On approximating shortest paths in weighted triangular tessellations

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Abstract

We study the quality of weighted shortest paths when a continuous 2-dimensional space is discretized by a weighted triangular tessellation. In order to evaluate how well the tessellation approximates the 2-dimensional space, we study three types of shortest paths: a weighted shortest path \( SP_w(s, t) \), which is a shortest path from \( s \) to \( t \) in the space; a weighted shortest vertex path \( SVP_w(s, t) \), which is a shortest path where the vertices of the path are vertices of the tessellation; and a weighted shortest grid path \( SGP_w(s, t) \), which is a shortest path whose edges are edges of the tessellation. The ratios \( \frac{\|SGP_w(s, t)\|}{\|SP_w(s, t)\|} \), \( \frac{\|SVP_w(s, t)\|}{\|SP_w(s, t)\|} \), and \( \frac{\|SVP_w(s, t)\|}{\|SGP_w(s, t)\|} \) provide estimates on the quality of the approximation.

Given any arbitrary weight assignment to the faces of a triangular tessellation, we prove upper and lower bounds on the estimates that are independent of the weight assignment. Our main result is that \( \frac{\|SGP_w(s, t)\|}{\|SP_w(s, t)\|} = \frac{2}{\sqrt{3}} \approx 1.15 \) in the worst case, and this is tight.

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1 Introduction

Geometric shortest path problems, where the goal is to find an optimal path in a geometric setting, are fundamental problems in computational geometry. In contrast to the classical shortest path problem in graphs, where the space of possible paths is discrete, in geometric settings the space is continuous: the source and target points can be anywhere within a certain geometric domain (e.g., a polygon, the plane, a surface), and the set of possible paths to consider has infinite size. Many variations of geometric shortest path problems exist, depending on the geometric domain, the objective function (e.g., Euclidean metric, link-distance, geodesic distance), or specific domain constraints (e.g., obstacles in the plane, or holes in polygons). Applications of geometric shortest path problems are ubiquitous, appearing in diverse areas such as robotics, video game design, or geographic information science. We refer to Mitchell [14] for a complete survey on geometric shortest path problems.

One of the most general settings for geometric shortest path problems arises when the cost of traversing the domain varies depending on the region. That is, the domain consists of a planar subdivision, that without loss of generality can be assumed to be triangulated. Each region $i$ of the subdivision has a weight $w_i$, that represents the cost per unit of distance of traveling in that region. Thus, the cost of traversing a region is typically given by the Euclidean distance traversed in the region, multiplied by the corresponding weight. The resulting metric is often called the weighted region metric, and the problem of computing a shortest path between two points under this metric is known as the weighted region problem (WRP) [12, 13]. The WRP is very general, since it allows to model many well-known variants of geometric shortest path problems. Indeed, having that all weights are equal makes the metric equivalent to the Euclidean metric (up to scaling), while using two different weight values, such as 1 and $\infty$, allows to model paths amidst obstacles.

Perhaps not surprisingly, the WRP turns out to be a challenging problem. The first algorithm for WRP was a $(1 + \varepsilon)$-approximation proposed by Mitchell and Papadimitriou [13], which runs in time $O(n^8 \log (\frac{n^2W}{w\varepsilon}))$, where $N$ is the maximum integer coordinate of any vertex of the subdivision, $W$ and $w$ are, respectively, the maximum finite and the minimum nonzero integer weight assigned to the regions. Substantial research has been devoted to studying faster approximation algorithms and different variants of the problem [1 2 3]. Approximation schemes for WRP are sophisticated methods that usually are based on variants of continuous Dijkstra, subdividing triangle edges in parts for which crossing shortest paths have the same combinatorial structure (e.g., [13]), or work by computing a discretization of the domain by carefully placing Steiner points (e.g., see [8] for the currently best method of this type). The lack of exact algorithms for WRP
is probably justified by algebraic reasons: WRP was recently shown to be impossible to solve in the Algebraic Computation Model over the Rational Numbers [9]. This is a model of computation where one can compute exactly any number that can be obtained from rational numbers by a finite number of basic operations. Efficient algorithms for WRP only exist for a few special cases, e.g., rectilinear subdivisions with $L_1$ metric [7], or weights restricted to $\{0, 1, \infty\}$ [10].

In applications where the WRP arises, like robotics, gaming or simulation, which usually require efficient and practical algorithms, the problem is simplified in two ways. First, the domain is approximated by using a (weighted) plane subdivision with a simpler structure. Secondly, optimal shortest paths in that simpler subdivision are approximated. The typical way to represent a 2D (or 3D) environment where shortest paths need to be computed is by using navigation meshes [17]. These are polygonal subdivisions together with a graph that models the adjacency between the regions. Path planning is then done first on the graph to obtain a sequence of regions to be traversed, and then within each region, for which a shortest geometric path is extracted. Triangles, convex polygons, disks or squares—of different sizes—are among the most frequently used region shapes [17]. Navigational meshes allow efficient path planning in large environments as long as the region weights are limited to $\{1, \infty\}$ (i.e., obstacles only). In case general weights are needed, the complexity of computing the shortest path inside each region requires the use of the simplest possible navigational mesh: regular grids. In 2D, the only three types of regular polygons that can be used to tessellate continuous environments are triangles, squares and hexagons. The drawback with a grid is that it imposes a fixed resolution, requiring in general a large number of cells or regions. Still, grids are often used as navigation meshes (even for weights $\{1, \infty\}$), since they are easy to implement, are a natural choice for environments that are grid-based by design (e.g., many game designs), and popular shortest path algorithms such as $A^*$ can be optimized for grids [15].

