BUMPY METRICS ON SPHERES AND MINIMAL INDEX GROWTH

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Dedicated to Paul Rabinowitz with best wishes

Abstract. The existence of two geometrically distinct closed geodesics on an $n$-dimensional sphere $S^n$ with a non-reversible and bumpy Finsler metric was shown independently by Duan & Long [7] and the author [25]. We simplify the proof of this statement by the following observation: If for some $N \in \mathbb{N}$ all closed geodesics of index $\leq N$ of a non-reversible and bumpy Finsler metric on $S^n$ are geometrically equivalent to the closed geodesic $c$ then there is a covering $c'$ of minimal index growth, i.e.

$$\text{ind}(c'^m) = m \text{ind}(c') - (m - 1)(n - 1)$$

for all $m \geq 1$ with $\text{ind}(c'^m) \leq N$. But this leads to a contradiction for $N = \infty$ as pointed out by Goresky & Hingston [13]. We also discuss perturbations of Katok metrics on spheres of even dimension carrying only finitely many closed geodesics. For arbitrarily large $L > 0$ we obtain on $S^2$ a metric of positive flag curvature carrying only two closed geodesics of length $< L$ which do not intersect.

1. Introduction

In this note we consider existence results for closed geodesics on spheres $S^n$ of dimension $n$ endowed with a Finsler metric $f$. The Finsler metric $f$ is called reversible, if $f(-X) = f(X)$ holds for all tangent vectors $X$. Otherwise we call the metric non-reversible. Note that two closed geodesics $c_1, c_2 : S^1 \to M$ of a non-reversible Finsler metric are geometrically equivalent if their images $c_1(S^1) = c_2(S^1)$ and their orientations coincide. In the reversible case $c_1, c_2$ are geometrically equivalent if and only if $c_1(S^1) = c_2(S^1)$. Closed geodesics which are not geometrically equivalent are called geometrically distinct. We call a closed geodesic prime if it is not the covering of a shorter closed curve. For $m \geq 1$ and a prime closed geodesic $c$ of a non-reversible Finsler metric the coverings $c^m$ defined by $c^m(t) = c(mt)$ are geometrically equivalent closed geodesics.

A closed geodesic $c : S^1 \to S^n$ is non-degenerate if its nullity vanishes. The nullity is the dimension of the kernel of the index form minus one, it
equals the dimension of periodic Jacobi fields orthogonal to the velocity field \( c' \) of the closed geodesic.

A Finsler metric \( f \) on a compact manifold \( M \) is called \textit{bumpy} if all closed geodesics are non-degenerate. In this case the energy functional \( E : \Lambda S^n \to \mathbb{R} \) defined on the free loop space \( \Lambda S^n \) carrying a canonical \( S^1 \)-action can be viewed as a Morse function whose critical set decomposes into a disjoint union of non-degenerate \( S^1 \)-orbits of closed geodesics. Following ideas by Birkhoff \cite[Sec.17]{Birkhoff}, \cite[p.135-139]{Birkhoff} the existence of a single closed geodesic on a compact and simply-connected manifold with a Finsler metric was shown by Lusternik & Fet \cite{Lusternik}. It turns out that existence results for several closed geodesics strongly depend on the reversibility of the metric. The sequence \( (\text{ind}(c^m))_{m \geq 1} \) of Morse indices of the coverings \( c^m \) of a closed geodesic \( c \) plays an important role in existence proofs for several closed geodesics. It is a result by Fet \cite{Fet} that a bumpy and reversible Finsler metric on a compact and simply-connected manifold carries at least two geometrically distinct closed geodesics. An analogous statement holds also for non-reversible Finsler metrics on spheres as shown independently by Duan-Long in \cite{Duan-Long} and the author in \cite{Raademacher}:

\textbf{Theorem 1.} \cite{Duan-Long}, \cite{Raademacher} A bumpy and non-reversible Finsler metric on a sphere \( S^n \) of dimension \( n \geq 3 \) carries two geometrically distinct closed geodesics.

