Non-rational $\hat{su}(2)$ cosets and Liouville field theory

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Abstract: We propose an $\hat{su}(2)$ WZNW model with a non-rational level and a continuous spectrum based on the non-unitary hermitian representations of the chiral algebra $\hat{su}(2)_\kappa$. It is conjectured that for this model the continuous spectra counterpart of the Goddard-Kent-Olive (GKO) coset construction yields the Liouville and the imaginary Liouville field theories. We support the conjecture by a number of nontrivial tests based on analytic calculations.

1 Introduction

It was conjectured several years ago that partition functions of $\mathcal{N} = 2$ superconformal SU($N$) gauge theories in four dimensions are directly related to correlation functions of the two-dimensional Liouville/Toda field theories [1, 2]. One of generalizations of the AGT correspondence was the relation between $\mathcal{N} = 2$ SU($N$) gauge theories on $\mathbb{R}^4/\mathbb{Z}_p$ and para-Liouville/Toda theories [3, 4]. It was in particular observed [5-8] that in the case $N = p = 2$ the blow-up formula for the Nekrasov partition function suggests the SL-LL relation of the schematic form

$$\text{free fermion } \otimes \mathcal{N} = 1 \text{ super-Liouville } \sim \text{ Liouville } \otimes_P \text{ Liouville}, \quad (1.1)$$

where the symbol $\otimes_P$ denotes a projected tensor product in which only selected pairs of conformal families are present.
An explanation of relations of this kind was given in [7]. It was motivated by old results [9–12] relating various rational models realized as quotients

$$V(p, m) \sim \frac{\tilde{su}(2)_m \times \tilde{su}(2)_p}{\tilde{su}(2)_{m+p}},$$

(1.2)

where $\tilde{su}(2)_p$ denotes the $su(2)$ Kac-Moody algebra of level $p$. Relation (1.1) can be seen as a non-rational counterpart of the relation between the Virasoro minimal models $V(m) = V(1, m)$ and the $\mathcal{N} = 1$ superconformal minimal models $SV(m) = V(2, m)$ [9–12]:

$$V(1) \otimes SV(m) \sim V(m) \otimes_P V(m + 1), \quad m = 1, 2, \ldots.$$  

Essential elements of the SL-LL relation in the Neveu-Schwarz sector were discussed in [8]. The proof has been recently completed in [13] and the extension to the Ramond sector was analyzed in [14].

The exact SL-LL correspondence raises the questions about other relations of this type. The main aim of the present paper is to formulate and justify the relations of the following schematic form

$$\tilde{su}(2)_\kappa \otimes \tilde{su}(2)_1 \sim \text{Liouville} \otimes_P \tilde{su}(2)_{\kappa+1},$$

$$\tilde{su}(2)_\kappa \otimes \tilde{su}(2)_1 \sim \text{imaginary Liouville} \otimes_P \tilde{su}(2)_{\kappa+1},$$

(1.3)

with continuous level $\kappa \in \mathbb{R}$ related to the central charge of the Liouville theory and with continuous spectra of $\tilde{su}(2)_\kappa$ and $\tilde{su}(2)_{\kappa+1}$ WZNW models. On the one hand side these relations can be seen as non-rational counterparts of the famous Goddard-Kent-Olive coset construction of minimal models [15] with the branching functions encoded in the projected tensor product. On the other hand they go beyond the standard coset construction. Being motivated by the SL-LL equivalence we propose to regard the relations above as exact equivalences of CFT$_2$ models. This in particular implies exact relations between structure constants and correlation functions of all the models involved.

Before entering the discussion of the $\tilde{su}(2)_\kappa$ WZNW theory it is instructive to place relation (1.3) in a slightly wider context. The Virasoro minimal models [16], the Dotsenko-Fateev (DF) models\footnote{By the DF model we mean a (not unitary) CFT with nonrational central charge and the infinite discrete diagonal spectrum consisting of all degenerate weights [19]. This model was recently discussed in [20] under the name generalized minimal model. As this term is frequently used for the imaginary Liouville theory we shall use the name proposed in [19].} [17, 18], the Liouville theory [21–24] and the imaginary Liouville theory (the generalized minimal model) [25–28] form a system of interrelated objects connected by analytic continuations of different structures. For instance the structure constants of the minimal models and the DF model are analytic continuations of the imaginary Liouville structure constants [20]. The (real) Liouville structure constants cannot be obtained in this way which can be seen as yet another manifestation of the $c = 1$ barrier [25–27]. They are however unique solutions to the equations obtained by the analytic continuation of the corresponding equations in the Liouville theory and the DF structure constants can be identified as their residues at the degenerate weights.

The underlying fundamental structure of the whole system is the fusion matrix for the Virasoro conformal blocks. Indeed the finite fusing matrices of the minimal models and of the DF model
can be obtained by analytic continuation of the fusion integral kernel of the Liouville theory. It was shown by Runkel [29, 30] in the case of minimal models and by Ponsot and Teschner [31] in the case of Liouville theory, that under certain assumptions all structure constants of these models can be reconstructed from their fusing matrices. In a more general context of non-rational theories the relation between structure constants and fusing matrices was discussed in [32]. It has been also recently clarified by Gaiotto that the fusing matrix is fundamental for the explicit construction of the Verlinde line operators in the Liouville theory [33].

In the case of the Kac-Moody algebra $\hat{\mathfrak{su}}(2)$ a counterpart of the picture above is less understood. There are however many elements already known. The relations between them were recently discussed in [20]. They are schematically shown on Fig.2. The box $\hat{\mathfrak{su}}(2)$ minimal models represents the unitary rational series of $\hat{\mathfrak{su}}(2)$ models with positive integer levels [34–36] and the non-unitary $\hat{\mathfrak{su}}(2)$ models with rational levels [37, 38] and spectra corresponding to the admissible representations [39].

The $\hat{\mathfrak{su}}(2)$ models with non-rational level were first analyzed in [40]. In the case of the diagonal spectrum of the degenerate representations the structure constants were found by Andreev [41]. The derivation was based on the Fateev-Zamolodchikov (FZ) relation [34] between the $\mathfrak{su}(2)$ Knizhnik-Zamolodchikov (KZ) equation [42] and the differential equations for a Virasoro degenerate field originally observed in the context of minimal models. The model has been recently discussed in [20] under the name generalized $\hat{\mathfrak{su}}(2)$ WZW model \footnote{We are using the name Andreev model instead as the term the generalized $\hat{\mathfrak{su}}(2)$ WZW model was already used for the $\hat{\mathfrak{su}}(2)$ counterpart of the imaginary Liouville theory [43].}

In the continuous spectra part of Fig.2 the counterpart of the (real) Liouville theory is called the (real) $\hat{\mathfrak{su}}(2)$ WZNW model. It is defined in the range of levels $\kappa < -2$ and its spectrum corresponds to principal unitary series of $\mathfrak{sl}(2, \mathbb{C})$ representations with $j \in -\frac{1}{2} + i \mathbb{R}$\footnote{This representations can be seen as non-unitary representations of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$.} Under the general duality between $G^C/G$ and $G$ WZNW models [44–47] the $\hat{\mathfrak{su}}(2)$ WZNW model at level $\kappa < -2$ is equivalent
imaginary $\hat{\mathfrak{su}}(2)$ WZNW model
( generalized $\hat{\mathfrak{su}}(2)$ minimal models )

$\hat{\mathfrak{su}}(2)$ minimal models

(res) $\hat{\mathfrak{su}}(2)$ WZNW model
( $H^+_3$ at level $-\kappa$ )

residues

Andreev model

$k > -2$

$k = -2$

$-2 > k$

Figure 2. $\hat{\mathfrak{su}}(2)$ models

to the $H^+_3 = SL(2, \mathbb{C})/SU(2)$ coset model at level $\kappa' = -\kappa > 2$.

The later one has been solved both by the conformal bootstrap
methods [47, 49–51] and by the path integral
techniques [52–55].

The crossing symmetry of the $H^+_3$ WZNW model was proven in
[56] with the help of the suitably extended FZ relation.

The $\hat{\mathfrak{su}}(2)_{\kappa < -2}$ structure constants do not admit an analytic
continuation to the range $\kappa > -2$. This
can be seen for instance from the relation between the $H^+_3$
and the Liouville correlators discovered in [57] and later re-derived
by the path integral techniques in [58]. Under this relation the $\kappa = -2$
barrier corresponds to the $c = 1$ in the Liouville theory. As in the case of the Virasoro models the
bootstrap difference equations can be analytically continued to the range $\kappa > -2$. This was observed
in [43] where also the corresponding structure constants were calculated. The model was called there
the generalized $\hat{\mathfrak{su}}(2)$ WZW model in parallel to the Virasoro case. We shall use an alternative name
the imaginary $\hat{\mathfrak{su}}(2)$ WZNW model to avoid confusion with [20]. The relations between different
$\hat{\mathfrak{su}}(2)$ models are in perfect analogy to the relations connecting the Virasoro models. Both the $\hat{\mathfrak{su}}(2)_{\kappa}$
minimal models and the A model structure constants are analytic continuations of the imaginary
$\hat{\mathfrak{su}}(2)$ WZNW model ones [20, 59]. On the other hand the residues of the (real) $\hat{\mathfrak{su}}(2)$ WZNW model
coincide with the A model structure constants [20].

The Virasoro system of fig.1 and its underlying fusing matrix are based on the special class of
representations of the Virasoro algebra - the Virasoro Verma modules. In the case of $\hat{\mathfrak{su}}(2)_{\kappa}$ symmetry
with nonrational level there are two possible choices which we shall briefly describe.

Let us first observe that any complex $\mathfrak{sl}(2, \mathbb{R})$ representation with the algebra generators $J^3, J^\pm$
is also an $\mathfrak{su}(2)$ representation with the generators $J^3, iJ^\pm$. This simple rescaling changes the
hermitian conjugation properties so the invariant hermitian forms are different. For instance the
principal unitary series representation $D_{j,\epsilon}, j \in \mathbb{C}, \epsilon = 0, \frac{1}{2}$ of $\mathfrak{sl}(2, \mathbb{R})$ [60, 61] is also a series of
$\mathfrak{su}(2)$ representations but with indefinite invariant hermitian forms. We denote by $\hat{D}^\kappa_{j,\epsilon}$ the relaxed

\[4\text{In the present context the equivalence was observed for instance in [48].}\]
module of the affine algebra \( \hat{\mathfrak{su}}(2) \) over the representation \( D_{j,\epsilon} \). The tensor product

\[
\hat{D}_{j,\epsilon} \otimes \hat{D}_{j,\epsilon}, \quad j \in -\frac{1}{2} + i\mathbb{R}, \epsilon = 0, \frac{1}{2},
\]

provides a representation of the direct sum \( \hat{\mathfrak{su}}(2) \oplus \hat{\mathfrak{su}}(2) \) of the left and the right chiral symmetries. This is the class of representation we are concerned with in the present paper.

The second possibility is to start with the principal continuous series representation \( P_j, j \in -\frac{1}{2} + i\mathbb{R} \) of \( \mathfrak{sl}(2, \mathbb{C}) \) which can be seen as a representation of \( \mathfrak{su}(2) \oplus \mathfrak{su}(2) \). One can then construct the relaxed \( \hat{\mathfrak{su}}(2) \oplus \hat{\mathfrak{su}}(2) \) module \( \hat{P}_j \) over it \([62]\). This is the class of representations used in the quantization of the coset model \( H^3_{\mathbb{C}} = \text{SL}(2, \mathbb{C})/\text{SU}(2) \) [49–51]. Most of the results mentioned above concerns models based on this class.

It should be emphasized that although similar in many respects these two classes of representations are essentially different. This implies two versions of the scheme of Fig.2. Both of them are very similar to the Virasoro system of Fig.1.\(^5\) To what extend the idea of systems of CFT models based on the same class of representations, conformal blocks and fusing matrices is an appropriate classifying concept depends on whether relations between individual models admit extensions to the whole systems. This is so for instance in the case of the FZ relation [34] originally found in the minimal model corner of the Virasoro and \( \hat{\mathfrak{su}}(2) \) systems and then successfully extended to their continuous spectra parts. The main idea of the present paper is to analyze whether the GKO construction has this property.

The organization of the paper is as follows. In Section 2 we introduce some basic material concerning hermitian \( \mathfrak{su}(2) \) representations \( D_{j,\epsilon} \). Most of the results are standard and based on the well developed theory of \( \mathfrak{sl}(2, \mathbb{R}) \) representations \([60, 61]\). This concerns the definition of the loop module (Subsection 2.1), the reducibility of the representations (Subsection 2.2), explicit constructions of the reflection map (Subsection 2.4) and of the bilinear invariants (Subsection 2.5) and the formulae for the 3-linear invariants (Clebsch-Gordan coefficients) in the spin basis (Subsection 2.8). A special attention has been payed to the izospin variables introduced in Subsection 2.3 and their role in constructing the bilinear (Subsection 2.6) and the 3-linear (Subsection 2.7) invariants. In Subsection 2.9 we discuss the conditions under which a hermitian \( \mathfrak{su}(2) \)-invariant pairing between representations \( D_{j,\epsilon} \) exists. The constructions of this subsection are not used in the present paper. They were included to complete the notion of the hermitian non-unitary \( \mathfrak{su}(2) \) representations. We close this section by a short analysis of the tensor product of representations relevant on the l.h.s of expected relations (1.3).

In Section 3 we formulate some basic structures of the \( \hat{\mathfrak{su}}(2) \) WZNW models with non-rational levels. They are based on representations (1.4) defined in Subsection 3.1. In Subsection 3.2 we introduce the izospin description of these representations and the associated highest weight modules. The spectrum of the model and the primary fields in the izospin variables are introduced in Subsection 3.3. Let us note that the \( J_0^3 \) eigenvalue of the highest weight state in the module related to the representation \( D_{-j,\epsilon} \) is \( j \). For this reason we denote the chiral primary field related

\(^5\)The same pattern emerges for the N=1 superconformal models as well.
to the representation $\mathcal{D}_{-1-j,\epsilon}$ by $\Phi_{j,\epsilon}$. The spectrum is diagonal and consists of representations (1.4). Since the representations $\mathcal{D}_{j,\epsilon}$ and $\mathcal{D}_{-1-j,\epsilon}$ are equivalent we declare an identification of the corresponding primary fields by means of the reflection map. The transformation properties of the primary fields are described in Subsection 3.4 in terms of the OPE’s with the $\mathfrak{su}(2)$ currents and the Ward identities.

In Subsection 3.5 the general form of the 2-point function of primary fields compatible with the symmetry and the reflection properties is analyzed. Due to the tensor product of the left and the right representations (1.4) the 2-point function factorizes into chiral parts. The same concerns the 3-point functions discussed in Subsection 3.6. The dependence on the field locations ($z$ variables) is determined by the Ward identities in the standard manner. Declaring compatibility with the reflection properties of primary fields we find an appropriate form of the 3-point invariants which fixes the dependence on the isospin $x$ variables. The remaining part depends only on $j$ parameters labeling the representations. We assume it is given by the $\mathbb{H}_3^+$ structure constants with the normalization chosen in [43] in the case of $\kappa > -2$ (being in line with the normalization used in the $\mathfrak{su}(2)$ minimal models). This is motivated by the mentioned above expectation that the $\hat{\mathfrak{su}}(2)_\kappa$ WZNW model at level $\kappa < -2$ should be equivalent to the $\mathbb{H}_3^+ = \text{SL}(2,\mathbb{C})/\text{SU}(2)$ coset model at level $\kappa' = -\kappa > 2$ [44–47]. Let us note however that the derivation of the structure constants given in [43] and based on the Teschner method [50, 51] assumes the complex principal $\mathfrak{sl}(2,\mathbb{C})$ representation rather then the tensor product of two principal $\mathfrak{sl}(2,\mathbb{R})$ representations. The exact proof that it also applies in the present case goes beyond the scope of this paper. We hope to come back to this point in subsequent publication.

Both the $z$-dependent part and the $x$-dependent part of the 3-point function naturally split into chiral components. Motivated by the freedom in the similar splitting of the Liouville structure constants [13] we define the chiral component of the $j$ dependent part by an appropriate splitting of the upsilon functions into the Barnes gamma function components. This choice of the chiral 3-point function is partially confirmed by its properties with respect to the chiral reflection.

As in the case of the SL-LL correspondence one has off-diagonal spectra in the projected tensor product on the r.h.s. of relations (1.3). In the Liouville and the imaginary Liouville theory the extension is the same as in the SL-LL equivalence. In the $\hat{\mathfrak{su}}(2)_\kappa+1$ WZNW model one has to include the representations

$$\hat{\mathcal{D}}_{j,\epsilon}^\kappa, \quad j = -\frac{1}{2} + n + is, \quad s \in \mathbb{R}, \quad \epsilon = 0, \frac{1}{2}, \quad n \in \frac{1}{2}\mathbb{Z}.$$ 

The spectrum is off-diagonal only with respect to the discrete variable $n$. The corresponding off-diagonal extension of the structure constants is based on the additional condition

$$n_1 + n_2 + n_3 \equiv 0,$$ 

where $\equiv$ denotes equality modulo 1. As the structure constants are already split into chiral components the extension is almost straightforward. Only the extension of the $x$-dependent part requires a special care in order to avoid square root ambiguities (Subsection 3.7).
WZNW models are on the opposite sides of the \( \kappa \) same real side of the \( \kappa \) Section 4 contains our main results. We start in Subsection 4.1 with the GSO construction of the \( \Phi \) representations on the right hand sides are generated by the highest unitary representations of \( \hat{\mathrm{su}}(2)_{\kappa=1} \). This yields the relations between the parameters of the theories involved. In the case of Liouville theory with the central charge parameterized by \( b < 1 \) the \( \mathrm{su}(2) \) WZNW models are on the opposite sides of the \( \kappa = -2 \) barrier and one gets relations (4.5). In the case of the imaginary Liouville theory parameterized by \( \hat{b} < 1 \) the \( \mathrm{su}(2) \) WZNW models are on the same real side of the \( \kappa = -2 \) barrier and the parameters are related by (4.6).

In Subsection 4.2 we analyze the decomposition of \( \hat{D}^\kappa_{-1-j, \epsilon} \otimes \hat{S}^{\kappa-1}_{j} \otimes \mathcal{V}_{\Delta, \epsilon} \) into irreducible representations of \( \hat{\mathrm{su}}(2)_{\kappa+1} \otimes \mathrm{Vir} \). Motivated by character decomposition (4.7) and calculations of low level states we conjecture the following decomposition of representations

\[
\hat{D}^\kappa_{-1-j, \epsilon} \otimes \hat{S}^{\kappa-1}_{j} = \bigoplus_{n \in \mathbb{Z}} \hat{D}^{\kappa-1}_{-1-j-n, \epsilon} \otimes \mathcal{V}_{\Delta, \epsilon}, \quad \hat{S}^{\kappa-1}_{j} = \bigoplus_{n \in \mathbb{Z} + \frac{1}{2}} \hat{D}^{\kappa-1}_{-1-j-n, \epsilon} \otimes \mathcal{V}_{\Delta, \epsilon}, \tag{1.5}
\]

where \( \bar{\epsilon} = \epsilon + \frac{1}{2} \) and \( \dot{+} \) denotes summation modulo 1 and \( \mathcal{V}_{\Delta, \epsilon} \) is the Virasoro Verma module with the highest weight \( \Delta \) and the central charge \( c \). In the case of the Liouville theory one has

\[
\Delta_n = \Delta_{j+\frac{n}{bQ}} = -Q^2 \left( j + \frac{n}{bQ} \right) \left( 1 + j + \frac{n}{bQ} \right), \quad c = c^L.
\]

In the izospin formulation the representations on the right hand sides are generated by the highest weight states \( |x\rangle_{j+n, \epsilon}^* \). The corresponding chiral fields \( \Phi_{j+n, \epsilon}^* \) are descendants for the algebra \( \hat{\mathrm{su}}(2)_{\kappa} \otimes \hat{\mathrm{su}}(2)_1 \) and are primaries with respect to the algebra \( \hat{\mathrm{su}}(2)_{\kappa+1} \otimes \mathrm{Vir} \). For \( n = 0, \pm \frac{1}{2}, \pm 1 \) they are explicitly calculated in Subsection 4.3 up to normalization factors.

Equivalences (1.3) can be formulated as equalities of all full correlation functions with an appropriate identification of fields on their opposite sides. In a slightly stronger version they state equalities of all chiral correlation functions. In the case of 3-point chiral fields and the Liouville theory we conjecture that one can always adjust the normalization of the fields such that the following relation holds:

\[
\langle \Phi_{j_3+n_3, \epsilon_3}^\ast(x_3, z_3) \Phi_{j_2+n_2, \epsilon_2}^\ast(x_2, z_2) \Phi_{j_1+n_1, \epsilon_1}^\ast(x_1, z_1) \rangle

= \langle \Phi_{j_3+n_3, \epsilon_3}^\ast(z_3) \Phi_{j_2+n_2, \epsilon_2}^\ast(z_2) \Phi_{j_1+n_1, \epsilon_1}^\ast(z_1) \rangle \langle \Phi_{j_3+n_3, \epsilon_3}(x_3, z_3) \Phi_{j_2+n_2, \epsilon_2}(x_2, z_2) \Phi_{j_1+n_1, \epsilon_1}(x_1, z_1) \rangle_{IS} \tag{1.6}
\]

where \( \Phi_{j+n, \epsilon}^\ast \) are Liouville fields normalized by setting the reflection amplitude to 1 and \( \Phi_{j+n, \epsilon}^\ast \) are primary fields in the \( \hat{\mathrm{su}}(2)_{\kappa+1} \) theory normalized by 2-point function (3.9).
Due to the simple form of the structure constants in the $\kappa = 1$ model (4.13) the condition $n_1 + n_2 + n_3 = 0$ is always satisfied in the 3-point functions. The r.h.s. of (1.6) can be calculated using shift properties of the Barnes gamma functions (3.14). The real difficulty is the calculation of the correlator of descendent fields on the l.h.s.

