Negative Screenings in Conformal Field Theory and 2D Gravity: The Braiding Matrix

Jørgen Rasmussen

Department of Theoretical Physics, Tata Institute of Fundamental Research
Homi Bhabha Road, Colaba, Mumbai 400 005, India

and

Jens Schnittger

Laboratoire de Mathématiques et Physique Théorique
Université de Tours, Parc de Grandmont, F-37200 Tours, France

Abstract

We consider an extension of the Coulomb gas picture which is motivated by Liouville theory and contains negative powers of screening operators on the same footing as positive ones. The braiding problem for chiral vertex operators in this extended framework is analyzed. We propose explicit expressions for the $R$-matrix with general integer screening numbers, which are given in terms of $4F_3$ $q$-hypergeometric functions through natural analytic continuations of the well-known expression for positive integer screenings. These proposals are subsequently verified using a subset of the Moore-Seiberg equations that is obtained by simple manipulations in the operator approach. Interesting new relations for $q$-hypergeometric functions (particularly of type $4F_3$) arise on the way.

PACS: 11.25.Hf
Keywords: Liouville theory; conformal field theory

1e-mail address: jorgen@theory.tifr.res.in
2e-mail address: schnittg@celfi.phys.univ-tours.fr
1 Introduction

The Coulomb gas picture of rational conformal field theory, and its various generalizations, has amply proved to be a very efficient tool for the computation of conformal blocks, structure constants and braiding properties in these theories \cite{1,2,3,4}. Interestingly, Coulomb gas techniques have turned out to describe (certain sectors of) irrational conformal field theories as well, such as WZNW theory and Liouville/Toda theory. Within the Gervais-Neveu quantization of Liouville theory, an efficient formulation of screened vertex operators, avoiding completely the manipulation of contours, was introduced long ago in \cite{5,6,7}.

The historical development of the exploration of Liouville theory in the operator framework was in fact such that for a long time, only primary fields corresponding to the Kac table were considered, in close analogy to the situation for minimal models. The next step of generalization \cite{8,9} consisted in formulating observables with arbitrary conformal dimensions, which however involved only a non-negative integer number of screenings. It was pointed out in ref. \cite{10} that while those observables (the general Liouville exponentials) are formally described by an infinite sum over positive integer powers of screening operators, these infinite sums do not permit any naive evaluation even in the simple context of three-point functions. Depending on the three-point function considered, negative screening powers can arise as a non-perturbative effect, establishing contact in this way with the Goulian-Li procedure \cite{11}. The analysis of the three-point functions furthermore suggests that there should exist a conformal algebra, closed under fusion and braiding, involving chiral vertex operators with both positive and negative integer powers of screenings. This would constitute a non-trivial generalization of the standard Coulomb gas picture and open very interesting perspectives towards an operator formulation of a new class of irrational conformal field theories.

A great virtue of the Gervais-Neveu approach consists in the fact that negative integer powers of screenings are just as well defined as positive ones so that their introduction does not require any analytic continuation procedure. Negative and positive screening powers are in fact related by a Weyl reflection \cite{12,13,14}, exchanging one of the two equivalent free fields of the Gervais-Neveu approach with the other. However, except for the cases considered in \cite{8,9}, so far very little has been known about the braiding and fusion algebras of vertex operators involving negative powers of screenings. It is clear that the $R$-matrix and the fusion matrix should be given by an appropriate analytic continuation of $q$-deformed $6j$-symbols, as was the case already in the generalization from the Kac table to arbitrary continuous spins \cite{15}.

The objective of the present paper is to establish the braiding algebra of chiral vertex operators with arbitrary integer screenings and continuous spins. The operator product is then determined as well by the general proportionality relation of Moore and Seiberg \cite{14} between fusion and braiding matrices.

The remaining part of the paper is organized as follows:

In Section 2 we introduce our notation and provide some well known background material.

In Section 3 we present our proposal (ansatz) for the analytic continuation of the $R$-matrix to negative screening numbers. The continuation procedure is remarkably
simple, and uses no more than a well-known transformation formula for (truncating) $q$-hypergeometric sums of type $4F_3$, which constitute the essential part of the analytic expression for the braiding matrix. Depending on the signs of the screening numbers of the vertex operators to be braided, the standard braiding matrix for positive integer screenings of refs. [4, 5] can always be brought into a form such that the continuation corresponds to a simple substitution of positive by negative screening numbers.

In Section 4 we derive, as a simple consequence of the polynomial equations of Moore and Seiberg [4], a system of determining equations for $R$-matrices with a mix of both positive and negative “ingoing” screening numbers. We verify that our proposal fulfills these equations, starting with the case where only one of the vertex operators to be braided has negative screening. In the process we need several new relations for $q$-hypergeometric functions, in particular of the type $4F_3$ and generalizations thereof. Proofs are outlined and will appear in more detail elsewhere [15]. We also discuss the delicate issue of uniqueness of the solution.

In Section 5 we turn to the class of $R$-matrices where both ingoing screening numbers are negative. In order to verify our analytic continuation proposal for this case, we will exploit the fact that it can be reduced to the ones treated in Section 4 by means of a concatenation procedure. Substantial evidence in favour of our proposal is then provided by checking explicitly the resulting identity for several non-trivial classes of examples.

In Section 6 we discuss the connection of our extended $R$-matrices with $6j$-symbols.

Section 7 describes the trivial generalization from only one type of screening charge to the case where the conjugate screening charge is included.

Finally, Section 8 is devoted to concluding remarks and a speculative outlook. We discuss in particular the non-chiral case in the context of Liouville theory, including the strong coupling regime.

Appendix A contains some useful observations on certain transformations of $q$-hypergeometric functions of type $4F_3$, whereas Appendix B is devoted to further considerations on the uniqueness of our proposal for the braiding matrix.

2 Coulomb Gas Picture and Liouville Theory

We start by introducing our notation and by recalling some elementary facts about the Coulomb gas picture as used in refs. [3, 4, 7, 10]. As usual, Coulomb gas vertex operators are written as a product of a free field vertex operator and some power of screening operators. A particularity of the formulation chosen in the above references is that the integration contour for the screenings is fixed once and for all, and does not depend on the correlator in question. Concretely, for a vertex operator of spin $J$ and $U(1)$ charge $m$ we have

$$U_m^{(J)}(\sigma) \equiv V_{-J}^{(J)}(\sigma) S^{J+m}(\sigma) \tag{1}$$

with

$$V_{-J}^{(J)}(\sigma) = e^{\alpha_{-J}X}(\sigma), \quad S(\sigma) = e^{2i(\sigma+1)} \int_0^\sigma dx V_1^{(-1)}(x) + \int_\sigma^{2\pi} dx V_1^{(-1)}(x) \tag{2}$$
Here, $X(\sigma)$ is a canonical free field and $\alpha_-$ is the “semi-classical” screening charge, related to the central charge $c$ of the theory in the usual way:

$$c = 1 + \frac{12}{\alpha_-^2} \left( 1 + \frac{\alpha_-^2}{2} \right)^2$$

(3)

Note that $c$ is arbitrary continuous, and $c > 25$ for $\alpha_-$ real. In terms of the deformation parameter $h$ ($q = e^{ih}$ is the deformation parameter relevant for the quantum group interpretation [16, 17, 18]) we have

$$\alpha_- = \sqrt{\frac{2h}{\pi}}, \quad c = 1 + \frac{6\pi}{h} \left( 1 + \frac{h}{\pi} \right)^2$$

(4)

Moreover, $\omega$ is essentially $i$ times the momentum zero mode of the free field, and is taken to be real in the following (this corresponds to the elliptic sector of Liouville theory [19], and is also the choice appropriate for instance for the description of minimal models in this framework [20]). Explicitly, one has

$$X(\sigma) = q_0 + p_0 \sigma + \sum_{n \neq 0} e^{-in\sigma} \frac{p_n}{n}$$

(5)

and $\omega = ip_0\sqrt{\frac{2\pi}{h}}$. The $U_m^{(J)}$ operators are characterized by their conformal weight $\Delta_J$, their $U(1)$ charge (momentum shift) $m$, and their normalization $I_m^{(J)}(\omega) \equiv \langle \omega|U_m^{(J)}(\sigma = 0)|\omega + 2m \rangle$:

$$\Delta_J = -J - \frac{h}{\pi} J(J+1)$$

$$U_m^{(J)} \omega = (\omega + 2m)U_m^{(J)}$$

$$I_m^{(J)}(\omega) \equiv \left( 2\pi\Gamma(1 + \frac{h}{\pi}) \right)^{J+m} e^{ih(J+m)(\omega - J + m)}$$

$$\cdot \prod_{\ell=1}^{J+m} \frac{\Gamma[1 + (2J - \ell + 1)h/\pi]}{\Gamma[1 + \ell h/\pi] \Gamma[1 - (\omega + 2m - \ell)h/\pi] \Gamma[1 + (\omega + \ell)h/\pi]}$$

(6)

The braiding properties and the operator product of the $U_m^{(J)}$ operators were derived in ref. [21] in the case of degenerate operators ($2J = 0, 1, 2, \ldots$ and $m = -J, -J+1, \ldots, J$) and generalized in refs. [7, 9] to the case of arbitrary $J$ and integer positive screening numbers $n = J + m$. Their algebra fulfills the Moore-Seiberg equations [14] of conformal field theory [4].

For later use, we introduce a special notation for the leading order operator product (fusion) of the two operators $U_m^{(J)}$ and $U_{m'}^{(J')}$:

$$U_m^{(J)}(\sigma) \otimes U_{m'}^{(J')}(\sigma) := \lim_{\sigma' \to \sigma} \frac{U_m^{(J)}(\sigma) \cdot U_{m'}^{(J')}(\sigma')}{(1 - e^{i(\sigma' - \sigma)})^{-\Delta_J - \Delta_{J'} + \Delta_{J + J'}}}$$

(7)

(as usual, $\sigma'$ has to be given a small positive imaginary part to make the above expression well defined). One has the simple multiplication law

$$U_m^{(J)}(\sigma) \otimes U_{m'}^{(J')}(\sigma) = q^{2J'(J+m)}U_{m+m'}^{(J+J')}(\sigma)$$

(8)
for any (positive or negative) integer values of the screenings.

Local observables (in the Liouville context they are the Liouville exponentials) can be constructed as bilinear combinations of the $U_m^{(J)}$ and their right moving counterparts $\overline{U}_m^{(J)}$, with $\varpi$-dependent coefficients $[3, 24, 7]$. Locality is a simple consequence of the orthogonality relations obeyed by the $q$-deformed $6j$-symbols that constitute the essential part of the chiral braiding matrices. In the present paper, we shall concentrate on the algebra of the chiral vertex operators; the application of the extended formalism with negative screenings to the construction of local observables, as considered in [10], is left for a future publication.

A crucial feature of the Gervais-Neveu approach is the introduction of a second free field, related to the one above by a quantum canonical transformation $[4]$. As pointed out in ref. [10], this is of direct relevance for the problem of constructing negative (integer) powers of screening operators: Negative powers of screening operators in terms of $X$ are nothing else than positive powers of screening operators in terms of $\tilde{X}$, the second free field. Therefore, there is no need of any analytic continuation procedure to define negative screenings, and the corresponding ambiguities are absent from the start. Concretely, we have (cf. ref. [10])

$$S^{-1}(\sigma) = \tilde{S}(\sigma) \frac{1}{I_{\frac{1}{2}}(\varpi + 1)I_{\frac{1}{2}}(-\varpi - 1)}$$

where $\tilde{S}$ is the screening operator constructed from $\tilde{X}$.

We remark here that, $X$ and $\tilde{X}$ being completely equivalent, the replacement of one by the other must not change any physical observable. From the quantum group point of view, this means that observables are invariant not only under infinitesimal $U_q(sl(2))$ transformations, but also under Weyl reflections $[12, 13]$. As discussed in [10], for Liouville exponentials outside the Kac table ($2J \neq 0, 1, 2, ...$), this is a highly non-trivial condition which is in conflict with naive charge conservation rules familiar from the standard Coulomb gas picture. It serves, in fact, as an important guideline for the correct non-perturbative evaluation of matrix elements of the Liouville exponentials as constructed within the Gervais-Neveu approach.

