Relative equilibria of four identical satellites

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We consider the Newtonian 5-body problem in the plane, where four bodies have the same mass $m$, which is small compared with the mass $M$ of the remaining body. We consider the (normalized) relative equilibria in this system and follow them to the limit when $m/M \to 0$. In some cases, two small bodies will coalesce at the limit. We call the other equilibria the relative equilibria of four separate identical satellites. We prove rigorously that there are only three such equilibria, all already known after the numerical researches by H. Salo and C. F. Yoder. Our main contribution is to prove that any equilibrium configuration possesses a symmetry, a statement indicated by J. Llibre as the missing key to proving that there is no other equilibrium.

Keywords: central configurations; symmetrical configurations; co-orbital satellites

1. Introduction

The recent work by Renner and Sicardy (Sicardy & Renner 2003; Renner 2004; Renner & Sicardy 2004) again drew the attention of astronomers to relative equilibria in the $N$-body problem, especially those with a big mass and several small masses. In such a relative equilibrium, the small bodies all describe almost the same circular orbit around the central body. Some of these relative equilibria are stable, and thus, may approximate the motion of small bodies in the solar system. Actually, according to a conjecture by Moeckel (1994), the existence of a dominant mass could be a necessary condition for the stability of a relative equilibrium in the $N$-body problem.

Let us recall briefly the previous work. Lagrange (1772) discovered the famous relative equilibrium with three bodies forming an equilateral triangle. Gascheau (1843) gave the condition for stability, which implies that the mass of one of the bodies should be at least 25 times the total mass of the other bodies. The Trojan asteroids were discovered in 1906, and, more recently, many other configurations in an equilateral triangle have been found (e.g. Renner & Sicardy 2004; Robutel & Souchay 2009). While studying Saturn’s rings, Maxwell (1859) proposed a configuration with many small bodies with equal masses orbiting on the same circle and forming a regular polygon, and proved its stability. He also examined the more realistic case where the small bodies have different masses.

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and the polygon is not regular, and claimed again the stability. This sort of thin discrete ring has not been observed yet. Maxwell’s stability claims were reconsidered and confirmed for the regular polygon with seven vertices or more (Moeckel 1994; Roberts 2000).

Several models were proposed (see Salo & Hänninen 1998; Namouni & Porco 2002) to explain the four or five co-orbital arcs of a ring around Neptune. Renner and Sicardy (see also the next to last paragraph in Namouni & Porco 2002) suggest that three or four little unknown satellites would orbit in a relative equilibrium configuration, and control the ends of the arcs. The equilibrium of the system composed of the arcs and the satellites would be stable enough and compatible with the close inner satellite Galatea (Renner 2004).

When speaking of stability above, we meant the linear stability of the relative equilibrium. A relative equilibrium is a motion where the distances between bodies remain the same. The relative motion of any body around any other is circular. All the bodies orbit in the same plane, but the linear stability includes small displacements of the positions and the velocities out of this plane.

When a configuration may have a relative equilibrium motion, it may also have a so-called homographic motion. Here the configuration changes size, keeping the same shape, each body being in the same plane and describing a Keplerian orbit around each other. So, one should discuss not only the stability of the circular motion, i.e. the relative equilibrium, but also the stability of the elliptic motion (see Danby 1964; Roberts 2002; Martínez et al. 2004; Meyer & Schmidt 2005; Hu & Sun submitted). It is good to give a name to the configurations allowed in relative equilibria and homographic motions. They are called the central configurations of the \( N \)-body problem (e.g. Wintner 1941; Saari 2005; Hampton & Moeckel 2006). The limiting configurations where the big mass is fixed, while the \( n = N - 1 \) small masses tend to zero together, their mutual ratios being fixed, are called the central configurations of the \( 1 + n \)-body problem.

Our work as well as several previous studies is dedicated to the non-coalescent planar central configurations of the \( 1 + n \)-body problem with small equal masses. All the words are required. Non-coalescent excludes the case where two small bodies coincide at the limit (see Xia 1991; Moeckel 1997). Planar should be indicated, because central configurations may also be non-planar. A special type of homographic motion, the homothetic motion, does not require that the configuration is in a plane. In a homothetic motion, the configuration is central and does not rotate.

We will avoid such a heavy terminology considering it as equivalent to relative equilibria of \( n \) separate identical satellites. A relative equilibrium is not only a configuration. Velocities should be attributed to the bodies in order to obtain a uniform rotation. But the choice of the velocities being mainly unique, the distinction between relative equilibrium and central configuration does not matter here.

