On the images of the Galois representations attached to genus 2 Siegel modular forms

Luis V. Dieulefait *

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Abstract

We address the problem of the determination of the images of the Galois representations attached to genus 2 Siegel cusp forms of level 1 having multiplicity one. These representations are symplectic. We prove that the images are as large as possible for almost every prime, if the Siegel cusp form is not a Maass spezialform and verifies two easy to check conditions.

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1 Introduction

In this article we apply the same techniques as in [5] to study the images of the Galois representations attached to genus 2 cuspidal Siegel modular forms on $\text{Sp}(4,\mathbb{Z})$ of even weight $k$ that are Hecke eigenforms. These four dimensional Galois representations have been constructed by Taylor and Weissauer (see [18], [22]).

We will restrict ourselves to the case where the automorphic representation corresponding to the Siegel modular form has multiplicity one. In this case the Galois representations are symplectic (cf. [22], [3]).

We will consider explicit examples of such Siegel modular forms whose first Fourier coefficients have been computed by Skoruppa (see [15], [16]) and we will show that the images are “as large as possible” (a certain symplectic group) for almost every prime. The examples of Skoruppa do not correspond to Maass spezialformen (Siegel modular forms in the image of

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the Saito-Kurokawa lift from $S_{2k-2}(\text{SL}_2(\mathbb{Z}))$, and in addition Eisenstein-Klingen series are excluded. For such cases it is known that the image is not maximal for any prime.

Our main tools will be the classification of maximal subgroups of $\text{PSp}(4, F_q)$ (see [10]); the description of the image of the inertia subgroup $I_\ell$ due to the work of Faltings (see [6], [2] and [20]) on Hodge-Tate decompositions via an application of Fontaine-Laffaille theory; and Serre’s conjecture (3.2.4?) (cf. [14]) for odd irreducible two-dimensional Galois representations.

Our main result shows, under the assumption of Serre’s conjecture (3.2.4?), that given a genus 2 Siegel cusp form for the full Siegel modular group of even weight, if it has multiplicity one, is not a Maass spezialform and verifies an additional condition (similar to the condition “without inner twists” imposed in [4] to classical modular forms), then the images of the corresponding Galois representations are “as large as possible” for almost every prime. In fact, a stronger statement is proved in the last section using the theory of pseudo-representations: the same result holds independently of Serre’s conjecture if we further impose the condition that for some prime $p$: $Q(d_p) = E$ and $\sqrt{d_p} \notin E$ where $E$ is the number field generated by the eigenvalues of the Siegel cusp form and $d_p$ is defined in section 4.8.

This condition is verified in the examples. Moreover, one can easily prove assuming Serre’s conjecture that this condition will always be satisfied.

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2 The setup

Let $f$ be a cuspidal weight $k$ Siegel modular form for $\text{Sp}(4, \mathbb{Z})$ which is a Hecke eigenform and denote $a_n$ its eigenvalues. Let

$$Z_f(s) := \zeta(2s - 2k + 4) \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

be the spinor zeta function. Then $Z_f$ has an Euler product of the form

$$Z_f(s) = \prod Q_p(p^{-s})^{-1},$$

where $Q_p$ is the polynomial:

$$Q_p(x) = 1 - a_p x + (a_p^2 - a_{p^2} - p^{2k-4})x^2 - a_p p^{2k-3} x^3 + p^{4k-6} x^4.$$ 

Then we have the following result of Taylor [18], completed by Weissauer [22] (see also [3]):
Theorem 2.1 Let \( f \) be a Siegel modular form of even weight \( k \) on the full Siegel modular group \( \text{Sp}(4, \mathbb{Z}) \) which is a cusp form and a simultaneous eigenform for all Hecke operators \( T(n) \). Furthermore, assume that the automorphic representation corresponding to \( f \) has multiplicity one. Let \( E \) be the number field generated by the eigenvalues \( a_n \). For any prime number \( \ell \) and any extension \( \lambda \) of \( \ell \) to \( E \) there exists a continuous Galois representation \( \rho_{f,\lambda} : G_{\mathbb{Q}} \to G\text{Sp}(4, E_{\lambda}) \) such that the following holds: the representation \( \rho_{f,\lambda} \) is unramified outside \( \ell \) and

\[
\det(Id - x\rho_{f,\lambda}(\text{Frob } p)) = Q_p(x)
\]

for every \( p \neq \ell \). If \( \rho_{f,\lambda} \) is absolutely irreducible, then it is defined over \( E_{\lambda} \).

For such an \( f \) and under the hypothesis of absolute irreducibility, the maximal possible image for \( \rho_{f,\lambda} \) is the group

\[
A_{\lambda}^k = \{ g \in G\text{Sp}(4, \mathcal{O}_{E_{\lambda}}) : \det(g) \in (\mathbb{Q}_\ell^*)^{4k-6} \},
\]

where \( \mathcal{O}_{E_{\lambda}} \) is the ring of integers of \( E_{\lambda} \).

If we consider the residual mod \( \lambda \) Galois representation \( \overline{\rho}_{f,\lambda} \) (see the remark below), then its image is contained in

\[
\{ g \in G\text{Sp}(4, \mathbb{F}_{\lambda}) : \det(g) \in (\mathbb{F}_\ell^*)^{4k-6} \}.
\]

The image of its projective image \( \mathbb{P}(\overline{\rho}_{f,\lambda}) \) is contained in \( \text{PSp}(4, \mathbb{F}_{\lambda}) \), if the residual degree of \( \lambda \) is even, and in \( \text{PGSp}(4, \mathbb{F}_{\lambda}) \) if it is odd (see [3]).

Remark: In these assertions the assumption of absolute irreducibility is not needed. This follows from the fact that \( E \) is the field generated by the coefficients of the characteristic polynomials of the images of the Frobenius elements and from a lemma on the reduction of \( \ell \)-adic representations proved in [3]. This lemma tells us that the mod \( \lambda \) residual representation obtained by reducing the coefficients of all the characteristic polynomials of the images of Frobenius elements “agrees” with the reduction of our \( \lambda \)-adic Galois representation that may not be defined over \( E_{\lambda} \); actually it is only thanks to this lemma that we can define the residual mod \( \lambda \) representation in general.

