Scattering in Relativistic Quantum Field Theory: Fundamental Concepts and Tools

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Abstract

We provide a brief overview of the basic tools and concepts of quantum field theoretical scattering theory. This article is commissioned by the Encyclopedia of Mathematical Physics, edited by J.-P. Françoise, G. Naber and T.S. Tsun, to be published by the Elsevier publishing house.

1 Physical Motivation and Mathematical Setting

The primary connection of relativistic quantum field theory to experimental physics is through scattering theory, \textit{i.e.} the theory of the collision of elementary (or compound) particles. It is therefore a central topic in quantum field theory and has attracted the attention of leading mathematical physicists. Although a great deal of progress has been made in the mathematically rigorous understanding of the subject, there are important matters which are still unclear, some of which will be indicated below.

In the paradigmatic scattering experiment, several particles, which are initially sufficiently distant from each other that the idealization that they are not mutually interacting is physically reasonable, approach each other and interact (collide) in a region of microscopic extent. The products of this collision then fly apart until they are sufficiently well separated that the approximation of noninteraction is again reasonable. The initial and final states of the objects in the scattering experiment are therefore to be modeled by states of noninteracting, \textit{i.e.} free, fields, which are mathematically represented on Fock space. Typically, what is measured in such experiments is the probability distribution (cross section) for the transitions from a specified state of the incoming particles to a specified state of the outgoing particles.

It should be mentioned that until the late 1950's, the scattering theory of relativistic quantum particles relied upon ideas from nonrelativistic quantum mechanical scattering theory (interaction representation, adiabatic limit, \textit{etc}), which were invalid in the relativistic context. Only with the advent of axiomatic quantum field theory did it become
possible to properly formulate the concepts and mathematical techniques which will be outlined here.

Scattering theory can be rigorously formulated either in the context of quantum fields satisfying the Wightman axioms \[10\] or in terms of local algebras satisfying the Haag–Kastler–Araki axioms \[6\]. In brief, the relation between these two settings may be described as follows: In the Wightman setting, the theory is formulated in terms of operator valued distributions $\phi$ on Minkowski space, the quantum fields, which act on the physical state space. These fields, integrated with test functions $f$ having support in a given region $O$ of space–time, $\int d^4x \, f(x)\phi(x)$, form under the operations of addition, multiplication and hermitian conjugation a polynomial *–algebra $P(O)$ of unbounded operators.

In the Haag–Kastler–Araki setting one proceeds from these algebras to algebras $A(O)$ of bounded operators which, roughly speaking, are formed by the bounded functions $A$ of the operators $\phi(f)$. This step requires some mathematical care, but these subtleties will not be discussed here. As the statements and proofs of the results in these two frameworks differ only in technical details, the theory is presented here in the more convenient setting of algebras of bounded operators ($C^*$–algebras).

Central to the theory is the notion of a particle, which, in fact, is a quite complex concept, the full nature of which is not completely understood, cf. below. In order to maintain the focus on the essential points, we consider in the subsequent sections primarily a single massive particle of integer spin $s$, i.e. a Boson. In standard scattering theory based upon Wigner’s characterization, this particle is simply identified with an irreducible unitary representation $U_1$ of the identity component $P^+_\uparrow$ of the Poincaré group with spin $s$ and mass $m > 0$. The Hilbert space $\mathcal{H}_1$ upon which $U_1(P^+_\uparrow)$ acts is called the one–particle space and determines the possible states of a single particle, alone in the universe. Assuming that configurations of several such particles do not interact, one can proceed by a standard construction to a Fock space describing freely propagating multiple particle states,

$$\mathcal{H}_F = \bigoplus_{n \in \mathbb{N}_0} \mathcal{H}_n,$$

where $\mathcal{H}_0 = \mathbb{C}$ and $\mathcal{H}_n$ is the $n$–fold symmetrized direct product of $\mathcal{H}_1$ with itself. This space is spanned by vectors $\Phi_1 \otimes \cdots \otimes \Phi_n$, where $\otimes$ denotes the symmetrized tensor product, which represent an $n$–particle state wherein the $k$–th particle is in the state $\Phi_k \in \mathcal{H}_1$, $k = 1, \ldots, n$. The representation $U_1(P^+_\uparrow)$ induces a unitary representation $U_F(P^+_\uparrow)$ on $\mathcal{H}_F$ by

$$U_F(\lambda) (\Phi_1 \otimes \cdots \otimes \Phi_n) \doteq U_1(\lambda)\Phi_1 \otimes \cdots \otimes U_1(\lambda)\Phi_n.$$  \hspace{1cm} (1)

In interacting theories, the states in the corresponding physical Hilbert space $\mathcal{H}$ do not have such an a priori interpretation in physical terms, however. It is the primary goal of scattering theory to identify in $\mathcal{H}$ those vectors which describe, at asymptotic times, incoming, respectively outgoing, configurations of freely moving particles. Mathematically, this amounts to the construction of certain specific isometries (generalized Møller operators) $\Omega^{in}$ and $\Omega^{out}$ mapping $\mathcal{H}_F$ onto subspaces $\mathcal{H}^{in} \subset \mathcal{H}$ and $\mathcal{H}^{out} \subset \mathcal{H}$, respectively, and intertwining the unitary actions of the Poincaré group on $\mathcal{H}_F$ and $\mathcal{H}$. The resulting vectors

$$(\Phi_1 \otimes \cdots \otimes \Phi_n)^{in/out} = \Omega^{in/out} (\Phi_1 \otimes \cdots \otimes \Phi_n) \in \mathcal{H}$$  \hspace{1cm} (2)

\[1\] Only four-dimensional Minkowski space $\mathbb{R}^4$ will be treated here.
are interpreted as incoming and outgoing particle configurations in scattering processes wherein the \( k \)-th particle is in the state \( \Phi_k \in H_1 \).

