On the slice genus of quasipositive knots in indefinite 4-manifolds

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Accepted: 22 June 2023 / Published online: 31 July 2023
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Abstract
Let $X$ be a closed indefinite 4-manifold with $b_+(X) = 3 \pmod 4$ and with non-vanishing mod 2 Seiberg–Witten invariants. We prove a new lower bound on the genus of a properly embedded surface in $X \setminus B^4$ representing a given homology class and with boundary a quasipositive knot $K \subset S^3$. In the null-homologous case our inequality implies that the minimal genus of such a surface is equal to the slice genus of $K$. If $X$ is symplectic then our lower bound differs from the minimal genus by at most 1 for any homology class that can be represented by a symplectic surface. Along the way, we also prove an extension of the adjunction inequality for closed 4-manifolds to classes of negative self-intersection without requiring $X$ to be of simple type.

Mathematics Subject Classification 57K10 · 57K41

1 Introduction
An important problem in the study of smooth 4-manifolds is to determine the minimal genus of an embedded surface representing a given homology class. The relative version of this problem for 4-manifolds with boundary has also received considerable attention [7, 8, 16, 18–20, 23, 24, 26, 28, 33].

Consider a closed, connected, oriented smooth 4-manifold $X$ and let $X_0$ be the 4-manifold with boundary obtained by removing an open ball from $X$. Given an oriented knot $K \subset S^3 = \partial X_0$ and a relative homology class $a \in H_2(X_0, \partial X_0; \mathbb{Z}) \cong H_2(X; \mathbb{Z})$, we seek the minimal genus of a properly embedded oriented surface $\Sigma \subset X$ with $\partial \Sigma = K$ and $[\Sigma] = a$. We define the slice genus of $K$ with respect to $(X, a)$ to be the minimum genus of such a surface and denote it by $g_4(K, X, a)$ or $g_4(K, X_0, a)$. Of particular interest is the case that $[\Sigma]$ is null-homologous. In this case we call the minimal genus of such a surface the $H$-slice genus of $K$ in $X$ and denote it by $g_H(K, X)$.

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or $g_H(K, X_0)$. In the case that $X = S^4$, $X_0 = B^4$ is the 4-ball and $g_H(K, X) = g_4(K)$ is the usual slice genus of $K$.

To motivate our results we first recall the following relative genus bound for surfaces in definite 4-manifolds:

**Theorem 1.1** (Ozsváth–Szabó [28]) Let $X_0$ be a smooth, compact, oriented, negative-definite 4-manifold with $b_1(X_0) = 0$ and $\partial X_0 = S^3$. For any smooth, properly embedded surface $\Sigma \subset X_0$ bounding a knot $K$, we have

$$2g(\Sigma) \geq [\Sigma]^2 + ||\Sigma|| + 2\tau(\Sigma).$$

Here $||\Sigma||$ is defined as $\sum_{i=1}^n \langle [\Sigma], e_1 \rangle$, where $e_1, \ldots, e_n$ is an orthonormal basis for $H^2(X_0; \mathbb{Z})$, (which exists by Donaldson’s diagonalisation theorem) and $\tau(K)$ is the Ozsváth–Szabó tau-invariant, defined in [28]. Taking $\Sigma$ to be null-homologous, we obtain a lower bound on the $H$-slice genus of $K$:

$$g_H(K, X_0) \geq \tau(K). \quad (1.1)$$

A surprising feature of this bound is that it does not depend on the 4-manifold $X_0$.

Suppose now that $K$ is quasipositive. This means that $K$ is the braid closure of a braid that is a product of the standard generators $\sigma_1, \ldots, \sigma_{n-1}$ and their conjugates [30]. It is shown in [15] that as a consequence of the inequality given in [29], one has $\tau(K) = g_4(K)$ for any quasipositive knot. For instance, if $T_{p,q}$ denotes the $(p, q)$-torus knot for positive coprime integers $p, q$, then $\tau(T_{p,q}) = g_4(T_{p,q}) = (p - 1)(q - 1)/2$. Thus for quasipositive knots the inequality (1.1) gives $g_H(K, X) \geq g_4(K)$. But we obviously have $g_H(K, X) \leq g_4(K)$ and so we deduce that the $H$-slice genus of any quasipositive knot in any negative definite 4-manifold with $b_1(X) = 0$ is equal to the slice genus.

In this paper we prove an analogue of Theorem 1.1 for quasipositive knots and links in a large class of indefinite 4-manifolds. Let $L \subset S^3 = \partial B$ be an oriented link. Define the Murasugi characteristic $\chi_4(L)$ of $L$ to be the maximum of the Euler characteristic of any smooth, properly embedded, oriented surface in $B$ bounding $L$ and having no closed components [4, 25]. If $L$ is a knot, then $\chi_4(L) = 1 - 2g_4(L)$, where $g_4(L)$ is the slice genus.

