Wave propagation in micromorphic anisotropic continua with an application to tetragonal crystals

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Abstract
We study the coupled macroscopic and lattice wave propagation in anisotropic crystals seen as continua with affine microstructure (or micromorphic). In the general case, we obtain qualitative information on the frequencies and the dispersion relations. These results are then specialized to crystals of the tetragonal point group for various propagation directions: exact representation for the acoustic and optic frequencies and for the coupled vibrations modes are obtained for propagation directions along the tetragonal c-axis.

Keywords
Anisotropic crystals, lattice vibrations, acoustic waves, optic waves, scintillating crystals

1. Introduction
Let me say from the beginning that this paper was motivated by second thoughts: indeed my research deals mostly with the photoelastic properties of scintillating crystals, that is crystals that convert ionizing radiation into photons within the visible range. Massive scintillating crystals were used to detect particle collisions in the CMS calorimeter at CERN, Geneva [1] and shall be used in the FAIR accelerator at GSI, Darmstadt [2] and can also be used in security and medical imaging devices. Amongst many other problems concerning quality control and efficiency, one of the major issues related to the prolonged use of these crystals is the radiation damage that displaces atoms and reduces crystal efficiency and the radiation/photons ratio (see, e.g., [3]).

In order to recover from the radiation damage many techniques were used: among these, one of the most promising is laser-induced ultrasound lattice vibration, which can be a really efficient way to recover the damage. It becomes mandatory, therefore, to study the problem of coupled bulk/lattice wave propagation in crystals in order to evaluate the frequency range of bulk and lattice vibrations, the amount of energy that is lost in the coupling between lattice and bulk and how much of the incoming energy makes the lattice vibrate and around which modes. The problem is remarkably complex because most scintillators exhibit strong anisotropy, such as monoclinic cerium-doped Lu$_2$Y$_{2-x}$SiO$_5$:Ce (LYSO) [4], hexagonal LaBr$_3$ [5] and tetragonal PbWO$_4$ (PWO) [6].

Such a problem can of course be studied with a classical lattice dynamics approach [7, 8]: however, because scintillators generally are massive crystals the size of which is in the range of decimeters, the continuum mechanics approach looks more suited to describe the interactions between the crystal lattice vibrations and...
the macroscopic vibrations of the crystal specimen. Thus, it seems natural to model the crystal as a *continuum with affine structure* [9] or *micromorphic continuum* [10], because it appears as a reasonable compromise between the microscopical aspects related to lattice vibrations and the macroscopic vibrations.

Micromorphic continua have attracted never-fading attention since the pioneering work of Mindlin [11], the treatises [9] and [10] and the revamped attention in the recent years, motivated by the study of metamaterials, by the means of both the classical approach as in [12] or the relaxed approach first proposed in [13]. The majority of these results, however, concern isotropic materials, with some limited exceptions. Here, for classical micromorphic continua, we extend to the general case of anisotropic materials the previously known results of wave propagation into isotropic material, and then specialize them to crystals of the tetragonal group.

The paper is organized as follows. In Section 2, we write directly the balance law as proposed in [11]; upon the assumption of linearized kinematics and by using linear constitutive relations as in [11], we arrive at the propagation condition for the macroscopic progressive waves coupled with microdistortion lattice waves. Such propagation condition, which depends on the wavenumber $\xi$ and on two dimensionless parameters that relate the various length scales of the problem, is completely described by a $12 \times 12$ Hermitian matrix whose blocks represent various kinds of generalized acoustic tensors and whose eigenvector represents macroscopic displacements coupled with lattice microdistortions. In the general case of triclinic crystals, we show that there always exist three acoustic and nine optic waves and we also give an insight into the structure of dispersion relations: moreover, by a suitable scaling in terms of the dimensionless parameters we show that the problem admits two physically meaningful limit cases, namely the *long-wavelength approximation*, which represents the propagation phenomena in a body in which we are “zooming out” away from the crystal lattice, and the *microwibration* case, where in contrast we are “zooming in” to the crystal lattice. These two limit cases first introduced into [11], besides representing two physical picture of the phenomena, give an insight into the general propagation problem with the cut-off optical frequencies given by the microvibration frequencies.

In Section 3 these general results are specialized to crystals of the tetragonal point group and the reason for such a choice is two-fold: first, we are interested in damage recovery in the tetragonal PWO crystals that are currently used in the FAIR accelerator [14]; second, the reduction of independent constitutive parameter for tetragonal micromorphic bodies allows for some explicit solutions of the propagation condition and to an at least qualitative representation of the dispersion relations. We study separately the low-symmetric tetragonal classes 4, 4, 4/$m$ and the high-symmetric classes 4$mm$, 422, 42$m$ and 4/$mm$, in the case of waves propagating along the tetragonal $c$-axis. For the other relevant case, that of waves propagating in directions orthogonal to the $c$-axis, we give an insight into the solution structure because along such directions the crystal behaves in a fully coupled manner as in triclinic crystal: hence, it is not possible to give exact closed-form solutions.

There exists a major criticism to such an approach: as was correctly pointed out in [15], such a formal treatment has some limitations because it depends indeed on a large number of parameters whose experimental identification can be both difficult and elusive. With respect to such a correct criticism, we can say that in any case we have a general framework to design correct experiments aimed at parameter identification and, moreover, as shown in two recent papers [16, 17], by using homogenization techniques we can estimate the micromorphic model constitutive parameters by the means of classical lattice dynamics.

In Section 4, the results for the full propagation condition are discussed; as far as we know, this is the most complete analysis of wave propagation in micromorphic tetragonal crystals up to now and it could be the starting point for both experiment design and homogenization techniques based on lattice dynamics.

As a final remark, we note that the results we obtained required a lot of tabular representations of higher-order tensors as well as pages of cumbersome algebra. Thus, in order to grant a reasonable length for this paper, we decided, following the suggestion of one of the reviewers, to maintain these calculations in the full-length, unabridged version of the paper which can be found at http://arxiv.org/abs/2009.09825, which shall be henceforth referred to in the text as reference “[1]” and to provide the readers with the present shortened version.

As far as the notation is concerned, let $V$ be the three-dimensional vector space whose elements we denote $v \in V$ and let $Lin$ be the space of the second-order tensors $A \in Lin$, $A : V \to V$. We denote by $A^T$ the transpose of $A$ such that $Au \cdot v = A^T v \cdot u$, $\forall u, v \in V$; we shall also denote by $Sym$ and $Skw$ the subspaces of $Lin$ of the symmetric ($A = A^T$) and skew-symmetric ($A = -A^T$) tensors, respectively. Let $Lin$ be the space of third-order tensors $P : Lin \to V$ and for all $P \in Lin$ we denote the transpose $P^T : V \to Lin$ as

$$P[A] \cdot v = P^T v \cdot A, \quad \forall v \in V, \forall A \in Lin;$$

we shall also make use of fourth-order tensors $C : Lin \to Lin$, fifth-order tensors $F : Lin \to Lin$ and sixth-order tensors $G : Lin \to Lin$ whose transpose are defined in a similar way as $C[A] \cdot B = C^T[B] \cdot A$,
\( \mathcal{F}[A] \cdot P = \mathcal{F}^T[P] \cdot A \) and \( \mathcal{S}[P] \cdot Q = \mathcal{S}^T[Q] \cdot P, \forall A, B \in \text{Lin} \) and \( \forall P, Q \in \mathcal{L}\text{lin} \). For \( \{e_k\}, k = 1, 2, 3 \) an orthonormal basis in \( \mathcal{V} \), we define the components of the aforementioned elements by

\[
\begin{align*}
v_k &= v \cdot e_k, \\
A_{ij} &= A e_i \cdot e_k = A \cdot e_k \otimes e_j, \\
P_{ijk} &= P[e_i \otimes e_k] \cdot e_l = P \cdot e_i \otimes e_k \otimes e_k, \\
C_{ijk} &= C[e_i \otimes e_k] \cdot e_l \otimes e_j, \\
F_{ijk} &= F[e_i \otimes e_k \otimes e_m] \cdot e_l \otimes e_j, \\
\mathcal{S}_{ijk} &= \mathcal{S}[e_i \otimes e_m \otimes e_p] \cdot [e_l \otimes e_j \otimes e_k].
\end{align*}
\]

We shall also make use of the orthonormal basis \( \{W_k\}, k = 1, \ldots, 9 \) in Lin:

\[
\begin{align*}
W_1 &= e_1 \otimes e_1, & W_2 &= e_2 \otimes e_2, & W_3 &= e_3 \otimes e_3, \\
W_4 &= e_2 \otimes e_3, & W_5 &= e_3 \otimes e_1, & W_6 &= e_1 \otimes e_2, \\
W_7 &= e_3 \otimes e_2, & W_8 &= e_1 \otimes e_3, & W_9 &= e_2 \otimes e_1,
\end{align*}
\]

whereas the orthogonal bases \( \{\hat{W}_k\}, k = 1, \ldots, 6 \) in Sym and \( \{\hat{W}_k\}, k = 4, 5, 6 \) in Skw are respectively defined as

\[
\begin{align*}
\hat{W}_k &= W_k, \quad k = 1, 2, 3, \quad \hat{W}_k = \frac{1}{2}(W_4 + W_7), \quad \hat{W}_k = \frac{1}{2}(W_5 + W_8), \quad \hat{W}_k = \frac{1}{2}(W_6 + W_9),
\end{align*}
\]

and

\[
\begin{align*}
\hat{W}_4 &= \frac{1}{2}(W_4 - W_7), \quad \hat{W}_5 = \frac{1}{2}(W_5 - W_8), \quad \hat{W}_6 = \frac{1}{2}(W_6 - W_9).
\end{align*}
\]

In terms of the bases (3)–(5) we can also represent the fourth-order tensors components with the so-called Voigt two-index notation, namely, for example for (3):

\[
C_{ij} = C[W_j] \cdot W_i, \quad i, j = 1, \ldots, 9.
\]

Finally, in order to describe the infinitesimal lattice vibrations, we shall make use of the following seven modes.

