NON-INVARIANCE OF THE BRAUER-MANIN OBSTRUCTION FOR SURFACES

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Abstract. In this paper, we study the properties of weak approximation with Brauer-Manin obstruction and the Hasse principle with Brauer-Manin obstruction for surfaces with respect to field extensions of number fields. We assume a conjecture of M. Stoll. For any nontrivial extension of number fields \( L/K \), we construct two kinds of smooth, projective, and geometrically connected surfaces defined over \( K \). For the surface of the first kind, it has a \( K \)-rational point, and satisfies weak approximation with Brauer-Manin obstruction off \( \infty_K \), while its base change by \( L \) does not so off \( \infty_L \). For the surface of the second kind, it is a counterexample to the Hasse principle explained by the Brauer-Manin obstruction, while the failure of the Hasse principle of its base change by \( L \) cannot be so. We illustrate these constructions with explicit unconditional examples.

1. Introduction

1.1. Background. For a proper algebraic variety \( X \) over a number field \( K \), if its \( K \)-rational points set \( X(K) \neq \emptyset \), then its adelic points set \( X(\mathbb{A}_K) \neq \emptyset \). The converse, as has been known, does not always hold. We say that \( X \) is a counterexample to the Hasse principle if the set \( X(\mathbb{A}_K) \neq \emptyset \) whereas the set \( X(K) = \emptyset \). Let \( S \subset \Omega_K \) be a finite subset. By the diagonal embedding, we always view \( X(K) \) as a subset of \( X(\mathbb{A}_K) \) (respectively of \( X(\mathbb{A}_K^K) \)). We say that \( X \) satisfies weak approximation (respectively weak approximation off \( S \)) if \( X(K) \) is dense in \( X(\mathbb{A}_K^K) \) (respectively in \( X(\mathbb{A}_K^K)^K \)). cf. [Sko01, Chapter 5.1]. Manin [Man71] used the Brauer group of \( X \) to define a closed subset \( X(\mathbb{A}_K^K) \subset X(\mathbb{A}_K^K) \), and showed that this closed subset can explain some failures of the Hasse principle and nondensity of \( X(K) \) in \( X(\mathbb{A}_K^K) \). The global reciprocity law gives an inclusion: \( X(K) \subset X(\mathbb{A}_K^K) \). We say that the failure of the Hasse principle of \( X \) is explained by the Brauer-Manin obstruction if the set \( X(\mathbb{A}_K^K) \neq \emptyset \) and the set \( X(\mathbb{A}_K^K)^{\text{Br}} = \emptyset \). We say that \( X \) satisfies weak approximation with Brauer-Manin obstruction (respectively with Brauer-Manin obstruction off \( S \)) if \( X(K) \) is dense in \( X(\mathbb{A}_K^K)^{\text{Br}} \) (respectively in \( pr^S X(\mathbb{A}_K^K)^{\text{Br}} \)). For a smooth, projective, and geometrically connected curve \( C \) defined over an number field \( K \), assume that the Tate-Shafarevich group and the rational points set of its Jacobian are both finite. By the dual sequence of Cassels-Tate, Skorobogatov [Sko01, Chapter 6.2] and Scharaschkin [Sch99] independently observed that \( C(K) = pr^\infty_K (C(\mathbb{A}_K^K)_{\text{Br}}) \). In particular, if this curve \( C \) is a counterexample to the Hasse principle, then this failure can be explained by the Brauer-Manin obstruction. Stoll [Sto07] generalized this observation, and made a conjecture that for any smooth, projective, and geometrically connected curve, it satisfies weak approximation with Brauer-Manin obstruction off \( \infty_K \): see Conjecture 3.0.1 for more details.

1.2. Questions. Let \( L/K \) be a nontrivial extension of number fields. Let \( S \subset \Omega_K \) be a finite subset, and let \( S_L \subset \Omega_L \) be the subset of all places above \( S \). Given a smooth, projective, and geometrically connected variety \( X \) over \( K \), let \( X_L = X \times_{\text{Spec} K} \text{Spec} L \) be its base change by \( L \). In this paper, we consider the following questions.

Question 1.2.1. If the variety \( X \) has a \( K \)-rational point, and satisfies weak approximation with Brauer-Manin obstruction off \( S \), must \( X_L \) also satisfy weak approximation with Brauer-Manin obstruction off \( S_L \)?

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Question 1.2.2. Assume that the varieties $X$ and $X_L$ are counterexamples to the Hasse principle. If the failure of the Hasse principle of $X$ is explained by the Brauer-Manin obstruction, must the failure of the Hasse principle of $X_L$ also be explained by the Brauer-Manin obstruction?

1.3. Main results. In this paper, we will construct smooth, projective, and geometrically connected surfaces to give negative answers to Questions 1.2.

1.3.1. A negative answer to Question 1.2.1. For any nontrivial extension of number fields $L/K$, assuming the Stoll’s conjecture, Liang [Lia18] found a quadratic extension $L$, and constructed a 3-fold to give a negative answer to Question 1.2.1. When $L = \mathbb{Q}(\sqrt{5})$ and $K = \mathbb{Q}$, using the construction method, he gave an unconditional example with explicit equations in loc. cit. The author [ Wu21] generalized his argument to any nontrivial extension of number fields. The varieties constructed there, are 3-folds. In this paper, we will generalize the results to smooth, projective, and geometrically connected surfaces.

For any nontrivial extension of number fields $L/K$, assuming the Stoll’s conjecture, we have the following theorem to give a negative answer to Question 1.2.1.

**Theorem 1.3.1.1 (Theorem 1.1.1).** For any nontrivial extension of number fields $L/K$, assuming the Stoll’s conjecture, there exists a smooth, projective, and geometrically connected surface $X$ defined over $K$ such that

- the surface $X$ has a $K$-rational point, and satisfies weak approximation with Brauer-Manin obstruction off $\infty_K$,
- the surface $X_L$ does not satisfy weak approximation with Brauer-Manin obstruction off $T$ for any finite subset $T \subset \Omega_L$.

When $K = \mathbb{Q}$ and $L = \mathbb{Q}(i)$, using the construction method given in Theorem 1.3.5, we give an explicit unconditional example in Subsection 5.2. The smooth, projective, and geometrically connected surface $X$ is defined by the following equations:

$$
\begin{align*}
(w_0w_2 + w_1^2 + 16w_2^2)(x_0^2 + x_1^2 - x_2^2) + (w_0w_1 + w_1w_2)(x_0^2 - x_1^2) &= 0 \\
w_1^2w_2 &= w_0^3 - 16w_2^3
\end{align*}
$$

in $\mathbb{P}^2 \times \mathbb{P}^2$ with bi-homogeneous coordinates $(w_0 : w_1 : w_2) \times (x_0 : x_1 : x_2)$.

1.3.2. A negative answer to Question 1.2.2. For any number field $K$, suppose that the Stoll’s conjecture holds. Assuming some conditions on the nontrivial extension $L$ over $K$, the author [ Wu21] constructed a 3-fold to give a negative answer to Question 1.2.1. Unconditional examples with explicit equations were given in loc. cit. The varieties constructed there, are 3-folds. In this paper, we will generalize the argument to smooth, projective, and geometrically connected surfaces.

For any nontrivial extension of number fields $L/K$, assuming the Stoll’s conjecture, we have the following theorem to give a negative answer to Question 1.2.2.

**Theorem 1.3.2.1 (Theorem 1.2.3).** For any nontrivial extension of number fields $L/K$, assuming the Stoll’s conjecture, there exists a smooth, projective, and geometrically connected surface $X$ defined over $K$ such that

- the surface $X$ is a counterexample to the Hasse principle, and its failure of the Hasse principle is explained by the Brauer-Manin obstruction,
- the surface $X_L$ is a counterexample to the Hasse principle, but its failure of the Hasse principle cannot be explained by the Brauer-Manin obstruction.

When $K = \mathbb{Q}$ and $L = \mathbb{Q}(i)$, using the construction method given in Theorem 1.2.8, we give an explicit unconditional example in Subsection 5.3. The smooth, projective, and geometrically connected surface $X$ is defined by the following two equations:

$$
\begin{align*}
(w_0w_2 + w_1^2 + 16w_2^2)(x_0^2 - 41x_1^2)(x_0^2 - 3x_1^2)(x_0^2 - 123x_1^2)(y_0^2 - 13y_1^2)(y_0^2 - 41y_1^2) \\
+ (w_0w_1 + w_1w_2)(x_0^2 - 17x_1^2)(x_0^2 - 13x_1^2)(x_0^2 - 221x_1^2)(y_0^2 - 53y_1^2)(y_0^2 - 53y_1^2) &= 0 \\
w_1^2w_2 &= w_0^3 - 16w_2^3
\end{align*}
$$

where $X_L$ is defined by the following equations:
in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$ with tri-homogeneous coordinates $(w_0 : w_1 : w_2) \times (x_0 : x_1) \times (y_0 : y_1)$.

