CLASSIFICATION OF SOLUTIONS TO A NONLOCAL EQUATION WITH DOUBLY HARDY-LITTLEWOOD-SOBOLEV CRITICAL EXPONENTS

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ABSTRACT. We consider the following nonlocal critical equation
\[- \Delta u = (I_{\mu_1} \ast |u|^{2^*_{\mu_1}})|u|^{2^*_{\mu_1} - 2} u + (I_{\mu_2} \ast |u|^{2^*_{\mu_2}})|u|^{2^*_{\mu_2} - 2} u, \quad x \in \mathbb{R}^N, \quad (1)\]
where \(0 < \mu_1, \mu_2 < N\) if \(N = 3\) or \(4\), and \(N - 4 \leq \mu_1, \mu_2 < N\) if \(N \geq 5\), \(2^{*}_i := \frac{N + \mu_i}{N - \mu_i} (i = 1, 2)\) is the upper critical exponent with respect to the Hardy-Littlewood-Sobolev inequality, and \(I_{\mu_i}\) is the Riesz potential
\[I_{\mu_i}(x) = \frac{\Gamma(\frac{N - \mu_i}{2})}{\Gamma(\frac{\mu_i}{2})\pi^{\frac{N}{2}} 2^{\mu_i} |x|^{N - \mu_i}}, \quad i = 1, 2,\]
with \(\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx\), \(s > 0\). Firstly, we prove the existence of the solutions of the equation (1). We also establish integrability and \(C^\infty\)-regularity of solutions and obtain the explicit forms of positive solutions via the method of moving spheres in integral forms. Finally, we show that the nondegeneracy of the linearized equation of (1) at \(U_0, V_0\) when \(\max\{\mu_1, \mu_2\} \to 0\) and \(\min\{\mu_1, \mu_2\} \to N\), respectively.

1. Introduction and the main results. In this paper we are concerned with the existence, regularity, classification and nondegeneracy of solutions of the following nonlocal critical equation
\[- \Delta u = (I_{\mu_1} \ast |u|^{2^*_{\mu_1}})|u|^{2^*_{\mu_1} - 2} u + (I_{\mu_2} \ast |u|^{2^*_{\mu_2}})|u|^{2^*_{\mu_2} - 2} u, \quad x \in \mathbb{R}^N, \quad (2)\]
where $0 < \mu_1, \mu_2 < N$ if $N = 3$ or 4, and $N - 4 \leq \mu_1, \mu_2 < N$ if $N \geq 5$, $2^* := \frac{N+\mu_i}{N-2}$ $(i = 1, 2)$ is the upper critical exponent with respect to the Hardy-Littlewood-Sobolev inequality, and $I_{\mu_i}$ is the Riesz potential

$$I_{\mu_i}(x) = \frac{\Gamma\left(\frac{N-\mu_i}{2}\right)}{\Gamma\left(\frac{\mu_i}{2}\right)} \frac{2}{\pi N^2} \int |x|^{N-\mu_i} \frac{dx}{|x|^{\mu_i}}$$

with $\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx$, $s > 0$.

Recently, the following nonlinear Choquard equation

$$-\Delta u + V(x)u = (I_{\mu_1} |u|^p) |u|^{p-2}u, \quad x \in \mathbb{R}^N, \quad (3)$$

has been investigated by many authors. In the case that $N = 3$, $\mu_2 = 2$, $p = 2$ and $V$ is a positive constant, the equation writes as

$$-\Delta u + Vu = (I_2 |u|^2)u, \quad x \in \mathbb{R}^3. \quad (4)$$

If $u$ solves (4) then the function $\psi$ defined by $\psi(t, x) = e^{it}u(x)$ is a solitary wave of the focusing time dependent Hartree equation

$$i\partial_t \psi = -\Delta \psi - (I_2 |\psi|^2)\psi.$$ 

Equation (4) arises as early as in 1954, in a work by Pekar describing the quantum mechanics of a polaron at rest [42]. In 1976, P. Choquard [30] used (4) to describe an electron trapped in its own hole, in an approximation of Hartree-Fock theory for a one-component plasma. In 1996, R. Penrose [37] proposed (4) as a model of self-gravitating matter, in a programme in which quantum state reduction is understood as a gravitational phenomenon.

To study the the uniqueness and nondegeneracy of the solutions for the nonlocal equation (2), we would like to recall some results for semilinear elliptic equation

$$-\Delta u = f(u), \quad x \in \mathbb{R}^N. \quad (5)$$

The uniqueness of positive solutions to equation (5) has been widely studied during the last thirty years. When $N = 3$ and $f(u) = u^3 - u$, it was initiated studied by Coffman in [12]. Later McLeod and Serrin [36] continued the study of Coffman and extended his result to functions with a certain convexity property for $f(u) = u^p - u$ where $1 < p \leq \frac{N}{N-2}$. Subsequently, Kwong [25] consider the following problem

$$-\Delta u + u = u^p, \quad x \in \mathbb{R}^N, \quad (6)$$

and established the uniqueness of the positive, radially symmetric solution by shooting method. For more general nonlinearity, Chen and Lin [7] adopted different from Kwong’s approach to study the uniqueness of equation (5). For other related results, we refer the readers to [14, 17, 18, 24, 40, 43, 45, 51, 52] and the references therein for the uniqueness of ground state solutions, positive radial solutions for semilinear elliptic equation.

In particular, semilinear elliptic equation with the Sobolev critical exponent have been studied extensively in recent years. For $N \geq 3$ and $f(u) = u^{\frac{N+2}{N-2}}$, the equation (5) writes as

$$-\Delta u = u^{\frac{N+2}{N-2}}, \quad x \in \mathbb{R}^N. \quad (7)$$

Classifying the solutions of the equation (7) and the related best Sobolev constant plays an important role in the Yamabe problem, the prescribed scalar curvature problem on Riemannian manifolds and the priori estimates in nonlinear equations. In [20], Gidas, Ni and Nirenberg proved the symmetry and uniqueness of the positive solutions to equation (7) respectively via moving plane method (see also [5]). This
method was initially invented by Alexanderov in [1]. Laterly, Chen et al. [8] and Li [27] simplified the proof of in [5] and [20]. Chen et al. [11] and Wei et at. [49] generalized the classification result to the solutions higher order conformally invariant equation

\[
(-\Delta)^{\frac{s}{2}} u = u^{\frac{N+2}{N-2}}, \quad x \in \mathbb{R}^N.
\]

The method of moving spheres was initially used by Padilla [41], Chen-Li [9] and Li-Zhu [29]. In [28], Li applied the method of moving spheres to establish the same classification result as that in [11]. For some cases, this method is more convenient than the method of moving planes. It can be applied to capture the explicit forms of solutions directly rather than going through the procedure of deriving radial symmetry of solutions and then classifying radial solutions.

The existence and uniqueness of solutions of Choquard equations were studied by some people. However, there is less works since Choquard equations are non-local. When the potential \( V \) is constant, E.H. Lieb [30] established the existence and uniqueness of the ground state of the equation (4) by using rearrangements technique. P.L. Lions [33] obtained the existence of a sequence of radially symmetric solutions of the equation (4) via variational methods. Moroz and Penrose et al. [37, 47] obtained the existence, uniqueness and decay properties of the positive ground state and changing-sign radial solutions of the equation (4) by ODE methods. Wei and Winter [48] proved the ground state is nondegenerate. Later, Ma and Zhao [34] employ the method of moving plane to prove the symmetry of positive radial ground state of the equation (3) with \( V(x) \equiv 1 \) and \( p = 2 \).

If \( V = 0 \) for equation (3), there are only few works deal with the following Choquard equation

\[
-\Delta u = (I_\mu \ast u^p) u^{p-1}, \quad x \in \mathbb{R}^N
\]

with \( p \in (\frac{N+\mu}{N}, \frac{N-\mu}{N-2}) \). Here \( \frac{N+\mu}{N} \) and \( \frac{N+\mu}{N-2} \) are called the lower and the upper critical exponents due to the Hardy-Littlewood-Sobolev inequality respectively (see Remark 1 below). For suitable range of the exponent \( p \), Moroz and Van Schaftingen [38] showed the regularity, positivity, radial symmetry of the ground states and derived the decay asymptotics at infinity of the groundstates. When \( p = \frac{N+\mu}{N} \), Moroz and Van Schaftingen [39] also proved the existence of one nontrivial solution to equation (3) if \( V(x) \) satisfies

\[
\liminf_{|x| \to +\infty} (1 - V(x))|x|^2 > \frac{N^2(N-2)}{4(N+1)}.
\]

As for \( p = \frac{N-\mu}{N+2} \), very recently, Du and Yang [16] or Guo et al. [22] considered independently the following nonlocal critical equation

\[
-\Delta u = (I_\mu \ast u^{2^*}) u^{2^*-1}, \quad x \in \mathbb{R}^N.
\]

The authors established the symmetry of positive solutions (10) using the moving plane method in integral forms, uniqueness of solutions and has the form

\[
u(x) = c \left( \frac{\xi}{\xi^2 + |x-\bar{x}|^2} \right)^{\frac{N-2}{2}},
\]

where \( c, \xi > 0, \bar{x} \in \mathbb{R}^N \).

Once we could establish the uniqueness of positive solutions of the equation, we can also show that the unique positive solution of the corresponding equation is nondegenerate (see (11) below). The uniqueness and nondegeneracy of ground states also play an important role in blow-up analysis for the corresponding standing
wave solutions in the corresponding time-dependent equations. Observe that, Aubin [2], Talenti [46] showed that best Sobolev constant $S$ can be achieved by the radial function

$$U_0(x) := (N(N - 2))^{\frac{N-2}{4}} \left( \frac{t}{t^2 + |x - \xi|^2} \right)^{\frac{N-2}{2}}. \quad (11)$$

Furthermore, Bianchi and Egnell [4], Bartsch, Weth and Willem [3] have proved the equation (7) has a $N + 1$-dimensional manifold of solution given by

$$\bar{Z} := \{ z_t, \xi = U_0(x) | \xi \in \mathbb{R}^N, t > 0 \},$$

which is nondegenerate critical manifold in the sense that the linearization of the equation (7) at $Z \in \bar{Z}$

$$-\Delta \varphi = Z^{\frac{N}{N-2}} \varphi, \quad x \in \mathbb{R}^N,$$

only admits solutions of the form

$$\psi = aD_t Z + b \cdot \nabla Z,$$

where $a \in \mathbb{R}, b \in \mathbb{R}^N$.

When $\mu_1 = \mu_2$ in equation (2), which goes back to equation (10). Particularly, Yang et al. [16, 19] also showed the equation (10) has a $N + 1$-dimensional critical manifold of solution given by

$$\tilde{Z}_\mu := \{ z_t, \xi = S^{\frac{N(2-N)}{4\mu+\mu^+}} (C_{N,\mu}^*)^{\frac{2-N}{2\mu+\mu^-}} U_0(x) | \xi \in \mathbb{R}^N, t > 0 \}$$

satisfying the nondegeneracy condition when $\mu$ is close to 0 or $N$, respectively. Using the approaches from [16, 19], then it is quite natural to ask if we can give unique minimizers for best constant that satisfies equation (2) and prove the nondegeneracy of this unique family of solutions. Unfortunately, we establish neither explicit minimizer $\tilde{U}_\mu(x)$ similar to equation (10) nor the concentration-compactness principle involving the doubly critical Choquard equation. So we need to think about it in a different perspective from equation (10). In addition, Yang et al [16] (see also [22]) showed that, if $(u, v) \in L^{\frac{2N}{N+\mu}}(\mathbb{R}^N) \times L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)$ is a positive solution of equation (10) in such a case, then $u$ and $v$ must be radially symmetric. However, they didn’t give the explicit forms of the solutions.

Inspired by the above literatures, we are led naturally to classify the positive solutions of the nonlinear Choquard equation (2) involving doubly upper critical exponents $2^*_\mu, := \frac{N+N+\mu}{N-2}(i = 1, 2)$ and give a confirm answer to the above question. Firstly we are going to study the existence of positive solutions of equation (2) and establish regularity of the solutions. Furthermore, we will apply the moving spheres method in integral forms to obtain the explicit forms of the positive solutions for the equation (2). Finally, we investigate the nondegeneracy of the linearied equation of (2) at $U_0, V_0$ (see (26) below) when $\mu_1, \mu_2 > 0$ is sufficiently close to 0 or close to $N$.

To understand the critical growth for nonlocal equation (2), we begin with giving sharp information on the best constant of Hardy-Littlewood-Sobolev inequality. This plays an important role in our analysis.

**Proposition 1.** (Hardy-Littlewood-Sobolev inequality [31, 32]) Let $\theta, r > 1$ and $0 < \mu < N$ with $\frac{1}{\theta} + \frac{1}{r} = 1 + \frac{\mu}{N}$. Let $f \in L^\theta(\mathbb{R}^N)$ and $g \in L^r(\mathbb{R}^N)$, there exists a sharp constant $C(\theta, \mu, r)$, independent of $f, g$, such that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^{N-\mu}} \, dx \, dy \leq C(\theta, r, \mu, N) \| f \|_\theta \| g \|_r. \quad (12)$$
If $\theta = r = \frac{2N}{N+\mu}$, then
\[ C(\theta, r, \mu, N) = C(N, \mu) = (\sqrt{\pi})^{N-\mu} \frac{\Gamma(\frac{N}{2})}{\Gamma(\frac{N-\mu}{2})^\mu} \]
and there is equality in (12) if and only if $f \equiv (\text{const.}) g$ and
\[ g(x) = A(\gamma^2 + |x - \tilde{a}|^2)^{-\frac{N+\mu}{2}} \]
for some $A \in \mathbb{C}$, $0 \neq \gamma \in \mathbb{R}$ and $\tilde{a} \in \mathbb{R}^N$.

Remark 1. For $F(\theta) = |\theta|^p$, $p > 0$. By the Hardy-Littlewood-Sobolev inequality,
\[ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(u(x))F(u(y))}{|x-y|^{N-\mu}} \, dx \, dy \]
is well defined if $F(u) \in L^\theta(\mathbb{R}^N)$ for $\theta > 1$ defined by
\[ \frac{2}{\theta} = 1 + \frac{\mu}{N} \].
Thus, for $u \in H^1(\mathbb{R}^N)$, there must hold
\[ \frac{N+\mu}{N} \leq p \leq \frac{N+\mu}{N-2} \].

In the following, we define the best constant $S^*_\mu_1,\mu_2$ corresponding to the double critical exponents $\mu_1$ and $\mu_2$ as
\[ S^*_\mu_1,\mu_2 := \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^{\frac{N-2}{2}} \left( \int_{\mathbb{R}^N} (I_{\mu_1} * |u|^{2\mu_1})|u|^{2\mu_1} \, dx \right)^{\frac{N-2}{2\mu_1}} \left( \int_{\mathbb{R}^N} (I_{\mu_2} * |u|^{2\mu_2})|u|^{2\mu_2} \, dx \right)^{\frac{N-2}{2\mu_2}} \] (13)

The following Lemma implies that $S^*_\mu_1,\mu_2$ is achieved.

Lemma 1.1. The constant $S^*_\mu_1,\mu_2$ defined in (13) is achieved by $u$ if and only if $u$ has the form
\[ u(x) = C \left( \frac{t}{t^2 + |x-\xi|^2} \right)^{\frac{N-2}{2}}, \quad t \in \mathbb{R} \setminus \{0\}, \quad \xi \in \mathbb{R}^N, \]
where $C > 0$ is a fixed constant. Moreover,
\[ S^*_\mu_1,\mu_2 = \left( (C^*_{N,\mu_1})^{\frac{N-2}{\mu_1}} + (C^*_{N,\mu_2})^{\frac{N-2}{\mu_2}} \right)^{-1} S, \quad i = 1, 2, \]
where $C^*_{N,\mu_i} = \left( \frac{\Gamma(N-\mu_i)}{\pi^\frac{N-\mu_i}{2}} \right)^{\frac{2}{\mu_i}} \left( \frac{\Gamma(N)}{\Gamma(\frac{N-\mu_i}{2})} \right)^{\frac{2}{\mu_i}}$.