If, in a theory, the equality \( H^\text{in} = H^\text{out} \) holds, then every incoming scattering state evolves, after the collision processes at finite times, into an outgoing scattering state. It is then physically meaningful to define on this space of states the scattering matrix, setting \( S = \Omega^\text{in} \Omega^\text{out*} \). Physical data such as collision cross sections can be derived from \( S \) and the corresponding transition amplitudes \( \langle (\Phi_1 \otimes \cdots \otimes \Phi_m)^\text{in}, (\Phi'_1 \otimes \cdots \otimes \Phi'_n)^\text{out} \rangle \), respectively, by a standard procedure. It should be noted, however, that neither the above physically mandatory equality of state spaces nor the more stringent requirement that every state has an interpretation in terms of incoming and outgoing scattering states, \( i.e. H = H^\text{in} = H^\text{out} \) (asymptotic completeness), has been fully established in any interacting relativistic field theoretic model so far. This intriguing problem will be touched upon in the last section of this article.

Before going into details, let us state the few physically motivated postulates entering into the analysis. As discussed, the point of departure is a family of algebras \( \mathcal{A}(O) \), more precisely a net, associated with the open subregions \( O \) of Minkowski space and acting on \( H \). Restricting attention to the case of Bosons, we may assume that this net is local in the sense that if \( O_1 \) is spacelike separated from \( O_2 \), then all elements of \( \mathcal{A}(O_1) \) commute with all elements of \( \mathcal{A}(O_2) \).\(^2\) This is the mathematical expression of the principle of Einstein causality. The unitary representation \( U \) of \( \mathcal{P}_+ \) acting on \( H \) is assumed to satisfy the relativistic spectrum condition (positivity of energy in all Lorentz frames) and, in the sense of equality of sets, \( U(\lambda) \mathcal{A}(O) U(\lambda)^{-1} = \mathcal{A}(\lambda O) \) for all \( \lambda \in \mathcal{P}_+ \) and regions \( O \), where \( \lambda O \) denotes the Poincaré transformed region. It is also assumed that the subspace of \( U(\mathcal{P}_+) \)-invariant vectors is spanned by a single unit vector \( \Omega \), representing the vacuum, which has the Reeh–Schlieder property, \( i.e. \) each set of vectors \( \mathcal{A}(O) \Omega \) is dense in \( H \). These standing assumptions will subsequently be amended by further conditions concerning the particle content of the theory.

2 Haag–Ruelle Theory

Haag and Ruelle were the first to establish the existence of scattering states within this general framework \(^8\); further substantial improvements are due to Araki and Hepp \(^1\). In all of these investigations, the arguments were given for quantum field theories with associated particles (in the Wigner sense) which have strictly positive mass \( m > 0 \) and for which \( m \) is an isolated eigenvalue of the mass operator (upper and lower mass gap). Moreover, it was assumed that states of a single particle can be created from the vacuum by local operations. In physical terms, these assumptions allow only for theories with short range interactions and particles carrying strictly localizable charges.

In view of these limitations, Haag–Ruelle theory has been developed in a number of different directions. By now, the scattering theory of massive particles is under complete control, including also particles carrying non–localizable (gauge or topological) charges and particles having exotic statistics (anyons, plektons) which can appear in theories in low spacetime dimensions. Due to page constraints, these results must go without further mention; we refer the interested reader to the articles \(^2\) \(^5\). Theories of massless particles and of particles carrying charges of electric or magnetic type (infraparticles) will

\(^2\)In the presence of Fermions, these algebras contain also fermionic operators which anti–commute.
be discussed in subsequent sections.

We outline here a recent generalization of Haag–Ruelle scattering theory presented in [1], which covers massive particles with localizable charges without relying on any further constraints on the mass spectrum. In particular, the scattering of electrically neutral, stable particles fulfilling a sharp dispersion law in the presence of massless particles is included (e.g. neutral atoms in their ground states). Mathematically, this assumption can be expressed by the requirement that there exists a subspace \( \mathcal{H}_1 \subset \mathcal{H} \) such that the restriction of \( U(P_1) \) to \( \mathcal{H}_1 \) is a representation of mass \( m > 0 \). We denote by \( P_1 \) the projection in \( \mathcal{H} \) onto \( \mathcal{H}_1 \).

To establish notation, let \( \mathcal{O} \) be a bounded spacetime region and let \( A \in \mathcal{A}(\mathcal{O}) \) be any operator such that \( P_1 A \mathcal{O} \neq 0 \). The existence of such localized (in brief, local) operators amounts to the assumption that the particle carries a localizable charge. That the particle is stable, i.e. completely decouples from the underlying continuum states, can be cast into a condition first stated by Herbst: For all sufficiently small \( \mu > 0 \)

\[
\| E_\mu (1 - P_1) A \mathcal{O} \| \leq c \mu^\eta,
\]

for some constants \( c, \eta > 0 \), where \( E_\mu \) is the projection onto the spectral subspace of the mass operator corresponding to spectrum in the interval \( (m - \mu, m + \mu) \). In the case originally considered by Haag and Ruelle, where \( m \) is isolated from the rest of the mass spectrum, this condition is certainly satisfied.

