Incoherent dictionaries and the statistical restricted isometry property

Shamgar Gurevich and Ronny Hadani

Abstract—In this article we present a statistical version of the Candès-Tao restricted isometry property (SRIP for short) which holds in general for any incoherent dictionary which is a disjoint union of orthonormal bases. In addition, under appropriate normalization, the eigenvalues of the associated Gram matrix fluctuate around \( \lambda = 1 \) according to the Wigner semicircle distribution. The result is then applied to various dictionaries that arise naturally in the setting of finite harmonic analysis, giving, in particular, a better understanding on a remark of Applebaum-Howard-Searle-Calderbank concerning RIP for the Heisenberg dictionary of chirp-like functions.

Index Terms—Incoherent dictionaries, Statistical RIP, Wigner semicircle distribution, deterministic examples, Heisenberg-Weil representation.

I. INTRODUCTION

Digital signals, or simply signals, can be thought of as complex valued functions on the finite field \( \mathbb{F}_p \), where \( p \) is a prime number. The space of signals \( \mathcal{H} = \mathbb{C}(\mathbb{F}_p) \) is a Hilbert space of dimension \( p \), with the inner product given by the standard formula

\[
(f, g) = \sum_{t \in \mathbb{F}_p} f(t) \overline{g(t)}.
\]

A dictionary \( \mathcal{D} \) is simply a set of vectors (also called atoms) in \( \mathcal{H} \). The number of vectors in \( \mathcal{D} \) can exceed the dimension of the Hilbert space \( \mathcal{H} \), in fact, the most interesting situation is when \( |\mathcal{D}| \gg p = \dim \mathcal{H} \). In this set-up we define a resolution of the Hilbert space \( \mathcal{H} \) via \( \mathcal{D} \), which is the morphism of vector spaces

\[
\Theta : \mathbb{C}(\mathcal{D}) \rightarrow \mathcal{H},
\]

given by \( \Theta(f) = \sum_{\varphi \in \mathcal{D}} f(\varphi) \varphi \), for every \( f \in \mathbb{C}(\mathcal{D}) \). A more concrete way to think of the morphism \( \Theta \) is as a \( p \times |\mathcal{D}| \) matrix with the columns being the atoms in \( \mathcal{D} \).

In the last two decades [11], and in particular in recent years [3], [4], [5], [6], [7], [8], resolutions of Hilbert spaces became an important tool in signal processing, in particular in the emerging theories of sparsity and compressive sensing.

II. THE RESTRICTED ISOMETRY PROPERTY

A useful property of a resolution is the restricted isometry property (RIP for short) defined by Candès-Tao in [7]. Fix a natural number \( n \in \mathbb{N} \) and a pair of positive real numbers \( \delta_1, \delta_2 \in \mathbb{R}_{>0} \).

Definition II.-1: A dictionary \( \mathcal{D} \) satisfies the restricted isometry property with coefficients \( (\delta_1, \delta_2, n) \) if for every subset \( S \subset \mathcal{D} \) such that \( |S| \leq n \) we have

\[
(1 - \delta_2) \|f\| \leq \|\Theta(f)\| \leq (1 + \delta_1) \|f\|,
\]

for every function \( f \in \mathbb{C}(\mathcal{D}) \) which is supported on the set \( S \).

Equivalently, RIP can be formulated in terms of the spectral radius of the corresponding Gram operator. Let \( G(S) \) denote the composition \( \Theta_S^* \Theta_S \) with \( \Theta_S \) denoting the restriction of \( \Theta \) to the subspace \( \mathcal{C}_S(\mathcal{D}) \subset \mathbb{C}(\mathcal{D}) \) of functions supported on the set \( S \). The dictionary \( \mathcal{D} \) satisfies \((\delta_1, \delta_2, n)\)-RIP if for every subset \( S \subset \mathcal{D} \) such that \( |S| \leq n \) we have

\[
\delta_2 \leq \|G(S) - I_{DS}\| \leq \delta_1,
\]

where \( I_{DS} \) is the identity operator on \( \mathcal{C}_S(\mathcal{D}) \).

It is known [2], [8] that the RIP holds for random dictionaries. However, one would like to address the following problem

1. [10], [9], [20], [21], [22], [23], [25], [24], [26], [27]:

   Problem II.-2: Find deterministic construction of a dictionary \( \mathcal{D} \) with \( |\mathcal{D}| \gg p \) which satisfies RIP with coefficients in the critical regime

\[
\delta_1, \delta_2 \ll 1 \quad \text{and} \quad n = \alpha \cdot p,
\]

for some constant \( 0 < \alpha < 1 \).