Once we have determined that for a prime \( \ell \) the image of \( \overline{\rho}_{f,\lambda} \) is as large as possible, an application of a lemma of Serre (cf. [13] and [3]) implies that the image of \( \rho_{f,\lambda} \) is \( A_{\lambda}^k \).
3 The tools

3.1 Maximal subgroups of $\text{PGSp}(4, \mathbb{F}_q)$

In [10], Mitchell gives the following classification of maximal proper subgroups $G$ of $\text{PSp}(4, \mathbb{F}_q)$ where $q = p^r$, $p$ odd prime, as groups of transformations of the projective space having an invariant linear complex:

1) a group having an invariant point and plane
2) a group having an invariant parabolic congruence
3) a group having an invariant hyperbolic congruence
4) a group having an invariant elliptic congruence
5) a group having an invariant quadric
6) a group having an invariant twisted cubic
7) a group $G$ containing a normal elementary abelian subgroup $E$ of order 16, with: $G/E \cong A_5$ or $S_5$
8) a group $G$ isomorphic to $A_6$, $S_6$ or $A_7$
9) a group conjugated under $\text{PSp}(4, \mathbb{F}_{p^r})$ with $\text{PSp}(4, \mathbb{F}_{p^k})$, where $r/k$ is an odd prime
10) a group conjugated under $\text{PGSp}(4, \mathbb{F}_{p^r})$ with $\text{PGSp}(4, \mathbb{F}_{p^k})$, where $r$ is even and $r/k = 2$

Cases 7) and 8) only occur if $r = 1$.

(For the relevant definitions see [8], see also [1] and [11] for cases 7 and 8). From this we obtain a classification of maximal proper subgroups $H$ of $\text{PGSp}(4, \mathbb{F}_q)$. It is similar to the one above, except that cases 7) and 8) change according to the relation between $H$ and $G$, given by the exact sequence:

$$1 \rightarrow G \rightarrow H \rightarrow \{\pm 1\} \rightarrow 1.$$ 

3.2 The image of inertia at $\ell$

In Weissauer’s construction, the representations $\rho_{f,\lambda}$ are realized in the tale cohomology of the Siegel variety $X_1$ of level 1. Applying results of Faltings and Chai-Faltings (see [6], [2]), Urban proves (see [20]) a result that combined with Fontaine-Laffaille theory (see [7]) gives:

**Proposition 3.1** Let $f$ be a Siegel cusp form as in theorem 2.1. Then $\rho_{f,\lambda}$ is crystalline and has Hodge-Tate weights $\{2k - 3, k - 1, k - 2, 0\}$. Moreover if $\ell - 1 > 2k - 3$, then we have the following possibilities for the action of
the inertia group at \( \ell \): 

\[
\tilde{\rho}_{f,\lambda}|_{I_\ell} \simeq \begin{pmatrix}
1 & * & * & * \\
0 & \chi^{k-2} & * & * \\
0 & 0 & \chi^{k-1} & * \\
0 & 0 & 0 & \chi^{2k-3}
\end{pmatrix},
\begin{pmatrix}
\psi^{2k-3} & 0 & * & * \\
0 & \psi^{(2k-3)\ell} & * & * \\
0 & 0 & \psi^{(k-2)+(k-1)\ell} & 0 \\
0 & 0 & 0 & \psi^{(k-1)+(k-2)\ell}
\end{pmatrix},
\begin{pmatrix}
1 & * & * & * \\
0 & \psi^{(2k-3)\ell} & * & * \\
0 & 0 & \chi^{2k-3} & * \\
0 & 0 & 0 & \psi^{(k-1)+(k-2)\ell}
\end{pmatrix},
\begin{pmatrix}
\psi^{2k-3} & 0 & * & * \\
0 & \psi^{(2k-3)\ell} & * & * \\
0 & 0 & \chi^{k-2} & * \\
0 & 0 & 0 & \chi^{k-1}
\end{pmatrix},
\begin{pmatrix}
1 & * & * & * \\
0 & \psi^{(k-2)+(k-1)\ell} & * & * \\
0 & 0 & \psi^{(k-1)+(k-2)\ell} & 0 \\
0 & 0 & 0 & \psi^{(k-1)+(k-2)\ell}
\end{pmatrix},
\]

where \( \psi \) denotes a fundamental character of level 2.

4 Study of the images

4.1 Reducible case: 1-dimensional constituent

Let \( f \) be a Siegel cusp form verifying all the conditions of theorem 2.1. Suppose that for a prime \( \lambda \) in \( E \) the representation \( \tilde{\rho}_{f,\lambda} \) is reducible with a 1-dimensional sub(or quotient) representation. The representation being unramified outside \( \ell \) we conclude from proposition 3.1 (assume \( \ell > 2k-2 \)) that this 1-dimensional constituent is: \( \chi^i \), \( i = 0, k-2, k-1 \) or \( 2k-3 \).

As in [5], we will use the fact that symplectic representations have "reciprocal" roots, i.e., that the roots of the characteristic polynomial of \( \rho_{f,\lambda}(\text{Frob} \, p) \) come in pairs \( \alpha, \beta = p^{2k-3}/\alpha \).

Remark: We have also used this fact in proposition 3.1 to discard many cases, in particular the cases involving fundamental characters of level 3 or 4.

Therefore we can assume that \( \chi^i \) is a root of the characteristic polynomial \( \text{Pol}_p(x) \) of \( \tilde{\rho}_{f,\lambda}(\text{Frob} \, p) \)

\[
\text{Pol}_p(x) = x^4 - a_p x^3 + (a_p^2 - a_p p^{2k-4}) x^2 - a_p p^{2k-3} x + p^{4k-6},
\]

for every \( p \neq \ell \), with \( i = 0 \) or \( k-1 \).

For \( i = 0 \) we obtain the following congruence:

\[
b_p - a_p (p^{2k-3} + 1) + p^{4k-6} + 1 \equiv 0 \pmod{\lambda} \quad (4.1)
\]

for every \( p \neq \ell \), where \( b_p = a_p^2 - a_p p^{2k-4} \) is the quadratic coefficient of \( \text{Pol}_p(x) \).

Weissauer proved (see [21]) that any genus 2 Siegel cusp form that is not
in the image of the Saito-Kurokawa lift (and more generally any irreducible cuspidal automorphic representations $\pi$ of $\text{GSp}(4, A)$ such that $\pi$ is not Cuspidal Associated to Parabolic and $\pi_\infty$ is a discrete series representation), i.e., not a Maass spezialform, verifies the generalized Ramanujan conjecture, so the roots of $Pol_p(x)$ have all the same absolute value $\sqrt{p^{2k-3}}$.

From now on we will only work with Siegel cusp forms that are not Maass spezialformen, with the twofold intention of obtaining large images and using Ramanujan’s conjecture. This is the case for the examples computed by Skoruppa (cf. [15], [16]), where the Siegel cusp forms that are not Maass spezialformen are called “interesting” (as we will call them).

Therefore, if $f$ is an interesting Siegel cusp form, using the bounds for the absolute values of $a_p$ and $b_p$ that follow from Ramanujan’s conjecture we conclude that congruence (4.1) is not an equality (for large enough $p$). Therefore, it can only hold for finitely many primes $\ell$.

