Bootstrapping Persistent Betti Numbers
and Other Stabilizing Statistics

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Abstract: The present contribution investigates multivariate bootstrap procedures
for general stabilizing statistics, with specific application to topological data analysis.
Existing limit theorems for topological statistics prove difficult to use in practice for
the construction of confidence intervals, motivating the use of the bootstrap in this
capacity. However, the standard nonparametric bootstrap does not provide for asymp-
totically valid confidence intervals in some situations. The smoothed bootstrap, instead,
is shown to give consistent estimation where the standard bootstrap fails. The present
work relates to other general results in the area of stabilizing statistics, including cen-
tral limit theorems for functionals of Poisson and Binomial processes in the critical
regime. Specific statistics considered include the persistent Betti numbers of Čech and
Vietoris-Rips complexes over point sets in \( \mathbb{R}^d \), along with Euler characteristics, and
the total edge length of the \( k \)-nearest neighbor graph. We further define a new type of
\( B \)-bounded persistent homology, and investigate its fundamental properties. Specific
emphasis is made to weakening the necessary conditions needed to establish bootstrap
consistency. In particular, the assumption of a continuous underlying density is not
required. A simulation study is provided to assess the performance of the bootstrap
for finite sample sizes. Data application is made to a cosmic web dataset from the
Sloan Digital Sky Survey (SDSS).

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1. Introduction

In recent years, a multitude of topological statistics have been developed to describe and an-
alyze the structure of data, achieving notable success. These methods have seen application
in astrophysics [1, 35, 36, 37], cancer genomics [3, 20, 10], medical imaging [17], materials
science [26], fluid dynamics [27] and chemistry [45], and other wide ranging fields.

The use of simplicial complexes to summarize the geometric and topological properties
of data culminates in the techniques of persistent homology. Summary statistics based in
persistent homology, persistent Betti numbers, persistence diagrams, and derivatives thereof
effectively extract essential topological properties from point cloud data. A broad introduc-
tion to the methods of topological data analysis can be found in [44, 14].

While the use of such statistics has seen broad success, very little is currently known
about the statistical properties of these topological summaries. An initial attempt at sta-
tistical analysis using persistent homology can be seen in [9], with the later introduction
of persistence landscapes in [8]. Likewise, central limit theorems have been developed for
persistence landscapes [12], Betti numbers [47] and persistent Betti numbers [24, 28] under
a variety of asymptotic settings. However, the form of these results is insufficient to provide for valid confidence intervals.

In the construction of asymptotically valid confidence intervals, subsampling and bootstrap estimation have proven successful. In [22], various techniques are given for constructing confidence sets for persistence diagrams and derived statistics, including persistence diagrams generated from sublevel sets of the density function, as well as for the Čech and Vietoris-Rips complexes of data constrained to a manifold in $\mathbb{R}^d$. In [12, 13], bootstrap consistency is established very generally for persistence landscapes drawn from independently generated point clouds in $\mathbb{R}^d$, assuming that the number of independent samples is allowed to grow. Finally, [38] has recently presented a bootstrap method for producing valid confidence sets in the large-sample limit for persistence diagrams generated by the sublevel sets of the density function. There the object of interest is the persistence diagram itself, and an approximation to the density sublevel diagram is made using the Čech complex.

However, even with these recent developments, the available techniques for constructing confidence sets using topological statistics remains severely limited. The bootstrap has proven one of the only effective tools, and the theoretical properties of bootstrap estimation applied to topological statistics are as of yet largely unestablished, relying instead on simulation results for justification. It is the goal of this work to provide the foundational theory for the bootstrap in this area. In the present work, the validity of the bootstrap in the multivariate setting is established, a key step towards an eventual process-level result. While motivated primarily by application to topological data analysis, the results presented here apply much more generally over a class of stabilizing statistics. As a more general application, we show convergence for the bootstrap applied to the total edge length of the $k$-nearest neighbor graph.

We also analyze the large-sample asymptotic properties of the bootstrap applied to the Čech and Vietoris-Rips complexes directly, where the underlying point cloud is a sample drawn iid from a common distribution on $\mathbb{R}^d$. In particular, we will show that the standard nonparametric bootstrap can fail to provide asymptotically valid confidence intervals. Via a smoothed bootstrap, however, we will construct multivariate confidence intervals for the mean persistent Betti numbers, which lie in bijection with the corresponding persistence diagram.

As defined in [32], a statistic stabilizes if the change in the function value induced by addition of new points to the underlying sample is at most locally determined. Applications of stabilization and its derivatives have allowed for, as previously mentioned, the development of central limit theorems for several topological statistics. [47] show that Betti numbers exhibit the stabilization property, and provide a central limit theorem for Betti numbers derived from a homogenous Poisson process with unit intensity. [24] considers persistent Betti numbers in the homogenous Poisson process case with arbitrary intensity. Most recently [28] established multivariate central limit theorems for persistent Betti numbers with an underlying point cloud coming from either a nonhomogenous Poisson or binomial process. For the results in the present contribution, we draw significant inspiration from this most recent work.

An application of our general consistency result is made to the persistent Betti numbers of a class of distance-based simplicial complexes, including the Čech and Vietoris-Rips complexes. We also define a new type of $B$-bounded persistent homology for which concrete stabilization properties and convergence rates for the bootstrap can be derived. A similar concept is found in [4]. Throughout, a focus is given towards weakening the necessary assumptions compared to previous results. Specifically, the theorems presented here apply for
distributions with unbounded support, unbounded density, and possible discontinuities. We assume only a bound for the \( L_p \)-norm of the underlying sampling density.

In Section 2 we will introduce the general theory of stabilization, and establish intermediate technical results in this context. We then present our general bootstrap consistency result as an implication of the discussed stabilization properties. In Section 3 we provide several simulations demonstrating the finite-sample properties of the smoothed bootstrap applied to Betti numbers and persistent Betti numbers. Likewise, the failure of the nonparametric bootstrap is shown by example. In Section 4 we illustrate our method with an application to a cosmic web dataset. In Section 5 we connect the general theory to the specific case of persistent homology and related statistics. We give an introduction to simplicial complexes and persistent homology, culminating in the definition of \( B \)-bounded persistent homology. In Section 6, the stabilization properties of persistent Betti numbers and their \( B \)-bounded analogs are analyzed, along with the Euler characteristic for general classes of distance-based simplicial complexes. Finally, we establish bootstrap consistency in the large-sample limit for each of these statistics, as well as for the total edge length of the \( k \)-nearest neighbor graph.

2. Stabilizing Statistics

2.1. Stabilization

To prove bootstrap convergence, we will utilize the property of stabilization found in [32], [33], [47], and [28]. Here, we extend and rephrase existing definitions to provide a more general and consistent statistical framework. Let \( \mathcal{A}(\mathbb{R}^d) \) denote the space consisting of multisets drawn from \( \mathbb{R}^d \) with no accumulation points, with the further restriction that no point in a given multiset may be counted more than finitely often. Any locally-finite point process on \( \mathbb{R}^d \) can be represented as a random element of \( \mathcal{A}(\mathbb{R}^d) \). Let \( \tilde{\mathcal{A}}(\mathbb{R}^d) \subset \mathcal{A}(\mathbb{R}^d) \) contain the finite multisets drawn from \( \mathbb{R}^d \). Let \( \psi: \tilde{\mathcal{A}}(\mathbb{R}^d) \to \mathbb{R} \) be a measurable function. Furthermore, for \( S, T \in \tilde{\mathcal{A}}(\mathbb{R}^d) \), define the addition cost of \( T \) to \( S \) as \( D(S, T; \psi) := \psi(S \cup T) - \psi(S) \). When \( T = \{z\} \) consists of a single point, we call \( D_z(S; \psi) := D(S, \{z\}; \psi) \) an add-one cost or the add-\( z \) cost.

If the centerpoint is \( z \in \mathbb{R}^d \), and the additional multiset is the singleton \( T = \{z\} \), the notation \( D = D_z \), \( D^\infty = D^\infty_z \), and \( \rho = \rho_z \) may be used. To simplify notation, we will also omit the explicit dependence on \( \psi \) and \( T \) in other situations.

We say that \( \psi \) stabilizes if the addition cost of a given \( T \) varies only on a bounded region. In the preceding literature, the terms “strong” and “weak” stabilization are very often used, with precise definitions changing based on circumstance. Let \( B_z(r) \) denote the closed euclidean ball centered at \( z \in \mathbb{R}^d \) with radius \( r \). In the interest of providing more explanatory terminology, we propose the following definitions:

**Definition 2.1** (Terminal Addition Cost). \( D^\infty: \tilde{\mathcal{A}}(\mathbb{R}^d) \to \mathbb{R} \) is a **terminal addition cost** for \( \psi \) centered at \( z \in \mathbb{R}^d \) if \( D^\infty(S) = \lim_{l \to \infty} D(S \cap B_z(l)) \) for any \( S \in \tilde{\mathcal{A}}(\mathbb{R}^d) \) such that the limit exists.

In the case of a finite multiset \( S \in \tilde{\mathcal{A}}(\mathbb{R}^d) \), the terminal addition cost centered at \( z \in \mathbb{R}^d \) is \( D^\infty(S) = D(S) \), because no further changes to the addition cost may occur once \( S \cap B_z(a) \) contains all of \( S \). For an infinite multiset this is not the case, motivating a separate definition.
**Definition 2.2** (Stabilization in Probability). For $S$ a point process taking values in $\mathcal{X}(\mathbb{R}^d)$, $\psi$ stabilizes on $S$ in probability if there exists a center point $z \in \mathbb{R}^d$ and a terminal addition cost $D^\infty$ for $\psi$ such that

$$\lim_{l \to \infty} \mathbb{P}^* [D(S \cap B_z (l)) \neq D^\infty (S)] = 0. \quad (2.1)$$

Here $\mathbb{P}^*$ denotes the outer probability of a set. Stabilization is said to occur in probability because, for any sequence of non-negative radii $(l_i)_{i \in \mathbb{N}}$ such that $l_i \to \infty$, $D(S \cap B_z (l_i)) \not\in \mathbb{P}^* D^\infty (S)$ whenever both quantities are measurable. The choice of $D^\infty$ is unique up to a null set. Definition 2.2 is difficult to show directly for many functions of interest. As such, we have the following:

**Definition 2.3** (Radius of Stabilization). $\rho: \mathcal{X}(\mathbb{R}^d) \to [0, \infty]$ is a radius of stabilization for $\psi$ centered at $z \in \mathbb{R}^d$ if, for any $S \in \mathcal{X}(\mathbb{R}^d)$ and $\infty > l \geq \rho(S)$,

$$D(S \cap B_z (l)) = D(S \cap B_z (\rho(S))). \quad (2.2)$$

In the case where $\lim_{l \to \infty} D(S \cap B_z (l))$ does not exist, $\rho(S) = \infty$ necessarily with the stabilization criterion satisfied vacuously. We have

$$D(S \cap B_z (l)) = D(S \cap B_z (\rho(S))) = D^\infty (S) \quad (2.3)$$

for all $\rho(S) \leq l < \infty$. In general, for any $\psi$, there exists a unique minimal radius of stabilization, defined by taking the pointwise minimum over all radii sharing the same centerpoint. Such a minimum exists because $\psi(S \cap B_z (l))$ is piecewise constant in $0 \leq l < \infty$, changing value only when a new point of $S$ is added, and because $S$ is locally finite.

**Definition 2.4** (Stabilization Almost Surely). For $S$ a point process taking values in $\mathcal{X}(\mathbb{R}^d)$, $\psi$ stabilizes on $S$ almost surely if there exists a radius of stabilization $\rho: \mathcal{X}(\mathbb{R}^d) \to [0, \infty]$ for $\psi$ centered at $z \in \mathbb{R}^d$ such that

$$\lim_{L \to \infty} \mathbb{P}^* [\rho(S) > L] = 0. \quad (2.4)$$

Mirroring our previous terminology, we say stabilization occurs almost surely because, for any sequence of nonnegative radii $(l_i)_{i \in \mathbb{N}}$ such that $l_i \to \infty$, $D(S \cap B_z (l_i)) \Rightarrow D^\infty (S)$ whenever both quantities are measurable. Here we use outer probability, because a radius of stabilization may not be a measurable function, specifically considering the unique minimal radius. Definition 2.4 is strictly stronger than Definition 2.2.

**Proposition 2.1.** For $S$ a simple point process taking values in $\mathcal{X}(\mathbb{R}^d)$, let $\psi$ stabilize on $S$ almost surely. Then $\psi$ stabilizes on $S$ in probability.

**Proof.** Let $\rho$ be a radius of stabilization satisfying Definition 2.4. Likewise, let $D^\infty$ be a corresponding terminal addition cost. For any $\rho(S) \leq l < \infty$, $D(S \cap B_z (l)) = D(S \cap B_z (\rho(S))) = D^\infty (S)$. Thus $\{D(S \cap B_z (l)) \neq D^\infty (S)\} \subseteq \{\rho(S) > l\}$, and consequently $\mathbb{P}^* [D(S \cap B_z (l)) \neq D^\infty (S)] \leq \mathbb{P}^* [\rho(S) > l] \to 0$. We see that $\psi$ stabilizes in probability on $S$ with terminal addition cost $D^\infty (S)$. \hfill \Box

It is often necessary to compare the stabilization properties of a function over a range of related point processes. For example, corresponding binomial and Poisson processes can be shown to have essentially equivalent local properties, while differing globally. As defined in Definition 2.3, a given radius of stabilization could feasibly show completely different behavior on each process type. This motivates the following:
Definition 2.5 (Locally Determined Radius of Stabilization). The radius of stabilization $\rho$ centered at $z \in \mathbb{R}^d$ is locally determined if for any $S, T \in \mathcal{X}(\mathbb{R}^d)$

$$T \cap B_z(\rho(S)) = S \cap B_z(\rho(S)) \implies \rho(T) = \rho(S).$$

With the local-determination criterion from Definition 2.5, we can assure that stabilization must occur simultaneously on any two point processes which are locally equivalent.

As defined here, almost-sure and locally-determined almost-sure stabilization correspond, respectively, to “weak” and “strong” stabilization, Definitions 3.1 and 2.1 in [32]. Here, we have generalized by accounting for possible measurability issues, however the definitions are essentially equivalent. As in the non-locally-determined case, there exists a unique minimal locally-determined radius of stabilization:

Proposition 2.2. For $\mathcal{R}$, the space of locally-determined radii of stabilization for $\psi$ centered at $z \in \mathbb{R}^d$, let $\rho^*: \mathcal{X}(\mathbb{R}^d) \rightarrow [0, \infty]$ such that $\rho^*(S) = \inf_{\rho \in \mathcal{R}} \rho(S)$. Then $\rho^*$ is a locally determined radius of stabilization for $\psi$ centered at $z$.

Proof. If all possible radii are infinite, the result follows trivially. Else for $S, T \in \mathcal{X}(\mathbb{R}^d)$ suppose $\rho^*(S) < \infty$ with $S \cap B_z(\rho^*(S)) = T \cap B_z(\rho^*(S))$. Since $S$ and $T$ have no accumulation points, for any $\epsilon > 0$ sufficiently small, we have $S \cap B_z(\rho^*(S) + \epsilon) = T \cap B_z(\rho^*(S) + \epsilon)$. There exists a locally determined radius of stabilization $\rho$ such that $\rho(S) \leq \rho^*(S) + \epsilon$. As $S \cap B_z(\rho^*(S) + \epsilon) = T \cap B_z(\rho^*(S) + \epsilon)$ with $\rho(S) \leq \rho^*(S) + \epsilon$, we have that $S \cap B_z(\rho(S)) = T \cap B_z(\rho(S))$. Thus $\rho(S) = \rho(T)$ by the local-determination criterion. Then $\rho^*(T) \leq \rho^*(S) + \epsilon$. Since the choice of $\epsilon$ was arbitrary, we have $\rho^*(T) \leq \rho^*(S)$. Thus, $S \cap B_z(\rho^*(T)) = T \cap B_z(\rho^*(T))$. By similar arguments, $\rho^*(S) \leq \rho^*(T)$. Combining, $\rho^*(S) = \rho^*(T)$ must hold, and the result follows.

To give an intuitive understanding of the stabilization properties found in this section, the contribution of a single data point to the overall function value depends only on a centered at $\rho$ of the function $\psi$, weakly-dependent pieces, whose interaction is controllable. Consequently, the overall value may be considered to be a sum of iid random variables, with the benefits of stabilization property is useful. It should be noted that the definitions presented here are not all-encompassing. See [33] for an alternative, but essentially similar, definition of “stabilization” which may prove useful in some settings.

2.2. Technical Results

In all of the following, $\mathcal{P}(\mathbb{R}^d)$ denotes the set of probability distributions over $\mathbb{R}^d$. $(Y_i)_{i=1}^{n} \sim G$ with $G \in \mathcal{P}(\mathbb{R}^d)$ and $Y' \sim G$ an independent copy. Let $Y_n$ be the multiset induced by the sample $(Y_i)_{i=1}^{n}$. This definition may be simply denoted by $Y_n := \{Y_i\}_{i=1}^{n} \sim G$. For a measurable function $\psi: \mathcal{X}(\mathbb{R}^d) \rightarrow \mathbb{R}$, define the following conditions:

(E1) For $\mathcal{C} \subseteq \mathcal{P}(\mathbb{R}^d)$ and some $a > 2$, there exists $E_a < \infty$ such that

$$\sup_{G \in \mathcal{C}} \sup_{n \in \mathbb{N}} \mathbb{E} \left[ |\psi(\sqrt{n}(Y_n \cup \{Y'\})) - \psi(\sqrt{n}Y_n)|^a \right] \leq E_a.$$

(2.5)
(E2) For some $a > 2$ and $R > 0$, there exist $U_a > 0$ and $u_a > 1$ satisfying the following property: For any $S \in \mathcal{X}(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$,

$$|\psi(S \cup \{y\}) - \psi(S)|^a \leq U_a \left(1 + \# \{S \cap B_y(R)\}^{u_a}\right).$$  \hfill (2.6)

(E1) requires a moment bound that holds uniformly in the sample size and distribution. Clearly, if (E1) is satisfied for $C$, it is also satisfied for any subset of $C$. In the context of the topological statistics considered in this work, (E1) is primarily useful for proof purposes, as (E1) is mainly established via (E2), as can be seen in Lemma 2.3. However, as will be seen with the case of the $k$-nearest neighbor distance, Theorem 6.13, there exist useful statistics which do not conform to (E2), and the more general condition must be used. (E1) is related to the “uniform bounded moments” condition, Definition 2.2 in [32]. Our version has been suitably generalized for the purposes of this work, the original definition considering only $a = 4$. Let $C_{p,M}(\mathbb{R}^d)$ denote the class of probability distributions $G \in \mathcal{P}(\mathbb{R}^d)$ admitting a density $g$ such that $\|g\|_p \leq M$. We have the following:

**Lemma 2.3.** For $p > 2$, let $\psi$ satisfy (E2) with $u_a \leq p - 1$ for some $a > 2$. Then for any $M < \infty$, $\psi$ satisfies (E1) for $C_{p,M}(\mathbb{R}^d)$.

For $d_{TV}$ the total variation distance between probability distributions and $B_F(\epsilon, d_{TV})$ the closed $\epsilon$-neighborhood of $F$ w.r.t. $d_{TV}$, we have the following stabilization conditions:

1. (S1) For a given $C \subseteq \mathcal{P}(\mathbb{R}^d)$, $F \in C$, $b > 0$, and some $(l_\epsilon)_{\epsilon > 0}$ such that $\lim_{\epsilon \to 0} l_\epsilon^{\epsilon b} = 0$,

   $$\lim_{\epsilon \to 0} \sup_{G \in C \cap B_F(\epsilon, d_{TV})} \sup_{n \in \mathbb{N}} \mathbb{P} \left[D_{\sqrt{n}\psi_{\epsilon,Y}} \left( (\sqrt{n}Y_n) \cap B_{\sqrt{n}\psi_{\epsilon,Y}}(l_\epsilon) \right) \neq D_{\sqrt{n}\psi_{\epsilon,Y}} \left( (\sqrt{n}Y_n) \right) \right] = 0.

2. (S2) For $G \in \mathcal{P}(\mathbb{R}^d)$, there exist locally-determined radii of stabilization $(\rho_z)_{z \in \mathbb{R}^d}$ for $\psi$ satisfying

   $$\lim_{L \to \infty} \sup_{n \in \mathbb{N}} \mathbb{P}^* \left[\rho_{\sqrt{n}\psi_{\epsilon,Y}} \left( (\sqrt{n}Y_n) > L \right) \right] = 0. \hfill (2.7)

(S1) and (S2) can be summarized as uniform stabilization conditions, either in probability or almost surely. (S1) is mainly useful for proof purposes only, and serves only to weaken the necessary conditions to provide for bootstrap consistency. As such, we have the following lemma linking (S1) and (S2).

**Lemma 2.4.** Let $\psi$ satisfy (S2) for $G = F \in C_{p,M}(\mathbb{R}^d)$. Then $\psi$ satisfies (S1) for $C_{p,M}(\mathbb{R}^d)$, $F$, $b = (p - 2) / (d(p - 1))$, and any $(l_\epsilon)_{\epsilon > 0}$ such that $\lim_{\epsilon \to 0} l_\epsilon^{(p-2)/(d(p-1))} = 0$ and $\lim_{\epsilon \to 0} l_\epsilon = \infty$.

