Calabi-Yau \((p + 1)\)-folds from \(p\)-folds

H. Lü†‡ and Zhao-Long Wang⋆

†China Economics and Management Academy
Central University of Finance and Economics, Beijing, 100081

‡Institute for Advanced Study, Shenzhen University, Nanhai Ave 3688, Shenzhen 518060

⋆Interdisciplinary Center for Theoretical Study,
University of Science and Technology of China, Hefei, Anhui 230026

ABSTRACT

We establish the general formalism for constructing metrics of Calabi-Yau \((p + 1)\)-folds in terms of that of a \(p\)-fold by adding a complex-line bundle. We present a few explicit low-lying examples. We further consider holomorphic linearization and obtain the six-dimensional analogue of the Gibbons-Hawking instanton. Whilst the Kähler potential for the Gibbons-Hawking instanton is given by the harmonic function of a three-dimensional flat space, for the generalized solution it is related to the harmonic functions of certain three-dimensional non-flat spaces that are direct products of \(\mathbb{R}\) and two-dimensional Kähler spaces.
1 Introduction

Six-dimensional Calabi-Yau (CY) manifolds [1, 2, 3] play an important role in string theory, since they provide natural internal compactifying spaces, giving rise to four-dimensional theories that preserve one quarter of the ten-dimensional supersymmetry. It is thus of great interest to construct explicit metrics on Calabi-Yau manifolds. It can be argued however that such metrics are typically either non-compact or singular. Recently metrics on Calabi-Yau 3-folds were constructed from a generic hyper-Kähler space in $D = 4$ by adding a complex-line bundle [4]. (See also [5].) In this follow-up paper we generalize the result to give the general recursive relation of constructing the metrics of Calabi-Yau $(p + 1)$-folds in terms of a $p$-fold. The metrics of Calabi-Yau $(p + 1)$-folds are obtained by deforming the Kähler potential of the Calabi-Yau $p$-fold with a function $G$ that satisfies a single basic equation. We establish the general formalism in section 2. In section 3, we present a few explicit low-lying examples including $p = 1, 2$ and 3.

A major addition in this paper is the systematical study of the holomorphic linearization suggested in [6, 5]. The basic equations for the function $G$ in the construction are increasingly non-linear and more difficult to solve for larger $p$. The case for $p = 1$ is special, leading to a Kähler potential that depends on a harmonic function of three-dimensional flat space. The resulting metric is the Gibbons-Hawking instanton. In section 4, we consider holomorphic linearization for $p = 2$, where the non-linear terms in the basic equation of $G$ vanish. This restricts the possible CY2 bases, which turn out to be determined by the solutions of the two-dimensional Liouville equation. For certain choices of special solutions, the function $G$ becomes a harmonic function of a certain three-dimensional non-flat space that is a direct product of $\mathbb{R}$ and a two-dimensional Kähler space. We conclude our paper in section 5. In the appendix, we present a rather general solution that unifies the two special examples presented in section 4.

2 The construction

In this section, we present the metric construction for the CY $(p + 1)$-folds from CY $p$-folds. This is the generalization of [4, 5] where $p = 2$ were discussed. Let us consider a generic CY $p$-fold with the complex coordinates $z^i, \bar{z}^i$ ($i = 1, \ldots, p$) and the Kähler potential $K_0(z^i, \bar{z}^i)$. The metric is given by

$$ds^2 = 2\tilde{g}_{ij}dz^id\bar{z}^j, \quad \tilde{g}_{ij} = \frac{1}{2}\partial_i\partial_j K_0 = \bar{g}_{ij}. \quad (1)$$
Since it is Ricci flat, the Ricci form \( R^{(1,1)} \) vanishes. In fact, the Ricci-form is given by

\[
R^{(1,1)} = i \bar{\partial} \bar{\partial} \log \sqrt{V} = 0,
\]

where \( V \equiv \det(\tilde{g}_{ij})^2 \) is the volume factor and \( \partial \) and \( \bar{\partial} \) are the Dolbeault 1-form differential operators defined by

\[
\partial \equiv dz^i \partial_{z^i}, \quad \bar{\partial} \equiv d\bar{z}^i \partial_{\bar{z}^i}.
\]

The equation (2) implies that \( \log V \) is the real part of a holomorphic function, or equivalently, \( V \) can be the norm of a holomorphic function. There is no unique choice for the complex coordinates and we can always make a holomorphic coordinate transformation \( z^i \to z'^i = f^i(z^j) \), under which the volume factor transforms as

\[
V \to |T|^{-4} V,
\]

where

\[
T = \det \left[ \frac{\partial(f^1, \ldots, f^p)}{\partial(z^1, \ldots, z^p)} \right].
\]

The Jacobian \( T \) can be any holomorphic function. Thus we can always set \( V = 1 \) by choosing appropriate complex coordinates. We shall do this for later convenience.

Let us assume that the complex vielbein for the CY \( p \)-fold are \( \tilde{e}^a, \tilde{\bar{e}}^a \) \((a = 1, \ldots, p)\), then the Kähler form and holomorphic \((p,0)\)-form are given by

\[
\tilde{j}^{(1,1)} = \frac{i}{2} \tilde{e}^a \wedge \tilde{\bar{e}}^a, \quad \tilde{\Omega}^{(p,0)} = \tilde{e}^1 \wedge \cdots \wedge \tilde{e}^p.
\]

We now use this structure to construct a CY \((p + 1)\)-fold. The metric ansatz is given by

\[
ds^2_{2(p+1)} = ds^2_{2p} + h^2 dy^2 + h^{-2}(d\alpha + A)^2 \\
= (\delta_{ab} + G_{ab}) \tilde{e}^a \tilde{\bar{e}}^b + h^2 dy^2 + h^{-2}(d\alpha + A)^2.
\]

The metric components are the functions of the \( y, z^i, \bar{z}^i \) coordinates, but are independent of the coordinate \( \alpha \), which is a manifest Killing direction. Thus the metric ansatz assumes at least one Killing direction. The metric components appearing in the \( ds^2_{2p} \) part are defined by

\[
\tilde{\partial} \tilde{\partial} G = dz^i \wedge d\bar{z}^j \partial_i \partial_j G = \tilde{e}^a \wedge \tilde{\bar{e}}^b \tilde{e}_a \tilde{\bar{e}}_b \partial_i \partial_j G = G_{ab} \tilde{e}^a \wedge \tilde{\bar{e}}^b,
\]

where \( \tilde{e}_a \) is the inverse complex vielbein. Note that if we replace \( G \) by \( K_0 \) in (8), we have \( \delta_{ab} \) instead of \( G_{ab} \). Thus the \( ds^2_{2p} \) in (7) is obtained by deforming the original Kähler potential \( K_0(z^i, \bar{z}^i) \) to \( K_0(z^i, \bar{z}^i) + G(y, z^i, \bar{z}^i) \).

3
The $ds^2_{2p}$ can be diagonalized by a local $SU(p)$ transformation $U_a^b(z^i, \bar{z}^i)$, namely

$$U \left( \delta_{ab} + G_{ab} \right) U^\dagger = \text{diag}\{\lambda_1(z^i, \bar{z}^i), \ldots, \lambda_p(z^i, \bar{z}^i)\}.$$  

(9)

We further suppose that the complex structure of the CY $p$-fold is part of complex structure of the CY $(p + 1)$-fold. This implies that the complex vielbein of the CY $(p + 1)$-fold is given by

$$e^b = \sum_a \sqrt{\lambda_b} \bar{e}^a (U^\dagger)_{a}^b, \quad e^{(p+1)} = e^{i\kappa} (h \, dy + i \, h^{-1} (d\alpha + A)).$$  

(10)

where $\kappa = \kappa(\alpha, y, z^i, \bar{z}^i)$ is a real function. Correspondingly, the Kähler form and the $(p + 1, 0)$-form for the CY $(p + 1)$-fold are given by

$$J^{(1, 1)} = \frac{i}{2}(e^1 \wedge \bar{e}^1 + \ldots + e^p \wedge \bar{e}^p + e^{(p+1)} \wedge \bar{e}^{(p+1)})$$

$$= \frac{i}{2}(\delta_{ab} + G_{ab}) e^a \wedge \bar{e}^b + dy \wedge (d\alpha + A),$$  

$$\Omega^{(p+1, 0)} = e^1 \wedge \ldots \wedge e^p \wedge e^{(p+1)} = e^{i\kappa} e^1 \wedge \ldots \wedge \bar{e}^p \wedge (h \, dy + i \, h^{-1} (d\alpha + A)),$$

