Stability theorem of depolarizing channels
for the minimal output quantum Rényi entropies

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We show that the stability theorem of the depolarizing channel holds for the output quantum $p$-Rényi entropy for $p \geq 2$ or $p = 1$, which is an extension of the well known case $p = 2$. As an application, we present a protocol in which Bob determines whether Alice prepares a pure quantum state close to a product state. In the protocol, Alice transmits to Bob multiple copies of a pure state through a depolarizing channel, and Bob estimates its output quantum $p$-Rényi entropy. By using our stability theorem, we show that Bob can determine whether her preparation is appropriate.

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I. INTRODUCTION

We extend the stability theorem of the depolarizing channel to the output quantum $p$-Rényi entropy for $p \geq 2$ or $p = 1$. The original stability theorem with the output purity is essentially equivalent to our stability theorem for the case $p = 2$ and was used in proving the equality $\text{QMA}(k) = \text{QMA}(2)$ for all $k \geq 2$ [1]. We generalize it to the output quantum $p$-Rényi entropy to create a more powerful tool, and we apply it to a type of polygraph test as discussed below. Generalization is accomplished by defining the notion of stability of a quantum channel with respect to any real valued continuous function. That is, if a state is close to achieving the minimal/maximal output value of a particular quantity (entropy function) through the channel, then it must be close to an input state giving the minimal/maximal value. In particular, we show that the depolarizing channel is stable with respect to the output quantum Rényi entropy.

Our theorem is constructed by generalizing the Taylor expansion of von Neumann entropy [2] to the quantum Rényi entropy. Whereas the original work employed the output purity [1], which is a relatively simpler function, we use a more general (and complicated) function, namely the output quantum Rényi entropy. The Taylor expansion of the output quantum Rényi entropy is the technique we use to prove the stability theorem for a depolarizing channel with respect to the the output quantum Rényi entropy. The protocol is described in [III] and provides us meaning and intuition for the stability theorem of the depolarizing channel. Furthermore, the protocol shows that our stability theorem has a benefit as our protocol has a smaller undecidable gap than the original case.

We organize our paper as follows. In [III] we provide some notions to define a stable channel clearly. Our main result appears in [IV] where we generalize the Taylor expansion of the von Neumann entropy to calculate the Taylor expansions of the quantum Rényi entropies. We use this result to show that the depolarizing channel is stable with respect to the output quantum $p$-Rényi entropies for $p \geq 2$ or $p = 1$. In [V] we introduce a polygraph test as an application of our stability theorem. Finally, in [VI] we conclude with discussion on our results.

II. STABLE CHANNELS

In this section, we define the notions of a quantity, an extremal state, an $\epsilon$-almost extremal state, an $\epsilon$-stable channel, and a stable channel. Except for this section, in the rest of the paper, we will be using these notions for only the depolarizing channel and the quantum Rényi entropy as a quantity.

Definition 1. Let $E : \mathcal{B}(\mathcal{H}_i) \to \mathcal{B}(\mathcal{H}_o)$ be a quantum channel (i.e., a trace preserving completely positive map) and let $Q$ be a real-valued continuous function on $\mathcal{B}(\mathcal{H}_o)$, where $\mathcal{B}(\mathcal{H}_i)$ and $\mathcal{B}(\mathcal{H}_o)$ are the sets of all states in the input space $\mathcal{H}_i$ and the output space $\mathcal{H}_o$, respectively.
For any $\epsilon > 0$, a state $\sigma \in B(H)$ is $\epsilon$-almost extremal with respect to the function $Q$ and the channel $\mathcal{E}$ if
\[ |Q(\mathcal{E}(\sigma)) - \text{ext}_\rho Q(\mathcal{E}(\rho))| \in O(\epsilon), \] (1)
where the extremal value, $\text{ext}_\rho$, refers to either the maximal value or the minimal value of $Q$ over all states $\rho$ in $B(H)$ according to a given quantity. A state $\sigma_0$ is said to be extremal with respect to $Q$ and $\mathcal{E}$ if
\[ Q(\mathcal{E}(\sigma_0)) = \text{ext}_\rho Q(\mathcal{E}(\rho)). \] (2)

Now we define a stable channel with respect to the function $Q$.

