Statistics of lattice points in thin annuli for
generic lattices

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Abstract
We study the statistical properties of the counting function of lattice points inside thin annuli. By a conjecture of Bleher and Lebowitz, if the width shrinks to zero, but the area converges to infinity, the distribution converges to the Gaussian distribution. If the width shrinks slowly to zero, the conjecture was proven by Hughes and Rudnick for the standard lattice, and in our previous paper for generic rectangular lattices. We prove this conjecture for arbitrary lattices satisfying some generic Diophantine properties, again assuming the width of the annuli shrinks slowly to zero.

One of the obstacles of applying the technique of Hughes-Rudnick on this problem is the existence of so-called close pairs of lattice points. In order to overcome this difficulty, we bound the rate of occurrence of this phenomenon by extending some of the work of Eskin-Margulis-Mozes on the quantitative Openheim conjecture.

1 Introduction
We consider a variant of the lattice points counting problem. Let \( \Lambda \subset \mathbb{R}^2 \) be a planar lattice, with \( \det \Lambda \) the area of its fundamental cell. Let

\[ N_\Lambda(t) = \{ x \in \Lambda : |x| \leq t \}, \]

denote its counting function, that is, we are counting \( \Lambda \)-points inside a disc of radius \( t \).
As well known, as \( t \to \infty \), \( N_\Lambda(t) \sim \frac{\pi}{\det \Lambda} t^2 \). Denoting the remainder or the error term
\[
\Delta_\Lambda(t) = N_\Lambda(t) - \frac{\pi}{\det \Lambda} t^2,
\]
it is a conjecture of Hardy that
\[
|\Delta_\Lambda(t)| \ll \epsilon t^{1/2+\epsilon}.
\]

Another problem one could study is the statistical behavior of the value distribution of \( \Delta_\Lambda \) normalized by \( \sqrt{t} \), namely of
\[
F_\Lambda(t) := \frac{\Delta_\Lambda(t)}{\sqrt{t}}.
\]

Heath-Brown \([\text{HB}]\) shows that for the standard lattice \( \Lambda = \mathbb{Z}^2 \), the value distribution of \( F_\Lambda \), weakly converges to a non-Gaussian distribution with density \( p(x) \). Bleher \([\text{BL3}]\) established an analogue of this theorem for a more general setting, where in particular it implies a non-Gaussian limiting distribution of \( F_\Lambda \), for any lattice \( \Lambda \subset \mathbb{Z}^2 \).

However, the object of our interest is slightly different. Rather than counting lattice points in the circle of varying radius \( t \), we will do the same for \emph{annuli}. More precisely, we define
\[
N_\Lambda(t, \rho) := N_\Lambda(t + \rho) - N_\Lambda(t),
\]
that is, the number of \( \Lambda \)-points inside the annulus of inner radius \( t \) and width \( \rho \). The "expected" value is the area \( \frac{\pi}{\det \Lambda} (2t\rho + \rho^2) \), and the corresponding normalized remainder term is
\[
S_\Lambda(t, \rho) := \frac{N_\Lambda(t + \rho) - N_\Lambda(t) - \frac{\pi}{\det \Lambda} (2t\rho + \rho^2)}{\sqrt{t}}.
\]

The statistics of \( S_\Lambda(t, \rho) \) vary depending to the size of \( \rho(t) \). Of our particular interest is the \emph{intermediate} or \emph{macroscopic regime}. Here \( \rho \to 0 \), but \( \rho t \to \infty \). A particular case of the conjecture of Bleher and Lebowitz \([\text{BL4}]\) states that \( S_\Lambda(t, \rho) \) has a Gaussian distribution. In 2004 Hughes and Rudnick \([\text{HR}]\) established the Gaussian distribution for the unit circle, under an additional assumption that \( \rho(t) \gg t^{-\epsilon} \) for every \( \epsilon > 0 \).

By a rotation and dilation (which does not essentially effect the counting function), we may assume, with no loss of generality, that \( \Lambda \) admits a basis one of whose elements is the vector \( (1,0) \), that is \( \Lambda = \langle 1, \alpha + i\beta \rangle \) (we make the natural identification of \( i \) with \( (0,1) \)). In a previous paper \([\text{W}]\) we already dealt with the problem of investigating the statistical properties of
the error term for rectangular lattice $\Lambda = \langle 1, i\beta \rangle$. We established the limiting Gaussian distribution for the "generic" case in this 1-parameter family.

Some of the work done in [W] extends quite naturally for the 2-parameter family of planar lattices $\langle 1, \alpha + i\beta \rangle$. That is, in the current work we will require algebraic independence of $\alpha$ and $\beta$ as well as the "strong Diophantinity" of the pair $(\alpha, \beta)$ (to be defined), rather than transcendence and strong Diophantinity of the aspect ratio of the ellipse, as in [W].

We say that a real number $\xi$ is strongly Diophantine, if for every fixed natural $n$, there exists $K_1 > 0$, such that for integers $a_j$ with $\sum_{j=0}^{n} a_j \xi^j \neq 0$,

$$\left|\sum_{j=0}^{n} a_j \xi^j\right| \gg n \frac{1}{\left(\max_{0 \leq j \leq n} |a_j|\right)^{K_1}}.$$

It was shown by Mahler [MAH], that this property holds for a "generic" real number. We say that a pair of numbers $(\alpha, \beta)$ is strongly Diophantine, if for every fixed natural $n$, there exists a number $K_1 > 0$, such that for every integral polynomial $p(x, y) = \sum_{i+j \leq n} a_{i,j} x^i y^j$ of degree $\leq n$, we have

$$|p(\alpha, \beta)| \gg n \frac{1}{\max_{i+j \leq n} |a_{i,j}|^{K_1}},$$

whenever $p(\alpha, \beta) \neq 0$. This holds for almost all real pairs $(\alpha, \beta)$, see section 2.2.

**Theorem 1.1.** Let $\Lambda = \langle 1, \alpha + i\beta \rangle$ where $(\alpha, \beta)$ is algebraically independent and strongly Diophantine pair of real numbers. Assume that $\rho = \rho(T) \to 0$, but for every $\delta > 0$, $\rho \gg T^{-\delta}$. Then for every interval $\mathcal{A}$,

$$\lim_{T \to \infty} \frac{1}{T} \text{meas}\left\{ t \in [T, 2T] : \frac{S_{\Lambda}(t, \rho)}{\sigma} \in \mathcal{A} \right\} = \frac{1}{\sqrt{2\pi}} \int_{\mathcal{A}} e^{-x^2} dx,$$

(1)

where the variance is given by

$$\sigma^2 := \frac{4\pi}{\beta} \cdot \rho.$$

(2)

**Remark:** Note that the variance $\sigma^2$ is $\alpha$-independent, since the determinant $\det(\Lambda) = \beta$.

One of the features of a rectangular lattice is that it is quite easy to show that the number of so-called close pairs of lattice points or pairs of
points lying within a narrow annulus is bounded by essentially its average (see lemma 5.2 of [W]). This particular feature of the rectangular lattices was exploited while reducing the computation of the moments to the ones of a smooth counting function (we call it “unsmoothing”). In order to prove an analogous bound for a general lattice, we extend a result from Eskin, Margulis and Mozes [EMM] for our needs to obtain proposition 3.1. We believe that this proposition is of independent interest.