In \cite{Duan-Long} the classification of the symplectic normal forms of the linearized Poincaré mapping of a closed geodesic and a case distinction is used. In \cite{Raademacher} the \textit{common index jump theorem} due to \cite[Thm.4.3]{Weinstein} is the main ingredient. The statement of Theorem 1 as well as the statement of the following Proposition also hold for compact and simply-connected manifolds \( M \) which are rationally homotopy equivalent to an \( n \)-sphere \( S^n \). In this note we show that one can use the following result about the index growth, i.e. the growth of the sequence \( (\text{ind}(c^m))_{m \geq 1} \) which also gives a short proof of Theorem 1:

\textbf{Proposition 1.} Let \( f \) be a bumpy and non-reversible Finsler metric on \( S^n \) and let \( N_1 \geq 5n \) be a number such that the following assumption holds: There is a prime closed geodesic \( c \) such that all closed geodesics \( d \) whose index satisfies \( \text{ind}(d) \leq N_1 + 2 \) are geometrically equivalent to \( c \). Let \( r = n \) if \( n \) is even and \( r = (n+1)/2 \) if \( n \) is odd. Let \( m_1 := \max \left\{ m \in \mathbb{N} : \text{ind}(c^m) \leq N_1 \right\} \). Then \( m_1 \geq 2 \) and the closed geodesic \( c' \) is of minimal index growth up to level \( m_1 \), i.e.

\[
(1) \quad \text{ind}(c'^m) = m \text{ind}(c') - (m - 1)(n - 1)
\]

holds for all \( m \leq m_1 \). The closed geodesic \( c' \) is of elliptic-parabolic type, i.e. its linearized Poincaré mapping decomposes into rotations.

We prove this Proposition in the next section. In the case \( n = 2, N_1 = \infty \) it was shown by Ziller in \cite[p.149]{Ziller} that Equation (1) holds with \( r = 2 \) and leads to a contradiction.

The closed geodesic \( c' \) satisfying Equality (1) is called of \textit{minimal index growth} up to level \( m_1 \). This is motivated by the following Inequality (2). On
the other hand the following Inequality (3) shows that Equation (1) can hold only for finitely many $m$ which shows that Proposition 1 implies Theorem 1 for $N_1 = \infty$.

**Proposition 2.** (cf. Goresky & Hingston [13, Prop.6.1]) Let $c$ be a closed geodesic of a bumpy Finsler metric. Then for all $m \geq 1$:

$$(2) \quad \text{ind}(c^{m}) \geq m \text{ind}(c') - (m - 1)(n - 1).$$

And there is an $N \in \mathbb{N}$ such that for all $m > N$:

$$(3) \quad \text{ind}(c^{m}) \geq m \text{ind}(c') - (m - 1)(n - 1) + 1.$$  

This Proposition is a particular case of [13, Prop.6.1]. It is a direct consequence of Bott’s formula for $\text{ind}(c^m)$, cf. [6, Thm.A]. The statement of the Proposition can also be shown without the assumption that the metric is bumpy.

It is a remarkable result by Bangert & Long [3] that for any non-reversible Finsler metric on $S^2$ there are two closed geodesics. There are a number of results for metrics in higher dimensions with and without non-degeneracy assumptions or with curvature assumptions, cf. for example [8], [9], [10], [11], [17] and [24].

There are non-reversible bumpy Finsler metrics of constant flag curvature depending on an irrational parameter on $S^{2n-1}$ resp. $S^{2n}$ with exactly $2n$ geometrically distinct closed geodesics. On the 2-sphere for any $N_1 \in \mathbb{N}$ there is a Katok metric satisfying the assumptions of Proposition 1. These metrics occurred in a paper by Katok [18], their geometry is investigated by Ziller in [26]. By construction these metrics are invariant under a rotation, the closed geodesics $c_1, c_1^{-1}, \ldots, c_n, c_n^{-1}$ occur in pairs differing by orientation and length. They are fixed point sets of reflections at a two-dimensional plane. In Section 3 we show that for a given arbitrarily large $L$ one can perturb these metrics on $S^{2n}$ to obtain a non-reversible Finsler metric with positive flag curvature carrying exactly $2n$ geometrically distinct closed geodesics $d_1, d_2, \ldots, d_{2n}$ with length $< L$, which pairwise do not intersect. Hence for these closed geodesics the closed curves $d_j^{-1}, j = 1, 2, \ldots, 2n$ with opposite orientation are not geodesics. These metrics are also invariant under a rotation and are obtained by breaking the symmetry with respect to the reflections at two-dimensional planes.

