Two-loop critical mass for Wilson fermions

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Abstract

We have redone a recent two-loop computation of the critical mass for Wilson fermions in lattice QCD by evaluating Feynman integrals with the coordinate-space method. We present the results for different types of infrared regularization. We confirm both the previous numerical estimates and the power of the coordinate-space method whenever high accuracy is needed.
1 Introduction

The lattice formulation is at present the best tool for a nonperturbative study of QCD. Its most widely used implementation is due to Wilson. Here, the lattice fermion doubling problem is solved by the introduction of a formally irrelevant second-order term that gives a mass of the order of the cut-off to the doublers. This term explicitly breaks chiral invariance and therefore the limit of zero mass can be obtained only by a fine tuning, which is to say that the bare and the renormalized quark mass no longer vanish together. Nowadays, there are new formulations in which a form of chiral invariance is preserved at the cost of a nonlocal action (see, for example, the recent reviews and references therein). Even though this represents a very important progress from a conceptual point of view, it is not clear if the new formulations can become practical for numerical simulations.

In the Wilson formulation, the breaking of chiral invariance gives rise to severe technical difficulties. Indeed, operators that have the correct chiral properties are obtained as sums of many different lattice bare operators multiplied by suitable renormalization constants. In principle, such constants should be computed nonperturbatively. In practice, this is often difficult and thus one resorts to perturbation theory.

Lattice perturbation theory is particularly complicated because of the loss of Lorentz invariance and at present a two-loop calculation represents a formidable task. For one-loop computations a completely general algebraic algorithm was introduced in Ref. for the pure gauge theory and extended in Ref. to include Wilson fermions. This algorithm allows to express every one-loop integral with gluon and Wilson-fermion propagators in terms of a small number of basic constants that can be computed with arbitrary precision. For two-loop calculations a powerful method was introduced by Lüscher and Weisz for the pure gauge theory, the so-called coordinate-space method. Its generalization to full QCD is again a nontrivial task. Some general ideas were presented in Ref. and will be extensively illustrated elsewhere.

Here, we present the first two-loop computation with the coordinate-space method in lattice QCD with fermions. The purpose is twofold. First, we wish to check that the additional complications that are present in the method in the presence of fermions do not spoil its performance. Second, we want to compare it with the alternative momentum-space method, in which one works directly in momentum space. For this purpose, we repeat
the calculation of the critical mass for Wilson fermions reported in Ref. [11]. It turns out that the coordinate-space method is quite precise, providing numerical expressions for the Feynman integrals with a precision of $10^{-10}$ or higher. The momentum-space method also works well: although not as precise as the coordinate-space method, still the final results have a relative precision of $10^{-5}$.

In this paper, we will consider Wilson fermions. The dressed inverse fermion propagator has the form

$$S^{-1}(p, m_B) = i \overline{p} + m_B + M_W(p) - \Sigma^L(p, m_B, g_0),$$

(1)

where, setting the lattice spacing equal to one,

$$\overline{p}_\mu = \sin p_\mu,$$

(2)

$$\hat{p}^2 = \sum_\mu \left(2 \sin \frac{p_\mu}{2}\right)^2,$$

(3)

$$M_W(p) = \frac{r_W}{2} \hat{p}^2.$$

(4)

The additive mass renormalization $\delta m_B$ is obtained by requiring $S^{-1}(0, \delta m_B) = 0$, i.e.

$$\Sigma^L(0, \delta m_B, g_0) = \delta m_B.$$ (5)

This equation can be solved in perturbation theory by expanding

$$\Sigma^L(0, m_B, g_0) = \sum_{n=1}^{\infty} g_0^{2n} \Sigma^{(n)}.$$ (6)

In the following we shall compute $\Sigma^{(1)}$ and $\Sigma^{(2)}$ for $r_W = 1$, gauge group $SU(N)$, and $N_f$ fermionic flavour species. We will work in the Feynman gauge.

