On Liouville-type theorems for the stationary MHD and the Hall-MHD systems in $\mathbb{R}^3$

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Abstract. In this paper, we prove a Liouville-type theorem for the stationary MHD and the stationary Hall-MHD systems. Assuming suitable growth condition at infinity for the mean oscillations for the potential functions, we show that the solutions are trivial. These results generalize the previous results obtained by two of the current authors in Chae and Wolf (J Differ Equ 295:233–248, 2021). To prove our main theorems, we use a refined iteration argument.

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1. Introduction

We consider the 3D stationary magnetohydrodynamics equations:

\[
\begin{cases}
-\Delta u + (u \cdot \nabla)u = -\nabla P + (B \cdot \nabla)B, \\
-\Delta B + (u \cdot \nabla)B = (B \cdot \nabla)u, \\
\text{div } u = \text{div } B = 0,
\end{cases}
\]

(1.1)

which physically describe the steady state of electrically conducting fluids, for example, plasmas. Here, $u = u(x) = (u_1, u_2, u_3)$ is the velocity field of the fluid, $B = B(x) = (B_1, B_2, B_3)$ is the magnetic field, and $P = P(x)$ is the pressure of the fluid. When $B \equiv 0$, the system reduces to the stationary Navier–Stokes system.

As the Liouville-type problem for the stationary Navier–Stokes equations has been studied extensively recently (see [1,3,4,8,9,11–15,18–20]), there are also many works on the Liouville-type problem on the MHD equations (see [16,22,23] and references therein). Here, we focus on the results under the assumptions in terms of potential functions. We let $\Phi \in L^1_{loc}(\mathbb{R}^3; \mathbb{R}^3 \times \mathbb{R}^3)$ and $\Psi \in L^1_{loc}(\mathbb{R}^3; \mathbb{R}^3 \times \mathbb{R}^3)$ be the potential functions for the vector fields $u \in L^1_{loc}(\mathbb{R}^3)$ and $B \in L^1_{loc}(\mathbb{R}^3)$, respectively, if $\text{div } \Phi = u$ and $\text{div } \Psi = B$. In [18], Seregin proved Liouville-type theorems for the stationary Navier–Stokes equations under the assumptions for the potential $\Phi \in BMO(\mathbb{R}^3)$ and for the velocity $v \in L^6(\mathbb{R}^3)$. (Later in [19], the velocity condition is dropped.) After that, Chae and Wolf improved the previous result (see [5]). For the MHD equations, Schulz obtained Liouville theorem in [17] under the conditions similar to those in [18]. Recently, Chae and Wolf in [6] proved the similar theorem but under relaxed conditions than [17]. Moreover, they also obtained similar Liouville-type result for the 3D Hall-MHD system

\[
\begin{cases}
-\Delta u + (u \cdot \nabla)u = -\nabla P + (B \cdot \nabla)B, \\
-\Delta B + (u \cdot \nabla)B = (B \cdot \nabla)u + \nabla \times ((\nabla \times B) \times B), \\
\text{div } u = \text{div } B = 0.
\end{cases}
\]

(1.2)
We note that the system (1.2) is a physically important generalization of (1.1). (See, e.g., [2] for preliminary mathematical results and physical motivation of the Hall-MHD system.)

Our aim in this paper is to generalize the main results in [5,6] (and hence [17]).

**Theorem 1.** Let \((u, B, P)\) be a smooth solution to (1.1). Assume that there exist \(\Phi, \Psi \in C^\infty(\mathbb{R}^3; \mathbb{R}^{3 \times 3})\) such that \(\nabla \cdot \Phi = u, \nabla \cdot \Psi = B,\) and

\[
\left( \frac{1}{|B(r)|} \int_{B(r)} |\Phi - \Phi_{B(r)}|^s \, dx \right)^{\frac{1}{s}} + \left( \frac{1}{|B(r)|} \int_{B(r)} |\Psi - \Psi_{B(r)}|^s \, dx \right)^{\frac{1}{s}} \leq C r^{\frac{s}{3} - \frac{s}{2}}, \quad r > 1 \tag{1.3}
\]

for some \(3 < s \leq 6\). Then, \(u \equiv B \equiv 0\).

**Remark 1.** In the case \(B \equiv 0\), Theorem 1 reduces to [5, Theorem 1.1].

**Theorem 2.** Let \((u, B, P)\) be a smooth solution to (1.2). Assume that there exist \(\Phi, \Psi \in C^\infty(\mathbb{R}^3; \mathbb{R}^{3 \times 3})\) such that \(\nabla \cdot \Phi = u, \nabla \cdot \Psi = B,\) and (1.3) for some \(3 < s \leq 6\). In addition,

\[
\left( \frac{1}{|B(r)|} \int_{B(r)} |B - B_{B(r)}|^p \, dx \right)^{\frac{1}{p}} = o\left(r^{\frac{3s}{2} - \frac{3}{4}}\right) \quad \text{as} \quad r \to \infty \tag{1.4}
\]

for some \(p > 3\). Then, \(u \equiv B \equiv 0\).

**Remark 2.** When \(s = 6\) and \(p = 6\), the condition (1.4) reduces to

\[
\frac{1}{r^3} \int_{B(r)} |B - B_{B(r)}|^6 \, dx \to 0 \quad \text{as} \quad r \to \infty,
\]

and we recover [6, Theorem 1.3].

### 2. Preliminary lemmas

In this section, we prove lemmas, which are essential to prove our main results.

**Lemma 1.** Let \(R > 1\) and \(f \in W^{1,2}(B(R); \mathbb{R}^3)\). For \(0 < \rho < R\), let \(\psi \in C^\infty_c(B(R))\) such that \(0 \leq \psi \leq 1\) and \(\nabla \psi \leq C/(R - \rho)\). Suppose that there exists \(F \in W^{2,2}(B(R); \mathbb{R}^{3 \times 3})\) with \(\nabla \cdot F = f\) and

\[
\left( \frac{1}{|B(r)|} \int_{B(r)} |F - F_{B(r)}|^s \, dx \right)^{\frac{1}{s}} \leq C r^{\frac{s}{3} - \frac{s}{2}}, \quad r > 1
\]

for some \(3 \leq s \leq 6\). Then, it holds

\[
\|\psi^2 f\|^2_{L^2(B(R))} \leq CR^{\frac{11}{2} - \frac{s}{2}} \|\psi \nabla f\|_{L^2(B(R))} + CR^{\frac{11}{2} - \frac{s}{2}} (R - \rho)^{-2} \tag{2.1}
\]

and

\[
\|\psi^3 f\|^3_{L^3(B(R))} \leq CR\|\psi \nabla f\|_{L^2(B(R))}^{\frac{18}{5}} + CR^{4 - \frac{s}{2}} (R - \rho)^{-3} + CR((R - \rho)^{-1} \|\psi^2 f\|_{L^2(B(R))})^{\frac{18}{5s}}. \tag{2.2}
\]

