ON THE UNIVERSAL ELLIPSITOMIC KZB CONNECTION

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Abstract. We construct a twisted version of the genus one universal Knizhnik–Zamolodchikov–Bernard (KZB) connection introduced by Calaque–Enriquez–Etingof, that we call the ellipsitomic KZB connection. This is a flat connection on a principal bundle over the moduli space of $\Gamma$-structured elliptic curves with marked points, where $\Gamma = \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$, and $M, N \geq 1$ are two integers. It restricts to a flat connection on $\Gamma$-twisted configuration spaces of points on elliptic curves, which can be used for proving the formality of some interesting subgroups of the pure braid group on the torus. We show that the universal ellipsitomic KZB connection realizes as the usual KZB connection associated with elliptic dynamical $r$-matrices with spectral parameter, and finally, also produces representations of cyclotomic Cherednik algebras.

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Introduction

In this paper, which fits in a series of works about universal Knizhnik–Zamolodchikov–Bernard (KZB) connections by different authors [6, 12], we focus on a twisted version of the genus 1 situation. In his seminal work [8], Drinfeld considers the monodromy representation of the universal Knizhnik–Zamolodchikov (KZ) equation which leads to the formality of the pure braid group (see reminder below) and the so-called theory of associators that makes the link between rich algebraic structures (such as braided monoidal categories) and the Grothendieck–Teichmüller group GT.

B. Enriquez generalizes in [9] Drinfeld’s work to the twisted (a-k-a trigonometric, or cyclotomic) situation and relates it to multiple polylogarithms at roots of unity. Namely, he uses the universal trigonometric KZ system to prove the formality of some subgroups of the pure braid
group on $\mathbb{C}^*$ and to emphasize relations between suitable algebraic structures (quasi-reflection algebras, or braided module categories) and analogues of the group $GT$.

The next step has been made by B. Enriquez, P. Etingof and the first author in [6], where a universal version of the elliptic KZB system (see [2]) is defined and used to:

- give a new proof (see [1] for the original one) of the filtered formality of the pure braid group on the torus,
- find a relation between the KZ associator and a generating series for iterated integrals of Eisenstein series (see also [11]),
- provide examples of elliptic structures on braided monoidal categories (see also [10]).

The main goal of the present paper is to introduce a twisted version of the universal elliptic KZB system, called the elliptomic KZB connection, and to derive from it the formality of some subgroups of the pure braid group on the torus. In a subsequent work, we will use it to emphasize a relation between generating series for values of multiple polylogarithms at roots of unity and values of elliptic multiple polylogarithms at torsion points.

Throughout the paper and unless otherwise specified, $k$ is a field of characteristic zero, $M, N$ are fixed positive integers, and $\Gamma := \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ and we write the composition of paths from left to right.

**Genus zero situation (rational KZ).** First observe that the holonomy Lie algebra of the configuration space

$$\text{Conf}(\mathbb{C}, n) := \{ \mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n | z_i \neq z_j \text{ if } i \neq j \}$$

of $n$ points on the complex line is isomorphic to the graded Lie $\mathbb{C}$-algebra $t_n$ generated by $t_{ij}$, $1 \leq i \neq j \leq n$, with relations

(S) $t_{ij} = t_{ji}$,
(L) $[t_{ij}, t_{kl}] = 0$ if $\# \{i, j, k, l\} = 4$,
(4T) $[t_{ij}, t_{ik} + t_{jk}] = 0$ if $\# \{i, j, k\} = 3$.

Then, on the one hand, denote by $\text{PB}_n$ the fundamental group of $\text{Conf}(\mathbb{C}, n)$, also known as the pure braid group with $n$ strands, and by $\text{pb}_n$ its Malcev Lie algebra (which is filtered by its lower central series, and complete). One can easily check that $\text{PB}_n$ is generated by elementary pure braids $P_{ij}$, $1 \leq i < j \leq n$, which satisfy (at least) the following relations:

(PB1) $(P_{ij}, P_{kl}) = 1$ if $\{i, j\}$ and $\{k, l\}$ are non crossing,
(PB2) $(P_{kj}P_{ij}P_{kl}^{-1}, P_{kl}) = 1$ if $i < k < j < l$,
(PB3) $(P_{ij}, P_{ik}P_{jk}) = (P_{jk}, P_{ij}P_{ik}) = (P_{ik}, P_{jk}P_{ij}) = 1$ if $i < j < k$.

We can depict the generator $P_{ij}$ in the following two equivalent ways:
Therefore one has a surjective morphism of graded Lie algebras \( p_n : t_n \to \text{gr}(\mathfrak{p}_n) \) sending \( t_{ij} \) to \( \sigma(\log(P_{ij})) \), \( i < j \) and \( \sigma : \mathfrak{p}_n \to \text{gr}(\mathfrak{p}_n) \) being the symbol map.

On the other hand, denote \( \exp(\hat{t}_n) \) the exponential group associated to the degree completion \( \hat{t}_n \) of \( t_n \). The universal KZ connection on the trivial \( \exp(\hat{t}_n) \)-principal bundle over \( \text{Conf}(\mathbb{C}, n) \) is then given by the holomorphic 1-form

\[
\omega_{KZ}^n := \sum_{1 \leq i < j \leq n} \frac{dz_i - dz_j}{z_i - z_j} t_{ij} \in \Omega^1(\text{Conf}(\mathbb{C}, n), t_n),
\]

which takes its values in \( t_n \). It is a fact that the connection associated to this 1-form is flat and descends to a flat connection on the moduli space \( \mathcal{M}_{0,n+1} \approx \text{Conf}(\mathbb{C}, n)/\text{Aff}(\mathbb{C}) \) of rational curves with \( n+1 \) marked points.

Firstly, the regularized holonomy of this connection along the real straight path from 0 to 1 in \( \mathcal{M}_{0,4} \approx \mathbb{P}^1 - \{0, 1, \infty\} \) gives an element \( \Phi_{KZ} \in \mathbb{C}(x_0, x_1) \), called the KZ associator, that is a generating series for values at 0 and 1 of multiple polylogarithms. Secondly, using the monodromy representation of the universal KZ connection, one obtains:

1. A morphism of filtered Lie algebras \( \mu_n : \mathfrak{p}_n \to \hat{t}_n \) such that \( \text{gr}(\mu_n) \circ p_n = \text{id} \). Hence one concludes that \( p_n \) and \( \mu_n \) are bijective. This proves that \( \mathfrak{p}_n \) is isomorphic to the degree completion of its associated graded, which is actually \( t_n \). This tells us that the group \( \text{PB}_n \) is 1-formal, meaning that its Malcev Lie algebra is isomorphic to the degree completion of a quadratic Lie algebra.

2. A system of relations (called Pentagon (\( P \)) and two Hexagons (\( H_\pm \))) satisfied by the KZ associator. Then, if \( k \) is a field of characteristic 0, one can define a set of \( k \)-associators \( \text{Ass}(k) \), for which the KZ associator will be a \( \mathbb{C} \)-point (showing at the same time that the set of such abstract \( \mathbb{C} \)-associators is indeed non-empty).

A twisted variant (trigonometric/cyclotomic KZ). Similarly, one can consider the configuration space

\[
\text{Conf}(\mathbb{C}^*, n) := \{ z = (z_1, \ldots, z_n) \in (\mathbb{C}^*)^n | z_i \neq z_j \text{ if } i \neq j \}
\]

of \( n \) points on \( \mathbb{C}^* \). Then \( \text{Conf}(\mathbb{C}^*, n) \approx \text{Conf}(\mathbb{C}, n+1)/\mathbb{C} \) and thus its fundamental group \( \text{PB}_n^1 \) is isomorphic to \( \text{PB}_{n+1} \). More generally, for any \( M \in \mathbb{Z} - \{0\} \) one can consider an \( M \)-twisted configuration space

\[
\text{Conf}(\mathbb{C}^*, n, M) := \{ z = (z_1, \ldots, z_n) \in (\mathbb{C}^*)^n | z_i^M \neq z_j^M \text{ for some } i \neq j \}.
\]
In [9] B. Enriquez proves, using the so-called universal trigonometric KZ connection, that one has an isomorphism $pb_n^M \to \exp(t_n^M)$, where $pb_n^M$ is the Malcev Lie algebra of the fundamental group $PB_n^M \subset PB_n^0$ of $Conf(C^n, n, M)$, and $t_n^M$ is the holonomy Lie algebra of $Conf(C^n, n, M)$. The monodromy of this connection along a suitable (non closed) path gives a universal pseudodotwist $\Psi^M_KZ \in \exp(t_n^M)$ that is a generating series for values of multiple polylogarithms at $M$th roots of unity, and satisfies relations with $\Phi_KZ$.

**Genus one situation (elliptic KZB).** The genus one universal Knizhnik–Zamolodchikov–Bernard (KZB) connection $\nabla_{1,n}^{KZB}$ was introduced in [6]. This is a flat connection over the moduli space of elliptic curves with $n$ marked points $M_{1,n}$, which was independently discovered by Levin–Racinet [24] in the specific cases $n = 1, 2$. It restricts to a flat connection over the configuration space

$$Conf(\mathbb{T}, n) := \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n | z_i - z_j \notin \Lambda, \text{ if } i \neq j\} / \Lambda^\circ_+$$

of $n$ points on an (uniformized) elliptic curve $E_\tau := \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z})$, for $\tau \in \mathfrak{h}$ and $\Lambda_\tau = \mathbb{Z} + \tau \mathbb{Z}$. More precisely, this connection is defined on a $G$-principal bundle over $M_{1,n}$ where the Lie algebra associated to $G$ has as components:

1. a Lie algebra $t_{1,n}$ related to $Conf(\mathbb{T}, n)$, somehow controlling the variations of the marked points: it has generators $x_i, y_i$, for $i = 1, \ldots, n$, corresponding to moving $z_i$ along the topological cycles generating $H_1(E_\tau)$;

2. a Lie algebra $\mathfrak{d}$ with as components the Lie algebra $\mathfrak{sl}_2$ with standard generators $e, f, h$ and a Lie algebra $\mathfrak{d}_+ := \text{Lie}(\{\delta_{2m} | m \geq 1\})$ such that each $\delta_{2m}$ is a highest weight element for $\mathfrak{sl}_2$. The Lie algebra $\mathfrak{d}$ somehow controls the variation of the curve in $M_{1,n}$ and is closely related to the one defined in [27].

Now, the connection $\nabla_{1,n}^{KZB}$ can be locally expressed as

$$\nabla_{1,n}^{KZB} := d - \Delta(z|\tau)d\tau - \sum_i K_i(z|\tau)dz_i$$

where

1. the term $K_i(-|\tau) : \mathbb{C}^n \to t_{1,n}$ is meromorphic on $\mathbb{C}^n$, having only simple poles on

$$\text{Diag}_{n,\tau} := \bigcup_{i \neq j} \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n | z_i - z_j \notin \Lambda_\tau\}.$$

It is constructed out of a function

$$k(x, z|\tau) := \frac{\theta(z + x|\tau)}{\theta(z|\tau)\theta(x|\tau)} - \frac{1}{x}.$$

This relates directly the connection $\nabla_{1,n}^{KZB}$ with Zagier’s work [29] on Jacobi forms (see Weil’s book [28]) and to Brown and Levin’s work [5].

2. the term $\Delta(z|\tau)$ is a meromorphic function $\mathbb{C}^n \times \mathfrak{h} \to \text{Lie}(G)$, with only simple poles on $\text{Diag}_n := \{(z, \tau) \in \mathbb{C}^n \times \mathfrak{h} | z \in \text{Diag}_{n,\tau}\}$. The coefficients of $\delta_{2m}$ in $\Delta(z|\tau)$ are Eisenstein series.

We also refer to Hain’s survey [22] and references therein for the Hodge theoretic and motivic aspects of the story.
Then, one can construct a holomorphic map sending each \( \tau \in \mathfrak{h} \) to a couple \( e(\tau) := (A(\tau), B(\tau)) \) where \( A(\tau) \) (resp. \( B(\tau) \)) is the regularized holonomy of the universal elliptic KZB connection along the straight path from 0 to 1 (resp. from 0 to \( \tau \)) in the once punctured elliptic curve \((C - \Lambda_\tau)/\Lambda_\tau \cong \text{Conf}(E_\tau, 2)/E_\tau\). B. Enriquez developed in [10] the general theory of elliptic associators, whose scheme is denoted Ell and for which the couple \( e(\tau) \) is an example of a \( \mathbb{C} \)-point. Some of the main features of the so-called elliptic KZB associators \( e(\tau) \) are the following:

- They satisfy algebraic and modularity relations.
- They satisfy a differential equation in the variable \( \tau \) expressed only in terms of iterated integrals of Eisenstein series, which will be called iterated Eisenstein in tegrals.
- When taking \( \tau \) to \( i\infty \) (which consists in computing the constant term of the \( q \)-expansion of the series \( A(\tau) \) and \( B(\tau) \), where \( q = e^{2\pi i \tau} \)), they can be expressed only in terms of the KZ associator \( \Phi_{KZ} \).
- They provide isomorphisms between the Malcev Lie algebra of the fundamental group \( \text{PB}_{1,n} \) of \( \text{Conf}(\mathbb{T}, n) \) and the degree completion of its associated Lie algebra \( t_{1,n} \).

Observe that, contrary to what happens in genus 0, \( \text{PB}_{1,n} \) (also known as the pure elliptic braid group) is not 1-formal (as \( t_{1,n} \) is not quadratic), but only filtered-formal according to the terminology of [25].

**Ellipsitomic KZB.** As we wrote above, the purpose of the present work is to define a twisted version of the genus one KZB connection introduced in [6]. This is a flat connection on a principal bundle over the moduli space of elliptic curves with a \( \Gamma \)-structure and \( n \) marked points. It restricts to a flat connection on the so-called \( \Gamma \)-twisted configuration space of points on an elliptic curve, which can be used for proving the filtered-formality of some interesting subgroups of the pure braid group on the torus.

In a subsequent work, we will define ellipsitomic KZB associators as renormalized holonomies along certain paths on a once punctured elliptic curve with a \( \Gamma \)-structure, and exhibit a relation between ellipsitomic KZB associators, the KZ associator [8] and the cyclotomic KZ associator [9]. Moreover, ellipsitomic associators can be regarded as a generating series for iterated Eisenstein integrals whose coefficients are elliptic multiple zeta values at torsion points. In the case \( M = N \) these coefficients are related to Goncharov’s work [19], and also to the recent work [4] of Broedel–Matthes–Richter–Schlotterer.

We finally prove that the universal KZB connection realizes as the usual KZB connection associated to elliptic dynamical \( \tau \)-matrices with spectral parameter, that should be compared with [15, 17].

It is worth mentioning the recent work [26], where Toledano-Laredo and Yang define a similar KZB connection. More precisely, they construct a flat KZB connection on moduli spaces of elliptic curves associated with crystallographic root systems. The type \( A \) case coincides with the universal elliptic KZB connection defined in [6], and we suspect that the type \( B \) case coincides with the connection of the present paper for \( M = N = 2 \). It is interesting to point out that a common generalization of their work and ours (for \( M = N \)) could be obtained by constructing a universal KZB connection associated with arbitrary complex reflection groups.
Plan of the paper. The paper is organized as follows:

- In Section 1, we introduce $\Gamma$-twisted configuration spaces on an elliptic curve and define the universal ellipsitomic KZB connection on them. It takes values in a the Lie algebra $t_{\Gamma,1,n}$ of infinitesimal ellipsitomic (pure) braids, that we also define.

- As in [6], the connection extends from the configuration space to the moduli space $\mathcal{M}_{1,\{n\}}^\Gamma$ of elliptic curves with a $\Gamma$-level structure and unordered marked points. This is proven in Section 3 using some technical definitions introduced in Section 2, involving derivations of the Lie algebra $t_{1,n}^\Gamma$ related to the twisted configuration space in genus 1. As in the untwisted case, the results of this section also apply to the “unordered marked points” situation as well.

- In Section 4, we provide a notion of realizations for the Lie algebras previously introduced, and show that the universal ellipsitomic KZB connection realizes to a flat connection intimately related to elliptic dynamical $r$-matrices with spectral parameter.

- In Section 5, we derive from the monodromy representation the filtered-formality of the fundamental group of the twisted configuration space of the torus, which is a subgroup of $\text{PB}_{1,n}$. As in the cyclotomic case, it extends to a relative filtered-formality result for the map $B_{1,n} \to \Gamma^n \rtimes S_n$.

- Finally, in Section 6, we construct a homomorphism from the Lie algebra $t_{\Gamma,1,n} \rtimes \delta^\Gamma$ to the twisted Cherednik algebra $H_{\Gamma}^n(k)$. This allows us to consider the twisted elliptic KZB connection with values in representations of the twisted Cherednik algebra. This study shall be closely related to the recent paper [3].

Acknowledgements. Both authors are grateful to Benjamin Enriquez, Richard Hain and Pierre Lochak for numerous conversations and suggestions. We also thank Nils Matthes for discussions about twisted elliptic MZVs. The first author acknowledges the financial support of the ANR project SAT and of the Institut Universitaire de France. This paper is part of the second author’s doctoral thesis [20] at Sorbonne Université, and part of this work has been done while the second author was visiting the Institut Montpelliéрайn Alexander Grothendieck, thanks to the financial support of the Institut Universitaire de France. The second author warmly thanks the Max-Planck Institute of Mathematics in Bonn, for its hospitality and excellent working conditions.
1. Bundles with flat connections on $\Gamma$-twisted configuration spaces

1.1. The Lie algebra of infinitesimal ellipsitomic braids. In this paragraph, $\Gamma$ can be replaced by any finite abelian group (with the additive notation).

For any finite set $I$ we define $\mathfrak{t}^1_{\mathcal{I}}(k)$ to be the bigraded $k$-Lie algebra with generators $x_i$ ($i \in I$) in degree $(1,0)$, $y_i$ ($i \in I$) in degree $(0,1)$, and $\ell^\alpha_{ij}$ ($\alpha \in \Gamma$, $i \neq j$) in degree $(1,1)$, and relations

\begin{align*}
(tS_{\mathcal{I}\ell}1) & \quad \ell^\alpha_{ij} = \ell^\alpha_{ji}, \\
(tS_{\mathcal{I}\ell}2) & \quad [x_i, y_j] = [x_j, y_i] = \sum_{\alpha \in \Gamma} \ell^\alpha_{ij}, \\
(tN_{\mathcal{I}\ell}) & \quad [x_i, x_j] = [y_i, y_j] = 0, \\
(tT_{\mathcal{I}\ell}) & \quad [x_i, y_i] = - \sum_{j \neq i} \sum_{\alpha \in \Gamma} \ell^\alpha_{ij}, \\
(tL_{\mathcal{I}\ell}1) & \quad [\ell^\alpha_{ij}, \ell^\beta_{kl}] = 0, \\
(tL_{\mathcal{I}\ell}2) & \quad [x_i, \ell^\alpha_{jk}] = [y_i, \ell^\alpha_{jk}] = 0, \\
(t4T_{\mathcal{I}\ell}1) & \quad [\ell^\alpha_{ij}, \ell^{\alpha+\beta}_{jk} + \ell^\beta_{jk}] = 0, \\
(t4T_{\mathcal{I}\ell}2) & \quad [x_i + x_j, \ell^\alpha_{ij}] = [y_i + y_j, \ell^\alpha_{ij}] = 0,
\end{align*}

where $1 \leq i, j, k, l \leq n$ are pairwise distinct and $\alpha, \beta \in \Gamma$. We will call $\mathfrak{t}^1_{\mathcal{I}f}(k)$ the $k$-Lie algebra of infinitesimal ellipsitomic braids. Observe that $\sum x_i$ and $\sum y_i$ are central in $\mathfrak{t}^1_{\mathcal{I}f}$. Then we denote by $\mathfrak{t}^1_{\mathcal{I}f}(k)$ the quotient of $\mathfrak{t}^1_{\mathcal{I}f}(k)$ by $\sum x_i$ and $\sum y_i$, and the natural morphism $\mathfrak{t}^1_{\mathcal{I}f}(k) \to \mathfrak{t}^1_{\mathcal{I}f}(k)$: $u \mapsto \bar{u}$.

There is an alternative presentation of $\mathfrak{t}^1_{\mathcal{I}f}(k)$ and $\mathfrak{t}^1_{\mathcal{I}f}(k)$:

**Lemma 1.1.** The Lie $k$-algebra $\mathfrak{t}^1_{\mathcal{I}f}(k)$ (resp. $\mathfrak{t}^1_{\mathcal{I}f}(k)$) can equivalently be presented with the same generators, and the following relations: $(tS_{\mathcal{I}\ell}1)$, $(tS_{\mathcal{I}\ell}2)$, $(tN_{\mathcal{I}\ell})$, $(tL_{\mathcal{I}\ell}1)$, $(tL_{\mathcal{I}\ell}2)$, $(t4T_{\mathcal{I}\ell}1)$, and, for every $i \in I$,

$$[\sum_j x_j, y_i] = [\sum_j y_j, x_i] = 0$$

(resp. $\sum x_j = \sum y_j = 0$).

**Proof.** If $x_i, y_i$ and $\ell^\alpha_{ij}$ satisfy the initial relations, then

$$[\sum_j x_j, y_i] = [x_i, y_i] + [\sum_j x_j, y_i] = - \sum_{j \neq i} \sum_{\alpha \in \Gamma} \ell^\alpha_{ij} + \sum_{j \neq i} \sum_{\alpha \in \Gamma} \ell^\alpha_{ij} = 0.$$

Now, if $x_i, y_i$ and $\ell^\alpha_{ij}$ satisfy the above relations, then relations $[\sum_j x_j, y_i] = 0$ and $[x_j, y_i] = \sum_{\alpha \in \Gamma} \ell^\alpha_{ij}$, for $i \neq j$, imply that $[x_i, y_i] = - \sum_{j \neq i} \sum_{\alpha \in \Gamma} \ell^\alpha_{ij}$. Now, relations $[\sum x_k, y_j] = 0$ and $[\sum x_k, x_i] = 0$ imply that $[\sum x_k, \sum_{\alpha \in \Gamma} \ell^\alpha_{ij}] = 0$. Thus, as $[x_i, \ell^\alpha_{jk}] = 0$ if $\text{card}\{i,j,k\} = 3$, we obtain relation $[x_i + x_j, \ell^\alpha_{ij}] = 0$, for $i \neq j$. In the same way we obtain $[y_i + y_j, \ell^\alpha_{ij}] = 0$, for $i \neq j$. $\square$

There is an action $\Gamma^T \to \text{Aut}(\mathfrak{t}^1_{\mathcal{I}f}(k))$ defined as follows:
• it leaves \(x_i\)'s and \(y_i\)'s invariant.
• for every \(i \in I\) and every \(\alpha \in \Gamma\), \(\alpha_i\) leaves \(t_{kl}^\beta\)'s invariant if \(k, l \neq i\), and sends \(t_{ij}^\beta\) to \(t_{ij}^{\beta+\alpha}\).

Here \(\alpha_i\) denotes the element of \(\Gamma^I\) whose only nonzero component is the \(i\)th one and is \(\alpha\).

This action descends to an action on \(\bar{t}^I_{\Gamma,I}(k)\).

**Proposition 1.2.** For any group morphism \(\rho : \Gamma_1 \to \Gamma_2\) we have a comparison morphism \(\phi : t^I_{\Gamma_1}(k) \to t^I_{\Gamma_2}(k)\) defined by \(x_i \mapsto x_i,\ y_i \mapsto y_i,\) and

\[
t_{ij}^{\alpha} \mapsto \frac{1}{\# \ker(\rho)} \sum_{\beta \in \ker(\rho)} \# \ker(\rho_j) t_{ij}^{\alpha}.
\]

**Proof.** Let us prove that relation \([x_i, y_j] = \sum_{\alpha \in \Gamma_1} t_{ij}^{\alpha}\) holds for every \(i \neq j\), so is preserved by \(\phi\). On the one hand \(\phi(x_i), \phi(y_j) = \sum_{\alpha \in \Gamma_1} t_{ij}^{\alpha}\). On the other hand

\[
\phi([x_i, y_j]) = \sum_{\alpha \in \Gamma_1} \phi(t_{ij}^{\alpha}) = \sum_{\alpha \in \Gamma_1} \frac{1}{\# \ker(\rho)} \sum_{\beta \in \ker(\rho)} t_{ij}^{\alpha} = \sum_{\alpha \in \Gamma_2} t_{ij}^{\alpha}.
\]

The last equality holds because \(\rho(\alpha)\) is in the image of \(\rho\) and \(\beta\) is not. The fact that the remaining relations are preserved is immediate. \(\square\)

When \(\rho\) is not surjective it depends on the choice of a section \(\coker(\rho) \to \Gamma_2\). Comparison morphisms commute with insertion-coproduct morphisms. Moreover, both are bigraded and pass to the quotient by \(\sum_i x_i, \sum_i y_i\).