1.3.3. Main ideas behind our constructions in the proof of theorems. Let $L/K$ be a nontrivial extension of number fields. We find a smooth, projective, and geometrically connected curve $C$ such that $C(K)$ and $C(L)$ are both finite, nonempty, and that $C(K) \neq C(L)$. Then we construct a pencil of curves parametrized by the curve $C$: $\beta : X \to C$ such that the fiber of each $C(K)$ point is isomorphic to one given curve $C_{\infty}$, and that the fiber of each $C(L) \setminus C(K)$ point is isomorphic to another given curve $C_0$. By combining some fibration arguments with the functoriality of Brauer-Manin pairing, the arithmetic properties of $C_{\infty}$ and $C_0$ will determine the arithmetic properties of $X$. We carefully choose the curves $C_{\infty}$ and $C_0$ to meet the needs for theorems.

2. Notation and preliminaries

Let $K$ be a number field, and let $\mathcal{O}_K$ be the ring of its integers. Let $\Omega_K$ be the set of all nontrivial places of $K$. Let $\infty_K \subset \Omega_K$ be the subset of all archimedean places, and let $\Omega_K^f = \Omega_K \setminus \infty_K$. Let $\infty_K^r \subset \infty_K$ be the subset of all real places, and let $2_K \subset \Omega_K$ be the subset of all $2$-adic places. For $v \in \Omega_K$, let $K_v$ be the completion of $K$ at $v$. For $v \in \infty_K$, let $\tau_v : K \hookrightarrow K_v$ be the embedding of $K$ into its completion. Given a finite subset $S \subset \Omega_K$, let $\mathbb{A}_K$ (respectively $\mathbb{A}_K^S$) be the ring of adeles (adeles without $S$ components) of $K$. We say an element is a prime element, if the ideal generated by this element is a prime ideal. For a prime element $p \in \mathcal{O}_K$, we denote its associated place by $v_p$. We always assume that a field $L$ is a finite extension of $K$. Let $S_L \subset \Omega_L$ be the subset of all places above $S$.

In this paper, a $K$-scheme will mean a reduced, separated scheme of finite type over $K$, and all geometric objects are $K$-schemes. A $K$-curve will mean a proper $K$-scheme such that every irreducible components are of dimension one. In particular, a $K$-curve may have more than one irreducible component, and may have singular points. We say that a $K$-scheme is a $K$-variety if it is geometrically integral. Be cautious that a integral $K$-scheme may be not a variety, i.e. it may have multiple geometrically irreducible components. Given a proper $K$-scheme $X$, if $X(\mathbb{A}_K) \neq \emptyset$, let $pr^S : X(\mathbb{A}_K) \to X(\mathbb{A}_K^S)$ be the projection induced by the natural projection $pr^S : \mathbb{A}_K \to \mathbb{A}_K^S$.

By combining the Čebotarev’s density theorem with global class field theory, we have the following lemma to choose prime elements. This lemma is a generalization of Dirichlet’s theorem on arithmetic progressions.

Lemma 2.0.1. Given an extension of number fields $L/K$, let $\mathfrak{I} \subset \mathcal{O}_K$ be a proper nonzero ideal. Let $x \in \mathcal{O}_K$. Suppose that the image of $x$ in $\mathcal{O}_K/\mathfrak{I}$ is invertible. Then there exists a prime element $p \in \mathcal{O}_K$ such that

1. $p \equiv x \mod \mathfrak{I}$,
2. $\tau_v(p) > 0$ for all $v \in \infty_K^r$,
3. additionally, if $x = 1$, then $p$ splits completely in $L$.

And the set of places associated to such prime elements has positive density.

Proof. Let $m_\infty$ be the product of all places in $\infty_K^r$, and let $m = 3m_\infty$ be a modulus of $K$. Let $K_m$ be the ray class field of modulus $m$. Let $I_m$ be the group of fractional ideals that are prime to $\mathfrak{I}$. Let $P_m \subset I_m$ be the subgroup of principal ideals generated by some $a \in K^\times$ with $a \equiv 1 \mod \mathfrak{I}$ and $\tau_v(a) > 0$ for all $v \in \infty_K^r$. Then by Artin reciprocity law (cf. [Neu99] Theorem 7.1 and Corollary 7.2), the classical Artin homomorphism $\theta$ gives an exact sequence:

$$0 \to P_m \to I_m \xrightarrow{\theta} \text{Gal}(K_m/K) \to 0.$$ 

Let $(x)$ be the ideal generated by $x$. For the image of $x$ in $\mathcal{O}_K/\mathfrak{I}$ is invertible, we have $(x) \in I_m$. Let $(\theta((x))) \subset \text{Gal}(K_m/K)$ be the subgroup generated by $\theta((x))$. By the Čebotarev’s density theorem (cf. [Neu99] Theorem 13.4), the set of places associated to the prime elements satisfying conditions 1 and 2, has density $\xi(\theta((x)))/[K_m : K]$. Let $M$ be a smallest Galois extension of $K$ containing $L$, then a place of $K$ splits completely in $L$ if
and only if it splits completely in $M$. Let $MK_m$ be a composition field of $M$ and $K_m$. If $x = 1$, then by the Čebotarev’s density theorem, the set of places associated to the prime elements satisfying all these conditions $[1]$, $[2]$ and $[3]$, has density $1/[MK_m : K]$. □

2.1. Hilbert symbol. For $a, b \in K_v^\times$ and $v \in \Omega_K$, we use Hilbert symbol $(a, b)_v \in \{\pm 1\}$. By definition, $(a, b)_v = 1$ if and only if the curve defined over $K_v$ by the equation $x_0^2 - ax_1^2 - bx_2^2 = 0$ in $\mathbb{P}^2$, has a $K_v$-point.

3. Stoll’s conjecture for curves

For a smooth, projective, and geometrically connected curve $C$ defined over an number field $K$, if the Tate-Shafarevich group and the rational points set of its Jacobian are both finite, then by the dual sequence of Cassels-Tate, Skorobogatov [Sko01, Chapter 6.2] independently observed that $C(K) = \text{pr}^{\infty_K}(C(A_K)^{Br})$. In particular, if this curve $C$ is a counterexample to the Hasse principle, then this failure can be explained by the Brauer-Manin obstruction. Stoll [Sto07, Theorem 8.6] generalized this observation. Furthermore, he [Sto07, Conjecture 9.1] made the following conjecture.

**Conjecture 3.0.1.**  [Sto07, Conjecture 9.1] For any smooth, projective, and geometrically connected curve $C$ defined over a number field $K$, suppose that Conjecture 3.0.1 holds for all smooth, projective, and geometrically connected curve defined over a number field $K$. Then there exists a smooth, projective, and geometrically connected curve $C$ defined over $K$ such that the triple $(C, K, L)$ is of type $I$.

The following definition and lemma have already been stated in the paper [Wn21]. We give them below for the convenience of reading.

**Definition 3.0.3.** ([Wn21, Definition 4.0.3]) Given a smooth, projective, and geometrically connected curve $C$ defined over an number field $K$, let $L/K$ be a nontrivial extension of number fields. We say that a triple $(C, K, L)$ is of type $I$ if

- $C(K)$ and $C(L)$ are both finite nonempty sets,
- $C(K) \neq C(L)$,
- the Stoll’s Conjecture $\text{3.0.1}$ holds for the curve $C$.

**Lemma 3.0.4.** ([Wn21, Lemma 4.0.4]) Let $L/K$ be a nontrivial extension of number fields. Suppose that Conjecture $\text{3.0.1}$ holds for all smooth, projective, and geometrically connected curves defined over $K$. Then there exists a smooth, projective, and geometrically connected curve $C$ defined over $K$ such that the triple $(C, K, L)$ is of type $I$.

The following lemma is a strong form of [Wn21, Lemma 6.1.3]. It will be used to choose a dominant morphism from a given curve to $\mathbb{P}^1$.

