THE GROUP GENERATED BY THE ROUND FUNCTIONS OF A GOST-LIKE CIPHER

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Abstract. We define a cipher that is an extension of GOST, and study the permutation group generated by its round functions. We show that, under minimal assumptions on the components of the cipher, this group is the alternating group on the plaintext space. This we do by first showing that the group is primitive, and then applying the O’Nan-Scott classification of primitive groups.

1. Introduction

When DES was about to be broken by brute force, and Triple DES was introduced as a replacement, Kaliski, Rivest and Sherman considered in [KRS88] the question, whether DES (that is, the set of transformations it defines) is a group. Had this been the case, Triple DES would have been no different from DES. They gave evidence for the fact that DES was indeed not a group, and also showed that if the group generated by a cipher is too small, then certain attacks based on the birthday paradox are possible. Note, however, that Murphy, Paterson and Wild [MPW94] have constructed a weak cipher that generates the whole symmetric group — therefore the latter requirement alone is not enough to guarantee the strength of the cipher.

Coppersmith and Grossman defined a set of functions which can be adapted for constructing a block cipher, and studied the permutation group generated by them [CG75]. Even and Goldreich defined certain DES-like functions, and proved that the permutation group generated by these functions is the alternating group [EG83]. Wernsdorf later showed that the group generated by the round functions of DES is the alternating group [Wer93], and Sparr and Wernsdorf showed that the same holds for KASUMI [SW15] and AES [SW08]. Since the group generated by a cipher (with independent round keys) is a normal subgroup of the group generated by the round functions, and the alternating group is a simple group, it follows that the former group is also alternating.
In [CDVS09b, CDVS09a, ACDVS14] another approach to these questions was taken, in that one first shows that the group generated by the round functions of an AES-like cipher is a primitive permutation group, provided the S-boxes satisfy some cryptographic assumptions, such as being weakly APN functions. This shows that the cipher has no imprimitivity trapdoor [Pat99]. And then the O’Nan-Scott classification of finite primitive groups [LPS88, Li03] is used to show that the group must be alternating or symmetric. In this paper we apply this point of view to an extension of the cipher GOST 28147-89 [Dol10], or GOST for short, and show that its round functions generate the alternating group. It might be noted that we require only minimal assumptions on the components of this cipher, basically only that the S-boxes are bijective, and that the rotation has the “right” extent. This appears to indicate that the Feistel structure plays an important role in guaranteeing that the group is large.

Oliynykov considered in [Oli11] ciphertext-only attacks on Feistel networks, and proved that the use of secret, non-bijective S-boxes allows for the introduction of trapdoors. In particular, the author applied his results to GOST.

In Section 2 we describe GOST. In Section 3 we introduce our extension of GOST. In Section 4 we show that the group generated by the round functions of this GOST-like cipher is primitive. In Section 5 we analyse the cases in the O’Nan-Scott classification, to conclude that the group generated by the round functions of our GOST-like cipher is the alternating group.

2. The group generated by the round functions of GOST

Consider the set $V^0 = \mathbb{F}_2^n$, for some $n > 1$. (Here $\mathbb{F}_2$ is the field with two elements, and see Remark 2.3 for the actual values in GOST of this, and the other parameters we are going to introduce in the following.) We consider two group structures on $V^0$. The first operation is the bitwise sum (XOR), which will be denoted by $\oplus$. The bitwise sum makes $V^0$ into a vector space over $\mathbb{F}_2$.

The second operation, denoted by $\boxplus$, is the sum modulo $2^n$. That is, we represent $a, b \in V^0$ as

$$a = (a_0, a_1, \ldots, a_{n-1}), \quad b = (b_0, b_1, \ldots, b_{n-1}),$$

with $a_i, b_i \in \{0, 1\}$ integers, and let

$$a \boxplus b = (c_0, c_1, \ldots, c_{n-1}),$$

where

$$(a_0 + a_12 + a_22^2 + \cdots + a_{n-1}2^{n-1}) + (b_0 + b_12 + b_22^2 + \cdots + b_{n-1}2^{n-1}) \equiv c_0 + c_12 + c_22^2 + \cdots + c_{n-1}2^{n-1} \pmod{2^n},$$

with $c_i \in \{0, 1\}$ integers. (Here $+$ denotes the ordinary sum of integers.) Therefore $V^0$ under $\boxplus$ is the same thing as the group $\mathbb{Z}_{2^n}$ of integers modulo $2^n$, and we will denote it by $(\mathbb{Z}_{2^n}, \boxplus)$. We use $\boxminus a$ to indicate the opposite of $a \in V^0$ with respect to $\boxplus$. 

We record a few elementary facts that we will be using repeatedly without further mention.

**Lemma 2.1.**

- The subgroups of \((\mathbb{Z}_{2^n}, \oplus)\) are linearly ordered; they are the \(\langle 2^q \rangle\), for \(0 \leq q \leq n\).
- The endomorphisms of \((\mathbb{Z}_{2^n}, \oplus)\) are of the form \(x \mapsto zx\), where \(z\) is an integer, \(0 \leq z < 2^n\). Such a map is an automorphism if and only if \(z\) is odd.
- Every subgroup of \((\mathbb{Z}_{2^n}, \oplus)\) is fully invariant (that is, it is sent into itself by any endomorphism of \((\mathbb{Z}_{2^n}, \oplus)\)) and thus characteristic (that is, it is sent onto itself by any automorphism of \((\mathbb{Z}_{2^n}, \oplus)\)).
- The element \(2^{n-1} = (0, 0, \ldots, 0, 1)\) is the only involution (that is, element of order 2) of \((\mathbb{Z}_{2^n}, \oplus)\). Therefore \(2^{n-1}\) is fixed by any automorphism of \((\mathbb{Z}_{2^n}, \oplus)\), and it is sent to zero by any endomorphism which is not an automorphism.

In GOST 28147-89 [Dol10] the plaintext space is \(V = V^1 \times V^2\), where \(V^1, V^2\) are two copies of \(V^0\), and the key space \(K\) is another copy of \(V^0\). Clearly \(V\) inherits both group structures componentwise from \(V^1, V^2\).

**Definition 2.2.** When considering a subset of \(V^i\), for \(i = 0, 1, 2\), we will call it

- a subspace if it is a subgroup (and thus a vector subspace) of \((\mathbb{F}_2^n, +)\), and
- a \(\oplus\)-subgroup, or simply a subgroup, if it is a subgroup of \((\mathbb{Z}_{2^n}, \oplus)\).

This terminology can be extended to the subsets of \(V\).

**Definition 2.3.** We will consider \(V^i\), for \(i = 0, 1, 2\), as the Cartesian product

\[
V^i = V^i_1 \times \cdots \times V^i_\delta = V^i_1 || \cdots || V^i_\delta
\]

of \(\delta > 1\) subspaces \(V^i_j\), all of the same dimension \(m > 1\). (Here || denotes concatenation of strings.)