(11)

(12)

where

$$f = \sqrt{\lambda_1 \ldots \lambda_p} = \sqrt{\det(\delta_{ab} + G_{ab})}.$$  

(13)

The requirement that the metric (7) be Calabi-Yau becomes the requirement that the above Kähler form and $(p + 1, 0)$-form are both closed. Analogous to the derivation in [4], we find that $dJ = 0$ implies that

$$A = -\frac{i}{4} (\bar{\partial} - \partial) \partial_y G + \lambda(y, z_i, \bar{z}_i) \, dy.$$  

(14)

Note that the vanishing of $d\Omega$ implies that $\lambda$ is a pure gauge and can be set to zero as shown in [4]. Let us denote $g \equiv f \, h^{-1}$ in the following. Two classes of solutions emerges for the Calabi-Yau $(p + 1)$-fold metrics, corresponding to taking either $\kappa = \alpha$ or $\kappa = 0$. They are summarized as follows

Class I:

\[
\begin{align*}
\kappa &= \alpha, \\
g &= \exp \left( -\frac{1}{4} \partial_y G \right), \\
\partial_y \left[ \exp \left( -\frac{1}{4} \partial_y G \right) \right] &= 2 \det(\delta_{ab} + G_{ab});
\end{align*}
\]

and

Class II:

\[
\begin{align*}
\kappa &= 0, \\
g &= 1, \\
\partial_y^2 G &= -4 \det(\delta_{ab} + G_{ab}).
\end{align*}
\]

(15)

(16)
3 Low-lying examples

3.1 $p = 1$

Locally, the CY1 metric is flat, namely

$$ds^2 = dz d\bar{z}.$$ \hfill (17)

The Kähler potential is given by $K_0 = z \bar{z}$. A proper complex vielbein is given by $\tilde{e}^1 = dz$ and $\tilde{\epsilon}^1 = d\bar{z}$.

We first considered the $\kappa = \alpha$ case. The system is determined solely by the following equation for $G$

$$\partial_y \left[ \exp \left( -\frac{1}{2} \partial_y G \right) \right] = 2(1 + G_{11}).$$ \hfill (18)

It is more convenient to use $g^2$ as the basic function in this case. The basic equation becomes

$$\partial_y^2(g^2) + 4 \partial_z \partial_{\bar{z}} \log(g^2) = 0.$$ \hfill (19)

The corresponding CY2 metric is given by

$$ds^2 = \frac{1}{2} \partial_y(g^2) dz d\bar{z} + \frac{1}{2} g^{-2} \partial_y(g^2) dy^2 + \frac{2}{g^{-2} \partial_y(g^2)} \left( d\alpha + \frac{i}{2} (\partial - \bar{\partial}) \log(g^2) \right)^2.$$ \hfill (20)

The general solution for (19) is unknown. A simple way to obtain a special solution is to consider the separation of variables. It is straightforward to show that the resulting metrics are either $\mathbb{R}^4$ or the Eguchi-Hanson metric.

We now demonstrate that the supersymmetric limit [7, 8, 9] of the Ricci-flat Plebanski metric [10] is contained in (19). The metric is given by

$$ds^2 = (\tilde{y} - \tilde{x}) d\tilde{x}^2 + \frac{(\tilde{y} - \tilde{x}) d\tilde{y}^2}{4 \Delta_x} + \frac{\Delta_x}{\tilde{y} - \tilde{x}} (d\psi - \tilde{y} d\phi)^2 + \frac{\Delta_y}{\tilde{y} - \tilde{x}} (d\psi - \tilde{x} d\phi)^2.$$ \hfill (21)

where

$$\Delta_x = -\tilde{x}^2 + \mu, \quad \Delta_y = \tilde{y}^2 - \nu.$$ \hfill (22)

The coordinate $x$ is compact lying within $[-\sqrt{\mu}, \sqrt{\mu}]$ and the coordinate $y$ is non-compact lying in $[\sqrt{\nu}, \infty)$. The metric is singular at $\tilde{y} = \tilde{x}$, since we have

$$\text{Riem}^2 = \frac{384(\mu - \nu)^2}{(\tilde{y} - \tilde{x})^6},$$ \hfill (23)

but this curvature singularity can be avoided by requiring $\mu < \nu$.

The complex vielbein is given by

$$e^1 = e^{i\psi} \left[ \sqrt{\frac{\tilde{y} - \tilde{x}}{4 \Delta_x}} d\tilde{x} + i \sqrt{\frac{\Delta_x}{\tilde{y} - \tilde{x}}} (d\psi - \tilde{y} d\phi) \right],$$
\[ \epsilon^2 = e^{i\psi} \left[ \sqrt{\frac{y - \bar{x}}{4\Delta_y}} \, dy + i \sqrt{\frac{\Delta_y}{y - \bar{x}}} \, (d\psi - \bar{x} d\phi) \right]. \] (24)

Note that the complex vielbein is defined up to a local \( SU(2) \) transformation \( \epsilon^a \rightarrow U^a_b \epsilon^b \). If we take
\[ U = \frac{1}{\sqrt{\Delta_x + \Delta_y}} \begin{pmatrix} e^{-i\psi} & 0 \\ 0 & e^{i\psi} \end{pmatrix} \begin{pmatrix} \sqrt{\Delta_y} & -\sqrt{\Delta_x} \\ \sqrt{\Delta_x} & \sqrt{\Delta_y} \end{pmatrix}, \] (25)
then we get an expression as following
\[ \epsilon^1 = \sqrt{(\bar{y} - \bar{x}) \Delta_x \Delta_y} \left[ \Delta_x \Delta_y \right] \left[ d\bar{x} + d\bar{y} \right] + i \left[ \Delta_x + \Delta_y \right] \left[ d\psi - \bar{x} \Delta_y \Delta_x \right] d\phi, \] (26)
This form can be directly related to our initial ansatz, namely
\[ g^2 = \frac{1}{4} \Delta_x \Delta_y, \quad y = \frac{\bar{x} + \bar{y}}{4}, \quad \alpha = 2\psi, \quad z = x_1 + ix_2, \quad x_2 = \phi, \]
\[ x_1 = -\frac{1}{2\sqrt{\mu}} \text{arctanh} \left( \frac{\bar{x}}{\sqrt{\mu}} \right) - \frac{1}{2\sqrt{\nu}} \text{arccoth} \left( \frac{\bar{y}}{\sqrt{\nu}} \right). \] (27)
It is now straightforward to verify that \( g^2 \) satisfies \( (29) \).

We now consider the second case, corresponding to \( \kappa = 0 \). The solution is determined solely by the following basic equation for \( G \)
\[ \partial_y^2 G + 4(1 + G_{11}) = 0. \] (28)
Note that this equation implies that \( (G + 2y^2) \) is a harmonic function of the flat three-dimensional space \( ds^2 = dy^2 + dzd\bar{z} \). Thus \( G \) can be solved completely, giving rise to the metric
\[ ds^2 = f^2 \, dzd\bar{z} + f^2 \, dy^2 + f^{-2} \left[ d\alpha - \frac{i}{4} (dz\partial_z - d\bar{z}\partial_{\bar{z}})\partial_y G \right]^2 \]
\[ = f^2 (dx_1^2 + dx_2^2 + dx_3^2) + f^{-2} \left[ d\alpha + \frac{1}{4} (dx_2\partial_{x_1} - dx_1\partial_{x_2})\partial_{x_3} G \right]^2, \] (29)
where \( z = x_1 + ix_2 \) and \( x_3 \equiv y \). This is exactly the Gibbons-Hawking instanton
\[ ds^2 = V^{-1} (d\alpha + A_i dx_i)^2 + V \, dx_i dx_i, \] (30)
with \( V = f^2 \), and the gauge fixing of \( A_3 = 0 \) by redefinition of \( \alpha \). It is straightforward to verify that
\[ \partial_i \partial_i V = 0, \quad \partial_i V = \epsilon_{ijk} \partial_j A_k. \] (31)
Thus the most general solution for the second case is the Gibbons-Hawking instanton.
3.2 $p = 2$

The general formalism and many examples of $p = 2$ was discussed in [4]. Here we shall demonstrate that a non-trivial example of cohomogeneity-two Calabi-Yau metric can be put into the form of the general ansatz. The local metric is given by [11, 12]

$$ds^2_6 = \frac{\tilde{x} + \tilde{y}}{4\Delta_x} d\tilde{x}^2 + \frac{\Delta_x}{\tilde{x} + \tilde{y}} (d\tau + \frac{\tilde{y}}{2\tilde{\alpha}} \sigma_3)^2 + \frac{\tilde{x} + \tilde{y}}{4\Delta_y} (d\tau - \frac{\tilde{x}}{2\tilde{\alpha}} \sigma_3)^2 + \frac{\tilde{x} \tilde{y}}{4\tilde{\alpha}} (\sigma_1^2 + \sigma_2^2).$$ (32)

where

$$\Delta_x = \tilde{x}(\tilde{x} + \tilde{\alpha}) - \frac{2\mu}{\tilde{x}}, \quad \Delta_y = \tilde{y}(\tilde{\alpha} - \tilde{y}) + \frac{2\nu}{\tilde{y}}.$$ (33)

It is analogous to the $D = 4$ Plebanski metric. It can be viewed [13] as the partial resolution of the $Y^{p,q}$ spaces [14].

