Analogues of Mathai-Quillen forms in sheaf cohomology
and applications to topological field theory

Richard S. Garavuso\textsuperscript{1}, Eric Sharpe\textsuperscript{2}

\textsuperscript{1} Harish-Chandra Research Institute
Chhatnag Road
Jhunsi, Uttar Pradesh 211019
India

\textsuperscript{2} Department of Physics
Robeson Hall, 0435
Virginia Tech
Blacksburg, VA 24061, USA

garavuso@hri.res.in, ersharpe@vt.edu

We construct sheaf-cohomological analogues of Mathai-Quillen forms, that is, holomorphic bundle-valued differential forms whose cohomology classes are independent of certain deformations, and which are believed to possess Thom-like properties. Ordinary Mathai-Quillen forms are special cases of these constructions, as we discuss. These sheaf-theoretic variations arise physically in A/2 and B/2 model pseudo-topological field theories, and we comment on their origin and role.

October 2013
| Section                                                                 | Page |
|------------------------------------------------------------------------|------|
| 4.2.1 A/2 model realization of first kernel construction               | 34   |
| 4.2.2 B/2 model realization of second kernel construction             | 38   |
| 4.3 Cokernels                                                           | 40   |
| 4.3.1 B/2 model realization of first cokernel construction           | 40   |
| 4.3.2 A/2 model realization of second cokernel construction           | 42   |
| 4.4 Cohomologies of short complexes                                   | 45   |
| 4.4.1 A/2 model realization of first short complex construction      | 45   |
| 4.4.2 B/2 model realization of second short complex construction      | 48   |
| 4.5 A note on anomalies                                                | 50   |
| 5 Conclusions                                                          | 51   |
| 6 Acknowledgements                                                    | 51   |
| References                                                             | 51   |
1 Introduction

The Mathai-Quillen formalism [1] provides mathematics with a by-now well-known realiza-
tion of Thom classes, which is to say, it allows for integrals of differential forms over total
spaces of vector bundles to be reduced to integrals over base spaces. Given a complex mani-
fold \( Y = \{ s = 0 \} \subset M \) for \( s \) a section of a holomorphic vector bundle \( G \rightarrow M \), say, a typical
application of a Mathai-Quillen form \( U(G, \nabla) \) is to use its pullback \( s^*U(G, \nabla) \) to relate an
integral of a differential form over \( Y \) to an integral over \( M \). Specifically, if \( \omega_{MQ} \propto s^*U(G, \nabla) \),
then
\[
\int_Y \alpha \propto \int_M \tilde{\alpha} \wedge \omega_{MQ}
\]
for a differential form \( \alpha \) and a suitable pullback \( \tilde{\alpha} \). In physics, Mathai-Quillen forms play a
further role in understanding topological field theories, see e.g. [2, 3, 4, 5, 6].

In this paper, we propose six analogues of Mathai-Quillen forms for sheaf cohomology
valued in locally-free sheaves, for various circumstances. Specifically, given a complex mani-
fold \( Y \) as above, with bundle \( \mathcal{E}' \) and isomorphism \( \text{det} \mathcal{E}'^* \cong K_Y \) (or \( \text{det} \mathcal{E}' \cong K_Y \)) so that
integrals of the form
\[
\int_Y \mathcal{O}_1 \wedge \cdots \wedge \mathcal{O}_n,
\]
for
\[
\mathcal{O}_i \in H^\bullet (Y, \wedge^\bullet \mathcal{E}'^*) \text{ or } H^\bullet (Y, \wedge^\bullet \mathcal{E}'),
\]
are well-defined, we propose sheaf cohomology classes \( \omega \) such that for systematically-defined
cohomology classes \( \tilde{\mathcal{O}}_i \) lifting the \( \mathcal{O}_i \),
\[
\int_Y \mathcal{O}_1 \wedge \cdots \wedge \mathcal{O}_n \propto \int_Z \tilde{\mathcal{O}}_1 \wedge \cdots \wedge \tilde{\mathcal{O}}_n \wedge \omega.
\]
Here, \( Z = M \) or \( Z \) is the total space \( X \) of a holomorphic vector bundle over \( M \).

Our constructions rest on a representation of sheaf cohomology valued in locally-free
sheaves as \( \partial \)-cohomology classes of bundle-valued differential forms, and on an analogous
representation of hypercohomology valued in a complex of locally-free sheaves.

In the special case that \( \mathcal{E}' \) is the tangent bundle to \( Y \), so that
\[
H^\bullet (Y, \wedge^\bullet \mathcal{E}'^*) = H^{\bullet \bullet} (Y),
\]
one of our six constructions will specialize to ordinary Mathai-Quillen forms describing \( Y \) as
the zero locus of a section of \( \mathcal{G} \).

We demonstrate that our bundle-valued forms define suitable elements of cohomology,
and have some of the same cohomological-invariance properties of ordinary Mathai-Quillen
forms. However, the demonstration that

$$\int_{Y} O_1 \wedge \cdots \wedge O_n \propto \int_{Z} \bar{O}_1 \wedge \cdots \wedge \bar{O}_n \wedge \omega.$$ 

is left for later work: it is a consequence of the physical origin of these analogues (specifically, it is a mathematical prediction of the renormalization group), but we do not offer a rigorous mathematical demonstration.

Our construction is motivated by physics. In much the same way that ordinary Mathai-Quillen forms enter ordinary topological field theories, the analogues we propose enter heterotic analogues of topological field theories, the A/2 and B/2 models, which is where they were first observed. (See for example [7, 8, 9, 10, 11, 12, 13] for further information on the A/2 and B/2 models.) Part of the purpose of this paper is to try to extract precise mathematical predictions about analogues of Mathai-Quillen forms from those heterotic constructions, and we devote the latter part of the paper to explaining physical origins and applications.

In particular, this paper was originally motivated by the desire to understand claims made in [10], regarding the dependence of correlation functions in the A/2 model on certain complex and bundle moduli. Our original hope was to establish their claims in a clean mathematical setting. Although we did succeed in establishing some of their claims, others are left for future work.

We begin in section 2 by briefly reviewing pertinent aspects of the Mathai-Quillen formalism. In section 3 we discuss various sheaf-theoretic generalizations. The basic format is that we take a bundle on $Y$ which can be realized as e.g. a kernel or cokernel on a larger space including $Y$, and lift the sheaf cohomology computation to that larger space. We give representatives of sheaf cohomology, $\bar{\partial}$-closed bundle-valued differential forms, whose cohomology classes are independent of certain deformations. In particular, in this section we observe how ordinary Mathai-Quillen forms can be understood as special cases of the kernel construction, and also discuss how the results of [10] can be understood in this context.

Finally in section 4 we discuss how these analogues of Mathai-Quillen forms arise in heterotic versions of topological field theories. Corresponding to each analogue of a Mathai-Quillen form discussed earlier, we give a Landau-Ginzburg model that renormalization group flows to a nonlinear sigma model. The analogues of Mathai-Quillen forms provide a mathematical mechanism for understanding how correlation functions in the (UV) Landau-Ginzburg theories can match those of (IR) nonlinear sigma models. See [14, 15] for a discussion of the pertinent Landau-Ginzburg models, whose descriptions are further elaborated upon here.

This paper could be viewed as a step in the further development of Landau-Ginzburg models over nontrivial spaces, further developing results in e.g. [14, 15, 16, 17, 18].

A mathematician solely interested in analogues of Mathai-Quillen forms can safely skip
section 4 as it is intended to give context to physicists reading this article.

2 Brief review of the Mathai-Quillen formalism

Consider a complex vector bundle $\mathcal{G} \xrightarrow{\pi} M$ with standard fiber $V$, over a complex manifold $M$. Suppose that $\mathcal{G}$ has fiber metric $(\cdot, \cdot)_G$ and compatible connection $\nabla$. Under these circumstances, the Mathai-Quillen formalism [1, 2, 3, 4, 5, 6, 19] provides an explicit representative $U(G, \nabla)$ of the Thom class of $G$. Furthermore, the pullback $s^* U(G, \nabla)$ of $U(G, \nabla)$ by any section $s: M \to \mathcal{G}$ of $\mathcal{G}$ is a representative of the top Chern class of $\mathcal{G}$. Let us review the formalism in more detail.

2.1 Conventions

Our conventions for $M$, $\mathcal{G}$, and the dual $\mathcal{G}^*$ of $\mathcal{G}$ are as follows. The exterior derivatives on $M$ and $\mathcal{G}$ are respectively denoted by $d$ and $d^\mathcal{G}$. We choose local coordinates $\phi^I$ on $M$. The connection on $\mathcal{G}$ is then given by $\nabla = \partial I \nabla^I$. In terms of this connection, the curvature 2-form on $\mathcal{G}$ is given by $R = \nabla^2$. We choose a local oriented orthonormal frame $\{e_A\}$ for $\mathcal{G}$ and let $\{f_A\}$ be the dual coframe. The section $s$ may thus be expressed as $s = e_A \rho_A$. Similarly, we write $\rho = \rho_A f^A$, where the $\rho_A$ are anticommuting orthonormal coordinates on $\mathcal{G}^*$. The dual pairing on $\mathcal{G}$ is denoted by $\langle \cdot, \cdot \rangle_G$. Finally, the metric on $\mathcal{G}^*$ is denoted by $\langle \cdot, \cdot \rangle_{G^*}$.

Now, consider the pullback bundle $\pi^* \mathcal{G} \to \mathcal{G}$. This bundle has fiber metric

$$(\cdot, \cdot)_{\pi^* \mathcal{G}} \equiv \langle \cdot, \cdot \rangle_{\pi^* \mathcal{G}},$$

compatible connection $\pi^* \nabla \equiv \tilde{\nabla}$, curvature 2-form $\pi^* R \equiv \tilde{R}$, local oriented orthonormal frame $\{\pi^* e_A\} \equiv \{\tilde{e}_A\}$, and tautological section $\tilde{s} = \tilde{x}^A \tilde{e}_A$. The dual bundle $(\pi^* \mathcal{G})^* \to \mathcal{G}$ has coframe $\{(\pi^*)^* f^A\} \equiv \{\tilde{f}^A\}$ and metric $\langle \cdot, \cdot \rangle_{(\pi^* \mathcal{G})^*}$. We write $\tilde{\rho} = \tilde{\rho}_A \tilde{f}^A$, where the $\tilde{\rho}_A \equiv (\pi^*)^* \rho_A$ are anticommuting orthonormal coordinates on $(\pi^* \mathcal{G})^*$. The dual pairing on $\pi^* \mathcal{G}$ is denoted by $\langle \cdot, \cdot \rangle_{\pi^* \mathcal{G}}$.

2.2 Definition

The Mathai-Quillen form $U(\mathcal{G}, \nabla)$ is defined to be proportional to

$$u(\mathcal{G}, \nabla) = \int d\tilde{\rho} \exp \left( -\tilde{A} \right),$$

(1)
where
\[
\tilde{A} = \frac{1}{2} \left( \tilde{x}, \tilde{x} \right)_{\pi^*\mathcal{G}} + \left\langle \tilde{\nabla}_{\tilde{x}}, \tilde{\rho} \right\rangle_{\pi^*\mathcal{G}} + \frac{1}{2} \left( \tilde{\rho}, \tilde{\mathcal{R}}\tilde{\rho} \right)_{(\pi^*\mathcal{G})^*}.
\] (2)

By construction,
\[
\tilde{A} \in \bigoplus_i \Omega^i (\mathcal{G}, \wedge^i (\pi^*\mathcal{G})^*),
\]
hence
\[
\exp \left( - \tilde{A} \right) \in \bigoplus_i \Omega^i (\mathcal{G}, \wedge^i (\pi^*\mathcal{G})^*),
\]
hence \( u(\mathcal{G}, \nabla) \in \Omega^r(\mathcal{G}) \). Moreover, the Mathai-Quillen form is closed:
\[
d^\mathcal{G} u(\mathcal{G}, \nabla) = 0.
\]

To show that it is closed, first note that since \( \tilde{\nabla} \) is compatible with the metric \((\cdot, \cdot)_{\pi^*\mathcal{G}}\), it follows that
\[
d^\mathcal{G} \int d\tilde{\rho} \exp \left( - \tilde{A} \right) = \int d\tilde{\rho} \tilde{\nabla} \tilde{\alpha},
\]
where \( \tilde{\alpha} \in \Omega (\mathcal{G}, \wedge (\pi^*\mathcal{G})^*) \). Furthermore,
\[
\left( \tilde{\nabla} + \tilde{x}_A \frac{\partial}{\partial \tilde{\rho}_A} \right) \tilde{A} = \left( \tilde{\nabla}_{\tilde{x}}, \tilde{x} \right)_{\pi^*\mathcal{G}} + \left\langle \tilde{\mathcal{R}}\tilde{x}, \tilde{\rho} \right\rangle_{\pi^*\mathcal{G}} - \frac{1}{2} \left( \tilde{\rho}, \tilde{\nabla}\tilde{\mathcal{R}}\tilde{\rho} \right)_{(\pi^*\mathcal{G})^*} + \left( \tilde{\nabla}_{\tilde{x}} \tilde{x} \right)_{\pi^*\mathcal{G}} - \left\langle \tilde{\mathcal{R}}\tilde{x}, \tilde{\rho} \right\rangle_{\pi^*\mathcal{G}},
\]
\[= 0, \] (3)
where we have used the Bianchi identity \( \tilde{\nabla} \tilde{\mathcal{R}} = 0 \). From these results, we obtain
\[
d^\mathcal{G} u(\mathcal{G}, \nabla) = d^\mathcal{G} \int d\tilde{\rho} \exp \left( - \tilde{A} \right),
\]
\[= \int d\tilde{\rho} \tilde{\nabla} \exp \left( - \tilde{A} \right),
\]
\[= \int d\tilde{\rho} \left( \tilde{\nabla} + \tilde{x}_A \frac{\partial}{\partial \tilde{\rho}_A} \right) \exp \left( - \tilde{A} \right),
\]
\[= \int d\tilde{\rho} \left[ - \left( \tilde{\nabla} + \tilde{x}_A \frac{\partial}{\partial \tilde{\rho}_A} \right) \tilde{A} \right] \exp \left( - \tilde{A} \right),
\]
\[= 0.
\]
Here, the third equality holds because \( \tilde{x}_A (\partial/\partial \tilde{\rho}_A) e^{-\tilde{A}} \) contributes nothing to the Grassmann integral.
The Mathai-Quillen form has additional properties, most importantly that

$$\int_V U(\mathcal{G}, \nabla) = 1.$$  

This implies that the Mathai-Quillen form is a representative of the Thom class\(^1\) of \(\mathcal{G}\). This fact plays an important role in physics; for example, as a Thom class it provides an understanding of renormalization group flow in Landau-Ginzburg models \([14, 15, 16]\). However, we will not establish analogues of this in this paper, we merely propose that they exist, and as the property is standard, we omit its derivation here.

### 2.3 Pullbacks of Mathai-Quillen forms

What will be more relevant for this paper is the pullback of the Mathai-Quillen form \(U(\mathcal{G}, \nabla)\) by any section \(s\) of \(\mathcal{G}\). We shall write

$$s^* u(\mathcal{G}, \nabla) = \int d\rho \exp (-A),$$ (4)

where

$$A = \frac{1}{2} (s, s)_{\mathcal{G}} + \langle \nabla s, \rho \rangle_{\mathcal{G}} + \frac{1}{2} (\rho, \mathcal{R} \rho)_{\mathcal{G}^*}. \quad (5)$$

The form \(s^* u(\mathcal{G}, \nabla)\) satisfies

(i) \(s^* u(\mathcal{G}, \nabla) \in \Omega^{rk \mathcal{G}}(M)\),

(ii) \(d s^* u(\mathcal{G}, \nabla) = 0\).

We see these properties as follows. The first follows immediately from the fact that, by construction,

$$A \in \oplus \Omega^i (\mathcal{G}, \wedge^i \mathcal{G}^*).$$

The proof of the second property is similar to that of the argument that \(u(\mathcal{G}, \nabla)\) is \(d\mathcal{G}\)-closed in the last section, and uses the results

$$d \int \int d\rho \alpha = \int d\rho \nabla \alpha,$$

where \(\alpha \in \Omega (\mathcal{G}, \wedge \mathcal{G}^*)\), and

$$\left( \nabla + s_A \frac{\partial}{\partial \rho_A} \right) A = 0.$$ (6)

\(^1\) A representative of the Thom class of \(\mathcal{G}\) is a \(d\mathcal{G}\)-closed differential form \(U(\mathcal{G}) \in \Omega^r(\mathcal{G})\) such that \(\int_V U(\mathcal{G}) = 1\), for \(r = rk \mathcal{G}\).
For our purposes, the most important mathematical property of the pullback $s^*u(G, \nabla)$ is that the $d$-cohomology class of $s^*u(G, \nabla)$ is independent of the section $s$. We can see this as follows. Let $s_\tau = s + \tau s'$ be an affine one-parameter family of sections of $G$ and let

$$A_\tau = \frac{1}{2} (s_\tau, s_\tau)_G + \langle \nabla s_\tau, \rho \rangle_G + \frac{1}{2} \langle \rho, \mathcal{R}_\rho \rangle_{g^*}.$$ 

Then

$$\frac{d}{d\tau} s_\tau^* u(G, \nabla) = \frac{d}{d\tau} \int d\rho \exp (-A_\tau),$$

$$= - \int d\rho \left[ (s', s_\tau)_G + \langle \nabla s', \rho \rangle_G \right] \exp (-A_\tau),$$

$$= - \int d\rho \left\{ \left[ \nabla + (s_\tau)_A \frac{\partial}{\partial \rho_A} \right] \langle s', \rho \rangle_G \right\} \exp (-A_\tau),$$

$$= - \int d\rho \left[ \nabla + (s_\tau)_A \frac{\partial}{\partial \rho_A} \right] \left[ \langle s', \rho \rangle_G \exp (-A_\tau) \right],$$

$$= -d \int d\rho \langle s', \rho \rangle_G \exp (-A_\tau).$$

It follows that

$$s_{\tau_2}^* u(G, \nabla) - s_{\tau_1}^* u(G, \nabla) = -d \int_{\tau_1}^{\tau_2} d\tau \int d\rho \langle s', \rho \rangle_G \exp (-A_\tau).$$

Thus, for arbitrary sections $s_{\tau_1}$ and $s_{\tau_2}$ of $G$, the $d$-closed forms $s_{\tau_1}^* u(G, \nabla)$ and $s_{\tau_2}^* u(G, \nabla)$ differ by a $d$-exact form and hence are cohomologous.

Mathematically, the form $s^*U(G, \nabla)$ is cohomologous to the differential form

$$\frac{1}{(2\pi)^{\frac{1-A}{2}}} \int d\rho \exp \left[ \frac{1}{2} \langle \rho, \mathcal{R}_\rho \rangle_{g^*} \right] = \text{Det} \left( \frac{\mathcal{R}}{2\pi} \right)$$

and hence is a representative of the top Chern class of $G$. We can recover this simply by choosing $s$ to be the zero section.

As an aside, if $s$ intersects the zero section of $G$ transversely, then $s^*U(G, \nabla)$ is Poincaré dual to $s^{-1}(0)$, i.e.

$$\int_{s^{-1}(0)} \alpha = \int_M \tilde{\alpha} \wedge s^*U(G, \nabla),$$

(7)

where $\tilde{\alpha} \in \Omega^{\dim{M} - \text{rk} G}(M)$ is $d$-closed. Also, when $\dim{M} = \text{rk} G$, integrating $s^*U(G, \nabla)$ over $M$ yields a representation of the integral of the top Chern class of $G$. 