Given a spin$^c$-structure $s$ on a compact, oriented smooth 4-manifold $X$ with $b_+(X) > 1$, we let $SW(X, s) \in \mathbb{Z}$ denote the Seiberg–Witten invariant of $(X, s)$. We also let

$$d(X, s) = \frac{c_1(s)^2 - \sigma(X)}{4} - 1 + b_1(X) - b_+(X)$$

denote the expected dimension of the Seiberg–Witten moduli space.

**Theorem 1.2** Let $X$ be a smooth, closed, oriented 4-manifold with $b_1(X) = 0$ and $b_+(X) = 3 \pmod{4}$. Suppose that there is a spin$^c$-structure $s$ with $SW(X, s) = 1 \pmod{2}$. Let $X_0$ be the 4-manifold with boundary $S^3$ obtained by removing an open
ball from $X$ and let $L \subset S^3$ be a quasipositive link. Then for any connected, smooth, oriented, properly embedded surface $\Sigma \subset X$ bounding $L$, we have

$$-\chi(\Sigma) \geq [\Sigma]^2 + |\langle [\Sigma], c_1(s) \rangle| - \chi_4(L).$$

In particular, if $L = K$ is a quasipositive knot, then

$$2g(\Sigma) \geq [\Sigma]^2 + |\langle [\Sigma], c_1(s) \rangle| + 2g_4(K).$$

Taking $\Sigma$ in Theorem 1.2 to be null-homologous and $L$ to be a knot, we obtain:

**Corollary 1.3** Let $X$ and $K$ be as in Theorem 1.2. Then $g_H(K, X) = g_4(K)$.

In the case that $a \in H_2(X; \mathbb{Z})$ can be represented by a closed surface $S$ for which the adjunction inequality is sharp in the sense that $2g(S) - 2 = a^2 + \langle a, c_1(s) \rangle$, we find that the lower bound given by Theorem 1.2 differs from the minimal genus by at most 1:

**Corollary 1.4** Let $X$, $s$ and $K$ be as in Theorem 1.2. If there exists a closed embedded surface $S \subset X$ representing the homology class $a \in H_2(X; \mathbb{Z})$ and satisfying $2g(S) - 2 = a^2 + \langle a, c_1(s) \rangle$, then we have

$$0 \leq g_4(K, X, a) - \frac{a^2 + \langle a, c_1(s) \rangle}{2} - g_4(K) \leq 1$$

for any quasipositive knot.

**Example 1.5** Consider the elliptic surface $X = E(2n)$. We have $b_+(X) = 4n - 1$ [13, Proposition 3.1.11], so $b_+(X) = 3 \pmod{4}$. Suppose that $a \in H_2(X; \mathbb{Z})$ is orthogonal to the canonical class and $a^2 \geq -2$. Then according to [14, Theorem 1.1], $a$ can be represented by a surface of genus $g$, where $2g - 2 = a^2$. Thus Corollary 1.4 gives

$$0 \leq g_4(K, E(2n), a) - \frac{1}{2}a^2 - g_4(K) \leq 1$$

for any quasipositive knot.

When $X$ is symplectic and the class $a \in H_2(X; \mathbb{Z})$ can be represented by a symplectic surface, we can apply the positive solution of the symplectic Thom conjecture [27] to Corollary 1.4:

**Theorem 1.6** Let $X$ be a smooth, closed symplectic 4-manifold with $b_1(X) = 0$ and $b_+(X) = 3 \pmod{4}$. If the homology class $a \in H_2(X; \mathbb{Z})$ can be represented by a closed embedded symplectic surface, then for any quasipositive knot $K$, we have

$$0 \leq g_4(K, X, a) - \frac{a^2 + \langle a, K_X \rangle}{2} - g_4(K) \leq 1,$$

where $K_X$ is the canonical class of $X$. 
Proof Let $s$ be the anti-canonical spin$^c$-structure, so $c_1(s) = K_X$. Recall that for a symplectic 4-manifold with $b_+(X) > 1$, one has $\text{SW}(X, s) = \pm 1$. Let $S \subset X$ be a closed symplectic surface representing $a$. Then by the adjunction formula, we have $2g(S) - 2 = a^2 + \langle a, K_X \rangle$. Hence we may apply Corollary 1.4. □

