Lattice Dirac Fermions in a non-Abelian Random Gauge Potential: Many Flavors, Chiral Symmetry Restoration and Localization

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Abstract

In the previous paper we studied Dirac fermions in a non-Abelian random vector potential by using lattice supersymmetry. By the lattice regularization, the system of disordered Dirac fermions is defined without any ambiguities. We showed there that at strong-disorder limit correlation function of the fermion local density of states decays algebraically at the band center. In this paper, we shall reexamine the multi-flavor or multi-species case rather in detail and argue that the correlator at the band center decays exponentially for the case of a large number of flavors. This means that a delocalization-localization phase transition occurs as the number of flavors is increased. This discussion is supported by the recent numerical studies on multi-flavor QCD at the strong-coupling limit, which shows that the phase structure of QCD drastically changes depending on the number of flavors. The above behaviour of the correlator of the random Dirac fermions is closely related with how the chiral symmetry is realized in QCD.

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1 Introduction

In the last several years, Dirac fermions in a random vector potential and/or with random mass have been studied by various methods. This system is closely related with the phase transition between plateaus in the quantum Hall states and quasiexcitations in the d-wave superconductor\cite{1, 2, 3, 4, 5, 6, 7, 8}. However it is still controversial if there exist extended states at the band center. In the previous paper\cite{9}, we studied the case of non-Abelian $U(N_c)$ random vector potential for large $N_c$. There both weak and strong disorder cases are investigated separately by using lattice supersymmetry(LSUSY) which was originally invented for the study on the SUSY gauge theory. Especially it was shown that correlation function of the fermion local density of states(DoS) decays algebraically at the band center. This result indicates the existence of extended states there.

The above random Dirac fermion system on the lattice is closely related with the lattice QCD for which there exist many useful and definite studies by numerical simulations. Among them, one of the most interesting studies shows that the phase structure of lattice QCD changes drastically depending on the number of flavors of quarks\cite{10}. For small number of flavors, quarks are confined and the chiral symmetry is spontaneously broken, whereas as the number of flavors is increased, quarks are deconfined and the chiral symmetry is restored. Then mass of “pions” is nonvanishing even for the vanishing bare quark mass for many flavors.

The above result of multi-flavor lattice QCD holds even at the strong-coupling limit which directly corresponds to the strong-disorder limit in the random Dirac fermions. In the previous study on the Dirac fermions in a random $U(N_c)$ vector potential\cite{9}, we implicitly assumed a small number of fermion flavors. In this paper we shall reexamine the system of multi-flavor or multi-species Dirac fermions. The flavor degrees of freedom is sometimes introduced in the replica tricks and one takes the limit $N_f (=the$ number of flavors) $\rightarrow 0$ after calculation. However in the present study, $N_f$ is kept finite and moreover we even consider the case like $N_f/N_c > 1$. In
the above lattice QCD, $N_c = 3$ and the critical value of $N_f$ is 7.

Plan of the present paper is as follows. In Sec.2, we shall study the strong-coupling limit of lattice QCD by deriving an effective action of $U(N_c)$ invariant excitations, which is a kind of antiferromagnetic spin model. From this effective action, we can study the vacuum structure, especially how the chiral symmetry is realized. For a small number of flavors, the chiral symmetry is spontaneously broken and (quasi)massless pions appear as a Nambu-Goldstone boson as we showed previously[11]. For a large number of flavors, on the other hand, we show that the chiral symmetry is restored and “pions” become massive even at the chiral-invariant limit. This result is of course in agreement with the numerical studies[10]. In Sec.3, we review the lattice model of the Dirac fermions in a $U(N_c)$ random vector potential. Especially we explain the LSUSY which is introduced in order to take ensemble average over the random vector potential. Then we shall derive an effective action by using similar techniques used for QCD. From the discussion on QCD in Sec.2, we show that there is a phase transition as the number of flavors is increased. This phase transition can be observed from the correlator of the fermion local DoS. Section 4 is devoted to conclusion.

2 QCD with many flavors and chiral symmetry restoration

Before going into details of the Dirac fermions in a random vector potential, we shall investigate the strong-coupling limit of the lattice QCD. We follow the discussion in Ref.[11] which is also applicable for the random fermion system rather straightforwardly[9].