We have checked relation (1.6) and its counterpart for the imaginary Liouville theory for $n = 0, \pm \frac{1}{2}, \pm 1$. In the simplest case of $n = 0$ this is done in Subsection 4.3. The proof is based on new identities for the Barnes gamma functions (4.17), (4.20) derived in Appendix B. This version of (1.6) fixes relative normalization of primary fields on both sides of the correspondence. It is interesting to observe that the spectrum of the imaginary Liouville theory implied by the quotient construction coincides with the spectrum recently proposed on the basis of numerical analysis of the bootstrap equations [28].

In Subsection 4.4 we show that in the general case relation (1.6) can be cast in the form of explicit expression (4.23) for the coset factor defined as

$$
\frac{\langle \Phi_{j_1+n_3,\epsilon_3+n_3}(x_3, z_3)\Phi_{j_2+n_2,\epsilon_2+n_2}(x_2, z_2)\Phi_{j_1+n_1,\epsilon_1+n_1}(x_1, z_1) \rangle_{\epsilon}}{\langle \Phi_{j_3,\epsilon_3}(x_3, z_3)\Phi_{j_2,\epsilon_2}(x_2, z_2)\Phi_{j_1,\epsilon_1}(x_1, z_1) \rangle_{\epsilon}},
$$

where $\Phi_{j,\epsilon}(x, z)$ are primary fields in the $\hat{\text{su}}(2)$ theory. An essential advantage of this form of the conjecture is that it is the same for the Liouville and for the imaginary Liouville theory and reduces calculations to the product of $\hat{\text{su}}(2)$ and $\hat{\text{su}}(2)$ WZNW models with $\kappa$ on the real side of the $\kappa = -2$ barrier.

Some explicit calculations of the coset factor involving $n = \pm \frac{1}{2}$ and $n = \pm 1$ fields are given in Subsections 4.5 and 4.6, respectively. We have also calculated all other coset factors with $n_1, n_2, n_3 \in \{\pm \frac{1}{2}, \pm 1\}$. In all the cases considered we got confirmation of the conjectured form (4.23).

These verifications are the main results of the present paper. They are based on the new nontrivial identities for the Barnes gamma functions with different parameters (4.17), (4.20), decomposition (1.5) and the properties of the 3-linear invariants of the representations involved. They provide strong evidence that the general relations (1.3) are correct. Some consequences of this result and possible extensions of this paper are briefly discussed in Section 5.

2 \textit{su}(2) hermitian representations

2.1 loop module

For any $j, \alpha \in \mathbb{C}$ we define the loop module $D_{j,\alpha}$ as an $\text{sl}(2, \mathbb{C})$ module with the basis $\{ | n + \alpha \rangle : n \in \mathbb{Z} \}$ and the algebra action given by [62]

$$
J^3 | n + \alpha \rangle = (n + \alpha) | n + \alpha \rangle,
J^+ | n + \alpha \rangle = (n + \alpha - j) | n + \alpha + 1 \rangle,
J^- | n + \alpha \rangle = (-n - \alpha - j) | n + \alpha - 1 \rangle.
$$
The loop module $D_{j,a}$ can be seen as a complex representation of the su(2) algebra:

\[
[J^3, J^\pm] = \pm J^\pm,
\]
\[
[J^+, J^-] = 2J^3,
\]

where $J^\pm = J^1 \pm iJ^2$, $[J^a, J^b] = i\epsilon_{abc}J^c$. The eigenvalue of the Casimir operator

\[
C = J^-J^+ + (J^3)^2 + J^3
\]

in this representation is $j(j + 1)$. The requirement of integrability leads to the condition

\[
e^{i4\pi J^3} = 1
\]

hence $\alpha \in \frac{1}{2}\mathbb{Z}$. Without loss of generality, one can assume $\alpha = \epsilon = 0, \frac{1}{2}$. In the following we shall restrict ourselves to the two cases $\alpha = \epsilon$ and $\alpha = j$. In the second case the module $D_{j,j}$ contains a highest weight submodule $E_j$ generated from the highest weight state $|j\rangle$

\[
J^+ |j\rangle = 0, \quad J^3 |j\rangle = j |j\rangle.
\]

### 2.2 reducibility of $D_{j,\epsilon}$ representations

The parameters $(j, \epsilon)$ are called integral if $2j$ and $2\epsilon$ are integers of the same parity. For all non-integral $(j, \epsilon)$ the representation $D_{j,\epsilon}$ is irreducible. For integral $(j, \epsilon)$, $n_+ + \epsilon = n_- - \epsilon = j$ is integer or half integer and will be denoted by $l$. One has

\[
J^+ |n_+ + \epsilon\rangle = 0, \quad J^3 |n_+ + \epsilon\rangle = n_+ |n_+ + \epsilon\rangle,
\]
\[
J^- |n_- + \epsilon\rangle = 0, \quad J^3 |n_- + \epsilon\rangle = n_- |n_- + \epsilon\rangle.
\]

There are two invariant subspaces

\[
D_l^+ = \text{span}\{| n + \epsilon : n \leq n_+\}, \quad D_l^- = \text{span}\{| n + \epsilon : n \geq n_-\}.
\]

If $n_- > n_+ (l < 0)$ the intersection $D_l^+ \cap D_l^-$ is null and the sum is an invariant subspace. Then the quotient $D_{l,\epsilon}/D_l^+ \cup D_l^-$ is equivalent to the finite-dimensional, spin $l$ representations $S_l$ of su(2). For $l = -\frac{1}{2}$, $D_{l,\epsilon} = D_l^+ \cap D_l^-$ hence the quotient representation is trivial $S_{-\frac{1}{2}} = \{0\}$. The representations $S_l^\pm$ induced on subspaces $D_l^\pm$ are irreducible. The simplest nontrivial quotient is

$S_{-\frac{1}{2}} = \{|\frac{1}{2}\rangle, | -\frac{1}{2}\rangle\}$

with the action of generators given by $^6$

\[
K^3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad K^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad K^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]

If $n_- < n_+ (l > 0)$ also the intersection $D_l^+ \cap D_l^-$ is invariant and is equivalent to the standard finite-dimensional, spin $l$ representations $S_l$ of su(2). The quotients $D_l^+/D_l^+ \cap D_l^-$ are isomorphic to the representations $S_l^\pm$, respectively. In the simplest case $S_{\frac{1}{2}} = \{|\frac{1}{2}\rangle, | -\frac{1}{2}\rangle\}$ the action of generators takes the form

\[
K^3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad K^+ = -\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad K^- = -\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]

$^6$For the sake of future convenience we shall denote the su(2) generators in the finite dimensional representations by symbol $K^n$ rather then $J^n$. 

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2.3 associated highest weight module and the isospin variables

We shall now describe a construction of the highest weight module $E_j$ associated to the module $D_{-1-j,\epsilon}$. To this end let us define a new set of generators

$$
J_{a}(x) = \begin{cases} 
J_{a}^{+}(x) = J_{a}^{+} - 2xJ_{a}^{3} - x^2J^{-} , & \\
J_{a}^{3}(x) = J_{a}^{3} + xJ_{a}^{-} , & \\
J_{a}^{-}(x) = J_{a}^{-} , & 
\end{cases}
$$

where $x$ is a complex parameter called the isospin variable $[34]$. For any $x$ they satisfy the commutation relations of the $su(2)$ algebra:

$$
\begin{aligned}
[J_{a}^{3}(x), J_{a}^{\pm}(x)] &= \pm J_{a}^{\pm}(x), \\
[J_{a}^{+}(x), J_{a}^{-}(x)] &= 2J_{a}^{3}(x).
\end{aligned}
$$

Consider now a formal series $[62]

$$
| x \rangle_{j,\epsilon} = \sum_{m=-\infty}^{\infty} x^{j-m-\epsilon} | m + \epsilon \rangle_{-1-j}
$$

which can be seen as a generating function for $J_{0}^{3}$ eigenstates in $D_{-1-j,\epsilon}$. This is the highest weight state with respect to the algebra $J_{a}(x)$:

$$
J_{a}^{+}(x) | x \rangle_{j,\epsilon} = 0, \quad J_{a}^{3}(x) | x \rangle_{j,\epsilon} = j | x \rangle_{j,\epsilon}.
$$

One also has

$$
\begin{aligned}
J_{a}^{+} | x \rangle_{j,\epsilon} &= \left( -x^2 \partial_{x} + 2jx \right) | x \rangle_{j,\epsilon} , \\
J_{a}^{3} | x \rangle_{j,\epsilon} &= \left( -x \partial_{x} + j \right) | x \rangle_{j,\epsilon} , \\
J_{a}^{-} | x \rangle_{j,\epsilon} &= \partial_{x} | x \rangle_{j,\epsilon} .
\end{aligned}
$$

An advantage of this construction is that it treats the whole $D_{-1-j,\epsilon}$ representation as a single highest weight state $| x \rangle_{j,\epsilon}$. This is also an efficient tool in analyzing multi-linear invariants.

2.4 equivalent representations and the reflection map

Representations $D_{j,\epsilon}, D'_{j',\epsilon'}$ are equivalent if there exists an operator $Q : D_{j,\epsilon} \rightarrow D'_{j',\epsilon'}$ such that $Q J_{a} = J_{a} Q$, $a = 3, \pm$. In both representations the operator $J_{a}^{3}$ is diagonal in the basis $| n + \epsilon \rangle$. As all the diagonal matrix elements of $J_{a}^{3}$ are different the operator $Q$ has to be diagonal as well:

$$
Q | n + \epsilon \rangle = q_{n+\epsilon} | n + \epsilon' \rangle .
$$
Then the equation $QJ^3 = J^3Q$ implies

$$q_{n+\epsilon}(n + \epsilon) = (n + \epsilon')q_{n+\epsilon}$$

hence $\epsilon = \epsilon'$. The equations $QJ^\pm = J^\pm Q$ take the form

$$q_{n+\epsilon+1}(-j + n + \epsilon) = (-j' + n + \epsilon)q_{n+\epsilon},$$
$$q_{n+\epsilon}(j + n + \epsilon + 1) = (j' + n + \epsilon + 1)q_{n+\epsilon+1}.$$ 

For non-vanishing $q_{n+\epsilon}$ they can be satisfied only if

$$j' = j, \text{ or } j' = -j - 1.$$ 

In the second case one gets nontrivial $Q_j$:

$$Q_jJ^a_j = J^a_{-1-j}Q_j,$$

$$q_n^{j+1} = \frac{j + 1 + n + \epsilon}{-j + n + \epsilon}q_n^{j+1}.$$ 

This determines $Q_{j,\epsilon}$ up to a $(j, \epsilon)$-dependent constant. We chose this constant by assuming that $Q_{j,\epsilon}$ is given by

$$q_n^{j+\epsilon} = \frac{\Gamma(1 + j + n + \epsilon)}{\Gamma(-j + n + \epsilon)}. \tag{2.3}$$ 

Let us observe that

$$q_n^{j+\epsilon} = \frac{1}{q_n^{1-j}}.$$ 

### 2.5 invariant bilinear forms

We say that a bilinear form $D$ is invariant on $\mathcal{D}_{j,\epsilon} \times \mathcal{D}_{j,\epsilon}$ if:

$$D(J^a f_1, f_2) + D(f_1, J^a f_2) = 0.$$ 

Let $\{F_n = |n + \epsilon\}\}, \{F'_n = |n + \epsilon'\}\}$ be the bases in $\mathcal{D}_{j,\epsilon}, \mathcal{D}_{j,\epsilon'}$, respectively. The condition for $J^3$ reads

$$D(J^3 F'_n, F_m) + D(F'_n, J^3 F_m) = (n + m + \epsilon + \epsilon')(F'_n, F_m) = 0.$$ 

Since $\epsilon, \epsilon' \in \{0, \frac{1}{2}\}$ and $n + m$ is an integer one gets $\epsilon = \epsilon'$ and

$$D(F'_n, F_m) = 0 \text{ for } n \neq -m - 2\epsilon.$$ 

The other two conditions

$$D(F'_n, J^+ F_n) + D(J^+ F'_m, F_n) = 0,$$
$$D(F'_n, J^- F_n) + D(J^- F'_m, F_n) = 0,$$
yield
\[ (n + \epsilon - j)D(F'_{n-1-2\epsilon}, F_{n+1}) + (-n - 1 - 2\epsilon + \epsilon - j')D(F'_{-n-2\epsilon}, F_n) = 0, \] 
\[ (n + 2\epsilon - \epsilon - j')D(F'_{-n-1-2\epsilon}, F_{n+1}) + (-n - 1 - \epsilon - j)D(F'_{-n-2\epsilon}, F_n) = 0, \] 
which implies \( j(1 + j) = j'(1 + j') \) and
\[ j' = j \quad \text{or} \quad j' = -j - 1. \] (2.5)

For non-integral \((j, \epsilon)\), in both cases the bilinear form \(D\) is determined by (2.4) up to an overall normalization:
\[ D_1(F'_{-n-1-2\epsilon}, F_{n+1}) = D_1(F'_{-n-2\epsilon}, F_n) \quad \text{for} \quad j' = -1 - j, \]
\[ D_{II}(F'_{-n-1-2\epsilon}, F_{n+1}) = \frac{n + 1 + \epsilon + j}{n + \epsilon - j}D_{II}(F'_{-n-2\epsilon}, F_n) \quad \text{for} \quad j' = j. \]

One easily checks that the forms are related by the reflection map \(Q_j : D_{j, \epsilon} \to D_{-1-j, \epsilon} : \)
\[ D_{II}(\cdot, \cdot) \propto D_1(Q_j \cdot, \cdot). \]

2.6 invariant bilinear forms in the izospin variables

Suppose \(D : D_{-1-j', \epsilon} \times D_{-1-j, \epsilon} \to \mathbb{C}\) is an invariant bilinear form:
\[ D(J^n F'_n, F_m) + D(F'_n, J^n F_m) = 0. \]

One can consider the formal double series
\[ D(x, y) \equiv D(|x\rangle_{j', \epsilon}, |y\rangle_{j, \epsilon}) = \sum_{n,m \in \mathbb{Z}} x^{j'-m-\epsilon} y^{j-n-\epsilon} D(F'_m, F_n). \] (2.6)

The formula for the coefficients
\[ D(F'_m, F_n) = \oint_{2\pi i} \oint_{2\pi i} \frac{dx \ dy}{2\pi i} x^{-1-j'+m+\epsilon} y^{-1-j+n+\epsilon} D(x, y) \] (2.7)
requires the integrand to be a well defined function on the product \(S^1 \times S^1\) of unit circles in the complex \(x, y\) variables.

It follows from (2.2) that \(D(x, y)\) should satisfy the equations
\[ \partial_x D(x, y) + \partial_y D(x, y) = 0, \]
\[ \left( -x \partial_x + j \right) D(x, y) + \left( -y \partial_y + j \right) D(x, y) = 0, \]
\[ \left( -x^2 \partial_x + 2j'x \right) D(x, y) + \left( -y^2 \partial_y + 2jy \right) D(x, y) = 0. \]
The Dirac delta function $\delta(x - y)$ solves these equations if $j' = -1 - j$. In this case formula (2.7) yields

$$D(F'_m, F_n) = \oint \frac{dx}{2\pi i} \frac{dy}{2\pi i} x^{-j' - 1 + m + \epsilon} y^{-j - 1 + n + \epsilon} \delta(x - y)$$

$$= \oint \frac{dx}{2\pi i} x^{-1 + m + n + 2\epsilon} = \delta_{m+n+2\epsilon,0}.$$

Thus

$$D_1(x, y) \propto \delta(x - y).$$

For $j' = j$ one has another solution

$$D(x, y) = (x - y)^{2j}.$$

Due to the complex values of the exponent this is a multi-valued function. The solution is determined up to a constant possibly depending on $j$. One has for instance another solution

$$(y - x)^{2j} = (-1)^{2j} (x - y)^{2j}.$$

In the $x$-formulation one needs a function (or a distribution) such that

$$x^{-j - 1 + m + \epsilon} y^{-j - 1 + n + \epsilon} D(x, y)$$

is well defined (single valued) on the product $S^1 \times S^1$ of unit circles in the complex $x, y$ variables. In order to analyze the problem we shall introduce a convenient parametrization of $S^1 \times S^1$:

$$M = (-\pi, \pi] \times (-\pi, \pi] \ni (\varphi_1, \varphi_2) \rightarrow (x, y) = (e^{i\varphi_1}, e^{i\varphi_2}) \in S^1 \times S^1.$$

In this parametrization

$$(x - y)^{2j} = (2i)^{2j} e^{i(\varphi_1 + \varphi_2)} j (\sin \frac{\pi}{2})^{2j}$$

and

$$x^{-j - 1 + m + \epsilon} y^{-j - 1 + n + \epsilon} (x - y)^{2j} = (2i)^{2j} e^{i(m + \epsilon - 1) \varphi_1} e^{i(n + \epsilon - 1) \varphi_2} (\sin \frac{\pi}{2})^{2j}.$$

This expression is singular along the line $\varphi_1 = \varphi_2$ which in the space $M$ of parameters separates two regions:

$$M_\varphi = \{ (\varphi_1, \varphi_2) \in M : \varphi_1 > \varphi_2 \}, \quad M_\varphi_\varphi = \{ (\varphi_1, \varphi_2) \in M : \varphi_1 < \varphi_2 \}.$$

Let $N_\varphi, N_\varphi_\varphi$ be the images of $M_\varphi, M_\varphi_\varphi$ in $S^1 \times S^1$. For the function

$$(x, y)^{2j} = \begin{cases} (x - y)^{2j} & \text{on } N_\varphi, \\ (-1)^{2j} (y - x)^{2j} & \text{on } N_\varphi_\varphi \end{cases}$$

one gets

$$x^{-j - 1 + m + \epsilon} y^{-j - 1 + n + \epsilon} S_{j, \epsilon}(x, y) = (2i)^{2j} \text{sign}^{2\epsilon} (\varphi_1 - \varphi_2) e^{i(m + \epsilon - 1) \varphi_1} e^{i(n + \epsilon - 1) \varphi_2} \sin \frac{\pi}{2} (\varphi_1 - \varphi_2)^{2j},$$

Thus

$$D_1(x, y) \propto \delta(x - y).$$

For $j' = j$ one has another solution

$$D(x, y) = (x - y)^{2j}.$$
which has periodic boundary conditions on the square $M$. Using the change of variables
\[ \eta = \varphi_1 + \varphi_2, \quad \theta = \varphi_1 - \varphi_2 \]
and the symmetry properties of $S_{j,\epsilon}(x, y)$ one obtains
\[
D(F_m, F_n) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} d\varphi_1 \int_{-\pi}^{\pi} d\varphi_2 (2i)^2 \text{sign}(\varphi_1 - \varphi_2) e^{i(m+\epsilon)\varphi_1} e^{i(n+\epsilon)\varphi_2} |\sin \frac{\varphi_1 - \varphi_2}{2}|^{2j}
\]
\[
= \frac{1}{(2\pi)^2} \int_{-2\pi}^{2\pi} d\eta \int_{-2\pi}^{2\pi} d\theta e^{i(m+\epsilon)\eta} e^{i\frac{m-n+2\epsilon}{2} - n\theta} |\sin \frac{\theta}{2}|^{2j}
\]
\[
= \delta_{m+n+2\epsilon, 0} (2i)^{2j} \frac{2\pi}{\pi} \int_{0}^{2\pi} d\theta e^{i(m+\epsilon)\theta} (\sin \frac{\theta}{2})^{2j}
\]
\[
= \delta_{m+n+2\epsilon, 0} C_{j,\epsilon} \frac{\Gamma(-j + m + \epsilon)}{\Gamma(1 + j + m + \epsilon)}
\]
\[
C_{j,\epsilon} = \frac{\pi}{\cos((j + \epsilon)\pi)} \Gamma(-2j)
\]
hence
\[
D_{II}(x, y) \propto S_{j,\epsilon}(x, y).
\]
Let us observe that $S_{j,\epsilon}(x, y)$ can be seen as an integral kernel of the reflection map
\[
Q_{-1-j} \mid x \rangle_{j,\epsilon} = \frac{1}{C_{j,\epsilon}} \int \frac{dy}{2\pi i} S_{j,\epsilon}(x, y) \mid y \rangle_{-1-j,\epsilon}.
\]
Indeed
\[
Q_{-1-j} \mid m + \epsilon \rangle_{-1-j} = \int \frac{dx}{2\pi i} x^{-j+m+\epsilon-1} Q_{-1-j} \mid x \rangle_{j,\epsilon}
\]
\[
= \frac{1}{C_{j,\epsilon}} \sum_n \int \frac{dx}{2\pi i} \frac{dy}{2\pi i} x^{-j+m+\epsilon-1} y^{-1-j-n-\epsilon} S_{j,\epsilon}(x, y) \mid n + \epsilon \rangle_j
\]
\[
= \frac{1}{(2\pi)^2 C_{j,\epsilon}} \sum_n \int_{-2\pi}^{2\pi} d\theta \int_{-2\pi}^{2\pi} d\eta (2i)^{2j} e^{i\frac{m-n+2\epsilon}{2}} e^{i\frac{m-n+2\epsilon}{2} \eta} |\sin \frac{\theta}{2}|^{2j} \mid n + \epsilon \rangle_j
\]
\[
= \frac{(2i)^{2j}}{\pi C_{j,\epsilon}} \int_{0}^{2\pi} d\theta e^{i(m+\epsilon)\theta} (\sin \frac{\theta}{2})^{2j} \mid m + \epsilon \rangle_j
\]
\[
= \frac{\Gamma(-j + m + \epsilon)}{\Gamma(1 + j + m + \epsilon)} \mid m + \epsilon \rangle_j
\]
in line with (2.3).
2.7 invariant three-linear forms in the isospin variables

