Asymptotics of a Mathieu-Gaussian series

R. B. PARIS
Division of Computing and Mathematics,
Abertay University, Dundee DD1 1HG, UK

Abstract
We consider the asymptotic expansion of the functional series

\[ S_{\mu, \gamma}(a; \lambda) = \sum_{n=1}^{\infty} n^\gamma e^{-\lambda n^2/a^2} \frac{n}{(n^2 + a^2)^\mu} \]

for real values of the parameters \( \gamma, \lambda > 0 \) and \( \mu \geq 0 \) as \( |a| \to \infty \) in the sector \(|\arg a| < \pi/4\).

For general values of \( \gamma \) the expansion is of algebraic type with terms involving the Riemann zeta function and a terminating confluent hypergeometric function. Of principal interest in this study is the case corresponding to even integer values of \( \gamma \), where the algebraic-type expansion consists of a finite number of terms together with a contribution comprising an infinite sequence of increasingly subdominant exponentially small expansions. This situation is analogous to the well-known Poisson-Jacobi formula corresponding to the case \( \mu = \gamma = 0 \). Numerical examples are provided to illustrate the accuracy of these expansions.

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1. Introduction
The functional series

\[ \sum_{n=1}^{\infty} \frac{n}{(n^2 + a^2)^\mu} \]

in the case \( \mu = 2 \) was introduced by Mathieu in his 1890 book [2] dealing with the elasticity of solid bodies. The asymptotic expansion for large \( a \) of more general functional series has been discussed in [4] and [11]. More recently, Gerhold and Tomovski [1] extended the asymptotic study of such Mathieu series by introducing in (1.1) the factor \( z^n \), where \( |z| \leq 1 \). From this result they were able to deduce, in particular, the large-\( a \) expansions of the trigonometric Mathieu series

\[ \sum_{n=1}^{\infty} \frac{n \sin nx}{(n^2 + a^2)^\mu} \quad \sum_{n=1}^{\infty} \frac{n \cos nx}{(n^2 + a^2)^\mu}. \]

Subsequently, the above trigonometric series were generalised to include the oscillatory Bessel functions \( J_\nu(x) \) and \( Y_\nu(x) \) with argument proportional to \( n/a \), and their large-\( a \) asymptotics determined in [6]. In addition, this last study also considered the inclusion of the modified Bessel function \( K_\nu(x) \) of similar argument, which contains the decaying exponential as a special case.

The asymptotic expansion we consider in this paper is the Mathieu series coupled with a Gaussian exponential of the form

\[ S_{\mu, \gamma}(a; \lambda) := \sum_{n=1}^{\infty} \frac{n^\gamma e^{-\lambda n^2/a^2}}{(n^2 + a^2)^\mu} \quad (\mu \geq 0, \lambda > 0) \]
for $|a| \to \infty$ in the sector $|\arg a| < \pi/4$. It will be supposed throughout that $\gamma$ is real, although the analysis is easily modified to incorporate complex $\gamma$. We shall employ the Mellin transform approach used in [3, 6, 11], where our interest will be primarily concerned with even integer values of $\gamma$ (positive or negative). We shall find that the asymptotic expansion of $S_{\mu, \gamma}(a; \lambda)$ with these parameter values for large complex $a$ in the sector $|\arg a| < \pi/4$ consists of a finite algebraic expansion together with an infinite sequence of increasingly subdominant exponentially small contributions.

It is interesting that the apparently simple series (1.2) should possess such an intricate asymptotic structure in the case of even integer values of $\gamma$. This is also found to be the case when $\lambda = 0$ in (1.2); see [3] for details. A well-known related series corresponding to $\mu = \gamma = 0$ is the Poisson-Jacobi formula [10, p. 124]

$$S_{0,0}(a; \lambda) = \sum_{n=1}^{\infty} e^{-\lambda n^2/a^2} = \frac{a}{2} \sqrt{\frac{\pi}{\lambda}} - \frac{1}{2} + \frac{\sqrt{\pi}}{\lambda} \sum_{n=1}^{\infty} e^{-\pi^2 n^2 a^2 / \lambda}. \quad (1.3)$$

This sum is also seen to consist of a finite algebraic contribution together with an infinite sum of exponentially small terms when $|a| \to \infty$ in the sector $|\arg a| < \pi/4$.

In the application of the Mellin transform method to the series in (1.2) and its alternating variant we shall require the following estimates for the gamma function and the Riemann zeta function. For real $\sigma$ and $t$, we have the estimates

$$\Gamma(\sigma \pm it) = O(t^{\sigma - \frac{1}{2}} e^{-\frac{1}{2} \pi t^2}), \quad |\zeta(\sigma \pm it)| = O(t^{\Omega(\sigma)} \log^a t) \quad (t \to +\infty), \quad (1.4)$$

where $\Omega(\sigma) = 0 \ (\sigma > 1), \frac{1}{2} - \frac{1}{2} \sigma \ (0 \leq \sigma \leq 1), \frac{1}{2} - \sigma \ (\sigma < 0)$ and $\alpha = 1 \ (0 \leq \sigma \leq 1), \alpha = 0$ otherwise [9, p. 95]. The zeta function $\zeta(s)$ has a simple pole of unit residue at $s = 1$ and the evaluations for positive integer $k$

$$\zeta(0) = -\frac{1}{2}, \quad \zeta(-2k) = 0, \quad \zeta(2k) = \frac{(2\pi)^{2k}}{2(2k)!} B_{2k} \quad (k \geq 1), \quad (1.5)$$

where $B_k$ are the Bernoulli numbers. Finally, we have the well-known functional relation satisfied by $\zeta(s)$ given by [3, p. 603]

$$\zeta(s) = 2^s \pi^{s-1} \zeta(1-s) \Gamma(1-s) \sin \frac{1}{2} \pi s. \quad (1.6)$$

2. An integral representation

The generalised Mathieu series defined in (1.2) can be written as

$$S_{\mu, \gamma}(a; \lambda) = a^{-\delta} \sum_{n=1}^{\infty} h(n/a), \quad h(x) := \frac{x^\gamma e^{-\lambda x^2}}{(1 + x^2)^\delta}, \quad \delta := 2\mu - \gamma, \quad (2.1)$$

where the parameter $\delta$ is real and $\lambda > 0$. We employ a Mellin transform approach as discussed in [7, Section 4.1.1]. The Mellin transform of $h(x)$ is $\mathcal{H}(s) = \int_0^\infty x^{s-1} h(x) \, dx$, where [3, (13.4.4)]

$$\mathcal{H}(s) = \int_0^\infty \frac{x^{\gamma+s-1} e^{-\lambda x^2}}{(1 + x^2)^\delta} \, dx = \frac{1}{2} \Gamma\left(\frac{\gamma + s}{2}\right) U\left(\frac{\gamma + s}{2}, 1 + \frac{s}{2} - \mu, \lambda\right) \quad (2.2)$$

in the half-plane $\Re(s) > -\gamma$, with $U(a, b, z)$ being the confluent hypergeometric function of the second kind. The transform $\mathcal{H}(s)$ can be represented alternatively in the form

$$\mathcal{H}(s) = \frac{1}{2} \{\mathcal{H}_1(s) + \mathcal{H}_2(s)\}, \quad (2.3)$$
where

\[ \mathcal{H}_1(s) = \frac{\Gamma\left(\frac{\mu+\gamma}{2}\right)\Gamma\left(\mu - \frac{\mu+\gamma}{2}\right)}{\Gamma(\mu)} \, _1F_1\left(\frac{\mu+\gamma}{2}; 1 + \frac{\mu+\gamma}{2} - \mu; \lambda; \right), \quad (2.4) \]

\[ \mathcal{H}_2(s) = \lambda^{\mu-(\gamma+s)/2}\Gamma\left(\frac{\mu+\gamma}{2} - \mu\right) \, _1F_1\left(\mu; 1 - \frac{\mu+\gamma}{2} + \mu; \lambda\right). \quad (2.5) \]