### 3 The Braiding Matrix

The braiding matrix $R$ describes the braiding of the two chiral fields $U_m^{(J)}(\sigma)$ and $U_{m'}^{(J')}(\sigma')$

$$U_m^{(J)}(\sigma)U_{m'}^{(J')}(\sigma') = \sum_{n_1, n_2} R(J, J'; \varpi; n_2, n_1) U_{m_2}^{(J')}(\sigma')U_{m_1}^{(J)}(\sigma)$$

The ordering of $\sigma$ and $\sigma'$ is implicit in this definition of $R$ and we shall deal with the case $0 < \sigma < \sigma' < 2\pi$ explicitly, reserving the notation $\overline{R}$ for the opposite ordering of $\sigma$ and $\sigma'$. The sums extend over the integers

$$n_1 = J + m_1, n_2 = J' + m_2$$

the ranges of which we shall discuss below. The parameters are subject to the condition

$$m_1 + m_2 = m + m'$$
so that there is really just one sum in Eq. \((10)\). In general, the combinations

\[ n = J + m \quad , \quad n' = J' + m' \]

(13)

are related to the screening numbers and have mainly been treated as non-negative integers in the literature. In particular, the braiding matrix for non-negative integers \( n \) and \( n' \) and arbitrary spins is known to be \([7]\)

\[ R(J, J'; \varpi)_{n,n'}^{n_1,n_1} = \exp \{-i\pi(\Delta_x + \Delta_{x+m+m'} - \Delta_{x+m2} - \Delta_{x+m})\} q^{2nJ' - 2nJ} \]

\[ \cdot \left[ 2x - 2J' + 2n_2 + 1 \right]_{n_1} \left[ 2x - 2J - 2J' + n + n_2 + 1 \right]_{n'} \left[ 2J' - n_2 + 1 \right]_{n-n_1} \]

(14)

\[ \cdot 4F_3 \left( -J + n, -J' + n_2, -n_1, -n' \right) \]

\[ -2x - n - n' - 1, n - n_1 + 1, 2x - 2J - 2J' + n + n_2 + 1 \quad ; \quad q, 1 \]

\[ = \exp \{-i\pi(\Delta_x + \Delta_{x+m+m'} - \Delta_{x+m2} - \Delta_{x+m})\} q^{2nJ' - 2nJ} \]

\[ \cdot \left[ 2x - 2J' + 2n_2 + 1 \right]_{n_1} \left[ 2x - 2J - 2J' + n + n_2 + 1 \right]_{n'} \left[ 2J' - n_2 + 1 \right]_{n-n_1} \]

\[ \cdot \sum_{l=0}^{\infty} \frac{(-2J + n)_{l+1}(-2J' + n_2)_{l}(-n_1)_{l}(-n')_{l}}{l! (2x - 2J - 2J' + n + n_2 + 1)_{l}} \]

with \( x \) defined by \( \varpi = 2x + 1 + \frac{\pi}{h} \). Here, the standard notation for the \( q \)-hypergeometric function has been used

\[ 4F_3 \left( \frac{a}{e}, \frac{b}{f}, \frac{c}{g}, \frac{d}{h} ; \frac{q}{\rho} \right) = \sum_{n=\infty} |a|_{n} b_{n} c_{n} d_{n} |e|_{n} f_{n} g_{n} h_{n} \rho^{n} \]

\[ |a|_{0} = 1 \quad , \quad |a|_{n} = |a| |a+1| \ldots |a+n-1| \]

(15)

It should be noted that in the standard case (\( n \) and \( n' \) non-negative integers) the summation in \((14)\) truncates after a finite number of terms. We shall use the convention \( |y| = \sin(yh)/\sin(h) \).

A \( q \)-hypergeometric function \((15)\) is balanced if \( g = a + b + c + d - e - f + 1 \) and Saalschutzian if furthermore \( d \) (or equivalently \( a, b \) or \( c \)) is a non-positive integer. A \( q \)-deformed Saalschutzian \( 4F_3 \) hypergeometric function satisfies the standard transformation (ST) rule \([8]\)

\[ 4F_3 \left( \frac{a}{e}, \frac{b}{f}, \frac{c}{g}, \frac{d}{h} ; \frac{q}{\rho} \right) = \frac{[f - c]_{-d} [e + f - a - b]_{-d}}{[f]_{-d} [e + f - a - b - c]_{-d}} \cdot 4F_3 \left( \frac{e - a}{e}, \frac{e - b}{e}, \frac{c}{f}, \frac{d}{h} ; \frac{q}{\rho} \right) \]

(16)

The present work is devoted to a discussion of the braiding matrix for the case where the ingoing screening numbers are arbitrary integers. We will assume that the outgoing screening numbers are also integers - see below for further comments on this point.
As the braiding matrix for non-negative integer screenings Eq. (14) is expressed in terms of a $q$-hypergeometric function, the most naive procedure for an extension to negative screenings would be to simply replace one or both of the ingoing screening numbers $n, n'$ by negative integers and simultaneously allow the outgoing screening numbers to be arbitrary integers. However, this would in general lead to $q$-hypergeometric sums that do not truncate, and such sums may well diverge for $|q| = 1$ since the denominators of individual terms cannot be bounded away from zero. The remedy is given by employing ST (14) before continuing the ingoing screening numbers after which a continuation results in a finite and well defined expression, as will be discussed below. We thus arrive at an explicit proposal for the $R$-matrix, which will subsequently be verified.

3.1 Analytic Continuation

From a mathematical point of view, the extension of the explicit expression (14) to arbitrary integer screenings is certainly not unique as we are trying to analytically continue a function given only on a discrete set of points. Let us make a division into subcases characterized by the signs of the ingoing screening numbers and treat them one by one.

The expressions for the $R$-matrices we obtain should of course only be considered as an ansatz. The evidence for their validity will be given in the following sections.

3.1.1 I: Positive-Positive Case, $n, n' \geq 0$

It is well known [7] that the set of chiral vertex operators corresponding to non-negative screening numbers closes under braiding, ensuring that both of the outgoing operators have non-negative screening numbers. It is easily verified algebraically that precisely then is the $R$-matrix (14) non-vanishing and has no poles in the limits where $2J$ or $2J'$ becomes an integer\(^{1}\). The finite summation range for the truncating $q$-hypergeometric function is given explicitly by

$$\max(0, n_1 - n) \leq l \leq \min(n_1, n')$$

and $n_1, n_2 \geq 0$. This analysis suggests that a more natural expression exists for $n_1 > n$, which allows to avoid cancellations of the form $[-1]/[-1]!$. Let us introduce the notation $R^I >$ and $R^I <$ for the type $I$ $R$-matrix ($n, n' \geq 0$) in the cases $n_1 > n$ and $n_1 \leq n$, respectively. By construction, $R^I <$ is given by (14) whereas $R^I >$ is naturally represented as

$$R^I > (J, J'; \varnothing)^{n_2, n_1}_{n, n'} = \exp\{-i\pi(\Delta_x + \Delta_{x+m+m'} - \Delta_{x+m_2} - \Delta_{x+m})\} q^{2nJ'-2n_2J} \cdot \left[\frac{2x - 2J' + n_2 + 1}{n_1 - n}\right]^{n' \atop n_1 - n} \cdot \left[\frac{2x + n_2 + 2}{n_1 - n}\right]^{n + n' + 1\atop n_1 - n} \cdot \left[\frac{2x - 2J' + n_2 + 1}{n_1 - n}\right]^{n_2 - 2J + n_1\atop n_1 - n} \cdot \binom{4F_{3}}{-2J + n + n' - n_2, -2J' + n', -n, -n_2}{2x - 2J - 2J' + n + n' + 1, -2x - n - n_2 - 1, n' - n_2 + 1; q, 1}$$

\(^{1}\)Such an analysis is similar in spirit to the discussion of fusion rules in ref. [4].
In the final case where both \( n \) and \( q \) follow, we shall initially focus on the non-negativity of a screening number is preserved under any braid ing; see also Section 4.

Accumulating the information from cases I and II, in complete analogy with the analysis of case II: Negative-Positive Case, \( n < 0 \leq n' \)

Inspection of Eqs. (14) and (18) shows that \( R_{II}^l \) may be represented by exactly the same mathematical expressions as \( R^l \) when in the latter \( n \) has been continued to negative integers:

\[
R_{II}^l = R^l_{\leq} \quad , \quad R_{II}^l = R^l_{>}
\]

For the second equality to make sense, it is crucial that Eq. (18) vanishes term by term for \( n < 0 \) due to the binomial prefactor. \( R_{II}^l \) also vanishes in this case. An alternative way to obtain Eq. (19) is the following: In order to have a well defined and non-vanishing \( R \)-matrix for \( n < 0 \leq n' \) and \( 2J \) and \( 2J' \) non-integer, we find that the summation variable \( l \) in Eq. (14) must satisfy

\[
0 \leq l \leq \min(n_1 - n - 1, n') \quad \text{or} \quad \max(0, n_1 - n) \leq l \leq n'
\]

It should be noted that there is no overlap between these two regimes. The first regime is excluded by our second demand that the \( R \)-matrix must be well defined in the limit where \( 2J' \) is an integer (analyzing the limit where \( 2J \) is an integer does not provide new information). This leaves us with

\[
\max(0, n_1 - n) \leq l \leq n' \quad \rightarrow \quad 0 \leq n_2
\]

The last inequality expresses a selection rule preserving the non-negativity of the screening number under braiding (from the left) with a negative screening vertex operator.

III: Positive-Negative Case, \( n' < 0 \leq n \)

In complete analogy with the analysis of case II we find that the type III \( R \)-matrix may be represented by the following single surviving range for the summation variable \( l \) (14)

\[
\max(0, n_1 - n) \leq l \leq n_1 \quad \rightarrow \quad 0 \leq n_1
\]

Accumulating the information from cases I, II and III, we observe that the property of non-negativity of a screening number is preserved under any braiding; see also Section 4.

IV: Negative-Negative Case, \( n, n' < 0 \)

In the final case where both \( n \) and \( n' \) are negative integers our construction goes as follows. We shall initially focus on the \( q \)-hypergeometric part of \( R^l \) which will be denoted \( I \). Employing ST once results in the rewriting

\[
I = _4F_3\left(\begin{array}{c}
-2J + n, & -2J' + n_2, & -n - n' + n_2, & -n' \\
-2x - n - n' - 1, & -n' + n_2 + 1, & 2x - 2J - 2J' + n + n_2 + 1 + q, 1
\end{array}\right)
\]

\[
= \frac{(-2x + 2J' - n - n' - n_2 - 1)_{n'}(-2x + 2J - n - n')_{n'}}{(-2x - n - n' - 1)_{n'}(-2x + 2J + 2J' - n - n' - n_2)_{n'}}
\]

\[
\cdot _4F_3\left(\begin{array}{c}
2J - n - n' + n_2 + 1, & n + 1, & -2J' + n_2, & -n' \\
-n' + n_2 + 1, & -2x + 2J - n - n', & 2x - 2J' + n + n_2 + 2 + q, 1
\end{array}\right)
\]
It should be stressed that a possible pole or zero in $I$ which is matched by an appropriate zero or pole, respectively, in the prefactors of $R^I$, must also be present in the right hand side. In this sense, the two expressions are completely equivalent. Now, the right hand side is also well defined after the substitution $n_i \rightarrow -n_i -1$. This is the core of our analytic continuation and we define the object $I_{-n^{-1}_{2},-n_{-1}^{-1}}$ by the right hand side in (23) after the substitution. The full type IV $R$-matrix is obtained by multiplying $I_{-n^{-1}_{2},-n_{-1}^{-1}}$ by the original prefactors in $R^I$ in which the substitution $n_i \rightarrow -n_i -1$ has been performed. The shift by $-1$ in the substitution $n_i \rightarrow -n_i -1$ is convenient because $R^{IV}$ is defined only for purely negative ingoing screening numbers. Nevertheless, let us state our proposal in the form

$$R^{IV}(J, J'; \omega)^{-n_{2},-n_{1}}_{-n_{2},-n_{1}} = \exp \{-i\pi(\Delta_x + \Delta_{x+m+m'} - \Delta_{x+m} - \Delta_{x+m'})\} q^{-2nJ+2nJ'}$$

$$\cdot \left[2x - 2J' - 2n_2 + 1\right] \left(\begin{array}{c} -n \\ -n_1 \end{array}\right)$$

$$\cdot \frac{\left[2x - n_2 + 2\right]_{n_2-1} \left[2x - 2J - 2J' - 2n - n' - n_2 + 2\right]_{n_1} \left[2J' + n_2 + 1\right]_{n'-n_2}}{\left[2x - 2J - 2n - n' + 2\right]_{n+n'-1}}$$

$$\cdot {}_4F_3 \left(-2J - n, -2J' - n_2, 1 - n_2, 1 - n, -2x, n' - n_2 + 1, 2x - 2J - 2J' - 2n - n' - n_2 + 2 ; q, 1\right)$$

(24)

where an additional ST has been employed. Here $n$ and $n'$ are positive integers and $-n_i = J_i + m_i$.

