Topological classification of Morse–Smale flows on 3-manifolds

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Abstract

We construct a topological invariant for a Morse–Smale flow on a 3-manifold and prove that the flows are topologically equivalent iff their invariants are same.

1 Introduction

In this paper we study the topological properties of Morse-Smale flows on closed oriented 3-manifolds. We give the topological classification of these flows by constructing complete topological invariant (distinguished graph) and proving the criteria of topological equivalence.

Let $M$ and $N$ are closed smooth manifold.

Two flows $X$ and $Y$ on $M$ are topologically equivalent, if there exists a homeomorphism $h : M \to M$ that carries trajectories of $X$ to those of $Y$ and preserves their direction.

In [11], Peixoto gave a topological classification of Morse–Smale flows on closed orientable 2-manifolds. There are many papers where this classification was improved and new approaches were proposed. For references, see, for example [2, 8, 9, 10, 12, 19].

Vlasenko [18] and, independently, Bonatti and Langevin [7] gave a classification of Morse-Smale diffeomorphisms on closed 2-manifolds.
A topological classification of Morse–Smale dynamical systems on 3-manifolds are known only for some classes of systems [6, 8, 13, 15, 16, 17].

For Morse–Smale flows without closed orbits, Smale constructed handle decompositions. For flows with closed orbits, Asimov constructed a round handle decomposition. For our classification of Morse–Smale flows, we use Asimov’s idea that it is possible to reconstruct the flow by using the decomposition on prime and round handles. However, it is necessary to note that there are infinitely many such decompositions for the same flow. Therefore, it is necessary to chose among them a canonical one. In the general case, a Morse–Smale flow has infinitely many trajectories in the intersection of 2-dimension stable and unstable manifolds. This difficulties are similar to difficulties that arise in the classification of Morse–Smale diffeomorphisms on surfaces. Besides, for dimension 3, it is necessary to introduce an additional invariant for an extension of homeomorphisms from the boundary of the round handles to their interior. We call this invariant a $\tau$-invariant.

2 Basic definitions

Let $M$ be a closed 3-manifold and $X$ a flow on it. $X$ is a Morse–Smale flow, if the following conditions hold:

1) $X$ has a finite number of critical elements (fixed points and closed orbits) and all of them are non-degenerate (hyperbolic);

2) stable and unstable manifolds of critical elements have transversal intersections;

3) the non-wandering set of $X$ is a union of fixed points and closed orbits.

Two flows $X$ and $Y$ on $M$ are topologically equivalent, if there exists a homeomorphism $h : M \to M$ that carries trajectories of $X$ to those of $Y$ and preserves their direction.

Denote by $v(x)$ and $u(x)$ stable and unstable manifolds of a critical element $x$. If $x$ and $y$ are critical elements such that $\dim v(x) = \dim u(y)$, then...
dim \( u(y) = 2 \) and \( v(x) \cap u(y) \neq \emptyset \), then [7] there is a sequence of saddle closed orbits \( \beta_1, ..., \beta_n \) satisfying

\[
v(x) \cap u(\beta_1) \neq \emptyset, v(\beta_k) \cap u(b_{k+1}) \neq \emptyset, v(\beta_n) \cap u(y) \neq \emptyset.
\]

Denote by \( m \) the maximal number of elements \( \beta_k \) in all such sequences and set \( \text{beh}(y|x) = m + 1 \). Thus \( \text{beh}(y|x) = 0 \), if \( v(x) \cap u(y) = \emptyset \). We say that a flow has \( \text{beh} \) equal \( s \), if \( s \) is a maximal value of \( \text{beh}(y|x) \).

The number \( \text{beh}(y) = \max_x \text{beh}(y, x) \) is called the height of the closed orbit of index 1. Thus, a closed orbit has height 0, if its stable manifolds do not intersect stable manifolds of closed orbits of index 1. A closed orbit of index 1 has height 1, if its stable manifolds intersect unstable manifolds of closed orbits of height 1 and do not intersect unstable manifolds of other closed orbits of index 1, etc.

Below we assume that \( X \) and \( X' \) are Morse–Smale flows on a closed oriented 3-manifold \( M \).

Let \( a_1, ..., a_k \) and \( a'_1, ..., a'_k \) be index 0 fixed points of \( X \) and \( X' \), respectively, and let \( b_1, ..., b_n \) and \( b'_1, ..., b'_n \) be index 1 fixed points of \( X \) and \( X' \), correspondingly. By \( \gamma^i_j, j = 1, \ldots, m(i) \), we denote closed orbits the height of which equals \( i \).

Let \( K \) be the union of the following stable manifolds:

1) fixed points of indices 0 and 1,
2) closed orbits of index 0.

Fixed points of index 1 are called lower points, and their unstable manifolds are called lower manifolds.

Consider a regular neighbourhood \( U(K) \) of \( K \). Let \( \Phi = \partial U(K) \) be the boundary of this neighbourhood of the flow \( X \) and let \( \Phi' \) denote the boundary for the flow \( X' \).

Fixed points of index 2 and closed orbits of index 1 are called upper critical elements; their stable manifolds are called upper manifolds. The curves that are intersections of \( \Phi \) and lower manifolds are called lower curves, and the intersection of \( \Phi \) and upper manifolds are called upper curves.
Note that all lower curves are closed (homeomorphic to circles), and the upper curves can be both closed and not closed. Moreover, infinitely many open curves can correspond to an upper critical element.

The surface Φ, together with the lower and upper curves, is called a diagram of the flow. Two diagrams are called homeomorphic, if there exists a homeomorphism of Φ that maps the lower and upper curves to lower and upper curves, respectively.

**Proposition 1** If flows are topologically equivalent, then their diagrams are homeomorphic.

**Proof.** Let \( \psi: M \to M' \) be a topological equivalence between \( X \) and \( X' \). Then the required homeomorphism of the diagrams is given by the formula

\[
\Phi \times \varphi(\psi(x)) \cap \Phi',
\]

where \( \varphi(y) \) is a trajectory of \( X' \) that contains \( y \).