In dimension 2 this produces for $k \geq 4$ in any $C^k$-neighborhood of the Katok metric examples of non-reversible Finsler metrics on $S^2$ of positive flag curvature with two simple closed geodesics, which do not intersect.

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2. **Proof of Proposition 1**

We denote by \( f : TS^n \to \mathbb{R} \) the Finsler metric defined on the tangent bundle \( TS^n \) of the \( n \)-sphere. We consider the free loop space \( \Lambda S^n \) of \( H^1 \)-maps \( \sigma : S^1 = [0, 1]/\{0, 1\} \to S^n \) on the sphere \( S^n \).

Closed geodesics are the critical points of the energy functional
\[
E : \Lambda S^n \to \mathbb{R} ; \quad E(\sigma) = \frac{1}{2} \int_0^1 f^2(\sigma'(t)) \, dt.
\]

There is a canonical \( S^1 \)-action \( (z, \sigma) \in S^1 \times \Lambda S^n \to z.\sigma \in \Lambda S^n \) defined by \( z.\sigma(t) = c(t+z) \) on the free loop space leaving the energy functional invariant.

The index form \( H_c \) of the closed geodesic can be identified with the hessian \( d^2E(c) \) of the energy functional at \( c \) by the second variational formula. If the metric is bumpy one can view the energy functional as a Morse function on the quotient space \( \Lambda S^n/S^1 \), for details see for example [22, Sec.2].

We assume that there is a prime closed geodesic \( c \) such that any other closed geodesic \( d \) with \( \text{ind}(d) \leq N_1 + 2 \) satisfies \( d = z.c^m \) for some \( z \in S^1 \) and \( m \geq 1 \).

Let \( \Lambda^0 S^n = \{ \sigma \in \Lambda S^n ; E(\sigma) = 0 \} \) be the set of point curves, which one can identify with the sphere \( S^n \). We denote by \( b_k := b_k (\Lambda S^n/S^1, \Lambda^0 S^n/S^1; \mathbb{Q}) \) the Betti numbers of the quotient space pair \( (\Lambda S^n/S^1, \Lambda^0 S^n/S^1) \) with rational coefficients.

Let \( \Lambda(c) := \{ \sigma \in \Lambda S^n ; E(\sigma) < E(c) \} \). Then the Betti numbers of the local homology produced by the covering \( c^m \) are given by:
\[
b_j(c^m) = \dim H_j ( (\Lambda(c^m) \cup S^1.c^m) / S^1, \Lambda(c^m)/S^1; \mathbb{Q})
\]
\[
= \begin{cases} 
1 & j = \text{ind}(c^m) \text{ and } m \equiv 1 \text{ or } \text{ind}(c^2) \equiv \text{ind}(c) \pmod{2} \\
0 & \text{otherwise}
\end{cases}
\]

For \( k \leq N_1 + 2 \):
\[
w_k = \#\{ m \in \mathbb{N} ; \text{ind}(c^m) = k, m \equiv 1 \text{ or } \text{ind}(c^2) \equiv \text{ind}(c) \pmod{2} \}
\]
gives the number of critical points of index \( k \) of the Morse function \( E : \Lambda S^n/S^1 \to \mathbb{R} \) producing non-trivial local homology.

Bott’s formula for the indices \( \text{ind}(c^m) \) of iterates \( c^m \) implies the following statements about the parity of \( \text{ind}(c^m), m \geq 1 \):
\[
\text{ind}(c^{2m+1}) \equiv \text{ind}(c) ; \quad \text{ind}(c^{2m}) \equiv \text{ind}(c^2) \pmod{2},
\]
cf. [22] Sec.1. This observation implies that \( w_k = 0 \) for all numbers \( k \equiv \text{ind}(c) + 1 \pmod{2} \) for all \( k \leq N_1 + 2 \). Hence the equality case holds in the Morse inequalities:
\[
w_k = b_k, k \leq N_1,
\]
cf. [22] (2.3)]. Since Bott’s formula also implies \( \text{ind}(c^m) \geq \text{ind}(c) ; m \geq 1 \) we conclude
\[
\text{ind}(c) = n - 1.
\]
Here we use that $b_0 = b_1 = \ldots = b_{n-2} = 0, b_{n-1} = 1$, cf. [22, Thm.2.4] resp. [15, p.104]. Equation (7) implies that the sequence $(\text{ind}(c^m))_{m \geq 1}$ is monotone increasing, i.e.

\begin{equation}
\text{ind}(c^{m+1}) \geq \text{ind}(c^m), \ m \geq 1,
\end{equation}

cf. the *successive index estimates* by Long & Zhu, cf. [20, 19, Sec. 10.2]. If \( \text{ind}(c^2) \equiv \text{ind}(c) + 1 \pmod{2} \) it follows from Inequality (8) and Equality (5) that

\begin{equation}
\text{ind}(c^{m+1}) \geq \text{ind}(c^m) + 1
\end{equation}

for all \( m \geq 1 \).

