta \), assume an initial momentum state \( \langle x|p \rangle = \exp(ipx/\hbar) \). This can be written in the form (11) as \( \exp(ipx) \exp(inx) \), where \( \beta \) and \( n \) are, respectively, the fractional and integer parts of \( p/\hbar \) [similarly, after replacing \( \hat{p} \) in Eq. (12) by \( \hat{L}, \hat{L} + \beta \hbar \) is the decomposition of the linear-momentum operator into an “integer” part, \( \hat{L} \), and a conserved “fractional” part, \( \beta \hbar \)]. Then, after \( t \) kicks, this state will evolve to a state having still the form (11) with the same \( \beta \): \( \langle x|p \rangle_{t} = \exp(i\beta x) \phi_\beta(x; t) \), where \( \phi_\beta(x; t) \) is \( 2\pi \)-periodic in \( x \). The usual quantum KR in Sec. IIB corresponds to \( \beta = 0 \). One can show that the eigenvalue property for (12) with general \( \beta \) is also equivalent to a tight-binding chain with an on-site potential that is pseudorandom for generic irrational \( \hbar/(2\pi) \). Dynamical localization is then expected to occur.

Consider now an arbitrary wave packet \( \Phi(x) \) of the kicked particle. This can be always expressed as a superposition of Bloch functions:

\[
\Phi(x) = \int_{0}^{1} d\beta \exp(i\beta x) \phi_\beta(x), \tag{13}
\]

where \( \phi_\beta(x) = \sum_n \tilde{\Phi}(n + \beta \hbar) \exp(inx)/\sqrt{2\pi} \overline{\Phi}(p) \) is the momentum representation of \( \Phi(x) \). Using (13) and the fact that \( \hat{p} \exp(i\beta x) = \exp(i\beta x)(\hat{p} + \beta \hbar) \), one then gets the basic relation

\[
\hat{U}^t \Phi(x) = \int_{0}^{1} d\beta \exp(i\beta x) \hat{U}^t_\beta \phi_\beta(x). \tag{14}
\]

Relation (14) connects the quantum dynamics of the kicked particle with that of all the \( \beta \)-KRs.

D. Quantum Resonance, Antiresonance, Diffusion, and Ratchet Accelerator

Classically, the KR map (2) is translationally invariant in \( L \) with period \( 2\pi \). Quantally, one may ask under which condition the quantum map (12) is invariant under a translation in \( \hat{p} = \hat{L} \). For a quantum rotor, such a translation is given by the operator \( \hat{T}_\beta = \exp(-i\beta \hat{\theta}) \), where \( \beta \) must be integer since \( \theta \) is an angle; thus, \( \hat{T}_\beta \) is a translation by \( \beta \hbar \) in \( L \) in accordance with the fact that \( \hat{L}/\hbar \) has integer eigenvalues. Using the last fact, the requirement \( \{\hat{U}_\beta, \hat{T}_\beta\} = 0 \), with \( \hat{x} \to \theta, \hat{p} \to \hat{L} \) in (12), leads to the conditions

\[
\frac{\hbar}{2\pi} = \frac{l}{q}, \tag{15}
\]

\[
\beta = \frac{r}{gl} - \frac{qg}{2} \mod(1), \tag{16}
\]

where \( l \) and \( q \) are coprime integers, \( g = \tilde{q}/q \) is integer, and \( r \) and \( ql \) are also coprime integers. For given \( (l, q) \), \( \beta \) in Eq. (16) can take any rational value \( \beta_\ell \) in \([0, 1)\). From (15) and (16), one can always pick \( g \) such that \( r = (\beta_\ell + qg/2)g \) is integer. Given \( \beta = \beta_\ell \), one chooses \( g \) as the smallest positive integer satisfying the latter requirement; in general, \( g > 1 \). For the usual KR \( (\beta_\ell = 0) \), \( g = 1 \) if \( lq \) is even and \( q = 2 \) if \( lq \) is odd; compare with previous works. One denotes \( \beta_\ell \) by \( \beta_{l,q} \), where the integer \( r = (\beta_\ell + qg/2)g \) labels all the different values of \( \beta_\ell \) for given minimal \( q \).

The QE states \( \psi_{\beta,\omega} \) for \( \beta = \beta_{l,q} \) can be chosen as simultaneous eigenstates of \( \hat{U}_\beta \) and \( \hat{T}_{g\beta} \): \( \hat{U}_\beta \psi_{\beta,\omega} = \psi_{\beta,\omega} \), \( \hat{T}_{g\beta} \psi_{\beta,\omega} = \exp(-ig\alpha) \psi_{\beta,\omega} \), where \( \alpha \) is a “quasiangle” varying in the “Brillouin zone” (BZ) \( 0 \leq \alpha < 2\pi/(gq) \). One may view the Bloch function \( \exp(i\beta x) \psi_{\beta,\omega}(x) \) as a state on the “quantum torus” \( 0 \leq x < 2\pi \), \( 0 \leq p < q\hbar \), with toral boundary conditions specified by \( (\alpha, \beta) \). Using standard methods, it is easy to show from the last two eigenvalue equations that at fixed \( \alpha \) one has precisely \( qg \) QE eigenvalues \( \omega_q(\alpha, \beta), b = 0, \ldots, qg - 1 \). Since \( g \) is minimal, the BZ is maximal for the given value of \( \beta = \beta_{r,q} \). Then, as \( \alpha \) is varied continuously in the BZ, the \( qg \) eigenvalues generically form \( qg \) QE bands with corresponding band eigenstates \( \psi_{\beta_{l,q}} \). This band continuous QE spectrum for \( \beta = \beta_{l,q} \) implies
an asymptotic ballistic increase in time of the expectation value of the kinetic energy in any initial wave-packet:
\[ \langle \frac{L^2}{2} \rangle_{\beta,t} \propto t^2. \]
This is the “quantum-resonance” (QR) phenomenon for rational \( h/(2\pi) \) and \( \beta \), in contrast to dynamical localization for irrational \( h/(2\pi) \). This QR generalizes the original QR for \( \beta = 0 \).\(^{21,22}\)

