Generalized Thomas hyperplane sections 
and relations between vanishing cycles

Morihiko Saito

Abstract. As is remarked by B. Totaro, R. Thomas essentially proved that the Hodge conjecture is inductively equivalent to the existence of a hyperplane section, called a generalized Thomas hyperplane section, such that the restriction to it of a given primitive Hodge class does not vanish. We study the relations between the vanishing cycles in the cohomology of a general fiber, and show that each relation between the vanishing cycles of type (0,0) with unipotent monodromy around a singular hyperplane section defines a primitive Hodge class such that this singular hyperplane section is a generalized Thomas hyperplane section if and only if the pairing between a given primitive Hodge class and some of the constructed primitive Hodge classes does not vanish. This is a generalization of a construction by P. Griffiths.

Introduction

Let $X$ be a smooth complex projective variety of dimension $2n$, and $\mathcal{L}$ be an ample line bundle on $X$. Let $k$ be a positive integer such that $\mathcal{L}^k$ is very ample. Let $S = |\mathcal{L}^k|$, and $\mathcal{X}$ be the universal family $\coprod_{s \in S} X_s$ over $S$ with the discriminant $D$. We assume that the vanishing cycle at a general point of $D$ does not vanish as in [8], XVIII, Cor. 6.4 (replacing $k$ if necessary). As is remarked by B. Totaro (see the last remark in §3 of [29]), R. Thomas essentially proved that the Hodge conjecture is inductively equivalent to the existence of a point $0$ of $D$ such that the restriction $\zeta|_{X_0}$ of a given primitive Hodge class $\zeta$ on $X$ does not vanish (replacing $k$ if necessary). Here $X_0$ is called a generalized Thomas hyperplane section. One may assume further that $X_0$ has only ordinary double points (see loc. cit.), and $X_0$ is called a Thomas hyperplane section in this case. Note that a generalized Thomas hyperplane section is a special kind of hyperplane section (e.g. it must be reducible if $n = 1$). It has been observed that an explicit construction of a generalized Thomas hyperplane section for a given primitive Hodge class is rather difficult.

M. Green and P. Griffiths [11] have introduced a notion of singularities of a normal function. This is the cohomology class of a normal function. They showed that non-vanishing of the singularity at $0 \in D$ of the normal function $\nu$ associated to $\zeta$ is equivalent to that $X_0$ is a Thomas hyperplane section associated to $\zeta$, see also [3]. Note that the value $\nu_s$ of the normal function at $s \in S^* := S \setminus D$ can be viewed as the restriction of $\zeta$ to $X_s$ in the derived category of mixed Hodge structures using [4]. (This is related to the ‘restriction’ of the Leray spectral sequence to a fiber in [21], (0.6), see also Remark (1.2)(i) below.)
Their result shows that the necessary information is not lost by using this ‘restriction’ even after restricting to a sufficiently small neighborhood in $S$ of $0 \in D$ in the classical topology. It implies for example that a Thomas hyperplane section must have at least two ordinary double points since the cohomology class of the associated normal function in the one-variable case is always torsion, see e.g. [22], 2.5.4. More generally, for a special fiber to be a generalized Thomas hyperplane section, there must be some relation between the vanishing cycles in the cohomology of a general fiber as is shown below.

Let $0 \in D$. To compare the cohomology of $X_0$ with that of $X_s$ for $s \in S^* := S \setminus D$, we choose a germ of an irreducible analytic curve on $S$ whose intersection with $D$ consists of 0. Let $C$ be the normalization of the curve. We assume that $C$ is an open disk. Let $f : Y \to C$ be the base change of $X \to S$ by $C \to S$. Let $t$ be a local coordinate of $C$ around 0. We first assume that $Y_0 (= X_0)$ has only isolated singularities to simplify the exposition. Then we have the following (see also [26], [27]):

**Proposition 1.** If $\text{Sing} \ Y_0$ is isolated, there is an exact sequence of mixed Hodge structures

\[ H^{2n-1}(Y_\infty) \xrightarrow{\text{can}} \bigoplus_{y \in \text{Sing} \ Y_0} H^{2n-1}(Z_{y, \infty}) \to H^{2n}(Y_0) \xrightarrow{\text{sp}_{2n}} H^{2n}(Y_\infty), \]

where $H^j(Z_{y, \infty}, \mathbb{Q})$ denotes the vanishing cohomology at $y \in \text{Sing} \ Y_0$, and $H^j(Y_\infty)$ is the cohomology of a general fiber of $f$ endowed with the limit mixed Hodge structure at $0 \in C$.

Taking the dual of (1.1), we have the dual exact sequence

\[ H_{2n-1}(Y_\infty) \xleftarrow{\text{can}} \bigoplus_{y \in \text{Sing} \ Y_0} H_{2n-1}(Z_{y, \infty}) \leftarrow H_{2n}(Y_0) \xleftarrow{\text{sp}_{2n}} H_{2n}(Y_\infty). \]