**Definition 2.** For a given $\epsilon > 0$, a channel $\mathcal{E}$ is $\epsilon$-stable with respect to a quantity $Q$ if, for all $\sigma$ $\epsilon$-almost extremal, an extremal state $\sigma_0$ exists with respect to $Q$ and $\mathcal{E}$ such that
\[ \|\sigma - \sigma_0\|_1^2 \in O(\epsilon), \] (3)
where $\|\cdot\|_1$ denotes the trace norm. A channel $\mathcal{E}$ is stable with respect to a quantity $Q$ if it is $\epsilon$-stable with respect to the quantity $Q$ for all $\epsilon > 0$.

We have provided some generalized definitions to establish the notion of a stable channel. In the next section, as our main result, we present the stability theorem of the depolarizing channel for the output quantum Rénnyi entropy and prove that the depolarizing channel is stable with respect to the quantum Rénnyi entropy.

### III. STABILITY OF THE DEPOLARIZING CHANNEL

In this subsection, we present and prove the stability theorem of the depolarizing channel with respect to the output quantum Rénnyi entropy. This section consists of two subsections. In the first subsection, we evaluate the Taylor expansion of the quantum Rénnyi entropy which is crucial to prove our main theorem in the second subsection.

#### A. The Taylor expansion of the quantum $p$-Rényi entropy

In this subsection, we the Taylor expansion technique for the von Neumann entropy [2] to calculate the Taylor expansion of the quantum $p$-Rényi entropy. This technique is key to prove the stability theorem of the depolarizing channel for the output quantum $p$-Rényi entropy.

For $p > 0$ ($p \neq 1$), the quantum $p$-Rényi entropy [3] of a state $\rho$ is
\[ S_p(\rho) := \frac{1}{1 - p} \log \text{Tr} \rho^p. \] (4)
The minimal output quantum $p$-Rényi entropy of a quantum channel $\mathcal{E}$ is defined as
\[ S^\text{min}_p(\mathcal{E}) := \min_\rho S_p(\mathcal{E}(\rho)), \] (5)
where the minimum is taken over all input states $\rho$ of $\mathcal{E}$. The quantum $p$-Rényi entropy converges to the von Neumann entropy as $p$ tends to one, and we can thus consider the quantum Rényi entropy as a generalization of the von Neumann entropy [3].

In order to obtain the Taylor expansion of the quantum Rényi entropy, we exploit the following lemma.

**Lemma 1** (Gour and Friedland [2]). Let $A = \text{diag}(p_1, \cdots, p_m) \in \mathbb{C}^{m \times m}$ be a diagonal square matrix, and $B = [b_{ij}] \in \mathbb{C}^{m \times m}$ be a complex square matrix. Let $f$ be a $C^2$ function defined on a real open interval $(a, b)$. Then
\[ f(A + tB) = f(A) + tL_A(B) + t^2Q_A(B) + O(t^3) \] (6)
for $L_A : \mathbb{C}^{m \times m} \to \mathbb{C}^{m \times m}$ a linear operator and $Q_A : \mathbb{C}^{m \times m} \to \mathbb{C}^{m \times m}$ a quadratic homogeneous non-commutative polynomial in $B$. For $i, j = 1, \cdots, m$, we have
\[ [L_A(B)]_{ij} = \Delta f(p_i, p_j) b_{ij} = \frac{f(p_i) - f(p_j)}{p_i - p_j} b_{ij}, \]
\[ [Q_A(B)]_{ij} = \sum_{k=1}^m \Delta^2 f(p_i, p_k, p_j) b_{ik} b_{kj}. \] (7)
In particular,

\[
\text{Tr} \left( L_A (B) \right) = \sum_{j=1}^{m} f' (p_j) b_{jj},
\]

\[
\text{Tr} \left( Q_A (B) \right) = \sum_{i,j=1}^{m} \frac{f' (p_i) - f' (p_j)}{2 (p_i - p_j)} b_{ij} b_{ji}.
\]

(8)

Now we use Lemma [1] to calculate the Taylor expansion of \( S_p (\rho (t)) \).