Hall (preprint) gave a precise proposition proving in particular that in a relative equilibrium of \( n \) satellites, the big body is at the centre of a circle which passes through the satellites. In other words, the satellites are co-orbital. A couple of years after Hall’s unpublished work, Salo & Yoder (1988) studied independently the relative equilibria of \( n \) separate identical satellites and discovered that they form strange sequences. When \( n = 2, 3, \ldots, 10, \ldots \), respectively, they found
2, 3, 3, 3, 3, 3, 1, 1, \ldots \text{ distinct such equilibria. Very probably, from } n = 9, \text{ there is only one equilibrium, which is the regular polygon. Numerical experiments by Simó confirming this fact for } n = 9, \ldots , 100 \text{ are mentioned in (Cors et al. 2004a, p. 326). All the equilibria in Salo & Yoder (1988) list have exactly one axis of symmetry (passing through the central body), except the regular polygons, which have more symmetry. The number of linearly stable relative equilibria in Salo & Yoder (1988) list is, again for } n = 2, 3, \ldots \text{, respectively, } 1, 1, 1, 1, 2, 2, 1, 1, \ldots .

The numerical experiments suggest several mathematical statements. Very few of them are proved. Hall proved rigorously that there are three relative equilibria of three separate identical satellites. He then claimed: ‘For \( n \geq 4 \) the equations become considerably more difficult to handle. We will give some (little) numerical description for \( n \geq 3 \) in section 7’.

Improving another result by Hall, Casasayas et al. (1994) proved that the regular polygon is the unique equilibrium if \( n > e^{73} \).

Cors et al. (2004a) proved that there are only three symmetric equilibria of four separate identical satellites. We prove here that any such equilibrium must be symmetric, a statement indicated in Cors et al. (2004b) as the missing key to completing the proof of the following theorem. The reader should refer to Albouy et al. (2008) for other problems of central configurations where the equality of some of the masses implies a symmetry of the configuration.

**Theorem 1.1.** There are exactly three relative equilibria of four separate identical satellites. In the corresponding configurations \( \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3 \), the small bodies are placed on a circle with the big body in the centre, in such a way that the four angular distances between consecutive bodies are, respectively, \((\pi/3, 2\pi/3, 2\pi/3, \pi/3)\), \((\theta_1, \theta_2, \theta_1, \theta_4)\), \((\pi/2, \pi/2, \pi/2, \pi/2)\), where \( \theta_1, \theta_2, \theta_4 \) satisfy \( 2\theta_1 + \theta_2 + \theta_4 = 2\pi \), \( 0 < \theta_2 < \theta_1 < \pi < \theta_4 \).

Let us discuss briefly the most natural extensions of the theorem, first removing the condition ‘separate’. According to Martínez & Simó (2003), there are exactly eight other relative equilibria of four identical satellites, counting configurations of indistinguishable satellites up to isometry. Three of them are two-dimensional and asymmetric, two are two-dimensional and have one axis of symmetry, and three are one-dimensional (Moulton configurations).

Renner & Sicardy (2004) removed the condition ‘identical’ and obtained many results, including mathematical results about the inverse problem: given a configuration of the satellites, find the small masses making it a relative equilibrium.

Finally, the non-coalescent three-dimensional central configurations of the \( 1 + 4 \)-body problem with equal small masses were also studied. Albouy & Llibre (2002) proved their symmetry, but left open the possibility of (infinitely) many configurations with exactly one plane of symmetry. Numerical studies show there is only one such central configuration. The configurations discussed in Albouy & Llibre (2002) being non-planar, they are not allowed in a motion of relative equilibrium. However, together with the planar central configurations classified by the theorem above, they would complete the classification of homothetic motions with a big body and four bodies with equal small enough masses (see Roberts 2000; Almeida Santos 2004 for examples of explicit versions of the ‘small enough’ condition).
2. Study of a function

The equations characterizing the relative equilibria of \( n \) satellites are presented in Hall (preprint), Salo & Yoder (1988) and Moeckel (1994). The masses of the satellites are \( m_i = \epsilon \mu_i \). One passes to the limit as \( \epsilon \to 0 \), keeping only the highest order terms. These authors first deduce that in such an equilibrium, the \( n \) bodies with zero mass (the satellites) lie on a circle centred at the body with non-zero mass. So one considers only the normalized configurations with the same property when formulating the equations. One assumes the satellites are separate, and denotes by \( \phi_1, \ldots, \phi_n \) their distinct positions on the oriented circle. The system has \( n \) equations, the \( i \)-th being

\[
0 = \sum_{j \neq i} \mu_j f(\phi_i - \phi_j),
\]

where

\[
f(\theta) = \sin \theta (1 - (2 - 2 \cos \theta)^{-3/2}). \tag{2.1}
\]

We pass to the case of four identical satellites. We use as main variables the four positive angular distances between two consecutive small bodies. These four angles \( \theta_1, \theta_2, \theta_3, \theta_4 \) satisfy \( \theta_1 + \theta_2 + \theta_3 + \theta_4 = 2\pi \). We set

\[
f_1 = f(\theta_1), \ldots, f_4 = f(\theta_4),
\]

\[
f_{12} = f(\theta_1 + \theta_2), f_{23} = f(\theta_2 + \theta_3), f_{34} = f(\theta_3 + \theta_4), f_{14} = f(\theta_1 + \theta_4).
\]

Obviously \( f_{14} = -f_{23}, f_{12} = -f_{34} \). The equations are now

\[
f_{34} = f_2 - f_3 = f_1 - f_4, \quad f_{23} = f_1 - f_2 = f_4 - f_3. \tag{2.2}
\]

In this section, we state some properties of \( f \) which, together with some simple inequalities involving \( f(\pi/6), f(\pi/2) \) and \( f'(\pi/2) \), are necessary and sufficient for us to prove the symmetry of the central configurations. As \( f^{(n)}(2\pi - \theta) = (-1)^{n+1}f^{(n)}(\theta) \), it is enough to focus on \( \theta \in ]0, \pi[ \).

