QCD Vacuum and Axions: What’s Happening?

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Abstract

A deeper understanding of the vacuum structure in QCD invites one to rethink certain aspects of the axion physics. The recent advances are mostly due to developments in supersymmetric gauge theories and the brane theory, in which QCD can be embedded. They include, but are not limited to, the studies of metastable vacua in multicolor gluodynamics, and the domain walls. We briefly review basics of the axion physics and then present a modern perspective on a rich interplay between the QCD vacuum structure and axion physics.

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1 Introduction

Almost 25 years elapsed since the axion was introduced in particle physics as a possible solution of the strong CP problem. Since then, it became a text-book and encyclopedia subject. For instance, Oxford University’s “A Dictionary of Science” defines axion as “a hypothetical elementary particle postulated to explain why there is no observed CP violation (see CP invariance) in the strong interaction (see fundamental interactions). Axions have not been detected experimentally, although it has been possible to put limits on their mass and other properties from the effects that they would have on some astrophysical phenomena (e.g. the cooling of stars). It has also been suggested that they may account for some or all of the missing mass in the universe.”

While at the early stages the axion physics was considered predominantly in the context of quantum chromodynamics, at present the center of gravity of the axion studies shifted in astrophysics. It was realized rather early that the axion was a viable dark matter candidate. The research on this aspect of the axion physics quickly picked up and never subsided. Extensive investigations were and are being carried out in the astrophysical community. At the same time, after the rapid advances in the 1980’s, the QCD practitioners seemingly lost interest in this subject. The reason is obvious: the progress in understanding the QCD vacuum structure was painfully slow. The prevailing impression was that “nothing happened here,” so there was no motivation for revisiting QCD-related aspects of the axion physics.

In this review we will try to argue that “something interesting happened here.” A substantial progress has been achieved in the recent years mainly due to insights in QCD obtained from supersymmetry and the brane theory. The existence of a multitude of (quasi) stable vacua at large $N_c$ and “abnormally” thin domain walls with “abnormally” heavy excitations are just a few topics of interest that should be mentioned in this context. A deeper understanding of the QCD vacuum structure requires a reassessment of a number of issues of direct relevance to axions. After a brief summary of basics of the axion physics we review these new developments.

2 The strong CP problem

2.1 The $\theta$ term

After the discovery of asymptotic freedom in QCD in 1973 for a short while it was believed that QCD possesses the same natural conservation laws as its more primitive predecessor, QED. The discovery that $P$ and $T$ conservation in QCD is not natural came as a shocking surprise. This fact was realized with the advent of instantons which demonstrated that the so-called $\theta$ term

$$\Delta L_\theta \equiv \frac{\theta}{32 \pi^2} G^a_{\mu\nu} \tilde{G}^a_{\mu\nu},$$

1
does not necessarily vanish. Here the dual field strength is defined as

$$\tilde{G}^a_{\mu\nu} \equiv (1/2) \varepsilon_{\mu\nu\alpha\beta} G^a_{\alpha\beta}.$$  

(The indices are assumed to be contracted via the flat space metric). The operator $G\tilde{G}$ has dimension four, it can and should be added to the QCD Lagrangian. The $\theta$ term (1) violates $P$ and $T$ invariance (and hence, it violates $CP$ since $CPT$ is preserved). Note that the analogous term $\Delta L \sim F\tilde{F}$ in QED has no impact on the theory whatsoever. What is the difference?

The $\theta$ term can be rewritten as follows

$$\Delta L_\theta = \theta \partial_\mu K_\mu, \tag{2}$$

where $K_\mu$ is the Chern-Simons current defined as

$$K_\mu = \frac{1}{16 \pi^2} \varepsilon_{\mu\nu\alpha\beta} \left( A^a_\nu \partial_\alpha A^a_\beta + \frac{1}{3} f^{abc} A^a_\nu A^b_\alpha A^c_\beta \right). \tag{3}$$

Being a total derivative, the $\theta$ term does not affect the equations of motion. At a naive level, one can discard in the action the integrals over full derivatives. This was a rationale behind the original belief that QCD naturally conserves $P$ and $T$.

The instantons revealed the fact that the vacuum structure in QCD is more complicated than that in QED. In particular, the field configurations with the instanton boundary conditions give rise to a nonvanishing

$$\left(\Delta S_\theta\right)_{\text{one inst}} = \int d^4x \left(\Delta L_\theta\right)_{\text{one inst}} = \theta. \tag{4}$$

The integral over the full derivative does not vanish 1. Therefore, $CP$-violating effects may be present in strong interactions.

We pause here to make an explanatory remark regarding Eq. (4). A key notion is the topological charge $\mathcal{V}$ of a gauge field configuration,

$$\mathcal{V} \equiv \int d^4x \partial_\mu K_\mu = \int d^3x K_0(x, t)|^{t=+\infty}_{t=-\infty}$$

$$\equiv \mathcal{K}(t = +\infty) - \mathcal{K}(t = -\infty), \tag{5}$$

where $\mathcal{K}$ is usually referred to as the Pontryagin number. The topological charge is zero for any perturbative gauge fields — such fields are said to have trivial topology. The instanton field configuration has a nontrivial topology. In the $A_0 = 0$ gauge it interpolates between $A_m(x, t \to -\infty) = 0$, $m = 1, 2, 3$, and

$$A_m(x, t \to +\infty) = U^+ \partial_m U, \tag{6}$$

We jumped here from the Minkowski to the Euclidean formulation of the theory. In passing from Minkowski to Euclidean, the $\theta$ term (1) acquires an “$i$” factor, so does the integration measure in the action. Since this is a text-book topic, we will pass freely from Minkowski to Euclidean and back making no explicit statements as to which space any given formula belongs to.
where the matrix $U$ is Poyakov’s hedgehog,

$$U(\vec{x}) = \exp \left( - \frac{i \pi \vec{x} \vec{\sigma}}{\sqrt{\vec{x}^2 + \rho^2}} \right).$$  \hfill (7)

The Pontryagin number for $U$ reduces to

$$\mathcal{K} = \frac{1}{24 \pi^2} \int d^3 x \varepsilon_{ijk} \text{Tr} \left( U^+ \partial^i U \right) \left( U^+ \partial^j U \right) \left( U^+ \partial^k U \right) = 1,$$  \hfill (8)

implying that the instanton topological charge is unity.

### 2.2 Superselection rule and $\theta$ sectors

As we pointed out in the previous section, the value of the parameter $\theta$ a priori can be arbitrary. The theories with different values of $\theta$ describe different worlds which do not “communicate” with each other. In other words, the worlds with different $\theta$ belong to distinct superselection sectors [6, 7].

To see that this is the case and to further elucidate the role of $\theta$ let us consider pure gluodynamics in the Hamilton gauge $A_0 = 0$. In this gauge the Lagrangian does not depend on $A_0$ and the Gauss’ law (which in other gauges could have been obtained by varying the action with respect to $A_0$) is imposed as a constraint on physical states

$$D^i G_{i0} |\text{Phys}\rangle = 0.$$  \hfill (9)

This gauge fixing does not eliminate, however, the gauge freedom completely. Purely spatial gauge transformations (independent of the time variable) are still allowed. The generator of these residual gauge transformations can be written as

$$\mathcal{G}(\alpha) \equiv \exp \left( i \int d^3 x \text{Tr} D^i G_{i0}(\vec{x}) \alpha(\vec{x}) \right),$$  \hfill (10)

where $\alpha = \alpha^a t^a$ and the trace $\text{Tr}$ runs over the color indices. The generator $\mathcal{G}(\alpha)$ acts on the spatial components of the gauge fields,

$$\mathcal{G}^+ A_k \mathcal{G} = U^+ \left( A_k + i \partial_k \right) U,$$  \hfill (11)

where $U \equiv \exp(i\alpha(x))$. Furthermore, it is straightforward to show that the operator $\mathcal{G}$ does not commute with $\mathcal{V}$ if the corresponding gauge transformations give rise to a nonzero right-hand side in Eq. (8) (the latter are called large gauge transformations),

$$[\mathcal{G} \mathcal{V}] \neq 0.$$  \hfill (12)

Therefore, an eigenstate of $\mathcal{V}$ cannot be a physical state. Instead, the physical state is defined as a superposition of the eigenstates of $\mathcal{V}$

$$|\theta\rangle = \sum_{n=-\infty}^{+\infty} e^{i\theta n} |n\rangle,$$  \hfill (13)
where \( \mathcal{V} | n \rangle = n | n \rangle \).

In other words, in the infinitely-dimensional space of fields there is one direction parametrized by the variable \( K = \int K_0 d^3x \) which forms a closed circle. The wave function (as a function of \( K \)) is the Bloch superposition \([8, 6, 7]\). The parameter \( \theta \) is nothing but a “quasimomentum” \([8, 4, 7]\). In QCD it is called the vacuum angle.

In this formulation the \( \theta \) angle enters as an arbitrary phase in \([13]\) and varies in the interval from 0 to \( 2\pi \). Physics must be \( 2\pi \) periodic in \( \theta \). At \( \theta \neq 0 \) or \( \pi \) one can expect \( P \) and \( T \) noninvariant effects.

It is straightforward to show that for any gauge invariant operator \( \hat{O} \)

\[
\langle \theta' | \hat{O} | \theta \rangle \sim \delta_{\theta' \theta} .
\] (14)

Therefore, no gauge invariant operator can transform a state of one \( \theta \) world into a state of another \( \theta \) world. The different \( \theta \) worlds are disconnected from each other.

### 2.3 Constraints on \( \theta \)

As was mentioned, nonvanishing \( \theta \) leads to \( CP \) violating observables in QCD. (The point \( \theta = \pi \) will be discussed separately). It is known that strong interactions conserve \( CP \). Hence, a natural question arises as to what are the experimental constraints on the value of \( \theta \).

In a full theory, with quarks, there is an additional contribution to the \( CP \) odd part of the Lagrangian. It comes from the imaginary phases of the quark mass matrix \( \mathcal{M} \). These phases can be rotated away from the mass matrix by chiral transformations of the quark fields. However, because of the axial anomaly \([8]\), which manifests itself as a noninvariance of the Feynman integral measure under the chiral transformations at the quantum level \([10]\), the quark mass matrix phases appear in front of the \( G\tilde{G} \) term in the QCD Lagrangian. Therefore, the actual parameter that sets the magnitude of \( CP \) violation in QCD is

\[
\theta + \arg (\det \mathcal{M}) .
\] (15)

In what follows, for simplicity of notation, we will denote this parameter by the same letter \( \theta \), implying that the part \( \arg (\det \mathcal{M}) \) is included by default.

Perhaps, the most pronounced effect of the \( \theta \) term is generation of a nonzero neutron electric dipole moment (nEDM). The latter is parametrized by the following effective Lagrangian

\[
\mathcal{L}_{nEDM} = \frac{d_n}{2} \bar{n} i \gamma_5 \sigma_{\mu\nu} n F^{\mu\nu} ,
\] (16)

where the photon field strength is \( F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \), and \( n \) stands for the neutron. Moreover, \( \sigma_{\mu\nu} \) is the antisymmetric product of two Dirac’s gamma matrices, \( \sigma_{\mu\nu} \equiv [\gamma_\mu \gamma_\nu]/2i \).
In the presence of the $\theta$ term, nEDM can be found from the following matrix element:

$$\langle n(p_f) \gamma(k) | e J_{\mu}^{\text{em}} A^\mu \cdot i \int d^4x \Delta L_{\theta} | n(p_i) \rangle = d_n^\gamma \bar{n}(p_f) \gamma_5 \sigma_{\mu\nu} n(p_i) k^\mu \epsilon^\nu(k),$$

where $J_{\mu}^{\text{em}}$ is the quark electromagnetic current. The momentum carried by the photon, $k^\mu = (p_f)_{\mu} - (p_i)_{\mu}$, equals to the difference between the final and initial momenta of the neutron while $\epsilon^\nu(k)$ denotes the photon polarization four-vector.

The matrix element on the left-hand side of (17) is a highly nonperturbative object. Its calculation in QCD is nontrivial. Nevertheless, there are a number of different methods by which nEDM had been estimated in the past. We will list them below. The bag model calculation was performed in Ref. [11]. The result is

$$d_n^\gamma \bigg|_{\text{Bag}} \simeq \theta 2.7 \cdot 10^{-16} e \cdot \text{cm}. \quad (18)$$

Shortly after Ref. [11], the chiral logarithms (CL) method was used [12] leading to the estimate

$$d_n^\gamma \bigg|_{\text{CL}} \simeq \theta 5.2 \cdot 10^{-16} e \cdot \text{cm}. \quad (19)$$

The method of chiral perturbation theory (ChPT) was further advanced in Ref. [13] with the following result

$$d_n^\gamma \bigg|_{\text{ChPT}} \simeq \theta 3.3 \cdot 10^{-16} e \cdot \text{cm}. \quad (20)$$

Finally, the most recent paper on QCD sum rule (SR) calculations [14] of $d_n^\gamma$ gives

$$d_n^\gamma \bigg|_{\text{SR}} \simeq \theta 1.2 \cdot 10^{-16} e \cdot \text{cm}. \quad (21)$$

All results above have a considerable uncertainty, of at least 50%, which reflects a variety of uncertainties inherent to nonperturbative QCD calculations. Even though the results scatter by a factor of several units, it is beyond any doubt that $d_n^\gamma \bigg|_{\text{theor}} \sim \theta 10^{-16} e \cdot \text{cm}$.

This number should be compared with the most recent experimental result for nEDM presented in Ref. [15]

$$|d_n^\gamma|_{\text{exp}} < 6.3 \cdot 10^{-26} e \cdot \text{cm}. \quad (22)$$

One gets a very strong constraint on the value of $\theta$,

$$|\theta| \lesssim 10^{-9}. \quad (23)$$

We see that if the value of $\theta$ is nonvanishing it has to be unnaturally small. There is no a priori reason why two terms in (15), the bare theta parameter and $\text{arg} \left( \text{det} \mathcal{M} \right)$, should cancel each other with such an extraordinary accuracy, of one
part in $10^9$ or better. A dynamical mechanism is needed to explain the unnatural smallness of the $\theta$ term.

Before proceeding further, let us mention that other $CP$ odd effects are induced by the $\theta$ term too. They impose less stringent bounds on $\theta$, however. For instance, a nonzero $\theta$ gives rise to a nonvanishing amplitude of a $CP$ violating decay $\eta \to \pi^+ \pi^-$ for which $\mathrm{Br}(\eta \to \pi^+ \pi^-) \simeq \theta^2 \cdot 1.8 \cdot 10^2$ [17, 13]. The experimental limit for this decay is $\mathrm{Br}_{\exp}(\eta \to \pi^+ \pi^-) < 1.5 \cdot 10^{-3}$. This yields a constraint $|\theta| < 3 \cdot 10^{-3}$, much weaker than (23).

