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ON THE CONTROLLABILITY OF RACING SAILING BOATS WITH FOILS

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ABSTRACT. The development of foils for racing boats has changed the strategy of sailing. Recently, the America’s cup held in San Francisco, has been the theatre of a tragicomic history due to the foils. During the last round, the New-Zealand boat was winning by 8 to 1 against the defender USA. The winner is the first with 9 victories. USA team understood suddenly (may be) how to use the control of the pitching of the main foils by adjusting the rake in real time in order to stabilize the ship. And USA won by 9 victories against 8 to the challenger NZ. Our goal in this paper is to point out few aspects which could be taken into account in order to improve this mysterious control law which is known as the key of the victory of the USA team. There are certainly many reasons and in particular the cleverness of the sailors and of all the engineering team behind this project. But it appears interesting to have a mathematical discussion, even if it is a partial one, on the mechanical behaviour of these extraordinary sailing boats. The numerical examples given here are not the true ones. They have just been invented in order to explain the theoretical developments concerning three points: the possibility of tacking on the foils for sailing upwind, the nature of foiling instabilities, if there are, when the boat is flying and the control laws.

FIGURE 1. Principle of the flying boat using foils

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1. Introduction. The control of foiling during the America’s cup appeared to be a determinant point in the success of Oracle Team USA (OTUSA). In particular during upwind legs, when the boat had to avoid the waves created by the wake of the preceding boat, the automatic stabilisation is a fundamental advantage that OTUSA exploited in a smart way and finally won the competition. Such situations are represented on the figures 2 taken from TV transmissions during the America’s cup in San Francisco (September 2013).

Only two movements of the ship are taken into account in this paper : the heaving which is a normal displacement to the surface of the sea, and the pitching which is the rotation around a horizontal axis transverse to the main direction of the ship. Hence, the yawing angle and the rolling are eliminated from our model. Obviously they are meaningful, but according to our mind, not necessary for the understanding of our purpose.

Our aim is to study the controllability of the boat modelling the skipper’s action on the foil by a control law. The inclination of the main foil should be manually driven but a hydraulic ram can be used for the control process (rules of the race) using the high pressure collected from a small hole at the front of the foil. Because the system is a second order one (with inertia, damping and stiffness), only a phase control can lead to optimal results. This driven angle is named the rake. It appears, in the numerical simulations, that the regulation law strongly depends on the ship velocity. Even if the experimental data that we introduce in our numerical model could be improved, they are sufficient in order to give an idea of how things work.

The rudder is supposed to be fixed, the control being an action only on the main foil. The foil handling is modelled by a rotation about a center $O$ and through an angle $\alpha$ : this angle is the control variable.

We study several controls laws. The first one is an exact control law. From a numerical point of view, the approach consists in an optimal control problem. From a theoretical point of view, the exact controllability will be obtain, under suitable conditions, by an asymptotic analysis from optimal control problems. This kind of controls are parts of the large family of the theory of ‘controls without constraint’. Unfortunately, this is not realistic in our case since it would be not acceptable to obtain an angle at the foil equal to $\frac{\pi}{2}$ for example!

The second type of control is an optimization problem with constraints : the magnitude of the control, which is still the angle of the main foil, being restricted to 2 or 3 degrees. The time of controllability and the magnitude of the constraint are then connected.
The third and last kind of control law is a sign feedback loop system. The law is commonly used by planes when they enter in a turbulence zone. This type of control is the one used by OTUSA team in 2013.

Let us notice that this paper rests on a previous one (see [7]) where we gave a simple and precise mathematical model and analysis of such boats in a bi-dimensional case which respects the following facts: the existence of the foiling velocity under which the boat can’t stand up on its foil, the possibility that the velocity of the boat can be greater than the wind velocity, and the possibility to discuss the stall flutter phenomenon of the foils. We consider the same model and we recall in a first section some results obtained in [7].

2. Dynamical model of the boat. The orthonormal basis of $\mathbb{R}^3$ is denoted by $(e_x, e_y, e_z)$.

The velocity of the boat is $-ue_x$ where $u > 0$ and therefore the velocity of the flow of the water in the basis connected to the boat is $ue_x$. As said before, the movement is assumed to be represented by two functions (see figure 3): the heaving $z$ and the pitching angle $\gamma$ in the plan $(e_z, e_x)$. The equilibrium is written at an arbitrary point - say $O$. For sake of convenience, it is chosen to be the center of rotation of the main foil.

The following notations are used:

**Characteristics of the boat, of the air and of the water**

- $\varrho_a$ mass density of the air,
- $\varrho_e$ mass density of the water,
- $g = 9.81 m/s^2$ is the gravity,
- $-ue_x$ velocity of the ship,
- $M$ is the mass of the ship,
- $G$ center of mass of the boat,
- $J_G$ is the inertia around the center of mass $G$ in the pitching movement,
- $J_O$ is the inertia around the center of mass $O$ in the pitching movement,
- $M_o$ is the pitching moment of the external forces at point $O$,
\(d_s = O'S\) is the length of the stick supporting the rudder steering, \(O'\) being the anchor point of the rear foil,

\(d_f = OF\) is the length of the foil in the depth direction,

\(S_s, S_f\) are respectively the cross sections of the foils at the extremities of the rudder and the main foil,

\(a\) (respectively \(b\)) is the distance between the center of mass and \(O\) (respectively \(O'\)),

\(h = a + b = OO'\),

\(d_{oq}\) is the distance from the rotation point of the foil to the center of mass of this foil.

**Variables for the description of the movement of the boat:**

\(z\) is the heaving,

\(\gamma\) is the pitching angle.

**For the angles, apparent velocities and forces:**

\(\alpha\) is the angle of attack of the mail foil,

\(\beta\) is the angle of attack of the rear foil and it is supposed to be fixed,

\(c_{zf}\) and \(c_{zs}\) are the lift hydrodynamic coefficients for the main foil and the rear foil.

They are continuous functions in their variables,

\(c_{mf}\) et \(c_{ms}\) are the pitching hydrodynamic coefficients at points \(F\) and \(S\). They are continuous functions in their variables,

\(L\) is an arbitrary length used in the definition of pitching hydrodynamic coefficients,

\(v\) is the absolute wind velocity: it is in the plane \((e_x, e_y)\),

\(V\) the modulus of \(v\), absolute wind velocity,

\(\theta = (v, -e_x)\) is the angle between the velocity of the wind and the direction in which the boat is moving forward,

\(V_a\) is the modulus of the apparent velocity of the wind,

\(V_{as}, V_{af}\) are respectively the apparent flow velocity at the two foils: one on the rudder and the other one which, is the main one, supported by the daggerboard,

\(u\) is the modulus of the velocity of the ship,

\(c_f\) and \(\xi\) are respectively the stiffness and the damping coefficient of the system used for the stabilisation of the main foil.

The forces applied to the ship and implying an evolution of these two previous functions, are those due the rear and the main foils. The local hydrodynamical coefficients \((c_z\) for the lift and \(c_m\ for the pitching moment) depend respectively on the apparent local angle of attack of each foil. It is denoted by \((\beta + \gamma)_a\) for the rear foil and by \((\alpha + \gamma)_a\ for the main one.

The variables are \(z\) and \(\gamma\). Let us recall that \(\beta\ is assumed to be fixed (equal to \(\beta_0\)), and the evolution of the pitching angle of the rear foil is only due to the global pitching of the boat -say \(\gamma\).