Property 2.1. We have \( f(\pi/3) = f(\pi) = 0 \). If \( \theta \in ]0, \pi/3[ \) then \( f(\theta) < 0 \). If \( \theta \in ]\pi/3, \pi[ \) then \( f(\theta) > 0 \).

Property 2.2. In \( ]0, \pi[ \) there is a unique \( \theta_c \) such that \( f'(\theta_c) = 0 \), satisfying \( \theta_c > \pi/3 \). If \( \theta \in ]0, \theta_c[ \), \( f'(\theta) > 0 \), if \( \theta \in ]\theta_c, \pi[ \), \( f'(\theta) < 0 \).

Property 2.3. We have \( f''(\pi) = 0 \), and \( f''(\theta) < 0 \) in \( ]0, \pi[ \).

Property 2.4 (Hall preprint). We have \( f'''(\theta) > 0 \) in \( ]0, \pi[ \).

Property 2.5. Consider \( g : f(0^+), f(\theta_c) \to ]0, \theta_c[ \) such that \( g(f(\theta)) = \theta \), which is well defined according to property 2.2. We have \( g''' > 0 \).

Proof. The identity \( 2 \sin^2(\theta/2) = 1 - \cos \theta \) suggests the introduction of some intermediate notation in the expression of \( f \). We write

\[
f(\theta) = \sin \theta (1 - (2s)^{-3}), \quad \text{with} \quad s = \sin \frac{\theta}{2}, \quad \theta \in ]0, 2\pi[. \tag{2.3}
\]
We may notice that if $\theta \in ]2\pi, 4\pi[$ formulae (2.1) and (2.3) disagree, because $s < 0$. But we are only interested in the interval $]0, 2\pi[$. We find

$$f'(\theta) = \frac{1}{4s^3} - \frac{1}{8s} + 1 - 2s^2, \quad f''(\theta) = \sin \theta \left( -\frac{3}{16s^5} + \frac{1}{32s^3} - 1 \right),$$

$$f'''(\theta) = \frac{3}{4s^5} - \frac{5}{8s^3} + \frac{1}{32s} - 1 + 2s^2.$$

The rational substitution

$$s = \frac{\alpha_0 + \alpha_1 y}{\beta_0 + \beta_1 y}$$

mapping $y \in ]0, \infty[ \mapsto s \in ]\alpha_0/\beta_0, \alpha_1/\beta_1[$ is often efficient in proving rigorously that a rational function in $s$ is sign definite in an interval. Taking $\alpha_0 = 0, \alpha_1 = 1, \beta_0 = 1, \beta_1 = 1$, this substitution reads $s = y/(1 + y)$ and gives

$$32(1 + y)^2 y^5 f''' = 37y^5 + 7y^6 + 275y^5 + 641y^4 + 740y^3 + 484y^2 + 168y + 24, \quad \text{so} \quad f'''$$

is positive when $y \in ]0, +\infty[$, i.e. $s \in ]0, 1[$, i.e. $\theta \in ]0, 2\pi[$. This proves property 2.4. We deduce immediately properties 2.1–2.3, after checking $f''(\pi) = f(\pi) = f(\pi/3) = 0$.

We consider now property 2.5. We differentiate twice the relation $g(f(\theta)) = \theta$, substitute $g'(f(\theta))$, differentiate again and find, after substituting $g''(f(\theta)) = -f'(f(\theta))f'''(\theta) + 3f'^2(\theta)$. We compute this expression, substitute $s = y/(1 + y)$ and get

$$256y^8(1 + y)^4 f''(\theta)g''(f(\theta)) = 259y^{12} + 7412y^{11} + 16934y^{10} + 32960y^9 + 42564y^8 + 39236y^7 + 35306y^6 + 32904y^5 + 24389y^4 + 12208y^3 + 3880y^2 + 720y + 60.$$

This shows that $g''(f(\theta))$ is positive if $\theta \in ]0, \theta_1[$.