2 The distribution of $\tilde{S}_{\Lambda, M, L}$

In this section, we are interested in the distribution of the smooth version of $S_{\Lambda}(t, \rho)$, denoted $\tilde{S}_{\Lambda, M, L}(t)$, where $L := \frac{1}{\rho}$ and $M$ is the smoothing parameter. Just as in [W] and [HR],

$$\tilde{S}_{\Lambda, M, L}(t) = \frac{\tilde{N}_{\Lambda, M}(t + \frac{1}{L}) - \tilde{N}_{\Lambda, M}(t) - \frac{\pi}{d}(\frac{2}{L} + \frac{1}{L^2})}{\sqrt{t}},$$

(3)

where $\tilde{N}_{\Lambda, M}$ is the smooth version of $N_{\Lambda}$, computed by means of convolution of the characteristic function of the unit ball with $\psi$, a smooth function with a compact support (see [HR] or [W] for details). We assume that for every $\delta > 0$, $L = L(T) = O(T^\delta)$, which corresponds to the assumption of theorem 1.1 regarding $\rho := \frac{1}{L}$.

Rather than drawing $t$ at random from $[T, 2T]$ with a uniform distribution, we prefer to work with smooth densities: introduce $\omega \geq 0$, a smooth function of total mass unity, such that both $\omega$ and $\hat{\omega}$ are rapidly decaying, namely

$$|\omega(t)| \ll \frac{1}{(1 + |t|)^A}, \quad |\hat{\omega}(t)| \ll \frac{1}{(1 + |t|)^A},$$

for every $A > 0$. Define the averaging operator

$$\langle f \rangle_T = \frac{1}{T} \int_{-\infty}^{\infty} f(t) \omega\left(\frac{t}{T}\right) dt,$$

and let $P_{\omega, T}$ be the associated probability measure:

$$P_{\omega, T}(f \in A) = \frac{1}{T} \int_{-\infty}^{\infty} 1_A(f(t)) \omega\left(\frac{t}{T}\right) dt.$$
Remark: In what follows, we will suppress the explicit dependency on $T$, whenever convenient.

**Theorem 2.1.** Suppose that $M(T)$ and $L(T)$ are increasing to infinity with $T$, such that $M = O(T^\delta)$ for all $\delta > 0$, and $L/\sqrt{M} \to 0$. Then if $(\alpha, \beta)$ is an algebraically independent strongly Diophantine pair, we have for $\Lambda = \langle 1, \alpha + i\beta \rangle$,

$$\lim_{T \to \infty} \mathbb{P}_{\omega, T} \left\{ \frac{\tilde{S}_{\Lambda, M, L}}{\sigma} \in \mathcal{A} \right\} = \frac{1}{\sqrt{2\pi}} \int_{\mathcal{A}} e^{-\frac{x^2}{2}} dx,$$

for any interval $\mathcal{A}$, where

$$\sigma^2 := \frac{4\pi}{\beta L}. \quad (4)$$

**Definition:** A tuple of real numbers $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ is called Diophantine, if there exists a number $K > 0$, such that for every integer tuple $\{a_i\}_{i=0}^n$,

$$\left| a_0 + \sum_{i=1}^n a_i \alpha_i \right| \gg \frac{1}{qK}, \quad (5)$$

where $q = \max_{0 \leq i \leq n} |a_i|$. Khintchine proved that almost all tuples in $\mathbb{R}^n$ are Diophantine (see, e.g. [S], pages 60-63).

Denote the dual lattice

$$\Lambda^* = \langle 1, -\frac{\alpha}{\beta} + i\frac{1}{\beta} \rangle.$$

We assume for the rest of current section that the set of squared norms of $\Lambda^*$ satisfy the Diophantine property, which means that $(\alpha^2, \alpha \beta, \beta^2)$ is a Diophantine triple of numbers. We may assume the Diophantinity of $(\alpha^2, \alpha \beta, \beta^2)$, since theorem 1.1 (and theorem 2.1) assume $(\alpha, \beta)$ is strongly Diophantine, which is obviously a stronger assumption.

We use the following approximation to $\tilde{N}_{\Lambda, M}(t)$ (see e.g [W], lemma 4.1):

**Lemma 2.2.** As $t \to \infty$,

$$\tilde{N}_{\Lambda, M}(t) = \frac{\pi t^2}{\beta} - \frac{\sqrt{t}}{\beta \pi} \sum_{\vec{k} \in \Lambda^* \backslash \{0\}} \frac{\cos \left( \frac{2\pi t |\vec{k}| + \frac{\pi}{4}}{4} \right)}{|\vec{k}|^\frac{3}{2}} \cdot \hat{\psi} \left( \frac{|\vec{k}|}{\sqrt{M}} \right) + O \left( \frac{1}{\sqrt{t}} \right), \quad (6)$$

where, again, $\Lambda^*$ is the dual lattice.
By the definition of $\tilde{S}_{\Lambda, M, L}$ in (3) and appropriately manipulating the sum in (6) we obtain the following

**Corollary 2.3.**

\[
\tilde{S}_{\Lambda, M, L}(t) = \frac{2}{\beta \pi} \sum_{\vec{k} \in \Lambda^* \setminus \{0\}} \sin \left( \frac{\pi |\vec{k}|}{L} \right) \sin \left( 2\pi (t + \frac{1}{2L})|\vec{k}| + \frac{\pi}{4} \right) \hat{\psi} \left( \frac{|\vec{k}|}{\sqrt{M}} \right) \tag{7}
\]

\[+ O \left( \frac{1}{\sqrt{t}} \right). \]

One should note that $\hat{\psi}$ being compactly supported means that the sum essentially truncates at $|\vec{k}| \approx \sqrt{M}$.

Unlike the standard lattice, clearly there are no nontrivial multiplicities in $\Lambda$, that is

**Lemma 2.4.** Let $\vec{a}_j = m_j + n_j(\alpha + i\beta) \in \Lambda$, $j = 1, 2$, with an irrational $\alpha$ such that $\gamma / \in \mathbb{Q}(\alpha)$. Then if $|\vec{a}_1| = |\vec{a}_2|$, either $n_1 = n_2$ and $m_1 = m_2$ or $n_1 = -n_2$ and $n_2 = -m_2$.