Even when a regular grid is used as a navigation mesh, in practice exact weighted shortest paths are not computed: instead, an approximation is obtained by computing a shortest path on a weighted graph associated to the grid. To this end, two different graphs have been considered in the literature [5], corner-vertex graph and $k$-corner grid graph, defined next. The baseline to analyze the quality of any approximate path is the weighted shortest path that takes into account the full geometry of each region, as in the WRP. A weighted shortest path will be denoted $SP_w(s, t)$. As already mentioned, exact weighted shortest paths for regions with weights in $\{0, 1, \infty\}$ were studied in [12] [13].

In a corner-vertex graph $G_{\text{corner}}$, the vertex set is the set of corners of the tessellation and every pair of vertices is connected by an edge. These graphs can be seen as the complete graphs over the set of vertices. Figure [15] depicts
some of the neighbors of a vertex $v$ in the corner-vertex graph. Note that in this graph some edges overlap. A path in this graph is called a \textit{vertex path}; a shortest vertex path between $s$ and $t$ will be denoted $SVP_w(s,t)$, where $w$ makes explicit that this path depends also on a particular weight assignment $w$.

In a $k$-corner grid graph $G_{k\text{corner}}$, which is a subgraph of a corner-vertex graph, the vertex set is the set of corners of the tessellation, and each vertex is connected by an edge to a predefined set of $k$ neighboring vertices, depending on the tessellation and other design decisions. See Figure 1b for the 6-corner grid graph in a triangular tessellation. (Analogous $k$-corner grid graphs can be defined for square and hexagonal tessellations.) A path in this graph is called a \textit{grid path}; a shortest grid path between $s$ and $t$ will be denoted $SGP_w(s,t)$.

Shortest vertex paths and shortest grid paths for the case of weights of the cells being 1 or $\infty$ have been previously studied in [16] and [4], respectively. In all cases, the weight of each graph edge is defined based on the cost of the associated line segment, depending on the weights of the regions that it goes through. More formally, let $T_i$ be a region in a subdivision with weight $\omega_i \in \mathbb{R}_{>0}$. The cost of a segment $\pi_i$ in the interior of a cell $T_i$ is given by $\omega_i \| \pi_i \|$, where $\| \cdot \|$ is the Euclidean norm. In the case where a segment $\pi$ lies in the boundary of two cells $T_j$ and $T_k$, the cost is $\min\{\omega_j, \omega_k\} \| \pi \|$.

Figure 2 shows an example, illustrating the three paths considered: the shortest path $SP_w(s,t)$ (blue), the shortest vertex path $SVP_w(s,t)$ (green), and the shortest grid path $SGP_w(s,t)$ (red) in a 6-corner grid graph. Note that in all figures in this work, cells that are not depicted are considered to have infinite weight.

1.1 Quality bounds for approximation paths

The goal of this work is to understand the relation between $SGP_w(s,t)$, $SVP_w(s,t)$, and the baseline $SP_w(s,t)$. Since $SVP_w(s,t)$ and $SGP_w(s,t)$ are approximations of $SP_w(s,t)$, a fundamental question is: what is the worst-case approximation factor that they can give?

In this paper we focus on weighted tessellations where every face is
an equilateral triangle (analog ideas can be used for square and hexagonal grids). In particular, we are interested in upper-bounding the ratios \( \frac{\|SGP_w(s,t)\|}{\|SP_w(s,t)\|} \) and \( \frac{\|SVP_w(s,t)\|}{\|SP_w(s,t)\|} \), since they indicate the approximation factor of the shortest grid and vertex path, respectively. The ratio \( \frac{\|SGP_w(s,t)\|}{\|SP_w(s,t)\|} \) is also studied, to see which approximation is better.

Almost all previous bounds on the ratio \( \frac{\|SGP_w(s,t)\|}{\|SP_w(s,t)\|} \) consider a limited set of weights for the cells. Nash [16] considered only weights in the set \( \{1, \infty\} \) and proved that the weight of \( SGP_w(s,t) \) in hexagonal \( G_{6\text{corner}} \), and \( G_{12\text{corner}} \), square \( G_{4\text{corner}} \), and \( G_{8\text{corner}} \), triangle \( G_{6\text{corner}} \), and \( G_{3\text{corner}} \) can be, respectively, up to \( \approx 1.15 \), \( \approx 1.04 \), \( \approx 1.41 \), \( \approx 1.08 \), \( \approx 1.15 \), and 2 times the weight of \( SP_w(s,t) \). When the weights of the cells are allowed to be in \( \mathbb{R}_{>0} \), the only result that we are aware of is for square tessellations and another type of shortest path, with vertices at the center of the cells, for which Jaklin [11] showed that \( \frac{\|SGP_w(s,t)\|}{\|SP_w(s,t)\|} \leq 2\sqrt{2} \).