When \(k = \mathbb{C}\) and \(I = \{1, \ldots, n\}\), we write \(t^I_{n,n} := t^I_{\Gamma,I}(\mathbb{C})\) and \(t^I_{1,n} := t^I_{\Gamma,I}(\mathbb{C})\).

### 1.2. Principal bundles over \(\Gamma\)-twisted configuration spaces.

Let \(E\) be an elliptic curve over \(\mathbb{C}\) and consider the connected unramified \(\Gamma\)-covering \(p : \tilde{E} \to E\) corresponding to the canonical surjective group morphism \(\rho : \pi_1(E) \cong \mathbb{Z}^2 \to \Gamma\) where \(\pi_1(E) \cong \mathbb{Z}^2\) is the natural choice of such an isomorphism. Let us then define the twisted configuration space

\[
\text{Conf}(\mathbb{T}, n, \Gamma) := \{z = (z_1, \ldots, z_n) \in \tilde{E}^n | p(z_i) = p(z_j)\text{ if }i \neq j\},
\]

and \(\text{C}(\mathbb{T}, n, \Gamma) := \text{Conf}(\mathbb{E}, n, \Gamma)/\tilde{E}\) its reduced version. Notice that \(\text{C}(\mathbb{E}, n, \Gamma)\) is just the inverse image of \(\text{C}(\mathbb{E}, n)\) under the surjection \(p^n : \tilde{E}^n \to \mathbb{E}^n\).

Let us fix a uniformization \(\tilde{E} \cong E_{\tau}\), where \(\tau \in \mathbb{H}\): \(E_{\tau} = \mathbb{C}/\Lambda_{\tau}\), with \(\Lambda_{\tau} = \mathbb{Z} + \tau\mathbb{Z}\). Then \(E = E_{\tau,\Gamma}\), where \(E_{\tau,\Gamma} = \mathbb{C}/\Lambda_{\tau,\Gamma}\) and \(\Lambda_{\tau,\Gamma} := (1/M)\mathbb{Z} \times (\tau/N)\mathbb{Z}\). Therefore

\[
\text{Conf}(\mathbb{E}, n, \Gamma) \cong (\mathbb{C}^n - \text{Diag}_{\tau,n,\Gamma}/\Lambda_{\tau}^n),
\]

where

\[
\text{Diag}_{\tau,n,\Gamma} := \{(z_1, \ldots, z_n) \in \mathbb{C}^n | z_{ij} = z_i - z_j \in \Lambda_{\tau,\Gamma} \text{ for some } i \neq j\}.
\]

We now define a principal \(\exp(t^I_{\Gamma,n})\)-bundle \(\mathcal{P}_{\tau,n,\Gamma}\) over \(\text{Conf}(\mathbb{E}, n, \Gamma)\) as the quotient

\[
\left((\mathbb{C}^n - \text{Diag}_{\tau,n,\Gamma}) \times \exp(t^I_{\Gamma,n})\right)/\Lambda_{\tau}^n.
\]

In other words, it is the restriction on \(\text{Conf}(\mathbb{E}, n, \Gamma)\) of the bundle over \(\mathbb{C}^n/\Lambda_{\tau}^n\) for which a section on \(U \subset \mathbb{C}^n/\Lambda_{\tau}^n\) is a regular map \(f : \pi^{-1}(U) \to \exp(t^I_{\Gamma,n})\) such that

• \(f(z + \delta_i) = f(z),\)
• \( f(z + \tau \delta_i) = e^{-2\pi i z_i} f(z) \).

Here \( \pi : \mathbb{C}^n \rightarrow \mathbb{C}^n/\Lambda_n^\ast \) is the canonical projection and \( \delta_i \) is the \( i \)th vector of the canonical basis of \( \mathbb{C}^n \).

Since the \( e^{-2\pi i z_i} \)'s in \( \exp(\hat{T}_{1,n}) \) pairwise commute and their product is 1, then the image of \( \mathcal{P}_{\tau,n,\Gamma} \) under the natural morphism \( \exp(\hat{T}_{1,n}) \rightarrow \exp(\hat{T}_{1,n}) \) is the pull-back of a principal \( \exp(\hat{T}_{1,n}) \)-bundle \( \mathcal{P}_{\tau,n,\Gamma} \) over \( C(E,n,\Gamma) \).

### 1.3. Variations

The first variation we are interested in concerns unordered configuration spaces. The symmetric group \( \mathfrak{S}_n \) acts freely by automorphisms of \( \text{Conf}(E,n,\Gamma) \) by

\[
\sigma \ast (z_1, \ldots, z_n) := (z_{\sigma^{-1}(1)}, \ldots, z_{\sigma^{-1}(n)}).
\]

This descends to a free action of \( \mathfrak{S}_n \) on \( C(E,n,\Gamma) \). We then defined the unordered twisted configuration spaces

\[
\text{Conf}(E,[n],\Gamma) := \text{Conf}(E,n,\Gamma)/\mathfrak{S}_n \text{ and } C(E,[n],\Gamma) := C(E,n,\Gamma)/\mathfrak{S}_n.
\]

The symmetric group \( \mathfrak{S}_n \) also obviously acts on the Lie algebra \( \hat{T}_{1,n} \). One can then define, keeping the notation of the previous paragraph, a principal \( \exp(\hat{T}_{1,n}) \times \mathfrak{S}_n \)-bundle \( \mathcal{P}_{\tau,[n],\Gamma} \) over \( \text{Conf}(E,[n],\Gamma) \): it is the restriction on \( \text{Conf}(E,[n],\Gamma) \) of the bundle over \( \mathbb{C}^n/\Lambda_n^\ast \times \mathfrak{S}_n \) for which a section on \( U \subset \mathbb{C}^n/\Lambda_n^\ast \times \mathfrak{S}_n \) is a regular map \( f : \pi^{-1}(U) \rightarrow \exp(\hat{T}_{1,n}) \times \mathfrak{S}_n \) such that

- \( f(z + \delta_i) = f(z) \),
- \( f(z + \tau \delta_i) = e^{-2\pi i z_i} f(z) \),
- \( f(\sigma \ast z) = \sigma f(z) \).

In more compact form:

\[
\mathcal{P}_{\tau,[n],\Gamma} = \left( \left( \mathbb{C}^n - \text{Diag}_{\tau,n,\Gamma} \right) \times \exp(\hat{T}_{1,n}) \times \mathfrak{S}_n \right)/\left( \Lambda_n^\ast \times \mathfrak{S}_n \right).
\]

**Remark 1.3.** As before, \( \mathcal{P}_{\tau,[n],\Gamma} \) descends to a principal \( \exp(\hat{T}_{1,n}) \times \mathfrak{S}_n \)-bundle \( \mathcal{P}_{\tau,[n],\Gamma} \) over the reduced unordered twisted configuration space \( C(E,[n],\Gamma) \).

The second variation concerns ordinary configuration spaces of the base \( E = E_{\tau,\Gamma} \) of the covering map \( E_{\tau,\Gamma} \rightarrow E_{\tau,\Gamma} \).

Recall from §1.1 that the group \( \Gamma\ast \) acts on \( \hat{T}_{1,n} \). Hence one has a principal \( \exp(\hat{T}_{1,n}) \times \Gamma\ast \)-bundle

\[
\mathcal{P}_{(\tau,\Gamma),n} := \left( \left( \mathbb{C}^n - \text{Diag}_{\tau,n,\Gamma} \right) \times \exp(\hat{T}_{1,n}) \times \Gamma\ast \right)/\Lambda_n^\ast,\Gamma.
\]

over \( \text{Conf}(E,n) \approx \left( \mathbb{C}^n - \text{Diag}_{\tau,n,\Gamma} \right)/\Lambda_n^\ast,\Gamma \). Here the action of \( \Lambda_n^\ast,\Gamma \) on \( \hat{T}_{1,n} \) is given by the morphism

\[
\Lambda_n \rightarrow \Gamma, \quad a + b\tau \mapsto (\hat{a},\hat{b}).
\]

**Remark 1.4.** In a similar way as before, the above bundle obviously descends to a principal \( \exp(\hat{T}_{1,n}) \times (\Gamma\ast/\Gamma) \)-bundle \( \mathcal{P}_{(\tau,\Gamma),n} \) over the reduced ordinary configuration space \( C(E,n) \).

In concrete terms, a section over \( U \subset \mathbb{C}^n/\Lambda_{\tau,\Gamma} \) of \( \mathcal{P}_{(\tau,\Gamma),n} \) is a regular map \( f : \pi^{-1}(U) \rightarrow \exp(\hat{T}_{1,n}) \times \Gamma\ast \) such that

- \( f(z + \delta_i/M) = (1,0) f(z) \),
- \( f(z + \tau \delta_i/N) = (0,1) e^{-2\pi i z_i} f(z) \).
Remark 1.5. We leave to the reader the task of combining the two variations.

1.4. Flat connections on $\mathcal{P}_{\tau,n,\Gamma}$ and its variants. A flat connection $\nabla_{\tau,n,\Gamma}$ on $\mathcal{P}_{\tau,n,\Gamma}$ is the same as an equivariant flat connection on the trivial $\exp(\mathfrak{h}_{\tau,n})$-bundle over $\mathbb{C}^n = \text{Diag}_{\tau,n,\Gamma}$, i.e., a connection of the form

$$\nabla_{\tau,n,\Gamma} := d - \sum_{i=1}^n K_i(z|\tau)dz_i,$$

where $K_i(-|\tau) : \mathbb{C}^n \to \mathfrak{h}_{\tau,n}$ are meromorphic with only poles at $\text{Diag}_{\tau,n,\Gamma}$, and such that for any $i,j$:

- (a) $K_i(z + \delta_j|\tau) = K_i(z|\tau)$,
- (b) $K_i(z + \tau\delta_j|\tau) = e^{-2\pi i \text{ad}(z_j)}K_i(z|\tau)$,
- (c) $[\partial_i - K_i(z|\tau), \partial_j - K_j(z|\tau)] = 0$.

Moreover, the image of $\nabla_{\tau,n,\Gamma}$ under $\mathcal{I}_{\Gamma,\tau} \to \mathcal{I}_{\Gamma,\tau}^n$ is the pull-back of a (necessarily flat) connection $\nabla_{\tau,n,\Gamma}$ on $\mathcal{P}_{\tau,n,\Gamma}$ if and only if:

- (d) $K_i(z|\tau) = \hat{K}_i(z + u \sum_{j} \delta_j|\tau)$ for any $u \in \mathbb{C}$ and $\sum_{j} K_i(z|\tau) = 0$.

Similarly, the image of $\nabla_{\tau,n,\Gamma}$ under $\mathcal{I}_{\Gamma,\tau}^n \to \mathcal{I}_{\Gamma,\tau}^n \times \mathfrak{g}_{n}$ is the pull-back of a (necessarily flat) connection $\nabla_{\tau,n,\Gamma}$ on $\mathcal{P}_{\tau,n,\Gamma}$ if and only if:

- (e) $K_i(z + \delta_j/|\tau) = (1,0)_{\tau} \cdot K_i(z|\tau)$,
- (f) $K_i(z + \tau\delta_j/|\tau) = (0,1)_{\tau} \cdot e^{\frac{\pi i}{2} \text{ad}(z_j)}K_i(z|\tau)$.

Remark 1.6. Observe that (e) implies (a), and that (f) implies (b).

Finally, the image of $\nabla_{\tau,n,\Gamma}$ under $\mathcal{I}_{\Gamma,\tau}^n \to \mathcal{I}_{\Gamma,\tau}^n \times \mathfrak{g}_n$ is the pull-back of a (necessarily flat) connection $\nabla_{\tau,[n],\Gamma}$ on $\mathcal{P}_{\tau,[n],\Gamma}$ if and only if:

- (g) $K_i((ij) * z) = (ij) \cdot K_i(z)$.

1.5. Constructing the connection. We now construct a connection satisfying properties (d) to (g). Let us take the same conventions for theta functions as in [6]. This is the unique holomorphic function $\mathbb{C} \times \mathfrak{h} \to \mathbb{C}$, $(z, \tau) \mapsto \theta(z|\tau)$, such that

- $\{ z|\theta(z|\tau) = 0 \} = \Lambda_{\tau}$,
- $\theta(z + 1|\tau) = -\theta(z|\tau)$,
- $\theta(z + \tau|\tau) = e^{-\pi i \tau}e^{-2\pi i \theta(z|\tau)}$,
- $\partial_{\tau}\theta(0|\tau) = 1$.

In particular, $\theta(z|\tau + 1) = \theta(z|\tau)$, while $\theta(-z|\tau) - 1/\tau = -(1/\tau)e^{(\pi i/\tau)^2}\theta(z|\tau)$. If $\eta(\tau) = q^{1/24}\prod_{n=1}(-1 - q^n)$ where $q = e^{2\pi i \tau}$, and if we set $\theta(z|\tau) := \eta(\tau)^3\theta(z|\tau)$, then $\partial_{\tau}\theta = (1/4\pi i)\partial_{z}^2\theta$.

Observe that for any $\tilde{\alpha} = (a_0, \tilde{\alpha}) \in \Gamma_{\tau}$, the term $e^{-2\pi i ax}(\theta(z - \tilde{\alpha} + x))/((\theta(z - \tilde{\alpha})\theta(x)))$ only depends on the class $\alpha = (a_0, \tilde{\alpha}) \in \Gamma$ of $\tilde{\alpha}$ mod $\Gamma$. Then we set

$$k_{\alpha}(z, \tau) := \frac{\theta(z - \tilde{\alpha} + x|\tau)}{\theta(z - \tilde{\alpha}|\tau)} - \frac{1}{x} = e^{-2\pi i ax}k(x, z - \tilde{\alpha}|\tau) + e^{-2\pi i ax} - \frac{1}{x},$$

where $k(x, z|\tau) := \frac{\theta(x + z)}{\theta(x)} - \frac{1}{x}$ (as in [6]), and

$$K_{ij}(z|\tau) = \sum_{\alpha \in \Gamma} k_{\alpha}(ad x_1, z|\tau)(l_{ij}^\alpha), \quad K_i(z|\tau) := -y_i + \sum_{j \neq i} K_{ij}(z_{ij}|\tau).$$
In the rest of the section we fix $\tau \in \mathcal{H}$ and drop it from the notation. Recall from [6] that $k(x, z \pm 1) = k(x, z)$ and

$$k(x, z \pm \tau) = e^{\frac{2\pi i x}{\tau} k(x, z)} + \frac{e^{\frac{2\pi i x}{\tau} - 1}}{x}.$$  

We then define the universal ellipsitomic KZB connection on $\mathcal{P}_{\tau, n, \Gamma}$ by

$$\nabla^{\text{KZB}} \colon d = \sum_{i=1}^{n} K_i(\alpha_i) d\alpha_i.$$  

**Proposition 1.7.** The $K_{ij}(z)$’s have the following equivariance properties:

(1) $K_{ij}(z + \frac{1}{M}) = (1, 0) \cdot (K_{ij}(z))$,

(2) $K_{ij}(z + \frac{\tau}{N}) = (0, -1) \cdot e^{-\frac{2\pi i}{N} \text{ad} x_j} \cdot (K_{ij}(z)) + (0, -1) \cdot (\sum_{\alpha \in \Gamma} e^{-\frac{2\pi i}{N} \text{ad} x_j - 1}(t_{ij}^\alpha)).$

**Proof.** Let us choose representatives $0 \leq u \leq M - 1$ and $0 \leq v \leq N - 1$ so that $\tilde{\alpha} = \frac{u}{M} + \frac{v}{N}$. The first equation comes from a straightforward verification. Let us show the second relation. On the one hand, we have

$$K_{ij}(z + \frac{\tau}{N}) = \sum_{\alpha \in \Gamma} k_{ij} \left( \text{ad} x_i, z + \frac{\tau}{N} \right) (t_{ij}^\alpha)$$  

$$= \left( \sum_{\alpha \in \Gamma} e^{-\frac{2\pi i \text{ad} x_i}{N} k(x, z + \frac{\tau}{N} - \tilde{\alpha}) + \frac{e^{-\frac{2\pi i \text{ad} x_i}{N} - 1}}{\text{ad} x_i} \right) (t_{ij}^\alpha)$$

$$= \left( \sum_{\alpha \in \Gamma} e^{-\frac{2\pi i \text{ad} x_i}{N} (v+1)} k(\text{ad} x_i, z - \tilde{\alpha}) + \frac{e^{-\frac{2\pi i \text{ad} x_i}{N} - 1}}{\text{ad} x_i} \right) (t_{ij}^\alpha)$$

$$= (0, -1) \cdot \left( \sum_{\alpha \in \Gamma} e^{-\frac{2\pi i \text{ad} x_i}{N} (v+1)} k(\text{ad} x_i, z - \tilde{\alpha}) + \frac{e^{-\frac{2\pi i \text{ad} x_i}{N} - 1}}{\text{ad} x_i} \right) (t_{ij}^\alpha).$$

On the other hand,

$$e^{-\frac{2\pi i}{N} \text{ad} x_j} K_{ij}(z) = e^{-\frac{2\pi i}{N} \text{ad} x_j} \left( \sum_{\alpha \in \Gamma} k_{ij} (\text{ad} x_i, z) \right) (t_{ij}^\alpha)$$

$$= e^{-\frac{2\pi i}{N} \text{ad} x_j} \left( \sum_{\alpha \in \Gamma} e^{-\frac{2\pi i \text{ad} x_i}{N} k(x, z - \tilde{\alpha}) + \frac{e^{-\frac{2\pi i \text{ad} x_i}{N} - 1}}{\text{ad} x_i} \right) (t_{ij}^\alpha)$$

$$= \left( \sum_{\alpha \in \Gamma} e^{-\frac{2\pi i \text{ad} x_i}{N} (v+1)} k(\text{ad} x_i, z - \tilde{\alpha}) + \frac{e^{-\frac{2\pi i \text{ad} x_i}{N} - 1}}{\text{ad} x_i} \right) (t_{ij}^\alpha),$$

so

$$\sum_{\alpha \in \Gamma} e^{-\frac{2\pi i (v+1)}{N} \text{ad} x_i} k(\text{ad} x_i, z - \tilde{\alpha})(t_{ij}^\alpha) = e^{-\frac{2\pi i}{N} \text{ad} x_j} K_{ij}(z)$$

$$- \sum_{\alpha \in \Gamma} e^{-\frac{2\pi i (v+1)}{N} \text{ad} x_i - \frac{2\pi i}{N} \text{ad} x_i} (t_{ij}^\alpha).$$
By putting these two equations together we finally get

\[
K_{ij} \left( z + \frac{\tau}{N} \right) = (0, -1)_i \cdot e^{-\frac{2\pi i}{N} ad x_j} K_{ij}(z) + \sum_{\alpha \in \mathbb{C}} \frac{e^{\frac{2\pi i}{N} ad x_i} + e^{\frac{2\pi i}{N}(z-\tau) ad x_i} - 1}{ad x_i} (t^\alpha_{ij})
\]

\[
= (0, -1)_i \cdot e^{-\frac{2\pi i}{N} ad x_j} K_{ij}(z) + (0, -1)_i \cdot \left( \sum_{\alpha \in \mathbb{C}} \frac{e^{\frac{2\pi i}{N} ad x_i} - 1}{ad x_i} (t^\alpha_{ij}) \right).
\]

Now recall that \( \frac{e^{\frac{2\pi i}{N} ad x_i} - 1}{ad x_i} = \frac{e^{-\frac{2\pi i}{N} ad x_j}}{ad x_j} \) and \( \frac{e^{-\frac{2\pi i}{N} ad x_j}}{ad x_j} (t^\alpha) = (1 - e^{\frac{2\pi i}{N} ad x_j}) (y_i) \). We thus have

\[
K_i \left( z + \frac{\tau}{N} \right) \delta_j = -y_i + \sum_{j' \neq i, j} K_{ij'} (z_{ij'}) + K_{ij} \left( z_{ij} + \frac{\tau}{N} \right)
\]

and therefore we get the announced relation

\[
K_i \left( z + \frac{\tau}{N} \delta_j \right) = (0, 1)_j \cdot e^{-\frac{2\pi i}{N} ad x_j} K_i(z).
\]

Consequently the \( K_i(z) \)'s satisfy conditions (e) and (f) above (and thus also (a) and (b)). Moreover, the \( K_i(z) \)'s also satisfy conditions (d). Indeed, the first part of (d) is immediate and \( k_{\alpha}(x, z) + k_{-\alpha}(-x, -z) = 0 \), therefore \( K_{ij}(z) + K_{ji}(-z) = 0 \), and thus \( \sum_i K_i(z) = -\sum_i y_i \).

Finally, from their very definition, the \( K_i(z) \)'s also satisfy condition (g).

In the next paragraph we show that the flatness condition (c) is satisfied.

1.6. Flatness of the connection.

**Proposition 1.8.** \( [\partial_i - K_i(z), \partial_j - K_j(z)] = 0 \), i.e., condition (c) is satisfied.

**Proof.** First we have

\[
[\partial_i(K_i(z)) - \partial_j(K_j(z))] = \partial_i K_{ji}(z_{ij}) - \partial_j K_{ij}(z_{ij}) = \partial_i(K_{ij}(z_{ij}) + K_{ji}(z_{ij})) = 0
\]

since \( K_{ij}(z) + K_{ji}(-z) = 0 \). Therefore we have to prove that \( [K_i(z), K_j(z)] = 0 \). As in [6] it follows from the universal classical dynamical Yang-Baxter equation:

(CDYBE)

\[-[y_i, K_{jk}] + [K_{ji}, K_{ki}] + c.p.(i, j, k) = 0,
\]

which we now prove (here \( K_{ij} := K_{ij}(z_{ij}) \)). For any \( f(x) \in \mathbb{C}[[x]] \) we have

\[
[y_k, f(ad x_i)(t^\alpha_{ij})] = \sum_{\beta \in \mathbb{C}} f(ad x_i) - f(-ad x_j) \frac{[t^\beta_{ki}, t^\alpha_{ij}]}{ad x_i + ad x_j},
\]

\[
[y_i, f(ad x_j)(t^\alpha_{ij})] = \sum_{\beta \in \mathbb{C}} f(ad x_j) - f(ad x_i + ad x_j) \frac{[t^\beta_{ij}, t^\alpha_{ij}]}{-ad x_i},
\]

\[
[y_j, f(ad x_k)(t^\alpha_{ij})] = \sum_{\beta \in \mathbb{C}} f(-ad x_i - ad x_j) - f(-ad x_i) \frac{[t^\beta_{j_k}, t^\alpha_{ij}]}{-ad x_j}.
\]
It follows that the l.h.s. of (CDYBE) is now
\[
\sum_{\alpha, \beta \in \Gamma} (k_\alpha(-\text{ad}x_j, z_{ij})k_\beta(-\text{ad}x_k, z_{ik}) - k_\alpha(\text{ad}x_i, z_{ij})k_{\beta-\alpha}(-\text{ad}x_k, z_{jk}) \\
+ k_\beta(\text{ad}x_i, z_{ik})k_{\beta-\alpha}(\text{ad}x_j, z_{jk}) + \frac{k_{\beta-\alpha}(\text{ad}x_j, z_{jk}) - k_{\beta-\alpha}(\text{ad}x_i + \text{ad}x_j, z_{jk})}{\text{ad}x_j} \\
+ \frac{k_\beta(\text{ad}x_i, z_{ik}) - k_\beta(\text{ad}x_i + \text{ad}x_j, z_{ik})}{\text{ad}x_i + \text{ad}x_j})[\tau_{ij}, \tau_{ik}^{\beta}],
\]
and thus (CDYBE) follows from the identity
\[
\begin{align*}
&k_\alpha(-v, z)k_\beta(u + v, z') - k_\alpha(u, z)k_{\beta-\alpha}(u + v, z' - z) + k_\beta(u, z')k_{\beta-\alpha}(v, z' - z) \\
+ &\frac{k_{\beta-\alpha}(v, z' - z) - k_{\beta-\alpha}(u + v, z' - z)}{v} + k_\beta(u, z') - k_\beta(u + v, z') \\
- &\frac{k_\alpha(u, z) - k_\alpha(-v, z)}{u + v} = 0.
\end{align*}
\]
This last identity can be written as
\[
\begin{align*}
&\left(k_\alpha(-v, z) - \frac{1}{v}\right)\left(k_\beta(u + v, z') + \frac{1}{u + v}\right) - \left(k_\alpha(u, z) + \frac{1}{u}\right)\left(k_{\beta-\alpha}(u + v, z' - z) + \frac{1}{u + v}\right) \\
&\quad + \left(k_\beta(u, z') + \frac{1}{v}\right)\left(k_{\beta-\alpha}(v, z' - z) + \frac{1}{v}\right) = 0,
\end{align*}
\]
which (taking into account that \(k_\alpha(x, z) + (1/x) = e^{-2\pi i a z} (k(x, z - a) + (1/x))\)) is a consequence of equation (3) of [6].

