An Achievable Rate for an Optical Channel with Finite Memory

K Gautam Shenoy, Vinod Sharma
Dept. of ECE, Indian Institute of Science, Bangalore-560012
Email: konchady,vinod@ece.iisc.ernet.in

Abstract—A fiber optic channel is modeled in a variety of ways; from the simple additive white complex Gaussian noise model, to models that incorporate memory in the channel. Because of Kerr nonlinearity, a simple model is not a good approximation to an optical fiber. Hence we study a fiber optic channel with finite memory and provide an achievable bound on channel capacity that improves upon a previously known bound.

Index Terms—Channel capacity, finite memory, Gaussian noise, optical fiber.

I. INTRODUCTION

The study of the fiber optic channel (FOC) from an information theoretic standpoint has been an interesting subject in recent times. In particular, estimating the channel capacity, has been a daunting task owing to several factors. Firstly, from [1]–[3], there is no agreed upon model which incorporates all the non-linear effects that occur in an optical fiber. Thus we have several models, each limited to an aspect under study. Secondly, for most of these models, the channel capacity is theoretically unknown and, while there may be good lower bounds available for these channels, only a few of them have useful upper bounds on channel capacity.

Nonlinearities of a FOC from an information theoretic standpoint were first studied by Stark and Mitra in [3]. Lower and upper bounds on channel capacity for free space optical intensity channels were obtained in [6] under peak and average power constraints. Input dependent Gaussian noise channels were studied in [7] where tight upper and lower bounds on channel capacity were derived under a variety of power constraints. The conditions for these bounds were further refined in [8]. In [9], a lower bound on the capacity of an additive white complex Gaussian noise (AWCGN) channel with input perturbed by phase noise was derived. Also, an upper bound on the cascade of nonlinear and noisy channels was derived in [2]. A survey of various types of channels for optical fibers is carried out in [10]. A more general fiber optical channel with memory (also known as the finite memory Gaussian noise model) was studied in [11]. The strategy used was that of block i.i.d. coding to mitigate the effects of memory.

A trait of most, if not all, of the models is that the channel is modeled as an AWCGN channel with the necessary modifications. The effects of Kerr nonlinearity introduce memory in the channel and it is essential that it be taken into consideration. However, the tradeoff here is that introducing memory makes the channel harder to analyze and, for this channel, the capacity is unknown. In this paper, we study the model of [11] and provide a closed-form, improved lower bound to the capacity of a FOC with finite memory. The technique used is that of comparison with a suitable auxiliary channel [12], [13]. Furthermore, we also use an algorithmic method [14], [15] to obtain even tighter lower bounds.

The paper is organized as follows. We provide the notation and channel model in Section II. A lower bound on the capacity for the FOC with a finite memory is stated in Section III and proved in Section IV. Section V describes an algorithm to compute a better achievable rate. Section VI compares the lower bound derived in Section IV with the rate computed via algorithm in Section V and the lower bound available in [11]. Section VII concludes the paper.

II. PRELIMINARIES

A. Notations

We use lower case letters (e.g., $x$) to refer to scalars and upper case letters (e.g., $X$) for random variables. Vectors are represented using boldface letters (e.g., $\mathbf{x}$) for random vectors. Matrices are represented using boldface upper case letters (e.g., $\mathbf{A}$). We use $\mathbb{N}$ to represent the set of natural numbers, $\mathbb{R}$ to represent the set of real numbers, $\mathbb{C}$ to represent the set of complex numbers, and $\mathbb{R}^n$ to represent the set of $n$-dimensional real vectors.

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A discrete time memoryless channel is represented by $W(x; y)$, $x \in X$, $y \in Y$, which for continuous alphabet, is the channel conditional probability density function. Given an input distribution $P$, we shall denote the corresponding output distribution by $PW(y) := \int_{x \in X} P(x)W(y|x)dx.$ (1)

The differential entropy of a continuous random variable $X$ is denoted by $h(X)$ and the mutual information between random variables $X$ and $Y$ will be denoted by $I(X; Y)$. If the distributions need to be emphasized, we will use the equivalent notation $I(P_X; W)$ for mutual information, where $P_X$ is the input distribution or density and the channel is $W$. These notations are taken from [16], [17]. For the memoryless channel, the capacity

$$C = \sup_{P_X} I(P_X; W),$$ (2)

where $P_X$ is the set of distributions that satisfy the needed constraints (e.g., average power constraints).
For any channel with finite memory, assuming it is information stable (see [18]), the channel capacity

\[ C = \lim_{n \to \infty} \sup_{P_{X_n}} \frac{I(X^n; Y^n)}{n}, \]

