SUBCRITICAL CATALYTIC BRANCHING RANDOM WALK
WITH FINITE OR INFINITE VARIANCE OF OFFSPRING NUMBER

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Abstract

Subcritical catalytic branching random walk on \(d\)-dimensional lattice is studied. New theorems concerning the asymptotic behavior of distributions of local particles numbers are established. To prove the results different approaches are used including the connection between fractional moments of random variables and fractional derivatives of their Laplace transforms. In the previous papers on this subject only supercritical and critical regimes were investigated assuming finiteness of the first moment of offspring number and finiteness of the variance of offspring number, respectively. In the present paper for the offspring number in subcritical regime finiteness of the moment of order \(1 + \delta\) is required where \(\delta\) is some positive number.

Keywords and phrases: branching random walk, subcritical regime, finite variance of offspring number, infinite variance of offspring number, local particles numbers, limit theorems, fractional moments, fractional derivatives.

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1 Introduction and main results

The paper is devoted to investigation of catalytic branching random walk (CBRW) on integer lattice \(\mathbb{Z}^d\), \(d \in \mathbb{N}\). This modification of branching random walk (BRW) with a single source of branching was proposed by V.A. Vatutin, V.A. Topchij and E.B. Yarovaya in [1] and it comprises symmetric BRW studied earlier (see, e.g., [2]).

Recall description of CBRW on \(\mathbb{Z}^d\). Let at the initial time \(t = 0\) there be a single particle on the lattice located at point \(x \in \mathbb{Z}^d\). If \(x \neq 0\) then the particle performs the continuous time random walk until the first hitting of the origin. We assume that the random walk outside the origin is specified by infinitesimal matrix \(A = (a(u, v))_{u, v \in \mathbb{Z}^d}\) and is symmetric, homogeneous, having a finite variance of jumps and irreducible (i.e. the particle passes from an arbitrary point \(u \in \mathbb{Z}^d\) to any point \(v \in \mathbb{Z}^d\) within finite time with positive probability). It means that

\[
a(u, v) = a(v, u), \quad a(u, v) = a(0, v - u) := a(v - u), \quad u, v \in \mathbb{Z}^d,
\]

\[
\sum_{v \in \mathbb{Z}^d} a(v) = 0 \quad \text{where} \quad a(0) < 0 \quad \text{and} \quad a(v) \geq 0 \quad \text{for} \quad v \neq 0, \quad \sum_{v \in \mathbb{Z}^d} \|v\|^2 a(v) < \infty
\]

(here \(\| \cdot \|\) denotes an arbitrary norm in \(\mathbb{R}^d\)). If \(x = 0\) or the particle has just hit the origin then it spends there random time distributed according to exponential law with parameter 1. Afterwards it either dies with probability \(\alpha\), producing just before the death a random offspring number \(\xi\) or leaves the origin with probability \(1 - \alpha\), so that the intensity of transition from the origin to point

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$v \neq 0$ equals $-(1 - \alpha)a(v)/a(0)$. At the origin the branching of particle is specified by probability generating function

$$f(s) := \mathbb{E}s^\xi = \sum_{k=0}^{\infty} f_k s^k, \quad s \in [0, 1].$$

At the birth moment the newborn particles are located at the origin. They evolve in accordance with the scheme described above independently of each other as well as of their parents history.

The natural objects of study in CBRW are total and local particles numbers. Denote by $\mu(t)$ the number of particles existing on the lattice $\mathbb{Z}^d$ at time $t \geq 0$. In a similar way we define local numbers $\mu(t; y)$ as quantities of particles located at separate points $y \in \mathbb{Z}^d$ at time $t$.

It was established in [3] that exponential growth (as $t \to \infty$) of both total and local particles numbers in CBRW on $\mathbb{Z}^d$ holds if and only if $\mathbb{E}\xi > 1 + h_d\alpha^{-1}(1 - \alpha)$. Here $h_d$ is probability of the event that a particle which has left the origin will never come back. Thus, the value $\mathbb{E}\xi = 1 + h_d\alpha^{-1}(1 - \alpha)$ is critical and, similarly to many types of branching processes (see, e.g., [4]), CBRW is classified as supercritical, critical or subcritical if the mean offspring number $\mathbb{E}\xi$ is greater, equal or less than $1 + h_d\alpha^{-1}(1 - \alpha)$, respectively. In view of the random walk properties such as recurrence or transience, $h_1 = h_2 = 0$, whereas $0 < h_d < 1$ for $d \geq 3$. Recall that $\mathbb{E}\xi = f'(1)$ and for classic Galton-Watson branching processes the critical value of $f'(1)$ is equal to 1.

