PERIODIC BILLIARDS IN ISOSCELES TRIANGLES

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Abstract. Any periodic trajectory on an isosceles triangle gives rise to a periodic trajectory on a right triangle obtained by identifying the halves of the original triangle. We examine the relationship between periodic trajectories on isosceles triangles and the trajectories on right triangles obtained in this manner, and the consequences of this relationship for the existence of stable trajectories on isosceles triangles and the properties of their orbit tiles.

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1. Introduction

We consider an idealized model of a billiard ball on a convex polygonal table, behaving as a point mass with no friction which collides perfectly elastically with the edges of the table. Such systems are referred to as “polygonal billiards” and are a common subject of study in the field of dynamics. A complete description of the system can be found in a survey of the subject by Boldrighini, Keane and Marchetti [1].

Many natural questions about polygonal billiards remain open. Most notably, it is unknown whether general polygonal billiards admit periodic trajectories, and whether generic polygonal billiards are ergodic. An account of the proven and conjectured properties of periodic trajectories can be found in Section 1 of Schwartz’s article [9]. Restricted cases have been shown to have periodic trajectories, including rational polygons as proven by Masur in [6], and triangles with maximal angles less than 100 degrees as proven by Schwartz in [9]. Discussion of ergodicity results can be found in a survey by Gutkin [3]. Recently there has been considerable interest in two closely related objects associated with periodic trajectories, the combinatorial type and the orbit tile.

Definition 1.1. Let f be a periodic trajectory. The combinatorial type of f is the sequence \( \sigma_1 \cdots \sigma_k \) of edges struck by f. Two combinatorial types are considered equivalent if they are identical or one is obtained by reversing the other.
Definition 1.2. Given a combinatorial type $C$, the orbit tile $O(C)$ of $C$ is the set of polygons which admit a periodic trajectory with combinatorial type $C$. The orbit tile of a periodic trajectory is the orbit tile of its combinatorial type.

Analysis of these objects has been successfully applied to the problem of finding periodic trajectories, most notably by Hooper and Schwartz in [5, 8, 9]. This work has suggested that stable periodic trajectories are of particular interest (see for example [4, 5]).

Definition 1.3. A periodic trajectory is stable if its orbit tile is open.

This paper focuses exclusively on the theory of periodic trajectories up to combinatorial type; we let $\sim$ denote this equivalence. We restrict our attention to the special case of an isosceles triangle $T$. Hooper and Schwartz showed in [5] that all isosceles triangles—and in fact triangles sufficiently close to being isosceles—possess periodic trajectories. One interesting property of periodic trajectories $f$ on $T$ is that, by identifying each point in $T$ with its reflection across the line of symmetry, we obtain a periodic trajectory $f'$ on the resulting right triangle $T'$. This paper studies the relationship between these trajectories, and in particular between their combinatorial types. This relationship turns out to be governed by whether $T$ satisfies the following condition.

Condition 1.4. The base angle of $T$ is either an irrational multiple of $\pi$ or of the form $\frac{a}{b}\pi$ with $a, b \in \mathbb{Z}$ and $b$ even.

In particular, we prove the following theorem.

Theorem 1.5. Let $f$ be a periodic trajectory on an isosceles triangle $T$ satisfying Condition [1.4]. If $g$ is a periodic trajectory on $T$ and $f' \sim g'$, then $f \sim g$.

Continuing in the tradition of Hooper and Schwartz, we interpret this result as a statement about orbit tiles. As usual, we regard the set of triangles as the set of pairs $(\theta_1, \theta_2)$ with $0 < \theta_1, \theta_2 < \frac{\pi}{2}$, which are interpreted as triangles with angles $\theta_1, \theta_2$ and $\pi - \theta_1 - \theta_2$.

Corollary 1.6. Any orbit tile which contains an isosceles triangle satisfying Condition [1.4] is symmetric across the line of isosceles triangles.

For isosceles triangles not satisfying Condition [1.4] Theorem [1.5] fails drastically. In fact, we prove:

Theorem 1.7. Let $T$ be an isosceles triangle with base angle $\frac{a}{b}\pi$, $b$ odd, $x \in T$ a point in the base other than its center, $\theta \in S^1$ and $\text{Orb}(x, \theta)$ the trajectory originating from $x$ with direction $\theta$. Then the set of $\theta$ such that $\text{Orb}(x, \theta)$ is periodic satisfies Theorem [1.5] is nowhere dense.