We have therefore proved:

**Theorem 1.9.** \(\nabla_{\tau,n,\Gamma}^\tau\) is a flat connection on \(P_{\tau,n,\Gamma}\), and its image under \(\hat{t}_{1,n}^\tau \to \hat{t}_{1,n}^\tau\) is the pull-back of a flat connection \(\nabla_{\tau,n,\Gamma}^\tau\) on \(P_{\tau,n,\Gamma}\).

\[
\square
\]

## 2. Lie algebras of derivations and associated groups

### 2.1. The Lie algebras \(\mathfrak{d}_0^\Gamma\) and \(\mathfrak{d}_1^\Gamma\).

Let \(\mathfrak{f}_\Gamma\) be the free Lie algebra with generators \(x, t^\alpha\) \((\alpha \in \Gamma)\). Let \(p, q > 0\). We define \(\mathfrak{d}_0^{p,q}\) to be the subspace of \(\mathfrak{f}_\Gamma \oplus (\mathfrak{f}_\Gamma)^{p,q}\) consisting of elements
\[
(D, C), \quad \text{where } C = (C_\alpha)_{\alpha \in \gamma},
\]
such that \(\deg_x(D) + \deg_t(D) = \deg_x(C_\alpha) + \deg_t(C_\alpha) = p\) and \(\deg_x(D) - 1 = \deg_t(C_\alpha) = q\) for every \(\alpha \in \Gamma\), and that satisfy the following of linear equations:

(i) \(C_\alpha(x, t^\beta) = C_{-\alpha}(x, t^{-\beta})\) in \(\mathfrak{f}_\Gamma\),

(ii) \([x, D(x, t^\beta)] + \sum_\alpha [t^\alpha, C_\alpha(x, t^\beta)] = 0\) in \(\mathfrak{f}_\Gamma\),

(iii) \([D(x_1, t_1^\beta), y_2] + c.p.(1, 2, 3) = 0\) in \(t_{1,3}\),

(iv) \([D(x_1, t_1^\beta) + D(x_1, t_2^\beta) - [C_\alpha(x_2, t_2^\beta), y_1], t_{2,3}^\alpha] = 0\) in \(t_{1,3}\),

(v) \([C_{\alpha}(x_1, t_1^\beta), t_1^{\alpha+\beta} + t_1^{\beta}] + [t_1^{\alpha+\beta}, C_{\alpha}(x_1, t_1^\beta)] + [t_1^{\beta}, C_{\alpha}(x_2, t_2^\beta)]\) commutes with \(t_{12}^\alpha\) in \(t_{1,3}\).

Remark that (i) and (ii) imply another relation

(vi) \(D(x, t^\beta) = -D(x, t^{-\beta})\),
which is very useful for computations. Then \( \tilde{\mathfrak{d}}^p_\Gamma := \oplus_{p,q} (\tilde{\mathfrak{d}}^p_\Gamma)^{p,q} \).

We then define a Lie bracket \( \langle , \rangle \) on \( \mathfrak{fr} \oplus (\mathfrak{fr})^{0} \) as follows:

\[
\langle (D, C), (D', C') \rangle := (\delta_C(D') - \delta_{C'}(D), [C, C'] + \delta_C(C') - \delta_{C'}(C)),
\]

where \( \delta_C \in \text{Der}(\mathfrak{fr}) \) is the derivation

- \( x \mapsto 0, t^\alpha \mapsto [t^\alpha, C_\alpha] \),
- \( \delta_C \) acts on \((\mathfrak{fr})^{0} \) componentwise on a direct sum : \( \delta_C(C'_\alpha) = \delta_C(C'_\alpha) \),
- the bracket is understood componentwise as well: \([C, C']_\alpha = [C_\alpha, C'_\alpha] \).

We let the reader check that \( \tilde{\mathfrak{d}}^p_\Gamma \) is stable under \( \langle , \rangle \), and becomes a bigraded Lie algebra\(^1\).

We now define \( \tilde{\mathfrak{d}}^F \) as the quotient of the free product \( \tilde{\mathfrak{d}}^p_\Gamma \ast \mathfrak{sl}_2 \) by the relations \([\tilde{e}, (D, C)] = 0, [\tilde{h}, (D, C)] = (p-q)(D, C), \) and \((\text{ad}^p \tilde{f})(D, C) = 0 \) if \((D, C) \in \tilde{\mathfrak{d}}^p_\Gamma\) is homogeneous of bidegree \((p, q)\).

Here

\[
\tilde{e} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tilde{h} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \tilde{f} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\]

form the standard basis of \( \mathfrak{sl}_2. \) If we respectively give degree \((1, -1), (0, 0)\) and \((-1, 1)\) to \( \tilde{e}, \tilde{h} \) and \( \tilde{f} \) then \( \tilde{\mathfrak{d}}^F \) becomes \( \mathbb{Z}^2 \)-graded.

We then define \( \tilde{\mathfrak{d}}^p_\Gamma := \ker(\tilde{\mathfrak{d}}^F \rightarrow \mathfrak{sl}_2) \), which is \((\mathbb{Z}_{p0})^2\)-graded. One observes that it is positively graded and finite dimensional in each degree. Thus, it is a direct sum of finite dimensional \( \mathfrak{sl}_2 \)-modules.

2.2. The Lie algebras \( \tilde{\mathfrak{d}}^p_0 \) and \( \tilde{\mathfrak{d}}^F \). We write \( \tilde{\mathfrak{d}}^p_0 \) for the free bigraded Lie algebra generated by \( \delta_{s,\gamma}'s \) \((s \geq 0, \gamma \in \Gamma)\) in degree \((s+1, s)\) with relations

\[
\delta_{s,\gamma} = (-1)^s \delta_{s,-\gamma},
\]

for all \( s \geq 0 \) and \( \gamma \in \Gamma \).

We then define \( \tilde{\mathfrak{d}}^F \) as the quotient of the free product \( \tilde{\mathfrak{d}}^p_0 \ast \mathfrak{sl}_2 \) by the relations

\[
[\tilde{e}, \delta_{s,\gamma}] = 0, \ [\tilde{h}, \delta_{s,\gamma}] = s\delta_{s,\gamma} \text{ and } \text{ad}^{s+1}(\tilde{f})(\delta_{s,\gamma}) = 0;
\]

and \( \tilde{\mathfrak{d}}^p_0 \) as the kernel of \( \tilde{\mathfrak{d}}^F \rightarrow \mathfrak{sl}_2. \) As above, we have \( \tilde{\mathfrak{d}}^F = \tilde{\mathfrak{d}}^p_0 \ast \mathfrak{sl}_2, \) and \( \tilde{\mathfrak{d}}^p_0 \) is positively graded (actually \((\mathbb{Z}_{p0})^2\)-graded).

We now give examples of elements in \( \tilde{\mathfrak{d}}^p_0 \) that are of some use below. For any \( s \in \mathbb{N} \) and \( \gamma \in \Gamma, \) we set

\[
D_{s,\gamma} := \sum_{p+q=s+1} \sum_{\beta \in \Gamma} [(\text{ad}x)^p t^\beta, \gamma, (\text{ad}x)^q t^\beta]
\]

and

\[
(C_{s,\gamma})_\alpha := (\text{ad}x)^s t^{\alpha - \gamma} + (\text{ad}x)^s t^{\alpha + \gamma}.
\]

Observe that \((D_{s,\gamma}, C_{s,\gamma}) = (-1)^s(D_{s,-\gamma}, C_{s,-\gamma}).\)

The following result tells us that \( \delta_{s,\gamma} \mapsto (D_{s,\gamma}, C_{s,\gamma}) \) defines a bigraded Lie algebra morphism \( \tilde{\mathfrak{d}}^p_0 \rightarrow \tilde{\mathfrak{d}}^p_0, \) that obviously extends to \( \tilde{\mathfrak{d}}^F \rightarrow \tilde{\mathfrak{d}}^F.\)

Proposition 2.1. \((D_{s,\gamma}, C_{s,\gamma}) \in (\tilde{\mathfrak{d}}^p_0)^{s+1,1}..\)

\(^1\)The proof is straightforward but quite long. We do not give it since we do use another simpler Lie algebra below.
Proof. First observe that relations (i) and (vi) are obviously satisfied.

To prove (ii) it suffices to notice that in the free Lie algebra with three generators \( x, t_1, t_2 \) we have

\[
[t_1, (\text{ad} x)^s t_2] + [t_2, (\text{ad} x)^s t_1] = \sum_{p+q=s-1} [x, [(-\text{ad} x)^q t_1, (\text{ad} x)^p t_2]].
\]

Let us prove (iii). In \( \mathfrak{t}_1^r, n \) we compute for \( \# \{i, j, k \} = 3 \),

\[
y_k, (\text{ad} x_i)^p t_{i,j}^p = -\sum\limits_{k+l=p-1} (\text{ad} x_i)^k [t_{i,j}^k, (\text{ad} x_i)^l t_{i,j}^l] = \sum\limits_{k+l=p-1} (\text{ad} x_i)^k (-\text{ad} x_j)^l [t_{i,j}^k, (\text{ad} x_i)^l t_{i,j}^l].
\]

Therefore, in \( \mathfrak{t}_1^r, j \), we have

\[
y_1, D(x_2, t_{23}^\gamma) = \sum\limits_{k+l+m-2=\alpha, \beta} \sum\limits_{\gamma} [([\text{ad} x_2])^k t_{23}^\gamma, (-\text{ad} x_3)^l t_{13}^{\alpha-\beta}] + (-\text{ad} x_3)^m t_{23}^{\alpha-\beta}]
\]

Then \( y_1, D(x_2, t_{23}^\gamma) \) + c.p. \((1, 2, 3) = 0 \) follows from the Jacobi identity.

Let us prove (iv). On the one hand we have

\[
[D(x_1, t_{12}^\gamma) + D(x_1, t_{13}^\gamma), t_{23}^\gamma] = \sum\limits_{p+q=s-1} \sum\limits_{\beta \neq \beta} \sum\limits_{\gamma} \sum\limits_{\text{ad} x_1}^p [t_{13}^{\alpha+\beta}, \gamma, \alpha^\gamma + \text{ad} x_3]^q t_{13}^{\alpha-\beta}, \alpha^\gamma + \text{ad} x_3]^m t_{23}^{\alpha-\beta}, \gamma^\beta]
\]

On the other hand, we have

\[
[C_\alpha(x_2, t_{23}^\gamma, y_1)] = ([\text{ad} x_2])^s t_{23}^{\alpha-\gamma} + (-\text{ad} x_2)^s t_{23}^{\alpha+\gamma}, y_1]
\]

Therefore (iv) is satisfied.

Let us prove (v). We have

\[
[C_\alpha(x_1, t_{12}^\gamma), t_{13}^{\alpha+\beta} + t_{23}^{\beta}] = ([\text{ad} x_1])^s t_{12}^{\alpha-\gamma} + (-\text{ad} x_1)^s t_{12}^{\alpha+\gamma}, t_{13}^{\alpha+\beta} + t_{23}^{\beta}]
\]

Then, by defining \( A = t_{23}^{\gamma}, B = t_{13}^{\alpha-\beta}, \alpha^{\alpha+\beta} \) we have

\[
[t_{12}^{\alpha}, [C_\alpha(x_1, t_{12}^\gamma), t_{13}^{\alpha+\beta} + t_{23}^{\beta}] = [t_{12}^{\alpha}, [t_{13}^{\alpha+\beta}, (\text{ad} x_2)^s A] + [t_{23}^{\beta}, (\text{ad} x_1)^s B]]
\]

Therefore, in \( \mathfrak{t}_1^r, n \) we have
implies that for any $\xi$ that
\[
=t^\alpha_{13} + (\text{ad } x_1)^* B, t^\alpha_{12}]
+[(t^\alpha_{13}, t^\beta_{12}), (\text{ad } x_3)^* A] + [(t^\alpha_{13}, (\text{ad } x_3)^* B, t^\alpha_{12}]]
+[(t^\alpha_{13}, t^\beta_{12}), (\text{ad } x_3)^* A, t^\alpha_{12}]]
= [(t^\alpha_{13}, (\text{ad } x_2)^* A) + [(t^\alpha_{13}, (\text{ad } x_1)^* B), t^\alpha_{12}]].
\]
This finishes the proof.

\[\square\]

**Remark 2.2.** We do not know if $\mathfrak{d}_0^\Gamma \rightarrow \mathfrak{d}_0^\Gamma$ is injective or not.

2.3. **Derivations of $\mathfrak{d}_{1,n}^\Gamma$ and $\mathfrak{d}_{1,n}^\Gamma$.**

**Lemma 2.3.** We have a bigraded Lie algebra morphism $\mathfrak{d}_0^\Gamma \rightarrow \text{Der}(\mathfrak{d}_{1,n}^\Gamma)$, taking $(D, C) \in \mathfrak{d}_0^\Gamma$ to the derivation $\xi_{(D,C)}$:

\[
x_i \mapsto 0,
\]
\[
y_i \mapsto \sum_{j \not= i} D(x_i, t^\alpha_{ij}),
\]
\[
t^\alpha_{ij} \mapsto [t^\alpha_{ij}, C_\alpha(x_i, t^\beta_{ij})].
\]

This induces a bigraded Lie algebra morphism $\mathfrak{d}_0^\Gamma \rightarrow \text{Der}(\mathfrak{d}_{1,n}^\Gamma)$.

**Proof.** We have to prove that defining relations of $\mathfrak{d}_{1,n}^\Gamma$ are preserved by $\xi := \xi_{(D,C)}$. First observe that relations $[x_i, x_j] = [x_i + x_j, t^\alpha_{ij}] = [x_i, t^\alpha_{ik}] = [t^\alpha_{ij}, t^\alpha_{ij}] = 0$ are obviously preserved. Then conditions (i) and (ii) respectively imply that $t^\alpha_{ij} = t^\alpha_{ij}$ and $[x_i, y_j] = \sum t^\alpha_{ij}$ are preserved. Condition (vi) implies that $[x_i, y_j] = [x_j, y_i]$ is preserved, and (vi) together with (iii) imply that $[y_i, y_j] = 0$ is preserved. Therefore it follows from the centrality of $\sum x_i$ and $\xi(\sum x_i) = 0$ that

\[
\xi([x_i, y_i]) = \xi(- \sum_{j \not= i} [x_j, y_i]) = \xi(- \sum_{j \not= i} t^\alpha_{ij}).
\]

Condition (iv) ensures that $[y_i, t^\alpha_{ij}] = 0$ is preserved, and together with (vi) it implies that

\[
[y_i + y_j, t^\alpha_{ij}] = 0.
\]

Finally condition (v) implies that the twisted infinitesimal braiding relations are preserved, and the first part of the statement follows.

For the second part of the statement it remains to prove that the centrality of $\sum y_i$ is preserved. This follows directly from the identity $\xi(\sum y_i) = 0$ that we now prove. Relation (vi) implies that for any $i \not= j$ one has $D(x_i, t^\beta_{ij}) = D(x_j, t^\beta_{ij}) = D(x_j, t^\beta_{ij})$ in $\mathfrak{d}_{1,n}^\Gamma$ (the last equality happens since $\deg_x(D) = \deg_x(C_\alpha) + 1 > 0$), and hence

\[
\xi(\sum y_i) = 0.
\]

We are done (the compatibility with bracket and grading are easy to check).

The last part of the statement is a consequence of the fact that $\xi(\sum y_i) = \xi(\sum x_i) = 0$, that we have already proved.

We now prove that this morphism extends to a Lie algebra morphism $\mathfrak{d}_0^\Gamma \rightarrow \text{Der}(\mathfrak{d}_{1,n}^\Gamma)$:
Proposition 2.4. We have a bigraded Lie algebra morphism \( \tilde{\mathfrak{F}} \rightarrow \text{Der}(\mathfrak{t}^\Gamma_{1,n}) \) taking \( (D,C) \in \mathfrak{t}^\Gamma_{1,n} \) to \( \xi \in \mathfrak{sl}_2 \) to the derivation

\[
\xi_{ij} : t^\alpha_{ij} \mapsto 0, \quad (x_i, y_i) \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]

This induces a bigraded Lie algebra morphism \( \tilde{\mathfrak{F}} \rightarrow \text{Der}(\mathfrak{t}^\Gamma_{1,n}) \).

In what follows we write \( \mathfrak{d} := \tilde{h}, \mathfrak{X} := \tilde{e} \) and \( \Delta_0 := \tilde{f} \) and \( \tilde{\mathfrak{d}} := \xi_h, \tilde{\mathfrak{X}} := \xi_e \) and \( \tilde{\Delta}_0 := \xi_f \).

Proof. It is obvious that for any \( g, g' \in \mathfrak{sl}_2 \), \( \xi_g \) defines a derivation of the same degree of \( \mathfrak{t}^\Gamma_{1,n} \), and that \( \xi_{[g,g']} = [\xi_g, \xi_{g'}] \). Hence we have a bigraded Lie algebra morphism \( \mathfrak{sl}_2 \rightarrow \text{Der}(\mathfrak{t}^\Gamma_{1,n}) \).

Let us prove that it factorizes through the quotient \( \tilde{\mathfrak{F}} \).

It is relatively clear that \( [\tilde{\mathfrak{X}}, \xi_{(D,C)}] = 0 \) and \( [\mathfrak{d}, \xi_{(D,C)}] = (p-q)(D,C) \) if \( (D,C) \in \tilde{\mathfrak{F}}^{p,q} \). Thus it remains to prove that \( (\text{ad}\tilde{\Delta}_0)^p(\xi_{(D,C)}) = 0 \) if \( (D,C) \in \tilde{\mathfrak{F}}^{p,q} \). We do this now. Let us write \( \xi := \xi_{(D,C)} \) and \( A := (\text{ad}\tilde{\Delta}_0)^p(\xi) \). Then after an easy computation one obtains on generators:

\[
A(x_i) = -p\tilde{\Delta}_0^{-1}\xi(y_i) = -p\tilde{\Delta}_0^{-1}\left( \sum_{j:j=i} D(x_i, t^\beta_{ij}) \right),
\]

\[
A(y_i) = \tilde{\Delta}_0^p \xi(y_i) = \tilde{\Delta}_0^p \left( \sum_{j:j=i} D(x_i, t^\beta_{ij}) \right),
\]

\[
A(t^\alpha_{ij}) = \tilde{\Delta}_0^p(\xi(t^\alpha_{ij})) = \tilde{\Delta}_0^p(\left[ t^\alpha_{ij}, C_{\alpha}(x_i, t^\beta_{ij}) \right]).
\]

Finally remark that we have an increasing filtration on \( \mathfrak{t}_{1,n}^\Gamma \) defined by \( \text{deg}(x_i) = 1 \) and \( \text{deg}(t^\alpha_{ij}) = \text{deg}(y_i) = 0 \). \( \Delta_0 \) decreases the degree by 1 and vanishes on degree zero elements. The result then follows from the fact that \( \text{deg}_{\mathfrak{d}}(\theta_\alpha) = p-q < p \) and \( \text{deg}_{\mathfrak{d}}(D) = p-q-1 < p-1 \).  

Now composing with \( \tilde{\mathfrak{d}}^\Gamma \rightarrow \tilde{\mathfrak{F}}^\Gamma \) (resp. \( \mathfrak{d}^\Gamma \rightarrow \mathfrak{F}^\Gamma \)) one obtains a Lie algebra morphism \( \tilde{\mathfrak{d}}^\Gamma \rightarrow \text{Der}(\mathfrak{t}_{1,n}^\Gamma) \) (resp. \( \mathfrak{d}^\Gamma \rightarrow \text{Der}(\mathfrak{t}_{1,n}^\Gamma) \). We write \( \xi_{s,\gamma} := \xi_{(D_{s,\gamma}, C_{s,\gamma})} \) for the image of \( \delta_{s,\gamma} \). We then have \( \mathfrak{t}_{1,n}^\Gamma \times \mathfrak{d}^\Gamma = (\mathfrak{t}_{1,n}^\Gamma \times \mathfrak{d}_{\gamma}^\Gamma) \times \mathfrak{sl}_2 \), with \( \mathfrak{t}_{1,n}^\Gamma \times \mathfrak{d}_{\gamma}^\Gamma \) positively graded (since both \( \mathfrak{t}_{1,n}^\Gamma \) and \( \mathfrak{d}_{\gamma}^\Gamma \) are \( \mathbb{Z}_{\geq 0} \)-graded) and a sum of finite dimensional \( \mathfrak{sl}_2 \)-modules. Therefore we can construct the semi-direct product group

\[
G_{1,n}^\Gamma := \exp(\mathfrak{t}_{1,n}^\Gamma \times \mathfrak{d}_{\gamma}^\Gamma)^\wedge \rtimes \text{SL}_2(C),
\]

where \( \exp(\mathfrak{t}_{1,n}^\Gamma \times \mathfrak{d}_{\gamma}^\Gamma)^\wedge \) is the exponential group associated to the degree completion of \( \mathfrak{t}_{1,n}^\Gamma \times \mathfrak{d}_{\gamma}^\Gamma \).

Similarly, we define \( \bar{G}_{1,n}^\Gamma := \exp(\mathfrak{t}_{1,n}^\Gamma \times \mathfrak{d}_{\gamma}^\Gamma)^\wedge \rtimes \text{SL}_2(C) \).

Notice that one can also define semi-direct product groups \( \bar{G}_{1,n}^\Gamma := \exp(\mathfrak{t}_{1,n}^\Gamma \times \mathfrak{d}_{\gamma}^\Gamma)^\wedge \rtimes \text{SL}_2(C) \) and \( \bar{G}_{1,n}^\Gamma := \exp(\mathfrak{t}_{1,n}^\Gamma \times \mathfrak{d}_{\gamma}^\Gamma)^\wedge \rtimes \text{SL}_2(C) \). We therefore have the following commutative diagram:

\[
\begin{array}{ccc}
G_{1,n}^\Gamma & \longrightarrow & \bar{G}_{1,n}^\Gamma \\
\downarrow & & \downarrow \\
G_{1,n}^\Gamma & \longrightarrow & \bar{G}_{1,n}^\Gamma 
\end{array}
\]
Lemma 2.5. The kernel of \( \delta_0^n \rightarrow \text{Der}(t_1^{1,n}) \) \((n \geq 2)\) is the space of elements \((0,C)\) for which \(C_\alpha\) is proportional to \(t^\alpha\), and \(\ker(\delta_0^n \rightarrow \text{Der}(t_1^{1,n})) = \mathbb{C} \delta_{0,0} \).

