Collision of rarefaction waves in Bose-Einstein condensates

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We consider the problem of expansion of Bose-Einstein condensate released from a box. On the contrary to the standard situation of release from a harmonic trap, in this case the dynamics is complicated by a process of collision of two rarefaction waves propagating to the center of the initially uniform distribution. Complete analytical solution of this problem is obtained by Riemann method in hydrodynamic dispersionless approximation and the results are compared with the exact numerical solution of the Gross-Pitaevskii equation.

PACS numbers: 03.75.-b, 67.85.-d, 67.85.De

I. INTRODUCTION

One of the basic problems in dynamics of Bose-Einstein condensates (BECs) is that of expansion after its release from a trap, because in many experimental situations measurements are performed at the state of BEC’s inertial expansion which properties are predetermined to much extend by the initial stage of evolution. The simplest approach to this problem was formulated and studied in Refs. [1,2] in hydrodynamic approximation for the case of harmonic traps when the initial state of BEC is described accurately enough by the Thomas-Fermi distribution. More detailed study of this problem with the use of classical approach of Talanov [3] was given in Refs. [4,5]. These solutions were self-similar and at every moment of time the space distribution of the density had the parabolic Thomas-Fermi time-dependent form. However, if the trap is not harmonic, then the expansion is not self-similar anymore and some characteristic features of the initial distribution can persist for quite long period of evolution to be noticeable experimentally. For example, this happens after release of BEC from a box-like trap with a uniform potential realized experimentally in Refs. [6,7] where the initial size 2l of a box plays the role of the parameter which determines the expansion dynamics for time $t \sim l/c_0$, where $c_0$ is the sound velocity at the initial uniform state of BEC. Indeed, the evolution begins with propagation of two rarefaction waves from the edges of BEC and these two waves collide at the center of the initial distribution at the moment $t_x = l/c_0$ after which the distribution of the density acquires quite complicated form different from parabolic self-similar distributions known from Refs. [1,2,4,5]. The aim of this paper is to study such an evolution in hydrodynamic approximation and to reveal its characteristic features. To solve this problem analytically, we use the powerful Riemann method developed in compressible fluid dynamics with quite general equation of state (see, e.g., [8,9]) which seems most suitable in BEC’s hydrodynamics case with its non-standard “adiabatic index” $\gamma = 2$ (see, e.g., Ref. [10], where the wave breaking problem was considered by this method in similar nonlinear optics context).

II. FORMULATION OF THE PROBLEM

To demonstrate specific features of evolution of BEC expansion after its release from a box-like trap, we consider one-dimensional situation where the dynamics is governed by the Gross-Pitaevskii (GP) equation

$$i\psi_t + \frac{1}{2}\psi_{xx} - |\psi|^2 \psi = 0, \quad (1)$$

written here in standard non-dimensional variables. Transition from the BEC wave function $\psi$ to more convenient in hydrodynamics variables density $\rho$ and flow velocity $u$ is performed by means of the substitution

$$\psi(x,t) = \sqrt{\rho(x,t)} \exp \left( i \int u(x',t) dx' \right), \quad (2)$$

so that the GP equation is cast to the system

$$\rho_t + (\rho u)_x = 0, \quad (3)$$

$$u_t + uu_x + \rho_x + \left[ \frac{\rho_x^2}{8\rho^2} - \frac{\rho_{xx}}{4\rho} \right] x = 0.$$  

The last term in the second equation describes the dispersive effects and in the hydrodynamic approximation it can be neglected since we consider evolution of BEC cloud with mainly smooth enough dependence of $\rho$ and $u$ on the space coordinate $x$. As a result, we arrive at the so-called “shallow water” equations

$$\rho_t + (\rho u)_x = 0, \quad u_t + uu_x + \rho_x = 0. \quad (4)$$

At the initial moment of time the distribution of density is uniform within the interval $-l \leq x \leq l$,

$$\rho(x,0) = \begin{cases} \rho_0, & |x| \leq l, \\ 0, & |x| > l. \end{cases} \quad (5)$$

Although this distribution cannot be considered as “smooth”, we shall show later by comparison of hydrodynamic approximation with the exact numerical solution of the GP equation (1), that if $l \gg 1$ (that if the size of the trap is much greater than the healing length), then...
deviations of the exact solution from its hydrodynamic approximation is negligibly small almost everywhere except small regions at the boundaries of the BEC cloud with vacuum.

