EXTENSION OF OSTROWSKI TYPE INEQUALITY VIA MOMENT GENERATING FUNCTION

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Abstract. In this paper, generalizations of weighted Ostrowski inequality are derived by using moment generating functions in bounded variation, $L_\infty$ and $L_p$ spaces. Applications to composite quadrature formulae are developed in which $\frac{1}{3}$ Simpson’s, $\frac{3}{8}$ Simpson’s, trapezoidal and midpoint inequalities are derived.

Keywords: moment generating function; Ostrowski inequality; numerical quadrature formulae

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1. INTRODUCTION

In 1938 Ostrowski developed an important inequality [11] which states that:

**Theorem 1.1.** Let $\phi : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be differentiable mapping on $I'$ the interior of interval $I$ such that $\phi' \in L[u,v]$, where $u, v \in I$ with $u < v$. If $|\phi'(y)| \leq M$, then the following inequality holds

$$\left| \phi(y) - \frac{1}{v-u} \int_u^v \phi(t)dt \right| \leq (v-u) \left[ \frac{1}{4} + \frac{(y-u+v)^2}{(v-u)^2} \right] M \quad (1.1)$$

which holds $\forall y \in [u,v]$ and $\frac{1}{4}$ is the best possible constant in a sense that it cannot be replaced by a smaller constant.

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In [6], Dragomir, et. al. proved the generalization of the Ostrowski inequality for $L_{\infty}$ space using some parameter $\lambda$.

**Proposition 1.2.** Let a function $\phi : [u, v] \rightarrow \mathbb{R}$ is continuous on $[u, v]$ and differentiable on $(u, v)$, assume that its derivative is bounded on $(u, v)$ and denote

$$
\|\phi'\|_{\infty} := \sup_{t \in [u, v]} |\phi'(t)| < \infty.
$$

Then

$$
\left| (v-u) \left[ \frac{\lambda \phi(u) + \phi(v)}{2} + (1-\lambda)\phi(y) \right] - \int_{u}^{v} \phi(t)dt \right| 
\leq \left[ \frac{(v-u)^2}{4} \left( \lambda^2 + (1-\lambda)^2 \right) + \left( y - \frac{u+v}{2} \right)^2 \right] \|\phi'\|_{\infty}
$$

\(\forall \lambda \in [0, 1]\) and the Peano kernel defined as:

$$
K(y,t) = \begin{cases} 
  t - (u + \lambda \frac{v-u}{2}) , & t \in [u,y], \\
  t - (v - \lambda \frac{v-u}{2}) , & t \in (y,v].
\end{cases}
$$

(1.3)

where $y \in [u + \lambda \frac{v-u}{2}, v - \lambda \frac{v-u}{2}]$.

We should also know the definition of bounded variation.

**Definition 1.3.** If a function $\phi : [a, b] \rightarrow \mathbb{R}$ and $[c, d]$ be any closed subinterval of $[a, b]$ and if the set

$$
S = \left\{ \sum_{i=1}^{n} |\phi(x_i) - \phi(x_{i-1})| : x_i : 1 \leq i \leq n \text{ is a partition of } [c, d] \right\}
$$

is bounded then the variation of $\phi$ on $[c, d]$ is defined to be $\mathcal{V}^d_c(\phi) = \sup S$. If $S$ is unbounded then the variation of $\phi$ is said to be infinite. A function $\phi$ is of bounded variation on $[c, d]$ if $\mathcal{V}^d_c(\phi)$.

Also according to [7, p.318]

**Definition 1.4.** Let $w : (u, v) \rightarrow [0, \infty)$ is integrable, i.e., $\int_{u}^{v} w(t)dt < \infty$. We denote the first two moments to be $m$ and $M$, where

$$
m(u,v) = \int_{u}^{v} w(t)dt, \quad M(u,v) = \int_{u}^{v} tw(t)dt
$$
Theorem 2.1. Let a function $\phi : [u, v] \to \mathbb{R}$ is bounded variation on $[u, v]$ and $y \in [\mu, \nu]$, then the following inequality holds

$$\left| \frac{\lambda \phi(u) + \phi(v)}{2} + (1 - \lambda)\phi(y) \right| - \frac{1}{v - u} \int_u^v \phi(t) dt$$

$$\leq \frac{1}{2(v - u)} \max \{\lambda(v - u), 2y - [(2 - \lambda)u + \lambda]v, [\lambda u + (2 - \lambda)v - 2y]\} \int_u^v \phi(t) dt$$

(2.1)



In this paper our aim is to give generalization of Preposition 1.2 by using moment generating function and to study three different cases namely $\phi$ is bounded variation, $\phi' \in L_\infty[u, v]$ and $\phi' \in L_p[u, v]$. We use first two moments of weighted function.