In the isospin variables the three-linear invariants satisfy the equations:

\[
\left( \partial_{x_1} + \partial_{x_2} + \partial_{x_3} \right) D_{j_3}^{j_2 j_1} = 0,
\]

\[
\left( -x_1 \partial_{x_1} + j_1 - x_2 \partial_{x_2} + j_2 - x_3 \partial_{x_3} + j_3 \right) D_{j_3}^{j_2 j_1} = 0,
\]

\[
\left( -x_1^2 \partial_{x_1} + 2 j_1 x_1 - x_2^2 \partial_{x_2} + 2 j_2 x_2 - x_3^2 \partial_{x_3} + 2 j_3 x_3 \right) D_{j_3}^{j_2 j_1} = 0.
\]

Up to a multiplicative, \((j, \epsilon)\)'s dependent constant the solution reads

\[
D_{j_3}^{j_2 j_1} = (x_1 - x_2)^{j_1}_2 (x_2 - x_3)^{j_2}_3 (x_3 - x_1)^{j_3}_1,
\]

where \(j_{mn}^k = j_m + j_n - j_k\). For the isospin representation one needs a function such that

\[
x_1^{-j_1 - 1 + m_1 + \epsilon_1} x_2^{-j_2 - 1 + m_2 + \epsilon_2} x_3^{-j_3 - 1 + m_3 + \epsilon_3} D_{j_3}^{j_2 j_1} = (e^{i\varphi_1}, e^{i\varphi_2}, e^{i\varphi_3}) \in S^1 \times S^1 \times S^1
\]

is well defined on the product \(S^1 \times S^1 \times S^1\) of unit circles. In the parametrization:

\[
M = (-\pi, \pi] \times (-\pi, \pi] \times (-\pi, \pi] \ni (\varphi_1, \varphi_2, \varphi_3) \rightarrow (x_1, x_2, x_3) = (e^{i\varphi_1}, e^{i\varphi_2}, e^{i\varphi_3}) \in S^1 \times S^1 \times S^1
\]

one has

\[
D_{j_3}^{j_2 j_1} = (x_1 - x_2)^{j_1}_2 (x_2 - x_3)^{j_2}_3 (x_3 - x_1)^{j_3}_1,
\]

\[
= (2i)^{j_{123}} e^{i(\varphi_1 j_1 + \varphi_2 j_2 + \varphi_3 j_3)}
\]

\[
\times (\sin \frac{\varphi_1 - \varphi_2}{2})^{j_1}_2 (\sin \frac{\varphi_2 - \varphi_3}{2})^{j_2}_3 (\sin \frac{\varphi_3 - \varphi_1}{2})^{j_3}_1,
\]

where \(j_{123} = j_1 + j_2 + j_3\) and

\[
x_1^{-j_1 - 1 + m_1 + \epsilon_1} x_2^{-j_2 - 1 + m_2 + \epsilon_2} x_3^{-j_3 - 1 + m_3 + \epsilon_3} D_{j_3}^{j_2 j_1} = (2i)^{j_{123}} e^{i(m_1 + \epsilon_1)\varphi_1} e^{i(m_2 + \epsilon_2)\varphi_2} e^{i(m_3 + \epsilon_3)\varphi_3}
\]

\[
\times (\sin \frac{\varphi_1 - \varphi_2}{2})^{j_1}_2 (\sin \frac{\varphi_2 - \varphi_3}{2})^{j_2}_3 (\sin \frac{\varphi_3 - \varphi_1}{2})^{j_3}_1.
\]

This expression is singular along the planes \(\varphi_1 = \varphi_2, \varphi_2 = \varphi_3, \varphi_1 = \varphi_3\) intersecting the parameter cube \(M\) into six regions:

\[
M_{321} = \{ \varphi_3 < \varphi_2 < \varphi_1 \}, \quad M_{312} = \{ \varphi_3 < \varphi_1 < \varphi_2 \},
\]

\[
M_{132} = \{ \varphi_1 < \varphi_3 < \varphi_2 \}, \quad M_{123} = \{ \varphi_1 < \varphi_2 < \varphi_3 \},
\]

\[
M_{213} = \{ \varphi_2 < \varphi_1 < \varphi_3 \}, \quad M_{231} = \{ \varphi_2 < \varphi_3 < \varphi_1 \}.
\]

Let \(N_{abc}\) denote the images of these regions in the 3-dimensional torus \(N \subset \mathbb{C}^3\). The singularity surface divides \(N\) into two disjoint parts:

\[
N_A = N_{321} \cup N_{132} \cup N_{213}, \quad N_B = N_{312} \cup N_{123} \cup N_{231}.
\]
This implies that the space of 3-linear invariant forms is 2-dimensional. For \( \epsilon_i \in \{0, \frac{1}{2}\} \) satisfying

\[ \epsilon_1 + \epsilon_2 + \epsilon_3 = 0, \]

we define

\[
S_A \left[ \begin{array}{ccc} j_3 & j_2 & j_1 \\ x_3 & x_2 & x_1 \end{array} \right] = \begin{cases} (-1)^{2(\epsilon_1+\epsilon_2)}(x_1 - x_2)^{j_1}2(x_2 - x_3)^{j_2}3(x_1 - x_3)^{j_3} & \text{on } N_{321} \\ (-1)^{2(\epsilon_1+\epsilon_2)}(x_2 - x_1)^{j_1}2(x_2 - x_3)^{j_2}3(x_3 - x_1)^{j_3} & \text{on } N_{132} \\ (-1)^{2(\epsilon_2+\epsilon_3)}(x_1 - x_2)^{j_1}2(x_3 - x_2)^{j_2}3(x_3 - x_1)^{j_3} & \text{on } N_{213} \end{cases}, \]

and the linear combinations

\[ S_\epsilon = (-1)^{2\epsilon}S_A + S_B, \quad \epsilon = 0, \frac{1}{2}. \quad (2.9) \]

The function

\[
x_1^{j_1-1+m_1+\epsilon_1}x_2^{j_2-1+m_2+\epsilon_2}x_3^{j_3-1+m_3+\epsilon_3}S_\epsilon \left[ \begin{array}{ccc} j_3 & j_2 & j_1 \\ x_3 & x_2 & x_1 \end{array} \right]
\]

\[
= (2\epsilon)^{j_1+j_2+j_3}e^{(m_1+\epsilon_1-1)\varphi_1}e^{(m_2+\epsilon_2-1)\varphi_2}e^{(m_3+\epsilon_3-1)\varphi_3}
\times \text{sign}(\varphi_1 - \varphi_2)^{2(\epsilon+\epsilon_1+\epsilon_2)}\sin \frac{\varphi_1 - \varphi_2}{2}^{j_2}
\times \text{sign}(\varphi_2 - \varphi_3)^{2(\epsilon+\epsilon_2+\epsilon_3)}\sin \frac{\varphi_2 - \varphi_3}{2}^{j_3}
\times \text{sign}(\varphi_3 - \varphi_1)^{2(\epsilon+\epsilon_3+\epsilon_3)}\sin \frac{\varphi_3 - \varphi_1}{2}^{j_3}
\]

is periodic in all variables \( \varphi_i \) for all \( \epsilon, \epsilon_i \) satisfying \( \epsilon_1 + \epsilon_2 + \epsilon_3 \in \mathbb{Z} \). Under this condition the following identity holds:

\[
\text{sign}(\varphi_1 - \varphi_2)^{2(\epsilon+\epsilon_1+\epsilon_2)}\text{sign}(\varphi_2 - \varphi_3)^{2(\epsilon+\epsilon_2+\epsilon_3)}\text{sign}(\varphi_3 - \varphi_1)^{2(\epsilon+\epsilon_3+\epsilon_3)}
= (-1)^{\epsilon+\epsilon_1+\epsilon_2+\epsilon_2+\epsilon_3+\epsilon_3+\epsilon_3+\epsilon_1}\text{sign}(\varphi_1 - \varphi_2)\text{sign}(\varphi_2 - \varphi_3)\text{sign}(\varphi_3 - \varphi_1).
\]

From the definition of \( S_\epsilon \) one gets the identity

\[
(x_3 - x_1)^{n_3}2(x_2 - x_3)^{n_3}2(x_1 - x_2)^{n_3}2S_\epsilon \left[ \begin{array}{ccc} j_3 & j_2 & j_1 \\ x_3 & x_2 & x_1 \end{array} \right] = S_{\epsilon_1}^{j_3}\frac{1}{N_{123}}S_{\epsilon_2}^{j_2}\frac{1}{N_{123}}S_{\epsilon_3}^{j_1}\frac{1}{N_{123}}\left[ \begin{array}{ccc} j_3+n_3 & j_2+n_2 & j_1+n_1 \\ x_3 & x_2 & x_1 \end{array} \right]. \quad (2.10)
\]
2.8 invariant three-linear forms in the spin basis

Let \( F^i_{n_i} = \{ n_i + \epsilon_i \} \) be the base in \( D_{j_i, \epsilon_i} \) \((i = 1, 2, 3)\). The 3-linear forms in these bases are given by

\[
S_{\epsilon}[-1-j_3, -1-j_2, -1-j_1] \quad \text{in the spin basis}
\]

\[
= \int \frac{dx_1}{2\pi i} \int \frac{dx_2}{2\pi i} \int_0^\pi \frac{dx_3}{2\pi i} \epsilon(i_{m_3} e_{m_2} e_{m_1}) x_1^{j_3} x_2^{j_2} x_3^{j_1} + \epsilon_3 \epsilon S_{\epsilon} \epsilon_{m_3} e_{m_2} e_{m_1} \epsilon
\]

\[
= (2\pi j_{123} x_1^{j_2} x_2^{j_1} x_3^{j_3} \epsilon(i_{m_3} e_{m_2} e_{m_1}) x_1^{j_3} x_2^{j_2} x_3^{j_1} + \epsilon_3 \epsilon S_{\epsilon} \epsilon_{m_3} e_{m_2} e_{m_1} \epsilon
\]

\[
\times (-1)^{\epsilon_1 \epsilon_2 \epsilon_3} \Gamma(1 + j_{12}) \Gamma(1 + j_{32}) \Gamma(1 + j_{13}) g^0(j_i, m_i + \epsilon_i)
\]

where \( j_{123} = j_1 + j_2 + j_3, j_{bc} = j_b + j_c - j_a \). This is the well known integral representation of the Clebsch-Gordan coefficients (CGC) for the principal continuous representations of \( SL(2, \mathbb{R}) \). It can be explicitly calculated \([61, 63, 64]\):

\[
S_{\epsilon}[-1-j_3, -1-j_2, -1-j_1] = -\frac{1}{\pi^2} \delta_{\epsilon_1 \epsilon_2 \epsilon_3} \epsilon_{0} i^{j_{123}} \Gamma(1 + j_{12}) \Gamma(1 + j_{32}) \Gamma(1 + j_{13}) g^0(j_i, m_i + \epsilon_i)
\]

and

\[
g^0(j_i, m_i + \epsilon_i) = s(\frac{1}{2} j_{13}^2 - \epsilon + \epsilon_2) s(j_1 - \epsilon_1) s(j_3 + \epsilon_3) \left( g^{10}(j_i, m_i + \epsilon_i) + (-1)^{\epsilon_1} g^{10}(j_i, m_i + \epsilon_i) \right)
\]

\[
g^{10}(j_i, m_i) = G^{10}(j_i, m_i) = G
\]

\[
G^{10}(j_i, m_i) = G^{10}(j_i, m_i) = G
\]

\[
G^{10}(j_i, m_i) = G \frac{\Gamma[a] \Gamma[b] \Gamma[c] \Gamma[f]}{\Gamma[e] \Gamma[f]} 3F_2[a_{b c} f | 1], \quad s(x) \equiv \sin(\pi x).
\]

The CGC coefficients satisfy the reflection relation\(^7\)

\[
\frac{\Gamma(-j_3 + m_3 + \epsilon_3)}{\Gamma(1 + j_3 + m_3 + \epsilon_3)} S_{\epsilon}[-1-j_3, -1-j_2, -1-j_1] = (-1)^{\epsilon_2 \epsilon_3} \frac{\Gamma(-j_{23})}{\Gamma(1 + j_{23})} \frac{s(-\frac{1}{2} - \frac{j_{23}^1}{j_{23}^2} + \epsilon + \epsilon_2)}{s(\frac{1}{2} j_{23}^2 - (\epsilon + \epsilon_3) + \epsilon_2)} S_{\epsilon + \epsilon_3}[-1-j_3, -1-j_2, -1-j_1]
\]

\[
= \frac{\Gamma(-j_{23})}{\Gamma(1 + j_{23})} \frac{s(-\frac{1}{2} - \frac{j_{23}^1}{j_{23}^2} + \epsilon + \epsilon_1)}{s(\frac{j_{23}^2}{2} - (\epsilon + \epsilon_3) + \epsilon_1)} S_{\epsilon + \epsilon_3}[-1-j_3, -1-j_2, -1-j_1]
\]

\(^7\)We derive this relation in Appendix A.
In the isospin variables it takes the form

\[
\frac{1}{C_j \epsilon_\delta} \oint dy S_{j_3,\epsilon_\delta}(x_3, y) S_{\epsilon \delta} \left[ \begin{array}{c} -j_3 - j_2 - j_1 \\ j_3 j_2 j_1 \end{array} \right] \left( \begin{array}{c} x_3 x_2 x_1 \\ x_3 x_2 x_1 \end{array} \right) \]

\[
= (-1)^{2\epsilon_\delta} \frac{\Gamma(-j_{13})}{\Gamma(1 + j_{13})} \frac{s(-1/2 - 1/2j_{13} + \epsilon + \epsilon_2)}{s(1/2j_{13} - (\epsilon + \epsilon_3) + \epsilon_2)} S_{\epsilon_\delta} \left[ \begin{array}{c} j_3 j_2 j_1 \\ x_3 x_2 x_1 \end{array} \right] \]

\[= \frac{\Gamma(-j_{13})}{\Gamma(1 + j_{13})} \frac{s(-1/2 - 1/2j_{13} + \epsilon + \epsilon_1)}{s(1/2j_{13} - (\epsilon + \epsilon_3) + \epsilon_1)} S_{\epsilon_\delta} \left[ \begin{array}{c} j_3 j_2 j_1 \\ x_3 x_2 x_1 \end{array} \right]. \tag{2.13}\]

2.9 Hermitian Adjoint Representations

Now we shall turn to the question of a Hermitian pairing between representations. The method of analysis is the same as in the case of \( \mathfrak{sl}(2, \mathbb{R}) \) but the conditions for the adjoint generators are different. We say that the \( \mathfrak{su}(2) \) representations \( D_{j,\epsilon}, D_{j',\epsilon'} \) are Hermitian adjoint if there exists a Hermitian form such that:

\[
(J^3 f, f') = (f, J^3 f'), \quad (J^\pm f, f') = (f, J^{\mp} f').
\]

This condition can be satisfied if and only if

\[
j' = \bar{j} \quad \text{or} \quad j' = -\bar{j} - 1. \tag{2.14}\]

In both cases the Hermitian pairing is determined up to an overall normalization:

\[
(\bar{j} - \epsilon - n)(F_{n+1}, F'_{n+1})_\Pi = (\bar{j} + \epsilon + n + 1)(F_n, F'_n)_I \quad \text{for} \quad j' = \bar{j},
\]

\[
(F_{n+1}, F'_{n+1})_I = -(F_n, F'_n)_I \quad \text{for} \quad j' = -1 - \bar{j}, \tag{2.15}\]

where \( \{F_n = |n + \epsilon\}\), \( \{F'_n = |n + \epsilon\}\) are the bases in \( \mathcal{D}_{j,\epsilon}, \mathcal{D}_{j',\epsilon'} \), respectively.

One easily checks that both Hermitian forms are related by the reflection map \( Q_j : \mathcal{D}_{j,\epsilon} \to \mathcal{D}_{-1-j,\epsilon} \):

\[
(\ldots, \ldots)_\Pi = (\ldots, Q_j^2 \ldots)_I.
\]

For non-integral \((j, \epsilon)\) and real \(j\) the form \((\ldots, \ldots)_\Pi\) defines an indefinite scalar product on \( \mathcal{D}_{j,\epsilon} \).

For non-integral \((j, \epsilon)\) and \(j\) of the form \(j = -\frac{1}{2} + is, \quad s \in \mathbb{R}\)

\((\ldots, \ldots)_I\) defines an indefinite scalar product on \( \mathcal{D}_{j,\epsilon} \). We denote by \((\ldots, \ldots)_{j,\epsilon}\) this scalar product normalized by the condition

\[
(F_0, F_0)_{j,\epsilon} = 1.
\]

This product is invariant with respect to the reflection map \( (2.3) \)

\[
(Q_{j,\epsilon} \chi, Q_{j,\epsilon} \chi')_{-1-j,\epsilon} = (\chi, \chi')_{j,\epsilon}.
\]
For integral \((l, \epsilon)\) the scalar product \((\cdot, \cdot)\) is positively defined. For instance in the cases of \(l = \frac{1}{2}, -\frac{3}{2}\) condition (2.15) reads
\[
(F_0, F_0) = (F_{-1}, F_{-1}) \quad \text{II}.
\]
For generic \(j \in \mathbb{C}\) one has the hermitian conjugate pairs of representations
\[
(j = -\frac{1}{2} + i s + \nu, \epsilon), \quad (\tilde{j} = -\frac{1}{2} + i s - \nu, \epsilon), \quad s, \nu \in \mathbb{R}.
\]
Except the case \(\nu = 0\) the eigenvalues of the Casimir operator for this representations are complex conjugate numbers
\[
j(j + 1) = (-\frac{1}{4} - s^2 + 2i s \nu + \nu^2), \quad \tilde{j}(\tilde{j} + 1) = (-\frac{1}{4} - s^2 - 2i s \nu + \nu^2).
\]

In order to construct a hermitian representations with \(\nu \neq 0\) it is necessary to extend the space of representation to the direct sum
\[
\mathcal{G}_{j, \epsilon} = \mathcal{D}_{j, \epsilon} \oplus \mathcal{D}_{-1 - \tilde{j}, \epsilon}.
\]
On the extended space the generators are defined by
\[
J^a = \begin{pmatrix} J^a_0 & 0 \\ 0 & J^a_0 \end{pmatrix}.
\]
This is a hermitian representation of \(su(2)\) with the indefinite scalar product defined by
\[
(F_n, F_m) = (\tilde{F}_n, \tilde{F}_m) = 0, \quad (F_n, \tilde{F}_m) = (\tilde{F}_m, F_n) = (F_n, \tilde{F}_m) = (-1)^n \delta_{m,n},
\]
where \(F_n \in \mathcal{D}_{j, \epsilon}, \tilde{F}_n \in \mathcal{D}_{-1 - \tilde{j}, \epsilon}\).

