Competing pairing channels in the doped honeycomb lattice Hubbard model

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Proposals for superconductivity emerging from correlated electrons in the doped Hubbard model on the honeycomb lattice range from chiral $d + id$ singlet to $p + ip$ triplet pairing, depending on the considered range of doping and interaction strength, as well as the approach used to analyze the pairing instabilities. Here, we consider these scenarios using large-scale dynamic cluster approximation (DCA) calculations to examine the evolution in the leading pairing symmetry from weak to intermediate coupling strength. These calculations focus on doping levels around the van Hove singularity (VHS) and are performed using DCA simulations with an interaction-expansion continuous-time quantum Monte Carlo cluster solver. We calculated explicitly the temperature dependence of different uniform superconducting pairing susceptibilities and found a consistent picture emerging upon gradually increasing the cluster size: while at weak coupling the $d + id$ singlet pairing dominates close to the VHS filling, an enhanced tendency towards $p$-wave triplet pairing upon further increasing the interaction strength is observed. The relevance of these systematic results for existing proposals and ongoing pursuits of odd-parity topological superconductivity are also discussed.

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I. INTRODUCTION

Many aspects of the fascinating physics of the low-energy Dirac electrons in graphene can be explored based on non-interacting tight-binding models on the honeycomb lattice, in particular close to charge neutrality, where the effects of the electronic interactions are delayed to the strong-coupling regime due to a vanishing density of states (DOS) at low energies. At finite doping, however, the presence of even weak interactions among the electrons is predicted by several studies to lead to new collective behavior. Of particular recent interest are interaction-driven instabilities towards unconventional superconductivity in doped honeycomb systems [1–17]. Some early studies concluded that superconductivity might not be stable with respect to charge or spin order for the basic Hubbard model on the honeycomb lattice [2, 3], and a possible quantum liquid state has been suggested recently for the van Hove singularity (VHS) filling [18]. In most of the recent theoretical studies, however, a general tendency towards some flavor of superconductivity upon doping the honeycomb lattice is indeed observed. However, various proposals on the nature of the emerging superconducting state and the stability range of competing pairing channels still lead to a mosaic of different scenarios. Several mean field theory and renormalization-group (RG) calculations predict chiral $d + id$ singlet superconductivity to emerge in the weak-coupling region upon doping towards or onto the VHS, which corresponds to electronic densities of $n = 3/4$ and $5/4$ for the Hubbard or related models with explicit spin exchange terms or extended interactions [4–10]. Variational Monte Carlo simulations [19] also showed a chiral $d$-wave solution over a wide range of doping. The $d$-wave pairing state in this scenario is related to enhanced antiferromagnetic fluctuations near half filling as well as the VHS-increased DOS.

On the other hand, a recent study using the variational cluster approximation (VCA) and cellular dynamical mean field theory (CDMFT) performed for larger values of the local repulsion, found a stable $p$-wave triplet pairing state for a weak nearest-neighbor repulsion [11], with possibly a coexisting Kekulé pattern [12, 13, 17]. A possible $p + ip$ pairing state was also reported at low filling from determinantal quantum Monte Carlo studies; however, the sign problem poses restrictions on the accessible system sizes, interaction strengths and temperature ranges. In addition, Grassmann tensor renormalization calculations have been performed [14, 15], and in Ref. [14], a $d + id$ state is reported for the $t – J$ model, while for infinite local repulsion a $p + ip$ superconducting state, coexisting with ferromagnetic order, has been proposed for the Hubbard model at low doping [15]. Hence, despite active pursuits, such deviations among the various proposals and employed methods show that a consistent picture of possible superconductivity even in the basic Hubbard model on the honeycomb lattice is still lacking, apparently due to competition among several possible low-energy states upon varying the doping or interaction strength. It thus appears promising and necessary to examine this problem from the perspective of a method that allows us to tune these parameters over a wide range while accounting for the growing local electronic correlations beyond the weak-coupling regime.