In conclusion, our two analytic continuation procedures result in the same expressions for the braiding matrices. In case IV, though, only one approach seems to apply directly. In [15] we shall return to the question of infinite sum representations of $q$-hypergeometric functions and braiding matrices.

We have seen that all four types may be constructed using well defined concatenations of ST followed by trivial continuations of one or two of the ingoing screening numbers. Thus, the approach based on repeated use of ST demonstrates the universality of the mathematical expression for the original braiding matrix. Of course, this statement pertains to our proposal only. In the following sections we shall argue for the validity of this proposal.

### 3.2 Weyl Reflection Symmetry

In this section we shall perform a first test on our proposal by comparing it with the braiding matrices obtainable by Weyl reflection symmetry from the well known type I. Since Weyl reflections act on the screening numbers as $n_i \rightarrow 2J_i - n_i$, subclasses of all four types are reachable, depending on the signs of $2J - n$ and $2J' - n'$. As we are not addressing the question of non-integer screening numbers we shall restrict ourselves to the case $2J, 2J' \in \mathbb{Z}$.

The explicit relation between $\tilde{S}^n$ and $S^{-n}$ is

$$\tilde{S}^n = S^{-n}K_n(\omega)$$

$$K_n(\omega) = K_1(\omega + 2(n - 1))K_1(\omega + 2(n - 2))...K_1(\omega)$$

(25)
where \( K_1(\omega) \) is given by the normalization factor in the right hand side of Eq. (3)

\[
K_1(\omega) = I_{1/2}^{(4)}(\omega + 1)I_{1/2}^{(4)}(-\omega - 1)
\]  

(26)

Weyl reflection dictates that

\[
R(J, J'; \omega)^{-k_2, -k_1}_{-n, -n'} = R(J, J'; -\omega)^{k_3, k_1}_{n, n'} I_J^{(j)}(\omega) I_{J'}^{(j')}(-\omega - 2n) \frac{I_J^{(j)}(\omega) I_{J'}^{(j')}(-\omega - 2n - 2k_2)}{I_J^{(j)}(\omega) I_{J'}^{(j')}(-\omega - 2n - 2k_2)}
\]

(27)

and one may show that this reduces to

\[
q^{2nJ^1 - 2n_2^1} R(J, J'; -\omega)^{2J^1 - n_2, 2J^1 - n_1}_{2J - n, 2J' - n'} = q^{-2n^1J^1 + 2n_1^1} R(J, J'; \omega)^{n_2, n_1}_{n, n'}
\]

(28)

This relation may be seen as boundary conditions on our proposal imposed by the reflection symmetry.

Due to the universality of the mathematical structure of the braiding matrix, it is almost straightforward to verify that our proposal satisfies (28). In principle, one should distinguish between the four resulting types of braiding matrices. However, we know from Appendix A that Weyl reflections for \( 2J, 2J' \in \mathbb{Z} \) and (concatenations of) the STs we employ commute so it is sufficient to consider only one type. In the representation of type I we have

\[
q^{2nJ - 2n_2^1} R^I(J, J'; -\omega)^{2J - n_2, 2J - n_1}_{2J - n, 2J' - n'} = \exp \{-i\pi(\Delta_x + \Delta_{x+m+n'} - \Delta_{x+m_2} - \Delta_{x+m})\} \left[2x - 2J + 2n_2 + 1\right] \left(\frac{2J - n}{2J - n_1}\right) \frac{\left[2x - 2J - 2J' + n + n' + 1\right]_{2J - n_1} \left[2x - 2J' + n + n' + n_2 + 2\right]_{2J' - n'} \left[n + 1\right]_{n - n_2}}{\left[2x - 2J - 2J' + n + n' + n_2 + 1\right]_{2J + 2J' - n - n' + 1}} F_3(-2J + n_1, -2J' + n', -n_2, -n, -2x - n - n_2 - 1, n' - n_2 + 1, 2x - 2J - 2J' + n + n' + 1; q, 1)
\]

where the last equality is due to a concatenation of STs.

### 4 Double Braiding Relations for the Type II and III \( R \)-matrices

In this section we shall set up simple relations for the \( R \)-matrices using one of the polynomial equations of Moore and Seiberg [14], namely the commutativity of fusion and braiding. The Moore-Seiberg equations were originally set up within the context of rational conformal field theory, but can be viewed as a set of consistency relations for conformal field theory in general. For the present operator algebra, which extends the standard
Coulomb gas in a rather non-trivial fashion, their validity may not seem self-evident - especially since both fusion and braiding matrices involve an infinite number of conformal blocks. We will take the commutativity of fusion and braiding as an axiomatic starting point of our analysis, though in principle our explicit operatorial construction should allow to discuss its validity. In any such discussion, manipulations with infinite sums over conformal operators will necessarily arise and one may speculate that the Moore-Seiberg equations will act as a defining principle for their treatment; clearly, further analysis will be necessary to elucidate this point.

There are several possibilities for choosing an appropriate equation system to study. It turns out that a convenient choice is the recursive system to be analyzed in the following.

### 4.1 Recursive Equation Systems

Let us first consider the product

\[ U^{(-J_1)}_{n_1}(\sigma) \odot U^{(J_2)}_{m_2}(\sigma) U^{(J_3)}_{m_3}(\sigma') \]  

(cf. Eq. (30)) and braid through from the right the operator \( U^{(J_3)}_{m_3}(\sigma') \). Demanding that the above expression can be evaluated either by braiding first and then fusing the operators at \( \sigma \), or vice versa, allows us to derive the double braiding relation

\[ \sum_{l=0}^{n_2+n_3} q^{2J_2(n_1-k)} R(J_2, J_3; \varpi - 2m_1)^{n_2+n_3-l,l} R(-J_1, J_3; \varpi)^{-n_1+n_2+n_3-l,k,-k} \]

\[ \cdot U^{(J_2)}_{m_2-n_1-1} U^{(-J_1+J_2)}_{n_3-n_2-1-l} U^{(-J_2-J_1)}_{1} U^{(J_3)}_{m_3-n_1} U^{(-J_3+J_2)}_{n_2-n_1-1} U^{(J_3)}_{m_3-n_1} U^{(-J_3+J_2)}_{n_2-n_1-1} U^{(J_3)}_{m_3-n_1} \]

\[ = \sum_i R(-J_1 + J_2, J_3; \varpi)^{-n_1+n_2+n_3-i,i} U^{(J_2)}_{n_2-n_1-1} U^{(-J_3+J_2)}_{n_2-n_1-1} U^{(J_3)}_{m_3-n_1} U^{(-J_3+J_2)}_{n_2-n_1-1} U^{(J_3)}_{m_3-n_1} \]  

\[ (30) \]  

\[ (31) \]

\( n_1 \) is a positive integer while \( n_2 \) and \( n_3 \) are non-negative ones. The sums over \( k \) and \( i \) could in principle contain non-integer values or even an integration, since they are related to outgoing screening numbers about which we have no a priori knowledge as soon as we leave the safe grounds of the standard Coulomb gas with positive screenings only. In Appendix B, we present an argument involving degenerate field techniques which suggests that non-integer values may actually occur, with the corresponding \( R \)-matrix elements decoupling from those for integer outgoing screenings. This is quite puzzling, as one would expect our explicit operator construction to select one particular \( R \)-matrix, removing the ambiguity. However, braiding problems with negative ingoing screenings involve infinite sums over products of outgoing vertex operators, the evaluation of which is expected to be delicate [10]; in particular, several equivalent representations with numerically different \( R \)-matrices may exist, as is suggested by the argument in Appendix B. These questions, though clearly important, go beyond the scope of this first analysis of the braiding problem with negative screenings. We will therefore discuss here only the \( R \)-matrix elements for integer outgoing screenings. We consider the case \( n_3 = 0 \) in Eq. (31), for which we may deduce the equation system

#### Triangular Equation System

\[ \sum_{l=0}^{n_2} q^{2J_2(n_1+l-1)} R(J_2, J_3; \varpi - 2m_1)^{n_2-l,l} R(-J_1, J_3; \varpi)^{-n_1+n_2-L-l} \]
by comparison of coefficients of like operators. For integer outgoing screenings, \( L \) is integer. Note that in any case there is no coupling between matrix elements for integer and non-integer outgoing screenings. Due to the “triangular” structure of this equation system, all braiding matrices of type \( \mathbf{II} \) are determined recursively in terms of \( R \)-matrices of type \( \mathbf{I} \) and the subclass of type \( \mathbf{II} \) consisting of \( R \)-matrices of the form \( R(J, J'; \varpi)^{n_2, -n_1}_{-n_0, 0} \). However, the latter may be determined by an additional and independent equation system and a selection rule. Consider the product

\[
U_{n/2}^{(n/2)}(\sigma) \odot S^{-n}(\sigma)V_{-J'}^{(J')}(\sigma')
\]

and braid through from the right the operator \( V_{-J'}^{(J')}(\sigma') \). One obtains the double braiding relation

\[
\sum_{n_1, n_1'} R(0, J'; \varpi + n)_{-n_0, 0}^{n_1, -n-n_1} R(n/2, J'; \varpi)_{n, n_1-n-n_1'}^{n_1-n-n_1} V_{-J'}^{(J')}(\sigma') S_{n_1-n-n_1'}^{n_1-n-n_1'}(\sigma') V_{-n/2}^{(n/2)}(\sigma) = e^{i\hbar n J'} V_{-J'}^{(J')}(\sigma') V_{-n/2}^{(n/2)}(\sigma)
\]

which we shall denote the “initial” one. This equation system fixes the remaining indeterminacy, if we impose the additional selection rule

**Initial Equation System**

\[
e^{-i\hbar n J'} \sum_{l=0}^{n} R(n/2, J'; \varpi)_{-n, n}^{K,l} R(0, J'; \varpi + n)_{-n, -n-K}^{K+l-n-l, -n-l-K} = \delta_{K,0}
\]

**Selection Rule**

\[
R(J_1, J_2; \varpi)_{-n, n}^{n_2, -n-n_2} = 0 \quad \text{for} \quad n > 0, \quad n' \geq 0, \quad n_2 < 0
\]

\[
R(J_1, J_2; \varpi)_{-n, n}^{n-n_1, n_1} = 0 \quad \text{for} \quad n \geq 0, \quad n' > 0, \quad n_1 < 0
\]

All screening numbers are taken to be integers. The selection rule means that positive screenings always remain positive screenings after braiding. This rule is certainly fulfilled by our analytic continuation of the \( R \)-matrix as already discussed, but we do not have, at present, an a priori derivation. Its validity can be verified in the case where one of the ingoing vertex operators is degenerate, cf. Appendix B. It is an interesting question whether it can be deduced in general by invoking other polynomial equations, or by using more information about the explicit structure of the negative screening vertex operators.

If we consider the case of integer \( K \) and take into account the selection rule, we can rewrite Eq. (35) as

\[
e^{-i\hbar n J'} \sum_{l=\max(0, K-n)}^{K} R(n/2, J'; \varpi)_{-n, n}^{K,n+n-l-K} R(0, J'; \varpi + n)_{-n, 0}^{l-n-l} = \delta_{K,0}
\]
This equation system now determines recursively all remaining unknowns, if we consider successively \( K = 0, 1, 2, \ldots \). We will now outline the proof that our explicit proposal is the unique solution of Triangular and Initial equation systems plus selection rule, if only integer outgoing screenings are admitted. The argument is based on certain new identities for \( q \)-hypergeometric functions and generalizations thereof. Detailed proofs of these identities will be presented elsewhere [15].

### 4.2 Verification of Proposal

Here we shall verify that the analytically continued expression (14) subject to (21) provides a solution to the two double braiding relations discussed above.

