COVARIANT PHASE DIFFERENCE OBSERVABLES

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ABSTRACT. Covariant phase difference observables are determined in two different ways, by a direct computation and by a group theoretical method. A characterization of phase difference observables which can be expressed as difference of two phase observables is given. Classical limit of such phase difference observables are determined and the Pegg-Barnett phase difference distribution is obtained from the phase difference representation. The relation of Ban’s theory to the covariant phase theories is exhibited.

1. Introduction

In quantum optical phase measurements like heterodyne and eight-port homodyne detections one can measure the phase difference between two single-mode input fields. However, if the second field, the reference field, can be considered as a classical field with well-known phase and (high) amplitude, then the theory reduces to a single-mode theory with one input beam. Under such conditions the heterodyne [1, 2] and the eight-port homodyne [3, 4, 5] detection schemes measure the single-mode phase observable

$$X \mapsto E_{|0\rangle}(X) := \frac{1}{\pi} \int_X \int_0^{\infty} |z\rangle\langle z| |d|dz| d(\text{arg } z)$$

defined in terms of the coherent states $|z\rangle := e^{-|z|^2/2} \sum_{n=0}^{\infty} z^n / \sqrt{n!} |n\rangle$. Here $|z\rangle\langle z|$ denotes the projection on the one-dimensional subspace spanned by $|z\rangle$ and $z = |z| \text{ arg } z$ is a complex number. The phase observable $E_{|0\rangle}$ is covariant with respect to the shifts generated by the single-mode number operator $N := \sum_{n=0}^{\infty} n|n\rangle\langle n|$, that is,

$$e^{i\theta N} E_{|0\rangle}(X)e^{-i\theta N} = E_{|0\rangle}(X + \theta)$$

for all (Borel) sets $X \subseteq [0,2\pi)$ and $\theta \in \mathbb{R}$, with $+$ denoting the addition modulo $2\pi$. This condition is a natural covariance condition for observables describing coherent state phase measurements, and one may define a (single-mode) phase observable as a phase shift covariant normalized positive operator measure [1, 2, 3, 4]. The structure of

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such observables is completely known and they can be characterized in at least four different ways in terms of phase matrices, sequences of unit vectors, sequences of generalized vectors, or using covariant trace-preserving operations, see e.g. [6, 8, 10, 11].

In this paper we consider the difference of two (single-mode) phase observables and we notice that it satisfies a natural covariance condition. We take this condition as the defining condition of (two-mode) phase difference observables. We give both a direct (Sect. 3) and a group theoretical (Sect. 4) characterization of such observables whereas in Section 5 we obtain a characterization of the phase difference observables which can be expressed as a difference of two phase observables. Section 6 puts the phase difference distribution of Barnett and Pegg [12, 13] in the present context. Section 7 studies the classical limit of the two-mode theory whereas Section 8 discusses the relation of Ban’s theory [14, 15, 16] to the covariant phase and phase difference theories. In the final section of the paper some historical remarks are due and the question of measurability of the phase difference is briefly reviewed.

2. Phase difference observables

Two phase observables can be joined in a natural way to a phase difference observable. Indeed, suppose that $E_1 : \mathcal{B}([0, 2\pi)) \to \mathcal{L}(\mathcal{H})$ and $E_2 : \mathcal{B}([0, 2\pi)) \to \mathcal{L}(\mathcal{H})$ are phase observables, where $\mathcal{B}([0, 2\pi))$ is the Borel $\sigma$-algebra of the phase interval $[0, 2\pi)$, $\mathcal{H}$ is a separable Hilbert space spanned by the number states $|n\rangle$, $n \in \mathbb{N}$, and $\mathcal{L}(\mathcal{H})$ is the set of bounded operators on $\mathcal{H}$. The product map $(X, Y) \mapsto E_1(X) \otimes E_2(Y)$ defines a unique operator measure

$$\tilde{E} : \mathcal{B}([0, 2\pi) \times [0, 2\pi)) \to \mathcal{L}(\mathcal{H} \otimes \mathcal{H}),$$

with the property

$$\tilde{E}(X \times Y) = E_1(X) \otimes E_2(Y).$$

Using the function

$$f : [0, 2\pi) \times [0, 2\pi) \to [0, 2\pi), \quad (x, y) \mapsto x - y \mod 2\pi$$

one gets from $\tilde{E}$ the observable which is the difference of the observables $E_1$ and $E_2$:

$$E^{\text{diff}} : \mathcal{B}([0, 2\pi)) \to \mathcal{L}(\mathcal{H} \otimes \mathcal{H}), \quad E^{\text{diff}}(X) := \tilde{E}(f^{-1}(X)).$$
Using the explicit form of a phase observable $E : \mathcal{B}([0, 2\pi)) \to \mathcal{L}(\mathcal{H})$,
\begin{equation}
E(X) = \sum_{n,m=0}^{\infty} \langle \varphi_n | \varphi_m \rangle \frac{1}{2\pi} \int_X e^{i(n-m)\theta} \, d\theta \, |n\rangle \langle m|,
\end{equation}
where $(\varphi_n)_{n \in \mathbb{N}} \subset \mathcal{H}$ is a sequence of unit vectors, one easily computes that the difference of $E_1$ and $E_2$ is
\begin{equation}
E^{\text{diff}}(X) = \sum_{n,m,l,k} \delta_{n-m,l-k} \langle \varphi_n^1 | \varphi_m^1 \rangle \langle \varphi_k^2 | \varphi_l^2 \rangle \frac{1}{2\pi} \int_X e^{i(n-m)\theta} \, d\theta \, |n,k\rangle \langle m,l|.
\end{equation}
Here $\delta$ is the Kronecker delta, $|n,k\rangle \langle m,l|$ stands for the rank one operator $\mathcal{H} \otimes \mathcal{H} \ni \psi \mapsto \langle m,l|\psi\rangle |n,k\rangle \in \mathcal{H} \otimes \mathcal{H}$, and, for instance, $|n,k\rangle$ is the short hand notation for the tensor product vector $|n\rangle \otimes |k\rangle$.