**Proof.** We first show (2.1) following the proof of [7, Lemma 2.1]. Using \(f = \nabla \cdot F\) and integration by parts, we have

\[
\int_{B(R)} |\psi^2 f|^2 \, dx = \int_{B(R)} \partial_i (F_{ij} - (F_{ij})_{B(R)}) f_j \psi^4 \, dx
\]

\[
= -\int_{B(R)} (F_{ij} - (F_{ij})_{B(R)}) \partial_i f_j \psi^4 \, dx - 4 \int_{B(R)} (F_{ij} - (F_{ij})_{B(R)}) f_j \psi^3 \partial_i \psi \, dx
\]
Thus, it follows from Young’s inequality that
\[ \int_{B(R)} |F - F_{B(R)}||ψ∇f||ψ^3| \, dx + 4 \int_{B(R)} |F - F_{B(R)}||ψ^2f||∇ψ||ψ| \, dx \]
\[ \leq \int_{B(R)} |F - F_{B(R)}||ψ∇f| \, dx + C(R - ρ)^{-1} \int_{B(R)} |F - F_{B(R)}||ψ^2f| \, dx. \]
\[ \int_{B(R)} |F - F_{B(R)}||ψ∇f| \, dx \leq \left( \frac{1}{|B(R)|} \int_{B(R)} |F - F_{B(R)}|^6 \, dx \right)^{\frac{1}{2}} \|ψ∇f\|_{L^2}|B(R)|^{\frac{1}{2}} \]
and
\[ \int_{B(R)} |F - F_{B(R)}||ψ^2f| \, dx \leq \left( \frac{1}{|B(R)|} \int_{B(R)} |F - F_{B(R)}|^6 \, dx \right)^{\frac{1}{2}} \|ψ^2f\|_{L^2}|B(R)|^{\frac{1}{2}}. \]
Thus, together with our assumption, we have
\[ \int_{B(R)} |ψ^2f|^2 \, dx \leq CR^{\frac{11}{6} - \frac{1}{2}} \|ψ∇f\|_{L^2} + C(R - ρ)^{-1} R^{\frac{11}{6} - \frac{1}{2}} \|ψ^2f\|_{L^2}. \]
Applying Young’s inequality to this inequality, we obtain (2.1).
Now, we show (2.2) following the proof of [7, Lemma 2.2]. Using \( f = ∇ \cdot F \) and integration by parts, we can estimate in a similar way as before
\[ \int_{B(R)} |ψ^3f|^3 \, dx \leq C \int_{B(R)} |F - F_{B(R)}||ψ∇f||ψ^3f| \, dx + C(R - ρ)^{-1} \int_{B(R)} |F - F_{B(R)}||ψ^3f|^2 \, dx. \]
We consider the second integral on the right-hand side. Using Hölder’s inequality and our assumption, we get
\[ C(R - ρ)^{-1} \int_{B(R)} |F - F_{B(R)}||ψ^3f|^2 \, dx \]
\[ \leq C(R - ρ)^{-1} \left( \frac{1}{|B(R)|} \int_{B(R)} |F - F_{B(R)}|^8 \, dx \right)^{\frac{1}{2}} \|ψ^3f\|^2_{L^4}|B(R)|^{\frac{1}{4}} \]
\[ \leq CR^{\frac{1}{2}} (R - ρ)^{-1} \|ψ^3f\|^2_{L^4}. \]
Thus, it follows from Young’s inequality that
\[ C(R - ρ)^{-1} \int_{B(R)} |F - F_{B(R)}||ψ^3f|^2 \, dx \leq CR^{4 - \frac{3}{2}} (R - ρ)^{-3} + \frac{1}{4} \int_{B(R)} |ψ^3f|^3 \, dx. \]
Next, we estimate the first integral on the right-hand side. From Hölder’s inequality and our assumption, we have
\[
\int_{B(R)} |F - F_{B(R)}| \psi \nabla f |\psi^3 f| \, dx \leq \left( \frac{1}{|B(R)|} \int_{B(R)} |F - F_{B(R)}|^s \, dx \right)^{\frac{1}{s}} \|\psi \nabla f\|_{L^2} \|\psi^3 f\|_{L^{\frac{2s}{3}}} |B(R)|^{\frac{1}{2}} \leq R^{\frac{3}{2} + \frac{2}{3}} \|\psi \nabla f\|_{L^2} \|\psi^3 f\|_{L^{\frac{2s}{3}}}.
\]

Since Gagliardo–Nirenberg interpolation inequality and Hölder’s inequality imply
\[
\|\psi^3 f\|_{L^{\frac{2s}{3}}} \leq C \left( \|\psi^3 \nabla f\|_{L^2} + (R - \rho)^{-1} \|\psi^2 f\|_{L^2} \right)^{\frac{6}{s} - 1} \|\psi^3 f\|_{L^2}^{\frac{2}{s}} - \frac{6}{s} \leq C \|\psi \nabla f\|_{L^2}^{\frac{6}{s} - 1} \|\psi^3 f\|_{L^2}^{\frac{2}{s}} + C \left( (R - \rho)^{-1} \|\psi^2 f\|_{L^2} \right)^{\frac{6}{s} - 1} \|\psi^3 f\|_{L^2}^{\frac{2}{s}},
\]
we can see with the application of Young’s inequality that
\[
\int_{B(R)} |F - F_{B(R)}| \psi \nabla f |\psi^3 f| \, dx \leq CR^{\frac{3}{2} + \frac{2}{3}} \|\psi \nabla f\|_{L^2}^{\frac{6}{s} - 1} \|\psi^3 f\|_{L^2}^{\frac{2}{s}} + CR^{\frac{3}{2} + \frac{2}{3}} \|\psi \nabla f\|_{L^2} \left( (R - \rho)^{-1} \|\psi^2 f\|_{L^2} \right)^{\frac{6}{s} - 1} \|\psi^3 f\|_{L^2}^{\frac{2}{s}} \leq CR \|\psi \nabla f\|_{L^2}^{\frac{6}{s} - 1} \|\psi^3 f\|_{L^2}^{\frac{2}{s}} + \frac{1}{4} \int_{B(R)} |\psi^3 f|^3 \, dx.
\]
Combining the above estimates, we obtain (2.2). This completes the proof.