- Non-uniform dilatation: \( D_1 = \alpha W_1 + \beta W_2 + \gamma W_3 \)
- Dilatation along \( e_1 \) and uniform plane strain in the plane orthogonal to \( e_3 \): \( D_2 = \alpha(I - W_3) + \gamma W_3 \)
- Traceless plane strain orthogonal to \( e_3 \): \( D_3 = W_1 - W_2 \)
- Shear in the plane orthogonal to \( e_3 \): \( S_1 = \alpha \hat{W}_6 \)
- Shear between \( e_1 \) and the direction \( e_\perp = \alpha e_1 + \beta e_2 \): \( S_2 = -\alpha \hat{W}_4 + \beta \hat{W}_5 \)
- Rigid rotation around the direction \( e_3 \): \( R_1 = \omega_1 \hat{W}_6 \)
- Rigid rotation around the direction \( e_\perp = \omega_1 e_1 + \omega_2 e_2 \): \( R_2 = \omega_2 \hat{W}_5 - \omega_1 \hat{W}_4 \)

2. Crystal as a micromorphic continuum

2.1. Balance laws, linearized kinematics and constitutive relations

Let \( B \) be a region of the Euclidean three-dimensional space we pointwise identify with the reference configuration of a crystal, and let \( x \) be a point of \( B \). We assume that at each point \( x \in B \) is defined as a crystal lattice \( \{a_1, a_2, a_3\} \) with \( a_1 \times a_2 \times a_3 \geq 1 \).

As in [11] we assume that at each point \( x \in B \) and at each time \( t \in [0, \tau] \) the motion can be well-defined by the means of the two fields \( y = y(x, t), \mathbf{G} = \mathbf{G}(x, t) \) provided \( y \) is locally injective and orientation-preserving and \( \mathbf{G} \) is orientation-preserving, namely det \( \mathbf{F} > 0, \mathbf{F}(x, t) = \nabla y(x, t), \det \mathbf{G} > 0 \), in such a way that at each point \( y \in B_t = y(B, t) \) the deformed crystal lattice \( \{\bar{a}_1, \bar{a}_2, \bar{a}_3\} \) is given by \( \bar{a}_k = \mathbf{G} a_k, k = 1, 2, 3 \).

We identify \( B \), endowed with the motion \( (y, \mathbf{G}) \), with a continuum with affine structure [9] or micromorphic [18]. There are many equivalent formulations for the balance laws: here we follow that proposed in [11],

\[
\begin{align*}
\text{div}(T + S) + b &= \rho \dot{v}, \quad \text{balance of macroforces} \\
\text{div} H + S + B &= \rho \mathbf{G} J \mathbf{G}^T, \quad \text{balance of microforces}
\end{align*}
\]
where $T$ is the symmetric part of the Cauchy stress, $S$ is the relative stress, $H$ is the third-order microstress tensor $b$ and $B$ are the volume macroforce and microforce densities, respectively, $\rho$ is the mass density, $v$ the material velocity and $J$ is the microinertia tensor.

We assume that both the deformation gradient and the lattice deformation can be decomposed additively into $F = I + \nabla u$ and $G = I + L$, where $u(x) = y(x) - x$ is the displacement vector and $L$ is the microdisplacement or microdistortion [15]. As is shown in [9, 10, 18], there are many appropriate kinematical measures for a constitutive theory of micromorphic continua; here we choose those proposed in [10, equation (1.5.11)],

$$E = \frac{1}{2}(F^T F - I), \quad M = F^T G^{-1} - I, \quad G = G^{-1} \text{ grad } G,$$

where $E$ is the Green–Lagrange deformation measure; if we assume that $\varepsilon = \sup(\|\nabla u\|, \|L\|)$, then (8) within higher-order terms in $\varepsilon$ reduce to

$$E = \text{sym } \nabla u, \quad M = \nabla u^T - L, \quad G = \nabla L;$$

the tensor $M$ being called the relative strain [11] or the relative distortion [15].

We assume a linear dependence of $T, S$ and $H$ on the linearized kinematical variables (9) and write (cf. [11, equation (5.3)])

$$T = C[E] + D[M] + L_c F[G],$$
$$S = D^T[E] + B[M] + L_c G[G],$$
$$H = L_c F^T[E] + L_c G^T[M] + L_c^2 \delta[G],$$

where:

- $C : \text{Sym} \to \text{Sym}, C = C^T$ is the fourth-order elasticity tensor, whose components obey $C_{ijkh} = C_{jikh} = C_{ijhk}$, with at most 21 independent components.
- the fourth-order tensor $B : \text{Lin} \to \text{Lin}, B = B^T$, whose independent components are at most 45 and obey $B_{ijkh} = B_{hijk}$;
- the fourth-order tensor $D : \text{Lin} \to \text{Sym},$ which has 54 independent components $D_{ijkh} = D_{jikh}, (D^T)_{ijhk} = D_{hjki}$;
- the fifth-order tensors $F : \text{Lin} \to \text{Sym}$ and $G : \text{Lin} \to \text{Lin}$ have 162 and 243 components, respectively, which obey $F_{ijkhlm} = F_{jikhlm}, (F^T)_{ijkhlm} = F_{hikhlm}$ and $(G^T)_{ijkhlm} = G_{hikhlm}$;
- the sixth-order tensor $\delta = \delta^T$ has at most 378 independent components $\delta_{ijkhlm} = \delta_{hijklm}$;
- $L_c > 0$ is the micromorphic correlation length, which makes all these tensorial quantities of the dimension of a stress (force/area).

The correlation length $L_c$ is the first length scale we need to introduce into the model and is related to the non-local effects associated with the gradient of the microdistorsion tensor. For $L_c \to 0$ we are considering large samples of crystals [19] whereas the limit $L_c \to \infty$ acts as a zoom into the microstructure (cf. [15]): we shall make these statements more rigorous in the next subsection.

As in [11], the requirement that the energy density is positive

$$2\varepsilon(E, M, G) = T \cdot E + S \cdot M + H \cdot G > 0,$$

implies that the block matrix $C$

$$C = \begin{bmatrix}
C & D & L_c F \\
D^T & B & L_c G \\
L_c F^T & L_c G^T & L_c^2 \delta
\end{bmatrix}$$

be positive-definite, a requirement which in turn implies the positive-definiteness of $C, B$ and $\delta$. In the most general case, that of crystals of the triclinic group, these constitutive relations require knowledge of 903 material constants, subject to the positive-definiteness of $C$, whereas in the simplest case of isotropic materials these constants reduce to 18 independent at most [11]. In the next subsection, we give a general and formal treatment of waves propagation in a crystal without any of the restrictions given by crystal symmetries.
2.2. Wave propagation

The balance laws (7) written in terms of the linearized kinematics (9) by the means of the constitutive relations (10) and zero volume macroforces and microforces:

\[
\text{div}(\mathbf{T} + \mathbf{S}) = \rho \ddot{\mathbf{u}}, \quad \text{div} \mathbf{H} + \mathbf{S} = \rho J \dot{\mathbf{L}},
\]

where \( J[\dot{\mathbf{L}}] = J \dot{\mathbf{L}}^T, \) \( J_{ijhk} = \delta_{ih} J_{jk}, \) are the starting point for the description of the microscopic crystal vibrations coupled with the macroscopic bulk vibrations. We seek for (13) progressive plane wave solutions of the form

\[
\mathbf{u}(x, t) = \mathbf{a} e^{i \mathbf{\sigma}^T \mathbf{x}}, \quad \mathbf{L}(x, t) = \mathbf{C} e^{i \mathbf{\sigma}^T \mathbf{x}}, \quad \mathbf{\sigma} = \xi \mathbf{x} \cdot \mathbf{m} - \omega t,
\]

where \( \omega \) is the frequency, \( \mathbf{m} \) the direction of propagation, \( \xi = \lambda^{-1} > 0 \) the wavenumber with \( \lambda \) the wavelength and where \( \mathbf{a} \in \mathcal{V} \) and \( \mathbf{C} \in \text{Lin} \) denote the displacement and microdistortion amplitudes, respectively, which in the general case are complex-valued.