The main contribution of this paper is the analysis of the quality of the three types of shortest paths for a triangular grid for \( G_{6\text{corner}} \), which is the most natural graph defined on a triangular grid. In contrast to previous work, we allow the weights \( \omega_i \) to take any value in \( \mathbb{R}_{>0} \), so the main challenge here is to obtain tight upper bounds that hold for any assignment of region weights. Surprisingly, we show that this is possible: the ratios are upper bounded by constants that are independent of the weights assigned to the regions in the tessellation. Our main result is that \( \frac{\|SGP_w(s,t)\|}{\|SP_w(s,t)\|} = \frac{2}{\sqrt{3}} \) in the worst case, for any (positive) weight assignment. This implies bounds for the other two ratios considered. Moreover, our upper bound for \( \frac{\|SGP_w(s,t)\|}{\|SP_w(s,t)\|} \) is tight, since it matches the lower bound claimed by Nash [16]. Table 1 summarizes our results, together with the previously known lower bounds.

2 \( \frac{\|SGP_w(s,t)\|}{\|SP_w(s,t)\|} \) ratio in \( G_{6\text{corner}} \) for triangular cells

This section is devoted to obtaining, for two vertices \( s \) and \( t \), an upper bound on the ratio \( \frac{\|SGP_w(s,t)\|}{\|SP_w(s,t)\|} \) in \( G_{6\text{corner}} \) in a triangular tessellation \( T \). We suppose,
Observation 1. Let $GP_w(s,t)$ and $SP_w(s,t)$ be, respectively, a weighted grid path, and a weighted shortest path, from $s$ to $t$. Let $u_i$ and $u_{i+1}$ be two consecutive points where $GP_w(s,t)$ and $SP_w(s,t)$ coincide; in case $GP_w(s,t)$ and $SP_w(s,t)$ share one or more segments, we define the corresponding points as the endpoints of each of these segments, see Figure 3 for an illustration. Observation 1 below is a special case of the mediant inequality.

**Observation 1.** Let $GP_w(s,t)$ and $SP_w(s,t)$ be, respectively, a weighted grid path, and a weighted shortest path, from $s$ to $t$. Let $u_i$ and $u_{i+1}$ be two consecutive points where $GP_w(s,t)$ and $SP_w(s,t)$ coincide. Then, the ratio $\frac{\|GP_w(s,t)\|}{\|SP_w(s,t)\|}$ is at most the maximum of all ratios $\frac{\|GP_w(u_i, u_{i+1})\|}{\|SP_w(u_i, u_{i+1})\|}$.

### 2.1 Crossing paths and weakly simple polygons

In the weighted version of the problem, conversely to the unweighted version, we need to take into account all the different weights of the regions intersected by $SP_w(s,t)$. In addition, we do not know the shape of the shortest paths $SP_w(s,t)$ and $SGP_w(s,t)$. To solve all these inconveniences, for each $SP_w(s,t)$ we will define a particular grid path called crossing path $X(s,t) \in G_{6\text{corner}}$, whose behavior will be easier to control. See orange path in Figure 3. Then, the key idea to prove the upper bound on the ratio $\frac{\|SP_w(s,t)\|}{\|X(s,t)\|}$ in $G_{6\text{corner}}$ will be to upper-bound it by the ratio $\frac{\|X(s,t)\|}{\|SP_w(s,t)\|}$. To do so, we will analyze the components resulting from the intersection between $SP_w(s,t)$ and $X(s,t)$. Each component will be a weakly simple polygon, which will be the basic unit that we will analyze to obtain our main result. Also, a relation between the weights of some cells intersected by $SP_w(s,t)$ and $X(s,t)$ will be obtained.

Let $(T_1, \ldots, T_n)$ be the ordered sequence of consecutive cells intersected by $SP_w(s,t)$ in the tessellation $T$. Let $v_1^i, v_2^i, v_3^i$ be the three consecutive corners of the boundary of $T_i$, $1 \leq i \leq n$. Let $(s = a_1, a_2, \ldots, a_{n+1} = t)$ be the sequence of consecutive points where $SP_w(s,t)$ changes cell in $T$. In particular, let $a_i$ and $a_{i+1}$ be, respectively, the points where $SP_w(s,t)$ enters and leaves $T_i$. In a triangular tessellation, the crossing path $X(s,t)$ from a vertex $s$ to a vertex $t$ is defined as follows:

![Table 1: Bounds on the quality of approximations of shortest paths in weighted triangular tessellations for $G_{6\text{corner}}$. The upper bound for the ratio $\frac{\|SGP_w(s,t)\|}{\|SP_w(s,t)\|}$, and the results in the last column are shown in the Appendix.](image-url)
Figure 3: Weighted shortest path $SP_w(s, t)$ (blue) and the crossing path $X(s, t)$ (orange) from $s$ to $t$ in a triangular tessellation.

Figure 4: Some of the positions of the intersection points between $SP_w(s, t)$ (blue) and a cell. The vertices of the crossing path $X(s, t)$ in a triangular cell are depicted in orange.

Definition 2. The crossing path $X(s, t) \in G_{6corner}$ between two vertices $s$ and $t$ in a triangular tessellation $T$ is defined by the sequence $(X_1, \ldots, X_n)$, where $X_i$ is a sequence of vertices determined by the pair $(a_i, a_{i+1})$, $1 \leq i \leq n$, as follows. Let $e_i^1 \in T_i$ be an edge containing $a_i$, then:

- If $a_{i+1} \in e_i^1$, let $[v, w]$ be the endpoints of $e_i^1$, where $a_i$ is encountered before $a_{i+1}$ when traversing $e_i^1$ from $v$ to $w$. Then $X_i = (v, w)$, see Figure 4a.