**Lemma 3.0.5.** Let $L/K$ be a nontrivial extension of number fields. Given a smooth, projective, and geometrically connected curve $C$ defined over $K$, suppose that the triple $(C, K, L)$ is of type $I$ (Definition $\text{3.0.1}$). For any finite $K$-scheme $R \subset \mathbb{P}^1$, there exists a dominant $K$-morphism $\gamma: C \to \mathbb{P}^1$ such that

1. $\gamma(C(K)) = \{\infty\} \subset \mathbb{P}^1(K)$,
2. $\gamma(C(L) \setminus C(K)) = \{0\} \subset \mathbb{P}^1(K)$,
3. $\gamma$ is étale over $R$.

**Proof.** The proof is along the same idea as the proof of [Wn21, Lemma 6.1.3], where the statement was shown for $R \subset \mathbb{P}^1 \setminus \{0, \infty\}$. We will put one more condition for choosing a rational function. Let $K(C)$ be the function field of $C$. For $C(K)$ and $C(L)$ are both finite nonempty and $C(L) \setminus C(K) \neq \emptyset$, by Riemann-Roch theorem, we can choose a rational function $\phi \in K(C)^\times \setminus K^\times$ such that

- the set of its poles contains $C(K)$,
- the set of its zeros contains $C(L) \setminus C(K)$,
all poles and zeros are of multiplicity one.

Then this rational function \( \phi \) gives a dominant \( K \)-morphism \( \gamma_0 : C \to \mathbb{P}^1 \) such that

- \( \gamma_0(C(L) \setminus C(K)) = \{0\} \subset \mathbb{P}^1(K) \),
- \( \gamma_0(C(K)) = \{\infty\} \subset \mathbb{P}^1(K) \),
- \( \gamma_0 \) is étale over \( \{0, \infty\} \).

Then the branch locus of \( \gamma_0 \) is contained in \( \mathbb{P}^1 \setminus \{0, \infty\} \). We can choose an automorphism \( \varphi_{\lambda_0} : \mathbb{P}^1 \to \mathbb{P}^1, (u : v) \mapsto (\lambda_0 u : v) \) with \( \lambda_0 \in K^\times \) such that the branch locus of \( \gamma_0 \) has no intersection with \( \varphi_{\lambda_0}(R) \). Let \( \lambda = (\varphi_{\lambda_0})^{-1} \circ \gamma_0 \). Then the morphism \( \lambda \) is étale over \( R \), and satisfies other conditions.

\[ \square \]

4. Main results

In this section, we will construct smooth, projective, and geometrically connected surfaces to give negative answers to Questions 1.2.

4.1. Non-invariance of weak approximation with Brauer-Manin obstruction. For any number field \( K \), assuming Conjecture [Wu21, Theorem 6.2.1] found a quadratic extension \( L \), and constructed a 3-fold to give a negative answer to Question 1.2.1. The author [Wu21, Theorem 6.2.1] generalized his result to any nontrivial extension of number fields. Although the strategies of these two papers are different, the methods used there are combining the arithmetic properties of Châtelet surfaces with a construction method from Poonen [Poo10]. Thus the varieties constructed there, are 3-folds. In this subsection, we will use the arithmetic properties of curves, whose irreducible components are projective lines. For any extension of number fields \( L/K \), assuming Conjecture [Wu21, Theorem 6.2.1] we will construct a smooth, projective, and geometrically connected surface to give a negative answer to Question 1.2.1

4.1.1. Preparation Lemmas. We state the following lemmas, which will be used for the proof of Theorem 4.1.3.

The following fibration lemma has already been stated in the paper [Wu21]. We give them below for the convenience of reading.

Lemma 4.1.1. ([Wu21, Lemma 6.1.1]) Let \( K \) be a number field, and let \( S \subset \Omega_K \) be a finite subset. Let \( f : X \to Y \) be a \( K \)-morphism of proper \( K \)-varieties \( X \) and \( Y \). Suppose that

1. the set \( Y(K) \) is finite,
2. the variety \( Y \) satisfies weak approximation with Brauer-Manin obstruction off \( S \),
3. for any \( P \in Y(K) \), the fiber \( X_P \) of \( f \) over \( P \) satisfies weak approximation off \( S \).

Then the variety \( X \) satisfies weak approximation with Brauer-Manin obstruction off \( S \).

The following fibration lemma can be viewed as a modification of [Wu21] Lemma 6.1.2 to fit into our context.

Lemma 4.1.2. Let \( K \) be a number field, and let \( S \subset \Omega_K \) be a finite subset. Let \( f : X \to Y \) be a \( K \)-morphism of proper \( K \)-schemes \( X \) and \( Y \). We assume that

1. the set \( Y(K) \) is finite,
2. there exists some \( P \in Y(K) \) such that the fiber \( X_P \) of \( f \) over \( P \) does not satisfy weak approximation with Brauer-Manin obstruction off \( S \).

Then the scheme \( X \) does not satisfy weak approximation with Brauer-Manin obstruction off \( S \).

Proof. By Assumption 2, take a \( P_0 \in Y(K) \) such that the fiber \( X_{P_0} \) does not satisfy weak approximation with Brauer-Manin obstruction off \( S \). Then there exist a finite nonempty subset \( S' \subset \Omega_K \) and a nonempty open subset \( L = \prod_{\lambda \in S'} U_{\lambda} \times \prod_{\lambda \in S'} X_{P_0}(K_{\lambda}) \subset X_{P_0}(\mathbb{A}_K) \) such that \( L \cap X_{P_0}(\mathbb{A}_K)^{Br} \neq \emptyset \), but that \( L \cap X_{P_0}(K) = \emptyset \). By Assumption 1, the
set \( Y(K) \) is finite, so we can take a Zariski open subset \( V_P^0 \subset Y \) such that \( V_P^0(K) = \{ P \} \). For any \( v \in S' \), since \( U_v \) is open in \( X_v^0(K_v) \subset f^{-1}(V_{P_v})(K_v) \), we can take an open subset \( W_v \) of \( f^{-1}(V_{P_v})(K_v) \) such that \( W_v \cap X_v^0(K_v) = U_v \). Consider the open subset \( N = \prod_{v \in S'} W_v \times \prod_{v \in S'} X_v^0(K_v) \subset X(K_v) \), then \( L \subset N \). By the factoriality of Brauer-Manin pairing, we have \( X_{P_v}^{\Br}(K_v) \subset X(K_v)^{\Br} \). So the set \( N \cap X(K) = N \cap X_{P_v}(K) = L \cap X_{P_v}(K) = \emptyset \), which implies that \( X \) does not satisfy weak approximation with Brauer-Manin obstruction on \( S \).

**Lemma 4.1.3.** Let \( K \) be a number field, and let \( S \subset \Omega_K \) be a finite subset. Let \( X \) be a \( K \)-scheme, which is not a \( K \)-variety, i.e. it has multiple geometrically irreducible components. Assume \( \prod_{v \in \Omega_K} X_v(K_v) \neq \emptyset \), then \( X \) does not satisfy weak approximation off \( S \).

**Proof.** Let \( X^0 \) be the smooth locus of \( X \). Claim that \( X^0 \subset X \) is an open dense subset. For \( X \) is reduced and \( \text{Char} \, K = 0 \), the scheme \( X \) is geometrically reduced. For any geometrically irreducible component of \( X \), by [Har97, Chapter II. Corollary 8.16], its smooth locus is open dense in this geometrically irreducible component. So the claim follows. From this claim, we have \( X \) and \( X^0 \) have the same number of geometrically irreducible components.

By assumption that \( X \) has multiple geometrically irreducible components, let \( X_1^0 \) and \( X_2^0 \) be two different geometrically irreducible components of \( X^0 \), defined over \( K_1 \) and \( K_2 \) respectively. By Lang-Weil estimate [LW54], the varieties \( X_1^0 \) and \( X_2^0 \) have local points for almost all places of \( K_1 \) and \( K_2 \) respectively. By the Čebotarev’s density theorem, we can take two different places \( v_1, v_2 \in \Omega_{K} \setminus S \) such that \( v_1, v_2 \) split in \( K_1 \) and also in \( K_2 \), and that \( X_1^0(K_{v_1}) \neq \emptyset \) and \( X_2^0(K_{v_2}) \neq \emptyset \). For \( \prod_{v \in \Omega_K} X_v(K_v) \neq \emptyset \), we consider a nonempty open subset \( L = X_1^0(K_{v_1}) \times X_2^0(K_{v_2}) \times \prod_{v \in \Omega_K \setminus \{ v_1, v_2 \}} X_v(K_v) \subset \prod_{v \in \Omega_K} X_v(K_v) \). For \( X^0 \) is smooth, and the varieties \( X_1^0, X_2^0 \) are different geometrically irreducible components, we have \( X_1^0(K_{v_1}) \cap X_2^0(K_{v_2}) = \emptyset \), which implies \( X(K) \cap L = \emptyset \). Hence \( X \) does not satisfy weak approximation off \( S \).

