Freund-Rubin Revisited

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Abstract

We utilise the duality between $M$ theory and Type IIA string theory to show the existence of Freund-Rubin compactifications of $M$ theory on 7-manifolds with singularities supporting chiral fermions. This leads to a concrete way to study phenomenologically interesting quantum gravity vacua using a holographically dual three dimensional field theory.
1. Introduction

In 1980, Freund and Rubin found a solution of d=11 supergravity which describes a compactification of the theory on a round seven sphere to four dimensional anti de Sitter space \((AdS_4)\) [1]. We now know that this is a vacuum of \(M\) theory which arises as the dual to the three dimensional conformal field theory of \(M2\)-branes in flat space [2]. More generally the Freund-Rubin ansatz is satisfied for any seven manifold \(X\) whose metric is Einstein with positive scalar curvature, and these compactifications are dual to more complicated 3d CFT’s [3].

Freund-Rubin vacua were originally of interest in a program to obtain realistic four dimensional physics by appropriately choosing \(X\), see [4] for a review. This program originally failed due to Witten’s no go theorem which states that chiral fermions do not arise when \(X\) is smooth [5]. Moreover, the subsequent discovery [6] that heterotic string compactifications do produce chiral fermions and a classically zero cosmological constant diverted attention away from d=11 supergravity compactifications.

More recently, we have learned in the context of compactifications of \(M\) theory on \(G_2\)-holonomy manifolds to flat space, that singularities of \(X\) can support non-Abelian gauge fields [7, 8, 9] and chiral fermions [10, 11]. These are membranes which have collapsed at the singularity. Many of the arguments which were used to show the existence of chiral fermions in \(G_2\)-holonomy compactifications were local to the singularity, and do not really depend upon \(X\) having a \(G_2\)-holonomy metric. Therefore, it is natural to ask if Freund-Rubin compactifications with singularities can also produce chiral fermions.

In this paper we will use the duality between \(M\) theory and Type IIA string theory to show that Freund-Rubin vacua with chiral fermions do indeed exist.

Aside from producing new \(M\) theory compactifications with potentially realistic particle physics, there are several other reasons why this result might be of interest. As mentioned above, Freund-Rubin compactifications are dual to three dimensional conformal field theories. This offers the attractive possibility of describing, say, a supergravity GUT using 3d CFT. Supersymmetry breaking in four dimensions could also be studied holographically. Another reason is that, as far as we can tell, Freund-Rubin vacua are not dual to compactifications on special holonomy manifolds such as the heterotic string on Calabi-Yau threefolds or \(M\) theory on \(G_2\)-manifolds. As such, they might be totally disconnected from these more intensively studied vacua and thus represent new points on the space of potentially realistic quantum gravity vacua. This clearly also has implications for the ideas proposed in [12], which pre-
supposes that the special holonomy vacua represent an ‘order one fraction’ of ‘realistic vacua’.

Of course, the vacua under discussion here are not realistic since they have a negative cosmological constant and unbroken supersymmetry. But the same is true of generic \( N = 1 \) string or \( M \) theory compactifications on special holonomy manifolds after accounting for appropriate fluxes and/or quantum corrections, for example see [13]. So, in most supersymmetric string or \( M \) theory compactifications we still need a mechanism for raising the cosmological constant to, or above, zero.

In the next section we describe some general properties of Type IIA duals of Freund-Rubin vacua. We then go on to describe a fairly general construction of such Type IIA backgrounds which are obtained by adding \( D6 \) and \( D2 \)-branes to \( G_2 \)-holonomy spacetimes. In section four we describe some examples of such Type IIA compactifications to \( AdS_4 \) which contain chiral fermions. We conclude with a discussion of the holographically dual description of these \( M \) theory compactifications.

2. IIA Duals of Freund-Rubin Vacua

Freund-Rubin compactifications of \( M \) theory, on a 7-manifold \( X \), arise as the large \( N \) limit of \( N M2 \)-branes residing at the tip of a cone with base \( X \). Specifically, the metric for \( N M2 \)-branes on a cone over \( X \) is given by

\[
g_{10+1} = H^{-\frac{2}{3}}(r) g_{2+1} + H^{\frac{1}{3}}(r) (dr^2 + r^2 g_7(X))
\]

where \( H = 1 + \frac{a^6}{r^6} \) and \( g_7(X) \) is Einstein with cosmological constant equal to 6, \( g_{2+1} \) is the d=3 Minkowski metric and \( a^6 \sim \text{Vol}(X)N \). The solution has non-zero \( G \)-flux \( G = d\text{Vol}_{2+1} \wedge dH^{-1} \). Supersymmetry requires that the cone \( g_8(C(X)) = dr^2 + r^2 g_7(X) \) is an 8-metric whose holonomy is a subgroup of Spin(7) [3].

In the large \( N \) (equivalently small \( r \)) limit the solution becomes the Freund-Rubin compactification on \( X \):

\[
g_{FR} = g(AdS_4) + a^2 g_7(X)
\]

where \( g(AdS_4) \) is the standard metric on \( AdS_4 \) of “radius” \( a \).