Since the Riemann method is not commonly used in theoretical physics, we shall provide in the next section the relevant basic information about it.

III. RIEMANN METHOD

For future convenience, we consider the compressible fluid dynamics equations with adiabatic equation of state, \( p = \rho^\gamma / \gamma \), where \( p \) denotes the pressure in the gas,

\[
\rho_t + (\rho u)_x = 0, \quad u_t + uu_x + \rho^{\gamma-2} \rho_x = 0,
\]

so that Eqs. (4) are reproduced for \( \gamma = 2 \). These equations can be cast into diagonal Riemann form by introduction of new variables, namely Riemann invariants

\[
r_\pm = \frac{u}{2} \pm \frac{1}{\gamma - 1} \rho^{x_{\pm}},
\]

for which we get the equations

\[
\frac{\partial r_+}{\partial t} + v_+(r_+,r_-) \frac{\partial r_+}{\partial x} = 0,
\]

\[
\frac{\partial r_-}{\partial t} + v_-(r_+,r_-) \frac{\partial r_-}{\partial x} = 0,
\]

where

\[
v_+ = \frac{1}{2} [(1 + \gamma)r_+ + (3 - \gamma)r_-],
\]

\[
v_- = \frac{1}{2} [(3 - \gamma)r_+ + (1 + \gamma)r_-].
\]

Riemann noticed that Eqs. (8) become linear with respect to the dependent variables if one considers \( x \) and \( t \) as functions of the Riemann invariants, \( x = x(r_+,r_-), t = t(r_+,r_-) \), and after this "hodograph transform" we arrive at the system

\[
\frac{\partial x}{\partial r_+} - v_+(r_+,r_-) \frac{\partial t}{\partial r_+} = 0,
\]

\[
\frac{\partial x}{\partial r_-} - v_-(r_+,r_-) \frac{\partial t}{\partial r_-} = 0.
\]

We look for the solution of this system in the form

\[
x - v_+(r_+,r_-)t = w_+(r_+,r_-),
\]

\[
x - v_-(r_+,r_-)t = w_-(r_+,r_-).
\]

Their substitution into Eqs. (10) and elimination of \( t \) yields with account of Eqs. (9)

\[
\frac{1}{w_+ - w_-} \frac{\partial w_+}{\partial r_+} = \frac{1}{v_+ - v_-} \frac{\partial v_+}{\partial r_+} = \frac{\beta}{r_+ - r_-},
\]

\[
\frac{1}{w_+ - w_-} \frac{\partial w_-}{\partial r_+} = \frac{1}{v_+ - v_-} \frac{\partial v_-}{\partial r_+} = \frac{\beta}{r_+ - r_-}.
\]

FIG. 1: (a) The initial data are given along the curve \( AB \) in the hodograph plane in the general formulation of the Riemann method. (b) The segments \( AC \) and \( CB \) form the "initial data curve" for the problem of collision of two rarefaction waves by the Riemann method.

\[
\text{where}
\]

\[
\beta = \frac{3 - \gamma}{2(\gamma - 1)}.
\]

This means that \( \partial w_+/\partial r_- = \partial w_-/\partial r_+ \) and, hence, we can represent \( w_\pm \) as

\[
w_+ = \frac{\partial W}{\partial r_+}, \quad w_- = \frac{\partial W}{\partial r_-},
\]

where \( W \) is a solution of the Euler-Poisson (EP) equation

\[
\frac{\partial^2 W}{\partial r_+ \partial r_-} - \frac{\beta}{r_+ - r_-} \left( \frac{\partial W}{\partial r_+} - \frac{\partial W}{\partial r_-} \right) = 0.
\]

The characteristics of this second order partial differential equation are the straight lines \( r_+ = \xi = \text{const}, \ r_- = \eta = \text{const} \) parallel to the coordinates axes in the hodograph plane. The Riemann method is based on the idea that one can find the solution of the EP equation in the form similar to d’Alembert solution of the wave equation with explicit account of the initial conditions given on some curve \( AB \) in the hodograph plane (see Fig.1(a)). These data are transferred along the characteristics into the domain of dependence \( D \), so that the function \( W \) can be found at any point \( P(\xi, \eta) \in D \).

