On the duality of space-trusses and plate structures of rigid plates and elastic edges

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January 12, 2022

Abstract

Dualities have been known to map space trusses and plate structures to each other since the 1980-s. Yet the computational similarity of the two has not been used to solve the unfamiliar plate structure with the methods of the well known truss. This paper gives a method to find the forces and displacements of a plate structure with rigid plates and elastic edges, using a dual truss. It is applicable for both statically determinate and indeterminate structures, subjected to both statical and kinematical loads.

1 Introduction

Dualities as an articulated projective geometrical concept have emerged in the field of statics with Maxwell\cite{1} and Cremona\cite{2}, and were expanded at that time by Klein and Wieghardt\cite{3}. While these methods were able to give one the forces of a (planar) truss, with the spread of algebraic methods they were almost forgotten in engineering circles. It was in the 1980-s when the renaissance of projective geometry in structural engineering started, discovering that the rigidity of a truss is a projective invariant \cite{4,5}. Investigation of the duality of engineering structures followed, both in 2D between plane trusses and grillages\cite{6,7,8} and in 3D between spatial trusses and plate (sheet) structures\cite{9}. Parallel to this scientific renaissance, danish architect Wester\cite{10,11} began researching and popularising plate structures for their efficiency and clarity ("pure plate action", in his words). While to this day very few homogeneous, cast concrete plate structures have been built, with the spread of automation into construction the use of smaller, prefabricated plates forming a spatial structure is getting more obvious and economical. This motivates researchers to study, for instance, glass\cite{12} and wooden\cite{13} plate structures. New ways are being tested to connect the prefabricated plates on-site. In case of glass this is usually done by glueing\cite{14,15,16}, while wooden elements easily allow the use of traditional finger joints and mechanical elements as well\cite{17,18}. In any case the on-site joints between the prefabricated elements are generally softer and they are generally responsible for the majority of the displacements occurring in the structure. This motivates the choice to model such structures with rigid plates connected by elastic joints.

The continuation of the research on projective geometry turned back to the path set by Maxwell, with its sight set on geometric representation of forces of spatial structures\cite{19,20} and the extension of such diagrams to kinematics of the structures\cite{21,22,23}. While La Magna et. al.\cite{24} mention the possibility of using a dual truss structure to compute the edge forces of a plate structure, a method to do this in the general case of a possibly statically indeterminate (hyperstatic) structure subjected to both statical and kinematical loads is hitherto undeveloped.
2 Notation and modelling

2.1 Trusses and plate structures

Here a quick review is given on the statistical and compatibility equations of trusses and plate structures, presenting the similarity of the two structures and to provide the objects of comparison.

2.1.1 The well known model of a truss

is comprised of pin- or ball-joints and elastic bars connecting them. Loads are restricted to concentrated forces acting on the joints. The system of statical and compatibility equations take the form\(^{[25, 26]}\) of:

\[
\begin{bmatrix}
0 & G' & [\epsilon] \\
G'^T & F' & [\phi'] \\
& & [\pi']
\end{bmatrix}
+ \begin{bmatrix}
0 \\
[\pi]'
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}.
\]

(1)

The upper block contains the equilibrium equations of the joints in global frames, while the lower block contains the compatibility equations of the bars in their local (1 dimensional) frames. In the case of a space-truss with \(n\) joints and \(l\) bars, \(G' \in \mathbb{R}^{3n \times l}\) is the geometrical matrix while matrix \(F' \in \mathbb{R}^{l \times l}\) is in fact diagonal and contains the stiffnesses of the bars on its main diagonal. The vector \(\epsilon \in \mathbb{R}^{3n}\) contains the displacements of the joints such that

\[
\epsilon = \begin{bmatrix}
\epsilon_1 \\
\vdots \\
\epsilon_n
\end{bmatrix}
\text{ and } 
\epsilon_i = \begin{bmatrix}
\epsilon_{i,1} \\
\epsilon_{i,2} \\
\epsilon_{i,3}
\end{bmatrix}.
\]

(2)

Here \(\epsilon_i\) is the 3 dimensional displacement of joint \(i\). Vector \(\pi'\) contains the loads of the vertices similarly. Vector

\[
\phi' = \begin{bmatrix}
\phi'_1 \\
\vdots \\
\phi'_l
\end{bmatrix}
\]

contains the bar forces, \(\phi_i\) being the force in bar \(i\).