The \( M2 \)-brane metric (for all \( N \)) has a dual description in Type IIA string theory if \( g_7(X) \) admits a \( U(1) \) action. In this case

\[
g_7(X) = h^2(Y)(d\tau + A(Y))^2 + g_6(Y)
\]

where \( h \) is a function on \( Y = X/U(1) \) and \( A \) is a 1-form on \( Y \). The dual IIA solution has a metric \( g_{IIA} \), dilaton \( \phi \) and \( RR \) gauge potential \( C \) determined.
from the $M$ theory metric by
\[ g_{10+1} = e^{\frac{4e}{3}}(d\tau + C)^2 + e^{-\frac{36}{5}}g_{IIA} \]  (4)

In particular, the IIA dual of the $M2$-brane metric is
\[ g_{IIA} = r h(Y) \left( H(r)^{-\frac{1}{2}}g_{2+1} + H(r)^{\frac{1}{2}}dr^2 + H(r)^{\frac{3}{2}}r^2g_6(Y) \right) \]  (5)
and the dilaton is given by
\[ e^{\frac{2e}{3}} = r h(Y)H(r)^{\frac{1}{6}} \]  (6)

At large $r$ the $M$ theory metric becomes the product of 3d Minkowski space and a Ricci flat cone with base $X$. The large $r$ IIA metric above is simply the IIA metric one obtains from the large $r$ $M$ theory metric using the formula (4).

The IIA $D2$-brane metric (5) is similar to the standard $D2$-brane metric except that in that case there is no overall factor of $rh(Y)$ and in the standard case $H(r) \sim 1 + r^{-5}$. The differences are crucial however, because the large $N$ limit of the $D2$-brane metric (5) is a compactification to $AdS_4$, whereas in the usual case this limit does not give an $AdS$-background. Specifically, the large $N$ limit of (5) gives
\[ g_{IIA}|_{N\to\infty} = a^3h(Y) \left( a^{-6}r^4g_{2+1} + r^{-2}dr^2 + g_6(Y) \right) \]  (7)

This is a warped metric on $AdS_4 \times Y$ with a warp factor determined by $h(Y)$. Because the large $N$ limit produces an $AdS$ vacuum, the worldvolume dynamics of the $D2$-branes is conformal in this limit.

Note however that at large $r$ the dilaton is growing, so we cannot use string perturbation theory to understand the physics. Similarly, at small $r$, in the AdS region, $e^{\frac{2e}{3}} \sim ah(Y)$, so the dilaton is finite, but the presence of RR 4-form flux through the AdS makes it difficult to quantise the theory.

In all the examples we describe in this paper we will actually see that the strong coupling at large $r$ is an artefact of the failure of the solution to properly describe the physics there. This will be illustrated shortly with a simple example.

Another important feature is that when $h$ vanishes along 3-dimensional submanifolds $Q_i$ of $Y$ the $M$ theory circle goes to zero and the IIA background has $D6$-branes wrapping $Q_i$. We will mainly be interested in such examples here. In these examples, the $M$ theory circle is contractible unlike, for instance, the case of pure $D2$-branes in flat space.
As a specific application of these results, which illustrates many features of the more interesting solutions we will find, we can consider the standard $M2$-brane solution, whose large $N$ limit is the original Freund-Rubin compactification on the round $S^7$. The metric in $M$ theory takes the form (2) with

$$g_7(X) = g(S^7)$$

We write the round metric on $S^7$ as

$$g(S^7) = d\beta^2 + \sin^2 \beta \, g(S^3) + \cos^2 \beta \, g'(S^3)$$

where $g(S^3)$ and $g'(S^3)$ are round metrics on $S^3$ with unit radius and $0 \leq \beta \leq \frac{\pi}{2}$. We now divide by the $U(1)$ which acts nontrivially on the first $S^3$ as the standard fixed point free Hopf action. In this case

$$g_6(Y) = d\beta^2 + \frac{1}{4} \sin^2 \beta \, g(S^2) + \cos^2 \beta \, g'(S^3)$$

where $g(S^2)$ is the round metric on $S^2$ of radius one, $h(Y) = \sin \beta$ and $A$ is the standard charge one Dirac monopole connection on $S^2$. Note that $g_6(Y)$ is a squashed metric on $S^6$ — the round metric does not have the factor of $\frac{1}{4}$.

In $S^7$ the $U(1)$ has a fixed point at $\beta = 0$ which is a copy of $S^3$. Since this is codimension four in $M$ theory, in Type IIA theory we have a $D6$-brane. Thus, our solution describes $N$ $D2$-branes on a cone over $S^6$ which has been deformed by the presence of a $D6$-brane wrapping a cone over $S^3$. In the large $N$ limit we get IIA on $S^6 \times AdS_4$ with a space filling $D6$-brane wrapping $S^3 \times AdS_4$. Note that since $S^3$ is contractible in $S^6$, the total $D6$-brane charge is zero: i.e. the wrapped $D6$ is dipolar.

Since a cone on $S^6$ is $\mathbb{R}^7$ we interpret the solution as the result of adding $N$ $D2$-branes to a $D6$-brane in flat space, with the $D2$-branes inside the $D6$-brane. The deviation of the metric on $S^6$ away from the round metric is precisely the back reaction of the metric in the presence of the $D6$-brane. Note, however, that this interpretation makes sense only near the branes, because $D$-branes in flat space should approach flat space at infinity, with a constant dilaton.