### 2.10 tensoring representations

Let us consider the tensor product of the \(su(2)\) representations: \(\mathcal{D}_{j, \epsilon} \otimes S_{\frac{1}{2}}\). The eigenvalues of \(J^3 = J^3 + K^3\) are double degenerate on \(\mathcal{D}_{j, \epsilon} \otimes S_{\frac{1}{2}}\):
\[
J^3 |n + \epsilon\rangle_j \otimes |\frac{1}{2}\rangle = (n + \epsilon + \frac{1}{2}) |n + \epsilon\rangle_j \otimes |\frac{1}{2}\rangle, \\
J^3 |n + 1 + \epsilon\rangle_j \otimes |\frac{1}{2}\rangle = (n + \epsilon + \frac{1}{2}) |n + 1 + \epsilon\rangle_j \otimes |\frac{1}{2}\rangle.
\]
Diagonalizing the Casimir operator on these 2-dim eigenspaces one gets the eigenvalues
\[
(j + \frac{1}{2})(j + \frac{3}{2}), \quad (j - \frac{1}{2})(j + \frac{1}{2})
\]
and the corresponding eigenvectors
\[
|n + \frac{1}{2} + \epsilon\rangle_{j + \frac{1}{2}} = (1 + j + n + \epsilon) |n\rangle_j \otimes |\frac{1}{2}\rangle + (j - n - \epsilon) |n + 1\rangle_j \otimes |\frac{1}{2}\rangle, \\
|n + \frac{1}{2} + \epsilon\rangle_{j - \frac{1}{2}} = |n\rangle_j \otimes |\frac{1}{2}\rangle - |n + 1\rangle_j \otimes |\frac{1}{2}\rangle.
\]
It follows that
\[
\mathcal{D}_{j, \epsilon} \otimes S_{\frac{1}{2}} \simeq \mathcal{D}_{j + \frac{1}{2}, \epsilon} \oplus \mathcal{D}_{j - \frac{1}{2}, \epsilon}.
\]
where $\bar{\epsilon} = \epsilon + \frac{1}{2}$. The same analysis for $D_{j,\epsilon} \otimes S_{-\frac{3}{2}}$ yields the same eigenvalues. The corresponding vectors are slightly different:

$$
|n + \frac{1}{2} + \frac{\epsilon}{2}\rangle_{j+\frac{1}{2}} = (1 + j + n + \epsilon)|n\rangle_j \otimes |\frac{1}{2}\rangle_{-\frac{3}{2}} + (-j + n + \epsilon)|n\rangle_j \otimes |\frac{1}{2}\rangle_{-\frac{3}{2}},
$$

$$
|n + \frac{1}{2} + \frac{\epsilon}{2}\rangle_{j-\frac{1}{2}} = |n\rangle_j \otimes |\frac{1}{2}\rangle_{-\frac{3}{2}} + |n + 1\rangle_j \otimes |\frac{1}{2}\rangle_{-\frac{3}{2}},
$$

but the tensor product decomposition takes the same form

$$
D_{j,\epsilon} \otimes S_{-\frac{3}{2}} \simeq D_{j+\frac{1}{2},\epsilon} \oplus D_{j-\frac{1}{2},\epsilon}.
$$

For the hermitian representations one thus gets

$$
G_{j,\epsilon} \otimes S_{\frac{1}{2}} \simeq G_{j,\epsilon} \otimes S_{-\frac{3}{2}} \simeq G_{j+\frac{1}{2},\epsilon} \oplus G_{j-\frac{1}{2},\epsilon}.
$$

3 Nonrational $\hat{\mathfrak{su}}(2)_{\kappa}$ WZNW model

3.1 relaxed $\hat{\mathfrak{su}}(2)_{\kappa}$ modules

The $\mathfrak{su}(2)_{\kappa}$ affine algebra at the level $\kappa$ is defined by:

$$
\begin{align*}
[J^3_m, J^3_n] &= \kappa \delta_{m+n,0}, \\
[J^\pm_m, J^\mp_n] &= \pm J^\pm_{m+n}, \\
[J^+_m, J^-_n] &= 2J^3_{m+n} + \kappa m \delta_{m+n,0}, \quad m, n \in \mathbb{Z}.
\end{align*}
$$

The Sugawara construction

$$
L_m = \frac{1}{2(\kappa + 2)} \sum_n (2 : J^3_n J^3_{m-n} : + : J^+_n J^-_{m-n} : + : J^-_n J^+_n : )
$$

yields the associate Virasoro algebra

$$
\begin{align*}
[L_m, L_n] &= (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \\
[L_m, J^3_n] &= -nJ^3_{m+n},
\end{align*}
$$

with the central charge

$$
c = \frac{3\kappa}{2 + \kappa}.
$$

Let $\{|m + \epsilon\rangle\}_{m \in \mathbb{Z}}$ be the canonical basis in the representation $D_{j,\epsilon}$ with the $\mathfrak{su}(2)$ generators denoted by $J^3_n, J^\pm_n$. The relaxed $\mathfrak{su}(2)_{\kappa}$ module $D_{j,\epsilon}^{\kappa}[62]$ is generated from the states $|m + \epsilon\rangle$ satisfying the annihilation conditions:

$$
J^3_n |m + \epsilon\rangle = J^+_n |m + \epsilon\rangle = J^-_n |m + \epsilon\rangle = 0, \quad \text{for } n > 0,
$$

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by the action of the generators: $J^3_n$, $J^n_m$, $n < 0$. It has a natural $\mathbb{Z}$ grading

$$\hat{D}^\kappa_{j,\epsilon} = \bigoplus_{n=0}^{\infty} \hat{D}^{\kappa,n}_{j,\epsilon}, \quad \hat{D}^{\kappa,0}_{j,\epsilon} = D_{j,\epsilon},$$

where $\hat{D}^{\kappa,n}_{j,\epsilon}$ are eigenspaces of the operator $L_0 - \frac{j(j+1)}{\kappa + \frac{1}{2}}$. For generic $j$ there is no hermitian, non-degenerate bilinear form on $\hat{D}^\kappa_{j,\epsilon}$. But for the modules $\hat{D}^\kappa_{j,\epsilon}$ and $\hat{D}^\kappa_{-1-j,\epsilon}$ there is a hermitian pairing defined on the zero level subspaces $D_{j,\epsilon}$, $D_{-1-j,\epsilon}$ by the form $(\cdot , \cdot)_I$ and extended to the whole modules by the hermitian conjugation rules:

$$(J^3_n)\dagger = J^3_{-n}, \quad (J^\pm_n)\dagger = J^\mp_{-n}. \quad (3.2)$$

### 3.2 associated highest weight module

We shall now describe a construction of the highest weight module $\mathcal{H}^\kappa_j$ associated to the module $\hat{D}^\kappa_{-1-j,\epsilon}$. To this end let us define a new set of generators

$$J^+_n(x) = J^+_n - 2xJ^3_n - x^2J^-_n, \quad J^-_n(x) = J^-_n + xJ^+_n, \quad J^0_n(x) = J^0_n,$$

(3.3)

where $x$ is a complex parameter. For any $x$ they satisfy the commutation relations of the $\hat{su}(2)$ affine algebra at level $\kappa$:

$$[J^3_m(x), J^3_n(x)] = \frac{\kappa}{2} m\delta_{m+n,0},$$

$$[J^3_m(x), J^\pm_n(x)] = \pm J^\pm_{m+n}(x),$$

$$[J^\pm_m(x), J^\pm_n(x)] = 2J^\pm_{m+n}(x) + \kappa m\delta_{m+n,0}, \quad m, n \in \mathbb{Z}.$$

The state $\langle x \rangle_{j,\epsilon}$ is the highest weight state with respect to the algebra $J^\alpha_n(x)$. Indeed it is annihilated by $J^\alpha_n(x)$, $n > 0$ and

$$J^+_0(x) \langle x \rangle_{j,\epsilon} = 0,$$

$$J^0_0(x) \langle x \rangle_{j,\epsilon} = j \langle x \rangle_{j,\epsilon}.$$

The construction above was first introduced in the case of finite dimensional representations $S_l$ [34] and is known as the $x$-representation in the WZNW models (with $x$ called the isospin variable). One has for instance

$$\langle x \rangle_{1/2} \equiv \langle x \rangle_{3/4} = \frac{1}{2} \langle -\frac{1}{2} \rangle_{-3/2} + x \langle -\frac{1}{2} \rangle_{-3/2},$$

and:

$$K^+_0(x) \langle x \rangle_{1/2} = (K^+_0 - 2xK^3_0 - x^2K^-_0)(\frac{1}{2} \langle -\frac{1}{2} \rangle_{-3/2} + x \langle -\frac{1}{2} \rangle_{-3/2}) = 0,$$

$$K^3_0(x) \langle x \rangle_{1/2} = (K^3_0 + xK^-_0)(\frac{1}{2} \langle -\frac{1}{2} \rangle_{-3/2} + x \langle -\frac{1}{2} \rangle_{-3/2}) = \frac{1}{2} \langle x \rangle_{3/4},$$

$$K^-_0(x) \langle x \rangle_{1/2} = K^-_0 \langle x \rangle_{1/2} = \langle -\frac{1}{2} \rangle_{-3/2}.$$

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3.3 spectrum and operator - state correspondence

We assume that the $\hat{su}(2)_\kappa$ WZNW model at non-rational level $\kappa$ is based on the representations

$$\hat{D}^\kappa_{j,\epsilon}, \quad j = -\frac{1}{2} + is, \quad s \in \mathbb{R}, \quad \epsilon = 0, \frac{1}{2}.$$  

The spectrum is diagonal and the space of states is the direct integral/sum of the tensor products

$$\hat{D}^\kappa_{j,\epsilon} \otimes \hat{D}^\kappa_{j,\epsilon}$$

with the indefinite scalar product $(.,.)_{j,\epsilon}$ defined by hermitian form (2.15) extended by hermitian conjugation relations (3.2).

The primary fields in the isospin variables can be introduced by the operator state correspondence:

$$\lim_{z \to 0} \Phi_{j,\epsilon}(x, \bar{x}; z, \bar{z}) |0\rangle = |x\rangle_{j,\epsilon} \otimes |\bar{x}\rangle_{j,\epsilon},$$

where $|x\rangle_{j,\epsilon}, |\bar{x}\rangle_{j,\epsilon}$ are the highest weight states defined in (2.1):

$$|x\rangle_{j,\epsilon} = \sum_{m = -\infty}^{\infty} x^{j-m-\epsilon} |m + \epsilon\rangle_{-1-j}.$$  

One can decompose $\Phi_{j,\epsilon}(x, \bar{x}; w, \bar{w})$ into the primary fields in the spin bases

$$\Phi_{j,\epsilon}(x, \bar{x}; z, \bar{z}) = \sum_{m, \bar{m} = -\infty}^{\infty} V_{m+\epsilon, \bar{m}+\epsilon}^{-1-j,\epsilon}(z, \bar{z}) x^{j-m-\epsilon} \bar{x}^{j-\bar{m}-\epsilon}$$

for which the operator-state correspondence takes the form

$$\lim_{z \to 0} V_{m+\epsilon, \bar{m}+\epsilon}^{-1-j,\epsilon}(z, \bar{z}) |0\rangle = |m + \epsilon\rangle_{-1-j} \otimes |\bar{m} + \bar{\epsilon}\rangle_{-1-j}.$$  

The properties of the primary fields listed above suggest that they can be seen as tensor products of their chiral parts

$$\Phi_{j,\epsilon}(x, \bar{x}; z, \bar{z}) = \Phi_{j,\epsilon}(x; z) \otimes \Phi_{j,\epsilon}(\bar{x}; \bar{z}),$$

$$V_{m+\epsilon, \bar{m}+\epsilon}^{-1-j,\epsilon}(z, \bar{z}) = V_{m+\epsilon}^{-1-j,\epsilon}(z) \otimes V_{\bar{m}+\bar{\epsilon}}^{-1-j,\bar{\epsilon}}(\bar{z}).$$

It should be emphasized that although $\bar{z}$ is complex conjugate to the complex variable $z$, $x$ and $\bar{x}$ are independent complex variables.

Since the representations $\mathcal{D}_{j,\epsilon}, \mathcal{D}_{-1-j,\epsilon}$ are equivalent we declare an identification of the corresponding primary fields. It means that in any correlation function the following relation holds

$$(Q_{-1-j,\epsilon} \otimes Q_{-1-j,\epsilon}) \Phi_{j,\epsilon}(x, \bar{x}; z, \bar{z}) = R_{j,\epsilon}^{2} \Phi_{1,\epsilon}(x, \bar{x}; z, \bar{z}).$$
In terms of the chiral primary fields one has

\[ Q_{-1-j}\Phi_{j,\epsilon}(x, z) = R_{j,\epsilon} \Phi_{j,\epsilon}(x; z), \]
\[ Q_{-1-j}\Phi_{j,\epsilon}(\bar{x}, \bar{z}) = R_{j,\epsilon} \Phi_{j,\epsilon}(\bar{x}; \bar{z}), \]

where \( Q_{-1-j} \) is reflection map (2.3) and the coefficient \( R_{j,\epsilon} \) is the chiral reflection amplitude. Since \( Q_j Q_{-1-j} = \text{id} \), it satisfies \( R_{-1-j}\epsilon R_{j,\epsilon} = 1 \). Using (2.8) one can write the identification in the form

\[ \frac{1}{C_{j,\epsilon}} \oint \frac{dy}{2\pi i} S_{j,\epsilon}(x, y) \Phi_{-1-j,\epsilon}(y, z) = R_{j,\epsilon} \Phi_{j,\epsilon}(x; z), \]
\[ \frac{1}{C_{j,\epsilon}} \oint \frac{dy}{2\pi i} S_{j,\epsilon}(\bar{x}, y) \Phi_{-1-j,\epsilon}(y, \bar{z}) = R_{j,\epsilon} \Phi_{j,\epsilon}(\bar{x}; \bar{z}). \]

\[ (3.4) \]

### 3.4 Ward identities

Relations (2.2) imply the following OPE of the \( \text{su(2)} \) currents with the primary fields

\[ J^+(z) \Phi_{j,\epsilon}(x, \bar{x}; w, \bar{w}) \sim \frac{(-x^2 \partial_x + 2 jx) \Phi_{j,\epsilon}(x, \bar{x}; w, \bar{w})}{z - w}, \]
\[ J^3(z) \Phi_{j,\epsilon}(x, \bar{x}; w, \bar{w}) \sim \frac{(-x \partial_x + j) \Phi_{j,\epsilon}(x, \bar{x}; w, \bar{w})}{z - w}, \]
\[ J^-(z) \Phi_{j,\epsilon}(x, \bar{x}; w, \bar{w}) \sim \frac{\partial_x \Phi_{j,\epsilon}(x, \bar{x}; w, \bar{w})}{z - w}, \]

and the corresponding relations for the right currents \( J^+(\bar{z}), J^3(\bar{z}), J^-(\bar{z}) \). It is convenient to introduce the \( \text{su(2)} \) currents with the isospin variables [34]:

\[ J^+(x, z) = J^+(z) - 2x J^3(z) - x^2 J^-(z), \]
\[ J^3(x, z) = J^3(z) + x J^-(z), \]
\[ J^-(x, z) = J^-(z). \]

The excited states are iteratively defined in the standard manner

\[ J_n^a(y) J_M^\Phi_{j,\epsilon}(x, z) = \oint z \frac{dw}{2\pi i} (w - z)^n J_n^a(y, w) J_M^\Phi_{j,\epsilon}(x, z), \]

where \( J_M^\Phi_{j,\epsilon}(x, z) \) denotes an arbitrary array of the operators \( J_n^a(y') \). The general Ward identities with the \( \text{su(2)} \) currents take the following form,

\[ \langle J_K \Phi_{j_3,\epsilon_3}(x_3, z_3) J_L \Phi_{j_2,\epsilon_2}(x_2, z_2) J_n^a \Phi_{j_1,\epsilon_1}(x_1, z_1) \rangle = -\sum_{p=0}^{\infty} (-1)^p (n+p-1) \frac{\langle J_p^a J_K \Phi_{j_3,\epsilon_3}(x_3, z_3) J_L \Phi_{j_2,\epsilon_2}(x_2, z_2) J_n^a \Phi_{j_1,\epsilon_1}(x_1, z_1) \rangle}{(z_3 - z_1)^{n+p}} \]
\[ -\sum_{p=0}^{\infty} (-1)^p (n+p-1) \frac{\langle J_p^a J_K \Phi_{j_3,\epsilon_3}(x_3, z_3) J_L \Phi_{j_2,\epsilon_2}(x_2, z_2) J_n^a \Phi_{j_1,\epsilon_1}(x_1, z_1) \rangle}{(z_2 - z_1)^{n+p}}. \]
In the special case of two primary fields and the $x$-dependent currents we have
\[
\langle \Phi_{j_3,\epsilon_3}(x_3, z_3)\Phi_{j_2,\epsilon_2}(x_2, z_2) J_n^0(x_1) J_M(x_1) \Phi_{j_1,\epsilon_1}(x_1, z_1) \rangle = \frac{\langle J_n^0(x_1) \Phi_{j_3,\epsilon_3}(x_3, z_3)\Phi_{j_2,\epsilon_2}(x_2, z_2) J_M(x_1) \Phi_{j_1,\epsilon_1}(x_1, z_1) \rangle}{(z_3 - z_1)^n} - \frac{\langle \Phi_{j_3,\epsilon_3}(x_3, z_3) J_n^0(x_1) \Phi_{j_2,\epsilon_2}(x_2, z_2) J_M(x_1) \Phi_{j_1,\epsilon_1}(x_1, z_1) \rangle}{(z_2 - z_1)^n},
\]
where the action of the zero modes can be derived from (3.5):
\[
\begin{align*}
J_0^+ (x) \Phi_{j,\epsilon}(y, w) &= -((x - y)^2 \partial_y + 2j(x - y)) \Phi_{j,\epsilon}(y, w), \\
J_0^0 (x) \Phi_{j,\epsilon}(y, w) &= ((x - y) \partial_y + j) \phi_j(y, w), \\
J_0^- (x) \Phi_{j,\epsilon}(y, w) &= \delta_y \Phi_{j,\epsilon}(y, w).
\end{align*}
\]
In the limit $z_3 \to \infty, z_2 \to z, z_1 \to 0$ one gets further simplification
\[
\langle \Phi_{j_3,\epsilon_3}(x_3, \infty)\Phi_{j_2,\epsilon_2}(x_2, z) J_n^0(x_1) J_M(x_1) \Phi_{j_1,\epsilon_1}(x_1, 0) \rangle = - \frac{\langle \Phi_{j_3,\epsilon_3}(x_3, \infty) J_n^0(x_1) \Phi_{j_2,\epsilon_2}(x_2, z) J_M(x_1) \Phi_{j_1,\epsilon_1}(x_1, 0) \rangle}{z^n}.
\]

### 3.5 2-point functions

The global Ward identities imply the following general form of the 2-point function
\[
\begin{align*}
\langle \Phi_{j_1,\epsilon_1}(x_1; \bar{x}_1; z_1, \bar{z}_1)\Phi_{j_2,\epsilon_2}(x_2; \bar{x}_2; z_2, \bar{z}_2) \rangle &= \langle \Phi_{j_1,\epsilon_1}(x_1; z_1)\Phi_{j_2,\epsilon_2}(x_2; z_2) \rangle \langle \Phi_{j_1,\epsilon_1}(\bar{x}_1; \bar{z}_1)\Phi_{j_2,\epsilon_2}(\bar{x}_2; \bar{z}_2) \rangle, \\
\langle \Phi_{j_1,\epsilon_1}(x_1; z_1)\Phi_{j_2,\epsilon_2}(x_2; z_2) \rangle &= (z_1 - z_2)^{-2\Delta_1} \delta_{\epsilon_1,\epsilon_2} \\
&\times \left[ A_{j_2,\epsilon_2} \delta_{-1-j_1} \delta(x_1 - x_2) + B_{j_2,\epsilon_2} \delta_{j_1} S_{j_2,\epsilon_2}(x_1, x_2) \right].
\end{align*}
\]

The consistency conditions with identification (3.4) read
\[
\begin{align*}
R_{j_1,\epsilon_1} \langle \Phi_{j_1,\epsilon_1}(x_1; z_1)\Phi_{j_2,\epsilon_2}(x_2; z_2) \rangle &= \frac{1}{C_{j_1,\epsilon_1}} \int \frac{dy}{2\pi} S_{j_1,\epsilon_1}(x_1, y) \langle \Phi_{-1-j_1,\epsilon_1}(y; z_1)\Phi_{j_2,\epsilon_2}(x_2; z_2) \rangle, \\
R_{j_2,\epsilon_2} \langle \Phi_{j_1,\epsilon_1}(x_1; z_1)\Phi_{j_2,\epsilon_2}(x_2; z_2) \rangle &= \frac{1}{C_{j_2,\epsilon_2}} \int \frac{dy}{2\pi} S_{j_2,\epsilon_2}(x_2, y) \langle \Phi_{j_1,\epsilon_1}(x_1; z_1)\Phi_{-1-j_2,\epsilon_2}(y; z_2) \rangle.
\end{align*}
\]
They are satisfied if
\[
A_{j,\epsilon} = R_{j,\epsilon} C_{j,\epsilon} B_{j,\epsilon},
\]
We chose the solution
\[
A_{j,\epsilon} = c, \quad B_{j,\epsilon} = \frac{c}{R_{j,\epsilon} C_{j,\epsilon}},
\]
which for a given $R_{j,\epsilon}$ fixes the normalization of fields up to an overall $(j, \epsilon)$-independent constant $c$. With this normalization the chiral 2-point function takes the form
\[
\langle \Phi_{j_1,\epsilon_1}(x_1; z_1)\Phi_{j_2,\epsilon_2}(x_2; z_2) \rangle = c (z_1 - z_2)^{-2\Delta_1} \delta_{\epsilon_1,\epsilon_2} \\
\times \left[ \delta_{-1-j_1} \delta(x_1 - x_2) + \frac{\delta_{j_1} \delta_{j_2}}{R_{j_2,\epsilon_2} C_{j_2,\epsilon_2}} S_{j_2,\epsilon_2}(x_1, x_2) \right].
\]
3.6 3-point functions