Setting \( A(x) = U(x)A U(x)^{-1} \), where \( U(x) \) is the unitary implementing the spacetime translation\(^3\) \( x = (x_0, \vec{x}) \), one puts, for \( t \neq 0 \),

\[
A_t(f) = \int d^3x \, g_t(x_0) f_{x_0}(\vec{x}) A(x).
\]

Here \( x_0 \mapsto g_t(x_0) = g((x_0 - t)/|t|^\kappa)/|t|^\kappa \) induces a time averaging about \( t \), \( g \) being any test function which integrates to 1 and whose Fourier transform has compact support, and \( 1/(1 + \eta) < \kappa < 1 \) with \( \eta \) as above. The Fourier transform of \( f_{x_0} \) is given by \( \hat{f}_{x_0}(\vec{p}) = \hat{f}(\vec{p}) e^{-i x_0 \omega(\vec{p})} \), where \( \hat{f} \) is some test function on \( \mathbb{R}^3 \) with \( \hat{f}(\vec{p}) \) having compact support, and \( \omega(\vec{p}) = (\vec{p}^2 + m^2)^{1/2} \). Note that \( (x_0, \vec{x}) \mapsto f_{x_0}(\vec{x}) \) is a solution of the Klein–Gordon equation of mass \( m \).

With these assumptions, it follows by a straightforward application of the harmonic analysis of unitary groups that in the sense of strong convergence \( A_t(f) \mathcal{O} \to P_1 A \mathcal{O} \) and \( A_t(f)^\ast \mathcal{O} \to 0 \) as \( t \to \pm \infty \), where \( A(f) = \int d^3x \, f(\vec{x}) A(0, \vec{x}) \). Hence, the operators \( A_t(f) \) may be thought of as creation operators and their adjoints as annihilation operators. These operators are the basic ingredients in the construction of scattering states. Choosing local operators \( A_k \) as above and test functions \( f^{(k)} \) with disjoint compact supports in Fourier space, \( k = 1, \ldots, n \), the scattering states are obtained as limits of the Haag–Ruelle approximants

\[
A_1(f^{(1)}) \cdots A_n(f^{(n)}) \mathcal{O}.
\]

Roughly speaking, the operators \( A_{kt}(f^{(k)}) \) are localized in spacelike separated regions at asymptotic times \( t \), due to the support properties of the Fourier transforms of the functions \( f^{(k)} \). Hence they commute asymptotically because of locality and, by the clustering properties of the vacuum state, the above vector becomes a product state of single particle states. In order to prove convergence one proceeds, in analogy to Cook’s method in

\(^3\)The velocity of light and Planck’s constant are put equal to 1 in what follows.
quantum mechanical scattering theory, to the time derivatives,

\[
\partial_t A_1(t)^{(1)} \cdots A_n(t)^{(n)} \Omega
\]

\[
= \sum_{k \neq l} A_{kl}((f^{(1)}) \cdots [\partial_k A_k(t)^{(k)}, A_l(t)^{(l)}] \cdots A_n(t)^{(n)} \Omega
\]

\[
+ \sum_k A_{1t}(f^{(1)}) \cdots \hat\lambda_k \cdots A_n(t)^{(n)} \partial_t A_k(t)^{(k)} \Omega
\]  

(6)

where \(\hat\lambda_k\) denotes omission of \(A_k(t)^{(k)}\). Employing techniques of Araki and Hepp, one can prove that the terms in the second line, involving commutators, decay rapidly in norm as \(t\) approaches infinity because of locality, as indicated above. By applying condition (3) and the fact that the vectors \(\partial_t A_k(t)^{(k)} \Omega\) do not have a component in the single particle space \(H_1\), the terms in the third line can be shown to decay in norm like \(|t|^{-\kappa(1+n)}\). Thus the norm of the vector \(6\) is integrable in \(t\), implying the existence of the strong limits

\[
\left( P_1 A_1(f^{(1)}) \Omega \otimes \cdots \otimes P_1 A_n(f^{(n)}) \Omega \right)_{\text{in/out}} = \lim_{t \to \mp \infty} A_1(t)^{(1)} \cdots A_n(t)^{(n)} \Omega
\]  

(7)

As indicated by the notation, these limits depend only on the single particle vectors \(P_1 A_k(f^{(k)}) \Omega \in H_1, k = 1, \ldots, n\), but not on the specific choice of operators and test functions. In order to establish their Fock structure, one employs results on clustering properties of vacuum correlation functions in theories without strictly positive minimal mass. Using this, one can compute inner products of arbitrary asymptotic states and verify that the maps

\[
\left( P_1 A_1(f^{(1)}) \Omega \otimes \cdots \otimes P_1 A_n(f^{(n)}) \Omega \right) \mapsto \left( P_1 A_1(f^{(1)}) \Omega \otimes \cdots \otimes P_1 A_n(f^{(n)}) \Omega \right)_{\text{in/out}}
\]  

(8)

extend by linearity to isomorphisms \(\Omega^{\text{in/out}}\) from the Fock space \(H_F\) onto the subspaces \(H^{\text{in/out}} \subset H\) generated by the collision states. Moreover, the asymptotic states transform under the Poincaré transformations \(U(P^+_\lambda)\) as

\[
U(\lambda) \left( P_1 A_1(f^{(1)}) \Omega \otimes \cdots \otimes P_1 A_n(f^{(n)}) \Omega \right)_{\text{in/out}}
\]

\[
= \left( U_1(\lambda) P_1 A_1(f^{(1)}) \Omega \otimes \cdots \otimes U_1(\lambda) P_1 A_n(f^{(n)}) \Omega \right)_{\text{in/out}}
\]  

(9)

Thus the isomorphisms \(\Omega^{\text{in/out}}\) intertwine the action of the Poincaré group on \(H_F\) and \(H^{\text{in/out}}\). We summarize these results, which are vital for the physical interpretation of the underlying theory, in the following theorem.

**Theorem 1** Consider a theory of a particle of mass \(m > 0\) which satisfies the standing assumptions and the stability condition (3). Then there exist canonical isometries \(\Omega^{\text{in/out}}\), mapping the Fock space \(H_F\) based on the single particle space \(H_1\) onto subspaces \(H^{\text{in/out}} \subset H\) of incoming and outgoing scattering states. Moreover, these isometries intertwine the action of the Poincaré transformations on the respective spaces.