We consider $d$-dimensional hypercubic lattice. Action of the QCD at the strong-
coupling limit is given by

\[ S_{QCD} = \frac{1}{2} \sum \left[ \bar{\psi}(x) \gamma_\mu U_\mu(x) \psi(x + \mu) - \bar{\psi}(x + \mu) \gamma_\mu U_\mu^\dagger(x) \psi(x) \right], \tag{2.1} \]

where \( x = (x_0, \ldots, x_{d-1}) \) denotes lattice site, \( \mu = (0, \ldots, d-1) \) is the direction index, \( U_\mu(x) \) is the field on the link \( (x, x + \mu) \) \( U_\mu(x) = \left(U_\mu(x)\right)_b^a \in U(N_c) \) and we set the lattice spacing \( a_L = 1 \). The Dirac fermion \( \psi \) carries not only the spinor and color indices \( s = 1, \ldots, 2^{d/2}, a = 1, \ldots, N_c \) but also flavor index \( I = 1, \ldots, N_f \), i.e., \( \psi = \psi_{a,I}^s \).

In Eq. (2.1), \( \gamma_\mu \) is the \( \gamma \)-matrices which work on the spinor index of \( \psi \). We add the following quark mass term in the action

\[ S_M = -M_B \sum \bar{\psi}(x) \psi(x), \tag{2.2} \]

where \( M_B \) is the quark bare mass. The gauge field action is vanishing because we are considering the strong-coupling limit. In the numerical studies\[10\], the Wilson fermion is used for defining definite number of flavor. Here we consider the naive lattice fermion. However it is quite plausible that a similar phase transition occurs as “the number of flavor” \( N_f \) is varied.

It is well-known that \( S_{QCD} \) in (2.1) can be rewritten more compact form by the following change of variables,

\[ \psi(x) = T(x) \chi(x), \quad \bar{\psi}(x) = \bar{\chi}(x) T^\dagger(x), \tag{2.3} \]

with \( T(x) = (\gamma_0)^{x_0} \cdots (\gamma_{d-1})^{x_{d-1}} \) and using the identities like \( (\gamma_\mu)^2 = 1 \) and \( (\gamma_0)^n \gamma_1 = (-)^n \gamma_1 (\gamma_0)^n \) (\( n \) is an integer), etc,

\[ S_{QCD} = \frac{1}{2} \sum \left[ \bar{\chi}(x) \eta_\mu(x) U_\mu(x) \chi(x + \mu) - \bar{\chi}(x + \mu) \eta_\mu(x) U_\mu^\dagger(x) \chi(x) \right] = \sum \bar{\chi}(x) \hat{D}\chi(x), \tag{2.4} \]

where \( \eta_0(x) = 1, \ \eta_1(x) = (-)^{x_0}, \ \eta_2(x) = (-)^{x_0+x_1}, \ \cdots \), and

\[ \hat{D}\chi(x) = \frac{1}{2} \sum_\mu \left[ \eta_\mu(x) U_\mu(x) \chi(x + \mu) - \eta_\mu(x - \mu) U_\mu^\dagger(x - \mu) \chi(x - \mu) \right]. \tag{2.5} \]
Similarly

\[ S_M = -M_B \sum \bar{\chi}(x)\chi(x). \]  

(2.6)

In Eqs. (2.4) and (2.6), the spinor indices of \( \chi \) and \( \bar{\chi} \) are diagonal and they play the same role as the flavor index.

Partition function of the system is given by

\[ Z_{QCD} = \int [DUD\bar{\chi}D\chi]e^{S_{QCD}+S_M}, \]  

(2.7)

where \([DU] = \prod_{\text{link}} dU_\mu(x)\) is the Haar measure of \( U(N_c) \). In Eq.(2.4), the integral over the gauge field \( U_\mu(x) \) can be performed by the \( 1/N_c \) expansion because it reduces to the one-link integral in the strong-gauge-coupling limit \[1\]. There are two “phases” in the one-link integral \[1\]. The present case corresponds to the “strong-coupling regime”. Details of the calculation can be seen in Ref.\[9, 11\] and here we simply give the final expression. To this end, we define the following color singlet quark bilinear operators,

\[ m_{\beta}^{\alpha}(x) = \frac{1}{N_c} \sum_a \chi^{a,\alpha}(x)\bar{\chi}_{a,\beta}(x), \]  