Similarly, for $i = k - 1$ we obtain the congruence:

$$p^{2k-4}(1 + p^2) - a_pp^{k-2}(1 + p) + b_p \equiv 0 \pmod{\lambda} \quad (4.2)$$

for every $p \neq \ell$.

Again, Ramanujan’s conjecture implies that this is not an equality for large enough $p$. Thus, the reducible case with 1-dimensional constituent can only hold for finitely many primes.

### 4.2 Reducible case: related 2-dimensional constituents

Suppose that, after semi-simplification, $\bar{\rho}_{f,\lambda}$ decomposes as the sum of two 2-dimensional irreducible Galois representations: $\bar{\rho}_{f,\lambda} \cong \pi_1 \oplus \pi_2$. Assume also that these two constituents are related, i.e., if $\alpha, \beta$ are the roots of the characteristic polynomial of $\pi_1(\text{Frob } p)$, then $p^{2k-3}/\alpha, p^{2k-3}/\beta$ are the roots of that of $\pi_2(\text{Frob } p)$. If not, then $\alpha = p^{2k-3}/\beta$, so det($\pi_1$) = det($\pi_2$) = $\chi^{2k-3}$; this case will be studied in the next subsection.

We apply proposition 3.1 (assume $\ell > 2k - 2$). There are two cases to consider:

- **Case 1**: $\det(\pi_1) = \chi^{k-1}$, $\det(\pi_2) = \chi^{3k-5}$.
- **Case 2**: $\det(\pi_1) = \chi^{k-2}$, $\det(\pi_2) = \chi^{3k-4}$.

- **Case 1**: In this case we have the factorization:

$$Pol_p(x) \equiv (x^2 - Ax + p^{3k-5}) \left( x^2 - \frac{Ax}{p^{k-2}} + p^{k-1} \right) \pmod{\lambda}.$$
Eliminating $A$ from the equation, we obtain:

$$(b_p - p^{k-1} - p^{3k-5}) (p^{k-2} + 1)^2 - a_p^2 p^{k-2} \equiv 0 \pmod{\lambda} \quad (4.3)$$

for every $p \neq \ell$.

From the bounds on the coefficients that follow from Ramanujan’s conjecture, we see that for large enough $p$ this is not an equality. Thus, only finitely many $\ell$ can satisfy (4.3).

• Case 2: This case is quite similar to the previous one. We start with:

$$Pol_p(x) \equiv (x^2 - Ax + p^{3k-4}) (x^2 - \frac{Ax}{p^{k-1}} + p^{k-2}) \pmod{\lambda}.$$  

From this:

$$(b_p - p^{k-2} - p^{3k-4}) (p^{k-1} + 1)^2 - a_p^2 p^{k-1} \equiv 0 \pmod{\lambda} \quad (4.4)$$

for every $p \neq \ell$.

From the bounds on the coefficients we conclude that for an interesting Siegel cusp form the reducible case with related two-dimensional constituents can only hold for finitely many primes.

4.3 The remaining reducible case with Serre’s conjecture

As explained above, in the remaining reducible case we have: $\bar{\rho}^\text{ss}_{f,\lambda} \cong \pi_1 \oplus \pi_2$ with $\det(\pi_1) = \det(\pi_2) = \chi^{2k-3}$. Assume $\ell > 2k - 2$. In proposition [X] we have given a description of $\bar{\rho}_{f,\lambda}|_{I_\ell}$. This gives for $\pi_1|_{I_\ell}$, $\pi_2|_{I_\ell}$:

$$\begin{pmatrix} 1 & \ast \\ 0 & \chi^{2k-3} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \psi^{2k-3} & 0 \\ 0 & \psi(2k-3)\ell \end{pmatrix}$$

and

$$\begin{pmatrix} \chi^{-k-2} & \ast \\ 0 & \chi^{-k-1} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \psi^{(k-2)+(k-1)\ell} & 0 \\ 0 & \psi^{(k-1)+(k-2)\ell} \end{pmatrix},$$

respectively. Besides, both two-dimensional representations are unramified outside $\ell$. At this point we invoke Serre’s conjecture (3.2.47) (see [14]) that gives us a control on $\pi_1$ and $\pi_2^{\prime} := \chi^{-k+2} \otimes \pi_2$. Both representations should be modular of weights $2k-2$ and 2, respectively, and level 1; i.e., there exist two cusp forms $f_1, f_2$ with:

$$\bar{\rho}_{f_1,\lambda} \cong \pi_1, \; \bar{\rho}_{f_2,\lambda} \cong \pi_2^{\prime}, \quad f_1 \in S_{2k-2}(1), \; f_2 \in S_2(1)$$

(we are assuming that $\pi_1$ and $\pi_2$ are irreducible; otherwise the results of section 4.1 apply). Both cusp forms have trivial nebentypus.
But $S_2(1) = 0$ and we obtain a contradiction.

We conclude, assuming Serre’s conjecture, that the reducible case with unrelated two dimensional constituents cannot happen if $\ell > 2k - 2$ for genus two Siegel cusp forms for the full Siegel modular group $\text{Sp}(4, \mathbb{Z})$ verifying the conditions of theorem [2.1].

**Remark:** In all reducible cases (sections 4.1, 4.2 and 4.3) we have considered reducibility over $\overline{F}_\lambda$.

### 4.4 Groups with a reducible index 2 subgroup

If $G_\lambda$, the image of $\bar{\rho}_{f,\lambda}$, corresponds to an irreducible subgroup inside (its projective image) some of the maximal subgroups in cases 3), 4) and 5) of Mitchell’s classification, there is a normal subgroup of index 2 of $G_\lambda$:

$$1 \to M_\lambda \to G_\lambda \to \{\pm 1\} \to 1,$$

where the subgroup $M_\lambda$ is reducible (not necessarily over $\overline{F}_\lambda$).

For $\ell > 2k - 2$ we apply the description of $\bar{\rho}_{f,\lambda}|_{I_\ell}$ given in proposition [3.1].

Observe that $\chi_{k-2}$ has order larger than 2 because $\ell - 1 > 2k - 3$, and the order of $\psi^{(k-2)+(k-1)\ell}$ is clearly a multiple of $\ell + 1$, therefore larger than 2.

Then we conclude that the image of $I_\ell$ is contained in $M_\lambda$.

Therefore, if we take the quotient $G_\lambda/M_\lambda$ we obtain a representation

$$G_\mathbb{Q} \to C_2$$

whose kernel is a quadratic field unramified everywhere; we thus obtain a contradiction.

We conclude that the image of $\bar{\rho}_{f,\lambda}$ never falls in this case if $\ell > 2k - 2$.

