Deriving constant coefficient linear recurrences for enumerating standard Young tableaux of periodic shape

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Abstract

The enumeration of several classes of standard Young tableaux are known (see [7]). However, there are standard Young tableaux of some shapes whose enumeration is still not yet as well known. In this paper, we establish a sufficient condition on the shape of a Young tableaux that implies that the standard Young tableaux of such shapes can be enumerated using constant coefficient linear recurrences.

Specifically, we introduce a generalization of the standard Young tableaux considered in [4], [2], and [5] that we call standard Young tableaux of periodic shape. Then, by generalizing an approach used in [6], we prove that all standard Young tableaux of this shape can be enumerated using matrix difference equations, then prove that these matrices have many identical rows. This generalizes constant coefficient linear recurrence results from [4] and [2].

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1 Notation and terminology

Throughout, we use English notation to denote standard Young tableaux. And cell \((i, j)\) will denote the cell in row \(i\) and column \(j\) of the tableau. All variables, unless otherwise stated, are positive integers, and given any finite set \(S\), \(|S|\) denotes the cardinality of \(S\). Also, if \(f\) and \(g\) are functions, then \(f \circ g\) denotes function composition: \((f \circ g)(x) = f(g(x))\).

Moreover, let \(\mathbb{N}\) denote the set of positive integers and \([n] = \{1, 2, \ldots, n\}\). Furthermore, for all \(p, q \in \mathbb{Z}^2\) where \(p = (p_1, p_2)\) and \(q = (q_1, q_2)\), define \(p + q = (p_1 + q_1, p_2 + q_2)\) and define \(p - q = (p_1 - q_1, p_2 - q_2)\). If \(S \subseteq \mathbb{Z}^2\) and \(p \in \mathbb{Z}^2\), then define \(p + S = S + p = \{p + q : q \in S\}\) and \(-p + S = S - p = \{q - p : q \in S\}\).

We will also identify cells of tableaux as elements of the integer lattice \(\mathbb{Z}^2\). For convenience, we will write coordinates of \(\mathbb{Z}^2\) using matrix notation; that is, write \((i, j) = (j, -i)_{\text{Cartesian}}\), where \((x, y)_{\text{Cartesian}}\) denotes a point being written in Cartesian coordinates. We write \(\mathbb{N}^2\) to denote the bottom-right quadrant of \(\mathbb{Z}^2\); i.e., \(\mathbb{N}^2 = \{(i, j) \in \mathbb{Z}^2 : i, j \in \mathbb{N}\}\).

Example 1. The set of cells \(\{(2, 1), (1, 3), (1, 4)\}\) is depicted as the following subset of \(\mathbb{Z}^2\) (here, \(\circ\) denotes the origin).

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If $S$ is a finite subset of $\mathbb{Z}^2$. Define a *partial tableau* $\tau$ of shape $S$ to be an injective map $f_\tau : S \to \mathbb{N}$ such that for all positive integers $i$ and $j$, if $(i, j), (i, j + 1) \in S$, then $f_\tau(i, j) < f_\tau(i, j + 1)$, and if $(i, j), (i + 1, j) \in S$, then $f_\tau(i, j) < f_\tau(i + 1, j)$. Moreover, if $(i, j) \in S$, say that $f_\tau((i, j))$ is the entry of $\tau$ contained in $(i, j)$. If $\tau$ is a standard Young tableau of shape $S$, and $v \in S$, we also write $\tau(v)$ to denote the entry of $\tau$ in cell $v$ of $\tau$.

Furthermore, define a *subtableau* of $\tau$ to be the restriction of $f_\tau : S \to \mathbb{N}$ to any subset of $S$. In particular, the shape of a subtableau of $\tau$ is a subset of the shape of $\tau$. Now, define a *standard Young tableau of shape* $S$ to be a partial tableau $\tau$ of shape $S$ such that for all $(i, j) \in S$, the entry of $\tau$ contained in $(i, j)$ is an integer in $\{1, 2, \ldots, |S|\}$. If $\tau$ is a standard Young tableau of shape $S$ and $X \subseteq S$, let $\tau|_X$ denote the restriction of $\tau$ to $X$.

Lastly, a finite subset $S \in \mathbb{Z}^2$ is connected if for any two distinct cells $u, v \in S$, there is a sequence of cells $u = a_0, a_1, \ldots, a_n = v$ in $S$ such that for all $i$, $a_{i+1} - a_i \in \{(1, 0), (0, 1), (-1, 0), (0, -1)\}$.

**Example 3.** Let $S = \{(2, 1), (1, 3), (1, 4)\} \subseteq \mathbb{Z}^2$ be the set of cells. An example of a standard Young tableau of shape $S$, which we denote by $\tau$, is

\[
\begin{array}{ccc}
  & 1 & 3 \\
2 &   &   \\
1 &   &   \\
2 &   &   \\
3 &   &   \\
2 &   &   \\
1 & & 3
\end{array}
\]

and three subtableaux of $\tau$ are

\[
\begin{array}{ccc}
  & 1 & 3 \\
2 &   &   \\
1 &   &   \\
2 &   &   \\
3 &   &   \\
2 &   &   \\
1 & & 3
\end{array}
\]

and

The above three subtableaux are partial tableaux but not standard Young tableaux. However, $\tau$ is also a partial tableau because it is a standard Young tableau.
2 Standard Young tableaux of periodic shape

When denoting a partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \), assume that \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \).

**Definition 1.** A periodic shape is a finite subset \( P \subseteq \mathbb{Z}^2 \) that contains \((1,1)\) and is the skew shape \( \lambda/\mu \) written in French notation for some partitions \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \) and \( \mu = (\mu_1, \mu_2, \ldots, \mu_n) \) (with \( n \geq m \)) such that the following conditions hold.

- \( \lambda/\mu \) is connected.
- If \( C \) is a column of \( P \) that is neither the left-most column nor the right-most column of \( P \), then there is a cell \((i_0, j_0) \in C \) such that \((i_0, j_0 - 1) \in P \) and \((i_0, j_0 + 1) \in P \).

**Definition 2.** Let \( P \) be a periodic shape and let \( w \) be a non-negative integer. Moreover, let \( a \in \mathbb{Z}^2 \) be the left-most cell in the top row of \( P \) and let \( b \in \mathbb{Z}^2 \) be the left-most cell in the bottom row of \( P \). Then for all \( m \), define

\[
Sh^w(P, m) = \bigcup_{i=0}^{m} i(b - a + (1, w)) + P.
\]

**Example 4.** If \( \lambda = (4, 4) \) and \( \mu = (1) \) so that

\[
P = \{(1,1), (1,2), (1,3), (1,4)(2,2), (2,3), (2,4)\},
\]

is the following shape

\[
\begin{array}{cccccc}
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & & \\
\end{array}
\]

then \( Sh^0(P, 2) \) is

\[
\begin{array}{cccccc}
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & & \\
\cdot & \cdot & \cdot & & \\
\cdot & \cdot & \cdot & & \\
\cdot & \cdot & \cdot & & \\
\end{array}
\]
and \( \text{Sh}^1(P, 2) \) is

\[
\begin{array}{ccccccc}
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\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \\
\bullet & \bullet & \bullet & \bullet & \\
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\end{array}
\]

The motivation for the following definition is due to the fact that \( \text{Sh}^w(P, m) \) need not be periodic and the following example

**Example 5.** Let \( P \) be a single row (so \( P = \{(1, j) : 1 \leq j \leq n\} \) for some \( n \)) and \( w = 0 \). Then the maximum number of cells in a column in \( \text{Sh}^w(P, n) \) is \( n \) and hence, not bounded for all \( n \).

**Definition 3.** Let \( P \) be a periodic shape and \( w \) be a non-negative integer. Then say that \( P \) and \( w \) are compatible if the following hold.

- \( \text{Sh}^w(P, n) \) is periodic for all \( n \).
- The maximum number of cells in a column in \( \text{Sh}^w(P, n) \) is bounded for all \( n \).

**Definition 4.** Let \( P \) be a periodic shape and \( w \) be a non-negative integer such that \( P \) and \( w \) are compatible. Moreover, let \( S = \text{Sh}^w(P, m) \) for some positive integer \( m \). Then call \( P \) a period of \( S \) and say that \( S \) is a periodic shape with period \( P \) and shift number \( w \). Moreover, if \( \tau \) is a standard Young tableau of shape \( S \), then call \( P \) a period of \( \tau \) and say that \( \tau \) is a standard Young tableau of periodic shape with period \( P \) and shift number \( w \).

