POSITIONAL VOTING, DOUBLY STOCHASTIC MATRICES, AND THE BRAID ARRANGEMENT

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ABSTRACT. We provide elementary proofs of results from [8] and [5] regarding the possible outcomes arising from a fixed profile within the class of positional voting systems. Our arguments enable a simple and explicit construction of paradoxical profiles, and we also demonstrate how to choose weights that realize desirable results from a given profile. The analysis ultimately boils down to considering the image of the fundamental chamber of the essentialization of the braid arrangement under linear maps described by doubly stochastic matrices.

1. Introduction

Suppose there are $n$ candidates running for a single office. There are many different social choice procedures one can use to select a winner. In this paper, we study a particular class called positional voting systems. A positional voting system is an electoral system in which each voter submits a ranked list of the candidates. Points are then assigned according to a fixed weighting vector that gives $w_i$ points to a candidate every time they appear in position $i$ on a ballot, and candidates are ranked according to the total number of points received. For example, plurality is a positional voting system with weighting vector $w = [1 \ 0 \ 0 \ \cdots \ 0]^T$: one point is assigned to each voter’s top choice, and the candidate with the most points wins. The Borda count is another common example of a positional voting system in which the weighting vector is given by $w = [n-1 \ n-2 \ \cdots \ 1 \ 0]^T$. Other examples include the systems used in the Eurovision Song Contest and to elect the Parliament of Nauru [1, 7].

By tallying points in this way, a positional voting system outputs not just a winner, but a complete ranking of all candidates, called the societal ranking. The societal ranking produced by a positional voting system depends not only on the set of ballots (called the profile) but also on the choice of weighting vector. With the freedom to choose different weighting vectors, one can achieve many different outcomes from the same profile. An immediate questions is “Given a profile, how many different societal rankings are possible?”

In [8], Donald Saari famously showed that for a given profile on $n$ alternatives, there are at most $n! – (n-1)!$ different possible outcomes depending on the choice of weighting vector. Moreover, this bound is sharp in that there exist profiles for which exactly this many different societal rankings are possible. In [5], Daugherty, Eustis, Minton, and Orrison provided a new approach to analyzing positional voting systems and extended some of Saari’s results to cardinal (as opposed to ordinal) rankings, as well as to partial rankings; see also [4]. Our paper serves to complement these works by providing alternative derivations which afford more explicit constructions, new perspectives, and more generally accessible arguments.

After establishing some conceptual foundations, we proceed by proving the main result in [5] using facts about doubly stochastic matrices (Theorem 3.3). With a little more linear algebra, we are then able to recover Saari’s findings (Theorems 3.6 and 3.9). An upshot of our methodology is that it gives a concrete
means of constructing ‘paradoxical profiles.’ It also enables us to provide a simple characterization of the possible outcomes resulting from a given profile (Theorem 4.2).

In addition, our work illustrates the utility of thinking about doubly stochastic matrices and hyperplane arrangements in problems related to social choice procedures. Doubly stochastic matrices arise very naturally in our analysis, and the braid arrangement is a useful setting for thinking about choice as it provides a geometric realization of rankings. As with the ‘algebraic voting theory’ from [5], the hope is that by formulating problems in terms of different mathematical constructs, new tools and perspectives become available. While the hyperplane connection has received some attention in previous works—for instance, Terao’s proof of Arrow’s impossibility theorem [10]—doubly stochastic matrices seem to have been given much less consideration in the context of voting.

Finally, and perhaps most importantly, we feel that our approach renders the proofs of the aforementioned results more broadly accessible. Whereas the arguments in [8] depend on some rather involved combinatorial and geometric reasoning about high-dimensional simplices and those in [5] are based on the representation theory of the symmetric group, our proofs should be comprehensible to a motivated student who has taken a first course in linear algebra.

2. Notation and Terminology

Throughout this paper, \( n \geq 3 \) is a fixed integer representing the number of candidates. The number of voters is \( N \), which we only assume to be rational (though Proposition 3.8 shows that we may take \( N \in \mathbb{N} \) if we are just interested in ordinal rankings). While \( n \) is arbitrary and given in advance, \( N \) may have some implicit constraints depending on the context. We work exclusively over the field \( \mathbb{Q} \) of rational numbers, and we write \( \mathbb{N}_0 \) for the set of nonnegative integers. Vectors are always written in boldface with the components of a generic \( n \)-dimensional vector \( \mathbf{v} \) denoted by \( v_1, v_2, \ldots, v_n \). We write \( \mathbf{1} \) for the vector of all ones and \( \mathbf{O} \) for the matrix of all ones, where the dimensions are clear from context.

Let \( \sigma \) be a permutation in \( S_n \) and define the following subsets of \( \mathbb{Q}^n \):

\[
V_0 = \{ \mathbf{x} \in \mathbb{Q}^n : \sum_{i=1}^n x_i = 0 \},
\]

\[
C_\sigma = \{ \mathbf{x} \in \mathbb{Q}^n : x_{\sigma(1)} > x_{\sigma(2)} > \cdots > x_{\sigma(n)} \},
\]

\[
W = C_{id} \cap V_0 = \{ \mathbf{x} \in \mathbb{Q}^n : x_1 > x_2 > \cdots > x_n, x_1 + \cdots + x_n = 0 \}.
\]

Thus, \( W \) is the set of weighting vectors with decreasing entries, normalized so the sum of a vector’s components is zero. Label the permutations in \( S_n \) lexicographically according to one-line notation, and let \( R_\ell \) be the \( n \times n \) permutation matrix corresponding to \( \sigma_\ell \), defined by \( R_\ell(i,j) = 1(\sigma_\ell(j) = i) \).