An element \(\gamma\) of the symmetric group \(\text{Sym}(V^i)\) on \(V^i\) is called a bricklayer transformation with respect to (2.1) if it preserves the direct product decomposition, that is, if there are S-boxes \(\gamma_j \in \text{Sym}(V^i_j)\) such that, writing \(v \in V^i\) as

\[
v = (v_1, \cdots, v_\delta),
\]

with \(v_j \in V^i_j\), we have

\[
v\gamma = (v_1\gamma_1, \cdots, v_\delta\gamma_\delta).
\]

We will refer to each \(V^i_j\) as a brick.

Let \(S = \gamma R \in \text{Sym}(V^i)\), where \(\gamma \in \text{Sym}(V^i)\) is a bricklayer transformation and \(R\) is the right rotation by \(r\) bits (we refer to \(r\) as the extent of the rotation), with \(m \leq r \leq (\delta - 1)m\), that is

\[
(a_0, \ldots, a_{n-1})R = (a_{n-r}, \ldots, a_{n-1}, a_0, \ldots, a_{n-r-1}).
\]
Remark 2.4. In the case of GOST, the actual values of the parameters are: \( n = 32, m = 4, \delta = 8 \) and \( r = 11 \).

For \( (k_1, k_2) \in V = V^1 \times V^2 \), consider the \( \boxplus \)-translation on \( V \) by \( (k_1, k_2) \)
\[
\rho_{(k_1, k_2)} : V_1 \times V_2 \to V_1 \times V_2,
\]
\[
(x_1, x_2) \mapsto (x_1 \boxplus k_1, x_2 \boxplus k_2).
\]

We now introduce a formal \( 2n \times 2n \) matrix, which implements the Feistel structure,
\[
\Sigma = \begin{bmatrix} 0 & 1 \\ 1 & S \end{bmatrix},
\]
where \( 0 \) and \( 1 \) are \( n \times n \) matrices. This acts (on the right) on \( (x_1, x_2) \in V = V^1 \times V^2 \) by
\[
(x_1, x_2)\Sigma = (x_2, x_1 + x_2S).
\]
Note that \( \Sigma \) has the formal inverse matrix
\[
\Sigma^{-1} = \begin{bmatrix} S & 1 \\ 1 & 0 \end{bmatrix}.
\]

A round function of GOST with respect to the round key \( k \in K \) can now be described as
\[
\tau_k = \rho_{(0, k)} \Sigma \rho_{(\boxplus k, 0)}.
\]
(As we let permutations act on the right, this is a left-to-right composition.) In fact
\[
(x_1, x_2)\tau_k = (x_1, x_2) \rho_{(0, k)} \Sigma \rho_{(\boxplus k, 0)}
\]
\[
= (x_1, x_2 \boxplus k) \Sigma \rho_{(\boxplus k, 0)}
\]
\[
= (x_2 \boxplus k, x_1 + (x_2 \boxplus k)S) \rho_{(\boxplus k, 0)}
\]
\[
= (x_2, x_1 + (x_2 \boxplus k)S).
\]

Thus the group generated by the round functions of GOST is
\[
\mathcal{G} = \langle \tau_k : k \in K \rangle.
\]

3. A LARGER GROUP

In our notation, in an actual GOST round \( \tau_k \) the key addition (\( \boxplus \)-translation) preceding \( \Sigma \), and that following \( \Sigma \), are related: the first one acts only on \( V^2 \), the second one only on \( V^1 \), and the extents of the two translations are one the \( \boxplus \)-opposite of the other. In this paper we will be studying a GOST-like system in which a round generalizes the one of GOST: we allow to \( \boxplus \)-sum two arbitrary (unrelated) pairs of keys before and after applying the Feistel transformation \( \Sigma \). So in our cipher the plaintext \( V \) is the same as that of GOST, while the key space is \( \mathcal{H} = K \times K = V \), and a round takes the form
with \( k, h \in \mathcal{H} \). Such a round operates on \((x_1, x_2) \in V = V^1 \times V^2\) by
\[
(x_1, x_2) \rho_k \Sigma \rho_h = (x_2 \boxplus k_2, x_1 \boxplus k_1 + (x_2 \boxplus k_2)S) \rho_h
= (x_2 \boxplus k_2 \boxplus h_1, (x_1 \boxplus k_1 + (x_2 \boxplus k_2)S) \boxplus h_2),
\]
where \( k_i, h_i \in V^i \).

The corresponding group will thus be
\[
\Gamma = \langle \rho_k \Sigma \rho_h : k, h \in \mathcal{H} \rangle
\]
Clearly our group \( \Gamma \) contains the group \( \mathcal{G} \) generated by the round functions of GOST.

We collect a couple of elementary observations.

(1) \( \Sigma \in \Gamma \). This follows from setting \( k = h = 0 \) in \( (3.1) \).

(2) For all \( k \in \mathcal{H} \), we have that \( \rho_k \in \Gamma \). It suffices to set \( h = 0 \) in \( (3.1) \) and then note that \( \rho_k = (\rho_k \Sigma) \Sigma^{-1} \) is in \( \Gamma \), as both factors are.

Therefore
\[
(3.2) \quad \Gamma = \langle \mathcal{T}, \Sigma \rangle,
\]
where
\[
\mathcal{T} = \{ \rho_k : k \in \mathcal{H} \}
\]
is the group of \( \boxplus \)-translations on \( V \). In particular, \( \Gamma \) acts transitively on \( V \).

We now state our main result.

**Theorem 3.1.** Let \( n = \delta m \), with \( \delta \geq 4 \) and \( m \geq 2 \). Consider the \( \mathbb{F}_2 \)-vector spaces \( V^i = \mathbb{F}_2^n \), for \( i = 1, 2 \), and \( V = V^1 \times V^2 \), under the operation \( + \). For \( i = 1, 2 \), write
\[
(3.3) \quad V^i = V^i_1 \times \cdots \times V^i_\delta,
\]
where each \( V^i_j \) is a subspace of dimension \( m \) over \( \mathbb{F}_2 \).

Let \( \boxplus \) be the operation on \( V^i, V \) defined in the previous Section, so that each \((V^i, \boxplus)\) is cyclic, of order \( 2^n \). Let \( \mathcal{T} \) be the group of \( \boxplus \)-translations \( \rho_k : x \mapsto x \boxplus k \) on \( V \), for \( k \in V \).

Consider

(1) A bricklayer transformation \( \gamma \) with respect to \( (3.3) \).

(2) The right rotation \( R \) by \( r \) bits on \( V^i \).

(3) \( S = \gamma R \).