The complex vielbein is given by

$$\begin{align*}
e^1 &= e^{i\tau} \sqrt{\frac{\tilde{x} \tilde{y}}{4\tilde{\alpha}}} (\sigma_1 + i \sigma_2), \\
e^2 &= e^{i\tau} \left[ \sqrt{\frac{\tilde{y} + \tilde{x}}{4\Delta_x}} d\tilde{x} + i \sqrt{\frac{\Delta_x}{\tilde{y} + \tilde{x}}} (d\tau + \frac{\tilde{y}}{2\tilde{\alpha}} \sigma_3) \right], \\
e^3 &= e^{i\tau} \left[ \sqrt{\frac{\tilde{y} + \tilde{x}}{4\Delta_y}} d\tilde{y} - i \sqrt{\frac{\Delta_y}{\tilde{y} + \tilde{x}}} (d\tau - \frac{\tilde{x}}{2\tilde{\alpha}} \sigma_3) \right].
\end{align*}$$ (34)

Note that the complex vielbein is defined up to a local $SU(3)$ transformation $e^a \to U^a_b e^b$. If we take

$$U = \frac{1}{\sqrt{\Delta_x + \Delta_y}} \begin{pmatrix} e^{-i\tau} & 0 & 0 \\
0 & e^{-i\tau} & 0 \\
0 & 0 & e^{2i\tau} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\
0 & \sqrt{\Delta_y} & \sqrt{\Delta_x} \\
0 & -\sqrt{\Delta_x} & \sqrt{\Delta_y} \end{pmatrix},$$ (35)

then we get an expression as following

$$\begin{align*}
e^1 &= \sqrt{\frac{\tilde{x} \tilde{y}}{4\tilde{\alpha}}} (\sigma_1 + i \sigma_2), \\
e^2 &= \sqrt{\frac{(\tilde{y} + \tilde{x})\Delta_x \Delta_y}{4(\Delta_x + \Delta_y)}} \left( \frac{d\tilde{x}}{\Delta_x} + \frac{d\tilde{y}}{\Delta_y} \right) + \frac{i}{2\tilde{\alpha}} \sqrt{\frac{(\tilde{y} + \tilde{x})\Delta_x \Delta_y}{\Delta_x + \Delta_y}} \sigma_3, \\
e^3 &= -e^{3i\tau} \left[ \sqrt{\frac{\tilde{y} + \tilde{x}}{4(\Delta_x + \Delta_y)}} (d\tilde{x} - d\tilde{y}) + i \sqrt{\frac{\Delta_x + \Delta_y}{\tilde{y} + \tilde{x}}} (d\tau - \frac{\tilde{x} \Delta_y - \tilde{y} \Delta_x}{2\tilde{\alpha} (\Delta_x + \Delta_y)} \sigma_3) \right].
\end{align*}$$ (36)

It is now straightforward to relate the solution to our ansatz. Making the following identification

$$d\rho = \frac{\bar{\alpha}}{2} \left( \frac{d\tilde{x}}{\Delta_x} + \frac{d\tilde{y}}{\Delta_y} \right), \quad \alpha = 3\tau, \quad y = \frac{\tilde{x} - \tilde{y}}{6},$$

7
\[ g^2 = \frac{\tilde{x}\tilde{y}\Delta_x\Delta_y}{9\tilde{\alpha}^3}, \quad \partial_\rho G + 8y = \frac{2\tilde{x}\tilde{y}}{\tilde{\alpha}}, \]  

we find that \((g, G)\) satisfy

\[ \partial_y g^2 = \frac{1}{8} \partial_\rho \left( (\partial_\rho G + 8y)^2 \right), \]
\[ \partial_y(\partial_\rho G + 8y) + 2\partial_\rho \log g^2 - 8 = 0. \]

It is straightforward now to verify that the basic equation [15] is satisfied. Thus the resolved \(Y^{p,q}\) cone is indeed a class I solution of our basic construction, although it is hard to obtain directly by solving the basic equation.

### 3.3 \( p = 3 \)

With the increasing value of \( p \), the basic equation becomes more and become non-linear and difficult to solve. Here we shall again only demonstrate that a non-trivial example of previously-known Calabi-Yau metric can indeed put into the form of the general ansatz.

The metric is cohomogeneity-2 and given by [11, 12]

\[ ds^2 = \frac{\tilde{x} + \tilde{y}}{4\Delta_x} d\tilde{x}^2 + \frac{\Delta_x}{\tilde{x} + \tilde{y}} \left( d\tau + \frac{\tilde{y}}{2\tilde{\alpha}} (d\beta + \gamma^2 \sigma_3) \right)^2 + \frac{\Delta_y}{\tilde{x} + \tilde{y}} \left( d\tau - \frac{\tilde{x}}{2\tilde{\alpha}} (d\beta + \gamma^2 \sigma_3) \right)^2 \]
\[ + \frac{\tilde{x}\tilde{y}}{\tilde{\alpha}} \left( \frac{d\gamma^2}{V_0} + \frac{1}{4} V_0 \gamma^2 \sigma_3^2 + \frac{1}{4} \gamma^2 (\sigma_1^2 + \sigma_2^2) \right). \]

where

\[ \Delta_x = \tilde{x}(\tilde{x} + \tilde{\alpha}) - \frac{2\mu}{\tilde{x}^2}, \quad \Delta_y = \tilde{y}(\tilde{\alpha} - \tilde{y}) + \frac{2\nu}{\tilde{y}^2}. \]

The complex vielbein is given by

\[ e^1 = e^{i\tau + i\frac{3}{2} \beta} \sqrt{\frac{\tilde{x}\tilde{y}}{\tilde{\alpha}}} \left( \frac{d\gamma}{\sqrt{V_0}} + \frac{i}{2} \gamma \sqrt{V_0} \sigma_3 \right), \]
\[ e^2 = e^{i\tau} \gamma \sqrt{\frac{\tilde{x}\tilde{y}}{4\tilde{\alpha}}} (\sigma_1 + i \sigma_2), \]
\[ e^3 = e^{i\tau} \sqrt{\frac{\tilde{y} + \tilde{x}}{4\Delta_x}} d\tilde{x} + i \sqrt{\frac{\Delta_x}{\tilde{y} + \tilde{x}}} \left( d\tau + \frac{\tilde{y}}{2\tilde{\alpha}} (d\beta + \gamma^2 \sigma_3) \right), \]
\[ e^4 = e^{i\tau} \sqrt{\frac{\tilde{y} + \tilde{x}}{4\Delta_y}} d\tilde{y} - i \sqrt{\frac{\Delta_y}{\tilde{y} + \tilde{x}}} \left( d\tau - \frac{\tilde{x}}{2\tilde{\alpha}} (d\beta + \gamma^2 \sigma_3) \right). \]