Given a weighting vector \( \mathbf{w} \in W \), define the \( n \times n! \) matrix \( T_\mathbf{w} = [\sigma_1 \mathbf{w} \ \sigma_2 \mathbf{w} \ \cdots \ \sigma_n \mathbf{w}] \) having \( \ell \)-th column \( \sigma_\ell \mathbf{w} := R_\ell \mathbf{w} = [w_{\sigma^{-1}_\ell(1)} \ w_{\sigma^{-1}_\ell(2)} \ \cdots \ w_{\sigma^{-1}_\ell(n)}]^T \). For a given profile \( \mathbf{p} \in \mathbb{Q}^{n!} \), the results vector for the positional voting procedure associated with \( \mathbf{w} \) is given by

\[
\mathbf{r} = T_\mathbf{w} \mathbf{p} = p_1 R_1 \mathbf{w} + p_2 R_2 \mathbf{w} + \cdots + p_{n!} R_{n!} \mathbf{w} = Q_\mathbf{p} \mathbf{w}
\]

where \( Q_\mathbf{p} = \sum_{\ell=1}^{n!} p_\ell R_\ell \).

Each \( \sigma \in S_n \) corresponds to the ranking of the candidates (labeled 1 through \( n \)) in which candidate \( \sigma(k) \) is the \( k \)-th favorite. The profile \( \mathbf{p} \) encodes preferences of the electorate so that \( p_k \) is the number of voters with preference \( \sigma_k \). If each voter assigns \( w_k \) points to their \( k \)-th favorite candidate, then \( r_j \) is the total number of points given to candidate \( j \). The societal ranking for this election procedure is \( \pi \in S_n \) with \( \mathbf{r} \in C_\pi \).
Note that we have given two different ways of computing the results vector $r$ in Equation (1). On one hand, we have $T_w$, an $n \times n!$ matrix that encodes all the possible permutations of the weighting vector, which can be combined with the $n!$-dimensional profile vector $p$ to yield the result. On the other hand, we have $Q_p$, an $n \times n$ matrix that encodes the number of votes each candidate receives in each place, which can be combined with the $n$-dimensional weighting vector $w$ to yield the result.

**Example 2.1.** Consider an election with 4 candidates and preferences described by the following table.

| Voting Preference | Number of Votes |
|-------------------|-----------------|
| (1, 3, 2, 4)      | 15              |
| (2, 4, 3, 1)      | 11              |
| (4, 1, 3, 2)      | 10              |
| (2, 4, 1, 3)      | 16              |
| (3, 1, 4, 2)      | 20              |
| (4, 3, 2, 1)      | 13              |

Then

$$Q_p = 15 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} + 11 \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} + 10 \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

+ 16 $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ + 20 $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ + 13 $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 15 & 30 & 16 & 24 \\ 27 & 0 & 28 & 30 \\ 20 & 28 & 21 & 16 \\ 23 & 27 & 20 & 15 \end{bmatrix}$

To implement the Borda count, we use the weighting vector $w = [1.5 \ 0.5 \ -0.5 \ -1.5]^T$, which is obtained from $[3 \ 2 \ 1 \ 0]^T$ by subtracting $\frac{1}{4}(3 + 2 + 1 + 0)1$. This yields

$$Q_pw = \begin{bmatrix} 15 & 30 & 16 & 24 \\ 27 & 0 & 28 & 30 \\ 20 & 28 & 21 & 16 \\ 23 & 27 & 20 & 15 \end{bmatrix} \cdot \begin{bmatrix} 1.5 \\ 0.5 \\ -0.5 \\ -1.5 \end{bmatrix} = \begin{bmatrix} -6.5 \\ -18.5 \\ 9.5 \\ 15.5 \end{bmatrix},$$

so that the societal ranking is (4, 3, 1, 2).

If we instead use the plurality method, $w = [0.75 \ -0.25 \ -0.25 \ -0.25]^T$, we find that

$$Q_pw = \begin{bmatrix} 15 & 30 & 16 & 24 \\ 27 & 0 & 28 & 30 \\ 20 & 28 & 21 & 16 \\ 23 & 27 & 20 & 15 \end{bmatrix} \cdot \begin{bmatrix} 0.75 \\ -0.25 \\ -0.25 \\ -0.25 \end{bmatrix} = \begin{bmatrix} -6.25 \\ 5.75 \\ -1.25 \\ 1.75 \end{bmatrix},$$

resulting in (2, 4, 3, 1). By changing the weights, we moved the last place candidate to first place!
Of course, it is also possible that \( r \notin C_\pi \) for any \( \pi \in S_n \) if there are ties between candidates. In this case, \( r \) lies on one or more of the hyperplanes \( H_{ij} = \{ x \in \mathbb{Q}^n : x_i = x_j \} \), \( 1 \leq i < j \leq n \), in the braid arrangement. The hyperplane \( H_{ij} \) consists of all results vectors for which candidates \( i \) and \( j \) are tied. Similarly, define the following subsets of \( \mathbb{Q}^n \):

\[
H^+_{ij} = \{ x \in \mathbb{Q}^n : x_i > x_j \}, \\
H^-_{ij} = \{ x \in \mathbb{Q}^n : x_i < x_j \}.
\]