**Lemma 2.6.** If a function $\phi$ satisfies $\phi'' > 0$ on the interval $]a, b[$, then for any $t_1, t_2, t_3, t_4, a \leq t_1 < t_2 < t_3 < t_4 \leq b$, we have

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ t_1 & t_2 & t_3 & t_4 \\ \phi(t_1) & \phi(t_2) & \phi(t_3) & \phi(t_4) \end{vmatrix} > 0.$$

**Remark 2.7.** A more general statement is stated and proved as Problem 5-99 in Pólya & Szegö (1976).

**Lemma 2.8.** Consider two horizontal chords drawn on the graph of $f$, whose projections on the horizontal axis are the two segments $[t^L_1, t^R_1], [t^L_2, t^R_2]$, with $t^L_1 < t^L_2 < \theta_c < t^R_2 < t^R_1$. Then $t^L_2 + t^R_1 < t^L_1 + t^R_2$, i.e. the segment from the middle point of one chord to the middle point of the other chord has a negative slope.

**Proof.** By property 2.4, $f$ satisfies the hypothesis of lemma 2.6, so, setting $f_1 = f(t^L_1) = f(t^R_1), f_2 = f(t^L_2) = f(t^R_2)$, we get

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ t^L_1 & t^L_2 & t^R_1 & t^R_2 \\ \phi(t^L_1) & \phi(t^L_2) & \phi(t^R_1) & \phi(t^R_2) \end{vmatrix} > 0.$$

The determinant factorizes as $(f_2 - f_1)(t^R_1 - t^L_1)(t^R_2 - t^L_2)(t^L_1 + t^R_2 - t^L_2 - t^R_1).$
Corollary 2.9. For any horizontal chord as in lemma 2.8, projecting on the segment \([t^L, t^R]\), \(t^L < \theta_c\), we have \(2\theta_c < t^L + t^R\).

Proof. This is lemma 2.8 when the highest chord tends to the horizontal tangent at \((\theta_c, f(\theta_c))\).

Lemma 2.10. Consider two chords drawn on the first increasing branch of the graph of \(f\), the arc delimited by the ‘inner’ chord being included in the arc delimited by the ‘outer’ one. If the segment from the middle of one chord to the middle of the other chord is horizontal, then the slope of the outer chord is less than the slope of the inner chord.

Proof. By property 2.5, we can apply lemma 2.6 to the reciprocal function \(g\) of \(f\). We take some notation. The outer chord projects horizontally on the segment \([t_1, t_4]\), the inner one on the segment \([t_2, t_3]\), with \(t_1 < t_2 < t_3 < t_4 < \theta_c\). We set \(f_i = f(t_i), \ i = 1, \ldots, 4\) and get

\[
\begin{vmatrix}
1 & 1 & 1 & 1 \\
1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4
\end{vmatrix} > 0
\]

By the hypothesis \(f_1 + f_4 = f_2 + f_3\), the determinant factorizes as \((f_4 - f_3)(f_3 - f_1)(t_4 - t_1)(f_3 - f_2) - (t_3 - t_2)(f_1 - f_4)\), so we have \((t_4 - t_1)(f_3 - f_2) - (t_3 - t_2)(f_4 - f_1) > 0\).

Remark 2.11. A result similar to lemma 2.10 was used in Almeida Santos & Vidal (2007) to prove the symmetry of central configurations in the 4+1-body problem, the four equal big masses forming a square.

Remark 2.12. We could consider the general homogeneous law of force instead of Newton’s force. Newton’s force would then be the particular case \(p = -3\) in the expressions

\[
\begin{align*}
  f(\theta) &= \sin \theta (1 - (2s)^p), \\
  f'(\theta) &= 1 - 2s^2 - (2s)^p(1 + p) + (2 + p)s^2(2s)^p, \\
  f''(\theta) &= \sin \theta (-p(1 + p)(2s)^{p-2} + (2 + p)^22^{p-2}s^p - 1).
\end{align*}
\]

The problem of point vortices on a plane (O’Neil 1987) would correspond to \(p = -2\). It may be asked if the results presented here are still true for other values of \(p\). It seems that there are always three relative equilibria for any negative \(p\). Property 2.2 requires \(p \in ]-\infty, -1[\). According to expression (2.7), property 2.3 is true on the interval \([-\infty, \rho]\), where \(\rho = -1.0022967\ldots\) is the root of the equation\(^1\) in \(p\)

\[
4(-p - 1)^p((2 + p)^2)^{2-p} = (8 - 4p)^{2-p}.
\]

\(^1\)Here we use an easy generalization of the classical discriminants \(b^2 - 4ac\) and \(4p^3 - 27q^2\). The expression \(aX^\alpha + bX^\beta + cX^\gamma\) has a double root if

\[
\left( \frac{a}{\gamma - \beta} \right)^{\gamma - \beta} \left( \frac{b}{\alpha - \gamma} \right)^{\alpha - \gamma} \left( \frac{c}{\beta - \alpha} \right)^{\beta - \alpha} = 1.
\]

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We exclude the second case in proposition 3.1 using properties 2.1–2.3 and property 2.5. Numerical studies indicate that property 2.5 is always true in the previous interval $[\infty, \rho]$. We exclude the first case using properties 2.1–2.4. Property 2.4 imposes further restrictions. According to numerical studies, we should restrict to $p \in ] -37.61045, -1.26766[$. To finish the classification of the relative equilibria, we will need in particular some polynomial calculus which can be carried out only for simple integer or rational values of $p$.