Remark 1.7 In addition to the smooth $H$-slice genus $g_H(K, X)$, one can also define the topological $H$-slice genus $g_H^\text{top}(K, X)$ of $K$ in $X$, defined as the minimal genus of a connected, properly embedded, topologically locally flat, null-homologous surface in $X_0 = X \setminus B^4$ bounding $K$. When $X = S^4$, $g_H^\text{top}(K, S^4) = g_4^\text{top}(K)$ is the topological 4-genus of $K$. For a torus knot $K = T_{p, q}$ where $p, q$ are positive coprime integers, it is known that $g_4^\text{top}(K) < g_4(K)$, except in the cases $K = T_{2, n}, T_{3, 4}$ and $T_{3, 5}$ [1, 31]. In fact, for such knots one has $g_4^\text{top}(K) \leq \frac{6}{7} g_4(K)$ [1]. From this and Corollary 1.3 we deduce the following: let $K$ be a torus knot other than $T_{2, n}, T_{3, 4}$ or $T_{3, 5}$ and let $X$ be a smooth, closed 4-manifold with $b_1(X) = 0, b_+(X) = 3 \mod 4$ and having a non-vanishing mod 2 Seiberg–Witten invariants. Then

$$g_H^\text{top}(K, X) \leq g_4^\text{top}(K) \leq \frac{6}{7} g_4(K) = \frac{6}{7} g_H(K, X).$$

In particular, for such an $X$ the difference $g_H(K, X) - g_H^\text{top}(K, X)$ can be arbitrarily large.

Taking $L$ to be the unknot, Theorem 1.2 also implies an extension of the adjunction inequality to classes of negative self-intersection without requiring $X$ to be of simple type:

Theorem 1.8 Let $X$ be a smooth, closed, oriented 4-manifold with $b_1(X) = 0$ and $b_+(X) = 3 \mod 4$. Let $s$ be a spin$^c$-structure on $X$ with $\text{SW}(X, s) = 1 \mod 2$. Suppose $\Sigma$ is a compact, connected, smoothly embedded surface in $X$ of genus $g(\Sigma)$. Then

$$2g(\Sigma) \geq [\Sigma]^2 + |\langle \Sigma, c_1(s) \rangle|.$$

Note that this is slightly weaker than the usual adjunction inequality when $[\Sigma]^2 > 0$, which takes the form $2g(\Sigma) - 2 \geq [\Sigma]^2 + |\langle \Sigma, c_1(s) \rangle|$. When $X$ is of simple type the adjunction inequality in the stronger form with $2g(\Sigma) - 2 \geq [\Sigma]^2 + |\langle \Sigma, c_1(s) \rangle|$ was proven in [27].

Lastly, applying the same type of argument used in the proof of Theorem 1.2 to a negative definite 4-manifold, we obtain an extension of the Ozsváth–Szabó inequality to quasipositive links.

Theorem 1.9 Let $X$ be a smooth, compact, oriented negative-definite 4-manifold with $b_1(X) = 0$ and $\partial X = S^3$. Let $L \subset S^3$ be a quasipositive link. Then for any connected, smooth, properly embedded surface $\Sigma \subset X$ bounding $L$, we have

$$-\chi(\Sigma) \geq [\Sigma]^2 + |\Sigma| - \chi(S(L)).$$
We note here that an extension of the Ozsváth–Szabó $\tau$ invariant for links was defined in [6] and one has that $-\chi(\Sigma) \geq 2\tau(L) - n$ for a smooth, connected, properly embedded surface $\Sigma$ in $X$ bounding $L$, where $n$ is the number of components of the link [6, Proposition 1.4]. This suggests that there should be a generalisation of the Ozsváth–Szabó inequality to arbitrary links taking the form $-\chi(\Sigma) \geq |\Sigma|^2 + |[\Sigma]| + 2\tau(L) - n$.

1.1 Remark on orientations

Throughout the paper knots and links are oriented. If $\Sigma_1$ is a properly embedded surface bounding a knot or link $L$, then we require $\Sigma_1$ to be oriented and that the induced orientation on $\partial \Sigma_1$ agrees with the given orientation on $L$.

1.2 Structure of the paper

In Sect. 2 we prove some results concerning surfaces in $\mathbb{CP}^2 \setminus B^4$ bounding the mirror $\overline{L}$ of a quasipositive link $L \subset S^3$, leading to Lemma 2.2. In Sect. 3 we first prove an extension of the Seiberg–Witten adjunction inequality to the case of non-vanishing Bauer–Furuta invariant (Proposition 3.1). We then use this and Lemma 2.2 to prove the main results of the paper.

2 Quasipositive links

Let $B_n$ denote the braid group on $n$ strands and $\sigma_1, \ldots, \sigma_{n-1}$ the standard generators. A link $L$ is said to be quasipositive if it can be realised as the braid closure of a braid $\beta$ which is a product of $\sigma_1, \ldots, \sigma_{n-1}$ and their conjugates [30]. Every quasipositive link can be realised as the transverse intersection of a plane algebraic curve $\Gamma \subset \mathbb{C}^2$ with a 3-sphere $S^3 = \partial B \subset \mathbb{C}^2$ bounding a ball $B \subset \mathbb{C}^2$ [30]. Conversely, any link $L \subset S^3$ constructed in this manner is quasipositive [5].