We can often greatly simplify the addition costs and radii of stabilization required in (S1) and (S2). For example, given a translation-invariant function $\psi$ and any $D_0, \rho_0$ for $\psi$ centered at 0, corresponding quantities can be constructed for any other center point. For $z \in \mathbb{R}^d$, $D_z: \mathcal{X}(\mathbb{R}^d) \to \mathbb{R}$ where $D_z(S) = D_0(S - z)$ is an add-$z$ cost for $\psi$ centered at $z$. Likewise $\rho_z: \mathcal{X}(\mathbb{R}^d) \to [0, \infty]$ where $\rho_z(S) = \rho_0(S - z)$ is a radius of stabilization for $\psi$ centered at $z$. In the following, $\mathbf{P}_\lambda$ denotes a homogeneous Poisson process on $\mathbb{R}^d$ with intensity $\lambda$. 
Lemma 2.5. Let $F \in C_{p,M}$ with $p > 2$ and $M < \infty$. Let $\rho_0$ be a locally-determined radius of stabilization for $\psi$ centered at 0. Suppose that for any given $a, b \in (0, \infty)$, and $\delta > 0$, there exists an $L_{a,b,\delta} < \infty$ and a measurable set $A_{a,b,\delta}$ with $\rho_0^{-1}((L_{a,b,\delta}, \infty)) \subseteq A_{a,b,\delta}$ such that

$$\sup_{\lambda \in [a,b]} \mathbb{P}^* \left[ \rho_0 (P_\lambda) > L_{a,b,\delta} \right] \leq \sup_{\lambda \in [a,b]} \mathbb{P} \left[ P_\lambda \in A_{a,b,\delta} \right] \leq \delta. \quad (2.8)$$

Then for any $\delta > 0$ there exists an $n_\delta < \infty$ and $L_\delta < \infty$ such that

$$\sup_{n \geq n_\delta} \mathbb{P}^* \left[ \rho_0 (X_n - X') > L_\delta \right] \leq \delta. \quad (2.9)$$

Lemma 2.5 provides a convenient tool for “de-Poissonizing” a locally determined radius of stabilization. Often it is easier to show stabilization properties for a homogeneous Poisson process than for a binomial process directly. Lemma 2.5 allows for the easy extension of homogeneous Poisson results to the binomial setting, as is required for many of the results in this work. Note that the conclusion is not the same as the statement of (S1), only applying for $n \geq n_\delta$. Usually some small effort is required to extend the result to include all $n \in \mathbb{N}$, depending on the specifics of the function $\psi$ considered. We come to the following important proposition, the main supporting result for our general bootstrap consistency theorem, Theorem 2.7.

Proposition 2.6. For $p > 2$ and $M < \infty$, let $\psi$ satisfy (E1) and (S1) for $C_{p,M} (\mathbb{R}^d)$, $F \in C_{p,M} (\mathbb{R}^d)$, and some $a > 2$. Then for any $G \in C_{p,M} (\mathbb{R}^d) \cap B_F (\epsilon, d_{TV})$, there exist iid coupled random variables $((X_i, Y_i))_{i \in \mathbb{N}}$ such that $X_n = \{X_i\}_{i=1}^n \overset{iid}{\sim} F$, $Y_n = \{Y_i\}_{i=1}^n \overset{iid}{\sim} G$, and

$$\sup_{n \in \mathbb{N}} \text{Var} \left[ \frac{1}{\sqrt{n}} (\psi \left( \sqrt{n}X_n \right) - \psi \left( \sqrt{n}Y_n \right)) \right] \leq \gamma_\epsilon. \quad (2.10)$$

The value $\gamma_\epsilon$ does not depend on $G$ and satisfies $\lim_{\epsilon \to 0} \gamma_\epsilon = 0$.

For $W_2$ the 2-Wasserstein metric between probability distributions and $\mathcal{L}$ denoting the law or distribution of a random variable, the variance given in the conclusion of Proposition 2.6 bounds above

$$W_2^2 \left( \mathcal{L} \left\{ \frac{1}{\sqrt{n}} (\psi \left( \sqrt{n}X_n \right) - \mathbb{E} \left[ \psi \left( \sqrt{n}X_n \right) \right]) \right\}, \mathcal{L} \left\{ \frac{1}{\sqrt{n}} (\psi \left( \sqrt{n}Y_n \right) - \mathbb{E} \left[ \psi \left( \sqrt{n}Y_n \right) \right]) \right\} \right). \quad (2.11)$$

Consequently, Proposition 2.6 shows that this $W_2$ distance can be made arbitrarily small uniformly over a neighborhood of distributions around $F$. An appropriately smoothed empirical distribution falls within such a small neighborhood with high probability, given sufficiently large sample sizes.

Furthermore, it can be seen that Proposition 2.6 extends easily to finite sums. Given any $(A_i)_{i=1}^k$ and $(B_i)_{i=1}^k$, we have that $\text{Var} \left[ \sum_{i=1}^k A_i - \sum_{i=1}^k B_i \right] \leq k \sum_{i=1}^k \text{Var} \left[ A_i - B_i \right]$. Thus, if the conclusion of Proposition 2.6 holds for any finite set of functions, $(\psi_i)_{i=1}^k$, it also holds for $\sum_{i=1}^k \psi_i$, with rate depending on the worst case $\psi_i$.

It should be noted that condition (S1) is slightly stronger than necessary to establish Proposition 2.6. As stated, the add-1 cost $D_{\sqrt{n}Y^*} ((\sqrt{n}Y_n) \cap B_{\sqrt{n}Y^*} (l_i))$ itself is compared
to the terminal addition cost \( D \hat{\pi}_{\mathcal{Y}'_n} (\sqrt[n]{Y_n}) \). As may be useful for some statistics, it is only required that an appropriate bound displays the desired stabilization property, see the provided proof for details.

2.3. Bootstrap

The bootstrap is an estimation technique used primarily for constructing approximate confidence intervals for a population parameter of interest. Bootstrap estimation is well-studied in the statistical literature, an introduction being provided in [34]. In this section, we will show consistency for the bootstrap in estimating the limiting distribution of a standardized stabilizing statistic, \( \psi \), in the multivariate setting. We describe the general procedure below:

Let \( X_n = \{X_i\}_{i=1}^n \overset{i.i.d.}{\sim} F \). In the bootstrap, we estimate the sampling distribution of

\[
\frac{1}{\sqrt{n}} \left( \psi (\sqrt[n]{X_n}) - \mathbb{E} \left[ \psi (\sqrt[n]{X_n}) \right] \right)
\]  

(2.12)

using a plug-in estimator for the underlying data distribution \( F \). In the standard nonparametric bootstrap, we estimate \( F \) by \( \hat{F}_n \), the empirical distribution giving probability to each unique value of \( \{X_i\}_{i=1}^n \), proportional to the number of repetitions within \( X_n \). We have the bootstrap statistic

\[
\frac{1}{\sqrt{m}} \left( \psi (\sqrt[m]{X_m^*}) - \mathbb{E} \left[ \psi (\sqrt[m]{X_m^*}) \right] \right),
\]

(2.13)

where \( X_m^* = \{X_i^*\}_{i=1}^m \overset{i.i.d.}{\sim} \hat{F}_n \), conditional on \( X_n \). The sampling distribution of the bootstrap version provides an estimate for the distribution of the original statistic, which in the ideal case converges to the truth in the large-sample limit.

However, as will be seen in Section 3, for some classes of statistics the standard bootstrap does not directly replicate the correct sampling distribution asymptotically. As such, for the theorems we will present, we instead estimate \( F \) by a smoothed empirical distribution. Such a smoothed bootstrap can be shown to provide consistent estimation, even when the standard nonparametric bootstrap fails.

For \( h > 0 \) and an appropriate kernel \( Q \), the kernel density estimator of \( f(x) \) based on the sample \( \{X_i\}_{i=1}^n \) is \( \hat{f}_{n,h}(x) := 1/(nh^d) \sum_{i=1}^n Q((x - X_i)/h) \). The probability distribution for which \( \hat{f}_{n,h} \) is a density is the smoothed empirical measure \( \hat{F}_{n,h} \). The following theorem establishes consistency for the smoothed bootstrap in the multivariate setting. We give the result for a vector of stabilizing statistics. In the context of the topological statistics introduced in Section 5, this can be the persistent Betti numbers or Euler characteristics evaluated at different filtration parameters.
Theorem 2.7. Let $F \in \mathcal{P}(\mathbb{R}^d)$ with density $f$ such that $\|f\|_p < \infty$ for some $p > 2$. Furthermore, let $F$, $Q$, and $(h_n)_{n \in \mathbb{N}}$ be such that $\|f_{n,h_n}(x) - f(x)\|_1 \to 0$ and $\|f_{n,h_n}(x) - f(x)\|_p \to 0$ in probability (resp. a.s.). Suppose $\tilde{\psi}: \tilde{\mathcal{X}}(\mathbb{R}^d) \to \mathbb{R}$ has component functions $\psi_j: \tilde{\mathcal{X}}(\mathbb{R}^d) \to \mathbb{R}$, $1 \leq j \leq k$ satisfying (E1) and (S1), as in Theorem 2.6. Suppose that the smoothed characteristic and persistent Betti numbers for a class of simplicial complexes.

The conditions under which $\frac{1}{\sqrt{n}} \left( \tilde{\psi}(\sqrt{n}X_n) - E \left[ \tilde{\psi}(\sqrt{n}X_n^*) \right] \right) \overset{d}{\to} G$ in probability (resp. a.s.) can be found in [19] and [23]. Notably, no conditions are placed on the density $f$ except $\|f\|_p \infty$. Strictly speaking, convergence to a limiting distribution is not required for the bootstrap to provide asymptotically valid confidence intervals. Proposition 2.6 gives that the cumulative distribution function $F_{\tilde{\psi}_n}$ of $(\tilde{\psi}(\sqrt{n}X_n) - E[\tilde{\psi}(\sqrt{n}X_n)])/\sqrt{n}$ has the property

$$\lim_{\delta \to 0} \lim_{n \to \infty} \sup_{x \in \mathbb{R}^d} \sup_{n \to \infty} \left| F_{\tilde{\psi}_n}(x + \delta) - F_{\tilde{\psi}_n}(x) \right| \to 0,$$

(2.14)

it can be shown that confidence intervals constructed from the bootstrap statistic still achieve the stated confidence level with high probability, given a sufficiently large sample. Convergence to a continuous limiting CDF is just one way of satisfying this condition. However, for the topological statistics considered here, the CDFs of the standardized statistics are poorly understood, thus it is not possible at this time to make such an extension.

In the later sections, we will show that the necessary moment and stabilization conditions for Theorem 2.7 are satisfied for several specific statistics of interest, chiefly the Euler characteristic and persistent Betti numbers for a class of simplicial complexes.

3. Simulation Study

In this section we present the results of a series of simulations illustrating the finite-sample properties of the bootstrap applied to persistent Betti numbers $\beta_{d,s}^r$. Precise definitions and an introduction to the properties of these statistics may be found in Section 5.

First, we will show that the standard nonparametric bootstrap may fail to reproduce the correct sampling distribution asymptotically. We justify this statement heuristically. For a wide class of simplicial complexes built over point sets in $\mathbb{R}^d$, repetitions within a set are ignored in calculating the corresponding persistence diagram or persistent Betti numbers.
Fig 1. The first plot shows the original data set of size $n = 10,000$, from which a single bootstrap sample is drawn. In the second plot, the persistence diagrams for both the original and bootstrap samples are shown, along with lines denoting the median birth and death in each diagram. The asymptotic bias can be clearly seen. The third plot shows the correspondence of the median birth and death times after a multiplicative correction of $\sqrt{1 - e^{-1}} \approx 0.795$ is applied to the bootstrap sample.

This behavior holds for both the Vietoris-Rips and Čech complexes, defined in Section 5.1. As any repetitions are ignored in statistic calculations, the standard bootstrap then becomes a subsampling technique, where the size of a given subsample is random, equal to the number of unique data points present in the corresponding bootstrap sample.

It can be shown using elementary arguments that the expected number of unique data points in a given bootstrap sample of size $n$ is $n(1 - (1 - 1/n)^n) \approx (1 - e^{-1}) n \approx 0.632n$. As such, a bootstrapped sample behaves effectively like a sample of size $0.632n$, but is not scaled accordingly. This discrepancy in scaling introduces an asymptotic bias. The effect is shown in Figure 1 for the Vietoris-Rips complex.

For $K_{VR}$ the Vietoris-Rips complex and $X_m = \{X_i\}_{i=1}^m \overset{iid}{\sim} F$, $m \approx 0.632n$ it has been shown in [28] that the following rescaled and standardized statistic converges to a limiting normal distribution:

$$
\frac{1}{\sqrt{m}} \left( \beta_q^{r,s} (K_{VR} (\sqrt{m}X_m)) - E \left[ \beta_q^{r,s} (K_{VR} (\sqrt{m}X_m)) \right] \right) \xrightarrow{d} N \left( 0, \sigma^2_{r,s} \right). \quad (3.1)
$$

If the scaling constants are changed, we obtain from the scaling properties of the Vietoris-Rips filtration

$$
\frac{1}{\sqrt{n}} \left( \beta_q^{r,s} (K_{VR} (\sqrt{n}X_m)) - E \left[ \beta_q^{r,s} (K_{VR} (\sqrt{n}X_m)) \right] \right) \approx \frac{\sqrt{0.632}}{\sqrt{m}} \left( \beta_q^{\sqrt{0.632}} (K_{VR} (\sqrt{m}X_m)) - E \left[ \beta_q^{\sqrt{0.632}} (K_{VR} (\sqrt{m}X_m)) \right] \right) \xrightarrow{d} N \left( 0, 0.632 \sigma^2_{r,s} \right). \quad (3.2)
$$

We see that scaling by the wrong factor can introduce a bias that persists asymptotically, not only in the scale of the variance, but also the filtration parameters. Since the number of unique points in a given bootstrap sample is random, and merely close to $(1 - e^{-1}) n$ with high probability, in reality there will be some deviation from the above. Rescaling by the observed (random) number of samples introduces another source of randomness whose


| Label | Description |
|-------|-------------|
| $F_1$ | Rotationally symmetric pole in $\mathbb{R}^2$, finite $L_2$ norm, infinite $L_{2+\epsilon}$ norm |
| $F_2$ | Rotationally symmetric pole in $\mathbb{R}^2$, finite $L_8$ norm, infinite $L_{8+\epsilon}$ norm |
| $F_3$ | $S^1$ embedded in $\mathbb{R}^2$, additive Gaussian noise |
| $F_4$ | Uniformly distributed over $B_0(1)$ in $\mathbb{R}^3$ |
| $F_5$ | 5 clusters in $\mathbb{R}^3$, additive exponential noise |
| $F_6$ | $S^2$ embedded in $\mathbb{R}^5$, additive Cauchy noise |
| $F_7$ | Pair of tangent coplanar copies of $S^1$ (figure-8) embedded in $\mathbb{R}^{10}$, additive Gaussian noise |

Table 1: Description of distributions considered in simulation study. For the distributions on manifolds, we first draw uniformly from the manifold, then apply additive noise.

The effect is not obvious. However, combined with the provided simulations, there is sufficient evidence to illustrate the general effect.

Next, we provide a series of simulations investigating the coverage probability of our bootstrapped confidence intervals for persistent Betti numbers and Euler characteristics of the Vietoris-Rips complex. Table 1 lists the data distributions considered, with descriptions. The results of the simulations are given in Table 2. For the persistent Betti numbers, a single choice of $(r, s)$ was made for each combination of distribution and feature dimension, chosen to lie within the body of features in the corresponding persistence diagram. For computational reasons, only feature dimensions $q = 1$ and $q = 2$ are considered. Bandwidth selection was done using the Silverman’s rule of thumb, see [39].

We see that coverage probability is generally lower than the stated level. Likewise, we also notice a steep dropoff in coverage as the data dimension increases. This is expected, as the kernel density estimator is known to suffer from a ‘curse of dimensionality’. Furthermore, as can be seen from the provided simulations and the proof of Proposition 2.6, approximation quality suffers with increasing feature dimension $q$.

Of particular interest is the case of distribution $F_1$. This distribution was specifically chosen to violate the assumptions of Theorem 6.6, yet the coverage probability does not appear to deviate far from the stated value. It is possible that a relaxation to the necessary conditions may be made to include this case. Alternatively it is possible that the simulations considered here do not consider large enough sample sizes to rule out consistency in that case. Further investigations are necessary.

4. Data Analysis

In this section we show how bootstrap estimation performs on a real dataset. We consider a selection of galaxies from the Sloan Digital Sky Survey [5], chosen from a selection of sky with right ascension values between $100^\circ$ and $270^\circ$ and declination between $-7^\circ$ and $70^\circ$. Three selections of galaxies were considered, separated by red-shift. The selections consists of galaxies with red-shift within $(0.027, 0.028)$, $(0.028, 0.029)$, and $(0.029, 0.030)$, respectively.

We apply the Vietoris-Rips complex to each of these selections, and calculate a selection of persistent Betti numbers in dimensions $q = 0$ and $q = 1$. As we are dealing with effectively 2-dimensional slices of the night sky, features of dimension greater than $q = 1$ were not considered. The datasets and rescaled persistence diagrams in dimension $q = 1$ can be seen in Figure 2. After properly rescaling the data, we consider the Betti numbers $\beta_0^r$ for $r = 1, \ldots, 10$. Likewise, we consider the Betti numbers $\beta_1^r$ for $r = 1, \ldots, 10$, and the persistent Betti numbers $\beta_1^{r,r+1}$ for $r = 1, \ldots, 10$. The persistent Betti numbers were chosen to lie close to the diagonal of the persistence diagram, excluding features with a lifetime less than 1.
### Table 2

Coverage proportions for 95% bootstrap confidence interval on the mean persistent Betti numbers. Mean coverage probabilities are calculated using $N = 100$ independent base samples with $B = 200$ bootstrap samples each. The true mean persistent Betti numbers are estimated using a large ($N = 10000$) number of independent samples. Coverage proportions marked in red illustrate where the bootstrap exhibited particularly poor performance.

| Sample Size | Distribution | $(r, s)$ Coverage | Sample Size | Distribution | $(r, s)$ Coverage |
|-------------|--------------|-------------------|-------------|--------------|-------------------|
| $n = 100$   | $F_1$        | $(4.2, 4.8)$      | $q = 1$     | $0.90$       | $q = 2$          |
|             | $F_2$        | $(6.6, 5)$        | $0.92$      | $-$          | $-$              |
|             | $F_3$        | $(3.3, 5)$        | $0.86$      | $-$          | $-$              |
|             | $F_4$        | $(1.9, 2)$        | $0.88$      | $0.91$       | $-$              |
|             | $F_5$        | $(1.3, 1.4)$      | $0.91$      | $0.26$       | $-$              |
|             | $F_6$        | $(1.6, 1.7)$      | $0.72$      | $0.18$       | $-$              |
|             | $F_7$        | $(1.2, 1.4)$      | $0.72$      | $0.03$       | $-$              |
| $n = 200$   | $F_1$        | $(4.2, 4.8)$      | $0.93$      | $-$          | $-$              |
|             | $F_2$        | $(6.6, 5)$        | $0.95$      | $-$          | $-$              |
|             | $F_3$        | $(3.3, 5)$        | $0.99$      | $-$          | $-$              |
|             | $F_4$        | $(1.9, 2)$        | $0.91$      | $0.85$       | $-$              |
|             | $F_5$        | $(1.3, 1.4)$      | $0.95$      | $0.16$       | $-$              |
|             | $F_6$        | $(1.6, 1.7)$      | $0.79$      | $0.99$       | $-$              |
|             | $F_7$        | $(1.2, 1.4)$      | $0.78$      | $0.00$       | $-$              |
| $n = 300$   | $F_1$        | $(4.2, 4.8)$      | $0.87$      | $-$          | $-$              |
|             | $F_2$        | $(6.6, 5)$        | $0.94$      | $-$          | $-$              |
|             | $F_3$        | $(3.3, 5)$        | $0.93$      | $-$          | $-$              |
|             | $F_4$        | $(1.9, 2)$        | $0.95$      | $0.83$       | $-$              |
|             | $F_5$        | $(1.3, 1.4)$      | $0.94$      | $0.92$       | $-$              |
|             | $F_6$        | $(1.6, 1.7)$      | $0.89$      | $0.82$       | $-$              |
|             | $F_7$        | $(1.2, 1.4)$      | $0.85$      | $0.00$       | $-$              |
| $n = 400$   | $F_1$        | $(4.2, 4.8)$      | $0.93$      | $-$          | $-$              |
|             | $F_2$        | $(6.6, 5)$        | $0.95$      | $-$          | $-$              |
|             | $F_3$        | $(3.3, 5)$        | $0.99$      | $-$          | $-$              |
|             | $F_4$        | $(1.9, 2)$        | $0.95$      | $0.83$       | $-$              |
|             | $F_5$        | $(1.3, 1.4)$      | $0.94$      | $0.92$       | $-$              |
|             | $F_6$        | $(1.6, 1.7)$      | $0.89$      | $0.82$       | $-$              |
|             | $F_7$        | $(1.2, 1.4)$      | $0.81$      | $0.02$       | $-$              |
The results of these statistics can be seen in Figure 3. We use bootstrap estimation to construct 95% confidence intervals for the population mean values, holding simultaneously within each regime across $r = 1, ..., 10$.

As can be seen, the bootstrap confidence intervals allow for significant variation in the persistent Betti numbers. From this analysis, we conclude that the topological properties of the three samples are consistent over the filtration parameters considered. Any difference in topological structure seen at the local level is within the margin of error provided by the bootstrap confidence intervals. It is here we note the special utility of bootstrap estimation. Without a measure of the inherent variability possible within the data, one might easily find erroneous patterns based only on the provided persistent Betti curves. At the moment, the bootstrap provides one of the only convenient measures of that inherent variability.

The consistency shown in this work for bootstrap estimation applies only for those features within the “body” of topological features. Features with large persistence or that appear at large diameter are not accounted for in this. As such, our analysis does not preclude differences among those topological features which exist at a large relative scale, being those that describe the global structure of the dataset. Furthermore, we have used a relatively simple version of the bootstrap, assuming an independent sample from a common density. It is likely that another type of bootstrap would more accurately represent the inherent dependencies in this data. However, these more detailed considerations lie outside the scope of this work.
5. Statistics of Simplicial Complexes Constructed over Point Clouds in $\mathbb{R}^d$

### 5.1. Simplicial Complexes

Let $\mathcal{K} = \{K^r\}_{r \in \mathbb{R}}$ be a filtration of simplicial complexes, with $K^r \subseteq K^t$ for $r < t$. Each complex is a collection of simplices, subsets of the vertex multiset, $V$. Here any repeated vertices are considered distinct. For a collection of simplices $K$ to be a simplicial complex, for any two simplices $S \subseteq V$ and $T \subseteq S$, $S \in K$ only if $T \in K$. Here a simplex is only included along with all of its subsets. For a given simplicial complex $K$, $K_q$ denotes the subset of $K$ consisting of all $q$-simplices. $q$-simplices are those simplices consisting of $q + 1$ vertices. Each $q$-simplex said to have dimension $q$. A graph or network refers to a simpicial complex consisting of only 1-simplices (edges) and 0-simplices (vertices).