If the flow doesn’t have closed orbits, the existence of a homeomorphism of the diagrams is also a sufficient condition for topological equivalence of the flows [15].

This is not true in general and, in addition, it is difficult to prove that there exists a homeomorphism between diagrams with an infinite number of noncompact curves.

In what follows, our aim would be to apply the diagram to the surface Φ with the embedded graph \( G \) and determine additional information which, together with the pair \((\Phi, G)\), would give a complete invariant of the flow. It is convenient to express this information in terms of the decomposition in simple and round handles.

A handle decomposition of \( M \) is a sequence \( D^3 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_k = M \), where \( M_i \) is obtained from \( M_{i-1} \) by adding a handle or a round handle.

We construct a handle decomposition such that the cores and the cocores lie on stable and unstable manifolds of the critical elements,
respectively. We order the critical elements in such a way that, for all $i$, the following holds: critical points of index $i$ are less than closed orbits of index $i$ which are still less than critical points of index $i + 1$; for closed orbits of index 1, $\gamma^i_j < \gamma^{i+1}_j$. Denote the critical elements by $x_0, x_1, \ldots, x_n$ ($x_0 < x_1 < \ldots < x_n$). Then we construct a decomposition in handles in the following way: the handle $v(x_0)$ is a closed tubular neighbourhood $U(x_0)$ of the least critical element $x_0$, $H_i = \text{cl}(U(v(x_i))) \cup \bigcup_{k<i} U_k$, $i = 1, \ldots, n$.

Note that the surface $\Phi$ can be considered as the boundary of the union of handles that correspond to critical elements of index 0 and critical points of index 1. Since stable and unstable manifolds intersect transversally, it is possible to choose the neighbourhood $U(v(x_i))$ to be so small that all the intersections of the core and the cocore with the bases of 2-handles and round 1-handles, as well as the side legs of the 1-handles and the round 1-handles, are parallel. This means that it is possible to introduce a structure of the direct products, $S^1 \times [-1,1]$, such that these intersections have the form $\{s_i\} \times [-1,1]$.

If the flow does not have closed orbits, then all decompositions in handles, which have parallel intersections, are isomorphic and can be given, up to an isomorphism, by the diagram. In case of a flow with closed orbits of height not smaller than 2, a decrease of neighbourhoods of their stable manifolds increases the number of intersections of the cores and cocores of the handles of smaller orders. The idea of our construction is to choose handles at each step to be the largest neighbourhoods for which the property of the intersections is preserved. If the constructed handle decompositions are isomorphic, then the complexes that consist of the boundaries of the handles are isomorphic and, also, the isomorphism maps meridians of the round handles into meridians.

As in Proposition 1, different choices of simple 0- and 1-handles and round 0-handles lead to isomorphic handles decompositions. We consider in more details the case of round 1-handles.
3 The structure of the flow in a neighbourhood of a closed 1-orbits

Closed orbits of index 1 can have trivial or twisted neighbourhoods (both of them are solid tori). In case of a trivial neighbourhood, it intersects stable and unstable manifolds in cylinders $S^1 \times I$ and the torus in a pair of circles. In case of twisted neighbourhoods, their intersections are Mobius bands and circles.

A trivial neighbourhood can be represented as the product $[1, 3] \times S^1 \times [-1, 1]$ with coordinates $\{\rho, \alpha, z\}$ (an analog of the cylindrical coordinates) [13]. Thus, if a trajectory intersects a torus in a point with $\rho = 2 \pm 1, \alpha = \alpha_0, z = z_0 \neq 0$, then its second intersection point with the torus has the coordinates

$$\rho = 2 \pm |z_0|, \alpha = \alpha_0 + \ln|z_0|, z = \text{sign } z_0. \quad (1)$$

A stable manifold is determined by the equation $z = 0$, an unstable manifold by $\rho = 2$.

We consider the torus $T$ as the union $T = T^- \cup T^+$, where

$$T^- = \{\rho \in \{1; 3\}, -1, 1\}$$

$$T^+ = \{13, z \in \{-1; 1\}\}$$

Thus, $T^- \subset \Phi$ is a set of the points incoming in torus and $T^+ \subset F$ is a set of outgoing points.

Formula (1) defines a map $g$ that maps points of $T^-$ with coordinates $\rho = 1, z > 0$ into points of $T^+$. Similar formulas hold if $\rho = 3$ and $z < 0$.

The intersection of the stable manifold of a closed 1-orbit and the torus consists of two circles, $\rho = 1, z = 0$ and $\rho = 3, z = 0$, which we denote by $v^1, v^2$. The unstable manifold of a closed 1-orbit intersects the torus in two circles, $\rho = 2, z = -1$ and $\rho = 2, z = 1$, denoted by $u^1, u^2$.

Since stable and unstable manifolds have transversal intersections, the curves $u_i$ intersect the circles $v^1, v^2$ also transversally.
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Thus, in a neighbourhood of an intersection point, \( u_i \cap v^1 \), the curve \( u_i \) is given by the equation \( \rho = 1, z = t, \alpha = tk(t) \), where \( k = k(t) \) is a smooth function. The image of this curve with respect to the map \( g \) is the curve

\[
\begin{align*}
x &= (2 - t) \cos(tk(t) + \ln t), \\
y &= (2 - t) \sin(tk(t) + \ln t), \\
z &= 1.
\end{align*}
\]

As \( t \to 0^+ \), this curve winds around \( u^1 = \{ \rho = 2, z = 1 \} \), and the tangent to the curve approaches the tangent to the circle. As \( t \to 0^- \), the same is true if \( z = 1 \) is replaced by \( z = -1 \) and \( t \) by \( |t| \).