**Proof of theorem 2.1.** We will show that the moments of $\tilde{S}_{\Lambda, M, L}$ corresponding to the smooth probability space converge to the moments of the normal distribution with zero mean and variance which is given by theorem 2.1. This allows us to deduce that the distribution of $\tilde{S}_{\Lambda, M, L}$ converges to the normal distribution as $T \to \infty$, precisely in the sense of theorem 2.1.

First, we show that the mean is $O \left( \frac{1}{\sqrt{T}} \right)$. Since $\omega$ is real,

\[
\left| \left\langle \sin \left( 2\pi (t + \frac{1}{2L})|\vec{k}| + \frac{\pi}{4} \right) \right\rangle \right| = \left| \Im \left\{ \hat{\omega}(-T|\vec{k}|) e^{i\pi (\frac{|\vec{k}|}{L} + \frac{1}{4})} \right\} \right| \ll \frac{1}{T^A |\vec{k}|^A}
\]

for any $A > 0$, where we have used the rapid decay of $\hat{\omega}$. Thus

\[
\left| \left\langle \tilde{S}_{\Lambda, M, L} \right\rangle \right| \ll \sum_{\vec{k} \in \Lambda^* \setminus \{0\}} \frac{1}{T^A |\vec{k}|^{A+3/2}} + O \left( \frac{1}{\sqrt{T}} \right) \ll O \left( \frac{1}{\sqrt{T}} \right),
\]

due to the convergence of $\sum_{\vec{k} \in \Lambda^* \setminus \{0\}} \frac{1}{|\vec{k}|^{A+3/2}}$, for $A > \frac{1}{2}$.

Now define

\[
\mathcal{M}_{\Lambda, m} := \left\langle \left( \frac{2}{\beta \pi} \sum_{\vec{k} \in \Lambda^* \setminus \{0\}} \frac{\sin \left( \frac{\pi |\vec{k}|}{L} \right)}{|\vec{k}|^{\frac{3}{2}}} \sin \left( 2\pi (t + \frac{1}{2L})|\vec{k}| + \frac{\pi}{4} \right) \hat{\psi} \left( \frac{|\vec{k}|}{\sqrt{M}} \right) \right)^m \right\rangle \tag{8}
\]
Then from \( (7) \), the binomial formula and the Cauchy-Schwartz inequality,

\[
\left\langle (\tilde{S}_{\Lambda,M,L})^m \right\rangle = \mathcal{M}_{\Lambda,m} + O \left( \sum_{j=1}^{m} \binom{m}{j} \frac{\sqrt{\mathcal{M}_{2m-2j}}}{T^{j/2}} \right)
\]

Proposition 2.5 together with proposition 2.8 allow us to deduce the result of theorem 2.1 for an algebraically independent strongly Diophantine \((\xi, \eta) := (-\frac{\alpha}{\beta}, \frac{1}{\beta})\). Clearly, \((\alpha, \beta)\) being algebraically independent and strongly Diophantine is sufficient.

\[
\begin{align*}
\text{Proposition 2.5.} & \quad \text{If } M = O \left( \frac{T^{1/(K+1/2+\delta)}}{\sqrt{M}} \right) \text{ for fixed } \delta > 0, \text{ then the variance of } \tilde{S}_{\Lambda,M,L} \text{ is asymptotic to } \\
& \quad \sigma^2 := \frac{4}{\beta^2 \pi^2} \sum_{\vec{k} \in \Lambda^* \setminus \{0\}} \frac{\sin^2 \left( \frac{\pi |\vec{k}|}{L} \right)}{|\vec{k}|^3} \psi^2 \left( \frac{|\vec{k}|}{\sqrt{M}} \right)
\end{align*}
\]

If \( L \to \infty \), but \( L/\sqrt{M} \to 0 \), then

\[
\sigma^2 \sim \frac{4\pi}{\beta L} \tag{9}
\]

\[\text{Proof.} \ \text{Expanding out } (8), \text{ we have} \]

\[
\mathcal{M}_{\Lambda,2} = \frac{4}{\beta^2 \pi^2} \sum_{\vec{k}, \vec{l} \in \Lambda^* \setminus \{0\}} \frac{\sin \left( \frac{\pi |\vec{k}|}{L} \right) \sin \left( \frac{\pi |\vec{l}|}{L} \right) \hat{\psi} \left( \frac{|\vec{k}|}{\sqrt{M}} \right) \hat{\psi} \left( \frac{|\vec{l}|}{\sqrt{M}} \right)}{|\vec{k}|^2 |\vec{l}|^2} \\
\times \left\langle \sin \left( 2\pi \left( t + \frac{1}{2L} \right) |\vec{k}| + \frac{\pi}{4} \right) \sin \left( 2\pi \left( t + \frac{1}{2L} \right) |\vec{l}| + \frac{\pi}{4} \right) \right\rangle \tag{10}
\]

2.1 The variance

The computation of the variance is done in two steps. First, we reduce the main contribution to the diagonal terms, using the assumption on the pair \((\alpha, \beta)\) (i.e. \((\alpha^2, \alpha \beta, \beta^2)\) is Diophantine). Then we compute the contribution of the diagonal terms. We sketch these steps, since they are very close to the corresponding one \[W\].

Suppose that the triple \((\alpha^2, \alpha \beta, \beta^2)\) satisfies (5).

\[\text{Proposition 2.5.} \quad \text{If } M = O \left( \frac{T^{1/(K+1/2+\delta)}}{\sqrt{M}} \right) \text{ for fixed } \delta > 0, \text{ then the variance of } \tilde{S}_{\Lambda,M,L} \text{ is asymptotic to} \]

\[
\sigma^2 := \frac{4}{\beta^2 \pi^2} \sum_{\vec{k} \in \Lambda^* \setminus \{0\}} \frac{\sin^2 \left( \frac{\pi |\vec{k}|}{L} \right)}{|\vec{k}|^3} \psi^2 \left( \frac{|\vec{k}|}{\sqrt{M}} \right)
\]
It is easy to check that the average of the second line of the previous equation is:

\[
\frac{1}{4} \left[ \hat{\omega} \left( T(|\vec{k}| - |\vec{l}|) \right) e^{i\pi(1/L)(|\vec{k}| - |\vec{l}|)} + \hat{\omega} \left( T(|\vec{l}| - |\vec{k}|) \right) e^{i\pi(1/L)(|\vec{l}| - |\vec{k}|)} + \hat{\omega} \left( T(|\vec{k}| + |\vec{l}|) \right) e^{-i\pi(1/2 + 1/L)(|\vec{k}| + |\vec{l}|)} - \hat{\omega} \left( - T(|\vec{k}| + |\vec{l}|) \right) e^{i\pi(1/2 + 1/L)(|\vec{k}| + |\vec{l}|)} \right]
\]

(11)

Recall that the support condition on \( \hat{\psi} \) means that \( \vec{k} \) and \( \vec{l} \) are both constrained to be of length \( O(\sqrt{M}) \). Thus the off-diagonal contribution (that is for \( |\vec{k}| \neq |\vec{l}| \)) of the first two lines of (11) is

\[
\ll \sum_{\vec{k}, \vec{l} \in \Lambda^\ast \setminus \{0\}} \frac{M^{A(K+1/2)}}{T_A} \ll \frac{M^{A(K+1/2)+2}}{T_A} \ll T^{-B},
\]

for every \( B > 0 \), by Diophantinity of \((\alpha, \alpha\beta, \beta^2)\).