**Definition 5.** A compatible pair is a pair \((P, w)\) where \( P \) is a periodic shape, \( w \) is a non-negative integer, and \( P \) and \( w \) are compatible.

**Definition 6.** If \( P \) is a periodic shape, then let \( \text{SYT}(P) \) denote the set of standard Young tableaux of shape \( P \).

**Example 6.** An example of a standard Young tableau of periodic shape with period \( \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3)\} \) and shift number 0 is as follows

\[
\begin{array}{ccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \\
\bullet & \bullet & \bullet & \bullet & \\
\bullet & \bullet & \\
\bullet & 
\end{array}
\]
Example 7. An example of a standard Young tableau of periodic shape with period \( \{(1,j): 3 \leq j \leq 8\} \) and shift number 2 is as follows:

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 6 & 9 \\
5 & 7 & 8 & 10 & 12 & 14 \\
11 & 13 & 15 & 16 & 17 & 18 \\
\end{array}
\]

Remark 1. In one direction, a generalization of standard Young tableau of periodic shape was introduced in \[2\] and are called Dispositional Digraphs.

In another direction, analysing constant coefficient linear recurrences for enumerating standard Young tableau of periodic shape with period \( \{(1,j): 1 \leq j \leq k\} \) and shift number 1, where \( k \) is fixed, was done in \[6\], \[2\], \[4\]. The research of \[2\], \[4\] rigorously proved the empirical recurrence relations \[5\] conjectured by R. H. Hardin and A. P. Heinz.

The most general analysis of such recurrences (for period \( \{(1,j): 1 \leq j \leq k\} \) and shift number 1) was done in \[2\], where they proved that all periodic standard Young tableaux of that period and shift number can be enumerated using constant coefficient linear recurrences and also determine bounds on the orders of such recurrences.

3 The poset induced by the 0-Hecke algebra action on tableaux

In this section, we adapt a series of definitions and properties that come from \[6\], which are used there to analyse standard reverse composition tableaux, to standard Young tableaux of periodic shape. To do this, we introduce shapes that we call normal shapes and regular shapes.
Definition 7. Call a finite subset $N \subseteq \mathbb{N}^2$ normal (or say that $N$ has normal shape) if $N$ is connected and

$$N = \text{conv } N \cap \mathbb{N}^2$$

where $\text{conv } N$ denotes the convex hull of $N$ in $\mathbb{R}^2$.

Definition 8. Call a finite subset $R \subseteq \mathbb{N}^2$ regular (or say that $R$ has regular shape) if for some $a \in \mathbb{Z}^2$, periodic shape $P$, and normal shape $N$, $R = a + S$, where $S = P \cap N$ and the bottom row of $N$ equals to the bottom row of $P$.

Remark 2. From the above definitions, we note that all periodic shapes are regular, and that all regular shapes are normal.

Definition 9. Let $R$ be a regular shape, and $\tau \in \text{SYT}(R)$. The column reading word $\text{col}(\tau)$ of $\tau$ is the permutation whose one-line notation equals to the string obtained from $\tau$ by reading the entries of each column from top to bottom and reading the columns from right to left.

Example 8. The column reading word of

$$
\begin{array}{cccccc}
1 & 3 & 6 \\
2 & 4 & 5 & 7 & 9 & 11 \\
8 & 10 & 12 & 13 & 14 & 15
\end{array}
$$

is the permutation represented in one-line notation by

$$15 14 13 11 12 9 10 7 6 8 5 3 4 1 2$$

and the column reading word of

$$
\begin{array}{ccc}
1 & 2 & 4 \\
3 & 6 \\
5 & 7 & 10 \\
8 & 11 \\
9 & 12 & 13 \\
14 & 15
\end{array}
$$

is the permutation represented in one-line notation by

$$15 13 14 12 11 10 9 8 7 6 4 5 3 2 1$$
Definition 10. A standard Young tableau \( \tau \) of normal shape \( N \) is the source tableaux of shape \( N \) if for any pair of adjacent cells in the same row of \( \tau \), their entries are consecutive. A standard Young tableau \( \tau \) of regular shape \( N \) is the sink tableaux of shape \( N \) if for any pair of adjacent cells in the same column of \( \tau \), their entries are consecutive.

Example 9. Let \( R = \{ (1,2), (1,3), (2,1), (2,2), (2,3), (2,4) \} \). Then the source tableau of shape \( R \) is

\[
\begin{array}{cccc}
1 & 2 \\
3 & 4 & 5 & 6
\end{array}
\]

and the sink tableau of shape \( R \) is

\[
\begin{array}{cccc}
2 & 4 \\
1 & 3 & 5 & 6
\end{array}
\]

Example 10. Let \( P = \{ (1,j) : 1 \leq j \leq 6 \} \) and \( w = 2 \). The source tableau of shape \( Sh^w(P,3) \) is

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
7 & 8 & 9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16 & 17 & 18
\end{array}
\]

and the sink tableau of shape \( Sh^w(P,3) \) is

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 5 & 7 & 10 \\
4 & 6 & 8 & 11 & 13 & 15 \\
9 & 12 & 14 & 16 & 17 & 18
\end{array}
\]

Example 11. Let \( P = \{ (1,1), (1,2), (1,3), (2,2), (2,3) \} \) and \( w = 0 \). The source tableaux of shape \( Sh^w(P,3) \) is

\[
\begin{array}{cccccccc}
1 & 2 & 3 \\
4 & 5 \\
6 & 7 & 8 \\
9 & 10 \\
11 & 12 & 13 \\
14 & 15
\end{array}
\]
and the sink tableaux of shape $Sh^w(P, 3)$ is

$$
\begin{array}{cccccc}
1 & 2 & 5 \\
3 & 6 \\
4 & 7 & 10 \\
8 & 11 \\
9 & 12 & 14 \\
13 & 15 \\
\end{array}
$$

**Proposition 1.** Let $\tau$ be a standard Young tableau of regular shape $R$ and $N = |R|$. Moreover, let $1 \leq i \leq N - 1$ be such that the cell containing $i$ is $(i_1, j_1)$, the cell containing $i + 1$ is $(i_2, j_2)$, $i_1 > i_2$, and $j_1 < j_2$. Then $(i \ i + 1)\tau \in SYT(R)$ where $(i \ i + 1)\tau$ is the tableau obtained from $\tau$ by switching entries $i$ and $i + 1$.

**Proof.** As $i$ and $i + 1$ are consecutive, switching these entries will not affect the order of the entries in any row or any column of $\tau$. From this, the proposition follows.

**Definition 11.** Let $\tau$ be a standard Young tableau of regular shape $R$, $N = |R|$, $i$ satisfy $1 \leq i \leq N - 1$, $(i_1, j_1)$ be the cell of $\tau$ that contains $i$, and $(i_2, j_2)$ be the cell of $\tau$ that contains $i + 1$. If $i_1 > i_2$ and $j_1 < j_2$, then let

$$
\pi_i(\tau) = \begin{cases} 
(i \ i + 1)\tau & \text{if } i \text{ is located to the left of } i + 1 \text{ in } col(\tau) \\
\tau & \text{if } i + 1 \text{ is located to the left of } i \text{ in } col(\tau).
\end{cases}
$$

And if $i_1 \leq i_2$ or $j_1 \geq j_2$, then let

$$
\pi_i(\tau) = 0,
$$

where $0$ is an element that is not in $SYT(R)$.

If $X$ is a set and $\mathcal{R}$ is a binary relation on $X$ (that is, $\mathcal{R} \subseteq X \times X$), then the transitive closure of $\mathcal{R}$ is the intersection of all transitive binary relations $\mathcal{R}'$ on $X$ that satisfy $\mathcal{R} \subseteq \mathcal{R}'$. Recall also that if $\mathcal{R}$ is a binary relation, then $a \mathcal{R} b$ denotes $(a, b) \in \mathcal{R}$.  

