ON SUMMABLE, POSITIVE POISSON-MEHLER KERNELS BUILT OF AL-SALAM–CHIHARA AND RELATED POLYNOMIALS

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Abstract. Using special technique of expanding ratio of densities in an infinite series of polynomials orthogonal with respect to one of the densities, we obtain simple, closed forms of certain kernels built of the so called Al-Salam—Chihara (ASC) polynomials. We consider also kernels built of some other families of polynomials such as the so called big continuous $q$–Hermite polynomials that are related to the ASC polynomials. The constructed kernels are symmetric and asymmetric. Being the ratios of the densities they are automatically positive. We expand also reciprocals of some of the kernels, getting nice identities built of the ASC polynomials involving 6 variables like e.g. formula (3.6). These expansions lead to asymmetric, positive and summable kernels. The particular cases (referring to $q=1$ and $q=0$) lead to the kernels build of certain linear combinations of the ordinary Hermite and Chebyshev polynomials.

1. Introduction

In many models of the so called $q$–oscillators considered in quantum physics, classical and noncommutative probability or generally in some branches of analysis appears a problem of summing and examining positivity of kernels built of certain families of orthogonal polynomials (see e.g. [8], [9], [1]). The kernels (more precisely the Poisson–Mehler kernels) are, generally speaking, expressions of the form $K(x, y) = \sum_{n \geq 0} a_n D_n(x) F_n(y)$, where $\{D_n\}_{n \geq 0}$, $\{F_n\}_{n \geq 0}$ are certain families of orthogonal polynomials and $x, y, \{a_n\}_{n \geq 0}$ are real numbers. Usually the numbers $a_n$ are of the form $t^n / \|D_n\| \|F_n\|$, $|t| < 1$ where $\|\cdot\|$ denotes certain (usually $L_2$) norm. Often $D_n = F_n$, then such kernels are called symmetric otherwise they are called asymmetric. Sometimes the symmetric kernels, with $a_n = t^n / \|D_n\|^2$ are called bilinear generating functions.

Notice that we have $\int K(x, y) D_n(x) d\delta(x) = a_n \|D_n\|^2 F_n(y)$ where $\delta(x)$ denotes the measure with respect to which polynomials $\{D_n\}$ are orthogonal. The property often exploited in various branches of analysis including integral equations.

One would like to express such kernels in closed, compact forms and what is more important give conditions under which such kernels are nonnegative for all $x, y$ from certain Cartesian product of intervals.

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Then for example an expression 
\[ h(x, y) = f(x)K(x, y)g(y) \]
with \( K \) being a positive kernel, \( f \) and \( g \) being probability densities such that the polynomials \( \{ D_n \} \) and \( \{ F_n \} \) are orthogonal with respect to the measures with these densities would give a Lancaster expansion of certain 2 dimensional probabilistic density. This so since \( \int \int h(x, y) \, dx \, dy = 1 \) and \( h(x, y) \geq 0 \). For the Lancaster expansions and their most important properties see e.g. [15].

The problem of summing and examining positivity of kernels is nontrivial and was solved only for a few families of polynomials including the \( q \)-Hermite polynomials. Thus it is important to extend the list of polynomials for which kernels are summable and nonnegative.

This paper extends such a list by adding to it two families of orthogonal polynomials (indexed by real parameter \( q \)). One family added partially are the so called big \( q \)-Hermite (bH) polynomials and the other family more completely added are the so called Al-Salam–Chihara (ASC) polynomials and of course the \( q = 1 \) and \( q = 0 \) special cases. These special cases simplify for \( q = 1 \) to respectively shifted and shifted and re-scaled Hermite polynomials and for \( q = 0 \) to certain combinations of the Chebyshev polynomials of the second kind.

The summation formulae for kernels built of bH and ASC polynomials have been found in 1996 in [3]. They were expressed with the help of the basic hypergeometric functions: \( \phi \) in the case of bH polynomials and \( \hat{\phi} \) (in fact \( \hat{W} \)) in the case of ASC polynomials under some conditions imposed on some parameters of considered families of polynomials. Hence they are 'inconsumable' for the reader not familiar with specialized branches of special functions theory. The positivity of obtained kernels was shown in [3] only for the very special cases.

We will present a compact, simple, summable form of those kernels for different (than in [3]) sets of restrictions imposed on the parameters and for special values of the parameter \( t \). In summing kernels we apply the original method of expanding the ratio of densities in a certain Fourier series of polynomials that are orthogonal with respect one of the densities. This technique was used successfully in [16] to present new proofs of compact forms of some kernels. Thus we obtain some new results and also known results by the new technique. Besides we obtain expansions of reciprocals of some of the kernels. This procedure leads to non-symmetric kernels that are naturally nonnegative.

To help those who are not familiar with the \( q \)-series theory we will present basic notation and basic notions used in this theory and also point out and discuss in detail two special cases. As stated above the case \( q = 1 \) leads to shifted and re-scaled Hermite polynomials and \( q = 0 \) leads to linear combinations of Chebyshev polynomials of the second kind. Hence, although focused mostly on the applications of \( q \)-series theory the paper can also be interesting for the people working in orthogonal polynomials.

The paper is organized as follows. In the next section we present basic notation and notions of the \( q \)-series theory, then we introduce and recall basic properties of families of polynomials discussed in the paper. Next section presents main results while the last section contains some longer, less interesting proofs.

2. Preliminaries

In order to formulate briefly the properties of bH and ASC polynomials we must introduce notation traditionally used in the so called \( q \)-series theory. \( q \) is a real
parameter $-1 < q \leq 1$. Having $q$ we define \([0]_q = 0; [n]_q = 1 + q + \ldots + q^{n-1}, [n]_q! = \prod_{i=1}^n [i]_q\), with \([0]_q! = 1, [n]_q! = \left\{ \frac{[n]_q!}{[n-k]_q![k]_q!}, n \geq k \geq 0, \text{ otherwise} \right. \). It will be useful to use the so called $q$–Pochhammer symbol for $n \geq 1$:

\[
(a; q)_n = \prod_{i=0}^{n-1} (1 - aq^i), \quad (a_1, a_2, \ldots, a_k; q)_n = \prod_{i=0}^{k-1} (a_i; q)_n,
\]

with $(a; q)_0 = 1$. Often $(a; q)_n$ as well as $(a_1, a_2, \ldots, a_k; q)_n$ will be abbreviated to $(a)_n$ and $(a_1, a_2, \ldots, a_k)_n$, if it will not cause misunderstanding.