(3)

where the supremum is taken over all input distributions that satisfy the power constraint. Now from the properties of mutual information, for any positive integer \( N \),

\[ \frac{I(X^{n+N}; Y^n)}{n} \geq \frac{I(X^n; Y^n)}{n}. \]

(4)

From this it follows that

\[ \lim_{n \to \infty} \sup_{P_{X_n}} \frac{I(X^{n+N}; Y^n)}{n} \geq \lim_{n \to \infty} \sup_{P_{X_n}} \frac{I(X^n; Y^n)}{n} = C. \]

(5)

We also have

\[ \frac{I(X^{n+N}; Y^n)}{n} \leq \frac{I(X^{n+N}; Y^{n+N})}{n + N} \cdot \frac{n + N}{n}, \]

(6)

which gives us

\[ \lim_{n \to \infty} \sup_{P_{X_n}} \frac{I(X^{n+N}; Y^n)}{n} \leq C. \]

(7)

Thus we get capacity

\[ C = \lim_{n \to \infty} \sup_{P_{X_n}} \frac{I(X^{n+N}; Y^n)}{n}. \]

(8)

This is the form we will use in this paper.

B. Kerr Nonlinearity in FOCs

Wave propagation in an optical fiber is described by the non-linear Schrödinger wave equation (see [3], [19])

\[ \frac{\partial x}{\partial z} + \frac{j \beta}{2} \frac{\partial^2 x}{\partial t^2} - j \gamma |x|^2 x = 0, \]

(9)

where \( x = x(t, z) \) is the complex envelope of the optical signal at time \( t \) and distance \( z \), \( \beta \) represents the group velocity dispersion and \( \gamma \) is the Kerr nonlinearity coefficient. While designing the channel model, Kerr non-linearity introduces memory in the channel and manifests itself as a third power of the superposition of current input as well as past and future inputs. Hence the channel model is non-causal.

Another phenomenon that occurs is that of Amplified Spontaneous Emission (ASE) in the optical fiber (see [3]). This is modeled as an additive noise (see Section II-C) in the channel.

C. Channel Model

The channel model we considered was proposed and studied in [11]. The model (see Fig. 1), after a suitable modification (see the remark below), is given by

\[ Y_i = X_i + A_i + Z_i z_{S_i}^3, \]

(10)

for \( 1 \leq i \leq n \) where

1) \( X_i \in \mathbb{C} \) is the input to the channel.
2) \( Y_i \in \mathbb{C} \) is the channel output.

Fig. 1. The finite memory optical channel.

3) \( A_i \in \mathbb{C} \) is the noise due to ASE modeled as circularly symmetric complex Gaussian random variable with zero mean and variance \( \sigma^2_A \). We assume \( \{A_i\} \) are independent and identically distributed (i.i.d.).
4) \( Z_i \in \mathbb{C} \) is a circularly symmetric complex normal random variable with zero mean and unit variance. We assume \( \{Z_i\} \) are i.i.d.
5) \( S_i \geq 0 \) models the memory in the channel and is given by

\[ S_i = \frac{1}{2N + 1} \sum_{k=-N}^{i+N} |X_k|^2, \]

(11)

where \( N \geq 0 \) is the memory in the channel. When \( i \leq N \), some of the indices \( k \) in the sum may be negative (we take those corresponding \( |X_k|^2 \) as 0). However this does not affect our capacity analysis and so we may assume henceforth that \( i > N \). Note that \( \{S_i\} \) are not independent for \( N \geq 1 \). \( \eta \) is a constant that depends on the non-linearity parameter \( \gamma \) and the design of the optical fiber. Both \( S_i \) and \( \eta \) together model the Kerr non-linearity.
6) \( \{X_i\}, \{A_i\} \) and \( \{Z_i\} \) are mutually independent.
7) We assume that the channel has an average input power constraint,

\[ \frac{1}{n} \sum_{i=1}^{n} |X_i|^2 \leq P \quad \text{a.s.} \]

(12)

where \( n \) is the codeword length.

We remark that this channel is functionally equivalent (i.e., has the same mutual information and capacity) as the following channel,

\[ Y_i = X_i + Z_i \sqrt{\sigma^2_A + \eta S_i^3}, \]

(13)

which was originally studied in [11]. This equivalence follows from the following lemma.