Since the scattering states have been identified with Fock space, asymptotic creation and annihilation operators act on \(H^{\text{in/out}}\) in a natural manner. This point will be explained in the following section.
3 LSZ Formalism

Prior to the results of Haag and Ruelle, an axiomatic approach to scattering theory was developed by Lehmann, Symanzik and Zimmermann (LSZ), based on time–ordered vacuum expectation values of quantum fields. The relative advantage of their approach with respect to Haag–Ruelle theory is that useful reduction formulae for the $S$-matrix greatly facilitate computations, in particular in perturbation theory. Moreover these formulae are the starting point of general studies of the momentum space analyticity properties of the $S$-matrix, as outlined elsewhere in this encyclopedia. Within the present general setting, the LSZ method was established by Hepp.

For simplicity of discussion, we consider again a single particle type of mass $m > 0$ and integer spin $s$, subject to condition (3). According to the results of the preceding section, one then can define asymptotic creation and annihilation operators on the scattering states, setting

$$A(f)_{\text{in/out}} \left( P_1 A_1(f^{(1)}) \Omega \otimes \cdots \otimes P_1 A_n(f^{(n)}) \Omega \right)^{\text{in/out}} = A_t(f) A_1 t(f^{(1)}) \cdots A_n t(f^{(n)}) \Omega. \quad \text{(10)}$$

From the latter expression one can proceed to

$$= \lim_{t \to \mp \infty} A_t(f) \left( P_1 A_1(f^{(1)}) \Omega \otimes \cdots \otimes P_1 A_n(f^{(n)}) \Omega \right)^{\text{in/out}}, \quad \text{(11)}$$

due to the fact that $A_t(f)$ is uniformly bounded in $t$ on states of finite energy. In a similar manner one can show that

$$A(f)_{\text{in/out}}^* \left( P_1 A_1(f^{(1)}) \Omega \otimes \cdots \otimes P_1 A_n(f^{(n)}) \Omega \right)^{\text{in/out}} = \lim_{t \to \mp \infty} A_t(f)^* \left( P_1 A_1(f^{(1)}) \Omega \otimes \cdots \otimes P_1 A_n(f^{(n)}) \Omega \right)^{\text{in/out}} = 0, \quad \text{(12)}$$

provided the Fourier transforms of the functions $f, f^{(1)}, \ldots f^{(n)}$ have disjoint supports.

We mention as an aside that, by replacing the time averaging function $g$ in the definition of $f_t$ by a delta function, the above formulae still hold. But the convergence is then to be understood in the weak Hilbert space topology. In this form the above relations were anticipated by Lehmann, Symanzik and Zimmermann (asymptotic condition).

It is straightforward to proceed from these relations to reduction formulae. Let $B$ be any local operator. Then one has, in the sense of matrix elements between outgoing and incoming scattering states,

$$B A(f)_{\text{in}} - A(f)_{\text{out}} B = \lim_{t \to \infty} (B A(f_{-t}) - A(f_t) B)$$

$$= \lim_{t \to \infty} \left( \int d^4x f_{-t}(x) B A(x) - \int d^4x f_t(x) A(x) B \right). \quad \text{(13)}$$

Because of the (essential) support properties of the functions $f_{\pm t}$, the contributions to the latter integrals arise, for asymptotic $t$, from spacetime points $x$ where the localization regions of $A(x)$ and $B$ have a negative timelike (first term), respectively positive timelike
(second term) distance. One may therefore proceed from the products of these operators to the time–ordered products \( T(BA(x)) \), where \( T(BA(x)) = A(x)B \) if the localization region of \( A(x) \) lies in the future of that of \( B \), and \( T(BA(x)) = BA(x) \) if it lies in the past. It is noteworthy that a precise definition of the time ordering for finite \( x \) is irrelevant in the present context — any reasonable interpolation between the above relations will do. Similarly, one can define time–ordered products for an arbitrary number of local operators. The preceding limit can then be recast into

\[
= \lim_{t \to \infty} \int d^4x \left( f_{-t}(x) - f_t(x) \right) T(BA(x)).
\]  

(14)

The latter expression has a particularly simple form in momentum space. Proceeding to the Fourier transforms of \( f_{\pm t} \) and noticing that, in the limit of large \( t \),

\[
\left( \tilde{f}_{-t}(p) - \tilde{f}_t(p) \right) / (p_0 - \omega(\vec{p})) \longrightarrow -2\pi i \tilde{f}(\vec{p}) \delta(p_0 - \omega(\vec{p})),
\]

(15)

one gets

\[
BA(f)^{in} - A(f)^{out} B = -2\pi i \int d^3\vec{p} \tilde{f}(\vec{p}) \left( p_0 - \omega(\vec{p}) \right) T(B\tilde{A}(-p)) \bigg|_{p_0 = \omega(\vec{p})}.
\]

(16)

Here \( T(B\tilde{A}(p)) \) denotes the Fourier transform of \( T(BA(x)) \), and it can be shown that the restriction of \( (p_0 - \omega(\vec{p})) T(B\tilde{A}(-p)) \) to the manifold \( \{ p \in \mathbb{R}^4 : p_0 = \omega(\vec{p}) \} \) (the “mass shell”) is meaningful in the sense of distributions on \( \mathbb{R}^3 \). By the same token, one obtains

\[
A(f)^{out*} B - BA(f)^{in*} = -2\pi i \int d^3\vec{p} \tilde{f}(\vec{p}) \left( p_0 - \omega(\vec{p}) \right) T(\tilde{A}^*(p)B) \bigg|_{p_0 = \omega(\vec{p})}.
\]

(17)

Similar relations, involving an arbitrary number of asymptotic creation and annihilation operators, can be established by analogous considerations. Taking matrix elements of these relations in the vacuum state and recalling the action of the asymptotic creation and annihilation operators on scattering states, one arrives at the following result, which is central in all applications of scattering theory.