Using the Mellin inversion theorem (see, for example, [7, p. 118]), we find

\[ S_{\mu,\gamma}(a; \lambda) = \frac{a^{-\delta}}{2\pi i} \sum_{n=1}^{\infty} \int_{c-i\infty}^{c+i\infty} \mathcal{H}(s)(n/a)^{-s} ds = \frac{a^{-\delta}}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{H}(s)\zeta(s)a^s ds, \quad (2.6) \]

where \( \zeta(s) \) is the Riemann zeta function and \( c > \max\{1, -\gamma\} \). The inversion of the order of summation and integration is justified by absolute convergence provided \( c \) satisfies this condition.

From the estimates in (1.1) and the fact that the confluent hypergeometric functions appearing in \( \mathcal{H}_1(s) \) and \( \mathcal{H}_2(s) \) are both \( O(1) \) as \( 3(s) \to \pm \infty \), the integral in (2.6) then defines \( S_{\mu,\gamma}(a; \lambda) \) for complex \( a \) in the sector \( |\arg a| < \pi/4 \). The integration path in (2.6) lies to the right of the simple pole of \( \zeta(s) \) at \( s = 1 \) and the poles of \( \Gamma((\gamma + s)/2) \) at \( s = -\gamma - 2k \) \( (k = 0, 1, 2, \ldots) \), these being the only poles of the integrand since the \( U \) function in (2.2) has no poles; see Appendix A for a demonstration of this fact.

We consider the integral in (2.6) taken round the rectangular contour with vertices at \( c \pm iT \) and \(-c' \pm iT\), where \( c' > 0 \) and \( T > 0 \). The contribution from the upper and lower sides of the rectangle \( s = \sigma \pm iT \), \( -c' \leq \sigma \leq c \), vanishes as \( T \to \infty \) provided \( |\arg a| < \pi/4 \), since from (1.4), the modulus of the integrand is controlled by \( O(T^{\Omega(s) + (\sigma - \delta - 1)/2} \log T e^{-\Delta T}) \), where \( \Delta = \pi/4 - |\arg a| \). Displacement of the integration path to the left over the pole at \( s = 1 \) and those of \( \mathcal{H}_1(s) \) at \( s = -2k - \gamma \) (when \( \mu > 0 \) then yields

\[ S_{\mu,\gamma}(a; \lambda) - a^{1-\delta} \mathcal{H}(1) \sim a^{-2\mu} \sum_{k=0}^{\infty} \frac{(-\gamma)^k (\mu - k)^k}{a^{2k}} \zeta(-2k - \gamma) \, _1F_1(-k; 1 - \mu - k; \lambda), \quad (2.7) \]

where

\[ \mathcal{H}(1) = \frac{1}{2} \Gamma(\frac{1+\gamma}{2}) \Gamma(\frac{1+\gamma}{2} + \frac{1+\gamma}{2} - \mu, \lambda). \quad (2.8) \]

When \( \mu = 0 \), we have \( \mathcal{H}_1(s) \equiv 0 \) and the poles from \( \mathcal{H}_2(s) \) at \( s = -2k - \gamma \) yield

\[ S_{0,\gamma}(a; \lambda) - \frac{1}{2} \left( \frac{a^2}{\lambda} \right)^{(\gamma+1)/2} \frac{\Gamma(\gamma+1/2)}{\Gamma(\gamma+1/2)} \sim \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!a^{2k}} \zeta(-2k - \gamma), \quad (2.9) \]

where we have used the fact that \( U(\alpha, 1 + \alpha, x) = x^{-\alpha} \). Both expansions (2.7) and (2.9) hold as \( |\arg a| \to \infty \) in \( |\arg a| < \pi/4 \). The hypergeometric functions appearing in the sum in (2.7) are polynomials in \( \lambda \) of degree \( k \) when \( \mu \) is non-integer; for positive integer \( \mu \) they can be expressed by Kummer’s theorem as \( e^\delta \) multiplied by a polynomial in \( \lambda \) of degree \( \mu - 1 \).

The expansion (2.7) holds for general values of \( \mu > 0 \) and \( \gamma \). If \( \gamma \) equals an odd negative integer, the pole at \( s = 1 \) has both a double pole contribution (resulting from \( \mathcal{H}_1(s) \)) and a simple pole contribution (resulting from \( \mathcal{H}_2(s) \)). An example of the expansion when \( \gamma = -1 \) is discussed in Appendix B. We remark that when \( \gamma = 0 \) in (2.9), the expansion for \( S_{0,0}(a; \lambda) \) correctly reduces to the first two terms in the Poisson-Jacobi formula (1.3), but does not account for the exponentially small contribution. Further consideration of this case is discussed at the end of Section 3.

Finally, we note that when \( \gamma = 2p \) is an even integer, there will be a finite number of poles of the integrand of the sequence \( s = -2k - 2p \) on account of the trivial zeros of \( \zeta(s) \) at \( s = -2, -4, \ldots \). This results in the number of terms in the asymptotic series in (2.7) and (2.9).

\footnote{The \( \Gamma \) function appearing in \( \mathcal{H}_1(s) \) can be written as \( e^\delta \Gamma(1 - \mu; 1 + 1/2(\gamma + s) - \mu; -\lambda) \) by Kummer’s transformation [3, p. 325].}
being either finite or zero. This situation is the main subject of this paper. We shall show that, in addition to a finite algebraic contribution, there is a sequence of increasingly subdominant exponentially small terms in the large-\(a\) limit. This is analogous to the exponentially small contribution appearing on the right-hand side of the Poisson-Jacobi formula \([13]\).

### 3. The exponentially small contribution to \(S_{\mu, \gamma}(a; \lambda)\) when \(\gamma = 2p\)

Let \(\gamma = 2p\) be an even integer and \(\mu \geq 0\). Then the quantity \(\delta\) defined in \([2.1]\) is \(\delta = 2(\mu - p)\). The number of poles of the sequence \(s = -2k - 2p\) is finite (when \(p \leq 0\)) or zero (when \(p \geq 1\)) on account of the trivial zeros of \(\zeta(s)\). Then we have upon displacement of the integration path

\[
S_{\mu, \gamma}(a; \lambda) = a^{1-\delta} \mathcal{H}(1) + H_{\mu, \gamma}(a; \lambda) + J(a; \lambda),
\]

where

\[
H_{\mu, \gamma}(a; \lambda) = \begin{cases} 
0 & \quad (p \geq 1) \\
-\frac{1}{2} a^{-\delta} & \quad (p = 0) \\
a^{-\delta} \sum_{k=0}^{\lfloor |p| \rfloor} \zeta(2k) R_k(\mu, |p|) a^{2k} & \quad (p \leq -1).
\end{cases}
\]

The quantity \(R_k(\mu, |p|)\) denotes the residue of \(\mathcal{H}(s)\) at \(s = 2k\), \(0 \leq k \leq |p|\) when \(p \leq -1\) given by

\[
R_k(\mu, |p|) = \frac{(-1)^{q-k}}{(q-k)!} U(k - q, 1 + k - q - \mu, \lambda) \quad (q = p, 0 \leq k \leq q).
\]