- If $a_i$ is an endpoint of $e_i^1$, let $p$ be the midpoint of the edge $e_i^2 \in T_i$ not containing $a_i$. If $a_{i+1} \in e_i^2$ is to the left of $\overrightarrow{a_ip}$, $X_i$ is $a_i$ and the endpoint of $e_i^2$ to the right of $\overrightarrow{a_ip}$, see Figure 4b. Otherwise, $X_i$ is $a_i$ and the endpoint of $e_i^2$ to the left of $\overrightarrow{a_ip}$.

- If $a_i$ is in the interior of $e_i^1$ and $a_{i+1}$ is a corner, $X_i = (a_{i+1})$, see Figure 4c.

- If $a_i$ and $a_{i+1}$ belong to the interior of two different edges $e_i^1$ and $e_i^2$, $X_i$ is the common endpoint of $e_i^1$ and $e_i^2$, see Figure 4d.

Let $(s = u_1, u_2, \ldots, u_\ell = t)$ be the sequence of consecutive points where $X(s, t)$ and $SP_w(s, t)$ coincide. The union of $SP_w(s, t)$ and $X(s, t)$ between
two consecutive points \( u_j \) and \( u_{j+1} \), for \( 1 \leq j < \ell \), induces a weakly simple polygon (see [6] for a formal definition). We distinguish six different types of weakly simple polygons, denoted \( P_1, \ldots, P_6 \), depending on the number of edges intersected by \( SP_w(u_j, u_{j+1}) \), see Figure 5. Observe that, by definition of \( X(s, t) \), these are the only weakly simple polygons that can arise.

The weakly simple polygons will be an important tool in our proof, since it will be enough to upper bound \( \| X(s, t) \| \| SP_w(s, t) \| \) for each of \( P_1, \ldots, P_6 \).

**Definition 3.** Let \( u_j \) and \( u_{j+1} \) be two consecutive points in a triangular tessellation, where \( X(s, t) \in G_{\text{corner}} \) and \( SP_w(s, t) \) coincide. Let \( p \) be a common endpoint of the edges of the tessellation that contain \( u_j \) and \( u_{j+1} \). A weakly simple polygon induced by \( u_j \) and \( u_{j+1} \) is of type \( P_k \), for \( 1 \leq k \leq 6 \), if the subpath \( SP_w(u_j, u_{j+1}) \) intersects \( k \) consecutive edges around \( p \).

### 2.2 Bounding the ratio for weakly simple polygons

We are now ready to upper bound the ratio \( \frac{\| X(u_j, u_{j+1}) \|}{\| SP_w(u_j, u_{j+1}) \|} \) for each of the six types of weakly simple polygons in \( G_{\text{corner}} \). We begin by observing that, by definition, for \( P_1 \) we have \( \frac{\| X(u_j, u_{j+1}) \|}{\| SP_w(u_j, u_{j+1}) \|} = 1 \).

Therefore the focus will be on bounding \( P_2, \ldots, P_6 \). We will begin from the simpler case of \( P_3, \ldots, P_6 \), and later we will consider \( P_2 \), which is substantially more involved.

First we make a geometric observation that will be needed later. Let \( p \) and \( q \) be two points that are in the interior of two different edges on the
Figure 6: Weighted shortest path $SP_w(s, t)$ (blue), crossing path $X(s, t)$ (orange), and shortcut path $\Pi_i(s, t)$ (purple) intersecting a weakly simple polygon $P_2$.

boundary of the same triangular cell. Then, the length of the subpath of the weighted shortest path between $p$ and $q$ is given in Observation 4, which can be proved using the law of cosines.

**Observation 4.** Let $T_i$ be a triangular cell, and let $(u, v, w)$ be the three vertices of $T_i$, in clockwise order. Let $p \in [u, v]$ and $q \in [v, w]$ be two points on the boundary of $T_i$. Then, $|pq| = \sqrt{|pv|^2 + |vq|^2 - |pv||vq|}$, see Figure 6a.

**Lemma 5.** Let $u_j, u_{j+1} \in P_k$, for $3 \leq k \leq 6$, be two consecutive points where a shortest path $SP_w(s, t)$ and the crossing path $X(s, t)$ coincide in a triangular tessellation $\mathcal{T}$. An upper bound on the ratio $\frac{\|X(u_j, u_{j+1})\|}{\|SP_w(u_j, u_{j+1})\|}$ in $G_{6\text{corner}}$ is $\frac{2}{\sqrt{3}}$.

The grid paths in $G_{6\text{corner}}$ are paths whose edges are edges of the triangular cells. Thus, the ratio $\frac{\|X(u_j, u_{j+1})\|}{\|SP_w(u_j, u_{j+1})\|}$ between two consecutive crossing points $u_j$ and $u_{j+1}$ depends on the weights of these regions. So, the next difficulty that we encountered, related to the crossing path, was that it is possible to find an instance where $SP_w(s, t)$ intersects a weakly simple polygon $P_2$ such that the ratio $\frac{\|X(s, t)\|}{\|SP_w(s, t)\|}$ is much larger than $\frac{\|SGP_w(s, t)\|}{\|SP_w(s, t)\|}$, see Figure 6b. However, between $s$ and $t$ there are other grid paths shorter than $X(s, t)$ that intersect a $P_2$. So, in order to obtain an upper bound when $SP_w(s, t)$ intersects a $P_2$, we will need a finer analysis.

Since the ratio $\frac{\|X(u_j, u_{j+1})\|}{\|SP_w(u_j, u_{j+1})\|}$ is at most $\frac{2}{\sqrt{3}}$ for weakly simple polygons $P_k$, $k \neq 2$, we will assume from now on, that the ratio is maximized when all weakly simple polygons are of type $P_2$. Otherwise, we are done.