Here, we employ such an approach by providing results from large-scale dynamic cluster approximation (DCA) calculations, with a focus on pairing susceptibilities to probe for uniform superconducting instabilities. Upon systematically increasing the cluster size, we find that a consistent picture starts to emerge for the leading pairing channels on the honeycomb lattice Hubbard model from small- to medium-sized local interactions: while at weak coupling, chiral $d + id$ singlet pairing dominates close to VHS filling, when the interaction becomes stronger, a tendency towards $p$-wave triplet pairing develops. Our calculations are performed with an interaction-expansion continuous-time quantum Monte Carlo (CT-INT) cluster solver [21–24], keeping up to 24 cluster sites (see Appendix A for details on the CT-INT approach). Before discussing our results, we provide details about the considered
model and the DCA computational framework for calculating the pairing susceptibilities.

II. MODEL AND METHOD

The Hubbard model on honeycomb lattice has the Hamiltonian

\[
\hat{H} = \hat{H}_0 + \hat{H}_I, \\
\hat{H}_0 = -t \sum_{\langle i,j \rangle} \hat{c}_i^\dagger \hat{c}_j - \mu \sum_i \hat{n}_i, \\
\hat{H}_I = U \sum_i (\hat{n}_i - \frac{1}{2})(\hat{n}_i - \frac{1}{2}),
\]

where \(t\) denotes the hopping amplitude between nearest neighbor sites \((i,j)\), \(\mu\) is the chemical potential that controls the electronic density, and \(\hat{n}_i = \hat{c}_i^\dagger \hat{c}_i\) is the number operator for spin flavor \(\sigma\) on the \(i\)-th lattice site. Furthermore, \(U\) denotes the onsite Coulomb repulsion. Longer-ranged interaction will not be considered here, and at finite doping, especially close to the VHS, screening plays an important role and cuts off the long-ranged tail of the Coulomb potential [10].

The DCA maps this original lattice model onto a periodic cluster, embedded into a self-consistently determined bath. Spatial correlations within the cluster are treated explicitly, while those at longer length scales are described at the dynamical mean-field level [20]. For this work, we have systematically employed three cluster sizes, shown in Fig. 1, with \(N_c = 3, 4\) and 12 unit cells. The lattice \(D_{6h}\) symmetry is enforced for the \(N_c = 4\) cluster (see the figure caption), while the \(N_c = 3\) and 12 clusters explicitly retain this symmetry. For the largest cluster, we are able to study inverse temperatures up to \(\beta t = 40\) at a coupling of \(U = 2t\). Compared to the widely studied square [21, 23, 25–30] or triangular lattices [22, 31, 32], the DCA formalism needs to be modified for the honeycomb lattice, which is a bipartite lattice with two sites per unit cell. In particular, the single particle Green’s function’s and the self-energy for spin flavour \(\sigma\) are \(2 \times 2\) matrices \(\Sigma_{\alpha,\beta}^{\sigma}(\mathbf{k}, \omega_n)\) and \(\Sigma_{\alpha,\beta}^{\sigma}(\mathbf{k}, \omega_n)\), with the band or orbital indices \(\alpha, \beta = 1, 2\). We developed a generic scheme for performing DCA calculations on such more complex lattices, with details provided in Appendixes B and C.

III. PAIRING SUSCEPTIBILITIES

In order to probe for superconductivity with respect to different pairing channels, we consider appropriate pairing order parameters in real space,

\[
\Delta_\eta(i) = \sum_l f_\eta(\delta_l) (\hat{c}_i^\dagger \hat{c}_{i+\delta_l} \pm \hat{c}_i \hat{c}_{i+\delta_l}^\dagger),
\]

where \(\eta\) denotes the different pairing channels: \(s, p, d, f\), \(p \pm ip\) and \(d \pm id\). Here, \(f_\eta(\delta_l)\) are form factors that correspond to the pairing symmetry \(\eta\) and are provided explicitly in Fig. 1; \(\delta_l\) indicates the pairing bonds (we restrict ourselves to nearest-neighbor pairings, \(l = 1, 2, 3\), and + and − denote triplet and singlet states, respectively. The possible pairing channels can be classified according to the irreducible representations of the \(D_{6h}\) point group of the honeycomb lattice [4, 9–11]. The corresponding uniform susceptibility for a pairing channel \(\eta\) is then obtained in the imaginary-time formulation as

\[
\chi^\eta(T) = \frac{1}{N} \int_0^\beta d\tau \sum_{ij} \langle T_\tau \Delta_\eta(i) \Delta_\eta^\dagger(j,0) \rangle.
\]