By a well-known result of Masur in [6], the set of $\theta$ such that the trajectory originating from $x$ with direction $\theta$ is periodic is dense. Thus Theorem [1.7] provides a converse to Theorem [1.5]. Because Theorem [1.5] holds on a dense set of isosceles triangles, Theorem [1.7] can be interpreted as a result on stable trajectories.

Corollary 1.8. Let $T$ be an isosceles triangle with base angle $\frac{a}{b}\pi$, $b$ odd, and $x \in T$ a point in the base other than its center. Then the set of $\theta$ such that $\text{Orb}(x, \theta)$ is periodic and stable is nowhere dense.
2. Preliminaries

In this section we work with arbitrary convex polygons rather than restricting our attention to triangles. Let \( P \subset \mathbb{C} \) be a convex \( n \)-gon with vertices \( v_i \), labeled counter-clockwise. Let \( e_i \) denote the edge from \( v_i \) to \( v_{i+1} \), with addition interpreted mod \( n \).

**Definitions 2.1.** A *periodic trajectory* is a function \( f : [0, 1] \to P \) with constant unit derivative, except at the edges of \( P \) where the derivative is reflected across the edge. A *cylinder of periodic trajectories* is a continuous function \( g : (0, 1) \times [0, 1] \to P \) such that fixing the first coordinate of \( g \) for any \( s \in (0, 1) \) gives a periodic trajectory \( g_s \), and such that \( \frac{dg}{dt} \big|_0 \) is independent of \( s \).

Recall that periodic trajectories give rise to lines in the *unfolding* of \( P \), where instead of reflecting the billiard ball off an edge, we reflect the polygon across the edge, as in Figure 2.1.

We will make heavy use of the unfolding, particularly in the rational case where it forms a compact Riemann surface (in fact, a translation surface). We develop the unfolding using the following notation.

**Notation 2.2.** Let \( r_i \) denote the (real) linear part of the reflection map across \( e_i \). Let \( G(P) \) be the group of such maps.

Recall that \( G(P) \) is finite iff \( P \) is rational, in which case we can make use of the following construction. This construction is essentially identical to the one found in the survey by Masur and Tabachnikov [7].

**Construction 2.3.** We construct the Riemann surface \( R(P) \) by gluing together the polygons \( \alpha P \) for \( \alpha \in G(P) \). We glue \( \alpha P \) and \( r_i \alpha P \) along \( e_i \) such that \( v_i \in \alpha P \) is identified with \( v_i \in r_i \alpha P \) and \( v_{i+1} \in \alpha P \) with \( v_{i+1} \in r_i \alpha P \). The inclusion maps \( \varphi_\alpha : \alpha P \hookrightarrow R(P) \) give us local parametrizations near every point in the interior of \( \alpha P \). For points in the edge \( \varphi_\alpha(\alpha v_i) \), we define local parametrizations via the inclusion map \( \alpha P \cup r_i \alpha P \hookrightarrow R(P) \). At the vertex \( \varphi_\alpha(\alpha v_i) \) the polygons

\[ \begin{array}{c}
\begin{array}{c}
\vdots
\end{array}
\end{array} \]
\[ \alpha P, r_{i-1} \alpha P, r_{i-1} \alpha P, r_{i-1} r_{i-1} \alpha P, \ldots, (r_{i} r_{i-1})^{m} \alpha P = \alpha P \text{ meet.} \]

We define a local parametrization near \( \varphi_{\alpha}(\alpha v_{i}) \) by

\[
\varphi(z) = \begin{cases} 
\varphi_{\alpha}(z^{k}) & \text{if } 0 \leq \text{Arg}(z) \leq \frac{\phi}{m} \\
\varphi_{r_{i-1}}(z^{k}) & \text{if } \frac{\phi}{m} \leq \text{Arg}(z) \leq \frac{2\phi}{m} \\
\vdots & \vdots \\
\varphi_{r_{i-1}(r_{i-1})^{m-1}}(z^{k}) & \text{if } \frac{(2m-1)\phi}{m} \leq \text{Arg}(z) \leq \frac{2m\phi}{m} 
\end{cases}
\]

where \( \phi = \frac{1}{m} \pi \) is the angle between \( e_{i-1} \) and \( e_{i} \). Together these maps form an atlas for \( R(P) \).