Obviously, the contribution to (10) of the two last lines of (11) is negligible both in the diagonal and off-diagonal cases, justifying the diagonal approximation of (10) in the first statement of the proposition, and we omit the rest of the proof.

\[\square\]

2.2 The higher moments

In order to compute the higher moments we will prove that the main contribution comes from the so-called diagonal terms (to be explained later). In order to bound the contribution of the off-diagonal terms, we will use the following theorem, which is a consequence of the work of Kleinbock and Margulis [KM]. The contribution of the diagonal terms is computed exactly in the same manner it was done in [W], and so we will omit it here.

**Theorem 2.6.** Let an integer \( n \) be given. Then almost all pairs of real numbers \((\xi, \eta) \in \mathbb{R}^2\) satisfy the following property: there exists a number \( K_1 \in \mathbb{N} \) such that for every integer polynomial of 2 variables \( p(x, y) = \sum_{i+j \leq n} a_{i,j} x^i y^j \) with degree \( \leq n \), we have

\[
\left| p(\xi, \eta) \right| \gg h^{-K_1},
\]

where \( h = \max_{i+j \leq n} |a_{i,j}| \) is the height of \( p \). The constant involved in the ” \( \gg \)” notation depends only on \( \xi, \eta \) and \( n \).
We will remark that theorem A in [KM] is much more general when the result we are using. As a matter of fact, we have the inequality
\[ \left| b_0 + b_1 f_1(x) + \ldots + b_n f_n(x) \right| \gg \epsilon \frac{1}{h^{\epsilon n + \epsilon}} \]
with \( b_i \in \mathbb{Z} \) and
\[ h := \max_{0 \leq i \leq n} |b_i|. \]
The inequality above holds for every \( \epsilon > 0 \) for a wide class of functions \( f_i : U \to \mathbb{R} \), for almost all \( x \in U \), where \( U \subset \mathbb{R}^m \) is an open subset. Here we use this inequality for the monomials.

**Definition:** We call the pairs \((\xi, \eta)\) which satisfy for all natural \( n \) the property of theorem 2.6 strongly Diophantine. Thus theorem 2.6 states that almost all real pairs of numbers are strongly Diophantine.

**Remark:** Simon Kristensen [KR] has recently shown, that the set of all pairs \((\xi, \eta) \in \mathbb{R}^2\) which fail to be strongly Diophantine has Hausdorff dimension 1.

Obviously, strong Diophantinity of \((\xi, \eta)\) implies Diophantinity of any \( n \)-tuple of real numbers which consists of any set of monomials in \( \xi \) and \( \eta \). Moreover, \((\xi, \eta)\) is strongly Diophantine iff \((-\frac{\alpha}{\beta}, \frac{1}{\beta})\) is such.

We have the following analogue of lemma 4.7 in [W], which will eventually allow us to exploit the strong Diophantinity of \((\alpha, \beta)\).

**Lemma 2.7.** If \((\xi, \eta)\) is strongly Diophantine, then it satisfies the following property: for any fixed natural \( m \), there exists \( K \in \mathbb{N} \), such that if
\[ z_j = a_j^2 + b_j^2 \xi^2 + 2a_j b_j \xi + b_j^2 \eta^2 \ll M, \]
and \( \epsilon_j = \pm 1 \) for \( j = 1, \ldots, m \), with integral \( a_j, b_j \) and if \( \sum_{j=1}^{m} \epsilon_j \sqrt{z_j} \neq 0 \), then
\[ \left| \sum_{j=1}^{m} \epsilon_j \sqrt{z_j} \right| \gg M^{-K}, \quad (12) \]
where the constant involved in the ”\( \gg \)” notation depends only on \( \eta \) and \( m \).

The proof is essentially the same as the one of lemma 4.7 from [W], considering the product \( Q \) of numbers of the form \( \sum_{j=1}^{m} \delta_j \sqrt{z_j} \) over all possible signs \( \delta_j \). Here we use the Diophantinity of the real tuple \((\xi, \eta)\) rather than of a single real number.
Proposition 2.8. Let \( m \in \mathbb{N} \) be given. Suppose that \( \Lambda = \langle 1, \alpha + i\beta \rangle \), such that the pair \((\xi, \eta) := (-\frac{\alpha}{\beta}, \frac{1}{\beta})\) is algebraically independent strongly Diophantine, which satisfy the property of lemma 2.7 for the given \( m \), with \( K = K_m \). Then if \( M = O\left( T^{-\frac{2}{K_m}} \right) \) for some \( \delta > 0 \), and if \( L \to \infty \) such that \( L/\sqrt{M} \to 0 \), the following holds:

\[
\mathcal{M}_{\Lambda,m} = \begin{cases} 
\frac{m!}{2^{m/2} \pi^{m/2}} + O\left( \frac{\log L}{L} \right), & m \text{ is even} \\
O\left( \frac{\log L}{L} \right), & m \text{ is odd}
\end{cases}
\]

Proof. Expanding out (8), we have

\[
\mathcal{M}_{\Lambda,m} = \frac{2^m}{\beta^{3m\pi m}} \sum_{\vec{k}_1, \ldots, \vec{k}_m \in \Lambda^* \setminus \{0\}} \prod_{j=1}^{m} \sin \left( \frac{\pi |\vec{k}_j|}{L} \right) \psi'(\frac{|\vec{k}_j|}{\sqrt{M}}) \left| \vec{k}_j \right|^{-\frac{3}{2}} \times \left( \prod_{j=1}^{m} \sin \left( 2\pi (t + \frac{1}{2L}) |\vec{k}_j| + \frac{\pi}{4} \right) \right)
\]

(13)

Now,

\[
\left\langle \prod_{j=1}^{m} \sin \left( 2\pi (t + \frac{1}{2L}) |\vec{k}_j| + \frac{\pi}{4} \right) \right\rangle
\]

\[
= \sum_{\epsilon_j = \pm 1} \prod_{j=1}^{m} \epsilon_j^{\frac{1}{2^{m} \pi m}} \omega \left( -T \sum_{j=1}^{m} \epsilon_j |\vec{k}_j| \right) e^{\frac{\pi i}{2} \sum_{j=1}^{m} \epsilon_j (1/L) |\vec{k}_j| + \pi/4}
\]

We call a term of the summation in (13) with \( \sum_{j=1}^{m} \epsilon_j |\vec{k}_j| = 0 \) diagonal, and off-diagonal otherwise. Due to lemma 2.7, the contribution of the off-diagonal terms is:

\[
\ll \sum_{\vec{k}_1, \ldots, \vec{k}_m \in \Lambda^* \setminus \{0\}} \left( \frac{T}{MK_m} \right)^{-A} \ll M^m T^{-A\delta},
\]

for every \( A > 0 \), by the rapid decay of \( \omega \) and our assumption regarding \( M \).