Transforming to momentum and frequency space and normalizing by the form factors, we obtain

\[
\chi^\eta(T) = \frac{1}{\beta} \frac{\sum_{pp'qq'} \langle \Phi^\eta(k) | \chi(p, p', q, q') | \Phi^\eta(k') \rangle}{\sum_k |\Phi^\eta(k)\rangle \langle \Phi^\eta(k) |}
\]

which contains the form factor, written in vector form as

\[
|\Phi^\eta(k)\rangle = \left( \sum_l f_\eta(\delta_l) e^{ik\delta_l} \pm \sum_l f_\eta(\delta_l^\dagger) e^{-ik\delta_l}^\dagger \right).
\]

Here, \(p = (\mathbf{k}, \omega_n)\), \(p' = (\mathbf{k}', \omega'_n)\) and \(q = (\mathbf{q}, \nu_m)\) are four-momenta containing both momentum and frequency. In the following, we restrict ourselves to uniform pairing states, corresponding to \(q = 0\) and \(\nu = 0\); that is, we focus here on the pairing channel with respect to only the point group symmetry. Directly comparing pairing susceptibilities \(\chi^\eta(T)\) of
different channels is not a practical way to numerically identify the leading pairing channel because, usually, the non-interacting pairing susceptibility masks the interaction effects in the pairing susceptibility within the temperature range where the CT-INT simulation can be performed. Two routes can be taken to overcome this issue. One possibility is to perform an eigenvalue analysis of the pairing vertex secular equation [23–25, 33], which usually requires high quality data for the irreducible vertices and very low temperatures, such that the momentum dependence of the leading eigenvector does indeed reflect the pairing symmetry of the superconducting ground state. In the other approach, one subtracts the decoupled part of the pairing susceptibility $\chi^0$ (the particle-particle bubble) from the interacting one, such that the effective pairing susceptibility $\chi_{\text{eff}} = \chi - \chi^0$, stemming from the electronic correlations, can be extracted [3, 34]. Here, we adopt the latter scheme and thus extract the effective pairing susceptibilities

$$\chi_{\text{eff}}^\eta(T) = \frac{1}{\beta} \sum_{k} \sum_{p'q'=0} \langle \Phi_{\eta}(k)|\chi(p,p',q) - \chi^0(p,p',q)|\Phi_{\eta}(k)\rangle$$

Further details on the calculation of the vertex function and the effective pairing susceptibilities within the DCA framework are provided in Appendices D and E. The latter also discusses the relation to the eigenvalue analysis of the pairing vertex.

IV. RESULTS

Before discussing interaction effects, it is useful to examine the bare ($U = 0$) pairing susceptibilities, which are shown in Fig. 2 at the VHS density $n = 3/4$. Due to the VHS in the bare DOS, the bare $\chi(T)$ diverge logarithmically as $T \to 0$, with the strongest divergence exhibited by the $p$-waves. This provides an important background to the pairing susceptibilities in the interacting case. Hence, at finite $U$, the decoupled part of the pairing susceptibility $\chi^0(T)$ is subtracted in order to make the effective pairing susceptibilities $\chi_{\text{eff}}(T)$ manifest, as mentioned above.

Turning then to the interacting case, we calculated both the temperature and the filling dependence of $\chi_{\text{eff}}$ for different values of $U$ and cluster sizes $N_c$. In the following, we present explicitly our results for $d$- and $p$-wave pairing, which we observe to be the most dominant channels. In Fig. 3(a), the temperature dependence of the effective pairing susceptibilities at $U = 2t$ and the VHS filling are shown for the $N_c = 3$ cluster. Here, we identify $d + id$ as the dominant pairing channel. However, for $U = 6t$ [Fig. 3(b)], the $p + ip$ triplet channel increases and also becomes positive, and furthermore exhibits a tendency to diverge. In order to monitor this behavior as a function of doping, the dependence of $\chi_{\text{eff}}$ on the density $n$ for different interaction strengths at $\beta t = 20$ is shown in Fig. 3(c). The dome-shaped behavior in the $d + id$ singlet pairing channel $\chi_{\text{eff}}$ indicates an optimal doping between $n = 0.75$ and $0.85$. At $U = 6t$, $p + ip$ also exhibits a dome-shaped $\chi_{\text{eff}}$-maximum, even though the amplitude is still lower than for $d + id$.