(2.8)

where \( \alpha \) and \( \beta \) are \textit{flavor-spinor} indices which take \( N_{fs} = N_f \times 2^{[d/2]} \) values. It is obvious that after integral over \( U_\mu(x) \), the partition function \( Z_{QCD} \) is expressed in terms of \( m_{\beta}^{\alpha}(x) \),

\[ Z_{QCD}(J) = \int [D\bar{\chi}D\chi]e^{S_{m}(\lambda)+S_M+J\cdot m}, \]  

(2.9)

where \( J \)'s are source fields and \( J \cdot m = \text{Tr}(Jm) = \sum J_{\beta}^{\alpha}m_{\alpha}^{\beta} \). In the leading-order of \( 1/N_c \)

\[ \frac{1}{N_c}S_{m}(\lambda) = \sum_{x,\mu} \text{Tr}\left[ g(\lambda_{\mu}(x)) \right], \]  

(2.10)

with

\[ \lambda_{\mu}(x) = m(x)m(x + \mu), \]
\[ g(\lambda) = 1 - (1 - \lambda)^{1/2} + \ln \left[ \frac{1}{2}(1 + (1 - \lambda)^{1/2}) \right], \]

(2.11)

\[ ^{1}\text{There is a systematic expansion in powers of } 1/N_c. \text{ However, essential feature of the effective action and the final results are not influenced by the higher-order terms.} \]
and Tr is the trace over the flavor-spinor indices.

The path-integral over $\chi$ and $\bar{\chi}$ in Eq.(2.9) can be performed by introducing elementary meson fields $\mathcal{M}_\beta^\alpha(x)$. In terms of $\mathcal{M}_\beta^\alpha(x)$, the partition function is expressed as

$$ Z_{QCD}(J) = \int [D\mathcal{M}] e^{S_{eff}(\mathcal{M}) + J \cdot \mathcal{M}}, $$

$$ \frac{1}{N_c} S_{eff}(\mathcal{M}) = \sum_{x,\mu} \text{Tr}[g(\hat{\lambda}_\mu(x))] - \sum_x \left[ \text{Tr} \ln \mathcal{M}(x) - M_B \text{Tr} \mathcal{M}(x) \right], \quad (2.12) $$

where $\hat{\lambda}_\mu(x) = \mathcal{M}(x) \mathcal{M}(x + \mu)$ and integral over $\mathcal{M}$ is defined by polar decomposing $\mathcal{M}$ as $\mathcal{M} = RV$ with positive-definite Hermitian matrix $R$ and $U(N_{fs})$ matrix $V$, and $\int d\mathcal{M} = \int dV$ (=the $U(N_{fs})$ Haar measure).

From the effective action (2.12), we can study the vacuum structure of the system.

“Expectation value” of $\mathcal{M}(x)$ is obtained by the effective potential which is obtained by substituting $\mathcal{M}_\beta^\alpha(x) = v \delta_\beta^\alpha$ in $S_{eff}$,

$$ \frac{1}{N_c N_{fs}} V_{eff}(v^2) = -dg(v^2) + \frac{1}{2} \ln v^2 - M_B v. \quad (2.13) $$

From (2.13),

$$ v = \sqrt{\frac{2d - 1}{d^2}} + O(M_B). \quad (2.14) $$

For vanishing bare quark mass $M_B = 0$, $S_{eff}(\mathcal{M})$ is a function of $\hat{\lambda}_\mu(x)$, and then the vacuum expectation value of the fields $\mathcal{M}(x)$ is given as

$$ \langle \mathcal{M}(x) \rangle = \begin{cases} vV_0 & \text{at even site} \\ vV_0^\dagger & \text{at odd site} \end{cases}, \quad (2.15) $$

where $V_0 \in U(N_{fs})$. From the above discussion, low-energy excitations are obtained by introducing “pions” $\pi_K(x)$ as

$$ \mathcal{M}(x) = \begin{cases} vV(x) = ve^i \sum_K \pi_K(x)^T \pi_K & \text{at even site} \\ vV^\dagger(x) = ve^{-i} \sum_K \pi_K(x)^T \pi_K & \text{at odd site} \end{cases}. \quad (2.16) $$

\footnote{We often omit the flavor-spinor indices on $m(x)$ etc.}
where $T^K$’s are generators of $U(N_{fs})$.