Proof. Notice that, for all $u \in D^{1,2}(\mathbb{R}^N)$, by the Hardy-Littlewood-Sobolev inequality implies that
\[ \int_{\mathbb{R}^N} (I_{\mu_i} * |u|^{2\mu_i})|u|^{2\mu_i} \, dx = \frac{\Gamma(N-\mu_i)}{\pi^\frac{N-\mu_i}{2}} \frac{\Gamma(\frac{N-\mu_i}{2})}{\Gamma(\frac{N-\mu_i}{2})} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u|^{2\mu_i} |u|^{2\mu_i} \, dx \, dy}{|x-y|^{N-\mu_i}} \]
\[ \leq C^*_{N,\mu_i} \left( \int_{\mathbb{R}^N} |u|^2 \, dx \right)^{\frac{N+\mu_i}{N}} \] (14)
for \( i = 1, 2 \), where \( 2^* := \frac{2N}{N-2} \) is the Sobolev critical exponent. Thus

\[
S^*_{\mu_1, \mu_2} \geq \frac{1}{\left( (C_{N, \mu_1})^{\frac{N-2}{N+\mu_1}} + (C_{N, \mu_2})^{\frac{N-2}{N+\mu_2}} \right) \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left( \int_{\mathbb{R}^N} |u|^{2^*} dx \right)^{\frac{2}{2^*}}}} = \left( (C_{N, \mu_1})^{\frac{N-2}{N+\mu_1}} + (C_{N, \mu_2})^{\frac{N-2}{N+\mu_2}} \right)^{-1} S.
\]

Moreover, there is equality in (5) holds if and only if

\[
\int_{\mathbb{R}^N} (I_{\mu_i} * |u|^{2^*_1})|u|^{2^*_1} dx = C_{N, \mu_i} \left( \int_{\mathbb{R}^N} |u|^{2^*_1} dx \right)^{\frac{N + \mu_i}{N-2}}, \quad i = 1, 2
\]

if and only if

\[
u(x) = C\left( \frac{t}{t^2 + |x - \xi|^2} \right)^{\frac{N-2}{2}}.
\]

Combining this with the definition of \( S^*_{\mu_1, \mu_2} \), we have

\[
S^*_{\mu_1, \mu_2} \leq \frac{1}{\left( (C_{N, \mu_1})^{\frac{N-2}{N+\mu_1}} + (C_{N, \mu_2})^{\frac{N-2}{N+\mu_2}} \right) \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left( \int_{\mathbb{R}^N} |u|^{2^*_2} dx \right)^{\frac{2}{2^*_2}}}} = \left( (C_{N, \mu_1})^{\frac{N-2}{N+\mu_1}} + (C_{N, \mu_2})^{\frac{N-2}{N+\mu_2}} \right)^{-1} S.
\]

From the above argument, we know that \( S^*_{\mu_1, \mu_2} \) is achieved if and only if \( u \) has the form

\[
u(x) = C\left( \frac{t}{t^2 + |x - \xi|^2} \right)^{\frac{N-2}{2}}.
\]

Thus \( S^*_{\mu_1, \mu_2} = \left( (C_{N, \mu_1})^{\frac{N-2}{N+\mu_1}} + (C_{N, \mu_2})^{\frac{N-2}{N+\mu_2}} \right)^{-1} S. \)

According to the fact that \( S^*_{\mu_1, \mu_2} \) is achieved, then we know that there exists some \( u \) which is a minimizer of the best constant \( S^*_{\mu_1, \mu_2} \). Consequently, we are going to study the regularity of solutions for equation (2). Moreover, we need apply moving spheres method to establish explicit forms of positive solutions of equation (2).

Thus, we need to consider the equivalent integral forms of equation (2). Similar as Theorem 4.5 of [11], then we know that equation (2) is equivalent to the following integral equation

\[
u(x) = \int_{\mathbb{R}^N} \frac{1}{|x - y|^{N-2}} \left( \int_{\mathbb{R}^N} \frac{|u(z_1)|^{2^*_1}}{|y - z_1|^{N-\mu_1}} dz_1 \right) |u(y)|^{2^*_1 - 2} u(y) dy + \int_{\mathbb{R}^N} \frac{1}{|x - y|^{N-2}} \left( \int_{\mathbb{R}^N} \frac{|u(z_2)|^{2^*_2}}{|y - z_2|^{N-\mu_2}} dz_2 \right) |u(y)|^{2^*_2 - 2} u(y) dy.
\]

Then we can rewrite the above equality (15) as the following equivalent system

\[
\begin{align*}
\left\{ \begin{array}{l}
u(x) = \int_{\mathbb{R}^N} \frac{v(y)|u(y)|^{2^*_1 - 2} u(y)}{|x - y|^{N-2}} dy + \int_{\mathbb{R}^N} \frac{w(y)|u(y)|^{2^*_2 - 2} u(y)}{|x - y|^{N-2}} dy, \\
v(x) = \int_{\mathbb{R}^N} \frac{|u(y)|^{2^*_1}}{|x - y|^{N-\mu_1}} dy, \\
w(x) = \int_{\mathbb{R}^N} \frac{|u(y)|^{2^*_2}}{|x - y|^{N-\mu_2}} dy.
\end{array} \right.
\end{align*}
\]
Theorem 1.2. Suppose that \( N \geq 3, 0 < \mu_1, \mu_2 < N \). Let \( u \in D^{1,2}(\mathbb{R}^N) \) be a positive solution of (2). Then

(C1) If \( N = 3, 0 < \mu_1, \mu_2 < 2 \), or \( N - 4 < \mu_1, \mu_2 < N \) while \( N \geq 4 \), then \( u \in L^p(\mathbb{R}^N) \) with

\[
p \in \left( \frac{N}{N-2}, +\infty \right).
\]

(C2) If \( N \geq 4, 0 < \mu_1, \mu_2 < \min\{2, N - 4\} \), then \( u \in L^p(\mathbb{R}^N) \) with

\[
p \in \left( \frac{N}{N-2}, \frac{2N}{N-\mu_1-4} \right) \cap \left( \frac{N}{N-2}, \frac{2N}{N-\mu_2-4} \right).
\]

(C3) If \( N = 3, 4, 5, 6, N - 4 < \mu_1, \mu_2 < N \) or \( 0 < \mu_1, \mu_2 < \frac{N-2}{2} \), or \( 2 < \mu_1, \mu_2 < N - 4 \) while \( N \geq 7 \), then \( u \in L^p(\mathbb{R}^N) \) with

\[
p \in \left( \frac{2N}{2N-\mu_1-2}, \frac{2N}{\mu_1-2} \right) \cap \left( \frac{2N}{2N-\mu_2-2}, \frac{2N}{\mu_2-2} \right).
\]

(C4) If \( N = 3, 4, 5, 6, \frac{N-2}{2} < \mu_1, \mu_2 < N \), or \( 2 < \mu_1, \mu_2 < N - 4 \) while \( N \geq 7 \), then \( u \in L^p(\mathbb{R}^N) \) with

\[
p \in \left( \frac{2N}{2N-\mu_1-2}, \frac{2N}{\mu_1-4} \right) \cap \left( \frac{2N}{2N-\mu_2-2}, \frac{2N}{\mu_2-4} \right).
\]

We can establish the integrability and \( C^{\infty} \)-regularity of the solutions, that is

Theorem 1.3. Suppose that \( N = 3, 0 < \mu_1, \mu_2 < 2 \), and \( N - 4 < \mu_1, \mu_2 < N \) while \( N \geq 4 \). Let \( u \in D^{1,2}(\mathbb{R}^N) \) be a positive solution of (2), then \( u \in L^\infty(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N \setminus \{0\}) \).

Next, we state the explicit form of the positive solutions of (2) as follows.

Theorem 1.4. Assume that \( 0 < \mu_1, \mu_2 < N \) if \( N = 3 \) or 4, and \( N - 4 < \mu_1, \mu_2 < 4 \) if \( N \geq 5 \), and let \( u \) be a positive classical solution of (2). Then, \( u(x) \) must has the form

\[
u(x) = \tilde{c}_0 \left( \frac{\xi}{\xi^2 + |x-\bar{x}|^2} \right)^{\frac{N-2}{2}} \]

for some \( \bar{x} > 0 \) and \( \bar{x} \in \mathbb{R}^N \), where \( \tilde{c}_0 \) satisfies

\[
\left[ \frac{2\mu_1+1}{2\mu_2+1} \tilde{c}_0 \frac{N-\mu_1}{\frac{N-2}{2}} I\left( \frac{N-\mu_1}{2} \right) + \frac{2\mu_1+1}{2\mu_2+1} \tilde{c}_0 \frac{N-\mu_2}{\frac{N-2}{2}} I\left( \frac{N-\mu_2}{2} \right) \right] I\left( \frac{N-2}{2} \right) = 1,
\]

and \( I(\tau) = \frac{\pi^{\frac{N-2}{2}}}{\Gamma(\frac{N-2}{2})} \) for \( 0 < \tau < \frac{N}{2} \).
Observe that, it is well known that a radial function $U_0(x)$ solves (7) then $U_0(x)$ achieves the best Sobolev constant for the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$, that is
\[
S := \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\|u\|^2}{\|\nabla u\|^2} = \frac{\|U_0\|^2}{\|\nabla U_0\|^2},
\] (19)
where $\|u\|^2 = \int_{\mathbb{R}^N} |\nabla u|^2 \, dx$. Moreover, for the equation
\[
- \Delta u = (I_\mu * |u|^{2\ast_\mu})|u|^{2\ast_\mu - 2}u, \quad x \in \mathbb{R}^N,
\] (20)
we definition
\[
S^*_{H,L} := \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \, dx}{(\int_{\mathbb{R}^N} (I_\mu * |u|^{2\ast_\mu})|u|^{2\ast_\mu} \, dx)^{\frac{1}{2\ast_\mu}}},
\] (21)
Combining with $\int_{\mathbb{R}^N} |\nabla U_0(x)|^2 \, dx = S^{\ast \frac{N}{2}}$, we know that
\[
\tilde{U}_\mu(x) = S^{\frac{N(2-N)}{4N+2\mu}}(C^*_{N,\mu})^{\frac{2-N}{4N+2\mu}} U_0(x)
\]
solves (20) and $\tilde{U}_\mu(x)$ is the unique minimizer for the constant $S^*_{H,L}$ (see [16]), where
\[
C^*_{N,\mu} = (2\sqrt{\pi})^{-\mu} \frac{\Gamma\left(\frac{\mu}{2}\right)}{\Gamma\left(\frac{N+2}{2}\right)} \left(\frac{\Gamma(N)}{\Gamma\left(\frac{N}{2}\right)}\right)^{\frac{N}{2}}.
\]
Notice that, for fixed $t$ and $\xi$, we have
\[
\tilde{U}_\mu(x) \to U_0 \text{ as } \mu \to 0.
\]
Thus, the limit equation of (20) as $\mu \to 0$ is
\[
- \Delta U_0 = |U_0|^{2\ast - 2}U_0, \quad x \in \mathbb{R}^N.
\] (22)

Obviously, the Euler-Lagrange equation of (2) with respect to $U_0$ is
\[
- \Delta U_0 = S^*_{\mu_1,\mu_2} \left[ (C^*_{N,\mu_1})^{\frac{2\mu_1-2}{N+2\mu_1}} S^{\frac{2\mu_1-2}{2}} (I_{\mu_1} * |U_0|^{2\ast_{\mu_1}})|U_0|^{2\ast_{\mu_1} - 2}U_0 
+ (C^*_{N,\mu_2})^{\frac{2\mu_2-2}{N+2\mu_2}} S^{\frac{2\mu_2-2}{2}} (I_{\mu_2} * |U_0|^{2\ast_{\mu_2}})|U_0|^{2\ast_{\mu_2} - 2}U_0 \right].
\] (23)

Note that, as $\max\{\mu_1, \mu_2\} \to 0$,
\[
C^*_{N,\mu_i} \to 1, \quad i = 1, 2.
\]

Consequently, equation (22) can be seen as the limit case (up to a scaling) of the equation (23) as $\max\{\mu_1, \mu_2\} \to 0$.

Hence, we next result establishes nondegeneracy of linearized equation of (2) at $U_0$ as follows.

**Theorem 1.5.** Suppose $0 < \mu_1, \mu_2 < N$ if $N = 3$ or 4, and $N - 4 < \mu_1, \mu_2 < N$ if $N \geq 5$, let $\max\{\mu_1, \mu_2\} \to 0$, then the linearized equation of (2) at $U_0$, given by
\[
- \Delta \psi - 2\mu_1^* (I_{\mu_1} * (U_0^{2\mu_1-1} \psi)U_0^{2\mu_1 - 1} - (2\mu_1 - 1)(I_{\mu_1} * U_0^{2\mu_1 - 2})U_0^{2\mu_1 - 2} \psi 
- 2\mu_2^* (I_{\mu_2} * (U_0^{2\mu_2-1} \psi)U_0^{2\mu_2 - 1} - (2\mu_2 - 1)(I_{\mu_2} * U_0^{2\mu_2 - 2})U_0^{2\mu_2 - 2} \psi = 0
\] (24)
only admits solutions in $D^{1,2}(\mathbb{R}^N)$ of the form
\[
\psi = aD_1 U_0 + b \cdot \nabla U_0,
\]
where $a \in \mathbb{R}$, $b \in \mathbb{R}^N$. 
On the other hand, because of the presence of the term $\Gamma(\frac{N-2}{2})$ in the coefficient of $I_\mu$, which implies that the functional defined $J_\mu$ of the equation (20) blows up when $\mu \to N$. Thus we need to get rid of this by taking a scaling $\bar{V}_\mu := S_{N,\mu} U_\mu$, where

$$S_{N,\mu} := \left( \frac{\Gamma(\frac{N-\mu}{2})}{\Gamma(\frac{N}{2})\pi^{\frac{N-2}{2}}} \right)^{\frac{1}{N-\mu}} \sim \left( \frac{1}{N-\mu} \right)^{\frac{1}{N-2}}$$

as $\mu \to N$.

Thus, to study the nondegeneracy of linearized equation (20) at $\bar{V}_\mu$, it suffices to study corresponding property of the solutions $\bar{V}_\mu$ for following equation

$$- \Delta v = \left( \frac{1}{|x|^{N-\mu}} \ast |v|^{2^*_\mu} \right) |v|^{2^*_\mu-2} v.$$  (25)

It is easy to see that the corresponding limit equation of (25) as $\mu \to N$ is the following

$$- \Delta v = \left( \int_{\mathbb{R}^N} |v|^{2^*_\mu} dx \right) |v|^{2^*_\mu-2} v, \ x \in \mathbb{R}^N.$$  (26)

Note that, from [19], we know

$$\bar{V}_\mu(x) \to V_0 := S^{\frac{2-N}{N(N+2)}} U_0 \text{ as } \mu \to N.$$  

Thus, we have

$$- \Delta V_0 = \left( \int_{\mathbb{R}^N} |V_0|^{2^*_\mu} dx \right) |V_0|^{2^*_\mu-2} V_0, \ x \in \mathbb{R}^N.$$  (27)

Now, to study nondegeneracy of linearized nonlocal critical equation (2), one only need to study the following equation as $\mu_1, \mu_2 > 0$ sufficiently close to $N$

$$- \Delta v = \left( \frac{1}{|x|^{N-\mu_1}} \ast |v|^{2^*_{\mu_1}} \right) |v|^{2^*_{\mu_1}-2} v + \left( \frac{1}{|x|^{N-\mu_2}} \ast |v|^{2^*_{\mu_2}} \right) |v|^{2^*_{\mu_2}-2} v, \text{ in } \mathbb{R}^N. \ (28)$$

Similarly, we use $\bar{S}^*_{\mu_1, \mu_2}$ denote the best constant defined by

$$\bar{S}^*_{\mu_1, \mu_2} := \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left( \int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_{\mu_1}}|u(y)|^{2^*_{\mu_1}}}{|x-y|^\frac{N-2}{2}} dx dy \right)^{\frac{N-2}{N-\mu_1}} + \left( \int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_{\mu_2}}|u(y)|^{2^*_{\mu_2}}}{|x-y|^\frac{N-2}{2}} dx dy \right)^{\frac{N-2}{N-\mu_2}}}.$$  (29)

Moreover, by Lemma 1.1, $\bar{S}^*_{\mu_1, \mu_2}$ is achieved, and

$$\bar{S}^*_{\mu_1, \mu_2} = \left( (C^*_{N,\mu_1})^{\frac{N-2}{N-\mu_1}} + (C^*_{N,\mu_2})^{\frac{N-2}{N-\mu_2}} \right)^{-1} S.$$  

Obviously, the Euler-Lagrange equation of (28) with respect to $V_0$ is

$$- \Delta V_0 = \bar{S}^*_{\mu_1, \mu_2} \left[ (C^*_{N,\mu_1})^{\frac{N-2}{N-\mu_1}} S^{\frac{\mu_1-2}{N-\mu_1}} \left( \frac{1}{|x|^{N-\mu_1}} \ast |V_0|^{2^*_{\mu_1}} \right) |V_0|^{2^*_{\mu_1}-2} V_0 

+ (C^*_{N,\mu_2})^{\frac{\mu_2-2}{N-\mu_2}} S^{\frac{\mu_2-2}{N-\mu_2}} \left( \frac{1}{|x|^{N-\mu_2}} \ast |V_0|^{2^*_{\mu_2}} \right) |V_0|^{2^*_{\mu_2}-2} V_0 \right].$$  (30)

Indeed, equation (27) also can be seen as the limit case (up to a scaling) of the equation (30) as $\min\{\mu_1, \mu_2\} \to N$.

We first established nondegeneracy of linearized equation of (26) at $V_0$ when $\mu_1, \mu_2$ is sufficiently close to $N$. See [19].
Theorem 1.6. For $0 < \mu_1, \mu_2 < N$, let $\min\{\mu_1, \mu_2\} \rightarrow N$, and $V_0$ be the unique positive solution of (26). Then the linearized equation of (26) at $V_0$, given by

$$- \Delta \psi - 2^* (\int_{\mathbb{R}^N} V_0^{2^* - 1} \psi dx) V_0^{2^* - 1} - (2^* - 1) (\int_{\mathbb{R}^N} V_0^{2^*} dx) V_0^{2^* - 2} \psi = 0$$

only admits solutions in $D^{1,2} (\mathbb{R}^N)$ of the form

$$\psi = a D_1 V_0 + b \cdot \nabla V_0,$$

where $a \in \mathbb{R}$, $b \in \mathbb{R}^N$.

