Linear and Chiral Superfields
are Usefully Inequivalent

Tristan Hübsch
Department of Physics and Astronomy
Howard University, Washington, DC 20059
thubsch@howard.edu

ABSTRACT
Chiral superfields have been used, and extensively, almost ever since
supersymmetry has been discovered. Complex linear superfields afford
an alternate representation of matter, but are widely misbelieved to be
‘physically equivalent’ to chiral ones. We prove the opposite is true.
Curiously, this re-enables a previously thwarted interpretation of the
low-energy (super)field limit of superstrings.

Brevity is the soul of wit.
— W. Shakespeare

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0. Introduction

Chiral superfields and their conjugates have been used for over two decades [1-5]. They are defined by a system of two homogeneous, simple, first order superdifferential equations:
\[
[D_{\dot{\alpha}}, \Phi] = 0, \quad [D_{\alpha}, \bar{\Phi}] = 0,
\]
written here in $N=1$ supersymmetric 4-dimensional spacetime notation. The complex linear superfield [3,7], $\Theta$, and its conjugate, $\bar{\Theta}$, are defined to obey the homogeneous, single, second order superdifferential equations:
\[
[D^2, \Theta] = 0, \quad [D^2, \bar{\Theta}] = 0,
\]
where [1]:
\[
D^2 = D^\alpha D_{\alpha} = \epsilon^{\alpha\beta} D_{\beta} D_{\alpha}, \quad \text{and} \quad \bar{D}^2 = D_{\dot{\alpha}} \bar{D}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}} \bar{D}_{\dot{\beta}} \bar{D}_{\dot{\alpha}}.
\]
The complex linear superfield is also called ‘non-minimal chiral superfield’, as its defining equation is closely related to those of the chiral superfield [1]; in retrospect, $\Phi$ ought to be called ‘minimal chiral’ [8]. For the purposes of this short note, we adopt the shorter, ‘chiral’ and ‘linear’ names, and moreover use only complex such fields.

The simple Lagrangian densities
\[
L_{\Phi}^{(0)} = \int d^4\theta \, \Phi \bar{\Phi}, \quad \text{and} \quad L_{\Theta}^{(0)} = -\int d^4\theta \, \Theta \bar{\Theta},
\]
rewritten in terms of the component fields are in a 1–1 correspondence after the elimination of auxiliary component fields [2]. Even in the current literature, this fact has been misinterpreted into a ‘proof’ that chiral and linear superfields are physically equivalent.

Herein, we dispell such and related delusions by offering two simple proofs, which rely only on the definition of these superfields, and focus on two very distinct aspects of the inequivalence between them. These proofs encompass, mutatis mutandis, all of the ‘minimal’ and ‘non-minimal’ haploid superfields of Ref. [8].

1. 1st Proof: Correlation Functions

It is a rather well exploited fact that chiral superfields form a ring. In fact, any analytic function of a collection of chiral superfields is also a chiral superfield.

This, rather obviously, is not the case with linear superfields: no nonlinear function of even a single linear superfield is linear, i.e., obeys Eqs. (2). That is, the set of linear superfields is not closed under multiplication: the product $\Theta_1 \Theta_2$ does not obey Eq. (2).

This simple mathematical fact has an extraordinary consequence when comparing any two models using chiral vs. linear superfields. Supersymmetric correlation functions of chiral superfields are always independent of the (spacetime) position of the superfields [3]:
\[
\frac{\partial}{\partial x_1^m} \langle \Phi_1 \Phi_2 \rangle = \langle \Phi_1 \frac{\partial \Phi_2}{\partial x_1^m} \Phi_2 \rangle = \sum_{\alpha} \gamma_{\delta}^{\alpha\dot{\alpha}} \langle \bar{\Phi}_{\dot{\alpha}} [Q_{\alpha}, \Phi_1] \Phi_2 | \Phi_2 \rangle \equiv 0,
\]
where $[Q_{\alpha}, \Phi_1] \Phi_2 + [Q_{\alpha}, \Phi_1] \Phi_2 \bar{\Phi}_{\dot{\alpha}} \bar{\Phi}_{\dot{\alpha}} \Phi_2 | \Phi_2 \rangle \equiv 0$.
where $|\ast\rangle$ is any supersymmetric vacuum, annihilated by all the supercharges, and we used the defining equations (1). Thus, models involving chiral superfields always have a ‘topological’ (more properly, rigid) sector among their correlation functions.