We will be looking at simplicial complexes constructed over point clouds in $\mathbb{R}^d$. The two major examples are the Čech and Vietoris-Rips complexes:

\begin{align}
K^C_\mathcal{C}(S) &= \{ \sigma \subseteq S : \exists z \in \mathbb{R}^d \text{ s.t. } \|z - x\| \leq r \ \forall x \in \sigma \} \\
K^C_{\mathcal{VR}}(S) &= \{ \sigma \subseteq S : \|x - y\| \leq 2r \ \forall x, y \in \sigma \} .
\end{align}

(5.1) (5.2)

Each of these complexes summarizes the geometric and topological properties within a given point cloud. The Vietoris-Rips complex can be considered a “completion” of the Čech complex, in so much that the Vietoris-Rips complex is the largest simplicial complex with the same edge set as the Čech complex. While the primary motivation for the results given here is application to the Čech and Vietoris-Rips complexes, our main theorems apply for a range of possible complexes. For example, for computational reasons it is often convenient to limit the number of simplices present within the final complex. As such, we have two...
approximations, the alpha complex and its completion

\[ K^\alpha_r (S) = \{ \sigma \subseteq S : \exists z \in \mathbb{R}^d \text{ s.t. } \|z - x\| \leq r \text{ and } \|z - y\| \leq \|z - y\| \quad \forall x \in \sigma, y \in S \} \]

\[ K^\alpha_r (S) = \{ x, y \} \in K^\alpha_r (S) \quad \forall x, y \in \sigma \} . \]

As can be seen from the definitions, these complexes avoid adding simplices between disparate points, controlling the total number of simplices present within the complex. It has been shown that the alpha and Čech complexes are both homotopy equivalent to a union of closed balls around the underlying point set, thus sharing equivalent homology groups. However, for the completion, denoted here as the alpha* complex, there is no such relationship. The alpha complex is a subcomplex of the Čech complex as well as the Delaunay complex

\[ K_D (S) = \{ \sigma \subseteq S : \exists z \in \mathbb{R}^d \text{ s.t. } \|z - x\| \leq \|z - y\| \quad \forall x \in \sigma, y \in S \} . \] (5.3)

### 5.2. Persistent Homology

Now, of chief interest are the topological properties for a given simplicial complex. Both the Čech and Vietoris-Rips complexes reflect the structure of an underlying point cloud. As such the topology of each provides an effective summary statistic for describing the structural properties of a dataset in \( \mathbb{R}^d \).

Define \( C(K) \) to be the free abelian group generated by the simplices in \( K \). Elements of \( C(K) \) are sums of the form \( \sum_{i \in I} a_i \sigma_i \), where \( \sigma_i \in K \) for \( a_i \) an appropriate group element. If we further allow the coefficients to come from a field, then \( C(K) \) is a vector space. For the purposes of this paper, coefficients are drawn from the two-element field \( \mathbb{F}_2 = \{0, 1\} \). \( C(K) \) is equipped with a linear boundary operator \( \partial : C(K) \rightarrow C(K) \) where \( \partial(\{x_1, \ldots, x_q+1\}) = \sum_{i=1}^q (-1)^i \{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_q+1\} \). As a fundamental property, \( \partial \circ \partial = 0 \). With coefficients in \( \mathbb{F}_2 \), the boundary of a simplex reduces to the sum of all its faces. \( C_q(K) = C(K)_q \) is the subspace spanned by the \( q \)-simplices of \( K \), with the image of \( C_q(K) \) under \( \partial \) lying in \( C_{q-1}(K) \). \( \partial_q : C_q(K) \rightarrow C_{q-1}(K) \) denotes the restriction of \( \partial \) to \( C_q(K) \).

We now construct the homology groups of \( K \). Let \( Z(K) = \ker(\partial) \) be the subspace of \( C(K) \) containing the cycles, those elements whose boundary under \( \partial \) is 0. \( Z_q(K) = Z(K)_q = \ker(\partial_q) \) is the restriction of \( Z(K) \) to dimension \( q \). Let \( B(K) = \text{im}(\partial) \) denote the subspace of boundaries in \( C(K) \). \( B_q(K) = B(K)_q = \text{im}(\partial_{q+1}) \) is the subspace consisting of the boundaries of elements in \( C_{q+1}(K) \), lying in \( C_q(K) \).

The homology groups are given by \( H_q(K) := Z_q(K) / B_q(K) \), the cycles \( Z_q \) in dimension \( q \) modulo the boundaries \( B_q \). In words, the elements of the homology groups represent “holes” within the simplicial complex, shown by closed loops whose interior is not filled by other elements in the complex. These homology groups provide a topological summary of the structure in the simplicial complex \( K \). As stated previously, because we assume field coefficients for \( C(K) \), each homology group is also a vector space. The Betti numbers of the complex represent the degree or dimension of each homology space. We denote the \( q \)-th Betti number of \( K \) by \( \beta_q(K) = \dim(Z_q(K) / B_q(K)) = \dim(Z_q(K)) - \dim(B_q(K)) \). Moving forward, Betti numbers and their like will be of primary interest.

Homology provides a topological invariant constructed from a single simplicial complex. For a filtration of nested simplicial complexes, persistent homology provides more detail. Given a filtration \( \mathcal{K} = \{K^r\}_{r \in \mathbb{R}} \), the homology groups for each complex, \( H_q(K^r) \), are
defined. However, due to the nested structure of the filtration, simplices are shared across complexes, and thus there exists a natural inclusion map between homology spaces. Cycles in \( Z_q(K^r) \) are also cycles in \( Z_q(K^t) \) if \( r < t \). The boundary spaces behave similarly. For a given element \( x + B_q(K^r) \in H_q(K^r) \), \( x + B_q(K^r) \rightarrow x + B_q(K^t) \) specifies the inclusion map from \( H_q(K^r) \) to \( H_q(K^t) \).

If a given element \( x \in H_q(K^r) \) maps to \( y \in H_q(K^t) \) upon inclusion, with \( y \neq 0 \), we say that \( x \) represents a persistent cycle across the filtration. Essentially the same underlying element is reflected in the homology groups over a range of simplicial complexes. The collection of homology groups and inclusion maps form a persistence module. A wide body of work exists on the properties of these persistence modules, see [48] for an introduction. For any feature in the complex, there is a well defined death time, being the smallest parameter level for which the given element lies in the kernel. The Betti numbers of a filtration form a function in the filtration parameter, \( r \).

It is a fundamental theorem of persistent homology that a sufficiently well-behaved persistence module can be represented by a persistence diagram. Each diagram is a multiset of points in \( \mathbb{R}^2 \times \mathbb{Z} \), with points \((b, d, q)\). Each point represents a single persistent feature in the module. \( b \) denotes the birth time of the feature, being the smallest parameter level for which that feature is represented in the homology groups. Likewise \( d \) gives the death time, and \( q \) the dimension of the feature. The collection of persistent features represented by the diagram forms a basis for the persistence module. This decomposition and representation allows the complex topological information present within a filtration of simplicial complexes to be condensed into a simple summary statistic. An example of a persistence diagram is shown in Figure 1.

### 5.3. Persistent Betti Numbers

We arrive at the main focus of this section. Define the persistent homology groups of a filtration \( K = \{K^r\}_{r \in \mathbb{R}} \) as

\[
H_q^{r, s}(K) := Z_q(K^r) / (B_q(K^s) \cap Z_q(K^r)).
\]

(5.4)

Nonzero elements in this group represent features born at or before time \( r \) which persist until at least time \( s \). The dimension of these spaces gives the persistent Betti numbers

\[
\beta_q^{r, s}(K) := \dim (Z_q(K^r) / B_q(K^s) \cap Z_q(K^r)) = \dim (Z_q(K^r)) - \dim (B_q(K^s) \cap Z_q(K^r)).
\]

(5.5)

(5.6)

Persistent Betti numbers are in one-to-one correspondence with the respective persistence diagram. Here \( \beta_q^{r, s}(K) \) counts the number of points in \( D_q(K) \) falling within \((−\infty, r] \times (s, \infty]\). When \( s = r \), we recover the regular Betti numbers, \( \beta_q^{r, r}(K) = \beta_q(K^r) \). An important result for persistent Betti numbers is given in the following lemma.

**Lemma 5.1** (Geometric Lemma). [Lemma (2.11) [24]] Let \( J = \{J^r\}_{r \in \mathbb{R}} \) and \( K = \{K^r\}_{r \in \mathbb{R}} \) be filtrations of simplicial complexes with with \( J^r \subseteq K^r \) for all \( r \in \mathbb{R} \). Then

\[
|\beta_q^{r, s}(K) - \beta_q^{r, s}(J)| \leq \max \{\# \{K^r_q \setminus J^r_q\}, \# \{K^r_{q+1} \setminus J^r_{q+1}\}\}
\]

\[
\leq \# \{K^r_q \setminus J^r_q\} + \# \{K^r_{q+1} \setminus J^r_{q+1}\}.
\]

(5.7)

(5.8)
The Geometric Lemma 5.1 relates the change in persistent Betti numbers between two filtrations to the additional simplices gained moving between them. As a brief explanation of the lemma, simplices can be divided into two classes, positive and negative. For two simplicial complexes $J \subset K$, if we imagine adding the additional $q$-simplices in $K$ to $J$ one by one, a positive $q$-simplex will increase the dimension of $Z_q$ by one, and a negative $q$-simplex will increase the dimension of $B_{q-1}$ by one. Either change can affect the persistent Betti numbers. This dichotomy is a basic result from persistent homology, see ?? . The bound given in the Geometric Lemma describes a worst case, when all $q$-simplices at time $r$ are positive or all $(q+1)$-simplices at time $s$ are negative. The Geometric Lemma will be critical moving forward, as it allows us to control the change in persistent Betti numbers by counting appropriate simplices.

5.4. B-Bounded Persistent Betti Numbers

It is at times convenient to place controls on the size of the possible cycles within a simplicial complex. As such, we present the following definitions. Let $S \in \mathcal{X}(\mathbb{R}^d)$ and $K = K(S)$ be a simplicial complex with vertices in $S$. For a given chain of simplices $\sum_{i=1}^{m} \sigma_i \in C(K(S))$, we have the diameter given by $\text{diam}(\sum_{i=1}^{m} \sigma_i) := \text{diam}(\bigcup_{i=1}^{m} \sigma_i)$. Let the space of $B$-bounded cycles of the complex $K$ be the vector space, denoted by $Z_{q,B}(K)$, spanned by cycles in $K$ with diameter no larger than $B$. We have $Z_{q,B}(K) := \text{span}\{x \in Z_q(K) \text{ s.t. diam}(x) \leq B\}$. Likewise let the space of $B$-bounded boundaries be $B_{q,B}(K) := \text{span}\{x \in B_q(K) \text{ s.t. diam}(x) \leq B\}$.

The definitions presented here are directly inspired by a previous concept under the name “$M$-bounded persistence” found in [4], and may be viewed as a generalization thereof. In this previous work, it was shown that a diameter bound of this type is sufficient for establishing functional central limit theorems for persistent Betti numbers, and thus an extension is desirable. The original definition given in [4] is based on a correspondence between loops and connected components in the complement space, and does not apply to arbitrary simplicial complexes and feature dimensions.

The change in naming effected here is not meant to draw a distinction between the two definitions, but purely to avoid overloading symbols within this paper. $M$ is used in this work to denote an upper bound for a density norm.

The $B$-bounded spaces obey many of the same properties as their original counterparts. We have $B_{q,M}(K) \subseteq Z_{q,M}(K)$. Thus we can define the $B$-bounded homology spaces as $H_{q,B}(K) = Z_{q,B}(K)/B_{q,B}(K)$. It should be noted that these definitions allow for chains of unbounded diameter, so long as there exists a decomposition into a sum of bounded chains. Furthermore, for $B_{q,B}(K)$, the diameter control is on the chains $x \in B_q(K)$, not on a corresponding $y \in C_{q+1}(K)$ with $x = \partial y$. It is possible to have a chain with arbitrarily high diameter, whose boundary has diameter less than $B$.

We next define the analog of Betti numbers and persistent Betti numbers over a filtration of simplicial complexes in the bounded context. Given a filtration $\mathcal{K} = \{K^r\}_{r \in \mathbb{R}}$, we have $B$-bounded analogs for the Betti numbers, persistent homology spaces, and persistent Betti
Lemma 5.2. Let \( J = \{ J^r \}_{r \in \mathbb{R}} \) and \( K = \{ K^r \}_{r \in \mathbb{R}} \) be filtrations of simplicial complexes with \( J^r \subseteq K^r \) for all \( r \in \mathbb{R} \). Then

\[
|\beta_{q,B}^{r,s}(K) - \beta_{q,B}^{r,s}(J)|\leq \max \left\{ \dim \left( \frac{Z_{q,B}(K^r)}{Z_{q,B}(J^r)} \right), \dim \left( \frac{B_{q,B}(K^s)}{B_{q,B}(J^s)} \right) \right\}
\]

\[
\leq \dim \left( \frac{Z_{q,B}(K^r)}{Z_{q,B}(J^r)} \right) + \dim \left( \frac{B_{q,B}(K^s)}{B_{q,B}(J^s)} \right).
\]

Proof.

\[
\begin{align*}
|\beta_{q,B}^{r,s}(K) - \beta_{q,B}^{r,s}(J)| &= \left| \dim \left( \frac{Z_{q,B}(K^r)}{Z_{q,B}(J^r)} \right) - \dim \left( \frac{Z_{q,B}(J^r)}{Z_{q,B}(J^r) \cap B_{q,B}(J^s)} \right) \right|
\leq & \max \left\{ \dim \left( \frac{Z_{q,B}(K^r) + B_{q,B}(K^s)}{Z_{q,B}(J^r) + B_{q,B}(K^s)} \right), \dim \left( \frac{Z_{q,B}(J^r) \cap B_{q,B}(K^s)}{Z_{q,B}(J^r) \cap B_{q,B}(J^s)} \right) \right\}
\leq & \max \left\{ \dim \left( \frac{Z_{q,B}(K^r)}{Z_{q,B}(J^r)} \right), \dim \left( \frac{B_{q,B}(K^s)}{B_{q,B}(J^s)} \right) \right\}
\leq & \dim \left( \frac{Z_{q,B}(K^r)}{Z_{q,B}(J^r)} \right) + \dim \left( \frac{B_{q,B}(K^s)}{B_{q,B}(J^s)} \right).
\end{align*}
\]
We make a note here about the difference between Lemma 5.2 and the Geometric Lemma 5.1. While drawn from the same fundamental inequality, in the persistent Betti number case, we reduce to counting the simplices that are added when moving from one complex to the other. This reduction cannot be made in the $B$-bounded case, and we must count the number of additional linearly independent loops and boundaries. Different combinatorial techniques will be needed when applying each Lemma, as can be seen in the proofs of Theorems 6.6, 6.7, and 6.8.

5.5. The Euler Characteristic

For a given simplicial complex $K$, the Euler characteristic is defined as

$$\chi(K) := \sum_{k=0}^{\infty} (-1)^k \# \{ K_k \}.$$  \hfill (5.16)

Furthermore, we define the $q$-truncated Euler characteristics as

$$\chi_q(K) := \sum_{k=0}^{q} (-1)^k \# \{ K_k \}.$$ \hfill (5.17)

It can be shown that the Euler characteristic has the following identity with the Betti numbers:

$$\chi(K) = \sum_{k=0}^{\infty} (-1)^k \beta_k(K).$$ \hfill (5.18)

This identity holds, provided there is an $m \in \mathbb{N}$ such that the Betti numbers $\beta_q(K)$ are 0 for all $q > m$ (as in (D4)). This relationship with the Betti numbers makes the Euler characteristic an important topological invariant of study. Applications of the Euler characteristic and derivatives may be found in [35, 37, 42].

5.6. $k$-Nearest Neighbor Graph

The $k$-nearest neighbor graph $K_{\text{NN},k}$ of a vertex set $S$ connects each point of $x$ of $S$ with the $k$ closest vertices to $x$ within $S \setminus x$. This graph may either be directed or undirected. Let the total length of the edges in this graph be denoted by $l_{\text{NN},k}$. The total length of the $k$-nearest neighbor graph is used primarily as a local measure of density for $S$.

6. Bootstrapping Statistics of Simplicial Complexes

6.1. General Conditions for Simplicial Complexes

The results we present here apply for a range of simplicial complexes constructed over point clouds in $\mathbb{R}^d$. In this section, we will explain the specific conditions used, and for which common simplicial complexes they apply. Let $K$ be a function taking as input $S \in \hat{X}(\mathbb{R}^d)$, giving as output a simplicial complex with vertices in $S$. We have the following conditions:

(K1) For any $S \in \hat{X}(\mathbb{R}^d)$ and $z \notin S$, $K(S) \subseteq K(S \cup \{ z \})$. Furthermore, $\sigma \in K(S \cup \{ z \}) \setminus K(S)$ only if $z \in \sigma$. 

(K2) For any $S \in \tilde{X}(\mathbb{R}^d)$ and $z \in \mathbb{R}^d$, $\sigma \in K(S)$ only if $\sigma - z \in K(S - z)$.

(D1) There exists $\phi < \infty$ such that for any $S \in \tilde{X}(\mathbb{R}^d)$, $\sigma \in K^r(S)$ only if $\text{diam}(\sigma) \leq \phi$.

(D2) There exists $\phi < \infty$ such that for any $S \in \tilde{X}(\mathbb{R}^d)$ and $z \in \mathbb{R}^d$, $\sigma \in K(S \cup \{z\}) \triangle K(S)$ only if $\sigma \in B_z(\phi)$.

(D3) There exists an $\eta > 0$ such that for any $S \in \tilde{X}(\mathbb{R}^d)$ and $x \in Z(K(S))$, $\text{diam}(x) \leq \eta$ only if $x \in B(K(S))$.

(D4) There exists an $m \in \mathbb{N}$ such that for any $k > m$ and $S \in \tilde{X}(\mathbb{R}^d)$, $Z_k(K(S)) = B_k(K(S))$.

(K1) means that the addition of a new point will not change the existing complex, only add new simplices. Furthermore, any new simplices gained must contain the added point as a vertex. (K2) gives that the complex is essentially translation invariant. (D1) sets a maximum diameter for any simplex in the complex. (D2) gives that the influence of a new point on the complex is confined to a local region around that point, within a fixed diameter. This condition allows for both the addition and removal of simplices from the complex, but only within the prescribed radius. It can be easily shown that if (D2) holds for $\phi$, (D1) holds for $2\phi$. Conversely if both (K1) and (D1) hold for $\phi$, (D2) also holds for $\phi$. Finally, (D3) gives that no small loops can exist with unfilled interiors, and (D4) gives that all Betti numbers in sufficiently high dimensions are 0.

Now, let $K = (K^r)_{r \in \mathbb{R}}$ be a function taking as input $S \in \tilde{X}(\mathbb{R}^d)$, giving as output a filtration of simplicial complexes with vertices in $S$. As a slight abuse, we will often refer to the function $K$ as a filtration of simplicial complexes, even though it is a function defining more than a single filtration, depending on the underlying point cloud. We say that a given condition is satisfied for $K$ if it is satisfied by $K^r$ for any $r \in \mathbb{R}$. In the cases of (D1), (D2), and (D3), $\phi$ and $\eta$ may depend on $r$ as increasing functions $\phi: \mathbb{R} \to [0, \infty)$ and $\eta: \mathbb{R} \to [0, \infty)$.

It can be shown that all of conditions (K1)-(D3) are satisfied for both the Vietoris-Rips and Čech complexes in $\mathbb{R}^d$ using $\phi(r) = \eta(r) = 2r$. The same functions apply for the alpha complex in $\mathbb{R}^d$ and its completion $\mathcal{K}_\alpha^\star$, with the notable exception that (K1) is violated. Finally, it is known that (D4) is satisfied by the alpha, Čech, and Delaunay complexes in $\mathbb{R}^d$ for $m = d - 1$.

While covering a wide class of distance-based simplicial complexes, there are several complexes used in practice that may fail to satisfy any or all of these. For example, the addition of a new point to the Delaunay complex, Gabriel graph, witness complex, or $k$-nearest neighbor graph can both add and remove simplices, violating (K1). Furthermore, there need not be any limit on the simplex diameter within any of these complexes, violating (D1). Likewise, the addition of a single point can alter simplices at arbitrarily large distances, violating (D2). As a special note, it is common in practice to consider the intersection of the Vietoris-Rips and Delaunay complexes, which unfortunately may violate all the assumptions here. It is unclear if an extension or special consideration could be made to incorporate these types of complexes.

6.2. Stabilization Results

To apply the general bootstrap theorem, here we provide several technical lemmas establishing locally-determined radii of stability for persistent Betti numbers, $B$-bounded persistent Betti numbers, and the Euler characteristic. The results given here apply for general classes...
of simplicial complexes constructed over subsets of \( \mathbb{R}^d \), using the conditions listed previously. Reiterating, \( \mathcal{C}_{p,M} (\mathbb{R}^d) \) is the class of distributions \( G \) on \( \mathbb{R}^d \) with densities \( g \) such that \( \|g\|_p \leq M \). We have the following lemmas:

**Lemma 6.1.** Let \( F \in \mathcal{C}_{p,M} (\mathbb{R}^d) \) for some \( p > 2 \) and \( M < \infty \), and let \( \mathcal{K} = \{ K^r \}_{r \in \mathbb{R}} \) be a filtration of simplicial complexes satisfying (K2), (D2), and (D3). Then for any \( r \in \mathbb{R} \), \( s \in \mathbb{R} \), and \( q \geq 0 \), \( \beta_{q,r}^r (\mathcal{K}) \) satisfies (S2) for \( F \).