9
Definition 12. Let $R$ be a regular shape, and let $N = |R|$. Define an order relation $\leq_*$ on $\text{SYT}(R)$ to be the transitive closure of the following binary relation on $\text{SYT}(R)$.

$$
\text{for all } \tau \in \text{SYT}(R), \tau \leq_* \tau
$$

$$
\text{for all } \tau \in \text{SYT}(R) \text{ and } 1 \leq i \leq N - 1, \text{ such that } \pi_i(\tau) \neq 0, \tau \leq_* \pi_i(\tau)
$$

We will write $(\text{SYT}(R), \leq_*)$ to denote the resulting poset; we define $\leq$ to be $\leq_*$. Whether $\leq$ denotes the order relation of $(\text{SYT}(R), \leq_*)$ or the order relation on the integers can easily be determined from context.

Proposition 2. Let $R$ be a regular shape. Then the poset $(\text{SYT}(R), \leq_*)$ has a maximum and a minimum element. Moreover, the maximum element of $(\text{SYT}(R), \leq_*)$ is the sink tableau of shape $R$ and the minimum element of $(\text{SYT}(R), \leq_*)$ is the source tableau of shape $R$.

Proof. Let $\tau \in \text{SYT}(R)$ and $N$ be the number of cells in $\tau$. Then the following hold.

$\tau$ is the sink tableau of shape $R$ if and only if it is impossible to find a pair of cells $(i_1, j_1), (i_2, j_2)$ in $\tau$ containing $i + 1$ and $i$ respectively for some $1 \leq i \leq N - 1$ such that $i_1 > i_2$ and $j_1 < j_2$.

$\tau$ is the source tableau of shape $R$ if and only if it is impossible to find a pair of cells $(i_1, j_1), (i_2, j_2)$ in $\tau$ containing $i$ and $i + 1$ respectively for some $1 \leq i \leq N - 1$ such that $i_1 > i_2$ and $j_1 < j_2$.

So from the definition of the operator $\pi_i$, the fact that $(\text{SYT}(R), \leq_*)$ is finite now implies the proposition.

4 The tableau graph transfer matrix

In this section, we introduce the notion of a tableau graph for partial tableaux, explain how they characterize standard Young tableau of periodic shape, then introduce a series of definitions which will ultimately lead to the definition of the tableau graph transfer matrix. Recall the notion of a partial tableau from section one.
Definition 13. Let $S$ be a finite subset of $\mathbb{Z}^2$, and let $\tau$ be a partial tableau of shape $S$. Then the tableau graph $Tb(\tau)$ of $\tau$ is the pair

$$(V(Tb(\tau)), E(Tb(\tau)))$$

where $V(Tb(\tau)) = S$ and $E(Tb(\tau))$ is the set of two element subsets $\{(i_1, j_1), (i_2, j_2)\} \subseteq S$ such that $i_1 > i_2$, $j_1 < j_2$, and the entry of $\tau$ contained in $(i_1, j_1)$ is less than the entry of $\tau$ contained in $(i_2, j_2)$.

If $G$ is a tableau graph, then call the elements of $V(G)$ the vertices of $G$ and call the elements of $E(G)$ the edges of $G$.

Example 12. The tableau graph of

$$
\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 6 & 9 \\
5 & 7 & 8 & 10 & 12 & 14 \\
11 & 13 & 15 & 16 & 17 & 18
\end{array}
$$

is

![Diagram of the tableau graph of the example]

and the tableau graph of

$$
\begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 \\
6 & 7 & 8 \\
9 & 10 \\
11 & 12 & 13 \\
14 & 15
\end{array}
$$

is
In fact, standard Young tableaux of regular shape can be identified with their tableau graphs as we will demonstrate. Let $R$ be a regular shape. Consider the set $NG(R)$ of tableau graphs of tableaux from $SYT(R)$; i.e., define $NG(R) = \{Tb(\tau) : \tau \in SYT(R)\}$. Moreover, define a partial order on $NG(R)$ as follows. For all $G, H \in NG(R)$, define $G \leq H$ if $E(G) \subseteq E(H)$.

**Proposition 3.** Let $R$ be a regular shape. Then the map $f : (SYT(R), \leq) \rightarrow (Tb(R), \leq)$ defined by $f(\tau) = Tb(\tau)$ for all $\tau \in SYT(R)$ is a bijective order preserving map.

**Proof.** Consider the left weak Bruhat order $\leq_L$ on the symmetric group $S_n$. We first adapt a property of column reading words from [6] to standard Young tableaux of regular shape.

It is well-known, see [6, Proposition 3.1.3], that for all permutations $\pi, \sigma \in S_n$, $\pi \leq_L \sigma$ if and only if $Inv(\pi) \subseteq Inv(\sigma)$, where for all $\alpha \in S_n$,

$$Inv(\alpha) = \{\{i, j\} \subseteq [n] : i < j \text{ and } \alpha(i) > \alpha(j)\}.$$ 

Recall the definition of the column reading word in the beginning of Section 3. The map $Col : (SYT(R), \leq) \rightarrow (S_n, \leq_L)$, defined by $Col(\tau) = col(\tau)$ is an injective order preserving map. To see this, observe that for all $\tau \in SYT(R)$ and $i$, where $1 \leq i \leq N - 1$ and $N = |R|$, that satisfy $\pi_i(\tau) \neq 0$, $Inv(\tau) \subseteq Inv(\pi_i(\tau))$.

Now, let $(X, \leq_L)$ denote the image of $Col : (SYT(R), \leq) \rightarrow (S_n, \leq_L)$. By Proposition 2, $(X, \leq_L)$ has a minimum element $\pi_0$ and a maximum element
π₁, and Inv(π₀) ⊆ Inv(π) ⊆ Inv(π₁) for all π ∈ X. We can define Tb′ : (X, ≤_L) → (P(Inv(π₁) \ Inv(π₀)), ⊆), where P(R) is the power-set of R and ⊆ is set inclusion, by

\[ Tb'(\pi) = Inv(\pi) \setminus Inv(\pi₀) \]

It can be seen that Tb′ is an injective order preserving map. Moreover, from the definition of the tableau graph of a standard Young tableau of periodic shape, it can be seen that the image of Tb′ is order isomorphic to (Tb′(R), ≤) (identify the possible edges of a tableau graph with the elements of Inv(π₁) \ Inv(π₀)). From this, the proposition follows.

The motivation for the above theorem is shown below. In fact, our goal is to generalize the following association made in [6].

Remark 3. In [6], it was proved that the number of standard Young tableaux of shape Sh¹(R, n), where R = \{(1, 1), (1, 2), (1, 3)\}, is 2^{n-1}. Their proof relied on making the following association, which they expressed using inversion sets from (\(S_N, \leq_L\)) \cong (\{Inv(\pi) : \pi \in S_N\}, \subseteq). The standard Young tableaux of shape Sh¹(R, n) can be identified with tableau graphs of the form (below, \(n = 4\) and \(R = \{(1, 1), (1, 2), (1, 3)\}\)).

Recall that a translation on \(\mathbb{Z}^2\) is a function \(f\) on \(\mathbb{Z}^2\) such that for some \(q \in \mathbb{Z}^2\), \(f(p) = p + q\) for all \(p \in \mathbb{Z}^2\).

Definition 14. Let G and H be the tableau graphs of two partial tableaux. Then a tableau graph isomorphism \(\phi : G \rightarrow H\) is a translation \(\phi\) on \(\mathbb{Z}^2\) such that \(\phi(V(G)) = H\) and for all \(a, b \in V(G)\), \(\{a, b\} \in E(G)\) if and only if \(\{\phi(a), \phi(b)\} \in E(H)\). If \(\phi : G \rightarrow H\) is an isomorphism, then say that G is isomorphic to H and write \(G \cong H\).
Now, we introduce a series of definitions that are needed to define the tableau graph transfer matrix.

**Definition 15.** Let $P$ be a periodic shape. Then define $\overline{P}$ to be the image of $P$ under a translation of $\mathbb{Z}^2$ such that for any non-negative integers $p$ and $q$ such that $p > 0$ or $q > 0$,

$$\overline{P} - (p, q) \notin \mathbb{N}^2.$$  

**Definition 16.** Let $(P, w)$ be a compatible pair. Then the shift vector $\nabla_w P$ corresponding to $P$ and $w$ is the vector defined by

$$\nabla_w P = b - a + (1, w)$$

where $a$ is the left-most cell in the top row of $P$ and $b$ is the left-most cell in the bottom row of $P$.