Let
\begin{align*}
\Sigma N &:= N \otimes I + I \otimes N \\
\Delta N &:= N \otimes I - I \otimes N
\end{align*}
denote the sum and the difference of the number operators of the two modes, and let $\Sigma N = \sum_{k \in \mathbb{N}} k P^\Sigma_k$ and $\Delta N = \sum_{k \in \mathbb{Z}} k P^\Delta_k$ be their respective spectral decompositions, with the spectral projections
\begin{align*}
P^\Sigma_k &= \sum_{n=0}^{k} |k-n,n\rangle \langle k-n,n|, \quad k \in \mathbb{N}, \\
P^\Delta_k &= \sum_{n \geq \max\{0,-k\}} |k+n,n\rangle \langle k+n,n|, \quad k \in \mathbb{Z}.
\end{align*}
Consider the unitary operators
\begin{align*}
V_\Sigma(\alpha) &= e^{i\alpha \Sigma N}, \quad \alpha \in \mathbb{R}, \\
V_\Delta(\beta) &= e^{i\beta \Delta N}, \quad \beta \in \mathbb{R}.
\end{align*}
The difference $E^{\text{diff}}$ of the phase observables $E_1$ and $E_2$ is invariant under $V_\Sigma$,
\begin{equation}
V_\Sigma(\alpha) E^{\text{diff}}(X) V_\Sigma(\alpha)^* = E^{\text{diff}}(X),
\end{equation}
for all $\alpha \in \mathbb{R}, X \in \mathcal{B}([0, 2\pi))$. This condition is equivalent to the commutativity of $\Sigma N$ and $E^{\text{diff}}$, that is,
\begin{equation}
P^\Sigma_k E^{\text{diff}}(X) = E^{\text{diff}}(X) P^\Sigma_k
\end{equation}
for all $k \in \mathbb{N}, X \in \mathcal{B}([0, 2\pi))$. Since the number sum is a projection valued observable $k \mapsto P^\Sigma_k$, the commutativity of $\Sigma N$ and $E^{\text{diff}}$ equals with their being (functionally) coexistent, that is, they have a joint
observable, see, for instance, [17]. It is another immediate observation that \( E_{\text{diff}} \) satisfies the following covariance condition under \( V_\Delta \):

\[
V_\Delta(\beta) E_{\text{diff}}(X) V_\Delta(\beta)^* = E_{\text{diff}}(X + 2\beta),
\]

for all \( \beta \in \mathbb{R} \), \( X \in \mathcal{B}([0, 2\pi]) \).

Let

\[
\Theta(\alpha, \beta) = e^{i\alpha N \otimes I + i\beta I \otimes N}, \quad \alpha, \beta \in \mathbb{R}.
\]

Since

\[
\Theta(\alpha, \beta) = V_\Sigma(\frac{\alpha}{2}) V_\Delta(\frac{\alpha}{2}) V_\Sigma(\frac{\beta}{2}) V_\Delta(-\frac{\beta}{2}),
\]

we observe that the invariance and covariance conditions (3) and (4) are equivalent with the condition

\[
\Theta(\alpha, \beta) E_{\text{diff}}(X) \Theta(\alpha, \beta)^* = E_{\text{diff}}(X \mp (\alpha - \beta))
\]

for all \( X \in \mathcal{B}([0, 2\pi]) \) and \( \alpha, \beta \in \mathbb{R} \).

These observations lead us to the following definition.

**Definition 1.** A phase difference observable is a normalized positive operator measure \( E : \mathcal{B}([0, 2\pi]) \to \mathcal{L}(H \otimes H) \) which satisfies the covariance condition

\[
\Theta(\alpha, \beta) E(X) \Theta(\alpha, \beta)^* = E(X \mp (\alpha - \beta))
\]

for all \( X \in \mathcal{B}([0, 2\pi]) \) and \( \alpha, \beta \in \mathbb{R} \).

In the next sections we characterize all phase difference observables and we also give a necessary and sufficient condition for a phase difference observable to be a difference of two (one-mode) phase observables.

### 3. Direct method

The following lemma simplifies the proof of the Theorem 1 below.

**Lemma 1.** Let \( q \in \mathbb{Z} \) and let \( \nu_q : \mathcal{B}([0, 2\pi]) \to \mathbb{C} \) be a \( \sigma \)-additive set function. Then \( \nu_q(X + \theta) = e^{iq\theta} \nu_q(X) \) for all \( X \in \mathcal{B}([0, 2\pi]) \) and \( \theta \in [0, 2\pi) \) if and only if \( \nu_q(X) = c_q e^{i2\pi \int_X e^{iq\theta} \, d\theta} \) for all \( X \in \mathcal{B}([0, 2\pi]) \), where \( c_q \in \mathbb{C} \).

**Proof.** The 'if'-part of the lemma is clear, so we have to prove 'only if'-statement. Assume that \( \nu_q(X + \theta) = e^{iq\theta} \nu_q(X) \) for all \( X \in \mathcal{B}([0, 2\pi]) \) and \( \theta \in [0, 2\pi) \). Since \( [0, 2\pi) \cup [0, 2\pi) = [0, 2\pi) \) it follows that

\[
\nu_q([0, 2\pi)) = \nu_q([0, 2\pi) \cup [0, 2\pi)) = c_0 \delta_{\theta, q},
\]

where \( c_0 \) is a complex constant. The rest of the proof is same as the proof of Lemma 1 in [10]. \( \square \)
Let \( E : \mathcal{B}([0,2\pi]) \to \mathcal{L}(\mathcal{H} \otimes \mathcal{H}) \) be an arbitrary operator measure, that is, an \( \mathcal{L}(\mathcal{H} \otimes \mathcal{H}) \)-valued map defined on \( \mathcal{B}([0,2\pi]) \) which is \( \sigma \)-additive with respect to the weak operator topology.

**Theorem 1.**

a) If the operator measure \( E : \mathcal{B}([0,2\pi]) \to \mathcal{L}(\mathcal{H} \otimes \mathcal{H}) \) satisfies the covariance condition \( (6) \), then for all \( X \in \mathcal{B}([0,2\pi]) \),

\[
E(X) = \sum_{n,m,k,l=0}^{\infty} c_{n,m,k,l} \frac{1}{2\pi} \int_X e^{i(n-m)\theta} \, d\theta \, |n,k\rangle \langle m,l|,
\]

where \( c_{n,m,k,l} \in \mathbb{C} \), and \( c_{n,m,k,l} = 0 \) if \( n - m \neq l - k \), for all \( n,m,k,l \in \mathbb{N} \).

b) If, in addition, \( E \) is positive, that is, \( E(X) \geq 0 \) for all \( X \in \mathcal{B}([0,2\pi]) \), then \( \sum_{n,m,k,l=0}^{N} c_{n,m,k,l} |n,k\rangle \langle m,l| \geq 0 \) for all \( N \in \mathbb{N} \), and

c) if \( E \) is normalized, that is, \( E([0,2\pi]) = I \), then \( c_{n,n,k,k} = 1 \) for all \( n,k \in \mathbb{N} \).

**Proof.**

a) Using the covariance condition we get

\[
\langle n,k|E(X + (\alpha - \beta))|m,l\rangle = e^{i\alpha(n-m)+i\beta(k-l)} \langle n,k|E(X)|m,l\rangle
\]

for all \( n,m,k,l \in \mathbb{N} \), \( \alpha, \beta \in \mathbb{R} \) and \( X \in \mathcal{B}([0,2\pi]) \). Choosing \( \alpha = \beta \) it follows that \( \langle n,k|E(X)|m,l\rangle = 0 \) if \( n - m \neq l - k \).

