LOGARITHMIC FORM OF LAGRANGE INVERSION FORMULA

A.S. DZHUMADIL’DAEV

Abstract. We give presentation of composition inverse of formal power serie in a logarithmic form.

1. Introduction

For a formal power serie \( f(x) = a_1 x + a_2 x^2 + a_3 x^3 + \cdots \in \mathbb{C}[[x]] \) its inverse under composition can be given by Lagrange inversion formula

\[
 f^{(-1)}(x) = \sum_{m \geq 1} \frac{d^{m-1}}{dx^{m-1}} \left( \frac{x}{f(x)} \right) \bigg|_{x=0} \frac{x^m}{m!}
\]

(see [1]). The aim of our paper is to prove the following logarithmic version of Lagrange inversion formula

**Theorem 1.**

\[
 f^{(-1)}(x) = \ln \sum_{m \geq 0} \left( \frac{1}{f'(x)} \frac{d}{dx} \right)^m (e^x) \bigg|_{x=0} \frac{x^m}{m!}.
\]

To prove this result we introduce three kinds of multiplications on differential operators, composition \( \odot \), white multiplication \( \circ \) and black multiplication \( \bullet \). They are partially associative and partially left-symmetric (Proposition 2). We give presentation of compositions of differential operators of first order in terms of these multiplications (Theorem 4). It allows us to use black multiplication Bell polynomials to construct powers of differential operators of first order.

2. Three kinds of multiplications on differential operators

We consider differential operators of a form \( \sum_{i=1}^n u_i \partial_i \), where \( u_i = u_i(x_1, \ldots, x_n) \) and \( \partial_i = \frac{d}{dx_i} \) are partial derivations. Sometimes differential operators of first order are called vector fields.

Let \( Z_0 \) be set of non-negative integers, and

\[
 Z^n_0 = \{ \alpha = (\alpha_1, \ldots, \alpha_n) | \alpha_i \in Z_0 \},
\]

\[
 x^\alpha = \prod_{i=1}^n x_i^{\alpha_i}, \quad \partial^\alpha = \prod_{i=1}^n \partial_i.
\]

Denote by \( Diff_n \) space of differential operators with \( n \) variables. Say that a differential operator \( X = \sum_{\alpha \in Z} u_\alpha \partial^\alpha \) has (differential) order \( k \) if \( u_\alpha = 0 \) as soon as \( |\alpha| = \sum \alpha_i \neq k \). Denote by \( Diff_{n,k} \) space of differential operators of order \( k \).

In case \( k = 1 \) we use special notation \( Vect(n) = Diff_{n,1} \).

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In our paper we study compositions of differential operators of first order with \( n \) variables. We introduce three kinds of multiplications on differential operators. The first one is a composition, \( \diamond \)-multiplication, defined by on generators
\[
\sum_{\gamma \in \mathbb{Z}^n_0} \left( \frac{\alpha}{\gamma} \right) u^{\alpha} v^{\beta - \gamma}.
\]
The second one is \( \circ \)-multiplication. It is defined by
\[
\sum_{\alpha} u^{\alpha} v^{\beta} = u^{\alpha} v^{\beta}.
\]
The third one is \( \bullet \)-multiplication. It is defined by
\[
\sum_{\alpha} u^{\alpha} v^{\beta} = uv^{\alpha + \beta}.
\]
Sometimes we will call \( \circ \) and \( \bullet \) as white and black multiplications.

**Proposition 2.** The multiplication \( \circ \) is associative. For any \( X, Y, Z \in \text{Diff}_n \),
\[
X \circ (Y \circ Z) = (X \circ Y) \circ Z,
\]
The multiplication \( \bullet \) is associative and commutative. For any \( X, Y, Z \in \text{Diff}_n \),
\[
X \bullet (Y \bullet Z) = (X \bullet Y) \bullet Z,
\]
\[
X \bullet Y = Y \bullet X.
\]
For any \( X \in \text{Vect}(n) \) and \( Y, Z \in \text{Diff}_n \),
\[
(X, Y, Z) \circ = (X \bullet Y) \circ Z,
\]
\[
(X, Y, Z) \circ = X \circ (Y \bullet Z) = (X \bullet Y) \circ Z + Y \bullet (X \circ Z),
\]
where \( (X, Y, Z) \circ = X \circ (Y \circ Z) - (X \circ Y) \circ Z \) is associator for the multiplication \( \circ \).
For any \( X, Y \in \text{Vect}(n) \), \( Z \in \text{Diff}_n \),
\[
(X, Y, Z) \circ = (Y, X, Z) \circ.
\]

**Proof.** Associativity of the multiplication \( \circ \) is well known. Associativity and commutativity of the black multiplication \( \bullet \) is evident.