We introduce several matrices of the system \((M, C\) et \(K)\ whose coefficients will be explicited in a next step:

\[
M = \begin{pmatrix}
M & -aM \\
-aM & J_0
\end{pmatrix}, \quad C = \begin{pmatrix}
C_{11} & C_{12} \\
C_{21} & C_{22}
\end{pmatrix}, \quad K = -\begin{pmatrix}
K_1 & T_1 \\
K_2 & T_2
\end{pmatrix},
\]

(1)

and the righthandsides \(B, E\) are defined by:

\[
B = \begin{pmatrix}
B_1 \\
B_2
\end{pmatrix}, \quad E = \begin{pmatrix}
E_1 \\
E_2
\end{pmatrix}.
\]

(2)
We proved in [7] that the linearized movement of the boat can be modelled by the following system at the equilibrium point: find \( X = \begin{pmatrix} z \\ \gamma \end{pmatrix} \) satisfying:

\[
\mathcal{M} \ddot{X} - \mathcal{C} \dot{X} + \mathcal{K} X = \delta \mathcal{B} + \dot{\delta} \mathcal{E},
\]

with initial data \( X(0) = X_0 \in \mathbb{R}^2 \) and \( \dot{X}(0) = X_1 \in \mathbb{R}^2 \). For \( \xi_1 \) and \( \xi_2 \) small enough, we set:

\[
\begin{align*}
    c_{zs}(\beta^0 + \xi) &= c_{zs}^0 + R_{zs} \xi + o(\xi), \\
    c_{zf}(\alpha^0 + \xi) &= c_{zf}^0 + R_{zf} \xi + o(\xi), \\
    c_{ms}(\beta^0 + \xi) &= c_{ms}^0 + R_{ms} \xi + o(\xi), \\
    c_{mf}(\alpha^0 + \xi) &= c_{mf}^0 + R_{ms} \xi + o(\xi),
\end{align*}
\]

The expressions in (3) at the equilibrium point are the following:

- the stiffness matrix is:

\[
\mathcal{K} = -\rho_e u^2 \begin{pmatrix} 0 & R_1 \\ 0 & R_2 \end{pmatrix}.
\]

where \( R_1 \) and \( R_2 \) are defined by:

\[
R_1 = \left( \frac{S_s R_{zs}}{2} + \frac{S_f R_{zf}}{2} \right),
\]

and

\[
R_2 = \frac{1}{2} [-S_s h R_{zs} + S_f d_f \sin(\alpha_0) R_{zf} + S_s L R_{ms} + S_f L R_{mf} + S_s d_s c_{zs}^0 + S_f d_f \cos(\alpha_0) c_{zf}^0].
\]

- the coefficients of \( C \) are given explicitly in the following table:

| \( C_{11} \) | \(-\frac{\rho_e}{2} u \left( S_s R_{zs} + S_f R_{zf} \right)\) |
| \( C_{12} \) | \( \rho_e u \left( S_s d_s c_{zs}^0 + S_f d_f \cos(\alpha_0) c_{zf} + \frac{S_s h}{2} R_{zs} - \frac{S_f d_f}{2} \sin(\alpha_0) R_{zf} \right) \) |
| \( C_{21} \) | \( \frac{\rho_e}{2} u \left( -S_s L R_{ms} - L S_f R_{mf} - S_f d_f \sin(\alpha_0) R_{zf} + S_s h R_{zs} \right) \) |
| \( C_{22} \) | \( -\frac{\rho_e}{2} S_s h u \left[ 2 d_s c_{zs}^0 + h R_{zs} \right] + \frac{\rho_e u S_f d_f^2}{2} \left[ \sin(2 \alpha_0) c_{zf}^0 - \sin(\alpha_0)^2 R_{zf} \right] \\
+ \frac{\rho_e u L S_s}{2} \left[ 2 d_s c_{ms}^0 + h R_{ms} \right] + \frac{\rho_e u L}{2} S_f d_f \left[ -\sin(\alpha_0) R_{mf} + 2 \cos(\alpha_0) c_{mf}^0 \right] \) |
• the righthandsides $\mathcal{B}$ and $\mathcal{E}$ are:

$$\mathcal{B} = \varrho e u^2 \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}$$

with

$$G_1 = \frac{S_f}{2} R_z f,$$

$$G_2 = \frac{S_f}{2} \left[ LR_m f + d_f \sin(\alpha_0) R_z f + d_f \cos(\alpha_0) c_{zf}^0 \right];$$

and

$$\mathcal{E} = \varrho e u \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}$$

with

$$H_1 = \frac{S_f}{2} \left[ 2d_f \cos(\alpha^0) c_{zf}^0 - d_f \sin(\alpha^0) R_z f \right],$$

$$H_2 = \frac{S_f d_f}{2} \left[ d_f (\sin(2\alpha^0) c_{zf}^0 - \sin(\alpha^0)^2 R_z f \right]$$

$$+ L (-\sin(\alpha^0) R_m f + 2 \cos(\alpha^0) c_{m f}^0).$$

We now turn to the controllability of the boat. There are two steps: the first one consists in defining an exact control without constraint. The second step takes into account the constraints on the control magnitude. In this second step appears the links between the control time -say $T$- and the bounds on the magnitude of the control (ie. the maximum of the magnitude of the rake). Let us set $\delta = \alpha - \alpha_0$, where $\alpha_0$ is a nominal position of $\alpha$ at the static equilibrium.

3. **Control of the dynamical behavior via the rake of the main foil without constraint.**

We first consider an optimal control problem depending on a small parameter $\varepsilon$. Exact controllability will be discussed to justify an asymptotic analysis when $\varepsilon$ tends to 0 and it will involve restrictions on the coefficients.

3.1. **The optimal control problem.** The control variable is the rake $\alpha$. In this study we don’t discuss the pratical feasibility of the optimal control law that we compute. It is just an indication of what should be done and it can be understood as an educational result, as far as the real driven law is manual. The strategy that we are using consist in defining a control criterion and to minimize it with respect to the control variable. This control criterion includes a norm of the gap between the observed state variables at a time $T$ (pitching and heaving) and the desired values of these functions. Furthermore we make use of a linearized approximation of the state equation recalled in section 2.

Therefore the control problem consists in defining a criterion depending on the state variables (ie. $z$ and $\gamma$) but also on the control $\delta = \alpha - \alpha_0$ and to minimize it with respect to $\delta$. Let us first define a delay -say $T$- corresponding to the reaction time required in the control process. Our goal is to define a control law such that at time $T$ the ship is back to the equilibrium:

$$z(T) = z_0, \quad \dot{z}(T) = \gamma(T) = \dot{\gamma}(T) = 0.$$  

Let us consider $a_0 > 0$, $b_0 > 0$ and $\varepsilon > 0$. Let $X_0 \in \mathbb{R}^2$ et $X_1 \in \mathbb{R}^2$ be the initial data. The criterion for the optimal control problem, is defined as follows:
Let us compute the gradient of $J_\varepsilon$. Let $X$ be the solution of
\[
\mathcal{M}\dddot{X} - C\dddot{X} + KX = B\delta + E\dot{\delta}, \quad X(0) = X_0, \quad \dot{X}(0) = X_1,
\]
and $Z$ be the derivative of $X$ with respect to $\delta$ in the direction $\omega$. $Z$ is solution of:
\[
\mathcal{M}\ddot{Z} - C\ddot{Z} + KZ = B\dot{\delta} + E\dot{\dot{\delta}}, \quad Z(0) = 0, \quad \dot{Z}(0) = 0.
\]

We write $X = \begin{pmatrix} z \\ \gamma \end{pmatrix}$ and $Z = \begin{pmatrix} \ddot{z} \\ \dot{\gamma} \end{pmatrix}$. The derivative of $J_\varepsilon$, denoted by $J_\varepsilon'$, is
\[
J_\varepsilon'(\delta, \omega) = (z(T) - z_0)\ddot{z}(T) + \dot{z}(T)\dddot{z}(T) + \gamma(T)\dot{\gamma}(T) + \dot{\gamma}(T)\dddot{\gamma}(T) + \varepsilon \int_0^T \left( a_0 \delta \omega + b_0 \dot{\delta} \dot{\omega} \right) dt.
\]

The optimal control is finally defined as the solution of the following problem:
\[
J_\varepsilon(\delta) = \min_{\omega \in H^1_0([0,T])} J_\varepsilon(\omega). \tag{14}
\]