(4) The formal matrix
\[
\Sigma = \begin{bmatrix} 0 & 1 \\ 1 & S \end{bmatrix},
\]
which operates on \( V = V^1 \times V^2 \) by
\[
(x_1, x_2) \Sigma = (x_2, x_1 + x_2S).
\]

Consider the GOST-like cipher with plaintext and key space \( V \), in which a round has the form
\[
\rho_k \Sigma \rho_h,
\]
for the round keys \( k, h \in V \).

Then the group generated by the round functions is
\[
\Gamma = \langle T, \Sigma \rangle,
\]
where \( T = \{ \rho_k : k \in V \} \) is the set of \( \oplus \)-translations on \( V \).

Assume that
1. the rotation extent \( r \) satisfies \( m \leq r \leq (\delta - 1)m \), and
2. the bricklayer transformation \( \gamma \) is bijective (equivalently, each S-box is bijective, or \( S \) is bijective).

Then
\[
\Gamma = \text{Alt}(V).
\]

Here \( \text{Alt}(V) \) is the alternating group, consisting of the even permutations on the set \( V \). We record the following

**Lemma 3.2.** All permutations of \( \Gamma \) are even, that is, \( \Gamma \leq \text{Alt}(V) \).

**Proof.** The group \( T \) of \( \oplus \)-translations is generated by \( \rho_{(0,1)} \) and \( \rho_{(1,0)} \). Both maps are even permutations, as each of them is the product of \( 2^n \) cycles of length \( 2^n \).

We now show that \( \Sigma \) is also an even permutation. \( \Sigma \) can be considered as the composition of two permutations of order 2 of \( V \). The first permutation
\[
(x_1, x_2) \mapsto (x_2, x_1),
\]
which exchanges the coordinates, has the \( 2^n \) fixed points \((x, x)\), for \( x \in V^0 \), and thus it is the product of an even number
\[
\frac{2^{2n} - 2^n}{2} = 2^{2n-1} - 2^{n-1}
\]
of 2-cycles, as \( n > 1 \). The second permutation
\[
(x_2, x_1) \mapsto (x_2, x_1 + x_2S)
\]
has also order 2, and has also \( 2^n \) fixed points, which correspond to the value \( x_2 = 0S^{-1} \), and thus it is also even. \( \square \)

**Remark 3.3.** The arguments of Section 4 could be extended to cover any rotation different from the identity. For the arguments of Subsection 5.2 to work with any rotation different from the identity, however, we would need to add extra hypotheses on the behaviour of the last S-box. Therefore we have preferred to stick to this setting, which requires only two natural assumptions on the cipher.
Let us consider a cipher consisting of a fixed number of rounds as in Theorem 3.1 with independent round keys. The group $\Gamma'$ generated by (the transformations of) this cipher will be a normal subgroup of $\Gamma$. (See Lemma 3.4 below.) Since the alternating group acting on at least 5 letters is simple, it follows $\Gamma'$ is also the alternating group on $V$.

**Lemma 3.4.** Let $\Gamma$ be a group generated by elements $g_i$, for some index set.

Let $N$ be a positive integer.

Let $\Gamma'$ be the subgroup of $\Gamma$ generated by all products

$$g_{i_1}g_{i_2} \cdots g_{i_N}.$$ 

Then $\Gamma'$ is a normal subgroup of $\Gamma$.

**Proof.** We have to show that for all choices of generators $g = g_{i_0}, g_{i_1}, g_{i_2}, \ldots, g_{i_N}$ of $\Gamma$, the conjugate $g^{-1}(g_{i_1}g_{i_2} \cdots g_{i_N})g$ lies in $\Gamma'$.

We have

$$g^{-1}(g_{i_1}g_{i_2} \cdots g_{i_N})g = (g^N)^{-1}(g^{N-1}g_{i_1})(g_{i_2} \cdots g_{i_N}g) \in \Gamma'.$$

Clearly our result for $\Gamma$ has no immediate implication about the size of the smaller group $G$ of GOST.

**4. Primitivity**

We recall a couple of basic properties of imprimitive groups. Let $G$ be a finite group acting transitively on a set $V$.

**Lemma 4.1.** A block (of imprimitivity) is of the form $vH$, for some $v \in V$, and some proper subgroup $H$ of $G$ which properly contains the stabiliser of $v$ in $G$.

**Lemma 4.2.** If $T$ is a transitive subgroup of $G$, then a block for $G$ is also a block for $T$.

In our case, $T$ is a transitive subgroup of $\Gamma$. We first record a trivial observation, which is an immediate consequence of the fact that the map $v \mapsto \rho_v$ is an isomorphism $(V, \ominus) \to T$.

**Lemma 4.3.** The subgroups of $T$ are of the form

$$\{ \rho_u : u \in U \},$$

where $U$ is a subgroup of $(V, \ominus)$.

We obtain

**Lemma 4.4.** If $\Gamma$ acting on $V$ has a block system, then this consists of the cosets of a $\ominus$-subgroup of $V$, that is, it is of the form

$$\{ W \ominus v : v \in V \}$$

where $W$ is a non-trivial, proper subgroup of $(V, \ominus)$. 
According to Lemma 4.4, to prove the primitivity of $\Gamma$ we have to show that no subgroup of $(V, \Box)$ is a block. Goursat has characterized \textmd{[Gou89, Sections 11–12]} the subgroups of the direct product of two groups in terms of suitable sections of the direct factors. (See also \textmd{[Pet09].})

**Theorem 4.5 (Goursat’s Lemma).** Let $(G_1, \Box)$ and $(G_2, \Box)$ be two groups. There exists a bijection between

1. the set of all subgroups of the direct product $G_1 \times G_2$, and
2. the set of all triples $(A/B, C/D, \psi)$, where
   - $A$ is a subgroup of $G_1$,
   - $C$ is a subgroup of $G_2$,
   - $B$ is a normal subgroup of $A$,
   - $D$ is a normal subgroup of $C$, and
   - $\psi: A/B \to C/D$ is a group isomorphism.