Critical and subcritical CBRW on $\mathbb{Z}^d$ are of special interest since in these cases there arise diverse kinds of limit (in time) behavior of the total and local particles numbers depending on dimension $d$. For instance, for $d = 1$ and $d = 2$ probability $\mathbb{P}_x(\mu(t) > 0)$ of non-extinction of the population tends to zero as time grows (index $x \in \mathbb{Z}^d$ denotes the starting point of CBRW), whereas for $d \geq 3$ this probability has a positive limit. Such effect is due to existence, on lattice $\mathbb{Z}^d$ for $d \geq 3$, of particles which with positive probability are ever alive and never hit the source of branching. The total particles number in critical CBRW on integer line $\mathbb{Z}$, i.e. for $d = 1$, was studied in the fundamental paper [1]. In papers [5] – [8] the investigation was continued for critical and subcritical CBRW on $\mathbb{Z}^d$ for any $d \in \mathbb{N}$.

The analysis of local particles numbers is much more hard. For critical CBRW on $\mathbb{Z}^d$ the limit distributions of local particles numbers were studied in the series of papers by V.A. Vatutin, V.A. Topchij, Y. Hu, E.B. Yarovaya and the author (see, e.g., [9] – [12]). It is necessary to note that in all these papers the finiteness of second moment of offspring number $\xi$ are less restrictive than those in the critical case. Namely, we find the asymptotic behavior of the mean local numbers $m(t; x, y) := \mathbb{E}_x\mu(t; y)$ for fixed $x, y \in \mathbb{Z}^d$ and $t \to \infty$. Assuming that $\mathbb{E}\xi^{1+\delta} < \infty$ for some $\delta \in (0, 1]$, the similar problem is solved for non-extinction probability $q(t; x, y) := \mathbb{P}_x(\mu(t; y) > 0)$ of local particles numbers. Moreover, under the same restriction on the moment $\mathbb{E}\xi^{1+\delta}$ the conditional limit theorems are proved for $\mu(t; y)$ as $t \to \infty$.

To formulate the main results let us introduce additional notation. Let

$$q(s, t; x, y) := 1 - \mathbb{E}_x s^{\mu(t; y)} \quad \text{and} \quad J(s; y) := \int_0^\infty \Phi(q(s, t; 0, y)) \, dt$$

where $\Phi(s) := \alpha(f(1 - s) - 1 + f'(1)s)$ and $s \in [0, 1]$, $t \geq 0$, $x, y \in \mathbb{Z}^d$. Consider transition probabilities $p(t; x, y)$, $t \geq 0$, $x, y \in \mathbb{Z}^d$, of the random walk generated by matrix $A$. According to [9] and Theorem 2.1.1 in [13], for fixed $x$ and $y$, as $t \to \infty$, the following asymptotic relations hold true

$$p(t; x, y) \sim \frac{\gamma_d}{t^{d/2}}, \quad p'(t; 0, 0) \sim -\frac{d \gamma_d}{2 t^{d/2 + 1}}, \quad p(t; 0, 0) - p(t; x, y) \sim \frac{\gamma_d (y - x)}{t^{d/2 + 1}}.$$

(1)
where \( \gamma_d := \left( (2\pi)^d |\det \phi''_{\theta \theta}(0)| \right)^{-1/2}, \phi(\theta) := \sum_{z \in \mathbb{Z}^d} a(0, z) \cos (z, \theta), \theta \in [-\pi, \pi]^d, \)

\[
\phi''_{\theta \theta}(0) := \left( \frac{\partial^2 \phi(\theta)}{\partial \theta_i \partial \theta_j} \right)_{i,j \in \{1, \ldots, d\}, i,j \in \{1, \ldots, d\}}, \quad \tilde{\gamma}_d(z) := \frac{1}{2(2\pi)^d} \int_{\mathbb{R}^d} (v, z)^2 e^{\frac{i}{2} (\phi''_{\theta \theta}(0)v, v)} dv, \quad z \in \mathbb{Z}^d,
\]

and \((\cdot, \cdot)\) denotes the inner product in \(\mathbb{R}^d\). Set \(G_\lambda(x, y) := \int_0^\infty e^{-\lambda t} p(t; x, y) dt, \lambda \geq 0, x, y \in \mathbb{Z}^d\), i.e. \(G_\lambda(x, y)\) is the Laplace transform of the transition probability \(p(\cdot; x, y)\). By virtue of (1) for \(d \geq 3\) the Green’s function \(G_0(x, y)\) is finite for all \(x, y \in \mathbb{Z}^d\), which means the transience of the random walk. However, for \(d = 1\) or \(d = 2\), we deal with recurrent random walk since \(\lim_{\lambda \to 0^+} G_\lambda(x, y) = \infty\).