The complete supergravity solution describing $N$ $D2$ branes inside a $D6$-brane was found in [14]. Near the branes it is exactly the solution described above.

Many of the features of this example will be present in all the examples we discuss later, so it is worthwhile summarising them. The first and most important for our purposes is that the IIA duals of Freund-Rubin compactifications are only good descriptions of configurations of $D2$-branes and $D6$-branes near the branes. The above simple example illustrates that — since
the metric of $D$-branes in flat spacetime should asymptote to flat spacetime — the $M2$-brane metric is not a good description of the Type IIA background away from the branes. Since our main interest is in any case Freund-Rubin compactification, we only require the solution near the branes\(^1\).

A second striking feature which will persist in later examples is that, in Type IIA on $S^6 \times AdS_4$, the $D6$-brane is wrapping a contractible submanifold, yet fills the entire $AdS$ space. This does not lead to a tadpole because the $D6$-brane charge is zero. However it raises an obvious stability issue. Namely, why doesn’t the brane simply slip off of the sphere and decay to the vacuum? This cannot happen either classically or semiclassically because of supersymmetry. A supersymmetric vacuum state with negative cosmological constant is classically stable against decay due to the Breitenlohner-Freedman bound \([15]\). Similarly, supersymmetry prohibits the existence of spherical instantons which mediate semi-classical tunneling processes out of the supersymmetric vacua \([16, 17, 18]\). See \([19]\) for some recent discussion of bubble nucleation in quantum gravity. Solutions describing branes wrapping contractible submanifolds have appeared in other $AdS$ contexts \([20]\).

### 3. IIA Duals from $G_2$-cones

One possible approach to obtaining compactifications with chiral fermions is to find dual Type IIA descriptions in which $D6$-branes intersect along $AdS_4$ in such a way that the open string stretched between them is a chiral fermion. This approach was certainly useful in the case of compactifications to Minkowski space; for the first supersymmetric chiral examples see \([21]\).

Since $D6$-branes which intersect along a copy of four dimensional Minkowski spacetime are known to typically give rise to chiral fermions \([22]\), one’s first guess is to generalise the previous example by adding an additional stack of $D6$-branes intersecting the first. Remarkably the supergravity solution for this system, even including a large number of $M2$-branes, is already known in the near horizon limit \([23]\) and we will rederive it here in a simpler way. Unfortunately, as we will explain, even though this spacetime does contain chiral fermions, it also contains their $C$-conjugates, so the total spectrum is non-chiral.

Consider a pair of flat $D6$-branes in flat spacetime, $C^3 \times R^{3,1}$ which intersect along the copy of $R^{3,1}$ at the origin of $C^3$ and are oriented in such a way that the three angles between them are all $\frac{2\pi}{3}$. This is a supersymmetric configuration and the massless open strings stretched between the two branes are described by a chiral supermultiplet containing a chiral fermion.

\(^1\)As an aside, our results do predict the existence of many new manifolds of $Spin(7)$ holonomy, which represent the metric transverse to the $M2$-branes in the full solution.
Near the intersection point this Type IIA background is known to be dual to $M$ theory on a $G_{2}$-holonomy cone over $\text{CP}^{3}$ [10]. In $M$ theory, the conical singularity of the $G_{2}$-holonomy metric

$$d\rho^2 + \rho^2 g(\text{CP}^{3})$$

(11)

corresponds to the intersection point of the two branes$^{2}$.

Since it has a Type IIA interpretation, the metric $g_{6}$ on $\text{CP}^{3}$ can be written as

$$g(\text{CP}^{3}) = f^{2}(S^{5})(d\tau + A)^{2} + g(S^{5})$$

(12)

i.e. as a circle fibration over $S^{5}$. Note that the induced metric on $S^{5}$ is not round, due to the presence of the branes. The metric can be given explicitly, but we will not require its details.

In $M$ theory, the eleven dimensional metric describing these branes is

$$g_{10+1} = g_{2+1} + dw^{2} + d\rho^{2} + \rho^{2} g(\text{CP}^{3})$$

(13)

with $w$ a coordinate on $\mathbb{R}$. Note that this spacetime can be written in the form

$$g_{10+1} = g_{2+1} + dr^{2} + r^{2}(d\alpha^{2} + \sin^{2}\alpha g(\text{CP}^{3}))$$

(14)

where $w = r \cos \alpha$ and $\rho = r \sin \alpha$, with $0 \leq \alpha \leq \pi$.

We can now add $N$ $M2$-branes at $r = 0$, giving a solution which takes the form of (1). The near horizon limit of this solution is

$$g_{10+1} = g(AdS_{4}) + a^{2}(d\alpha^{2} + \sin^{2}\alpha g(\text{CP}^{3}))$$

(15)

This is a Freund-Rubin compactification on the 7-manifold with metric of the form $g_{7}(X) = d\alpha^{2} + \sin^{2}\alpha g_{6}$. This Freund-Rubin solution was recently found in [23]. Notice that near $\alpha = 0$ and $\pi$ the metric has conical singularities isomorphic to those of the original $G_{2}$-cone we began with. Since we already know that such singularities support chiral fermions in $M$ theory [10] this Freund-Rubin solution has two chiral fermions. However, these two chiral fermions are $C$-conjugates of each other because the two singularities have opposite orientation. Hence the full spectrum is non-chiral. We now discuss the IIA dual.