**Remark 1.** The choice of the space $H^1_0([0,T])$ has been done in order to have a finite cost for the control $\delta$. The boundary conditions could be different (at $t = 0$ and $t = T$). For instance, with free edge conditions, one would obtain more degrees of freedom for the control. But the discontinuity at the extremities, is not always a good strategy in the practical implementation. Nevertheless, this could be useful in particular case where the controllability condition is not satisfied. This point is discussed in the following. \hfill \Box

The optimization problem (14) is very classical (see [3] or [14]): there exists a unique solution $\delta \in H^1_0([0,T])$ solution of (14). The first point is to formulate the optimality relation which is obtained in this case by writing that $J_\varepsilon'(\delta) = 0$. It is quite classical to introduce the adjoint state $P$ solution of the following differential system. Let $X$ be the solution at the optimum value and $P \in H^2_0([0,T])^2$ (adjoint state) be the solutions of:
\[
\mathcal{M}\dddot{X} - C\dddot{X} + KX = B\delta + E\dot{\delta}, \quad X(0) = X_0, \quad \dot{X}(0) = X_1,
\]
and
\[
\begin{aligned}
\mathcal{M}\dddot{P} + C\dddot{P} + K\dot{P} = 0 & \quad \text{on } ]0,T[, \\
\mathcal{M}P(T) = \begin{pmatrix} \dot{z}(T) \\ \dot{\gamma}(T) \end{pmatrix}, \quad \mathcal{M}\dot{P}(T) = - \begin{pmatrix} z(T) - z_0 \\ \gamma(T) \end{pmatrix} - \dot{C}P(T).
\end{aligned} \tag{16}
\]

The optimality relation can be written (see [11]): (\cdot denotes the scalar product in $\mathbb{R}^2$)
\[
\forall \omega \in H^1_0([0,T]), \quad \int_0^T \left( B\omega + E\dot{\omega} \right) P dt + \varepsilon \int_0^T \left( a_0 \delta \omega + b_0 \dot{\delta} \dot{\omega} \right) dt = 0.
\]
Since $\omega \in H^1_0([0,T])$, this equation leads to:

$$\begin{cases} 
\dot{t}B \dot{P} - \dot{t}E \dot{P} + \varepsilon [a_0 \delta - b_0 \ddot{\delta}] = 0 \quad \text{on } [0,T], \\
\delta(0) = \delta(T) = 0.
\end{cases} \quad (17)$$

3.1.1. Computation of the optimum : first formula. Let us introduce the Hilbert basis $(w_n)_n$ of $L^2([0,T])$:

$$w_n(t) = \sqrt{\frac{2}{T}} \sin\left(\frac{n\pi t}{T}\right) \quad \text{for } n \geq 1$$

and let us look for:

$$\delta(t) = \sum_{p \geq 1} A_p w_p(t) \quad \text{where } \forall p, \ A_p \in \mathbb{R}.$$ 

With $r = \sqrt{\frac{a_0}{b_0}}$ and $f(t) = \frac{1}{\varepsilon b_0} [tB \dot{P} - \dot{t}E \dot{P}]$, we obtain

$$\ddot{\delta} - r^2 \delta = f \quad \text{on } [0,T] \quad (18)$$

and

$$A_n = - \frac{b_0}{(a_0 + b_0 \frac{n^2 \pi^2}{T^2})} \int_0^T f(s) w_n(s) ds.$$ 

Hence

$$\delta(t) = \frac{2}{T \varepsilon} \sum_{n \geq 1} \left( \int_0^T \frac{tE \dot{P}(s) - tB \dot{P}(s)}{a_0 + b_0 \frac{n^2 \pi^2}{T^2}} \sin\left(\frac{n\pi s}{T}\right) ds \right) \sin\left(\frac{n\pi t}{T}\right). \quad (19)$$

3.1.2. Computation of the optimum : second formula. We present another way to compute $\delta$ which is numerically more stable. With the notation of the previous paragraph, and considering (18), we write:

$$\delta(t) = A(t) \cosh(rt) + B(t) \sinh(rt)$$

with

$$\begin{cases} 
\dot{A} \cosh(rt) + \dot{B} \sinh(rt) = 0 \\
\dot{A} \sinh(rt) + \dot{B} \cosh(rt) = \frac{f(t)}{r}.
\end{cases}$$

We easily get

$$\delta(t) = A_0 \cosh(rt) + B_0 \sinh(rt) + \frac{1}{r} \int_0^t f(s) \sinh(r(t-s)) ds.$$ 

Since $\delta(0) = \delta(T) = 0$, we must have:

$$A_0 = 0 \quad \text{et} \quad rB_0 \sinh(rT) + \int_0^T f(s) \sinh(r(T-s)) ds = 0.$$ 

The second expression of the optimum is therefore:

$$\delta(t) = \frac{1}{r} \left( \frac{\sinh(rt)}{\sinh(rT)} \int_0^T f(s) \sinh(r(T-s)) ds + \int_0^t f(s) \sinh(r(t-s)) ds \right). \quad (20)$$

Our expressions of the optimal control depends on the small parameter $\varepsilon$. In what follows, we write $\delta_\varepsilon$, $P_\varepsilon$ and $X_\varepsilon$ the optimum data of the functional $J_\varepsilon$ and we are now interested in the asymptotic analysis when $\varepsilon$ tends to 0.
3.2. **Asymptotic analysis.** Let us introduce the set of admissible controls:

\[ U_{\text{ex}} = \left\{ \omega \in H_0^1([0,T]) \mid \text{the solution } X \text{ of (15) satisfies } \dot{X}(T) = \frac{\epsilon}{2}(z_0, 0) \text{ and } \dot{X}(T) = \frac{\epsilon}{2}(0, 0) \right\}. \]

There is no chance to control exactly if (and only if) \( U_{\text{ex}} = \emptyset \). In the contrary, if \( U_{\text{ex}} \neq \emptyset \), it’s a close and convex set in \( H_0^1([0,T]) \). We prove:

**Theorem 3.1.** Assume \( U_{\text{ex}} \neq \emptyset \). The sequence \((\delta_\epsilon)_\epsilon\) strongly converges in \( H_0^1([0,T])\) to the unique point \( \delta \in U_{\text{ex}} \) such that:

\[
\int_0^T \left( a_0 \delta_\epsilon^2 + b_0 \dot{\delta}_\epsilon^2 \right) dt = \min_{\omega \in U_{\text{ex}}} \int_0^T \left( a_0 \omega^2 + b_0 \dot{\omega}^2 \right) dt \tag{21}
\]

**Proof.** Let \( \delta_\epsilon \) be the minimum argument of \( J_\epsilon \) in \( H_0^1([0,T]) \) and \( X_\epsilon \) be the solution of (15) associated to \( \delta_\epsilon \). Let \( \omega_0 \in U_{\text{ex}} \) (independent on \( \epsilon \)). For every \( \epsilon > 0 \):

\[
J_\epsilon(\delta_\epsilon) \leq J_\epsilon(\omega_0) \quad \text{and} \quad J_\epsilon(\omega_0) = \frac{\epsilon}{2} \int_0^T \left( a_0 \omega_0^2 + b_0 \dot{\omega}_0^2 \right) dt.
\]

On the one hand, we deduce that

\[
\forall \epsilon > 0, \quad \int_0^T \left( a_0 \delta_\epsilon^2 + b_0 \dot{\delta}_\epsilon^2 \right) dt \leq \int_0^T \left( a_0 \omega_0^2 + b_0 \dot{\omega}_0^2 \right) dt, \tag{22}
\]

and on the other hand:

\[
|X_\epsilon(T) - \frac{\epsilon}{2}(z_0, 0)|^2 + |\dot{X}_\epsilon(T)|^2 = \mathcal{O}(\epsilon). \tag{23}
\]