In this bijection, each subgroup of $G_1 \times G_2$ can be uniquely written as

\begin{equation}
U_\psi = \{ (a, c) \in A \times C : (a \Box B)\psi = c \Box D \}.
\end{equation}

Let us consider the case when $G_1 = G_2 = \mathbb{Z}_{2^n}$, with operation $\Box$. Then $A = \langle 2^s \rangle$ and $C = \langle 2^t \rangle$ for some $s, t$, with $0 \leq s, t \leq n$. Assume first that $s \leq t$. Therefore there is an odd integer $z \geq 1$ such that

\begin{equation}
(2^s \Box B)\psi = z2^t \Box D.
\end{equation}

Let us consider the endomorphism $\varphi: x \mapsto z2^{t-s}x$ of $\mathbb{Z}_{2^n}$. Since $2^s \varphi = z2^t$, we have that $\varphi$ induces $\psi$, that is, for $a \in A$

\begin{equation}
(a \Box B)\psi = a\varphi \Box D.
\end{equation}

If $t \leq s$, we have similarly that for the endomorphism $\varphi: x \mapsto z2^{s-t}x$ of $\mathbb{Z}_{2^n}$ one has

\begin{equation}
(c \Box D)\psi^{-1} = c\varphi \Box B.
\end{equation}

We claim

**Lemma 4.6.** In the above notation, we have

\begin{equation}
U_\psi = \{ (a, a\varphi \Box d) : a \in A, d \in D \} \quad \text{when } s \leq t,
\end{equation}

\begin{equation}
U_\psi = \{ (c\varphi \Box b, c) : c \in C, b \in B \} \quad \text{when } t \leq s.
\end{equation}

**Proof.** We will prove only the first equality, the proof of the other being analogous.

Note first that the right-hand side of \textmd{(4.4)} is contained in $U_\psi$, since for $a \in A$ and $d \in D$ we have

\begin{equation}
(a \Box B)\psi = a\varphi \Box D = a\varphi \Box d \Box D,
\end{equation}

that is, $(a, a\varphi \Box d) \in U_\psi$. 

We now prove that $U_\psi$ is contained in the right-hand side of (4.4). If $(a, c) \in U_\psi$ we have, using (4.2)
$$a \varphi \boxplus D = (a \boxplus B)\psi = c \boxplus D,$$
so that $c = a \varphi \boxplus d$ for some $d \in D$. \hfill □

We now show that no subgroup $U$ of $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$ is a block. By Lemma 4.4, we
have to prove the following

Lemma 4.7. There is no nontrivial, proper $\boxplus$-subgroup $U$ of $V$, and $(v_1, v_2) \in V$
such that

\begin{equation}
U \Sigma = U \boxplus (v_1, v_2).
\end{equation}

Proof. By Theorem 4.5 and Lemma 4.6, there is $\varphi \in \text{End}(\mathbb{Z}_{2^n})$ such that

\begin{equation}
U = \{ (a, a \varphi \boxplus d) : a \in A, d \in D \}
\end{equation}

for some $A \leq \mathbb{Z}_{2^n}$ and $D \leq A \varphi$, or

\begin{equation}
U = \{ (c \varphi \boxplus b, c) : c \in C, b \in B \}
\end{equation}

for some $C \leq \mathbb{Z}_{2^n}$ and $B \leq C \varphi$.

Suppose first that $U$ satisfies (4.6) and (4.7). By the definition (2.2) and (2.3)
of $\Sigma$, we have

$$ (a, a \varphi \boxplus d)\Sigma = (a \varphi \boxplus d, a + (a \varphi \boxplus d)S). $$

Setting $a = d = 0$, we see that $(0, 0)\Sigma = (0, 0S)$ so that we can take $v_1 = 0$ and
$v_2 = 0S$. We have thus that for any $a \in A, d \in D$, there are $x \in A, y \in D$ such that

\begin{equation}
(a \varphi \boxplus d, a + (a \varphi \boxplus d)S) = (x, x \varphi \boxplus y \boxplus 0S) \in U \boxplus (0, 0S),
\end{equation}

that is, $x = a \varphi \boxplus d$ and $y \boxplus 0S = a + (a \varphi \boxplus d)S \boxplus (a \varphi \boxplus d)\varphi$, and so

\begin{equation}
a + (a \varphi \boxplus d)S \boxplus (a \varphi \boxplus d)\varphi \in 0S \boxplus D.
\end{equation}

Note that in the equation $x = a \varphi \boxplus d, a$ and $x$ range in $A$ while $d$ ranges in $D$.
Since $D \leq A \varphi$, we obtain that $A \varphi = A$, and so $s = t$, and $\varphi$ is an automorphism
of $\mathbb{Z}_{2^n}$.

Setting $a = 0$ in (4.10), we see that $DS \subseteq 0S \boxplus D$. Since $S$ is bijective, we have
$|DS| = |D| = |0S \boxplus D|$, so that

\begin{equation}
DS = 0S \boxplus D.
\end{equation}

When $D = \mathbb{Z}_{2^n}$, since $\varphi$ is an automorphism of $\mathbb{Z}_{2^n}$, we have also $C = B = A = \mathbb{Z}_{2^n}$ in Theorem 4.5 so that $U = V$, a trivial block.

In Subsection 4.1 (see Corollary 4.15) we will show that for $D < \mathbb{Z}_{2^n}$, the
identity (4.11) can only hold when $D = \{ 0 \}$. Then in Subsection 4.2 we deal with
the case $D = \{ 0 \}$, that is,

$$ U = \{ (a, a \varphi) : a \in A \}. $$
It remains to deal with case (4.8). Recalling that \( U^\Sigma = U \boxplus (0, 0S) \), we argue as in the first case and deduce that for \( c \in C, b \in B \), there are \( x \in C, y \in B \) such that

\[
(c, (c \varphi \boxplus b) + cS) = (x \varphi \boxplus y, x \boxplus 0S).
\]

Setting \( y = 0 \), we obtain \( C = C \varphi \), and so \( \varphi \) is an automorphism. But then, by (4.3), we have \( |B| = |D| \) and so \( A = C \) and \( B = D \). Setting \( a = c \varphi \boxplus b \) in (4.8), we obtain

\[
U = \{ (a, (a \boxplus b) \varphi^{-1}) : a \in A, b \in B \} = \{ (a, a \varphi^{-1} \boxplus b \varphi^{-1}) : a \in A, b \in B \} = \{ (a, a \varphi^{-1} \boxplus d) : a \in A, d \in D \},
\]

so that we have reduced to the previous case.

\[ \square \]

**4.1. The case \( DS = 0S \boxplus D \), with \( D \neq \{ 0 \} \).** For \( v \in F_2^n \), we denote by \( v[h,k] \) the string of bits consisting of the bits of \( v \) from the \( h \)-th bit to the \( k \)-th bit. (We start counting from 0.) For example if \( v = (0,1,1,0) \), then \( v[1,3] = (1,1,0) \). For any \( W \subseteq F_2^n \) we denote by \( W_{[h,k]} \) the set \( \{ v[h,k] : v \in W \} \).

According to Lemma 2.1, a subgroup \( D \) of \( \mathbb{Z}_{2^n} \) is of the form \( \langle 2^q \rangle \), for some \( 0 \leq q < n \). So the representation of each element of \( D = \langle 2^q \rangle \) as an element of \( F_2^n = F_2^q \times F_2^{n-q} \) is of the form \( 0_{[0,q-1]} \parallel d_{[q,n-1]} \) with \( d_{[q,n-1]} \in F_2^{n-q} \). Recall that \( F_2^n = F_2^m \parallel \cdots \parallel F_2^m \).