**Theorem 2** Consider the theory of a particle of mass \( m > 0 \) subject to the conditions stated in the preceding sections and let \( f^{(1)}, \ldots, f^{(n)} \) be any family of test functions whose Fourier transforms have compact and non–overlapping supports. Then

\[
\langle (P_1 A_1(f^{(1)})\Omega \otimes \cdots \otimes P_1 A_k(f^{(k)})\Omega)^{out} , (P_1 A_{k+1}(f^{(k+1)})\Omega \otimes \cdots \otimes P_1 A_n(f^{(n)})\Omega)^{in} \rangle = (-2\pi i)^n \int \cdots \int d^3p_1 \cdots d^3p_n \tilde{f}^{(1)}(\vec{p}_1) \cdots \tilde{f}^{(k)}(\vec{p}_k) \tilde{f}^{(k+1)}(\vec{p}_{k+1}) \cdots \tilde{f}^{(n)}(\vec{p}_n) \times
\]

\[
\times \prod_{i=1}^n \left( p_{i0} - \omega(\vec{p}_i) \right) \langle \Omega , T(\tilde{A}_1^*(p_1) \cdots \tilde{A}_k^*(p_k) \tilde{A}_{k+1}(p_{k+1}) \cdots \tilde{A}_n(-p_n)\Omega) \rangle \bigg|_{p_{j0} = \omega(\vec{p}_j)},
\]

(18)

in an obvious notation.

Thus the kernels of the scattering amplitudes in momentum space are obtained by restricting the (by the factor \( \prod_{i=1}^n (p_{i0} - \omega(\vec{p}_i)) \) amputated Fourier transforms of the vacuum expectation values of the time–ordered products to the positive and negative mass shells, respectively. These are the famous LSZ reduction formulae, which provide a convenient link between the time–ordered (Green’s) functions of a theory and its asymptotic particle interpretation.
4 Asymptotic Particle Counters

The preceding construction of scattering states applies to a significant class of theories; but even if one restricts attention to the case of massive particles, it does not cover all situations of physical interest. For an essential input in the construction is the existence of local operators interpolating between the vacuum and the single particle states. There may be no such operators at one’s disposal, however, either because the particle in question carries a non–localizable charge, or because the given family of operators is too small. The latter case appears, for example, in gauge theories, where in general only the observables are fixed by the principle of local gauge invariance, and the physical particle content as well as the corresponding interpolating operators are not known from the outset. As observables create from the vacuum only neutral states, the above construction of scattering states then fails if charged particles are present. Nevertheless, thinking in physical terms, one would expect that the observables contain all relevant information in order to determine the features of scattering states, in particular their collision cross section. That this is indeed the case was first shown by Araki and Haag [1].

In scattering experiments the measured data are provided by detectors (e.g. particle counters) and coincidence arrangements of detectors. Essential features of detectors are their lack of response in the vacuum state and their macroscopic localization. Hence, within the present mathematical setting, a general detector is represented by a positive operator \( C \) on the physical Hilbert space \( \mathcal{H} \) such that \( C \Omega = 0 \). Because of the Reeh–Schlieder Theorem, these conditions cannot be satisfied by local operators. However, they can be fulfilled by “almost local” operators. Examples of such operators are easy to produce, putting \( C = L^* L \) with

\[
L = \int d^4 x f(x) A(x),
\]

where \( A \) is any local operator and \( f \) any test function whose Fourier transform has compact support in the complement of the closed forward lightcone (and hence in the complement of the energy momentum spectrum of the theory). In view of the properties of \( f \) and the invariance of \( \Omega \) under translations, it follows that \( C = L^* L \) annihilates the vacuum and can be approximated with arbitrary precision by local operators. The algebra generated by these operators \( C \) will be denoted by \( \mathcal{C} \).

When preparing a scattering experiment, the first thing one must do with a detector is to calibrate it, i.e. test its response to sources of single particle states. Within the mathematical setting, this amounts to computing the matrix elements of \( C \) in states \( \Phi \in \mathcal{H}_1 \):

\[
\langle \Phi, C \Phi \rangle = \int \int d^3 p d^3 q \ \Phi(\vec{p}) \Phi(\vec{q}) \langle \vec{p} | C | \vec{q} \rangle.
\]

(20)

Here \( \vec{p} \mapsto \Phi(\vec{p}) \) is the momentum space wave function of \( \Phi \), \( \langle \cdot | C | \cdot \rangle \) is the kernel of \( C \) in the single particle space \( \mathcal{H}_1 \), and we have omitted (summations over) indices labelling internal degrees of freedom of the particle, if any. The relevant information about \( C \) is encoded in its kernel. As a matter of fact, one only needs to know its restriction to the diagonal, \( \vec{p} \mapsto \langle \vec{p} | C | \vec{p} \rangle \). It is called the sensitivity function of \( C \) and can be shown to be regular under quite general circumstances [1] [2].