Near the branes the Type IIA metric is

$$g_{IIA} = a^{3} f(S^{5}) \sin \alpha (a^{-6} r^{4} g_{2+1} + r^{-2} dr^{2} + d\alpha^{2} + \sin^{2}\alpha g(S^{5}))$$

(16)

which describes a compactification to $AdS_{4}$ on a non-round $S^{6}$ with two $D6$-branes. The $M$ theory circle vanishes when the function $f$ is zero, and this

$^{2}$ $g(\text{CP}^{3})$ is not the Fubini-Study metric [27].
consists of two copies of $S^3$. Thus the $D6$-branes each wrap different $S^3$’s in $S^6$ and meet each other at $\alpha = 0$ and $\pi$.

This is also clear because the space surrounding the $D2$-brane is $S^6$, with or without $D6$-branes. The two $S^3$’s are simply the intersections of $R^4$’s in $R^7$ with $S^6$. We can thus interpret the solution as describing the result of adding a large number of $D2$-branes to the two intersecting $D6$-branes in flat spacetime. Again we emphasise that the solution is only valid near the $D6$-branes.

Near each intersection point the geometry is that of the intersecting branes in flat space, so again we see that there are two ($C$-conjugate) chiral fermions in this description, consistent with the $M$ theory interpretation.

Much more generally, if

$$d\rho^2 + \rho^2 g_6(W)$$

is any $G_2$-holonomy cone, then

$$g_7(W) = d\alpha^2 + \sin^2 \alpha g_6(W)$$

is a compact Einstein manifold with positive cosmological constant [23] which provides a supersymmetric Freund-Rubin solution of $M$ theory.

Hence, we can write an $M2$-brane solution whose near horizon limit is a Freund-Rubin compactification with a pair of $C$-conjugate conical singularities. Many examples of such $M$ theory vacua can be produced by simply considering Type IIA on flat $C^3 \times R^{3,1}$ with collections of $D6$-branes spanning linearly embedded $R^3$’s through the origin in $C^3$. If the $R^3$’s are chosen at $SU(3)$ angles to one another, then near the intersection point the $M$ theory description is a conical singularity in a $G_2$-manifold.

For instance, if $W = SU(3)/U(1)^2$ with its natural metric, then the corresponding $G_2$-holonomy cone is the $M$ theory description of 3 $D6$-branes at $SU(3)$ angles [10]. If $W = WCP^3_{ppqq}$ then the IIA background has $p$ $D6$-branes intersecting $q$ $D6$-branes again at $SU(3)$ angles [10], although in this case the metric on $W$ is not known.

There is another simple reason why beginning with supersymmetric $D6$-branes in flat space does not produce chiral fermions in $M$ theory when we add a large number of $M2$-branes. This is because in the $AdS$ region, the $D6$-branes are wrapping 3-manifolds in $S^6$. Such a 3-manifold has trivial homology class and hence the intersection number of a pair of such 3-manifolds, which is what counts the net number of chiral fermions, is zero. Hence, in order to produce genuinely chiral examples we need to find Type IIA duals in which the $S^6$ gets replaced with a 6-manifold $Z$ with non-zero third homology group. This can be achieved by replacing the $R^7$ transverse to the
$D2$-branes in flat space by a more complicated manifold. We will in fact replace $R^7$ by a $G_2$-holonomy cone over a 6-manifold $Z$, for which $H_3(Z)$ is non-trivial.

Note that now there are two $G_2$-holonomy cones playing distinct roles in these spacetimes. One has a base $W$ and represents the $M$ theory singularity close to the intersection of the $D6$-branes. The second has base $Z$ and represents the space transverse to the $D2$-branes and which the $D6$-branes wrap.

The basic overall picture of the Type IIA backgrounds we will henceforth consider may be summarised as follows. We begin with Type IIA on a $G_2$-holonomy cone $C(Z)$ with base $Z$ times 3d Minkowski spacetime. We then add supersymmetric configurations of $D2$-branes and $D6$-branes. The $D2$-branes span the Minkowski space and reside at the tip of the cone over $Z$. Each set of $D6$-branes wrap the Minkowski space times a supersymmetric 4-manifold $N_i$ in $C(Z)$. We additionally require that $N_i$ is itself a cone $C(Q_i)$ with $Q_i$ a compact 3-manifold in $Z$. The complete supergravity solution describing this background is difficult to write, but it is clear that it should share many features of the the previous backgrounds we have discussed, in which $Z = S^6$. In particular, supersymmetry guarantees its existence.

Specifically, the spacetime will asymptote to $C(Z)$ times Minkowski space, without branes. Near the tip of the cone we will find a Type IIA compactification on $Z$ to $AdS_4$. Additionally, there will be $D6$-branes which fill the $AdS$ space and wrap the 3-manifolds $Q_i$ in $Z$. In order to obtain chiral fermions in this background we require the intersection numbers of the $Q_i$ to be non-zero.