As in the case of the 2-point function the global Ward identities imply factorization into chiral parts up to \( j, \epsilon \)-dependent constants

\[
\langle \Phi_{j_3, \epsilon_3}(x_3, \bar{x}_3; z_3, \bar{z}_3) \Phi_{j_2, \epsilon_2}(x_2, \bar{x}_2; z_2, \bar{z}_2) \Phi_{j_1, \epsilon_1}(x_1, \bar{x}_1; z_1, \bar{z}_1) \rangle = c[\Delta(j_1); z_1] c[\Delta(j_2); \bar{z}_2] \sum_{\epsilon, \epsilon' = 0, 1} S_{\epsilon} \left[ \begin{array}{ccc} j_3 & j_2 & j_1 \\ \epsilon_3 & \epsilon_2 & \epsilon_1 \\ x_3 & x_2 & x_1 \end{array} \right] S'_{\epsilon'} \left[ \begin{array}{ccc} j_3 & j_2 & j_1 \\ \epsilon_3 & \epsilon_2 & \epsilon_1 \\ x_3 & x_2 & x_1 \end{array} \right] C^{\epsilon, \epsilon'}[j_1; \epsilon_1],
\]

where

\[
c[\Delta; z_1] = (z_2 - z_1)^{\Delta_3 - \Delta_2 - \Delta_1} (z_3 - z_1)^{\Delta_2 - \Delta_1} (z_3 - z_2)^{\Delta_1 - \Delta_3 - \Delta_2}
\]

and \( S_{\epsilon} \) is 3-linear \( su(2) \) invariant (2.9). We assume the following diagonal form of the constants \( C^{\epsilon, \epsilon'}[j_1; \epsilon_1] \):

\[
C^{\epsilon, \epsilon'}[j_1; \epsilon_1] = \delta_{\epsilon, \epsilon'} s_{\epsilon}[j_1; \epsilon] C[j_3, j_2, j_1],
\]

where \( s_{\epsilon}[j_1; \epsilon] \) is a classical part independent of \( \kappa \) and \( C[j_3, j_2, j_1] \) is a quantum part independent of \( \epsilon \)'s. With this assumption the 3-point function takes the form

\[
\langle \Phi_{j_3, \epsilon_3}(x_3, \bar{x}_3; z_3, \bar{z}_3) \Phi_{j_2, \epsilon_2}(x_2, \bar{x}_2; z_2, \bar{z}_2) \Phi_{j_1, \epsilon_1}(x_1, \bar{x}_1; z_1, \bar{z}_1) \rangle = c[\Delta(j_1); z_1] c[\Delta(j_2); \bar{z}_2] \sum_{\epsilon = 0, 1} S_{\epsilon} \left[ \begin{array}{ccc} j_3 & j_2 & j_1 \\ \epsilon_3 & \epsilon_2 & \epsilon_1 \\ x_3 & x_2 & x_1 \end{array} \right] S_{\epsilon} \left[ \begin{array}{ccc} j_3 & j_2 & j_1 \\ \epsilon_3 & \epsilon_2 & \epsilon_1 \\ x_3 & x_2 & x_1 \end{array} \right] s_{\epsilon}[j_1; \epsilon_1] C[j_3, j_2, j_1].
\]

As it was mentioned in the introduction the \( su(2) \) WZNW model at level \( \kappa < -2 \) should be equivalent to the \( H_3^+ = SL(2, \mathbb{C})/SU(2) \) coset model at level \( \kappa' = -\kappa > 2 \). This suggests that the quantum part \( C[j_3, j_2, j_1] \) should be given by the \( H_3^+ \) structure constants. With the normalization chosen in [43] in the case of \( \kappa > -2 \) (being in line with the normalization used in the \( su(2) \) minimal models) one has

- for \( \kappa < -2 \), \(-\kappa + 2 = b^{-2} > 0 \)

\[
C^b_b[j_3, j_2, j_1] = \frac{M^b_b \sqrt{\prod_{a=1}^3 \Upsilon_b(-b(2j_a + 1)) \Upsilon_b(-b(2j_a + 1))}}{\Upsilon_b(-b(2j_{123} + 1)) \Upsilon_b(-b(j_{12}^2) \Upsilon_b(-b(j_{12}^2))},
\]

- for \( \kappa > -2 \), \(-\kappa + 2 = -\hat{b}^{-2} < 0 \)

\[
C^\hat{b}_b[j_3, j_2, j_1] = \frac{M^\hat{b}_b \Upsilon_{\hat{b}}(\hat{b}(2j_{123} + 2)) \Upsilon_{\hat{b}}(\hat{b}(2j_a + 1)) \Upsilon_{\hat{b}}(\hat{b}(2j_{12}^2) + 1) \Upsilon_{\hat{b}}(\hat{b}(2j_{12}^2) + 1))}{\sqrt{\prod_{a=1}^3 \Upsilon_{\hat{b}}(\hat{b}(2j_a + 1)) \Upsilon_{\hat{b}}(\hat{b}(2j_a + 1))}})
\]

Using the shift relations

\[
\Upsilon_b(x + b) = \gamma(bx)b^{-2bx} \Upsilon_b(x), \quad \Upsilon_b(x + b^{-1}) = \gamma(b^{-1}x)b^{-1+2b^{-1}x} \Upsilon_b(x), \quad \Upsilon_b(Q - x) = \Upsilon_b(x),
\]

\[
- \frac{\gamma}{\sqrt{\Upsilon_b(b(2j_a + 1))}}
\]

\[
- \frac{\gamma}{\sqrt{\Upsilon_b(b(2j_a + 1))}}
\]
one gets the reflection properties
\[ C_b^a(-1 - j_3, j_2, j_1) = \sqrt{\gamma(-2j_3 - 1)\gamma(-2j_3)} \gamma(j_2^3 + 1) \gamma(j_1^3 + 1) C_b^a(j_3, j_2, j_1). \]
\[ C_b^{\bar{a}}(-1 - j_3, j_2, j_1) = \sqrt{\gamma(-2j_3 - 1)\gamma(-2j_3)} \gamma(j_1^3 + 1) \gamma(j_2^3 + 1) C_b^{\bar{a}}(j_3, j_2, j_1). \]

If we chose (3.10) with
\[ s_{[i; i]} = \frac{1}{s(\frac{1}{2}j_i^3 + \epsilon + \epsilon_3)s(\frac{1}{2}j_3^3 + \epsilon + \epsilon_2)s(\frac{1}{2}j_2^3 + \epsilon + \epsilon_1)s(\frac{1}{2} + \frac{1}{2}j_1^3 + \epsilon)}, \]
then (2.13) implies that 3-point functions (3.11) satisfy the simple reflection rule
\[ \frac{1}{C_{j_3, e_3}^2} \int \frac{dy}{2\pi i} S_{j_3, e_3}(x_3, y) \int \frac{dy}{2\pi i} S_{j_1, e_1}(\bar{x}_3, \bar{y}) \times \langle \Phi_{-1-j_3, e_3}(x_3, \bar{x}_2; z_3, \bar{z}_3) \Phi_{j_2, e_2}(x_2, \bar{x}_2; z_2, \bar{z}_2) \Phi_{j_1, e_1}(x_1, \bar{x}_1; z_1, \bar{z}_1) \rangle = \sqrt{\gamma(-2j_3 - 1)\gamma(-2j_3)} \langle \Phi_{j_3, e_3}(x_3, \bar{x}_2; z_3, \bar{z}_3) \Phi_{j_2, e_2}(x_2, \bar{x}_2; z_2, \bar{z}_2) \Phi_{j_1, e_1}(x_1, \bar{x}_1; z_1, \bar{z}_1) \rangle \).

For the purposes of the off-diagonal extension we shall introduce the chiral 3-point functions
\[ \langle \Phi_{j_3, e_3}(x_3; z_1) \Phi_{j_2, e_2}(x_2; z_2) \Phi_{j_1, e_1}(x_3; z_3) \rangle^A \]
\[ = c[\Delta(j_i); z_i] \tilde{S}_e[j_3, j_2, j_1] C^A[j_3, j_2, j_1], \]
\[ \langle \Phi_{j_3, e_3}(\bar{x}_3; \bar{z}_1) \Phi_{j_2, e_2}(\bar{x}_2; \bar{z}_2) \Phi_{j_1, e_1}(\bar{x}_3; \bar{z}_3) \rangle^A \]
\[ = c[\Delta(j_i); z_i] \tilde{S}_e[j_3, j_2, j_1] \bar{C}^A[j_3, j_2, j_1], \quad A = s, \bar{s}, \]
where
\[ \tilde{S}_e[j_3, j_2, j_1] = S_e[j_3, j_2, j_1] \sqrt{s_{[i; i]}}, \quad (3.15) \]
and branches of the square root are chosen in such a way that formula (3.11) for the full 3-point function is reproduced. The chiral splitting of the j-dependent parts of the structure constants is motivated by the similar splitting in the Liouville theory [13]:
\[ C_b^a(j_3, j_2, j_1) = \sqrt{M_b^a} \Gamma_b(-b(j_3 + 1)) \Gamma_b(-bj_3^3) \Gamma_b(-bj_2^3) \sqrt{\prod_{i=1}^3 \Gamma_b(-2b_i)} \Gamma_b(-2(b_i + 1)) \quad (3.16) \]
\[ C_b^{\bar{a}}(j_3, j_2, j_1) = \sqrt{M_b^{\bar{a}}} \Gamma_b(Q + b(j_3 + 1)) \Gamma_b(Q + bj_3^3) \Gamma_b(Q + bj_2^3) \sqrt{\prod_{i=1}^3 \Gamma_b(Q + 2b_i)} \Gamma_b(Q + 2(b_i + 1)), \]
\[ C_b^{a}(j_3, j_2, j_1) = \sqrt{M_b^a} \Gamma_b(\frac{1}{b} - b(j_3 + 1)) \Gamma_b(\frac{1}{b} - bj_3^3) \Gamma_b(\frac{1}{b} - bj_2^3) \sqrt{\prod_{i=1}^3 \Gamma_b(\frac{1}{b} - 2b_i)} \Gamma_b(\frac{1}{b} - 2(b_i + 1)), \]
\[ C_b^{a}(j_3, j_2, j_1) = \sqrt{M_b^a} \Gamma_b(b(j_3 + 1)) \Gamma_b(b + bj_3^3) \Gamma_b(b + bj_2^3) \sqrt{\prod_{i=1}^3 \Gamma_b(b(2i + 1))} \Gamma_b(b(2i + 2)), \]
\[ C_b^{a}(j_3, j_2, j_1) = \sqrt{M_b^a} \Gamma_b(b + (j_3 + 1)) \Gamma_b(b + bj_3^3) \Gamma_b(b + bj_2^3) \sqrt{\prod_{i=1}^3 \Gamma_b(b(2i + 1))} \Gamma_b(b(2i + 2)). \]
As in the case of the SL-LL correspondence there is no canonical splitting into chiral parts. Our choice is motivated by the relations

\[ \bar{C}^\kappa_b(j_3, j_2, j_1) = r(j_i) C^\kappa_b(-j_3 - 1, -j_2 - 1, -j_1 - 1), \]

\[ \bar{C}^\kappa_b(j_3, j_2, j_1) = r(j_i) C^\kappa_b(-j_3 - 1, -j_2 - 1, -j_1 - 1), \]  

where

\[ r(j_i) = \sqrt{\frac{2\pi}{\prod_{i=1}^3 \Gamma (2j_i + 1) \Gamma (2j_i + 2)}} \]

which reduce the calculations to one sector. Relations (3.17) can be easily derived using the shift formulae

\[ \Gamma_b(x + b) = \sqrt{2\pi b^{bx - \frac{1}{2}}} \frac{\Gamma_b(x)}{\Gamma(b)}, \quad \Gamma_b(x + b^{-1}) = \sqrt{2\pi b^{-b^{-1}x + \frac{1}{2}}} \frac{\Gamma_b(x)}{\Gamma(b^{-1}x)}. \]  

3.7 off-diagonal extension

As we shall see the coset construction requires an off-diagonal extension of the \( \hat{\text{su}}(2) \) WZW model to the following class of representations

\[ \hat{D}^\kappa_{j,\epsilon}, \quad j = -\frac{1}{2} + n + is, \quad s \in \mathbb{R}, \quad \epsilon = 0, \frac{1}{2}, \quad n \in \frac{1}{2}\mathbb{Z}. \]

The space of states is the direct integral/sum of the tensor products of the left and the right relaxed modules from this class

\[ \hat{D}^\kappa_{jL,\epsilon} \otimes \hat{D}^\kappa_{jR,\epsilon}, \quad j_L - j_R = n_L - n_R \in \mathbb{Z}. \]

Let us note that the spectrum is off-diagonal only with respect to the discrete variable \( n \).

For \((n_L, n_R) \neq (0, 0)\) there is no hermitian invariant form on \( \hat{D}^\kappa_{jL,\epsilon} \otimes \hat{D}^\kappa_{jR,\epsilon} \). One can however construct such form on the direct sum

\[ \left( \hat{D}^\kappa_{jL,\epsilon} \otimes \hat{D}^\kappa_{jR,\epsilon} \right) \oplus \left( \hat{D}^\kappa_{-1-jL,\epsilon} \otimes \hat{D}^\kappa_{-1-jR,\epsilon} \right) \]

by analogy with the construction of the hermitian \( \text{su}(2) \) representations given in subsection 2.9.

In the case of the diagonal spectrum 2- and 3-point functions were introduced as products of their left and right chiral components. The extension of the chiral 3-point function in the \( z \)- and the \( j \)-dependent parts is straightforward. If we assume the condition

\[ n_1 + n_2 + n_3 = 0 \]

the arguments of all the Barnes gamma functions involved are shifted by integer multiplicities of \( b \) or \( \hat{b} \) and can be explicitly calculated by shift relations (3.19). In the following we assume condition (3.20). As we shall see it is always satisfied in relations (1.3) if we restrict ourselves to the diagonal spectra on their left hand sides.
The extension of the $x$-dependent part is more subtle. One has to preserve its role of the generating function for the structure constants in the spin basis. This requires definite periodicity conditions on the torus $S^1 \times S^1 \times S^1$. Using (2.10) and the identities for the sine function

$$s \left( x + \frac{1}{2} n_{123} - n_i + (\epsilon + \frac{1}{2} n_{123}) \right) = (-1)^{\frac{1}{2} n_{123} + (1 + 4 \epsilon) \delta + n_i - (1 + 4 \epsilon) \delta} s \left( x + \epsilon + \epsilon_i \right),$$

$$s \left( x + \frac{1}{2} n_{123} + (\epsilon + \frac{1}{2} n_{123}) \right) = (-1)^{\frac{1}{2} n_{123} + (1 + 4 \epsilon) \delta} s \left( x + \epsilon \right),$$

where $\delta = \frac{1}{2} n_{123} + 0, \delta_i = \epsilon_i + 0$, one gets the identity for the invariant $S_\epsilon \sqrt{s_\epsilon}$:

$$\left( x_1 - x_2 \right)^{n_{12}} \left( x_3 - x_1 \right)^{n_{31}} \left( x_2 - x_3 \right)^{n_{32}} S_\epsilon \left[ \begin{array}{ccc} j_3 & j_2 & j_1 \\ \epsilon_3 & \epsilon_2 & \epsilon_1 \end{array} \right] \sqrt{s_\epsilon[j_1; \epsilon_1]}$$

$$= \sqrt{(-1)^{n_{123} - \sum (4 \epsilon_i + 1) \delta_i}} S_\epsilon \left[ \begin{array}{ccc} j_1 + n_3 & j_2 + n_2 & j_1 + n_1 \\ \epsilon_1 + n_1 & \epsilon_2 + n_2 & \epsilon_3 + n_3 \end{array} \right] \sqrt{s_\epsilon[j_1 + n_i; \epsilon_i + \epsilon_n]}.$$  

(3.21)

We use this formula as a motivation for the definition of the off-diagonal extension of $x$-dependent part (3.15)

$$\tilde{S}_{\epsilon + \frac{1}{2} n_{123}} \left[ \begin{array}{ccc} j_3 + n_3 & j_2 + n_2 & j_1 + n_1 \\ \epsilon_3 + n_3 & \epsilon_2 + n_2 & \epsilon_1 + n_3 \end{array} \right] = (-1)^{\eta(n_3, n_2, n_1)} \left( x_1 - x_2 \right)^{n_{12}} \left( x_3 - x_1 \right)^{n_{31}} \left( x_2 - x_3 \right)^{n_{32}} S_\epsilon \left[ \begin{array}{ccc} j_3 & j_2 & j_1 \\ \epsilon_3 & \epsilon_2 & \epsilon_1 \end{array} \right],$$

(3.22)

where condition (3.20) and the prescription given below (3.15) are assumed. We also assume that all square root ambiguities are hidden in the sign factor $(-1)^{\eta(n_3, n_2, n_1)}$. The general form of the function $\eta(n_3, n_2, n_1)$ is not known. In principle it could be derived from expected equivalence (1.6) and the general form of highest weight states (4.11). The results of explicit calculations in a number of cases of low lying states are presented in Appendix 3.

### 3.8 Liouville and imaginary Liouville structure constants

In the symmetric normalization $\Phi_\alpha = \Phi_{Q-\alpha}$ the DOZZ structure constants [21, 22] for primary fields $\Phi_\alpha$ of conformal dimension $\Delta_\alpha = \alpha(Q - \alpha)$ take the form

$$C^\text{DOZZ}_{\alpha_3, \alpha_2, \alpha_1} = M_b^\alpha \prod_i \sqrt{\frac{Y_b(Q - 2 \alpha_i) Y_b(-Q + 2 \alpha_i)}{Y_b(\alpha_{i23} - Q) Y_b(\alpha_{i3}^2) Y_b(\alpha_{i3}^1) Y_b(\alpha_{i3}^2)}},$$  

(3.23)

where $\alpha_{123} = \alpha_1 + \alpha_2 + \alpha_3$, $\alpha_{12}^3 = \alpha_1 + \alpha_2 - \alpha_3$, etc. and the real parameters $b, Q = b + b^{-1}$ are related to the central charge $c$ of the Liouville theory by

$$c^L = 1 + 6Q^2 \quad Q = b + b^{-1}.$$  

For the purpose of the present paper it is convenient to parameterize the Liouville primary fields in terms of

$$j = -\frac{\alpha}{Q}, \quad \Delta_j^L = -Q^2 j(1 + j),$$

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rather then $\alpha$. Using the freedom in decomposing $C^\text{DOZZ}_b$ into chiral structure constants [13] we define

$$C^\mu_b[j_3, j_2, j_1] = C^\mu_b(j_3, j_2, j_1)\overline{C}^\mu_b(j_3, j_2, j_1),$$

$$C^\mu_b(j_3, j_2, j_1) = \frac{\sqrt{M^2}}{\prod^{3}_{i=1} \Gamma_b(-Q(j_{123} + 1)) \Gamma_b(-Qj_{12}^3) \Gamma_b(-Qj_{13}^2) \Gamma_b(-Qj_{23}^1)} \cdot \prod^{3}_{i=1} \Gamma_b(-2Qj_i) \Gamma_b(-Q(2j_i + 1)),$$

$$\bar{C}^\mu_b(j_3, j_2, j_1) = \frac{\sqrt{M^2}}{\prod^{3}_{i=1} \Gamma_b(Q + Q(j_{123} + 1)) \Gamma_b(Q + Qj_{12}^3) \Gamma_b(Q + Qj_{13}^2) \Gamma_b(Q + Qj_{23}^1)} \cdot \prod^{3}_{i=1} \Gamma_b(Q + 2Qj_i) \Gamma_b(2Q + 2Qj_i).$$

Let us note that $C^\mu_b(-j_3 - 1, -j_2 - 1, -j_1 - 1) = \bar{C}^\mu_b(j_3, j_2, j_1)$.

