Cartesian trees and Lyndon trees

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Abstract

The article describes the structural and algorithmic relations between Cartesian trees and Lyndon Trees. This leads to a uniform presentation of the Lyndon table of a word corresponding to the Next Nearest Smaller table of a sequence of numbers. It shows how to efficiently compute runs, that is, maximal periodicities occurring in a word.

1 Cartesian and Lyndon trees

The Cartesian tree, introduced by Vuillemin [15] is a binary tree associated with a sequence of numbers that label its nodes. It is both a heap, with the smallest element at the root, and the sequence is recovered during a symmetric traversal of the tree.

Cartesian tree have a series of applications in addition to that introduced by Vuillemin [15] on two-dimensional images. To quote a few of them, they are used for range searching to implement range minimum queries in a sequence of numbers through the help of Lowest Common Ancestor queries in the Cartesian tree of the sequence [8]. They are also part of sorting methods that want to take advantage of partially sorted subsequences (see for example [12]).

Lyndon trees are associated with Lyndon words, words that are lexicographically smaller than all their proper non-empty suffixes (see [13] and [2]). They also have several interesting algorithmic applications and attracted much interest in connection with the detection of runs (maximal periodicities) in words. The notion of Lyndon roots of runs, introduced for cubic runs in [5], has led to the property that there is linear number of square runs in a word. Originally conjectured by Kolpakov and Kucherov [11], it has eventually been proved by Bannai et al. [1]. They also show how to compute efficiently all the runs using implicitly the notion of Lyndon table (array), which is a side product of the Lyndon tree construction.

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This article may be viewed as a follow-up of the publication by Hohlweg and Reutenauer [10] in which they show the link between the two types of trees. The bridge between them is a key property (stated in Proposition 1) that relates a local condition on the factors of the word to a global condition on its suffixes. It implies the structure of a Lyndon tree is the same as the Cartesian tree of ranks of the associated word suffixes.

2 Cartesian tree

Let \( x = (x[0], x[1], \ldots, x[n-1]) \) be a sequence of numbers of length \( n \). Below is a standard algorithm for computing its associated Cartesian tree. Nodes of the tree are identified with positions of numbers on the sequence and are labelled by the numbers with \( X \). To simplify the algorithm we insert a sentinel into the original sequence, i.e., we add a number \( x[n] = -\infty \). The purpose of this number is that it is smaller than any other number that already exists in the sequence.