An important case is that of the main QRs, \( h/(2\pi) = l \) \((q = 1)\). In this case, several additional exact results can be derived\(^{27}\) since the evolution operator \( U \) (with \( \hat{x} \to \theta, \hat{p} \to \hat{L} \)) reduces, up to a nonrelevant constant phase factor, to
\[
\hat{U}_\beta = \exp \left( -i\frac{h_\beta}{\hbar} \hat{L} \right) \exp \left[ -ikV(\theta)/\hbar \right],
\]
where \( h_\beta = \pi l(2\beta + 1) \). We identify \( \hat{U}_\beta \) in Eq. (17) as the one-period evolution operator for the well-known linear KR\(^{24}\), which is exactly solvable for arbitrary potential \( V(\theta) = \sum_m V_m \exp(-im\theta) \). The following exact results were obtained\(^{19,25}\), First, the \( gg \) = \( g \) QE bands for \( \beta = \beta_0 \) are all nonflat (have finite width) only if there is at least one nonzero Fourier coefficient \( V_m \) of \( V(\theta) \) with \( m \) multiple of \( g \) (including \( m = \pm g \)); then, QR indeed takes place. Otherwise, all the \( g \) bands are flat (infinitely degenerate) and QR is replaced by a diametrically opposite phenomenon, “quantum antiresonance” (QAR), a bounded periodic time evolution of \( \langle \frac{L^2}{2} \rangle_{\beta,t} \). For the origin of the term “antiresonance”, see Sec. III. Second, the expectation value of the kinetic energy \( \langle \hat{p}^2/2 \rangle_{\beta,t} \) of the kicked particle in any initial wave-packet grows linearly (“diffusively”) in time: \( \langle \hat{p}^2/2 \rangle_{\beta,t} \propto t \), see also other works.\(^{18,25}\) A similar growth in time is exhibited by the kinetic energy of an incoherent mixture of \( \beta \)-KRs:
\[
\int_0^1 d\beta F(\beta) \langle \frac{L^2}{2} \rangle_{\beta,t} \propto t,
\]
where \( F(\beta) \) is a generic normalized distribution, \( \int_0^1 d\beta F(\beta) = 1 \). These phenomena have been experimentally observed for QRs of order \( q = 1, 2, 3 \) using atom-optics methods with Bose-Einstein condensates (BECs)\(^{28}\).

In view of the QR ballistic behavior \( \langle \frac{L^2}{2} \rangle_{\beta,t} \propto t^2 \), one may ask whether, under some conditions, \( \langle \hat{L} \rangle_{\beta,t} \propto t \).

The latter behavior may be viewed as a quantum ratchet acceleration since the force \( -dV/d\theta \) is unbiased, i.e., it obviously has zero average, \( \int_0^{2\pi} d\theta dV/d\theta = 0 \) (see also Sec. IVD). Ratchet effects can arise only if some symmetry is broken. The asymmetry is usually associated with the system, i.e., \( V(\theta) \) is asymmetric. However, a different new kind of asymmetry was proposed\(^{22}\). Assume that both \( V(\theta) \) and the initial-state amplitude \( |\phi_0(\theta)\rangle \) have inversion symmetry around different symmetry centers; then, the noncoincidence of the symmetry centers is a relative asymmetry which can produce a ratchet effect. For example, in the simple case of \( V(\theta) = \cos(\theta - \gamma) \) and \( \phi_0(\theta) = \left[ 1 + \exp(-i\theta) \right]/\sqrt{4\pi} \), the symmetry centers of \( V(\theta) \) and \( \phi_0(\theta) \) are located at \( \theta = \gamma \) and \( \theta = 0 \), respectively. One then finds for the main QRs \((q = 1)\) that \( \Delta \langle \hat{L} \rangle_{\beta,t} = \langle \hat{L} \rangle_{\beta,t} - \langle \hat{L} \rangle_{\beta,0} \) is given by\(^{28}\),
\[
\Delta \langle \hat{L} \rangle_{\beta,t} = \frac{k \sin(h_\beta/2)}{2 \sin(h_\beta/2)} \sin[h_\beta(t + 1/2 - \gamma)].
\]

Now, if \( \beta \) takes any of the \( l \) resonant values \( \beta_r = \frac{r}{l} - \frac{1}{2} \mod(1), r = 0, \ldots, l - 1 \) [from Eq. \( \ref{eq:19} \) with \( q = 1 \), the only resonant value of \( q \) in the case of \( V(\theta) = \cos(\theta - \gamma) \)], one gets from Eq. \( \ref{eq:15} \) that \( \Delta \langle \hat{L} \rangle_{\beta,t} = k \sin(\gamma)/2 \). The latter QR ratchet effect was experimentally observed by atom-optics methods for \( l = 1, \beta_r = 0.5 \) using BECs with quasimomentum width \( \Delta \beta \approx 0.1 \). Because of this width causes a saturation of the linear increase of \( \Delta \langle \hat{L} \rangle_{\beta,t} \). Theoretical results concerning this saturation effect were also experimentally confirmed\(^{28}\).