In the imaginary Liouville theory [25, 26] with purely imaginary parameter $b = -ib$ and the central charge

$$e^{i\nu} = 1 - 6\hat{Q}^2 \quad \hat{Q} = \hat{b}^{-1} - \hat{b}$$

it is convenient to use the parametrization

$$j = -\frac{\alpha}{\hat{Q}}, \quad \Delta^\nu_j = \hat{Q}^2 j(1 + j).$$

The corresponding structure constants are given by

$$C^\mu_n[j_3, j_2, j_1] = C^\mu_b[j_3, j_2, j_1]\overline{C}^\mu_b[j_3, j_2, j_1],$$

$$C^\mu_n[j_3, j_2, j_1] = \frac{\sqrt{M^2}}{\prod^{3}_{a=1} \Gamma_b(\hat{b} - 2\hat{Q}j_a) \Gamma_b(\hat{b} - \hat{Q}(2j_a + 1))} \cdot \prod^{3}_{a=1} \Gamma_b(\hat{b} - Q(j_{123} + 1)) \Gamma_b(\hat{b} - \hat{Q}j_{12}^3) \Gamma_b(\hat{b} - \hat{Q}j_{13}^2) \Gamma_b(\hat{b} - \hat{Q}j_{23}^1),$$

$$\bar{C}^\mu_n[j_3, j_2, j_1] = \frac{\sqrt{M^2}}{\prod^{3}_{a=1} \Gamma_b(\hat{b}^{-1} + 2\hat{Q}j_a) \Gamma_b(\hat{b}^{-1} + \hat{Q}(2j_a + 1))} \cdot \prod^{3}_{a=1} \Gamma_b(\hat{b}^{-1} + Q(j_{123} + 1)) \Gamma_b(\hat{b}^{-1} + \hat{Q}j_{12}^3) \Gamma_b(\hat{b}^{-1} + \hat{Q}j_{13}^2) \Gamma_b(\hat{b}^{-1} + \hat{Q}j_{23}^1).$$

4 Nonrational $\widehat{su}(2)_\kappa$ cosets

4.1 GKO construction

The GKO construction [15] is based on the observation that any representation of the algebra $\widehat{su}(2)_\kappa \oplus \widehat{su}(2)_1$ is also a representation of the two mutually commuting algebras: $\widehat{su}(2)_{\kappa+1}, \widehat{Vir}_c$. The corresponding generators are given by

$$J^a_n = J^a_n + K^a_n, \quad a = \pm, 3,$$

$$L^\nu_n = \frac{1}{\kappa + 1} L^\nu_n + \frac{1}{\kappa + 3} L^m_n - \frac{1}{\kappa + 3} A_m,$$

where

$$A_m = \sum_{r = -\infty}^{\infty} \left( J^+_m - K^-_r + J^-_m - K^+_r + 2J^3_m - K^3_r \right).$$
and $L^\kappa_m, L^1_m$ are the Virasoro generators related by the Sugawara construction to the algebras $J^a_n$ and $K^a_n$:

$$L^\kappa_m = \frac{1}{\kappa + 2} \sum_{r=-\infty}^{\infty} (:\ J^+_m - rJ^-_m : + : J^+_m - rJ^+_m : + 2J^3_{m-r}J^3_r :),$$

$$L^1_m = \frac{1}{6} \sum_{n=-\infty}^{\infty} (:\ K^+_m - rK^-_m : + : K^+_m - rK^+_m : + 2K^3_{m-r}K^3_r :).$$

The Virasoro generators related to the algebra $J^a_n$ can be expressed as

$$L^{\kappa+1}_m = \frac{\kappa + 2}{\kappa + 3} L^\kappa_m + \frac{3}{\kappa + 3} L^1_m + \frac{1}{\kappa + 3} A_m,$$

and they satisfy the following relations with the other Virasoro generators

$$L^{\kappa+1}_m + L^\nu_m = L^\kappa_m + L^1_m,$$

$$L^{\kappa+1}_m - (\kappa + 2)L^\nu_m = \frac{3 - \kappa^2 - 2\kappa}{\kappa + 3} L^1_m + A_m.\tag{4.3}$$

The first equation is equivalent to the condition for the central charges of the corresponding algebras

$$c^{\kappa+1} + c^\nu = c^{\kappa} + 1,\tag{4.4}$$

where $c^\kappa = \frac{3\kappa}{\kappa + 2}$. In this paper we will consider the case where $c^\nu$ is the central charge of the Liouville theory

$$c^\nu = 1 + 6 \left( b + b^{-1} \right)^2,$$

or the imaginary Liouville theory with purely imaginary parameter $b = -i\hat{b}$

$$c^{\nu} = 1 - 6 \left( \hat{b} - \hat{b}^{-1} \right)^2.$$

Matching condition (4.4) implies the relation between the level $\kappa$ and the Liouville parameter $b$,

$$\kappa_1 = \frac{3b + 2b^{-1}}{b + b^{-1}}, \quad \kappa_2 = \frac{3b^{-1} + 2b}{b + b^{-1}}.$$

We choose $\kappa = \kappa_2$ and assume $b < 1$. Then the levels of the $\hat{su}(2)_\kappa$ and $\hat{su}(2)_{\kappa+1}$ are on the opposite sides of the $\kappa = -2$ barrier:

$$\kappa < -2 < \kappa + 1.$$

We shall parameterize the theories by $b^A$ and $\hat{b}^B$, respectively

$$(b^A)^2 = \frac{1}{\kappa + 2} = 1 + b^2, \quad (\hat{b}^B)^2 = \frac{1}{\kappa + 3} = 1 + b^{-2},$$

$$b^A = \sqrt{bQ}, \quad \hat{b}^B = \sqrt{b^{-1}Q}, \quad Q = b + b^{-1}.\tag{4.5}$$

In the case of the imaginary Liouville theory, for $\hat{b} < 1$,

$$\kappa = -\frac{3\hat{b}^{-1} - 2\hat{b}}{\hat{b} - 1} < -3.$$
The corresponding $\mathfrak{su}(2)$ theories are parameterized by

$$
(b^A)^2 = -\frac{1}{\kappa + 2} = 1 - \hat{b}^2, \quad (b^B)^2 = -\frac{1}{\kappa + 3} = \hat{b}^{-2} - 1,
$$

(4.6)

$$
b^A = \sqrt{\hat{b}\hat{Q}}, \quad b^B = \sqrt{\hat{b}^{-1}\hat{Q}}, \quad \hat{Q} = \hat{b}^{-1} - \hat{b}.
$$

4.2 decomposing representations

In this paper we restrict ourselves to the real spectrum of the $\hat{\mathfrak{su}}(2)_\kappa$ model on the l.h.s of the expected relations

$$
\hat{\mathfrak{su}}(2)_\kappa \otimes \hat{\mathfrak{su}}(2)_1 \sim \text{Liouville} \otimes \mathfrak{p} \hat{\mathfrak{su}}(2)_{\kappa + 1},
$$

$$
\hat{\mathfrak{su}}(2)_\kappa \otimes \hat{\mathfrak{su}}(2)_1 \sim \text{imaginary Liouville} \otimes \mathfrak{p} \hat{\mathfrak{su}}(2)_{\kappa + 1}.
$$

The aim is therefore to decompose the representations

$$
\hat{D}_{\kappa, j, \epsilon} \otimes \hat{S}_{-1}, \quad \hat{D}_{\kappa, j, \epsilon} \otimes \hat{S}_{-\frac{3}{2}}, \quad j = -\frac{1}{2} + is, \ s \in \mathbb{R},
$$

of the algebra $\hat{\mathfrak{su}}(2)_\kappa \oplus \hat{\mathfrak{su}}(2)_1$ into irreducible representations of the algebra $\hat{\mathfrak{su}}(2)_{\kappa + 1} \oplus \mathfrak{vir}_c$.

The general form of these decompositions can be derived by analyzing the characters of the representations involved. For the representations $\hat{D}_{\kappa, j, \epsilon}, \hat{S}_{-1}, \hat{S}_{-\frac{3}{2}}$ and for the Virasoro Verma module with the weight $\Delta$ and the central charge $c$ they are given by

$$
\chi^\kappa_{\kappa, j, \epsilon}(q, y) = (\eta(q))^{-3} q^{-\frac{j}{4} + j(1+j)} \sum_{m \in \mathbb{Z}} q^{m+\epsilon},
$$

$$
\chi_{-1}(q, y) = \frac{1}{\eta(q)} \sum_{n \in \mathbb{Z}} q^n y^n,
$$

$$
\chi_{-\frac{3}{2}}(q, y) = \frac{1}{\eta(q)} \sum_{n \in \mathbb{Z}} q^{(n-\frac{1}{2})^2} y^{n-\frac{1}{2}},
$$

$$
\chi^{\mathfrak{vir}}(\Delta, c)(q) = \frac{1}{\eta(q)} q^{-\frac{\Delta - \frac{1}{2}}{c + 1}},
$$

respectively. One has the following decompositions

$$
\chi^\kappa_{\kappa, j, \epsilon}(q, y) \chi_{-1}(q, y) = \eta(q)^{-4} q^{\frac{j^2 + j(j+1)}{k+2}} \sum_{n \in \mathbb{Z}} q^{n^2} \sum_{p \in \mathbb{Z}} q^{p+\epsilon}
$$

$$
= \sum_{n \in \mathbb{Z}} \chi^{\mathfrak{vir}}_{\Delta_n, c}(q) \chi^{\kappa + 1}_{\kappa, j, \epsilon}(q, y),
$$

$$
\chi^\kappa_{\kappa, j, \epsilon}(q, y) \chi_{-\frac{3}{2}}(q, y) = \eta(q)^{-4} q^{\frac{j^2 + j(j+1)}{k+2}} \sum_{n \in \mathbb{Z} - \frac{1}{2}} q^{n^2} \sum_{p \in \mathbb{Z}} q^{p+\epsilon - \frac{1}{2}}
$$

$$
= \sum_{n \in \mathbb{Z} - \frac{1}{2}} \chi^{\mathfrak{vir}}_{\Delta_n, c}(q) \chi^{\kappa + 1}_{\kappa, j, \epsilon}(q, y),
$$

(4.7)
where the parameters $\Delta_n, j_n$ are subject to the condition

$$\Delta_n + \frac{j_n(1 + j_n)}{\kappa + 3} = \frac{j(1 + j)}{\kappa + 2} + n^2, \quad n \in \mathbb{Z}. \quad (4.8)$$

It follows from (4.3) that $n^2$ and $n^2 - \frac{1}{4}$ are the levels with respect to the operator $L_n^0 + L^1_n$ in $\mathcal{D}^{\kappa}_{j, \epsilon} \otimes \hat{S}_{1/2}$ and $\mathcal{D}^{\kappa}_{j, \epsilon} \otimes \hat{S}_{1/2}$, respectively. We conjecture that the relevant solution to (4.8) is given by

$$j_n = j + n, \quad (4.9)$$

which implies

$$\Delta_n = \Delta_{j+\frac{n}{bQ}} = -Q^2 \left( j + \frac{n}{bQ} \right) \left( 1 + j + \frac{n}{bQ} \right),$$

for the Liouville weights, and

$$\Delta_n = \Delta_{j+\frac{n}{bQ}} = Q^2 \left( j + \frac{n}{bQ} \right) \left( 1 + j + \frac{n}{bQ} \right),$$

for the imaginary Liouville weights.

In the case of Liouville theory decomposition of characters (4.7) and conjecture (4.9) lead to the following decomposition of representations

$$\mathcal{D}^{\kappa}_{-1-j, \epsilon} \otimes \hat{S}_{-1} = \bigoplus_{n \in \mathbb{Z}} \mathcal{D}^{\kappa+1}_{-1-j-n, \epsilon} \otimes \mathcal{V}_{\Delta_n} \otimes \mathcal{E}_{c}, \quad \mathcal{D}^{\kappa}_{-1-j, \epsilon} \otimes \hat{S}_{-1} = \bigoplus_{n \in \mathbb{Z} + \frac{1}{2}} \mathcal{D}^{\kappa+1}_{-1-j-n, \epsilon} \otimes \mathcal{V}_{\Delta_n} \otimes \mathcal{E}_{c},$$

where we used the relation $\Delta_{1-j-n, \epsilon} = \Delta_{1-j-n, \epsilon}$ and $\mathcal{V}_{\Delta_n, c}$ denotes the Virasoro Verma module with the weight $\Delta_n$ and the central charge $c$. The subspaces $\mathcal{D}^{\kappa+1}_{-1-j-n, \epsilon} \otimes \mathcal{V}_{\Delta_n} \otimes \mathcal{E}_{c}$ are generated by the algebras $\mathcal{J}_{k}^{\alpha}$ from the subspaces $\mathcal{D}_{-1-j-n, \epsilon} \otimes \mathcal{V}_{\Delta_n} \otimes \mathcal{E}_{c}$ which can be described in terms of families $\{| m + \epsilon \rangle_{-1-j-n} \}$ of $\mathcal{J}_{0}^3$ eigenstates such that

$$\mathcal{J}_{k}^\alpha | m + \epsilon \rangle_{-1-j-n} = L_{k}^\alpha | m + \epsilon \rangle_{-1-j-n} = 0, \quad k > 0,$$

$$L_{0}^\alpha | m + \epsilon \rangle_{-1-j-n} = \Delta_{-1-j-n} | m + \epsilon \rangle_{-1-j-n},$$

$$\mathcal{J}_{0}^\alpha | m + \epsilon \rangle_{-1-j-n} = (m + \epsilon + j + 1 - n) | m + \epsilon + 1 \rangle_{-1-j-n},$$

$$\mathcal{J}_{0}^\alpha | m + \epsilon \rangle_{-1-j-n} = (m - \epsilon + j + 1 - n) | m - \epsilon + 1 \rangle_{-1-j-n}. \quad (4.10)$$

A family with these properties can be obtained by looking for the states

$$| x \rangle_{j+n, \epsilon}^* = \sum_{m \in \mathbb{Z}} x^{j+n-m-\epsilon} | m + \epsilon \rangle_{-1-j-n}^*,$$

satisfying

$$\mathcal{J}_{0}^\alpha (x) | x \rangle_{j+n, \epsilon}^* = \mathcal{J}_{k}^\alpha (x) | x \rangle_{j+n, \epsilon}^* = L_{k}^\alpha | x \rangle_{j+n, \epsilon}^* = 0, \quad k > 0,$$

$$\mathcal{J}_{0}^\alpha (x) | x \rangle_{j+n, \epsilon}^* = (j + n) | x \rangle_{j+n, \epsilon}^*, \quad L_{0}^\alpha | x \rangle_{j+n, \epsilon}^* = \Delta_{n}^\alpha | x \rangle_{j+n, \epsilon}^*. \quad (4.11)$$

\(^8\text{In the present paper we use this conjecture for } n = 0, \pm \frac{1}{2}, \pm 1. \text{ It is checked in these cases by explicit calculations. The proof of the general case seems to require more advanced techniques. We hope to come back to this point in a separate publication.}\)
For $n = 0$ the state in question is simply given by

$$| x \rangle_{j, \epsilon}^* = | x \rangle_{j, \epsilon} \otimes | x \rangle_0.$$  

For $n = \pm \frac{1}{2}, \pm 1$ these states can be obtained directly from the definition. For higher $n$ the calculations become prohibitively lengthy and we are not aware of any general construction of such states. Let us first consider the state

$$| x \rangle_{j, \epsilon}^* = | x \rangle_{j, \epsilon} \otimes | x \rangle_{\frac{1}{2}}$$

$$= \sum x^{j+\frac{1}{2} - m - \epsilon} \left( | m - \epsilon \rangle_{-1-j} \otimes | \frac{1}{2} \rangle_{-\frac{3}{2}} + | m + 1 - \epsilon \rangle_{-1-j} \otimes | -\frac{1}{2} \rangle_{-\frac{3}{2}} \right).$$

This is a highest weight state with respect to the algebra $\mathcal{J}_k^a(x) = J^a_k(x) + K^a_k(x)$ and the Virasoro algebra $L_k^1$ such that

$$\mathcal{J}_k^a(x) \mid x \rangle_{j, \epsilon}^* = (j + \frac{1}{2}) \mid x \rangle_{j, \epsilon}^*$$

$$L_k^1 \mid x \rangle_{j, \epsilon}^* = \Delta_{\frac{1}{2}} \mid x \rangle_{j, \epsilon}^*.$$

One checks that the family

$$| m + \epsilon \rangle_{-1-j - \frac{1}{2}} = | m - \epsilon \rangle_{-1-j} \otimes | \frac{1}{2} \rangle_{-\frac{3}{2}} + | m + 1 - \epsilon \rangle_{-1-j} \otimes | -\frac{1}{2} \rangle_{-\frac{3}{2}}$$

satisfies conditions (4.10) in agreement with our previous calculations (2.16).

One can also easily verify that the state

$$| x \rangle_{j, \epsilon}^* = [J_0^- (x) - 2j K_0^- (x)] \mid x \rangle_{j, \epsilon}^*$$

$$= J_0^- \mid x \rangle_{j, \epsilon} \otimes \mid x \rangle_{\frac{1}{2}} - 2j \mid x \rangle_{j, \epsilon} \otimes K_0^- \mid x \rangle_{\frac{1}{2}}$$

is a highest weight state satisfying

$$\mathcal{J}_k^a(x) \mid x \rangle_{j, \epsilon}^* = (j - \frac{1}{2}) \mid x \rangle_{j, \epsilon}^*$$

$$L_k^1 \mid x \rangle_{j, \epsilon}^* = \Delta_{-\frac{1}{2}} \mid x \rangle_{j, \epsilon}^*.$$

In the case of $n = \pm 1$ the solutions to conditions (4.11) take the form

$$| x \rangle_{j+1, \epsilon}^* = \left( J^+_{-1}(x) \mid x \rangle_{j, \epsilon} - (\kappa - 2j) K^+_{-1}(x) \right) \mid x \rangle_{j, \epsilon} \otimes \mid x \rangle_0,$$

$$| x \rangle_{j-1, \epsilon}^* = \left( J^+_{-1}(x) (J_0^-)^2 + 2(2j - 1) J^3_{-1}(x) J_0^- - 2j(2j - 1) J_{-1}^- \right) \mid x \rangle_{j, \epsilon} \otimes \mid x \rangle_0$$

$$- (\kappa + 2j + 2) \left( K^+_{-1}(x)(J_0^-)^2 + 2(2j - 1) K^3_{-1}(x) J_0^- - 2j(2j - 1) K^-_{-1} \right) \mid x \rangle_{j, \epsilon} \otimes \mid x \rangle_0.$$  

The chiral fields corresponding to states (4.11) have a general form of descendants in the product theory of $\mathfrak{su}(2)_\epsilon$ and $\mathfrak{su}(2)_1$

$$\Phi_{j+n, \epsilon+n}(x, z) = M(j, \epsilon, n) O_n(J^a(x), K^a(x)) \left( \Phi_{j, \epsilon}^a \otimes \Phi_{1}^a \right)(x, z),$$
where \( c' = n \) and \( M(j, \epsilon, n) \) is a normalization. In particular, for \( n_i = 0, \pm \frac{1}{2}, \pm 1 \) they read

\[
\Phi^*_j, (x, z) = \left( \Phi^A_{j, \epsilon} \otimes \Phi^B_0 \right) (x, z), \quad \Phi^*_{j+\frac{1}{2}, \epsilon, z} (x, z) = M(j, \epsilon, \frac{1}{2}) \left( \Phi^A_{j, \epsilon} \otimes \Phi^\frac{1}{2} \right) (x, z), \tag{4.12}
\]

\[
\Phi^*_{j-\frac{1}{2}, \epsilon, z} (x, z) = M(j, \epsilon, -\frac{1}{2}) \left( J_0 (x) - 2 J K^0 (x) \right) \left( \Phi^A_{j, \epsilon} \otimes \Phi^\frac{1}{2} \right) (x, z), \quad \Phi^*_{j+1, \epsilon, z} (x, z) = M(j, \epsilon, 1) \left( J_+^0 (x) - (\kappa - 2j) K^+ (x) \right) \left( \Phi^A_{j, \epsilon} \otimes \Phi^0 \right) (x, z), \quad \Phi^*_{j-1, \epsilon, z} (x, z) = M(j, \epsilon, -1) \left( - J_+^0 (x) J_0 (x) - 2(2j - 1) J_+^0 (x) J_0 - 2j(2j - 1) J_+^0 \right) \left( \Phi^A_{j, \epsilon} \otimes \Phi^0 \right) (x, z).
\]

The normalization in the \( su(2)_1 \) theory is chosen such that

\[
\langle \Phi^1_0 (x_3, z_3) \Phi^1_0 (x_2, z_2) \Phi^1_0 (x_1, z_1) \rangle = 1,
\]

\[
\langle \Phi^1_0 (x_3, z_3) \Phi^1 \frac{1}{2} (x_2, z_2) \Phi^1 \frac{1}{2} (x_1, z_1) \rangle = (x_1 - x_2) (z_2 - z_1)^{-\frac{1}{2}},
\]

\[
\langle \Phi^1 \frac{1}{2} (x_3, z_3) \Phi^1_0 (x_2, z_2) \Phi^1 \frac{1}{2} (x_1, z_1) \rangle = (x_3 - x_1) (z_3 - z_1)^{-\frac{1}{2}},
\]

\[
\langle \Phi^1 \frac{1}{2} (x_3, z_3) \Phi^1 \frac{1}{2} (x_2, z_2) \Phi^1_0 (x_1, z_1) \rangle = (x_2 - x_3) (z_3 - z_2)^{-\frac{1}{2}}. \tag{4.13}
\]

### 4.3 checks for \( n = 0 \)

We conjecture the relation between chiral correlators in the product theory of \( su(2)_\kappa \) and \( su(2)_1 \) on one side, and chiral correlators of \( su(2)_{\kappa+1} \) and Liouville theory on the other side, (1.6):

\[
\langle \Phi^*_{j_3+n_3, \epsilon_3+n_3} (x_3, z_3) \Phi^*_{j_2+n_2, \epsilon_2+n_2} (x_2, z_2) \Phi^*_{j_1+n_1, \epsilon_1+n_1} (x_1, z_1) \rangle = \langle \Phi^A_{j_3+n_3} (z_3) \Phi^B_{j_2+n_2} (z_2) \Phi^A_{j_1+n_1} (z_1) \rangle \times \langle \Phi^B_{j_3+n_3, \epsilon_3+n_3} (x_3, z_3) \Phi^B_{j_2+n_2, \epsilon_2+n_2} (x_2, z_2) \Phi^B_{j_1+n_1, \epsilon_1+n_1} (x_1, z_1) \rangle \overline{S}^L \tag{4.14}
\]

In the case of all \( n_i = 0 \) the conjecture takes its simplest form

\[
\langle \Phi^*_{j_3, \epsilon_3} (x_3, z_3) \Phi^*_{j_2, \epsilon_2} (x_2, z_2) \Phi^*_{j_1, \epsilon_1} (x_1, z_1) \rangle = \langle \Phi^A_{j_3} (z_3) \Phi^B_{j_2} (z_2) \Phi^A_{j_1} (z_1) \rangle \overline{S}^L = \langle \Phi^B_{j_3, \epsilon_3} (x_3, z_3) \Phi^B_{j_2, \epsilon_2} (x_2, z_2) \Phi^B_{j_1, \epsilon_1} (x_1, z_1) \rangle \overline{S}^L. \tag{4.15}
\]

