The Projectivization Matroid of a $q$-Matroid

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Abstract

In this paper, we investigate the relation between a $q$-matroid and its associated matroid called the projectivization matroid. The latter arises by projectivizing the groundspace of the $q$-matroid and considering the projective space as the groundset of the associated matroid on which is defined a rank function compatible with that of the $q$-matroid. We show that the projectivization map is a functor from categories of $q$-matroids to categories of matroids, which allows to prove new results about maps of $q$-matroids. We furthermore show the characteristic polynomial of a $q$-matroid is equal to that of the projectivization matroid. We use this relation to establish a recursive formula for the characteristic polynomial of a $q$-matroid in terms of the characteristic polynomial of its minors. Finally we use the projectivization matroid to prove a $q$-analogue of the critical theorem in terms of $\mathbb{F}_{q^m}$-linear rank metric codes and $q$-matroids.

Keywords: Projectivization matroid, $q$-matroids, characteristic polynomial, strong maps, weak maps, rank metric code, critical theorem.

1 Introduction

In recent years, $q$-matroids, the $q$-analogue of matroids, have been intensively studied due to their connection to linear rank metric codes. They were first studied by Jurrius and Pellikaan in [14], who showed that an $\mathbb{F}_{q^m}$-linear rank metric code induces a $q$-matroid. It was shown later on that matrix linear rank metric codes induce a $q$-polymatroid, a generalization of $q$-matroids (see [9, 10]). Since then, many results on $q$-(poly)matroids, and how they relate to rank metric codes, have been established, see for example [2, 5, 6, 8, 9, 10, 13]. Because of their $q$-analogue nature, many of the newly discovered properties of $q$-matroids turn out to be analogues of well established matroid theory results. See [2, 11, 15, 17] for more information on matroid theory. It has therefore been of interest to determine which notions and properties of matroid theory can be generalized to $q$-matroids.

In [6], the authors show that, similarly to matroids, there exists a variety of cryptomorphic definitions for $q$-matroids. In this paper, we define $q$-matroids via a rank function on the lattice of subspace of a finite dimensional vector space over a finite field, and occasionally use the flat cryptomorphism. In [5], Byrne and co-authors, define the notion of a characteristic polynomial for $q$-polymatroids and use it to establish a $q$-analogue of the Assmus-Mattson Theorem. Furthermore

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they show that the characteristic polynomial of a \(q\)-polymatroid induced by a linear rank metric code determines the weight distribution of the code. Maps between \(q\)-matroids are defined and studied in [5], which allows the authors to consider \(q\)-matroids from a category theory perspective. They introduce the notions of weak and strong maps, which respectively respect the rank structure and the flat structure of \(q\)-matroids. Although those maps are defined in an analogous way than the weak and strong maps between matroids (see [12, 17]), substantial differences occur when comparing categories of \(q\)-matroids with categories of matroids. In fact, the authors show that, unlike for categories of matroids, a coproduct does not always exist in the category of \(q\)-matroids with strong maps but always exists when the morphisms are linear weak maps.

In [13], Johnsen and co-authors make the connection between matroids and \(q\)-matroids more apparent by showing that a \(q\)-matroid induces a matroid, called the projectivization matroid. Furthermore, they show that the lattice of flats of the \(q\)-matroid is isomorphic to the lattice of flats of its projectivization matroid. This allows them to express the generalized rank weights of an \(F_{q^m}\)-linear rank metric code in terms of the Betti numbers of the dual of the projectivization matroid.

In this paper, we further investigate the construction introduced in [13] and use the projectivization matroid as a tool to study maps between \(q\)-matroids and the characteristic polynomial of a \(q\)-matroid. We define and study properties of the projectivization matroid in Section 3. In Section 4, we show that the projectivization map from a vector space to its projective space is a functor from the category of \(q\)-matroids with weak (resp. strong) maps to the category of matroids with weak (resp. strong) maps. We use the relation between those categories to show that strong maps between \(q\)-matroids are weak maps. Although Section 4 shows how the projectivization matroid can be used in a more category theory approach, results therein are not used in later sections. The reader interested in the relation between the characteristic polynomial of a \(q\)-matroid and that of its projectivization matroid may skip Section 4 on a first reading. We then proceed in Section 5 to study the characteristic polynomial of \(q\)-matroids. We start by showing that the characteristic polynomial is identically 0 if the \(q\)-matroid contains a loop, and is fully determined by the lattice of flats otherwise. We use this fact to show that the characteristic polynomial of a \(q\)-matroid is equal to that of its projectivization matroid. This in turn, allows us to find a recursive formula for the characteristic polynomial of a \(q\)-matroid in terms of the characteristic polynomial of its minors. Finally, in Section 6, we consider the projectivization matroid of \(q\)-matroids induced by an \(F_{q^m}\)-linear rank metric code. In [1], Alfarano et. al. associate a linear block code with the Hamming metric to an \(F_{q^m}\)-linear rank metric code \(C\). This code, called a Hamming-metric code associated to \(C\), induces a matroid that turns out to be equivalent to the projectivization matroid of the \(q\)-matroid associated with \(C\). This connection allows us to prove a \(q\)-analogue of the critical theorem in terms of \(F_{q^m}\)-linear rank metric codes and \(q\)-matroids.

**Notation:** Throughout \(\mathbb{F}_q\) denotes a finite field of order \(q\). \(E\) denotes a finite dimensional vector space over \(\mathbb{F}_q\), and \(\mathcal{L}(E)\) denotes the lattice of subspace of \(E\). \(S\) and \(T\) are finite sets, \(2^S\) is the power set of \(S\) and \(\{n\} := \{1, \ldots, n\}\) for \(n \in \mathbb{N}_0\). Furthermore given a set \(S\) and \(A \subseteq S\), let \(S - A := \{e \in S : e \notin A\}\). Finally, \(q\)-matroids will be denoted by the script letters \(\mathcal{M}, \mathcal{N}\), whereas matroids will be denoted by the capital letters \(M, N\).
2 Basic Notions of Matroids and \( q \)-Matroids

In this section we review well-known notions of matroids and \( q \)-matroids that will be used throughout the paper. For more details about matroids and \( q \)-matroids the reader may refer to [6, 14, 15, 17].

**Definition 2.1.** A matroid is an ordered pair \( M = (S, r) \), where \( S \) is a finite set and \( r \) is a function \( r : 2^S \to \mathbb{N}_0 \) such that for all \( A, B \in 2^S \):

1. **Boundedness:** \( 0 \leq r(A) \leq |A| \).
2. **Monotonicity:** If \( A \subseteq B \) then \( r(A) \leq r(B) \).
3. **Submodularity:** \( r(A \cup B) + r(A \cap B) \leq r(A) + r(B) \).

\( S \) is called the *groundset* of \( M \) and \( r \) its *rank function*.

Throughout, identify \( \{e\} \) with \( e \) and \( \{v\} \) with \( v \). Two matroids \( M = (S, r_M) \) and \( N = (T, r_N) \) are *equivalent*, denoted \( M \cong N \), if there exists a bijection between the groundsets, \( \psi : S \to T \), such that \( r_M(A) = r_N(\psi(A)) \) for all \( A \subseteq S \). Given a matroid \( M = (S, r) \), \( e \in S \) is a *loop* of \( M \) if \( r(e) = 0 \). \( M \) is said to be *loopless* if it does not contain any loops. A subset \( F \subseteq S \) is a *flat* if \( r(F \cup v) > r(F) \) for all \( v \notin F \). It is well known that the collection of flats, denoted \( \mathcal{F}_M \), forms a geometric lattice. For any \( F_1, F_2 \in \mathcal{F}_M \), the meet and join are defined as follow \( F_1 \wedge F_2 := F_1 \cap F_2 \) and \( F_1 \vee F_2 := \text{cl}_M(F_1 \cup F_2) \), where \( \text{cl}_M(A) = \{ v \in S : r(A \cup v) = r(A) \} = \bigcap \{ F \in \mathcal{F}_M : A \subseteq F \} \).

Given \( F_1, F_2 \in \mathcal{F}_M \), we say \( F_2 \) covers \( F_1 \) if for all \( F \in \mathcal{F}_M \) such that \( F_1 \subseteq F \subseteq F_2 \) then \( F = F_1 \) or \( F = F_2 \). When discussing \( \mathcal{F}_M \), we interchangeably use the terms collection of flats and lattice of flats. The flats of a matroid satisfy three axiomatic properties that fully determine the matroid.

**Proposition 2.2.** [15, Sec. 1.4 Prob 11.] Let \( M = (S, r_M) \) be a matroid and \( \mathcal{F}_M \) its collection of flats. Then \( \mathcal{F}_M \) satisfies the following:

1. \( S \in \mathcal{F}_M \).
2. If \( F_1, F_2 \in \mathcal{F}_M \) then \( F_1 \cap F_2 \in \mathcal{F}_M \).
3. If \( F \in \mathcal{F}_M \) and \( v \notin F \), then there exists a unique \( F' \in \mathcal{F}_M \) covering \( F \) such that \( F \cup v \subseteq F' \).

Furthermore, \( r_M \) is uniquely determined by \( \mathcal{F}_M \) and \( r_M(A) = h(\text{cl}_M(A)) \) for \( A \subseteq S \), where \( h(F) \) denotes the height of \( F \) in the lattice \( \mathcal{F}_M \).

We now turn to \( q \)-matroids, which are defined in an analogous way. Recall that \( E \) denotes a finite dimensional vector space over \( \mathbb{F}_q \) and \( \mathcal{L}(E) \) is the collection of subspace of \( E \).

**Definition 2.3.** A *\( q \)-matroid* is an ordered pair \( \mathcal{M} = (E, \rho) \), where \( \rho \) is a function \( \rho : \mathcal{L}(E) \to \mathbb{N}_0 \) such that for all \( V, W \in \mathcal{L}(E) \):

1. **Boundedness:** \( 0 \leq \rho(V) \leq \dim(V) \).
2. **Monotonicity:** If \( V \subseteq W \) then \( \rho(V) \leq \rho(W) \).
3. **Submodularity:** \( \rho(V + W) + \rho(V \cap W) \leq \rho(V) + \rho(W) \).
$E$ is called the groundspace of $\mathcal{M}$ and $\rho$ its rank function.

Two q-matroids $\mathcal{M} = (E_1, \rho_{\mathcal{M}})$ and $\mathcal{N} = (E_2, \rho_{\mathcal{N}})$ are equivalent, denoted $\mathcal{M} \cong \mathcal{N}$, if there exists a linear isomorphism $\psi : E_1 \rightarrow E_2$ such that $\rho_{\mathcal{M}}(V) = \rho_{\mathcal{N}}(\psi(V))$ for all $V \subseteq E_1$. Given a q-matroid $\mathcal{M} = (E, \rho)$ we say $\langle e \rangle$, where $e \in E - \{0\}$, is a loop if $\rho(\langle e \rangle) = 0$, and $\mathcal{M}$ is loopless if it does not contain any loops. A subspace $F \subseteq E$ is a flat of $\mathcal{M}$ if $\rho(F + \langle v \rangle) > \rho(F)$ for all $v \notin F$. Furthermore the collection of flats of a q-matroid, denoted $\mathcal{F}_\mathcal{M}$, forms a geometric lattice as well.

The meet and join operation are given by $F_1 \wedge F_2 := F_1 \cap F_2$ and $F_1 \vee F_2 := cl_M(F_1 + F_2)$, where $cl_M(V) := \{v : \rho(V + \langle v \rangle) = \rho(V)\} = \bigcap\{F \in \mathcal{F}_\mathcal{M} : V \leq F\}$. The notion of cover for the lattice of flats of q-matroids is identical to that of matroids. The collection of flats of a q-matroid also satisfies three axiomatic properties that fully determine the q-matroid.

**Proposition 2.4.** Let $\mathcal{M} = (E, \rho_{\mathcal{M}})$ be a q-matroid and $\mathcal{F}_\mathcal{M}$ be its collection of flats. Then $\mathcal{F}_\mathcal{M}$ satisfies the following:

(qF1) $E \in \mathcal{F}_\mathcal{M}$.

(qF2) If $F_1, F_2 \in \mathcal{F}_\mathcal{M}$ then $F_1 \cap F_2 \in \mathcal{F}_\mathcal{M}$.

(qF3) Let $F \in \mathcal{F}_\mathcal{M}$ and $e \notin F$, then there exists a unique $F' \in \mathcal{F}_\mathcal{M}$ covering $F$ such that $F + \langle e \rangle \leq F'$.

Furthermore, $\rho_{\mathcal{M}}$ is uniquely determined by $\mathcal{F}_\mathcal{M}$ and $\rho_{\mathcal{M}}(V) = h(cl_M(V))$ for all $V \leq E$, where $h(F)$ denotes the height of $F$ in the lattice $\mathcal{F}_\mathcal{M}$. Thus, we may denote this q-matroid as $\mathcal{M} = (E, \mathcal{F}_\mathcal{M})$.

For both matroids and q-matroids, there exists a notion of duality, defined respectively with complements of sets and orthogonal spaces.

**Definition 2.5.** Let $M = (S, r)$ be a matroid. The dual matroid $M^* = (S, r^*)$ is defined via the rank function

$$r^*(A) = |A| - r(S) + r(S - A).$$

Duality for q-matroids depends on a choice of non-degenerate symmetric bilinear form (NSBF). Let $E$ be a vector space over $\mathbb{F}_q$ and $\langle \cdot, \cdot \rangle : E \times E \rightarrow \mathbb{F}_q$ be a NSBF. The orthogonal space of $V \leq E$ w.r.t $\langle \cdot, \cdot \rangle$ is the space $V^\perp := \{w \in E : \langle w, v \rangle = 0 \text{ for all } v \in V\}$.

**Definition 2.6.** Let $\mathcal{M} = (E, \rho)$ be a q-matroid and $\langle \cdot, \cdot \rangle$ be a NSBF on $E$. The dual q-matroid $\mathcal{M}^* = (E, \rho^*)$, w.r.t the chosen NSBF, is defined via the rank function

$$\rho^*(V) = \dim(V) - \rho(E) + \rho(V^\perp).$$

It was shown in [9] Thm 2.8 that given two NSBFs $\langle \cdot, \cdot \rangle_1, \langle \cdot, \cdot \rangle_2$, the respective dual q-matroids $\mathcal{M}^{*1}$ and $\mathcal{M}^{*2}$ of $\mathcal{M}$ are equivalent. For both matroids and q-matroids, an element $v$ of the groundset, respectively groundspace, is a coloop if $e$ is loop in the dual matroid, respectively dual q-matroid.

We now define the operations of deletion and contraction for matroids and q-matroids.

