A model for ripple instabilities in granular media

Orestis Terzidis, Philippe Claudin and Jean-Philippe Bouchaud
Service de Physique de l’Etat Condensé, C.E. Saclay
Orme des Merisiers, 91191 Gif-sur-Yvette Cedex, France

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Abstract

We extend the model of surface granular flow proposed in [1] to account for the effect of an external ‘wind’, which acts as to dislodge particles from the static bed, such that a stationary state of flowing grains is reached. We discuss in detail how this mechanism can be described in a phenomenological way, and show that a flat bed is linearly unstable against ripple formation in a certain region of parameter space. We focus in particular on the (realistic) case where the migration velocity of the instability is much smaller than the grains’ velocity. In this limit, the full dispersion relation can be established. We find that the critical wave vector is of the order of the saltation length. We provide an intuitive interpretation of the instability.

1 Motivation

Common observations suggest that flat sand surfaces can become unstable when subjected to moving air or water. After some time regular patterns appear, as can be observed on desert dunes, underwater sand, ‘dry’ snow, etc. These patterns resemble surface waves; however their physics is completely different since in the case of sand there is no surface tension. Following Bagnold ([2], chap. 11) these patterns can be classified into ripples, ridges and dunes. The repetition distance of ridges varies with time, whereas ripples exhibit a stationary wavelength after some transient. Early qualitative arguments by Bagnold [2] suggested that the ripple wavelength \( \lambda \) is related to the typical path length of the blown grains, called the ‘saltation length’ \( \xi \). A more quantitative ‘two-species’ model was proposed by Anderson [3], which describes the coupling between the moving grains and the static bed.
Such a model predicts that a flat surface is unstable for all wavelengths, with a faster growing mode indeed comparable to the typical jump length of the grains. However, this model is incomplete: while the dynamics of the static bed is treated exactly, the description of the moving phase is highly simplified. Alternatively, there are also several numerical models for ripple formation. In this paper, we extend Anderson’s theoretical model of ripple formation, by adapting the phenomenological equations for surface flow introduced in [1] in the context of avalanches, and further discussed in [4, 5, 6, 7, 8].

It is worth recalling, after Bagnold [2], some basic facts about the motion of the grains and the formation of these patterns: (i) There are two qualitatively distinct transport mechanisms for the grains, saltation and surface creep. The trajectories of grains in saltation is determined by the velocity profile of the wind, the air friction limiting the grain velocity and by the initial energy of the grain when first expelled from the sand bed. One of the characteristic features of the path is the flat angle of incidence which varies between 10° and 15°. (ii) The saltation has two effects on the surface: it either rebounds and/or ejects grains leading to a new saltation or it produces surface creep. There is however no sharp boundary between these two processes, since the energy of the ejected grains varies continuously. Both saltation and creep lead to a net flow of grains in the direction of the wind. (iii) The time scale of ripple formation is much larger than that of saltation. (iv) The migration velocity of the ripples is much smaller than a mean transport velocity (averaged over saltation and creep).

2 A ‘two-species’ model with wind

The phenomenological approach we consider in the following is based on the observation that two different species of grains enter the problem: moving grains and grains at rest. We will not distinguish between grains in saltation and creep, but introduce an appropriately averaged quantity describing grains that are convected by either of the two mechanisms, which we call the moving grain density \( R(x, t) \), where \( x \) is the coordinate in the direction of the wind and \( t \) the time. (We will assume that the problem is

\[ R(x, t) = \frac{1}{v} \int R(x, v, t) dv \]

\[ 1 \text{To which one should also add ‘suspension’, corresponding to very small grains flying high in the air.} \]

\[ 2 \text{In principle, one should consider a density } R(x, v, t) \text{ which depends on the instantaneous velocity of the grains. } R(x, t) \text{ is the average of } R(x, v, t) \text{ over all velocities.} \]
translationally invariant in the direction transverse to the wind; see [8] for an extension to two dimensions. The grains at rest contribute to the local height $h(x, t)$ of the static bed. The dynamical equations for $R$ and $h$ read, in the hydrodynamical (long wavelength) limit:

$$\begin{align*}
\partial_t R &= -V \partial_x R + D_1 \partial_x^2 R + \Gamma[R, h] \\
\partial_t h &= -\Gamma[R, h]
\end{align*}$$