2.4 Can QCD solve the strong $CP$ problem itself?

Thus, theorists’ task is to try to find a mechanism which would make $CP$ conservation in strong interactions natural. Two alternative approaches are logically possible. One can invoke physics beyond QCD (this approach will be discussed in the bulk of this review) or one can try a minimalistic standpoint and ask whether QCD could solve the strong $CP$ problem itself, with no new physics.

An obvious solution of the latter type exists: were one of the quarks massless, e.g., $m_u = 0$, then all $\theta$ effects would be unobservable. In this case there is a global $U_A(1)$ symmetry of the chiral rotations of the $u$ quark field, $u_R \to \exp(i\beta) u_R$, $u_L \to \exp(-i\beta) u_L$, which can eliminate $\theta$ altogether. However, $m_u = 0$ does not go through on phenomenological grounds [18], and at present this scenario may be safely discarded.

A more intricate solution could exist if confinement itself were to ensure the effects of the $\theta$ term to be screened. As far as we know, this question was first raised by A. Polyakov shortly after the discovery of the strong $CP$ problem. His argumentation was as follows. The $\theta$ term in the action is the integral over the full derivative, which can be operative only if there are long-range components of the gauge fields. In the quasiclassical approximation such components are certainly present, as is evident from instanton calculations. However, this approximation misses the most salient feature of QCD, color confinement, which might eliminate long-range interactions (i.e. “screen” color) and make the integral over the full derivative vanish. That’s exactly what happens in the (1+2)-dimensional Polyakov model of color confinement [19].

This issue was studied in Ref. [17], with the negative conclusion. The problem is that there are two effects in QCD which depend on the above “screening”: the heaviness of the $\eta'$ mass (the so-called U(1) problem) and $CP$ conservation/violation. Even though we do not know precisely how exactly the confinement mechanism works in QCD, we know for a fact that $\eta'$ is split from the octet of the Goldstone bosons. This knowledge (plus some reasonable arguments regarding the value of chiral and/or $1/N_c$ corrections) is sufficient to show that confinement in QCD does not eliminate the $\theta$ dependence and thus does not solve the strong $CP$ problem. We briefly outline this line of reasoning below.

The flavor singlet meson, the $\eta'$, is significantly heavier than the flavor octet
Goldstones, \( m_{\eta'} \approx 958 \text{ MeV} \). As was shown by Weinberg in the pre-QCD era, were the \( \eta' \) a Goldstone, its mass would be constrained by \( m_{\eta'} < \sqrt{3} m_{\pi} \). This suggests that, unlike the octet of the Goldstone bosons, the \( \eta' \) is not massless in the chiral limit (unless chiral expansion is invalid). This is the only fact we will need.

An extra contribution to the \( \eta' \) mass comes from nonperturbative effects due to the axial anomaly, as was exemplified \([20]\) by the same instantons.

To quantify the effect on the theoretical side, let us introduce the correlator of the topological charge densities (in the literature it is referred to as the topological susceptibility)

\[
\mathcal{X} = -i \int d^4x \langle 0 | T Q(x) Q(0) | 0 \rangle ,
\]

where

\[
Q \equiv \frac{1}{32\pi^2} G^a_{\mu\nu} \tilde{G}^a_{\mu\nu} .
\]

According to the low-energy theorem derived in Refs. \([21, 22]\) in the leading approximation of the \( 1/N_c \) expansion, the quantity \( \mathcal{X} \) is saturated by the \( \eta' \) contribution implying the following formula for the \( \eta' \) mass:

\[
m_{\eta'}^2 = \frac{6\mathcal{X}}{f^2} + \mathcal{O}(m_q) + \mathcal{O}\left(\frac{1}{N_c^2}\right) ,
\]

where the topological susceptibility \( \mathcal{X} \) on the right-hand side is evaluated in pure gluodynamics, the Yang-Mills theory with no light quarks. In order for the \( \eta' \) mass to be nonzero in the chiral limit, \( \mathcal{X} \) should be nonzero in pure gluodynamics.

A substantial amount of theoretical evidence is accumulated in the last 20 years showing that the topological susceptibility in pure Yang-Mills does not vanish. The lattice \([23]\) and the QCD sum rule studies \([24]\) yield \( \mathcal{X} \simeq (180 \text{ MeV})^4 \neq 0 \). This successfully takes care of the \( U(1) \) problem.

Having nonzero topological susceptibility in pure gluodynamics means, *per se*, that there is a sensitivity to the parameter \( \theta \) in this theory. Indeed, \( \mathcal{X} \) is nothing but the second derivative of the vacuum energy with respect to \( \theta \) taken at \( \theta = 0 \)

\[
\mathcal{X} = -\frac{\partial^2}{\partial \theta^2} \left( \frac{\ln Z_\theta}{V} \right) \bigg|_{\theta=0} .
\]

In the theory with the light quarks included, the topological susceptibility can be calculated by applying the chiral perturbation theory (see e.g. \([17]\)). As expected on general grounds, in this case \( \mathcal{X} \propto m_q \) provided the \( \eta' \) is split from the octet of the Goldstones; otherwise the \( \eta' \) contribution cancels the \( \mathcal{O}(m_q) \) term in the topological susceptibility. However, in the theory with the light quarks it is much more instructive to calculate directly \( CP \) odd decay rates, for instance the rate of \( \eta \to \pi^+ \pi^- \). This amplitude is forbidden by \( CP \). In the same way as with the
topological susceptibility, the chiral low-energy theorem yields \[17\] a nonvanishing amplitude \(O(m_\eta)\) provided the \(\eta'\) is split from the octet of the Goldstones.

Therefore, since QCD does solve the U(1) problem — it does split the \(\eta'\) from the octet Goldstones — it cannot solve the strong CP problem \[17\] without help from outside.

### In search of a solution beyond QCD

#### 3.1 Peccei-Quinn mechanism

The first dynamical mechanism solving the strong \(CP\) problem was proposed by Peccei and Quinn \[27\]. The main observation of Ref. \[27\] is as follows: if there is a U(1) axial symmetry in the theory

\[
q_L \rightarrow e^{i\alpha} q_L, \quad q_R \rightarrow e^{-i\alpha} q_R,
\]

then the \(\theta\) term can be removed from the Lagrangian, much like in the case of one massless quark discussed in the previous section. Below this symmetry will be referred to as \(U(1)_{PQ}\).

To see whether this symmetry is present in the Standard Model with one Higgs doublet let us consider the Yukawa sector and restrict ourselves to the first generation quarks (consideration in the general case is quite similar),

\[
\lambda_u \bar{Q}_L \phi u_R + \lambda_d \bar{Q}_L \phi_c d_R + \text{H.c.} + V(\phi^+ \phi),
\]

\[29\]

2 The situation seems to be quite clear in this respect, nevertheless, an attempt to develop models of confinement that would solve the strong CP problem are not abandoned. Let us comment on a proposal of Ref. \[25\] where two distinct “topological susceptibilities” are defined: the local and global ones. Let \(V_c\) denote the volume at which the confinement effects take place, and \(V\) be the total volume of space time \(V_c \ll V \rightarrow \infty\). According to \[25\], the local topological susceptibility is \(X_{\text{loc}} = \int_{V_c} d^4x \langle 0 \mid T Q(x) Q(0) \mid 0 \rangle\), while the global one is \(X_{\text{glob}} = \int_{V} d^4x \langle 0 \mid T Q(x) Q(0) \mid 0 \rangle\).

The author claims that the solution of the U(1) problem requires that \(X_{\text{loc}} \neq 0\), while the solution of the strong \(CP\) implies \(X_{\text{glob}} = 0\), and both conditions may be dynamically compatible, so that both the strong \(CP\) and U(1) problems could be solved simultaneously. The underlying dynamics outlined in \[25\] is a special interaction between instantons which “screens” them.

There are a number of objections to this suggestions. First, it is the global topological susceptibility that enters in the Witten-Veneziano relation and determines the \(\eta'\) mass modulo \(1/N_c\) corrections. Even if we forget about theoretical calculations of this quantity demonstrating that it does not vanish, we know that the \(\eta'\) is split from the Goldstone octet because the Weinberg relation \(m_{\eta'} < \sqrt{3}m_\pi\) is grossly violated. (Of course, the validity of the chiral expansion is assumed; otherwise the Weinberg relation is meaningless.) If the \(\eta'\) is split, there is no way out \[17\]: \(\theta\)-induced effects are observable.

In terms of the model suggested in \[25\] this means that if one were able to complete the calculations at the level of physical observables, one would find that the \(\eta'\) is not split from the octet of the Goldstones or, more likely, that the required interaction between instantons is not sustainable. The latter variant was advocated in \[24\].
where $Q_L$ is the left-handed SU(2)$_W$ quark doublet, $(\phi_c)_i = \epsilon_{ij} \phi^* j$ is the charge conjugate Higgs field, and $V$ denotes the Higgs potential. Although the first term in this expression is invariant under the transformations

$$q_L \rightarrow e^{i\alpha} q_L, \quad q_R \rightarrow e^{-i\alpha} q_R, \quad \phi \rightarrow e^{2i\alpha} \phi,$$

there is a second term which is not invariant under (30) since $\phi_c$ transforms as conjugate to $\phi$. Therefore, the one Higgs doublet SM is not invariant under U(1)$_{PQ}$ and the strong CP cannot be solved.

However, as was pointed out in Ref. [27], the required U(1)$_{PQ}$ symmetry is present in SM with two Higgs doublets — let us call them $\phi$ and $\chi$. In this case the Yukawa sector for the first generation quarks reads

$$\lambda_u \bar{Q}_L \phi u_R + \lambda_d \bar{Q}_L \chi^* d_R + \text{H.c.} + V(\phi^+ \phi, \chi^+ \chi, (\phi^+ \chi)(\chi^+ \phi)).$$

It is invariant under

$$q_L \rightarrow e^{i\alpha} q_L, \quad q_R \rightarrow e^{-i\alpha} q_R, \quad \phi \rightarrow e^{2i\alpha} \phi, \quad \chi \rightarrow e^{-2i\alpha} \chi.$$  

This fact can be used to solve the strong CP problem [27]. The symmetry (32) is explicitly broken by the axial anomaly. As a result, instantons induce an effective potential for the $\theta$ term. The potential can be calculated in a certain approximation [27]. The crucial model-independent fact is that the resulting potential is minimized at a zero value of the CP violating phase [27].

3.2 Weinberg-Wilczek axion

When the Higgs fields develop vacuum expectation values (VEV’s) the local electroweak symmetry group is spontaneously broken. This gives masses to the intermediate $W^\pm$ and $Z$ vector bosons. Simultaneously, the global U(1)$_{PQ}$ is spontaneously broken too. Spontaneous breaking of the global symmetry leads to the emergence of a massless Goldstone boson, the axion in the present case [1, 2]. In SM with two Higgs doublets the axion is given by the following superposition

$$a \equiv \frac{1}{v} (v_\phi \text{Im}\phi_0 - v_\chi \text{Im}\chi_0),$$

where $\phi_0$ and $\chi_0$ denote the neutral components of the Higgs doublets. Moreover, $v \equiv \sqrt{v_\phi^2 + v_\chi^2} \simeq 250$ GeV, and $v_\phi$ and $v_\chi$ are the vacuum expectation values of $\phi$ and $\chi$, respectively. In this approximation the axion is massless. However, as we mentioned above, nonperturbative QCD effects (such as instantons) give rise to a potential for the axion. Hence, the axion acquires a nonzero mass which can be estimated as follows [1, 2]

$$m_a \simeq \frac{f_\pi m_\pi}{v} \simeq 100 \text{KeV}.$$  

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Moreover, the axion decay constant is $1/v$. Therefore, we see that the Weinberg-Wilczek (WW) axion mass and decay constant are tied to the electroweak symmetry breaking scale $v$. This turns out to be too much of a constraint, and, as we will discuss in Sec. 3.5, the WW axion is excluded on the basis of existing experimental data.

### 3.3 KSVZ axion

If the scale of PQ symmetry breaking is much higher than the electroweak scale, then according to (34), the axion is much lighter and its decay constant is much smaller. Such an “invisible” axion would not be in conflict with experimental data.

A scenario with the harmless axion was first proposed in Refs. [28] and [17] (the KSVZ axion). In the latter paper it was called *phantom axion*. Needless to say that to untie the axion from the electroweak scale one has to decouple the corresponding scalar fields from the known quarks and couple them to hypothetical (very) heavy fermion fields carrying color.

In more detail, one introduces a complex scalar field $\Phi$ coupled to a hypothetical electroweak singlet, a quark field $Q$ in the fundamental representation of color $SU(3)$,

$$
\Delta L = \Phi \bar{Q}_R Q_L + H.c.
$$

(35)

The modulus of $\Phi$ is assumed to develop a large vacuum expectation value $f/\sqrt{2}$, while the argument of $\Phi$ becomes the axion field $a$, modulo normalization,

$$
a(x) = f \alpha(x) , \quad \alpha(x) \equiv \text{Arg}\Phi(x) , \quad f \gg \Lambda .
$$

(36)

Then the low-energy coupling of the axion to the gluon field is

$$
\Delta L = \frac{1}{f} a \frac{1}{32\pi^2} G^a_{\mu\nu} \tilde{G}^a_{\mu\nu} ,
$$

(37)

so that the QCD Lagrangian depends on the combination $\theta + \alpha(x)$.

In general, one could introduce more than one fundamental field $Q$, or introduce them in a higher representation of the color group. Then, the axion-gluon coupling (37) acquires an integer multiplier $N$,

$$
\Delta L' = \frac{1}{f} a N \frac{1}{32\pi^2} G^a_{\mu\nu} \tilde{G}^a_{\mu\nu} .
$$

(38)

This factor $N$ (not to be confused with the number of colors $N_c$ nor with $N$ of extended supersymmetry) is sometimes referred to as the axion index. The minimal axion corresponds to $N = 1$. In the general case the QCD Lagrangian depends on the combination $\theta + N\alpha(x)$. As previously, nonperturbative QCD effects generate a potential for $\theta + N\alpha(x)$. The latter is minimized at the value $\theta + N\alpha_{vac} = 0$, i.e., the strong $CP$ problem is automatically solved.
3.4 ZDFS axion

An alternative way to introduce an “invisible” axion was proposed in Refs. [29] and [30], (the ZDFS axion). In this proposal one maintains the PQ symmetry of the two doublet SM but separates the scales of the PQ and electroweak breaking [29, 30]. To this end the SM Lagrangian is extended – a scalar SM singlet field $\Sigma$ is added,

$$
\begin{align*}
\lambda_d \bar{Q}_L \phi u_R + \lambda_d \bar{Q}_L \chi^* d_R + \text{H.c.} + \\
V(\phi^+ \phi, \chi^+ \chi, (\phi^+ \chi)(\chi^+ \phi), \Sigma^+ \Sigma, (\phi^+ \chi)\Sigma^2).
\end{align*}
$$

One notes that this expression is invariant under the following axial transformations

$$
q_L \to e^{i\alpha} q_L, \quad q_R \to e^{-i\alpha} q_R, \quad \phi \to e^{2i\alpha} \phi, \quad \chi \to e^{-2i\alpha} \chi, \quad \Sigma \to e^{2i\alpha} \Sigma.
$$

Upon spontaneous breaking of this symmetry the Goldstone particle, an axion, emerges as a superposition

$$
a \equiv \frac{1}{V} \left( v_\phi \text{Im} \phi_0 - v_\chi \text{Im} \chi_0 + v_\Sigma \text{Im} \Sigma \right),
$$

where $V \equiv \sqrt{v_\phi^2 + v_\chi^2 + v_\Sigma^2}$, and $v_\phi$, $v_\chi$, and $v_\Sigma$ are the vacuum expectation values of $\phi$, $\chi$, and $\Sigma$, respectively. The vacuum expectation value of $\Sigma$ does not have to be related to the electroweak symmetry breaking scale. In fact, it can be as large as the GUT scale. If so, the axion is light and its decay constant is tiny. We will discuss experimental bounds on these quantities in the next section.