Set

\[
\begin{align*}
E(Y_0) &= \text{Ker} \big( \text{sp}_{2n} : H^{2n}(Y_0, \mathbb{Q}(n)) \to H^{2n}(Y_\infty, \mathbb{Q}(n)) \big), \\
R(Y_0) &= \text{Ker} \big( \text{can}^\vee : \bigoplus_{y \in \text{Sing} \ Y_0} H_{2n-1}(Z_{y, \infty}, \mathbb{Q}(n)) \to H_{2n-1}(Y_\infty, \mathbb{Q}(n)) \big),
\end{align*}
\]

where $H_{2n-1}(Y_\infty, \mathbb{Q}(n)) = H^{2n-1}(Y_\infty, \mathbb{Q}(n))^\vee$ and similarly for $H_{2n-1}(Z_{y, \infty}, \mathbb{Q}(n))$. Let

\[
E^\vee(Y_0) := E(Y_0)^\vee = \text{Coker} \big( \text{sp}_{2n} : H_{2n}(Y_\infty, \mathbb{Q}(n)) \to H_{2n}(Y_0, \mathbb{Q}(n)) \big),
\]

where $^\vee$ denotes the dual. By [3] there is a canonical isomorphism

\[ E(Y_0) = \mathcal{H}^1(j_!* \mathbf{H}_\mathbb{Q}), \]

where $\mathbf{H}$ is a variation of Hodge structure on $S^* := S \setminus D$ defined by $H^{2n-1}(X_s)(n)$ for $s \in S^*$, and $j_*$ is the intermediate direct image by the inclusion $j : S^* \to S$, see [1]. We denote the unipotent monodromy part of $R(Y_0)$ by $R(Y_0)_1$. For $H = E(Y_0), E^\vee(Y_0), R(Y_0)_1$, set

\[ H^{(0, 0)} := \text{Hom}_{\text{HMS}}(\mathbb{Q}, G_{\mu_0}^W H) (\supset \text{Hom}_{\text{MHS}}(\mathbb{Q}, H)). \]

This is compatible with the dual. We say that $E^\vee(Y_0)^{(0, 0)}$ (resp. $E(Y_0)^{(0, 0)}$) is the space of extra Hodge cycles (resp. cocycles) on $Y_0$. An element of $R(Y_0)_1^{(0, 0)}$ is called a global
relation between the local vanishing cycles of type \((0,0)\) with unipotent monodromy around \(Y_0\). (In the non-isolated singularity case, we will omit ‘local’ and ‘global’.)

**Theorem 1.** (i) The restriction of a primitive Hodge class \(\zeta\) to \(Y_0\) defines an extra Hodge cocycle on \(Y_0\), i.e. an element of \(E(Y_0)^{(0,0)}\). The latter space is canonically isomorphic to the dual of \(R(Y_0)^{(0,0)}\), i.e. there is a canonical isomorphism

\[
(0.4) \quad R(Y_0)^{(0,0)} = E^\vee(Y_0)^{(0,0)}.
\]

(ii) If \(\gamma_\beta\) denotes the image of \(\beta \in R(Y_0)^{(0,0)}\) in \(H_{2n}(X, \mathbb{Q}(n))^{\text{prim}}\) by the composition of (0.4) with the canonical morphism

\[
(0.5) \quad E^\vee(Y_0) \to H_{2n}(X, \mathbb{Q}(n))^{\text{prim}},
\]

then \(Y_0\) is a generalized Thomas hyperplane section for a primitive Hodge class \(\zeta\) if and only if \(\langle \zeta, \gamma_\beta \rangle \neq 0\) for some \(\beta \in R(Y_0)^{(0,0)}\).

This is closely related to recent work of M. Green and P. Griffiths [11]. We are informed that the construction of \(\gamma_\beta\) was found by P. Griffiths ([12], p. 129) in the ordinary double point case, and the Hodge property of \(\gamma_\beta\) has been considered by H. Clemens (unpublished).

In the general case, using the vanishing cycle functor \(\varphi\) in [8], XIII and XIV, we have

**Theorem 2.** Proposition 1 and Theorem 1 hold without assuming \(\text{Sing} Y_0\) is isolated if we replace respectively

\[
\bigoplus_{y \in \text{Sing} Y_0} H^{2n-1}(Z_y, \mathbb{Q}(n)) \quad \text{and} \quad \bigoplus_{y \in \text{Sing} Y_0} H_{2n-1}(Z_y, \mathbb{Q}(n))
\]

by

\[
H^{2n-1}(Y_0, \varphi_{f^*t}Q_Y(n)) \quad \text{and} \quad H^{2n-1}(Y_0, \varphi_{f^*t}Q_Y(n))^\vee.
\]

By (0.3), the dimension \(r(Y_0)\) of \(R(Y_0)^{(0,0)}\) or \(E(Y_0)^{(0,0)}\) is independent of \(C\). So we may assume \(C\) smooth for the calculation of \(R(Y_0)^{(0,0)}\) and \(E(Y_0)^{(0,0)}\), see Remark (2.8)(i). As a corollary of Theorem 1, \(Y_0\) cannot be a generalized Thomas hyperplane section if \(r(Y_0) = 0\). In the ordinary double point case, the relations are all of type \((0,0)\) with unipotent monodromy, see Theorem 3 below. In the isolated singularity case we have a rather explicit construction of \(\gamma_\beta\) (which is essentially the same as Griffiths construction in [12], p. 129), see (2.5) below. The rank of \(\gamma_\beta\) may depend on the position of the singularities, see Thm. (4.5) in [9], p. 208 and also [10], (3.5).

In the isolated singularity case we have moreover

**Proposition 2.** If the singularities of \(Y_0\) are isolated, then these are isolated complete intersection singularities, \(\tilde{H}^j(Z_y, \infty) = 0\) for \(j \neq 2n - 1\), and \(\tilde{H}^{2n-1}(Z_y, \infty)\) is independent of \(C\) except for the monodromy.