(1)

where $V$ and $D_1$ are the average velocity of the grains and the dispersion constant, related to the fact that grains do not all move with the same velocity $V$. $\Gamma$ describes the rate with which a grain at rest is converted into a moving grain (or vice versa) and depends both on $R$ and on the local surface profile. For simplicity we have defined $R$ to have the same dimension as $h$, and it can be thought of as the width of grains which has been removed from the static bed. Correspondingly, $\Gamma$ has the dimension of a velocity. The construction of $\Gamma$ is based on phenomenological arguments [1], and encodes different physical processes:

- Due to the presence of wind, grains can be ‘spontaneously’ ejected from the surface, even in the absence of already moving grains. The rate at which this occurs depends on the local wind velocity (or rather velocity gradient at the surface). Since the wind velocity tends to be larger when the local slope is facing the wind, we write:

$$\Gamma_{sp} = \alpha_0 + \alpha_1 \partial_x h - \alpha_2 \partial_x^2 h$$

(2)

where the coefficients $\alpha$ are positive or zero. We have also included the second derivative contribution with a minus sign, since grains are harder to dislodge in troughs than at the top of a crest. Note that all these coefficients are expected to depend on the external wind velocity. In particular, as shown by Bagnold himself, the coefficient $\alpha_0$ is only non-zero above a certain critical wind velocity, noted $V_{fluid}^\ast$.

- When hitting the ground, a moving grain can either be captured or transfer a part of its kinetic energy to other static grains and provide new moving particles. The rate at which both these process occur is proportional to $R$ (at least for small enough $R$ – see below), and also depends on the wind velocity and on the local slope. For example, flying grains have a larger probability to land on a surface facing the wind, rather than in the wind.

These terms can be understood, more generally, as the long-wavelength limit of a more general non-local convection term of the kind $\int K(x - x') R(x', t) \, dx'$. 
shadow. This suggests to write the *stimulated* conversion rate as:

\[ \Gamma_{st} = -R[\gamma_0 + \gamma_1 \partial_x h + \gamma_2 \partial_x^2 h] \]  

(3)

The sign of \( \gamma_0 \) depends on the strength of the wind; for small wind velocity, one expects capture to be more important than emission, and thus that \( \gamma_0 > 0 \). As again shown by Bagnold, a localized source of moving grains tends to die away when the wind velocity is less than a certain \( V_{\text{impact}}^* < V_{\text{fluid}}^* \), whereas a steady saltation is found for larger velocities, suggesting that \( \gamma_0 < 0 \) for \( V > V_{\text{impact}}^* \). In this case, however, it is easy to see that \( R \) increases exponentially, and that higher order terms are needed to describe the stationary situation. One can think of several non-linear effects: for example, collision between flying grains leads to dissipation and hence to a poorer efficiency of the impacts on the static bed. Also, the presence of a layer of moving grains screens the hydrodynamical flow, which in turn reduces the energy transfer between the wind and the saltating grains. These effects can be described by a term proportionnal to \( -\beta R^2 \) in \( \Gamma_{st} \).

If trapping dominates (as is the case for underwater ripples) one expects \( \gamma_1 > 0 \) because more grains fall on the slope facing the convective flow. For the same reason, if stimulated emission dominates, as is the case for wind blown sand, one expects that \( \gamma_1 < 0 \). Finally, \( \gamma_2 \) is positive since, again, grains are easier to dislodge at the top of a bump.

The total conversion rate \( \Gamma \) is obtained as the sum of \( \Gamma_{sp} \) and \( \Gamma_{st} \), while the model proposed in \[1\] did not contain the wind induced contribution proportionnal to \( \alpha \), nor the non-linear term. The equation for \( h \) thus reads:

\[ \partial_t h = (R\gamma_0 - \alpha_0) + \beta R^2 + (R\gamma_1 - \alpha_1) \partial_x h + (R\gamma_2 + \alpha_2) \partial_x^2 h \]  

(4)

The gradient term can be interpreted as a translation of the surface profile with time, at velocity \( \alpha_1 - R\gamma_1 \). The direct action of the wind (\( \alpha_1 \)) is indeed to erode grains from the windward slope of a bump and transport them in the direction of the wind. The other contribution (\( R\gamma_1 \)), however, moves the bumps ‘backwards’ since grains are effectively deposited on the windward slope, contributing to a translation of the bump against the wind. (A similar discussion can be found in \[2\], \[3\].)

