Coloring clique-hypergraph of $K_5$-minor-free graphs

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Abstract

A clique-coloring of a graph $G$ is a coloring of the vertices of $G$ so that no maximal clique of size at least two is monochromatic. The clique-hypergraph, $\mathcal{H}(G)$, of a graph $G$ has $V(G)$ as its set of vertices and the maximal cliques of $G$ as its hyperedges. A (vertex) coloring of $\mathcal{H}(G)$ is a clique-coloring of $G$. The clique-chromatic number of $G$ is the least number of colors for which $G$ admits a clique-coloring. Every planar graph has been proved to be 3-clique-colorable (Electr. J. Combin. 6 (1999), #R26). Recently, we showed that every claw-free planar graph, different from an odd cycle, is 2-clique-colorable (European J. Combin. 36 (2014) 367-376). In this paper we generalize these results to $\{\text{claw, } K_5\text{-minor}\}$-free graphs.

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1 Introduction

A hypergraph $\mathcal{H}$ is a pair $(V, \mathcal{E})$ where $V$ is a finite set of vertices and $\mathcal{E}$ is a family of non-empty subsets of $V$ called hyperedges. A $k$-coloring of $\mathcal{H}$ is a function $\phi : V \rightarrow \{1, 2, \ldots, k\}$ such that for each $S \in \mathcal{E}$, with $|S| \geq 2$, there exist $u, v \in S$ with $\phi(u) \neq \phi(v)$, that is, there is no monochromatic hyperedge of size at least two. If such a function exists we say that $\mathcal{H}$ is $k$-colorable. The chromatic number $\chi(\mathcal{H})$ of $\mathcal{H}$ is the smallest $k$ for which $\mathcal{H}$ admits a $k$-coloring.
In other words, a $k$-coloration of $\mathcal{H}$ is a partition $\mathcal{P}$ of $V$ into at most $k$ parts such that no hyperedge of cardinality at least 2 is contained in some $P \in \mathcal{P}$.

Here we consider hypergraphs arising from graphs: for an undirected simple graph $G$, we call clique-hypergraph of $G$ (or hypergraph of maximal cliques of $G$) the hypergraph $\mathcal{H}(G) = (V(G), E)$ which has the same vertices as $G$ and whose hyperedges are the maximal cliques of $G$ (a clique is a complete induced subgraph of $G$, and it is maximal if it is not properly contained in any other clique). A $k$-coloring of $\mathcal{H}(G)$ is also called a $k$-clique-coloring of $G$, and the chromatic number $\chi(\mathcal{H}(G))$ of $\mathcal{H}(G)$ is called the clique-chromatic number of $G$, denoted by $\chi_C(G)$. If $\mathcal{H}(G)$ is $k$-colorable we say that $G$ is $k$-clique-colorable.

Note that what we call $k$-clique-coloration here is also called weak $k$-coloring by Andreae, Schughart and Tuza in [1, 3] or strong $k$-division by Hoáng and McDiarmid in [11]. Clearly, any (vertex) $k$-coloring of $G$ is a $k$-clique-coloring of $G$, so $\chi_C(G) \leq \chi(G)$. On the other hand, note that if $G$ is triangle-free (contains no a clique on three vertices), then $\mathcal{H}(G) = G$, which implies $\chi_C(G) = \chi(G)$. Since the chromatic number of triangle-free graphs is known to be unbounded [17], we get that the same is true for the clique-chromatic number.

The clique-hypergraph coloring problem was posed by Duffus et al. [8]. In general, clique-coloring can be a very different problem from ordinary vertex coloring [2]. Clique-coloring is harder than ordinary vertex coloring: it is coNP-complete even to check whether a 2-clique-coloring is valid [2]. The complexity of 2-clique-colorability is investigated in [13], where they show that it is NP-hard to decide whether a perfect graph is 2-clique-colorable. However, it is not clear whether this problem belongs to NP. Recently, Marx [15] prove that it is $\Sigma_2^p$-complete to check whether a graph is 2-clique-colorable. On the other hand, Bacsó et al. [2] proved that almost all perfect graphs are 3-clique-colorable. A necessary and sufficient condition for $\chi_C \leq k$ on line graphs was given [1]. Recently, Campos et al. [4] showed that powers of cycles is 2-clique-colorable, except for odd cycles of size at least five, that need three colours, and showed that odd-seq circulant graphs are 4-clique-colorable. Many papers focus on finding the classes of graphs with $\chi_C = 2$. Claw-free perfect graphs and claw-free graphs without an odd hole are 2-clique-colorable [2]. Claw-free graphs of maximum degree at most four, other than an odd cycle, are 2-clique-colorable [3]. Many subclasses of odd-hole-free graphs have been studied and showed to be 2-clique-colorable [5, 6, 8]. Other works considering the clique-hypergraph coloring problem in classes of graphs can be found in the literature [10, 11, 12, 14].

For planar graphs, Mohar and Škrekovski [16] have shown that every planar graph is 3-clique-colorable, and Kratochvíl and Tuza [13] proposed a polynomial-time algorithm to decide if a planar graph is 2-clique-colorable (the set of cliques is given in the input).

Mohar and Škrekovski [16] proved the following theorem.
Theorem 1.1 (Mohar and Škrekovski [16]). Every planar graph is 3-clique-colorable.