Lemma 1. Consider two channels

1) \( Y_i = X_i + Z_i \sqrt{\sigma^2_A + U_i} \)
2) \( T_i = X_i + A_i \sigma_A + V_i \sqrt{U_i} \),

where \( X_i \) is the input to the channel, \( Z_i \), \( A_i \) and \( V_i \) are i.i.d. circularly symmetric complex Gaussian random variables with zero mean and unit variance and are mutually independent of each other. Let \( U_i \) be a non-negative random variable which is independent of \( Z_i \), \( A_i \) and \( V_i \) but is dependent on \( \{X_k\}_{k=-N}^{i+N} \). Then it follows that for any \( n \geq 1 \),

\[ I(X^{n+N}; Y^n) = I(X^{n+N}; T^n). \]

(14)
Moreover, this implies that the two channels have the same capacity.

Proof. We shall first prove the result for real valued random variables, replacing circularly symmetric Gaussian with standard Gaussian. To show that the mutual informations are equal, it suffices to show that

$$\{X_{1}, \ldots, X_{n}\} \sim \mathcal{G}(\mu, \Sigma),$$

where \(\mathcal{G}(\mu, \Sigma)\) denotes equal in distribution. As we are working with vectors, we will introduce some useful shorthand notation. We shall write

$${\{[Y] \leq [y]\}} = \{Y_1 \leq y_1, Y_2 \leq y_2, \ldots, Y_n \leq y_n\}$$

and

$${\{|X| \leq |x|\}} = \{X_1 \leq x_1, \ldots, X_n \leq x_n\}$$

Hence we have

$$\frac{n}{2} \log \left( \frac{1}{P \sum_{i=1}^{n} (|Y_i| - |x_i|)^2} \right)$$

where

$$P = \frac{1}{n} \sum_{i=1}^{n} \max_{y_i \in [x_i]} \min_{u_i \in [y_i]} \left( |Z_i\sqrt{\sigma_A^2 + u_i} - y_i - x_i| \right).$$

We now state the main theorem of this paper which we will prove in the next section.

**Theorem 1.** The capacity \(C\) of the channel described in (10), under average power constraint \(P\) is lower bounded by

$$\log_2 \left( \frac{1}{\sigma_A^2 + \eta P} \right)$$

where \(\eta P\) is as given in (21). and

$$P_N^* = \left( \frac{\sigma_A^2 (2N + 1)^2}{2\eta(2N + 3)(2N + 2)} \right)^{1/3}.$$

A quick look at the bound reveals that it may be the capacity of an equivalent AWCGN channel of some sort. This is exactly the strategy we use to obtain the lower bound.

**IV. PROOF OF THEOREM**

We first introduce a lemma that appeared in papers by Arimoto [12] and Blahut [13].

**Lemma 2** (Auxiliary Channel Lower Bound (ACLB)). Let \(W\) be the original channel and \(\bar{W}\) any other channel with input and output alphabet of \(W\). Then given any input distribution \(P_X\), we have

$$I(P_X; W) \geq \mathbb{E} \left[ \log \left( \frac{\bar{W}(Y|X)}{P_X(W(y|x))} \right) \right],$$

where the expectation is with respect to \(P_X(x)W(y|x)\).
A proof for the above lemma may be found in [15]. The advantage of this lemma is that the capacity of a “difficult” channel may be lower bounded by the capacity of a “manageable” channel. The accuracy of this approximation, depends on the choice of the auxiliary channel.

Consider a new channel \( \overline{W} \), where the output \( \overline{Y}_i \) is given by
\[
\overline{Y}_i = X_i + A_i + Z_i \sqrt{\eta S(P - \delta)},
\]
where \( \eta S(P - \delta) \) is now a constant, given \( P, \delta \) and \( N \). Unlike \([10]\), this is a memoryless AWCGN channel and it is one of the simplest channels to analyze, making it a good choice for an auxiliary channel. Before we proceed, we will prove the following lemma.

**Lemma 3.** Under \( \{X_i\} \) i.i.d. complex Gaussian inputs as defined in Section [II] we have
1. \( \mathbb{E}[Y_i] = \mathbb{E}[\overline{Y}_i] = 0 \),
2. \( \mathbb{E}[|Y_i|^2] = \mathbb{E}[|\overline{Y}_i|^2] = P - \delta + \sigma_A^2 + \eta S(P - \delta) \),
3. \( \mathbb{E}[Y_i - X_i]^2 = \mathbb{E}[|Y_i - X_i|^2] = \sigma_A^2 + \eta S(P - \delta) \).