We first consider the triangular equation system (32). Insertion of our proposal yields the equation system

\[
\sum_{l=0}^{n_2} [2x + 2J_1 - 2J_3 - 2n_1 + 2n_2 - 2l + 1] \left( \frac{n_2}{l} \right) \tilde{R}(-J_1, J_3; \varpi)^{-n_1+n_2-L,-l+L}_{-n_1,n_2-l} \\
\cdot \frac{[2x + 2J_1 - 2n_1 + n_2 - l + 2]_l [2J_3 - n_2 + l + 1]_{n_2-l} [2x + 2J_1 - 2J_3 - 2n_1 + n_2 - l + 1]_{n_2+1}}{[2x - 2J_3 - n_1 + n_2 - L + 1]_{n_1+n_2+1}}
\]

(38)

where

\[
\tilde{R}(-J_1, J_3; \varpi)^{-n_1+n_2-L,-l+L}_{-n_1,n_2-l} = \frac{\Theta(-n_1 + n_2 - L \geq 0) [2x - 2J_3 - 2n_1 + 2n_2 - 2L + 1](-1)^L \frac{[n_1 - n_2]_L}{[L]!} [2x - n_1 + n_2 - L + 1]_{n_1+n_2+1}}{[2x - 2J_3 - n_1 + n_2 - L + 1]_{n_1+n_2+1}}
\]

(39)

\( \tilde{R} \) is merely the \( R \)-matrix (14) subject to (21) without the explicit phases and powers of \( q \) which cancel out their analogues in the other \( R \)-matrices. Our proof will be by induction in \( n_2 \) and it is recalled that \( n_1 \geq 1 \). Introduce the integer

\[
N = -n_1 + n_2 - L
\]

(40)

Due to the selection rule, the equation system becomes trivial for \( N < 0 \), so in the following we may assume that \( N \geq 0 \).

It is easily verified that the Triangular equation system (38) is satisfied for the initial value \( n_2 = 0 \). For general \( n_2 \), the left hand side in (38) will be denoted \( \mathcal{L} \) whereas the
right hand side will be denoted $\mathcal{R}$. The finite double summation in $\mathcal{L}$ may be expressed as

$$\mathcal{L} = \sum_{j \geq 0} S_j$$

where

$$S_j = (-1)^{n_2} [2x - 2J_3 + 2N + 1] \frac{[n_2]! [n_1 + N - j - 1]! [2J_3 - n_2 - N + j + 1]_{n_2+N-j}}{[n_2 - j]! [n_1 - 1]! [j]! [N - j]! [-2x - N - 1]_{n_1+N-j}} \cdot \frac{[-2J_1 + n_1 - n_2 + j + 1]_{n_2-j}}{[2J_3 - n_2 - N + j + 1]_{n_2-j}} \cdot \frac{[2J_3 - n_2 - N + j + 1]_{n_2-j}}{[2J_3 - n_2 - N + j + 1]_{n_2-j}}$$

We also introduce

$$S_j\{-p\} = S_j(2x + p, 2J_1, 2J_3 - p, n_1, n_2 - p, N - p)$$

where the parameters $n_2$, $N$, $-2x$ and $2J_3$ all have been subtracted $p$ while the remaining parameters $n_1$ and $2J_1$ are left unchanged. $\mathcal{R}\{-p\}$ is defined analogously where the unshifted $\mathcal{R}$ is given explicitly by

$$\mathcal{R} = -\frac{[2x - 2J_3 + 2N + 1] n_1 - n_2 + N - 1]! [2J_3 - N + 1]_{N} [-2x + 2J_3 - N]_{n_1-n_2-1}}{[N]! [n_1 - n_2 - 1]! [-2x - N - 1]_{n_1-n_2+N}}$$

We shall make a case distinction and consider $N = 0$ first in which case

$$\mathcal{L} = S_0$$

Lemma 1

$$S_0 = \mathcal{R}(2x, 2J_1, 2J_3, n_1, n_2, N = 0)$$

For $N > 0$, let $M_k$ denote the following sum

$$M_k = (-1)^{n_2} [2x - 2J_3 + 2N + 1] \frac{[n_1 + N - 1]!}{[n_1 - 1]! [k]! [N]! [N - k - 1]!} \cdot \frac{[2J_3 - n_2 - N + k + 1]_{n_2+N-k}}{[2J_3 - n_2 - N + k + 1]_{n_2-k}} \cdot \frac{[-2x - 2J_1 + 2J_3 + 2n_1 - 2n_2 + k - 1]_{n_2-k+1}}{[-2x + 2J_3 + n_1 - n_2 - N - 1]_{n_2+N-k+1}}$$

13
\[
\sum_{i=0}^{n_2-k} \frac{[2x + 2J_1 - 2J_3 - 2n_1 + 2n_2 - 2l + 1 - k]}{[l]![-2J_1 + n_1 - n_2 + k + 1]_l} \cdot \frac{[-2x - 2J_1 + 2n_1 - n_2 - 1]_l}{[2J_3 - n_2 + N + k + 1]_l[-2x - 2J_1 + 2J_3 + 2n_1 - n_2]_l}
\]

where \(M_0 = S_0\). The following lemma determines a relation between \(S_j\) and \(M_k\).

**Lemma 2**

For \(k < n_2\) we have

\[
M_k - M_{k+1} = \sum_{m=0}^{k+1} A^m_k S_m(\{-k - 1 + m\})
\]

where

\[
A^m_k = (-1)^{k+m} \frac{[n_1 - n_2 + N - 1]_{k+1-m} [2J_3 - k + m]_{k+1-m}}{[k + 1 - m]![-2x + 2J_3 - N - k - 1 + m]_{k+1-m}}
\]

and in particular \(A^{k+1}_k = -1\).

It then follows that

\[
\mathcal{L} = \sum_{k=0}^{\min(N,n_2)-1} \sum_{m=0}^{k} A^m_k S_m(\{-k - 1 + m\}) + \Theta(n_2 < N)M_{n_2}
\]

from which \((38)\) may be deduced, thus completing the proof of the Triangular equation system \((37)\).

Now we turn to the Initial equation system \((37)\). Insertion of our proposal yields the equation system

\[
\delta_{K,0} = [2x - 2J' + 2K + 1](-1)^n \frac{[n]_{n-K} [2x + K + 2]_{n-K} [-2J']_K [2x - 2J' + 2]_{n-1}}{[n - K]! [2x - 2J' + K + 1]_{n+1} [2x + 2]_n}
\]

\[
\cdot \sum_{l=\max(0,K-n)}^{K} \frac{[2x - 2J' + n + 2l + 1] \frac{[-K]_l [2x - 2J' + K + 1]_l}{[l]! [n - K + 1]_l}}{[2x - 2J' + n + K + 2]_l [2x - 2J' + 2]_l}
\]

It is immediate that the right hand side vanishes for \(K < 0\) and that it reduces to 1 for \(K = 0\). We may therefore assume \(K > 0\). After a case distinction into \(0 < K \leq n\) and \(0 \leq n < K\) the identity follows by induction in \(K\) and \(n\), respectively, thus completing the proof of the Initial equation system \((37)\). Details on the proofs of both the Triangular and the Initial equation systems will appear elsewhere \([15]\).
4.3 The Type III R-matrix

So far we have only verified our proposal for R-matrices of type II. Type III R-matrices may be obtained from the latter essentially by hermitian conjugation. We have

\[
\left( V^{(J)}_{-J'}(\sigma)S^n(\sigma)V^{(J')}_{-J}(\sigma')S^{-n'}(\sigma') \right)^\dagger = S^{-n\dagger}(\sigma')V^{(J')}_{-J}(\sigma')S^{n\dagger}(\sigma)V^{(J)}_{-J'}(\sigma) \\
= \tilde{S}^{-n'}(\sigma')V^{(J')}_{-J}(\sigma')\tilde{S}^n(\sigma)V^{(J)}_{-J'}(\sigma)
\]

(52)

Here we have used that for real \( \varpi \), the two free fields are just hermitian conjugates of each other \([3]\) \((\tilde{V}^{(J')}_{-J'})\) and \( \tilde{S} \) denote operators constructed from the field \( \tilde{X} \). The order of screening operators and free field exponentials can be reverted to the standard form at the expense of simple phase factors, so that

\[
\left( V^{(J)}_{-J'}(\sigma)S^n(\sigma)V^{(J')}_{-J}(\sigma')S^{-n'}(\sigma') \right)^\dagger = q^{2(Jn-J'n')}\tilde{V}^{(J')}_{-J}(\sigma')\tilde{S}^{-n'}(\sigma')\tilde{V}^{(J)}_{-J'}(\sigma)\tilde{S}^n(\sigma)
\]

(53)

The expression on the right hand side is of the type II already discussed. Braiding the operators on the left hand side with the unknown type III R-matrix, we obtain

\[
R^*(J, J'; \varpi - 2n_1 + 2n_2 + 2(J + J'))^{n_2,n_1}_{n,-n'} = q^{2(Jn_1-J'n_2)}\tilde{R}^*(J', J; -\varpi - 2n_1 + 2n_2 + 2(J + J'))^{n_1,-n_2}_{n',n}
\]

(54)

The bar on the R-matrix on the right hand side indicates that this R-matrix braids an operator at \( \sigma' \) with an operator at \( \sigma \) (cf. comment following Eq. \([3]\)), contrary to the one on the left hand side. Notice also the replacement \( \varpi \rightarrow -\varpi \) in this R-matrix which reflects the fact that it applies to the braiding of “tilded” operators. The final result for the type III braiding matrix thus becomes

\[
R(J, J'; \varpi)^{n_2,n_1}_{n,-n'} = q^{-2[(Jn_1-J'n_2)]}\tilde{R}^*(J', J; -\varpi - 2n_1 + 2n_2 + 2(J + J'))^{n_1,-n_2}_{n',n}
\]

(55)

In the case of braiding of positive screening vertex operators, it is well known that \( \tilde{R}(J, J'; \varpi)^{n_2,n_1}_{n,-n'} \) differs from \( R(J, J'; \varpi)^{n_2,n_1}_{n,-n'} \) only by a phase factor \([21]\), and it is evident from our analytic continuation procedure that the type II R-matrix will keep this property. Therefore, we can write Eq. \((55)\) as a relation between R-matrices corresponding to the same position ordering:

\[
R(J, J'; \varpi)^{n_2,n_1}_{n,-n'} = q^{-2[(Jn_1-J'n_2)]}e^{-2i\pi(\Delta_x+\Delta_x+m'x-m-\Delta_x+m)} \cdot \tilde{R}^*(J', J; -\varpi - 2n_1 + 2n_2 + 2(J + J'))^{n_1,-n_2}_{n',n}
\]

(56)

where \( m' = -n' - J' \) and \( m_2 = -n_2 - J' \). Eq. \((56)\) determines the type III R-matrix in terms of the type II one, which we have already verified. We show that our analytic continuation fulfills Eq. \((56)\) by considering separately the two cases \( n \geq n_1 \) and \( n < n_1 \). From our formula for the type II R-matrix, we obtain for the right hand side (up to explicit phases and powers of \( q \), cf. comment following Eq. \([3]\))

\[
\tilde{R}(J', J; -\varpi - 2n_1 + 2n_2 + 2(J + J'))^{n_1,-n_2}_{n',n} = [-2x + 2J' + 2n_2 - 1]^{n_2}_n \cdot \left[ -2x + 2J' - n_1 + 2n_2 \right]^{n_2}_n \cdot \sum_{l=\max(0,n'-n_2)}^{n} [l!]|2x - 2J - 2J' + n_1 - n_2 + l|2J - n_1 + 1|n_2 - n'|_l|n_2 - n' + 1|_l[-2x - n_2]_l
\]

(57)
In the first case \( n \geq n_1 \), the summation in (57) may be written
\[
\sum_{l=n' - n_2}^{n} \frac{|-n|_{n' - n_2} |n_2|_{n' - n_2}}{[n' - n_2]! [n_2 - n' + 1]_{n' - n_2}} \left[ -2J - n' \right]_{n' - n_2} \left[ -2J + n_1 \right]_{n' - n_2} = \frac{2x - 2J - 2J' + n_1 - n - 1}{[2x - 2J - 2J' + n - n_2 - 1]_{n - 1} \[n_1 - 1\]} (58)
\]

and by re-inserting it we recognize the left hand side of Eq. (58). In the second case \( n < n_1 \), the summation on the left hand side of Eq. (58) may be written
\[
\sum_{l=0}^{n_1} \frac{|-n_1|_{n_1 - n} |n'|_{n_1 - n}}{[n_1 - n]! [n_1 - n + 1]_{n_1 - n}} \left[ -2J + n_1 \right]_{n_1 - n} \left[ -2J' - n_2 \right]_{n_1 - n} = \frac{2x - n + n' - 1}{[2x - 2J - 2J' + n - n_2 + 1]_{n_1 - n} \[n_1 - 1\]} (59)
\]

and by re-inserting it we recognize the expression (57). This concludes the verification of our proposal for the type III \( R \)-matrix.