Proof. Let us first prove it for \(n = 2\). Recall that \( t_1^{1,2} = t_1^{1,2}/(x_1 + x_2, y_1 + y_2) \), so it is the Lie algebra generated by \(x\) (the class of \(x_1\)), \(y\) (the class of \(y_1\)) and \(t^\alpha\)'s (classes of \(t_1^{1,2}\)'s) with the relation \([x,y] = \sum_{\alpha \in \Gamma} t^\alpha \). Then the derivation \(\xi_{(D,C)}\) associated to \((D,C) \in \delta_0^n\) is given by

\[
x \mapsto 0, y \mapsto D(x,t^\beta), t^\alpha \mapsto [t^\alpha,C_\alpha(x,t^\beta)].
\]

This derivation vanishes if and only if \(D = 0\) and \(C_\alpha\) is proportional to \(t^\alpha\). Finally, the result for \(n \geq 2\) follows from the fact that

\[
\xi_{(D,C)}^{(2)} = (u \mapsto u^{1,2,\ldots,n}) \circ \xi_{(D,C)}^{(n)} \circ (u \mapsto u^{1,\ldots,n}),
\]

where \(\xi_{(D,C)}^{(n)}\) denotes the derivation of \( t_1^{1,n} \) associated to \((D,C)\). \(\Box\)

2.4. Comparison morphisms. Let \(\rho : \Gamma_1 \rightarrow \Gamma_2\) a group morphism. We have a comparison morphism \(\delta_0^n \rightarrow \tilde{\delta}_0^{t_1^n}\), \((D,C) \mapsto (D^\rho,C^\rho)\) defined by

\[
D^\rho := D \left( x, \sum_{\gamma \in \ker(\rho)} \frac{\rho(\gamma)}{\# \ker(\rho)} \right), \quad (C^\rho)_\alpha := C_\alpha \left( x, \sum_{\gamma \in \ker(\rho)} \frac{\rho(\gamma)}{\# \ker(\rho)} \right).
\]

When \(\rho\) is not surjective it depends on the choice of a section \(\text{coker}(\rho) \rightarrow \Gamma_2\). It extends to \(\tilde{\delta}_1^{\Gamma_1} \rightarrow \tilde{\delta}_1^{\Gamma_2}\) by sending the generators of \(\mathfrak{sl}_2\) to themselves. These comparison morphisms are compatible with the morphisms \(\tilde{\delta}_i^{\Gamma_1} \rightarrow \text{Der}(t_1^{1,n})\), for \(i = 1,2\). Namely, there is a commutative diagram

\[
\begin{array}{ccc}
\tilde{\delta}_1^{\Gamma_1} \times \mathfrak{t}_1^{1,n} & \rightarrow & \mathfrak{t}_1^{1,n} \\
\downarrow & & \downarrow \\
\tilde{\delta}_1^{\Gamma_2} \times \mathfrak{t}_1^{1,n} & \rightarrow & \mathfrak{t}_1^{1,n}
\end{array}
\]

Finally, we have comparison morphisms for the corresponding groups that fit into a commutative diagram

\[
\begin{array}{ccc}
\tilde{G}_n^{\Gamma_1} & \rightarrow & \tilde{G}_n^{\Gamma_2} \\
\downarrow & & \downarrow \\
\tilde{G}_n^{\Gamma_1} & \rightarrow & \tilde{G}_n^{\Gamma_2}
\end{array}
\]

Notice that the image of \((D_{x,\gamma},C_{x,\gamma})\) under a comparison morphism is no longer of this form except if \(\rho\) is injective. In this case (and in this case only) we have a comparison morphism \(\mathfrak{t}_1^{1,n} \times \mathfrak{t}_1^{1,n} \rightarrow \mathfrak{t}_1^{1,n} \times \mathfrak{t}_1^{1,n}\) taking \(x^i, y^i\) to themselves, and \(t^\alpha_{ij}\) to \(\sum_{\beta \in \ker(\rho)} \rho(\beta) + \beta\) and \(\delta_{x,\gamma}\) to \(\sum_{\beta \in \ker(\rho)} \delta_{x,\beta} + \beta\). In particular we have a canonical natural inclusion \(G_n^0 \rightarrow G_n^\Gamma\) (which descends to an inclusion \(G_n^0 \rightarrow \tilde{G}_n^\Gamma\)).
3. Bundles with flat connections on moduli spaces

3.1. On some subgroups of $\text{SL}_2(\mathbb{Z})$ and moduli spaces. Consider the group $\Gamma := \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ and consider the following (finite index) subgroup of $\text{SL}_2(\mathbb{Z})$:

$$\text{SL}_2^\Gamma(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid a \equiv 1 \mod M, d \equiv 1 \mod N, b \equiv 0 \mod N \text{ and } c \equiv 0 \mod M \right\}.$$ 

We write $Y(\Gamma)$ for the set of equivalence classes of pairs $(E, \phi)$ where $E$ is an elliptic curve and $\phi: \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \to E$ is an injective group morphism that is orientation preserving i.e. such that the basis $(\frac{\phi(1,0)}, \frac{\phi(0,1)})$ of $T_0E$ is direct. Then, one can see that $Y(\Gamma) = \mathfrak{f}/\text{SL}_2^\Gamma(\mathbb{Z})$ and therefore inherits the structure of a complex orbifold.

**Remark 3.1.** The biggest congruence subgroup on which the connection we will construct in this section is well defined and flat is the subgroup $\text{SL}_2^\Gamma(\mathbb{Z})$ of $\text{SL}_2(\mathbb{Z})$ consisting of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ such that $Mb \equiv 0 \mod N$ and $Nc \equiv 0 \mod M$. Nevertheless, in order to retrieve the twisted elliptic KZB connection defined at the level of configuration spaces, it suffices to consider the usual congruence subgroup $\text{SL}_2^\Gamma(\mathbb{Z}) \subset \text{SL}_2(\mathbb{Z})$.

Recall the following standard group actions:

- The group $\text{SL}_2(\mathbb{Z})$ acts on $\mathbb{C}^n \times \tilde{\mathfrak{f}}$:

  $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z|\tau) := \begin{pmatrix} z \\ \frac{a\tau + b}{c\tau + d} \end{pmatrix}.$$ 

  This obviously descends to an action of $\text{SL}_2(\mathbb{Z})$ on $(\mathbb{C}^n \times \tilde{\mathfrak{f}})/\mathbb{C}$, where $\mathbb{C}$ acts diagonally on $\mathbb{C}^n$: $u \cdot (z|\tau) := (z + u \sum \delta_i|\tau)$.

- The group $(\mathbb{Z}^n)^2$ acts on $\mathbb{C}^n \times \tilde{\mathfrak{f}}$:

  $$(m, n) \cdot (z|\tau) := (z + m + \tau n|\tau).$$ 

  It obviously descends to an action of $(\mathbb{Z}^n)^2/\mathbb{Z}^2$ on $\mathbb{C}^n \times \tilde{\mathfrak{f}}/\mathbb{C}$, where $\mathbb{Z}^2$ is the diagonal subgroup in $(\mathbb{Z}^n)^2 = (\mathbb{Z}^2)^n$.

- Finally, there is a right action of $\text{SL}_2(\mathbb{Z})$ on $(m, n) \in \mathbb{Z}^2$ by automorphisms:

  $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (m, n) \mapsto (n, m) \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$ 

We can thus form the semi-direct products $(\mathbb{Z}^n)^2 \rtimes \text{SL}_2(\mathbb{Z})$ and $((\mathbb{Z}^n)^2/\mathbb{Z}^2) \rtimes \text{SL}_2(\mathbb{Z})$.

A few observations are then in order:

- The above actions are compatible in the sense that we have a left action of $(\mathbb{Z}^n)^2 \rtimes \text{SL}_2(\mathbb{Z})$ on $\mathbb{C}^n \times \tilde{\mathfrak{f}}$, which descends to an action of $((\mathbb{Z}^n)^2/\mathbb{Z}^2) \rtimes \text{SL}_2(\mathbb{Z})$ on $(\mathbb{C}^n \times \tilde{\mathfrak{f}})/\mathbb{C}$, where $\mathbb{Z}^2$ is embedded in $(\mathbb{Z}^n)^2$ via the diagonal map. One can think of translation by $\mathbb{C}$ as a left or right action as it commutes with the $((\mathbb{Z}^n)^2 \rtimes \text{SL}_2(\mathbb{Z}))$-action.

- The action of $(\mathbb{Z}^n)^2$ preserves the subset $\text{Diag}_{n, \Gamma} := \{(z|\tau) \in \mathbb{C}^n \times \tilde{\mathfrak{f}}| \exists z \in \text{Diag}_{\tau, n, \Gamma}\}$. 


The action of the subgroup \( \text{SL}_2(\mathbb{Z}) \subset \text{SL}_2(\mathbb{Z}) \) also preserves \( \text{Diag}_{n,\Gamma} \).

We are thus ready to define several variants of \( Y(\Gamma) \) “with marked points”:

- We define the quotient
  \[
  \mathcal{M}_{1,n}^\Gamma := (\mathbb{Z}^n)^2 \times \text{SL}_2(\mathbb{Z}) \backslash \left( \left( \mathbb{C}^n \times \delta \right) - \text{Diag}_{n,\Gamma} \right) / \mathbb{C}
  \]
  and call it the \textit{moduli space of} \( \Gamma \)-\textit{structured elliptic curves with} \( n \) \textit{ordered marked points}.
- It has a non-reduced variant
  \[
  p : \mathcal{M}_{1,n}^\Gamma := (\mathbb{Z}^n)^2 \times \text{SL}_2(\mathbb{Z}) \backslash \left( \left( \mathbb{C}^n \times \delta \right) - \text{Diag}_{n,\Gamma} \right) \to \mathcal{M}_{1,n}^\Gamma.
  \]
- One can also define the \textit{moduli space of} \( \Gamma \)-\textit{structured elliptic curves with} \( n \) \textit{unordered marked points}
  \[
  \mathcal{M}_{1,[n]}^\Gamma := \mathcal{M}_{1,n}^\Gamma / \mathbb{S}_n
  \]
  and its non-reduced variant
  \[
  \mathcal{M}_{1,[n]}^\Gamma := \mathcal{M}_{1,n}^\Gamma / \mathbb{S}_n - \text{Diag}_{n,\Gamma}.
  \]

\textbf{Remark 3.2.} We have \( \mathcal{M}_{1,1}^\Gamma = \mathcal{M}_{1,[1]}^\Gamma = Y(\Gamma) \), and \( \mathcal{M}_{1,1}^\Gamma = \mathcal{M}_{1,[1]}^\Gamma \) is the universal curve over it. The fiber of \( \mathcal{M}_{1,n}^\Gamma \to Y(\Gamma) \) (resp. \( \mathcal{M}_{1,n}^\Gamma \to Y(\Gamma) \)) at \( \{ \tau \} \) is precisely the twisted (resp. reduced twisted) configuration space \( \text{Conf}(E, \Gamma, n, \Gamma) \) (resp. \( C(E, \Gamma, n, \Gamma) \)). Moreover, the map

\[
\delta : \mathcal{M}_{1,2}^\Gamma \to \mathcal{M}_{1,1}^\Gamma
\]

factors through (and is open in) \( \mathcal{M}_{1,1}^\Gamma \). We can interpret \( \mathcal{M}_{1,2}^\Gamma \) as the \( \Gamma \)-punctured universal curve over \( Y(\Gamma) \).

\textbf{3.2. Principal bundles over} \( \mathcal{M}_{1,n}^\Gamma \) \textbf{and} \( \mathcal{M}_{1,n}^\Gamma \). In this §, \( \mathcal{G}_n^\Gamma \) is defined as in (4) and we define a principal \( \mathcal{G}_n^\Gamma \)-bundle \( \mathcal{P}_{n,\Gamma} \) over \( \mathcal{M}_{1,n}^\Gamma \) whose image under the natural morphism \( \mathcal{G}_n^\Gamma \to \mathcal{G}_n^\Gamma \) is the pull-back of a principal \( \mathcal{G}_n^\Gamma \)-bundle \( \mathcal{P}_{n,\Gamma} \) over \( \mathcal{M}_{1,n}^\Gamma \). Let us fix the notation first: for \( u \in \mathbb{C}^n \) and \( v, u_i \in \mathbb{C} \) \( (i = 1, \ldots, n) \),

\[
u^d := \left( \begin{array}{c} u \\ 0 \\ u^{-1} \end{array} \right), \quad e^{uX} := \left( \begin{array}{cc} 1 & v \\ 0 & 1 \end{array} \right).
\]

Since \( [X, x_i] = 0 \) then it makes sense to define \( e^{uX} \Sigma_i w_i x_i = e^{X} \Sigma_i w_i x_i \). In particular, we have \( \text{Ad}(u^d)(x_i) = u x_i \) and \( \text{Ad}(u^d)(y_i) = y_i / u \) \( (\forall i) \), \( \text{Ad}(u^d)(X) = u^2 X \) and \( \text{Ad}(u^d)(\Delta_0) = \Delta_0 / u^2 \).

Let \( \pi : \mathbb{C}^n \times \delta \to \mathcal{M}_{1,n} \) be the canonical projection.

\textbf{Proposition 3.3.} There exists a unique principal \( \mathcal{G}_n^\Gamma \)-bundle \( \mathcal{P}_{n,\Gamma} \) over \( \mathcal{M}_{1,n}^\Gamma \) for which a section on \( U \subset \mathcal{M}_{1,n}^\Gamma \) is a function \( f : \pi^{-1}(U) \to \mathcal{G}_n^\Gamma \) such that

\[
\begin{align*}
f(z + \delta_i | \tau) &= f(z | \tau), \\
f(z + \tau \delta_i | \tau) &= e^{\frac{2\pi i}{n}} f(z | \tau), \\
f(z, \tau + 1) &= f(z | \tau), \\
f\left( \frac{z}{\tau} - \frac{1}{n} \right) &= \tau^d e^{\frac{2\pi i}{n} (X + \Sigma_i x_i)} f(z | \tau).
\end{align*}
\]
Moreover, the image of $P_{\alpha, \Gamma}$ under $G^\Gamma_n \to \bar{G}^\Gamma_n$ is the pull-back of a unique principal $\bar{G}^\Gamma_n$-bundle $P_{\alpha, \Gamma}$ over $\mathcal{M}_{1, n}$ on which a section on $U \subset \mathcal{M}_{1, n}$ is a function $f : (p \circ \sigma)^{-1}(U) \to \mathcal{M}_{1, n}$ satisfying the above conditions (with $x_i$’s replaced by $\tilde{x}_i$’s) and such that $f(z + v \sum_i \delta_i \sigma) = f(z \sigma)$ for any $v \in \mathbb{C}$.

Proof. First recall that for $\Gamma = 0$ this is precisely [6, Proposition 3.4]. Then observe that we have an obvious map $i : \mathcal{M}_{1, n} \to \mathcal{M}_{0, n}$. Therefore we define $P_{\alpha, \Gamma}$ (resp. $\bar{P}_{\alpha, \Gamma}$) to be the image under the natural inclusion $G^\Gamma_n \to \bar{G}^\Gamma_n$ (resp. $G^\Gamma_n \to \bar{G}^\Gamma_n$) of $i^* P_{\alpha, 0}$ (resp. $i^* \bar{P}_{\alpha, 0}$).

We thus proved existence. Unicity is obvious. \qed

In other words, there exists a unique non-abelian 1-cocycle $(c_g)_{g \in (\mathbb{Z}^n)^2 \ast \text{SL}_2(\mathbb{Z})}$ on $\mathbb{C}^n \times \mathfrak{H}$ with values in $G^\Gamma_n$ such that $c(\delta, 0) = 1$, $c(0, \delta) = e^{-2\pi i x \cdot \delta}$, $c_S = 1$ and

$$c_T(z|\tau) = e^{2\pi i (\tau \cdot \Sigma_2 z^2 z)} = e^{2\pi i ((\tau \cdot \Sigma_2 z^2 z) \cdot \tau \cdot d},$$

where $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ are the generators of $\text{SL}_2(\mathbb{Z})$. Here cocycle means (as in [6]) that $c_g$’s are holomorphic functions $\mathbb{C}^n \times \mathfrak{H} \to G^\Gamma_n$ satisfying the cocycle condition $c_g g' \cdot (z|\tau) = c_g(g' \cdot (z, \tau))c_{g'}(z|\tau)$.

Remark 3.4. Notice that we do have a $(\mathbb{Z}^n)^2 \ast \text{SL}_2(\mathbb{Z})$-cocycle (since our bundle is defined as the pull-back of a bundle on $\mathcal{M}_{0, 1}$) but the cocycle defining $P_{\alpha, \Gamma}$ is its restriction to $(\mathbb{Z}^n)^2 \ast \text{SL}_2(\mathbb{Z})$.

3.3. Connections on $P_{\alpha, \Gamma}$ and $\bar{P}_{\alpha, \Gamma}$. A connection on $P_{\alpha, \Gamma}$ is the same as an equivariant connection on the trivial $G^\Gamma_n$-bundle over $\mathbb{C}^n \times \mathfrak{H} - \text{Diag}_{\alpha, \Gamma}$. Namely, it is of the form $\nabla_{n, \Gamma} := d - \eta(z|\tau)$, where $\eta$ is a $t_{1, n} \ast \text{Ad}_\Gamma$-valued meromorphic one-form on $\mathbb{C}^n \times \mathfrak{H}$ with only poles on $\text{Diag}_{n, \Gamma}$, and the equivariance condition reads: for any $g \in (\mathbb{Z}^n)^2 \ast \text{SL}_2(\mathbb{Z})$,

$$g^* \eta = (dc_g(z|\tau))c_g(z|\tau)^{-1} + \text{Ad}(c_g(z|\tau))(\eta(z|\tau)).$$

We now construct such a connection. For any $\gamma \in \Gamma$ we define $g_\gamma(x, z|\tau) := \partial_x k_\gamma(x, z|\tau)$,

$$\varphi_\gamma(x|\tau) = \sum_{x \in \mathbb{Z}} A_{\gamma, \gamma}(x) x^\alpha := g_{-\gamma}(x, 0|\tau).$$

Then we set

$$\Delta(z|\tau) := -\frac{1}{2\pi i} \left( \Delta_0 + \frac{1}{2} \sum_{s, \gamma \in \Gamma} A_{s, \gamma}(\tau) \delta_{s, \gamma} - \sum_{i, j} g_{ij}(z|\tau) \right),$$

where $g_{ij}(z|\tau) := \sum_{a \in \mathfrak{H}} g_{\alpha}(ad x_i, z|\tau)(t_{ij}^\alpha)$. And finally, with $K_{\gamma}(z|\tau)$’s as in §1.4, we define

$$\eta(z|\tau) := \Delta(z|\tau)d\tau + \sum_{\gamma} K_{\gamma}(z|\tau)dz.$$}

Remark 3.5. One can see that $\varphi_0(x) = (\theta'/\theta)'(x) + 1/x^2$ and that for any $\gamma \in \Gamma - \{0\}$

$$\varphi_\gamma(x) = \partial_x \left( e^{2\pi i x} \frac{\theta(\tilde{\gamma} + x)}{\theta(\tilde{\gamma})} - \frac{1}{x} \right),$$

where $\tilde{\gamma} = (c_0, c) \in \Lambda_{\tau, \Gamma} - \Lambda_\tau$ is any lift of $\gamma$.

Proposition 3.6. The equivariance identity (7) is satisfied for any $g \in (\mathbb{Z}^n)^2 \ast \text{SL}_2(\mathbb{Z})$. 

Before proving this statement, let us notice that the SL\(_2(\mathbb{Z})\)-equivariance is stronger than what we need (the SL\(_2(\mathbb{C})\)-equivariance), but easier to prove. The action of SL\(_2(\mathbb{Z})\) moves the poles while SL\(_2(\mathbb{C})\) fixes them. In both cases, it makes sense to prove this proposition for meromorphic forms on \(\mathbb{C}^n \times \mathfrak{h}\).

**Proof.** For \(g = (\delta_j,0)\), the identity translates into \(K_i(z + \delta_j|\tau) = K_i(z|\tau)\) (i = 1, ..., n) and \(\Delta(z + \delta_j|\tau) = \Delta(z|\tau)\), which are immediate.

For \(g = (0, \delta_j)\), the identity translates into \(K_i(z + \tau \delta_j|\tau) = e^{-2\pi i \text{ad}(\delta_j)} K_i(z|\tau)\) (\(\forall i\)) and

\[
\Delta(z + \tau \delta_j|\tau) + K_j(z + \tau \delta_j|\tau) = e^{-2\pi i \text{ad}(\delta_j)} \Delta(z|\tau).
\]

The first equality is proved in \(\S 1.4\), and we prove the second one now. First remember that for any \(\tau \in \mathfrak{H}\), \(z \in \mathbb{C} - \left(\frac{1}{2\tau} \mathbb{Z} + \frac{1}{2\tau} \mathbb{Z}\right)\) and \(\alpha \in \Gamma\), we have the following identity in \(\mathbb{C}[[z]]\):

\[
e^{-2\pi i z} (g_\alpha(x,z) - 1/\tau^2) + 1/\tau^2 - 2\pi i (k_\alpha(x,z + \tau) + 1/\tau) = g_\alpha(x,z + \tau).
\]

Then we can compute \(2\pi i (K_j(z + \tau \delta_j|\tau) - e^{-2\pi i \text{ad}(\delta_j)} \Delta(z|\tau))\) is equal to

\[
2\pi i \left(\sum_{k,k+j} k_\alpha(\text{ad}x_j, z_{jk} + \tau) - y_j\right) + \Delta_0 + \frac{1 - e^{-2\pi i \text{ad}x_j}}{\text{ad}x_j}(y_j) + \frac{1}{2} \sum_{\gamma \in \Gamma} A_{x,\gamma} \delta_{x,\gamma} - e^{-2\pi i \text{ad}x_j} \sum_{k < l} g_{kl}(z_{kl}),
\]

and therefore using

\[
\frac{1 - e^{-2\pi i \text{ad}x_j}}{\text{ad}x_j}(y_j) - 2\pi i y_j = \left(\frac{e^{-2\pi i \text{ad}x_j} - 1}{(\text{ad}x_j)^2}\right) + \frac{2\pi i}{\text{ad}x_j} \left(\sum_{\alpha \in \Gamma} \sum_{k,k+j} t_{jk}^\alpha\right)
\]

together with (9) we obtain

\[
\Delta_0 + \frac{1}{2} \sum_{s \geq 0, \gamma \in \Gamma} A_{x,\gamma} \delta_{x,\gamma} - \sum_{k < l} g_{kl}(z_{kl}) - \sum_{k,k+j} g_{kl}(\text{ad}x_j, z_{jk} + \tau) (t_{jk}^\alpha),
\]

which is precisely equal to \(-2\pi i \Delta(z + \tau \delta_j)\).

For \(g = S\), the identity translates into \(K_i(z|\tau + 1) = K_i(z)\) (\(\forall i\)) and \(\Delta(z|\tau + 1) = \Delta(z)\). Both equalities obviously follow from \(\theta(z|\tau + 1) = \theta(z|\tau)\).