**Proof.** We have
\[
\mathbb{E}[Y_i] = \mathbb{E}\left[X_i + A_i + Z_i \sqrt{\eta S_i}\right] = \mathbb{E}[X_i] + \mathbb{E}[A_i] + \mathbb{E}[Z_i] \mathbb{E}\left[\sqrt{\eta S_i}\right]
= 0,
\]
where in (36) we used the fact that \( Z_i \) and \( S_i \) are independent. Similarly, we can show that \( \mathbb{E}[\overline{Y}_i] = 0 \).

For the second part, we observe that even though \( X_i \) and \( Z_i \sqrt{\eta S_i} \) are dependent variables, they are uncorrelated. They are both zero mean and
\[
\mathbb{E}[X_iZ_i^* \sqrt{\eta S_i}] = \mathbb{E}[Z_i^*] \mathbb{E}[X_iZ_i \sqrt{\eta S_i}] = 0.
\]
Hence we have
\[
\mathbb{E}[|Y_i|^2] = \text{Var}(Y_i) = \mathbb{E}[|X_i|^2] + \mathbb{E}[|A_i|^2] + \eta \mathbb{E}[|Z_i|^2] \mathbb{E}[|S_i|] = P - \delta + \sigma_A^2 + \eta S(P - \delta).
\]
Similarly, we get the same value for \( \mathbb{E}[|\overline{Y}_i|^2] \).

The proof of the third part is the same as the proof of the second part with \( X_i = 0 \) in (39).

Now we consider \( I(X^{n+N}; Y^n) \). Extending Lemma 2 to vectors, we get
\[
I(X^{n+N}; Y^n) \geq \mathbb{E}\left[\frac{\overline{W}(Y^n|X^{n+N})}{P_X \overline{W}(Y^n)}\right],
\]
where the expectation is taken with respect to the joint density of the input distribution and the original channel density. Note that we have
\[
\overline{W}(y^n|x^{n+N}) = \overline{W}(y^n|x^n) = \prod_{i=1}^{n} \overline{W}(y_i|x_i)
\]
and
\[
P_X \overline{W}(y^n) = \prod_{i=1}^{n} P_X \overline{W}(y_i),
\]
due to the iid nature of inputs and because \( \overline{W} \) is a memoryless channel. Substituting these values in (40), yields
\[
I(X^{n+N}; Y^n) \geq n \log_2\left(1 + \frac{P - \delta}{\sigma_A^2 + \eta S(P - \delta)}\right) - \mathbb{E}\left[\frac{||Y^n - X^n||^2}{(\sigma_A^2 + \eta S(P - \delta))}\right] \log_2(e)
+ \mathbb{E}\left[||Y^n||^2\right] \log_2(e).
\]
We have by Lemma 3
\[
\mathbb{E}[||Y^n||^2] = \sum_{k=1}^{n} \mathbb{E}[|Y_i|^2] = n(P - \delta + \sigma_A^2 + \eta S(P - \delta)).
\]
and
\[
\mathbb{E}[||Y^n - X^n||^2] = n(\sigma_A^2 + \eta S(P - \delta)).
\]
Putting it all together and after taking limits as \( n \to \infty \), we take \( \delta \to 0 \). Thus we get
\[
C \geq \lim_{n \to \infty} I(X^{n+N}; Y^n) \geq \log_2\left(1 + \frac{P}{\sigma_A^2 + \eta S(P)}\right).
\]

The function on RHS, as a function of \( P \), attains its maximum at \( P_N^* \) given in (33). For \( P < P_N^* \), it is a monotone increasing function of \( P \). However after \( P_N^* \), it decreases towards 0. Given powers \( P_1 \) and \( P_2 \) such that \( P_1 < P_2 \), we note that a codeword satisfying average power constraint \( P_1 \) for this channel, also satisfies the constraint for \( P_2 \). Hence an achievable rate for \( P_1 \) is also achievable under \( P_2 \). This means that for this channel, the capacity is monotone non-decreasing as a function of average power constraint \( P \). Fig. 2 illustrates this property for the example provided in Section VII.