It is easy to notice that $(q)_n = (1 - q)^n [n]_q!$ and thus that:

\[
\frac{n}{k} \equiv \left\{ \frac{(q)_n}{(q)_{n-k}(q)_k}, n \geq k \geq 0, \frac{0}{0}, \text{ otherwise} \right. \).
\]

Notice that for $n \geq k \geq 0$ we have: $[n]_1 = n, [n]_1! = n!$, $[n]_k = \binom{n}{k} (a; 1)_n = (1 - a)^n$ and $[n]_0 = \left\{ \begin{array}{ll} 1 \quad & n \geq 1 \\ 0 \quad & n = 0 \end{array} \right.$, $[n]_0! = 1$, $[n]_0! = 1$; $(a; 0)_n = \left\{ \begin{array}{ll} 1 - a \quad & n = 0 \\ a \quad & n \geq 1 \end{array} \right.$.

The continuous $q$–Hermite polynomials $h_n (x|q)$ are defined by the following 3–term recurrence

\[
h_{n+1} (x|q) = 2xh_n (x|q) - (1 - q^n)h_{n-1} (x|q),
\]

with $h_{-1} (x|q) = 0$, $h_0 (x|q) = 1$, for $|q| < 1$. Many facts are known about these polynomials. The most important recent references are [11] and [13].

We will consider the modified version of these polynomials mostly because these modifications have nice probabilistic interpretations and applications see e.g. [5], [4], [6], [13], [17], [16]. Namely we will consider the polynomials $\{H_n (x|q)\}$ defined by the relationship for $q \neq 1$:

\[
H_n (x) = h_n \left( x \sqrt{1 - q/2}q \right) / (1 - q)^{n/2}; n \geq 0
\]

and $H_n (x|1) = H_n (x)$, where $\{H_n (x)\}$ are the so called probabilistic Hermite polynomials that is polynomials orthogonal with respect to the measure that has density equal to exp $(-x^2/2) / \sqrt{2\pi}$. To see that really $H_n (x|q) = H_n (x)$ one has to refer to 3–term recurrence satisfied by the polynomials $\{H_n (x|q)\}$ that in this case has the following simple form for $n \geq 0$:

\[
(2.1)
\]

\[
H_{n+1} (x|q) = xH_n (x|q) - [n]_q H_{n-1} (x|q),
\]

with $H_{-1} (x|q) = 0$, $H_0 (x|q) = 1$. Symbol $[n]_q$ was explained above and also above we remarked that $[n]_1 = n$ hence for $q = 1$ (2.1) reduces to well known (see [11] or [2]) 3–term recurrence satisfied by ’probabilistic’ Hermite polynomials i.e. the ones that are orthogonal with respect to measure with the density exp $(-x^2/2) / \sqrt{2\pi}$.

From (2.1) it follows also that $H_n (x|0) = U_n (x/2)$, where $\{U_n (x)\}_{n \geq 0}$ are the so called Chebyshev polynomials of the second kind. It is so since $[n]_0 = 1$ for $n \geq 1$. More precisely polynomials $\{U_n \}$ satisfy the following 3-term recurrence:

\[
U_{n+1} (x) = 2xU_n (x) - U_{n-1} (x),
\]
with \( U_{-1} (x) = 0, U_0 (x) = 1 \). To learn more about Chebyshev polynomials the reader is referred either to [2] or to [11]. It is known in particular that the characteristic function of the polynomials \( \{ H_n \} \) for \( |q| < 1 \) is given by the formula:

\[
\sum_{n=0}^{\infty} \frac{t^n}{\|q\|^n} H_n (x|q) = \varphi (x|t, q),
\]

where

\[(2.2) \quad \varphi (x|t, q) = \frac{1}{\prod_{k=0}^{\infty} (1 - (1 - q)x^k + (1 - q)t^2 q^{2k})}.
\]

Convergence here is absolute for \( |\sqrt{1 - qt}| < 1 \). For the proof of this formula see e.g. [11]. \( (2.2) \) will be justified as a simple corollary of our main result presented below. Besides because of the fact that \( H_n (x|1) = H_n (x) \) we set

\[\varphi (x|t, 1) = \exp (xt - t^2 / 2).\]

Notice also that:

\[\varphi (x|t, 0) = \frac{1}{1 - xt + t^2}.
\]

It is also known (again see [11]) that for \(-1 < q < 1\), these polynomials are orthogonal with respect to the positive measure defined by the following density:

\[(2.3) \quad f_N (x|q) = \frac{\sqrt{1 - q} (q)^n}{2\pi \sqrt{4 - (1 - q)x^2}} \prod_{k=0}^{\infty} ((1 + q^k)^2 - (1 - q)x^2 q^{2k}) I_S (q) (x),
\]

\( x \in \mathbb{R} \), where \( S (q) = [-2/\sqrt{1 - q}, 2/\sqrt{1 - q}] \) and \( I_A (x) = \left\{ \begin{array}{ll} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{array} \right. \). That is

\[
\int_{S (q)} H_n (x|q) H_m (x|q) f_N (x|q) \, dx = [n|q| \delta_{m,n},
\]

where \( \delta_{n,m} = 0 \) if \( n \neq m \) and 1 for \( m = n \). The case \( q = 1 \) was presented above. We set then \( f_N (x|1) = \exp (-x^2 / 2) / \sqrt{2\pi} \). One can also easily notice that

\[f_N (x|0) = \frac{1}{2\pi} \sqrt{4 - x^2 I_{[-2,2]} (x)}. \]

The big continuous \( q \)-Hermite polynomials (briefly continuous \( bH \) polynomials) are defined as the polynomials satisfying the following 3-term recurrence (see e.g. [11]):

\[(2.4) \quad 2xh_n (x|a, q) = h_{n+1} (x|a, q) + aq^n h_n (x|a, q) + (1 - q^n) h_{n-1} (x|a, q).
\]

One can immediately notice that for \( a = 0 \) we have \( h_n (x|a, q) = h_n (x|q) \). One proves (see e.g. [8]) that

\[(2.5) \quad h_n (x|a, q) = \sum_{i=0}^{n} \left[ \begin{array}{l} n \\ i \end{array} \right] q \, (-1)^i q^{i(2)} a^i h_{n-i} (x|q), \quad \text{and} \quad h_n (x|q) = \sum_{i=0}^{n} \left[ \begin{array}{l} n \\ i \end{array} q \right] a^i h_{n-i} (x|a, q),
\]

where \( \{ h_i (x|q) \} \) denote the continuous \( q \)-Hermite polynomials.

