Multi-Execution Lattices Fast and Slow

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ABSTRACT
Methods for automatically, soundly, and precisely guaranteeing the noninterference security policy are predominantly based on multi-execution. All other methods are either based on undecidable theorem proving or suffer from false alarms. The multi-execution mechanisms, meanwhile, work by isolating security levels during program execution and running multiple copies of the target program, once for each security level with carefully tailored inputs that ensure both soundness and precision. When security levels are hierarchically organised in a lattice, this may lead to an exponential number of executions of the target program as the number of possible ways of combining security levels grows. In this paper we study how the lattice structure for security levels influences the runtime overhead of multi-execution. We additionally show how to use Galois connections to gain speedups in multi-execution by switching from lattices with high overhead to lattices with low overhead. Additionally, we give an empirical evaluation that corroborates our analysis and shows how Galois connections have potential to speed up multi-execution.

1 INTRODUCTION

Language-based information flow control (IFC) is a set of techniques for controlling the way that information flows inside a program [42]. The techniques in this field are generally aimed at ensuring the noninterference security policy [20, 23]: a program is noninterfering if its secret inputs cannot influence its public outputs. Traditional enforcement mechanisms for IFC, whether static [10, 41, 45] or dynamic [8, 24, 31, 48], suffer either from high false-alarm rates [28, 46] or undecidability [11, 12].

To address this issue, IFC researchers have introduced a family of enforcement mechanisms collectively known as multi-execution [9, 21, 27, 35, 43]. Multi-execution guarantees transparency: if $E$ is a transparent enforcement mechanism and $p$ is a noninterfering program, then $E[p](x) = p(x)$ for all $x$. Consequently, multi-execution cannot introduce any false alarms.

How it works is in the name, programs are executed multiple times to ensure that output at each security level can only depend on input visible to that level. A consequence of this strategy is that in the worst case, multi-execution introduces unmanageable performance overhead [2, 4, 9, 43]. In fact, under certain assumptions, it is impossible to construct a transparent enforcement mechanism that introduces less than exponential worst-case overhead in execution time on secure programs [2].

To be secure and transparent, multi-execution has to respect every way a program combines data of different security levels [2]. If the program $p$ combines data sensitive to Alice, Bob, and Charlie, multi-execution is forced to run $p$ eight times, once for every subset of {Alice, Bob, Charlie}. However, if $p$ only combines public and secret data, multi-execution only needs to run $p$ twice, once for secret and once for public data, as the combination of public and secret data is secret data.

In general, we consider the case where the security levels are drawn from a lattice [20]. As demonstrated above, the shape of the lattice matters. A powerset lattice over principals (like Alice, Bob, and Charlie) introduces an exponential number of combinations of security levels, but a total order (like the two-point lattice with levels for public and secret data) doesn’t suffer from this issue. This relationship is the core of what this paper is about.

There are a number of practical takeaways from this insight. Applications that do not require the combinatorial power of powerset lattices do well under multi-execution. For example, Alpernas et al. [5] use three lattices in their case studies of IFC in a serverless setting that are all “wide and short”. In their lattices, each user has a security label, but the applications do not distinguish between arbitrary combinations of users, and the lattices have a flat structure. For example, their $gg$ lattice has least and greatest elements $\bot$ and $\top$ and a set of incomparable elements {$Alice, Bob, \ldots$} where the combination of any two such elements is $\top$. At most $n + 2$ executions need to be performed when multi-executing in this lattice, one execution for each principal that appears in the input, one for $\bot$, and one for $\top$. This demonstrates that some applications admit lattices that accommodate the security requirements of the application while introducing low overhead.

The primary goal of this paper, then, is to answer the following question: Given some lattice $L$ and input elements that are associated with $N$ distinct labels in $L$, what is the time overhead that multi-execution imposes with respect to $N$?

To answer this question, we describe a new perspective on multi-execution that comes in two parts. Firstly, we provide the tools necessary to analyse security lattices to quantify how much overhead they introduce in multi-execution. Secondly, we show how to pair the fact that some lattices introduce less overhead than others with the insight that Galois connections give rise to natural translations between security lattices in multi-execution.

Concretely, we make the following contributions:

(1) We characterise the connection between the choice of security lattice and the worst-case runtime of black-box multi-execution enforcement (Sections 3 and 4).

(2) We present a theory for computing bounds on multi-execution overhead for different lattices (Section 3).

(3) We show how Galois connections reduce the overhead of multi-execution by executing in one lattice while observing the results in another (Section 5).

(4) We give a method for specifying optimal Galois connections for multi-execution (Section 5).

(5) We present a Haskell implementation of our techniques and empirically evaluate our predictions (Section 6).
2 REVIEW OF THE MULTI-EXECUTION FRAMEWORK

A (join semi-)lattice \( \mathcal{L} \) is a set \( \mathcal{L} \) with a transitive, reflexive, and antisymmetric order \( \sqsubseteq \) that has a least element \( \bot \) and is such that any two elements \( \ell, j \in \mathcal{L} \) have a least upper bound \( \ell \sqcup j \in \mathcal{L} \). For a finite subset \( S \subseteq \mathcal{L} \) we write \( \mathcal{L} \supseteq S \) for the least upper bound of all elements in \( S \). For example, the two-point lattice has \( \mathcal{L} = \{ \bot, H \} \), \( \bot \) denotes public information and \( H \) denotes secret information. Public information can flow to secret information so \( \sqsubseteq \) is the smallest reflexive relation such that \( \mathcal{L} \sqsubseteq H \). Finally, this means that \( L \sqcup L = L \) and \( \ell \sqcup j = H \) if either \( \ell \) or \( j \) is \( H \).

Following Algehed and Flanagan [2] we consider batch-job programs from labeled sets to labeled sets and let \( P \) denotes public information and \( H \) denotes secret information. A label \( \ell \in \mathcal{L} \) is noninterfering when for all \( \ell \), \( x \sim y \), and \( \ell \) projections are the same:

\[
\ell \sim \ell \iff (\ell \sqcup x, j \in \ell) \text{ if and only if their }\ell \text{-projections are the same:} \]

\[
x \sim \ell y \iff x \downarrow x \ell = y \downarrow x \ell
\]

The projection \( x \downarrow \ell \) of \( x \) at \( \ell \) is precisely all the information in \( x \) that is visible to \( \ell \). Likewise, this means that if two sets \( x \) and \( y \) look the same to \( \ell \), then they are \( \ell \)-equivalent. The definition of noninterference meanwhile is that \( p \) is noninterfering if it does not reveal more about its inputs than what one can know by looking at the input. In other words, if two inputs \( x \) and \( y \) differ only in values that are secret to an observer at level \( \ell \), they are \( \ell \)-equivalent, then \( p(x) \) and \( p(y) \) should also be \( \ell \)-equivalent.

**Definition 2.2 (Noninterference).** We say that program \( p : \mathcal{P}(I \times \mathcal{L}) \rightarrow \mathcal{P}(O \times \mathcal{L}) \) is noninterfering if it preserves \( \ell \)-equivalence. Concretely, \( p \) is noninterfering when for all \( \ell, x, y \) such that

1. \( x \sim \ell y \)
2. \( p(x) \) and \( p(y) \) are both defined,

it is the case that \( p(x) \sim \ell p(y) \).

Note that this definition of noninterference is a partial correctness criterion, it says that \( \ell \)-equivalence only has to be preserved up to termination, known as Termination Insensitive Noninterference (TINI) [25]. The theory of termination sensitivity in this setting is rich [2]. However, termination is orthogonal to our development and we omit it here.

**Example 2.3.** The program \( \text{secure} \) below is noninterfering (we write \( |x| \) for the size of the set \( x \)):

\[
\text{secure}(x) \triangleq \{ |x \downarrow \ell| \mid \ell \in \{ L, H \} \}
\]

Conversely, the program \( \text{insecure} \) is not noninterfering:

\[
\text{insecure}(x) \triangleq \{ |x| L \}
\]

Before we dive into more examples of how this framework works, we introduce a core lattice for this paper, the powerset lattice \( \mathcal{P}(A) \) over some set \( A \) of atoms or principals. A label \( \ell \in \mathcal{P}(A) \) is a subset \( \ell \subseteq A \) and labels are ordered by set inclusion, \( \ell \subseteq \ell' \) if and only if \( \ell \subseteq \ell' \). The least element of \( \mathcal{P}(A) \) is the empty set and the least upper bound of two labels \( \ell \sqcup \ell' \) is their union \( \ell \cup \ell' \). We usually write singleton labels, like \{Alice\}, without the brackets as Alice.

Finally, given \( x \in \mathcal{P}(V \times \mathcal{L}) \) we define the context labels of \( x \) as:

\[
\mathcal{L}(x) \triangleq \{ \ell \mid a' \in x \}
\]

**Example 2.4.** We present our running examples. First is the program \( \text{badSum}, \) that takes the sum of its inputs and labels the output with the least upper bound of the labels in the input.

\[
\text{badSum}(x) \triangleq \{ \sum a' \sqcup \mathcal{L}(x) \}
\]

This program is not noninterfering, as \( \emptyset \sim \{ \text{Alice} \} \) but:

\[
\text{badSum}(\emptyset) = \{ 1 \} \uparrow \downarrow \{ \text{Alice} \} = \text{badSum}(\{ \text{Alice} \})
\]

The problem is that the definition of noninterference is presence-sensitive, input at a label is considered sensitive information that may not leak. To address the problem with \( \text{badSum} \) we define \( \text{goodSum}_L \), a family of programs indexed by a set of labels \( L \) that are required to be in the input, and which form the levels for which the sum is taken; and consequently \( \text{goodSum}_L \) is noninterfering:

\[
\text{goodSum}_L(x) \triangleq \{ \sum a' \in x, \ell \in L \} \sqcup \mathcal{L}(x)
\]

The next noninterfering program combines input data pairwise.

\[
\text{pairwise}(x) \triangleq \{ \max(a, b) \sqcup L | a' \in x, b' \in x \}
\]

We want enforcement mechanisms for noninterference not to alter the semantics of programs that are already noninterfering. This type of enforcement is known as transparent IFC enforcement [55], and has been extensively studied in the literature [2, 4, 9, 15, 18, 19, 21, 27, 32, 35, 36, 39, 40, 43, 55]. The common denominator of all these is that they are based on the idea of multi-execution [21]. Figure 1 (originally appearing in [2]) illustrates multi-execution in the setting of public and secret data. Multi-execution runs the program \( p \) twice to produce \( \text{ME}[p] \), once with only public input (this is called the "public run") and once with both public and private input (this is called the "private run"). The final public outputs come from the public run, and the private outputs from the private run.

This guarantees noninterference: the output in the public run cannot depend on the secret input. Similarly, if \( p \) is noninterfering then multi-execution preserves its extensional behaviour. The secret output of \( \text{ME}[p] \) is the same as the secret output of \( p \), and the public output of \( \text{ME}[p] \) is the same as the public output of \( p \) by virtue of \( p \) being noninterfering.

To formalise multi-execution we first re-state Algehed and Flanagan’s definitions of some auxiliary functions:

**Definition 2.5.** Given a finite subset \( S \subseteq \mathcal{L} \) of \( \mathcal{L} \) we define the context set of \( S \) as:

\[
\mathcal{L}(S) \triangleq \{ \bigcup S' \mid S' \subseteq S \}
\]
This is the set of all combinations of levels in \( S \), and as we have seen it corresponds to the runs that multi-execution will have to do when \( S \) are the labels in the input to the program. The next notion we define is the \( \uparrow \)-set of \( \ell \) in \( S \) as:

\[
\ell \uparrow S \triangleq \{ j \mid j \subseteq j, \forall i \in S. \ i \subseteq j \Rightarrow i \subseteq \ell \}
\]

This is a technical notion that captures all the labels that "see the same view" of an input. It is what allows multi-execution to correctly propagate outputs from the target program that are not strictly combinations of the security levels in the input. Finally, given an \( x \subseteq V \times L \) and an \( L \subseteq L \) we define the selection of \( x \) at \( L \):

\[
x@L \triangleq \{ a^\ell \mid a^\ell \in x, \ell \in L \}
\]

Next we formalise multi-execution:

**Definition 2.6 (Multi-Execution [2]).**

\[
\text{MEF}[p](x) = \bigcup \{ p(x \downarrow \ell)@\ell \uparrow C(L(x)) \mid \ell \in C(L(x)) \}
\]

The definition of MEF is superficially different from the overview in Figure 1. Specifically, MEF\([p](x)\) runs \( p \) for every \( \ell \) in \( C(L(x)) \) rather than \( L \). This is because the set of all projections \( x \downarrow \ell \) for \( \ell \in L \) is equal to the set of all projections \( x \downarrow \ell \) for \( \ell \in C(L(x)) \). Consequently, MEF\([p](x)\) runs \( p(x \downarrow \ell) \) for all the \( \ell \) necessary to have every "view" of \( x \).