2. IF $\phi$ IS FUNCTION OF BOUNDED VARIATION

Proof. We now use the fact from [1], for a continuous function $p : [c, d] \to \mathbb{R}$ and a function $f : [c, d] \to \mathbb{R}$ of bounded variation, the following inequality holds

$$\left| \int_c^d p(t) f(t) dt \right| \leq \sup_{t \in [c, d]} |p(t)| \int_c^d |f(t)| dt.$$ 

Applying the above inequality for $p(t) = K(y,t)$ and $f(t) = \phi(t)$, we have

$$\left| \int_u^v K(y,t) \phi(t) dt \right| \leq \left| \int_u^v K(y,t) \phi(t) dt \right| + \left| \int_y^v K(y,t) \phi(t) dt \right|$$

Now we use the fact to get

$$\sup_{t \in [u, v]} K(y,t) \int_u^v \phi(t) dt \leq \sup_{t \in [u, v]} K(y,t) \int_u^v \phi(t) dt + \sup_{t \in [u, v]} K(y,t) \int_y^v \phi(t) dt$$

$$\left| \int_u^v K(y,t) \phi(t) dt \right| \leq \max \left\{ \lambda \frac{v - u}{2}, y - \frac{(2 - \lambda)u + \lambda v}{2} \right\} \int_u^v \phi(t) dt$$

$$+ \max \left\{ \lambda \frac{v - u}{2}, \frac{\lambda u + (2 - \lambda)v}{2} - y \right\} \int_y^v \phi(t) dt$$

$$\leq \frac{1}{2(v - u)} \max \{\lambda(v - u), 2y - [(2 - \lambda)u + \lambda v]v, [\lambda u + (2 - \lambda)v - 2y]\} \int_u^v \phi(t) dt.$$
Lemma 2.2. Let a function $\phi : [u, v] \to \mathbb{R}$ is absolutely continuous and let weighted function $w : [u, v] \to [0, \infty)$ is integrable and $\int_u^v w(s) ds = m(u, v) < \infty$ then

$$
\int_u^v P_w(y, t) \phi'(t) dt = m(\mu, \vartheta) \phi(y) + m(\vartheta, v) \phi(v) - m(\mu, u) \phi(u) - \int_u^v w(t) \phi(t) dt \tag{2.2}
$$

where

$$
P_w(y, t) = \begin{cases}
\int_{\mu}^t w(s) ds = m(\mu, t), & t \in [u, y], \\
\int_{\vartheta}^t w(s) ds = m(\vartheta, t), & t \in (y, v].
\end{cases} \tag{2.3}
$$

and

$$
\mu = u + \lambda \frac{v - u}{2}, \quad \vartheta = v - \lambda \frac{v - u}{2}
$$

for

$$
u \leq \mu \leq y \leq \vartheta \leq v
$$

Proof. Use Integration-by-parts on kernel (2.3), we get

$$
\int_u^y m(\mu, t) \phi'(t) dt = m(\mu, y) \phi(y) - m(\mu, u) \phi(u) - \int_u^y \phi(t) d(t) \tag{2.4}
$$

and

$$
\int_y^v m(\vartheta, t) \phi'(t) dt = m(\vartheta, v) \phi(v) - m(\vartheta, u) \phi(u) - \int_y^v \phi(t) d(t) \tag{2.5}
$$

By adding equations (2.4) and (2.5), we get (2.2).

Theorem 2.3. If $\phi : [u, v] \to \mathbb{R}$ is a function of bounded variation on $[u, v]$ and $y \in [u + \lambda \frac{v - u}{2}, v - \lambda \frac{v - u}{2}]$, then the following weighted inequality holds

$$
\left| \phi(y)m(\mu, \vartheta) + \phi(u)m(u, \mu) + \phi(v)m(\vartheta, v) - \int_u^v \phi(t) w(t) dt \right| \leq \max \left\{ m(u, \mu), m(\mu, y), m(y, \vartheta), m(\vartheta, v) \right\} \sqrt{\phi} \tag{2.6}
$$

Proof. By using the same fact we used in the previous theorem

$$
\left| \int_c^d p(t) df(t) \right| \leq \sup_{t \in [c, d]} |p(t)| \sqrt{f}.
$$
Applying the above inequality for \( p(t) = P_w(y, t) \) and \( f(t) = \phi(t) \), we have

\[
\left| \int_u^v P_w(y, t) d\phi(t) \right| \\
\leq \left| \int_u^y P_w(y, t) d\phi(t) \right| + \left| \int_y^v P_w(y, t) d\phi(t) \right|
\]