We find it at this point mandatory to introduce, in addition to the characteristic length scale \( L_c \), two further length scales: the macroscopic length \( L_m > 0 \) associated with the displacement amplitude and the lattice length \( L_l > 0 \) associated with the microinertia and that allow us to write

\[
\mathbf{a} = L_m \mathbf{a}_o, \quad J = L_l^2 J_o,
\]

with \( \mathbf{a}_o \) a dimensionless vector and \( J_o \) a dimensionless fourth-order tensor [15]. If we now define the two dimensionless parameters

\[
\xi_1 = \frac{L_c}{L_m}, \quad \xi_2 = \frac{L_l}{L_m},
\]

then for \( \xi_1 \to 0 \) we are considering large samples of crystals and in the limit \( \xi_2 \to \infty \) we are zooming into the microstructure. In the case \( \xi_2 \to 0 \) we are instead neglecting the contribution of microinertia with respect to the macroscopic inertia.

Since \( \nabla \mathbf{u} = i \xi L_m \mathbf{a}_o \otimes \mathbf{m} e^{i \mathbf{\sigma}^T \mathbf{x}}, \) \( \ddot{\mathbf{u}} = -\omega^2 L_m \mathbf{a}_o \otimes \mathbf{m} e^{i \mathbf{\sigma}^T \mathbf{x}}, \) \( \nabla \mathbf{L} = i \xi \mathbf{C} \otimes \mathbf{m} e^{i \mathbf{\sigma}^T \mathbf{x}} \) and \( \dot{\mathbf{L}} = -\omega^2 \mathbf{C} e^{i \mathbf{\sigma}^T \mathbf{x}}, \) then we are led by (9), (10), (13) and (16) to the propagation conditions that can be written as

\[
(A - \omega^2 J) \mathbf{w} = 0,
\]

where the 12 × 12 Hermitian block matrix \( A(\xi) = A^*(\xi), \) which we call the acoustic matrix, the 12 × 12 symmetric block matrix \( J = J^T \) and the 12-dimensional eigenvector \( \mathbf{w} \) are defined by

\[
A(\xi) = \begin{bmatrix}
\xi^2 \mathbf{A} & \xi^2 \mathbf{P} + i \xi L_m^{-1} \mathbf{Q} \\
\xi^2 \mathbf{P}^T - i \xi L_m^{-1} \mathbf{Q}^T & \xi^2 \mathbf{A} + L_m^{-2} \mathbf{B}
\end{bmatrix}, \quad J = \begin{bmatrix} 1 & 0 \\ 0 & \xi_2^2 J_o \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} \mathbf{a}_o \\ \mathbf{C} \end{bmatrix}.
\]

The elements of (18) are defined as

\[
\mathbf{A}(\mathbf{m})\mathbf{a} = \rho^{-1}(C[a \otimes \mathbf{m}] + D[m \otimes \mathbf{a}] + D^T[a \otimes \mathbf{m}] + B[m \otimes \mathbf{a}]), \\
\mathbf{P}(\mathbf{m})[\mathbf{C}] = \rho^{-1}(F + G)[C \otimes \mathbf{m}]\mathbf{m}, \\
\mathbf{Q}(\mathbf{m})[\mathbf{C}] = \rho^{-1}(D + B)[C]\mathbf{m}, \\
\mathbf{A}(\mathbf{m})[\mathbf{C}] = \rho^{-1} \delta_l[C \otimes \mathbf{m}]\mathbf{m}, \\
\mathbf{B} = \rho^{-1} B;
\]

we note that because the symmetry of \( C, B \) and \( \delta_l \) then we have both \( \mathbf{A} \) and \( \mathbf{A} \) symmetric: we call \( \mathbf{A}(\mathbf{m}) \) the generalized acoustic tensor and \( \mathbf{A}(\mathbf{m}) \) the microacoustic tensor.

We require that \( A \) be positive-definite for all \( \xi > 0 \) (the semidefiniteness as in [19] being required when \( \xi \to 0 \)) and then, because \( \mathbf{A} \) and \( \mathbf{B} \) are positive-definite by (12) and the definition (19), the positive-definiteness of \( A \)
implies that also $\mathbf{A}(\mathbf{m})$ be positive-definite; accordingly the eigenvalue problem (17) admits the 12 eigencouples with real eigenvalues\(^2\)

\[
(\omega_h, w_h), \quad \langle \mathbf{J} w_h, w_h \rangle = \delta_{hk}, \quad h, k = 1, \ldots, 12.
\]  

(20)

The characteristic equation associated with the propagation condition (17) is

\[
\det(\mathbf{A}(\xi) - \omega^2 \mathbf{J}) = 0,
\]  

(21)

and because the components of $\mathbf{M}$ are functions of the wavenumber $\xi$, then also the eigencouples are

\[
\xi \mapsto (\omega_k(\xi), w_k(\xi)), \quad k = 1, \ldots, 12.
\]  

(22)

The functional dependences between $\omega$ and $\xi$ are called the dispersion relations and in terms of these relations we can define the phase $v_p^k$ and group velocity $v_g^k$ as

\[
v_p^k(\xi) = \frac{\omega_k(\xi)}{\xi}, \quad v_g^k(\xi) = \frac{d\omega_k(\xi)}{d\xi}, \quad k = 1, \ldots, 12.
\]  

(23)

As pointed out into [21], waves in micromorphic continua can be classified into:

- **acoustic waves**, whose frequencies $\omega_k(\xi)$ go to zero for $\xi \to 0$;
- **optic waves** for which the limit for $\xi \to 0$ is different from zero; the limit $\omega_k(0)$ is called the cut-off frequency with group velocities $v_g^k(0) = 0$;
- **standing waves**, those associated with imaginary values $\xi = \pm ik$, $k > 0$ for some frequencies; these waves do not propagate, but keep oscillating increasing or decreasing in a given, limited, region within the crystal.

In view of this classification, we begin to study the behaviour of the eigencouples of (17) as $\xi \to 0$ by using some results obtained in [22].

First, provided the components of $\mathbf{A}$ are analytic functions of $\xi \geq 0$, there exists a neighbourhood of $\xi = \xi_0$ where the eigenvalues $\omega_k(\xi)$ are regular and where their derivatives are well-defined. We use this result, in the case of simple eigenvalues, to give an explicit formula for the group velocities [22, Theorem 5]:

\[
v_p^k(\xi_0) = \frac{1}{2\omega_k(\xi_0)} \left( \frac{d\mathbf{A}}{d\xi} \bigg|_{\xi = \xi_0} w_k(\xi_0), w_k(\xi_0) \right);
\]  

(24)

moreover, the eigenvectors $w_k(\xi)$ are differentiable functions of the wavenumber.

Now, because for $\xi \to 0$ the matrix $\mathbf{A}$ is real with three multiple eigenvalues $\omega = 0$, nine non-zero (simple) eigenvalues $\hat{\omega}_j$ and twelve real eigenvectors $\hat{w}_k$, then we can use Theorem 2 of [22] to show that (21) admits three and only three zero eigenvalues as $\xi$ approaches zero: moreover, by the differentiability of eigenvalues and eigenvectors we have that

\[
\hat{\omega}_k = \lim_{\xi \to 0} \omega_k(\xi), \quad \hat{w}_k = \lim_{\xi \to 0} w_k(\xi),
\]  

(25)

and for the non-zero eigenvalues we have from (24)

\[
v_g^j(0) = \frac{1}{2\hat{\omega}_j} \left( \frac{d\mathbf{A}}{d\xi} \bigg|_{\xi = 0} \hat{w}_j, \hat{w}_j \right) = 0.
\]  

(26)

Therefore, by the application of the results obtained in [22] to our case, we obtain that in any anisotropic crystal:

- there exist three acoustic waves and nine optic waves;
- the cut-off frequencies for the optic waves are the limit as $\xi \to 0$ of the eigenvalues of (17);
- the frequencies for the acoustic waves are the eigenvalues of (17), which goes to zero in the limit for $\xi \to 0$;
- the eigenvectors for $\xi \to 0$ are real.
As far as the standing waves are concerned, we note that in this case it would be easy to show that \( A \) would be not positive-definite and therefore we can conclude that no standing waves are possible within this anisotropic model. Indeed, as was observed in [13, 21, 23, 24] for the isotropic case, standing waves are associated with band-gap material and are not possible within the classical micromorphic model: they appear instead in the relaxed micromorphic model proposed in [13].

The solutions of the eigenvalues problem (17) depend, in addition to the parameter \( \xi \), also on the three parameters \( L_m, \xi_1 \) and \( \xi_2 \) whose limiting values, as we have already remarked, corresponds to different physical scales: therefore, in addition to the complete condition given by (17), we shall study in some details these two limit cases.