On both sides of this relation:

\[
\langle \Phi^*_{j_3, \epsilon_3} (x_3, z_3) \Phi^*_{j_2, \epsilon_2} (x_2, z_2) \Phi^*_{j_1, \epsilon_1} (x_1, z_1) \rangle = \langle \Phi^A_{j_3} (x_3, z_3) \Phi^B_{j_2} (x_2, z_2) \Phi^A_{j_1} (x_1, z_1) \rangle \overline{S}^L = c [\Delta^A(j_i) ; z_i] \overline{S}^L \left[ j_3 \atop j_2 \atop j_1 \atop x_3 \atop x_2 \atop x_1 \right] \overline{C}^S_{j_3, j_2, j_1},
\]

and

\[
\langle \Phi^A_{j_3} (z_3) \Phi^B_{j_2} (z_2) \Phi^A_{j_1} (z_1) \rangle \overline{S}^L = \langle \Phi^B_{j_3, \epsilon_3} (x_3, z_3) \Phi^B_{j_2, \epsilon_2} (x_2, z_2) \Phi^B_{j_1, \epsilon_1} (x_1, z_1) \rangle \overline{S}^L = c [\Delta^L(j_i) + \Delta^B(j_i) ; z_i] \overline{S}^L \left[ j_3 \atop j_2 \atop j_1 \atop x_3 \atop x_2 \atop x_1 \right] \overline{C}^S_{j_3, j_2, j_1} \overline{C}^S_{j_3, j_2, j_1} \overline{C}^S_{j_3, j_2, j_1}.
\]
we have the same 3-linear $\mathfrak{su}(2)$ invariant $\tilde{S}_i$. The $z_i$-dependent terms agree due to \((4.5)\) and equation \((4.15)\) reduces to the relation between $j$-dependent parts

$$C_{\lambda B}^s[j_3, j_2, j_1] = C_{\lambda B}^{11}[j_3, j_2, j_1] C_{\lambda A}^{13}[j_3, j_2, j_1]. \quad (4.16)$$

Using the identity for Barnes gamma functions

$$\frac{\Gamma_{\lambda A}(-b^A j) \Gamma_{\lambda A} \left( \frac{1}{\Gamma_{b}(-Q j)} \right)}{\Gamma_{\lambda A}(b^A) \Gamma_{\lambda A} \left( \frac{1}{\Gamma_{b}(-Q j)} \right)} = \frac{\Gamma_{\lambda A}(b^A) \Gamma_{\lambda A} \left( \frac{1}{\Gamma_{b}(-Q j)} \right)}{\Gamma_{\lambda A} \left( \frac{1}{\Gamma_{b}(-Q j)} \right)} \left( b^{-1} Q \right)^{\frac{Q_j(j+1)}{4}} b^{-\frac{Q_j}{2} Q_j(j+1)-\frac{j}{2}+\frac{1}{2}}, \quad (4.17)$$

one can show that \((4.16)\) indeed holds if the relative normalization is given by

$$M_{\lambda A}^S M_{\lambda B}^L = \left( \frac{\Gamma_{\lambda A}(b^A) \Gamma_{\lambda A} \left( \frac{1}{\Gamma_{b}(-Q j)} \right)}{\Gamma_{\lambda A} \left( \frac{1}{\Gamma_{b}(-Q j)} \right)} \right)^2 \left( b^{-1} Q \right)^{-\frac{Q_j^2}{2} b^{Q_j(j+1)+\frac{j}{2}+\frac{1}{2}}}. \tag{4.18}$$

Under the same condition equations \((4.1)\) and \((3.17)\) imply the relation for the right structure constants

$$\bar{C}_{\lambda A}^s[j_3, j_2, j_1] = \bar{C}_{\lambda B}^{11}[j_3, j_2, j_1] \bar{C}_{\lambda A}^{13}[j_3, j_2, j_1]. \tag{4.19}$$

In the case of imaginary Liouville theory the counterpart of relation \((4.15)\) reads

$$\langle \Phi_{j_3, \epsilon_3}(x_3, z_3) \Phi_{j_2, \epsilon_2}(x_2, z_2) \Phi_{j_1, \epsilon_1}(x_1, z_1) \rangle = \langle \Phi_{j_3, \epsilon_3}(x_3, z_3) \Phi_{j_1, \epsilon_1}(x_1, z_1) \rangle_{\lambda B} \langle \Phi_{j_2, \epsilon_2}(x_2, z_2) \Phi_{j_1, \epsilon_1}(x_1, z_1) \rangle_{\lambda A}^{13}. \tag{4.20}$$

For parameters of the theories as in \((4.6)\), the l.h.s is the same as in \((4.15)\), while on the r.h.s one has correlators from the imaginary Liouville theory and the $\mathfrak{su}(2)_{\kappa+1}$ WZNW model with $\kappa < -3$. By the same argument as in the Liouville case, relation \((4.18)\) reduces to the relation between $j$-dependent parts of the structure constants

$$C_{\lambda A}^s[j_3, j_2, j_1] = C_{\lambda B}^{11}[j_3, j_2, j_1] C_{\lambda A}^{13}[j_3, j_2, j_1]. \tag{4.19}$$

Using another identity

$$\frac{\Gamma_{b^B}(-b^B j)}{\Gamma_{b^A}(-b^A j) \Gamma_{b^B}(b^B - Q j)} = \frac{\Gamma_{b^B}(b^B) \Gamma_{b^B}(b^B - Q j)}{\Gamma_{b^A}(b^A) \Gamma_{b^B}(b^B - Q j)} \left( b^{-1} Q \right)^{\frac{Q_j(j+1)}{4}} b^{-\frac{Q_j}{2} Q_j(j+1)+\frac{j}{2}+\frac{1}{2}}, \quad (4.20)$$

one can show that equality \((4.19)\) and its right counterpart hold if the normalizations of fields are related by

$$M_{\lambda A}^s = \left( \frac{\Gamma_{b^B}(b^B) \Gamma_{b^B}(b^B - Q j)}{\Gamma_{b^A}(b^A) \Gamma_{b^B}(b^B - Q j)} \right)^2 \left( b^{-1} Q \right)^{-\frac{Q_j^2}{2} b^{-2} + \frac{3}{2}. \tag{4.20}$$

We assume in the following that the relative normalizations are fixed by relations \((4.16)\) and \((4.19)\). One can therefore safely drop the normalization constants $M_{\lambda A}^s, M_{\lambda B}^L, M_{\lambda B}^S, M_{\lambda B}^{11}$ in subsequent formulæ. Identities \((4.17)\), \((4.20)\) are derived in Appendix B.
4.4 reformulation of the general case

In the general case the r.h.s. of (4.14) can be calculated explicitly using shift relations (3.14)

\[
\left(\Phi_{j_1+\frac{n_1}{\hat{b}Q}}(z_1)\Phi_{j_2+\frac{n_2}{\hat{b}Q}}(z_2)\Phi_{j_1+\frac{n_1}{\hat{b}Q}}(z_1)\right)_L^I
\times \left(\Phi_{j_3+j_3+\frac{n_3}{\hat{b}Q}}(x_3, z_3)\Phi_{j_2+j_2+\frac{n_2}{\hat{b}Q}}(x_2, z_2)\Phi_{j_3+j_3+\frac{n_3}{\hat{b}Q}}(x_1, z_1)\right)^{IS}_{\epsilon+\frac{1}{2}n_{123}}
\]

\[
= c \left[ \Delta^L(j_i + \frac{n_i}{\hat{b}Q}) + \Delta^R(j_i + n_i - \epsilon) \right] \tilde{S}^\epsilon_{\epsilon+\frac{1}{2}n_{123}} \left[ \prod_{x_1}^b \left[ \frac{j_3+n_3}{x_3} \frac{j_2+n_2}{x_2} \frac{j_1+n_1}{x_1} \right] \right]
\times C^I_{j_1} \left[ j_3 + n_3, j_2 + n_2, j_1 + n_1 \right] C^L_{j_2} \left[ j_3, j_2, j_1 \right]
\]

where

\[
l(x, n) = \begin{cases} 
\prod_{p=2}^{n} \prod_{q=1}^{p-1} (x - p(\kappa + 2) + q(\kappa + 3)) & , n > 1, \\
1 & , n = 0, 1, \\
\prod_{p=0}^{n-1} \prod_{q=0}^{p} (x + p(\kappa + 2) - q(\kappa + 3)) & , n < 0,
\end{cases}
\]

\[
N(j, n) = (-1)^{n(2n-1)} (l(2j, 2n) l(2j + 1, 2n))^{-\frac{1}{2}}. 
\]

Applying (3.21) and (4.16) one can rewrite the r.h.s. of (4.14) in the following form

\[
\left(\Phi_{j_3+\frac{n_3}{\hat{b}Q}}(z_3)\Phi_{j_2+\frac{n_2}{\hat{b}Q}}(z_2)\Phi_{j_1+\frac{n_1}{\hat{b}Q}}(z_1)\right)_L^I
\times \left(\Phi_{j_3+j_3+\frac{n_3}{\hat{b}Q}}(x_3, z_3)\Phi_{j_2+j_2+\frac{n_2}{\hat{b}Q}}(x_2, z_2)\Phi_{j_3+j_3+\frac{n_3}{\hat{b}Q}}(x_1, z_1)\right)^{IS}_{\epsilon+\frac{1}{2}n_{123}}
\]

\[
= (-1)^{n(n_3, n_2, n_1)} \left( \prod_{i=1}^{3} N(j_i, n_i) \right)
\times l(j_{123} + 1, n_{123}) l(j_{12}^3, n_{12}^3) l(j_{13}, n_{13}^1) l(j_{23}, n_{23}^1)
\times (z_2 - z_1)^n_{n_3^2-n_3^2-n_2^2-n_2^2} (z_3 - z_1)^n_{n_2^2-n_2^2-n_3^2-n_3^2} (z_3 - 2z_2)^n_{n_2^2-n_2^2-n_3^2-n_3^2}
\times (x_1 - x_2)^n_{n_1+n_2-n_3} (x_3 - x_1)^n_{n_1+n_3-n_2} (x_2 - x_3)^n_{n_2+n_3-n_1}
\times \left(\Phi_{j_3, e_3}^\Lambda(x_3, z_3)\Phi_{j_2, e_2}^\Lambda(x_2, z_2)\Phi_{j_1, e_1}^\Lambda(x_1, z_1)\right)^{IS}_{\epsilon}.
\]
As it was mentioned in the Introduction we call the object on the l.h.s of (4.14) the coset factor.\footnote{Its counterpart in the SL-LL correspondence is called the blow up factor for reason coming from the 4-dim side of the AGT relation.}

In the case of the imaginary Liouville theory the counterpart of conjecture (4.14) takes the form

\begin{equation}
\langle \Phi_{j_1+n_1,\epsilon^1+n_1}(x_3, z_3) \Phi_{j_2+n_2,\epsilon^2+n_2}(x_2, z_2) \Phi_{j_1+n_1,\epsilon^1+n_1}(x_1, z_1) \rangle = \langle \Phi_{j_1+n_1,\epsilon^1+n_1}(x_3, z_3) \Phi_{j_2+n_2,\epsilon^2+n_2}(x_2, z_2) \Phi_{j_1+n_1,\epsilon^1+n_1}(x_1, z_1) \rangle^{\eta(n_3, n_2, n_1)} \prod_{i=1}^{3} N(j_i, n_i) + \text{other terms} \tag{4.24}
\end{equation}

One can follow the same steps as in the Liouville case using (4.19) instead of (4.16). The result is exactly the same as in (4.23). The only difference is that in the Liouville case \(\kappa\) is in the range \(-3 < \kappa < -2\) while for the imaginary Liouville \(\kappa < -3\). In both cases the \(\mathfrak{su}(2)_k\) WZNW model is on the real side of the \(\kappa = -2\) barrier so the analytic form of the coset factor should be the same in agreement with conjectured relation (4.23). Checking (4.23) verifies therefore both relations (1.3).

4.5 checks for \(n = \pm \frac{1}{2}\)

We choose locations of fields \(z_3 = \infty, z_2 = z, z_1 = 0\) for which the Ward identities take their simple form (3.8). Due to the general condition \(\epsilon_1 + \epsilon_2 + \epsilon_3 \equiv 0\) there are three subcases of the 3-point function containing two fields with \(j \pm \frac{1}{2}\) and one primary field. The simplest case reads

\begin{equation}
\langle \Phi_{j_1+n_1,\epsilon^1+n_1}(x_3, \infty) \Phi_{j_2+n_2,\epsilon^2+n_2}(x_2, z) \Phi_{j_1+n_1,\epsilon^1+n_1}(x_1, 0) \rangle = M(j_1, \epsilon_1, \frac{1}{2}) M(j_2, \epsilon_2, \frac{1}{2}) \langle \Phi_{j_1+n_1,\epsilon^1+n_1}(x_3, \infty) \Phi_{j_1+n_1,\epsilon^1+n_1}(x_1, 0) \rangle^{\epsilon_1 \epsilon_2} \langle \Phi_{j_2+n_2,\epsilon^2+n_2}(x_2, z) \Phi_{j_1+n_1,\epsilon^1+n_1}(x_1, 0) \rangle^{\epsilon_2} \tag{4.25}
\end{equation}

The proof of (4.14) reduces therefore to showing that one can chose normalizations of fields such that the following relation holds

\begin{equation}
\frac{\langle \Phi_{j_1+n_1,\epsilon^1+n_1}(x_3, z_3) \Phi_{j_2+n_2,\epsilon^2+n_2}(x_2, z_2) \Phi_{j_1+n_1,\epsilon^1+n_1}(x_1, z_1) \rangle}{\langle \Phi_{j_1+n_1,\epsilon^1+n_1}(x_3, z_3) \Phi_{j_2+n_2,\epsilon^2+n_2}(x_2, z_2) \Phi_{j_1+n_1,\epsilon^1+n_1}(x_1, z_1) \rangle^\epsilon} = (-1)^{\eta(n_3, n_2, n_1)} \prod_{i=1}^{3} N(j_i, n_i) \times l(j_{123} + 1, n_{123}) \times (n_{3} - z_1)^{n_{2} - n_{1}^2 - n_{3}^2} (z_3 - z_1)^{n_{2}^2 - n_{1}^2 - n_{3}^2} (z_3 - z_2)^{n_{1}^2 - n_{2}^2 - n_{3}^2} \times (x_1 - x_2)^{n_{1} + n_{2} + n_{3}} (x_3 - x_1)^{n_{1} + n_{2} - n_{3}} (x_2 - x_3)^{n_{2} + n_{3} - n_{1}}. \tag{4.23}
\end{equation}

As it was mentioned in the Introduction we call the object on the l.h.s of (4.23) the coset factor.\footnote{Its counterpart in the SL-LL correspondence is called the blow up factor for reason coming from the 4-dim side of the AGT relation.}
In the calculation of the next one
\[
\langle \Phi_{j_3,\epsilon_3}^\ast (x_3, \infty) \Phi_{j_2,\epsilon_2}^\ast \big( x_2, z \big) \Phi_{j_1,\epsilon_1}^\ast \big( x_1, 0 \big) \rangle_\epsilon
\]
\[
= M(j_2, \epsilon_2, -\frac{1}{2}) M(j_1, \epsilon_1, \frac{1}{2}) \left[ \langle \Phi_{j_3,\epsilon_3}^\ast (x_3, \infty) J_0^\ast \Phi_{j_2,\epsilon_2}^\ast (x_2, z) \Phi_{j_1,\epsilon_1}^\ast (x_1, 0) \rangle_\epsilon \langle \Phi_{j_2,\epsilon_2}^\ast (x_2, z) \Phi_{j_1,\epsilon_1}^\ast (x_1, 0) \rangle_\epsilon \right]
\]
\[
- 2j_2 \langle \Phi_{j_3,\epsilon_3}^\ast (x_3, \infty) \Phi_{j_2,\epsilon_2}^\ast (x_2, z) \Phi_{j_1,\epsilon_1}^\ast (x_1, 0) \rangle_\epsilon \langle K_0^- \Phi_{j_2,\epsilon_2}^\ast (x_2, z) \Phi_{j_1,\epsilon_1}^\ast (x_1, 0) \rangle_\epsilon
\]
\[
= -M(j_2, \epsilon_2, -\frac{1}{2}) M(j_1, \epsilon_1, \frac{1}{2}) (j_2 + j_3 - j_1) \left( \frac{x_3 - x_1}{x_2 - x_3} \right) - \frac{1}{2} \langle \Phi_{j_3,\epsilon_3}^\ast (x_3, \infty) \Phi_{j_2,\epsilon_2}^\ast (x_2, z) \Phi_{j_1,\epsilon_1}^\ast (x_1, 0) \rangle_\epsilon
\]

one uses the action of the zero modes (3.7) and the identity
\[
\partial_{x_2} \tilde{S}_\epsilon \left[ \frac{j_2 + j_3 - j_1}{x_2 - x_3} \right. - \frac{j_1 + j_2 - j_3}{x_1 - x_2} \left. \right] \tilde{S}_\epsilon \left[ \frac{j_2 + j_3 - j_1}{x_3 - x_2} \right. \frac{j_1 + j_2 - j_3}{x_3 - x_1} \left. \right]
\]
The third case is slightly more complicated:
\[
\langle \Phi_{j_3,\epsilon_3}^\ast (x_3, \infty) \Phi_{j_2,\epsilon_2}^\ast \big( x_2, z \big) \Phi_{j_1,\epsilon_1}^\ast \big( x_1, 0 \big) \rangle_\epsilon = M(j_2, \epsilon_2, -\frac{1}{2}) M(j_1, \epsilon_1, \frac{1}{2}) \times \left[ \langle \Phi_{j_3,\epsilon_3}^\ast (x_3, \infty) J_0^\ast \Phi_{j_2,\epsilon_2}^\ast (x_2, z) \Phi_{j_1,\epsilon_1}^\ast (x_1, 0) \rangle_\epsilon \langle \Phi_{j_2,\epsilon_2}^\ast (x_2, z) \Phi_{j_1,\epsilon_1}^\ast (x_1, 0) \rangle_\epsilon \right]
\]
\[
- 2j_2 \langle \Phi_{j_3,\epsilon_3}^\ast (x_3, \infty) \Phi_{j_2,\epsilon_2}^\ast (x_2, z) \Phi_{j_1,\epsilon_1}^\ast (x_1, 0) \rangle_\epsilon \langle K_0^- \Phi_{j_2,\epsilon_2}^\ast (x_2, z) \Phi_{j_1,\epsilon_1}^\ast (x_1, 0) \rangle_\epsilon
\]
\[
- 2j_1 \langle \Phi_{j_3,\epsilon_3}^\ast (x_3, \infty) J_0^- \Phi_{j_2,\epsilon_2}^\ast (x_2, z) \Phi_{j_1,\epsilon_1}^\ast (x_1, 0) \rangle_\epsilon \langle \Phi_{j_2,\epsilon_2}^\ast (x_2, z) K_0^- \Phi_{j_1,\epsilon_1}^\ast (x_1, 0) \rangle_\epsilon
\]
\[
= M(j_2, \epsilon_2, -\frac{1}{2}) M(j_1, \epsilon_1, \frac{1}{2}) \left( \frac{j_1 + j_2 + j_3 + 1}{x_1 - x_2} - \frac{j_1 + j_2 - j_3}{x_1 - x_2} \right) \langle \Phi_{j_3,\epsilon_3}^\ast (x_3, \infty) \Phi_{j_2,\epsilon_2}^\ast (x_2, z) \Phi_{j_1,\epsilon_1}^\ast (x_1, 0) \rangle_\epsilon
\]
Choosing the fields normalization equal to the factor (4.21),
\[
M(j, \epsilon, \pm \frac{1}{2}) = N(j, \pm \frac{1}{2})
\]
one gets the agreement with the conjecture (4.23) in all the cases.