Let $L \subset S^3$ be a quasipositive link. Hence we can realise $L$ as the transverse intersection $L = \partial B \cap \Gamma$ of a plane algebraic curve $\Gamma \subset \mathbb{C}^2$ with an open ball $B \subset \mathbb{C}^2$. Let $\overline{\Gamma} \subset \mathbb{CP}^2$ be the projective completion of $\Gamma$. Here we identify $\mathbb{C}^2$ as the complement $\mathbb{C}^2 = \mathbb{CP}^2 \setminus L_\infty$ of a projective line $L_\infty$ in $\mathbb{CP}^2$.

**Lemma 2.1** There exists a non-singular sextic curve $\Sigma \subset \mathbb{CP}^2$ which is disjoint from $\overline{B} \cup Sing(\overline{\Gamma})$ and which meets $\overline{\Gamma}$ transversally.

**Proof** Let $P = \mathbb{P}(H^0(\mathbb{CP}^2, O(1)))$ be the dual projective space of $\mathbb{CP}^2$, the space of lines in $\mathbb{CP}^2$. Let $U \subset P$ be the set of lines disjoint from $\overline{B}$. This is an open subset of $P$ and is non-empty, since the line at infinity is disjoint from $\overline{B}$. For each $x \in Sing(\overline{\Gamma})$, let $\ell_x \subset P$ be the set of lines passing through $x$. Then $\ell_x$ is a projective line in $P$. Now since $Sing(\overline{\Gamma})$ is finite, it follows that $V = P \setminus \left( \bigcup_{x \in Sing(\overline{\Gamma})} \ell_x \right)$ is a dense open subset of $P$. Hence $W = U \cap V$ is non-empty. So there exists a non-empty open subset $W$ of lines in $\mathbb{CP}^2$ disjoint from $\overline{B}$ and $Sing(\overline{\Gamma})$. 

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Now consider the linear system \( R = \mathbb{P}(H^0(\mathbb{C}P^2, O(6))) \) of sextic curves in \( \mathbb{C}P^2 \). Let \( U_1 \subset R \) denote the open subset of sextic curves disjoint from \( \overline{B} \cup \text{Sing}(\Gamma) \). Then \( U_1 \) is non-empty. Indeed, let \( N \) be a projective line in \( \mathbb{C}P^2 \) disjoint from \( \overline{B} \) and \( \text{Sing}(\Gamma) \). Then the sextic with divisor \( 6N \) belongs to \( U_1 \).

Choose a point \( p \in \mathbb{C}P^2 \) not lying \( \Gamma \). The space of lines in \( \mathbb{C}P^2 \) through \( p \) can be identified with \( \mathbb{C}P^1 \) and this determines a regular map \( \rho : \Gamma \rightarrow \mathbb{C}P^1 \) which sends a point \( q \in \Gamma \) to the line joining \( p \) and \( q \). Let \( T \subset \mathbb{C}P^1 \) be the finite set \( \rho(\text{Sing}(\Gamma)) \). Then \( \rho : \Gamma \setminus \rho^{-1}(T) \rightarrow \mathbb{C}P^1 \setminus T \) is a branched covering. Any point in \( \mathbb{C}P^1 \setminus T \) which is not a branch point of \( \rho \) corresponds to a line in \( \mathbb{C}P^2 \) such that each point of intersection with \( \overline{\Gamma} \) has multiplicity 1. Let \( U_2 \subset R \) be the set of sextics whose intersection multiplicities with \( \overline{\Gamma} \) all equal 1. Then \( U_2 \) is a non-empty open subset of \( R \) because we can take a sextic which is the union of six lines corresponding to six distinct non-branch points of \( \rho \) in \( \mathbb{C}P^1 \setminus T \). So \( U_1 \cap U_2 \) is a non-empty open subset of \( R \). Bertini’s theorem implies that there exists a non-singular sextic which belongs to \( U_1 \cap U_2 \).

Let \( S \subset \mathbb{C}P^2 \) be a non-singular sextic curve which is disjoint from \( \overline{B} \cup \text{Sing}(\Gamma) \) and which meets \( \overline{\Gamma} \) transversally. Note that \( S \) is connected, since any two distinct components of \( S \) would intersect in a singular point. Suppose \( \overline{\Gamma} \) has degree \( d \). Then \( \overline{\Gamma} \) and \( S \) meet in exactly \( 6d \) points, by Bézout’s theorem. Let \( W = \mathbb{C}P^2 \setminus B \). Then \( W \) is a 4-manifold with boundary \( \partial W = S^3 \). Consider \( \Gamma_0 = \overline{\Gamma} \setminus B \). Then \( \Gamma_0 \) is a properly embedded surface in \( X \) meeting \( \partial X \) in the link \( \overline{L} = \partial W \cap \Gamma_0 \) (we obtain the mirror \( \overline{L} \) of \( L \), because the orientation on \( S^3 = \partial W \) is opposite to the orientation obtained by viewing \( S^3 \) as the boundary of \( B \)).