## 2 One-loop result

In Ref. [6] we already reported the analytic one-loop expression for the fermionic self-energy $\Sigma^L$. The first computation for the Wilson action in Feynman gauge was given in Ref. [12] and it was subsequently corrected in Ref. [13]. Our result is expressed in terms of three purely bosonic
|     | \(Z_0\) | 0.15493339023106021408437208 |
|-----|---------|-----------------------------|
|     | \(Z_1\) | 0.10778131353987400134391550 |
|     | \(F_0\) | 4.369225233874758          |

Table 1: Numerical values of the three constants \(Z_0\), \(Z_1\), and \(F_0\).

| \(\mathcal{F}(1,0)\) | 0.08539036359532067914 |
| \(\mathcal{F}(1,-1)\) | 0.46936331002699614475 |
| \(\mathcal{F}(1,-2)\) | 3.39456907367713000586 |
| \(\mathcal{F}(2,-1)\) | 0.05188019503901136636 |
| \(\mathcal{F}(2,-2)\) | 0.23874773756341478520 |
| \(\mathcal{F}(3,-2)\) | 0.03447644143803223145 |
| \(\mathcal{F}(3,-3)\) | 0.13202727122781293085 |
| \(\mathcal{F}(3,-4)\) | 0.75167190030295682254 |

Table 2: Numerical values of the constants appearing in the fermionic integrals.

\[ \mathcal{B}(p; n_x, n_y, n_z, n_t) = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{\hat{k}_{n_x}^2 \hat{k}_{n_y}^2 \hat{k}_{n_z}^2 \hat{k}_{n_t}^2}{D_B(k, m_b)^p}, \]

\[ \mathcal{F}(p, q; n_x, n_y, n_z, n_t) = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{\hat{k}_{n_x}^{2n_x} \hat{k}_{n_y}^{2n_y} \hat{k}_{n_z}^{2n_z} \hat{k}_{n_t}^{2n_t}}{D_F(k, m_f)^p D_B(k, m_b)^q}. \]

Notice, however, that Eq. (3.15) in Ref. [13] contains a misprint: the correct result is given in Eq. (10b) of Ref. [12].
where
\[ D_B(k, m_b) = \hat{k}^2 + m_b^2, \]
\[ D_F(k, m_f) = \hat{k}^2 + M_W(k)^2 + m_f^2. \] (8)

When one of the \( n \)'s vanishes, we shall not write it. Then
\[ Z_0 = B(1)|_{m_b=0} \] (9)
\[ Z_1 = \frac{1}{4} B(1; 1, 1)|_{m_b=0} \] (10)
\[ F_0 = \lim_{m_b \to 0} [16\pi^2 B(2) + \log m_b^2 + \gamma_E], \] (11)

while
\[ Y_0 = \lim_{m \to 0} \left[ F(2, 0) + \frac{1}{16\pi^2} \left( \log m^2 + \gamma_E - F_0 \right) \right] \] (12)
\[ Y_1 = \frac{1}{8} F(1, 1; 1, 1)|_{m=0} \]
\[ Y_2 = \frac{1}{16} F(1, 1; 1, 1, 1)|_{m=0} \]
\[ Y_3 = \frac{1}{16} F(1, 2; 1, 1, 1)|_{m=0}, \] (13)

where we have set \( m \equiv m_f = m_b \). Also, in Table 2 the numerical values refer to the massless case \( m = 0 \).

At one-loop order
\[ \Sigma^{(1)} = \frac{N^2 - 1}{N} \sum_{i=1}^{2} c_i^{(1)} \] (14)

where \( c_i^{(1)} \) are the contributions of the two diagrams illustrated in Fig. 1. In terms of the basic integrals defined above, they are given by
\[ c_1^{(1)} = -Z_0 \]
\[ c_2^{(1)} = \frac{Z_0}{2} - F(1, 0). \]

The numerical values are reported in Table 3 and compared with the results of Ref. [1], obtained by using the momentum-space method. The agreement is excellent. Summing up the two contributions we obtain
\[ \sum_{i=1}^{2} c_i^{(1)} = -[Z_0 + 2F(1, 0)] \approx -0.16285705871085078618. \] (15)
Figure 1: One-loop diagrams contributing to $\Sigma^L$. Wavy (solid) lines represent gluons (fermions).

| $i$ | $c_i^{(1)}$ |
|-----|-------------|
| 1   | $-0.15493339023106$ |
|     | $-0.15493339023106021408$ |
| 2   | $-0.007923668480(2)$ |
|     | $-0.00792366847979057210$ |

Table 3: Coefficients $c_i^{(1)}$. For each of them we report in the first line the result of Ref. [11], obtained by means of a momentum-space integration, and in the second line our result, obtained by means of the coordinate-space method.