Since \( m \) is constant, this allows us to reduce the sum to the diagonal terms. In order to be able to sum over all the diagonal terms we need the following analogue of a well-known theorem due to Besicovitch [BS] about incommensurability of square roots of integers.

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Proposition 2.9. Suppose that $\xi$ and $\eta$ are algebraically independent, and
\[ z_j = a_j^2 + 2a_jb_j\xi + b_j^2(\xi^2 + \eta), \] (14)
such that $(a_j, b_j) \in \mathbb{Z}_+^2$ are all different primitive vectors, for $1 \leq j \leq m$. Then \( \{\sqrt{z_j}\}_{j=1}^m \) are linearly independent over $\mathbb{Q}$.

The last proposition is an immediate consequence of a theorem proved in the appendix of [BL2].

Computing the contribution of the diagonal terms is done literally the same way it was done in [W] and thus it is omitted here. In order to be able to sum over the diagonal terms, we use here proposition 2.9 rather than proposition 3.2 in [W].

\[ \square \]

3 Bounding the number of close pairs of lattice points

Roughly speaking, we say that a pair of lattice points, $n$ and $n'$ is close, if $||n| - |n'||$ is small. We would like to show that this phenomenon is rare. This is closely related to the Oppenheim conjecture, as $|n|^2 - |n'|^2$ is a quadratic form on the coefficients of $n$ and $n'$. In order to establish a quantitative result, we use a technique developed in a paper by Eskin, Margulis and Mozes [EMM].

3.1 Statement of the results

The ultimate goal of this section is to establish the following

Proposition 3.1. Let $\Lambda$ be a lattice and denote
\[ A(R, \delta) := \{(\vec{k}, \vec{l}) \in \Lambda : R \leq |\vec{k}|^2 \leq 2R, |\vec{k}|^2 \leq |\vec{l}|^2 \leq |\vec{k}|^2 + \delta\}. \] (15)

Then if $\delta > 1$, such that $\delta = o(R)$, we have
\[ \#A(R, \delta) \ll R\delta \cdot \log R \]

In order to prove this result, we note that evaluating the size of $A(R, \delta)$ is equivalent to counting integer points $\vec{v} \in \mathbb{R}^4$ with $T \leq ||\vec{v}|| \leq 2T$ such that
\[ 0 \leq Q_1(v) \leq \delta, \]
where \( Q_1 \) is a quadratic form of signature \((2, 2)\), given explicitly by

\[
Q_1(\vec{v}) = (v_1 + v_2 \alpha)^2 + (v_2 \beta)^2 - (v_3 + v_4 \alpha)^2 - (v_4 \beta)^2.
\]

For a fixed \( \delta > 0 \) and a large \( R \), this situation was considered extensively by Eskin, Margulis and Mozes [EMM]. We will examine how the constants involved in their result depend on \( \delta \), and find out that there is a linear dependency, which is what we essentially need. The author wishes to thank Alex Eskin for his assistance with this matter.

**Remark:** For our purposes we need a weaker result:

\[
\#A(R, \delta) \ll \epsilon R \delta \cdot R^\epsilon,
\]

for every \( \epsilon > 0 \). If \( \Lambda \) is a rectangular lattice (i.e. \( \alpha = 0 \)), then this result follows from properties of the divisor function (see e.g. [BL], lemma 3.2).

Theorem 2.3 in [EMM] considers a more general setting than proposition 3.1. We state here theorem 2.3 from [EMM] (see theorem 3.2). It follows from theorem 3.3 from [EMM], which will be stated as well (see theorem 3.3). Then we give an outline of the proof of theorem 2.3 of [EMM], and inspect the dependency on \( \delta \) of the constants involved.

### 3.2 Theorems 2.3 and 3.3 from [EMM]

Let \( \Delta \) be a lattice in \( \mathbb{R}^n \). We say that a subspace \( L \subset \mathbb{R}^n \) is \( \Delta \)-rational, if \( L \cap \Delta \) is a lattice in \( L \). We need the following definitions:

**Definitions:**

\[
\alpha_i(\Delta) := \sup \left\{ \frac{1}{d_\Delta(L)} \right\} \quad \text{where} \quad L \text{ is a } \Delta - \text{rational subspace of dimension } i,
\]

where

\[
d_\Delta(L) := vol(L/(L \cap \Delta)).
\]

Also

\[
\alpha(\Delta) := \max_{0 \leq i \leq n} \alpha_i(\Delta).
\]

Since the space of unimodular lattices is canonically isomorphic to \( SL(n, \mathbb{R})/SL(n, \mathbb{Z}) \), the notation \( \alpha(g) \) makes sense for \( g \in G := SL(n, \mathbb{R}) \).

For a bounded function \( f : \mathbb{R}^n \to \mathbb{R} \), with \( |f| \leq M \), which vanishes outside a ball \( B(0, R) \), define \( \tilde{f} : SL(n, \mathbb{R}) \to \mathbb{R} \) by the following formula:

\[
\tilde{f}(g) := \sum_{v \in \mathbb{Z}^n} f(gv).
\]
Lemma 3.1 in [S2] implies that
\[ \tilde{f}(g) < c\alpha(g), \]  
(17)

where \( c = c(f) \) is an explicit constant
\[ c(f) = c_0 M \max(1, R^n), \]
for some constant \( c_0 = c_0(n) \), independent on \( f \). In section 3.4 we prove a stronger result, assuming some additional information about the support of \( f \).

Let \( Q_0 \) be a quadratic form defined by
\[ Q_0(\vec{v}) = 2v_1v_n + \sum_{i=2}^{p} v_i^2 - \sum_{i=p+1}^{n-1} v_i^2. \]

Since
\[ v_1v_n = \frac{(v_1 + v_n)^2 - (v_1 - v_n)^2}{2}, \]
\( Q_0 \) is of signature \( p, q \). Obviously, \( G := SL(n, \mathbb{R}) \) acts on the space of quadratic forms of signature \( (p, q) \), and discriminant \( \pm 1 \), \( \mathcal{O} = \mathcal{O}(p, q) \) by:
\[ Q^g(v) := Q(gv). \]

Moreover, by the well known classification of quadratic forms, \( \mathcal{O} \) is the orbit of \( Q_0 \) under this action.