In the ordinary double point case we show

**Proposition 3.** With the notation of Theorem 1, assume the singularities of \(Y_0\) are ordinary double points. Then the singularities of the total space \(Y\) are of type \(A_k\).
Using this, we get the following

**Theorem 3.** With the notation and the assumption of Proposition 3, the constant sheaf on $Y$ is the intersection complex up to a shift, i.e. $Y$ is a rational homology manifold. Moreover, the vanishing cohomology at each singular point of $Y_0$ is $\mathbb{Q}(-n)$ as a mixed Hodge structure, and has a unipotent monodromy.

Combined with [19], Lemma 5.1.4, the first assertion of Theorem 3 implies

**Corollary 1.** With the notation and the assumption of Proposition 3, let $T$ be the local monodromy around 0. Then for $c \in C \setminus \{0\}$ sufficiently near 0

$$\text{Ker can} = \text{Ker}(T - \text{id}) \text{ on } H^{2n-1}(Y_c, \mathbb{Q}).$$

This may be useful in the last section of [3]. Note that Theorem 3 and Corollary 1 do not hold if the fibers $Y_c$ are even-dimensional with $k$ odd, see Remark (2.8)(ii) below.

In Section 1 we review some recent development in the theory of normal functions, and show certain assertions related to Theorem 1. In Section 2 we prove the main theorems.

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### 1. Normal functions

**1.1. Normal functions associated to primitive Hodge classes.** With the notation of Introduction let $H$ be a variation of Hodge structures of weight $-1$ on $S^* = S \setminus D$ defined by $H^{2n-1}(X_s, \mathbb{Z}(n))$ $(s \in S^*)$. This gives a family of intermediate Jacobians $\coprod_{s \in S^{*}} J^n(X_s)$ containing a constant subfamily $J^n(X)$. Take a primitive Hodge class

$$\zeta \in \text{Hdg}^n(X)_{\text{prim}} \subset H^{2n}(X, \mathbb{Z}(n))_{\text{prim}}.$$

By lifting it to an element of Deligne cohomology and restricting to $X_s$, it defines an admissible normal function [22]

$$\nu \in \text{NF}(S^*, H)^{\text{ad}}.$$

This is identified with an extension class of $\mathbb{Z}_{S^*}$ by $H$ as admissible variations of mixed Hodge structures ([15], [28]), and also with a holomorphic section of $\coprod_{s \in S^{*}} J^n(X_s)$. It is well-defined up to a constant section with values in $J^n(X)$. Let $j : S^* \to S$ denote the inclusion. The normal function $\nu$ has the cohomology class

$$\gamma(\nu) \in H^1(S^*, H),$$

using the underlying extension class of local systems. It induces at each $0 \in D$

$$\gamma_0(\nu) \in (R^1j_*H)_0.$$

This is independent of the ambiguity of the normal function.
On the other hand, $\zeta$ induces by restriction

$$\zeta|_{X_0} \in H^{2n}(X_0, \mathbb{Q}(n)).$$

Using the functorial morphism $id \to Rj_*j^*$, it induces further an element of $(R^1j_*\mathbb{H}_Q)_0$. By P. Brosnan, H. Fang, Z. Nie and G. J. Pearlstein [3] (extending the theory of M. Green and P. Griffiths [11]) we have the commutativity of the diagram

$$Hdg^n(X)^{\text{prim}} \quad \longrightarrow \quad \mathbf{N}F(S^*, \mathbb{H})^{\text{ad}}/J^n(X)$$

$$\downarrow \alpha \quad \quad \quad \downarrow$$

$$H^{2n}(X_0, \mathbb{Q}(n)) \quad \quad \beta \quad \quad (R^1j_*\mathbb{H}_Q)_0$$

and the restriction of $\beta$ to the image of $\alpha$ is injective.

1.2. Remarks. (i) The value $\nu_s$ of the normal function $\nu$ at $s \in S^*$ may be viewed as the restriction of a primitive Hodge class $\zeta$ to $X_s$ in the derived category of mixed Hodge structures (using [4]). The above commutative diagram (1.1.1) asserts that the restriction of $\zeta$ to $X_0$ can be calculated by using these ‘restrictions’ of $\zeta$ to $X_s$ for $s \in S^*$ sufficiently near $s$. This implies that the necessary information is not lost by using this ‘restriction’ even after restricting to a small neighborhood of 0 in the classical topology. (Note that maximal information will be preserved if we can use the restriction as algebraic cycles. This situation is similar to the ‘restriction’ of the Leray spectral sequence to a fiber in [21], (0.6).)

(ii) M. de Cataldo and L. Migliorini [6] have proposed a theory of singularities for primitive Hodge classes using the decomposition theorem [1] but without normal functions. For the moment, it is not very clear how to calculate the image of $\zeta$ in $(R^1j_*\mathbb{H}_Q)_0$ without using the normal functions as in Remark (i) above.

(iii) A key observation in Thomas argument [29] is that the algebraic cycle classes coincide with the Hodge classes if and only if for any Hodge class there is an algebraic cycle class such that their pairing does not vanish. For a primitive Hodge class $\zeta$, the condition that $\zeta|_{X_0} \neq 0$ for some $0 \in D$ implies the existence of an algebraic cycle such that their pairing does not vanish as in Remark (iv) below. Note, however, that this condition does not immediately imply the algebraicity of $\zeta$ (unless it is satisfied for any $\zeta$) since this is insufficient to show the coincidence of the algebraic and Hodge classes.