**Definition 6** determines another class of grid paths called **shortcut paths** that gives a tighter upper bound on the ratio $\frac{\|SGP_w(s, t)\|}{\|SP_w(s, t)\|}$ when a weakly simple polygon $P_2$ is intersected by $SP_w(s, t)$.

Let $\{v_1, \ldots, v_n\}$ be a sequence of corners of a triangular tessellation. Then, the grid path $\Pi(s, [v_1, \ldots, v_n], t)$ is defined as the path $X(s, v_1) \cup \pi(v_1, \ldots, v_n) \cup X(v_n, t)$, where $\pi(v_1, \ldots, v_n)$ is the grid path through the vertices $v_1, \ldots, v_n$ in that order. We now define shortcut paths.
Definition 6. Let \((u, v, w)\) be the ordered sequence of vertices of a cell \(T_i \in \mathcal{T}\). If \(X(s, t) \in G_{6\text{corner}}\) contains the subpath \((u, v, w)\), the shortcut path \(\Pi_i(s, t)\) is defined as the grid path \(\Pi(s, [u, w], t)\), see purple path in Figure 6a.

Now, we have all the tools needed to obtain an upper bound on the ratio \(\frac{\|X(u_i, u_{i+1})\|}{\|SP_w(s, t)\|}\) for \(P_2\). By using the shortcut path \(\Pi_i(s, t)\), we will be able to obtain a relation between the weights of the cells adjacent to \(T_i \in \mathcal{T}\) intersected by the crossing path \(X(s, t)\). This relation is given in the next lemma.

Lemma 7. Let \((T_k, \ldots, T_m)\) be the sequence of consecutive cells for which there exists a shortcut path \(\Pi_i(s, t)\), \(k \leq i \leq m\), for a given assignment of weights \(w\) to the cells of the triangular tessellation. The ratio \(\frac{\|SGP_w(s, t)\|}{\|SP_w(s, t)\|}\) is maximized when \(\|X(s, t)\| = \|\Pi_i(s, t)\|\) in \(G_{6\text{corner}}\).

Proof. Consider an instance for which the ratio \(\frac{\|SGP_w(s, t)\|}{\|SP_w(s, t)\|}\) is maximized. Recall this instance just contains weakly simple polygons of type \(P_2\). We will argue that if there is a grid path \(GP_w(s, t)\) among \(X(s, t), \Pi_i(s, t), k \leq i \leq m\), that is strictly shorter than the other grid paths, then this instance cannot maximize \(\frac{\|SGP_w(s, t)\|}{\|SP_w(s, t)\|}\).

Suppose that there is one grid path \(GP_w(s, t)\) among \(X(s, t), \Pi_i(s, t), k \leq i \leq m\), that is strictly shorter than the other grid paths in the set. Since \(GP_w(s, t)\) is a grid path, the ratio \(\frac{\|SGP_w(s, t)\|}{\|SP_w(s, t)\|}\) is upper bounded by \(\frac{\|GP_w(s, t)\|}{\|SP_w(s, t)\|}\). The objective of the proof is to find another assignment of weights \(w'\) for the cells, such that \(GP_{w'}(s, t)\) is still a shortest grid path among the grid paths in the set, and \(\frac{\|SP_w(s, t)\|}{\|SP_{w'}(s, t)\|} > \frac{\|SP_{w'}(s, t)\|}{\|SP_{w'}(s, t)\|}\).

Let \(u_j, u_{j+1}\) be two consecutive points where \(GP_w(s, t)\) and \(SP_w(s, t)\) coincide. Let \(T_i\) be the cell that shares the edge of \(\Pi_i(s, t)\) with \(T_i\), see Figure 7. We first set to infinity the weight of all the cells that are not traversed by \(SP_w(s, t)\). This way, we ensure that when modifying the weights of some cells, the combinatorial structure of the shortest path is preserved. The weight of the crossing path \(X(s, t)\) along the edges of \(T_i\) is \(2\min\{\omega_{i-1}, \omega_i\} + 2\min\{\omega_i, \omega_{i+1}\}\), and the weight of the shortcut path \(\Pi_i(s, t)\) along the edges of \(T_i\) is \(2\min\{\omega_i, \omega_{i+1}\} = 2\omega_i\) (because \(\omega_{i+1} = \infty\)). Let \([p, q]\), and \([p', q']\) be, respectively, the edges containing \(u_j\) and \(u_{j+1}\), where \(p, p' \in T_i\).

- If \(GP_w(s, t) = X(s, t)\) then \(\|X(s, t)\| < \|\Pi_i(s, t)\|\), and we have that
  \[\min\{\omega_{i-1}, \omega_i\} + \min\{\omega_i, \omega_{i+1}\} < \omega_i,\]
  \[\text{(1)}\]
  - Suppose \(\omega_i \leq \omega_{i-1}\), then \(\omega_i + \min\{\omega_i, \omega_{i+1}\} < \omega_i\), which is not possible since \(\min\{\omega_i, \omega_{i+1}\} > 0\). Hence, \(\omega_i > \omega_{i-1}\).
  - Suppose \(\omega_i \leq \omega_{i+1}\), then \(\min\{\omega_{i-1}, \omega_i\} + \omega_{i+1} < \omega_i\), which is not possible since \(\min\{\omega_{i-1}, \omega_i\} > 0\). Hence, \(\omega_i > \omega_{i+1}\).
These two facts together with Equation (1) imply that $\omega_{i-1} + \omega_{i+1} < \omega_i$. We also have that