Recently, we proved the following result in [19].

Theorem 1.2 (Shan, Liang and Kang [19]). Every claw-free planar graph, different from an odd cycle, is 2-clique-colorable.

The purpose of this paper is to generalize the above results to $K_5$-minor-free graphs. Section 2 gives some notation and terminology. In Section 3, we first show that every edge-maximal $K_5$-minor-free graph is 3-clique-colorable and every edge-maximal $K_4$-minor-free graph is 2-clique-colorable. Secondly, we prove that every \{claw, $K_5$-minor\}-free graph $G$, different from an odd cycle, is 2-clique-colorable and a 2-clique-coloring can be found in polynomial time.

2 Preliminaries

Let $G$ be an undirected simple graph with vertex set $V(G)$ and edge set $E(G)$. If $H$ is a subgraph of $G$, then the vertex set of $H$ is denoted by $V(H)$. For $v \in V(G)$, the open neighborhood $N(v)$ of $v$ is $\{u : uv \in E(G)\}$, and the closed neighborhood $N[v]$ of $v$ is $N(v) \cup \{v\}$. The degree of the vertex $v$, written $d_G(v)$ or simply $d(v)$, is the number of edges incident to $v$, that is, $d_G(v) = |N(v)|$. The maximum and minimum degrees of $G$ are denoted by $\Delta(G)$ and $\delta(G)$, respectively. For a subset $S \subseteq V(G)$, the subgraph induced by $S$ is denoted by $G|S$. As usual, $K_{m,n}$ denotes a complete bipartite graph with classes of cardinality $m$ and $n$; $K_n$ is the complete graph on $n$ vertices, and $C_n$ is the cycle on $n$ vertices. The graph $K_{1,3}$ is also called a claw, and $K_3$ a triangle. The graph $K_4 - e$ (obtained from $K_4$ by deleting one edge) is called a diamond. A graph $G$ is claw-free if it does not contain $K_{1,3}$ as an induced subgraph. A graph $H$ is a minor of a graph $G$ if $H$ can be obtained from $G$ by deleting edges, deleting vertices, and contracting edges. A graph $G$ is $H$-minor-free, if $G$ has no minor which is isomorphic to $H$. The family of $K_5$-minor-free graphs is a generalization of the planar graphs. For a family $\{F_1, \ldots, F_k\}$ of graphs, we say that $G$ is $\{F_1, \ldots, F_k\}$-free if it is $F_i$-free for all $i$.

For an integer of $k$, a clique of size $k$ of a graph $G$ is called a $k$-clique of $G$. The largest such $k$ is the clique number of $G$, denoted $\omega(G)$. A subset $I$ of vertices of $G$ is called an independent set of $G$ if no two vertices of $I$ are adjacent in $G$. The maximum cardinality of an independent set of $G$ is the independence number $\alpha(G)$ of $G$. A set $D \subseteq V(G)$ is called a clique-transversal set of $G$ if $D$ meets all cliques of $G$, i.e., $D \cap V(C) \neq \emptyset$ for every clique $C$ of $G$. The clique-transversal number, denoted by $\tau_C(G)$, is the cardinality of a minimum clique-transversal set of $G$. The notion of clique-transversal set in graphs can be regarded as a special case of the transversal set in hypergraph theory. Erdős et al. [9] have proved that the problem of finding a
minimum clique-transversal set for a graph is NP-hard. It is therefore of interest to determine bounds on the clique-transversal number of a graph. In [9] Erdős et al. proposed to find sharp estimates on the clique-transversal number $\tau_C$ for particular classes of graphs (planar graphs, perfect graphs, etc.).

We call $G$ a plane triangulation if every face of a planar $G$ (including the outer triangulation face) is bounded by a triangle. Let $G$ be a planar graph and $C$ a cycle of $G$. The interior $\text{Int}(C)$ of $C$ denotes the subgraph of $G$ consisting of $C$ and all vertices and edges in the disk bounded by $C$. Similarly, $\text{Ext}(C) \subseteq G$ is the exterior of $C$. Obviously, $\text{Int}(C) \cap \text{Ext}(C) = C$.

3 $K_5$-Minor-free graphs

In this section, we first show that every edge-maximal $K_4$-minor-free graph is 2-clique-colorable and every edge-maximal $K_5$-minor-free graph is 3-clique-colorable. Secondly, we show that every $\{\text{claw}, K_5\text{-minor}\}$-free graph, different from an odd cycle, is 2-clique-colorable. As an immediate corollary, we prove that every $\{\text{claw}, K_5\text{-minor}\}$-free graph, different from an odd cycle, has the clique-transversal number bounded above by half of its order.

**Lemma 3.1.** ([7]) A graph with at least 3 vertices is edge-maximal without a $K_4$-minor if and only if it can be constructed recursively from triangles by pasting along $K_2$’s.