**Definition 17.** Let $(P, w)$ be a compatible pair and let $S = Sh_w (P, m)$ for some positive integer $m$. Then define $v_{S,P}$ to be the left-most cell of the top-most row of $P + (m - 1)\nabla_w P$.

**Definition 18.** Let $P$ be a periodic shape, and let $v \in S$ where $v = (i_0, j_0)$. Then the corner of $P$ generated by $v$ is the shape

$$\Delta^P v = \{(i, j) \in P : i \leq i_0 \text{ and } j > j_0\}.$$  

**Definition 19.** Let $(P, w)$ be a compatible pair. Moreover, let $S = Sh_w (P, m)$ for some positive integer $m$ and assume that $m$ is sufficiently large.

Let $u$ be the cell in the bottom row of $S$ such that

$$u - \nabla_w P = v_{S,P} - (1, 0).$$

The $w^{th}$ index shape of $P$ is

$$I(P, w) = \Delta^S u$$

and the $w^{th}$ coefficient shape of $P$ is

$$C(P, w) = A \cup (P + (m - 1)\nabla_w P)$$

where

$$A = \Delta^S u \cup \Delta^S (u - \nabla_w P).$$
We illustrate the above definitions with some examples.

**Example 13.** Let \( P = \{(1, j) : 1 \leq j \leq 6\} \) and \( w = 1 \). Then \( P \) and \( w \) are compatible, \( I(P, w) \) is the following figure depicted with black dots

\[
\begin{array}{c}
\circ \circ \circ \circ \circ \\
\circ \circ \circ \circ \circ \\
\circ \circ \circ \circ \circ \\
\circ \circ \circ \circ \circ \\
\circ \circ \circ \circ \circ \\
\circ \circ \circ \circ \circ \\
\end{array}
\]

, and \( C(P, w) \) is the following figure also depicted with black dots

\[
\begin{array}{c}
\circ \circ \circ \circ \circ \\
\circ \circ \circ \circ \circ \\
\circ \circ \circ \circ \circ \\
\circ \circ \circ \circ \circ \\
\circ \circ \circ \circ \circ \\
\circ \circ \circ \circ \circ \\
\end{array}
\]

**Example 14.** Let \( P = \{(1, j) : 1 \leq j \leq 6\} \) and \( w = 2 \). Then \( P \) and \( w \) are compatible, \( I(P, w) \) is the following figure depicted with black dots

\[
\begin{array}{c}
\circ \circ \circ \circ \circ \\
\circ \circ \circ \circ \circ \\
\circ \circ \circ \circ \circ \\
\circ \circ \circ \circ \circ \\
\circ \circ \circ \circ \circ \\
\circ \circ \circ \circ \circ \\
\end{array}
\]

, and \( C(P, w) \) is the following figure also depicted with black dots

\[
\begin{array}{c}
\circ \circ \circ \circ \circ \\
\circ \circ \circ \circ \circ \\
\circ \circ \circ \circ \circ \\
\circ \circ \circ \circ \circ \\
\circ \circ \circ \circ \circ \\
\circ \circ \circ \circ \circ \\
\end{array}
\]

**Example 15.** Let \( P = \{(1,1), (1,2), (1,3), (2,2), (2,3)\} \) and \( w = 0 \). Then \( I(P, w) \) is the following figure depicted with black dots

\[
\begin{array}{c}
\circ \circ \circ \\
\circ \circ \\
\circ \\
\end{array}
\]

, and \( C(P, w) \) is the following figure depicted with black dots

\[
\begin{array}{c}
\circ \circ \circ \circ \circ \\
\circ \circ \circ \circ \circ \\
\circ \circ \circ \circ \circ \\
\circ \circ \circ \circ \circ \\
\end{array}
\]
In the above definition and examples, we see that $C(P, w)$ contains the union of two translates of $I(P, w)$. This is formalized in the following definition.

**Definition 20.** Let $(P, w)$ be a compatible pair and $\tau \in \text{SYT}(C(P, w))$. The top index tableau of $\tau$ is the subtableau $\tau^T$ of $\tau$ that is tableau graph isomorphic to some tableau in $\text{SYT}(I(P, w))$ and is such that

- the top row of $V(Tb(\tau^T))$ is contained in the top row of $\tau$,
- and the left-most column of $V(Tb(\tau^T))$ is contained in the left-most column of $\tau$.

The bottom index tableau of $\tau$ is the subtableau $\tau^B$ of $\tau$ that is tableau graph isomorphic to some tableau in $\text{SYT}(I(P, w))$ and is such that

- the bottom row of $V(Tb(\tau^T))$ is contained in the bottom row of $\tau$,
- and the right-most column of $V(Tb(\tau^T))$ is contained in the right-most column of $\tau$.

Moreover, let $C(P, w)^T$ denote the shape of $\tau^T$ and let $C(P, w)^B$ denote the shape of $\tau^B$.

In order for us to apply the above definition to standard Young tableaux of periodic shape, define the following.

**Definition 21.** Let $(P, w)$ be a compatible pair, $S = \text{Sh}^w(P, m)$ for some positive integer $m$, $\tau \in \text{SYT}(S)$, and assume that $m$ is sufficiently large.

The index tableau $\tau^I$ of $\tau$ is the subtableau of $\tau$ such that the tableau graph $Tb(\tau^I)$ is tableau graph isomorphic to $Tb(\xi)$ for some $\xi \in \text{SYT}(I(P, w))$, the
bottom row of $\tau^I$ is contained in the bottom row of $\tau$, and the right-most column of $\tau^I$ is contained in the right-most column of $\tau$. Also, let $S^I$ denote the shape of $\tau^I$.

The coefficient tableau $\tau^C$ of $\tau$ is the subtableau of $\tau$ such that the tableau graph $Tb(\tau^C)$ is tableau graph isomorphic to $Tb(\xi)$ for some $\xi \in SYT(C(P, w))$, the bottom row of $\tau^C$ is contained in the bottom row of $\tau$, and the right-most column of $\tau^C$ is contained in the right-most column of $\tau$. Also, let $S^C$ denote the shape of $\tau^C$.

The following definition will be helpful.

**Definition 22.** Let $(P, w)$ be a compatible pair. Then define

$$\dim(P, w) = |SYT(I(P, w))|.$$ 

At last, we define the tableau graph transfer matrix whose rows and columns are both indexed by $SYT(I(P, w))$ (in particular, the matrix has $\dim(P, w)$ rows and $\dim(P, w)$ columns). In order for such a matrix to be well-defined, we have to order the rows and the columns which we will do as follows.

Define a total order $\leq_T$ on $SYT(P)$ by declaring that $\tau_1 \leq_T \tau_2$ if

$$col(\tau_1) \leq_{lex} col(\tau_2)$$

where $\leq_{lex}$ is the lexicographic order. So if $\tau_1, \tau_2 \in \dim(P)$ and $\tau_1 <_T \tau_2$, then row $\tau_1$ is above row $\tau_2$ and column $\tau_1$ is to the left of column $\tau_2$.

Given a matrix, $X$, let $X(i, j)$ be the entry in the $i^{th}$ row and $j^{th}$ column of $X$.

**Definition 23.** Let $(P, w)$ be a compatible pair. Then the $w^{th}$ tableau graph transfer matrix $M_{P, w}$ of $P$ is the matrix with $\dim(P, w)$ rows and $\dim(P, w)$ columns such that for all $\tau_1, \tau_2 \in SYT(I(P, w))$,

$$M_{P, w}(\tau_2, \tau_1) = |\{ \tau \in SYT(C(P, w)) : \tau^T \cong \tau_1 \text{ and } \tau^B \cong \tau_2 \}|.$$
5 Deriving constant coefficient linear recurrences

Let $P$ be a periodic shape and $w$ be a non-negative integer such that $P$ and $w$ are compatible. We would like to determine constant coefficient recurrence relations for the sequence $P_w(1), P_w(2), P_w(3), \ldots$ defined by

$$P_w(n) = |SYT(Sh^w(P, n))|.$$  

We will prove that for any fixed $P$ and $w$, where $P$ and $w$ are compatible, the sequence $P_w(n)$ for $n \geq N(P, w) + \dim(P, w)$ satisfies a constant coefficient linear recurrence of order at most $\dim(P, w)$ where $N(P, w)$ is the smallest value of $m$ such that $I(P, w) + v \subseteq Sh^w(P, m)$ for some $v \in \mathbb{Z}^2$. Afterwards, we explain how these results can be applied to the recurrences analysed in [4], [2], and [5].