To prove Theorem 2, it is important to estimate the Hall term in (1.2), \( \nabla \times ((\nabla \times B) \times B) \), carefully. Here, we collect some simple inequalities which will help it. For a measurable set \( E \subset \mathbb{R}^3 \) and a cutoff function \( \varphi \in C_c^\infty (\mathbb{R}^3) \) with \( 0 \leq \varphi \leq 1 \), we use the notations
\[
f_E := \frac{1}{|E|} \int_E f \, dx, \quad f_\varphi := \left( \int_{\mathbb{R}^3} \varphi \, dx \right)^{-1} \int_{\mathbb{R}^3} f \varphi \, dx
\]
for \( f \in L^1_{loc}(\mathbb{R}^3; \mathbb{R}^3) \).

**Lemma 2.** Let \( R > 1 \) and \( \varphi \in C_c^\infty (B(R)) \) satisfy \( 0 \leq \varphi \leq 1 \) and \( \|\varphi\|_{L^1(B(R))} \geq c|B(R)| \) for some \( c > 0 \). Then, for a measurable set \( E \subset \mathbb{R}^3 \) with \( \text{supp} (\varphi) \subset E \subset B(R) \), we have
\[
\|f - f_\varphi\|_{L^p(E)} \leq \frac{c + 1}{c} \|f - f_E\|_{L^p(E)}, \quad 1 \leq p \leq \infty.
\]

**Proof.** Noting that \( f_\varphi - \tau = (f - \tau)_\varphi \) for all \( \tau \in \mathbb{R} \), we have
\[
\|f - f_\varphi\|_{L^p(E)} \leq \|f - f_E\|_{L^p(E)} + \|(f - f_E)_\varphi\|_{L^p(E)}.
\]
Since Hölder’s inequality implies that
\[
|(f - f_E)_\varphi| \leq \frac{1}{c|B(R)|} \|f - f_E\|_{L^p(E)} \|\varphi\|_{L^p(E)} \leq \frac{|E|^{1 - \frac{1}{p}}}{c|B(R)|} \|f - f_E\|_{L^p(E)}, \quad 1 \leq p \leq \infty,
\]
it follows that
\[
\|f - f_\varphi\|_{L^p(E)} \leq \left( 1 + \frac{|E|}{c|B(R)|} \right) \|f - f_E\|_{L^p(E)}.
\]
This completes the proof. □

**Lemma 3.** Let $B \in C^\infty(\mathbb{R}^3;\mathbb{R}^3)$ and $\Psi \in C^\infty(\mathbb{R}^3;\mathbb{R}^{3 \times 3})$ satisfy $\nabla \cdot \Psi = B$ and (1.3). Let $R > 1$ and $\varphi \in C^\infty_c(B(R))$ satisfy $0 \leq \varphi \leq 1$, $\|\varphi\|_{L^1(B(R))} \geq c|B(R)|$, and $|\nabla \varphi| \leq C/R$ for some $c > 0$, $C > 0$. Then, we have

$$|B_\varphi| \leq \frac{C}{c} R^{-\frac{2}{3} - \frac{1}{2}}.$$  

**Proof.** We clearly have that

$$|B_\varphi| \leq \frac{1}{c|B(R)|} \int_{B(R)} B \varphi \, dx \leq \frac{1}{c|B(R)|} \int_{B(R)} \left| \nabla \cdot (\Psi - \Psi_{B(R)}) \varphi \right| \, dx.$$  

Since integration by parts yields

$$\frac{1}{c|B(R)|} \int_{B(R)} \left| \nabla \cdot (\Psi - \Psi_{B(R)}) \varphi \right| \, dx \leq \frac{1}{c|B(R)|} \int_{B(R)} \left| \Psi - \Psi_{B(R)} \right| \left| \nabla \varphi \right| \, dx,$$

applying the Hölder’s inequality with $|\nabla \varphi| \leq C/R$ we see that

$$\frac{1}{c|B(R)|} \int_{B(R)} \left| \nabla \cdot (\Psi - \Psi_{B(R)}) \varphi \right| \, dx \leq \frac{C}{cR} \left( \frac{1}{|B(R)|} \int_{B(R)} \left| \Psi - \Psi_{B(R)} \right| \, dx \right)$$

$$\leq \frac{C}{cR} \left( \frac{1}{|B(R)|} \int_{B(R)} \left| \Psi - \Psi_{B(R)} \right| \, dx \right)^{\frac{3}{2}}.$$

Thus, by (1.3) we complete the proof. □

At the end of this section, we define a family of cutoff functions for simplicity. For $0 < \tau < \tau'$, we let $\zeta = \zeta_{\tau, \tau'} \in C^\infty_c(B(\tau'))$ be a radially non-increasing scalar function such that $\zeta = 1$ on $B(\tau)$, $|\nabla \zeta| < 2/(\tau' - \tau)$, and $|\nabla^2 \zeta| < 4/(\tau' - \tau)^2$.

### 3. Proof of Theorem 1

Firstly, we show that

$$\int_{\mathbb{R}^3} (|\nabla u|^2 + |\nabla B|^2) \, dx < \infty.$$  

(3.1)

Let $\varphi$ be a cutoff function in $C^\infty_c(\mathbb{R}^3)$. We multiply (1.1)$_1$ and (1.1)$_2$ by $u \varphi$ and $B \varphi$, respectively, and integrate over $\mathbb{R}^3$. Then, integration by parts with the divergence-free conditions yields that

$$\int_{\mathbb{R}^3} (|\nabla u|^2 + |\nabla B|^2) \varphi \, dx = \frac{1}{2} \int_{\mathbb{R}^3} (|u|^2 + |B|^2) \Delta \varphi \, dx + \frac{1}{2} \int_{\mathbb{R}^3} (|u|^2 + |B|^2) u \cdot \nabla \varphi \, dx$$

$$- \int_{\mathbb{R}^3} (u \cdot B) (B \cdot \nabla) \varphi \, dx + \int_{\mathbb{R}^3} (P - P_{B(\tau)}) u \cdot \nabla \varphi \, dx,$$  