We shall use the following compact notation:

1. A white box \( \begin{array}{c} \hline \end{array} \) denotes a subset of \( F_2^m \) of cardinality 1;

2. A ruled box \( \begin{array}{c} |\hline|\hline|\hline|\hline| \end{array} \) denotes a subset of \( F_2^m \) of cardinality \( 1 < t < 2^m \);

3. A black box \( \begin{array}{c} \hline \end{array} \) denotes the full set \( F_2^m \).

We will say that a box has white, ruled or black *type*.

We will also speak of

4. A riddle box \( \begin{array}{c} ? \end{array} \) which is any of the above.
Definition 4.8. Let $D$ be a subset of 
$$
\mathbb{F}_2^n = V_1 \times V_2 \times \cdots \times V_\delta,
$$
where each subspace $V_i$ has dimension $m$. We shall say that $D$ has a type if 
$$
D = (D \cap V_1) \times (D \cap V_2) \times \cdots \times (D \cap V_\delta).
$$
If $D$ has a type, the type of $D$ will be a sequence of $\delta$ white, ruled or black boxes, 
where the $i$-th box represents the set $D \cap V_i$.

Remark 4.9. A subgroup $D = \langle 2^q \rangle$ of $\mathbb{Z}_{2^n}$ has one of the following two types.

1. When $q \equiv 0 \pmod{m}$, the subgroup has type:

   
   $q$
   
   Here there are no ruled boxes, and the $q$-th bit occurs as the first bit of a 
   black box. Note that there are no white boxes when $q = 0$ (the subgroup is 
   the full group $\mathbb{Z}_{2^n}$), and there are no black boxes when $q = 2^n$ (the subgroup 
   is $\{0\}$).

2. When $q \not\equiv 0 \pmod{m}$, there is a ruled box:

   $q$

   where the $q$-th bit is inside the ruled box.

Definition 4.10. A subgroup of $\mathbb{Z}_{2^n}$ of the first type of Remark 4.9 will be called 
a whole subgroup.

In the next Lemma we consider the behaviour of the bitwise sum with respect 
to types.

Lemma 4.11. If $D$ is a subset of $\mathbb{Z}_{2^n}$ having a type and $v \in \mathbb{Z}_{2^n}$, then $D$ and 
v + $D$ have the same type.

Proof. Since $D$ has a type, 
$$
D = D_1 \times \cdots \times D_\delta,
$$
where $D_i = D \cap V_i$ for each 
i $\in \{1, \ldots, \delta\}$. Writing 
v = $(v_1, \cdots, v_\delta)$, clearly we have 
$$
D + v = (D_1 + v_1) \times \cdots \times (D_\delta + v_\delta)
$$
and so $D + v$ has a type. Since $|D_i| = |D_i + v_i|$, the two types coincide.

The behaviour of the modular sum $\boxplus$ with respect to types is more complex and 
can be described easily only for subgroups, as in the following lemma.
Lemma 4.12. If \( D \) is a subgroup of \( \mathbb{Z}_{2^n} \) and \( v \in \mathbb{Z}_{2^n} \), then \( D \) and \( v \Box D \) have the same type.

Proof. The binary representation of an element \( d \) of \( D = \langle 2^q \rangle \) has the form

\[
    d = 0_{[0,q-1]} \parallel d_{[q,n-1]},
\]

where \( 0_{[0,q-1]} \) is a zero vector of length \( q \). Write \( v = v_{[0,q-1]} \parallel v_{[q,n-1]} \). Then an element \( v \Box d \) of \( v \Box D \) can be written as

\[
    v \Box d = (v_{[0,q-1]} \parallel v_{[q,n-1]}) \Box (0_{[0,q-1]} \parallel d_{[q,n-1]}) = (v_{[0,q-1]} \Box 0_{[0,q-1]} \parallel (v_{[q,n-1]} \Box d_{[q,n-1]}) = v_{[0,q-1]} \parallel (v_{[q,n-1]} \Box d_{[q,n-1]}).
\]

As \( d_{[q,n-1]} \) ranges in \( \mathbb{F}_2^{n-q} \), so does \( v_{[l,n-1]} \Box d_{[q,n-1]} \). Therefore \( D \) and \( v \Box D \) have the same type. \( \square \)

Clearly a bricklayer transformations will map any set having a type to another set having the same type, since each S-box is a bijection.

Lemma 4.13. If \( D \) is a subgroup of \( \mathbb{Z}_{2^n} \), then \( D \), \( D\gamma \) and \( 0\gamma \Box D \) have the same type, for any bricklayer transformation \( \gamma \in \text{Sym}(V) \).

Moreover, if \( D \) is whole, then \( D\gamma = 0\gamma \Box D \).

Proof. Clearly \( D \) and \( D\gamma \) share the same type and by Lemma 4.12 this is the same type as \( 0\gamma \Box D \).

If \( D \) is a whole subgroup, then

\[
    D = 0_{[0,m-1]} \parallel \cdots \parallel 0_{[(l-1)m,lm-1]} \parallel D_l \parallel \cdots \parallel D_\delta
\]

for some \( l \leq \delta \), and thus

\[
    D\gamma = 0_{\gamma l} \parallel \cdots \parallel 0_{\gamma l-1} \parallel D_l \gamma_l \parallel \cdots \parallel D_\delta \gamma_\delta.
\]

Since \( D_i = \mathbb{F}_{2^m} \) for any \( i \in \{l, \ldots, \delta\} \), \( D\gamma = 0\gamma \Box D \). \( \square \)

Lemma 4.14. If \( D \) is a proper, nontrivial subgroup of \( \mathbb{Z}_{2^n} \), then \( DS \) and \( D \) have different types.

Proof. By Lemma 4.13 we know that \( D \) and \( D\gamma \) have the same type. We will now prove that an application of \( R \) changes the type, which will yield the claim. According to Remark 4.9 we distinguish the following three possibilities for the type of \( D\gamma \):
As in Definition 4.8 we count the boxes from 1 to \( \delta \).

- Consider first case a), when we have both black and white boxes, and the riddle box can be of any type.
  
  If \( r = 2m \), then the white box preceding the riddle box is sent by \( R \) onto the black box following the riddle box, a contradiction.

  Similarly, if \( r = (\delta - 1)m \), the first white box is sent by \( R \) onto the last black box. For later use, we regard this as \( R^{-1} \) sending the last black box onto the first white box.

  Now note first that every \( m \)-bit box that is contained in the stretch of white boxes will be white, even if it is not aligned with one of the bricks \( V_{ij} \). This is simply because all bits in this stretch take a single value each. Similarly, every \( m \)-bit box that is contained in the stretch of black boxes will be black, even if it is not aligned with one of the bricks. This is because all bits in this stretch take two values each, independent of one another.

