SEQUENTIAL MULTIPLE TESTING WITH GENERALIZED ERROR CONTROL: AN ASYMPTOTIC OPTIMALITY THEORY

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The sequential multiple testing problem is considered under two generalized error metrics. Under the first one, the probability of at least $k$ mistakes, of any kind, is controlled. Under the second, the probabilities of at least $k_1$ false positives and at least $k_2$ false negatives are simultaneously controlled. For each formulation, the optimal expected sample size is characterized, to a first-order asymptotic approximation as the error probabilities go to 0, and a novel multiple testing procedure is proposed and shown to be asymptotically efficient under every signal configuration. These results are established when the data streams for the various hypotheses are independent and each local log-likelihood ratio statistic satisfies a certain Strong Law of Large Numbers. In the special case of i.i.d. observations in each stream, the gains of the proposed sequential procedures over fixed-sample size schemes are quantified.

1. Introduction. In the early development of multiple testing, the focus was on procedures that control the probability of at least one false positive, i.e., falsely rejected null [16, 17, 27]. As this requirement can be prohibitive when the number of hypotheses is large, the emphasis gradually shifted to the control of less stringent error metrics, such as (i) the expectation [4] or the quantiles [21] of the false discovery proportion, i.e., the proportion of false positives among the rejected nulls, and (ii) the generalized familywise error rate, i.e., the probability of at least $k \geq 1$ false positives [18, 21]. During the last two decades, various procedures have been proposed to control the above error metrics [5, 15, 32, 33]. Further, the problem of maximizing the number of true positives subject to a generalized control on false positives has been studied in [22, 29, 37, 38], whereas in [6] the false negatives are incorporated into the risk function in a Bayesian decision theoretic framework.

In all previous references it is assumed that the sample size is deterministic. However, in many applications data are collected in real time and a
reliable decision needs to be made as quickly as possible. Such applications fall into the framework of *sequential hypothesis testing*, which was introduced in the ground-breaking work of Wald [42] and has been studied extensively since then (see, e.g., [39]).

When testing simultaneously *multiple* hypotheses with data collected from a different stream for each hypothesis, there are two natural generalizations of Wald’s sequential framework. In the first one, sampling can be terminated earlier in some data streams [1, 3, 26]. In the second, which is the focus of this paper, sampling is terminated at the same time in all streams [8, 9]. The latter setup is motivated by applications such as multichannel signal detection [41], multiple access wireless network [30] and multisensor surveillance systems [14], where a centralized decision maker needs to make a decision regarding the presence or absence of signal, e.g., an intruder, in multiple channels/areas monitored by a number of sensors. This framework is also motivated by online surveys and crowdsourcing tasks [20], where the goal is to find “correct” answers to a fixed number of questions, e.g., regarding some product or service, by asking the smallest necessary number of people.

In this paper we focus on two related, yet distinct, generalized error metrics. The first one is a generalization of the usual mis-classification rate [24, 26], where the probability of at least \( k \geq 1 \) mistakes, of any kind, is controlled. The second one controls generalized familywise error rates of both types [1, 10], i.e., the probabilities of at least \( k_1 \geq 1 \) false positives and at least \( k_2 \geq 1 \) false negatives.

Various sequential procedures have been proposed recently to control such generalized familywise error rates [1, 2, 3, 8, 9, 10]. To the best of our knowledge, the efficiency of these procedures is understood only in the case of *classical* familywise error rates, i.e., when \( k_1 = k_2 = 1 \). Specifically, in the case of independent streams with i.i.d. observations, an asymptotic lower bound was obtained in [35] for the optimal expected sample size (ESS) as the error probabilities go to 0, and was shown to be attained, under any signal configuration, by several existing procedures. However, the results in [35] do not extend to *generalized* error metrics, since the technique for the proof of the asymptotic lower bound requires that the probability of not identifying the correct subset of signals goes to 0. Further, as we shall see, existing procedures fail to be asymptotically optimal, in general, under generalized error metrics.

The lack of an optimality theory under such generalized error control also implies that it is not well understood how the best possible ESS depends on the user-specified parameters. This limits the applicability of generalized error metrics, as it is not clear for the practitioner how to select the number
of hypotheses to be “sacrificed” for the sake of a faster decision.

In this paper, we address this research gap by developing an asymptotic optimality theory for the sequential multiple testing problem under the two generalized error metrics mentioned above. Specifically, for each formulation we characterize the optimal ESS as the error probabilities go to 0, and propose a novel, feasible sequential multiple testing procedure that achieves the optimal ESS under every signal configuration. These results are established under the assumption of independent data streams, and require that the log-likelihood ratio statistic in each stream satisfies a certain Strong Law of Large Numbers. Thus, even in the case of classical familywise error rates, we extend the corresponding results in [35] by relaxing the i.i.d. assumption in each stream.

Finally, whenever sequential testing procedures are utilized, it is of interest to quantify the savings in the ESS over fixed-sample size schemes with the same error control guarantees. In the case of i.i.d. data streams, we obtain an asymptotic lower bound for the gains of sequential sampling over any fixed-sample size scheme, and also characterize the asymptotic gains over a specific fixed-sample size procedure.

In order to convey the main ideas and results with the maximum clarity, we first consider the case that the local hypotheses are simple, and then extend our results to the case of composite hypotheses. Thus, the remainder of the paper is organized as follows: in Section 2 we formulate the two problems of interest in the case of simple hypotheses. The case of generalized mis-classification rate is presented in Section 3, and the case of generalized familywise error rates in Section 4. In Section 5 we present two simulation studies under the second error metric. In Section 6 we extend our results to the case of composite hypotheses. We conclude and discuss potential extensions of this work in Section 7. Proofs are presented in the Appendix, where we also present more simulation studies and a detailed analysis of the case of composite hypotheses. For convenience, we list in Table 1 the procedures that are considered in this work.

| Procedure               | Metric     | Section | Main results | Conditions for AO |
|-------------------------|------------|---------|--------------|-------------------|
| Sum-Intersection†       | GMIS       | 3.1     | Thrm 3.3     | (8)               |
| Leap†                   | GFWER      | 4.2     | Thrm 4.3     | (8)               |
| Asym. Sum-Intersection† | GFWER      | 4.1     | Cor 4.4      | (8) + (11) + (12) |
| Intersection            | both       | 2.2     | Cor 3.4/ 4.4 | (8) + (11) / (12) |
| MNP (fixed-sample)      | both       | 2.3     | Thrm 3.5/ 4.5| Not optimal       |

Procedures marked with † are new. Procedures in bold font are asymptotically optimal (AO) without requiring a special testing structure. GMIS is short for generalized mis-classification rate, and GFWER for generalized familywise error rates.
2. Problem formulation. Consider independent streams of observations, $X^j := \{X^j(n) : n \in \mathbb{N}\}$, where $j \in [J] := \{1, \ldots, J\}$ and $\mathbb{N} := \{1, 2, \ldots\}$. For each $j \in [J]$, we denote by $P^j$ the distribution of $X^j$ and consider two simple hypotheses for it,

$$H^j_0 : P^j = P^j_0 \quad \text{versus} \quad H^j_1 : P^j = P^j_1. \quad (1)$$

We denote by $P_A$ the distribution of $(X^1, \ldots, X^J)$ when $A \subset [J]$ is the subset of data streams with signal, i.e., in which the alternative hypothesis is correct. Due to the assumption of independence among streams, $P_A$ is the following product measure:

$$P_A := \bigotimes_{j=1}^J P^j; \quad P^j = \begin{cases} P^j_0, & \text{if } j \notin A \\ P^j_1, & \text{if } j \in A. \end{cases} \quad (2)$$

Moreover, we denote by $\mathcal{F}^j_n$ the $\sigma$-field generated by the first $n$ observations in the $j$-th stream, i.e., $\sigma(X^j(1), \ldots, X^j(n))$, and by $\mathcal{F}_n$ the $\sigma$-field generated by the first $n$ observations in all streams, i.e., $\sigma(\mathcal{F}^j_n, j \in [J])$, where $n \in \mathbb{N}$.

Assuming that the data in all streams become available sequentially, the goal is to stop sampling as soon as possible, and upon stopping to solve the $J$ hypothesis testing problems subject to certain error control guarantees. Formally, a sequential multiple testing procedure is a pair $\delta = (T, D)$ where $T$ is an $\{\mathcal{F}_n\}$-stopping time at which sampling is terminated in all streams, and $D$ an $\mathcal{F}_T$-measurable, $J$-dimensional vector of Bernoullis, $(D^1, \ldots, D^J)$, so that the alternative hypothesis is selected in the $j$-th stream if and only if $D^j = 1$. With an abuse of notation, we also identify $D$ with the subset of streams in which the alternative hypothesis is selected upon stopping, i.e., $\{j \in [J] : D^j = 1\}$.

We consider two kinds of error control, which lead to two different problems. Their main difference is that the first one does not differentiate between false positives, i.e., rejecting the null when it is correct, and false negatives, i.e., accepting the null when it is false. Specifically, in the first one we control the generalized mis-classification rate, i.e., the probability of committing at least $k$ mistakes, of any kind, where $k$ is a user-specified integer such that $1 \leq k < J$. When $A$ is the true subset of signals, a decision rule $D$ makes at least $k$ mistakes, of any kind, if $D$ and $A$ differ in at least $k$ components, i.e., $|A \triangle D| \geq k$, where for any two sets $A$ and $D$, $A \triangle D$ is their symmetric difference, i.e. $(A \setminus D) \cup (D \setminus A)$, and $|\cdot|$ denotes set-cardinality. Thus, given tolerance level $\alpha \in (0, 1)$, the class of multiple testing procedures of interest in this case is

$$\Delta_k(\alpha) := \left\{ (T, D) : \max_{A \subseteq [J]} P_A(|A \triangle D| \geq k) \leq \alpha \right\}. \quad (3)$$
Then, the first problem is formulated as follows:

**Problem 2.1.** Given a user-specified integer \( k \) in \([1, J]\), find a sequential multiple testing procedure that (i) controls the generalized mis-classification rate, i.e., it can be designed to belong to \( \Delta_k(\alpha) \) for any given \( \alpha \), and (ii) achieves the smallest possible expected sample size,

\[
N_A^*(k, \alpha) := \inf_{(T, D) \in \Delta_k(\alpha)} E_A[T],
\]

for every \( A \subset [J] \), to a first-order asymptotic approximation as \( \alpha \to 0 \).