**Lemma 2.2** There exists a smoothly embedded, connected, oriented surface \( C_0 \subset W \) having the following properties:

1. \( C_0 \) meets \( \partial W \) transversally in \( \overline{L} \),
2. \( C_0 \) meets \( S \) transversally in \( 6d \) points,
3. \( \chi(C_0) = 3d - d^2 - \chi_4(L) \),
4. The homology class of \( C_0 \) in \( H_2(W, \partial W; \mathbb{Z}) \cong \mathbb{Z} \) is \( d[\ell] \), where \( [\ell] \) is the class of a projective line,
5. Each connected component of \( C_0 \) meets \( \overline{L} \).

**Proof** First note that it is enough to find a smoothly embedded, connected oriented surface \( C_0 \subset W \) satisfying (2)–(5) and such that \( C_0 \) meets \( \partial W \) transversally in a link which is isotopic to \( \overline{L} \), for then we can perform an isotopy on \( C_0 \) supported in a neighbourhood of \( \partial W \) so as to satisfy (1) while still maintaining conditions (2)–(5).

Let \( C \subset \mathbb{C}P^2 \) be a non-singular algebraic curve of degree \( d \), obtained by deforming \( \overline{\Gamma} \) within the space of degree \( d \) algebraic curves. In particular, \( C \) is connected. By choosing \( C \) to be sufficiently close to \( \overline{\Gamma} \), we can assume that the intersection of \( C \) with \( S^3 = \partial B \) remains transverse and that \( C \cap \partial B \) is isotopic to \( L \). Similarly, we can assume that \( C \) meets \( S \) transversally. Now take \( C_0 = C \setminus B \). Then \( C_0 \) meets \( \partial W \) transversally in a link which is isotopic to \( \overline{L} \).

We claim that \( C_0 \) satisfies (5). For if not, \( C_0 \) contains a connected component \( U \) which has no boundary. Then \( U \) is a complex submanifold of \( \mathbb{C}P^2 \). Chow’s theorem implies that \( U \) is algebraic, so \( U \) is an irreducible component of \( C \). But \( C \) is irreducible, which implies \( C = U \). But this is impossible as \( C \) has non-empty intersection with \( \overline{B} \).
C is an algebraic curve of degree $d$, so $C$ and $S$ meet in $6d$ points. Since $S$ is disjoint from $\overline{B}$, it follows that $C_0$ satisfies (2). Moreover, this implies (4), since $S$ represents $6[\ell]$ and all the intersections of $S$ and $C_0$ are positive.

As $C$ is an algebraic curve of degree $d$, we have $\chi(C) = 3d - d^2$ by the degree-genus formula and hence $\chi(C_0) = 3d - d^2 - \chi(C \cap \overline{B})$. So to prove (3) it remains to show that $\chi(C \cap \overline{B}) = \chi_4(L)$. In fact, this proven in [32].

3 Adjunction inequalities

We will need an extension of the adjunction inequality to the case of non-vanishing Bauer–Furuta invariants. Recall that to a compact oriented smooth manifold $M$ with $b_1(M) = 0$ and a spin$^c$-structure $s$, one may define an invariant called the Bauer–Furuta invariant, which takes values in a certain equivariant stable cohomotopy group [2]. In what follows, we will concern ourselves only with the corresponding non-equivariant Bauer–Furuta invariant

$$BF(M, s) \in \pi_{d(M, s)+1}^{st},$$

where $\pi_k^{st}$ denotes the $k$th stable homotopy group of spheres and

$$d(M, s) = \frac{c_1(s)^2 - \sigma(M)}{4} - 1 + b_+(M)$$

is the expected dimension of the Seiberg–Witten moduli space.

The following result is the adjunction inequality for Bauer–Furuta invariants. Special cases of this result have appeared in [10–12, 23].

**Proposition 3.1** Let $M$ be a smooth, compact oriented 4-manifold with $b_1(M) = 0$ and $b_+(M) > 1$. Let $s$ be a spin$^c$-structure on $M$ for $BF(M, s) \neq 0$. Suppose $\Sigma$ is a compact, connected, smoothly embedded surface in $M$ and that $[\Sigma]^2 \geq 0$. Then:

1. If $g \geq 1$ then

$$2g(\Sigma) - 2 \geq [\Sigma]^2 + |\langle [\Sigma], c_1(s) \rangle|$$

2. If $g = 0$, then $[\Sigma]$ is a torsion class.

**Proof** The proof is similar to the case where the Seiberg–Witten invariant of $(M, s)$ is non-zero. The main point is that since $BF(M, s) \neq 0$, it follows that the Seiberg–Witten equations for $(M, s)$ admits a solution for any metric and any perturbation.