Riemann showed (see, e.g., Refs. [8, 9]) that \( W(P) \) can be represented in the form

\[
W(P) = \frac{1}{2} (R W)_A + \frac{1}{2} (R W)_B \int_A^B (V dr_+ + U dr_-),
\]

where

\[
U = \frac{1}{2} \left( R \frac{\partial W}{\partial r_-} - W \frac{\partial R}{\partial r_-} \right) - \frac{\beta}{r_+ - r_-} WR,
\]

\[
V = \frac{1}{2} \left( W \frac{\partial R}{\partial r_+} - R \frac{\partial W}{\partial r_+} \right) - \frac{\beta}{r_+ - r_-} WR,
\]

\( \overline{W} \) in the right-hand sides represents values of \( W \) along the boundary arc \( AB \) in the hodograph plane (see
Fig. 1(a)), and \( R \) if the Riemann function which satisfies the equation
\[
\begin{align*}
\frac{\partial^2 R}{\partial r_+ \partial r_-} + \frac{\beta}{r_+ - r_-} \left( \frac{\partial R}{\partial r_+} - \frac{\partial R}{\partial r_-} \right) - \frac{2\beta R}{(r_+ - r_-)^2} &= 0,
\end{align*}
\]
and its solution case can be expressed in terms of hypergeometric function \( F(a, b; c; z) \) (see [8])
\[
R = \left( \frac{r_+ - r_-}{\xi - \eta} \right)^{\beta} F(\beta, 1 - \beta; 1; z),
\]
\[
z = \frac{(r_+ - \xi)(r_- - \eta)}{(r_+ - r_-)(\xi - \eta)},
\]
where \( \xi \) and \( \eta \) are the coordinates of the point \( P \) in the hodograph plane \((r_+, r_-)\). Now we can turn to our problem of collision of two rarefaction waves in BEC.