For $N_c/N_f > 1$, we can expect that fluctuation of $\pi_K(x)$ is not large as $S_{\text{eff}}(\mathcal{M}) \propto N_c$ and topologically nontrivial configurations do not contribute to the partition function. Therefore we expand $V(x)$ as

$$V(x) = 1 + i \sum_K \pi_K(x) T^K + \cdots, \quad (2.17)$$

and then

$$\frac{1}{N_c} S_{\text{eff}}(\mathcal{M}) \sim -v^2 \sum_{x,\mu} \sum_K \left( \nabla_\mu \pi_K(x) \right)^2 - M_B v \sum_x \pi^2_K(x). \quad (2.18)$$

Interaction between the pions is suppressed by $1/N_c$. In this case, the chiral symmetry is spontaneously broken and $\pi_K(x)$’s are quasi-Nambu-Goldstone bosons\cite{11}.

What happens for $N_c/N_{fs} \ll 1$. Numerical studies indicate that the “pions” become massive even in the limit $M_B \to 0$. This means that chiral symmetry is restored in this case. Actually this is not so difficult to see this behavior in many-flavor case from $S_{\text{eff}}$. The effective action (2.12) can be regarded as a system of $U(N_{fs})$ antiferromagnets.\footnote{More direct relationship between the lattice QCD at strong coupling and antiferromagnets can be seen in the Hamiltonian formalism\cite{13}.} One can expect that as degrees of freedom of the $U(N_{fs})$ spin, i.e., $V_\alpha^\beta$ in the present system, becomes large, the long-range order is destroyed by fluctuations. For example in spatial 2 dimensions, ordered phase exists in the Ising model whereas there is no ordered phase in the $O(3)$ spin model. Therefore we can naturally expect that the chiral symmetry is restored for $N_{fs}/N_c \gg 1$. Without the spontaneous breaking of the chiral symmetry, the expansion (2.17), which assumes small fluctuation of $\pi_K(x)$, is not justified.

Let us see the above chiral phase transition more closely. To this end, we shall introduce a more tractable model which is expected to belong to the same universality class of $S_{\text{eff}}$. Let us notice that as $M_B \to 0$, $S_{\text{eff}}(\mathcal{M})$ is a sum of the one-link term like

$$S_{\text{link}} = N_c \, \text{Tr} \left[ g(\hat{\lambda}_\mu(x)) - \frac{1}{2d} \ln \hat{\lambda}_\mu(x) \right]. \quad (2.19)$$
Then we consider the following model which consists of a similar one-link term,

\[ S_V = \frac{1}{g_0} \sum_{x,\mu} \text{Tr} [V^\dagger(x)V(x+\mu) + \text{H.c.}] , \tag{2.20} \]

where the parameter \( \frac{1}{g_0} \sim N_c v^2 \). Actually (a continuum version of) the action \( S_V \) is derived directly from the QCD action (2.1) by using Zirnbauer’s color-flavor transformation [14] (see also Ref. [15]), and therefore it is quite plausible that \( S_{\text{eff}} \) and \( S_V \) belong to the same universality class.

The system \( S_V \) (2.20) can be easily studied by the mean-field theory (MFT). To this end, let us assume the expectation value of the field \( V(x) \in U(N_{fs}) \) as

\[ \langle V^\alpha_\beta(x) \rangle = w \delta^\alpha_\beta, \tag{2.21} \]

where we have used the \( U(N_{fs}) \) symmetry and we assume that \( w \) is real.⁴ We employ the following one-site action as MF action [16],

\[ S_{\text{MF}} = \frac{w}{g_0} \text{Tr} [V + V^\dagger] - \frac{1}{g_0} N_{fs} w^2, \tag{2.22} \]

and the partition function of the MFT is given by

\[ Z_{\text{MF}} = e^{-V_{\text{MF}}(w)} = \int dV \ e^{S_{\text{MF}}}. \tag{2.23} \]

From Eqs. (2.22) and (2.23), it is obvious that the stationary condition of \( V_{\text{MF}}(w) \) gives the “gap” equation which determines the expectation value \( w \).