The $N_c = 4$ cluster results, shown in Fig. 4, are to a large extent similar to the $N_c = 3$ data: For weak interactions ($U = 2t$ and $4t$), and close to the VHS, the dominant pairing channel is also $d + id$. Upon increasing the interaction strength to $U = 6t$, the $p$-wave effective susceptibility again increases. However, we also find differences between the results for $N_c = 3$ and $N_c = 4$. As shown in Fig. 1(b), the cluster momenta for $N_c = 4$ are the $\Gamma$ point and the $\tilde{M}$ points. In the non-interacting band structure, these four cluster momenta are below the Fermi surface for densities $n > 0.75$. Thus, the $N_c = 4$ cluster does not capture charge fluctuations about (below and above) the Fermi surface for densities $n > 0.75$. This is reflected by the effective pairing susceptibilities. For example, for $U = 4t$ the effective pairing susceptibility at $T/t = 0.025$ in the $(d + id)$-wave channel for $N_c = 3$ is 0.154, while for $N_c = 4$, it is 0.074, i.e.,

![Graph](image_url)
about half the \( N_c = 3 \) value. Similarly, due to the deficits of the \( N_c = 4 \) cluster, the \( p \)-wave channel does not exhibit a tendency to increase for \( U = 4t \) at the VHS filling [see Fig. 4(b)]. In Fig. 4(c), we still observe a narrow density regime within which the effective pairing susceptibility for the \( p_y \) channel is positive, but it is smaller than that of the \( d + id \) channel. The observed trends suggest that only upon further increasing the interaction strength might the \( p_y \) channel possibly diverge more rapidly than the \( d + id \) channel. We note, that the breaking of the degeneracy among the different \( p \)-wave channels is due to finite-size effect; that is, the corresponding form factors have different gap sizes on the finite cluster momenta.

Within the DCA approach, one must study the systematic behavior upon increasing \( N_c \) in order to draw conclusions about the thermodynamic limit. As demonstrated by comparing the \( N_c = 3 \) and \( N_c = 4 \) results, it is also important to consider clusters that capture the low-energy fluctuations. We thus also employed the \( N_c = 12 \) cluster within the DCA/CT-INT framework, which has cluster momenta that include the \( \Gamma \) point, the two \( K \) points, the three \( M \) points and six other momenta [see Fig. 1(c)]. It provides a more detailed structure of the pairing symmetry than the \( N_c = 3 \) and 4 clusters. Unfortunately, the minus-sign problem becomes much more severe for \( N_c = 12 \), and we cannot access large values of \( U \) for \( N_c = 12 \). Nevertheless, we can draw interesting observations from the accessible parameter range: In Fig. 5(a), for \( U = t \), the dominant pairing channel is still \( d + id \) near the VHS filling, while in Fig. 5(b), we find for \( U = 2t \) that \( p \)-wave starts to increase at low temperatures. The \( p \)-wave channels (in particular \( p_y \)) tend to increase more rapidly upon cooling than the \( d \)-wave channels. This trend indicates that the \( p \)-wave channels compete strongly with \( d + id \) pairing at this interaction strength, such that in the intermediate interaction range, there is an enhanced tendency for \( p \)-wave triplet pairing to eventually dominate over \( d \)-wave singlet pairing in the thermodynamic limit. To analyze this trend in more detail, the density dependence of the leading effective pairing susceptibilities is shown in Fig. 5(c). The optimal doping range for \( d + id \) pairing is consistent with the \( N_c = 3 \) and 4 results. In addition, the enhancement of the \( p \)-wave channels upon increasing the interaction strength is quite pronounced on the \( N_c = 12 \) cluster. Hence, although the minus-sign problem renders us unable to make a definitive statement about whether \( p \)-wave triplet pairing will eventually replace \( d \)-wave singlet pairing, the available \( N_c \) = 12 data up to \( U = 2t \) suggest such a scenario.

V. DISCUSSIONS AND CONCLUSIONS

Our DCA/CT-INT results are consistent with previous reports that in the weak coupling limit the dominant pairing is the chiral \( d + id \) singlet channel. However, we find upon increasing the interaction strength (e.g., for \( N_c = 12, U = 2t \)) a clear tendency towards a competing \( p \)-wave triplet pairing. This finding is consistent with several recent findings. For example, it has been reported [11], in the presence of both on-site interaction and nearest neighbor repulsion, for a wide range of doping around the VHS, the dominant pairing is a \( p \)-wave triplet. In Refs. 14 and 15, where the infinite-\( U \) limit was considered, a \( p + ip \) wave superconducting ground state was proposed. Our calculation focused on the range of small and medium strength interactions, and indeed suggested the possibility that the dominant pairing channel changes from \( d + id \) to \( p \)-wave upon increasing the interaction strength.