The algorithm proceeds from right to left, instead of left to right as usual, to fit with the Lyndon tree construction. One step \( i \) is to go up the leftmost path of the tree from \( i + 1 \) to find where to insert the node \( i \). (The artificial node \( n \) acts as a sentinel to simplify the design.) During the traversal, going to the parent of node \( S \) is like going to the next nearest value smaller than \( x[i] \).

```plaintext
CartesianTree\( (x \text{ non-empty sequence of numbers of length } n) \)
1. \( x[n] \leftarrow -\infty \)
2. \( X[n] \leftarrow x[n] \)
3. \( n.\text{LeftChild} \leftarrow \text{Null} \)
4. for \( i \leftarrow n - 1 \) downto 0 do
5.   \( S \leftarrow i + 1 \)
6.   while \( x[i] < X[S] \) do
7.     \( S \leftarrow S.\text{Parent} \)
8.     \( i.\text{rightchild} \leftarrow S.\text{leftchild} \)
9.     \( S.\text{leftchild} \leftarrow i \)
10. return labelled built tree
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The number of comparisons executed at line 6 is linear in \( n \). Any comparison that yields \( x[i] \geq X[S] \) means that the while fails and therefore occurs at most once for each \( i \). Moreover the comparisons that yield \( x[i] < X[S] \) for some position \( j \), i.e., \( x[j] = X[S] \) implies that position \( j \) will no longer be involved in a latter comparison. An alternative view of this process is a that the consequent \( S \leftarrow S.\text{Parent} \) assignment moves upward on the rightmost branch of the current tree. Thus, the running time is \( O(n) \).

The picture displays the Cartesian tree of the sequence of numbers: \((7, 15, 12, 4, 10, 1, 5, 13, 6, 14, 11, 3, 9, 0, 2, 8, -\infty)\).
The next picture exemplifies the CartesianTree Algorithm by inserting the number 5 into an Cartesian tree. At each step the algorithm considers the leftmost branch of the tree, highlighted by the arrows in the picture. These arrows represent the parent pointers that are used by the inner cycle while \( x[i] < X[S] \). In this example the while guard is true twice when 5 < 13 and 5 < 6. The final comparison yields 5 > 3 and therefore the cycle stops, notice that the parent pointers of 13 and 13 are not represented by arrows in the second tree, as they are no longer part of the leftmost branch.
This moving upwards process computed by the inner cycle, corresponds to finding the Next nearest smaller value. In our example, when given the number 5 we searched the sequence until we reached the number 3, note that by moving upwards on the tree this process is faster than computing linear scan from right to left, in particular we did not compare with the numbers 14 and 11.

**Next nearest smaller table** The Next nearest smaller table NNS of a (non-empty) sequence $y$ of numbers is defined as follows. For a position $i$ on $x$, $i = 0, \ldots, |x| - 1$, NNS[$i$] is the smallest position $j > i$ of an element $x[j] < x[i]$, or $n$ if none exist:

$$NNS[i] = \min\{j \mid x[j] < x[i]\} \cup \{n\}.$$
The following picture shows the NNS table illustrated over the Cartesian tree. We show the corresponding table below the tree. Moreover each node also shows an arrow to point to the corresponding Next nearest smaller node. When a node is a left child of its parent the arrows are simply the parent pointers. However when the node is a right child of its parent then the arrows are shown with dashed lines and point to an ancestor of the node that is to its right.

It is interesting to notice that the algorithm used for constructing Cartesian trees can be adapted to compute the NNS values. As illustrated by the picture when a node is a left child then the NNS value is actually a pointer to its parent on the tree. Now recall the CartesianTree Algorithm and notice that whenever a value is inserted in the tree it is always a left child and therefore its parent, when it gets inserted, is the corresponding NNS value. Recall our example when the value of \( x[6] = 5 \) is inserted into the tree it becomes the left child of node 11, with \( x[11] = 3 \), therefore NNS[6] = 11. Note also in this example that when \( x[6] \) is processed we have that the node 8 with \( x[8] = 6 \) was a left child of node 11 with \( x[11] = 3 \) before the insertion but becomes a right child after the insertion. Still the value NNS[8] = 11 is not altered by this procedure.

The following modification of the CartesianTree Algorithm uses this information to obtain the NNS values. Likewise it also runs in linear time.
NextNearestSmaller(\(x\) non-empty sequence of numbers of length \(n\))

1. \((x[n], NNS[n - 1]) \leftarrow (-\infty, n)\)
2. for \(i \leftarrow n - 2\) downto 0 do
   3. \(j \leftarrow x[i + 1]\)
   4. while \(x[i] < x[j]\) do
      5. \(j \leftarrow NNS[j]\)
   6. \(NNS[i] \leftarrow j\)
3. return \(NNS\)

3 Lyndon tree

Lyndon trees are associated with Lyndon words. Recall that a Lyndon word is a non-empty word lexicographically smaller than all its proper non-empty suffixes. The Lyndon tree of a Lyndon word \(y\) corresponds recursively to the following suffix (or standard) factorisation of \(y\) when not reduced to a single letter: \(y\) can be written \(uv\) where \(v\) is chosen as the smallest proper non-empty suffix of \(y\). The word \(u\) is then also a Lyndon word (see [13]).