**Theorem 2.** A nonsingular density matrix

\[
\rho (t) = \rho + t \gamma_0 + t^2 \gamma_1 + O(t^3),
\]

with \( \rho \) diagonal, \( \gamma_0 \) all zeroes along the diagonal and \( \gamma_1 \) having zero trace, has quantum \( p \)-Rényi entropy

\[
S_p (\rho (t)) = S_p (\rho) + \frac{1}{1-p} t^2 \left( p \frac{\text{Tr} (\rho^{p-1} \gamma_1)}{\text{Tr} (\rho^p)} + p \frac{\text{Tr} (Q_\rho (\gamma_0))}{\text{Tr} (\rho^p)} \right) + O(t^3).
\]

(10)

**Remark.** As \( p \) tends to one, Theorem 2 implies the Taylor expansion of the von Neumann entropy, hence generalizes the von Neumann entropy.

**Proof.** As \( \rho \) is nonsingular, \( \mathbb{I} - \rho (t) < \mathbb{I} \) for small \( t \). Thus, we can employ the Taylor expansion with respect to \( t \).

From the following Taylor expansion

\[
\rho^p (t) = [\mathbb{I} - (\mathbb{I} - \rho (t))]^p = \sum_{n=0}^{\infty} \binom{p}{n} (-1)^n (\mathbb{I} - \rho (t))^n,
\]

(11)

we obtain

\[
\text{Tr} (\rho^p (t)) = \sum_{n=0}^{\infty} \binom{p}{n} (-1)^n \text{Tr} [(\mathbb{I} - \rho (t))^n].
\]

(12)

Expanding the trace term in the right-hand side of Eq. (12) up to second order in \( t \) yields

\[
\text{Tr} [(\mathbb{I} - \rho (t))^n] = \text{Tr} [(\mathbb{I} - \sigma (t))^n] - t^2 n \text{Tr} [(\mathbb{I} - \rho)^{n-1} \gamma_1] + O(t^3),
\]

(13)

where \( \sigma (t) = \rho + t \gamma_0 \). From Eq. (12) and Eq. (13),

\[
\text{Tr} (\rho^p (t)) = \text{Tr} (\sigma^p (t)) + p t^2 \text{Tr} (\rho^{p-1} \gamma_1) + O(t^3).
\]

(14)

As Lemma [1] yields the equality

\[
\text{Tr} (\sigma^p (t)) = \text{Tr} (\rho^p) + t p \text{Tr} (\rho^{p-1} \gamma_0) + t^2 \text{Tr} (Q_\rho (\gamma_0)) + O(t^3),
\]

(15)

and \( \gamma_0 \) is zero along the diagonal, we obtain

\[
\text{Tr} (\rho^p (t)) = \text{Tr} (\rho^p) + t^2 (p \text{Tr} (\rho^{p-1} \gamma_1) + \text{Tr} (Q_\rho (\gamma_0))) + O(t^3).
\]

(16)

Using the Taylor expansion of the logarithm function,

\[
\log (1 + x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n},
\]

(17)

we obtain

\[
\log \text{Tr} (\rho^p (t)) = \log \left[ \text{Tr} (\rho^p) \left( 1 + t^2 \left( p \frac{\text{Tr} (\rho^{p-1} \gamma_1)}{\text{Tr} (\rho^p)} + p \frac{\text{Tr} (Q_\rho (\gamma_0))}{\text{Tr} (\rho^p)} \right) + O(t^3) \right) \right]
\]

\[
= \log \text{Tr} (\rho^p) + t^2 \left( p \frac{\text{Tr} (\rho^{p-1} \gamma_1)}{\text{Tr} (\rho^p)} + p \frac{\text{Tr} (Q_\rho (\gamma_0))}{\text{Tr} (\rho^p)} \right) + O(t^3).
\]

(18)

Therefore, by definition of the quantum \( p \)-Rényi entropy in Eq. (4), the equality (10) can be readily obtained from Eq. (18). This completes the proof.