Denote

\[
\nu_{n,m,k}(X) := \langle n,k|E(X)|m,k+n-m\rangle.
\]

We get

\[
\nu_{n,m,k}(X + (\alpha - \beta)) = \langle n,k|E(X + (\alpha - \beta))|m,k+n-m\rangle = e^{i\alpha(n-m)+i\beta(m-n)} \langle n,k|E(X)|m,k+n-m\rangle = e^{i(\alpha-\beta)(n-m)} \nu_{n,m,k}(X).
\]

Taking \( q = n - m \), Lemma \ref{Lemma}\( \) now gives Equation \( (7) \).

b) If \( \sum_{n,m,k,l=0}^{N} c_{n,m,k,l} \langle \psi|n,k\rangle \langle m,l|\psi\rangle < 0 \) for some \( N \in \mathbb{N} \) and \( \psi \in \mathcal{H} \otimes \mathcal{H} \) then, due to the continuity of the density function, one may choose an \( \epsilon > 0 \) such that \( \langle P_N\psi|E([0,\epsilon])P_N\psi\rangle < 0 \), where \( P_N := \sum_{n,k=0}^{N} |n,k\rangle \langle n,k| \). This is a contradiction.

c) This is a direct check.

\( \square \)

To prove the converse of the above theorem, consider a positive normalized operator measure \( \tilde{E} : \mathcal{B}([0,2\pi] \times [0,2\pi]) \to \mathcal{L}(\mathcal{H} \otimes \mathcal{H}) \) and
a set of complex numbers $\tilde{c} := (\tilde{c}_{n,m,k,l})_{n,m,k,l \in \mathbb{N}}$. We say that $\tilde{E}$ is $\Theta$-covariant if

$$\Theta(\alpha, \beta) \tilde{E}(Z) \Theta(\alpha, \beta)^* = \tilde{E}(Z + (\alpha, \beta))$$

for all $Z \in \mathcal{B}([0, 2\pi) \times [0, 2\pi))$, $\alpha, \beta \in \mathbb{R}$, with $+$ meaning (componentwise) addition mod $2\pi$, and we say that $\tilde{c}$ is normalized positive semidefinite if

$$\tilde{c}_{n,n,m,m} = 1,$$

$$\sum_{n,m,k,l=0}^{N} \tilde{c}_{n,m,k,l} |n, k\rangle \langle m, l| \geq O,$$

for all $n, m, N \in \mathbb{N}$.

With the above notations the following theorem is then obtained. Its proof is essentially the same as in the one-dimensional case [8] so that we omit it here.

**Theorem 2.**

a) If $\tilde{E}$ is $\Theta$-covariant, then there is a normalized positive semidefinite $\tilde{c}$ such that for any $Z \in \mathcal{B}([0, 2\pi) \times [0, 2\pi))$,

$$\tilde{E}(Z) = \sum_{n,m,k,l=0}^{\infty} \tilde{c}_{n,m,k,l} \int_{Z} e^{i[(n-m)x+(k-l)y]} \frac{dx}{2\pi} \frac{dy}{2\pi} |n, k\rangle \langle m, l|.$$

(8)

b) If $\tilde{c}$ is normalized positive semidefinite, then formula (8) defines (weakly) a $\Theta$-covariant normalized positive operator measure $\tilde{E} : \mathcal{B}([0, 2\pi) \times [0, 2\pi)) \rightarrow \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$.

By the above theorem, given a normalized positive semidefinite set of complex numbers $\tilde{c}$ we get a $\Theta$-covariant normalized positive operator measure $\tilde{E}$. Consider again the function $f(x, y) = x - y \ (\text{mod} \ 2\pi)$, defined on the rectangle $[0, 2\pi) \times [0, 2\pi)$. Then the map $\mathcal{B}([0, 2\pi)) \ni X \mapsto \tilde{E}^f(X) := \tilde{E}(f^{-1}(X)) \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$ is a phase difference observable with the structure given in Equation (7), where now

$$c_{n,m,k,l} = \tilde{c}_{n,m,k,l} \delta_{n,m,l-k}.$$

(9)

**Remark 1.** Let $(\psi_{n,k})_{n,k \in \mathbb{N}}$ be a set of unit vectors in $\mathcal{H}$. It is clear that defining

$$\tilde{c}_{n,m,k,l} = \langle \psi_{n,k} | \psi_{m,l} \rangle,$$

$\tilde{c}$ is normalized positive semidefinite. Also the converse is true, any normalized positive semidefinite set of complex numbers is of the form (10). Construction of a set of unit vectors $\{\psi_{n,k}\}_{n,k \in \mathbb{N}}$ for a given $\tilde{c}$ is similar than the one given in [10], section II.B. We note also that if $\tilde{c}$ is
positive semidefinite defined by unit vectors \((\psi_{n,k})_{n,k \in \mathbb{N}}\) then \(c\) defined as in Eq. (9) is positive semidefinite since one may choose a sequence \((\psi_{n,k} \otimes |n + k\rangle)_{n,k \in \mathbb{N}}\) of unit vectors to define \(c\).

**Remark 2.** Equation (9) shows that \(\Theta\)-covariant observables \(\tilde{E}\) form a "wider" class of observables than phase difference observables of Definition 1 in the sense that there are many \(\Theta\)-covariant observables which give the same phase difference observable, and any phase difference observable with \(c\) defines a \(\Theta\)-covariant observable which has, for instance, the same \(c\) as its structure unit. One may define an equivalence relation between \(\Theta\)-covariant observables as follows: two \(\Theta\)-covariant observables with \(\tilde{c}\) and \(\tilde{d}\) are equivalent if \(\tilde{c}_{n,m,k,l} = \tilde{d}_{n,m,k,l}\) for all \(n,m,k,l \in \mathbb{N}, n - m = l - k\), that is, if they define the same phase difference observable.

**Remark 3.** Using (7), it easy to see that any phase difference observable \(E\) has a uniform distribution in states where one mode is in a number state. For example, if \(\psi := \varphi \otimes |s\rangle, \varphi \in \mathcal{H}, \|\psi\| = 1, s \in \mathbb{N}\), then
\[
\langle \psi | E(X) | \psi \rangle = \frac{1}{2\pi} \int_X d\theta, \quad n, k \in \mathbb{N}, X \in \mathcal{B}([0, 2\pi])
\]
Moreover, one may also witness that there is no projection valued phase difference observable. For example,
\[
\langle 0, 0 | E(X)^2 | 0, 0 \rangle = \left| \frac{1}{2\pi} \int_X d\theta \right|^2
\]
and choosing \(X = [0, \pi)\) we get \(\langle 0, 0 | E([0, \pi])^2 | 0, 0 \rangle = \frac{1}{4}\). Compared to \(\langle 0, 0 | E([0, \pi]) | 0, 0 \rangle = \frac{1}{2}\), this shows that a phase difference observable cannot be a spectral measure.