**Lemma 6.2.** Let \( \mathcal{K} \) satisfy (K1). Then for any \( B \geq 0 \), \( r \in \mathbb{R} \), \( s \in \mathbb{R} \), \( q \geq 0 \), and \( z \in \mathbb{R}^d \), \( \rho_z = 2B \) is a locally determined radius of stabilization for \( \beta_{q,r}^r (\mathcal{K}) \) centered at \( z \).

**Lemma 6.3.** Let \( \mathcal{K} \) satisfy (D2). Then for any \( B \geq 0 \), \( r \in \mathbb{R} \), \( s \in \mathbb{R} \), \( q \geq 0 \), and \( z \in \mathbb{R}^d \), \( \rho_z = 2\max \{ \phi (r), \phi (s) \} + 2B \) is a locally determined radius of stabilization for \( \beta_{q,B}^r (\mathcal{K}) \) centered at \( z \).

Since in both Lemma 6.2 and Lemma 6.3 the radius of stabilization is a deterministic constant, (S2) is satisfied for any distribution \( G \). The same is true in the following results for the truncated Euler characteristics.

**Lemma 6.4.** Let \( \mathcal{K} \) satisfy (K1) and (D1). Then for any \( z \in \mathbb{R}^d \) and \( q \geq 0 \), \( \rho_z = \phi \) is a locally determined radius of stabilization for \( \chi_q (\mathcal{K}) \) centered at \( z \).

**Lemma 6.5.** Let \( \mathcal{K} \) satisfy (D2). Then for any \( z \in \mathbb{R}^d \) and \( q \geq 0 \), \( \rho_z = 2\phi \) is a locally determined radius of stabilization for \( \chi_q (\mathcal{K}) \) centered at \( z \).

### 6.3. Bootstrap Results

Here we present the main applied theorems of this paper. Each is derived from Theorem 2.7 and the corresponding stabilization lemmas. For given vectors of birth and death times, \( \vec{r} = (r_i)_{i=1}^k \) and \( \vec{s} = (s_i)_{i=1}^k \), let \( \beta_{q,r}^{r,s} = (\beta_{q,r}^{r,s})_{i=1}^k \) denote the multivariate function whose components are the persistent Betti numbers evaluated at each pair of birth and death times. Likewise, let \( \beta_{q,B}^{r,s} = (\beta_{q,B}^{r,s})_{i=1}^k \) be the corresponding \( B \)-bounded analog. For a vector of filtration times \( \vec{r} = (r_i)_{i=1}^k \), let \( \chi_{q,r}^r \) denote the multivariate function giving the \( q \)-truncated Euler characteristic at each time \( r_i \), with \( \chi_{q,r}^r (\mathcal{K}) := (\chi_q (K^{r_i}))_{i=1}^k \). Similarly, let \( \chi_{q,B}^{r,s} := (\chi_q (K^{r_i}))_{i=1}^k \) denote the multivariate Euler characteristic.

The following apply for \( F \in \mathcal{P} (\mathbb{R}^d) \) with density \( f \) such that \( \|f\|_p < \infty \) for some \( p > 2 \), as specified. \( F \), \( Q \) and \( (h_n)_{n \in \mathbb{N}} \) are such that \( \|\hat{f}_{n,h_n} - f\|_1 \to 0 \) and \( \|\hat{f}_{n,h_n} - f\|_p \to 0 \) in probability (resp. a.s.). Let \( X_n = \{ X_i \}_{i=1}^n \overset{iid}{\sim} F \) and \( (m_n)_{n \in \mathbb{N}} \) such that \( \lim_{n \to \infty} m_n = \infty \). \( X_{m_n}^* = \{ X_i^* \}_{i=1}^{m_n} \overset{iid}{\sim} F_{n,h_n} | X_n \) is a bootstrap sample and \( G \) a multivariate distribution. Recalling the conclusion of Theorem 2.7, for a multivariate statistic \( \vec{\psi} \):

\[
\frac{1}{\sqrt{n}} \left( \vec{\psi} (\sqrt{n} X_n) - \mathbb{E} \left[ \vec{\psi} (\sqrt{n} X_n) \right] \right) \overset{d}{\to} G
\]

(6.1)

if and only if

\[
\frac{1}{\sqrt{m_n}} \left( \vec{\psi} (\sqrt{m_n} X_{m_n}^*) - \mathbb{E} \left[ \vec{\psi} (\sqrt{m_n} X_{m_n}^*) \right] \right) \overset{d}{\to} G \text{ in probability (resp. a.s.)}
\]
Theorem 6.6 (Persistent Betti Numbers). Let $q \geq 0$ and $p > 2q + 3$. Let $\mathcal{K}$ be a filtration of simplicial complexes satisfying (K1), (K2), (D1), and (D3). Then for any given $\vec{r}, \vec{s}$, (6.1) holds for $\beta_{q}^{\vec{r},\vec{s}}$.

Theorem 6.7 (Persistent Betti Numbers – Alt.). Let $q \geq 0$ and $p > 2q + 5$. Let $\mathcal{K}$ be a filtration of simplicial complexes satisfying (K2), (D2), and (D3). Then for any given $\vec{r}, \vec{s}$, (6.1) holds for $\beta_{q}^{\vec{r},\vec{s}}$.

The only differences between the above theorems are the conditions satisfied by the underlying simplicial complex and the necessary norm bound on the density. The corresponding results for the Betti numbers follow as special cases of Theorem 6.6 and Theorem 6.7, when the given birth and death parameters are equal ($\beta_{q}^{r} = \beta_{q}^{r,r}$). Also, although the statements of Theorem 6.6 and Theorem 6.7 are given in terms of a fixed feature dimension $q$, a direct extension exists if $q$ is allowed to differ for each $(r_i, s_i)$. The form as given shows the dependence of the density norm assumption on the chosen feature dimension.

The higher value of $p$ required in Theorem 6.7 compared to 6.6 can be motivated intuitively based on the assumptions used. For the persistent Betti numbers, the main quantity controlling convergence is the expected number of simplices altered or introduced when a new datapoint is added to the sample. (D2) ensures that these simplices fall within a small ball around the new data point. The stated density norm conditions control the expected number of points, and by extension possible simplices, that can lie within that small ball. Introducing (K1) further controls the number of possible simplices, and allows for a weakening of the necessary norm condition. (K1) requires that, as the sample grows by a single point, any additional simplices must contain the new point as a vertex, and no deletion of simplices is possible. This means that every added simplex has one less “free” vertex, and a weaker norm condition is required for control. The same intuition applies to Theorems 6.8-6.12 below, whenever (K1) is assumed.

In the specific case of the alpha complex, both of the above Theorems 6.6 and 6.7 apply. While the alpha complex does not satisfy (K1), it has equal persistent Betti numbers to the Čech complex, which does. Thus, the weaker conditions of Theorem 6.6 are sufficient in this unique case.

Theorem 6.8 (B-Bounded Persistent Betti Numbers). Let $q \geq 0$ and $p > 2q + 3$. Let $\mathcal{K}$ be a filtration of simplicial complexes satisfying (K1). Then for any given $\vec{r}, \vec{s}$, and $B > 0$, (6.1) holds for $\beta_{q,B}^{\vec{r},\vec{s}}$.

Note the difference in necessary conditions between Theorem 6.6 and Theorem 6.8. Theorem 6.6 notably requires a translation-invariant simplicial complex, along with the elimination of small loops via (D3). Theorem 6.8 imposes relatively few assumptions on the underlying simplicial complex. As a general statement, it can be seen that the $B$-bounded persistent Betti numbers defined here are better behaved than the unbounded persistent Betti numbers. However, it is suspected that some of the simplicial complex assumptions can be relaxed in the persistent Betti number case, but the extent to which this is possible is still unknown. Furthermore, the $B$-bounded persistent Betti numbers allow for an explicit rate calculation for the 2-Wasserstein metric in Proposition 2.6.

Theorem 6.9 (Truncated Euler Characteristic). Let $m < \infty$ and $p > 2m + 3$. Let $\mathcal{K}$ be a filtration of simplicial complexes satisfying (K1), (D1), and (D4) for $m$. Then for any given $\vec{r}$, (6.1) holds for $\chi^{\vec{r}}$. 
Theorem 6.10 (Truncated Euler Characteristic - Alt.). Let $m < \infty$ and $p > 2m + 5$. Let $K$ be a filtration of simplicial complexes satisfying (D2) and (D4) for $m$. Then for any given $\vec{r}$, (6.1) holds for $\chi^{\vec{r}}$.

Proof. We prove together Theorems 6.9 and 6.10. Recall that the Euler characteristic $\chi$ can be written as an alternating (finite) sum of the Betti numbers when (D4) holds. As mentioned after the proposition statement, since Proposition 2.6 holds for the Betti numbers in dimensions $0 \leq q \leq m$ under the assumed conditions (see the proofs of Theorems 6.6 and 6.7), then the same holds for their (alternating) sum, namely the Euler characteristic. The proof of Theorem 2.7 applies without alteration. \qed

Theorem 6.11 (Euler Characteristic). Let $q \geq 0$ and $p > 2q + 1$. Let $K$ be a filtration of simplicial complexes satisfying (K1) and (D1). Then for any given $\vec{r}$, (6.1) holds for $\chi^{\vec{r}}_q$.

Theorem 6.12 (Euler Characteristic - Alt.). Let $q \geq 0$ and $p > 2q + 3$. Let $K$ be a filtration of simplicial complexes satisfying (D2). Then for any given $\vec{r}$, (6.1) holds for $\chi^{\vec{r}}_q$.

As stated, Theorems 6.6-6.12 only establish consistency of the bootstrap in the multivariate setting, not any convergence rates to the limiting distributions. Proposition 2.6 does allow for a rate calculation in the 2-Wasserstein distance between the bootstrap and true sampling distributions in the case of truncated Euler characteristics and $B$-bounded persistent Betti numbers. However, an overall rate calculation for the bootstrap requires knowledge of the convergence rate to the asymptotic distribution $G$ for the original statistic. For persistent Betti numbers in the multivariate setting, general central limit theorems have been shown in [28], but little is known at this time about the rate of convergence to the limiting multivariate normal. As such, we may only conclude consistency of the bootstrap for the functions considered.

In the following, let $D_{\gamma,r_0}(C)$ be the class of distributions $G$ with support on a bounded $C \subset \mathbb{R}^d$ such that $\int_{B_r(x)} dG \geq \gamma r^d$ for all $r \leq r_0$ and $x \in C$.

Theorem 6.13 (Total Edge Length of the $k$-Nearest Neighbor Graph). Let $p > 2$. Furthermore, let $F \in D_{\gamma,r_0}(C)$ and $1 \left\{ \hat{F}_n,h_n \in D_{\gamma,r_0}(C) \right\} \rightarrow 1$ in probability (resp. a.s.). Then (6.1) holds for $l_{NN,k}$.

The conditions of Theorem 6.13 are in particular satisfied when $C$ is known and convex, with $f$ bounded below on $C$ by a constant, provided further that $\|f_n,h_n - f\|_\infty \rightarrow 0$ in probability (resp. a.s.). We include this final theorem to demonstrate the utility of stabilization as a general tool for proving bootstrap convergence theorems outside of topological data analysis. As the $k$-nearest neighbor graph does not fall in the general simplicial complex framework provided in Section 6.1, special treatment is needed to show the required stabilization and moment conditions. Here we rely on previous results from the literature.

7. Discussion

In this work we have shown the large-sample consistency of multivariate bootstrap estimation for a range of stabilizing statistics. This includes the persistent Betti numbers, $B$-bounded persistent Betti numbers, the Euler characteristic, and the total edge length of the $k$-nearest neighbor graph. However, many open questions still remain.

The results presented here apply only in the multivariate setting, the obvious extension being to stochastic processes. However, at this time it is unclear how to proceed for the persistent Betti numbers. Essential to a result of this type would be a convenient tail bound for
the radius of stabilization, which is yet unavailable. In the case of persistent Betti numbers, there is a strong relationship between the persistent Betti function and an empirical CDF in two dimensions. As such, there is much established theory in that regard which may be applied once stochastic equicontinuity is established.

In practice it is common that data comes not from a density in $\mathbb{R}^d$, but instead from a manifold. It is suspected that a version of the results in this paper could apply in the manifold setting. However, this requires a bootstrap that adapts to a possibly unknown manifold structure, similar to that found in [25]. Combined with the inherent challenges of working with manifolds, this extension presents many technical hurdles.

Also, we have considered in this work only one type of smoothed bootstrap, with deterministic choice of bandwidth. Several extensions could be made, using data-dependent distribution estimates. Furthermore, in this work we focused on consistency, not on rates of convergence to the limiting distribution. Extra work is needed in this regard. In the case of the $B$-bounded persistent Betti numbers and the Euler characteristic, we did derive rates of convergence in the 2-Wasserstein distance. The same is not true for the persistent Betti numbers. Several parameters within the proofs provided here can be tuned to optimize the rate of convergence, provided extra assumptions on the underlying density. For rates of convergence in the persistent Betti numbers case, again a tail bound for the radius of stabilization is essential.

Finally, there are several statistics of interest, including those based on the Delaunay complex, which do not fit into the specific frameworks provided here. However, it may be that these statistics may still satisfy Theorem 2.7 in the general case, by techniques others than those provided here. More work in this vein is necessary.

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**Appendix**

Throughout these proofs, we will make ample use of the Hölder, Jensen, and Minkowsky inequalities, along with the following. For brevity, these inequalities may be used implicitly and in combination. For \( m \in \mathbb{N} \), \( \{x_i\}_{i=1}^m \subset \mathbb{R} \), and \( k \geq 1 \),

\[
\left| \sum_{i=1}^{m} x_i^k \right| \leq m^{k-1} \cdot \left( \sum_{i=1}^{m} |x_i|^k \right). \tag{A.1}
\]

Likewise for \( 0 \leq k \leq 1 \)

\[
\left| \sum_{i=1}^{m} x_i^k \right| \leq \sum_{i=1}^{m} |x_i|^k. \tag{A.2}
\]

Next, for any density \( f \) and \( 1 \leq j \leq k \)

\[
\|f\|_{j}^{k} \leq \|f\|_{k}^{(j-1)/k}. \tag{A.3}
\]

Finally, for any set \( A \subseteq \mathbb{R}^d \) with \( |A| \) the Lebesgue measure of \( A \) and \( k \geq 1 \)

\[
\left( \int_{A} f(x) \, dx \right)^k \leq |A|^{k-1} \cdot \int_{A} |f(x)|^k \, dx. \tag{A.4}
\]

Furthermore, in each of the following, we use the simplified notation \( H_n(S, T) = \psi (\sqrt{n}S) - \psi (\sqrt{n}T) \) for the change in the statistic \( \psi \) when the underlying scaled point cloud is altered. In the multivariate case, given \( \psi = (\psi_j)_{j=1}^k \) we use the notation \( \tilde{H}_n(S, T) = (H_{n,j}(S, T))_{j=1}^k \), where \( H_{n,j} = \psi_j (\sqrt{n}S) - \psi_j (\sqrt{n}T) \).

**A.1. Proofs of Section 2.2**

**Lemma A.1** (Lemma 2.3). For \( p > 2 \), let \( \psi \) satisfy (E2) with \( u_a \leq p - 1 \) for some \( a > 2 \). Then for any \( M < \infty \), \( \psi \) satisfies (E1) for \( C_{p,M} (\mathbb{R}^d) \).

**Proof.** Let \( R > 0 \) and \( a > 2 \) be as given such that \( u_a \leq p - 1 \). Define \( I_n := \# \{ Y_n \cap B_{Y'} (R/\sqrt{n}) \} = \# \{ (\sqrt{n}Y_n) \cap B_{Y'} (R) \} \). Conditional on \( Y' \), \( I_n \) follows a binomial distribution with expectation \( n \int_{B_{Y'}} (R/\sqrt{n}) g(y) \, dy \), where \( g \) is a density of \( G \). By (E2), we have that

\[
\mathbb{E} \left[ \left| \psi\left(\sqrt{n}(Y_n \cup \{Y'\})\right) - \psi\left(\sqrt{n}Y_n\right)\right|^a \right] \tag{A.5}
\]

\[
\leq \mathbb{E} \left[ U_a (1 + I_n^u) \right] \tag{A.6}
\]

\[
\leq U_a (1 + \mathbb{E} [I_n^u]). \tag{A.7}
\]
Via Corollary (3) in [29], there is a universal constant $K$ such that the conditional $u_a$-th moment of $I_n$ is at most

$$
\left( \frac{K}{\log(u_a)} \right)^{u_a} \max \left\{ n \int_{B_{\sqrt{n}}} g(y) \, dy, \left( n \int_{B_{\sqrt{n}}} g(y) \, dy \right)^{u_a} \right\}
$$

$$
\leq \left( \frac{K}{\log(u_a)} \right)^{u_a} \left( \int_{B_{\sqrt{n}}} g(y) \, dy + \left( n \int_{B_{\sqrt{n}}} g(y) \, dy \right)^{u_a} \right).
$$

Removing the conditioning on $Y'$, for $V_d$ the volume of a unit ball in $\mathbb{R}^d$, we have

$$
\int_{\mathbb{R}^d} n \left( \int_{B_{\sqrt{n}}} g(y) \, dy \right)^{u_a} g(x) \, dx
$$

$$
= \int_{\mathbb{R}^d} \int_{B_d(R)} g \left( x + \frac{t}{\sqrt{n}} \right) g(x) \, dt \, dx
$$

$$
= \int_{B_d(R)} \int_{\mathbb{R}^d} g \left( x + \frac{t}{\sqrt{n}} \right) g(x) \, dx \, dt
$$

$$
\leq V_d R^d \|g\|_p^2
$$

$$
\leq V_d R^d \|g\|_{\infty}^p
$$

and

$$
\int_{\mathbb{R}^d} \left( n \int_{B_{\sqrt{n}}} g(y) \, dy \right)^{u_a} g(x) \, dx
$$

$$
= \int_{\mathbb{R}^d} \left( \int_{B_d(R)} g \left( x + \frac{t}{\sqrt{n}} \right) \, dt \right)^{u_a} g(x) \, dx
$$

$$
\leq (V_d R^d)^{u_a} \|g\|_{u_a+1}^{u_a+1}
$$

$$
\leq (V_d R^d)^{u_a} \|g\|_p^{u_a}
$$

$$
\leq \left( V_d R^d M^\frac{p}{p-1} \right)^{u_a}. \quad (A.8)
$$

Combining, we have

$$
\mathbb{E} \left[ \left| \psi \left( \sqrt{n} (Y_n \cup \{Y'\}) \right) - \psi \left( \sqrt{n} Y_n \right) \right| \right]^{u_a}
$$

$$
\leq U_a \left( 1 + \left( K \frac{u_a}{\log(u_a)} \right)^{u_a} \left( V_d R^d M^\frac{p}{p-1} + \left( V_d R^d M^\frac{p}{p-1} \right)^{u_a} \right) \right). \quad (A.9)
$$

Since this bound does not depend on $G$ or $n$, (E1) is satisfied by $\psi$ for $C_{p,M}(\mathbb{R}^d)$. \hfill \Box
**Lemma A.2** (Lemma 2.4). Let \( \psi \) satisfy (S2) for \( F \in C_{p,M} (\mathbb{R}^d) \). Then \( \psi \) satisfies (S1) for \( C_{p,M} (\mathbb{R}^d) \), \( F, b = (p - 2) / (d (p - 1)) \), and any \((l_\varepsilon)_{\varepsilon>0}\) such that \( \lim_{\varepsilon \to 0} l_\varepsilon \varepsilon^{(p-2)/(d(p-1))} = 0 \) and \( \lim_{\varepsilon \to 0} l_\varepsilon = \infty \).

**Proof.** Let \( \{X_i\}_{i \in \mathbb{N}} \overset{\text{id}}{\sim} F \) with \( X' \sim F \) an independent copy. Likewise, for \( G \in C_{p,M} (\mathbb{R}^d) \cap B_F (\varepsilon; d_{TV}) \), let \( \{Y_i\}_{i \in \mathbb{N}} \overset{\text{id}}{\sim} G \) with \( Y' \sim G \) an independent copy. Denote \( X_n := \{X_i\}_{i=1}^n \).

As \( d_{TV} (F,G) \leq \varepsilon \), it may be assumed that \( \{(X_i,Y_i)\}_{i \in \mathbb{N}} \) are iid with \( \mathbb{P} [X_i \neq Y_i] \leq \varepsilon \) for all \( i \in \mathbb{N} \).

Let \((l_\varepsilon)_{\varepsilon>0}\) be such that \( \lim_{\varepsilon \to 0} l_\varepsilon \varepsilon^{(p-2)/(d(p-1))} = 0 \). Define the following sets:

\[
A_Y := \{Y' = X'\} \tag{A.11}
\]

\[
B_{Y,l_\varepsilon} := \left\{ Y_n \cap B_{X'} \left( \frac{l_\varepsilon}{\sqrt{n}} \right) = X_n \cap B_{X'} \left( \frac{l_\varepsilon}{\sqrt{n}} \right) \right\} \tag{A.12}
\]

\[
C_{l_\varepsilon} := \{ \rho \psi_{X'} \left( \sqrt{n}X_n \right) \leq l_\varepsilon \}. \tag{A.13}
\]

By the local-definition criterion (2.5), we have \( A_Y \cap B_{Y,l_\varepsilon} \cap C_{X,l_\varepsilon} \subseteq \{ \rho \psi_{X'} \left( \sqrt{n}Y_n \right) \leq l_\varepsilon \} \). Then

\[
\mathbb{P}^* \left[ \rho \psi_{X'} \left( \sqrt{n}Y_n \right) > l_\varepsilon \right] \leq \mathbb{P}^* \left[ A_Y \cup B_{Y,l_\varepsilon} \cup C_{l_\varepsilon} \right] \leq \mathbb{P} [A_Y] + \mathbb{P} \left[ B_{Y,l_\varepsilon} \right] + \mathbb{P}^* \left[ C_{l_\varepsilon} \right]. \tag{A.14}
\]

Bounding each piece, \( \mathbb{P} [A_Y] = \mathbb{P} [X' \neq Y'] \leq \varepsilon \). Likewise, by (S2) we have \( \mathbb{P}^* \left[ C_{l_\varepsilon} \right] = \mathbb{P}^* \left[ \rho \psi_{X'} \left( \sqrt{n}X_n \right) > l_\varepsilon \right] \leq p_\varepsilon \), with \( p_\varepsilon \) not depending on \( G \) or \( n \) such that \( \lim_{\varepsilon \to 0} p_\varepsilon = 0 \). It thus remains to be shown that \( B_{Y,l_\varepsilon} \) occurs with small probability, uniformly in \( n \) and \( G \).