**Lemma 4.1.4.** Let \( K \) be a number field, and let \( S \subset \Omega_K \) be a finite subset. Let \( C \) be a curve defined over \( K \) by a homogeneous equation: \( x_0^2 - x_1^2 = 0 \) with homogeneous coordinates \( (x_0 : x_1 : x_2) \in \mathbb{P}^2 \). Then the curve \( C \) does not satisfy weak approximation with Brauer-Manin obstruction off \( S \).

**Proof.** For \( C \) is a union of two projective lines meeting at one point, the argument that \( \text{Br}(K) \cong \text{Br}(C) \), follows from [HS14 Corollary 1.5]. For the curve \( C \) has \( K \)-rational points and two irreducible components, by Lemma 4.1.3, this lemma follows.

**Theorem 4.1.5.** For any nontrivial extension of number fields \( L/K \), assuming that Conjecture 3.0.1 holds over \( K \), there exists a smooth, projective, and geometrically connected surface \( X \) defined over \( K \) such that

- the surface \( X \) has a \( K \)-rational point, and satisfies weak approximation with Brauer-Manin obstruction off \( \infty_K \),
- the surface \( X_L \) does not satisfy weak approximation with Brauer-Manin obstruction off \( T \) for any finite subset \( T \subset \Omega_L \).

**Proof.** We will construct a smooth, projective, and geometrically connected surface \( X \). Let \( C_\infty \) be a projective line defined over \( K \) by a homogeneous equation: \( x_0^2 + x_1^2 - x_2^2 = 0 \) with homogeneous coordinates \( (x_0 : x_1 : x_2) \in \mathbb{P}^2 \). Let \( C_0 \) be a curve defined over \( K \) by a homogeneous equation: \( x_0^2 - x_1^2 = 0 \) with homogeneous coordinates \( (x_0 : x_1 : x_2) \in \mathbb{P}^2 \). Let \( (u_0 : u_1) \times (x_0 : x_1 : x_2) \) be the coordinates of \( \mathbb{P}^1 \times \mathbb{P}^2 \), and let \( s' = u_0(x_0^2 + x_1^2 - x_2^2) + u_1(x_0^2 - x_1^2) \in (\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}(1, 2)) \). Let \( X' \) be the locus defined by \( s' = 0 \) in \( \mathbb{P}^1 \times \mathbb{P}^2 \). By Jacobian criterion, the locus \( X' \) is smooth. Let \( R \) be the locus over which the composition \( X' \to \mathbb{P}^1 \times \mathbb{P}^1 \) is not smooth. Then it is finite over \( K \). By the assumption that Conjecture 3.0.1 holds over \( K \), and Lemma 3.0.3, we can take a smooth, projective, and geometrically connected curve \( C \) defined over \( K \) such that the triple \( (C, K, L) \) is of type I. By Lemma 4.0.5, we can choose a \( K \)-morphism \( \gamma : C \to \mathbb{P}^1 \) such that \( \gamma(C(L) \setminus C(K)) = \)
\{0\} \subset \mathbb{P}^1(K), \gamma(C(K)) = \{\infty\} \subset \mathbb{P}^1(K), and that \gamma is étale over \text{R}. Let \text{B} = \mathbb{C} \times \mathbb{P}^2. Let \mathcal{L} = (\gamma, id)^*\mathcal{O}(1, 2), and let s = (\gamma, id)^*(s') \in \Gamma(B, \mathcal{L}). Let \text{X} be the zero locus of \text{s} in \text{B}. For \gamma is étale over the locus \text{R}, the surface \text{X} is smooth. Since \text{X} is defined by the support of the global section \text{s}, it is an effective divisor. The invertible sheaf \mathcal{O}(X') on \mathbb{P}^1 \times \mathbb{P}^2 is isomorphic to \mathcal{O}(1, 2), which is a very ample sheaf on \mathbb{P}^1 \times \mathbb{P}^2. And (\gamma, id) is a finite morphism, so the pull back of this ample sheaf is again ample, which implies that the invertible sheaf \mathcal{O}(X) on \mathbb{C} \times \mathbb{P}^2 is ample. By [Hat97 Chapter III, Corollary 7.9], the surface \text{X} is geometrically connected. So the surface \text{X} is smooth, projective, and geometrically connected. Let \beta : \text{X} \hookrightarrow \text{B} = \mathbb{C} \times \mathbb{P}^2 \subset \text{C} be the composition morphism.

Next, we will check that the surface \text{X} has the properties. We will show that \text{X} has a \text{K}-rational point. For any \text{P} \in C(\text{K}), the fiber \beta^{-1}(P) \cong C_{\infty}. The projective line \text{C}_{\infty} has a \text{K}-rational point, so the set \text{X}(\text{K}) \neq \emptyset. We will show that \text{X} satisfies weak approximation with Brauer-Manin obstruction off \text{C}_{\infty}. Since the projective line \text{C}_{\infty} satisfies weak approximation, we consider the morphism \beta, then Assumption (3) of Lemma 4.1.1 holds. Since Conjecture 3.0.1 holds for the curve \text{B}, then \text{Assumption (3) of Lemma 4.1.1 holds}. Since Conjecture 3.0.1 holds for the curve \text{C}, using Lemma 4.2.1 for the morphism \beta, the surface \text{X} satisfies weak approximation with Brauer-Manin obstruction off \text{C}_{\infty}.

For any finite subset \text{T} \subset \Omega_L, we will show that \text{X}_L does not satisfy weak approximation with Brauer-Manin obstruction off \text{T}. Take a point \text{Q} \in C(L) \setminus C(\text{K}), by the choice of the curve \text{C} and morphism \beta, the fiber \beta^{-1}(\text{Q}) \cong C_{\text{O}_L}. By Lemma 4.1.2, the curve \text{C}_L does not satisfy weak approximation with Brauer-Manin obstruction off \text{T} \cup \infty_L. By Lemma 4.1.2, the surface \text{X}_L does not satisfy weak approximation with Brauer-Manin obstruction off \text{T} \cup \infty_L. So it does not satisfy weak approximation with Brauer-Manin obstruction off \text{T}.

4.2. Non-invariance of the failures of the Hasse principle explained by the Brauer-Manin obstruction. For an extension of number fields \text{L}/\text{K}, assuming that the degree \text{[L : K]} is odd, or that the field \text{L} has one real place, also assuming Conjecture 3.0.1, the author [Wu21] Theorem 6.3.1 and Theorem 6.3.2 constructed \text{3}-folds to give negative answers to Question 1.2.2. The method used there is combining the arithmetic properties of Châtelet surfaces with a construction method from Poonen [Poo10]. Thus the varieties constructed there, are \text{3}-folds. For any extension of number fields \text{L}/\text{K}, assuming Conjecture 3.0.1 in this subsection, we will construct a smooth, projective, and geometrically connected surface to give a negative answer to Question 1.2.2.

4.2.1. Preparation lemmas. We state the following lemmas, which will be used for Choosing curves.

**Lemma 4.2.1.** Given a number field \text{K}, let \text{p}_1, \text{p}_2 be two odd prime elements and \text{v}_\text{p}_1 \neq \text{v}_\text{p}_2. If \text{v}_{\text{p}_1}(\text{p}_1) = 1, then \text{p}_2 \in \text{K}_{v_{\text{p}_1}}. Otherwise, if \text{v}_{\text{p}_1}(\text{p}_1) = -1, then \text{p}_2 \notin \text{K}_{v_{\text{p}_1}}.

*Proof.* Consider the case (\text{p}_1, \text{p}_2)_{v_{\text{p}_1}} = 1. By definition, the equation \text{x}_1^2 - \text{p}_1 \text{x}_2^2 - \text{p}_2 \text{x}_3^2 = 0 has a nontrivial solution in \text{K}_{v_{\text{p}_1}}. Let (x, y, z) be a primitive solution of this equation. Then \text{v}_{\text{p}_1}(x) = \text{v}_{\text{p}_1}(z) = 0. So \text{x}^2 - \text{p}_2 \text{z}^2 \equiv 0 \mod \text{p}_1. By Hensel’s lemma, we have \text{p}_2 \in \text{K}_{v_{\text{p}_1}}. □

**Lemma 4.2.2.** Given a number field \text{K}, let \text{v} \in \Omega^1_L. Then there exists a proper nonzero ideal \text{I} \subset \mathcal{O}_\text{K} such that for any \text{a} \in \mathcal{O}_\text{K}, if \text{a} \equiv 1 \mod \text{I}, then \text{a} \notin \text{K}_v^{\times 2}.