Next we obtain the following result involving the nondegeneracy of linearized equation of (28) at $V_0$ when $\mu_1, \mu_2$ is sufficiently close to $N$.

Theorem 1.7. For $0 < \mu_1, \mu_2 < N$, let $\min\{\mu_1, \mu_2\} \rightarrow N$, then the linearized equation of (28) at $V_0$, given by

$$- \Delta \psi - 2^* \mu_1 \left( \frac{1}{|x|^{N-\mu_1}} \ast (V_0^{2^* \mu_1 - 1} \psi) \right) V_0^{2^* \mu_1 - 1} - (2^* - 1) \left( \frac{1}{|x|^{N-\mu_1}} \ast V_0^{2^* \mu_1} \right) V_0^{2^* - 2} \psi$$

$$- 2^* \mu_2 \left( \frac{1}{|x|^{N-\mu_2}} \ast (V_0^{2^* \mu_2 - 1} \psi) \right) V_0^{2^* \mu_2 - 1} - (2^* - 1) \left( \frac{1}{|x|^{N-\mu_2}} \ast V_0^{2^* \mu_2} \right) V_0^{2^* - 2} \psi = 0$$

only admits solutions in $D^{1,2} (\mathbb{R}^N)$ of the form

$$\psi = a D_1 V_0 + b \cdot \nabla V_0,$$

where $a \in \mathbb{R}$, $b \in \mathbb{R}^N$.

This paper is organized as follows. In section 2, we considered the regularity of solutions to the equivalent integral forms of the equation (2). In section 3, we derive the explicit form of the positive solutions of the equation (2) by the method of moving spheres in integral forms. In section 4, we devoted to complete the proof of the nondegeneracy of linearized equation of the equation (2). In section 5, we give the further remarks and discussions involving nonlocal critical and local critical equation with combined nonlinearities.

2. Regularity. We now consider the regularity of solutions for equation (2). According to (16), we know that equation (2) is equivalent to the following system

$$\begin{align*}
  u(x) &= \int_{\mathbb{R}^N} \frac{v(y) |u(y)|^{2^* \mu_1 - 2} u(y)}{|x-y|^{N-\mu_1}} dy + \int_{\mathbb{R}^N} \frac{w(y) |u(y)|^{2^* \mu_2 - 2} u(y)}{|x-y|^{N-\mu_2}} dy,
  \\
  v(x) &= \int_{\mathbb{R}^N} \frac{|u(y)|^{2^* \mu_1}}{|x-y|^{N-\mu_1}} dy,
  \\
  w(x) &= \int_{\mathbb{R}^N} \frac{|u(y)|^{2^* \mu_2}}{|x-y|^{N-\mu_2}} dy.
\end{align*}$$

To conclude Theorems 1.2-1.3, it suffices to study the regularity of positive solutions to system (33). Consequently, we next discuss the integrability and $C^\infty$ regularity of positive solutions of system (33).

Theorem 2.1. Suppose that $N \geq 3$, $0 < \mu_1, \mu_2 < N$. Let $(u, v, w) \in L^{2N}_{loc} (\mathbb{R}^N) \times L^{2N}_{loc} (\mathbb{R}^N) \times L^{2N}_{loc} (\mathbb{R}^N)$ be a pair of positive solutions of system (33).
(C1). If $N = 3$ and $0 < \mu_1, \mu_2 < 2$, or $N - 4 < \mu_1, \mu_2 < N$ while $N \geq 4$, then $(u, v, w) \in L^p(\mathbb{R}^N) \times L^q(\mathbb{R}^N) \times L^l(\mathbb{R}^N)$ with

$$p \in \left(\frac{2N}{N + 2}, +\infty\right), \quad q \in \left(\frac{2N}{2N - \mu_1 - 2}, \frac{2N}{2 - \mu_1}\right) \quad \text{and} \quad l \in \left(\frac{2N}{2N - \mu_2 - 2}, \frac{2N}{2 - \mu_2}\right).$$

(C2). If $N \geq 4$ and $0 < \mu_1, \mu_2 < \min\{2, N - 4\}$, then $(u, v) \in L^p(\mathbb{R}^N) \times L^q(\mathbb{R}^N) \times L^l(\mathbb{R}^N)$ with

$$p \in \left(\frac{2N}{N + 2}, \frac{2N}{N - \mu_1 - 2}\right) \cap \left(\frac{2N}{N - \mu_2 - 2}, \frac{2N}{2 - \mu_2}\right),$$

$q \in \left(\frac{2N}{2N - \mu_1 - 2}, \frac{2N}{N - \mu_1}\right)$ \quad \text{and} \quad l \in \left(\frac{2N}{2N - \mu_2 - 2}, \frac{2N}{N - \mu_2}\right).$

(C3). If $N = 3, 4, 5, 6, N - 4 < \mu_1, \mu_2 < N$ or $0 < \mu_1, \mu_2 < \frac{N - 2}{2},$ or $2 < \mu_1, \mu_2 < N - 4$ while $N \geq 7$, then $(u, v, w) \in L^p(\mathbb{R}^N) \times L^q(\mathbb{R}^N) \times L^l(\mathbb{R}^N)$ with

$$p \in \left(\frac{2N}{2N - \mu_1 - 2}, \frac{2N}{N - \mu_1}\right) \cap \left(\frac{2N}{2N - \mu_2 - 2}, \frac{2N}{N - \mu_2}\right),$$

$q \in \left(\frac{2N}{N - \mu_1}, +\infty\right)$ \quad \text{and} \quad l \in \left(\frac{2N}{N - \mu_2}, +\infty\right).$

(C4). If $N = 3, 4, 5, 6, \frac{2N}{N - 2} < \mu_1, \mu_2 < N$ or $2 < \mu_1, \mu_2 < N - 4$ while $N \geq 7,$ then $(u, v, w) \in L^p(\mathbb{R}^N) \times L^q(\mathbb{R}^N) \times L^l(\mathbb{R}^N)$ with

$$p \in \left(\frac{2N}{2N - \mu_1 - 2}, \frac{2N}{N - \mu_1}\right) \cap \left(\frac{2N}{2N - \mu_2 - 2}, \frac{2N}{N - \mu_2}\right),$$

$q \in \left(\frac{2N}{N - \mu_1}, \frac{2N}{N - 2\mu_2 - 2}\right)$ \quad \text{and} \quad q \in \left(\frac{2N}{N - \mu_2}, \frac{2N}{N - 2\mu_2 - 2}\right).$

Theorem 2.2. Suppose that $N \geq 3$ and $0 < \mu_1, \mu_2 < N$. Let $(u, v, w) \in L^\frac{2N}{N+2}(\mathbb{R}^N) \times L^\frac{2N}{N-\mu_1}(\mathbb{R}^N) \times L^\frac{2N}{N-\mu_2}(\mathbb{R}^N)$ be a pair of positive solutions of system (33), then $v(x), w(x) \in L^\infty(\mathbb{R}^N)$ and $u(x) \in L^\infty(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N \setminus \{0\}).$

Next, we need the following regularity lifting Lemma [35]. Let $V$ be a topological vector space. Suppose there are two extended norms defined on $V$. $(\|\cdot\|_X, \|\cdot\|_Y : V \to [0, \infty])$. Set

$$X := \{f \in V : \|f\|_X < \infty\}, \quad Y := \{f \in V : \|f\|_Y < \infty\}.$$

Theorem 2.3. Assume that $T$ be a contraction map form $X \to X$ and $Y \to Y$, $f \in X$ and there exists a function $g \in X \cap Y$ such that $f = Tf + g$ in $X$. Then $f \in X \cap Y$.

For any $A > 0$, we define

$$u_A(x) = \begin{cases} u(x), & |u(x)| > A \text{ or } |x| > A, \\ 0, & \text{otherwise}; \end{cases}$$

and $u_B(x) = u(x) - u_A(x)$. Define

$$T_1(r, s, t) = \int_{\mathbb{R}^N} \frac{|u_A(y)|^{2\mu_1 - 1}s(y)}{|x - y|^{N - 2}}dy + \int_{\mathbb{R}^N} \frac{|u_A(y)|^{2\mu_2 - 1}t(y)}{|x - y|^{N - 2}}dy$$

and

$$T_2(r, s, t) = \int_{\mathbb{R}^N} \frac{|u_A(y)|^{2\mu_1 - 1}r(y)}{|x - y|^{N - \mu_1}}dy, \quad T_3(r, s, t) = \int_{\mathbb{R}^N} \frac{|u_A(y)|^{2\mu_2 - 1}r(y)}{|x - y|^{N - \mu_2}}dy.$$
Set

\[
F_{u_B}(x) = \int_{\mathbb{R}^N} \frac{|u_B(y)|^{2^*_p-1}v(y)}{|x-y|^{N-2}}dy + \int_{\mathbb{R}^N} \frac{|u_B(y)|^{2^*_p-1}w(y)}{|x-y|^{N-2}}dy,
\]

\[
G_{u_B}(x) = \int_{\mathbb{R}^N} \frac{|u_B(y)|^{2^*_p-1}u(y)}{|x-y|^{N-\mu_1}}dy \quad \text{and} \quad H_{u_B}(x) = \int_{\mathbb{R}^N} \frac{|u_B(y)|^{2^*_p-1}u(y)}{|x-y|^{N-\mu_2}}dy.
\]

Denote the norm on the product space \(L^p(\mathbb{R}^N) \times L^q(\mathbb{R}^N) \times L^l(\mathbb{R}^N)\) by

\[
\|(r, s, t)\|_{L^p(\mathbb{R}^N) \times L^q(\mathbb{R}^N) \times L^l(\mathbb{R}^N)} = \|r\|_{L^p(\mathbb{R}^N)} + \|s\|_{L^q(\mathbb{R}^N)} + \|t\|_{L^l(\mathbb{R}^N)},
\]

and define an mapping \(T : L^p(\mathbb{R}^N) \times L^q(\mathbb{R}^N) \times L^l(\mathbb{R}^N) \to L^p(\mathbb{R}^N) \times L^q(\mathbb{R}^N) \times L^l(\mathbb{R}^N)\) by

\[
T(r, s, t) = (T_1(r, s, t), T_2(r, s, t), T_3(r, s, t)).
\]

By the above definition and the system (33), then there holds

\[
(u, v, w) = T(u, v, w) + (F_{u_B}(x), G_{u_B}(x), H_{u_B}(x)).
\]

**Lemma 2.4.** Suppose that \(p, q\) satisfy

\[
p \in \left(\frac{N}{N-2} + \infty, \frac{2N}{2N-\mu_1-2}, +\infty\right) \cap \left(\frac{2N}{2N-\mu_2-2}, +\infty\right),
\]

\[2N > p(\mu_i - 2)(i = 1, 2), \quad \frac{1}{p} - \frac{1}{q} = \frac{\mu_1}{2N} \quad \text{and} \quad \frac{1}{p} - \frac{1}{l} = \frac{\mu_2}{2N}.\]

For \(A\) sufficiently large, \(T\) is a contraction map from \(L^p(\mathbb{R}^N) \times L^q(\mathbb{R}^N) \times L^l(\mathbb{R}^N)\) to itself.

**Proof.** For any \((r, s, t) \in L^p(\mathbb{R}^N) \times L^q(\mathbb{R}^N) \times L^l(\mathbb{R}^N),\) using the Hardy-Littlewood-Sobolev inequality with Riesz potential and the Hölder inequality, then for

\[
\frac{Np}{2p+N} > 1, \quad \frac{(N+2p)\mu_i}{2p+N(\mu_i-1)} > 1(i = 1, 2), \quad \frac{Np}{2p+N}(\frac{\mu_1-2}{2N} + \frac{1}{q} + \frac{2}{N}) = 1,
\]

and \(\frac{Np}{2p+N}(\frac{\mu_2-2}{2N} + \frac{1}{l} + \frac{2}{N}) = 1,\)

one has

\[
\|T_1(r, s, t)\|_{L^p(\mathbb{R}^N)} = \left\| \int_{\mathbb{R}^N} \frac{|u_A(y)|^{2^*_p-1}s(y)}{|x-y|^{N-2}}dy + \int_{\mathbb{R}^N} \frac{|u_A(y)|^{2^*_p-1}t(y)}{|x-y|^{N-2}}dy \right\|_{L^p(\mathbb{R}^N)}
\]

\[
\leq C_1\|u_A(y)|^{2^*_p-1}s\|_{L^{2^*_p}(\mathbb{R}^N)} + C_2\|u_A(y)|^{2^*_p-1}t\|_{L^{2^*_p}(\mathbb{R}^N)}
\]

\[
\leq C_1\|u_A\|_{L^{2^*_p}(\mathbb{R}^N)}\|s\|_{L^q(\mathbb{R}^N)} + C_2\|u_A\|_{L^{2^*_p}(\mathbb{R}^N)}\|t\|_{L^l(\mathbb{R}^N)},
\]

(34)

and for \(\frac{2Np}{2N+p(\mu_i+2)} > 1,\)

\[
\|T_2(r, s, t)\|_{L^q(\mathbb{R}^N)} = \left\| \int_{\mathbb{R}^N} \frac{|u_A(y)|^{2^*_p-1}r(y)}{|x-y|^{N-\mu_1}}dy \right\|_{L^q(\mathbb{R}^N)}
\]

\[
\leq C_3\|u_A\|_{L^{2^*_p}(\mathbb{R}^N)}\|r\|_{L^{\frac{2Np}{2N+p(\mu_i+2)}}(\mathbb{R}^N)},
\]

(35)

\[
\leq C_3\|u_A\|_{L^{2^*_p}(\mathbb{R}^N)}\|r\|_{L^p(\mathbb{R}^N)}.
\]
Similarly, for $\frac{2Np}{2N+p(\mu_2+2)} > 1$, 
\[
\|T_3(r, s, t)\|_{L^1(\mathbb{R}^N)} = \left\| \frac{|u_A(y)|^{2\nu_2-1}r(y)}{|x-y|^{N-\mu_2}} \right\|_{L^1(\mathbb{R}^N)}
\leq C_4 \|u_A\|_{L^{\frac{2Np}{2N+p(\mu_2+2)}}(\mathbb{R}^N)}^{2\nu_2-1} \|r\|_{L^{\frac{2Np}{2N+p(\mu_2+2)}}(\mathbb{R}^N)} 
\leq C_4 \|u_A\|_{L^{2^*}(\mathbb{R}^N)}^{2\nu_2-1} \|r\|_{L^p(\mathbb{R}^N)},
\] 
(36)

Since $u \in L^{2^*}(\mathbb{R}^N)$, we may choose sufficiently large $A$ and constant $\tilde{C}_1 = \max\{C_1, C_3\}, \tilde{C}_2 = \max\{C_2, C_4\}$ such that
\[
\tilde{C}_1 \|u_A\|_{L^{2^*}(\mathbb{R}^N)} \leq \frac{1}{4} \quad \text{and} \quad \tilde{C}_2 \|u_A\|_{L^{2^*}(\mathbb{R}^N)} \leq \frac{1}{4}.
\]

Hence, combining (34) with (36), we obtain 
\[
\|T(r, s, t)\|_{L^p \times L^s \times L^t} = \|T_1(r, s, t)\|_{L^p} + \|T_2(r, s, t)\|_{L^s} + \|T_3(r, s, t)\|_{L^t}
\leq 1 - 2(\|r\|_{L^p(\mathbb{R}^N)} + \|s\|_{L^s(\mathbb{R}^N)} + \|t\|_{L^t(\mathbb{R}^N)}),
\]
which implies that $T$ is a contraction map from $L^p(\mathbb{R}^N) \times L^s(\mathbb{R}^N) \times L^t(\mathbb{R}^N)$ to itself.