**Definition 2.7.** Let $M = (S, r)$ be a matroid and let $A \subseteq S$. 


• The matroid \( M \setminus A = (S - A, r_{M \setminus A}) \), where \( r_{M \setminus A}(B) = r(B) \) for all \( B \subseteq S - A \), is called the deletion of \( A \) from \( M \).

• The matroid \( M/A = (S - A, r_{M/A}) \), where \( r_{M/A}(B) = r(B \cup A) - r(A) \) for all \( B \subseteq S - A \), is called the contraction of \( A \) from \( M \).

The following well-known facts about the deletion and contraction of matroids will be needed. Refer to [15, Prop 3.1.25] for a proof.

**Proposition 2.8.** Let \( M = (S, r) \) be a matroid. Let \( A, B \subseteq S \) be disjoint sets. Then

\[
\begin{align*}
& (M \setminus A) \setminus B = M \setminus (A \cup B) = (M \setminus B) \setminus A, \\
& (M/A)/B = M/(A \cup B) = (M/B)/A, \\
& (M \setminus A)/B = (M/B) \setminus A.
\end{align*}
\]

To avoid the surplus of parenthesis, we omit them if there is no risk of confusion.

At this point we make a brief comment about notation. The notation \( \setminus \) always denotes the deletion operation and set exclusion is denoted by the \( - \) sign. However, the notation \( / \) is used to denote both the contraction of \((q)\)-matroids and quotient space (i.e \( E/V \)). The reader should therefore use context in order to differentiate between the latter two.

For \((q)\)-matroids, the operations of deletion and contraction are defined in an analogous way.

**Definition 2.9.** Let \( \mathcal{M} = (E, \rho) \) be a \((q)\)-matroid, let \( V \leq E \) and fix a NSBF on \( E \). Furthermore let \( \pi : E \rightarrow E/V \) be the canonical projection.

- The \((q)\)-matroid \( M \setminus V = (V^\perp, r_{M \setminus V}) \), where \( r_{M \setminus V}(W) = \rho(W) \) for all \( W \leq V^\perp \) is called the deletion of \( V \) from \( \mathcal{M} \).
- The \((q)\)-matroid \( M/V = (E/V, \rho_{M/V}) \), where \( \rho_{M/V}(W) = \rho(\pi^{-1}(W)) - \rho(V) \) for all \( W \leq E/W \), is called the contraction of \( V \) from \( \mathcal{M} \).

It is worth noting that for both matroids and \((q)\)-matroids, deletion and contraction are dual operations, i.e. \( \mathcal{M}^* \setminus V \cong (\mathcal{M}/V)^* \) (equality rather than equivalence holds for matroids only). A proof of this fact for \((q)\)-matroids can be found in [9, Thm 5.3] and in [15, Sect. 3] for matroids. A matroid \( N \) (resp. \((q)\)-matroid \( \mathcal{N} \)) is a minor of \( M \) (resp. \( \mathcal{M} \)) if it can be obtained from \( M \) (resp. \( \mathcal{M} \)) by a sequence of deletion and contraction.

For both matroids and \((q)\)-matroids, the flats of a contraction can be characterized in terms of the flats of the original \((q)\)-matroid.

**Proposition 2.10.** Let \( M = (S, r_M) \) be a matroid, \( \mathcal{M} = (E, \rho_M) \) be a \((q)\)-matroid and \( \mathcal{F}_M, \mathcal{F}_\mathcal{M} \) their respective lattice of flats. Let \( A \subseteq S, V \leq E \) and consider \( M/A \) and \( \mathcal{M}/V \). Then

1. \( \mathcal{F}_{M/A} = \{ F \subseteq S - A : F \cup A \in \mathcal{F}_M \} \),
2. \( \mathcal{F}_{\mathcal{M}/V} = \{ F : \pi^{-1}(F) \in \mathcal{F}_\mathcal{M} \} \), where \( \pi : E \rightarrow E/V \).

Furthermore \( A \) (resp. \( V \)) is a flat of \( M \) (resp. \( \mathcal{M} \)) if and only if \( M/A \) (resp. \( \mathcal{M}/V \)) is loopless.
Proof. (1) is shown in [15] Prop 3.3.7. For (2), first let \( F \in \mathcal{F}_{M/V} \) and consider the space \( W := \pi^{-1}(F) \leq E \). Let \( x \notin W \). Then \( \rho_M(W \oplus \langle x \rangle) = \rho_{M/V}(F \oplus \langle \pi(x) \rangle) + \rho_M(V) > \rho_{M/V}(F) + \rho_M(V) = \rho_M(W) \), where the inequality holds because \( F \in \mathcal{F}_{M/V} \) and \( \pi(x) \notin F \). Since this is true for all \( x \notin W \) then \( W \notin \mathcal{F}_M \).

Now let \( F \leq E/V \) such that \( \pi^{-1}(F) \in \mathcal{F}_M \). Let \( \langle x \rangle \leq F/V \) such that \( x \notin F \). Then \( \rho_{M/V}(F \oplus \langle x \rangle) = \rho_M(\pi^{-1}(F) + \pi^{-1}(\langle x \rangle)) - \rho_M(V) > \rho_M(\pi^{-1}(F)) - \rho_M(V) = \rho_{M/V}(F) \). Once again, since this is true for all \( x \notin F \) then \( F \in \mathcal{F}_{M/V} \).

We show the second part of the statement for matroids, and note the proof for \( q \)-matroid is analogous to it. Consider \( M/A \) with \( A \in \mathcal{F}_M \) and let \( e \in S - A \). Then \( r_{M/A}(e) = r_M(e \cup A) - r_M(A) > 0 \) since \( A \) is a flat. Since this holds for all \( e \in S - A \), then \( M/A \) is loopless. Now assume \( A \notin \mathcal{F}_M \) then \( A \subseteq cl_M(A) \) and let \( e \in cl_M(A) - A \subseteq S - A \). Then \( r_{M/A}(e) = r_M(A \cup e) - r(A) = 0 \) since \( e \in cl_M(A) \). Hence \( M/A \) contains a loop. \[ \square \]

The last matroid operation we discuss is that of the single element extension by adjoining a loop, which we refer to as loop extension. The loop extension will play an important role in section \( 4 \) when defining maps between matroids. The reader can refer to [15 Sect. 7.2] and [17 Chap. 8] for proofs and a more detailed discussion of the single element extension.

**Proposition 2.11.** Let \( M = (S, r) \) be a matroid and \( \{o_M\} \) denotes a symbol disjoint from \( S \). Let \( S_o := S \cup \{o_M\} \) and \( r_o : 2^{S_o} \rightarrow \mathbb{N}_0 \) be such that \( r_o(A) = r(A - \{o_M\}) \), for all \( A \subseteq S_o \). Then \( M_o := (S_o, r_o) \) is a matroid, and \( \{o_M\} \) is a loop in \( M_o \). Furthermore \( M_o \) is called a loop extension of \( M \).

The subscript of the added loop may be omitted if it is clear from context in which matroid the loop is contained. The next proposition relates the flats \( \mathcal{F}_{M_o} \) and \( \mathcal{F}_M \). Furthermore, we recall that two lattices are isomorphic (denoted by \( \cong \)) if there exists an order preserving bijection between the lattices that preserves meets and joins.

**Proposition 2.12.** Let \( M \) be a matroid, \( M_o \) a loop extension of \( M \), and \( \mathcal{F}_M, \mathcal{F}_{M_o} \) their respective collection of flats. Then
\[
\mathcal{F}_{M_o} = \{ F \cup \{o\} : F \in \mathcal{F}_M \}.
\]
and \( \mathcal{F}_M \cong \mathcal{F}_{M_o} \) as lattices.

**Remark 2.13.** Note that \( M_o \setminus \{o\} = M \). This deletion can be seen as identifying the element \( \{o\} \) with the empty set of \( M \), and does not change the overall structure of the matroid.

## 3 The Projectivization Matroid

In [13], Johnsen and co-authors showed that a \( q \)-matroid \( \mathcal{M} \) with groundspace \( E \) induces a matroid \( P(\mathcal{M}) \) with groundset the projective space of \( E \). This induced matroid, called the projectivization matroid of \( \mathcal{M} \) turns out to be an interesting object to study. In fact, it was shown in that same paper, that the projectivization preserves the flat structure of \( \mathcal{M} \). It therefore becomes a useful tool when studying properties of \( q \)-matroids that depend only on flats.

For completeness, we reintroduce the construction of the projectivization matroid. The following notation will be used. Given a finite dimensional vector space \( E \) over \( \mathbb{F}_q \), let \( P E := \{ \langle v \rangle_{\mathbb{F}_q} : v \in E - \}

\{0\} \) be the projective space of \( E \). The map, \( \hat{P} : (E - \{0\}) \to \mathbb{P}E \), \( v \mapsto \langle v \rangle_{\mathbb{F}_q} \) induces a lattice map \( P : \mathcal{L}(E) \to 2^{\mathbb{P}E} \), where \( P(\{0\}) = \emptyset \) and \( P(V) = \{ \hat{P}(v) : v \in V - \{0\} \} = \{ P(v) : v \in V - \{0\} \} \) for \( V \leq E \). We call the lattice map \( P \) the projectivization map. Usually, \( \hat{P} \) is called the projectivization map, however for our purposes, it is more convenient to consider the projectivization as a lattice map. Note that \( P \) is inclusion preserving and that \( P(V \cap W) = P(V) \cap P(W) \) for all \( V, W \in \mathcal{L}(E) \). For any \( S \subseteq \mathbb{P}E \) let \( P^\perp(S) := \{ v \in E : \hat{P}(v) \in S \} = \{ v \in E : P(v) \in S \} \). Note that \( (P^\perp \circ P)(V) = V \) for all \( V \leq E \). Finally let \( \langle S \rangle := \langle P^\perp(S) \rangle_{\mathbb{F}_q} \) for any \( S \subseteq \mathbb{P}E \). We say \( S \subseteq \mathbb{P}E \) contains a basis of \( E \) if \( \langle S \rangle = E \). We can now introduce the projectivization matroid.

**Theorem 3.1.** ([I3 Def.14, Prop. 15]) Let \( \mathcal{M} = (E, \rho) \) be a \( q \)-matroid and let \( r : 2^{\mathbb{P}E} \to \mathbb{N}_0 \) such that for all \( S \subseteq \mathbb{P}E \),
\[
r(S) = \rho(\langle S \rangle).
\]
Then \( P(\mathcal{M}) := (\mathbb{P}E, r) \) is a matroid, and is called the projectivization matroid of \( \mathcal{M} \).

We now turn towards the relation between the flats of a \( q \)-matroid \( \mathcal{M} \) and those of its projectivization matroid \( P(\mathcal{M}) \). In the following result, the meet and join refers to those of the lattice of flats defined in Section 1.

**Lemma 3.2.** ([I3 Lem.16, Prop.21]) Let \( \mathcal{M} \) be a \( q \)-matroid, \( P(\mathcal{M}) \) its projectivization matroid, and \( \mathcal{F}, \mathcal{F}_{P(\mathcal{M})} \) their respective lattice of flats. Furthermore, let \( P(\mathcal{F}) := \{ P(F) : F \in \mathcal{F} \} \). Then the following hold:

1) \( \mathcal{F}_{P(\mathcal{M})} = P(\mathcal{F}) \).

2) \( P(F_1 \cup F_2) = P(F_1) \cup P(F_2) \) and \( P(F_1 \cap F_2) = P(F_1) \cap P(F_2) \), for all \( F_1, F_2 \in \mathcal{F} \).

Therefore \( \mathcal{F}_{P(\mathcal{M})} \cong \mathcal{F} \) as lattices.

The next result shows when a matroid with groundset \( \mathbb{P}E \) is the projectivization matroid of a \( q \)-matroid with groundspace \( E \).

**Theorem 3.3.** Let \( \mathcal{M} = (\mathbb{P}E, r) \) be a matroid and \( \mathcal{F} \) its lattice of flats. Furthermore let \( P^{-1}(\mathcal{F}) := \{ P^{-1}(F) \cup \{0\} : F \in \mathcal{F} \} \). If \( P^{-1}(F) \cup \{0\} \) is a subspace of \( E \) for all \( F \in \mathcal{F} \), then \( \mathcal{M} = (E, P^{-1}(\mathcal{F})) \) is \( q \)-matroid. Furthermore \( \mathcal{F} \cong \mathcal{F}_{P^{-1}(\mathcal{F})} \).

**Proof.** We show \( \mathcal{F} := P^{-1}(\mathcal{F}) \) is a collection of flats of a \( q \)-matroid by showing it satisfies (qF1)-(qF3) of Proposition 2.4. Throughout the proof we use the fact that \( \mathcal{F} \) is the collection of flats of a matroid, and hence satisfies (F1)-(F3) of Proposition 2.2. Since \( \mathcal{F} \) satisfies (F1), \( \mathbb{P}E \in \mathcal{F} \), and therefore \( P^{-1}(\mathbb{P}E) \cup \{0\} = E \in \mathcal{F} \). This shows (qF1). Let \( V_1 := P^{-1}(F_1) \cup \{0\}, V_2 := P^{-1}(F_2) \cup \{0\} \in \mathcal{F} \). Since \( F_1, F_2 \in \mathcal{F} \) then \( F_1 \cap F_2 \in \mathcal{F} \). Furthermore, \( P(V_1 \cap V_2) = P(V_1) \cap P(V_2) = F_1 \cap F_2 \in \mathcal{F} \). Hence \( V_1 \cap V_2 = P^{-1}(F_1 \cap F_2) \cup \{0\} \in \mathcal{F} \), showing (qF2).

Finally for (qF3), let \( V := P^{-1}(F) \cup \{0\} \in \mathcal{F} \) and \( w \notin V \). Since \( P \) is inclusion preserving \( P(\langle w \rangle) \notin P(V) = F \). Hence there exists a unique flat \( F' \in \mathcal{F} \) covering \( F \) such that \( F \cup P(\langle w \rangle) \subseteq F' \). Let \( V' := P^{-1}(F') \cup \{0\} \). By definition \( V' \in \mathcal{F} \) and since \( V' \) is a subspace containing \( V \cup w \) then \( V \oplus \langle w \rangle \subseteq V' \). To show \( V' \) covers \( V \), assume there exists \( W \in \mathcal{F} \) such that \( V \leq W \leq V' \). Applying \( P \) and using the fact that \( P \) is inclusion preserving, we get \( F = P(V) \subseteq P(W) \subseteq P(V') = F' \). However because \( W \in \mathcal{F} \) then \( P(W) \in \mathcal{F} \). But \( F' \) covers \( F \) hence we must have that \( P(W) = F' \).
and therefore $W = V'$. This implies $V'$ is a cover of $V$ and shows $\mathcal{F}$ is the collection of flats of a $q$-matroid.