In view of the same formula (1), the function \(\lim_{\lambda \to 0^+} (G_\lambda(0, 0) - G_\lambda(x, y))\) is finite for all \(d \in \mathbb{N}\) and \(x, y \in \mathbb{Z}^d\). So we may introduce the function

\[
\rho_d(z) := \begin{cases} 
(1 - \alpha) a^{-1} - \beta \int_0^\infty (p(t; 0, 0) - p(t; 0, z)) dt & \text{if } z \neq 0, \\
1 & \text{if } z = 0
\end{cases}
\]

where for the sake of convenience we set \(a := -a(0)\) and \(\beta := \alpha(f'(1) - 1)\). As shown in [9], \(h_d = (aG_0(0, 0))^{-1}\). It implies that in subcritical regime \(\beta < (1 - \alpha)(aG_0(0, 0))^{-1}\) and, therefore, \(\rho_d(\cdot)\) is a strictly positive function for all \(d \in \mathbb{N}\). Note also that in accordance with formula (2.1.15) in [13] inequality \(p(t; 0, 0) > p(t; x, y), t \geq 0\), is valid and the function \(p(\cdot; x, y)\) is symmetric and homogeneous in \(x, y \in \mathbb{Z}^d\).

For \(d = 1\) and \(x, y \in \mathbb{Z}\), we define

\[
C_1(0, y) := \frac{1 - \alpha}{2a\gamma_1\pi^2} \rho_1(y), \quad C_1(x, y) := \frac{1}{2\gamma_1\pi^2} \rho_1(x) \rho_1(y) + \tilde{\gamma}_1(x) + \tilde{\gamma}_1(y) - \tilde{\gamma}_1(y - x), \quad x \neq 0.
\]

In a similar way, for \(d = 2\) and \(x, y \in \mathbb{Z}^2\) set

\[
C_2(0, y) := \frac{1 - \alpha}{a\gamma_2\beta^2} \rho_2(y), \quad C_2(x, y) := \frac{1}{\gamma_2\beta^2} \rho_2(x) \rho_2(y), \quad x \neq 0.
\]

At last, for \(d \geq 3\), specify functions \(C_d(x, y), x, y \in \mathbb{Z}^d\), by way of

\[
C_d(0, y) := \frac{(1 - \alpha) a \gamma_d}{(1 - \alpha - a \beta G_0(0, 0))^2} \rho_d(y), \quad C_d(x, y) := \frac{a^2 \gamma_d}{(1 - \alpha - a \beta G_0(0, 0))^2} \rho_d(x) \rho_d(y), \quad x \neq 0.
\]

**Theorem 1** Let \(\xi_1 < 1 + h_d\alpha^{-1}(1 - \alpha)\). Then, as \(t \to \infty\), for each \(x, y \in \mathbb{Z}^d\), the following relations are true

\[
m(t; x, y) \sim C_1(x, y) \frac{t^{3/2}}{t^{3/2}}, \quad d = 1,
\]

\[
m(t; x, y) \sim C_2(x, y) \frac{t^{3/2}}{t^{3/2}}, \quad d = 2,
\]

\[
m(t; x, y) \sim C_d(x, y) \frac{t^{3/2}}{t^{3/2}}, \quad d \geq 3,
\]

and the functions \(C_d(\cdot, \cdot), d \in \mathbb{N}\), introduced above, are strictly positive.

The statement of Theorem 1 generalizes the corresponding results of Chapter 5 in [13] concerning the asymptotic behavior of the first moments of local particles numbers in subcritical symmetric BRW on \(\mathbb{Z}^d\). Here, in contrast to [13], we use the approaches such as representation of complex-valued measures in terms of Banach algebras (see [9]) and Tauberian theorems for derivatives of Laplace transform (see Section 7.3 in [14]).
Theorem 2 If $\xi < 1 + h_d \alpha^{-1}(1 - \alpha)$ and there exists $\xi^{1+\delta}$ for some $\delta \in (0, 1]$ then, for fixed $x, y \in \mathbb{Z}^d$, as $t \to \infty$, one has

$$q(t; x, y) \sim \frac{C_1(x, y) - C_1(x, 0)J(0; y)}{t^{3/2}}, \quad d = 1,$$

$$q(t; x, y) \sim \frac{C_2(x, 0)(\rho_2(y) - J(0; y))}{t \ln^2 t}, \quad d = 2,$$

$$q(t; x, y) \sim \frac{C_d(x, 0)(\rho_d(y) - J(0; y))}{t^{d/2}}, \quad d \geq 3,$$

where $C_1(x, y) - C_1(x, 0)J(0; y) > 0$ for all $x, y \in \mathbb{Z}$ and $J(0; y) < \rho_d(y)$ for $d \geq 2$ and $y \in \mathbb{Z}^d$.