3. Symmetry result

**Proposition 3.1.** Consider any solution of equation (2.2), given by the four positive angles $\theta_i$ between the adjacent bodies, which satisfy $\theta_1 + \theta_2 + \theta_3 + \theta_4 = 2\pi$. At least one of the following situations occurs: a diagonal of the quadrilateral is a diameter of the circle (i.e. $\theta_1 + \theta_2 = \pi$ or $\theta_2 + \theta_3 = \pi$), or two angles among $\theta_1$, $\theta_2$, $\theta_3$ and $\theta_4$ are equal.

**Proof.** We assume the conclusion is not satisfied and will derive a contradiction. We can take without loss of generality the labelling convention such that $\theta_4 < \theta_2 < \theta_1$ and $\theta_3 < \theta_1$. This implies $\theta_3 + \theta_4 < \theta_1 + \theta_2$, which is $\pi < \theta_1 + \theta_2$. The remaining assumption is $\theta_2 + \theta_3 \neq \pi$, i.e. we are either in the first case $\theta_1 + \theta_4 < \pi$ or in the second case $\theta_2 + \theta_3 < \pi$.

**First case.** Here $\theta_1 < \theta_1 + \theta_4 < \pi$. As $\theta_1$ is the greatest angle, $\pi/2 < \theta_1$, thus $f_1 > 0$ and $f_{14} > 0$. From the last inequality and the equations (2.2), we get $f_3 > f_1$ and $f_2 > f_1$, the latter with $\theta_2 < \theta_1$ implying $\theta_3 < \theta_1$ (figure 1).

By the relation $f_{14} - f_{11} = f_1 - f_3$, the expression $A(\theta) = f(\theta_1 + \theta) - f(\theta_3 + \theta)$ is such that $A(\theta_4) = -A(0)$. We have, for $\theta \in [0, \theta_4]$, $A'(\theta) = f'(\theta_1 + \theta) - f'(\theta_3 + \theta) < 0$, by property 2.3. So $A(0) > 0$, i.e. $f_1 > f_3$. Together with what we obtained, this gives $f_2 > f_1 > f_3 > f_4$. 

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From $\theta_1 > \theta_3$ and $f_1 > f_3$, we know that $\theta_3 < \theta_c$. The order $\theta_2 < \theta_3$ is then impossible, as it would imply $f_2 < f_3$. Also our case hypothesis is $\theta_1 + \theta_4 < \theta_2 + \theta_3$, or $\theta_1 - \theta_2 < \theta_3 - \theta_4$. Putting these arguments together, we get the order $\theta_1 < \theta_3 < \theta_2 < \theta_1$.

We know that $f_2 > f_1 > 0$ and $\theta_c < \theta_1$. Let us call $\theta^L_1 < \theta_c$ the unique angle satisfying this inequality such that $f(\theta^L_1) = f_1$. We call $\theta^L_2$ and $\theta^R_2$, $\theta^L_2 < \theta_c < \theta^R_2$, the two angles such that $f(\theta^L_2) = f(\theta^R_2) = f(\theta_2) = f_2$. So we have either $\theta_2 = \theta^L_2$ or $\theta_2 = \theta^R_2$. We get

$$0 < \frac{f_2 - f_1}{\theta^L_2 - \theta^L_1} < \frac{f_3 - f_4}{\theta_3 - \theta_4} < \frac{f_2 - f_1}{\theta_1 - \theta_2} \leq \frac{f_2 - f_1}{\theta^R_1 - \theta^R_2}. \quad (3.1)$$

All the numerators take the same value $f_2 - f_1 = f_3 - f_1 > 0$. The first inequality between slopes is due to $f'' < 0$. The second is our case hypothesis. The last is simply $\theta_3 < \theta_2 < \theta_1$. Now, the inequality resulting from equation (3.1), namely $\theta_1 - \theta^R_2 < \theta^L_2 - \theta^L_1$, or $\theta^L_1 + \theta_1 < \theta^L_2 + \theta^R_2$, contradicts lemma 2.8.