Consider first the case that $g \geq 1$ and $[\Sigma]^2 = 0$. The standard neck stretching argument of Kronheimer–Mrowka [22] implies that there exists a translation invariant solutions to the Seiberg–Witten equations on $\Sigma \times \mathbb{R}$, where $\Sigma$ is given a constant scalar curvature metric and $\mathbb{R}$ the standard Euclidean metric. As in [22], this implies that $2g(\Sigma) - 2 \geq |\langle [\Sigma], c_1(s) \rangle|$. Reversing orientation on $\Sigma$ if necessary, we obtain $2g(\Sigma) - 2 \geq |\langle [\Sigma], c_1(s) \rangle|$, which proves the result in this case.
The case $g = 0$ was proven in [23, Theorem 4.5] (note however that the condition $[\Sigma] \neq 0$ in the statement of [23, Theorem 4.5] should be replaced with the stronger condition that $[\Sigma]$ is non-torsion. This can be seen by examination of the proof of [9, Lemma 5.1], on which the proof of [23, Theorem 4.5] is based).

Lastly, suppose that $g \geq 1$ and $[\Sigma]^2 = n > 0$. By possibly reversing orientation on $\Sigma$, we can assume that $(|\Sigma|, c_1(s)) \geq 0$. Let $M_n$ be manifold obtained by blowing up $M$ at $n$ points disjoint from $\Sigma$. So $M_n$ is diffeomorphic to $M\#^n \mathbb{CP}^2$. Let $\tilde{s}_i$ be a spin$^c$-structure on the $i$th copy of $\mathbb{CP}^2$ such that $\tilde{s}_i = c_1(\tilde{\sigma}_i)$ represents the $i$th exceptional divisor and let $\Sigma_n$ denote the connected sum of $\Sigma$ with the 2-spheres representing $-S_1, \ldots, -S_n$. Then $\Sigma_n$ has the same genus as $\Sigma$ and $[\Sigma_n]^2 = 0$. Define a spin$^c$-structure $\tilde{s}$ on $M_n$ by gluing together the spin$^c$-structures $s_i, s_1, \ldots, s_n$. Then $c_1(\tilde{\sigma}) = c + S_1 + \cdots + S_n$. Since $\mathbb{CP}^2$ is negative definite, has vanishing first Betti number and $d(\mathbb{CP}^2, s_i) = -1$, it follows that $BF(\mathbb{CP}^2, s_i) \in \pi_0^{st}$ is the identity map. The connected sum formula for Bauer–Furuta invariants [3, Theorem 1.1], implies that $BF(M_n, \tilde{s}) \neq 0$. Therefore, we are in the self-intersection zero case, so as shown above we have

$$2g(\Sigma) - 2 \geq |(\Sigma_n, c_1(s))| = |(\Sigma, c_1(s))| + n = |(\Sigma, c_1(s))| + [\Sigma]^2,$$

where we used $(|\Sigma|, c_1(s)) \geq 0$ and $[\Sigma]^2 = n$. This is the adjunction inequality. \hfill \Box

**Theorem 3.2** Let $X$ be a smooth, compact, oriented 4-manifold with $b_1(X) = 0$ and $b_+(X) = 3 \pmod 4$. Suppose that there is a spin$^c$-structure $s$ with $SW(X, s) = 1 \pmod 2$. Let $X_0$ be the 4-manifold with boundary $S^3$ obtained by removing an open ball from $X$. Let $L \subset S^3$ be a quasipositive link. Then for any connected, oriented, smooth, properly embedded surface $\Sigma \subset X_0$ bounding $L$, we have

$$-\chi(\Sigma) \geq [\Sigma]^2 + |(\Sigma, c_1(s))| - \chi_4(L).$$

**Proof** First note that by blowing up $X$ at points disjoint from $\Sigma$, we can assume $d(X, s) = 0$. Doing this does not alter the value of $[\Sigma]^2 + |(\Sigma, c_1(s))|$. Let $Y = X\# \mathbb{CP}^2$ be the connected sum of $\mathbb{CP}^2$ and $X$. More precisely, we obtain $Y$ by identifying $W = \mathbb{CP}^2 \setminus B$ and $X_0$ along their boundary $S^3$. Here we orient $S^3$ so that it is an ingoing boundary of $\mathbb{CP}^2$ and an outgoing boundary of $X_0$. Since $S$ is disjoint from $\overline{B}$, we may regard it as an embedded surface in $Y$. Let $\pi : \overline{Y} \to Y$ be the double cover of $Y$ branched over $S$. To see that the branched cover exists, first consider the double cover $\pi_Z : Z \to \mathbb{CP}^2$ branched along $S$. In fact $Z$ is a $K3$ surface [13, Corollary 7.3.25] ($Z$ is a compact complex surface with $b_1(Z) = 0$ and using the adjunction formula, one finds that the canonical bundle of $Z$ is trivial). Then $\pi_Z^{-1}(B)$ consists of two balls in $Z$. By removing these balls and gluing in two copies of $X_0$, we obtain $\overline{Y}$. Further, this shows that $\overline{Y}$ is diffeomorphic to the connected sum $Z\#X\#X$ of a $K3$ surface and two copies of $X$.