We have evaluated the Taylor expansion of the quantum Rényi entropy. In the next subsection, we use this result to prove the stability theorem of the depolarizing channel for the output quantum Rényi entropy.
B. The stability theorem of the depolarizing channel for the output quantum Rényi entropy

In this subsection, we prove our main theorem, namely the stability theorem of the output quantum \( p \)-Rényi entropy for the depolarizing channel for \( p \geq 2 \). First, we present the following lemma, which is crucial to prove the theorem.

**Lemma 3.** For \( p \geq 2 \), \( r > 1 \) and \( d \geq 2 \),

\[
f_p(x) := \frac{p}{1 - p} \left[ \left( \frac{(r^x)^{p-1} - 1}{r^x - 1} \right) \left( \frac{(r - 1)^2}{(d + r - 1)^2(r^p + (d - 1))} \right)^x + \left( \frac{r^{p-1} + r + (d - 2)}{r^p + (d - 1)} \right)^x - 1 \right]
\]  
(19)
is monotonically increasing on \([2, \infty)\).

**Remark.** Let \(|\psi\rangle\) be an \( n \)-qudit pure state satisfying \(|\langle \psi|\phi\rangle|^2 = 1 - t^2\) for an \( n \)-qudit product state \(|\phi\rangle\). Then the function \( f_p \) is the coefficient of the second order term in the Taylor expansion of \( S_p \left( D_\lambda^n |\psi\rangle \langle \psi| \right) \).

**Proof.** Observe that

\[
f_p'(x) = A_p \left[ B_p \log \left( \frac{r^p + d - 1}{r^{p-1}(r - 1)^2} \right) + C_p \log r \right],
\]
(20)
where

\[
A_p = \frac{p}{1 - p} \frac{(r - 1)^2}{(r^x - 1)^2(r^p + d - 1)},
\]
\[
B_p = (r^x - 1)(1 - (r^x)^{p-1}),
\]
\[
C_p = - (r^x)^p + p(r^x - 1) + 1.
\]  
(21)

As \( r > 1 \), straightforward calculations yield \( A_p \leq 0 \), \( C_p \leq B_p \leq 0 \) for \( p \geq 2 \). Thus, we obtain the inequality

\[
f_p'(x) \geq A_p \left[ B_p \log \left( \frac{r^p + d - 1}{r^{p-1}(r - 1)^2} \right) + B_p \log r \right]
= A_p B_p \log \left( \frac{r^{p+1} + (d - 1)r}{r^{p-1}(r - 1)^2} \right).
\]  
(22)

Here the right-hand side of the inequality (22) is clearly nonnegative as the inequality

\[
\log \left( \frac{r^{p+1} + (d - 1)r}{r^{p-1}(r - 1)^2} \right) > 0
\]  
(23)
can be easily proved due to the inequality

\[2r^{p-1} - r^{p-2} + (d - 1) > 0.\]  
(24)

Therefore, the function \( f_p(x) \) is monotonically increasing. \( \square \)

We now present one more lemma, which tells us that the minimal output quantum \( p \)-Rényi entropy of the depolarizing channel is achieved for product state inputs. The lemma can be readily obtained by the additivity of the minimal output quantum \( p \)-Rényi entropy \([3]\).

**Lemma 4.** For the \( n \)-partite product depolarizing channel \( D_\lambda^n \), the quantum \( p \)-Rényi entropy of the output state is minimized for product state inputs and furthermore has the same value for all product state inputs; that is,

\[S_p^{\min} \left( D_\lambda^n \right) = S_p \left( D_\lambda^n |\phi\rangle \langle \phi| \right),\]
(25)
for any \( n \)-partite pure product state \(|\phi\rangle\).