**4. Group theoretical solution**

In [10] all phase observables were calculated using a generalized imprimitivity theorem due to Cattaneo [18]. Here we follow the same method to give an alternative way to derive the structure of phase difference observables. In using group theoretical methods, it is convenient to work in the torus \(\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}\), instead of phase interval \([0, 2\pi)\) where addition is to be taken modulo \(2\pi\). We regard \(\mathbb{T}\) as a compact (second countable) Abelian group and we let \(\mu\) denote its Haar measure. The product group \(\mathbb{T} \times \mathbb{T}\) has a unitary representation \(U\) on \(L^2(\mathbb{T} \times \mathbb{T}, \mu \times \mu)\), defined by
\[
(11) \quad [U(a, b)f](z_1, z_2) = f(az_1, bz_2).
\]
To solve the covariance condition \((12)\), we will first characterize all positive normalized operator measures \(F : \mathcal{B}(\mathbb{T}) \to \mathcal{L}(L_2(\mathbb{T} \times \mathbb{T}, \mu \times \mu))\) that satisfy

\[
U(a, b)F(X)U(a, b)^* = F(ab^{-1}X)
\]

for all \(X \in \mathcal{B}(\mathbb{T})\), \(a, b \in \mathbb{T}\). The canonical spectral measure \(F_{\text{can}}\) satisfying this condition is of the form

\[
[F_{\text{can}}(X)f](z_1, z_2) = \chi_X(z_1^{-1}z_2)f(z_1, z_2),
\]

where \(\chi_X\) is the characteristic function of the set \(X\).

Notice that \(U(a, b) = U(a, 1)U(1, b)\), so that the covariance conditions

\[
U(a, 1)F(X)U(a, 1)^* = F(aX), \quad a \in \mathbb{T}, X \in \mathcal{B}(\mathbb{T}),
\]

and

\[
U(1, b^{-1})F(X)U(1, b^{-1})^* = F(bX), \quad b \in \mathbb{T}, X \in \mathcal{B}(\mathbb{T}),
\]

taken together are equivalent with the condition \((12)\). We will denote the representation \(a \mapsto U(a, 1)\) as \(U_1\) and the representation \(a \mapsto U(1, a^{-1})\) as \(U_2\).

Covariance condition \((12)\) can be solved by looking the action \(z \mapsto ab^{-1}z\) of \(\mathbb{T} \times \mathbb{T}\) on \(\mathbb{T}\) and noting that the stability subgroup is \(\mathbb{T}\) \(\mathbb{T}\). Here we proceed in a different way. We characterize the normalized positive operator measures satisfying separately conditions \((14)\) and \((15)\). Then we combine the results to obtain operator measures satisfying condition \((12)\). Finally, we go back to the original Hilbert space \(\mathcal{H}\) and to the phase interval \([0, 2\pi)\) to get all the phase difference observables.

Let \(\tilde{F}\) be a normalized positive operator measure satisfying condition \((14)\). Since the action \(z \mapsto az\) of \(\mathbb{T}\) on itself is transitive, \((U_1, F)\) is a transitive system of covariance based on \(\mathbb{T}\) and, hence, \((U_1, F)\) is described by \([8, \text{Proposition 2}]\). In order to apply the cited result, let us notice the following facts. The stability subgroup of any point of \(\mathbb{T}\) is the trivial subgroup \(\{1\}\). The trivial representation \(\sigma\) of \(\{1\}\) acting on \(L_2(\mathbb{T}, \mu)\) contains all the (trivial) representations of \(\{1\}\) and the corresponding imprimitivity system \((R, P)\) for \(\mathbb{T}\) based on \(\mathbb{T}\) induced by \(\sigma\) acts on \(L^2(\mathbb{T}, \mu, L_2(\mathbb{T}, \mu)) \simeq L_2(\mathbb{T} \times \mathbb{T}, \mu \times \mu)\) as

\[
(R(a)\varphi)(z_1, z_2) = \varphi(a^{-1}z_1, z_2),
\]

\[
(P(X)\varphi)(z_1, z_2) = \chi_X(z_1)\varphi(z_1, z_2),
\]

where \(\varphi \in L_2(\mathbb{T} \times \mathbb{T}, \mu \times \mu), a \in \mathbb{T}, X \in \mathcal{B}(\mathbb{T})\) and \(z_1, z_2 \in \mathbb{T}\).
Proposition 2 of [18] shows that, given a normalized covariant positive operator measure $F$, there exists an isometry

$$W_1 : L_2(\mathbb{T} \times \mathbb{T}, \mu \times \mu) \to L_2(\mathbb{T} \times \mathbb{T}, \mu \times \mu),$$

which intertwines the action $U_1$ with $R$ and such that

$$F(X) = W_1^* P(X) W_1, \quad X \in B(\mathbb{T}).$$

Conversely, given an intertwining isometry $W_1$ from $L_2(\mathbb{T} \times \mathbb{T}, \mu \times \mu)$ to $L_2(\mathbb{T} \times \mathbb{T}, \mu \times \mu)$, equation (18) defines a positive normalized operator measure $F$ satisfying equation (14).

Hence, to classify all normalized positive operator measures satisfying condition (14), one has to determine all the isometric mappings $W_1$ such that

$$W_1 U_1(a) = R(a) W_1, \quad a \in \mathbb{T}.$$  

To perform this task, observe that the monomials $e_n, \ n \in \mathbb{Z}$, $e_n(z) = z^n, \ z \in \mathbb{T}$, form an orthonormal basis of $L_2(\mathbb{T}, \mu)$. Similarly, the product vectors

$$(e_n e_k)(z_1, z_2) = e_n(z_1) e_k(z_2) = z_1^n z_2^k, \ n, k \in \mathbb{Z},$$

form an orthonormal basis of $L_2(\mathbb{T} \times \mathbb{T}, \mu \times \mu)$.

The action of $U_1$ in this base is

$$U_1(a)(e_n e_k) = a^n (e_n e_k)$$

and the action of $R$ is simply

$$R(a)(e_n e_k) = a^{-n}(e_n e_k).$$

From equation (19) we get

$$R(a)W_1(e_n e_k) = W_1 U_1(a)(e_n e_k) = a^n W_1(e_n e_k)$$

for all $n, k \in \mathbb{Z}$. It follows that $W_1(e_n e_k)$ must be in the vector space $\text{span}\{(e_{-n} e_j)\}_{j \in \mathbb{Z}} \simeq L_2(\mathbb{T}, \mu)$. This means that $W_1(e_n e_k) = (e_{-n} \psi_{n,k})$, where $\psi_{n,k}$ is some unit vector in $L_2(\mathbb{T}, \mu)$.