E. Staggered-Ladder QE Spectra and their Quantum-Transport Manifestations

We have seen in Sec. II D that under the QR conditions \( \beta_0 \) and \( \beta_l \) (with minimal \( q \)) the QE spectrum of the \( \beta \)-KR consists of \( gg \) bands \( \omega_0(\alpha, \beta), b = 0, \ldots, gg - 1 \). It was recently shown\(^{29}\) that this set of bands is actually a “staggered ladder”, i.e., the superposition of \( g \) equally-spaced ladders, each consisting of \( g \) bands. Writing \( b = (c, d) \), where \( c, e = 1, \ldots, g \), labels the \( g \) ladders and \( d = 0, \ldots, g - 1 \), labels the \( g \) bands in each ladder, ladder \( c \) is given by
\[
\omega_{c,d}(\alpha, \beta) = \omega_{c,0}(\alpha, \beta) + 2\pi dl(\beta + q/2) \mod(2\pi) \tag{\ref{eq:19}}
\]
This staggered-ladder QE spectrum is another consequence of the translational invariance of the classical map\(^{2}\) in the \( L \) direction with period \( 2\pi \). In the limit of irrational \( \beta (g \to \infty) \), each of he \( g \) ladders\(^{19}\) covers densely the entire QE range \([0, 2\pi] \). We notice that for the usual KR \((\beta = 0)\), there are either no ladders \((g = 1 \text{ for } lq \equiv 0)\) or trivial ladders with spacing \( \Delta \omega = \pi (g = 2 \text{ for } lq \equiv 1) \). Thus, \( \beta = 0 \) is a nongeneric case for rational \( h/(2\pi) \). For the main QRs \((q = 1)\), the QE spectrum is just one ladder which, for potentials \( V(\theta) \) with a finite number of harmonics and for sufficiently high-order rational \( \beta \), consists of \( g \) flat bands\(^{34}\) (see also Sec. II D). The regularity of the generic staggered-ladder QE spectrum is fundamentally different from that of the Poisson QE spectrum for irrational \( h/(2\pi) \) (see Sec. II B).

The spectra\(^{19}\) have several quantum-transport manifestations\(^{25}\): (a) A suppression of QRs for rational \( \beta \) as \( g \) increases. (b) A dynamical localization for irrational \( \beta \) which is basically different from that for irrational \( h/(2\pi) \); for example, its time evolution features traveling-wave components in position space and a staggered-ladder frequency spectrum symmetric around a central ladder which is independent of the nonintegrability strength \( k \). Most of these phenomena were shown to persist when averaged over realistic quasimomentum widths \( \Delta \beta \) of BECs and should therefore be experimentally observable.
III. MODULATED KICKED ROTOR AND QUANTUM ANTIRESONANCE

A significant extension of the KR was introduced, following previous papers in which a special case of this extension was studied. This is the modulated KR (MKR), defined by a generalization of the Hamiltonian

\[ H = \frac{L^2}{2} + kV(\theta) \sum_{j=0}^{M-1} c_j \Delta(t' - t_j), \]  

(20)

where \( \Delta(t') = \sum_{j=0}^{\infty} \delta(t' - t_j), c_j (j = 0, \ldots, M - 1) \) are arbitrary coefficients, and \( t_j (j = 0, \ldots, M - 1) \) are arbitrary except of the conditions \( 0 \leq t_j < t_{j+1} \leq 1 \) and \( t_0 = 0 \); one also defines \( t_M = 1 \). The classical map for (20) can be easily written and gives rise to KAM scenario. The classical map for (20) can be written as

\[ \tau = \frac{L}{2} + \sum_{j=0}^{M-1} c_j \Delta(t' - t_j), \]  

(21)

where \( \tau_j = t_{j+1} - t_j \) and the factors under the product sign in (21) are arranged from right to left in order of increasing \( j \). Thus, \( \hat{U} \) in Eq. (21) is the composition of \( M \) quantum maps \( \hat{U}_j \), each corresponding to a KR with time period \( \tau_j \) and kick strength \( c_j \). Now, the condition for the main QRs of this KR is:

\[ \hbar \tau_j = 4\pi m_j, \]  

(22)

where \( m_j \) is an arbitrary positive integer. Since \( \hat{L}/\hbar \) has integer eigenvalues, condition (22) implies that \( \exp[-i\tau_j \hat{L}^2/(2\hbar)] = 1 \) is identically satisfied, so that \( \hat{U}_j = \exp[-i\epsilon_{cj} V(\theta)/\hbar], \) which indeed leads to QR for KR \( j \). Then, the MKR is described by the evolution operator

\[ \hat{U} = \prod_{j=0}^{M-1} \exp[-i\epsilon_{cj} V(\theta)/\hbar] = \exp \left[ -i \sum_{j=0}^{M-1} c_j V(\theta)/\hbar \right], \]  