Let us define re-scaled continuous \( bH \) polynomials by defining for \( n \geq 0 \):

\[(2.6) \quad H_n (x|a, q) = h_n \left( x\sqrt{1 - q}/2a \sqrt{1 - q} q \right) / (1 - q)^{n/2}.
\]

We will call them the big \( q \)-Hermite (briefly \( bH \)) polynomials.
We immediately have:

**Proposition 1.**

\[(2.7) \quad H_n(x|a,q) = \sum_{i=0}^{n} \binom{n}{i} q^{(i)}(-a)^i H_{n-i}(x|q),\]

\[(2.8) \quad H_n(x|q) = \sum_{i=0}^{n} \binom{n}{i} a^i H_{n-i}(x,a,q)\]

**Proof.** It is trivial in view of (2.5).

One can also rewrite (2.4) in terms of the polynomials \(H_n(x|a,q)\):

\[(2.9) \quad xH_n(x|a,q) = H_{n+1}(x|a,q) + aq^nH_n(x|a,q) + [n]_q H_{n-1}(x|a,q)\]

In [10] (formula 3.18.2) we can also find the density of the measure with respect to which polynomials \(h_n(x|a,q)\) are orthogonal. One can easily rewrite this density for the polynomials \(H_n(x|a,q)\). We have then:

\[(2.10) \quad f_{0N}(x|a,q) = f_N(x|q) \varphi(x|a,q).\]

Notice that here we have to assume \(|a\sqrt{1-q}| < 1\), since then the series \(\sum_{n=0}^{\infty} \frac{a^n}{[n]_q} H_n(x|q)\) is absolutely convergent. By the way one shows (see [10]) that for \(|a\sqrt{1-q}| > 1\) polynomials \(H_n\) are orthogonal with respect to a mixed measure. That is the measure that apart from absolutely continuous part has also several (depending on the size of parameter \(a\)) discrete "mass" points. However since we are interested in the expansions of ratios of the densities we will consider the absolutely continuous case only.

The cases \(q = 0\) and \(q = 1\) are treated in the following Remark.

**Remark.** i) \(\forall n \geq 0 : H_n(x|a,1) = H_n(x-a)\)

ii) \(\forall n \geq 0 : H_n(x|a,0) = U_n(x/2) - aU_{n-1}(x/2)\)

**Proof.** i) Follows (2.7) and the well known formula concerning Hermite polynomials:

\(H_n(x-a) = \sum_{i=0}^{n} \binom{n}{i}(-a)^i H_{n-i}(x)\).

ii) Follows (2.7) and the fact that \(q^{(i)}\) for \(q = 0\) is nonzero only for \(i = 0\) and \(i = 1\).

Now let us introduce the so called Al-Salam–Chihara (ASC) polynomials \(P_n(x|y,\rho,q)\) that are defined by the following 3-term recurrence:

\[(2.11) \quad P_{n+1}(x|y,\rho,q) = (x-\rho y q^n) P_n(x|y,\rho,q) - [n]_q (1-\rho^2 q^{n-1}) P_{n-1}(x|y,\rho,q),\]

with \(P_{-1}(x|y,\rho,q) = 0, P_0(x|y,\rho,q) = 1\). In fact this is again modified version of the other family of polynomials. Namely in the literature (see e.g. [10] or [11]) better known (under the name of ASC) are polynomials defined by the following 3-term recurrence:

\[(2.12) \quad p_{n+1}(x|a,b,q) = (2x-(a+b)q^n)p_n(x|a,b,q) - (1-q^n)(1-abq^{n-1})p_{n-1}(x|a,b,q),\]

with \(p_{-1}(x|a,b,q) = 0, p_0(x|a,b,q) = 1\). One can easily deduce that if we take \(a\) and \(b\) defined by \(ab = \rho^2\) and \(a + b = \rho q\sqrt{1-q}\), then:

\[(2.13) \quad P_n(x|y,\rho,q) = p_n(x\sqrt{1-q}/2|a,b,q)/(1-q)^{n/2}.\]
We will assume that \( y \in S(q) \) and \(|\rho| < 1\). Notice that then the parameters \( a, b \) are complex but forming the conjugate pair and also that such a choice of parameters \( a, b \) is reasonable since from the Favard’s theorem (e.g. [11]) it follows that for \(|ab| \leq 1\) the measure with respect to which the polynomials \( p_n \) are orthogonal is positive. Again polynomials \( P_n \) have nice probabilistic interpretation (again see for example [6], [18], [17], [19], [20]) that is why we will use them.

Again we have a remark concerning special cases \( q = 0 \) and \( q = 1 \).

**Remark 2.** i) \( \forall n \geq 0 : P_n (x|y, \rho, 1) = H_n \left( \frac{x - \rho y}{\sqrt{1 - \rho^2}} \right) (1 - \rho^2)^{n/2} \).

ii) \( \forall n \geq 1 : P_n (x|y, \rho, 0) = U_n (x/2) - \rho y U_{n-1} (x/2) + \rho^2 U_{n-2} (x/2) \), with \( P_0 (x|y, \rho, 0) = 1 \).

**Proof.** i) For \( q = 1 \) let us denote

\[
\tilde{P}_n(x) = H_n \left( \frac{x - \rho y}{\sqrt{1 - \rho^2}} \right) (1 - \rho^2)^{n/2}.
\]

Using 3-term recurrence (2.1) for \( q = 1 \) and remembering that \( [n]_1 = 1 \) we see that polynomials \( \left\{ H_n \left( \frac{x - \rho y}{\sqrt{1 - \rho^2}} \right) \right\} \) satisfy the following 3-term recurrence:

\[
H_{n+1} \left( \frac{x - \rho y}{\sqrt{1 - \rho^2}} \right) = H_n \left( \frac{x - \rho y}{\sqrt{1 - \rho^2}} \right) - nH_{n-1} \left( \frac{x - \rho y}{\sqrt{1 - \rho^2}} \right),
\]

which after multiplying both sides \((1 - \rho^2)^{(n+1)/2}\) and applying definition of \( \tilde{P}_n \) reduces to

\[
(2.14) \hspace{1cm} \tilde{P}_{n+1} (x) = (x - \rho y) \tilde{P}_n (x) - n(1 - \rho^2) \tilde{P}_{n-1} (x),
\]

with initial conditions \( P_{-1} (x) = 0, \tilde{P}_0 (x) = 1 \). Now let us consider equation (2.11) for \( q = 1 \) (it is a polynomial in \( q \)). It is elementary to notice that \( P_n (x|y, \rho, 1) \) satisfies 3-term recurrence (2.14) with the same initial conditions.

ii) Follows formula (2.20) presented below and the properties of polynomials \( B_n \) also listed below. In particular the fact that \( B_n (x|0) = 0 \) for \( n \geq 3 \).