In $M$ theory the complete spacetime will be asymptotic to $S^1 \times C(Z)$ times 3d Minkowski space. Without the $M2$-branes the point at which all the $D6$-branes meet — the tip of $C(Z)$ — is described in $M$ theory by a conical singularity in a manifold with $Spin(7)$-holonomy. Hence, near the $M2$-branes the $M$ theory metric will approach a Freund-Rubin solution of the form (2), in which $X$ is a compact 7-manifold with a $U(1)$ quotient which is $Z$. The $U(1)$ fibers will degenerate to zero size on each of the $Q_i$’s. At the intersection points between $Q_i$ and $Q_j$, $X$ will have a conical singularity with base $W_{ij}$. Near the intersection points the cone on $W_{ij}$ also has $G_2$-holonomy. More generally there could be multiple intersection points.

Supersymmetry requires that the complete 8-manifold $V$ transverse to the $M2$-branes is a manifold with $Spin(7)$-holonomy. We are thus predicting the existence of many new metrics of $Spin(7)$-holonomy. $Spin(7)$-manifolds interpolating between $S^3 \times C(Z)$ and cones on Freund-Rubin manifolds $X$ are known [24, 25] for some choices of $Z$ and $X$, though not in the cases we will study here.
An important new feature of these more general backgrounds is that the $Q_i$ are incontractible, so we cannot wrap $D6$-branes on arbitrary $Q_i$, since $Z$ is compact. Cancellation of tadpoles requires that the total $D6$-brane charge is zero, i.e.

$$\sum_i k_i [Q_i] = 0$$

(19)

where $k_i$ is the number of $D6$-branes wrapping $Q_i$ whose homology class is $[Q_i]$. In compactifications to flat space, for example on a Calabi-Yau, this condition can never be satisfied non-trivially in a supersymmetric fashion because there the $Q_i$ are calibrated by a closed 3-form. The way around this problem there is to introduce orientifold planes which have negative charge. As we will see in compactifications to $AdS_4$ on $Z$ one can find non-trivial solutions to (19) without orientifolds, and in this case $Q$ is not calibrated by a closed 3-form. These examples are presumably related to the “generalised” calibrations studied in [26].

Like the solution in section two, these configurations, since they have zero $D6$-brane charge are in principle non-perturbatively unstable to decay to an $AdS$ vacuum without $D6$-branes, but as discussed there, these vacua are, at the very least, semi-classically stable.

The only known example of a $G_2$-holonomy cone $C(Z)$ in which $Z$ has non-trivial 3-cycles is $S^3 \times S^3$ (or its discrete quotients). Happily, as we will see, supersymmetric compactifications of Type IIA on $S^3 \times S^3$ with $D6$-branes exist which contain chiral fermions.

4. Chiral Fermions in $S^3 \times S^3 \times AdS_4$

The following metric has $G_2$-holonomy [27]

$$ds^2 = dr^2 + \frac{r^2}{9} (\omega_a^2 + \bar{\omega}_a^2 - \omega_a \bar{\omega}_a)$$

(20)

This is a cone with base $Z = S^3 \times S^3$, where the induced metric on $Z$ is $G/H$, with $G = SU(2)^3$ and $H$ the diagonal $SU(2)$ subgroup. The $\omega_a$ are left invariant 1-forms on the first $S^3$, viewed as the $SU(2)$ group manifold, and $\bar{\omega}_a$ are left invariant 1-forms on the second $S^3$. The metric is invariant under the order six group generated by exchanging the two $S^3$’s which acts on the one forms as

$$\alpha : \begin{pmatrix} \omega_a \\ \bar{\omega}_a \end{pmatrix} \rightarrow \begin{pmatrix} \bar{\omega}_a \\ \omega_a \end{pmatrix} \quad (21)$$

and by cyclic permutations of the three $SU(2)$’s in $G$, which has the effect

$$\beta : \begin{pmatrix} \omega_a \\ \bar{\omega}_a \end{pmatrix} \rightarrow \begin{pmatrix} -\bar{\omega}_a \\ \omega_a - \bar{\omega}_a \end{pmatrix} \quad (22)$$
The metric on $S^3 \times S^3$ has an almost complex structure defined by introducing the complex frames

$$\eta_a \equiv \frac{1}{3}(\omega_a + \tau \tilde{\omega}_a)$$

(23)

where $\tau = e^{2\pi i/3}$. The metric is thus

$$\frac{1}{9}(\omega_a^2 + \omega_a^2 - \omega_a \tilde{\omega}_a) = \eta_a \bar{\eta}_a$$

(24)

The associated Kahler and holomorphic volume form are defined as

$$\omega \equiv i \frac{1}{2} \eta_a \wedge \bar{\eta}_a$$

(25)

$$\Omega \equiv \eta_1 \wedge \eta_2 \wedge \eta_3$$

(26)

and obey the equations

$$d\omega = -3\text{Im} \Omega$$

(27)

$$d\text{Re} \Omega = -2\omega \wedge \omega$$

(28)