Finally to show $\mathcal{F}_M$ and $\mathcal{F}$ are isomorphic as lattices, note that $P(\mathcal{F}) = \mathcal{F}_M$ hence by Theorem 3.2 the isomorphism follows.

We now show that the lattice of flats of the $q$-matroid $\mathcal{M}$ contracted by a flat $F$ is isomorphic to the lattice of flats of $P(\mathcal{M})/P(F)$.

**Theorem 3.4.** Let $\mathcal{M}$ be a $q$-matroid, $P(\mathcal{M})$ its projectivization matroid and $\mathcal{F}_M, \mathcal{F}_{P(\mathcal{M})}$ their respective lattice of flats. Then $\mathcal{F}_{M/F} \cong \mathcal{F}_{P(\mathcal{M})/P(F)}$ (as lattices) for any $F \in \mathcal{F}_M$.

*Proof.* Throughout let $F'_1, F'_2 \in \mathcal{F}_{M/F}$ and $V_i = \pi^{-1}(F'_i)$, where $\pi : E \to E/F$. By Proposition 2.10 and Lemma 3.2 $F'_i \in \mathcal{F}_{M/F} \iff V_i \in \mathcal{F}_M \iff P(V_i) \in \mathcal{F}_{P(\mathcal{M})} \iff P(V_i) - P(F) \in \mathcal{F}_{P(\mathcal{M})/P(F)}$. Furthermore, $F'_1 = F'_2 \iff V_1 = V_2 \iff P(V_1) - P(F) = P(V_2) - P(F)$. Hence there is a one-to-one correspondence between $\mathcal{F}_{M/F}$ and $\mathcal{F}_{P(\mathcal{M})/P(F)}$ described by the map $\psi : \mathcal{F}_{M/F} \to \mathcal{F}_{P(\mathcal{M})/P(F)}$, where $\psi(F'_i) = P(V_i) - P(F)$. Since the lattices of flats are finite, to show $\psi$ is a lattice isomorphism, we need only to show $\psi$ preserves meets. Recall that the meet of flats in either lattice is the intersection of the flats.

$$\psi(F'_1 \cap F'_2) = P(V_1 \cap V_2) - P(F)$$
$$= (P(V_1) \cap P(V_2)) - P(F)$$
$$= (P(V_1) - P(F)) \cap (P(V_2) - P(F))$$
$$= \psi(F'_1) \cap \psi(F'_2),$$

which completes the proof. 

The next few properties about projectivization matroids, although not difficult to prove, will be useful in following sections.

**Proposition 3.5.** Let $\mathcal{M} = (E, \rho)$ be a $q$-matroid and $P(\mathcal{M}) = (\mathbb{P}E, r)$ its projectivization matroid. Then $\mathcal{M}$ contains a loop if and only if $P(\mathcal{M})$ contains a loop.

*Proof.* Let $\langle e \rangle \leq E$ be a 1-dimensional subspace. By definition, $r(P(\langle e \rangle)) = \rho(\langle e \rangle))$. Hence $\langle e \rangle$ is a loop in $\mathcal{M}$ if and only if $P(\langle e \rangle)$ is a loop in $P(\mathcal{M})$. 

**Proposition 3.6.** Let $\mathcal{M} = (E, \rho)$ be a $q$-matroid and $P(\mathcal{M}) = (\mathbb{P}E, r)$ its projectivization matroid. Let $A \subseteq \mathbb{P}E$ be such that $A$ contains a basis of $E$. Then $r(A) = r(\mathbb{P}E)$.

*Proof.* Since $A$ contains a basis of $E$ then $\langle A \rangle = E$. Hence $r(A) = \rho(\langle A \rangle) = \rho(E) = r(\mathbb{P}E)$. 

We conclude the section by studying the relation between minors of a $q$-matroid and minors of its projectivization matroid. To do so we introduce the following notation.

**Notation 3.7.** Let $V \leq E$.

- $\mathcal{Q}_V := \{ \langle w \rangle \in \mathbb{P}E : \langle w \rangle \not\leq V \} = \mathbb{P}E - P(V)$. 

8
\begin{itemize}
\item \( Q^A_V := Q_V - A \) for \( A \subseteq Q_V \).
\end{itemize}

Note that \( \mathbb{P}E - Q_V = \mathbb{P}V \). Furthermore for spaces \( W, V \leq E \) such that \( W \oplus V = E \), every element in \( E/W \) can be written as \( v + W \) for a unique \( v \in V \). Thus the map \( \psi : E/W \rightarrow V, \ v + W \rightarrow v \) is a well-defined vector space isomorphism and induces a bijection on projective spaces. By slight abuse of notation we use \( \psi \) as both the vector space isomorphism and the projective space bijection. It can then easily be shown that \( \langle \psi(A) \rangle = \psi(\langle A \rangle) \) for all \( A \subseteq \mathbb{P}(E/W) \).

\begin{Theorem}
Let \( \mathcal{M} = (E, \rho) \) be a \( q \)-matroid and let \( W, V \leq E \) such that \( W \oplus V = E \). Let \( V^\perp \) be the orthogonal space of \( V \) w.r.t a fix NSBF. Furthermore let \( S = \{ \langle w_1 \rangle, \ldots, \langle w_t \rangle \} \subseteq \mathbb{P}E \) such that \( \{ w_1, \ldots, w_t \} \) is a basis of \( W \). Then
\[
\begin{align*}
P(\mathcal{M}/W) &\cong (P(\mathcal{M})/S) \setminus Q^S_V \\
P(\mathcal{M} \setminus V^\perp) &\cong P(\mathcal{M}) \setminus Q_V.
\end{align*}
\]
\end{Theorem}

\begin{proof}
Let \( N := P(\mathcal{M})/S \setminus Q^S_V \). Note \( N \) has groundset \( \mathbb{P}(E - Q_V) = \mathbb{P}V \) whereas \( P(\mathcal{M}/W) \) has groundset \( \mathbb{P}(E/W) \). Let \( \psi : \mathbb{P}(E/W) \rightarrow \mathbb{P}V \) be the bijection described previously. To show \( N \cong P(\mathcal{M}/W) \), we must show \( r_{P(\mathcal{M}/W)}(A) = r_N(\psi(A)) \) for all \( A \subseteq \mathbb{P}(E/W) \). Let \( \pi : E \rightarrow E/W \) be the canonical projection. Since \( S \) is a basis of \( W \) then \( \pi^{-1}(\langle A \rangle) = \langle \psi(A) \rangle + W = \langle \psi(A) \cup S \rangle \). Furthermore by Theorem 3.1 \( r_{P(\mathcal{M})}(S) = \rho(W) \). Hence we get:
\[
\begin{align*}
r_{P(\mathcal{M}/W)}(A) &= \rho_{\mathcal{M}/W}(\langle A \rangle) \\
&= \rho_{\mathcal{M}}(\pi^{-1}(\langle A \rangle)) - \rho_{\mathcal{M}}(W) \\
&= \rho_{\mathcal{M}}(\langle \psi(A) \cup S \rangle) - \rho_{\mathcal{M}}(\langle S \rangle) \\
&= r_{P(\mathcal{M})}(\psi(A) \cup S) - r_{P(\mathcal{M})}(S) \\
&= r_{P(\mathcal{M})/S}(\psi(A)) \\
&= r_N(\psi(A)),
\end{align*}
\]
where the last equality holds because \( \psi(A) \subseteq \mathbb{P}V \) which is the groundset of \( N \).

Moving on to the second equivalence. Both matroid \( P(\mathcal{M} \setminus V^\perp) \) and \( P(\mathcal{M}) \setminus Q_V \) have groundset \( \mathbb{P}V \). Hence we need only to show that the rank functions of both matroids are equal. Let \( A \subseteq \mathbb{P}V \).
\[
\begin{align*}
r_{P(\mathcal{M} \setminus V^\perp)}(A) &= \rho_{\mathcal{M} \setminus V^\perp}(\langle A \rangle) \\
&= \rho_{\mathcal{M}}(\langle A \rangle) \\
&= r_{P(\mathcal{M})}(A) \\
&= r_{P(\mathcal{M}) \setminus Q_V}(A),
\end{align*}
\]
where the last equality follows because \( A \subseteq \mathbb{P}V \) which is the groundset of \( P(\mathcal{M}) \setminus Q_V \).
\end{proof}
4 Maps of Matroids and \(q\)-Matroids

Maps between matroids were first introduced to study matroids from a category theory approach. The reader may refer to \([12, 17]\) for more details. Maps between \(q\)-matroids were defined for the same purposes, but were only recently introduced in \([8]\). In this section we focus on the relation between maps of \(q\)-matroids and maps of matroids, and show that the projectivization map is a functor from categories of \(q\)-matroids to categories of matroids. This, in turn, provides a new approach to study maps between \(q\)-matroids. For both matroids and \(q\)-matroids, there exists two main types of maps that preserve the matroid structure: weak and strong maps. To avoid confusion between maps of matroids and maps of \(q\)-matroids, we use the terms weak and strong maps for the former, and, \(q\)-weak and \(q\)-strong maps for the latter. We recall the definitions of weak and strong maps between matroids and some of their properties.

The reader should note that the loop extension (Definition 2.11) is needed to define maps between matroids. As the next definition will show a map between matroids is a map defined on the groundset of the loop extension matroids. By remark 2.13 the added loop can be seen as an element representing the empty set of the matroid. Hence, mapping an element to the added loop of the codomain can be seen as mapping an element to the empty set.

**Definition 4.1.** Let \(M = (S, r_M)\) and \(N = (T, r_N)\) be matroids and \(M_o, N_o\) be their respective loop extension matroids. A map \(\sigma: M \to N\) is a map between the groundset of the loop extension matroids, i.e. \(\sigma: S_o \to T_o\), such that \(\sigma(o_M) = o_N\). Furthermore \(\sigma\) is said to be:

- **weak** if \(r_{N_o}(\sigma(A)) \leq r_{M_o}(A)\) for all \(A \subseteq S_o\).
- **strong** if \(\sigma^{-1}(F) \in F_{M_o}\) for all \(F \in F_{N_o}\).

It is well known (see \([17\text{, Chap. 8, Lem. 8.1.7]}\)) that strong maps are weak maps. Furthermore a map \(\sigma: M \to N\) induces a map \(\sigma^\#: F_{M_o} \to F_{N_o}\), where \(\sigma^\#(F) = cl_{N_o}(\sigma(F))\) for all \(F \in F_{M_o}\). Using Proposition 2.12 one can alternatively define \(\sigma^\#: F_M \to F_N\). As the following theorem shows, the induced map \(\sigma^\#\) provides an alternative definition for strong maps.

**Theorem 4.2.** (\([17\text{, Prop 8.1.3]}\))

A map \(\sigma: M \to N\) is a strong map if and only if the following hold:

1. For all \(F_1, F_2 \in F_M\),
   \[
   \sigma^\#(F_1 \lor F_2) = \sigma^\#(F_1) \lor \sigma^\#(F_2)
   \]
2. \(\sigma^\#\) sends atoms of \(F_M\) to atoms or to the zero of \(F_N\).

The main result of this section is the analogue of Theorem 4.2 for \(q\)-matroids. We turn to the definitions of maps between \(q\)-matroids, as introduced in \([8]\). Similarly to matroids, maps between \(q\)-matroids are maps between groundspaces that send subspaces to subspaces.

**Definition 4.3.** A map \(\sigma: E_1 \to E_2\) is an \(L\)-map if \(\sigma(V) \in \mathcal{L}(E_2)\) for all \(V \in \mathcal{L}(E_1)\). An \(L\)-map \(\sigma\) induces a map \(\sigma_L: \mathcal{L}(E_1) \to \mathcal{L}(E_2)\). Two \(L\)-maps \(\sigma, \psi\) from \(E_1\) to \(E_2\) are \(L\)-equivalent, denoted by \(\sigma \sim_L \psi\), if \(\sigma_L = \psi_L\).
We consider the following two types of maps between \( q \)-matroids.

**Definition 4.4.** Let \( \mathcal{M} = (E_1, \rho_\mathcal{M}) \) and \( \mathcal{N} = (E_2, \rho_\mathcal{N}) \) be \( q \)-matroids. A map \( \sigma : \mathcal{M} \to \mathcal{N} \) is an \( \mathcal{L} \)-map between the groundspaces of \( \mathcal{M} \) and \( \mathcal{N} \), i.e \( \sigma : E_1 \to E_2 \). Furthermore \( \sigma \) is said to be

(a) \( q \)-weak if \( \rho_\mathcal{N}(\sigma(V)) \leq \rho_\mathcal{M}(V) \) for all \( V \leq E_1 \).

(b) \( q \)-strong if \( \sigma^{-1}(F) \in \mathcal{F}_\mathcal{M} \) for all \( F \in \mathcal{F}_\mathcal{N} \).

To study the relation between maps of matroids and maps of \( q \)-matroids we need the following notation. Given a vector space \( E \), define the *extended projective space* of \( E \) as \( \mathbb{P}_oE = \mathbb{P}E \cup \{o\} \), where \( \{o\} \) is an arbitrary element disjoint from \( \mathbb{P}E \). Let \( P_o : E \to \mathbb{P}_oE \), where \( P_o(0) = o \) and \( P_o(v) = \hat{P}(v) \) for \( v \neq 0 \) and \( \hat{P} : E - \{0\} \to \mathbb{P}E \) is as introduced in the previous section. We call \( P_o \) the *extended projectivization map*. Note, unlike the projectivization map, we do not consider \( P_o \) as lattice map but as a map between a vector space to its extended projective space. Given a \( q \)-matroid \( \mathcal{M} = (E, \rho) \) and the loop extension of its projectivization matroid \( P(\mathcal{M})_o = (\mathbb{P}_oE, r_o) \), the map \( P_o \) can be viewed as a map between the groundspace \( E \) to the groundset \( \mathbb{P}_oE \) such that \( \rho(V) = r_o(P_o(V)) \) for all \( V \leq E \). Furthermore for any \( A \subseteq \mathbb{P}_oE \) let \( \langle A \rangle := (P_o^{-1}(A))_{\hat{P}_q} \). It can easily be shown that \( r_o(A) = \rho(\langle A \rangle) \).