The \( \uparrow \) construction is responsible for reconstructing the outputs at each level in \( L \). The intuition for this construction is that \( \ell \uparrow C(L(x)) \) is the set of levels \( \{ j_1, \ldots, j_n \} \) such that the execution of \( p(x \downarrow \ell) \) is responsible for computing the output at levels \( j_i \). Formally, \( \ell \uparrow C(L(x)) \) is the set of all \( j_i \) such that \( x \downarrow j_i = x \downarrow \ell \). This is because the set of all projections \( x \downarrow \ell \) for \( \ell \in C(L(x)) \) is responsible for computing the output at levels \( j_i \).

For example, consider what happens when we run

\[
\text{MEF}[\text{goodSum}_L]((\{1\text{Alice}, 2\text{Charlie}\}))
\]

for \( L = \{ \text{Alice}, \text{Bob} \} \). We have that

\[
C((\{\text{Alice}, \text{Charlie}\})) = \{ \bot, \text{Alice}, \text{Charlie}, \{\text{Alice}, \text{Charlie}\} \}.
\]

This means that we have four runs of \( \text{goodSum} \):

\[
\begin{align*}
\text{goodSum}_L((\{1\text{Alice}, 2\text{Charlie}\}) \downarrow \bot) &= \{0\{\text{Alice}, \text{Bob}\}\} \\
\text{goodSum}_L((\{1\text{Alice}, 2\text{Charlie}\}) \downarrow \text{Alice}) &= \{1\{\text{Alice}, \text{Bob}\}\} \\
\text{goodSum}_L((\{1\text{Alice}, 2\text{Charlie}\}) \downarrow \text{Charlie}) &= \{0\{\text{Alice}, \text{Bob}\}\} \\
\text{goodSum}_L((\{1\text{Alice}, 2\text{Charlie}\}) \downarrow \{\text{Alice}, \text{Charlie}\}) &= \{1\{\text{Alice}, \text{Bob}\}\}
\end{align*}
\]

To determine the final output of \( \text{MEF}[\text{goodSum}_{\{\text{Alice}, \text{Bob}\}}] \), we need to decide which of these outputs we preserve to the final output. Figure 2 shows the up-sets of \( \bot, \text{Alice}, \text{Charlie}, \text{Alice, Charlie} \) in the powerset lattice for three principals Alice, Bob, and Charlie when \( L(x) = \{ \text{Alice, Charlie} \} \). We see that \( \{ \text{Alice, Bob} \} \) falls in the up-set of Alice, and so we have that the final result is:

\[
\text{MEF}[\text{goodSum}_{\{\text{Alice}, \text{Bob}\}}]((\{1\text{Alice}, 2\text{Charlie}\})) \Rightarrow \{1\{\text{Alice}, \text{Bob}\}\}
\]

MEF enjoys both noninterference and transparency [2].

**Theorem 1 (Security).** \( \text{MEF}[p] \) is noninterfering.

**Theorem 2 (Transparency).** If \( p \) is noninterfering, then:

\[
\text{MEF}[p](x) = p(x)
\]

Note that MEF\([p](x)\) works by "looping over" \( C(L(x)) \). In general, \( |C(L(x))| \) can be exponential in \( |L(x)| \) and so the runtime overhead of MEF is substantial. In the worst case with a powerset lattice, MEF\([p](x)\) is exponentially slower than \( p(x) \), even when \( p \) is a noninterfering program.

More generally, the runtime of MEF\([p](x)\) is bounded by the runtime of \( p \) times the size of \( C(L(x)) \) (and some auxiliary computations that we return to in Section 4). Consequently, the choice of lattice \( L \) makes a big difference to performance as it decides the number of executions of \( p \).

**Example 2.7.** Consider the pairwise program from Example 2.4.

In the example, the lattice used in pairwise is left unspecified, so we consider two cases.

The first lattice we consider is the discrete lattice \( D(A) \) over a set of principals

\[
A = \{ \text{Alice, Bob, Charlie, \ldots} \}.
\]
While a good version of the function, that similarly to goodSum, we get different values for $C(L(x))$ depending on what lattice the labels Alice, Bob, and Charlie are drawn from. In the first lattice, $D(A)$, we have:

$$C(L(x)) = \{ \bot, \text{Alice}, \text{Bob}, \text{Charlie}, \top \}$$

While in $P(A)$ we have:

$$C(L(x)) = \{ \emptyset, \text{Alice}, \text{Bob}, \text{Charlie}, \{\text{Alice}, \text{Bob}\}, \{\text{Alice}, \text{Charlie}\}, \{\text{Bob}, \text{Charlie}\}, \{\text{Alice}, \text{Bob}, \text{Charlie}\}\}$$

In other words, if $\text{pairwise}$ is implemented with the powerset lattice then $\text{MEF[\text{pairwise}]}(x)$ runs $\text{pairwise}$ eight times, compared to five when $\text{pairwise}$ is implemented with the discrete lattice. However, the trade-off is that the powerset lattice allows more fine-grained control over security levels; in the discrete lattice all combinations of input levels collapse to $\top$, whereas the powerset lattice allows the user to see more fine-grained labels for such combined data like $\{\text{Alice}, \text{Bob}\}$. In the end of Section 3, we introduce a family of truncated powerset lattices that allows us to fine-tune this trade-off.

Before we dive into how lattice shape influences runtime for multi-execution, we discuss other possible data representations. Firstly, we have seen the data representation of Algehed and Flanagan [2]: inputs and outputs are sets $S \in P(V \times L)$. Another possibility is the notion of a facetized value $\text{Fac}$ over some set $V$ [4, 43, 44]:

$$f \in \text{Fac}(V) := o \in V | (f ? f) \in f$$

A facetized value is like a decision tree of labels and their meaning can be given by the selection of a facetized tree at a particular label:

$$v@f = o$$

$$\langle j ? f_0 : f_1 \rangle@f = \begin{cases} f_0@f & \text{if } j \subseteq f \\ f_1@f & \text{otherwise} \end{cases}$$

With facetized values in mind we can think of a different notion of computation:

$$p : \text{List Fac}(I) \rightarrow \text{Fac}(O)$$

In this setting, an insecure sum function is:

$$\text{badListSum}_1(i) = \sum_{f \in i} f@\top$$

It leaks the sum of the most secret view of all its inputs to a public (non-facetized) output. Likewise, the following sum function has the same security leak as the $\text{badSum}$ function above, the output label depends on the presence of labels in the input:

$$\text{badListSum}_2(i) = \langle \bigcup_{f \in i} L(f) \rangle \sum_{f @ \top : 0}$$

While a good version of the function, that similarly to $\text{goodSum}$ picks a security-level a-priori, is:

$$\text{goodListSum}_1(i) = \langle t \sum_{f @ t : 0} \rangle$$

These examples demonstrate that the same kind of functions that one can write in the $P(I \times L) \rightarrow P(O \times L)$ setting can be re-created in the faceted setting. In fact, for the same reasons one needs to multi-execute for all levels in $C(L(x))$ in our setting, one needs to multi-execute faceted functions in an analogous manner [4, 9, 21, 15]. Consequently, the choice of setting does not decide the overhead of multi-execution, rather it is still bounded by $|C(L(x))|$.

## 3 Great and Small

In this section, we explore how the overhead of multi-execution differs with the choice of lattice.

**Example 3.1.** Following Example 2.7, consider the lattice $D(\mathbb{N})$. If we take some $S_n \subseteq D(\mathbb{N})$ of size $n$, what is the largest we can make $C(S_n)$? To answer this, consider some $L \subseteq S_n$, what are the possible values for $\bigcup L$? It can be only one of three possible things, either $\bigcup L = \bot$, $\bigcup L = \top$, or $\bigcup L = i$ for some $i \in \mathbb{N}$. However, if $\bigcup L = i$, then $i \in L$ as the only way to get two elements of $D(\mathbb{N})$ to join to $i$ is for at least one of them to be $i$ in the first place. Consequently, we have that $C(S_n) \subseteq S_n \cup \{\bot, \top\}$. This in turn means that:

$$|C(S_n)| \leq |S_n| + 2 = n + 2$$

In other words, $|C(S_n)|$ grows no faster than $n$.

To contrast, in the lattice $P(\mathbb{N})$ the size of $C(S_n)$ grows more quickly. Consider $S_n = \{\{i\}| i \in [1..n]\}$, the set of singleton sets $\{i\}$ for $i$ in the interval $1 \leq i \leq n$. The set $S_n$ has size $n$, but the closure set of $S_n$ is much bigger:

$$C(S_n) = \{L|L \subseteq [1..n]\}$$

$C(S_n)$ is the set of all subsets of $[1..n]$ and has size $2^n$. In other words, the size of closure sets in $D(\mathbb{N})$ grows linearly with the size of the input set, while the size of the closure sets in $P(\mathbb{N})$ grows exponentially.

We begin to formalise the intuition in Example 3.1 by reminding the reader of a few notions from complexity theory. Specifically, Definition 3.2 formalizes the key notions of upper and lower bounds.

**Definition 3.2 (Lower and Upper bounds).** If $f, g : \mathbb{N} \rightarrow \mathbb{N}$ we say that:

- $f(n)$ is $O(g(n))$ if $f(n)$ grows no faster than $g(n)$ if and only if there exists an $N_0 \in \mathbb{N}$ and a $C \in \mathbb{Q}^+$ such that for all $n \geq N_0$ it is the case that $f(n) \leq g(n)C$.
- $f(n)$ is $\Omega(g(n))$ if $f(n)$ grows no slower than $g(n)$ if and only if there exists an $N_0 \in \mathbb{N}$ and a $C \in \mathbb{Q}^+$ such that for all $n \geq N_0$ it is the case that $f(n) \geq g(n)C$.
- $f(n)$ is $\Theta(g(n))$ if $f(n)$ grows like $g(n)$ if and only if $f(n)$ is $O(g(n))$ and $\Omega(g(n))$.

While standard, Definition 3.2 warrants breaking down slightly. The definition of $f(n)$ being $O(g(n))$ says that there is some point, $N_0$, after which all $n$ are such that $g(n)$ bigger than or equal to $f(n)$ up to a constant factor independent of $n$. Likewise, the definition of $f(n)$ being $\Omega(g(n))$ says that eventually, as $n \geq N_0$, $f(n)$ is bigger than or equal to $g(n)$ up to a constant factor. The constant factor provides the generality necessary to allow us to say things
like “the function \( f(n) = 2n^2 + n \) grows like \( n^2 \)” as it allows us to formally ignore both the factor 2 and the addition of \( n \).

Next, we translate these bounds from functions to the size of a lattice’s closure sets.

**Definition 3.3.** Given a lattice \( \mathcal{L} \) define its closure-size as:

\[
CS_L(n) = \max\{ |C(S)| \mid S \subseteq \mathcal{L}, |S| \leq n \}
\]

To measure the size of the closures in \( \mathcal{L} \), \( CS_L(n) \) gives us the size of the biggest closure set that \( L \) can produce for a set \( S \subseteq L \) of size at most \( n \). Consequently, \( CS_L(n) \) measures the worst-case number of executions of \( p \) that \( \text{MEF}(p)(x) \) does if \( |x| = n \) when the given lattice is \( \mathcal{L} \).

Next we lift the definition of bounds from functions to lattices to introduce a convenient terminology for lattices.

**Definition 3.4.** The lattice \( \mathcal{L} \) is:

- \( O(f(n)) \) if and only if \( CS_L(n) = O(f(n)) \).
- \( \Omega(f(n)) \) if and only if \( CS_L(n) = \Omega(f(n)) \).
- \( \Theta(f(n)) \) if and only if \( CS_L(n) = \Theta(f(n)) \).

In Section 4 we will see how these bounds on lattices translate to worst-case time complexity for multi-execution. Specifically:

- \( O(f(n)) \) translates to an upper-bound on the worst-case overhead of multi-execution, and
- \( \Omega(f(n)) \) translates to a lower-bound on the worst-case overhead, and
- \( \Theta(f(n)) \) gives a tight bound on worst-case overhead.

Next we re-visit Examples 2.7 and 3.1 using our new terminology.

**Example 3.5.** The discrete lattice \( D(N) \) is \( \Theta(n) \) as we know from Example 3.1 that \( CS_{D(\mathbb{N})}(n) = n + 2 \). Furthermore, the \( P(\mathbb{N}) \) lattice is \( O(2^n) \) as we know from Example 3.1 that \( CS_{P(\mathbb{N})}(n) = 2^n \). Finally, we note that the linear (total order) lattice \( \mathbb{N} \) ordered in the standard way is \( \Theta(n) \), which is demonstrated by the fact that \( CS_{\mathbb{N}}(n) = n + 1 \) as \( C([1 \ldots n]) = [0 \ldots n] \).