The matrix elements of $F$ in the basis $\{(e_n e_k)\}_{n,k \in \mathbb{N}}$ are thus:

$$\langle (e_n e_k) | F(X) | (e_m e_l) \rangle = \langle (e_n e_k) | W_1^* P(X) W_1(e_m e_l) \rangle = \langle W_1(e_n e_k) | P(X) W_1(e_m e_l) \rangle = \langle (e_{-n} \psi_{n,k}) | P(X) (e_{-m} \psi_{m,l}) \rangle = \langle \psi_{n,k} | \psi_{m,l} \rangle \int_X z^{n-m} \, d\mu(z).$$

We consider next condition (15). Like in the previous case, $U_2$ and a normalized positive operator measure $F$ satisfying (14), form a transitive system of covariance based on $\mathbb{T}$. The corresponding imprimitivity
The action of \( U_R e_k \) in the basis \( \{ e_n \}_{n \in \mathbb{N}} \) is
\[
U_2(a)(e_n e_k) = a^{-k} (e_n e_k).
\]

If \( W_2 \) is an isometry intertwining representations \( U_2 \) and \( R \), then
\[
R(a) W_2(e_n e_k) = W_2 U_2(a)(e_n e_k) = a^{-k} W_2(e_n e_k).
\]

Thus \( W_2(e_n e_k) \) must be in the vector space \( \text{span}\{ (e_k e_j) \}_{j \in \mathbb{Z}} \simeq L_2(\mathbb{T}, \mu) \) and \( W_2(e_n e_k) = e_k \varphi_{n,k} \) for some unit vector \( \varphi_{n,k} \in L_2(\mathbb{T}, \mu) \).

Matrix elements are now:
\[
\langle (e_n e_k) | F(X)(e_m e_l) \rangle = \langle (e_n e_k) | W_2^* P(X) W_2(e_m e_l) \rangle = \langle W_2(e_n e_k) | P(X) W_2(e_m e_l) \rangle = \langle (e_k \varphi_{n,k}) | P(X)(e_l \varphi_{m,l}) \rangle = \langle \varphi_{n,k} | \varphi_{m,l} \rangle \int_X z^{-k} \, d\mu(z).
\]

Assume now that \( F \) is a normalized positive operator measure that satisfy condition (12), or, equivalently, conditions (14) and (15). This means that the matrix elements (20) and (22) are the same:
\[
\langle \psi_{n,k} | \psi_{m,l} \rangle \int_X z^{n-m} \, d\mu(z) = \langle \varphi_{n,k} | \varphi_{m,l} \rangle \int_X z^{-k} \, d\mu(z)
\]
for all \( n, m, k, l \in \mathbb{Z} \) and \( X \in \mathcal{B}(\mathbb{T}) \). From this we get \( n - m = l - k \) and \( \langle \psi_{n,k} | \psi_{m,l} \rangle = \langle \varphi_{n,k} | \varphi_{m,l} \rangle \).

We summarize the above construction in the following theorem.

**Theorem 3.** Any normalized positive operator measure \( F : \mathcal{B}(\mathbb{T}) \to L_2(\mathbb{T} \times \mathbb{T}, \mu \times \mu) \) satisfying covariance condition (12) is of the form
\[
F(X) = \sum_{n,m,k,l \in \mathbb{Z}} \delta_{n-m,l-k} \langle \psi_{n,k} | \psi_{m,l} \rangle \int_X z^{n-m} \, d\mu(z) \, |e_n e_k\rangle \langle e_m e_l|
\]
for some set \( \{ \psi_{n,k} \}_{n,k \in \mathbb{Z}} \subset L_2(\mathbb{T}, \mu) \) of unit vectors.

We note that in (24) only the inner products of the vectors \( \psi_{n,k} \) are relevant. Thus two set of unit vectors \( \{ \psi_{n,k} \}_{n,k \in \mathbb{Z}} \) and \( \{ \eta_{n,k} \}_{n,k \in \mathbb{Z}} \) define the same positive operator measure exactly when
\[
\delta_{n-m,l-k} \langle \psi_{n,k} | \psi_{m,l} \rangle = \delta_{n-m,l-k} \langle \eta_{n,k} | \eta_{m,l} \rangle
\]
for all \( n, m, k, l \in \mathbb{Z} \).

**Example 1.** The canonical spectral measure \( F_{\text{can}} \) of Equation (13) written in the above form is simply
\[
F_{\text{can}}(X) = \sum_{n,m,k,l \in \mathbb{Z}} \delta_{n-m,l-k} \int_X z^{n-m} \, d\mu(z) \, |e_n e_k\rangle \langle e_m e_l|,
\]
showing that $F$ can be defined by a set $(\psi_{n,k})_{n,k \in \mathbb{Z}}$, where $\psi_{n,k} = \psi$ for all $n, k \in \mathbb{Z}$ and $\psi$ is any unit vector.

We are now ready to solve the covariance condition (3). Let $\mathcal{H}$ be a complex separable Hilbert space with an orthonormal basis $\{|n\rangle\}_{n \in \mathbb{N}}$, and $T : \mathcal{H} \otimes \mathcal{H} \rightarrow L_2(\mathbb{T} \times \mathbb{T}, \mu \times \mu)$ be a linear isometry with the property

$$T|n, m\rangle = e_n e_m, \text{ for all } n, m \in \mathbb{N}.$$  

If $[0, 2\pi)$ is identified with $\mathbb{T}$ by the mapping $\alpha \mapsto e^{i\alpha}$, then $\Theta$ can be regarded as a unitary representation of $\mathbb{T} \times \mathbb{T}$. Clearly, $T$ intertwines representations $\Theta$ and $U$, $T\Theta = UT$. If $\tilde{F} : B(\mathbb{T}) \rightarrow \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$ satisfies the equation

$$(25) \quad \Theta(a, b)\tilde{F}(X)\Theta(a, b)^* = \tilde{F}(ab^{-1}X)$$

for all $a, b \in \mathbb{T}$, $X \in B(\mathbb{T})$, then $F(X) := T\tilde{F}(X)T^*$ is a normalized positive operator measure having property (12). Moreover, if $F : B(\mathbb{T}) \rightarrow \mathcal{L}(L_2(\mathbb{T} \times \mathbb{T}, \mu \times \mu))$ satisfies condition (12), then $X \mapsto T^*F(X)T$ is a normalized positive operator measure acting in $\mathcal{L}(\mathcal{H} \otimes \mathcal{H})$ and satisfying (25). Using theorem 3, one thus has the following result.

**Theorem 4.** A normalized positive operator measure $E : B([0, 2\pi)) \rightarrow \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$ is a phase difference observable if and only if

$$(26) \quad E(X) = \sum_{n,m,k,l \in \mathbb{N}} \delta_{n-m,l-k}\langle \xi_{n,k} | \xi_{m,l} \rangle \frac{1}{2\pi} \int_X e^{i(n-m)\theta} d\theta \ket{n, k} \bra{m, l},$$

for some set of unit vectors $(\xi_{n,k})_{n,k \in \mathbb{N}}$ of $\mathcal{H}$.