We deduce from (22) that \((\delta_\epsilon)_\epsilon\) is bounded in \( H_0^1([0,T]) \). Hence, there exists a subsequence (still denoted by \((\delta_\epsilon)_\epsilon\)) weakly converging in \( H_0^1([0,T]) \) to a function \( \delta \in H_0^1([0,T]) \). By linearity of the model, \((X_\epsilon)_\epsilon\) weakly converges in \( H^1([0,T]) \) to the solution \( X \) of (15) associated to \( \delta \). Furthermore, \( \mathcal{M} \) does not depend on \( \epsilon \) and is invertible thus \((X_\epsilon)_\epsilon\) is then bounded in \( H^2([0,T]) \). After a second extraction of a subsequence, \((X_\epsilon)_\epsilon\) strongly converges in \( C^1([0,T]) \). Assertion (23) leads to \( X(T) = \frac{\epsilon}{2}(z_0, 0) \) and \( \dot{X}(T) = \frac{\epsilon}{2}(0,0) \). We have proved that a subsequence of \((\delta_\epsilon)_\epsilon\) converges to an exact control \( \delta \in U_{\text{ex}} \). Since:

\[
\int_0^T \left( a_0 \delta_\epsilon^2 + b_0 \dot{\delta}_\epsilon^2 \right) dt \leq \lim_{\epsilon \to 0} \inf \int_0^T \left( a_0 \delta^2 + b_0 \dot{\delta}^2 \right) dt,
\]

we have that (22), which is valid for every \( \omega_0 \in U_{\text{ex}} \), implies (21). Because the functional

\[
J(\omega) = \int_0^T \left( a_0 \omega^2 + b_0 \dot{\omega}^2 \right) dt
\]

is strictly convex and continuous on \( U_{\text{ex}} \) (close and convex), its reaches its minima at a unique point. Therefore, the (weak) limit point of the sequence \((\delta_\epsilon)_\epsilon\) is unique and thus all the sequence converges to \( \delta \). Finally, let’s take \( \omega_0 = \delta \) in (22). We get

\[
J(\delta) \leq \liminf_{\epsilon \to 0} J(\delta_\epsilon) \leq \limsup_{\epsilon \to 0} J(\delta_\epsilon) \leq J(\delta)
\]

and we deduce that \((||\delta_\epsilon||\mathcal{H}_0^1([0,T]))_\epsilon\) converges to \( ||\delta||\mathcal{H}_0^1([0,T]) \). The weak convergence in \( H_0^1([0,T]) \) and the convergence of the \( H_0^1([0,T]) \)-norms imply the strong convergence of \((\delta_\epsilon)_\epsilon\) to \( \delta \) in \( H_0^1([0,T]) \), unique minima of \( J \) on \( U_{\text{ex}} \). The theorem is proved. \( \square \)
Remark 2. The exact controllability depends on the data and on the control time \( T \). The more \( T \) is small the more the amplitude of the control is large. But, in our case, the amplitude of the control is limited by the angles at the foil. This is why, it is particularly interested to study the controllability with constraint. This is done in the future sections.

We now study condition \( U_{ex} \neq \emptyset \) and the computation of the exact control when it exists. We end this section with numerical simulations.

3.3. The exact controllability and condition \( U_{ex} \neq \emptyset \). A formal development with respect to \( \varepsilon \) leads to another way to solve exact controllability. This approach explicit the unique continuation property underlying the condition \( U_{ex} \neq \emptyset \). Let us set a priori:

\[
\left\{ \begin{array}{l}
X_\varepsilon = X^0 + \varepsilon X^1 + \varepsilon^2 X^2 + \ldots, \\
P_\varepsilon = P^0 + \varepsilon P^1 + \varepsilon^2 P^2 + \ldots, \\
\delta_\varepsilon = \delta^0 + \varepsilon \delta^1 + \varepsilon^2 \delta^2 + \ldots
\end{array} \right.
\] (24)

By introducing a priori these expressions into the system (15), (16) et (17) and by equating the terms of same power in the resulting expressions, one obtains:

- at the order 0:

\[
\left\{ \begin{array}{l}
M \ddot{X}^0 - C \dot{X}^0 + \kappa X^0 = B \delta^0 + \mathcal{E} \delta^0, \\
M \ddot{P}^0 + iC \dot{P}^0 + i\kappa P^0 = 0,
\end{array} \right.
\] (25)

\[
M P^0(T) = \dot{X}^0(T), \\
M \dot{P}^0(T) = -\left( \begin{array}{c} z^0(T) - z_0 \\
\gamma^0(T) \end{array} \right) - i\mathcal{C} P^0(T),
\]

\[
\forall t \in [0, T], \quad iB P^0(t) - i\mathcal{E} \dot{P}^0(t) = 0.
\]

- at the order 1:

\[
\left\{ \begin{array}{l}
M \ddot{X}^1 - C \dot{X}^1 + \kappa X^1 = B \delta^1 + \mathcal{E} \delta^1, \\
M \ddot{P}^1 + iC \dot{P}^1 + i\kappa P^1 = 0,
\end{array} \right.
\] (26)

\[
M P^1(T) = \dot{X}^1(T), \\
M \dot{P}^1(T) = -X^1(T) - i\mathcal{C} P^1(T),
\]

\[
[a_0 \delta^0 - b_0 \delta^0] + iB P^1 - i\mathcal{E} \dot{P}^1 = 0,
\]

and others orders are similar to first one.

Our first point is to prove that with reasonable assumptions, one has \( P^0(t) = 0 \) on \([0, T]\). This is in fact an exact controllability result for \( z^0(T), \dot{z}^0(T), \dot{\gamma}^0(T) \) and \( \gamma^0(T) \) because it will imply that \( z^0(T) = z_0, \dot{z}^0(T) = \dot{\gamma}^0(T) = \gamma^0(T) = 0 \).

Let us now turn to a controllability result which is a more adapted version of the general Bellman’s result [1]. We assume that \( B, \mathcal{E} \) are linearly independent. Let \( B^*, \mathcal{E}^* \) be the dual basis of \( B, \mathcal{E} \) defined by:

\[
B^*, B = 1, \quad B^*, \mathcal{E} = 0, \quad \mathcal{E}^*, \mathcal{E} = 1, \quad \mathcal{E}^*, B = 0.
\]
Let \( H, L, Z_1, Z_2 \) and \( Z_3 \) be the following matrices or vectors:
\[
\begin{align*}
H &= \mathcal{M}^{-1} \mathcal{C}, \\
L &= \mathcal{M}^{-1} \mathcal{K}, \\
Z_1 &= H\mathcal{B}^*, \\
Z_2 &= H\mathcal{E}^*, \\
Z_3 &= L\mathcal{B}^*, \\
Z_4 &= L\mathcal{E}^*.
\end{align*}
\]

We prove:

**Theorem 3.2.** Assume \( \mathcal{B}, \mathcal{E} \) are linearly independent and that \( Z_4\mathcal{E} \neq 0 \). Let \( r_i \) (\( i = 1, 2 \)) be the roots of:
\[
\begin{align*}
r^2Z_4\mathcal{E} + r\left[ Z_4\mathcal{E}Z_1\mathcal{B} - Z_4\mathcal{B}Z_1\mathcal{E}\right] \\
+ \left[ Z_4\mathcal{E}Z_2\mathcal{B} - Z_4\mathcal{B}Z_2\mathcal{E} + Z_4\mathcal{E}Z_3\mathcal{B} - Z_4\mathcal{B}Z_3\mathcal{E}\right] &= 0.
\end{align*}
\]
The unique continuation property \( P^0 = 0 \) is valid under the hypotheses:
1. if \( r_1 \neq r_2 \) (single roots), and for \( i = 1, 2 \):
\[
r_i^2[1 + Z_1\mathcal{E}] + r_i\left[ Z_2\mathcal{E} + Z_3\mathcal{E}\right] + Z_4\mathcal{E} \neq 0.
\]
2. if \( r_1 = r_2 \) (double roots), it satisfies:
\[
r_1^2[1 + Z_1\mathcal{E}] - Z_4\mathcal{E} \neq 0.
\]