These three bounds have practical implications. Firstly, both the \textit{gg} and \textit{Feature Extraction} lattices of Alpernas et al. [5] for describing the security concerns of multi-user serverless applications are discrete lattices over the set of users, for which we expect worst-case \( O(n) \) overhead. Secondly, mashup lattices of Magazinius et al. [30] that allow a website to arbitrarily combine data from third-party domains is a powerset lattice and we expect worst-case \( O(2^n) \) overhead for multi-execution in their setting. Finally, the linear lattice \( \mathbb{N} \) is a generalization of the traditional "military lattice" with levels like \( L \subseteq M \subseteq H \) discussed as early as Denning’s seminal work introducing lattice-based IFC [20].

Table 1 summarises the results in this section, describing the complexity of various lattices, including products \( \times \), two different sum operations \( \uparrow \) and \( \circ \), exponentiation, and a few other examples. For example, if \( L_0 \) and \( L_1 \) have their closure sets up bounded by \( u_0(n) \) and \( u_1(n) \) respectively, then \( L_0 \times L_1 \) is upper bounded by \( u_0(n) \times u_1(n) \).

Next we provide a set of tools for and examples of how to analyse lattice shape. We present a number of basic facts about lattice shape, and continue to present the analysis that underlies the results in Table 1. Proofs that are not in the body of the paper are found in the appendices.

**Lemma 1 (\( \Omega \) families).** The lattice \( L \) is \( \Omega(f(n)) \) if and only if there exists a family \( \{S_n\}_{n \in \mathbb{N}} \) of sets such that:

1. \( \forall n, S_n \subseteq L \)
2. \( \forall n, |S_n| \leq n \)
3. \( |C(S_n)| = \Omega(f(n)) \)

This lemma gives us the basic building block for proving lower bounds. An analogous reasoning principle can be established for upper bounds, \( L \) is \( O(f(n)) \) if there is a family of sets \( L_n \) of size \( |L_n| \leq f(n) \) such that \( C(S) \subseteq L_n \) for each \( S \) of size less than or equal to \( n \). Furthermore, there is a global upper bound on all lattices.

**Theorem 3 (Global Bounds).**

1. All lattices are \( O(2^n) \).
2. If \( L \) is non-finite, then \( L \) is \( \Omega(n) \), otherwise it is \( O(1) \).

The second item in Theorem 3 highlights that we treat all finite lattices the same way, they introduce a constant amount of overhead in multi-execution. This is true because our analysis is
asymptotic, if the lattice \( L \) is finite then the maximum number of multi-executions is constant at \( |L| \). If the lattice is tiny, like the two-point lattice, then treating the overhead as constant is accurate. If the lattice is finite but large it is impractical to multi-execute for every lattice label and the overhead will be dominated by the asymptotic behaviour of “adaptive” multi-execution like MEF or faceted execution. Finally, we note that potentially unbounded lattices are common in the IFC literature, e.g. in DC-labels [47], the DLM [34], and FLAM [6].

If the lattice \( L \) is contained in \( L' \), we expect that \( L' \) is at least as big as \( L \). To make this formal, we define the notion of a lattice homomorphism and giving us an embedding.

**Definition 3.6.** A lattice homomorphism from \( L \) to \( L' \) is a function \( h : L \to L' \) such that:

1. \( h(\bot_L) = \bot_{L'} \)
2. \( h(f \sqcup f') = h(f) \sqcup h(f') \)

If \( h \) is injective we say that \( h \) is an embedding of \( L \) in \( L' \) and that \( L \) can be embedded in \( L' \).

**Theorem 4 (Embedding Complexity).** If \( L \) can be embedded in \( L' \) then:

- If \( L \) is \( \Omega(l(n)) \) then \( L' \) is \( \Omega(l(n)) \) and
- If \( L' \) is \( O(u(n)) \) then \( L \) is \( O(u(n)) \)

The following example illustrates the usefulness of embeddings for proving bounds.

**Example 3.7.** Free boolean algebra over a set of principals \( A \), the set of propositional logic formulas with atomic propositions from \( A \) ordered by implication, form the basis of a number of security lattices in the literature, notably Disjunction Category Labels (DC Labels) [47] and the Flow Limited Authorization Model (FLAM) [7]. The powerset lattice can be embedded into any such free boolean algebra by the embedding:

\[
\text{embed}(a_0, \ldots, a_n) = a_0 \lor \ldots \lor a_1
\]

By Theorems 4 and 3 we have that the free boolean algebra is \( \Omega(2^n) \) and \( O(2^n) \) respectively, giving us the tight bound of \( \Theta(2^n) \).

Next, we explore the way that bounds interact with a few methods for forming lattices from smaller lattices and introduce the \( k \)-truncated powerset lattice \( P_k(A) \). Specifically, the next three subsections establish results in Table 1 and the reader is free to skip them on first reading, while the final subsection is important to understand later examples.

### 3.1 Product Lattices

The first lattice formation method we consider is the product lattice.

**Definition 3.8 (Product Lattice).** The lattice \( L_0 \times L_1 \) is called the product of lattices \( L_0 \) and \( L_1 \), and has pairs \((t_0, t_1)\) as elements where \( t_i \in L_i \) and has order:

\[
(t_0, t_1) \sqsubseteq (s_0, s_1) \iff \forall i \in \{0, 1\}, t_i \sqsubseteq s_i
\]

The least-upper-bound of \((t_0, t_1)\) and \((s_0, s_1)\) is the pair of least-upper-bounds:

\[
(t_0, t_1) \sqcup (s_0, s_1) = (t_0 \sqcup s_0, t_1 \sqcup s_1)
\]

The IFC literature has many examples of product lattices, many of which simultaneously track both confidentiality and integrity. For example, an element of the DC-labels lattice [47] is formed by taking a pair of CNF formulas over principals; one represents confidentiality requirements on data and the other integrity requirements.

Next we begin to establish bounds for these product lattices.

**Theorem 5.** If \( L \) is \( O(u(n)) \) and \( L' \) is \( O(u'(n)) \), then \( L \times L' \) is \( O(u(n)u'(n)) \).

This theorem says that upper bounds multiply in the product lattice, what about lower bounds? One might expect that if \( L \) and \( L' \) are \( \Theta(l(n)) \) and \( \Theta(l'(n)) \) respectively, then \( L \times L' \) is \( \Theta(l(n)l'(n)) \). However, we know that \( \mathcal{P}(A) = \Theta(2^{|A|}) \) for non-finite \( A \); if lower bounds multiply we would have that \( \mathcal{P}(A) \times \mathcal{P}(A) = \Theta(2^{|A|}2^{|A|}) \). But \( \mathcal{P}(A) \times \mathcal{P}(A) \), like all lattices, is \( O(2^{|A|}) \) and consequently \( 2^{|A|}2^{|A|} \) would also be upper bounded by \( O(2^{|A|}) \), which it is not.

**Example 3.9.** The lattice \( N \times N \) is \( \Theta(n^2) \). That \( N \times N \) is \( O(n^2) \) follows from Theorem 5. To see that \( N \times N \) is \( \Omega(n^2) \) we construct the family:

\[
S_n = [0 \ldots \lfloor \frac{n}{2} \rfloor - 1] \times [0] \cup [0] \times [0 \ldots \lfloor \frac{n}{2} \rfloor - 1]
\]

To see the construction of \( S_n \) visually, see Figure 4.

Clearly, \( |S_n| \leq n \) and so it remains to show that \( C(S_n) = \Omega(n^2) \). It suffices to show that \( [0 \ldots \lfloor \frac{n}{2} \rfloor - 1]^2 \subseteq C(S_n) \) as \( [0 \ldots \lfloor \frac{n}{2} \rfloor - 1]^2 \) is \( \Omega(n^2) \). If \( i, j \in [0 \ldots \lfloor \frac{n}{2} \rfloor - 1] \), then \( i, j \in [0 \ldots \lfloor \frac{n}{2} \rfloor - 1] \) and consequently, \( i, j = \lfloor \frac{n}{2} \rfloor \). Note that \( C(S_n) \) is \( \Omega(n^2) \) and, by Lemma 1, that \( N \times N \) is \( \Omega(n^2) \).

The construction in the example above can be generalised to show that \( \mathbb{N}^k = \Theta(n^k) \) for any \( k \). The same generalisation allows us to prove the following theorem.

**Theorem 6.** If \( L \) and \( L' \) are \( \Omega(l(n)) \) and \( \Omega(l'(n)) \) for strictly positive functions \( l(n) \) and \( l'(n) \), then \( L \times L' \) is \( \Omega(l(\lfloor \frac{|A|}{2} \rfloor)l'(\lfloor \frac{|B|}{2} \rfloor)) \).

Note that the insight that we can divide the two halves of the set between \( L \) and \( L' \) can be generalised. Specifically, any split between \( L \) and \( L' \) works, which allows us to give other bounds, such as \( \Omega(l(\lfloor \frac{|A|}{2} \rfloor)l'(\lfloor \frac{|B|}{2} \rfloor)) \) and \( \Omega(\max_{0 \leq k \leq |A|} l(k)l'(n - k)) \).

This insight may be used to achieve tighter bounds than Theorem 6. For example, Theorem 6 gives a lower bound of \( \Omega(\frac{n}{2} \frac{2^{|A|}}{2}) \) for

![Figure 4: An illustration of S_{10}.](image)
We define two types of sums to capture disconnected parts of a lattice.

\[ \mathcal{L} \circ \mathcal{L}' = \begin{cases} \mathcal{L} & \mathcal{L} \subseteq \mathcal{L}' \\ \mathcal{L}' & \mathcal{L}' \subseteq \mathcal{L} \end{cases} \quad \mathcal{L} \uparrow \mathcal{L}' = \begin{cases} \mathcal{L}' & \mathcal{L}' \subseteq \mathcal{L} \end{cases} \]

Figure 5: Sum Lattices

the lattice \( \mathbb{N} \times \mathcal{P}(\mathbb{N}) \). However, taking \( S_n = \{0\} \times \{i\} | i \in [0..n] \) gives the lower-bound \( O(2^n) \).

### 3.2 Sum Lattices

We define two types of sums to capture disconnected parts of a lattice.

**Definition 3.10.** The vertical sum of \( \mathcal{L} \) and \( \mathcal{L}' \), written \( \mathcal{L} \uparrow \mathcal{L}' \), has elements in \( \mathcal{L} \uplus \mathcal{L}' \) (defined as \( \{0\} \times \mathcal{L} \cup \{1\} \times \mathcal{L}' \)) where \( \ell \subseteq \ell' \) if and only if:
- \( \ell = (0, j) \) and \( \ell' = (1, j') \) or
- \( \ell = (i, j) \), \( \ell' = (i, j') \) and \( j \subseteq j' \).

The horizontal sum of \( \mathcal{L} \) and \( \mathcal{L}' \), written \( \mathcal{L} \circ \mathcal{L}' \), has elements in \( \{0,1\} \cup (\mathcal{L} \times \mathcal{L}') \) where \( \ell \subseteq \ell' \) if and only if either \( \ell = 0 \), \( \ell' = 1 \), or \( \ell = (i, j) \), \( \ell' = (i, j') \) and \( j \subseteq j' \).

In other words, \( \mathcal{L} \uparrow \mathcal{L}' \) is putting \( \mathcal{L}' \) on top of \( \mathcal{L} \) and \( \mathcal{L} \circ \mathcal{L}' \) is putting \( \mathcal{L} \) and \( \mathcal{L}' \) next to each other and gluing 0 to the bottom and 1 to the top of the two lattices. Figure 5 contains a graphical rendition of lattice sums.

These kind of structures appear in the literature in the form of lattices that incorporate two disjoint parts of an organisation or application. For example, the *Hello Retail!* lattice of Alpernas et al. is similar to a horizontal sum lattice [5] and the Zone Hierarchies of Yip et al. also form a horizontal sum [54].

We can establish bounds on the size of the closure sets for these lattice sums. Specifically, because there is no complex interaction between \( \mathcal{L} \) and \( \mathcal{L}' \) in either \( \mathcal{L} \uparrow \mathcal{L}' \) nor \( \mathcal{L} \circ \mathcal{L}' \) both sums have closure sets that scale like the closure sets of \( \mathcal{L} \) and \( \mathcal{L}' \) taken in isolation.

**Theorem 7.** If \( \mathcal{L} \) and \( \mathcal{L}' \) are \( \Theta(f(n)) \) and \( \Theta(g(n)) \) respectively, then \( \mathcal{L} \uparrow \mathcal{L}' \) and \( \mathcal{L} \circ \mathcal{L}' \) are both \( \Theta(f(n) + g(n)) \).

### 3.3 Exponential Lattices

**Definition 3.11.** Given the lattice \( \mathcal{L} \) we define the exponential lattice \( 2^{\mathcal{L}} \) as the lattice whose elements are subsets of \( \mathcal{L} \) and where:
- \( \ell \subseteq \ell' \iff \forall j \in \ell \exists j' \in \ell' \). \( j \subseteq j' \)

We also require that the set of labels in \( 2^{\mathcal{L}} \) is additionally quotiented by the equivalence relation \( \sim \) given by:
- \( \ell \sim \ell' \iff \ell \subseteq \ell' \land \ell' \subseteq \ell \)

The last requirement of Definition 3.11 is a technical necessity to make \( \subseteq \) antisymmetric (i.e. that \( \ell \subseteq \ell' \subseteq \ell \implies \ell = \ell' \)). If \( \mathcal{L} \) is not quotiented by \( \sim \) and there exists \( \ell, \ell' \subseteq \mathcal{L} \) such that \( \ell \subseteq \ell' \), we have that \( \ell' \subseteq \{\ell, \ell'\} \subseteq \{\ell'\} \) but \( \{\ell, \ell'\} \neq \{\ell'\} \).