Recall from Definition 4.3 that an \( \mathcal{L} \)-map \( \sigma : E_1 \to E_2 \) induces a map on the lattices of subspaces \( \sigma_\mathcal{L} : \mathcal{L}(E_1) \to \mathcal{L}(E_2) \). By restricting \( \sigma_\mathcal{L} \) to the 1-dimensional spaces and the 0 of \( E_1 \), \( \sigma_\mathcal{L} \) can be viewed as map between the extended projective spaces \( \mathbb{P}_oE_1 \) and \( \mathbb{P}_oE_2 \), i.e \( \sigma_\mathcal{L} : \mathbb{P}_oE_1 \to \mathbb{P}_oE_2 \). As the next proposition shows, \( \sigma \) and \( \sigma_\mathcal{L} \) commute with the extended projectivization map.

**Proposition 4.5.** Let \( \sigma : E_1 \to E_2 \) be an \( \mathcal{L} \)-map, \( \sigma_\mathcal{L} : \mathbb{P}_oE_1 \to \mathbb{P}_oE_2 \) its induced map on the extended projective spaces, and \( P_o : E_i \to \mathbb{P}_oE_i \) be the extended projectivization map. Then

\[
P_o \circ \sigma = \sigma_\mathcal{L} \circ P_o.
\]

**Proof.** Let \( v \in E_1 \). Since \( \sigma \) is an \( \mathcal{L} \)-map then \( \langle \sigma(v) \rangle = \sigma(\langle v \rangle) = \sigma_\mathcal{L}(\langle v \rangle) \). But note \( \langle \sigma(v) \rangle = P_o(\sigma(v)) \) and \( \langle v \rangle = P_o(v) \). Hence the wanted equality follows. \( \square \)

We now consider the case when an \( \mathcal{L} \)-map \( \sigma \) is a map between \( q \)-matroids. The induced map \( \sigma_\mathcal{L} \) between the extended projective spaces turns out to be a map between projectivization matroids. Furthermore \( \sigma \) being \( q \)-weak or \( q \)-strong is fully determined by whether \( \sigma_\mathcal{L} \) is weak or strong, and vice versa.

**Theorem 4.6.** Let \( \mathcal{M} = (E_1, \rho_\mathcal{M}) \), \( \mathcal{N} = (E_2, \rho_\mathcal{N}) \) be \( q \)-matroids and \( P(\mathcal{M}) = (\mathbb{P}E_1, r_{P(\mathcal{M})}) \), \( P(\mathcal{N}) = (\mathbb{P}E_2, r_{P(\mathcal{N})}) \) be their projectivization matroids. Let \( \sigma : \mathcal{M} \to \mathcal{N} \) be an \( \mathcal{L} \)-map. Then \( \sigma_\mathcal{L} : P(\mathcal{M}) \to P(\mathcal{N}) \) is a map between the projectivization matroids and the following holds:

- \( \sigma \) is \( q \)-weak if and only if \( \sigma_\mathcal{L} \) is weak
- \( \sigma \) is \( q \)-strong if and only if \( \sigma_\mathcal{L} \) is strong.

**Proof.** To start note that \( \sigma_\mathcal{L} \) is a map between the groundsets of the loop extension matroid \( P(\mathcal{M})_o \) and \( P(\mathcal{N})_o \) with \( \sigma_\mathcal{L}(o_{P(\mathcal{M})}) = o_{P(\mathcal{N})} \). Thus \( \sigma_\mathcal{L} : P(\mathcal{M}) \to P(\mathcal{N}) \) is well defined.
We first prove \( \sigma \) is \( q \)-weak if and only if \( \sigma_L \) is weak. Assume \( \sigma \) is weak. Let \( A \subseteq P_o E_1 \) and \( V := (A) \). Both \( P_o \) and \( \sigma \) are inclusion preserving, hence, \( (P_o \circ \sigma)(P_o^{-1}(A)) \subseteq (P_o \circ \sigma)(V) \). Using Proposition 4.5 on the first term of the previous inclusion gives us \( (\sigma_L \circ P_o)(P_o^{-1}(A)) = \sigma_L(A) \subseteq (P_o \circ \sigma)(V) \). Furthermore, by the monotonicity property of the rank functions and because \( \sigma \) is weak, we get

\[
r_{P(N)_o}(\sigma_L(A)) \leq r_{P(N)_o}((P_o \circ \sigma)(V)) = \rho_N(\sigma(V)) \leq \rho_M(V) = r_{P(M)_o}(A).
\]

Because \( A \subseteq P_o E_1 \) was arbitrarily chosen, then \( \sigma_L \) is weak.

Now assume \( \sigma_L \) is weak. Let \( V \leq E_1 \) and recall \( \rho_M(V) = r_{P(M)_o}(P_o(V)) \). Since \( \sigma_L \) is weak \( r_{P(N)_o}((\sigma_L \circ P_o)(V)) \leq r_{P(M)_o}(P_o(V)) \). Hence by Proposition 4.5, \( r_{P(N)_o}((P_o \circ \sigma)(V)) \leq r_{P(M)_o}(P_o(V)) \). This implies \( \rho_N(\sigma(V)) \leq \rho_M(V) \) and shows \( \sigma \) is \( q \)-weak.

We now show that \( \sigma \) is \( q \)-strong if and only if \( \sigma_L \) is strong. From Proposition 2.12 and Lemma 3.2, \( F \in \mathcal{F}_M \iff P(F) \in \mathcal{F}_{P(M)} \iff P(F) \cup \{o\} \in \mathcal{F}_{P(M)_o} \). A similar chain of equivalence holds for \( \mathcal{F}_N \) and \( \mathcal{F}_{P(N)_o} \). Furthermore all flats of \( P(N)_o \) are of the form \( P_o(F) = P(F) \cup \{o\} \) for some flat in \( N \).

Therefore \( \sigma_L \) is strong if and only if \( \sigma_L^{-1}(P_o(F)) \in \mathcal{F}_{P(M)} \) for all \( P_o(F) \in \mathcal{F}_{P(N)_o} \) iff \( \sigma_L \circ P_o \) is weak. Hence, \( \sigma_L \circ P_o \) is weak. Let \( F \in \mathcal{F}_N \) and \( P_o \circ \sigma^{-1}(P_o(F)) = \sigma^{-1}(F) \in \mathcal{F}_M \) for all \( F \in \mathcal{F}_N \) iff \( \sigma \) is \( q \)-strong.

\( \square \)

From the above theorem it can easily be seen that the projectivization map is a functor from the category of \( q \)-matroids with \( q \)-weak (resp. \( q \)-strong) maps to the category of matroids with weak (resp. strong) maps. We now turn towards a proof of the analogue of Theorem 4.2. We first define the analogue of the map \( \sigma^\# \).

**Definition 4.7.** Let \( M \) and \( N \) be \( q \)-matroids with respective groundspaces \( E_1, E_2 \) and \( \sigma : M \to N \) be an \( L \)-map. Define \( \sigma^\# : \mathcal{F}_M \to \mathcal{F}_N \) such that

\[
\sigma^\#(F) = cl_N(\sigma(F)).
\]

The next useful Lemma shows that the induced maps \( \sigma^\# \) and \( \sigma^#_L \) commute with the extended projectivization map.

**Lemma 4.8.** Let \( M, N \) be \( q \)-matroids, \( \mathcal{F}_M, \mathcal{F}_N \) their lattices of flats and \( P(M), P(N) \) their projectivization matroids. Furthermore let \( \sigma : M \to N \) be an \( L \)-map, \( \sigma_L : P(M) \to P(N) \) its induced map and \( P_o : E_i \to \mathbb{P}_o E_i \) the extended projectivization map. Then

\[
P_o \circ \sigma^\# = \sigma^#_L \circ P_o
\]

**Proof.** First recall, \( F \in \mathcal{F}_M \iff P_o(F) \in \mathcal{F}_{P(M)_o} \). Let \( F \in \mathcal{F}_M \), then \( \sigma(F) \subseteq \sigma^\#(F) \) and since \( P_o \) is inclusion preserving \( (P_o \circ \sigma)(F) \subseteq (P_o \circ \sigma^\#)(F) \). By Proposition 4.5, the above containment implies \( (\sigma_L \circ P_o)(F) \subseteq (P_o \circ \sigma^\#)(F) \). Applying the closure operator of \( P(N)_o \), we get

\[
(\sigma^#_L \circ P_o)(F) = cl_{P(N)_o}((\sigma_L \circ P_o)(F)) \subseteq cl_{P(N)_o}((P_o \circ \sigma^\#)(F)) = (P_o \circ \sigma^\#)(F),
\]

where the final equality holds because \( \sigma^\#(F) \in \mathcal{F}_N \) and therefore \( (P_o \circ \sigma^\#)(F) \in \mathcal{F}_{P(N)_o} \). Assume,
We now state and show the analogue of Theorem 4.2 for $q$-matroids. By considering their preimage under $P_o$ and because $\sigma_L \circ P_o = P_o \circ \sigma$, we get

$$
(\sigma_L \circ P_o)(F) \subseteq F' \subseteq (P_o \circ \sigma^#)(F).
$$

By considering their preimage under $P_o$ and because $\sigma_L \circ P_o = P_o \circ \sigma$, we get

$$
\sigma(F) \subseteq P_o^{-1}(F') \subsetneq \sigma^#(F).
$$

However since $F' \in \mathcal{F}_{P(N)}$, then $P_o^{-1}(F') \in \mathcal{F}_N$. Therefore $P_o^{-1}(F')$ must contain $\text{cl}_N(\sigma(F)) = \sigma^#(F)$, a contradiction. Hence

$$
(\sigma_L^# \circ P_o)(F) = (P_o \circ \sigma^#)(F)
$$

$\square$

In the statement of the previous Lemma, one can replace the extended projectivization map $P_o$ by the projectivization map $P$ introduced in the previous section. In fact, as previously remarked, the map $\sigma_L^#$ can be considered as map between the lattices of flats $\mathcal{F}_{P(M)}$ to $\mathcal{F}_{P(N)}$. Furthermore the projectivization map can also be restricted to a map between the lattice of flats of a $q$-matroid and its projectivization matroid. In the following Lemma, $P$ refers to the projectivization map restricted to the lattice of flats $\mathcal{F}_M$ and $\mathcal{F}_N$.

**Lemma 4.9.** Let the data be as in Lemma 4.8 and let $P$ be the projectivization map on the lattices of flats $\mathcal{F}_M$ and $\mathcal{F}_N$. Then

$$
P \circ \sigma^# = \sigma_L^# \circ P.
$$

**Proof.** Recall $\mathcal{F}_{P(M)} = \{F' \setminus \{o\} : F' \in P(M)\setminus \{o\} = \{P_o(F) \setminus \{o\} : F \in \mathcal{F}_M\} = \{P(F) : F \in \mathcal{F}_M\}$ and that the same holds for $P(N)$. From the above chain of equality and Lemma 4.8 equality follows straightforwardly. $\square$

We now state and show the analogue of Theorem 4.2 for $q$-strong maps.

**Theorem 4.10.** Let $\mathcal{M}$, $\mathcal{N}$ be $q$-matroids. An $\mathcal{L}$-map $\sigma : \mathcal{M} \to \mathcal{N}$ is a $q$-strong map if and only if the following holds:

1. for all $F_1, F_2 \in \mathcal{F}_M$,

$$
\sigma^#(F_1 \lor F_2) = \sigma^#(F_1) \lor \sigma^#(F_2)
$$

2. $\sigma^#$ sends of $\mathcal{F}_M$ atoms to atoms or to the zero of $\mathcal{F}_N$.

**Proof.** $(\Rightarrow)$ Let $\sigma : \mathcal{M} \to \mathcal{N}$ be a $q$-strong map, which implies by Theorem 4.6 that $\sigma_L : P(M) \to P(N)$ is a strong map. By Lemma 3.2(1), $F \in \mathcal{F}_M \iff P(F) \in \mathcal{F}_{P(M)}$. Furthermore, from Theorem 4.2 we obtain $\sigma_L^#(P(F_1) \lor P(F_2)) = \sigma_L^#(P(F_1)) \lor \sigma_L^#(P(F_2))$ for all $F_1, F_2 \in \mathcal{F}_{P(M)}$. By Lemma 3.2(2), $P(F_1) \lor P(F_2) = P(F_1 \lor F_2)$, hence $\sigma_L^#(P(F_1 \lor F_2)) = \sigma_L^#(P(F_1)) \lor \sigma_L^#(P(F_2))$. Applying
Lemma 4.9 on the above equalities gives us

\[(P \circ \sigma^#)(F_1 \lor F_2) = (P \circ \sigma^#)(F_1) \lor (P \circ \sigma^#)(F_2) = P(\sigma^#(F_1) \lor \sigma^#(F_2)).\]

Finally since \(P\) is an isomorphism on the lattice of flat, the above equality implies

\[\sigma^#(F_1 \lor F_2) = \sigma^#(F_1) \lor \sigma^#(F_2),\]

which shows \(\sigma\) satisfies property (1) for all \(F_1, F_2 \in \mathcal{F}_M\).

To show \(\sigma\) satisfies property (2), let \(F \in \mathcal{F}_M\) be an atom. Since \(P\) is a lattice isomorphism then \(P(F)\) is an atom of \(\mathcal{F}_{P(M)}\). Moreover \(\sigma_L\) is a strong map, hence by Theorem 4.2 \((\sigma_L^# \circ P)(F)\) must be an atom or the zero of \(\mathcal{F}_{P(N)}\). But by Lemma 4.9 \((\sigma_L^# \circ P)(F) = (P \circ \sigma^#)(F)\), which implies \(\sigma^#(F)\) must be an atom or the zero of \(\mathcal{F}_N\) because, once again, \(P\) is a lattice isomorphism. This concludes that \(\sigma^#\) satisfies the wanting properties.

\(\Leftrightarrow\) Let \(\sigma^#\) satisfy properties (1) and (2). We show that \(\sigma\) is a \(q\)-strong map by showing that \(\sigma_L\) is a strong map. To do so we show that \(\sigma_L^#\) satisfies Proposition 4.2.

Let \(P(F_1), P(F_2) \in \mathcal{F}_{P(M)}\). By Lemma 3.2 \((\sigma_L^#(P(F_1) \lor P(F_2))) = \sigma_L^#(P(F_1 \lor F_2))\). Using Lemma 4.9 and the fact that \(\sigma_L^#\) satisfies property (1), we get

\[(\sigma_L^# \circ P)(F_1 \lor F_2) = (P \circ \sigma^#)(F_1 \lor F_2)
= P(\sigma^#(F_1) \lor \sigma^#(F_2))
= (P \circ \sigma^#)(F_1) \lor (P \circ \sigma^#)(F_2)
= (\sigma_L^# \circ P)(F_1) \lor (\sigma_L^# \circ P)(F_2),\]

where the second to last equality follows from Lemma 3.2 (2). Hence \(\sigma_L^#\) satisfies property (1) of Prop 4.2.

