Study of the fundamental interactions in the bounded states of the systems: Earth-Moon, Proton-Electron (hydrogen atom), positronium (process e− e+→e− e+), and quarkonium, ( QQ)

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Abstract. All The bound states of two particles such as the earth-moon system, bound by the gravitational interaction, proton-electron pe− hydrogen atom, positron-electron (e+ e−) positronium linked by the electromagnetic interaction and quark-antiquark (q̅q) quarkonium linked by strong interaction, provide physicists with a natural laboratory to study these interactions. In this paper we review the classical theories formulated by Newton and relativist due to Einstein of gravitational interactions to study the earth-moon system. As regards the bound states (pe−), (e+ e−) and (q̅q) we used Schrödinger equation to calculate the energy spectrum of these systems, as well as the Feynman diagrams at tree order to determine in some detail the Hamiltonian of these systems, known as Breit-Fermi potential. The similarities and differences between the hydrogen atom and the positronium are analyzed. The quarkonio is studied in the framework of the non-relativistic quark model (MQNR), heavy quark, where the relativistic effects are negligible, confronting the theoretical and experimental results to observe its validity.

Introduction

Thanks to the rapid development of technology and the art of building high-energy particle accelerators and detectors, it is possible to confront the validity of the theories that today explain the functioning of the universe and a better understanding of the structure of matter than for physicists Today, all known ordinary matter is composed of atoms, which in turn consist of a nucleus composed of protons and neutrons, surrounded by a cloud of electrons. Protons and neutrons, meanwhile, are formed by quarks. Quarks are fundamental particles found in groups of three or four, or in quark-antiquark pairs, linked by particles called gluons and by such incredibly powerful forces that no isolated quark has ever been observed [1]. For them, there are four types of actions: gravitational, electromagnetic, weak and strong. The bound states: earth-moon (gravitational interaction), proton-electron (hydrogen atom) and positronium (e+ e−) (electromagnetic interaction), while quarkonium (q̅q) is strong and provides physicists with an important natural laboratory for their study, which is the object of study in this article.

For the description of the gravitational interactions there are two formulations: the Newtonian one, in which the gravitational force on a body of mass \( m \) (gravitational mass) in a given position is proportional to the gravitational field \( g \) produced by the gravitational potential \( \Phi \) in that position and Einsteinian, in which gravity is considered as a manifestation of the curvature of spacetime, caused by
the presence of matter. Therefore, according to this theory, when a certain amount of matter occupies a region of space-time, it causes it to become deformed and since all objects move in space-time, when it deforms, the trajectory of these deviates producing an acceleration we call force of gravity.

The bound states \((p^+ e^-)\) hydrogen atom and \((e^+ e^-)\) positronium were studied initially in the framework of quantum mechanics and subsequently by quantum electrodynamics (QED) developed in the mid-twentieth century. Another two-particle system of special interest for this work is the quark-antiquark \((q\bar{q})\) bound states known as quarkonium which, as in the positronium, the constituent particles have the same mass and orbit around a common center. In principle, the non-relativistic quark model (MQNR) could be applicable for heavy quarks systems, where the relativistic effects are very small and the coupling constant becomes small.

Theoretically, the systems \(q\bar{q}\) have been essentially undertaken in two ways: On the one hand it has been proven to extract information from QCD \([2,3]\) and on the other to set a potential by adjusting a few parameters and trying to predict other quantities. From QCD an idea is to obtain a Coulomb potential near the origin modified by the strong coupling constant; of the lattice gauge models \([4]\) it is suggested that the potential between quarks could grow as the distances between them. Parallel to these two directions of research a few years ago a more general approach was initiated, in this case a phenomenological potential is set and the general properties that this potential must satisfy will be investigated.

**INTERACTIONS**

**GRAVITATIONAL:** After enunciating the laws that govern the movement of bodies, Newton's other great contribution and perhaps one of the most impressive achievements made by man, is the Law of Universal Gravitation, which establishes

\[
f = m_g \mathbf{g} = -m_g \nabla \Phi(r)
\]

where the gravitational potential \(\Phi(r)\) is given by the Poisson equation

\[
\nabla^2 \Phi(r) = 4\pi G \rho(r)
\]

where \(\rho(r)\) is the gravitational mass density and \(G\) is Newton's gravitational constant. Equation 2 is Newton's field equation, from which the classical formula of gravitational force can be deduced

\[
F = G \frac{m_1 m_2}{r^2}
\]

Note that Equation 2 does not depend explicitly on time, which implies that both the potential and the gravitational force respond immediately to any variation in mass density, that is, it is inconsistent with the special theory of relativity.