We now use Theorem 2 and the above lemmas to obtain the stability theorem of the output quantum \( p \)-Rényi entropy for the depolarizing channel.
Lemma 5. Let \( p \geq 2, \epsilon > 0 \) and \( |\psi\rangle \in (\mathbb{C}^d)^n \) be a state. Then
\[
S_p (D^\otimes n |\psi\rangle \langle \psi|) < S_p^{\min} (D^\otimes n) + 2\epsilon p \frac{r - 1}{p - 1} \frac{1}{(r^p + dr + d - 2)} + O(\epsilon^{3/2})
\]
holds only if a pure product state \(|\phi\rangle\) exists such that \(|\psi\rangle\) satisfies
\[
(|\langle \psi| \phi\rangle|^2 \geq 1 - \epsilon.
\]

Proof. We prove the contrapositive of the theorem. Let
\[
\epsilon_0 = 1 - \max \left\{ (|\langle \psi| \phi_1, \cdots, \phi_n\rangle|^2 : |\phi_i\rangle \in \mathbb{C}^d) \right\} > \epsilon.
\]
Without loss of generality, we may assume that one of the states achieving the maximum in Eq. (28) is \(|0^n\rangle = |0\rangle\). We then have
\[
|\psi\rangle = \sqrt{1 - \epsilon_0} |0\rangle + \sqrt{\epsilon_0} |\phi\rangle
\]
for some state \(|\phi\rangle\) such that \(|\langle 0| \phi\rangle = 0\); that is, \(|\phi\rangle = \sum_{x \neq 0} \alpha_x |x\rangle\) for some \(\alpha_x\) such that \(\sum_{x \neq 0} |\alpha_x|^2 = 1\). We can write explicitly
\[
|\psi\rangle\langle \psi| = (1 - \epsilon_0) |0\rangle \langle 0| + \sqrt{\epsilon_0} (1 - \epsilon_0) (|0\rangle \langle 0| + |\phi\rangle \langle 0| + \epsilon_0 |\phi\rangle \langle \phi|).
\]

Therefore, we have
\[
D^\otimes n |\psi\rangle \langle \psi| = D^\otimes n |0\rangle \langle 0| + \sqrt{\epsilon_0} (1 - \epsilon_0) D^\otimes n (|0\rangle \langle 0| + |\phi\rangle \langle 0|) + \epsilon_0 D^\otimes n (|\phi\rangle \langle 0| - |0\rangle \langle 0|)
\]
\[
= D^\otimes n |0\rangle \langle 0| + \sqrt{\epsilon_0} D^\otimes n (|0\rangle \langle 0| + |\phi\rangle \langle 0|) + \epsilon_0 D^\otimes n (|\phi\rangle \langle 0| - |0\rangle \langle 0|) + O(\epsilon_0^{3/2}).
\]

For the last equality in Eq. (31), we use the Taylor expansion
\[
\sqrt{1 - x} = 1 - \frac{1}{2} x + O(x^2)
\]
for all \(|x| < 1\). Now we use
\[
\rho = D^\otimes n |0\rangle \langle 0|,
\gamma_0 = D^\otimes n (|0\rangle \langle 0| + |\phi\rangle \langle 0|),
\gamma_1 = D^\otimes n (|\phi\rangle \langle 0| - |0\rangle \langle 0|),
\]
\[
t = \sqrt{\epsilon_0},
\rho (t) = \rho + t \gamma_0 + t^2 \gamma_1 + O(t^3).
\]

Then Theorem 2 implies that
\[
S_p (D^\otimes n |\psi\rangle \langle \psi|) = S_p (\rho (t)) = S_p (\rho) + \frac{1}{1 - p} t^2 \left( p \frac{\text{Tr} (\rho^{p-1} \gamma_1)}{\text{Tr} (\rho^p)} + \frac{\text{Tr} (Q_\rho (\gamma_0))}{\text{Tr} (\rho^p)} \right) + O(t^3).
\]

For convenience, we let
\[
a = (1 + (d - 1) \lambda) / d, \ b = (1 - \lambda) / d.
\]