If \( r \) lies in \( H^+_{ij} \), then candidate \( i \) finished ahead of candidate \( j \), and if \( r \) lies in \( H^-_{ij} \), then candidate \( j \) finished ahead of candidate \( i \). Non-strict rankings correspond to lower dimensional faces of the braid arrangement, where a face is any nonempty intersection of the form \( G = \bigcap_{1 \leq i < j \leq n} H^+_{ij} \). The faces correspond to ordered set partitions of \([n] \) with equal to \([B_1, \ldots, B_m]\) where \( G \) is the set of all \( x \in \mathbb{Q}^n \) such that \( x_i > x_j \) if and only if there exist \( k < \ell \) with \( i \in B_k \) and \( j \in B_\ell \). If a results vector \( r \) is in the face \( G \sim [B_1, \ldots, B_m] \), then \( B_k \) is the set of candidates tied for \( k^{th} \)-place. Strict rankings correspond to the \( n \)-dimensional faces (called chambers) which are of the form \( C_\pi \sim \{ \{\pi(1)\}, \ldots, \{\pi(n)\}\} \).

Finally, we observe that if \( w \in V_0 \), then \( \sigma w \in V_0 \) for all \( \sigma \in S_n \), so \( r \) as defined in Equation (1) is a linear combination of sum-zero vectors and thus lies in \( V_0 \) as well. Also, the decomposition \( Q_p = \sum_{\ell=1}^{n!} p_{\ell} R_{\ell} \) shows that every row and column of \( Q_p \) sums to \( N = \sum_{\ell=1}^{n!} p_{\ell} \), the total number of ballots cast. Thus the condition that \( w \in V_0 \) is not much of a restriction since any \( y \in C_{id} \) can be decomposed as \( y = y^+ + a_y 1 \) where \( y^+ \in W \) and \( a_y = \frac{1}{n} \sum_{i=1}^{n} y_i \). Moreover, \( Q_p y \) lies in the same face as \( Q_p y + N a_y 1 \) because

\[
Q_p y = Q_p y^+ + a_y Q_p 1 = Q_p y + N a_y 1.
\]

Indeed, \( \bigcap_{1 \leq i < j \leq n} H_{ij} = \{ c 1 : c \in \mathbb{Q} \} \), so it is natural to project the braid arrangement onto the orthogonal complement, \( V_0 \). This is called its essentialization.

3. Main Results

Before stating our first result, we pause to record the following crucial observation, which is well-known but included for the sake of completeness.

**Proposition 3.1.** Let \( P \) be any collection of \((n-1)^2 + 1\) linearly independent \( n \times n \) permutation matrices, and let \( M_n \) be the vector space of \( n \times n \) matrices over \( \mathbb{Q} \) with all row and columns sums equal. Then every matrix in \( M_n \) can be written as a linear combination of matrices in \( P \).

**Proof.** Suppose \( S \) is an \( n \times n \) matrix with all row and column sums equal to \( t \), and let \( m = \max_{(i,j) \in [n] \times [n]} |S(i,j)| \). Then \( P = (mn + t)^{-1}(S + mO) \) is doubly stochastic (all entries nonnegative and all row and column sums equal to 1). The Birkhoff-von Neumann theorem [2, 11] shows that \( P \) can be written as a convex combination of permutation matrices, \( P = \sum_{\ell=1}^{n!} \lambda_{\ell} R_{\ell} \). Similarly, \( \frac{1}{n} O = \sum_{\ell=1}^{n!} \kappa_{\ell} R_{\ell} \) as it too is doubly stochastic. It follows that \( S = (mn + t)P - mO \) is a linear combination of permutation matrices. Since every linear combination of permutation matrices has all row and column sums equal, \( M \) is precisely the linear span of the permutation matrices.

To see that the dimension of \( M_n \) is \((n-1)^2 + 1\), define \( E_{i,j} \) to be the \( n \times n \) matrix with 1’s in positions \((i,j)\) and \((n,n)\), -1’s in positions \((i,n)\) and \((n,j)\), and 0’s elsewhere for each \((i,j) \in [n-1]^{2}\). If \( Z = [z_{i,j}]_{i,j=1}^{n-1} \) is any matrix with all row and column sums equal to 0, then it is easy to see that \( Z = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} z_{i,j} E_{i,j} \).

Now let \( S \) be any matrix with all row and column sums equal to \( t \). Then \( S - tI \) has all row and column sums zero, hence \( S \) can be expressed as a linear combination of the \( E_{i,j} \)’s and \( I \). As these \((n-1)^2 + 1\) matrices are clearly linearly independent, the assertion follows. \( \square \)
Remark 3.2. Proposition 3.1 can also be proved without invoking Birkhoff-von Neumann by showing that the collection of permutation matrices
\[ B = \{ R_{(i,j,n)} : i, j \in [n-1] \text{ are distinct} \} \cup \{ R_{(i,n)} : i \in [n-1] \} \cup \{ I \} \]
is a basis for \( M_n \). (The subscripts represent permutations in cycle notation and \( R_x \) is as previously defined.) Linear independence follows by looking at the final rows and columns of the matrices and \( M_n = \text{span}(B) \) follows from the above argument upon observing that \( E_{i,j} = I - R_{(i,n)} - R_{(j,n)} + R_{(j,i,n)} \) for distinct \( i, j < n \) and \( E_{k,k} = I - R_{(k,n)} \) for \( k < n \).