(23)

leading to QR for the entire MKR system unless

\[ \sum_{j=0}^{M-1} c_j = 0, \]  

(24)

implying that

\[ \hat{U} = 1 \]  

(25)

identically. Eq. (25) means that all the QE spectrum consists of one infinitely degenerate level (flat band) \( \omega = 0 \) and no wave-packet moves. This phenomenon is diametrically opposite to that of the QRs exhibited by the individual KRs “composing” the MKR according to Eq. (23). This is the quantum antiresonance (QAR) phenomenon already considered in Sec. IID. As far as we are aware, the term “antiresonance” was first coined in a work studying a different class of systems (to be considered in Sec. IV), when referring to a yet unpublished paper. This term reflects the “cancellation” of QRs of sub-systems composing a given system, due to some condition like (24). Such a cancellation effect can be shown to be responsible also to QARs associated with more than one infinitely degenerate QE level, such as the QAR considered in Sec. IID. One should also mention that the QAR for the \( M = 2 \) MKR with \( \tau_0 = \tau_1 = 1/2, \) \( c_0 = -c_1 = 1, \) and \( V(\theta + \pi) = -V(\theta) \) was shown to be exactly equivalent to a well-known period-2 QAR occurring in the KR.

It is natural to ask about the behavior of the MKR in the immediate vicinity of QAR, i.e., when \( \hbar \to \hbar(1 + \epsilon) \) in Eq. (22) and condition (24) still holds. It was shown that this perturbation of \( \hbar \) removes the infinite degeneracy of the single QE level \( \omega = 0, \) and the QE spectrum is then given by \( \omega = \epsilon \tilde{\omega}; \) here \( \tilde{\omega} \) are the eigenvalues of a one-dimensional Schrödinger equation for the integrable system of a “pendulum” with a potential \( k_{\text{eff}}^2 (dV/d\theta)^2, \) where \( k_{\text{eff}} \) is multiplied by some quantity dependent on the coefficients \( c_j. \) The corresponding QE eigenstates are then exponentially localized in angular-momentum space with a localization length \( \xi, \) where \( \xi^{-1} \) is not smaller than the smallest distance of a singularity of \( dV/d\theta \) in the complex \( \theta \) plane from the real axis.

One can expect that the MKRs should exhibit a rich variety of classical and quantum transport phenomena associated with many different choices of the large number \( 2M \) of parameters \( \tau_j \) and \( c_j, \) especially when these parameters are not subjected to conditions like (22) and/or (24). In fact, several new phenomena were discovered by Paul Brumer and co-workers, using just MKRs and modified MKRs whose \( \tau_j \) values are all the same (\( \tau_j = 1/M \)) and with \( c_j = \pm 1. \) These phenomena are:

(a) A quantum diffusion, taking place over long time scales, which is faster than the classical anomalous one (superdiffusion due to accelerator-mode islands).

(b) Controlled enhancement of the dynamical localization length.

(c) Classical chaotic ratchet acceleration, with clear quantum-transport manifestations, exhibited by an asymmetric MKR with accelerator-mode islands (see also Sec. IVD).

The \( M = 2 \) MKR with arbitrary \( \tau_0 = \tau_1 = 1, c_0, \) and \( c_1 \) is the so-called “double KR”, extensively studied by several groups during the last decade. A detailed review of the many results concerning this and related systems is beyond the scope of the present paper. We refer the interested reader to representative sets of works on this subject.
IV. NON-KAM SYSTEMS AND KICKED HARPER MODELS

A. Classical Kicked Charges in a Magnetic Field and Transport on Stochastic Webs

We now consider a class of kicked systems fundamentally different from those in Secs. II and III. These are charged particles periodically kicked in a direction perpendicular to a uniform magnetic field $B$ \(^{39,40}\). We follow here the general approach introduced in our work \(^{40}\) assuming, for definiteness and without loss of generality, that the particles have unit mass and unit charge. Using dimensionless scaled coordinates and notation similar to that in Eq. (1), the Hamiltonian of the system is:

$$H = \frac{\Pi^2}{2} + kV(x) \sum_{t=-\infty}^{\infty} \delta(t'-t),$$  

(26)

where $\Pi = (\Pi_x, \Pi_y) = \mathbf{p} - \mathbf{B} \times \mathbf{r}/(2c)$ is the kinetic momentum, $\mathbf{p} = (p_x, p_y)$ is the canonical momentum, $\mathbf{B}$ is in the $z$ direction, $\mathbf{r} = (x, y)$, and $V(x)$ is a general $2\pi$-periodic potential. It is well known \(^{41}\) that the natural degrees of freedom in a uniform magnetic field are the conjugate pairs $(\Pi_x, \Pi_y)$ and $(x_c, y_c)$ (coordinates of the center of a cyclotron orbit). Defining $u = \Pi_x/\Omega$ and $v = \Pi_y/\Omega$, where $\Omega = B/c$ is the cyclotron angular velocity, one has the relation $x_c = x + v$, easily derivable from simple geometry. The Hamiltonian (26) can thus be written as:

$$H = \frac{\Omega^2}{2} (u^2 + v^2) + kV(x_c - v) \sum_{t=-\infty}^{\infty} \delta(t'-t).$$  

