REAL AND COMPLEX ANALYSIS

On a Scale of Criteria on $n$-Dependence

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Abstract—In this paper we prove that a planar set $\mathcal{X}$ of at most $mn - 1$ points, where $m \leq n$, is $k$-dependent if and only if there exists a number $r$, $1 \leq r \leq m - 1$, and an essentially $k$-dependent subset $\mathcal{Y} \subset \mathcal{X}$, $\# \mathcal{Y} \geq rs$, where $r + s - 3 = k$, belonging to an algebraic curve of degree $r$, and not belonging to any curve of degree less than $r$. Moreover, if $\# \mathcal{Y} = rs$, then the set $\mathcal{Y}$ coincides with the set of intersection points of some two curves of degrees $r$ and $s$, respectively. Let us mention that the first three criteria of the scale, for $m = 1, 2, 3$, are well-known results.

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1. INTRODUCTION, $n$-INDEPENDENCE

Denote by $\Pi_n$ the space of bivariate algebraic polynomials of total degree less than or equal to $n$. Its dimension is given by

$$N := \dim \Pi_n = \binom{n + 2}{2}.$$

A plane algebraic curve is the zero set of some bivariate polynomial. To simplify notation, we shall use the same letter $p$, say, to denote the polynomial $p$ and the curve given by the equation $p(x, y) = 0$. More precisely, suppose $p$ is a polynomial without multiple factors. Then the plane curve defined by the equation $p(x, y) = 0$ shall also be denoted by $p$. So lines, conics, and cubics are equivalent to polynomials of degree 1, 2, and 3, respectively. Suppose a set of $k$ distinct points is given:

$$\mathcal{X}_k = \{(x_i, y_i) : i = 1, 2, \ldots, k\} \subset \mathbb{C}^2.$$

The problem of finding a polynomial $p \in \Pi_n$ which satisfies the conditions

$$p(x_i, y_i) = c_i, \quad i = 1, \ldots, k,$$

is called interpolation problem. We denote this problem by $(\Pi_n, \mathcal{X})$. The polynomial $p$ is called interpolating polynomial.

Definition 1.1. The set of points $\mathcal{X}_k$ is called $n$-poised if for any data $(c_1, \ldots, c_k)$, there is a unique polynomial $p \in \Pi_n$ satisfying the conditions (1.1).

By a linear algebra argument a necessary condition for $n$-poisedness is

$$k = \# \mathcal{X}_k = \dim \Pi_n = N.$$

Definition 1.2. The interpolating problem $(\Pi_n, \mathcal{X}_k)$ is called $n$-solvable if for any data $(c_1, \ldots, c_k)$ there exists a (not necessarily unique) polynomial $p \in \Pi_n$ satisfying the conditions (1.1).

A polynomial $p \in \Pi_n$ is called $n$-fundamental polynomial of a point $A \in \mathcal{X}$ if $p(A) = 1$ and $p|_{\mathcal{X}\setminus\{A\}} = 0$, where $p|_{\mathcal{X}}$ means the restriction of $p$ to $\mathcal{X}$. We shall denote such a polynomial by $p_A^\star, \mathcal{X}$.

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Sometimes we call \( n \)-fundamental also a polynomial from \( \Pi_n \) that just vanishes at all the points of \( X \) but \( A \), since such a polynomial is a nonzero constant multiple of \( p^*_A \). A fundamental polynomial can be described as a plane curve containing all but one point of \( X \).

Next we consider an important concept of \( n \)-independence and \( n \)-dependence of sets (see [1, 2, 4].

**Definition 1.3.** A set of points \( X \) is called \emph{\( n \)-independent} if each its point has an \( n \)-fundamental polynomial. Otherwise, it is called \emph{\( n \)-dependent}.

Since the fundamental polynomials are linearly independent, we get that \( \#X \leq N \) is a necessary condition for \( n \)-independence.

**Proposition 1.1.** A set \( X \) is \( n \)-independent if and only if the interpolation problem \((\Pi_n, X)\) is \( n \)-solvable.

**Proof.** Suppose \( X := X_k \). In the case of \( n \)-independence we have the following Lagrange formula for a polynomial \( p \in \Pi_n \) satisfying interpolating conditions (1.1):

\[
p = \sum_{i=1}^{k} c_i p^*_i.
\]

On the other hand, if the interpolation problem is \( n \)-solvable, then for each point \((x_i, y_i), i = 1, \ldots, k\), there exists an \( n \)-fundamental polynomial. Indeed, it is the solution of the interpolation problem (1.1), where \( c_i = 1 \) and \( c_j = 0 \) \( \forall j \neq i \). \( \square \)

**Definition 1.4.** A set of points \( X \) is called \emph{essentially \( n \)-dependent} if none of its points has an \( n \)-fundamental polynomial.