By using the same fact

\[
\leq \sup_{t \in [u, y]} P_w(y, t) \sqrt[\phi]{(\phi)} + \sup_{t \in [y, v]} P_w(y, t) \sqrt[\phi]{(\phi)}
\]

\[
\leq \max \{m(u, \mu), m(\mu, y)\} \sqrt[\phi]{(\phi)} + \max \{m(y, \vartheta), m(\vartheta, v)\} \sqrt[\phi]{(\phi)}
\]

\[
\leq \max \{m(u, \mu), m(\mu, y), m(y, \vartheta), m(\vartheta, v)\} \sqrt[\phi]{(\phi)}
\]

which implies

\[
\left| \phi(y)m(\mu, \vartheta) + \phi(u)m(u, \mu) + \phi(v)(\vartheta, v) - \int_u^v \phi(t)w(t)dt \right| \\
\leq \max \{m(u, \mu), m(\mu, y), m(y, \vartheta), m(\vartheta, v)\} \sqrt[\phi]{(\phi)}.
\]

**Corollary 2.4.** By replacing \( w(s) = \frac{1}{v-u} \), and the values of \( \mu \) and \( \vartheta \) in (2.6), we will get the inequality (2.1).

**Corollary 2.5.** By replacing \( \lambda = 0 \) in (2.1), we will get the following inequality

\[
\left| \phi(y)m(u, v) - \int_u^v w(t)\phi(t)dt \right| \leq \max \{m(u, y), m(v, y)\} \sqrt[\phi]{(\phi)}.
\] (2.7)

**Remark 2.6.** By replacing \( w(s) = \frac{1}{v-u} \) in (2.7), we will get the following inequality

\[
\left| \phi(y) - \frac{1}{v-u} \int_u^v \phi(t)dt \right| \leq \frac{1}{v-u} \left[ \max \{(y-u), (v-y)\} \sqrt[\phi]{(\phi)} \right].
\] (2.8)
Corollary 2.7. By replacing $y = \frac{u + v}{2}$ in (2.7), we will get the following inequality

$$\left| \left[ \lambda \phi(u) + \phi(v) + (1 - \lambda) \phi \left( \frac{u + v}{2} \right) \right] - \frac{1}{v - u} \int_{u}^{v} \phi(t) dt \right|$$

$$\leq \frac{1}{2} \left[ \max \{ \lambda, (1 - \lambda) \} \int_{u}^{v} \phi(t) dt \right]$$

$$= \frac{1}{2} \left[ \left( \frac{1}{2} + \left| \lambda - \frac{1}{2} \right| \right) \int_{u}^{v} \phi(t) dt \right]$$

(2.9)

the inequality (2.9) is the result of Corollary 1 of [1].

Corollary 2.8. In (2.9)

1. by replacing $\lambda = 0$, we will get midpoint inequality

$$\left| \phi \left( \frac{u + v}{2} \right) - \frac{1}{v - u} \int_{u}^{v} \phi(t) dt \right| \leq \frac{1}{2} \int_{u}^{v} \phi(t) dt$$

(2.10)

2. by replacing $\lambda = \frac{1}{4}$, we will get $\frac{3}{8}$ Simpson’s inequality

$$\left| \frac{3}{8} \left[ \frac{\phi(u) + \phi(v)}{3} + 2 \phi \left( \frac{u + v}{2} \right) \right] - \frac{1}{v - u} \int_{u}^{v} \phi(t) dt \right| \leq \frac{3}{8} \int_{u}^{v} \phi(t) dt$$

(2.11)

3. by replacing $\lambda = \frac{1}{3}$, we will get $\frac{1}{5}$ Simpson’s inequality

$$\left| \frac{1}{6} \left[ \phi(u) + 4 \phi \left( \frac{u + v}{2} \right) + \phi(v) \right] - \frac{1}{v - u} \int_{u}^{v} \phi(t) dt \right| \leq \frac{1}{5} \int_{u}^{v} \phi(t) dt$$

(2.12)

4. by replacing $\lambda = \frac{1}{2}$, we will get perturbad trapezoidal inequality

$$\left| \frac{1}{4} \left[ \phi(u) + 2 \phi \left( \frac{u + v}{2} \right) + \phi(v) \right] - \frac{1}{v - u} \int_{u}^{v} \phi(t) dt \right| \leq \frac{1}{4} \int_{u}^{v} \phi(t) dt$$

(2.13)