Then we obtain the following three facts.
(i) As \( \rho = D^\otimes n |0\rangle \langle 0| \) can be rewritten as \( \sum_y a^{n-|y|} b^{|y|} |y\rangle \langle y| \),
\[
\text{Tr} (\rho^p) = \text{Tr} \left( \sum_y (a^p)^{n-|y|} (b^p)^{|y|} |y\rangle \langle y| \right) = (a^p + (d - 1) b^p)^n,
\]
for \(|y|\) denoting Hamming weight of an \(n\)-bit string \(y\).
(ii) We can evaluate
\[
\text{Tr} (\gamma_1 \rho^{p-1}) = \text{Tr} \left( D_{\gamma}^{\otimes n} |\phi\rangle \langle \phi| \right) (D_{\gamma}^{\otimes n} |0\rangle \langle 0|)^{p-1} - \text{Tr} (\rho^p)
\]
\[
= \sum_{x, x' \neq 0, y} \alpha_x \alpha_{x'}^* \left( (a^{p-1})^{|x|} (b^{p-1})^{|x'|} \right) \frac{1 - \lambda (1 - d)^{\delta_{x,y} - 1}}{d} - \text{Tr} (\rho^p)
\]
\[
= \sum_{x \neq 0} |\alpha_x|^2 (a^{p-1})^{n-|y|} (b^{p-1})^{|y|} \prod_{i=1}^n \frac{1 - \lambda (1 - d)^{\delta_{x,y} - 1}}{d} - \text{Tr} (\rho^p)
\]
\[
= \sum_{x \neq 0} |\alpha_x|^2 (a^p + (d - 1)b)^n \left( \frac{a^{p-1}b + ab^{p-1} + (d - 2)b^p}{a^p + (d - 1)b^p} \right)^{|x|} - \text{Tr} (\rho^p).
\] (37)

(iii) For $n$-bit strings $j$ and $k$, let
\[
g_{jk} := \frac{(a^{n-|j|}|k|)^{p-1} - (a^{n-|k|}|j|)^{p-1}}{2 (a^{n-|j|}|j| - a^{n-|k|}|k|)}.
\] (38)

Then we write
\[
\text{Tr} (Q_{\rho} (\gamma_0)) = p \left( \sum_{jk} g_{jk} \right) \left( D_{\gamma}^{\otimes n} |0\rangle \langle \phi| \right)_{jk}^2
\]
\[
= p \sum_{x \neq 0} |\alpha_x|^2 \left( \frac{(a^{|x|})^{p-1} - (b^{|x|})^{p-1}}{|a^{|x|} - b^{|x|}|} \right) (a^p + (d - 1)b^p)^n - |x|.
\] (39)

Here $Q_{\rho}$ (39) is a polynomial defined in Eq. (7), and all equalities can be proved by tedious but straightforward calculations except the last equality in Eq. (37), which can be shown by mathematical induction on $n$.

Combining the above facts, we have
\[
S_p (\rho (t)) - S_p (\rho) = t^2 \sum_{x \neq 0} |\alpha_x|^2 f_p (|x|) + O(t^3)
\] (41)

with
\[
f_p (|x|) = \frac{p - 1}{p} \left( \frac{(a^{|x|})^{p-1} - (b^{|x|})^{p-1}}{|a^{|x|} - b^{|x}|} \right) \left( \frac{\lambda^2}{a^p + (d - 1)b^p} \right)^{|x|}
\]
\[
+ \frac{p}{p - 1} \left[ \frac{(a^{p-1}b + ab^{p-1} + (d - 2)b^p)^{|x|}}{a^p + (d - 1)b^p} \right] - 1.
\] (42)

Here it can be shown that the function $f_p$ (42) is equal to the function $f_p$ defined in Eq. (19) by taking $r = a/b$. Hence, Lemma 3 implies that $f_p (|x|)$ is monotonically increasing on $|x|$ for all $p \geq 2$. As $|\phi\rangle$ does not have any weight-one components $4$; that is, $\alpha_x = 0$ for $|x| < 2$, from Theorem 2 and Lemma 4 we can finally obtain the following inequality, and thereby complete the proof.