To prove Theorem 2 we apply the Hölder inequality combined with results on connection between fractional moments of random variables and fractional derivatives of their Laplace transforms (see, e.g., [15]). Theorem 3 can be considered as a corollary of Theorem 2.

Theorem 3 Let $\xi < 1 + h_d \alpha^{-1}(1 - \alpha)$ and $\xi^{1+\delta}$ be finite for some $\delta \in (0, 1]$. Then, for fixed $x, y \in \mathbb{Z}^d$ and each $s \in [0, 1]$, the following equalities are valid

$$\lim_{t \to \infty} \mathbb{E}_x \left( s^{\mu(t; y)} \mid \mu(t; y) > 0 \right) = \frac{s \mu(x, y) - \mu(x, 0)J(0; y) - \mu(s; y)}{\mu(x, y) - \mu(x, 0)J(0; y)} = 1,$$

$$\lim_{t \to \infty} \mathbb{E}_x \left( s^{\mu(t; y)} \mid \mu(t; y) > 0 \right) = \frac{s \rho_d(y) - J(0; y) - \rho_d(y)}{\rho_d(y) - J(0; y)}, \quad d \geq 2.$$

Remark. Comparing formulations of Theorems 1 and 3 and the results on local particles numbers in critical CBRW on $\mathbb{Z}^d$ (see, e.g., [16] and [17]) the following conclusion suggests itself. Local particles numbers in critical CBRW demonstrate “subcritical” behavior in the case of random walk on integer lattice. Moreover, the scheme of proofs of Theorems 1 and 3 is easily carried over to the case of critical CBRW on $\mathbb{Z}^d$. Consequently, we may state that the results of [16] and [17], concerning the local numbers in critical CBRW on $\mathbb{Z}^d$, are valid under less restrictive conditions on the moments of offspring number. Namely, it is sufficient to require finiteness of $\xi^{1+\delta}$ for some $\delta \in (0, 1]$ instead of condition $\xi^2 < \infty$.

Concluding the first part of the paper let us note the close relation between CBRW and superprocesses, namely, catalytic super-Brownian motion with a single point of catalysis (see, e.g., [18] and references therein). It is of interest that in view of [11] CBRW may be considered as a queueing system with a random number of independent servers. This gives an opportunity for wide applications of the established results.

2 Proof of Theorem 1

Similarly to the proof of Theorem 1 in [17], we use backward and forward integral equations for the family of functions $\{m(\cdot; x, y)\}_{x, y \in \mathbb{Z}^d}$. These equations coincide with equations (8) and (9) in [17], derived for the mean local particles numbers in critical CBRW on $\mathbb{Z}^d$, upon the replacement...
of critical value \( \beta_c = (1 - \alpha)a^{-1}G_0^{-1}(0, 0) \) by value \( \beta \), that is,

\[
m(t; x, y) = p(t; x, y) + \left(1 - \frac{a}{1 - \alpha}\right) \int_0^t p(t - u; x, 0)m(u; 0, y) \, du
\]
\[
+ \frac{a\beta}{1 - \alpha} \int_0^t p(t - u; x, 0)m(u; 0, y) \, du,
\]
\[
m(t; x, y) = p(t; x, y) + \left(1 - \frac{\alpha}{a} - 1\right) \int_0^t m(u; x, 0)p'(t - u; 0, y) \, du
\]
\[
+ \beta \int_0^t m(u; x, 0)p(t - u; 0, y) \, du.
\]

Relations (2) and (3), as well as equations (8) and (9) in [17], are obtained by means of the variation of constants formula applied to backward and forward differential equations (5) and (6) in [17] established in Banach space \( l_\infty(\mathbb{Z}^d) \).

We also need the following auxiliary statement proved similarly to Lemma 3.3.5 in [13] and Lemma 1 in [17].

**Lemma 1** For each \( y \in \mathbb{Z}^d \) the function \( m(t; y, y) \) does not increase in variable \( t \).

Now we turn directly to the proof of Theorem 1. At first let us consider the case \( x = y = 0 \). Apply the Laplace transform to both sides of equality (3) and use the obtained relation to express function \( \hat{m}(\lambda) := \int_0^\infty e^{-\lambda t}m(t; 0, 0) \, dt, \lambda \geq 0 \). Then

\[
\hat{m}(\lambda) = \frac{G_\lambda(0, 0)}{1 - ((1 - \alpha)a^{-1} - 1) \int_0^\infty e^{-\lambda t}p'(t; 0, 0) \, dt - \beta G_\lambda(0, 0)}.
\]