Let
\[
X = \sum_i u_i \partial_i,\ Y = \sum_{\alpha} v_{\alpha} \partial^{\alpha},\ Z = \sum_{\beta} w_{\beta} \partial^{\beta}.
\]
Then
\[
X \circ (Y \circ Z) =
\sum_{i,\alpha,\beta} u_i \partial_i (v_{\alpha} \partial^{\alpha} (w_{\beta})) \partial^{\beta} =
\sum_{i,\alpha,\beta} u_i \partial_i (v_{\alpha} \partial^{\alpha} (w_{\beta})) \partial^{\beta} + u_i v_{\alpha} \partial^{\alpha} (w_{\beta}) \partial^{\beta} =
\sum_{i,\alpha,\beta} u_i \partial_i (v_{\alpha} \partial^{\alpha} (w_{\beta})) \partial^{\beta}.
\]
Therefore,
\[
(X, Y, Z) \circ = \sum_{i,\alpha,\beta} u_i v_{\alpha} \partial^{\alpha} (w_{\beta}) \partial^{\beta} = (L \bullet M) \circ R.
\]
Another application of this fact. If \( X,Y \in \text{Vect}(n) \), by commutativity of the multiplication \( \bullet \),
\[
(X,Y,Z) = (X \circ Y) \circ Z = (Y \circ X) \circ Z = (X,Y,Z). 
\]

**Corollary 3.** If \( X \) be differential operator of first order, then
\[
X \circ (Y \circ Z) = (X \circ Y) \circ Z. 
\]

**Proof.** Note that for differential operator of first order \( X \) the composition \( X \circ Y \) can be presented as
\[
X \circ Y = X \circ Y + X \bullet Y. 
\]
Therefore by Proposition 2
\[
X \circ (Y \circ Z) = (X \circ Y) \circ Z + (X \bullet Y) \circ Z = (X \circ Y) \circ Z. 
\]

3. **Composition of differential operators**

Let \( L_k = \sum_{i=1}^m u_{k,j} \partial_j \), \( k = 1,2,\ldots,m \), be differential operators of first order. Let \( [m] = \{1,2,\ldots,m\} \) and \( A \) is subset of \( [m] \). Suppose that \( A = \{i_1,\ldots,i_s\} \), where \( i_1 < i_2 < \cdots < i_s \). Let us denote \( i_1 = \min(A) \) as \( h(A) \) and set \( b(A) = \{i_2,\ldots,i_s\} \). Denote by \( L_A^\circ \) composition of differential operators
\[
L_A^\circ = L_{i_1} \circ \cdots \circ L_{i_s}. 
\]
Let
\[
L_A = (L_{i_1} \circ \cdots \circ L_{i_s}) \circ L_{i_1}. 
\]
In other words,
\[
L_A = L_{b(A)}^\circ \circ L_{h(A)}. 
\]
Recall that system of non-empty subsets \( \pi = \{A_1,\ldots,A_k\} \) is called partition of \( [m] \), if \( [m] = A_1 \cup \cdots \cup A_k \), and \( A_i \cap A_j = \emptyset \), for \( i \neq j \). Denote by \( \Pi(m) \) set of partitions of \( [m] \). For a partition \( \pi = \{A_1,\ldots,A_k\} \in \Pi(m) \) we set
\[
L_\pi^\bullet = L_{A_1} \bullet \cdots \bullet L_{A_k}. 
\]

**Theorem 4.** If \( L_1,\ldots,L_m \) are differential operators of first order, then
\[
L_m \circ \cdots \circ L_1 = \sum_{\pi \in \Pi(m)} L_\pi^\bullet. 
\]

**Proof.** We use induction on \( m \). For \( m = 1 \) nothing is to prove.

Let \( A \) be some subset of \( [m] \) and \( A = \{i_1,\ldots,i_s\} \), such that \( i_1 < \cdots < i_s \). Let us join element \( m+1 \) to \( A \) and denote obtained set \( A' \). Then \( A' \subseteq [m+1] \).

Let us prove that
\[
(1)
\]
\[
L_{m+1} \circ L_A = L_{A'} 
\]
Note that \( b(A') = \{i_2,\ldots,i_s,m+1\} = (m+1) \cup b(A) \) and \( h(A') = i_1 = h(A) \).