For the future, it would be interesting, to allow also for inhomogeneous pairing states within the DCA calculations, in light of several recent proposals of superconductivity coexisting with Kekulé patterns [12, 13, 17]. On a more general note, the effect of Hund’s coupling and spin-orbit coupling could be included in the DCA calculations, given the Hund’s coupling induced triplet pairing scenario of Ref. [35] as well as recent NMR experiments on Cu,Bi2Se3, a spin-orbital coupled topological material with moderate electron correlations, which suggest an odd-parity, spin-rotation symmetry breaking triplet pairing state [36].
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Appendix A: Interaction expansion

The partition function for the CT-INT can be obtained as

$$Z = \text{Tr} \left[ e^{-\beta \hat{H}_0} e^{-i \int_0^\beta d\tau \hat{H}_1(\tau)} \right]$$

$$= \sum_k (-1)^k \int_0^\beta d\tau_1 \cdots \int_0^\beta d\tau_k \text{Tr} \left[ e^{-\beta \hat{H}_1(\tau_k) \cdots \hat{H}_1(\tau_1)} \right]$$

$$= \sum_{\phi_k} \frac{-U}{2} \frac{1}{k!} \prod_{\sigma} \left[ \text{Tr} (\hat{\eta}_1 \sigma - \alpha_{1\sigma})(\hat{\eta}_2 \sigma - \alpha_{2\sigma}) \cdots (\hat{\eta}_{k \sigma} - \alpha_{k\sigma}) \right]$$

$$= \sum_{\phi_k} \frac{-U}{2} \frac{1}{k!} \prod_{\sigma} \text{det} D^\sigma(k)$$

(A1)

where we have written the interaction part $\hat{H}_1$ of the Hamiltonian Eq. (1) in the main text as $U/2 \sum_{i \neq j} \prod_\sigma (\hat{\eta}_i \sigma - \alpha_\sigma(s))$, with $\alpha_\sigma(s) = 1/2 + \sigma s(1/2 + 0^+)$.

The honeycomb lattice has two sites per unit cell, and the DCA self-consistent loop requires more steps for such a complex lattice than for simple lattices such as square or triangular lattice. In this appendix, we describe the general DCA scheme we have developed for such more complex lattices.

We define the cluster excluded Green’s function in matrix form $g^0(R, \tau)$, whose matrix elements are $g^0_{\alpha \beta}(R, \tau)$ with $R$ being the distance vector between unit cells, and $\alpha$, $\beta$ being indices for the two sublattices, $A$ and $B$.

The DCA loop starts from a non-interacting cluster self-energy $\Sigma_c(K, i\omega_n) = 0$ or the self-energy obtained from second-order perturbation theory. One then uses the cluster self-energy to approximate the lattice self-energy, and the lattice Green’s function is

$$g^{latt}(k, i\omega_n) = g^{latt}(K + \hat{k}, i\omega_n)$$

$$= \frac{1}{(i\omega_n + \mu) 1 - H_0(K + \hat{k}) - \Sigma_c(K, i\omega_n)}$$

where

$$H_0(k) = \left( \begin{array}{cc} 0 & -i \sum_{l=1}^3 e^{i \hat{k} \delta_l} \\
-t \sum_{l=1}^3 e^{-i \hat{k} \delta_l} & 0 \end{array} \right)$$

(B1)

and $\delta_l$ are the three nearest neighbor vectors, $\delta_1 = (0, -e^\gamma, 0)$, $\delta_2 = (1, 1, e^\gamma)$, and $\delta_3 = (-1, 1, -e^\gamma)$. Note that for the honeycomb lattice $g^{latt}(k, i\omega_n)$ and $\Sigma_c(K, i\omega_n)$ are 2 x 2 matrices with sublattice indices. One then needs to prepare the cluster excluded Green’s function $g^0(R, \tau)$ for the CT-INT impurity solver.