### 4.5 The stabilizer of a twisted cubic

In this case all upper-triangular matrices are of the form:

$$
\begin{pmatrix}
a^3 & * & * & * \\
0 & a^2d & * & * \\
0 & 0 & ad^2 & * \\
0 & 0 & 0 & d^3
\end{pmatrix}.
$$

Let us compare this with the four possibilities for the image of the inertia subgroup at $\ell$ for $\ell > 2k - 2$ given in proposition [3.1].

In the first case we see that this inertia subgroup has the required form only if: $\chi^{2k-3} \equiv \chi^{3k-6}$, $\chi^{2k-3} \equiv \chi^{3k-3}$, $\chi^{3k-4} \equiv 1$ or $\chi^{3k-4} \equiv \chi^{6k-9}$. But this
implies $\ell - 1 \mid k - 3$, $\ell - 1 \mid k$, $\ell - 1 \mid 3k - 5$ or $\ell - 1 \mid 3k - 4$, respectively. With the restriction $\ell > 2k - 2$, we see that this is impossible.

In the second case, we see that the inertia subgroup has the required form only if:

$$\psi^{(3k-4)+(3k-5)\ell} \equiv \psi^{6k-9}$$ or $$\psi^{(6k-9)\ell},$$
or

$$\psi^{(2k-3)+(4k-6)\ell} \equiv \psi^{(3k-3)+(3k-6)\ell}$$ or $$\psi^{(3k-6)+(3k-3)\ell}. $$

But this implies $\ell + 1 \mid 3k - 5$, $\ell + 1 \mid 3k - 4$, $\ell + 1 \mid k$ or $\ell + 1 \mid k - 3$, respectively. Again, this is impossible for $\ell > 2k - 2$.

In the third case, we see that the inertia subgroup has the required form only if:

$$\psi^{(2k-3)(1+2\ell)} \equiv \chi^{3k-6}$$ or $$\chi^{3k-3},$$
or

$$\chi^{3k-4} \equiv \psi^{6k-9}$$ or $$\psi^{(6k-9)\ell}.$$ 

The first congruence implies $\ell + 1 \mid 2k - 3$, which is impossible for $\ell > 2k - 2$.

The second congruence implies:

$$\ell^2 - 1 \mid (-3k + 5) + (3k - 4)\ell,$$ \(\dagger\)

which is impossible for every $\ell$ because $\ell - 1 \mid (-3k + 4) + (3k - 4)\ell$.

In the fourth and last case, if the inertia subgroup has the required form then:

$$\psi^{(3k-4)+(3k-5)\ell} \equiv 1$$ or $$\psi^{6k-9},$$
or

$$\chi^{2k-3} \equiv \psi^{(3k-6)+(3k-3)\ell}$$ or $$\psi^{(3k-3)+(3k-6)\ell}. $$

The first congruence again gives ($\dagger$).

The second implies $\ell + 1 \mid (3k - 6) + (3k - 3)\ell$, and from this: $\ell + 1 \mid 3$.

We conclude that the image does not fall inside the stabilizer of a twisted cubic for any $\ell > 2k - 2$.

**Remark:** We are working with even weight. For the case of weight 3, the image of inertia may fall inside the stabilizer of a twisted cubic.

### 4.6 The exceptional groups

We call exceptional groups those appearing in cases 7) and 8) of the classification. In these cases, comparing the exceptional group $H \subseteq \text{PGSp}(4,F_\lambda)$ or $G \subseteq \text{PSp}(4,F_\lambda)$, its order and structure, with the fact that the image of
\( \mathbb{P}(\bar{\rho}_{f,\lambda}) \) contains the image of \( \mathbb{P}(\bar{\rho}_{f,\lambda}|_{l^e}) \), and applying proposition 3.1 (assuming \( \ell > 2k-2 \)) we conclude that these cases can never happen for any \( \ell > 2k-2 \). If \( k < 6 \) we have to demand also that \( \ell > 7 \), but there is no interesting Siegel cusp forms for the full Siegel modular group \( \text{Sp}(4, \mathbb{Z}) \) of weight smaller than 20.

4.7 Smaller symplectic groups

To obtain the maximal possible image we need to impose the following condition:

**Definition 4.1** We say that a genus two Siegel cusp form \( f \) is untwisted if there exists a prime \( p \) such that \( a_p \neq 0 \) and \( \mathbb{Q}(b_p) = E \), where \( b_p = a_p^2 - a_{p^2} - p^{2k-4} \).

Alternatively, we can ask the number field \( E \) to be generated by \( a_p^2 \) for some prime \( p \) (*), both conditions are enough to exclude for almost every prime the case where the image is in a smaller symplectic group. This second condition is essentially the same as that required in the case of classical modular forms to exclude the case of inner twists (cf. [12], [4]).

Suppose that the Siegel cusp form \( f \) is untwisted and let \( p \) be a prime as in definition 4.1. Let \( \ell \) be a prime different from \( p \) such that \( \ell \nmid \text{Norm}(a_p) \) and \( \ell \nmid \text{disc}(b_p) \). (4.5)

Let \( \lambda \) be a prime of \( E \) lying above \( \ell \). We will show that for such a \( \lambda \) the projective image of \( \bar{\rho}_{f,\lambda} \) cannot fall in a proper symplectic subgroup of \( \text{PGSp}(4, \mathbb{F}_\lambda) \), i.e., it cannot be a subgroup as in cases 9) and 10) of the classification.

Assume the contrary. Then, there exists \( c \in \mathbb{F}_\lambda^\ast \) such that the polynomial

\[
x^4 - ca_p x^3 + c^2 b_p x^2 - c^3 a_p p^{2k-3} x + c^4 p^{4k-6} \mod \lambda
\]

has its coefficients in \( \mathbb{F}_\lambda \) with \( [\mathbb{F}_\lambda : \mathbb{F}_\lambda^\ast] > 1 \).

This implies that \( c^3 a_p / ca_p = c^2 \in \mathbb{F}_\lambda^\ast \) (recall that \( \ell \nmid \text{Norm}(a_p) \)). Thus, \( c^2 b_p / c^2 = b_p \in \mathbb{F}_\lambda^\ast \), contradicting the assumption that \( \mathbb{Q}(b_p) = E \) and \( \ell \nmid \text{disc}(b_p) \). This contradiction proves that the image does not fall in a proper symplectic group, if \( \ell \) verifies (4.5).