In our case the signature is \( (p, q) = (2, 2) \) and \( n = 4 \). We fix an element \( h_1 \in G \) with \( Q^{h_1} = Q_1 \), where \( Q_1 \) is given by (16). There exists a constant \( \tau > 0 \), such that for every \( v \in \mathbb{R}^4 \),
\[ \tau^{-1}\|v\| \leq \|h_1v\| \leq \tau\|v\|. \]  
(18)

We may assume, with no loss of generality that \( \tau \geq 1 \).

Let \( H := \text{Stab}_{Q_0}(G) \). Then the natural morphism \( H \setminus G \rightarrow \mathcal{O}(p, q) \) is a homeomorphism. Define a 1-parameter family \( a_t \in G \) by:
\[ a_t e_i = \begin{cases} e^{-t}e_1, & i = 1 \\ e_i, & i = 2, \ldots, n - 1 \\ e^t e_n, & i = n \end{cases}. \]

Clearly, \( a_t \in H \). Furthermore, let \( \tilde{K} \) be the subgroup of \( G \) consisting of orthogonal matrices, and denote \( K := H \cap \tilde{K} \).
Let \((a, b) \in \mathbb{R}^2\) be given and let \(Q : \mathbb{R}^n \to \mathbb{R}\) be any quadratic form. The object of our interest is:

\[
V_{(a, b)}(Z) = V^Q_{(a, b)}(Z) = \{x \in \mathbb{Z}^n : a < Q(x) < b\}.
\]

Theorem 2.3 states, in our case:

**Theorem 3.2** (Theorem 2.3 from [EMM]). Let \(\Omega = \{v \in \mathbb{R}^4 \|v\| < \nu(v/\|v\|)\}\), where \(\nu\) is a nonnegative continuous function on \(S^3\). Then we have:

\[
\#V^Q_{(a, b)}(Z) \cap T\Omega < cT^2 \log T,
\]

where the constant \(c\) depends only on \((a, b)\).

The proof of theorem 3.2 relies on theorem 3.3 from [EMM], and we give here a particular case of this theorem:

**Theorem 3.3** (Theorem 3.3 from [EMM]). For any (fixed) lattice \(\Delta\) in \(\mathbb{R}^4\),

\[
\sup_{t > 1} \frac{1}{t} \int_K \alpha(a_t k\Delta)dm(k) < \infty,
\]

where the upper bound is universal.

### 3.3 Outline of the proof of theorem 3.2:

**Step 1:** Define

\[
J_f(r, \zeta) = \frac{1}{r^2} \int_{\mathbb{R}^2} f(r, x_2, x_3, x_4)dx_2dx_3,
\]

where

\[
x_4 = \frac{\zeta - x_2^2 + x_3^2}{2r}
\]

Lemma 3.6 in [EMM] states that \(J_f\) is approximable by means of an integral over the compact subgroup \(K\). More precisely, there is some constant \(C > 0\), such that for every \(\epsilon > 0\),

\[
|C \cdot e^{2t} \int_K f(a_kv)\nu(k^{-1}e_1)dm(k) - J_f(\|v\|e^{-t}, Q_0(v))\nu(\frac{v}{\|v\|})| < \epsilon
\]

with \(e^t, \|v\| > T_0\) for some \(T_0 > 0\).
Step 2: Choose a continuous nonnegative function $f$ on $\mathbb{R}^4_+ = \{ x_1 > 0 \}$ which vanishes outside a compact set so that

$$J_f(r, \zeta) \geq 1 + \epsilon$$
on $[\tau^{-1}, 2\tau] \times [a, b]$. We will show later, how one can choose $f$.

Step 3: Denote $T = e^t$, and suppose that $T \leq \|v\| \leq 2T$ and $a \leq Q_0(h_1v) \leq b$. Then by (18), $J_f(\|h_1v\|T^{-1}, Q_0(h_1v)) \geq 1 + \epsilon$, and by (20), for a sufficiently large $t$,

$$C \cdot T^2 \int_{\mathcal{K}} f(a_tkh_1v)dm(k) \geq 1,$$

(21)

for $T \leq \|v\| \leq 2T$ and

$$a \leq Q_0^x(v) \leq b.$$

(22)

Step 4: Summing (21) over all $v \in \mathbb{Z}^4$ with (22) and $T \leq \|v\| \leq 2T$, we obtain:

$$\#V_{(a, b)}(\mathbb{Z}) \cap [T, 2T]S^3 \leq \sum_{v \in \mathbb{Z}^n} C \cdot T^2 \int_{\mathcal{K}} f(a_tkh_1v)dm(k)$$

$$= C \cdot T^2 \int_{\mathcal{K}} \tilde{f}(a_tkh_1)dm(k),$$

(23)

using the nonnegativity of $f$.

Step 5: By (17), (23) is

$$\leq C \cdot c(f) \cdot T^2 \int_{\mathcal{K}} \alpha(a_tkh_1)dm(k).$$

Step 6: The result of theorem 2.3 is obtained by using theorem 3.5 on the last expression.

3.4 $\delta$-dependency:

In this section we assume that $(a, b) = (0, \delta)$, which suits the definition of the set $A(R, \delta)$, (15). One should notice that there only 3 $\delta$-dependent steps:
• Choosing $f$ in step 2, such that $J_f \geq 1 + \epsilon$ on $[\tau^{-1}, 2\tau] \times [0, \delta]$. We will construct a family of functions $f_\delta$ with an universal bound $|f_\delta| \leq M$, such that $f_\delta$ vanishes outside of a compact set which is only slightly larger than

$$V(\delta) = [\tau^{-1}, 2\tau] \times [-1, -1]^2 \times [0, \frac{\delta \tau}{2}].$$  

(24)

This is done in section 3.4.1.

• The dependency of $T_0$ of step 3, so that the usage of lemma 3.6 in [EMM] is legitimate. For this purpose we will have to examine the proof of this lemma. This is done in section 3.4.2.

• The constant $c$ in (17). We would like to establish a linear dependency on $\delta$. This is straightforward, once we are able to control the number of integral points in a domain defined by (24). This is done in section 3.4.3.

3.4.1 Choosing $f_\delta$:

Notation: For a set $U \subset \mathbb{R}^n$, and $\epsilon > 0$, denote

$$U_{\epsilon} := \{x \in \mathbb{R}^n : \max_{1 \leq i \leq n} |x_i - y_i| \leq \epsilon, \text{ for some } y \in U\}.$$

Choose a nonnegative continuous function $f_0$, on $\mathbb{R}^4_+$, which vanishes outside a compact set, such that its support, $E_{f_0}$, slightly exceeds the set $V(1)$. More precisely, $V(1) \subset E_{f_0} \subset V(1)_{\delta_0}$ for some $\delta_0 > 0$. By the uniform continuity of $f$, there are $\epsilon_0, \delta_0 > 0$, such that if $\max_{1 \leq i \leq 4} |x_i - x_0^i| \leq \delta_0$, then $f(x) > \epsilon_0$, for every $x_0 = (x_0^1, 0, x_0^3, x_0^4) \in V(1)$.