Let $R$ be a regular shape and consider the poset $(SYT(R), \leq)$ from section 3. We prove some properties of this poset below.

The following lemma makes it easier to analyse the tableau graphs of tableaux.

**Lemma 1.** (Manipulation Rules) Let $N$ be a normal shape, and let $\tau \in SYT(N)$. Consider two distinct cells $u, v \in N$ where $u = (u_1, u_2)$, $v = (v_1, v_2)$, $u_1 > v_1$, and $u_2 < v_2$. And let $w \in N$ be such that $w = (w_1, w_2)$. Then the following hold.

If $\{u, v\} \in E(Tb(\tau))$, $w_1 \leq u_1$, $w_2 \leq u_2$, and $w_1 > v_1$, then $\{w, v\} \in E(Tb(\tau))$.

If $\{u, v\} \in E(Tb(\tau))$, $w_1 \geq v_1$, $w_2 \geq v_2$, and $w_1 < u_1$, then $\{u, w\} \in E(Tb(\tau))$.

*Proof.* (S. van Willigenburg) First assume that $\{u, v\} \in E(Tb(\tau))$, $w_1 \leq u_1$, $w_2 \leq u_2$, and $w_1 > v_1$. Then $w_1 > v_1$ and $w_2 < v_2$. Moreover, as $\tau$ is a standard Young tableau of shape $N$, $\tau(w) \leq \tau(u)$, and, as $\{u, v\} \in E(Tb(\tau))$, $\tau(u) < \tau(v)$. So, $\tau(w) \leq \tau(u) < \tau(v)$. Combining this with the fact that $w_1 > v_1$ and $w_2 < v_2$ now implies $\{w, v\} \in E(Tb(\tau))$. 

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Next, assume that \( \{u, v\} \in E(Tb(\tau)), w_1 \geq v_1, w_2 \geq v_2, \) and \( w_1 < u_1 \). Then \( w_1 < u_1 \) and \( w_2 > u_2 \). Moreover, as \( \tau \) is a standard Young tableau of shape \( N \), \( \tau(v) \leq \tau(w) \), and, as \( \{u, v\} \in E(Tb(\tau)), \tau(u) < \tau(v) \). So, \( \tau(u) < \tau(v) \leq \tau(w) \). Combining this with the fact that \( w_1 < u_1 \) and \( w_2 > u_2 \) now implies \( \{u, w\} \in E(Tb(\tau)) \).

Next, we prove the following strong sufficient condition involving chains of the poset \((SYT(R), \leq)\). If \( S \) is a finite subset of \( \mathbb{N}^2 \) and \( \tau \in SYT(R) \), say that a cell \( v \in V(Tb(\tau)) \) is incident to an edge \( e \in e \) if \( v \in e \).

**Lemma 2. (Chain Commutativity Lemma)** Let \( R \) be a regular shape and consider the poset \((SYT(R), \leq)\). Moreover, let \( \tau_0, \tau_1, \tau_2 \in SYT(R) \) be such that \( \tau_0 \leq \tau_1 \leq \tau_2 \). Assume that for all edges \( e \in E(Tb(\tau_2)) \setminus E(Tb(\tau_1)) \) and \( h \in E(Tb(\tau_1)) \setminus E(G(\tau_0)) \), there is no cell \( v \in R \) that is incident to both \( e \) and \( h \). Then, there exists a tableau \( \tau \in SYT(R) \) such that \( \tau_0 \leq \tau \leq \tau_2 \) and

\[
E(Tb(\tau)) = E(Tb(\tau_0)) \cup (E(Tb(\tau_2)) \setminus E(Tb(\tau_1))).
\]

**Proof.** Let \( N = |R| \). As \( \tau_0 \leq \tau_1 \leq \tau_2 \), let \( i_1, i_2, \ldots, i_U \in [N] \) and \( j_1, j_2, \ldots, j_V \in [N] \) be two sequences such that

\[
\tau_1 = \pi_{i_U} \ldots (\pi_{i_2} (\pi_{i_1} (\tau_0)))
\]

and

\[
\tau_2 = \pi_{j_V} \ldots (\pi_{j_2} (\pi_{j_1} (\tau_1))).
\]

Because for all edges \( e \in E(Tb(\tau_2)) \setminus E(Tb(\tau_1)) \) and \( h \in E(Tb(\tau_1)) \setminus E(G(\tau_0)) \), there is no cell \( v \in R \) that is incident to both \( e \) and \( h \), \( f = \pi_{i_U} \circ \ldots \circ \pi_{i_2} \circ \pi_{i_1} \) and \( g = \pi_{j_V} \circ \ldots \circ \pi_{j_2} \circ \pi_{j_1} \) permute disjoint subsets of \([N]\). Hence, the tableau \( \tau = g(\tau_0) \) satisfies \( \tau \in SYT(R), \tau_0 \leq \tau \leq \tau_2 \) and

\[
E(Tb(\tau)) = E(Tb(\tau_0)) \cup (E(Tb(\tau_2)) \setminus E(Tb(\tau_1))).
\]

\( \square \)

It is convenient to partition standard Young tableaux of regular shape into two parts and to connect this with chains in the poset \((SYT(R), \leq)\) defined in section 3.
We use the following notation and terminology. If \( N \) is a normal shape, write the sink tableau of shape \( N \) as \( 1_N \). Let \( e \in E(Tb(1_N)) \) and \( e = \{u, v\} \) where \( u = (u_1, u_2) \) and \( v = (v_1, v_2) \). Say that \( u \) is the bottom cell of \( e \) and that \( v \) is the top cell of \( e \) if \( u_1 > v_1 \). Moreover, let \( u, v \in \mathbb{N}^2 \) where \( u = (u_1, u_2) \) and \( v = (v_1, v_2) \). Say that \( u \) is strictly south-west of \( v \) if \( u_1 > v_1 \), and \( u_2 < v_2 \). Furthermore, say that \( u \) is weakly north-west of \( v \) if \( u_1 \leq v_1 \) and \( u_2 \leq v_2 \).

For the following four lemmas, assume the following. Let \( R \) be regular and \( R = A \cup B \), where the union is disjoint, both \( A \) and \( B \) are non-empty, \( A \) is normal, \( B \) is regular, the bottom row of \( B \) is contained in the bottom row of \( R \), the right-most column of \( B \) equals to the right-most column of \( R \).

**Lemma 3.** (Tableau Bipartition Lemma Part 1) Let \( \tau \in SYT(R) \). Then there are tableaux \( \tau_A \in SYT(A) \) and \( \tau_B \in SYT(B) \) such that \( Tb(\tau|_A) = Tb(\tau_A) \) and \( Tb(\tau|_B) = Tb(\tau_B) \).

**Proof.** To show that \( \tau_A \) exists, let \( M = |A| \), let \( N_A \) denote the set of entries of \( \tau|_A \) and write \( N_A = \{a_i : i = 1, 2, \ldots, M\} \) where \( a_1 < a_2 < \cdots < a_M \). Then, let \( f_A : N_A \to [M] \), be the order isomorphism from \( a_1 < a_2 < \cdots < a_M \) to \( 1 < 2 < \cdots < M \). Now, define \( \tau_A \) to be the unique tableaux of shape \( A \) such that for all \( v \in A \),

\[
\tau_A(v) = f_A(\tau(v)).
\]

As \( f_A \) is an order isomorphism, \( \tau_A \in SYT(A) \) and \( Tb(\tau_A) = Tb(\tau|_A) \). The proof that \( \tau_B \) exists follows similarly.