4.6 checks for \( n = \pm 1 \)

Let us first consider correlators containing one field \( \Phi_{j,\pm 1,\epsilon}^\ast \). For \( n = 1 \), using Ward identities (3.8), one obtains
\[
\langle \Phi_{j_3,\epsilon_3}^\ast (x_3, \infty) \Phi_{j_2,\epsilon_2}^\ast (x_2, z) \Phi_{j_1,\epsilon_1}^\ast (x_1, 0) \rangle_\epsilon
\]
\[
= M(j_1, \epsilon_1, 1) \langle \Phi_{j_3,\epsilon_3}^\ast (x_3, \infty) \Phi_{j_2,\epsilon_2}^\ast (x_2, z) J_{+1}^\ast (x_1) \Phi_{j_1,\epsilon_1}^\ast (x_1, 0) \rangle_\epsilon
\]
\[
= -\frac{1}{z} M(j_1, \epsilon_1, 1) \langle \Phi_{j_3,\epsilon_3}^\ast (x_3, \infty) J_{-1}^\ast (x_2) \Phi_{j_2,\epsilon_2}^\ast (x_2, z) \Phi_{j_1,\epsilon_1}^\ast (x_1, 0) \rangle_\epsilon
\]
\[
= -M(j_1, \epsilon_1, 1) (j_2 + j_3 - j_1) \left( \frac{x_3 - x_1}{x_2 - x_3} \right) - \frac{1}{z} \langle \Phi_{j_3,\epsilon_3}^\ast (x_3, \infty) \Phi_{j_2,\epsilon_2}^\ast (x_2, z) \Phi_{j_1,\epsilon_1}^\ast (x_1, 0) \rangle_\epsilon,
\]
where the last equation is based on the identity
\[
((x_1 - x_2)^2 \partial_{x_2} + 2j_2 (x_1 - x_2)) \tilde{S}_1^{[j_3/j_2/j_1]}_{[x_3/x_2/x_1]} = (j_2 + j_3 - j_1) \frac{(x_1 - x_2)(x_1 - x_3)}{(x_2 - x_3)} \tilde{S}_1^{[j_3/j_2/j_1]}_{[x_3/x_2/x_1]},
\]
Calculations in the case of \( n = -1 \) are more tedious
\[
\langle \Phi^*_{j_3, \epsilon_3} (x_3, \infty) \Phi^*_{j_2, \epsilon_2} (x_2, z) \Phi^*_{j_1 - 1, \epsilon_1} (x_1, 0) \rangle_{\epsilon} = M(j, \epsilon, -1) \left[ - (\Phi^A_{j_3, \epsilon_3} (x_3, \infty) \Phi^A_{j_2, \epsilon_2} (x_2, z) J^+_{-1}(J^-_0) \Phi^A_{j_1, \epsilon_1} (x_1, 0))_{\epsilon}^S 
\right. \\
-2(2j_1 - 1) (\Phi^A_{j_3, \epsilon_3} (x_3, \infty) \Phi^A_{j_2, \epsilon_2} (x_2, z) \partial_{x_1} \Phi^A_{j_1, \epsilon_1} (x_1, 0))_{\epsilon}^S \\
\left. +2(2j_1 - 1) (\Phi^A_{j_3, \epsilon_3} (x_3, \infty) J^-_0 (x_1) \Phi^A_{j_2, \epsilon_2} (x_2, z) \partial_{x_1} \Phi^A_{j_1, \epsilon_1} (x_1, 0))_{\epsilon}^S 
-2j_1 (2j_1 - 1) (\Phi^A_{j_3, \epsilon_3} (x_3, \infty) J^-_0 (x_1) \Phi^A_{j_2, \epsilon_2} (x_2, z) \Phi^A_{j_1, \epsilon_1} (x_1, 0))_{\epsilon}^S \right]
= -M(j_1, \epsilon_1, -1) (j_1 + j_2 - j_3)(j_1 - j_2 + j_3)(1 + j_1 + j_2 + j_3) \frac{(x_2 - x_3)}{(x_1 - x_2)(x_3 - x_1)} z^{-1} \\
\times \langle \Phi^A_{j_3, \epsilon_3} (x_3, \infty) \Phi^A_{j_2, \epsilon_2} (x_2, z) \Phi^A_{j_1, \epsilon_1} (x_1, 0) \rangle_{\epsilon}^S.
\]
The relation relevant at the last step reads
\[
((x_1 - x_2)^2 \partial_{x_2} + 2j_2 (x_1 - x_2)) \partial_{x_1}^2 - 2(2j_1 - 1) ((x_1 - x_2) \partial_{x_2} + j_2) \partial_{x_1} - j_1 \partial_{x_2}) \tilde{S}_1^{[j_3/j_2/j_1]}_{[x_3/x_2/x_1]} = - (j_1 + j_2 - j_3)(j_1 - j_2 + j_3)(1 + j_1 + j_2 + j_3) \frac{(x_2 - x_3)}{(x_1 - x_2)(x_3 - x_1)} \tilde{S}_1^{[j_3/j_2/j_1]}_{[x_3/x_2/x_1]},
\]
The calculations above are in perfect agreement with conjecture (4.23) if we assume:
\[
M(j, \epsilon, \pm 1) = N(j, \pm 1).
\]
Finally, one can consider the correlators of the form
\[
\langle \Phi^*_{j_3 \pm 1/2, \epsilon_3} (x_3, \infty) \Phi^*_{j_2, \epsilon_2} (x_2, z) \Phi^*_{j_1 \pm 1/2, \epsilon_1} (x_1, 0) \rangle,
\]
when the Ward identities for \( \widehat{su}(2) \) currents yield a nontrivial contribution and the already established normalizations are assumed. We checked all examples of this type. The calculations are similar to these already presented, but more lengthy. With the sign factor given by the tables in Appendix C they agree with conjecture (4.23) in each case.

5 Conclusions

In this paper we have formulated exact relations between \( \widehat{su}(2) \) WZNW model with non-rational level on the one hand side and the Liouville or the imaginary Liouville field theories on the other.
These relations can be seen as continuous spectra counterparts of the GKO construction of the minimal models. We have found strong evidences that they are correct. There are, however, still many questions left open.

First of all one would like to prove decomposition (1.5) and find an explicit construction of the excited states generating representations on the r.h.s. of (1.5). In an analogous problem in the SL-LL correspondence the free field realization of the Virasoro and super Virasoro Verma modules proved to be helpful. Whether the Wakimoto realization might be relevant in the present context is still not clear to us. Another approach is to explore the relation between these excited states and the null states in the relaxed (spin basis) and the highest weight (izospin variables) modules and its behavior under the reflection map. The next step would be to calculate the general form of the coset factor (4.23). This point was the most difficult part of the proof of the SL-LL relation and one may expect similar difficulties in the present case. The last step - the proof of equivalence of the 3-point functions for descendants of the fields $\Phi^*_{j+n,\epsilon}$ with respect to the algebra $\hat{\mathfrak{su}}(2)_{\kappa+1} \otimes \mathfrak{Vir}$ is then possible by slight generalizations of the arguments used in the SL-LL case.

Extending the equivalence to n-point correlation functions ($n > 3$) on the sphere requires a better understanding of the $\hat{\mathfrak{su}}(2)_{\kappa}$ model introduced in Section 3. The main conjecture of the present paper seems to be a good motivation for further investigations of this model even though it is based on non-unitary representations. There are some obvious steps in this direction: the analysis of the 4-point invariants and the structure of the 4-point conformal blocks, the derivation of the $j$-dependent part of the structure constants by Teschner’s method and the factorization properties of the 4-point functions. In a slightly more general context an analytic expression for the fusion matrix would be an essential step in a deeper understanding of the scheme of fig.2.

Another group of questions concerns relations of the model of Section 3 to other WZNW models with continuous spectra like the $H^+_3$ coset model based on different class of representations. It would be also interesting to explore the relation to the $\hat{\mathfrak{sl}}(2,\mathbb{R})$ model based on the principal unitary series of $\mathfrak{sl}(2,\mathbb{R})$ representations which differs from the present one only by the invariant hermitian form.

There are interesting problems related to the GSO construction itself. Motivated by some aspects of various generalizations of the AGT relation [4, 7] and the coset constructions of rational CFT one can look for non-rational counterparts of (1.2) with integer $p$ bigger then 1. This leads to the following conjectures

\begin{align*}
\hat{\mathfrak{su}}(2)_{\kappa} \otimes \hat{\mathfrak{su}}(2)_2 & \sim N = 1 \text{ super-Liouville} \otimes_p \hat{\mathfrak{su}}(2)_{\kappa+2}, \\
\hat{\mathfrak{su}}(2)_{\kappa} \otimes \hat{\mathfrak{su}}(2)_p & \sim \text{ para-Liouville} \otimes_p \hat{\mathfrak{su}}(2)_{\kappa+p}, \quad p > 2.
\end{align*}

As the structure constants of the $\mathfrak{N}=1$ super-Liouville [65, 66] and the para-Liouville theories [67] are already known this is a perfect ground to test the coset construction.

Even more interesting would be a generalization to the symmetry algebras $\hat{\mathfrak{su}}(N), N > 2$. In this
case one should expect the relations involving Toda [68, 69] and the para-Toda field theories

\[
\hat{su}(N)_k \otimes \hat{su}(N)_1 \sim \text{Toda} \otimes_P \hat{su}(N)_{k+1}, \\
\hat{su}(N)_k \otimes \hat{su}(N)_p \sim \text{para-Toda} \otimes_P \hat{su}(N)_{k+p}, \quad p > 1.
\]

A challenging problem is to analyze whether the recently proposed Toda field theory structure constants [70, 71] fit the general scheme of Fig.1 and Fig.2. This however requires the \( \hat{su}(N) \) WZNW structure constants which to our knowledge are not yet known.

Let us observe that for \( n_i = 0 \) equation (1.6) relates the \( \hat{su}(2) \) WZNW and the Liouville structure constants. If the conjecture holds for higher point functions as well it would allow to calculate \( n \)-point functions of primary fields of the Liouville theory in terms of correlators of the \( \hat{su}(2) \) WZNW model:

\[
\langle \Phi_{j_k}(z_k) \cdots \Phi_{j_2}(z_2)\Phi_{j_1}(z_1) \rangle^L = \frac{\langle \Phi_{j_k}^{\Lambda}(x_k, z_k) \cdots \Phi_{j_2}^{\Lambda}(x_2, z_2)\Phi_{j_1}^{\Lambda}(x_1, z_1) \rangle^S}{\langle \Phi_{j_k}^{B}(x_k, z_k) \cdots \Phi_{j_2}^{B}(x_2, z_2)\Phi_{j_1}^{B}(x_1, z_1) \rangle^I},
\]

where all the variables of the right sector are suppressed. In this particular case the symmetries on both sides allow for independent calculations. There are however situations when the symmetry of the coset is not strong enough to fix the structure constants of the theory (e.g. the Toda field theories) or it is not known at all (e.g. most of the para-Liouville field theories). Solving the \( \hat{su}(N) \) WZNW model would automatically provide solutions to its various cosets.

The coset construction for non-rational CFT models seems to be a powerful tool for analyzing basic common structures of large classes of models. This provides a strong motivation to investigate the \( \hat{su}(2) \) WZNW models with non-rational levels and their generalizations for other groups.

A reflection relation

Using the property of the \( G \)-function defined in (2.11)

\[
G \left[ \begin{array}{ccc} a & b & c \\ e & f & c \end{array} \right] = \frac{\Gamma(b)\Gamma(c)\Gamma(e+f-a-b-c)}{\Gamma(a)\Gamma(b)\Gamma(e-c)} G \left[ \begin{array}{ccc} a & e-b & e-c \\ e & f-b- \end{array} \right] .
\]

one can find the transformations of functions \( g^{31}, g^{13} \) under the reflection \( j_3 \rightarrow -j_3 - 1 \)

\[
G \left[ \begin{array}{ccc} -j_1 - m_1 & 1+j_3 & 1+j_3 + m_3 \\ 1+j_2 - j_1 + m_3 & 2+j_2 + j_3 - m_1 \end{array} \right] = \frac{\Gamma(1+j_2)}{\Gamma(2+j_{13})} G \left[ \begin{array}{ccc} -j_1 - m_1 & -j_3 + m_3 & -j_{13} \\ 1+j_2 - j_1 + m_3 & 1-j_3 + j_2 - m_1 & -j_{13} \end{array} \right],
\]

\[
G \left[ \begin{array}{ccc} -j_1 + m_1 & 1+j_3 & 1+j_3 - m_3 \\ 1+j_2 - j_1 - m_3 & 2+j_2 + j_3 + m_1 \end{array} \right] = \frac{\Gamma(1+j_{13})}{\Gamma(2+j_{13})} G \left[ \begin{array}{ccc} -j_1 + m_1 & -j_3 - m_3 & -j_{13} \\ 1+j_2 - j_1 - m_3 & 1-j_3 - j_2 + m_1 & -j_{13} \end{array} \right].
\]

This leads to the relation

\[
g'(j_3 -1, j_2, j_1; m_i) = \left( \frac{\Gamma(1+j_2)}{\Gamma(-j_{13})} \frac{\Gamma(1+j_{13})}{\Gamma(2+j_{13})} \right)^{s(\frac{1}{2} - \frac{1}{2}j_{13} - \epsilon + \epsilon_2)} s(j_1 - \epsilon_1) s(-j_2 - 1 + \epsilon_2) \\
\times \left( \frac{\Gamma(1+j_3 - m_3)}{\Gamma(-j_3 - m_3)} g^{31} + (-1)^{2\epsilon} \frac{\Gamma(1+j_3 + m_3)}{\Gamma(-j_3 + m_3)} g^{13} \right).
Taking into account the identity
\[ \Gamma(-j_3 + m_3 + \epsilon_3) \Gamma(1 + j_3 - m_3 - \epsilon_3) = (-1)^{2\epsilon_3}, \]
one gets the first equation of (2.12)

\[
\frac{\Gamma(-j_3 + m_3 + \epsilon_3)}{\Gamma(1 + j_3 + m_3 + \epsilon_3)} S_{\epsilon} \left[ \frac{j_3 - 1 - j_2 - 1 - j_1}{m_3 - m_2 - m_1} \right] = (-1)^{2\epsilon_3} \frac{\Gamma(-j_3 + m_3 + \epsilon_3)}{\Gamma(1 + j_3 + m_3 + \epsilon_3)} S_{\epsilon} \left[ \frac{j_3 - 1 - j_2 - 1 - j_1}{m_3 - m_2 - m_1} \right].
\]

One checks that the coefficient has expected properties with respect to the exchange 1 \(\leftrightarrow\) 2:

\[
\frac{\Gamma(-j_3 + m_3 + \epsilon_3)}{\Gamma(1 + j_3 + m_3 + \epsilon_3)} S_{\epsilon} \left[ \frac{j_3 - 1 - j_2 - 1 - j_1}{m_3 - m_2 - m_1} \right] = (-1)^{2\epsilon_3} \frac{\Gamma(-j_3 + m_3 + \epsilon_3)}{\Gamma(1 + j_3 + m_3 + \epsilon_3)} S_{\epsilon} \left[ \frac{j_3 - 1 - j_2 - 1 - j_1}{m_3 - m_2 - m_1} \right],
\]
which yields the second equation of (2.12).

**B Gamma Barnes identities**

We shall start from the identities for the Barnes gamma function relevant in the Liouville case

\[
\frac{\Gamma_b(-b^A j) \Gamma_b \left( \frac{1}{b} - j b^A \right)}{\Gamma_b(-Q j)} = a_1(j),
\]

\[
\frac{\Gamma_b(-b^A j) \Gamma_b \left( \frac{1}{b} + b^A (j + 1) \right) \Gamma_b \left( b^A (j + 1) \right)}{\Gamma_b(Q (j + 1))} = a_1(j),
\]

\[
a_1(j) = \frac{\Gamma_b(b^A) \Gamma_b \left( \frac{1}{b} + b^A \right)}{\Gamma_b(Q)} \left( b^{-1} Q \right)^{2 (j + 1)} b^{-2 b Q j (j+1) - 2 j - \frac{1}{2}},
\]

where

\[ b^A = \sqrt{b Q}, \quad b^B = \sqrt{b^{-1} Q}, \quad Q = b + b^{-1}. \]

Using the shift relations

\[
\Gamma_b(x + b) = \sqrt{2 \pi} b^{b x - \frac{1}{2}} \Gamma^{-1} \left( b x \right) \Gamma_b(x),
\]

\[
\Gamma_b \left( x + b^{-1} \right) = \sqrt{2 \pi} b^{-x/b + \frac{1}{2}} \Gamma^{-1} \left( x/b \right) \Gamma_b(x),
\]

\[
\frac{\Gamma_b(x + b) \Gamma_b \left( \frac{1}{b} + b \right)}{\Gamma_b(Q)} \left( b^{-1} Q \right)^{2 (j + 1)} b^{-2 b Q j (j+1) - 2 j - \frac{1}{2}},
\]
one checks that all the left hand sides of equations (B.1) have the same properties with respect to the shift \( j \to j - \frac{b}{Q} \):

\[
\frac{\Gamma_{b^\Lambda}(-b^\Lambda j + b^\Lambda - \frac{1}{b^\Lambda}) \Gamma \bar{b}^\Lambda (\frac{1}{b^\Lambda} - \hat{b}^\Lambda j)}{\Gamma_b(-Qj + b)} = b^{2j + \frac{2}{b}Q} \left( \sqrt{1 + b^2} \right)^{-\frac{1}{2} - j\frac{Q}{b}} \frac{\Gamma_{b^\Lambda}(-b^\Lambda j) \Gamma \bar{b}^\Lambda (\frac{1}{b^\Lambda} - \hat{b}^\Lambda j)}{\Gamma_b(-Qj)},
\]

\[
\frac{\Gamma_{b^\Lambda}(-b^\Lambda j + b^\Lambda) \Gamma_{b\bar{b}} (b^\Lambda + b^\Lambda j - \frac{1}{b^\Lambda})}{\Gamma_b(Q + Qj - b)} = b^{2j + \frac{2}{b}Q} \left( \sqrt{1 + b^2} \right)^{-\frac{1}{2} - j\frac{Q}{b}} \frac{\Gamma_{b^\Lambda} (-b^\Lambda j) \Gamma_{b\bar{b}} (b^\Lambda + b^\Lambda j)}{\Gamma_b(Q + Qj)},
\]

\[
a_1 \left( j - \frac{b}{Q} \right) = b^{2j + \frac{2}{b}Q} \left( \sqrt{1 + b^2} \right)^{-\frac{1}{2} - j\frac{Q}{b}} a_1(j),
\]

and to the shift \( j \to j - \frac{1}{bQ} \):

\[
\frac{\Gamma_{b^\Lambda}(-b^\Lambda j + \frac{1}{b^\Lambda}) \Gamma_{b\bar{b}} (-\hat{b}^\Lambda j + \hat{b}^\Lambda)}{\Gamma_b(-Qj + b^{-1})} = b^{\frac{1}{2} + j} \left( \sqrt{1 + b^2} \right)^{-\frac{1}{2} - j\frac{Q}{b}} \frac{\Gamma_{b^\Lambda} (-b^\Lambda j) \Gamma \bar{b}^\Lambda (\frac{1}{b^\Lambda} - \hat{b}^\Lambda j)}{\Gamma_b(-Qj)},
\]

\[
\frac{\Gamma_{b^\Lambda} (b^\Lambda + b^\Lambda j) \Gamma_{b\bar{b}} (b^\Lambda j + \frac{1}{b^\Lambda})}{\Gamma_b(Q + Qj - \frac{1}{b})} = b^{\frac{1}{2} + j} \left( \sqrt{1 + b^2} \right)^{-\frac{1}{2} - j\frac{Q}{b}} \frac{\Gamma_{b^\Lambda} (b^\Lambda + b^\Lambda j) \Gamma_{b\bar{b}} (b^\Lambda + b^\Lambda j)}{\Gamma_b(Q + Qj)},
\]

\[
a_1 \left( j - \frac{1}{bQ} \right) = b^{\frac{1}{2} + j} \left( \sqrt{1 + b^2} \right)^{-\frac{1}{2} - j\frac{Q}{b}} a_1(j).
\]

For non-rational \( b \) it yields the proof of (B.1) up to \( j \)-independent factors. They can be found calculating the sides of (B.1) at \( j = -1 \) and \( j = 0 \), respectively.

Multiplying the first two equations of (B.1) side by side one gets the identity for the upsilon functions

\[
\frac{Y_{b\Lambda}(-b^\Lambda j) Y_{\bar{b}^\Lambda} (\frac{1}{b^\Lambda} - \hat{b}^\Lambda j)}{Y_b(-Qj)} = \frac{Y_{b\Lambda} (b^\Lambda) Y_{\bar{b}^\Lambda} (\frac{1}{b^\Lambda} + \hat{b}^\Lambda)}{Y_b (\frac{1}{b} + b)} \left( b^{-1} Q \right)^{\frac{a_2(j)}{2}} \frac{Q^2 (j + 1)}{(b - b Q (j + 1)^{-j - 1})}.
\]

Using the same method one can prove the identities

\[
\frac{\Gamma_{b\bar{b}} (-b^\Lambda j)}{\Gamma_{b\bar{b}} (-b^\Lambda j) \Gamma_b (-Qj + b)} = \frac{\Gamma_{b\bar{b}} (\frac{1}{b^\Lambda} + b^\Lambda (j + 1))}{\Gamma_{b\bar{b}} (\frac{1}{b^\Lambda} + b^\Lambda (j + 1)) \Gamma_b (Q(j + 1) + b)} = a_2(j),
\]

\[
a_2(j) = \frac{\Gamma_{b\bar{b}} (b^\Lambda)}{\Gamma_{b\bar{b}} (b^\Lambda) \Gamma_b (\frac{1}{b})} \left( b^{-1} Q \right)^{\frac{a_2(j)}{2}} \frac{Q^2 (j + 1)}{(b - b Q (j + 1)^{j + 1} + \frac{b}{2} (j + 1)^{j + \frac{1}{2}})}.
\]
where

\[ \mathbf{b}^A = \sqrt{\mathbf{b}Q}, \quad \mathbf{b}^B = \sqrt{\mathbf{b}^{-1}Q}, \quad \mathbf{Q} = \mathbf{b}^{-1} - \mathbf{b}. \]

They imply the identity

\[
\frac{\Upsilon_{b^A}(-b^A j) \Upsilon_{b^B}(-\mathbf{Q}j + \mathbf{b})}{\Upsilon_{b^B}(-b^B j)} = \frac{\Upsilon_{b^A}((a^A) \Upsilon_{b^B}((a^B) (j+1))}{\Upsilon_{b^B}((a^B) (j+1))} \mathbf{b}^{2(j+1)} \mathbf{Q}^{j(j+1)+1}.
\]

| $n_1$ | $n_2$ | $n_3$ | $\eta(n_3, n_2, n_1)$ |
|-------|-------|-------|---------------------|
| $1/2$ | $1/2$ | $0$   | $0$                 |
| $0$   | $1/2$ | $1/2$ | $0$                 |
| $1/2$ | $-1/2$ | $0$  | $1$                |
| $1/2$ | $0$   | $-1/2$ | $1$                |
| $-1/2$ | $0$   | $1/2$ | $1$                |
| $-1/2$ | $1/2$ | $0$   | $0$                |
| $1/2$ | $0$   | $-1/2$ | $0$                |
| $-1/2$ | $-1/2$ | $0$  | $0$                |
| $0$   | $-1/2$ | $-1/2$ | $0$                |
| $-1/2$ | $0$   | $-1/2$ | $0$                |

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