The standard 1-form \( dz \) on \( \mathbb{C} \) induces a holomorphic 1-form \( dz_{P} \) via the inclusions \( \varphi_{\alpha} \). It is easy to see that this is nonzero except at the vertices \( \varphi_{\alpha}(\alpha v_{i}) \), where it has a zero of order \( k-1 \).

We have a natural action of \( G(P) \) on \( R(P) \) where \( \beta \in G(P) \) sends \( \varphi_{\alpha}(\alpha P) \) to \( \varphi_{\beta \alpha}(\beta \alpha P) \).

Illustrations of Construction 2.3 can be found in Figures 3.2 and 3.3. While it is common to square \( dz_{P} \) to produce a quadratic differential, we shall restrict our attention to the 1-form.

**Definitions 2.4.** Trajectories on \( R(P) \) are paths \( f \) which do not contain any vertex such that \( dz_{P} \circ df = e^{i\theta} \) for some fixed \( \theta \), called the argument of \( f \). A cylinder of periodic trajectories is defined similarly.

The requirement that trajectories not contain a vertex is crucial, because one way we show that two trajectories are not in the same cylinder is by showing that any cylinder containing both trajectories contains a vertex. This is especially useful in light of the following standard results, the first of which can be found in Section 3 of [1].

**Lemma 2.5.** For any \( x \in P \), the set of angles \( \theta \in S^{1} \) such that \( \text{Orb}(x, \theta) \) comes arbitrarily close to vertices of \( P \) has full measure.

Clearly any two periodic trajectories in the same cylinder will have the same combinatorial type. In fact, the converse of this holds as well. The proof of this fact is obvious to those familiar with billiards, but is included here for completeness.

**Lemma 2.6.** If \( f \sim g \), then up to reversal \( f \) and \( g \) lie in the same cylinder of periodic trajectories.

**Proof.** Up to reversal, \( f \) and \( g \) strike the same sequence of edges. They also must have the same initial angle, as otherwise by considering the unfolding it is clear that they would eventually strike different edges. Let \( h : [0, 1] \times [0, 1] \rightarrow P \) be the cylinder obtained by translating from \( f \) to \( g \). Clearly \( h_{s}(1) = h_{s}(0) \) for all \( s \in [0, 1] \), so the only possible obstruction to this being extendable to a cylinder of periodic trajectories is that its image contains a vertex of \( P \). But then we would have some smallest \( t \) such that there exists some \( s \in [0, 1] \) for which \( h_{s}(t) \) is a vertex of \( P \), and thus at this point \( f(t) = h_{0}(t) \) and \( g(t) = h_{1}(t) \) lie on different edges, contradicting the fact that they strike the same sequence of edges. \( \square \)

**Remark 2.7.** Since the copies of \( P \) in \( R(P) \) are indistinguishable from the edge labeling, if \( f \) and \( g \) are regarded as trajectories on \( R(P) \) then in order for Lemma 2.6 to hold we must allow an action of \( G(P) \). Similarly, it was necessary to allow
reversal in the definition of combinatorial type since there is no canonical choice of orientation for $R(P)$ (as some elements of $G(P)$ are orientation-reversing).

The following lemmas provide a useful characterization of orbit tiles. A version of this lemma for triangles is proven in Section 2.5 of [9]; however that version is more suited for computations and less useful for our purposes.

**Lemma 2.8.** Let $\mathcal{P}$ denote the set of polygons $P$ with any of the standard metrics. Let $j_1, \ldots, j_k \in \{1, 2, \ldots, n\}$ with $j_1 = j_k$ and let $\theta_i$ denote the angles of $P$. Then there exist

1. a continuous, analytic a.e. function $\Omega : \mathcal{P} \times (0, 1) \times S^1 \to \mathbb{R}$ such that $\text{Orb}(P, xv_{j_1} + (1 - x)v_{j_1+1}, \theta)$ strikes $e_{j_1}, \ldots, e_{j_k}$ in order iff $\Omega(P, x, \theta) > 0$,
2. an analytic function $D : \mathcal{P} \to S^1$ such that, assuming $\text{Orb}(P, xv_{j_1} + (1 - x)v_{j_1+1}, \theta)$ strikes $e_{j_1}, \ldots, e_{j_k}$, it is periodic iff $D(P) = \theta$, and
3. if $k$ is even, a linear function $\Theta(\theta_1, \ldots, \theta_n)$ with even integer coefficients which gives the change in direction of any trajectory striking $e_{j_1}, \ldots, e_{j_k}$.