**Remark:** In a similar way, one can prove under condition (*) that the image does not fall in a proper symplectic group, for every \( \ell \) such that \( \ell \nmid \text{Norm}(a_p) \) and \( \ell \nmid \text{disc}(a_p^2) \).
4.8 Unconditional results without Serre’s conjecture

Without using Serre’s conjecture we can show that the images are “as large as possible” for large density sets of primes, applying tricks similar to those used in [5]. For λ a prime in E, let r be its residue class degree: \( \mathbb{F}_\ell^r = \mathbb{F}_\lambda \). Serre’s conjecture was used to eliminate the following two possibilities:

i) The image of \( \overline{\rho}_{f,\lambda} \) is contained in
\[
\{ A \times B \in \text{GL}(2, \mathbb{F}_\lambda) \times \text{GL}(2, \mathbb{F}_\lambda) : \det(A) = \det(B) = \chi^{2k-3} \}. \tag{4.6}
\]
i) The image of \( \overline{\rho}_{f,\lambda} \) is contained in
\[
\{ M \in \text{GL}(2, \mathbb{F}_\ell^r) : \det(M) = \chi^{2k-3} \}. \tag{4.7}
\]
The inclusion of this group in \( \text{GSp}(4, \mathbb{F}_\lambda) \) is given by the map:
\[
M \rightarrow \text{diag}(M, M \text{Frob}), \text{ where Frob is the non-trivial element in Gal(} \mathbb{F}_\ell^r / \mathbb{F}_\ell^r\text{}).
\]

• Case i): We will use the standard factorization (see [15]):
\[
\text{Pol}_p(x) = (x^2 - (a_p/2 + \sqrt{d_p})x + p^{2k-3})(x^2 - (a_p/2 - \sqrt{d_p})x + p^{2k-3}),
\]
where \( d_p = -3/4a_p^2 + a_p^2 + p^{2k-4} + p^{2k-3} \).
We impose the condition:
\[
\mathbb{Q}(d_p) = E \text{ and } \sqrt{d_p} \notin E. \tag{4.8}
\]
Then, if we restrict to the primes \( \lambda \) in E such that
\[
\ell \nmid \text{disc}(d_p) \text{ and } d_p \notin (\mathbb{F}_\lambda)^2 \tag{4.9}
\]
case i) cannot hold, because the matrices \( A \) and \( B \) in (4.6) would have their traces in \( \mathbb{F}_\ell^r \setminus \mathbb{F}_\lambda \).

• Case ii): Assume \( \ell > 2k-2 \) and apply proposition [3.1]. Because \( M = M^{\text{Frob}} \) for any \( M \in \text{GL}(2, \mathbb{F}_\lambda) \), if the matrices in the image of inertia were in case ii) it should hold:
\[
\begin{align*}
\{1, \chi^{2k-3}\} &= \{\chi^{k-2}, \chi^{k-1}\}, \\
\{\psi^{2k-3}, \psi^{(2k-3)\ell}\} &= \{\psi^{(k-2)+(k-1)\ell}, \psi^{(k-1)+(k-2)\ell}\}, \\
\{\psi^{2k-3}, \psi^{(2k-3)\ell}\} &= \{\chi^{k-2}, \chi^{k-1}\} \text{ or} \\
\{1, \chi^{2k-3}\} &= \{\psi^{(k-2)+(k-1)\ell}, \psi^{(k-1)+(k-2)\ell}\}.
\end{align*}
\]
The first of these equalities implies \( \ell - 1 \mid k - 2 \) or \( \ell - 1 \mid k - 1 \). The second implies \( \ell + 1 \mid k - 2 \) or \( \ell + 1 \mid k - 1 \). The third implies \( \ell + 1 \mid 2k - 3 \), and the fourth implies \( \ell + 1 \mid (k - 2) + (k - 1)\ell \), impossible for every \( \ell \). We conclude that no \( \ell > 2k - 2 \) can verify any of these equalities, so case ii) can never happen for such an \( \ell \).
4.9 Conclusion

Having gone through all cases of the classification (see section 3.1), we conclude that under certain conditions on a Siegel cusp form \( f \) the images of the attached Galois representations are “as large as possible” for almost every prime, to be more precise:

**Theorem 4.2** Let \( f \) be a Siegel cusp form (and Hecke eigenform) of even weight \( k \) for the full Siegel modular group \( \text{Sp}(4, \mathbb{Z}) \) verifying the conditions:

- Multiplicity one.
- Not a Maass spezialform.
- Untwisted (see definition 4.1).

Assume Serre’s conjecture \((3.2.4)\). Then the images of the Galois representations \( \rho_{f, \lambda} \) are \( A^k_{\lambda} \), for almost every \( \lambda \).

Without assuming Serre’s conjecture, take a prime \( p \) such that \((4.8)\) holds. Then for all but finitely many primes verifying \((4.9)\) the image of \( \rho_{f, \lambda} \) is \( A^k_{\lambda} \).

**Remark:** We are using the fact (see section 2) that the maximality of the images of \( \rho_{f, \lambda}, \bar{\rho}_{f, \lambda} \) and \( \mathbb{P} (\bar{\rho}_{f, \lambda}) \) are equivalent. In section 6 we will prove, without assuming Serre’s conjecture, that the above result applies to almost every prime.

The result, with or without the assumption of Serre’s conjecture, is effective, and we can compute the exceptional primes given a Siegel cusp form verifying the above conditions.

In fact, assuming Serre’s conjecture, all exceptional primes (more exactly, a finite set containing all exceptional primes) are computed using equations \((4.1), (4.2), (4.3), (4.4)\) and \((4.5)\) (this last condition implies in particular that \( \ell \) is unramified in \( E/\mathbb{Q} \)) applied to more than one \( \text{Pol}_p(x) \) with the restrictions \( \ell > 2k - 2 \).

Without assuming Serre’s conjecture, we also compute several \( d_p \) such that \((4.8)\) holds and find the finitely many exceptional primes among the primes verifying \((4.9)\) for any of these \( d_p \).

5 Examples

We start by applying our method to an example of Skoruppa (see [15], [16]), the example already investigated in [3]. This example consists of a Siegel Hecke eigenform \( \Upsilon \) of weight \( k = 28 \) for the full Siegel modular group \( \text{Sp}(4, \mathbb{Z}) \) such that the space spanned by the Galois conjugates of \( \Upsilon \) is the complement of the space spanned by the Eisenstein-Klingen series and the
Maass spezialformen, so the automorphic representation corresponding to \( \Upsilon \) has multiplicity one.

Using the results of [15] and the tables in [16], we can compute the characteristic polynomials \( \text{Pol}_p(x) \) for \( p = 2, 3 \) and 5 as in [3].

The field \( E \) generated by the eigenvalues of \( \Upsilon \) is the cubic field generated by some root \( \alpha \) of \( x^3 - x^2 - 294086x - 59412960 \).

The required Fourier coefficients of \( \Upsilon \):

\begin{align*}
a(1, 1, 1); a(2, 2, 2); a(3, 3, 3); a(4, 4, 4); a(5, 5, 5); a(1, 1, 7); a(3, 3, 7); a(1, 1, 19)
\end{align*}

are given in [16] in terms of \( \alpha \). From these coefficients, using the formulas in [15], the first eigenvalues of \( \Upsilon \): \( a_2, a_4, a_3, a_9, a_5, a_{25} \) are computed (see [3]).