In the second problem of interest in this work, we control generalized familywise error rates of both types, i.e., the probabilities of at least \( k_1 \) false positives and at least \( k_2 \) false negatives, where \( k_1, k_2 \geq 1 \) are integers such that \( k_1 + k_2 \leq J \). When the true subset of signals is \( A \), a decision rule \( D \) makes at least \( k_1 \) false positives when \( |D \setminus A| \geq k_1 \) and at least \( k_2 \) false negatives when \( |A \setminus D| \geq k_2 \). Thus, given tolerance levels \( \alpha, \beta \in (0, 1) \), the class of procedures of interest in this case is

\[
\Delta_{k_1,k_2}(\alpha, \beta) := \{(T, D) : \max_{A \subset [J]} P_A(|D \setminus A| \geq k_1) \leq \alpha \text{ and } \max_{A \subset [J]} P_A(|A \setminus D| \geq k_2) \leq \beta\}.
\]

(3)

Then, the second problem is formulated as follows:

**Problem 2.2.** Given user-specified integers \( k_1, k_2 \geq 1 \) such that \( k_1 + k_2 \leq J \), find a sequential multiple testing procedure that (i) controls generalized familywise error rates of both types, i.e., it can be designed to belong to \( \Delta_{k_1,k_2}(\alpha, \beta) \) for any given \( \alpha, \beta \in (0, 1) \), and (ii) achieves the smallest possible expected sample size,

\[
N_A^*(k_1, k_2, \alpha, \beta) := \inf_{(T, D) \in \Delta_{k_1,k_2}(\alpha, \beta)} E_A[T],
\]

for every \( A \subset [J] \), to a first-order asymptotic approximation as \( \alpha \) and \( \beta \) go to 0, at arbitrary rates.

2.1. **Assumptions.** We now state the assumptions that we will make in the next two sections in order to solve these two problems. First of all, for each \( j \in [J] \) we assume that the probability measures \( P_0^j \) and \( P_1^j \) in (1) are mutually absolutely continuous when restricted to \( \mathcal{F}_n^j \), and we denote the corresponding log-likelihood ratio (LLR) statistic as follows:

\[
\lambda^j(n) := \log \frac{dP_1^j}{dP_0^j}(\mathcal{F}_n^j), \quad \text{for } n \in \mathbb{N}.
\]
For $A, C \subset [J]$ and $n \in \mathbb{N}$ we denote by $\lambda^{A,C}(n)$ the LLR of $P_A$ versus $P_C$ when both measures are restricted to $\mathcal{F}_n$, and from (2) it follows that
\begin{equation}
\lambda^{A,C}(n) := \log \frac{dP_A}{dP_C}(\mathcal{F}_n) = \sum_{j \in A \setminus C} \lambda^j(n) - \sum_{j \in C \setminus A} \lambda^j(n).
\end{equation}

In order to guarantee that the proposed multiple testing procedures terminate almost surely and satisfy the desired error control, it will suffice to assume that
\begin{equation}
P^j_1 \left( \lim_{n \to \infty} \lambda^j(n) = \infty \right) = P^j_0 \left( \lim_{n \to \infty} \lambda^j(n) = -\infty \right) = 1 \quad \forall \ j \in [J].
\end{equation}

In order to establish an asymptotic lower bound on the optimal ESS for each problem, we will need the stronger assumption that for each $j \in [J]$ there are positive numbers, $I^j_1, I^j_0$, such that the following Strong Laws of Large Numbers (SLLN) hold:
\begin{equation}
P^j_1 \left( \lim_{n \to \infty} \frac{\lambda^j(n)}{n} = I^j_1 \right) = P^j_0 \left( \lim_{n \to \infty} \frac{\lambda^j(n)}{n} = -I^j_0 \right) = 1.
\end{equation}

When the LLR statistic in each stream has independent and identically distributed (i.i.d.) increments, the SLLN (6) will also be sufficient for establishing the asymptotic optimality of the proposed procedures. When this is not the case, we will need an assumption on the rate of convergence in (6). Specifically, we will assume that for every $\epsilon > 0$ and $j \in [J]$,
\begin{equation}
\sum_{n=1}^{\infty} P^j_1 \left( \left| \frac{\lambda^j(n)}{n} - I^j_1 \right| > \epsilon \right) < \infty, \quad \sum_{n=1}^{\infty} P^j_0 \left( \left| \frac{\lambda^j(n)}{n} + I^j_0 \right| > \epsilon \right) < \infty.
\end{equation}

Condition (7) is known as complete convergence [19], and is a stronger assumption than (6), due to the Borel-Cantelli lemma. This condition is satisfied in various testing problems where the observations in each data stream are dependent, such as autoregressive time-series models and state-space models. For more details, we refer to [39, Chapter 3.4].

To sum up, the only distributional assumption for our asymptotic optimality theory is that the LLR statistic in each stream
\begin{itemize}
  \item either has i.i.d. increments and satisfies the SLLN (6),
  \item or satisfies the SLLN with complete convergence (7).
\end{itemize}

**Remark 2.1.** If (6) (resp. (7)) holds, the normalized LLR, $\lambda^{A,C}(n)/n$, defined in (4), converges almost surely (resp. completely) under $P_A$ to
\begin{equation}
\mathcal{T}^{A,C} := \sum_{i \in A \setminus C} \mathcal{T}^i_1 + \sum_{j \in C \setminus A} \mathcal{T}^j_0.
\end{equation}
The numbers $I_A^C$ and $I_C^A$ will turn out to determine the inherent difficulty in distinguishing between $P_A$ and $P_C$ and will play an important role in characterizing the optimal performance under $P_A$ and $P_C$, respectively.

2.2. The Intersection rule. To the best of our knowledge, Problem 2.2 has been solved only under the assumption of i.i.d. data streams and only in the case of classical error control, that is when $k_1 = k_2 = 1$ [35]. An asymptotically optimal procedure in this setup is the so-called “Intersection” rule, $\delta_I := (T_I, D_I)$, proposed in [8, 9], where

$$T_I := \inf \left\{ n \geq 1 : \lambda^j(n) \notin (-a, b) \text{ for every } j \in [J] \right\},$$
$$D_I := \left\{ j \in [J] : \lambda^j(T_I) > 0 \right\},$$

and $a, b$ are positive thresholds. This procedure requires the local test statistic in every stream to provide sufficiently strong evidence for the sampling to be terminated. The Intersection rule was also shown in [10] to control generalized familywise error rates, however its efficiency in this setup remains an open problem, even in the case of i.i.d. data streams. Our asymptotic optimality theory in the next sections will reveal that the Intersection rule is asymptotically optimal with respect to Problems 2.1 and 2.2 only when the multiple testing problem satisfies a very special structure.

**Definition 2.1.** We say that the multiple testing problem (1) is

(i) symmetric, if for every $j \in [J]$ the distribution of $\lambda^j$ under $P^j_0$ is the same as the distribution of $-\lambda^j$ under $P^j_1$,

(ii) homogeneous, if for every $j \in [J]$ the distribution of $\lambda^j$ under $P^j_i$ does not depend on $j$, where $i \in \{0, 1\}$.

It is clear that when the multiple testing problem is both symmetric and homogeneous, we have

$$T_0^j = T_1^j = T \text{ for every } j \in [J].$$

In the next sections we will show that the Intersection rule is asymptotically optimal for Problem 2.1 when (11) holds, whereas its asymptotic optimality with respect to Problem 2.2 will additionally require that the user-specified parameters satisfy the following conditions:

$$k_1 = k_2 \text{ and } \alpha = \beta.$$
2.3. Fixed-sample size schemes. Let $\Delta_{\text{fix}}(n)$ denote the class of procedures for which the decision rule depends on the data collected up to a deterministic time $n$, i.e.,
\[
\Delta_{\text{fix}}(n) := \{(n, D) : D \subset [J] \text{ is } \mathcal{F}_n\text{-measurable}\}.
\]
For any given integers $k, k_1, k_2 \geq 1$ with $k, k_1 + k_2 < J$ and $\alpha, \beta \in (0, 1)$, let
\[
(n^*(k, \alpha)) := \inf\left\{ n \in \mathbb{N} : \Delta_{\text{fix}}(n) \cap \Delta_k(\alpha) \neq \emptyset \right\},
\]
\[
n^*(k_1, k_2, \alpha, \beta) := \inf\left\{ n \in \mathbb{N} : \Delta_{\text{fix}}(n) \cap \Delta_{k_1, k_2}(\alpha, \beta) \neq \emptyset \right\},
\]
denote the minimum sample sizes required by any fixed-sample size scheme under the two error metrics of interest. In the case of i.i.d. observations in the data streams, we establish asymptotic lower bounds for the above two quantities as the error probabilities go to 0. To the best of our knowledge, there is no fixed-sample size procedure that attains these bounds. For this reason, we also study a specific procedure that runs a Neyman-Pearson test at each stream. Formally, this procedure is defined as follows:
\[
\delta_{\text{NP}}(n, h) := \left(n, D_{\text{NP}}(n, h)\right), \quad D_{\text{NP}}(n, h) := \{j \in [J] : \lambda_j^n(n) > nh_j\},
\]
where $h = (h_1, \ldots, h_J) \in \mathbb{R}^J$, $n \in \mathbb{N}$, and we refer to it as multiple Neyman-Pearson (MNP) rule. In the case of Problem 2.1, we characterize the minimum sample size required by this procedure,
\[
n_{\text{NP}}(k, \alpha) := \inf\{n \in \mathbb{N} : \exists h \in \mathbb{R}^J, \delta_{\text{NP}}(n, h) \in \Delta_k(\alpha)\},
\]
to a first-order approximation as $\alpha \to 0$. In the case of Problem 2.2, for simplicity of presentation we further restrict ourselves to homogeneous, but not necessarily symmetric, multiple testing problems, and characterize the asymptotic minimum sample size required by the MNP rule that utilizes the same threshold in each stream, i.e.,
\[
\hat{n}_{\text{NP}}(k_1, k_2, \alpha, \beta) := \inf\{n \in \mathbb{N} : \exists h \in \mathbb{R}, \delta_{\text{NP}}(n, h \mathbf{1}_J) \in \Delta_{k_1, k_2}(\alpha, \beta)\},
\]
where $\mathbf{1}_J \in \mathbb{R}^J$ is a $J$-dimensional vector of ones.