3.5 Constraints on the axion mass

As we discussed in the previous sections, the PQ symmetry is explicitly broken by the axial anomaly. Therefore, the axion is a pseudo Goldstone boson. Nonperturbative QCD effects induce the axion mass. For further discussions it is convenient to parametrize the axion mass as follows

$$
m_a \simeq 0.6 \text{eV} \frac{10^7 \text{GeV}}{f},
$$

where $f$ is the axion decay constant determined by the PQ breaking scale.

In general, while discussing phenomenological constrains on the axion mass, one should distinguish between the KSVZ and ZDFS cases. The axion couplings to matter are different in these two scenarios. In particular, the KSVZ axion has no tree-level couplings to the standard model quarks and leptons. However, the aim of the present section is to summarize briefly an order of magnitude constraints on the axion mass. For this goal the effects which distinguish between the KSVZ and ZDFS axions will not be important (for detailed studies see Ref. [31] and citations therein).
The quantity $1/f$ sets the strength of the axion coupling. Light axions can be produced in stars and a part of the energy of a star can be carried away by those axions. Stars can lose energy due to the production of light axions in the following possible processes

(i) Nucleon-nucleon bremsstrahlung: $N + N \rightarrow N + N + a$;

(ii) The Primakoff process: $\gamma \leftrightarrow a$ conversion in the electromagnetic field of a nucleus;

(iii) Photoproduction on an electron: $\gamma + e^- \rightarrow e^- + a$;

(iv) Electron bremsstrahlung on a nucleus: $e^- + (A, Z) \rightarrow e^- + a + (A, Z)$;

(v) Photon fusion: $\gamma + \gamma \rightarrow a$.

Unless $1/f$ is really small, the emission of axions that are produced in the above reactions would lead to unacceptable energy loss by the star. This leads to the following lower bound on the axion decay constant $f \gtrsim 10^9 \text{GeV}$.

It is remarkable that cosmology puts an upper bound on $f$ [32, 33, 34]. The latter comes about as follows. If $f$ is too large then the axion coupling $1/f$ is very small. As a result, during the course of cosmological evolution of the universe axions decouple early and begin to oscillate coherently. There are two major mechanisms by which the energy density stored in these oscillations can be dissipated – the Hubble expansion of the universe and the particle production by axions. However, if $f \gtrsim 10^{12} \text{GeV}$, neither of these mechanisms are effective (the axion coupling is too small). As a result, at some point of the evolution the axion energy density exceeds the critical energy density and over-closes the universe. In order for this not to happen one should impose the constraint $f \lesssim 10^{12} \text{GeV}$.

Summarizing, we obtain the following order of magnitude bounds on $f$ and $m_a$:

$$10^9 \text{GeV} \lesssim f \lesssim 10^{12} \text{GeV}, \quad 10^{-6} \text{eV} \lesssim m_a \lesssim 10^{-3} \text{eV}. \quad (43)$$

For further details see, e.g., Ref. [31].

4 The vacuum structure in large $N_c$ gluodynamics

The early studies [35] of the chiral Ward identities in QCD revealed that the vacuum energy density depends on the vacuum angle $\theta$ through the ratio $\theta/N_f$, where $N_f$ is the number of quarks with mass $m_q \ll \Lambda$. Shortly after it was shown in Refs. [36] and [37] that this structure occurs naturally, provided that there exist $N_f$ states in the theory such that one of them is the true vacuum, while others are local extrema; all are intertwined in the process of “the $\theta$ evolution.” Namely, in passage from $\theta = 0$ to $\theta = 2\pi$, from $\theta = 2\pi$ to $\theta = 4\pi$, and so on, the roles of the above states interchange: one of the local extrema becomes the global minimum and vice versa. This would imply, with necessity, that at $\theta = k\pi$ (where $k$ is an odd integer) there are two degenerate vacuum states. Such a group of intertwined states will be referred to as the “vacuum family.” The crossover at $\theta = \pi, 3\pi, \text{etc.}$ is called the Dashen phenomenon [38].
This picture was confirmed by a detailed examination of effective chiral Lagrangians [36, 37, 39, 40] (for a recent update see [41]). For two and three light quarks with equal masses it was found that the vacuum family consists of two or three states respectively; one of them is a global minimum of the potential, while others are local extrema. At $\theta = \pi$ the levels intersect. Thus, Crewther’s dependence [35] on $\theta/N_f$ emerges.

On the other hand, the examination of the effective chiral Lagrangian with the realistic values of the quark masses, $m_d/m_u \sim 1.8$, $m_s/m_d \sim 20$, yields [36, 37, 41] a drastically different picture – the vacuum family disappears (shrinks to one state); the crossover phenomenon at $\theta = \pi$ is gone as well.

This issue remained in a dormant state for some time. Recently arguments were given that the “quasivacua” (i.e. local minima of the energy functional), which together with the true vacuum form a vacuum family, is an indispensable feature of gluodynamics. The first argument in favor of this picture derives [42] from supersymmetric gluodynamics, with supersymmetry softly broken by a gluino mass term. The same conclusion was reached in Ref. [43] based on a D-brane construction in the limit of large $N_c$. One can see that in both approaches the number of states in the vacuum family scales as $N_c$. In fact, in Ref. [43] the expression for the theta dependence of the vacuum energy in the large $N_c$ pure Yang-Mills (YM) model was derived from a D-brane construction. It has the following form [43]:

$$E(\theta) = C \min_k (\theta + 2\pi k)^2 + \mathcal{O}\left(\frac{1}{N_c}\right),$$

(44)

where $C$ is some constant independent of $N_c$ and $k$ stands for an integer number. This expression has a number of interesting features which might seem a bit puzzling from the field theory point of view. Indeed, in the large $N_c$ limit there are $N_c^2$ degrees of freedom in gluodynamics, thus, naively, one would expect that the vacuum energy density in this theory scales as $\sim N_c^2$. However, the leading term in Eq. (44) scales as $\sim 1$. As a natural explanation, one could conjecture that there should be a colorless massless excitation which saturates the expression for the vacuum energy density (44). However, pure gluodynamics generates a mass gap and there are no physical massless excitations in the model. Thus, the origin of Eq. (44) seems to be a conundrum. We will discuss and resolve this puzzle in the next section.

Note, that an additional argument in favor of the vacuum family may be found in a cusp structure which develops once one sums up sub-leading in $1/N_c$ terms in the effective $\eta'$ Lagrangian. At large $N_c = \infty$ the states from the vacuum family are stable, and so are the domain walls interpolating between them [43, 45].

When $N_c < \infty$ the degeneracy and the vacuum stability is gone, strictly speaking. It is natural to ask what happens if one switches on the axion field. This

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3 We stress that the states from the vacuum family need not necessarily lie at the minima of the energy functional. As was shown by Smilga [41], at certain values of $\theta$ some may be maxima. Those which intersect at $\theta = k\pi$ ($k$ odd) are certainly the minima at least in the vicinity of $\theta = k\pi$.

4 It has been conjectured long time ago in [36].
generically leads to the formation of the axion domain walls. The axion domain wall \[46\] presents an excellent set-up for studying the properties of the QCD vacuum under the $\theta$ evolution. Indeed, inside the axion wall, the axion field (which, in fact, coincides with an effective $\theta$) changes slowly from zero to $2\pi$. The characteristic length scale, determined by the inverse axion mass $m_a^{-1}$, is huge in the units of QCD, $\Lambda^{-1}$. Therefore, by visualizing a set of spatial slices parallel to the axion wall, separated by distances $\gg \Lambda^{-1}$, one obtains a chain of QCD laboratories with distinct values of $\theta_{\text{eff}}$ slowly varying from one slice to another. In the middle of the wall $\theta_{\text{eff}} = \pi$.

Intuitively, it seems clear that in the middle of the axion wall, the effective value of $\theta_{\text{eff}} = \pi$. Thus, in the central part of the wall the hadronic sector is effectively in the regime with two degenerate vacua, which entails a stable gluonic wall as a core of the axion wall. In fact, we deal here with an axion wall “sandwich.” Its core is the so-called D-wall, see \[47\].

Below we will discuss this idea more thoroughly. We also address the question whether this phenomenon persists in the theory with light quarks, i.e., in real QCD. Certainly, in the limit $N_c = \infty$ the presence of quarks is unimportant, and the axion wall will continue to contain the D-wall core. As we lower the number of colors, however, below some critical number it is inevitable that the regime must change, the gluonic core must disappear as a result of the absence of the crossover. The parameter governing the change of the regimes is $\Lambda/N_c$ as compared to the quark mass $m_q$. At $m_q \ll \Lambda/N_c$, even if one forces the axion field to form a wall, effectively it is screened by a dynamical phase whose origin can be traced to the $\eta'$, so that in the central part of the axion wall the hadronic sector does not develop two degenerate vacua. The D-walls cannot be accessed in this case via the axion wall.

The issue of hadronic components of the axion wall in the context of a potential with cusps \[44\] were discussed in \[48, 49, 50\]. However, the gluonic component of the axion walls was not studied. The $\eta'$ component in the axion walls was discussed in \[51, 48\].

### 4.1 Arguments from supersymmetric gluodynamics

First we will summarize arguments in favor of the existence of a nontrivial vacuum family in pure gluodynamics.

The first indication that the crossover phenomenon may exist in gluodynamics comes \[12\] from supersymmetric Yang-Mills theory, with supersymmetry being broken by a gluino mass term. The same conclusion was reached in Ref. \[13\] based on a D-brane construction in the limit of large $N_c$. In both approaches the number of states in the vacuum family is $N_c$.

The Lagrangian of softly broken supersymmetric gluodynamics is

$$L = \frac{1}{g^2} \left\{ -\frac{1}{4} G_{\mu\nu}^a G_{\mu\nu}^a + i \bar{\lambda}_\alpha^a D^{\dot{\alpha}\alpha} \lambda^a_\alpha - (m_\lambda^a \lambda^{a\alpha} + \text{H.c.}) \right\}$$
\[ + \theta \frac{1}{32\pi^2} G^a_{\mu\nu} \tilde{G}^a_{\mu\nu}. \quad (45) \]

where \( m \) is the gluino mass which is assumed to be small, \( m \ll \Lambda \) (here we rescaled the gluon and gluino fields so that \( 1/g^2 \) appears as a common multiplier in the Lagrangian).

There are \( N_c \) distinct chirally asymmetric vacua, which (in the \( m = 0 \) limit) are labeled by

\[ \langle \lambda^2 \rangle_\ell = N_c \Lambda^3 \exp \left( i \frac{\theta + 2\pi \ell}{N_c} \right), \quad \ell = 0, 1, \ldots, N_c - 1. \quad (46) \]

At \( m = 0 \) there are stable domain walls interpolating between them \([52] \). Setting \( m \neq 0 \) we eliminate the vacuum degeneracy. To first order in \( m \) the vacuum energy density in this theory is

\[ \mathcal{E} = \frac{m}{g^2} \langle \lambda^2 \rangle + \text{H.c.} = -m N_c^2 \Lambda^3 \cos \frac{\theta + 2\pi \ell}{N_c}. \quad (47) \]

Degeneracy of the vacua is gone. As a result, all the metastable vacua will decay very quickly. Domain walls between them, will be moving toward infinity because of the finite energy gradient between two adjacent vacua. Eventually one ends up with a single true vacuum state in the whole space.

For each given value of \( \theta \) the ground state energy is given by

\[ \mathcal{E}(\theta) = \min_\ell \left\{ -m N_c^2 \Lambda^3 \cos \frac{\theta + 2\pi \ell}{N_c} \right\}. \quad (48) \]

At \( \theta = \pi, 3\pi, \ldots \), we observe the vacuum degeneracy and the crossover phenomenon. If there is no phase transition in \( m \), this structure will survive, qualitatively, even at large \( m \) when the gluinos disappear from the spectrum, and we recover pure gluodynamics.

Based on a D-brane construction Witten showed \([43] \) that in pure SU\((N_c)\) (non-supersymmetric) gluodynamics in the limit \( N_c \to \infty \) a vacuum family does exist, the theory has an infinite group of states (one is the true vacuum, others are non-degenerate metastable "vacua") which are intertwined as \( \theta \) changes by \( 2\pi \times \) (integer), with a crossover at \( \theta = \pi \times \) (odd integer). The energy density of the \( k \)-th state from the family is

\[ \mathcal{E}_k(\theta) = N_c^2 \Lambda^4 F \left( \frac{\theta + 2\pi k}{N_c} \right), \quad (49) \]

---

\( ^5 \)This was shown in Ref. \([43] \) assuming that there is no phase transition in a certain parameter of the corresponding D-brane construction. In terms of gauge theory, this assumption amounts of saying that there is no phase transition as one interpolates to the strong coupling constant regime. Thus, the arguments of \([43] \) have the same disadvantage as those of SUSY gluodynamics where one had to assume the absence of the phase transition in the gluino mass.
where $F$ is some $2\pi$-periodic function, and the truly stable vacuum for each $\theta$ is obtained by minimizing $\mathcal{E}_k$ with respect to $k$,

$$\mathcal{E}(\theta) = N_c^2 \Lambda^4 \min_k F \left( \frac{\theta + 2\pi k}{N_c} \right),$$

much in the same way as in Eq. (48).

At very large $N_c$ Eq. (50) takes the form

$$\mathcal{E}(\theta) = \Lambda^4 \min_k (\theta + 2\pi k)^2 + \mathcal{O}\left(\frac{1}{N_c}\right).$$

The energy density $\mathcal{E}(\theta)$ has its absolute minimum at $\theta = 0$. At $N_c = \infty$ the “vacua” belonging to the vacuum family are stable but non-degenerate. To see that the lifetime of the metastable “vacuum” goes to infinity in the large $N_c$ limit one can consider the domain walls which separate these vacua [43, 53]. These walls are seen as wrapped D-branes in the construction of [43], and they indeed resemble many properties of the QCD D-branes on which a QCD string could end. We refer to them as D-walls because of their striking similarity to D2-branes. The consideration of D-walls has been carried out [45] and leads to the conclusion that the lifetimes of the quasivacua go to infinity as $\exp(\text{const} N_c^4)$.