### IV. COLLISION OF RAREFACTION WAVES

In BEC we have \( \gamma = 2 \) and, consequently, \( \beta = 1/2 \), so Eqs. [19] for the Riemann invariants \( r_{\pm} = \pm u/2 \pm \sqrt{\rho} \) take the form
\[
\begin{align*}
\frac{\partial r_+}{\partial t} + \frac{1}{2}(3r_+ + r_-) \frac{\partial r_+}{\partial x} &= 0, \\
\frac{\partial r_-}{\partial t} + \frac{1}{2}(r_+ + 3r_-) \frac{\partial r_-}{\partial x} &= 0.
\end{align*}
\]
Before the moment of the collision, i.e., for \( t < l/c_0 \), \( c_0 = \sqrt{\rho_0} \), the rarefaction waves are given by the simple wave solutions of the system (19),
\[
\begin{align*}
r_+ &= c_0, & x - \frac{1}{2}(c_0 + 3r_-)t &= \frac{\partial W}{\partial r_-} = l, \\
l - c_0 t &\leq x \leq l + 2c_0 t; \\
r_- &= -c_0, & x - \frac{1}{2}(3r_+ - c_0) t &= \frac{\partial W}{\partial r_+} = -l, \\
-l - 2c_0 t &\leq x \leq -l + c_0 t,
\end{align*}
\]
and the condensate remains at rest with the density \( \rho_0 \) in the region \((-l + c_0 t) \leq x \leq (-l - c_0 t)\). After the moment \( l/c_0 \) the region of the general solution of the system (19) appears in the interval \( x_L(t) \leq x \leq x_R(t) \) where both Riemann invariants change with time and space coordinate, and this general solution matches with the simple waves (20) at the points \( x_L(t) \) and \( x_R(t) \). This means that in the hodograph plane the function \( W \) must satisfy the boundary conditions
\[
\begin{align*}
\frac{\partial W}{\partial r_-} &= l \text{ at } r_+ = c_0, \\
\frac{\partial W}{\partial r_+} &= -l \text{ at } r_- = -c_0,
\end{align*}
\]
that is
\[
W = -l(r_+ - r_-)
\]
on the sides \( AC \) and \( CB \) of the rectangle \( ACBP \) defined in Fig. 1(b). If we solve the EP equation (14) \( (\beta = 1/2) \) with this boundary condition, then we can find \( W \) at the point \( P(\xi, \eta) \), and then the values \( r_+ = \xi, r_- = \eta \) of the Riemann invariants at this point are related with \( x \) and \( t \) by the formulae (11), that is
\[
x - \frac{1}{2}(3\xi + \eta)t = \frac{\partial W}{\partial \xi}, \quad x - \frac{1}{2}(\xi + 3\eta)t = \frac{\partial W}{\partial \eta}.
\]
To find the solution of Eq. (11), we use the Riemann formula (15) for \( \beta = 1/2 \) the Riemann function (18) can be transformed to more convenient expression
\[
R = \frac{2}{\pi} \frac{r_+ - r_-}{\sqrt{(r_+ - \eta)(\xi - r_-)}} K(m),
\]
\[
m = \frac{(r_+ - \xi)(\eta - r_-)}{(r_+ - \eta)(\xi - r_-)}, \quad 0 \leq m \leq 1,
\]
where \( K(m) \) is the elliptic integral of the first kind. Substitution of Eq. (22) into Eq. (15) followed by integration by parts with account of (16) yields
\[
W(P) = (RW)_C + \frac{3l}{2} \left\{ \int_0^{r_0} Rdr_+ + \int_{-r_0}^{-\eta} Rdr_- \right\},
\]
where the first integral is taken along the side \( AC \) of the rectangle in Fig. 1(b), the second one along the side \( CB \), and at the point \( C \) we have \( r_+ = c_0, r_- = -c_0, m = m_0 \),
\[
m_0 = \frac{(c_0 - \xi)(c_0 + \eta)}{(c_0 + \xi)(c_0 - \eta)}.
\]
Changing integration over \( r_+ \) and \( r_- \) to integration over corresponding specification of the variable \( m \) yields the final expression
\[
W(\xi, \eta) = -\frac{8l\rho_0^2}{\pi} \frac{K(m_0)}{\sqrt{(c_0 + \xi)(c_0 - \eta)}} + \frac{3l}{\pi} \sqrt{\xi - \eta} \int_0^{m_0} F(\xi, \eta, m) dm,
\]
where
\[
F(\xi, \eta, m) = \left\{ \left( \frac{\rho_0 + \xi}{\rho_0 + \eta} \right)^{3/2} \left( \frac{\rho_0 + \eta - (\rho_0 + \xi)m^{5/2}}{(\rho_0 + \xi - (\rho_0 - \eta)m^{5/2})} \right)^{3/2} \right\} (1 - m) K(m).
\]
These formulae together with Eqs. (25) define implicitly the Riemann invariants as functions of \( x \) and \( t \) and, consequently, the values of the density and the flow velocity,
\[
\rho = \frac{1}{4}(\xi - \eta)^2, \quad u = \xi + \eta.
\]
and the evolution time \( t = 300 \). As one can see, the agreement is very good almost everywhere except for the edges of the wave near the boundaries with vacuum where the regions of small oscillations appear. Such oscillations are generated due to dispersion effects and they originate from the sharp dependence of the initial distribution \( [5] \) of the density on \( x \) at the edges \( x = \pm l \). Thus, the hydrodynamic approximation gives accurate enough description of the wave resulting from collision of two rarefaction waves in BEC.

Although the above formulæ provide the complete solution of our problem, its analysis can be considerably simplified by the following remark. The compatibility condition \( \partial^2 x / \partial \xi \partial \eta = \partial^2 x / \partial \eta \partial \xi \) of the equations (see Eqs. [10])

\[
\frac{\partial x}{\partial \xi} - \frac{1}{2} (\xi + 3\eta) \frac{\partial t}{\partial \xi} = 0, \quad \frac{\partial x}{\partial \eta} - \frac{1}{2} (3\xi + \eta) \frac{\partial t}{\partial \eta} = 0.
\]

yields the Euler-Poisson equation for the function \( t = t(\xi, \eta) \),

\[
\frac{\partial^2 t}{\partial \xi \partial \eta} - \frac{3}{2} \frac{\partial t}{\partial \xi} \left( \frac{\partial t}{\partial \xi} - \frac{\partial t}{\partial \eta} \right) = 0.
\]

Its solution satisfying the necessary boundary conditions can be found by the same method as Eq. [17] is solved (see [8]) and it is given by the formula

\[
t = \frac{8l c_0^2}{(c_0 + \xi)^3/2(c_0 - \eta)^3/2} F\left[\frac{3}{2}, \frac{3}{2}; 1; \frac{(c_0 - \xi)(c_0 + \eta)}{(c_0 + \xi)(c_0 - \eta)}\right],
\]

(32)

where \( F \) is again the hypergeometric function. This formula gives the dependence of time \( t \) on \( \xi \) and \( \eta \) in the whole region of the general solution.