(iv) As is remarked by B. Totaro (see the last remark in §3 of [29]), Thomas argument is extended to the case of arbitrary singularities by using the injectivity of

$$G_{W}^{2n}H^{2n}(X_0, \mathbb{Q}) \to H^{2n}(\widetilde{X}_0, \mathbb{Q}),$$

where $\widetilde{X}_0 \to X_0$ is a desingularization. (This injectivity follows from the construction of mixed Hodge structure using a simplicial resolution [7]). If $\zeta|_{X_0} \neq 0$ for a primitive Hodge class $\zeta$, then we can apply the Hodge conjecture for $\widetilde{X}_0$ as an inductive hypothesis to construct an algebraic cycle on $X_0$ whose pairing with $\zeta$ does not vanish, using the
above injectivity (together with the strict compatibility of the weight filtration $W$). This point is the only difference between the general case and the ordinary double point case in Thomas argument [29], and the hypothesis on ordinary double points is not used in the other places (as far as the proof of the Hodge conjecture is concerned).

1.3. Cohomology classes of normal functions. Let $S$ be a complex manifold, and $S^*$ be an open subset such that $D := S \setminus S^*$ is a divisor. Let $H$ be a polarizable variation of Hodge structure of weight $-1$ on $S^*$. Let

$$\nu \in \text{NF}(S^*, H_Q)_{S}^{ad} := \text{NF}(S^*, H)_{S}^{ad} \otimes \mathbb{Z} Q.$$ 

It is an extension class of $Q$ by $H_Q$ as admissible variations of mixed $Q$-Hodge structures ([15], [28]), and is identified with an extension class as shifted mixed Hodge modules on $S$

(1.3.1) $$Q_S \rightarrow Rj_* H_Q[1].$$

Here $Q_S$ and $Rj_* H_Q$ are mixed Hodge modules up to a shift of complex by $n$ since $D$ is a divisor. Let $j_1, H_Q$ be the intermediate direct image, i.e. the intersection complex up to a shift of complex by $n$, see [1]. Then (1.3.1) factors through $(j_1, H_Q)[1]$ by the semisimplicity of the graded pieces of mixed Hodge modules since the weight of $H$ is $-1$, see [3], [18].

Let $i_0 : \{0\} \rightarrow S$ denote the inclusion. Then (1.3.1) induces a morphism of mixed Hodge structures

$$Q \rightarrow H^1 i_0^* Rj_* H_Q,$$

factorizing through $H^1 i_0^* j_1^* H_Q$. The image of $1 \in Q$ by this morphism is called the cohomology class of $\nu$ at 0. We get thus the morphisms

(1.3.2) $$\text{NF}(S^*, H_Q)_{S}^{ad} \rightarrow \text{Hom}_{MHS}(Q, H^1 i_0^* j_1^* H_Q) \rightarrow \text{Hom}_{MHS}(Q, H^1 i_0^* Rj_* H_Q).$$

Here the injectivity of the last morphism easily follows from the support condition on the intersection complexes, see [3] (and also (1.4) below for the normal crossing case).

1.4. Intersection complexes in the normal crossing case. With the above notation, assume that $S$ is a polydisk $\Delta^n$ with coordinates $t_1, \ldots, t_n$, $S^* = (\Delta^n)^n$, and the local monodromies $T_i$ around $t_i = 0$ are unipotent. Let $H$ be the limit mixed Hodge structure of $H$, see [24]. Set $N_i = \log T_i$. The functor $i_0^*$ between the derived category of mixed Hodge modules [20] is defined in this case by iterating the mapping cones of $\text{can} : \psi_{t_i} \rightarrow \varphi_{t_i}$.

So $H^1 i_0^* Rj_* H_Q$ is calculated by the cohomology at degree 1 of the Koszul complex

$$K^*(H; N_1, \ldots, N_n) := [0 \rightarrow H \oplus \bigoplus_i N_i \rightarrow \bigoplus_i H(-1) \rightarrow \bigoplus_i H(-2) \rightarrow \cdots],$$

where $H$ is put at the degree 0. Moreover, it is known (see e.g. [5]) that $H^1 i_0^* j_1^* H_Q$ is calculated by the cohomology at degree 1 of the subcomplex

$$I^*(H; N_1, \ldots, N_n) := [0 \rightarrow H \oplus \bigoplus_i N_i \rightarrow \bigoplus_i \text{Im} N_i \rightarrow \bigoplus_i \text{Im} N_i N_j \rightarrow \cdots].$$
Define \((\bigoplus_i \text{Im } N_i)^0 = \ker(\bigoplus_i \text{Im } N_i \to \bigoplus_{i \neq j} \text{Im } N_i N_j)\) so that

\[(1.4.1) \quad (\bigoplus_i \text{Im } N_i)^0 / \text{Im}(\bigoplus_i N_i) = H^1 I^* (H; N_1, \ldots, N_n) = H^1 i_0 j_1^* \mathbb{H}_Q.\]

**1.5. Remark.** With the notation and the assumption of (1.4), assume \(H\) is a nilpotent orbit. Then it is easy to show (see e.g. [23]) that (1.3.2) induces a surjective morphism

\[(1.5.1) \quad \text{NF}(S^*, \mathbb{H}_Q)_{ad}^S \to \text{Hom}_{\text{MHS}}(\mathbb{Q}, H^1 i^*_0 j_1^* H_Q).\]

**1.6. Remark.** If \(H\) is not a nilpotent orbit, let \(\tilde{H}\) denote the associated nilpotent orbit. The target of (1.5.1) does not change by replacing \(H\) with \(\tilde{H}\). However, the image of (1.5.1) can change in general (see [23]).

The following is closely related to Theorem 1 in the case where \(X_0\) has only ordinary double points and \(D\) is a divisor with normal crossings around \(0 \in S\) (since \(\text{Im } N_i\) is generated by a vanishing cycle via the Picard-Lefschetz formula, see [8], XV, Th. 3.4). In the geometric case, this is due to [11].