The following theorem was proved in [5] using facts about the representation theory of \( S_n \). Our proof is based on the same general reasoning—essentially, that one can write \( T_{w_r}p = Q_pw \)—but uses only linear algebra.

Theorem 3.3. Given any linearly independent \( w_1, \ldots, w_{n-1} \in W \) and any \( r_1, \ldots, r_{n-1} \in V_0 \), there are infinitely many \( p \in Q^{n!} \) with \( T_{w_r}p = r_k \) for \( k = 1, \ldots, n-1 \).

Proof. Define \( r_0 = w_0 = 1 \) and set \( F = [w_0 \ w_1 \ \cdots \ w_{n-1}], R = [r_0 \ r_1 \ \cdots \ r_{n-1}], \) and \( Q = RF^{-1}. \) (\( F \) is invertible since \( w_1, \ldots, w_{n-1} \) are linearly independent and all orthogonal to \( w_0. \)) Then \( Qw_k = r_k \) for \( k = 0, \ldots, n-1 \) since
\[ [Qw_0 \ Qw_1 \ \cdots \ Qw_{n-1}] = QF = R = [r_0 \ r_1 \ \cdots \ r_{n-1}]. \]
The condition \( Qw_0 = r_0 \) implies that the rows of \( Q \) sum to 1. To see that the columns sum to 1, we first observe that
\[ w_0^TR = \left[ \langle w_0, r_0 \rangle \ \langle w_0, r_1 \rangle \ \cdots \ \langle w_0, r_{n-1} \rangle \right] = [n \ 0 \ 0 \ 0] \]
since \( w_0 = r_0 = 1 \) is orthogonal to each of \( r_1, \ldots, r_{n-1} \) by assumption. Accordingly,
\[ w_0^TQ = w_0^TRF^{-1} = [n \ 0 \ 0 \ 0]F^{-1} = n\left[ F^{-1}(1,1) \ F^{-1}(1,2) \ \cdots \ F^{-1}(1,n) \right] = nf \]
where \( f \) is the first row of \( F^{-1}. \) Since \( F^{-1}F = I, \) it must be the case that \( \langle f^T, w_0 \rangle = 1 \) and \( \langle f^T, w_k \rangle = 0 \) for \( k = 1, \ldots, n-1. \) The latter condition implies that \( f = C1^T \) for some \( C, \) so the former implies \( 1 = \langle f^T, w_0 \rangle = C(1,1) = nC. \) Therefore, \( w_0^TQ = nf = w_0^T, \) so the columns of \( Q \) sum to 1 as well.

Since \( Q \) has all rows and columns summing to 1, it follows from Proposition 3.1 that it is a linear combination of permutation matrices. This means that there exists \( p \in Q^{n!} \) such that \( Q = \sum_{\ell=1}^{n!} p_{\ell}R_{\ell}. \) Accordingly, \( T_{w_r}p = Qw_k = r_k \) for \( k = 1, \ldots, n-1. \) In fact there are infinitely many such \( p \) since there are \( n! \) permutation matrices and the space of doubly stochastic matrices is \( (n^2 - 2n + 2)-\text{dimensional}. \)

The preceding proof works just as well if one takes the weighting vectors to lie in \( W, \) the closure of \( W. \) This allows for voting schemes in which the same point value can be assigned to multiple candidates, such as \( w = [1 \ 1 \ 0 \ \cdots \ 0]^T. \) One may impose additional constraints like all weighting vectors having the same positions tied by restricting to some lower dimensional face \( F \subset W, \) but then the linear independence condition dictates that there are only \( d = \dim(F) \) weighting/results vectors. To treat this case, take \( w_1, \ldots, w_d \) to be linearly independent vectors in \( F \) and \( r_1, \ldots, r_d \) to be the desired results vectors in \( V_0. \) Then choose \( w_{d+1}, \ldots, w_{n-1} \) to be any vectors in \( W \) for which \( w_1, \ldots, w_{n-1} \) are linearly independent and let \( r_{d+1}, \ldots, r_{n-1} \) be any results vectors in \( V_0. \)

Also, observe that \( Q = RF^{-1} \) is explicit and may be easily realized as a linear combination of doubly stochastic matrices; the entries of \( Q \) may be negative, so one must add an appropriate multiple of the all ones matrix and rescale to obtain a doubly stochastic matrix \( P \) as in the proof of Proposition 3.1. As there
are algorithms for finding a Birkhoff-von Neumann decomposition of any doubly stochastic matrix [6], our method actually provides a construction of the paradoxical profile. (A brute force algorithm proceeds as follows: Choose $R_t$ so that $P(i, \sigma_t^{-1}(i)) \neq 0$ for all $i \in [n]$. This is possible by Hall’s marriage theorem and the fact that $P$ is doubly stochastic. Next, subtract $\min_i P(i, \sigma_t^{-1}(i)) R_t$ from $P$. The resulting matrix is a positive multiple of a doubly stochastic matrix, so the previous steps can be repeated, at most $n^2$ times, to produce a decomposition.)