2.4. The i.i.d. case. As mentioned earlier, our asymptotic optimality theory will apply whenever condition (8) holds, thus, beyond the case of i.i.d. data streams. However, our analysis of fixed-sample size schemes will rely on large deviation theory [11] and will be focused on the i.i.d. case. Thus, it is useful to introduce some relevant notations for this setup.
Specifically, when for each \( j \in [J] \) the observations in the \( j \)-th stream are independent with common density \( f^j \) relative to a \( \sigma \)-finite measure \( \nu^j \), the hypothesis testing problem (1) takes the form
\[
H_0^j : f^j = f_0^j \quad \text{versus} \quad H_1^j : f^j = f_1^j,
\]
and \( I_1^j, I_0^j \) correspond to the Kullback-Leibler divergences between \( f_1^j \) and \( f_0^j \), i.e.,
\[
I_k^j = \int \log \left( \frac{f_k^j}{f_{1-k}^j} \right) f_k^j \, d\nu^j.
\]
In this case, each LLR statistic \( \lambda^j \) has i.i.d. increments, and (8) is satisfied as long as \( I_1^j \) and \( I_0^j \) are both positive and finite. For each \( j \in [J] \), we further introduce the convex conjugate of the cumulant generating function of \( \lambda^j \)
\[
\Phi^j(z) := \sup_{\theta \in \mathbb{R}} \{ z\theta - \Psi^j(\theta) \}, \quad \text{where} \quad \Psi^j(\theta) := \log E_0^j \left[ e^{\theta \lambda^j(1)} \right].
\]
The value of \( \Phi^j \) at zero is the Chernoff information [11] for the testing problem (15), and we will denote it as \( C^j \), i.e., \( C^j := \Phi^j(0) \).

Finally, we will illustrate our general results in the case of testing normal means. Hereafter, \( \mathcal{N} \) denotes the density of the normal distribution.

**Example 2.1.** If \( f_0^j = \mathcal{N}(0, \sigma_j^2) \) and \( f_1^j = \mathcal{N}(\mu_j, \sigma_j^2) \) for all \( j \in [J] \), then
\[
\lambda^j(1) = \theta_j^2 \left( \bar{X}^j(1)/\mu_j - 1/2 \right), \quad \text{where} \quad \theta_j := \mu_j/\sigma_j.
\]
Consequently the multiple testing problem is symmetric and
\[
I^j := I_0^j = I_1^j = \theta_j^2/2, \quad \Phi^j(z) = (z + I^j)^2/(4I^j) \quad \text{for any} \quad z \in \mathbb{R}.
\]

2.5. **Notation.** We collect here some notations that will be used extensively throughout the rest of the paper: \( C_k^J \) denotes the binomial coefficient \( \binom{J}{k} \), i.e., the number of subsets of size \( k \) from a set of size \( J \); \( a \lor b \) represents \( \max\{a, b\} \); \( x \sim y \) means that \( \lim_{y \to 1} x/y = 1 \) and \( x(b) = o(1) \) that \( \lim_{b \to 0} x(b) = 0 \), with \( y, b \to 0 \) or \( \infty \). Moreover, we recall that \( | \cdot | \) denotes set-cardinality, \( \mathbb{N} := \{1, 2, \ldots\} \), \( [J] := \{1, \ldots, J\} \), and that \( A \triangle B \) is the symmetric difference, \( (A \setminus B) \cup (B \setminus A) \), of two sets \( A \) and \( B \).

3. **Generalized mis-classification rate.** In this section we consider Problem 2.1 and carry out the following program: first, we propose a novel procedure that controls the generalized mis-classification rate. Then, we establish an asymptotic lower bound on the optimal ESS and show that it is attained by the proposed scheme. As a corollary, we show that the Intersection rule is asymptotically optimal when condition (11) holds. Finally, we make a comparison with fixed-sample size procedures in the i.i.d. case (15).
3.1. **Sum-Intersection rule.** In order to implement the proposed procedure, which we will denote \( \delta_S(b) := (T_S(b), D_S(b)) \), we need at each time \( n \in \mathbb{N} \) prior to stopping to order the absolute values of the local test statistics, \( |\lambda^j(n)|, j \in [J] \). If we denote the corresponding ordered values by

\[
\tilde{\lambda}^1(n) \leq \ldots \leq \tilde{\lambda}^J(n),
\]

we can think of \( \tilde{\lambda}^1(n) \) (resp. \( \tilde{\lambda}^J(n) \)) as the least (resp. most) “significant” local test statistic at time \( n \), in the sense that it provides the weakest (resp. strongest) evidence in favor of either the null or the alternative. Then, sampling is terminated at the first time the sum of the \( k \) least significant local LLRs exceeds some positive threshold \( b \), and the null hypothesis is rejected in every stream that has a positive LLR upon stopping, i.e.,

\[
T_S(b) := \inf \left\{ n \geq 1 : \sum_{j=1}^{k} \tilde{\lambda}^j(n) \geq b \right\}, \quad D_S(b) := \{ j \in [J] : \lambda^j(T_S(b)) > 0 \}.
\]

The threshold \( b \) is selected to guarantee the desired error control. When \( k = 1 \), \( \delta_S(b) \) coincides with the Intersection rule, \( \delta_I(b, b) \), defined in (10). When \( k > 1 \), the two rules are different but share a similar flavor, since \( \delta_S(b) \) stops the first time \( n \) that all sums \( \sum_{j \in B} |\lambda^j(n)| \) with \( B \subset [J] \) and \( |B| = k \) are simultaneously above \( b \). For this reason, we refer to \( \delta_S(b) \) as **Sum-Intersection rule**. Hereafter, we typically suppress the dependence of \( \delta_S(b) \) on threshold \( b \) in order to lighten the notation.

3.2. **Error control of the Sum-Intersection rule.** For any choice of threshold \( b \), the Sum-Intersection rule clearly terminates almost surely, under every signal configuration, as long as condition (5) holds. In the next theorem we show how to select \( b \) to guarantee the desired error control. We stress that no additional distributional assumptions are needed for this purpose.

**Theorem 3.1.** Assume (5) holds. For any \( \alpha \in (0, 1) \) we have \( \delta_S(b_\alpha) \in \Delta_k(\alpha) \) when

\[
(19) \quad b_\alpha = |\log(\alpha)| + \log(C_k^J).
\]

**Proof.** The proof can be found in Appendix B.1. \( \square \)

The choice of \( b \) suggested by the previous theorem will be sufficient for establishing the asymptotic optimality of the Sum-Intersection rule, but may be conservative for practical purposes. In the absence of more accurate approximations for the error probabilities, we recommend finding the value of
for which the target level is attained using Monte Carlo simulation. This means simulating off-line, i.e., before the sampling process begins, for every $A \subset [J]$ the error probability $P_A(|A \triangle D_S(b)| \geq k)$ for various values of $b$, and then selecting the value for which the maximum of these probabilities over $A \subset [J]$ matches the nominal level $\alpha$.

This simulation task is significantly facilitated when the multiple testing problem has a special structure. If the problem is symmetric, for any given threshold $b$ the error probabilities of the Sum-Intersection rule coincide for all $A \subset [J]$; thus, it suffices to simulate the error probability under a single measure, e.g., $P_\emptyset$. If the problem is homogeneous, the error probabilities depend only on the size of $A$, not the actual subset; thus, it suffices to simulate the above probabilities for at most $J + 1$ configurations. Similar ideas apply in the presence of block-wise homogeneity.

Moreover, it is worth pointing out that when $b$ is large, importance sampling techniques can be applied to simulate the corresponding “small” error probabilities, similarly to [34].

### 3.3. Asymptotic lower bound on the optimal performance

We now obtain an asymptotic (as $\alpha \to 0$) lower bound on $N^*_A(k, \alpha)$, the optimal ESS for Problem 2.1 when the true subset of signals is $A$, for any given $k \geq 1$. When $k = 1$, from [40, Theorem 2.2] it follows that when (6) holds, such a lower bound is given by $\left| \log(\alpha) \right| / \min_{C \neq A} I^{A,C}$, where $I^{A,C}$ is defined in (9). Thus, the asymptotic lower bound when $k = 1$ is determined by the “wrong” subset that is the most difficult to be distinguished from $A$, where the difficulty level is quantified by the information numbers defined in (9).

The techniques in [40] require that the probability of selecting the wrong subset goes to 0; thus, they do not apply to the case of generalized error control ($k > 1$). Nevertheless, it is reasonable to conjecture that the corresponding asymptotic lower bound when $k > 1$ will still be determined by the wrong subset that is the most difficult to be distinguished from $A$, with the difference that a subset will now be “wrong” under $P_A$ if it differs from $A$ in at least $k$ components, i.e., if it does not belong to

$$U_k(A) := \{ C \subset [J] : |A \triangle C| < k \}.$$ 

This conjecture is verified by the following theorem.

**Theorem 3.2.** Fix $k \geq 1$. If (6) holds, then for any $A \subset [J]$, as $\alpha \to 0$,

$$N^*_A(k, \alpha) \geq \frac{|\log(\alpha)|}{D_A(k)} (1 - o(1)), \quad \text{where} \quad D_A(k) := \min_{C \notin U_k(A)} I^{A,C}. \quad (20)$$
The proof in the case of the classical mis-classification rate \((k = 1)\) is based on a change of measure from \(P_A\) to \(P_{A^*}\), where \(A^*\) is chosen such that (i) \(A\) is a “wrong” subset under \(P_{A^*}\), i.e., \(A \neq A^*\) and (ii) \(A^*\) is “close” to \(A\), in the sense that \(I^{A,A^*} \leq I^{A,C}\) for every \(C \neq A\) (see, e.g., [40, Theorem 2.2]).

When \(k \geq 2\), there are more than one “correct” subsets under \(P_A\). The key idea in our proof is that for each “correct” subset \(B \in U_k(A)\) we apply a different change of measure \(P_A \to P_{B^*}\), where \(B^*\) is chosen such that (i) \(B\) is a “wrong” subset under \(P_{B^*}\), i.e., \(B \notin U_k(B^*)\), and (ii) \(B^*\) is “close” to \(A\), in the sense that \(I^{A,B^*} \leq I^{A,C}\) for every \(C \notin U_k(A)\). The existence of such \(B^*\) is established in Appendix B.2, and the proof of Theorem 3.2 is carried out in Appendix B.3.

3.4. Asymptotic optimality. We are now ready to establish the asymptotic optimality of the Sum-Intersection rule by showing that it attains the asymptotic lower bound of Theorem 3.2 under every signal configuration.

Theorem 3.3. Assume (8) holds. Then, for any \(A \subset [J]\) we have as \(b \to \infty\) that

\[
E_A[T_S(b)] \leq \frac{b}{D_A(k)} (1 + o(1)).
\]

When in particular \(b\) is selected such that \(\delta_S \in \Delta_k(\alpha)\) and \(b \sim |\log(\alpha)|\), e.g. as in (19), then for every \(A \subset [J]\) we have as \(\alpha \to 0\)

\[
E_A[T_S] \sim \frac{|\log \alpha|}{D_A(k)} \sim N_A^*(k, \alpha).
\]

Proof. If (21) holds and \(b\) is such that \(\delta_S \in \Delta_k(\alpha)\) and \(b \sim |\log(\alpha)|\), then \(\delta_S\) attains the asymptotic lower bound in Theorem 3.2. Thus, it suffices to prove (21), which is done in the Appendix B.4. \(\square\)

The asymptotic characterization of the optimal ESS, \(N_A^*(k, \alpha)\), illustrates the trade-off among the ESS, the number of mistakes to be tolerated, and the error tolerance level \(\alpha\). Specifically, it suggests that, for “small” values of \(\alpha\), tolerating \(k - 1\) mistakes reduces the ESS by a factor of \(D_A(k)/D_A(1)\), which is at least \(k\) for every \(A \subset [J]\). To justify the latter claim, note that if we denote the ordered information numbers \(\{I^j_1, j \in A\} \cup \{I^j_0, j \notin A\}\) by \(\widetilde{T}^{(1)}(A) \leq \ldots \leq \widetilde{T}^{(J)}(A)\), then

\[
D_A(k) = \sum_{j=1}^{k} \widetilde{T}^{(j)}(A).
\]
In the following corollary we show that the Intersection rule is asymptotically optimal when (11) holds, which is the case for example when the multiple testing problem is both symmetric and homogeneous.