A label in the \( 2^{\mathcal{L}} \) lattice represents the "most liberal" extension of a collection of labels \( \mathcal{L} \subseteq \mathcal{L} \) to a security label. It allows us to extend a lattice by introducing additional least upper bounds, and so it considers more programs secure than the underlying \( \mathcal{L} \) lattice. However, as demonstrated by the following theorem this naturally introduces additional overhead.

**Theorem 8.** If there is a non-finite \( \mathcal{L} \subseteq \mathcal{L} \) such that \( \forall \ell, \ell' \in \mathcal{L}, \ell \subseteq \ell' \implies \ell = \ell' \) then \( \mathcal{L} \) is \( \Theta(2^n) \).

### 3.4 k-Truncated Powersets

The final lattice we explore is the \( k \)-truncated powerset lattice. This lattice is like the powerset lattice, with the exception that it is truncated, all sets of size greater than \( k \) are replaced by \( \top \).

**Definition 3.12.** The lattice \( \mathcal{P}_k(A) \) is the lattice of all subsets of \( A \), ordered by inclusion, with cardinality less than or equal to \( k \) and \( \bot \) joined with a distinguished greatest element \( \top \).

A graphical rendition of the 2-truncated powerset lattice can be found in Figure 6. There are four principals \( A \) through \( D \) in the lattice, and a further six combinations of at most two principals that form the upper bound of any two singleton labels, for example \( AD \) or \( BC \). However, all two-principals labels have upper bound \( \top \), unlike the standard powerset lattice there are no labels \( ABC \) or \( BCD \) in \( \mathcal{P}_2([A, B, C, D]) \).

To establish bounds on \( \mathcal{P}_k(A) \) we note that the number of subsets of size \( k \) of an \( n \) element set is exactly equal to \( n \) choose \( k \):

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}
\]

Two observations about this function are necessary to get convenient upper and lower bounds for \( \mathcal{P}_k(A) \):

\[
\frac{n^k}{k!} \leq \frac{n!}{k!(n-k)!} \leq \frac{n^k}{k!}
\]

Because \( k \) and consequently \( k! \) and \( n! \) are constants, we can establish a tight bound of \( \Theta(n^{k^2}+\ldots+n^k) = \Theta(n^k) \) for the closure-size of \( \mathcal{P}_k(A) \).

**Theorem 9.** If \( A \) is non-finite then \( \mathcal{P}_k(A) \) is \( \Theta(n^k) \).

\(^1\)See Lemma 11 in the Appendix.
4 FAST AND SLOW

In the previous section, we established bounds for the number of executions required to multi-execute the program $p$ given the lattice $L$. However, it is not sufficient that $L$ has small closure sets for multi-execution to be efficient. Specifically, MEF has to do a number of things other than executing $p$. Recall the definition of MEF[$p$]:

$$\text{MEF}[p](x) = \bigcup \{ p(x \downarrow \ell) @ (\ell \uparrow C(L(x))) \mid \ell \in C(L(x)) \}$$

There are three computations that we may classify as overhead:

1. Enumerating the elements of $C(L(x))$.
2. Computing $x \downarrow \ell$ for each $\ell \in C(L(x))$.
3. Computing membership of $\ell \uparrow C(L(x))$ for $p(x \downarrow \ell) @ (\ell \uparrow C(L(x)))$.

In this analysis, we conservatively assume that all lattice operations (like $\subseteq$ and $\downarrow$) take constant time. Clearly, (2) above is computable in $O(|x|)$ time. In this section we show that both (1) and (3) are also computable with reasonable bounds.

To address (1), the following lemma and theorem allow us to give an algorithm for efficiently computing (or enumerating) $C(L)$ given that we have an upper bound on the elements of $C(L)$.

**Lemma 4.1.** Let $S \subseteq \ell = \{ \ell' \in S \mid \ell' \subseteq \ell \}$, then

$$\bigcup \{ S \downarrow \ell \} = \ell \iff \ell \in C(S)$$

**Proof.** Lemma 4.1 suggests a procedure for enumerating $C(S)$ given $S$ and $f(S)$:

```
    for \ell \in f(S) do
        if \bigcup \{ S \downarrow \ell \} = \ell then
            emit \ell
        end if
    end for
```

This procedure takes $O(t(|S|) + |f(S)||S|)$ time and by Lemma 4.1 we can see that it enumerates $C(S)$. \qed

Next we tackle (3). Lemma 2 lets us convert our upper bounds for the elements of $C(S)$ into upper bounds on the time it takes to compute $C(S)$. For example, if $S \subseteq L_0 \times L_1$ we know that $C(S) \subseteq C(S_0) \times C(S_1)$ for $S_i = \{ \ell_i \mid (\ell_0, \ell_1) \in S \}$ and so the time it takes to compute $C(S)$ is bounded by $|S||C(S_0)||C(S_1)|$ and the time it takes to compute $C(S_0) \times C(S_1)$. Similarly, as we saw in Examples 2.7 and 3.1 in the discrete lattice $D(A)$ we have that $C(L) \subseteq L \cup \{ \bot, \top \}$.

Furthermore, there is an equivalent formulation of membership in $\ell \uparrow C(L(x))$ that can be read as a linear-time algorithm. This is a novel formulation that does not depend on the size of $C(L(x))$, in contrast to the formulation in Section 2.

**Lemma 3.** Given $\ell \in C(L)$ it is possible to compute $j \in \ell \uparrow C(L)$ in $O(|L|)$ time.

Proof. It suffices to check the condition $P(j, \ell)$ defined as:

$$P(j, \ell) \iff \ell \subseteq j \wedge \forall i \in L. i \subseteq j \Rightarrow i \subseteq \ell.$$ 

To see that $P(j, \ell)$ implies $j \subseteq \ell \uparrow C(L)$, consider that if $P(j, \ell)$ then $\ell$ is an upper bound on any subset $L'$ of $L$ such that $\bigcup L' \subseteq j$ and so $j \in \ell \uparrow C(L)$. Likewise, if $j \in \ell \uparrow C(L)$ then $\ell \subseteq j$ and if $i \in L$ and $i \subseteq j$ then $\bigcup \{ i \} \subseteq j$ and so $\bigcup \{ i \} \subseteq \ell$ by the definition of $j \in \ell \uparrow C(L)$. \qed

Finally, we put these lemmas together to give an upper bound on the execution time of MEF[$p$](x).

**Theorem 10** (Time Complexity of Multi-Execution). Assume:

1. That the lattice $L$ is $O(s_L(n))$.
2. That $p(x)$ can be computed in $O(t_p(|x|))$.
3. A function $f$ that is computable in $O(t_f(n))$ time and for all $S, C(S) \subseteq f(S)$ and $|f(S)| = O(s_L(|S|))$.

Then the elements of $\text{MEF}[p](x)$ can be enumerated in time:

$$O(t_f(|x|) + s_L(|x|) t_p(|x|) |x|)$$

5 THROUGH THE LOOKING GLASS

If we find that the lattice used by some application is causing unacceptably high performance overheads, how do we switch to a different lattice? The trick is Galois connections [14].

5.1 Galois Connections

To understand how Galois connections relate to information flow control, consider translating data labeled in one lattice $L$ to another lattice $L'$. To be secure, this needs to be done with a monotonic function $F : L \rightarrow L'$ that translates each label in $L$ to a new label in $L'$. Suppose additionally that we want to securely back-translate labels from $L'$ to $L$ using a function $G : L' \rightarrow L$. The goal is then to find $F$ and $G$ such that “re-labeling” using $F$ and “back-labeling” using $G$ composes to a secure function.

For example, consider the re-labeling function $F$ defined as:

$$F(\ell) = \text{if } \ell \subseteq \{ \text{Alice, Bob} \} \text{ then } \bot \text{ else } \top$$

between the powerset and two-point lattices. To find a reasonable back-labeling for $F$, consider using $F$ to translate the labels in the following set:

$$\{ 1_{\text{Alice}} , 2_{\text{Bob}} , 3_{\text{Charlie}} , 4_{\text{Dave}} \} \mapsto F \{ 1^\bot, 2^\bot, 3^\top, 4^\top \}$$

How ought we back-translate this set from the two-point lattice to the powerset lattice? Back-translating $\bot$ to either Alice or Bob would be wrong, as doing so would leak the value from one to the other. Likewise, back-translating $\top$ to either Charlie or Dave would be wrong for the same reason. Fortunately, the following back-translation works well:

$$G(\ell) = \text{if } \ell = \top \text{ then } \top \text{ else } \{ \text{Alice, Bob} \}$$

The round-trip we get is then:

$$\{ 1_{\text{Alice}}, 2_{\text{Bob}}, 3_{\text{Charlie}}, 4_{\text{Dave}} \} \mapsto F \rightarrow G \mapsto \{ 1^\bot, 2^\bot, 3^\top, 4^\top \} \mapsto F \rightarrow G \mapsto \{ 1_{\text{Alice}, \text{Bob}}, 2_{\text{Alice, Bob}}, 3^\top, 4^\top \}$$

For the interested reader there is a significant body of work on the efficiency of lattice operations [1, 16, 33].
Furthermore, two Galois connections \( \leftrightarrow \) leisure using a textbook of their choice (for example [14]). DC-labels lattice over a set \( \mathcal{L} \) and \( \mathcal{L}' \) is a pair \( F : \mathcal{L} \rightarrow \mathcal{L}' \) and \( G : \mathcal{L}' \rightarrow \mathcal{L} \) of functions such that:

\[
F(t) \sqsubseteq_{L'} j \iff G(j) \sqsubseteq_{L} t
\]

The usual formal definition of Galois connections is the following.

**Definition 5.1.** A Galois connection \( F \triangleright G \) between \( \mathcal{L} \) and \( \mathcal{L}' \) is a pair \( F : \mathcal{L} \rightarrow \mathcal{L}' \) and \( G : \mathcal{L}' \rightarrow \mathcal{L} \) of functions such that:

\[
F(t) \sqsubseteq_{L'} j \iff G(j) \sqsubseteq_{L} t
\]

Galois connections have a number of useful theoretical properties. For example, given a Galois connection \( F \triangleright G \), \( G \circ F \) is a closure operator, meaning that:

\[
F(t) \sqsubseteq_{L'} j \iff G(j) \sqsubseteq_{L} t
\]

Furthermore, two Galois connections \( F \triangleright G \) between \( \mathcal{L} \) and \( \mathcal{L}' \) and \( F' \triangleright G' \) between \( \mathcal{L}' \) and \( \mathcal{L}'' \) compose to form a Galois connection \( (F' \circ F) \triangleright (G' \circ G) \) between \( \mathcal{L} \) and \( \mathcal{L}'' \). The list goes on and the interested reader is encouraged to explore these structures at their leisure using a textbook of their choice (for example [14]).

**Example 5.2.** We can derive a Galois connection between the DC-labels lattice over a set \( A \) of principals and the \( k \)-truncated powerset lattice \( \mathcal{P}_k(A) \). The idea is that we map the DC-label \( t \) to the label of at most \( k \) principals who either have confidentiality concerns registered in or who can vouch for data labeled by \( t \). For example, the label:

\[
(\text{Alice}, \text{Alice} \land (\text{Bob} \lor \text{Charlie}))
\]

labels data with Alice’s confidentiality that either Alice alone or both Bob and Charlie together can vouch for and this label maps to the label \( \{\text{Alice, Bob, Charlie}\} \) in \( \mathcal{P}_k(A) \) if \( k \geq 3 \). The label \( \{\text{Alice, Bob, Charlie}\} \) meanwhile, maps back to the DC-label:

\[
(\text{Alice} \land \text{Bob} \land \text{Charlie}, \text{Alice} \lor \text{Bob} \lor \text{Charlie}).
\]

The outline of the Galois connection is:

\[
\text{DCLabels} \leftrightarrow \mathcal{P}(A) \times \mathcal{P}(A) \leftrightarrow \mathcal{P}(A) \leftrightarrow \mathcal{P}_k(A)
\]

The first step maps a DC label to the collection of principals in the label:

\[
(\text{Alice}, \text{Alice} \land (\text{Bob} \lor \text{Charlie})) \mapsto (\{\text{Alice}\}, \{\text{Alice, Bob, Charlie}\})
\]

The second step unites the confidentiality and integrity principals:

\[
(\{\text{Alice}\}, \{\text{Alice, Bob, Charlie}\}) \mapsto \{\text{Alice, Bob, Charlie}\}
\]

Finally, we map \( \mathcal{P}(A) \) to \( \mathcal{P}_k(A) \) by a Galois connection we call \( \text{truncate}_k + \text{embed} \):

\[
\text{truncate}_k(S) = \begin{cases} \text{true} & \text{if } |S| \leq k \text{ then } S \text{ else } \top \\ \text{else embed}(S) \end{cases}
\]

We refer to the \( \text{truncate}_k + \text{embed} \) Galois connection as \( t + e \) when \( k \) is clear from the context. In our example where \( k = 3 \), truncating the set does nothing.