For \(g = T\), the identity translates into

\[
\frac{1}{\tau} K_i\left(\frac{z}{\tau}\right) - \frac{1}{\tau} = \text{Ad} (c_T(z|\tau)) (K_i(z|\tau)) + 2\pi i x_i
\]

for all \(i \in \{1, \ldots, n\}\) and

\[
\frac{1}{\tau^2} \left(\Delta(\frac{z}{\tau}) - \frac{1}{\tau}\right) - \sum_i z_i K_i\left(\frac{z_i}{\tau} - \frac{1}{\tau}\right) = \text{Ad} (c_T(z|\tau)) (\Delta(z|\tau)) + \frac{d}{\tau} - 2\pi i X.
\]

Let us check (10) first. \(\text{Ad} (e^{2\pi i \sum_i z_i x_j} (y_i/\tau)) + 2\pi i x_i\) equals

\[
- \text{Ad} (e^{2\pi i \sum_i z_i x_j} (y_i/\tau)) = \frac{y_i}{\tau} - \frac{e^{2\pi i \text{ad}(\sum_j z_j x_j)} - 1}{\text{ad}(\sum_j z_j x_j)} \left(\sum_j z_j x_j, \frac{y_i}{\tau}\right)
\]

\[
= \frac{y_i}{\tau} \frac{e^{2\pi i \sum_j z_j x_j} - 1}{\sum_j z_j \text{ad}x_j} \left(\sum_{j \neq i} \frac{y_i}{\tau} \frac{x_i^j}{\text{ad}x_j} \right) = \frac{y_i}{\tau} \sum_{j \neq i} \frac{e^{2\pi i z_j x_j \text{ad}x_i}}{z_j \text{ad}x_i} \left(\sum_{\alpha \in \Gamma} \frac{z_j}{\tau} t_{ij}^\alpha\right).
\]
Therefore we have
\[(12)\quad -\frac{y_i}{\tau} = \text{Ad}(c_T(z|\tau))(-y_i) + 2\pi i x_i - \sum_{j:j\neq i} e^{2\pi i z_{ij} \text{ad}x_i} (\sum_{\alpha \in \Gamma} t_{ij}^\alpha - \frac{t_i^\alpha}{\tau}).\]

Now substituting \((x, z) = (\text{ad}x_j, z_j)\) in
\[(13)\quad \frac{1}{\tau}(k_\alpha(x, \frac{z}{\tau} \cdot 1 - \frac{1}{\tau}) = e^{2\pi i x z} k_\alpha(\tau x, z|\tau) + \frac{e^{2\pi i x z} - 1}{\tau},\]
then applying to \(t_{ij}^\alpha\), summing over \(j \neq i\) and \(\alpha \in \Gamma\), and adding up (12) we obtain (10) by using that
\[
e^{2\pi i z_{ij} \text{ad}x_i} k_\alpha(\tau x, z_{ij}|\tau)(t_{ij}^\alpha) = \text{Ad}(e^{2\pi i (\tau x + \Sigma_j z_j x_j)})(k_\alpha(\text{ad}x_i, z_{ij}|\tau)(t_{ij}^\alpha)).\]

We now check (11). Differentiating (13) w.r.t. \(x\) and dividing by \(\tau\), we get
\[
\frac{1}{\tau^2} g_\alpha(x, \frac{z}{\tau} \cdot 1 - \frac{1}{\tau}) = e^{2\pi i x z} g_\alpha(\tau x, z|\tau) + \frac{2\pi i z}{\tau^2} k_\alpha(x, \frac{z}{\tau} \cdot 1 - \frac{1}{\tau}) + \frac{1 + 2\pi i x z - e^{2\pi i x z}}{\tau^2 x^2}.
\]

Now substituting \((x, z) = (\text{ad}x_i, z_{ij})\), applying to \(t_{ij}^\alpha\), and summing over \(\alpha \in \Gamma\) we obtain
\[
\frac{1}{\tau^2} g_\alpha(x, \frac{z}{\tau} \cdot 1 - \frac{1}{\tau}) = \text{Ad}(c_T(z|\tau))(g_{ij}(z|\tau)) + \frac{2\pi i z_{ij}}{\tau^2} K_{ij}(\frac{z_{ij}}{\tau} - \frac{1}{\tau}) + \left(1 + 2\pi i z_{ij} \text{ad}x_i - e^{2\pi i z_{ij} \text{ad}x_i}\right)(\sum_{\alpha \in \Gamma} t_{ij}^\alpha).
\]

Then taking the sum over \(i < j\) one gets
\[(14)\quad \frac{1}{\tau^2} \sum_{i < j} g_{ij}(\frac{z}{\tau} \cdot 1 - \frac{1}{\tau}) = \text{Ad}(c_T(z|\tau))\left(\sum_{i < j} g_{ij}(z|\tau)\right) + \frac{2\pi i}{\tau^2} \sum_i z_i K_i(\frac{z_i}{\tau} - \frac{1}{\tau}) + B(z),\]
where
\[
B(z) := \sum_i 2\pi i z_i y_i \frac{1}{\tau^2} + \sum_{i < j} \left(1 + 2\pi i z_{ij} \text{ad}x_i - e^{2\pi i z_{ij} \text{ad}x_i}\right)(\sum_{\alpha \in \Gamma} t_{ij}^\alpha).
\]

**Lemma 3.7.** \(\text{Ad}(c_T(z|\tau))(\Delta_0) = \frac{\Delta_0 + 2\pi i d}{\tau} - (2\pi i)^2 \left(\frac{1}{\tau} \sum_i z_i x_i + X\right) + B(z).\)

**Proof of the lemma.** We first compute
\[
\text{Ad}(c_T(z|\tau))(\Delta_0) =\text{Ad}(e^{2\pi i (\tau x + \Sigma_i z_i x_i)})(\frac{\Delta_0}{\tau^2}) = \text{Ad}(e^{2\pi i \Sigma_i z_i x_i})(\frac{\Delta_0}{\tau^2} + \frac{2\pi i d}{\tau} - (2\pi i)^2 X)
\]
\[
= \text{Ad}(e^{2\pi i \Sigma_i z_i x_i})(\frac{\Delta_0}{\tau^2}) + \frac{2\pi i d}{\tau} - (2\pi i)^2 \left(\frac{1}{\tau} \sum_i z_i x_i + X\right).
\]

It remains to show that \(\text{Ad}(e^{2\pi i \Sigma_i z_i x_i})(\frac{\Delta_0}{\tau^2}) = \frac{\Delta_0 + 2\pi i d}{\tau} + B(z)\). The proof of this fact goes along the same lines of computation as in [6, pp.16-17].

Using the above lemma and equation (14), one sees that equation (11) follows from
\[
\text{Ad}(c_T(z|\tau))(\sum_{s,\gamma} A_{s,\gamma}(\tau)\delta_{s,\gamma}) = \sum_{s,\gamma} A_{s,\gamma}(\frac{\tau - 1}{\tau}) \delta_{s,\gamma}.
\]
This last equality is proved using \([x_1, \delta_{s,\gamma}] = 0 = [X, \delta_{s,\gamma}], [d, \delta_{s,\gamma}] = s\delta_{s,\gamma}\), and, since
\[
\varphi_\gamma(x) - \frac{1}{\tau} = \tau^2 \varphi_\gamma(\tau x | \tau),
\]
we get \(A_{s,\gamma}(-\frac{1}{\tau}) = \tau^{s+2} A_{s,\gamma}(\tau)\).

We therefore have:

**Theorem 3.8.** \(\nabla_{n,\Gamma}\) defines a connection on \(\mathcal{P}_{n,\Gamma}\). Moreover, its image under \(G_n^\Gamma \to G_n^\Gamma\) is the pull-back of a connection \(\nabla_{n,\Gamma}\) on \(\mathcal{P}_{n,\Gamma}\).

**Proof.** The first part follows from Proposition 3.6 above. For the second part, we need to prove the three following identities:

\begin{itemize}
  \item \(\sum \bar{K}_i(z | \tau) = 0\);
  \item \(\bar{K}_i(z + u \sum \delta_j | \tau) = \bar{K}_i(z | \tau)\), for all \(i\);
  \item \(\bar{\Delta}(z + u \sum \delta_j | \tau) = \bar{\Delta}(z | \tau)\).
\end{itemize}

The first two equalities have already been proven, and the last one is obvious.

\[\square\]

\[\]~

3.4. Flatness. In this paragraph we prove the flatness of \(\nabla_{n,\Gamma}\) (and thus of \(\nabla_{n,\Gamma}\)).

**Proposition 3.9.** For any \(i \in \{1, \ldots, n\}\) we have \([\bar{\partial}_i - \Delta(z | \tau), \partial_i - K_i(z | \tau)] = 0\).

In what follows, we often drop \(\tau\) from the notation when it does not lead to any confusion.

**Proof.** Let us first prove that \(\partial_i K_i(z) = \partial_i \Delta(z)\). This follows from the identity \(\partial_i g_\alpha(x, z) = 2\pi i \partial_i k_\alpha(x, z)\), which is proved as follows (here \(\tilde{\alpha} = (a_0, a)\) is any lift of \(\alpha\)):

\[
\partial_i g_\alpha(x, z) = \partial_i \partial_x k_\alpha(x, z) = \partial_x \partial_i \left(e^{-2\pi i ax} k(x, z - \tilde{\alpha}) + e^{-2\pi i ax} - 1\right) \\
= e^{-2\pi i ax} \partial_x \partial_i k(x, z - \tilde{\alpha}) - 2\pi i a e^{-2\pi i ax} \partial_x k(x, z - \tilde{\alpha}) \\
= 2\pi i e^{-2\pi i ax} \partial_x k(x, z - \tilde{\alpha}) - 2\pi i a e^{-2\pi i ax} \partial_x k(x, z - \tilde{\alpha}) \\
= 2\pi i \partial_x e^{-2\pi i ax} k(x, z - \tilde{\alpha}) = 2\pi i \partial_x k_\alpha(x, z).
\]

It remains to prove that \([\Delta(z), K_i(z)] = 0\).

Let us first prove it in the case \(n = 2\). Namely, we will prove that

\[
[\Delta_0 + \frac{1}{2} \sum_{s,\gamma} A_{s,\gamma} \delta_{s,\gamma} - \sum_{\alpha \in \Gamma} g_\alpha(ad \cdot t_1, z)(t_{12}^\Delta), y_2 + \sum_{\beta \in \Gamma} k_\beta(ad \cdot t_1, z)(t_{12}^\Delta)] = 0.
\]

One the one hand,

\[
[\Delta_0 + \frac{1}{2} \sum_{s,\gamma} A_{s,\gamma} \delta_{s,\gamma} - \sum_{\alpha \in \Gamma} g_\alpha(ad \cdot t_1, z)(t_{12}^\Delta), y_2] = [y_1, \sum_{\alpha \in \Gamma} g_\alpha(ad \cdot t_1, z)(t_{12}^\Delta)] - \frac{1}{2} \sum_{\alpha, \gamma} \sum_{p, q} a_{p, q}^\gamma [ad^p x_1(t_{12}^{\alpha, \gamma}), ad^q x_1(t_{12}^{\alpha, \gamma})],
\]

where

\[
\varphi_\gamma(u) - \varphi_\gamma(v) = \sum_{p, q} a_{p, q}^\gamma u^p v^q.
\]

\[\]~
On the other hand, we have

\[
\Delta_0, \sum_{\beta} k_\beta(\text{ad}\, x, z)\left(t^\beta_{12}\right) = [y_1, \sum_{\beta} g_\beta(\text{ad}\, x_1, z)\left(t^\beta_{12}\right)] \\
+ \sum_{p,q,\alpha,\beta \in \Gamma} b^\alpha_{p,q}(z) [\text{ad}^p x_1(t^\alpha_{12}), \text{ad}^q x_1(t^\beta_{12})],
\]

where the series \( \sum_{p,q} b^\alpha_{p,q}(z) u^pv^q \) is given by

\[
\frac{1}{2} \left( \frac{1}{v^2} (k_\beta(u + v, z) - k_\beta(u, z) - v\partial_u k_\beta(u, z)) - \frac{1}{u^2} (k_\alpha(u + v, z) - k_\alpha(u, z) - u\partial_u k_\alpha(v, z)) \right).
\]

Therefore the l.h.s. of (15) equals

\[
\frac{1}{2} \left( \sum_{p,q,\alpha,\beta \in \Gamma} c^\alpha_{p,q}(z) [\text{ad}^p x_1(t^\alpha_{12}), \text{ad}^q x_1(t^\beta_{12})] \right),
\]

where \( \sum_{p,q} c^\alpha_{p,q} u^pv^q(z) \) is given by

\[
\frac{1}{v^2} (k_\beta(u + v, z) - k_\beta(u, z) - v\partial_u k_\beta(u, z)) - \frac{1}{u^2} (k_\alpha(u + v, z) - k_\alpha(u, z) - u\partial_u k_\alpha(v, z)) \\
\quad + \frac{\varphi_{\alpha - \beta}(u) - \varphi_{\alpha - \beta}(v)}{u + v} k_\alpha(u + v, z) \varphi_{\alpha - \beta}(v) - k_\beta(u + v, z) \varphi_{\alpha - \beta}(u) \\
\quad + k_\beta(u, z) g_\alpha(v, z) - g_\beta(u, z) k_\alpha(v, z),
\]

which can be rewritten as

\[
\begin{aligned}
&\left( g_{\beta - \alpha}(u, z - z') - \frac{1}{u^2} \right) \left( k_{\alpha}(u + v, z') + \frac{1}{u + v} \right) - \left( g_{\alpha - \beta}(v, z' - z) - \frac{1}{v^2} \right) \left( k_{\beta}(u + v, z) + \frac{1}{u + v} \right) \\
&\quad + \left( g_{\alpha}(v, z') - \frac{1}{u^2} \right) \left( k_{\beta}(u, z) + \frac{1}{u} \right) - \left( g_{\beta}(u, z) - \frac{1}{v^2} \right) \left( k_{\alpha}(v, z') + \frac{1}{v} \right)
\end{aligned}
\]

with \( z = z' \). Thus to end the proof of equation (15) the following lemma is sufficient:

**Lemma 3.10.** Expression (16) equals zero.

**Proof of the lemma.** The case \( \alpha = \beta = 0 \) follows from an explicit computation. Then we chose lifts \( \tilde{\alpha} = (a_0, a) \) and \( \tilde{\beta} = (b_0, b) \) of \( \alpha \) and \( \beta \), respectively. One has

\[
k_{\alpha}(x, z) + 1/2 = e^{-2i\pi x} (k(x, z - \tilde{\alpha}) + 1/2) \quad \text{and} \\
g_{\alpha}(x, z) - 1/2 = e^{-2i\pi x} (g(x, z - \tilde{\alpha}) - 1/2x^2) - 2i\pi b (k_{\alpha}(x, z) + 1/2x).
\]

Therefore (16) equals

\[
\begin{aligned}
&\quad -2\pi(a - b) \left( \left( k_{\alpha}(v, z') + \frac{1}{v} \right) \left( k_{\beta}(u, z) + \frac{1}{u} \right) + \left( k_{\beta - \alpha}(u, z - z') + \frac{1}{u} \right) \left( k_{\alpha}(u + v, z') + \frac{1}{u + v} \right) \\
&+ \left( k_{\alpha - \beta}(v, z' - z) - \frac{1}{v} \right) \left( k_{\beta}(u + v, z) + \frac{1}{u + v} \right) \right) \\
&\quad = 0,
\end{aligned}
\]

which vanishes because of (3).

Let us now assume that \( n > 2 \).

Let \( t^\alpha_{i,j} \subset t^\alpha_{1,n} \) be the subalgebra generated by \( x_i, t^\alpha_{j,k} \) (\( i, j, k = 1, \ldots, n, j \neq k, \alpha \in \Gamma \)).
We have functions \( E_{ij}(z) \) with values in \( t^{i}_{a,n} \), defined by \( E_{ij}(z) = [\Delta_0, k_{ij}] - [y_i, g_{ij}] \), which decomposes as \( e_{ij}(z) + \sum_{k+i,j} e_{ijk}(z) \), where \( e_{ij}(z) \) takes its values in
\[
\text{Span}_{p,q,a,b}[(\text{ad}x_i)^p(t^{\alpha}_{ijk}), (\text{ad}x_j)^q(t^{\beta}_{ijk})]
\]
and \( e_{ijk}(z) \) takes its values in \( \text{Span}_{a,b} C[\text{ad}x_i, \text{ad}x_j][t^\alpha_{ijk}, t^\beta_{ijk}] \). Explicitly,
\[
e_{ij}(z) = \sum_{a,b} \sum_{p,q} b^{a,b}_{p,q}(z_{ij}) [\text{ad}^p x_i(t^\alpha_{ij}), \text{ad}^q x_i(t^\beta_{ij})],
\]
where \( b^{a,b}_{p,q}(z) \) is as before, and
\[
e_{ijk}(z) = \sum_{a,b} \left( k^a_{ij}(\text{ad}x_i, z_{ij}) - k^a_{ij}(-\text{ad}x_j, z_{ij}) - g^a_{ij}(-\text{ad}x_j, z_{ij}) \right) \frac{([\text{ad}x_i, \text{ad}x_j])}{\text{ad}x_i + \text{ad}x_j} [t^\alpha_{ijk}, t^\beta_{ijk}].
\]

On the other hand, we have \( Y_{ijk}(z) \in t^{i}_{a,n} \), defined by \( Y_{ijk}(z) = [y_i, g_{jk}] \). It takes its values in \( \text{Span}_{a,b} C[\text{ad}x_i, \text{ad}x_j][t^\alpha_{ijk}, t^\beta_{ijk}] \). Explicitly,
\[
Y_{ijk}(z) = -\sum_{a,b} g^a_{ij}(\text{ad}x_j, z_{jk}) - g^b_{ij}(\text{ad}x_k, -z_{jk}) \frac{([\text{ad}x_j, \text{ad}x_k])}{\text{ad}x_j + \text{ad}x_k} [t^\alpha_{ijk}, t^\beta_{ijk}]
\]
(remember that \( g^a_{ij}(u, z) = g^b_{-a}(-u, -z) \)). We have
\[
[\Delta(z), K_1(z)] = \sum_{i \neq 1} \left( [\Delta_0, k_{11}] - [y_i, g_{11}] + \frac{1}{2} \sum_{\alpha} \delta_{\varphi_{\alpha}}(y_i) - [g_{1i}, k_{11}] - \frac{1}{2} \sum_{\alpha} \delta_{\varphi_{\alpha}}(y_i) \right)
- \sum_{1 < i < j} \left( [g_{1i}, k_{1j}] + [g_{1i}, k_{11} + k_{1j}] \right)
(17)
\]
where \( \{\}_{1i} \) is the natural morphism \( t^{i}_{1,2} \to t^{i}_{1,n} \), \( u_1 \mapsto u_1, u_2 \mapsto u_1, u_3 \mapsto u_1 \) \((u = x, y)\), \( t^i_{1i} \mapsto t^i_{1i} \). It is easy to see that the line \((17)\) equals \( \sum_{i \neq 1} ([\Delta(z_{1i}), K_1(z_{1i})])_{1i} \), which is zero as we have seen before (case \( n = 2 \)).

Therefore \([\Delta(z), K_1(z)]\) equals
\[
\sum_{1 < i < j} \sum_{a,b} \left( \frac{k^a_{ij}(\text{ad}x_j, z_{ij}) - k^a_{ij}(-\text{ad}x_i, z_{ij}) - g^a_{ij}(-\text{ad}x_i, z_{ij}) \text{ad}x_i + \text{ad}x_j)}{\text{ad}x_i + \text{ad}x_j} [t^\alpha_{ijk}, t^\beta_{ijk}] 
- \frac{k^b_{ij}(\text{ad}x_i, z_{ij}) - k^b_{ij}(-\text{ad}x_i, z_{ij}) - g^b_{ij}(-\text{ad}x_i, z_{ij}) \text{ad}x_i + \text{ad}x_j)}{\text{ad}x_i + \text{ad}x_j} [t^\alpha_{ijk}, t^\beta_{ijk}]
\]
and is zero because of Lemma 3.10.

We have therefore proved (Proposition 1.8 and Proposition 3.9 above):
Theorem 3.11. The connection $\nabla_{n,\Gamma}$ is flat, and thus so is $\overline{\nabla}_{n,\Gamma}$. \qed

Let us now show how the universal KZB connexion over moduli spaces coincides with the one defined over configuration spaces.

Remark 3.12. The connection $\nabla_{n,\Gamma}$ defined above is an extension to the twisted moduli space $\mathcal{M}_{1,n}^\Gamma$ of the connection $\nabla_{1,n,\Gamma}$ defined over the twisted configuration space $\text{Conf}(E_{\tau,\Gamma}, n, \Gamma)$ from Subsection 1.4.

Indeed, the pull-back of the principal $G_n^\Gamma$-bundle with flat connection $(\mathcal{P}_{n,\Gamma}, \nabla_{n,\Gamma})$ along the inclusion

$$\text{Conf}(E_{\tau,\Gamma}, n, \Gamma) \hookrightarrow \mathcal{M}_{1,n}^\Gamma$$

of the fiber at (the class of) $\tau$ in $Y(\Gamma)$ admits a reduction of structure group to

$$\exp(t_{1,n}^\Gamma) \subset G_n^\Gamma,$$

and one easily sees from our explicit formulæ that it coincides with $(\mathcal{P}_{\tau,\Gamma}, \nabla_{\tau,n,\Gamma})$ constructed in Subsection 1.4.

Similarly, the connection $\nabla_{n,\Gamma}$ is an extension to the twisted moduli space $\mathcal{M}_{1,n}^\Gamma$ of the connection $\nabla_{1,n,\Gamma}$ defined over the reduced twisted configuration space $C(E_{\tau,\Gamma}, n, \Gamma)$.

3.5. Variations. Let us first consider the unordered variants

$$\mathcal{M}_{1,[n]}^\Gamma := \mathcal{M}_{1,n}^\Gamma/\mathfrak{S}_n \quad \text{and} \quad \mathcal{M}_{1,[n]}^\Gamma := \mathcal{M}_{1,n}/\mathfrak{S}_n,$$

where, as before, the action of $\mathfrak{S}_n$ is again by permutation on $C^n$.

Proposition 3.13. 1. There exists a unique principal $G_{n}^\Gamma \times \mathfrak{S}_n$-bundle $\mathcal{P}_{[n],\Gamma}$ over $\mathcal{M}_{1,[n]}^\Gamma$, such that a section over $U \subset \mathcal{M}_{1,[n]}^\Gamma$ is a function

$$f : \tilde{\pi}^{-1}(U) \to G_{n}^\Gamma \times \mathfrak{S}_n$$

satisfying the conditions of Proposition 3.3 as well as $f(\sigma z|\tau) = \sigma f(z|\tau)$ for $\sigma \in \mathfrak{S}_n$ (here $\tilde{\pi} : (C^n \times \tilde{\Delta}) - \text{Diag}_{n,\Gamma} \to \mathcal{M}_{1,[n]}^\Gamma$ is the canonical projection).

2. There exists a unique flat connection $\nabla_{[n],\Gamma}$ on $\mathcal{P}_{[n],\Gamma}$, whose pull-back to $(C^n \times \tilde{\Delta}) - \text{Diag}_{n,\Gamma}$ is the connection

$$d - \Delta(z|\tau) d\tau - \sum_i K_i(z|\tau) d z_i$$
on the trivial $G_n^\Gamma \times \mathfrak{S}_n$-bundle.

3. The image of $(\mathcal{P}_{[n],\Gamma}, \nabla_{[n],\Gamma})$ under $G_n^\Gamma \times \mathfrak{S}_n \to G_n^\Gamma \times \mathfrak{S}_n$ is the pull-back of a flat principal $G_n^\Gamma \times \mathfrak{S}_n$-bundle $(\mathcal{P}_{[n],\Gamma}, \nabla_{[n],\Gamma})$ on $\mathcal{M}_{1,[n]}^\Gamma$.

Proof. For the proof of the first point, one easily checks that $\sigma c_{\tilde{\gamma}}(z|\tau) \sigma^{-1} = c_{\tilde{\gamma}\sigma^{-1}}(\sigma^{-1} z)$, where $\tilde{\gamma} \in (C^n)^2 \times \text{SL}_2(Z)$, $\sigma \in \mathfrak{S}_n$. It follows that there is a unique cocycle $c_{(\tilde{\gamma},\sigma)} : C^n \times \tilde{\Delta} \to G_n^\Gamma \times \mathfrak{S}_n$ such that $c_{(\tilde{\gamma},1)}(z) = c_{\tilde{\gamma}}$ and $c_{(1,\sigma)}(z|\tau) = \sigma$.

For the proof of the second point, taking into account Theorem 3.11, one only has to show that this connection is $\mathfrak{S}_n$-equivariant. We have already mentioned that $\sum_i K_i(z|\tau) d z_i$ is equivariant, and $\Delta(z|\tau)$ is also checked to be so.

The third point is obvious. \qed
For every (class of) $\tau$ in $Y(\Gamma)$, one has an action of $\Gamma^n$ on the fiber $\text{Conf}(E_{\tau,\Gamma}, n, \Gamma)$ at $\tau$ of $\mathcal{M}_{1,n}^\Gamma \to Y(\Gamma)$, resp. an action of $\Gamma^n/\Gamma$ on the fiber $\mathcal{C}(E_{\tau,\Gamma}, n, \Gamma)$ at $\tau$ of $\mathcal{M}_{1,n}^\Gamma \to Y(\Gamma)$. 

Recall that 

$$\text{Conf}(E_{\tau,\Gamma}, n, \Gamma)/\Gamma^n = \text{Conf}(E_{\tau,\Gamma}, n) \quad \text{and} \quad \mathcal{C}(E_{\tau,\Gamma}, n, \Gamma)/(\Gamma^n/\Gamma) = \mathcal{C}(E_{\tau,\Gamma}, n).$$

This action depends holomorphically of $\tau$, so that we have an action of $\Gamma^n$ on $\mathcal{M}_{1,n}$, resp. an action of $\Gamma^n/\Gamma$ on $\mathcal{M}_{1,n}$.

**Proposition 3.14.** 1. There exists a unique principal $G_1^n \times \Gamma^n$-bundle $P_{(\Gamma), n}$ over $\mathcal{M}_{1,n}^\Gamma/\Gamma^n$, such that a section over $U \subset \mathcal{M}_{1,n}^\Gamma/\Gamma^n$ is a function 

$$f : \tilde{\pi}^{-1}(U) \to G_1^n \times \Gamma^n$$

satisfying the following conditions:

$$f(z + \frac{\delta_i}{M}|\tau) = (1, 0), f(z|\tau),$$

$$f(z + \frac{\delta_j}{N}|\tau) = e^{2\pi i \frac{\delta_j}{N}}(0, 1), f(z|\tau),$$

$$f(z, \tau + 1) = f(z|\tau),$$

$$f(z, \tau) + 1) = \tau^k e^{\sum_{i} z_i} f(z|\tau).$$

Here, $\tilde{\pi} : (\mathbb{C}^n \times \mathfrak{g}) - \text{Diag}_{n, \Gamma} \to \mathcal{M}_{1,n}^\Gamma/\Gamma^n$ is the canonical projection. 