This nice interpretation presented in the Remark 2 is the second reason why we prefer to work with the polynomials \( P_n \).

It is known (again e.g. [6], [11] or [17]) that

\[
(2.15) \hspace{1cm} \sum_{i=0}^{\infty} \frac{t^i}{[i]_q!} P_i (x|y, \rho, q) = \frac{\varphi (x|t, q)}{\varphi (y|\rho t, q)},
\]

where function \( \varphi (x|t, q) \) is given by (2.2) and that

\[
(2.16) \hspace{1cm} \int_{S(q)} P_n (x|y, \rho, q) P_m (x|y, \rho, q) f_{CN} (x|y, \rho, q) \, dx = (\rho^2)^n \, [n]_q! \delta_{m,n},
\]

where

\[
(2.17) \hspace{1cm} f_{CN} (x|y, \rho, q) = f_N (x|q) \prod_{k=0}^{\infty} w (x, y, \rho q^k|q),
\]

with \( w (x, y, \rho|q) \) defined by:

\[
(2.18) \hspace{1cm} w (x, y, \rho|q) = (1 - \rho^2)^2 - (1 - q) \rho(1 + \rho^2)xy + (1 - q) \rho^2(x^2 + y^2).
\]
In (2.15) and (2.17) we assume $|q| < 1$, $x, y \in S(q)$, $|\rho| \leq 1$, $|\sqrt{1-q}| < 1$ assuring that the series and infinite product are absolutely convergent.

We have also the following relationship between the Poisson kernel built of $q$–Hermite polynomials and the density $f_{CN}$:

$$(2.19) \quad \prod_{k=0}^{\infty} w(x, y, \rho \rho^k|q) = \sum_{i \geq 1} \frac{\rho^i}{|i|!} H_i(x|q) H_i(y|q).$$

There are many proofs of (2.19). For some see e.g. [11], [7], [16]. Convergence here is absolute for $|\rho| < 1$, $x, y \in S(q)$. (2.19) is known under the name the Poisson–Mehler expansion formula. Generally discussion of the convergence conditions for expressions involving $q$–Hermite polynomials is done in [10] or [18].

Again as before we have the following remark where special values of parameter $q$ are considered.

**Remark 3.** i) For $q = 1$ we have $f_{CN}(x|y, \rho, 1) = \exp \left( \frac{-1}{2} \rho^2 \sqrt{1-q} \right)$ (so called Kesten–McKay density).

ii) For $q = 0$ we have $f_{CN}(x|y, \rho, 0) = \frac{1}{2\pi w(x, y, \rho|0)}$.

**Proof.** i) Rigorous proof that $f_{CN}(x|y, \rho, q) \to \exp \left( \frac{-1}{2} \rho^2 \sqrt{1-q} \right)$ as $q \to 1$ can be found in [12]. Less formal and more intuitive argument is the following. By Remark (2.1) we deduce that all moments tend (as $q \to 1$) to the moments of the limiting distribution, hence distribution with the density $f_{CN}$ must tend weakly to a distribution with the density $\exp \left( \frac{-1}{2} \rho^2 \right)$.

ii) We simply set $q = 0$ in (2.17). \( \square \)

Furthermore we will need some auxiliary polynomials that are in fact related to $H_n(x|q^{-1})$, $-1 < q \leq 1$. Namely let us consider polynomials $\{B_n\}_{n \geq -1}$ satisfying the following 3-term recurrence

$$B_{n+1}(x|q) = -q^n x B_n(x|q) + q^{n-1}[n]_q B_{n-1}(x|q), \quad (n \geq 0)$$

with the usual initial conditions $B_{-1} = 0$, $B_0 = 1$. To support intuition, let us remark (following [6]), that

$$B_n(x|1) = i^n H_n(ix),$$

and $B_n(x|0) = 0$, for $n \geq 3$, with $B_0(x|0) = 1$, $B_1(x|0) = -x$, $B_2(x|0) = 1$.

It was shown in [6] that:

$$(2.20) \quad \forall n \geq 0 : P_n(x|y, \rho, q) = \sum_{i=0}^{n} \binom{n}{i} \rho^{n-i} B_{n-i}(y|q) H_i(x|q).$$

We will use this formula to extend definition of ASC polynomials for $|\rho| \geq 1$. Of course for $|\rho| \geq 1$ ASC polynomials are not orthogonal with respect to a positive measure.

Now let us recall the idea of expansion of the ratio of two densities in a series of polynomials orthogonal with respect to the one of them presented with many examples in [16]. More precisely let us assume that we have two positive probability measures $d\alpha$ and $d\beta$ with densities respectively $A(x)$ and $B(x)$. Moreover let us
assume that \( \{a_n(x)\}_{n \geq 0} \) and \( \{b_n(x)\}_{n \geq 0} \) are sets of monic polynomials orthogonal with respect to \( da \) and \( d\beta \) respectively. Besides assume that \( \text{supp} \beta = \text{supp} \alpha \).

Further let us also assume that we know the so called ‘connection coefficients’ between set \( \{b_n\} \) and \( \{a_n\} \). More precisely let us assume that we know the numbers \( \gamma_{k,n} \) such that for every \( n \geq 0 \):

\[
a_n(x) = \sum_{k=0}^{n} \gamma_{k,n} b_k(x)
\]

for every \( x \in \mathbb{R} \). Besides assume that \( \int_{\text{supp}(B)} (B^2(x)/A^2(x))da(x) < \infty \), then we have

\[
(2.21) \quad B(x) = A(x) \sum_{n=0}^{\infty} \frac{\gamma_{0,n}}{\hat{a}_n} \hat{b}_n(x),
\]

where \( \hat{a}_n = \int_{\text{supp}(A)} a_n(x)^2 da(x) \), similarly for the polynomials \( b_n \). Convergence in \( (2.21) \) is \( L^2 \) convergence, however if coefficients \( \frac{\gamma_{0,n}}{\hat{a}_n} \) are such that \( \sum_{n \geq 0} \left( \frac{\gamma_{0,n}}{\hat{a}_n} \right)^2 \log^2 n < \infty \), then by the Rademacher-Menshov theorem we have almost pointwise, absolute convergence. In most cases interesting in the \( q \)-series theory this condition is trivially satisfied hence all expansions we are going to consider will be absolutely almost pointwise convergent.