Making the following \( SU(4) \) transformation

\[ U = \frac{1}{\sqrt{\Delta_x + \Delta_y}} \begin{pmatrix} e^{-i\tau} & 0 & 0 & 0 \\ 0 & e^{-i\tau} & 0 & 0 \\ 0 & 0 & e^{-i\tau} & 0 \\ 0 & 0 & 0 & e^{3i\tau} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{\Delta_y} & \sqrt{\Delta_x} \\ 0 & 0 & \sqrt{-\Delta_x} & \sqrt{-\Delta_y} \end{pmatrix}, \]
we get an expression as following

\[ \epsilon^1 = e^{\frac{3}{2}i\beta} \sqrt{\frac{\bar{x} y}{\alpha}} \left( \frac{d\gamma}{\sqrt{V_0}} + \frac{i}{2} \gamma \sqrt{V_0} \sigma_3 \right) , \]

\[ \epsilon^2 = \sqrt{\frac{\bar{x} y}{4\alpha}} (\sigma_1 + i \sigma_2) , \]

\[ \epsilon^3 = \sqrt{\frac{\bar{y} + \bar{x}}{4(\Delta_x + \Delta_y)} \left( \frac{d\bar{x}}{\Delta_x} + \frac{d\bar{y}}{\Delta_y} \right) + \frac{i}{2\alpha} \sqrt{\frac{(\bar{y} + \bar{x})\Delta_x \Delta_y}{\Delta_x + \Delta_y}} (d\beta + \gamma^2 \sigma_3) , \] \quad (43)

\[ \epsilon^4 = -e^{4i\tau} \left[ \sqrt{\frac{\bar{y} + \bar{x}}{4(\Delta_x + \Delta_y)}} (d\bar{x} - d\bar{y}) + i \sqrt{\frac{\Delta_x + \Delta_y}{\bar{y} + \bar{x}}} \left( d\tau - \frac{\bar{x} \Delta_y - \bar{y} \Delta_x}{2\alpha (\Delta_x + \Delta_y)} (d\beta + \gamma^2 \sigma_3) \right) \right] . \]

To relate to our original anzatz, we make the following identification

\[ d\rho = \frac{\alpha}{2} \left( \frac{d\bar{x}}{\Delta_x} + \frac{d\bar{y}}{\Delta_y} \right) , \quad \alpha = 4\tau , \quad y = \frac{\bar{x} - \bar{y}}{8} , \]

\[ g^2 = \frac{\bar{x} \bar{y}^2 \Delta_x \Delta_y}{16 \alpha^4} , \quad \partial_\rho G + 12y = \frac{2 \bar{x} \bar{y}}{\alpha} . \] \quad (44)

We also find the following relations

\[ \partial_y g^2 = \frac{1}{24} \partial_\rho \left( (\partial_\rho G + 12y)^3 \right) , \]

\[ \partial_y (\partial_\rho G + 12y) + 2\partial_\rho \log g^2 - 12 = 0 . \] \quad (45)

Then it is easy to verify that \( g \) and \( G \) satisfy the basic equation \((15)\). Thus this non-trivial CY3 metric is indeed a class I solution of our basic construction, although it is hard to obtain directly by solving the basic equation.

### 4 The holomorphic linearization

The main obstacle of solving the basic equations \((15)\) and \((16)\) is their non-linearity. As shown in section 3.1, for \( \kappa = 0 \) and \( p = 1 \), the equation becomes linear and it can be solved completely, giving rise to the Gibbons-Hawking instanton. We are now looking for a subset of CY \( p \)-folds such that the non-linear terms of the resulting basic equations for \( (p+1) \)-folds vanish. We shall focus our attention on \( p = 2 \), for which, when \( \kappa = 0 \), the basic equation is given by

\[ \partial_y^2 G + 4(1 + G_{11} + G_{22} + G_{11}G_{22} - G_{12}G_{21}) = 0 . \] \quad (46)

If we have

\[ G_{11}G_{22} - G_{12}G_{21} = \text{constant} , \] \quad (47)

we shall be left with a linear equation just like the one in the \( p = 1 \) case.
An immediate example is the holomorphic linearization discussed in [6]. (See also [5].) In this case, the function \( G \) is of the form \( G = G(\omega, \bar{\omega}, y) \) where \( \omega = \omega(z_1, z_2) \) is an arbitrary holomorphic function. Then it is easy to show that \( G_{11}G_{22} - G_{12}G_{21} = 0 \) by noting that \( G_{ij} = \omega_i \bar{\omega}_j G_{\omega \bar{\omega}} \), where \( \omega_i = \partial_{z_i} \omega \). The basic equation now takes the form

\[
\partial_y^2 G = -4(1 + G_{11} + G_{22}) = -4 \left( 1 + \sum \bar{z}_i \bar{z}_j \partial_{\theta_i} \partial_{\bar{\theta}_j} G \right) = -4 \left( 1 + \bar{g}^{ij} \partial_{\theta_i} \partial_{\bar{\theta}_j} G \right) = -4 \left( 1 + \Delta_4 G \right) \equiv -4 \left( 1 + \bar{g}^{11} \partial_{\theta_1} \partial_{\bar{\theta}_1} G \right),
\]

where \( \Delta_4 \) is the Laplacian on the four dimensional base and we have used \( G = G(\omega, \bar{\omega}, y) \) at the last step. We can always take a new complex coordinate system \( \{ \bar{z}_1, \bar{z}_2 \} \) where \( \bar{z}_1 = \omega \), and then drop off the tilde. Therefore, the holomorphic linearization is equivalent to take \( G = G(z_1, \bar{z}_1, y) \) in our general construction. In this coordinate system, the basic equation takes the form

\[
\partial_y^2 G = -4 \left( \bar{g}^{11} \partial_{\theta_1} \partial_{\bar{\theta}_1} G \right).
\]

The consequence is that the base CY2 space is now restricted so that we have \( \bar{g}^{11} = \bar{g}^{11}(z_1, \bar{z}_1) \). As shown in section 2, we can impose \( V = 1 \) without loss of generality, for which, we have \( \bar{g}_{22} = \bar{g}^{11} \). It follows from (1) that the Kähler potential for the four-dimensional base must take the following form

\[
K_0 = K_0^{(1)}(z_1, \bar{z}_1) + z_2 \bar{z}_2 K_0^{(2)}(z_1, \bar{z}_1) + \tilde{K}_0^{(3)}(z_1, \bar{z}_1, z_2) + \tilde{K}_0^{(3)}(\bar{z}_1, z_1, \bar{z}_2).
\]

The condition \( V = 1 \) becomes

\[
(\partial_{\theta_1} \partial_{\bar{\theta}_1} K_0^{(1)} + \partial_{\theta_1} \partial_{\bar{\theta}_1} K_0^{(3)} + \partial_{\theta_1} \partial_{\bar{\theta}_1} \tilde{K}_0^{(3)} + z_2 \bar{z}_2 \partial_{\theta_1} \partial_{\bar{\theta}_1} K_0^{(2)}) \tilde{K}_0^{(2)} = (\partial_{\theta_2} \partial_{\bar{\theta}_2} K_0^{(3)} + z_2 \bar{z}_2 \partial_{\theta_1} \partial_{\bar{\theta}_1} \tilde{K}_0^{(3)} + \bar{z}_2 \partial_{\theta_1} \tilde{K}_0^{(2)}) = 1.
\]

Performing \( \partial_2 \partial_2 \) on the both sides, we have

\[
K_0^{(2)} \partial_{\theta_1} \partial_{\bar{\theta}_1} K_0^{(2)} - \partial_{\theta_2} \partial_{\bar{\theta}_2} K_0^{(3)} \partial_{\theta_2} \partial_{\bar{\theta}_2} K_0^{(3)} - \partial_{\theta_1} \tilde{K}_0^{(2)} \partial_{\theta_1} K_0^{(2)} = 0.
\]

It implies that

\[
K_0^{(3)}(z_1, \bar{z}_1, z_2) = z_2^2 k_2(z_1, \bar{z}_1) + z_2 k_1(z_1, \bar{z}_1).
\]