The constant is in excellent agreement with the result of Ref. [11], $\sum_i c_i^{(1)} = 0.162857058711(2)$.

3 Two-loop result

At two loops there are 26 diagrams that have been drawn in Fig. 2. They are numbered as in Ref. [11] in order to simplify the comparison. The $i$-th diagram gives a contribution of the form

$$D_i \equiv (N^2 - 1) \left[ c_{1,i}^{(2)} + \frac{1}{N^2} c_{2,i}^{(2)} + \frac{N_f}{N} c_{3,i}^{(2)} \right].$$

In Table 4 we report the results given in Ref. [11] and those obtained here by using the configuration-space method. When we have not reported an error, the precision we achieve is much higher than the reported digits. This occurs in general when the diagram is the product of one-loop integrals. All results are in agreement with what had been presented in Ref. [11]. Only for
Figure 2: Two-loop diagrams contributing to $\Sigma^L$. Wavy (solid, dotted) lines represent gluons (fermions, ghosts). Crosses denote vertices stemming from the measure part of the action; the solid circle is the one-loop fermion-mass counterterm.
| $\ell$ | $c_{\ell 1}(2)$ | $c_{\ell 2}(2)$ | $c_{\ell 3}(2)$ |
|-------|----------------|----------------|----------------|
| 3     | $0.002000362950707492$ | $-0.0030005444260612375$ | 0 |
| 4     | $0.002000362950707492$ | $-0.0030005444260612480722$ | 0 |
| 5     | 0 | 0 | 0 |
| 6     | $0.0000488991(8)$ | $0.000097778(2)$ | 0 |
| 7, 8, 9, 10, 11 | $0.013927(3)$ | $0.014525(2)$ | 0 |
| 12, 13 | 0 | 0 | 0 |
| 14, 15, 16, 17, 18 | $0.007553(1)$ | $0.0058323(7)$ | 0 |
| 19, 20 | 0 | 0 | 0 |
| 21, 22, 23 | $0.0000696768(4)$ | $0.0000696768(4)$ | 0 |
| 24     | 0 | 0 | 0 |
| 25     | $0.0000762(1)$ | $0.00015524(3)$ | 0 |
| 26     | $0.0000762(1)$ | $0.000155226212(8)$ | 0 |
| 27     | 0 | $0.00006522(1)$ | 0 |
| 28     | $0.000078482(5)$ | $0.0001569613054406589$ | 0 |
| Total  | $-0.007537(3)$ | $0.00118618(8)$ | 0 |

Table 4: Coefficients $c^{(2)}_{\ell 1}$, $c^{(2)}_{\ell 2}$, and $c^{(2)}_{\ell 3}$. For each of them we report in the first line the result of Ref. [11], obtained by means of a momentum-space integration, and in the second line our result, obtained by means of the coordinate-space method.
diagram 6 there is apparently a (very) small underestimation of the error, which is negligible in the sum of all contributions.

In Table 4 diagrams are grouped together in order to obtain infrared-convergent results. This is necessary for the numerical implementation of the momentum-space method. In our case, however, we have followed a different strategy considering an infrared regularization. This allows us to compute each Feynman diagram separately. We have used four different infrared regularizations:

(a) We simply introduce a regulator in the propagators as in Eq. (17). Explicitly, for the gluon \( \Delta_B(k) \) and for the fermion \( \Delta_F(k) \) propagator we use:

\[
\Delta_B(k) = \frac{1}{D_B(k, m)},
\]

\[
\Delta_F(k) = \frac{-i\bar{k}_\mu \gamma_\mu + M_W(k)}{D_F(k, m)}.
\]

(b) We regularize the gluon propagator as in Eq. (17), but use instead the correct Wilson-fermion propagator

\[
\Delta_F(k) = \frac{-i\bar{k}_\mu \gamma_\mu + M_W(k) + m\bar{k}^2 + (M_W(k) + m)^2}{k^2 + (M_W(k) + m)^2}.
\]

(c) We regularize the Wilson fermion as in (a), but use the massless propagator for the gluon.