Given a state \( \Psi \in \mathcal{H} \) for which the expectation value \( \langle \Psi, C(x) \Psi \rangle \) differs significantly from 0, one concludes that this state deviates from the vacuum in a region about \( x \). For finite \( x \), this does not mean, however, that \( \Psi \) has a particle interpretation at \( x \). For
that spacetime point may be just the location of a collision center, for example. Yet if one proceeds to asymptotic times, one expects, in view of the spreading of wave packets, that the probability of finding two or more particles in the same spacetime region is dominated by the single particle contributions. It is this physical insight which justifies the expectation that the detectors \( C(x) \) become particle counters at asymptotic times. Accordingly, one considers for asymptotic times \( t \) the operators

\[
C_t(h) = \int d^3x \ h(\vec{x}/t) C(t, \vec{x}),
\]

(21)

where \( h \) is any test function on \( \mathbb{R}^3 \). The role of the integral is to sum up all single particle contributions with velocities in the support of \( h \) in order to compensate for the decreasing probability of finding such particles at asymptotic times about the localization center of the detector. That these ideas are consistent was demonstrated by Araki and Haag, who established the following result [1].

**Theorem 3** Consider, as before, the theory of a massive particle. Let \( C^{(1)}, \ldots, C^{(n)} \in \mathcal{C} \) be any family of detector operators and let \( h^{(1)}, \ldots, h^{(n)} \) be any family of test functions on \( \mathbb{R}^3 \). Then, for any state \( \Psi_{\text{out}} \in \mathcal{H}_{\text{out}} \) of finite energy,

\[
\lim_{t \to \infty} \langle \Psi_{\text{out}}, C^{(1)}_t(h^{(1)}) \cdots C^{(n)}_t(h^{(n)}) \Psi_{\text{out}} \rangle = \int \cdots \int d^3p_1 \cdots d^3p_n \langle \Psi_{\text{out}}, \rho_{\text{out}}(\vec{p}_1) \cdots \rho_{\text{out}}(\vec{p}_n) \Psi_{\text{out}} \rangle \prod_{k=1}^n h(\vec{p}_k/\omega(\vec{p}_k)) \langle \vec{p}_k | C^{(k)} | \vec{p}_k \rangle,
\]

(22)

where \( \rho_{\text{out}}(\vec{p}) \) is the momentum space density (the product of creation and annihilation operators) of outgoing particles of momentum \( \vec{p} \), and (summations over) possible indices labelling internal degrees of freedom of the particle are omitted. An analogous relation holds for incoming scattering states at negative asymptotic times.

This result shows, first of all, that the scattering states have indeed the desired interpretation with regard to the observables, as anticipated in the preceding sections. Since the assertion holds for all scattering states of finite energy, one may replace in the above theorem the outgoing scattering states by any state of finite energy, if the theory is asymptotically complete, i.e. \( \mathcal{H} = \mathcal{H}_{\text{in}} = \mathcal{H}_{\text{out}} \). Then choosing, in particular, any incoming scattering state and making use of the arbitrariness of the test functions \( h^{(k)} \) as well as the knowledge of the sensitivity functions of the detector operators, one can compute the probability distributions of outgoing particle momenta in this state, and thereby the corresponding collision cross sections.

The question of how to construct certain specific incoming scattering states by using only local observables was not settled by Araki and Haag, however. A general method to that effect was outlined in [3]. As a matter of fact, for that method only the knowledge of states in the subspace of neutral states is required. Yet in this approach one would need for the computation of, say, elastic collision cross sections of charged particles the vacuum correlation functions involving at least eight local observables. This practical disadvantage of increased computational complexity of the method is offset by the conceptual advantage of making no appeal to quantities which are a priori nonobservable.
5 Massless particles and Huygens’ principle

The preceding general methods of scattering theory apply only to massive particles. Yet taking advantage of the salient fact that massless particles always move with the speed of light, Buchholz succeeded in establishing a scattering theory also for such particles [6]. Moreover, his arguments lead to a quantum version of Huygens’ principle.

As in the case of massive particles, one assumes that there is a subspace \( \mathcal{H}_1 \subset \mathcal{H} \) corresponding to a representation of \( U(P^+_{\pm}) \) of mass \( m = 0 \) and, for simplicity, integer helicity; moreover, there must exist local operators interpolating between the vacuum and the single particle states. These assumptions cover, in particular, the important examples of the photon and of Goldstone particles. Picking any suitable local operator \( A \) interpolating between \( \Omega \) and some vector in \( \mathcal{H}_1 \), one sets, in analogy to (11),

\[
A_t = \int d^4x \ g_t(x_0) \left(-1/2\pi\right) \varepsilon(x_0) \delta(x_0^2 - \bar{x}^2) \partial_0 A(x). \tag{23}
\]

Here \( g_t(x_0) = (1/|\ln t|) g((x_0 - t)/|\ln t|) \) with \( g \) as in (11), and the solution of the Klein–Gordon equation in (4) has been replaced by the fundamental solution of the wave equation; furthermore, \( \partial_0 A(x) \) denotes the derivative of \( A(x) \) with respect to \( x_0 \). Then, once again, the strong limit of \( A_t \Omega \) as \( t \to \pm \infty \) is \( P_1 A \Omega \), with \( P_1 \) the projection onto \( \mathcal{H}_1 \).

In order to establish the convergence of \( A_t \) as in the LSZ approach, one now uses the fact that these operators are, at asymptotic times \( t \), localized in the complement of some forward, respectively backward, light cone. Because of locality, they therefore commute with all operators which are localized in the interior of the respective cones. More specifically, let \( \mathcal{O} \subset \mathbb{R}^4 \) be the localization region of \( A \) and let \( \mathcal{O}_\pm \subset \mathbb{R}^4 \) be the two regions having a positive, respectively negative, timelike distance from all points in \( \mathcal{O} \). Then, for any operator \( B \) which is compactly localized in \( \mathcal{O}_\pm \), respectively, one obtains

\[
\lim_{t \to \pm \infty} A_t B \Omega = \lim_{t \to \pm \infty} B A_t \Omega = B P_1 A \Omega. \tag{24}
\]

This relation establishes the existence of the limits

\[
A^{\text{in/out}} = \lim_{t \to \mp \infty} A_t
\]

on the (by the Reeh–Schlieder property) dense sets of vectors \( \{ B \Omega : B \in \mathcal{A}(\mathcal{O}_\mp) \} \subset \mathcal{H} \). It requires some more detailed analysis to prove that the limits have all of the properties of a (smeared) free massless field, whose translates \( x \mapsto A^{\text{in/out}}(x) \) satisfy the wave equation and have \( c \)-number commutation relations. From these free fields one can then proceed to asymptotic creation and annihilation operators and construct asymptotic Fock spaces \( \mathcal{H}^{\text{in/out}} \subset \mathcal{H} \) of massless particles and a corresponding scattering matrix as in the massive case. The details of this construction can be found in the original article, cf. [2].

