The ADM Formulation of the SME Gravity

Carlos M. Reyes
Centro de Ciencias Exactas, Universidad del Bío-Bío
Avda. Andrés Bello 720, Chillán, 3800708, Chile

The Hamiltonian formulation of the gravitational sector of the Standard-Model Extension (SME) with nondynamical fields $u$ and $s_{\mu \nu}$ is studied. We provide the relevant Hamiltonians that describe the constrained phase space and the dynamics of the induced metric on the ADM hypersurface. The generalization of the Gibbons-Hawking-York boundary term has been crucial to preventing second time-derivatives of the metric tensor in the Hamiltonians. By extracting the dynamics and constraints from the Einstein equations we have proved the equivalence between the Lagrangian and Hamiltonian formulations.

1. Introduction

Einstein’s theory of General Relativity (GR) has profoundly shaped our understanding of the physical world at large and small scales. GR tells us that spacetime is a four-dimensional manifold on which a metric solving Einstein’s equations defines how matter moves, and matter can deform the spacetime geometry.

Soon after the advent of Quantum Field Theory and the success of the perturbative method for the Standard Model (SM) of particles, it was explored whether the perturbative techniques could be applied on the gravitational field. However, it was shown that pure gravity has nonrenormalizable divergences at two loops and becomes more singular when coupling to matter, in particular for the electromagnetic and Dirac fields.

The incompatibility of SM and GR suggest that gravity is a low-energy approximation of a fundamental theory. New physics in the form of Lorentz and CPT symmetry breaking has been postulated as one possible effect that could be detected at low energies. The Standard-Model Extension (SME) is a comprehensive effective-field theory framework developed to study departures from CPT and Lorentz symmetry in particle physics and diffeomorphism and local Lorentz violations in gravity.
2. Hamiltonian form of the SME gravity

We start with the gravitational action\(^6\) without a cosmological constant

\[ S = \int_M d^4x \frac{\sqrt{-g}}{2\kappa} \left[ (1-u)R + s^{\mu\nu} R_{\mu\nu} \right], \]  

(1)

with \(\kappa = 8\pi G_N\), \(G_N\) the Newton constant, the Ricci tensor \(R_{\mu\nu}\) and the associated Ricci scalar \(R := \text{tr}(R_{\mu\nu})\) of the four-dimensional spacetime manifold \(M\) with metric tensor \(g_{\mu\nu}\) and \(\sqrt{|\det(g_{\mu\nu})|}\). Furthermore, \(u = u(x)\) and \(s^{\mu\nu} = s^{\mu\nu}(x)\) are nondynamical background fields having a generic spacetime dependence.

The modified Einstein equations of motion for the Lagrangian (1) are

\[ 0 = (1-u)G_{\mu\nu} + \frac{1}{2} (\nabla^\mu \nabla^\nu u + \nabla^\nu \nabla^\mu u) - g^{\mu\nu} \Box u - \frac{1}{2} \left( s^{\alpha\beta} R_{\alpha\beta} - g^{\mu\nu} \nabla_\alpha s_{\mu\nu} - g^{\mu\nu} \nabla_\beta s^{\alpha\beta} \right), \]  

(2)

where \(G_{\mu\nu}\) is the Einstein tensor, \(\Box = \nabla^\mu \nabla_\mu\), \(\nabla_\mu\) is the covariant derivative and we have used that the backgrounds have zero fluctuations.

The Hamiltonian formulation by definitions needs a time variable and, hence, our next step is to decompose spacetime. We consider the 3 + 1 decomposition of spacetime due to Arnowitt, Deser and Misner (ADM).\(^7\) The ADM decomposition of the metric tensor \(g_{\mu\nu}\) turns to be

\[ g_{00} = -N^2 + q_{ij} N_i N_j, \quad g_{0i} = N_i \quad \text{and} \quad g_{ij} = q_{ij}, \]  

(3)

where \(q_{ij}\) is the induced metric on the hypersurface \(\Sigma_t\), and \(N, N^i\) are called the lapse and the shift, respectively.

The Legendre transformation of the Lagrangian (1) leads to the canonical Hamiltonians described in Ref. 8. To give a general idea, we focus on the derivation of the Hamiltonian in the \(u\) sector. One finds the conjugate momentum

\[ \pi^{ij} = \frac{\sqrt{q}}{2\kappa} \left[ (1-u)(K^{ij} - q^{ij} K) + \frac{1}{N} q^{ij} \mathcal{L}_{m} u \right], \]  

(4)

and the Hamiltonian

\[ H_u = \int_{\Sigma_t} d^3x \left[ -\frac{\sqrt{q}}{2\kappa} N \left( (1-u)R + 2D^i D_i u \right) + \frac{\mathcal{L}_{m} u}{1-u} \left( \pi - \frac{3}{4} \sqrt{q} N \mathcal{L}_{m} u \right) \right. \]  

\[ + \left. 2\kappa N \frac{\sqrt{q}}{\sqrt{(1-u)}} \left( \pi^{ij} \pi_{ij} - \frac{\pi^2}{2} \right) - 2(D_i \pi^{ij}) N^i \right]. \]  