The integral over \( V \) in (2.23) with the \( U(N_{fs}) \) Haar measure can be performed by the same techniques used before for integrating over the gauge field \( U_\mu(x) \),

\[ V_{\text{MF}}(w) = -N_{fs} g \left( \frac{4}{N_{fs}} \left( \frac{w}{g_0} \right)^2 \right) + \frac{N_{fs}}{g_0} w^2. \tag{2.24} \]

Especially for small \( w \),

\[ V_{\text{MF}}(w) = - \left( \frac{w}{g_0} \right)^2 + \frac{N_{fs}}{g_0} w^2 + O(w^4). \tag{2.25} \]

⁴Please do not confuse \( w \) with \( v \). In the terminology of the spin model, \( v \) is the magnitude of spin, whereas \( w \) is the magnetization. Therefore nonvanishing \( w \) means the long-range order.
From Eqs. (2.24) and (2.25), it is obvious that for \( N_{fs} > \frac{1}{g_0} = N_c v^2 \), \( w = 0 \) is the stable minimum, whereas for \( N_{fs} < \frac{1}{g_0} = N_c v^2 \), \( w = 0 \) becomes unstable and \( w \) has a finite value at the minimum of \( V_{MF} \). Then this simple consideration using \( S_V \) and the MFT supports our expectation, i.e., for \( N_{fs}/N_c \ll 1 \) the vacuum is chiral symmetric and there are no Nambu-Goldstone bosons, whereas for \( N_c/N_{fs} \ll 1 \) the chiral symmetry is spontaneously broken and gapless excitations exist.

In the chiral-symmetric case, physical quantities are calculated by using the \( U(N_{fs}) \) Haar measure for \( V(x) \). We are interested in the meson-meson correlation functions like \( \langle M_{\alpha \beta}(x) M_{\gamma \omega}(0) \rangle \). The integral over \( V(x) \) can be performed by using the character-expansion for \( V(x)V^\dagger(x + \mu) \in U(N_{fs}) \), and each link term of the action can be written as:

\[
\exp(S_{\text{link}}) = \sum_{\rho} F_{\rho}(N_c, v) \chi_{\rho}(V^\dagger(x)V(x + \mu)), \tag{2.26}
\]

where \( \rho \) refers to irreducible representations of \( U(N_{fs}) \), \( \chi_{\rho} \) is the character and \( F_{\rho}(N_c, v) \) is a function of \( N_c \) and \( v \).

Integral over \( V \in U(N_{fs}) \) can be performed by the following formula:

\[
\int dV V^\dagger_{\alpha \beta} V_{\gamma \omega} = \frac{1}{N_{fs}} \delta^\alpha_\omega \delta^\gamma_\beta. \tag{2.27}
\]

From Eqs. (2.26) and (2.27), the meson correlation function is calculated for the simplest case, i.e., one-dimensional system with free boundary condition. In this case the \( V^\dagger V \) correlator is obtained as follows:

\[
\langle V^\dagger_{\beta \alpha}(x) V_{\gamma \omega}(0) \rangle \sim \left( \frac{F_f(N_c, v)}{N_{fs} F_0(N_c, v)} \right)^{|x|} \delta^\alpha_\omega \delta^\gamma_\beta = e^{-m_\pi |x|} \delta^\alpha_\omega \delta^\gamma_\beta, \tag{2.28}
\]

where \( f \) and 0 refer to the fundamental and trivial representations of \( U(N_{fs}) \), respectively, and \( m_\pi = \ln(N_{fs} F_0/F_f) \). As the dimension of the group \( U(N_{fs}) \) becomes larger, the mass \( m_\pi \) is getting larger as we expected.

The above considerations naturally explain the numerical calculations of the multi-flavor QCD. In the following section, we shall apply the above observation to the
Dirac fermions in a random vector potential. We shall begin with reviewing the previous studies using LSUSY [9].