**Proof of Theorem 2.1.** By Theorem 2.3, we know that, for $A$ is sufficiently large, $T$ is a contraction map from $L^p(\mathbb{R}^N) \times L^s(\mathbb{R}^N) \times L^t(\mathbb{R}^N)$ to itself. Next we will prove that $(F_{a_B}, G_{a_B}, H_{a_B}) \in L^p(\mathbb{R}^N) \times L^s(\mathbb{R}^N) \times L^t(\mathbb{R}^N)$. We consider estimate of $F_{a_B}, G_{a_B}$ and $H_{a_B}$. By Sobolev embedding $u \in L^{\frac{2N}{N-2}}(\mathbb{R}^N)$, and Hardy-Littlewood-Sobolev inequality, we know that $v \in L^{\frac{2N}{2N-2}}(\mathbb{R}^N), w \in L^{\frac{2N}{2N-2}}(\mathbb{R}^N)$. Note that by the Hardy-Littlewood-Sobolev inequality with Riesz potential and the Hölder inequality, we have 
\[
\|F_{a_B}(x)\|_{L^p(\mathbb{R}^N)} = \left\| \int_{\mathbb{R}^N} \frac{|u_B(y)|^{2\nu_1-1}v(y)}{|x-y|^{N-\mu_1}} dy \right\|_{L^p(\mathbb{R}^N)} 
\leq C_1 \|u_B(y)|^{2\nu_1-1}|v\|_{L^{\frac{2Np}{2N+p(\mu_1+2)}}(\mathbb{R}^N)} + C_2 \|u_B(y)|^{2\nu_1-1}|w\|_{L^{\frac{2Np}{2N+p(\mu_1+2)}}(\mathbb{R}^N)} 
\leq C_1 \|u_B(y)|^{2\nu_1-1}\|_{L^{\frac{2Np}{2N+p(\mu_1+2)}}(\mathbb{R}^N)} \|v\|_{L^{\frac{2N}{2N-2}}(\mathbb{R}^N)} 
+ C_2 \|u_B(y)|^{2\nu_1-1}\|_{L^{\frac{2Np}{2N+p(\mu_1+2)}}(\mathbb{R}^N)} \|w\|_{L^{\frac{2N}{2N-2}}(\mathbb{R}^N)},
\]
and for $2N + p(\mu_1 + 2) > p(N - 2)$, 
\[
\|G_{a_B}(x)\|_{L^s(\mathbb{R}^N)} = \left\| \int_{\mathbb{R}^N} \frac{|u_B(y)|^{2\nu_1-1}u(y)}{|x-y|^{N-\mu_2}} dy \right\|_{L^s(\mathbb{R}^N)} 
\leq C_3 \|u_B(y)|^{2\nu_1-1}\|_{L^{\frac{2Np}{2N+p(\mu_1+2)}}(\mathbb{R}^N)} \|u\|_{L^{\frac{2N}{2N-2}}(\mathbb{R}^N)}.
\]
Therefore, it follows from Theorem 2.3 that, if \( p > \frac{N}{N - 2} \), \( p > \frac{2N}{2N - \mu_1 - 2} \), \( 2N > p(\mu_1 + 2) > p(N - 2) \),

\[
\|H_{u,v}(x)\|_{L^r(\mathbb{R}^N)} = \left\| \int_{\mathbb{R}^N} \frac{|u_B(y)|^{2\mu_2 - 1}u(y)}{|x-y|^{N-\mu_2}}dy \right\|_{L^r(\mathbb{R}^N)}
\leq C_3\|u_B^{2\mu_2 - 1}u\|_{L^{2Np/(N + 2\mu_2 + 2)}(\mathbb{R}^N)}
\leq C_3\|u_B^{2\mu_2 - 1}\|_{L^{2Np/(N + 2\mu_2 + 2)}(\mathbb{R}^N)}\|u\|_{L^{2N/(N+2\mu_2)}(\mathbb{R}^N)}.
\]

Therefore, it follows from Theorem 2.3 that, if \( p \) and \( q \) satisfy

\[
p > \frac{N}{N - 2}, \quad p > \frac{2N}{2N - \mu_1 - 2}, \quad 2N > p(\mu_1 + 2) > p(N - 2),
\]

\((u, v, w) \in (L^{2N/(N - 2)}(\mathbb{R}^N) \times L^{2N/(N - 2)}(\mathbb{R}^N) \times L^{2N/(N - 2)}(\mathbb{R}^N)) \cap (L^p(\mathbb{R}^N) \times L^q(\mathbb{R}^N) \times L^r(\mathbb{R}^N)).\]

Next, if \( \frac{1}{p} = \frac{1}{q} = \frac{\mu_1 - 2}{2N} \) and \( \frac{1}{l} = \frac{\mu_2 - 2}{2N} \), then we have the following classification of the regularity lifting for any pair of solution \((u, v, w)\) of system (33):

\((C_1)\) If \( N = 3, 0 < \mu_1, \mu_2 < 2, N - 4 < \mu_1, \mu_2 < N \) while \( N \geq 4 \), then \((u, v, w) \in L^p(\mathbb{R}^N) \times L^q(\mathbb{R}^N) \times L^r(\mathbb{R}^N)\) with

\[
p \in \left( \frac{N}{N - 2}, +\infty \right), \quad q \in \left( \frac{2N}{2N - \mu_1 - 2}, \frac{2N}{2N - \mu_2 - 2} \right), \quad l \in \left( \frac{2N}{2N - \mu_2 - 2}, \frac{2N}{2N - \mu_2 - 2} \right).
\]

\((C_2)\) If \( N \geq 4, 0 < \mu_1, \mu_2 < \min\{2, N - 4\} \), then \((u, v) \in L^p(\mathbb{R}^N) \times L^q(\mathbb{R}^N) \times L^r(\mathbb{R}^N)\) with

\[
p \in \left( \frac{N}{N - 2}, \frac{2N}{N - \mu_1 - 4} \right) \cap \left( \frac{N}{N - 2}, \frac{2N}{N - \mu_2 - 4} \right), \quad q \in \left( \frac{2N}{2N - \mu_1 - 2}, \frac{2N}{2N - \mu_2 - 2} \right) \cap \left( \frac{2N}{2N - \mu_2 - 2}, \frac{2N}{2N - \mu_2 - 2} \right).
\]

\((C_3)\) If \( N = 3, 4, 5, 6, N - 4 < \mu_1, \mu_2 < N \) or \( 0 < \mu_1, \mu_3 < \frac{N - 2}{2}, \) and \( 2 < \mu_1, \mu_2 < N - 4 \) while \( N \geq 7 \), then \((u, v, w) \in L^p(\mathbb{R}^N) \times L^q(\mathbb{R}^N) \times L^r(\mathbb{R}^N)\) with

\[
p \in \left( \frac{2N}{2N - \mu_1 - 2}, \frac{2N}{\mu_1 - 2} \right) \cap \left( \frac{2N}{2N - \mu_2 - 2}, \frac{2N}{\mu_2 - 2} \right), \quad q \in \left( \frac{N}{N - \mu_1}, +\infty \right) \text{ and } l \in \left( \frac{N}{N - \mu_2}, +\infty \right).
\]

\((C_4)\) If \( N = 3, 4, 5, 6, N - 2 < \mu_1, \mu_2 < N \) and \( 2 < \mu_1, \mu_2 < N - 4 \) while \( N \geq 7 \), then \((u, v, w) \in L^p(\mathbb{R}^N) \times L^q(\mathbb{R}^N) \times L^r(\mathbb{R}^N)\) with

\[
p \in \left( \frac{2N}{2N - \mu_1 - 2}, \frac{2N}{N - \mu_1 - 4} \right) \cap \left( \frac{2N}{2N - \mu_2 - 2}, \frac{2N}{N - \mu_2 - 4} \right), \quad q \in \left( \frac{2N}{N - \mu_1}, \frac{2N}{N - 2\mu_1 - 2} \right) \text{ and } q \in \left( \frac{2N}{N - \mu_2}, \frac{2N}{N - 2\mu_2 - 2} \right).
\]

Since the above integrability results of system (33) are equivalent to Theorem 1.2, thus completed the proof of Theorem 1.2.

**Proof of Theorem 2.2.** Firstly, we prove that \( v(x), w(x) \in L^\infty(\mathbb{R}^N) \). By (33), then for any \( r > 0 \), we have

\[
v(x) = \int_{\mathbb{R}^N} \frac{|u(y)|^{2\mu_2 - 1}u(y)}{|x-y|^{N-\mu_2}}dy \leq \int_{B_r(0)} \frac{|u(y)|^{2\mu_2}u(y)}{|x-y|^{N-\mu_2}}dy + \int_{\mathbb{R}^N \setminus B_r(0)} \frac{|u(y)|^{2\mu_2}u(y)}{|x-y|^{N-\mu_2}}dy.
\]
Thus, we can get that
\[ x \text{ and integrability imply that } x \]
On the one hand, for \( x \in \mathbb{R}^N \setminus B_r(0) \), we have
\[ \int_{B_r(0)} \frac{|u(y)|^{2^{*}_1-1}}{|x-y|^{N-2}} dy < \infty, \]
Thus, we can get that
\[ v(x) \in L^\infty(\mathbb{R}^N). \]  
(37)

Similar to the above argument, we have
\[ w(x) \in L^\infty(\mathbb{R}^N). \]  
(38)

Secondly, we prove that \( u(x) \in L^\infty(\mathbb{R}^N) \). By (33) and (38), for any \( r > 0 \), we have
\[ |u(x)| \leq \int_{\mathbb{R}^N} \frac{|u(y)||u(y)|^{2^{*}_1-1}}{|x-y|^{N-2}} dy + \int_{\mathbb{R}^N} \frac{|w(y)||u(y)|^{2^{*}_2-1}}{|x-y|^{N-2}} dy \]
\[ \leq \|v(y)\|_{L^\infty(\mathbb{R}^N)} \left\{ \int_{B_r(0)} \frac{|u(y)|^{2^{*}_1-1}}{|x-y|^{N-2}} dy + \int_{\mathbb{R}^N \setminus B_r(0)} \frac{|u(y)|^{2^{*}_1-1}}{|x-y|^{N-2}} dy \right\} \]
\[ + \|w(y)\|_{L^\infty(\mathbb{R}^N)} \left\{ \int_{B_r(0)} \frac{|u(y)|^{2^{*}_2-1}}{|x-y|^{N-2}} dy + \int_{\mathbb{R}^N \setminus B_r(0)} \frac{|u(y)|^{2^{*}_2-1}}{|x-y|^{N-2}} dy \right\}. \]
On the one hand, for \( x \in \mathbb{R}^N \setminus B_{2r}(0) \) and \( 1 < k < \frac{N}{N-2} \), by the Hölder inequality and integrability imply that
\[ \int_{B_r(0)} \frac{|u(y)|^{2^{*}_1-1}}{|x-y|^{N-2}} dy \leq \|u\|_{L^{(2^{*}_1-1)k}(\mathbb{R}^N)} \int_{B_r(0)} \frac{1}{|y|^{(N-2)k}} dy < \infty, \]
Moreover, for \( x \in B_{2r}(0) \), we have
\[ \int_{B_r(0)} \frac{|u(y)|^{2^{*}_1-1}}{|x-y|^{N-2}} dy < \infty. \]
On the other hand, combined the Hölder inequality with integrability, we have
\[ \int_{\mathbb{R}^N \setminus B_r(0)} \frac{|u(y)|^{2^{*}_1-1}}{|x-y|^{N-2}} dy = \int_{(\mathbb{R}^N \setminus B_r(0)) \cap B_s(0)} \frac{|u(y)|^{2^{*}_1-1}}{|x-y|^{N-2}} dy + \int_{(\mathbb{R}^N \setminus B_s(0)) \cap (\mathbb{R}^N \setminus B_r(0))} \frac{|u(y)|^{2^{*}_1-1}}{|x-y|^{N-2}} dy < \infty. \]
Similarly, we have
\[ \int_{B_r(0)} \frac{|u(y)|^{2^{*}_2-1}}{|x-y|^{N-2}} dy + \int_{\mathbb{R}^N \setminus B_r(0)} \frac{|u(y)|^{2^{*}_2-1}}{|x-y|^{N-2}} dy < \infty. \]
Therefore, combining with the above argument, we know
\[ u(x) \in L^\infty(\mathbb{R}^N). \]
Now, we prove that \( u(x) \in C^\infty(\mathbb{R}^N \setminus \{0\}) \). For any \( x \in \mathbb{R}^N \setminus \{0\} \), by (33), we know
\[ u(x) = \int_{\mathbb{R}^N} \frac{v(y)|u(y)|^{2^{*}_1-2}u(y)}{|x-y|^{N-2}} dy + \int_{\mathbb{R}^N} \frac{w(y)|u(y)|^{2^{*}_2-2}u(y)}{|x-y|^{N-2}} dy. \]
Notice that, by [32], there exists \( r < \frac{|x|}{2} \) and for any \( \delta < 2 \) such that
\[
\int_{B_{2r}(x)} \frac{v(y)u(y)^{2r_1-2}u(y)}{|x-y|^{N-2}} \, dy + \int_{B_{2r}(x)} \frac{w(y)u(y)^{2r_2-2}u(y)}{|x-y|^{N-2}} \, dy \in C^\delta(\mathbb{R}^N \setminus \{0\}).
\]  
(39)
Next, we need to show that
\[
\int_{\mathbb{R}^N \setminus B_{2r}(x)} \frac{v(y)u(y)^{2r_1-2}u(y)}{|x-y|^{N-2}} \, dy + \int_{\mathbb{R}^N \setminus B_{2r}(x)} \frac{w(y)u(y)^{2r_2-2}u(y)}{|x-y|^{N-2}} \, dy \in C^\infty(\mathbb{R}^N \setminus \{0\}).
\]

Firstly, we denote by
\[
\varphi(x) = \int_{\mathbb{R}^N} \frac{v(y)u(y)^{2r_1-2}u(y)}{|x-y|^{N-2}} \chi_{(\mathbb{R}^N \setminus B_{2r}(x))} \, dy + \int_{\mathbb{R}^N} \frac{w(y)u(y)^{2r_2-2}u(y)}{|x-y|^{N-2}} \chi_{(\mathbb{R}^N \setminus B_{2r}(x))} \, dy,
\]
similar to the argument of [15], we can claim that \( \varphi(x) \in C^1(\mathbb{R}^N \setminus \{0\}) \). Continuing this process, we can improve \( \varphi(x) \in C^\infty(\mathbb{R}^N \setminus \{0\}) \). Combining with (39), we obtain \( u(x) \in C^\infty(\mathbb{R}^N \setminus \{0\}) \). Finally, by the classical bootstrap technique [32], we deduce \( u(x) \in C^\infty(\mathbb{R}^N \setminus \{0\}) \). The above \( C^\infty \) regularity and integrability is equivalent to Theorem 1.3 for equation (2). Completed the proof.

3. Uniqueness. We will employ the method of moving spheres in integral forms to prove Theorem 1.4, that is, it will give explicit form of positive solutions equation (2). This method is different from the moving planes method in integral forms. Thus it is not necessary to prove the symmetry of solutions beforehand as in the method of moving planes. To apply the method of moving spheres, we first introduce the following notations. Let \( x_0 \) be any point in \( \mathbb{R}^N \). For \( x \in B_\lambda(x_0) \), we denote by
\[
x^\lambda = \frac{\lambda^2|x - x_0|^2}{|x - x_0|^{2}} + x_0,
\]
the inversion point of \( x \) about the sphere \( S_\lambda(x_0) \equiv \{x| |x - x_0| = \lambda \} \). We define the Kelvin transform of the function \( u, v, w \) centered at \( x_0 \) as follows,
\[
u_\lambda(x) = \left( \frac{\lambda}{|x - x_0|} \right)^{N-2} u(x^\lambda), \quad v_\lambda(x) = \left( \frac{\lambda}{|x - x_0|} \right)^{N-2} v(x^\lambda),
\]
\[
w_\lambda(x) = \left( \frac{\lambda}{|x - x_0|} \right)^{N-2} w(x^\lambda).
\]
Without loss of generality, here we take \( x_0 = 0 \). Thus, we will write
\[
x^\lambda = \frac{\lambda^2x}{|x|^2}, \quad u_\lambda(x) = \left( \frac{\lambda}{|x|} \right)^{N-2} u(x^\lambda), \quad v_\lambda(x) = \left( \frac{\lambda}{|x|} \right)^{N-2} v(x^\lambda),
\]
\[
w_\lambda(x) = \left( \frac{\lambda}{|x|} \right)^{N-2} w(x^\lambda), \quad \text{for each } \lambda > 0,
\]
and we also set
\[
B^u_\lambda = \{x \in B_\lambda \setminus \{0\} : u(x) > u_\lambda(x)\}; \quad B^v_\lambda = \{x \in B_\lambda \setminus \{0\} : v(x) > v_\lambda(x)\};
\]
\[
B^w_\lambda = \{x \in B_\lambda \setminus \{0\} : w(x) > w_\lambda(x)\}.
\]
Note that, for all \( x, y \in \mathbb{R}^N \setminus \{0\} \), we know
\[
dx^\lambda = \left( \frac{\lambda}{|y|} \right)^{2N} dy, \quad \left( \frac{|x|}{\lambda} \right) \left( \frac{|y|}{\lambda} \right) |x^\lambda - y^\lambda| = |x - y|.
\]  
(40)
Next, we have the following Lemma.

**Lemma 3.1.** For each positive solution \((u, v, w)\) of system \((16)\), we have

\[
u(x) - u_\lambda(x) = \int_{B_\lambda} \frac{1}{|x-y|^{N-2}} - \frac{|y|}{|x-y|^{N-2}} \left[ \left( |u(y)|^{2\mu_1-2}u(y) - \left( \frac{\lambda}{|y|} \right)^\alpha |u_\lambda(y)|^{2\mu_1-2}u_\lambda(y) \right) + \left( |u(y)|^{2\mu_2-2}w(y) - \left( \frac{\lambda}{|y|} \right)^\beta |u_\lambda(y)|^{2\mu_2-2}w_\lambda(y) \right) \right] dy,
\]

where \(\alpha = N + \mu_1 - 2\mu_1 (N-2)\), \(\beta = N + \mu_2 - 2\mu_2 (N-2)\).