**Theorem 3.1.** Every edge-maximal $K_4$-minor-free graph is 2-clique-colorable.

**Proof.** Let $G$ be an edge-maximal $K_4$-minor-free graph. For $|V(G)| = 2$, the assertion is trivial. So we may assume that $|V(G)| \geq 3$. Suppose that $A$ is a subgraph isomorphic to $K_2$ in $G$ and $\phi$ is a (not necessarily proper) coloring of $A$. We show by induction on $|V(G)|$ that $\phi$ can be extended to a 2-clique-coloring of $G$. For $|V(G)| = 3$, since $G$ is a triangle, the assertion is obvious. For $|V(G)| \geq 4$, by Lemma 3.1 we have $G = G_1 \cup G_2$ such that $G_1 \cap G_2 \cong K_2$ where $G_1, G_2$ are proper subgraphs of $G$. Clearly $A$ is a subgraph of $G_1$ or $G_2$. Without loss of generality, let $A$ be a subgraph of $G_1$. By the induction hypothesis applied to $G_1$, $\phi$ can be extended to a 2-clique-coloring of $G_1$. This coloring induces a (not necessarily proper) coloring of $G_1 \cap G_2$, and by the induction hypothesis applied to $G_2$, the coloring of $G_1 \cap G_2$ further can be extended to a 2-clique-coloring of $G_2$. The union of these 2-clique-colorings of $G_1$ and $G_2$ forms a 2-clique-coloring of $G$. The assertion follows. □

**Remark 1.** The condition “edge-maximal” in Theorem 3.1 is the best possible, because the graph exhibited in Figure 5 is a $K_4$-minor-free graph and we see that it is not 2-clique-colorable.

Next we shall show that every edge-maximal $K_5$-minor-free graph is 3-clique-colorable. For this purpose, we need the following lemmas.
Lemma 3.2. (Škrekovski [20]) Every $K_5$-minor-free graph is 5-choosable.

Lemma 3.3. Let $G$ be a $K_5$-minor-free graph with at least one edge such that each edge of $G$ is contained in some triangle of $G$. Then $G$ has a 3-clique-coloring such that no triangle of $G$ is monochromatic.

Proof. By Lemma 3.2 there is a 5-coloring $\phi$ of $G$. For $i = 1, \ldots, 5$, let $V_i \subseteq V(G)$ be the set of vertices colored $i$. Now, let $\phi(v) = 1$ if $v \in V_1 \cap V_2$, let $\phi(v) = 2$ if $v \in V_3 \cup V_4$, and let $\phi(v) = 3$ if $v \in V_5$. Since every maximal clique $K$ in $G$ contains at least 3 vertices, $K$ uses at least 3 colors in the 5-coloring of $G$, and hence $\phi$ uses at least both colors on $K$. Therefore, $\phi$ is a 3-clique-coloring of $G$ and no triangle of $G$ is monochromatic. $\square$

Lemma 3.4. (Mohar and Škrekovski [16]) Let $G$ be a connected planar graph whose outer cycle $C$ is a triangle. Let $\phi : V(C) \to \{1, 2, 3\}$ be a coloring of $\mathcal{H}(C)$. Then $\phi$ can be extended to a 3-clique-coloring of $G$ and no triangle of $G$ is monochromatic.

![Figure 1: Two different representations of the Wagner graph.](image)

The Wagner graph, denoted by $V_8$, is a graph constructed from an 8-cycle (we call it the outer cycle) by connecting the antipodal vertices (these edges will be called the diagonal edges). The Wagner graph is depicted in Figure 1. Note that the Wagner graph is triangle-free and 3-colorable (because it is cubic).

The following easy lemma about the Wagner graph is obtained by Naserasr et al. in [18].

Lemma 3.5. ([18]) If $e$ is an edge of the Wagner graph $V_8$, then $V_8 - e$ admits a 3-coloring such that the end vertices of $e$ receive the same color.

Theorem 3.2. (Wagner [21]) If $G$ is an edge-maximal $K_5$-minor-free graph with at least 4 vertices, then $G$ can be constructed recursively, by pasting along $K_2$'s and $K_3$'s, from plane triangulations and copies of the Wagner graph.

In order to make our arguments easier to follow we use the following notations in [18]: Let $\mathcal{T} = T_1, T_2, \ldots, T_r$ be a sequence of graphs where each $T_i$ is either a plane triangulation or a
copy of the Wagner graph $V_8$. By $T$, we construct another sequence $\mathcal{G} = G_1, G_2, \ldots, G_r$ of graphs as follows: $G_1 = T_1$, $G_i$ is obtained from $G_{i-1}$ and $T_i$ by pasting $T_i$ to $G_{i-1}$ along a $K_2$ or a $K_3$.

Given an edge-maximal $K_5$-minor-free graph $G$, the sequence $\mathcal{T}$ is said to be a Wagner sequence of the graph $G$, if $G = G_r$ for some sequence $\mathcal{G}$ constructed from $\mathcal{T}$. Note that each edge-maximal $K_5$-minor-free graph has a Wagner sequence by Theorem 3.2.

We actually can prove the following somewhat stronger result.

**Theorem 3.3.** If $G$ is an edge-maximal $K_5$-minor-free graph, then $G$ has a 3-clique-coloring such that no triangle of $G$ is monochromatic.

**Proof.** If $|V(G)| \leq 3$, the assertion is trivial. So let $|V(G)| \geq 4$. As we have seen, $G$ has a Wagner sequence $\mathcal{T}$. Let $\mathcal{T} = T_1, T_2, \ldots, T_r$ be a Wagner sequence of $G$. We proceed by induction the length $r$ of $\mathcal{T}$. When $r = 1$, $G$ is either the Wagner graph or a plane triangulation. If $G$ is the Wagner graph, then the assertion is obvious, since the Wagner graph is 3-colorable. If $G$ is a plane triangulation, then the assertion follows directly from Lemma 3.3. So assume that $r \geq 2$.