2. There exists a unique flat connection on this bundle whose pull-back to $(\mathbb{C}^n \times \mathfrak{g}) - \text{Diag}_{n, \Gamma}$ is the connection 

$$d - \Delta(z|\tau) d \tau - \sum_i K_i(z|\tau) d z_i$$

on the trivial $G_1^n \times \Gamma^n$-bundle. 

3. The image of the above flat bundle under $G_1^n \times \Gamma^n \to \mathcal{G}^\Gamma_1 \times (\Gamma^n/\Gamma)$ is the pull-back of a flat principal $G_1^n \times (\Gamma^n/\Gamma)$-bundle on $\mathcal{M}_{1,n}^\Gamma/(\Gamma^n/\Gamma)$.

**Proof.** The first assertion is left to the reader. Assertion 3 is evident. Let us prove assertion 2. By Proposition 1.7, we know that the $K_i$ satisfy 

- $K_i(z + \frac{\delta_j}{M}|\tau) = (1, 0), K_i(z|\tau),$
- $K_i(z + \frac{\delta_j}{N}|\tau) = (0, 1), e^{\frac{2\pi i}{\gamma_j}} K_i(z|\tau).$

The fact that $\Delta(z + \frac{\delta_j}{N}|\tau) = (1, 0, 1) \cdot \Delta(z|\tau)$ is immediate. Thus, it remains to show that 

$$\Delta(z + \frac{\delta_j}{M}|\tau) = e^{2\pi i \frac{\delta_j}{N}}(0, 1, (\Delta(z|\tau) - K_j(z|\tau))$$

which is proved in Lemma 3.15 below. \(\square\)

**Lemma 3.15.** We have

$$\Delta(z + \frac{\delta_j}{M}|\tau) = e^{2\pi i \frac{\delta_j}{N}}(0, 1, (\Delta(z|\tau) - K_j(z|\tau)).$$

**Proof.** On the one hand, we have 

$$-2\pi i \Delta(z + \frac{\delta_j}{N}) = \Delta_0 + \frac{1}{z} \sum_{x \in \text{red} \Gamma} A_x, \gamma \delta_{x, \gamma} - \sum_{k \in \Gamma} g_{k, i} z_k - \sum_{k \in \Gamma} g_{k, j} z_k + \frac{\tau}{N}(t_0^n).$$
On the other hand, as

\[ e^{-2\pi i \text{ad}(x_j)} (\Delta_0) = (1 - (1 - e^{-2\pi i \text{ad}(x_j)})) (\Delta_0) = (\Delta_0) + \frac{1 - e^{-2\pi i \text{ad}(x_j)}}{\text{ad}x_j} (y_j) \]

and the \( \delta_{\alpha, \gamma} \) commute with the \( x_j \), we compute

\[ 2\pi i \left( K_j (z + \frac{\tau}{N} \delta_j \tau) - e^{-2\pi i \text{ad}(x_j)} (0, 1) \cdot \Delta(z) \right) \]

\[ = 2\pi i \left( (0, -1) \cdot K_j (z + \frac{\tau}{N} \delta_j \tau) - e^{-2\pi i \text{ad}(x_j)} \Delta(z) \right) \]

\[ = 2\pi i (\bar{0}, -1) \left( \sum_{k \neq j} k_\alpha (\text{ad}x_j, z_{jk} + \frac{\tau}{N}) - y_j \right) + \frac{1}{2} \sum_{\gamma \in \Gamma} A_{\alpha, \gamma} \delta_{\alpha, \gamma} - e^{-2\pi i \text{ad}(x_j)} \sum_{k \leq l} g_{kl} (z_{kl}). \]

Next, by combining

\[ K_{ij} (z - \frac{\tau}{N}) = e^{-2\pi i \text{ad}(x_i)} (0, -1)_i \cdot (K_{ij} (z)) + (0, -1)_i \cdot \left( \sum_{\alpha \in \Gamma} e^{-2\pi i \text{ad}(x_i)} - \frac{1}{\text{ad}x_i} (t_{ij}^\alpha) \right), \]

and equations

\[ g_{\alpha}(x, z) - 1/x^2 = e^{-2\pi i \alpha x} \left( g(x, z - \alpha) - 1/x^2 \right) - 2i \pi b (k_{\alpha}(x, z) + 1/x). \]

We can follow the same lines as in the proof of relation (8) to obtain the wanted equation. \( \square \)

We also leave to the reader the task of combining several variants.

4. Realizations

4.1. Realizations of \( L_{\tau, n}^\Gamma \) and \( L_{\tau, n}^\Gamma \). Let \( g \) be a Lie algebra and \( t_\theta \in S^2(g)^\theta \) be nondegenerate. Assume that we have a group morphism \( \Gamma \rightarrow \text{Aut}(g, t_\theta) \) and set \( L := g^\Gamma \) and \( :\oplus_{\chi \in \Gamma} g_\chi \),

where \( g_\chi \) is the eigenspace of \( g \) corresponding to the character \( \chi : \Gamma \rightarrow \mathbb{C}^\ast \). Then we have \( g = l \oplus u \) with \([1, u] \subset u\), and \( t = t_1 + t_2 \) with \( t_1 \in S^2(l)^l \) and \( t_2 \in S^2(u)^l \). We denote by \((a, b) \mapsto (a, b)\) the invariant pairing on \( \mathfrak{l} \) corresponding to \( t_1 \) and write \( t_l = \sum_{\nu \in \nu} e_{\nu} \otimes e_{\nu}. \)

Let \( \text{Diff}(\mathfrak{l}) \) be the algebra of algebraic differential operators on \( \mathfrak{l} \). It has generators \( x_l, \partial_l \) \((l \in \mathfrak{l})\) and relations \( x_{l, \nu} = t x_l + x_{l, \nu}, \partial_{l, \nu} = t \partial_l + \partial_{l, \nu}, [x_l, x_{l, \nu}] = 0 = [\partial_l, \partial_{l, \nu}] \) and \([\partial_l, x_{l, \nu}] = \langle l, \nu \rangle\). Moreover, one has a Lie algebra morphism \( \mathfrak{l} \rightarrow \text{Diff}(\mathfrak{l}); l \mapsto X_l := \sum_{\nu \in \nu} x_l(x_{l, \nu}) \partial_{e_{\nu}} \).

We denote by \( \text{ad}_{\mathfrak{l}} \) the image of the induced morphism

\[ I \mapsto Y_l := X_l \otimes 1 + 1 \otimes \sum_{l = 1}^n l^{(i)}(I) \otimes U_{\otimes} U(g)^\otimes \]

and define \( H_n(g, \mathfrak{l}) \) as the Hecke algebra of \( A_n := \text{Diff}(\mathfrak{l}) \otimes U(g)^\otimes \) with respect to \( \text{ad}_{\mathfrak{l}} \).

Namely, \( H_n(g, \mathfrak{l}) := (A_n)^f(A_n, t_{\text{ad}_{\mathfrak{l}}}^f). \) It acts in an obvious way on \((O_{\nu} \otimes (\otimes_{j=1}^n V_j)^{\otimes})\) if \((V_j)_{1 \leq j \leq n}\) is a collection of \( g \)-modules.
Let us set $x_\nu := x_{e_\nu}$ and $\partial_\nu := \partial_{e_\nu}$, and write $\alpha(i)$, for the action of $\alpha \in \Gamma$ on the $i$-th component in $U(g)^{\otimes n}$.

**Proposition 4.1.** There is a unique Lie algebra morphism $\rho_g : \overline{t}_{1,n} \to H_n(g, l^\ast)$ defined by

\[\begin{align*}
\bar{x}_i & \mapsto M \sum_{\nu} x_\nu \otimes e^{(i)}_\nu, \\
\bar{y}_i & \mapsto -N \sum_{\nu} \partial_\nu \otimes e^{(i)}_\nu, \\
\bar{t}_{ij} & \mapsto 1 \otimes (\alpha(i) \cdot t_g)^{(ij)}.
\end{align*}\]

**Proof.** Let us use the presentation of $\overline{t}_{1,n}$ coming from Lemma 1.1. The only non trivial check is that the relation $\sum_j \bar{x}_j = 0$ is preserved. We have

\[\rho_g \left( \sum_{i=1}^n x_i \right) = M \sum_{\nu} x_\nu \otimes \sum_{i=1}^n e^{(i)}_\nu = M \sum_{\nu} (x_\nu \otimes 1) \left( \sum_{i=1}^n e^{(i)}_\nu \right) \]

\[\equiv M \sum_{\nu} (x_\nu \otimes 1) (Y_\nu - X_\nu \otimes 1) \]

\[\equiv M - \sum_{\nu} x_\nu X_\nu \otimes 1 = M \sum_{\nu_1, \nu_2} x_{\nu_1} x_{[\nu_1, \nu_2]} \partial_{\nu_2} \otimes 1 = 0\]

as $x_{\nu_1}$ commutes with $x_{[\nu_1, \nu_2]}$ and $t_l$ is invariant. Here the sign $\equiv$ means that both terms define the same equivalence class in $H_n(g, l)$.

The proof that $\sum_j \bar{y}_j = 0$ is preserved is a consequence of the fact that $\rho_g \left( \sum_j \bar{y}_j \right) = 0$, which was proven in [6, Proposition 6.1]. \(\square\)

Let $\overline{t}_{n,+} \subset \overline{t}_{1,n}$ be the Lie subalgebra generated by $\bar{x}_i$’s and $\bar{t}_{ij}$’s. Then the restriction of $\rho_g$ to $\overline{t}_{n,+}$ lifts to a Lie algebra morphism $\overline{t}_{n,+} \to (\mathcal{O}_{l^*} \otimes U(g)^{\otimes n})^l$. Moreover, $(\mathcal{O}_{l^*} \otimes U(g)^{\otimes n})^l$ is a subalgebra of $H_n(g, l^\ast)$ that is a Lie ideal for the commutator, and one has a commutative diagram

\[\begin{array}{ccc}
\overline{t}_{n,+} \times \overline{t}_{n,+} & \xrightarrow{(u,v) \mapsto [u,v]} & \overline{t}_{n,+} \\
\downarrow & & \downarrow \\
H_n(g, l^\ast) \times (\mathcal{O}_{l^*} \otimes U(g)^{\otimes n})^l & \xrightarrow{} & (\mathcal{O}_{l^*} \otimes U(g)^{\otimes n})^l.
\end{array}\]

**4.2. Realizations of $\overline{t}_{1,n} \rtimes \varGamma$.** Let us write $t_g = \sum a_u \otimes a_u$. 

Proposition 4.2. The Lie algebra morphism $\rho_\eta$ of Proposition 4.1 extends to a Lie algebra morphism $\Omega^1_{\eta,n} \times \Omega^\ast \to H^\ast_{n}(\mathfrak{g},\Omega^\ast)$ defined by

\[
\begin{align*}
\mathbf{d} & \mapsto -\frac{1}{2} \left( \sum_{\nu} x_{\nu} \partial_\nu + \partial_\nu x_\nu \right) \otimes 1, \\
\mathbf{X} & \mapsto \frac{1}{2} \left( \sum_{\nu} x_\nu^2 \right) \otimes 1, \\
\Delta_0 & \mapsto -\frac{1}{2} \left( \sum_\nu \partial_\nu^2 \right) \otimes 1,
\end{align*}
\]

\[
\xi_{s,\gamma} \mapsto \frac{1}{|\Gamma|} \sum_{\nu_1,\ldots,\nu_s,\nu_\gamma} x_{\nu_1} \cdots x_{\nu_s} x_\gamma \otimes \sum_{i=1}^n \left( \text{ad}(e_{\nu_i}) \cdots \text{ad}(e_{\nu_s}) (a_u) \otimes (\gamma \cdot a_u) \right)^{(i)}.
\]

Here $\otimes$ denotes the symmetric product: $A \otimes B := AB + BA$.

Proof. Since $t_\eta$ is invariant under the commuting actions of $\Gamma$ and $\Gamma$ then the relation $\xi_{s,\gamma} = (-1)^s \xi_{s,-\gamma}$ is also preserved. This invariance argument also implies that $[\rho_{\eta}(\xi_{s,\gamma}), \rho_{\eta}(\bar{\xi}_i)]$ equals

\[
\frac{1}{|\Gamma|} \sum_{\nu_1,\ldots,\nu_s,\nu_\gamma} x_{\nu_1} \cdots x_{\nu_s} x_\gamma \otimes \sum_{i=1}^n \left( \text{ad}(e_{\nu_i}) \cdots \text{ad}(e_{\nu_s}) (a_u) \otimes (\gamma \cdot a_u) \right)^{(i)},
\]

which is zero since the first and second factors are respectively symmetric and antisymmetric in $(\nu, \nu_s)$. Let us now prove that the relation $[\xi_{s,\gamma}, t_{ij}^\ast] = [t_{ij}^\ast, (\text{ad}x_i^\ast) (t_{ij}^\ast\gamma) + (\text{ad}x_j^\ast) (t_{ij}^\ast\gamma)]$ is preserved. It is sufficient to do it for $n = 2$:

\[
\rho_{\eta}(\xi_{s,\gamma}) + (\text{ad}x_1^\ast) (t_{12}^\ast\gamma) + (\text{ad}x_2^\ast) (t_{12}^\ast\gamma) = \sum_{\nu_1,\ldots,\nu_s} x_{\nu_1} \cdots x_{\nu_s} \otimes (\gamma \cdot a_u),
\]

where $\Delta$ is the standard coproduct of $U_{\mathfrak{g}}$ and $B_{\nu_1,\ldots,\nu_s} := \sum u \text{ad}(e_{\nu_s}) \cdots \text{ad}(e_{\nu_s}) (a_u) \otimes (\gamma \cdot a_u)$; therefore $[\rho_{\eta}(\xi_{s,\gamma}) + (\text{ad}x_1^\ast) (t_{12}^\ast\gamma) + (\text{ad}x_2^\ast) (t_{12}^\ast\gamma)]$ commutes with $\rho_{\eta}(t_{ij}^\ast)$. Hence it remains to prove that the relation $[\xi_{s,\gamma}, t_{ij}^\ast] = \sum_{i,j,s} D_{s,\gamma} (t_{ij}^\ast, \gamma)$ is preserved. For this we compute $[\rho_{\eta}(\xi_{s,\gamma}), \rho_{\eta}(t_{ij}^\ast)]$:

\[
\frac{1}{|\Gamma|} \sum_{\nu_1,\ldots,\nu_s,\nu_\gamma} \left( \sum_{j=1}^n \left[ \partial_\nu x_{\nu_1} \cdots x_{\nu_s} x_\gamma \otimes \epsilon_j^0 \right] \right) \text{ad}(e_{\nu_i}) \cdots \text{ad}(e_{\nu_s}) (a_u) \otimes (\gamma \cdot a_u) \right)^{(i)}.
\]

The term corresponding to $j = i$ is the linear map $S^{s-1}(\mathfrak{g}) \to U(\mathfrak{g})^\otimes n$ such that for $x \in \mathfrak{g}$

\[
x^{s-1} \mapsto x^{s-1} \frac{1}{|\Gamma|} \sum_{\nu,\nu_1,\ldots,\nu_s} \left[ \text{ad}(x) \text{ad}(e_{\nu}) \text{ad}(x) (a_u) \otimes (\gamma \cdot a_u) \right]^{(i)}.
\]

Using $I$-invariance of $\sum_u a_u \otimes (\gamma \cdot a_u)$ one obtains that this last expression equals

\[
\frac{1}{|\Gamma|} \sum_{\nu,\nu_1,\ldots,\nu_s} \left( \text{ad}(x) \text{ad}(e_{\nu}) \text{ad}(x) (a_u) \otimes (\gamma \cdot a_u) \right) \approx 0.
\]
which is zero from the $l$-invariance of $t_1 = \sum, e_v \otimes e_v$. The term corresponding to $j \neq i$ is the linear map $S^{l+1}(l) \to U(g)^n$ such that for $x \in l$

\[
x^{s-1} \mapsto \frac{1}{|\Gamma|} \sum_{p+q=s-1, \nu, u} (ad(x)^p ad(e_v) ad(x)^q (a_u) \otimes (\gamma \cdot a_u)^{(s)} e_v^{(i)} - (i \leftrightarrow j))
\]

\[
= \frac{1}{|\Gamma|} \sum_{p+q=s-1, \nu, u} (ad(x)^p ([e_v, a_u]) \otimes (-ad(x))^q (\gamma \cdot a_u)^{(s)} e_v^{(i)} - (i \leftrightarrow j))
\]

\[
= \frac{1}{|\Gamma|} \sum_{p+q=s-1, \nu, u} (-1)^q (ad(x)^p ([e_v, a_u]) \otimes (ad(x))^q (\gamma \cdot a_u)^{(s)} e_v^{(i)} - (i \leftrightarrow j))
\]

\[
= \frac{1}{|\Gamma|^2} \sum_{\beta \in \Gamma} \sum_{p+q=s-1, \nu, u} (-1)^q (ad(x)^p (\beta \cdot a_u) \otimes ad(x)^q (\gamma \cdot a_u)^{(s)} e_v^{(i)} - (i \leftrightarrow j))
\]

\[
= \frac{1}{|\Gamma|^2} \sum_{\beta \in \Gamma} \sum_{p+q=s-1, \nu, u} (-1)^q (ad(x)^p (\beta \cdot a_u) \otimes ad(x)^q ((\beta + \gamma) \cdot a_u)^{(s)} [a_u, a_v]^{(s)} - (i \leftrightarrow j))
\]

\[
= \frac{1}{|\Gamma|^2} \sum_{\beta \in \Gamma} \sum_{p+q=s-1, \nu, u} (-1)^q (ad(x)^p (\beta \cdot a_u) \otimes ad(x)^q ((\beta + \gamma) \cdot a_u)^{(s)} [a_u, a_v]^{(s)} - (i \leftrightarrow j))
\]

which coincides with the image of

\[
D_{s, \gamma} \left( \frac{x_i}{M}, \frac{ij}{|\Gamma|} \right) = \sum_{p+q=s-1, \beta \in \Gamma} \left( \sum_{\nu, u} (a_u x_i)^p \frac{ij}{|\Gamma|} \right) \left( -ad x_i^{(s)} \right)^q \frac{ij}{|\Gamma|}
\]

under $\rho_\beta$. In conclusion we get the relation

\[
\rho_\beta \left( \left[ \xi_{s, \gamma}, \frac{\nu}{\lambda} \right] \right) = \rho_\beta \left( \xi_{s, \gamma}, \frac{\nu}{\lambda} \right).
\]

A direct computation shows that the commutation relations of $[X, \xi_{s, \gamma}] = 0$, $[d, \xi_{s, \gamma}] = s\xi_{s, \gamma}$ and $ad^{s+1}(\Delta_0)(\xi_{s, \gamma}) = 0$ are preserved, which finishes the proof.

4.3. **Reductions.** Assume that $I$ is finite dimensional and we have a reductive decomposition $I = \mathfrak{h} \oplus \mathfrak{m}$, i.e. $\mathfrak{h} \subset I$ is a subalgebra and $\mathfrak{m} \subset I$ is a vector subspace such that $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. We also assume that $t_1 = t_\mathfrak{h} + t_\mathfrak{m}$ with $t_\mathfrak{h} = \sum, e_v \otimes e_v \in S^2(\mathfrak{h})^\mathfrak{h}$ and $t_\mathfrak{m} \in S^2(\mathfrak{m})^\mathfrak{h}$, and that for a generic $h \in \mathfrak{h}$, $ad(h)_{\mathfrak{m}} \in \text{End}(\mathfrak{m})$ is invertible. This last condition means that $P(\lambda) := det(ad(\lambda^\gamma))_{\mathfrak{m}} \in S^{dim(\mathfrak{m})}(\mathfrak{h})$ is nonzero, where $\lambda^\gamma := (\lambda \otimes \text{id})(t_\mathfrak{h})$ for any $\lambda \in \mathfrak{h}^*$. 
We now define $H_n(g, h_{reg}^\ast)$. As in the previous paragraph, $\text{Diff}(h^\ast)$ has generators $\bar{x}_h, \bar{\partial}_h$ ($h \in h$) and relations

$$\bar{x}_{th+ht'} = t\bar{x}_h + \bar{x}_{h'}, \\
\bar{\partial}_{th+ht'} = t\bar{\partial}_h + \bar{\partial}_{h'}, \\
[\bar{x}_h, \bar{x}_{h'}] = 0 = [\bar{\partial}_h, \bar{\partial}_{h'}], \\
[\bar{\partial}_h, \bar{x}_{h'}] = (h, h'),$$

and $\text{Diff}(h_{reg}^\ast) = \text{Diff}(h^\ast)[\frac{1}{m}]$ with $[\bar{\partial}_h, \frac{1}{m}] = -\frac{[\partial_t, P]}{m}$. One has a Lie algebra morphism $h \mapsto \text{Diff}(h^\ast); h \mapsto \bar{X}_h := \sum_x \bar{x}_{[h,x]} \partial_{ex}$.

We denote by $h^{\text{diag}}$ the image of the map

$$h \mapsto \bar{Y}_h := \bar{X}_h \oplus \sum_{i=1}^n f(i) \in \text{Diff}(h_{reg}^\ast) \otimes U(g)^{\otimes n} =: B_n,$$

and define $H_n(g, h_{reg}^\ast)$ as the Hecke algebra of $B_n$ with respect to $h^{\text{diag}}$:

$$H_n(g, h_{reg}^\ast) := (B_n)^h/(B_n h^{\text{diag}})^h.$$

It acts in an obvious way on $(\mathcal{O}_{h_{reg}^\ast} \otimes (\otimes_{i=1}^n V_i))^h$ if $(V_i)_{1 \leq i \leq n}$ is a collection of $g$-modules. Finally, let us set, for $\lambda \in h^\ast$,

$$r(\lambda) := (\text{id} \otimes (\text{ad} \lambda)^{-1})(\ell_m).$$

Then, following [13], $r : h_{reg}^\ast \rightarrow \lambda^2(m)$ is an $h$-equivariant map satisfying the classical dynamical Yang-Baxter equation (CDYBE)

$$\sum_i c_{\lambda}(i) \partial_{e(i)} + [r(12), r(13)] + c.p.(1, 2, 3) = 0,$$

and we write $r = \sum_i a_i \otimes b_i \otimes \ell_i \in (m^{\otimes 2} \otimes S(h)[1/P])^h$.

**Proposition 4.3.** There is a unique Lie algebra morphism $\rho_{g,h} : \bar{U}_{1,n} \rightarrow H_n(g, h_{reg}^\ast)$ given by

$$\bar{x}_i \mapsto M \sum_{\nu} \bar{x}_{\nu} \otimes h_{\nu}^{(i)}, \\
\bar{y}_i \mapsto -N \sum \bar{\partial}_{\nu} \otimes h_{\nu}^{(i)} + \sum_j \ell_j \otimes a_j^{(i)} b_j^{(i)}, \\
\bar{t}_{ij}^\alpha \mapsto 1 \otimes (a^{(i)} \cdot t_{ij}^{(j)}).$$

**Proof.** First of all, the images of the above elements are all $h$-invariant. As in [6], we will imply summation over repeated indices, and adopt the following conventions: $\partial_{e(i)} = \bar{\partial}_h$, $\bar{x}_{e(i)} = \bar{x}_h$, and $1 \otimes -$’s and $- \otimes 1$’s may be dropped from the notation.

In particular, $\rho_{g,h}(\bar{x}_i) = h_{\nu}^{(i)} \bar{x}_{\nu}$, $\rho_{g,h}(\bar{y}_i) = -h_{\nu}^{(i)} \bar{\partial}_h + \sum_{j=1}^n r(\lambda)^{(i)}$ (here, for $x \otimes y \in g^{\otimes 2}$, $(x \otimes y)^{(i)} := x^{(i)} y^{(i)}$).