2.1.2 A common model of a plate structure

is comprised of plates, subjected only to loads in their planes. In the following, the plates are considered to be rigid against forces acting in their planes, while their joints with other plates are considered to be elastic. The plates are considered to be so soft perpendicular to their planes that deformations in this direction happen without significant resistance. Hence only the displacements happening in the plane of the plates are relevant (cause forces) and the forces between two plates act such that they are in the common plane of both the plates they join.

Proposition 1. With the proper choice of equations and parameters the system of equilibrium and compatibility equations of a plate structure can be cast in the form:

\[
\begin{bmatrix}
0 & G' & [\delta] \\
G'^T & F & [\phi] \\
& & [\pi]
\end{bmatrix}
+ \begin{bmatrix}
0 \\
[\pi]'
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}.
\]

(4)

The top block contains the moment equations of the plates (in a coordinate system specified later) and the bottom block contains the compatibility equations of the edges in their local 1 dimensional coordinate systems. Here also \(G\) is the geometrical matrix, \(F\) is the stiffness matrix of the elastic joins, \(\phi\) contains the edge-forces with which the plates support each other, \(\delta\) contains the small displacements (rotations) of the plates and \(\pi\) contains the loads. While \(G\) is more intuitive, \(F\) might need to be clarified. If along edge \(k\) two plates are joined with relative deformation \(t_k\) (measured in the direction of the edge as translation) then the arising force \(\phi_k\) acting on both plates is such that \(t_k = F_{k,k} |\phi_k|\) holds. The comparison of the compatibility conditions in case of a truss and the plate structure can be seen in Figure 1.

The similarity of (1) and (4) suggests that analyzing a truss or a plate structure are mathematically equivalent. This enables one to determine the behaviour of a plate structure by investigating a properly chosen truss. This paper will show how this choice can be methodized with the canonical duality of projective geometry. The steps of the method are

1. Determine the geometry and stiffness of the dual truss
2. Determine the dual loads on the dual truss
3. Solve the dual truss
4. Transform the solution back to the plate structure

While projective geometry naturally gives duals of points, lines and planes, the transformation of forces, displacements and the stiffness matrix needs to be worked out. In this paper transformation matrices $\Delta$, $\Phi$ and $\Pi$ will be presented satisfying

$$\delta = \Delta \epsilon$$
$$\phi = \Phi \phi'$$
$$\pi = \Pi \pi'$$
$$F = \Phi^{-1} F' \Phi^{-1}.$$  

2.2 Review of projective geometry and screw theory

We will make use of concepts from projective geometry as well as screw theory. For the sake of clarity, they are provided here. Let us start by defining the following equivalence relation on a vector space $\mathcal{V}$:

$$x \sim y \iff y \in \{ \lambda x \mid \lambda \in \mathbb{R} \setminus \{0\} \} \quad (x, y \in \mathcal{V}).$$

Let us denote the arising equivalence class of $x$ with $x_\sim$. This notion enables us to assign (homogenous) coordinates to points, planes and lines of the projective space as follows.

2.2.1 Homogenous coordinates of planes and points

can be attained by setting the dimension of the vectors in (9) to be 4. Each equivalence class uniquely represents a point, and all points are represented this way. Although the projective space does not discriminate between its points, in human thought we often think of the three dimensional projective space ($PG(3)$) as a union of Euclidian space ($\mathbb{E}^3$) and the ideal plane at infinity (which is $PG(2)$ essentially). We will use the convention that if $p \in \mathbb{E}^3$ we will use...
the representant of form \((1, p)\), while in case of ideal points all representants are of form \((0, \lambda p)\). If no distinction is to be made we will use the form \((p_0, p)\)~.