Routine calculations show that when \(p = -1, -2\), for example, we have

\[
R_0(\mu, 1) = -(\mu + \lambda), \quad R_1(\mu, 1) = 1, \\
R_0(\mu, 2) = \frac{1}{2} \mu(1 + \mu) + \mu \lambda + \frac{1}{2} \lambda^2, \quad R_1(\mu, 2) = -(\mu + \lambda), \quad R_2(\mu, 2) = 1.
\]

The integral \(J(a; \lambda)\) appearing in \([3.1]\) is defined by

\[
J(a; \lambda) = a^{-\delta} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{H}(-s) \zeta(-s) a^{-s} ds \quad (c > 0),
\]

where we have replaced \(s\) by \(-s\). Use of the functional relation for \(\zeta(s)\) in \([10]\) followed by expansion of \(\zeta(1 + s)\) (permissible since \(c > 0\)) leads to

\[
J(a; \lambda) = -a^{-\delta} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{H}(-s) \zeta(1 + s) \Gamma(1 + s) \frac{\sin \frac{\pi s}{2\pi}}{\pi} (2\pi a)^{-s} ds
\]

\[
= -a^{-\delta} \frac{1}{\pi} \sum_{k \geq 1} \frac{1}{k} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{H}(-s) \Gamma(1 + s) \sin \frac{\pi s}{2\pi} (2\pi ka)^{-s} ds.
\]

From \([2.2]\) and an application of Kummer’s transformation \([3.13.2.40]\) we have

\[
\mathcal{H}(-s) = \frac{1}{2} \Gamma(p - \frac{1}{2}s) \lambda^{p-s/2} U(\mu, 1 + \mu - p + \frac{1}{2}s, \lambda),
\]

whence

\[
J(a; \lambda) = \frac{(-1)^p a^{-\delta}}{2\lambda^{p-\mu}} \sum_{k \geq 1} \frac{1}{k} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2^{-s} \Gamma(1 + s)}{\Gamma(1 - p + \frac{1}{2}s)} X_k^{-s/2} U(\mu, 1 + \mu - p + \frac{1}{2}s, \lambda) ds,
\]

where

\[
X_k := \frac{\pi^2 k^2 a^2}{\lambda}.
\]

We first consider the case \(p = 0, 1, 2, \ldots\). Since there are no poles of the integrand in \([3.3]\) in \(\Re(s) > 0\) the integration path can be displaced as far to the right as we please such that \(|s|\) is
everywhere large on the new path. The quotient of gamma functions in the integrand can then be expanded as \[7\] p. 53, \[3\] (5.11.19)

\[
\frac{2^{-s} \sqrt{\pi} \Gamma(1 + s)}{\Gamma(1 - p + \frac{1}{2}s)} = \frac{\Gamma(\frac{1}{2} + \frac{1}{s}) \Gamma(1 + \frac{1}{2}s)}{\Gamma(1 - p + \frac{1}{2}s)} = \sum_{j=0}^{p} (-)^j c_j \Gamma(\frac{1}{2}s + \theta - j),
\]

(3.5)

where \(\theta = p + \frac{1}{2}\). The coefficients are given explicitly by

\[
c_j = \frac{1}{j!} (-p)_j (-p + \frac{1}{2})_j = \frac{(-2p)_j}{2^j j!}.
\]

(3.6)

We observe that \(c_j = 0\) for \(j > p\) so that the above sum of gamma functions terminates and so is exact. Substitution of the expansion (3.5) in (3.3), combined with the integral representation \[3\] (13.4.4)

\[
U(\mu, 1 + \mu - p + \frac{1}{2}s, \lambda) = \frac{1}{\Gamma(\mu)} \int_{0}^{\infty} e^{-\lambda t^{\mu-1}(1 + t)^{-p-s/2}} dt \quad (\mu > 0),
\]

then shows that, provided \(\mu > 0\),

\[
J(a; \lambda) = \frac{(-)^p a^{-\delta} \lambda^{p-\mu}}{2\sqrt{\pi} \Gamma(\mu)} \sum_{k \geq 1} \frac{1}{k} \sum_{j=0}^{p} (-)^j c_j \int_{0}^{\infty} e^{-\lambda t^{\mu-1}} \left( \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(\frac{1}{2}s + \theta - j) \left( \frac{X_k}{1+t} \right)^{-s/2} ds \right) dt.
\]

The inner integral appearing in \(J(a; \lambda)\) can be evaluated by making use of the well-known result

\[
\frac{1}{2\pi i} \int_{L} \Gamma(s + \alpha) z^{-s} ds = z^\alpha e^{-z} \quad (|\arg z| < \frac{1}{2}\pi),
\]

(3.7)

where \(L\) is a path parallel to the imaginary \(s\)-axis lying to the right of all the poles of \(\Gamma(s + \alpha)\); see, for example, \[7\] Section 3.3.1. Evaluation of the inner integral when \(|\arg a| < \pi/4\) then produces the final exact result

\[
J(a; \lambda) = \frac{(-)^p a^{2p} e^{\lambda}}{2\sqrt{\pi} \lambda^p} \sum_{k \geq 1} \frac{1}{k} \sum_{j=0}^{p} (-)^j c_j \left( \frac{\lambda}{\pi^2 k^2 a^2} \right)^j I_{jk},
\]

where

\[
I_{jk} = \int_{0}^{\infty} t^{\mu-1} e^{-\psi(t)} dt, \quad \psi(t) := \lambda(1+t) + \frac{X_k}{1+t} - 2\pi k a.
\]

(3.8)

In the special case \(\mu = 0\), we have from (3.3) (since \(U(0, b, z) = 1\)) that

\[
J(a; \lambda) = \frac{(-)^p a^{2p}}{2\sqrt{\pi} \lambda^p} \sum_{k \geq 1} \frac{1}{k} \sum_{j=0}^{p} (-)^j c_j \cdot \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(\frac{1}{2} + \theta - j) X_k^{-s/2} ds.
\]

Evaluation of the integral by means of (3.7) then produces

\[
J(a; \lambda) = \frac{(-)^p}{2\sqrt{\pi} \lambda^{p+1/2}} \sum_{k \geq 1} k^2 e^{-\pi^2 k^2 a^2/\lambda} \sum_{j=0}^{p} (-)^j c_j \left( \frac{\lambda}{\pi^2 k^2 a^2} \right)^j \quad (\mu = 0).
\]

Then we have the following theorem:
Theorem 1. For $\mu \geq 0$, $\lambda > 0$, $\delta = 2\mu - \gamma$ and $\gamma = 2p$, where $p$ is a non-negative integer, we have when $|\arg a| < \pi/4$

$$S_{\mu, \gamma}(a; \lambda) = a^{1-\delta}H(1) - \frac{1}{2}a^{-\delta}\delta_{0p} + J(a; \lambda),$$

where $H(1)$ is defined in (2.3) and $\delta_{0p}$ is the Kronecker symbol. The exponentially small contribution $J(a; \lambda)$ is given exactly by the double sums

$$J(a; \lambda) = \left(-\frac{p\pi^\mu}{\Gamma(\mu)}\right)^{-2p-1/2} \sum_{k \geq 1} k^{2p} e^{-2\pi ka^2} \sum_{j=0}^p (-)^j c_j \left(\frac{\lambda}{\pi^2k^2a^2}\right)^j I_{jk} \quad (\mu > 0), \quad (3.9)$$

and

$$J(a; \lambda) = (-)^p \left(\frac{\lambda}{\pi a^2}\right)^{-2p-1/2} \sum_{k \geq 1} k^{2p} e^{-\pi^2k^2a^2/\lambda} \sum_{j=0}^p (-)^j c_j \left(\frac{\lambda}{\pi^2k^2a^2}\right)^j \quad (\mu = 0), \quad (3.10)$$

where the coefficients $c_j = (-2p)!/(2^j j!)$ and the integrals $I_{jk}$ are defined in (3.8).

