Information transfer and orthogonality speed via pulsed-driven qubit

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Abstract

We investigate the transfer and exchange information between a single qubit system excited by a rectangular pulse. The dynamics of the system is treated within and outside rotating wave approximation (RWA). The initial state of the qubit plays an important role for sending information with high fidelity. Within RWA, and as the fidelity of the transformed information increases the exchange information with the environment increases. For increasing values of atomic detuning, the fidelity decreases faster and the exchange information has an upper limit. Outside RWA, the fidelity of the transformed information increases as one increases the perturbation parameter. However the exchange information is very high compared with that within RWA.

The orthogonality speed of the travelling qubit is investigated for different classes of initial atomic state settings and field parameters.

Keywords: Driven qubit, Exchange information, Fidelity, Orthogonality

1 Introduction

Within the context of quantum information theory, quantum objects are considered as carriers of information [1]. A two-state system, such as single 2-level atom, called single qubit is one basic example of quantum information unit. Qubits represent a fundamental aspect of quantum computer [1][2]. Dynamical properties of single qubits have been investigated in different directions. Very recently, single control qubit Shor algorithm for the case of static imperfections induced by residual couplings between qubits was formulated [3].

Cryptographic applications of single-qubit rotations in quantum public-key cryptography has been discussed in references [4]-[6]. The use of single photon qubit as a quantum encoder for single-photon qubits was reported in
The authors in [8] have used the charged qubit pair to perform quantum teleportation. The speed of communication using single qubit has been treated in classical and quantum framework [9, 10, 11].

Dynamical and spectral properties of a short pulsed-driven qubit in the absence of dissipation processes has been studied by many researchers (14-16 and refs therein). In [17], the authors have investigated the fluorescence spectrum of a rectangular pulsed-driven single qubit within and outside the rotating wave approximation (RWA, where fast oscillatory terms are dropped) for different initial atomic states. Specifically, for initial atomic coherent state, the transient fluorescence spectrum exhibits asymmetric Rabi splitting which turns to "ringing" for large pulse area. The ringing behavior is attributed to the initial finite coherent dispersion and interference process between amplitude spectra of the atomic inversion and polarization variables. Properties of single qubit rotation operations using simple RF pulses have been investigated in [18]. Also, measure of the errors in single qubit rotations by pulsed electron paramagnetic resonance has been reported in [19].

In the present work, we investigate the dynamics of the coded information in a single qubit subject to a rectangular pulse. We quantify the exchange information between the qubit and the environment (i.e the pulse). Also, we quantify the speed of quantum communication by evaluating the speed of orthogonality of the density operator.

The paper is organized as follows. In Sec.2, we present the model and its exact operator solution within the RWA and its iterative solution outside RWA [17]. The fidelity and the exchange information is the subject of Sec.3. The speed of orthogonality and hence the speed of transfer information is discussed in Sec.4. Finally, a summary is given in Sec.5.

2 The Model and its solutions

We consider a single 2-level atom (qubit) of transition frequency $\omega_a$ interacts with a rectangular pulse of circular frequency $\omega_l$ in the absence of any atomic dissipation. The full Hamiltonian of this system outside the RWA (in units of $\hbar = 1$) is given by [17],

$$
\hat{H} = \omega_a \hat{S}_z + \frac{\Omega(t)}{2} \left\{ (\hat{S}_+ + \hat{S}_-) (e^{-i\omega_l t} + e^{i\omega_l t}) \right\},
$$

(1)
where the spin-$\frac{1}{2}$ operators $\hat{S}_x, \hat{S}_z$ obey the $SU(2)$ algebra,

$$[\hat{S}_+, \hat{S}_-] = 2\hat{S}_z, \quad [\hat{S}_z, \hat{S}_\pm] = \pm \hat{S}_\pm. \quad (2)$$