With the above argument, we have finally proved (32).
A. Optimality on choice of variance

While using Lemma 2 the auxiliary channel was chosen by replacing \( Z_i \sqrt{\eta S_i^3} \) with \( Z_i \sqrt{nS_i(P - \delta)} \). Now what if there is a constant \( V \) such that replacing \( S(P - \delta) \) with \( V \) leads to a better lower bound? It turns out that \( S(P - \delta) \) is the optimal choice for that constant. To see this, substitute \( S(P - \delta) \) with \( V \) in (43). We get

\[
h(V) = n \log_2 \left( 1 + \frac{P - \delta}{\sigma_A^2 + \eta V} \right) - \mathbb{E} \left[ \frac{\|Y^n - X^n\|^2}{(\sigma_A^2 + \eta V)} \right] \log_2(e) + \mathbb{E} \left[ \frac{\|Y^n\|^2}{(P + \sigma_A^2 + \eta V)} \right] \log_2(e). \tag{47}
\]

Using Lemma 3 we get

\[
h(V) = n \log_2 \left( 1 + \frac{P - \delta}{\sigma_A^2 + \eta V} \right) - \frac{\sigma_A^2 + \eta S(P - \delta)}{\sigma_A^2 + \eta V} \log_2(e) + \frac{P - \delta + \sigma_A^2 + \eta S(P - \delta)}{P - \delta + \sigma_A^2 + \eta V} \log_2(e). \tag{48}
\]

Differentiating with respect to \( V \) and equating to 0 gives us \( V = S(P - \delta) \) as the maximizer.

B. Variation of lower bound with memory

Consider \( S(P) \) as defined in (31). For fixed \( P \), \( S(P) \) decreases as memory \( N \) increases, which causes an overall increase in the lower bound. Hence with increasing memory, our lower bounds increase for a fixed \( P \). This is illustrated in Fig. 3 for the example in Section VI.

If we were to consider the scenario where i.i.d. inputs (not necessarily Gaussian but with \( \mathbb{E}[X_i^2] = P \)) are provided to the channel, then

\[
S_i = \frac{1}{2N + 1} \sum_{k=-N}^{i+N} |X_k|^2 \rightarrow P, \text{ a.s.} \tag{49}
\]

as \( N \to \infty \), by the strong law of large numbers [20].

Consider the channel

\[
Y_i = X_i + A_i + Z_i \sqrt{\eta P_i^3}, \tag{50}
\]

with average input power constraint \( P \). This is nothing but the finite memory channel where \( S_i \) is replaced with \( P \) and is known as the Gaussian noise (GN) model for fiber optic channels (see [11]) . This channel is often considered (see [21], [22] and [23]) while modeling non-linearities without considering the effects of memory. As this happens to be an AWGN channel, we may apply Shannon’s channel capacity theorem [16] to obtain capacity as

\[
C_G = \log_2 \left( 1 + \frac{P}{\sigma_A^2 + \eta P^3} \right). \tag{51}
\]

We see that this capacity function first increases as a function of \( P \), reaches its maximum value at

\[
P_G^* = \left( \frac{\sigma_A^2}{2\eta} \right)^{1/3} \tag{52}
\]

and then decreases to 0. Recalling the definition of \( P_N^* \) in (33), we see that

\[
\lim_{N \to \infty} P_N^* = P_G^*. \tag{53}
\]

Hence for \( P \leq P_G^* \), we deduce that the lower bound tends to the capacity of the GN model with increasing memory. Fig. 3 illustrates this effect for the example in Section VI. For \( P > P_G \), we cannot conclude anything as in this regime, the capacity of the finite memory model may exceed that of the GN model.

V. ALGORITHM FOR CHANNEL CAPACITY LOWER BOUND

We now describe an algorithm that calculates a lower bound on channel capacity for the channel with memory. The algorithm follows principles discussed in [14], [15]. This gives an achievable bound better than the theoretical lower bound we described. For simplicity, we will assume that the memory \( N = 1 \) throughout our discussion.

A. Message passing algorithm for i.i.d. inputs

To compute the mutual information, we need to compute \( h(Y^n) \) and \( h(Y^n | X^{n+1}) \). We note that

\[
h(Y^n | X^{n+1}) = \sum_{i=1}^{n} \mathbb{E} \left[ \log \left( \pi e \left( \sigma_A^2 + \eta S_i^3 \right) \right) \right]. \tag{54}
\]

However due to the i.i.d. nature of the inputs, all terms in the summation (except \( i = 1 \)) will be equal. Hence this expectation can be efficiently evaluated using Monte Carlo integration methods. We focus on computing \( h(Y^n) \).
We suitably modify the message passing algorithm given in [14], [15] to describe the channel in [10]. The algorithm essentially computes a Monte Carlo estimate of mutual information.