Remark 1. When $\mu = p = 0$, we find from (2.8), (3.1) and (3.10) (since $U(\frac{1}{2}, \frac{3}{2}, \lambda) = \lambda^{-1/2}$) the result

$$S_{0,0}(a; \lambda) = a^{\frac{3}{2}} \sqrt{\frac{\pi}{\lambda}} - 1 + a \sqrt{\frac{\pi}{\lambda}} \sum_{k \geq 1} e^{-\pi^2ka^2/\lambda},$$

which is the Poisson-Jacobi formula stated in (1.3).

When $p = 1$, we have $U(\frac{3}{2}, \frac{5}{2}, \lambda) = \lambda^{-3/2}$ and

$$S_{0,2}(a; \lambda) = a^3 \sqrt{\frac{\pi}{4\lambda^{3/2}}} - \left(\frac{\pi a^2}{\lambda}\right)^{5/2} \sum_{k \geq 1} \left(k^2 - \frac{\lambda}{2\pi^2a^2}\right) e^{-\pi^2ka^2/\lambda}.$$

We observe that this last case can also be obtained by differentiation of the Poisson-Jacobi formula with respect to $\lambda$, since

$$S_{0,2p}(a; \lambda) = (-)^p a^{2p} \frac{\partial^p}{\partial \lambda^p} S_{0,0}(a; \lambda) \quad (p \geq 1).$$

Remark 2. When $p = -1, -2, \ldots$, the expansion (3.5) does not terminate and becomes an inverse factorial expansion. Then we have the exponentially small contribution given by

$$J(a; \lambda) \sim \left(-\frac{p\pi^\mu}{\Gamma(\mu)}\right)^{-2p-1/2} \sum_{k \geq 1} k^{2p} e^{-2\pi ka^2} \sum_{j=0}^\infty (-)^j c_j \left(\frac{\lambda}{\pi^2k^2a^2}\right)^j I_{jk} \quad (\mu > 0) \quad (3.11)$$

as $|a| \to \infty$ in $|\arg a| < \pi/4$.

4. Alternative form of expansion for positive integer values of $\mu$

Let $\mu = m$ be a positive integer and $\gamma = 2p$. We split $H(s)$ into its two constituent parts given by (2.3) and write $J(a; \lambda) = J_1(a; \lambda) + J_2(a; \lambda)$ in an obvious manner.

4.1 Evaluation of $J_1(a; \lambda)$. Then we have

$$J_1(a; \lambda) = \frac{a^{-\delta}}{2\pi} \sum_{k \geq 1} \frac{1}{k} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{H_1(-s)\Gamma(1+s)\sin \frac{\pi}{2} s (2\pi ka)^{-s}}{s} ds,$$
where from [2,4] with an application of Kummer’s transformation

\[ \mathcal{H}_1(-s) = \frac{(-)^m e^{\pi \eta_1 \lambda}}{\Gamma(m) \sin \frac{\pi}{2} s} \Gamma(p - \frac{1}{2} s) \Gamma(1 + s + m - p - \frac{3}{2} s) \frac{1}{\Gamma(1 - m + p - \frac{3}{2} s)} \frac{1}{\Gamma(1 - m + p - \frac{1}{2} s)} F_1(1 - m; 1 - m + p - \frac{1}{2} s; -\lambda) \]
\[ = \frac{(-)^m e^{\pi \eta_1 \lambda}}{\Gamma(m) \sin \frac{\pi}{2} s} \sum_{r=0}^{m-1} \frac{1}{r!} \frac{1}{\Gamma(1 - m + p - \frac{1}{2} s + r)} \Gamma(p - \frac{1}{2} s) \Gamma(1 - m + p - \frac{3}{2} s + r) \]
\[ = \frac{(-)^{p-1} e^{\pi \eta_1 \lambda}}{\Gamma(m) \sin \frac{\pi}{2} s} \sum_{r=0}^{m-1} \frac{1}{r!} \frac{1}{\Gamma(1 - m + p - \frac{1}{2} s + r)} \Gamma(m - p - r + \frac{1}{2} s) \Gamma(1 - p + \frac{1}{2} s). \]

If we make the change of summation index \( r \to m - 1 - \ell \) and use the fact that \( (1 - m)_{m-1-\ell} = (-)^{m-1+\ell} \Gamma(m)! \), we find

\[ \mathcal{H}_1(-s) = \frac{(-)^m e^{\pi \eta_1 \lambda}}{\sin \frac{\pi}{2} s} \sum_{\ell=0}^{m-1} A_\ell (1 - p + \frac{1}{2} s)_\ell, \quad A_\ell := (-)^{\ell} \frac{\lambda^{m-1-\ell}}{(m-1-\ell)!}. \quad (4.1) \]

The Pochhammer symbol appearing in (4.1) can be written in the form

\[ (1 - p + \frac{1}{2} s)_\ell = \sum_{r=0}^{\ell} B_{\ell r} (s + 1)_r, \]

where

\[ B_{00} = 1, \quad B_{01} = \frac{1}{2} - p, \quad B_{11} = \frac{1}{2}, \]
\[ B_{02} = \frac{1}{2} - p, \quad B_{12} = \frac{3}{8} - p, \quad B_{22} = \frac{1}{4}, \]
\[ B_{03} = \frac{1}{2} - p, \quad B_{13} = \frac{3}{8}(5 - 10p + 4p^2), \quad B_{23} = \frac{3}{8}(1 - p), \quad B_{33} = \frac{1}{8}, \ldots. \quad (4.2) \]

Then we obtain

\[ J_1(a; \lambda) = (-)^{k-p} e^{\pi \eta_1 \lambda} a^{-\delta} \sum_{\ell=0}^{m-1} \sum_{r=0}^{\ell} A_\ell B_{\ell r} \frac{1}{s} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(1 + s + r)(2\pi ka)^{-s} ds. \quad (4.3) \]

The integrals appearing in \( J_1(a; \lambda) \) can be evaluated by (4.7) to produce the final exact result

\[ J_1(a; \lambda) = (-)^{m-p} e^{2\pi a} \sum_{r=0}^{m-m-1} \sum_{\ell=r} A_\ell B_{\ell r} \frac{\sigma_r(a)(2\pi ka)^r}{a^{2m-r}} \quad (4.4) \]

for positive integer \( m \). Here we have defined the sums

\[ \sigma_r(a) := e^{2\pi a} \sum_{k \geq 1} k^r e^{-2\pi ka}, \quad (4.5) \]

which have the evaluations for \( 0 \leq r \leq 3 \)

\[ \sigma_0(a) = \frac{e^{2\pi a}}{2 \sinh \pi a}, \quad \sigma_1(a) = \frac{e^{2\pi a}}{4 \sinh^2 \pi a}, \quad \sigma_2(a) = \frac{e^{2\pi a} \cosh \pi a}{4 \sinh^3 \pi a}, \]
\[ \sigma_3(a) = \frac{e^{2\pi a}(2 + \cosh \pi a)}{8 \sinh^5 \pi a}, \ldots. \]

Note that \( J_1(a; \lambda) \equiv 0 \) when \( m = 0 \), since \( \mathcal{H}_1(s) \) vanishes for these values.