Recall that $\pi_1^{st} \cong \mathbb{Z}_2$ and is generated by the Hopf map $\eta : S^3 \to S^2$. According to [3, Proposition 4.4], if $b_+(M) = 3 \pmod 4$ and $SW(M, s) = 1 \pmod 2$, then $BF(M, s) = \eta \in \pi_1^{st}$. In particular, this is the case for $(X, s)$ and also for the $K3$
surface $Z$, equipped with the unique spin$^c$ structure $s_Z$ which comes from a spin-structure. Let $s_{\tilde{Y}}$ denote the spin$^c$-structure on $\tilde{Y} = Z\#X\#X$ obtained by gluing together the spin$^c$-structures $s_Z, s, s$. The connected sum formula for Bauer–Furuta invariants implies that $BF(\tilde{Y}, s_{\tilde{Y}}) = \eta^3 \in \pi_3^s$. Now since $\eta^3 \neq 0$, we have that the Bauer–Furuta invariant of $(\tilde{Y}, s_{\tilde{Y}})$ is non-zero.

Let $C_0 \subset W$ be as in Lemma 2.2 and let $\Sigma' = \Sigma \cup L C_0$ be the surface in $Y$ obtained by attaching $\Sigma$ and $C_0$ along their boundaries. We have that $\Sigma'$ is connected, since $\Sigma$ is connected and every component of $C_0$ meets $L$. Let $\hat{\Sigma} = \pi^{-1}(\Sigma')$. Since $C_0$ meets $S$ transversally, it follows that $\hat{\Sigma}$ is a smooth, compact, embedded surface in $\tilde{Y}$ and that the restriction of $\pi : \hat{\Sigma} \to \Sigma$ is a branched double cover with $6d$ branch points.

We have that

$$[\hat{\Sigma}]^2 = 2[\Sigma']^2 = 2[C_0]^2 + 2[\Sigma] = 2d^2 + 2[\Sigma]^2.$$ 

Now we observe that for any given quasipositive link $L$, we can choose the algebraic curve $\Gamma$ to have arbitrarily large degree. Indeed, we can replace $\Gamma$ by the union of $\Gamma$ with any number of lines which are disjoint from $B$. Thus we can take $d$ large enough that $[\hat{\Sigma}]^2 = 2d^2 + 2[\Sigma]^2 > 0$. Having chosen such a $\Gamma$, we may apply Proposition 3.1 to obtain

$$-\chi(\hat{\Sigma}) \geq [\hat{\Sigma}]^2 + |\langle [\hat{\Sigma}], c_1(s_{\tilde{Y}}) \rangle|$$

$$\geq 2[\Sigma]^2 + 2d^2 + 2|\langle [\Sigma], c_1(s) \rangle|.$$ 

Furthermore, by Riemann–Hurwitz, we have

$$\chi(\hat{\Sigma}) = 2\chi(\Sigma') - 6d = 2\chi(\Sigma) + 2\chi(C_0) - 6d = 2\chi(\Sigma) - 2d^2 - 2\chi_4(L),$$

where we used that $\chi(C_0) = 3d - d^2 - \chi_4(L)$ from Lemma 2.2. Putting these together, we get

$$-\chi(\Sigma) + d^2 + \chi_4(L) \geq [\Sigma]^2 + d^2 + |\langle [\Sigma], c_1(s) \rangle|,$$

which gives the result. \hspace{1cm} \Box

**Remark 3.3** The proof of Theorem 3.2 can be thought of as a generalisation of the proof of [21, Corollary 1.3].

**Corollary 3.4** Let $X$ be a smooth, compact, oriented 4-manifold with $b_1(X) = 0$ and $b_+(X) = 3 \pmod 4$. Suppose that there is a spin$^c$-structure $s$ with $d(X, s) = 0$ and $SW(X, s) = 1 \pmod 2$. Let $X_0$ be the 4-manifold with boundary $S^3$ obtained by removing an open ball from $X$. Let $K \subset S^3$ be a quasipositive knot. Then for any connected, oriented, smooth, properly embedded, homologically trivial surface $\Sigma \subset X_0$ bounding $K$, we have $g(\Sigma) \geq g_4(K)$. Thus $g_4(K)$ is the minimal genus of a homologically trivial surface in $X$ bounding $K$.

**Remark 3.5** We make some comments on the proof of Theorem 3.2:
In the proof of Theorem 3.2 it is essential that we pass to the branched cover of $Y = X \# \mathbb{CP}^2$. This is because $b_+(\mathbb{CP}^2) = 1 \pmod{4}$ and so the adjunction inequality Proposition 3.1 cannot be directly applied to $Y$.