The sample pairs which contribute to \( X_n \cap B_{X'} \left( \frac{l_\varepsilon}{\sqrt{n}} \right) \) but not \( Y_n \cap B_{X'} \left( \frac{l_\varepsilon}{\sqrt{n}} \right) \) are those \((X_i,Y_i)\) for which \( X_i \neq Y_i \) and either \( \|X_i - X\| \leq l_\varepsilon / \sqrt{n} \) or \( \|Y_i - X\| \leq l_\varepsilon / \sqrt{n} \). Conditional on \( X' \), their count follows a binomial distribution with expectation at most \( n \mathbb{P} [X_i \neq Y_i] \int_{B_{X'} \left( \frac{l_\varepsilon}{\sqrt{n}} \right)} \hat{f} (y) + \hat{g} (y) \ dy \). Here \( \hat{f} \) and \( \hat{g} \) are the densities of \( X_i \) and \( Y_i \) conditional on the event \( \{X_i \neq Y_i\} \). These densities can be shown to exist via the absolute continuity of \( F \) and \( G \) with respect to the Lebesgue measure on \( \mathbb{R}^d \). Subsequently, we have that \( \| \hat{f} \|_p \leq \| \hat{f} \|_p / \mathbb{P} [X_i \neq Y_i] \leq M / \mathbb{P} [X_i \neq Y_i] \) and \( \| \hat{g} \|_p \leq M / \mathbb{P} [X_i \neq Y_i] \). Removing the conditioning on \( X' \), via Hölder’s inequality the expected number of pairs which contribute
to $X_n \triangle Y_n$ within $B_{X'}(l_\epsilon/\sqrt{n})$ is at most
\[
\int_{\mathbb{R}^d} \left( nP [X_i \neq Y_i] \int_{B_{\epsilon}(1/\sqrt{n})} f(y) + \bar{g}(y) \ dy \right) f(x) \ dx
\]
\[
= P [X_i \neq Y_i] \int_{\mathbb{R}^d} \int_{B_{\epsilon}(l_\epsilon)} \left( f(x + t/\sqrt{n}) + \bar{g}(x + t/\sqrt{n}) \right) f(x) \ dx \ dt
\]
\[
= P [X_i \neq Y_i] \int_{\mathbb{R}^d} \int_{B_{\epsilon}(l_\epsilon)} \left( f(x + t/\sqrt{n}) + \bar{g}(x + t/\sqrt{n}) \right) f(x) \ dx \ dt
\]
\[
\leq P [X_i \neq Y_i] V_d l_\epsilon^d \left( \| \bar{f} \|_{L_p} + \| \bar{g} \|_{L_p} \right) \| f \|_p
\]
\[
\leq P [X_i \neq Y_i] M V_d l_\epsilon^d \left( \| \bar{f} \|_{L_p} + \| \bar{g} \|_{L_p} \right)
\]
\[
\leq 2P [X_i \neq Y_i] M V_d l_\epsilon^d \left( \frac{M}{P [X_i \neq Y_i]} \right)^{1/\gamma}
\]
\[
\leq 2 M P^{\frac{p}{2}} V_d l_\epsilon^d e^{\frac{\epsilon^2}{2}}.
\]  

This also provides a bound on the probability that $X_n$ and $Y_n$ coincide within $B_{X'}(l_\epsilon/\sqrt{n})$. Thus we have that $P [B_{Y'_{l_\epsilon}}] \leq 2 M P^{\frac{p}{2}} V_d l_\epsilon^d e^{\frac{\epsilon^2}{2}}$. The bound does not depend on $G$ or $n$, with
\[
\lim_{\epsilon \to 0} 2 M P^{\frac{p}{2}} V_d l_\epsilon^d e^{\frac{\epsilon^2}{2}} = 2 M P^{\frac{p}{2}} V_d \left( \lim_{\epsilon \to 0} l_\epsilon e^{\frac{\epsilon^2}{1-(p-1)}} \right)^d = 0.
\]  

Finally, by the definition of a radius of stabilization we have that
\[
P \left[ D_{\sqrt{n}Y'} \left( (\sqrt{n}Y_n) \cap B_{\sqrt{n}} (l_\epsilon) \right) \neq D_{\sqrt{n}Y'} (Y_n) \right]
\]
\[
\leq P^* \left[ \rho \sqrt{n}Y', (\sqrt{n}Y_n) > l_\epsilon \right]
\]
\[
\leq \epsilon + p_\epsilon + 2 M P^{\frac{p}{2}} V_d l_\epsilon^d e^{\frac{\epsilon^2}{2}}.
\]  

Here the final quantity does not depend on $G$ or $n$, and goes to 0 as $\epsilon \to 0$. Thus (S1) is satisfied.

\[\square\]

**Lemma A.3** (Theorem 2.5). Let $F \in C_{p,M}$ with $p > 2$ and $M < \infty$. Let $\rho_0$ be a locally-determined radius of stabilization for $\psi$ centered at 0. Suppose that for any given $a,b \in (0,\infty)$, and $\delta > 0$, there exists an $L_{a,b,\delta} < \infty$ and a measurable set $A_{a,b,\delta}$ with $\rho_0^{-1}((L_{a,b,\delta}, \infty)) \subseteq A_{a,b,\delta}$ such that
\[
\sup_{\lambda \in [a,b]} P^* \left[ \rho_0 (P_\lambda) > L_{a,b,\delta} \right] \leq \sup_{\lambda \in [a,b]} P \left[ P_\lambda \in A_{a,b,\delta} \right] \leq \delta.
\]  

Then for any $\delta > 0$ there exists an $n_\delta < \infty$ and $L_\delta < \infty$ such that
\[
\sup_{n \geq n_\delta} P^* \left[ \rho_0 (X_n - X') > L_\delta \right] \leq \delta.
\]  

**Proof.** We consider $n \geq n_0$. Define two independent sets of random variables $(U_i)_{i=1}^\infty \overset{iid}{\sim} F$ and $(U_i')_{i=1}^\infty \overset{iid}{\sim} F$. For $N \sim \text{Pois} (n)$, denote by $P_n$ the Poisson process given by $\{U_1\}_{i=1}^N$, having intensity $nf$ over $\mathbb{R}^d$. We will couple this Poisson process to $X_n$. \{$U_i\}_{i=1}^N \cap \{U_i'\}_{i=1}^{(n-N)+}$
has the same distribution as \( X_n \), thus we assume that the two random variables are equal. For a given random variable \( U_i \) or \( U^*_i \) and \( L > 0 \), the probability of falling within \( B_{X'} (L/\sqrt{n}) \) is bounded, as shown below. Applying the Cauchy-Schwartz inequality, we have

\[
\int_{\mathbb{R}^d} \int_{B_{x'}(\frac{L}{\sqrt{n}})} f(y) f(x) \, dy \, dx \\
= \int_{\mathbb{R}^d} \int_{B_0(\frac{L}{\sqrt{n}})} f(x+t) f(x) \, dt \, dx \\
= \int_{B_0(\frac{L}{\sqrt{n}})} \int_{\mathbb{R}^d} f(x+t) f(x) \, dx \, dt \\
\leq \frac{V_d L^d}{n} \|f\|_2^2 \\
\leq \frac{V_d L^d}{n} \|f\|_{\frac{p}{p-1}}^p \\
\leq \frac{V_d L^d M_{\frac{p}{p-1}}}{n}.
\]

The expected number of points within \( B_{X'} (L/\sqrt{n}) \) that contribute to \( P_n \triangle X_n \) is then at most

\[
\mathbb{E} \left[ |N - n| \frac{V_d L^d M_{\frac{p}{p-1}}}{n} \right] \leq \frac{V_d L^d M_{\frac{p}{p-1}}}{n} \sqrt{\text{Var} [N]} \leq \frac{M_{\frac{p}{p-1}} V_d L^d}{\sqrt{n}}. \tag{A.23}
\]

As the number of differing points is an integer-valued random variable, this expectation bounds the probability that \( X_n \) and \( P_n \) differ within \( B_{X'} (L/\sqrt{n}) \). For a fixed value of \( L \) and sufficiently large \( n \), the bound can be made arbitrarily small.

Next, we will couple the Poisson process \( P_n \) with a conditionally homogeneous approximation. We construct the following coupling: Let \( T \) be a homogeneous Poisson process on \( \mathbb{R}^d \times [0, \infty) \) with unit intensity. The point process given by \( \{U_i \text{ s.t. } (U_i, T_i) \in T, T_i \leq n f(U_i)\} \) is then a nonhomogeneous Poisson process with intensity \( n f \). We can safely assume that this process equals \( P_n \). Define the point process \( H_n := \{U_i \text{ s.t. } (U_i, T_i) \in T \text{ and } T_i \leq n f(X')\} \).

Conditional on \( X' \), \( H_n \) is a homogeneous Poisson process with intensity \( n f(X') \). The number of observations within \( B_{X'} (L/\sqrt{n}) \) that contribute to \( P_n \triangle H_n \) follows a Poisson distribution with rate parameter

\[
\int_{B_{X'} \left(\frac{L}{\sqrt{n}}\right)} |n f(y) - n f(X')| \, dy \tag{A.24}
\]

Removing the conditioning on \( X' \), the expected number is

\[
\int_{\mathbb{R}^d} \left( n \int_{B_{x'}(\frac{L}{\sqrt{n}})} |f(y) - f(x)| \, dy \right) f(x) \, dx \tag{A.25}
\]

As the expectation above is an upper bound for the probability that \( P_n \) and \( H_n \) fail to coincide within \( B_{X'} (L/\sqrt{n}) \), we show that this quantity can be made arbitrarily small. Consider \( C \), the set of Lebesgue points of \( f \). We have that \( C^c \) has Lebesgue measure 0 by
the Lebesgue differentiation theorem. By the definition of a Lebesgue point, we may write

$$C = \bigcap_{\gamma > 0} \bigcup_{\Delta > 0} \bigcap_{\delta \leq \Delta} \left\{ x \in \mathbb{R} \text{ s.t. } \frac{\int_{B_x(\delta)} |f(y) - f(x)| \, dy}{V_d \delta^d} \leq \gamma \right\}$$  \quad (A.26)

Here $V_d$ denotes the volume of a unit ball in $\mathbb{R}^d$. Now as $f$ is a density, it may be shown that $\int_{B_x(\delta)} |f(y) - f(x)| \, dy/V_d \delta^d$ is a jointly continuous function of $x$ and $\delta$, and therefore it is measurable. Via the continuity with respect to $\delta$, we need only consider rational $\delta \leq \Delta$, because the rationals are dense in the reals. Thus,

$$C_{\Delta,\gamma} := \bigcap_{\delta \leq \Delta} \left\{ x \in \mathbb{R} \text{ s.t. } \frac{\int_{B_x(\delta)} |f(y) - f(x)| \, dy}{V_d \delta^d} \leq \gamma \right\}$$

is a countable intersection of measurable sets. Finally, by the Archimedean principle and other standard calculus arguments, we may assume $\gamma$ and $\Delta$ also come from a countable class, $\{1/n : n \in \mathbb{N}\}$, for example. Let $C_{\gamma} := \cup_{\Delta > 0} C_{\Delta,\gamma}$. We have that $C_{\delta,\gamma}$ and $C_{\gamma}$ are measurable with $\lim_{\Delta \to 0} C_{\delta,\gamma}^c = C_{\gamma}^c$ and $\lim_{\gamma \to 0} C_{\gamma}^c = C^c$. By continuity of measure, the Lebesgue measure of $C_{\gamma}$ must go to 0, as well for $\int_{C_{\gamma}^c} f(x) \, dx$. We decompose the integral in (A.25) as follows. For any integer $1 < a \leq p - 1$, an application of Hölder’s inequality gives

\[
\int_{\mathbb{R}^d} \left( \frac{1}{n} \int_{B_x(\frac{t}{\sqrt{n}})} |f(y) - f(x)| \, dy \right)^{\frac{a}{p}} f(x) \, dx \\
= \int_{\mathbb{R}^d} \left( \frac{1}{n} \int_{B_x(\frac{t}{\sqrt{n}})} |f(y) - f(x)| \, dy \right) f(x) \mathbb{1} \left\{ x \in \frac{C}{\sqrt{n}, \gamma} \right\} f(x) \, dx \\
+ \int_{\mathbb{R}^d} \left( \frac{1}{n} \int_{B_x(\frac{t}{\sqrt{n}})} |f(y) - f(x)| \, dy \right) f(x) \mathbb{1} \left\{ x \in C_{\Delta,\gamma}^c \right\} f(x) \, dx \\
\leq \gamma V_d L^d \\
+ \left( \int_{\mathbb{R}^d} \left( \int_{B_0(L)} \left| x + \frac{t}{\sqrt{n}} \right| - f(x) \right)^{\frac{a}{p}} f(x) \, dx \right)^{\frac{p}{a}} P \left[ X' \in \frac{C_{\Delta,\gamma}}{\sqrt{n}, \gamma} \right]^{1 - \frac{p}{a}}. \quad (A.27)
\]
For the integral above
\[
\int_{\mathbb{R}^d} \left( \int_{B_0(L)} \left| f \left( x + \frac{t}{\sqrt{n}} \right) - f (x) \right| \, dt \right)^a \, dx \\
\leq (VdL^d)^{a-1} \int_{\mathbb{R}^d} \int_{B_0(L)} \left| f \left( x + \frac{t}{\sqrt{n}} \right) - f (x) \right|^a \, dt \, dx \\
= (VdL^d)^{a-1} \int_{B_0(L)} \int_{\mathbb{R}^d} \left| f \left( x + \frac{t}{\sqrt{n}} \right) - f (x) \right|^a \, dt \, dx \\
\leq 2^{a-1} (VdL^d)^{a-1} \int_{B_0(L)} \int_{\mathbb{R}^d} \left( f \left( x + \frac{t}{\sqrt{n}} \right) + f (x)^a \right) f (x) \, dt \, dx \\
\leq (2VdL^d)^a \| f \|_{p+1}^{a} \\
\leq (2VdL^d)^a \| f \|_{p}^{a} \\
\leq \left( 2VdL^d M^{\frac{p}{p-1}} \right)^a.
\]
Thus (A.27) is at most
\[
\gamma VdL^d + 2VdL^d M^{\frac{p}{p-1}} \mathbb{P} \left[ X' \in C^c_{\frac{V}{\sqrt{n}} \gamma} \right]^{1-\frac{a}{2}} \quad (A.28)
\]
\[
\leq \gamma VdL^d + 2VdL^d M^{\frac{p}{p-1}} \mathbb{P} \left[ X' \in C^c_{\frac{V}{\sqrt{n}} \gamma} \right]^{\frac{a}{p-1}}. \quad (A.29)
\]

This provides a bound for the probability that \( P_n \) and \( H_n \) fail to coincide within \( B_{X'} (L/ \sqrt{n}) \). The bound holds in the limiting \( p = \infty \) case and can be made arbitrarily small for \( \gamma \) sufficiently small and \( n \) sufficiently large. \( \gamma \) can be chosen as a function of \( F, L, \) and \( n \) to provide the tightest bound, but this requires specific knowledge of \( f \). Combining with the previous steps, we have coupled \( X_n \) and \( H_n \) to be equal with arbitrarily high probability.

Now for \( \eta, \zeta > 0 \) define \( D_{*, \eta} = f^{-1} ([\eta, \infty)) \) and \( D_{\zeta}^* = f^{-1} ([0, \zeta]) \). For \( \eta \) sufficiently small and \( \zeta \) sufficiently large, \( \mathbb{P} \left[ X' \in D_{*, \eta}^c \right] \) and \( \mathbb{P} \left[ X' \in D_{\zeta}^c \right] \) can be made arbitrarily small.

By assumption, for any given \( \eta, \zeta, \) and \( \nu > 0 \) there is an \( L_{\eta, \zeta, \nu} \) and a measurable set \( A_{\eta, \zeta, \nu} \) such that for any homogeneous Poisson process \( Q_\lambda \) on \( \mathbb{R}^d \) with intensity \( \lambda \) bounded between \( \eta \) and \( \zeta \), be have \((\rho_0)^{-1} ([L_{\eta, \zeta, \nu}, \infty)) \leq A_{\eta, \zeta, \nu} \) and \( \mathbb{P}^* [\rho_0 (Q_\lambda) > L_{\eta, \zeta, \nu}] \leq \mathbb{P} [Q_\lambda \in A_{\eta, \zeta, \nu}] \leq \nu \). \( L_{\eta, \zeta, \nu} \) is possibly increasing as \( \eta \to 0 \), \( \zeta \to \infty \), and \( \nu \to 0 \).

As \( \sqrt{n} (H_n - X') \) is a homogeneous Poisson process, conditional on \( X' \in D_{*, \eta} \cup D_{\zeta}^c \), we have
\[
\mathbb{P}^* \left[ \rho_0 (\sqrt{n} (H_n - X')) > L_{\eta, \zeta, \nu} \right] X' \in D_{*, \eta} \cup D_{\zeta}^c \] \quad (A.30)
\[
\leq \mathbb{P} \left[ \sqrt{n} (H_n - X') \in A_{\eta, \zeta, \nu} \mid X' \in D_{*, \eta} \cup D_{\zeta}^c \right] \quad (A.31)
\[
= \mathbb{E} \left[ \mathbb{P} \left[ \sqrt{n} (H_n - X') \in A_{\eta, \zeta, \nu} \mid X' \right] X' \in D_{*, \eta} \cup D_{\zeta}^c \right] \quad (A.32)
\[
\leq \nu. \quad (A.33)
\]
Lemma A.4 (Proposition 2.6). For $p > 2$ and $M < \infty$, let $\psi$ satisfy (E1) and (S1) for $C_{p,M}(\mathbb{R}^d)$, $F \in C_{p,M}(\mathbb{R}^d)$, and some $a > 2$. Then for any $G \in C_{p,M}(\mathbb{R}^d) \cap B_F(\epsilon, d_{TV})$, there exist iid coupled random variables $\{(X_i, Y_i)\}_{i \in \mathbb{N}}$ such that $X_n = \{X_i\}_{i=1}^n \overset{iid}{\sim} F$, $Y_n = \{Y_i\}_{i=1}^n \overset{iid}{\sim} G$, and

$$\sup_{n \in \mathbb{N}} \text{Var} \left[ \frac{1}{\sqrt{n}} \left( \psi \left( \sqrt{n}X_n \right) - \psi \left( \sqrt{n}Y_n \right) \right) \right] \leq \gamma_\epsilon.$$  

$\gamma_\epsilon$ is a value not depending on $G$ such that $\lim_{\epsilon \to 0} \gamma_\epsilon = 0$.