2.2.1. The long-wavelength approximation: macroscopic waves. If we let \( \xi_1 \to 0 \) and \( \xi_2 \to 0 \) we are at the same time “zooming-out” from the crystal and disregarding the microinertia; such an approximation is called the long-wavelength approximation [15]. The propagation conditions (17) then reduce explicitly, to within higher-order terms in \( \xi_1 \) and \( \xi_2 \), to the two equations

\[
\begin{align*}
\xi^2 A(m)a_o + i\xi L_m^{-1}Q(m)[C] &= \omega^2 a_o, \\
-\omega & - i\xi Q^T(m)a_o + L_m^{-1}B[C] = 0;
\end{align*}
\]  
(27)

because \( B \) is positive-definite, then from (27) we obtain \( C = iL_m\xi B^{-1}Q^T(m)a_o \) and therefore from (27) we recover the classical continuum propagation condition

\[
\hat{A}(m)a_o = c^2 a_o,
\]
where the acoustic tensor \( \hat{A}(m) \) (which is independent of the macroscopic length \( L_m \)) is defined as

\[
\hat{A}(m)a_o = A(m)a_o - Q(m)B^{-1}Q^T(m)a_o = (C - C_{\text{micro}})[a_o \otimes m]m.
\]  
(29)

In this definition \( C \) is the elasticity tensor from (10), whereas by using (19) we can represent the positive-definite microelasticity tensor \( C_{\text{micro}} \) as

\[
C_{\text{micro}} = DB^{-1}D^T, \quad C_{\text{micro}} = C_{\text{micro}}^T, \quad C_{\text{micro}} : \text{Sym} \to \text{Sym};
\]  
(30)

by one of the consequences of (12) \( C \) is positive-definite and, hence, the acoustic tensor \( \hat{A}(m) \) is positive-definite too.

The three eigencouples of (28) represent acoustic waves; however, in this approximation \( \hat{A}(m) \) is not the acoustic tensor of the linear elasticity and the presence of the microstructure makes the propagation velocities smaller than in linearly elastic bodies: moreover, we also have three microdistortions associated with the eigenvectors of (28) by means of the expression of \( C \); these microdistortions are purely imaginary and depend on the ratio \( L_m \xi = L_m/\lambda \) between the macrosopic scale and the wavelength.

2.2.2. The limit \( L_c \to 0 \): microvibrations. If we let \( L_c \to \infty \), for fixed \( L_m \approx L_0 \), then we are zooming into the crystal; into the limit the constitutive relations (10)_1,2 remain finite for any choice of material only if \( \nabla L = 0 \). In this case, from (14)_2 we have that \( \xi_1 = 0 \) and the propagation condition (17)_1 leads to a solution \( \omega = 0 \) with multiplicity three. As, by (14)_1, \( u \) reduces to a rigid motion, then without loss of generality we can set \( u(x, t) = 0 \) and \( L(t) = C e^{i\omega t} \); then from the propagation condition (17) we are led to the characteristic equation

\[
\det(B - \rho \omega^2 J) = 0.
\]  
(31)

For \( L_c \to \infty \) we therefore recover the microvibration solutions of (17), which was studied in detail in [11] for isotropic materials (see also [21]); the propagation condition (31) admits nine eigencouples \( (\omega_k, C_k), k = 1, \ldots, 9 \), with real eigentensors and whose eigenvalues represent the cut-off frequencies for the propagation condition (17), as shown in Section 2.3.
3. Wave propagation in tetragonal crystals

As we have already remarked, an explicit solution for the propagation condition (17) and an explicit determination for the dispersion relations (22) is not possible in the general anisotropic case, nor it would be particularly useful, because the associated kinematics would be fully coupled. However, many of the components of both the acoustic tensors \( A \) and \( \Lambda \), as well as of \( P \) and \( Q \), may vanish according to both the crystal symmetry group and the propagation direction \( m \) and it could make sense to obtain explicit solutions for special cases of symmetry and directions of propagation.

The simplest case of isotropic material (which depends on only 18 constitutive parameters) was studied in full length in [11] and further analysed and extended to a relaxed micromorphic model in [13, 19]. In this section, we study the wave propagation condition (17) for crystals belonging to the tetragonal symmetry group.4

We took \( e_3 \) directed as the tetragonal \( c \)-axis, hence the acoustic tensors have the representation given in Section 5.4 of the Appendix of [A]. At a glance, by looking at (158), (159), (162), (164), (166), (170), (173) and (174) of [A], for a generic propagation direction \( m \), the matrix \( A \) does not simplify enough and the problem maintains the same complexity as for crystal of the triclinic group.

However, for the two relevant cases of propagation direction either parallel (\( m \times e_3 = 0 \)) or orthogonal (\( m \cdot e_3 = 0 \)) to the tetragonal \( c \)-axis, many of the components of \( A \) vanish and the number of the independent components reduces too, allowing for an explicit solution of (17) whose associated kinematics can be understood more easily. Therefore, we study these two propagation direction and we begin with the two limit propagation problems we obtained in Section 2.2.1 (long-wavelength approximation) and Section 2.2.2 (microvibrations).

3.1. Long-wavelength approximation

For the propagation direction \( m = e_3 \) (i.e. along the tetragonal \( c \)-axis: henceforth we shall use \( c \) and \( e_3 \) as synonymous when we describe the material symmetry) the tensor (29) reduces for all classes to the isotropic-like representation:

\[
\hat{A}(e_3) = \frac{1}{\rho} \left( \hat{C}_{44}(I - W_3) + \hat{C}_{33} W_3 \right),
\]

and we have a longitudinal and two transverse acoustic waves with frequencies:

\[
\omega_{1,2}(\xi) = \frac{\xi}{\sqrt{\frac{\hat{C}_{44}}{\rho}}}, \quad a_1 = \cos \beta e_1 + \sin \beta e_2, \quad a_2 \cdot a_1 = 0,
\]

\[
\omega_3(\xi) = \frac{\xi}{\sqrt{\frac{\hat{C}_{33}}{\rho}}}, \quad a_3 = e_3,
\]

These macroscopic displacements are accompanied by a microdistortion associated with the longitudinal wave of frequency (33)

\[
C_3 = iL_m \xi \hat{B}^{-1} Q^T(e_3) e_3 = \alpha_3 (I - W_3) + \beta_3 W_3 + \gamma_3 \bar{W}_6,
\]

where the dependence of the coefficients \( \alpha_3, \beta_3 \) and \( \gamma_3 \) on \( B^{-1} \) and \( Q \) is provided in detail in equation (78) of [A]. The microdistortions associated with the longitudinal waves are therefore a combination of the modes \( D_2 \) and \( R_1 \).

The microdistortions accompanied to the transverse waves which propagates along \( m = e_3 \) are instead given by

\[
C_k = iL_m \xi \hat{B}^{-1} Q^T(e_3) a_k = \alpha_k \hat{W}_4 + \beta_k \hat{W}_5 + \gamma_k \bar{W}_4 + \delta_k \bar{W}_5, \quad k = 1, 2,
\]

where the coefficients are given explicitly in [A]. Therefore each transverse wave \( a_k \) which propagates along \( m = e_3 \) generates a combination of the modes \( S_2 \) and \( R_2 \).

Remark 1. (Classes 4mm, 422, 4/mmm, 42m) The only noticeable difference with the results thus far obtained is that \( \gamma_3 = 0 \) into (34) and hence \( C_3 \) reduces to the mode \( D_2 \). See [A] for full details.
Whenever the propagation direction is orthogonal to the $c$–axis, say $m = \cos \theta e_1 + \sin \theta e_2$ then we have, for all $\theta$, a transverse wave which is directed as the $c$–axis:

$$\omega_1(\xi) = \xi \sqrt{\hat{C}_{44}/\rho}, \quad a_1 = e_3$$

and two waves which in general are neither transverse nor longitudinal:

$$\omega_{2,3}(\xi) = \xi \sqrt{\frac{1}{2\rho} \left( a \pm \sqrt{b \cos^2 2\theta + c \sin^2 2\theta + 2d \sin 2\theta \cos 2\theta} \right)},$$

$$a_2 = \cos \beta e_1 + \sin \beta e_2, \quad a_3 \cdot a_2 = 0,$$

with

$$\tan \beta = \frac{a - \sqrt{b \cos^2 2\theta + c \sin^2 2\theta + 2d \sin 2\theta \cos 2\theta}}{2\hat{C}_{16} \cos 2\theta + (\hat{C}_{66} + 2\hat{C}_{12}) \sin 2\theta},$$

and where, for the classes 4, $\bar{4}$ and $4/m$,

$$a = \hat{C}_{11} + \hat{C}_{66}, \quad b = (\hat{C}_{11} - \hat{C}_{66})^2 + 4\hat{C}_{16}^2,$$

$$c = \hat{C}_{16}^2 + (\hat{C}_{66} + 2\hat{C}_{12})^2, \quad d = \hat{C}_{16}(\hat{C}_{11} + \hat{C}_{66} + 4\hat{C}_{12}).$$

We may search for the angle $\theta$ such that $a_2$ is a longitudinal wave with frequency $\omega_2$ and $a_3$ is a transverse wave with frequency $\omega_3$, which is equivalent to requiring either that $a_2 \times m = 0$ and hence $\theta = \beta$ or that $A_{12} = 0$ which gives

$$\tan 2\theta = -\frac{2\hat{C}_{16}}{\hat{C}_{12} + 2\hat{C}_{66}}.$$  \hspace{1cm} (40)

For the classes 4mm, 422, $\bar{4}/mm$ and $\bar{4}m$ with $\hat{C}_{16} = 0$, then

$$b = (\hat{C}_{11} - \hat{C}_{66})^2, \quad c = (\hat{C}_{66} + 2\hat{C}_{12})^2, \quad d = 0,$$

and then from (40) and (38) with $\beta = \theta$ it is easy to see that for $\theta \in \{0, \pi/4, \pi/2\}$ we have a longitudinal ($a_2 = m$) and a transverse ($a_3 = e_3 \times m$) wave whose frequencies are given by (37) when we use (41) in place of (39).