Note that the above set of equations is non-linear, so that non-trivial dynamics is expected. Some essential features of the model can be investigated by linearizing the system in the vicinity of the situation where the

\[ \text{In principle, the dependence of } V \text{ on } R \text{ should also be taken into account. We do not consider this here, since this does not affect the linear instability analysis.} \]
Figure 1: Stability diagram, showing $\frac{dR}{dt}$ as a function of $R$ in an homogeneous situation. The case $\gamma_0 < 0$ corresponds to blown sand with $V > V^{*}_{\text{impact}}$, where stimulated emission is very efficient, and where $R_0$ can be non-zero even if $\alpha_0 = 0$ (i.e. when $V < V^{*}_{\text{fluid}}$). The situation where capture dominates ($\gamma_0 > 0$) is probably relevant for sand under water.

surface is flat ($h_0 = 0$). The moving grain density is then equal to (see Fig 1):

$$R_0 = \frac{1}{2\beta} \left[ -\gamma_0 + \sqrt{\gamma_0^2 + 4\alpha_0\beta} \right].$$

(5)

3 Stability analysis

We will perform a stability analysis, i.e. investigate whether a small perturbation is amplified or dies out with time. Therefore we consider $R = R_0 + \bar{R}$, $h = h_0 + \bar{h}$ and neglect second order terms of the kind $\bar{R}\bar{h}$, $\bar{R}^2$ and $\bar{h}^2$. For simplicity of notation we drop the bars; the linearized equations then read

$$\partial_t R = -\tilde{\gamma}_0 R - V \partial_x R + D_1 \partial_x^2 R + W \partial_x h - D_2 \partial_x^2 h + ...$$

$$\partial_t h = \tilde{\gamma}_0 R - W \partial_x h + D_2 \partial_x^2 h + ...$$

(6)
with an effective velocity $W = \alpha_1 - R_0 \gamma_1$ and an effective diffusion constant $D_2 = R_0 \gamma_2 + \alpha_2 > 0$. $\tilde{\gamma}_0$ is equal to $\gamma_0 + 2 \beta R_0$ and is thus always positive. A Fourier analysis of the linearized equations leads to

$$\begin{pmatrix} -i \omega - \tilde{\gamma}_0 - ikV - k^2 D_1 & ikW + k^2 D_2 \\ \tilde{\gamma}_0 & -i \omega - ikW - k^2 D_2 \end{pmatrix} \begin{pmatrix} \tilde{R} \\ \tilde{h} \end{pmatrix} = 0 \quad (7)$$

where the tilde denotes the Fourier transforms. This system has a non-trivial solution if the determinant of the above matrix is zero, leading to the relation

$$\omega^2 + \omega(a + ib) + (c + id) = 0. \quad (8)$$

The coefficients read

$$a = (V + W) k$$
$$b = -[\tilde{\gamma}_0 + (D_1 + D_2)k^2]$$
$$c = VWk^2 - D_1 D_2 k^4$$
$$d = -(D_1 W + D_2 V) k^3; \quad (9)$$

they are functions of the wave vector $k$ and of the system’s parameters ($V$, $W$, $D_1$, $D_2$, $\tilde{\gamma}_0$).

Equation (8) establishes a dispersion relation $\omega(k)$ with two branches corresponding to the two solutions of the quadratic equation, where $\omega$ has to be considered as a complex variable. (Writing down the corresponding equations for the real and the imaginary part of $\omega$ leads to quartic equations.) In the context of a stability analysis we are interested in the imaginary part of $\omega(k)$: as long as it is positive $e^{i \omega t}$ will decay exponentially, while a negative imaginary part does lead to an instability. This imaginary part is given by:

$$2 \text{ Im}(\omega) = -b \pm \frac{1}{\sqrt{2}} \left[-(a^2 - b^2 - 4c) + \left[(a^2 - b^2 - 4c)^2 + (2ab - 4d)^2\right]^{1/2}\right]^{1/2} \quad (10)$$

which is a function of $k$. A critical wave vector $k^*$ can be defined such that $\text{Im}(\omega)$ is exactly zero, which leads to $d^2 - abd + b^2 c = 0$. Inserting the explicit expressions (9), one finds a cubic equation for $k^{*2}$. Whenever this equation admits a positive solution, there will be a finite band of wave vectors $[0, k^*]$ which are unstable (see Figure 2).
It is instructive to study the asymptotic behaviour of the functions $\text{Im}(\omega_-)$. One finds:

$$\text{Im}(\omega_-) = \begin{cases} -\frac{m^2}{\eta_0} k^2 + \ldots & \text{for } k \ll \tilde{\gamma}_0/V \\ D_2 k^2 + \ldots & \text{for } k \gg \tilde{\gamma}_0/V \end{cases}$$

(11)

with $\eta = W/V$. The transport velocity $V$ (by convention) and the diffusion constants are positive; the main control parameter remaining is the relative migration velocity $\eta$. One sees that for $0 < \eta < 1$ there is indeed a band of instable wave vectors (one can see from the asymptotic solutions that the sign changes for large $k$’s). The situation where $\eta < 0$ is stable and $\eta > 1$ (i.e. a bump moving faster than the transport velocity) does not seem physical. The second branch $\text{Im}(\omega_+)$ is always positive and is thus of no importance for our stability considerations.