**Proof.** For $\Omega$, consider the unfolding of $P$ along the edges $e_{j_1}, \ldots, e_{j_k}$. Note that for each $1 \leq i < k$, any trajectory which passes through $e_{j_1}, \ldots, e_{j_i}$ passes through $e_{j_{i+1}}$ if it passes through $v_{j_{i+1}}$ and $v_{j_{i+1}+1}$. The position of each vertex and the slope of each edge is an analytic function of $P$. Thus the intersection of $\text{Orb}(xv_{j_1} + (1 - x)v_{j_1+1}, \theta)$ with each edge $e_{j_i}$ is an analytic function of $P, x$ and $\theta$. Call these functions $h_i(P, x, \theta)$. Let $p_i : \mathbb{R}^2 \to \mathbb{R}$ be the functional which takes the component of a vector in the direction of $e_{j_i}$. We have shown that $\text{Orb}(xv_{j_1} + (1 - x)v_{j_1+1}, \theta)$ is as desired iff $p_i(v_{j_i}) < p_i(h_i(P, x, \theta))$ and $p_i(v_{j_i+1}) > p_i(h_i(P, x, \theta))$ for all $i$. Thus if we let

$$\Omega(P, x, \theta) = \min \left( \min \{p_i(h_i(P, x, \theta) - v_{j_i})\}, \max \{p_i(v_{j_i+1} - h_i(P, x, \theta))\} \right)$$

we get the desired function.

For $D$, note that in order for the trajectory to return to its starting point, it must be parallel to the line connecting the first and last vertex described above. The direction of this line is easily seen to be an analytic function.

For $\Theta$, note that upon striking an the edge $e_{j_i}$ and then $e_{j_{i+1}}$, the angle of a trajectory has been rotated by twice the angle between them, which is one of the $\theta_j$. Since $k$ is even, this gives us the desired function. \hfill $\square$

For periodic trajectories, we can always assume $k$ is even as we can have the trajectory repeat. We will use an easy but important fact for triangles that follows from Lemma 2.8. This is stated in Section 2.1 of [9], but not proven. A less precise version is given by Lemma 7.1 in [8].

**Corollary 2.9.** An orbit tile of triangles is either an open set or an open subset of a line (in the subspace topology).

**Proof.** Let $C$ be a combinatorial type, and note that $T \in \mathcal{O}(C)$ iff we have some $x \in (0, 1)$ such that $\Omega(T, x, D(T)) > 0$ and $\Theta(T) = 0$. Let $\Lambda_x(T) = \Omega(T, x, D(T))$. Thus

$$\mathcal{O}(C) = \ker \Theta \cap \left( \bigcup_{x \in (0, 1)} \Lambda_x^{-1}((0, \infty)) \right)$$

and since $\Theta$ is affine of rank 0 or 1, $\ker \Theta$ can either be the entire space or a line. Since $\Lambda_x$ is continuous for all $x \in (0, 1)$ and $(0, \infty)$ is open, the union of the preimages is open and the result follows. \hfill $\square$
Before proceeding to the proofs of our main theorems, we must point out two peculiarities of our definitions.

Remark 2.10. Some authors also consider combinatorial types equivalent if they differ by cyclic permutations, to allow for different parametrizations of the trajectories. We will also require that trajectories originate from the base of the triangle, which is justified by the fact that a periodic trajectory must strike all three edges. It should be clear from the proof of Theorem 1.5 that the theorem also holds with these two peculiarities removed. However, we do not know whether Theorem 1.7 holds if we consider combinatorial types differing by cyclic permutations equivalent. On the other hand, if we do not consider such combinatorial types equivalent, we must require that trajectories originate on the base in order for Theorem 1.5 to hold. Fortunately these peculiarities do not impact the corollaries.

3. Main Theorems

From this point on, $T$ denotes an isosceles triangle and $T'$ the right triangle obtained by identifying the points in $T$ with their reflection across the line of symmetry. The labeling of vertices is shown in Figure 3.1.