\[
S_p (D_{\gamma}^{\otimes n} |\psi\rangle \langle \psi|) - S_p (D_{\gamma}^{\otimes n}) = S_p (\rho (t)) - S_p (\rho)
\]
\[
\geq \varepsilon_0 f_p (2) + O(\varepsilon^{3/2})
\]
\[
\geq \frac{p}{p - 1} \left[ \frac{2\lambda (1 - \lambda)(a^{p-1} - b^{p-1})(2(a^p - b^p) + db^{p-1}(a + b))}{(2 + (d - 2)\lambda)(a^p + (d - 1)b^p)^2} \right] + O(\varepsilon^{3/2})
\]
\[
= -2\varepsilon \frac{p}{p - 1} \left[ \frac{(r - 1)(r^{p-1} - 1)(2r^p + dr + d - 2)}{(r + 1)(r^p + d - 1)^2} \right] + O(\varepsilon^{3/2}).
\] (43)

Remark. Although we have not yet established the stability theorem for $1 < p < 2$, we can show that it still holds for $p = 1$: that is, the stability theorem for the von Neumann entropy of the depolarizing channel holds by using a
similar method to what we have used. (We have numerically checked that the same result holds for the several cases of $p$ with $1 < p < 2$.) Let us see the proof for the case of $p = 1$.

We calculate the Taylor expansion of the von Neumann entropy to get the difference between the output von Neumann entropy and its minimum value as follows.

$$S\left(D_{\lambda}^{\otimes n}|\psi\rangle\langle\psi|\right) - S_{p}^{\min}\left(D_{\lambda}^{\otimes n}\right) = \varepsilon \sum_{x \neq 0} |\alpha_x|^2 f(|x|) + O(\varepsilon^{3/2}),$$

(44)

where the function $f$ is defined as

$$f(|x|) = |x| (a - b) \log \frac{a}{b} - (a - b)^2 |x| \left( \frac{\log a^{(|x|)} - \log b^{(|x|)}}{a^{(|x|)} - b^{(|x|)}} \right),$$

(45)

which is the limit of $f_p(|x|)$ for $p$ tends to one. As it is easier than Lemma 3 to prove that $f(|x|)$ is monotonically increasing, we can easily obtain the almost same result as our stability theorem.

**Theorem 6.** The depolarizing channel is stable with respect to the output quantum $p$-Rényi entropy for $p \geq 2$ or $p = 1$.

**Proof.** Let $\varepsilon > 0$ be given, and let the quantity $Q$ be $S_p$. Let $|\psi\rangle\langle\psi|$ be an $\varepsilon$-almost extremal state with respect to $S_p$ and $D_{\lambda}^{\otimes n}$. Then

$$|S_p\left(D_{\lambda}^{\otimes n}|\psi\rangle\langle\psi|\right) - S_{p}^{\min}\left(D_{\lambda}^{\otimes n}\right)| \leq O(\varepsilon),$$

(46)

by Lemma 3 there exists some extremal state $|\phi\rangle\langle\phi|$ with respect to $S_p$ and $D_{\lambda}^{\otimes n}$ such that

$$||\psi\rangle\langle\psi| - |\phi\rangle\langle\phi||_1^2 \leq \varepsilon.$$

(47)

Thus, the depolarizing channel $D_{\lambda}^{\otimes n}$ is $\varepsilon$-stable with respect to the quantum $p$-Rényi entropy $S_p$ for any $\varepsilon > 0$, and hence it is stable with respect to $S_p$. \square

**Remark.** We obtain the stability theorem of the depolarizing channel with respect to the output purity as a corollary of Theorem 6.

Furthermore, we can similarly show that if $n$-qudit pure state is close to be product then its output quantum $p$-Rényi entropy is close to the minimal output quantum $p$-Rényi entropy with a specific precision as follows.

**Theorem 7.** Let $p \geq 2$, $\varepsilon > 0$ and $|\psi\rangle \in (\mathbb{C}^d)^{\otimes n}$ be a state. Then

$$S_p\left(D_{\lambda}^{\otimes n}|\psi\rangle\langle\psi|\right) \geq S_{p}^{\min}\left(D_{\lambda}^{\otimes n}\right) + \frac{p}{p - 1} + O(\varepsilon^{3/2})$$

implies

$$|\langle\psi|\phi\rangle|^2 < 1 - \varepsilon$$

(49)

for any product state $|\phi\rangle$.