9
3 Sheaf-cohomological Mathai-Quillen analogues

In this section, we will propose six analogues of Mathai-Quillen forms for sheaf cohomology. The general prototype is as follows. We propose analogues of Mathai-Quillen forms whose insertions relate integrals of cup products of sheaf cohomology classes

\[ H^\bullet (N, \wedge \mathcal{E}^*) \text{ or } H^\bullet (N, \wedge \mathcal{E}) , \]

(or suitable hypercohomology classes, depending upon the analogue,) where \( \mathcal{E} \) is a locally-free sheaf on \( N \), to corresponding integrals of cup products of sheaf cohomology classes

\[ H^\bullet (Y, \wedge \mathcal{E}'^*) \text{ or } H^\bullet (Y, \wedge \mathcal{E}') \]

over \( Y \subset N \), where \( \mathcal{E}' \) is a locally-free sheaf on \( Y \) constructed in part from the data in the analogue of the Mathai-Quillen form.

The first analogue we will discuss, the first kernel construction, will specialize to (pull-backs of) ordinary Mathai-Quillen forms in the case that \( \mathcal{E}' = TY \), as we shall discuss.

3.1 Kernels

3.1.1 First kernel construction

Suppose that one is interested in computing integrals over some space \( Y \equiv \{ s = 0 \} \subset M \) \((s \in \Gamma(\mathcal{G}))\) of sheaf cohomology classes

\[ \mathcal{O} \in H^\bullet (Y, \wedge \mathcal{E}'^*) , \]

where \( \mathcal{E}' \) is a holomorphic vector bundle on \( Y \). Such integrals will be of the form

\[ \int_Y \mathcal{O}_1 \wedge \cdots \wedge \mathcal{O}_n \]

and will be well-defined if \( \det \mathcal{E}'^* \cong K_Y \), and one picks a particular isomorphism.

We propose that if \( \mathcal{E}' \) is given as the restriction to \( Y \) of the kernel of a smooth surjective map \( \tilde{F} : \mathcal{F}_1 \to \mathcal{F}_2 \) (\( \mathcal{F}_1, \mathcal{F}_2 \) holomorphic vector bundles on \( M \)), whose restriction to \( Y \) is holomorphic, then at least for some \( \mathcal{O}' \)'s it is possible to write the integral in the different form

\[ \int_Y \mathcal{O}_1 \wedge \cdots \wedge \mathcal{O}_n \propto \int_M \tilde{\mathcal{O}}_1 \wedge \cdots \wedge \tilde{\mathcal{O}}_n \wedge \omega_{K_1} , \]

where \( \tilde{\mathcal{O}}_i \) (when it exists) is an element of \( H^\bullet (M, \wedge \mathcal{F}_1^*) \) ‘lifting’ \( \mathcal{O}_i \), in a manner we shall describe shortly, and \( \omega_{K_1} \) is a \( \overline{\partial} \)-closed analogue of a Mathai-Quillen form,

\[ \omega_{K_1} \in H^9 \left( M, \wedge \mathcal{F}_1^* \otimes \det \mathcal{G}^* \otimes \det \mathcal{F}_2 \right) \quad \text{(8)} \]
(f_i = \text{rk } F_i, \ g = \text{rk } G), \text{ where one has an isomorphism}
\[ K_M \cong \det F_1^* \otimes \det F_2 \otimes \det G^* \]
that restricts to the isomorphism \( \det E'^* \cong K_Y \) appearing above.

As a consistency check, notice that
\[ \tilde{O}_1 \wedge \cdots \wedge \tilde{O}_n \in H^{\dim M - g}(M, \wedge^{f_1 - f_2} F_1^*) \]
so that the cohomology class of \( \omega_{K1} \) is correct for the integrand of \( M \) to be a top-form.

We propose an expression for \( \omega_{K1} \) below, and check its properties.

First, let us describe the relation between
\[ \mathcal{O} \in H^\bullet(Y, \wedge^\bullet E') \text{ and } \tilde{\mathcal{O}} \in H^\bullet(M, \wedge^\bullet F_1^*). \]
Let \( i : Y \hookrightarrow M \) denote inclusion. Then, given \( \tilde{\mathcal{O}} \), we have
\[ i^* \tilde{\mathcal{O}} \in H^\bullet(Y, \wedge^\bullet F_1^{|Y}). \]
Next, dualizing the short exact sequence
\[ 0 \rightarrow \mathcal{E}' \rightarrow F_1|_Y \rightarrow F_2|_Y \rightarrow 0 \]
to
\[ 0 \rightarrow F_2^*|_Y \rightarrow F_1^*|_Y \rightarrow \mathcal{E}'^* \rightarrow 0, \]
we see that there is a surjective map
\[ \wedge^\bullet F_1^*|_Y \rightarrow \wedge^\bullet \mathcal{E}'^* \]
which induces
\[ j_* : H^\bullet(Y, \wedge^\bullet F_1^*|_Y) \rightarrow H^\bullet(Y, \wedge^\bullet \mathcal{E}'^*). \]
Hence, a pair \( \mathcal{O}, \tilde{\mathcal{O}} \), when they exist, are related as
\[ \mathcal{O} = j_* i^* \tilde{\mathcal{O}}. \]

Our proposal for \( \omega_{K1} \) is given by the Grassmann integral
\[ \omega_{K1} = \int \prod d\lambda^r d\chi^s \exp(-A_{K1}), \]
where
\[ A_{K1} = h^{x\pi} s_{x\pi} + \chi^r \lambda^s \tilde{F}_{r\pi} + \lambda^s \lambda^\pi \tilde{F}_{r\gamma} + F_{r\pi\gamma} \chi^r \lambda^\pi \lambda^\gamma, \]
where $x$ indexes local coordinates along the fibers of $G$, $\gamma$ indexes local coordinates along the fibers of $F_1$, $r$ indexes local coordinates (denoted $p$) along the fibers of $F_2^*$, and $i$ indexes local coordinates on $M$. The curvature term

$$F_{\tau \tau \gamma \lambda} \chi^r \lambda^\gamma$$

is defined by the condition\(^2\)

$$\bar{\partial}_\tau \hat{F}_{r\gamma} = h^{x\tau} s_x F_{r\gamma \tau} = -h^{x\tau} s_x F_{\tau r \gamma}$$

(10)

and defines an element of

$$H^1 (M, F_1^* \otimes F_2 \otimes G^*) .$$

(Physically, $F$ arises as part of the curvature of a holomorphic vector bundle, hence is $\bar{\partial}$-closed by virtue of the Bianchi identity. Similar considerations are the reason that the curvature of a bundle defines the Atiyah class as an element of sheaf cohomology.)

Now, let us explain some aspects of $\omega_{K1}$ in more detail. In $A_{K1}$, every $\lambda^\tau$ is accompanied by a $\chi^r$, so integrating out the $\lambda^\tau$’s should result in $\text{rk} \ G = g \chi^r$’s, hence a degree $g$ cohomology class. Similarly, each $\chi^r$ is accompanied by a $\lambda^\gamma$, so integrating out the $\chi^r$’s should result in coefficients $\wedge f_2 F_1^*$. Furthermore, the Grassmann integral measure makes $\omega_{K1}$ couple to $\det G^* \otimes \det F_2$. Thus, $\omega_{K1}$ is a form of the type indicated in equation (8). We shall show it is $\bar{\partial}$-closed momentarily.

Now, we can argue formally that the analogue of a Mathai-Quillen form defined above is $\bar{\partial}$-closed. The central point is that

$$\left( \bar{D} + h^{x\tau} s_x \frac{\partial}{\partial \lambda^\tau} \right) A_{K1} = \chi^r \lambda^\gamma \left( \bar{\partial}_\tau \hat{F}_{r\gamma} + h^{x\tau} s_x F_{\tau r \gamma} \right),$$

$$= 0$$

using equation (10), where

$$\bar{D} = \chi^r \bar{\partial}_r .$$

---

\(^2\) This constraint is imposed physically by supersymmetry. Mathematically, it can be shown that one can always find $F_{\tau \gamma}$ satisfying this condition [20]. For example, if $G$ is a line bundle, then this curvature term is the coboundary of

$$\hat{F}_{|Y} \in H^0 (Y, F_1^{|Y} \otimes F_2|Y)$$

in the long exact sequence derived from tensoring

$$0 \to \mathcal{O}(-Y) \to \mathcal{O} \to \mathcal{O}_Y \to 0$$

by $F_1^* \otimes F_2$. In this special case, the constraint above is merely the specification of the coboundary map. For another special case, when one specializes to ordinary Mathai-Quillen forms, $\hat{F}_{ix} = D_i s_x$, and the constraint becomes

$$\bar{D}_\tau D_j s_x = [\bar{D}_\tau, D_j] s_x = R_{\tau j x} h^{y \tau} s_y,$$

a standard result. As another consistency check, note that along the locus $\{s = 0\}$, the constraint (10) becomes the statement that $\hat{F}_{|Y}$ is holomorphic.
Given the result above, it follows that

$$
\overline{\partial} \omega_{K1} = \int \prod d\lambda d\chi \text{exp}(-A_{K1}),
$$

$$
= \int \prod d\lambda d\chi \left( D + h^T s_x \frac{\partial}{\partial \lambda_T} \right) \exp(-A_{K1}),
$$

$$
= 0,
$$

and so $\omega_{K1}$ defines a $\overline{\partial}$-closed form.

Next, we will argue that the cohomology class of $\omega_{K1}$ is unchanged by antiholomorphic deformations of the section $s$. In other words, consider the one-parameter family

$$
A_{K1,\tau} = h^T s_x (s_T + \tau \bar{t}), \quad \chi^r \lambda^\gamma D_T (s_T + \tau \bar{t}) + \chi^r \lambda^\gamma \bar{F}_{r\gamma} + F_{T\gamma} \lambda^\gamma \lambda^\gamma.
$$

Then,

$$
\frac{d}{d\tau} \omega_{K1,\tau} = \frac{d}{d\tau} \int \prod d\lambda d\chi \text{exp}(-A_{K1,\tau}),
$$

$$
= - \int \prod d\lambda d\chi \left( h^T s_x \bar{t} + \chi^r \lambda^\gamma D_T \bar{t} \right) \exp(-A_{K1,\tau}),
$$

$$
= \int \prod d\lambda d\chi \left( D + h^T s_x \frac{\partial}{\partial \lambda_T} \right) (-\lambda^T \bar{t}) \exp(-A_{K1,\tau}),
$$

$$
= \overline{\partial} \int \prod d\lambda d\chi (-\lambda^T \bar{t}) \exp(-A_{K1,\tau}),
$$

thus demonstrating the desired result.

As we will see later in section 4.2.1, this analogue of a Mathai-Quillen form appears in the $\Lambda/2$ model pseudo-topological field theories [15], where it plays a role analogous to that of the Mathai-Quillen form in some ordinary topological field theories. $\Lambda/2$ model correlation functions amount to sheaf cohomology computations, so the fact that this deformed Mathai-Quillen form defines an $\overline{\partial}$-cohomology class is precisely what is needed to correlate with its physical role.

One simple special case is that in which $G = 0$, so that our proposed analogue of a Mathai-Quillen form simply reduces to $M$ rather than some complete intersection inside $M$. In this case, $A_{K1}$ becomes simply

$$
\chi^r \lambda^\gamma \bar{F}_{r\gamma}.
$$

Intuitively, its role is clear: the Grassmann quantity $\lambda^\gamma$ above annihilates those $\lambda$ which are not in the kernel of $F$, thus reducing sheaf cohomology valued in $F_1$ to sheaf cohomology valued in the kernel of $\bar{F} : F_1 \to F_2$. 

13
3.1.2 Specialization to ordinary Mathai-Quillen and its deformations

An important special case of the construction above is to the ordinary Mathai-Quillen form and its deformations.

First, let us outline a family of deformations of (pullbacks of) ordinary Mathai-Quillen forms. Consider the pullback, $s^*U(\mathcal{G}, \nabla) \propto s^*u(\mathcal{G}, \nabla)$, defined earlier. Assume that $\mathcal{G}$ is a holomorphic vector bundle. Consider deforming $s^*u(\mathcal{G}, \nabla)$ to

$$\omega_{\delta s}(\mathcal{G}, \nabla) = \int d\rho \exp (-\mathcal{A}_{\delta s}),$$

$$\mathcal{A}_{\delta s} = \mathcal{A} + \left( \rho^p f_p', d\phi^i (\delta s)_{ip} e^p \right).$$

Here, $\mathcal{A}$ is given by (21) and

$$(\delta s)_{ip} \in \Gamma (\pi^* \mathcal{G} \otimes \pi^* TM).$$

These deformations (and pullbacks of ordinary Mathai-Quillen forms themselves, in the case $\delta s = 0$) are special cases of the sheaf-cohomological analogue of the previous section. Specifically, this corresponds to the special case that $\mathcal{F}_1 = TM$, $\mathcal{F}_2 = \mathcal{G}$, and with map $\tilde{F} : \mathcal{F}_1 \to \mathcal{F}_2$ defined by

$$F_{ip} = D_i s_p + (\delta s)_{ip},$$

where $s_p$ is a holomorphic section of $\mathcal{G}$. Then, note that

$$\overline{\nabla}_s (D_j s_p + (\delta s)_{jp}) = \left[ \overline{\nabla}_s, D_j \right] s_p = R_{s_{jp} q} e^p s_q$$

so that the curvature term, now an element of

$$H^1 (M, \Omega^1_M \otimes \mathcal{G} \otimes \mathcal{G}^*)$$

is determined by the curvature of $\mathcal{G}$, specifically, the Atiyah class of $\mathcal{G}^*$, exactly as needed to match ordinary Mathai-Quillen forms.

In terms of sheaf cohomology, this is the special case in which $\mathcal{E}'$ is a deformation of $TY$, with deformation determined by $\delta s$. If $\delta s = 0$, then $\mathcal{E}' = TY$, and this sheaf-cohomological analogue of a Mathai-Quillen form is relating

$$H^\bullet (M, \wedge^* T^* M) = H^{**}(M)$$

to

$$H^\bullet (Y, \wedge^* T^* Y) = H^{**}(Y),$$

just as expected.
Applying previous results, we know that
\[ \bar{\partial} \omega_{\delta s}(G, \nabla) = 0. \]

(If instead we deformed the original pullback of the Mathai-Quillen form by \( (\delta s) \) analogously, the result would similarly be a \( \partial \)-closed differential form.)

We also know, from specializing previous results, that the cohomology class of this deformation of the pullback of the Mathai-Quillen form is invariant under “antiholomorphic” deformations of \( s \), for the notion of antiholomorphic deformation defined earlier.

One formal consequence of the result above is that, by rescaling \( \tau \) to zero, \( \omega_{\delta s} \) can be written in the purely holomorphic form
\[
\int d\rho \exp \left( - \left( \rho^{p'} f^{p'}, (D s_p + d\phi^i(\delta s)_{ip})e_p^i \right)_G + \frac{1}{2} (\rho, R\rho)_G \right),
\]
which is in the same \( \bar{\partial} \)-cohomology class.

The \( \bar{\partial} \)-cohomology class of this deformation does seem to depend upon the \( (\delta s)_{ip} \), at least naively. Let \( \delta s = (\delta s)_{ip} + \tau (\delta s)_{ip} \) and \( A_{\delta s, \tau} = A + \langle \rho^{p'} f^{p'}, d\phi^i(\delta s)_{ip} e^p \rangle_G \). Then
\[
\frac{d}{d\tau} \omega_{\delta s, \tau}(G, \nabla) = \frac{d}{d\tau} \int d\rho \exp \left( - A_{\delta s, \tau} \right),
\]
\[
= - \int d\rho \langle \rho^{p'} f^{p'}, d\phi^i(\delta s)_{ip} e^p \rangle_G \exp \left( - A_{\delta s, \tau} \right).
\]
It follows that
\[
\omega_{\delta s, \tau_2}(G, \nabla) - \omega_{\delta s, \tau_1}(G, \nabla) = - \int_{\tau_1}^{\tau_2} d\tau \int d\rho \langle \rho^{p'} f^{p'}, d\phi^i(\delta s)_{ip} e^p \rangle_G \exp \left( - A_{\delta s, \tau} \right),
\]
which is at least not obviously \( \bar{\partial} \)-exact. We will comment on the physical meaning of this result in the next subsection.

### 3.1.3 Application to work of Melnikov-McOrist

In [10], it was argued, based on physical properties of gauged linear sigma models, that \( A/2 \) correlation functions for deformations of the tangent bundle should be independent of the deformation \( \delta s \). One of the original hopes of this work was to see that result explicitly in a cleaner setting.

\(^3\) Physically, actually taking such a limit is more subtle than we have indicated, because for example this removes the bosonic potential which adds new scalar zero mode directions.
Although it is true that the deformed object $\omega_{\delta s}$ given by (11) is independent of anti-holomorphic deformations of the section $s$, as noted earlier, we have not found a simple explanation for the claim above in this context.

Implicitly in this paper we are discussing mathematics motivated by Landau-Ginzburg models, which are related to the gauged linear sigma models of [10] via renormalization group flow. It is entirely possible that their results are only visible in gauged linear sigma models, that the renormalization group flow obscures the result in question.

### 3.1.4 Second kernel construction

Another construction exists for kernels. Suppose that one is interested in computing integrals over some space $Y \equiv \{s = 0\} \subset M$ ($s \in \Gamma(G)$) of sheaf cohomology classes

$$\mathcal{O} \in H^\bullet(Y, \wedge^\bullet \mathcal{E}'),$$

where $\mathcal{E}'$ is a holomorphic vector bundle on $Y$.

We propose that if $\mathcal{E}'$ is given as the restriction to $Y$ of the kernel of a surjective holomorphic map $\tilde{F} : F_1 \to F_2$ ($F_1$, $F_2$ holomorphic vector bundles on $M$), then at least for some $\mathcal{O}$'s it is possible to write the integral in a different form

$$\int_Y \mathcal{O}_1 \wedge \cdots \wedge \mathcal{O}_n \propto \int_X \tilde{\mathcal{O}}_1 \wedge \cdots \wedge \tilde{\mathcal{O}}_n \wedge \omega_{K2},$$

where $X$ is the total space of $\pi : F_2^* \to M$, $\tilde{\mathcal{O}}_i$ (when it exists) is an element of hypercohomology

$$\mathbb{H}^\bullet(X, \cdots \to \wedge^2 \pi^* F_1 \to \pi^* F_1 \to \mathcal{O}_X)$$

(with maps given by contraction with $p\tilde{F}$) ‘lifting’ $\mathcal{O}_i$, in a fashion we shall describe momentarily, and $\omega_{K2}$ is hypercohomology class in

$$\mathbb{H}^g(X, \pi^* \det G^* \otimes \cdots \to \wedge^2 \pi^* F_1 \to \pi^* F_1 \to \mathcal{O}_X),$$

where $g = \text{rk} G$ and one has an isomorphism

$$K_X \cong \pi^* \det G^* \otimes \pi^* \mathcal{F}_1$$

that restricts to the isomorphism $\det \mathcal{E}' \cong K_Y$ needed to define the corresponding integral on $Y$.

We propose an expression for $\omega_{K2}$, and check its properties.