Differentiate each side of the last relation in \( \lambda \). Consequently, taking into account the identity \( \int_0^\infty e^{-\lambda t}p'(t; 0, 0) \, dt = \lambda G_\lambda(0, 0) - 1 \) one has

\[
\hat{m}'(\lambda) = \frac{(1 - \alpha)a^{-1}G'_\lambda(0, 0) + ((1 - \alpha)a^{-1} - 1)G^2_\lambda(0, 0)}{((1 - \alpha)a^{-1} - ((1 - \alpha)a^{-1} - 1)\lambda G_\lambda(0, 0) - \beta G_\lambda(0, 0))^2}.
\]

According to Tauberian Theorem 2 in §5 of Chapter XIII in [19] along with Corollary 43 in [14], relation (11) implies

\[
G_\lambda(0, 0) \sim \frac{\gamma_1\sqrt{\pi}}{\sqrt{\lambda}}, \quad G'_\lambda(0, 0) \sim -\frac{\gamma_1\sqrt{\pi}}{2\lambda^{3/2}}, \quad d = 1,
\]
\[
G_\lambda(0, 0) \sim \gamma_2 \ln \frac{1}{\lambda}, \quad G'_\lambda(0, 0) \sim -\frac{\gamma_2}{\lambda}, \quad d = 2,
\]

as \( \lambda \to 0^+ \). Substituting these asymptotic equalities into (5) for \( d = 1 \) and \( d = 2 \), respectively, we find

\[
\hat{m}'(\lambda) \sim -\frac{1 - \alpha}{2a \gamma_1 \sqrt{\pi} \beta^2 \sqrt{\lambda}}, \quad d = 1,
\]
\[
\hat{m}'(\lambda) \sim -\frac{1 - \alpha}{a \gamma_2 \beta^2 \lambda \ln^2 \lambda}, \quad d = 2,
\]

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as \( \lambda \to 0^+ \). Applying Corollary 43 in [14] to the above relations we come to the assertion of Theorem 1 when \( x = y = 0 \) and \( d = 1 \) or \( d = 2 \).

For \( d \geq 3 \), we employ another approach, namely, the representation of complex-valued measures in terms of Banach algebras. In view of (1) two cumulative distribution functions, having the Laplace transforms \( G_{\lambda}(0, 0) \) and \( \int_0^\infty e^{-\lambda t}p(t; 0, 0) \, dt \), respectively, possess tails equivalent to constants (the second constant being zero) multiplied by the same function \( t^{1-d/2} \). Hence, due to Lemma 6 in [9] and formula (1) the following asymptotic equality holds true

\[
\int_t^\infty m(u; 0, 0) \, du \sim \frac{2(1-\alpha) a \gamma_d}{(d-2)(1-\alpha - a \beta G_0(0, 0))} t^{d/2-1}, \quad t \to \infty.
\]

Whence by Lemma 1 and the classical results on differentiating of asymptotic formulae (see, e.g., [20], Chapter 7, Section 3) we get the assertion of Theorem 1 when \( x = y = 0 \) and \( d \geq 3 \).

Let us consider the case \( x \neq 0 \) and \( y = 0 \). Integration by parts permits to rewrite the family of equations (2) as follows

\[
m(t; x, 0) = \frac{a}{1-\alpha} p(t; x, 0) + \left(1 - \frac{a}{1-\alpha}\right) \int_0^t m(t-u; 0, 0) p'(u; x, 0) \, du + \frac{a \beta}{1-\alpha} \int_0^t p(t-u; x, 0) m(u; 0, 0) \, du,
\]

(6)

\[
\frac{a}{1-\alpha} m(t; 0, 0) = \frac{a}{1-\alpha} p(t; 0, 0) + \left(1 - \frac{a}{1-\alpha}\right) \int_0^t m(t-u; 0, 0) p'(u; 0, 0) \, du + \frac{a \beta}{1-\alpha} \int_0^t p(t-u; 0, 0) m(u; 0, 0) \, du.
\]

(7)

Subtracting equation (6) from (7) we come to

\[
\frac{a}{1-\alpha} m(t; 0, 0) - m(t; x, 0) = \frac{a}{1-\alpha} \left( p(t; 0, 0) - p(t; x, 0) \right) + \left(1 - \frac{a}{1-\alpha}\right) \int_0^t m(t-u; 0, 0) \left( p'(u; 0, 0) - p'(u; x, 0) \right) \, du + \frac{a \beta}{1-\alpha} \int_0^t m(t-u; 0, 0) \left( p(u; 0, 0) - p(u; x, 0) \right) \, du.
\]

(8)

Employing the results on differentiating of asymptotic formulae once again (see, e.g., [20], Chapter 7, Section 3), as well as relation (1) and inequality \( p''(t; 0, 0) \geq p''(t; x, 0), \quad t \geq 0 \), implied by (2.1.15) in [13], we find that

\[
p'(t; 0, 0) - p'(t; x, 0) \sim -\frac{(d+2) \gamma_d(x)}{2 t^{d/2+2}}, \quad t \to \infty.
\]