Substituting this into (51), we obtain a polynomial of \( z_2 \) and \( \bar{z}_2 \) that vanishes. The vanishing of the coefficients of all the powers of \( z_2 \) and \( \bar{z}_2 \) implies that

\[
K_0^{(2)} \partial_{\theta_1} \partial_{\bar{\theta}_1} K_0^{(1)} - \partial_{\theta_1} \partial_{\bar{\theta}_1} k_1 = 1,
\]

\[
K_0^{(2)} \partial_{\theta_1} \partial_{\bar{\theta}_1} k_1 - \partial_{\theta_1} K_0^{(2)} \partial_{\bar{\theta}_1} k_1 = 2 \partial_{\theta_1} k_2 \partial_{\bar{\theta}_1} k_1,
\]

\[
K_0^{(2)} \partial_{\theta_1} \partial_{\bar{\theta}_1} k_1 - \partial_{\theta_1} K_0^{(2)} \partial_{\bar{\theta}_1} k_2 = 4 \partial_{\theta_1} k_2 \partial_{\bar{\theta}_1} k_2,
\]

\[
K_0^{(2)} \partial_{\theta_1} \partial_{\bar{\theta}_1} k_2 - 2 \partial_{\theta_1} K_0^{(2)} \partial_{\bar{\theta}_1} k_2 = 0.
\]
One class of solutions is that \(k_2 \equiv 0\), for which, the equation (56) can be solved as follows
\[
K_0^{(2)} = F(z_1)\bar{F}(\bar{z}_1).
\] (58)

The equation (55) then implies that \(\partial_1 k_1/K_0^{(2)}\) is an anti-holomorphic function. It can be shown that up to a holomorphic gauge transformation for the Kähler potential, we can also write \(k_1/K_0^{(2)} = \bar{h}(\bar{z}_1)\). By using the holomorphic coordinate transformation \(z_1 \to z_1\) and \(z_2 \to z_2 - h(z_1)\), we can further set \(K_0^{(3)} = 0\) while preserving our ansatz. By using (54), we find
\[
K_0 = H(z_1)\bar{H}(\bar{z}_1) + z_2\bar{z}_2 F(z_1)\bar{F}(\bar{z}_1), \quad H(z_1) = \int \frac{1}{F(z_1)} dz_1.
\] (59)

Note that the condition \(G = G(z_1, \bar{z}_1, y)\) is equivalent to the condition \(G = G(H, \bar{H}, y)\). Thus we can always perform the transformation \(\tilde{z}_1 = H(z_1)\) and \(\tilde{z}_2 = z_2 F(z_1)\) without violating the ansatz. Therefore, the base space is just the \(\mathbb{R}^4\) and the corresponding CY3-fold is simply the direct product of the Gibbons-Hawking instanton and \(\mathbb{R}^2\). This case was discussed in detail in [6, 5].

We shall now focus our attention on the new case with non-vanishing \(k_2\). The solution to (57) is given by
\[
\partial_1 k_2 = \bar{h}_2(\bar{z}_1)(K_0^{(2)})^2.
\] (60)

By the holomorphic coordinate transformation \(\tilde{z}_1(z_1) = \int h_2(z_1)dz_1\) we can absorb \(h_2\). It is necessary to make a coordinate transformation \(\tilde{z}_2 = z_2/h_2(z_1)\) in order to preserve \(V = 1\). However this has no effect on the equations (54)-(57). Thus this transformation is equivalent to set \(h_2 = 1\). Then (56) and (55) become
\[
\partial_1 \partial_1 \log K_0^{(2)} = 4(K_0^{(2)})^2, \quad \partial_1 \left(\frac{\partial_1 k_1}{K_0^{(2)}}\right) = 2\partial_1 k_1.
\] (61)

Note that \(K_0^{(1)}\) can then be determined by (54).

The first equation of (61) is precisely the two-dimensional Liouville equation. Once the \(K_0^{(2)}\) is solved, the remaining functions follow straightforwardly. Thus, the corresponding CY3 space is governed in essence by the solutions of the Liouville equation. The general solutions to the Liouville equation are not known. In [15], many special solutions were given. Here, we examine two examples in detail.

\footnote{The two examples discussed in Sec7.1 & 7.2 of [6] are indeed the same one. The solution in section 7.2 of [6] corresponds to take \(F = \sqrt{z_1}\) in our above discussion.}
The first example of the special solutions of the Liouville equation is given by \[15\]

\[
(K_0^{(2)})^2 = \frac{F''(z_1)\bar{F}'(\bar{z}_1)}{4(F(z_1) + F(\bar{z}_1))^2}.
\] (62)

(Note that in \[15\] the solution is more general in that the \(\bar{F}\) is replaced by an unrelated anti-holomorphic function to \(F\). Here we chose it to be \(\bar{F}\) so that \(K_0^{(2)}\) is real.) Consequently we have

\[
k_2 = -\frac{F'}{4(F + \bar{F})}.
\] (63)

Up to a gauge transformation of the Kähler potential, the second equation of \[64\] implies

\[
\partial_1 k_1 = 2K_0^{(2)}\bar{K}_1 = \frac{(F'\bar{F}')^{\frac{3}{2}}}{F + \bar{F}k_1}.
\] (64)

It can be shown further that we can always fix \(k_1 = 0\) by choosing proper complex coordinates. Thus we have

\[
\partial_1 \partial_1 K_0^{(1)} = \frac{2(F + \bar{F})}{(F'\bar{F}')^{\frac{3}{2}}},
\]

\[
z_2\bar{z}_2 K_0^{(2)}(z_1, \bar{z}_1) + K_0^{(3)}(z_1, \bar{z}_1, z_2) + \bar{K}_0^{(3)}(\bar{z}_1, z_1, \bar{z}_2) = \frac{\bar{z}_2 z_2 (F'\bar{F}')^{\frac{3}{2}}}{2(F + \bar{F})} - \frac{\bar{z}_2^2 F' + \bar{z}_2 \bar{F}'}{4(F + \bar{F})}.
\] (65)

Making the coordinate transformation \(\tilde{z}_1(z_1) = \int F^{\prime-\frac{1}{2}}dz_1\) and \(\tilde{z}_2 = z_2 F^{\prime\frac{1}{2}}\), and then dropping off the tildes, the Kähler potential becomes

\[
K_0 = 2(H_1 + \bar{H}_1)(z_1 + \bar{z}_1) - \frac{(z_2 - \bar{z}_2)^2}{4(F + \bar{F})}, \quad H_1(z_1) = \int Fdz_1.
\] (66)

The corresponding CY2 base is given by

\[
ds^2 = \left[\frac{2(F + \bar{F})}{(F'\bar{F}')^{\frac{3}{2}}} - \frac{(z_2 - \bar{z}_2)^2 F'\bar{F}'}{2(F + \bar{F})^3}\right]dz_1 d\bar{z}_1 + \frac{1}{2(F + \bar{F})}dz_2 d\bar{z}_2
\]

\[-\frac{(z_2 - \bar{z}_2)F'}{2(F + \bar{F})^2}dz_1 d\bar{z}_2 + \frac{(z_2 - \bar{z}_2)\bar{F}'}{2(F + \bar{F})^2}dz_2 dz_1.
\] (67)

There is a curvature singularity at \(F + \bar{F} = 0\). The proper complex vielbein is given by

\[
\tilde{e}^1 = \frac{\sqrt{2}(F + \bar{F})^\frac{3}{2}}{2(F + \bar{F})^\frac{3}{2}}dz_1,
\]

\[
\tilde{e}^2 = -\frac{(z_2 - \bar{z}_2)F'}{\sqrt{2}(F + \bar{F})^\frac{3}{2}}dz_1 + \frac{1}{\sqrt{2}(F + \bar{F})^\frac{3}{2}}dz_2.
\] (68)

The metric \(67\) is in fact a special case of the Gibbons-Hawking solutions. This can be seen from the fact that the metric has a Killing direction \((z_2 + \bar{z}_2)\). To see this in detail, let \(F(z_1) = u + iv, z_2 = \alpha + i\omega\) and \(\bar{y} = \frac{\alpha}{i\omega}\). Then we find

\[
\tilde{e}^1 = 2u^\frac{3}{2}(dx_1 + i dx_2),
\]

\[
\tilde{e}^2 = -i(v - \bar{y})^\frac{3}{2}dz_1 + \frac{1}{i(v - \bar{y})^\frac{3}{2}}dz_2.
\]
by the holomorphic function $\hat{G}$ is determined by the property of the two-dimensional K"ahler space, and in particular is given by

$$
\hat{G} = -8\hat{y}u^2 - z_1\bar{z}_1 + 2\int u^2 dz_1 d\bar{z}_1 .
$$

Having obtained the CY2 base, we now only need to solve the function $G$ in order to obtain the corresponding CY3. By construction, the basic equation for $G$ is linear and given by

$$
\partial^2_y G + 4 + 2(F + \bar{F})^{-1}\partial_1\partial_1 G = 0 .
$$

This means that $(G + 2y^2)$ is the harmonic function on the space

$$
ds^2_3 = dy^2 + 2(F + \bar{F})dz_1 d\bar{z}_1 .
$$

Note that the three-space is a direct product of $\mathbb{R}$ associated with the coordinate $y$ and a two-dimensional K"ahler space associated with the coordiantes $(z_1, \bar{z}_1)$. The K"ahler potential is given by $K^{(1)}_0$, i.e. it is given by the first term of $K_0$ given in (66). Thus the nature of the CY3 is determined by the property of the two-dimensional K"ahler space, and in particular by the holomorphically function $F$. The CY3 metric is given by

$$
\begin{align*}
\hat{e}^2 & = -\frac{2i}{\sqrt{2}(2u)^{\frac{3}{2}}} (du + i dv) + \frac{1}{\sqrt{2}(2u)^{\frac{3}{2}}} (d\alpha + i dw) \\
& = \frac{1}{2u^2} (d\alpha + \frac{1}{2u} dv) + i 2u^2 dy .
\end{align*}
$$