First, let us describe the relation between

$$\mathcal{O} \in H^\bullet(Y, \wedge^\bullet \mathcal{E}') \quad \text{and} \quad \tilde{\mathcal{O}} \in \mathbb{H}^\bullet(X, \cdots \to \wedge^2 \pi^* F_1 \to \pi^* F_1 \to \mathcal{O}_X).$$
Briefly, we can use the fact that

\[ \mathbb{H}^\bullet \left( X, \cdots \longrightarrow \wedge^2 \pi^* \mathcal{F}_1 \longrightarrow \pi^* \mathcal{F} \longrightarrow \mathcal{O}_X \right) \cong H^\bullet \left( M, \wedge^\bullet \mathcal{E}' \right) \]

(a consequence of a minor variation of an argument given in \cite{15} appendix A). If we let \( i : Y \hookrightarrow M \) denote the inclusion, then the relation between the pair \( \mathcal{O}, \tilde{\mathcal{O}} \) is simply

\[ \mathcal{O} = i^* \tilde{\mathcal{O}}, \]

using the isomorphism above.

Our proposal for \( \omega_{K2} \) is given by the Grassmann integral

\[ \omega_{K2} = \int \prod d\lambda \exp(-A_{K2}), \]

where

\[ A_{K2} = h^s_{x} s_{x}^\sigma + h^s_{\sigma} p^r \bar{F}_{\sigma} F_{r} + \chi^s \theta_{x} F_{\sigma} h^r + \chi^s \theta_{\sigma} D_{x} \bar{F}_{\sigma} h^r, \]

where \( x \) indexes local coordinates along the fibers of \( G \), \( \gamma \) indexes local coordinates along the fibers of \( \mathcal{F}_1 \), \( r \) indexes local coordinates (denoted \( p \)) along along the fibers of \( \mathcal{F}_2^* \), and \( i \) indexes local coordinates on \( M \). Note that since each \( \lambda^r \) is paired with a \( \chi^r \), integrating out \( \lambda^r \)'s results in \( g \) factors of \( \chi^r \), interpreted as a degree \( g \) form, as advertised.

Now, we can argue formally that the analogue of a Mathai-Quillen form defined above is an element of hypercohomology. The central point is that

\[
\left( \overline{D} + h^s_{x} s_{x} \frac{\partial}{\partial \lambda^r} \right) A_{K2} = p^r F_{r} \chi^s \bar{F}_{r} F_{r} + p^r F_{r} \bar{F}_{r} \chi^s \theta_{x} \bar{F}_{r} h^r + \chi^s \theta_{x} \bar{F}_{r} D_{x} \bar{F}_{r} h^r,
\]

\[
= -p^r F_{r} \chi^s \bar{F}_{r} \theta_{x} A_{K2},
\]

where

\[ \overline{D} = \chi^s \bar{F}_{r} + \chi^s \theta_{x} \bar{F}_{r} \]

and we have used the fact that

\[ \chi^s \chi^s \theta_{x} \bar{F}_{r} \theta_{x} \left( h^r \bar{F}_{r} D_{x} \bar{F}_{r} h^r \right) = 0, \]

which follows from the fact that \( \mathcal{F}_1, \mathcal{F}_2 \) are holomorphic. Given the contribution to the coefficients from the Grassmann integral itself, it is now straightforward to show that \( \omega_{K2} \) is an element of the aforementioned hypercohomology group.

\[ ^4 \text{This section works in the language of hypercohomology, despite this isomorphism, because the pertinent analogue of a Mathai-Quillen form seems most straightforwardly expressed in the language of hypercohomology.} \]
Next, we will argue that the cohomology class of $\omega_{K^2}$ is unchanged by antiholomorphic deformations of the section $s$. As the details are somewhat more complicated than the argument in the previous subsection, we sketch the details here. Consider the one-parameter family

$$A_{K^2, \tau} = h^x s_x (\bar{s}_x + \tau \bar{t}_x) + h^\gamma r^\gamma \bar{F}_r \bar{F}_r - h^x \lambda^x (D^x \bar{s}_x + \tau D^x \bar{t}_x)
+ \chi^\gamma \theta_\gamma \bar{F}_r \bar{F}_r - h^x \gamma + \chi^\gamma \theta_\gamma \bar{F}_r \bar{F}_r h^\gamma,$$

so that

$$\frac{d}{d\tau} \omega_{K^2, \tau} = \frac{d}{d\tau} \int \prod d\lambda^x \exp (-A_{K^2, \tau}),$$

$$= - \int \prod d\lambda^x (h^x s_x t_x + \chi^\gamma \lambda^x \bar{D}^x \bar{t}_x) \exp (-A_{K^2, \tau}),$$

$$= \int \prod d\lambda^x (\partial D + h^x s_x \frac{\partial}{\partial \lambda^x} + r^\gamma \bar{F}_r \gamma \frac{\partial}{\partial \theta_\gamma}) (-\lambda^x \bar{t}_x) \exp (-A_{K^2, \tau}),$$

$$= \left( \partial + r^\gamma \bar{F}_r \gamma \frac{\partial}{\partial \theta_\gamma} \right) \int \prod d\lambda^x (-\lambda^x \bar{t}_x) \exp (-A_{K^2, \tau}),$$

establishing the desired result.

The relevance of this construction to physics will be discussed in section 4.2.2.

### 3.2 Cokernels

#### 3.2.1 First cokernel construction

Analogous constructions exist for cokernels. Suppose that one is interested in computing integrals over some space $Y \equiv \{s = 0\} \subset M$ ($s \in \Gamma(G)$) of sheaf cohomology classes

$$O_i \in H^\bullet(Y, \wedge^\bullet E'),$$

where $E'$ is a holomorphic vector bundle on $Y$, as before.

We propose that if $E'$ is given as the restriction to $Y$ of the cokernel of a smooth injective map $\bar{E} : F_1 \to F_2$ ($F_1$, $F_2$ holomorphic vector bundles on $M$), whose restriction to $Y$ is holomorphic, then at least for some $O_i$’s it is possible to write

$$\int_Y O_1 \wedge \cdots \wedge O_n \propto \int_M \bar{O}_1 \wedge \cdots \wedge \bar{O}_n \wedge \omega_{CK^1},$$

where $\bar{O}_1$ (when it exists) is an element of $H^\bullet(M, \wedge^\bullet F_2)$ ‘lifting’ $O_i$, in a fashion we shall discuss momentarily, and $\omega_{CK^1}$ is a $\bar{\partial}$-closed analogue of a Mathai-Quillen form,

$$\omega_{CK^1} \in H^g (M, \wedge^1 F_2 \otimes \det G^* \otimes \det F_1^*)$$

(13)
\((f_1 = \text{rk } F_1, g = \text{rk } G)\), where one has an isomorphism

\[ K_M \cong \det F_1^* \otimes \det F_2 \otimes \det G^* \]

that restricts to the isomorphism \(\det E' \cong K_Y\) needed to define the corresponding integral on \(Y\).

As a consistency check, notice that

\[ \mathcal{O}_1 \wedge \cdots \wedge \mathcal{O}_n \in H^{\dim M - g}(M, \wedge^{f_2 - f_1} F_2) \]

so that the cohomology class of \(\omega_{CK1}\) is correct for the integrand of \(M\) to be a top-form.

We propose an expression for \(\omega_{CK1}\) below, and check its properties.

First, let us explain the relationship between

\[ O \in H^\bullet(Y, \wedge^\bullet E') \text{ and } \bar{O} \in H^\bullet(M, \wedge^\bullet F_2). \]

Let \(i : Y \hookrightarrow M\) denote inclusion, so

\[ i^*\bar{O} \in H^\bullet(Y, \wedge^\bullet F|_Y). \]

Next, from the short exact sequence

\[ 0 \longrightarrow F_1|_Y \longrightarrow F_2|_Y \longrightarrow E' \longrightarrow 0 \]

we have a surjective map

\[ \wedge^\bullet F|_Y \longrightarrow \wedge^\bullet E' \]

which induces

\[ j_* : H^\bullet(Y, \wedge^\bullet F|_Y) \longrightarrow H^\bullet(Y, \wedge^\bullet E'). \]

Then, the pair \(O, \bar{O}\), when they exist, are related by

\[ O = j_* i^* \bar{O}. \]

Our proposal for \(\omega_{CK1}\) is given by the Grassmann integral

\[ \omega_{CK1} = \int \prod d\lambda^s d\chi^m \exp(-A_{CK1}), \]

where

\[ A_{CK1} = h_{x\gamma} s^x \bar{s}^{\gamma} + \chi^\gamma \chi^\gamma f_{\gamma} s^x h_{x\gamma} + \chi^m \theta_\gamma \bar{E}_m h_{\gamma\gamma} + F_{m\alpha\gamma} \chi^\gamma \lambda^s \theta_\gamma h_{\gamma\gamma}, \]

where \(x\) indexes local coordinates along the fibers of \(G\), \(m\) indexes local coordinates (denoted \(q\)) along the fibers of \(F_1\), \(\gamma\) indexes local coordinates along the fibers of \(F_2\), \(i\) indexes local
coordinates on $M$, and $s^x$ now denotes a component of a holomorphic section of $\mathcal{G}$ (rather than $s_x$ as was used in the discussion of kernels, for reasons of notational sanity). The curvature term

$$F_{tm \tau \pi} \chi^x \chi^m \lambda^\theta \gamma \eta^\tau$$

is fixed to solve

$$h_{\gamma \tau} \tilde{E}_m = s^x F_{tm \tau \pi} = -s^x F_{tm \tau}$$

and defines an element of

$$H^1(M, F^*_1 \otimes F^*_2 \otimes G^*) .$$

(Physically, $F$ arises as part of the curvature of a holomorphic vector bundle, hence is $\overline{\partial}$-closed by virtue of the Bianchi identity, much as in the closely related discussion in the kernels section.)

Now, let us explain some aspects of $\omega_{CK1}$ in more detail. In $A_{CK1}$ above, every $\lambda^x$ is accompanied by a $\chi^x$, so integrating out the $\lambda^x$'s should result in $\text{rk} \mathcal{G} = g$ of $\chi^x$'s, hence a degree $g$ cohomology class. Similarly, each $\chi^m$ is accompanied by a $\lambda^\tau$, so integrating out the $\chi^m$'s should result in coefficients $\wedge^f \tilde{F}_2$. The Grassmann integrals are responsible for a $\det \mathcal{G}^* \otimes \det F^*_1$ factor in the coefficients. Thus, $\omega_{CK1}$ is a form of the type indicated in equation (13). We shall show it is $\overline{\partial}$-closed momentarily.

Now, we can argue formally that the analogue of a Mathai-Quillen form defined above is $\overline{\partial}$-closed. The central point is that

$$\left( \overline{D} + s^x \frac{\partial}{\partial \lambda^x} \right) A_{CK1} = \chi^x \chi^m \theta \gamma \left( \overline{D}_{\tau} \tilde{E}_m + s^x F_{tm \tau \pi} h^\gamma \right) ,$$

using equation (14), where

$$\overline{D} = \chi^x \overline{\partial}_{\tau} .$$

Proceeding in the same fashion as before, it is simple to show that $\overline{\partial} \omega_{CK1} = 0$.

Next, we will argue that the cohomology class of $\omega_{CK1}$ is unchanged by ‘antiholomorphic’ deformations of $s$. To that end, consider the one-parameter family

$$A_{CK1, \tau} = h_{\lambda \tau} s^x \left( \tilde{s} \tau^\tau \right) + \chi^x \lambda^\theta \gamma \left( \overline{D}_{\tau} \tilde{E}_m + \tau \overline{\partial}_{\tau} \tilde{E}_m \right) h_{\tau \tau \pi} + \chi^m \theta \gamma \tilde{E}_m h_{\gamma \tau} + F_{tm \tau \pi} \chi^x \lambda^\theta \gamma h^\tau \gamma .$$

Solutions exist for reasons closely analogous to those in the analogous discussion in the kernels section. For example, if $\mathcal{G}$ is a line bundle, then this curvature term is the coboundary of

$$\tilde{E}|_Y \in H^0(Y, F^*_1|_Y \otimes F^*_2|_Y)$$

in the long exact sequence derived from tensoring

$$0 \rightarrow \mathcal{G}^* \rightarrow \mathcal{O} \rightarrow \mathcal{O}_Y \rightarrow 0$$

by $F^*_1 \otimes F^*_2$. In this special case, the condition above amounts to the definition of the coboundary map.
Then,
\[
\frac{d}{d\tau} \omega_{CK1,\tau} = \frac{d}{d\tau} \int \prod \, d\lambda^x d\chi^m \exp \left( -\mathcal{A}_{CK1,\tau} \right),
\]
\[
= -\int \prod \, d\lambda^x d\chi^m \left( h_x \pi^x T^x + \chi^x h_x \pi^x D^x \pi^x \right) \exp \left( -\mathcal{A}_{CK1,\tau} \right),
\]
\[
= \int \prod \, d\lambda^x d\chi^m \left( D + s^x \frac{\partial}{\partial \lambda^x} \right) \left( -h_x \pi^x T^x \right) \exp \left( -\mathcal{A}_{CK1,\tau} \right),
\]
\[
= \overline{\mathcal{D}} \int \prod \, d\lambda^x d\chi^m \left( -h_x \pi^x T^x \right) \exp \left( -\mathcal{A}_{CK1,\tau} \right),
\]
from which the desired result follows.

In passing, an important special case to which this cokernels construction is relevant is Euler sequences and generalizations describing tangent bundles of toric varieties. In general, the tangent bundle of a toric variety $Z$ is the cokernel
\[
0 \rightarrow \mathcal{O} \otimes V \overset{\tilde{E}}{\longrightarrow} \oplus D \mathcal{O}(D) \longrightarrow TZ \longrightarrow 0,
\]
where $V$ is a vector space, and the $D$’s in the middle entry are toric divisors. By deforming $\tilde{E}$, one can deform $TZ$ to a different holomorphic vector bundle on $Z$.

The physical relevance of this construction will be discussed in section 4.3.1.

### 3.2.2 Second cokernel construction

Suppose that one is interested in computing integrals over some space $Y \equiv \{ s = 0 \} \subset M$ ($s \in \Gamma(\mathcal{G})$) of sheaf cohomology classes
\[
\mathcal{O} \in H^\bullet(Y, \wedge^\bullet \mathcal{E}'),
\]
where $\mathcal{E}'$ is a holomorphic vector bundle on $Y$, as before, but the sheaf cohomology classes are valued in powers of $\mathcal{E}'^*$ instead of $\mathcal{E}'$.

We propose that if $\mathcal{E}'$ is given as the restriction to $Y$ of the cokernel of a holomorphic injective map $\tilde{E} : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ ($\mathcal{F}_1, \mathcal{F}_2$ holomorphic vector bundles on $M$), then at least for some $\mathcal{O}$’s it is possible to write the integral in the different form
\[
\int_Y \mathcal{O}_1 \wedge \cdots \wedge \mathcal{O}_n \propto \int_X \tilde{\mathcal{O}}_1 \wedge \cdots \wedge \tilde{\mathcal{O}}_n \wedge \omega_{CK2},
\]
where $\tilde{\mathcal{O}}_i$ (when it exists) is an element of hypercohomology
\[
\mathbb{H}^\bullet \left( X, \cdots \rightarrow \wedge^2 \pi^* \mathcal{F}^*_2 \rightarrow \pi^* \mathcal{F}^*_2 \rightarrow \mathcal{O}_X \right)
\]
(with maps given by inclusion along $q \tilde{E}$) ‘lifting’ $\mathcal{O}_i$, in a fashion we shall explain momentarily,

$$X = \text{Tot} \left( \mathcal{F}_1 \xrightarrow{\pi} M \right),$$

and $\omega_{CK2}$ is a $\mathcal{D}$-closed analogue of a Mathai-Quillen form,

$$\omega_{CK2} \in \mathbb{H}^g \left( X, \pi^* \det \mathcal{F}_1^* \otimes (\cdots \rightarrow \wedge^2 \pi^* \mathcal{F}_2^* \rightarrow \pi^* \mathcal{F}_2^* \rightarrow \mathcal{O}_X) \right)$$

(15)

($g = \text{rk} \mathcal{G}$), where one has an isomorphism

$$K_X \cong \pi^* \det \mathcal{F}_1^* \otimes \pi^* \det \mathcal{F}_2^*$$

that restricts to the isomorphism $\det \mathcal{E}^* \cong K_Y$ needed to define the corresponding integral on $Y$. Note that in this section, unlike the last, $\tilde{E}$ must be holomorphic everywhere.

We propose an expression for $\omega_{CK2}$ below, and check its properties.

First, let us discuss the relationship between $\mathcal{O} \in \mathbb{H}^\bullet (Y, \wedge \mathcal{E}^*)$ and $\tilde{\mathcal{O}} \in \mathbb{H}^\bullet \left( X, \cdots \rightarrow \wedge^2 \pi^* \mathcal{F}_2^* \rightarrow \pi^* \mathcal{F}_2^* \rightarrow \mathcal{O}_X \right)$. Briefly, we can use the isomorphism

$$\mathbb{H}^\bullet \left( X, \cdots \rightarrow \wedge^2 \pi^* \mathcal{F}_2^* \rightarrow \pi^* \mathcal{F}_2^* \rightarrow \mathcal{O}_X \right) \cong H^\bullet (M, \wedge \mathcal{E}^*)$$

discussed in [15][appendix A]. Let $i : Y \hookrightarrow M$ denote inclusion, then the pair $\mathcal{O}, \tilde{\mathcal{O}}$, when it exists, is related by

$$\mathcal{O} = i^* \tilde{\mathcal{O}}$$

using the isomorphism above.

Our proposal for $\omega_{CK2}$ is given by the Grassmann integral

$$\omega_{CK2} = \int \prod d\lambda^x \exp(-\mathcal{A}_{CK2}),$$

where

$$\mathcal{A}_{CK2} = h_{x_{\mathcal{S}}} s_{\mathcal{S}} \overline{s_{\mathcal{S}}} + h_{x_{\mathcal{S}}} q_{m_{\mathcal{S}}} \overline{q_{m_{\mathcal{S}}}} \overline{\mathcal{E}_{m_{\mathcal{S}}} \mathcal{E}_{m_{\mathcal{S}}}} + \chi_{x_{\mathcal{S}}} \lambda_{\mathcal{S}} \overline{\mathcal{D}_{\mathcal{S}} \mathcal{F}_{\mathcal{S}}} \mathcal{h}_{x_{\mathcal{S}}} + \chi_{x_{\mathcal{S}}} \lambda_{\mathcal{S}} \mathcal{E}_{m_{\mathcal{S}}} \mathcal{h}_{x_{\mathcal{S}}} + \chi_{x_{\mathcal{S}}} \lambda_{\mathcal{S}} \mathcal{D}_{\mathcal{S}} \overline{\mathcal{E}_{m_{\mathcal{S}}}} \mathcal{h}_{x_{\mathcal{S}}}$$

where $x$ indexes local coordinates along the fibers of $\mathcal{G}$, $m$ indexes local coordinates (denoted $q$) along the fibers of $\mathcal{F}_1$, $\gamma$ indexes local coordinates along the fibers of $\mathcal{F}_2$, $i$ indexes local coordinates on $M$, and $(m, i)$ index local coordinates on $X$.