(d) We regularize the Wilson fermion as in (b) and the massless propagator for the gluon.

The result for each diagram will be indicated by \( D_i^{(x)} \) where \( (x) \) refers to the chosen infrared regularization.

In Table 5 we report the result for each diagram computed in the regularization (a). As expected, the divergences cancel when the appropriate combinations of integrals are considered. The finite results are reported in Table 4.

Then, we consider regularization (b). In Table 6 we report, for two groups of graphs, the difference \( D_i^{(b)} - D_i^{(a)} \), where the superscripts refer to the chosen regularization. Note that, if we use the correct fermion propagator there are
Table 5: Results for the infrared regularization (a) at two loops. Graphs whose divergent contributions sum up to zero have been grouped together.

|   | Divergent Part | $a_1^{[2]}$ | $a_2^{[2]}$ | $a_3^{[2]}$ |
|---|----------------|-------------|-------------|-------------|
| 7 | $(\frac{1}{256}) \log m^2 + \frac{5}{32} \log \frac{m^2}{\pi^2}$ | -0.01347759288130718 | 0.0145250053341618950704 | 0 |
| 8 | $(\frac{1}{1024}) \log m^2 + \frac{5}{32} \log \frac{m^2}{\pi^2}$ | 0.001000549213311037 | 0 | 0 |
| 9 | $(\frac{1}{1024}) \log m^2 + \frac{5}{32} \log \frac{m^2}{\pi^2}$ | -0.00347141693(2) | 0 | 0 |
| 10 | $(\frac{1}{256}) \log m^2 + \frac{5}{32} \log \frac{m^2}{\pi^2}$ | 0.000020884775(4) | 0 | 0 |
| 11 | $(\frac{1}{1024}) \log m^2 + \frac{5}{32} \log \frac{m^2}{\pi^2}$ | -0.000110949599457269 | 0 | 0 |
| 12 | $N_f \left( \frac{1}{32} \log \frac{m^2}{\pi^2} + \frac{3}{8} \log \frac{m^2}{\pi^2} \right)$ | 0 | 0 | 0.000056962(2) |
| 13 | $N_f \left( \frac{1}{32} \log \frac{m^2}{\pi^2} + \frac{3}{8} \log \frac{m^2}{\pi^2} \right)$ | 0 | 0 | 0.000056962(2) |
| 14 | $\left( \frac{1}{1024} \right) \log m^2 + \frac{5}{32} \log \frac{m^2}{\pi^2}$ | -0.0005494345058160373 | 0.005832127004694453 | 0 |
| 15 | $\left( \frac{1}{1024} \right) \log m^2 + \frac{5}{32} \log \frac{m^2}{\pi^2}$ | -0.00028296107(1) | 0 | 0 |
| 16 | $\left( \frac{1}{1024} \right) \log m^2 + \frac{5}{32} \log \frac{m^2}{\pi^2}$ | -0.00005544799728634 | 0 | 0 |
| 17 | $\left( \frac{1}{1024} \right) \log m^2 + \frac{5}{32} \log \frac{m^2}{\pi^2}$ | 0.0001912446811(2) | 0 | 0 |
| 18 | $\left( \frac{1}{1024} \right) \log m^2 + \frac{5}{32} \log \frac{m^2}{\pi^2}$ | -0.000110949599457269 | 0 | 0 |
| 19 | $\left( \frac{1}{1024} \right) \log m^2 + \frac{5}{32} \log \frac{m^2}{\pi^2}$ | 0 | 0 | -0.00040789541774414 |
| 20 | $\left( \frac{1}{1024} \right) \log m^2 + \frac{5}{32} \log \frac{m^2}{\pi^2}$ | 0 | 0 | -0.00043346(4) |
| 21 | $\left( \frac{1}{1024} \right) \log m^2 + \frac{5}{32} \log \frac{m^2}{\pi^2}$ | 0.001606284825541242 | -0.001606284825541242 | 0 |
| 22 | $\left( \frac{1}{1024} \right) \log m^2 + \frac{5}{32} \log \frac{m^2}{\pi^2}$ | 0.0005015205(2) | -0.0005015205(2) | 0 |
| 23 | $\left( \frac{1}{1024} \right) \log m^2 + \frac{5}{32} \log \frac{m^2}{\pi^2}$ | -0.000210104645066014 | 0.000210104645066014 | 0 |
\[
\begin{align*}
\text{Table 6: Difference } D_i^{(b)} - D_i^{(a)}. & \text{ Graphs whose contributions sum up to zero are grouped together.} \\