**Proof.** Suppose that

$$1 - \varepsilon_1 = \max\left\{|\langle\psi_1|\cdots\langle\psi_n\rangle|^2 : |\psi_1\rangle \in \mathbb{C}^d\right\} \geq 1 - \varepsilon.$$

(50)

From the same arguments in Theorem 6 we obtain the same equality as in Eq. (40). Then

$$S_p\left(D_{\lambda}^{\otimes n}|\psi\rangle\langle\psi|\right) - S_{p}^{\min}\left(D_{\lambda}^{\otimes n}\right) = \varepsilon_1 \sum_{x \neq 0} |\alpha_x|^2 f_p(|x|) + O(\varepsilon_1^{3/2})$$

$$< \varepsilon_1 \frac{p}{1 - p} + O(\varepsilon_1^{3/2})$$

$$< \varepsilon_1 \frac{p}{1 - p} + O(\varepsilon^{3/2}),$$

(51)

where the first inequality is obtained due to the monotonicity of $f_p(|x|)$ and the second inequality results from Eq. (40). \square

**Remark.** The coefficient of $\varepsilon$ in Eq. (49) is smaller than the coefficient of $\varepsilon$ in Eq. (45), which means that some gap exists between them even though it is close to zero for sufficiently small $\varepsilon$. Furthermore, for a sufficiently large $p$, the gap can be smaller than the gap for the case of $p = 2$ as we will see in IV.
IV. AN APPLICATION: A POLYGRAPH TEST

In this section, we introduce a polygraph test as an application of our main results, namely Theorems 5 and 7. Let us consider the following protocol wherein sender Alice transmits multiple copies of an $n$-qudit state through depolarizing channels to receiver Bob.

1. Bob informs Alice of a small enough $\epsilon > 0$ chosen as an error bound.
2. Alice prepares an $n$-qudit pure state that is close to a product state with fidelity at least $1 - \epsilon$, as in Eq. (27) and sends multiple copies to Bob through depolarizing channels.
3. Bob estimates its output quantum Rényi entropy.
4. Bob determines whether Alice’s preparation satisfies the requirement or not, and our results help Bob make the correct decision as discussed below.

Accept: If Bob’s estimate of the output quantum Rényi entropy satisfies Inequality (26) then Alice definitely prepared a correct state according to Theorem 5.

Reject: If Bob’s estimate of the output quantum Rényi entropy satisfies Inequality (48) then Theorem 7 guarantees that Alice’s preparation fails the requirement.

Remark. Some gap exists between the coefficients of $\epsilon$ in Eq. (26) and of $\epsilon$ in Eq. (48), which means that Bob cannot detect Alice’s lie when neither Eqs. (26) nor (48) holds for the output quantum Rényi entropy of the state Alice sent. However, the probability that Alice cheats Bob can be forced to be close to zero if Bob chooses small enough $\epsilon$. Thus, the gap problem can be resolved in this way.

We have introduced the polygraph test as an application of our main theorem, and we have proved that the protocol for our stability theorem has a smaller undecidable gap than for the protocol in the original stability theorem.
V. CONCLUSIONS

We have shown that the stability theorem of the depolarizing channel holds for the output quantum $p$-Rényi entropy for $p \geq 2$, which was one of the open questions in Ref. [1]. Furthermore, we have also proved that the stability theorem of the depolarizing channel holds for the output von Neumann entropy ($p = 1$), and have numerically checked that the stability theorem holds for the several cases $p$ with $1 < p < 2$. Therefore, we expect that the stability theorem holds for all $p \geq 1$, and leave this for future work.

As an application of our main results, we have introduced a polygraph test and have presented its protocol. The original stability theorem can be also applied to the polygraph test, as the original one is essentially equivalent to our stability theorem when $p = 2$. In the protocol, Bob determines whether Alice prepares a pure quantum state close to a product state. However, Bob cannot perfectly decide whether her preparation is proper, that is, there is an undecidable gap in which he can decide nothing. We have shown that the undecidable gap of our protocol can be smaller than the original case. Therefore, our results improve the original stability theorem as well as generalize it.

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