Now, let us explain some aspects of $\omega_{CK2}$ in more detail. In $\mathcal{A}_{CK2}$ above, every $\lambda^x$ is accompanied by a $\chi^x$, so integrating out the $\lambda^x$'s should result in $\text{rk} \mathcal{G} = g \chi^x$'s, hence a degree
$g$ cohomology class. Furthermore, the Grassmann integral measure makes $\omega_{CK2}$ couple to $\det G^*$. Thus, $\omega_{CK2}$ is a form of the type indicated in equation (15). We shall show it defines an element of hypercohomology momentarily.

Now, we can argue formally that the analogue of a Mathai-Quillen form defined above represents an element of hypercohomology. The argument is a simple variation of that seen previously. The central point is that for

$$
\bar{D} = \chi^\tau \partial_\tau + \chi^m \partial_m
$$

we have

$$
\left( \bar{D} + s^x \frac{\partial}{\partial \lambda^x} \right) A_{CK2} = \chi^m h_{\gamma \tau} q^m \tilde{E}_m \frac{\partial}{\partial \lambda^{\gamma}} + \chi^\tau q^m \tilde{E}_m \tilde{D}_\gamma \left( h_{\gamma \tau} \tilde{E}_m \right),
$$

$$
= -q^m \tilde{E}_m \frac{\partial}{\partial \lambda^{\gamma}} A_{CK2}.
$$

Given this result, we see that although $\omega_{CK2}$ is not $\bar{D}$-closed, it does define an element of hypercohomology of the following sequence on $X$:

$$
\pi^* \det G^* \otimes (\cdots \to \wedge^2 \pi^* F^*_2 \to \pi^* F^*_2 \to O_X)
$$

with maps given by inclusion along $q\tilde{E}$, and the $\pi^* \det G^*$ factor determined by the Grassmann integral, as desired.

Next we will argue that the cohomology class of $\omega_{CK2}$ is unchanged by antiholomorphic deformations of the section $s$. As the details are somewhat more complicated than the argument in section 3.3, we sketch the details here. Consider the one-parameter family

$$
A_{CK2, \tau} = h_{x\tau} \tilde{s}^x \tilde{\tau}^x + h_{\gamma \tau} q^m \tilde{E}_m \frac{\partial}{\partial \lambda^{\gamma}} (\tilde{s}^x + \tau \tilde{t}^x) h_{x\tau}
$$

so that

$$
\frac{d}{d\tau} \omega_{CK2, \tau} = \frac{d}{d\tau} \int \prod \lambda^x \exp (-A_{CK2, \tau}),
$$

$$
= -\int \prod \lambda^x \left( h_{x\tau} \tilde{s}^x \tilde{t}^x + \chi^\tau \tilde{D}_\tau \tilde{t}^x h_{x\tau} \right) \exp (-A_{CK2, \tau}),
$$

$$
= \int \prod \lambda^x \left( \bar{D} + s^x \frac{\partial}{\partial \lambda^x} + q^m \tilde{E}_m \frac{\partial}{\partial \lambda^{\gamma}} \left( -h_{x\tau} \tilde{t}^{\tau} \right) \exp (-A_{CK2, \tau}),
$$

$$
= \left( \bar{D} + q^m \tilde{E}_m \frac{\partial}{\partial \lambda^{\gamma}} \right) \int \prod \lambda^x \left( -h_{x\tau} \tilde{t}^{\tau} \right) \exp (-A_{CK2, \tau}),
$$

from which the result follows.
In the special case that $G = 0$, we should note that it was shown in [15][appendix A] that
\[
\mathbb{H}^\bullet (X, \cdots \longrightarrow \wedge^2 \pi^* \mathcal{F}_2^* \longrightarrow \pi^* \mathcal{F}_2^* \longrightarrow \mathcal{O}_X) = H^\bullet (M, \wedge^\bullet \mathcal{E}'^*) ,
\]
where $\mathcal{E}'$ is the cokernel, exactly as expected for the hypercohomology described in this section to be related to ordinary sheaf cohomology.

As we will see in section 4.3.2, this analogue of a Mathai-Quillen form appears in the A/2 model pseudo-topological field theories [15], just as the analogues in the last section.

### 3.3 Cohomologies of short complexes

In this section, we will consider two constructions that will relate sheaf cohomology on
\[
Y \equiv \{s = 0\} \subset M
\]
to sheaf cohomology on the total space of a bundle over $M$, where the coefficients in question are given as the cohomology of a short complex.

#### 3.3.1 First short complex construction

Suppose we want to compute integrals of sheaf cohomology classes
\[
\mathcal{O} \in \mathcal{H}^\bullet (Y, \wedge^\bullet \mathcal{E}'^*)
\]
where $\mathcal{E}'$ is a holomorphic vector bundle on $Y$. Suppose that
\[
\mathcal{E}' = \frac{\ker \tilde{F}|_Y}{\text{im } \tilde{E}|_Y},
\]
where $\tilde{E} : \mathcal{F}_1 \to \mathcal{F}_2$ is an injective map between two holomorphic vector bundles on $M$, and $\tilde{F} : \mathcal{F}_2 \to \mathcal{F}_3$ is a surjective map between two holomorphic vector bundles on $M$, where $\tilde{E}$ is holomorphic on all of $M$ but $\tilde{F}$ is only holomorphic along $Y \subset M$, and whose restrictions to $Y$ form a complex:
\[
0 \longrightarrow \mathcal{F}_1|_Y \xrightarrow{\tilde{E}} \mathcal{F}_2|_Y \xrightarrow{\tilde{F}} \mathcal{F}_3|_Y \longrightarrow 0.
\]
(The composition $\tilde{F} \circ \tilde{E}$ vanishes everywhere on $M$.)

---

6 Experts will note that this is not the most general possibility allowed by physics. We leave more general cases for future work.
Then, we propose that, for those $\mathcal{O}$ such that lifts $\tilde{O}$ exist,

$$\int_Y \mathcal{O}_1 \wedge \cdots \wedge \mathcal{O}_n \propto \int_X \tilde{\mathcal{O}}_1 \wedge \cdots \wedge \tilde{\mathcal{O}}_n \wedge \omega_{MON},$$

where

$$X \equiv \text{Tot} \left( \mathcal{F}_1 \rightarrow \pi \rightarrow M \right),$$

the lifts $\tilde{O}$ are elements of hypercohomology

$$\mathbb{H}^\bullet \left( X, \cdots \rightarrow \wedge^2 \pi^* \mathcal{F}_2^* \rightarrow \pi^* \mathcal{F}_2^* \rightarrow \mathcal{O}_X \right)$$

with maps given by inclusion with $q^m \tilde{E}_m$, and

$$\omega_{MON} \in \mathbb{H}^{g+f_3} \left( X, \pi^* \det G^* \otimes \pi^* \det \mathcal{F}_3 \otimes \left( \cdots \rightarrow \wedge^2 \pi^* \mathcal{F}_2^* \rightarrow \pi^* \mathcal{F}_2^* \rightarrow \mathcal{O}_X \right) \right)$$

(16) $(g = \text{rk } G, f_i = \text{rk } F_i)$ given by

$$\omega_{MON} = \int \prod d\lambda^x d\chi^r \exp (-\mathcal{A}_{MON}),$$

where

$$\mathcal{A}_{MON} = h^{x\tau} s_x \tilde{E}_\tau + \chi^x \lambda^x \tilde{E}_m \tilde{E}_m + \chi^x \lambda^x \tilde{F}_{r\gamma} + F_{r\gamma} \lambda^x \lambda^x \lambda^\gamma$$

$$+ h_{r\gamma} q^m \tilde{E}_m \tilde{E}_m + \chi^x \lambda^x \tilde{F}_{m\gamma} h_{r\gamma} + \chi^x \lambda^x \tilde{F}_{m\gamma} h_{r\gamma}.$$

In the expression above, $x$ indexes local coordinates along the fibers of $G$, $m$ indexes local coordinates along the fibers of $F_1$, $\gamma$ indexes local coordinates along the fibers of $F_2$, $r$ indexes local coordinates along the fibers of $\mathcal{F}_3^*$, and $i$ indexes local coordinates on $M$. The curvature term

$$F_{r\gamma} \lambda^x \lambda^x \lambda^\gamma$$

represents the pullback of an element of

$$H^1 \left( M, \mathcal{F}_2^* \otimes \mathcal{F}_3 \otimes G^* \right)$$

which is defined by the condition

$$\overline{\partial}_r \tilde{F}_{r\gamma} = h^{x\tau} s_x F_{r\gamma} = -h^{x\tau} s_x F_{r\gamma} \quad (17)$$

(much as in the earlier discussion of kernels), and in addition, we assume that the curvature defined by $F$ annihilates the image of $\tilde{E}$:

$$q^m \tilde{E}_m F_{r\gamma} = 0.$$

Finally, there is an isomorphism

$$K_X \cong \pi^* \det G^* \otimes \pi^* \det \mathcal{F}_2^* \otimes \pi^* \det \mathcal{F}_3$$
which restricts to the isomorphism needed to define the integrals of sheaf cohomology classes on $Y$.

As before, we will check some elementary properties of $\omega_{MON1}$.

First, let us explain the relationship between $$\mathcal{O} \in H^\bullet (Y, \wedge^\bullet \mathcal{E}'^\ast) \quad \text{and} \quad \tilde{\mathcal{O}} \in H^\bullet (X, \cdots \rightarrow \wedge^2 \pi^\ast \mathcal{F}_2^\ast \rightarrow \pi^\ast \mathcal{F}_2^\ast \rightarrow \mathcal{O}_X).$$

Define $S$ to be the cokernel

$$0 \rightarrow \mathcal{F}_1 \xrightarrow{\tilde{E}} \mathcal{F}_2 \rightarrow S \rightarrow 0$$

We will use the isomorphism \[15][\text{appendix A}]

$$H^\bullet (X, \cdots \rightarrow \wedge^2 \pi^\ast \mathcal{F}_2^\ast \rightarrow \pi^\ast \mathcal{F}_2^\ast \rightarrow \mathcal{O}_X) \cong H^\bullet (M, \wedge^\bullet S^\ast).$$

Let $i : Y \hookrightarrow M$ denote the inclusion, and note that the injective map

$$\mathcal{E}' = \frac{\ker \tilde{F}|_Y}{\text{im} \tilde{E}|_Y} \rightarrow \frac{\mathcal{F}_2|_Y}{\text{im} \tilde{E}|_Y} = i^\ast S$$

defines a map

$$j_* : H^\bullet (Y, \wedge^\bullet \mathcal{E}'^\ast) \rightarrow H^\bullet (Y, \wedge^\bullet i^\ast S^\ast).$$

Then, using the isomorphism above, a pair $\mathcal{O}, \tilde{\mathcal{O}}$ (when it exists) is related by

$$j_* \mathcal{O} = i^\ast \tilde{\mathcal{O}}.$$

Next, we will argue that $\omega_{MON1}$ above defines an element of hypercohomology \[16]. The argument is a variation of that repeated previously. As before, the central point is that for

$$\overline{D} = \chi^r \overline{\partial}_r + \chi^m \overline{\partial}_m$$

we have

$$(\overline{D} + h^{x\pi} s_x \frac{\partial}{\partial \lambda^x} + q^m \tilde{E}_{m}^\gamma \frac{\partial}{\partial \lambda^\gamma}) A_{MON1}$$

$$= -\chi^r q^m \tilde{E}_{m}^\gamma \tilde{F}_{r\gamma} + \chi^r \chi^r \chi^\gamma \left( \overline{\partial}_r \tilde{F}_{r\gamma} + h^{x\pi} s_x F_{r\pi \gamma} \right)$$

$$- q^m \tilde{E}_{m}^\gamma F_{r\pi \gamma} \chi^r \chi^\gamma,$$

using the conditions discussed above. Combined with the fact that the Grassmann integration measure couples to

$$\pi^\ast \mathcal{G}^\ast \otimes \pi^\ast \mathcal{F}_3$$
and the fact that integrating over $\lambda^x$ brings down $g = \text{rk} \mathcal{G}$ factors of $\chi^r$, $\chi^r$ brings down $f_3 = \text{rk} \mathcal{F}_3$ factors of $\lambda^\gamma$, we see that $\omega_{\text{MON1}}$ can be interpreted as an element of the hypercohomology group (16).

Next, we demonstrate that the cohomology class of $\omega_{\text{MON1}}$ is independent of antiholomorphic deformations of $s$. Consider the one-parameter family

$$A_{\text{MON1},\tau} = h^x s_x \tau_x + \chi^r \lambda^x (D_x s_x \tau_x + D_x \tau_x) + \chi^r \lambda^\gamma \tilde{F}_{r\gamma} + F_{r\tau} \chi^r \lambda^x \lambda^\gamma + h_{\gamma\tau} q^m \bar{E}_m \tilde{E}_m^\gamma + \chi^r \lambda^\gamma \tilde{E}_m \tilde{E}_m \lambda_{\gamma\tau},$$

so that

$$\frac{d}{d\tau} \omega_{\text{MON1},\tau} = \frac{d}{d\tau} \int \prod d\lambda^x d\chi^r \exp (-A_{\text{MON1},\tau}),$$

$$= -\int \prod d\lambda^x d\chi^r \left(h^x s_x \tau_x + \chi^r \lambda^x D_x \tau_x\right) \exp (-A_{\text{MON1},\tau}),$$

$$= \int \prod d\lambda^x d\chi^r \left(D + h^x s_x \frac{\partial}{\partial \lambda^x} + q^m \bar{E}_m \frac{\partial}{\partial \lambda^\gamma}\right) \left(-\lambda^\tau \tau_x\right) \exp (-A_{\text{MON1},\tau}),$$

$$= \left(D + q^m \bar{E}_m \frac{\partial}{\partial \lambda^\gamma}\right) \int \prod d\lambda^x d\chi^r \left(-\lambda^x \tau_x\right) \exp (-A_{\text{MON1},\tau}),$$

from which the result follows.

The relevance of this construction to physics will be discussed in section 4.4.1.

### 3.3.2 Second short complex construction

Suppose we want to compute integrals of sheaf cohomology classes

$$\mathcal{O} \in H^\bullet (Y, \wedge^\bullet \mathcal{E}'),$$

where $\mathcal{E}'$ is a holomorphic vector bundle on $Y$, of the form

$$\mathcal{E}' = \frac{\ker \tilde{F}|_Y}{\text{im} \tilde{E}|_Y},$$

where $\tilde{E} : \mathcal{F}_1 \to \mathcal{F}_2$ is an injective map between two holomorphic vector bundles on $M$, and $\tilde{F} : \mathcal{F}_2 \to \mathcal{F}_3$ is a surjective map between two holomorphic vector bundles on $M$, where $\tilde{F}$ is holomorphic on all of $M$ but $\tilde{E}$ need only be holomorphic along $Y \subset M$, and whose restrictions form a complex

$$0 \to \mathcal{F}_1|_Y \xrightarrow{\tilde{E}} \mathcal{F}_2|_Y \xrightarrow{\tilde{F}} \mathcal{F}_3|_Y \to 0.$$
The composition $\tilde{F} \circ \tilde{E}$ vanishes everywhere on $M$.

Then, we propose that, for those $\mathcal{O}$ such that lifts $\tilde{\mathcal{O}}$ exist,

$$
\int_Y \mathcal{O}_1 \wedge \cdots \wedge \mathcal{O}_n \propto \int_X \tilde{\mathcal{O}}_1 \wedge \cdots \wedge \tilde{\mathcal{O}}_n \wedge \omega_{MON2},
$$

where

$$
X \equiv \text{Tot} \left( \mathcal{F}_3^* \xrightarrow{\pi} M \right)
$$

the lifts $\tilde{\mathcal{O}}$ are elements of hypercohomology

$$
\mathbb{H}^* (X, \cdots \rightarrow \wedge^2 \pi^* \mathcal{F}_2 \rightarrow \pi^* \mathcal{F}_2 \rightarrow \mathcal{O}_X)
$$

with maps given by inclusion with $p^* \tilde{F}_{r\gamma}$, and

$$
\omega_{MON2} \in \mathbb{H}^{g+f_1} (X, \pi^* \det G^* \otimes \pi^* \det \mathcal{F}^*_1 \otimes (\cdots \rightarrow \wedge^2 \pi^* \mathcal{F}_2 \rightarrow \pi^* \mathcal{F}_2 \rightarrow \mathcal{O}_X))
$$

(18)

($g = \text{rk } G$, $f_i = \text{rk } \mathcal{F}_i$) given by

$$
\omega_{MON2} = \int \prod d\lambda^x d\chi^m \exp (-A_{MON2}),
$$

where

$$
A_{MON2} = h_x^x s^x \tilde{E}_x + h^{\gamma \gamma} p^\gamma p^\gamma \tilde{E}_{r\gamma} \tilde{E}_{r\gamma} + \chi^\gamma \theta \gamma h^{\gamma \gamma} \tilde{F}_{r\gamma} + \chi^\gamma \theta \gamma h^{\gamma \gamma} p^\gamma \tilde{D}_{\gamma} \tilde{F}_{r\gamma}
$$

$$
+ \chi^\gamma \gamma D_{\gamma} h_x^x + \chi^m \theta \gamma \tilde{E}_m^\gamma + \mathcal{F}^m_{mx \chi} \chi^m \lambda^x \theta \gamma \gamma.
$$

In the expression above, $x$ indexes local coordinates along the fibers of $\mathcal{G}$, $m$ indexes local coordinates along the fibers of $\mathcal{F}_1$, $\gamma$ indexes local coordinates along the fibers of $\mathcal{F}_2$, $r$ indexes local coordinates along the fibers of $\mathcal{F}_3^*$, and $i$ indexes local coordinates on $M$. The curvature term

$$
\mathcal{F}^m_{mx \chi} \chi^m \lambda^x \theta \gamma \gamma
$$

represents the pullback of an element of

$$
H^4 (M, \mathcal{F}_1^* \otimes \mathcal{F}_2 \otimes \mathcal{G}^*)
$$

and solves the equation

$$
h_{r\gamma} \tilde{D}_{r\gamma} \tilde{E}_m^\gamma = s^x \mathcal{F}^m_{mx \gamma} = -s^x \mathcal{F}^m_{mx \gamma}. \quad (19)
$$

Its existence and properties were discussed in the earlier cokernels section. In addition, we assume the curvature defined by $F$ is in the kernel of $\tilde{F}$:

$$
p^\gamma \tilde{F}_{r\gamma} h^{\gamma \gamma} \mathcal{F}^m_{mx \gamma} = 0.
$$

7 More general cases are left for future work.
Finally, there is an isomorphism

\[ K_X \cong \pi^* \det G^* \otimes \pi^* \det F_1^* \otimes \pi^* \det F_2 \]

that restricts to the isomorphism that makes the integrals on \( Y \) well-defined.

As before, we will check some elementary properties of \( \omega_{MON2} \).