Note that $T_1, T_2, \ldots, T_{r-1}$ is a Wagner sequence of the subgraph $G_{r-1}$ of $G$. By the induction hypothesis, $G_{r-1}$ has a 3-clique-coloring such that no triangle of $G_{r-1}$ is monochromatic. Let $\phi$ be such a 3-clique-coloring of $G_{r-1}$. It suffices to show that $\phi$ can be extended to a 3-clique-coloring of $G_r$ such that no triangle of $G_r$ is monochromatic. Suppose that $T_i$ is pasted to $G_{r-1}$ along a triangle $T$. Clearly the 3-clique-coloring $\phi$ of $G_{r-1}$ induces a coloring of $\mathcal{H}(T)$, since $T$ is not monochromatic. We can easily extend the coloring of $\mathcal{H}(T)$ to $\text{Int}(A)$ and to $\text{Ext}(A)$ (respectively) by applying Lemma 3.4. So the assertion follows. Suppose that $T_r$ is pasted to $G_{r-1}$ along a $K_2$, say $A$. Then $\phi$ induces a (not necessarily proper) coloring of $A$. If $A$ is a maximal clique of $T_r$, then $T_r = V_8$. By using the 3-colorability of the Wagner graph $V_8$ and Lemma 3.5, we are done. Finally, if $A$ is not a maximal clique of $T_r$, we first extend the coloring of $A$ to a 3-coloring of $\mathcal{H}(T)$ where $T$ is the triangle of $T_r$ containing $A$, and then extend the coloring of $\mathcal{H}(T)$ to $\text{Int}(T)$ and to $\text{Ext}(T)$ (respectively) in $T_r$ by Lemma 3.4. Thus we obtain a 3-clique-coloring of $G_r$ such that no triangle of $G_r$ is monochromatic. □

**Remark 2.** By Theorem 3.3, we know that every edge-maximal $K_5$-minor-free graph $G$ is 3-clique-colorable. Furthermore, we conjecture that this assertion is true for general $K_5$-minor-free graphs.

We now turn our attention to the claw-free graphs without $K_5$-minors. Let $C_n + K_1$ be the graph obtained from the disjoint union of $C_n$ and $K_1$ by joining the single vertex of $K_1$ to all the vertices of $C_n$. The graph $C_n + K_1$ is also called a $n$-wheel, denoted by $W_n$, and the vertex
in $K_1$ is known as the *hub* of $W_n$.

For claw-free graphs $G$ without 4-cliques, we observe the following simple property of the graph $G$ by the Ramsey number $R(3, 3) = 6$, its proof is similar to that of Lemma 8 in [19].

**Lemma 3.6.** If $G$ is a claw-free graph without 4-cliques, then $\Delta(G) \leq 5$ and $G|N[v]$ is a 5-wheel $W_5$ for each vertex $v$ of degree 5.

In [19] we proved that for a claw-free planar graph $G$, any 2-clique-coloring of $G - v$ can be extended to a 2-clique-coloring of $G$, where $v$ is a vertex of degree 5 in $G$. By Lemma 3.6, we can generalize this result to {claw, $K_5$-minor}-free graphs. Its proof resembles that of Lemma 9 in [19], and is omitted.

**Lemma 3.7.** Let $G$ be a {claw, $K_5$-minor}-free graph without 4-cliques and let $v$ be a vertex of degree 5 in $G$. If $G - v$ is 2-clique-colorable, then the same is true for $G$.

**Lemma 3.8.** Every {claw, $K_5$-minor}-free graph has maximum degree at most 6.

**Proof.** Let $G$ be a {claw, $K_5$-minor}-free graph. Suppose, to the contrary, that $\Delta(G) \geq 7$. Let $v \in V(G)$ such that $d_G(v) \geq 7$. Since $G$ contains no $K_5$-minor and claw, $G|N(v)$ contains no $K_4$-minor, and so $\alpha(G|N(v)) = 2$. To obtain a contradiction, we consider the graph $G|N(v)$.