It also follows from these arguments that the asymptotic fields \( A^{\text{in/out}} \) of massless particles emanating from a region \( \mathcal{O} \), i.e. for which the underlying interpolating operators \( A \) are localized in \( \mathcal{O} \), commute with all operators localized in \( \mathcal{O}_\mp \), respectively. This result may be understood as an expression of Huygens’ principle. More precisely, denoting by \( \mathcal{A}^{\text{in/out}}(\mathcal{O}) \) the algebras of bounded operators generated by the asymptotic fields \( A^{\text{in/out}} \), respectively, one arrives at the following quantum version of Huygens’ principle.

**Theorem 4** Consider a theory of massless particles as described above and let \( \mathcal{A}^{\text{in/out}}(\mathcal{O}) \) be the algebras generated by massless asymptotic fields \( A^{\text{in/out}} \) with \( A \in \mathcal{A}(\mathcal{O}) \). Then

\[
\mathcal{A}^{\text{in}}(\mathcal{O}) \subset \mathcal{A}(\mathcal{O}_-)’ \quad \text{and} \quad \mathcal{A}^{\text{out}}(\mathcal{O}) \subset \mathcal{A}(\mathcal{O}_+’)’. \tag{25}
\]

Here the prime denotes the set of bounded operators commuting with all elements of the respective algebras (i.e. their commutants).
6 Beyond Wigner’s Concept of Particle

There is by now ample evidence that Wigner’s concept of particle is too narrow in order to cover all particle–like structures appearing in quantum field theory. Examples are the partons which show up in non–abelian gauge theories at very small spacetime scales as constituents of hadrons, but which do not appear at large scales due to the confining forces. Their mathematical description requires a quite different treatment, which cannot be discussed here. But even at large scales, Wigner’s concept does not cover all stable particle–like systems, the most prominent examples being particles carrying an abelian gauge charge, such as the electron and the proton, which are inevitably accompanied by infinite clouds of (“on shell”) massless particles.

The latter problem was discussed first by Schroer, who coined the term infraparticle for such systems. Later, Buchholz showed in full generality that, as a consequence of Gauss’ law, pure states with an abelian gauge charge can neither have a sharp mass nor carry a unitary representation of the Lorentz group, thereby uncovering the simple origin of results found by explicit computations, notably in quantum electrodynamics \[9\]. Thus one is faced with the question of an appropriate mathematical characterization of infraparticles which generalizes the concept of particle invented by Wigner. Some significant steps in this direction were taken by Fröhlich, Morchio and Strocchi, who based a definition of infraparticles on a detailed spectral analysis of the energy–momentum operators. For an account of these developments and further references, cf. \[6\].

We outline here an approach, originated by Buchholz, which covers all stable particle–like structures appearing in quantum field theory at asymptotic times. It is based on Dirac’s idea of improper particle states with sharp energy and momentum. In the standard (rigged Hilbert space) approach to giving mathematical meaning to these quantities one regards them as vector–valued distributions, whereby one tacitly assumes that the improper states can coherently be superimposed so as to yield normalizable states. This assumption is valid in the case of Wigner particles but fails in the case of infraparticles. A more adequate method of converting the improper states into normalizable ones is based on the idea of acting on them with suitable localizing operators. In the case of quantum mechanics, one could take as a localizing operator any sufficiently rapidly decreasing function of the position operator. It would map the improper “plane wave states” of sharp momentum into finitely localized states which thereby become normalizable. In quantum mechanics, these two approaches can be shown to be mathematically equivalent. The situation is different, however, in quantum field theory.

In quantum field theory, the appropriate localizing operators \(L\) are of the form \(19\). They constitute a (non–closed) left ideal \(\mathcal{L}\) in the \(C^*\)-algebra \(\mathcal{A}\) generated by all local operators. Improper particle states of sharp energy–momentum \(p\) can then be defined as linear maps \(|\cdot\rangle_p : \mathcal{L} \to \mathcal{H}\) satisfying\(^4\)

\[
U(x) |L\rangle_p = e^{ipx} |L(x)\rangle_p, \quad L \in \mathcal{L}.
\]

In theories of massive particles one can always find localizing operators \(L \in \mathcal{L}\) such that their images \(|L\rangle_p \in \mathcal{H}\) are states with a sharp mass. This is the situation covered in Wigner’s approach. In theories with long range forces there are, in general, no such operators, however, since the process of localization inevitably leads to the production of

\(^4\)It is instructive to (formally) replace here \(L\) by the identity operator, making it clear that this relation indeed defines improper states of sharp energy–momentum
low energy massless particles. Yet improper states of sharp momentum still exist in this situation, thereby leading to a meaningful generalization of Wigner’s particle concept.