First, let us explain the relationship between

\[ \mathcal{O} \in H^\bullet(Y, \wedge \mathcal{E}') \text{ and } \tilde{\mathcal{O}} \in \mathbb{H}^\bullet(X, \cdots \to \wedge^2 \pi^* F_2 \to \pi^* F_2 \to \mathcal{O}_X). \]

Define \( S \) to be the kernel

\[ 0 \to S \to F_2 \to F_3 \to 0 \]

on \( M \), and let \( i : Y \hookrightarrow M \) denote the inclusion. We will use the isomorphism [15][appendix A]

\[ \mathbb{H}^\bullet(X, \cdots \to \wedge^2 \pi^* F_2 \to \pi^* F_2 \to \mathcal{O}_X) \cong H^\bullet(M, \wedge^\bullet S). \]

The map

\[ i^* S = \ker \tilde{F}|_Y \to \frac{\ker \tilde{F}|_Y}{\text{im } E|_Y} = \mathcal{E}' \]

defines a map

\[ j_* : H^\bullet(Y, \wedge^\bullet i^* S) \to H^\bullet(Y, \wedge^\bullet \mathcal{E}'). \]

Then, the pair \( \mathcal{O}, \tilde{\mathcal{O}} \), when it exists, is related by

\[ \mathcal{O} = j_* i^* \tilde{\mathcal{O}}. \]

Next, we show that \( \omega_{MON2} \) above defines an element of the hypercohomology group [18]. The argument is a variation of that repeated several times already. As before, the central point is that for

\[ \overline{D} = \chi^\tau \partial_\tau + \chi^\sigma \partial_\sigma \]

we have

\[
\left( \overline{D} + s^x \frac{\partial}{\partial \lambda^x} + p^r \tilde{F}_{r \gamma} \frac{\partial}{\partial \theta_\gamma} \right) \mathcal{A}_{MON2} = -\chi^m p^r \tilde{E}_{r \gamma} \tilde{F}_{r \gamma} + \chi^r \chi^m \theta_\gamma \left( \overline{D} \tilde{E}_{r \gamma} + s^\tau F_{mx \tau} h^{\gamma \tau} \right) = 0, \]

using the conditions discussed above. The Grassmann integration measure contributes a factor of \( \pi^* \det G^* \otimes \pi^* \det F_1^* \) to the coefficients. Each \( \lambda^x \) is accompanied by a \( \chi^x \), and each
is accompanied by a \( \theta_{\gamma} \), so the Grassmann integration yields a result of degree \( g + f_1 \) in \( \chi, \theta_{\gamma} \), determining the degree.

Next, we will demonstrate that the cohomology class of \( A_{\text{MON2}} \) is independent of antiholomorphic deformations of \( s \). Consider the one-parameter family

\[
A_{\text{MON2}, \tau} = h \chi^x \left( s^x + \tau \tilde{t}^x \right) + h^x p^x \tilde{F}_{\gamma} \tilde{T}_{\gamma} \tau + \chi^x \theta_{\gamma} h^x p^x \tilde{T}_{\gamma} \tau + \chi^x \theta_{\gamma} h^x p^x \tilde{D}_{\gamma} \tilde{F}_{\gamma} \tau \\
+ \chi^x \lambda^x \left( \tilde{D}_{\gamma} \tilde{s}^x + \tau \tilde{D}_{\gamma} \tilde{t}^x \right) h \chi^x + \chi^m \theta_{\gamma} \tilde{E}_m + F_{\gamma} \chi^m \lambda^x \theta_{\gamma} h^x \tilde{m},
\]

so that

\[
\frac{d}{d\tau} \omega_{\text{MON2}, \tau} = \frac{d}{d\tau} \int \prod d\lambda^x d\chi^m \exp \left( -A_{\text{MON2}, \tau} \right),
\]

\[
= - \int \prod d\lambda^x d\chi^m \left( h \chi^x \left( s^x + \tau \tilde{t}^x \right) + \chi^x \lambda^x \tilde{D}_{\gamma} \tilde{t}^x h \chi^x \right) \exp \left( -A_{\text{MON2}, \tau} \right),
\]

\[
= \int \prod d\lambda^x d\chi^m \left( \tilde{D} + s^x \frac{\partial}{\partial \lambda^x} + p^x \tilde{F}_{\gamma} \frac{\partial}{\partial \theta_{\gamma}} \right) \left( -h \chi^{x} \lambda^{x} \tilde{t}^{x} \right) \exp \left( -A_{\text{MON2}, \tau} \right),
\]

\[
= \left( \tilde{D} + p^x \tilde{F}_{\gamma} \frac{\partial}{\partial \theta_{\gamma}} \right) \int \prod d\lambda^x d\chi^m \left( -h \chi^x \lambda^x \tilde{t}^x \right) \exp \left( -A_{\text{MON2}, \tau} \right),
\]

from which the result follows.

The relevance of this construction to physics will be discussed in section 4.4.2.

4 Applications in topological field theory

The original Mathai-Quillen form [1] has appeared in topological field theories in several ways. One of its original uses was as a route to define topological field theories (see e.g. [2, 3, 4, 5, 6]), but we are more concerned with a more recent application to A-twisted Landau-Ginzburg models [14], and heterotic generalizations thereof [15].

In comparing expressions from heterotic strings and mathematics, we will have to perform a convention switch. Standard heterotic string conventions result in \( \partial \)-closed forms, whereas standard mathematics conventions involve \( \bar{\partial} \)-closed forms. Rather than use nonstandard conventions for either, we will simply complex conjugate whenever we wish to compare heterotic string results to mathematics.
4.1 Ordinary A-twisted Landau-Ginzburg models

4.1.1 (2,2) locus

The paper [14] studied examples of A-twisted Landau-Ginzburg models which RG flow to nonlinear sigma models. A prototypical example is the Landau-Ginzburg model on

\[ X = \text{Tot} (\pi : G^* \to M), \]

with superpotential

\[ W = p\pi^* s, \]

where \( p \) is a fiber coordinate and \( s \) a section of \( G \). This model RG flows to a nonlinear sigma model on \( Y \equiv \{ s = 0 \} \subset M \).

In A-twists of this Landau-Ginzburg model, the structure of a Mathai-Quillen form naturally arises, whose effect is to give a mathematical understanding of the effect of the renormalization group in this case. In other words, correlation functions in the A-twisted Landau-Ginzburg theory look like wedge products of differential forms on \( M \), but with an insertion of the Mathai-Quillen form, which makes them equivalent to computations on \( Y \subset M \).

Briefly, the action for the Landau-Ginzburg theory is of the form

\[
S^{(2,2)} = 2t \int d^2 z \left[ \frac{1}{2} (g_{\mu\nu} + iB_{\mu\nu}) \partial_{\mu} \phi^{a} \bar{\partial}_{\nu} \phi^{a} + ig_{\mu a} \bar{\psi}_{+}^{a} \partial_{\mu} \phi^{a} + ig_{\partial a} \bar{\psi}_{-}^{a} D_{z} \psi_{+}^{a} \right. \\
\left. + R_{\alpha a b} \psi_{+}^{a} \psi_{+}^{b} \psi_{-}^{c} + g^{\alpha a} \partial_{\alpha} W \bar{\partial}_{\beta} W + \psi_{+}^{a} \psi_{-}^{b} \partial_{a} W \partial_{b} W + \left( \phi^{a} \bar{\psi}_{+}^{a} D_{z} \phi^{b} \bar{\psi}_{-}^{b} \right) \right], \quad (20)
\]

where \( \phi^{a} = (p, \phi^{i}) \), \( p \) a fiber coordinate, \( \phi^{i} \) coordinates on \( M \), and in the A-twisted theory,

\[
\begin{aligned}
\psi_{+}^{a} &\equiv \chi^{a}_{+} \in \Gamma \left( (\phi^{*} (T^{1,0} M))^{*} \right), & \psi_{-}^{a} &\equiv \chi^{a}_{-} \in \Gamma \left( (K_{\Sigma} \otimes (\phi^{*} (T^{0,1} M))^{*}) \right), \\
\psi_{+}^{a} &\equiv \bar{\chi}_{+}^{a} \in \Gamma \left( K_{\Sigma} \otimes (\phi^{*} (T^{1,0} M))^{*} \right), & \psi_{-}^{a} &\equiv \bar{\chi}_{-}^{a} \in \Gamma \left( (K_{\Sigma} \otimes (\phi^{*} (T^{0,1} M))^{*}) \right), \\
\psi_{+}^{p} &\equiv \psi_{+}^{\pi} \in \Gamma \left( K_{\Sigma} \otimes (\phi^{*} T^{1,0})^{*} \right), & \psi_{-}^{p} &\equiv \psi_{-}^{\pi} \in \Gamma \left( (\phi^{*} T^{0,1})^{*} \right), \\
\psi_{+}^{\bar{\pi}} &\equiv \chi_{+}^{\bar{\pi}} \in \Gamma \left( (\phi^{*} T^{1,0})^{*} \right), & \psi_{-}^{\bar{\pi}} &\equiv \chi_{-}^{\bar{\pi}} \in \Gamma \left( (K_{\Sigma} \otimes (\phi^{*} T^{0,1}))^{*} \right),
\end{aligned}
\]

where \( K_{\Sigma} \) is the canonical bundle on \( \Sigma \) and \( T^{\pi} \) is the relative tangent bundle of the projection \( \pi : G^* \to M \). To make sense of the A-twist of this theory, it was also necessary to twist some of the bosons, specifically,

\[
p \equiv p_{z} \in \Gamma \left( K_{\Sigma} \otimes (\phi^{*} T^{1,0}) \right), \quad \bar{p} \equiv \bar{p}_{\pi} \in \Gamma \left( K_{\Sigma} \otimes (\phi^{*} T^{0,1}) \right),
\]

and the \( \phi^{i} \) remain untwisted.

Although \( \chi^{i}, \chi^{\bar{\pi}}, \chi^{p} \), and \( \chi^{\bar{\pi}} \) are all scalars, it can be shown (following [14]) that only \( \chi^{i}, \chi^{\bar{\pi}} \) are BRST-invariant. Furthermore, as the \( p \) fields are twisted, scalar zero modes lie
along \( \{ p = 0 \} = M \), and so correlation functions take the form of integrals over \( M \) of wedge products of observables of the form

\[
f(\phi^i)\chi^i \cdots \chi^i_n \bar{\chi}^i \cdots \bar{\chi}^i_m
\]

with an insertion of an exponential of zero mode interactions we shall discuss momentarily.

We can see the relevance of pullbacks of Mathai-Quillen forms as follows. If, for example, we restrict to degree zero maps on a genus zero worldsheet, then from restricting to zero modes we recover the following interactions on the zero modes:

\[
g^p s_p \bar{s}_p + \chi^i \chi^p D_i s_p + \chi^p \chi^i \bar{D}_i \bar{s}_p + R_{\bar{\tau}p} \chi^i \chi^p \chi^i \bar{\tau}.
\]

If we now complex conjugate this expression to relate it to standard mathematics conventions, we obtain

\[
g^p s_p \bar{s}_p + \chi^i \chi^p \bar{D}_i \bar{s}_p + \chi^p \chi^i D_i s_p + R_{\bar{\tau}p} \chi^i \chi^p \chi^i \tau
\]

\[= g^p s_p \bar{s}_p + \rho^p D_s \bar{s} + \bar{D} s_p \rho^p + \rho^p R_{sp} \rho^p,\]

\[= (s_p e^p, \bar{s}_p e^p)_{\mathcal{G}} + \left( \rho^p f^p, D_s \bar{s}_p e^p \right)_{\mathcal{G}} + \left( \bar{D} \bar{s}_p \rho^p, \rho^p f^p \right)_{\mathcal{G}} + \left( \rho^p f^p, f^p (R_{sp} f^p, \rho^p f^p) \right)_{\mathcal{G}^*},\]

\[= \frac{1}{2} (s, s)_{\mathcal{G}} + (\nabla s, \rho)_{\mathcal{G}} + \frac{1}{2} (\rho, R \rho)_{\mathcal{G}^*},\]

\[= \mathcal{A}, \quad (21)\]

where

\[
d = \partial + \bar{\partial} = d\phi^i \partial_i + d\phi^i \bar{\partial}_i = \chi^i \partial_i + \chi^i \bar{\partial}_i,
\]

\[
\nabla = D + \bar{D} = d\phi^i D_i + d\phi^i \bar{D}_i = \chi^i D_i + \chi^i \bar{D}_i,
\]

\[s = s_p e^p + \bar{s}_p e^p,\]

\[\rho = \rho^p f^p + \rho^p \bar{f}^p = \chi^p f^p + \chi^p \bar{f}^p.
\]

Thus, (21) is minus the exponential of the pullback of a Mathai-Quillen form, giving a mathematical understanding of the behavior of RG flow in this model.

In this language, the fact that \( A \) model correlators in nonlinear sigma models are independent of the complex structure is a consequence of the fact that the \( d \)-cohomology class of the pullback of the Mathai-Quillen form by \( s \) is independent of \( s \).

### 4.1.2 A/2 deformation

Now, let us turn to the A/2 model for a deformation of the model above, describing a deformation of the tangent bundle. Mathematically, the tangent bundle to \( Y \equiv \{ s = 0 \} \),

\[32\]
are required to obey

\[0 \rightarrow TY \rightarrow TM|_Y \xrightarrow{(D_{s})_{yp}} G|_Y \rightarrow 0.\]

A deformation of the tangent bundle above is defined by

\[0 \rightarrow \mathcal{E}' \rightarrow TM|_Y \xrightarrow{(D_{s} + (\delta s)_{yp})} G|_Y \rightarrow 0,\]

where the \((\delta s)_{yp}\) define the deformation.

A commonly-discussed special case of this involves deformations of tangent bundles of hypersurfaces in projective spaces. In such cases, in homogeneous coordinates \(z^i\), the \((\delta s)_{yp}\) are required to obey

\[z^i(\delta s)_{yp} = 0.\]

In affine coordinates, this instead becomes the statement that, across coordinate patches, there are several different \((\delta s)_{yp}\), but on any one given coordinate patch, one is determined by the others.

The action of the A/2 twist of the heterotic Landau-Ginzburg model that RG flows to a nonlinear sigma model with tangent bundle deformation above is given by

\[S^{(0, 2)} = 2t \int_\Sigma d^2z \left[ \frac{1}{2} \left( g_{\mu\nu} + iB_{\mu\nu} \right) \partial_\mu \phi^a \partial_\nu \phi^b + i\bar{g}_{\bar{\alpha}\bar{\beta}} \bar{\psi}^\alpha \bar{\partial}_\bar{\alpha} \psi^\beta + i\bar{g}_{\bar{\alpha}\bar{\beta}} \bar{\lambda}^\alpha D_\alpha \psi^\beta \right.\]

\[+ R_{\alpha\beta\gamma\delta} \bar{\psi}^{\alpha} \psi^\beta \lambda^\gamma_\delta + g^{a\bar{c}} F_a \bar{\Gamma}_{\bar{\alpha} \bar{\beta}} + \psi^\alpha_+ D_\alpha F_+ + \psi^\alpha_- D_\alpha F_- + \psi^\beta_+ \bar{\lambda}^\beta_\delta D_\delta F^\alpha_+ \right], \quad (23)\]

with target space

\[X = \text{Tot} \left( G^* \xrightarrow{\pi} M \right)\]

and gauge bundle \(\mathcal{E} = TX\), where

\[F_a = (F_p, F_i) = (s_p, p(D_{s} + (\delta s)_{yp}))\], \[\bar{\Gamma}_{\bar{\alpha} \bar{\beta}} = \left( \bar{\Gamma}_{\bar{\alpha} \bar{\beta}} \right) \]

\[D_a F_b = \partial_a F_b - \Gamma^c_{ab} F_c, \quad \bar{\Gamma}_{\bar{\alpha} \bar{\beta}} = \left( \bar{\Gamma}_{\bar{\alpha} \bar{\beta}} \right), \]

\[\bar{D}_\alpha \psi^a_+ = \bar{\partial}_\alpha \psi^a_+ - \bar{\partial}_\alpha \phi^b \Gamma^c_{bc} \psi^a_+, \quad D_\alpha \bar{\lambda}^- = \partial_\alpha \bar{\lambda}^- + \partial_\alpha \phi^\beta \Gamma^c_{\beta \alpha} \bar{\lambda}^c_- \]

and

\[\psi^i_+ \equiv \chi^i \in \Gamma \left( \phi^* (T^{1, 0} M) \right), \quad \lambda^+_- \equiv \lambda^i_- \in \Gamma \left( K \otimes \left( \phi^* (T^{0, 1} M) \right) \right) \]

\[\psi^\alpha_+ \equiv \psi^\alpha \in \Gamma \left( K \otimes \phi^* (T^{1, 0} M) \right), \quad \lambda^\alpha_- \equiv \lambda^\alpha \in \Gamma \left( K \otimes \phi^* (T^{0, 1} M) \right) \]

\[\psi^\beta_+ \equiv \psi^\beta \in \Gamma \left( K \otimes \phi^* T^{1, 0}_\pi \right), \quad \lambda^\beta_- \equiv \lambda^\beta \in \Gamma \left( K \otimes \phi^* T^{0, 1}_\pi \right) \]

\[p \equiv p_z \in \Gamma \left( K \otimes \phi^* T^{1, 0}_\pi \right), \quad \bar{p} \equiv \bar{p}_\pi \in \Gamma \left( K \otimes \phi^* T^{0, 1}_\pi \right) \].
Proceeding as in the last example, to illustrate the relevance of the deformed object \( \omega_{\delta s} \) given by (11), if for example we restrict to zero modes on a genus zero worldsheet, in the degree zero sector we find the following interactions among zero modes:

\[
\begin{align*}
&g^{\mu \nu} F_{\mu} F_{\nu} + \chi^i \lambda^p D_i F_{\nu} + \chi^i \lambda^i \bar{\nabla}_\mu F_{\nu} + R_{\mu \nu \rho \sigma} \lambda^p \lambda^\sigma, \\
&= g^{\mu \nu} \bar{s}_p s_p + \chi^i \lambda^p D_i s_p + \chi^i \lambda^i (\bar{\nabla}_\mu \bar{s}_p + (\delta \bar{s}_p)) + R_{\mu \nu \rho \sigma} \lambda^p \lambda^\sigma.
\end{align*}
\]

If we now complex conjugate so as to relate the heterotic expression above to standard mathematics conventions, we find

\[
\begin{align*}
&g^{\mu \nu} s_p \bar{s}_p + \chi^i \lambda^p \bar{D}_\nu \bar{s}_p + \chi^i \lambda^i (D_i s_p + (\delta s)_i) + R_{\mu \nu \rho \sigma} \lambda^p \lambda^\sigma, \\
&= g^{\mu \nu} s_p \bar{s}_p + \rho^p D_s + \bar{D}_\nu \rho^p + \rho^p \bar{R}_{\nu \rho} + (\rho^p \rho^i (\delta s)_i), \\
&= (s_p e^p, \bar{s}_p e^p) + \langle \rho^p f_{\rho'}, D s_p e^p \rangle + \langle \bar{D} \bar{s}_p e^p, \rho^p f_{\rho} \rangle + \langle \rho^p f_{\rho'}, \rho^p f_{\rho} \rangle + \langle \rho^p f_{\rho'}, d \phi^i (\delta s)_i e^p \rangle, \\
&= A + \langle \rho^p f_{\rho'}, d \phi^i (\delta s)_i e^p \rangle, \\
&= A_{\delta s}, \quad (24)
\end{align*}
\]

which is minus the exponent of (11).

4.2 Kernels

4.2.1 A/2 model realization of first kernel construction

We can write down an A/2 model describing a kernel as follows. Suppose we wish to build a (0,2) Landau-Ginzburg model that RG flows to a nonlinear sigma model on \( Y \equiv \{ s = 0 \} \subset M \) with gauge bundle \( \mathcal{E}' \) defined by the kernel of the restriction of a surjective map \( \tilde{F} : \mathcal{F}_1 \rightarrow \mathcal{F}_2 \) to \( \{ s = 0 \} \). The map \( \tilde{F} \) is surjective everywhere on \( M \). The restriction of \( \tilde{F} \) to \( Y \) is holomorphic; however, over the rest of \( M \), \( \tilde{F} \) need be merely smooth.