Algorithm LyndonTree builds the Lyndon tree of a Lyndon word \(y\). The hypothesis on \(y\) is not a significant restriction because any word can be turned into a Lyndon word by prepending to it a letter smaller than all letters occurring in it. Otherwise, since any word factorises uniquely into Lyndon words, the algorithm can produce the forest of Lyndon trees of the factors.

The algorithm proceeds naturally from right to left on \(y\) to find the longest Lyndon word starting at each position \(i\). It applies a known property: if \(u\) and \(v\) are Lyndon words and \(u < v\) then \(uv\) is also a Lyndon word with \(u < uv < v\).

To facilitate the presentation, variable \(u\) stores a phrase, that is, the occurrence of a Lyndon factor of \(y\) though the position of the factor is not explicitly given, and \(T(u)\) is the Lyndon tree associated with this occurrence. Idem for \(v\).

LyndonTree(\(y\) Lyndon word of length \(n\))

1. \((v, T(v)) \leftarrow (y[n - 1], (y[n - 1]))\)
2. for \(i \leftarrow n - 2\) downto 0 do
   3. \((u, T(u)) \leftarrow (y[i], (y[i]))\)
   4. while \(u < v\) do
      5. \(T(uv) \leftarrow (\text{new node}, T(u), T(v))\)
      6. \(u \leftarrow uv\)
      7. \(v \leftarrow \text{next phrase, empty word if none}\)
3. return \(T(y)\)

If the comparison \(u < v\) at line 4 is done by mere letter comparisons, the algorithm may run in quadratic time, for example if applied on \(y = a^kba^k\) (each factor \(a^i b\) is compared with the prefix \(a^{i+1}\) of \(a^k\) or with \(a^k\) itself).

However the algorithm can be implemented to run in linear time if the test \(u < v\) at line 4 is done in constant time because each execution of instructions at lines [3,7] decreases the number of Lyndon phrases, which goes from \(n\) to 1.

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The Lyndon table Lyn of a (non-empty) word $y$ is defined as follows. For a position $i$ on $y$, $i = 0, \ldots, |y| - 1$, $\text{Lyn}[i]$ is the length of the longest Lyndon factor of $y$ starting at $i$:

$$\text{Lyn}[i] = \max\{\ell \mid y[i \ldots i + \ell - 1] \text{ is a Lyndon word}\}.$$
The computation of the Lyndon table is an offspring of the previous algorithm, like the computation of the Next nearest smaller table for Algorithm CartesianTree. Algorithm LongestLyndon computes Lyn using the same right-to-left detection of Lyndon factors as above.

\begin{verbatim}
LONGESTLYNDon(y non-empty word of length n)
  for i ← n – 1 downto 0 do
    (Lyn[i], j) ← (1, i + 1)
    while j < n and y[i ... j – 1] < y[j ... j + Lyn[j] – 1] do
      (Lyn[i], j) ← (Lyn[i] + Lyn[j], j + Lyn[j])
  return Lyn
\end{verbatim}

4 Key property

It is clear that the previous algorithms all share the same algorithmic structure. The link between the trees or their reduced versions is even tighter when the running time of the Lyndon tree construction is concerned. Indeed, the comparison between two consecutive phrases of the factorisation of y at line 4 in LyndonTree or at line 3 in LongestLyndon comes back to considering the ranks of suffixes in alphabetic order. This is shown by the next proposition where the local comparison between two phrases is shown to be equivalent to the comparison of their associated suffixes.

In addition, the next statement also leads to prove that the Lyndon tree of y, possibly reduced to its internal nodes, has the same structure than the Cartesian tree built from the ranks of the word suffixes, which has been first noticed by Hohlweg and Reutenauer in [10].

**Proposition 1** Let u be a Lyndon word and v ⋅ v_1 ⋅ v_2 ⋅ ⋅ ⋅ v_m be the Lyndon factorisation of a word w. Then u < v iff uw < w.

**Proof.** Let us consider the different cases.

Assume first u < v. If u ≪ v then uw ≪ vv_1v_2 ⋅ ⋅ ⋅ v_m = w.

Consider the case where u is a proper prefix of v. Let e > 0 be the largest integer for which v = u^ez. Since v is a Lyndon word, z is not empty and we have u^e < z. Since u is not a prefix of z (by definition of e) nor z a prefix of u (because v is border-free) we have u ≪ z. This implies u^{e+1} ≪ u^ez = v and then uw < w.

Then assume v ≤ u. If v ≪ u we have obviously w < uw.

It remains to consider the situations where v is a prefix of u. If it is a proper prefix, u writes vz for a non-empty word z. We have v < z because u is a Lyndon word. The word z cannot be a prefix of t = v_1v_2 ⋅ ⋅ ⋅ v_m because v would not be the longest Lyndon prefix of w, a contradiction with a property of the factorisation. Thus, either t ≤ z or z ≪ t. In the first case, if t is a prefix
of z, \( w = vt \) is a prefix of u and then of uw, that is, \( w < uw \). In the second case, for some suffix \( z' \) of z and some factor \( v_k \) of t we have \( z' \ll v_k \). The factorisation implies \( v_k \leq v \). Therefore, the suffix \( z' \) of u is smaller than its prefix v, a contradiction with the fact that u is a Lyndon word.

For each position \( i \) on y, \( i = 0, \ldots, |y| − 1 \), let \( \text{Rank}[i] \) be the rank of the suffix \( x[i \ldots |y| − 1] \) is the increasing alphabetic list of all non-empty suffixes of y (ranks run from 0 to \(|y| - 1\)).

\[
\begin{array}{cccccccccccccccc}