That this characterization of particles covers all situations of physical interest can be justified in the general setting of relativistic quantum field theory as follows. Picking $g_t$ as in (4) and any vector $\Psi \in \mathcal{H}$ with finite energy, one can show that the functionals $\rho_t$, $t \in \mathbb{R}$, given by
\[
\rho_t(L^*L) = \int d^4x g_t(x_0) \langle \Psi, (L^*L)(x)\Psi \rangle, \quad L \in \mathcal{L},
\]
are well defined and form an equicontinuous family with respect to a certain natural locally convex topology on the algebra $\mathcal{C} = \mathcal{L}^*\mathcal{L}$. This family of functionals therefore has, as $t \to \pm \infty$, weak-*–limit points, denoted by $\sigma$. The functionals $\sigma$ are positive on $\mathcal{C}$ but not normalizable. (Technically speaking, they are weights on the underlying algebra $\mathcal{A}$.) Any such $\sigma$ induces a positive semi–definite scalar product on the left ideal $\mathcal{L}$ given by
\[
\langle L_1 | L_2 \rangle = \sigma(L_1^*L_2), \quad L_1, L_2 \in \mathcal{L}.
\]
After quotienting out elements of zero norm and taking the completion, one obtains a Hilbert space and a linear map $|\cdot\rangle$ from $\mathcal{L}$ into that space. Moreover, the spacetime translations act on this space by a unitary representation satisfying the relativistic spectrum condition.

It is instructive to compute these functionals and maps in theories of massive particles. Making use of relation (22) in Section II one obtains, with a slight change of notation,
\[
\langle L_1 | L_2 \rangle = \int d\mu(p) \langle p | L_1^*L_2|p \rangle,
\]
where $\mu$ is a measure giving the probability density of finding at asymptotic times in state $\Psi$ a particle of energy–momentum $p$. Once again, possible summations over different particle types and internal degrees of freedom have been omitted here. Thus, setting $|L\rangle_p = L |p\rangle$, one concludes that the maps $|\cdot\rangle$ can be decomposed into a direct integral $|\cdot\rangle = \int \oplus d\mu(p)^{1/2} |\cdot\rangle_p$ of improper particle states of sharp energy–momentum. It is crucial that this result can also be established without any a priori input about the nature of the particle content of the theory, thereby providing evidence of the universal nature of the concept of improper particle states of sharp momentum.

**Theorem 5** Consider a relativistic quantum theory satisfying the standing assumptions. Then the maps $|\cdot\rangle$ defined above can be decomposed into improper particle states of sharp energy–momentum $p$,
\[
|\cdot\rangle = \int \oplus d\mu(p)^{1/2} |\cdot\rangle_p,
\]
where $\mu$ is some measure depending on the state $\Psi$ and the respective time limit taken.

It is noteworthy that whenever the space of improper particle states corresponding to fixed energy-momentum $p$ is finite dimensional (finite particle multiplets), then in the corresponding Hilbert space there exists a continuous unitary representation of the little group of $p$. This implies that improper momentum eigenstates of mass $m = (p^2)^{1/2} > 0$ carry definite (half)integer spin, in accordance with Wigner’s classification. However, if $m = 0$, the helicity need not be quantized, in contrast to Wigner’s results.
Though a general scattering theory based on improper particle states has not yet been developed, some progress has been made in \[3\]. There it is outlined how inclusive collision cross sections of scattering states, where an undetermined number of low energy massless particles remains unobserved, can be defined in the presence of long range forces, in spite of the fact that a meaningful scattering matrix may not exist.

7 Asymptotic completeness

Whereas the description of the asymptotic particle features of any relativistic quantum field theory can be based on an arsenal of powerful methods, the question of when such a theory has a complete particle interpretation remains open to date. Even in concrete models there exist only partial results, cf. \[7\] for a comprehensive review of the current state of the art. This situation is in striking contrast to the case of quantum mechanics, where the problem of asymptotic completeness has been completely settled.

One may trace the difficulties in quantum field theory back to the possible formation of superselection sectors \[6\] and the resulting complex particle structures, which cannot appear in quantum mechanical systems with a finite number of degrees of freedom. Thus the first step in establishing a complete particle interpretation in a quantum field theory has to be the determination of its full particle content. Here the methods outlined in the preceding section provide a systematic tool. From the resulting data one must then reconstruct the full physical Hilbert space of the theory comprising all superselection sectors. For theories in which only massive particles appear, such a construction has been established in \[2\], and it has been shown that the resulting Hilbert space contains all scattering states. The question of completeness can then be recast into the familiar problem of the unitarity of the scattering matrix. It is believed that phase space (nuclearity) properties of the theory are of relevance here \[6\].

However, in theories with long range forces, where a meaningful scattering matrix may not exist, this strategy is bound to fail. Nonetheless, as in most high energy scattering experiments, only some very specific aspects of the particle interpretation are really tested — one may think of other meaningful formulations of completeness. The interpretation of most scattering experiments relies on the existence of conservation laws, such as those for energy and momentum. If a state has a complete particle interpretation, it ought to be possible to fully recover its energy, say, from its asymptotic particle content, \textit{i.e.} there should be no contributions to its total energy which do not manifest themselves asymptotically in the form of particles. Now the mean energy–momentum of a state \(\Psi \in \mathcal{H}\) is given by \(\langle \Psi, P\Psi \rangle\), \(P\) being the energy–momentum operators, and the mean energy–momentum contained in its asymptotic particle content is \(\int d\mu(p) p\), where \(\mu\) is the measure appearing in the decomposition (30). Hence, in case of a complete particle interpretation the following should hold:

\[
\langle \Psi, P\Psi \rangle = \int d\mu(p) p.
\]

(31)

Similar relations should also hold for other conserved quantities which can be attributed to particles, such as charge, spin \textit{etc.} It seems that such a weak condition of asymptotic completeness suffices for a consistent interpretation of most scattering experiments. One may conjecture that relation (31) and its generalizations hold in all theories admitting a local stress energy tensor and local currents corresponding to the charges.

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