Then, we consider a Landau-Ginzburg model on

\[
X = \text{Tot} \left( \mathcal{F}_2^* \xrightarrow{\pi} M \right)
\]
with gauge bundle $\mathcal{E}$ given by an extension

$$0 \rightarrow \pi^* \mathcal{G}^* \rightarrow \mathcal{E} \rightarrow \pi^* \mathcal{F}_1 \rightarrow 0. \quad (25)$$

To specify the physical theory, we need to specify both the extension $\mathcal{E}$ and a holomorphic section of $\mathcal{E}^*$. The details of the extension class are, except for certain special cases, largely not relevant to this paper, so let us give the holomorphic section first, and then we shall outline pertinent facts on the extension class. Dualizing the extension above to

$$0 \rightarrow \pi^* \mathcal{F}_1^* \rightarrow \mathcal{E}^* \rightarrow \pi^* \mathcal{G} \rightarrow 0,$$

it is straightforward to see that a holomorphic section of $\mathcal{E}^*$ uniquely determines a holomorphic section of $\pi^* \mathcal{G}$, call it $\pi^* s$. Furthermore, a holomorphic section of $\mathcal{E}^*$ noncanonically determines a smooth section of $\pi^* \mathcal{F}_1^*$ which is holomorphic over $\{s = 0\}$. (Alternatively, we could work with holomorphic sections in local trivializations, but in this paper it will be more convenient to work with a global smooth section.) To define the physical theory, we will pick a holomorphic section of $\mathcal{E}^*$ determined by the pullback of $s \in \Gamma(\mathcal{G})$, determining $Y$, and we will take the smooth section of $\pi^* \mathcal{F}_1^*$ to be given by $p\tilde{F}$, where $p$ denotes fiber coordinates on $X$.

As an aside, the four-fermi term in $\mathcal{A}_{K_1}$ in the Mathai-Quillen analogue associated to this theory, defined by an element of

$$H^1(M, \mathcal{F}_1^* \otimes \mathcal{G}^* \otimes \mathcal{F}_2)$$

is related to the extension class of (25) as follows. One computes

$$\text{Ext}_X^1(\pi^* \mathcal{F}_1, \pi^* \mathcal{G}^*) = H^1(X, \pi^* \mathcal{F}_1^* \otimes \pi^* \mathcal{G}^*),$$

$$= H^1(M, \pi_* \pi^* (\mathcal{F}_1^* \otimes \mathcal{G}^*)), $$

$$= H^1(M, \mathcal{F}_1^* \otimes \mathcal{G}^* \otimes \text{Sym}^\bullet \mathcal{F}_2),$$

since $\pi$ is affine, by Leray. The four-fermi term can be understood as living in one of the sheaf cohomology groups above.

In the special case that $\mathcal{E}'$ is the restriction of a bundle on $M$ to $Y$ (i.e. the map $\tilde{F}$ is globally holomorphic, not just smooth), the extension will be trivial: $\mathcal{E} = \pi^* \mathcal{G}^* \otimes \pi^* \mathcal{F}_1$. Readers familiar with Distler-Kachru models can derive the same result physically by thinking about the bundle defined by the fermi superfields. In general, however, the extension need not be trivial. For example, if $\mathcal{E}' = TY$, as will be the case for ordinary Mathai-Quillen forms, then the extension will be nontrivial. In this special case, $\mathcal{E} = TX$ (as appropriate for a $(2,2)$ supersymmetric theory), which is realized via the nontrivial extension

$$0 \rightarrow \pi^* \mathcal{G}^* \rightarrow \mathcal{E} \rightarrow \pi^* TM \rightarrow 0,$$

\footnote{The smooth section depends upon a choice of splitting of the smooth bundle $\mathcal{E}^*$, forgetting the holomorphic structure.}
where $\mathcal{F}_1 = TM$. More generally, for readers familiar with Distler-Kachru models, if one integrates out fermionic shift symmetries, the extension will be nontrivial.

The action of the A/2 twisted Landau-Ginzburg model that RG flows to the A/2 twist of the nonlinear sigma model above is given in local coordinates by [15]

$$S = 2t \int_{\Sigma} d^2z \left[ \frac{1}{2} (g_{\mu \nu} + iB_{\mu \nu}) \partial_z \phi^\mu \bar{D}_\tau \phi^\nu + ig_{\alpha \beta} \psi_+^\beta \bar{D}_\tau \psi_+^\alpha + ig_{\alpha} \lambda_\alpha D_z \lambda_\alpha \right. $$

$$+ F_{\alpha \beta \gamma} \psi_+^\alpha \psi_+^\beta \lambda_\gamma \bar{\lambda}_\gamma + h^{\gamma \rho} s_x \bar{\gamma}_\rho + h^{\gamma \rho} p^\rho \bar{F}_{\gamma \rho} \bar{F}_{\gamma \rho} $$

$$+ \psi_+^i \lambda_x D_i s_x + \psi_+^i \lambda_\gamma \bar{F}_{\gamma \gamma} + \psi_+^i \lambda_\gamma p^\gamma D_i \bar{F}_{\gamma \gamma} $$

$$+ \psi_+^r \lambda_x D_i \bar{\gamma}_\rho + \psi_+^r \lambda_\gamma \bar{F}_{\gamma \gamma} + \psi_+^r \lambda_\gamma p^\gamma D_i \bar{F}_{\gamma \gamma} \right],
$$

where $x$ indexes local coordinates along the fibers of $\mathcal{G}$, $\gamma$ indexes local coordinates along the fibers of $\mathcal{F}_1$, $r$ indexes local coordinates (denoted $p$) along along the fibers of $\mathcal{F}_2^*$, $i$ indexes local coordinates on $M$, $a \sim (r, i)$ indexes local coordinates on $X$, and $\alpha \sim (x, \gamma)$ indexes local coordinates along the fibers of $\mathcal{E}$. In the notation of [15], in local coordinates, $(F_\alpha) = (s_x, p^\gamma \bar{F}_{\gamma \gamma})$.

Note that the space of vacua is given by $\{s = 0\} \cap \{p = 0\}$: the first condition follows from the potential term $|s|^2$, the second from the potential term $|p\bar{F}|^2$ and the fact that $F$ is surjective everywhere on $M$. (In a (2,2) theory, if the space becomes singular so that $p$ gets a vev, the result seems to have an interpretation in terms of cotangent complexes [21], with $\mathbb{C}^\times$ rotations of the fibers of $X$ providing a grading. In (0,2) theories, if surjectivity of $\bar{F}$ breaks down and $p$ gets a vev, then sometimes, under certain circumstances that are not well-understood, the theory will still be well-behaved, but typically it will be singular [22].

The fermions and bosons are twisted as follows:

$$\psi_+^i \equiv \chi^i \in \Gamma(\phi^*(TM)), \quad \psi_+^r \equiv \psi_+^r \in \Gamma(K_\Sigma \otimes \phi^*T^*M), \quad \psi_+^x \equiv \psi_+^x \in \Gamma((\phi^* T^{1,0}_G)^*), \quad \psi_+^r \equiv \chi^r \in \Gamma((\phi^* T^{1,0}_G)^*),$$

$$\lambda_x^\alpha \equiv \lambda_x^\alpha \in \Gamma((\phi^* T^{1,0}_G)^*), \quad \lambda_\gamma^\alpha \equiv \lambda_\gamma^\alpha \in \Gamma(K_\Sigma \otimes \phi^* T^{0,1}_G), \quad \lambda_\gamma^\alpha \equiv \lambda_\gamma^\alpha \in \Gamma((\phi^* T^{0,1}_G)^*), \quad \lambda_\gamma^\alpha \equiv \lambda_\gamma^\alpha \in \Gamma((\phi^* T^{1,0}_G)^*),$$

$$p \equiv p_x \in \Gamma(K_\Sigma \otimes \phi^* T^{1,0}_G), \quad p \equiv p_x \in \Gamma(K_\Sigma \otimes \phi^* T^{1,0}_G).$$

Anomalies constrain the theory above. Specifically, one must require that

$$\det \mathcal{G}^* \otimes \det \mathcal{F}_1^* \cong \det \mathcal{F}_2^* \otimes K_M, \quad \text{ch}_2(\mathcal{E}) = \text{ch}_2(T_X).$$

(Not that the first condition is slightly different than merely $\det \mathcal{E}^* \cong K_X$, ultimately because the twist acts differently on various contributions to $\mathcal{E}$ and $TX$.) One can show that anomaly-freedom in the UV implies anomaly-freedom in the IR.
The effective interactions can be obtained by truncating to fermi zero modes. In the
degree zero sector, they are

\[ h^x s_x \bar{s}_x + \chi^i \lambda^x D_i s_x + \chi^\gamma \lambda^x \bar{T}_{i\gamma} + F_{i\pi x} \chi^i \lambda^x \lambda^\gamma. \]

The reader should recognize the expression above as the complex conjugate of equa-
tion (9). The complex conjugation is necessary because of a difference in standard con-
ventions: standard heterotic string conventions yield \( \partial \)-closed operators, whereas standard
mathematics conventions in this context involve \( \bar{\partial} \)-closed operators. The two are related by
a simple complex conjugation.

Although \( \chi^i, \chi^\gamma, \lambda^x \), and \( \lambda^\gamma \) are all scalars, it can be shown following [15] that only \( \chi^i \),
\( \lambda^\gamma \) are BRST-invariant. Furthermore, since the \( p \) fields are twisted, the bosonic zero modes
lie along \( M \). Restricting to zero modes, observables are then of the form

\[ f(\phi^i)\chi^i \ldots \chi^n \lambda^{\gamma_1} \ldots \lambda^{\gamma_m}, \]

which after a complex conjugation are naively interpreted in terms of

\[ H^\bullet (M, \wedge^\bullet F^*_1). \]

(The more nearly correct interpretation of the chiral ring is in terms of a restriction to
\( \{s = 0\} \) of the cohomology above, but this is not essential for this discussion.) Correlation
functions then are of the form of integrals over \( M \) of observables times the exponential of
the zero mode interactions.

Since this theory flows under RG to an A/2-twisted nonlinear sigma model, correlation
functions of the observables above should coincide with correlation functions in the nonlinear
sigma model, and this is the root of the claims in this paper regarding analogues of Mathai-
Quillen forms.

Ordinary (2,2) Landau-Ginzburg models, and deformations thereof, are special cases of
this construction. In a (2,2) Landau-Ginzburg model on \( X \), \( \mathcal{E} = TX \) and the \( (F_a) \) are
given by derivatives of a superpotential \( W \). By taking \( \mathcal{F}_1 = TM, \mathcal{F}_2 = \mathcal{G} \), we can take the extension

\[ 0 \rightarrow \pi^* \mathcal{G}^* \rightarrow \mathcal{E} \rightarrow \pi^* \mathcal{F}_1 \rightarrow 0 \]
to coincide with the tangent bundle \( TX \),

\[ 0 \rightarrow \pi^* \mathcal{G}^* \rightarrow TX \rightarrow \pi^* TM \rightarrow 0, \]

where \( X \) is the total space of \( \pi : \mathcal{G}^* \rightarrow M \). The (2,2) superpotential \( W = ps \), \( s \) here
the pullback of a section of \( \mathcal{G} \), so we take \( (F_a) = (s, pD_i s) \). (A deformation of the (2,2)
locus would be described by the same bundles but with \( (F_a) = (s, p(D_i s + (\delta s)_i)) \). It
is straightforward to check that the Lagrangian in this special case corresponds with the Lagrangian given earlier for the (2,2) theory.

In the special case of a (2,2) theory, the curvature term appearing in the zero mode interactions, now an element of

\[ H^1(M, \Omega^1_M \otimes G \otimes G^*) \]

can be shown to coincide with the Atiyah class of \( G \), from the fact that the extension defined is \( TX \). Specifically, from earlier work one has

\[ \text{Ext}^1_X(\pi^*TM, \pi^*G^*) = H^1(M, \Omega^1_M \otimes G^* \otimes \text{Sym}^*G) . \]

Now, there is a natural \( \mathbb{C}^\times \) scaling action on the fibers of \( \pi : X \to M \), which induces an action on tangent bundles. The original extension class has weight 0 under this \( \mathbb{C}^\times \), \( \Omega^1_M \) has weight 0, \( G^* \) has weight 1, and \( \text{Sym}^*G \) has weight \(-\cdot\), so the original extension in \( H^1(X, \pi^*(\Omega^1_M \otimes G^*)) \) is an element of \( H^1(M, \Omega^1_M \otimes G^* \otimes G) \), and one can check in local trivializations that this element is the Atiyah class of \( G \).

In any event, the fact that the curvature term so determined coincides with the Atiyah class of \( \mathcal{G} \), means that on the (2,2) locus we exactly reproduce the curvature term appearing in standard Mathai-Quillen forms.

### 4.2.2 B/2 model realization of second kernel construction

We can also write down a B/2 model describing a kernel as follows. Suppose we wish to build a (0,2) Landau-Ginzburg model that RG flows to a nonlinear sigma model on

\[ Y \equiv \{ s = 0 \} \subset M \]

with gauge bundle defined by the kernel of the restriction of a surjective holomorphic map

\[ \tilde{F} : \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \]

to \( \{ s = 0 \} \). As before, \( \tilde{F} \) is surjective everywhere on \( M \), and in addition we also impose the condition that \( \tilde{F} \) be holomorphic everywhere on \( M \), not just the restriction to \( Y \).

Then, we consider a Landau-Ginzburg model on

\[ X = \text{Tot} \left( \mathcal{F}_2^* \xrightarrow{\pi} M \right) \]

with gauge bundle \( \mathcal{E} = \pi^*\mathcal{G}^* \oplus \pi^*\mathcal{F}_1 \).

---

\(^9\) We would like to thank T. Pantev for a discussion of this point.
The action of the B/2 twisted Landau-Ginzburg model that RG flows to the B/2 twist of the nonlinear sigma model above is of the same form as that discussed previously for the A/2 twist:

\[
S = 2t \int_{\Sigma} d^2z \left[ \frac{1}{2} (g_{\mu\nu} + iB_{\mu\nu}) \partial_\mu \phi^* \partial_\nu \phi^* + ig_{\alpha\beta} \psi_+^\alpha \overline{\partial}_\sigma \psi_+^\beta + ig_{\alpha\beta} \lambda^\alpha_+ D_+ \lambda^\beta_+ \\
+ F_{\alpha\beta\gamma} \psi_+^\alpha \overline{\lambda}_-^\beta \lambda^\gamma_+ + h^\gamma p^r \overline{F}_{r\gamma} \overline{F}_{r\gamma} + \psi_+^i \lambda_+^x D_i s_x + \psi_+^i \lambda_+^\gamma F_\gamma + \psi_+^i \lambda_+^x D_i \overline{F}_{r\gamma} + \psi_+^i \lambda_+^\gamma D_i \overline{F}_{r\gamma}
\]

where \( x \) indexes local coordinates along the fibers of \( \mathcal{G} \), \( \gamma \) indexes local coordinates along the fibers of \( \mathcal{F}_1 \), \( r \) indexes local coordinates (denoted \( p \)) along along the fibers of \( \mathcal{F}_2 \), \( i \) indexes local coordinates on \( M \), \( a \sim (r, i) \) indexes local coordinates on \( X \), and \( \alpha \sim (x, \gamma) \) indexes local coordinates along the fibers of \( \mathcal{E} \). In the notation of [15], in local coordinates, \( (F_a) = (s_x, p^r \overline{F}_{r\gamma}) \).

The fermions are twisted as follows:

\[
\psi_+^i \equiv \chi^i \in \Gamma(\phi^* (TM)), \quad \psi_+^r \equiv \psi^r \in \Gamma(K_X \otimes \phi^* T^* M), \\
\psi_+^x \equiv \chi^x \in \Gamma(\phi^* T^1_0), \quad \psi_+^\gamma \equiv \psi^\gamma \in \Gamma(K_X \otimes (\phi^* T^1_0)^*), \\
\lambda_+^x \equiv \lambda^x \in \Gamma((\phi^* \mathcal{G})^*), \quad \lambda_+^\gamma \equiv \lambda^\gamma \in \Gamma(\overline{\mathcal{F}_1} \otimes \phi^* T^0_{\mathcal{G}}), \\
\lambda_-^x \equiv \lambda^x \in \Gamma((\phi^* \mathcal{F}_1)^*), \quad \lambda_-^\gamma \equiv \lambda^\gamma \in \Gamma(\overline{\mathcal{F}_1} \otimes \phi^* T^1_{\mathcal{F}_1}).
\]

In the B/2 twisted theory, no bosons need to be twisted.

Anomalies constrain the theory as follows:

\[
K_X \cong \pi^* \det \mathcal{G}^* \otimes \pi^* \det \mathcal{F}_1, \quad \mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(T \Sigma X).
\]

In the IR, the first condition becomes \( \det \mathcal{E}' \cong K_Y \), as needed to define the B/2 twist of the nonlinear sigma model.

The effective interactions can be obtained by truncating to fermi zero modes. In the degree zero sector, they are

\[
h^\gamma p^r \overline{F}_{r\gamma} \overline{F}_{r\gamma} + \chi^i \lambda^x D_i s_x + \chi^\gamma \lambda^\gamma_+ F_\gamma + \chi^i \lambda^\gamma p^r D_i \overline{F}_{r\gamma}.
\]

Of the scalars \( \chi^i, \chi^r, \lambda^x, \lambda^\gamma, \chi^i, \chi^r \) are BRST invariant, and if we define

\[
\theta_\gamma \equiv h_{\gamma} \lambda^x, \quad \theta_\gamma \equiv h_{\gamma} \lambda^\gamma.
\]

39
then it can be shown
\[ Q \cdot \theta = \propto \tilde{s}, \quad Q \cdot \theta = \propto \tilde{p} \tilde{F} \tilde{p}. \]

Observables built from \( \chi^i, \chi^r, \theta \) are then elements of hypercohomology
\[ \mathbb{H}^\bullet \left( X, \cdots \rightarrow \wedge^2 \mathcal{F}_1 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{O}_X \right) \]
(with maps given by contraction with \( \tilde{p} \tilde{F} \)). Under RG flow these become observables in a B/2-twisted nonlinear sigma model, and the zero-mode interactions define an analogue of a Mathai-Quillen form.

### 4.3 Cokernels

#### 4.3.1 B/2 model realization of first cokernel construction

In this section we will describe a B/2 twisted (0,2) Landau-Ginzburg model for a cokernel. This will yield another analogue of a Mathai-Quillen form.

Suppose we wish to build a (0,2) Landau-Ginzburg model that RG flows to a nonlinear sigma model on
\[ Y \equiv \{ s = 0 \} \subset M \]
(where \( s \in \Gamma(\mathcal{G}) \)), and with gauge bundle given by the cokernel of the restriction of an injective map
\[ \tilde{E} : \mathcal{F}_1 \rightarrow \mathcal{F}_2 \]
to \( Y \). The map \( \tilde{E} \) is injective everywhere on \( M \). The restriction of \( \tilde{E} \) to \( Y \) is holomorphic; however, over the rest of \( M \), \( \tilde{E} \) need be merely smooth.