Thus for $(r, \zeta) \in [\tau^{-1}, 2\tau] \times [0, \delta]$, the contribution of $[-\delta_0, \delta_0]^2$ to $J_{f_0}$ is $\geq \epsilon_0 \cdot (2\delta_0)^2$. Multiplying $f_0$ by a suitable factor, and by the linearity of $J_{f_0}$, we may assume that this contribution is at least $1 + \epsilon$.

Now define $f_\delta(x_1, \ldots, x_4) := f_0(x_1, x_2, x_3, \frac{x_4}{\delta})$. We have for $\delta \geq 1$

$$\frac{\zeta - x_2^2 + x_3^2}{2r\delta} = \frac{\zeta/2r}{\delta} - \frac{(x_2/\sqrt{\delta})^2}{2r} + \frac{(x_3/\sqrt{\delta})^2}{2r}.$$

Thus for $\delta \geq 1$, if $(r, \zeta) \in [\tau^{-1}, 2\tau] \times [0, \delta]$ and for $i = 2, 3$, $|x_i| < \delta_0$, $f_\delta$ satisfies:

$$f_\delta(r, x_2, x_3, x_4) > \epsilon_0,$$

and therefore the contribution of this domain to $J_{f_\delta}$ is

$$\geq \epsilon_0(2\delta)^2 \geq 1 + \epsilon$$

by our assumption.

By the construction, the family $\{f_\delta\}$ has a universal upper bound $M$ which is the one of $f_0$. 
3.4.2 How large is $T_0$

The proof of lemma 3.6 from [EMM] works well along the same lines, as long as

$$f(a_t x) \neq 0$$

implies that for $t \to \infty$, $x/\|x\|$ converges to $e_1 = (1, 0, 0, 0)$. Now, since $a_t$ preserves $x_1x_4$, (25) implies for the particular choice of $f = f_\delta$ in section 3.4.1:

$$|x_1x_4| = O(\delta); \quad x_1 \gg T.$$  

Thus

$$\|x\| = x_1 + O(\frac{\delta}{T}) + O(1),$$

and so, as long as $\delta = o(T)$, $x/\|x\|$ indeed converges to $e_1$.

3.4.3 Bounding integral points in $V_\delta$:

**Lemma 3.4.** Let $V(\delta)$ defined by

$$V(\delta) = [\tau^{-1}, 2\tau] \times [-1, -1]^{n-2} \times [0, \frac{\delta \beta}{2}].$$

for some constant $\tau$ and $n \geq 3$. Let $g \in SL(n, \mathbb{R})$ and denote

$$N(g, \delta) := \#V(\delta) \cap g\mathbb{Z}^n.$$  

Then for $\delta \geq 1$,

$$\left| N(g, \delta) - \frac{2^{n-2}(2\tau - \tau^{-1})\delta}{\det g} \right| \leq c_5\delta \sum_{i=1}^{n-1} \frac{1}{\text{vol}(L_i/(g\mathbb{Z}^n \cap L_i))}$$

for some $g$-rational subspaces $L_i$ of $\mathbb{R}^4$ of dimension $i$, where $c_5 = c_5(n)$ depends only on $n$.

A direct consequence of lemma 3.4 is the following

**Corollary 3.5.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative function which vanishes outside a compact set $E$. Suppose that $E \subset V_\epsilon(\delta)$ for some $\epsilon > 0$. Then for $\delta \geq 1$, (17) is satisfied with

$$c(f) = c_3 \cdot M\delta,$$

where the constant $c_3$ depends on $n$ only.

In order to prove lemma 3.4 we shall need the following:
Lemma 3.6. Let $\Lambda \subset \mathbb{R}^n$ be a $m$-dimensional lattice, and let

$$A_t = \begin{pmatrix} 1 & \ldots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \ldots & t \end{pmatrix}$$

(27)

an $n$-dimensional linear transformation. Then for $t > 0$ we have

$$\det A_t \Lambda \leq t \det \Lambda.$$  

(28)

Proof. We may assume that $m < n$, since if $m = n$, we obviously have an equality. Let $v_1, \ldots, v_m$ the basis of $\Lambda$ and denote for every $i$, $u_i \in \mathbb{R}^{n-1}$ the vector, which consists of first $n-1$ coordinates of $v_i$. Also, let $x_i \in \mathbb{R}$ be the last coordinate of $v_i$. By switching vectors, if necessary, we may assume $x_1 \neq 0$. We consider the function

$$f(t) := (\det A_t \Lambda)^2,$$

as a function of $t \in \mathbb{R}$.

Obviously,

$$f(t) = \det \left( < u_i, u_j > + x_i x_j t^2 \right)_{1 \leq i, j \leq m}.$$

Substracting $\frac{x_i}{x_1}$ times the first row from any other, we obtain:

$$f(t) = \left| \begin{array}{c} < u_1, u_j > + x_1 x_j t^2 \\ < u_2, u_j > - \frac{x_2}{x_1} < u_1, u_j > \\ \vdots \\ < u_m, u_j > - \frac{x_m}{x_1} < u_1, u_j > \end{array} \right|,$$

and by the multilinearity property of the determinant, $f$ is a linear function of $t^2$. Write

$$f(t) = a(t^2 - 1) + bt^2.$$

Thus

$$b = f(1); \quad a = -f(0),$$

and so $b = \det \Lambda$, and $a = -\det < u_i, u_j > \leq 0$, being minus the determinant of a Gram matrix. Therefore,

$$(\det A_t \Lambda)^2 - t^2 \det \Lambda = a(t^2 - 1) \leq 0$$

for $t \geq 1$, implying (27).
Proof of lemma 3.4. We will prove the lemma, assuming $\beta = 2$. However, it implies the result of the lemma for any $\beta$, affecting only $c_5$. Let $\delta > 0$. Trivially,

$$N(g, \delta) = N(g_0, 1),$$

where $g_0 = A_\delta^{-1}g$ with $A_\delta$ given by (27). Let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be the successive minima of $g_0$, and pick linearly independent lattice points $v_1, \ldots, v_n$ with $\|v_i\| = \lambda_i$. Denote $M_i$ the linear space spanned by $v_1, \ldots, v_i$ and the lattice $\Lambda_i = g_0\mathbb{Z}^n \cap M_i$.