(9)

Therefore, on account of Lemma 5.1.2 (“lemma on convolutions”) in [13] along with formula (1) and the proved part of Theorem 1 we deduce from (8) that

\[
m(t; x, 0) \sim m(t; 0, 0) \left( 1 - \frac{a \beta}{1-\alpha} \int_0^\infty \left( p(u; 0, 0) - p(u; x, 0) \right) \, du \right) - \frac{a \gamma_d(x)}{(1-\alpha) t^{d/2+1}} \left( 1 + \beta \int_0^\infty m(u; 0, 0) \, du \right), \quad t \to \infty.
\]

(10)
However by virtue of already established part of Theorem 1 we see that \( \int_0^\infty m(u; 0, 0) \, du < \infty \) for all \( d \in \mathbb{N} \) and, moreover, according to 11 for \( \lambda = 0 \) one gets
\[
\int_0^\infty m(u; 0, 0) \, du = -\beta^{-1} \quad \text{if} \quad d = 1 \quad \text{or} \quad d = 2.
\]
Hence we conclude that only the first summand in the right-hand side of (10) makes contribution to the asymptotic behavior of \( m(t; x, 0) \). So, the assertion of Theorem 1 for \( x \neq 0 \) and \( y = 0 \) is entailed by relation (10) and the proved part of this theorem when \( x = y = 0 \).

Let now \( x \in \mathbb{Z}^d \) and \( y \neq 0 \). In view of (8) one has
\[
m(t; x, 0) - m(t; x, y) = p(t; x, 0) - p(t; x, y) + \left( \frac{1 - \alpha}{a} - 1 \right) \int_0^t m(t-u; x, 0) (p'(u; 0, 0) - p'(u; 0, y)) \, du \]
\[
+ \beta \int_0^t m(t-u; x, 0) (p(u; 0, 0) - p(u; 0, y)) \, du.
\]
Then taking into account formulae (11) and (9) along with Lemma 5.1.2 in [13] and the proved part of Theorem 1 we find that
\[
m(t; x, y) \sim m(t; x, 0) \left( \frac{1 - \alpha}{a} - \beta \int_0^\infty (p(u; 0, 0) - p(u; 0, y)) \, du \right) + \frac{\tilde{\gamma}_d(x) - \tilde{\gamma}_d(y-x)}{t^{d/2+1}} - \frac{\beta \tilde{\gamma}_d(y)}{t^{d/2+1}} \int_0^\infty m(u; x, 0) \, du, \quad t \to \infty.
\]
By the established part of Theorem 1 applied one again we come to inequality \( \int_0^\infty m(u; x, 0) \, du < \infty \). Moreover, similarly to the verification of equality (11) we check that \( \int_0^\infty m(u; x, 0) \, du = -\beta^{-1} \) for \( d = 1 \) or \( d = 2 \). Thus, the statement of Theorem 1 for \( x \in \mathbb{Z}^d \) and \( y \neq 0 \) is implied by relation (12) and the proved part of Theorem 1 for \( x \in \mathbb{Z}^d \) and \( y = 0 \).

To complete the proof of Theorem 1 one has to make sure only that functions \( C_d(\cdot, \cdot), d \in \mathbb{N} \), are strictly positive. It is easy except for the case \( d = 1 \) when \( x \neq 0 \) and \( y \neq 0 \). Let us show that in this case \( C_1(x, y) > 0 \) as well. For this purpose we turn to function \( H_{x,0}(t) \), \( t \geq 0 \), which is a cumulative distribution function of time from leaving point \( x \) till the first hitting point \( 0 \) in the framework of the random walk generated by matrix \( A \). Clearly,
\[
p(t; x, y) - \int_0^t p(t-u; 0, y) \, dH_{x,0}(u) \geq 0, \quad t \geq 0.
\]
Then by virtue of the evident identity \( p(t; x, 0) = \int_0^t p(t-u; 0, 0) \, dH_{x,0}(u) \) one has
\[
p(t; x, y) - \int_0^t p(t-u; 0, y) \, dH_{x,0}(u) = p(t; x, y) - p(t; x, 0) + \int_0^t (p(t-u; 0, 0) - p(t-u; 0, y)) \, dH_{x,0}(u).
\]
Since with the help of relation (11), Lemma 5.1.2 in [13] and Lemma 3 in [11] one can find the asymptotic behavior of the right-hand side of the latter equality, we establish that, as \( t \to \infty \),
\[
p(t; x, y) - \int_0^t p(t-u; 0, y) \, dH_{x,0}(u) \sim \frac{\tilde{\gamma}_1(x) + \tilde{\gamma}_1(y) - \tilde{\gamma}_1(y-x)}{t^{3/2}} + \frac{(1 - \alpha - a \rho_1(x))(1 - \alpha - a \rho_1(y))}{2 a^2 \gamma_1 \pi \beta^2 t^{3/2}}.
\]
Hence, it follows from (13) that

$$\frac{\rho_1(x)\rho_1(y)}{2\gamma_1 \pi \beta^2} + \tilde{\gamma}_1(x) + \tilde{\gamma}_1(y) - \tilde{\gamma}_1(y-x) \geq \frac{1 - \alpha}{2 \alpha \gamma_1 \pi \beta^2} \left( \rho_1(x) + \rho_1(y) - \frac{1 - \alpha}{a} \right).$$