**1.7. Proposition.** With the notation and the assumption of (1.4), assume \(N_i N_j = 0\) for any \(i, j\), and \(\text{Im } N_i \subset H(-1)\) is a direct sum of 1-dimensional mixed Hodge structures for any \(i\). Let \(r\) be the dimension of the relations between the \(\text{Im } N_i\), i.e.

\[(1.7.1) \quad r = \dim \ker(\bigoplus_{i=1}^n \text{Im } N_i \to H(-1)).\]

Then

\[(1.7.2) \quad \dim \text{Hom}_{\text{MHS}}(\mathbb{Q}, (R^1 j_* \mathbb{H}_Q)_0) = \dim H^1 (j_1^* \mathbb{H}_Q)_0 = r.\]

**Proof.** Since \(N_i^2 = 0\), the weight filtration of \(\text{Im } N_i \subset H(-1)\) is given by the monodromy filtration for \(\sum_{j \neq i} N_j\), see [5] and the references there. Then \(\text{Im } N_i\) has type \((0, 0)\) (using the hypotheses on \(N_i\)), and the first isomorphism of (1.7.2) follows.

There is a nondegenerate pairing \(\langle *, * \rangle\) of \(H\) giving a polarization of mixed Hodge structure; in particular \(\langle N_i u, v \rangle = -\langle u, N_i v \rangle\). It is well-known (see e.g. loc. cit.) that there is a nondegenerate pairing \(\langle *, * \rangle_i\) of \(\text{Im } N_i\) defined by

\[\langle N_i u, N_i v \rangle_i := \langle N_i u, v \rangle = -\langle u, N_i v \rangle.\]

Then the morphisms

\[\bigoplus_i N_i : H \to \bigoplus_i \text{Im } N_i, \quad \bigoplus_i \text{Im } N_i \to H(-1)\]

are identified with the dual of each other, and \(\bigoplus_i \text{Im } N_i = (\bigoplus_i \text{Im } N_i)^0\) in the notation of (1.4.1). So the assertion follows.
2. Vanishing cycles

2.1. Proof of Proposition 1 in the general case. We show Proposition 1 without assuming \( \text{Sing} \ Y_0 \) is isolated as in Theorem 2. Forgetting the mixed Hodge structure, this is more or less well-known, see [8], XIII and XIV. For the compatibility with the mixed Hodge structure, we can argue as follows. (If \( \text{Sing} \ Y_0 \) is isolated, we can use [26], [27].)

Since \( f \) is projective and \( C \) can be replaced by a sufficiently small open disk, we may assume that \( Y \) is an intersection of divisors on \( \mathbb{P}^m \times C \). Then \( Q_Y \) is defined in the derived category of mixed Hodge modules, see e.g. the proof of Cor. 2.20 in [20]. (In this case, \( Y \) is a complete intersection and \( Q_Y[2n] \) is a perverse sheaf so that it underlies a mixed Hodge module.) Let \( t \) be a local coordinate around \( 0 \in C \), and \( i : Y_0 \to Y \) be the inclusion. Then there is a distinguished triangle in the derived categories of mixed Hodge modules on \( Y_0 \)

\[
i^* Q_Y \to \psi_{f^*} Q_Y \to \varphi_{f^*} Q_Y \to 1.
\]

Taking the direct image of this triangle by the morphism \( Y_0 \to pt \), the assertion follows.

2.2. Proof of Proposition 2. This follows from the theory of versal flat deformations of complete intersections with isolated singularities in the category of analytic spaces (see [14], [30]) using the base change of Milnor fibrations. (The vanishing for \( j \neq 2n-1 \) follows also from the fact that \( Q_Y[2n] \) and \( \varphi_{f^*} Q_Y[2n-1] \) are perverse sheaves since \( Y \) is a complete intersection.)

For each singular point \( y_i \), we see that \( (Y_0, y_i) \) is a complete intersection since \( X \) is smooth, and hence there is a versal flat deformation of \( (Y_0, y_i) \)

\[(2.2.1) \quad h_i : (\mathbb{C}^{m_i}, 0) \to (\mathbb{C}^{m_i}, 0),\]

such that \( (Y, y_i) \to (C, 0) \) is isomorphic to the base change of \( h_i \) by a morphism

\[\rho_i : (C, 0) \to (\mathbb{C}^{m_i}, 0).\]

Let \( B_i, B_i' \) be open balls in \( \mathbb{C}^{m_i}, \mathbb{C}^{m_i} \) with radius \( \varepsilon_i \) and \( \varepsilon_i' \) respectively. Let \( D_i' \subset B_i' \) be the discriminant of \( h_i \). For \( 1 \gg \varepsilon_i \gg \varepsilon_i' > 0 \), consider the restriction of \( h_i \)

\[B_i \cap h_i^{-1}(B_i' \setminus D_i') \to B_i' \setminus D_i'.\]

This is a \( C^\infty \) fibration, and the fiber \( B_i \cap h_i^{-1}(s) \) for \( s \in B_i' \setminus D_i' \) is topologically independent of \( 1 \gg \varepsilon_i \gg \varepsilon_i' > 0 \). We have moreover for \( s \in B_i' \setminus D_i' \) (see [13], [16])

\[\tilde{H}^j(B_i \cap h_i^{-1}(s), \mathbb{Q}) = 0 \quad \text{for} \quad j \neq 2n - 1.\]

Using the base change of this fibration by \( \rho_i \), the assertion follows.