**Proof.** Writing \( P^0(t) = \xi_1(t)\mathcal{B}^* + \xi_2(t)\mathcal{E}^* \) in (25), we get on \( [0, T] \):
\[
\begin{align*}
\dot{\xi}_1 + \xi_1Z_1\mathcal{B} + \xi_2Z_2\mathcal{B} + \xi_1Z_3\mathcal{B} + \xi_2Z_4\mathcal{B} &= 0, \\
\ddot{\xi}_2 + \dot{\xi}_1Z_1\mathcal{E} + \xi_2Z_2\mathcal{E} + \dot{\xi}_1Z_3\mathcal{E} + \xi_2Z_4\mathcal{E} &= 0, \\
\xi_1 &= \dot{\xi}_2,
\end{align*}
\]
and thus (still on \( [0, T] \))
\[
\begin{align*}
\ddot{\xi}_1 + \dot{\xi}_1Z_1\mathcal{B} + \xi_1\left[ Z_2\mathcal{B} + Z_3\mathcal{B}\right] + \xi_2Z_4\mathcal{B} &= 0, \\
\dot{\xi}_1[1 + Z_1\mathcal{E}] + \xi_1\left[ Z_2\mathcal{E} + Z_3\mathcal{E}\right] + \xi_2Z_4\mathcal{E} &= 0, \\
\xi_1 &= \dot{\xi}_2.
\end{align*}
\]
Let us multiply the first equation by \( Z_4\mathcal{E} \), the second one by \( Z_4\mathcal{B} \) and by adding, we get:
\[
\begin{align*}
\dot{\xi}_1Z_4\mathcal{E} + \dot{\xi}_1\left[ Z_4\mathcal{E}Z_1\mathcal{B} - Z_4\mathcal{B}Z_1\mathcal{E} - Z_4\mathcal{B}\right] \\
+ \xi_1\left[ Z_4\mathcal{E}(Z_2\mathcal{B} + Z_3\mathcal{B}) - Z_4\mathcal{B}(Z_2\mathcal{E} + Z_3\mathcal{E})\right] &= 0, \\
\dot{\xi}_1[1 + Z_1\mathcal{E}] + \xi_1\left[ Z_2\mathcal{E} + Z_3\mathcal{E}\right] + \xi_2Z_4\mathcal{E} &= 0, \\
\xi_1 &= \dot{\xi}_2.
\end{align*}
\]
The solutions are \( \xi_1(t) = A_1e^{r_1t} \) where \( r_i \) is a root of
\[
\begin{align*}
r^2Z_4\mathcal{E} + r\left[ Z_4\mathcal{E}Z_1\mathcal{B} - Z_4\mathcal{B}Z_1\mathcal{E} - Z_4\mathcal{B}\right] \\
+ \left[ Z_4\mathcal{E}Z_2\mathcal{B} - Z_4\mathcal{B}Z_2\mathcal{E} + Z_4\mathcal{E}Z_3\mathcal{B} - Z_4\mathcal{B}Z_3\mathcal{E}\right] &= 0.
\end{align*}
\]
Case 1- Assume that the roots are single. We get \( \xi_2(t) = \frac{A_i}{r_i} e^{r_i t} + c_0 \) where \( c_0 \) is constant. With this expression in the second equation of (27), we obtain for each set of solutions \( (\xi_1, \xi_2) \) and for \( i = 1, 2 \):

\[
\begin{align*}
    i &= 1, 2 : \quad r_i^2 [1 + Z_1. E] + r_i [Z_2. E + Z_3. E] + Z_4. E = 0,

    \text{and}

    Z_4. E = 0.
\end{align*}
\]

Finally, if none of these relations is true, we must have:

\[
\forall t \in [0, T] : \quad \xi_1(t) = \xi_2(t) = 0 \text{ and thus } \forall t \in [0, T] : \quad P^0(t) = 0.
\]

This ends the proof in the case of single roots.

Case 2- Let us assume that there exists a double roots: \( r_1 = r_2 \). The solutions of (27) are:

\[
\begin{align*}
    \xi_1(t) &= A e^{r_1 t}, \xi_2(t) = \frac{A}{r_1} e^{r_1 t} \text{ and } \xi_1(t) = B t e^{r_1 t}, \xi_2(t) = B (r_1 t - 1) e^{r_1 t}.
\end{align*}
\]

The following condition is needed for the unique continuation property:

\[
\begin{align*}
    r_1^2 [1 + Z_1. E] - Z_4. E \neq 0.
\end{align*}
\]

The theorem is proved \( \square \)

Let us now consider special cases for which the condition can be simplified.

3.3.1. The case without kinetic coupling. We consider that the kinetic coupling can be neglected and thus \( C \) is zero. If \( C = 0 \) then \( E = 0 \) in the system (25). The unique continuation property becomes:

\[
\forall t \in [0, T] : \quad P_0(t) = 0 \text{ on } [0, T] \Rightarrow P^0(t) = 0 \text{ on } [0, T].
\]

We finally prove:

**Theorem 3.3.** A sufficient and necessary condition is:

\[
\begin{align*}
    \alpha_0 \cdot B \neq 0 \quad \text{and} \quad \det(K \begin{pmatrix} 0 \\ 1 \end{pmatrix}, B) \neq 0.
\end{align*}
\]

This condition is a very simple test to do.

**Remark 3.** We write the second column of \( K \):

\[
K \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -K_s + K_f,
\]

where \( -K_s \) is the contribution of the rudder and \( K_f \) the foil’s one. We have \( K_f = -B \) and thus \( K_s = -B - K \begin{pmatrix} 0 \\ 1 \end{pmatrix} \). The condition \( \det(K \begin{pmatrix} 0 \\ 1 \end{pmatrix}, B) \neq 0 \) is the same as \( \det(K_s, B) \neq 0 \) and this means that the two contributions (rudder and foil) are not collinear.

**Proof:** We consider a unit vector \( D \) of \( \mathbb{R}^2 \) orthogonal to \( B \). If \( B \cdot P^0(t) = 0 \text{ on } [0, T] \), we get

\[
P^0(t) = \xi(t)D \text{ on } [0, T].
\]

System (25) leads to:

\[
M^{-1} K D B_\xi(t) = 0 \text{ on } [0, T].
\]
The unique continuation property is then:

\[
\mathcal{M}^{-1} \mathcal{K}\mathcal{D}.B \neq 0.
\]

As said before, the matrix \( \mathcal{K} \) is made of two contributions: one of the rudder and one of the foil which is \( -\mathcal{B} \). We write \( -\mathcal{K}_s \) the rudder’s one, and we obtain:

\[
\mathcal{K} = \begin{pmatrix} 0 & 0 \\ -\mathcal{B} & -\mathcal{K}_s \end{pmatrix}.
\]

Furthermore, one has:

\[
\mathcal{M}^{-1} = \frac{1}{MJ_G} \begin{pmatrix} J_0 & aM \\ aM & M \end{pmatrix}.
\]

Since \( \mathcal{B}.D = 0 \), we get

\[
\mathcal{M}^{-1} \mathcal{K}\mathcal{D}.B = -\frac{1}{J_G} \mathcal{K}_s.D \begin{pmatrix} a \\ 1 \end{pmatrix}.\mathcal{B}.
\]

The condition \( \mathcal{K}_s.D \neq 0 \) being equivalent to \( \mathcal{K}_s \) non parallel to \( \mathcal{B} \), this ends the proof of the theorem.