5 Connection with 6j-symbols

The braiding matrix for vertex operators from the Kac table was shown in [21] to coincide with \( q \)-deformed 6j-symbols for \( U_q(sl(2)) \), up to normalization factors that were determined explicitly. This result continues to be true for general positive screening vertex operators [7, 9]. We will not show here that our expression for the \( R \)-matrix in the presence of negative screenings can indeed be interpreted consistently as a \( q \)-deformed 6j-symbol - this would require verifying all of the defining properties of the latter - but rather offer a most natural generalization of the standard relation between \( R \)-matrix and 6j-symbol, as given in ref. [8]:
\[
R(J, J', \omega)_{n_2,n_1} = e^{-i\pi(\Delta_c + \Delta_b - \Delta_e - \Delta_f)} \kappa_{ab}^{c} \kappa_{de}^{f} \{a b \mid |e\} \{d c \mid |f\} (60)
\]

The arguments of the \( q \)-deformed 6j-symbol are
\[
a = J, \quad b = x + m + m', \quad c = x \\
d = J', \quad e = x + m_2, \quad f = x + m
\]
and the coefficients \( \kappa_{J_1,1}^{J_2,2} \) are given by
\[
\kappa_{J_1,1}^{J_2,2} = \left( \frac{h e^{-i h + \pi}}{2 \pi \Gamma(1 + h / \pi) \sin h} \right)^{J_1 + J_2 - J_{12}} e^{i h (J_1 + J_2 - J_{12})(J_1 - J_2 - J_{12})} \prod_{k=1}^{J_1 + J_2 - J_{12}} \left[ \frac{1 + 2J_1 - k}{k} \right] \left[ 1 + 2J_2 - k \right] \left[ (1 + 2J_{12} + k) \right] (62)
\]
The last equation makes sense for arbitrary \( J_1, J_2, J_{12} \) such that \( J_1 + J_2 - J_{12} \) is a non-negative integer, but is readily extended to negative screening numbers if we define products with negative upper limits as usual by

\[
\prod_{j=1}^{-n} f(j) := \frac{1}{\prod_{j=1}^{n} f(1-j)} \tag{63}
\]

As our method provides us directly with the braiding matrix, rather than the \( 6j \)-symbol, Eq. 63 should be read as our definition of (or proposal for) the \( q \)-deformed \( 6j \)-symbol, extended to arbitrary integer screening numbers.

An extension to the purely negative case where all (ingoing and outgoing) screening numbers are negative, has been provided in ref. 3. Here we shall show that our definition (63) based on our proposal for the braiding matrix, reproduces (and generalizes 3) the result of 3 in the purely negative case. The idea is to verify invariance of our proposal under the replacement \( J_i \rightarrow -J_i - 1 \) (implying in particular that screening numbers are transformed according to \( n_i \rightarrow -n_i - 1 \))

\[
\begin{align*}
\{J \atop J'}_{x-J'-x+n+n' \atop x-J+n} &= \left\{ -J-1 \atop -J-1' \atop -x+J+J'-n-n'-1 \atop 1 \atop -x+J-n-1 \right\} \tag{64}
\end{align*}
\]

which is the defining property for the extension in ref. 3. In order to do that let us write down explicitly the well known \( q \)-deformed \( 6j \)-symbol

\[
\begin{align*}
\{J \atop J'}_{x-J'-x+n+n' \atop x-J+n} &= \left( \frac{2J - n + 1}{2J' - n + 1} \right)_{n-n'-1} \frac{[2x - 2J - 2J' + n + n' + n_2 + 1]_{n-n_2}}{[2x + n + 2]_{n+n_2}} \\
&\cdot \left( \frac{[2x - 2J' + n_2 + 1]_{n+n'+1} [2x - 2J + 2n + 1]_{n_1} [n_1]! [n_1']!}{[2x - 2J + n + 1]_{n+n'+1} [2x - 2J' + 2n_2 + 1] [n]! [n']!} \right)^{\frac{1}{2}} \\
&\cdot R'(J, J'; \omega)_{n,n_1}^{n_2,n_1} \tag{65}
\end{align*}
\]

and our proposal in the purely negative case

\[
\begin{align*}
\{J \atop J'}_{x-J'-x+n+n' \atop x-J+n} &= \left( \frac{-2J + n}{-2J' + n_2} \right)_{n-n_2} \frac{[-2x + 2J + 2J' - n - n' - n_2]_{n_2-n}}{[-2x - n_2 - 1]_{n-n_2}} \\
&\cdot \left( \frac{[-2x + 2J' - n_2]_{n-n'-1} [-2x + 2J - 2n - 1] [-n_1 - 1]! [-n_2 - 1]!}{[-2x + 2J - n]_{n-n'-1} [-2x + 2J' - 2n_2 - 1] [-n-1]! [-n'-1]!} \right)^{\frac{1}{2}} \\
&\cdot R^{IV} (-J - 1, -J'-1; -\omega)_{-n_1-n_2-1}^{-n_2-1,n_1-1} \tag{66}
\end{align*}
\]

Employing \( ST \) six times, it is now straightforward to verify the invariance (64).

In ref. 3 it is observed that the \( q \)-deformed \( 6j \)-symbol possesses the simple symmetry

\[
\{J' \atop J}_{x-J'-x+n+n' \atop x-J+n} = \{J \atop J'}_{x-J'-x+n+n' \atop x-J+n} \tag{67}
\]

\footnote{In contrast to ref. 3, we do not assume that the outgoing screenings are negative.}
An explicit proof based on (65) is immediate, so let us conclude this section by stating the equivalence of (67) in terms of $R$-matrices

$$q^{2nJ-2nJ'} R^I(J',J;\varpi)_{n_2,n_1} \equiv \frac{2x-2J+2n+1|n_1|!|n_2|!(2x-2J-2J'+n+n'+n_2+1)}{[2x-2J'+2n+1|n'|!(2x+n_2+2)]_{n-n_2}} \cdot \frac{2J-n+1|n_2-n'|2x-2J+n+1}{[2J'-n_2+1|n_2-n'2x-2J+n+1]_{n+n'}} q^{-2nJ'+2nJ} R^I(J,J';\varpi)_{n_2,n_1} \quad (68)$$

6 Construction of the Type IV $R$-matrix by Concatenation

In this section we shall argue for the validity of our proposal for the type IV $R$-matrix by explicit comparison with an expression obtained by a 3 step concatenation.

Step 1 is a direct verification that our proposal for $J=J'=0$ is identical to the result of imposing Weyl reflection symmetry on the corresponding type I $R$-matrix. Eq. (25) readily ensures that. Step 2 is then the generalization to $J'=0$ and $J$ generic, using the result of Step 1. Finally, Step 3 completes the generalization to $J$ and $J'$ generic. In both Step 2 and Step 3, we consider the braiding of $V^{(J)}_J S^{-n}(\sigma)$ with $V^{(J')}_{J'} S^{-n'}(\sigma')$, for $n, n' > 0$.

6.1 Step 2

The $R$-matrix we want to establish is given by

$$V^{(J)}_J(\sigma)S^{-n}(\sigma)S^{-n'}(\sigma') = \sum_{n_2} R(J,0;\varpi)_{-n_2,-n_1} S^{-n_2}(\sigma')V^{(J)}_J(\sigma)S^{-n_1}(\sigma) \quad (69)$$

where $n_1 \equiv n + n' - n_2$. On the other hand we also have $(k_2 \equiv n + n' - k_1)$

$$V^{(J)}_J(\sigma)S^{-n}(\sigma)S^{-n'}(\sigma') = V^{(J)}_J(\sigma) \sum_{k_1} R(0,0;\varpi)_{-k_2,-k_1} S^{-k_2}(\sigma')S^{-k_1}(\sigma) \quad (70)$$

where $0 \leq k_1, k_2 \leq n + n'$ by reflection symmetry (whereby the sum over $k_1$ is seen to be finite). The right hand side can be further rewritten as $(l_1 \equiv l_2 - k_2)$

$$\sum_{k_1} R(0,0;\varpi-2J)^{-k_2,-k_1} \sum_{l_2} R(J,0;\varpi)^{-l_2,l_1} S^{-l_2}(\sigma')V^{(J)}_J(\sigma)S^{l_1-k_1}(\sigma) \quad (71)$$

leading to the equation

**Step 2 Equation**

$$R(J,0;\varpi)^{-n_2,n_2-n-n'} = \sum_{k=0}^{\min(n_2,n+n')} R(0,0;\varpi-2J)^{-k,-n-n'} R(J,0;\varpi)^{-n_2,n_2-k} \quad (72)$$

The right hand side is fully known as the type III braiding has been solved. The upper summation limit $n_2$ is due to the selection rule and could also be inferred by a “fusion rule analysis” (see Section 3.1) of the left hand side. Though we do not have a complete proof of (72) we do have substantial evidence since we have been able to prove it for $n_2 = 0, 1$ or 2.
6.2 Step 3

It is firstly noted that

\[ V_{-j}^{(J)}(\sigma)S^{-n}(\sigma)V_{-j'}^{(J')}(\sigma')S^{-n'}(\sigma') = q^{2J'n'}V_{-j}^{(J)}(\sigma)S^{-n}(\sigma)V_{-j'}^{(J')}(\sigma') \]

(73)

We will consider this last expression, assuming that our proposal for \( R(J, 0; \omega)_{-n, n'}^{-n_2, n_2} \) has already been verified (Step 2). We then have

\[
V_{-j}^{(J)}(\sigma)S^{-n}(\sigma)V_{-j'}^{(J')}(\sigma') \sum_{k_2 \geq 0} R(J, J'; \omega)_{-n, -n'}^{-k_2, -n - n'} \sum_{l_2} R(J, J'; \omega)_{k_2, -n - n', 0}^{l_2, l_1} V_{-j'}^{(J')}(\sigma')S^{-n}(\sigma)S^{-n'}(\sigma')V_{-j'}^{(J')}(\sigma') \]

(74)

in addition to

\[
V_{-j}^{(J)}(\sigma)S^{-n}(\sigma)V_{-j'}^{(J')}(\sigma')S^{-n'}(\sigma') = \sum_{n_2} R(J, J'; \omega)_{-n, -n'}^{-n_2, -n_1} V_{-j'}^{(J')}(\sigma')S^{-n_2}(\sigma')V_{-j'}^{(J')}(\sigma')S^{-n_1}(\sigma) \]

(75)

with \( l_1, n_1 \) again defined by conservation of total screening number. By comparing the two results for the braiding in question, we find

**Step 3 Equation**

\[
R(J, J'; \omega)_{-n, -n'}^{-n_2, n_2} = \sum_{k_2 \geq 0} q^{2J'(n'-k)} R(J, J'; \omega)_{-n, -n'}^{-k, -n' - n'} R(J, J'; \omega - 2k)_{k, -n', 0}^{k-n_2, n_2 - n'-n'} \]

(76)

By analyzing the second factor on the right hand side we find that the summation over \( k \) is restricted as

\[
\begin{align*}
n_2 & \leq k \quad \text{for} \quad n + n' \leq n_2 \\
n_2 & \leq k \leq n + n' - 1 \quad \text{for} \quad n_2 \leq n + n' - 1
\end{align*}
\]

(77)

An explicit proof of Eq. (76) is quite laborious, but we have checked it explicitly in the case \( n = 1 \) and \( n_2 \leq n' \), thus presenting further evidence in favour of our proposal.

A further discussion on the type IV \( R \)-matrix will appear elsewhere [15].