Proof. The proof is inspired by that of Proposition (5.4) in [28]. We expand using a martingale difference sequence (MDS). Let $\{(X_i, Y_i)\}_{i=1}^\infty$ be iid such that $\{X_i\}_{i=1}^\infty \overset{iid}{\sim} F$ and $\{Y_i\}_{i=1}^\infty \overset{iid}{\sim} G$. Each pair $(X_i, Y_i)$ can be identically coupled such that $P[X_i \neq Y_i] = d_{TV}(F, G) \leq \epsilon$. For $X := \{X_i\}_{i=1}^\infty$ and $Y := \{Y_i\}_{i=1}^\infty$ let $\mathcal{F}_j := \sigma\{X_j, Y_j\}$ with $\mathcal{F}_0 := \{\Omega, \emptyset\}$. For $(X', Y')$ an independent copy of the $(X_i, Y_i)$, let

$$X'_{n,j} := \{X_1, \ldots, X_{j-1}, X', X_{j+1}, \ldots, X_n\}$$
$$Y'_{n,j} := \{Y_1, \ldots, Y_{j-1}, Y', Y_{j+1}, \ldots, Y_n\}.$$
Using the orthogonality of a MDS and the conditional Jensen inequality, we have

\[
\text{Var} \left[ \frac{1}{\sqrt{n}} H_n (X_n, Y_n) \right] = \frac{1}{n} \text{Var} \left[ H_n (X_n, Y_n) \right]
\]

\[
= \frac{1}{n} \sum_{j=1}^{n} E \left[ \left( H_n (X_n, Y_n) - H_n (X'_n, Y'_n) \right)^2 \right]
\]

\[
= \frac{1}{n} \sum_{j=1}^{n} E \left[ \left( H_n (X_n, Y_n) - H_n (X'_n, Y'_n) \left| F_j \right. \right)^2 \right]
\]

\[
\leq \frac{1}{n} \sum_{j=1}^{n} \text{Var} \left[ \left. H_n (X_n, Y_n) - H_n (X'_n, Y'_n) \right| F_j \right]
\]

\[
= \frac{1}{n} \sum_{j=1}^{n} \text{Var} \left[ \left. H_n (X_n, Y_n) - H_n (X'_n, Y'_n) \right| F_j \right]
\]

\[
\leq \frac{1}{n} \sum_{j=1}^{n} \left[ H_n (X_n, Y_n) - H_n (X'_n, Y'_n) \right]^2
\]

\[
= E \left[ \left. H_n (X_n, Y_n) - H_n (X'_n, Y'_n) \right| F_j \right]^2.
\]

The above holds for any \(1 \leq j \leq n\). We have

\[
E \left[ \left. H_n (X_n, Y_n) - H_n (X'_n, Y'_n) \right| F_j \right]^2
\]

\[
\leq 2E \left[ \left. H_n (X_n \cup X', X_n) - H_n (Y_n \cup Y', Y_n) \right| F_j \right]^2
\]

\[
+ 2E \left[ \left. H_n (X_n \cup X', X'_n) - H_n (Y_n \cup Y', Y'_n) \right| F_j \right]^2
\]

\[
= 4E \left[ \left. H_n (X_n \cup X', X_n) - H_n (Y_n \cup Y', Y_n) \right| F_j \right]^2.
\]

We will decompose the expectation in (A.43) using the stabilization of \(\psi\). Let \(L > 0\), and define the following sets. Note that when all three are satisfied, \(H_n (X_n \cup X', X_n) = H_n (Y_n \cup Y', Y_n)\).

\[
A_Y := \{Y' = X'\}
\]

\[
B_{Y,L} := \left\{ Y_n \cap B_{X'} \left( \frac{L}{\sqrt{n}} \right) = X_n \cap B_{X'} \left( \frac{L}{\sqrt{n}} \right) \right\}
\]

\[
C_{X,L} := \left\{ D_{\sqrt{n}} \psi_{\pi X} \left( \sqrt{n} X_n \right) \cap B_{\psi_{\pi X}} \left( L \right) = D_{\psi_{\pi X}} \left( \sqrt{n} X_n \right) \right\}
\]

\[
C_{Y,L} := \left\{ D_{\sqrt{n}} \psi_{\pi Y} \left( \sqrt{n} Y_n \right) \cap B_{\psi_{\pi Y}} \left( L \right) = D_{\psi_{\pi Y}} \left( \sqrt{n} Y_n \right) \right\}.
\]

Let \(C_{X,L} \subseteq C_{X,L}\) and \(C_{Y,L} \subseteq C_{Y,L}\) be measurable. We decompose the expectation in (A.43) along these events:
$$E \left[ H_n (X_n \cup X', X_n) - H_n (Y_n \cup Y', Y_n) \right]$$

$$= E \left[ H_n (X_n \cup X', X_n) - H_n (Y_n \cup Y', Y_n) \right] \mathbb{1} \{ A_Y \cap B_{Y,L} \cap C_{X,L*} \cap C_{Y,L*} \}$$

$$+ E \left[ H_n (X_n \cup X', X_n) - H_n (Y_n \cup Y', Y_n) \right] \mathbb{1} \{ A_Y^c \cup B_{Y,L}^c \cup C_{X,L*}^c \cup C_{Y,L*}^c \}$$

$$= E \left[ H_n (X_n \cup X', X_n) - H_n (Y_n \cup Y', Y_n) \right] \mathbb{1} \{ A_Y^c \cup B_{Y,L}^c \cup C_{X,L*}^c \cup C_{Y,L*}^c \}.$$ 

Let $a > 2$ satisfy (E1). Hölder’s inequality gives

$$E \left[ H_n (X_n \cup X', X_n) - H_n (Y_n \cup Y', Y_n) \right] \mathbb{1} \{ A_Y^c \cup B_{Y,L}^c \cup C_{X,L*}^c \cup C_{Y,L*}^c \}$$

$$\leq \left\| H_n (X_n \cup X', X_n) - H_n (Y_n \cup Y', Y_n) \right\| \frac{2}{2} \left[ A_Y^c \cup B_{Y,L}^c \cup C_{X,L*}^c \cup C_{Y,L*}^c \right]^{1-\frac{2}{2}}.$$ 

As the choice of $C_{X,L*}$ and $C_{Y,L*}$ was arbitrary, 

$$E \left[ H_n (X_n \cup X', X_n) - H_n (Y_n \cup Y', Y_n) \right]$$

$$\leq \left\| H_n (X_n \cup X', X_n) - H_n (Y_n \cup Y', Y_n) \right\| \frac{2}{2} \left[ A_Y^c \cup B_{Y,L}^c \cup C_{X,L*}^c \cup C_{Y,L*}^c \right]^{1-\frac{2}{2}}$$

$$\leq \max \left\{ \mathbb{P} [ A_Y^c ] + \mathbb{P} [ B_{Y,L}^c ] + \mathbb{P} [ C_{X,L*}^c ] + \mathbb{P} [ C_{Y,L*}^c ], 1 \right\}^{1-\frac{2}{2}}.$$ 

Consider the norm above:

$$\left\| H_n (X_n \cup X', X_n) - H_n (Y_n \cup Y', Y_n) \right\| \frac{2}{2}$$

$$\leq 2 \left( \left\| H_n (X_n \cup X', X_n) \right\| \frac{2}{2} + \left\| H_n (Y_n \cup Y', Y_n) \right\| \frac{2}{2} \right)$$

$$\leq 4E \frac{2}{2}.$$ 

This bound does not depend on $\epsilon$, $G$, or $n$. It thus remains to show that, for a certain choice of $L$ and as $\epsilon \to 0$, that each of the events $A_Y^c$, $B_{Y,L}^c$, $C_{X,L*}^c$, and $C_{Y,L*}^c$ can be made to occur with small outer probability, uniformly in $G$ and $n$. For $A^c$, this is satisfied because $\mathbb{P} [ X' \neq Y' ] \leq \epsilon$. We then consider $B_{Y,L}^c$. As seen in the proof of Lemma 2.4, the probability that $X_n$ and $Y_n$ coincide within $L$ is at most 

$$2M \frac{\beta}{2} \eta d \epsilon \frac{2}{2}.$$ 

Let $(l_c)_{c>0}$ satisfy (S1) and $L = l_c$. We have that $\mathbb{P} [ B_{Y,l_c}^c ] \leq 2M \frac{\beta}{2} \eta d \epsilon \frac{2}{2} \to 0.$ 

By (S1), both $\mathbb{P} [ C_{X,l_c}^c ]$ and $\mathbb{P} [ C_{X,l_c}^c ]$ are bounded above by a quantity $p_c$ such that $\lim_{c \to 0} p_c = 0.$ 

Let $\delta_c = \min \left\{ \epsilon + 2M \frac{\beta}{2} \eta d \epsilon \frac{2}{2} + 2p_c, 1 \right\}$ be the derived upper bound for 

$$\mathbb{P} [ A_Y^c \cup B_{Y,l_c}^c \cup C_{X,l_c}^c \cup C_{X,l_c}^c ].$$
We achieve a final upper bound for (A.40) of

\[ 16E_a^{\frac{2}{3}} b_t^1 \delta_t^{1 - \frac{2}{3}}. \]  

(A.51)

If (S1) is satisfied for many \((l_\epsilon)_{\epsilon>0}\) such that \(\lim_{\epsilon\to0} l_\epsilon (p-2)/(d(p-1)), \) \(l_\epsilon\) can be further chosen to optimize the rate of \(\delta_t\), provided a rate for \(p_\epsilon\). Furthermore, if (E1) is satisfied for more than one \(a > 2\), \(a = a_\epsilon\) may be chosen to optimize the final rate as \(\epsilon \to 0\). Such considerations depend on the specifics of the statistic \(\psi\) and the density assumptions used.

\[ \square \]

### A.2. Proofs of Section 2.3

**Theorem A.5** (Theorem 2.7). Let \(F \in \mathcal{P}(\mathbb{R}^d)\) with density \(f\) such that \(\|f\|_p < \infty\) for some \(p > 2\). Furthermore, let \(F, \, Q, \) and \((h_n)_{n\in\mathbb{N}}\) be such that \(\|\hat{f}_{n,h_n}(x) - f(x)\|_1 \to 0\) and \(\|\hat{f}_{n,h_n}(x) - f(x)\|_p \to 0\) in probability (or a.s.). Suppose \(\tilde{\psi}: \hat{X}(\mathbb{R}^d) \to \mathbb{R}^k\) has component functions \(\psi_j: \hat{X}(\mathbb{R}^d) \to \mathbb{R},\ 1 \leq j \leq k\) satisfying (E1) and (S1) for \(C_{p,M}(\mathbb{R}^d),\ M > \|f\|_p\), \(F\) and \(b = (p-2)/(d(p-1))\). Then for a sample \(X_n = \{X_i\}_{i=1}^n \overset{\text{iid}}{\sim} F,\ (m_n)_{n\in\mathbb{N}}\) such that \(\lim_{n\to\infty} m_n = \infty\), a bootstrap sample \(X_{m_n}^* = \{X_i^*\}_{i=1}^{m_n} \overset{\text{iid}}{\sim} \hat{F}_{n,h_n}\), and a multivariate distribution \(G\),

\[
\frac{1}{\sqrt{m_n}} \left( \tilde{\psi} \left( \frac{1}{\sqrt{m_n}} X_{m_n}^* \right) - \mathbb{E} \left[ \tilde{\psi} \left( \frac{1}{\sqrt{m_n}} X_{m_n}^* \right) \right] \right) \overset{d}{\to} G
\]

if and only if

\[
\frac{1}{\sqrt{m_n}} \left( \tilde{\psi} \left( \frac{1}{\sqrt{m_n}} X_{m_n}^* \right) - \mathbb{E} \left[ \tilde{\psi} \left( \frac{1}{\sqrt{m_n}} X_{m_n}^* \right) \right] \right) \overset{d}{\to} G \text{ in probability (or a.s.)}
\]

**Proof.** For any bounded, Lipschitz function \(v: \mathbb{R}^k \to \mathbb{R}\), consider the functional given by \(V_{m_n} := \mathbb{E} \left[ v \left( \hat{H}_{m_n}(Y_{m_n}) \right) \right]\), where \(Y_{m_n} = \{Y_i\}_{i=1}^{m_n}\) is an iid sample, and the functional takes as input the shared distribution of the \(Y_i\). Let \(v\) be bounded within \([-L, L]\) with a Lipschitz constant of \(L\). First assuming that \(\hat{H}_{m_n}(X_{m_n}) \overset{d}{\to} G\), we have \(V_{m_n}(F) \overset{\text{TV}}{\to} \int_{\mathbb{R}} v \, dG\).

Now, let \(X_{m_n}' = \{X_{m_n,i}'\}_{i=1}^{m_n} \overset{\text{iid}}{\sim} F\) be an independent copy of \(X_{m_n} = \{X_i\}_{i=1}^{m_n}\). Furthermore, as in the proof of Proposition 2.6, \(X_{m_n}'\) and \(X_{m_n}^*\) can be coupled so that \(\mathbb{P} [X_{m_n,i}' \neq X_{m_n,i}^*] = d_{TV} \left( F, \hat{F}_{n,h} \right) = \|\hat{f}_{n,h} - f\|_1/2\), conditional on \(X_n\). Via Proposition 2.6 and Chebyshev’s
inequality, for \( \delta > 0 \) we have almost surely that

\[
V_{m_n} \left( \hat{f}_{n,h} \right) \\
= E \left[ v \left( \bar{H}_{m_n} \left( X_{m_n}^* \right) \right) \right] X_n \\
= E \left[ v \left( \bar{H}_{m_n} \left( X_{m_n}^* \right) \right) \mathbb{1} \left\{ \left\| \bar{H}_{m_n} \left( X_{m_n}^* \right) - \bar{H}_{m_n} \left( X_{m_n}' \right) \right\| \leq \delta \right\} \right] X_n \\
+ E \left[ v \left( \bar{H}_{m_n} \left( X_{m_n}^* \right) \right) \mathbb{1} \left\{ \left\| \bar{H}_{m_n} \left( X_{m_n}^* \right) - \bar{H}_{m_n} \left( X_{m_n}' \right) \right\| > \delta \right\} \right] X_n \\
\leq E \left[ v \left( \bar{H}_{m_n} \left( X_{m_n}^* \right) \right) \right] + \delta |X_n| \\
+ L \left( 2 - \delta \right) \sum_{j=1}^{m_n} \mathbb{P} \left[ |H_{m_n,j} \left( X_{m_n}^* \right) - H_{m_n,j} \left( X_{m_n}' \right) | > \delta \right] \mathbb{1} \{ \| \hat{f}_{n,h} \|_p \leq M \} \\
+ L \left( 2 - \delta \right) \mathbb{1} \{ \| \hat{f}_{n,h} \|_p > M \} \\
\leq E \left[ v \left( \bar{H}_{m_n} \left( X_{m_n}^* \right) \right) \right] + \delta \\
+ L \left( 2 - \delta \right) \left( \sum_{j=1}^{m_n} \frac{k \gamma \| \hat{f}_{n,h} - f \|_1}{\delta^2} \right) \mathbb{1} \{ \| \hat{f}_{n,h} \|_p \leq M \} + \mathbb{1} \{ \| \hat{f}_{n,h} \|_p > M \} . \tag{A.54}
\]

Here \( \gamma \| \hat{f}_{n,h} - f \|_1/2 \) is as given in Proposition 2.6 applied to \( \psi_j \) for \( \epsilon = \| \hat{f}_{n,h} - f \|_1/2 \). Similarly, almost surely

\[
V_{m_n} \left( \hat{f}_{n,h} \right) \\
\geq E \left[ v \left( \bar{H}_{m_n} \left( X_{m_n}^* \right) \right) \right] - \delta \tag{A.53} \\
- L \left( 2 - \delta \right) \left( \sum_{i=1}^{k \gamma \| \hat{f}_{n,h} - f \|_1}{\delta^2} \right) \mathbb{1} \{ \| \hat{f}_{n,h} \|_p \leq M \} + \mathbb{1} \{ \| \hat{f}_{n,h} \|_p > M \} . \tag{A.54}
\]

As \( \| \hat{f}_{n,h} - f \|_p \to 0 \) and \( M > \| f \|_p \), we have that the lower bound for \( V_{m_n} \left( \hat{f}_{n,h} \right) \) converges to \( \int_R v \ dG - L\delta \) and the upper bound converges to \( \int_R v \ dG + L\delta \), either in probability or a.s., depending on assumptions. Since this holds for any \( \delta > 0 \), we have that \( V_{m_n} \left( \hat{f}_{n,h} \right) \to \int_R v \ dG \) in probability (or a.s.).

Now we will show the converse direction. Let \( X_{m_n}^* \) and \( X_{m_n}' \) be as previously defined.
We have

$$V_{m,n}(F)$$

$$= \mathbb{E} \left[ \mathbb{E} \left[ v \left( \hat{H}_{m,n} \left( X_{m,n}^* \right) \right) \left| X_n \right. \right] \right]$$

$$\leq \mathbb{E} \left[ \mathbb{E} \left[ v \left( \hat{H}_{m,n} \left( X_{m,n}^* \right) \right) \left| X_n \right. \right] + L \delta$$

$$+ L(2 - \delta) \mathbb{E} \left[ \min \left\{ \sum_{i=1}^{k} \frac{k \gamma_{|f_{n,h}|}^{1/f_{n,h} - 1/f_{n,h}^*}}{\delta^2}, 1 \right\} 1 \left\{ \| \hat{f}_{n,h} \|_F \leq M \right\} + 1 \left\{ \| \hat{f}_{n,h} \|_F > M \right\} \right]$$

and

$$V_{m,n}(F)$$

$$\geq \mathbb{E} \left[ \mathbb{E} \left[ v \left( \hat{H}_{m,n} \left( X_{m,n}^* \right) \right) \left| X_n \right. \right] - L \delta$$

$$- L(2 - \delta) \mathbb{E} \left[ \min \left\{ \sum_{i=1}^{k} \frac{k \gamma_{|f_{n,h}|}^{1/f_{n,h} - 1/f_{n,h}^*}}{\delta^2}, 1 \right\} 1 \left\{ \| \hat{f}_{n,h} \|_F \leq M \right\} + 1 \left\{ \| \hat{f}_{n,h} \|_F > M \right\} \right] .$$

Each expectation involves only bounded random variables, thus the lower bound converges to \( \int_R v \, dG - L \delta \) and the upper bound to \( \int_R v \, dG + L \delta \), assuming \( \mathbb{E} \left[ v \left( \hat{H}_{m,n} \left( X_{m,n}^* \right) \right) \left| X_n \right. \right] \rightarrow \int_R v \, dG. \) This holds if the assumed convergence is either in probability or almost sure. Since this holds for any \( \delta > 0 \), we have \( V_{m,n}(F) \rightarrow \int_R v \, dG. \) Since our initial choice of \( v \) was arbitrary, the desired result follows.

\[\square\]

**A.3. Proofs of Section 6.2**

**Lemma A.6** (Lemma 6.1). Let \( F \in C_{p,M}(\mathbb{R}^d) \) for some \( p > 2 \) and \( M < \infty \), and let \( K = \{K_r\}_{r \in \mathbb{R}} \) be a filtration of simplicial complexes satisfying (K2), (D2), and (D3). Then for any \( r \in \mathbb{R}, s \in \mathbb{R}, \) and \( q \geq 0, \beta_{q}^* (K) \) satisfies (S2) for \( F. \)

**Proof.** We start by defining a crude locally-determined radius of stabilization. Let \( K \) be either \( K^r \) or \( K^s. \) Denote \( \phi = \max \{ \phi(r), \phi(s) \} \) as given by (D2). For \( z \in \mathbb{R}^d, S \in X(\mathbb{R}^d), \) and \( a > \phi, \) consider the connected components in \( K (S \cap B_z(a)) \) and \( K ((S \cap B_z(a)) \cup \{ z \}) \) with at least one simplex entirely contained within \( B_z(\phi). \) By (D2), if these components are entirely contained within \( B_z(a - \phi), \) no simplices will be added or removed from them within \( K (S \cap B_z(b)) \) or \( K ((S \cap B_z(b)) \cup \{ z \}) \) for any \( b > a. \) This property holds for both \( K^s \) or \( K^r. \) The persistent Betti numbers are additive with respect to connected components, thus the add-\( z \) cost is entirely defined by those components altered by the inclusion of \( z, \) which necessarily must include one simplex within \( B_z(\phi). \) As such, \( a \) is a locally determined radius of stabilization for \( S \) in this case. Any changes to the simplices outside of \( a \) must contribute to different connected components, and thus do not influence the add-\( z \) cost.

Now, \( X_n \) contains \( n \) total points. Including one point within \( B_z(\phi), \) the longest possible chain of \( n \) connected points reaches at most a radius of \( n \phi. \) Therefore, \( \rho_{\phi X_n} (\sqrt{n}X_n) = (n + 1) \phi \) is a locally-determined radius of stabilization on \( \sqrt{n}X_n \) centered at \( \sqrt{n}X', \) as shown in the previous paragraph. However, since this radius grows with \( n, \) it alone cannot provide for the desired result.
Given (D2) and (D3), by Theorem 4.3 in [28] and the proof thereof, there exists a locally-determined radius of stabilization $\rho_0^*$ for $\beta_{q,B}^*(K)$ centered at 0 such that the conditions of Lemma 2.5 are satisfied. It must be noted that the original statement of the referenced lemma does not give this result directly. However, a careful analysis of the provided proof yields this more general result with minimal additions, and is not restated here. By (K2), we may define a radius of stabilization $\rho_0^*$ for $\beta_{q,B}^*(S)$ centered at $z \in \mathbb{R}^d$ with $\rho_0^*(S) = \rho_0^*(S-z)$. Thus, for any $\delta > 0$, there exists an $L_\delta < \infty$ and $n_\delta < \infty$ such that $\mathbb{P}[\rho_0^*(\sqrt{n}X_n) \leq \delta]$ for all $n \geq N_\delta$.

Denote by $P_z (S)$ the union of all connected components in either $K(S)$ or $K(S \cup \{0\})$ with at least one simplex entirely contained within $B_\delta (\phi)$. For any center point $z \in \mathbb{R}^d$, define $\rho_z : \mathcal{X} \to [0, \infty]$ with $\rho_z (S) = \min \{ \text{diam} (P_z (S)) + \phi, \rho^\star (S-z) \}$. We have that $\rho_z$ is a locally-determined radius of stabilization. Furthermore, for $n < n_\delta$, we have that $\rho_{\sqrt{n}X_n} (\sqrt{n}X_n) \leq (n_\delta + 1) \phi$ almost surely. Likewise, for $n \geq n_\delta$, we have $\rho_{\sqrt{n}X_n} (\sqrt{n}X_n) \leq \rho_{\sqrt{n}X_n} (\sqrt{n}X_n) \leq L_\delta$ with probability at least $1 - \delta$. Therefore, $\sup_{n \in \mathbb{N}} \mathbb{P}[\rho_{\sqrt{n}X_n} (\sqrt{n}X_n) > \max \{ L_\delta, (n_\delta + 1) \phi \}] \leq \delta$, and the result follows.  

**Lemma A.7 (Lemma 6.2).** Let $K$ satisfy (K1). Then for any $B \geq 0$, $r \in \mathbb{R}$, $s \in \mathbb{R}$, $q \geq 0$, and $z \in \mathbb{R}^d$, $\rho_z = 2B$ is a locally determined radius of stabilization for $\beta_{q,B}^*(K)$ centered at $z$.

**Proof.** Let $a \geq 2B$ and $S \in \mathcal{X} (\mathbb{R}^d)$. We decompose $Z_{q,B} (K^\ast ((S \cap B_z (a)) \cup \{ z \}))$ into three spaces. Let $U_z$ be spanned by the generators of $Z_{q,B} (K^\ast ((S \cap B_z (a)) \cup \{ z \}))$ with $z$ as a vertex. Let $U_a$ be spanned by the generators with vertices within $B_z (a) \setminus B_z (2B)$. Finally, let $U_s$ be spanned by the generators without $z$ as a vertex and with no vertices within $B_z (a) \setminus B_z (2B)$. Since the generators of $Z_{q,B} (K^\ast ((S \cap B_z (a)) \cup \{ z \}))$ have diameter at most $B$, there are no generating cycles with vertices both at $z$ and in $B_z (a) \setminus B_z (2B)$.

By (K1) we have $Z_{q,B} (K^\ast ((S \cap B_z (a)) \cup \{ z \})) = U_z + U_a + U_s$. Similarly, $Z_{q,B} (K^\ast ((S \cap B_z (a))) = U_a + U_s$, and $Z_{q,B} (K^\ast ((S \cap B_z (2B))) = U_a$. Now for any cycle within $U_z$, the associated vertex set must lie within $B_z (B)$. Likewise, for any cycle in $U_a$, the associated vertex set must lie within $B_z (a) \setminus B_z (B)$. These vertex sets cannot intersect, thus $U_z \cap U_a = \emptyset$.