3.3.2. Kinetic coupling from the rudder. We assume that the kinetic coupling from the foil is null. No hypothesis is made on the rudder coupling. In that case, \( \mathcal{E} = 0 \). The unique continuation property becomes:

\[
\mathcal{B}.P_0(t) = 0 \quad \text{on} \quad ]0, T[ \Rightarrow P_0(t) = 0 \quad \text{on} \quad ]0, T[.
\]

Setting:

\[
\begin{align*}
r_1 &= \mathcal{M}^{-1} \mathcal{K}\mathcal{D}.D, \\
r_2 &= \mathcal{M}^{-1} \mathcal{K}\mathcal{D}.B, \\
r_3 &= \mathcal{M}^{-1} \mathcal{K}\mathcal{D}.B, \\
r_4 &= \mathcal{M}^{-1} \mathcal{K}\mathcal{D}.\mathcal{B},
\end{align*}
\]

we prove the following result:

**Theorem 3.4.** A sufficient and necessary condition is

\[
(r_2^2 - r_1r_4r_3 + r_2r_3^2) \neq 0.
\]

**Proof.-** Let \( \mathcal{D} \) be a unit vector of \( \mathbb{R}^2 \) orthogonal to \( \mathcal{B} \). If \( \mathcal{B}.P_0(t) = 0 \quad \text{on} \quad ]0, T[ \), then

\[
P_0(t) = \xi(t) D \quad \text{on} \quad ]0, T[.
\]

With (25), we obtain:

\[
\begin{cases}
\dot{\xi} + \mathcal{M}^{-1} \mathcal{K}\mathcal{D}.D\dot{\xi}(t) + \mathcal{M}^{-1} \mathcal{K}\mathcal{D}.B\xi(t) = 0 \quad \text{on} \quad ]0, T[, \\
\mathcal{M}^{-1} \mathcal{K}\mathcal{D}.B\xi(t) + \mathcal{M}^{-1} \mathcal{K}\mathcal{D}.B\dot{\xi}(t) = 0 \quad \text{on} \quad ]0, T[.
\end{cases}
\]

The four reals numbers \( r_i \ (i = 1 \cdots 4) \) introduced are the coefficients of the system and they do not depend on \( t \). In that case, the unique condition is:

\[
(r_2^2 - r_1r_4r_3 + r_2r_3^2)\xi(t) = 0 \quad \text{on} \quad ]0, T[.
\]

and Theorem 3.4 is proved.

We end this section with the explicit computation of the exact control.
3.3.3. Computation of the exact control. The control $\delta^0$ can be computed in different ways as what we have done for the optimal control. A Fourier expansion leads to:

$$\delta^0(t) = \frac{2}{T} \sum_{n \geq 1} A_n \sin\left(\frac{n\pi t}{T}\right),$$

with:

$$A_n = -\frac{1}{a_0 + b_0 \frac{n^2\pi^2}{T^2}} \int_0^T \left[ B.P^1(s) - \mathcal{E} \dot{P}^1(s) \right] \sin\left(\frac{n\pi s}{T}\right) ds.$$  \hfill (31)

Once $\delta^0$ explicit, we write using Fourier decomposition:

$$\begin{align*}
\Phi &= (P^1(T), \dot{P}^1(T)), \\
\delta \Phi &= (Q(T), \dot{Q}(T)), \\
G_1(s) &= B.Q^1(s) - \mathcal{E} \dot{Q}^1(s), \\
B_n &= -\int_0^T G_1(s) \sin\left(\frac{n\pi s}{T}\right) ds, \\
\Lambda(\Phi, \delta \Phi) &= \frac{2}{T} \sum_{n \geq 1} A_n B_n, \\
L(\delta \Phi) &= M\dot{X}^0(0).Q(0) - M\dot{X}^0(0).\dot{Q}(0) - C\dot{X}^0(0).Q(0), \\
\end{align*}$$

and we look after $\Phi$ solution of:

$$\Phi \in \mathbb{R}^2 \times \mathbb{R}^2, \quad \forall \delta \Phi \in \mathbb{R}^2 \times \mathbb{R}^2, \quad \Lambda(\Phi, \delta \Phi) = L(\delta \Phi).$$  \hfill (33)

**Remark 4.** In order to avoid Gibbs phenomenon, it can be judicious to explicit $\delta^0$ in the following way:

$$\delta^0(t) = A_0 \cosh(rt) + B_0 \sinh(rt) + \int_0^t f(s) \sinh(r(t-s)) ds,$$

with:

$$r = \sqrt{\frac{a_0}{b_0}}, \quad B_0 = -\frac{\int_0^T f(s) \sinh(r(t-s)) ds}{\sinh(rT)}, \quad A_0 = 0,$$

and $f(s) = \frac{B.P^1(s) - \mathcal{E} \dot{P}^1(s)}{b_0 r}$.

This is the way used for our numerical tests that we present now.

3.3.4. Numerical Tests. They concern the exact control for both heaving and pitching movements. The values of the parameters $a_0$ and $b_0$ are $a_0 = b_0 = 1$. For each figure, the upper graph concerns the pitching movement, the middle one concerns the heaving movement and the bottom one is the graph of the exact control. Angles are always expressed in radians. Figures 4, 5 and 6 concern the exact controllability of an initial heaving of 10 cm. The initial pitching and its velocity are equal to zero. The velocity of the boat is 10 m/s, 15 m/s or 20 m/s. We observe that an exact control is acceptable for a 10 m/s speed boat but not in the other cases. Indeed, the heaving or the angle at the main foil would be
too large for realistic models. If the speed of the boat is 10m/s, the angle at the main foil can reach 0.4 radian (20 degrees) which seems to us a very extreme case.

Figures 7, 8 and 9 concern the exact controllability of an initial pitching equal to \(-2\) degrees with null initial pitching velocity and null heaving. The velocity of the boat is equal to 10m/s, 15m/s or 20m/s. The same remarks as before apply in this case.

**Figure 4.** Exact control with an heaving excitation - \(V = 10\text{m/s}\)

**Figure 5.** Exact control with an heaving excitation - \(V = 15\text{m/s}\).
4. Control with constraints. In practice the magnitude of the control is restricted. Therefore the minimisation of the criterion $J^\varepsilon$ should be performed over the convex and closed bounded set:

$$K = \{ q \in H^1_0([0, T]) \mid |q| \leq q_{\text{max}} \},$$

where $q_{\text{max}} > 0$.

The new optimal control problem is now:

$$J^\varepsilon(\delta^\varepsilon) = \min_{q \in K} J^\varepsilon(q).$$

(36)
From classical results in optimisation, the optimal solution is characterized by:
\[
\begin{align*}
&\delta_c \in K, \\
&\forall q \in K, \ J'_c(q - \delta_c) \geq 0 \ \text{a.e. on } [0,T].
\end{align*}
\] (37)

With (13), we get:
\[
\forall q \in K, \ \int_0^T \left( \varepsilon [a_0 \delta_c - b_0 \dot{\delta}_c] + \dot{P}B - \dot{P} \right)(q - \delta_c) \geq 0.
\] (38)
For $t_0 \in ]0, T[$, we deduce that:

\[ |\delta_\varepsilon(t_0)| \leq q_{max} \Rightarrow \varepsilon |a_0 \delta_\varepsilon - b_0 \delta_\varepsilon| + b \| \dot{P} \| - \varepsilon \dot{E} \dot{P} = 0 \text{ in a neighborhood of } t_0, \]
\[ \delta_\varepsilon(t_0) = q_{max} \Rightarrow \varepsilon |a_0 \delta_\varepsilon - b_0 \delta_\varepsilon| + b \| \dot{P} \| - \varepsilon \dot{E} \dot{P} \leq 0 \text{ in a neighborhood of } t_0, \]
\[ \delta(t_0) = -q_{max} \Rightarrow \varepsilon |a_0 \delta_\varepsilon - b_0 \delta_\varepsilon| + b \| \dot{P} \| - \varepsilon \dot{E} \dot{P} \geq 0 \text{ in a neighborhood of } t_0. \]

For $\varepsilon \to 0$ one can define a limit control problem (see [6]) following the same idea as in the previous section, but it is not always exact. Let us consider two following possibilities.