**Corollary 3.4.** (i) Assume (5) holds. For any $\alpha \in (0,1)$ we have $\delta_I(b, b) \in \Delta_k(\alpha)$ when $b$ is equal to $b_\alpha/k$, where $b_\alpha$ is defined in (19).
(ii) Suppose $b$ is selected such that $\delta_I(b, b) \in \Delta_k(\alpha)$ and $b \sim |\log \alpha|/k$, e.g., as in (i). If (8) holds, then

$$E_A [T_I] \leq \frac{|\log \alpha|}{kD_A(1)} (1 + o(1)).$$

If also (11) holds, then for any $A \subset [J]$ we have as $\alpha \to 0$ that

$$E_A [T_I] \sim \frac{|\log \alpha|}{kL} \sim N_A^*(k, \alpha).$$

**Proof.** The proof can be found in Appendix B.5.

**Remark 3.1.** When (11) is violated, the Intersection rule fails to be asymptotically optimal. This will be illustrated with a simulation study in Appendix A.2.

### 3.5. Fixed-sample size rules.

Finally, we focus on the i.i.d. case (15) and consider procedures that stop at a deterministic time, selected to control the generalized mis-classification rate. We recall that $C^j$ is the Chernoff information in the $j^{th}$ testing problem, and we denote by $B(k)$ the sum of the smallest $k$ local Chernoff informations, i.e.,

$$B(k) := \sum_{j=1}^{k} C^{(j)},$$

where $C^{(1)} \leq C^{(2)} \leq \ldots \leq C^{(J)}$ are the ordered values of the local Chernoff information numbers $C^j, j \in [J]$.

**Theorem 3.5.** Consider the multiple testing problem with i.i.d. streams defined in (15) and suppose that the Kullback-Leibler numbers in (16) are positive and finite. For any user-specified integer $1 \leq k \leq (J + 1)/2$ and $A \subset [J]$, we have as $\alpha \to 0$

$$\frac{D_A(k)}{B(2k - 1)} (1 - o(1)) \leq \frac{n^*(k, \alpha)}{N_A^*(k, \alpha)} \leq \frac{n_{NP}(k, \alpha)}{N_A^*(k, \alpha)} \sim \frac{D_A(k)}{B(k)}.$$
Proof. The proof can be found in Appendix B.6. □

Remark 3.2. Since any fixed time is also a stopping time, the lower bound is relevant only when \( D_A(k) > B(2k - 1) \) for some \( A \subset [J] \).

We now specialize the results of the previous theorem to the testing of normal means, introduced in Example 2.1 (a Bernoulli example is presented in Appendix B.7). In this case, \( C^j = T^j/4 \) for every \( j \in [J] \), which implies \( D_A(k) = 4B(k) \) for every \( A \subset [J] \), and by Theorem 3.5 it follows that

\[
n_{NP}(k, \alpha) \sim 4 N_\ast_A(k, \alpha) \quad \forall \ A \subset [J].
\]

That is, for any \( k \in [1, (J+1)/2] \), when utilizing the MNP rule instead of the proposed asymptotically optimal Sum-Intersection rule, the ESS increases by roughly a factor of 4, for small values of \( \alpha \), under every configuration. From Theorem 3.5 it also follows that for any \( A \subset [J] \) we have

\[
\liminf_{\alpha \to 0} \frac{n_\ast(k, \alpha)}{N_\ast_A(k, \alpha)} \geq \frac{4B(k)}{B(2k - 1)}.
\]

If in addition the hypotheses have identical information numbers, i.e., (11) holds, this lower bound is always larger than 2, which means that any fixed-sample size scheme will require at least twice as many observations as the Sum-Intersection rule, for small error probabilities.

4. Generalized familywise error rates of both kinds. In this section we study Problem 2.2. While we follow similar ideas and the results are of similar nature as in the previous section, the proposed procedure and the proof of its asymptotic optimality turn out to be much more complicated.

To describe the proposed multiple testing procedure, we first need to introduce some additional notations. Specifically, we denote by

\[
0 < \hat{\lambda}^1(n) \leq \ldots \leq \hat{\lambda}^p(n)
\]

the order statistics of the positive LLRs at time \( n \), \( \{\lambda^j(n) : \lambda^j(n) > 0, \ j \in [J]\} \), where \( p(n) \) is the number of the strictly positive LLRs at time \( n \). Similarly, we denote by

\[
0 \leq \hat{\lambda}^1(n) \leq \ldots \leq \hat{\lambda}^q(n)
\]

the order statistics of the absolute values of the non-positive LLRs at time \( n \), i.e., \( \{-\lambda^j(n) : \lambda^j(n) \leq 0, \ j \in [J]\} \), where \( q(n) := J - p(n) \). We also adopt the following convention:

\[
(22) \quad \hat{\lambda}^j(n) = \infty \text{ if } j > p(n), \quad \text{and} \quad \hat{\lambda}^j(n) = \infty \text{ if } j > q(n).
\]
Moreover, we use the following notation

\[
\hat{\lambda}_{ij}(n) = \tilde{\lambda}_j(n), \quad \forall j \in \{1, \ldots, p(n)\},
\]

\[
\tilde{\lambda}_{ij}(n) = -\hat{\lambda}_j(n), \quad \forall j \in \{1, \ldots, q(n)\},
\]

for the indices of streams with positive and non-positive LLRs at time \(n\), respectively. Thus, stream \(i_1(n)\) (resp. \(i_1(n)\)) has the least significant positive (resp. negative) LLR at time \(n\).

4.1. Asymmetric Sum-Intersection rule. We start with a procedure that has the same decision rule as the Sum-Intersection procedure (Subsection 3.1), but a different stopping rule that accounts for the asymmetry in the error metric that we consider in this section. Specifically, we consider a procedure \(\delta_0(a, b) \equiv (\tau_0, D_0)\) that stops as soon as the following two conditions are satisfied simultaneously: (i) the sum of the \(k_1\) least significant positive LLRs is larger than \(b > 0\), and (ii) the sum of the \(k_2\) least significant negative LLRs is smaller than \(-a < 0\). Formally,

\[
\tau_0 := \inf \left\{ n \geq 1 : \sum_{j=1}^{k_1} \hat{\lambda}_j(n) \geq b \text{ and } \sum_{j=1}^{k_2} \tilde{\lambda}_j(n) \geq a \right\},
\]

\[
D_0 := \{ j \in [J] : \lambda_j(\tau_0) > 0 \} = \{ \hat{i}_1(\tau_0), \ldots, \hat{i}_{p(\tau_0)}(\tau_0) \}.
\]

We refer to this procedure as asymmetric Sum-Intersection rule. Note that similarly to the Sum-Intersection rule, this procedure does not require strong evidence from every individual stream in order to terminate sampling. Indeed, upon stopping there may be insufficient evidence for the hypotheses that correspond to the \(k_1 - 1\) least significant positive statistics and the \(k_2 - 1\) least significant negative statistics, turning them into the anticipated false positives and false negatives, respectively, which we are allowed to make.

We will see that while the asymmetric Sum-Intersection rule can control generalized familywise error rates of both types, it is not in general asymptotically optimal. To understand why this is the case, let \(A\) denote true subset of streams with signals and suppose that there is a subset \(B\) of \(\ell\) streams with noise, i.e., \(B \subset A^c\) with \(|B| = \ell\), such that \(\ell < k_1\) and

\[
I_j^1 \gg I_0^1 \gg I_0^2, \quad \forall j \in A, \quad i_1 \in A^c \setminus B, \quad i_2 \in B,
\]

i.e., the hypotheses in streams with signal are much easier than in streams with noise, and the hypotheses in \(B\) are much harder than in the other streams with noise. In this case, the first stopping requirement in \(\tau_0\) will
be easily satisfied, but not the second one, since the streams in $B$ will slow down the growth of the sum of the $k_2$ least significant negative LLRs.

These observations suggest that the performance of $\delta_0$ can be improved in the above scenario if we essentially “give up” the testing problems in $B$, presuming that we will make $\ell$ of the $k_1 - 1$ false positives in these streams. This can be achieved by (i) ignoring the $\ell$ least significant negative statistics in the second stopping requirement of $\tau_0$, and asking the sum of the next $k_2$ least significant negative statistics to be small upon stopping, and (ii) modifying the decision rule to reject the nulls not only in streams with positive LLR, but also in the $\ell$ streams with the least significant negative LLRs upon stopping. However, if we modify the decision rule in this way, we have spent from the beginning $\ell$ of the $k_1 - 1$ false positives we are allowed to make. This implies that we need to also modify the first stopping requirement in $\tau_0$ and ask the sum of the $k_1 - \ell$ least significant positive LLRs to be large upon stopping. If we denote by $\hat{\delta}_\ell := (\hat{\tau}_\ell, \hat{D}_\ell)$ the procedure that incorporates the above modifications, then

$$\hat{\tau}_\ell := \inf \left\{ n \geq 1 : \sum_{j=1}^{k_2-\ell} \hat{\lambda}_j(n) \geq a \right\},$$

$$\hat{D}_\ell := \{\hat{i}_{\ell+1}(\hat{\tau}_\ell), \ldots, \hat{i}_{p(\hat{\tau}_\ell)}(\hat{\tau}_\ell)\} \bigcup \{\hat{i}_1(\hat{\tau}_\ell), \ldots, \hat{i}_{\ell}(\hat{\tau}_\ell)\},$$

where we omit the dependence on $a, b$ in order to lighten the notation.

By the same token, if there are $\ell < k_2$ streams with signal in which the testing problems are much harder than in other streams, it is reasonable to expect that $\delta_0$ may be outperformed by a procedure $\tilde{\delta}_\ell := (\tilde{\tau}_\ell, \tilde{D}_\ell)$, where

$$\tilde{\tau}_\ell := \inf \left\{ n \geq 1 : \sum_{i=\ell+1}^{\ell+k_1} \tilde{\lambda}_i(n) \geq b \text{ and } \sum_{j=1}^{k_2-\ell} \tilde{\lambda}_j(n) \geq a \right\},$$

$$\tilde{D}_\ell := \{\tilde{i}_{\ell+1}(\tilde{\tau}_\ell), \ldots, \tilde{i}_{p(\tilde{\tau}_\ell)}(\tilde{\tau}_\ell)\}.$$

Fig. 1 provides a visualization of these stopping rules.