  i & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
  y[i] & a & b & b & a & a & b & a & a & b & b & a & a & b & a & b & a & b \\
  \text{Lyn}[i] & 3 & 1 & 1 & 2 & 1 & 8 & 5 & 1 & 1 & 3 & 1 & 1 & 2 & 1 & 3 & 2 & 1 \\
  \text{Rank}[i] & 7 & 15 & 12 & 4 & 10 & 1 & 5 & 13 & 6 & 14 & 11 & 3 & 9 & 0 & 2 & 8 \\
\end{array}
\]

Applying the above property to update line 3 Algorithm LONGESTLYNДRewrites as follows, where tables Lyn and Rank concern the input word y.

LONGESTLYNД(y non-empty word of length n)

1. \[ i \leftarrow n - 1 \text{ downto } 0 \text{ do} \]
2. \[ (\text{Lyn}[i], j) \leftarrow (1, i + 1) \]
3. \[ \text{while } j < n \text{ and } \text{Rank}[i] < \text{Rank}[j] \text{ do} \]
4. \[ (\text{Lyn}[i], j) \leftarrow (\text{Lyn}[i] + \text{Lyn}[j], j + \text{Lyn}[j]) \]
5. \[ \text{return } \text{Lyn} \]

As for the running time, when the table Rank is precomputed, the comparison of words at line 3 can be realised in constant time. And since the number of comparisons is no more than \( 2|x| - 2 \) (exactly \( n - 1 \) negative comparisons that stop the while loop and no more than \( n - 1 \) positive comparisons since each reduces the number of Lyndon factors in the overall factorisation of y), the total running time is linear.

Note the Lyndon factorisation of y can be recovered by following the longest decreasing sequence of ranks from the first rank. It is \( (7, 4, 3, 1, 0) \) in the above example, corresponding to positions \( (0, 3, 5, 7, 15) \) and to the Lyndon factorisation \( abb \cdot ab \cdot ab \cdot aababbab \cdot a \).

Also note the relation between Lyn and NNS: \( \text{NNS}[i] = i + \text{Lyn}[i] \), since Lyn[i] is the smallest distance to a next rank value smaller than Rank[i].

5 Computing runs

Algorithm LONGESTLYNД extends to an algorithm for computing efficiently all runs occurring in a word.

Recall that a run in the word y is an occurrence of a factor, say \( y[i \ldots j] \), whose length is at least twice its (smallest) period. The main result in [1] shows that a run can be identified with a special position s on y for which Lyn[s] is the period of \( y[i \ldots j] \) and \( 2 \times \text{Lyn}[s] \leq j - i + 1 \), considering some alphabet ordering or its inverse.

To compute all runs of the word y, we just have to check if the longest Lyndon factor starting at \( i \) produces a special position of a run. This is done by
extending the Lyndon factor to the left and to the right according to the period of the resulting factor and using Longest common extensions. This is done by computing \( r = \text{LCE}_R(i, i + \text{Lyn}[i]) \) and \( \ell = \text{LCE}_L(i - 1, i + \text{Lyn}[i] - 1) \) when appropriate and verifying if \( \ell + r \geq \text{Lyn}[i] \). If the inequality holds a run can be reported. In the algorithm below we assume \( \ell \) to be set to null if \( i = 0 \) and \( r \) to null also if \( i + \text{Lyn}[i] = n \).

\[
\begin{align*}
\text{RUNS}(y \text{ non-empty word of length } n) \\
1 & \quad \text{for } i \leftarrow n - 1 \text{ downto } 0 \text{ do} \\
2 & \quad (\text{Lyn}[i], j) \leftarrow (1, i + 1) \\
3 & \quad \text{while } j < n \text{ and } \text{Rank}[i] < \text{Rank}[j] \text{ do} \\
4 & \quad \quad (\text{Lyn}[i], j) \leftarrow (\text{Lyn}[i] + \text{Lyn}[j], j + \text{Lyn}[j]) \\
5 & \quad \quad (\ell, r) \leftarrow (\text{LCE}_L(i - 1, i + \text{Lyn}[i] - 1), \text{LCE}_R(i, i + \text{Lyn}[i])) \\
6 & \quad \quad \text{if } \ell + r \geq \text{Lyn}[i] \text{ then} \\
7 & \quad \quad \quad \text{output run } x[i - \ell \ldots i + \text{Lyn}[i] + r - 1]
\end{align*}
\]

To locate all runs, Algorithm RUNS has to be executed twice, for the tables corresponding to some alphabet ordering and for the tables corresponding to the inverse alphabet ordering.

**Running time of RUNS** Algorithm RUNS can be implemented to run in linear time \( O(|y|) \) when the alphabet is linearly-sortable.

Indeed, with the hypothesis, it is known that suffixes of \( y \) can be sorted in linear time (see for example [3]). Then also the table Rank that is just the inverse of the sorted list of starting positions of the suffixes.

Again with the hypothesis, LCE queries at line 5 can be executed in constant time after a linear-time preprocessing. The reader can refer to the review by Fischer and Heun [6] concerning LCE queries. More advanced techniques to implement them over a general alphabet and to compute runs can be found in [9, 4] and references therein.

Therefore the whole algorithm RUNS runs in linear time when the alphabet is linearly-sortable.

## 6 Concluding remarks

The relation between suffix sorting, part of the suffix array, and Lyndon factorisation is examined by Mantaci, Restivo, Rosone and Sciortino in [14]. Franek, Islam, Rahman and Smyth present several algorithms to compute the Lyndon table in [7].

The structure of the Cartesian tree with its nodes labelled by numbers is richer than the structure of the Lyndon tree because it seems difficult to recover the labels without completely sorting the ranks of suffixes. This question is certainly related to the application of Cartesian to sorting (see for example [12]).
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