We can reduce Theorem 1.5 to the rational case by the following lemma.

**Lemma 3.1.** Any orbit tile containing an irrational isosceles triangle $T$ contains a rational isosceles triangle with base angle $\frac{a}{b}\pi$ where $b$ is even.

**Proof.** Recall the function $\Theta$ from Lemma 2.8. By the lemma, the zeros of $\Theta$ satisfy $x\theta_1 + y\theta_2 + z\pi = 0$ for some $x, y, z \in \mathbb{Z}$, and since $\theta_1 = \theta_2 = \theta$ we get $(x+y)\theta + z\pi = 0$. If $x + y = 0$ then $z = 0$ as well. Thus either $\Theta$ is identically 0, in which case the orbit tile is open, or ker $\Theta$ is the line of isosceles triangles so by Corollary 2.9 the orbit tile is an open subset of this line. Since the set of isosceles triangles with base angle $\frac{a}{b}\pi$ where $b$ is even is dense in the line of isosceles triangles, the orbit tile must contain some such triangle. If $x + y \neq 0$ then $\theta = \frac{z}{x+y}\pi$ and the third angle is $\pi + \frac{2z}{x+y}\pi$, contradicting the fact that $T$ is irrational. \(\square\)

Note that Lemma 3.1 does not show that every periodic trajectory on an irrational isosceles triangle is stable. Indeed, it is an easy exercise to show that the orbit tile of the combinatorial type 3132 contains all isosceles triangles, but no other triangles, hence is associated with an unstable trajectory.

3.1. **Theorem 1.5.** In this section, we assume $b$ is even. The bulk of Theorem 1.5 is contained in the following lemma, which is illustrated by Figure 3.2.
Lemma 3.2. There exists a double-cover \( \pi : \mathcal{R}(T) \to \mathcal{R}(T') \) of Riemann surfaces such that \( d\pi_T = d\pi_{T'} \circ d\pi \). Furthermore, the deck transformation \( \lambda \) preserves combinatorial types.

Proof. Note that \( G(T) = G(T') \), since \( r_1 \) and \( r_3 \) agree in both groups while \( r_3(r_1r_3)^{b/2} \in G(T) \) equals \( r_2 \in G(T') \). Let \( r \) denote these maps. Let \( \varphi_\alpha \) denote the inclusion map into \( \mathcal{R}(T) \) and \( \varphi'_\alpha \) the inclusion map into \( \mathcal{R}(T') \). For \( z \) in the image of \( \varphi_\alpha \), define \( \pi : \mathcal{R}(T) \to \mathcal{R}(T') \) by

\[
\pi(z) = \begin{cases} 
\varphi'_\alpha(\varphi^{-1}_\alpha(z)) & \text{if } \Re(\alpha^{-1}\varphi^{-1}_\alpha(z)) \geq 0 \\
\varphi'_{\alpha'}(r\varphi^{-1}_\alpha(z)) & \text{if } \Re(\alpha^{-1}\varphi^{-1}_\alpha(z)) \leq 0
\end{cases}
\]

A simple check shows that this is well-defined, locally biholomorphic and that \( d\pi_T = d\pi_{T'} \circ d\pi \). Intuitively we are mapping each copy of \( T \) in \( \mathcal{R}(T) \) onto a pair of copies of \( T' \) in \( \mathcal{R}(T') \) identified along \( e_2 \). Since each copy of \( T \) is mapped onto two copies of \( T' \), \( \pi \) is a double-cover.

For \( z \) in the image of \( \varphi_\alpha \), it is easy to verify that the deck transformation \( \lambda \) is given by \( \lambda(z) = \varphi_{\alpha'}(r\varphi^{-1}_\alpha(z)) \). Let \( f \) be a periodic trajectory on \( \mathcal{R}(T) \). Clearly \( d\lambda \) is linear, thus \( \lambda \circ f \) is also a periodic trajectory. In order to show that \( \lambda \circ f \) has the same combinatorial type as \( f \), it suffices to show that \( \lambda \circ f \) is the reversal of \( (r_1r_3)^{b/2} \circ f \). Note that \( \lambda \circ f(0) = (r_1r_3)^{b/2} \circ f(0) \) as \( f(0) \in e_3 \). Since \( d\lambda = dr = -(r_1r_3)^{b/2} \) we have \( d(\lambda \circ f) = -(r_1r_3)^{b/2} \circ f \), thus \( \lambda \circ f(t) = (r_1r_3)^{b/2} \circ f(1-t) \). \( \square \)