Hence finding connection coefficients between two sets of orthogonal polynomials becomes crucial.

Returning to the big \( q \)-Hermite and ASC polynomials we have the following easy lemma.

**Lemma 1.** i) \( \forall n \geq 0, x, y, b, a, q \in \mathbb{R} : \)

\[
H_n(x|a,q) = \sum_{j=0}^{n} \binom{n}{j} P_j \left( \frac{x+a}{b}, q \right) \left( \frac{a}{b} \right)^{n-j} H_{n-j}(y|b,q).
\]

ii) \( \forall n \geq 0, x, y, \rho, a, q \in \mathbb{R} : \)

\[
P_n(x|y, \rho, q) = \sum_{i=0}^{n} \binom{n}{i} \rho^{n-i} B_{n-i}(y|a/\rho, q) H_i(x|a,q),
\]

where we denoted \( B_n(x|b,q) \) \( \equiv \sum_{j=0}^{m} \binom{m}{j} b^{m-j} B_j(x|q) \).

iii) \( \forall n \geq 0, x, y, \rho_1, \rho_2, q \in \mathbb{R} : \)

\[
P_n(x|y, \rho_1, \rho_2, q) = \sum_{i=0}^{n} \binom{n}{i} \rho_1^{n-i} P_i(x|z, \rho_1, q) P_{n-i}(z|y, \rho_2, q).
\]

iv) \( \forall n \geq 1, |q| < 1 : \)

\[
\max_{x \in S(q)} |H_n(x|q)| \leq (1-q)^{-n/2} r_n(1|q),
\]

\[
\max_{x \in S(q)} |H_n(x|a,q)| \leq \left( -|a|\sqrt{1-q} \right)^n (1-q)^{-n/2} r_n(1|q),
\]

where \( r_n(x|q) \) is given by \( r_n(x|q) = \sum_{i=0}^{n} \binom{n}{i} x^i \).
v) $\forall |q| < 1$, $x, y \in S(q) : 0 < \frac{(\rho^2)}{(|\rho|)_n^2} \leq f_{CN}(x|y, \rho, q) \leq \frac{(\rho^2)}{(|\rho|)_n^2}$

Easy and not very interesting proof of this lemma has been moved to Section 4.

3. Main results

Now using the described above idea of density expansion and Lemma 1 we have the following theorem.

**Theorem 1.** For $|b| > |a|, |q| < 1$, $x, y \in S(q)$ we have

1) (3.1)

$$0 \leq \sum_{n \geq 0} \frac{a^n}{|n| q^n} H_n(x|a, q) H_n(y|b, q) = \left(\frac{a^2}{b^2}\right)^{\infty} \prod_{k=0}^{\infty} \frac{1 - (1 - q)aq^k + (1 - q)a^2q^{2k}}{w(x, y, q^k a/b | q)}.$$

2) (3.2)

$$1/\sum_{n \geq 0} \frac{a^n}{|n| q^n} H_n(x|a, q) H_n(y|b, q) = \sum_{n \geq 0} \frac{a^n}{|n| q^n (a^2/b^2)_n} B_n(y|b, q) P_n(x|y, a/b, q).$$

3) For $|\rho_1|, |\rho_2|, |q| < 1$, $x, y, z \in S(q)$

$$0 \leq \sum_{n \geq 0} \frac{\rho_1^n \rho_2^n}{|n| q^n (\rho_1^2 \rho_2^2)^n} P_n(x|y, \rho_1 \rho_2, q) P_n(z|y, \rho_2, q) = \left(\frac{\rho_1^2}{\rho_2^2} \right)^{\infty} \prod_{k=0}^{\infty} \frac{w(x, y, q^k \rho_1 \rho_2 | q)}{w(x, z, q^k \rho_1 | q)}.$$

Proof. i) Let us apply assertion of the Lemma 1 with $\rho = a/b$. We get then:

$$H_n(x|a, q) = \sum_{i=0}^{n} \frac{n!}{i! q^n} P_i(x|y, a/b, q) (a/b)^{n-i} H_{n-i}(y|b, q).$$

Hence coefficient $\gamma_{0,n} = (a/b)^n H_n(y|b, q)$. Thus applying described above idea of expanding ratio of two densities we get

(3.3) $f_{CN}(x|y, a/b, q) = f_{CN}(x|q) \varphi(x|a, q) \sum_{n \geq 0} \frac{(a/b)^n}{|n| q^n} H_n(x|a, q) H_n(y|b, q).$

Since we have $\int_{S|q} H_n^2(x|a, q) f_{N}(x|q) \varphi(x|a, q) dx = |n| q!$ we apply (2.17).

ii) To get ii) we reason in the similar way this time using $P_n(x|y, a/b, q) = \sum_{i=0}^{n} \frac{n!}{i! q^n} (a/b)^{n-i} B_{n-i}(y|b, q) H_i(x|a, q)$ and keeping in mind (2.10). As far as convergence of both series is concerned we see that for $|\rho|, |q| < 1$ and $x, y \in S_q$ function $g(x|y, \rho, q) = f_{CN}(x|y, \rho, q) / f_{N}(x|q)$ is both bounded and 'cut away from zero' (compare (2.10) and (2.17)). Hence its square as well as reciprocal of this square are integrable on compact interval $S_q$.

iii) We reason in the similar way using assertion iii) of the Lemma 1. This time coefficient $\gamma_{0,n} = \rho_1^n P_n(z|y, \rho_2, q)$. Since we have (2.16) we can write

(3.4) $f_{CN}(x|z, \rho_1, q) = f_{CN}(x|y, \rho_1 \rho_2, q) \sum_{n \geq 0} \frac{\rho_1^n}{|n| q^n (\rho_1^2 \rho_2^2)^n} P_n(x|y, \rho_1 \rho_2, q) P_n(z|y, \rho_2, q).$