The microdistortion that corresponds to the transverse wave (36) is $C_1 = iL_m \xi \hat{B}^{-1} Q^T(\theta)e_3$ and we obtain again (35) with the coefficients given by equation (88) of [A]. Accordingly, this transverse wave is associated with a shear between $m$ and $e_3$ and a rigid rotation about $e_3 \times m$.

When we turn our attention to the other two waves, which are neither transverse nor longitudinal, we have two associated microdistortions

$$C_2 = iL_m \xi \hat{B}^{-1} Q^T(\theta)a_2 = iL_m \xi (\cos \beta B_1 + \sin \beta B_2),$$

$$C_3 = iL_m \xi \hat{B}^{-1} Q^T(\theta)a_3 = iL_m \xi (-\sin \beta B_1 + \cos \beta B_2);$$

where the two tensors $B_k = \hat{B}^{-1} Q^T(\theta)e_k, k = 1, 2$, have the common structure

$$B_k = \alpha_k W_1 + \beta_k W_2 + \gamma_k W_3 + \delta_k \dot{W}_6 + \epsilon_k \ddot{W}_6,$$  \hspace{1cm} (43)

with the coefficients given explicitly by equation (92) of [A].

Interestingly enough for both the waves with amplitudes $a_2$ and $a_3$, the corresponding microdistortions are a combination of the modes $D_1, S_1$ and $R_1$. We remark that this situation is maintained even when the propagation direction is given by (40) and the two waves becomes one transverse and the other longitudinal.

**Remark 2.** (Classes 4mmm, 422, 4/mm, and $\bar{4}m$) For these classes we have that formula (43) holds with $\alpha_k = \beta_k$ and $\delta_k = \epsilon_k$ and therefore the mode $D_1$ changes into $D_2$. 
3.2. Microvibrations

3.2.1. Classes 4, 3 and 4lm. We begin with the lower-symmetry tetragonal classes: when we consider the propagation condition (31) with $B$ and $J$ given by equations (147) and (150) of [A], respectively, we note that both tensors are reduced by the two subspaces of $\text{Lin}$, $\mathcal{U}_1$ and $\mathcal{U}_2$ with $\text{Lin} = \mathcal{U}_1 \oplus \mathcal{U}_2$:

$$\mathcal{U}_1 \equiv \text{span}\{W_1, W_2, W_3, W_6, W_9\}, \quad \mathcal{U}_2 \equiv \text{span}\{W_4, W_5, W_7, W_8\},$$

that is $\mathbb{B}[C_\alpha] \in \mathcal{U}_\alpha$, $\mathbb{J}[C_\alpha] \in \mathcal{U}_\alpha$, $\forall C_\alpha \in \mathcal{U}_\alpha$, $\alpha = 1, 2$. The eigencouples split accordingly into two groups: the first $(\omega_k, C_k)$, $k = 1, \ldots, 5$, with $C_k \in \mathcal{U}_1$ and whose eigentensors are a combination of the modes $D_k$, $S_1$ and $R_1$; the second $(\omega_j, C_j)$, $j = 6, \ldots, 9$, with $C_j \in \mathcal{U}_2$ and whose eigentensors are a combination of the modes $S_2$ and $R_2$.

We define the normalized components of $\mathbb{B}$ as follows:

$$a = \frac{\mathbb{B}_{11}}{\rho J_{11}}, \quad b = \frac{\mathbb{B}_{33}}{\rho J_{33}}, \quad d = \frac{\mathbb{B}_{12}}{\rho \sqrt{J_{11}J_{33}}}, \quad e = \frac{\mathbb{B}_{13}}{\rho \sqrt{J_{11}J_{33}}},$$

$$c = \frac{\mathbb{B}_{66}}{\rho J_{11}}, \quad f = \frac{\mathbb{B}_{69}}{\rho J_{11}}, \quad g = \frac{\mathbb{B}_{16}}{\rho J_{11}}, \quad h = \frac{\mathbb{B}_{26}}{\rho J_{11}}, \quad l = \frac{\mathbb{B}_{36}}{\rho \sqrt{J_{11}J_{33}}},$$

$$m = \frac{\mathbb{B}_{44}}{\rho J_{33}}, \quad n = \frac{\mathbb{B}_{55}}{\rho J_{11}}, \quad p = \frac{\mathbb{B}_{45}}{\rho \sqrt{J_{11}J_{33}}}, \quad q = \frac{\mathbb{B}_{47}}{\rho \sqrt{J_{11}J_{33}}},$$

and we begin our analysis with the subspace $\mathcal{U}_1$. We note that the algebraic fifth-order characteristic equation in $\omega^2$ can be factorized into

$$(\omega^4 - B\omega^2 + C)(\omega^6 + D\omega^4 + E\omega^2 + F) = 0,$$

where

$$B = a + c + f - d,$$

$$C = (a - d)(c + f) - (g - h)^2,$$

$$D = a + b + c + d - f,$$

$$E = 2a(b + c) + bc - f(a + b + d) - 2(e^2 + f^2) - (g + h)^2,$$

$$F = (c - b)^2 - (g + h)^2 - 2d^2 - 9e^2 + f^2 + 18f,$$

$$+ 3d(b + c) - 2f(2b + c + d) - 3gh - 4bc.$$

The eigenvalues are thus given by

$$\omega_{1,2}^2 = \frac{1}{2}(B \mp \sqrt{B^2 - 4C}),$$

with $\omega_1 < \omega_2$ and, by the means of Cardano’s formulae [28], by

$$\omega_3^2 = \frac{1}{3} \left( D + 2\sqrt{P \cos \frac{\theta}{3}} \right), \quad \omega_{4,5}^2 = \frac{1}{3} \left( D + 2\sqrt{P \cos \frac{\theta \pm 2\pi}{3}} \right),$$

with either $\omega_4 < \omega_3 < \omega_5$ or $\omega_4 > \omega_3 > \omega_5$ and where

$$P = D^2 - 3E, \quad Q = D^3 - 9DE - 27F, \quad \theta = \cos^{-1} \frac{Q}{2\sqrt{P}}.$$

The five associated eigentensors $C_k$, $k = 1, \ldots, 5$, combine the modes $D_1, S_1, R_1$ in all but in the case of $C_3$, when it combines $D_2$ and $R_1$ (for the explicit expressions see equation (101) of [A]).

When we turn our attention to the subspace $\mathcal{U}_2$, then we obtain two eigenvalues of multiplicity two:

$$\omega_{6,8}^2 = \omega_{7,9}^2 = \frac{1}{2}(m + n) \pm \sqrt{\left(\frac{m - n}{2}\right)^2 + p^2 + q^2},$$
whose four eigentensors $C_j$, $j = 6, 7, 8, 9$, represent a combination of the modes $S_2$ and $R_2$ (see equation (103) from [A] for the explicit expressions); from (51) we have $\omega_6 = \omega_7 < \omega_8 = \omega_9$. All together, we have

$$\omega_1 < \omega_2, \quad \omega_4 < \omega_3 < \omega_5, \quad (\text{or } \omega_4 > \omega_3 > \omega_5), \quad \omega_6 = \omega_7 < \omega_8 = \omega_9,$$

and in addition to this we cannot give a complete ordering between these frequencies without knowledge of the numerical values of components of $B$.

3.2.2. Classes 4mm, 422, $\overline{4}2m$ and 4/mm. For these classes the matrix $B$ is reduced by the three subspaces $Z_1 \equiv \text{span}\{W_1, W_2, W_3\}$, $Z_2 \equiv \text{span}\{W_6, W_9\}$ and $Z_3 = U_2$, with $\text{Lin} \equiv Z_1 \oplus Z_2 \oplus Z_3$.

To find the solution in $Z_1$ we note that because $\omega^2 = a - d$ is a root for the cubic characteristic equation, then we can easily obtain

$$\omega_1^2 = a - d, \quad \omega_{2,3}^2 = \frac{a + b + d}{2} \mp \sqrt{(\frac{a + d - b}{2})^2 + 2e^2},$$

with $\omega_2 < \omega_3$, $\omega_1 < \omega_3$. The corresponding eigentensors are $C_1 = D_3$ and $C_{2,3} = D_1$, the explicit expression given by equations (109) and (110) of [A].

In the subspace $Z_2$ we have two eigencouples associated with a shear in the plane orthogonal to the $c$-axis ($C_4$) and a rotation about the propagation direction ($C_5$):

$$(\omega_4^2 = c - f, \quad C_4 = \overline{W}_6), \quad (\omega_5^2 = c + f, \quad C_5 = \hat{W}_6),$$

whereas the solutions on $Z_3$ are given by (51) with $p = 0$; again we have $\omega_6 = \omega_7 < \omega_8 = \omega_9$ and for $B_{69} > 0$ also $\omega_4 < \omega_5$, the inequality being reversed when $B_{69} < 0$.

The kinematics of microdistortions for these classes is represented in $Z_1$ as either $D_1$ or the traceless plane strain $D_3$; in $Z_2$ we have instead a shear in the plane orthogonal to the $c$-axis and a rigid microrotation about the same direction whereas in $Z_3$ we have the same kinematics as in $U_2$.