We are now able to prove Theorem 1.5 and Corollary 1.6.}

Proof of Theorem 1.5. Note that the trajectories \( f' \) and \( g' \) on \( T' \) obtained from \( f \) and \( g \) are \( \pi \circ f \) and \( \pi \circ g \). Since these have the same combinatorial type, by Lemma 2.6 up to reversal we have some \( \alpha \in G(T') \) and some cylinder of periodic trajectories \( \hat{h} : (0,1) \times [0,1] \to \mathcal{R}(T') \) such that for some \( u, v \in (0,1) \), \( \pi \circ f = h_u \) and \( \alpha \circ \pi \circ g = h_v \). Lifting this to a function \( \hat{h} : (0,1) \times [0,1] \to \mathcal{R}(T) \), we get that \( \hat{h}_u \) has constant argument for each \( s \in (0,1) \) since \( d\pi_T = d\pi_{T'} \circ d\pi \). Since \( \hat{h} \) can be chosen such that \( \hat{h}_u = f \), with this choice of \( \hat{h} \) we see that \( \hat{h}_u(0) = \hat{h}_u(1) \), thus \( \hat{h}_u(0) = \hat{h}_u(1) \) for all \( s \in (0,1) \) as the lift of the line \( \hat{h}_u(1) : (0,1) \to \mathcal{R}(T') \) is determined by \( \hat{h}_u(1) \), thus \( \hat{h} \) is a cylinder of periodic trajectories. It is easy to see that \( \alpha \circ \pi = \pi \circ \alpha \), thus we get that \( \hat{h} \) is a lift of \( \alpha \circ g \). Since \( \lambda \) preserves combinatorial types, it follows that \( \hat{h}_u \) has the same combinatorial type as \( \alpha \circ g \). Since \( f \) and \( \hat{h}_u \) lie in the same cylinder of periodic trajectories, \( f \sim \hat{h}_u \). Thus \( f \sim g \) as well. \( \square \)

Proof of Corollary 1.6. Let \( f \) be a periodic trajectory on an isosceles triangle \( T \) satisfying Condition 1.4. Let \( g \) be the trajectory obtained by reflecting \( f \) across the line of symmetry of \( T \). Since the roles of the two congruent edges are reversed, it is clear that the orbit tile of \( g \) is the reflection of that of \( f \) across the line of isosceles triangles. But clearly \( f' \sim g' \), thus by Theorem 1.5 \( f \sim g \), hence the orbit tiles of \( f \) and \( g \) agree. It follows that the orbit tile of \( f \) is symmetric across the line of isosceles triangles. \( \square \)
3.2. **Theorem 1.7.** We now assume $b$ is odd. In this case, instead of a double cover of $R(T')$ by $R(T)$, we get a biholomorphism, as shown in Figure 3.3. This is a result of the fact (which can be easily verified) that when $b$ is odd, $r_2 \in G(T')$ does not correspond to any element of $G(T)$, so $G(T)$ is a proper subgroup of $G(T')$. Note that the under our embedding of $T$, the requirement that $x$ not be the center of the base is equivalent to $x \neq 0$.

**Lemma 3.3.** $R(T)$ and $R(T')$ are biholomorphic, and the 1-forms $dz_T$ and $dz_{T'}$ agree under this biholomorphism.

**Proof.** It is easy to see that the map $\pi$ defined in Lemma 3.2 is also a covering in this case. Since $G(T)$ is a proper subgroup of $G(T')$, we can see that in this case $\pi$ is injective, thus a biholomorphism. Since $dz_T = dz_{T'} \circ d\pi$ by the lemma, the 1-forms agree under $\pi$. 

This biholomorphism allows us to interpret the action of $G(T')$ on $R(T')$ as an action on $R(T)$. In particular, $r_2 \in G(T')$ acts distinctly from any element of $G(T)$. This gives us a way to construct trajectories violating Theorem 1.5. Combining this with Lemma 2.5, we are able to prove Theorem 1.7 and Corollary 1.8.