Then, we consider a Landau-Ginzburg model on
\[ X = \text{Tot} \left( \mathcal{F}_1 \xrightarrow{\pi} M \right), \]
with gauge bundle \( \mathcal{E} \rightarrow X \) given by an extension
\[ 0 \rightarrow \pi^* \mathcal{F}_2 \rightarrow \mathcal{E} \rightarrow \pi^* \mathcal{G} \rightarrow 0. \]  \quad (26)

To uniquely determine the physics, we must specify which extension, and also a holomorphic section of \( \mathcal{E} \), such that the resulting (0,2) Landau-Ginzburg theory will renormalization-group flow to the (0,2) nonlinear sigma model above.

Let us begin with the choice of holomorphic section of \( \mathcal{E} \). This determines a holomorphic section of \( \pi^* \mathcal{G} \) together with a smooth section of \( \pi^* \mathcal{F}_2 \) that becomes holomorphic over the
vanishing locus of the section of $\pi^*G$. We will take the holomorphic section of $\pi^*G$ to be the pullback of $s$ (whose vanishing locus is $Y$), and the smooth section of $\pi^*F_2$ to be $q\bar{E}$, where $q$ is a fiber coordinate on $X$.

The choice of extension class is largely not relevant to the purpose of this paper, so we shall not describe it in detail. Suffice it to say, in general, the extension will be nontrivial, with certain exceptions. In the special case that $E'$ is the restriction of a bundle on $M$ to $Y$ (i.e. the map $E$ is globally holomorphic), the extension will be trivial: $E = \pi^*G \otimes \pi^*F_2$.

The action of the B/2 twisted Landau-Ginzburg model that RG flows to the B/2 twist of the nonlinear sigma model above is given in local coordinates by \[15\]

$$S = 2t \int_{\Sigma} d^2z \left[ \frac{1}{2}(g_{\mu\nu} + iB_{\mu\nu}) \partial_\nu \phi^i \partial^\mu \phi^i + ig_{aa} \bar{\psi}_m \gamma^a \bar{\psi}_m^a + ig_{aa} \lambda_\alpha \partial_\alpha \lambda_\alpha \right. $$

$$ + \left. F_{\alpha\beta\gamma} \psi_+^a \lambda_\alpha \lambda_\beta \gamma + h_{x\gamma} s^x \bar{s}^\gamma + h_{\gamma\gamma} q^m \bar{q}^m \bar{E}_m \bar{E}_m^\gamma + \psi_+^l \bar{\lambda}^x D_l s^x h_{x\gamma} + \psi_+^m \bar{\lambda}^\gamma \bar{E}_m h_{\gamma\gamma} + \psi_+^q \bar{\lambda}^\gamma \bar{q}^m \left( D_l \bar{E}_m^\gamma \right) h_{\gamma\gamma} \right] ,$$

where, much as in the last section, $x$ indexes local coordinates along the fibers of $G$, $m$ indexes local coordinates (denoted $q$) along the fibers of $F_1$, $\gamma$ indexes local coordinates along the fibers of $F_2$, $i$ indexes local coordinates on $M$, $\alpha \sim (m, i)$ indexes local coordinates on $X$, and $\alpha \sim (x, \gamma)$ indexes local coordinates along the fibers of $E$, and $s^x$ now denotes a component of a holomorphic section of $G$ (rather than $s_x$ as was used elsewhere, because of the different way $G$ appears in $E$). In the notation of \[15\], in local coordinates, $(E^\alpha) = (s^x, q^m \bar{E}_m^\gamma)$.

The space of vacua is of the form $\{s = 0\} \cap \{q = 0\}$. The first condition is a result of the bosonic potential $|s|^2$, and the second is a result of the bosonic potential $|q\bar{E}|^2$ plus the fact that $\bar{E}$ is injective everywhere.

The fermions and bosons are twisted as follows:

$$\psi_+^i \equiv \chi^i \in \Gamma(\phi^*(TM)),$$

$$\psi_+^m \equiv \psi_+^m \in \Gamma(K_{\Sigma} \otimes \phi^*T^1_{\pi})$$

$$\lambda_\alpha^x \equiv \lambda_\alpha^x \in \Gamma(K_{\Sigma} \otimes (\phi^*T^0_{\pi})^*)$$

$$\lambda_\alpha^{\gamma} \equiv \lambda_\alpha^{\gamma} \in \Gamma((\phi^*T^0_{\pi})^*)$$

$$q \equiv q \in \Gamma(K_{\Sigma} \otimes \phi^*T^1_{\pi})$$

Anomalies constrain the theory as follows:

$$K_M \equiv \det G^* \otimes \det F_1^* \otimes \det F_2, \quad \text{ch}_2(\mathcal{E}) = \text{ch}_2(TX).$$

41
The effective interactions can be obtained by truncating to fermi zero modes. In the
degree zero sector, they are
\[ h_x s^x \bar{s}^x + \chi^i \lambda^\gamma D_i s^x h_x \bar{s}^x + \chi^m \chi^\gamma \tilde{E}^m \tilde{F}^m h_{\gamma \gamma} + F_{\pi m \gamma} \chi^i \chi^m \lambda^\gamma \chi^\gamma. \]
It will be useful to define \( \theta^\gamma \equiv \tilde{E}^\gamma \tilde{F}^\gamma \), so that the effective interactions become
\[ h_x s^x \bar{s}^x + \chi^i \lambda^\gamma D_i s^x h_x \bar{s}^x + \chi^m \theta^\gamma \tilde{E}^m + F_{\pi m \gamma} \chi^i \chi^m \lambda^\gamma \theta^\gamma h_{\gamma \gamma}. \]

It can be shown that the scalars \( \chi^i \) and \( \theta^\gamma \) are BRST invariant, and the other scalars \( \chi^m \), \( \lambda^\gamma \) are not. Furthermore, since the \( q \) fields are twisted, the bosonic zero modes lie along \( M \). Restricting to zero modes, observables are then of the form
\[ f(\phi^i \chi^1 \ldots \chi^n \theta^1 \ldots \theta^m) \]
which after a complex conjugation are naively interpreted in terms of
\[ H^* (M, \wedge^* F_2). \]
(The more nearly correct interpretation of this chiral ring is in terms of a restriction to \( \{ s = 0 \} \) of the cohomology above, but this is not essential for this discussion.)

Correlation functions then are of the form of integrals over \( M \) of observables times the
exponential of the zero mode interactions.

As an aside, the four-fermi term in the zero mode interactions above, defined by an
element of \( H^1 (M, F_1^* \otimes G^* \otimes F_2) \) is related to the extension class of (26) as follows. One computes
\[
\text{Ext}^1_X (\pi^* G, \pi^* F_2) = H^1 (X, \pi^* G^* \otimes \pi^* F_2),
\]
\[
= H^1 (M, \pi_* \pi^* (G^* \otimes F_2)),
\]
\[
= H^1 (M, G^* \otimes F_2 \otimes \text{Sym}^* F_1^*),
\]
as previously, as the four-fermi term lives in one of the sheaf cohomology groups above.

4.3.2 A/2 model realization of second cokernel construction

We can also write down an A/2 model describing a cokernel. Our description will be dual
to the description of kernels in section 4.2.2 but for completeness, we give the details here.
Suppose we wish to build a (0,2) Landau-Ginzburg model that RG flows to a nonlinear sigma
model on
\[ Y \equiv \{ s = 0 \} \subset M \]
(where $s \in \Gamma(G)$), and with gauge bundle given by the cokernel of the restriction of an injective map

$$
\tilde{E} : \mathcal{F}_1 \longrightarrow \mathcal{F}_2
$$

to $Y$, where $\tilde{E}$ is both injective and holomorphic everywhere on $M$.

Then, we consider a Landau-Ginzburg model on

$$
X = \text{Tot} \left( \mathcal{F}_1 \xrightarrow{\pi} M \right),
$$

with gauge bundle $\mathcal{E} \to X$ given as the sum

$$
\mathcal{E} = \pi^*\mathcal{F}_2 \oplus \pi^*\mathcal{G}.
$$

To uniquely determine the physics, we must specify which extension, and also a holomorphic section of $\mathcal{E}$, such that the resulting (0,2) Landau-Ginzburg theory will renormalization-group flow to the (0,2) nonlinear sigma model above.

Let us begin with the choice of holomorphic section of $\mathcal{E}$. This is determined by holomorphic sections of each of $\mathcal{F}_2$, $\mathcal{G}$. We will take the holomorphic section of $\pi^*\mathcal{G}$ to be the pullback of $s$ (whose vanishing locus is $Y$), and the holomorphic section of $\pi^*\mathcal{F}_2$ to be $q\tilde{E}$, where $q$ is a fiber coordinate on $X$.

The action of the A/2 twisted Landau-Ginzburg model that RG flows to the A/2 twist of the nonlinear sigma model above is given in local coordinates by [15]

$$
S = 2t \int_{\Sigma} d^2z \left[ \frac{1}{2} (g_{\mu\nu} + iB_{\mu\nu}) \partial_\nu \phi^* \overline{\partial_\tau \phi}^\nu + ig_{\alpha\beta} \psi_+^\alpha \overline{D_\tau \psi_+^\beta} + ig_{\alpha\beta} \lambda_+^\alpha D_\tau \lambda_-^\beta \\
+ F_{a\beta\gamma} \psi_+^a \psi_+^\beta \lambda_-^\gamma + h_\gamma \psi_+^a \overline{\psi_+^\beta} \overline{\psi_+^\alpha} \overline{D_\tau \psi_+^\beta} + h_\gamma q^m \overline{q^m} \overline{\tilde{E}_m} \overline{\tilde{E}_m} \\
+ \psi_+^a \lambda_+^\beta \overline{D_\tau \psi_+^\beta} h_\gamma + \psi_+^a \overline{\lambda_+^\beta} \tilde{E}_m h_\gamma + \psi_+^a \lambda_+^\beta q^m \left( D_\gamma \tilde{E}_m \right) h_\gamma \\
+ \psi_+^a \lambda_+^\beta \overline{D_\gamma \tilde{E}_m} h_\gamma + \psi_+^a \overline{\lambda_+^\beta} \tilde{E}_m h_\gamma + \psi_+^a \lambda_+^\beta \overline{q^m} \left( \overline{D_\gamma \tilde{E}_m} \right) h_\gamma \right],
$$

where, much as in the last section, $x$ indexes local coordinates along the fibers of $\mathcal{G}$, $m$ indexes local coordinates (denoted $q$) along the fibers of $\mathcal{F}_1$, $\gamma$ indexes local coordinates along the fibers of $\mathcal{F}_2$, $i$ indexes local coordinates on $M$, $a \sim (m,i)$ indexes local coordinates on $X$, and $\alpha \sim (x,\gamma)$ indexes local coordinates along the fibers of $\mathcal{E}$. In the notation of [15], in local coordinates, $(E^\alpha) = (s^x, q^m \tilde{E}_m^\gamma)$. 

43
The fermions are twisted as follows:

$$\psi^+_i \equiv \chi^i \in \Gamma(\phi^*(TM)),$$

$$\psi^+_m \equiv \chi^m \in \Gamma(\phi^*T^{1,0}_{\pi}),$$

$$\lambda^x \equiv \lambda^x \in \Gamma(K_{\Sigma} \otimes (\phi^*T^{0,1}_G)^*),$$

$$\lambda^\gamma \equiv \lambda^\gamma \in \Gamma(K_{\Sigma} \otimes (\phi^*T^{0,1}_F)^*).$$

Unlike the A/2 kernels theory, in the A/2 cokernels case there is no need to twist bosons.

Anomalies constrain the theory, as follows:

$$K_M \cong \text{det} G^* \otimes \text{det} F_1 \otimes \text{det} F_2^*, \quad \text{ch}_2(\mathcal{E}) = \text{ch}_2(TX).$$

For later use, the first condition implies

$$K_X \cong \pi^* \text{det} G^* \otimes \pi^* \text{det} F_2^*.$$

One can show that anomaly-freedom in the UV implies anomaly-freedom in the IR.

The effective interactions can be obtained by truncating to fermi zero modes. In the degree zero sector, they are

$$h^x \pi_x \overline{\pi} + h_m q^m \overline{q}^m \overline{E} \overline{E} + \chi^i \lambda^x D_i s^x h_x \pi + \chi^m \lambda^\gamma E^\gamma_m h_m \pi + \chi^i \lambda^\gamma q^m \left(D_i \tilde{E}^\gamma\right) h_m \pi,$$

which descend to define an analogue of a Mathai-Quillen form.

Although $\chi^i$, $\chi^m$, $\lambda^x$, and $\lambda^\gamma$ are all scalars, it can be shown following [15] that only $\chi^i$, $\chi^m$ are genuinely BRST-invariant, and the others merely nearly BRST invariant:

$$Q \cdot \lambda^x \propto \overline{\pi}^x, \quad Q \cdot \lambda^\gamma \propto \overline{q}^m \overline{E}.$$

Furthermore, since the $q$ fields are not twisted, the bosonic zero modes lie along $X$.

Restricting to zero modes, observables are then of the form

$$f(\phi^i, \phi^m) \chi^i \cdot \chi^m \cdot \lambda^x \cdot \lambda^\gamma,$$

which after a complex conjugation are naively interpreted in terms of the hypercohomology on $X$ of a complex of the form

$$\cdots \longrightarrow \wedge^2 \pi^* F_2 \longrightarrow \pi^* F_2^* \longrightarrow O_X$$

with maps given by inclusion along $q \tilde{E}$. Note that in order for this interpretation to hold, $\tilde{E}$ must be holomorphic everywhere on $M$, which is the reason we restricted to that case in this section.

It was shown in [15][section 4.1, appendix A] that the hypercohomology on $X$ of the sequence above is isomorphic to

$$H^\bullet \left(M, \wedge^\bullet \mathcal{E}' \right),$$

where $\mathcal{E}'$ is the cokernel of the map $\tilde{E} : F_1 \rightarrow F_2$. Correlation functions then are of the form of integrals over $X$ of observables times the exponential of the zero mode interactions.
4.4 Cohomologies of short complexes

Suppose we want to build a Landau-Ginzburg model that flows to a nonlinear sigma model on

\[ Y \equiv \{ s = 0 \} \subset M, \]

as before, with gauge bundle given by

\[ \frac{\ker \tilde{F}|_Y}{\text{im} \tilde{E}|_Y}, \]

where \( \tilde{E} : \mathcal{F}_1 \rightarrow \mathcal{F}_2 \) is injective everywhere on \( M \), \( \tilde{F} : \mathcal{F}_2 \rightarrow \mathcal{F}_3 \) is surjective everywhere on \( M \).

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathcal{F}_1 & \xrightarrow{\tilde{E}} & \mathcal{F}_2 & \xrightarrow{\tilde{F}} & \mathcal{F}_3 & \rightarrow & 0
\end{array}
\]

The restrictions of both \( \tilde{E} \) and \( \tilde{F} \) to \( Y \) are holomorphic, but only one need be holomorphic on all of \( M \). Furthermore, the composition \( \tilde{F} \circ \tilde{E} \) should vanish everywhere on \( M \), making the restriction into a complex:

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathcal{F}_1|_Y & \xrightarrow{\tilde{E}|_Y} & \mathcal{F}_2|_Y & \xrightarrow{\tilde{F}|_Y} & \mathcal{F}_3|_Y & \rightarrow & 0.
\end{array}
\]

(Experts will note that this is not the most general possibility allowed by physics; we leave such more general cases for future work.)

Depending upon whether \( \tilde{E} \) or \( \tilde{F} \) is holomorphic on \( M \), one gets two slightly different Landau-Ginzburg models that renormalization-group flow to the nonlinear sigma model described above. We will describe each in turn.

4.4.1 A/2 model realization of first short complex construction

Suppose that \( \tilde{E} \) is holomorphic on all of \( M \), where \( \tilde{F} \) is only holomorphic after restriction to \( Y \subset M \). Corresponding to this nonlinear sigma model is a Landau-Ginzburg model on

\[ Z = \text{Tot} \left( \mathcal{F}_1 \oplus \mathcal{F}_3^* \xrightarrow{\hat{\pi}} M \right) \]

with gauge bundle

\[
\begin{array}{cccccc}
0 & \rightarrow & \tilde{\pi}^* \mathcal{G}^* & \rightarrow & \mathcal{E} & \rightarrow & \tilde{\pi}^* \mathcal{F}_2 & \rightarrow & 0.
\end{array}
\]

Physically, following [15], to specify this theory, we must specify a holomorphic section of \( \mathcal{E} \) and a holomorphic section of \( \mathcal{E}^* \), whose compositions vanish, along with the precise extension above.

We will begin by specifying the sections. Let us denote fiber coordinates on \( \mathcal{F}_1, \mathcal{F}_3^* \) by \( q, p \), respectively.
A holomorphic section of $E$ determines a holomorphic section of $\tilde{\pi}^*F_2$ and a smooth section of $\tilde{\pi}^*G^*$, holomorphic over the vanishing locus of the section of $\tilde{\pi}^*F_2$. We will take the holomorphic section of $\tilde{\pi}^*F_2$ to be $q\tilde{E}$, and the smooth section of $\tilde{\pi}^*G^*$ to be identically zero.

Using the dual sequence

$$0 \longrightarrow \tilde{\pi}^*F_2^* \longrightarrow E^* \longrightarrow \tilde{\pi}^*G \longrightarrow 0,$$

a holomorphic section of $E^*$ determines a holomorphic section of $\tilde{\pi}^*G$ and a smooth section of $\tilde{\pi}^*F_2^*$ which is holomorphic over the vanishing locus of the section of $\tilde{\pi}^*G$. We will take the holomorphic section of $\tilde{\pi}^*G$ to be the pullback of $s$ (whose vanishing locus is $Y$), and the smooth section of $\tilde{\pi}^*F_2^*$ to be $p\tilde{F}$.

Consistency requires the composition of these sections to vanish, and indeed, $\tilde{F}\tilde{E} = 0$ and $(s)(0) = 0$.

The data above – a smooth not-necessarily holomorphic $\tilde{F}$ and a globally holomorphic $\tilde{E}$ – are only compatible with the $A/2$ twist in general.

The action of the $A/2$ twisted Landau-Ginzburg model that RG flows to the $A/2$ twist of the nonlinear sigma model above is given in local coordinates by [15]

$$S = 2t \int d^2z \left[ \frac{1}{2} (g_{\mu \nu} + i B_{\mu \nu}) \partial_\nu \phi^\mu \overline{\partial}_\tau \phi^\nu + ig_{mn} \psi^m_+ \overline{D}_x \psi^m_+ + i g_{\alpha \beta} \lambda^\alpha D_x \lambda^\beta 
+ F_{0mn} \psi^m_+ \psi^m_+ \lambda^\alpha \lambda^\beta 
+ h^m x_k s_k \overline{s}_x + h^{n\tau} p^{\tau \tau} \tilde{F}_{\tau \gamma} \overline{\tilde{F}_{\tau \gamma}} 
+ \psi^m_+ \lambda^\alpha D_x \tilde{F}_{\tau \gamma} + \psi^m_+ \lambda^\beta D_x \tilde{F}_{\tau \gamma} 
+ \psi^m_+ \lambda^\alpha \overline{\tilde{F}_{\tau \gamma}} + \psi^m_+ \lambda^\beta \overline{\tilde{F}_{\tau \gamma}} 
+ h_{\tau \gamma} q^{m\overline{m}} \tilde{E}_m \overline{\tilde{E}_m} 
+ \psi^m_+ \lambda^\alpha \tilde{E}_m h_{\tau \gamma} + \psi^m_+ \lambda^\beta q^{m} D_i \tilde{E}_m h_{\tau \gamma} 
+ \psi^m_+ \lambda^\alpha \tilde{E}_m \overline{h_{\tau \gamma}} + \psi^m_+ \lambda^\beta q^{m} \overline{D}_r \tilde{E}_m \overline{h_{\tau \gamma}} \right],$$

where $x$ indexes local coordinates along the fibers of $G$, $m$ indexes local coordinates along the fibers of $F_1$, $\gamma$ indexes local coordinates along the fibers of $F_2$, $r$ indexes local coordinates along the fibers of $F_3^*$, $i$ indexes local coordinates on $M$, $a \sim (m,r,i)$ indexes local coordinates on $X$, and $\alpha \sim (x,\gamma)$ indexes local coordinates along the fibers of $E$. 