In all three cases convergence is absolute and almost sure since we have assertion v) of Lemma 1.

\[\square\]

\[1\] This form of the lower bound was suggested by the referee.
Notice that the right hand side of (3.2) can be written as \( f_{CN}(x|z, \rho_1, q)/f_{CN}(x|y, \rho_1 \rho_2, q) \) which would enable to pass easily with \( q \) to 1\(^{-}\). Keeping this in mind we have the following statement as a corollary:

**Corollary 1.** i) Setting \( q = 1 \) we get for \( x, y \in \mathbb{R} \) and \(|a| < |b|\):

\[
\sum_{n \geq 0} \frac{\rho^n}{n!} H_n(x - a) H_n(y - b) = \frac{1}{\sqrt{1 - \rho^2}} \exp\left( \frac{(x - \rho y)^2}{2(1 - \rho^2)} + \frac{(x - a)^2}{2} \right).
\]

ii) Setting \( q = 0 \) we get for \( x, y \in [-2, 2] \), \( 1 > |b| > |a| \):

\[
\sum_{n \geq 0} \frac{\rho^n}{n!} (U_n(x/2) - aU_{n-1}(x/2)) (U_n(y/2) - bU_{n-1}(y/2)) = (1 - \rho^2) (1 - ax + a^2) \frac{w(x, y, \rho|0)}{w(x, y, \rho|0)}.
\]

In both formulae above we denoted for simplicity \( \rho = a/b \).

iii) Setting \( q \to 1^- \) in (3.2) we get for \( x, y, z \in \mathbb{R} \) and \(|\rho_1|, |\rho_2| < 1\):

\[
\sum_{n=0}^{\infty} \frac{\rho^n}{n!} \frac{(1 - \rho_2)^{n/2}}{1 - \rho_1^2} \frac{(1 - \rho_2)^{n/2}}{1 - \rho_1^2} H_n\left( \frac{x - \rho_1 \rho_2 y}{\sqrt{1 - \rho_1^2 \rho_2^2}} \right) H_n\left( \frac{z - \rho_2 y}{\sqrt{1 - \rho_1^2 \rho_2^2}} \right) = \sqrt{1 - \rho_1^2 \rho_2^2} \frac{1}{\rho_1^2 \rho_2^2} \exp\left( -\frac{(x - \rho_1 z)^2}{2(1 - \rho_1^2)} + \frac{(x - \rho_1 \rho_2 y)^2}{2(1 - \rho_1 \rho_2)} \right).
\]

iv) Setting \( q = 0 \) in (3.2) we get for \( x, y, z \in [-2, 2] \) and \(|\rho_1|, |\rho_2| < 1\):

\[
1 + \frac{1}{(1 - \rho_1^2 \rho_2^2)} \sum_{n \geq 1} \rho_n P_n(x|y, \rho_1 \rho_2, 0) P_n(z|y, \rho_2, 0) = (1 - \rho_1^2) \frac{w(x, y, \rho_1 \rho_2|0)}{w(x, z, \rho_1|0)}.
\]

**Proof.** i) First we consider (2.11) for \( q = 1 \) and notice that \( \varphi(x|a, 1) = \exp(ax - a^2/2), f_N(x) = \exp(-x^2/2)/\sqrt{2\pi} \). By Lemma 1 i) and Remark 2 i) we have:

\[
H_n(x - a) = \sum_{j=0}^{n} \binom{n}{j} P_j(x|y, a/b, 1) \left( \frac{a}{b} \right)^{n-j} H_{n-j}(y - b).
\]

Now we apply (2.21) which proves (3.3) for \( q = 1 \) and consequently implies assertion. ii) We use (3.1) and Remark 2 i) iii) First we use Lemma 1 iii) with \( q = 1 \) and Remark 2 i). Then we use (2.21) and show (3.4) which implies the assertion. iv) Again we use (3.2) to get left hand side while (3.3) and the second assertion of Remark 2 to get the right hand side. \( \square \)

**Remark 4.** In [3] the following formula (14.14) (expressed in terms of polynomials \( h_n(x|a, q) \) with slightly different definition of big \( q \)-Hermite polynomials) was given:

\[
\sum_{n \geq 0} \left( \frac{t}{b} \right)^{n} \frac{1}{(q)_n} h_n(x|a, q) h_n(y|b, q) = \frac{1}{(a e^{i(\theta + \phi)}/b, t a e^{i\theta} - t a e^{-i\theta})}_\infty \times \\
3 \begin{phi} \left( \frac{t}{a^2 b^2}, a e^{i(\theta + \phi)}/b, a e^{i\theta} - a e^{-i\theta} ; q, a e^{-i\theta} \right), \right.
\]
with $x = \cos\theta$ and $y = \cos\phi$ convergent for $|x|, |y|, |t| \leq 1$ and $|q| < 1$. If $|b| < |a|$, $t = 1$ we get

$$\phi_2 \left(\frac{t}{a^2 b^2}, \frac{a \sin \phi}{b} e^{i\theta}; q, a e^{-i\theta} \right)_{t=1} = 1,$$

hence when polynomials $h_n$ are replaced by polynomials $H_n$, $a, b, x$ and $y$ by $a\sqrt{1-q}$, $b\sqrt{1-q}$, $x\sqrt{1-q}/2$ and $y\sqrt{1-q}/2$ respectively, we get left hand side of (3.1). To get right hand side of (3.1) we must use the fact that $\cos\theta$ and $\cos\phi$ should be replaced $x\sqrt{1-q}/2$ and $y\sqrt{1-q}/2$ respectively and also the following observations:

$$(1 - ae^{i\theta}\sqrt{1-q}q^k)(1 - ae^{-i\theta}\sqrt{1-q}q^k) = 1 - \sqrt{1-q}q^k \cos \theta + (1-q)a^2,$$

$$\frac{(1 - a^2e^{i(\theta+\phi)q^k})(1 - ab e^{-i(\theta-\phi)q^k})}{1 - b^2 e^{-i(\theta+\phi)q^k}}(1 - \frac{a}{b} e^{-i(\theta+\phi)q^k}) = w(x, y, qk\frac{a}{b}q).$$

However our proof of (3.1) is much simpler than the proof presented in [3]. Besides we obtain asymmetric kernel expansion of the reciprocal of (3.1).