(27)

Since the conjugate mate $y_c$ of $x_c$ is absent in (27), $x_c$ is a constant of the motion. Then, since also $(u, v)$ are conjugate, (27) is just the Hamiltonian of a harmonic oscillator periodically kicked by a potential $V(x_c - v)$ dependent on the “parameter” $x_c$. The classical map on the $(u, v)$ phase plane for (27), from $t' = t - 0$ to $t' = t + 1 - 0$, can easily be derived from Hamilton equations:

$$\dot{u} = \Omega^{-1} \partial H/\partial v, \quad \dot{v} = -\Omega^{-1} \partial H/\partial u,$$

(28)

$$z_{t+1} = [z_t + k f(x_c - v_t)] e^{-i\Omega t},$$

(29)

where $z = u + iv$ and $f(x) = -\Omega^{-1} Dv/dx$. The map (28) in the special case of $x_c = 0$ and $V(x) = \cos(x)$ was first presented by Zaslavsky \textit{et al.} \(^{39}\) and it is known as the “Zaslavsky map” or “web map”. It was later generalized to the form (28) by Dana and Amit. \(^{40}\) The main motivation for this generalization is the sensitivity of the dynamics to the value of $x_c$. Before discussing this sensitivity, we first consider some more basic aspects of the system. The harmonic oscillator is a degenerate system since $\partial^2 H_0/\partial J^2 = 0$; here $H_0 = \Omega^2 (u^2 + v^2)/2 = \Omega J$ is the unperturbed Hamiltonian in (27) and $J$ is the action. This in contrast with $H_0 = L^2/2$ in (1), with $\partial^2 H_0/\partial J^2 = 1 \neq 0 (J = L)$. Thus, unlike (1), the KAM theorem cannot be applied to (27), which is therefore a non-KAM system. One then expects that global chaos, i.e., unbounded chaotic motion of $(u, v)$, may exist for arbitrarily small $k$ ($k_c = 0$). In fact, such an unbounded motion under the map (28) for all $k$ is observed to take place diffusively on a “stochastic web” for resonance (rational) values of $\Omega = 2\pi l/n$ ($l$ and $n$ are coprime integers). For $n = 3, 4, 6$, the web has crystalline symmetry (triangular, square, hexagonal) while for other values of $n > 4$ it has quasicrystalline symmetry.

The chaotic diffusion on the stochastic web for $\Omega = 2\pi l/n$ is characterized by the diffusion coefficient \(^{40}\)

$$D(x_c) = \lim_{t \to \infty} \frac{\langle |z_{mt} - z_0|^2 \rangle}{2nt},$$

(29)

where $\langle \cdot \rangle$ denotes average over an ensemble $\{(u_0, v_0)\}$ of initial conditions in the stochastic web. Analytical and numerical results for crystalline webs indicate a strong dependence of $D(x_c)$ on $x_c$. Since a general ensemble of charged particles exhibits all values of $x_c$, a (weighted) average of $D(x_c)$ over $x_c$ is usually necessary. It was shown \(^{42}\) that such averaging removes much of the rich structure (e.g., oscillations) of $D$ versus $k$ at fixed $x_c$.

For crystalline webs, featuring translational invariance in the $(u, v)$ phase plane, “accelerator-mode” periodic orbits exist for sufficiently large $k$: $u_{mn} = u_0 + 2\pi m$, $v_{mn} = v_0 + 2\pi n$, where $m$ is the minimal period and $(m, n)$ are integers, not both zero. As in the case of KAM systems (see Sec. II A), stable accelerator-mode orbits cause the anomalous chaotic transport of superdiffusion \(^{42}\): $\langle |z_{mnt} - z_0|^2 \rangle \sim \mu(x_c)^t, 1 < \mu(x_c) < 2$; the anomalous-diffusion exponent $\mu(x_c)$ is again strongly dependent on $x_c$. It has been suggested \(^{42}\) that the strong variation of $D(x_c)$ or $\mu(x_c)$ with $x_c$ may be used to “filter” from a general ensemble of charged particles a subensemble having any desired well-defined value of $x_c$.

B. Generalized Kicked Harper Models as Realistic Systems

We now consider a well known quantum-chaos system, the kicked Harper model (KHM) \(^{39,43,44}\) whose most generalized version is described by the Hamiltonian \(^{47}\): \(^{39}\)

$$H_{\text{KHM}} = kV_1(u) + kV_2(u) \sum_{t=-\infty}^{\infty} \delta(t'/2 - t),$$

(30)

where $V_1(u)$ and $V_2(u)$ are general (not necessarily periodic) functions of the phase-space variables $u$ and $v$ defined above. The original version of the KHM, which appeared in the paper by Zaslavsky \textit{et al.} \(^{39}\), is the very special case of (30) with $V_1(u) = \cos(u)$ and $V_2(u) = \cos(u)$. This KHM and its asymmetric variant with $V_2(u) = A \cos(u) (A \neq 1)$ were later studied as kicked-rotor systems \(^{44,48}\) i.e., by viewing $u$ as an angle and $v$ as an angular momentum. Such systems were found to exhibit a variety of quantum-transport phenomena for different values of the parameters $k, A$, and a scaled Planck...
constant. For irrational values of the latter constant, these phenomena include dynamical localization, quantum diffusion (see also Sec. IVC), and ballistic quantum motion. The latter two phenomena are not exhibited by the quantum kicked rotor (see Sec. IIIB).