Moreover, it was argued [44, 47] that the width of these wall scales as $1/N_c$ both, in SUSY and pure gluodynamics. To reconcile this observation with the fact that masses of the glueball mesons scale as $N_c^0$, we argued [47] that there should exist heavy (glue) states with masses $\propto N_c$ out of which the walls are built. The D-brane analysis [55], effective Lagrangian arguments and analysis of the wall junctions [56], support this interpretation. These heavy states resemble properties of D0-branes. The analogy is striking, as D0-branes make D2-branes from the standpoint of the M(atrix) theory [57], so these QCD “zero-branes” make QCD D2-branes (domain walls) [4]. The distinct vacua from the vacuum family differ from each other by a restructuring of these heavy degrees of freedom. They are essentially decoupled from the glueballs in the large $N_c$ limit.

Now we switch on the axion

$$\Delta \mathcal{L} = \frac{1}{2} f^2 (\partial_\mu \alpha)(\partial^\mu \alpha) + \frac{\alpha}{32\pi^2} G^a_{\mu\nu} \tilde{G}^a_{\mu\nu},$$

with the purpose of studying the axion walls. The potential energy $\mathcal{E}(\theta)$ in Eq. (50) or (51) is replaced by $\mathcal{E}(\theta + \alpha)$.

Since the hadronic sector exhibits a nontrivial vacuum family and the crossover at $\theta = \pi, 3\pi$, etc., strictly speaking, it is impossible to integrate out completely the hadronic degrees of freedom in studying the axion walls. If we want to resolve the cusp, near the cusp we have to deal with the axion field plus those hadronic degrees.

---

6See also closely related discussions in Ref. [58].
7For nonminimal axions, with $N \geq 2$, the crossover occurs at $\alpha = k\pi/N$. 

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of freedom which restructure. In the middle of the wall, at \( \alpha = \pi \), it is mandatory to jump from one hadronic vacuum to another – only then the energy of the overall field configuration will be minimized and the wall be stable. Thus, in gluodynamics the axion wall acquires a D-wall core by necessity.

One can still integrate out the heavy degrees of freedom everywhere except a narrow strip (of a hadronic size) near the middle of the wall. Assume for simplicity that there are two states in the hadronic family. Then the low-energy effective Lagrangian for the axion field takes the form (83). The domain wall profile will also exhibit a cusp in the second derivative. The wall solution takes the form:

\[
\alpha(z) = \begin{cases} 
8 \arctan \left( e^{m_a z} \tan \frac{\pi}{8} \right), & \text{at } z < 0 \\
-2\pi + 8 \arctan \left( e^{m_a z} \tan \frac{3\pi}{8} \right), & \text{at } z > 0,
\end{cases}
\] (53)

at \( N_c = 2 \) (the wall center is at \( z = 0 \)). Examining this cusp with an appropriately high resolution one would observe that it is smoothed on the hadronic scale, where the hadronic component of the axion wall “sandwich” would become visible. The cusp carries a finite contribution to the wall tension which cannot be calculated in the low-energy approximation but can be readily estimated, \( T_{\text{core}} \sim \Lambda^3 N_c \). In subsequent sections we will examine this core manifestly in a toy solvable model.

### 4.2 Arguments from D-brane construction

The theta dependent vacuum energy (44) is related to the correlator measuring the vacuum fluctuations of the topological charge. The question which arises here is whether this can be seen from the original string theory computation [43]. We are going to discuss below how the string theory calculation suggests that the vacuum energy (44) should indeed be related to the vacuum fluctuations of the topological charge. In fact, we argue that this is related to the instantons carrying \( D0 \)-brane charge in the Type IIA fourbrane construction of the four-dimensional YM model.

In general, a great deal of information can be learned on nonperturbative phenomena in four-dimensional gauge theories by obtaining these models as a low-energy realization of certain D-brane configurations [53], and/or using a duality of large \( N \) superconformal gauge theories and string theory compactified on certain spaces (see Refs. [60] and [81, 21]). This duality, being a powerful technique, has also been generalized for the case of non-supersymmetric models [63]. This was applied to study various dynamical issues in large \( N_c \) pure Yang-Mills theory [64, 65, 66, 67, 68, 69].

To begin with let us recall how the theta dependent vacuum energy appears in the brane construction of the four-dimensional YM model [63]. One starts with Type IIA superstring theory on \( \mathcal{M} \equiv R^4 \times S^1 \times R^5 \), with \( N_c \) coincident \( D4 \)-branes [83]. The \( D4 \)-brane worldvolume is assumed to be \( R^4 \times S^1 \) and the fermion boundary conditions on \( S^1 \) are chosen in such a way that the low-energy theory on the
worldvolume is pure non-supersymmetric $U(N_c)$ YM theory \[63\]. In the dual description, the large $N_c$ limit of the $SU(N_c)$ part of this theory can be studied by string theory on a certain background \[60, 61, 62, 63\]. It was shown in Ref. \[43\] that the theta dependent vacuum energy \[44\] arises in the dual string description due to the $U(1)$ gauge field $B_M$, $M = 1, \ldots, 5$. To find out what this corresponds to in the original gauge theory language recall that this $U(1)$ field is nothing but the Ramond-Ramond (RR) one-form of Type IIA theory. Furthermore, once the gauge theory is realized in the Type IIA fourbrane construction, the Wess-Zumino-Witten (WZW) term present in the worldvolume effective action defines the correspondence between the gauge theory operators on one side and the string theory Ramond-Ramond fields on the other side. In the case at hand the worldvolume WZW term looks as follows:

$$S_{WZW} = \frac{1}{8\pi^2} \int_\Omega B \wedge \text{Tr} G \wedge G, \quad (54)$$

where $\Omega$ denotes the worldvolume of a wrapped fourbrane, $\Omega \equiv R^4 \times S^1$. In accordance with the general principles of the large $N_c$ AdS/CFT correspondence \[60, 61, 62\] the classical action for the RR one-form on the string theory side defines the YM correlation functions of the composite operator $G\hat{G}$ (since this is the operator which couples to the corresponding RR field in \(54\)). Thus, it is not surprising that the theta dependent vacuum energy which is defined by the RR one-form in the string theory calculation is related to the nonzero value of the topological susceptibility in the gauge theory studies. The physical reason for this correspondence, as we have mentioned above, is the special property of the gauge theory instantons in the fourbrane construction. Indeed, in accordance with \(54\) the RR one-form couples to the topological charge density $G\hat{G}$, on the other hand the RR one-form couples by definition to $D0$-branes. Thus, the gauge theory instantons in this case carry zerobrane charge. This is the physical reason for the correspondence discussed above.

### 4.3 Derivation of vacuum energy in QCD

The aim of this section is to derive Eq. \(14\) in pure YM model and, in particular, to identify the degrees of freedom which are responsible for the theta dependent vacuum energy density.

In the quasi-classical approach the theta dependence can be calculated using instantons \[5\]. In a simplest approximation of non-interacting instantons the theta angle enters the Euclidean space partition function in the following form:

$$\exp \left( - \frac{8\pi^2}{g^2} \pm i\theta \right) \equiv \exp \left( - N_c \frac{8\pi^2}{\lambda} \pm i\theta \right), \quad (55)$$

where $g$ stands for the strong coupling constant. $\lambda$ denotes the ’t Hooft’s coupling $\lambda \equiv N_c g^2$ which is kept fixed in the large $N_c$ limit. The expression above vanishes in the large $N_c$ limit, so does the theta dependence in \(55\). However, this conclusion
The limit of the expression \( (55) \) prevent one to do so. Indeed, the quasi-classical approximation is valid in the limit of small coupling constant (see, for instance, discussions in Ref. \[70\]). Once quantum corrections are taken into account the coupling constant \( g^2 \) in \( (55) \) becomes a scale dependent quantity. In fact, it will depend upon an instanton size \( g_2^2 = g^2(\rho) \). For small size instantons the running coupling is small and the quasi-classical approximation in \( (55) \) holds. However, for large size instantons, i.e. large couplings, it is not even clear whether the notion of a single instanton is a legitimate approximation. The overlap between instantons can be big in this case and some more complicated field configurations should be relevant for the description of physical phenomena \[71\]. In any event, the expression \( (55) \) is no longer reliable in the strong coupling limit. Thus, the conclusion that the theta dependence goes away in the large \( N_c \) limit cannot be justified. One way to study the infrared region is to look for some appropriate composite colorless excitations for which the notion of an asymptotic state can be used. We will start by searching for these excitations in pure Yang-Mills theory. To proceed, let us recall that the topological susceptibility, \( \mathcal{X} \), is a nonzero number in pure gluodynamics (we rewrite it in the following form)

\[
\mathcal{X} = -i \int \partial^\mu \partial^\nu \langle 0| T K_\mu(x) K_\nu(0)|0 \rangle d^4x \neq 0 . \tag{56}
\]

Here, \( K_\mu \) as before denotes the Chern-Simons current. As we discussed earlier, the value of \( \mathcal{X} \) in large \( N_c \) pure YM theory determines the \( \eta' \) meson mass in full QCD with massless quarks via the Witten-Veneziano formula \( m_{\eta'}^2 f_{\eta'}^2 \propto \mathcal{X} \), with \( f_{\eta'} \) being the \( \eta' \) meson decay constant \[21, 22\].

In what follows it will prove convenient to introduce a new variable by rewriting the expression for the topological charge density \( Q \) in terms of a four-index (four-form) tensor field \( H^{\mu\nu\alpha\beta} \):

\[
Q = \frac{\varepsilon_{\mu\nu\alpha\beta} \mathcal{H}^{\mu\nu\alpha\beta}}{4!} \tag{57}
\]

where the four-form field \( H^{\mu\nu\alpha\beta} \) is the field strength for the three-form potential \( C_{\mu\nu\alpha} \):

\[
H^{\mu\nu\alpha\beta} = \partial_\mu C_{\nu\alpha\beta} - \partial_\nu C_{\mu\alpha\beta} - \partial_\alpha C_{\mu\nu\beta} - \partial_\beta C_{\mu\nu\alpha}. \tag{58}
\]

The \( C_{\mu\nu\alpha} \) field is defined as a composite operator of the gluon fields \( A_\mu^a \):

\[
C_{\mu\nu\alpha} = \frac{1}{16\pi^2} (A_\mu^a \partial_\nu A_\alpha^a - A_\nu^a \partial_\mu A_\alpha^a - A_\alpha^a \partial_\nu A_\mu^a + 2 f_{abc} A_\mu^a A_\nu^b A_\alpha^c). \tag{59}
\]

Here, \( f_{abc} \) denote the structure constants of the corresponding \( SU(N_c) \) gauge group. The right-left derivative in this expression is defined as \( A \partial B \equiv A(\partial B) - (\partial A)B \).
Notice, that the \( C_{\nu\alpha\beta} \) field is not a gauge invariant quantity; if the gauge transformation parameter is \( \Lambda^a \), the three-form field transforms as
\[
C_{\nu\alpha\beta} \rightarrow C_{\nu\alpha\beta} + \partial_\nu \Lambda_{\alpha\beta} - \partial_\alpha \Lambda_{\nu\beta} - \partial_\beta \Lambda_{\alpha\nu},
\]
where \( \Lambda_{\alpha\beta} \propto A_a^\alpha \partial_\beta A^a - A_a^\beta \partial_\alpha A^a \). However, one can check that the expression for the field strength \( H_{\mu\nu\alpha\beta} \) is gauge invariant.

It has been known for some time\(^7\) that the \( C_{\nu\alpha\beta} \) field propagates long-range correlations if the topological susceptibility is nonzero in the theory. The easiest way to see this is to turn to the notion of the Kogut-Susskind pole\(^8\). Let us consider the correlator of the vacuum topological susceptibility at a nonzero momentum. In this case \( \mathcal{X} \) is defined as the zero momentum limit of the correlator of two Chern-Simons currents multiplied by two momenta:
\[
\mathcal{X} = -i \lim_{q \to 0} q^\mu q^\nu \int e^{i q x} \langle 0 | T K_\mu(x) K_\nu(0) | 0 \rangle d^4 x. \tag{60}
\]
Since this expression is nonzero, it must be that the correlator of two Chern-Simons currents develops a pole as the momentum vanishes, the Kogut-Susskind pole\(^7\). Given that the correlator of two Chern-Simons currents has a pole and that the Chern-Simons current and the three-form \( C_{\nu\alpha\beta} \) field are related, one concludes that the \( C_{\nu\alpha\beta} \) field also has a nonzero Coulomb propagator\(^7\). Thus, the \( C_{\nu\alpha\beta} \) field behaves as a massless collective excitation propagating a long-range interaction\(^7\).

These properties, in the large \( N_c \) limit, can be summarized in the following effective action for the \( C_{\nu\alpha\beta} \) field:
\[
S_{\text{eff}} = -\frac{1}{2 \cdot 4!} \frac{1}{\mathcal{X}} \int H_{\mu\nu\alpha\beta}^2 d^4 x - \frac{\theta}{3!} \int_{\partial \Gamma} C_{\nu\alpha\beta} dx^\nu \wedge dx^\alpha \wedge dx^\beta + \text{High dim.} \tag{61}
\]
The first term in this expression yields the correct Coulomb propagator for the three-form \( C_{\nu\alpha\beta} \) field. The second term is just the usual CP odd \( \theta \) term of the initial YM action written as a surface integral at spatial infinity \( \partial \Gamma \). Notice that higher dimensional terms are not explicitly written in this expression. There might be two types of higher dimensional contributions in (61). First of all, there are terms with higher powers of derivatives of the fields. These terms are suppressed by momenta of the “massless” three-form field and do not contribute to the zero momentum vacuum energy of the system. In addition, there might be higher dimensional terms with no additional derivatives. In the next section we will present some of these contributions and show that they are suppressed by higher powers of \( 1/N_c \).

In what follows we are going to study the large \( N_c \) effective action given in Eq. (61). In particular, we will calculate the ground state energy of the system in the large \( N_c \) limit using the effective action (61). In fact, we will derive Eq. (44).

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\(^7\)The action (61) is not an effective action in the Wilsonian sense. It is rather related to the generating functional of one-particle-irreducible diagrams of the composite field in the large \( N_c \) limit. The effective action in Eq. (61) is not to be quantized and loop diagrams of that action are not to be taken into account in calculating higher order Green’s functions. The analogous effective action for the CP even part of the theory was constructed in Refs. \(^7\), \(^8\).
Before we turn to this calculation let us mention that Maxwell’s equations for a free four-form field-strength $H_{\mu\nu\alpha\beta}$ yield only a constant solution in $(3 + 1)$-dimensional space-time [76]. The reason is as follows. A four-form potential has only one independent degree of freedom in four-dimensional space-time, let us call it $\Sigma$. Then, the four Maxwell’s equations written in terms of the $\Sigma$ field require that this field is independent of the all four space-time coordinates, thus the solution can only be a space-time constant. As a result, the free $H_{\mu\nu\alpha\beta}$ field propagates no dynamical degrees of freedom in $(3 + 1)$-dimensions. However, this field can be responsible for a positive vacuum energy density in various models of Quantum Field Theory (see Ref. [77]). Thus, studying classical equations of motion for the $H_{\mu\nu\alpha\beta}$ field one can determine the value of the ground state energy given by configurations of $H_{\mu\nu\alpha\beta}$. We are going to solve explicitly the classical equations of motion for the effective action (61). Then, the energy density associated with those solutions will be calculated.