Suppose that $G|N(v)$ contains a diamond $D$. Let $V(D) = \{u_1, u_2, u_3, u_4\}$ with $d_D(u_1) = d_D(u_2) = 2$ and let $G_1 = G|N(v) - V(D)$. Then $|V(G_1)| \geq 3$. Since $\alpha(G|N(v)) = 2$, $G_1$ contains at most two components and each vertex of $G_1$ is adjacent to $u_1$ or $u_2$. Clearly, all vertices in each component of $G_1$ is adjacent to only one of $u_1$ and $u_2$, for otherwise $G|N(v)$ would contain a $K_4$-minor. If $G_1$ consists of precisely one component. Without loss of generality, we may assume that all vertices of $G_1$ is adjacent to $u_1$. Thus $u_2$ is not adjacent to any vertex of $G_1$. Observe that $G|V(G_1) \cup \{u_1\}$ is not a complete subgraph in $G|N(v)$, since $|V(G_1)| \geq 3$ and $G|N(v)$ contains no $K_4$-minor. Thus there exist vertices $u_5, u_6 \in V(G_1)$ such that $u_5u_6 \in E(G)$. But then $\{u_2, u_5, u_6\}$ is an independent set of $G|N(v)$, contradicting the fact that $\alpha(G|N(v)) = 2$. If $G_1$ consists of precisely two component $O_1$ and $O_2$, and assume that $|V(O_1)| \geq |V(O_2)|$, then $|V(O_1)| \geq 2$. As we have observed above, all vertices of $O_i$ is adjacent to exactly one of $u_1$ and $u_2$. Without loss of generality, let us suppose that all vertices of $O_1$ is adjacent to $u_1$. So $u_2$ is not adjacent to any vertex of $O_1$. By $\alpha(G|N(v)) = 2$, we see that $G|V(O_1) \cup \{u_1\}$ is complete, and thus $|V(O_1)| = 2$, since $G|N(v)$ is $K_4$-minor-free. Hence $O_1 = K_2$. On the other hand, we claim that all vertices of $O_2$ is adjacent to $u_2$. Indeed, if not, we take $c_i \in V(O_i)$ for $i = 1, 2$, then $\{c_1, c_2, u_2\}$ is an independent set of size 3 of $G|N(v)$, a contradiction. By the $K_4$-minor-freeness of $G|N(v)$, it is easy to see that one of $u_3$ and $u_4$, say $u_3$, is not adjacent to any vertex of $O_2$. $\alpha(G|N(v)) = 2$ implies that $u_3$ is adjacent to all vertices of $O_1$. But then $G|V(O_1) \cup \{u_1, u_3\}$ is a $K_4$-minor in $G|N(v)$, a contradiction.
Suppose that $G|N(v)$ contains no diamond. Let $I = \{u_1, u_2\}$ be a maximum independent set of $G|N(v)$, and let $N_1 = N(v) - \{u_1, u_2\}$. As we have observed, each vertex of $N_1$ is adjacent to at least one of $u_1$ and $u_2$ by $\alpha(G|N(v)) = 2$. If $G|N_1$ contains a triangle, then one of $u_1$ and $u_2$ is adjacent to at least two vertices of this triangle. So $G|N(v)$ contains a diamond, contradicting our assumption. Thus $G|N_1$ contains no triangle. By Ramsey number $R(3, 3) = 6$, we conclude that $|N_1| = 5$ and $\alpha(G|N_1) = 2$, and so $|N(v)| = 7$. Clearly, $G|N_1$ is isomorphic to the cycle $C_5$. Note that either $u_1$ or $u_2$ is adjacent to at least three vertices on the cycle $G|N_1$. But then $G|N(v)$ has a $K_4$-minor, a contradiction. □

Lemma 3.9. Let $G$ be a $\{claw, K_5$-minor$\}$-free graph with $\omega(G) = 4$ and let $v$ be a vertex which lies in a 4-clique of $G$. If there exists a 2-clique-coloring of $G - v$, then the same is true for $G$.

Proof. The proof is by contradiction. Suppose that $G$ has no 2-clique-coloring. Let $\phi'$ be a 2-clique-coloring of $G - v$ with colors red and green. Then the extension of the coloring $\phi'$ of $G - v$ is impossible. Consequently, $G$ contains two maximal cliques $K$ and $L$ such that $V(K) \cap V(L) = v$. Let $Q := K - v$ and $R := L - v$. Without loss of generality, we may assume that the vertices of $Q$ are red in $\phi'$, while those of $R$ are green. Thus we cannot color $v$ neither red nor green in any extension of $\phi'$. Since $\phi'$ is a 2-clique-coloring of $G - v$, there exist two cliques (not necessarily maximal) $Q'$ and $R'$ in $G - v$ such that $Q' = Q + q_1$ and $R' = R + q_2$ with $q_1 \notin V(Q)$, $\phi'(q_1) = green$ and $q_2 \notin V(R)$, $\phi'(q_2) = red$, since otherwise $\phi'$ would not be a proper 2-clique-coloring of $G - v$.

Suppose that one of $K$ and $L$ is a 4-clique of $G$. Without loss of generality, let $K$ be a 4-clique of $G$. By the $K_5$-minor-freeness of $G$, clearly $q_1$ is not adjacent to $v$. Let us consider the graph $G - V(Q)$. If there is a path $P$ between $q_1$ and $v$ in $G - V(Q)$, then the vertices of $V(P) \cup V(Q)$ would contain a $K_5$-minor of $G$, contradicting our assumption. Hence $q_1$ and $v$ lie in the distinct components of $G - V(Q)$. Let $C$ be the component containing $v$. Let us now define a vertex coloring $\phi$ of $G$ as follows: we color $v$ green and change the colors of the vertices in $C - v$, and assign colors in $\phi'$ to all other vertices. We claim that $\phi$ is a 2-clique-coloring of $G$. Suppose not, let $M$ be a maximal clique of $G$ which is monochromatic in $\phi$. Then $M$ must contain at least one vertex, say $k$, of $Q$, and some vertices of $C$. This implies that $M$ is red, and thus $v \notin V(M)$. Hence $M$ contains at least a vertex $c$ that is not adjacent to $v$. One can easily see that $c \in V(C)$. Note that there is no path between $q_1$ and $v$ in $G - V(Q)$, so $c$ is not adjacent to $q_1$. But then we find a claw induced by $\{k, q_1, v, c\}$ centered at $k$, a contradiction. Thus, we may assume that neither $K$ nor $L$ is a 4-clique of $G$.