(3.2)

where $\tau$ is any positive number. We set $R > \rho > 1$ and $\overline{R} = (R + \rho)/2$ and then take $\varphi = \zeta_{\rho, \overline{R}}$ in (3.2). From the properties of $\zeta_{\rho, \overline{R}}$, Hölder’s inequality and Young’s inequality, we infer that
\[
\int_{B(\rho)} (|\nabla u|^2 + |\nabla B|^2) \, dx \leq C(R - \rho)^{-2} \int_{B(\rho) \setminus B(\pi)} (|u|^2 + |B|^2) \, dx
\]
\[
+ C(R - \rho)^{-1} \int_{B(\pi) \setminus B(\rho)} (|u|^3 + |B|^3) \, dx + C(R - \rho)^{-1} \int_{B(\pi)} |P - P_{B(\pi)}|^\frac{3}{2} \, dx.
\]

Let \( \psi = \zeta_{\pi, R} \). Since \( \psi = 1 \) on \( B(R) \), it follows
\[
\int_{B(\rho)} (|\nabla u|^2 + |\nabla B|^2) \, dx \leq C(I_1 + I_2 + I_3)
\]
where
\[
I_1 := (R - \rho)^{-2} \int_{B(R)} (|\psi^2 u|^2 + |\psi^2 B|^2) \, dx,
\]
\[
I_2 := (R - \rho)^{-1} \int_{B(R)} (|\psi^3 u|^3 + |\psi^3 B|^3) \, dx,
\]
\[
I_3 := (R - \rho)^{-1} \int_{B(\pi)} |P - P_{B(\pi)}|^\frac{3}{2} \, dx.
\]

To estimate \( I_3 \) first, we consider the functional \( F \in W^{-1, \frac{3}{2}}(B(\pi)) \) such that
\[
\langle F, \varphi \rangle = \int_{B(\pi)} (\nabla u - u \otimes u + B \otimes B) : \nabla \varphi \, dx, \quad \varphi \in W^{1,3}_0(B(\pi)).
\]

Due to that \((u, B, P)\) solves Eq. (1.1), we can verify \( F = -\nabla (P - P_{B(\pi)}) \) and \( \langle F, \varphi \rangle = 0 \) for \( \varphi \in W^{1,3}_0(B(\pi)) \) with \( \text{div} \varphi = 0 \). Thus, we can apply \([21, \text{Lemma 2.1.1}]\) and see that
\[
\int_{B(\pi)} |P - P_{B(\pi)}|^\frac{3}{2} \, dx \leq C\| F \|_{W^{-1, \frac{3}{2}}(B(\pi))}^\frac{3}{2}.
\]

With the following estimate
\[
\| F \|_{W^{-1, \frac{3}{2}}(B(\pi))}^\frac{3}{2} \leq \| \nabla u - u \otimes u + B \otimes B \|_{L^\frac{3}{2}(B(\pi))}^\frac{3}{2}
\]
\[
\leq CR^\frac{3}{2} \left( \int_{B(\pi)} |\nabla u|^2 \, dx \right)^\frac{3}{4} + C \int_{B(\pi)} (|u|^3 + |B|^3) \, dx,
\]
we can obtain
\[
I_3 \leq CR(R - \rho)^{-1} \left( \int_{B(R)} |\nabla u|^2 \, dx \right)^\frac{3}{4} + CI_2.
\]

Hence,
\[
I_3 \leq \epsilon \int_{B(R)} |\nabla u|^2 \, dx + CR^4(R - \rho)^{-4} + CI_2, \quad \epsilon > 0.
\]
Before estimating $I_2$, we notice that $\psi = \zeta_{\pi, R}$ satisfies $0 \leq \psi \leq 1$ and $|\nabla \psi| \leq 4/(R - \rho)$; thus, with (1.3), Lemma 1 is applicable. We use (2.2) and have

\[
(R - \rho)^{-1} \int_{B(R)} |\psi^3 u|^2 \, dx \leq CR(R - \rho)^{-1} \|\psi \nabla u\|^\frac{18}{11} \|u\|^\frac{11}{18} + CR^{4 - \frac{2}{3}} (R - \rho)^{-4} \\
+ CR(R - \rho)^{-1} ((R - \rho)^{-1} \|\psi^2 u\|_{L^2(B(R))})^\frac{10}{7}. 
\]

By $R > 1$, $(R - \rho)^{-1} > 1$, and Young’s inequality, it follows

\[
(R - \rho)^{-1} \int_{B(R)} |\psi^3 u|^3 \, dx \leq \epsilon \|\psi \nabla u\|^2_{L^2(B(R))} + C(\epsilon) R^\frac{4+6}{7} (R - \rho)^{-\frac{2+6}{7}} + I_1, \quad \epsilon > 0. 
\]

Repeating the above process for $B$ instead of $u$, we can have

\[
(R - \rho)^{-1} \int_{B(R)} |\psi^3 B|^3 \, dx \leq \epsilon \|\psi \nabla B\|^2_{L^2(B(R))} + C(\epsilon) R^\frac{4+6}{7} (R - \rho)^{-\frac{2+6}{7}} + I_1, \quad \epsilon > 0. 
\]

Therefore, we obtain that

\[
I_2 \leq \epsilon \int_{B(R)} (|\nabla u|^2 + |\nabla B|^2) \, dx + C(\epsilon) R^\frac{4+6}{7} (R - \rho)^{-\frac{2+6}{7}} + I_1, \quad \epsilon > 0. 
\]

We continue estimating $I_1$. Using (2.1) with $R > 1$, we clearly have

\[
(R - \rho)^{-2} \int_{B(R)} |\psi^2 u|^2 \, dx \leq CR^\frac{11}{18} (R - \rho)^{-\frac{2}{3}} \|\psi \nabla u\|_{L^2(B(R))} + CR^{4 - \frac{2}{3}} (R - \rho)^{-4} \\
\leq CR^2 (R - \rho)^{-2} \|\psi \nabla u\|_{L^2(B(R))} + CR^4 (R - \rho)^{-4}. 
\]

Then, by Young’s inequality,

\[
(R - \rho)^{-2} \int_{B(R)} |\psi^2 u|^2 \, dx \leq \epsilon \|\psi \nabla u\|^2_{L^2(B(R))} + C(\epsilon) R^4 (R - \rho)^{-4}, \quad \epsilon > 0. 
\]

Similarly, we also obtain

\[
(R - \rho)^{-2} \int_{B(R)} |\psi^2 B|^2 \, dx \leq \epsilon \|\psi \nabla B\|^2_{L^2(B(R))} + C(\epsilon) R^4 (R - \rho)^{-4}, \quad \epsilon > 0. 
\]

Thus,

\[
I_1 \leq \epsilon \int_{B(R)} (|\nabla u|^2 + |\nabla B|^2) \, dx + C(\epsilon) R^4 (R - \rho)^{-4}, \quad \epsilon > 0. 
\]