Following the intuition that the ripples move much more slowly than the grains are transported, we will assume in the sequel that $0 < \eta \ll 1$, which we attribute to the fact that the $\alpha$ coefficients are small compared to $V$.  

Figure 2: Rescaled damping rate as a function of the rescaled wave vector. The plot shows data for $\eta = 0.1, \tilde{\gamma}_0D_1 = V^2$ and $D_2/D_1 = 0.1$. 
Since $D_2 \propto \alpha$, this suggests that the diffusion constants $D_1$ and $D_2$ are in the same ratio, so we write: $D_2 = \delta \eta D_1$, where $\delta$ is of the order of one. These assumptions make it possible to simplify the algebra and to find the solution:

\[ \Im(\omega) = \eta \frac{k^2 \left[ -\tilde{\gamma}_0 V^2 + D_1 \delta (\tilde{\gamma}_0 D_1 + V^2) k^2 + \delta D_1^2 k^4 \right]}{\tilde{\gamma}_0^2 + k^2(2\tilde{\gamma}_0 D_1 + V^2) + k^4 D_1^2} + o(\eta^2). \tag{12} \]

which is plotted in Figure 2. As we discuss now, three relevant facts can be verified with this formula: (i) the critical wave vector is of the order of the inverse mean saltation length, (ii) the ripple velocity is of the order of $\eta V$ and (iii) the time scale of ripple formation is much larger than the saltation time scale.

Let us first give some arguments for (i). Since the saltation trajectories result from some random initial vertical velocity of the grains, the saltation lengths will also be random, with both short jumps (actually corresponding to creep) and long jumps. It is reasonable to assume that the width of the saltation length distribution is of the same order as its mean $\xi$ (a similar assumption is discussed in [3]). In this situation, the ‘Péclet’ number defined as $\text{Pe} = V \xi / D_1$ is of order one: convective and diffusive effects are of the same order of magnitude. In the case where the jump length distribution is sharply peaked around $\xi$, one would rather have $\text{Pe} \gg 1$.

Defining $\xi = V \tau$, where $\tau$ is the typical saltation time, one finds that the the zero of (12) is located at:

\[ k^* = \frac{\text{Pe}}{2 \xi^2} \left\{ \sqrt{(\tilde{\gamma}_0 \tau + \text{Pe})^2 + 4 \tilde{\gamma}_0 \tau / \delta - (\text{Pe} + \tilde{\gamma}_0 \tau)} \right\} \tag{13} \]

Since $\tilde{\gamma}_0$ is the (renormalized) rate of sticking, it is reasonable to assume that $\tilde{\gamma}_0 \tau \sim 1$, thereby confirming that $k^* \sim \xi^{-1}$ for $\text{Pe} \sim 1$. On the other hand, for weakly dissipative collisions (hard grains) one expects that $\tilde{\gamma}_0 \tau \ll 1$, leading to larger unstable wavelengths $\sim \xi \sqrt{\delta / \tilde{\gamma}_0 \tau}$.

The ripple velocity is given by the corresponding dispersion relation, i.e. the real part of $\omega(k)$. One finds:

\[ 2\Re(\omega) = \eta \frac{k^3 V [V^2 + \tilde{\gamma}_0 D_1 (1 + \delta) + D_1^3 k^2]}{\tilde{\gamma}_0^2 + k^2(2\tilde{\gamma}_0 D_1 + V^2) + k^4 D_1^2} + o(\eta^2). \tag{14} \]

The formula shows that for $k \sim k^*$, both phase and group velocities are indeed of the order of $\eta V$, establishing thus (ii).

Finally knowing the fastest growing wave vector, one finds that the ripple formation time $t_{\text{ripple}}$ (determined by the depth of the minimum in figure 2)
is a factor $1/\eta$ larger than say $1/\tilde{\gamma}_0$ or $\tau$, i.e. that ripple formation occurs on much slower time scales than any microscopic process. The ratio of formation time and microscopic time scales should indeed be roughly the same as that between migration and convection velocity (iii).