## 3 LSUSY and localization

Let us start with reviewing lattice formulation of random Dirac fermions. Our model is motivated by the network model for the quantum Hall phase transition in two-dimensional electron systems in a strong magnetic field [19]. There electrons move along equipotential line of the random impurity potential and acquire Aharonov-Bohm phase from the strong magnetic field. Then we shall consider $d = 2$ case in this section. Fermionic (electronic) part of the action is given as in the lattice QCD in the previous section;

$$S_D = \sum \bar{\chi}(x) \hat{D} \chi(x),$$

(3.1)

where $\chi(x)$ and $\bar{\chi}(x)$ are two-component Grassmann fields with the color $U(N_c)$ and flavor $U(N_{fs})$ indices as before. The bare mass $M_B$ in Sec.2 corresponds to the energy measured from the band center in the present electron system and we set $M_B = 0$ as we consider states at the band center. Corresponding to the Grassmann fields, we introduce complex scalar fields $\phi(x)$ and $\phi^\dagger(x)$ with the color and flavor degrees of freedom as follows [18],

$$S_\phi = \sum \hat{\phi}^\dagger(x) \hat{D} \phi(x).$$

(3.2)

It can be shown that the system $S_D + S_\phi$ possesses LSUSY,

$$\delta \phi = \bar{\epsilon}_+ \chi_- + \bar{\epsilon}_- \chi_+,$$

$$\delta \phi^\dagger = \bar{\chi}_- \epsilon_+ + \bar{\chi}_+ \epsilon_-,$$

$$\delta \chi_\pm = -\hat{D} \phi \epsilon_\pm,$$

$$\delta \bar{\chi}_\pm = -\hat{D} \phi^\dagger \bar{\epsilon}_\pm,$$

(3.3)

where $\chi_\pm$ is the chiral decomposition, i.e., $\gamma_5 \chi_\pm = \pm \chi_\pm$ and $\epsilon_\pm$ are anticommuting spinor variables with chirality $\gamma_5 \epsilon_\pm = \pm \epsilon_\pm$. From the LSUSY (3.3), the partition
function is just unity, i.e., the determinant from the $\chi$ integral and that from $\phi$ integral cancell with each other. Parameter $g$ which controls randomness of the vector potential can be introduced as follows,

$$P[U] = \prod_x \exp \left[ \frac{N_c}{g} \text{tr}(U_\mu(x) + U_\mu(x)^\dagger) \right]. \quad (3.4)$$

Ensemble average with respect to the random vector potential for calculating correlators etc can be performed readily by the LSUSY:

$$\langle O \rangle = \int \left[ DUD\bar{\chi}D\phi^\dagger D\phi \right] P[U] e^{-S_D - S_\phi} \mathcal{O}$$

$$= \int \left[ DUD\bar{\chi}D\chi \right] P[U] \frac{e^{-S_D}}{\det(\hat{D}^\dagger \hat{D})} \mathcal{O}$$

$$= \int \left[ DUD\bar{\chi}D\chi \right] P[U] \frac{e^{-S_D}}{\langle \chi; U|\chi; U \rangle} \mathcal{O}, \quad (3.5)$$

where $\mathcal{O}$ is physical observable of the Dirac fermion and $|\chi; U \rangle$ is the lowest-energy state of $\chi(x)$ in the background of $U_\mu(x)$. From Eq. (3.5), the utility of the LSUSY is obvious. In the rest of the discussion, we consider the strong-disorder limit $g \to \infty$ and therefore there is no distribution weight for the gauge field $U_\mu(x)$ as in the strong-coupling limit of QCD.