**Proof.** By \((16)\) and \((40)\), we have

\[
u(x) = \int_{B_\lambda} \frac{v(y)|u(y)|^{2\mu_1-2}u(y)}{|x-y|^{N-2}} dy + \int_{B_\lambda} \frac{w(y)|u(y)|^{2\mu_2-2}u(y)}{|x-y|^{N-2}} dy
\]

\[
+ \int_{B_\lambda^c} \frac{v(y)|u(y)|^{2\mu_1-2}u(y)}{|x-y|^{N-2}} dy + \int_{B_\lambda^c} \frac{w(y)|u(y)|^{2\mu_2-2}u(y)}{|x-y|^{N-2}} dy
\]

\[
= \int_{B_\lambda} \frac{v(y)|u(y)|^{2\mu_1-2}u(y)}{|x-y|^{N-2}} \left( \frac{\lambda}{|y|} \right)^\alpha |u_\lambda(y)|^{2\mu_1-2}u_\lambda(y) v_\lambda(y) \right) dy
\]

\[
+ \int_{B_\lambda} \frac{w(y)|u(y)|^{2\mu_2-2}u(y)}{|x-y|^{N-2}} \left( \frac{\lambda}{|y|} \right)^\beta |u_\lambda(y)|^{2\mu_2-2}u_\lambda(y) w_\lambda(y) \right) dy,
\]

which leads to

\[
u_\lambda(x) = \int_{B_\lambda} \frac{\left( \frac{\lambda}{|y|} \right)^\alpha v_\lambda(y)|u_\lambda(y)|^{2\mu_1-2}u_\lambda(y)}{|x-y|^{N-2}} + \frac{|u(y)|^{2\mu_1-2}u(y)v(y)}{|x-y|^{N-2}} dy
\]

\[
+ \int_{B_\lambda} \frac{\left( \frac{\lambda}{|y|} \right)^\beta w_\lambda(y)|u_\lambda(y)|^{2\mu_2-2}u_\lambda(y)}{|x-y|^{N-2}} + \frac{|u(y)|^{2\mu_2-2}u(y)w(y)}{|x-y|^{N-2}} dy.
\]

From \((41)\) and \((42)\), we then get first identity. By a similar argument, we can also prove others.

**Step 1.** We claim that for \(\lambda\) sufficiently small, we have

\[
u(x) \leq u_\lambda(x), \ v(x) \leq v_\lambda(x) \text{ and } w(x) \leq w_\lambda(x), \text{ for all } x \in B_\lambda \setminus \{0\}.
\]

For completing the proof, we need the following lemma.
Lemma 3.2. For any \( x \in B^u_\lambda \), there exists a constant \( C > 0 \) such that
\[
\| u(x) - u_\lambda(x) \|_{L^{2^*}(B^u_\lambda)} 
\leq C \left[ \| u \|_{L^{\frac{2^*_\lambda}{N-2}}(B^u_\lambda)}^{2^*_\lambda-2} \| v \|_{L^{\frac{2N}{N+2}}(B^u_\lambda)} + \| u \|_{L^{2^*_\lambda}(B^u_\lambda)}^{2^*_\lambda-1} \| u \|_{L^{2^*_\lambda}(B^u_\lambda)}^{2^*_\lambda-1} 
+ \| u \|_{L^{\frac{2^*_\mu}{N-2}}(B^u_\lambda)}^{2^*_\mu-2} \| v \|_{L^{\frac{2N}{N+2}}(B^u_\lambda)} + \| u \|_{L^{2^*_\mu}(B^u_\lambda)}^{2^*_\mu-1} \| u \|_{L^{2^*_\mu}(B^u_\lambda)}^{2^*_\mu-1} \right] \| u - u_\lambda \|_{L^{2^*}(B^u_\lambda)}.
\]

Proof. By Lemma 3.1, we have
\[
\begin{aligned}
&u(x) - u_\lambda(x) \leq \int_{B^u_\lambda} \left( \frac{1}{|x - y|^{N-2}} - \frac{1}{|x - \lambda y|^{N-2}} \right) \left[ |u(y)|^{2^*_\lambda-1} v(y) 
- |u_\lambda(y)|^{2^*_\lambda-1} v_\lambda(y) \right] dy 
+ \int_{B^u_\lambda \setminus \{ u^* \lambda - 1 > u_\lambda^* \lambda - 1 \}} |x - y|^{N-2} \left( |u|^2 - |u_\lambda|^2 \right) dy 
\leq (2^*_\mu - 1) \int_{B^u_\lambda} \frac{w u^{2^*_\lambda-2}(u - u_\lambda)}{|x - y|^{N-2}} dy + \int_{B^u_\lambda} \frac{u^{2^*_\lambda-1}(v - v_\lambda)}{|x - y|^{N-2}} dy 
+ (2^*_\mu - 1) \int_{B^u_\lambda} \frac{w u^{2^*_\mu-2}(u - u_\lambda)}{|x - y|^{N-2}} dy + \int_{B^u_\lambda} \frac{u^{2^*_\mu-1}(w - w_\lambda)}{|x - y|^{N-2}} dy.
\end{aligned}
\]

Hence, by Hölder inequality and the Hardy-Littlewood-Sobolev inequality [16], we have
\[
\begin{aligned}
\left( \frac{2^*_\mu}{N-2} \int_{B^u_\lambda} \frac{w u^{2^*_\mu-2}(u - u_\lambda)}{|x - y|^{N-2}} dy \right)_{L^{2^*_\lambda}(B^u_\lambda)} 
\leq C \| u \|_{L^{\frac{2^*_\mu}{N-2}}(B^u_\lambda)}^{2^*_\mu-2} \| v \|_{L^{\frac{2N}{N+2}}(B^u_\lambda)} + \| u \|_{L^{2^*_\lambda}(B^u_\lambda)}^{2^*_\lambda-1} \| u \|_{L^{2^*_\lambda}(B^u_\lambda)}^{2^*_\lambda-1} 
\end{aligned}
\] (45)
and
\[
\begin{aligned}
\left( \int_{B^u_\lambda} \frac{u^{2^*_\mu-1}(u - u_\lambda)}{|x - y|^{N-2}} dy \right)_{L^{2^*_\mu}(B^u_\lambda)} 
\leq C \| u \|_{L^{\frac{2^*_\mu}{N-2}}(B^u_\lambda)}^{2^*_\mu-2} \| u \|_{L^{\frac{2N}{N+2}}(B^u_\lambda)} + \| u \|_{L^{2^*_\mu}(B^u_\lambda)}^{2^*_\mu-1} \| u \|_{L^{2^*_\mu}(B^u_\lambda)}^{2^*_\mu-1} \| u - u_\lambda \|_{L^{2^*}(B^u_\lambda)}.\n\end{aligned}
\] (46)

Similarly, we have
\[
\begin{aligned}
\left( \frac{2^*_\mu}{N-2} \int_{B^u_\lambda} \frac{w u^{2^*_\mu-2}(u - u_\lambda)}{|x - y|^{N-2}} dy \right)_{L^{2^*_\mu}(B^u_\lambda)} + \int_{B^u_\lambda} \frac{u^{2^*_\mu-1}(u - u_\lambda)}{|x - y|^{N-2}} dy \right)_{L^{2^*}(B^u_\lambda)} 
\leq C \left[ \| u \|_{L^{\frac{2^*_\mu}{N-2}}(B^u_\lambda)}^{2^*_\mu-2} \| v \|_{L^{\frac{2N}{N+2}}(B^u_\lambda)} + \| u \|_{L^{2^*_\mu}(B^u_\lambda)}^{2^*_\mu-1} \| u \|_{L^{2^*_\mu}(B^u_\lambda)}^{2^*_\mu-1} \right] \| u - u_\lambda \|_{L^{2^*}(B^u_\lambda)}.
\end{aligned}
\] (47)

Thus, combining with (45)-(47), we can complete the proof.

Now we show that the claim (43). According to Lemma 3.2 and \( u, v \) and \( w \) are integrable, there exists \( \varepsilon \in (0, 1) \) small enough, such that for \( \lambda > 0 \) sufficient small,
we have

\[ C \left[ \|u\|_{L^{2^*_N}(B_0^N)}^{2^*_N-2} \|v\|_{L^\frac{2N}{N-2p}+}(B_0^N) + \|u\|_{L^{2^*_N}(B_0^N)}^{2^*_N-1} \|u\|_{L^2(B_0^N)}^{2^*_N-1} \right] \\
+ \|u\|_{L^{2^*_N}(B_0^N)}^{2^*_N-2} \|v\|_{L^\frac{2N}{N-2p}+}(B_0^N) + \|u\|_{L^{2^*_N}(B_0^N)}^{2^*_N-1} \|u\|_{L^2(B_0^N)}^{2^*_N-1} \right] \\
\leq C \left[ \|u\|_{L^{2^*_N}(B_\lambda)}^{2^*_N-2} \|v\|_{L^\frac{2N}{N-2p}+}(B_\lambda) + \|u\|_{L^{2^*_N}(B_\lambda)}^{2^*_N-1} \|u\|_{L^2(B_\lambda)}^{2^*_N-1} \right] \\
+ \|u\|_{L^{2^*_N}(B_\lambda)}^{2^*_N-2} \|v\|_{L^\frac{2N}{N-2p}+}(B_\lambda) + \|u\|_{L^{2^*_N}(B_\lambda)}^{2^*_N-1} \|u\|_{L^2(B_\lambda)}^{2^*_N-1} \right] \\
= C \left[ \|u\|_{L^{2^*_N}(B_\lambda)}^{2^*_N-2} \left( \int_{\mathbb{R}^N} \chi(B_\lambda^c)(x) |v|^{\frac{2N}{N-2p}+} dx \right)^{\frac{N-p}{2N}} \\
+ \|u\|_{L^{2^*_N}(B_\lambda)}^{2^*_N-1} \left( \int_{\mathbb{R}^N} \chi(B_\lambda^c)(x) |u|^{2^*_N} dx \right)^{\frac{N-p}{2N}} \\
+ \|v\|_{L^{2^*_N}(B_\lambda)}^{2^*_N-2} \left( \int_{\mathbb{R}^N} \chi(B_\lambda^c)(x) |u|^{\frac{2N}{N-2p}+} dx \right)^{\frac{N-p}{2N}} \\
+ \|v\|_{L^{2^*_N}(B_\lambda)}^{2^*_N-1} \left( \int_{\mathbb{R}^N} \chi(B_\lambda^c)(x) |u|^{2^*_N} dx \right)^{\frac{N-p}{2N}} \right] \\
\leq \frac{1}{2} \quad (48) \\

Consequently, by Lemma 3.2 imply that

\[ \|u - u_\lambda\|_{L^{2^*_N}(B_\lambda)} = 0. \]

This implies that the set \( B_\lambda^N \) is empty. Thus we derive \( u(x) \leq u_\lambda(x) \) in \( B_\lambda \setminus \{0\} \).

From inequality (44), we can also prove that \( v(x) \leq v_\lambda(x) \) and \( w(x) \leq w_\lambda(x) \) in \( B_\lambda \setminus \{0\} \). This verifies (43).

\textbf{Step 2.} Step 1 provides a starting point to carry out the method of moving spheres for any given center \( x_0 \in \mathbb{R}^N \). Then we will continuously increase the radius \( \lambda \) of the sphere \( S_\lambda(x_0) \equiv \partial B_\lambda(x_0) \) such that the inequality

\[ u(x) \leq u_\lambda(x), \quad v(x) \leq v_\lambda(x) \quad \text{and} \quad w(x) \leq w_\lambda(x) \]

holds, for every \( x \in B_\lambda(x_0) \setminus \{x_0\} \).

Let

\[ \lambda_1 = \sup\{\lambda | u(x) \leq u_\eta(x), v(x) \leq v_\eta(x), w(x) \leq w_\eta(x), x \in B_\eta \setminus \{0\}, \eta \leq \lambda \}. \]

Then we must have \( \lambda_1 < \infty \). In this part, we need the following Lemma.

\textbf{Lemma 3.3.} If \( \lambda_1 < \infty \), we have

\[ u(x) \equiv u_\lambda(x), \quad v(x) \equiv v_\lambda(x) \quad \text{and} \quad w(x) \equiv w_\lambda(x), \quad \text{for every} \quad x \in B_\lambda \setminus \{0\}. \]

\textbf{Proof.} Suppose \( u(x) \neq u_\lambda(x) \) in \( B_\lambda \setminus \{0\} \). By Lemma 3.1, we obtain that in the interior of \( B_\lambda \), there holds

\[ u(x) < u_\lambda(x). \quad (49) \]

Now we are going to show that there exist \( C > 0 \) and \( \zeta > 0 \) such that

\[ u_\lambda(x) - u(x) \geq C, \quad \text{in} \quad B_\zeta \setminus \{0\}. \quad (50) \]
Indeed, by (44), (49) and Fatou’s Lemma, we know
\[
\liminf_{|x| \to 0} (u_{\lambda_1}(x) - u(x)) \\
\geq \liminf_{|x| \to 0} \int_{B_{\lambda_1}} \left( \frac{1}{|x-y|^{N-2}} - \frac{1}{|\lambda_1| |y|^{N-2}} \right) \left[ \left( |u_{\lambda_1}(y)|^{2\nu_1-1} v_{\lambda_1}(y) - |u(y)|^{2\nu_1-1} w_{\lambda_1}(y) \right) \right. \\
- \left. |u(y)|^{2\nu_1-1} v(y) + \left( |u_{\lambda_1}(y)|^{2\nu_1-1} w_{\lambda_1}(y) - |u(y)|^{2\nu_1-1} w(y) \right) \right] \, dy \\
\geq \int_{B_{\lambda_1}} \left( \frac{1}{|y|^{N-2}} - \frac{1}{|\lambda_1| |y|^{N-2}} \right) \left[ \left( |u_{\lambda_1}(y)|^{2\nu_1-1} v_{\lambda_1}(y) - |u(y)|^{2\nu_1-1} v(y) \right) \\
+ \left( |u_{\lambda_1}(y)|^{2\nu_1-1} w_{\lambda_1}(y) - |u(y)|^{2\nu_1-1} w(y) \right) \right] \, dy \\
> 0.
\]
Thus, there exists some \( C > 0 \) such that \( u_{\lambda_1}(x) - u(x) \geq C \) for \( \zeta \) sufficiently small in \( B_\zeta \setminus \{0\} \). Similarly, we also have other cases for \( \zeta \) small enough. This proves (50).

Furthermore, by (50), there exists some constant \( m_0 > 0 \) and fix \( 0 < \delta_0 < \frac{\alpha}{2} \) sufficiently small, such that
\[
u_{\lambda_1}(x) - v(x) \geq m_0, \text{ in } B_{\lambda_1-\delta_0} \setminus \{0\}.
\]
Similarly, \( v_{\lambda_1}(x) - v(x) \) and \( w_{\lambda_1}(x) - w(x) \) also hold in \( B_{\lambda_1-\delta_0} \setminus \{0\} \). Observe that, \( u, v \) and \( w \) are uniformly continuous on an arbitrary set, thus there exists some \( 0 < \delta_1 < \delta_0 \) such that for any \( \lambda \in (\lambda_1, \lambda_1 + \delta_1) \), it holds that
\[
u_{\lambda}(x) - u(x) \geq \frac{m_0}{2}, \text{ in } B_{\lambda_1-\delta_0} \setminus \{0\}.
\]
Similarly, we also have \( v_{\lambda}(x) - v(x) \geq \frac{m_0}{2} \), \( w_{\lambda}(x) - w(x) \geq \frac{m_0}{2} \), in \( B_{\lambda_1-\delta_0} \setminus \{0\} \).
Hence, we have \( B_{\lambda_1}^\alpha, B_{\lambda_1}^\nu, B_{\lambda_1}^w \subset B_{\lambda_1-\delta_0} \setminus B_{\lambda_1+\delta_0} \).
Notice that, we can choose \( \delta_0 \in (0, \frac{\alpha}{2}) \) sufficiently small such that
\[
C \left[ \left\| u \right\| L^{2\nu_1-2}_{\lambda_1}(B_{\lambda_1-\delta_0} \setminus B_{\lambda_1+\delta_0}) \right\| L^{2\nu_2-2}_{\lambda_1}(B_{\lambda_1-\delta_0} \setminus B_{\lambda_1+\delta_0}) \right] \\
\times \left[ \left\| v \right\| L^{2\nu_1-1}_{\lambda_1}(B_{\lambda_1-\delta_0} \setminus B_{\lambda_1+\delta_0}) \right\| L^{2\nu_2-1}_{\lambda_1}(B_{\lambda_1-\delta_0} \setminus B_{\lambda_1+\delta_0}) \right] \\
\times \left[ \left\| w \right\| L^{2\nu_1-1}_{\lambda_1}(B_{\lambda_1-\delta_0} \setminus B_{\lambda_1+\delta_0}) \right\| L^{2\nu_2-1}_{\lambda_1}(B_{\lambda_1-\delta_0} \setminus B_{\lambda_1+\delta_0}) \right] \\
\leq 1.
\]
Thus, by (51) and Lemma 3.2, we have
\[
\left\| u_{\lambda} - u \right\| L^{2\nu}(B_{\lambda}^\nu) = 0.
\]
Similarly, we also can derive \( \left\| v_{\lambda} - v \right\| L^{2\nu}(B_{\lambda}^\nu) = 0 \), and \( \left\| w_{\lambda} - w \right\| L^{2\nu}(B_{\lambda}^\nu) = 0 \). This implies that \( B_{\lambda}^\nu, B_{\lambda}^\nu \) and \( B_{\lambda}^\nu \) must be a set with zero measure, hence must be empty up to a set with zero measure. So, we assert that \( u_{\lambda} \geq u, v_{\lambda} \geq v \) and \( w_{\lambda} \geq w \) for all \( \lambda \in (\lambda_1, \lambda_1 + \delta_1) \) \( B_{\lambda} \setminus \{0\} \). This contradicts with the definition of \( \lambda_1 \). Completed the proof.