7 Case of Two Screening Charges: \( \alpha_+ \) and \( \alpha_- \)

Up to now, we have restricted ourselves to vertex operators involving only screenings corresponding to the “semi-classical” screening charge \( \alpha_- \). However, it is straightforward to include the screenings corresponding to the conjugate charge \( \alpha_+ \), with \( \alpha_+ \alpha_- = 2\pi \). The
corresponding deformation parameter is $\hat{h} \equiv \frac{\alpha_-}{h}$. Denoting by $\hat{S}$ the screening operators corresponding to $\alpha_+$, one may consider the general vertex operators $U_{m^e}^{(J^e)}$ given by

$$U_{m^e}^{(J^e)} := V_{-}^{(J^e)} S^n \hat{S} \hat{n}$$

with $n, \hat{n}$ integer and $m^e = n + \hat{n} \frac{\alpha_+}{\alpha_-} - J^e$. One may then follow exactly the same reasoning as in ref. [8] to obtain the braiding matrix for $U_{m^e}^{(J^e)}$ from that of the single screening vertex operators; the essential observation here is that screening operators of opposite types commute, $[S(\sigma), \hat{S}(\sigma')] = 0$. One thus obtains, exactly as in the positive screening case,

$$R(J^e, J^{e'}; \omega)_{m^e, m^{e'}}^{n, n'} = q^{-J^e J^{e'} \hat{q}^{-J^e J^{e'}}} R(J^e, J^{e'}; \omega)_{n, n'}^{n, n'} \hat{R}(\hat{J}^e, \hat{J}^{e'}; \hat{\omega})_{\hat{n}, \hat{n}'}^{\hat{n}, \hat{n}'}$$

The conventions here are the following: $n = (n, \hat{n})$ represents the two screening numbers corresponding to $\alpha_-$ and $\alpha_+$, $\hat{J}^e = J^e \frac{h}{\pi} = J^e \frac{\alpha_-}{\alpha_+}$, $\hat{\omega} = \omega \frac{h}{\pi}$ and $\hat{q} \equiv e^{i \hat{h}}$. Finally, $\hat{R}$ is the same function of its arguments as $R$, except that $\alpha_-$ is replaced by $\alpha_+$ everywhere (or equivalently $h$ by $\hat{h}$).

8 Conclusions and Outlook

What have we learnt from the present analysis? First of all, we believe to have presented convincing evidence that there is a closed exchange algebra for a set of generalized Coulomb gas vertex operators involving positive and negative powers of screenings alike, and we have obtained the explicit form of the braiding matrix in this general context, through natural analytic continuations of the braiding matrix for positive screenings. The basic lesson is that the additional structure generated by the inclusion of negative screenings is essentially determined by the algebra for vertex operators with positive screenings alone, through fundamental consistency conditions of conformal field theory. This is because negative screening operators are, in a rather precise sense, inverses of positive screening operators. Our results can be expressed, as in the positive screening case, in terms of $4F3$ $q$-hypergeometric functions which truncate.

As pointed out in Appendix B, the question of uniqueness of our solution is more delicate than might have been naively expected. Whether there is a deeper meaning to the formal possibility of introducing, in our analysis, $R$-matrix elements for non-integer outgoing screenings merits further investigation. Likewise, the origin of the selection rule of Section 4.1 should be elucidated. The technique employed in ref. [7] (cf. footnote in Section 7), which uses detailed input from the operatorial construction of the vertex operators, may allow to shed some light on both of these questions.

To complete the description of the conformal algebra, we also need the fusion matrix. While it should be given in general in terms of the braiding matrix through one of the

3It is quite remarkable that the technique used in refs. [7, 8] to derive the braiding matrix continues to work in the presence of negative screenings. We have been able to show that the fundamental equations (2.17) of ref. [7] and (5.7) of ref. [8] continue to hold, when using the standard continuation (63) to products with negative upper limits. These equations could in fact have been used as an alternative starting point for our analysis.
Moore-Seiberg relations, one could also think of an independent derivation along similar lines as for the braiding matrix. It appears that this is in fact possible, and one obtains recursion relations from the associativity ($\mathcal{F}\mathcal{F}\mathcal{F} = \mathcal{F}\mathcal{F}$) relation. We hope to come back to this question in a future publication.

The growing body of results thus indicates strongly that on the level of the chiral operator algebra, there exists a sequence of inclusions, each representing a consistent solution of the Moore-Seiberg equations; see Fig. 1. The situation will, however, be much more complicated on the non-chiral level, as will be discussed below.

Another interesting direction for future investigations would be an analysis of the quantum group aspect of the generalized Coulomb gas picture advocated here. It is well known [24, 16, 17] that there exists another basis of chiral vertex operators $\xi^{(J)}_M$ which renders the underlying $U_q(sl(2))$ symmetry of the algebra manifest; braiding and fusion symbols for these vertex operators are given by universal $R$-matrix and $3j$-symbols for $U_q(sl(2))$, respectively. The “covariant” vertex operators are related to those of the Coulomb gas picture by a basis transformation with $\varpi$-dependent coefficients $|J, \varpi\rangle^m_M$.

In succession of our work ref. [10], we were able to carry out formally an analysis similar to the one above within the quantum group covariant basis, with the role of the screening number $n$ played by $N = J + M$. It is not hard to obtain the braiding matrix for the case where one or both of $N, N'$ are negative (integer). The analysis is in fact much simpler than in the Coulomb gas basis as the algebraic expression for the $R$-matrix in the

---

4See ref. [9] for a discussion on how the Kac table is embedded into the larger shells from the point of view of the polynomial equations.
covariant basis is essentially just a ratio of $q$-factorials, and the generalized $R$-matrix can essentially be obtained by naive analytic continuation. Similar selection rules as for the Coulomb gas basis hold. However, the covariant vertex operators for generic $2J$ are given by infinite sums over Coulomb gas vertex operators, the precise meaning of which is not a priori clear. In particular, by replacing one free field by the other, one would seem to obtain two different copies of the same quantum group representation. It is interesting to speculate whether non-perturbative mechanisms along the lines of ref. [10] will render these two copies equivalent. We hope to return to the study of these questions in the future.

8.1 Non-chiral Case and Liouville Theory

Our analysis so far has addressed the chiral operator algebra only; let us now make some more speculative remarks on the conformal field theories that could be obtained by combining the two chiralities, and in particular about the relevance of our analysis for Liouville theory. The general rule for constructing the non-chiral theory is that operators of the latter should have crossing-symmetric correlators, and obey a consistent closed fusion and braiding algebra. To find the general answer to this question is, obviously, a formidable task. However, within the operator approach to Liouville theory, two mechanisms for producing local fields have been identified. In the standard “weak coupling” ($c > 25$) regime, locality arises from the orthogonality relations for the $q$-deformed $6j$-symbols. In fact, one writes the local fields - the Liouville exponentials - as diagonal combinations of left and right moving vertex operators, both in the sense of conformal weight and $U(1)$ quantum number $m$:

$$e^{-J\alpha - \Phi} = \sum_{n \geq 0} a_n^{(J)}(\bar{\omega}) U_m^{(J)}(u) \overline{U_m^{(J)}}(v)$$

(80)

with $u := \tau + \sigma$, $v := \tau - \sigma$. The coefficients $a_n^{(J)}(\bar{\omega})$ can actually be absorbed in the normalization of the vertex operators, upon which the braiding of the latter will be given in terms of $q$-deformed $6j$-symbols alone [8]. Mutual locality of two Liouville exponentials then becomes the statement that the contraction of left and right moving $R$-matrices over the $m$-indices yields the unit matrix. When written in terms of the quantum group covariant basis, Eq. (80) simply becomes the singlet formed from left and right moving spin $J$ representations of highest and lowest weight, respectively. Remarkably, this mechanism reproduces in particular the minimal models when one continues to central charges $c < 1$; that is, there is a description in terms of Liouville exponentials of the observables of these models [20]. For $c > 1$ (in fact, $c > 25$), the closest analog of the minimal models is Liouville theory restricted to the Kac table. The diagonal sums are finite (the $n$-sum in Eq. (80) truncates at $n = 2J$), and closure of the chiral and non-chiral algebra become essentially equivalent.

When we leave the Kac table, the situation changes rather drastically: it is no longer possible to give a unique representation of the Liouville exponentials in terms of a sum over chiral vertex operators. As pointed out in ref. [10], this is due to two reasons: First, the formal expansion Eq. (80) cannot be evaluated term by term, using naive charge conservation rules, when the sum is infinite (see also ref. [25] for a recent discussion on
this fact from another point of view). Second, when $2J$ is not an integer, the expansion Eq. (80) suffers from a multi-valuedness problem as regards its zero mode dependence, and therefore does not represent accurately the “true” Liouville exponentials even classically. According to the ideas of ref. [4], adapted to the Liouville context in [10], this does not mean that such expansions cannot be used; rather, one needs to introduce a new (continuous) parameter $\beta$ which controls the monodromy properties of the expansion with respect to the zero mode. One arrives then at generalized expansions of the form

$$\sum_n a_{m;\beta}^{(J)}(\varpi) V_{-J}^{(J)} U_{J}^{(J)} S^{\beta+n} S^{3+n}.$$ (81)

For the three point function, with operators situated at $0, 1, \infty$, all $\beta$ parameters are fully determined, up to integers. For the operator at $z = 1$, not replaced by a highest weight state, one has to choose $\beta$ such that the total monodromy becomes trivial, i.e. the multi-valuedness disappears. One can in particular consider the set of three point functions where $\beta$ is of the form $n + \hat{n}_+ \alpha_+ (n, \hat{n} \text{ integer})$, so that the exponential can be described by Eq. (80) or its generalization to both screenings $\alpha_+, \alpha_-$. This was done in ref. [10]. In this context, one already needs the negative screening operators discussed in the present paper, because they are generated non-perturbatively by a careful evaluation of the expansion Eq. (80). The non-perturbative contributions can be rendered perturbative in terms of an “effective” representation of the Liouville exponentials that can be evaluated using naive charge conservation rules:

$$e^{-J_{\alpha_-} \Phi} = \sum_{n=-\infty}^{\infty} a^{(J)}_n(\varpi) U_{J}^{(J)}(u) \bar{U}_{J}^{(J)}(v)$$ (82)

It is an interesting question whether Eq. (82) is more than just a recipe for the computation of a class of three point functions. In ref. [8] it was shown that Eq. (80) is compatible with locality in the sense that any two Liouville exponentials of the form (80) will commute at equal times, order by order in powers of screening operators (i.e. perturbatively). It was also observed that in the same sense, the Liouville equation is fulfilled by the exponential with $J = -1$ (see also [23]).

It is natural to ask whether the same properties will obtain non-perturbatively, i.e. when taking into account the additional negative screening contributions of Eq. (82). While this is essentially trivial for the equations of motion, non-perturbative locality requires that

$$[e^{-J_{\alpha_-} \Phi}(\tau, \sigma), e^{-J_{\alpha_-} \Phi}(\tau, \sigma')] = 0$$ (83)

Here, both exponentials are given by Eq. (80) and understood to be treated perturbatively, but the second one is constructed in terms of the free field $\bar{X}$. The results of the present paper provide all the necessary tools in order to answer this question. One has to show that there is a new orthogonality relation between $q$-deformed $6j$-symbols corresponding to positive and negative screenings, respectively; work on this problem is in progress. If locality holds both in the perturbative and the non-perturbative sense, at least for integer (positive or negative) $2J$, this would be a hint that there exist consistent conformal field theories described by positive integer, and arbitrary integer screening powers, respectively.
Some counter-evidence, however, seems to be provided by the analysis of the “perturbative” operator product of two Liouville exponentials. While it formally closes on the set of Liouville exponentials with integer positive screenings, there appear arbitrarily strong short-distance singularities \(^5\) that render the result dubious:

\[
e^{-J_1 \alpha \cdot \Phi(z_1, \bar{z}_1)} e^{-J_2 \alpha \cdot \Phi(z_2, \bar{z}_2)} = \sum_{J_{12} = -\infty}^{J_1 + J_2} \sum_{\nu, \{\nu\}} e^{-J_{12} \alpha \cdot \Phi(\nu, \{\nu\})(z_2, \bar{z}_2)} \cdot |z_1 - z_2|^2(\Delta_{J_{12}} - \Delta_{J_1} - \Delta_{J_2}) \langle \varpi_{J_{12}}, \varpi_{J_{12}}; \nu, \{\nu\} | e^{-J_1 \alpha \cdot \Phi(z_1 - z_2, \bar{z}_1 - \bar{z}_2)} | \varpi_{J_2}, \varpi_{J_2} \rangle \quad (84)
\]

where \(\varpi_J := \varpi_0 + 2J \equiv 1 + \frac{\pi}{h} + 2J\), and \(\nu\) denotes descendant contributions. The matrix element on the right hand side represents the operator product coefficient as well as the descendant contributions to the conformal block given by \(J_{12}\); see ref. \([7]\) for details. The contributions from very large negative \(J_{12}\) are non-vanishing in general and so there appear unwanted short-distance singularities of arbitrarily high order as \(\Delta_{J_{12}} \to -\infty\).