  To deal with the intermediate cases \( 2m \leq r \leq (\delta - 1)m \), start with the case \( r = 2m \), and shift the black box next to the riddle box right by one bit. As just noted, this will still be black, and for \( r = 2m + 1 \), the rotation \( R \) will take the white box next to the riddle box onto the shifted black box, a contradiction.

  We keep shifting the black box to the right one bit at a time, until we hit the rightmost black box. In this way we will have covered all rotations \( R \), for \( 2m \leq r \leq \vartheta m \), where \( \delta - \vartheta + 1 \) is the position of the riddle box.

  To cover the remaining rotations, start with the last black box, which for \( r = \vartheta m \) is taken by the left rotation \( R^{-1} \) onto the white box adjacent to the riddle box. Shift the latter white box left by one bit. By the remark above, this will still be white, and the left rotation \( R^{-1} \) by \( r = \vartheta m + 1 \) bits will take the last black box onto it, a contradiction.

  Shifting bit by bit the white box to the left, until it overlaps completely the first white box, we see that for \( 2m \leq r \leq (\delta - 1)m \), one or both of the following possibilities will have occurred.
(1) The rotation $R$ sends a white box onto a black box, or over two adjacent black boxes. Since in a white box all bits take a single value, while in a black box each bit takes two values, independent of one another, this is a contradiction.

(2) The left rotation $R^{-1}$ sends a black box onto a white box, or over two adjacent white boxes. This is again a contradiction.

If $m \leq r < 2m$, then $R$ sends the last black box, either onto the first white box, or in any case to overlap the first white box in the last $2m - r > 0$ bits of the latter. Once more, this is a contradiction.

- Consider now case b). Here we do not have black boxes and the ruled box is the rightmost one, at position $\delta$. Under the rotation to the right by $r$ bits, the ruled box is moved onto a white box, or comes to overlap two adjacent white boxes. This implies that all bits of the ruled box take a single value each, so that the ruled box is a singleton, that is, it is also white, a contradiction.

- Finally, in case c) we do not have white boxes, and the ruled box is the leftmost one. Applying a rotation to the right by $r$ bits, the ruled box is moved onto a black box, or comes to overlap two adjacent black boxes. Since concatenation of boxes means concatenation of strings, and in a black box each bit takes two values, independent of one another, this would make the ruled box black, a contradiction.

Corollary 4.15. If $D \neq \{0\}$ is a subgroup of $\mathbb{Z}_{2^n}$, then $DS \neq 0S \boxplus D$.

Proof. It follows from Lemma 4.14 and Lemma 4.12.

4.2. The diagonal case $D = \{0\}$. Here we deal with the case when a subgroup of the form

$$U = \{(a, a\varphi) : a \in A \},$$

for some $0 \neq A \leq \mathbb{Z}_{2^n}$ and $\varphi \in \text{Aut}(\mathbb{Z}_{2^n})$, is a block. Since

$$(a, a\varphi)\Sigma = (a\varphi, a + a\varphi S),$$

as in the discussion following Lemma 4.7 we have

$$U\Sigma = U \boxplus (0, 0S).$$

Therefore for each $a \in A$ there is $x \in A$ such that

$$(a\varphi, a + a\varphi S) = (x, x\varphi \boxplus 0S),$$

so that $x = a\varphi$, and substituting

$$(a\varphi)^2 = (a + a\varphi S) \boxminus 0S.$$  

Since $\varphi$ is an automorphism, we have $2^{n-1}\varphi = 2^{n-1}$. Now for any $y$ it is easy to see that

$$y + 2^{n-1} = y \boxplus 2^{n-1},$$

□
as in both cases we are just changing the most significant bit of \( y \). Therefore, setting \( a = 2^{n-1} \in A \) in (4.12), we obtain
\[
2^{n-1} = 2^{n-1} \oplus 2^{n-1}S \ominus 0S,
\]
or in other words
\[
2^{n-1}S = 0S,
\]
contradicting the fact that \( S \) is a bijection.

5. O’Nan-Scott

We have shown in the previous section that the subgroup \( \Gamma \) of \( \text{Sym}(V) \) is primitive. We may thus prove Theorem 3.1 by appealing to the O’Nan-Scott classification of primitive groups [LPS88]. However, since by (3.2) \( \Gamma \) contains the group \( T \) of translations, which is an abelian subgroup acting regularly on \( V \), we are able to appeal to a particular case of the O’Nan-Scott classification, obtained by Li [Li03, Theorem 1.1], which describes the primitive groups containing an abelian regular subgroup. In the particular case when \( \Gamma \) acts on a set whose order is a power of 2, Li’s result can be stated as follows.

**Theorem 5.1** ([Li03], Theorem 1.1). Let \( \Gamma \) be a primitive group acting on a set \( V \) of cardinality \( 2^b \), with \( b > 1 \). Suppose \( \Gamma \) contains a regular abelian subgroup \( T \).

Then \( \Gamma \) is one of the following.

1. **Affine**, \( \Gamma \leq \text{AGL}(b, 2) \).
2. **Wreath product**, that is
\[
\Gamma \cong (K_1 \times \cdots \times K_l).O.P,
\]
   with \( 2^b = c^l \) for some \( c \) and \( l > 1 \). Here \( T = T_1 \times \cdots \times T_l \), with \( T_i \leq K_i \) and \( |T_i| = c \) for each \( i \), \( K_1 \cong \ldots \cong K_l \), \( O \leq \text{Out}(K_1) \times \cdots \times \text{Out}(K_l) \), \( P \) permutes transitively the \( K_i \), and either \( K_i = \text{Sym}(c) \) or \( K_i = \text{Alt}(c) \).
3. **Almost simple**, i.e., \( K \leq \Gamma \leq \text{Aut}(K) \) for a nonabelian simple group \( K \).

Here the notation \( S.T \) denotes an extension of the group \( S \) by the group \( T \).

Case (2) is the case of the (wreath product in) **product action**. In dealing with this, we will be supplementing Li’s statement with the information from [LPS88].

In the next three subsections we will examine the three cases of Theorem 5.1 and show that only the almost simple case can hold, with \( \Gamma = \text{Alt}(V) \).

Recall that in our case \( |V| = 2^b \), with \( b = 2n \), \( n = \delta m \) with \( \delta \geq 4 \) and \( m \geq 2 \). These conditions imply that \( b \geq 16 \) and \( n \geq 8 \).

5.1. **The affine case.** Suppose case (1) of Theorem 5.1 holds, that is, \( \Gamma \leq \text{AGL}(2n, 2) \). Then \( \text{AGL}(2n, 2) \) should contain the cyclic subgroup \( \mathbb{Z}_{2^n} \).