At the right boundary between the general solution and the rarefaction wave we have \( \xi = c_0 \), hence Eq. (32) simplifies to

\[
t = \frac{2\sqrt{2l c_0^{1/2}}}{(c_0 - \eta)^{3/2}},
\]

(33)

and elimination of \( \eta \) from this equation and the first formula (20) (with \( r_- = \eta \)) for the rarefaction wave gives the law of motion of this boundary,

\[
x_R(t) = l + 2c_0 t - 3t \left( \frac{c_0 t}{l} \right)^{1/3}.
\]

(34)

Fig. 3 demonstrates good agreement of this analytical formula with the numerical results.

At the center of the distribution we have \( \eta = -\xi \) so that Eq. (32) yields in implicit form the dependence of \( \xi \) on \( t \),

\[
t = \frac{8l c_0^2}{(c_0 + \xi)^3/2(c_0 - \eta)^3/2} F\left[\frac{3}{2}, \frac{3}{2}; 1; \frac{(c_0 - \xi)}{(c_0 + \xi)}\right].
\]

(35)

Since the flow velocity \( u \) vanishes here and, consequently, \( \rho = \xi^2 \), this formula gives the dependence of the density \( \rho \) on time \( t \) at the center of the wave at \( x = 0 \). For asymptotically large time \( t \gg l/c_0 \) we get with logarithmic accuracy

\[
\rho \approx \rho_0 \left( \frac{2}{\pi} \cdot \frac{l}{c_0(t-t_0)} \right) + \frac{1}{\pi^2} \left( \frac{l}{c_0(t-t_0)} \right)^2 \ln \frac{c_0 t}{l},
\]

(36)

where

\[
t_0 = \frac{l}{2\pi c_0} (7 - 4\gamma - 5 \ln 2 - 4\psi(3/2) + \ln \pi + \psi(3/2)) \approx 0.353988(l/c_0),
\]

\[
\psi(3/2) \approx 0.9189385332.
\]
FIG. 4: Dependence of the density $\rho$ of BEC at the center of the wave at $x = 0$ on time $t$. The initial parameters are equal to $\rho_0 = 1$, $l = 1$. Exact numerical solution is shown by a solid blue line and analytical approximation by a red dashed line.

$\gamma \approx 0.577216$ is the Euler constant, $\psi(z) = \Gamma'(z)/\Gamma(z)$. Even the first term here give very good approximation to the exact expression $\left(\frac{1}{e} - 1\right)$ in the whole region $t > 1$.

The analytical dependence $\rho(0, t)$ obtained in the dispersionless approximation is compared with numerical solution of the GP equation in Fig. 4. We have chosen here the initial length of the distribution equal to $l = 1$ to show that there is some difference between the analytical theory and the exact solution at the initial stage of evolution, if the initial size of the box is about one healing length. For larger time $t \gtrsim 5$ this difference disappears and if $l \sim 10$ it is negligibly small for all values of time $t > l/c_0$.

V. CONCLUSION

Thus, the considered in this paper problem demonstrates that the process of free expansion of BEC released from a trap can be more complicated than such an expansion in the case of harmonic potential traps, and the solution obtained here provides the method of analytical description of such a process. For scales greater than the healing length, the hydrodynamic dispersionless approximation to the GP equation is a very convenient tool for analytical investigations for the following reasons. First, we have in our disposal a very well developed apparatus of the compressible gas dynamics which can be successfully applied to concrete problems, as it is demonstrated in this paper for the problem of collision of two rarefaction waves. Second, the analytical solution provides the main characteristic parameters of the wave as, for example, the size of the BEC cloud or its density at the center, at any moment of time, what may be useful for quantitative estimates and comparison with experiment. At last, the dispersionless solution can be part of more complicated wave structures as it happens, for example, in experiments with formation of dispersive shock waves in BEC [11] or in similar experiments in nonlinear optics [12].

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