Let us start with the equations of motion. Taking the variation of the action (61) with respect to the $C_{\nu\alpha\beta}(z)$ field one gets

$$\partial^\mu H_{\mu\nu\alpha\beta}(z) = \theta \mathcal{X} \int_{\partial T} \delta^{(4)}(z - x) \, dx_\nu \wedge dx_\alpha \wedge dx_\beta. \quad (62)$$

This equation can be solved exactly in four-dimensional space-time [76]. The solution is the sum of a particular solution of the inhomogeneous equation and a general solution of the corresponding homogeneous equation:

$$H_{\mu\nu\alpha\beta}(z) = \theta \mathcal{X} \int \delta^{(4)}(z - x) \, dx_\mu \wedge dx_\nu \wedge dx_\alpha \wedge dx_\beta + b \varepsilon_{\mu\nu\alpha\beta}. \quad (63)$$

The integration constant $b$, if nonzero, induces an additional CP violation beyond the existed $\theta$ angle. However, periodicity of the $\theta$ angle with respect to shifts by $2\pi \times$ (integer) allows for some nonzero $b$ proportional to $2\pi \mathbb{Z}$. As a result, the general solution to the equation of motion reads as follows:

$$H_{\mu\nu\alpha\beta} = -(\theta + 2\pi k) \mathcal{X} \varepsilon_{\mu\nu\alpha\beta}. \quad (64)$$

Thus, the different vacua are labeled by the integer $k$ and the order parameter for these vacua in the large $N_c$ limit can be written as:

$$\langle GG \rangle_k = (\theta + 2\pi k) \mathcal{X}. \quad (65)$$

As a next step let us compute the vacuum energy associated with the solution given in Eq. (64). The density of the energy-momentum tensor for the action (61) takes the form

$$\Theta_{\mu\nu} = -\frac{1}{3! \mathcal{X}} \left( H_{\mu\alpha\beta\tau} H^\alpha\beta\tau - \frac{1}{8} g_{\mu\nu} H_{\rho\alpha\beta\tau}^2 \right). \quad (66)$$

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Using the expression (64) one calculates the corresponding energy density\(^{10}\) \(\mathcal{E}_k\)

\[
\mathcal{E}_k(\theta) = \frac{1}{2} (\theta + 2\pi k)^2 \mathcal{X}.
\] (67)

Since the \(H_{\mu\nu\alpha\beta}\) field does not propagate dynamical degrees of freedom the expression above is the total energy density of the system given by the action (61) \(^{11}\).

Before we go further let us stop here to discuss some of the consequences of Eq. (67). First of all, let us notice that the result (67), as well as Eq. (44), is only valid in the limit of infinite \(N_c\). Below we will calculate subleading order corrections to Eqs. (44, 67) and argue that these expressions can also be used as a good approximation for large but finite \(N_c\). The constant \(C\) emerging in (44) is related to the topological susceptibility as follows:

\[
\mathcal{X}|_{N_c \to \infty} = 2C.
\]

Thus, the vacuum energy (44,67) is defined by vacuum fluctuations of the topological charge measured by \(\mathcal{X}\).

The crucial feature of (67) is that it defines an infinite number of vacua. The true vacuum is obtained by minimizing (67) with respect to \(k\) as in (44):

\[
\mathcal{E}_0(\theta) = \frac{1}{2} \mathcal{X} \min_k (\theta + 2\pi k)^2.
\]

This expression is periodic with respect to shifts of \(\theta\) by \(2\pi\mathbb{Z}\) and is also a smooth function of \(\theta\) except for \(\theta = \pi\) \(^{13}\) (see also discussions below). Thus, there are an infinite number of the false vacua in the theory \(^{13}\). The fate of these states will be discussed in section 5.

Let us now consider full QCD with three quark flavors. We are going to write down a low-energy effective Lagrangian for this case and then gradually decouple quarks by taking the quark masses to infinity. The resulting effective Lagrangian should be giving the energy density for pure Yang-Mills theory.

In the large \(N_c\) expansion the effective Lagrangian of QCD with three flavors takes the form \(^{39, 40}\):

\[
\mathcal{L}(U, U^*, Q) = \mathcal{L}_0(U, U^*) + \frac{1}{2} i Q(x) \text{Tr} \left( \ln U - \ln U^* \right) + \frac{1}{2\mathcal{N}} Q^2(x) + \theta Q(x) + B \frac{2}{\sqrt{2}} \text{Tr} (MU + M^* U^*) + \ldots ,
\] (68)

\(^{10}\) Notice that the total YM energy density should contain some negative constant related to the nonzero value of the gluon condensate \(^3\). This constant is subtracted from the expression for the energy discussed in this work. The energy density (67) is normalized as \(\mathcal{E}_0(\theta = 0) = 0\), and for \(k = 0\) was discussed in \(^{38}\).

\(^{11}\) One might wonder whether the same result is obtained if one treats \(\theta\) not as a constant multiplying \(Q\) in the Lagrangian, but as the phase that the states acquire under a topologically non-trivial gauge transformations. In this case the arbitrary integration constant in Eq. (63) has to be chosen in such a way which would guarantee a proper \(\theta\) dependence of the VEV of the topological charge density. This would leave the results of our discussion without modifications.
where $U$ denotes the flavor group matrix of pseudoscalar mesons, $\mathcal{L}_0$ denotes the part of the Lagrangian which contains the meson fields only \cite{79, 39, 40}. $B$ is some constant related to the quark condensate, and $M$ stands for the meson mass matrix (for recent discussions of the effective chiral Lagrangian approach see Ref. \cite{80}). Higher order terms in (68) are suppressed by quadratic and higher powers of $1/N_c$.

In order to study vacuum properties, we concentrate on the low-momentum approximation. The Lagrangian presented above can be used to solve the $U(1)$ problem \cite{21, 22}. Indeed, the field $\bar{Q}$ enters the Lagrangian in a quadratic approximation and can be integrated out. As a result, the flavor singlet meson, the $\eta'$, gets an additional contribution into its mass term. This leads to the Witten-Veneziano relation and the solution of the $U(1)$ problem without instantons \cite{21, 22}. In the present case we would like to follow an opposite way. Namely, we are going to make quarks very heavy and integrate them out keeping the field $Q$ in the Lagrangian. In the limit $m_q \to \infty$ one finds that $M \to \infty$. Thus, the low-energy effective Lagrangian which is left after the mesons are integrated out will take the form:

$$
\mathcal{L}_{\text{eff}}(Q) = \frac{1}{2} \mathcal{X} Q^2(x) + \theta \; Q(x) + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}^2}{M^2}, \frac{\partial^2 Q^2}{\Lambda_{\text{QCD}}^{10}}, \frac{1}{N_c^2}\right).
$$

(69)

Rewriting the field $Q$ in terms of the “massless” tensor $C_{\alpha\beta\gamma}$ as in the previous section, one finds that the expression (69) is nothing but the Lagrangian presented in (61). Thus, the higher order terms neglected in (61) which could contribute to the vacuum energy at zero momenta would correspond to higher corrections in $1/N_c$. In fact, the subleading corrections to the effective Lagrangian (68) can also be found \cite{81}. These terms are proportional (with the corresponding dimensionful coefficients) to the following expressions:

$$
\text{const.} \frac{1}{N_c^2} Q^2 \; \text{Tr} (\partial_\mu U \; \partial_\nu U^*), \quad \frac{1}{N_c^2} \; Q^4.
$$

(70)

The terms listed above are suppressed in the effective Lagrangian by the factor $1/N_c^2$. As a next step, we can include the terms (70) into the full effective Lagrangian and then integrate the heavy meson fields out. The net result of this procedure is that the terms proportional to $Q^4$ appear in the effective Lagrangian for pure YM theory. This, in its turn, modifies the equation of motion for the single component of $H_{\mu\nu\alpha\beta}$ considered in the previous section. Performing the calculation of the vacuum energy in the same manner as discussed above we find the following result for the energy density:

$$
\mathcal{E}(\theta) = \frac{1}{2} \; \mathcal{X} \left(\theta + 2\pi k\right)^2 + \frac{\text{const.}}{N_c^2} \; \mathcal{X} \left(\theta + 2\pi k\right)^4 + \mathcal{O}\left(\frac{1}{N_c^3}\right).
$$

(71)

In this expression a constant emerges as a result of integration of the equation of motion\cite{12}. Notice that the topological susceptibility in the expression above is

\footnote{The numerical value of this constant was recently calculated on the lattice \cite{82}.}
also defined in the corresponding order in the large $N_c$ expansion: $\mathcal{X} = 2C + \mathcal{X}_1/N_c + \mathcal{X}_2/N_c^2$. Thus, the expressions (67,71) could in principle give a reasonable approximation for big enough but otherwise finite $N_c$. The true vacuum energy density, $\mathcal{E}_0(\theta)$, can be obtained by minimizing the expression (71) with respect to $k$ as in (71). Then, $\mathcal{E}_0(\theta)$ satisfies the relation $\partial^2_\theta \mathcal{E}_0(\theta) \mid_{\theta=0} = \mathcal{X}$, no matter what is the value of the constant in (71).

### 5 Dynamics of false vacua

In this section we will discuss the dynamics of the false vacua present in the theory. In accordance with (67,71) there are an infinite number of vacua for any given value of the theta angle. Clearly, not all of these are degenerate. As we discussed, the true vacuum state is defined by minimizing the expressions (67,71) with respect to $k$. All the other states are false vacua with greater values of the energy density. There is a potential barrier that separates a given false vacuum state from the true one. Thus, a false vacuum can in general decay into the true state through the process of bubble nucleation. The decay rates for these vacua were evaluated in Ref. [45]. In this section we analyze the fate of the false vacua for different realizations of the initial conditions in which the system is placed. For the sake of simplicity we will be discussing transitions between the vacuum states labeled by $k'$ and $k$ for different values of these integers. The first two cases considered in this section were studied in Refs. [43] and [45], the remaining of the section follows Ref. [53].

#### 5.1 False vacua with $k' \sim 1$

In this subsection we consider the system which in its initial state exists in a false vacuum with $k'$ of order $\sim 1$. Let us start with the case when $N_c$ is a large but finite number so that the formula (71) (or (74)) is still a good approximation. Since there exists the true vacuum state with less energy, the false vacuum can “decay” into the true one via the bubble nucleation process. That is to say, there is a finite probability to form a bubble with the true vacuum state inside. The shell of the bubble is a domain wall which separates the false state from the true one. The dynamical question we discuss here is whether it is favorable energetically to create and expand such a bubble. Let us study the energy balance for the case at hand. While creating the shell of the bubble one looses the amount of energy equal to the surface area of the bubble multiplied by the tension of the shell. On the other hand, the true vacuum state is created inside the bubble, thus, one gains the amount of energy equal to the difference between the energies of the false and true states. The

\[\text{This decay can go through the Euclidean “bounce” solution [84]. Though the existence of the bounce for this case is not easy to understand within the field theory context, nevertheless, one could be motivated by the brane construction where this object appears as a sixbrane bubble wrapped on a certain space [43].}\]
energy balance between these two effects defines whether the bubble can be formed, and, whether the whole false vacuum can transform into the true one by expanding this bubble to infinity. Let us start with the volume energy density. The amount of the energy density which is gained by creating the bubble is

$$\Delta E = \frac{1}{2} (\theta + 2\pi)^2 - \theta^2 = 2\pi \times (\theta + \pi).$$

Thus, $\Delta E$ scales as $\sim 1$ in the large $N_c$ limit as long as the volume of the bubble is finite. Let us now turn to the surface energy which is lost. This energy is defined as:

$$E_s = T_D \times \text{(surface area)}.$$

The tension of the wall between the adjacent vacua, $T_D$, should scale as $T_D \sim N_c$ in the large $N_c$ limit. Hence, the surface energy will also scale as $\sim N_c$. Thus, the process of creation of a finite volume bubble in the large $N_c$ limit is not energetically favorable. Indeed, the amount of energy which is lost while creating the shell is bigger than the amount which is gained. In terms of the false vacuum decay width this means that the width of this process is suppressed in the large $N_c$ limit [45]:

$$\frac{\Gamma}{\text{Volume}} \propto \exp \left(-aN_c^4\right),$$

where $a$ stands for some positive constant [45]. Thus, one concludes that in the limit $N_c \to \infty$ the false vacua with $k' \sim 1$ are stable [33, 35].

### 5.2 False vacua with $k' \sim N_c$

Here we study the fate of the false vacua with $k' \sim N_c$. We discuss a possibility that these vacua can decay into a state $k$ with $k' - k \sim N_c$ and $k' + k \sim N_c$. As in the previous subsection, we are going to study the energy balance for the bubble nucleation process. The amount of the volume energy density which is gained by creating such a bubble in the large $N_c$ limit scales as follows:

$$\Delta E \propto \mathcal{X} N_c^2.$$

Thus, the volume energy which is gained increases as $\sim N_c^2$. Let us now turn to the surface energy which is lost while nucleating a bubble. This is defined as $E'_s = T'_D \times \text{(surface area)}$, where $T'_D$ denotes the tension of the domain wall interpolating between the vacua labeled by $k'$ and $k$. Since $k' - k \sim N_c$ these vacua are not neighboring ones. Thus, in general, there is no reason to expect that the tension of the wall interpolating between these vacua scales as $\sim N_c$. $T'_D$ might scale as $\sim N_c^2$ at most (as the energy of a generic configuration in a model with $N_c^2$ degrees

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14In this subsection we assume that $\theta \neq \pm \pi$. The case $\theta = \pi$ will be considered below.
of freedom). However, even in the case when $T_D' \sim N_c^2$ the volume energy which is gained is at least of the same order as the surface energy which is lost. Hence, it is energetically favorable to increase the radius of such a bubble (since the volume energy scales as a cubic power of the radius while the surface energy scales only as a quadratic power of the effective size). Thus, the bubble nucleation process will not be suppressed and the false vacua with $k' \sim N_c$ will eventually decay into the true ground state. Note, that the state $k' = N_c$ can as well decay into the neighboring vacuum $k = N_c - 1$ which subsequently is allowed to turn into the ground state.