However the left-hand side of this inequality appears to be $C_1(x, y)$, whereas the right-hand side is strictly positive since $\rho_1(z) > (1 - \alpha)a^{-1}$ for $z \neq 0$. In its turn, the last inequality is satisfied due to definition of function $\rho_1(\cdot)$ and negativity of $\beta$ in subcritical regime for $d = 1$.

Therefore, Theorem [1] is proved completely. □

3 Proofs of Theorems 2 and 3

It is not difficult to verify (following the scheme in [17]) that in subcritical CBRW on $\mathbb{Z}^d$, as well as in critical CBRW, for all $x, y \in \mathbb{Z}^d$, $s \in [0, 1]$ and $t \geq 0$, the non-linear integral equations hold true

$$q(s, t; x, y) = (1 - s)m(t; x, y) - \int_0^t m(t-u; x, 0)\Phi(q(s, u; 0, y)) \, du. \quad (14)$$

So, the following upper estimate for $q(s, t; x, y)$ ensues in view of non-negativeness of functions $m(\cdot; x, 0)$ and $\Phi(\cdot)$

$$q(s, t; x, y) \leq (1 - s)m(t; x, y). \quad (15)$$

**Lemma 2** If $\mathbb{E} \xi < 1 + h \alpha^{-1}(1 - \alpha)$ and $\mathbb{E}^{1+\delta} < \infty$ for $\delta \in (0, 1]$ then for some positive constants $K_1$ and $K_2$ the inequalities are valid

$$\Phi(s) \leq K_1 s^{1+\delta}, \quad s \in [0, 1], \quad (16)$$

$$\mathbb{E}_x \mu(t; y)^{1+\delta} \leq K_2 m(t; x, y), \quad t \geq t_0(x, y), \quad x, y \in \mathbb{Z}^d \quad (17)$$

with a certain non-negative function $t_0(\cdot; \cdot)$.

**Proof.** At first we consider the case $0 < \delta < 1$. Let us take advantage of the connection between fractional moments of random variables and fractional derivatives of their Laplace transforms. For the first time such results were obtained in [15] where the traditional notion of Riemann-Liouville fractional derivative was used. However it is more convenient to involve the up-to-date counterpart of these results, namely, Lemma 2.1 in [21], which gives

$$\mathbb{E} \xi^{1+\delta} = \frac{\delta(1+\delta)}{\Gamma(1-\delta)} \int_0^\infty \frac{\Phi(1-e^{-v}) + \alpha f'(1)(e^{-v}-1+v)}{\alpha v^{2+\delta}} \, dv. \quad (18)$$

In view of finiteness of $\mathbb{E} \xi^{1+\delta}$ the latter equality entails $\int_0^1 v^{-2-\delta}\Phi(v) \, dv < \infty$. Taking into account that $\Phi'(s) \geq 0, s \in [0, 1]$, integration by parts implies relation (16) for $\delta \in (0, 1)$.

Turn to verification of (17) when $0 < \delta < 1$. Applying Lemma 2.1 in [21] once again we come to the following equality

$$\mathbb{E}_x \mu(t; y)^{1+\delta} = \frac{\delta(1+\delta)}{\Gamma(1-\delta)} \int_0^\infty \frac{vm(t; x, y) - q(e^{-v}, t; x, y)}{v^{2+\delta}} \, dv.$$
Substituting formula (14) into the latter relation we get

\[
E_x\mu(t; y)^{1+\delta} = m(t; x, y)\frac{\delta(1+\delta)}{\Gamma(1-\delta)} \int_0^\infty \frac{e^{-v} - 1 + v}{v^{2+\delta}} \, dv \\
+ \frac{\delta(1+\delta)}{\Gamma(1-\delta)} \int_0^\infty \frac{1}{v^{2+\delta}} \int_0^t m(t-u; x, 0)\Phi(q(e^{-v}, u; 0, y)) \, du \, dv.
\]

(19)