If a point set \( X \) is \( n \)-dependent, then for some \( A \in X \), there is no \( n \)-fundamental polynomial, which means that for any polynomial \( p \in \Pi_n \) we have that

\[
p|_{X \setminus \{A\}} = 0 \quad \Rightarrow \quad p(A) = 0.
\]

Thus a set \( X \) is essentially \( n \)-dependent means that any plane curve of degree \( n \) containing all but one point of \( X \) contains all of \( X \).

In the proof of the main result, we will need the following

**Proposition 1.2** ([5], Cor. 2.2). Suppose a set \( X \) is given. Denote by \( Y \) the subset of \( X \) that have \( n \)-fundamental polynomials with respect to \( X \). Then the set \( X \setminus Y \) is essentially \( n \)-dependent.

**Corollary 1.1.** Any \( n \)-dependent point set has an essentially \( n \)-dependent subset.

Set \( d(n, k) := \dim \Pi_n - \dim \Pi_{n-k} \). It is easily seen that \( d(n, k) = (n+1) + n + \cdots + (n-k+2) = \frac{1}{2}k(2n-k+3) \) if \( k \leq n \).

In the sequel we will need the following well-known proposition (see, e.g., [7], Proposition 3.1).

**Proposition 1.3.** Let \( q \) be a curve of degree \( k \) without multiple components and \( k \leq n \). Then the following assertions hold:

(i) any set of more than \( d(n, k) \) points located on the curve \( q \) is \( n \)-dependent;
(ii) any set \( X \) of \( d(n, k) \) points located on the curve \( q \) is \( n \)-independent if and only if

\[
p \in \Pi_n, \quad p|_X = 0 \Rightarrow p =fq, \quad \text{where} \quad f \in \Pi_{n-k}.
\]

**Corollary 1.2.** The following assertions hold:

(i) any set of at least \( n+2 \) points located on a line is \( n \)-dependent;
(ii) any set of at least \( 2n+2 \) points located on a conic is \( n \)-dependent;
(iii) any set of at least \( 3n+1 \) points located on a cubic is \( n \)-dependent.
2. SOME KNOWN RESULTS

Let us start with the three known results which coincide with the first three items of the scale established in this paper, respectively.

**Theorem 2.1** ([8]). Any set \( X \) consisting of at most \( n + 1 \) points is \( n \)-independent.

**Theorem 2.2** ([1, Proposition 1]). A set \( X \) with no more than \( 2n + 2 \) points on the plane is \( n \)-dependent if and only if either \( n + 2 \) of them are collinear or \( \#X = 2n + 2 \) and all the \( 2n + 2 \) points belong to a conic.

**Theorem 2.3** ([3, Theorem 5.1]). A set \( X \) consisting of at most \( 3n \) points is \( n \)-dependent if and only if at least one of the following conditions hold:

1. \( n + 2 \) points are collinear;
2. \( 2n + 2 \) points belong to a (possibly reducible) conic;
3. \( \#X = 3n \) and there exist \( \sigma_3 \in \Pi_3 \) and \( \sigma_n \in \Pi_n \) such that \( X = \sigma_3 \cap \sigma_n \).

The following two results describe some properties of essentially dependent point sets lying in a curves of certain degrees.

**Proposition 2.1** ([6, Proposition 3.3]). Suppose that \( m \leq n \). If a set \( X \) of at most \( mn \) points is essentially \( \kappa \)-dependent then all the points of \( X \) lie in a curve of degree \( m \).

We say that a curve \( \sigma \) is not empty with respect to a set \( X \) if \( X \cap \sigma \neq \emptyset \).

**Theorem 2.4** ([6, Theorem 3.4]). Assume that \( \sigma_m \) is a curve of degree \( m \), which is either irreducible or reducible such that all its irreducible components are not empty with respect to a set \( X \subset \sigma_m \), where \( X \) is essentially \( \kappa \)-dependent and \( m \leq n + 2 \). Then we have \( \#X \geq mn \).

The next result states a necessary and sufficient conditions for a set of \( mn \) points to coincide with the set of the intersection points of some two plane algebraic curves of degrees \( m \) and \( n \), respectively.