Similarly, each plane is represented with a single equivalence class and each equivalence class represents a unique plane. Denoting the standard inner product with \(\langle \cdot , \cdot \rangle\), the incidence relations take the following form: point \(P\) with coordinates \(p_\sim\) lies in plane \(S\) with coordinates \(s_\sim\) if and only if \(\langle p_\sim, s_\sim \rangle = 0\) holds. As it can be seen, different choice of representant from the same equivalence class does not affect this relation.

### 2.2.2 Homogenous coordinates of lines

are attained in a slightly different way. Let us have \(l = (l_1, l_2, l_3)\), \(\bar{l} = (l_4, l_5, l_6)\) such that the tuple \((l, \bar{l}) \in \mathbb{R}^6\) is not a 0 vector. The equivalence class \((l, \bar{l})_\sim\) represents a line of \(PG(3)\) if and only if

\[
\langle l, \bar{l} \rangle = l_1 l_4 + l_2 l_5 + l_3 l_6 = 0 \quad \tag{10}
\]

holds, and all lines of \(PG(3)\) are represented this way. This way of representation is called the Plücker-coordinates of a line and equation (10) the Plücker-identity.

Here it is convenient to present a pair of additional identities, see Pottmann and Wallner [27] for more details:

Coordinates of the line passing through points \((p_0, p)\)~ and \((q_0, q)\)~ are given by:

\[
(l, \bar{l})_\sim = (p_0 q - q_0 p, p \times q)_\sim \quad \tag{11}
\]

This returns 0, iff the two points coincide. Similarly, the common line of planes \((u_0, u)\)~ and \((v_0, v)\)~ are given by

\[
(l^*, \bar{l}^*)_\sim = (u \times v, u_0 v - v_0 u)_\sim \quad \tag{12}
\]

which also gives 0 iff the two planes are identical.

Observing equations (11) and (12) shows that any line passing through a given point arise from the linear combination of 3 distinct lines passing through the point, and any line lying in a plane arises as a linear combination of 3 distinct lines lying in the plane.

### 2.2.3 Dualities

are incidence preserving one-to one maps between points and planes of \(PG(3)\). With the interchange of incidence relations (line joining points - line in which planes intersect) they induce a corresponding one to one mapping on lines. Dualities with period 2 are called polarities and have often been used in graphi-}

\[
\text{cal statics before}[1, 2]. \text{ The effect of any duality on homogenous coordinates can be represented with an invertible matrix equivalence class } M_\sim, \text{ such that the dual of point } (p_0, p)\sim \text{ is plane } (p_0, p)M_\sim, \text{ and the dual of plane } (s_0, s)\sim \text{ is point } (p_0, p)M_\sim^T. \text{ A corresponding linear map can be constructed acting on the Plücker-coordinates of lines.}

The ‘canonical’ duality is the one represented with the 4 dimensional identity matrix (its equivalence class), mapping point \((p_0, p)\)~ into plane \((p_0, p)\)~ and plane \((p_0, p)\)~ into point \((p_0, p)\)~. As it can be seen, it has period 2 thus it is a polarity as well. (In a more geometric approach it can be thought of as being represented by the purely complex unit sphere in the complex projective space.) The induced line-to line mapping in this case is given as \((l, \bar{l})_\sim \mapsto (\bar{l}, l)_\sim\).

So far we have homogenous coordinates, but we would like to do mechanics with metric quantities. In order to have some, let us introduce the notion of the oriented line segment (an idea is dating back to at least Klein[28]).