We will use the same presentation of $\bar{U}_{1,n}$ as in Lemma 1.1. The relations $[\bar{x}_i, \bar{x}_j] = 0$ and $\bar{t}_{ij}^\alpha = \bar{t}_{ji}^{-\alpha}$ are obviously preserved.
Let us check that \([\bar{x}_i, \bar{y}_j] = \sum \bar{t}^{ij}_{\alpha}\) is preserved. We have for \(i \neq j\),
\[
\frac{1}{MN} [\rho_{\bar{g}, h}(\bar{x}_i), \rho_{\bar{g}, h}(\bar{y}_j)] = -\sum_{\nu_1,\nu_2} [\bar{x}_{\nu_1}, \partial_{\nu_2}] h^{(ij)}_{\nu_1} h^{(ij)}_{\nu_2} + \sum_{\nu, \delta, k} \bar{x}_{\nu} \partial_{\delta} h^{(ij)}_{\nu} \otimes a^{(j)\delta}_{\nu} \]
\[
= t^{(ij)}_{\nu} + t^{(ij)}_{\mu} = t^{(ij)}_{\nu} = \frac{1}{MN} \sum_{\nu} \alpha_{(i)}, t^{(ij)}_{\nu}
\]
by the same argument as in Proposition 4.1.

Let us check that \(\sum \bar{x}_i = \sum \bar{y}_i = 0\) are preserved. We have \(\sum \rho_{\bar{g}, h}(\bar{x}_i) = 0\) and \(\sum \rho_{\bar{g}, h}(\bar{y}_i) = \sum_{\nu,\delta, k} \bar{x}_{\nu} \partial_{\delta} h^{(ij)}_{\nu} \) (by the antisymmetry of \(r\)), which equals zero as in Proposition 4.1.

The fact that the relation \([\bar{y}_i, \bar{y}_j] = 0\) is satisfied for \(i \neq j\) is a consequence of the dynamical Yang-Baxter equation (this follows from the exact same argument as in the proof of [6, Proposition 63]).

Next, \([\bar{x}_i, \bar{t}^{ij}_{\alpha}] = 0\) is preserved \((i, j, k\) distinct). Indeed, we have
\[
[\rho_{\bar{g}, h}(\bar{x}_i), \rho_{\bar{g}, h}(\bar{t}^{ij}_{\alpha})] = \sum_{\nu} \bar{x}_{\nu} [h^{(ij)}_{\nu}, \alpha_{(i)}, t^{(j)\alpha}_{\nu}] = 0.
\]

Finally \([\bar{y}_i, \bar{t}^{ij}_{\alpha}] = 0\) is preserved \((i, j, k\) distinct): we have
\[
[\rho_{\bar{g}, h}(\bar{y}_i), \rho_{\bar{g}, h}(\bar{t}^{ij}_{\alpha})] = [-\sum_{\nu} h^{(ij)}_{\nu} \partial_{\nu} + \sum_{\nu} r^{(ij)}_{\nu} \alpha_{(i)}, t^{(j)\alpha}_{\nu}]
\]
\[
= [r(\lambda)^{(ij)}_{\nu}, r(\lambda)^{(j)\alpha}_{\nu}] = 0,
\]
where the last equality follows the the \(t^r\)-invariance of \(t^r\). \(\square\)

**Remark 4.4.** We expect that there is Lie algebra morphism \(\text{red}_{\Gamma,h} : H_n(\bar{g}, \Gamma) \rightarrow H_n(\bar{g}, \bar{h}^{reg})\) such that the following diagram commutes
\[
\begin{array}{ccc}
\Gamma_{1,n} & \xrightarrow{\rho_{\bar{g}}} & H_n(\bar{g}, \Gamma) \\
\rho_{\bar{g},n} \downarrow & & \downarrow \text{red}_{\Gamma,h} \\
H_n(\bar{g}, \bar{h}^{reg}) & & \\
\end{array}
\]

### 4.4. Elliptic dynamical \(r\)-matrix systems as realizations of the universal \(\Gamma\)-KZB system on twisted configuration spaces

Let \(K(z)\) be a meromorphic function on \(\mathbb{C}\) with values in the subalgebra \(\bar{\mathfrak{t}}_{\mathbb{C},+} \subset \bar{\mathfrak{h}}_{1,2}\) generated by \(x_1, x_2, t^{(i)\alpha}_{12} (\alpha \in \Gamma)\), such that \(K(-z) = -K(z)^{2,1}\) and satisfying the universal CDYBE with a spectral parameter
\[
-[y_1, K(z_{23})]^{2,3} + [K(z_{12})]^{1,2}, [K(z_{13})]^{1,3} + c.p.(1,2,3) = 0.
\]

On the one hand, it follows from §4.1 that the image \(r(x, z) = \rho_{\bar{g}}(K(z))\) of \(K(z)\) under \(\rho_{\bar{g}} : \bar{\mathfrak{t}}_{2,2} \rightarrow (\bar{\mathfrak{c}}_1 \otimes \bar{\mathfrak{g}}_{\text{A}2})\) is a dynamical \(r\)-matrix\(^2\) with spectral parameter, i.e. a solution of the CDYBE with a spectral parameter for the pair \((l, g)\)
\[
\sum_{\nu} c_{\nu}^{(1)} \partial_{\nu} r(x, z_{23})^{(23)} + [r(x, z_{12})^{(12)}, r(x, z_{13})^{(13)}] + c.p.(1,2,3) = 0,
\]
\(^2\)Remember that \(\bar{\mathfrak{c}}_1 := \mathfrak{s}(l)\) and \(\bar{\mathfrak{c}}_2 := \hat{\mathfrak{s}}(l)\).
which satisfies \( r(x, -z) = -r(x, z) \)
\(^{(21)}\). On the other hand, the image of \( K(z) \) under \( \rho_{g,b} : \mathfrak{h}^* \to (\hat{\mathcal{O}}_{\mathfrak{h}^*} \otimes \mathfrak{g}^{\otimes 2})^b \) is precisely equal to the restriction \( \rho_{g}(K(z))_b \in (\hat{\mathcal{O}}_{\mathfrak{h}^*} \otimes \mathfrak{g}^{\otimes 2})^b \) of \( \rho_{b}(K(z)) \) to \( \mathfrak{h}^* \). Then applying [13, Proposition 0.1], we conclude that
\[
\tilde{r}(\bar{x}, z) := \rho_{g,b}(K(z)) + r(\lambda)
\]
is a solution of the CDYBE with spectral parameter for \((\mathfrak{h}, \mathfrak{g})\):
\[
\sum_{\nu} e_\nu^{(1)} \partial_\nu \tilde{r}(\bar{x}, z_{23})^{(23)} + [\tilde{r}(\bar{x}, z_{12})^{(12)}, \tilde{r}(\bar{x}, z_{13})^{(13)}] + c.p.(1,2,3) = 0.
\]

Then for any \( n \)-tuple \( \mathbf{V} = (V_1, \ldots, V_n) \) of \( \mathfrak{g} \)-modules one has a flat connection \( \nabla^{\mathbf{V}}_{\tau,n,r} \) on the trivial vector bundle over \( \mathbb{C}^n - \text{Diag}_{\tau,n,r} \) with fiber \((\mathcal{O}_{\mathfrak{h}^*} \otimes (\otimes_i V_i))^b\), defined by the following compatible system of first order differential equations:
\[
\partial_\nu F(\bar{x}, z) = \sum_{\nu} e_\nu^{(1)} \cdot \partial_\nu F(\bar{x}, z) + \sum_{j \neq i} \tilde{r}^{(ij)}(\bar{x}, z_{ij}) \cdot F(\bar{x}, z).
\]
Here \( z \mapsto F(\bar{x}, z) \) is a function with values in \((\mathcal{O}_{\mathfrak{h}^*} \otimes (\otimes_i V_i))^b\).

Starting from \( K(z) = K_{12}(z) \) as in §1.4, it would be interesting to know if one can recover (up to gauge equivalence), using the above realization morphisms, the generalization of Felder’s elliptic dynamical \( r \)-matrices [17] constructed in [15, 16].

5. Formality of subgroups of the pure braid group on the torus

5.1. Relative formality. Let \( G \) and \( S \) be two affine groups over \( k \) and let \( \varphi : G \to S \) be a surjective group morphism with finitely generated kernel \( \text{Ker} \varphi \). We then consider the category of pro-algebraic groups \( G' \) under \( G \), together with a surjective morphism \( \varphi' : G' \to S \) with \( k \)-pronipotent kernel. This category has an initial object, denoted \( \varphi(k) : G \to G(\varphi, k) \), which we call the relative (\( k \)-pronipotent) completion of \( G \) with respect to \( \varphi \). One can easily check that the kernel \( \text{Ker}(\varphi(k)) \) of \( \varphi(k) \) is the usual \( k \)-pronipotent completion \( (\text{Ker} \varphi)(k) \) of the kernel of \( \varphi \), which we can therefore unambiguously denote \( \text{Ker} \varphi(k) \).

Observe that this coincides with the partial completion defined [9, §1.1], and with the relative completion defined in [21] (which is somehow slightly more general).

**Lemma 5.1.** If \( S \) is finite then the extension
\[
1 \to \text{Ker} \varphi(k) \to G(\varphi, k) \to S \to 1
\]
splits.

**Proof.** We consider the filtration \((F_i)_i\), given by the lower central series of \( \text{Ker} \varphi(k) \), and prove by induction by induction that
\[
1 \to \text{Ker} \varphi(k)/F_i \to G(\varphi, k)/F_i \to S \to 1
\]
splits.
Initial step \((i = 2)\): Recall that \(F_1 = \ker \varphi (k)\), and that \(F_1/F_2\) is abelian and finitely generated, so that

\[
1 \rightarrow \ker \varphi (k)/F_2 \rightarrow G(\varphi, k)/F_2 \rightarrow S \rightarrow 1
\]
splits as every extension of a finite group by a finite dimensional representation splits (this is because the cohomology of a finite group with coefficients in a divisible module vanishes).

**Induction step:** We have a (surjective) morphism of extensions

\[
\begin{array}{ccc}
1 & \rightarrow & \ker \varphi (k)/F_{i+1} \\
& & \downarrow \\
1 & \rightarrow & \ker \varphi (k)/F_i \\
& & \downarrow \\
& & \rightarrow \\
& & \rightarrow \\
& & 1
\end{array}
\]

Assuming (by induction) that the bottom extension splits, we have that the corresponding obstruction class in the first non-abelian cohomology \(H^1(S, \ker \varphi (k)/F_i)\) is trivial. Hence, by exactness of

\[
H^1(S, F_i/F_{i+1}) \rightarrow H^1(S, \ker \varphi (k)/F_{i+1}) \rightarrow H^1(S, \ker \varphi (k)/F_i)
\]
we get that the obstruction class for the splitting of the top extension lies in the image of

\[
H^1(S, F_i/F_{i+1}) \rightarrow H^1(S, \ker \varphi (k)/F_{i+1}).
\]

We conclude by using the vanishing of group cohomology of a finite group in a finite dimensional representation. \(\square\)

The above Lemma tells us in particular that \(G(\varphi, k) \equiv \ker (\varphi)(k) \times S\), and justifies the following definition from \([9, \S 1.2]\)\(^3\).

**Definition 5.2.** If \(S\) is finite, we say that the surjective group morphism \(\varphi : G \rightarrow S\) with finitely generated kernel is (relatively) filtered-formal if there exists a group isomorphism

\[
G(k, \varphi) \equiv \exp (\mathfrak{gr} \ker \varphi (k)) \times S
\]

over \(S\). This is equivalent to having an \(S\)-equivariant formality isomorphism

\[
\ker \varphi (k) \equiv \mathfrak{gr} \ker \varphi (k).
\]

**Example 5.3.** The surjective morphism \(B_n \rightarrow \mathfrak{S}_n\), where \(B_n\) is the standard \(n\) strands braid group is filtered-formal. This morphism, or rather the exact sequence

\[
1 \rightarrow PB_n \rightarrow B_n \rightarrow \mathfrak{S}_n \rightarrow 1,
\]
can be deduced from the covering map \(\text{Conf}(\mathbb{C}, n) \rightarrow \text{Conf}(\mathbb{C}, n)/\mathfrak{S}_n\). It is interesting to say that this relative filtered-formality result follows from \([23]\) when \(k = \mathbb{C}\), and from \([8]\) for \(k = \mathbb{Q}\). We also refer to \([21, \text{Example 1.5}]\) for interesting considerations about this example. More precisely, one has an \(\mathfrak{S}_n\)-equivariant isomorphism \(PB_n(k) \cong \exp (\hat{1}_n)\).

---

\(^3\)In \([9]\), Enriquez speaks about *relative formality*. We prefer to speak about *relative filtered-formality* in order to remain consistent with our conventions in the absolute case \(S = 1\) (recall that we were following the convention from \([25]\) in the absolute case).
Example 5.4. Let $M \in \mathbb{N}$ be a positive integer. From the covering map $\text{Conf}(C^*, n, M) \to \text{Conf}(C^*, n)/\mathfrak{S}_n$, one also gets an exact sequence

$$1 \to \text{PB}_n^M \to B_n^1 \to S \to 1,$$

where $S := (\mathbb{Z}/M\mathbb{Z})^n \rtimes \mathfrak{S}_n$. It follows from [9, §1.3–1.6] that the surjective morphism $B_n^1 \to S$ is filtered-formal. More precisely, Enriquez proves the existence of a $S$-equivariant isomorphism $\text{PB}_n^M(k) \cong \exp(\hat{t}^M)$.

5.2. Relation between relative completion and completion of groupoids. In this paragraph we briefly compare the notion of relative $k$-prominipotent completion with the $k$-prominipotent completion for groupoids.

There is a functor that goes

- from the category of surjective morphisms $G \to S$ with finitely generated kernel and with $S$ a finite group.
- to the category of groupoids.

This functor sends $\varphi : G \to S$ to the groupoid $\mathcal{G}(\varphi)$ defined as follows:

- the set of objects of of $\mathcal{G}(\varphi)$ is $S$.
- for $s, s' \in S$,
  $$\text{Hom}_{\mathcal{G}(\varphi)}(s, s') := \{g \in G | \varphi g = s^{-1} s'\}$$
- the multiplication of arrows in $\mathcal{G}(\varphi)$ is the multiplication in $G$.

Example 5.5. It is easy to check that $\mathcal{G}(B_n \to \mathfrak{S}_n)$ is the colored braid groupoid $\text{CoB}(n)$ from [18, §5.2.8]. Similarly, one can define:

- the groupoid
  $$\text{CoB}^N(n) := \mathcal{G}(B_n^1 \to (\mathbb{Z}/N\mathbb{Z})^n \rtimes \mathfrak{S}_n)$$
  of the colored twisted (or cyclotomic) braids.
- the groupoid
  $$\text{CoB}_{\text{ell}}(n) := \mathcal{G}(B_{1,n} \to \mathfrak{S}_n)$$
  of colored elliptic braids.
- the groupoid
  $$\text{CoB}_{\text{ell}}^\Gamma(n) := \mathcal{G}(B_{1,n} \to (\Gamma^n/\Gamma) \rtimes \mathfrak{S}_n)$$
  of colored ellipsitomic braids.

We let the reader prove that $\mathcal{G}(\varphi)(k) \cong \mathcal{G}(\varphi(k))$, where $\mathcal{G}(\varphi)(k)$ is the $k$-prominipotent completion of the groupoid $\mathcal{G}(\varphi)$. 
5.3. **Subgroups of** $B_{1,n}$. For $\tau \in A_1$ let $U_{\tau,n,\Gamma} \subset C^n - \text{Diag}_{\tau,n,\Gamma}$ be the open subset of all $z = (z_1, \ldots, z_n)$ of the form $z_i = a_i + \tau b_i$, where $0 < a_n < \cdots < a_1 < 1/M$ and $0 < b_n < \cdots < b_1 < 1/N$. If $z_0 \in U_{\tau,n,\Gamma}$ then it both defines a point in the $\Gamma$-twisted configuration space $\text{Conf}(E_{\tau,\Gamma}, n, \Gamma)$ and in the (non twisted) unordered configuration space $\text{Conf}(E_{\tau,\Gamma}, [n])$.

Recall that the map

$$\text{Conf}(E_{\tau,\Gamma}, n, \Gamma) \rightarrow \text{Conf}(E_{\tau,\Gamma}, [n])$$

is a covering map with structure group $\Gamma^n * S_n$. Hence we get a short exact sequence

$$1 \rightarrow \text{PB}^\Gamma_{1,n} \rightarrow B_{1,n} \xrightarrow{\varphi_n} \Gamma^n * S_n \rightarrow 1,$$

where $\text{PB}^\Gamma_{1,n} := \pi_1(\text{Conf}(E_{\tau,\Gamma}, n, \Gamma), z_0)$ and $B_{1,n} := \pi_1(\text{Conf}(E_{\tau,\Gamma}, [n]), z_0)$.

We will also consider $\text{PB}^\Gamma_{1,n} = \pi_1(\text{Conf}(E_{\tau,\Gamma}, n), z_0)$, and the short exact sequence

$$1 \rightarrow \text{PB}^\Gamma_{1,n} \rightarrow \text{PB}_{1,n} \rightarrow \Gamma^n \rightarrow 1$$

associated with the $\Gamma^n$-covering map

$$\text{Conf}(E_{\tau,\Gamma}, n, \Gamma) \rightarrow \text{Conf}(E_{\tau,\Gamma}, n).$$

Our main aim in this Section is to prove that the surjective morphism

$$B_{1,n} \rightarrow \Gamma^n * S_n$$

is relatively filtered-formal, which in turns implies the relative filtered-formality of $\text{PB}_{1,n} \rightarrow \Gamma^n$, and the filtered-formality of $\text{PB}^\Gamma_{1,n}$.

Moreover, we will have an explicit description of the relative completion in terms of the Lie algebra $t^\Gamma_{1,n}$.

5.4. **The monodromy morphism** $B_{1,n} \rightarrow \exp(t^\Gamma_{1,n}) \rtimes (\Gamma^n * S_n)$. The monodromy of the flat $\exp(t^\Gamma_{1,n}) \rtimes (\Gamma^n * S_n)$-bundle $(P_{(\tau,\Gamma),[n]}, \nabla_{(\tau,\Gamma),[n]})$ on $\text{Conf}(E_{\tau,\Gamma}, [n])$ provides us with a group morphism

$$\mu_{z_0, (\tau,\Gamma),[n]} : B_{1,n} \rightarrow \exp(t^\Gamma_{1,n}) \rtimes (\Gamma^n * S_n).$$

This actually fits into a morphism of short exact sequences

$$1 \rightarrow \text{PB}^\Gamma_{1,n} \rightarrow B_{1,n} \rightarrow \Gamma^n \rtimes S_n \rightarrow 1,$$

$$1 \rightarrow \exp(t^\Gamma_{1,n}) \rightarrow \exp(t^\Gamma_{1,n}) \rtimes (\Gamma^n \rtimes S_n) \rightarrow \Gamma^n \rtimes S_n \rightarrow 1,$$

where the first vertical morphism is the monodromy morphism

$$\mu_{z_0, \tau,n,\Gamma} : \text{PB}^\Gamma_{1,n} \rightarrow \exp(t^\Gamma_{1,n})$$

of associated with the flat $\exp(t^\Gamma_{1,n})$-bundle $(P_{\tau,n,\Gamma}, \nabla_{\tau,n,\Gamma})$ on $\text{Conf}(E_{\tau,\Gamma}, n, \Gamma)$.

Indeed, this comes from the fact that $\nabla_{(\tau,\Gamma),[n]}$ is obtained by descent, from $\nabla_{\tau,n,\Gamma}$ and using its equivariance properties (see §1.3). More precisely, the monodromy of $\nabla_{(\tau,\Gamma),[n]}$ along a loop $\gamma$ based at $z_0$ in $\text{Conf}(E_{\tau,\Gamma}, [n])$ can be computed along the following steps:
• First consider the unique lift $\tilde{\gamma}$ of $\gamma$ departing from $z_0 \in \text{Conf}(E_{\tau,n}, \Gamma)$. Note that it ends at $g \cdot z_0 = g \in \Gamma_n \times S_n$. If $g = (g_1, \ldots, g_n) \in \Gamma_n$ and $z_0 = (z_1, \ldots, z_n)$ we will simply write $g \cdot z_0 := (z_1^{g_1}, \ldots, z_n^{g_n})$.

• Then compute the holonomy of $\nabla_{\tau,n,\Gamma}$ along $\tilde{\gamma}$: this is an element in $\exp(\hat{t}_{1,n})$, as $\nabla_{\tau,n,\Gamma}$ is defined on a principal $\exp(\hat{t}_{1,n})$-bundle obtained as a quotient of the trivial one on $\mathbb{C}^n - \text{Diag}_{\tau,n,\Gamma}$ (see §1.2), that we abusively denote $\mu_{z_0,\tau,n,\Gamma}(\tilde{\gamma})$.

• Finally, $\mu_{z_0,\tau,n,\Gamma}(\gamma) = g\mu_{z_0,\tau,n,\Gamma}(\tilde{\gamma})$.

Having such a morphism of exact sequences guaranties that it factors through a morphism

$$
\begin{array}{cccc}
1 & \longrightarrow & PB^\Gamma_{1,n}(\mathbb{C}) & \longrightarrow & B_{1,n}(\varphi_n, \mathbb{C}) & \longrightarrow & \Gamma_n \times S_n & \longrightarrow & 1 \\
1 & \longrightarrow & \exp(\hat{t}_{1,n}) & \longrightarrow & \exp(\hat{t}_{1,n}) \times (\Gamma_n \times S_n) & \longrightarrow & \Gamma_n \times S_n & \longrightarrow & 1
\end{array}
$$

where $\hat{B}_{1,n}(\varphi_n, \mathbb{C})$ is the relative pronipotent completion of the morphism $B_{1,n} \to \Gamma_n \times S_n$, and $\hat{PB}^\Gamma_{1,n}(\mathbb{C})$ is the pronipotent completion of $PB^\Gamma_{1,n}$.

We will call the vertical maps the completed monodromy morphisms.

In the remainder of this Section we will prove that these completed monodromy morphisms are isomorphisms, which implies in particular the relative filtered-formality of $B_{1,n} \to \Gamma_n \times S_n$.

**Theorem 5.6.** The completed monodromy morphism

$$\hat{B}_{1,n}(\varphi_n, \mathbb{C}) \longrightarrow \exp(\hat{t}_{1,n}) \times (\Gamma_n \times S_n)$$

is an isomorphism. Equivalently, the completed monodromy morphism

$$PB^\Gamma_{1,n}(\mathbb{C}) \longrightarrow \exp(\hat{t}_{1,n})$$

is an isomorphism.

Our aim now is to prove that Theorem 5.6, namely that the completed monodromy morphism

$$\hat{\mu}_{z_0,\tau,n,\Gamma}(\cdot) : \hat{PB}^\Gamma_{1,n}(\mathbb{C}) \longrightarrow \exp(\hat{t}_{1,n})$$

is an isomorphism. For this we will prove that the induced morphism on Malcev Lie algebras

$$\text{Lie}(\hat{\mu}_{z_0,\tau,n,\Gamma}) : \text{pb}^\Gamma_{1,n} \longrightarrow \hat{t}_{1,n}$$

is an isomorphism of filtered Lie algebras.

**5.5. A morphism $t^\Gamma_{1,n} \to \text{gr}(\text{pb}^\Gamma_{1,n})$.** Let us start with a few algebraic facts about $PB_{1,n}$ and $PB^\Gamma_{1,n}$.

The group $PB_{1,n}$ is generated by the $X_i$'s and $Y_i$'s ($i = 1, \ldots, n$), where $X_i$ (resp. $Y_i$) is the class of the path given by $[0,1] \ni t \mapsto z_0 + t\delta_i/M$ (resp. $[0,1] \ni t \mapsto z_0 + t\tau\delta_i/N$). One
sees very easily that $X_i^M$ (resp. $Y_i^N$) is the class of the path given by $[0, 1] \ni t \mapsto z_0 + t\delta_i$ (resp. $[0, 1] \ni t \mapsto z_0 + t\tau\delta_i$), so that $X_i^M$ and $Y_i^N$ are elements of $\text{PB}^\Gamma_{1,n}$.

One has an obvious inclusion $\text{PB}_n \hookrightarrow \text{PB}^\Gamma_{1,n}$ coming from the identification of $\mathbb{C}$ with the fundamental domain $\{z = a + b\tau \in \mathbb{C} | 0 < a < \frac{1}{M}, 0 < b < \frac{1}{N}\}$ of $E_{\tau, \Gamma}$.