The chain going back to DC-labels is similar:

\[
\{\text{Alice, Bob, Charlie}\} \mapsto (\{\text{Alice, Bob, Charlie}\}, \{\text{Alice, Bob, Charlie}\}) \mapsto (\text{Alice} \land \text{Bob} \land \text{Charlie}, \text{Alice} \lor \text{Bob} \lor \text{Charlie})
\]

If there are more than \( k \) principals, like the label \( t = (\text{Alice} \land \text{Bob}, \text{Charlie} \lor \text{Dave}) \) for \( k = 3 \), we lose information when going all the way from DC-labels to \( \mathcal{P}_k(A) \). Specifically, \( t \) maps to \( \top \) in \( \mathcal{P}_k(A) \), which maps back to \( \top \).

Next we use the insight that Galois connections can collapse large lattices into smaller ones by defining a variant of the MEF enforcement mechanism. If \( F : A \rightarrow B \) and \( S \subseteq A \) let \( F^*(S) = \{ F(a) \mid a \in S \} \).

**Definition 5.3.** Given a Galois connection \( F \triangleright G \) between \( \mathcal{L} \) and \( \mathcal{L}' \), we define:

\[
C_{F,G}(S) = G^*(C(F^*(S)))
\]

\[
\text{MEF}^{F,G}[p](x) = \bigcup \{ p(x) \mid t \in C_{F,G}(\mathcal{L}(x)) \}
\]

Note that \( \text{MEF}^{F,G} \) is essentially the same as MEF, except that instead of enumerating the labels in \( C(\mathcal{L}(x)) \) we enumerate the labels in \( C_{F,G}(\mathcal{L}(x)) \). This means that we only take the closure over the labels in the target lattice of the \( F \triangleright G \) Galois connection.

Recall the \textit{goodSum} program from Example 2.4:

\[
\text{goodSum}_l(x) = \left\{ \sum \{ a \mid a \in x, t \in L \} \right\}
\]

This program is noninterfering and only produces output at level \( \bigcup L \). By providing a Galois connection between \( \mathcal{P}(A) \) and the two-point lattice \( \{\bot, \top\} \) we can optimize \( \text{MEF[goodSum]} \). The Galois connection in question, which we call \text{specify} \_l + \text{unspecify} \_l, is outlined in Figure 7 for \( t = AB \) and is defined as:

\[
\text{specify}_l(j) \triangleq \begin{cases} \bot & \text{if } j \in t \text{ then } \bot \text{ else } \top \\ \text{unspecify}_l(j) \triangleq \begin{cases} \bot & \text{if } j = \top \text{ then } \bot \text{ else } t \end{cases}
\]

The reader is free to verify that these two form a Galois connection. In the rest of the paper, we abbreviate \text{specify} \_l + \text{unspecify} \_l.
as $s \triangleq u$ when $t$ is clear from context. Now, note that $\mathcal{L}(x) = C_{\text{specify}(x)} \cup \mathcal{L}(x)$ for all $x$, and so

$$\text{MEF}^{\text{specify}}[{\text{goodSum}}_1](x) = \text{MEF}[\text{goodSum}_1](x) = \text{goodSum}_1(x)$$

as $\text{goodSum}_1(x \triangleq \mathcal{L}(x))$ will always be executed by MEF and its result included in the final output. While MEF needs to run $O(2^n)$ executions of $\text{goodSum}_1$, MEF$^{\text{specify}}$ needs one execution of $\text{goodSum}_1$ in the best case, and two in the worst, as the size of $C_{\text{specify}}(\mathcal{L}(x))$ scales as the target lattice, which in this case has only two elements. Specifically, if $\mathcal{L}(x) \subseteq \mathcal{L}$ then $\text{goodSum}_1(\mathcal{L}(x)) = \{\mathcal{L} \}$ and so there will only be one execution of $\text{goodSum}_1$, while if there is some $t \in \mathcal{L}(x)$ such that $t \not\subseteq \mathcal{L}$ we consequently have that $\text{goodSum}_1(\mathcal{L}(x)) = \{\mathcal{L} \}$ and so we get two executions of $\text{goodSum}_1$.

For a practical example, consider the COWL system [49] that provides an IFC framework for the web. One application of COWL is so-called "mashups", sites that include content from many different websites and present it in aggregate. For example, a mashup can collect and compare price information to display purchase recommendations from multiple online retailers to the user. In this mashup, individual retailers need their own security label to manage sensitive data, and the mashup needs a label that collects data to provide the price recommendation.

In other words, the mashup site needs the discrete lattice $\mathcal{D}(S)$ where $S$ is the set of sites. If we use a general label and container system like COWL and attempt to apply multi-execution to this example, we would need a Galois connection between DC-labels, the native labels in COWL, and $\mathcal{D}(S)$. Fortunately, $\mathcal{D}(S)$ is isomorphic to $\mathcal{P}_1(S)$, and so the Galois connection from Example 5.2 with $k = 1$ is sufficient. This brings the number of executions of our hypothetical site down from $O(2^n)$ for the DC-labels lattice, to $O(n)$ in the $\mathcal{P}_1(S)$ lattice. We can establish noninterference for MEF$^{s\!G}$.

**Theorem 11.** MEF$^{s\!G}[p]$ is noninterfering.

Next we tackle transparency. The Galois connection changes the behaviour of MEF$^{s\!G}$ and there are Galois connections for which MEF$^{s\!G}[p](x) \neq p(x)$ even for noninterfering $p$. In fact, we encourage the reader to come up with an example of such a Galois connection. However, if a Galois connection accurately captures the behaviour of a noninterfering program $p$, such as the case in the example of $\text{goodSum}_1$ with $s \triangleq u$ above, then we expect that the semantics of $p$ is preserved by MEF$^{s\!G}$.

**Theorem 12. If $p$ is noninterfering and $j \in (G \circ F)^*(\mathcal{L})$ then:**

$$\text{MEF}^{s\!G}[p](x)@j = p(x)@j$$

The second precondition of Theorem 12 can be read as stating a condition on $F$ and $G$ relative to $p$. In effect, it says that a transparent Galois connection $F \circ G$ for the program $p$ is such that $\forall x. \mathcal{L}(p(x)) \subseteq (G \circ F)^*(\mathcal{L})$. Consider the noninterfering example programs in Example 2.4, we see that $\text{specify}(x \triangleq \mathcal{L})$ is a transparent Galois connection for $\text{goodSum}_1$.

However, $\text{truncate}_2 \triangleq \text{embed}$ is not a transparent Galois connection for pairwise! To understand why, consider that:

$$\{\text{Alice, Bob, Charlie}\} \notin \text{pairwise}(\{\text{Alice, Bob}\}, \{\text{Charlie}\})$$

But $\{\text{Alice, Bob, Charlie}\} \notin \text{pairwise}(\{\text{Bob}\}, \{\text{Charlie}\})$ and consequently $\text{truncate}_2 \triangleq \text{embed}$ misses this output of pairwise. In other words, if there are elements of the input to pairwise where the labels have more than one element, this label gets forgotten by the $\text{truncate}_2 \triangleq \text{embed}$ Galois connection. However, this does not preclude this Galois connection from being useful. For example, we can multi-execute pairwise on the input $x$ with the $\text{truncate}_2 \triangleq \text{embed}$ Galois connection.

### 5.2 Execution Time of MEF$^{s\!G}$

The factors influencing the execution time of MEF discussed in Section 4 are also present for MEF$^{s\!G}$. Firstly, if $F \circ G$ goes between $\mathcal{L}$ and $\mathcal{L}'$, the size of $C_{F \circ G}(\mathcal{L}(x))$ and so the number of executions of $p$ grows as $CS_{\mathcal{L}}$. In other words, if $\mathcal{L}'$ is $O(f(n))$ then the number of executions of $p$ is too.

Secondly, following Lemma 3, next we give a similar lemma relating to computing membership of $\mathcal{L}(x)$ that gives us a polynomial time algorithm for this piece of overhead as well.

**Lemma 4.** Given $x \in \mathcal{L}$ and assuming $F \circ G$ such that both $F$ and $G$ are time constant, it is possible to compute $j \in G(F(x)) \cap C_{F \circ G}(\mathcal{L}(x))$ in $O(|\mathcal{L}|)$ time by computing:

$$G(F(t)) \subseteq j \land \forall t \in F^*(\mathcal{L}), G(t) \subseteq j \Rightarrow G(t) \subseteq G(F(t))$$

With this lemma, we have all the pieces we need to find the execution time bound on MEF$^{s\!G}[p](x)$. It is essentially the same as the bound in Theorem 10, where the lattice $\mathcal{L}$ is the target lattice of the $F \circ G$ Galois connection.

**Theorem 13 (Time Complexity of Multi-Execution).** Assume:

1. That $F \circ G$ is a Galois connection between $\mathcal{L}$ and $\mathcal{L}'$.
2. That the lattice $\mathcal{L}'$ is $O(n)$.
3. That $p(x)$ can be computed in $O(|\mathcal{L}'|(x))$.
4. A function $f$ that is computable in $O(f(n))$ time and for all $S, C(S) \subseteq f(S)$ and $|f(S)|$ is $O(|\mathcal{L}'|)$ for the lattice $\mathcal{L}'$.

Then the elements of MEF$^{s\!G}[p](x)$ can be enumerated in time:

$$O(f(|\mathcal{L}'|(x)) + \mathcal{L}'(x)|H_p(x)|x)$$

### 5.3 Finding Galois Connections

Next we address the issue of finding the right Galois connection. Specifically, we show that there is a specification for a most coarse-grained Galois connection for each program. This can be used to show that, for some programs, no transparent Galois connection can reduce overhead while for other programs. We do this by using the closure operator $(G \circ F)$ of a Galois connection $F \circ G$. Specifically, we obtain a Galois connection from any closure operator $k$.

**Definition 5.4.** If $\mathcal{L}$ is a lattice we say that a function $k : \mathcal{L} \rightarrow \mathcal{L}$ is a closure operator if and only if it satisfies:

1. Extensivity: $\mathcal{L} \subseteq k(\mathcal{L})$
2. Monotonicity: $\mathcal{L} \subseteq \mathcal{J} \Rightarrow k(\mathcal{L}) \subseteq k(\mathcal{J})$
3. Idempotence: $k(k(\mathcal{L})) = k(\mathcal{L})$

Alternatively, $k$ is a closure operator if and only if $\mathcal{L} \subseteq k(\mathcal{J}) \Rightarrow k(\mathcal{L}) \subseteq k(\mathcal{J})$. 

---

1. Hint: what happens if the target lattice has only a single element?
Theorem 14 (From [14]). If \( k \) is a closure operator then \( \mathcal{L} \setminus k \) forms a lattice with equivalence classes up to \( k \) for elements and the order inherited from \( \mathcal{L} \).

Theorem 15 (From [14]). If \( k \) is a closure operator then \( k + \lambda \ell \cdot \ell \) is a Galois connection between \( \mathcal{L} \) and the quotient lattice \( \mathcal{L} / \{(\ell, j) \mid k(\ell) = k(j)\} \), where elements are equivalence classes up to \( k \) ordered by \( \subseteq \) on representative elements (fixpoints of \( k \)).

The key corollary of Theorem 15 is that if \( k \) differentiates between all labels that are in the range of \( p \), then \( k \) gives rise to a transparent Galois connection that can be used together with \( \text{MEF}^{F,G} \) to multi-execute \( p \).

Corollary 5.5. Given a closure operator \( k \) on \( \mathcal{L} \) such that \( \mathcal{L}(p(x)) \subseteq k^*(\mathcal{L}) \) for all \( x \) we have that \( k + \lambda \ell \cdot \ell \) is a transparent Galois connection between \( \mathcal{L} \) and \( \mathcal{L} \setminus \{(\ell, j) \mid k(\ell) = k(j)\} \) for \( p \).

Now, we can finally give a detailed specification of a closure operator for lattices that have greatest lower bounds.

Definition 5.6. Given program \( p : \mathcal{P}(I \times \mathcal{L}) \rightarrow \mathcal{P}(O \times \mathcal{L}) \) for a lattice \( \mathcal{L} \) with meets we define:

\[
k_p(\ell) = \bigcap \{ j \mid \exists x \; j \in \mathcal{L}(p(x)) \land \ell \subseteq j \}
\]

Theorem 16. \( k_p \) is a closure operator.