For these classes it is not possible to give a complete ordering between all the nine eigenvalues in the absence of the numerical values for the components of $B$.

3.3. Micromorphic continua

3.3.1. Propagation along the tetragonal $c$-axis.

Classes 4, $\overline{4}$ and 4/m. We begin with the lower-symmetry classes; in this case the blocks of the matrix $A$ have the following non-null components

$$A \equiv \begin{bmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{bmatrix}.$$
with the independent components given by equations (158), (166), (162) and (173) of \([A]\) evaluated for \(m_1 = m_2 = 0\) and \(m_3 = 1\).

The matrix (55) is reduced by the pairs \(\mathcal{M}_1 \equiv \text{span}\{e_3\} \oplus \mathcal{U}_1\) and \(\mathcal{M}_2 \equiv \text{span}\{e_1, e_2\} \oplus \mathcal{U}_2\) were \(\mathcal{U}_1\) and \(\mathcal{U}_2\) are defined by (44).

We begin with the subspace \(\mathcal{M}_1\) and define the normalized components:

\[
a(\xi) = \xi^2 A_{33}, \\
b(\xi) = \xi^2 \frac{P_{311} L_1^2}{L_m \sqrt{J_{11}}} + i \xi \frac{Q_{311}}{\sqrt{J_{11}}}, \\
c(\xi) = \xi^2 \frac{P_{333} L_2^2}{L_m \sqrt{J_{33}}} + i \xi \frac{Q_{333}}{\sqrt{J_{33}}}, \\
d(\xi) = \xi^2 \frac{A_{666} L_2^2}{J_{11}} + \frac{B_{66}}{\rho J_{11}}, \\
e(\xi) = \xi^2 \frac{A_{69} L_2^2}{J_{11}} + \frac{B_{69}}{\rho J_{11}}, \\
f(\xi) = \xi^2 \frac{A_{11} L_2^2}{J_{11}} + \frac{B_{11}}{\rho J_{11}}, \\
g(\xi) = \xi^2 \frac{A_{33} L_2^2}{J_{33}} + \frac{B_{33}}{\rho J_{33}}, \\
h(\xi) = \xi^2 \frac{A_{12} L_2^2}{\sqrt{J_{11} J_{33}}} + \frac{B_{12}}{\rho J_{11} J_{33}}, \\
l(\xi) = \xi^2 \frac{A_{13} L_2^2}{\sqrt{J_{11} J_{33}}} + \frac{B_{13}}{\rho J_{11} J_{33}}, \\
m(\xi) = \xi^2 \frac{A_{16} L_2^2}{J_{11}} + \frac{B_{16}}{\rho J_{11}}, \\
p(\xi) = \xi^2 \frac{A_{26} L_2^2}{J_{11}} + \frac{B_{26}}{\rho J_{11}}, \\
q(\xi) = \xi^2 \frac{P_{312} L_2^2}{L_m \sqrt{J_{11}}} + i \xi \frac{Q_{312}}{\sqrt{J_{11}}};
\]

then, because the characteristic equation can be factorized into a second- and a fourth-grade algebraic equations, we obtain by means of Cardano’s formulae for the fourth-grade algebraic equations [28], the explicit representation of the six eigenvalues

\[
\omega_{1,2} = \frac{1}{2} \left( F \mp \sqrt{F^2 - 4G} \right), \\
\omega_{3,4} = \frac{1}{4} A + \frac{1}{2} \sqrt{\frac{1}{4} A^2 + \frac{2}{3} B + W} \\
+ \frac{1}{2} \sqrt{-\frac{1}{2} A^2 + \frac{4}{3} B - W - \frac{A^3 + 4AB - 8C}{4 \sqrt{\frac{1}{4} A^2 + \frac{2}{3} B + W}}}, \\
\omega_{5,6} = \frac{1}{4} A + \frac{1}{2} \sqrt{\frac{1}{4} A^2 + \frac{2}{3} B + W} \\
+ \frac{1}{2} \sqrt{-\frac{1}{2} A^2 + \frac{4}{3} B - W - \frac{A^3 + 4AB - 8C}{4 \sqrt{\frac{1}{4} A^2 + \frac{2}{3} B + W}}},
\]

where

\[
A = a + d - e + f + g + h, \\
B = cc^* + 2(bb^* + l^2 + m^2 + qq^*) + (n + p)^2 - (d - e)(a + f + g + h) \\
- (f + h)(a + g) - ag, \\
C = 2bb^*(d + e + g) + 2l^2(a + d - e) + 2m^2(a + f + h) + 2qq^*(f + g + h) \\
+ (a + g)(n + p)^2 - (f + h)(d - e) + (cc^* - ag)(d - e + f + h) \\
- 4((lm + qb^*)(n + p) + (lb + mq)c^*), \\
D = 2bb^*(m^2 - g(d - e)) + 2qq^*(2l^2 - g(f + h)) + 2l(2bc^* - al)(d - e) \\
+ 2m(2cq^* - am)(f + h) + ((n + p)^2 - (f + h)(d - e))(cc^* - ag) \\
+ 4(n + p)(m(cb^* + al) + q(gb^* - c^*l)) - 8lmbq^*.
\]
Figure 1. Schematic of the dispersion relations in $\mathcal{M}_1$. The dotted line represents the linearly elastic longitudinal wave. AL, acoustic longitudinal wave; OL$_{1,2,4,5}$, optic longitudinal waves, modes D$_1$, S$_1$, R$_1$; OL$_3$, optic longitudinal wave, modes D$_2$, R$_1$. The frequencies on the $\omega$-axis are the cut-off frequencies (48) and (49).

\[ F = d + e + f - h, \]
\[ G = (d + e)(f - h) - (n - p)^2, \]
\[ P = -2B^3 - 9ACB + 72DB + 27C^2 + 27A^2D, \]
\[ Q = B^2 + 3AC + 12D, \]

and

\[ W = \frac{\sqrt{P + \sqrt{P^2 - 4Q^3}}}{3\sqrt{2}} + \frac{\sqrt{2}Q}{3\sqrt{P + \sqrt{P^2 - 4Q^3}}}. \]

By looking at (57), by (56) and (58) we note that the frequencies $\omega_{1,2}$ depend solely on the components of $A$ and $B$, whereas the others depend also on the acoustic tensors $A$, $P$ and $Q$.

The associated eigenvectors represent a longitudinal wave combined either with the modes D$_1$, S$_1$, R$_1$ or with the modes D$_2$ and R$_1$; their explicit expressions are given by equation (119) of [A].

For $\xi \to 0$, because $a$, $b$, $c$ and $q$ vanish, then from (57) and the results of Section 5.5.2 of [A] we have that

\[
(\omega_{1,2}(\xi), w_{1,2}) \to (\omega_{1,2}, [0, C_{1,2}]),
\]
\[(\omega_0(\xi), w_0) \to (\omega_0, [0, C_0]),
\]
\[(\omega_{4,5}(\xi), w_{4,5}) \to (\omega_{4,5}, [0, C_{4,5}]),
\]
\[(\omega_3(\xi), w_3) \to (0, [e_3, 0]),
\]

with the cut-off frequencies $\omega_{1,2}$ and $\omega_{4,3,4,5}$ given by (48) and (49), respectively, and where the microdistortions $C_{1-5}$ are given by equation (101) of [A]; therefore, in the subspace $\mathcal{M}_1$ there are one acoustic longitudinal wave and five optic waves with cut-off frequencies (48) and (49).

A qualitative graph of the dispersion relations $\omega = \omega(\xi)$ for the solutions in the subspace $\mathcal{M}_1$, which can be obtained for arbitrary values of the components, is given in Figure 1.

In the subspace $\mathcal{M}_2$, provided that we define the normalized components,

\[
a(\xi) = \xi^2 A_{11}, \quad b(\xi) = \xi^2 A_{12},
\]
\[c(\xi) = \xi^2 \frac{P_{113L^2_c}}{L_m \sqrt{J_{11}}} + i\xi \frac{Q_{113}}{\sqrt{J_{11}}}, \quad d(\xi) = \xi^2 \frac{P_{131L^2_c}}{L_m \sqrt{J_{11}}} + i\xi \frac{Q_{131}}{\sqrt{J_{11}}},
\]
\[e(\xi) = \xi^2 \frac{A_{44L^2_c}}{J_{33}} + \frac{B_{44}}{\rho J_{33}}, \quad f(\xi) = \xi^2 \frac{A_{55L^2_c}}{J_{11}} + \frac{B_{33}}{\rho J_{11}},
\]
yields four optic waves (with multiplicity two) with cut-off frequencies (51) and eigenvectors

\( \begin{align*}
(\mathbf{OL1,2,4,5}) & \text{Four optic waves associated with a macroscopic displacement along } \mathbf{L}.
\end{align*} \)

In this case, all the frequencies depend on all the components of the block matrix \( \mathbf{A} \). The eigenvectors are concerned, the kinematics they represent is formed by a macroscopic transverse wave coupled with a combination of the modes \( \mathbf{S}_2 \) and \( \mathbf{R}_2 \) (see equation (125) of [A]).