For a rectangular laser pulse of duration $T$, the Rabi frequency $\Omega(t) = \Omega f(t)$ with $f(t) = 1$ through the interval of time $t \in [0, T]$ and $f(t) = 0$ otherwise, with $\Omega$ taken real. The pulse duration $T$ is much shorter than the lifetime of the atomic upper state, hence atomic damping can be discarded. Introducing the rotating frame operators

$$\hat{\sigma}_\pm(t) = \hat{S}_\pm(t)e^{\mp i\omega_l t}, \quad \hat{\sigma}_z(t) \equiv \hat{S}_z(t), \quad (3)$$

where the $\hat{\sigma}$ operators obey the same algebraic form of Eq.(2), Heisenberg equations for the atomic operators $\hat{\sigma}_x, \hat{\sigma}_z$ according to (1) are of the form,

$$\dot{\hat{\sigma}}_+ = i\Delta \hat{\sigma}_+ - i\Omega(t){\hat{\sigma}}_z(1 + e^{-2i\omega_l t}) = \left[\hat{\sigma}_-, \hat{\sigma}_+\right] \dagger,$$

$$\dot{\hat{\sigma}}_z = -i\frac{\Omega(t)}{2}\left[\hat{\sigma}_+(1 + e^{2i\omega_l t}) - \hat{\sigma}_-(1 + e^{-2i\omega_l t})\right], \quad (4)$$

where $\Delta = \omega_a - \omega_l$, is the atomic detuning. Not that the terms in $e^{\pm2i\omega_l t}$ represent the effect of the interaction of the atom with the pulse outside RWA (note that, the issue of RWA is only associated with linearly polarized light; cf. [14]).

(a) Exact solution within RWA.

Within RWA we discard the terms $e^{\pm2i\omega_l t}$ in (4) and assume the qubit initially in the coherent state,

$$|\theta, \phi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{-i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle, \quad (5)$$

where $0 \leq \phi \leq 2\pi, 0 \leq \theta \leq \pi$ and $|0\rangle, |1\rangle$ are the bare ground and excited atomic states respectively. Using the notation,

$$u_i(0) = \langle \theta, \phi | \hat{\sigma}_i(0) | \theta, \phi \rangle, \quad i = x, y, z \quad (6)$$

for the initial Bloch vector where $\hat{\sigma}_\pm = \hat{\sigma}_x \pm i \hat{\sigma}_y$, the exact solution of Eq.(4) in terms of the Bloch vector $u_i(t)$ within RWA are written in the following form [17],

$$\begin{pmatrix} u_x(0) \\ u_y(0) \\ u_z(0) \end{pmatrix} = \begin{pmatrix} \alpha_x(0) & \alpha_y(0) & \alpha_z(0) \\ \beta_x(0) & \beta_y(0) & \beta_z(0) \\ \gamma_x(0) & \gamma_y(0) & \gamma_z(0) \end{pmatrix} \begin{pmatrix} u_x(0) \\ u_y(0) \\ u_z(0) \end{pmatrix}, \quad (7)$$
where
\[
\begin{align*}
\alpha_x^{(0)} &= \frac{1}{2}\left[\left(\frac{\Omega}{\Omega_1}\right)^2 + \left(\frac{\Delta^2 + \Omega_1^2}{\Omega_1^2}\right) \cos \Omega_1 t + \left(\frac{\Omega}{\Omega_1}\right)^2 (1 - \cos \Omega_1 t) \right], \\
\alpha_y^{(0)} &= -\left(\frac{\Delta}{\Omega_1}\right) \sin \Omega_1 t, \\
\alpha_z^{(0)} &= \left(\frac{\Delta \Omega}{\Omega_1^2}\right) (1 - \cos \Omega_1 t), \\
\beta_x^{(0)} &= -\alpha_y^{(0)}, \\
\beta_y^{(0)} &= \frac{1}{2}\left[\left(\frac{\Omega}{\Omega_1}\right)^2 + \left(\frac{\Delta^2 + \Omega_1^2}{\Omega_1^2}\right) \cos \Omega_1 t - \left(\frac{\Omega}{\Omega_1}\right)^2 (1 - \cos \Omega_1 t) \right], \\
\beta_z^{(0)} &= -\left(\frac{\Omega}{\Omega_1}\right) \sin \Omega_1 t, \\
\gamma_x^{(0)} &= \alpha_z^{(0)}, \\
\gamma_y^{(0)} &= -\beta_z^{(0)}, \\
\gamma_z^{(0)} &= \left(\frac{\Omega}{\Omega_1}\right)^2 \left[\cos \Omega_1 t + \left(\frac{\Delta}{\Omega}\right)^2\right],
\end{align*}
\] (8)