4.1. Case of an exact control in $K$. We assume that there is an exact control -say $\delta^0 \in K$ for the initial conditions and the time control $T$. The solution of (15) satisfies at the final time $T$: $X(T) - (z_0, 0)$ et $\dot{X}(T) = 0$. We have $\delta_\varepsilon \in K$ and

\[ \forall q \in K, J(\delta_{\varepsilon}) \leq J(q). \]

Let's input $q = \delta^0$, we get the functional:

\[ a_0 \| \delta_\varepsilon \|_{L^2[0, T]}^2 + b_0 \| \dot{\delta}_\varepsilon \|_{L^2[0, T]}^2 \leq a_0 \| \delta^0 \|_{L^2[0, T]}^2 + b_0 \| \dot{\delta}^0 \|_{L^2[0, T]}^2. \]

The functions $\delta_\varepsilon$ are bounded in $H^1([0, T])$ and up to a subsequence, we get

\[ \lim_{\varepsilon \to 0} \delta_{\varepsilon} = \delta^* \text{ in } H^1_0([0, T]) - \text{ weak} \]

and thus $\delta^* \in K$. In a similar way, $X_\varepsilon$ are bounded in $H^2([0, T])$. A subsequence strongly converges in $H^1([0, T])$ to the solution $X^*$ of (15) with righthandside $\delta^*$. Furthermore, with (39), we obtain:

\[ \lim_{\varepsilon \to 0} \left( \| X_\varepsilon(T) - X_d \|_2^2 + \| \dot{X}_\varepsilon(T) \|_2^2 \right) = 0. \]

We deduce that $\delta^*$ is an exact control in $K$ and:

\[ a_0 \| \delta^* \|_{L^2[0, T]}^2 + b_0 \| \dot{\delta}^* \|_{L^2[0, T]}^2 = \min_{\delta^0 \in U_{ad} \cap K} \left( a_0 \| \delta^0 \|_{L^2[0, T]}^2 + b_0 \| \dot{\delta}^0 \|_{L^2[0, T]}^2 \right). \]

The function $\delta^*$ is thus an exact control of minimum norm because $\delta^0$ is an arbitrary exact control. Since this minimum is unique because of the strict convexity of the norm in $H^1_0([0, T])$, there is a unique limit point and the full sequence converges to $\delta^*$. It is easy to see that the adjoint state $P_\varepsilon$, which is solution of (16), converges to 0.

4.2. If there is no exact control in $K$. A strategy is to weaken the admissible control set. Let us consider the functional:

\[ J_{\varepsilon}(q) = \frac{1}{2} \left( |X(T) - (z_0, 0)|^2 + |\dot{X}(T)|^2 + \varepsilon \int_0^T a_0 q(t)^2 dt \right) \]

on the set

\[ K_0 = \{ q \in L^2([0, T]), |q(t)| \leq q_{max} \text{ a.e on } [0, T] \}, \]

where $X$ is solution of

\[ \begin{cases} \mathcal{M} \ddot{X} - C \dot{X} + KX = Bq & \text{sur } [0, T] \\ X(0) = X_0, \ \dot{X}(0) = X_1. \end{cases} \]

(40)
$J_\varepsilon$ reaches its minimum at $\delta_\varepsilon \in K_0$ and we get

$$\forall q \in K_0, \quad J'(\delta_\varepsilon)(q - \delta_\varepsilon) \geq 0. \quad (41)$$

Following the steps of the previous section, we consider $X_\varepsilon$ the solution of (40) associated to $\delta_\varepsilon$ and the adjoint state $P_\varepsilon$ solution of

$$\begin{cases} \mathcal{M}\dot{P}_\varepsilon + \mathcal{C}\dot{P}_\varepsilon + \mathcal{K}P_\varepsilon = 0 \quad \text{on } [0, T] \\ \mathcal{M}P_\varepsilon(T) = \dot{X}_\varepsilon(T) \quad \mathcal{M}\dot{P}_\varepsilon(T) = -(X_\varepsilon(T) - \deltaT z_0, 0)) - \mathcal{C}P_\varepsilon(T). \end{cases} \quad (42)$$

We get

$$J'_\varepsilon(\delta_\varepsilon)(q) = \int_0^T (P_\varepsilon Bq + \varepsilon a_0 \delta_\varepsilon q).$$

and thus

$$\forall q \in K_0, \quad \int_0^T \left( \mathcal{B} P_\varepsilon + \varepsilon a_0 \delta_\varepsilon \right)(q - \delta_\varepsilon) \geq 0.$$

Since $\delta_\varepsilon \in K_0$, these functions are bounded in $L^\infty([0, T])$. Up to a subsequence, they converge in $L^2([0, T])$ weak and in $L^\infty([0, T])$ weak-* to $\delta^* \in K_0$. Furthermore $(X_\varepsilon)_\varepsilon$ weakly converges in $H^2([0, T])$ to the solution $X \in H^2(0, T)$ of (40) associated to $\delta^*$ and we can assume that the convergence of $X_\varepsilon$ to $X$ is strong in $C^1([0, T])$. We deduce that $(P_\varepsilon)_\varepsilon$ strongly converges in $C^1([0, T])$ to the solution $P^*$ of the adjoint limit problem. Finally, we get

$$\forall q \in K_0, \quad \int_0^T \mathcal{B} P^*(q - \delta_\varepsilon) \geq 0. \quad (43)$$

Since $(J_\varepsilon)_\varepsilon$ pontually converges to

$$J(q) = \frac{1}{2} \left( |X(T) - \deltaT z_0, 0|^2 + |\dot{X}(T)|^2 \right),$$

condition (43) can be written

$$\forall q \in K_0, \quad J'(\delta^*)(q - \delta^*) \geq 0.$$

This is the optimality condition of $J$ on $K_0$ and we deduce that $\delta^*$ satisfies

$$J(\delta^*) = \min_{q \in K_0} J(q).$$

By loss of strict convexity, we lose the unique character of the limit point and of the optimal point. Condition (43) is equivalent to

$$\delta^*(t) = -q_{\max} \text{sign}(\mathcal{B} P^*)(t) \quad \text{sur} \quad \mathcal{B} P^* \neq 0,$$

and this is what we call a bang-bang control : $\delta^*$ takes only the two optimal possible values. The lack of convergence and the non uniqueness of the optimal control explain numerical instabilities. Therefore it is worth to choose $T$ in order to ensure that there exists an exact control, otherwise some instabilities could appear in the computation of $\delta$ (with $\varepsilon > 0$), even if it is unique (and in the space $H^2_0([0, T])$) but for any $\varepsilon$ small enough.

**Remark 5.** Usually the R. Bellman condition [1] and the fact that the real part of the eigenvalues of the characteristic equation of the linear system are negative, strictly negative or simply negative and simple for those which are purely imaginative, are sufficient conditions to ensure that there exists an exact control but with a control time which can be larger than the one used in the computation of the exact control without constraint. The minimum time $T_{\min}$ for an exact control is the boundary between the existence and the non-existence of an exact control in the set $K$. Nevertheless the bang-bang control should be avoided in this case (as for an aircraft) because it is not robust and implies shocks which can be at
the origin of unwanted perturbations. Therefore, knowing the maximum amplitude $q_{\text{max}}$ of the control, it is possible to compute the minimum control time -say $T_{\text{min}}$- and therefore to ensure that the exact control law is computed with a control time $T$ sufficiently larger than $T_{\text{min}}$.

4.3. **Numerical tests.** The data of the tests are the same as in the case of exact control without constraint. We accept foil angle under $\pm 3$ degré. We remark that the control time is longer than before.

![Figure 10](image1.png)  
**Figure 10.** Control (with contraints) with an heaving excitation - $V = 10m/s$  

![Figure 11](image2.png)  
**Figure 11.** Control (with contraints) with an heaving excitation - $V = 15m/s$.  

5. **Feed-back system.**

5.1. **A simple proportional feed-back system.** As we have pointed out in the previous sections, it is better to introduce the term $\dot{\delta}$ in the control law. A control which only include $\delta$ would be unstable because of the vanishing stiffness in the direction of the heaving movement. Furthermore the absolute referential for $\dot{\delta} = \alpha - \alpha_0$ is not obvious in the practical implementation. But the pitching velocity is easier to detect as far as a flexible rotation is allowed at the jonction between the foil and the bow of the boat. In fact this is shown on figure ?? . The flexibility can be stiffened using a spring and damped by an hydraulic ram.
This is a simple feed-back loop based on prescribing the rake through this ram. Let us try to explain why this method, even if it is not optimal, is a good one for the stabilisation of the oversea flight.