$$\frac{\|X(s, t)\|}{\|SP_w(s, t)\|} = \frac{\|X(s, p)\| + 2(\omega_{i-1} + \omega_{i+1}) + \|X(p', t)\|}{\|SP_w(s, u_j)\| + |u_j u_{j+1}| \omega_i + \|SP_w(u_{j+1}, t)\|},$$

being $|u_j u_{j+1}| > 0$, so if we decrease the weight $\omega_i$ until $\omega_{i-1} + \omega_{i+1} = \omega_i$, the denominator $\|SP_w(s, t)\|$ will decrease, and the numerator $\|X(s, t)\|$ will remain. Hence, the ratio $\frac{\|X(s, t)\|}{\|SP_w(s, t)\|}$ will increase, and we found another weight assignment $w'$ such that $\frac{\|X(s, t)\|}{\|SP_w(s, t)\|} > \frac{\|X(s, t)\|}{\|SP_w(s, t)\|}$ and $\|X(s, t)\| = \|\Pi_i(s, t)\|$.

- Otherwise, if $GP_w(s, t) = \Pi_i(s, t)$ then $\|\Pi_i(s, t)\| < \|X(s, t)\|$, and we have that $\omega_i < \min\{\omega_{i-1}, \omega_i\} + \min\{\omega_i, \omega_{i+1}\}$. We also have that

$$\frac{\|\Pi_i(s, t)\|}{\|SP_w(s, t)\|} = \frac{\|\Pi_i(s, p)\| + 2\omega_i + \|\Pi_i(p', t)\|}{\|SP_w(s, u_j)\| + |u_j u_{j+1}| \omega_i + \|SP_w(u_{j+1}, t)\|},$$

given $|u_j u_{j+1}| < 2$. If we increase the weight $\omega_i$ until $\omega_i = \omega_{i-1} + \omega_{i+1}$, the numerator $\|\Pi_i(s, t)\|$ will increase faster than the denominator $\|SP_w(s, t)\|$. Hence, the ratio $\frac{\|\Pi_i(s, t)\|}{\|SP_w(s, t)\|}$ will increase, so we found another weight assignment $w'$ such that $\frac{\|\Pi_i(s, t)\|}{\|SP_w(s, t)\|} > \frac{\|\Pi_i(s, t)\|}{\|SP_w(s, t)\|}$ and $\|X(s, t)\| = \|\Pi_i(s, t)\|$.

We are now ready to prove the upper bound on the ratio $\frac{\|X(u_j, u_{j+1})\|}{\|SP_w(u_j, u_{j+1})\|}$ in a $P_2$. Lemma 8 presents an upper bound on the ratio $\frac{\|X(u_j, u_{j+1})\|}{\|SP_w(u_j, u_{j+1})\|}$, where $u_j, u_{j+1} \in T_i$ are two consecutive points where $X(s, t) \in G_{6corner}$ and $\Pi_i(s, t)$ coincide. Lemma 7 implies that the ratio $\frac{\|X(s, t)\|}{\|SP_w(s, t)\|}$ in $G_{6corner}$ is maximized when $\|X(s, t)\| = \|\Pi_i(s, t)\|$ for each $i$ such that the shortcut path $\Pi_i(s, t)$ exists. Thus, the ratio is obtained in a weakly simple polygon $P_2$ when $\|X(s, t)\| = \|\Pi_i(s, t)\|$. Since the exact shape of $SP_w(s, t)$ is unknown, when calculating the ratio in the following Lemma 8, we will maximize the ratio for any position of the points $u_j$ and $u_{j+1}$ where $SP_w(s, t)$ and $X(s, t)$ coincide.
Lemma 8. Let $u_j, u_{j+1} \in P_2$ be two consecutive points in a triangular tessellation $T$, where a shortest path $SP_w(s,t)$ and the crossing path $X(s,t)$ coincide. Let $u_j, u_{j+1} \in T_i$ and $\|X(s,t)\| = \|\Pi_i(s,t)\|$, then an upper bound on the ratio $\frac{\|X(u_j,u_{j+1})\|}{\|SP_w(u_j,u_{j+1})\|}$ in $G_{6\text{corner}}$ is $\frac{2}{\sqrt{3}}$.

Finally, we have all the pieces to prove our main result.

Theorem 9. In $G_{6\text{corner}}$, an upper bound on the ratio $\frac{\|SGP_w(s,t)\|}{\|SP_w(s,t)\|}$ is $\frac{2}{\sqrt{3}}$.

Figure 8 provides an illustration of the lower bound $\frac{2}{\sqrt{3}}$ on the ratio between the weighted shortest grid path $SGP_w(s,t) \in G_{6\text{corner}}$ (red) and the weighted shortest path $SP_w(s,t)$ (blue) claimed by Nash [16]. Hence, the upper bound in Theorem 9 is tight for $G_{6\text{corner}}$.

3 Discussion and future work

We presented upper bounds on the ratio between the lengths of three types of weighted shortest paths in a triangular tessellation. The fact that a compact grid graph such as $G_{6\text{corner}}$ guarantees an error bound of $\approx 15\%$, regardless of weights used, justifies its widespread use in applications in areas such as gaming and simulation, where performance is a priority over accuracy.