with \(\Omega_1 = \sqrt{\Omega^2 + \Delta^2}\) and we have used the notations \(u_{x,y,z}^{(0)}(t)\) are used for the exact RWA solution.

(b) Iterative solution outside RWA.

Outside RWA, Eq.(4) has an iterative solution to \(\mathcal{O}(\lambda)\); \(\lambda = \frac{\Omega}{\omega_0}\) at exact resonance \((\Delta = 0)\) and is given by [17],
\[
\begin{pmatrix}
\alpha_x^{(1)}(t) \\
\alpha_y^{(1)}(t) \\
\alpha_z^{(1)}(t)
\end{pmatrix} =
\begin{pmatrix}
\alpha_x^{(1)} & \alpha_y^{(1)} & \alpha_z^{(1)} \\
\beta_x^{(1)} & \beta_y^{(1)} & \beta_z^{(1)} \\
\gamma_x^{(1)} & \gamma_y^{(1)} & \gamma_z^{(1)}
\end{pmatrix}
\begin{pmatrix}
u_x(0) \\
u_y(0) \\
u_z(0)
\end{pmatrix},
\] (9)

where,
\[
\begin{align*}
\alpha_x^{(1)} &= 1 + \lambda \left[\left(2 - \frac{1}{4} \sin 2\omega_0 t \right) \sin \Omega t + \left(1 - \frac{1}{4} \cos 2\omega_0 t \cos \Omega t \right) \right], \\
\alpha_y^{(1)} &= -\frac{\lambda}{4} \left[\left(1 - \cos 2\omega_0 t \right) \sin \Omega t - \sin 2\omega_0 t \cos \Omega t \right], \\
\alpha_z^{(1)} &= \frac{\lambda}{2} \left(\cos 2\omega_0 t \cos \omega t - 1 \right), \\
\beta_x^{(1)} &= \frac{\lambda}{4} \left[\left(1 - \cos 2\omega_0 t \right) \sin \Omega t + \sin 2\omega_0 t \cos \Omega t \right],
\end{align*}
\]
\( \beta_y^{(1)} = \cos \Omega t - \frac{\lambda}{4} \left[ \sin 2\omega t \sin \Omega t + (4 - \cos 2\omega t) \cos \Omega t \right], \)

\( \beta_z^{(1)} = -\sin \Omega t - \frac{\lambda}{2} \sin 2\omega t \cos \Omega t, \)

\( \gamma_x^{(1)} = \lambda(2 - \cos 2\omega t - \cos \Omega t), \)

\( \gamma_y^{(1)} = \sin \Omega t + \lambda \sin 2\omega t, \)

\( \gamma_z^{(1)} = \cos \Omega t. \)  

and have used the notations \( u_{x,y,z}^{(1)}(t) \) for the iterative solution outside RWA. Note, for \( \lambda = 0 \), the solutions (9) with (10) reduce to those in (7) with (8) at \( \Delta = 0 \).

### 3 Information transfer

Let us assume that we have coded information in the initial state of the qubit. The initial density operator takes the form,

\[ \rho(0) = \frac{1}{2} \left( 1 + u_x(0)\sigma_x + u_y(0)\sigma_y + u_z(0)\sigma_z \right). \]  

Due to the interaction with the pulse the coded information may be exchanged between the pulse and the initial state in (11). Our aim is to find how information evolves with time. We measure the transfer information using the fidelity, \( F = \text{tr}\{\rho(t)\rho(0)\} \), while the entropy exchange, \( S_e = -\text{tr}\{\rho \ln \rho\} \) is used as a measure of the information exchanged between the system and the environment [21]. For this purpose we plot some figures for different initial state settings within and outside RWA.