Similarly, for arcs in the intersections \( u_i \cap v^2 \) with \( \rho = 3 \), the image of the arc of \( u_i \) winds around the same circles \( u^1 = \{ \rho = 2, z = 1 \} \) and \( u^2 = \{ \rho = 2, z = -1 \} \).

For the intersections \( v_i \cap u^1 \) and \( v_i \cap u^2 \) from the upper and lower bases, the cluster sets of the pre-image \( g^{-1}(v_i) \) will contain the circles \( v^1 \) and \( v^2 \).

By reducing the neighbourhood of the closed orbit, it is possible to achieve that the following two properties hold:

1) Each arc in the intersection \( u_i \cap T \) intersects \( v^1 \) or \( v^2 \) transversally in one point. Each arc in the intersections \( v_i \cap T \) intersects \( u^1 \) or \( u^2 \) transversally in one point.

2) If an orientation is fixed on each arc in the intersections \( u_i \cap T \) and \( v_j \cap T \) and on the torus \( T \), then all the intersections \( g(u_i) \cap v_j \) have the same sign of intersection. Similarly, all the intersections \( g^{-1}(v_j) \cap u_i \) also have the same sign.

Here the sign of the intersection is positive, if the orientations of the intersecting curves determine an orientation of the torus such that it coincides with the given one; the sign is negative in the opposite case.

A neighbourhood of the closed orbit satisfying these properties is called standard.
For a twisted neighbourhood, the concept of a standard neighbourhood is introduced in a similar way. It can also be obtained from a standard trivial neighbourhood by slitting it into the disks $\alpha = \text{const}$ and subsequent pastings a pair of the obtained disks together via the central symmetry map.

Let us consider these neighbourhoods as round 1-handles. The orientation of the closed orbit introduces a parallel orientation in the corner $T^- \cap T^+$. 

4 The scheme of the flow

Let us consider the diagram of a flow. On the lower and upper circles and the arcs that correspond to closed orbits, we set an orientation to be parallel to the direction in which the corresponding closed orbit moves. We pair these oriented circles and sets of arcs together in such a way that the circles or the sets of the arcs from one pair correspond to a single closed orbits. Since the set of arcs can be considered as cuts of the circle, there is a natural cyclic order on them.

We consider the upper circle $u_0^i$ which corresponds to the closed orbit $\gamma^0_i$ of height 0. From the previous paragraph it follows that there is a sufficiently small neighbourhood $U$ of the circles $u_0^i$ on $\Phi$ satisfying the following property: it can be cut along a curve that is transversal to all upper circles and arcs so that the obtained set is homeomorphic to $[0,1] \times [0,1]$, and the cut of the upper arcs in $U$ has the form $\{t\} \times [0,1]$. A neighbourhood with such properties is called trivial. If $u_0^i$ intersects the lower circles, then we take the arcs of these circles to be a transversal curve. We demand that the boundary of $U$ consist of arcs of the diagram. A trivial neighbourhood $U$ is called maximal (MTN), if there are no greater neighbourhoods that possess the above properties Fig. 4).

For circles of height more than 0, a maximal trivial neighbourhood is defined similarly. It is easy to see that a maximal neighbourhood always exists, except for the case when there is an annulus.
$S^1 \times [0, 1]$ such that its boundary consists of upper circles and all the arcs are wound around $S^1 \times \{0\}$ with one end and around $S^1 \times \{1\}$ with another end (see Fig. 5). In this case, we take the maximal neighborhood to be any trivial not intersecting neighborhood, the complement to which has in $S^1 \times [0, 1]$ the minimal number of points of intersection of the lower circles, the upper circles, and the arcs.

Consider now the set of arcs $\{u\}$ whose diagrams correspond to a closed orbit of height 1. Since its stable manifold intersects unstable manifolds of dimension 2 in a finite number of trajectories, this set consists of a finite number of arcs. Let $U$ be a neighborhood of the set of arcs on $\Phi$ in the complement of a maximal trivial neighborhood of circles of height 0. Such a neighborhood is the union of products $[-1, 1] \times [0, 1]$. It is called trivial, if all its intersections with arcs of height not less 1 are of the form $\{t\} \times [0, 1]$. The boundary is also considered as consisting of arcs. Besides, we require that all the arcs $\{\pm 1\} \times [0, 1]$ correspond to a single closed orbit, and the the relation of cyclic order on them agrees with the relation on $\{u\}$. A trivial neighborhood is called maximal if there is no a greater maximal neighborhood possessing these properties.

Similarly to the case of circles, there may not exist a maximal neighbourhood. In such a case, the maximal neighbourhood is taken to be an arbitrary neighborhood whose complement does not contain points of intersection of arcs and circles.

Consider a collection of arcs $\{w\}$ that correspond to a closed orbit of height 2. If the arcs of height 1 are considered as a circle cut in a finite number of points, then $\{w\}$ can be considered as a finite number of arcs winded around these circles. Cutting the circle with a transversal curve leads to cutting the winding arc into an infinite number of arcs. Hence, $\{w\}$ consists of an infinite number of arcs. Nevertheless, there is only a finite number of them that do not lie in the maximal neighbourhoods constructed above. For these arcs, a construction of a maximum trivial neighbourhood is same as for arcs of height 1.

Again, there is only a finite number of arcs of height 3, not lying
in maximal neighbourhoods, and arcs of heights 0–2. For them, we construct maximal neighbourhoods, and we continue this process until maximal neighbourhoods are constructed for all sets of the arcs.