5. by replacing $\lambda = 1$, we will get trapezoidal inequality

$$\left| \phi(u) + \phi(v) \right| - \frac{1}{v - u} \int_{u}^{v} \phi(t) dt \right| \leq \frac{1}{2} \int_{u}^{v} \phi(t) dt$$

(2.14)

The inequalities (2.10), (2.12), (2.13) and (2.14) are the results of Corollary 2 of [1].
3. FOR THE CASE $\phi' \in L_\infty[u,v]$

**Theorem 3.1.** Let a function $\phi : [u,v] \to \mathbb{R}$ is absolutely continuous and $\phi'$ is bounded on $[u,v]$ i.e.,

$$\|\phi'\|_\infty = \sup_{t \in [u,v]} |\phi'(t)| < \infty.$$  

Then the following inequality holds

$$\left| \phi(y)m(\mu, \vartheta) + \phi(u)m(u, \mu) + \phi(v)m(\vartheta, v) - \int_u^v \phi(t)w(t)dt \right|$$

$$\leq [um(\mu, u) + y(m(\mu, y) + m(\vartheta, y)) + vm(\vartheta, v) + M(u, \mu) + M(y, \vartheta) + M(v, \vartheta)] \|\phi'\|_\infty. \quad (3.1)$$

**Proof.** By using the identity (2.2), kernel (2.3) and the fact

$$\int_a^d \left( \int_a^d w(s)ds \right) dt = \int_a^d t m(a, t) dt = tm(a, t)|_c^d - M(c, d)$$

we have

$$\left| \phi(y)m(\mu, \vartheta) + \phi(u)m(u, \mu) + \phi(v)m(\vartheta, v) - \int_u^v \phi(t)w(t)dt \right|$$

$$= \left| \int_u^v P_w(y, t)\phi'(t)dt \right|$$

$$\leq \int_u^v |P_w(y, t)|dt \|\phi'\|_\infty$$

$$\leq \left[ - \int_u^\mu m(\mu, t)dt + \int_\mu^y m(\mu, t)dt - \int_\vartheta^v m(\vartheta, t)dt + \int_\vartheta^v m(\vartheta, t)dt \right] \|\phi'\|_\infty$$

$$\leq [um(\mu, u) + y(m(\mu, y) + m(\vartheta, y)) + vm(\vartheta, v) + M(u, \mu) + M(y, \vartheta) + M(v, \vartheta)] \|\phi'\|_\infty.$$ 

**Corollary 3.2.** By replacing $w(s) = \frac{1}{v-a}$, $\mu$ and $\vartheta$ in (3.1), we will get the inequality of Proposition 1.2.

**Corollary 3.3.** By replacing $\lambda = 0$, we will get the following inequality

$$\left| \phi(y)m(u, v) - \int_u^v w(t)\phi(t)dt \right| \leq [ym(u, y) + M(y, u) + M(y, v)] \|\phi'\|_\infty. \quad (3.2)$$
Remark 3.4. By replacing \( w(s) = \frac{1}{v-u} \) in (3.2), we will get the following inequality

\[
\left| \phi(y) - \frac{1}{v-u} \int_u^v \phi(t)dt \right| \leq \frac{1}{v-u} \left[ \left( y - \frac{u+v}{2} \right)^2 + \frac{(v-u)^2}{4} \right] \| \phi' \|_\infty. \tag{3.3}
\]

Corollary 3.5. By replacing \( y = \frac{u+v}{2} \) in Corollary 3.2, we will get following inequality

\[
\left| \left[ \lambda \frac{\phi(u) + \phi(v)}{2} + (1-\lambda) \phi \left( \frac{u+v}{2} \right) \right] - \frac{1}{v-u} \int_u^v \phi(t)dt \right| \leq \frac{(v-u)}{4} [\lambda^2 + (\lambda - 1)^2] \| \phi' \|_\infty \tag{3.4}
\]

This inequality is a result of Corollary 4 in [1].

Corollary 3.6. In (3.4)

1. by replacing \( \lambda = 0 \), we get mid-point inequality

\[
\left| \phi \left( \frac{u+v}{2} \right) - \frac{1}{v-u} \int_u^v \phi(t)dt \right| \leq \frac{1}{4} (v-u) \| \phi' \|_\infty \tag{3.5}
\]

2. by replacing \( \lambda = \frac{1}{4} \), we get \( \frac{3}{8} \) Simpson’s inequality

\[
\left| \frac{3}{8} \left[ \frac{\phi(u) + \phi(v)}{3} + 2\phi \left( \frac{u+v}{2} \right) \right] - \frac{1}{v-u} \int_u^v \phi(t)dt \right| \leq \frac{5}{32} (v-u) \| \phi' \|_\infty \tag{3.6}
\]