Now, consider any vector spaces $X$, $Y$, and $Z$ such that $X \cap Y = \{ 0 \}$. Because $X \cap Y \cap Z$ is a subspace of $X \cap Y$, it is also the trivial space $\{ 0 \}$. We have

$$\dim (X+Y+Z) - \dim (Y+Z) = \dim (X) - \dim (X \cap Y) - \dim (X \cap Z) + \dim (X \cap Y \cap Z)$$

$$= \dim (X) - \dim (X \cap Z) = \dim (X+Z) - \dim (Z).$$

We use this result in each of the following. We have

$$\dim (Z_{q,B} (K^\ast ((S \cap B_z (a)) \cup \{ z \})) = \dim (Z_{q,B} (K^\ast ((S \cap B_z (a))))$$

$$= \dim (U_z + U_a + U_s) - \dim (U_a + U_s)$$

$$= \dim (Z_{q,B} (K^\ast ((S \cap B_z (2B))) \cup \{ z \})) - \dim (Z_{q,B} (K^\ast ((S \cap B_0 (2B))))).$$

A similar result holds for the boundaries. Let $V_z$, $V_a$, and $V_s$ be defined similarly to $U_z$, $U_a$, and $U_s$, respectively, instead using the generators of $B_{q,B} (K^\ast ((S \cap B_z (a)) \cup \{ z \})).$
Similarly $B_{q,B}(K^*((S \cap B_z(a)) \cup \{z\})) = V_z + V_\alpha + V_\gamma$, $B_{q,B}(K^*((S \cap B_z(2B)) \cup \{z\})) = V_z + V_\gamma$, and we conclude $V_z \cap V_\alpha = \{0\}$. Furthermore, we have $U_z \cap V_\alpha = V_z \cap U_\alpha = \{0\}$ by similar vertex-based arguments. Then $(U_z + V_z) \cap (U_\alpha + V_\alpha) = \{0\}$. We have
\[
\dim (Z_{q,B}(K^*((S \cap B_z(a)) \cup \{z\})) \cap B_{q,B}(K^*((S \cap B_z(a)) \cup \{z\})) - \dim (Z_{q,B}(K^*(S \cap B_z(a))) \cap B_{q,B}(K^*(S \cap B_z(a)))) = \dim ((U_z + U_\alpha + V_z) \cap (V_z + V_\alpha + V_\gamma)) - \dim ((U_\alpha + U_z) \cap (V_z + V_\alpha + V_\gamma)) = \dim ((U_\alpha + U_z) \cap (V_z + V_\alpha + V_\gamma) - \dim (U_\alpha + U_z + V_\alpha + V_z + V_\gamma) = \dim (Z_{q,B}(K^*(S \cap B_z(2B)) \cup \{z\})) \cap B_{q,B}(K^*((S \cap B_z(2B)) \cup \{z\})) - \dim (Z_{q,B}(K^*(S \cap B_z(2B))) \cap B_{q,B}(K^*(S \cap B_z(2B)))).
\]
Combining these pieces, the $B$-bounded persistent Betti numbers must stabilize after a constant radius of $\rho_z = 2B$. 

**Lemma A.8** (Lemma 6.3). Let $K$ satisfy (D2). Then for any $B \geq 0$, $r \in \mathbb{R}$, $s \in \mathbb{R}$, $q \geq 0$, and $z \in \mathbb{R}^d$, $\rho_z = 2\max \{\phi(r), \phi(s)\} + 2B$ is a locally determined radius of stabilization for $\beta^{a,r}_{q,B}(K)$ centered at $z$.

**Proof.** Denote by $\phi := \max(\phi(r), \phi(s))$. Let $S \in \mathcal{X}(\mathbb{R}^d)$. Furthermore, let $T$ be any finite multiset of points in $\mathbb{R}^d$ with $S \cap B_z(2\phi + 2B) \subseteq T$ and $y \notin B_z(2\phi + 2B)$. We have the following partition. Let $U_z$, $U_y$, and $U_\alpha$, respectively, be spanned by the generators of $Z_{q,B}(K^*(T))$ having: a simplex within $B_z(\phi)$, a simplex within $B_y(\phi)$, or neither. Let $U_z^\#$ be spanned by the generators of $Z_{q,B}(K^*(T \cup \{z\}))$ have a simplex in $B_z(\alpha)$). Finally, let $U_y^\#$ be spanned by the generators of $Z_{q,B}(K^*(T \cup \{y\}))$ have a simplex in $B_y(\phi)$.

By (D2) we have $Z_{q,B}(K^*(T)) = U_z + U_\alpha + U_y$, $Z_{q,B}(K^*(T \cup \{z\})) = U_z^\# + U_\alpha + U_y$, $Z_{q,B}(K^*(T \cup \{y\})) = U_z^\# + U_z + U_y^\#$, and $Z_{q,B}(K^*(T \cup \{z, y\})) = U_z^\# + U_\alpha + U_y^\#$.

Now for any cycle within $U_z$, the associated vertex set must lie within $B_z(\phi + B)$. Likewise, for any cycle in $U_y$, the associated vertex set must lie within $B_y(\phi + B)$. Because $|y - z| > 2\phi + 2B$, these vertex sets cannot intersect, thus $U_z \cap U_y = \{0\}$. Likewise $U_z \cap U_y^\# = U_z^\# \cap U_y = U_z^\# \cap U_y^\# = \{0\}$.

Now, for any vector spaces $X$, $X^*$, $Y$, and $Z$ such that $X \cap Y^* = X^* \cap Y^* = \{0\}$, we have
\[
\dim (X^* + Y^* + Z) - \dim (X + Y^* + Z) = \dim (X) - \dim (X^*) + \dim (X \cap Y^*) + \dim (X \cap Z) - \dim (X^* \cap Y^*) - \dim (X^* \cap Z) + \dim (X^* \cap Y^* \cap Z) - \dim (X \cap Y^* \cap Z) = \dim (X) - \dim (X^*) + \dim (X \cap Z) - \dim (X^* \cap Z) = \dim (X + Z) - \dim (X^* + Z).
\]

Thus we have
\[
\dim (Z_{q,B} (K^* (T \cup \{z, y\}))) - \dim (Z_{q,B} (K^* (T \cup \{y\})))
\]
\[
= \dim (U^*_z + U_z + U^*_y) - \dim (U_z + U_y + U^*_y)
\]
\[
= \dim (U^*_z + U_z) - \dim (U_z + U_z)
\]
\[
= \dim (U^*_z + U_z + U_y) - \dim (U_z + U_y + U_y)
\]
\[
= \dim (Z_{q,B} (K^* (T \cup \{z\}))) - \dim (Z_{q,B} (K^* (T)))
\]

A similar result holds for the boundaries. Let \( V_z, V^*_z, V_y, V^*_y \), and \( V \) be defined similarly to \( U_z, U^*_z, U_y, U^*_y \), and \( U \), respectively, instead using the generators of \( B_{q,B} (K^* (T)) \), \( B_{q,B} (K^* (T \cup \{z\})) \), and \( B_{q,B} (K^* (T \cup \{y\})) \). Similarly \( B_{q,B} (K^* (T)) = V_z + V_y + V^*_y \), \( B_{q,B} (K^* (T \cup \{z\})) = V_z + V_y + V^*_y \), and \( B_{q,B} (K^* (T \cup \{y\})) = V_z + V_y + V^*_y \). We conclude \( V_z \cap V_y = V_z \cap V_y = V_z \cap V_y = V^*_z \cap V^*_y = \{0\} \). Furthermore, we have \( U_z \cap V_y = U_z \cap V_y = U^*_z \cap V_y = U^*_z \cap V^*_y = \{0\} \) by similar vertex-based arguments. Thus \( U_z + V_z \cup (U_y + V_y) = (U_z + V_z) \cap (U^*_y + V^*_y) = (U_z + V_z) \cap (U_y + V_y) = (U_z + V^*_z + V^*_y + V_y) = \{0\} \). We have
\[
\dim (Z_{q,B} (K^* (T \cup \{z, y\}))) \cap B_{q,B} (K^* (T \cup \{z, y\}))
\]
\[
= \dim (Z_{q,B} (K^* (T \cup \{z\}))) \cap B_{q,B} (K^* (T \cup \{y\}))
\]
\[
= \dim ((U^*_z + U_z) \cap (V^*_z + V_y + V^*_y)) - \dim ((U^*_z + U_z) \cap (V_z + V_y + V^*_y))
\]
\[
= \dim (U^*_z + U_z + V^*_z + V_y + V^*_y) - \dim (U^*_z + U_z + V^*_z + V_y + V^*_y)
\]
\[
= \dim (U^*_z + U_z + U_y) - \dim (U_z + U_y + V^*_z + V_y) + \dim (U_z + U_z + U^*_y + V_z + V_y)
\]
\[
= \dim ((U^*_z + U_z + U_y) \cap (V^*_z + V_y + V^*_y)) - \dim ((U_z + U_y) \cap (V_z + V_y + V^*_y))
\]
\[
= \dim (Z_{q,B} (K^* (T \cup \{z\}))) \cap B_{q,B} (K^* (T \cup \{z\}))
\]
\[
= \dim (Z_{q,B} (K^* (T \cup \{y\}))) \cap B_{q,B} (K^* (T \cup \{y\}))
\]

Thus, the addition of \( y \) to \( T \) does not change the add-\( z \) cost. We proceed inductively. Starting with \( S \cap B_z (2\phi + 2B) \), for any \( a > 2\phi + 2B \), the finitely many points of \( (S \cap B_z (a)) \setminus (S \cap B_z (2\phi + 2B)) \) may be added one at a time, while leaving the add-\( z \) cost unchanged. Thus, the \( B \)-bounded persistent Betti numbers must stabilize after a constant radius of \( \rho_z = 2\phi + 2B \). \( \Box \)

**Lemma A.9** (Lemma 6.4). Let \( K \) satisfy (K1) and (D1). Then for any \( z \in \mathbb{R}^d \) and \( q \geq 0 \), \( \rho_z = \phi \) is a locally determined radius of stabilization for \( \chi_q (K) \) centered at \( z \).

**Proof.** Let \( a \geq \phi \). By (K1) and (D1), we can partition \( K ((S \cap B_z (a)) \cup \{z\}) \) into the sets
\[
U := \{ \sigma \in K ((S \cap B_z (a)) \cup \{z\}) \text{ s.t. } z \in \sigma \} \quad \text{(A.59)}
\]
\[
V := \{ \sigma \in K ((S \cap B_z (a)) \cup \{z\}) \text{ s.t. } \sigma \subset B_z (\phi) \setminus \{z\} \} \quad \text{(A.60)}
\]
\[
W := \{ \sigma \in K ((S \cap B_z (a)) \cup \{z\}) \text{ s.t. } \sigma \cap B_z (a) \setminus B_z (\phi) \neq \emptyset \} \quad \text{(A.61)}
\]

Condition (D1) gives that no simplices may simultaneously have \( z \) as a vertex and intersect \( B_z (a) \setminus B_z (\phi) \), thus \( U, V, \) and \( W \) indeed partition \( K ((S \cap B_z (a)) \cup \{z\}) \). Condition
Lemma A.10 (Lemma 6.5) gives that the addition of \( \{ z \} \) and \( S \cap (B_z(a) \setminus B_z(\phi)) \) to \( S \cap B_z(\phi) \) may only introduce simplices to \( K (S \cap B_z(\phi)) \) with vertices somewhere within \( S \cap (B_z(a) \setminus B_z(\phi)) \cup \{ z \} \), and thus not included in \( V \). Therefore, \( V \subseteq K (S \cap B_z(\phi)) \). Furthermore, since \( S \cap B_z(\phi) \subseteq (S \cap B_z(a)) \cup \{ z \}, K (S \cap B_z(\phi)) \subseteq V \). Thus we have \( V = K (S \cap B_z(\phi)) \). Using similar arguments, condition (K1) also gives \( K ((S \cap B_z(\phi))\cup \{ z \}) = U \cup V \) and \( K (S \cap B_z(a)) = V \cup W \).

For \( U_k, V_k, \) and \( W_k \) denoting the set of \( k \)-simplices contained in \( U, V, \) and \( W, \) respectively, the add-\( z \) cost for the \( q \)-truncated Euler characteristic becomes

\[
\begin{align*}
\chi_q (K ((S \cap B_z(a)) \cup \{ z \})) - \chi_q (K (S \cap B_z(a))) &= \\
= & \sum_{k=0}^{q} (-1)^k \# \{ K_k ((S \cap B_z(a)) \cup \{ z \}) \} - \sum_{k=0}^{q} (-1)^k \# \{ K_k (S \cap B_z(a)) \} \\
= & \sum_{k=0}^{q} (-1)^k \# \{ U_k \} + \# \{ V_k \} + \# \{ W_k \} - \sum_{k=0}^{q} (-1)^k \# \{ V_k \} + \# \{ W_k \} \\
= & \sum_{k=0}^{q} (-1)^k \# \{ U_k \} + \# \{ V_k \} - \sum_{k=0}^{q} (-1)^k \# \{ V_k \} \\
= & \sum_{k=0}^{q} (-1)^k \# \{ K_k ((S \cap B_z(\phi)) \cup \{ z \}) \} - \sum_{k=0}^{q} (-1)^k \# \{ K_k (S \cap B_z(\phi)) \} \\
= & \chi_q (K ((S \cap B_z(\phi)) \cup \{ z \})) - \chi_q (K (S \cap B_z(\phi))).
\end{align*}
\]

We see that \( \chi_q (K) \) stabilizes after a constant radius of \( \rho_z = \phi \), thus the local-determination criterion is immediately satisfied.

\[\square\]

**Lemma A.10 (Lemma 6.5).** Let \( K \) satisfy (D2). Then for any \( z \in \mathbb{R}^d \) and \( q \geq 0, \rho_z = 2\phi \) is a locally determined radius of stabilization for \( \chi_q (K) \) centered at \( z \).

**Proof.** Let \( z \in \mathbb{R}^d \) and \( S \in \mathcal{X} (\mathbb{R}^d) \). Furthermore, let \( T \) be a finite multiset of points in \( \mathbb{R}^d \) such that \( S \cap B_z(2\phi) \subseteq T \). Let \( y \notin B_z(2\phi) \). Consider the partition

\[
\begin{align*}
U := \{ \sigma \in K (T) \text{ s.t. } \sigma \subset B_z(\phi) \} \\
U^* := \{ \sigma \in K (T \cup \{ z \}) \text{ s.t. } \sigma \subset B_z(\phi) \} \\
V := \{ \sigma \in K (T) \text{ s.t. } \sigma \subset B_y(\phi) \} \\
V^* := \{ \sigma \in K (T \cup \{ y \}) \text{ s.t. } \sigma \subset B_y(\phi) \} \\
W := \{ \sigma \in K (T) \text{ s.t. } \sigma \nsubseteq B_z(\phi) \text{ and } \sigma \nsubseteq B_y(\phi) \}.
\end{align*}
\]

Condition (D2) limits the influence of a single additional point on the complex to the ball of radius \( \phi \) around it. \( B_z(\phi) \cap B_y(\phi) = \emptyset \) because \( \|y - z\| > 2\phi \). Thus we have \( K(T) = U \cup W \cup V, K(T \cup \{ z \}) = U^* \cup W \cup V, K(T \cup \{ y \}) = U \cup W \cup V^*, \) and \( K(T \cup \{ y, z \}) = U^* \cup W \cup V^* \).

For \( U_k, U_k^*, V_k, V_k^*, \) and \( W_k \) denoting the set of \( k \)-simplices contained in \( U, U^*, V, V^*, \)
and $W$, respectively, the add-$z$ cost for the $q$-truncated Euler characteristic becomes

$$\chi_q(K(T \cup \{y, z\})) - \chi_q(K(T \cup \{y\}))$$

$$= \sum_{k=0}^{q} (-1)^k \# \{K_k(T \cup \{y, z\})\} - \sum_{k=0}^{q} (-1)^k \# \{K_k(T \cup \{y\})\}$$

$$= \sum_{k=0}^{q} (-1)^k (\# \{U_k^*\} + \# \{W_k\} + \# \{V_k^*\}) - \sum_{k=0}^{q} (-1)^k (\# \{V_k^*\} + \# \{W_k\})$$

$$= \sum_{k=0}^{q} (-1)^k \# \{K_k(T \cup \{z\})\} - \sum_{k=0}^{q} (-1)^k \# \{K_k(T)\}$$

We see that the addition of $\{y\}$ does not change the add-$z$ cost. Starting with $T = S \cap B_\delta(\phi)$, for any radius $\alpha > 2\phi$, $S \cap (B_\delta(\alpha) \setminus B_\delta(\phi))$ consists of finitely many points, which may be added to $S \cap B_\delta(\phi)$ in succession while leaving the add-$z$ cost unchanged. We conclude that $\rho_z = 2\phi$ is a radius of stabilization for $\chi_q(K)$, and is locally-determined by virtue of being constant.

\[\square\]

A.4. Proofs of Section 6.3

**Theorem A.11** (Theorem 6.6). Let $q \geq 0$ and $p > 2q + 3$. Let $K$ be a filtration of simplicial complexes satisfying (K1), (K2), (D1), and (D3). Then for any given $\vec{r}, \vec{s}$, (6.1) holds for $\beta^{\vec{r}, \vec{s}}_q$.

**Proof.** For given $r, s \in \mathbb{R}$, we will verify that assumption (E2) is satisfied for $\psi = \beta^{\vec{r}, \vec{s}}_q(K)$. Let $Y_n = \{Y_i\}_{i=1}^n$ be iid and $Y'$ an independent copy. By the Geometric Lemma (5.1), a bound for the change in persistent Betti numbers when $\{\sqrt[n]{Y'}\}$ is added to $\sqrt[n]{Y_n}$ is given by the number of new simplices introduced to the corresponding complexes. By (K1), (D1), it suffices to count the number of points within $\phi := \max\{\phi(r), \phi(s)\}$ of $\sqrt[n]{Y'}$, the combinations of which include any possible new simplices. Let $I_n = \sum_{i=1}^n 1\{\|Y_i - Y'\| \leq \phi/\sqrt[n]{\chi}\}$ for any $\alpha > 2$ we have

\[\beta^{\vec{r}, \vec{s}}_q(K(\sqrt[n]{Y_n} \cup \{Y'\})) - \beta^{\vec{r}, \vec{s}}_q(K(\sqrt[n]{Y_n}))\]

\[\leq \# \{K^*_q(Y_n) \setminus K^*_q(\sqrt[n]{Y_n})\}\]

\[\leq \left| \binom{I_n}{q} + \binom{I_n}{q+1} \right|^a\]

\[\leq \frac{1}{((q+1))} (I_n + 1)^a(q+1)\]

\[\leq 2^{a(q+1)-1} \left( I_n^{a(q+1)} + 1 \right) .\]
In this case $R = \phi$, $U_a < 2^{a(q+1) - 1}/((q + 1)!)^a$ and $u_a = a(q + 1)$. (E1) then follows from Lemma 2.3 for $p \geq a(q + 1) + 1 > 2q + 3$. As (K1) and (D1) together imply (D2), (S2) is satisfied as shown in Lemma 6.1. Then (S1) follows from Lemma 2.4. Finally an application of Theorem 2.7 gives the desired result.

Referring to Proposition 2.6 and the proof thereof, for $p < \infty$, using $a = (p - 1)/(q + 1)$ we achieve an optimal rate for $\gamma_\epsilon$ of

$$O \left( \delta_\epsilon^{1 - \frac{2q + 2}{p}} \right). \quad (A.73)$$

Details of the calculation are omitted here. For $p = \infty$, using $a = a_\epsilon = 2 - \log(\delta_\epsilon)$ we achieve an optimal rate of

$$O \left( \delta_\epsilon \left( \frac{- \log(\delta_\epsilon)}{\log(- \log(\delta_\epsilon))} \right)^{2q + 2} \right). \quad (A.74)$$

Both of these rates depend on $\delta_\epsilon$, the upper bound for the total probability found in the proof of Proposition 2.6. The techniques found in the proofs of Lemma 2.3 and Proposition 2.6 allow for a bound on $\delta_\epsilon$, provided a tail bound for $\sup_{n\in\mathbb{N}} \rho_0(\sqrt{n}(Y_n - Y'))$. At this time, such a bound is unavailable, thus no explicit rate calculation is possible. \hfill \Box

**Theorem A.12** (Theorem 6.7). Let $q \geq 0$ and $p > 2q + 5$. Let $K$ be a filtration of simplicial complexes satisfying (K2), (D2), and (D3). Then for any given $\bar{r}, \bar{s}$, (6.1) holds for $\beta_{\bar{r}, \bar{s}}$.

**Proof.** The proof follows exactly that of Theorem 6.6, thus we will omit many replicated details. Let $Y_n = \{Y_i\}_{i=1}^n$ be iid and $Y'$ an independent copy. Define $\phi := \max\{\phi(r), \phi(s)\}$.