5.3 Parallel domain walls

In this subsection we consider the special case when all the vacua are present simultaneously in the initial state of the model. This can be achieved, for instance, by placing in space an infinite number of parallel domain walls separating different vacua from each other. It is rather convenient to picture these walls as parallel planes. Each vacuum state is sandwiched between the corresponding two domain walls (two planes) separating this state from the neighboring vacua. Each domain is labeled by $k$ and in accordance with (67, 71) is characterized by the corresponding value of the vacuum energy. Furthermore, the order parameter $\langle G \bar{G} \rangle$ takes different values in these vacua in accordance with (65). Let us turn to the true vacuum state. For simplicity we assume that this state is given by $k = 0$ (which corresponds to $|\theta|$ being less than $\pi$). The corresponding vacuum energy is the lowest one. Consider the two states which are adjacent to the true vacuum. These states have the energy density bigger than that of the true vacuum. Thus, there is a constant pressure acting on the domain walls separating the true vacuum from the adjacent false ones. This pressure will tend to expand the domain of the true vacuum. In fact, for large but finite $N_c$, the pressure will indeed expand the spatial region of the true vacuum by moving apart the centers of the domain walls sandwiching this state. The very same effect will be happening between any two adjacent vacua. Indeed, let us calculate the jump of the energy density between the two vacua labeled by $k'$ and $k$:

$$\Delta \mathcal{E}_{k'k} = 2\pi X (k' - k) \left( \theta + \pi (k' + k) \right).$$

(74)

As far as $N_c$ is large but finite, the walls will start to accelerate. Farther the wall is located from the true vacuum (i.e. larger the sum $k' + k$), bigger the initial acceleration of the wall is going to be; i.e., the walls will start to move apart from each other with the following initial acceleration:

$$a_{k'k} \propto \Lambda_{YM} \frac{(k' - k) [\theta + \pi (k' + k)]}{N_c}.$$  

(75)

For finite $N_c$ all the walls will be moving to spatial infinity and the whole space will eventually be filled with the true vacuum state. On the other hand, when $N_c \to \infty$ the picture is a bit different. There are a number of interesting cases to consider:
First of all let us set \( k' - k = 1 \) and \( k' , k \sim 1 \). Then, in the limit \( N_c \to \infty \) the acceleration \( a_{k'k} \to 0 \). Thus, the neighboring walls stand still if they had no initial velocity. The physical reason of this behavior is as follows. Although there is a constant pressure of order \( \sim 1 \) acting on the wall, nevertheless, the wall cannot be moved because the mass per unit surface area of the wall tends to infinity in the limit \( N_c \to \infty \).

The second interesting case would be when the constant pressure produced by the energy jump between some neighboring vacua is of order \( \sim N_c \). In this case it will be possible to accelerate these walls up to the speed of light and send them to spatial infinity. Indeed, if \( k' - k = 1 \) but \( k' + k \sim N_c \), then the wall between these two vacua starts moving with a non-vanishing acceleration which scales as follows:

\[
a_{k'k} \propto \Lambda_{YM} \frac{\pi(k' + k)}{N_c} \sim O(1). \tag{76}
\]

Thus, these walls will eventually be approaching spatial infinity with a speed of light even in the limit of infinite \( N_c \).

In addition to the effects emphasized above there might also be decays of the false vacua happening in each particular domain. As we discussed in the previous subsections, for large but finite \( N_c \) all the false vacua will be nucleating bubbles with energetically favorable phases inside and expanding these bubbles to infinity. Thus, for large but finite \( N_c \), there are two effects which eliminate the false vacua: The moving walls are sweeping these states to infinity, and, in addition, these vacua are decaying via bubble nucleation processes.

What happens for an infinite \( N_c \)? As we learned above there are an infinite number of domains which will stay stable in that limit and the corresponding false vacua would not decay because of the exponential suppression. Thus, there are an infinite number of inequivalent spatial regions which are separated by domain walls. Consider one of the regions sandwiched between two domain walls. The three-form field \( C_{\mu\nu\alpha} \) will couple to the walls and the large \( N_c \) effective action for this case will look as follows:

\[
\tilde{S} = S_{\text{eff}} + \sum_{i=k, k+1} \mu_i \int_{W_i} C_{\mu\nu\alpha} \, dx^\mu \wedge dx^\nu \wedge dx^\alpha, \tag{77}
\]

where \( S_{\text{eff}} \) is defined in (61), \( \mu_i \) stands for the coupling of the three-form potential to a corresponding domain wall; \( W_i \) denotes the worldvolume of the wall. In this case the domain wall can be regarded as a source of the corresponding three-form potential. This is reminiscent to what happens in the large \( N_c \) supersymmetric YM model [85].

### 5.4 Domain walls at \( \theta = \pi \)

If \( \theta = \pi \), the initial classical Lagrangian is CP invariant. Indeed, under CP transformations \( \theta = \pi \) goes into \(-\pi\). Since \( \pi \) and \(-\pi\) angles are equivalent, CP is a
symmetry of the Lagrangian. However, in accordance with (65), this symmetry is spontaneously broken by the vacuum of the theory. Thus, one finds the following two degenerate true vacua:

$$\mathcal{E}_{k=0} = \mathcal{E}_{k=-1} = \frac{1}{2} \mathcal{X} \pi^2.$$  
(78)

These two vacua are labeled by the order parameter (65). In the $k = 0$ state $\langle \tilde{G} G \rangle = \pi \mathcal{X}$ and in the $k = -1$ state $\langle \tilde{G} G \rangle = -\pi \mathcal{X}$. As a result of the spontaneous breaking of a discrete symmetry there should be a domain wall separating these two vacua. Let us consider the case discussed in the previous section. Namely, let us choose the initial condition of the system as a state where all the possible vacua are simultaneously realized in space. That is, there are an infinite number of domain walls (parallel planes) dividing space into an infinite number of domains with different values of the vacuum energy density labeled by $k$. As we mentioned above, only two of these domains have equal minimal energy density given in (78). The domain wall separating these two vacua, as we will see below, is somewhat special. In accordance with the discussions in the previous subsection, for large but finite $N_c$, all the walls merging with the false vacua will tend to rush to spatial infinity. The final stable state of the model can be described as a space separated into two parts by a single domain wall. To the left (right) of the wall one discovers the phase with $k = -1$ with the corresponding order parameter $\langle \tilde{G} G \rangle = -\pi \mathcal{X}$, and, to the right (left) of the wall one finds the state with $k = 0$ and $\langle \tilde{G} G \rangle = \pi \mathcal{X}$. Vacuum energies of these two states are degenerate.

In the case of infinite $N_c$ the picture is slightly different. As elucidated in the previous subsection, there will be an infinite number of stable vacua. The domain wall separating the two true vacua can be regarded in this case as the fixed plane under $\mathbb{Z}_2$ transformations of the coordinate transverse to the plane. The three-form field $C_{\mu \nu \alpha}$ will be able to couple to this wall in a manner discussed in the previous subsection.

## 6 Axions and vacuum structure in gluodynamics

### 6.1 Two scenarios

The invisible axion is very light. Integrating out all other degrees of freedom and studying the low-energy axion effective Lagrangian must be a good approximation. The axion effective potential in QCD can be of two distinct types.

Assuming that for all values of $\theta$ the QCD vacuum is unique one arrives at the axion effective Lagrangian of the form

$$\mathcal{L}_a = f^2 \left[ \frac{1}{2} (\partial_{\mu} \alpha)^2 + m_a^2 (\cos(\alpha + \theta) - 1) \right].$$  
(79)

The axion potential does not have to be (and generically is not) a pure cosine; it may have higher harmonics. In the general case it is a smooth periodic function
of $\alpha + \theta$, with the period $2\pi$. For illustration we presented the potential as a pure cosine. This does not change the overall picture in the qualitative aspect.

As we will see below, a smooth effective potential of the type (79) emerges even if the (hadronic) vacuum family is non-trivial, but the transition between the distinct hadronic vacua does not occur inside the axion wall. This is the case with very light quarks, $m_q \ll \Lambda/N_c$. In the opposite limit, one arrives at the axion potential with cusps, considered below.

In the theory (79) one finds the axion walls interpolating between the vacuum state at $\alpha = -\theta$ and the same vacuum state at $\alpha = -\theta + 2\pi$,

$$\alpha(z) + \theta = 4 \arctan(e^{m_a z}), \quad (80)$$

where the wall is assumed to lie in the $xy$ plane, so that the wall profile depends only on $z$. This is the most primitive “$2\pi$ wall.”

The tension of this wall is obviously of the order of

$$T_1 \sim f^2 m_a. \quad (81)$$

Taking into account that $f^2 m_a^2 \sim X$ where $X$ is the topological susceptibility of the QCD vacuum, we get

$$T_1 \sim X/m_a. \quad (82)$$

The inverse proportionality to $m_a$ is due to the fact that the transverse size of the axion wall is very large.

Let us now discuss the axion effective potential of the second type. In this case the potential has cusps, as is the case in pure gluodynamics, where the axion effective Lagrangian is of the form

$$\mathcal{L}_a = \frac{f^2}{2} (\partial_\mu \alpha)^2 + \min_{\ell} \left\{ N_c^2 \Lambda^4 \cos \frac{\alpha + 2\pi \ell}{N_c} \right\}, \quad (83)$$

(see more detailed discussions below). Here the $\theta$ angle was included in the definition of the axion field. The axion wall interpolates between $\alpha = 0$ and $\alpha = 2\pi$.

What is the origin of this cusp? The cusps reflect a restructuring in the hadronic sector. When one (adiabatically) interpolates in $\alpha$ from 0 to $2\pi$ a gluonic order parameter, for instance $\langle \tilde{G}G \rangle$, necessarily experiences a restructuring in the middle of the wall corresponding to the restructuring of heavy gluonic degrees of freedom. In other words, one jumps from the hadronic vacuum which initially (at $\alpha = 0$) had $\langle \tilde{G}G \rangle = 0$ into the vacuum in which initially $\langle \tilde{G}G \rangle \neq 0$. Upon arrival to $\alpha = 2\pi$, we find $\langle \tilde{G}G \rangle = 0$ again. This implies that the central part of such an axion wall is dominated by a gluonic wall. Thus, the cusp at $\alpha = \pi$ generically indicates the formation of a hadronic core, the D-wall [47] in the case at hand.

Returning to the question of the tension we note that

$$X \sim \Lambda^4 N_c^0, \quad m_a \sim \Lambda^2 N_c^0 f^{-1} \quad \text{in pure gluodynamics},$$

$$X \sim \Lambda^2 N_c m_q, \quad m_a \sim \Lambda^{3/2} m_q^{1/2} N_c^{1/2} f^{-1} \quad \text{in QCD with light quarks}, \quad (84)$$
which implies, in turn,

\[
T_1 \sim \begin{cases} 
  f \Lambda^2 N_c^0 & \text{in pure gluodynamics} \\
  f \Lambda^{3/2} m_q^{1/2} N_c^{1/2} & \text{in QCD with light quarks} .
\end{cases}
\]

(85)

Here \( m_q \) is the light quark mass.

The presence of the large parameter \( f \) in \( T_1 \) makes the axion halo the dominant contributor to the wall tension. The contribution of the hadronic component contains only hadronic parameters, although it may have a stronger dependence on \( N_c \). Examining the cusp with an appropriately high resolution one would observe that it is smoothed on the hadronic scale, where the hadronic component of the axion wall “sandwich” would become visible. The cusp carries a finite contribution to the wall tension which cannot be calculated in the low-energy approximation \[86\]. To this end one needs to consider the hadronic core explicitly. The tension of the core \( T_{\text{core}} \sim \Lambda^3 N_c \), while the tension of the axion halo \( T_{\text{halo}} \sim f \Lambda^2 \) (in pure gluodynamics).

We pause here to make a comment on the literature. The consideration of the axion walls in conjunction with hadrons dates back to the work of Huang and Sikivie, see Ref. \[46\]. This work treats the Weinberg-Wilczek \( N = 2 \) axion in QCD with two light flavors, which is replaced by a chiral Lagrangian for the pions, to the leading order (quadratic in derivatives and linear in the light quark masses). It is well-known \[36, 37, 41\] that in this theory the crossover phenomenon takes place at \( m_u = m_d \). In the realistic situation, \( (m_d - m_u)/(m_d + m_u) \sim 0.3 \) considered in Ref. \[46\], there is no crossover. The pions can be integrated over, leaving one with an effective Lagrangian for the axion of the type (79) (with \( \alpha \rightarrow 2\alpha \)). The potential is not pure cosine, higher harmonics occur too. The axion halo exhausts the wall, there is no hadronic core in this case.

At the same time, Huang and Sikivie (see Ref. \[46\]) found an explicit solution for the “\( \pi^0 \)” component of the wall. In fact, this is an illusion. The Huang-Sikivie (HS) solution refers to the bare \( \pi^0 \) field. To find the physical \( \pi^0 \) field one must diagonalize the mass matrix at every given value of \( \alpha \) (the bare \( f_\alpha \) is the physical axion field up to small corrections \( \sim f_\alpha^2 / f^2 \) where \( f_\pi \) stands for the pion decay constant). Once this is done, one observes that the physical pion field, which is a combination of the bare pion and \( f_\alpha \), is not excited in the HS solution. The equation (2.16) in the HS paper is exactly the condition of vanishing of the physical pion in the wall profile. This explains why the wall thickness in the HS work is of order \( m_{\pi}^{-1} \), with no traces of the \( m_{\pi}^{-1} \) component. The crossover of the hadronic vacua at \( \alpha = \pi/2 \) (remember, this is \( N = 2 \) model) could be recovered in the Huang-Sikivie analysis at \( m_u = m_d \). However, the chiral pion Lagrangian predicts in this case the vanishing of the pion mass in the middle of the wall, for accidental reasons. This is explained in detail by A. Smilga, Ref. \[11\].
6.2 An Illustrative model

To find the axion walls with D-wall core one has to solve QCD, which is way beyond our possibilities. Our task is more modest. We would like to obtain a qualitative description of the axion wall sandwich which, with luck, can become semi-quantitative. To this end we want to develop toy models. An obvious requirement to any toy model is that it must qualitatively reproduce the basic features of the vacuum structure which we expect in QCD. In SUSY gluodynamics it was possible to write down a toy model with a $Z_{N_c}$ symmetry [87] which “integrates in” the heavy degrees of freedom and allows one to investigate the BPS domain walls in the large $N_c$ limit [54] (see also [88]). We will suggest a similar model in (nonsupersymmetric) QCD, then switch on axions, and study the axion domain walls in a semi-realistic setting. In this model we will be able to find exact solutions for D-walls and axion walls.

Here we suggest a simple toy model which has a proper vacuum structure. In what follows an appropriate (complex) glue order parameter is denoted by $\Phi$. The modulus and phase of this field describe respectively the $0^{++}$ and $0^{-+}$ channels of the theory.

Our toy model Lagrangian is

$$\mathcal{L} = N_c^2 (\partial_\mu \Phi)^* (\partial_\mu \Phi) - V(\Phi, \Phi^*)$$

$$V = V_0 + V_1,$$

$$V_0 = N_c^2 A^2 \left| 1 - \Phi e^{-i\theta} \right|^2,$$

$$V_1 = \left\{-\frac{\mathcal{X} N_c^2}{2} \Phi \left[ 1 + \frac{1}{N_c} (1 - \Phi e^{-i\theta}) \right] + \frac{\mathcal{X} N_c^2}{2} \right\} + \text{H.c.}.$$  \(86\)

Here $A$ is a numerical constant of order one, and $\mathcal{X}$ is the vacuum topological susceptibility in pure gluodynamics (note that $\mathcal{X}$ is independent of $N_c$). The scale parameter $\Lambda$ is set to unity.