At the end of the 19th century, it was observed that the orbit of Mercury presented a precession of its perihelion, that is, an advance of the orbital axis of about 9.55 arc minutes for every 100 Earth years, a disturbance that was supposed to be due to gravitational effects of the other planets or another planet in orbit around the Sun closer than Mercury. But the search for the other planets proved unsuccessful, and the calculations within the framework of Newtonian theory gave a lower value than that observed in approximately 43.1 arc seconds per century.

These facts led Einstein to reformulate the classical theory of gravitation, with his general theory of relativity, which in addition to generalizing the principle of restricted relativity, to include non-inertial
reference systems, reviews the problem of gravitational interaction, reinterpreting gravity as a geometric effect of mass on spacetime. Therefore, according to this theory, when a certain amount of matter occupies a region of space-time, it causes it to become deformed and since all objects move in space-time, as this trajectory deforms, it deviates producing an acceleration that we call force of gravity. The Einsteinian field equation that describes not only the geometric properties of space, but also contains the density and pressure of matter is

\[ G_{\mu\nu} = \frac{8\pi}{c^2} T_{\mu\nu} \]

where:

- \( G_{\mu\nu} \) is the Einstein curvature tensor,
- \( T_{\mu\nu} \) is the energy moment tensor and
- \( c \) the speed of light in a vacuum, which in terms of the curvature tensor \( R_{\mu\nu} \), the Ricci \( R \) scalar, the cosmological constant \( \Lambda \) and the metric tensor \( g_{\mu\nu} \) is rewritten as

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi}{c^2} T_{\mu\nu} \]

equation that in the classical limit is reduced to equation (2).

For an elliptical orbit, such as that of the moon around the earth and considering that the gravitional field is weak, it is possible to estimate the total and potential energy of it by means of the Newtonian formulation of gravitation, given by the equations

\[ E = K \left( \frac{r}{r_1} \right) \left( \frac{\varepsilon-1}{\varepsilon+1} \right) \]

And

\[ U = -\frac{2K}{1+\varepsilon} \left( \frac{r}{r_1} \right) \]

Where \( K \) is the kinetic energy, \( r \) mean orbital radius, \( r_1 \) polar radius and \( \varepsilon \) eccentricity of the orbit.

ELECTROMAGNETIC: The study of the spectrum of the hydrogen atom has proved to be of vital importance for modern physics, because once its spectrum was deciphered and understood it was possible to understand the spectra of other elements. The effort made by many physicists to explain this spectrum allowed to formulate many laws of quantum mechanics, today these laws not only apply to atoms, but also to molecules and other physical systems. Energy levels for hydrogen atoms are obtained by solving the Schrödinger equation

\[ H\psi(r) = E\psi(r) \]

where \( H \) is the Hamiltonian system given by:

\[ H = \frac{p^2}{2\mu} + V(r) \]

where \( V(r) \) is Coulomb's potential given by

\[ V(r) = -\frac{1}{4\pi\varepsilon_0} + \frac{2e^2}{r} \]

Detailed solutions of equation (8) found in most quantum mechanics texts. The energetic values of energy are obtained through the equation:
\[ E_n = -\mu c^2 \left( \frac{Z e^2}{2n^2} \right), n = 1, 2, 3, \ldots \]  

where \( \mu \) is the reduced mass, \( n = n_r + \ell + 1 \), \( n_r \) is radial quantum number, \( \ell \) is the orbital angular momentum and \( \alpha \) is the fine structure constant, which accounts for the fine splitting of energy levels, given by

\[ \alpha = \frac{1}{4\pi \varepsilon_0} \frac{e^2}{\hbar c} \approx \frac{1}{137} \]  

Note that equation (11) predicts the degeneration of energy levels, however very precise measures from atomic physics and spectroscopy demonstrate the existence of some effects (fine and hyperfine splitting of the spectral lines that cannot be derived from the Hamiltonian given by the equation (9). The splitting of the spectral lines (fine and hyperfine structure) is due to relativistic effects, to analyze these effects it is necessary to find a wave function for the orbiting electron that satisfies the requirements of the special theory of special relativity and quantum mechanics. The most rigorous way to obtain these relativistic corrections is to solve the Dirac equation, whose solution for the hydrogen atom is

\[ E_{n,f} = mc^2 \left\{ 1 + \left( \frac{\alpha}{n - (\frac{1}{2})^2 + \sqrt{\left( \frac{1}{2} \right)^2 - \alpha^2}} \right)^2 \right\}^{-1/2} - 1 \]  

Another way is to use the Feynman diagrams which, in order, a tree for the hydrogen atom are illustrated in Figure 1.