In view of Remark 1, this result is the same as the one obtained in Section 3. Since $T : \mathcal{H} \otimes \mathcal{H} \rightarrow L_2(\mathbb{T} \times \mathbb{T}, \mu \times \mu)$ is not surjective, there is no projection valued phase difference observable.

**Remark 4.** The moment operators $E^{(r)}, r \in \mathbb{N}$, of the phase difference observable $E$ are defined as

$$E^{(r)} := \int_0^{2\pi} \theta^r dE(\theta)$$

and they are bounded self-adjoint operators. By direct calculation we get

$$\langle n, k | E^{(1)} | m, l \rangle = \delta_{n-m,l-k}\langle \xi_{n,k} | \xi_{m,l} \rangle \frac{i}{m - n}$$

when $n \neq m$. For $n = m$ one gets

$$\langle n, k | E^{(1)} | n, l \rangle = \pi \delta_{k,l}.$$
Thus the phase difference observable $E$ is uniquely determined by its first moment operator $E^{(1)}$. This is notable since $E$ is not projection valued. The same result holds also for phase observables, see \[20\] for a further discussion of this conundrum.

Similarly, the $r$th cyclic moment operator of $E$ is defined as the operator $C^{(r)}_E$,

$$C^{(r)}_E := \int_0^{2\pi} e^{ir\theta} dE(\theta), \quad r \in \mathbb{N}.\$$

They are easily determined to be

$$C^{(r)}_E = \sum_{n,l=0}^{\infty} \langle \xi_{n,l+r} \xi_{n+r,l} \rangle |n, l+r \rangle \langle n+r, l|.$$

Since $C^{(1)}_E |0,0 \rangle = 0$, the first cyclic moment is not unitary. This is another way to see the already mentioned fact that there is no projection valued phase difference observable.

5. Phase difference observable vs. difference of phase observables

Till now we have characterized in two different ways the phase difference observables, and we have also constructed explicitly the difference of two phase observables. The following proposition characterizes those phase difference observables which are, that is, can be expressed as, the difference of two phase observables. It’s proof is a direct comparison of formulas (2) and (26).

**Proposition 1.** Let $E : \mathcal{B}([0, 2\pi)) \to \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$ be a phase difference observable, characterized by a set $(\xi_{n,k})_{n,k \in \mathbb{N}}$. Observable $E$ is a difference of two phase observables if and only if there are sequences $(\varphi^1_n)_{n \in \mathbb{N}}$ and $(\varphi^2_n)_{n \in \mathbb{N}}$ of unit vectors in $\mathcal{H}$ such that

$$\delta_{n-m,l-k} \langle \xi_{n,k} | \xi_{m,l} \rangle = \delta_{n-m,l-k} \langle \varphi^1_n | \varphi^1_m \rangle \langle \varphi^2_k | \varphi^2_l \rangle$$

for all $n, k, m, l \in \mathbb{N}$.

The next example shows that there are phase difference observables that are not difference of two phase observables. It also opens the question of finding physically meaningful conditions for Proposition 1.

**Example 2.** Fix an arbitrary unit vector $\psi \in \mathcal{H}$ and let $\theta_j, j = 1, 2, 3, 4$, be real numbers. Define $\xi_{0,2} = e^{i\theta_1} \psi$, $\xi_{2,2} = e^{i\theta_2} \psi$, $\xi_{0,4} = e^{i\theta_3} \psi$, and...
\[ \xi_{2,4} = e^{i \theta_4} \psi, \quad \xi_{n,k} = \psi \quad \text{otherwise.} \]

Assume now that there are sequences \((\varphi^1_n)_{n \in \mathbb{N}}\) and \((\varphi^2_n)_{n \in \mathbb{N}}\) such that equation (27) holds. Then

\[
\begin{align*}
e^{i \theta_4} &= \langle \xi_{3,3} | \xi_{2,4} \rangle = \frac{\langle \varphi^1_3 | \varphi^1_2 \rangle \langle \varphi^2_3 | \varphi^2_4 \rangle}{\langle \varphi^1_1 | \varphi^1_0 \rangle \langle \varphi^2_1 | \varphi^2_2 \rangle} \\
&= \frac{\langle \xi_{3,1} \xi_{2,2} \rangle \langle \xi_{1,3} | \xi_{0,4} \rangle}{\langle \xi_{1,1} | \xi_{0,2} \rangle} = e^{i (\theta_2 + \theta_3 - \theta_1)}.
\end{align*}
\]

Choosing the numbers \(\theta_j\) in such a way that \(e^{i \theta_4} \neq e^{i (\theta_2 + \theta_3 - \theta_1)}\) we thus get a contradiction.

From equation (27) it is also clear that two different pairs of phase observables may define the same phase difference observable.

We close this section with a terminological choice. We say that a phase difference observable is \textit{canonical} if it is the difference of two canonical phase observables and we denote it by \(E_{\text{can}}^{\text{diff}}\). Since the canonical phase observable \(E_{\text{can}}\) has the structure

\[
E_{\text{can}}(X) = \sum_{n \in \mathbb{N}} \frac{1}{2\pi} \int_X e^{i(n-m)\theta} |n\rangle \langle m|,
\]

the explicit form of \(E_{\text{can}}^{\text{diff}}\) can be read from both (2) and (24) with the involved inner products equal to one in each case. Some properties of canonical phase difference observable are discussed in sections 6 and 8.

6. \textsc{Radon-Nikodým derivatives and the phase difference representation}

Let \(T\) be a state (positive trace-one operator) on \(\mathcal{H} \otimes \mathcal{H}\), let \(E\) be a phase difference observable with \(c\), and let \(\tilde{E}\) be a \(\Theta\)-covariant observable with \(\tilde{c}\). Using similiar methods as in \[11, \text{Sec. V}\], one can show that

\[
\begin{align*}
\text{tr}(TE(X)) &= \frac{1}{2\pi} \int_X g_T^E(\theta) \, d\theta, \quad X \in \mathcal{B}([0,2\pi]), \\
\text{tr}(T\tilde{E}(Z)) &= \frac{1}{(2\pi)^2} \int_Z g_T^E(x,y) \, dx \, dy, \quad Z \in \mathcal{B}([0,2\pi] \times [0,2\pi]).
\end{align*}
\]
where

\[ g^E_T(\theta) = \sum_{n,m,k,l=0}^{\infty} c_{n,m,k,l} e^{i(n-m)\theta} \langle m,l|T|n,k \rangle, \]

\[ \tilde{g}^E_T(x,y) = \sum_{n,m,k,l=0}^{\infty} \tilde{c}_{n,m,k,l} e^{i(n-m)x+i(k-l)y} \langle m,l|T|n,k \rangle, \]

for \( d\theta \)-almost all \( \theta \in \mathbb{R} \) and for \( dx dy \)-almost all \( (x,y) \in \mathbb{R}^2 \). The above notations \( \sum_{n,m,k,l=0}^{\infty} \) mean that for some increasing subsequences \((s_t)_{t \in \mathbb{N}} \subseteq \mathbb{N}\), \( \sum_{n,m,k,l=0}^{\infty} = \lim_{t \to \infty} \sum_{n,m,k,l=0}^{s_t} \). It is easy to see that if \( E \) is constructed from \( \tilde{E} \) (that is, Eq. (9) holds) then

\[ g^E_T(\theta) = \frac{1}{2\pi} \int_0^{2\pi} \tilde{g}^E_T(x + \theta, x) dx. \]