Obviously, in equality (19) the first integral converges. Let us estimate the double integral. According to (15) and inequality \(\Phi(\kappa s) \leq \kappa \Phi(s), \kappa, s \in [0, 1]\), guaranteed by convexity property of function \(\Phi(\cdot)\), one has

\[
\int_0^\infty \frac{1}{v^{2+\delta}} \int_0^t m(t-u; x, 0)\Phi(q(e^{-v}, u; 0, y)) \, du \, dv \leq \int_0^t m(t-u; x, 0)m(u; 0, y) du \int_0^\infty \frac{\Phi(1-e^{-v})}{v^{2+\delta}} \, dv.
\]

At the right-hand side of the above inequality the integral in variable \(u\) is equivalent (up to a constant factor) to the function \(m(t; x, y)\), as \(t \to \infty\), on account of Theorem 1 and Lemma 5.1.2 in [13]. Furthermore, in the same inequality the integral in variable \(v\) converges by virtue of formula (18) combined with finiteness of \(E\xi^{1+\delta}\). Therefore, this arguing together with relation (19) lead to equality (17) with \(0 < \delta < 1\).

Now we consider the case \(\delta = 1\). The identity \(f''(1) = E\xi(\xi - 1)\) and the Lemma conditions imply the existence of \(f''(1)\) and, consequently, the existence of \(\Phi''(0)\). Moreover, since \(\Phi(0) = 0\) and \(\Phi'(0) = 0\), inequality (16) for \(\delta = 1\) is proved. Let us take the second left derivatives at \(s = 1\) for each side of equality (14). Using relations \(m(t; x, y) = -\partial_s q(s, t; x, y)|_{s=1}\) and \(E_x\mu(t; y)(\mu(t; y) - 1) = -\partial_s^2 q(s, t; x, y)|_{s=1}\), we obtain

\[
E_x\mu(t; y)(\mu(t; y) - 1) = \alpha f''(1) \int_0^t m(t-u; x, 0) (m(u; 0, y))^2 \, du.
\]

In accordance with Theorem 1 and Lemma 5.1.2 in [13], in the latter equality the integral behaves as \(m(t; x, 0)\) up to a constant factor, as \(t \to \infty\). Hence, on account of Theorem 1 this entails the desired inequality (17) when \(\delta = 1\). Lemma 2 is proved completely. "}

Let us turn to proving Theorem 2. It is easily verified that due to formulae (15) and (16) along with Theorem 1 and the proof scheme of Lemma 4 in [16], one gets

\[
\int_0^t m(t-u; x, 0)\Phi(q(s, u; 0, y)) \, du \sim m(t; x, 0)J(s; y), \quad t \to \infty.
\]

(20)

Observe that the function \(J(\cdot; \cdot)\), appearing in formulations of Theorems 2 and 3 is defined correctly in view of upper estimate (15) for \(q(s, t; x, y)\). Let us find the lower estimate for \(q(t; x, y)\). By the Hölder inequality one has

\[
E_x\mu(t; y) = E_x\mu(t; y)\mathbb{I}(\mu(t; y) > 0) \leq (E_x\mu(t; y)^{1+\delta})^{1/(1+\delta)} (E_x\mathbb{I}(\mu(t; y) > 0)^{1+\delta)/\delta)}^{\delta/(1+\delta)}
\]

where \(\mathbb{I}(B)\) denotes the indicator of set \(B\). Rewrite the last inequality as follows

\[
q(t; x, y) \geq \left(\frac{m(t; x, y)^{1+\delta}}{E_x\mu(t; y)^{1+\delta})^{1/\delta}}, \quad t > 0.
\]
Now employing (17) we come to the desired lower estimate for $q(t; x, y)$

$$q(t; x, y) \geq K_2^{-1/\beta} m(t; x, y), \quad t \geq t_0(x, y), \; x, y \in \mathbb{Z}^d.$$  \hspace{1cm} (21)

Combining formulae (14) and (20), when $s = 0$, with estimate (21) we conclude that, as $t \to \infty$,

$$q(t; x, y) \sim m(t; x, y) - m(t; x, 0)J(0; y) \quad \text{and} \quad J(0; y) < \lim_{t \to \infty} \frac{m(t; x, y)}{m(t; x, 0)}$$

for fixed $x, y \in \mathbb{Z}^d$. Just these relations amount to validity of Theorem 2. \hfill □

Let us prove Theorem 3. Applying formulae (14) and (20) once again we have

$$q(s, t; x, y) \sim (1 - s)m(t; x, y) - m(t; x, 0)J(s; y), \quad t \to \infty, \; x, y \in \mathbb{Z}^d.$$  

Then with the help of the identity

$$\lim_{t \to \infty} \mathbb{E}_x \left( s^{\mu(t; y)} \mid \mu(t; y) > 0 \right) = 1 - \lim_{t \to \infty} \frac{q(s, t; x, y)}{q(t; x, y)}$$

and Theorem 2 we obtain the assertion of Theorem 3. \hfill □

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