3. by replacing \( \lambda = \frac{1}{3} \), we get \( \frac{1}{3} \) Simpson’s inequality

\[
\left| \frac{1}{6} \left[ \phi(u) + 4\phi \left( \frac{u+v}{2} \right) + \phi(v) \right] - \frac{1}{v-u} \int_u^v \phi(t)dt \right| \leq \frac{5}{36} (v-u) \| \phi' \|_\infty \tag{3.7}
\]

4. by replacing \( \lambda = \frac{1}{2} \), we get perturbed trapezoidal inequality

\[
\left| \frac{1}{4} \left[ \phi(u) + 2\phi \left( \frac{u+v}{2} \right) + \phi(v) \right] - \frac{1}{v-u} \int_u^v \phi(t)dt \right| \leq \frac{1}{8} (v-u) \| \phi' \|_\infty \tag{3.8}
\]

5. by replacing \( \lambda = 1 \), we get trapezoidal inequality

\[
\left| \frac{1}{2} [\phi(u) + \phi(v)] - \frac{1}{v-u} \int_u^v \phi(t)dt \right| \leq \frac{1}{4} (v-u) \| \phi' \|_\infty. \tag{3.9}
\]

The inequalities (3.5), (3.7), (3.8) and (3.9) are the results of Corollary 5 of [1].
4. For the Case $\phi' \in L^p[u, v]$

Theorem 4.1. Let $\phi : I \subset \mathbb{R} \to \mathbb{R}$ is absolutely continuous mapping on $I^0$, the interior of the interval $I$, where $u, v \in I$ with $u < v$. If $\phi' \in L^p[u, v]$, $p > 1$. Then the following inequality holds

$$\left\| \left[ \frac{\lambda v - u}{2} + (1 - \lambda) \phi(y) \right] - \frac{1}{v - u} \int_u^v \phi(t) dt \right\| \leq \frac{1}{v - u} \left\| \frac{1}{(q + 1) \lambda} \left[ 2 \left( \frac{\lambda v - u}{2} \right)^{q+1} + \left( y - \frac{(2 - \lambda)u + \lambda v}{2} \right)^{q+1} \right] \right\| \| \phi' \|_p$$

(4.1)

$\forall \lambda \in [0, 1]$ and $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$ and

$$u + \lambda \frac{v - u}{2} \leq y \leq v - \lambda \frac{v - u}{2}.$$

Proof. We have

$$\left\| \left[ \frac{\lambda v - u}{2} + (1 - \lambda) \phi(y) \right] - \frac{1}{v - u} \int_u^v \phi(t) dt \right\| = \left\| \frac{1}{v - u} \int_u^v K(y, t) \phi'(t) dt \right\|$$

Applying Hölder’s Inequality

$$\leq \frac{1}{v - u} \left( \int_u^v |K(y, t)|^q dt \right)^{\frac{1}{q}} \left( \int_u^v |\phi'(t)|^p dt \right)^{\frac{1}{p}}$$

$$\leq \frac{1}{v - u} \left[ \int_u^v \left| t - \left( u + \frac{\lambda v - u}{2} \right) \right|^q dt + \int_y^v \left| t - \left( v - \frac{\lambda v - u}{2} \right) \right|^q dt \right]^{\frac{1}{q}} \| \phi' \|_p$$

$$\leq \frac{1}{v - u} \frac{1}{(q + 1) \lambda} \left[ 2 \left( \frac{\lambda v - u}{2} \right)^{q+1} + \left( y - \frac{(2 - \lambda)u + \lambda v}{2} \right)^{q+1} \right]^{\frac{1}{q}} \| \phi' \|_p.$$
Theorem 4.2. Let a function $\phi : [u,v] \rightarrow \mathbb{R}$ is absolutely continuous and $\phi'$ is bounded on $[u,v]$.

If $\phi'$ belongs to $L_p[u,v], p > 1$, then the following inequality holds

$$
\left| \phi(y)m(\mu, \vartheta) + \phi(u)m(a, \mu) + \phi(v)m(\vartheta, v) - \int_u^v \phi(t)w(t)dt \right|
\leq \frac{1}{(q+1)^\frac{1}{q}} \left[ \int_u^\mu [m(t, \mu)]^q dt + \int_y^v [m(\mu, t)]^q dt + \int_y^\vartheta [m(t, \vartheta)]^q dt + \int_u^\vartheta [m(\vartheta, t)]^q dt \right]^\frac{1}{q} \|\phi'\|_p.