Collecting the estimates for $I_1$, $I_2$, and $I_3$, we have with $(R - \rho)^{-1} > 1$ that

\[
\int_{B(R)} (|\nabla u|^2 + |\nabla B|^2) \, dx \leq \epsilon \int_{B(R)} (|\nabla u|^2 + |\nabla B|^2) \, dx + C(\epsilon) R^\frac{4+6}{7} (R - \rho)^{-\frac{2+6}{7}}, \quad \epsilon > 0. 
\]

We fix $\epsilon < 1$. Then, thanks to the iteration Lemma in [10, V.Lemma 3.1], we can deduce

\[
\int_{B(R)} (|\nabla u|^2 + |\nabla B|^2) \, dx \leq CR^\frac{4+6}{7} (R - \rho)^{-\frac{2+6}{7}}. 
\]

Taking $R = 2\rho$ and passing $\rho \to \infty$, we have (3.1).
Secondly, we show that
\[
    r^{-1} \int_{B(2r) \setminus B(r)} (|u|^3 + |B|^3) \, dx \to 0 \quad \text{as} \quad r \to \infty. \tag{3.4}
\]

Let \( R > \rho > R/4 > 1 \) and \( \psi = \zeta_{\rho,R} - \zeta_{\rho/4,R/4} \). Since \( \psi \) satisfies the assumptions for Lemma 1, with the assumption (1.3) we use Lemma 1 and have
\[
\int_{B(R)} \psi^3 u^3 \, dx \leq CR \| \psi \nabla u \|_{L^2(B(R))}^{18} + CR^{1-\frac{2}{5}} (R - \rho)^{-3} \]
\[
+ CR ((R - \rho)^{-2} R^{11} \| \psi \nabla u \|_{L^2(B(R))} + R^{11-\frac{2}{5}} (R - \rho)^{-4}) \frac{2}{5}.
\]

Applying Young’s inequality, we have
\[
\int_{B(R)} |\psi^3 u|^3 \, dx \leq CR \| \psi \nabla u \|_{L^2(B(R))}^{18} + CR^{1-\frac{2}{5}} (R - \rho)^{-3} \]
\[
+ CR \left( \| \psi \nabla u \|_{L^2(B(R))}^2 + R^{11-\frac{2}{5}} (R - \rho)^{-4} \right)^{\frac{2}{5}}.
\]

And by taking \( \rho = 2r \) and \( R = 4r \) for \( r > 1 \), we can deduce that
\[
    r^{-1} \int_{B(2r) \setminus B(r)} |u|^3 \, dx \leq C \| \nabla u \|_{L^2(B(4r) \setminus B(r/2))}^{18} + Cr^{-\frac{2}{5}}.
\]

Note that we can show the above inequality for \( B \). Thus, we obtain (3.4) due to (3.1).

Thirdly, we show
\[
    r^{-1} \int_{B(r)} (|u|^3 + |B|^3) \, dx \leq C, \quad r > 1. \tag{3.5}
\]

By direct computation, we see that
\[
    r^{-1} \int_{B(r)} (|u|^3 + |B|^3) \, dx = \sum_{j=1}^{\infty} 2^{-j} (2^{-j} r)^{-1} \int_{B(2^{-j-1} r) \setminus B(2^{-j} r)} (|u|^3 + |B|^3) \, dx
\]
\[
\leq \sup_{1/2 \leq \rho \leq r/2} \rho^{-1} \int_{B(2\rho) \setminus B(\rho)} (|u|^3 + |B|^3) \, dx + \int_{B(1)} (|u|^3 + |B|^3) \, dx.
\]

Hence, (3.4) implies (3.5).

Now, we conclude \( u \equiv B \equiv 0 \). Let \( r > 1 \). Inserting \( \varphi = \zeta_{r,2r} \) into (3.2), we infer that
\[
\int_{B(r)} (|\nabla u|^2 + |\nabla B|^2) \, dx \leq Cr^{-2} \int_{B(2r) \setminus B(r)} (|u|^2 + |B|^2) \, dx
\]
\[
+ Cr^{-1} \int_{B(2r) \setminus B(r)} (|u|^3 + |B|^3) \, dx + Cr^{-1} \left( \int_{B(2r)} |P - P_{B(2r)}|^\frac{3}{2} \, dx \right)^{\frac{2}{3}} \left( \int_{B(2r) \setminus B(r)} |u|^3 \, dx \right)^{\frac{1}{3}}.
\]

Hölder’s inequality implies
\[
    r^{-2} \int_{B(2r) \setminus B(r)} (|u|^2 + |B|^2) \, dx \leq C r^{-\frac{2}{5}} \left( r^{-1} \int_{B(2r) \setminus B(r)} (|u|^3 + |B|^3) \, dx \right)^{\frac{2}{5}}.
\]
By (3.5), we have
\[ r^{-2} \int_{B(2r) \setminus B(r)} (|u|^2 + |B|^2) \, dx \to 0 \quad \text{as} \quad r \to \infty. \]

Also recall that (3.4)
\[ r^{-1} \int_{B(2r) \setminus B(r)} (|u|^3 + |B|^3) \, dx \to 0 \quad \text{as} \quad r \to \infty. \]

As in the way we estimated \( I_3 \), we can show that
\[ \int_{B(2r)} |P - P_{B(2r)}|^3 \, dx \leq C r^{\frac{3}{2}} \left( \int_{B(2r)} |
abla u|^2 \, dx \right)^{\frac{3}{4}} + C \int_{B(2r)} (|u|^3 + |B|^3) \, dx. \]

From (3.1) and (3.5), we see that
\[ r^{-1} \int_{B(2r)} |P - P_{B(2r)}|^\frac{3}{2} \, dx \leq C, \quad r > 1. \]

Hence, it follows
\[ r^{-1} \left( \int_{B(2r)} |P - P_{B(2r)}|^\frac{3}{2} \, dx \right)^{\frac{2}{3}} \left( \int_{B(2r) \setminus B(r)} |u|^3 \, dx \right)^{\frac{1}{3}} \leq C \left( r^{-1} \int_{B(2r) \setminus B(r)} |u|^3 \, dx \right)^{\frac{1}{3}}. \]

And according to (3.4), we have
\[ r^{-1} \left( \int_{B(2r)} |P - P_{B(2r)}|^\frac{3}{2} \, dx \right)^{\frac{2}{3}} \left( \int_{B(2r) \setminus B(r)} |u|^3 \, dx \right)^{\frac{1}{3}} \to 0 \quad \text{as} \quad r \to \infty. \]

This shows that
\[ \int_{B(r)} (|\nabla u|^2 + |\nabla B|^2) \, dx \to 0 \quad \text{as} \quad r \to \infty, \]

which implies that \( u \) and \( B \) must be constants. Thanks to (3.4), we finally obtain \( u \equiv B \equiv 0 \).