2.3. Proof of Proposition 3. This follows from the theory of versal flat deformations explained in (2.2). Indeed, by the assumption that the singularities of \( Y_0 \) are ordinary double points, we have \( m_i = 1 \) and \( h_i \) in (2.2.1) is given by

\[(2.3.1) \quad h : (\mathbb{C}^{2n}, 0) \ni (x_1, \ldots, x_{2n}) \mapsto \sum_{i=1}^{2n} x_i^2 \in (\mathbb{C}, 0).\]
If the degree of \( \rho_i : (C, 0) \to (C, 0) \) is \( k_i + 1 \) with \( k_i \in \mathbb{N} \), then \( (Y, y_i) \) is locally isomorphic to a hypersurface defined by
\[
\sum_{i=1}^{2n} x_i^2 = t^{k_i+1},
\]
where \( t \) is a local coordinate of \( C \). So it has a singularity of type \( A_{k_i} \) if it is singular.

2.4. Proof of Theorem 1 in the general case. We show Theorem 1 in the general case as in Theorem 2. The first assertion follows from the hypothesis that \( \zeta \) is Hodge and primitive. By (0.2) modified as in Theorem 2, \( R(Y_0)^{(0,0)} \) is canonically isomorphic to \( E^\vee(Y_0)^{(0,0)} \), and this is the dual of \( E(Y_0)^{(0,0)} \). Thus Theorem 1 (i) is proved in the general case.

For \( \beta \in R(Y_0)^{(0,0)} \), let \( \gamma' \) be the corresponding element in \( E^\vee(Y_0)^{(0,0)} \). We have the canonical morphism (0.5) using the Lefschetz decomposition for \( X \) since the image of \( H^j(Y_\infty, \mathbb{Q}(n)) \) is contained in the non-primitive part. We define \( \gamma \) by (0.5). Here the pairing with \( \zeta \) does not change by taking only the primitive part, since the pairing between the primitive part and the non-primitive part vanishes. So we get Theorem 1 (ii) in the general case using Theorem 1 (i).

2.5. Construction of \( \gamma \) in the isolated singularity case. We can construct \( \gamma \) in Theorem 1 rather explicitly in this case as follows (forgetting the mixed Hodge structure). For \( c \in C \) sufficiently near \( 0 \in C \), let \( \rho : Y_c \to Y_0 \) be a good retraction inducing an isomorphism over \( Y_0 \setminus \text{Sing} Y_0 \). (This can be constructed by taking an embedded resolution and composing it with a good retraction for the resolution, see also [8], XIV.) Set
\[
Z_c = \bigcup_{y \in \text{Sing} Y} \rho^{-1}(y) \cap Y_c.
\]
Since \( H^j(Y_c, Z_c) = H^j_c(Y_c \setminus Z_c) \), there are isomorphisms
\[
\rho^* : H^j(Y_0, Z_0) \xrightarrow{\sim} H^j_c(Y_c, Z_c)
\]
for any \( j \), and \( H^j(Y_0, Z_0) = H^j(Y_0) \) for \( j \geq 2 \). So the exact sequence (0.1) is identified with
\[
H^{2n-1}(Y_c) \to H^{2n-1}(Z_c) \to H^{2n}(Y_c, Z_c) \to H^{2n}(Y_c),
\]
and similarly for the dual. Take a topological relative cycle \( \gamma' \in H_{2n}(Y_c, Z_c) \) whose image in \( H_{2n-1}(Z_c) \) is \( \beta \). Then
\[
\rho_\ast \gamma' \in H_{2n}(Y_0, Z_0) = H_{2n}(Y_0),
\]
and \( \gamma \) in Theorem 1 is the primitive part of its image in \( H_{2n}(X) \). This construction is essentially the same as the one found by P. Griffiths ([12], p. 129) in the ordinary double point case.

2.6. Remark. In case \( n = 1 \), the above construction is quite intuitive since we get a topological 2-chain bounded by vanishing cycles on a nearby fiber \( Y_c \), which gives an algebraic cycle supported on the singular fiber \( Y_0 \) by taking the direct image by \( \rho \). However, this does not immediately imply the Hodge conjecture for this case since the problem seems to be converted to the one studied in [29] using the pairing between Hodge classes.
and algebraic cycles. The situation may be similar for \( n \geq 2 \) if one assumes the Hodge conjecture for a desingularization of \( Y_0 \).

**2.7. Proof of Theorem 3.** A hypersurface singularity is a rational homology manifold if and only if 1 is not an eigenvalue of the Milnor monodromy. In the isolated singularity case this follows from the Wang sequence, see e.g. [17]. It is also well-known (see loc. cit.) that the eigenvalues of the Milnor monodromy of an even-dimensional \( A_k \)-singularity are

\[
\exp \left( \frac{2\pi i p}{(k + 1)} \right) \quad \text{with} \quad p = 1, \ldots, k.
\]

(This is a simple case of the Thom-Sebastiani formula [25].) So the first assertion follows.

For the last assertion, recall that the weight filtration on the unipotent monodromy part of \( \varphi_{f*} \mathbb{Q}_Y[2n-1] \) is the monodromy filtration shifted by \( 2n \) so that the middle graded piece has weight \( 2n \), see [20]. Using the base change of the Milnor fibration by \( p_i \), we see that the vanishing cohomology is 1-dimensional and has a unipotent monodromy in this case. So the vanishing cohomology is pure of weight \( 2n \), and the assertion follows.