**Example 3.4.** Suppose that

$$w_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ -3 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \quad w_3 = \begin{bmatrix} 4 \\ 3 \\ -8 \\ -11 \end{bmatrix}, \quad r_1 = \begin{bmatrix} 9 \\ 4 \\ -2 \\ -11 \end{bmatrix}, \quad r_2 = \begin{bmatrix} 5 \\ 4 \\ 2 \\ -6 \end{bmatrix}, \quad r_3 = \begin{bmatrix} 13 \\ -2 \\ -5 \\ -6 \end{bmatrix}.$$  

Then

$$Q = \begin{bmatrix} 1 & 9 & 5 & 13 \\ 1 & 4 & 4 & -2 \\ 1 & -2 & 2 & -5 \\ 1 & -11 & -11 & -6 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 1 & 1 & 1 & 3 \\ 1 & 0 & -1 & 1 \\ 1 & -3 & -1 & -8 \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 72 & -36 & -24 & 0 \\ 75 & -45 & -37 & 19 \\ -9 & 27 & -17 & 11 \\ -126 & 66 & 90 & -18 \end{bmatrix}$$

and

$$P = (4 \cdot \frac{126}{12} + 1)^{-1} (Q + \frac{126}{12} O) = \frac{1}{516} \begin{bmatrix} 198 & 90 & 102 & 126 \\ 201 & 81 & 89 & 145 \\ 117 & 153 & 109 & 137 \\ 0 & 192 & 216 & 108 \end{bmatrix}$$

is doubly stochastic.

According to the computer algebra system Sage [9], a Birkhoff-von Neumann decomposition of $P$ is given by

$$P = \frac{3}{56} R_{(1,2,3,4)} + \frac{45}{516} R_{(1,2,4,3)} + \frac{145}{516} R_{(1,3,4,2)} + \frac{1}{516} R_{(2,1,3,4)} + \frac{2}{126} R_{(2,3,4,1)} + \frac{17}{56} R_{(2,4,1,3)} + \frac{17}{56} R_{(2,4,3,1)} + \frac{69}{516} R_{(3,1,2,4)} + \frac{7}{126} R_{(3,2,4,1)}.$$  

Here the subscripts represent permutations in one-line notation.

Since $O = R_{(1,2,3,4)} + R_{(2,1,4,3)} + R_{(3,4,1,2)} + R_{(4,3,2,1)}$, we see that

$$Q = -9 R_{(1,2,3,4)} + \frac{35}{12} R_{(1,2,4,3)} + \frac{145}{12} R_{(1,3,4,2)} + \frac{1}{12} R_{(2,1,3,4)} - \frac{21}{2} R_{(2,1,4,3)} + \frac{2}{3} R_{(2,3,4,1)} + \frac{17}{2} R_{(2,4,1,3)} + \frac{17}{2} R_{(2,4,3,1)} + \frac{69}{12} R_{(3,1,2,4)} + \frac{7}{3} R_{(3,2,4,1)} - \frac{21}{2} R_{(3,4,1,2)} - \frac{21}{2} R_{(4,3,2,1)}.$$  

Thus one profile for which the weight $w_k$ produces the result $r_k$ consists of $-9$ votes for 1 above 2 above 3 above 4, $\frac{35}{12}$ votes for 1 above 2 above 4 above 3, and so forth.

In typical settings, we are concerned with ordinal rather than cardinal rankings, and Theorem 3.3 can then be used to generate significantly many more paradoxes. To facilitate the ensuing argument, we record the following simple lemma.

**Lemma 3.5.** For any $w \in W$, $x \in V_0$, there is some rational number $\eta_0 > 0$ such that $\eta w + x \in W$ for all $\eta \geq \eta_0$.

**Proof.** Let $m = \min_{1 \leq k \leq n-1} (w_k - w_{k+1})$ and $M = \max_{1 \leq k \leq n} |x_k|$, and set $\eta_0 = 3M/m$. Then the successive entries of $\eta w$ differ by at least $3M$, so adding $x$ does not change their relative order. $\square$
Our next theorem implies the result from [8] that there exist profiles from which \((\frac{n-1}{n})n!\) different ordinal rankings can be obtained by judicious choices of weighting vectors in \(W\).

**Theorem 3.6.** There exist infinitely many profiles \(p \in \mathbb{Q}^n!\) such that for every \(\pi \in S_n\) satisfying \(\pi(n) \neq 1\), there is some \(w(\pi) \in W\) with \(T_{w(\pi)}p \in C_\pi\).

**Proof.** To begin, let \(w_1, \ldots, w_{n-1}\) be any linearly independent weighting vectors in \(W\), with \(w_1\) scaled so that \(w_1 - n\binom{n+1}{2}w_k \in W\) for \(k = 2, \ldots, n-1\). (This is possible by Lemma 3.5.) Set \(f_k = e_k - e_{k+1}\) for \(k = 1, \ldots, n-1\) where \(e_1, \ldots, e_n\) are the standard basis vectors in \(\mathbb{Q}^n\), and let \(p\) be as in Theorem 3.3 with each \(r_k = f_k\). Fix \(\pi \in S_n\) with \(\pi(n) \neq 1\), and write \(b = \pi(n)\). The result will follow if we can find \(\alpha_1, \ldots, \alpha_{n-1}\) so that

\[
w(\pi) = \sum_{k=1}^{n-1} \alpha_k w_k \in W
\]

and

\[
s(\pi) = T_w p = Q_p w = \sum_{k=1}^{n-1} \alpha_k f_k \in C_\pi.
\]