(a) Within RWA.

In Fig.(1), we investigate the effect of the initial phase angle \( \phi \), on the fidelity of the transfer and exchange information. Fig.(1a) shows the periodic dynamical behavior of the fidelity in the resonant case i.e \( \Delta = 0 \). The initial state of the coded information is chosen such that \( |\psi(0)\rangle = e^{-i\phi}|1\rangle \). It is clear that, for \( \phi = \frac{\pi}{2} \) the fidelity, \( F \) decreases with time smoothly and completely vanishes for the first time round at \( \Omega t \simeq 2.5 \). As one decreases \( \phi \) the fidelity decreases but does not completely vanish and the minimum values of the fidelity \( F \), decrease as \( \phi \) decreases .

In Fig.(1b), we investigate the amount of information exchanged between the state and the environment for the same parameters as in Fig.(1a). As the
Figure 1: The Fidelity $\mathcal{F}$ and the exchange information $E_n$, (a), (b) respectively against the scaled time $\tau = \Omega t$ for the system (within RWA), $\lambda = 0$ with $\Delta = 0$ and $\theta = \frac{\pi}{2}$. The solid, dash and dot curves are for $\phi = \frac{\pi}{2}, \frac{\pi}{3}$ and $\frac{\pi}{4}$ respectively.

The fidelity decreases the exchange information increases but does not reach the maximum value i.e there is still coherent information in the state which carries these coded information. Also, with the fidelity $\mathcal{F}$ reaches to its minimum value there is no exchange information. This means that, for some interval of time one can use the driven qubit safely in quantum communication.

In Fig.(2), we consider different initial states setting for $\theta$, where the phase angle is fixed at $\phi = \frac{\pi}{4}$. In this case the initial state of the driven qubit is defined by $|\psi(0)\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{-i\pi/4}|1\rangle$. In general the behavior of the fidelity is the same as that depicted in Fig.(1a), but the the fidelity does not completely vanish as in Fig.(1a) for some classes of initial states. As $\theta$ decreases, the minimum value of $\mathcal{F}$ decreases. Also, the exchange information, Fig.(2b), increases for large fidelity and vice-versa. At the minimum values of the fidelity the exchange almost vanishes.

The effect of the detuning parameter, $\Delta$ is seen in Fig.(3), where we consider the choice for $\theta = \frac{\pi}{3}$ and $\phi = \frac{\pi}{4}$ which maximize the fidelity and minimize the change information. In this case the initial state in which we coded the information is given by $|\psi(0)\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}e^{-i\pi/4}|1\rangle$. The general behavior of the fidelity is the same as shown in Fig.(1) and Fig.(2). For a very small value of the detuning $\Delta = 0.1$, the minimum value of $\mathcal{F}$ is lesser than the resonance case: see the solid curve in Fig.(2a) and Fig.(3a). As one increases the value of $\Delta$, the curves shift to the left. This means that the fidelity decreases faster at earlier time. Fig.(3b) describes the dynamics
Figure 2: The same as Fig.(1), but for fixed $\phi = \frac{\pi}{4}$ and different $\theta = \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}$ for the solid, dashed and dot curves respectively.

Figure 3: The same as Fig.(1) but for $\theta = \frac{\pi}{3}, \phi = \frac{\pi}{4}$. The solid, dashed and dot curves for $\Delta = 0.1, 0.5$ and 0.8 respectively.
Figure 4: (a) The fidelity $F$ and (b) the information exchange $E_n$ against $\tau$ for the system treated outside the RWA ($\lambda \neq 0$). The initial state of the qubit is described by $|\theta = \phi = \pi/2\rangle$ and $\Omega = \omega = 1$. The solid, dashed and dot curves for $\lambda = 0.01, 0.2$ and $0.4$ respectively.

(b) Outside the RWA.