### 4.2 Evaluation of \( J_2(a; \lambda) \)

We have

\[ J_2(a; \lambda) = \frac{a^{-\delta}}{2\pi} \sum_{k \geq 1} \frac{1}{k} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{H}_2(-s) \Gamma(1 + s) \sin \frac{\pi}{2} (2\pi ka)^{-s} ds, \]
where from (2.5)

\[ \mathcal{H}_2(s) = (-)^{m-p-\lambda} \frac{\pi^{m-p-s/2}}{\sin \frac{s}{2}} \frac{1}{\Gamma(1+m-p+\frac{s}{2})} \]

\[ = (-)^{m-p} \frac{\pi^{m-p-s/2}}{\sin \frac{s}{2}} \sum_{r=0}^{\infty} \frac{(m)_r \lambda^r}{r! \Gamma(1+m-p+r+\frac{s}{2})}. \]

Then we obtain

\[ J_2(a; \lambda) = \frac{(-\lambda)^{m-p} a^{-\delta}}{\sqrt{\pi}} \sum_{k \geq 1} \frac{1}{\sqrt{k}} \sum_{r=0}^{\infty} \frac{(m)_r \lambda^r}{r!} K_{kr}, \]  

(4.6)

where

\[ K_{kr} = \frac{\sqrt{\pi}}{4\pi i} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{2^{-s} \Gamma(1+s)}{\Gamma(1+m-p+r+\frac{s}{2})} X_k^{-s/2} ds \]  

(4.7)

with \( X_k \) defined in (3.4). Since there no poles of the integrand in \( \Re(s) > 0 \) the integration path in (4.7) can be displaced as far to the right as we please such that \( |s| \) is everywhere large on the new path. The quotient of gamma functions in the integrand can then be expanded in a manner similar to that in (3.5) to find [7 p. 53]

\[ \frac{2^{-s} \sqrt{\pi} \Gamma(1+s)}{\Gamma(1+m-p+r+\frac{s}{2})} = \frac{\Gamma(\frac{1}{2} + \frac{s}{2}) \Gamma(1 + \frac{s}{2})}{\Gamma(1+m-p+r+\frac{s}{2})} \]

\[ = \sum_{j=0}^{M-1} (-)^j \hat{c}_j(r) \Gamma(\frac{1}{2} + \vartheta - j) + \rho_M(s) \Gamma(\frac{1}{2} s + \vartheta - M), \]

where \( M \) is a positive integer, \( \vartheta = \frac{1}{2} + p - m - r \) and \( \rho_M(s) = O(1) \) as \( |s| \to \infty \) in \( |\arg s| < \pi \).

The coefficients are given explicitly by

\[ \hat{c}_j(r) = \frac{1}{j!} (m-p+r)_j (m-p+r+\frac{1}{2})_j = \frac{(2m-2p+2r)_{2j}}{2^j j!}. \]

(4.8)

When \( m = r = 0 \), the coefficients \( \hat{c}_j(r) \) reduce to \( c_j \) in (3.8). Then

\[ K_{kr} = \sum_{j=0}^{M-1} (-)^j c_j(r) \cdot \frac{\sqrt{\pi}}{4\pi i} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\Gamma(\frac{1}{2} + \vartheta - j) X_k^{-s/2}}{\Gamma(1+m-p+r+\frac{s}{2})} ds + R_{M,r} \]

\[ = X_k^0 e^{-X_k} \left\{ \sum_{j=0}^{M-1} (-)^j c_j(r) X_k^{-j} + O(X_k^{-M}) \right\} \]

(4.9)

by (5.1), where the remainder term

\[ R_{M,r} = \frac{1}{4\pi i} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \rho_M(s) \Gamma(\frac{1}{2} s + \vartheta - j) X_k^{-s/2} ds = O(X_k^{\vartheta-M} e^{-X_k}) \]

as \( |a| \to \infty \) in \( |\arg a| < \pi/4 \) by Lemma 2.7 in [7 p. 71].

From (4.6) and (4.9), it then follows that

\[ J_2(a; \lambda) \sim (-)^{m-p} \left( \frac{\lambda}{\pi a^2} \right)^{\delta-1/2} \sum_{k \geq 1} \frac{e^{-\pi^2 k^2 a^2/\lambda}}{k^\delta} \sum_{r=0}^{\infty} \sum_{j=0}^{\infty} (-)^j c_j(r) \frac{(m)_r \lambda^r}{r!} \left( \frac{\lambda}{\pi^2 k^2 a^2} \right)^{r+j} \]

\[ = (-)^{m-p} \left( \frac{\lambda}{\pi a^2} \right)^{\delta-1/2} \sum_{r=0}^{\infty} (-)^r C_r \left( \frac{\lambda}{\pi^2 a^2} \right)^r \sum_{k \geq 1} \frac{e^{-\pi^2 k^2 a^2/\lambda}}{k^{2r+\delta}}, \]

(4.10)
where the double sum over \( r \) and \( j \) has been summed ‘diagonally’ (see [3] p. 58) and the coefficients \( C_r \) are given by

\[
C_r \equiv C_r(m, p, \lambda) = \sum_{n=0}^{r} \frac{(-\lambda)^n(m)_n}{n!} \hat{c}_{r-n}(n).
\]

Use of (4.8) shows that the \( C_r \) may be expressed in terms of a terminating \( 2F_2 \) hypergeometric series:

\[
C_r = \sum_{n=0}^{r} \frac{(-\lambda)^n(m)_n}{n!(r-n)!} \frac{2^{2n-2r}(2m-2p)_{2r}}{(2m-2p)_{2n}} = \frac{(2m-2p)_{2r}}{2^{2r}r!} \sum_{n=0}^{r} \frac{(-r)_n(m)_n\lambda^n}{n!(m-p)_n(m-p+\frac{1}{2})n}
\]

(4.11)

where we have made use of the results \((a+2n)_{2r-2n} = (a)_{2r}/(a)_{2n}\) and \((2a)_{2n} = 2^{2n}(a)_n(a+\frac{1}{2})_n\).

Then we have the following theorem:

**Theorem 2.** Let \( \mu = m \) and \( \gamma = 2p \), where \( m \geq 1, p \) are integers, and \( \delta = 2(m-p) \) with \( \lambda > 0 \). Then we have the representation

\[
S_{\mu, \gamma}(a; \lambda) = a^{1-\delta}H(1) + H_{\mu, \gamma}(a; \lambda) + J(a; \lambda),
\]

(4.12)

where \( H(1) \) and \( H_{\mu, \gamma}(a; \lambda) \) are defined in (2.8) and (3.2). The exponentially small contribution \( J(a; \lambda) \) has the expansion

\[
J(a; \lambda) + (-)^{m-p}m!a^{2p+1}e^{2\pi a^2} \sum_{r=0}^{m-1} \frac{A_r B_{r+1} \sigma_r(a)(2\pi)^k a^{2m-r}}{r!}
\]

\[
\sim (-)^{m-p}m!a^{2p+1}e^{2\pi a^2} \sum_{r=0}^{\infty} (-)^r C_r \left( \frac{\lambda}{2\pi a^2} \right)^r e^{-\pi^2 k^2 a^2/\lambda} \sum_{k=1}^{\infty} \frac{e^{-\pi^2 k^2 a^2/\lambda}}{k^{2r+\delta}}
\]

(4.13)

as \( |a| \to \infty \) in \( \arg a < \pi/4 \). The coefficients \( A_r, B_{r+1} \) and \( C_r \) are defined in (4.1), (4.2) and (4.11) and the functions \( \sigma_r(a) \) are given by (4.5). When \( m = 0 \) the double sum on the left-hand side of (4.13) vanishes.