Let \(P(F) \in \mathcal{F}_{P(M)}\) be an atom which, since \(P\) is a lattice isomorphism, implies \(F\) is an atom of \(\mathcal{F}_M\). Once again we use \((\sigma_L^# \circ P)(F) = (P \circ \sigma^#)(F)\). Because \(\sigma^#\) satisfies property (2) then \(\sigma^#(F)\) is an atom or the zero of \(\mathcal{F}_N\). Finally since \(P\) is a lattice isomorphism between \(\mathcal{F}_N\) and \(\mathcal{F}_{P(N)}\) then \(P(\sigma^#(F)) = \sigma_L^#(P(F))\) must be an atom or the zero of \(\mathcal{F}_{P(N)}\) which show \(\sigma_L^#\) satisfies property (2) of Proposition 4.2. Therefore \(\sigma_L\) is a strong map and by Theorem 4.6 we get that \(\sigma\) is \(q\)-strong, concluding the proof.

We conclude the section by showing that \(q\)-strong maps are \(q\)-weak maps.

**Corollary 4.11.** Let \(M = (E_1, \rho_M), N = (E_2, \rho_N)\) be \(q\)-matroids and \(\sigma : M \to N\) be a \(q\)-strong map. Then \(\sigma\) is a \(q\)-weak map.

**Proof.** Let \(V \subseteq E_1\), such that \(\rho_M(V) = s\) and let \(F := \text{cl}_M(V)\). Note that \(V \subseteq F\) and \(\rho_M(F) = \rho_M(V) = h(F)\), where \(h\) is the height function of the lattice of flats as in Proposition 2.4. Because \(\mathcal{F}_M\) is a geometric lattice, and \(F\) has height \(s\), then \(F = \bigvee_{i=1}^s a_i\) where \(a_i\) are atoms of \(\mathcal{F}_M\). Since \(\sigma\) is a \(q\)-strong map, by Theorem 4.10 we get, \(\sigma^#(F) = \sigma^#(\bigvee_{i=1}^s a_i) = \bigvee_{i=1}^s \sigma^#(a_i)\). Because \(a_i\) are atoms of \(\mathcal{F}_M\), by Theorem 4.10 \(\sigma^#(a_i)\) must be an atom or the zero of \(\mathcal{F}_N\) and that for all \(1 \leq i \leq s\).
Hence $\sigma^\#(F)$ is be the join of at most $s$-atoms, which implies $\rho_N(\sigma^\#(F)) = h(\sigma^\#(F)) \leq s$. Finally, since $V \leq F$ and $\sigma$ is an $L$-map, then $\sigma(V) \leq \sigma(F) \leq \sigma^\#(F)$, where the last containment follows from the definition of $\sigma^\#$. Therefore by the monotonicity property of $\rho_N$ we get $\rho_N(\sigma(V)) \leq \rho_N(\sigma^\#(F)) \leq s = \rho_M(V)$. Since this holds true for all $V \leq E_1$, $\sigma$ must be $q$-weak.

5 The Characteristic Polynomial

The characteristic polynomial is a useful invariant for both matroids and $q$-matroids. For the former it was intensively studied over the years, see for example [16, 19]. The latter was more recently introduced for $q$-polymatroids [5], and was used to establish a weaker version of the Assmus-Mattson Theorem. However, in this paper, we are only interested in the characteristic polynomial of $q$-matroids.

Before defining the characteristic polynomial, we recall the definition of the Möbius function which will often be used throughout the section.

**Definition 5.1.** Let $(P, \leq)$ be a finite partially ordered set. The Möbius function for $P$ is defined via the recursive formula

$$
\mu_P(x, y) := \begin{cases} 
1 & \text{if } x = y, \\
- \sum_{x \leq z \leq y} \mu_P(x, z) & \text{if } x < y, \\
0 & \text{otherwise.}
\end{cases}
$$

We use the subscript of $\mu$ to distinguish between the Möbius functions of different posets. If the underlying poset is clear, the subscript may be omitted. We now define the characteristic polynomial of a matroid.

**Definition 5.2.** Let $M = (S, r)$ be a matroid. The characteristic polynomial of $M$ is defined as follow:

$$
\chi_M(x) = \sum_{A \subseteq S} (-1)^{|A|} x^{r(S) - r(A)}.
$$

It is well known that if a matroid $M$ contains a loop, its characteristic polynomial is identically 0. On the other hand if $M$ is loopless, then the characteristic polynomial of $M$ is fully determined by the lattice of flats. Furthermore, one can recursively define the characteristic polynomial of a matroid in terms of the characteristic polynomial of its minors. We summarize this in the following theorem. Proofs can be found in [16, Sec.3] and [19, Sec.7.1].

**Theorem 5.3.** Let $M = (S, r)$ be a matroid and $F$ be its lattice of flats. If $M$ contains a loop then $\chi_M(x) = 0$. If $M$ has no loops, then

$$
\chi_M(x) = \sum_{F \in F} \mu_F(0, F) x^{r(S) - r(F)}.
$$

Furthermore for $e \in S$,

$$
\chi_M(x) = \begin{cases} 
\chi_M\setminus e(x)\chi_M/e(x) & \text{if } e \text{ is a coloop,} \\
\chi_M\setminus e(x) - \chi_M/e(x) & \text{otherwise.}
\end{cases}
$$
Similarly to matroids, the characteristic polynomial of a $q$-matroid $\mathcal{M}$ is identically 0 if $\mathcal{M}$ contains a loop, and is fully determined by the lattice of flats otherwise. This was in fact shown by Whittle in [18], where the author generalized the result to any weighted lattice endowed with a closure operator. However for self containment purposes proofs of those facts will be included in this paper. Furthermore, we use the projectivization matroid to find a recursive formula for the characteristic polynomial of $q$-matroids. The characteristic polynomial of a $q$-matroid is defined in the following way.

**Definition 5.4.** [5, Def. 22] Let $\mathcal{M} = (E, \rho)$ be a $q$-matroid and $\mathcal{L}(E)$ be the subspace lattice of $E$. The characteristic polynomial is defined as

$$\chi_\mathcal{M}(x) := \sum_{V \leq E} \mu_{\mathcal{L}(E)}(0, V)x^{\rho(E) - \rho(V)}.$$

We state a few straightforward lemmas that will be useful later on.

**Lemma 5.5.** Let $\mathcal{M} = (E, \rho)$ be a $q$-matroid, $\mathcal{F}_\mathcal{M}$ its lattice of flats, and $L := \{e \in E \mid \rho(\langle e \rangle) = 0\}$. Then $L$ is a subspace and $\rho(L) = 0$. It is called the the subspace of loops of $\mathcal{M}$. Furthermore $L \leq F$ for all $F \in \mathcal{F}_\mathcal{M}$.

**Proof.** The first statement was proven in [14, Lemma 11]. For the second statement let $V \leq E$. By the monotonicity and submodularity properties of the rank function, $\rho(V) \leq \rho(V + L) \leq \rho(V) + \rho(L) - \rho(V \cap L) \leq \rho(V)$. Hence equality holds throughout and $L \leq \text{cl}_\mathcal{M}(V)$. Since this is true for all $V \leq E$ then $L \leq F$ for all flats $F$ of $\mathcal{M}$. \qed

**Lemma 5.6.** Let $\mathcal{M} = (E, \rho)$ be a $q$-matroid, $\mathcal{F}$ its lattice of flats and $\mathcal{L}$ the subspace lattice of $E$. Then

$$\chi_\mathcal{M}(x) = \sum_{F \in \mathcal{F}} \sum_{V : \text{cl}_\mathcal{M}(V) = F} \mu_{\mathcal{L}}(0, V)x^{\rho(E) - \rho(F)}.$$

**Proof.** Let $V \leq E$, then $\text{cl}_\mathcal{M}(V) \in \mathcal{F}$ and $\rho(V) = \rho(\text{cl}_\mathcal{M}(V))$. Hence we get

$$\chi_\mathcal{M}(x) = \sum_{V \leq E} \mu_{\mathcal{L}}(0, V)x^{\rho(E) - \rho(V)}$$

$$= \sum_{F \in \mathcal{F}} \sum_{V : \text{cl}_\mathcal{M}(V) = F} \mu_{\mathcal{L}}(0, V)x^{\rho(E) - \rho(F)}. \quad \Box$$

We can now show that if a $q$-matroid contains a loop its characteristic polynomial is identically 0, whereas if it is loopless then the characteristic polynomial is determined by the lattice of flats.

**Theorem 5.7.** [see also [18, Prop. 3.4]] Let $\mathcal{M}$ be a $q$-matroid such that $\mathcal{M}$ contains a loop then $\chi_\mathcal{M}(x) = 0$.

**Proof.** From Lemma 5.6 we know

$$\chi_\mathcal{M}(x) = \sum_{F \in \mathcal{F}} \sum_{V : \text{cl}(V) = F} \mu_{\mathcal{L}}(0, V)x^{\rho(E) - \rho(F)}.$$
We show that for all flat $F \in \mathcal{F}$,
\[
\sum_{V : c(V) = F} \mu(L)(0, V) = 0. \quad (5.1)
\]

We proceed by induction on the rank value of flats. Let $F \in \mathcal{F}_M$ such that $\rho(F) = 0$, i.e. $F = cL_M(0)$. Since $\mathcal{M}$ contains a loop, $\{0\} \subseteq F$. Hence by Definition 5.1
\[
\sum_{V : c(V) = F} \mu(L)(0, V) = \sum_{0 \leq V \leq F} \mu(L)(0, V) = 0.
\]

Assume (5.1) holds for all $F \in \mathcal{F}$ such that $\rho(F) \leq k - 1$. Fix $F \in \mathcal{F}$ such that $\rho(F) = k$. Then
\[
0 = \sum_{V \leq F} \mu(L)(0, V)
= \sum_{V : c(V) = F} \mu(L)(0, V) + \sum_{F' \subseteq F} \sum_{V : c(V) = F'} \mu(L)(0, V)
= \sum_{V : c(V) = F} \mu(L)(0, V),
\]
where the last equality follows by induction hypothesis. Therefore (5.1) holds, and $\chi_{\mathcal{M}}(x) = 0$. \qed

**Theorem 5.8.** [see also [18, Thm. 3.2]] Let $\mathcal{M} = (E, \rho)$ be a loopless $q$-matroid and $\mathcal{F}$ its lattice of flats. Then
\[
\chi_{\mathcal{M}}(x) = \sum_{F \in \mathcal{F}} \mu(L)(0, F)x^{\rho(E) - \rho(F)}.
\]

**Proof.** Let $\mathcal{L} = \mathcal{L}(E)$. By Lemma 5.6 we know
\[
\chi_{\mathcal{M}}(x) = \sum_{F \in \mathcal{F}} \sum_{V : c(V) = F} \mu(L)(0, V)x^{\rho(E) - \rho(F)}.
\]

Hence, for all $F \in \mathcal{F}$ we must show
\[
\sum_{V : c(V) = F} \mu(L)(0, V) = \mu(L)(0, F). \quad (5.2)
\]

We once again proceed by induction on the rank of the flats of $\mathcal{M}$. Since $\mathcal{M}$ is loopless, $\{0\} \in \mathcal{F}$. Let $F = \{0\}$, then (5.2) follows trivially from Definition 5.1. Now assume (5.2) holds true for all $F \in \mathcal{F}$ such that $\rho(F) \leq k - 1$. Fix a flat $F \in \mathcal{F}$ with $\rho(F) = k$. Then
\[
\mu_{\mathcal{F}}(0, F) = - \sum_{F' \leq F, F' \in \mathcal{F}} \mu_{\mathcal{F}}(0, F') \\
= - \sum_{F' \leq F, F' \in \mathcal{F}} \sum_{V : d(V) = F'} \mu_{\mathcal{L}}(0, V) \\
= - \sum_{V : d(V) \leq F} \mu_{\mathcal{L}}(0, V) \\
= \sum_{V : d(V) = F} \mu_{\mathcal{L}}(0, V),
\]

where the second equality follows from the induction hypothesis, and the last equality follows from Definition 5.1. This completes the proof.

As the next theorem shows, defining the characteristic polynomial in terms of the lattice of flats of the \(q\)-matroid allows us to link the characteristic polynomial of a \(q\)-matroid with that of its projectivization matroid.

**Theorem 5.9.** Let \(\mathcal{M}\) be a \(q\)-matroid and \(P(\mathcal{M})\) be its projectivization matroid. Then

\[\chi_{\mathcal{M}}(x) = \chi_{P(\mathcal{M})}(x).\]

**Proof.** By Proposition 3.5, \(\mathcal{M}\) contains a loop if and only if \(P(\mathcal{M})\) contains a loop. Furthermore, by Theorem 3.2, we know \(\mathcal{F}_{\mathcal{M}} \cong \mathcal{F}_{P(\mathcal{M})}\) as lattices. Due to Propositions 2.2 and 2.4, \(\rho_{\mathcal{M}}(F) = h_{\mathcal{F}_{\mathcal{M}}}(F) = h_{\mathcal{F}_{P(\mathcal{M})}}(P(F)) = r_{P(\mathcal{M})}(P(F))\) and this for all flats \(F \in \mathcal{F}_{\mathcal{M}}\) and \(P(F) \in \mathcal{F}_{P(\mathcal{M})}\). Therefore, the result follows directly from Theorems 5.3, 5.7 and 5.8.

We furthermore get the following result when considering the contraction of \(\mathcal{M}\) by a subspace \(V \leq E\).

**Proposition 5.10.** Let \(\mathcal{M} = (E, \rho)\) be a \(q\)-matroid and \(P(\mathcal{M}) = (P(E), r)\) its projectivization matroid. Then for all \(V \leq E\),

\[\chi_{\mathcal{M}/V}(x) = \chi_{P(\mathcal{M})/P(V)}(x).\]

**Proof.** Let \(V \leq E\), then \(V \in \mathcal{F}_{\mathcal{M}} \iff P(V) \in \mathcal{F}_{P(\mathcal{M})}\). If \(V \notin \mathcal{F}_{\mathcal{M}}\) then by Proposition 2.10, \(\mathcal{M}/V\) and \(P(\mathcal{M})/P(V)\) contain loops, therefore \(\chi_{\mathcal{M}/V}(x) = 0 = \chi_{P(\mathcal{M})/P(V)}(x)\). If \(V \in \mathcal{F}_{\mathcal{M}}\), by Theorem 3.4, \(\mathcal{F}_{\mathcal{M}/V} \cong \mathcal{F}_{P(\mathcal{M})/P(V)}\) as lattices and, by Proposition 2.10, both matroids are loopless. Hence, Theorems 5.3 and 5.8 imply, \(\chi_{\mathcal{M}/V}(x) = \chi_{P(\mathcal{M})/P(V)}(x)\).