In the first step, we coarse grain the lattice Green’s function

$$g^{latt}(k, i\omega_n) = \frac{1}{N_k} \sum_k g^{latt}(K + \hat{k}, i\omega_n)$$

(B2)

with $N_c$ being the number of the cluster size (the number of unit cells in a cluster), and $N_k$ being the number of $K$ points within each $K$ patch.

Then, by using the Dyson equation, the cluster excluded Green’s function in $(K, i\omega_n)$ space can be obtained as

$$g^0(K, i\omega_n) = \left( g^{latt}(K, i\omega_n) + \Sigma_c(K, i\omega_n) \right)^{-1}.$$  

(B3)

Finally, $g^0(K, i\omega_n)$ needs to be transformed to $g^0(R, \tau)$ to provide the input for the impurity solver as

$$g^0(K, i\omega_n) \rightarrow g^0(R, \tau) \rightarrow g^0(R, \tau) \rightarrow g^0(R, \tau)$$

(B4)

For simple lattices, $g^0(K, i\omega_n)$ and $g^0(R, \tau)$ are connected by a Fourier transformation. But for more complex lattices, this requires more steps. We explain steps 1, 2 and 3 in Eq. (B4) below.

In step 1, we perform an infinite Matsubara frequency summation to obtain $n_c = 1000$ in our code.

Then

$$g^0(K, \tau) \in (0, \beta) = -\frac{1}{\beta} \sum_{n = -n_c}^{n_c-1} g^0(K, i\omega_n) e^{-i\omega_n \tau},$$

(B5)

and

$$g^0(K, \tau = 0^+) = -\frac{1}{\beta} \sum_{n = -n_c}^{n_c-1} \left( g^0(K, i\omega_n) - \frac{1}{i\omega_n} \right) e^{-i\omega_n \tau} +$$

$$-\frac{1}{\beta} \sum_{n = -\infty}^{n_c-1} e^{-i\omega_n \tau}$$

$$= -\frac{1}{\beta} \sum_{n = -n_c}^{n_c-1} g^0(K, i\omega_n) + \frac{1}{2},$$

(B6)
and the periodic boundary condition in the time axis gives
\[ g^0(K, \tau = \beta) = 1 - g^0(K, \tau = 0^+), \] (B6)
and
\[ g^0(K, \tau(\in [-\beta, 0])) = -g^0(K, \tau + \beta). \] (B7)

In step 2, which leads from \( g^0_i(K, \tau) \) to \( g^0_{i,\alpha,\beta}(\tau) \), we need to perform a modified Fourier transformation
\[ g^0_{i,\alpha,\beta}(\tau) = \frac{1}{N^6} \sum_K g^0_{\alpha,\beta}(K, \tau) e^{i K \tau} e^{-i K R_i}, \] (B8)
using inner cell coordinates in the phase, i.e., \( R_i = R_i + t_{\alpha(\beta)} \), where \( R_i \) is the unit cell coordinate and \( t_{\alpha(\beta)} \) is the inner cell coordinate: \( t_{\alpha} = (0, 0) \) and \( t_{\beta} = (0, 1/\sqrt{3}) \), for the two sublattices.

In step 3, leading from \( g^0_{i,\alpha,\beta}(\tau) \) to \( g^0_{\alpha,\beta}(R, \tau) \), we make use of translational symmetry and perform a constrained summation
\[ g^0_{\alpha,\beta}(R, \tau) = \frac{1}{N^6} \sum_{R_i - R} g^0_{i,\alpha,\beta}(\tau). \] (B9)

**Appendix C: Two particle Green’s function**

To calculate correlation functions, we need to evaluate the two-particle Green’s functions within the CT-INT, \( \langle G_{\alpha\beta}(P_1, P_1' P_2, P_2') \rangle \), where \( G_{\alpha\beta}(P, P') \) is defined as
\[ G_{\alpha\beta}(P, i \omega_n, P' (K', i \omega_n')) = g^0_{\alpha,\beta}(K, i \omega_n) \delta_{K,K'} \delta_{\omega_n,\omega_n'} \] (C1)

with
\[ \Gamma_{\gamma,\eta}(K, i \omega_n; K', i \omega_n') = -\frac{T}{N^6} \sum_{i,j} e^{-i K R_i} e^{i \omega_n \tau_j} M(k)_{i,\alpha,\beta} e^{-i \omega_n' \tau_j} e^{i K' R_j} \] (C2)

where \( M(k) = D(k)^{-1} \) and \( R_i = R_i + t_{\alpha(\beta)} \), where \( R_i \) is the unit cell coordinate and \( t_{\alpha(\beta)} \) is the inner cell coordinate.