The metric corresponds to the following solution of (28) for the function $\hat{G}$

$$
\hat{G} = -8\hat{y}u^2 - z_1\bar{z}_1 + 2\int u^2 dz_1 d\bar{z}_1 .
$$

We can rewrite the new CY3 metric in the following form

$$
\begin{align*}
\Delta_3 V &= 0 , \\
dV &= *_3 dA .
\end{align*}
$$

Note that in (73) the component $A_3$ is fixed to be zero by appropriate shifting of the $\alpha$ coordinate.

The metric (74) is the six-dimensional analogue of Gibbons-Hawking instanton. It can also be viewed as an $\mathbb{R}^2$ bundle over a four-space $X_4$, where the bundle coordinates are
$(z_2, \bar{z}_2)$ and the base-space coordinates are $(\alpha, y, z_1, \bar{z}_1)$. The base space $X_4$ depends on two functions. One is an arbitrary holomorphic function $F$ of coordinate $z_1$. The other is the harmonic function $V$ on a generically non-flat three-dimensional space with the metric $ds_3^2$. When $F$ is a constant, the base metric is the Gibbons-Hawking instanton, and corresponding CY3 is the instanton appended by an $\mathbb{R}^2$.

For non-constant $F$, as a simple demonstrative example, we may take $F = z_1$, the analysis of $z_1 + \bar{z}_1=$constant surface suggests that this CY2 metric describes the $a \to \infty$ limit of the Eguchi-Hanson space. The basic equation (49) becomes

$$\partial_y^2 G + 4 + 2(z_1 + \bar{z}_1)^{-1}\partial_1 \partial_\bar{1} G = 0. \tag{75}$$

The general solution is given by

$$G = -2y^2 + \sum_{p_0, p_2} e^{i(p_0 y + p_2 x_2)} \left( \lambda^{(1)}_{p_0, p_2} \text{Ai}(u) + \lambda^{(2)}_{p_0, p_2} \text{Bi}(u) \right),$$

$$u = p_0^4 x_1 + p_2^2 p_0 - 4, \quad z_1 = x_1 + i x_2, \tag{76}$$

where $\text{Ai}(u)$ and $\text{Bi}(u)$ are the Airy functions.

Another example of the special solutions to the Liouville equation is given by \[15\]

$$(K_0^{(2)})^2 = \frac{F'(z_1)\bar{F}'(\bar{z}_1)}{4(1 - F(z_1)\bar{F}(\bar{z}_1))^2}. \tag{77}$$

Then we have

$$k_2 = \frac{F'\bar{F}}{4(1 - FF)}. \tag{78}$$

Up to a gauge transformation of the Kähler potential, the second equation of \[61\] implies

$$\partial_1 k_1 = 2k_0^{(2)} \bar{k}_1 = \frac{(F'\bar{F})^{\frac{1}{2}}}{1 - FF} \bar{k}_1. \tag{79}$$

We take a simple solution of $k_1 = 0$, it follows that we have

$$\partial_1 \partial_\bar{1} K_0^{(1)} = \frac{2(1 - F\bar{F})}{(F'\bar{F})^{\frac{1}{2}}}, \tag{80}$$

$$z_2 \bar{z}_2 K_0^{(2)}(z_1, \bar{z}_1) + K_0^{(3)}(z_1, \bar{z}_1, z_2) + \bar{K}_0^{(3)}(\bar{z}_1, z_1, \bar{z}_2) = \frac{z_2 \bar{z}_2 (F'\bar{F})^{\frac{1}{2}}}{2(1 - FF)} + \frac{z_2^2 F'\bar{F} + \bar{z}_2^2 F\bar{F}'}{4(1 - FF)}. \tag{81}$$

After the coordinate transformation $\tilde{z}_1(z_1) = \int F^{-\frac{1}{2}} dz_1$ and $\bar{z}_2 = z_2 F'^{\frac{1}{2}}$ and dropping off the tildes afterwards, the Kähler potential becomes

$$K_0 = 2\left( z_1 \bar{z}_1 - H_1 \bar{H}_1 \right) + \frac{z_2^2 \bar{F} + \bar{z}_2^2 F + 2z_2 \bar{z}_2}{4(1 - FF)}, \quad H_1(z_1) = \int F dz_1. \tag{81}$$
The corresponding CY2 base is given by

\[
\begin{align*}
    ds^2 &= \left[ 2(1 - FF) + \frac{z_2^{2} F + z_2\bar{z}_2(1 + FF)}{2(1 - FF)^{3}} F'\bar{F}' \right] dz_1 d\bar{z}_1 + \frac{1}{2(1 - FF)} dz_2 d\bar{z}_2 \\
    &= \frac{z_2\bar{z}_2}{2(1 - FF)^2} F'dz_1 d\bar{z}_2 + \frac{z_2 + z_2\bar{F}}{2(1 - FF)^2} F'dz_2 d\bar{z}_1.
\end{align*}
\] (82)

There is a curvature singularity at \( F\bar{F} = 1 \). The proper complex vielbein is given by

\[
\begin{align*}
    \tilde{e}^1 &= \sqrt{2} \left( 1 - FF \right)^{\frac{1}{2}} dz_1, \\
    \tilde{e}^2 &= \frac{(z_2\bar{F} + z_2 F')}{\sqrt{2} (1 - FF)^{\frac{1}{2}}} dz_2 + \frac{1}{\sqrt{2} (1 - FF)^{\frac{1}{2}}} dz_2.
\end{align*}
\] (83)

The metric (82) does not have a Killing direction and hence lies outside the construction of \( p = 1 \) case discussed in section 3. To obtain the corresponding CY3 solution, we need to solve the basic equation

\[
\partial_y^2 G + 4 + 2(1 - FF)^{-1} \partial_1 \partial_1 G = 0.
\] (84)

It means that \((G + 2y^2)\) is the harmonic function on the space

\[
ds_3^2 = dy^2 + 2(1 - FF) d\bar{z}_1 d\bar{z}_2.
\] (85)

Again this metric is a direct product of \( \mathbb{R} \) and a two-dimensional Kähler space whose Kähler potential is given by the first term of \( K_0 \) given by (81). The corresponding CY3 metric is given by

\[
\begin{align*}
    ds^2 &= f^2 ds_3^2 + f^{-2} \left[ d\alpha + \frac{1}{4} (dx_2 \partial_\alpha - dx_1 \partial_\alpha - dx_2 \partial_\alpha) \partial_\alpha G \right]^2 \\
    &\quad + \frac{z_2^{2} F + z_2\bar{z}_2(1 + FF)}{2(1 - FF)^{3}} F'\bar{F}' dz_1 d\bar{z}_1 + \frac{1}{2(1 - FF)} dz_2 d\bar{z}_2 \\
    &\quad + \frac{z_2\bar{z}_2}{2(1 - FF)^2} F' dz_1 d\bar{z}_2 + \frac{z_2 + z_2\bar{F}}{2(1 - FF)^2} F' d\bar{z}_2 d\bar{z}_1, \\
    f^2 &= 1 + \frac{1}{2(1 - FF)} \partial_1 \partial_1 G = -\frac{1}{4} \partial_y^2 G.
\end{align*}
\] (86)