4.2. The Leap rule. The previous discussion suggests that the asymmetric Sum-Intersection rule, defined in (23), may be significantly outperformed by some of the procedures, $\{\hat{\delta}_\ell, 0 \leq \ell < k_1\}$ and $\{\tilde{\delta}_\ell, 1 \leq \ell < k_2\}$, under some signal configurations, when the multiple testing problem is asymmetric and/or inhomogeneous. In this case, we propose combining the above procedures, i.e., stop as soon as any of them does so, and use the corresponding decision rule upon stopping. If multiple stopping criteria are sat-
Fig 1: Set $J = 7, k_1 = 3, k_2 = 2$. Suppose at time $n$, $p(n) = 4, q(n) = 3$. Each rule stops when the sum of the terms with solid underline exceeds $b$, and at the same time the sum of the terms with dashed underline is below $-a$. Upon stopping, the null hypothesis for the streams in the bracket are rejected. Note that by convention (22), $\hat{\lambda}_4(n) = \infty$, which makes the stopping rule $\hat{\tau}_2$ have only one condition to satisfy.

isfied at the same time, we then use the decision rule that rejects the most null hypotheses.

Formally, the proposed procedure $\delta_L := (T_L, D_L)$ is defined as follows:

$$T_L := \min \left\{ \min_{0 \leq \ell < k_1} \hat{\tau}_\ell, \min_{1 \leq \ell < k_2} \tilde{\tau}_\ell \right\},$$

$$D_L := \left( \bigcup_{0 \leq \ell < k_1; \hat{\tau}_\ell = T_L} \hat{D}_\ell \right) \bigcup \left( \bigcup_{1 \leq \ell < k_2; \tilde{\tau}_\ell = T_L} \tilde{D}_\ell \right),$$

and we refer to it as “Leap rule”, because $\hat{\delta}_\ell$ (resp. $\tilde{\delta}_\ell$) “leaps” across the $\ell$ least significant negative (resp. positive) LLRs.

4.3. Error control of the Leap rule. We now show that the Leap rule can control generalized familywise error rates of both types.

**Theorem 4.1.** Assume (5) holds. For any $\alpha, \beta \in (0, 1)$ we have that $\delta_L \in \Delta_{k_1, k_2}(\alpha, \beta)$ when the thresholds are selected as follows:

$$a = |\log(\beta)| + \log(2^{k_2}C_{k_2}^J), \quad b = |\log(\alpha)| + \log(2^{k_1}C_{k_1}^J).$$

**Proof.** The proof can be found in Appendix C.1. □

The above threshold values are sufficient for establishing the asymptotic optimality of the Leap rule, but may be conservative in practice. Thus, as in
the previous section, we recommend using simulation to find the thresholds that attain the target error probabilities. This means simulating for every $A \subset [J]$ the error probabilities of the Leap rule, $P_A(\lvert D_L(a, b) \setminus A \rvert \geq k_1)$ and $P_A(\lvert A \setminus D_L(a, b) \rvert \geq k_2)$, for various pairs of thresholds, $a$ and $b$, and selecting the values for which the maxima (with respect to $A$) of the above error probabilities match the nominal levels, $\alpha$ and $\beta$, respectively.

As in the previous section, this task is facilitated when the multiple testing problem has a special structure. Specifically, when it is symmetric and the user-specified parameters are selected so that $\alpha = \beta$ and $k_1 = k_2$, i.e., when condition (12) holds, we can select without any loss of generality the thresholds to be equal ($a = b$). Moreover, if the multiple testing problem is homogeneous, the discussion following Theorem 3.1 also applies here.

4.4. Asymptotic optimality. For any $B \subset [J]$ and $1 \leq \ell \leq u \leq J$, we denote by

$$I_1^{(1)}(B) \leq \ldots \leq I_1^{(|B|)}(B)$$

the increasingly ordered sequence of $I_1^j, j \in B$, and by

$$I_0^{(1)}(B) \leq \ldots \leq I_0^{(|B|)}(B)$$

the increasingly ordered sequence of $I_0^j, j \in B$, and we set

$$D_1(B; \ell, u) := \sum_{j=\ell}^{u} I_1^j(B), \quad \text{where} \quad I_1^j(B) = \infty \quad \text{for} \quad j > |B|,$$

$$D_0(B; \ell, u) := \sum_{j=\ell}^{u} I_0^j(B), \quad \text{where} \quad I_0^j(B) = \infty \quad \text{for} \quad j > |B|.$$

The following lemma provides an asymptotic upper bound on the expected sample size of the stopping times that compose the stopping time of the Leap rule.

**Lemma 4.2.** Assume (8) holds. For any $A \subset [J]$ we have as $a, b \to \infty$

$$E_A[\hat{\tau}_1] \leq \max \left\{ \frac{b(1 + o(1))}{D_1(A; 1, k_1 - \ell)}, \frac{a(1 + o(1))}{D_0(A^c; \ell + 1, \ell + k_2)} \right\}, \quad 0 \leq \ell < k_1,$$

$$E_A[\hat{\tau}_2] \leq \max \left\{ \frac{b(1 + o(1))}{D_1(A; \ell + 1, \ell + k_1)}, \frac{a(1 + o(1))}{D_0(A^c; 1, k_2 - \ell)} \right\}, \quad 0 \leq \ell < k_2.$$

**Proof.** The proof can be found in Appendix C.2. \qed
If thresholds are selected according to (25), then the upper bounds in the previous lemma are equal (to a first-order asymptotic approximation) to
\[
\hat{L}_A(\ell; \alpha, \beta) := \max \left\{ \left| \log \alpha \right| \frac{D_1(A; \ell, k_1 - \ell)}{D_0(A^c; \ell + 1, \ell + k_2)} , \left| \log \beta \right| \frac{D_0(A^c; \ell + 1, \ell + k_2)}{D_1(A; \ell, k_1 - \ell)} \right\}
\]
for \(\ell < k_1\),
\[
\bar{L}_A(\ell; \alpha, \beta) := \max \left\{ \left| \log \alpha \right| \frac{D_1(A; \ell + 1, \ell + k_1)}{D_0(A^c; 1, k_2 - \ell)} , \left| \log \beta \right| \frac{D_0(A^c; 1, k_2 - \ell)}{D_1(A^c; \ell + 1, \ell + k_2)} \right\}
\]
for \(\ell < k_2\), and from the definition of Leap rule in (24) it follows that as \(\alpha, \beta \to 0\) we have
\[
E_A[T_L] \leq L_A(k_1, k_2, \alpha, \beta) \left(1 + o(1)\right),
\]
where
\[
L_A(k_1, k_2, \alpha, \beta) := \min \left\{ \min_{0 \leq \ell < k_1} \hat{L}_A(\ell; \alpha, \beta) , \min_{0 \leq \ell < k_2} \bar{L}_A(\ell; \alpha, \beta) \right\}.
\]

In the next theorem we show that it is not possible to achieve a smaller ESS, to a first-order asymptotic approximation as \(\alpha, \beta \to 0\), proving in this way the asymptotic optimality of the Leap rule.

**Theorem 4.3.** Assume (8) holds and that the thresholds in the Leap rule are selected such that \(\delta_L \in \Delta_{k_1,k_2}(\alpha, \beta)\) and \(a \sim |\log(\beta)|, b \sim |\log(\alpha)|\), e.g. according to (25). Then, for any \(A \subset [J]\) we have as \(\alpha, \beta \to 0\),
\[
E_A[T_L] \sim L_A(k_1, k_2, \alpha, \beta) \sim N^*_A(k_1, k_2, \alpha, \beta).
\]

**Proof.** In view of the discussion prior to the theorem, it suffices to show that for any \(A \subset [J]\) we have as \(\alpha, \beta \to 0\) that
\[
N^*_A(k_1, k_2, \alpha, \beta) \geq L_A(k_1, k_2, \alpha, \beta) \left(1 - o(1)\right).
\]

For the proof of this asymptotic lower bound we employ similar ideas as in the proof of Theorem 3.2 in the previous section. The change-of-measure argument is more complicated now, due to the interplay of the two kinds of error. We carry out the proof in Appendix C.4.

**Remark 4.1.** When \(k_1 = k_2 = 1\), the asymptotic optimality of the Intersection rule was established in [35] only in the i.i.d. case. Since the Leap rule coincides with the Intersection rule when \(k_1 = k_2 = 1\), Theorem 4.3 generalizes this result in [35] beyond the i.i.d. case.

We motivated the Leap rule by the inadequacy of the asymmetric Sum-Intersection rule, \(\delta_0\), in the case of asymmetric and/or inhomogeneous testing problems. In the following corollary we show that \(\delta_0\) is asymptotically
optimal when (i) condition (11) holds, which is the case when the multi-
ple testing problem is symmetric and homogeneous, and also (ii) the user-
specified parameters are selected in a symmetric way, i.e., when (12) holds.
In the same setup we establish the asymptotic optimality of the Intersection
rule, $\delta_I$, defined in (10).

**Corollary 4.4.** Suppose (8) and (11)–(12) hold and consider the asym-
metric Sum-Intersection rule $\delta_0(b, b)$ with $b = b_\alpha$ and the Intersection rule
$\delta_I(b, b)$ with $b = b_\alpha/k_1$, where $b_\alpha$ is defined in (19) with $k = k_1$. Then
$\delta_0, \delta_I \in \Delta_{k_1, k_1}(\alpha, \alpha)$, and for any $A \subset [J]$ we have as $\alpha \to 0$ that

$$
E_A[\tau_0] \sim E_A[T_I] \sim \frac{|\log(\alpha)|}{k_1 I} \sim N_A^*(k_1, k_1, \alpha, \alpha).
$$

**Proof.** The proof can be found in Appendix C.5.

**Remark 4.2.** In Section 5.2 we will illustrate numerically that when condi-
tion (11) is violated, both $\delta_0$ and $\delta_I$ fail to be asymptotically optimal.

4.5. **Fixed-sample size rules.** We now focus on the i.i.d. case (15) and
consider procedures that stop at a deterministic time, which is selected to
control the generalized familywise error rates.

For simplicity of presentation, we restrict ourselves to homogeneous test-
ning problems, i.e., there are densities $f_0$ and $f_1$ such that

$$
(27) \quad f_j^0 = f_0, \quad f_j^1 = f_1 \quad \text{for every } j \in [J].
$$

This assumption allows us to omit the dependence on the stream index $j$
and write $I_0 := I_0^j$, $I_1 := I_1^j$ and $\Phi := \Phi^j$, where $\Phi^j$ is defined in (17).
Moreover, without loss of generality, we apply the MNP rule (14) with the
same threshold for each stream.