\(\square\)

**Lemma 4.** (Tableau Bipartition Lemma Part 2) Consider the poset \( (SYT(R), \leq) \). Assume that \( B \) also satisfies the following property. For all edges \( e \in E(Tb(1_R)) \) where \( e = \{u, v\} \) and \( u \) is strictly southwest of \( v \), \( u, v \in A, u, v \in B \), or the following occurs: \( u \in B \) and \( v \in A \). Then for all \( \tau \in SYT(R) \), there is a tableau \( \tau_1 \in SYT(R) \) such that

\[
\tau_1 \leq \tau, \ Tb(\tau_1|_A) = Tb(\tau|_A), \text{ and } E(Tb(\tau_1)) \subseteq E(Tb(1_A)).
\]

**Proof.** (C., S. van Willigenburg) Assume that \( \tau \in SYT(R) \) is such that \( E(Tb(\tau)) \not\subseteq E(Tb(1_A)) \). Then there exists two cells \( u, v \in R \) where \( u \) is strictly south-west of \( v \), \( \tau(u) < \tau(v) \), and \( u \in B \). As the entries of any row
of \( \tau \) are increasing when read from left to right and as the entries of any column of \( \tau \) are increasing when read from top to bottom, the additional assumption on \( B \) made in this lemma implies that for some positive integer \( i \) satisfying \( \tau(u) \leq i < i + 1 \leq \tau(v) \), the cell \( u' \) of \( \tau \) that contains \( i \) is in \( B \) and \( u' \) is strictly south-west of \( v' \) where \( v' \) is the cell of \( \tau \) that contains \( i+1 \).

Hence, as \( \{u', v'\} \in E(Tb(\tau)) \), there is a tableau \( \tau' \in SYT(R) \) such that \( \pi_i(\tau') = \tau \) and \( \{\{u', v'\}\} = E(Tb(\tau')) \setminus E(Tb(\tau)) \). So as \( E(Tb(\tau)) \) is finite, repeating the above argument a finite number of times implies the lemma.

\[ \square \]

**Lemma 5.** (Tableau Bipartition Lemma Part 3) Consider the poset \((SYT(R), \leq)\). Assume that \( B \) also satisfies the following property. For all edges \( e \in E(Tb(1_R)) \) where \( e = \{u, v\} \) and \( u \) is strictly southwest of \( v \), \( u, v \in A \), \( u, v \in B \), or the following occurs: \( u \in A \) and \( v \in B \). Then for all \( \tau \in SYT(R) \), there is a tableau \( \tau_1 \in SYT(R) \) such that

\[
\tau_1 \geq \tau, \quad E(Tb(\tau_1)) \supseteq E(Tb(1_R)) \setminus E(Tb(1_B)), \text{ and } Tb(\tau_1|_B) = Tb(\tau|_B).
\]

**Proof.** The proof is dual to the proof of part 2 of the Tableau Bipartition Lemma. Assume that \( \tau \in SYT(R) \) is such that \( E(Tb(\tau)) \not\supset E(Tb(1_R)) \setminus E(Tb(1_B)) \). Then there exists two cells \( u, v \in R \) where \( u \) is strictly south-west of \( v \), \( \tau(u) > \tau(v) \), and \( u \in A \). As the entries of any row of \( \tau \) are increasing when read from left to right and as the entries of any column of \( \tau \) are increasing when read from top to bottom, the additional assumption on \( B \) made in this lemma implies that for some positive integer \( i \) satisfying \( \tau(v) \leq i < i + 1 \leq \tau(u) \), the cell \( u' \) of \( \tau \) containing \( i + 1 \) is in \( A \), and \( u' \) is strictly south-west of \( v' \) where \( v' \) is the cell of \( \tau \) containing \( i \).

Hence, as \( \{u', v'\} \in E(Tb(\tau)) \), there is a tableau \( \tau' \in SYT(R) \) such that \( \pi_i(\tau') = \tau' \) and \( \{\{u', v'\}\} = E(Tb(\tau')) \setminus E(Tb(\tau)) \). So as \( E(Tb(1_R)) \) is finite, repeating the above argument a finite number of times implies the lemma.

\[ \square \]

The following lemma won’t be used in this section, but it will be needed for section 6.

**Lemma 6.** (Tableau Bipartition Lemma Part 4) Consider the poset \((SYT(R), \leq)\). Assume that \( B \) also satisfies the following property. For all edges \( e \in \)
where \( e = \{u, v\} \) and \( u \) is strictly south-west of \( v \), \( u, v \in A \), \( u, v \in B \), or the following occurs: \( u \in B \) and \( v \in A \).

Moreover, let \( B' \subset B \) be such that the following conditions hold

- \( B' \) is regular, the bottom row of \( B' \) is contained in the bottom row of \( R \), and the right-most column of \( B' \) is contained in the right-most column of \( R \).
- For all edges \( e \in E(Tb(1_R)) \) where \( e = \{u, v\} \) and \( u \) is strictly south-west of \( v \), \( u, v \notin B' \); \( u, v \in B \), or the following occurs: \( u \in B' \) and \( v \notin B' \).
- For all edges \( e \in E(Tb(1_R)) \) where \( e = \{u, v\} \), \( u \) is strictly south-west of \( v \), \( u \in B' \), and \( v \in A \), there is an edge \( h \in E(Tb(1_R)) \) where \( h = \{u', v\} \), \( u' \) is strictly south-west of \( v \), and \( u' \) is weakly north-west of \( u \).

Lastly, set \( A' = R B' \). Then for all \( \tau \in SYT(R) \), there is a tableaux \( \tau_1 \in SYT(R) \) such that

\[
\tau_1 \leq \tau, \ Tb(\tau_1|A') = Tb(\tau|A'), \text{ and } E(Tb(\tau_1)) \subseteq E(Tb(1_A')).
\]

Proof. Let \( \tau \in SYT(R) \) and assume that \( E(Tb(\tau)) \notin E(Tb(1_A')) \). By part 2 of the Tableau Bipartition Lemma, there is a tableau \( \tau'_1 \in SYT(R) \) such that \( \tau'_1 < \tau \) and a saturated chain \( \tau' = \tau_0 < \tau_1 < \cdots < \tau_M = \tau \) of \( (SYT(R), \leq) \). The saturated chain can be identified with the following sequence of edges

\[
e_1, e_2, \ldots, e_M
\]

where \( e_i \) is the unique edge satisfying \( e_i \in E(Tb(\tau_i))\setminus E(Tb(\tau_{i-1})) \). For all \( i \), the top cell of \( e_i \) is in \( A \), and the bottom cell of \( e_i \) is in \( B \). Observe the following. If \( e_i \) and \( e_j \) are such that the bottom cell of \( e_i \) is in \( B' \) and the bottom cell of \( e_j \) is not in \( B' \), then the top cell of \( e_i \) is different from the top cell of \( e_j \), or the bottom cell of \( e_j \) is weakly north-west of the bottom cell of \( e_i \). In the first case, \( e_i \cap e_j = \emptyset \), and in the second case, the Manipulation Rules imply that \( e_j \) must appear to the left of \( e_i \) in the above sequence of edges. Hence, by the Chain Commutativity Lemma, there is a permutation \( \sigma \) of \([M]\) such that the resulting sequence of edges

\[
e_1, e_2, \ldots, e_M
\]
\[ h_1, h_2, \ldots, h_M \]
defined by \( h_i = e_{\sigma(i)} \) satisfies two properties.

- If \( h_i \) and \( h_j \) are two edges such that the bottom edge of \( h_i \) is in \( B' \) and the bottom edge of \( h_j \) is not in \( B' \), then \( j < i \).
- There exists a saturated chain \( \tau' = \xi_0 < \xi_1 < \cdots < \xi_M = \tau \) of \((\text{SYT}(R), \leq)\) such that for all \( 1 \leq i \leq M \), \( h_i \) is the unique edge in \( E(Tb(\xi_i)) \setminus E(Tb(\xi_{i-1})) \).

Let \( j \) be the largest index such that the bottom edge of \( h_j \) is not in \( B' \). Then setting \( \tau_1 = \xi_j \) gives the lemma.

\[ \Box \]

Let \((P, w)\) be a compatible pair, and let \( v_{P,w} \in \mathbb{R}^{\dim(P,w)} \) be defined as follows. The rows of \( v_{P,w} \) are indexed by \( \text{SYT}(I(P,w)) \), and for all \( \tau \in \text{SYT}(I(P,w)) \), the entry in row \( \tau \) of \( v_{P,w} \) is

\[ |\{ \xi \in Sh^w(P,N(P,w)) : \xi^I \equiv \tau \}|. \]

Moreover, order the entries of the vector \( v_{P,w} \) like so. Let \( \leq_T \) be as defined as in the paragraph before Definition 21. If \( \tau_1 \leq_T \tau_2 \), then the entry of \( v_{P,w} \) in row \( \tau_1 \) of \( v_{P,w} \) is above the entry of \( v_{P,w} \) in row \( \tau_2 \) of \( v_{P,w} \).