The constructed maximal neighbourhoods are used for constructing round handles. These neighbourhoods will be intersections of the corresponding round handles and \( \Phi \). We construct the round handles starting with height 0. Round handles of height 0 are neighbourhoods of stable manifolds in the way that their bases are the maximal neighbourhoods. The arbitrariness in the choice of the side walls is not important, since the first return map defines a homeomorphism between them (and our constructions are, actually, carried out up to a homeomorphism). Since the lower bases are trivial neighbourhoods, it follows from Section 2 that the intersections of stable manifolds of height 0 and side walls make parallel curves. In side walls of 1-handles, consider regular neighbourhoods of parallel curves of height 1, which extend maximal trivial neighbourhoods of height 1. This means that these neighbourhoods contain parallel curves of height more than 1 and that their ends lie in MTN of height 1. The union of these neighbourhoods and the MTN of height 1 are bases of round handles of height 1. Similarly to the case of handles of height 0, various choices of side walls lead to homeomorphic constructions. The stable manifold of height more 1, again, intersects these side walls in parallel curves. In the constructed side walls of heights 0 and 1, we choose neighbourhoods of parallel curves of height 2 that would extend the MTN of height 2 to the bases of round handles of height 2; round handles of all other heights are constructed similarly.

We thus have constructed some two-dimensional compact stratified set \( S \), which is the union of the surface \( \Phi \) and the boundaries of round handles of index 1. In this construction, the union of 0- and 1-dimensional strata consists of the following closed curves: 1) b-spheres of 1-handles, 2) a- and b-spheres of round 1-handles, 3) a-sphere of 2-handles, 4) boundaries of the bases of round 1-handles.
On strata of the second type, the orientation is defined in correspondence with the motion on the corresponding closed orbit of index 1. It also follows from the construction that the diagram of the flow can be recovered uniquely (up to a homeomorphism) from the stratified set. Thus, existence of an isomorphism of the stratified sets is a necessary condition for topological equivalence of the flows.

On each torus, which is the boundary of a round handle, we single out a closed curve in $S$, which bounds a disk that intersects the corresponding closed orbit transversely in a single point.

**Definition.** A scheme of the flow is a stratified set $S$, together with the following: 1) the set of 2-dimensional subsets that are boundaries of round handles and their indices, 2) a partition of 1-strata into 4 types, 3) orientations of cycles of the 2nd type and partitions of some of them into pairs.

Two schemes are called isomorphic if there is a homeomorphism of the stratified sets, which preserves types of the 1-stratas and the partition of the cycles into pairs and orientations (on those cycles where the orientations are given).

**Theorem 1** There is a topological equivalence of two Morse–Smale flows $X$ and $X'$. preserving the order of critical elements if and only if there is an isomorphism between their schemes mapping chosen curves of the first scheme into curves that are homotopic to the chosen curves of the second scheme and preserving the chosen orientations on these curves.

**Proof.** Necessity. Let flows $X$ and $X'$ be topologically equivalent. Then there is a homeomorphism $h$ that maps trajectories of $X$ into trajectories of $X'$. The image of the scheme of $X$ is a scheme of $X'$. Thus, it is necessary to show that two different schemes of the same flow are isomorphic. Indeed, the first return map defines a homeomorphism between the diagrams and, also, a homeomorphism between the stratified sets. Thus, since another choice of chosen curves yields homotopic curves by the construction, we see that the first return map is an isomorphism between the schemes.
Sufficiency. Let an isomorphism between the schemes of Morse–Smale flows be given. Let us show how to construct another isomorphism between the schemes such that the homeomorphism $f : S \to S'$ can be extended to a homeomorphism of the manifolds. By using the construction and the fact that the bases of round 1-handles are trivial neighbourhoods, it is possible to restore the diagram from the scheme, and the scheme isomorphism induces a diagram isomorphism $\Phi \to \Phi$.

The first step. By given isomorphism of the scheme we fix homeomorphisms of angles of the round 1-handle. Consider round handles $H_k^i$ of height $k$ for closed orbit $\gamma_k^i$. Stable and unstable 2-dimensional manifolds intersect the boundary of the handle in curves of two types: 1) transversal intersected stable or unstable manifold in one point, 2) noncompact. Consider the first intersection map $g$ from the bases to the side walls. It map curves of one type to the curves of other type (see Fig. 6). The isomorphism of the scheme give one to one correspondence between the intersections of the curves. We extend it to homeomorphisms of noncompact curves of the bases of $H_0^i$ coordinated with homeomorphisms of the angles. Using the first intersection map $g$ we construct the homeomorphisms of transversal curves $H_1^i$, then we extend it to homeomorphisms of noncompact curves of the bases of $H_1^i$ and so on. By analogy, we construct homeomorphisms of noncompact curves of side walls and transversal curves of bases beginning from the handles of biggest height. We extend the homeomorphisms of curves to regions bounded by them on bases of $H_0^i$. Using $g$ construct homeomorphisms of region of side walls of $H_0^i$. The construct homeomorphisms of regions of $H_1^i$ and so on. Thus we construct the homeomorphisms of the boundaries of round 1-handles.

Since the chosen cycles are homotopic we can extend the constructed homeomorhism to the disk having one transversal intersection with the closed orbit and chosen cycles as it boundary.

Now it is possible to extend the homeomorphism of tori to homeomorphisms of solid tori.
The latter is equivalent to the condition that each circle of the torus $T_i$, which bounds a disk in the solid torus, is mapped into a circle that has the same property. Then we can extend the homeomorphism of the circle to a homeomorphism of the disk so that it will be compatible with the first return map from the torus to the disk. Furthermore, the complement of this disk in the solid torus is a three-dimensional disk and, using compatibility of the homeomorphism of the torus and the map $g$, we can extend the homeomorphism from the boundary of the 3-disk inside it along the trajectories. To do this, fix a Riemannian metric on the solid torus. We consider parts of the trajectories that lie in the 3-disk and have their ends in its boundary. Then, by the construction of the homeomorphism on boundary, it maps each pair of ends of one trajectory into a similar pair. We define a homeomorphism between these parts of the trajectories in such a way that all arcs of these trajectories have proportional lengths. So the constructed homeomorphism of parts of the trajectories defines a homeomorphism of the solid tori.