We further assume that user-specified parameters are selected as follows

$$
(28) \quad k_1 = k_2, \quad \alpha = \beta^d \quad \text{for some } d > 0,
$$

and that for each $d > 0$ there exists some $h_d \in (-I_0, I_1)$ such that

$$
(29) \quad \Phi(h_d)/d = \Phi(h_d) - h_d.
$$

When $d = 1$, condition (28) reduces to (12) and $h_d$ is equal to 0. However,
when $d \neq 1$, we allow for an asymmetric treatment of the two kinds of error.
Theorem 4.5. Consider the multiple testing problem (27) and assume that the Kullback-Leibler numbers in (16) are positive and finite. Further, assume that (28) and (29) hold. Then as $\beta \to 0$,

$$\frac{d (1 - o(1))}{(2k_1 - 1)\Phi(h_d)} \leq \frac{n^*(k_1, k_1, \beta^d, \beta)}{|\log(\beta)|} \leq \frac{\hat{n}_{NP}(k_1, k_1, \beta^d, \beta)}{|\log(\beta)|} \sim \frac{d}{k_1 \Phi(h_d)}.$$ \hfill \Box

Proof. The proof is similar to that of Theorem 3.5, but requires a generalization of Chernoff’s lemma [11, Corollary 3.4.6] to account for the asymmetry of the two kinds of error. This generalization is presented in Lemma G.1 and more details can be found in Appendix C.6.

Theorem 4.5, in conjunction with Theorem 4.3, allows us to quantify the performance loss that is induced by stopping at a deterministic time. Specifically, in the case of testing normal means (Example 2.1), by (18) we have $\mathcal{I} = \mathcal{I}_1 = \mathcal{I}_0$ and for any $d \geq 1$

$$h_d = \frac{\sqrt{d} - 1}{\sqrt{d} + 1} \mathcal{I}, \quad \Phi(h_d) = \frac{d}{(1 + \sqrt{d})^2} \mathcal{I}.$$\hfill \hfill \hfill

Thus, by Theorem 4.3 it follows that as $\beta \to 0$,

$$N^*_A(k_1, k_1, \beta^d, \beta) \leq \hat{L}_A(0; \beta^d, \beta) = \begin{cases} \frac{|\log(\beta)|}{k_1^2}, & \text{if } |A| < k_1 \\ \frac{|d \log(\beta)|}{k_1^2}, & \text{if } |A| \geq k_1 \end{cases}.$$\hfill \hfill \hfill

When in particular $d = 1$, i.e., $\alpha = \beta$, for any $A \subset [J]$ we have

$$2N^*_A(k_1, k_1, \beta, \beta)(1 - o(1)) \leq n^*(k_1, k_1, \beta, \beta)$$

$$\leq \hat{n}_{NP}(k_1, k_1, \beta, \beta) \sim 4N^*_A(k_1, k_1, \beta, \beta),$$

which agrees with the corresponding findings in Subsection 3.5.

5. Simulations for generalized familywise error rates. In this section we present two simulation studies that complement our asymptotic optimality theory in Section 4 for procedures that control generalized familywise error rates. In the first study we compare the Leap rule (24), the Intersection rule (10) and the asymmetric Sum-Intersection rule (23), in a symmetric and homogeneous setup where conditions (11) and (12) hold and all three procedures are asymptotically optimal. In the second study we compare the same procedures when condition (11) is slightly violated, and only the Leap rule enjoys the asymptotic optimality property.
In both studies we consider the testing of normal means (Example 2.1), with $\sigma_j = 1$ for every $j \in [J]$. This is a symmetric multiple testing problem, where the Kullback-Leibler information in the $j$-th testing problem is $I^j = \mu_j^2 / 2$. Moreover, we assume that condition (12) holds, i.e., $\alpha = \beta$ and $k_1 = k_2$. This implies that we can set the thresholds in each sequential procedure to be equal, i.e., $a = b$, and as a result the two types of generalized familywise error rates will be the same. Finally, in both studies we include the performance of the fixed-sample size multiple Neyman-Pearson (MNP) rule (14), for which the choice of thresholds depends crucially on whether the problem is homogeneous or not.

In what follows, the “error probability (Err)” is the generalized familywise error rate of false positives (3), i.e., the maximum probability of $k_1$ false positives, with the maximum taken over all signal configurations. Thus, Err does not depend on the true subset of signals $A \subset [J]$.

5.1. Homogeneous case. In the first simulation study we set $\mu_j = 0.25$ for each $j \in [J]$. In this homogeneous setup, the expected sample size (ESS) of all procedures under consideration depend only on the number of signals, and we can set the thresholds in the MNP rule, defined in (14), to be equal to 0. Moreover, it suffices to study the performance when the number of signals is no more than $J/2$. We consider $J = 100$ in Fig. 2 and $J = 20$ in Fig. 3.

In Fig. 2a, we fix $k_1 = 4$ and evaluate the ESS of the Leap rule for four different cases regarding the number of signals. We see that, for any given Err, the smallest possible ESS is achieved in the boundary case of no signals ($|A| = 0$). This is because some components in the Leap rule only have one condition to be satisfied in the boundary cases (e.g. $\tilde{\tau}_2$ in Fig. 1).

In Fig. 2b, we fix the number of signals to be $|A| = 50$ and evaluate the Leap rule for different values of $k_1$. We observe that there are significant savings in the ESS as $k_1$ increases and more mistakes are tolerated.

In Fig. 2c and 2d, we fix $k_1 = 4$ and compare the four rules for $|A| = 0$ and 50, respectively. In this symmetric and homogeneous setup, where (11) and (12) both hold, we have shown that all three sequential procedures are asymptotically optimal. Our simulations suggest that in practice the Leap rule works better when the number of signals, $|A|$, is close to 0 or $J$, but may perform slightly worse than the asymmetric Sum-Intersection rule, $\delta_0$, when $|A|$ is close to $J/2$.

In Fig. 2c, 2d and 3a, we also compare the performance of the Leap rule with the MNP rule. Further, in Fig. 2e, 2f, 3b and 3c, we show the histogram of the stopping time of the Leap rule at particular error levels. From these
figures we can see that the best-case scenario for the MNP is when both the number of hypotheses, \( J \), and the error probabilities, \( \text{Err} \), are large. Note that this does not contradict our asymptotic analysis, where \( J \) is fixed and we let \( \text{Err} \) go to 0.

![Graphs](image)

5.2. **Non-homogeneous case.** In the second simulation study we set \( J = 10, \mu_j = 1/6, j = 1, 2, \mu_j = 1/2, j \geq 3 \), so that the first two hypotheses are much harder than others. Specifically, \( \mathcal{I}_j = 1/72 \) for \( j = 1, 2 \), and \( \mathcal{I}_j = 1/8 \) for \( j \geq 3 \).

When the true subset of signals is \( A^* = \{6, \ldots, 10\} \), the optimal asymptotic performance, (26), is equal to 8|log(Err)|. In Fig. 4a, we plot the ESS against |log₁₀(Err)|, and the ratio of ESS over 8|log(Err)| in Fig. 4b. For the (asymptotically optimal) Leap rule, this ratio tends to 1 as \( \alpha \to 0 \). In contrast, the other rules have a different “slope” from the Leap rule in Fig. 4a, which indicates that they fail to be asymptotically optimal in this context.

Finally, we note that in such a non-homogeneous setup, the choice of thresholds for the MNP rule (14) is not obvious. We found that instead of
Fig 3: Homogeneous case: \( J = 20, k_1 = 2 \). In (a), the x-axis is \(|\log_{10}(\text{Err})|\) and the y-axis is the ESS under \( P_A \). In (b) and (c) are the histogram of the stopping time of the Leap rule with \( \text{Err} = 5\% \) and \( 1\% \).

setting \( h_j = 0 \) for every \( j \in [J] \), it is much more efficient to take advantage of the flexibility of generalized familywise error rates, as we did in the construction of the Leap rule in Subsection 4.2, and set \( h_1 = -\infty \), \( h_2 = \infty \) and \( h_j = 0 \) for \( j \geq 3 \). This choice “gives up” the first two “difficult” streams by always rejecting the null in the first one and accepting it in the second. The error constraints can then still be met as long as we do not make any mistakes in the remaining “easy” streams. In fact, we see that while the MNP rule behaves significantly worse than the asymptotically optimal Leap rule, it performs better than the Intersection rule, which requires strong evidence from each individual stream in order to stop.

Fig 4: Non-homogeneous case: \( J = 10, k_1 = k_2 = 2, A^* = \{6, \cdots, 10\} \). The x-axis in both graphs is \(|\log_{10}(\text{Err})|\). The y-axis in (a) is the ESS under \( P_{A^*} \), and in (b) is the ratio of the ESS over \( 8|\log(\text{Err})|\).
6. Extension to composite hypotheses. We now extend the setup introduced in Section 2, allowing both the null and the alternative hypothesis in each local testing problem to be composite. Thus, for each $j \in [J]$, the distribution of $X^j$, the sequence of observations in the $j$-th stream, is now parametrized by $\theta^j \in \Theta^j$, where $\Theta^j$ is a subset of some Euclidean space, and the hypothesis testing problem in the $j$-th stream becomes

$$H^j_0 : \theta^j \in \Theta^j_0 \quad \text{versus} \quad H^j_1 : \theta^j \in \Theta^j_1,$$

where $\Theta^j_0$ and $\Theta^j_1$ are two disjoint subsets of $\Theta^j$. When $A \subset [J]$ is the subset of streams in which the alternative is correct, we denote by $\Theta_A$ the subset of the parameter space $\Theta := \Theta^1 \times \ldots \times \Theta^J$ that is compatible with $A$, i.e.,

$$\Theta_A := \{(\theta^1, \ldots, \theta^J) \in \Theta : \theta^i \in \Theta^i_0, \theta^j \in \Theta^j_1 \quad \forall i \notin A, j \in A\}.$$

We denote by $P^j_\theta$ the distribution of the $j$-th stream when the value of its local parameter is $\theta^j$. Moreover, we denote by $P_{A, \theta}$ the underlying probability measure when the subset of signals is $A$ and the parameter is $\theta = (\theta^1, \ldots, \theta^J) \in \Theta_A$, and by $E_{A, \theta}$ the corresponding expectation. Due to the independence across streams, we have $P_{A, \theta} = P^1_\theta \otimes \ldots \otimes P^J_\theta$.