Now we prove the first two main theorems of this paper. Given a column matrix \( X \), let \( X^T \) denote the transpose of \( X \).

**Theorem 1.** Let \( P \) be a periodic shape \( w \) be a non-negative integer such that \( P \) and \( w \) are compatible, and let \( M_{P,w} \) be the \( w \)th tableau graph transfer matrix of \( P \). Then for any \( n \geq N(P,w) \),

\[ P_w(n) = u_{P,w}^T M_{P,w}^{n-N(P,w)} v_{P,w}, \]

where \( u_{P,w} \in \mathbb{R}^{\dim(P,w)} \) is the column matrix with all entries equal to 1 and \( v_{P,w} \in \mathbb{R}^{\dim(P,w)} \) is as defined above.
Proof. Let \( n \geq N(P, w) \). For all tableaux \( \tau_1 \in SYT(Sh^w(P, n + 1)) \), part 1 of the Tableau Bipartition Lemma implies that there are tableaux \( \tau_2 \in SYT(Sh^w(P, n)) \) and \( \tau_3 \in SYT(C(P, w)) \) such that \( Tb(\tau_2) = Tb(\tau_1|_X) \) and \( Tb(\tau_3) \cong Tb(\tau_1|_Y) \), where \( X = Sh^w(P, n) \) and \( Y = WC \), where \( W = Sh^w(P, n + 1) \).

So it is enough to prove the following. For all tableaux \( \tau_0 \in SYT(Sh^w(P, n)) \) and \( \xi \in SYT(WC) \) (where \( W = Sh^w(P, n + 1) \)) such that \( Tb(\xi^T) = Tb(\tau_0^T) \), there is a tableau \( \tau \in SYT(Sh^w(P, n + 1)) \) such that

\[
E(Tb(\tau)) = E(Tb(\tau_0)) \cup E(Tb(\xi)).
\]

Throughout, we consider the poset \((SYT(Sh^w(P, n)), \leq)\). Define three sets of edges \( E_0, E_1, \) and \( E_2 \) as follows.

\[
E_0 = E(Tb(\tau_0)),
\]

\[
E_1 = E(Tb(1_{Sh^w(P, n)})) \setminus E_0,
\]

and

\[
E_2 = E(Tb(\xi)) \cap \left( E(Tb(1_X)) \setminus E(Tb(1_Y)) \right)
\]

where \( X = WC \) (with \( W = Sh^w(P, n + 1) \)) and \( Y = X \setminus (P + n \nabla_w P) \).

Now, define \( \tau_0' \) to be the tableau of \( SYT(Sh^w(P, n + 1)) \) such that \( \tau_0'(v) = \tau_0(v) \) for all \( v \in Sh^w(P, n) \) and the entries of any pair of adjacent cells in the same row of \( \tau_0'|_{P+n\nabla_w P} \) are consecutive. It can be seen that \( \tau_0' \in SYT(Sh^w(P, n + 1)) \) and that

\[
E(Tb(\tau_0')) = E(Tb(\tau_0)) = E_0.
\]

By part 3 of the Tableau Bipartition Lemma, there is a tableau \( \tau_1 \in SYT(Sh^w(P, n + 1)) \) such that \( \tau_1 \geq \tau_0' \) and

\[
E(Tb(\tau_1)) = E_0 \cup E_1.
\]

Next, define \( \xi' \) to be the tableau of \( SYT(Sh^w(P, n + 1)) \) such that \( \xi'(v) = \xi(v) + N \), where \( N = |Sh^w(P, n + 1)| - |C(P, w)| \), and the entries of any pair of adjacent cells in the same column of \( \xi'|_X \), where \( X = Sh^w(P, n + 1) \setminus YC \) and
\( Y = Sh^w(P, n+1) \), are consecutive. It can be seen that \( \xi' \in SYT(Sh^w(P, n+1)) \) and that

\[
E(Tb(\xi')) = E_0 \cup E_1 \cup E_2.
\]

Moreover, by part 2 of the Tableau Bipartition Lemma, \( \xi' \geq \tau_1 \). So, \( \tau'_0 \leq \tau_1 \leq \xi' \).

Now, we observe the following. The Manipulation Rules implies that no cell of \( Sh^w(P, w) \) is incident to an edge in \( E_1 \) and an edge in \( E_2 \). Hence, by the Chain Commutativity Lemma, there is a tableau \( \tau \in SYT(Sh^w(P, n+1)) \) such that \( \tau'_0 \leq \tau \leq \xi' \) and

\[
E(Tb(\tau)) = E_0 \cup E_2 = E(Tb(\tau_0)) \cup E(Tb(\xi)).
\]

This completes the proof.

The above theorem implies the following

**Theorem 2.** Let \( P \) be a periodic shape and \( w \) be a non-negative integer such that \( P \) and \( w \) are compatible. Then the sequence \( P_w^n \) for \( n \geq N(P, w) + \dim(P, w) \) satisfies a constant coefficient linear recurrence of order at most \( \dim(P, w) \).

**Proof.** Let \( M_{P,w}, v_{P,w}, \) and \( u_{P,w} \) be as in the preceding theorem. Consider the sequence \( v_{P,w}, M_{P,w}v_{P,w}, M^2_{P,w}v_{P,w}, \ldots \), and let \( \chi_{P,w} \) be the characteristic polynomial of \( M_{P,w} \). By the Cayley-Hamilton Theorem,

\[
\chi_{P,w}(M_{P,w}) = 0.
\]

In particular, for all non-negative integers \( p \),

\[
M_{P,w}^p\chi_{P,w}(M_{P,w}) = 0.
\]

\[
M_{P,w}^p\chi_{P,w}(M_{P,w})v_{P,w} = 0
\]

and,

\[
u_{P,w}^T M_{P,w}^p\chi_{P,w}(M_{P,w})v_{P,w} = 0
\]
this completes the proof.

For any positive integer $p$ and non-negative integer $w$, let $M_{p,w}$ denote $M_{R,w}$, where $R$ is a single row with $p$ cells and $(1,1) \in R$. Interestingly, computing the characteristic polynomials for $M_{4,1}$, $M_{5,1}$, and $M_{6,1}$ gives us three of Hardin and Alois’ Recurrence relations as explained in the following corollaries.

**Corollary 1** ([4], [2]). Let $T(n,k) = |Sh^1(R,n)|$, where $R$ is a single row with $k$ cells and $(1,1) \in R$. Then

$$T(n,4) = 6T(n-1,4) - T(n-2,4)$$

$$T(n,5) = 24T(n-1,5) - 40T(n-2,5) - 8T(n-3,5)$$

**Proof.** The 1st transfer matrix of $R$ when $|R| = 4$ is

$$M_{4,1} = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$$

and its characteristic polynomial is $x^2 - 6x + 1$. The 1st transfer matrix of $R$ when $|R| = 5$ is

$$M_{5,1} = \begin{pmatrix} 4 & 5 & 5 & 6 & 6 & 7 & 7 \\ 3 & 4 & 4 & 5 & 5 & 6 & 6 \\ 3 & 4 & 4 & 5 & 5 & 6 & 6 \\ 2 & 3 & 3 & 4 & 4 & 5 & 5 \\ 2 & 3 & 3 & 4 & 4 & 5 & 5 \\ 0 & 0 & 2 & 0 & 3 & 0 & 4 \\ 0 & 0 & 2 & 0 & 3 & 0 & 4 \end{pmatrix}$$

and its characteristic polynomial is $x^7 - 24x^6 + 40x^5 + 8x^4$. Moreover, Hadrian and Alois have computed $T(n,4)$ and $T(n,5)$ for the first 30 values of $n$ - the above recurrences can be verified for those numbers. From this, the recurrences follow.

Computing the tableau graph transfer matrix $M_{6,1}$, which turns out to be a $66 \times 66$ matrix, gives us the following characteristic polynomial
Hence, we have the following result as $N(R, 1) = 5$ (the below recurrence was proved for all values of $n$ in [2]).