4. Proof of Theorem 2

In this section, we use the notation
\[ G(r) := \int_{B(r)} (|\nabla u|^2 + |\nabla B|^2) \, dx \]

and
\[ \Theta(r) := \frac{1}{r^{\left(\frac{4}{p+1} + 1\right)\left(\frac{1}{3} - \frac{1}{p}\right)}} \left( \frac{1}{|B(r)|} \int_{B(r)} |B - B_{B(r)}|^p \, dx \right)^{\frac{1}{p}} \]

for \( r > 1 \). Notice that by means of the condition (1.4), it holds
\[ \lim_{r \to \infty} \Theta(r) = 0. \]  
(4.1)
We first show (3.1). For a cutoff function $\varphi \in C_c^\infty(\mathbb{R}^3)$, as we obtained (3.2) we have from (1.2) that
\[
\int_{\mathbb{R}^3} (|\nabla u|^2 + |\nabla B|^2) \varphi \, dx = \frac{1}{2} \int_{\mathbb{R}^3} (|u|^2 + |B|^2) \Delta \varphi \, dx + \frac{1}{2} \int_{\mathbb{R}^3} (|u|^2 + |B|^2) u \cdot \nabla \varphi \, dx
\]
\[\quad - \int_{\mathbb{R}^3} (u \cdot B)(B \cdot \nabla) \varphi \, dx - \int_{\mathbb{R}^3} ((\nabla \times B) \times B) \cdot B \times \nabla \varphi \, dx + \int_{\mathbb{R}^3} (P - P_{B(\tau)}) u \cdot \nabla \varphi \, dx
\]
for any $\tau > 0$. Let $R > \rho > 1$ and $\overline{\mathcal{R}} = (R + \rho)/2$ and insert $\varphi = \zeta_{\rho, \overline{\mathcal{R}}}$ into (4.2). Then, we can infer that
\[
G(\rho) \leq C(R - \rho)^{-2} \int_{B(\overline{\mathcal{R}}) \setminus B(\rho)} (|u|^2 + |B|^2) \, dx + C(R - \rho)^{-1} \int_{B(\overline{\mathcal{R}}) \setminus B(\rho)} (|u|^3 + |B|^3) \, dx
\]
\[\quad + C(R - \rho)^{-1} \int_{B(\overline{\mathcal{R}})} |P - P_{B(\overline{\mathcal{R}})}|^{\frac{3}{2}} \, dx + C(R - \rho)^{-1} \int_{B(\overline{\mathcal{R}})} |\nabla B||B|^2 \, dx.
\]
We define
\[
I_4 := (R - \rho)^{-1} \int_{B(\overline{\mathcal{R}})} |\nabla B||B|^2 \, dx.
\]
With the notations (4.3), we have that
\[
\int_{B(\rho)} (|\nabla u|^2 + |\nabla B|^2) \, dx \leq C(I_1 + I_2 + I_3 + I_4).
\]
In the proof of Theorem 1, we already showed
\[
I_1 + I_2 + I_3 \leq \epsilon G(R) + C(\epsilon) R^{\frac{d+6}{2}} (R - \rho)^{-\frac{d+6}{2}}, \quad \epsilon > 0.
\]
We estimate $I_4$ by considering the cases $3 < p \leq 4$ and $p > 4$ separately.

(i) $3 < p \leq 4$ case:
Using Hölder’s inequality, we have
\[
I_4 \leq (R - \rho)^{-1} \|B\|_{L^p(B(R))} \|B\|_{L^{\frac{3(4-p)}{4}}(B(R))} G(R)^{\frac{1}{2}}.
\]
Let $\varphi = \zeta_{R/2, R}$. Then, since it is satisfied regardless of $R > 1$ that $0 \leq \varphi \leq 1$, $\|\varphi\|_{L^1} \geq |B(R)|/8$, $|\nabla \varphi| \leq 4/R$, we can see by (2.3) and (2.4) that
\[
\int_{B(R)} |B|^p \, dx \leq C \int_{B(R)} |B - B_\varphi|^p \, dx + C|B_\varphi|^p |B(R)|
\]
\[\quad \leq C \int_{B(R)} |B - B_{B(R)}|^p \, dx + CR^{-\frac{2p}{3} - \frac{p}{3} + 3}.
\]
Thus, it follows
\[
\int_{B(R)} |B|^p \, dx \leq CR^{\frac{4p}{3} + \frac{1}{3}} (R - 1)^{\frac{1}{3}} \Theta(R)^p + CR^{-\frac{2p}{3} - \frac{p}{3} + 3}.
\]
Continuously, using $R^{\frac{4p}{3} + \frac{1}{3}} (R - 1)^{\frac{1}{3}} \Theta(R)^p + CR^{-\frac{2p}{3} - \frac{p}{3} + 3} \leq R^{6-p}$, we obtain
\[
\int_{B(R)} |B|^p \, dx \leq CR^{6-p} (R^{\frac{4p}{3} + \frac{1}{3}} (R - 1)^{\frac{1}{3}} \Theta(R)^p + 1).
\]
As above, we can also have
\[ \int_{B(R)} |B|^6 \, dx \leq C \int_{B(R)} |B - B_{B(R)}|^6 \, dx + CR^{-1 - \frac{6}{p}}. \]

By Poincaré–Sobolev inequality with \( R^{-1 - \frac{6}{p}} \leq 1 \),
\[ \int_{B(R)} |B|^6 \, dx \leq C \left( \int_{B(R)} |\nabla B|^2 \, dx \right)^3 + C. \tag{4.5} \]

Combining the above estimates, we can infer
\[ I_4 \leq CR(R - \rho)^{-1} \left( R^{2\beta + 4} (\frac{\nu}{\rho - 1}) \Theta(R)^p + 1 \right) \frac{1}{p - \beta} \left( G(R) + 1 \right)^{\frac{3(4 - p)}{2(6 - p)} + \frac{3}{2}}. \]

Noting that
\[ \frac{3(4 - p)}{2(6 - p)} + \frac{1}{2} < 1, \quad p > 3, \]
with the Young’s inequality and \( R(R - \rho)^{-1} > 1 \), we deduce that
\[ I_4 \leq \epsilon G(R) + C(\epsilon) R^{\frac{6 - p}{p - 3}} (R - \rho)^{-\frac{6 - p}{p - 3}} \left( R^{2\beta + 4} (\frac{\nu}{\rho - 1}) \Theta(R)^p + 1 \right) \frac{1}{p - \beta}, \quad \epsilon > 0. \]