Now for \(1 \leq i < j \leq n\), define \(f_{ij} = e_i - e_j = \sum_{k=i}^{j-1} f_k\) and \(w_{ij} = \sum_{k=i}^{j-1} w_k \in W\). Then for any collection of numbers \(\{\beta_{ij}\}_{i < j},\)

\[
\sum_{i < j} \beta_{ij} f_{ij} = \sum_{i < j} \beta_{ij} \sum_{k=i}^{j-1} f_k = \sum_{k=1}^{n-1} \alpha_k f_k
\]

and

\[
\sum_{i < j} \beta_{ij} w_{ij} = \sum_{k=1}^{n-1} \alpha_k w_k
\]

with \(\alpha_k = \sum_{i < j} \beta_{ij} 1\{i \leq k < j\}\). Accordingly, it suffices to construct \(w = \sum_{i < j} \beta_{ij} w_{ij} \in W\) with \(s = Q_p w = \sum_{i < j} \beta_{ij} f_{ij} \in C_\pi\). Note that each \(w_{ij}\) is in \(W\), so \(w\) will be as well whenever the \(\beta_{ij}\)'s are nonnegative (and not all 0) since \(W\) is closed under nontrivial conical combinations.

Next, we observe that if \(b = n\), then we can take \(\beta_{kn} = n - \pi^{-1}(k) \geq 0\) for \(k = 1, \ldots, n-1\) and \(\beta_{ij} = 0\) for \(j \neq n\) as this gives the \(k\)th place candidate \(s_{\pi(k)} = n - \pi^{-1}(\pi(k)) = n-k > 0\) points for \(k = 1, \ldots, n-1\) and gives \(-\binom{2}{2} = 0\) points to candidate \(n\). As all \(\beta_{ij}\) are nonnegative, \(w(\pi) = \sum_{i < j} \beta_{ij} w_{ij} \in W\).

If \(b \neq n\), let \(\tilde{\pi}\) be the permutation formed from \(\pi\) by moving \(n\) to last place (so \(\tilde{\pi}(i) = \pi(i)\) for \(i < \pi^{-1}(n)\), \(\tilde{\pi}(i) = \pi(i+1)\) for \(\pi^{-1}(n) \leq i < n\), and \(\tilde{\pi}(n) = n\)), and let \(\tilde{w} = w(\tilde{\pi}), \tilde{s} = s(\tilde{\pi})\) be constructed as above.

Now set \(w = \tilde{w} - \gamma_{bn} w_{bn}\) with \(\gamma_{bn} = \binom{n}{2} + n - \pi^{-1}(n) + \frac{1}{2}\). Then \(s = Q_p w = \tilde{s} - \gamma_{bn} f_{bn} \in C_\pi\) as candidate \(n\) now has \(n - \pi^{-1}(n) + \frac{1}{2}\) points, candidate \(b\) has a negative number of points, and all other candidates have the same point values as in \(\tilde{s}\). Also, \(1 < b < n\) and \((n-b)\gamma_{bn} = (n-b)\left[\binom{n+1}{2} - \pi^{-1}(n) + \frac{1}{2}\right] < n\binom{n+1}{2},\) so

\[
w_1 - \gamma_{bn} w_{bn} = w_1 - \gamma_{bn} \sum_{k=b}^{n-1} w_k = \frac{1}{n-b} \sum_{k=b}^{n-1} \left(w_1 - (n-b)\gamma_{bn} w_k\right) \in W.
\]

As \(\tilde{w} = aw_1 + \sum_{k=2}^{n-1} (n-\tilde{\pi}^{-1}(k))w_k\) with \(a = n - \tilde{\pi}^{-1}(1) \geq 1,\)

\[
w = \tilde{w} - \gamma_{bn} w_{bn} = (\beta_{1n} - 1)w_1 + \sum_{k=2}^{n-1} (n-\tilde{\pi}^{-1}(k))w_k + (w_1 - \gamma_{bn} w_{bn})
\]

is a conical combination of vectors in \(W\) and so belongs to \(W\). This completes the proof. \(\square\)
Remark 3.7. The \( b = n \) part of the above argument can be interpreted as saying that if we have \( n - 1 \) ‘serious candidates,’ then by introducing ‘dummy candidate’ \( n \), who is assured to lose, there are profiles which achieve any relative ordering of the serious candidates by choosing appropriate weighting vectors.

It is often more convenient to work with \( Q_p \) than \( T_w \), so we take a moment to observe that if one only cares about the ordinal rankings of candidates, then it can always be assumed that each profile consists of nonnegative integers or that the matrix \( Q_p \) is doubly stochastic.

**Proposition 3.8.** Using the notation from above,

1. For any \( p \in \mathbb{Q}^{n!} \), there exists \( \tilde{p} \in \mathbb{N}_0^n \) with \( Q_p \tilde{p} \) lying in the same face as \( Q_p w \) for all \( w \in \mathbb{Q}^{n!} \).
2. For any \( p \in \mathbb{Q}^{n!} \), there exists \( \tilde{p} \in \mathbb{Q}^{n!} \) such that \( \tilde{p} \geq 0 \) for all \( \ell \), \( \sum_{\ell=1}^{n!} \tilde{p}_\ell = 1 \), and \( Q_p \tilde{p} \) lies in the same face as \( Q_p w \) for all \( w \in \mathbb{Q}^{n!} \).

**Proof.** For the first claim, set \( x = \max_\ell \{ p_\ell \} \), \( d = \text{lcm}(p_1, \ldots, p_{n!}) \), and \( \tilde{p} = d(p + x1) \in \mathbb{N}_0^n \). If \( Q_p w = T_w p = r \), then

\[
Q_p \tilde{p} = T_w d(p + x1) = dT_w p + dxT_w 1 = dr
\]

since \( T_w 1 = 0 \). Thus \( Q_p \tilde{p} \) is a positive multiple of \( Q_p w \) and so lies in the same face.