**2.8. Remarks.**

(i) In the isolated singularity case, we can choose a curve \( C \subset S \) passing through 0 and such that the base change \( Y \) of \( X \) is smooth by using a linear system spanned by \( X_0 \) and \( X_s \) such that \( X_s \) does not meet any singular points of \( X_0 \) (as is well-known). In this case Proposition 1 follows from the theory of Steenbrink [26]. However, it is sometimes desirable to show Proposition 1 for \( C \subset S \) such that \( Y \) is not smooth, e.g. when \( C \) is the image of a curve on a resolution of singularities of \( (S, D) \), see the last section of [3].

(ii) Theorem 3 and Corollary 1 do not hold if the fibers are \( 2n \)-dimensional and if the singularities are of type \( A_k \) with \( k \) odd. In this case \( \mathbb{Q}_Y[\dim Y] \) is not an intersection complex, and the monodromy \( T \) on \( H^{2n}(Y_c, \mathbb{Q}) \) is the identity since the \( k \) are odd. However, we have non-vanishing of the canonical morphism

\[
can : H^{2n}(Y_c, \mathbb{Q}) \to \bigoplus_{y \in \operatorname{Sing} Y_0} \mathbb{Q}(-n),
\]

for example, if it is obtained by the base change under a double covering \( C \to C' \) of a morphism \( Y' \to C' \) with \( Y' \) smooth.

(iii) It is known that the rank of the morphism \( \text{can} \) in Proposition 1 may depend on the position of the singularities, see e.g. Thm. (4.5) in [9], p. 208 and also [10], (3.5). Here the examples are hypersurfaces in \( \mathbb{P}^{2n} \). One can construct a hypersurface \( X \) in \( \mathbb{P}^{2n+1} \) whose hyperplane section is a given hypersurface \( Y \) as follows.

Let \( f \) be an equation of \( Y \), which is a homogeneous polynomial of degree \( d \). Let \( g = \sum_{i=0}^{d} g_i \), where \( g_i \) is a homogeneous polynomial of degree \( i \), and \( g_d = f \). Let \( X \) be the closure of \( \{g = 0\} \subset \mathbb{C}^{2n+1} \) in \( \mathbb{P}^{2n+1} \). Then \( X \) is smooth along its intersection with the divisor at infinity \( \mathbb{P}^{2n} \) if \( \{g_{d-1} = 0\} \) does not meet the singularities of \( Y = \{g_d = 0\} \).

As for the intersection of \( X \) with the affine space \( \mathbb{C}^{2n+1} \), it is defined by \( g \), and is smooth if \( g_0 \) is sufficiently general since the critical values of \( g \) are finite. (It does not seem easy to construct \( X \) having two given hyperplane sections. If we consider a pencil defined by a linear system spanned by two hypersurfaces we get a pencil of the projective space embedded by \( \mathcal{O}(d) \) in a projective space.)
Using Remark (2.8)(i) above we can show the following (which would be known to specialists).

2.9. Proposition. For an ordinary double point \(x\) of \(X_0\), let \((\Sigma, x)\) be the critical locus near \(x\), and \((D_x, 0)\) be its image in \(S\). Then \((\Sigma, x)\) is isomorphic to \((D_x, 0)\) and they are smooth.

Proof. By [14], [30], there is a morphism

\[ g_x : (S, s) \to (C, 0), \]

such that \((\mathcal{X}, x) \to (S, s)\) is isomorphic to the base change of \(h_i\) in (2.2.1) by \(g_x\). Then we have \(D_x = g_x^{-1}(0)\). Let \(i : (C, 0) \to (S, s)\) be a curve in Remark (2.8)(i). The composition \(g_x \circ i\) has degree 1 since the base change of \(h\) by it has otherwise a singularity. So \(g_x\) has a section and hence \(g_x\) and \(D_x\) are smooth. Then \((\Sigma, x)\) is also smooth since \((\Sigma, x) \to (D_x, s)\) is bijective. Thus the assertion is proved.

2.10. Remarks. (i) Let \(x_1, \ldots, x_m\) be ordinary double points on \(X_0\). Then we have a morphism

\[ G : (S, 0) \to (C^m, 0), \]

whose composition with the \(i\)-th projection \(pr_i : C^m \to C\) coincides with \(g_{x_i}\) in the proof of Proposition (2.9). It is not easy to calculate \(G\) although \(g_{x_i} = pr_i \circ G\) is smooth by Proposition (2.9).

(ii) Assume \(X_0 = Y_0\) has only ordinary double points as singularities, and let \(\tilde{Y}_0 \to Y_0\) be the resolution of singularities obtained by blowing up along all the singular points \(x_i\) \((i = 1, \ldots, m)\). In this case we have by [11]

\[ r(Y_0) = h^{n,n-1}(\tilde{Y}_0) - h^{n,n-1}(Y_\infty) + m, \]

\[ = \dim Hdg^{n-1}(\tilde{Y}_0) - \dim Hdg^{n-1}(X) + (1 - \delta_{n,1})m. \]

Since \(H^{2n-2}(Y_0) = H^{2n-2}(Y_\infty) = H^{2n-2}(X)\) and \(H^{2n-2}(E_i)(n-1) \cong \mathbb{Q}^2\), they are closely related to the exact sequences

\[ 0 \to H^{2n-2}(Y_0) \to H^{2n-2}(\tilde{Y}_0) \oplus \bigoplus_{i=1}^m H^{2n-2}(E_i) \to H^{2n-1}(Y_0) \to H^{2n-1}(\tilde{Y}_0) \to 0, \]

\[ 0 \to H^{2n-1}(Y_0) \to H^{2n-1}(Y_\infty) \oplus \bigoplus_{i=1}^m \mathbb{Q}(-n) \to E(Y_0)(-n) \to 0, \]

since these two imply also \(h^{n,n-1}(Y_0) = h^{n,n-1}(\tilde{Y}_0)\) and

\[ \dim \text{Gr}_{2n-2} H^{2n-1}(Y_0) = \dim \text{Gr}_{2n} H^{2n-1}(Y_\infty) = m - r(Y_0). \]