It is well known that if \( p \) is a prime, then the exponent of the \( p \)-Sylow subgroup of \( \text{GL}(2n, p) \) is the smallest power \( p^k \) such that \( p^k \geq 2n \). In our case the exponent
of the 2-Sylow subgroup of $GL(2n, 2)$ is the smallest power $2^k \geq 2n$, so that $k \geq \log_2(n) + 1$, and

$$k = \lceil \log_2(n) + 1 \rceil = \lceil \log_2(n) \rceil + 1.$$ 

Since $AGL(2n, 2)$ is the extension of an elementary abelian group by $GL(2n, 2)$, the exponent of the 2-Sylow subgroup of $AGL(2n, 2)$ can only increase by a factor of two with respect to that of $GL(2n, 2)$. Therefore if there is an element of order $2^n$ in $AGL(2n, 2)$, then

$$\lceil \log_2(n) \rceil + 2 \geq n,$$

which fails for $n > 5$. (Recall that we have $n \geq 8$.)

5.2. The wreath product case. This is case III(b) (wreath product in product action) of [LPS88]. Therefore

$$V = W_1 \times \cdots \times W_l,$$

with $K_i$ acting transitively on the subsets $W_i$, each of the latter having cardinality $c > 1$. Since $T_i$ is a subgroup of order $c$ of the $\oplus$-translation group $T \cong \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$, and $T = T_1 \times \cdots \times T_l$, it follows that $l = 2$, $c = 2^n$, and $T_i \cong \mathbb{Z}_{2^n}$. By Lemma 4.3, $T_i = \{ \rho_u : u \in U_i \}$, where the $U_i$ are subgroups of $V$. Since $T$ acts regularly on $V$, we have $W_i = 0$ for $K_i = 0$, so that the $W_i$ are subgroups of $V$.

Since $\Gamma = \langle T, \Sigma \rangle$, $T$ is contained in the normal subgroup $K_1 \times K_2$, and $P$ permutes the $K_i$ by conjugation, it follows that

$$\Sigma^{-1} K_1 \Sigma = K_2.$$

We have thus

$$W_1 \Sigma = 0 K_1 \Sigma = 0 \Sigma K_2 = 0 \Sigma T_2 = (0, 0S) \oplus W_2.$$ 

(Here and in the following, recall (2.2) and (2.3).)

We now prove that (5.1) cannot hold, with arguments similar to those of Section 4.2.

We appeal once again to Goursat’s Lemma to describe the subgroups $W_1, W_2$ of $V = V^1 \times V^2$. Note that, in the notation of Theorem 4.5, the subgroup $U_\psi$ of the direct product contains $B \times D$. Since $W_1 \cong \mathbb{Z}_{2^n} \cong W_2$ are indecomposable, one of $B$ and $D$ must be trivial. In (4.4) of Lemma 4.6, $D$ is the image of $B$ under an endomorphism, and in (4.5) $B$ is the image of $D$ under an endomorphism. It follows that in the notation of Lemma 4.6 $W_1, W_2$ are of one of the two forms

$$\{ (x, x \sigma) : x \in \mathbb{Z}_{2^n} \}, \quad \{ (y \tau, y) : y \in \mathbb{Z}_{2^n} \},$$

where $\sigma, \tau \in \text{End}(\mathbb{Z}_{2^n})$. There are four cases to consider.

The first case is

$$W_1 = \{ (x, x \sigma) : x \in \mathbb{Z}_{2^n} \}, \quad W_2 = \{ (y, y \tau) : y \in \mathbb{Z}_{2^n} \},$$
for $\sigma, \tau \in \text{End}(\mathbb{Z}_{2^n})$. In this case (5.1) states that for each $y \in \mathbb{Z}_{2^n}$ there is a unique $x \in \mathbb{Z}_{2^n}$ such that

$$(x\sigma, x + xS) = (y, y\tau \oplus 0S).$$

Therefore $y = x\sigma$, and $\sigma \in \text{Aut}(\mathbb{Z}_{2^n})$. Set $x = 2^{n-1}$. We get

$$2^{n-1} + 2^{n-1}S = 2^{n-1}\tau \oplus 0S.$$ 

If $\tau$ is also an automorphism, we get $2^{n-1}S = 0S$, a contradiction to the fact that $S$ is bijective. If $\tau$ is a proper endomorphism, that is, an endomorphism which is not an automorphism, we get

(5.2) $$2^{n-1} + 2^{n-1}S = 0S.$$

Regarding this as an identity in $V^0$, it states that $0S$ and $2^{n-1}S$ differ only in the last bit. Clearly $0$ and $2^{n-1}$ differ only in the last bit, so that $0_\gamma$ and $2^{n-1}_\gamma$ differ only in their component in $V^0_\delta$. But then, once one applies the right rotation $R$ by $r$ bits, with $m \leq r \leq (\delta - 1)m$, we have that the components in $V^0_\delta$ of $0S = 0_\gamma R$ and $2^{n-1}S = 2^{n-1}_\gamma R$ coincide, contradicting (5.2).

The second case is

$$W_1 = \{ (x, x\sigma) : x \in \mathbb{Z}_{2^n} \}, \quad W_2 = \{ (y\tau, y) : y \in \mathbb{Z}_{2^n} \},$$

for $\sigma, \tau \in \text{End}(\mathbb{Z}_{2^n})$. We thus have that for each $x \in \mathbb{Z}_{2^n}$ there is a unique $y \in \mathbb{Z}_{2^n}$ such that

$$(x\sigma, x + xS) = (y\tau, y \oplus 0S).$$

Setting $x = 0$, we see that $\tau$ is an automorphism, and similarly $\sigma$ is an automorphism. Setting $x = 2^{n-1}$, we have also $y = 2^{n-1}$, so that we get once more

$$2^{n-1}S = 0S.$$

The third case is

$$W_1 = \{ (x\sigma, x) : x \in \mathbb{Z}_{2^n} \}, \quad W_2 = \{ (y, y\tau) : y \in \mathbb{Z}_{2^n} \},$$

for $\sigma, \tau \in \text{End}(\mathbb{Z}_{2^n})$. Thus we have that for each $x \in \mathbb{Z}_{2^n}$ there is a unique $y \in \mathbb{Z}_{2^{n-1}}$ such that

$$(x, x\sigma + xS) = (y, y\tau \oplus 0S).$$

Therefore $x = y$, and for each $y \in \mathbb{Z}_{2^n}$ we have

$$y\sigma + yS = y\tau \oplus 0S.$$ 

If $\sigma, \tau$ are both automorphisms, or both proper endomorphisms, setting $y = 2^{n-1}$ we get once more $2^{n-1}S = 0S$, a contradiction. If one of $\sigma, \tau$ is an automorphism, and the other is a proper endomorphism, then setting $y = 2^{n-1}$ we get as above

$$2^{n-1}S + 2^{n-1}S = 0S,$$

a contradiction.