**Proof of Theorem 1.7.** Let $f = \text{Orb}(x, \theta)$ and assume $f$ is periodic. Suppose $f$ satisfies Theorem 1.5. Then $f \sim r_2 \circ f$. Thus we have some $\alpha \in G(T)$ such that $f$ and $\alpha \circ r_2 \circ f$ lie in the same cylinder of periodic trajectories, up to reversal. Hence their arguments must agree up to negation, so $df$ lies in the $\pm 1$-eigenspace.
of \(d(\alpha \circ r_2) = \alpha r_2\). Since this cannot be the identity, either it is a reflection across a line of angle a multiple of \(\frac{2\pi}{b}\) or it is a nontrivial rotation. In the first case, \(\theta\) is of the form \(\frac{n\pi}{b}\) or \(\frac{\pi}{2} + \frac{n\pi}{b}\), and the set of such angles is finite.

Let \((a, b)\) be an interval in \(S^1\). By Lemma \(2.5\), we have some vertex \(v\) of a copy of \(T\) in \(R(T)\) lying along a trajectory passing through 0 with argument \(\phi \in (a, b)\) which is not of the form \(\frac{n\pi}{b}\) or \(\frac{\pi}{2} + \frac{n\pi}{b}\). Suppose \(\theta = \phi + \epsilon\), with \(|\epsilon|\) sufficiently small that \(\phi + \epsilon\) is not of the form \(\frac{n\pi}{b}\) or \(\frac{\pi}{2} + \frac{n\pi}{b}\). Thus \(\alpha r_2\) must be a nontrivial rotation with real eigenvalue, hence is a rotation by \(\pi\), so \(\alpha = r_3\) and \(f\) lies in the same cylinder of periodic trajectories as the reversal of \(r_3 r_2 \circ f\). Let \(h\) be the cylinder containing \(f\) and the reversal of \(r_3 r_2 \circ f\), say with \(h_u(t) = f(t)\) and \(h_v(t) = r_3 r_2 \circ f(1 - t)\). Then \(h_u(0)\) lies in the \(e_3\) and connects \(x\) and \(r_2 x\), thus \(h\) has width at least \(|x - r_2 x|\) which is positive since \(x \neq 0\). Since the cylinder varies continuously with \(\epsilon\), if \(|\epsilon|\) is sufficiently small the cylinder contains the trajectory connecting 0 and \(v\), thus contains \(v\), a contradiction. Hence the set of such \(\theta\) is not dense in \((a, b)\), so this set is nowhere dense. \(\square\)

**Proof of Corollary 1.8.** It suffices to note that Theorem 1.5 is a statement about combinatorial types, which are constant on orbit tiles. Thus if \(f\) is a stable trajectory originating from \(x\) with argument \(\theta\), Theorem 1.5 holds for \(f\) hence \(\theta\) lies in the nowhere dense set from Theorem 1.7. \(\square\)

The same line of reasoning shows that Theorem 1.7 provides a converse to Corollary 1.6 as well. If the orbit tile of a trajectory were equal to its reflection across the line of isosceles triangles, then by Lemma 2.9 either it is an open set or an open subset of the line of isosceles triangles. In either case, the tile contains an isosceles triangle satisfying Condition 1.4.

The assumption that \(x \neq 0\) is crucial for Theorem 1.7, as it is easy to see that when \(x = 0\), \(r_2 \circ f\) is the reversal of \(f\) and thus Theorem 1.5 holds for \(f\). It is unclear whether it is necessary for Corollary 1.8 in fact it is possible that Corollary 1.8 holds for all polygons.

We close with a remark regarding further applications of these results.

**Remark 3.4.** Let \(T\) be an isosceles triangle with base angle \(\frac{\pi}{b}\) where \(b\) is odd. The proof of Theorem 1.7 together with Corollary 1.6 gives a classification of the stable trajectories on \(R(T)\). Either they have argument 0 or \(\pi/2\) up to the action of \(G(T)\), or they lie in the same cylinder as some trajectory passing through the center of
the base of $T$. These latter trajectories are precisely the “mirror trajectories” first defined and studied by Galperin and Zvonkin in [2]. This classification may be of use in attacking open problems concerning the behavior of stable trajectories on isosceles triangles, such as the conjecture in [5] that no finite collection of orbit tiles covers any neighborhood of certain isosceles triangles with the property of Veech.

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