First, assume that $\lambda_n \leq \sqrt{\tau^2 + (n - 1)} =: r$. Now, by Gauss’ argument,

$$\left| N(g_0, 1) - \frac{2^{n-1}(2\tau - \tau^{-1})\delta}{\det g} \right| \leq \frac{1}{\det g_0} \text{vol}(\Sigma),$$

where

$$\Sigma := \{ x : \text{dist}(x, \partial V(1)) \leq n\lambda_n \}.$$ 

Now, for $\lambda_n \leq r$,

$$\text{vol}(\Sigma) \ll \lambda_n,$n

where the constant implied in the “$\ll$“-notation depends on $n$ only (this is obvious for $\lambda_n \leq \frac{1}{2n}$, and trivial otherwise, since for $\lambda_n \leq r$, $\text{vol}(\Sigma) = O(1)$).

Thus,

$$\left| N(g_0, 1) - \frac{2^{n-1}(2\tau - \tau^{-1})\delta}{\det g} \right| \ll \frac{\lambda_n}{\det g_0} \ll \frac{1}{\det \Lambda_{n-1}} \leq \frac{1}{\text{vol}(M_{n-1}/M_{n-1} \cap g_0\mathbb{Z}^n)} \leq \frac{\delta}{\text{vol}(A_\delta M_{n-1}/A_\delta M_{n-1} \cap g\mathbb{Z}^n)}.$$ 

Next, suppose that $\lambda_n > r$. Then,

$$V(\delta) \cap g_0\mathbb{Z}^n \subset V(\delta) \cap \Lambda_{n-1}.$$ 

Thus, by the induction hypothesis, the number of such points is:

$$\leq c_4 \sum_{i=0}^{k-1} \frac{1}{\det(A_i)} = \sum_{i=0}^{k-1} \frac{1}{\text{vol}(M_i/M_i \cap g_0\mathbb{Z}^n)} \leq \delta \sum_{i=0}^{k-1} \frac{1}{\text{vol}(A_\delta M_i/A_\delta M_i \cap g\mathbb{Z}^n)}.$$ 

Since $\lambda_n > r$, we have

$$\frac{1}{\det g} = \frac{1}{\lambda_n \det g/\lambda_n} \ll \frac{1}{\det g/\lambda_n} \ll \frac{1}{\lambda_1 \ldots \lambda_n},$$

and we’re done by defining $L_i := A_\delta M_i$. \qed
4 Unsmoothing

4.1 An asymptotic formula for $N_\Lambda$

We need an asymptotic formula for the sharp counting function $N_\Lambda$. Unlike the case of the standard lattice, $\mathbb{Z}^2$, in order to have a good control over the error terms we should use some Diophantine properties of the lattice we are working with. We adapt the following notations:

Let $\Lambda$ be a lattice and $t > 0$ a real variable. Denote the set of squared norms of $\Lambda$ by

$$SN_\Lambda = \{ |\vec{n}|^2 : n \in \Lambda \}.$$

Suppose we have a function $\delta_\Lambda : SN_\Lambda \to \mathbb{R}$, such that given $\vec{k} \in \Lambda$, there are no vectors $\vec{n} \in \Lambda$ with $0 < |\vec{n}|^2 - |\vec{k}|^2 < \delta_\Lambda(|\vec{k}|^2)$. That is,

$$\Lambda \cap \{ \vec{n} \in \Lambda : |\vec{k}|^2 - \delta_\Lambda(|\vec{k}|^2) < |\vec{n}|^2 < |\vec{k}|^2 + \delta_\Lambda(|\vec{k}|^2) \} = A_{|\vec{k}|},$$

where

$$A_y := \{ \vec{n} \in \Lambda : |\vec{n}| = y \}.$$

Extend $\delta_\Lambda$ to $\mathbb{R}$ by defining $\delta_\Lambda(x) := \delta_\Lambda(|\vec{k}|^2)$, where $\vec{k} \in \Lambda$ minimizes $|x - |\vec{k}|^2|$ (in the case there is any ambiguity, that is if $x = \frac{|\vec{n}_1|^2 + |\vec{n}_2|^2}{2}$ for vectors $\vec{n}_1, \vec{n}_2 \in \Lambda$ with consecutive increasing norms, choose $\vec{k} := \vec{n}_1$). We have the following lemma:

**Lemma 4.1.** For every $a > 0$, $c > 1$,

$$N_\Lambda(t) = \frac{\pi t^2}{\beta \pi} - \frac{\sqrt{t}}{\beta N} \sum_{\vec{\kappa} \in \Lambda \setminus \{0\}} \frac{\cos \left( \frac{2\pi t|\vec{\kappa}|}{|\vec{\kappa}|^2} \right)}{|\vec{\kappa}|^2} + O(N^a)$$

$$+ O\left( \frac{t^{2c-1}}{\sqrt{N}} \right) + O\left( \frac{t}{\sqrt{N}} \cdot \log t + \log(\delta_\Lambda(t^2)) \right)$$

$$+ O\left( \log N + \log(\delta_* (t^2)) \right)$$

As a typical example of such a function, $\delta_\Lambda$, for $\Lambda = \langle 1, \alpha + i\beta \rangle$, with a Diophantine $(\alpha, \alpha^2, \gamma^2)$, we may choose $\delta_\Lambda(y) = \frac{c}{y^k}$, where $c$ is a constant. In this example, if $\Lambda \ni \vec{k} = (a, b)$, then by lemma 2.4 $A_{|\vec{k}|} = \pm (a, b)$, provided that $\gamma$ is irrational.

The proof of this lemma is essentially the same as the one of lemma 5.1 in [W], starting from

$$\mathcal{Z}_\Lambda(s) := \frac{1}{4} \sum_{\vec{k} \in \Lambda \setminus \{0\}} \frac{1}{|\vec{k}|^{2s}} = \sum_{(m, n) \in \mathbb{Z}_+^2 \setminus \{0\}} \frac{1}{((m + n\alpha)^2 + (\beta n)^2)^s}.$$
Proposition 4.2. Let a lattice $\Lambda = \langle 1, \alpha + i\beta \rangle$ with a Diophantine triple of numbers $(\alpha^2, \alpha\beta, \beta^2)$ be given. Suppose that $L \to \infty$ as $T \to \infty$ and choose $M$, such that $L/\sqrt{M} \to 0$, but $M = O(T^\delta)$ for every $\delta > 0$ as $T \to \infty$. Suppose furthermore, that $M = O(L^{s_0})$ for some (fixed) $s_0 > 0$. Then

$$\left( \left| S_\Lambda(t, \rho) - \tilde{S}_{\Lambda, M, L}(t) \right|^2 \right) \ll \frac{1}{\sqrt{M}}$$

The proof of proposition 4.2 proceeds along the same lines as the one of proposition 6.1 in [W], using again an asymptotic formula for the sharp counting function, given by lemma 4.1. The only difference is that here we use proposition 3.1 rather than lemma 6.2 from [W].

Once we have proposition 4.2 in our hands, the proof of our main result, namely, theorem 1.1 proceeds along the same lines as the one of theorem 1.1 in [W].

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