Second case. Here $\theta_2 + \theta_3 < \pi$. From the combination $f_3 - f_2 = f_2 - f_1$, exactly as we did in the previous case, we deduce $f_2 > f_1$ and $f_3 > f_2$. The latter combined with $\theta_3 + \theta_4 < \theta_2 + \theta_3$ implies $\theta_3 < \theta_2 + \theta_3$. As a particular result, we have $f_2 > 0$, and so, $f_1 > f_2$ and $f_4 > f_3$. Putting together, we get $f_1 > f_2 > f_4 > f_3$. This fits with the conventions $\theta_3 < \theta_1$ and $\theta_4 < \theta_2 < \theta_1$ only if $\theta_3 < \theta_c$ and $\theta_4 < \theta_2 < \theta_c$. It is then obvious from $f_3 > f_4$ that $\theta_3 < \theta_4$. This gives the order $\theta_1 < \theta_2 < \theta_3 < \theta_1$. Let $\theta^L_1 < \theta_c$ be the unique angle satisfying this inequality such that $f(\theta^L_1) = f_1$. We have $\theta_3 < \theta_4 < \theta_2 < \theta^L_1 < \theta_c$. According to equation (2.2), lemma 2.10 may be applied, giving $(\theta^L_1 - \theta_3)(f_2 - f_3) - (\theta_2 - \theta_4)(f_1 - f_3) > 0$, and the inequality persists if we change $\theta^L_1$ to $\theta_1$. As $f_1 - f_3 = f_2 - f_4 + 2f_23$, it becomes $(\theta_1 - \theta_3 - \theta_2 + \theta_4)(f_2 - f_3) - 2(\theta_2 - \theta_4)f_23 > 0$, which may be written as

$$\frac{f_23}{\pi - (\theta_2 + \theta_3)} < \frac{f_2 - f_4}{\theta_2 - \theta_4}, \quad (3.2)$$

because $\theta_2 - \theta_4 > 0$ and $2\pi - 2(\theta_2 + \theta_3) = \theta_1 - \theta_3 - \theta_2 + \theta_4 > 0$. On the other hand,

$$\frac{f(\pi) - f_23}{\pi - (\theta_2 + \theta_3)} < \frac{f_23 - f_34}{(\theta_2 + \theta_3) - (\theta_3 + \theta_4)} = \frac{f_2 - f_4}{\theta_2 - \theta_4}$$

due to the concavity of $f$, which, as $f(\pi) = 0$, contradicts equation (3.2).

**Proposition 3.2.** If in a solution of equation (2.2) two consecutive angles are equal, then the two remaining angles are equal. The configuration has an axis of symmetry.

**Proof.** We assume $\theta_1 = \theta_2$ and $\theta_4 > \theta_3$, and deduce a contradiction. By equation (2.2), $\theta_1 = \theta_2$ implies $-f_14 = f_23 = f_3 = f_4 = 0$. But there are only three roots of $f(\theta) = 0$, namely $\pi/3$, $\pi$ and $5\pi/3$. As $\theta_2 + \theta_3 = 2\pi - (\theta_1 + \theta_4)$ and $\theta_1 > \theta_3$, the only possibility is $\theta_1 + \theta_4 = 5\pi/3$ and $\theta_2 + \theta_3 = \pi/3$. The last equality gives us three complementary cases, i.e. either $\theta_1 = \theta_2 = \theta_3 = \pi/6$, or $\theta_3 < \pi/6 < \theta_1 = \theta_2 < \pi/3$, or $\theta_1 = \theta_2 < \pi/6 < \theta_3 < \pi/3$. In the first case, we have $\theta_4 = 3\pi/2$ and
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$$4(f_1 - f_3) = 3(2(\sqrt{2} - 1) + \sqrt{6}) > 0,$$ therefore the configuration is not central.\footnote{Interestingly, if we take the interaction corresponding to $p = -0.101834199\ldots$ in equation (2.5), this configuration is central and is the continuation of $E_2$.}

The second case is impossible because we have both $f_2 > f_3$ due to property 2.2, and $f_3 - f_2 = f_{12} > 0$ due to both equation (2.2) and property 2.2. In the third case, these two inequalities are reversed, giving again a contradiction. ■

**Proposition 3.3.** If in a solution of equation (2.2) a diagonal of the quadrilateral formed by the four small bodies passes through the centre of the circumcircle, this diagonal is an axis of symmetry for the configuration.

**Proof.** We assume $\theta_1 + \theta_4 = \theta_2 + \theta_3 = \pi$. Obviously, the four $\theta_i$’s belong to $[0, \pi]$, and we have by (2.2) $f_1 - f_2 = f_4 - f_3 = f_{23} = 0$, i.e. both $f_1 = f_2$ and $f_3 = f_4$. Therefore, if we have $\theta_1 \neq \theta_2$, then $\theta_3 \neq \theta_4$, and by the corollary 2.9 of lemma 2.8, we have both $\theta_1 + \theta_2 > 2\theta_c$ and $\theta_3 + \theta_4 > 2\theta_c$. We check that $f'(\pi/2) > 0$, which implies by property 2.3 that $\pi/2 < \theta_c$, which in turn leads to the contradiction $2\pi = \theta_1 + \theta_2 + \theta_3 + \theta_4 > 2\pi$. ■

**Remark 3.4.** Proposition 3.3 corresponds to a case of proposition 11 in Cors et al. (2004a). Our method of proof is faster. Our three propositions prove that any solution of equation (2.2), i.e. any relative equilibrium of four separate identical satellites, possesses some symmetry. One possibility is the equality of two non-adjacent angles, namely $\theta_1 = \theta_3$ or $\theta_2 = \theta_4$. In this case, there is an axis of symmetry that contains only the central body, i.e. the centre of the circumcircle. The other possibilities left after proposition 3.1 are the hypotheses of propositions 3.2 and 3.3, and they both imply the existence of an axis of symmetry passing through the central body and two of the four small bodies. In the next section we recall, after Cors et al. (2004a), that there are two configurations in both classes of symmetry, one of them, the square, being common to both classes.