For the second claim, set \( m = \max_{i,j} |Q_p(i,j)| \), \( c = (mn + \sum_{\ell=1}^{n!} p_\ell)^{-1} > 0 \), and \( \tilde{Q}_p = c(Q_p + mO) \) where \( O \) is the all ones matrix. Then \( \tilde{Q}_p \) is nonnegative with rows and columns summing to 1, so Birkhoff-von Neumann guarantees the existence of a nonnegative \( \tilde{p} \in \mathbb{Q}^{n!} \) with entries summing to 1 that satisfies \( Q_p \tilde{p} = \tilde{Q}_p \). The claim follows since

\[
\tilde{Q}_p w = c(Q_p + mO) w = cQ_p w + mcO w = cQ_p w
\]
lies in the same face as \( Q_p w \). \( \square \)

The first part of Proposition 3.8 is only helpful inasmuch as one might object to having negative or fractional votes cast. The second is useful from a more mathematical perspective: Since \( W \) is a cone, its image under \( Q_p \) is the same as its image under the doubly stochastic matrix \( Q_p \). Thus we can study possible ordinal outcomes of positional voting procedures by looking at the image of the convex cone \( W \) (or its closure \( \overline{W} \)) under doubly stochastic matrices. An example of the utility of this observation is that, by Proposition 3.1, any set of \( (n-1)^2 + 1 \) linearly independent permutation matrices can give rise to all paradoxical profiles for positional voting procedures. In other words, one needs only this many distinct preferences amongst the electorate. In fact, the construction from Proposition 3.8 shows that one may take the doubly stochastic matrix to have at least one entry equal to zero, and a result from [3] then shows that a Birkhoff-von Neumann decomposition of size at most \( (n-1)^2 \) exists. This is the content of Theorem 4 in [8].

The final result of this section uses the doubly stochastic matrix perspective to show that a profile can give rise to at most \( \frac{(n-1)!}{n!} \) strict societal rankings; Theorem 3.6 shows that this is sharp. This result was first proved in [8] by analyzing certain simplicial constructs geometrically. Our argument relies on a different approach which we feel is a bit more transparent.

**Theorem 3.9.** For any \( p \in \mathbb{Q}^{n!} \), there are at most \( n! - (n-1)! \) permutations \( \pi \in S_n \) such that \( T_w p \in C_\pi \) for some \( w \in \mathbb{Q}^{n!} \).

**Proof.** Given any profile \( p \in \mathbb{Q}^{n!} \), there is a doubly stochastic matrix \( Q \) such that the possible strict societal rankings arising from \( p \) are precisely those \( \pi \in S_n \) for which \( C_\pi \cap QW \neq \emptyset \). Equivalently, since doubly
Proof. Clearly for every given any \( w \) \( \pi \in C \cap V_0 \), \( \pi \) is a possible ranking for \( p \) iff \( C \cap W \neq \emptyset \).

Now \( T(W) \) is the linear image of a convex set and thus is convex. Also, \( T(W) \) is a proper subset of \( V_0 \) because if \( T \) is not bijective then \( T(W) \subseteq T(V_0) \subseteq V_0 \), and if \( T \) is bijective then \( T(W) \cap -T(W) = T(W) \cap T(W) = \{0\} \). This means that there is some \( v \in V_0 \setminus T(W) \), hence \( \sigma \in V_0 \setminus T(W) \) for every \( \varepsilon > 0 \), so \( 0 \in \partial T(W) \). Accordingly, the supporting hyperplane theorem shows that there is a hyperplane \( H \) through the origin in \( V_0 \) with \( T(W) \) contained entirely in the associated closed positive half-space. In particular, there is a nonzero \( h \in V_0 \) such that \( \pi \) is a possible ranking for \( p \) only if there is an \( r \in C \cap V_0 \) such that \( \langle r, h \rangle \geq 0 \).

We will establish the result by showing that this constraint precludes \((n - 1)!\) rankings from being possible outcomes associated with \( p \). To this end, fix \( \sigma' \in S_{n-1} \) and define \( \sigma \in S_n \) by \( \sigma(i) < \sigma(j) \) if \( \sigma'(i) < \sigma'(j) \) and \( \sigma^{-1}(k) < \sigma^{-1}(n) \) if \( h_k \leq 0 \) for \( i, j, k < n \). Since \( h_n = -\sum_{k=1}^{n-1} h_k \), we see that for every \( r \in C \cap V_0 \),

\[
\langle r, h \rangle = \sum_{k=1}^{n-1} r_k h_k + r_n h_n = \sum_{k=1}^{n-1} (r_k - r_n) h_k < 0,
\]

where the final inequality is due to the fact that \( (r_k - r_n) h_k \leq 0 \) for each \( k < n \) with at least one inequality strict since \( h \neq 0 \). It follows that \( \sigma \) is not a possible ranking associated with \( p \). As this construction produces a different impossible ranking for each \( \sigma' \in S_{n-1} \), there are at most \( n! - (n - 1)! \) possible rankings. \( \square \)

4. Choosing Weighting Vectors

In this final section, we characterize the possible results vectors arising from a given profile \( p \) as the conical hull of a set of vectors constructed from the columns of \( Q_p \). The construction is surprisingly simple and provides a straightforward procedure to reverse engineer an election by selecting desirable weights for a given profile.