Therefore, by Corollary 3
\[
L_{m+1} \circ L_A = L_{m+1} \circ (L_{b(A)}^\circ \circ L_{h(A)}) = (L_{m+1} \circ L_{b(A)}^\circ) \circ L_{h(A)} = L_{b(A')}^\circ \circ L_{h(A')} = L_{A'}. 
\]
So, (1) is established.

Note that partitions of \( [m+1] \) can be constructed by partitions of \( [m] \) in two ways: either \( [m+1] \) generates separate block or the element \( m+1 \) is joined to some block of a partition of \( [m] \). For a partition \( \pi \in \Pi(m+1) \) say that \( \pi \) has type \( t \) if the block that contains \( m+1 \) has \( t \) elements. Denote by \( \Pi(m+1)(t) \)
set of partitions of \([m + 1]\) of type 1 any by \(\Pi(m + 1)^{(>1)}\) set of partitions of type \(t > 1\).

Suppose that for \(m\) our statement is true,

\[
L_m \diamond \cdots \diamond L_1 = \sum_{\pi \in \Pi(m)} L_\pi.
\]

Let \(\pi\) be some partition of \([m]\) and \(\pi = A_1 \cup \cdots \cup A_k\). By (1) for any \(r = 1, \ldots, k\),

\[
L_{m+1} \circ L_{A_r} = L_{A'_r}.
\]

Therefore, by Proposition 2

\[
L_{m+1} \circ L_\pi = \sum_{r=1}^{k} L_{A_1} \circ \cdots \circ L_{A_{r-1}} \circ L_{A'_r} \circ L_{A_{r+1}} \circ \cdots \circ L_{A_k} = \sum_{r=1}^{k} L_{\pi'(r)},
\]

where \(\pi'(r)\) is a partition of \([m + 1]\) constructed by partition \(\pi = \{A_1, \ldots, A_k\} \in \Pi(m)\), by the rule:

\[
\pi'(r) = B_1 \cup \cdots \cup B_k,
\]

\[
B_1 = A_1, \ldots B_{r-1} = A_{r-1}, B_r = \{m + 1\} \cup A_r, B_{r+1} = A_{r+1}, \ldots, B_k = A_k.
\]

Therefore

\[
(2) \quad \sum_{\pi \in \Pi(m)} L_{m+1} \circ L_\pi = \sum_{B \in \Pi(m+1)^{(>1)}} L_B
\]

Note that

\[
(3) \quad \sum_{B \in \Pi(m+1)^{(1)}} L_B = \sum_{\pi \in \Pi(m)} L_{m+1} \circ L_\pi
\]

By Proposition 2

\[
L_{m+1} \circ (L_m \circ \cdots \circ L_1) = \sum_{\pi \in \Pi(m)} L_{m+1} \circ L_\pi + L_{m+1} \circ L_\pi.
\]

Hence, by (2) and (3)

\[
L_{m+1} \circ (L_m \circ \cdots \circ L_1) = \sum_{B \in \Pi(m+1)} L_B.
\]

So, inductive step is possible. Our Theorem is proved.

**Example.** There are 5 partitions of \([3]\) = \{1, 2, 3\},

\[
\pi_1 = 1 - 2 - 3, \quad L_{\pi_1} = L_3 \circ L_2 \circ L_1,
\]

\[
\pi_2 = 12 - 3, \quad L_{\pi_2} = L_3 \circ (L_2 \circ L_1),
\]

\[
\pi_3 = 13 - 2, \quad L_{\pi_3} = L_2 \circ (L_3 \circ L_1),
\]

\[
\pi_4 = 1 - 23, \quad L_{\pi_4} = (L_3 \circ L_2) \circ L_1,
\]

\[
\pi_5 = 123, \quad L_{\pi_5} = (L_3 \circ L_2) \circ L_1.
\]

Therefore,

\[
L_3 \circ L_2 \circ L_1 = L_3 \circ L_2 \circ L_1 + L_3 \circ (L_2 \circ L_1) + L_2 \circ (L_3 \circ L_1) + (L_3 \circ L_2) \circ L_1 + (L_3 \circ L_2) \circ L_1.
\]
4. Power of vector fields in terms of Bell polynomials

If otherwise is not stated below we use notation $L^m$ for a power of differential operator of first order under composition, $L^m = L \circ \cdots \circ L$. As far as powers of degree $m$ under multiplication $\circ$ and $\bullet$ we use the following notations $L^m = (L \circ L \cdots) \circ L$ and $L^m = L \bullet \cdots \bullet L$.