This model has the vacuum family composed of $N_c$ states. Indeed, the minima of the energy are determined from the equations

$$\left. \frac{\partial V}{\partial \Phi} \right|_{\text{vac}} = \left. \frac{\partial V}{\partial \Phi^*} \right|_{\text{vac}} = 0,$$  \(87\)

which have the following solutions (we put temporarily $\Lambda = 1$):

$$\Phi_{\ell \text{vac}} = \exp \left( i \frac{\theta + 2\pi \ell}{N_c} \right), \quad \ell = 0, 1, ..., N_c - 1.$$  \(88\)

In the $\ell$-th minimum $V_0$ vanishes, while $V_1$ produces a non-vanishing vacuum energy density,

$$\mathcal{E}_\ell = \mathcal{X} N_c^2 \left\{ 1 - \cos \left( \frac{\theta + 2\pi \ell}{N_c} \right) \right\}.$$  \(89\)
For each given $\theta$ the genuine vacuum is found by minimization,

$$
\mathcal{E}(\theta) = N_c^2 \mathcal{X} \min_{\ell} \left\{ 1 - \cos \left( \frac{\theta + 2\pi \ell}{N_c} \right) \right\}.
$$

(90)

The remaining $N_c - 1$ minima are quasivacua. Once the heavy field $\Phi$ is integrated out, the vacuum energy is given by the expression (90); it has cusps at $\theta = \pi, 3\pi$ and so on. Needless to say that the potential (86) has no cusps.

We will first consider the model (86) without the axion field, at $\theta = 0$, in the limit $N_c \to \infty$. In this limit the false vacua from the vacuum family are stable.

The classical equation of motion defining the wall is

$$
N_c^2 \Phi''' = \frac{\partial V}{\partial \Phi},
$$

(91)

where primes denote differentiation with respect to $z$ (we look for a solution which depends on the $z$ coordinate only).

This is a differential equation of the second order. It is possible, however, to reduce it to a first order equation. Indeed, Eq. (91) has an obvious “integral of motion” (“energy”),

$$
N_c^2 \Phi'' \Phi' - V = \text{Const} = 0,
$$

(92)

where the second equality follows from the boundary conditions. In the large $N_c$ limit one can parametrize the field $\Phi$ as follows ($\rho \sim 1$):

$$
\Phi \equiv 1 + \frac{\rho}{N_c}.
$$

(93)

Taking the square root of Eq. (92), substituting Eq. (93) and neglecting the terms of the subleading order in $1/N_c$ we arrive at

$$
\dot{\rho}' = iAN_c \left( 1 - \exp(\rho) \right).
$$

(94)

The phase on the right-hand side can be chosen arbitrarily. The choice in Eq. (94) is made in such a way as to make it compatible with the boundary conditions for the wall interpolating between $\Phi_{\text{vac}} = 1$ and $\Phi_{\text{vac}} = \exp(2\pi i/N_c)$. This is precisely the expression that defines the domain walls in SUSY gluodynamics [54, 88]. It is not surprising that the same equation determines the D-walls in non-SUSY gluodynamics – the fermion-induced effects are not important for D-walls in the large $N_c$ limit.

The solution of this equation was obtained in [88]. In the parametrization $\rho = \sigma + i\tau$ the solution takes the form:

$$
\cos \tau = (\sigma + 1) \exp(-\sigma),
$$

$$
\int_{\sigma(0)}^{\sigma(z)} \left[ \exp(2t) - (1 + t)^2 \right]^{-1/2} dt = -AN_c |z|.
$$

(95)
The real part of $\rho$ is a bell-shaped function with an extremum at zero; it vanishes at $\pm \infty$. The imaginary part of $\rho$, on the other hand, changes its value from 0 to $2\pi$. This determines a D-wall in the large $N_c$ gluodynamics. The width of the wall scales as $1/N_c$.

The solution presented above is exactly the same as in SUSY gluodynamics. This is not surprising since the ansatz (93) implies that $V_1$ does not affect the solution – its impact is subleading in $1/N_c$, while $V_0$ is exactly the same as in the SUSY-gluodynamics-inspired model of Ref. [54]. Moreover, for the same reason the domain wall junctions emerging in this model will be exactly the same as in the SUSY-gluodynamics-inspired model [47]. Inclusion of $V_1$ in the subleading order makes the wall to decay.

Inclusion of the $N = 1$ axion field amounts to the replacement

$$\theta \to \theta + \alpha$$

in Eq. (86), plus the axion kinetic term

$$L_{\text{kin}} = \left( \frac{f^2 + 2\Phi^* \Phi}{2} \right) (\partial_\mu \alpha)^2 + iN_c (\partial_\mu \alpha) (\Phi^* \partial_\mu \Phi - \Phi \partial_\mu \Phi^*).$$

(96)

The occurrence of the mixing between $\alpha$ and the phase of $\Phi$ is necessary, as is readily seen from the softly broken SUSY gluodynamics. (To get the potential of the type (89) in this model, one must eliminate the $G\tilde{G}$ term by a chiral rotation. Then $m \to m \exp((\theta + \alpha)/N_c)$ and, additionally one gets $\partial_\mu \alpha \times [\text{the gluino axial current}]$.)

The term $2\Phi^* \Phi$ in the brackets has to be included to reproduce the correct mass for the axion after the physical heavy state is integrated out. The presence of this term signals that QCD dynamics generates not only the potential for the axion but also modifies its kinetic term. On the other hand, since $\Phi^* \Phi \leq \Lambda^2$ and, moreover, $\Lambda << f$, this term can be neglected for all practical purposes.

We are interested in the configuration with $\alpha$ interpolating between 0 and $2\pi$. The phase of $\Phi$ will first adiabatically follow $\alpha/N_c$, then at $\alpha \approx \pi$, when the phase of $\Phi$ is close to $\pi/N_c$, it will very quickly jump by $-2\pi/N_c$, and then it will continue to grow as $\alpha/N_c$, so that when $\alpha$ reaches $2\pi$ the phase of $\Phi$ returns to zero. This jump is continuous, although it occurs at a scale much shorter than $m_a^{-1}$. This imitates the D-wall core of the axion wall. One cannot avoid forming this core, since otherwise the interpolation would not connect degenerate states – on one side of the wall we would have (hadronic) vacuum, on the other side an excited state.

In the large $N_c$ limit one can be somewhat more quantitative. Indeed, in this approximation the model admits the exact solutions. The gluonic core of the wall has the same form as before, Eq. (93), but the phase $\tau$ is now substituted by the superposition $\tau - (\alpha + \theta)$ since the axion field is mixed with the phase of the $\Phi$ field.

This very narrow core is surrounded by a diffused axion halo. The axion field is described in this halo by the solution to the Lagrangian (83). This takes the form:

$$\theta + \alpha(z) = -2\pi + 4N_c \arctan \left( e^{m_\alpha z} \tan \frac{\pi}{4N_c} \right), \quad z < 0,$$

33
\[ \theta + \alpha(z) = -4N_c \arctan\left(e^{-m_a z} \tan \frac{\pi}{4N_c}\right), \quad z > 0. \tag{97} \]

Thus, we find explicitly the stable axion wall with a D-wall core. Note that this is a usual “2\pi” wall as it separates two identical hadronic vacua. As we discussed in the introduction, this wall can decay quantum mechanically. However, its lifetime is infinite for all practical purposes. Moreover, this wall is harmless cosmologically. It will be produced bounded by global axion strings in the early universe. Bounded walls shrink very quickly by decaying into axions and hadrons.

7 Axions and vacuum structure in QCD with light quarks

So far we discussed pure gluodynamics with the axion. Our final goal is to study QCD with \( N_f = 3 \). There are two, physically distinct regimes to be considered in this case. In real QCD

\[ m_u, m_d \ll m_s \sim \frac{\Lambda}{N_c}, \quad m_u, m_d, m_s \ll \Lambda. \tag{98} \]

In this regime the consideration of the chiral Lagrangians \[36, 37, 41\], does not exhibit the vacuum family. We will comment on why the light quarks screen the vacuum family of the glue sector, so that the axion domain wall provides no access to it. In the limit (98) the effects due to the D-walls will be marginal.

On the other hand, in the genuinely large \( N_c \) limit

\[ \frac{\Lambda}{N_c} \ll m_u, m_d \ll m_s \ll \Lambda, \tag{99} \]

physics is rather similar to that of pure gluodynamics. The light quarks are too heavy to screen the vacuum family of the glue sector.

In what follows we study the axion walls and their hadronic components in the limits (98) and (99), separately.

7.1 One light quark

To warm up, let us start from the theory with one light quark. In the limit of large \( N_c \) this introduces a light meson, “\( \eta' \)”. An appropriate effective Lagrangian can be obtained by combining the vacuum energy density of gluodynamics with what remains from the chiral Lagrangian at \( N_f = 1 \),

\[ \mathcal{L} = \frac{F^2}{2} \left( \partial_\mu \beta \right)^2 - V(\beta), \]

\[ V = -m_q \Lambda^3 N_c \cos \beta + \min_{\ell} \left\{ -N_c^2 \Lambda^4 \cos \frac{\beta + \theta + 2\pi \ell}{N_c} \right\}. \tag{100} \]
Here $\beta$ is the phase of $U \sim \bar{q}_L q_R$, while $F^2 \sim \Lambda^2 N_c$ is the “$\eta'$” coupling constant squared. The product $F\beta$ is the “$\eta'$” field. The first term in $V$ corresponds to the quark mass term, $\mathcal{M}U + h.c.$ At $N_c = \infty$ the second term in $V$ becomes $(\beta + \theta)^2$. It corresponds to $(\ln \det U + \theta)^2$ in Eq. (11) in [30]. The subleading in $1/N_c$ terms sum up into a $2\pi$ periodic function of the cosine type, with the cusps. It is unimportant that we used cosine in Eq. (100). Any $2\pi$ periodic function of this type would lead to the same conclusions. The second term in Eq. (100) differs from the vacuum energy density in gluodynamics by the replacement $\theta \rightarrow \beta + \theta$.

If $m_q \ll \Lambda/N_c$, the first term in $V$ is a small perturbation; therefore, in the vacuum, $\beta + \theta = 2\pi k$, and, hence, the $\theta$ dependence of the vacuum energy is

$$\mathcal{E}_{\text{vac}}(\theta) = -m_q \Lambda^3 N_c \cos \theta.$$  

This is smooth, $2\pi$ periodic and proportional to $m_q$ as it should be on general grounds in the theory with one light quark.

The condition $m_q \ll \Lambda/N_c$ precludes us from sending $N_c \rightarrow \infty$. The would be “$2\pi$” wall in the variable $\beta$ is expected to be unstable. This is due to the fact that at $N_c \sim 3$ the absolute value of the quark condensate $\bar{\psi} \psi$ is not “harder” than the phase of the condensate $\beta$, and the barrier preventing the creation of holes in the “$2\pi$” wall is practically absent.

If one closes one’s eyes on this instability one can estimate that the tension of the “$\eta'$” wall is proportional to $\Lambda^3 N_{c1/2}$, with a small correction $m_q \Lambda^2 N_{c3/2}$ from the quark mass term. The tension of the D wall core is, as previously, $\Lambda^3 N_c$.

In the opposite limit

$$m_q \gg \frac{\Lambda}{N_c}, \quad \text{but } m_q \ll \Lambda,$$  

the situation is trickier. Now the first term in $V$ is dominant, while the second is a small perturbation. There are $N_c$ distinct vacua in the theory,

$$\beta_\ell = -\frac{2\Lambda}{m_q N_c} (\theta + 2\pi \ell).$$  

Then the $\theta$ dependence of the vacuum energy density is

$$\mathcal{E}_{\text{vac}}(\theta) = \Lambda^4 \min_{\ell} (\theta + 2\pi \ell)^2,$$

this is similar to that in the theory without light quarks (i.e., the same as in gluodynamics). The “$\eta'$” wall is stable at $N_c \rightarrow \infty$, with a D-wall core in its center. The $\eta'$ wall is a “$2\pi$” wall.

From this standpoint the quark with the mass (102) is already heavy, although the “$\eta'$” is still light on the scale of $\Lambda$,

$$M_{\eta'} \sim m_q^{1/2} \Lambda^{1/2} \ll \Lambda.$$
So far the axion was switched off. What changes if one includes it in the theory? The Lagrangian now becomes
\[ \mathcal{L} = F^2 (\partial_\mu \beta)^2 + f^2 (\partial_\mu \alpha)^2 - V(\beta, \alpha), \]
where \( V(\beta, \alpha) \) is given by
\[ V = -m_q \Lambda^3 N_c \cos \beta + \min_\ell \left\{ -N_c^2 \Lambda^4 \cos \frac{\beta + \alpha + 2\pi \ell}{N_c} \right\}, \]
with the \( \theta \) angle absorbed in the definition of the axion field.

The bare \( \eta' \) mixes with the bare axion. It is easy to see that in the limit \( m_q \ll \Lambda / N_c \) the physical \( \eta' \) is proportional to \( \beta + \alpha \), rather than to \( \beta \). Therefore, even if we force the axion wall to develop, (i.e. \( \alpha \) to evolve from 0 to \( 2\pi \)) the \( \eta' \) wall need not develop. It is energetically expedient to have \( \beta + \alpha = 0 \). Thus, the effect of the axion field on the hadronic sector is totally screened by a dynamical phase \( \beta \) coming from the quark condensate. In other words, the axion wall with the lowest tension corresponds to the frozen physical \( \eta' \),
\[ \beta + \alpha = 0. \]

There is no hadronic core. The tension of this wall is determined from the term \( \propto m_q \Lambda^3 N_c \).

(If one wishes, one could add the \( \eta' \) wall to the axion wall. Then the \( \eta' \) wall, with the D-wall core will appear in the middle of the axion wall, but they are basically unrelated. This will be a secondary phenomenon, and the D wall core will be, in fact, the core of the \( \eta' \) wall rather than the axion wall.)

If the quark mass is such that (102) applies, then the axion field \( \alpha \) cannot be screened, since we cannot freeze \( \beta + \alpha \) everywhere in the axion wall profile at zero – at \( m_q \gg \Lambda / N_c \), \( \beta \) is proportional to the physical \( \eta' \) and is much heavier than the axion field. Thus, in this case the axion wall will be described by the Lagrangian (83) and will have a D-wall core. One may also add, on top of it, the \( \eta' \) wall. This will cost \( m_q^{1/2} \Lambda^{5/2} N_c \) in the wall tension – still much less than \( \Lambda^3 N_c \) of the D-wall core of the axion wall.

The limit (102) is unrealistic. Moreover, in this limit the D walls taken in isolation, without the axion walls, are stable by themselves, although they interpolate between nondegenerate states [45].

### 7.2 Three Light Quarks

Let us turn the case of three light flavors. The physical picture is quite similar to that of the one-flavor case, see Sec. 7.1.