The close connection between the characteristic polynomial of a \(q\)-matroid \(\mathcal{M}\) and that of its projectivization matroid gives a new approach to study the former. In fact, we use this approach to find a recursive formula for the characteristic polynomial of a \(q\)-matroid in terms of the characteristic polynomial of its minors. Because we will be using the recursive formula defined in Theorem 5.3 on the projectivization matroid \(P(\mathcal{M})\) and its minors, we need to pay a particular attention on whether \(P(\mathcal{M})\) and its minors contain coloops. We thus need the following few lemmas.

**Lemma 5.11.** Let \(\mathcal{M} = (S, r)\) be a matroid and \(\mathcal{M} = (E, \rho)\) be a \(q\)-matroid. Then
(a) $w \in S$ is a coloop of $M$ $\iff$ $r^*(w) = 0 \iff r(S - w) = r(S) - 1$

(b) $\langle w \rangle \subseteq E$ is a coloop of $M$ $\iff$ $\rho^*(\langle w \rangle) = 0 \iff \rho(\langle w \rangle^\perp) = \rho(E) - 1$

Proof. Statement (a) can be found in [13, Section 1.6, Exercise 6]. Statement (b) follows from Definition 2.6.

Recall the notation $Q_V = \{\langle w \rangle : \langle w \rangle \not\subseteq V\}$, and $Q^e_v = Q_V - \{\langle e \rangle\}$ for $\langle e \rangle \in Q_V$ introduced in Notation 3.7. To make the results and proofs easier to read, we may omit the brackets to denote 1-dimensional spaces and we let $v^\perp := \langle v \rangle^\perp$ for $v \in E$.

Lemma 5.12. Let $M = (E, \rho)$ be a $q$-matroid, $P(M) = (\mathbb{P}E, r)$ its projectivization matroid and $e, v \in \mathbb{P}E$ such that $\langle e \rangle \oplus \langle v \rangle^\perp = E$. Let $A \subseteq Q^e_v$, then:

(a) for all $w \in Q^e_v \setminus A$, the element $w$ is not a coloop of the matroid $P(M) \setminus A$.

(b) for all $w \in Q^e_v$ and $z \in Q^e_w - (A \cup w)$, the element $z$ is not a coloop of $P(M) \setminus A/w$.

(c) for all $w_1, w_2 \in Q^e_v - A$, the matroid $P(M) \setminus A/\{w_1, w_2\}$ contains a loop.

(d) $e$ is a coloop of the matroid $P(M) \setminus Q^e_v$ if and only if $\langle v \rangle$ is a coloop of $M$.

Proof. Throughout, let $Q := Q^e_v$, $H := v^\perp$ and $\{h_1, \ldots, h_{n-1}\}$ be a basis of $H$. Furthermore let $A \subseteq Q$, $w \in Q - A$ and consider the matroid $N := P(M) \setminus A = (\mathbb{P}E - A, r_N)$. Note that $r(S) = r_N(S)$ for all $S \subseteq \mathbb{P}E - A$.

Statement (a). Since $e \not\in H$ and $\dim H = \dim E - 1$, the set $B := \{h_1, \ldots, h_{n-1}, e\}$ is a basis of $E$, and $B \subseteq \mathbb{P}E - Q \subseteq \mathbb{P}E - (A \cup w) \subseteq \mathbb{P}E - A$. Hence by Proposition 3.6 $r(\mathbb{P}E - Q) = r(\mathbb{P}E - (A \cup w)) = r(\mathbb{P}E - A) = r(\mathbb{P}E)$. Because $r_N(S) = r(S)$ for all $S \subseteq \mathbb{P}E - A$, we have $r_N(\mathbb{P}E - (A \cup w)) = r_N(\mathbb{P}E - A)$ so, by Lemma 5.11 $w$ is not a coloop of $N$, proving statement (a).

Statement (b). Let $B := \{h_1, \ldots, h_{n-1}, w\}$ which is a basis of $E$. We need only to show $r^*_N/\{w\}(z) \neq 0$ for all $z \in Q^e_w - (A \cup w)$.

\[
r^*_N/\{w\}(z) = |z| + r_N/\{w\}(\mathbb{P}E - (A \cup w \cup z)) - r_N/\{w\}(\mathbb{P}E - (A \cup w))
\]
\[
= 1 + r_N((\mathbb{P}E - (A \cup w \cup z)) \cup w) - r_N(w) - r_N((\mathbb{P}E - (A \cup w)) \cup w) + r_N(w)
\]
\[
= 1 + r(\mathbb{P}E - (A \cup z)) - r(\mathbb{P}E - A)
\]
\[
= 1 + \rho(E) - \rho(E) = 1
\]

where the last equality follows from Proposition 3.6 because $B$ is a subset of $\mathbb{P}E - (A \cup z)$ and $\mathbb{P}E - A$.

Statement (c). Let $w_1, w_2 \in Q^e_v - A$ and $W = \langle w_1, w_2 \rangle$. Clearly $\dim W = 2$ and $\dim (W \cap \langle v \rangle^\perp) = 1$. Hence there exists $z \in W$ such that $\langle z \rangle \not\subseteq Q^e_v$. We show $z$ is a loop of $\langle w_1, w_2 \rangle$, i.e. $r_N/\{w_1, w_2\}(z) = 0$.

\[
r_N/\{w_1, w_2\}(z) = r(z \cup \{w_1, w_2\}) - r(\{w_1, w_2\})
\]
\[
= \rho(\{z, w_1, w_2\}) - \rho(\{w_1, w_2\})
\]
\[
= \rho(W) - \rho(W) = 0.
\]
Statement (d). Let \( N' := P(M) \setminus Q = (PE - Q, r_{N'}) \). By Lemma 5.11 (v) is a coloop of \( M \) if and only if \( \rho(H) = \rho(E) - 1 \). Moreover, \( \rho(H) = r(PH) = r_{N'}(PH) \) and \( \rho(E) = r(PE) = r(PE - Q) = r_{N'}(PE - Q) \). Hence \( (v) \) is a coloop of \( M \) if and only if \( r_{N'}(PH) = r_{N'}(PH \cup e) - 1 \) if and only if \( e \) is a coloop of \( N' \).

With those results in place, we are now ready to consider the first step of our main theorem. For the next results we use the following notation. Given \( Q_{v^2} \), fix an ordering of its elements. Define \( S_0 := \emptyset \) and \( S_i := \{w_1, \ldots, w_i\} \) where \( w_j \) is the \( j \)th element of \( Q_{v^2}^n \). Note furthermore that \( |S_i| = i \) and \( S_i^{q-1} = Q_{v^2} \). Moreover, in the proofs of the remaining results in this section, Proposition 2.8 and Theorem 5.3 may be used without mention.

**Proposition 5.13.** Let \( M = (E, \rho) \) be a q-matroid, \( P(M) = (PE, r) \) its projectivization matroid, \( e, v \in PE \) such that \( (e) \oplus (v)^\perp = E \). Then

\[
\chi_M(x) = \begin{cases} 
\chi_{M\setminus e}(x) - \sum_{i=0}^{q^n-2} \chi_{P(M)\setminus S_i \setminus w_i+1}(x) & \text{if } v \text{ is a coloop of } M \\
\chi_{M\setminus v}(x) - \chi_{M\setminus e}(x) - \sum_{i=0}^{q^n-2} \chi_{P(M)\setminus S_i \setminus w_i+1}(x) & \text{otherwise}
\end{cases}
\]

**Proof.** We first use an induction argument on \( k := |S_k| \) to show that for all \( 1 \leq k \leq q^n - 1 \),

\[
\chi_{P(M)}(x) = \chi_{P(M)\setminus S_k}(x) - \sum_{i=0}^{k-1} \chi_{P(M)\setminus S_i \setminus w_i+1}(x). \quad (5.3)
\]

We prove the base case when \( k = 1 \). By Lemma 5.12 (a), \( w_1 \) is not a coloop of \( P(M) \) hence, by Theorem 5.3 \( \chi_{P(M)}(x) = \chi_{P(M)\setminus S_1}(x) - \chi_{P(M)\setminus S_0 w_1}(x) \), where recall \( S_0 = \emptyset \) and \( S_1 = \{w_1\} \).

Now assume (5.3) holds for \( k = q^n - 2 \). Then

\[
\chi_{P(M)}(x) = \chi_{P(M)\setminus S_k}(x) - \sum_{i=0}^{k-1} \chi_{P(M)\setminus S_i \setminus w_i+1}(x)
\]

\[
= \chi_{P(M)\setminus (S_k \cup w_{k+1})}(x) - \chi_{P(M)\setminus S_k \setminus w_{k+1}}(x) - \sum_{i=0}^{k-1} \chi_{P(M)\setminus S_i \setminus w_i+1}(x)
\]

\[
= \chi_{P(M)\setminus S_{k+1}}(x) - \sum_{i=0}^{k} \chi_{P(M)\setminus S_i \setminus w_{i+1}}(x).
\]

The second equality holds true by Theorem 5.3 because \( w_{k+1} \) is not a coloop of \( P(M) \setminus S_k \) by 5.12 (a). This establishes (5.3).

Because \( S_i^{q^n-1} = Q_{v^2}^i \), we conclude the proof by using Theorem 5.3 on \( \chi_{P(M)\setminus S_i^{q^n-1}}(x) \) with the element \( e \in Q_{v^2}^i - S_i^{q^n-1} \). We therefore consider two cases: when \( e \) is a coloop of \( P(M) \setminus Q_{v^2}^i \) or not. By Lemma 5.12 (d), those two cases correspond exactly to when \( (v) \) is a coloop of \( M \) or not. First assume \( e \) is a coloop of \( P(M) \setminus Q_{v^2}^i \), and therefore \( (v) \) is a coloop of \( M \). Then by Theorem 5.3

\[
\chi_{P(M)\setminus Q_{v^2}^i}(x) = \chi_{P(M)\setminus Q_{v^2}^i}(x) \chi_{P(M)\setminus Q_{v^2}^i}(x).
\]

By Theorem 5.5 \( P(M) \setminus Q_{v^2}^i = P(M \setminus v) \) and \( P(M) \setminus Q_{v^2}^i \equiv P(M/e) \) hence their respective characteristic polynomials are equal. Furthermore by Theorem 5.9 \( \chi_{P(M\setminus v)}(x) = \chi_{M\setminus v}(x) \)

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and \( \chi_{P(M/e)}(x) = \chi_{M/e}(x) \). Therefore, \( \chi_{P(M)\setminus S_{p-1}}(x) = \chi_{P(M)\setminus Q_{+}^{e}}(x) = \chi_{M\setminus \bar{v}}(x) \cdot \chi_{M/e}(x) \), which when substituted in (5.3) gives us the wanted equality.

If \( e \) is a not coloop of \( P(M) \setminus Q_{+}^{e} \), and therefore \( \langle v \rangle \) is not a coloop of \( M \). Then

\[
\chi_{P(M)\setminus Q_{+}^{e}}(x) = \chi_{P(M)\setminus Q_{+}^{e}}(x) = \chi_{M\setminus \bar{v}}(x) \cdot \chi_{M/e}(x).
\]

Once again using Theorems 3.8 and 5.9 the wanted equality follows.

At this point, note that the characteristic polynomial \( \chi_{M}(x) \) depends on both the characteristic polynomial of minors of the \( q \)-matroid \( M \) and the characteristic polynomial of minors of \( P(M) \). In the following Theorem, we rewrite all characteristic polynomials of minors of \( P(M) \) in terms of characteristic polynomials of minors of \( M \).

**Theorem 5.14.** Let \( M = (E, \rho) \) be a \( q \)-matroid and \( e, v \in E \) such that \( \langle e \rangle \oplus \langle v \rangle = E \). Then

\[
\chi_{M}(x) = \begin{cases} 
\chi_{M\setminus \bar{v}}(x) \cdot \chi_{M/e}(x) - \sum_{w \in Q_{+}^{e}} \chi_{M/w}(x) & \text{if } v \text{ is a coloop of } M, \\
\chi_{M\setminus \bar{v}}(x) - \sum_{w \in Q_{+}^{e}} \chi_{M/w}(x) & \text{otherwise.}
\end{cases}
\]

**Proof.** Given the equation of Proposition 5.13 we show that

\[
\sum_{i=0}^{q^n-2} \chi_{P(M)\setminus S_{i+w_1}}(x) = \sum_{w \in Q_{+}^{e}} \chi_{M/w}(x). \tag{5.4}
\]

Fix \( 0 \leq i \leq q^n-2 \) and consider \( \chi_{P(M)\setminus S_{i+w_1}}(x) \). For all \( A \subseteq Q_{+}^{e} \) contains a loop which implies its characteristic polynomial is 0. Hence \( \chi_{P(M)\setminus S_{i+w_1}}(x) = \chi_{P(M)\setminus S_{i+w_1}}(x) \). Now let \( |A| = k \) and let \( w \in A \). Since \( |A-w| = k-1 \) by induction hypothesis we get \( \chi_{P(M)\setminus S_{i+w_1}}(x) = \chi_{P(M)\setminus (S_{i+w_1})}(x) \). Once again by Lemma 5.12(b), \( w \) is not a coloop of \( P(M) \setminus S_{i+w_1} \). Therefore

\[
\chi_{P(M)\setminus S_{i+w_1}}(x) = \chi_{P(M)\setminus (S_{i+w_1})}(x) = \chi_{P(M)\setminus (S_{i+w_1})}(x).
\]

By Lemma 5.12(c), \( P(M) \setminus S_{i+w_1} \) contains a loop which implies its characteristic polynomial is 0. Hence \( \chi_{P(M)\setminus S_{i+w_1}}(x) = \chi_{P(M)\setminus S_{i+w_1}}(x) \). Now let \( |A| = k \) and let \( w \in A \). Since \( |A-w| = k-1 \) by induction hypothesis we get \( \chi_{P(M)\setminus S_{i+w_1}}(x) = \chi_{P(M)\setminus (S_{i+(A-w)})}/w_{i+1}(x) \). Once again by Lemma 5.12(b), \( w \) is not a coloop of \( P(M) \setminus (S_{i+(A-w)})/w_{i+1} \) hence

\[
\chi_{P(M)\setminus (S_{i+(A-w)})}/w_{i+1}(x) = \chi_{P(M)\setminus (S_{i+(A-w)})}/w_{i+1}(x) = \chi_{P(M)\setminus (S_{i+(A-w)})}/w_{i+1}(x).
\]

By Lemma 5.12(c) \( P(M) \setminus (S_{i+(A-w)})/w_{i+1} \) contains a loop and therefore its characteristic polynomial is 0. This completes the proof of (5.5), which if \( A = Q_{+}^{e} \) - \( (S_{i} \cup w_{i+1}) \) shows that

\[
\chi_{P(M)\setminus S_{i+w_1}}(x) = \chi_{P(M)\setminus Q_{+}^{e}}(x).
\]

Finally by Theorems 3.8 (where \( S = \{w_{i+1}\} \)) and 5.9 we get

\[
\chi_{P(M)\setminus S_{i+w_1}}(x) = \chi_{M/w_{i+1}}(x). \tag{5.6}
\]
Since the above induction holds true for any \( i \) chosen, then (5.6) holds for all \( 0 \leq i \leq q^n - 1 \) and

\[
q^n - 1 - 2 \sum_{i=0}^{q^n - 1 - 2} \chi_{P(M) \setminus S_i}(x) = q^n - 1 - 2 \sum_{i=0}^{q^n - 1 - 2} \chi_M(w_i + 1)(x) = \sum_{w \in Q^*_{v_1}} \chi_M(w + 1)(x).
\]

Substituting (5.4) into the equation of Proposition 5.13 gives the desired result. \( \square \)

6 Rank Metric and Linear Block Codes

The study of matroids and \( q \)-matroids plays an important role in coding theory. In fact, it is well known that \( \mathbb{F}_{q^m} \)-linear rank metric codes induce \( q \)-matroids, and that linear block codes with the Hamming metric give rise to matroids. Furthermore, many of the code invariants can be determined from the induced (\( q \))-matroid. In [1], Alfarano and co-authors showed that an \( \mathbb{F}_{q^m} \)-linear rank metric code \( C \) induces a linear block code that shares similar parameters. We show in this section how the projectivization matroid of a \( q \)-matroid relates to the matroid associated to that linear block code. Furthermore, we use this relation and results from Section 5 to show the \( q \)-analogue of the critical theorem in terms of \( q \)-matroids and \( \mathbb{F}_{q^m} \)-linear rank metric codes.