Recall that a sequence of non-decreasing integers $\lambda = (\lambda_1, \ldots, \lambda_k)$ is called partition of $m$ and denoted $\lambda \vdash m$, if $\lambda_1 + \cdots + \lambda_k = m$. If $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k$, we say that $k = \text{length}(\lambda)$ is length of partition $\lambda$. Suppose that among components of $\lambda$ there are $l_1$ elements 1, $l_2$ elements equal to 2, etc. There is another way to define partitions. A sequence of non-negative integers $(l_1, \ldots, l_m)$ generates partition of $m$, if $1l_1 + 2l_2 + \cdots + ml_m = m$. We call $1l_12l_2 \cdots m_l_m$ multiplicity form or $m$-form of partition $\lambda$.

We recall definition of Bell polynomial $Y_m(x_1, \ldots, x_m)$. For $\lambda = 1l_1 \cdots mL_m \vdash m$, set

$$k_{l_1, \ldots, l_m} = \frac{m!}{l_1!l_2!\cdots l_m!}.$$  

Then

$$Y_m(x_1, \ldots, x_m) = \sum_{\lambda \vdash m} k_{l_1, \ldots, l_m} x_1^{l_1} \cdots x_m^{l_m}.$$  

Bell polynomials are defined over associative commutative algebra with generators $x_1, x_2, \ldots$. In particular we can construct Bell polynomials on differential operators algebra under black multiplication. Instead of $x_i$ we can consider a differential operator of first order $L^{-i} \circ L$. Let us denote $Y_m^\bullet(L, L \circ L, \ldots, L^{m-1} \circ L)$ Bell polynomial $Y_m(x_1, \ldots, x_m)|_{x_i \rightarrow L^{-i} \circ L}$, where by associative commutative multiplication we understand the multiplication $\bullet$. For example,$$Y_3(x_1, x_2, x_3) = x_1^3 + 3x_1x_2 + x_3,$$
and,$$Y_3^\bullet(L, L \circ L, L^2 \circ L) = L^3 + 3L \bullet L^2 + L^2 \circ L.$$

**Theorem 5.** For any differential operator of first order $L$, and for any non-negative integer $m$,

$$L^m = Y_m^\bullet(L, L \circ L, \ldots, L^{m-1} \circ L).$$  

**Proof.** It is known that number of partitions of $[m]$ as union of $l_i$ subsets with $i$ elements, $i = 1, \ldots, m$, is equal to $k_{l_1, \ldots, l_m}$. Therefore, by Theorem 4

$$L^m = \sum_{1^{l_1} \cdots m^{l_m} \vdash m} k_{l_1, \ldots, l_m} (L^{-1} \circ L) \cdots (L^{-1} \circ L) \cdots (L^{m-1} \circ L) \cdots (L^{m-1} \circ L).$$  

Theorem is proved.

**Example.** $L^4 = L^4 + 6 L^2 \bullet L^2 + 4 L \bullet (L^2 \circ L) + 3 (L^2 \circ L) \bullet + L^3 \circ L$.

Bell polynomials $Y_m(x_1, \ldots, x_m)$ have many interesting properties. Theorem 5 allows us re-write these properties in terms of powers vector fields. Let us give some of such results.

Exponential generating function for Bell polynomials is

$$1 + \sum_{m \geq 1} Y_m(x_1, \ldots, x_m) \frac{z^m}{m!} = \exp \sum_{m \geq 1} x_m \frac{z^m}{m!}. $$
Consider this identity on algebra of differential operators under multiplication \( \cdot \) and take \( x_i = L^{i-1} \circ L \). We obtain the following result.

**Theorem 6.** For any differential operator of first order \( L \),

\[
\sum_{m \geq 0} \frac{L^m z^m}{m!} = \exp \sum_{m \geq 1} \frac{L^{m-1} \circ L z^m}{m!}
\]

In other words,

\[
\exp L z = \sum_{k \geq 0} \frac{\left(\sum_{m \geq 1} \frac{L^{m-1} \circ L z^m/m!}{m!}\right)^k}{k!}.
\]

Another formulation of Theorem 6

\[
\sum_{m \geq 1} \frac{L^{m-1} \circ L z^m}{m!} = \ln \left(1 + \sum_{m \geq 1} \frac{L^m z^m}{m!}\right).
\]