As in the previous example, we can rewrite the metric in the following form

\[
\begin{align*}
    ds^2 &= V^{-1} (d\alpha + A_1 dx_1)^2 + V ds_3^2 \\
    &\quad + \frac{z_2^{2} F + z_2\bar{z}_2(1 + FF)}{2(1 - FF)^{3}} F'\bar{F}' dz_1 d\bar{z}_1 + \frac{1}{2(1 - FF)} dz_2 d\bar{z}_2 \\
    &\quad + \frac{z_2\bar{z}_2}{2(1 - FF)^2} F' dz_1 d\bar{z}_2 + \frac{z_2 + z_2\bar{F}}{2(1 - FF)^2} F' d\bar{z}_2 d\bar{z}_1, \\
    \Delta_3 V &= 0, \\
    dV &= *_3 dA.
\end{align*}
\] (87)

Analogous to the previous example, the metric is a generalization of the Gibbons-Hawking instanton. The metrics depends on two functions \( F \) and \( V \). \( F \) is an arbitrary holomorphic
function and \( V \) is the harmonic function in \( ds^2 \). For constant \( F \), the metric is a direct product of Gibbons-Hawking instanton and \( \mathbb{R}^2 \). For non-constant \( F \), as a demonstrative example, we consider \( F = z_1 \). Then we have

\[
G = -2y^2 + \sum_{p_0, p_2} e^{i(p_0 y + p_2 \theta)} 2 e^{p_2} r e^{-i\theta} \left( \lambda^{(1)}_{p_0, p_2} W_1(p_0, p_2, r) + \lambda^{(2)}_{p_0, p_2} W_2(p_0, p_1, r) \right),
\]

\[
W_1(p_1, p_2, r) = U \left( 4(p_2 + 1) - i\sqrt{2}p_0, p_2 + 1, \frac{ip_0 r^2}{\sqrt{2}} \right),
\]

\[
W_2(p_1, p_2, r) = L_{p_2}^2 - 4(p_2 + 1) + i\sqrt{2}p_0 \left( \frac{ip_0 r^2}{\sqrt{2}} \right), \quad z_1 = re^{i\theta},
\]

where \( U \) is the confluent hypergeometric function and \( L \) is the generalized Laguerre polynomial.

It turns out that the above two examples can be unified to give rise to a more general solution. The metric becomes more complicated, but the essential properties remain the same. We present the detail discussion in the appendix.

There are still some other special solutions of the Liouville equation which are known as functional separable solutions. The explicit forms are \([15]\)

\[
(K_0^{(2)})^2 = \frac{1}{(c_1 e^{x_1} \pm 4 \cos x_2)^2},
\]

\[
(K_0^{(2)})^2 = \frac{c_2^2 - c_1^2}{16(c_1 \cosh x_1 + c_2 \sin x_2)^2},
\]

\[
(K_0^{(2)})^2 = \frac{c_2^2 + c_1^2}{16(c_1 \sinh x_1 + c_2 \cos x_2)^2}.
\]

where \( c_1 \) and \( c_2 \) are arbitrary constants and \( x_1 + ix_2 = z_1 \). These Liouville solutions will lead to different CY2 bases and CY3 metrics. We shall not analyze them further.

### 5 Conclusions

In this paper, we establish the general formalism for constructing metrics of Calabi-Yau \((p + 1)\)-folds in terms of that of a \( p \)-fold by adding a complex-line bundle. We present a few explicit low-lying examples. The metrics are determined by the basic equations for the function \( G \), given by \([15]\) or \([16]\). The obstacle to solve these equations is the higher non-linearity for higher \( p \). For \( p = 1 \), the equation for \( G \) in \([16]\) becomes linear and the resulting solution is the Gibbons-Hawking instanton.

For \( p = 2 \), we consider holomorphic linearization and focus on subset of solutions where the non-linear terms in \([16]\) vanish. This restricts the possible CY2 base spaces, which turn out to be governed by a two-dimensional Liouville equation. The Kähler potential for
the CY3 is then given by a harmonic function in certain three-dimensional non-flat spaces, depending on the specific solution of the Liouville equation. We provide detail analysis for two such special solutions of the Liouville equation. In both of these examples, the metric can be viewed as a complex-line bundle over a four-dimensional space $X_4$ which can be viewed as an $U(1)$ bundle over a three dimensional space that is a direct product of $\mathbb{R}$ and a two-dimensional Kähler space. The two-dimensional Kähler space is determined by an arbitrary holomorphic function. When the holomorphic function is a pure constant, the three-space becomes flat and $X_4$ becomes precisely the Gibbons-Hawking instanton. Alternatively, by our construction, these solutions are complex-line bundle over CY2 bases. In particular, the CY2 metrics (82) and (99) we obtained are highly non-trivial in that they do not have Killing direction and lie outside our construction for $p=1$.

It is of great interest to investigate further our new solutions for generic holomorphic functions and examine the global structure of the resulting CY2 bases and corresponding CY3 metrics. We shall study this in a future publication.

We expect the linearization procedure can be generalized further to higher-dimensions and give rise to the higher-dimensional analogue of Gibbons-Hawking instanton.

**Appendix: more general CY2 and CY3 metrics**

In section 4, we demonstrate that for some special choice of the CY2 base space, the function $G$ is governed by a linear equation. The key property of the base space is that it is determined by the solutions of the two-dimensional Liouville equation, namely

$$\partial_1 \partial \log K_0^{(2)} = 4(K_0^{(2)})^2,$$

(92)

Consulting the mathematics reference book [15], we presented two special solutions and discuss the resulting CY2 and CY3 metrics in details. In this appendix, we show that these two solutions can in fact be unified into one more general solution; it is given by

$$(K_0^{(2)})^2 = \frac{F_1'(z_1)F_2'(\bar{z}_1)}{4} \left(\frac{a}{(1-aF_1(z_1)F_2(\bar{z}_1))^2} + \frac{1-ab^2}{(b+F_1(z_1)+F_2(\bar{z}_1)+abF_1(z_1)F_2(\bar{z}_1))^2}\right)$$

$$= \frac{F_1'F_2'(aF_1^2 + 2abF_1 + 1)(aF_2^2 + 2abF_2 + 1)}{4(1-aF_1F_2)^2(b+F_1+F_2+abF_1F_2+1)^2}.$$  

(93)

In order for the solution to be real, we shall take $a,b$ to be real and $F_1 = F_2 \equiv F$. For $a = 0$, the solution reduces to the special solution (62). When $ab^2 = 1$, we obtain the other special solution (77).

It follows from

$$\partial_1 k_2 = (K_0^{(2)})^2,$$

(94)

17
that we have

\[ k_2 = \frac{a F' F}{4 (1 - a F F)} - \frac{F'(1 + a b F)}{4 (b + F + F + a b F F)} = \frac{F'(2a^2 b F F^2 + a F^2 + 2a F \bar{F} - 1)}{4 (1 - a F F)(b + F + F + a b F F)}. \]  

(95)

Up to a gauge transformation of the Kähler potential, the second equation of (61) implies

\[ \partial_1 k_1 = 2K_0^{(2)} \bar{k}_1. \]  

(96)

We take simplest solution of \( k_1 = 0 \), it follows that we have

\[ \partial_1 \partial_1 K_0^{(1)} = \frac{2}{(F' F)^{\frac{1}{2}} (a F^2 + 2 a b F + 1)^{\frac{1}{2}}(a F^2 + 2 a b F + 1)^{\frac{1}{2}}} \frac{(1 - a F \bar{F})(b + F + \bar{F} + a b F \bar{F})}{2 (1 - a F F)(b + F + F + a b F F)} \]

\[ + \frac{z_2 \bar{z}_2 (F' F')^{\frac{1}{2}} F^2 (2a^2 b F F^2 + a F^2 + 2a F \bar{F} - 1)}{4 (1 - a F F)(b + F + F + a b F F)} + \frac{z_2 \bar{z}_2 (2a^2 b F F^2 + a F^2 + 2a F \bar{F} - 1)}{4 (1 - a F F)(b + F + F + a b F F)} \]  

(97)