We start the section by recalling some coding theory concepts. Throughout, let \( \mathbb{F}_{q^m} \) be a field extension of \( \mathbb{F}_q \) of degree \( m \). Furthermore, let \( \Gamma \) be a basis of the vector space \( \mathbb{F}_{q^m} \) over \( \mathbb{F}_q \). For all \( v = (v_1, \ldots, v_n) \in \mathbb{F}_{q^m}^n \), let \( \Gamma(v) \) be the \( n \times m \) matrix such that the \( i \)th row of \( \Gamma(v) \) is the coordinate vector of \( v_i \) with respect to the basis \( \Gamma \). Let \( \text{rk}_F(-), \text{colsp}_F(-), \text{rowsp}_F(-) \) respectively denote the rank, column space and row space of a matrix over the field \( F \). Moreover, throughout the section, we fix the NSBF on \( \mathbb{F}_q^n \) to be the standard dot product.

We define the following two weight functions.

**Definition 6.1.** For all \( v \in \mathbb{F}_{q^m} \), the **Hamming weight** \( \omega_H \), and the **rank weight** \( \omega_R \) of \( v \) are defined as follow:

\[
\omega_H(v) := \# \text{non-zero entries of } v
\]

\[
\omega_R(v) := \text{rk}_{\mathbb{F}_q}(\Gamma(v)).
\]

It is well known that the rank weight is independent of the basis \( \Gamma \) chosen. Both weight functions induce a metric on \( \mathbb{F}_{q^m}^n \), where \( d_\Delta(v, w) = \omega_\Delta(v - w) \) for \( v, w \in \mathbb{F}_{q^m} \) and \( \Delta \in \{ H, R \} \). A linear block code is a subspace of the metric space \( (\mathbb{F}_{q^m}^n, d_H) \), and an \( \mathbb{F}_{q^m} \)-linear rank metric code is a subspace of the metric space \( (\mathbb{F}_{q^m}^n, d_R) \). Given a code \( C \subseteq \mathbb{F}_{q^m}^n \), let \( d_H(C) \), respectively \( d_R(C) \), denote the minimum weight over all non zero elements of the code. If \( \dim C = k \), then \( d_\Delta(C) \leq n - k + 1 \) for \( \Delta \in \{ H, R \} \). For each metric, the above bound is called the **Singleton bound**. \( C \) is said to be **maximum distance separable**, respectively **maximum rank distance** if the Hamming-metric, respectively the rank-metric, Singleton bound is achieved. For both metrics, the weight distribution of the code is defined as follows.
\textbf{Definition 6.2.} Let $C \leq \mathbb{F}_q^n$ be a code. For $\Delta \in \{H,R\}$, let

$$W_{\Delta}^{(i)}(C) := |\{v \in C : \omega_{\Delta}(v) = i\}|.$$ 

Furthermore let $W_\Delta(C) := (W_{\Delta}^{(i)}(C) : 0 \leq i \leq n)$. $W_H(C)$ is called the Hamming weight distribution of $C$ and $W_R(C)$ is called the rank weight distribution of $C$.

We now introduce two notions of support for elements of $\mathbb{F}_q^n$.

\textbf{Definition 6.3.} Let $v = (v_1, \ldots, v_n) \in \mathbb{F}_q^n$ and $V \subseteq \mathbb{F}_q^n$.

$$S_H(v) = \{i : v_i \neq 0\} \quad \text{and} \quad S_H(V) = \bigcup_{v \in V} S_H(v)$$

$$S_R(v) = \text{colsp}_{\mathbb{F}_q}(\Gamma(v)) \quad \text{and} \quad S_R(V) = \sum_{v \in V} S_R(v)$$

$S_H$, respectively $S_R$, are called the \textit{Hamming support} and \textit{rank support} of $v \in \mathbb{F}_q^n$ or $V \subseteq \mathbb{F}_q^n$.

Once again, the rank support of an element $v \in \mathbb{F}_q^n$, or subset $V \subseteq \mathbb{F}_q^n$, is independent of the basis $\Gamma$ chosen. Furthermore, a linear block code, respectively $\mathbb{F}_q^n$-linear rank metric code, $C \leq \mathbb{F}_q^n$, is said to be \textit{non-degenerate} if $S_H(C) = [n]$, respectively $S_R(C) = \mathbb{F}_q^n$. Given a code $C \leq \mathbb{F}_q^n$, it is of interest to consider the set of elements of $C$ with a given support.

\textbf{Definition 6.4.} Let $C \leq \mathbb{F}_q^n$ be a code, $A \subseteq [n]$ and $V \leq \mathbb{F}_q^n$. Let

$$C_H(A) := \{v \in C : S_H(v) = A\}$$

$$C_R(V) := \{v \in C : S_R(v) = V\}$$

We now make the connection between codes and $(q)$-matroids. Note that linear block codes and $\mathbb{F}_q^n$-linear rank metric codes can be represented via a generating metric $G \in \mathbb{F}_q^{k \times n}$, where $C := \text{rows}_{\mathbb{F}_q}(G)$. Both a matroid and a $q$-matroid can be induced from the generating matrix. The following construction is well known for matroids (see \cite[Sec.6]{15}) and has been established in \cite[Sec. 5]{14} for $q$-matroids.

\textbf{Proposition 6.5.} Let $C \leq \mathbb{F}_q^n$ be a code and $G \in \mathbb{F}_q^{k \times n}$ a generating matrix of $C$. For $i \in [n]$, let $e_i \in \mathbb{F}_q$ denote the $i$th standard basis vector and for $V \leq \mathbb{F}_q^n$ let $Y_V \in \mathbb{F}_q^n$ such that $\text{colsp}_{\mathbb{F}_q}(Y_V) = V$. Define $r : [n] \to \mathbb{N}_0$ and $\rho : \mathcal{L}(\mathbb{F}_q^n) \to \mathbb{N}_0$ such that:

$$r(A) = \text{rk}_{\mathbb{F}_q}(G \cdot [e_{i_1} \cdots e_{i_a}]) \quad \text{for all} \ A \subseteq [n]$$

$$\rho(V) = \text{rk}_{\mathbb{F}_q}(G \cdot Y_V) \quad \text{for all} \ V \leq \mathbb{F}_q^n.$$ 

Then $M_C := ([n], r)$ is a matroid and $M_C = (\mathbb{F}_q^n, \rho)$ is a $q$-matroid and are called the matroid (resp. $q$-matroid) associated with $C$.

Note that neither $M_C$ nor $M_C$ depend on the choice of generating matrix for $C$. Furthermore a matroid or $q$-matroid is $\mathbb{F}_q^n$-representable if it is induced by a linear block code or rank metric code $C \leq \mathbb{F}_q^n$. The $(q)$-matroid induced by a code is a useful tool to determine some of the code’s
invariants. In the rest of this section, we consider invariants of the code that are determined by the characteristic polynomial of the induce \((q)\)-matroid. We first recall the notion of weight enumerator of a \((q)\)-matroid. The weight enumerator of the \(q\)-matroid was defined in [5, Def. 43] and a similar concept was established in [11] for matroids.

**Definition 6.6.** Let \(M = (S, r)\) be a matroid and \(M = (E, \rho)\) be a \(q\)-matroid, with \(|S| = n = \dim E\). Let

\[
A_M^{(i)}(x) = \sum_{A \subseteq S, |A| = i} \chi_{M/(S-A)}(x),
\]

\[
A_M^{(i)}(x) = \sum_{V \leq E, \dim V = i} \chi_{M/V^\perp}(x).
\]

The weight enumerator of the matroid, respectively \(q\)-matroid, is the list \(A_M := (A_M^{(i)}(x) : 1 \leq i \leq n)\), respectively \(A_M := (A_M^{(i)}(x) : 1 \leq i \leq n)\).

Note in the above equations that if \(S-A\) or \(V^\perp\) are not flats of their respective matroid or \(q\)-matroid, then \(\chi_{M/(S-A)}(x) = 0 = \chi_{M/V^\perp}(x)\). Hence the summands of the weight enumerator can be restricted to the complement, respectively the orthogonal space, of flats. Thus we have the following result, which is well-known for matroids (see [11, Prop 3.3]) and was hinted at in [5] for \(q\)-matroids.

**Theorem 6.7.** Let \(M = (S, r)\) be a matroid and \(M = (E, \rho)\) be a \(q\)-matroid, with \(|S| = n = \dim E\). Then

\[
A_M^{(i)}(x) = \sum_{F \in \mathcal{F}_M, |F| = n-i} \chi_{M/F}(x),
\]

\[
A_M^{(i)}(x) = \sum_{F \in \mathcal{F}_M, \dim F = n-i} \chi_{M/F}(x).
\]

It was shown in [5, Lem 49] that the weight enumerator of a representable \(q\)-matroid is closely related to the weight distribution of its associated code. For matroids a similar relation holds and was established in [11, Prop 3.2].

**Theorem 6.8.** Let \(C \subseteq \mathbb{F}_q^n\) be a code, \(M\) and \(\mathcal{M}\) be, respectively, the matroid and \(q\)-matroid induced by \(C\). Let \(A \subseteq [n]\) and \(V \subseteq \mathbb{F}_q^n\). Then

\[
(1) \quad \chi_{M/A}(q^m) = |C_H([n]-A)| \quad \text{and} \quad W^{(i)}_H(C) = A_M^{(i)}(q^m).
\]

\[
(2) \quad \chi_{\mathcal{M}/V}(q^m) = |C_R(V^\perp)| \quad \text{and} \quad W^{(i)}_R(C) = A_M^{(i)}(q^m).
\]

**Remark 6.9.** The above Theorem together with Proposition [6.1] tells us that if \(|C_H([n]-A)|\) and \(|C_R(V^\perp)|\) are non-zero then \(A\) and \(V\) are flats in \(M\) and \(\mathcal{M}\), respectively.

Because the weight enumerator of a \((q)\)-matroid can be expressed in terms of flats, we get the following relation between the weight enumerator of a \(q\)-matroid and that of its projectivization matroid.
Proposition 6.10. Let \( M = (E, \rho) \) be a \( q \)-matroid and \( P(M) = (\mathbb{P}E, r) \) its projectivization matroid. Then
\[
A^{(j)}_{P(M)}(x) = \begin{cases} 
A^{(i)}_M(x) & \text{if } j = \frac{q^n-q^{n-i}}{q-1}, \\
0 & \text{otherwise}.
\end{cases} \tag{6.1}
\]

Proof. First, recall by Lemma 3.2, \( \mathcal{F}_{P(M)} = \{ P(F): F \in \mathcal{F}_M \} \) and \( \dim F = n - i \iff |P(F)| = \frac{q^n-q^{n-i}}{q-1} \Rightarrow |\mathbb{P}E - P(F)| = \frac{q^n-q^{n-i}}{q-1}. \) Furthermore, by Proposition 5.10, if \( X \subseteq \mathbb{P}E \) is not a flat, then \( \chi_{P(M)/X}(x) = 0. \) So for all \( 1 \leq j \leq \frac{q^n-1}{q-1}, \) such that \( j \neq \frac{q^n-q^{n-i}}{q-1} \) for some \( 1 \leq i \leq n, \) we get \( A^{(j)}_{P(M)}(x) = 0. \) Now assume \( j = \frac{q^n-q^{n-i}}{q-1} \) for some \( 1 \leq i \leq n. \) Then
\[
A^{(i)}_M(x) = \sum_{F \in \mathcal{F}_M, \dim F = n - i} \chi_{M/F}(x)
= \sum_{F \in \mathcal{F}_M, \dim F = n - i} \chi_{P(M)/P(F)}(x)
= \sum_{P(F) \in \mathcal{F}_{P(M)}, |P(F)| = \frac{q^n-1}{q-1}} \chi_{P(M)/P(F)}(x)
= A^{(j)}_{P(M)}(x),
\]
where the second equality follows from Proposition 5.10. \( \square \)

With the above setup, we now discuss the linear block code induced by an \( \mathbb{F}_{q^n} \)-linear rank metric code, as introduced in [1]. In their paper, the authors use \( q \)-systems and projective systems to introduce the Hamming-metric code associated to a rank metric code. We use a slightly different approach to introduce the associated Hamming metric code that does not require the previously stated notions.