Since \( \mathcal{H} \) is isomorphic with the Hardy space \( H^2 \) of the unit circle spanned by the vectors \( e_n, n \in \mathbb{N} \), one can consider any \( \psi \in \mathcal{H} \) as an element of \( H^2 \), that is, as a function (or equivalence class of functions). Using this interpretation, for any \( \varphi, \psi \in \mathcal{H} \) and \( X \in \mathcal{B}([0,2\pi]) \), one may write

\[ (28) \quad \langle \varphi \otimes \psi | E_{\text{can}} \otimes E_{\text{can}}(X) | \varphi \otimes \psi \rangle = \frac{1}{2\pi} \int_X \frac{1}{2\pi} \int_0^{2\pi} |\varphi(x + \theta)|^2 |\psi(x)|^2 dx d\theta. \]

This phase difference distribution was first suggested by Barnett and Pegg [12, 13].

### 7. Classical Limit

Like in the one-mode case [9], it is easy to show that for any operator measure \( E : \mathcal{B}([0,2\pi]) \to \mathcal{L}(\mathcal{H} \otimes \mathcal{H}) \) the condition

\[ \langle z_1 e^{i\alpha}, z_2 e^{i\beta} | E(X) | z_1 e^{i\alpha}, z_2 e^{i\beta} \rangle = \langle z_1, z_2 | E(X + (\alpha - \beta)) | z_1, z_2 \rangle, \]

for \( z_1, z_2 \in \mathbb{C}, \alpha, \beta \in \mathbb{R}, X \in \mathcal{B}([0,2\pi]) \), equals the covariance condition (8) where \( |z_1, z_2\rangle := |z_1\rangle \otimes |z_2\rangle \) is a two-mode coherent state.

Suppose that \( E_{\text{diff}} \) is the difference of phase observables \( E_1 \) and \( E_2 \) with \( (c^1_{n,m}) \) and \( (c^2_{n,m}) \), respectively. If, for example, \( \lim_{n \to \infty} c^2_{n,n+k} = e^{ik\alpha} \) for all \( k \geq 1, \alpha \in [0,2\pi] \), then for any continuos function \( g : [0,2\pi] \to \mathbb{C} \)

\[ \lim_{|z| \to \infty} \int_0^{2\pi} g(x) d\langle z | E_2(x) | z \rangle = g(\arg z - \alpha) \]
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(see, [21, Th. 7.1]). Let $g^{E_n}_{|z⟩} : [0, 2\pi] \to [0, \infty)$ be a continuous Radon-Nikodým derivative of the probability measure $X \mapsto \langle z|E_n(X)|z⟩$, $n = 1, 2$. Then

$$\lim_{|z_2| \to \infty} \frac{1}{2\pi} \int_0^{2\pi} g^{E_1}_{|z_1⟩}(x + \theta)g^{E_2}_{|z_2⟩}(x)dx = g^{E_1}_{|z_1⟩}(\theta + \arg z_2 - \alpha)$$

which implies the following proposition:

**Proposition 2.** For any $X \in \mathcal{B}([0, 2\pi])$,

$$\lim_{|z_2| \to \infty} \langle z_1, z_2|E^{\text{diff}}(X)|z_1, z_2⟩ = \langle z_1|E_1(X + \arg z_2 - \alpha)|z_1⟩.$$ 

This means that, in the classical limit $|z_2| \to \infty$ of the second mode, the two-mode theory reduces to a single-mode theory. Moreover, if also $\lim_{n \to \infty} c^1_{n,n+k} = e^{ik\alpha'}$ for all $k \geq 1$, then

$$\lim_{|z_1|,|z_2| \to \infty} \langle z_1, z_2|E^{\text{diff}}(X)|z_1, z_2⟩ = \delta_{\arg z_1 - \arg z_2 - \alpha' + \alpha}(X)$$

where $\delta_p$ is a Dirac measure concentrated on the point $p$. This is the classical limit of the two-mode system.

**Remark 5.** It is known from the theory of homodyne detection [22, 23] that when the reference mode is in a large amplitude coherent state $|z⟩$, $|z| \gg 0$, the lowering operator $a$ of the reference mode can be replaced with the "classical" observable $zI$ in practical calculations. This means that the energy and the phase of the reference field are well known and fixed. A similar result also holds for the difference of phase observables, as well.

Let $\alpha \in [0, 2\pi)$ and define a **fixed-phase observable**

$$F_\alpha : \mathcal{B}([0, 2\pi)) \to \mathcal{L}(\mathcal{H}), \ X \mapsto \delta_\alpha(X)I$$

where $\delta_\alpha$ is the Dirac measure concentrated on $\alpha$. The fixed-phase observable $F_\alpha$ is the spectral measure of a self-adjoint operator $\alpha I$ and, thus, it is not a phase observable. If we choose the phase observable $E_2$ to be the fixed-phase observable $F_\alpha$ (this can be done similarly as in the case of two phase observables although $F_\alpha$ is not covariant), then $E^{\text{diff}}(X) = E_1(X + \alpha) \otimes I$, that is, the "phase difference" $E^{\text{diff}}$ and the single-mode phase $E_1$ are practically the same observables (up to unitary equivalence or the choice of the reference phase $\alpha$).
8. Ban's theory

In the series of papers [14, 15, 16] Ban has proposed a unitary two-mode phase operator in relation to the number difference. To discuss Ban’s theory in the present context, consider the number difference \( \Delta N = \sum_{k \in \mathbb{Z}} k \Delta P_k \) defined in section 2. All the eigen spaces \( H_k := \Delta P_k (\mathcal{H} \otimes \mathcal{H}) \) are infinite dimensional and the vectors \( \{|k+n,n\rangle\}_{n \geq \max\{0,-k\}} \) constitute an orthonormal basis of \( H_k \). One may thus define a unitary operator \( D \) on \( \mathcal{H} \otimes \mathcal{H} \) so that, for each \( k \in \mathbb{Z} \), \( D(H_k) = H_{k-1} \). To exhibit such an operator we rename the basis vectors using the notation of Ban:

\[
|k,n\rangle := \begin{cases} 
|n+k,n\rangle, & k \geq 0 \\
|n,n-k\rangle, & k < 0
\end{cases}
\]