We assume the mass matrix \( \mathcal{M} \) in the meson Lagrangian to be diagonal. Therefore, we will looking for a diagonal \( U(3) \) meson matrix which minimizes the potential,
\[ U = \text{diag}\left( e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3} \right). \]
The potential takes the form

\[ V = - \sum_i m_i \Lambda^3 N_c \cos \phi_i + \min_\ell \left\{ -N_c^2 \Lambda^4 \cos \frac{\sum_i \phi_i + \theta + 2\pi \ell}{N_c} \right\}. \]  

(107)

As before, we will consider two limiting cases, (98) and (99).

Let us switch off the axion field first. In the limit of genuinely light quarks, Eq. (98), when the second term in the potential (107) is dominant, the solutions for \( \phi \)'s were found in [36, 37]. They satisfy to the relation \( \phi_3 \simeq 0 \) and \( \phi_1 + \phi_2 = -\theta \). The corresponding expression for the vacuum energy density is

\[ E_{\text{vac}}(\theta) = - N_c \Lambda^3 \sqrt{m_u^2 + m_d^2 + 2m_u m_d \cos \theta}. \]  

(108)

As in Sec. 7.1, we deal here with a smooth single-valued function of \( \theta \). The inclusion of the axion replaces \( \theta \to \theta + \alpha \to \alpha \). The physical \( \eta' \) field is given by the sum \( \sum_i \phi_i + \alpha \). It is energetically favorable to freeze this state. Thus, the situation is identical to that in the one-flavor case: even if the axion wall is forced to develop, the physical \( \eta' \) wall (which is now the \( \sum_i \phi_i + \alpha \) wall) does not have to occur. Effectively, the vacuum angle is screened, and there is no D-wall core in the axion wall.

If, nonetheless, the \( \eta' \) wall is formed due to some cosmological initial conditions, it will have a D-wall core (albeit the \( \eta' \) wall is unstable in the limit at hand and cannot be considered in the static approximation). The would-be \( \eta' \) wall is independent of the axion wall; its effect on the axion wall formation is rather irrelevant.

In addition to this, a “\( 2\pi \)” wall could develop for nonsinglet mesons at certain values of the quark masses. There is nothing new we could add to this issue which is decoupled from the issue of the vacuum family in the glue sector and D-walls.

We now pass to the opposite limit (99), when the first term in the potential (107) is dominant. As in Sec. 7.1, there are \( N_c \) distinct vacua with the energy given by (104). It is straightforward to show that the potential for the axion in this case is of the form (83), with the cusps which signal the presence of the D-wall core. This is similar to what happens in gluodynamics. One cannot avoid having an \( \eta' \) wall in the middle of the axion wall, which entails a D-wall too. The D-walls separate the degenerate vacua. Since they “live” in the middle of the axion wall, they are perfectly stable.

(In addition, there can be “\( 2\pi \)” walls in either of \( \phi \)'s or their linear combinations. However, these latter are unstable and do not appear in the physical spectrum of the theory.)

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References

[1] S. Weinberg, Phys. Rev. Lett. 40, 223 (1978).
[2] F. Wilczek, Phys. Rev. Lett. 40, 279 (1978).
[3] D. J. Gross and F. Wilczek, Phys. Rev. Lett. 30 (1973) 1343.
[4] H. D. Politzer, Phys. Rev. Lett. 30 (1973) 1346.
[5] A. A. Belavin, A. M. Polyakov, A. S. Shvarts and Y. S. Tyupkin, Phys. Lett. B 59 (1975) 85.
[6] R. Jackiw and C. Rebbi, Phys. Rev. Lett. 37, 172 (1976).
[7] C. G. Callan, R. F. Dashen and D. J. Gross, Phys. Lett. B 63, 334 (1976).
[8] V.N. Gribov, 1975, unpublished.
[9] S. L. Adler, Phys. Rev. 177, 2426 (1969).
J. S. Bell and R. Jackiw, Nuovo Cim. A 60, 47 (1969).
[10] K. Fujikawa, Phys. Rev. Lett. 42, 1195 (1979).
[11] V. Baluni, Phys. Rev. D 19, 2227 (1979).
[12] R. J. Crewther, P. Di Vecchia, G. Veneziano and E. Witten, Phys. Lett. B 88, 123 (1979) [Erratum-ibid. B 91, 487 (1980)].
[13] A. Pich and E. de Rafael, Nucl. Phys. B 367, 313 (1991).
[14] M.A. Shifman, A.I. Vainshtein, V.I. Zakharov, Nucl. Phys. B147 (1979) 385.
[15] M. Pospelov and A. Ritz, Phys. Rev. Lett. 83, 2526 (1999) [arXiv:hep-ph/9904453].
[16] P. G. Harris et al., Phys. Rev. Lett. 82 (1999) 904.
[17] M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B 166, 493 (1980).
[18] H. Leutwyler, Nucl. Phys. Proc. Suppl. 94, 108 (2001) [arXiv:hep-ph/0011048].
[19] A. M. Polyakov, Nucl. Phys. B 120, 429 (1977).
[20] G. ’t Hooft, Phys. Rev. Lett. 37, 8 (1976).
[21] E. Witten, Nucl. Phys. B156 (1979) 269.
[22] G. Veneziano, Nucl. Phys. B 159, 213 (1979).
[23] B. Alles, M. D’Elia, A. Di Giacomo, Nucl. Phys. B494 (1997) 281; [hep-lat/9605013].

[24] G. Grunberg, Phys. Rev D30 (1984) 1570.

[25] S. Samuel, Mod. Phys. Lett. A 7, 2007 (1992).

[26] N. J. Dowrick and N. A. McDougall, Nucl. Phys. B399, 426 (1993).

[27] R. D. Peccei and H. R. Quinn, Phys. Rev. Lett. 38, 1440 (1977). R. D. Peccei and H. R. Quinn, Phys. Rev. D 16, 1791 (1977).

[28] J. E. Kim, Phys. Rev. Lett. 43, 103 (1979).

[29] A. R. Zhitnitsky, Sov. J. Nucl. Phys. 31 (1980) 260 [Yad. Fiz. 31 (1980) 497].

[30] M. Dine, W. Fischler and M. Srednicki, Phys. Lett. B104, 199 (1981).

[31] G. G. Raffelt, Ann. Rev. Nucl. Part. Sci. 49, 163 (1999) [arXiv:hep-ph/9903472].

[32] J. Preskill, M. B. Wise and F. Wilczek, Phys. Lett. B 120, 127 (1983).

[33] L. F. Abbott and P. Sikivie, Phys. Lett. B 120, 133 (1983).

[34] M. Dine and W. Fischler, Phys. Lett. B 120, 137 (1983).

[35] R. J. Crewther, Phys. Lett. B70, 349 (1977); Acta Phys. Austriaca Suppl. 19, 47 (1978).

[36] E. Witten, Ann. Phys. 128, 363 (1980).

[37] P. Di Vecchia and G. Veneziano, Nucl. Phys. B171, 253 (1980).

[38] R. Dashen, Phys. Rev. D3, 1879 (1971).

[39] C. Rosenzweig, J. Schechter and C. G. Trahern, Phys. Rev. D21, 3388 (1980).

[40] R. Arnowitt, Pran Nath, Phys. Rev. D23 (1981) 473; Nucl. Phys. B209 (1982) 234, 251;

[41] M. Creutz, Phys. Rev. D52, 2951 (1995); [hep-th/9505112];
A. V. Smilga, Phys. Rev. D59, 114021 (1999) [hep-ph/9805214];
M. H. Tytgat, Phys. Rev. D 61, 114009 (2000) [arXiv:hep-ph/9909532];

[42] M. Shifman, Prog. Part. Nucl. Phys. 39, 1 (1997); [hep-th/9704114];
N. Evans, S. D. Hsu and M. Schwetz, Phys. Lett. B404, 77 (1997); [hep-th/9703197].

[43] E. Witten, Phys. Rev. Lett. 81, 2862 (1998); [hep-th/9807109].
[44] I. Halperin and A. Zhitnitsky, Phys. Rev. D58, 054016 (1998) [hep-ph/9711398]; Mod. Phys. Lett. A13, 1955 (1998) [hep-ph/9707286]; I. Halperin and A. Zhitnitsky, Nucl. Phys. B539, 166 (1999) [hep-th/9802093]; I. Halperin and A. Zhitnitsky, Phys. Rev. Lett. 81, 4071 (1998) [hep-ph/9803301]; A. R. Zhitnitsky, Nucl. Phys. Proc. Suppl. 73 (1999) 647.

[45] M. Shifman, Phys. Rev. D59, 021501 (1999) [hep-th/9809184].

[46] P. Sikivie, Phys. Rev. Lett. 48, 1156 (1982); P. Sikivie, UF-TP-83-6; Based on lectures given at 21st Schladming Winter School, Schladming, Austria, Feb 26 - Mar 6, 1982; M. C. Huang and P. Sikivie, Phys. Rev. D32, 1560 (1985); S. Chang, C. Hagemann and P. Sikivie, Phys. Rev. D59, 023505 (1999) [hep-ph/9807374].

[47] G. Gabadadze and M. Shifman, Phys. Rev. D61, 075014 (2000); [hep-th/9910050].

[48] T. Fugleberg, I. Halperin and A. Zhitnitsky, Phys. Rev. D59, 074023 (1999); [hep-ph/9808469].

[49] I. Halperin and A. Zhitnitsky, Phys. Lett. B440, 77 (1998) [hep-ph/9807333].

[50] M. M. Forbes and A. R. Zhitnitsky, Phys. Rev. Lett. 85, 5268 (2000) [arXiv:hep-ph/0004051].

[51] N. Evans, S. D. Hsu, A. Nyffeler and M. Schwetz, Nucl. Phys. B494, 200 (1997) [hep-ph/9608490].

[52] G. Dvali and M. Shifman, Phys. Lett. B396, 64 (1997); [hep-th/9612128], (E) B407, 4521 (1997).

[53] G. Gabadadze, Nucl. Phys. B552, 194, (1999); [hep-th/9902191].

[54] G. Dvali, G. Gabadadze, Z. Kakushadze, Nucl. Phys. B562, 158 (1999); [hep-th/9901032].

[55] J. Polchinski and M. J. Strassler, [hep-th/0003130]; A. R. Frey, JHEP 0012, 020 (2000) arXiv:hep-th/0007125.

[56] A. Gorsky and M. Shifman, Phys. Rev. D61, 085001 (2000); [hep-th/9909015].

[57] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, Phys. Rev. D55, 5112 (1997) [hep-th/9610043].

[58] G. Gabadadze and Z. Kakushadze, Mod. Phys. Lett. A15, 293 (2000) [hep-th/9905198]; Mod. Phys. Lett. A14, 2151 (1999) [hep-th/9908039].

40
[59] For a review, see:
  A. Giveon and D. Kutasov, Rev. Mod. Phys. 71, 983 (1999) [arXiv:hep-th/9802067].

[60] J. M. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998) [Int. J. Theor. Phys. 38, 1113 (1999)] [arXiv:hep-th/9711200].

[61] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B 428, 105 (1998) [arXiv:hep-th/9802109].

[62] E. Witten, Adv. Theor. Math. Phys. 2, 253 (1998) [arXiv:hep-th/9802150].

[63] E. Witten, Adv. Theor. Math. Phys. 2, 505 (1998) [arXiv:hep-th/9803131].

[64] D. J. Gross and H. Ooguri, Phys. Rev. D 58, 106002 (1998) [arXiv:hep-th/9805129].

[65] C. Csaki, H. Ooguri, Y. Oz and J. Terning, JHEP 9901, 017 (1999) [arXiv:hep-th/9806021];
   R. de Mello Koch, A. Jevicki, M. Mihailescu and J. P. Nunes, Phys. Rev. D 58, 105009 (1998) [arXiv:hep-th/9806125];
   M. Zyskin, Phys. Lett. B 439, 373 (1998) [arXiv:hep-th/9806128].

[66] H. Ooguri, H. Robins and J. Tannenhauser, Phys. Lett. B 437, 77 (1998) [arXiv:hep-th/9806171].

[67] I. R. Klebanov, “From threebranes to large N gauge theories,” [arXiv:hep-th/9901018].

[68] I. R. Klebanov and A. A. Tseytlin, Nucl. Phys. B 546, 155 (1999) [arXiv:hep-th/9811035].

[69] J. A. Minahan, JHEP 9901, 020 (1999) [arXiv:hep-th/9811150].

[70] S. Coleman, “Aspects of Symmetry”, Cambridge University Press, 1985.

[71] E.V. Shuryak, Nucl.Phys. B203 (1982) 93;
   D.I. Diakonov, V.Y. Petrov, Nucl.Phys. B245 (1984) 259;
   T. DeGrand, A. Hazenfratz, T.G. Kovacs, Nucl. Phys. B505 (1997) 417; [hep-lat/9705009].

[72] M. Lüscher, Phys. Lett. 78B (1978) 465.

[73] J. Kogut, L. Susskind, Phys. Rev. D11 (1975) 3594.

[74] J. Schechter, Phys. Rev. D21 (1980) 3393.

[75] A.A. Migdal, M.A. Shifman, Phys. Lett. 114B (1982) 445.
[76] A. Aurilia, Phys. Lett. **81B** (1979) 203.

[77] A. Aurilia, H. Nicolai, P.K. Townsend, Nucl. Phys. **B176** (1980) 509.

[78] G. Gabadadze, Phys. Rev. **D58** (1998) 094015; hep-ph/9710402.

[79] P. Di Vecchia, G. Veneziano, Nucl. Phys. **B171** (1980) 253.

[80] A. Pich, Rept. Prog. Phys. **58** (1995) 563; hep-ph/9502366.

[81] P. Di Vecchia, F. Nicodemi, R. Pettorino, G. Veneziano, Nucl. Phys. **B181** (1981) 318.

[82] L. Del Debbio, H. Panagopoulos and E. Vicari, arXiv:hep-th/0204125.

[83] M. B. Voloshin, I. Yu. Kobzarev, L. B. Okun’, Yad. Fiz. **20** (1974) 1229 (Sov. J. Nucl. Phys. **20** (1975) 644); S. Coleman, Phys. Rev **D15** (1977) 2929; Erratum-ibid. **D16** (1977) 1248.

[84] S. Coleman, Phys. Rev **D15** (1977) 2929; Erratum-ibid. **D16** (1977) 1248; C. Callan, S. Coleman, Phys. Rev. **D16** (1977) 1762.

[85] G. R. Dvali, G. Gabadadze and Z. Kakushadze, Nucl. Phys. B **562**, 158 (1999) arXiv:hep-th/9901032.

[86] I. I. Kogan, A. Kovner and M. Shifman, Phys. Rev. **D57**, 5195 (1998); hep-th/9712040.

[87] G. Gabadadze, Nucl. Phys. **B544**, 650 (1999); hep-th/9808005.

[88] G. Dvali and Z. Kakushadze, Nucl. Phys. **B537**, 297 (1999); hep-th/9807140.