Due to the unusual form of the “kinetic energy” $kV(x)$ in (37) (a generally non-quadratic function of the “momentum” $v$), one may ask to what extent the KHM represents a realistic system. The original KHM in the paper of Zaslavsky et al. was claimed to describe only approximately the system (27) in the case of $\Omega = \pi/2$ (square crystalline web), $x_c = 0$, and $V(x) = \cos(x)$. Actually, it was later shown\textsuperscript{12} that the generalized KHM (30) is exactly related, both classically and quantally, to the system (27) with $\Omega = \pi/2$, arbitrary $x_c$, and $V(x)$ replaced by a time-periodic potential $V(x,t')$ with period $T = 4$ and satisfying some conditions. To see the classical relation, consider the map $M(k)$ for the latter system, from $t' = t - 0$ to $t' = t + 4 - 0$. This is the composition of four maps, $M(k) = M_2M_1M_0$, where, similarly to (28) for $\Omega = \pi/2$,\textsuperscript{12}

$$M_1: \quad v_{t+1} = -[u_t + k f(x_c - v_t,t)], \quad u_{t+1} = v_t, \quad (31)$$

with $f(x,t') = -\Omega^{-1}\partial V(x,t')/\partial x$. The map $M_{\text{KHM}}(k)$ for (30), from $t' = 2t - 0$ to $t' = 2(t + 1) - 0$, is:

$$M_{\text{KHM}}(k): \quad v_{t+1} = v_t + k f_2(u_t), \quad u_{t+1} = u_t - k f_1(v_{t+1}), \quad (32)$$

where $f_2(x) = -\Omega^{-1}dV_j(x)/dx$, $j = 1, 2$. Then, a straightforward but tedious calculation\textsuperscript{12} shows that the maps $M(k)$ and $M_{\text{KHM}}(k)$ are exactly related,

$$M(k) = M_{\text{KHM}}^{-2}(-k), \quad (33)$$

provided the following conditions are satisfied:

$$V(x_c - v,0) = V(x_c + v,2) = V_1(v), \quad (34)$$

$$V(x_c + u,1) = V(x_c - u,3) = V_2(u). \quad (35)$$

The quantum version of the exact relation (33) turns out to be\textsuperscript{12}

$$\hat{U}(k) = -\hat{U}_{\text{KHM}}^{-2}(-k), \quad (36)$$

where

$$\hat{U}(k) = -e^{-iK\hat{V}(x-u,3)/\hbar} e^{-iK\hat{V}(x+v,2)/\hbar} \times e^{-i\hat{K}\hat{V}(x+u,1)/\hbar} e^{-i\hat{K}\hat{V}(x-v,0)/\hbar} \quad (37)$$

is the evolution operator from $t' = t - 0$ to $t' = t + 4 - 0$ for the system of kicked charges defined above and $\hat{U}_{\text{KHM}}(k) = \exp[-iK\hat{V}_1(v)/\hbar] \exp[-iK\hat{V}_2(u)/\hbar]$ is the evolution operator from $t' = t - 0$ to $t' = t + 2 - 0$ for (30). The conditions for the validity of (33) are again (34) and (35). The minus sign after the equality sign in (33) and (37) is of pure quantum origin and is physically irrelevant. If one wishes to consider only time-independent potentials $V(x,t') = V(x)$, conditions (34) and (35) imply that $V(x)$ must be an even function around $x = x_c$; for example, $V(x) = \cos(x - \gamma)$ with $\gamma = x_c$. The KHM is then symmetric, $V_1(x) = V_2(x)$. It is clear from (34) and (35) that one can always find a time-periodic potential $V(x,t')$ which realizes any given generalized KHM. The kicked harmonic oscillator with $V(x) = \cos(x)$ has already been experimentally realized using atom-optics methods with BECs\textsuperscript{38} but the parameters used do not correspond to $\Omega = \pi/2$, i.e., to the symmetric KHM.

C. Quantum Antiresonance and Diffusion

Quantum transport has been studied in general systems (27)\textsuperscript{52,53} not related to KHM\textsuperscript{s}, i.e., $\Omega \neq \pi/2$ and/or conditions (34) and (35) are not satisfied. A first rigorous result is as follows.\textsuperscript{33} Let us denote by $\hat{U}_0$ the one-period evolution operator for (27) from $t' = t - 0$ to $t' = t + 1 - 0$ and by $\rho = [\hat{u},\hat{v}]/(2\pi\Omega)$ a dimensionless Planck constant. We ask under precisely which conditions the system will exhibit quantum antiresonance (QAR), i.e., $\hat{U}_0^n$ is identically equal to a phase factor for some finite power $m$ [compare with the $m = 1$ case of Eq. (26) in Sec. III]. The answer is that QAR will occur if and only if three conditions are satisfied: (a) The potential $V(x_c - v)$ is odd, $V(x_c + v) = -V(x_c - v)$, up to some additive constant. (b) There is classical “crystalline” resonance, i.e., $\Omega = 2\pi/n$ (see Sec. IVA) with either $n = 4$ (square crystalline case) or $n = 6$ (hexagonal crystalline case); in both cases, the power $m = n$. (c) $\rho$ is integer for $n = 4$ while $\sqrt{3}\rho/2$ is integer for $n = 6$. It was also shown\textsuperscript{33} that for $n = 4$ the QAR is due to the “cancellation” of the main QRs of two KHM\textsuperscript{s}. This is analogous to the QAR of MKRs\textsuperscript{30} arising from the cancellation of the main QRs of KRs (see Sec. III).