Definition 6.11. Let \( C \leq \mathbb{F}_{q^n} \) be an \( \mathbb{F}_{q^n} \)-linear rank metric code and let \( G \in \mathbb{F}_{q^n}^{k \times n} \) be a generating matrix of \( C. \) Furthermore let \( H \in \mathbb{F}_{q}^{n \times \frac{q^n-1}{q-1}} \) where each column of \( H \) is a representative of a distinct element of \( \mathbb{P} \mathbb{F}_{q^n}. \) We call the matrix \( C^H := G \cdot H \) an \( \mathbb{F}_{q^n} \)-decomposition of \( G \) via \( H \) and \( C^H := \text{rowsp}_{\mathbb{F}_{q^n}}(G^H) \) is called a Hamming-metric code associated to \( C \) via \( H \)

Remark 6.12. Given a non-degenerate \( \mathbb{F}_{q^n} \)-rank metric code \( C, \) the code \( C^H \) of Definition 6.11 is a Hamming-metric code associated with \( C \) as in [1] Def. 4.5. In fact, it is easy to show the projective system induced by the columns of \( C^H \) is a representative of the equivalence class \( (\text{Ext}^H \circ \Phi)([C]) \) as introduced in [1]. Furthermore note that unlike the construction in [1], Definition 6.11 does not depend on \( q \)-systems and projective systems hence we do not require \( C \) to be non-degenerate code. Finally, as noted in [1], a Hamming metric code associated to an \( \mathbb{F}_{q^n} \)-linear rank metric code \( C \) is not unique. In our case, \( C^H \) depends on the choice of the matrix \( H \) of the \( \mathbb{F}_{q^n} \)-decomposition. However, all Hamming-metric codes associated with \( C \) are monomially equivalent.

Because \( C^H \) is a linear block code, it induces a matroid \( M_{C^H}. \) It turns out that \( M_{C^H} \) is equivalent to the projectivization matroid \( P(M_C) \) of the \( q \)-matroid induced by \( C. \)
Theorem 6.13. Let $\mathcal{C} \leq \mathbb{F}_{q}^{n}$ be a rank metric code, $\mathcal{M}_{\mathcal{C}}$ its associated $q$-matroid, and $P(\mathcal{M}_{\mathcal{C}})$ its projectivization matroid. Furthermore let $\mathcal{C}^{H}$ be a Hamming-metric code associated to $\mathcal{C}$ via $H$ and $\mathcal{M}_{\mathcal{C}^{H}}$ its induced matroid. Then

$$P(\mathcal{M}_{\mathcal{C}}) \cong \mathcal{M}_{\mathcal{C}^{H}} \quad \text{as matroids.}$$

Proof. Let $G \in \mathbb{F}_{q}^{k \times n}$ be a generator matrix of $\mathcal{C}$ and let $G^{H} = G \cdot H$ be a $\mathbb{F}_{q}$-decomposition of $G$ via $H$. Let $h_{i}$ be the $i^{th}$ column of $H$, hence $\mathbb{F}_{q}^{n} = \{ \langle h_{i} \rangle : i \in \left[ \frac{q^{n} - 1}{q - 1} \right] \}$. Define the bijection $\psi : \mathbb{F}_{q}^{n} \rightarrow \left[ \frac{q^{n} - 1}{q - 1} \right]$, where $\psi(\langle h_{i} \rangle) = i$. Furthermore, let $A \subseteq \mathbb{F}_{q}^{n}$, $\psi(A) = \{ i_{1}, \cdots i_{a} \}$ and $Y_{A} := [e_{i_{1}} \cdots e_{i_{a}}] \in \mathbb{F}_{q}^{\frac{q^{n} - 1}{q - 1} \times |A|}$, where $e_{j}$ is the $j^{th}$ standard basis element of $\mathbb{F}_{q}^{\frac{q^{n} - 1}{q - 1}}$. By Proposition 6.3 we get

$$r_{\mathcal{M}_{\mathcal{C}^{H}}}(\psi(A)) = \text{rk}_{q^{m}}(G^{H} \cdot [e_{i_{1}} \cdots e_{i_{a}}])$$
$$= \text{rk}_{q^{m}}(G \cdot H \cdot [e_{i_{1}} \cdots e_{i_{a}}])$$
$$= \text{rk}_{q^{m}}(G \cdot [h_{i_{1}} \cdots h_{i_{a}}])$$
$$= \rho_{\mathcal{M}_{C}}(\langle h_{i_{1}}, \cdots, h_{i_{a}} \rangle_{\mathbb{F}_{q}})$$
$$= \text{rk}(P(\mathcal{M}_{\mathcal{C}}) \cdot A),$$

where the last equality follows from Theorem 3.1.

Remark 6.14. The above theorem allows us to relabel the groundset $\left[ \frac{q^{n} - 1}{q - 1} \right]$ of the matroid $\mathcal{M}_{\mathcal{C}^{H}}$ in terms of the elements of the projective space $\mathbb{P}_{\mathbb{F}_{q}^{n}}$. Precisely if $G \cdot H$ is the $\mathbb{F}_{q}$-decomposition associated to $\mathcal{C}^{H}$, relabel $i \in \left[ \frac{q^{n} - 1}{q - 1} \right]$ by $\langle h_{i} \rangle \in \mathbb{P}_{\mathbb{F}_{q}^{n}}$, where $h_{i}$ is the $i^{th}$ column of $H$.

We get the following result as an immediate corollary of Theorem 6.13:

Corollary 6.15. If $\mathcal{M}$ is $\mathbb{F}_{q^{m}}$-representable then its projectivization matroid $P(\mathcal{M})$ is $\mathbb{F}_{q^{m}}$-representable.

In [1, Theorem 4.8], it was established that the rank-weight distribution of a rank metric code $\mathcal{C}$ is closely related to the Hamming-weight distribution of any Hamming-metric code associated to $\mathcal{C}$. By Theorem 6.8, Proposition 6.10 and Theorem 6.13 we arrive at the same result from a purely matroid/$q$-matroid approach.

Theorem 6.16. [1, Thm 4.8] Let $\mathcal{C}$ be an $\mathbb{F}_{q^{m}}$-linear rank metric code and $\mathcal{C}^{H}$ be a Hamming-metric code associated to $\mathcal{C}$. Then

$$W_{H}^{(j)}(\mathcal{C}^{H}) = \begin{cases} W_{H}^{(i)}(\mathcal{C}) & \text{if } j = \frac{q^{n} - q^{n-i}}{q-1}, \\ 0 & \text{otherwise.} \end{cases}$$

We conclude the paper by showing the $q$-analogue of the critical Theorem. The critical Theorem, introduced by Crapo and Rota [7, Thm 1], states that the characteristic polynomial of the matroid $\mathcal{M}_{\mathcal{C}}$ induced by the linear block code $\mathcal{C}$ determines the number of multisets of codewords with a given support. It was nicely restated in [3, Thm 2] in terms of coding theory terminology (recall Definition 6.3).
Theorem 6.17. Let $C \leq \mathbb{F}_q^n$ be a linear block code and $M = ([n], r)$ its induced matroid. For all $A \subseteq [n]$, the number of ordered $t$-tuples $V = (v_1, \cdots, v_t)$, where $v_j \in C$ for all $1 \leq j \leq t$, such that $S_H(V) = A$ is given by $\chi_{M/([n]-A)}(q^t)$.

For our last result, we show an analogous statement for $\mathbb{F}_q^n$-linear rank metric codes and $q$-matroids by using the projectivization matroid. It is worth mentioning that Alfarano and Byrne were able to show an analogue of the critical theorem for $q$-polymatroids and matrix rank metric codes by using a different approach involving the Möbius inversion formula [4].

For the next results we make use of Remark 6.14, and relabel the elements of the groundset of the matroid induced by $C^H$ in terms of elements of $\mathbb{P}\mathbb{F}_q^n$. Following this relabeling, we can also describe the support of a codeword of $C^H$ in terms of the elements of $\mathbb{P}\mathbb{F}_q^n$. More precisely, if $C^H$ is induced by the $F_q$-decomposition $G \cdot H$, for any $v \in C^H$, let $S_H(v) = \{\langle h_i \rangle \in \mathbb{P}\mathbb{F}_q^n : v_i \neq 0\}$, where $h_i$ and $v_i$ are respectively the $i$th column of $H$ and the $i$th component of $v$. Furthermore, we need the following well-known result for which we include a proof for self-containment. For two vectors $v, w$ we let $v \cdot w$ denote the standard dot-product.

Lemma 6.18. Let $v \in \mathbb{F}_q^n$, $S_R(v) = W \leq \mathbb{F}_q^n$ and $w \in \mathbb{F}_q^n$. Then $v \cdot w = 0$ if and only if $w \in W^\perp$.

Proof. Let $\Gamma := \{\gamma_1, \cdots, \gamma_m\}$ be a basis of $\mathbb{F}_q^n$ over $\mathbb{F}_q$, and let $Y := \Gamma(v) \in \mathbb{F}_q^n \times m$, where $W := \text{colsp}_{\mathbb{F}_q}(Y)$. Then $v \cdot w = 0 \iff \sum_{i=1}^n v_i w_i = 0 \iff \sum_{i=1}^n \left(\sum_{j=1}^m \gamma_j v_{ij}\right) w_i = 0 \iff \sum_{i=1}^m \gamma_j \left(\sum_{i=1}^n v_{ij} w_i \right) = 0$. Since $\Gamma$ is a basis of $\mathbb{F}_q^n$ over $\mathbb{F}_q$ and $v_{ij} w_i \in \mathbb{F}_q$, the previous equality holds if and only if $\sum_{i=1}^n v_{ij} w_i = 0$ for all $1 \leq j \leq m$. But note $\sum_{i=1}^n v_{ij} w_i = v^{(j)} \cdot w$, where $v^{(j)}$ is the $j$th column of $Y$. Hence $v \cdot w = 0 \iff v^{(j)} \cdot w = 0$ for all $1 \leq j \leq m \iff w \in \text{colsp}_{\mathbb{F}_q}(Y)^\perp \iff w \in W^\perp$.

The following Lemma relates the rank support of elements of the code $C$ with the Hamming support of elements of the associated Hamming-metric code $C^H$.

Lemma 6.19. Let $C \leq \mathbb{F}_q^n$ be a rank metric code, $C^H$ be a Hamming-metric code associated to $C$ via $H$. Furthermore, let $V = \{v_1, \cdots, v_t\}$ be a subset of $C$ and $V \cdot H := \{v_1 \cdot H, \cdots, v_t \cdot H\}$. Then $S_R(V) = W \iff S_H(V \cdot H) = \mathbb{P}\mathbb{F}_q^n - P(W^\perp)$.

Moreover if $M$ and $P(M)$ are the $q$-matroid and projectivization matroid induced respectively by $C$ and $C^H$ then $W^\perp$ and $P(W^\perp)$ are, respectively, flats of $M$ and $P(M)$.

Proof. Consider the subset $V := \{v_1, \cdots, v_t\} \subseteq C$. By definition, $S_R(V) = \sum_{j=1}^t S_R(v_j) =: W$. Let $W_j := S_R(v_j)$. For all $1 \leq j \leq t$, by Lemma 6.18 $v_j \cdot w = 0$ if and only if $w \in W_j^\perp$. Hence for all columns $h_i$ of $H$, it follows that $v_j \cdot h_i = 0$ if and only if $h_i \in W_j^\perp$. By definition, this is true if and only if $S_H(v_j \cdot H) = \mathbb{P}\mathbb{F}_q^n - P(W_j^\perp)$. Hence $S_H(V \cdot H) = \bigcup_{i=1}^t S_H(v_j \cdot H) = \bigcup_{i=1}^t (\mathbb{P}\mathbb{F}_q^n - P(W_j^\perp))$, where the first equality follows by definition. Therefore $S_R(V) = W \iff S_H(V \cdot H) = \bigcup_{i=1}^t (\mathbb{P}\mathbb{F}_q^n - P(W_j^\perp)) = \mathbb{P}\mathbb{F}_q^n - (\bigcap_{i=1}^t P(W_j^\perp)) = \mathbb{P}\mathbb{F}_q^n - P(W^\perp)$. Finally, $W^\perp$ and $P(W^\perp)$ are flats of $M$ and $P(M)$ respectively because of Remark 6.9.

We are now ready to show the critical theorem for $\mathbb{F}_q^n$-linear rank metric codes and $q$-matroids.
Theorem 6.20. Let $C \subseteq \mathbb{F}_q^n$ be an $\mathbb{F}_q^m$-linear rank metric code and $M$ its induced $q$-matroid. For all $W \subseteq \mathbb{F}_q^n$, the number of ordered $t$-tuples $V = (v_1, \ldots, v_t)$, where $v_j \in C$ for all $1 \leq j \leq t$, such that $S_R(V) = W$ is given by $\chi_{M/W}(q^{mt})$.

Proof. Let $C^H$ be the Hamming-metric code associated with $C$ via $H$ and $P(M)$ be its associated matroid. Note that every element of $C^H$ is of the form $v \cdot H$ for some $v \in C$. Hence every tuple of elements of $C^H$ is of the form $V \cdot H$ for some $V \subseteq C$. Furthermore, since $H$ has full-row rank, there is a bijection between elements of $C$ and $C^H$ and hence a bijection between $t$-tuples $V \subseteq C$ and $t$-tuples $V \cdot H \subseteq C^H$. By Theorem 6.17, the number of $t$-tuple $V \cdot H \subseteq C^H$ such that $S_H(V \cdot H) = \mathbb{P}E - P(W^\perp)$ is given by $\chi_{P(M)/P(W^\perp)}(q^{mt})$. Moreover, by Proposition 5.10, $\chi_{P(M)/P(W^\perp)}(q^{mt}) = \chi_{M/W}(q^{mt})$. Finally, by Lemma 6.19, $S_R(V \cdot H) = \mathbb{P}E - P(W^\perp)$ if and only if $S_R(V) = W$ and therefore $\chi_{M/W}(q^{mt})$ counts the number of $t$-tuples $V \subseteq C$ such that $S_R(V) = W$. $\square$

The above result, and its proof, shows a close connection between the critical theorem for matroids and that for $q$-matroids. Furthermore it can easily been seen from the above approach that the critical problem for $q$-matroids, that is finding the smallest power of $q$ that makes the characteristic polynomial of a $q$-matroid non-zero, is a specific case of the critical problem for matroids.

Further questions

Here are a few questions that arise from our work:

- If a projectivization matroid $P(M)$ is $\mathbb{F}_q^m$-representable then is the $q$-matroid $M$ representable?

- It is well known that the characteristic polynomial of a matroid can be derived from the Tutte polynomial of that matroid. Hence the characteristic polynomial of a $q$-matroid $M$ can also be derived from the Tutte polynomial of the projectivization matroid $P(M)$. Can invariants of $\mathbb{F}_q^m$-rank metric codes be determined from the Tutte polynomial of the projectivization matroid associated to the code.

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