**Corollary 2 ([2]).** Let $T(n, k) = \left| Sh^1(R, n) \right|$, where $R$ is a single row with $k$ cells and $(1, 1) \in R$. Then for $n \geq 76$,

\[
T(n, 6) = 120T(n - 1, 6) - 1672T(n - 2, 6) + 544T(n - 3, 6) - 6672T(n - 4, 6) + 256T(n - 5, 6)
\]

6 Symmetries in the tableau graph transfer matrix

In this section, we show that there are many repeated rows in the tableau graph transfer matrices. This implies, by the matrix compression method of [3], that these transfer matrices can be compressed. As an application of this, we will show that if $R$ is a single row and $w$ is a shift number such that $R$ and $w$ are compatible, then for $n \geq N(R, w)$, the sequence $A^w_P(n) = \left| SYT(Sh^w(R, n)) \right|$ satisfies a constant coefficient linear recurrence relation of order at most $n - 2w$ if $|R| \leq 3w + 1$.

We define the following equivalence relation on $I(P, w)$, which depends on which subset of $I(P, w)$ is chosen.

**Definition 24.** Let $(P, w)$ be a compatible pair. Call $S$ a redundant subset of $I(P, w)$ if $S \subseteq I(P, w)$, and $S$ satisfies the following properties

- $A = C(P, w)$, $B = P + (N(P, w) - 1)\nabla_w P$, and $B' = S + v$ (where $I(P, w) + v = W^B$ and $W = C(P, w)$) satisfy the conditions assumed in part 4 of the Tableau Bipartition Lemma.
- If $e$ is an edge of $E(Tb(1_X)) \setminus E(Tb(1_Y))$, where $X = C(P, w)$ and $Y = W^B$ (where $W = C(P, w)$), if $u$ is the top cell of $e$ and the top cell of an edge $h$ whose bottom cell $v$ is in $S$, then the bottom cell of $e$ is weakly north-west of $v$.  

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Definition 25. Let \((P, w)\) be a compatible pair, consider \(I(P, w)\), and let \(S\) be a compatible subset of \(I(P, w)\). For all \(\tau_1, \tau_2 \in I(P, w)\), define \(\tau_1 \equiv_S \tau_2\) if

\[
Tb(\tau_1|_{I(P, w)\setminus S}) = Tb(\tau_2|_{I(P, w)\setminus S})
\]

Now, we prove the last main theorem of this paper. Afterwords, we prove a nice recurrence result as an application.

Theorem 3. Let \(P\) be a periodic shape and \(w\) be a non-negative integer such that \(P\) and \(w\) are compatible. Moreover, let \(S\) be a redundant subset of \(I(P, w)\). Consider the matrix \(M_{P,w}\). If \(\tau_1, \tau_2 \in I(P, w)\) and \(\tau_1 \equiv_S \tau_2\), then rows \(\tau_1\) and \(\tau_2\) of \(M_{P,w}\) are identical.

Proof. We prove this theorem similarly to how Theorem 1 was proved. Let \(S' = S + v\), where \(I(P, w) + v = W^B\) and \(W = C(P, w)\). And let \(I'(P, w) = X^B\), where \(X = C(P, w)\).

For all tableaux \(\tau_1 \in SYT(C(P, w))\), part 1 of the Tableau Bipartition Lemma implies that there are tableaux \(\tau_2 \in SYT(C(P, w)\setminus S')\) and \(\tau_3 \in SYT(I'(P, w))\) such that \(Tb(\tau_1|_{C(P, w)\setminus S'}) = Tb(\tau_2)\), and \(Tb(\tau_1|_{I'(P, w)}) = Tb(\tau_3)\).

So it is enough to prove the following. For all tableaux \(\tau_0 \in SYT(C(P, w)\setminus S')\) and \(\xi \in SYT(I'(P, w))\) such that \(Tb(\tau_0|_{X \cap Y}) = Tb(\xi|_{X \cap Y})\) \((X = C(P, w)\setminus S'\) and \(Y = SYT(I'(P, w))\)), there is a tableau \(\tau \in SYT(C(P, w))\) such that \(E(Tb(\tau)) = E(Tb(\tau_0)) \cup E(Tb(\xi))\).

Throughout, we consider the poset \((SYT(C(P, w)), \leq)\). Define three sets of edges \(E_0, E_1,\) and \(E_2\) as follows.

\[
E_0 = E(Tb(\tau_0))
\]

\[
E_1 = E(Tb(1_{C(P, w)\setminus S'})) \setminus E_0
\]

\[
E_2 = E(Tb(\xi)) \cap (E(Tb(1_X)) \setminus E(Tb(1_Y)))
\]

where \(X = I'(P, w)\) and \(Y = X \setminus S'\).
Consider any tableau $\tau''_0 \in SYT(C(P, w))$ such that $\tau''_0(v) = \tau''_0(v)$ for any $v \in C(P, w) \setminus S'$. Applying part 4 of the Tableau Bipartition Lemma $\tau''_0$ implies that there is a tableau $\tau'_0 \in SYT(C(P, w))$ such that $E(Tb(\tau'_0)) \subseteq E(Tb(1_{C(P, w)})) \setminus E(Tb(1'S))$ and

$$E(Tb(\tau'_0)) = E(Tb(\tau_0)) = E_0.$$ 

By part 3 of the Tableau Bipartition Lemma, there is a tableau $\tau_1 \in SYT(C(P, w))$ such that $\tau_1 \geq \tau'_0$ and

$$E(Tb(\tau_1)) = E_0 \cup E_1.$$ 

Next, define $\xi'$ to be the tableau of $SYT(C(P, w))$ such that $\xi'(v) = \xi(v) + N$, where $N = |C(P, w)| - |I'(P, w)|$, and the entries of any pair of adjacent cells in the same column of $\xi'|_X$, where $X = C(P, w) \setminus I'(P, w)$, are consecutive. It can be seen that $\xi' \in SYT(C(P, w))$ and that

$$E(Tb(\xi')) = E_0 \cup E_1 \cup E_2.$$ 

By part 4 of the Tableau Bipartition Lemma, $\xi' \geq \tau_1$. So, $\tau_0' \leq \tau_1 \leq \xi'$. 

Now, we observe the following. By how redundant subsets of $I(P, w)$ are defined, no cell of $C(P, w) \setminus S'$ is incident to an edge in $E_1$ and an edge in $E_2$. Hence, by the Chain Commutativity Lemma, there is a tableau $\tau \in SYT(C(P, w))$ such that $\tau_0' \leq \tau \leq \xi'$ and

$$E(Tb(\tau)) = E_0 \cup E_2 = E(Tb(\tau_0)) \cup E(Tb(\xi)).$$

This completes the proof. 

For any compatible pair $(P, w)$ and redundant subset $S \subseteq I(P, w)$, define

$$\dim^S(P, w) = |SYT(I(P, w))| / \equiv_s |.$$ 

Using Theorem 3, it is not hard to use the matrix compression technique from [3] to deduce the following result.

**Corollary 3.** Let $P$ be a periodic shape and $w$ be a non-negative integer such that $P$ and $w$ are compatible. Then the sequence $P_w(n)$ for $n \geq$
\(N(P, w) + \dim'(P, w)\) satisfies a constant coefficient linear recurrence of order at most \(\dim'(P, w)\).

As a nice application of Theorem 3, we prove the following.

**Proposition 4.** Let \(R\) be a single row containing \((1, 1)\), let \(w\) be a non-negative integer such that \(R\) and \(w\) are compatible. Then for all \(n \geq N(R, w)\), the sequence

\[A^P_w(n) = |SYT(Sh^w(R, n))|\]

satisfies a constant coefficient linear recurrence relation of order at most \(n - 2w\) if \(|R| \leq 3w + 1\).

**Proof.** For any redundant subset \(S \subseteq I(R, w)\),

\[|SYT(I(R, w))/_S| \leq |SYT(I(R, w)\setminus S)|.\]

So we examine the shape \(I(R, w)\setminus S\). Let \(S\) be the right-most \(n - w - 1\) cells in the bottom row of \(I(R, w)\). Then \(S\) is a redundant subset and the shape \(SYT(I(R, w))\setminus S\) is the union of a cell and a row of cells where the row of cells has \(n - 2w - 1\) elements. For an illustration, consider the following example.

**Example 16.** Let \(|R| = 13\) and \(w = 4\). Note that \(|R| \leq 3w + 1\). Then \(I(R, w)\) is depicted below and \(I(P, w)\setminus S\) is depicted with the black cells.

\[
\begin{array}{cccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\end{array}
\]

From this, it can be seen that

\[|SYT(I(P, w)\setminus S)| \leq (n - 2w - 1) + 1 = n - 2w\]

completing the proof by Corollary 3.
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