To complete the proof, we have the following claim.

Claim 1 Both $K$ and $L$ are 3-cliques of $G$, and the induced subgraph $G|N[v] \cup \{q_1, q_2\}$ is
isomorphic to $F_1$ or $F'_1$ (see Figure 2).

We first show that both $K$ and $L$ are 3-cliques of $G$. Indeed, if not, without loss of generality, we may assume that $K$ be a 2-clique of $G$ and let $V(K) = \{v, x\}$. Obviously, $x$ is not adjacent to any neighbor of $v$ by the maximality of $K$. According to our assumption, the vertex $v$ lies in a 4-clique, say $W$, of $G$. So $x$ is not adjacent to any vertex of $L - v$ ($= R$) and $W - v$. On the other hand, since $L$ and $W$ are two distinct cliques of $G$, there exists vertices $y \in V(L) - V(W)$ and $u \in V(W) - V(L)$ such that $yu \notin E(G)$. This implies that $\{v, x, u, y\}$ induces a claw centered at $v$, a contradiction. Therefore, $K$ and $L$ are 3-cliques of $G$.

![Figure 2: The graphs $F_1$ and $F'_1$.](image)

Let $V(K) = \{v, x_1, x_2\}$, $V(L) = \{v, y_1, y_2\}$ and let $W$ be a 4-clique that contains the vertex $v$. Note that $v \in V(W) \cap V(K) \cap V(L)$, so $1 \leq |V(W) \cap (V(K) \cup V(L))| \leq 3$, for otherwise $V(W) \supseteq V(K)$ or $V(W) \supseteq V(L)$, contradicting the fact that $K$ and $L$ are maximal cliques of $G$. Thus $V(W) - (V(K) \cup V(L)) \neq \emptyset$. Let $u \in V(W) - (V(K) \cup V(L))$. Clearly, $K$ (respectively $L$) contain at least a vertex that is not adjacent to $u$. Without loss of generality, let $x_1 \in V(K), y_1 \in V(L)$ such that $x_1u \notin E(G)$ and $y_1u \notin E(G)$. Since $\{v, x_1, y_1, u\}$ does not induce a claw centered at $v$, it immediately follows that $x_1y_1 \in E(G)$. This implies that $x_1y_2 \notin E(G)$.

![Figure 3: The graph $H_1$.](image)
and \( x_2y_1 \notin E(G) \) by the maximality of \( K \) and \( L \). Since \( \{v, x_1, y_2, u\} \) and \( \{v, x_2, y_1, u\} \) can not induce a claw centered at \( v \), we have \( wx_2, uy_2 \in E(G) \). Now we show that \( G[N[v] \cup \{q_1, q_2\}] \) is isomorphic to either \( F_1 \) or \( F'_1 \). If \( N(v) = \{x_1, x_2, y_1, y_2, u\} \), then \( x_2y_2 \in E(G) \) since \( v \) lies in a 4-clique of \( G \). Hence \( v \) lies in the 4-clique induced by \( \{v, u, x_2, y_2\} \). Furthermore, we claim that \( uw_i \notin E(G) \) for \( i = 1, 2 \). If not, let \( uw_i \in E(G) \), then \( G[N[v] \cup \{q_1\}] \) would contain a \( K_5 \)-minor. Therefore, \( G[N[v] \cup \{q_1, q_2\}] \) is isomorphic to the graph \( F_1 \). If \( N(v) \neq \{x_1, x_2, y_1, y_2, u\} \), we have \( d_G(v) = \Delta(G) = 6 \) by Lemma 3.8. Let \( N(v) = \{x_1, x_2, y_1, y_2, u, w\} \). Since \( \{v, x_1, y_2, w\} \) does not induce a claw centered at \( v \), \( wx_1 \in E(G) \) or \( wy_2 \in E(G) \). If \( wy_2 \in E(G) \), then \( wy_1 \notin E(G) \) by the maximality of \( L \). So \( wx_2 \in E(G) \) (see the graph \( H_1 \) in Figure 3), for otherwise \( \{v, x_2, y_1, w\} \) would induce a claw centered at \( v \). This implies that \( wx_1 \notin E(G) \) by the maximality of \( K \). As we have seen, \( u \) is not adjacent to \( x_1 \), it follows that \( uw \in E(G) \), since otherwise \( \{v, x_1, u, w\} \) induces a claw centered at \( v \). But now \( G[N[v]] \) contains a \( K_5 \)-minor, a contradiction. If \( wx_1 \in E(G) \), then \( wx_2 \notin E(G) \) by the maximality of \( K \). To avoid a claw induced by \( \{v, x_2, y_1, w\} \) centered at \( v \), we have \( wy_1 \in E(G) \). Thus \( wy_2 \notin E(G) \) by the maximality of \( L \). By the \( K_5 \)-minor-freeness of \( G \), it is easy to see that \( wu \notin E(G) \). Note that \( \{v, w, x_2, y_2\} \) can not induce a claw centered at \( v \), so \( x_2y_2 \in E(G) \). Finally, one easily see that \( uw_i \notin E(G) \) and \( wq \notin E(G) \) by the \( K_5 \)-minor-freeness of \( G \). Consequently, \( G[N[v] \cup \{q_1, q_2\}] \) is isomorphic to the graph \( F'_1 \). □

For convenience, let us denote by \( W \) and \( W' \) (if exists) the 4-cliques \( G \{x_2, y_2, u, v\} \) and \( G \{x_1, y_1, v, w\} \), that contain the vertex \( v \), in \( F_1 \) or \( F'_1 \).