In a second work\textsuperscript{52} basic aspects of the QE spectrum and quantum transport were studied as functions of $x_c$ and $\rho$ for $n = 4$ and $V(x) = -\cos(x)$. It was shown that if the parameter $\varepsilon = k\sin(\pi\rho)/(2\pi\rho)$ is sufficiently small, $\varepsilon \ll 1$ and $|1 - \varepsilon/2\sin(x_c)| \ll 1$ (i.e., $x_c$ is not very close to $\pi/2$), the QE spectrum at fixed $x_c$ and $k/\rho$ is approximately the spectrum of a (symmetric) Harper Hamiltonian\textsuperscript{54} $\hat{H} = -2\cos(x_c)[\cos(\hat{u}) + \cos(\hat{v})]$. The latter spectrum as function of $\rho$, $0 \leq \rho \leq 1$, forms the famous “Hofstadter butterfly” (HB)\textsuperscript{60} However, for $x_c = \pi/2$, one has $\hat{H} = 0$: the potential $V(x_c - v) = -\sin(v)$ is odd and QAR occurs for $\rho = 1$ (see above), with an infinitely degenerate QE spectrum. In fact, the QE spectrum for $x_c = \pi/2$ at fixed $k/\rho$ shrinks to a point as $\rho \to 0$ or $\rho \to 1$. After scaling the QEs by the $\rho$-dependent factor $\varepsilon^{-1}$, the spectral structure as function of $\rho$ at fixed small value of $k/\rho$ becomes very close to that of a “double HB”.

It is known\textsuperscript{58} that the spectrum of the Harper Hamiltonian is fractal (a Cantor set) for generic, irrational $\rho$ and that such spectrum leads to a “quantum diffusion” of the expectation value $\langle \hat{u}^2 \rangle_t$ or $\langle \hat{v}^2 \rangle_t$ for asymptotically large times $t$. This should be then also the approximate behavior of the expectation value of the kinetic energy in (27), at least for sufficiently small values of the parameter.
\[ \langle \Omega^2 \rangle_t \approx D_q(x_c)t, \]  

where \( D_q(x_c) \) is the quantum-diffusion coefficient. The asymptotic behavior \( \langle \cdot \rangle_t \) was extensively verified numerically and the approximate formula \( D_q(x_c) \approx D_q(0)\cos(x_c) \), valid at least in simple cases, was derived.\(^{32}\) As expected, \( D_q(x_c) \to 0 \) in the QAR limit of \( x_c \to \pi/2 \). The quantum-transport behavior \( \langle \cdot \rangle_t \) for the non-KAM system \( \langle x \rangle_t \) is in sharp contrast with the dynamical localization for KAM (KR) systems (see Secs. IIB and III). The quantum diffusion of a general ensemble of charged particles, exhibiting all values of \( x_c \), is characterized by a (weighted) average of \( D_q(x_c) \) over \( x_c \), in analogy to the classical case (see Sec. IVA).

It was already noted in the work mentioned above\(^{31}\) (see Fig. 1 there) that while QAR occurs in a strong quantum regime of non-small \( \rho \geq 1 \), it has a distinct classical analogue: For small \( k \), the chaotic diffusion on the crystalline stochastic web for odd potential \( V(x) = -\cos(x) \) and \( x_c = \pi/2 \) is much slower than that for even potential \( V(x) = -v \) (say, \( V(x) = -\cos(x) \) and \( x_c = 0 \)). This classical phenomenon was explained in a later work in the case of \( n = 4\).\(^{52}\) For even potential, relation \( \langle x \rangle_t \) holds, so that the map \( M(k) \) is, like the map \( \mathcal{M}_{\text{KHM}}(\varepsilon) \) in \( \langle x \rangle_t \), a perturbation of order \( O(k) \) of the identity map \( u_{t+4} = u_t, v_{t+4} = v_t \); on the other hand, for odd potential, the map \( M(k) \) turns out to be a much smaller perturbation, of order \( O(k^2) \), of the identity map.

**D. Quantum-Chaotic Ratchet Accelerators**

Classical Hamiltonian ratchets are systems in which a directed current can emerge in the chaotic region from an unbiased force, i.e., a force whose phase-space and/or time average is zero. This current is a mean position velocity (usual velocity) or momentum velocity (acceleration). In the latter case, one has a ratchet accelerator, first introduced by Gong and Brunner using MKRs in a strong-chaos regime (see Secs. IIIB and III; see also the very recent ratchet accelerator for arbitrarily weak chaos).\(^{58}\) For a persistent Hamiltonian ratchet effect to occur, it is necessary that the system is asymmetric and possesses transporting stability islands (accelerator-mode islands in the case of ratchet accelerators) whose total flux is non-zero due to the asymmetry. Thus, such an effect cannot arise in a fully chaotic regime (with no stability islands) even if the system is asymmetric.