Our analysis techniques, presented here for triangular grids, can also be applied to obtain upper bounds for the same ratios in the other two types of regular tessellations, square and hexagonal. The main differences lie in the exact definition of the crossing paths and the weakly simple polygons. Our techniques can also be used to derive upper bounds for another type of grid graphs, where the vertices are cell centers instead of corners (see, e.g., [11, 16]).

For future work, it would be interesting to close the gap for $\frac{\|SVP_w(s,t)\|}{\|SP_w(s,t)\|}$. It is an intriguing question whether the seemingly richer graph $SVP_w(s,t)$ can actually guarantee a better quality factor than $G_{6\text{corner}}$. 
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Figure 9: Subpaths of the grid path $\Pi_i(s,t)$ (purple), the crossing path $X(s,t)$ (orange), and $SP_w(s,t)$ (blue) in a triangular tessellation.

A Proof of Lemma 5

Lemma 10. Let $u_j, u_{j+1} \in P_k$, for $3 \leq k \leq 6$, be two consecutive points where a shortest path $SP_w(s,t)$ and the crossing path $X(s,t)$ coincide in a triangular tessellation $T$. An upper bound on the ratio $\frac{\|X(u_j,u_{j+1})\|}{\|SP_w(u_j,u_{j+1})\|}$ in $G_{6\text{corner}}$ is $\frac{2}{\sqrt{3}}$.

Proof. Let $u_j$ be the point where $SP_w(s,t)$ enters a cell $T_i$ from $T_{i-1}$, and $u_{j+1}$ be the point where $SP_w(s,t)$ leaves $T_{i+k-2}$, and enters $T_{i+k-1}$, see Figure 9a for $k = 3$. Let $(v_1^i, v_2^i, p)$ be the ordered sequence of consecutive vertices of the boundary of $T_i$, and $(v_1^{i+k-2}, p, v_2^{i+k-2})$ be the ordered sequence of consecutive vertices of the boundary of $T_{i+k-2}$. Let $x \in [v_2^i, p]$ be the point where $SP_w(s,t)$ enters $T_{i+1}$, and let $y \in [v_1^{i+k-2}, p]$ be the point where $SP_w(s,t)$ enters $T_{i+k-2}$. Let $a, b, c, d, b', e$ be the lengths $|u_jp|, |px|, |u_jx|, |u_{j+1}p|, |py|$, and $|u_{j+1}y|$, respectively. Hence, an upper bound on the ratio $\frac{\|X(u_j,u_{j+1})\|}{\|SP_w(u_j,u_{j+1})\|}$ in $P_k$ is

$$\frac{\|X(u_j,u_{j+1})\|}{\|SP_w(u_j,u_{j+1})\|} = \frac{a \min\{\omega_{i-1}, \omega_i\} + d \min\{\omega_{i+k-2}, \omega_{i+k-1}\}}{a \omega_i + \|SP_w(x,y)\| + e \omega_{i+k-2}} \leq \frac{a \omega_i + d \omega_{i+k-2}}{c \omega_i + e \omega_{i+k-2}} \leq \frac{a \omega_i + d \omega_{i+k-2}}{\sqrt{3}a \omega_i + \sqrt{3}d \omega_{i+k-2}} = \frac{a \omega_i + d \omega_{i+k-2}}{\sqrt{3}} \leq \frac{2}{\sqrt{3}}.$$
B Proof of Lemma 8

Lemma 11. Let \( u_j, u_{j+1} \in P_2 \) be two consecutive points in a triangular tessellation \( T \), where a shortest path \( SP_w(s, t) \) and the crossing path \( X(s, t) \) coincide. Let \( u_j, u_{j+1} \in T_i \) and \( \|X(s, t)\| = \|\Pi_i(s, t)\| \), then an upper bound on the ratio \( \frac{\|X(u_j, u_{j+1})\|}{\|SP_w(u_j, u_{j+1})\|} \) in \( G_{6\text{corner}} \) is \( \frac{2}{\sqrt{3}} \).

Proof. Let \( (v_1^1, v_2^1, v_3^1) \) be the ordered sequence of vertices of the boundary of a cell \( T_i \). Let \( u_j \in [v_1^1, v_2^1] \) be the point where \( SP_w(s, t) \) enters \( T_i \) from cell \( T_{i-1} \), and \( u_{j+1} \in [v_2^1, v_3^1] \) be the point where \( SP_w(s, t) \) leaves \( T_i \) and enters cell \( T_{i+1} \), see Figure 9b. Let \( a, b, c \) be the lengths \( |u_jv_1^1|, |v_2^1u_{j+1}|, \) and \( |u_ju_{j+1}| \), respectively. According to Observation 4 \( c = \sqrt{a^2 + b^2 - ab} \). Then we want to maximize the ratio \( \frac{\|X(u_j, u_{j+1})\|}{\|SP_w(u_j, u_{j+1})\|} \) for all weight assignments.