These values determine all the coefficients of the characteristic polynomials \( \text{Pol}_p(x) \) for \( p = 2, 3 \) and 5. From now on let us impose \( \ell > 2k - 2 = 54 \).

We compute the exceptional primes falling in a reducible case using formulas (4.1) to (4.4) with \( p = 2 \) and 3. To do so just take the norm in these equations and discard the primes \( \ell \) dividing the greatest common divisor of the norms obtained for \( p = 2 \) and 3. In the example we found no exceptional primes \( \ell > 54 \) at this step.

As a matter of fact, in this and other steps, we have to exclude the “primes dividing denominators” from consideration. These are the primes dividing the denominators of the norm of \( a_2, a_3, a_5, a_4, a_9 \) or \( a_{25} \). In the example, they form the set:

\[ \mathbb{D} = \{2, 3, 17, 2063, 8841304187, 1646767084367711\} . \]

We use equation (4.5) for \( p = 2, 3 \) to bound the set of exceptional primes falling in a smaller symplectic group \( (a_2 \text{ and } a_3 \text{ are not zero and } \mathbb{Q}(b_2) = \mathbb{Q}(b_3) = E) \). Excluding the primes in \( \mathbb{D} \), we know that any such exceptional prime \( \ell \) has to divide the greatest common divisor of the two numbers:

\[ \text{numerator}(\text{Norm}(a_p)) \cdot \text{numerator}(\text{disc}(b_p)), \quad p = 2, 3. \]

As the only possible exceptional primes we obtain the primes that ramify in \( E \):

\[ \mathbb{R} = \{5, 13, 73693, 1418741\} . \]

Therefore, we have the following result:

**Theorem 5.1** Let \( \Upsilon \) be the Siegel cusp form of weight 28 computed by Skoruppa. It is untwisted and has multiplicity one. Let \( \rho_{\Upsilon, \lambda} \) be the Galois representations attached to \( \Upsilon \) constructed by Weissauer. Then, if we assume Serre's conjecture (3.2.4?), it follows that for every \( \ell > 53, \ell \notin \mathbb{D} \cup \mathbb{R} \)
and every \( \lambda \) in \( E \) above \( \ell \) the image of \( \rho_{\Upsilon, \lambda} \) is \( A_{28}^{\lambda} \). In particular, except for the finitely many primes excluded, the groups \( \text{PSp}(4, \mathbb{F}_r) \), if \( r \) is even, and \( \text{PGSp}(4, \mathbb{F}_r) \), if \( r \) is odd, (where \( r \) denotes the residual degree of \( \lambda \)) are realized as Galois group over \( \mathbb{Q} \), and the corresponding Galois extension only ramifies at \( \ell \).

Without assuming Serre’s conjecture, we want to obtain an effective unconditional result for the images of the \( \rho_{\Upsilon, \lambda} \). With this aim, we compute the values \( d_p \), for \( p = 2, 3 \) and 5, and check that (4.8) holds for the three of them. Therefore, applying (4.9) three times, we obtain:

**Theorem 5.2** Let \( \Upsilon, \rho_{\Upsilon, \lambda} \) be as in theorem 5.1. Then, for every \( \ell > 53 \), \( \ell \notin \mathbb{D} \cup \mathbb{R} \) and every \( \lambda \in E \) above \( \ell \) such that:

\[
\ell \nmid \text{disc}(d_p) \quad \text{and} \quad d_p \notin (\mathbb{F}_\lambda)^2, \quad p = 2, 3, \text{ or } 5,
\]

the image of \( \rho_{\Upsilon, \lambda} \) is \( A_{28}^{\lambda} \) (with the same consequences for inverse Galois theory as in theorem 5.1).

For the primes \( \ell \) inert in \( E/\mathbb{Q} \) it is easy to see that if

\[
\ell \nmid \text{disc}(d_p) \quad \text{and} \quad \left( \frac{\text{Norm}(d_p)}{\ell} \right) = -1
\]

for \( p = 2, 3 \) or 5, then \( d_p \notin (\mathbb{F}_\lambda)^2 \). This gives an easier criterion for these primes.

The first primes \( \ell > 53 \) inert in \( E \) are:

\[
59, 67, 71, 101, 103, 137, 151, 157, 181, 191, 197.
\]

We apply the above result to them and we verify that except for 151, the other ten primes in this list verify the required conditions for some \( p \leq 5 \). Therefore we obtain the following corollary:

**Corollary 5.3** The groups \( \text{PGSp}(4, \mathbb{F}_{\ell_3}) \) are Galois groups over \( \mathbb{Q} \) for \( \ell = 59, 67, 71, 101, 103, 137, 157, 181, 191, 197 \).

### 5.1 The first interesting Siegel cusp form

Now we apply the algorithm to \( \Upsilon_{20} \), the first example of interesting Siegel cusp form for the full Siegel modular group. \( \Upsilon_{20} \) has weight \( k = 20 \) and rational Fourier coefficients. Its first eigenvalues were computed by Skoruppa.
in [15]. The dimension of the space of interesting Siegel cusp forms of weight 20 is 1, therefore \( \Upsilon_{20} \) has multiplicity one. We use the values of the \( a_p, a_{p^2} \) for \( p \leq 7 \).

Applying equation (4.1) with \( p = 2 \) and 3 we find no exceptional primes at this step.

Applying equation (4.2) with \( p = 2 \) and 3 we obtain as (possible) exceptional primes \( \ell = 2, 3, 5, 7, 11 \). According to the results of Skoruppa (cf. [15]) these primes are in fact exceptional because \( \Upsilon_{20} \) is congruent modulo these \( \ell \) (and only them) to a Maass specialform, so equation (4.2) holds, for these \( \ell \), for every \( p \).

Applying equation (4.3) with \( p = 2 \), 3, 5 and 7, we obtain as (possible) exceptional primes \( \ell = 2, 3, 5, 29, 71 \). According to the results of Skoruppa (cf. [15]) these primes are in fact exceptional because \( \Upsilon_{20} \) is congruent modulo these \( \ell \) (and only them) to an Eisenstein-Klingen series, so equation (4.3) holds, for these \( \ell \), for every \( p \).

Applying equation (4.4) with \( p = 2 \), 3, 5 only gives \( \ell = 2, 3, 5 \) as possible exceptional primes.