**Theorem 2.5** ([6, Theorem 3.1]). A set \( X \) with \( \#X = mn \), \( m \leq n \), is the set of intersection points of some two plane curves of degrees \( m \) and \( n \), respectively, if and only if the following two conditions are satisfied:

1. the set \( X \) is essentially \((m + n - 3)\)-dependent;
2. no curve of degree less than \( m \) contains all of \( X \).

3. MAIN RESULT

By combining Proposition 2.1 and Theorem 2.4, we readily get the following

**Proposition 3.1.** Suppose that \( X \) is an essentially \( \kappa \)-dependent point set with \( \#X \leq mn - 1 \), where \( m \leq n \), and \( \kappa = m + n - 3 \). Then there exists a number \( r \), \( 1 \leq r \leq m - 1 \), and a curve \( \sigma_r \) of degree \( r \) such that the following conditions hold:

1. \( \#X \geq rs \), where \( r + s = \kappa \);
2. \( \sigma_r \) contains all of \( X \);
3. there is no curve of degree less than \( r \) containing all of \( X \).

**Proof.** We obtain from Proposition 2.1 that the set of points \( X \) lies in a curve \( \sigma \) of degree at most \( m \). Without loss of generality, we may assume that \( \sigma \) is either irreducible or reducible such that all its irreducible components are not empty with respect to the set \( X \). Then, notice that the degree of the curve does not equal \( m \), since in that case, in view of Theorem 2.4, we would have \( \#X \geq mn \). Finally, consider such a curve \( \sigma_r \) of the smallest possible degree \( 1 \leq r \leq m - 1 \). Note that \( r < \kappa \). Now, Theorem 2.4 implies that \( \#X \geq rs \), where \( r + s = \kappa \).

Now, we are in a position to formulate the main result of the paper:

**Theorem 3.1.** Suppose that \( X \) is a set of points such that \( \#X \leq mn - 1 \), where \( m \leq n \). Then \( X \) is \( \kappa \)-dependent, where \( \kappa = m + n - 3 \), if and only if there exist a number \( r \), \( 1 \leq r \leq m - 1 \), and an essentially \( \kappa \)-dependent subset \( Y \subset X \), \( \#Y \geq rs \), where \( r + s = \kappa \), belonging to a curve of degree \( r \) and not belonging to any curve of degree less than \( r \).

Moreover, if \( \#Y = rs \), then \( Y \) coincides with the set of intersection points of some two plane curves of degrees \( r \) and \( s \), respectively.

**Proof.** The sufficiency part is obvious. If some set has a \( \kappa \)-dependent subset, then the set itself is \( \kappa \)-dependent. Now let us prove the part of necessity.
4. SOME SPECIAL CASES OF THEOREM 3.1

In this section we verify that Theorem 3.1 is a generalization of Theorems 2.1, 2.2, and 2.3. For this purpose, let us formulate Theorem 3.1 in the special cases with $m = 1, 2, 3,$ and 4.

**Case m = 1.** A set $\mathcal{X}$ of at most $\kappa + 1$ points is never $\kappa$-dependent. This is equivalent to Theorem 2.1.

**Case m = 2.** A set $\mathcal{X}$ of at most $2\kappa + 1$ points is $\kappa$-dependent if and only if $\kappa + 2$ points of $\mathcal{X}$ are collinear.

**Case m = 3.** A set $\mathcal{X}$ of at most $3\kappa - 1$ points is $\kappa$-dependent if and only if one of the following conditions hold:

(i) $\kappa + 2$ points of $\mathcal{X}$ belong to a line;

(ii) $2\kappa + 2$ points of $\mathcal{X}$ belong to a conic.

Clearly, the statement in Case $m = 3$ is a generalization of Theorem 2.2.

**Case m = 4.** A set $\mathcal{X}$ of at most $4\kappa - 5$ points is $\kappa$-dependent if and only if one of the following conditions hold:

(i) $\kappa + 2$ points of $\mathcal{X}$ belong to a line;

(ii) $2\kappa + 2$ points of $\mathcal{X}$ belong to a conic;

(iii) $\#\mathcal{X} = 3\kappa$ and $\mathcal{X}$ coincides with an intersection points of two algebraic curves of degrees 3 and $\kappa$;

(iv) more than $3\kappa$ points of $\mathcal{X}$ belong to a cubic.

Finally, this statement generalizes Theorem 2.3.

Note that in above statements we used Corollary 1.2.

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