46
The fermions and bosons are twisted as follows:

\[
\begin{align*}
\psi_i^+ &\equiv \chi_i^+ \in \Gamma(\phi^*(TM)), & \psi_+^m &\equiv \psi_+^m \in \Gamma(K_\Sigma \otimes \phi^*T^*M), \\
\psi_+^m &\equiv \psi_+^m \in \Gamma(\phi^*T^1_{\overline{F}_1}), & \psi_+^m &\equiv \psi_+^m \in \Gamma(K_\Sigma \otimes (\phi^*T^1_{\overline{F}_1})^*), \\
\psi^m &\equiv \psi^m \in \Gamma(\mathcal{K}_\Sigma \otimes \phi^*T^1_{\overline{F}_1}), & \psi_+^m &\equiv \psi_+^m \in \Gamma((\phi^*T^1_{\overline{F}_1})^*), \\
\lambda^\gamma_x &\equiv \lambda^\gamma_x \in \Gamma((\phi^*T^0_{\overline{G}})^*), & \lambda^\gamma_x &\equiv \lambda^\gamma_x \in \Gamma(K_\Sigma \otimes \phi^*T^0_{\overline{G}}), \\
p &\equiv p \in \Gamma(K_\Sigma \otimes \phi^*T^0_{\overline{G}}), & p &\equiv p \in \Gamma(K_\Sigma \otimes \phi^*T^0_{\overline{G}}).
\end{align*}
\]

The bosons \(q, \overline{q}\) are untwisted.

Anomalies constrain this theory as follows:

\[
K_M \cong \det \mathcal{G} \otimes \det \mathcal{F}_1 \otimes \det \mathcal{F}_2^* \otimes \det \mathcal{F}_3, \quad \chi_2(Z) = \chi_2(\mathcal{E}).
\]

The effective interactions can be obtained by truncating to fermi zero modes. In the degree zero sector, they are

\[
\begin{align*}
&h^{x\gamma}s_x s_x + \chi^i\lambda^x D_x s_x + \chi^i\lambda^7 T_{\gamma} + F_{\gamma\pi\tau} \chi^i \lambda^x \lambda^7 \\
&+ h_{\gamma\gamma} q^m \overline{q}^m \bar{E}_m \bar{E}_m + \chi^m \lambda^x \bar{E}_m h_{\gamma\gamma} + \chi^i \lambda^7 q^m D_i \bar{E}_m h_{\gamma\gamma} + F_{m\pi\tau\gamma} \chi^m \lambda^x \lambda^7. 
\end{align*}
\]

The curvature term

\[F_{m\pi\tau\gamma} \chi^m \lambda^x \lambda^7\]

will always vanish, so we omit it from further discussion.

Since the \(q\)'s are untwisted, the resulting analogue of a Mathai-Quillen form will live on a bundle over \(M\). Specifically, they live on

\[X \equiv \text{Tot} \left( \mathcal{F}_1 \xrightarrow{\pi} M \right).
\]

The scalars \(\psi^i, \psi^m\) are BRST-invariant. Similarly,

\[Q \cdot \lambda^7 \propto \overline{q}^m \bar{E}_m^7\]

so, after complex conjugation, we interpret observables built from \(\chi^i, \chi^m, \lambda^7\) in terms of hypercohomology

\[H^\bullet \left( X, \cdots \to \wedge^2 \pi^*\mathcal{F}_2^* \to \pi^*\mathcal{F}_2^* \to \mathcal{O}_X \right)\]

with maps given by inclusion with \(q^m \bar{E}_m^\gamma\).
4.4.2 B/2 model realization of second short complex construction

Suppose that $\tilde{E}$ is holomorphic only after restriction to $Y \subset M$, whereas $\tilde{F}$ is holomorphic everywhere on $M$. Corresponding to this nonlinear sigma model is a Landau-Ginzburg model on

$$Z = \text{Tot} \left( \mathcal{F}_1 \oplus \mathcal{F}_3^* \to \tilde{\pi} M \right)$$

with gauge bundle

$$0 \to \tilde{\pi}^* \mathcal{F}_2 \to \mathcal{E} \to \tilde{\pi}^* \mathcal{G} \to 0.$$

Physically, following [15], to specify this theory, we must specify a holomorphic section of $\mathcal{E}$ and a holomorphic section of $\mathcal{E}^*$, whose compositions vanish, along with the precise extension above.

We will begin by specifying the sections. Let us denote fiber coordinates on $\mathcal{F}_1, \mathcal{F}_3^*$ by $q, p$, respectively.

A holomorphic section of $\mathcal{E}$ determines a holomorphic section of $\tilde{\pi}^* \mathcal{G}$ and a smooth section of $\tilde{\pi}^* \mathcal{F}_2$, holomorphic over the vanishing locus of the section of $\tilde{\pi}^* \mathcal{G}$. We will take the holomorphic section of $\tilde{\pi}^* \mathcal{G}$ to be the pullback of $s$ (whose vanishing locus is $Y$), and the smooth section of $\tilde{\pi}^* \mathcal{F}_2$ to be $p\tilde{E}$.

Using the dual sequence

$$0 \to \tilde{\pi}^* \mathcal{G}^* \to \mathcal{E}^* \to \tilde{\pi}^* \mathcal{F}_2^* \to 0,$$

a holomorphic section of $\mathcal{E}^*$ determines a holomorphic section of $\tilde{\pi}^* \mathcal{F}_2^*$ and a smooth section of $\tilde{\pi}^* \mathcal{G}^*$ which is holomorphic over the vanishing locus of the section of $\tilde{\pi}^* \mathcal{F}_2^*$. We will take the holomorphic section of $\tilde{\pi}^* \mathcal{F}_2^*$ to be $q\tilde{F}$, and the smooth section of $\tilde{\pi}^* \mathcal{G}^*$ to be identically zero.

Consistency requires the composition of these sections to vanish, and indeed, $\tilde{F}\tilde{E} = 0$, $(0)(s) = 0$.

The data above – a smooth not-necessarily holomorphic $\tilde{E}$ and a globally holomorphic $\tilde{F}$ – are only compatible with the B/2 twist in general.

The action of the B/2 twisted Landau-Ginzburg model that RG flows to the B/2 twist
of the nonlinear sigma model above is given in local coordinates by [15]

\[
S = 2t \int_{\Sigma} d^2z \left[ \frac{1}{2} (g_{\mu\nu} + iB_{\mu\nu}) \partial_\xi \phi^\mu \partial_\eta \phi^\nu + ig_{\alpha\alpha} \psi^\alpha_+ \partial_\eta \psi^\alpha_+ + ig_{\alpha\alpha} \lambda^\alpha_+ D_\alpha \lambda^\alpha_-
\right.
\]
\[
+ F_{\alpha\alpha\alpha\alpha} \psi^\alpha_+ \psi^\alpha_- \lambda^\alpha_- + h^{\gamma\eta} p^\gamma \bar{F}_{\gamma\gamma} \bar{F}_{\eta\eta}
\]
\[
+ \psi^\gamma_+ \lambda^\gamma_+ \bar{F}_{\gamma\gamma} + \psi^\gamma_+ \lambda^\gamma_- p^\gamma D_\gamma \bar{F}_{\gamma\gamma}
\]
\[
+ \psi^\gamma_- \lambda^\gamma_- \bar{F}_{\gamma\gamma} + \psi^\gamma_- \lambda^\gamma_+ p^\gamma D_\gamma \bar{F}_{\gamma\gamma}
\]
\[
+ h_{x_{\alpha\beta}} s^\alpha \bar{s}^\beta + h_\gamma q^\gamma \bar{q}^\gamma \bar{E}_{\alpha\beta} \bar{E}_{\alpha\beta}
\]
\[
+ \psi^\gamma_+ \lambda^\gamma_- D_\gamma s^\gamma h_{x\gamma} + \psi^\gamma_- \lambda^\gamma_+ \bar{E}_{\alpha\beta} h_{x\gamma}
\]
\[
+ \psi^\gamma_- \lambda^\gamma_- \bar{E}_{\alpha\beta} h_{x\gamma} + \psi^\gamma_+ \lambda^\gamma_+ \bar{E}_{\alpha\beta} h_{x\gamma}
\],
\]

where \(x\) indexes local coordinates along the fibers of \(G\), \(m\) indexes local coordinates along the fibers of \(F_1\), \(\gamma\) indexes local coordinates along the fibers of \(F_2\), \(r\) indexes local coordinates along the fibers of \(F_3\), \(i\) indexes local coordinates on \(M\), \(a \sim (m, r, i)\) indexes local coordinates on \(X\), and \(\alpha \sim (x, \gamma)\) indexes local coordinates along the fibers of \(E\).

The fermions and bosons are twisted as follows:

\[
\psi^i_+ \equiv \chi^i_- \in \Gamma(\phi^*(TM)), \quad \psi^{\gamma}_+ \equiv \psi^{\gamma}_- \in \Gamma(K_\Sigma \otimes \phi^*T^*M),
\]
\[
\psi^m_+ \equiv \psi^m_- \in \Gamma(K_\Sigma \otimes \phi^*T_\pi^{1,0}), \quad \psi^{\gamma m}_+ \equiv \lambda^{\gamma m} \in \Gamma((\phi^*T^{1,0}_\pi)^*),
\]
\[
\psi^r_+ \equiv \chi^r_- \in \Gamma(\phi^*T_\pi^{1,0}), \quad \psi^{\gamma r}_+ \equiv \psi^{\gamma r}_- \in \Gamma(K_\Sigma \otimes (\phi^*T^{1,0}_\pi)^*),
\]
\[
\lambda^\alpha_+ \equiv \lambda^\alpha_- \in \Gamma(K_\Sigma \otimes (\phi^*T^{0,1}_G)^*), \quad \lambda^{\gamma \alpha}_+ \equiv \lambda^{\gamma \alpha}_- \in \Gamma(\phi^*T^{0,1}_G),
\]
\[
\lambda^\gamma_- \equiv \chi^\gamma_- \in \Gamma((\phi^*T^{0,1}_F)^*), \quad \lambda^{\gamma \gamma}_- \equiv \lambda^{\gamma \gamma}_+ \in \Gamma(K_\Sigma \otimes \phi^*T^{0,1}_F),
\]
\[
q \equiv q_z \in \Gamma(K_\Sigma \otimes \phi^*T_\pi^{1,0}), \quad \bar{q} \equiv \bar{q}_\pi \in \Gamma(K_\Sigma \otimes \phi^*T^{0,1}_\pi).
\]

In the B/2 twisted theory, \(p, \bar{p}\) are not twisted.

Anomalies constrain the theory as follows:

\[
K_M \cong \det G^* \otimes \det F_1^* \otimes \det F_2^* \otimes \det F_3^*, \quad \text{ch}_2(TZ) = \text{ch}_2(E).
\]

The effective interactions can be obtained by truncating to fermi zero modes. In the degree zero sector, they are

\[
h_{x_{\alpha\beta}} s^\alpha \bar{s}^\beta + h^{\gamma\eta} p^\gamma \bar{F}_{\gamma\gamma} \bar{F}_{\eta\eta} + \chi^\gamma \lambda^\gamma \bar{F}_{\gamma\gamma} + \chi^\gamma \lambda^\gamma p^r D_\gamma \bar{F}_{\gamma\gamma}
\]
\[
+ \chi^\gamma \lambda^\gamma D_\gamma s^\gamma h_{x\gamma} + \chi^\gamma \lambda^\gamma \bar{E}_{\alpha\beta} h_{x\gamma} + F_{m\pi\gamma} \chi^\gamma \lambda^\gamma \lambda^\gamma + F_{m\pi\gamma} \chi^\gamma \lambda^\gamma \lambda^\gamma,
\]
and they descend to define an analogue of a Mathai-Quillen form. The curvature term
\[ F_{\gamma \lambda} \chi^r \chi^\pi \lambda^\gamma \]
will always vanish, so we omit it from further discussion.

Since the \( p \)'s are untwisted, the observables live on
\[ X \equiv \text{Tot} \left( \mathcal{F}^* \to^\pi M \right). \]

The scalars \( \chi^i, \chi^r \), are BRST-invariant, and if we define \( \theta_\gamma \equiv h_{\gamma \lambda} \lambda^\gamma \), then
\[ Q \cdot \lambda^\gamma \propto \pi^\gamma, Q \cdot \theta_\gamma \propto \bar{p}^\gamma \bar{F}_{r\gamma}. \]
Observables built from \( \psi^i_+, \psi^r_+, \theta_\gamma \) can then, after complex conjugation, be interpreted in terms of hypercohomology
\[ \mathbb{H}^\bullet \left( X, \cdots \to \wedge^2 \pi^* \mathcal{F}_2 \to \pi^* \mathcal{F}_2 \to \mathcal{O}_X \right) \]
with maps given by inclusion with \( p^r \bar{F}_{r\gamma} \).

### 4.5 A note on anomalies

In the A/2 and B/2 models, we have encountered two different anomalies, one a condition on determinants of bundles, the second a condition on second Chern characters. The second condition is the standard Green-Schwarz condition; the first is specific to the A/2 and B/2 models.

The condition on determinants of bundles is applied mathematically to give well-defined integrals of products of sheaf cohomology classes. However, the first condition, the Green-Schwarz condition, did not appear in the mathematical discussion.

Although it is possible that a more detailed examination of the analogues of Mathai-Quillen forms we have proposed will require the Green-Schwarz condition, it is our belief that they will not. The reason is the manner in which the Green-Schwarz condition appears in e.g. quantum sheaf cohomology computations. There, its role is to ensure that in worldsheet instanton sectors, corresponding integrals of sheaf cohomology classes over moduli spaces of instantons are well-defined. In other words, its role is to help provide an analogue of the determinants condition over moduli spaces of instantons. As worldsheet instantons are not discussed in this paper, as they do not arise in our constructions of Mathai-Quillen analogues, it is our suspicion that Green-Schwarz is not relevant to the mathematics of the constructions presented here.
5 Conclusions

In this paper we have presented some sheaf-cohomological analogues of Mathai-Quillen forms, which is to say, $\partial$-closed bundle-valued differential forms which generalize Mathai-Quillen forms. We have shown that the cohomology classes of these forms are invariant under certain deformations, and we have conjectured (based on their physical origin relating UV and IR theories via renormalization group flow) that these analogues have Thom-form-like properties, though we have not given a mathematical argument to justify that assertion.

One of the original hopes of this work was to give a mathematical understanding of some claims of Melnikov and McOrist [10] regarding A/2 correlation functions and their dependence (or lack thereof) on certain complex and bundle moduli. Unfortunately, we were not able in this work to explicitly confirm more than a part of their claims, but neither have we disproven them, their verification remains an open problem.

6 Acknowledgements

We would like to thank R. Donagi, S. Katz, V. Mathai, and T. Pantev for useful conversations. R.G. thanks the Department of Mathematical and Statistical Sciences at the University of Alberta for hospitality during the initial stages of this project. E.S. thanks the Aspen Center for Physics for hospitality while this work was completed, under its NSF grant PHYS-1066293. E.S. was partially supported by NSF grant PHY-1068725.

References

[1] V. Mathai, D. Quillen, “Superconnections, Thom classes, and equivariant differential forms,” Topology 25 (1986) 85-110.

[2] J. Kalkman, “BRST model for equivariant cohomology and representatives for the equivariant Thom class,” Commun. Math. Phys. 153 (1993) 447-463.

[3] M. Blau, “The Mathai-Quillen formalism and topological field theory,” J. Geom. Phys. 11 (1993) 95-127, hep-th/9203026.

[4] S. Wu, “On the Mathai-Quillen formalism of topological sigma models,” J. Geom. Phys. 17 (1995) 299-309, hep-th/9406103.

[5] S. Cordes, G. Moore, S. Ramgoolam, “Lectures on 2D Yang-Mills theory, equivariant cohomology and topological field theories,” Nucl. Phys. Proc. Suppl. 41 (1995) 184-244, hep-th/9411210.
[6] S. Wu, “Mathai-Quillen formalism,” hep-th/0505003.

[7] S. Katz, E. Sharpe, “Notes on certain (0,2) correlation functions,” Comm. Math. Phys. 262 (2006) 611-644, hep-th/0406226.

[8] A. Adams, J. Distler, M. Ernebjerg, “Topological heterotic rings,” Adv. Theor. Math. Phys. 10 (2006) 657-682, hep-th/0506263.

[9] E. Sharpe, “Notes on certain other (0,2) correlation functions,” Adv. Theor. Math. Phys. 13 (2009) 33-70, hep-th/0605005.

[10] J. McOrist, I.V. Melnikov, “Summing the instantons in half-twisted linear sigma models,” JHEP 02 (2009) 026, arXiv:0810.0012.

[11] R. Donagi, J. Guffin, S. Katz, E. Sharpe, “A mathematical theory of quantum sheaf cohomology,” arXiv:1110.3751.

[12] R. Donagi, J. Guffin, S. Katz, E. Sharpe, “Physical aspects of quantum sheaf cohomology for deformations of tangent bundles of toric varieties,” arXiv:1110.3752.

[13] I. Melnikov, S. Sethi, E. Sharpe, “Recent developments in (0,2) mirror symmetry,” SIGMA 8 (2012) 068, arXiv:1209.1134.

[14] J. Guffin, E. Sharpe, “A-twisted Landau-Ginzburg models,” J. Geom. Phys. 59 (2009) 1547-1580, arXiv:0801.3836.

[15] J. Guffin, E. Sharpe, “A-twisted heterotic Landau-Ginzburg models,” J. Geom. Phys. 59 (2009) 1581-1596, arXiv:0801.3955.

[16] M. Ando, E. Sharpe, “Elliptic genera of Landau-Ginzburg models over nontrivial spaces,” Adv. Theor. Math. Phys. 16 (2012) 1087-1144, arXiv:0905.1285.

[17] P. Clarke, “Duality for toric Landau-Ginzburg models,” arXiv:0803.0447.

[18] M. Bertolini, I. Melnikov, R. Plessner, “Hybrid conformal field theories,” arXiv:1307.7063.

[19] N. Berline, E. Getzler, M. Vergne, Heat kernels and Dirac operators, Grund. der Math. Wiss. 298, Springer-Verlag, Berlin-Heidelberg-New York, 1992.

[20] R. Donagi, private communication.

[21] D. Ben-Zvi, private communication.

[22] J. Distler, B. Greene, D. Morrison, “Resolving singularities in (0,2) models,” Nucl. Phys. B481 (1996) 289-312, hep-th/9605222.