The second step. Let us extend the homeomorphism from $\Phi$ to $U(K)$.

For each lower closed orbit $\alpha_i$ of $X$, we choose a neighbourhood as in Section 2 and a sufficiently small neighbourhood of $u_i$ satisfying $T_i^+ \subset \Phi$. For the flow $X'$, these neighbourhoods are chosen so that $T_i^{+'} = f(T_i^+)$. Then there exist natural homeomorphisms of such neighbourhoods so that they coincide with $f$ on $T_i$. Since $\alpha_i$ is a lower closed orbit, we can assume that $T_i^+$ does not intersect $u_j$ except for $u_1^1$ and $u_1^2$.

For each fixed point of index 1 there is a neighbourhood which is homeomorphic to the cylinder $\{\rho 1\}$. Assume that the side walls of such cylinders are mapped into sufficiently small neighbourhoods of the curves $u_i$ on $\Phi$. For the flow $X'$, let us construct cylinders so that their side walls coincide with the images of the side walls under the mapping $f$. There are natural homeomorphisms between the cylinders, which coincide with $f$ on the side surfaces.
If we remove, from $U(K)$, the constructed cylinders, which are neighbourhoods of fixed points of index 1, and solid toruses, which are neighbourhoods of the lower closed orbits, we obtain a union of 3-disks each of which contains one source. Having a homeomorphisms of the boundaries of these disks, we extend it inside along the trajectories. We thus obtain the required homeomorphism $U(K)$.

Extend the homeomorphism to the rest of the manifold $K$ in the same way as to $U(K)$. This gives the required homeomorphism of the manifold $K$.

5 On an extension of homeomorphisms from the torus to the solid torus.

The condition that the chosen curves are homotopic for the isomorphism of the schemes is not constructive, — if the homotopy condition is omitted in the definition of the isomorphism of the schemes, then there are infinitely many different isomorphisms of the schemes. Thus, it is impossible find among them the one that satisfies the homotopy condition in a finite number of steps. On the other hand, there are infinitely many different ways to construct, in relation to other curves on the torus, a chosen cycle in the general case. To overcome these difficulties, we introduce a $\tau$-invariant and replace the homotopy condition imposed on chosen cycles by the condition that they have equal $\tau$-invariants, which is easy to check.

Let $T$ be the boundary of a round handle and $G = T \cap S^{(1)}$, where $S^{(1)}$ is the 1-skeleton of the stratified set. We fix an orientation on $T$. Let $Z = \{z_i\}$ be the set of non-self-intersecting cycles of the graph $G$. Set an orientation on each of these cycles and on a chosen cycle $w$ in $T$. Define a correspondence between each oriented cycle $z_i$ and an integer $\alpha_i$ that equals the algebraic number of points where it intersects $w$. Set $\beta_i$ to be equal to the algebraic number of points in which $z_i$ intersects $v$, taken modulo $\alpha_i$, where $v$ is a parallel on $T$. Thus, we have a pair of maps, $\alpha, \beta : Z \rightarrow \mathbb{Z}$. If an isomorphism
of the graphs is given, $\phi : G_1 \rightarrow G_2$, then it induces a pair of maps $\phi^*(\alpha_1), \phi^*(\beta_1) : Z_2 \rightarrow \mathbb{Z}$ from the pair of maps $\alpha_1, \beta_1 : Z_1 \rightarrow \mathbb{Z}$.

**Lemma 1** Let $T_1$ and $T_2$ be boundaries of solid tori, $G_i \subset T_i$, $i = 1, 2$, be the embedded graphs. Suppose that the isomorphism of the graphs, $\phi : G_1 \rightarrow G_2$, is extended to a homeomorphism that preserves the orientation of the tori. Then it can be extended (probably by using another homeomorphism of the tori) to a homeomorphism of the solid tori if and only if $\phi^*(\alpha_1) = \alpha_2, \phi^*(\beta_1) = \beta_2$.

**Proof.** The necessity is obvious. To prove sufficiency, we construct homeomorphisms of the tori which map chosen cycles into the chosen cycles. There are three essentially different cases that are possible:

1) all cycles are homotopic to 0. This is equivalent to the graphs being homotopic to 0. Then, for every $i = 1, 2$, one of the components of $T_i \setminus G_i$ has genus 1. Fix a chosen cycle in each component. Then it is easy to construct a homeomorphism between these components which would map the chosen cycles into the chosen cycles (cutting a component along a chosen cycle we then need to extend the homeomorphism from the boundary of the sphere with holes to its interior). On other components of $T_i \setminus G_i$, we define the homeomorphisms that satisfy the conditions of the lemma. Since all these homeomorphisms coincide on the edges, they generate a homeomorphism of the torus mapping the chosen cycle into the chosen cycle. The existence of such a homeomorphism is a sufficient condition for existence of extensions of the homeomorphism inside a solid torus.

2) There exist cycles that are not homotopic to 0 but all of them are homotopic to each other. This means that there also exists a closed curve $\gamma$ that is homotopic to these cycles and not intersecting the graphs. The pair $\alpha, \beta$ is then defined by the number of points where $\gamma$ intersects the chosen cycle and the parallel. Since the image of a chosen cycle of $T_1$ and a chosen cycle on $T_2$ have the same intersection numbers under the homeomorphism of the tori, $h : T_1 \rightarrow T_2$, they differ from each other, up to a homotopy, in a
Dehn twist along $\gamma$. Since this Dehn twist is a homeomorphism leaving the graphs fixed, we have a homeomorphism of toruses mapping a chosen cycle into a chosen cycle. Hence, it can be extended to a homeomorphism of solid toruses.

3) There are two or more cycles that are not homotopic to each other and not homotopic to 0. It is sufficient to consider two such cycles and the corresponding numbers $\alpha_i$. Using this, the chosen cycle can be reconstructed uniquely up to a homotopy. Since the numbers $\alpha_i$ of the corresponding cycles are same, the homeomorphism of the tori maps the chosen cycle into a chosen cycle, which was to be shown.