4. The symmetric configurations

**Proposition 4.1 (Cors et al. 2004a).** There are exactly two relative equilibria (labelled $E_1$ and $E_3$ in figure 2) of four separate identical satellites which are symmetric with respect to a diagonal of the quadrilateral of the small bodies.

This is proposition 12 of Cors et al. (2004a). The next result is also stated in Cors et al. (2004a), but we decided to give here a complete proof for several reasons. First, in several papers from Albouy (1996), the final arguments are only sketched. Some floating point approximations are used to discard some solutions, and the authors do not make clear if they consider that the work left to get a proof is just a routine work, or if they are satisfied with this degree of rigour. Second, as this method of proof may be tested on more and more complex situations, it is important to present what we consider as the most efficient way to produce a rigorous proof. For example, we choose instead of Sturm’s algorithm, the method of rational substitutions, inspired from Vincent’s method, as advised to us by Eduardo Leandro (2001). Third, compared with Cors et al. (2004a), we could reduce the degree of the key polynomial from degree 312 to degree 78.
Figure 2. The three relative equilibria of four separate identical satellites.

Proposition 4.2 (Cors et al. 2004a). There are exactly two relative equilibria (labelled $E_2$ and $E_3$ in figure 2) of four separate identical satellites which are symmetric with respect to an axis which is not a diagonal of the quadrilateral of the small bodies.

Proof. We suppose the symmetry $\theta_1 = \theta_3$. As we have the relation $2\theta_1 + \theta_2 + \theta_4 = 2\pi$, we need two angles to get the configuration. If we exchange the values of $\theta_2$ and $\theta_4$, we get essentially the same configuration. So it is a good idea to choose angles which behave well under this exchange. We choose $\sigma = (\theta_2 + \theta_4)/4$ and $\nu = (\theta_2 - \theta_4)/4$, so that $\theta_1 = \pi - 2\sigma$, $\theta_2 = 2(\sigma + \nu)$, $\theta_4 = 2(\sigma - \nu)$, $\theta_1 + \theta_2 = \pi + 2\nu$. A good expression for $f$ is
\[
f(\theta) = \frac{1}{4} \cos \frac{\theta}{2} \sin^{-2} \frac{\theta}{2} \left( 8 \sin^3 \frac{\theta}{2} - 1 \right).
\]
This is equation (2.3) rather than equation (2.1), so this expression is not correct if $\theta \in ]2\pi, 4\pi[$. We will simply ignore the solutions corresponding to this interval. Setting $C = \cos \sigma$, $S = \sin \sigma$, $c = \cos \nu$, $s = \sin \nu$, we get the rational expressions
\[
4f_1 = SC^{-2}(8C^3 - 1), \quad 4f_{12} = -sc^{-2}(8c^3 - 1),
\]
\[
4f_2 = (Cc - Ss)(Sc + Cs)^{-2}(8(Sc + Cs)^3 - 1),
\]
\[
4f_4 = (Cc + Ss)(Sc - Cs)^{-2}(8(Sc - Cs)^3 - 1).
\]
The two equations $f_2 + f_1 - 2f_1 = 0$ and $2f_{12} + f_2 - f_4 = 0$ characterize the central configurations. We denote by $P$ and $sQ$ the respective numerators of the fractions $f_2 + f_1 - 2f_1$ and $2f_{12} + f_2 - f_4$. The reader may use a computer to deduce expressions of $P$ and $Q$, and continue the computations using these expressions. We mention here simplified expressions which are not easy to obtain directly from a computer, due to our opportunistic use of the relations $C^2 + S^2 = 1$, $c^2 + s^2 = 1$
\[
P = S(C^2 - c^2)^2(1 - 16s^2C^3) - C^3c(s^2 + S^2),
\]
\[
Q = (C^2 - c^2)^2(1 - 16S^2c^3) + Sc^2(C^2 + c^2).
\]
Our system is \( P = sQ = 0, \) \( c^2 + s^2 = C^2 + S^2 = 1. \) This system possesses many complex solutions. We are interested in the solutions such that \( S, C, s, c \) are real. Then the \( \theta_i \) are real. As we said, we should furthermore impose \( 0 < \theta_i < 2\pi. \) This condition gives \( \sigma > 0 \) and, together with \( \theta_1 = \pi - 2\sigma, \sigma < \pi/2, \) i.e. \( S > 0 \) and \( C > 0. \)