Traversing cell \( T_i \) there is also the grid path \( \Pi_i(s, t) \). Let \( T_0 \) be the cell that shares the edge of \( \Pi_i(s, t) \) with \( T_i \). Since \( \|X(s, t)\| = \|\Pi_i(s, t)\| \Rightarrow 2 \min\{\omega_i, \omega_j\} + \min\{\omega_i, \omega_{j+1}\} = \min\{\omega_i, \omega_j\} \). Thus,

\[
\begin{align*}
\text{if } b \leq a & \quad \min\{\omega_i, \omega_{j+1}\} = \min\{\omega_i, \omega_j\} - \min\{\omega_i, \omega_{j+1}\} \\
\text{otherwise} & \quad \min\{\omega_i, \omega_{j+1}\} = \min\{\omega_i, \omega_j\} - \min\{\omega_i, \omega_{j+1}\}
\end{align*}
\]

Then, an upper bound on the ratio \( R = \frac{\|X(u_j, u_{j+1})\|}{\|SP_w(u_j, u_{j+1})\|} \) in \( P_2 \) is

\[
R = \frac{a \min\{\omega_i, \omega_j\} + b \min\{\omega_i, \omega_{j+1}\}}{c \omega_i} \leq \begin{cases} 
\frac{a \min\{\omega_i, \omega_j\} + (b-a) \min\{\omega_i, \omega_{j+1}\}}{c \omega_i} & \text{if } b \leq a \\
\frac{b \min\{\omega_i, \omega_j\} + (a-b) \min\{\omega_i, \omega_{j+1}\}}{c \omega_i} & \text{if } a < b
\end{cases}
\]

\[
\leq \begin{cases} 
\frac{a \min\{\omega_i, \omega_j\}}{c \omega_i} & \leq \frac{2}{\sqrt{3}} \\
\frac{b \min\{\omega_i, \omega_j\}}{c \omega_i} & \leq \frac{2}{\sqrt{3}}
\end{cases}
\]

where the last inequality in the two ratios is obtained by maximization over the values of \( a \in [0, 2] \) and \( b \in [0, 2] \).

\[\square\]

C Proof of Theorem 9

Theorem 1. In \( G_{6\text{corner}} \), an upper bound on the ratio \( \frac{\|SGP_w(s, t)\|}{\|SP_w(s, t)\|} \) is \( \frac{2}{\sqrt{3}} \).

Proof. Let \( SP_w(s, t) \) be a weighted shortest path between two corners \( s \) and \( t \) in a triangular tessellation. Let \( X(s, t) \in G_{6\text{corner}} \) be the crossing path from \( s \) to \( t \) obtained from \( SP_w(s, t) \). By Observation 6, \( \frac{\|X(s, t)\|}{\|SP_w(s, t)\|} \leq \frac{\|X(u_j, u_{j+1})\|}{\|SP_w(u_j, u_{j+1})\|} \), over all pairs \( (u_j, u_{j+1}) \) of consecutive points where \( SP_w(s, t) \) and \( X(s, t) \) coincide. As already mentioned, for \( P_1 \) we have \( \frac{\|X(u_j, u_{j+1})\|}{\|SP_w(u_j, u_{j+1})\|} = \frac{2}{\sqrt{3}} \).
1. According to Lemmas 5 and 8, an upper bound for $\|SVP_w(s,t)\|/\|SP_w(s,t)\|$ in the weakly simple polygons of type $P_k$, $2 \leq k \leq 6$, is $\frac{2\sqrt{7\sqrt{3}-12}}{(7-4\sqrt{3})(6\sqrt{2}+\sqrt{7\sqrt{3}-12})}$. Since $\|SGP_w(s,t)\| \leq \|SVP_w(s,t)\|$, we have that $\|SGP_w(s,t)\|/\|SP_w(s,t)\| \leq \frac{2}{\sqrt{3}}$.

D Ratios $\|SGP_w(s,t)\|/\|SVP_w(s,t)\|$ in $G_{6\text{corner}}$ and $\|SVP_w(s,t)\|/\|SP_w(s,t)\|$ for triangular cells

The length of a weighted shortest vertex path $SVP_w(s,t)$ is an upper bound for the length of a weighted shortest path $SP_w(s,t)$, so the upper bound on the ratio $\|SGP_w(s,t)\|/\|SP_w(s,t)\|$ obtained in Theorem 9 is an upper bound for $\|SGP_w(s,t)\|/\|SVP_w(s,t)\|$.

Corollary 2. In $G_{6\text{corner}}$, an upper bound on the ratio $\|SGP_w(s,t)\|/\|SVP_w(s,t)\|$ is $\frac{2}{\sqrt{3}}$.

When the weights of the cells are in the set $\{1, \infty\}$, the ratio $\|SGP_w(s,t)\|/\|SVP_w(s,t)\|$ given in [16] for $G_{6\text{corner}}$ is $\frac{2}{\sqrt{3}}$. Thus, for general (positive) weights this value is a lower bound on the ratio $\|SGP_w(s,t)\|/\|SVP_w(s,t)\|$ for $G_{6\text{corner}}$.

Finally, as a corollary of previous Theorem 9, we obtain Corollary 3. The result comes from the fact that $\|SVP_w(s,t)\|$ is a lower bound for $\|SGP_w(s,t)\|$. Note that these shortest paths do not use $G_{6\text{corner}}$.

Corollary 3. In a regular triangular tessellation, an upper bound on the ratio $\|SVP_w(s,t)\|/\|SP_w(s,t)\|$ is $\frac{2}{\sqrt{3}}$.

Observation 4. Since the green path in Figure 10 is a weighted shortest vertex path $SVP_w(s,t)$ between vertices $s$ and $t$, we can lower bound the ratio $\|SVP_w(s,t)\|/\|SP_w(s,t)\|$ by $\frac{2\sqrt{7\sqrt{3}-12}}{(7-4\sqrt{3})(6\sqrt{2}+\sqrt{7\sqrt{3}-12})}$ in a regular triangular tessellation.