1: Generate $N_s + 1$ samples of $\{x_k\}$, quantized complex Gaussian or any test input distribution $p_X$ that satisfies the input power constraints. $N_s$ is the number of iterations we will run the algorithm to get satisfactory convergence.
2: Compute the corresponding output samples $\{y_k\}$ via (10).
3: 
   
   $\mu_1(x_1, x_2) \leftarrow p_X(x_1)p_X(x_2|y_1, x_1, x_2)$ 

   $\forall (x_1, x_2)$.
4: $\lambda_1 \leftarrow \sum x_1, x_2 \mu_1(x_1, x_2)$.
5: $\mu_1(x_1, x_2) \leftarrow \mu_1(x_1, x_2)/\lambda_1 \forall (x_1, x_2)$.
6: for $k = 2$ to $N_s$ do
7: 
   
   $\mu_k(x_k, x_{k+1}) \leftarrow \sum_{x_{k-1}} \mu_{k-1}(x_{k-1}, x_k)p_X(x_k)$

   $\times p_{Y_k|X_{k-1}, X_k, X_{k+1}}(y_k|x_{k-1}, x_k, x_{k+1})$

   $\forall (x_k, x_{k+1})$.
8: $\lambda_k \leftarrow \sum_{x_k, x_{k+1}} \mu_k(x_k, x_{k+1})$.
9: $\mu_k(x_k, x_{k+1}) \leftarrow \mu_k(x_k, x_{k+1})/\lambda_k$.
10: end for
11: $h(Y^n) = -\sum_{i=1}^{N_s} \log(\lambda_i)/N_s$.

Analogous extensions are possible for $N > 1$. We use this algorithm to numerically evaluate a lower bound for channel capacity of [10] in Section VI.

VI. Numerical Results

We compare our results with those provided in [11]. We use the same system parameters as in [11]. Hence we choose the nonlinearity parameter $\eta = 7244$ W$^{-2}$ and the ASE noise variance as $\sigma^2 = 4.1 \times 10^{-6}$ W. Fig. 3 is a plot of our results using (32) and Fig. 4 plots the lower bounds from [11].

We infer from the plots that our bounds fare better than the bounds in [11] for $N \geq 1$. Moreover, unlike in [11], our bounds increase with increasing memory $N$ as explained in Section IV-B.

We also evaluate the lower bound with i.i.d. quantized complex Gaussian inputs for varying input power constraints via the algorithm in Section V. To do this, first generate a Gaussian random variable $g$ with variance $V$. Next, given $0 < \varepsilon < 1$, we split the interval $[-x_T, x_T]$ into $2(M-1)$ equal sub intervals, along with the intervals $[x_T, \infty)$ and $(-\infty, -x_T]$; where $x_T = 2VQ^{-1}(\varepsilon/2)$, where $Q^{-1}$ is the inverse Q function. Then if $g$ belongs to one of the subintervals, say $[a, b]$, then output $a$ if $g \geq 0$ or $b$ otherwise. It follows that the variance of the samples generated in this way will have variance less than $V$. To generate a complex quantized Gaussian sample $cg$, generate $g_1$ and $g_2$ quantized Gaussian

with variance $V/2$ and use $cg = g_1 + ig_2$. We take $M = 10$ and $\varepsilon = 10^{-5}$ for quantization of the Gaussian input and $N_s = 10000$. Fig. 5 compares the algorithmic bound (with monotone extension) with the theoretical results derived earlier. We see that the algorithmic bound significantly outperforms all known bounds.

This example shows that the lower bounds (32) and in [11] are not tight. The bound (32) is tighter than that in [11] and unlike the bound in [11], is in closed form. Also, although the algorithm in Section V provides much better rates, its complexity is quite high even for $N = 1$. It rapidly increases for higher $N$. Furthermore, unlike (32), it does not provide any structural or qualitative/quantitative insights. Thus it will be more useful to have better closed form lower and upper bounds.
VII. Conclusion

We have derived an improved lower bound in closed form on the channel capacity of an optical channel with memory. We have also shown that the bound improves with increase in memory. Next, we used an algorithm to compute an even better lower bound on the capacity.

The next step would be to derive a bound that is monotonically increasing with $P$ (and not just extended as is the case with our result). This would require us to exploit the memory in the channel to aid transmission or decoding and not just circumvent it. Moreover if a useful, computable upper bound on channel capacity could be derived, it would help in checking the accuracy of the lower bounds. Unfortunately, upper bounds are usually much harder to derive than lower bounds as they need to be optimized for any permissible input distribution and even now, we do not have any known upper bounds on the capacity of this channel.

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