4 Physical discussion and open questions

Let us finally give an intuitive interpretation of the instability. Imagine a flat surface with a finite number of moving grains above it (i.e. the stationary solution). Now imagine a small perturbation of this situation, say a small hump. The term $\partial_t R \sim \partial_x h$ in the linearized equations (6) increases locally the concentration of the moving grains thus producing a ‘cloud’ at the windward side of the hump. This cloud is convected with the velocity $V$ and after a time unit of $1/\tilde{\gamma}_0$ the cloud has moved a distance $\xi$ where the cloud starts to ‘rain’ (i.e. moving grains are converted into grains at rest). If the position of the hump has in the same time moved in the same direction, its height will increase, leading to an instability. (Conversely, if the bump moves backward – i.e. if $W < 0$ – the ‘rain’ will rather fill the hole and smear out the bump). The presence of the diffusive processes counterbalances the amplification for small distances and some optimum wavelength of the order of $\xi$ (corresponding with the minimum in figure 2) becomes visible.

Summarizing, we have thus shown that equations (1), which are phenomenological, but motivated by clear physical processes, indeed show an instability which is consistent with some essential features of ripple formation. It is worth noting that our analysis, which concentrated on the linearized system in the vicinity of the stationary solution, is universal in the sense that a whole class of models behaves in an analogous way (with some possible redefinition of the coefficients). For example, a non-linear dependence of the velocity $V$ on $R$ does not modify the above analysis, up to a redefinition of $V$. Note also that all phenomenological coefficients are, at least in principle, measurable in situations independent from ripple formation (since they are diffusion constants, convection velocities, deposition rates etc.). In this sense it should be possible to check experimentally for the consistency of the above description.

Our conclusions are very similar to those reached by Anderson [3], on the basis of a simplified model where the flowing phase (what we have called $R$ above) is assumed to be in equilibrium from the outset, and where a rather arbitrary distinction is made between ‘saltating grains’ which are
never captured by the bed, and ‘reptating’ grains which are captured after exactly one jump. Correspondingly, the structure of the dispersion relations differ in the two approaches. Furthermore, it is difficult to extend Anderson’s model beyond the linear instability analysis while our model, in principle, can account for non-linear effects.

Finally, there are several open questions which we would like to mention and leave for future work: (i) Can one establish some precise relations between the ‘microscopic’ coefficients (like wind velocity, polydispersity, elasticity etc.) and the phenomenological parameters? (ii) How are the above results modified if one considers two spatial dimensions? Is there an instability corresponding to the transverse wavelike shape of the ripples known from field observation? (iii) What is the ripple shape and height predicted from a non-linear analysis of the equations? (iv) Is it important to consider a non-local convection term, rather than the hydrodynamical form written in (1)? The question arises since the relevant wavelength is precisely of the same order as (and not much larger than) the jump length $\xi$.

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References

[1] J.P. Bouchaud, M.E. Cates, R. Prakash, S.F. Edwards: J. Phys. France 4 (1994) 1383, Phys. Rev. Lett. 74, (1995) 1982

[2] R. A. Bagnold: The physics of blown sand and desert dunes (1941), Reprinted by Chapman and Hall (1981)

[3] R.S. Anderson: Sedimentology 34 (1987), 943-956, Earth Science Reviews 29 (1990), 77-96

[4] P.G. de Gennes, Comptes Rendus Académie des Sciences, 321 II (1995) 501, Lecture Notes, Varenna Summer School on Complex Systems, July 1996.

[5] T. Boutreux, P.G. de Gennes, J. Phys. I France, 6 (1996) 1295.
[6] H. A. Makse, S. Havlin, P. R. King, H. E. Stanley, Nature (London) 386 (1997) 379. H. A. Makse, P. Cizeau, H. E. Stanley, Phys. Rev. Lett. 78 (1997) 3298.

[7] J.P. Bouchaud, M. E. Cates, Proceedings of ‘Dry Granular Matter’, held in Cargese (Sept. 1997), to appear (Springer Verlag).

[8] J.P. Bouchaud, M. E. Cates, preprint cond-mat/9801132 submitted to ‘Granular Matter’.

[9] W. Landry, B.T. Werner: Physica D77 (1994), 238-260

[10] O. Terzidis, Ph. Claudin, F. Rioual, A. Valance and J.P. Bouchaud, in preparation.