As we saw earlier, \( \phi'(x_1) = \phi'(x_2) = \text{red}, \phi'(y_1) = \phi'(y_2) = \text{green}, \) and \( \phi'(q_1) = \text{green} \) and \( \phi'(q_2) = \text{red} \). We give a 2-clique-coloring \( \phi \) of \( G \) as follows: we exchange the colors of \( x_2 \) and \( y_2 \), and assign red or green to \( v \), and let \( \phi(x) = \phi'(x) \) for all the vertices \( x \in V(G) \setminus \{v, x_2, y_2\} \). We claim that \( \phi \) is a 2-clique-coloring of \( G \). Suppose not, let \( M \) be a monochromatic maximal clique of \( G \) in \( \phi \). Then \( M \) must contain exactly one vertex of \( \{x_2, y_2\} \), and at least one vertex, say \( k \), of \( V(G) \setminus (N[v] \cup \{q_1, q_2\}) \). By the symmetry between \( x_2 \) and \( y_2 \) in \( F_1 \) or \( F'_1 \), we may assume that \( x_2 \) is in \( M \). This implies that \( M \) is green. By Lemma 3.8, \( d_G(x_2) \leq \Delta(G) \leq 6 \). If \( |V(M)| \geq 3 \), then all the vertices in \( V(M) \setminus \{k\} \) are in \( N[v] \cup \{q_1, q_2\} \); otherwise we have

\[
d_G(x_2) = |N(x_2)| \geq |\{x_1, y_2, u, v, q_1\}| + |V(M) \setminus (N[v] \cup \{q_1, q_2\})| \geq 7,
\]

which is a contradiction. We claim that \( |V(M)| \leq 3 \). Indeed, if \( |V(M)| = 4 \), by Claim 1, \( G[N[v] \cup \{q_1, q_2\}] \cong F_1 \) or \( F'_1 \), then \( k \) is not adjacent to \( v \). Since \( M \) is a maximal green 4-clique of \( G \), we have \( V(M) \setminus \{k\} = \{x_2, u, q_1\} \). But now we can find four vertex-disjoint paths linking \( x_1 \) to all vertices in the 4-clique \( W \). Thus \( G \) contains a \( K_5 \)-minor, contradicting our assumption. Consequently, \( M \) is either a green 2-clique or a green 3-clique of \( G \). We consider the following two cases.

**Case 1:** \( M \) is a 2-clique of \( G \), that is, \( M \) is the maximal 2-clique induced by \( \{x_2, k\} \). Obviously,
we have \( x_1k \notin E(G) \), \( y_2k \notin E(G) \) by the maximality of \( M \). By Claim 1, we know that \( x_1y_2 \notin G \). This implies that \( \{x_1, y_2, k, x_2\} \) induces a claw centered at \( x_2 \), a contradiction.

\[ \text{Figure 4: The graphs } F_2 \text{ and } F_3. \]

**Case 2:** \( M \) is a 3-clique of \( G \), and let \( V(M) = \{x_2, k, l\} \).

As we have seen, \( l \in N[v] \cup \{q_1, q_2\} \), and \( k \) is not adjacent to \( v \) by Claim 1. Since \( M \) is a maximal green 3-clique of \( G \), we have \( l = u \) or \( q_1 \).

If \( l = u \), that is, \( ku \in E(G) \), then \( \phi(u) = \text{green} \) (see the graph \( F_2 \) in Figure 4). This implies that \( y_2k \notin E(G) \) by the maximality of \( M \). Thus \( x_1k \in E(G) \) for avoiding the claw \( G\{x_1, x_2, y_2, k\} \) at \( x_2 \). Hence we can now find four vertex-disjoint paths linking \( x_1 \) to all vertices in the 4-clique \( W \), and so a \( K_5 \)-minor occurs in \( G \), a contradiction.

If \( l = q_1 \), that is, \( q_1k \in E(G) \), then \( V(M) = \{x_2, k, q_1\} \) (see the graph \( F_3 \) in Figure 4). By the maximality of \( M \), we have \( x_1k \notin E(G) \). By Claim 1, we know that \( x_1y_2 \notin E(G) \) and \( x_1u \notin E(G) \). Note that \( \{x_1, x_2, y_2, k\} \) and \( \{x_1, x_2, u, k\} \) can not induce claws at \( x_2 \), so \( y_2k, uk \in E(G) \), so we find four vertex-disjoint paths linking \( k \) to all vertices in the 4-clique \( W \). But now this produces a \( K_5 \)-minor in \( G \), a contradiction. \( \square \)

**Lemma 3.10** (Bacsó and Tuza [3]). *Every connected claw-free graph of maximum degree at most four, other than an odd cycle, is 2-clique-colorable. Moreover, a 2-clique-coloring can be found in polynomial time.*

Finally, we have the following result.