**Example.** Let \( L = x \partial \). Then \( L^{i-1} \circ L = L \), for any \( i \geq 1 \). Further, \( L^k = x^k \partial^k \). Since Touchard polynomial \( \sum_{k=0}^m S(m, k) x^k \) can be expressed as the value of Bell polynomial on all arguments being \( x \),

\[
\sum_{k=0}^m S(m, k) x^k = Y_m(x, x, \ldots, x),
\]

by Theorem 5 we obtain that

\[
(x \partial)^m = L^m = Y_m(L, L, \ldots, L) = \sum_{k=0}^m S(m, k) L^k = \sum_{k=0}^m S(m, k) x^k \partial^k.
\]

where \( S(m, k) \) are Stirling numbers of second kind. Since

\[
\sum_{m \geq 1} \frac{L^{m-1} \circ L z^m}{m!} = \sum_{m \geq 1} \frac{L z^m}{m!} = L \sum_{m \geq 1} \frac{z^m}{m!} = L(e^z - 1),
\]

by Theorem 6 we have also

\[
\exp x \partial z = \exp \sum_{i \geq 0} \frac{(x \partial)^i (e^z - 1)^i}{i!} = \sum_{i \geq 0} \frac{x^i \partial^i (e^z - 1)^i}{i!}.
\]

## 5. Logarithmic form for Lagrange inversion formula

Let \( \mathcal{C}[[x]] \) be algebra of formal power series, \( \mathcal{C}_1[[x]] \) be its subspace generated by formal power series of a form \( f(x) = a_1 x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \cdots \), where \( a_1 \neq 0 \). Then \( \mathcal{C}_1[[x]] \) forms a group under composition \( f \circ g(x) = f(g(x)) \). Unit is the identity function \( x \). Let \( f^{(-1)}(x) \) be inverse under composition for \( f(x) \in \mathcal{C}_1[[x]] \), \( f \circ f^{(-1)}(x) = x \). Let \( f^{(-1)}(x) = b_1 x + b_2 \frac{x^2}{2!} + b_3 \frac{x^3}{3!} + \cdots \), where \( b_1, b_2, b_3, \ldots \in \mathcal{C} \). Lagrange inversion formula \([1]\) states that

\[
b_n = \frac{d^{n-1}}{dx^{n-1}} \left( \frac{x}{f(x)} \right)^n \bigg|_{x=0}, \quad n \geq 1.
\]

Let us construct another form for Lagrange inversion formula.
Lemma 7. For any $n \geq 1$,
\[
  b_n = \left( \frac{1}{f'(x)} \frac{d}{dx} \right)^{n-1} \left( \frac{1}{f'(x)} \right) \bigg|_{x=0}.
\]

Proof. Let $g(x) = f^{-1}(x)$. Let us prove that
\[
  g^{(n)}(f(x)) = \left( \frac{1}{f'(x)} \frac{d}{dx} \right)^{n-1} \left( \frac{1}{f'(x)} \right).
\]
By chain rule
\[
g\left( f(x) \right) = x \Rightarrow g'(x)f'(x) = 1 \Rightarrow g'(f(x)) = \frac{1}{f'(x)}.
\]
So, (4) is true for $n = 1$. Suppose that (4) is valid for $n - 1 > 0$. By chain rule
\[
  (g^{(n-1)}(f(x)))' = g^{(n)}(f(x))f'(x).
\]
Therefore,
\[
g^{(n-1)}(f(x)) = \left( \frac{1}{f'(x)} \frac{d}{dx} \right)^{n-2} \left( \frac{1}{f'(x)} \right) \Rightarrow
\]
\[
g^{(n)}(f(x)) = \left( \frac{1}{f'(x)} \frac{d}{dx} \right)^{n-1} \left( \frac{1}{f'(x)} \right).
\]
So, induction by $n$ is valid, and (4) is established.

Since $f(0) = 0$, by (4)
\[
g^{(n)}(0) = \left( \frac{1}{f'(x)} \frac{d}{dx} \right)^{n-1} \left( \frac{1}{f'(x)} \right) \bigg|_{x=0}.
\]

Lemma 7 is proved.

Proof of Theorem 1. Follows from Theorem 6 and Lemma 7.

Example. Let $f(x) = xe^{-x}$. Then ([1], Example 5.4.4)
\[
f^{-1}(x) = \sum_{m \geq 1} m^{m-1} \frac{x^m}{m!}.
\]
Note that $f'(x) = e^{-x}(1 - x)$ and
\[
  \left( \frac{e^x}{1 - x} \frac{d}{dx} \right)^m (e^x) \bigg|_{x=0} = (m + 1)^{m-1}.
\]
Therefore,
\[
f^{-1}(x) = \ln \sum_{m \geq 0} (m + 1)^{m-1} \frac{x^m}{m!}.
\]

References
[1] R.Stanley, Enumerative combinatorics, vol.2, Cambridge Univ.Press, 2001.