The fourth case is

$$W_1 = \{ (x\sigma, x) : x \in \mathbb{Z}_{2^n} \}, \quad W_2 = \{ (y, y\tau) : y \in \mathbb{Z}_{2^n} \},$$
for $\sigma, \tau \in \text{End}(\mathbb{Z}_2^n)$. We thus have that for each $x \in \mathbb{Z}_2^n$ there is a unique $y \in \mathbb{Z}_2^n$ such that

$$(x, x\sigma + xS) = (y\tau, y 0S).$$

It follows that $\tau$ is an automorphism, and $y = x\tau^{-1}$. Thus for each $x \in \mathbb{Z}_2^n$ one has

$$x\sigma + xS = x\tau^{-1} 0S,$$

so this case reduces to the previous one.

5.3. **The almost simple case.** In the almost simple case (3) of Theorem 5.1 note that $K$ is a transitive subgroup of the primitive group $\Gamma$, so the intersection of a one-point stabiliser in $\Gamma$ with $K$ is a proper subgroup of $K$ of index $2^b$, with $b \geq 16$. By Theorem 1 and Section (3.3) in [Gur83], there are two possibilities for $K$.

The first possibility is for $K$ to be the group $\text{PSL}_\alpha(\beta)$, where in our case

(i) $\beta^{\alpha - 1}/(\beta - 1) = 2^b$,

(ii) $\beta$ is a power $\pi^e$ of a prime $\pi$;

(iii) $\alpha$ is a prime such that if $\alpha > 2$ then $\pi \equiv 1$ (mod $\alpha$).

(i) implies that $\beta$, and thus $\pi$, are odd. Hence

$$2^b = (\beta^{\alpha - 1}/(\beta - 1) = \beta^{\alpha - 1} + \beta^{\alpha - 2} + \cdots + \beta + 1 \equiv \alpha \pmod{2},$$

so that $\alpha = 2$. Thus

$$\pi^e = \beta = 2^b - 1 = (2^n - 1)(2^n + 1),$$

where both factors of the last term are greater than 1, as $n > 1$. If $e = 1$, this contradicts the fact that $\pi$ is a prime. If $e > 1$, then $\pi$ divides both $2^n - 1$ and $2^n + 1$, which contradicts the fact that these two numbers are coprime.

The other possibility is for $K$ to be the alternating group $\text{Alt}(2^b)$ of degree $2^b$. Since the automorphism group of $\text{Alt}(2^b)$ is $\text{Sym}(2^b)$, we obtain that $\Gamma$ is either $\text{Alt}(2^b)$ or $\text{Sym}(2^b)$. In view of Lemma 3.1, we have $\Gamma = \text{Alt}(\mathcal{V})$, as claimed.

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References

[ACDVS14] R. Aragona, A. Caranti, F. Dalla Volta, and M. Sala, *On the group generated by the round functions of translation based ciphers over arbitrary finite fields*, Finite Fields Appl. 25 (2014), 293–305. MR 3130605

[CDVS09a] A. Caranti, F. Dalla Volta, and M. Sala, *An application of the O’Nan-Scott theorem to the group generated by the round functions of an AES-like cipher*, Des. Codes Cryptogr. 52 (2009), no. 3, 293–301. MR 2506729 (2010a:94053)
[CDVS09b] On some block ciphers and imprimitive groups, Appl. Algebra Engrg. Comm. Comput. 20 (2009), no. 5-6, 339–350. MR 2564408 (2010k:94046)
[CG75] Don Coppersmith and Edna Grossman, Generators for certain alternating groups with applications to cryptography, SIAM J. Appl. Math. 29 (1975), no. 4, 624–627. MR 0495175 (58 #13909)
[Dol10] V. Dolmatov, GOST 28147-89: Encryption, decryption, and message authentication code (MAC) algorithms, Tech. report, 2010, http://tools.ietf.org/html/rfc5830
[EG83] Shimon Even and Oded Goldreich, DES-like functions can generate the alternating group, IEEE Trans. Inform. Theory 29 (1983), no. 6, 863–865. MR 733194 (85d:94010)
[Gou89] Edouard Goursat, Sur les substitutions orthogonales et les divisions régulières de l'espace, Ann. Sci. École Norm. Sup. (3) 6 (1889), 9–102. MR 1508819
[Gur83] Robert M. Guralnick, Subgroups of prime power index in a simple group, J. Algebra 81 (1983), no. 2, 304–311. MR 700286 (84m:20007)
[KRS88] Burton S. Kaliski, Jr., Ronald L. Rivest, and Alan T. Sherman, Is the data encryption standard a group? (Results of cycling experiments on DES), J. Cryptology 1 (1988), no. 1, 3–36. MR 935899 (89f:94017)
[Li03] Cai Heng Li, The finite primitive permutation groups containing an abelian regular subgroup, Proc. London Math. Soc. (3) 87 (2003), no. 3, 725–747. MR 2005881 (2004i:20003)
[LPS88] Martin W. Liebeck, Cheryl E. Praeger, and Jan Saxl, On the O'Nan-Scott theorem for finite primitive permutation groups, J. Austral. Math. Soc. Ser. A 44 (1988), no. 3, 389–396. MR 929529 (89a:20002)
[MPW94] Sean Murphy, Kenneth Paterson, and Peter Wild, A weak cipher that generates the symmetric group, J. Cryptology 7 (1994), no. 1, 61–65. MR 1258720 (94i:94017)
[Oli11] Roman Oliynykov, Cryptanalysis of symmetric block ciphers based on the Feistel network with non-bijective S-boxes in the round function, Cryptology ePrint Archive, Report 2011/685, 2011, https://eprint.iacr.org/2011/685.pdf
[Pet09] J. Petrillo, Goursat's other theorem, The College Mathematics Journal 40 (2009), no. 2, 119–124.
[Pat08] Rüdiger Sparr and Ralph Wernsdorf, Group theoretic properties of Rijndael-like ciphers, Discrete Appl. Math. 156 (2008), no. 16, 3139–3149. MR 2462120 (2010d:94114)
[SW15] The round functions of KASUMI generate the alternating group, J. Math. Cryptol. 9 (2015), no. 1, 23–32. MR 3318545
[Wer93] Ralph Wernsdorf, The one-round functions of the DES generate the alternating group, Advances in cryptology—EUROCRYPT ’92 (Balatonfured, 1992), Lecture Notes in Comput. Sci., vol. 658, Springer, Berlin, 1993, pp. 99–112. MR 1243663 (94g:94031)