**5. Examples.**

We present some examples of the expansion of \( S_{\mu, \gamma}(a; \lambda) \) stated in Theorem 2.

**Example 1.** Let \( \mu = 0 \) and \( \gamma = 2p \). Then from (4.12) and (4.13) we have, for \( p = 1, 2, \ldots \),

\[
S_{0, 2p}(a; \lambda) = \sum_{n=1}^{\infty} n^{2p}e^{-\lambda n^2/a^2} \sim \frac{1}{2} \left( \frac{\lambda}{a^2} \right)^{-p-1/2} \Gamma(p + \frac{1}{2})
\]

\[
+ (-)^p \left( \frac{\lambda}{2\pi a^2} \right)^{-2p-1/2} \sum_{r=0}^{\infty} (-)^r C_r \left( \frac{\lambda}{2\pi a^2} \right)^r \sum_{k=1}^{\infty} \frac{e^{-\pi^2 k^2 a^2/\lambda}}{k^{2r+2p}}
\]

(5.1)

as \( |a| \to \infty \) in \( \arg a < \pi/4 \), where from (4.11) \( C_r = (-2p)_{2r}/(2^{2r}r!) \). The case \( p = 0 \) is covered in Remark 1.

In the case \( p \leq -1 \) we let \( p = -q \) and note that \( H(s) = \frac{1}{2} \lambda^{s/2} \Gamma\left( \frac{1}{2} s - q \right) \) when \( \mu = 0 \). The residue of \( H(s) \) at \( s = 2q - 2k \) \((k = 0, 1, 2, \ldots)\) is given by \((-k)! k!\). Then

\[
S_{0, -2q}(a; \lambda) = \sum_{n=1}^{\infty} \frac{e^{-\lambda n^2/a^2}}{n^{2q}} \sim \frac{1}{2} \left( \frac{\lambda}{a^2} \right)^{-q-1/2} \Gamma\left( \frac{1}{2} q - q \right) + \sum_{k=0}^{q} \frac{(-k)!}{k!} \left( \frac{\lambda}{a^2} \right)^{k} \zeta(2q - 2k)
\]
Example 3. Let \( \delta = 2 \) and \( \gamma = 0 \). Then we have

\[
S_{0,0}(a; \lambda) = \frac{\pi e^\lambda}{2a} \operatorname{erfc}\sqrt{\lambda - \frac{1}{2a^2}} + J(a; \lambda),
\]

(5.3)

since [3] (13.6.8)

\[
\mathcal{H}(1) = \frac{1}{\pi} \sqrt{\pi} U\left(\frac{1}{2}, \frac{1}{2}, \lambda\right) = \frac{1}{\pi} e^\lambda \operatorname{erfc}\sqrt{\lambda},
\]

where erfc is the complementary error function. From (4.13), the exponentially small contribution is

\[
J(a; \lambda) = -\frac{\pi e^\lambda}{2a} e^{-\pi a} \frac{\lambda}{\pi a^2} \left[ \sum_{r=0}^{\infty} (-)^r C_r \left( \frac{\lambda}{\pi a^2} \right)^r \sum_{k \geq 1} \frac{e^{-\pi^2 k^2 a^2 / \lambda}}{k^{2r+2}} \right]
\]

(5.4)

as \(|a| \to \infty|\arg a| < \pi/4\), where from (4.11) the coefficients \( C_r \) are given by

\[
C_r = \frac{2 \Gamma(r + \frac{3}{2})}{\sqrt{\pi}} \, _1 F_1 (-r; \frac{5}{2}; \lambda).
\]

The first few \( C_r \) are therefore

\[
C_0 = 1, \quad C_1 = \frac{3}{2} - \lambda, \quad C_2 = \frac{15}{4} - 5\lambda + \lambda^2,
\]

\[
C_3 = \frac{105}{8} - \frac{105}{4} \lambda + \frac{39}{4} \lambda^2 - \lambda^3.
\]

We remark that the case \( \mu = 1, \gamma = 2 \) can be obtained directly from Theorem 2, but also follows from the Poisson-Jacobi formula ([1], together with (5.3) and (5.4), since

\[
S_{1,2}(a; \lambda) = \sum_{n=1}^{\infty} \frac{n^2 e^{-\lambda n^2 / a^2}}{n^2 + a^2} = \sum_{n=1}^{\infty} \left( 1 - \frac{a^2}{n^2 + a^2} \right) e^{-\lambda n^2 / a^2} = S_{0,0}(a; \lambda) - a^2 S_{1,0}(a; \lambda).
\]

Example 3. Let \( \delta = 2 \) and \( \gamma = 0 \) (so that \( \delta = 4 \)). Then we have

\[
S_{0,0}(a; \lambda) = \sum_{n=1}^{\infty} \frac{e^{-\lambda n^2 / a^2}}{(n^2 + a^2)^2} = \sqrt{\frac{\pi}{2a^3}} U\left(\frac{1}{2}, -\frac{1}{2}, \lambda\right) - \frac{1}{2a^4} + J(a; \lambda).
\]

(5.5)

From (4.1) and (4.2), the coefficients \( A_0 = \lambda, A_1 = -1, B_{00} = 1, B_{01} = B_{11} = \frac{1}{2} \), so that

\[
J_1(a; \lambda) = -\frac{\pi e^{\lambda - \frac{\pi a}{2}}}{2a^3 \sinh \frac{\pi a}{2}} \left( \lambda - \frac{1}{2} - \frac{\pi a e^{\pi a}}{2 \sinh \frac{\pi a}{2}} \right)
\]

and

\[
J_2(a; \lambda) \sim \left( \frac{\lambda}{\pi a^2} \right)^{7/2} \sum_{r=0}^{\infty} (-)^r C_r \left( \frac{\lambda}{\pi a^2} \right)^r \sum_{k \geq 1} \frac{e^{-\pi^2 k^2 a^2 / \lambda}}{k^{2r+4}}.
\]

From (4.11) the coefficients \( C_r \) are given by

\[
C_r = \frac{4(r + 1)}{3\sqrt{\pi}} \Gamma(r + \frac{5}{2}) \, _1 F_1 (-r; \frac{5}{2}; \lambda)
\]
so that the first few coefficients are therefore

\[ C_0 = 1, \quad C_1 = 5 - 2\lambda, \quad C_2 = \frac{105}{4} - 21\lambda + 3\lambda^2, \]

\[ C_3 = \frac{315}{2} - 189\lambda + 54\lambda^2 - 4\lambda^3. \]

Then the exponentially small contribution is

\[ J(a; \lambda) + \frac{\pi e^{\lambda - \pi a}}{2a^3 \sinh \pi a} \left( \lambda - \frac{1}{2} - \frac{\pi a e^{\pi a}}{2 \sinh \pi a} \right) \]

\[ \sim \left( \frac{\lambda}{\pi a^2} \right)^{7/2} \sum_{r=0}^{\infty} (-1)^r C_r \left( \frac{\lambda}{\pi^2 a^2} \right)^r \sum_{k \geq 1} \frac{e^{-\pi^2 k^2 a^2 / \lambda}}{k^{2r+4}} \]

(5.6)

as \( |a| \to \infty \) in \( |\arg a| < \pi/4 \).