Now we discuss four cases depending on the parity of the dimension \( n \) and the parity of \( \text{ind}(c^2) - \text{ind}(c) \).

**Case 1:** Assume \( n \) even, then

\begin{equation}
b_k = \begin{cases} 
2 & k = (2j + 1)(n - 1), j \geq 1 \\
1 & k \geq n - 1, k \text{ odd}, k \neq (2j + 1)(n - 1), j \geq 1 \\
0 & \text{otherwise}
\end{cases}
\end{equation}

cf. [22, Thm.2.4] resp. [15, p.104].

**Case 1.1:** Assume in addition that \( \text{ind}(c^2) \equiv \text{ind}(c) \pmod{2} \), i.e. \( \text{ind}(c^m) \equiv n - 1 \pmod{2} \) for all \( m \geq 1 \). Then Equations (6), (7) and (8) imply for \( k \leq N_1 \):

\begin{equation}
w_k = \# \{m \in \mathbb{N} ; \text{ind}(c^m) = k \} = b_k,
\end{equation}

and

\begin{equation}
(\text{ind}(c^m))_{m \leq m_1} = (n - 1, n + 1, \ldots, 3n - 5, 3n - 3, 3n - 1, \ldots, 5n - 7, 5n - 5, 5n - 3, 3n - 3, \ldots). 
\end{equation}

In particular we obtain for all \( m \leq m_1 \):

\begin{equation}
\text{ind}(c^{nm}) = (2m + 1)(n - 1).
\end{equation}

Since \( \text{ind}(c^{2n}) = 5n - 5 < N_1 \) we obtain \( m_1 \geq 2 \). And Equation (11) implies

\[ \text{ind}(c^{nm}) - m \text{ ind}(c^n) = -(m-1)(n-1) \]

for all \( m \leq m_1 \), hence Equation (11) holds for \( r = n \).

**Case 1.2:** Assume in addition that \( \text{ind}(c^2) \equiv \text{ind}(c) + 1 \pmod{2} \), i.e. \( \text{ind}(c^m) \equiv m \pmod{2} \). Then Equations (6), (7) and (8) imply for \( k \leq N_1 \):

\begin{equation}
w_k = \# \{l \in \mathbb{N} ; \text{ind}(c^{2l-1}) = k \} = b_k,
\end{equation}

and \( \text{ind}(c^{2m-1}) = \text{ind}(c^{2m+1}) = 3n - 3 \) contradicting Inequality (9). Therefore this case cannot occur.
Therefore it follows from \cite[Lemma 3.1(ii)]{2} that $c$ and \text{ind}(c) \equiv 0 \pmod{2}$.

In particular we obtain for all $m \leq m_1$,
\begin{equation}
\text{ind}(e^m)_{m \leq m_1} = (n - 1, n + 1, \ldots, 2n - 4, 2n - 2, 2n - 2, 2n, \ldots, 3n - 5, 3n - 3, 3n - 3, 3n - 1, \ldots).
\end{equation}

Case 2.1: Assume in addition that $\text{ind}(e^2) \equiv \text{ind}(c) \pmod{2}$, i.e. $\text{ind}(e^m) \equiv 0 \pmod{2}$ for all $m \geq 1$. Then Equations (6), (7) and (8) imply for $k \leq N_1$:
\begin{equation}
w_k = \#\{l \in \mathbb{N} ; \text{ind}(e^l) = k\} = b_k,
\end{equation}
i.e. $\text{ind}(e^m)_{m \leq m_1} = (n - 1, n + 1, \ldots, 2n - 4, 2n - 2, 2n - 2, 2n, \ldots, 3n - 5, 3n - 3, 3n - 3, 3n - 1, \ldots)$.