**Figure 1.** Feynman diagrams for the hydrogen atom process \( p e \rightarrow p e \). From which the dispersion matrix is obtained and the interaction potential is calculated by taking the Fourier transform from it [7]. The diagrams corresponding to the positronium are shown in Figure 2.

**Figure 2** Feynman diagrams for positronium, process \( e^+ e^- \rightarrow e^+ e^- \).

The first diagram is analogous to the hydrogen atom, the second shows the probability of annihilation. The interaction potential for the process \( (p e^- \rightarrow p e^-) \) that is obtained from the corresponding Feynman diagram is
\[ H = H_0 + H_1 \]

where \( H_0 \) (undisturbed Hamiltonian given by Eq. (9) and \( H_1 \) (disturbed Hamiltonian), given by:

\[ H_1 = H_{rel} + H_{SO} + H_D + H_K + H_{SS} + H_{ten} + H_R \]

Where \( H_{rel} \) corrections to kinetic energy derived from the expansion of total energy, \( H_{SO} \) spin-orbit correction, \( H_D \) is Thomas's precession responsible for fine splitting. The delay potential \( H_K \) for \( \ell = 0 \) contributes only to the shifts of energy levels (without affecting the sub-levels of fine and hyperfine structure), since it only depends on the main quantum number \( n \), while for \( \ell \neq 0 \) it contributes to the runs of fine structure and \( H_{SS}, H_{ten} \) are the spin-spin and tensor corrections that account for the hyperfine splitting of energy levels. For the positronium to the Hamiltonian given by equation (15) a new term is added due to the probability of annihilation that exists between the constituent particles.

Table 1 summarizes the energy results including the corrections of fine and hyperfine structure, for \( n = 1, 2 \) obtained from the proposed model.

![Feynman diagram in order process tree](image)

In this section, we examine the interaction for this system in a tree order for the non-relativistic case, this is the non-relativistic quark model (MQNR), using the theory of strong interactions, quantum chromodynamics (QCD). Although QCD is a non-disturbing theory, there is a region of high energies (greater than 3 GeV) where the model is expected to work, since the value of the coupling constant \( \alpha_S \) (strong structure constant) can be considered and the relativistic effects are very small. In this region of asymptotic freedom, perturbation theory techniques (with certain limitations) can be applied to the linked states \( q\bar{q} \), such as Charmonium (\( c\bar{c} \)), Bottomonium (\( b\bar{b} \)) and the heavy inn (\( b\bar{c} \)) [9-15].

The mass spectrum is calculated using the Schrödinger equation:

\[ H\psi_{\text{nf}} = E_{\text{nf}}\psi_{\text{nf}}, \]

Where

\[ H = m_1 + m_2 + H_0 + H_1 \]

With

\[ H_0 = \frac{p^2}{2\mu} + V(r) \]

Where \( \mu = m_1 m_2 / (m_1 + m_2) \), \( m_1, m_2 \) they are the masses of the constituent quarks, \( V(r) \) is Cornell's potential [2, 3, 4] which consists of a vector contribution and a scalar given by

\[ V(r) = c + V_{\text{exch}}(r) + V_{\text{conf}}(r) = c - \frac{4\alpha_S(r)}{3r} + Fr \]
Where the constant \( c \) is determined phenomenologically and is the perturbative Hamiltonian that is derived from the Fourier transform of the dispersion matrix, given by

\[
H_1 = -\frac{1}{8} \left( \frac{1}{m_i} + \frac{1}{m_j} \right) p^4 + H_{ij}^{SS} + H_{ij}^{SO} + H_{ij}^{ten} + H_{ij}^{eff} + H_{ij}^{A}
\]

\( H_{ij}^{SS} \) (spin-spin), \( H_{ij}^{SO} \) (spin-orbit), \( H_{ij}^{ten} \) (tensor) and (annihilation potential) are the spin-dependent interactions that give rise to the hyperfine structure and \( H_{ij}^{A} \) is a spin-independent term and describes the interactions associated with the orbital movement of the particles that make up the system. Table 2 summarizes the results obtained for the charmonium using the proposed model.