Using Theorem 16 we can re-construct the transparency of the \( \text{specify}_L \) \* \( \text{unspecify}_L \) Galois connection for \( \text{goodSum}_L \). Specifically, \( k_{\text{goodSum}_L} \) is isomorphic to \( \text{specify}_L \) \* \( \text{unspecify}_L \):

\[
k_{\text{goodSum}_L}(\ell) = \begin{cases} \ell & \text{if } \ell \subseteq \bigcap \mathcal{L} \text{ then } \mathcal{L}(\ell) \subseteq \bigcap \mathcal{L} \text{ else } \emptyset \end{cases}
\]

Having seen that the \( k_p \) closure operators allow us to compute Galois connections from programs, we next turn to the question of universality. Specifically, we prove that the \( k_p \) closure operator is the most coarse-grained closure operator that gives rise to a transparent Galois connection. This coupled with the fact that each Galois connection can be determined up to isomorphism from some closure operator and vice versa [14] then gives a roadmap to determine for a given program if there is some Galois connection that allows for efficient multi-execution.

Lemma 5 (Canonicity of \( k_p \)). Given \( k \) such that \( \mathcal{L}(p(x)) \subseteq k^*(\mathcal{L}) \) for all \( x \), if \( k_p(\ell) \neq k_p(j) \) then \( k(\ell) \neq k(j) \).

A consequence of this Lemma is that the overhead for multi-execution on \( \text{pairwise} \) can not be reduced below \( O(2^n) \) for the powerset lattice in the worst-case using a Galois connection to a smaller lattice. Specifically, this is because \{ \( \ell \mid \exists x. \; \ell \in \mathcal{L}(\text{pairwise}(x)) \} \) \( \subseteq \mathcal{L} \) and so \( k_{\text{pairwise}}(\ell) = \ell \) for all \( \ell \). Consequently, if \( \mathcal{L} = \mathcal{P}(A) \) then every transparent Galois connection \( F + G \) for \( \text{pairwise} \) has to preserve the structure of the powerset lattice, and thus preserving the \( O(2^n) \) bound on \( \mathcal{C}(\mathcal{L}(x)) \) in \( \mathcal{C}_{F,G}(\mathcal{L}(x)) \).

However, the following variant of \( \text{pairwise} \) that only works on singleton or empty labels admits a transparent Galois connection:

\[
\text{pairwise}_1(x) \equiv \{ \max(a,b)^{f_{a,b}} \mid a^f \in x, b^f \in x, |f| = |j| \leq 1 \}
\]

Specifically, we can construct the set \( S \) of all labels in its co-domain:

\[
S = \{ \ell \mid \exists x. \; \ell \in \mathcal{L}(\text{pairwise}_1(x)) \} = \{ \ell \mid |\ell| \leq 2 \}
\]

Figure 8: The Execution Times of MEF and MEF”\(^{\text{au}} \) on \( \text{goodSum}_L \).

From which we get:

\[
k_{\text{pairwise}_1}(\ell) = \begin{cases} 0 & \text{if } |\ell| \leq 2 \text{ then } \ell \text{ else } \top \end{cases}
\]

To formalise this reasoning, the final theorem of this Section shows that every transparent Galois connection for \( p \) introduces at least as many executions in \( \text{MEF}^{F,G} \) as the Galois connection given by \( k_p \).

Theorem 17 (Canonicity of Galois Connections). If \( F + G \) is a transparent for \( p \), then \( |\mathcal{C}_{F,G}(\mathcal{L})| \geq \|k_p(C(L))\| \).

With this theory in place, we see a clear path for future work to take our analysis of lattice shape and bring it into practice. One recipe for harnessing this theory is the following:

1. Propose a procedure \( K(p) \) to approximate \( k_p \).
2. Find the complexity of \( K \) of and of the resulting lattice.
3. Show that the overhead of \( K(p) \) and \( \text{MEF}^{K(p) + \lambda \ell \cdot \ell} \) is less than that of \( \text{MEF}[p](x) \).

This opens up a new research direction for Galois-Multi-Execution that we hope will incorporate insights from across the static and dynamic program analysis literature.

6 EMPIRICAL RESULTS

To validate our theoretical results empirically, we have implemented the framework used in this paper as a small\(^5\) Haskell [26] library\(^6\). Our enforcement mechanisms MEF and \( \text{MEF}^{F,G} \) are implemented as higher order functions:

\[
\text{mef} :: (\text{Lattice } L, \text{ Ord } l, \text{ Ord } b) \Rightarrow (\text{Set } (a, l) \rightarrow \text{Set } (b, l)) \Rightarrow (\text{Set } (a, l) \rightarrow \text{Set } (b, l))
\]

\[
\text{mefGalois} :: (\text{Lattice } L, \text{ Lattice } L', \text{ Ord } l, \text{ Ord } l', \text{ Ord } b) \Rightarrow \text{Galois } L \rightarrow (\text{Set } (a, l) \rightarrow \text{Set } (b, l)) \Rightarrow (\text{Set } (a, l) \rightarrow \text{Set } (b, l))
\]

\(^5\)50 lines of code, including all our experiments
\(^6\)That is available as supplementary material to this paper
The type signature for \texttt{mef} comes in three parts, line by line:

1. \((\text{Lattice } 1, \text{Ord } 1, \text{Ord } b)\) are constraints that require 1 to be a type that forms a lattice \((\text{Lattice } 1)\) that additionally has a total order \((\text{Ord } 1)\), note that this does not require the lattice ordering \(\subseteq\) on 1 to be total, and that \(b\) has a total order \((\text{Ord } b)\). The total order constraints are necessary in order to efficiently represent the inputs and outputs of \texttt{mef} as sets (implemented as e.g. AVL or red-black trees).

2. \((\text{Set } (a, 1) \to \text{Set } (b, 1))\) is a higher-order argument, a function \(p\) that takes sets of labeled as as input and produces labeled bs as output.

3. \((\text{Set } (a, 1) \to \text{Set } (b, 1))\) means that \texttt{mef} \(p\) is also a function from \texttt{Set } \((a, 1)\) to \texttt{Set } \((b, 1)\).

The difference between \texttt{mef} and \texttt{mefGalois} is that \texttt{mefGalois} additionally requires two lattices and a Galois connection between them as input (where a Galois connection is a pair of functions).

Figure 8 contains a teaser of our empirical results. It contains the log of runtime for \(\text{MEF}\{\text{goodSum}_{L}\}\), \(\text{MEF}_{\text{specify-unspecify}}\{\text{goodSum}_{L}\}\), and \(\text{goodSum}_{L}\) plotted against 100 inputs of size ranging from 0 to 100. As can be seen in the Figure, going from using the powerset lattice to the two-point lattice reduces the running time from exponential to polynomial.

In the experiments an input of size \(n\) is the set \(\{p_1, \ldots, p_n\}\) where each principal \(p_i\) is unique. The definition of \(L\) for \(\text{goodSum}_{L}\) for input \(n\) is \(L = \{p_1, \ldots, p_n\}\). \(\text{MEF}^{\text{specify-unspecify}}\) uses the Galois connection \(\text{specify}(p_1, \ldots, p_n) \to \text{unspecify}(p_1, \ldots, p_n)\) where the “specified” element of the powerset lattice is precisely \(\{p_1, \ldots, p_n\}\).

From Figure 8 we see that \(\text{MEF}\{\text{goodSum}_{L}\}\) takes exponential time. \(\text{MEF}^{\text{specify-unspecify}}\), meanwhile, has linear-time performance. This is because \(\text{MEF}^{\text{specify-unspecify}}\) introduces at most two executions of each program, one for \(\{p_1, \ldots, p_n\}\) and one for \(\top\), and so the running-time is proportional to the running time of \(\text{goodSum}_{L}\). Figure 8 also contains linear fit lines for \(\text{goodSum}_{L}\) and \(\text{MEF}^{\text{specify-unspecify}}\{\text{goodSum}_{L}\}\).

In a second experiment, reported in Figure 9, we compare the execution time of \texttt{MEF} and \texttt{MEF}^{\text{specify-unspecify}} on the \texttt{pairwise} function. The \(t \in e\) Galois connection here refers to the canonical Galois connection between \(P(A)\) and \(P_{\text{A}}(A)\). The input is the same as in the other experiment. We also test the timing results for running \texttt{MEF}^{\text{specify-unspecify}}{}\texttt{pairwise} and \texttt{pairwise} to polynomials of the shape \(ax^2 + bx + c\) using the Gnuplot \([53]\) implementation of the Levenberg-Marquardt algorithm \([29]\). The results can be seen as fit lines in Figure 9. The fits are good when the input size is greater than 40, showing us that quadratic order execution time is a good asymptotic fit.

7 RELATED WORK

There is a growing body of work on transparent IFC enforcement \([2, 4, 9, 13, 15, 18, 19, 21, 27, 32, 35–37, 39, 40, 43, 55]\) and the efficiency of multi-execution as an enforcement mechanism has been studied in various settings going back to the beginning \([21]\).

Theoretical Work on Multi-Execution Performance. In the introduction of the first paper on multi-execution Devriese and Piessens \([21]\) remark that:

“One obvious disadvantage of multi-execution is its cost in terms of CPU time and memory use.”

Some theoretical effort has gone into using the multiple-facets \([9]\) framework for reducing the number of superfluous runs of parts of programs under multi-execution \([4, 35]\). Algehed et al. \([4]\) limit the number of runs of a program under multi-execution by pruning what output levels in the security lattice are used by providing a primitive for pruning the lattice using a boolean algebra over labels. Their modified transparency criteria is similar to our notion of transparency up to a Galois connection in Theorem 12. The \(\Omega(|C(L(x))|)\) lower bound on the overhead of black-box transparent enforcement has been informally discussed in the literature \([4, 9, 35, 43]\), and was recently formally proven \([2]\).

Empirical Work on Multi-Execution Performance. The first empirical measurements of the performance overhead of multi-execution are in the original paper, where the authors study the timing overhead of SME for the two-point lattice on a number of small but realistic benchmarks \([21]\). Additionally, the “real-world” overhead of SME has been studied in the setting of the FlowFox IFC browser \([18]\). The first experiment to study how multi-execution scales with the number of security levels that we are aware of was in the work on the multiple facets (MF) version of multi-execution \([9]\). Extending MF, Schmitz et al. \([43]\) present Faceted Secure multi-execution (FSME) that unifies MF and SME and study trade-offs between time and memory use in these two formulations of multi-execution. Finally, Algehed et al. \([4]\) empirically evaluate the effect of filtering the views (akin to executions in multi-execution) of faceted values that appear in MF by selecting executions that may lead to observable outputs and ignoring ones that do not and find that it presents similar speedups to the ones presented in this paper.

Implementations of IFC in Haskell. There is a significant body of work on embedding IFC in Haskell \([3, 17, 22, 27, 38, 41, 43, 48, 50–52]\). Most of which falls into the category of “monadic” IFC libraries in which the code that is subject to IFC enforcement is written using a specialized interface exported by the library. This differs from our implementation, which works on non-monadic, native, code. On the other hand, because our setting only applies to batch-job
8 CONCLUSIONS

In this paper, we have presented a framework for reasoning about upper and lower bounds on the time overhead of multi-execution. We have shown that the choice of lattice alters this overhead; lattices that allow the programmer to express many different combinations of security levels result in large overheads. We also show how to use Galois connections to switch between different lattices, thus allowing programmers to switch from a lattice with high overhead to one with low overhead. This switching potentially comes at the cost of altering the behaviour of the target program, but for many programs it is possible to reduce overhead without affecting semantics. We show that a canonical Galois connection that is both as coarse-grained as possible (reducing the overhead as much as possible) and does not alter program semantics exists for every lattice with greatest lower bounds. Finally, we empirically evaluate our performance predictions on a small implementation of our framework in Haskell and find that the theory matches our empirical results.

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Proof. Assume a family \( \{ S_n \}_{n \in \mathbb{N}} \) with the required properties. By definition, \( CS_L(n) \geq |C(S_n)| \). Therefore \( CS_L(n) = \Omega(|C(S_n)|) \). Consequently, \( CS_L(n) \) is also \( \Omega(f(n)) \). For the other direction, assume \( CS_L(n) = \Omega(f(n)) \), then by the definition of \( CS_L \) we have that for each \( n \) there is a \( S_n \subseteq L \) of size \( n \) such that \( |C(S_n)| = CS_L(n) \). As \( CS_L(n) = \Omega(f(n)) \), so is \( |C(S_n)| \), and so \( S_n \) is the required family.

\[ \Box \]

**Theorem 3 (Global Bounds).**

(1) All lattices are \( O(2^n) \).

(2) If \( L \) is non-finite, then \( L = \Omega(n) \), otherwise it is \( O(1) \).

Proof.

(1) For all \( S \subseteq L \), \( \{ S^t | S^t \subseteq S \} = 2^{|S|} \), so \( |C(S)| \leq 2^{|S|} \), consequently \( CS_L(n) = O(2^n) \).