**Definition.** An oriented line segment \((l, \bar{l})_\sim\) is a sliding-vector bound to line \((l, \bar{l})_\sim\), with length \(\| (l, \bar{l}) \|\) and a direction, which are given by:

- if \(\| l \| > 0\), then \(\| (l, \bar{l}) \| = \| l \|\) and the direction is given by \(l\) (finite line).
- if \(\| l \| = 0\), then \(\| (l, \bar{l}) \| = \| \bar{l} \|\) and the direction is given by \(\bar{l}\) (ideal line).

We can now give the effect of the duality on this metric quantity similarly to the line, as:

\[
(l, \bar{l})_\sim \mapsto (\bar{l}, l)_\sim\quad \tag{13}
\]
2.2.4 Screw theory

was originally proposed by Sir Robert Ball[29], providing a connection between Plücker coordinates of lines and force/velocity/displacement systems. Here a quick excerpt is presented, the reader may find the modern (engineering) interpretation in Davidson and Hunt[30] or Gallardo-Alvarado[31]. For the mathematical minded, there is Pottmann and Wallner[27] and even Felix Klein[28].

The effect of system of forces and moments in $\mathbb{E}^3$ can be given (after a choice of coordinate system) as the vector couple $(R,T)$, where $R$ is a force acting at the chosen origin and $T$ is a moment both containing the original moments and accounting for the translation of the elements of the force system into the chosen origin. Generally this vector couple is called a dyname or wrench. In the special case of the translation of the elements of the force system containing the original moments and accounting for acting at the chosen origin and as the vector couple can be given (after a choice of coordinate system) providing a connection between Plücker coordinates of lines and force/velocity/displacement systems.

For the mathematical minded, there is Pottmann and Wallner[27] and even Felix Klein[28].

The following operation will be used to describe the relation of forces and the displacements they cause:

$$\langle l, \bar{l} \rangle \circ (m, \bar{m}) := \langle l, \bar{m} \rangle + (\bar{l}, m).$$ (16)

From the mathematical point of view, this is an indefinite inner product, satisfying linearity, commutativity but not positive definiteness. Also, (different) lines $(l, \bar{l})$ and $(m, \bar{m})$ intersect if and only if $(l, \bar{l}) \circ (m, \bar{m}) = 0$. From the physical point of view this operation can represent many things. The relevant ones are:

- The quantity $\frac{\|F\|}{\|(l, \bar{l})\|} (l, \bar{l}) \circ (m, \bar{m})$ is the moment of force $\frac{\|F\|}{\|(l, \bar{l})\|} (l, \bar{l})$ to the axis $(m, \bar{m})$.
- Given a rigid body rotated with $\frac{\alpha}{\|\bar{m}, m\|} (m, \bar{m})$, the quantity $\frac{\|\bar{m}, m\|}{\|(l, \bar{l})\|} (m, \bar{m}) \circ (l, \bar{l})$ is the $(l, \bar{l})$ directional component of the displacement of all points of the body lying on $(l, \bar{l})$.

Note how the canonical duality satisfies the following: Given two line segments $(l, \bar{l})$ and $(m, \bar{m})$, as well as their duals $(l, \bar{l})'$ and $(m, \bar{m})'$, we have:

$$(l, \bar{l}) \circ (m, \bar{m}) = (l, \bar{l})' \circ (m, \bar{m})'$$ (17)

which will greatly simplify our analysis.

2.3 Bases and dual bases

Due to the linear nature of the problem, the effect of the duality can be described with its effect on appropriately chosen bases. First a plate in plane $P$ and joint at dual point $P'$ will be considered. Let us have an orthonormal base in the finite part of $PG(3)$, given by $b_1, b_2$ and $b_3$, such that the origin is neither
$\mathcal{P}'$ nor lying in $\mathcal{P}$. With this restriction both point and plane can be represented with the same vector: $(1, n)$. We will also consider the additional condition of $b_1 \parallel n$, and we will take note of what it changes. Now two pairs of bases can be created, bound to $\mathcal{P}$ and $\mathcal{P}'$ respectively.