III. INCOHERENT DICTIONARIES

Fix a positive real number \( \mu \in \mathbb{R}_{>0} \). The following notion was introduced in [9], [12] and was used to study similar problems in [26], [27]:

Definition III.-3: A dictionary \( \mathcal{D} \) is called incoherent with coherence coefficient \( \mu \) (also called \( \mu \)-coherent) if for every pair of distinct atoms \( \varphi, \phi \in \mathcal{D} \)

\[
|\langle \varphi, \phi \rangle| \leq \frac{\mu}{\sqrt{p}}.
\]

In this article we will explore a general relation between RIP and incoherence. Our motivation comes from three examples of incoherent dictionaries which arise naturally in the setting of finite harmonic analysis:

- The first example [18], [19], referred to as the Heisenberg dictionary \( \mathcal{D}_H \), is constructed using the Heisenberg representation of the finite Heisenberg group \( H(\mathbb{F}_p) \). The Heisenberg dictionary is of size approximately \( p^2 \) and its coherence coefficient is \( \mu = 1 \).
- The second example [15], [16], [17], which is referred to as the oscillator dictionary \( \mathcal{D}_O \), is constructed using the Weil representation of the finite symplectic group.

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The oscillator dictionary is of size approximately $p^5$ and its coherence coefficient is $\mu = 4$.

- The third example [15], [16], [17], referred to as the extended oscillator dictionary $D_{EO}$, is constructed using the Heisenberg-Weil representation [28], [13] of the finite Jacobi group, i.e., the semi-direct product $J(\mathbb{F}_p) = SL_2(\mathbb{F}_p) \ltimes H(\mathbb{F}_p)$. The extended oscillator dictionary is of size approximately $p^5$ and its coherence coefficient is $\mu = 4$.

The three examples of dictionaries we just described constitute reasonable candidates for solving Problem [17-2]. They are large in the sense that $|\mathcal{D}| \gg p$, and empirical evidences suggest (see [1] for the case of $\mathcal{D}_H$) that they might satisfy RIP with coefficients in the critical regime [17-1]. We summarize this as follows:

**Problem III.4:** Do the dictionaries $\mathcal{D}_H$, $\mathcal{D}_O$ and $\mathcal{D}_{EO}$ satisfy the RIP with coefficients $\delta_1, \delta_2 \ll 1$ and $n = \alpha \cdot p$, for some $0 < \alpha < 1$?

### IV. MAIN RESULTS

In this article we formulate a relaxed statistical version of RIP, called statistical isometry property (SRIP for short) which holds for any incoherent dictionary $\mathcal{D}$ which is, in addition, a disjoint union of orthonormal bases:

$$\mathcal{D} = \bigcup_{x \in \mathcal{X}} B_x, \quad (IV-.2)$$

where $B_x = \{b_{x1}, \ldots, b_{xp}\}$ is an orthonormal basis of $\mathcal{H}$, for every $x \in \mathcal{X}$.

#### A. The statistical restricted isometry property

Let $\mathcal{D}$ be an incoherent dictionary of the form [IV-.2]. Roughly, the statement is that for $S \subset \mathcal{D}$, $|S| = n$ with $n = p^{1-\varepsilon}$, for $0 < \varepsilon < 1$, chosen uniformly at random, the operator norm $\|G(S) - I_{D_S}\|$ is small with high probability. Precisely, we have

**Theorem IV-A.1 (SRIP property [14]):** For every $k \in \mathbb{N}$, there exists a constant $C(k)$ such that the probability

$$\Pr\left(\|G(S) - I_{D_S}\| \geq p^{-\varepsilon/2}\right) \leq C(k) p^{-\varepsilon}\cdot. \quad (IV-A.1)$$

The above theorem, in particular, implies that probability $\Pr\left(\|G(S) - I_{D_S}\| \geq p^{-\varepsilon/2}\right) \to 0$ as $p \to \infty$ faster then $p^{-l}$ for any $l \in \mathbb{N}$.

#### B. The statistics of the eigenvalues

A natural thing to know is how the eigenvalues of the Gram operator $G(S)$ fluctuate around 1. In this regard, we study the statistical properties of the normalized error term

$$E(S) = (p/n)^{1/2} \left( G(S) - I_{D_S}\right).$$

Let $\rho_{E}(S) = n^{-1} \sum_{i=1}^{n} \lambda_i$ denote the spectral distribution of $E(S)$ where $\lambda_i$, $i = 1, \ldots, n$, are the real eigenvalues of the Hermitian operator $E(S)$. The following theorem asserts that $\rho_{E}$ converges in probability as $p \to \infty$ to the Wigner semicircle distribution $\rho_{SC}$.

**Theorem IV-B.1 (Semicircle distribution [14]):** We have

$$\lim_{p \to \infty} \rho_{E}(S) \to \rho_{SC}. \quad (IV-B.1)$$

**Remark IV-B.2:** A limit of the form (IV-B.1) is familiar in random matrix theory as the asymptotic of the spectral distribution of Wigner matrices. Interestingly, the same asymptotic distribution appears in our situation, albeit, the probability spaces are of a different nature (our probability spaces are, in particular, much smaller).

In particular, Theorems IV-A.1, IV-B.1 can be applied to the three examples $\mathcal{D}_H$, $\mathcal{D}_O$ and $\mathcal{D}_{EO}$, which are all of the appropriate form [IV-.2]. Finally, our result gives new information on a remark of Applebaum-Howard-Searle-Calderbank [1] concerning RIP of the Heisenberg dictionary.

**Remark IV-B.3:** For practical applications, it might be important to compute explicitly the constants $C(k)$ which appears in (IV-A.1). This constant depends on the incoherence coefficient $\mu$, therefore, for a fixed $p$, having $\mu$ as small as possible is preferable.

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