**Theorem 3.4.** *Every {claw, \( K_5 \)-minor}-free graph \( G \) of order \( n \), different from an odd cycle, is 2-clique-colorable.*

**Proof.** We proceed by induction on \( n \). For \( n \leq 4 \), clearly the assertion holds. Now let \( n > 4 \),
and assume that the assertion holds for smaller values than \( n \). If \( \Delta(G) \leq 4 \), by Lemma 3.10, \( G \) is 2-clique-colorable. So we may assume that \( \Delta(G) \geq 5 \).

Suppose that \( G \) has no 4-cliques. Then \( \Delta(G) \leq 5 \) by Lemma 3.6 and so \( \Delta(G) = 5 \). Let \( v \) be a vertex of degree 5 in \( G \). By Lemma 3.4 we have \( G[N[v]] \) is a 5-wheel. If \( G = G[N[v]] \), then \( G \) can easily be clique-colored in two colors. If \( G \neq G[N[v]] \), then clearly \( G - v \) is still a \{claw, \( K_5 \)-minor\}-free graph, not an odd cycle. Therefore, by the induction hypothesis, \( G - v \) is 2-clique-colorable. It follows from Lemma 3.7 that \( G \) is 2-clique-colorable.

Suppose that \( G \) has 4-cliques. Let \( v \) is a vertex of \( G \) that lies in a 4-clique. Obviously \( G - v \) is still a \{claw, \( K_5 \)-minor\}-free graph, not an odd cycle. By the induction hypothesis, \( G - v \) is 2-clique-colorable. So \( G \) is also 2-clique-colorable by Lemma 3.9.

\[ \square \]

Figure 5: Example of a \( K_5 \)-minor-free graph containing claws that is not 2-clique-colorable.

Remark 3. The condition \{claw, \( K_5 \)-minor\}-free in Theorem 3.4 cannot be dropped. For example, the graph shown in Figure 4 contains a claw and its clique-chromatic number is 3. The line graph \( L(K_6) \) of \( K_6 \) contains the complete graph \( K_5 \), and is not 2-clique-colorable by Ramsey number \( R(3, 3) = 6 \).

Note that if \( \phi \) is a 2-clique-coloring of a graph, then \( \phi^{-1}(r) \) and \( \phi^{-1}(g) \) are clique-transversal sets of \( G \). By Theorem 3.4 we immediately obtain an upper bound on the clique-transversal number for \{claw, \( K_5 \)-minor\}-free graphs.

Corollary 3.1. Every \{claw, \( K_5 \)-minor\}-free graph, different from an odd cycle, has the clique-transversal number bounded above by half of its order.

Finally, we present a polynomial-time algorithm to find a 2-clique-coloring of \{claw, \( K_5 \)-minor\}-free graphs. In [23] Bacsó and Tuza proposed the polynomial-time algorithm CLQCOL for 2-clique-coloring problem on claw-free graphs of maximum degree at most four, other than an odd hole.

Clearly, if \( G \) is a \{claw, \( K_5 \)-minor\}-free graph, not an odd cycle, then so is the graph \( G - v \), and \( G - v \) has fewer vertices. Based on the algorithm CLQCOL and Lemmas 3.4-3.7, we provide the
following algorithm to find a 2-clique-coloring on \{claw, \textit{K}_5\text{-minor}\}-free graphs different from odd cycles.

**Algorithm A.** 2-Clique-coloring of \{claw, \textit{K}_5\text{-minor}\}-free graphs.

*Input:* \{Claw, \textit{K}_5\text{-minor}\}-free graph \textit{G}, not an odd cycle.

*Output:* 2-Clique-coloring \(\phi: V(\textit{G}) \to \{r, g\}\).

*Step 1:* If \(\Delta(\textit{G}) \leq 4\), then CLQCOL(\textit{G})(see, Ref. [3]), stop the algorithm. If not, turn to Step 2.

*Step 2:* If \(\Delta(\textit{G}) = 5\) and there is no 4-clique in \textit{G}, turn to Step 3. If not, turn to Step 4.

*Step 3:* If \textit{G} is a 5-wheel, give a 2-clique-coloring directly. If not, find a vertex \(v\) with degree 5. Then \(A(\textit{G} - v)\). Extend the 2-clique-coloring of \textit{G} - \(v\) to a 2-clique-coloring of \textit{G}. Stop the algorithm.

*Step 4:* Find a vertex \(v\) of degree \(\leq 5\) in \textit{G} such that \(v\) lies in a 4-clique. Then \(A(\textit{G} - v)\). Extend the 2-clique-coloring of \textit{G} - \(v\) to a 2-clique-coloring of \textit{G}. Stop the algorithm.

By Lemmas 3.7, 3.9 and 3.10 the validity of Algorithm \(A\) can be easily verified. It is easy to check that the loop of every step is performed at most \(n\) times. In Step 1, the running time of Algorithm CLQCOL is \(O(n^2)\) and is carried out only once. In Steps 2, 3 and 4, the running time is at most \(O(n)\). Thus the overall time is at most \(O(n^2)\). Consequently, we obtain the following result.

**Theorem 3.5.** Algorithm \(A\) is a polynomial-time algorithm in \(O(n^2)\) for the 2-clique-coloring of \{claw, \textit{K}_5\text{-minor}\}-free graphs, different from odd cycles.

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