On the $G_2$-holonomy cone $dr^2 + r^2 \eta_a \bar{\eta}_a$ there exists a covariantly constant 3-form $\varphi$ and 4-form $*\varphi$. These forms may be written in terms of $\omega$ and $\Omega$ as follows

$$\varphi = r^2 dr \wedge \omega - r^3 \text{Im} \Omega$$

(29)

and

$$*\varphi = r^3 \text{Re} \Omega \wedge dr + \frac{r^4}{2} \omega \wedge \omega$$

(30)

Note that in this basis

$$\alpha : \eta_a \rightarrow \tau \bar{\eta}_a$$

(31)

and $\beta$ acts holomorphically as multiplication by $\tau$. Clearly, extending these discrete symmetries trivially to the cone,

$$\alpha : \varphi \rightarrow -\varphi$$

(32)

leaving $*\varphi$ invariant, whereas $\beta$ preserves both $\varphi$ and $*\varphi$.

We are interested in four dimensional, supersymmetric submanifolds $N$ of the $G_2$-holonomy cone. These are defined by the condition

$$*\varphi|_N = \pm d\text{Vol}(N)$$

(33)

i.e. that the restriction of $*\varphi$ to $N$ is its volume form, up to orientation. If $N_1$ and $N_2$ are calibrated with opposite orientation, then a configuration of
branes wrapping $N_1$ and anti-branes wrapping $N_2$ is supersymmetric. Equivalently, a configuration of branes on $N_1$ and branes on $N_2$ is supersymmetric, where $N_2$ is $N_2$ with the opposite orientation. Note that, up to the overall orientation of $N$, (33) is equivalent to

$$\varphi|_N = 0$$  \hspace{1cm} (34)

Examples of supersymmetric $N$'s may be obtained as the fixed point sets of orientation reversing isometries which preserve $\ast \varphi$ but reverse the sign of $\varphi$. Clearly $\alpha$ is just such an isometry. In $S^3 \times S^3$, the fixed point set of $\alpha$ is the ‘diagonal’ $S^3$. Therefore in the cone on $S^3 \times S^3$ the $\alpha$ fixed points are a cone over this $S^3$, which is a copy of $R^4$. The induced metric on this $R^4$ is not flat.

From the formula for $\ast \varphi$, one can explicitly check that the restriction to this $R^4$ is the volume form, up to orientation. Thus, after taking the large $N$ limit, we learn that in Type IIA on $Z \times AdS_4$, the diagonal $S^3$ is supersymmetric.

Furthermore, since $\varphi$ and $\ast \varphi$ are $\beta$ invariant, the images of the diagonal $S^3$ under $\beta$ and $\beta^2$ are also supersymmetric. So we have found three supersymmetric $S^3$'s in Type IIA on $S^3 \times S^3 \times AdS_4$. Moreover, the union of all three of these $S^3$'s is supersymmetric. Notice that the $S^3$'s are indeed ‘calibrated’ by $\Re \Omega$ which is not closed and that they each come in a three dimensional family parametrised by $SU(2)$. Denote these three families of supersymmetric $S^3$'s by $Q_\alpha$, $Q_{\beta \alpha}$ and $Q_{\beta^2 \alpha}$.

Before we go on to discuss wrapped $D6$-branes we need to consider the third homology of $Z = S^3 \times S^3$, $H_3(Z, Z) \cong Z \oplus Z$. Hence the third homology is a rank two lattice. The homology can be generated by $(1,0)$ and $(0,1)$ and explicit representatives of the generators may simply be taken to be $S^3 \times pt$ and $pt \times S^3$. Since $\alpha$ and $\beta$ are symmetries of $Z$, they must also act naturally on $H_3(Z, Z)$. This action is readily seen to be represented by the matrices

$$\alpha^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$ \hspace{1cm} (35)

and

$$\beta^* = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$ \hspace{1cm} (36)

Because of these symmetries $H_3(Z, Z)$ may naturally be regarded as the root lattice of the $SU(3)$ Lie algebra, i.e. the metric on $Z$ induces the lattice with a natural metric and complex structure. When regarded as a lattice in $C$, the metric is

$$dzd\bar{z}$$ \hspace{1cm} (37)
where $z = x + \tau y$. Then $\beta$ acts by $\frac{2\pi}{3}$ rotations and $\alpha$ by complex conjugation.

The fixed point set of $\beta$ acting on $H_3(Z, Z)$ is empty. Therefore if we take any 3-cycle and its two images under $\beta$, their sum is zero in homology. This means that the homology classes of the three mutually supersymmetric $S^3$’s add up to zero. Hence they are either given by $[Q_\alpha] = (1, -1)$, $[Q_\beta\alpha] = (1, 0)$, $[Q_\beta^2\alpha] = (0, 1)$ or by $[Q_\alpha] = (1, 1)$, $[Q_\beta\alpha] = (-1, 0)$, $[Q_\beta^2\alpha] = (0, -1)$. There are thus two different collections of mutually supersymmetric $S^3$’s. The difference between these cases can be described physically as the difference between branes and anti-branes.

Finally we discuss the intersection form between 3-cycles. If $[Q_1] = (a, b)$ and $[Q_2] = (c, d)$ are two three-cycles, then their intersection number is

$$[Q_1] \cdot [Q_2] = -[Q_2] \cdot [Q_1] = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(38)

Note that the pairwise intersection numbers between the three supersymmetric $S^3$’s is $\pm 1$.