In this case the flexibility of the foil is ensured around a rotational axis fixed on the bow of the ship or simply on the daggerboard case. Therefore a new flexibility is introduced. It is a rotation around this axis. Furthermore, two additional mechanical devices are introduced: one is a damper and the other is a spring. The damping coefficient is $\xi$, the stiffness of the spring is $c_f$ and the equations of the model are a little bit changed. We introduce three new
Figure 16. The notations used in this section

coefficients: one is the mass of the foil -say $m_f$- the second one is the inertia of the foil (alone) around point $o$ -say $J_f$ and the last one is the distance between the point $o$ and the center of mass of the foil (alone) -say $d_{og}$-; they are represented on figure 16.

The equations of the movement are now (we use the same notations as before excepted that $J_o$ and $M$ are respectively the inertia and the mass of the ship around point $o$ without the main foil):

\[
\begin{align*}
(M + m_f)\ddot{z} - aM\dot{\gamma} + d_{og}\sin(\alpha_0)m_f\ddot{\alpha} - c_{11}\dot{z} - c_{12}(\dot{\gamma} + \dot{\alpha}) &+ \frac{\rho Su^2}{2}R_1(\gamma + \alpha) = 0, \\
-aM\ddot{z} + J_0\dot{\gamma} - c_{21}\dot{z} - c_{22}(\dot{\gamma} + \dot{\alpha}) &+ \frac{\rho Su^2}{2}R_2(\gamma + \alpha) = 0, \\
d_{og}\sin(\alpha_0)m_f\ddot{z} + J_f\ddot{\alpha} - c_{21}\dot{z} - c_{22}\dot{\gamma} + (\xi - c_{22})\dot{\alpha} + \frac{\rho Lu^2}{2}R_2(\gamma + \alpha) + c_f\ddot{\alpha} = 0. 
\end{align*}
\]
Let us introduce new notations for the different matrices:

\[
\mathcal{M} = \begin{pmatrix}
M + m_f & -aM & d_{og} \sin(\alpha_0) m_f \\
-aM & J_0 & 0 \\
d_{og} \sin(\alpha_0) m_f & 0 & J_f
\end{pmatrix}
\]

\[
\mathcal{C} = \begin{pmatrix}
c_{11} & c_{12} & c_{12} \\
c_{21} & c_{22} & c_{22} \\
c_{21} & c_{22} & c_{22} - \xi
\end{pmatrix}
\]

\[
\mathcal{K} = \begin{pmatrix}
0 & \frac{gSu^2}{2} R_1 & \frac{gSu^2}{2} R_1 \\
0 & \frac{gLSu^2}{2} R_2 & \frac{gLSu^2}{2} R_2 \\
0 & \frac{gLSu^2}{2} R_2 & \frac{gLSu^2}{2} R_2 + c_f
\end{pmatrix}
\]

The equation (45) can therefore be written as follows (with initial conditions):

\[
\mathcal{M} \ddot{X} - \mathcal{C} \dot{X} + \mathcal{K} X = 0.
\]

The strong stability of the system is obtained as far as the imaginary part of the solution to the following eigenvalue problem are positive:

\[
\det(-\lambda^2 \mathcal{M} - i\lambda \mathcal{C} + \mathcal{K}) = 0
\]

(46)

In fact, the larger is the smallest imaginary part of the solution to (46), the best is the stabilisation of the system. The evolution of the imaginary part of these solutions are plotted on figure 17 for $\xi = 10^{-2} c_f N/s/m$ and $c_f = 10^8 N/m$. We observed that the results are not significantly dependent on these values. Let us notice that the stability is assured since real parts of eigenvalues are negative. For information, the dashed line is the horizontal axis on the upper figure. In the report published by the American team for America’s cup, they mention that this spring was not used.
5.2. The system used by Oracle Team USA. Let us consider the simple equation, just for explaining the strategy used, but clearly it has to be adjusted at purpose in practice:

\[ J_\ell \ddot{\alpha} + c_f \alpha = -\alpha_{\text{max}} \text{sign}(\dot{\alpha}), \quad \alpha(0) = \alpha_0, \quad \dot{\alpha}(0) = \alpha_1. \tag{47} \]

This is a very classical control strategy which is often used for stabilizing aircraft. It is not an exact control and the result depends on the value of the maximum amplitude \( \alpha_{\text{max}} \). But this can be adjusted easily by a computer on an aircraft and in the case of the America’s cup boat, it requires a simple education of the helmsman. The remaining of the control loop is purely automatic and based on simple mechanical devices. The sign function is detected by the rocker switch (see figure ??).

The solution have been plotted on figure 18 for six values of \( \alpha_{\text{max}} \). One can see that this is an efficient method. The time delay required for the stabilisation depends on both the initial data and the maximum magnitude \( \alpha_{\text{max}} \) of the control.

![Figure 18](image)

**Figure 18.** Several trajectories in the phase diagramm for different values of \( \alpha_{\text{max}} \) starting from the same initial condition. One can observe that \( \alpha_{\text{max}} \) should be adjusted in order to obtain the right control. This is why the helmsman has a control box on the steering wheel which enables him to adjust \( \alpha_{\text{max}} \) by steps of \( \pm 5^0 \).

5.3. Comparison between the exact control, OTUSA control and exact control with upper bound. For the same boat and foil-rudder set, we compare the obtained results. The results are given on the figures 4-9 for exact controls, 10-15 for controls with constraints and 19-24 for OTUSA control. The time of control that we choose in the two first cases is 10 seconds which seems to be large enough.

For the heaving control, the initial data used are: \( z^0 = 0.1 \) (10 cm), \( z^1 = \gamma^0 = \gamma^1 = 0 \).

For the pitching control, the initial data are: \( z^0 = z^1 = \gamma^1 = 0 \) and \( \gamma^0 = -2 \) degré or \( \gamma^0 = -0.035 \) radian. The desired final states are null.
We can conclude that exact controllability is not realistic in 10° even for small velocities. Indeed, the angle must be larger than 0.4 rad for heaving control and 0.2 for pitching control. These values are not acceptable in practical applications.

The control with constraint on the magnitude of the foil angle of 3 degrees is efficient but some values of the heaving seem dangerous for large velocity (larger than 0.25 m at 20 m/s). This appears in both cases of initial pitching or heaving in the control process.
One can see that the OTUSA system is very efficient at the beginning but the exact control is more precise for small perturbations. The ideal solution is to couple OTUSA control law at the beginning and the exact control when the magnitude of the oscillation are small enough. In this case the mechanical manufacturing of the system can be done using a steering box with an additional button compared to the one of OTUSA. But this is another study not included in this analysis. The use of a system of masses, springs and dashpots can lead to a suitable approximation of the exact feed-back. Because the matrix leading
to the exact control with respect to the perturbation is only dependent on the ship. In this study, it is a $4 \times 4$ symmetrical matrix which therefore depends on only 10 coefficients. A least-square method can be applied in order to give a fine approximation of the coefficients using a proportional-integral-derivative (P.I.D) composed of several mechanical blocks.

6. **Conclusion.** In this paper, we have suggested a formulation for a simple model in order to check the controllability of flying boats in a reduced configuration. In fact, only two
degrees of freedom have been considered. The control law is derived from optimal control for a vanishing cost parameter ($\varepsilon \to 0$). The controllability is discussed using the theory of R. Bellman [1]. But a careful study has been necessary because the control system is a part of the stiffness and of the damping matrices. It has been explained and justified that both the rake angle and its time derivative are involved in the control process.

The optimal control laws found in subsections 3.1 are theoretical and can only be exactly applied as far as an electronic device coupled with a mechanical actuator, is used. The comparison with the control system used by OTUSA shows the theoretical advantages of this exact control. But from a practical point of view, it is not obvious that these advantages would be significant. Because of the regulations in the America’s cup, only mechanical devices can be used and the manufacturing of a mechanical system reproducing the exact control law is not discussed in this paper.

Therefore the strategy used by the American Team which is fully compatible with the rules, even if it is not optimal, is an interesting alternative. Nevertheless, much better results would be obtained by using slightly different manufacturing of the foil and the rudder. We do not explicit these improvements in this paper which are still to be discussed and checked. May be they will be discussed in a future work.

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