This concludes the application of the algorithm to this example, proving in particular that the image of the Galois representations \( \rho_\ell \) attached to \( \Upsilon_{20} \) are “as large as possible” for every \( \ell > 71 \), under the assumption of Serre’s conjecture. To obtain an unconditional result, we use the first values of \( d_p \):

\[
\begin{align*}
d_2 &= 2^{14} \cdot 3^2 \cdot 7 \cdot 13 \cdot 19 \cdot 241, \\
d_3 &= 2^6 \cdot 3^{10} \cdot 19 \cdot 47 \cdot 150628997, \\
d_5 &= 2^8 \cdot 3^2 \cdot 5^6 \cdot 19 \cdot 47 \cdot 1396135808326877, \\
d_7 &= 2^8 \cdot 3^6 \cdot 7^6 \cdot 29 \cdot 1097 \cdot 4171309430662453.
\end{align*}
\]

**Theorem 5.4** Let \( \Upsilon_{20} \) be the Siegel cusp form of weight 20 computed by Skoruppa. It has multiplicity one and \( E = \mathbb{Q} \). Let \( \rho_\ell \) be the Galois representations attached to \( \Upsilon_{20} \) constructed by Weissauer. Then, if we assume Serre’s conjecture (3.2.4?) it follows that for every \( \ell > 37, \ell \neq 71 \) the image of \( \rho_\ell \) is \( A_{20}^\ell \). In particular for every \( \ell > 37, \ell \neq 71 \), the groups \( \operatorname{PGSp}(4, F_\ell) \) are realized as Galois group over \( \mathbb{Q} \), and the corresponding Galois extension of \( \mathbb{Q} \) only ramifies at \( \ell \).

The same conclusion holds unconditionally (without assuming Serre’s conjecture) if we also impose \((\frac{\ell}{7}) = -1 \) for \( p = 2, 3, 5 \) or 7.

**Remark:** In particular, we have proved that the images are “as large as possible” for an explicitly given set of primes of Dirichlet density \( \frac{15}{16} \).
6 Improving theorem 4.2

6.1 Pseudo-representations

We reproduce Wiles’ definition of pseudo-representations and some of its properties (cf. [23], [19]). This is a tool used to construct odd two-dimensional representations.

Let $A$ be a ring, $G$ a group, and $c \in G$ an element of order two.

Definition 6.1 A pseudo-representation of $G$ defined over $A$ is a quadruple $\tau = (a,d,t,x)$; $a,d,t : G \to A$, $x : G \times G \to A$ satisfying the conditions:

(i) $2a_g g' = a_g a_{g'} + x_{g,g'}$, $2d_g g' = d_g d_{g'} + x_{g',g}$
(ii) $a_g = t_g + t_{c g}$, $d_g = t_g - t_{c g}$
(iii) $t_1 = 2$, $t_c = 0$, $x_{g,c} = x_{c,g} = 0$
(iv) $x_{g,g'} x_{h,h'} = x_{g,h'} x_{h,g'}$,
$4x_{g,h',g'} = a_g a_{h'} x_{h,g'} + a_{h'} d_h x_{g,g'} + a_g d_{g'} x_{h,h'} + d_h d_{g'} x_{g,h'}$.

Define the trace of $\tau$ by $\text{trace}(\tau) = t$, and the determinant by $\text{det}(\tau)(g) = a_g d_g - x_{g,g}$.

Remarks: (i) If $G$ and $A$ have a topology, all maps are assumed to be continuous.
(ii) If $\rho : G \to \text{GL}(2,A)$, $\rho = (\alpha, \beta, \gamma, \delta)$ is a representation defined over $A$, with $\rho(c) = (1,0,0,-1)$ (an odd representation) it gives rise to a pseudo-representation $\tau = (a,d,t,x)$, where $a = 2\alpha$, $d = 2\delta$, $t = \alpha + \delta$, $x_{g,g'} = 4\beta_g \gamma_{g'}$.
(iii) A pseudo-representation is determined by its trace. If $A' \subset A$ is a subring, if $\tau$ is defined over $A$ and $\text{trace}(\tau)$ takes values in $A'$, then $\tau$ is defined over $A'$.

The following are two basic properties that we will use in the sequel (cf. [23], [19]):

Lemma 6.2 (“patching lemma”) Let $(U_i)_{i>0}$ be a sequence of ideals in $A$ such that $\bigcap_i U_i = \{0\}$. For each $i$, let $\tau_i$ be a pseudo-representation defined over $A/U_i$. Assume that there exist a dense subset $\Sigma$ of $G$ and a function $T : \Sigma \to A$ such that for any $i$, $T \mod U_i = \text{trace}(\tau_i)$. Then, there exists a pseudo-representation $\tau$ defined over $A$ such that for any $i$, $\tau \mod U_i = \tau_i$.

Theorem 6.3 If $A$ is a field of characteristic different from 2, any pseudo-representation $\tau$ corresponds to an odd two dimensional representation $\rho$. 

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6.2 From residual to $\lambda$-adic reducibility

Let $f$ be a genus two Siegel cusp form verifying the conditions of theorem 4.2 and consider the Galois representations $\rho_\lambda := \rho_{f,\lambda}$. Suppose that for an infinite set of primes $\{\lambda_i\}_{i=1}^\infty = \Lambda$ of $E$ the residual representations $\bar{\rho}_\lambda$ are reducible with irreducible two dimensional constituents of the same determinant.

Taking a subset of $\Lambda$ if necessary, we assume that the density of the set $L$ of rational primes $\ell_i$ such that $\lambda_i | \ell_i$ is 0.

We want to prove that with this hypothesis of residual reducibility the whole family of $\lambda$-adic representations $\rho_\lambda$ has to be reducible (and a fortiori residually reducible). We assume for the moment that the decompositions:

$$\bar{\rho}_\lambda^{ss} = \sigma_i \oplus \pi_i$$

induce a decomposition on the set $\{a_p\}$ of traces of the family $\rho_\lambda$: $a_p = u_p + v_p (\bigcirc)$ for every $p \not\in L$ that is independent of $\lambda_i$. That is to say, there exists such a decomposition (fixed) with $\{u_p \mod \lambda_i\}$ equal to the set of traces of $\sigma_i$ (restricted to $p \not\in L$) and $\{v_p \mod \lambda_i\}$ equal to the set of traces of $\pi_i$ (restricted to $p \not\in L$), for every $\lambda_i \in \Lambda$.

Fix a prime $\lambda$ of $E$. As usual, let $\ell$ be the rational prime such that $\lambda | \ell$.

For any set $S$ of rational primes let us denote by $Q^S$ the maximal extension of $\mathbb{Q}$ unramified outside $S$ and $G^S = \text{Gal}(Q^S/\mathbb{Q})$. We will work only with Galois representations unramified outside $L+\ell := L \cup \{\ell\}$, therefore we take the quotient $G^{L+\ell}$ of $G_Q$ and work with representations of this quotient group.

Consider the subgroup $W_\lambda$ of $G^{L+\ell}$ algebraically generated by the elements $\{\text{Frob } p\}_{p \not\in L+\ell}$.