A first branch of solutions corresponds to \( s = 0. \) We get \( c = \pm 1, \) and \( P = S^2(S^3 - cC^3) = 0. \) The inequalities above allow only the solution \( S = C = 1/\sqrt{2}, \ c = 1. \) This gives \( \theta_1 = \theta_2 = \theta_3 = \theta_4 = \pi/2, \) the square solution \( E_3. \)

Now we consider the branch \( s \neq 0. \) We first notice that there are only even powers of \( s \) in the simplified system. The reason is that exchanging \( \theta_2 \) and \( \theta_4 \) changes the sign of \( s, \) keeping \( c \) and \( s^2 \) invariant. We replace \( s^2 \) by \( 1 - c^2, \) set \( C = (1 - t^2)/(1 + t^2), \) \( S = 2t/(1 + t^2) \) and consider the numerators \( P_6(t, c) = -32t(t^2 - 1)^3(1 + t^2)^4c^6 + \cdots \) and \( Q_7(t, c) = -64t^2(1 + t^2)^4c^7 + \cdots \) of \( P \) and \( Q. \) Their resultant in \( c \) is a polynomial in \( t \) with integer coefficients, \( R = -5242885^5(t^2 - 1)^{16}(1 + t^2)^{32}R_{78}. \) The irreducible monic factor \( R_{78} \) is

\[
R_{78} = t^{78} + 12t^{77} + 77t^{76} + 464t^{75} - 64992t^{74} + \cdots - 104961536141944t^{39} + \cdots + 77790t^4 - 464t^3 - 77t^2 - 12t - 1.
\]

As \( t = \tan(\sigma/2) \) and \( 0 < \sigma < \pi/2, \ 0 < t < 1 \) and the acceptable roots of \( R(t) \) are in the interval \( ]0, 1[. \) Substituting successively

\[
t = \frac{y}{1 + 3y}, \quad t = \frac{1 + 7y}{3 + 20y}, \quad t = \frac{7(1 + y)}{10(2 + y)}, \quad t = \frac{7 + y}{10 + y},
\]

in \( R_{78}, \) we find numerators with respectively 0, 1, 1, 0 variations of sign, which according to Descartes rule of sign proves that \( R_{78} \) has only two real roots \( t_1 \) and \( t_2 \) in \( ]0, 1[. \) satisfying \( \frac{1}{3} < t_1 < \frac{7}{20} < t_2 < \frac{7}{10}. \) In order to prove that \( t_1 \) does not correspond to a real central configuration, we need to express the variable \( c \) as a function of \( t. \) There is a unique expression of \( c \) as a polynomial in \( t \) of degree lower than 78 and with rational coefficients, but this expression is not suitable for practical computations. Many expressions of \( c \) as a rational function of \( t \) with integer coefficients behave well. We can compute the Sylvester matrix of \( P_6(t, c) \) and \( Q_7(t, c) \) in \( c. \) It is a 13 \( \times \) 13 matrix whose entries are polynomials in \( t \) with integer coefficients, and whose determinant is the resultant \( R(t). \) We can set for example \( c = F(t) = -M_{1,7}/M_{1,8}, \) where \( M_{i,j} \) is the minor determinant of the Sylvester matrix obtained by removing the \( i \)th line and the \( j \)th column. We get

\[
F(t) = \frac{(t + 1)(t^2 + 1)^4N(t)}{8t^2(t - 1)^2D(t)},
\]

\[
N(t) = t^{56} + 8t^{55} + 42t^{54} + 248t^{53} - 48461t^{52} + \cdots + 284953591140t^{28} + \cdots + 47947t^{4} - 42t^{3} - 8t - 1,
\]

\[
D(t) = 15t^{59} + 7t^{58} + 83t^{57} + \cdots + 807713099949204t^{20} + \cdots - 141t^2 - 25t - 17.
\]

The function \( F(t) \) is well defined and decreasing from 1/4 to 1/2, as may be proved by applying the substitution \( t = (1 + y)/(4 + 2y), \) respectively, to \( D(t) \) and to the numerator of \( F'(t). \) So on the interval \( ]1/3, 7/20[ \) we have \( c = F(t) > F(7/20) > 1. \)
But \( c = \cos(v) \leq 1 \) so the root \( t_1 \) should be rejected. Finally, there is only one solution in this branch satisfying the reality conditions, which corresponds to the root \( t_2 \). This solution is \( \mathcal{E}_2 \).

The reader may use the above technique to prove rigorously many inequalities concerning the solution \( \mathcal{E}_2 \), for example, the inequalities in the theorem. He can check first that \( t_2 \in ]3/5, 7/10[ \). On this interval \( F(t) \) is increasing, from \( F(3/5) > 0 \) to \( F(7/10) < 1 \).

The solution \( \mathcal{E}_2 \) was described by Hall (preprint) and Salo & Yoder (1988), who both gave the estimates \( \theta_1 = \theta_3 = 41.5^\circ, \theta_2 = 37.4^\circ, \theta_4 = 239.6^\circ \). It was identified as a linearly stable relative equilibria in Salo & Yoder (1988) and Moeckel (1994).

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