To obtain a bound on $\chi(\Sigma)$, one may consider a simpler strategy of capping off $X_0$ with a 4-ball $B^4$ and closing up $\Sigma$ with a surface in $B^4$ with Euler characteristic equal to $\chi_4(L)$. This will give

$$-\chi(\Sigma) \geq [\Sigma]^2 + \langle [\Sigma], c_1(s) \rangle + \chi_4(L)$$

provided $[\Sigma]^2 \geq 0$ and $[\Sigma]$ is non-torsion. Notice that this differs from Theorem 3.2 in that the right hand side has $+\chi_4(L)$ whereas in Theorem 3.2 we have $-\chi_4(L)$. This inequality is usually a much weaker bound on $\chi(\Sigma)$ than Theorem 3.2. For example, if $L$ is a knot then $\chi_4(L) = 1 - 2\nu_4(L)$, so $-\chi_4(L) \geq \chi_4(L)$, except when $g_4(L) = 0$.

Our inequality may be compared with similar adjunction-type inequalities in [23]. For each spin$^c$-structure for which the Ozsváth–Szabó mixed invariant $\Phi_{X,s}$ is non-zero, we get

$$2g(\Sigma) - 2 \geq [\Sigma]^2 + \langle [\Sigma], c_1(s) \rangle - 2\nu^+(\overline{K})$$

provided either $[\Sigma]^2 \geq 2\nu^+(\overline{K})$ or $X$ has Ozsváth–Szabó simple type [23, Theorem 1.1]. Here $\nu^+(\overline{K})$ is the concordance invariant constructed by Hom and Wu [17]. Note that $\nu^+$ is always non-negative, so this inequality is weaker than Theorem 3.2, unless $2 - 2\nu^+(\overline{K}) > 2g_4(K)$ which can only happen if $K$ is slice.

In the proof of Theorem 3.2, it is crucial that $L$ is quasipositive. This ensures that the number of intersection points of $\Sigma' = \Sigma \cup_L C_0$ with $S$ is exactly $6d = \langle [\Sigma'], [S] \rangle$. For non-quasipositive knots, Lemma 2.2 must generally fail since there are knots which do not satisfy Theorem 3.2. For example, the left handed trefoil $K = \overline{T_{2,3}}$ is $H$-slice in $K\mathbb{Z}$ [23, Example 2.5], but $g_4(K) = 1$. This implies that Lemma 2.2 fails for $K$.

**Proof of Corollary 1.4** Suppose $a$ can be represented by a surface $S \subset X$ satisfying $2g(S) - 2 = a^2 + \langle a, c_1(s) \rangle$. Now let $X_0 = X \setminus B$, where the ball $B$ is chosen with $\partial B$ disjoint from $S$. Let $K \subset S^3$ be a quasipositive knot. Choose a connected, smooth, properly embedded surface $\Sigma \subset X_0$ bounding $K$, supported in a collar neighbourhood of $\partial X_0$ and having genus equal to $g_4(K)$. Let $\Sigma' \subset X_0$ be the surface obtained by attaching a handle joining $\Sigma$ to $S$. Then $g(\Sigma') = g(S) + g_4(K)$ and $\Sigma'$ is a properly embedded surface representing the class $a$ and bounding $K$. Therefore

$$g_4(K, X, a) \leq g(\Sigma') = g(S) + g_4(K) = \frac{a^2 + \langle a, c_1(s) \rangle}{2} + g_4(K) + 1.$$ 

The inequality $g_4(K, X, a) \geq \frac{a^2 + \langle a, c_1(s) \rangle}{2} + g_4(K)$ is obtained by applying Theorem 3.2. \qed
**Proof of Theorem 1.9** Let $s$ be a spin$^c$-structure on $X$ such that $d(X, s) = -1$. Since $X$ is negative definite this is equivalent to $c_1(s)^2 = -b_2(X)$. Consider the 4-manifold $X' = K3 \# X$ with spin$^c$-structure $s' = s_0 \# s$, where $s_0$ is the unique spin$^c$-structure on $K3$ coming from the spin-structure. Then $b_+(X') = 3$ and $SW(X', s') = SW(K3, s_0) = 1$, by the blowup formula for Seiberg–Witten invariants. So we may apply Theorem 3.2 to deduce the inequality

$$-\chi(\Sigma) \geq [\Sigma]^2 + |\langle [\Sigma], c_1(s) \rangle| - \chi_4(L),$$

for any spin$^c$-structure $s$ on $X$ for which $c_1(s)^2 = -b_2(X)$. By Donaldson’s diagonalisation theorem, the intersection form on $X$ is diagonalisable. From this, it is easily seen that the maximum of $|\langle [\Sigma], c_1(s) \rangle|$ over all such spin$^c$-structures on $X$ equals $|[\Sigma]|$. Thus we find that

$$-\chi(\Sigma) \geq [\Sigma]^2 + |[\Sigma]| - \chi_4(L).$$

\[\square\]

**Acknowledgements** We thank Hokuto Konno for comments on a draft of this paper.

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