2.11. Remarks. (i) Let \(\tilde{X} \to \mathbb{P}^1\) be a Lefschetz pencil where \(\pi : \tilde{X} \to X\) is the blow-up along the intersection of two general hyperplane sections. Let \(X_t\) be a general fiber
with the inclusion \( i_t : X_t \to \tilde{X} \). If \( 2p < \dim X \), then the Leray spectral sequence for the Lefschetz pencil induces an exact sequence

\[
0 \to H^{2p-2}(X_t, \mathbb{Q})(-1) \xrightarrow{(i_t)_*} H^{2p}(\tilde{X}, \mathbb{Q}) \xrightarrow{i_t^*} H^{2p}(X_t, \mathbb{Q}).
\]

This can be used to solve a minor problem in an argument in [29]. Indeed, by a Hilbert scheme argument (using the countability of the irreducible components of the Hilbert scheme), one can construct an algebraic cycle class \( \xi \) with rational coefficients on \( \tilde{X} \) whose restriction to \( X_t \) coincides with the restriction to \( X_t \) of a given primitive Hodge class \( \zeta \) on \( X \) where \( t \in \mathbb{P}^1 \) is quite general. However, it is not very clear whether \( \xi = \pi^*\zeta \) in loc. cit. This problem can be solved by considering the difference \( \pi^*\zeta - \xi \) since it is a Hodge class and belongs to the image of \((i_t)_*\) by (2.11.1) so that the inductive hypothesis on the Hodge conjecture applies. (This argument seems to be simpler than the one given by M. de Cataldo and L. Migliorini [6].)

(ii) The Hilbert scheme argument in [29] can be replaced by ‘spreading out’ of cycles (a technique initiated probably by S. Bloch [2], see also [31]). Indeed, let \( k \) be an algebraically closed subfield of \( C \) which has finite transcendence degree and over which the Lefschetz pencil \( \tilde{X}_k \to \mathbb{P}^1_k \) is defined. Let \( U \) be a dense open subvariety of \( \mathbb{P}^1_k \) over which the fibers are smooth. Let \( t \) be a \( k \)-generic point of \( \mathbb{P}^1_k \). Using the inductive hypothesis, the restriction of a Hodge cycle \( \zeta \) to \( X_t \) is represented by an algebraic cycle with rational coefficients \( \xi_t \). This \( \xi_t \) is defined over a subfield \( K \) of \( C \) which contains \( k(t) \) and is finitely generated over \( k \). Let \( R \) be a finitely generated \( k \)-subalgebra of \( K \) whose quotient field is \( K \) and such that \( \xi_t \) is defined over \( R \). Let \( \tilde{X}_{k,V} \) denote the base change of \( \tilde{X}_k \to \mathbb{P}^1_k \) by \( V := \text{Spec} \ R \to \mathbb{P}^1_k \), where we may assume that \( V \to \mathbb{P}^1_k \) factors through \( U \). Then \( \xi_t \) is defined on \( \tilde{X}_{k,V} \), and its cycle class is defined as a global section of the local system on \( V_C \), and coincides with the pull-back of the global section \( \tilde{\zeta} \) on \( U_C \subset \mathbb{P}^1_C \) which is defined by the restrictions \( \zeta|_{X_{t'}} \) for \( t' \in U_C \). (Indeed, \( V \) and \( V_C \) are irreducible, and the two global sections on \( V_C \) coincide at the point of \( V_C \) determined by the inclusion \( R \to C \)). Taking a curve \( C \) on \( V \) which is dominant over \( \mathbb{P}^1_k \), and using the direct image by the base change of \( C \to U \) (and dividing it by the degree of \( C \to U \)), we get a cycle on \( \tilde{X}_{k,U} \subset \tilde{X}_k \) whose cycle class coincides with \( \tilde{\zeta} \) as global sections on \( U_C \), where we may assume that \( C \) is finite over \( U \) replacing \( U \) and \( C \) if necessary. Then we can extend it to a cycle on \( \tilde{X}_k \) by taking the closure.

(iii) The above argument is essentially explained in Remarks (1.3)(ii) and (1.10)(ii) of [21], where it is noted that if \( \text{HC}(X, p) \) denotes the Hodge conjecture for cycles of codimension \( p \) on a smooth projective variety \( X \), then \( \text{HC}(X, p) \) for \( p > \dim X/2 \) is reduced to \( \text{HC}(Y, p - 1) \) for a smooth hyperplane section \( Y \) (using the Gysin morphism together with the weak Lefschetz theorem), and for \( p < \dim X/2 \), it is reduced to \( \text{HC}(Y, p) \) and \( \text{HC}(Y, p - 1) \) for a quite general hyperplane section \( Y \) (using a Lefschetz pencil \( \tilde{X} \to \mathbb{P}^1 \) and spreading out as above). Moreover, the problem in Remark (2.11)(i) above is also mentioned at the end of Remark (1.3)(ii) in loc. cit. (i.e. \( \text{HC}(Y, p - 1) \) is necessary in the second case).
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RIMS Kyoto University, Kyoto 606-8502 Japan
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