Our presentation in the case of composite hypotheses will focus on the control of generalized familywise error rates; the corresponding treatment of the generalized mis-classification rate will be similar. Thus, given $k_1, k_2 \geq 1$ and $\alpha, \beta \in (0, 1)$, the class of procedures of interest now is

$$\Delta_{k_1, k_2}^{\text{comp}}(\alpha, \beta) := \{(T, D) : \max_{A, \theta} P_{A, \theta}(|D \setminus A| \geq k_1) \leq \alpha \quad \text{and} \quad \max_{A, \theta} P_{A, \theta}(|A \setminus D| \geq k_2) \leq \beta\},$$

and the goal is the same as the one in Problem 2.2 with $N^*_A(k_1, k_2, \alpha, \beta)$ being replaced by

$$N^*_{A, \theta}(k_1, k_2, \alpha, \beta) := \inf_{(T, D) \in \Delta_{k_1, k_2}^{\text{comp}}(\alpha, \beta)} E_{A, \theta}[T],$$

and the asymptotic optimality being achieved for every $A \subset [J]$ and $\theta \in \Theta_A$.

6.1. Leap rule with adaptive log-likelihood ratios. The proposed procedure in this setup is the Leap rule (24), with the only difference that the local LLR statistics are replaced by statistics that account for the composite nature of the two hypotheses. To be more specific, for every $j \in [J]$ and $n \in \mathbb{N}$ we denote by $\ell^j(n, \theta^j)$ the log-likelihood function (with respect to
some $\sigma$-finite measure $\nu^j_0$) in the $j$-th stream based on the first $n$ observations, i.e.,
\[
\ell^j_i(n, \theta^j) := \ell^j_i(n - 1, \theta^j) + \log \left( p_{\theta^j_n}^{\ell^j_i}(X^j(n) | F^j_{n-1}) \right); \quad \ell^j_i(0, \theta^j) := 0,
\]
where $p_{\theta^j_n}^{\ell^j_i}(X^j(n) | F^j_{n-1})$ is the conditional density of $X^j(n)$ given the previous $n - 1$ observations in the $j$-th stream. Moreover, for every stream $j \in [J]$ and time $n \in \mathbb{N}$ we denote by $\ell^j_i(n)$ the corresponding \textit{generalized} log-likelihood under $H^j_i$, i.e.,
\[
\ell^j_i(n) := \sup \left\{ \ell^j_i(n, \theta^j) : \theta^j \in \Theta^j_i \right\}, \quad i = 0, 1.
\]
Further, at each $n \in \mathbb{N}$, we select an $\mathcal{F}_n$-measurable estimator of $\Theta, \hat{\theta}_n = (\hat{\theta}^1_n, \ldots, \hat{\theta}^J_n) \in \Theta$, and define the \textit{adaptive log-likelihood} statistic for the $j$-th stream as follows:
\[
(30) \quad \ell^j_s(n) := \ell^j_s(n - 1) + \log \left( p_{\theta^j_n}^{\ell^j_s}(X^j(n) | F^j_{n-1}) \right); \quad \ell^j_s(0) = 0,
\]
where $\hat{\theta}_0 := (\hat{\theta}^0_0, \ldots, \hat{\theta}^J_0) \in \Theta$ is some deterministic initialization. The proposed procedure in this context is the Leap rule (24), where each LLR statistic $\lambda^j_s(n)$ is replaced by the following \textit{adaptive} log-likelihood ratio:
\[
(31) \quad \lambda^j_s(n) := \begin{cases} 
\ell^j_s(n) - \ell^j_0(n), & \text{if } \ell^j_0(n) < \ell^j_1(n) \text{ and } \ell^j_0(n) < \ell^j_s(n) \\
-(\ell^j_s(n) - \ell^j_1(n)), & \text{if } \ell^j_1(n) < \ell^j_0(n) \text{ and } \ell^j_1(n) < \ell^j_s(n) \\
\text{undefined}, & \text{otherwise ,}
\end{cases}
\]
with the understanding that there is no stopping at time $n$ if $\lambda^j_s(n)$ is undefined for some $j$. Clearly, large positive values of $\lambda^j_s$ support $H^j_1$, whereas large negative values of $\lambda^j_s$ support $H^j_0$. We denote this modified version of the Leap rule by $\delta^*_L(a, b) = (T^*_L, D^*_L)$.

In the next subsection we establish the asymptotic optimality of $\delta^*_L$ under general conditions. In Appendix D.5 we discuss in more detail the above adaptive statistics, as well as other choices for the local statistics. In Appendix D.4 we demonstrate with a simulation study that if we replace the LLR $\lambda^j$ by the adaptive statistic $\lambda^j_s$ (31) in the \textit{Intersection rule (10)} and the \textit{asymmetric Sum-Intersection rule (23)}, then these procedures fail to be asymptotically optimal \textit{even in the presence of special structures}. Finally, we should point out that the gains over fixed-sample size procedures are also larger compared to the case of simple hypotheses, as sequential methods are more adaptive to the unknown parameter.
6.2. Asymptotic optimality. First of all, for each \( j \in [J] \) we generalize condition (7) and assume that for any distinct \( \theta^j, \tilde{\theta}^j \in \Theta^j \) there exists a positive number \( I^j(\theta^j, \tilde{\theta}^j) \) such that

\[
\frac{1}{n} \left( \ell^j(n, \theta^j) - \ell^j(n, \tilde{\theta}^j) \right) \xrightarrow{p^j_{\theta^j} \text{ completely}} I^j(\theta^j, \tilde{\theta}^j).
\]

Second, we require that the null and alternative hypotheses in each stream are separated, in the sense that if for each \( j \in [J] \) and \( \theta^j \in \Theta^j \) we define

\[
I^j_0(\theta^j) := \inf_{\hat{\theta}^j \in \Theta^j_0} I^j(\theta^j, \hat{\theta}^j) \quad \text{and} \quad I^j_1(\theta^j) := \inf_{\hat{\theta}^j \in \Theta^j_1} I^j(\theta^j, \hat{\theta}^j),
\]

then

\[
I^j_0(\theta^j) > 0 \quad \forall \theta^j \in \Theta^j_0 \quad \text{and} \quad I^j_1(\theta^j) > 0 \quad \forall \theta^j \in \Theta^j_1.
\]

Finally, we assume that for each \( j \in [J] \) and \( \epsilon > 0 \),

\[
\sum_{n=1}^{\infty} P^j_{\theta^j} \left( \frac{\ell^j_*(n) - \ell^j_0(n)}{n} - I^j_0(\theta^j) < -\epsilon \right) < \infty \quad \text{for every} \quad \theta^j \in \Theta^j_0,
\]

\[(35) \quad \sum_{n=1}^{\infty} P^j_{\theta^j} \left( \frac{\ell^j_*(n) - \ell^j_0(n)}{n} - I^j_1(\theta^j) < -\epsilon \right) < \infty \quad \text{for every} \quad \theta^j \in \Theta^j_1.
\]

We now state the main result of this section, the asymptotic optimality of \( \delta_L^* \) under the above conditions. The proof is presented in Appendix D.

**Theorem 6.1.** Assume (32), (34) and (35) hold. Further, assume the thresholds in the Leap rule are selected such that \( \delta_L^*(a, b) \in \Delta^\text{comp}_{k_1, k_2}(\alpha, \beta) \) and \( a \sim |\log(\beta)|, b \sim |\log(\alpha)| \), e.g. according to (25). Then, for any \( A \subset [J] \) and \( \theta \in \Theta_A \), we have as \( \alpha, \beta \to 0 \),

\[
E_{A, \theta} [T_L] \sim L_{A, \theta}(k_1, k_2, \alpha, \beta) \sim N_{A, \theta}^*(k_1, k_2, \alpha, \beta),
\]

where \( L_{A, \theta}(k_1, k_2, \alpha, \beta) \) is a quantity defined in Appendix D.1 that characterizes the asymptotic optimal performance.

While conditions (32) and (34) are easily satisfied and simple to check, the one-sided complete convergence condition (35) is not as apparent. It is known [39, p. 278-280] that when \( \hat{\theta}_n^j \) is selected to be the Maximum Likelihood estimator (MLE) of \( \theta^j \), condition (35) is satisfied when testing a normal mean with unknown variance, as well as when testing the coefficient of a first-order autoregressive model. In Appendix E we further show that condition (35) is satisfied when (i) the data in each stream are i.i.d. with some multi-parameter exponential family distribution, and (ii) the null and the alternative parameter spaces are compact.
7. Conclusion. In this paper we have considered the sequential multiple testing problem under two error metrics. In the first one, the goal is to control the probability of at least $k$ mistakes, of any kind. In the second one, the goal is to control simultaneously the probabilities of at least $k_1$ false positives and at least $k_2$ false negatives. Assuming that the data for the various hypotheses are obtained sequentially in independent streams, we characterized the optimal performance to a first-order asymptotic approximation as the error probabilities vanish, and proposed the first asymptotically optimal procedure for each of the two problems. Procedures that are asymptotically optimal under classical error control ($k = 1, k_1 = k_2 = 1$) were found to be suboptimal under generalized error metrics. Moreover, in the case of i.i.d. data streams we quantified the asymptotic savings in the expected sample size relative to fixed-sample size procedures.

There are certain questions that remain open. First, we conducted a first-order asymptotic analysis, ignoring higher-order terms in the approximation to the optimal performance. The latter however appears to be non-negligible in practice (see Fig. 4b). Thus, it is an open problem to obtain a more precise characterization of the optimal performance, as well as to examine whether the proposed rules enjoy a stronger optimality property. Second, the number of streams is treated as constant in our asymptotic analysis, but can be very large in practice. It is interesting to consider an enhanced asymptotic regime, where the number of streams also goes to infinity as the error probabilities vanish. Third, although simulation techniques can be used to determine threshold values that guarantee the error control, it is desirable to have closed-form expressions for less conservative threshold values.

Finally, there are several interesting generalizations in various directions. One direction is to relax the assumption that the streams corresponding to the different testing problems are independent. Another direction is to allow for early stopping in some streams, in which case the goal may be to minimize the total number of observations in all streams. Finally, it is interesting to study the corresponding problems with FDR-type error control.

APPENDIX A: SIMULATIONS FOR GENERALIZED MIS-CLASSIFICATION RATE

In this section we present two simulation studies that complement our asymptotic optimality theory for procedures that control the generalized mis-classification rate (Section 3). Specifically, our goal is to compare the proposed Sum-Intersection rule (Subsection 3.1) and the Intersection rule (10) in two setups. The first one is a symmetric and homogeneous setup, in which (11) holds and both rules are asymptotically optimal. The second one
is a non-homogeneous setup, where condition (11) is (slightly) violated and the Intersection rule fails to be asymptotically optimal. In each setup, we also include the performance of the multiple Neyman-Pearson rule (MNP) (14), which is a fixed-sample size procedure.