**Appendix D: Details on calculating the pairing susceptibility**

In the DCA formalism, we need to distinguish the cluster pairing susceptibility and the lattice pairing susceptibility. The real physical quantities, the lattice susceptibilities \( \chi \), are obtained with
\[ \chi = \frac{\chi_0}{1 - \Gamma \chi_0} \] (D1)

where \( \chi_0 \) is the coarse-grained non-interacting susceptibility, and \( \Gamma \) the irreducible vertex. Within the DCA approximation, the irreducible vertex \( \Gamma \) in the lattice susceptibility and the \( \Gamma_e \) in the cluster susceptibility are equivalent once they are coarse-grained to the cluster level, so that
\[ (\chi_0)^{-1} - (\chi)^{-1} = \Gamma_e = (\chi_e)^{-1} - (\chi_e)^{-1}. \] (D2)

The cluster pairing susceptibility matrix \( \chi_e(P, P', Q = 0) \) is defined as
\[ \begin{pmatrix}
\langle G_{11}^1(-P_1, -P_1') G_{12}^1(P_1, P_1') \rangle
& \langle G_{12}^1(-P_1, -P_1') G_{11}^1(P_1, P_1') \rangle

\langle G_{21}^1(-P_2, -P_1') G_{22}^1(P_1, P_1') \rangle
& \langle G_{22}^1(-P_2, -P_1') G_{21}^1(P_1, P_1') \rangle
\end{pmatrix} \] (D3)

with
\[ \chi_e(P, P', Q = 0) = \langle G_{11}^1(-P_1, -P_1') G_{12}^1(P_1, P_1') \rangle \langle G_{21}^1(-P_2, -P_1') G_{22}^1(P_1, P_1') \rangle \] (D4)

In the above equations, 1 and 2 denote sublattice indices and \( P = (K, i \omega) \), \( P' = (K', i \omega') \) and \( Q = (Q, i \nu) \) are four-vectors of cluster momentum and Matsubara frequency. Based on \( \chi_e \) and \( \chi_0 \), by using Eq. (D2), we then get the irreducible vertex \( \Gamma_e \), equivalently, \( \Gamma \). Then using Eq. (D1), we obtain the lattice pairing susceptibility \( \chi \). Finally, we get the effective pairing susceptibility from
\[ \chi_{eff}(T) = \frac{1}{\beta_{P,P',Q=0}} \sum_{P,P',Q=0} \langle \Phi_\eta(K) | \chi(P, P', Q = 0) - \chi_0(P, P', Q = 0) | \Phi_\eta(K') \rangle \] (D5)

where \( \Phi_\eta(K) \) and \( \Phi_\eta(K') \) are coarse-grained form factors as shown in the Fig. 1(d) in the main text.

**Appendix E: Relation to the pairing vertex eigenvalue analysis**

An eigenvalue analysis of the pairing vertex secular equation requires us to solve the eigenvalue problem
\[ \Delta \chi_0 | \phi_i \rangle = \lambda_i | \phi_i \rangle. \] (E1)

In terms of the pairing vertex, the effective pairing susceptibility is given as
\[ \chi_{eff} = \chi - \chi_0 = \chi_0 (1 - \Gamma \chi_0)^{-1} - \chi_0. \] (E2)

Inserting the diagonal representation of \( \Delta \chi_0 \) into the above equation, one obtains
\[ \chi_{eff} = \chi_0 \sum_i \frac{\lambda_i}{1 - \lambda_i} | \phi_i \rangle \langle \phi_i |. \] (E3)

The pairing susceptibility is similarly found to be given as
\[ \chi = \chi_0 \sum_i \frac{1}{1 - \lambda_i} | \phi_i \rangle \langle \phi_i |. \] (E4)
When the largest eigenvalue of the pairing vertex secular equation approaches 1, both the pairing susceptibility and the effective pairing susceptibility diverge. Usually, a positive effective pairing susceptibility indicates an enhancement of the pairing correlations due to the interaction vertex \[37\]. Comparing the effective pairing susceptibilities is thus equivalent to comparing the strength of the vertex corrections, and allows one to identify the dominant pairing channel.

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