(4.2)

Proof. By using the identity (2.2) and kernel (2.3) and applying Hölder’s Inequality

$$
\left| \int_u^v P_w \phi'(t)dt \right|
\leq \left[ \int_u^v |P_w(y,t)|^q dt \right]^\frac{1}{q} \|\phi'\|_p
\leq \left[ \int_u^\mu |m(\mu, t)|^q dt + \int_y^v |m(\vartheta, t)|^q dt \right]^\frac{1}{q} \|\phi'\|_p
\leq \left[ \int_u^\mu [m(t, \mu)]^q dt + \int_y^v [m(\mu, t)]^q dt + \int_y^\vartheta [m(t, \vartheta)]^q dt + \int_u^\vartheta [m(\vartheta, t)]^q dt \right]^\frac{1}{q} \|\phi'\|_p
$$

which completes the proof.

Corollary 4.3 By replacing $w(s) = \frac{1}{v-u}$ and the values of $\mu$ and $\vartheta$ in (4.1), we will get the following inequality

$$
\left| \frac{\lambda}{2} [\phi(u) + \phi(v)] + (1-\lambda)\phi(y) - \frac{1}{v-u} \int_u^v \phi(t)dt \right|
\leq \frac{1}{(q+1)^\frac{1}{q}(v-u)^\frac{q+1}{q}} \left[ \left( y - \frac{(2-\lambda)u + \lambda v}{2} \right)^{q+1} + 2 \left( \frac{\lambda}{2} \frac{v-u}{2} \right)^{q+1} + \left( \frac{\lambda u + (2-\lambda)v}{2} - y \right)^{q+1} \right] \|\phi'\|_p.

(4.3)

Corollary 4.4 By replacing $\lambda = 0$ in (4.1), we will get the following inequality

$$
\left| \phi(y)m(u,v) - \int_u^v w(t)\phi(t)dt \right|
\leq \left[ \int_u^\mu (m(u,t))^q dt + \int_y^v (m(t,v))^q dt \right]^\frac{1}{q} \|\phi'\|_p.

(4.4)

Remark 4.5 By replacing $w(s) = \frac{1}{v-u}$ in (4.4), we will get the following inequality

$$
\left| \phi(y) - \frac{1}{v-u} \int_u^v \phi(t)dt \right|
\leq \frac{1}{v-u} \frac{1}{(q+1)^\frac{1}{q}} \left[ (y-u)^{q+1} + (v-y)^{q+1} \right]^\frac{1}{q} \|\phi'\|_p

(4.5)
**Corollary 4.6** By replacing \( y = \frac{u+v}{2} \) in (4.3), we will get the following inequality

\[
\left| \frac{1}{2} \frac{\phi(u) + \phi(v)}{2} + (1 - \lambda) \frac{\phi(u + v)}{2} - \frac{1}{v - u} \int_{u}^{v} \phi(t) \, dt \right| \leq \frac{1}{2} \left( \frac{1 - \lambda}{q+1} + \lambda \frac{q+1}{q+1} \right)^{\frac{1}{q}} (v-u)^{\frac{1}{q}} \left\| \phi' \right\|_{p} \tag{4.6}
\]

The inequality (4.6) is the result of Corollary 7 in [1].

**Corollary 4.7** In (4.6)

1. by replacing \( \lambda = 0 \), we get mid-point inequality

\[
\left| \phi \left( \frac{u+v}{2} \right) - \frac{1}{v-u} \int_{u}^{v} \phi(t) \, dt \right| \leq \frac{1}{2} \left( \frac{1}{q+1} \right)^{\frac{1}{q}} (v-u)^{\frac{1}{q}} \left\| \phi' \right\|_{p} \tag{4.7}
\]

2. by replacing \( \lambda = \frac{1}{4} \), we get \( \frac{3}{8} \) Simpson’s inequality

\[
\left| \frac{3}{8} \left[ \frac{\phi(u) + \phi(v)}{3} + 2 \phi \left( \frac{u+v}{2} \right) \right] - \frac{1}{v-u} \int_{u}^{v} \phi(t) \, dt \right| \leq \frac{1}{8} \left( \frac{3q+1}{4(q+1)} \right)^{\frac{1}{q}} (v-u)^{\frac{1}{q}} \left\| \phi' \right\|_{p} \tag{4.8}
\]