**Remark 5.** In [3] there are formulae 14.5, and 14.8 (expressed in terms of polynomials $p_n(x)\alpha, b, q$) compare (2.12) and (2.13), leads to the condition $\alpha\beta = ab$ and of course $|t| \leq 1$. It was expressed in terms of the basic hypergeometric function $\phi\beta$. Positivity of those kernels was shown only in some particular special cases. Slightly more general form of kernels involving ASC polynomials are presented in [13].

Our kernel (3.2) is different. We do not need assumption $\alpha\beta = ab$ which, as indicated above (compare (2.12) and (2.13)), leads to the condition $\rho_1^2\rho_2^2 = \rho_2^2$. Recall that in our setting parameters $a, b$ are related to parameters $\rho, y \sqrt{1-q}$ and $a + b = \rho y\sqrt{1-q}$. We also assume that parameters $a, b, \alpha, \beta$ are related to one another however in a different way namely $(a + b) / \sqrt{ab} = (\alpha + \beta) / \sqrt{\alpha\beta}$. Thus our result although connected to known results is different and new, it was obtained by much simpler argument. Besides we have positivity of our kernels again 'for free'.

**Remark 6.** Notice that putting $\rho_2 = 0$ in (3.2) we get the Poisson–Mehler formula (i.e. formula (2.19)) since $P_n(x|y, 0, q) = H_n(x|q)$. Hence Theorem 1 provides yet another (and very simple, elementary) proof of the Poisson–Mehler formula. Similarly passing with $a$ and $b$ to zero in such a way that $a/b \to \rho$ we see that (3.1) is another generalization of the Poisson–Mehler formula.

**Remark 7.** If one formally extends the definition of ASC polynomials by using assertion ii) of Lemma 1 say with $a = 0$ for $|\rho| \geq 1$ (thus loosing the fact that they are orthogonal with respect to a positive measure) then assertion iii) of Theorem 1 can be rewritten in a more symmetric way (after redefining $\rho_1$ and $\rho_2$) in the following form: for $|\rho_1|, |\rho_2|, |q| < 1$, $x, y, z \in S(q)$

$$(3.5)\quad 0 \leq \sum_{n \geq 0} \frac{\rho_1^n}{n! q^n (\rho_2^n)} P_n(x|y, \rho_2, q) P_n(z|y, \rho_2, q) = \frac{(\rho_1^n)_{\infty}}{(\rho_2^n)_{\infty}} \prod_{k=0} w(x, y, qk^{|q|}),$$

from which directly follows the following inversion of the kernel formula: for $|\rho_1|, |\rho_2|, |q| < 1$, $x, y, z \in S(q)$

$$(3.6)\quad 1 = \sum_{n \geq 0} \frac{\rho_1^n}{n! q^n (\rho_2^n)} P_n(x|y, \rho_2, q) P_n(z|y, \rho_2, q) \times \sum_{n \geq 0} \frac{\rho_2^n}{n! q^n (\rho_1^n)} P_n(x|z, \rho_1, q) P_n(y|z, \rho_1, q).$$
Remark 8. Applying just for fun of checking the idea of expansion of the ratio of densities to the formulae (2.7) and (2.8) treated as the 'connection coefficient' formulae we get respectively:

\[
\begin{align*}
\phi(x|a, q) &= \sum_{n \geq 0} (-a)^n[n]_q q^n H_n(x|a, q), \\
\varphi(x|a, q) &= \sum_{n \geq 0} a^n[n]_q H_n(x|q).
\end{align*}
\]

Thus having yet another proof of the formula defining generating function of \(q\)-Hermite polynomials (3.8). As far as the formula (3.7) is concerned in [10] we find formula (3.18.14) which gives the sum

\[
\sum_{n \geq 0} (-t)^n(n)_q q^n H_n(x|a, q)
\]

in terms of hypergeometric function \(_1\phi_1\). Thus (3.7) gives special value of this generating function for \(t = a\).

4. Proofs

Proof of Lemma 7 We shall prove our identities for \(|q|, |\rho| < 1\). Then since both the left hand side and the right hand side of the above mentioned identity are polynomials in \(q\) and \(\rho\) of order at most \(n\) we will deduce that the identity is true for all \(q\) and \(\rho\). The proofs below are so to say by a 'direct method'.

However they can also be derived by 'characteristic function method' if one noticed the following:

\[
\begin{align*}
\varphi_{bH}(x|t, a, q) &= \sum_{j \geq 0} \frac{t^j}{[n]_q n!} H_n(x|a, q) = \varphi(x|t, q)(1-q) at_\infty, \\
\varphi_B(x|t, q) &= \sum_{j \geq 0} \frac{t^j}{[n]_q n!} B_n(x|q) = \frac{1}{\varphi(x|t, q)} \varphi_p(x|t, y, \rho, q) = \frac{\varphi(x|t, q)}{\varphi'(y|\rho t, q)}, \\
\varphi_{bB}(x|t, a, q) &= \sum_{j \geq 0} \frac{t^j}{[n]_q n!} B_n(x|a, q) = \frac{1}{\varphi(x|t, q)} (1-q) at_\infty
\end{align*}
\]

which are modified versions of the characteristic functions given in the literature: \(\varphi_{bH}(x|t, a, q)\) is a modification of appropriate formula from [10] and \(\varphi_B(x|t, q)\) from [6]. Then for example assertion i) follows identity

\[
\varphi_p(x|t, y, \frac{a}{\rho}, q) \varphi_{bH}(x|\frac{a}{\rho} t, a, q) = \varphi_{bH}(x|t, a, q).
\]

Similarly for the other assertions.

i) Recall that in [14] formula (4.7) it is a 'connection coefficient' formula between polynomials \(\{h_n\}\) and \(\{p_n\}\), which can be easily rewritten in terms of \(q\)-Hermite and ASC polynomials using (2.13). Formula (4.7) of [14] thus now reads:

\[
H_n(x|q) = \sum_{i=0}^{n} \binom{n}{i}_q \rho^{n-i} H_{n-i}(y|q) P_i(x|y, \rho, q).
\]
Let us use also (2.7). Then we get:

\[ H_n(x|a, q) = \sum_{i=0}^{n} \left[ \begin{array}{c} n \\ i \end{array} \right] (-1)^i q \binom{i}{q} a^i H_{n-i} (x|q) \]

\[ = \sum_{i=0}^{n} \left[ \begin{array}{c} n \\ i \end{array} \right] \rho^{n-i} \sum_{j=0}^{n-i} \left[ \begin{array}{c} n-i \\ j \end{array} \right] \rho^{n-i-j} H_{n-i-j} (y|q) P_j (x|y, \rho, q) \]

\[ = \sum_{j=0}^{n} \left[ \begin{array}{c} n \\ j \end{array} \right] P_j (x|y, \rho, q) \rho^{n-j} \sum_{i=0}^{n-j} \left[ \begin{array}{c} n-j \\ i \end{array} \right] (-1)^i q \binom{i}{q} H_{n-i-j} (y|q) \]

\[ = \sum_{j=0}^{n} \left[ \begin{array}{c} n \\ j \end{array} \right] P_j (x|y, \rho, q) \rho^{n-j} H_{n-j} (y|a/\rho|q). \]