**Lemma 3.4.** If \( \alpha + \beta > 0 \), Then \( \lambda_1 = \infty \).
Proof. Suppose $\alpha > 0$ and $\lambda_1 < \infty$. Then by (44) imply that

$$u(x) - u_{\lambda_1}(x) < \int_{B_{\lambda_1}} \left( \frac{1}{|x-y|^{N-2}} - \frac{1}{|\lambda_1^2|x-y|^{N-2}} \right) \left( |u(y)|^{2^*_{\lambda_1} - 1}v(y) - |u_{\lambda_1}(y)|^{2^*_{\lambda_1} - 1}v_{\lambda_1}(y) \right) dy \leq 0.$$  \hspace{1cm} (52)

From Lemma 3.1, we know

$$v(x) - v_{\lambda_1}(x) \leq \int_{B_{\lambda_1}} \left( \frac{1}{|x-y|^{N-\mu_1}} - \frac{1}{|\lambda_1^2|x-y|^{N-\mu_1}} \right) (u_{\lambda_1}^{2^*_{\lambda_1}}(y) - u_{\lambda_1}^{2^*_{\lambda_1}}(y)) dy < 0$$

and

$$w(x) - w_{\lambda_1}(x) \leq \int_{B_{\lambda_1}} \left( \frac{1}{|x-y|^{N-\mu_2}} - \frac{1}{|\lambda_1^2|x-y|^{N-\mu_2}} \right) (u_{\lambda_1}^{2^*_{\lambda_1}}(y) - u_{\lambda_1}^{2^*_{\lambda_1}}(y)) dy < 0.$$  \hspace{1cm}

Therefore, we derive $u(x) \leq u_{\lambda_1}(x), v(x) \leq v_{\lambda_1}(x)$ and $w(x) \leq w_{\lambda_1}(x)$ in $B_{\lambda_1} \setminus \{0\}$. Furthermore, repeating the argument of Lemma 3.3, this will lead to a contradiction. Hence completed the proof. \(\square\)

To prove our main results, we need the following Lemma ([29]):

Lemma 3.5. (A1) If for every $x_0 \in \mathbb{R}^N$, for all $\lambda \in \mathbb{R}$, it holds that

$$\left( \frac{\lambda}{|x-x_0|} \right)^{N-2} u \left( \frac{\lambda^2(x-x_0)}{|x-x_0|^2} + x_0 \right) \geq u(x), \forall x \in B_\lambda(x_0) \setminus \{x_0\},$$

then $u$ is constant.

(B1) If for every $x_0 \in \mathbb{R}^N$, there exists $\lambda_{x_0} \in (0, +\infty)$ which depends on $x_0$ such that

$$\left( \frac{\lambda}{|x-x_0|} \right)^{N-2} u \left( \frac{\lambda^2(x-x_0)}{|x-x_0|^2} + x_0 \right) \equiv u(x), \forall x \in B_{\lambda_{x_0}}(x_0) \setminus \{x_0\},$$

then there exist $\xi > 0$, $\bar{x} \in \mathbb{R}^N$ and some $c_0 \in \mathbb{R}$ such that

$$u(x) = c_0 \left( \frac{\xi}{\xi^2 + |x-\bar{x}|^2} \right)^{N-2}.$$  \hspace{1cm}

Step 3. The statement of main results can be reduced to the following: if there exist $x_0, y_0 \in \mathbb{R}^N$ such that the corresponding critical scale critical scale $\lambda_{x_0} = \infty$ and $\lambda_{y_0} < \infty$, then for every $\lambda_{x_0} \in \mathbb{R}^N$ the corresponding critical scale $\lambda_{y_0} = \infty$ or $\lambda_{x_0} < \infty$, respectively. Now we consider the several cases.

Firstly, if $\lambda_{x_0} = \infty$ for $x_0, y_0 \in \mathbb{R}^N$. Then for any $\lambda > 0$, we have

$$\left( \frac{\lambda}{|x-x_0|} \right)^{N-2} u \left( \frac{\lambda^2(x-x_0)}{|x-x_0|^2} + x_0 \right) \geq u(x), \ x \in B_\lambda(x_0) \setminus \{x_0\}.$$  \hspace{1cm}

Thus, we have

$$\lambda^{N-2} u \left( \frac{\lambda^2(x-x_0)}{|x-x_0|^2} + x_0 \right) \leq |x-x_0|^{N-2} u(x), \ x \in B_\lambda^c(x_0).$$

Multiplying the above by $|y|^{-2}$ and sending $|y|$ to infinity yields

$$\lim_{|y| \to \infty} |y|^{-2} u(y) \geq \lambda_{y_0}^{N-2} u(y_0).$$
In virtue of $\lambda$ can be chosen arbitrarily, we have
\[ \lim_{|y| \to \infty} |y|^{N-2} u(y) = \infty. \] (53)

On the other hand, if $\lambda_{y_0} < \infty$, thus by Lemma 3.3, we have
\[ \left( \frac{\lambda_{y_0}}{|x - y_0|} \right)^{N-2} u \left( \frac{\lambda_{y_0}^2 (x - y_0)}{|x - y_0|^2} + y_0 \right) = u(x), \quad x \in B_\delta^\circ(x_0). \]
This implies that
\[ \lim_{|y| \to \infty} |y|^{N-2} u(y) = \lambda_{y_0}^{N-2} u(y_0) < \infty. \]

Then above inequality contradicts (53). Thus, this case cannot happen.

Secondly, if critical scale $\lambda_{x_0} = \infty$ for every $x_0 \in \mathbb{R}^N$. Then we have
\[ \left( \frac{\lambda_{x_0}}{|x - x_0|} \right)^{N-2} u \left( \frac{\lambda_{x_0}^2 (x - x_0)}{|x - x_0|^2} + x_0 \right) \geq u(x), \quad \forall x \in B_x \setminus \{x_0\}. \]
Thus, by Lemma 3.5-(A1), we know $u$ is a constant. Hence this case cannot happen.

Finally, we consider for every $x_0 \in \mathbb{R}^N$ the corresponding critical scale $\lambda_{x_0} < \infty$. Then, by Lemma 3.3, we have for every $x \in \mathbb{R}^N \setminus \{x_0\}$
\[ \left( \frac{\lambda}{|x - x_0|} \right)^{N-2} u \left( \frac{\lambda^2 (x - x_0)}{|x - x_0|^2} + x_0 \right) \equiv u(x). \]
Thus, according to Lemma 3.5, for some $c_0, \xi > 0$, we have
\[ u(x) = c_0 \left( \frac{\xi}{\xi^2 + |x - \bar{x}|^2} \right)^{\frac{N-2}{2}}. \]

Moreover, for any $0 < \tau < \frac{N}{2}$, it follows from [13] that
\[ \int_{\mathbb{R}^N} \frac{1}{|x - y|^{2\tau}} \left( \frac{1}{1 + |y|^2} \right)^{N-2} dy = I(\tau) \left( \frac{1}{1 + |x|^2} \right)^\tau, \]
where $I(\tau) = \frac{\pi^{N-1} (N-\tau)^{\tau}}{\Gamma(N-\tau)}$. Thus, by above the identity and taking $\tau = \frac{N-\mu_1}{2}, \tau = \frac{N-\mu_2}{2}$, respectively, we have
\[ v(x) = \int_{\mathbb{R}^N} \frac{|u(y)|^{2^{\mu_1}}}{|x - y|^{N-\mu_1}} dy = c_0^{2^{\mu_1}} I \left( \frac{N - \mu_1}{2} \right) \left( \frac{\xi}{\xi^2 + |x - \bar{x}|^2} \right)^{\frac{N-\mu_1}{2}} \] (54)
and
\[ w(x) = \int_{\mathbb{R}^N} \frac{|u(y)|^{2^{\mu_2}}}{|x - y|^{N-\mu_2}} dy = c_0^{2^{\mu_2}} I \left( \frac{N - \mu_2}{2} \right) \left( \frac{\xi}{\xi^2 + |x - \bar{x}|^2} \right)^{\frac{N-\mu_2}{2}} \] (55)

Consequently, combining (54)-(55) with (16), we know
\[ u(x) = \int_{\mathbb{R}^N} v(y) \frac{|u(y)|^{2^{\mu_1}}}{|x - y|^{N-2}} u(y) dy + \int_{\mathbb{R}^N} w(y) \frac{|u(y)|^{2^{\mu_2}}}{|x - y|^{N-2}} u(y) dy \]
\[ = \left[ c_0^{2^{\mu_1} + 2^{\mu_2}} I \left( \frac{N - \mu_1}{2} \right) + c_0^{2^{\mu_2} + 2^{\mu_2}} I \left( \frac{N - \mu_2}{2} \right) \right] I \left( \frac{N - 2}{2} \right) \left( \frac{\xi}{\xi^2 + |x - \bar{x}|^2} \right)^{\frac{N-2}{2}}. \]

Therefore, we have
\[ c_0 = \left[ c_0^{2^{\mu_1} + N-2} I \left( \frac{N - \mu_1}{2} \right) + c_0^{2^{\mu_2} + N-2} I \left( \frac{N - \mu_2}{2} \right) \right] I \left( \frac{N - 2}{2} \right). \]

This completes the proof.
4. **Nondegeneracy.** In this section, we will establish that the nondegeneracy of for equation (2) when \( \mu_1, \mu_2 \) is sufficiently close to 0 or close to \( N \). We first introduce the following equivalent form of the Hardy-Littlewood-Sobolev inequality with Riesz potential.

**Proposition 2.** Let \( 1 \leq r < s < \infty \) and \( 0 < \mu < N \) satisfy
\[
\frac{1}{r} - \frac{1}{s} = \frac{\mu}{N}.
\]
Then for \( \mu \) sufficient close to \( N \), there exists constant \( C_{N, \mu, r} > 0 \) such that for any \( f \in L^r(\mathbb{R}^N) \), there holds
\[
\| \frac{1}{|x|^{N-\mu}} \ast f \|_{L^s(\mathbb{R}^N)} \leq K(N, \mu, r) \| f \|_{L^r(\mathbb{R}^N)}.
\]
Where \( K(N, \mu, r) \) satisfies
\[
\limsup_{\mu \to N} (N-\mu)K(N, \mu, r) \leq \frac{2}{r(r-1)} |\mathbb{S}^{N-1}|.
\]
The following from proposition 2 and the fact \( \Gamma(\frac{N}{2}) \sim \frac{1}{\mu} \) as \( \mu \to 0 \).

**Corollary 1.** Let \( r, s \) satisfy the assumption in proposition 2. Then for \( \mu \) sufficient close to 0, there exists \( C_{N, \mu, r} > 0 \) such that for any \( f \in L^r(\mathbb{R}^N) \)
\[
\| I_\mu \ast f \|_{L^s(\mathbb{R}^N)} \leq C_{N, \mu, r} \| f \|_{L^r(\mathbb{R}^N)}.
\]

4.1. **Nondegeneracy for \( \mu_1, \mu_2 \) near 0.** Firstly, we are going to show that the nondegeneracy of \( U_0 \) to (2) when \( \max\{\mu_1, \mu_2\} \to 0 \) for \( N = 3 \) or \( N = 4 \).

**Proof of Theorem 1.4.** Define a finite dimensional vector space
\[
M_\mu := \text{span}\{\partial_1 U_0, \partial_2 U_0, \partial_3 U_0, \partial_4 U_0\}.
\]
Arguing indirectly, for each \( j \), if there exists a sequence \( \max\{\mu_1^j, \mu_2^j\} \to 0 \) such that (24) have nontrivial solution \( \psi_j \) in the topological complement of \( M_{\mu^j} \) in \( L^2(\mathbb{R}^3) \).

We may assume that \( \psi_j \) is \( L^2(\mathbb{R}^3) \) orthogonal to \( M_\mu \). We claim that any \( L^2(\mathbb{R}^3) \) solution \( \psi \) of (24) automatically belongs to \( D^{1,2}(\mathbb{R}^3) \).

We define the operator
\[
L[\psi] := 2^{*}_{\mu_1}(I_{\mu_1} \ast (U_0^{\nu_{\mu_1} - 1} \psi))U_0^{\nu_{\mu_1} - 1} - (2^{*}_{\mu_1} - 1)(I_{\mu_1} \ast U_0^{\nu_{\mu_1}})U_0^{2^{*}_{\mu_1} - 2} \psi - 2^{*}_{\mu_2}(I_{\mu_2} \ast (U_0^{\nu_{\mu_2} - 1} \psi))U_0^{\nu_{\mu_2} - 1} - (2^{*}_{\mu_2} - 1)(I_{\mu_2} \ast U_0^{\nu_{\mu_2}})U_0^{2^{*}_{\mu_2} - 2} \psi.
\]

By elliptic regularity theory, it is enough to show that \( L[\psi] \in (D^{1,2}(\mathbb{R}^3))^* \) denote the dual space of \( D^{1,2}(\mathbb{R}^3) \). Then for any \( \varphi \in D^{1,2}(\mathbb{R}^3) \), we have
\[
|\langle L[\psi], \varphi \rangle| \leq 2^{*}_{\mu_1}|(I_{\mu_1} \ast (U_0^{\nu_{\mu_1} - 1} \psi))U_0^{\nu_{\mu_1} - 1} \varphi| + (2^{*}_{\mu_1} - 1)|I_{\mu_1} \ast U_0^{\nu_{\mu_1}}|U_0^{2^{*}_{\mu_1} - 2} \psi \varphi| + 2^{*}_{\mu_2}|(I_{\mu_2} \ast (U_0^{\nu_{\mu_2} - 1} \psi))U_0^{\nu_{\mu_2} - 1} \varphi| + (2^{*}_{\mu_2} - 1)|I_{\mu_2} \ast U_0^{\nu_{\mu_2}}|U_0^{2^{*}_{\mu_2} - 2} \psi \varphi|.
\]

For brevity, we denote \( |\langle L_1[\psi], \varphi \rangle| := 2^{*}_{\mu_1}|(I_{\mu_1} \ast (U_0^{\nu_{\mu_1} - 1} \psi))U_0^{\nu_{\mu_1} - 1} \varphi| + (2^{*}_{\mu_1} - 1)|I_{\mu_1} \ast U_0^{\nu_{\mu_1}}|U_0^{2^{*}_{\mu_1} - 2} \psi \varphi| := \hat{B} + \hat{C} \). We will only give the proof for \( |\langle L_1[\psi], \varphi \rangle| \), same argument of other parts. Here we denote \( \langle \cdot, \cdot \rangle \) as the inner product in \( D^{1,2}(\mathbb{R}^N) \).
Then by Hölder inequality and Corollary 1 imply that for any $\varphi \in D^{1,2}(\mathbb{R}^3)$

\[
\hat{B} = 2^*_{\mu_1} \left| \int_{\mathbb{R}^3} (I_{\mu_1} \ast (U_0^2 \psi^{-1}) U_0^{2^*_{\mu_1} - 1} \varphi) dx \right| \\
\leq C \| I_{\mu_1} \ast (U_0^2 \psi^{-1}) \|_{L^6(\mathbb{R}^3)} \| U_0^{2^*_{\mu_1} - 1} \|_{L^2(\mathbb{R}^3)} \| \varphi \|_{L^{2^*}(\mathbb{R}^3)} \\
\leq C \| U_0^{2^*_{\mu_1} - 1} \|_{L^{\frac{6}{\mu_1}}(\mathbb{R}^3)} \| U_0^{2^*_{\mu_1} - 1} \|_{L^2(\mathbb{R}^3)} \| \varphi \|_{L^{2^*}(\mathbb{R}^3)} \\
\leq C \| U_0^{2^*_{\mu_1} - 1} \|_{L^{\frac{3}{2}}(\mathbb{R}^3)} \| \psi \|_{L^{2^*}(\mathbb{R}^3)} \| U_0^{2^*_{\mu_1} - 1} \|_{L^{\frac{3}{2}}(\mathbb{R}^3)} \| \varphi \|_{L^{2^*}(\mathbb{R}^3)} \\
\leq C \| U_0 \|_{L^{\frac{3}{2}(2+\mu_1)}(\mathbb{R}^3)} \| U_0 \|_{L^{\frac{3}{2}(2+\mu_1)}(\mathbb{R}^3)} \| \psi \|_{L^{2^*}(\mathbb{R}^3)} \| \varphi \|_{L^{2^*}(\mathbb{R}^3)},
\]

and

\[
\check{C} = (2^*_{\mu_2} - 1) \left| \int_{\mathbb{R}^3} (I_{\mu_2} \ast U_0^{2^*_{\mu_2}} U_0^{2^*_{\mu_2} - 2} \psi \varphi) dx \right| \\
\leq C \| I_{\mu_2} \ast U_0^{2^*_{\mu_2}} \|_{L^3(\mathbb{R}^3)} \| U_0^{2^*_{\mu_2} - 2} \|_{L^3(\mathbb{R}^3)} \| \psi \|_{L^{2^*}(\mathbb{R}^3)} \| \varphi \|_{L^{2^*}(\mathbb{R}^3)} \\
\leq C \| U_0 \|_{L^{\frac{3}{2}(3+\mu_1)}(\mathbb{R}^3)} \| U_0 \|_{L^{\frac{3}{2}(3+\mu_1)}(\mathbb{R}^3)} \| \psi \|_{L^{2^*}(\mathbb{R}^3)} \| \varphi \|_{L^{2^*}(\mathbb{R}^3)}.
\]

In virtue of (57)-(58), we get

\[
\| (L_1[\psi], \varphi) \| \leq C \left[ \| U_0 \|_{L^{\frac{3}{2}(2+\mu_1)}(\mathbb{R}^3)} \| U_0 \|_{L^{\frac{3}{2}(2+\mu_1)}(\mathbb{R}^3)} + \| U_0 \|_{L^{\frac{3}{2}(3+\mu_1)}(\mathbb{R}^3)} \| U_0 \|_{L^{\frac{3}{2}(3+\mu_1)}(\mathbb{R}^3)} \right] \\
\times \| \psi \|_{L^{2^*}(\mathbb{R}^3)} \| \varphi \|_{L^{2^*}(\mathbb{R}^3)}.
\]

Similarly argument the rest part. Thus, we have

\[
\| (L[\psi], \varphi) \| \leq \]

\[
C \left[ \| U_0 \|_{L^{\frac{3}{2}(2+\mu_1)}(\mathbb{R}^3)} \| U_0 \|_{L^{\frac{3}{2}(2+\mu_1)}(\mathbb{R}^3)} + \| U_0 \|_{L^{\frac{3}{2}(3+\mu_1)}(\mathbb{R}^3)} \| U_0 \|_{L^{\frac{3}{2}(3+\mu_1)}(\mathbb{R}^3)} \right] \\
\times \| \psi \|_{L^{2^*}(\mathbb{R}^3)} \| \varphi \|_{L^{2^*}(\mathbb{R}^3)}.
\]

Thus we can find that the functional $L[\psi] \in (D^{1,2}(\mathbb{R}^3))^*$. Since $-\Delta \psi \in (D^{1,2}(\mathbb{R}^3))^*$, then we achieve that $\psi \in D^{1,2}(\mathbb{R}^3)$.