Recall that we write the composition of paths from left to right. Then one can check (by simply drawing) that the following relations are satisfied in $\text{PB}^\Gamma_{1,n}$:

1. $(X_i, X_j) = 1 = (Y_i, Y_j)$ ($i < j$),
2. $(X_i, Y_i^{-1}) = P_{ij} = (X_i, Y_j^{-1})$ ($i < j$),
3. $(X_i, Y_1) = P_{12} \cdots P_{1n}$,
4. $(X_i, P_{jk}) = 1 = (Y_i, P_{jk})$ ($\forall i, j < k$),
5. $(X_i, Y_j, P_{ij}) = 1 = (Y_i, Y_j, P_{ij})$ ($i < j$).

In particular $\text{PB}_n$ identifies with the subgroup of commutators in $\text{PB}^\Gamma_{1,n}$. Moreover, one observes that $X_1 \cdots X_n$ and $Y_1 \cdots Y_n$ are central in $\text{PB}^\Gamma_{1,n}$.

Now it follows from the geometric description of $\text{PB}^\Gamma_{1,n}$ that it is generated by $X_i^M$, $Y_i^N$ ($i = 1, \ldots, n$) and $P_{ij}^{(\alpha)} := X_j^\alpha Y_j^{-\alpha} P_{ij} Y_j^\alpha X_j^\alpha$ ($i < j$, $1 \leq p \leq M$, $1 \leq q \leq N$ and $\alpha = (\bar{p}, \bar{q})$). One can for instance represent lifts of $X_3$, $Y_3$ and $P_{12}^{(1,1)}$ in $\text{Conf}(E_{\tau, \Gamma}, n, \Gamma)$ as follows.

Observe that the standard descending filtration on $\mathfrak{t}^\Gamma_{1,n}$ coincides with the descending filtration coming from the grading of $\mathfrak{t}^\Gamma_{1,n}$ defined in §1.1.
Proposition 5.7. There is a surjective graded Lie algebra morphism \( p_n : \mathfrak{t}^\Gamma_{1,n} \rightarrow \mathfrak{gr}(\mathfrak{p}^\Gamma_{1,n}) \), sending

- \( x_i \mapsto \sigma\left(\log(X^M_i)\right) \) for \( i = 1, \ldots, n \),
- \( y_i \mapsto \sigma\left(\log(Y^N_i)\right) \) for \( i = 1, \ldots, n \),
- \( t^\alpha_{ij} \mapsto \sigma\left(\log(P^\sigma_{ij})\right) \) for \( i < j \),
- \( t^\alpha_{ij} \mapsto \sigma\left(\log(P^{-\sigma}_{ij})\right) \) for \( j < i \),

where \( \sigma \) denotes the symbol map \( \mathfrak{p}^\Gamma_{1,n} \rightarrow \mathfrak{gr}(\mathfrak{p}^\Gamma_{1,n}) \).

Proof. It is sufficient to check that the defining relations of \( \mathfrak{t}^\Gamma_{1,n} \) are preserved by the above assignment.

The relation \([x_i, x_j] = 0 = [y_i, y_j] \) is obviously preserved. Now using (T2) and the relation

\[
(X^M, Y^N) = \prod_{i=0}^{M-1} X^{M-i+1}_i \left( \sum_{j=0}^{N-1} Y^j(X, Y) Y^{-j} \right) X^{1-M-1}
\]

(which is true in the free group \( F_2 \), and thus in any group) with \( X = X_i \) and \( Y = Y_j \) \((i \neq j)\), one obtains that \([x_i, y_j] = [x_j, y_i] = \sum_t t^\alpha_{ij} \) is preserved. Using (T3) one also obtains that \([x_1, y_1] = -\sum_{i,j} t^\alpha_{ij} \) is preserved. Now it is obvious that the centrality of \( \sum_i x_i \) and \( \sum_j y_j \) is preserved, and thus it follows that \([x_i, y_j] = -\sum_{i,j} t^\alpha_{ij} \) is preserved for any \( i \in \{1, \ldots, n\} \). For any \( \alpha = (p, q) \) we compute

\[
\begin{align*}
(X^M_i, P^\alpha_{jk}) &= X^M_i X^p X^M_j P^q_{jk} Y^p Y^M_j X^q_i P^q_{jk} Y^q_i P^q_{jk} Y^q_i X^p_i \\
&= X^p_i (X^M_i, Y^q_j) X^q_j P^q_{jk} X^M_j Y^p_j X^q_j P^q_{jk} Y^q_j X^p_j \\
&= X^p_i (X^M_i, Y^q_j) Y^q_j P^q_{jk} Y^p_j (X^M_i, Y^q_j) - Y^q_j P^q_{jk} Y^q_j X^p_j \\
&= X^p_i (X^M_i, Y^q_j) Y^q_j P^q_{jk} Y^p_j (X^M_i, Y^q_j) - Y^q_j P^q_{jk} Y^q_j X^p_j \\
&= X^p_i (X^M_i, Y^q_j) Y^q_j P^q_{jk} Y^p_j (X^M_i, Y^q_j) - Y^q_j P^q_{jk} Y^q_j X^p_j.
\end{align*}
\]

One sees that the log of the l.h.s. lies in \( \mathfrak{p}^\Gamma_{1,n} \) and its symbol is equal to \( [\sigma(\log(X^M_i)), \sigma(\log(P^\alpha_{jk}))] \), and that the log of the r.h.s. lies in \( \mathfrak{p}^\Gamma_{1,n} \). Hence one obtains that \([x_i, t^\alpha_{ij}] = 0 \) is preserved. The proof that \([y_i, t^\alpha_{ij}] = 0 \) is preserved is identical, and the proof that \([x_i + x_j, t^\alpha_{ij}] = 0 = [y_i + y_j, t^\alpha_{ij}] \) \([t^\alpha_{ij}, t^\beta_{kl}] = 0 \) and \([t^\alpha_{ij}, t^\beta_{ik} + t^\beta_{jk}] = 0 \) are preserved is very similar. \(\square\)

5.6. The filtered-formality of \( \mathfrak{PB}^\Gamma_{1,n} \) (end of the proof of Theorem 5.6). To prove that \( \text{Lie}(\mu_{x_0, \tau, n, \Gamma}) \) is an isomorphism, it is sufficient to prove that it is an isomorphism on associated graded. According to Proposition 5.7, we simply have to prove that \( \phi := \text{grLie}(\mu_{x_0, \tau, n, \Gamma}) \circ p_n \) is an isomorphism of graded Lie algebras.

We will actually be more specific on prove the following:

Lemma 5.8. We have \( \phi(x_i) = -y_i \), \( \phi(y_i) = 2\pi i x_i - \tau y_i \), and \( \phi(t^\alpha_{ij}) = 2\pi i t^\alpha_{ij} \). In particular, \( \phi \) is an automorphism.

Proof. Recall that \( \mu_{x_0, \tau, n, \Gamma} \) can be computed as follows. Let \( F_{x_0} : U \rightarrow \exp(t_{1,n}^\Gamma) \) be such that

\[
\begin{cases}
(\partial / \partial z_i) F_{x_0}(z) = K^0_{i}(z) F_{x_0}(z), \\
F_{x_0}(z_0) = 1.
\end{cases}
\]
Then consider
\[ H_{\tau,n}^\Gamma := \left\{ z = (z_1, \ldots, z_n) | z_i = a_i + \tau b_i, 0 < a_n < \ldots < a_1 < \frac{1}{M} \right\} \]
and
\[ V_{\tau,n}^\Gamma := \left\{ z = (z_1, \ldots, z_n) | z_i = a_i + \tau b_i, 0 < b_n < \ldots < b_1 < \frac{1}{N} \right\} . \]

Let \( F_{z_0}^{H^\Gamma} \) (resp. \( F_{z_0}^{V^\Gamma} \)) be the analytic prolongations of \( F_{z_0} \) to \( H_{\tau,n}^\Gamma \) (resp. \( V_{\tau,n}^\Gamma \)). Then
\[ F_{z_0}^{H^\Gamma} (z + \delta_i) = F_{z_0}^{H^\Gamma} (z)\mu_{z_0,\tau,n,\Gamma}(X_i^M) \quad \text{and} \quad e^{2\pi i \alpha_i} F_{z_0}^{V^\Gamma} (z + \tau \delta_i) = F_{z_0}^{V^\Gamma} (z)\mu_{z_0,\tau,n,\Gamma}(Y_i^N). \]

Knowing that \( \log F_{z_0}^{H^\Gamma} (z) = - \sum_i (z_i - z_i^0) y_i + \text{terms of degree } \geq 2 \), we get
\[ \log \mu_{z_0,\tau,n,\Gamma}(X_i^M) = -y_i + \text{terms of degree } \geq 2 \]
and
\[ \log \mu_{z_0,\tau,n,\Gamma}(Y_i^N) = 2\pi i x_i - \tau y_i + \text{terms of degree } \geq 2 . \]

This gives us that \( \phi(x_i) = -y_i \) and \( \phi(y_i) = 2\pi i x_i - \tau y_i \).

In order to compute \( \log \mu_{z_0,\tau,n,\Gamma}(P_{ij}) \), which is also equal to \( \log \mu_{z_0,\tau,n,\Gamma}(P_{ij}) \), we will need to compute \( \mu_{z_0,\tau,n,\Gamma}(X_i) \), \( \mu_{z_0,\tau,n,\Gamma}(Y_i) \) and \( \mu_{z_0,\tau,n,\Gamma}(P_{ij}) \):

- As usual, we have
  \[ \mu_{z_0,\tau,n,\Gamma}(P_{ij}) = \exp(2\pi i t_{ij}^0 + \text{terms of degree } \geq 3) , \]
  where \( 0 = (0,0) \).
- We also have
  \[ F_{z_0}^{H^\Gamma} (z + \delta_i) = (1,0) F_{z_0}^{H^\Gamma} (z)\mu_{z_0,\tau,n,\Gamma}(X_i), \]
  which implies that
  \[ \mu_{z_0,\tau,n,\Gamma}(X_i) \in (-1,0), \exp(t_{ij}^\Gamma) . \]
- We finally have
  \[ e^{2\pi i \frac{\alpha_i}{g}} F_{z_0}^{V^\Gamma} (z + \tau \delta_i) = (0,1) F_{z_0}^{V^\Gamma} (z)\mu_{z_0,\tau,n,\Gamma}(Y_i), \]
  which implies that
  \[ \mu_{z_0,\tau,n,\Gamma}(Y_i) \in (0,-1), \exp(t_{ij}^\Gamma) . \]

Hence, if \( \alpha = (\bar{p}, \bar{q}) \in \Gamma \), then
\[ \mu_{z_0,\tau,n,\Gamma}(X_i^{-p}Y_j^{-q}) = g(\bar{p},0)i, (0,q)j , \]
with \( g \in \exp(t_{ij}^\Gamma) \), and
\[ \mu_{z_0,\tau,n,\Gamma}(Y_j^qX_i^p) = (0,-q)_j (-\bar{p},0)_i g^{-1} . \]

Therefore
\[ \mu_{z_0,\tau,n,\Gamma}(P_{ij}^\alpha) = g(\bar{p},0)i, (0,q)j \exp(2\pi i t_{ij}^0 + \text{terms of degree } \geq 3)(0,-q)_j (-\bar{p},0)_i g^{-1} \]
\[ = g \exp(2\pi i t_{ij}^\alpha + \text{terms of degree } \geq 3)g^{-1} . \]
This shows that $\log \mu_{s_0, \tau, n}(P_{ij}^\tau) = 2\pi i t_{ij}^\tau + \text{terms of degree } \geq 3$, so that $\phi(t_{ij}^\tau) = 2\pi i t_{ij}^\tau$. This ends the proof of the Lemma.

Finally, if we denote $\hat{\mathbb{PB}}_{1,n}(\mathbb{C}) := \hat{\pi}_1(C(E_{\tau, \Gamma}, n, \Gamma), \mathbb{C}_0)(\mathbb{C})$, where $\mathbb{C}_0$ is the image of $\mathbb{C}$ under the projection $\text{Conf}(E_{\tau, \Gamma}, n) \rightarrow C(E_{\tau, \Gamma}, n)$, then the isomorphism $\hat{\mu}_{\mathbb{C}_0, \tau, n, \Gamma}(\mathbb{C})$ descends to an isomorphism $\hat{\mu}_{\mathbb{C}_0, \tau, n, \Gamma} := \hat{\mu}_{\mathbb{C}_0, \tau, n, \Gamma}(\mathbb{C}) : \hat{\mathbb{PB}}_{1,n}(\mathbb{C}) \rightarrow \text{exp}(\hat{t}_{ij}^\tau)$.

Now let $\hat{\mathbb{B}}_{1,n}$ be the fundamental group $\pi_1(C(E_{\tau, \Gamma}, [n]), [\mathbb{C}_0])$. By considering the short exact sequence

$$1 \rightarrow \hat{\mathbb{PB}}_{1,n} \rightarrow \hat{\mathbb{B}}_{1,n} \xrightarrow{\varphi_n} (\Gamma^n/\Gamma) \rtimes \mathfrak{S}_n \rightarrow 1,$$

we deduce that the map

$$\hat{\mathbb{B}}_{1,n}(\varphi_n, \mathbb{C}) \rightarrow \text{exp}(\hat{t}_{ij}^\tau) \times ((\Gamma^n/\Gamma) \rtimes \mathfrak{S}_n)$$

is also relatively filtered-formal. In conclusion, we obtain the summarizing commutative cube

6. Representations of Cherednik algebras

6.1. The Cherednik algebra of a wreath product. In this paragraph $\Gamma$ is any finite group such that $\Gamma \subset \text{Aut}(\mathbb{C})$, $k = (k_\alpha)_\alpha \in \mathbb{C}^\Gamma$ is such that $k_\alpha = \overline{k_{-\alpha}}$ and $G := \Gamma \rtimes \mathfrak{S}_n$. We define the Cherednik algebra $H_n^\Gamma(k)$ as the quotient of the algebra $\mathbb{C}(x_1, \ldots, x_n, y_1, \ldots, y_n) \rtimes \mathbb{C}[G]$ by the relations

- $\sum_i x_i = \sum_i y_i = 0$
- $[x_i, x_j] = [y_i, y_j] = 0$,
- $[x_i, y_j] = \frac{1}{n} \sum_{\alpha \in \Gamma} k_{\alpha} s^\alpha_{ij}$ ($i \neq j$),

where $s^\alpha_{ij} = (\alpha_i - \alpha_j)s_{ij}$, and $s_{ij}$ is the permutation of $i$ and $j$.

Remark 6.1. As $\Gamma \subset \text{Aut}(\mathbb{C})$, $H_n^\Gamma(k)$ admits a geometric construction. Define $X := \{z \in \mathbb{C}^n | \sum_i z_i = 0\}$ and consider the following action of $G$ on it: $\mathfrak{S}_n$ acts in an obvious way and

$$\alpha_i(z) = (\alpha(i) - \frac{1}{n} \sum_j \alpha(j))(z),$$
where $\alpha^{(k)}$ is the action of $\alpha \in \Gamma$ on the $k$-th factor of $\mathbb{C}^n$. Following [14] one can construct a Cherednik algebra $H_{1,k=0}(X,G)$ on $X/G$. It can be defined as the subalgebra of $\text{Diff}(X) \rtimes \mathbb{C}[G]$ generated by the function algebra $\mathcal{O}_X$, the group $G$ and the Dunkl-Opdam operators $D_i - D_j$, where

$$D_i = \partial_{z_i} + \sum_{j \neq k \in \Gamma} k_{\alpha} \frac{1 - s_{ij}^{\alpha}}{(-\alpha)(z_i) - \alpha(z_j)}.$$  

One can then prove that there is a unique isomorphism of algebras $H_{n}^\Gamma(k) \to H_{1,k=0}(X,G)$ defined by

$$x_i \mapsto z_i,$$

$$y_i \mapsto D_i - \frac{1}{n} \sum_j D_j,$$

$$G \ni g \mapsto g.$$  

6.2. Morphisms from $\overline{\Gamma}_{1,n}$ to the Cherednik algebra.

**Proposition 6.2.** For any $a, b \in \mathbb{C}$ there is a morphism of Lie algebras $\phi_{a,b} : \overline{\Gamma}_{1,n} \to H_{n}^\Gamma(k)$ defined by

$$\bar{x}_i \mapsto ax_i,$$

$$\bar{y}_i \mapsto by_i,$$

$$\bar{t}_{ij}^\alpha \mapsto ab \left( \frac{1}{n} - k_{\alpha}s_{ij}^{\alpha} \right).$$

**Proof.** Straightforward from the alternative presentation of $\overline{\Gamma}_{1,n}$ in Lemma 1.1. \hfill $\Box$

Hence any representation $V$ of $H_{n}^\Gamma(k)$ yields a family of flat connections $\nabla_{a,b}^{(V)}$ over the configuration space $C(E, [n], \Gamma)$.

6.3. Monodromy representations of Hecke algebras. Let $E$ be an elliptic curve and $\hat{E} \to E$ the $\Gamma$-covering as in §1.2. Define $X = \hat{E}^\Gamma/\Gamma$ and $G = (\Gamma : \mathfrak{S}_n)/\Gamma_{\text{diag}}$. Then the set $X' \subset X$ of points with trivial stabilizer is such that $X'/\Gamma = C(E, [n], \Gamma)$.

Let us recall from [14] the construction of the Hecke algebra $\mathcal{H}_{n}^\Gamma(q, \mathcal{L})$ of $X/G$. It is the quotient of the group algebra of the orbifold fundamental group $\mathcal{B}_{1,n}$ of $C(E, [n], \Gamma)$ by the additional relations $(T_a - q^{-1}t_a)(T_a + q^{-1}t_a) = 0$, where $T_a$ is an element of $\mathcal{B}_{1,n}$ homotopic as a free loop to a small loop around the divisor $Y_a := \cup_{i \neq j}(z_i = \alpha \cdot z_j)$ in $X/G$, in the counterclockwise direction.\(^4\)

Let us consider the flat connection $\nabla_{a,b}^{(V)}$ and set

$$q = e^{-2\pi i ab/n}, \quad t_a = e^{-2\pi i k_a ab}.$$  

Then the monodromy representation $\mathcal{B}_{1,n}^\Gamma \to GL(V)$ of $\nabla_{a,b}^{(V)}$ obviously gives a representation of $\mathcal{H}_{n}^\Gamma(q, \mathcal{L})$ either if $V$ is finite dimensional or if $a, b$ are formal parameters. In particular,

\(^4\)Here the subgroup of $G$ acting trivially on $Y_a$ is the order 2 cyclic subgroup generated by $s_{ij}^{\alpha}$. 
taking \( a = b \) a formal parameter and \( V = H^e_n(k) \), one obtains an algebra morphism
\[
\mathcal{H}^e_n(q, t) \rightarrow H^e_n(k)[[a]].
\]
We do not know if this morphism is an isomorphism upon inverting \( a \).

6.4. The modular extension of \( \phi_{a,b} \). Now assume that \( a, b \neq 0 \).

**Proposition 6.3.** The Lie algebra morphism \( \phi_{a,b} \) can be extended to the algebra \( \mathcal{U}(\bar{t}_1 \Gamma_1, n \rtimes \bar{d}_1 \Gamma_1) \rtimes G \) by the following formulæ:
\[
\begin{align*}
\phi_{a,b}(s^{ij}_{n}) &= s^{ij}_{n}, \\
\phi_{a,b}(d) &= \frac{1}{2} \sum_i (x_i y_i + y_i x_i), \\
\phi_{a,b}(X) &= \frac{-1}{2} ab^{-1} \sum_i x_i^2, \\
\phi_{a,b}(\Delta_0) &= \frac{1}{2} ba^{-1} \sum_i y_i^2, \\
\phi_{a,b}(\xi_{s,\gamma}) &= -a^{-1} b^{-1} \sum_{i<j} (\gamma \cdot (x_i - x_j))^s.
\end{align*}
\]
Thus, the flat connections \( \nabla_{a,b}^\Gamma \) extend to flat connections on \( \mathcal{M}^\Gamma_{1,[n]} \).

**Proof.** The proof is a straightforward calculation. \( \square \)
### Lists of notation

#### Groups.
- $\text{PB}_{1,n}$: Pure braid group on the complex plane. 2
- $\text{PB}^M_{1,n}$: $M$-decorated pure braid group on the cylinder. 3
- $G^T_n$: Structure group of the principal bundle over $\mathcal{M}_1^T,n$. 17
- $G^\Gamma_n$: Structure group of the principal bundle over $\mathcal{M}_1^\Gamma,n$. 17
- $\text{SL}_2(\mathbb{Z})$: $\Gamma$-level principal congruence subgroup of $\text{SL}_2(\mathbb{Z})$. 18
- $\text{PB}^\Gamma_{1,n}$: $\Gamma$-decorated pure braid group on the torus. 38
- $B_{1,n}$: Braid group on the torus. 38
- $\text{PB}^\Gamma_{1,n}$: Pure braid group on the torus. 38

#### Spaces.
- $\text{Conf}(\mathbb{C},n)$: Configuration space of $n$ points in $\mathbb{C}$. 2
- $\text{Conf}(\mathbb{C}^*,n)$: Configuration space of $n$ points in $\mathbb{C}^*$. 3
- $\text{Conf}(\mathbb{C}^*,n,M)$: $M$-decorated configuration space of $n$ points in $\mathbb{C}^*$. 3
- $\text{Conf}(\mathbb{T},n)$: Configuration space of $n$ points in $\mathbb{T}$. 4
- $\text{Conf}(\mathbb{T},n,\Gamma)$: $\Gamma$-decorated configuration space of $n$ points in $\mathbb{T}$. 8
- $\mathcal{C}(\mathbb{T},n,\Gamma)$: Reduced $\Gamma$-decorated configuration space of $n$ points in $\mathbb{T}$. 8
- $\mathcal{M}^T_{1,n}$: Reduced moduli space of $\Gamma$-structured $n$-marked elliptic curves. 20
- $\mathcal{M}^T_{1,n}$: Non-reduced moduli space of $\Gamma$-structured $n$-marked elliptic curves. 20
- $\mathcal{M}^T_{1,[n]}$: Reduced moduli space of $\Gamma$-structured unorderly $n$-marked elliptic curves. 20
- $\mathcal{M}^T_{1,[n]}$: Non-reduced moduli space of $\Gamma$-structured unorderly $n$-marked elliptic curves. 20

#### Lie and associative algebras.
- $t^\mathbb{C}_n^M$: $M$-cycloctomic Kohno-Drinfeld Lie $\mathbb{C}$-algebra. 3
- $t^\mathbb{C}_{1,n}$: Elliptic Kohno-Drinfeld Lie $\mathbb{C}$-algebra. 4
- $t^\mathbb{C}_{1,T}(\mathbb{k})$: $\Gamma$-ellipsitomic Kohno-Drinfeld Lie $\mathbb{k}$-algebra. 7
- $\delta^T$: Intermediate twisted derivations Lie algebra. 14
- $\delta^\Gamma$: Twisted derivations Lie algebra. 14
- $H_n(\mathfrak{g},\Gamma^*)$: Hecke algebra of the pair $(\mathfrak{g},\Gamma)$. 29
- $H_n(\mathfrak{g},\mathfrak{h}^*_{\text{reg}})$: Reduced Hecke algebra of the pair $(\mathfrak{g},\mathfrak{h})$. 32

#### Bundles.
- $\mathcal{P}_{\tau,n,\Gamma}$: Principal $\exp(t^\mathbb{C}_{1,n})$-bundle over $\text{Conf}(E,n,\Gamma)$. 8
- $\mathcal{P}_{\tau,\{n\},\Gamma}$: Principal $\exp(t^\mathbb{C}_{1,n})$-bundle over $\text{Conf}(E,\{n\},\Gamma)$. 9
- $\mathcal{P}_{\tau,\Gamma,n}$: Principal $\exp(t^\mathbb{C}_{1,n}) \times \Gamma^n$-bundle over $\text{Conf}(E,n)$. 9
- $\mathcal{P}_{\Gamma,n,\Gamma}$: Principal $G^\Gamma_{n}$-bundle over $\mathcal{M}^T_{1,n}$. 20
- $\mathcal{P}_n,\Gamma$: Principal $G^\Gamma_{n}$-bundle over $\mathcal{M}^T_{1,n}$. 20
- $\mathcal{P}_{\{n\},\Gamma}$: Principal $G^\Gamma_{n}$-bundle over $\mathcal{M}^T_{1,n}$. 20
- $\mathcal{P}_{\Gamma,n}$: Principal $G^\Gamma_{n} \times \Gamma^n$-bundle over $\mathcal{M}^T_{1,n}/\Gamma^n$. 28
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