Since the homology classes of the three supersymmetric cycles add up to zero, the tadpole cancellation condition may be satisfied by wrapping an equal number $k$ of $D6$-branes on each 3-cycle.

There is a simple model of these $D6$-branes in flat space which is a good local description of the brane configuration. For this, replace $Z = S^3 \times S^3$ with $\mathbb{C}^3$ and with flat metric

$$dz_\alpha d\bar{z}_\alpha$$

(39)

where the complex structure is defined by $z_\alpha = x_\alpha + \tau y_\alpha$. We first add a $D6$-brane along the 3-plane at $y_\alpha = 0$. This is the local analog of the $(1, 0)$ brane on $Z$. Now we act with the center of $SU(3)$ on this 3-plane and put a $D6$-brane at the images. The center of $SU(3)$ is generated by $\tau \mathbf{1}$. This gives a configuration of 3 $D6$-branes which are supersymmetric because they are related by $SU(3)$ rotations. The $\tau$ and $\tau^2$ images are the analogs of the images of $(1, 0)$ under $\beta$. The image under $\tau$ is the analog of $(0, 1)$ and the image under $\tau^2$ is the analog of $(-1, -1)$. The minus sign is because $SU(3)$ preserves orientation. Notice that the $\tau^2$ brane may equally well be regarded as an anti-brane at 60 degrees to the other two.

In fact one may check that the three supersymmetric $S^3$’s in $Z$ meet at precisely 120 degrees in all directions, so the flat space model is indeed a good description of the branes locally.

Near the intersection point of the three sets of branes the $M$ theory description is that of a conical singularity with base $W$. In this case $W$ is a $\mathbb{Z}_k$-orbifold of $SU(3)/U(1)^2$ [10].
The spectrum of light charged particles in this model is simple to deduce. The gauge group is $U(k)^3$ and there are three sets of bifundamental fields in the representation $(k, \bar{k}, 1) \oplus (1, k, \bar{k}) \oplus (\bar{k}, 1, k)$. The reader might be worried that we have extrapolated results for the spectrum based upon those for which the branes intersect in flat spacetime. However, at extremely large $N$ the IIA spacetime is almost flat and the net number of chiral fermions is independent of $N$.

Apart from the central $U(1)^3$, this gauge group and matter content is anomaly free. This is expected because the brane configuration is free from tadpoles, at least in the supergravity approximation. As far as the anomalous $U(1)$’s are concerned we expect that they are in fact massive in the quantum theory, as is the case in all known string compactifications with anomalous $U(1)$’s at tree level [28].

There may be many other supersymmetric configurations of $D6$-branes on $Z$. For example, let $(p_i, q_i)$ be three 3-cycles. We may then wrap $k_i$ $D6$-branes on $(p_i, q_i)$ if

$$\sum_i k_i (p_i, q_i) = (0, 0) \quad (40)$$

This is readily seen to give an anomaly free spectrum of chiral fermions, charged under the gauge group $SU(k_1) \times SU(k_2) \times SU(k_3)$. The matter spectrum here is given by $n_{12}(k_1, \bar{k}_2, 1) \oplus n_{23}(1, k_2, \bar{k}_3) \oplus n_{31}(\bar{k}_1, 1, k_3)$, where

$$n_{ij} = p_i q_j - p_j q_i \quad (41)$$

Unfortunately, we have not been able to construct supersymmetric 3-cycles whose classes are not multiples of $(1, 0)$ and its images under $\beta$. However if they existed one might be able to find more phenomenologically interesting supersymmetric models in this way. We think it is also likely that many other Freund-Rubin compactifications exist which give rise to more phenomenologically interesting models of particle physics. These may or may not be dual to Type IIA vacua.

4.1. Compactifications?

An important point concerning typical Freund-Rubin compactifications such as those on round spheres or other homogeneous spaces is that there is no consistent low energy limit in which one can ignore the Kaluza-Klein modes. The reason is simply that in such examples the scale of $X$ is of the same order as that of the $AdS$ spacetime and hence fluctuations of the supergravity multiplet have masses of the same order as the Kaluza-Klein modes. This problem can be avoided if the volume of $X$ computed in the metric
$g_7(X)$ in equation (2) is parametrically smaller than that of the round $S^7$ of unit radius, whilst keeping the cosmological constant equal to 6. Note that examples of Freund-Rubin 7-manifolds obeying this criterium, but preserving $\mathcal{N} = 2$ spacetime supersymmetry, may be found in [29], and we presume that they also exist in the $\mathcal{N} = 1$ cases of interest here.

In the examples dual to Type IIA vacua with branes that we presented in this section this criterium is probably not obeyed, although, since we have not obtained the full supergravity solution in these examples we cannot be sure.