As mentioned in the previous example, the sums with \( \mu = 2 \) and \( p = 2, 4 \) can be obtained directly from Theorem 2, but also from the identities

\[ S_{2,1}(a; \lambda) = S_{1,0}(a; \lambda) - a^2 S_{2,0}(a; \lambda) \]

\[ S_{2,2}(a; \lambda) = S_{0,0}(a; \lambda) - 2a^2 S_{2,1}(a; \lambda) - 3a^4 S_{2,0}(a; \lambda). \]

6. Numerical results and concluding remarks

The expansion of the exponentially small contribution (when \( \gamma = 2p \)) given in Theorem 1 is exact for \( \mu \geq 0 \). It is possible to employ an asymptotic expansion for the integrals \( I_{jk} \), but this would necessarily introduce an error. However, we have evaluated these integrals to high numerical precision and have thereby verified the expansion (5.10) of \( J(a; \lambda) \) for several parameter values to 50 decimal precision.

We present some numerical examples of the large-\( a \) expansion of \( S_{\mu,\gamma}(a; \lambda) \) given in Theorem 2 to demonstrate the accuracy of our results. We subtract from the sum \( S_{\mu,\gamma}(a; \lambda) \) the finite terms appearing in (4.12) by defining

\[ \hat{S}_{\mu,\gamma}(a; \lambda) := S_{\mu,\gamma}(a; \lambda) - \{ a^{1-4\delta} \mathcal{H}(1) + H_{\mu,\gamma}(a; \lambda) + J_1(a; \lambda) \} \]

(6.1)

and comparing it with the exponentially small asymptotic expansion \( J_2(a; \lambda) \) in (4.10). We stress that the contribution \( J_1(a; \lambda) \) in (4.4) is an exact result when \( \mu \) is an integer. In Table 1 we show the values of the absolute relative error in the high-precision computation of \( \hat{S}_{\mu,\gamma}(a; \lambda) \) from (6.1) using the asymptotic expansion for \( J_2(a; \lambda) \) for different truncation index \( r \). The values of \( \mu \) and \( \gamma \) chosen correspond to the examples given in Section 5. The final entry in each column gives the value of \( \hat{S}_{\mu,\gamma}(a; \lambda) \). It is seen that the exponentially small contribution to \( S_{\mu,\gamma}(a; \lambda) \) when \( \gamma \) is an even integer agrees well with the expansion given in Theorem 2.

It is worth mentioning that \( J(a; \lambda) \) given in Theorems 1 and 2 appears to comprise two different types of exponentially small terms, namely \( \exp(-2\pi ka) \) in Theorem 1 and both \( \exp(-2\pi ka) \) and \( \exp(-\pi^2 k^2 a^2 / \lambda) \), \( k \geq 1 \) in Theorem 2. However, a closer examination of the integrals \( I_{jk} \) appearing in Theorem 1 reveals that they also contain the more subdominant terms \( \exp(-\pi^2 k^2 a^2 / \lambda) \).

To see this we consider

\[ e^{-2\pi ka} I_{jk} = e^{-2\pi ka} \int_0^\infty \frac{t^{\mu-1} e^{-\psi(t)}}{(1+t)^\beta} dt, \quad \beta = 2p - j + \frac{1}{2}. \]

The phase function \( \psi(t) \) in (3.8) has a saddle point at \( t = t_s \), where \( 1+t_s = \sqrt{X_k / \lambda} = \pi ka / \lambda \) and \( \psi(t_s) = 0 \). For large complex \( a \) in the sector \( |\arg a| < \pi/4 \), the integration path is chosen to emanate from the origin in the direction \( \arg t = \phi \).
Table 1: The absolute relative error in the computation of $\hat{S}_{\mu,\gamma}(a;\lambda)$ from (6.1) for different $\mu$, $\gamma$ and truncation index $r$ in the asymptotic expansion $J_\lambda(a;\lambda)$ when $\lambda = 2$ and $a = 3$.

| $r$ | $\mu = 0$, $\gamma = -2$ | $\mu = 1$, $\gamma = 0$ | $\mu = 2$, $\gamma = 0$ |
|-----|----------------------------|--------------------------|--------------------------|
| 0   | $3.307 \times 10^{-02}$    | $1.007 \times 10^{-02}$  | $2.464 \times 10^{-02}$  |
| 1   | $1.823 \times 10^{-03}$    | $1.070 \times 10^{-03}$  | $1.572 \times 10^{-03}$  |
| 2   | $1.408 \times 10^{-04}$    | $5.900 \times 10^{-05}$  | $3.764 \times 10^{-04}$  |
| 5   | $2.438 \times 10^{-07}$    | $7.124 \times 10^{-08}$  | $3.325 \times 10^{-07}$  |
| 10  | $9.421 \times 10^{-11}$    | $3.101 \times 10^{-12}$  | $7.198 \times 10^{-10}$  |
| 15  | $3.138 \times 10^{-13}$    | $6.596 \times 10^{-14}$  | $2.317 \times 10^{-12}$  |
| 20  | $4.678 \times 10^{-15}$    | $4.335 \times 10^{-16}$  | $5.392 \times 10^{-14}$  |

$\hat{S}_{\mu,\gamma} = -9.3737097 \times 10^{-22}$

$\hat{S}_{\mu,\gamma} = -9.7822227 \times 10^{-22}$

$+4.7287147 \times 10^{-24}$

to the singularity at $t = -1$ and thence along the path of steepest descent through $t_s$ to infinity in $\Re(t) > 0$. The contribution from the saddle is controlled by

$$2e^{-2\pi k a} \sqrt{\frac{\pi}{2\theta^m(t_s)}} \left(\frac{t_s^u}{1 + t_s^u}\right)^{\mu - 1/2} e^{-2\pi k a}$$

while that from the neighbourhood of the origin is approximately

$$e^{-X_k + i\mu \phi} \int_0^\infty e^{-|X_k|^2} r^{\mu - 1} dr = O(X_k^{-\mu} e^{-X_k}),$$

which produces the more subdominant exponential terms.

Finally, we note that the alternating version of (1.1) can be expressed in terms of $S_{\mu,\gamma}(a;\lambda)$ since

$$\sum_{n=1}^\infty \frac{(-)^n \gamma}{(n^2 + a^2)^\mu} e^{-\lambda n^2 / a^2} = S_{\mu,\gamma}(a;\lambda) - 2^{1-\delta} S_{\mu,\gamma}(1/2 a;\lambda).$$

Application of Theorems 1 and 2 then enables the large-$a$ expansion of the alternating series to be determined.