The behavior of the fidelity $F$ and the information exchange $E_n$ for the system given by (9) where the system is treated outside RWA is shown in Fig.(4). In this figure we set $\theta = \phi = \pi/2$ (the case described by the solid curve in Fig.(1a), within RWA). The dynamics of the fidelity is displayed in Fig.(4a) for different values of the parameter $\lambda$. It is clear that for small value of $\lambda (= 0.01)$, the behavior is almost the same as that within RWA. However, for larger values of $\lambda$, the minimum value of the fidelity increases. This means that the loss of the coded information decreases. Fig.(4b) describes the dynamics of the exchange information between the travelling qubit and the environment for the same parameter as that used in Fig.(1a). This figure shows that the amount of information exchange between the qubit and the environment is much larger compared with that depicted in Fig.(1b) (solid curve within RWA). Also, the minimum value of $E_n$ increases for larger values of $\lambda$. The effect of the excitation angle $\theta$ and the phase $\phi$ is investigated in Fig.(5), where we set the iteration parameter $\lambda = 0.2$. The dynamics of the
fidelity $\mathcal{F}$ is shown in Fig.(5a) for different values of the excitation angle $\theta$. At $t = 0$ the fidelity $\mathcal{F} < 1$, due to the effect of the $\lambda$-parameter $\lambda$. However, as one decreases the value of $\theta$, the upper and lower values of the fidelity show slight change. Fig.(5b), displays the behavior of $\mathcal{F}$ for different phases. As the phase decreases the minimum value of the fidelity is much larger and consequently the exchange information between the travelling qubit and the environment decreases. So, by decreasing the value of the phase one can overcome the negative effect of the iteration parameter $\lambda$.

Therefore, from our preceding results the fidelity of the transmitted information, which is coded in an initial state described by small excitation angle $\theta$ and phase $\phi$ decreases faster but the maximum value of the exchange information with the environment is smaller. However, the minimum value of fidelity loss can be improved for larger values of the angle $\theta$ and the phase $\phi$. If we consider a small value of the detuning parameter, we can send information with high fidelity, while the exchanged information is always upper bounded. Outside the RWA the fidelity of the travelling qubit could improve, but at the the same time increases the possibility of information exchange with the environment. The effect of the excitation angle $\theta$ outside RWA has a slight effect and the fidelity doesn’t reach its maximum value. However, for a smaller phase angle $\phi$, the value of $\mathcal{F} = 1$.

Figure 5: (a) The fidelity $\mathcal{F}$ against $\tau$ outside RWA ($\lambda = 0.2$) for fixed $\phi = \frac{\pi}{2}$ and different $\theta = \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}$ (solid, dashed and dot curves respectively) (b) The same as (a) but for fixed $\theta = \frac{\pi}{2}$ and different $\phi = \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}$ (solid, dashed and dot curves respectively)
4 Orthogonality Speed

In the previous section, we showed how the coded information evolves from one location (node) to another [9]. In a quantum computer, it is important to know the speed of sending information from one node to another. Since the information is coded in a density operator, therefore we seek how fast the density operator will change its orthogonality. In other words, we search for a minimum time needed for a quantum system to pass from one orthogonal state to another [9, 10].

Let us assume that the eigenvectors of the initial state are given by $u_1(0)$, $u_2(0)$ and for the final state are given by $u_1(t)$, $u_2(t)$. The speed of quantum computation is defined by the maximum number of orthogonal states that the system can pass through per unit time. The orthogonality is given by the scalar product of the vectors [12, 13]

$$S_{pij} = \langle u_i(0)|u_j(t) \rangle, \quad i, j = 1, 2.$$  \hspace{1cm} (12)

(a) Within RWA.

In Fig.(6), we have plotted $|S_{pij}|$, the amplitude values of $S_{pij}$ against the scaled time to display its behavior for different initial atomic state settings. From our results in Sec.3, we know that for small values of $\theta$ and $\phi$, one can transmit the coded information with high degree of fidelity.