(ii) \( p \geq 4 \) case:
Using Hölder’s inequality with \( \psi = \zeta_{\mathcal{R}, R} \), we have
\[ I_4 \leq (R - \rho)^{-1} \| B \|_{L^p(B(R))} \| \psi^3 B \|_{L^\beta(B(R))} G(R)^{\frac{3}{2}}. \]

Due to \( p \geq 4 \), we can verify \( R^{2\beta + 1} (\frac{\nu}{\rho - 1}) + 3 \leq R^{p - 2} R^{2\beta + 4} (\frac{\nu}{\rho - 1}) \) and \( R^{2\beta - \frac{\nu}{\rho - 1} + 3} \leq R^{p - 2} \). Applying it to (4.4) yields
\[ \int_{B(R)} |B|^p \, dx \leq CR^{p - 2} \left( R^{2\beta + 4} (\frac{\nu}{\rho - 1}) \Theta(R)^p + 1 \right). \]

On the other hand, recalling the definition of \( I_2 \), we have
\[ \int_{B(R)} |\psi^3 B|^3 \, dx \leq (R - \rho) I_2. \]

Hence, with the Young’s inequality we can infer that
\[ I_4 \leq CR^{\frac{p - 2}{p - 3}} (R - \rho)^{-\frac{2p - 2}{p - 3}} \left( R^{2\beta + 4} (\frac{\nu}{\rho - 1}) \Theta(R)^p + 1 \right) \frac{1}{p - \beta} G(R)^{\frac{3}{2}} I_2^{\frac{p - 1}{p - 3}} \]
\[ \leq CR(R - \rho)^{-1} \left( R^{2\beta + 4} (\frac{\nu}{\rho - 1}) \Theta(R)^p + 1 \right) \frac{1}{p - \beta} G(R)^{\frac{p - 3}{p - 2}} + I_2. \]

Noting that \( (p - 3)/(p - 2) < 1 \), we use Young’s inequality again to have
\[ I_4 \leq \epsilon G(R) + C(\epsilon) R^{p - 2} (R - \rho)^{-p - 2} \left( R^{2\beta + 4} (\frac{\nu}{\rho - 1}) \Theta(R)^p + 1 \right) + I_2, \quad \epsilon > 0. \]

Therefore, for any case we have
\[ I_4 \leq \epsilon G(R) + C(\epsilon) R^\beta (R - \rho)^{-\beta} \left( R^{2\beta + 4} (\frac{\nu}{\rho - 1}) \Theta(R)^p + 1 \right)^\gamma + I_2, \quad \epsilon > 0 \]
where
\[ \beta := \max \left\{ \frac{6 - p}{p - 3}, p - 2 \right\}, \quad \gamma := \max \left\{ \frac{1}{p - 3}, 1 \right\}. \]
Taking $\epsilon$ sufficiently small and redefining $\beta$ to satisfy $\beta \geq (s + 6)/(s - 3)$ also, we deduce

$$G(\rho) \leq \frac{1}{2} G(R) + C R^\beta (R - \rho)^{-\beta} (R^{\frac{2s+4}{s-4}} (\xi-1) \Theta(R)^p + 1)^\gamma.$$ 

We set $r > 1$ and $r \leq \rho < R \leq 2r$. Since we clearly have

$$R^{\frac{2s+4}{s-4}} (\xi-1) \Theta(R)^p + 1 \leq C \left( r^{\frac{2s+4}{s-4}} (\xi-1) \sup_{\tau \in [r, 2r]} \Theta(\tau)^p + 1 \right)^\gamma,$$

employing the iteration Lemma in [10, Lemma 3.1], we can see

$$G(\rho) \leq C R^\beta (R - \rho)^{-\beta} \left( r^{\frac{2s+4}{s-4}} (\xi-1) \sup_{\tau \in [r, 2r]} \Theta(\tau)^p + 1 \right)^\gamma.$$ 

(4.6)

And by letting $\rho = r$ and $R = 2r$, (4.1) gives that

$$G(r) \leq C \left( r^{\frac{2s+4}{s-4}} (\xi-1) \sup_{\tau \in [r, 2r]} \Theta(\tau)^p + 1 \right)^\gamma \leq C r^\gamma (\frac{2s+4}{s-4})$$

(4.7)

for sufficiently large $r > 1$. To finish showing (3.1), we estimate $I_4$ in another way. Using Hölder’s inequalities with $\psi = \zeta_{R, R}$, we can have

$$I_4 \leq (R - \rho)^{-1} \|B\|_{L^p(B(R))} \|B\|_{L^q(B(R))} \|\psi^3 B\|_{L^3(B(R))} G(R)^\frac{1}{2}.$$ 

Applying $R^{\frac{2s+4}{s-4}} (\frac{1}{2} - \frac{1}{s}) > 1$ to (4.4) provides us with

$$\|B\|_{L^p(B(R))} \leq CR^{\frac{2s+4}{s-4}} (\frac{1}{2} - \frac{1}{s}) \Theta(R) + CR^{-\frac{s}{2} - \frac{1}{2} + \frac{1}{p}} \leq CR^{\frac{2s+4}{s-4}} (\frac{1}{2} - \frac{1}{s}) \left( \Theta(R) + R^{-\frac{s}{2} - \frac{1}{2}} \right).$$

From (2.1) and (2.2), we see that

$$\|\psi^3 B\|_{L^3(B(R))} \leq CR^\frac{1}{2} G(R)^{\frac{2s}{2s-4}} + CR^\frac{1}{2} (\frac{11}{12} - \frac{1}{s}) \frac{n}{2s} (R - \rho)^{-\frac{2s}{2s-4}} G(R)^{\frac{2}{2s-4}} + CR^\frac{1}{2} (\frac{11}{12} - \frac{1}{s}) \frac{n}{2s} (R - \rho)^{-\frac{2s}{2s-4}}.$$

Using Young’s inequality along with $R(R - \rho) > 1$ and $R > 1$, we deduce

$$\|\psi^3 B\|_{L^3(B(R))} \leq CR^\frac{1}{2} G(R)^{\frac{2s}{2s-4}} + CR^\frac{1}{2} (\frac{11}{12} - \frac{1}{s}) \frac{n}{2s} (R - \rho)^{-\frac{2s}{2s-4}}.$$