Now we may assume that $\psi_j$ is a sequence of unit solution for the linearized equation at $U_0$. Thus there exists $\psi_0 \in D^{1,2}(\mathbb{R}^3)$ such that $\psi_j \rightharpoonup \psi_0$ in $D^{1,2}(\mathbb{R}^3)$ as $j \to \infty$. Then for any $\varphi \in D^{1,2}(\mathbb{R}^3)$, we have

\[
\int_{\mathbb{R}^3} \nabla \psi_j \cdot \nabla \varphi dx = \\
2^*_{\mu_1} \int_{\mathbb{R}^3} (I_{\mu_1} \ast (U_0^{2^*_{\mu_1} - 1} \psi_j) U_0^{2^*_{\mu_1} - 1} \varphi) dx + (2^*_{\mu_1} - 1) \int_{\mathbb{R}^3} (I_{\mu_1} \ast U_0^{2^*_{\mu_1}}) U_0^{2^*_{\mu_1} - 1} \psi_j \varphi dx \\
+ 2^*_{\mu_2} \int_{\mathbb{R}^3} (I_{\mu_2} \ast (U_0^{2^*_{\mu_2} - 1} \psi_j) U_0^{2^*_{\mu_2} - 1} \varphi) dx + (2^*_{\mu_2} - 1) \int_{\mathbb{R}^3} (I_{\mu_2} \ast U_0^{2^*_{\mu_2}}) U_0^{2^*_{\mu_2} - 1} \psi_j \varphi dx.
\]
We claim that, \( \max \{ \mu_1', \mu_2' \} \to 0 \) as \( j \to \infty \),
\[
2^*_{\mu_1'} \int_{\mathbb{R}^3} (I_{\mu_1'} * (U_0^{\mu_1} \psi)) U_0^{\mu_1} \varphi dx \to 3 \int_{\mathbb{R}^3} U_0^4 \psi_0 \varphi dx
\] (60)
and
\[
(2^*_{\mu_1'} - 1) \int_{\mathbb{R}^3} (I_{\mu_1'} * U_0^{\mu_1}) U_0^{\mu_1} \psi_j \varphi dx \to 2 \int_{\mathbb{R}^3} U_0^4 \psi_0 \varphi dx.
\] (61)
Indeed,
\[
\int_{\mathbb{R}^3} (I_{\mu_1'} * (U_0^{\mu_1} \psi_j)) U_0^{\mu_1} \varphi dx = \int_{\mathbb{R}^3} (I_{\mu_1'} * (U_0^{\mu_1} \psi_j - U_0^2 \psi_0)) U_0^{\mu_1} \varphi dx + \int_{\mathbb{R}^3} (I_{\mu_1'} * (U_0^2 \psi_0)) U_0^{\mu_1} \varphi dx + \int_{\mathbb{R}^3} U_0^2 U_0^{\mu_1} \psi_0 \varphi dx.
\]
\[
\int_{\mathbb{R}^3} (I_{\mu_1'} * (U_0^{\mu_1} \psi_j - U_0^2 \psi_0)) U_0^{\mu_1} \varphi dx \leq C \|U_0^{\mu_1} \psi_j - U_0^2 \psi_0\|_{L^{\frac{6}{\mu_1'(\epsilon+2)}}(\mathbb{R}^3)} \|U_0\|_{L^{\frac{3}{\mu_1'(\epsilon+2)}}(\mathbb{R}^3)} \|\varphi\|_{L^6(\mathbb{R}^3)}.
\]
Furthermore,
\[
\|U_0^{\mu_1} \psi_j - U_0^2 \psi_0\|_{L^{\frac{6}{\mu_1'(\epsilon+2)}}(\mathbb{R}^3)} \\
\leq \|U_0^{\mu_1} \psi_j - U_0^2 \psi_0\|_{L^6(\mathbb{R}^3)} + \|U_0^2 \psi_0 - U_0^2 \psi_j\|_{L^{\frac{6}{\mu_1'(\epsilon+2)}}(\mathbb{R}^3)} \to 0,
\]
as \( j \to \infty \). Next by the Hölder inequality, we obtain
\[
\int_{\mathbb{R}^3} (I_{\mu_1'} * (U_0^2 \psi_0)) U_0^{\mu_1} \varphi dx \leq \|I_{\mu_1'} * (U_0^2 \psi_0)\|_{L^2(\mathbb{R}^3)} \|U_0\|_{L^{\frac{6}{\mu_1'(\epsilon+2)}}(\mathbb{R}^3)} \|\varphi\|_{L^6(\mathbb{R}^3)} \to 0,
\]
as \( j \to \infty \). This proves (60). By a similar argument, we can obtain (61). Similarly, as \( \max \{ \mu_1', \mu_2' \} \to 0 \), we derive
\[
2^*_{\mu_2'} \int_{\mathbb{R}^3} (I_{\mu_2'} * (U_0^{\mu_2} \psi_j)) U_0^{\mu_2} \varphi dx \to 3 \int_{\mathbb{R}^3} U_0^4 \psi_0 \varphi dx
\] (62)
and
\[
(2^*_{\mu_2'} - 1) \int_{\mathbb{R}^3} (I_{\mu_2'} * U_0^{\mu_2}) U_0^{\mu_2} \psi_j \varphi dx \to 2 \int_{\mathbb{R}^3} U_0^4 \psi_0 \varphi dx.
\] (63)
Hence, it follows from (60)-(63) and as \( j \to \infty \) that
\[
\int_{\mathbb{R}^3} \nabla \psi_0 \nabla \varphi dx = 10 \int_{\mathbb{R}^3} U_0^4 \psi_0 \varphi dx
\]
for any \( \varphi \in D^{1,2}(\mathbb{R}^3) \). Therefore, \( \psi_0 \) satisfies
\[
-\Delta \psi_0 = 10 U_0^4 \psi_0.
\]
In virtue of the nondegeneracy of \( U_0 \), we have
\[
\psi_0 \in \text{span} \{ \partial_1 U_0, \partial_2 U_0, \partial_3 U_0, \partial_t U_0 \}.
\] (64)
Next, a similar argument of [16], we prove that \( \psi_0 \neq 0 \). We take \( \varphi = \psi_j \) in equation (59) to get
\[
\int_{\mathbb{R}^N} |\nabla \psi_j|^2 dx = \left\{ 2^{*}_{\mu_1} \int_{\mathbb{R}^3} \left( I_{\mu_1} * \left( U_0^{2^*_{\mu_1} - 1} \psi_j \right) \right) U_0^{2^*_{\mu_1} - 1} \psi_j dx \right\}
+ (2^{*}_{\mu_1} - 1) \int_{\mathbb{R}^N} \left( I_{\mu_1} * U_0^{2^*_{\mu_1}} \right) U_0^{2^*_{\mu_1} - 2} \psi_j^2 dx
+ \left\{ 2^{*}_{\mu_2} \int_{\mathbb{R}^N} \left( I_{\mu_2} * \left( U_0^{2^*_{\mu_2} - 1} \psi_j \right) \right) U_0^{2^*_{\mu_2} - 1} \psi_j dx \right\}
+ (2^{*}_{\mu_2} - 1) \int_{\mathbb{R}^N} \left( I_{\mu_2} * U_0^{2^*_{\mu_2}} \right) U_0^{2^*_{\mu_2} - 2} \psi_j^2 dx := l_j^1 + l_j^2.
\]
However, on the one hand, we normalize \( \psi_j \) as
\[
\int_{\mathbb{R}^N} |\nabla \psi_j|^2 dx = 1.
\]
On the other hand, we can repeat the arguments in (60)-(63) to get
\[
l_j^1 + l_j^2 \to 10 \int_{\mathbb{R}^3} U_0^3 \psi_0 \varphi dx \text{ as } j \to \infty,
\]
which implies that \( \psi_0 \neq 0 \).

Since we know
\[
\psi_j = \text{span}\{\partial_1 U_0, \partial_2 U_0, \partial_3 U_0, \partial_t U_0\}^\perp.
\]
Then for any
\[
\psi_j \in \tilde{\eta}_0 = a D_t U_0 + b \cdot \nabla U_0 = \text{span}\{\partial_1 U_0, \partial_2 U_0, \partial_3 U_0, \partial_t U_0\},
\]
where \( a \in \mathbb{R} \), \( b = (b_1, b_2, b_3) \in \mathbb{R}^3 \). Thus we have
\[
\langle \psi_n, \tilde{\eta}_0 \rangle = 0.
\]
However, \( \langle \psi_0, \tilde{\eta}_0 \rangle = 0 \) contradicts with the linearized nondegeneracy of \( U_0 \) that (64), since we proved that \( \psi_0 \neq 0 \). Thus, we know any solution satisfies (24) must belong to \( \bar{M}_\mu \) in the space \( L^2(\mathbb{R}^N) \) and completes the proof.

4.2. Nondegeneracy for \( \mu_1, \mu_2 \) near \( N \). We need to study a limit problem for the equation (26) as \( \min \{\mu_1, \mu_2\} \to N \) first.

**Proof of Theorem 1.5.** It is similar to that of Proposition 2.1 from [19], so is omitted.

Finally, we are going to prove that the nondegeneracy of linearized equation of (28) at \( V_0 \) when \( \min \{\mu_1, \mu_2\} \to N \).

**Proof of Theorem 1.6.** Define a finite dimensional vector space
\[
\bar{M}_\mu := \text{span}\{\partial_1 V_0, \partial_2 V_0, \cdots, \partial_N V_0, \partial_t V_0\}.
\]
The proof of Theorem 1.7 follows the same line with the proof of Theorem 1.5. The one we only need to prove is that \( \bar{L}[\psi] \in (D^{1,2}(\mathbb{R}^N))^* \) when \( \psi \in L^2(\mathbb{R}^N) \) is a solution to (32). We define the operator
\[
\bar{L}[\psi] := 2^{*}_{\mu_1} \left( \frac{1}{\|x\|^{N-\mu_1}} * (V_0^{2^*_{\mu_1} - 1} \psi) \right) V_0^{2^*_{\mu_1} - 1} + (2^{*}_{\mu_1} - 1)(\frac{1}{\|x\|^{N-\mu_1}} * V_0^{2^*_{\mu_1}}) V_0^{2^*_{\mu_1} - 2} \psi
+ 2^{*}_{\mu_2} \left( \frac{1}{\|x\|^{N-\mu_2}} * (V_0^{2^*_{\mu_2} - 1} \psi) \right) V_0^{2^*_{\mu_2} - 1} + (2^{*}_{\mu_2} - 1)(\frac{1}{\|x\|^{N-\mu_2}} * V_0^{2^*_{\mu_2}}) V_0^{2^*_{\mu_2} - 2} \psi.
\]
Then for any $\varphi \in D^{1,2}(\mathbb{R}^N)$, we have
\[
|\langle \tilde{L}[\psi], \varphi \rangle| \leq \frac{1}{2^{\mu_1}} \int_{\mathbb{R}^N} \left( \frac{1}{|x|^{N-\mu_1}} \ast (V_{0}^{2^{\mu_1}-1} \psi) \right) V_{0}^{2^{\mu_1}-1} \varphi \, dx \\
+ (2^{\mu_1} - 1) \int_{\mathbb{R}^N} \left( \frac{1}{|x|^{N-\mu_2}} \ast (V_{0}^{2^{\mu_2}-1} \psi) \right) V_{0}^{2^{\mu_2}-1} \varphi \, dx \\
+ 2^{\mu_2} \int_{\mathbb{R}^N} \left( \frac{1}{|x|^{N-\mu_2}} \ast (V_{0}^{2^{\mu_2}-1} \psi) \right) V_{0}^{2^{\mu_2}-1} \varphi \, dx \\
+ (2^{\mu_2} - 1) \int_{\mathbb{R}^N} \left( \frac{1}{|x|^{N-\mu_2}} \ast (V_{0}^{2^{\mu_2}-1} \psi) \right) V_{0}^{2^{\mu_2}-2} \varphi \, dx.
\]
For brevity, we denote $|\langle \tilde{L}[\psi], \varphi \rangle| := 2^{\mu_1} \int_{\mathbb{R}^N} \left( \frac{1}{|x|^{N-\mu_1}} \ast (V_{0}^{2^{\mu_1}-1} \psi) \right) V_{0}^{2^{\mu_1}-1} \varphi \, dx + (2^{\mu_1} - 1) \int_{\mathbb{R}^N} \left( \frac{1}{|x|^{N-\mu_1}} \ast (V_{0}^{2^{\mu_1}-1} \psi) \right) V_{0}^{2^{\mu_1}-2} \varphi \, dx := \tilde{B} + \tilde{C}$. Thus, by Hölder inequality and Proposition 2 imply that
\[
\tilde{B} = 2^{\mu_1} \int_{\mathbb{R}^N} \left( \frac{1}{|x|^{N-\mu_1}} \ast (V_{0}^{2^{\mu_1}-1} \psi) \right) V_{0}^{2^{\mu_1}-1} \varphi \, dx \\
\leq C \| \frac{1}{|x|^{N-\mu_1}} \ast (V_{0}^{2^{\mu_1}-1} \psi) \|_{L^{\frac{N}{\mu_1+N}}(\mathbb{R}^N)} \| V_{0}^{2^{\mu_1}-1} \|_{L^{\frac{N}{\mu_1+N}}(\mathbb{R}^N)} \| \varphi \|_{L^{2^*}(\mathbb{R}^N)} \leq C \| V_{0}^{2^{\mu_1}-1} \|_{L^{\frac{N}{\mu_1+N}}(\mathbb{R}^N)} \| \varphi \|_{L^{2^*}(\mathbb{R}^N)},
\]
and
\[
\tilde{C} = (2^{\mu_1} - 1) \int_{\mathbb{R}^N} \left( \frac{1}{|x|^{N-\mu_1}} \ast V_{0}^{2^{\mu_1}} \right) V_{0}^{2^{\mu_1}-2} \varphi \, dx \\
\leq C \| \frac{1}{|x|^{N-\mu_1}} \ast V_{0}^{2^{\mu_1}} \|_{L^{\frac{N}{\mu_1+N}}(\mathbb{R}^N)} \| V_{0}^{2^{\mu_1}-2} \|_{L^{\frac{N}{\mu_1+N}}(\mathbb{R}^N)} \| \varphi \|_{L^{2^*}(\mathbb{R}^N)} \leq C \| V_{0}^{2^{\mu_1}-2} \|_{L^{\frac{N}{\mu_1+N}}(\mathbb{R}^N)} \| \varphi \|_{L^{2^*}(\mathbb{R}^N)},
\]
In virtue of (66)-(67), similar to the above argument for other parts, we know
\[
|\langle \tilde{L}[\psi], \varphi \rangle| \leq C \left[ \| V_{0} \|_{L^{\frac{2^{\mu_1}}{N+2^{\mu_1}}}(\mathbb{R}^N)} \| V_{0} \|_{L^{\frac{2^{\mu_2}}{N+2^{\mu_2}}}(\mathbb{R}^N)} \| V_{0} \|_{L^{\frac{2^{\mu_2}}{N+2^{\mu_2}}}(\mathbb{R}^N)} \right] \| \varphi \|_{L^{2^*}(\mathbb{R}^N)} \| \varphi \|_{D^{1,2}(\mathbb{R}^N)},
\]
which shows that $L[\psi] \in (D^{1,2}(\mathbb{R}^N))^*$. Since $-\Delta \psi \in (D^{1,2}(\mathbb{R}^N))^*$, then $\psi \in D^{1,2}(\mathbb{R}^N)$.