By stark contrast and as a consequence of their differing definition only, no such rigidity obtains for supersymmetric correlation functions of products of linear superfields: models built with linear superfields in place of chiral ones have no such ‘topological’ sector.

As correlation functions determine a (quantum) model, it follows that models with linear superfields are fundamentally inequivalent to models with chiral superfields.

2. 2nd Proof: Gauge Interactions, in 2 Dimensions

Recall that Eqs. (1) and (2) hold also in 2-dimensions, but the Lorentz group now becoming $SL(2, \mathbb{C}) \rightarrow SO(1, 1) \approx U(1)$, we drop the over-dots on the spinorial indices.

Consider coupling these superfields to a gauge symmetry. As shown in Ref. [10], 2-dimensional supersymmetric theories admit a twisted version of the usual gauge vector multiplet the covariant (super)derivatives of which satisfy (6):

$$\{\nabla_\alpha, \nabla_\beta\} = 4i g \gamma^3_{\alpha\beta} P, \quad \{\nabla_\alpha, \nabla_\beta\} = 2i \gamma^\alpha_\beta \nabla \hat{m},$$

where $P$ is the Lie algebra valued superfield the superderivatives of which provide the twisted versions of the usual fermionic and bosonic field strengths and the auxiliary field:

$$[\nabla_\alpha, P] = \tilde{W}_\alpha, \quad [\nabla_\alpha, \tilde{W}_\beta] = \epsilon_{\alpha\beta}(\tilde{F} + i\tilde{D}).$$

In the presence of a gauge symmetry, the definitions (1) and (2) need to be modified into

$$[\nabla_\alpha, \Phi] = 0, \quad [\nabla_\alpha, \Phi] = 0,$$

and

$$[\nabla^2, \Theta] = 0, \quad [\nabla^2, \tilde{\Theta}] = 0.$$

Note that, as defined above, $P$ automatically satisfies the latter of Eqs. (6); its conjugate, $\tilde{P}$ is then chiral, whence we call $\{P, \tilde{W}_\alpha, \tilde{F}, \tilde{D}\}$ the ‘chiral vector multiplet’. The dimensionally reduced standard gauge multiplet is likewise determined from a twisted chiral superfield $[\Sigma, \tilde{W}_\alpha, \tilde{F}, \tilde{D}]$ the ‘twisted gauge vector multiplet’.

There is an ‘integrability’ condition one can derive for the chiral superfields $[\hat{P}, \hat{\tilde{P}}, \hat{\tilde{W}_\alpha}, \hat{\tilde{F}}, \hat{\tilde{D}}]$ by anticommuting $[\nabla_\alpha, \Phi]$ in (5)with $\nabla_\beta$, and antisymmetrizing:

$$0 = \{\nabla_\alpha, [\nabla_\beta, \Phi]\} + \{\nabla_\beta, [\nabla_\alpha, \Phi]\} = \{[\nabla_\alpha, \nabla_\beta],\Phi\} \overset{\text{def}}{=} 4i g \gamma^3_{\alpha\beta}[P, \Phi].$$

This implies that covariantly chiral superfields (5) must be chargeless with respect to the gauge group generated by the chiral gauge vector multiplet $\{P, \hat{W}_\alpha, \hat{F}, \hat{D}\}$.

Again, as a consequence of their differing definition only, no such restriction obtains for the covariantly linear superfield (5).

The ‘mirror’ of this argument proves that superfields which are covariantly twisted-chiral with respect to the twisted gauge multiplet must also be chargeless with respect to this gauge symmetry.
To sum up, covariantly chiral superfields couple to the twisted but not to the chiral
gauge multiplet (1), while covariantly twisted-chiral superfields couple to the chiral but not
to the twisted vector multiplet. By stark contrast, both linear superfields and their ‘mirror’-
twisted brethren couple indiscriminately to both chiral and twisted gauge multiplets, and
so cannot possibly be equivalent to the discriminately coupling (twisted-)chiral ilk.

3. An Old Proof Opens New Possibilities

An unassuming remark in Ref. [2], p. 200, states: “it is not possible to introduce arbitrary
mass and nonderivative self-interaction terms” for the linear complex superfield.