Since we do not assume (K1) in this case, the addition of $\sqrt{n}Y'$ to the complex may both add and remove simplices, but only within $B_{\sqrt{n}Y'}(\phi)$ by (D2). Any additional simplices may have $\sqrt{n}Y'$ as a vertex, whereas any removed simplices may only have vertices within $\sqrt{n}Y_n$. For $I = \sum_{i=1}^n \mathbb{1} \{||Y_i - Y'|| \leq \phi/\sqrt{n}\}$, via the Geometric Lemma 5.1 we have

$$\left| \beta_{q,s}^r(K(\sqrt{n}(Y_n \cup \{Y'\})) - \beta_{q,s}^r(K(\sqrt{n}Y_n)) \right|$$

$$\leq \left| \beta_{q,s}^r(K(\sqrt{n}(Y_n \cup \{Y'\})) \cup K(\sqrt{n}Y_n)) - \beta_{q,s}^r(K(\sqrt{n}(Y_n \cup \{Y'\}))) \right|$$

$$+ \left| \beta_{q,s}^r(K(\sqrt{n}(Y_n \cup \{Y'\})) \cap K(\sqrt{n}Y_n)) - \beta_{q,s}^r(K(\sqrt{n}Y_n)) \right|$$

$$\leq \# \left\{ K_q^r(\sqrt{n}Y_n) \setminus K_q^r(\sqrt{n}(Y_n \cup \{Y'\})) \right\}$$

$$+ \# \left\{ K_{q+1}^r(\sqrt{n}Y_n) \setminus K_{q+1}^r(\sqrt{n}(Y_n \cup \{Y'\})) \right\}$$

$$+ \# \left\{ K_q^r(\sqrt{n}(Y_n \cup \{Y'\})) \setminus K_q^r(\sqrt{n}Y_n) \right\}$$

$$+ \# \left\{ K_{q+1}^r(\sqrt{n}(Y_n \cup \{Y'\})) \setminus K_{q+1}^r(\sqrt{n}Y_n) \right\}$$

$$\leq \binom{I_n}{q + 1} + \binom{I_n}{q + 2} + \binom{I_n + 1}{q + 1} + \binom{I_n + 1}{q + 2}$$

$$\leq 2 \left( \frac{I_n + 2}{q + 2} \right)^{q + 3} \binom{I_n + 1}{q + 2}.$$

Thus for any $a > 2$,

$$\left| \beta_{q,s}^r(K(\sqrt{n}(Y_n \cup \{Y'\}))) - \beta_{q,s}^r(K(\sqrt{n}Y_n)) \right|^a \leq \left( \frac{2^{a+1}(q + 2)}{(q + 2)!} \right)^{a} \left( \frac{I_n}{(q + 2)!} \right)^{a} \left( I_n(q + 2) + 1 \right).$$
Proof. Let \( Y_n = \{ Y_i \}_{i=1}^n \) be iid and \( Y' \) an independent copy. For a given \( q \geq 0, B \geq 0, \) and \( r, s \in \mathbb{R}, \) we will show that \( B^r_{q,B}(K) \) satisfies assumption (E2).

Applying Lemma 5.2, we must bound above the number of linearly independent \( B \)-bounded \( q \)-cycles and \( q \)-boundaries added when \( \sqrt{n}Y' \) is included with the sample \( \sqrt{n}Y_n \). We start by considering the cycles. By (K1) the addition of \( \sqrt{n}Y' \) will only introduce simplices to the complex having \( \sqrt{n}Y' \) as a vertex. As such, any \( B \)-bounded cycles in \( Z_q(K^r(\sqrt{n}(Y_n \cup \{Y'\}))) \) not having \( \sqrt{n}Y' \) as a vertex must already be in \( Z_q(K^r(\sqrt{n}Y_n)) \), and thus in \( Z_{q,B}(K^r(\sqrt{n}Y_n)) \). Thus, we must only bound the possible number of linearly independent \( B \)-bounded cycles within \( K^r(\sqrt{n}(\cup \{Y'\})) \) which have \( \sqrt{n}Y' \) as a vertex.

Let \( I_n := \sum_{i=1}^n 1 \{ ||Y_i - Y'|| \leq B/\sqrt{n} \} \) be the number of sample points falling within \( B \) of \( \sqrt{n} \).

We will construct a worst-case scenario. For any simplicial complexes \( J \subseteq K \), we have that \( Z_{q,B}(J) \subseteq Z_{q,B}(K) \). The addition of more simplices to \( K^r(\sqrt{n}(Y_n \cup \{Y'\}))) \) having \( \sqrt{n}Y' \) as a vertex may increase the dimension of \( Z_{q,B}(K^r(\sqrt{n}(Y_n \cup \{Y'\}))) \), but will not alter \( Z_{q,B}(K^r(\sqrt{n}Y_n)) \). As a worst case, we assume \( K^r \) is such that all possible simplices containing \( \sqrt{n}Y' \) are included. Thus, for any simplex \( \sigma \in K^r(\sqrt{n}Y_n) \) such that \( \sigma \subseteq (\sqrt{n}Y_n) \cap B\sqrt{n}Y' \) and \( \dim(\sigma) \leq B, \sigma(\sqrt{n}Y') \) has diameter at most \( B \), contains \( \sqrt{n}Y' \) as a vertex, and is a cycle within \( \sqrt{n}Y_n \).

Let \( U := \{ \partial(\sigma \cup \{ \sqrt{n}Y' \}) \} \quad \text{s.t.} \quad \sigma \subseteq (\sqrt{n}Y_n) \cap B\sqrt{n}Y' \) and \# \{ \sigma \} = q + 1 \} .

Now consider \( x \) to be any cycle in \( Z_q(K^r(\sqrt{n}(Y_n \cup \{Y'\}))) \) with diameter at most \( B \) and a vertex at \( \sqrt{n}Y' \). For every simplex \( \sigma \) of \( x \) not containing \( \sqrt{n}Y' \) as a vertex, we add \( \partial(\sigma \cup \{ \sqrt{n}Y' \}) \) to \( x \), \( \partial(\sigma \cup \{ \sqrt{n}Y' \}) \) necessarily having diameter less than \( B \). This operation cannot add any new vertices to \( x \), and thus cannot increase the total cycle diameter. What remains after completing these additions is either 0 or a cycle \( x' \) whose simplices all contain \( \sqrt{n}Y' \) as a vertex, the latter being an impossibility. Thus, any \( B \)-bounded cycle in \( Z_q(K^r(\sqrt{n}(Y_n \cup \{Y'\}))) \) having a vertex at \( \sqrt{n}Y' \) can be written as a linear combination of \( B \)-bounded elements from \( U \). For \( I_n := \sum_{i=1}^n 1 \{ ||Y_i - Y'|| \leq B/\sqrt{n} \}, \) we arrive at a worst case bound of

\[
\text{dim} \left( \frac{Z_{q,B}(K^r(\sqrt{n}(Y_n \cup \{Y'\})))}{Z_{q,B}(K^r(\sqrt{n}Y_n))} \right) \leq \# \{ U \} = \left( I_n \right)_{q+1}.
\] (A.77)
A similar argument for the boundaries yields

\[
\text{dim} \left( \frac{B_{q,B} \left( K^* \left( \sqrt{n} \left( Y_n \cup \{Y'\} \right) \right) \right)}{B_{q,B} \left( K^* \left( \sqrt{n} \tilde{Y}_n \right) \right) \right) \leq \# \{ U \} = \left( \frac{I_n}{q + 1} \right). \tag{A.78}
\]

For any \( a > 2 \), via Lemma 5.2 we have

\[
\left| \beta_{q,B}^r \left( K \left( \sqrt{n} \left( Y_n \cup \{Y'\} \right) \right) \right) - \beta_{q,B}^r \left( K \left( \sqrt{n} Y_n \right) \right) \right|^a \leq \max \left\{ \text{dim} \left( \frac{Z_{q,B} \left( K^r \left( \sqrt{n} \left( Y_n \cup \{Y'\} \right) \right) \right)}{Z_{q,B} \left( K^r \left( \sqrt{n} \tilde{Y}_n \right) \right) \right) \right)^a, \text{dim} \left( \frac{B_{q,B} \left( K^* \left( \sqrt{n} \left( Y_n \cup \{Y'\} \right) \right) \right)}{B_{q,B} \left( K^* \left( \sqrt{n} \tilde{Y}_n \right) \right) \right) \right)^a \right\} \leq \left( \frac{I_n}{q + 1} \right)^a \leq \frac{1}{((q + 1)!)^a} \left( \frac{I_n}{q + 1} \right)^a (I_n^{(q + 1)} + 1). \tag{A.79}
\]

Here \( R = B, U_a = 1 / ((q + 1)!)^a \), and \( u_a = a (q + 1) \). (E1) is then satisfied via Lemma 2.3. (S2) is satisfied via Lemma 6.2, in this case with a constant radius of stability of \( 2B \). An application of Theorem 2.7 gives the desired result.

In this case, given that the radius of stabilization is a known constant, an explicit rate for \( \gamma_\epsilon \) in Proposition 2.6 can be calculated. Details omitted, from the proof of Proposition 2.6 we have

\[
\delta_\epsilon \leq B^d \left( \frac{2}{q + 2} \right)^p \left( 1 + B^d \left( \frac{2q + 2}{p - q + 2} \right) \right) \left( \frac{2q + 2}{p - q + 2} \right) \left( \frac{2q + 2}{p - q + 2} \right). \tag{A.80}
\]

Theorem A.14 (Theorem 6.11). Let \( q \geq 0 \) and \( p > 2q + 1 \). Let \( K \) be a filtration of simplicial complexes satisfying (K1) and (D1). Then for any given \( r' \), \( (6.1) \) holds for \( \chi^r \).

Proof. We will show that assumption (E2) is satisfied for \( \psi = \chi^r \): \( (6.1) \) holds for \( \chi^r \). We will show that assumption (E2) is satisfied for \( \psi = \chi^r \): \( (6.1) \) holds for \( \chi^r \). We will show that assumption (E2) is satisfied for \( \psi = \chi^r \): \( (6.1) \) holds for \( \chi^r \).
vertex at $\sqrt{n}Y'$. Let $I_n = \sum_{i=1}^n 1\{\|Y_i - Y'\| \leq \phi(r)/\sqrt{n}\}$. For any $a > 2$, we have

$$
\left| \chi_{r}^{q/2} \left( \mathcal{K} \left( \sqrt{n} (Y_n \cup \{Y'\}) \right) \right) - \chi_{r}^{q/2} \left( \mathcal{K} \left( \sqrt{n} Y_n \right) \right) \right|^{a}
= \left| \sum_{k=0}^{q} (-1)^k \# \left\{ K_k^+ \left( \sqrt{n} (Y_n \cup \{Y'\}) \right) \right\} - \sum_{k=0}^{q} (-1)^k \# \left\{ K_k^+ \left( \sqrt{n} Y_n \right) \right\} \right|^{a}
= \left| \sum_{k=0}^{q} (-1)^k \# \left\{ K_k^+ \left( \sqrt{n} (Y_n \cup \{Y'\}) \right) \right\} - \# \left\{ K_k^+ \left( \sqrt{n} Y_n \right) \right\} \right|^{a}
 \leq \left( \frac{q I_n^2}{k!} \right)^{a}
 \leq \left( \frac{q I_n^2}{k!} \right)^{a}
 \leq \left( \frac{q I_n^2}{k!} \right)^{a}
 \leq (eI_n^2)^{a}
 \leq e^a (1 + I_n^{aq})
.$$

Here $R = \phi(r)$, $U_a = e^a$, and $u_a = aq$, satisfying (E2). (E1) then follows from Lemma 2.3 for $p \geq qa + 1 > 2q + 1$. (S2) is satisfied via Lemma 6.4 with a constant radius of stabilization $\phi(r)$. (S1) is satisfied via Lemma 2.4. An application of Theorem 2.7 gives the final result.

For the rate in Proposition 2.6, for $p < \infty$ we have that $\delta \epsilon = \epsilon^{-2/q}$ up to constant factors. Using $a = (p - 1)/q$ we achieve a final rate for $\gamma$ of

$$
O \left( \epsilon^{-2/q} (1 - \frac{2q}{p}) \right). \tag{A.81}
$$

For $p = \infty$, using $a = a_\epsilon = 2 - \log(\epsilon)$ we achieve a final rate of

$$
O \left( \epsilon^{-2/q} \left( \frac{-\log(\epsilon)}{\log(-\log(\epsilon))} \right)^{2q} \right). \tag{A.82}
$$

**Theorem A.15** (Theorem 6.12). Let $q \geq 0$ and $p > 2q + 3$. Let $\mathcal{K}$ be a filtration of simplicial complexes satisfying (D2). Then for any given $r$, (6.1) holds for $\chi_{r}^{q}$.

**Proof.** The proof follows exactly that of Theorem 6.11. Let $Y_n = \{Y_i\}_{i=1}^n$ be an iid sample in $\mathbb{R}^d$, with $Y'$ an independent copy.

By (D2) it suffices to consider simplices within $B_{\sqrt{n}Y'}(\phi(r))$. Let

$$
I_n = \sum_{i=1}^n 1\{\|Y_i - Y'\| \leq \phi(r)/\sqrt{n}\}.
$$
For any \( a > 2 \), we have
\[
\chi_q^r (\mathcal{K} ( \sqrt{n} (Y_n \cup \{Y\}))) - \chi_q^r (\mathcal{K}( \sqrt{n}Y_n)) \\
= \sum_{k=0}^q (-1)^k \# \{ K_k^r ( \sqrt{n} (Y_n \cup \{Y\}))) - \sum_{k=0}^q (-1)^k \# \{ K_k^r (\sqrt{n}Y_n) \} \\
= \sum_{k=0}^q (-1)^k \# \{ K_k^r (\sqrt{n} (Y_n \cup \{Y\}))) - \# \{ K_k^r (\sqrt{n}Y_n) \} \\
= \sum_{k=0}^q (-1)^k \# \{ K_k^r (\sqrt{n} (Y_n \cup \{Y\}))) \setminus K_k^r ( \sqrt{n}Y_n) \} \\
- \sum_{k=0}^q (-1)^k \# \{ K_k^r (\sqrt{n}Y_n) \setminus K_k^r (\sqrt{n} (Y_n \cup \{Y\}))) \}.
\]

Any simplices added by the inclusion of \( \sqrt{n}Y \) may contain \( \sqrt{n}Y \) as a vertex, and any removed simplices must only have vertices within \( \sqrt{n}Y_n \). We bound the possible simplices in each dimension. Thus for any \( a > 2 \)
\[
\left| \chi_q^r (\mathcal{K} ( \sqrt{n} (Y_n \cup \{Y\}))) - \chi_q^r (\mathcal{K}( \sqrt{n}Y_n)) \right|^a \\
\leq \left( \sum_{k=0}^q \left( \frac{I_n}{k+1} + \frac{I_n+1}{k+1} \right) \right)^a \\
\leq \left( \frac{2}{k=0} \sum \frac{(I_n+1)^{k+1}}{(k+1)!} \right)^a \\
\leq \left( \frac{2}{k=0} \sum \frac{(I_n+1)^{q+1}}{(k+1)!} \right)^a \\
\leq (2^a (e-1)^a (I_n+1)^{a(q+1)} \\
\leq 2^{a(q+2)-1} (e-1)^a \left( 1 + I_n^{a(q+1)} \right).
\]

Here \( R = \phi (r) \), \( U_a \leq 2^{a(q+2)-1} (e-1)^a \), and \( u_a = a(q+1) \), satisfying (E2). (E1) is then satisfied via Lemma 2.3 for \( p \geq a(q+1) + 1 > 2q+3 \). (S2) is satisfied via Lemma 6.4 with a constant radius of stabilization \( \phi (r) \). (S1) is satisfied via Lemma 2.4. An application of Theorem 2.7 gives the final result.

For the rate in Proposition 2.6, for \( p < \infty \) we have that \( \delta = \epsilon^{\frac{p-2}{p-1}} \) up to constant factors. Using \( a = (p-1)/(q+1) \) we achieve a final rate for \( \gamma \epsilon \) of
\[
O \left( \epsilon^{\frac{p-2}{p-1}(1 - \frac{2a+2}{p-1})} \right).
\] (A.83)

For \( p = \infty \), using \( a = a_c = 2 - \log (\epsilon) \) we achieve a final rate of
\[
O \left( \epsilon \left( \frac{-\log (\epsilon)}{\log (\log (\epsilon))} \right)^{2q+2} \right).
\] (A.84)
Theorem A.16 (Theorem 6.13). Let $p > 2$. Furthermore, let $F \in D_{\gamma,r_0}(C)$ and $1 \{ \hat{F}_{n,h,n} \in D_{\gamma,r_0}(C) \} \to 1$ in probability (resp. a.s.). Then (6.1) holds for $l_{NN,k}$.

Proof. First, we will show that $E \left[ l_{NN,k} \left( \sqrt{\frac{1}{n}} (Y_n \cup \{Y'\}) \right) - l_{NN,k} \left( \sqrt{\frac{1}{n}} Y_n \right) \right] \leq \lambda_0$ uniformly for $G \in D_{\gamma,r_0}(C)$ and $Y', Y_1, \ldots, Y_n \overset{\text{iid}}{\sim} G$. Denote by $A_{k+1}$ the $k+1$ nearest neighbors of $\sqrt{\frac{1}{n}} Y'$ in $\sqrt{\frac{1}{n}} Y_n$. Denote by $B_k$ the set of points in $\sqrt{\frac{1}{n}} Y_n$ for which $\sqrt{\frac{1}{n}} Y'$ is among the $k$ nearest neighbors.

It may be shown that $\# \{ B_k \} \leq C_{d,k}$, where $C_{d,k}$ is a constant depending only on the dimension $d$ and $k$. To show this, consider a cone of angle $\pi/6$ whose point lies on $\sqrt{\frac{1}{n}} Y'$. For $y_1, \ldots, y_k$ the $k$ closest points of $B_k$ to $\sqrt{\frac{1}{n}} Y'$ within the cone, it follows from basic geometric arguments that any point lying within the cone, but outside a radius of $\max \{ ||y_i - \sqrt{\frac{1}{n}} Y'|| \}_{i=1}^k$ must be closer to each of $y_1, \ldots, y_k$ than to $\sqrt{\frac{1}{n}} Y'$. Thus, any cone of this type may contain at most $k$ points of $B_n$. Since $\mathbb{R}^d$ may be covered by finitely many of these cones, there must exist the required bound $C_{d,k}$.

Now, consider the points of $A_{k+1}$ and $B_k$. Let $R_{k+1,n} := \max \{ \|y - \sqrt{\frac{1}{n}} Y'\| : y \in A_n \}$. For any point $y$ in $B_n$, the distance to each point of $A_n$ is at most $\|y - \sqrt{\frac{1}{n}} Y'\| + R_{k+1,n}$ by the triangle inequality. In this case, the introduction of $\sqrt{\frac{1}{n}} Y'$ to the sample may reduce the contribution to $l_{NN,k}$ from the points in $B_n$ by at most

$$l_{NN,k} \left( \sqrt{\frac{1}{n}} Y_n \right) - l_{NN,k} \left( \sqrt{\frac{1}{n}} (Y_n \cup \{Y'\}) \right) \leq C_{d,k} R_{k+1,n}.$$ 

Likewise, the contribution of $\sqrt{\frac{1}{n}} Y'$ is bounded by

$$l_{NN,k} \left( \sqrt{\frac{1}{n}} (Y_n \cup \{Y'\}) \right) - l_{NN,k} \left( \sqrt{\frac{1}{n}} Y_n \right) \leq k R_{k,n} \leq k R_{k+1,n}.$$

Thus, we proceed by bounding $E \left[ R_{k+1,n}^a \right]$. For any $G \in D_{\gamma,r_0}$

$$E \left[ \int_0^\infty \mathbb{P} \left[ \|Y_j - Y'\| > \sqrt{\frac{1}{n}} Y' \right]^n \right] \leq \text{diam} (C) \left( 1 - \gamma \rho_0 \right)^n.$$

We apply a bound similar to Theorem 7 in [40]. In the statement of the referenced theorem, it is assumed that the above quantity is bounded by $C_T/n$ for an appropriate constant $C_T$. Here, we may improve that to an exponential bound. Consequently, we have

$$E \left[ R_{k+1,n}^a \right] \leq \left( \frac{k+1}{\gamma} \right)^\frac{n}{2} + \text{diam} (C) n^\frac{n}{2} \left( 1 - \gamma \rho_0 \right)^n + \frac{a (e/ (k+1))^{k+1}}{d (\gamma)^\frac{n}{2}} \int_0^\infty e^{-y \rho_0 + \frac{a}{d}} dy. \tag{A.85}$$

For any $a < \infty$, this quantity limits to a constant with $n \to \infty$, thus admitting a constant upper bound which holds for all $n \in \mathbb{N}$, satisfying (E1).

The required stabilization properties (2.4) are first established for a unit-intensity homogeneous Poisson process via Lemma 6.1 in [32]. Let $\rho$ denote the minimal locally-determined radius of stabilization for $l_{NN,k}$. Let $\mathbb{P}_\lambda$ denote a homogeneous Poisson process with intensity $\lambda$. By the scaling properties of $l_{NN,k}$, we have $\rho_0 (\mathbb{P}_\lambda) = \rho_0 \left( \mathbb{P}_1 / \sqrt{\lambda} \right) = \rho_0 (\mathbb{P}_1) / \sqrt{\lambda}$. Thus,
\[ P^* [\rho_0 (P_\lambda) > L] = P^* [\rho_0 (P_1) > \sqrt{\lambda L}] \]. For any \( \lambda > 1 \), \( P^* [\rho_0 (P_\lambda) > L] \leq P^* [\rho_0 (P_1) > L] \).

Likewise, for any \( \lambda_* < 1 \), we may choose \( L_\delta \) such that \( P^* [\rho_0 (P_\lambda) > \sqrt[\lambda]{\lambda_* L_\delta}] \leq \delta \). Then \( P [\rho_0 (P_\lambda) > L_\delta] \leq \delta \) for all \( \lambda \in [\lambda_*, \infty) \). Stabilization then extends to the binomial sampling setting via Lemma 2.5 and the translation invariance of \( l_{NN,k} \). We have for any \( \delta > 0 \) that there exists an \( n_\delta < \infty \) and \( L_\delta^* < \infty \) such that \( P^* [\rho_{\psi V'} (\sqrt{\psi Y_n} > L_\delta^*)] \leq \delta \). Both quantities do not depend specifically on \( G \).

When restricted to \( C \), we have an absolute upper bound of \( \text{diam} (C) \sqrt{\psi} \) for the radius of stabilization, as all points will fall inside of \( C \) almost surely. We set \( L_\delta = \max\{\text{diam} (C) \sqrt{\psi}, L_\delta^*\} \). Then \( P^* [\rho_{\psi V'} (\sqrt{\psi Y_n} > L_\delta)] \leq \delta \) for all \( n \in \mathbb{N} \), satisfying (S2).

We now have the required pieces to prove bootstrap convergence. Although \( C_{p,M} \cap D_{\gamma,r_0} (C) \) is only a subset of \( C_{p,M} \), the proof and conclusion of Proposition 2.6 still apply. Likewise, the proof of Theorem 2.7 is easily altered to include the additional condition \( \mathbb{I} \{ \hat{F}_{n,h_n} \in D_{\gamma,r_0} (C) \} \to 1 \). We omit details here.