Case 2.2: Assume in addition that $\text{ind}(e^2) \equiv n$ i.e. $\text{ind}(e^m) \equiv m \pmod{2}$. Then Equations (6), (7) and (8) imply for $k \leq N_1$:
\begin{equation}
w_k = \#\{l \in \mathbb{N} ; \text{ind}(e^{2l - 1}) = k\} = b_k,
\end{equation}
and $\text{ind}(e^n) = \text{ind}(e^{n + 2}) = 2n - 2$ contradicting Inequality (9). Therefore this case cannot occur.

In Case 1.1 and Case 2.1 $m_1 \geq 2$ and $\text{ind}(e^{2r}) - 2\text{ind}(c^r) = -(n - 1)$. Therefore it follows from \cite[Lemma 3.1(ii)]{2} that $c$ is of elliptic-parabolic type. And the linearized Poincaré mapping decomposes into $(n - 1) (2 \times 2)$ blocks conjugate to rotations.

3. Example

For $\epsilon > 0, \eta > 0$ with $\eta + \epsilon < 1/2$ let $h = h_{\epsilon, \eta} : \mathbb{R} \rightarrow [0, 1]$ be a smooth function satisfying
\begin{equation}
h_{\epsilon, \eta}(t) = \begin{cases} 
1 & t \in [-\eta, \eta] \\
0 & |t| \geq \eta + \epsilon/4
\end{cases}
\end{equation}
Let $H_{\epsilon} : \mathbb{R} \rightarrow \mathbb{R}$ be the smooth function with $H_{\epsilon}(t) = \epsilon h_{\epsilon, \epsilon/4}(t)$. And let $a = a_{\epsilon, \eta} : \mathbb{R} \rightarrow [-1, 1]$ be the smooth function with $a_{\epsilon, \eta}(t + 2) = -a_{\epsilon, \eta}(t)$ and $a_{\epsilon, \eta}(1 + t) = h_{\epsilon, \eta}(t)$ for all $-1 \leq t \leq 1$.

Let $p_1, p_2, \ldots, p_n$ be numbers which are relatively prime and let $p = p_1 p_2 \cdots p_n$. Let $e_0, e_1, e_2, \ldots, e_{2n}$ be an oriented orthonormal basis of $\mathbb{R}^{2n+1}$. 
For the standard Riemannian metric on $S^{2n} \subset \mathbb{R}^{2n+1}$ an isometric $S^1$-action $\phi : S^{2n} \times S^1 \rightarrow S^{2n}$ is defined by $\phi(e_0, t) = e_0$ and

\begin{equation}
\phi(e_{2j-1}, t) = \cos(2\pi pt/p_j)e_{2j-1} + \sin(2\pi pt/p_j)e_{2j},
\end{equation}

\begin{equation}
\phi(e_{2j}, t) = -\sin(2\pi pt/p_j)e_{2j-1} + \cos(2\pi pt/p_j)e_{2j}.
\end{equation}

with $j = 1, 2, \ldots, n$. We assume that there is a non-reversible Finsler metric $f$ invariant under $\phi$ for which there are only finitely many geometrically distinct closed geodesics $c_1, c_1^{-1}, c_2, c_2^{-1}, \ldots, c_n, c_n^{-1}$ which occur in pairs differing only by orientation.

The geodesic $c_j, j = 1, 2, \ldots, n$ is the intersection of the 2-plane generated by $e_{2j-1}, e_{2j}$ invariant under $\phi$, i.e. $c_j(t) = \cos(2\pi t)e_{2j-1} + \sin(2\pi t)e_{2j}$. The closed geodesics $c_j, j = 1, 2, \ldots, n$ do not intersect pairwise.

This assumptions are satisfied by the Katok metrics $N_\alpha$ on $S^{2n}$ with irrational $\alpha \in (0, 1)$ discussed in detail by Ziller [26, Sec.1]. Here the lengths $L(c_j), L(c_j^{-1})$ of the closed geodesics $c_j, c_j^{-1}$ depend on the parameter $\alpha$.

**Proposition 3.** Let $f$ be a bumpy Katok metric on $S^{2n}$ invariant under the $S^1$-action $\phi$ by Equation (14) with only 2$n$ geometrically distinct closed geodesics $c_1, c_1^{-1}, \ldots, c_n, c_n^{-1}$ and let $L_1 := \max\{L(c_j), L(c_j^{-1})\}, j = 1, 2, \ldots, n$. The $n$ closed geodesics $c_1, \ldots, c_n$ are invariant under $\phi$ and do not intersect pairwise.