The first one will be spanned by $d_1$, $d_2$ and $d_3$, where

$$d_j := (\langle n, b_j \rangle n, n \times b_j) \quad j = 1 \ldots 3.$$  

(18)

All line segments perpendicular to $\mathcal{P}$ can be given as a linear combination of these, since they all pass through $(0, n)$ and are linearly independent. Thus any rotation (or translation) of $\mathcal{P}$ in ideal points, thus pure translations of the plate in the plane are represented with linear combinations where $\mu_1 = 0$.

The next base will be spanned by $f_1$, $f_2$ and $f_3$, where

$$f_j := (n \times b_j, b_j) \quad j = 1 \ldots 3.$$  

(20)

Any oriented line segment lying in plane $\mathcal{P}$ can be given as a linear combination of these. The dual base for the displacements of the joint at $\mathcal{P}'$ will be \{d'_j\} where

$$d'_j := (n \times b_j, \langle n, b_j \rangle n) \quad j = 1 \ldots 3.$$  

(23)

They all lie in a plane passing through the origin, perpendicular to $n$. Any displacement of $\mathcal{P}$ can be expressed with the help of these, such that:

$$\sum_{j=1 \ldots 3} \frac{\mu_j}{\|d'_j\|} d'_j \mid \mu_j \in \mathbb{R}.$$  

(24)

To see that this indeed spans $\mathbb{E}^3$ and how the displacements scale one can express the translation caused at $\mathcal{P}'$ as

$$\sum_{k=1 \ldots 3} b_k \sum_{j=1 \ldots 3} \frac{\mu_j}{\|d'_j\|\|f'_k\|} d'_j \circ f'_k.$$  

(25)

Where

$$d'_j \circ f'_k = (\langle n \times b_j, (n \times b_k) \rangle + \langle n, b_j \rangle \langle n, b_k \rangle - \langle n, b_j \rangle \langle n, b_k \rangle + \langle n, b_j \rangle \langle n, b_k \rangle).$$  

(26)

(27)

This, the orthonormality of \{b_j\} and the fact that $\forall j \|f'_k\| = 1$ shows that the translation of $\mathcal{P}$ from the rotation described in (24) is

$$\|n\|^2 \sum_{j=1 \ldots 3} \frac{\mu_j}{\|d'_j\|} b_j.$$  

(29)

With this and (17) we have also discovered which base vector (line segment) intersects which one. Here the satisfaction of the additional condition of $b_1 \parallel n$ implies that $d'_1$ lies at infinity. A drawing of the bases is presented in Figure 2 in the general case and in Figure 3 if $b_1 \parallel n$ is satisfied.

Remark. The restriction of the origin of \{b_j\} not being a vertex of the truss or in a plane of a plate is a necessary condition of having two finite dual structures.
Figure 2: left: Bases associated with plane \((1, \bm{n})\). Elements of \(\{f_j\}\) are lying in it while elements of \(\{d_j\}\) are perpendicular to it. right: Dual bases, associated with point \((1, \bm{n})\). Elements of \(\{f'_j\}\) are passing through it, while elements of \(\{d'_j\}\) lie on a plane passing through the origin of \(\{b_j\}\).

Figure 3: Bases an dual bases if \(b_1 \parallel \bm{n}\) is satisfied. Some line segments are drawn twice, their apparently opposite direction is attributed to the fact that all lines are loops in \(PG(3)\). Left: bases associated with plane \((1, \bm{n})\). Line segment \(f_1\) is along the ideal line of the plane, accounting for the moments in the plane. Line segments \(d_2\) and \(d_3\) also lie in infinity, representing pure translations of the plane. Right: dual bases, associated with point \((1, \bm{n})\). Line segment \(d'_1\) lies along an ideal line.
2.4 Extension to multiple plates and joints

Trusses and plate structures consist of multiple plates/joints and edges/bars. The vector representing plane \( P_i \) and point \( P_i' \) will be denoted with \((1, n_i)\). Consequently there will be multiple bases attached to points and planes, for instance the \( j \)-th element of the base for forces attached to \( P_i \) will be denoted with \( f_{i,j} \).