Now denote \( a/\rho = b \).

ii) We will use (2.20) and (2.8). We have

\[ P_n (x|y, \rho, q) = \sum_{i=0}^{n} \left[ \begin{array}{c} n \\ i \end{array} \right] \rho^{n-i} B_{n-i} (y|q) H_i (x|q) \]

\[ = \sum_{i=0}^{n} \left[ \begin{array}{c} n \\ i \end{array} \right] \rho^{n-i} B_{n-i} (y|q) \sum_{k=0}^{i} \left[ \begin{array}{c} i \\ k \end{array} \right] \rho^{i-k} H_k (x|a, q) \]

\[ = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] H_k (x|a, q) \sum_{i=k}^{n} \left[ \begin{array}{c} n-k \\ i-k \end{array} \right] \rho^{n-i} B_{n-i} (y|q) \rho^{i-k} \]

\[ = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] H_k (x|a, q) \rho^{n-k} \sum_{m=0}^{n-k} \left[ \begin{array}{c} n-k \\ m \end{array} \right] (a/\rho)^m B_{n-k-m} (y|q) \]

iii) Keeping in mind (2.20) and (4.1) we get:

\[ P_n (x|y, \rho, q) = \sum_{i=0}^{n} \left[ \begin{array}{c} n \\ i \end{array} \right] \rho^{n-i} B_{n-i} (y|q) H_i (x|q) \]

\[ = \sum_{i=0}^{n} \left[ \begin{array}{c} n \\ i \end{array} \right] \rho^{n-i} B_{n-i} (y|q) \sum_{j=0}^{i} \left[ \begin{array}{c} i \\ j \end{array} \right] \rho^{i-j} H_{i-j} (z|q) P_j (x|z, r, q) \]

\[ = \sum_{j=0}^{n} \left[ \begin{array}{c} n \\ j \end{array} \right] P_j (x|z, r, q) \sum_{i=j}^{n-j} \left[ \begin{array}{c} n-j \\ i-j \end{array} \right] \rho^{n-i} \rho^{i-j} B_{n-i} (y|q) H_{i-j} (z|q) \]

\[ = \sum_{j=0}^{n} \left[ \begin{array}{c} n \\ j \end{array} \right] P_j (x|z, r, q) \rho^{n-j} \sum_{s=0}^{n-j} \left[ \begin{array}{c} n-j \\ s \end{array} \right] \rho^{j-s} r^s B_{n-j-s} (y|q) H_s (z|q). \]

Let us extend definition of polynomials \( P_n (x|y, t, q) \) for \(|t| \geq 1\) by defining \( P_n (x|y, t, q) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] t^{n-k} B_{n-k} (y|q) H_k (x|q). \)

Thus we have:

\[ P_n (x|y, \rho, q) = \sum_{j=0}^{n} \left[ \begin{array}{c} n \\ j \end{array} \right] r^{n-j} P_j (x|z, r, q) P_n-j (z|y, \rho/r, q). \]

Now let us redefine \( \rho \) and \( r \) by selecting them so that \(|r| > |\rho|\) and changing notation \( \rho/r \longrightarrow \rho_1 \), \( r \longrightarrow \rho_2 \), consequently \( \rho \longrightarrow \rho_1 \rho_2 \).
iv) \[
|H_n(x|a,q)| \leq \sum_{i=0}^{n-1} \sum_{i=0}^{n} \left[ \frac{n}{i} \right] q^{i} |a|^i |H_{n-i}(x|q)|
\]
\[
\leq \sum_{i=0}^{n} \left[ \frac{n}{i} \right] q^{i} |a|^i r_{n-i}(1|q) / (1-q)^{(n-i)/2}
\]
\[
\leq (1-q)^{-n/2} \sum_{i=0}^{n} \left[ \frac{n}{i} \right] q^{i} \left| a \sqrt{1-q} \right|^i r_{n-i}(1|q)
\]
\[
\leq (1-q)^{-n/2} r_n(1|q) \left| -a \sqrt{1-q} \right|^n
\]
\[
\leq \left( -|a| \right) \frac{1}{\sqrt{1-q}} \left( 1-q \right)^{-n/2} r_n(1|q).
\]

v) An easy proof based on Carlitz formulae presented e.g. in Exercise 12.2(b) and 12.2(c) of [11] was given in [18]. It contained however two misprints. Once concerning constant \( \rho \). Namely in all estimates one should take \( |\rho| \) naturally. The second one concerned the following estimation:

\[
w(x,y,\rho y^{k}|q) \leq (1-\rho^2 q^{2k})^2 + 2\sqrt{1-q(1+\rho^2 q^{2k})}|y\rho y^{k}| + 4\rho^2 q^{2k} + (1-q)\rho^2 y^2 q^{2k}
\]

instead of

\[
w(x,y,\rho y^{k}|q) \leq (1-\rho^2 q^{2k})^2 + 2(1-q)(1+\rho^2 q^{2k})|y\rho y^{k}| + 4\rho^2 q^{2k} + (1-q)\rho^2 y^2 q^{2k}.
\]

Now

\[
(1-\rho^2 q^{2k})^2 + 2\sqrt{1-q(1+\rho^2 q^{2k})}|y\rho y^{k}| + 4\rho^2 q^{2k} + (1-q)\rho^2 y^2 q^{2k}
\]

\[
= (1+\rho^2 q^{2k} + \sqrt{1-q|y\rho y^{k}|})^2 \leq (1+|\rho| q^{k})^4
\]

since \( \sqrt{1-q|y|} \leq 2 \) for \( y \in S(\rho) \). Notice that \( \prod_{k \geq 0} (1+|\rho| q^{k})^4 = (\rho^2)_\infty \). The last inequality was suggested by the referee. \( \square \)

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