On the other hand, quantum-ratchet-acceleration effects can generically occur also in systems whose classical phase space is fully chaotic.\(^{59,61}\) This was first shown by Gong and Brunner using an asymmetric KHM which is fully chaotic and does not exhibit a classical ratchet effect. For generic irrational values of a scaled Planck constant in a semiclassical regime, a strong quantum ratchet acceleration is observed using a spatially uniform initial state (a zero-momentum state). This acceleration is due to the existence of cantori (broken KAM tori) in the “vertical” (momentum \( v \)) direction whose broken phase-space structures (gaps in the KAM tori) have typical size smaller than the scaled Planck constant. Thus, quantumly, the gaps appear as “closed” and the cantori act effectively as vertical KAM tori which cause, already classically, a ratchet acceleration. This strong quantum effect was found to be robust to noise and should therefore be experimentally realizable using kicked harmonic oscillators that are exactly equivalent to KHMs (see Sec. IVB).

In two recent papers,\(^ {59,61}\) the quantum ratchet acceleration in fully chaotic KHMs was studied for small rational values of the dimensionless Planck constant \( \rho \) (using the notation in Sec. IVC), \( \rho = 1/N \) \((N \gg 1)\). This corresponds to high-order QRs in a deep semiclassical regime. The most important difference between this study and the one above of Gong and Brunner is that the initial states used are maximally uniform in phase space (MUPS), not only in ordinary space. Each such state corresponds to a “von-Neumann” lattice in phase space whose unit cell \((0 \leq u < a, 0 \leq v < b)\) has Planck area \(ab = \hbar/4\pi^2\rho = 4\pi^2/N^2\); the lattice origin is labeled by phase-space quasicoordinates \( (u_1, w_2) \) ranging in the Planck cell, \( 0 \leq w_1 < a, 0 \leq w_2 < b \). It was shown that a MUPS state \( |w_1, w_2\rangle \) leads to a QR ratchet acceleration \( I(w_1, w_2) \) much stronger than that obtained by using pure momentum states, already in completely symmetric KHMs. As in the case of the symmetric QR ratchets described in Sec. IID, this effect is due to the relative asymmetry associated with the noncoincidence of the symmetry centers of the KHM and the MUPS state \( |w_1, w_2\rangle \). It was also shown that the distribution of \( I(w_1, w_2) \) over \( |w_1, w_2\rangle \) is a Gaussian with mean \( \langle I \rangle = 0 \) and variance \( \langle I^2 \rangle = 2D/N^2 \), where \( D \) is the classical chaotic-diffusion coefficient in the momentum \( (v) \) direction. Besides \( |w_1, w_2\rangle \), other initial states were considered which approximate \( |w_1, w_2\rangle \) to some arbitrary order denoted by \( B \) \((0 \leq B \leq \infty)\); here \( B = 0 \) and \( B = \infty \) correspond, respectively, to pure momentum states and MUPS states \( |w_1, w_2\rangle \). It was found that the quantum ratchet acceleration over the approximating states has zero mean, \( \langle I \rangle_B = 0 \), and variance \( \langle I^2 \rangle \) increasing monotonically with \( B \): \( \langle I^2 \rangle_B \) for pure momentum states is significantly smaller than \( \langle I^2 \rangle_\infty = 2D/N^2 \) for the MUPS states. For sufficiently low order \( B \), the approximating states should be experimentally realizable.

**V. CONCLUSIONS**

In this brief review, we have focused on representative classes of 1D kicked systems with a periodic time dependence associated entirely with the kicking potential. These systems have been extensively investigated during the last three decades. Less is known about kicked systems in more than one dimension and/or with a nonperiodic time dependence. For the benefit of the interested reader, we conclude by considering very briefly two important examples of the latter systems. The first example is a rotor kicked by a potential quasiperiodic in
time with three incommensurate frequencies (including the frequency of the periodic delta function). For irrational values of a scaled Planck constant, the QE problem for this system is equivalent to that of a 3D pseudorandom tight-binding model (compare with the 1D case in Sec. IIB). Then, as a nonintegrability parameter is increased, there occurs a transition from pseudorandom Anderson localization to unbounded diffusion (delocalized regime) at a critical value of the parameter. This transition and related phenomena have been experimentally observed using atom-optics methods. The second example is that of a 1D periodically kicked particle subjected to an additional constant force or linear potential. The theoretical interest in this system began following the experimental discovery of the “quantum accelerator modes” (QAMs) in the free-falling frame (FFF) of periodically kicked atoms falling under gravity. In the FFF Hamiltonian, the linear potential does not appear but the momentum, in the kinetic-energy term, increases linearly in time, making the relevant time dependence of the system non-periodic. The QAMs in the FFF were theoretically explained as associated with a vicinity of the main QRs, i.e., with values $\hbar = 2\pi l + \epsilon (l$ an integer, see Sec. IID) of the scaled Planck constant $\hbar$. It was shown that this vicinity defines a “quasiclassical” regime in which $\epsilon$ plays the role of a fictitious Planck constant and the quantum evolution in the FFF can be approximately described by a classical map. Then, a wave packet initially trapped in an accelerator-mode island of this map “accelerates”; this is a QAM. The experimentally observed robustness of QAMs under relatively large deviations $\epsilon$ from $\hbar = 2\pi l$ was explained, in the framework of the quasiclassical approximation, as a “mode-locking” phenomenon. Theoretical predictions were verified by several experiments.

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