We begin by providing a convenient description of the space of (nonstrict) weighting vectors \( W \). Define \( v_1, \ldots, v_{n-1} \in \mathbb{Q}^n \) by \( v_k = \frac{k}{n} - \sum_{j=k+1}^{n} \frac{1}{n} e_j \), so that

\[
\begin{align*}
v_1 &= \left[ \frac{1}{n} \cdots \frac{1}{n} - \frac{n-1}{n} \right]^T \\
v_2 &= \left[ \frac{2}{n} \cdots \frac{2}{n} - \frac{n-2}{n} - \frac{n-2}{n} \right]^T \\
\vdots \\
v_{n-2} &= \left[ \frac{n-2}{n} \frac{n-2}{n} - \frac{2}{n} \cdots \frac{2}{n} \right]^T \\
v_{n-1} &= \left[ \frac{n-1}{n} - \frac{1}{n} \cdots - \frac{1}{n} \right]^T.
\end{align*}
\]

Proposition 4.1. Let \( v_1, \ldots, v_{n-1} \) be as above. Then

\[
W = \{ c_1 v_1 + \cdots + c_{n-1} v_{n-1} : c_1, \ldots, c_{n-1} \geq 0 \}.
\]

Proof. Clearly \( v_k \in W \) for \( k = 1, \ldots, n \), and thus so is any conical combination of the \( v_k \)’s. Conversely, given any \( w = [w_1 \ w_2 \ \cdots \ w_n]^T \in W \), we have

\[
w = \sum_{k=1}^{n-1} a_k v_k,
\]

where

\[
a_k = w_{n-k} - w_{n-k+1} \geq 0.
\]
Indeed, the $i^{th}$ coordinate of $\sum_{k=1}^{n-1} a_k v_j$ is
\[
\frac{1}{n} \sum_{j=1}^{n-1} j a_j - \frac{1}{n} \sum_{j=n-i+1}^{n-1} a_j (n-j) = \frac{1}{n} \sum_{j=1}^{n-1} j (w_{n-j} - w_{n-j+1}) - \frac{1}{n} \sum_{j=n-i+1}^{n-1} (w_{n-j} - w_{n-j+1}) = \frac{1}{n} (n-1) w_1 - \sum_{k=2}^{n} w_k - \sum_{k=1}^{i-1} (w_k - w_{k+1}) = \frac{1}{n} (nw_1 - \sum_{k=1}^{n} w_k) - (w_1 - w_i) = w_i - (w_1 - w_i) = w_i. \tag*{□}
\]

Now for any profile $p \in \mathbb{Q}^n$, there is an $n \times n$ matrix $Q$ with all row and column sums equal to $N$ such that the possible ordinal outcomes are those whose associated faces intersect $QW$. As the vectors in $W$ are precisely the conical combinations of $v_1, \ldots, v_{n-1}$, $QW$ consists of the conical combinations of $s_1, \ldots, s_{n-1}$ where $s_k = Q v_k$. Writing $Q = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix}$, we see that
\[
s_k = Q \left( \frac{k}{n} \mathbf{1} - \sum_{j=n-k+1}^{n} e_j \right) = \frac{k}{n} Q \mathbf{1} - \sum_{j=n-k+1}^{n} Q e_j = \frac{k}{n} Q \mathbf{1} - \sum_{j=n-k+1}^{n} q_j.
\]

Since adding multiples of $\mathbf{1}$ will not change the face a vector lies in, it suffices to consider conical combinations of
\[
t_k = s_{n-k} + \left( \frac{k}{n} + N - 1 \right) \mathbf{1} = N \mathbf{1} - \sum_{j=k+1}^{n} q_j = \sum_{j=1}^{k} q_j, \quad k = 1, \ldots, n-1.
\]

Also, scaling by a positive constant has no effect on which face a vector lies in, so the preceding observations can be stated as

**Theorem 4.2.** Let $p \in \mathbb{Q}^n$ be any profile and let $Q = \sum_{\ell=1}^{n} p_\ell R_\ell$ be given in column form by $Q = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix}$. Define $t_k = \sum_{j=1}^{k} q_j$. Then the possible outcomes for a positional voting procedure with input $p$ are those whose corresponding faces intersect the convex hull of $t_1, \ldots, t_{n-1}$.

This suggests a way for a nefarious election official to obtain the most desirable possible outcome for themselves: Given the preferences of the electorate, construct the matrix $Q$ and take $t_k$ to be the sum of its first $k$ columns for $k = 1, \ldots, n-1$. Then choose the most preferable outcome whose face intersects the convex hull of $t_1, \ldots, t_{n-1}$, pick a point $r$ in this intersection, and decompose it as $r = \sum_{k=1}^{n-1} b_k t_k$. The favored outcome is assured by declaring the weighting vector to be $w = \sum_{k=1}^{n-1} b_k v_{n-k}$.

In practice, this could be accomplished by repeatedly generating a random probability vector $b = [b_1 \cdots b_{n-1}]^T$ and recording the ranking corresponding to the face containing $s = T b$, $T = \begin{bmatrix} t_1 & \cdots & t_{n-1} \end{bmatrix}$. (This is just a matter of keeping track of the indices when $s$ is sorted in descending order.) After a sufficiently large number of iterations, one should have a nearly exhaustive list of possible rankings to choose from and can construct the desired weighting vector from the vector $b$ corresponding to the favorite.

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