This is clear from the fact that the definition of a chiral superfield (1) ensures that its
most general Lagrangian density takes the form:

\[ L_\Phi = \int d^4\theta \ K(\Phi, \bar{\Phi}) + \int d^2\theta \ W(\Phi) + \text{h.c.} \] (11)

whereas the definition of a linear superfield (2) restricts its most general Lagrangian to:

\[ L_\Theta = \int d^4\theta \ f(\Theta, \bar{\Theta}) + \text{h.c.} \] (12)

Besides the sophisticated consequences for renormalization\(^1\), an important difference be-
tween (11) and (12) is based on dimensional grounds only: when restricted to be renor-
malizable, in 4 dimensions, \(L_\Theta\) admits no mass parameter and must be quadratic!

This proves that renormalizable models with (only) linear superfields cannot describe
massive fields with Yukawa interactions, whereas renormalizable models with chiral super-
fields can. The two then cannot possibly be equivalent—even in the free field limit!

This brings about an interesting possibility. Recall that the effective low-energy (su-
per)field limit of superstring theories [11,12] is identified based only on physical (propagat-
ing) degrees of freedom. It is then logically possible that the limit in fact ought to involve
linear instead of chiral superfields. If so, matter fields descending from superstrings would
be massless and have only gauge interactions. This is precisely as needed for the models
discussed in Ref. [13], which when using chiral superfields end up thwarted mainly by
Yukawa interactions.

Note also that the combined use of chiral and linear superfields enables the description
of phenomena not describable by the use of either one of the superfields alone [14]. Indeed,
owing to no self-interaction, linear superfields are well adapted to describe the flatness of (co)tangent spaces. This is not unlike the richness in 2-dimensional (super)spacetime,
made possible by the joint use of chiral and twisted-chiral superfields in Ref. [15], and a
far greater richness made possible by the use of many other types of superfields [8].

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\(^1\) \(\int d^4\theta\)-terms notoriously undergo renormalization, and generally in a complicated fashion,
whereas \(\int d^2\theta\)-terms typically do not: see p. 358 of Ref. [2].
References

[1] J. Wess and J. Bagger: *Supersymmetry and Supergravity* (Princeton University Pub., Princeton NJ, 1983).

[2] S.J. Gates, Jr., M.T. Grisaru, M. Roček and W. Siegel: *Superspace* (Benjamin/Cummings Pub. Co., Reading, Massachusetts, 1983).

[3] P. West: *Introduction to Supersymmetry and Supergravity* (World Scientific, Singapore, 2nd ext, ed.: 1990).

[4] H. Muller-Kirsten and A. Wiedmann: *Supersymmetry: An Introduction With Conceptual and Calculational Details* (World Scientific, Singapore, 1987).

[5] I.L. Buchbinder and S.M. Kuzenko: *Ideas and Methods of Supersymmetry and Supergravity : Or a Walk Through Superspace*, (IOP, Bristol, 1998).

[6] S.J. Gates, Jr. and W. Siegel: *Nucl. Phys.* B187 (1981)389.

[7] S.J. Gates, Jr. and B.B. Deo: *Nucl. Phys.* B254 (1985)187-200.

[8] T. Hübsch: Haploid (2,2)-Superfields In 2-Dimensional Spacetime. hep-th/9901038.

[9] D. Amati, K. Konishi, Y. Meurice, G.C. Rossi and G. Veneziano: *Phys. Rep.* 162C (1988)170, and references therein.

[10] S.J. Gates, Jr.: *Phys. Lett.* B352 (1995)43–49.

[11] M.B. Green, J.H. Schwarz and E. Witten: *Superstring Theory II* (Cambridge University Press, Cambridge, 1987).

[12] T. Hübsch: *Calabi-Yau Manifolds—A Bestiary for Physicists* (World Scientific, Singapore, 1992).

[13] T. Hübsch, H. Nishino and J.C. Pati: *Phys. Lett.* 163B (1985)111.

[14] S.J. Gates, Jr. and Sergei M. Kuzenko: The CNM-Hypermultiplet Nexus. hep-th/9810137.

[15] S.J. Gates, Jr., C.M. Hull and M. Roček: *Nucl. Phys.* B248 (1984)157.