In Fig.(6a), in the resonant case ($\Delta = 0$) and $\theta = \pi/2$ and a very small value of $\phi = \pi \times 10^{-3}$, it is clear that $|S_{pij}|$ vanishes periodically at some specific time, $\Omega t$. At these times, the initial and final states are orthogonal and the information which is coded in the initial state is completely transferred to the final state. As one increases the phase angle $\phi = \pi/10$, the speed of orthogonality decreases (Fig.(6b)), where the number of vanishing $|S_{pij}|$ is lesser than that for smaller phase (see Fig.(6a)). Fig.(6c), is plotted for different class of initial state settings, where we consider $\theta = \pi/4$. It is seen that the number of orthogonality decreases and consequently the speed of computations is smaller than that depicted in Fig.(6a). The non-resonant case is shown in Fig.(6d), for $\Delta = 0.7$ where $|S_{pij}|$ does not vanish periodically as that predicted for the resonant case (see Fig.(6a)). As an example, around $\omega t \simeq 25$ and 30, $|S_{pij}| \neq 0$. This means that the information has not completely transferred to the final state and consequently there are some information gained by the environment.
Figure 6: The speed of orthogonality of the qubit as a function of the scaled time $\tau$. (a) For $\theta = \frac{\pi}{2}, \phi = 10^{-3}\pi, \Delta = 0$. (b) The same as (a), but $\phi = 10^{-1}\pi$. (c) The same as (a), but $\theta = \frac{\pi}{4}$. (d) The same as (a), but for $\Delta = 0.7$.

(b) Outside RWA.

The dynamics of the orthogonality speed for a system treated outside RWA is displayed in Fig.(7). The effect of the iteration parameter $\lambda$ is shown in Figs.(7a&7b), where we set $\lambda = 10^{-4}$ and $0.08$ respectively. It is clear that, the orthogonality speed $|Sp_{ij}|$ increases for small value of $\lambda$. As one decreases the excitation angle $\theta(=\frac{\pi}{8})$, with other parameters kept the same as in Fig.(7a), the speed of orthogonality decreases (see Fig.(7c)). However for large value of the phase angle, the orthogonality speed decreases- compare Figs.(7a&7d).
Figure 7: The orthogonality speed for the qubit treated outside the RWA ($\lambda \neq 0$) vs time, $\tau$. (a) For $\theta = \frac{\pi}{2}, \phi = \frac{\pi}{8}$ and $\lambda = 10^{-4}$. (b) The same as Fig. (a), but $\lambda = 0.08$. (c) The same as (a), but $\theta = \frac{\pi}{4}$ and $\phi = \frac{\pi}{8}$. (d) The same as (a), but $\theta = \phi = \frac{\pi}{4}$.

5 Conclusion

The dynamics of the coded information in a pulsed-driven qubit is investigated within and outside the rotating wave approximation (RWA). Within RWA, the initial atomic state setting plays the central role for the fidelity of sending information from one location to another. The fidelity and the exchange information are increased by decreasing the phase $\phi$ and the polarized angle $\theta$ of the initial atomic coherent state $|\theta, \phi\rangle$. The sensitivity of the fidelity of the transmitted information to the detuning parameter is discussed, where for large values of detuning, the fidelity decreases faster. However, the maximum value of the exchange information between the qubit and the environment is slightly affected as one increases the detuning parameter. Also,
we have investigated the effect of the initial state setting and the detuning on the speed orthogonality where it is shown that, the driven qubit could be used to achieve quantum computations much faster.

Outside RWA, we have examined the effect of the iteration parameter $\lambda$ on the dynamics of the exchange information and the fidelity of the transmitted information. Although the minimum value of the fidelity increases as one increases the value of $\lambda$, the exchange information between the travelling information and the environment increases. The excitation angle ($\theta$) has a small effect on the dynamics of the fidelity of the travelling information. However, for smaller value of the phase ($\phi$), the minimum value of the travelling information’s fidelity increases and its maximum value reaches one. The speed of orthogonality, for the system treated outside RWA is investigated for different values of $\lambda$, excitation and phase angles ($\theta, \phi$). We show that, the orthogonality speed decreases for large values of $\lambda$ and the initial phase $\phi$. However the speed of orthogonality decreases as one decreases the excitation angle ($\theta$).

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