*Proof.* Let \text{p} be the prime number such that \text{v}|p in \text{K}. Let \text{J} be the ideal generated by \text{p}^3. Then by Hensel’s lemma, we have \text{1} + \text{p}^3\mathcal{O}_\text{K} \subset \text{K}_v^{\times 2}, which implies this lemma. □

**Lemma 4.2.3.** Given a number field \text{K}, let \text{p}_1, \text{p}_2 be two odd prime elements and \text{v}_\text{p}_1 \neq \text{v}_\text{p}_2. Let \text{I} be the ideal generated by \text{p}_1 \text{p}_2. Then there exists an element \text{x} \in \mathcal{O}_\text{K} such that

- the image of \text{x} in \mathcal{O}_\text{K}/\text{I} is invertible,
- for any \text{a} \in \mathcal{O}_\text{K}, if \text{a} \equiv x \mod \text{I}, then \text{(p}_1, \text{a})_{v_{\text{p}_1}} = -1 and \text{(p}_2, \text{a})_{v_{\text{p}_2}} = 1.

□
4.2.2. Choosing two curves with respect to an extension. Given an extension of number fields \(L/K\), by Lemmas 2.0.1 and 4.2.2 we can choose an odd prime element \(p_1 \in \mathcal{O}_K\) satisfying the following conditions:
- \(\tau_v(p_1) > 0\) for all \(v \in \infty_K\),
- \(p_1 \in K_v^{\times 2}\) for all \(v \in 2_K\),
- \(p_1\) splits in \(L\).

Similarly, by Lemmas 2.0.1 and 4.2.2 we can choose an odd prime element \(p_2 \in \mathcal{O}_K\) satisfying the following conditions:
- \((p_1, p_2)_{v_p} = 1\),
- \(p_2\) splits in \(L\),
- \(v_{p_2} \neq v_{p_1}\).

Let \(L' = \sqrt{L} / \sqrt{p_1} \cdot \sqrt{p_2}\). By Lemma 2.0.1 we can choose an odd prime element \(p_3 \in \mathcal{O}_K\) such that \(v_{p_3} \notin \{v_{p_1}, v_{p_2}\}\), and that \(v_{p_3}\) splits in \(L'\). Let \(f(x_0, x_1; y_0, y_1) = (x_0^2 - p_1 x_1^2)(x_0^2 - p_2 x_1^2)(x_0^2 - p_3 x_1^2)(y_0^2 - p_3 y_1^2)(y_0^2 - p_3 y_1^2)\) be a bi-homogeneous polynomial, and let \(Z_f'\) be the zero locus of \(f\) in \(\mathbb{P}^1 \times \mathbb{P}^1\) with coordinates \((x_0 : x_1) \times (y_0 : y_1)\). With the notation, we have the following lemmas.

**Lemma 4.2.4.** Let \(Z_f' \subset \mathbb{P}^1 \times \mathbb{P}^1\) be the zero locus defined over \(K\) by the bi-homogeneous polynomial \(f(x_0, x_1; y_0, y_1)\). Then \(Z_f'\) and \(Z_L'\) violate the Hasse principle.

**Proof.** By the condition that the prime elements \(p_1, p_2,\) and \(p_3\), split in \(L\), the set \(Z_f'(K) = Z_f'(L) = \emptyset\). It will be suffice to prove that for any \(v \in \Omega_K\), the equation \((x_0^2 - p_1 x_1^2)(x_0^2 - p_2 x_1^2)(x_0^2 - p_3 x_1^2) = 0\) has a \(K_v\)-point in \(\mathbb{P}^1\) with coordinates \((x_0 : x_1)\).

Suppose that \(v \in \infty_K \cup 2_K\). Then, by the choice of \(p_1\), we have \(p_1 \in K_v^{\times 2}\), so \(x_0^2 - p_1 x_1^2 = 0\) has a \(K_v\)-point in \(\mathbb{P}^1\).

Suppose that \(v = v_{p_1}\). Then, by the choice of \(p_2\), we have \((p_1, p_2)_v = 1\). By Lemma 4.2.4 we have \(p_2 \in K_v^{\times 2}\). Hence \(x_0^2 - p_2 x_1^2 = 0\) has a \(K_v\)-point in \(\mathbb{P}^1\).

Suppose that \(v = v_{p_2}\). Using the product formula \(\prod_{v \in \Omega_K} (p_1, p_2)_v = 1\), we have \((p_1, p_2)_v = 1\). By Lemma 4.2.4 we have \(p_1 \in K_v^{\times 2}\). Hence \(x_0^2 - p_1 x_1^2 = 0\) has a \(K_v\)-point in \(\mathbb{P}^1\).

If \(v \in \Omega_K \setminus \infty_K \cup 2_K \cup \{v_{p_1}, v_{p_2}\}\), then, by the quadratic reciprocity law, at least one of equations: \(x_0^2 - p_1 x_1^2 = 0\), \(x_0^2 - p_2 x_1^2 = 0\), \(x_0^2 - p_1 p_2 x_1^2 = 0\), has a \(K_v\)-point in \(\mathbb{P}^1\).

So \(Z_f'(K_v) = \emptyset\).

**Lemma 4.2.5.** The natural morphism \(\text{Br}(L) \to \text{Br}(Z_L')\) is an isomorphism.

**Proof.** This follows from [HST13, Proposition 3.1].

Similar to the choice of \(p_1\), we can choose an odd prime element \(p_4 \in \mathcal{O}_K\) satisfying the following conditions:
- \(\tau_v(p_4) > 0\) for all \(v \in \infty_K\),
- \(p_4 \in K_v^{\times 2}\) for all \(v \in 2_K\),
- \(p_4\) splits in \(L\),
- \(v_{p_4} \notin \{v_{p_1}, v_{p_2}, v_{p_3}\}\).

By Lemmas 2.0.1 and 4.2.3 we choose an odd prime element \(p_5 \in \mathcal{O}_K\) satisfying the following conditions:
- \((p_4, p_5)_{v_{p_4}} = -1\),
- \(v_{p_5} \notin \{v_{p_1}, v_{p_2}, v_{p_3}, v_{p_4}\}\).

Similarly, by Lemmas 2.0.1 and 4.2.3 we choose an odd prime element \(p_6 \in \mathcal{O}_K\) satisfying the following conditions:
Let $g(x_0, x_1, y_0, y_1) = (x_0^2 - p_4 x_1^2)(x_0^2 - p_4 p_5 x_1^2)(y_0^2 - p_4 y_1^2)(y_0^2 - p_4 y_1^2)$ be a bi-homogeneous polynomial, and let $Z^g$ be the zero locus of $g$ in $\mathbb{P}^1 \times \mathbb{P}^1$ with coordinates $(x_0 : x_1) \times (y_0 : y_1)$. With the notation, we have the following lemma.

**Lemma 4.2.6.** Let $Z^g \subset \mathbb{P}^1 \times \mathbb{P}^1$ be the zero locus defined over $K$ by the bi-homogeneous polynomial $g(x_0, x_1; y_0, y_1)$. Then $Z^g(K_{v_p}) \neq \emptyset$ but $Z^g(K_{v_p}) = \emptyset$.

**Proof.** Suppose that $v \in \infty_K \cup 2K$. Then, by the choice of $p_4$, we have $p_4 \in K_v^\times$, so $x_0^2 - p_4 x_1^2 = 0$ has a $K_v$-point in $\mathbb{P}^1$ with coordinates $(x_0 : x_1)$.

Suppose that $v = v_p$. Then, by the choice of $p_6$, we have $(p_5, p_6)_v = 1$. By Lemma 4.2.1, we have $p_6 \in K_v^\times$. So $y_0^2 - p_6 y_1^2 = 0$ has a $K_v$-point in $\mathbb{P}^1$ with coordinates $(y_0 : y_1)$.

If $v \in \Omega_K \setminus \infty_K \cup 2K \cup \{v_p, v_{p_6}\}$, then, by the quadratic reciprocity law, at least one of equations: $x_0^2 - p_4 x_1^2 = 0$, $x_0^2 - p_4 p_5 x_1^2 = 0$, $x_0^2 - p_4 x_1^2 = 0$, $y_0^2 - p_4 y_1^2 = 0$ has no $K_v$-point in $\mathbb{P}^1$ with coordinates $(x_0 : x_1)$ and $(y_0 : y_1)$ respectively. By the choice of $p_5$, $p_6$, we have $(p_4, p_3, p_4)_{v_p} = -1$ and $(p_4, p_6)_{v_p} = -1$. By Lemma 4.2.1 we have $p_5 \notin K_{v_p}$ and $p_6 \notin K_{v_p}$. So $x_0^2 - p_4 x_1^2 = 0$ and $y_0^2 - p_4 y_1^2 = 0$ has no $K_v$-point in $\mathbb{P}^1$ with coordinates $(x_0 : x_1)$ and $(y_0 : y_1)$ respectively. So $Z^g(K_{v_p}) = \emptyset$.