(2) If \( L \) is non-finite observe that for each \( S = L \) we have that \( S \subseteq C(S) \) and so \( |S| \leq |C(S)| \) and because \( L \) is non-finite for each \( n \in \mathbb{N} \) there exists an \( S_n \subseteq L \) such that \( |S_n| = n \). This defines a family \( \{ S_n \}_{n \in \mathbb{N}} \) that is \( \Omega(n) \), consequently by Lemma 1 so is \( L \). If \( L \) is finite observe that \( 0 < CS_L(n) \leq |L| \) and so we pick \( C = |L| \) and \( N_0 = 0 \), we get that for all \( n \geq 0 = N_0 \), \( CS_L(n) \leq |C| = 1C \).

\[ \Box \]

**Theorem 4 (Embedding Complexity).** If \( L \) can be embedded in \( L' \) then:

- If \( L = \Omega(l(n)) \) then \( L' = \Omega(l(n)) \) and
- If \( L' = O(u(n)) \) then \( L = O(u(n)) \).

Proof.

If \( h : L \rightarrow L' \) is an embedding of \( L \in L' \), then clearly \( h(C(S)) = C(h(S)) \) as \( h \) preserves joins. Furthermore, \( h \) is injective and so \( |h(S)| = |S| \) for all \( S \). This means that \( CS_L(n) \leq CS_{L'}(n) \) as for every \( S \subseteq L \) such that \( |S| = n \) and \( |C(S)| = m \), it is the case that \( h(S) \subseteq L' \), \( |h(S)| = n \), and \( |C(h(S))| = |h(C(S))| = |C(S)| = m \). From this the required bounds follow trivially.

\[ \Box \]

**Lemma 8.** If \( L = \{ (t_1, t'_1), \ldots, (t_k, t'_k) \} \) where \( t_i \in L \) and \( t'_i \in L' \) for all \( i \), then \( |L| = \{|(t_1, \ldots, t_k), (t'_1, \ldots, t'_k)|\} \).

Proof. We prove this by induction on \( k \). In the case when \( k = 0 \), we have that \( \emptyset = \emptyset = \emptyset = (\emptyset, \emptyset) \). In the case when \( k = k_0 + 1 \) we have, by the induction hypothesis, that \( \cup_1^{k_0} = \{ (t_1, \ldots, t_{k_0}), \cup_1^{k_0} \} \) and \( L_0 = \{ (t_1, \ldots, t_{k_0}), \cup_1^{k_0} \} \).

If \( L = L_0 \cup \{ (t, t'_k) \} \) we have that

\[ L = \]

\[ L \cup (t, t'_k) = \]

\[ (\{ t_1, \ldots, t_{k_0} \}, \{ t'_1, \ldots, t'_k \}) \cup (t, t'_k) = \]

\[ (\{ t_1, \ldots, t_k \}, \{ t'_1, \ldots, t'_k \}) \]

which completes the proof.

\[ \Box \]

**Theorem 6.** If \( L \) and \( L' \) are \( \Omega(l(n)) \) and \( \Omega(l'(n)) \) for strictly positive functions \( l(n) \) and \( l'(n) \), then \( L \times L' = \Omega(l(l(n), l'(n))) \).

Proof. By Lemma 1 we have that both \( L \) and \( L' \) admit families \( S_n \) and \( S'_n \) where \( |C(S_n)| \) and \( |C(S'_n)| \) are \( \Omega(l(n)) \) and \( \Omega(l'(n)) \) respectively. As a consequence, \( |C(S_n(S'_n))| \) and \( |C(S'_n(S_n))| \) are \( \Omega(l(l(n), l'(n))) \) respectively. Let \( Z_n = S_{n 1/2} \) and \( Z'_n = S_{n 1/2} \). We
now construct the family $P_n = Z_n \times \bigcup L \cup_2 \times Z'_n$ in $L \times L'$. It is the case that $|P_n| = |Z_n| + |Z'_n| \leq 2 \frac{2^n}{n}$ for all $n$. Finally, it remains to show that $C(P_n) \supseteq C(Z_n) \times C(Z'_n)$, which gives us the lower bound that $|C(P_n)| \geq \Omega(1 \frac{2^n}{n})^{\Omega(1 \frac{2^n}{n})}$ as both $|L|$ and $|L'|$ are strictly positive functions. Let $k = \left\lceil \frac{2^n}{n} \right\rceil$, if $(t, t') \in C(Z_n) \times C(Z'_n)$ then by the definition of closure sets $(t, t') = (\bigcup L, \bigcup L')$ for $L = \{t_1, \ldots, t_k\} \subseteq Z_n$ and $L' = \{t'_1, \ldots, t'_k\} \subseteq Z'_n$. By Lemma 8 we have that $\bigcup \{(t_1, t'_1), \ldots, (t_k, t'_k)\} = (L, L')$ and so $(t, t') \in C(P_n)$. Finally, $(t, t'_1), \ldots, (t_k, t'_k) \in P_n$ and so $(t, t') \in C(P_n)$. Consequently, $C(P_n) \supseteq C(Z_n) \times C(Z'_n)$ and so $|C(P_n)| \geq \Omega(1 \frac{2^n}{n})^{\Omega(1 \frac{2^n}{n})}$. \qed

**Theorem 5.** If $L$ is $O(u(n))$ and $L'$ is $O(u'(n))$, then $L \times L'$ is $O(u(n)u'(n))$.

Proof. We show that for all:

$$S = \{(t_1, t'_1), \ldots, (t_n, t'_n)\} \subseteq L \times L'$$

We have that:

$$C(S) \subseteq C(t_1, t'_1) \times C(t_2, t'_2)$$

Let $L = \{t_1, \ldots, t_n\}$ and $L' = \{t'_1, \ldots, t'_n\}$. Now, pick any $(t, t') \in C(S)$, by definition of $C(S)$ we have that:

$$(t, t') = \bigcup \{(j, j'), (k, k')\}$$

For some $J = \{j_1, \ldots, j_k\} \subseteq L$ and $J' = \{j'_1, \ldots, j'_k\} \subseteq L'$. Furthermore, by Lemma 8 we have that $(t, t') = \bigcup (J, J')$. This gives us $\bigcup J \subseteq C(L)$ and $\bigcup J' \subseteq C(L')$. In other words, $(t, t') \in C(L \times C(L')$ and so $C(S) \subseteq C(L \times C(L')$. This immediately lets us conclude that:

$$C_{S \times L'}(n) \leq C_{S}(n)CS_{L'}(n)$$

Giving us that $L \times L'$ is $O(u(n)u'(n))$. \qed

**Lemma 9.** If $L$ and $L'$ are $\Omega(f(n))$ and $\Omega(g(n))$ respectively, then $L \times L'$ and $L \circ L'$ are both $\Omega(f(n) + g(n))$.

Proof. Call $S_n$ and $S'_n$ the respective $\Omega$-families of $L$ and $L'$ given by Lemma 1. We construct the family $F_n$ by, for each $n$, picking $F_n = \{0\} \times S_n$ if $|C(S_n)| = |C(S'_n)|$ and $F_n = \{1\} \times S'_n$ otherwise. Clearly, $|C(F_n)| \geq \max(|C(S_n)|, |C(S'_n)|)$ in both $L \times L'$ and $L \circ L'$ and so $|C(F_n)| \geq \Omega(\max(f(n), g(n)))$. By Lemmas 6 and 1 we have that both $L \times L'$ and $L \circ L'$ are $\Omega(f(n) + g(n))$. \qed

**Lemma 10.** If $L$ and $L'$ are $O(f(n))$ and $O(g(n))$ respectively, then $L \circ L'$ and $L \times L'$ are both $O(f(n) + g(n))$.

Proof. For $L \circ L'$ consider that $S \subseteq L \times L'$ means that there exists $L \subseteq L$ and $L' \subseteq L'$ such that $S = L \circ L'$. Next, we show that $C(S) \subseteq C(L \cup C(L')$, by observing that for any $J \subseteq L \cup L'$ there are two cases, either there are elements $(1, t')' \in J$ such that $t' \in L'$ or there are not. In the first case, $J = \{1\} \{t' \in J, (1, t') \in J\}$, as $(0, t) \cup (1, t') = t'$ and so $\bigcup J \subseteq C(L \cup C(L'))$. In the second case, $J = \{0\} \{t \in J, (0, t) \in J\}$ and so $\bigcup J \subseteq C(L \cup C(L'))$. Because $C(L \cup C(L')) \subseteq C(L \cup C(L'))$, we also have that $|C(L \cup L')| \leq |C(L) \cup C(L')| = |C(L)| + |C(L')|$. Because $C_{L}$ and $C_{L'}$ are both monotone functions, this means that if $|L \cup L'| \leq n$ then $|C(L) \cup C(L')| \leq CS_{L}(n) + CS_{L'}(n)$, which gives us our bound on $CS_{L \circ L'}(n)$ of $O(f(n) + g(n))$.

To see that $L \times L'$ is $O(f(n) + g(n))$ observe that if $S \subseteq L \times L'$ then $C(S) \subseteq \{0, 1\} \cup (C(L) \cup C(L'))$ for $L$ and $L'$ such that $S = L \times L' \cup J$ for $J \subseteq \{0, 1\}$. A similar observation as above then immediately gives the upper bound. \qed

**Theorem 7.** If $L$ and $L'$ are $\Theta(f(n))$ and $\Theta(g(n))$ respectively, then $L \times L'$ and $L \circ L'$ are both $\Theta(f(n) + g(n))$.

Proof. Follows immediately from Lemmas 9 and 10. \qed

**Theorem 8.** If there is a non-finite $L \subseteq \mathbb{L}$ such that $\forall \ell, t' \in L \subseteq \bigcup \{t \in L \mid t' \Rightarrow t = \ell \}$, then $L \subseteq \mathbb{L}$ is $\Theta(2^n)$.

Proof. We construct the $\Omega$-family $S_n = \{\{t_1, \ldots, t_n\}\}$ where $t_i \in L$ and $t_i = t_j \Rightarrow i = j$. Clearly, $C(S_n)$ is the powerset of $\{t_1, \ldots, t_n\}$ and so has size $2^n$. Using Lemma 1 this gives us our lower bound on $2^n$ of $\Omega(2^n)$. Together with Theorem 3 we get that $L$ is $\Theta(2^n)$. \qed

**Lemma 11.**

$$\frac{n^k}{k!} \leq \left( \begin{array}{c} n \\end{array} \right)$$

Proof. For $k = 1$ we have $\left( \begin{array}{c} n \\end{array} \right) = n = \frac{n!}{1!}$. For $k > 1$ and $0 < x < k \leq n$ we have:

$$\frac{n - x}{x} \leq \frac{n - x}{k - x} \leq \frac{x}{k} \geq 0$$

Giving us that $\frac{n^k}{k!} \geq \frac{n^k}{k!}$ and hence:

$$\frac{n^k}{k!} \leq \frac{n - 1}{k - 1} \cdot \frac{n - 1}{k - 1} \leq \ldots \leq \left( \begin{array}{c} n \\end{array} \right)$$

\qed

**Theorem 9.** If $A$ is non-finite then $\mathbb{P}_k(A)$ is $\Theta(n^k)$.

Proof. In the case when $k = 0 \mathbb{P}_k(A)$ is a finite lattice consisting of $\emptyset$ and $\top$ and so it is $\Theta(1)$. In the case when $k > 0$ we prove the upper and lower bound separately. For the upper bound, without loss of generality consider any $S = \{t_1, \ldots, t_n\}$ such that no $t_i = 0$, it is the case that:

$$C(S) \subseteq \{\top\} \cup \bigcup_{0 \leq i \leq k} \{\bigcup S' \mid S' \subseteq S, |S'| = i\}$$

In other words, each element of $C(S)$ is either $\top$, or a set of size at most $k$ that can be constructed by taking the supremum of some $S' \subseteq S$ of size at most $k$. Consequently, we get the following inequality for the size of $C(S)$:

$$|C(S)| \leq 1 + \sum_{0 \leq i \leq k} \left| \bigcup S' \mid S' \subseteq S, |S'| = i\right| \leq 1 + \sum_{0 \leq i \leq k} |S|^i$$

Which in turn means that $\mathbb{P}_k(A)$ is $O(n^k)$. For the lower bound, assume $A$ is non-finite and let $S_n = \{a_1, \ldots, a_n\}$. Such that all $a_i$ are distinct, which gives us:

$$C(S_n) \supseteq \bigcup_{0 \leq i \leq k} \{\bigcup S' \mid S' \subseteq S_n, |S'| = i\}$$

Giving us:

$$|C(S_n)| \geq \sum_{0 \leq i \leq k} \left| \bigcup S' \mid S' \subseteq S_n, |S'| = i\right| \geq \sum_{0 \leq i \leq k} n^i \geq \frac{n^k}{k!}$$

Which is sufficient to establish that $\mathbb{P}_k(A)$ is $\Omega(n^k)$, and so $\mathbb{P}_k(A)$ is $\Theta(n^k)$. \qed
B FAST AND SLOW

Lemma B.1. Let \( S \downarrow \ell = \{ t' \in S \mid t' \subseteq \ell \} \), then
\[
\bigcup (S \downarrow \ell) = \ell \iff \ell \in C(S)
\]

Proof. Left to right is trivial, \( S \downarrow \ell \subseteq S \) and so \( \bigcup S \downarrow \ell \subseteq C(S) \). For the right to left direction, consider that if \( \ell \in C(S) \) then there exists an \( L \subseteq S \) such that \( \bigcup L = \ell \). Note that \( L \subseteq S \downarrow \ell \) and that if \( L' \subseteq S \downarrow \ell \) then \( \bigcup L \subseteq \bigcup \{ L \cup L' \} \subseteq \ell \) as \( \ell \) is an upper-bound of \( L' \) and \( \bigcup \) is monotone with respect to \( \subseteq \). Consequently, we have that \( \ell = \bigcup L \subseteq \bigcup \{ L \cup (S \downarrow \ell - L) \} \subseteq \ell \) which means that \( \ell \subseteq S \downarrow \ell \subseteq \ell \), giving us \( S \downarrow \ell = \ell \). \( \square \)

Theorem 10 (Time Complexity of Multi-Execution). Assume:

1. That the lattice \( L \) is \( O(s_L(n)) \).
2. That \( p(x) \) can be computed in \( O(p_0(|x|)) \).
3. A function \( f \) is computable in \( O(f(|x|)) \) time and for all \( S, C(S) \subseteq f(S) \) and \( f(S) \) is \( O(s_L(|S|)) \).