**Remark.** It is not necessary to calculate the map $\beta$ in the third case.

There is a principal difference between the first two cases and the third one. It consists in extending the homeomorphism of tori to a homeomorphism of the solid in the third case as oppose to replacing the homeomorphism of the tori with another one in first two cases and then extending it to solid toruses, under the conditions of the lemma.

In the first case, the graphs divides the torus into parts, one of which has genus 1 (a torus with holes). Denote it by $L$. Then $L$ is simultaneously a part of the boundary of a round 0-handle and a part of the boundary of a round 2-handle (for others types of handles, their boundaries consist of strata of genus 0). For a meridian of a round 2-handle with respect to a round 0-handle, define the pair of numbers $(\alpha, \beta)$ as above. Then this pair is an invariant of the flow. For convenience, we assume that $\alpha = 0$ and $\beta = 0$ for other types of intersections of round 0- and 2-handles.

In the second case, it is possible to make a Dehn twist along $\gamma$ before extending the homeomorphism of the torus. In such a case, it is possible that $\gamma$ will not be homotopic to a 0-curve in the other torus, which means that the Dehn twist along it changes the homeomorphism of this torus, which is not desirable. Therefore, it is
necessary to introduce an additional invariant that would indicate this situation. If $\gamma$ is homotopic to 0 or to a chosen curve in the second torus, then the Dehn twist along it does not influence the possibility of extending the homeomorphism of the second torus. Hence, we will consider the situations in which the algebraic numbers of intersection points of $\gamma$ with both meridians are not equal to 0. We choose, on each of the two tori, a closed curve $\omega$ that intersects $\gamma$ in one point and such that, for the homotopy class of a meridian $m$, we have $[m] = k[\gamma] + l[\omega]$, where $0 < k < |l|$. Then the condition that the meridians are homotopic can be replaced by the condition that the curves $\omega$ are homotopic and the pairs of numbers $(k,l)$ are equal. Note that $m = \omega$ for round 1-handles.

If there are two homotopic curves $\gamma$ and $\gamma'$ in different components of $T\setminus G$ on the boundary of the round handle, then simultaneous Dehn twists along $\gamma$ and along $\gamma'$ with opposite orientations does not change the homotopy class of $m$ and $\omega$. Furthermore, the algebraic numbers of intersection points of $\omega$ with $\omega$ in other toruses are changed, but the sum of these numbers remains constant. Consider the graph $L$ whose vertexes correspond to the usual and round handles containing domains $D$ that are homeomorphic to rings in case 2. The edges of the graph correspond to these domains. The ordering of handles defines a natural orientation of the edges of $L$. To each edge of $L$, we put into the correspondence a number $\mu$ that equals the algebraic number of intersection points of the curves $\omega$ in the corresponding domain, or $\infty$, if the curve $\gamma$ is homotopic to 0 in the boundary of one of the two handles to which it belongs. Here, we take the curve $\omega$ to be the first if the order of its handle is less. The operation of simultaneous twisting on the curves $\gamma$ and $\gamma'$ generate an equivalence ration on the set of numbers $\mu$, which will be considered in more detail in the following section.

We fix one point by one component of $\partial D$. We demand that curves $\omega$ intersect $\partial D$ only in these points. For two curves $\omega_1$ and $\omega_2$ that intersect $\partial D$ we use its isotopy to curves with such property: if we chance parallel orientations on components of $\partial D$ and ori-
6 Equivalence of framed graphs

In this section, we consider oriented graphs in which one of the ends of each edge is a source or a sink and we call them MS-graphs. Except for the situation described above about the graph $L$, such graphs are a main invariant of a Morse–Smale flow without closed orbits on closed surfaces. Their vertexes are fixed points and their edges are separatrices. Vertexes of the MS-graph, which are neither sources nor sinks, are called saddles. In the graph $L$, each round 0-handle corresponds to a source of $L$ and round 2-handles correspond to a sink, and each saddle of $L$ corresponds to a round 1-handle (a round 1-handle can also correspond to a source or a sink of $L$).

An oriented graph are called framed, if each edge is put into a correspondence with an integer number or $\infty$. This number or $\infty$ are called a framing of the edge and the set of them is called a framing of the graph. Thus, a framing of a graph is a map of the set of edges into $\mathbb{Z} \cup \{\infty\}$. A framed graph is similar to a graph with colored edges.

Two framings of an MS-graph are called equivalent, if one of them can be obtained from another by a sequence of the following operations:

1) Simultaneous addition of an integer $k$ to the framing of two edges of the graph, provided they have the property that the origin of one is an endpoint of the other and, hence, this point is a saddle. Edges with such property will be called sequential.

2) Addition of an integer $k$ to the framing of one edge and $-k$ to the framing of the incident edge, provided that these edges are not sequential.

**Lemma 2** If each of the two framings of an MS-graph contains edges with the framing $\infty$, then they are equivalent if and only if
their sets of edges with framing $\infty$ coincide.

**Proof.** Necessity. The operations 1) and 2) do not change the set of edges with framing $\infty$.

Sufficiency. Let us show that, under conditions of the lemma, any edge with a finite framing $n$ can be changed to any integer $k$ without changing the framings of the remaining edges. Indeed, in view of the connectedness of the graph, there is a path $\{e_1, \ldots, e_m\}$ that starts at this edge and ends at an edge with framing $\infty$. Applying subsequently the operations 1) or 2) with numbers $k$ and $-k$ to pairs of the edges $(e_1, e_2), (e_2, e_3), \ldots, (e_{m-1}, e_m)$ we obtain the necessity statement.

The main aim of this section is to establish criteria for an equivalence of framed MS-graphs. Assume that the graphs are connected and do not contain edges with framing $\infty$. We subdivide all MS-graphs into 3 types and formulate criteria for equivalence for each type.