**Example 4.2.7.** For $K = \mathbb{Q}$ and $L = \mathbb{Q}(i)$, let prime elements $(p_1, p_2, p_3, p_4, p_5, p_6) = (17, 13, 53, 41, 13, 13)$. Then they satisfy all chosen conditions. They will be used for construction of our explicit unconditional example.

**Theorem 4.2.8.** For any nontrivial extension of number fields $L/K$, assuming that Conjecture 3.0.1 holds over $K$, there exists a smooth, projective, and geometrically connected surface $X$ defined over $K$ such that

- the surface $X$ is a counterexample to the Hasse principle, and its failure of the Hasse principle is explained by the Brauer-Manin obstruction,
- the surface $X_L$ is a counterexample to the Hasse principle, but its failure of the Hasse principle cannot be explained by the Brauer-Manin obstruction.

**Proof.** We will construct a smooth, projective, and geometrically connected surface $X$. For the extension $L/K$, we choose odd prime elements $p_1, p_2, p_3, p_4, p_5, p_6 \in \mathcal{O}_K$ as in Subsubsection 4.2.2. Let $f(x_0, x_1; y_0, y_1) = (x_0^2 - p_1 x_1^2)(x_0^2 - p_2 x_1^2)(x_0^2 - p_1 p_2 x_1^2)(y_0^2 - p_3 y_1^2)(y_0^2 - p_3 y_1^2)$ be a bi-homogeneous polynomial, and let $Z^f$ and $Z^g$ be the zero loci of $f$ and $g$ respectively in $\mathbb{P}^1 \times \mathbb{P}^1$ with coordinates $(x_0 : x_1) \times (y_0 : y_1)$. Let $(u_0 : u_1)(x_0 : x_1) \times (y_0 : y_1)$ be the coordinates of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, and let $s' = u_0 f(x_0, x_1; y_0, y_1) + u_1 f(x_0, x_1; y_0, y_1) \in \Gamma(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1, 1, 1))$. Let $X'$ be the locus defined by $s' = 0$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. For $Z^f$ and $Z^g$ meet transversally, the locus $X'$ is smooth. Let $R$ be the locus over which the composition $X' \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ is not smooth. Then it is finite over $K$.

By the assumption that Conjecture 3.0.1 holds over $K$, and Lemma 3.0.4, we can take a smooth, projective, and geometrically connected curve $C$ defined over $K$ such that the triple $(C, K, L)$ is of type $I$. By Lemma 3.0.3, we can choose a $K$-morphism $\gamma: C \to \mathbb{P}^1$ such that $\gamma(C(K) \cap C(L)) = \{0\} \subset \mathbb{P}^1(K)$, $\gamma(C(K)) = \{\infty\} \subset \mathbb{P}^1(K)$, and that $\gamma$ is étale over $R$.

Let $B = C \times \mathbb{P}^1 \times \mathbb{P}^1$. Let $\mathcal{L} = \gamma(id)^* \mathcal{O}(1, 5, 6)$, and let $s = (\gamma, id)^*(s') \in \Gamma(B, \mathcal{L})$. Let $X$ be the zero locus of $s$ in $B$. By the same argument as in the proof of Theorem 4.1.5, the surface $X$ is smooth, projective, and geometrically connected. Let $\beta: X \hookrightarrow B = C \times \mathbb{P}^1 \times \mathbb{P}^1$ be the composition morphism.
Next, we will check that the surface $X$ has the properties.

We will show $X(A_K) \neq \emptyset$. For any $P \in C(K)$, the fiber $\beta^{-1}(P) \cong \mathbb{Z}^g$. By Lemma 4.2.6, the set $Z^g(\mathbb{A}_K^{\text{proj}}) \neq \emptyset$. So $X(A_K^{\text{proj}}) \neq \emptyset$. For $v_{p_4}$ splits in $L$, take a place $v' \in \Omega^K_L$ above $v_{p_4}$ such that $K_{v_{p_4}} = L_v$. By Lemma 4.2.4, the set $Z_{\upsilon}^f(\mathbb{A}_L) \neq \emptyset$. Take a point $Q \in C(L) \backslash C(K)$, then the fiber $\beta^{-1}(Q) \cong Z_{\upsilon}^f$. We have $X(K_{v_{p_4}}) = X_L(L_{v'}) \supset \beta^{-1}(Q)(L_{v'}) \cong Z_{\upsilon}^f(L_{v'}) \neq \emptyset$. So $X(\mathbb{A}_K) \neq \emptyset$.

We will show $X(\mathbb{A}_K)^Br = \emptyset$. By Conjecture 3.0.1 the set $C(K)$ is finite, and $C(K) = \prod_{i \geq 0} (\mathbb{A}_K^{Br})$. By the functoriality of Brauer-Manin pairing, we have $\prod_{i \geq 0} (\mathbb{A}_K^{Br}) \subset \bigcup_{i \geq 0} (\mathbb{A}_K^{Br}) = \mathbb{Z}^g(\mathbb{A}_K^{\text{proj}}) \times C(K)$. But by Lemma 4.2.6, the set $Z_{\upsilon}^f(K_{v_{p_4}}) = \emptyset$, so we have $\prod_{i \geq 0} (\mathbb{A}_K^{Br}) \subset \bigcup_{i \geq 0} (\mathbb{A}_K^{Br}) = \emptyset$, which implies that $X(\mathbb{A}_K)^Br = \emptyset$.

So, the surface $X$ is a counterexample to the Hasse principle, and its failure of the Hasse principle is explained by the Brauer-Manin obstruction.

We will show $X(L(A_L)^Br) \neq \emptyset$. Take a point $Q \in C(L) \backslash C(K)$. By Lemma 4.2.3, the set $Z_{\upsilon}^f(L(A_L)^Br) \neq \emptyset$. By Lemma 4.2.4, it is nonempty. By the functoriality of Brauer-Manin pairing, the set $X_L(L(A_L)^Br)$ contains $\beta^{-1}(Q)(L(A_L)^Br) \cong Z_{\upsilon}^f(L(A_L)^Br)$, so it is nonempty. We will show $X(L) = \emptyset$. By Lemma 4.2.6 and the condition that $v_{p_4}$ splits in $L$, we have $Z^g(L(A_L) = \emptyset$, so $Z^g(L) = \emptyset$. By Lemma 4.2.4, the set $Z_{\upsilon}^f(L) = \emptyset$. Since each $L$-rational fiber of $\beta$ is isomorphic to $Z_{\upsilon}^f$, the set $X(L) = \emptyset$. So, the variety $X_L$ is a counterexample to the Hasse principle, but its failure of the Hasse principle cannot be explained by the Brauer-Manin obstruction. 

\[ \square \]

5. Explicit unconditional examples

In this section, let $K = \mathbb{Q}$ and $L = \mathbb{Q}(i)$. For this extension $L/K$, we will give explicit examples without assuming Conjecture 3.0.1 for Theorem 4.1.5 and Theorem 4.2.8.

5.1. Choosing an elliptic curve and a dominant morphism. For the extension $L/K$, as in the proof of Theorem 4.1.5 and Theorem 4.2.8 we can choose a common elliptic curve over $K$ for these examples.

5.1.1. Choosing an elliptic curve. For the extension $L/K$, we will choose an elliptic curve such that the triple $(E, K, L)$ is of type $I$. Let $E$ be an elliptic curve over $\mathbb{Q}$ defined by a homogeneous equation:

$$w_1^2w_2 = w_0^3 - 16w_2^3$$

in $\mathbb{P}^2$ with homogeneous coordinates $(w_0 : w_1 : w_2)$. Its quadratic twist $E^{(-1)}$ is isomorphic to an elliptic curve defined by a homogeneous equation: $w_1^2w_2 = w_0^3 + 16w_2^3$. The elliptic curves $E$ and $E^{(-1)}$ are of analytic rank 0. Then the Tate-Shafarevich group $\text{III}(E, \mathbb{Q})$ is finite, so $E$ satisfies weak approximation with Brauer-Manin obstruction off $K$. The Mordell-Weil groups $E(K)$ and $E^{(-1)}(K)$ are finite, so $E(L)$ is finite. Indeed, the Mordell-Weil group $E(K) = \{0 : 1\}$ and $E(L) = \{0 : \pm 4i : 1\}$, $(0 : 1 : 0)$. So the triple $(E, K, L)$ is of type $I$.