Fix $k \geq 4$. For any $L > L_1$ there is a non-reversible Finsler metric $F$ invariant under $\phi$ of positive flag curvature and $C^k$-arbitrarily close to $f$ with only $2n$ geometrically distinct closed geodesics $d_1, d_2, \ldots, d_{2n}$ of length $< L$. These $2n$ closed geodesics are invariant under $\phi$ and do not intersect pairwise.

Fix the closed geodesic $c = c_1$ and $m = m_1 = p/p_1$. Then there is a $S^1$-invariant tubular neighborhood $U(c)$ of the closed geodesic $c_1$ with coordinates

$$
(x, t) \in D^{2n-1}_\epsilon \times_{\mathbb{Z}_m} S^1 \hookrightarrow \phi \left( \cos \left( \|x\| \right) e_1 + \sin \left( \|x\| \right) \frac{x}{\|x\|}, t \right) \in U(c)
$$

Here $D^{2n-1}_\epsilon = \{x \in \mathbb{R}^{2n-1}; \|x\| < \epsilon\}$ is a disc of radius $\epsilon$ on the Euclidean space generated by $e_0, e_3, e_4, \ldots, e_{2n}$ which we identify via the exponential map of the standard Riemannian metric on $S^{2n}$ with an open neighborhood of $c(0) = e_1$ on the totally geodesic hypersphere through $c(0)$ orthogonal to the $S^1$-action.

The $S^1$-action induces an isometric $\mathbb{Z}_m$-action on $D^{2n-1}_\epsilon$, and therefore a diagonal action on the product $D^{2n-1}_\epsilon \times S^1$ with quotient space $D^{2n-1}_\epsilon \times_{\mathbb{Z}_m} S^1$. Then there are induced coordinates $(x, t, \xi, \tau) \in \left( D^{2n-1}_\epsilon \times_{\mathbb{Z}_m} S^1 \right) \times \mathbb{R}^{2n-1} \times \mathbb{R}$ on the tangent bundle $TU(c)$ restricted to the tubular neighborhood $U(c)$ of $c$. Because of the $S^1$-symmetry the Finsler metric $f$ in the coordinates $(x, t, \xi, \tau)$ does not depend on $t$, i.e. $f(x, t, \xi, \tau) = f(x, \xi, \tau)$.

Choose $\eta \in (0, 1/4)$ and a sufficiently small $\epsilon > 0$. For a sufficiently small $s > 0$ the mapping $\Psi = \Psi_{s, \epsilon, \eta} : TU(c) - T^0U(c) \rightarrow TU(c) - T^0U(c)$ of the
closed geodesics of Sec. 4]. And we obtain that the flag curvature, degree 1
\[ T_U(x, t, \xi, \tau) = \left( x + s H_e \left( \|x\| \right) a_{\eta, \epsilon} \left( \frac{\tau}{\sqrt{\tau^2 + \|\xi\|^2}} \right) \right) e_0, t, \xi, \tau \]
is a diffeomorphism. For a fixed \( k \geq 4 \) one can choose \( s > 0 \) sufficiently small such that the diffeomorphism \( \Psi_{s, \epsilon, \eta} \) is arbitrarily close to the identity map in the \( C^k \)-norm. The mapping is well-defined since \( e_0 \) is fixed under \( \phi \).

The diffeomorphism \( \Psi \) extends by the identity onto the complement of \( TU(c) \) on \( TS^{2n} \). And the diffeomorphism \( \Psi \) is positively homogeneous of degree 1, i.e. \( \Psi(x, t, \lambda \xi, \lambda \tau) = \lambda \Psi(x, t, \xi, \tau) \) for \( \lambda > 0 \).