Now, we shall complete the proof of Theorem 1.7. We may assume that $\tilde{\psi}_j$ is a sequence of unit solution for the linearized equation (32) at $V_0$. Thus there exists
\( \tilde{\psi}_0 \in D^{1,2}(\mathbb{R}^N) \) such that \( \tilde{\psi}_j \to \tilde{\psi}_0 \) in \( D^{1,2}(\mathbb{R}^N) \) as \( j \to \infty \). Therefore for any \( \varphi \in D^{1,2}(\mathbb{R}^N) \),

\[
\int_{\mathbb{R}^N} \nabla \tilde{\psi}_j \nabla \varphi dx = \\
2^* \int_{\mathbb{R}^N} \left( \frac{1}{|x|} \ast (V_0^{\mu_j} \tilde{\psi}_j) \right) V_0^{\mu_j - 1} \varphi dx \\
+ (2^* - 1) \int_{\mathbb{R}^N} \left( \frac{1}{|x|} \ast (V_0^{\mu_j} \tilde{\psi}_j) \right) V_0^{\mu_j - 2} \varphi dx \\
+ 2^* \int_{\mathbb{R}^N} \left( \frac{1}{|x|} \ast (V_0^{\mu_j} \tilde{\psi}_j) \right) V_0^{\mu_j - 1} \varphi dx \\
+ (2^* - 1) \int_{\mathbb{R}^N} \left( \frac{1}{|x|} \ast (V_0^{\mu_j} \tilde{\psi}_j) \right) V_0^{\mu_j - 2} \varphi dx.
\]

Now, \( \min\{\mu_1, \mu_2\} \to N \) as \( j \to \infty \), we claim that

\[
2^* \int_{\mathbb{R}^N} \left( \frac{1}{|x|} \ast (V_0^{\mu_j} \tilde{\psi}_j) \right) V_0^{\mu_j - 1} \varphi dx \\
\to 2^* \left( \int_{\mathbb{R}^N} V_0^{\mu_j - 1} \tilde{\psi}_0 dx \right) \left( \int_{\mathbb{R}^N} V_0^{\mu_j - 1} \varphi dx \right).
\]

In fact,

\[
\int_{\mathbb{R}^N} \left( \frac{1}{|x|} \ast (V_0^{\mu_j} \tilde{\psi}_j) \right) V_0^{\mu_j - 1} \varphi dx \\
= \int_{\mathbb{R}^N} \left( \frac{1}{|x|} \ast (V_0^{\mu_j} \tilde{\psi}_j - V_0^{\mu_j - 1} \tilde{\psi}_0) \right) V_0^{\mu_j - 1} \varphi dx \\
+ \int_{\mathbb{R}^N} \left( \frac{1}{|x|} \ast (V_0^{\mu_j - 1} \tilde{\psi}_0) \right) (V_0^{\mu_j - 2} - V_0^{\mu_j - 2} \tilde{\psi}_0) \varphi dx \\
+ \int_{\mathbb{R}^N} \left( \frac{1}{|x|} \ast V_0^{\mu_j - 1} \tilde{\psi}_0 \right) V_0^{\mu_j - 1} \varphi dx := (A) + (B) + (C).
\]

Firstly, for \( 0 < \varepsilon < 1 \) small enough, combined Hölder inequality with Proposition 2 imply that

\[
(A) \leq \\
\| \frac{1}{|x|} \ast (V_0^{\mu_j - 1} \tilde{\psi}_j - V_0^{\mu_j - 1} \tilde{\psi}_0) \|_{L^{\frac{2N}{N-2\mu_j-2}}(\mathbb{R}^N)} \||V_0^{\mu_j - 1} \|_{L^{\frac{N}{N-2\mu_j-2}}(\mathbb{R}^N)} \||\varphi||_{L^{2^*}(\mathbb{R}^N)}.
\]

By the decay estimates of \( V_0 \) and the local compact embedding, and the Dominated Convergence Theorem imply that

\[
\|V_0^{\mu_j - 1} \tilde{\psi}_j - V_0^{\mu_j - 1} \tilde{\psi}_0\|_{L^{\frac{2N}{N-2\mu_j-2}}(\mathbb{R}^N)} \\
\leq \|V_0^{\mu_j - 1} \tilde{\psi}_j - V_0^{\mu_j - 1} \tilde{\psi}_0\|_{L^{\frac{2N}{N-2\mu_j-2}}(\mathbb{R}^N)} + \|V_0^{\mu_j - 1} \tilde{\psi}_j - V_0^{\mu_j - 1} \tilde{\psi}_0\|_{L^{\frac{2N}{N-2\mu_j-2}}(\mathbb{R}^N)} \\
\leq \|V_0^{\mu_j - 1} \tilde{\psi}_j - V_0^{\mu_j - 1} \tilde{\psi}_0\|_{L^{\frac{2N}{N-2\mu_j-2}}(\mathbb{R}^N)} \to 0.
\]
as \( j \to \infty \), which implies that
\[
\int_{\mathbb{R}^N} \left( \frac{1}{|x|^{N-\mu_1^j}} \ast (V_0^{2^*+1} - V_0^{2^*+1} \bar{\psi}_j) \right) V_0^{2^*_j-1} \varphi \, dx \to 0
\]
as \( j \to \infty \). By a similar argument, we derive \((B) \to 0\).

Finally, we consider the part \((C)\). Observe that
\[
\left( \frac{1}{|x|^{N-\mu_1^j}} \ast (V_0^{2^*+1} \bar{\psi}_0) \right)(x)
\leq \int_{B_1(x)} \frac{1}{|x-y|^{N-\mu_1^j}} |V_0^{2^*-1} \bar{\psi}_0(y)| \, dy + \int_{B_1^c(x)} \frac{1}{|x-y|^{N-\mu_1^j}} |V_0^{2^*-1} \bar{\psi}_0(y)| \, dy
\]
\[
\leq C \left( \int_{B_1^c(x)} \frac{1}{|y|^{(N-\mu_1^j)(2^*-1)}} \, dy + 1 \right) \| \bar{\psi}_0 \|_{L^2(\mathbb{R}^N)},
\]
which implies that \( \frac{1}{|x|^{N-\mu_1^j}} \ast (V_0^{2^*-1} \bar{\psi}_0) \) is uniformly bounded for \( \{\mu_1^j\} \) sufficiently close to \( N \). Furthermore, similar as the proof of the Proposition 2.5 in [44], we have
\[
\lim_{n \to \infty} \| \frac{1}{|x|^{N-\mu_1^j}} \ast (V_0^{2^*-1} \psi_0) - \int_{\mathbb{R}^N} \bar{V}_0^{2^*-1} \psi_0 \, dx \|_{L^\infty(K)} = 0
\]
for any compact set \( K \subset \mathbb{R}^N \). Hence, proved the claim \((69)\).

Similarly, as \( \min \{\mu_1^j, \mu_2^j\} \to N \), we claim that
\[
(2^*_j - 1) \int_{\mathbb{R}^N} \left( \frac{1}{|x|^{N-\mu_1^j}} \ast V_0^{2^*_j} \right) V_0^{2^*_j-2} \bar{\psi}_j \varphi \, dx \to (2^* - 1) \int_{\mathbb{R}^N} V_0^{2^*-2} \bar{\psi}_0 \varphi \, dx
\]
as \( j \to \infty \).

In fact, note that
\[
\int_{\mathbb{R}^N} \frac{1}{|x|^{N-\mu_1^j}} \ast V_0^{2^*_j} V_0^{2^*_j-2} \bar{\psi}_j \varphi \, dx
\]
\[
= \int_{\mathbb{R}^N} \left( \frac{1}{|x|^{N-\mu_1^j}} \ast (V_0^{2^*_j} - V_0^{2^*-1}) \right) V_0^{2^*_j-2} \bar{\psi}_j \varphi \, dx + \int_{\mathbb{R}^N} \left( \frac{1}{|x|^{N-\mu_1^j}} \ast V_0^{2^*-1} \right) V_0^{2^*_j-2} \bar{\psi}_j \varphi \, dx,
\]
then we can estimate the integral in two parts.

Firstly, similar to the proof of the part \((A)\), for \( 0 < \varepsilon < 1 \) small enough, we apply Dominated Convergence Theorem imply that
\[
\int_{\mathbb{R}^N} \left( \frac{1}{|x|^{N-\mu_1^j}} \ast (V_0^{2^*_j} - V_0^{2^*-1}) \right) V_0^{2^*_j-2} \bar{\psi}_j \varphi \, dx
\]
\[
\leq \left\| \frac{1}{|x|^{N-\mu_1^j}} \ast (V_0^{2^*_j} - V_0^{2^*-1}) \right\| \left\| V_0^{2^*_j-2} \bar{\psi}_j \right\|_{L^\infty(K)} \left\| \left( \frac{1}{|x|^{N-\mu_1^j}} \ast V_0^{2^*-1} \right) \varphi \right\|_{L^2(\mathbb{R}^N)} \to 0.
\]

In addition, similar the above \((B)\), we have
\[
\int_{\mathbb{R}^N} \left( \frac{1}{|x|^{N-\mu_1^j}} \ast (V_0^{2^*-1}) \right) V_0^{2^*_j-2} \bar{\psi}_j \varphi \, dx \to \int_{\mathbb{R}^N} V_0^{2^*-2} \bar{\psi}_0 \varphi \, dx,
\]
which prove the claim \((70)\).
Similarly, when \( \min\{\mu_1^j, \mu_2^j\} \to N \) as \( j \to \infty \), we have
\[
2^* \mu_2^j \int_{\mathbb{R}^N} \left( \frac{1}{|x|^{N-\mu_2^j}} \ast (V_0^{2^*-1} \tilde{\psi}_j) \right) V_0^{2^*-1} \tilde{\psi}_j dx \to 2^* \left( \int_{\mathbb{R}^N} V_0^{2^*-1} \tilde{\psi}_0 dx \right) \left( \int_{\mathbb{R}^N} V_0^{2^*-1} \varphi dx \right)
\]
and
\[
(2^* \mu_2^j - 1) \int_{\mathbb{R}^N} \left( \frac{1}{|x|^{N-\mu_2^j}} \ast V_0^{2^* \mu_2^j} \psi_j \right) \psi_j dx \to (2^* - 1) \left( \int_{\mathbb{R}^N} V_0^{2^*} dx \right) \left( \int_{\mathbb{R}^N} V_0^{2^*} \tilde{\psi}_0 \varphi dx \right).
\]
(71)

(72)

Now, we may take the limit as \( j \to \infty \) in (68), and imply that
\[
\int_{\mathbb{R}^N} \nabla \tilde{\psi}_0 \nabla \varphi dx
= 2 \left( 2^* \int_{\mathbb{R}^N} V_0^{2^*-1} \tilde{\psi}_0 dx \right) \left( \int_{\mathbb{R}^N} V_0^{2^*-1} \varphi dx \right)
+ \left( (2^* - 1) \int_{\mathbb{R}^N} V_0^{2^*} dx \right) \left( \int_{\mathbb{R}^N} V_0^{2^*} \tilde{\psi}_0 \varphi dx \right).
\]
(73)

Thus, it follows from the above argument and for any \( \varphi \in D^{1,2}(\mathbb{R}^N) \) imply that \( \psi_0 \) satisfies
\[
-\Delta \tilde{\psi}_0 - 2 \left[ 2^* \left( \int_{\mathbb{R}^N} V_0^{2^*-1} \tilde{\psi}_0 dx \right) V_0^{2^*-1} - (2^* - 1) \left( \int_{\mathbb{R}^N} V_0^{2^*} dx \right) V_0^{2^*} \right] \tilde{\psi}_0 = 0.
\]
(74)

In virtue of the nondegeneracy of \( V_0 \), we have
\[
\psi_0 \in \text{span}\{\partial_1 V_0, \partial_2 V_0, \ldots, \partial_N V_0, \partial_t V_0\}.
\]
(75)

Finally, we can show that \( \tilde{\psi}_0 \neq 0 \). Indeed, we take \( \varphi = \tilde{\psi}_j \) in equation (68) to get
\[
\int_{\mathbb{R}^N} |\nabla \tilde{\psi}_j|^2 dx = \left\{ 2^* \mu_1^j \int_{\mathbb{R}^N} \left( \frac{1}{|x|^{N-\mu_1^j}} \ast (V_0^{2^* \mu_1^j} \tilde{\psi}_j) \right) V_0^{2^* \mu_1^j} \tilde{\psi}_j dx \right\}
+ \left\{ (2^* - 1) \int_{\mathbb{R}^N} \left( \frac{1}{|x|^{N-\mu_1^j}} \ast V_0^{2^* \mu_1^j} \tilde{\psi}_j \right) \tilde{\psi}_j dx \right\}
+ \left\{ 2^* \mu_2^j \int_{\mathbb{R}^N} \left( \frac{1}{|x|^{N-\mu_2^j}} \ast (V_0^{2^* \mu_2^j} \tilde{\psi}_j) \right) V_0^{2^* \mu_2^j} \tilde{\psi}_j dx \right\}
+ \left\{ (2^* - 1) \int_{\mathbb{R}^N} \left( \frac{1}{|x|^{N-\mu_2^j}} \ast V_0^{2^* \mu_2^j} \tilde{\psi}_j \right) \tilde{\psi}_j dx \right\} := P_1 + P_2.
\]

However, on the one hand, we normalize \( \tilde{\psi}_j \) as
\[
\int_{\mathbb{R}^N} |\nabla \tilde{\psi}_j|^2 dx = 1.
\]

On the other hand, we can repeat the arguments in claims (69)-(72) to get
\[
\tilde{I}_1 + \tilde{I}_2 \to 2 \left[ 2^* \int_{\mathbb{R}^N} V_0^{2^*-1} \tilde{\psi}_0 dx \right]^2 + \left( (2^* - 1) \int_{\mathbb{R}^N} V_0^{2^*} dx \right) \left( \int_{\mathbb{R}^N} V_0^{2^*} \tilde{\psi}_0 \varphi dx \right),
\]
which implies that \( \tilde{\psi}_0 \neq 0 \).

Since we know
\[
\tilde{\psi}_j = \text{span}\{\partial_1 V_0, \partial_2 V_0, \ldots, \partial_N V_0, \partial_t V_0\},
\]

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Then for any 
\[ \tilde{\psi}_j, \tilde{\eta}_b = aD_i V_0 + b \cdot \nabla V_0 = \text{span}\{ \partial_i V_0, \partial_2 V_0, \cdots, \partial_N V_0, \partial_1 V_0 \}, \]
where \( a \in \mathbb{R}, \ b = (b_1, b_2, \cdots, b_N) \in \mathbb{R}^N \). Thus we have 
\[ \langle \tilde{\psi}_j, \tilde{\eta}_b \rangle = 0 \]
However, \( \langle \tilde{\psi}_0, \tilde{\eta}_b \rangle = 0 \) contradicts with the linearized nondegeneracy of \( V_0 \) (75), since we proved that \( \tilde{\psi}_0 \neq 0 \). Therefore, we know any solution satisfies (32) must belong to \( M_\mu \) in the space \( L^2(\mathbb{R}^N) \). This completes the proof of Theorem 1.7.

5. Further remarks. We can also consider the following case with doubly critical exponents

\[ -\Delta u = (I_{\mu_1} \ast |u|^{2^*_{\mu_1}})|u|^{2^*_{\mu_1}} - 2u + (I_{\mu_2} \ast |u|^{2^*_{\mu_2}})|u|^{2^*_{\mu_2}} - 2u + |u|^{2^* - 2}u, \quad x \in \mathbb{R}^N, \ (76) \]

where \( 0 < \mu_1, \mu_2 < N \) if \( N = 3 \) or \( 4 \), \( 2^*_i := \frac{N + \mu_i}{N - 2} (i = 1, 2) \) is the upper critical exponent with respect to the Hardy-Littlewood-Sobolev inequality, \( 2^* = \frac{2N}{N - 2} \) is Sobolev critical exponent. If \( \max \{ \mu_1, \mu_2 \} \to 0 \), then we know the following equation plays the role of limit equation

\[ -\Delta u = 3|u|^{2^* - 2}u, \quad x \in \mathbb{R}^N. \quad (77) \]

In particular, we can rewrite the above equation (76) as the following equivalent system

\[
\begin{align*}
    u(x) &= \int_{\mathbb{R}^N} \frac{v(y)|u(y)|^{2^*_{\mu_1}} - 2u(y)}{|x - y|^{N - 2}} dy + \int_{\mathbb{R}^N} \frac{w(y)|u(y)|^{2^*_{\mu_2}} - 2u(y)}{|x - y|^{N - 2}} dy + \int_{\mathbb{R}^N} \frac{|u(y)|^{2^* - 2}u(y)}{|x - y|^{N - 2}} dy, \\
v(x) &= \int_{\mathbb{R}^N} \frac{|u(y)|^{2^*_{\mu_1}}}{|x - y|^{N - 2}} dy, \\
w(x) &= \int_{\mathbb{R}^N} \frac{|u(y)|^{2^*_{\mu_2}}}{|x - y|^{N - 2}} dy.
\end{align*}
\]

In fact, we may repeat the arguments in Sections 2-4 to prove the symmetry, uniqueness and nondegeneracy properties. But we will not outline the proof due to the length of the article.

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