The bosonic action $S_\phi$ can be rewritten into more compact form,

$$S_\phi \Rightarrow S_{\omega\varphi} = \sum \left[ \omega^\dagger \hat{D}\varphi + \varphi^\dagger \hat{D}\omega \right], \quad (3.6)$$

where $\omega$ and $\varphi$ are complex boson fields. Then the one-link integral of $U_\mu(x)$ in the previous section can be applied straightforwardly. Fermion-fermion(FF) sector is given by $S_m(\lambda)$ in Eq. (2.11). Boson-boson(BB) sector is also written in terms of the following composite fields like

$$\frac{1}{N_c} \sum_a \varphi^{a,\alpha}(x)\varphi_{a,\beta}^\dagger(x), \quad \frac{1}{N_c} \sum_a \omega^{a,\alpha}(x)\omega_{a,\beta}^\dagger(x),$$

etc., but the sign of the action is just opposite to the FF sector. There are additional terms which include fermionic composites like $\chi^a(x)\varphi_a^\dagger(x)$ but they contributions to the FF and BB sectors are negligible in the leading order of $1/N_c$.

From the above arguments and investigation on QCD in Sec.2, it is straightforward to discuss physical properties of the random Dirac fields at the band center which
corresponds to the vanishing bare mass $M_B \to 0$. We are interested in the correlation function of the local fermion DoS $m_\psi(x) = \sum_a \bar{\psi}_a(x)\psi^a(x)$, which is nothing but the “pion” field in QCD. Therefore in the case of a small number of flavors, one expects that the operator $m_\psi(x)$ condenses and has a nonvanishing expectation value. However as we are studying the 2-dimensional system, the genuine long-range order does not exist and the correlator of $m_\psi(x)$ decays algebraically (quasi-long-range order) instead

$$\langle \bar{\psi}\psi(x)\bar{\psi}\psi(0) \rangle \sim |x|^{-1/N_c}. \quad (3.7)$$

The power-decaying behavior in (3.7) comes from the fluctuation of the “pions” $\pi_K(x)$. It should be remarked here that in the present lattice formulation topologically non-trivial configurations of the random vector potential are all included. This formulation is motivated by the network model by Chalker and Coddington\cite{19} and it is in contrast to other formulations like that using the conformal field theory. Therefore it is not so surprising even if the decay power of the correlator (3.7) is different from that obtained in other methods\cite{8}. The power-decay behaviour (3.7) means the existence of the extended states anyway.

Let us turn to the many flavors case. The discussion in Sec.2 can be applied straightforwardly to the random system in this section. As the system is 2-dimensional, the disordered phase is enhanced in the $U(N_{fs})$ antiferromagnets. The correlator of the local fermion DoS behaves as

$$\langle \bar{\psi}\psi(x)\bar{\psi}\psi(0) \rangle \sim e^{-M_l|x|}, \quad (3.8)$$

though we cannot estimate the “inverse localization length” $M_l$ analytically. The above is a new and main result in this paper which indicates that extended states at the band center disappear as the number of flavors is increased.

Recently certain related model is studied in Ref.\cite{20}, and a similar phase transition is predicted, though the model is defined in the continuum space instead of on the

\footnote{Here we do not show explicitly the flavor-spinor indices as they are obvious as in \ref{2.28}.}
lattice. In the present lattice model, which is motivated by the network model for the quantum Hall transition, topological nontrivial configurations of the gauge field are included. Therefore it is quite nontrivial that a similar transition or crossover occurs in the both cases.

4 Conclusion

In this paper we studied Dirac fermions in the random $U(N_c)$ vector potential by using lattice formulation. Especially we showed that physical properties at the band center drastically change as the number of flavors is increased. For a small number of flavors, the correlator of the local fermion DoS exhibits algebraic decay, whereas for many flavors, it decays exponentially. This result suggests that extended states at the band center disappear as $N_f \to \text{large}$. Numerical studies of QCD with many flavors\cite{10} supports this result.

The above result is quite interesting and it is desired to study the same problem by other methods. Recently a closely related model, the random flux model, was studied by a lattice formulation\cite{21}. There a mathematical tool invented by Zirnbauer\cite{14} was used and the DoS was calculated for the \textit{single-flavor} case. Then it is interesting to calculate the DoS for many flavors and to see if qualitative difference appears as the number of flavors is increased.

In the present studies we cannot estimate the critical value of $N_f$ for the delocalization-localization transition(DLT). Closely related and tractable model is a quantum $U(N_f)$ antiferromagnetic Heisenberg model\cite{22}. There a phase transition, which corresponds to the DLT, is predicted and the critical value of $N_f$ is estimated.
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