3. by replacing \( \lambda = \frac{1}{3} \), we get \( \frac{1}{3} \) Simpson’s inequality

\[
\left| \frac{1}{6} \left[ \phi(u) + 4 \phi \left( \frac{u+v}{2} \right) + \phi(v) \right] - \frac{1}{v-u} \int_{u}^{v} \phi(t) \, dt \right| \leq \frac{1}{6} \left( \frac{2q+1}{3(q+1)} \right)^{\frac{1}{q}} (v-u)^{\frac{1}{q}} \left\| \phi' \right\|_{p} \tag{4.9}
\]

4. by replacing \( \lambda = \frac{1}{2} \), we get perturbed trapezoidal inequality

\[
\left| \frac{1}{4} \left[ \phi(u) + 2 \phi \left( \frac{u+v}{2} \right) + \phi(v) \right] - \frac{1}{v-u} \int_{u}^{v} \phi(t) \, dt \right| \leq \frac{1}{4} \left( \frac{1}{q+1} \right)^{\frac{1}{q}} (v-u)^{\frac{1}{q}} \left\| \phi' \right\|_{p} \tag{4.10}
\]

5. by replacing \( \lambda = 1 \), we get trapezoidal inequality

\[
\left| \frac{1}{2} \left[ \phi(u) + \phi(v) \right] - \frac{1}{v-u} \int_{u}^{v} \phi(t) \, dt \right| \leq \frac{1}{2} \left( \frac{1}{q+1} \right)^{\frac{1}{q}} (v-u)^{\frac{1}{q}} \left\| \phi' \right\|_{p} \tag{4.11}
\]

The inequalities (4.7), (4.9), (4.10) and (4.11) are the results of Corollary 8 in [1].

**5. Application to Quadrature Formula**

If \( I_{n} : u = y_{0} < y_{1} < y_{2} < \ldots < y_{n} = v \) be a partition of the interval \([u, v]\) and let \( h_{i} = y_{i+1} - y_{i} \) for \( i \in \{0, 1, 2, \ldots, n-1\} \)

Consider a general Quadrature Formula

\[
Q_{n}(I_{n}, \phi) := \sum_{i=0}^{n-1} \left[ \phi(\xi_{i})m(\mu_{i}, \vartheta_{i}) + \phi(y_{i})m(y_{i}, \mu_{i})ds + \phi(y_{i+1})m(\vartheta_{i}, y_{i+1}) \right] \tag{5.1}
\]
∀ λ ∈ [0, 1] and
\[ \mu_i = y_i + \lambda \frac{h_i}{2} \leq \xi_i \leq y_{i+1} - \lambda \frac{h_i}{2} = \vartheta_i \]
and
\[ R_n(I_n, \phi) = \int_u^v \phi(t)w(t)dt - Q_n(I_n, \phi) \]
which yields following theorems.

**Theorem 5.1.** Let φ be as defined in Theorem 3.1 and we have
\[ \int_u^v \phi(t)w(t)dt = R_n(I_n, \phi) + Q_n(I_n, \phi) \]
where \( Q_n(I_n, \phi) \) is given in (5.1), then the remainder satisfies the estimation
\[ |R_n(I_n, \phi)| \leq \sum_{i=0}^n \max \{ m(y_i, \mu_i), m(\mu_i, \xi_i), m(\xi_i, \vartheta_i), m(\vartheta_i, y_{i+1}) \} \left[ \frac{y_{i+1}}{y_i} \right] \sqrt{(\phi)}. \]  

**Proof.** Applying inequality (2.1) on the interval \([y_i, y_{i+1}]\), we get
\[ R_i(I_i, \phi) = \frac{1}{h_i} \int_{y_i}^{y_{i+1}} \phi(t)w(t)dt - \sum_{i=0}^n \left[ \phi(\xi_i)m(\mu_i, \vartheta_i) + \phi(y_i)m(y_i, \mu_i)ds + \phi(y_{i+1})m(\vartheta_i, y_{i+1}) \right]. \]
Sum the equalities presented above over \( i \) from 0 to \( n \), we get
\[ R_n(I_n, \phi) = \sum_{i=0}^n \frac{1}{h_i} \int_{y_i}^{y_{i+1}} \phi(t)w(t)dt \]
\[ - \sum_{i=0}^n \left[ \phi(\xi_i)m(\mu_i, \vartheta_i) + \phi(y_i)m(y_i, \mu_i)ds + \phi(y_{i+1})m(\vartheta_i, y_{i+1}) \right] \]
which implies
\[ |R_n(I_n, \phi)| = \left| \sum_{i=0}^n \frac{1}{h_i} \int_{y_i}^{y_{i+1}} \phi(t)w(t)dt \right| \]
\[ - \sum_{i=0}^n \left[ \phi(\xi_i)m(\mu_i, \vartheta_i) + \phi(y_i)m(y_i, \mu_i)ds + \phi(y_{i+1})m(\vartheta_i, y_{i+1}) \right] \]
\[ \leq \sum_{i=0}^n \max \{ m(y_i, \mu_i), m(\mu_i, \xi_i), m(\xi_i, \vartheta_i), m(\vartheta_i, y_{i+1}) \} \left[ \frac{y_{i+1}}{y_i} \right] \sqrt{(\phi)}. \]