To summarize, for the tetragonal classes 4, 4, 4/m, for a propagation direction \( \mathbf{m} \) along the tetragonal c-axis, we have three acoustic and nine optic waves that depend on 43 independent components of \( \mathbf{A} \) (3 components of \( \mathbf{A}, 13 \) for both \( \mathbf{A} \) and \( \mathbf{B} \), 7 for both \( \mathbf{P} \) and \( \mathbf{Q} \)).

(\text{AL}) One acoustic wave associated with a macroscopic displacement along \( \mathbf{c} \) and a combination of the modes \( \mathbf{D}_1, \mathbf{S}_1 \) and \( \mathbf{R}_1 \), which for \( \xi = 0 \) reduced to a macroscopic longitudinal wave.

(\text{AT1,2}) Two acoustic waves associated with a macroscopic displacement orthogonal to \( \mathbf{c} \), coupled with a combination of the modes \( \mathbf{S}_2 \) and \( \mathbf{R}_2 \). For \( \xi = 0 \) these waves reduce to two macroscopic orthogonal transverse waves.

(\text{OL1,2,4,5}) Four optic waves associated with a macroscopic displacement along \( \mathbf{c} \) and with a microdistortion that combines the modes \( \mathbf{D}_1, \mathbf{S}_1 \) and \( \mathbf{R}_1 \) which for \( \xi = 0 \) reduces to the pure microdistortions (101)\text{1,2,4,5} of [A].

\[
\begin{align*}
g(\xi) &= \xi^2 \frac{\hbar^2 L_c^2}{\sqrt{J_{11}J_{33}}} + \frac{E}{\rho \sqrt{J_{11}J_{33}}}, \quad h(\xi) = \xi^2 \frac{\hbar^2 L_c^2}{\sqrt{J_{11}J_{33}}} + \frac{E}{\rho \sqrt{J_{11}J_{33}}}, \\
m(\xi) &= \xi^2 \frac{P_{123} L_c^2}{L_m \sqrt{J_{11}}} + i \xi Q_{123}, \quad n(\xi) = \xi^2 \frac{P_{132} L_c^2}{L_m \sqrt{J_{11}}} + i \xi Q_{132},
\end{align*}
\]

we note that the characteristic equation can be factorized into two cubic equations in \( \omega^2 \). Accordingly the eigenvalues can be obtained by using Cardano’s formula twice:

\[
\begin{align*}
\omega_1^2 &= \frac{1}{3} \left( A_1 + 2 \sqrt{P_1 \cos \frac{\theta_1}{3}} \right), \quad \omega_2^2 = \frac{1}{3} \left( A_1 + 2 \sqrt{P_1 \cos \frac{\theta_1 + 2\pi}{3}} \right), \\
\omega_3^2 &= \frac{1}{3} \left( A_2 + 2 \sqrt{P_2 \cos \frac{\theta_2}{3}} \right), \quad \omega_4^2 = \frac{1}{3} \left( A_2 + 2 \sqrt{P_2 \cos \frac{\theta_2 + 2\pi}{3}} \right),
\end{align*}
\]

where

\[
\begin{align*}
A_1 &= a + b + c + d + e + f, \\
B_1 &= 2a^3 - 3a^2(b + c + d)
\end{align*}
\]

\[
+ 3a(2b^2 + 3(c^2 + dd^*) - (e - f)^2 - 6(g^2 + h^2) + 3(m^* + n^*)
\]

\[
+ 2b(e + f) + 4ef) \pm 2(b^3 \pm e^3 \pm f^3)
\]

\[
+ 3b(c + dd^* - e^2 - f^2) + 9(g^2 + h^2) + 9(b + d + e + f)
\]

\[
+ 9mm^*(b \pm e + f) + 9nn^*(b \pm f \mp e) - 3e(b^2 + f^2 - 3cc^* + 6dd^*)
\]

\[
- 3f(b^2 + 6cc^* - 3dd^* + e^2 + 4be) + 54(g(c^d + mn^*) + h(dm^* - cn^*))
\]

\[
C_1 = \pm cc^* - (dd^* + g^2 + h^2 + m^* + n^* + (a \pm b)(e + f) + ef,
\]

and

\[
\theta_\alpha = \cos^{-1} \frac{B_\alpha}{2\sqrt{P_\alpha}}, \quad P_\alpha = A_\alpha^2 - 3C_\alpha, \quad \alpha = 1, 2.
\]

In this case, all the frequencies depend on all the components of the block matrix \( \mathbf{A} \); as far as the corresponding eigenvectors are concerned, the kinematics they represent is formed by a macroscopic transverse wave coupled with a combination of the modes \( \mathbf{S}_2 \) and \( \mathbf{R}_2 \) (see equation (125) of [A]).

The behaviour of the frequencies (61), for \( \xi \to 0 \), because in such a case \( a = b = c = d = m = n = 0 \), yields four optic waves (with multiplicity two) with cut-off frequencies (51) and eigenvectors

\[
\begin{align*}
w_7(0) &= w_8(0) = \{0; C_{6,7}\}, \\
w_{10}(0) &= w_{11}(0) = \{0; C_{8,9}\},
\end{align*}
\]

and two acoustic waves with eigenvectors

\[
\begin{align*}
w_9(0) &= \{e_1 - e_2; 0\}, \\
w_{12}(0) &= \{e_1 + e_2; 0\}.
\end{align*}
\]
Figure 2. Schematic of the dispersion relations in \( \mathcal{M}_2 \). The dotted line represents the linearly elastic transverse waves. \( \text{AT}_{1,2} \), acoustic transverse wave; \( \text{OT}_{7,8,10,11} \), optic transverse waves, modes \( S_2, R_2 \). The frequencies on the \( \omega \)-axis are the cut-off frequencies (51).

(OL_3) One optic wave associated with a macroscopic displacement along \( c \) and with a combination of the modes \( D_2 \) and \( R_1 \).

(OT_{7,8,10,11}) Four optic waves associated with a macroscopic displacement orthogonal to \( c \) coupled with a combination of the \( S_2 \) and \( R_2 \) modes, which for \( \xi = 0 \) reduce to the shear microdistortion (103)_2 of \([A]\) and to the rigid rotation (103)_1 of \([A]\).

Classes 4\( mm \), 422, \( \bar{4}2m \) and 4/\( mm \). When we deal with the highly symmetric tetragonal classes, for a propagation direction along the tetragonal \( c \)-axis with \( \mathbf{m} = e_3 \), the matrix \( A \) has the following non-null components, the independent ones given by relations (159), (165), (170) and (174) of the appendix in \([A]\), evaluated for \( m_1 = m_2 = 0 \) and \( m_3 = 1 \):

\[
A \equiv \begin{bmatrix}
\vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\end{bmatrix}
\]

'accordingly \( A \) is reduced by the three subspaces \( \mathcal{N}_1 \equiv \text{span}\{e_1\} \oplus \mathbb{Z}_1, \mathcal{N}_2 \equiv \mathbb{Z}_2 \) and \( \mathcal{N}_3 \equiv \text{span}\{e_1, e_2\} \oplus \mathbb{Z}_3 \).

We begin our analysis with the subspace \( \mathcal{N}_1 \) and note that the eigenvalues can be obtained from those in \( \mathcal{M}_1 \) when we set \( d = e = 0 \) and \( m = n = p = q = 0 \) into (56). Then the characteristic equation admits the root \( \omega^2 = f - h \) and, thus, by means of Cardano’s formulae (see, e.g., [28]) we obtain the four eigenvalues (for once
we write one of them in terms of components and characteristic length):

\[
\omega_1^2(\xi) = \xi^2 \left( \frac{A_{11}}{J_{11}} + \frac{A_{12}}{\sqrt{J_{11}J_{33}}} \right) L_c^2 + \frac{B_{11}}{\rho J_{11}} + \frac{B_{12}}{\rho \sqrt{J_{11}J_{33}}},
\]

\[
\omega_2^2(\xi) = \frac{1}{3} \left( A + 2\sqrt{P} \cos \frac{\theta}{3} \right),
\]

\[
\omega_3^2(\xi) = \frac{1}{3} \left( A + 2\sqrt{P} \cos \frac{\theta \pm 2\pi}{3} \right),
\]

where

\[
P = A^2 - 3B, \quad Q = 2A^3 - 9AB - 27C, \quad \theta = \cos^{-1} \frac{Q}{2\sqrt{P}},
\]

and \(A, B\) and \(C\) are obtained by setting \(d = e = m = n = p = q = 0\) in (58). When we look at the corresponding eigenvectors we note first that the frequencies \(\omega_{2,3,4}(\xi)\) are associated with the mode \(D_2\) coupled with a macroscopic longitudinal wave; the frequency \(\omega_0(\xi)\) is instead associated uniquely with the traceless real microdistortion \(D_3\) (full details given in equation (132) of [A]).

In the limit \(\xi \to 0\), because \(a, b\) and \(c\) vanish, then the three frequencies \(\omega_0(\xi)\) and \(\omega_{2,3}(\xi)\) reduce to (53) with eigenvectors

\[
w_1(0) = \{0; C_1\}, \quad w_2(0) = \{0; C_2\}, \quad w_3 = \{0; C_3\},
\]

with the three microdistortions given by equation (109) of [A]: these are optic frequencies, the values (53) being the associated cut-off values; the frequency \(\omega_4(\xi)\) vanishes instead for \(\xi \to 0\) with

\[
w_4(0) = \{e_2; 0\},
\]

which represents a purely macroscopic acoustic longitudinal wave.