The extension of the additional condition on base directions takes the form of \( b_{i,1} \parallel n_i \) for all \( i \). Moreover, a line segment along edge \( e_k \) and dual bar \( e'_k \) will be denoted with \( e_k \) and \( e'_k \) respectively.

Now the validity of Proposition 1 can be shown. If edge force \( \phi_k \) acts along edge \( e_k \), its moment to axis \( d_{i,j} \) is given by

\[
\phi_k \frac{e_k \circ d_{i,j}}{\|e_k\| \|d_{i,j}\|}.
\]

Similarly, the \( e_k \) directional component from the small rotation \( \delta_{i,j} \) around axis \( d_{i,j} \) is

\[
\delta_{i,j} \frac{e_k \circ d_{i,j}}{\|e_k\| \|d_{i,j}\|}.
\]

It is apparent that if we want to use the same matrix (although transposed) in both the equilibrium and the compatibility equations, the required equilibrium equations are the moment equations around axes \( d_{i,j} \). By introducing the notation \( r = 3(i - 1) + j \), the elements of \( G \) can be given as

\[
G_{r,k} = \begin{cases} \frac{e_k \circ d_{i,j}}{\|e_k\| \|d_{i,j}\|} & \text{if } e_k \text{ is an edge of the plate } P_i \\ 0 & \text{otherwise.} \end{cases}
\]

In the following we will describe how statics and kinematics of a plate structure translate to statics and kinematics of the dual truss.

3 Finding the transformation matrices

3.1 Transformation of forces

Consider plate \( P_i \) acted upon by force

\[
\sum_{j=1}^{3} \pi_{i,j} \frac{f_{i,j}}{\|f_{i,j}\|}
\]

and supported along its \( e_k \) edges with \( \phi_k \) edge forces \((k \in \mathcal{I} \text{ some index-set})\). The equilibrium of forces is captured with the equation:

\[
\sum_{j=1}^{3} \pi_{i,j} \frac{f_{i,j}}{\|f_{i,j}\|} + \sum_{k \in \mathcal{I}} \phi_k \frac{e_k}{\|e_k\|} = 0.
\]

In case of the dual vertex at point \( P'_i \) the equilibrium of the dual forces and the dual load gives the equation:

\[
\sum_{j=1}^{3} \pi'_{i,j} \frac{f'_{i,j}}{\|f'_{i,j}\|} + \sum_{k \in \mathcal{I}} \phi'_{k} \frac{e'_k}{\|e'_k\|} = 0.
\]

Both (34) and (35) are linear combinations where the corresponding vectors \((f_{i,j} \text{ to } f'_{i,j} \text{ and } e_k \text{ to } e'_k)\) differ only in a common permutation. This implies that given the additional constraints

\[
\pi_{i,j} \frac{f_{i,j}}{\|f_{i,j}\|} = \pi'_{i,j} \frac{f'_{i,j}}{\|f'_{i,j}\|} \text{ and } \phi_k \frac{e_k}{\|e_k\|} = \phi'_{k} \frac{e'_k}{\|e'_k\|}
\]

the two equilibrium equations are equivalent. Since this is precisely what we want, we determine \( \Pi \) and \( \Phi \) such that (36) is satisfied. Both \( \Phi \) and \( \Pi \) are diagonal, the \( k,k \)-th element of \( \Phi \) is

\[
\Phi_{k,k} = \frac{\|e_k\|}{\|e'_k\|},
\]

while \( \Pi \) is more conveniently given in block-diagonal form as

\[
\Pi = \begin{bmatrix} \Pi_1 & \cdots & \Pi_i & \cdots & \Pi_n \end{bmatrix}
\]
Similarly to the case of the forces, these are linear combinations in which the vectors differ only in a common permutation. Furthermore, the method we want to create is good if it maps unit displacement to unit displacement as:

$$\frac{1}{\|d_{i,j}\|} \rightarrow \frac{1}{\|d'_{i,j}\|}$$

implying

$$\delta_{i,j} = \frac{\|d_{i,j}\|}{\|n_{i}\|^2} \epsilon_{i,j}. \tag{46}$$

From this we see that the matrix \(\Delta\) can also be given in block-diagonal form, as

$$\Delta = \begin{bmatrix} \Delta_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Delta_n \end{bmatrix} \tag{47}$$

where

$$\Delta_i = \frac{1}{\|n_{i}\|^2} \begin{bmatrix} \|d_{i,1}\| \\ \|d_{i,2}\| \\ \vdots \\ \|d_{i,3}\| \end{bmatrix}. \tag{48}$$

While we see that \(\Phi\) is determined only by the geometry of the structure and the center of the duality (the origin of the coordinate system in this case), \(\Pi\) and \(\Delta\) are sensitive to directions of \(b_{i,j}\). More precisely, if the additional condition of \(b_{i,1} \parallel n_i\) holds, then \(\Pi^{-1} = \Delta\) is true.

### 3.3 Transformation of the system of equations

Now we can examine how and when the systems of equations can be transformed into each other. In case of the equilibrium equations, we can write
\[ G\phi + \pi = 0 \]  
\[ G\Phi\phi' + \Pi\pi' = 0 \]  
\[ \Pi^{-1}G\Phi\phi' + \pi' = 0 \]

implying
\[ G' = \Pi^{-1}G\Phi. \]  

In case of the compatibility equations, this is
\[ G^T\delta + F\phi = 0 \]  
\[ G^T\Delta\epsilon + F\Phi\phi' = 0 \]  
\[ \Phi G^T\Delta\epsilon + \Phi F\Phi\phi' = 0, \]

which would imply
\[ G' = \Delta G\Phi \]  
\[ F' = \Phi F\Phi. \]

Two things are noteworthy. Firstly, the connection between \( F \) and \( F' \) is determined only by the geometry of the structure and the center of the polarity, not the choice of bases. Secondly, if the bases are chosen such that \( b_{i,1} \parallel n_i \) (or \( b_{i,j} \parallel n_i \) for all \( i \) and any \( j \)) is satisfied, then there is no contradiction between (52) and (56) and the whole system of equations can be transformed.

This condition however is not necessary to use this method. In fact it might be useful to use bases given by assuming \( b_{i,j} = b_{k,j} \forall i, k \in \{1 \ldots n\} \), resulting in the frame at each joint having the same orientation. The matrices \( \Pi, \Delta \) and \( \Phi \) can be constructed as given, and they can be used to transform the loads and the stiffness matrix. The dual truss can be solved with whatever equations one likes, and the resulting forces and displacements can be transformed back.

4 A numerical example

Consider the plate structure seen in Figure 4. The vertical plates are fixed in their planes supporting plate 5 in a statically indeterminate way. The force \( \pi_5 \) is of magnitude 1 kN, and there is an additional support displacement: plate 1 is rotated around the \( x \) axis with 1/10 degrees (0.0017 radians).

The geometry of the plates is given by vectors:
\[ (1, n_1) = (1, -1, 0, 0) \]  
\[ (1, n_2) = (1, -1, -1, 0) \]  
\[ (1, n_3) = (1, 1, -1, 0) \]  
\[ (1, n_4) = (1, 1, 0, 0) \]  
\[ (1, n_5) = (1, 0, -0.2, -0.4). \]

The plates are assumed to be glued together with polyurethane glue in 1 mm thickness and in 2 cm width. By assuming the glue has shear modulus of 1 N/mm² and after working out the edge lengths the stiffness matrix takes the form of
\[ F = \begin{bmatrix} 0.4472 & 0 & 0 & 0 \\ 0 & 0.3333 & 0 & 0 \\ 0 & 0 & 0.3333 & 0 \\ 0 & 0 & 0 & 0.4472 \end{bmatrix} \times 10^{-7} \text{ m kN}^{-1}. \]  