**Results and Discussion**

The total (mooring) and potential energy values for the earth-moon system calculated using equations (6) and (7) are:

\[
E = -3.865 \times 10^{28} \text{ J} = -2.41261 \times 10^{-47} \text{ eV}
\]

\[
U = 7.71454 \times 10^{28} \text{ J} = -4.79462 \times 10^{-47} \text{ eV}
\]

The energy of the earth-moon system is \( 10^{46} \) times the energy of the electron in the fundamental state of the hydrogen atom, which shows the stability of the system. In addition, it would be sufficient to supply Colombia and the world for \( 2.2 \times 10^{11} \) and \( 7.73 \times 10^{14} \) years, respectively if the expected consumption is maintained in the World Consumption of Primary Energy by Energy Type and Selected Country Groups, 1980-2004 »(XLS). Energy Information Administration, U.S. Department of Energy (July 31, 2006).

| State \( n\ell \) | Corrected energy \( E_{nj} \) (eV) | Corrected energy \( E_{nj} \) (est. Hyperfine) | Dirac \( E_{nj} \) (eV) |
|------------------|---------------------------------|---------------------------------|-----------------|
| \( 1s_{1/2} \)  | -3.60659903                     | -13.60720359                    | -13.605902      |
| \( 2s_{1/2} \)  | -3.40325                        | -3.403268                       | -3.4014869      |
| \( 2p_{1/2} \)  | -3.40325                        | -3.403268                       | -3.4014869      |
| \( 2p_{3/2} \)  | -3.403279                       | -3.403287                       | -3.4014461      |
| \( 1s_{1/2} \)  | -3.60659903                     | -13.60720359                    | -13.605902      |

The results obtained for the hydrogen atom through perturbation theory and Feynman diagrams agree quite well with those obtained with the Dirac equation, which shows the appropriateness of the model to address the solution to the problem of the energy spectrum for this system.
Table 2. Charmonium states \((c\bar{c})\) with \(b = 1.065, c = -0.255728 \text{ GeV},\)
\(m_c = 1.275 \text{ GeV} \) and \(F = 0.18 \text{ GeV}^2\)

| Particle | MQNR \(m_{\text{MQNR}}^\alpha\) \(\text{GeV}\) | \(m_{\text{Teo.}}^\alpha\) \(\text{GeV}\) | Disc. \(\%\) |
|----------|---------------------------------|----------------|-----|
| \(J/\psi(1s)\) | \((1^3s_1)\) | 3.09688(4) | 3.00003 | 3.23 |
| \(\psi(2s)\) | \((2^3s_1)\) | 3.6860(1) | 3.50848 | 5.06 |
| \(\psi(4040)\) | \((3^3s_1)\) | 4.04(1) | 3.88922 | 3.88 |
| \(\psi(4160)\) | \((4^3s_1)\) | 4.16(2) | 4.21111 | 1.21 |
| \(\psi(4415)\) | \((4-5^3s_1)\) | 4.415(6) | 4.42617 | 0.25 |
| \(\eta_b(1s)\) | \((1^1s_0)\) | 2.980(2) | 2.87649 | 3.60 |
| \(\eta_b(2s)\) | \((2^1s_0)\) | 3.594(5) | 3.42136 | 5.05 |
| \(\psi(3770)\) | \((1^3d_1)\) | 3.770(3) | 3.71703 | 1.43 |
| \(h_1(p)\) | \((1^1p_1)\) | 3.5261(2) | 3.41748 | 3.18 |
| \(\chi_{c1}(1p)\) | \((1^3p_0)\) | 3.415(1) | 3.37118 | 1.30 |
| \(\chi_{c1}(1p)\) | \((1^1p_1)\) | 3.5105(1) | 3.41491 | 2.80 |
| \(\chi_{c1}(1p)\) | \((1^3p_2)\) | 3.5562(1) | 3.42828 | 3.73 |

Note that the discrepancy between theoretically estimated and experimental masses does not exceed 6%, which is an indication that the MQNR model is suitable for these systems.

The highest perturbative corrections are presented for the states with orbital angular momentum \(\ell = 0\), where the coulombian part predominates and, which go as \(1/r, 1/r^2\) and \(1/r^3\) which partly explains these results, in addition to the values obtained for these justify because perturbation theory and a non-relativistic treatment of the problem were used.

CONCLUSIONS

Cornell's potential sets a model that despite its simplicity reproduces quite well the spectra of the mesons constituted by heavy quarks. It is possible that incorporating modifications to the coulombian potential could improve the MQNR estimates. In general, for the positronium system \((\text{process } e^+e^- \rightarrow e^+e^-)\) the energy contributions due to the perturbative term \(H_1\) are less than 1% for the ns states and of the order of 1 per 1000 for the np, states, which justifies the use of perturbation theory.

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