1. The graphs of the first type are MS-graphs without saddles. They allow only the second operation that, as easily seen, does not change the sum of the framings of all the edges.

**Lemma 3** Two framings of the graph of the first type are equivalent if and only if they have the same sum of the framings of all edges.

**Proof.** The necessity is obvious.

Sufficiency. Fix an edge $e$. Similarly to Lemma 1, for any edge $e_1$, by choosing a path between this edge and the edge $e$, we can change the framing of $e_1$ by an arbitrary integer $n$ so that the framing of $e$ will be changed by $-n$ with framings of the other edges preserved. Then using operation 2) for the first framing graph we successively change the framing of each edge to the necessary value modifying the value of the frame of the edge $e$ by the used difference. As a result, we have that it is possible to obtain, from the first framing, a framing that coincides with second one except for, probably, the
edge $e$. Since the sum of the framings of the edges are the same, these framings also coincide on $e$.

**Corollary 1** A framing of a graph of the first type is equivalent to a framing of a graph in which all edges have framing 0 save for a fixed edge $e$.

2. MS-graphs $G$ belong to the second type if there is a cycle (probably nonoriented) that has an odd number of saddles ($G$ is considered as an oriented graph). It is clear that the saddles of the cycle are saddles of the MS-graph (the inverse, generally, is not true). Existence of such a cycle is equivalent to existence of a self non-intersecting cycle that has an odd number of saddles.

**Lemma 4** Two framed graphs of the second type are equivalent if and only if the sums of their framings of all edges are equal modulo 2.

**Proof.** *Necessity.* Each of the operations 1) and 2) does not change the sum of framings of all edges modulo 2.

* Sufficiency. Let $e$ be an edge of a self non-intersected cycle $\omega = \{e, e_1, \ldots, e_m\}$ with an odd number of saddles. As in the previous lemmas, by using operations 1) and 2), we can pass from the first framing to a framing that coincides with the second one on all edges except for the edge $e$, for which the difference of these framings is even. A subsequent modification of the framings by 1 or $-1$ applying admissible operations to the pairs of edges $\{e, e_1\}, \{e_1, e_2\}, \ldots, \{e_m, e\}$ changes the framing of the edge $e$ by 2 leaving the framings of other edges unchanged. Thus, we can make the framing of $e$ equal to the second framing of the graph.

3. The remaining MS-graphs are of the third type. We cut the graphs into saddles. Since there are no cycles of the second type, the components of the obtained graph can be subdivided into two groups in such a way that, after the slitting, any two subsequent
edges belong to components from distinct groups. One of groups will be called the first one, another one the second. The sum of framings of the edges in a group will be called the total framing of the group.

**Lemma 5** Two framings of the third type are equivalent if and only if they have the same differences of total framings of the first and second groups.

**Proof.** *Necessity.* If two adjacent edges belong to one group, then the operation 2) can be applied to them. This operation does not change the sum of the framings of the group. If the adjacent edges belong to different groups, then the operation 1) can be applied to them; this operation does not change the difference of the total framings of the groups.

*Sufficiency.* Similarly to the previous lemmas, we can use admissible operation to obtain, from the first framing, a framing that coincides with the second one, except for, possibly, one edge. The conditions of the lemma guarantee that the framings of this edge coincide.

In the case of disconnected graphs, the framings are equivalent if the framings are equivalent for each connected component.

### 7 The main theorem

In the previous section, we have constructed a set of invariants that indicate a possibility of extending homeomorphisms from the boundaries of round handles to their interior. We call this set a $\tau$-invariant. Thus, a $\tau$-invariant consists of the following data:

1) pairs of numbers $(\alpha, \beta)$ for each pair of round 0- and 2-handles that intersect on a surface of the genus 1 with the orientation of this surface as in the boundary of the round 0-handle;
2) pairs of numbers \((\alpha, \beta)\) for meridians of round handles of type 2, pairs \((k, l)\) for the curves \(\omega\), and the equivalence class of the graph \(L\);

3) the numbers \(\alpha_i\) for every non-self-intersecting cycle on the boundaries of the round handles of type 3.

**Theorem 2** Let \(X\) and \(X'\) be Morse–Smale flows on a 3-manifold \(M\). There is a topological equivalence of two Morse–Smale flows \(X\) and \(X'\) preserving the order of critical elements if and only if there is an isomorphism of their schemes preserving the \(\tau\)-invariant, chosen curves, and their orientations.

**Note.** As opposed to Theorem 1, this theorem is constructive, that is, it allows, in a finite time, to make a test on topological equivalence of two flows.

**Proof.** *Necessity.* It follows from the construction and the results of the previous paragraph.

* Sufficiency. As in the proof of Theorem 1, we extend the homeomorphism of the complexes \(K\), possibly fixing it, to a homeomorphism of the manifold \(K^3\) without neighborhoods of closed orbits of index 0 and 2. Let us show that, if the \(\tau\)-invariants are equivalent, then the homeomorphisms from the tori that are boundaries of neighborhoods of closed orbits of indices 0 and 2 can be extended inside.

In case 1 of the definition of the \(\tau\)-invariant, by using the map \(g\), the homeomorphism between the tori can be changed as in Theorem 1, since the images of the boundaries of the disks, which intersect the closed orbit transversally in a single point, are homotopic to curves with the same properties. Here we use the fact that the corresponding numbers (the \(\tau\) invariants) are equal. This means that these boundaries differ by Dehn twists along the line of intersection of the stable manifolds of the lower orbit and the torus. The Dehn twist is performed by modifying the homeomorphism of the diagram with a contraction or expansion that is isotopic to the intersection of unstable manifolds of this lower orbit and the surface
\[ \Phi. \text{ Thus, as in the cases 2 and 3, boundaries of disks are mapped into curves that are homotopic to boundaries of disks, hence they are also boundaries such disks.} \]

Using the first return map, extend the homeomorphisms of the tori to disks and then, along the trajectories, to the whole solid torus. This gives a needed homeomorphism between the manifolds.