5.1.2. Choosing a dominant morphism. We choose the following dominant morphism from the elliptic curve $E$ to $\mathbb{P}^1$, which satisfies all conditions of Lemma 3.0.5:

Let $\mathbb{P}^2 \setminus \{(1 : 0 : 0), (-16 : 0 : 1), (-1 : \pm \sqrt{15}i : 1)\} \to \mathbb{P}^1$ be a morphism over $\mathbb{Q}$ given by $w_0 : w_1 : w_2 \rightarrow (w_0w_2 + w_1^2 + 16w_2^2 : w_0w_1 + w_1w_2)$. Composite with the natural inclusion $E \hookrightarrow \mathbb{P}^2 \setminus \{(1 : 0 : 0), (-16 : 0 : 1), (-1 : \pm \sqrt{15}i : 1)\}$, then we get a morphism $\gamma : E \rightarrow \mathbb{P}^1$, which is a dominant morphism of degree 6. The dominant morphism $\gamma$ maps $E(K)$ to $\{0\}$, and maps $(0 : \pm 4i : 1)$ to $0 := (0 : 1)$. To calculate the branch locus of $\gamma : E \rightarrow \mathbb{P}^1$, we use Jacobian criterion for the intersection of two curves $E$ and $(w_0w_2 + w_1^2 + 16w_2^2)u_1 = (w_0w_1 + w_1w_2)u_0$, here $\mathbb{P}^1$ has coordinates $(u_0 : u_0)$. By Bézout’s Theorem [Har97] Chapter I. Corollary 7.8 and calculation, the branch locus is contained in
\( \mathbb{P}^1 \setminus \{ \infty \} \). Let \( u_1 = 1 \) and \( w_2 = 1 \) to dehomogenize these two curves. By Jacobian criterion, the branch locus satisfies the following equations:

\[
\begin{align*}
  w_1^2 &= w_1^0 - 16 \\
  w_1^2 + w_0 + 16 &= w_1(w_0 + 1)u_0 \\
  3(2w_1 - w_1u_0 - u_0)u_0^2 + 2w_1(1 - w_1u_0) &= 0
\end{align*}
\]

Then the branch locus equals

\[
\left\{ (u_0 : 1) | u_0^{12} + \frac{60627u_0^{10}}{4913} + \frac{159828u_0^8}{4913} - \frac{3505917u_0^6}{19652} - \frac{4205761u_0^4}{58956} + \frac{76076u_0^2}{14739} - \frac{4112}{132651} = 0 \right\}.
\]

Let \( (u_0 : 1) \) be a branch point, then the degree \(|\mathbb{Q}(u_0) : \mathbb{Q}\)| is 12.

### 5.2. An explicit unconditional example for Theorem 4.1.5

For \( K = \mathbb{Q} \) and \( L = \mathbb{Q}(i) \), in the subsection, we will construct a smooth, projective, and geometrically connected surface having properties of Theorem 4.1.5.

#### 5.2.1. Construction of a smooth, projective, and geometrically connected surface

We will construct a smooth, projective, and geometrically connected surface \( X \) as in Theorem 4.1.5. Let \( (u_0 : v_1) \times (x_0 : x_1 : x_2) \) be the coordinates of \( \mathbb{P}^1 \times \mathbb{P}^2 \), and let \( s' = u_0(x_0^2 + x_1^2 - x_2^2) + u_1(x_0^2 - x_1^2) \in \Gamma(\mathbb{P}^1 \times \mathbb{P}^2, \mathcal{O}(1, 2)) \). The locus \( X' \) defined by \( s' = 0 \) in \( \mathbb{P}^1 \times \mathbb{P}^2 \) is smooth.

Let \( R \) be the locus over which the composition \( X' \rightarrow \mathbb{P}^1 \times \mathbb{P}^2 \rightarrow \mathbb{P}^1 \) is not smooth. By calculation, the locus \( R = \{ (0 : 1), (\pm 1 : 1) \} \). Let \( B = E \times \mathbb{P}^2 \). Let \( L = (\gamma, id)^* \mathcal{O}(1, 2) \), and let \( s = (\gamma, id)^*(s') \in \Gamma(B, L) \). Let \( X \) be the zero locus of \( s \) in \( B \). For the locus \( R \) does not intersect with the branch locus of \( \gamma : E \rightarrow \mathbb{P}^1 \), the surface \( X \) is smooth. So it is smooth, projective, and geometrically connected. By our construction, the surface \( X \) defined by the following equations:

\[
\begin{align*}
  (w_0w_2 + w_1^2 + 16u_0^2)(x_0^2 + x_1^2 - x_2^2) + (w_0w_1 + w_1w_2)(x_0^2 - x_1^2) &= 0 \\
  u_1^2w_2 &= u_0^3 - 16u_0^2
\end{align*}
\]

in \( \mathbb{P}^2 \times \mathbb{P}^2 \) with bi-homogeneous coordinates \((w_0 : w_1 : w_2) \times (x_0 : x_1 : x_2) \). For this surface \( X \), we have the following proposition.

**Proposition 5.2.1.** For \( K = \mathbb{Q} \) and \( L = \mathbb{Q}(i) \), the smooth, projective, and geometrically connected surface \( X \) has the following properties.

- The surface \( X \) has a \( K \)-rational point, and satisfies weak approximation with Brauer-Manin obstruction off \( \infty_K \).
- The surface \( X_L \) does not satisfy weak approximation with Brauer-Manin obstruction off \( T \) for any finite subset \( T \subset \Omega_L \).

**Proof.** This is the same as in the proof of Theorem 4.1.5. \( \square \)

### 5.3. An explicit unconditional example for Theorem 4.2.8

For \( K = \mathbb{Q} \) and \( L = \mathbb{Q}(i) \), in the subsection, we will construct a smooth, projective, and geometrically connected surface having properties of Theorem 4.2.8.

#### 5.3.1. Construction of a smooth, projective, and geometrically connected surface

We choose odd prime elements \((p_1, p_2, p_3, p_4, p_5) = (17, 13, 53, 41, 3, 13)\) as in Example 4.2.7. Then they satisfy all chosen conditions of Subsection 4.2.2. Let \( f(x_0, x_1; y_0, y_1) = x_0^2 - 17x_0^2(x_0^2 - 13x_0)(x_0^2 - 221x_0^2)(y_0^2 - 53y_0^2)(y_0^2 - 53y_0^2) \) and \( g(x_0, x_1; y_0, y_1) = x_0^2 - 41x_0^2(x_0^2 - 3x_0^2)(y_0^2 - 13y_0^2)(y_0^2 - 41y_0^2) \) be two bi-homogeneous polynomials, and let \( Z_f \) and \( Z_g \) be the zero loci of \( f \) and \( g \) respectively in \( \mathbb{P}^3 \times \mathbb{P}^4 \) with coordinates \((x_0 : x_1) \times (y_0 : y_1) \).

Let \((u_0 : u_1) \times (x_0 : x_1) \times (y_0 : y_1) \) be the coordinates of \( \mathbb{P}^3 \times \mathbb{P}^4 \times \mathbb{P}^4 \), and let \( s' = u_0g(x_0, x_1; y_0, y_1) + u_1f(x_0, x_1; y_0, y_1) \in \Gamma(\mathbb{P}^3 \times \mathbb{P}^4 \times \mathbb{P}^4, \mathcal{O}(1, 5, 6)) \). The locus \( X' \) defined by \( s' = 0 \) in \( \mathbb{P}^3 \times \mathbb{P}^4 \times \mathbb{P}^4 \) is smooth. Let \( R \) be the locus over which the composition \( X' \rightarrow \mathbb{P}^3 \times \mathbb{P}^4 \rightarrow \mathbb{P}^4 \) is not smooth. It is finite over \( \mathbb{Q} \). We can use computer to calculate this locus, and we give the calculation in Appendix 4. Let \( B = \mathbb{P}^3 \times \mathbb{P}^4 \). Let \( L = (\gamma, id)^* \mathcal{O}(1, 5, 6) \), and set \( s = (\gamma, id)^*(s') \in \Gamma(B, L) \). Let \( X \) be the zero locus of \( s \) in
For the locus $R$ does not intersect with the branch locus of $\gamma: E \to \mathbb{P}^1$, the surface $X$ is smooth. So it is smooth, projective, and geometrically connected. By our construction, the surface $X$ is defined by the following two equations:

\[
\begin{aligned}
(w_0w_2 + u^2 + 16v^2)(x_0^2 - 41x_2^2)(x_0^2 - 3x_2^2)(x_0^2 - 123x_2^2)(y_0^2 - 13y_2^2)(y_0^2 - 41y_2^2) \\
+ (w_0w_1 + w_1w_2)(x_0^2 - 17x_2^2)(x_0^2 - 13x_2^2)(x_0^2 - 221x_2^2)(y_0^2 - 53y_2^2)(y_0^2 - 53y_2^2) = 0
\end{aligned}
\]

in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$ with tri-homogeneous coordinates $(w_0: w_1: w_2) \times (x_0: x_1) \times (y_0: y_1)$. For this surface $X$, we have the following proposition.