Even if there is a fully consistent non-chiral algebra involving only integer screenings, we should be very careful as to its interpretation as a restriction of Liouville theory. In fact, the operator product of Liouville exponentials is expected \([27]\) to depend on all operators entering in the correlator, and to involve “hyperbolic” states corresponding to purely imaginary \(\varpi\). Crossing symmetry, or locality, should then involve the same intermediate channels, and the interpretation within Liouville theory of locality properties based on integer screenings only is a priori not very clear. On the other hand, general \(\beta\)-dependent expansions of type Eq. (81) would actually allow for the introduction of hyperbolic intermediate states. Consider a four point function with operators situated at 0, 1, \(\infty\) and \(z\), represented as

\[
\langle \varpi_4 | e^{-J_3 \alpha \cdot \Phi(z, \bar{z})} e^{-J_2 \alpha \cdot \Phi(1)} | \varpi_1 \rangle \quad (85)
\]

The parameters \(\beta_3, \beta_2\) appearing in the representation of the exponentials in the middle are coupled by the monodromy condition. For a given conformal block, they have a unique value. The question of what range to choose for the one free \(\beta\) parameter is thus tantamount to the factorization problem, and imaginary \(\beta\) will in fact produce hyperbolic intermediate states. While in Liouville theory restricted to the Kac table, locality and conformal invariance are sufficient to fully determine the operator construction of the theory, we see that in the present, much larger context this may well be an illusion. The simple correspondence between chiral and non-chiral algebras seems to be lost, and additional information is needed for the construction of non-chiral correlation functions. A careful analysis of the zero mode dependence of the exponentials, combined with group-theoretical arguments as in refs. \([28, 29]\), seems to be a step in the right direction. The presence of a context-dependent, floating \(\beta\) parameter as we advocate it here, means that general Liouville exponentials will invoke arbitrary quantum group representations, and not just highest or lowest weight ones as one could have naively expected from the classical picture. This would resemble the situation in affine \(SL(2)\) current algebra where continuous representations of intertwining operators appear for non-integrable weights.

\(^5\) The formula (84) corrects a misprint in ref. [8]. Note also that we are working here in Euclidean coordinates on the sphere where \(z := e^{\tau + i\sigma}\).
In any case, it seems clear that a deeper understanding of the full Liouville dynamics will have to pass through a study of the algebra of the chiral vertex operators, and it seems desirable to continue the program towards the analysis of arbitrary non-integer screenings. The study of correlation functions of the chiral vertex operators may be interesting in its own right, and is expected to lead to new mathematical functions already on the integer (including negative ones) screening level.

8.2 Application to Strong Coupling Liouville Theory

In a series of works [30, 31, 32, 1], Gervais and collaborators have proposed a conformal field theory with $1 < c < 25$ which they interpret as a candidate for Liouville theory in the forbidden, “strong coupling” region $1 < c < 25$. This theory exists for certain discrete values of the central charge ($c = 7, 13, 19$), and on a spectrum of highest weight states specified by the unitary truncation theorems of ref. [31]. Locality arises in a rather different way; local observables become products of sums of chiral vertex operators rather than sums of products, and each chiral factor is local up to a phase factor. The local operators can be divided into two sets, one with negative and one with positive conformal weights. The latter operators involve negative screenings; they can be written in the form

$$\chi^{(J)}_+(u) = \sum_{p \in \mathbb{N}_0} (-1)^{(2-s)p} \left[ 2(p+\frac{s-1}{2})+\frac{p+1}{4} \right] g^{x}_{J,x-J-1+p(\frac{s}{2}-1)} V^{(J)}_{p(\frac{s}{2}-1)-1-J}(u)$$

(86)

Here, $x := \frac{1}{2}(\omega - \omega_0)$ with $\omega_0 := 1 + \frac{s}{2}$, and $s = 0, \pm 1$ determines the central charge ($c = 1 + 6(s + 2)$). The operators $V^{(J)}_m$ are nothing but normalized $U^{(J)}_m$ fields, $V^{(J)}_m \equiv \frac{1}{r^{(J)}_m(\omega)} U^{(J)}_m$. Finally, the coupling constants $g^{(J)}_{J_1,J_2}$ appear as prefactors of the 6j-symbols in the fusion and braiding relations of the $V^{(J)}_m$ operators and were determined in [21].

The values that $\omega$ is allowed to take are not arbitrary continuous, but discrete:

$$\omega = (\frac{\pi}{h} - 1)(l + 1 + \frac{r}{2-s})$$

(87)

with $l \in \mathbb{Z}$, $r = 0, ..., 1 - s$. Similarly, $J$ is restricted to be

$$J = (\frac{\pi}{h} - 1) (l' + \frac{r'}{1-s}) - 1$$

(88)

with $l' \in \mathbb{Z}$, $r' = 0, ..., 2 - s$. The screening number with respect to $\alpha_-$ is $-p - 1$, while for $\alpha_+$ it is $p$. Thus indeed, negative screenings are involved, and one can obtain a second analogous set with the roles of $\alpha_+$ and $\alpha_-$ interchanged.

In the absence of an algebraic control over vertex operators with negative screenings, the properties of the $\chi$ fields were described using the formal symmetry $J \rightarrow -J - 1$ of the $q$-deformed 6j-symbols already mentioned above. The present analysis not only allows to further corroborate this discussion (at least as far as the braiding is concerned) by establishing a direct derivation from the operator construction, but also opens the road towards an investigation of new local operators with negative screenings which are not accessible by the $J \rightarrow -J - 1$ analytic continuation technique.
The results of the present analysis and its possible future extensions should also find direct applications in theories closely related to Liouville, such as \( SL(2,\mathbb{R}) \) or \( SL(2,\mathbb{C})/SU(2) \) WZNW theory \([28]\). From a larger perspective, this program can be viewed as a kind of bootstrap approach to irrational conformal field theory in the simplest context where the chiral symmetry algebra is just Virasoro. It is an interesting question whether all solutions can be identified either with (subsectors of) weak or strong coupling Liouville theory.

Acknowledgment

We would like to thank J. Teschner for useful discussions and comments. J.R. gratefully acknowledges the partial financial support from the Danish Natural Science Research Council, contract no. 9700517. He also thanks Laboratoire de Mathématiques et Physique Théorique, Université de Tours and The Niels Bohr Institute, where parts of this paper were written down, for their kind hospitality.

A Concatenation of Standard Transformations

Here we shall present some techniques involving ST \([16]\) relevant for the many manipulations of \( q \)-deformed Saalschutzian \( _4F_3 \) hypergeometric functions employed in the main body of the paper.

We are interested in the case where two of the four upper entries are integers and one of the three lower entries is an integer. Since our main focus shall be on the integer entries, let

\[
\begin{pmatrix}
A, & B \\
C
\end{pmatrix}
\]

represent such a situation. The remaining 4 entries are generic only subject to the requirement of the \( q \)-hypergeometric function being balanced. In order to keep the 3 integer entry structure, there are three classes of ST applicable to (89). The first class does not affect the three integers. In the language of (15) and (16), it amounts to

\[
c = A, \ d = B, \ e = C
\]

(90)

By construction, either \( A \) or \( B \) must be non-positive and here we have assumed \( B \leq 0 \). The second class leaves only \( A \) and \( B \) unchanged. Again we may assume that \( B \leq 0 \) in which case the second class of transformations is characterized by

\[
c = A, \ d = B, \ f = C
\]

(91)

The third class only affects one of the two upper entries, and assuming \( B \leq 0 \) we have

\[
a = A, \ d = B, \ e = C
\]

(92)

A simple inspection reveals that 48 different configurations may be obtained by naive (and repeated) applications of ST. The lower entry may take on the 6 different values \( C \), \( 1 + A + B - C \), \( 1 - A + B \) and 2 minus either of these. However, a priori not all of the
configurations thus obtained are well defined since we must have \( d \leq 0 \). Nevertheless, a further inspection shows that all well defined configurations, the number of which depends on the relations between \( A, B \) and \( C \), may be obtained from one another by well defined concatenations only.

Let us illustrate the above by considering the well defined concatenation of STs transforming (14) (subject to (17)) into (18). We have

\[
\begin{pmatrix}
-n_1, -n' \\
1 - n' + n_2
\end{pmatrix}
\rightarrow
\begin{pmatrix}
-n_1, -n' \\
n - n'
\end{pmatrix}
\rightarrow
\begin{pmatrix}
-n_2, -n' \\
n - n'
\end{pmatrix}
\rightarrow
\begin{pmatrix}
-n, -n_2 \\
n - n'
\end{pmatrix}
\rightarrow
\begin{pmatrix}
-n, -n_2 \\
1 + n' - n_2
\end{pmatrix}
\]  

(93)

Finally, a ST of the first kind may be needed in order to match all the entries of the \( q \)-hypergeometric function. It is then straightforward to verify that multiplying all prefactors encountered in this repeated use of (15) with the original ones in (14), produces the ones in (18).

An interesting property of ST is that (modulo extra ST) it commutes with all well defined substitutions of the parameters

\[
a, ..., g \rightarrow a', ..., g' = a' + b' + c' + d' - c' - f' + 1
\]  

(94)

keeping the integer structure as in (99). That is, \( A', B' \) and \( C' \) are integers and at least \( A' \) or \( B' \) is non-positive. Based on our classification above, the proof is almost immediate. Case 1 may be characterized by (90). For \( B' \leq 0 \) the statement is trivial whereas for \( A' \leq 0 \) it becomes equivalent to

\[
\frac{[f - d]_{-c}[e + f - a - b]_{-c}}{[f]_{-c}[e + f - a - b - d]_{-c}} = \frac{[f - c]_{-d}[e + f - a - b]_{-d}}{[f]_{-d}[e + f - a - b - c]_{-d}}
\]  

(95)

which is readily verified. Case 2 may be characterized by (91) and again the statement is trivial for \( B' \leq 0 \). For \( A' \leq 0 \) it becomes equivalent to (95). Case 3 may be characterized by (92) and the statement is trivial for \( B' \leq 0 \). For \( A' \leq 0 \) we need an additional ST in order to prove our assertion. Namely, employing ST twice results in

\[
\binom{a, b, c, d}{e, f, g}; q, 1 = \frac{[f - c]_{-d}[e + f - a - b]_{-d}}{[f]_{-d}[e + f - a - b - c]_{-d}} \cdot \frac{[b + c + d - e - f + 1]_{-d}[a + d - f + 1]_{-d}}{[c + d - f + 1]_{-d}[a + b + d - e - f + 1]_{-d}}
\]  

(96)

and the statement becomes equivalent to

\[
\frac{[f - d]_{-a}[e + f - b - c]_{-a}}{[f]_{-a}[e + f - b - c - d]_{-a}} = \frac{[f - c]_{-d}[e + f - a - b]_{-d}}{[f]_{-d}[e + f - a - b - c]_{-d}} \cdot \frac{[b + c + d - e - f + 1]_{-d}[a + d - f + 1]_{-d}}{[c + d - f + 1]_{-d}[a + b + d - e - f + 1]_{-d}}
\]  

(97)

which is readily verified.
B Remarks on Uniqueness

We start from the following simple recursion relation

\[
\delta_{l_2,n'} = e^{i\alpha_1 - i\alpha_0} R(-1/2, J'; \varpi - 1)^{l_2-1,n'-l_2} R(1/2, J'; \varpi)_{l_1, l_2}^{1,0} + R(-1/2, J'; \varpi - 1)^{l_2',-l_2} R(1/2, J'; \varpi)_{l_1, l_2}^{1,1}
\]