The internal edges will be numbered such that edge \( k \) joins plate \( k \) with plate 5, this way the edges can
be represented with
\[ e_1 = (0, -0.4, 0.2, 1, -0.2, -0.4) \]  
\[ e_2 = (0.4, -0.4, 0.2, 1, 0.8, -0.4) \]  
\[ e_3 = (0.4, 0.4, -0.2, -1, 0.8, -0.4) \]  
\[ e_4 = (0, 0.4, -0.2, -1, -0.2, -0.4), \]

and from this the matrix relating bar and edge forces is:
\[ \Phi = \begin{bmatrix} 0.4082 & 0 & 0 & 0 \\ 0 & 0.4472 & 0 & 0 \\ 0 & 0 & 0.4472 & 0 \\ 0 & 0 & 0 & 0.4082 \end{bmatrix}. \]  

From this one can have the stiffness matrix of the dual truss as:
\[ F' = \begin{bmatrix} 0.7454 & 0 & 0 & 0 \\ 0 & 0.6667 & 0 & 0 \\ 0 & 0 & 0.6667 & 0 \\ 0 & 0 & 0 & 0.7454 \end{bmatrix} 10^{-8} \text{ m kN}. \]  

The decomposition of the loads into the appropriate bases gives
\[ \delta_1 = \begin{bmatrix} 0.0017 \\ 0 \\ 0 \end{bmatrix} \text{ rad and } \pi_5 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \text{ kN}, \]

which can be transformed to truss loads as
\[ \pi_5' = \Pi_5^{-1} \pi_5 = \begin{bmatrix} -2.2361 \\ 0 \\ 0 \end{bmatrix} \text{ kN} \]

and
\[ \epsilon_1 = \Delta_1^{-1} \delta_1 = \begin{bmatrix} 0.0017 \\ 0 \\ 0 \end{bmatrix} \text{ m}. \]

Now everything is given in the problem of the dual truss, which can be seen in Figure 5, along with the directions of the arising bar forces as well.

Solving the dual truss gives
\[ \epsilon_5 = \begin{bmatrix} 0.0005 \\ -0.0009 \\ -0.0017 \end{bmatrix} \text{ m and } \phi' = \begin{bmatrix} -4.5643 \\ 5.5899 \\ -4.5643 \end{bmatrix} 10^4 \text{ kN}, \]

which can be transformed back to the plate structure giving
\[ \delta_5 = \begin{bmatrix} 0.0025 \\ -0.0010 \\ -0.0078 \end{bmatrix} \text{ and } \phi = \begin{bmatrix} -1.8634 \\ 2.4999 \\ -2.4999 \\ 1.8634 \end{bmatrix} 10^4 \text{ kN}. \]
Since the bases and dual bases are not normed, the actual rotation of $P_5$ is not $\|\delta_5\|$, but one has to compute

$$d_5 = 0.0175d_{5,1} - 0.0195d_{5,2} - 0.1561d_{5,3}$$

$$= (0, -0.0007, -0.0013, 0.0012, -0.0010, 0.0005) .$$

(86)

The norm of this can be taken as defined in case of a line segment, returning the magnitude of the rotation of plate 5 to be 0.0015 radians or 0.0850 degrees. The axis and direction of this is given by the line segment $d_5$ represents.

## 5 Summary

A metric correspondence was provided between forces and small displacements of a plate structure (made of plates rigid in their planes, soft perpendicular to their planes, joined by elastic edges) and those of its dual: a truss with elastic bars. Since the computational methods of trusses are well developed and widely known, this ability to turn unfamiliar problems into familiar ones may give a useful tool in the hands of structural engineers or researchers. Especially now, when automation propels the spread of prefabricated plate structures, where the joining edges are usually softer then the plates itself.

## Acknowledgments

The author wishes to thank professors P.L. Várkonyi and T. Tarnai for useful discussions on the subject.

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