i & \quad - \frac{N_f}{N} \left( \frac{1}{16m^2} - \frac{\mathcal{F}(1,-1)}{32\pi^2} - \frac{\mathcal{F}(1,1)}{8m^2} - \frac{\mathcal{F}(1,0)}{4\pi^2} \right) \\
19 & \quad - \frac{N_f}{N} \\
20 & \quad \frac{N_f}{N} \left( \frac{1}{16m^2} - \frac{\mathcal{F}(1,-1)}{32\pi^2} - \frac{\mathcal{F}(1,1)}{8m^2} - \frac{\mathcal{F}(1,0)}{4\pi^2} \right) \\
21 & \quad \left( 1 - \frac{1}{N^2} \right) \frac{Z_0}{16\pi^2} \\
22 & \quad \left( 1 - \frac{1}{N^2} \right) \frac{-Z_0}{32\pi^2} + \frac{\mathcal{F}(1,0)}{16\pi^2} \\
23 & \quad \left( 1 - \frac{1}{N^2} \right) \frac{-Z_0}{32\pi^2} - \frac{\mathcal{F}(1,0)}{16\pi^2} \\
\end{align*}
\]

\[
\begin{align*}
\text{Table 7: Difference } D_i^{(x)} - D_i^{(a)}. & \text{ Regularization } (x) = (c) \text{ is obtained by taking } \mathcal{P}_{\text{wil}} = 0, \text{ regularization } (x) = (d) \text{ by taking } \mathcal{P}_{\text{wil}} = 1. \\

i & \quad \left( 1 - \frac{1}{N^2} \right) \left( \frac{11Z_0}{96\pi^2} + \frac{5\mathcal{P}_{\text{wil}} Z_0}{96\pi^2} \right) \\
21 & \quad \left( 1 - \frac{1}{N^2} \right) \left( \frac{-11Z_0}{192\pi^2} + \frac{11\mathcal{F}(1,0)}{96\pi^2} + \mathcal{P}_{\text{wil}} \left( - \frac{5Z_0}{192\pi^2} + \frac{5\mathcal{F}(1,0)}{96\pi^2} \right) \right) \\
22 & \quad \left( 1 - \frac{1}{N^2} \right) \left( \frac{-11Z_0}{192\pi^2} - \frac{11\mathcal{F}(1,0)}{96\pi^2} - \mathcal{P}_{\text{wil}} \left( \frac{5Z_0}{192\pi^2} + \frac{5\mathcal{F}(1,0)}{96\pi^2} \right) \right) \\
\end{align*}
\]

additional divergences. As expected, the sum of the terms reported in Table 6 vanishes.

As a last check, we have repeated the calculation of diagrams 21, 22, 23 using the regularizations (c) and (d). In Table 7 we give the differences \( D_i^{(x)} - D_i^{(a)} \) for \( x = c, d \). The case \( x = c \) is obtained by setting \( \mathcal{P}_{\text{wil}} = 0 \), while \( x = d \) is obtained by setting \( \mathcal{P}_{\text{wil}} = 1 \). Again, we note that the individual diagrams are different, but the sum of the contributions vanishes.

In conclusion, the coordinate-space method is a very efficient tool for the computation of (infrared-finite and infrared-divergent) two-loop Feynman integrals also in the presence of Wilson fermions. The method proposed in Ref. [9] really works.
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