(98)

where \(e^{i\alpha_0}\) and \(e^{i\alpha_1}\) are phase factors that will be specified below. The \(R\)-matrices with spins \(-1/2\) and \(J'\) are considered as unknowns, while the ones with spins \(1/2\) and \(J'\) are known since the braiding of a degenerate field with any other primary is determined by null vector decoupling equations \([22]\). Eq. (98), as a two-term recursion relation, has an infinity of solutions that can be labelled for instance by an initial condition \(R(-1/2, J'; \varpi)^{l_2-1,n'-l_2}\). Note that while we assume that \(n'\) is a positive integer, we admit \(l_2\)-values of the form \(l_2 = l_2 + p\), with \(l_2\) a fixed real number and \(p\) an arbitrary integer. We can trivially rewrite Eq. (98) as

\[
\delta_{l_2,n'} = \sum_{n_2} e^{i\alpha_1 - i\alpha_{1+n_2}} R(-1/2, J'; \varpi - 1)^{n_2,n' - 1,n_2} R(1/2, J'; \varpi)_{l_1, l_2}^{1,1+n_2} \sum_{l_1=0}^{1} R(1/2, J'; \varpi)_{l_1, l_2}^{1+n_2-l_1, l_1} V_{-J'}^{(J')} S_{l_1, l_2}^{n_2+n_2-l_1, l_1} S_{l_1, l_2}^{n_2-1+n_2+l_1}(\sigma),
\]

(99)

since \(1 + n_2 - l_2\) can take the values 0 and 1 only (degenerate fields braid into degenerate fields). Now we multiply both sides by \(V_{-J'}^{(J')} S_{l_2}^{1}(\sigma') S_{n_2-1,l_2}^{n_2-1}(\sigma)\) from the right and sum over \(l_2 = l_2, l_2 + 1, l_2 + 2, \ldots\), or equivalently over \(l_1 = 1 + n_2 - l_2\). When \(l_2\) is integer, we get

\[
V_{-J'}^{(J')} S_{l_2}^{n_2}(\sigma') = \sum_{n_2} e^{i\alpha_1 - i\alpha_{1+n_2}} R(-1/2, J'; \varpi - 1)^{n_2,n' - 1,n_2} \sum_{l_1=0}^{1} R(1/2, J'; \varpi)_{l_1, l_2}^{1+n_2-l_1, l_1} V_{-J'}^{(J')} S_{l_1, l_2}^{1+n_2-l_1, l_1} S_{l_1, l_2}^{n_2-1+n_2+l_1}(\sigma).
\]

(100)

otherwise the left hand side vanishes. For definiteness of notation, we will write down the equations for \(l_2\) integer in the following. Let us now rewrite (somewhat artificially) \(S_{n_2-1,n_2+l_1}(\sigma)\) as

\[
S_{n_2-1,n_2+l_1}(\sigma) = V_{-1/2}^{(1/2)} S_{l_1}^{1}(\sigma) \otimes V_{1/2}^{(1/2)} S_{n_2-1,n_2}(\sigma) e^{i\alpha_1};
\]

(101)

Here, the renormalized product of Eq. (7) reappears, and the phase factor is given by Eq. (8), i.e. \(e^{i\alpha_1} = q^2\). Thus, Eq. (100) becomes

\[
V_{-J'}^{(J')} S_{n_2}(\sigma') = \sum_{n_2} e^{i\alpha_1} R(-1/2, J'; \varpi - 1)^{n_2,n' - 1,n_2} \sum_{l_1=0}^{1} R(1/2, J'; \varpi)_{l_1, l_2}^{1+n_2-l_1, l_1} \cdot V_{-J'}^{(J')} S_{l_1, l_2}^{1+n_2-l_1, l_1} V_{-1/2}^{(1/2)} S_{l_1}^{1}(\sigma) \otimes V_{1/2}^{(1/2)} S_{n_2-1,n_2}(\sigma),
\]

(102)

Here we recognize that the sum over \(l_1\) in front of the renormalized product is nothing but the result of the braiding of \(V_{-1/2}^{(1/2)} S(\sigma)\) with \(V_{-J'}^{(J')} S_{n_2}(\sigma')\). Pulling out the first of these two operators to the left, we have using the second line of Eq. (7),

\[
V_{-J'}^{(J')} S_{n_2}(\sigma') = \lim_{\sigma_1 \rightarrow \sigma} \sum_{n_2} SV_{-1/2}^{(1/2)} (\sigma_1) R(-1/2, J'; \varpi - 1)^{n_2,n' - 1,n_2} \cdot V_{-J'}^{(J')} S_{n_2}(\sigma') V_{1/2}^{(1/2)} S_{n_2-1,n_2}(\sigma) \frac{1}{(1 - e^{i(\sigma_1 - \sigma)})^{-\Delta_{1/2} - \Delta_{-1/2}}},
\]

(103)
Note that the remaining phase factor $e^{i\alpha_1}$ has been removed by reordering $S(\sigma)$ and $V_{-1/2}(\sigma)$. Now we can remove the factor $SV_{-1/2}(\sigma)$ on the right hand side by multiplying both sides by $V_{1/2}(-1/2)S^{-1}(\sigma)$, and we arrive finally at

$$V_{1/2}(-1/2)S^{-1}(\sigma)V^{(J')}_{-\sigma'}S^{n'}(\sigma') = \sum_{n_2} \frac{V^{(J')}_{-\sigma'}S^{n_2}(\sigma')V_{1/2}(-1/2)S^{n-n_2}}{R(-1/2, J'; \varpi)}$$

(104)

when $l_2$ is integer. This is the defining relation for the $R$-matrix $R(-1/2, J'; \varpi)$ and thus appears to be a triviality. However, in our derivation, $R(-1/2, J'; \varpi)$ is, a priori, not an $R$-matrix but a solution to Eq. (104). In particular, it is not unique, as long as we do not impose our selection rule. So the surprising conclusion is that there is an infinity of equivalent $R$-matrices for the braiding problem Eq. (104). This conclusion is not specific to the spin $J = -1/2$ and $n = -1$ but applies to type II and III,IV in general, as can be seen immediately by concatenation type arguments. When the starting value $l_2$ is not integer, the left hand side of Eq. (104) vanishes. This means that to any $R$-matrix with integer outgoing screening numbers, we can add contributions with non-integer outgoing screenings, if the latter are solutions to Eq. (104).

Of course, due to the presence of infinite sums one may question the rigorousness of various steps in the above argument, but it certainly poses an interesting problem, which merits further investigation. Note that except possibly for the step from Eq. (103) to Eq. (104), the argument can also be read backwards.

An interesting special case arises when $V^{(J')}_{-\sigma'}S^{n'}(\sigma')$ is degenerate ($2J'$ positive integer, $0 \leq n' \leq 2J'$) so that the null vector decoupling equations control the braiding problem Eq. (104). The resulting braiding matrix has integer outgoing screenings and is non-vanishing only for a finite number of $n_2$-values. This braiding matrix is the only solution to Eq. (104) with this property, and it is also the one that is produced by imposing our selection rule - a welcome verification of the latter. However, Eq. (104), even when restricted to integer $l_2$, also allows for solutions corresponding to infinite braiding sums in Eq. (104), with $n_2$ bounded from above but not from below. Indeed, the vanishing of $R(1/2, J'; \varpi)\frac{V^{(J')}_{-\sigma'}S^{n_2}(\sigma')V_{1/2}(-1/2)S^{n-n_2}}{R(-1/2, J'; \varpi)}$ for $l_2 = 2J' + 1$ implies that $R(-1/2, J'; \varpi)\frac{V^{(J')}_{-\sigma'}S^{n_2}(\sigma')V_{1/2}(-1/2)S^{n-n_2}}{R(-1/2, J'; \varpi)} = 0$ for $i = 0, 1, 2, ..., 2J'$ positive integer, $0 \leq n' \leq 2J'$. However, Eq. (104) does not enforce any truncation of these matrix elements at large negative $i$, though the equation at $l_2 = n'$ allows for such a truncation to occur through the vanishing of the $R$-matrix element $R(-1/2, J'; \varpi)\frac{V^{(J')}_{-\sigma'}S^{n-n_2}}{R(-1/2, J'; \varpi)}$ on the right hand side.

From the point of view of the monodromy analysis for solutions of the null vector decoupling equations, the appearance of additional solutions seems to be consistent only if there exist formal infinite linear combinations of $q$-hypergeometric functions (for $J' = 1/2$) and their generalizations, with parameters varying as a function of $n_2$, that vanish. The study of these questions is left for further analysis.

References

[1] B.L. Feigin and D.B. Fuchs, Lect. Notes in Math. 1060 (1984) 230.
[2] Vl. Dotsenko and V. Fateev, Nucl. Phys. B 240 (1984) 312; Nucl. Phys. B 251 (1985) 691.

[3] G. Felder, J. Fröhlich and J. Keller, Commun. Math. Phys. 124 (1989) 647.

[4] J.L. Petersen, J. Rasmussen and M. Yu, Nucl. Phys. B 457 (1995) 309; Nucl. Phys. B 481 (1996) 577; J. Rasmussen, Ph.D. thesis, October 1996, The Niels Bohr Institute, Copenhagen, hep-th/9610167.

[5] J.-L. Gervais and A. Neveu, Nucl. Phys. B 224 (1983) 329.

[6] D. Luest and J. Schnittger, Int. Jour. Mod. Phys. A6 (1991) 3625; J. Schnittger, Ph.D. thesis, Munich 1990.

[7] J.-L. Gervais and J. Schnittger, Phys. Lett. B 315 (1993) 258.

[8] J.-L. Gervais and J. Schnittger, Nucl. Phys. B 431 (1994) 273.

[9] J.-L. Gervais and J.-F. Roussel, Nucl. Phys. B 426 (1994) 140; J.-F. Roussel, Ph.D. thesis, March 1995 Ecole Normale Supérieure, Paris.

[10] J. Schnittger, Nucl. Phys. B 471 (1996) 521.

[11] M. Goulian and M. Li, Phys. Rev. Lett. 66 (1991) 2051.

[12] A. Bilal and J.-L. Gervais, Nucl. Phys. B 318 (1989) 579.

[13] A. Anderson, B.E.W. Nilsson, C.N. Pope and K.S. Stelle, Nucl. Phys. B 430 (1994) 107.

[14] G. Moore and N. Seiberg, Phys. Lett. B 212 (1988) 454; Nucl. Phys. B 313 (1989) 16; Commun. Math. Phys. 123 (1989) 177.

[15] J. Rasmussen and J. Schnittger, New Identities for $q$-hypergeometric Functions from Analyzing the Moore-Seiberg Equations, preprint in preparation.

[16] J.-L. Gervais, Commun. Math. Phys. 130 (1990) 257.

[17] E. Cremmer, J.-L. Gervais and J.-F. Roussel, Commun. Math. Phys. 161 (1994) 597.

[18] E. Cremmer, J.-L. Gervais and J. Schnittger, Commun. Math. Phys. 183 (1997) 609.

[19] J.-L. Gervais and A. Neveu, Nucl. Phys. B 209 (1982) 125.

[20] J.-L. Gervais, Nucl. Phys. B 391 (1993) 287.

[21] E. Cremmer, J.-L. Gervais and J.-F. Roussel, Nucl. Phys. B 413 (1994) 244.

[22] J.-L. Gervais and A. Neveu, Nucl. Phys. B 238 (1984) 125.
[23] L.C. Slater, *Generalized Hypergeometric Functions* (Cambridge University Press, 1966).

[24] O. Babelon, Phys. Lett. **B 215** (1988) 523.

[25] L. O’Raifeartaigh, J. Pawłowski and V. Sreedhar, *Duality in Quantum Liouville Theory*, hep-th/9811090.

[26] H.J. Otto and G. Weigt, Z. Phys. **C 31** (1986) 219.

[27] A.B. Zamolodchikov and Al.B. Zamolodchikov, Nucl. Phys. **B 477** (1996) 577.

[28] J. Teschner, *On Structure Constants and Fusion Rules in the SL(2,C)/SU(2) WZNW model*, hep-th/9712250.

[29] J. Teschner, *The Minisuperspace Limit of the SL(2,C)/SU(2) WZNW model*, hep-th/9712258.

[30] J.-L. Gervais and A. Neveu, Phys. Lett. **B 151** (1985) 271.

[31] J.-L. Gervais, Commun. Math. Phys. **138** (1991) 301.

[32] J.-L. Gervais and J.-F. Roussel, Phys. Lett. **B 338** (1994) 437; Nucl. Phys. **B 478** (1996) 245.