After the coordinate transformation \( \tilde{z}_1(z_1) = \int F' \frac{dz_1}{a F^2 + 2 a b F + 1} \frac{1}{(a F^2 + 2 a b F + 1)^{\frac{1}{2}}} \) and dropping off the tildes afterwards, the Kähler potential of the base space becomes

\[ K_0 = 2 H + z_2 \bar{z}_2 \frac{(a F^2 + 2 a b F + 1)^{\frac{1}{2}}}{2(1 - a F F)(b + F + F + a b F F)} + \frac{z_2 \bar{z}_2 (2a^2 b F F^2 + a F^2 + 2a F \bar{F} - 1)}{4 (1 - a F F)(b + F + F + a b F F)} + \frac{z_2 \bar{z}_2 (2a^2 b F F^2 + a F^2 + 2a F \bar{F} - 1)}{4 (1 - a F F)(b + F + F + a b F F)} \]

\[ H(z_1, \bar{z}_1) = \int dz_1 d\bar{z}_1 \frac{(a F^2 + 2 a b F + 1)^{\frac{1}{2}}}{(a F^2 + 2 a b F + 1)^{\frac{1}{2}}(a F^2 + 2 a b F + 1)^{\frac{1}{2}}} \]  

(98)

The corresponding CY2 base is given by

\[ ds^2 = \left[ \frac{(2(1 - a F \bar{F})(b + F + \bar{F} + a b F \bar{F})}{H_2^{\frac{1}{2}} H_2^{\frac{1}{2}}} + \frac{|z_2 H_1 H_2^{\frac{1}{2}} H_2^{\frac{1}{2}} + \bar{z}_2 H_2 H_2^{\frac{1}{2}} H_2^{\frac{1}{2}}|^2}{2(1 - a F F)^3(b + F + F + a b F F)} \right] dz_1 d\bar{z}_1 \]

\[ + \frac{2(1 - a F F)(b + F + F + a b F F)}{H_2^{\frac{1}{2}} H_2^{\frac{1}{2}}} d\bar{z}_2 \]

\[ + \frac{z_2 H_1 H_2^{\frac{1}{2}} H_2^{\frac{1}{2}}}{2(1 - a F F)^3(b + F + F + a b F F)} d F' d z_1 d \bar{z}_2 \]

\[ + \frac{z_2 H_1 H_2^{\frac{1}{2}} H_2^{\frac{1}{2}}}{2(1 - a F F)^3(b + F + F + a b F F)} d F' d z_2 \bar{z}_1. \]  

(99)

where

\[ H_1 = (a F^3 + 3 a b F^2 + 3 F)(a^2 b F^2 + a \bar{F}) + a F^2 + a b \bar{F} + a b^2 - 1, \]

\[ H_2 = a F^2 + 2 a b F + 1. \]  

(100)
The proper complex vielbein is given by

\[
\tilde{e}^1 = \sqrt{2}(1 - a F \bar{F})^{\frac{1}{2}} (b + F + \bar{F} + a b F \bar{F})^{\frac{1}{2}} dz_1,
\]

\[
\tilde{e}^2 = \frac{(z_2 H_1 H_2^{-1} H_2^2 + z_2 H_2^2 H_2^{-1} F')}{\sqrt{2}(1 - a F \bar{F})^{\frac{1}{2}} (b + F + \bar{F} + a b F \bar{F})^{\frac{1}{2}}} dz_1
\]

\[
+ \frac{H_2^\frac{1}{2} H_2^{-\frac{1}{2}}}{\sqrt{2}(1 - a F \bar{F})^{\frac{1}{2}} (b + F + \bar{F} + a b F \bar{F})^{\frac{1}{2}}} dz_2.
\]

To obtain the corresponding CY3 solution, we need to solve the basic equation

\[
\partial_y^2 G + 4 + \frac{2 H_2^\frac{1}{2} H_2^{-\frac{1}{2}}}{(1 - a FF)(b + F + F + a F \bar{F})} \partial_1 \partial_1 G = 0.
\]

By construction, the equation is linear. In fact, it implies that \((G + 2y^2)\) is the harmonic function on the space

\[
d s_3^2 = dy^2 + \frac{2(1 - a F \bar{F})(b + F + \bar{F} + a b F \bar{F})}{H_2^\frac{1}{2} H_2^{-\frac{1}{2}}} dz_1 d\bar{z}_1,
\]

which is a direct product of \(\mathbb{R}\) and a Kähler 2-space. The corresponding CY3 metric is given by

\[
d s^2 = f^2 d s_3^2 + f^{-2} \left[ d\alpha + \frac{1}{4} (dx_2 \partial_2 z_2 - dx_1 \partial_1 z_2) \partial_2 x_3 G \right]^2 + \tilde{e}^2 \tilde{e}'^2,
\]

\[
f^2 = 1 + \frac{H_2^\frac{1}{2} H_2^{-\frac{1}{2}}}{2(1 - a F \bar{F})(b + F + F + a F \bar{F})} \partial_1 \partial_1 G = -\frac{1}{4} \partial_y^2 G.
\]

As in the previous special examples, we can rewrite the metric in the following form

\[
d s^2 = V^{-1} (d\alpha + A_0 dx_1)^2 + V d s_3^2 + \tilde{e}^2 \tilde{e}'^2,
\]

\[
\quad \quad \triangle_3 V = 0, \quad dV = \ast_3 dA.
\]

Written in this form, the metric is complex line bundle over \(\mathcal{X}_4\), where \(\mathcal{X}_4\) is a \(U(1)\) bundle over \(d s_3^2\), which is a direct product \(\mathbb{R}\) associated with the coordinate \(y\) and the two-dimensional Kähler space associated with coordinates \((z_1, \bar{z}_1)\). When \(d s_3^2\) becomes flat, \(\mathcal{X}_4\) describes the Gibbons-Hawking instanton. This decomposition of the CY3 metric is different from the original construction, where the metric is viewed as a complex line bundle, associated with the coordinate \((y, \alpha)\), over the CY2 base metric, given by (99). The CY2 metric (99) is highly non-trivial in that it has no Killing vector and depends on all the coordinates; it thus lies outside of our construction for \(p = 1\).
References

[1] E. Calabi, *The space of Kähler metrics*, Proc. Internat. Congress Math. Amsterdam, (1954) 206.

[2] E. Calabi, *On Kähler manifolds with vanishing canonical class*, Algebraic geometry and topology: A symposium in honor of S. Lefschetz, Princeton University Press, (1957) 78.

[3] S.T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I*, Communications on Pure and Applied Mathematics 31 (3) (1978) 339.

[4] H. Lü, Y. Pang and Z.L. Wang, *Constructing Calabi-Yau Metrics From Hyperkähler Spaces*, [arXiv:0911.1385 [hep-th]].

[5] O.P. Santillan, *New non compact Calabi-Yau metrics in D = 6*, [arXiv:0909.1718 [hep-th]].

[6] A. Fayyazuddin, *Calabi-Yau 3-folds from 2-folds*, Class. Quant. Grav. 24, 3151 (2007) [arXiv:hep-th/0702135].

[7] M. Cvetič, H. Lü, D.N. Page and C.N. Pope, *New Einstein-Sasaki spaces in five and higher dimensions*, Phys. Rev. Lett. 95, 071101 (2005) [arXiv:hep-th/0504225].

[8] D. Martelli and J. Sparks, *Toric Sasaki-Einstein metrics on $S^2 \times S^3$*, Phys. Lett. B 621, 208 (2005) [arXiv:hep-th/0505027].

[9] M. Cvetič, H. Lü, D.N. Page and C.N. Pope, *New Einstein-Sasaki and Einstein spaces from Kerr-de Sitter*, JHEP 0907, 082 (2009) [arXiv:hep-th/0505223].

[10] J.F. Plebanski, *A class of solutions of Einstein-Maxwell equations*, Ann. Phys. 90, 196 (1975).

[11] W. Chen, H. Lü and C.N. Pope, *Kerr-de Sitter black holes with NUT charges*, Nucl. Phys. B 762, 38 (2007) [arXiv:hep-th/0601002].

[12] W. Chen, H. Lü and C.N. Pope, *General Kerr-NUT-AdS metrics in all dimensions*, Class. Quant. Grav. 23, 5323 (2006) [arXiv:hep-th/0604125].

[13] H. Lü and C.N. Pope, *Resolutions of cones over Einstein-Sasaki spaces*, Nucl. Phys. B 782, 171 (2007) [arXiv:hep-th/0605222].
[14] J.P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, *Sasaki-Einstein metrics on $S^2 \times S^3$*, Adv. Theor. Math. Phys. 8, 711 (2004) [arXiv:hep-th/0403002].

[15] A. Polyanin, V. F. Zaitsev, *Handbook of Nonlinear Partial Differential Equations*, Chapman & Hall/CRC (2003).