**Proposition 5.3.1.** For $K = \mathbb{Q}$ and $L = \mathbb{Q}(i)$, the smooth, projective, and geometrically connected surface $X$ has the following properties.

1. The surface $X$ is a counterexample to the Hasse principle, and its failure of the Hasse principle is explained by the Brauer-Manin obstruction.
2. The surface $X_L$ is a counterexample to the Hasse principle, but its failure of the Hasse principle cannot be explained by the Brauer-Manin obstruction.

**Proof.** This is the same as in the proof of Theorem 4.2.5. \qed

### 6. Appendix

#### 6.1. The locus $R$ in Example 5.3

Let $f(x_0, x_1, y_0, y_1) = (x_0^2 - 17x_2^2)(x_0^2 - 13x_2^2)(x_0^2 - 221x_2^2)(y_0^2 - 53y_2^2)(y_0^2 - 53y_2^2)$ be two bi-homogeneous polynomials. Let $X'$ be the locus defined by $u_0g(x_0, x_1, y_0, y_1) + u_1f(x_0, x_1, y_0, y_1) = 0$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with tri-homogeneous coordinates $(u_0: u_1) \times (x_0: x_1) \times (y_0: y_1)$. Let $R$ be the locus over which the composition $X' \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is not smooth. We will calculate this finite locus $R$. Firstly, we have $(0, 1, 1) \in R$. Next, let $u_1 = 1$. We consider affine pieces of $X'$.

Let $x_1 = 1$ and $y_1 = 1$. Then this gives an affine piece of $X'$ by $u_0g(x_0, 1, y_0) + f(x_0, 1, y_0, 1) = 0$ in $\mathbb{A}^3$ with coordinates $(u_0, x_0, y_0)$. For fixed $u_0$, we use Jacobian criterion to calculate the singularity. Then $u_0$ satisfies the following equations:

\[
\begin{aligned}
&u_0^3 + 442306822591 \begin{pmatrix} 11646015109 \\ 1 \end{pmatrix} + 15378563320976329 \\
&+ 88370249860153892 \\
&869161492638775 \\
&12499774409799225 \\
&592598691902 \\
&8525128146319119 \\
&404350781347597245907 \\
&234844291675869605461668927183 \\
&831122761765787726757012609996 \\
&5904171562466693207916031862479997 \\
&19025741278675416402123 \\
&81813419206188414529371556601 \\
&828412992413205 \\
&3122232345643934184218989753 \\
&22 - 92208729473355199292709492986694814828151 \\
&21 - 309968174534574592092727212802015211111 \\
&- 152062202204223160072845465564491660 \\
&- 7560152438190879715566030 \\
&4521211648352228373337686696241283085972943456148354951 \\
&- 1018751538313517658709723518941564898960938437834467752826797 \\
&19297265473174250194890767224345181742315195962874646185875693719364006755599 \\
&1018380280214629922211445955289223377845067484220026175243595167082425045000843 \\
&13757496560944177757217452516351998992835052582655414327000000000 \\
&427658452585861930640480850861547578671020008806518095789276807250275062595000000 \\
&- 1018384407466650496811584713281020654717426730023336590000000000000000000000000000 \\
&18409117070639492878050444049217573238489960560626602019715960 \\
&2127278400690574770555491625797244574047936226057875626609 \\
&13 - 21500228521217615828072074888050689019178258354222885616485649260 \\
&46006909655363382846545126701417230670986108561209 \\
&12 - 437508858647929635519779083411909995188849417138884136935622 \\
&201768964316034773941854111559241922515770252794766919973243288138465321058758450491 \\
&158867397231611066020558544216069019965121419247628263531138925014574770587248 \\
&- 12702662435169319509850810699990132444285013530555584178517856715177727898 \\
&2666552467620696756012519371559552757996872839029976126400995554328966126919774442264 \\
&- 1926601689583545780380598061910474850062638187197265625 \\
\end{aligned}
\]
Let $x_1$ and $y_0 = 1$. Then this gives an affine piece of $X'$ by \( u_0 g(x_0; 1; 1, y_1) + f(x_0; 1, 1, y_1) = 0 \) in \( \mathbb{A}^3 \) with coordinates \((x_0, x_1, y_1)\). For fixed \( x_0 \), we use Jacobian criterion to calculate the singularity. Then \( u_0 \) satisfies the following equations:

Using computer to calculate, we have \( u_0 = -48841/15129 \) or satisfies one of the following three equations:

\[
u_0^4 + 15/4760599 - 13/4990209 - 3/24885438 + 312/85824 = 0,
\]

\[
u_0^6 - 79/5806806 - 79/5806806 - 143/29906 - 345/28714 + 41/85824 = 0,
\]

\[
u_0^8 - 11/606431 - 7230/28200 + 7230/28200 - 143/29906 - 831/12600 = 0.
\]

Then this gives an affine piece of $X'$ by \( u_0 g(x_0; 1, 1; y_1) + f(x_0; 1, 1; y_1) = 0 \) in \( \mathbb{A}^3 \) with coordinates \((x_0, x_1, y_1)\). For fixed \( x_0 \), we use Jacobian criterion to calculate the singularity. Then \( u_0 \) satisfies the following equations:

Using computer to calculate, we have \( u_0 = -48841/15129 \) or satisfies one of the following three equations:

\[
u_0^4 + 15/4760599 - 13/4990209 - 3/24885438 + 312/85824 = 0,
\]

\[
u_0^6 - 79/5806806 - 79/5806806 - 143/29906 - 345/28714 + 41/85824 = 0,
\]

\[
u_0^8 - 11/606431 - 7230/28200 + 7230/28200 - 143/29906 - 831/12600 = 0.
\]
Using computer to calculate, we have \( u_0 = 0 \), or \(-2809/533\) or satisfies one of the following three equations:

\[
u_0^0 + \frac{181400069}{7609} + \frac{53731423}{22747} + \frac{261386156}{62828} + \frac{417060978}{75809} + \frac{72447774097252}{22747} + \frac{7067503817}{62828} = 0,
\]

\[
u_0^1 + \frac{16289590}{7609} - \frac{35731423}{22747} + \frac{417060978}{75809} + \frac{29904292900}{22747} + \frac{7067503817}{62828} = 0,
\]

\[
u_0^2 + \frac{13281249912320}{22747} + \frac{3122325466941087428899043}{22747} + \frac{32298872948473515909279049289668418283151}{22747} = -\frac{2809}{533}.
\]

Let \( x_0 = 1 \) and \( y_0 = 1 \). This gives an affine piece of \( X' \) by \( u_0g(x_1, y_1) + f(x_1, y_1) = 0 \) in \( \mathbb{A}^3 \) with coordinates \((u_0, x_1, y_1)\). For fixed \( u_0 \), we use Jacobian criterion to calculate the singularity. Then \( u_0 \) satisfies the following equations:

\[
u_0^0 + \frac{181400069}{7609} + \frac{53731423}{22747} + \frac{261386156}{62828} + \frac{417060978}{75809} + \frac{72447774097252}{22747} + \frac{7067503817}{62828} = 0,
\]

\[
u_0^1 + \frac{16289590}{7609} - \frac{35731423}{22747} + \frac{417060978}{75809} + \frac{29904292900}{22747} + \frac{7067503817}{62828} = 0,
\]

\[
u_0^2 + \frac{13281249912320}{22747} + \frac{3122325466941087428899043}{22747} + \frac{32298872948473515909279049289668418283151}{22747} = -\frac{2809}{533}.
\]
Let \((u_0 : 1)\) be a point in \(R\), then the degree \([Q(u_0) : Q]\) \(\in \{1, 4, 6, 24\}.

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