(5)

The Hamiltonians in the \(s^{\mu\nu}\) sector follows by considering the ADM decomposition of the background field \(s^{\mu\nu}\), which is

\[ s^{\alpha\beta} = q^{\alpha}_{\mu} q^{\beta}_{\nu} s^{\mu\nu} - (q^{\alpha}_{\mu} n^{\mu} + q^{\beta}_{\nu} n^{\nu}) s^{\alpha\beta} s^{\mu\nu} + n^{\alpha} n^{\beta} s^{\mu\nu}, \]  

(6)
where \(q_{\mu}^{\nu}\) projects a tensor or a part of it into \(\Sigma_t\). We have three more sectors the \(s^{ij} := q_{\mu}^{\nu} q_{\nu}^{\mu} s^{\mu\nu}\) as the purely spacelike sector of \(s^{\mu\nu}\) that lives in \(\Sigma_t\) entirely, the \(s^{\nu} := q_{\mu}^{\nu} n_{\nu} s^{\mu\nu}\) be the vector-valued piece and \(s^{nn} := n_{\mu} n_{\nu} s^{\mu\nu}\) the scalar part. The details can be found in Ref. 8, 9 and a similar treatment in Ref. 10.

3. Dynamics and constraints

To find the Hamiltonians we have introduced the extended Gibbons-Hawking-York term

\[
S_{\text{GHY}}^\text{ext} = \frac{\varepsilon}{2\kappa} \oint_{\partial M} d^3y \sqrt{|q|} \left[ 2(1-u)K - s^{nn} K + K_{ij} s^{ij} \right],
\]

where the parameter \(\varepsilon = \mp 1\) for a spacelike (timelike) boundary \(\partial M\) of the spacetime manifold \(M\), \(K_{ij}\) is the extrinsic curvature, \(q_{\mu}^{\nu}\) is the trace and the integral runs over the coordinates \(y^i\) defined on this boundary. In Ref. 9, we introduce a second boundary term for \(u^\nu\) and \(s^{nn}\) that is of plainly different nature compared to that of (6)

\[
S_{\partial \Sigma} = -\frac{1}{2\kappa} \oint_{\partial \Sigma_t} d^2z \sqrt{|q|} r_l \left[ N D^l (2u + s^{nn}) \right].
\]

Consider the equation of motion in the \(u\) sector

\[
Q^{\mu\nu} := (1-u)^4 G^{\mu\nu} + \nabla^\mu \nabla^\nu u - g^{\mu\nu} \Box u = 0.
\]

Now, the projected equation of motion into \(\Sigma_t\) is

\[
(q^j Q)^{ij} = \frac{2k}{N \sqrt{q}} \pi^{ij} + (1-u) \left( R^{ij} - \frac{R}{2} \delta^{ij} \right) + \frac{1}{N} \left( q^{ij} D_k D^k [(1-u)N] - D^i D^j (1-u)N \right) + q^{ij} a^k D_k u - (a^i D^i u + a^j D^j u) + \frac{4N^2}{q(1-u)} \left( \pi^{ij} - \frac{\pi^2}{2} \right) q^{ij} - \frac{4k^2}{q(1-u)} N q^{ij} L_m u \right], \]

the purely orthogonal projection given by

\[
2Q(n, n) = (1-u) R + 2 D_i D^i u - \frac{4k^2}{q(1-u)} \left( \pi^{ij} \pi^{ij} - \frac{\pi^2}{2} \right) - \frac{3(L_m u)^2}{2(1-u)N^2},
\]
and the mixed projection by

\[ 2Q^k(q(.), n) = \frac{2\kappa}{\sqrt{T}} \left[ 2D_i\pi^{ik} + \frac{\pi}{1-u} D^k u \right] - \frac{3L_{\mu}}{(1-u)N} D^k u. \]  

(11)

We have defined

\[ T(n, n) := n_\mu n_\nu T^{\mu\nu}, \quad T^q(q(.), n) := q^\rho_\mu n_\nu T^{\mu\nu}, \]

(12)

and

\[ (q^T)^{\alpha_1...\alpha_s}_{\beta_1...\beta_t} := q^{\alpha_1 \gamma_1} ... q^{\alpha_s \gamma_s} q^{\delta_1 \beta_1} ... q^{\delta_t \beta_t} T^{\gamma_1...\gamma_s}_{\delta_1...\delta_t}. \]  

(13)

In Ref. 9 we have shown the equivalence between the Lagrangian and Hamiltonian formulations.

4. Conclusions

We have performed the ADM decomposition of the gravitational sector of the SME. A crucial part of the derivation was to extend the Hawking-Gibbons-York boundary term in order to avoid second time-derivatives on the metric in the Hamiltonians. We have extracted the dynamics and constraints from the Einstein equation and compared with those obtained in the Hamiltonian formulation. In addition, the ADM formalism has stimulated the works on cosmological applications.\(^{11,12}\)

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