**Appendix A: The pole structure of $H(s)$**

The function $H(s)$ defined in (2.3), (2.4) and (2.5) has poles at $s = -2k - \gamma$, $k = 0, 1, 2, \ldots$ and apparent poles at $s = \pm 2k + \delta$, $\delta = 2\mu - \gamma$. We shall show in this appendix that $H(s)$ is regular at these last points. We have

$$H(s) = \frac{\pi G(s)}{2 \sin \pi (\mu - \frac{\gamma}{2})},$$

where

$$G(s) := \frac{\Gamma(\frac{s+\mu}{2})}{\Gamma(\mu)} F(\frac{s+\mu}{2}; 1 + \frac{s+\mu}{2} - \mu; \lambda) - \lambda^{\mu-(\gamma+s)/2} F(\mu; 1 - \frac{s+\mu}{2} + \mu; \lambda).$$

Here $F$ denotes the normalised confluent hypergeometric function defined by

$$F(a; b, z) = \frac{1}{\Gamma(b)} \frac{1}{1} F_1(a; b; z),$$

which is defined for all values of the parameter $b$. 
Let \( s_k = 2k + \delta \) so that \( \frac{1}{2}(\gamma + s_k) = k + \mu \). Then

\[
G(s_k) = (\mu)_k F(\mu + k; k + 1; \lambda) - \lambda^{-k} \sum_{r=k}^{\infty} \frac{(\mu)_r \lambda^r}{r! \Gamma(1 + r - k)}
\]

\[
= (\mu)_k F(\mu + k; k + 1; \lambda) - \sum_{r=0}^{\infty} \frac{(\mu + k)_r \lambda^r}{r! \Gamma(1 + k + r)}
\]

\[
= (\mu)_k F(\mu + k; k + 1; \lambda) - (\mu)_k \sum_{r=0}^{\infty} \frac{(\mu + k)_r \lambda^r}{r! \Gamma(1 + k + r)} \equiv 0.
\]

Hence \( \mathcal{H}(s) \) is regular at \( s_k = 2k + \delta \).

A similar argument when \( s_k = -2k + \delta \) shows that

\[
G(s_k) = \frac{\Gamma(\mu - k)}{\Gamma(\mu)} \sum_{r=k}^{\infty} \frac{(\mu - k)_r \lambda^r}{r! \Gamma(1 - k + r)} - \lambda^k F(\mu; 1 + k; \lambda)
\]

\[
= \frac{\Gamma(\mu - k)}{\Gamma(\mu)} \sum_{r=0}^{\infty} \frac{(\mu - k)_r + k \lambda^r}{r! \Gamma(1 + k + r)} - \lambda^k F(\mu; 1 + k; \lambda)
\]

\[
= \lambda^k \sum_{r=0}^{\infty} \frac{(\mu)_r \lambda^r}{r! \Gamma(1 + k + r)} - \lambda^k F(\mu; 1 + k; \lambda) \equiv 0,
\]

so that \( \mathcal{H}(s) \) is also regular at the points \( s_k = -2k + \delta \).

### Appendix B: The expansion in the case \( \gamma = -1 \)

We consider the large-\( a \) expansion of \( S_{\mu, \gamma}(a; \lambda) \) given in (2.7) in the special case \( \gamma = -1 \) when the singularity of the integrand in (2.6) at \( s = 1 \) is a double pole. We set \( s = 1 + \epsilon \), with \( \epsilon \to 0 \). Then

\[
\mathcal{H}_1(s) = \frac{\Gamma((\frac{1}{2})\epsilon) \Gamma(\mu - \frac{1}{2} \epsilon)}{2 \Gamma(\mu)} F_1(\frac{1}{2} \epsilon; 1 - \mu + \frac{1}{2} \epsilon; \lambda)
\]

\[
= \frac{1}{\epsilon} \left\{ 1 + \frac{1}{2} \epsilon (\psi(1) - \psi(\mu)) + O(\epsilon^2) \right\} F_1(\frac{1}{2} \epsilon; 1 - \mu + \frac{1}{2} \epsilon; \lambda),
\]

where

\[
F_1(\frac{1}{2} \epsilon; 1 - \mu + \frac{1}{2} \epsilon; \lambda) = 1 + \frac{\epsilon \lambda}{2(1 - \mu)} \left\{ 1 + \frac{11 \lambda}{(2 - \mu) 2!} + \frac{2! \lambda^2}{(2 - \mu) 3!} + \cdots \right\} + O(\epsilon^2)
\]

\[
= 1 + \frac{\epsilon \lambda}{2(1 - \mu)} F_2(1, 1; 2, 2 - \mu; \lambda) + O(\epsilon^2).
\]

Using the fact that \( \zeta(1 + \epsilon) = e^{-\epsilon}(1 + \gamma_E \epsilon + O(\epsilon^2)) \) and \( \psi(1) = -\gamma_E \), where \( \gamma_E \) is the Euler-Mascheroni constant, we obtain the residue resulting from \( \mathcal{H}_1(s) \) at \( s = 1 \) given by

\[
a \left\{ \log a + 1 + \frac{1}{2} \gamma_E - \frac{1}{2} \psi(\mu) + \frac{\lambda}{2(1 - \mu)} F_2(1, 1; 2, 2 - \mu; \lambda) \right\} \quad (\mu \neq 1, 2, \ldots).
\]

The residue resulting from \( \mathcal{H}_2(s) \) is

\[
a \mathcal{H}_2(1) = \frac{1}{2} \lambda \Gamma(1 - \mu) F_1(\mu; 1 + \mu; \lambda) \quad (\mu \neq 1, 2, \ldots).
\]

Hence, provided \( \mu \neq 1, 2, \ldots \),

\[
S_{\mu, -1}(a; \lambda) = \sum_{n=1}^{\infty} e^{-\lambda n^2/a^2} \frac{n(n^2 + a^2)^\mu}{n(n^2 + a^2)^\mu} \sim a^{-1-\delta} \left\{ \log a + 1 + \frac{1}{2} \gamma_E - \frac{1}{2} \psi(\mu) + \frac{\lambda}{2(1 - \mu)} F_2(1, 1; 2, 2 - \mu; \lambda) \right\}
\]
\[+a^{1-\delta} \sum_{k=1}^{\infty} \frac{(-)^k(\mu)_k}{k!} \zeta(1-2k)_{1F1}(-k; 1-\mu-k; \lambda)a^{-2k}\]  
(B.1)

as \(a \to \infty\) in \(|\arg a| < \pi/4\).

When \(\mu\) is a positive integer a limiting process is required. To illustrate, we consider only the case \(\mu = 1\). We find that

\[
H_1(1+\epsilon) = e^{\lambda/\epsilon} + O(\epsilon^2)
\]

and

\[
H_2(1+\epsilon) = e^{-\lambda/\epsilon} \sum_{n=1}^{\infty} \frac{\lambda^n}{(2)_n} + O(\epsilon^2), \quad \tau(n) := \sum_{r=1}^{n} \frac{1}{r+1}.
\]

Then we obtain the expansion

\[
S_{1,-1}(a; \lambda) = \sum_{n=1}^{\infty} \frac{e^{-\lambda n^2/a^2}}{n(n^2+a^2)} \sim \frac{a^{1-\delta}}{2} \left\{ e^{\lambda(\log \lambda + \gamma_E - 1) + \log (a^2/\lambda) + \gamma_E} + 1 - \lambda \sum_{n=1}^{\infty} \frac{\lambda^n \tau(n)}{(2)_n} \right\} + a^{1-\delta} \sum_{k=1}^{\infty} \frac{(-)^k(\mu)_k}{k!} \zeta(1-2k)e_k(\lambda)a^{-2k}
\]

(B.2)

as \(a \to \infty\) in \(|\arg a| < \pi/4\), where

\[
e_k(\lambda) := 1_{1F1}(-k; -k; \lambda) = \sum_{n=0}^{k} \frac{\lambda^n}{n!}.
\]

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