Let us consider the function $F : W_\lambda \rightarrow \overline{\mathbb{Q}}$ defined on the generators of $W_\lambda$ by: $F(\text{Frob } p) = u_p$.

Recall that the traces $a_p$ of $\rho_\lambda$ are the eigenvalues of the Siegel cusp form $f$. They generate the number field $E$. The decomposition of the characteristic polynomials $Pol_p$ introduced in section 4.8 shows that it must hold: $u_p = a_p/2 \pm \sqrt{d_p}$, so we have:

$$E' := \mathbb{Q}(\{u_p\}) = E(\{\sqrt{d_p}\}).$$

Moreover, from the definition of $d_p$ (see section 4.8) and standard properties of the eigenvalues of $f$ it is known that all $a_p$ and $a_p^2$ belong to $\mathcal{O}_E[1/N]$ where $N$ is a finite product of primes and all $u_p$ belongs to $\mathcal{O}' := \mathcal{O}_E[2/N]$, where
\( O_X \) denotes “ring of algebraic integers of \( X \).

Therefore, the image of \( F \) is inside \( O' \):

\[
F : W_\lambda \to O'. \tag{6.1}
\]

For each \( \lambda_i \in \Lambda \) we take a prime \( \lambda'_i \) of \( E' \) above \( \lambda_i \). Composing the function \( F \) with the reduction modulo \( \lambda'_i \) for every \( \lambda_i \in \Lambda \) gives the functions:

\[
F_{\lambda'_i} : W_\lambda \to F_{\lambda'_i}
\]

with \( F_{\lambda'_i}(\text{Frob } p) = u_p \mod \lambda'_i \), for every \( p \not\in \mathbb{L} + \ell \). This is the restriction to \( W_\lambda \) of the pseudo-representation \( \tilde{\sigma}_i \) associated to \( \sigma_i \), therefore \( F_{\lambda'_i} \) is a 2-dimensional pseudo-representation of \( W_\lambda \) for every \( \lambda_i \in \Lambda \).

**Remark:** For every \( \lambda_i \), \( \sigma_i \) gives a representation of \( G_{\mathbb{L} + \ell} \) because it is unramified outside \( \ell \).

Now we apply the patching lemma (see lemma 6.2) and we conclude that \( F \) as in (6.1) is a 2-dimensional pseudo-representation. We have already seen (see for example section 4.3) that the representations \( \sigma_i \) have to be odd. An application of theorem 6.3 proves that there exists a representation

\[
\sigma : W_\lambda \to \text{GL}(2, E')
\]

with \( \text{trace}(\sigma) = F \), i.e.,

\[
\text{trace}(\sigma(\text{Frob } p)) = u_p
\]

for every \( p \not\in \mathbb{L} + \ell \).

The same argument shows that by patching the representations \( \pi_i \) we construct a representation:

\[
\pi : W_\lambda \to \text{GL}(2, E')
\]

with

\[
\text{trace}(\pi(\text{Frob } p)) = v_p
\]

for every \( p \not\in \mathbb{L} + \ell \).

Now let us compare the semisimplification \((\rho_\lambda|_{W_\lambda})^{ss}\) with \( \sigma \oplus \pi \), as representations of \( W_\lambda \) with image in \( \text{GSp}(4, \mathbb{E}) \).

**Remark:** \( \rho_\lambda \) gives a representation of \( G_{\mathbb{L} + \ell} \) and \( W_\lambda \) because it is unramified outside \( \ell \).

Both representations have the same trace at every element of \( W_\lambda \), so they are conjugated. We conclude that \( \rho_\lambda|_{W_\lambda} \) is reducible for every \( \lambda \).

But the Cebotarev density theorem implies that the subgroup \( W_\lambda \) is dense.
in $G^{2+\ell}$, because $L$ has density 0. Therefore, being continuous, the representation $\rho_\lambda$ has to be reducible for every prime $\lambda$ of $E$, of the form

$$\rho_{f\lambda}^{ss} = \sigma_\lambda \oplus \pi_\lambda$$

with $\det(\sigma_\lambda) = \det(\pi_\lambda) = \chi^{2k-3}$.

It remains to remove the hypothesis (□). To do this, one should consider the tensor product representations $\sigma_i \otimes \pi_i$ for every $\lambda_i \in \Lambda$. These four dimensional irreducible representations can be patched as in the above construction because each of them has trace at Frobenius $p$ equal to $\{u_p \mod \lambda_i\}$ (here we need Taylor’s version of the theory of pseudo-representations and in particular of lemma 6.2 valid for the four dimensional case, see [17]). The resulting characteristic 0 representation will be tensor-decomposable: $\sigma \otimes \pi$, and using the uniqueness of tensor decompositions in the irreducible case, one concludes that for the two dimensional representations $\sigma$ and $\pi$ it holds:

$$\text{trace}(\sigma(\text{Frob } p)) + \text{trace}(\pi(\text{Frob } p)) = u_p + v_p$$

and the proof continues as before. Thus, the following result holds:

**Theorem 6.4** Let $f$ be a genus 2 Siegel cusp form verifying the conditions of theorem 4.4. Let $\rho_{f,\lambda}$ be the corresponding family of Galois representations. Suppose that the residual representations $\bar{\rho}_{f,\lambda}$ are reducible for infinitely many primes $\lambda_i$ of $E$. Then, for every prime $\lambda$ the $\lambda$-adic representation $\rho_{f,\lambda}$ is reducible.

Let $f$ be a Siegel cusp form verifying the conditions of theorem 4.2. Assume that we are in the situation of theorem 6.4. Then we have: $\rho_{f,\lambda}^{ss} = \sigma_\lambda \oplus \pi_\lambda$ for every $\lambda$, with $\det(\sigma_\lambda) = \det(\pi_\lambda) = \chi^{2k-3}$. Suppose that condition (4.8) is verified by some prime $p$. As was shown in section 4.8, this implies that for infinitely many $\lambda$ with

$$\ell \nmid \text{disc}(d_p) \quad \text{and} \quad d_p \notin (\mathbb{F}_\lambda)^2$$

the representation $\bar{\rho}_{f,\lambda}$ cannot be reducible. Thus we obtain a contradiction with theorem 6.4.

Combining this with the results of section 4 we conclude:

**Theorem 6.5** Let $f$ be a genus 2 Siegel cusp form and Hecke eigenform of even weight $k$ for the Siegel modular group $\text{Sp}(4, \mathbb{Z})$ with multiplicity one, untwisted and not a Maass specialform. Suppose that there is a prime $p$ with $\mathbb{Q}(d_p) = E$ and $\sqrt{d_p} \notin E$. Then the image of the Galois representation $\rho_{f,\lambda}$ is $A_k^\times$ for almost every $\lambda$. 

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Remark: This result applies in particular to the examples considered in the previous section.

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