Then define the Finsler metric
\[ \mathcal{T}(x, t, \xi, \tau) = \mathcal{T}_{s, \epsilon, \eta}(x, t, \xi, \tau) = f \left( \Psi_{s, \epsilon, \eta}^{-1}(x, t, \xi, \tau) \right). \]
Let \( V_\pm = \left\{ (x, t, \xi, \tau); \|x\| < \epsilon/4, \pm \tau/\sqrt{\tau^2 + \|\xi\|^2} > 1 - \eta \right\} \) and
\[ V_1 = \left\{ (x, t, \xi, \tau); |\tau|/\sqrt{\tau^2 + \|\xi\|^2} \leq 1 - \eta - \epsilon/4 \right\}. \]
Then
\[ (15) \quad \mathcal{T}_{s, \epsilon, \eta}(x, t, \xi, \tau) = \begin{cases} f(x, t, \xi, \tau) & ; (x, t, \xi, \tau) \in V_1 \\ f(x \mp \text{see} e_0, t, \xi, \tau) & ; (x, t, \xi, \tau) \in V_\pm \end{cases}. \]
Therefore we obtain that \( d_1 : t \mapsto (\text{see} e_0, t) \) and \( d_2 : t \mapsto (-\text{see} e_0, -t) \) are two closed geodesics of \( \mathcal{T}_{s, \epsilon, \eta} \) in \( U(c) \) which do not intersect.

The flag curvature of the Katok metric is constant with value 1, cf. [23 Sec. 4]. And we obtain that the flag curvature \( \mathcal{K}(y, Y, \sigma) \) of a flag \( (y, Y, \sigma) \) with \( y \in S^{2n}, Y \in T_y S^{2n} \) and \( \sigma \) a two-dimensional subspace of \( T_y S^{2n} \) containing \( y \) with respect to the perturbed Finsler metric \( \mathcal{T} = \mathcal{T}_{s, \epsilon, \eta} \) can be different from 1 only if \( y \in U(c) \) and if the coordinates \( (x, t, \xi, \tau) \) of the tangent vector \( Y \) satisfy: \( (x, t, \xi, \tau) \notin V_1 \cup V_\pm \cup V_- \). The Finsler metric \( \mathcal{T}_{s, \epsilon, \eta} \) is invariant under \( \phi \). But the symmetry with respect to the reflection at the \((e_1, e_2)\)-plane is broken.

We can use this perturbation inductively in tubular neighborhoods for all closed geodesics \( c_2, \ldots, c_n \) and obtain a non-reversible Finsler metric \( F = F_{s, \epsilon, \eta} \) on \( S^{2n} \) invariant under \( \phi \) with closed geodesics \( d_3, d_4, \ldots, d_{2n} \) also invariant under \( \phi \). Any two of these closed geodesics do not intersect.

Now fix \( L > L_1 \). Since \( f \) is bumpy there is a sufficiently small \( s > 0 \) such that all closed geodesics of the Finsler metric \( F_{s, \epsilon, \eta} \) of length \( < L \) are geometrically equivalent to \( d_1, d_2, \ldots, d_{2n} \), for this argument compare for example [11 p.12, 13]. Here we use the strong \( C^k \)-topology with \( k \geq 4 \) (instead of \( k \geq 2 \) in the Riemannian case) since in contrast to the Riemannian case the geodesic coefficients depend on fourth derivatives of the Finsler metric.

Since by Equation \( (15) \) the Finsler metrics \( f \) and \( F = F_{s, \epsilon, \eta} \) are isometric in open neighborhoods of the velocity fields \( c'_j \) and \( d_{2j-1}' \) resp. \( -c'_j \) and \( d_{2j}' \) for \( j = 1, 2, \ldots, n \) the flag curvature is constant in these neighborhoods and the lengths \( L(c_j) = L(d_{2j-1}), L(c_j^{-1}) = L(d_{2j}), j = 1, 2, \ldots, n \) coincide.
Therefore the picture from Morse theory of $f$ produced by $c_1, c_1^{-1}, \ldots, c_n, c_n^{-1}$ and $F$ produced by $d_1, d_2, \ldots, c_{2n}$ up to length $L$ coincide.

For $n = 2$ it follows from [14, Sec.6] that also the perturbed metric $F$ on $S^2$ is dynamically convex. Therefore there are either infinitely many geometrically distinct closed geodesics or there are only two geometrically distinct closed geodesics.

For $n = 2$ we obtain non-reversible and rotationally invariant Finsler metrics in any neighborhood of the Katok metric with two closed geodesics $d_1, d_2$ also invariant under the $S^1$-action which do not intersect. For existence results for closed geodesics invariant under isometries cf. [23, Prop.2, Prop.3].

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