The solutions in the subspace \(N_2\) are the same as those in \(Z_2\) with frequencies

\[
\omega_{5,6}^2(\xi) = \xi^2 L_c^2 \frac{A_{66} + A_{69}}{J_{11}} + \frac{B_{66} + B_{69}}{\rho J_{11}},
\]

and accordingly describe optic waves with purely microdistortion amplitudes and whose cut-off frequencies are given by (54); the corresponding real eigenvectors are

\[
w_5(0) = \{0; V_3 = C_4\}, \quad w_6(0) = \{0; V_6 = C_5\},
\]

with the microdistortions given by (54).

We finish with the subspace \(N_3\) where the solutions are obtained by setting \(m = n = h = 0\) into (60) which yield six eigenvalues whose expression is the same as (61) provided we put \(m = n = h = 0\) into (62), the eigenvectors following accordingly and describing a kinematics formed by a macroscopic transverse wave coupled a combination of the modes \(S_2\) and \(R_2\).

The eigenvalues in \(N_3\) and their associated eigenvectors become, because for \(\xi \to 0\) we have \(a = b = c = d = 0\),

\[
w_{7,8,9,10}(0) = \{0; C_6,7,8,9\},
\]

which correspond to four optic waves (with multiplicity two) with cut-off frequencies (51) with \(B_{45} = 0\) and two acoustic waves:

\[
w_{11}(0) = \{e_1 - e_2; 0\}, \quad w_{12}(0) = \{e_1 + e_2; 0\}.
\]

As far as the eigenvalues ordering is concerned, we can only say that

\[
\omega_5 < \omega_6,
\]

\[
\omega_3 < \omega_2 < \omega_4, \quad \text{(or } \omega_3 > \omega_2 > \omega_4),
\]

\[
\omega_8 < \omega_7 < \omega_9, \quad \text{(or } \omega_8 > \omega_7 > \omega_9),
\]

\[
\omega_{11} < \omega_{10} < \omega_{12}, \quad \text{(or } \omega_{11} > \omega_{10} > \omega_{12}).
\]

To summarize, in the highly symmetric tetragonal classes \(4mm, 422, 42m\) and \(4/m\), for the propagation direction along the tetragonal \(c\)-axis we have the three acoustic and nine optic waves, which depend on the 25 independent components of \(A\) (three of \(A\), seven each of \(A\) and \(B\) and four each of \(P\) and \(Q\)).
components and, more importantly, it is reduced by the pairs it depends on the components of an odd tensor and for the class 4 do not find it useful to pursue the matter any further.

for the eigencouples, these relations will be nearly useless given the total kinematical coupling. Therefore, we microvibrations problem of Section 3.2 and that, despite the fact that we can still write the explicit expressions is that there will be three acoustic and nine optic waves with cut-off frequencies given by the solution of the propagation condition is coupled as in the case of any crystal of the triclinic group. Hence, the best we can say problem fully coupled: further, for all but two classes, there is no great difference between the classes and the 3.3.2. Propagation orthogonal to the tetragonal c-axis. When the propagation direction is orthogonal to the tetragonal c-axis, we may assume in the constitutive relations (158), (159), (162), (164), (166), (170), (173) and (174) of [A] that \( m_1 = \cos \theta \), \( m_2 = \sin \theta \) and \( m_3 = 0 \). Inspection of these relations, however, shows that the tensor \( A(\theta) \) has 6 independent components, the tensor \( \tilde{A}(\theta) \) has 26 independent components and the tensor \( \tilde{\epsilon}^2 P + i \tilde{\epsilon} Q \) has 27 independent components for a generic value of \( \theta \). It is indeed this last block of \( A \) that makes the problem fully coupled: further, for all but two classes, there is no great difference between the classes and the propagation condition is coupled as in the case of any crystal of the triclinic group. Hence, the best we can say is that there will be three acoustic and nine optic waves with cut-off frequencies given by the solution of the microvibrations problem of Section 3.2 and that, despite the fact that we can still write the explicit expressions for the eigencouples, these relations will be nearly useless given the total kinematical coupling. Therefore, we do not find it useful to pursue the matter any further.

However, for the two centrosymmetric classes 4/m and 4/mmm the tensor \( P \) vanishes altogether because it depends on the components of an odd tensor and for the class 4/mmm the matrix \( A \) has only 41 independent components and, more importantly, it is reduced by the pairs \( L_1 = \text{span}(e_3) \oplus U_1 \) and \( L_2 = \{e_1, e_2\} \oplus U_1 \). In the subspace \( L_1 \) the fifth-order characteristic equation in \( \omega^2 \) depends on 15 independent components (which may reduce to 13 when either \( \theta = 0 \) or \( \theta = \pi/2 \)): therefore, rather than give the explicit solutions (which shall depend on relations more cumbersome than those we already obtained), we limit ourselves in this case to a qualitative analysis. The kinematics described by the five eigenvectors is composed by a macroscopic displacement directed as \( e_3 \) and a combination of the modes \( S_2 \) and \( R_3 \). In the limit \( \xi \to 0 \), we have a transverse acoustic wave and the microdeformations (103) of \([A]\).

Likewise, in the subspace \( L_2 \) we have seven eigenvectors representing a macroscopic displacement orthogonal to \( e_3 \) coupled with a combination of the \( D_1 \), \( S_1 \) and \( R_1 \) modes, which in the limit \( \xi \to 0 \) reduce to two acoustic waves (nor longitudinal, neither transverse) and to the microdeformations (101) of \([A]\) (or the corresponding highly symmetric classes). In fact, in the propagation orthogonal to the c-axis the role of the subspaces \( U_1 \) and \( U_2 \) associated with the microvibrations is exchanged with respect of the case \( m = e_3 \).

4. Conclusions

For a bulk three-dimensional crystalline body we modelled the interaction between macroscopic and lattice waves by means of a continuum with affine structure, or micromorphic. The propagation condition we obtained is described by a 12-dimensional Hermitian acoustic matrix \( A \), the eigenvector being elements of \( V \oplus \text{Lin} \) that represent three macroscopic displacements and nine lattice microdistortions. We showed that for such a propagation condition it admits three acoustic and nine optic waves for any crystalline symmetry; moreover, it depends on three length scales representing the crystal lattice inhomogeneity, the size of the bulk crystal and the effect of lattice inertia. In terms of these lengths and their ratios, we obtained two limit problems that recover the long-wavelength approximation and the microvibration problems, well-known and studied in detail for isotropic materials.
Then we investigated in full detail the propagation condition for crystals of the tetragonal point group: the matrix $A$ is reduced by two or three subspaces of $V \oplus \text{Lin}$, depending on the classes, whereas in the microvibrations limit case the propagation condition is determined by $B$, which is, in turn, reduced by two or three subspaces of Lin. It is remarkable that in the long-wavelength approximation the macroscopic displacements are real whereas the lattice microdistortions are purely imaginary: conversely, in the microvibration problem the matrix $A$ reduces to a real one and the lattice microdistortions are real. In both cases we find that the lattice modes can be described by the means of three kind of dilatations, two shear deformations and two rigid rotations.

When we deal with the complete propagation condition in the case of waves propagating along the tetragonal $c$-axis, in all but three cases the modes are complex and fully coupled macroscopic displacements with lattice microdistortions: only for the highly symmetric classes do we have three fully optic modes with real microdistortions without macroscopic displacements.

We finish by giving an insight for waves propagating in directions orthogonal to the $c$-axis, which, in a remarkable difference with the previous case, yields a fully coupled problem as in the fully anisotropic case.

As far as we know, this is the most complete analysis for the wave propagation problem in classical tetragonal micromorphic continua: of course, to make such an analysis a predictive tool for experimental applications, we need to know a complete set of parameters (43 for the low-symmetry classes 4, 4 and 4/m and 25 for the high-symmetry classes). These parameters, which are very difficult to obtain and evaluate by means of a set of simple experiments, can be obtained, at present, only by two viable approaches, that is the numerical homogenization approach used in [27] for relaxed micromorphic tetragonal continua undergoing plane strain, or by homogenization and identification methods from lattice dynamics, as was done for cubic diamond crystals and silicon into [16]. Such a technique shall be applied in a future paper to evaluate the components of the acoustic matrix $A$ for the tetragonal 4/m crystals of PbWO$_4$ (PWO), widely used in high-energy physics as ionizing radiation detectors.

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**Notes**

1. Indeed, in the absence of the microstructure, (19)$_1$ reduces to the acoustic tensor for linearly elastic bodies, see, e.g., [20, Section 70].
2. With the notation

\[ \langle a, b \rangle = \sum_{h=1}^{m} a_h b_h^*, \]

we denote the Euclidean inner product on an $m$-dimensional complex space $C^m$.
3. For the solution of (28) for the various symmetry groups, one can refer to [25] or [26].
4. The propagation problem in tetragonal materials was previously studied in [27]: however, their analysis, which concerns a relaxed micromorphic model rather than the classical model, was limited to two-dimensional plane strain; their main focus was indeed the parameter identification by means of numerical homogenization (see the comments at the end of Section 4).
5. See [A] for the tabular form.

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