Then, for any \( k \in \mathbb{Z} \), the spectral projection \( \Delta P_k \) can be expressed as \( \Delta P_k = \sum_{n \in \mathbb{N}} |k,n\rangle \langle k,n| \), and one may choose, for instance,

\[
D = \sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{N}} |k-1,n\rangle \langle k,n|.
\]

This is Ban’s proposal for a phase operator. Writing

\[
D = \int_0^{2\pi} e^{i\theta} \, dB(\theta),
\]

the spectral measure \( B \) of \( D \) has the form

\[
B(X) = \sum_{k,l \in \mathbb{Z}} \sum_{n \in \mathbb{N}} \frac{1}{2\pi} \int_X e^{i(k-l)\theta} \, d\theta \, |k,n\rangle \langle l,n|.
\]

Clearly, \( B \) is not phase shift covariant so that it is not a phase observable in the sense of [3] or [7]. However, the spectral measure \( B \) fulfills the covariance condition

\[
V_{\Delta}(\beta)B(X)V_{\Delta}(\beta)^* = B(X + \beta)
\]

for all \( \beta \in \mathbb{R}, X \in \mathcal{B}([0, 2\pi]) \). This differs from the covariance condition [4] by the factor 2. Thus \( B \) is neither a phase difference observable in the sense of Definition [11]. The difference by the factor 2 in the covariance conditions satisfied by \( B \) and \( E_{\text{diff}}^{(1)} \) is also reflected in the commutation properties of \( D \) and \( C_{E_{\text{diff}}^{(1)}} \) with \( \Delta N \). Indeed, for all \( k \in \mathbb{Z} \) and \( n \in \mathbb{N} \),

\[
[D, \Delta N]|k,n\rangle = D|k,n\rangle
\]

whereas

\[
[C_{E_{\text{diff}}^{(1)}}, \Delta N]|k,n\rangle = 2C_{E_{\text{diff}}^{(1)}}|k,n\rangle.
\]
Notice also that the first cyclic moment \( \mathcal{C}_{E_{\text{can}}}^{(1)} \) of the canonical phase observable \( E_{\text{can}} \) satisfies

\[
[\mathcal{C}_{E_{\text{can}}}^{(1)}, N]|n\rangle = \mathcal{C}_{E_{\text{can}}}^{(1)}|n\rangle
\]

for all \( n \in \mathbb{N} \). The factor 2 in the covariance condition (4) and the commutation relation (31) is natural for a phase difference observable. It is also worth to note that condition (4) has a projection valued solution [19]. The corresponding unitary operator is

\[
\sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{N}} |k - 2, n\rangle \langle k, n|.
\]

Compared to (29), here is again 2 instead of 1.

Although spectral measure \( B \) is neither a phase observable nor a phase difference observable, it has the following relation to canonical phase observable. When the second mode is in the vacuum state and the first mode is in an arbitrary state \( T \), then

\[
\text{tr} (T \otimes |0\rangle \langle 0| B(X)) = \text{tr} (T E_{\text{can}}(X)), \quad X \in \mathcal{B}([0, 2\pi]).
\]

9. Discussion

The first phase difference operators studied in the literature were suggested by Sussking and Glogover [24] (see also [25, 26]). Their operators were the so-called cosine and sine phase difference operators which can be represented as \( C_{12} = \int_0^{2\pi} \cos \theta \, dE_{\text{diff}}^{\text{can}}(\theta) \) and \( S_{12} = \int_0^{2\pi} \sin \theta \, dE_{\text{diff}}^{\text{can}}(\theta) \), respectively. The operators \( C_{12} \) and \( S_{12} \) do not commute, and their spectra are the interval \([-1, 1]\), including a countable dense set of eigenvalues \([24, 25, 26] \).

Lévy-Leblond [27] defined the relative exponential phase operator \( \int_0^{2\pi} e^{i\theta} dE_{\text{can}}(\theta) \otimes \int_0^{2\pi} e^{-i\theta} dE_{\text{can}}(\theta) = \int_0^{2\pi} e^{i\theta} dE_{\text{diff}}^{\text{can}}(\theta) \) by analogy with the classical expression \( e^{i(\theta_1 - \theta_2)} = e^{i\theta_1}e^{-i\theta_2} \). The operator \( \int_0^{2\pi} e^{i\theta} dE_{\text{diff}}^{\text{can}}(\theta) \) is not unitary but it is associated with the polar decomposition of \( a \otimes a^* \) in the following way: using \( |a \otimes a^*| = \sqrt{N} \otimes (N + I) \),

\[
a \otimes a^* = \int_0^{2\pi} e^{i\theta} dE_{\text{can}}^{\text{diff}}(\theta) \sqrt{N} \otimes (N + I).
\]

We can add an extra operator \( T := \sum_{n=0}^{\infty} |n\rangle \langle 0| \otimes |0\rangle \langle n| \) to \( \int_0^{2\pi} e^{i\theta} dE_{\text{can}}^{\text{diff}}(\theta) \) and it still satisfies the polar decomposition relation of \( a \otimes a^* \). When doing this we get a unitary operator \( \mathcal{E}_{12} := \int_0^{2\pi} e^{i\theta} dE_{\text{can}}^{\text{diff}}(\theta) + T \) and, thus, a self-adjoint operator \( \Phi_{12} \) such that \( \mathcal{E}_{12} = e^{i\Phi_{12}} \). Obviously, the operator \( \Phi_{12} \) is not the first moment operator of a covariant phase difference observable. Luis and Sánchez-Soto have shown [28] that the point spectrum of \( \Phi_{12} \) consists of eigenvalues.
\[ \{2\pi r/(n + 1) \mid n \in \mathbb{N}, \ r = 0, 1, \ldots, n \} \subset [0, 2\pi) \], the closure of this set being \([0, 2\pi]\). When the second mode is in a large amplitude coherent state \(|z\rangle\), the spectral measure of \(\Phi_{12}\) gives essentially the same results as \(E_{\text{can}}\) (or the difference of \(E_{\text{can}}\) and \(F_{\text{arg} z}\)) \[23, 24\].

Finally, we note that in an eight-port homodyne detection the measurement data is always discrete. Only in the limit of large intesity of the known fixed-phase reference oscillator the data becomes (essentially) "continuous" giving rise to the phase observable \(E_{\langle 0 \rangle}\). Thus, strictly speaking eight-port homodyne detection cannot be described as a measurement of the phase difference observable in \textit{two} arbitrary signal fields. However, using two eight-port homodyne detectors with the same large amplitude fixed-phase reference field one can measure the difference of the two phase observables \(E_{\langle 0 \rangle}\) and \(E_{\langle 0 \rangle}\) \[30\].

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