One method for reducing the volume of $X$ whilst preserving its cosmological constant is to consider discrete quotients of $X$, since if $X/\Gamma$ is a discrete quotient of $X$, then $\text{Vol}(X/\Gamma) = \text{Vol}(X)/|\Gamma|$, where $|\Gamma|$ is the order of $\Gamma$. Note however that we require $\Gamma$ to be such that $X/\Gamma$ is smaller than $X$ in all directions in order to increase the mass of all Kaluza-Klein modes. In the examples presented in this section, $Z$ has $SU(2)^3$ symmetry group and we can take $\Gamma$ to be any discrete subgroup. The largest freely acting subgroup that $Z$ has, which reduces the volume equally in all directions, is the product of two copies of the binary icosahedral group $I$, which acts in such a way that $Z/I = S^3/I \times S^3/I$. Since $|I| = 120$ the volume of the Type IIA spacetime is reduced by a factor of $120^2$, thus reducing the typical radius by a factor of about 5. This accounts for six of the seven dimensions of $X$. The last dimension is the $U(1)$ fiber which vanishes on the $D6$-branes. Since we have a $U(1)$ symmetry we can gauge a discrete subgroup of order $M$ and this reduces the radius of this dimension by $M$. Note that this increases the number of $D6$-branes by a factor of $M$. A factor of 5 is just about enough to give a gap in the spectrum between the Kaluza-Klein modes and the supergravity, gauge and matter multiplets.

In this case one obtains supersymmetric, chiral Type IIA compactifications with the branes wrapping copies of $S^3/I$. One then has interesting additional possibilities for breaking the $SU(k)$ gauge groups by discrete Wilson lines, which might be worthy of further study.

5. Holographic Duals

One of the most interesting aspects of Freund-Rubin compactifications is that they are dual to three dimensional field theories. For example, if $X$ is such that its singularities support a copy of the supersymmetric standard model, then there exists a 3d field theory on the worldvolume of the $N$ $M2$-branes which is the holographic dual of $M$ theory on $X \times \text{AdS}_4$. It is clearly of importance then to understand in detail these three dimensional field theories. In the examples of section three we can actually say what this
3d theory is in the UV.

Those examples have dual Type IIA descriptions as $D2$-branes in the background of intersecting $D6$-branes in flat spacetime, and in those cases the 3d theory is simply the worldvolume theory of the $D2$-branes. In fact, in these cases, the 3d theory on the $D2$-branes has already been studied in [30]. Since we have found $AdS$ duals of these theories at large $N$, all of these theories must flow in the infrared to renormalisation group fixed points. An interesting point for further study is that in some of these examples, the gauge dynamics in the bulk of $AdS_4$ can be strong at low energies. For example in the case that we begin with a set of $p$ $D6$-branes intersecting a single $D6$-brane at $\frac{2\pi}{3}$ angles, the gauge group and matter content supported at the singularities of $X$ is $SU(p)$ with one flavour. In flat space (which here is the large $N$ limit), this gauge theory is known to generate a non-perturbative superpotential [31]. If there is no mass term for the quarks and their superpartners this gauge theory has no vacuum. This model already raises many questions when embedded into a Freund-Rubin compactification of $M$ theory: is a mass term generated by quantum effects such as membrane instantons? Can the non-perturbative superpotential be understood in terms of membrane instantons? Since the background has a classical cosmological constant the superpotential is classically non-zero, so what is the vacuum state in $M$ theory?

In the examples of section four which contain chiral fermions in $AdS_4$, it is more difficult to understand the $D2$-brane theory. However, in principle one can quantise open strings in the $G_2$-holonomy background perturbatively and hence understand the $D2$-brane field theory in the UV. There are some obvious properties that this theory has.

The gauge group is $U(N)$. Since the three sets of $k$ $D6$-branes each support $SU(k)$ gauge groups, the d=3 theory has a $G = SU(k)^3$ global symmetry. Furthermore, the strings stretched between the $D2$-branes and $D6$-branes are massless in the UV and give rise to three sets of $k$ component fundamentals of $U(N)$, transforming in the obvious way under $G$. The UV theory is supersymmetric, so all these fields are in matter supermultiplets of d=3 $\mathcal{N} = 1$ supersymmetry.

Holography might also provide an interesting way to look at supersymmetry breaking. For, instance, if $X$ has singularities supporting a gauge theory which is known to spontaneously break supersymmetry, then how is that manifested in the holographically dual field theory? The UV/IR correspondence suggests that if supersymmetry is restored at high energies in the 4d bulk (at extremely large $N$), then the dual field theory has supersymmetry in the IR limit, but is broken in the UV. For example, the dual theory might be a supersymmetric theory perturbed in the UV by non-supersymmetric
irrelevant operators.

Finally, if the quantum dynamics of $M$ theory on $X$ is such that, after supersymmetry breaking is correctly accounted for, the vacuum is de Sitter space, we might be fortunate enough to learn something about the holographic duals of inflationary cosmologies. One possible context where this idea might be realised is if the singularities of $X$ produce strong gauge dynamics which spontaneously break supersymmetry at a scale $m_{\text{susy}}$. If $m_{\text{susy}}^4 \geq -\Lambda_{\text{AdS}} \sim m_p^2 a^{-2}$ then the effective cosmological constant will be $\geq 0$. For example, if $m_{\text{susy}} \sim 10^{11}\text{GeV}$ then the original $\text{AdS}$ mass scale $a^{-1}$ is of order 1 TeV.

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