Thus, with (4.5) we have

$$I_4 \leq C (R - \rho)^{-1} \left( R^{\frac{2s+4}{s-4}} (\frac{1}{2} - \frac{1}{s}) \left( \Theta(R) + R^{-\frac{s}{2} - \frac{1}{2}} \right) \right)^{\frac{2p}{2p-(3p+1)\beta}} G(R)^\frac{1}{2} + 1 \left( R^{\frac{2s+4}{s-4}} (\frac{1}{2} - \frac{1}{s}) \right)^{\frac{2p}{2p-(3p+1)\beta}} G(R)^\frac{1}{2}.$$ 

We also set $r \leq \rho < R \leq 2r$ for $r > 1$. By $R(R - \rho) > 1$ again, we can rewrite it as

$$I_4 \leq CR (R - \rho)^{-1} \left( \Theta(R) + R^{-\frac{s}{2} - \frac{1}{2}} \right) \frac{n}{2s} (R - \rho)^{-\frac{2s}{2s-4}} \right) \left( \Theta(R) + R^{-\frac{s}{2} - \frac{1}{2}} \right) \frac{n}{2s} (R - \rho)^{-\frac{2s}{2s-4}} \right) \leq CR (R - \rho)^{-1} \left( \Theta(R) + r^{-\frac{s}{2} - \frac{1}{2}} \right) \frac{n}{2s} (R - \rho)^{-\frac{2s}{2s-4}} \right) \leq CR \frac{r^{\frac{2}{s}}} (R - \rho)^{-\frac{2s}{2s-4}} \right).$$ 

We used in the last inequality that

$$\sup_{\tau \in [r, 2r]} \Theta(\tau) + r^{-\frac{s}{2} - \frac{1}{2}} \leq 1$$
which is true for sufficiently large $r > 1$. Recalling (4.3) and taking $\epsilon > 0$ sufficiently small, we arrive at

$$G(\rho) \leq \frac{1}{2}G(R) + CR^{\frac{\gamma+\sigma}{\gamma}}(R - \rho)^{-\frac{\sigma}{\gamma}} + CR(R - \rho)^{-1}\left(\sup_{\tau \in [r, 2r]} \Theta(\tau) + r^{-\frac{2}{\gamma} - \frac{1}{2}}\right)^{\frac{3p}{3p - 3\gamma + \gamma}} G(2r).$$

According to the iteration Lemma in [10, Lemma 3.1], it follows that

$$G(\rho) \leq CR^{\frac{\gamma+\sigma}{\gamma}}(R - \rho)^{-\frac{\sigma}{\gamma}} + CR(R - \rho)^{-1}\left(\sup_{\tau \in [r, 2r]} \Theta(\tau) + r^{-\frac{2}{\gamma} - \frac{1}{2}}\right)^{\frac{3p}{3p - 3\gamma + \gamma}} G(2r).$$

In particular, for $\rho = r$ and $R = 2r$,

$$G(r) \leq \kappa\left(\sup_{\tau \in [r, 2r]} \Theta(\tau) + r^{-\frac{2}{\gamma} - \frac{1}{2}}\right)^{\frac{3p}{3p - 3\gamma + \gamma}} G(2r) + \kappa$$

for some $\kappa > 0$. We consider $r > 1$ sufficiently large such that

$$\kappa\left(\sup_{\tau \in [r, 2r]} \Theta(\tau) + r^{-\frac{2}{\gamma} - \frac{1}{2}}\right)^{\frac{3p}{3p - 3\gamma + \gamma}} \leq 2^{-2\gamma(\frac{2}{\gamma} + 4)(\frac{\gamma}{2} - 1)}.$$

Then, iterating (4.8) $n$-times yields

$$G(r) \leq 2^{-2n\gamma(\frac{2}{\gamma} + 4)(\frac{\gamma}{2} - 1)}G(2^n r) + \kappa \sum_{j=0}^{n-1} 2^{-2j\gamma(\frac{2}{\gamma} + 4)(\frac{\gamma}{2} - 1)}.$$

Thus, (4.7) implies that

$$G(r) \leq 2^{-n\gamma(\frac{2}{\gamma} + 4)(\frac{\gamma}{2} - 1)} \gamma(\frac{2}{\gamma} + 4)(\frac{\gamma}{2} - 1) + C.$$

After letting $n \to \infty$, we pass $r \to \infty$, which shows (3.1).

Following the way we obtained (3.4) in the proof of Theorem 1, we clearly have from (3.1) the same conclusion, and (3.5) also follows.

Now, we are ready to finish the proof. Let $r > 1$ and $\varphi = \zeta_{r, 2r}$. From (4.2), we have

$$\int_{B(r)} (|\nabla u|^2 + |\nabla B|^2) \, dx \leq Cr^{-2} \int_{B(2r) \setminus B(r)} (|u|^2 + |B|^2) \, dx$$

$$+ Cr^{-1} \int_{B(2r) \setminus B(r)} (|u|^3 + |B|^3) \, dx + Cr^{-1} \left( \int_{B(2r)} |P - P_{B(2r)}|^\frac{3}{2} \, dx \right)^\frac{2}{3} \left( \int_{B(2r) \setminus B(r)} |u|^3 \, dx \right)^\frac{1}{3}$$

$$+ Cr^{-1} \int_{B(2r) \setminus B(r)} |B|^2 |\nabla B| \, dx.$$

To obtain

$$\int_{B(r)} (|\nabla u|^2 + |\nabla B|^2) \, dx \to 0 \quad \text{as} \quad r \to \infty,$$

it suffices to show that

$$r^{-1} \int_{B(2r) \setminus B(r)} |B|^2 |\nabla B| \, dx \to 0 \quad r \to \infty.$$
Using Hölder’s inequality yields
\[
\int_{B(2r) \setminus B(r)} |B|^2 |\nabla B| \, dx \leq C r^{-\frac{1}{2}} \left( \int_{B(2r)} |B|^6 \, dx \right)^{\frac{1}{2}} \left( \int_{B(2r) \setminus B(r)} |\nabla B|^2 \, dx \right)^{\frac{1}{2}}.
\]
By (4.5), we have
\[
\int_{B(2r) \setminus B(r)} |B|^2 |\nabla B| \, dx \leq C r^{-\frac{1}{2}} \left\{ \left( \int_{B(2r)} |\nabla B|^2 \, dx \right)^{\frac{3}{2}} + 1 \right\}^{\frac{1}{3}} \left( \int_{B(2r) \setminus B(r)} |\nabla B|^2 \, dx \right)^{\frac{1}{2}}.
\]
From (3.1), we obtain what we desired. This completes the proof.

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