8 The distinguishing graph of the flow

Now we construct an invariant of the diagrams, which is complete up to a (strict) isomorphism. It consists of a graph with additional information.

The graphs \( G \) is formed by the curves \( u_i \) and \( v_j \), the curves in the intersections \( \Phi \cap F \cap T_i \), and the curves that define the \( \tau \)-cycles. The vertexes of the graph \( G \) are intersections points of these curves; the edges are arcs between them. If a curve does not intersect other curves, then it forms a loop on the graph (one vertex and one edge with the ends in it). We denote the vertexes by \( A_i \) and the edges by \( B_i \). Fix an arbitrary orientation on the edges on which it is not given.

By cutting the complex \( K \) along the graph \( G \), we obtain a set of surfaces \( F_i \) with boundaries on which we fix orientations. For each component of the boundary, passing it along its orientation that agrees with the orientation of the surface \( F_i \), we write a word that consists of the letters \( B_j^{\pm 1} \) corresponding to the edges passed. The letter has power +1 if the orientation of the corresponding arc coincides with the orientation of the circle, and \(-1\) in the opposite case. Two words are called equivalent if one can be obtained from the other by a cyclic permutation of the letters. This corresponds to another choice for the beginning of the walk on the circle. The words are called inverse if one is obtained from the other by rewriting the letters in the inverse order reversing the powers and, possibly, making a cyclic permutation. This corresponds to a walk on the
circle in the direction opposite to the orientation.

For each surface $F_i$, we form a list that consists of the following:

1) the number $n_i$ that equals the genus of the surface $F_i$, if the surface is oriented, and $-n_i$, otherwise;

2) the words that are written when passing the boundary of the surface along the orientation.

Such two lists are called equivalent if the numbers $n_i$ are equal and there is a bijection between the words such that the corresponding words are equivalent or inverse to each other.

Thus, we have constructed a set of lists of words (SLW) in such a way that each list corresponds to one surface $F_i$. Two such sets are called equivalent, if there is a bijective correspondence between the lists such that the corresponding lists are equivalent.

Form a set of lists of words for each boundary of the round handle and, also, a subset of the incoming and outgoing sets for the round 1-handles.

We also make lists for the following curves:

1) the curves $u_i$ broken into pairs;

2) the curves $v_j$ broken into pairs;

3) the chosen cycles for the mean orbits;

4) the cycles and circles included in the $\tau$-invariant in case 2;

5) the curves included in the $\tau$-invariant in case 3.

For the $\tau$-invariant in case 1, we assign the corresponding number to the regions that lie on the torus.

A graph $G$ with SLW and lists 1) -5) is called a distinguishing graph of the flow.

Two distinguishing graphs are called equivalent if there is an isomorphism of the graphs mapping the SLW of first graph into an SLW that is equivalent to the SLW of the second graph and the lists 1) -5) into equivalent lists preserving the $\tau$-invariant.

**Theorem 3** Two Morse–Smale flows on a closed 3-manifold are topologically equivalent if and only if they have minimal diagrams whose distinguishing graphs are equivalent.
Proof. Taking into consideration the reasoning used in Section 7, it is necessary to prove that the equivalence of the distinguishing graphs is equivalent to stable equivalence of the stably minimal diagrams. The proof of this is similar to the proof of Theorem 1 in [8] with replacing the surface by the complex $K$.

9 Examples

Consider Morse–Smale flows with one closed 1-orbit and the minimal number of other critical elements on $S^3$.

First consider a flow with three closed orbits and no singularities. A closed 1-orbit can have a trivial or a twisted neighborhood. In the case of a trivial neighborhood, the round 1-handle is attached to a round 1-handle by two annuli such that the core of one of them is homotopic to 0 and another one is not. So the distinguishing graph consists of 4 loops, $a, b, c, d$. $SLW = \{L_1 = \{0, a, b^{-1}\}, L_2 = \{0, c, d^{-1}\}, L_3 = \{0, a^{-1}, b, c^{-1}\}, L_4 = \{0, d\}, L_5 = \{0, a, d^{-1}\}, L_6 = \{0, b, c^{-1}\}\}$. The boundary of the round handles are $T_0 = \{L_1, L_2, L_3, L_4\}$, $T_1 = \{\{L_1, L_2\}, \{L_5, L_6\}\}$, $T_2 = \{L_3, L_4, L_5, L_6\}$. The $\tau$-invariant is non-trivial for the intersection of $b$ with the meridians of the round 0- and 2-handles. The algebraic number of such intersections is equal to $\pm 1$. Hence, 4 cases are possible.

Thus, there are 4 different flows. One can be obtained from another by replacing the orientation of the closed 0- or 2-orbit and the flow in its neighborhood.

In the case of a twisted 1-orbit, the round 1-handles are regular neighborhoods of $2n + 1$ $\pi$-twisted bands in the complement of the round 0-handle. The distinguishing graph is the loop $a, b$. $SLW = \{L_1 = \{0, a, b^{-1}\}, L_2 = \{0, a, b^{-1}\}, L_3 = \{0, a, b^{-1}\}\}$. The algebraic number of the intersection of the meridian of the round 0-handles and $a$ is equal to $2n + 1$. It is a unique non-trivial part of the $\tau$-invariant. Hence, there are infinitely many such flows and they are given by $n$. The pairs of numbers for $\omega$ of round 0-, 1- and
2-handles are equal to \((1, 2n + 1), (1, 2)\) and \((0, 1)\), correspondingly. Total framing of \(L\) can be equal 0 or 1 mod 2. It depend of the orientations of the closed orbits.

If, in these examples, we replace each closed 0- and 2-orbit by two singularities, then we obtain a flow with infinitely many singular trajectories.

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