Then the elements of \( MEF[p](x) \) can be enumerated in time:
\[
O(t_f(|x|) + s_L(|x|)p_0(|x|)|x|)
\]

Proof. Firstly, Lemma 2 means it takes \( t_f(n) + |f(L(x))|n \) to enumerate \( C(L(x)) \). Secondly, there are \( O(s_L(n)p_0(n)) \) runs of \( p_0 \) each of which produces outputs bounded in size by \( O(p_0(|x|)) \) and for each such output Lemma 3 gives the time taken to compute membership of the respective up-set as \( O(n) \). Thirdly, \( |f(L(x))| \) is \( O(s_L(L(x))) \) and so can be ignored. Finally, putting these bounds together gives the time taken to enumerate all elements \( \ell \in C(L(x)) \), computing \( p(x \downarrow \ell) \) and filtering them by \( \ell \uparrow C(L(x)) \). \( \square \)

C THROUGH THE LOOKING GLASS

Lemma 12. If \( L \preceq \ell' \) then \( CF_{FG}(L) \preceq \ell' \).

Proof. Consider \( J \subseteq \ell' \) such that \( J \in CF_{FG}(L) \). There exists an \( S \subseteq L \) such that \( J = G(\bigcup F(S)) \). Because \( F \) is \( G \) and \( G \circ F \) are both monotone and so for any \( \ell \in S \) it is the case that:
\[
\ell \in G(F(\ell)) \subseteq \bigcup F(S) = J
\]
This means that \( J \) is an upper bound of \( S \subseteq L \), which by \( L \preceq \ell' \) means that \( S \subseteq L' \). In other words, \( CF_{FG}(L) \downarrow \ell \subseteq CF_{FG}(L') \downarrow \ell \). By symmetry of \( \preceq \) we have that \( CF_{FG}(\ell') \downarrow \ell \subseteq CF_{FG}(\ell) \downarrow \ell \) and so \( CF_{FG}(\ell') \preceq \ell' \).

Theorem 11. \( MEF^{FG}[p] \) is noninterfering.

Proof. Consider, \( \ell, x \sim y \) and \( a' \) such that \( a' \in MEF^{FG}[p](x) \downarrow \ell \). The definitions of \( \downarrow \) and \( MEF^{FG} \) give us that there is some \( \ell \in CF_{FG}(L(x)) \) such that \( \ell \subseteq J \) and \( \ell \in p(x) \downarrow \ell \). However, because \( \ell \subseteq J \) we also have that \( \ell \in CF_{FG}(L(y)) \), by Lemma 12. Furthermore, if \( J \not\subseteq \ell' \) then there is some other \( \ell' \in CF_{FG}(L(y)) \) such that \( \ell' \not\subseteq J \), but this is impossible because \( x \sim y \) and so \( a' \in p(y) \downarrow \ell' \) by \( CF_{FG}(L(y)) \). By definition of \( MEF^{FG} \) this gives us \( a' \in MEF^{FG}[p](y) \downarrow \ell \) and so \( MEF^{FG}[p](x) \downarrow \ell \subseteq MEF^{FG}[p](y) \downarrow \ell \). Symmetry of \( \sim \) means that \( MEF^{FG}[p](y) \downarrow \ell \subseteq MEF^{FG}[p](x) \downarrow \ell \) and so \( MEF^{FG}[p] \) is noninterfering. \( \square \)

Lemma 13. Take a Galois connection \( F \circ G \) between \( L \) and \( L' \) and a set \( S \subseteq L \). For any \( j \in (G \circ F)^*(L) \) there is some \( \ell \in CF_{FG}(L) \) such that \( j = \ell' \uparrow CF_{FG}(L) \).

Proof. Let \( \ell = G(\bigcup F(S)) \), \( \ell' \in CF_{FG}(L) \). We have that \( \ell = G(F(\bigcup F(S))) = \bigcup(F(S)) \subseteq j \) and so \( J \subseteq G(F(j)) = J \). Additionally, \( \ell' \in CF_{FG}(L) \) and \( \ell' \not\subseteq j \) then for some \( L' \) it is the case that \( G(F(L')) = G(F(S')) \subseteq \ell' \) and consequently \( \bigcup L' \subseteq G(F(S')) \subseteq j \) and so \( \bigcup L' \subseteq \bigcup L \) and so \( \ell' \subseteq \ell \). Giving us \( j \subseteq \ell \uparrow CF_{FG}(L) \). \( \square \)

Theorem 12. If \( p \) is noninterfering and \( j \in (G \circ F)^*(L) \) then:
\[
MEF^{FG}[p](x)@j = p(x)@j
\]

Proof. \[
MEF^{FG}[p](x)@j = \bigcup ( p(x \downarrow \ell)@[\ell' \uparrow CF_{FG}(L(x))]) \subseteq \bigcup ( p(x)@[\ell' \uparrow CF_{FG}(L(x))])
\]
(1) and \( x \downarrow \ell = x \downarrow j \) for \( j \subseteq \ell \uparrow CF_{FG}(L(x)) \)
\[
\bigcup ( p(x)@[\ell' \uparrow CF_{FG}(L(x))]) \subseteq \bigcup ( p(x)@[\ell' \uparrow CF_{FG}(L(x))])@j
\]
(2), Lemma 13, and Definition of \( \uparrow \)
\[
p(x)@j
\]
\( \square \)

Lemma 4. Given \( \ell \in L \) and assuming \( F \circ G \) such that both \( F \) and \( G \) are constant time it is possible to compute \( j \in G(F(\ell)) \uparrow CF_{FG}(L) \) in \( O(|L|) \) time by computing:
\[
G(F(\ell)) \subseteq j \wedge \forall i \in F(S), G(i) \subseteq j \Rightarrow G(i) \subseteq G(F(i))
\]

Proof. Call the condition in the Lemma statement \( P(j, \ell) \). To see that \( P(j, \ell) \Rightarrow P(j, \ell) \) consider that if \( P(j, \ell) \) and some \( i \in G(F^{\prime}(S)) \) is such that \( i \subseteq j \) then either \( L' \) is in which case \( i \subseteq G(F(\ell)) \) by monotonicity of \( G \), or \( F(\ell) \) such that \( F^{\prime}(S) \) \( \subseteq \bigcup F(S) \) \( \subseteq \bigcup F(\ell) \) \( \subseteq G(F(i)) \) \( \subseteq G(G(i)) \) \( \subseteq G(F(i)) \). Consequently, \( \bigcup L' \subseteq F(S) \) \( \subseteq G(F^{\prime}(S)) \) \( \subseteq F(S) \) \( \subseteq F(\ell) \) \( \subseteq G(F(i)) \) \( \subseteq G(F(i)) \) \( \subseteq G(F(i)) \). For the other direction, if \( j \subseteq G(F(\ell)) \uparrow CF_{FG}(L) \) then clearly \( G(F(\ell)) \subseteq j \) and if \( i \in F(\ell) \) then \( G(i) \subseteq j \) then \( G(i) \subseteq G(F(\ell)) \) by the definition of \( j \subseteq G(F(\ell)) \uparrow CF_{FG}(L) \). \( \square \)

Theorem 13 (Time Complexity of Multi-Execution). Assume:

1. That \( F \circ G \) is a Galois connection between \( L \) and \( L' \).
2. That the lattice \( L' \) is \( O(s_L(n)) \).
3. That \( p(x) \) can be computed in \( O(p_0(|x|)) \).
4. A function \( f \) is computable in \( O(f_0(|x|)) \) time and for all \( S, C(S) \subseteq f(S) \) and \( f(S) \) is \( O(s_L(|S|)) \) for the lattice \( L' \).

Then the elements of \( MEF^{FG}[p](x) \) can be enumerated in time:
\[
O(t_f(|x|) + s_L(|x|)p_0(|x|)|x|)
\]
Proof. Firstly, Lemma 2 means it takes \(t_f(n) + |f(L(x))|n\) to enumerate \(G^*(C(F^*(L(x))))\). Secondly, there are \(O(s_L(n)t_p(n))\) runs of \(p\), each of which produces outputs bounded in size by \(O(t_p(n))\) and for each such output Lemma 4 gives the time taken to compute membership of the respective \(\uparrow\text{-set}\) as \(O(n)\). Thirdly, \(|f(L(x))|\) is \(O(s_L(L(x)))\) and so can be ignored. Finally, putting these bounds together gives the time taken to enumerate all elements \(\ell \in C(L(x))\), computing \(p(x \downarrow \ell)\) and filtering them by \(\ell \uparrow C(L(x))\).

Theorem 16. \(k_p\) is a closure operator.

Proof. We have three proof obligations:

1. Extensivity: \(\ell \subseteq k_p(\ell)\)
2. Monotonicity: \(\ell \subseteq \j \Rightarrow k_p(\ell) \subseteq k_p(\j)\)
3. Idempotence: \(k_p(\ell) = k_p(k_p(\ell))\)

First we let \(S_p(\ell) = \{ j | \exists x. j \in L(p(x)) \land \ell \subseteq j \}\) and note that \(k_p(\ell) = \bigcap S_p(\ell)\). Proof obligations in order:

1. \(\ell\) is a lower bound of \(S_p(\ell)\) and so \(\ell \subseteq \bigcap S_p(\ell) = k_p(\ell)\).
2. If \(\ell \subseteq j\) then \(S_p(\ell) \supseteq S_p(j)\) and so \(k_p(\ell) \subseteq k_p(\j)\).
3. \(S_p(k_p(\ell)) = S_p(\ell)\) and so \(k_p(\ell) = k_p(k_p(\ell))\).

Lemma 5 (Canonicity of \(k_p\)). Given \(k\) such that \(L(p(x)) \subseteq k^*(L)\) for all \(x\), if \(k_p(\ell) \neq k_p(\j)\) then \(k(\ell) \neq k(\j)\).

Proof. If \(k_p(\ell) \neq k_p(\j)\) then without loss of generality we can assume that there is some \(i \in L(p(x))\) for some \(x\) such that \(\ell \subseteq i\) but \(\j \nsubseteq i\). Consequently, \(\ell \cup \j \nsubseteq i\). Assume for a contradiction that \(k(\ell) = k(\j)\). By monotonicity of \(k\) we know that \(\ell \subseteq k(\ell)\) and \(\j \subseteq k(\j)\) and so \(\ell \cup \j \subseteq k(\ell)\). However, monotonicity of \(k\) also gives us that because \(\ell \subseteq i\) we have that \(k(\ell) \subseteq k(i)\). But \(i \in L(p(x))\) for some \(x\) so \(k(i) = i\) and so \(k(\ell) \subseteq i\). Putting everything together gives us \(j \nsubseteq \ell \cup \j \subseteq k(\ell) \subseteq i\) which contradicts \(j \nsubseteq i\), so \(k(\ell) \neq k(\j)\).

Theorem 17 (Canonicity of Galois Connections). If \(F \vdash G\) is a transparent for \(p\), then \(|C_{FG}(L)| \geq |k_p^*(C(L))|\).

Proof. We have that \(L(p(x)) \subseteq (G \circ F)^*(L)\) for all \(x\) as \(F \vdash G\) is transparent. By Lemma 5 we have that:

\[k_p(\ell) \neq k_p(\j) \Rightarrow G(F(\ell)) \neq G(F(\j))\]

Therefore, any two elements in \(C(L)\) that are distinguished by \(k_p\) are distinguished by \(G \circ F\). Consequently, as:

\[C_{FG}(L) = \{G(F(S)) | S \subseteq L\}\]

\[= \{G(F \uplus S) | S \subseteq L\} = (G \circ F)^*(C(L))\]

this means that \(C_{FG}(L)\) has at least as many elements as \(k_p^*(C(L))\).

\[\square\]