**Theorem 5.2.** Let φ be as defined in Theorem 4.1 and we have
\[ \int_u^v \phi(t)w(t)dt = R_n(I_n, \phi) + Q_n(I_n, \phi) \]
where $Q_n(I_n, \phi)$ is given in (5.1), then the remainder satisfies the estimation

$$|R_n(I_n, \phi)| \leq \left| y_i m(\mu_i, y_i) + \xi_i (m(\mu_i, \xi_i) + m(\vartheta_i, \xi_i)) + y_i+1 m(\vartheta_i, y_i+1) + M(y_i, \mu_i) + M(\xi_i, \mu_i) + M(\xi_i, \vartheta_i) + M(y_i+1, \vartheta_i) \right| \|\phi'\|_\infty. \quad (5.3)$$

**Proof.** By using the similar technique used in Theorem 5.1, we get

$$\left| R_n(I_n, \phi) \right| = \left| \sum_{i=0}^{n} \frac{1}{h_i} \int_{y_i}^{y_{i+1}} \phi(t) w(t) \, dt \right|$$

$$- \sum_{i=0}^{n} \left[ \phi(\xi_i) m(\mu_i, \vartheta_i) + \phi(y_i) m(y_i, \mu_i) ds + \phi(y_i+1) m(\vartheta_i, y_i+1) \right]$$

$$\leq \left| y_i m(\mu_i, y_i) + \xi_i (m(\mu_i, \xi_i) + m(\vartheta_i, \xi_i)) + y_i+1 m(\vartheta_i, y_i+1) + M(y_i, \mu_i) + M(\xi_i, \mu_i) + M(\xi_i, \vartheta_i) + M(y_i+1, \vartheta_i) \right| \|\phi'\|_\infty.$$

**Theorem 5.3.** Let $\phi$ be as defined in Theorem 5.1 and we have

$$\int_{u}^{v} \phi(t) w(t) \, dt = R_n(I_n, \phi) + Q_n(I_n, \phi)$$

where $Q_n(I_n, \phi)$ is given in (5.1), then the remainder satisfies the estimation

$$|R_n(I_n, \phi)| \leq \frac{1}{(q+1)^{\frac{1}{q}}} \left[ \int_{y_i}^{\mu_i} (m(y_i, \mu_i))^{q} \, dt + \int_{\mu_i}^{\xi_i} (m(\mu_i, \xi_i))^{q} \, dt \right. \left. + \int_{\xi_i}^{\vartheta_i} (m(\xi_i, \vartheta_i))^{q} \, dt + \int_{\vartheta_i}^{y_{i+1}} (m(\vartheta_i, y_{i+1}))^{q} \, dt \right]^{\frac{1}{q}} \|\phi'\|_p. \quad (5.4)$$

**Proof** By using the similar technique used in Theorem 5.1, we get

$$\left| R_n(I_n, \phi) \right| = \left| \sum_{i=0}^{n} \frac{1}{h_i} \int_{y_i}^{y_{i+1}} \phi(t) w(t) \, dt \right|$$

$$- \sum_{i=0}^{n} \left[ \phi(\xi_i) m(\mu_i, \vartheta_i) + \phi(y_i) m(y_i, \mu_i) ds + \phi(y_i+1) m(\vartheta_i, y_i+1) \right]$$

$$\leq \frac{1}{(q+1)^{\frac{1}{q}}} \left[ \int_{y_i}^{\mu_i} (m(y_i, \mu_i))^{q} \, dt + \int_{\mu_i}^{\xi_i} (m(\mu_i, \xi_i))^{q} \, dt \right. \left. + \int_{\xi_i}^{\vartheta_i} (m(\xi_i, \vartheta_i))^{q} \, dt + \int_{\vartheta_i}^{y_{i+1}} (m(\vartheta_i, y_{i+1}))^{q} \, dt \right]^{\frac{1}{q}} \|\phi'\|_p.
6. Conclusion

We have derived three different versions of Ostrowski type inequality, namely for bounded variation, $L_p$ and $L_\infty$ space involving weights in terms of moment generating functions and by using them we also discussed their few applications in numerical integration.

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Conflict of Interests

The author(s) declare that there is no conflict of interests.

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