For these comparisons, we consider the testing of normal means, introduced in Example 2.1. As discussed in Example 2.1, this problem is symmetric. As a result, we set $h = 0$ in the MNP rule (14), and further the performance of each rule under consideration is the same for any subset of signals. Thus we do not need to specify the actual subset of signals.

A.1. Homogeneous case. In the first study we set $\mu_j = 0.25, \sigma_j = 1$ for every $j \in [J]$. We consider $J = 100$ in Fig. 5 and $J = 20$ in Fig. 6.

In Fig. 5a, we study the performance of the Sum-Intersection rule for different values of $k$. We observe that there are significant savings in the ESS as $k$ increases and more mistakes are tolerated. In Fig. 5b, we compare the three rules for $k = 4$. Although both sequential rules enjoy the asymptotic optimality property in this setup, we observe that the Sum-Intersection rule clearly outperforms the Intersection rule.

In Fig. 5b and 6a, we also compare the Sum-Intersection rule with the MNP rule. Further, in Fig. 5c, 6b and 6c, we show the histogram of the Sum-Intersection at particular error levels. From these figures we observe that the advantage of sequential procedures over the MNP rule increases as Err decreases and decreases as $J$ increases.

Fig 5: Homogeneous case: $J = 100$. In (a) and (b), the x-axis is $|\log_{10}(\text{Err})|$ and the y-axis represents the ESS. In (c), we plot the histogram of the stopping time of the Sum-Intersection rule with Err = 5%.
Fig 6: Homogeneous case: $J = 20$. In (a), the x-axis is $|\log_{10}(\text{Err})|$ and the y-axis represents the ESS. In (b) and (c), we plot the histogram of the stopping time of the Sum-Intersection rule with Err = 5% and 1%.

A.2. Non-homogeneous case. In the second study we have injected a slight violation of homogeneity. Specifically, we set $J = 10$, $k = 2$ and

$$f_j^0 = \mathcal{N}(0,1) \quad \forall \ j \in [J], \quad f_j^1 = \begin{cases} \mathcal{N}(1/6,1) & \text{if } j = 1 \\ \mathcal{N}(1/2,1) & \text{if } j \geq 2 \end{cases}.$$  

Thus, all testing problems are identical apart from the first one, which is much harder. Indeed, $\mathcal{I}_0 = \mathcal{I}_1 = \mathcal{I}$ and $\mathcal{I} = 1/72$ for $j = 1$ and $\mathcal{I} = 1/8$ for $j \geq 2$. Since $k = 2$, the optimal asymptotic performance in this problem is determined by the two most difficult hypotheses and is equal to $7.2|\log(\text{Err})|$. In Fig. 7a we plot the expected sample size (ESS) against $|\log_{10}(\text{Err})|$ and in Fig. 7b we plot the ratio of ESS over $7.2|\log(\text{Err})|$. We observe that this ratio tends to 1 for the asymptotically optimal Sum-Intersection rule, whereas this is not the case for the other two rules. In particular, as predicted by Theorem 3.5, the ratio for the MNP rule tends to 4 as $\text{Err} \to 0$.

APPENDIX B: PROOFS REGARDING THE GENERALIZED MIS-CLASSIFICATION RATE

B.1. Proofs of Theorem 3.1.

PROOF. It suffices to show that for any $b > 0$ and $A \subset [J]$ we have

$$P_A(|A \triangle D_S(b)| \geq k) \leq C_k e^{-b}.$$  

Fix $A \subset [J]$ and $b > 0$. Observe that the event $\{|A \triangle D_S| \geq k\}$ occurs if and only if there exist $B_1 \subset A$ and $B_2 \subset A^c$ such that $|B_1| + |B_2| = k$ and
the following event occurs:

\[ \Gamma(B_1, B_2) := \{ D^i_S = 0, D^j_S = 1, \forall i \in B_1, j \in B_2 \}. \]

Since there are \( C^J_k \) such pairs, due to Boole’s inequality it suffices to show that the probability of each of these events is bounded by \( e^{-b} \). To this end, fix \( B_1 \subset A, B_2 \subset A^c \) such that \( |B_1| + |B_2| = k \) and consider the set \( C = (A \setminus B_1) \cup B_2 \). Then, with the change of measure \( \mathbb{P}_A \rightarrow \mathbb{P}_C \), we have

\[
\mathbb{P}_A(\Gamma(B_1, B_2)) = \mathbb{E}_C \left[ \exp \left\{ \lambda^{A,C}(T_S) \right\} ; \Gamma(B_1, B_2) \right].
\] (36)

For \( i \in B_1 \) we have \( D^i_S = 0 \), which implies \( \lambda^i(T_S) \leq 0 \), and for \( j \in B_2 \) we have \( D^j_S = 1 \), which implies \( \lambda^j(T_S) > 0 \). Thus, on the event \( \Gamma(B_1, B_2) \),

\[
\lambda^{A,C}(T_S) = \sum_{i \in B_1} \lambda^i(T_S) - \sum_{j \in B_2} \lambda^j(T_S)
\]

\[
= -\sum_{i \in B_1} |\lambda^i(T_S)| - \sum_{j \in B_2} |\lambda^j(T_S)| \leq -\sum_{i=1}^k \tilde{\lambda}^i(T_S) \leq -b,
\] (37)

where the first equality is due to (4), the first inequality follows from the definition of \( \tilde{\lambda}^i \)’s, and the second from the definition of the stopping time \( T_S \). Thus, the proof is complete in view of (36).

**B.2. An important Lemma.** The following lemma is crucial in establish Theorem 3.2.
**Lemma B.1.** Let $A, B \subset [J]$. Then there exists $B^* \subset [J]$ such that
\[
(i) \quad B \notin \mathcal{U}_k(B^*), \quad (ii) \quad I^{A,B^*} \leq D_A(k).
\]

To show Lemma B.1, we start with a lemma about sets.

**Lemma B.2.** Let $A, B, \Gamma \subset [J]$. There exists $B^* \subset [J]$ such that
\[
A \triangle B^* \subset \Gamma \subset B \triangle B^*.
\]

**Proof.** Define the following disjoint sets:
\[
B_1 = B \cap \Gamma, \quad B_2 = B^c \cap \Gamma, \quad A_1 = A \cap \Gamma^c, \quad A_2 = A^c \cap \Gamma^c.
\]

Clearly, $\Gamma = B_1 \cup B_2$, and $\Gamma^c = A_1 \cup A_2$. Let $B^* = B_2 \cup A_1$.

On one hand, if $j \in B_1$, then $j \in B$ and $j \notin B^*$; if $j \in B_2$, then $j \notin B$ and $j \in B^*$. It implies $\Gamma = B_1 \cup B_2 \subset B \triangle B^*$.

On the other, if $j \in A_1$, then $j \notin A$ and $j \notin B^*$; if $j \in A_2$, then $j \notin A$ and $j \notin B^*$. Thus $\Gamma^c = A_1 \cup A_2 \subset (A \triangle B^*)^c$, which implies $A \triangle B^* \subset \Gamma$.

Now we are ready to prove Lemma B.1.

**Proof.** Let $C^* \notin \mathcal{U}_k(A)$ such that $D_A(k) = I^{A,C^*}$ and set $\Gamma = A \triangle C^*$. Then, clearly $|\Gamma| \geq k$. By Lemma B.2, there exists a set $B^* \subset [J]$ such that
\[
A \triangle B^* \subset \Gamma = A \triangle C^* \subset B \triangle B^*.
\]

From the second inclusion it follows that $|B \triangle B^*| \geq |\Gamma| \geq k$, which proves (i). From the first inclusion it follows that $A \triangle B^* \subset A \triangle C^*$ and $B^* \triangle A \subset C^* \triangle A$, therefore from (4) we conclude that
\[
I^{A,B^*} = \sum_{i \in A \backslash B^*} I_i + \sum_{j \in B^* \backslash A} T_j \leq \sum_{i \in A \backslash C^*} I_i + \sum_{j \in C^* \backslash A} T_j = I^{A,C^*},
\]

which proves (ii).

**B.3. Proof of Theorem 3.2.**

**Proof.** Fix $A \subset [J]$, $k \in [J]$, and set
\[
\ell_\alpha := |\log(\alpha)|/D_A(k), \quad \alpha \in (0, 1).
\]

By Markov’s inequality, for any stopping time $T$, $\alpha \in (0, 1)$ and $q > 0$,
\[
E_A[T] \geq q\ell_\alpha P_A(T \geq q\ell_\alpha).
\]
Thus, it suffices to show for every $q \in (0, 1)$ we have

\begin{equation}
\liminf_{\alpha \to 0} \inf_{(T,D) \in \Delta_k(\alpha)} P_A(T \geq q\ell_\alpha) \geq 1,
\end{equation}

as this will imply $\liminf_{\alpha \to 0} N_A^*(k, \alpha)/\ell_\alpha \geq q$, and the desired result will follow by letting $q \to 1$.

In order to prove (38), let us start by fixing arbitrary $\alpha, q \in (0, 1)$ and $(T, D) \in \Delta_k(\alpha)$. Then,

\begin{equation}
1 - \alpha \leq P_A(D \in U_k(A)) = \sum_{B \in U_k(A)} P_A(D = B).
\end{equation}

Now, consider an arbitrary $B \in U_k(A)$, and let $B^* \subset [J]$ be a set that satisfies the two conditions in Lemma B.1. Then, $|B^* \triangle B| \geq k$, and consequently

\begin{equation}
P_{B^*}(D = B) \leq \alpha.
\end{equation}

We can now decompose the probability $P_A(D = B)$ as follows:

\[ P_A\left(\lambda^{A,B^*}(T) < \log \left(\frac{\eta}{\alpha}\right); D = B\right) + P_A\left(\lambda^{A,B^*}(T) \geq \log \left(\frac{\eta}{\alpha}\right); D = B\right), \]

where $\eta$ is an arbitrary constant in $(0, 1)$. We denote the first term by $I$ and second by $\Pi$. For the first term, by a change of measure $P_A \rightarrow P_{B^*}$ we have

\[ I = E_{B^*}\left[\exp\{\lambda^{A,B^*}(T)\}; \lambda^{A,B^*}(T) < \log \left(\frac{\eta}{\alpha}\right), D = B\right] \leq \frac{\eta}{\alpha} P_{B^*}(D = B) \leq \eta, \]

where the second inequality follows from (40). For the second term, we have

\[ \Pi \leq P_A\left(T \leq q\frac{\log \alpha}{I^{A,B^*}}, \lambda^{A,B^*}(T) \geq \log \left(\frac{\eta}{\alpha}\right)\right) + P_A(T \geq q\ell_\alpha, D = B). \]

By construction, $B^*$ satisfies $I^{A,B^*} \leq D_A(k)$; thus the first term in the right-hand side is bounded above by

\[ \epsilon_{\alpha,B^*}(T) := P_A\left(T \leq q\frac{\log \alpha}{I^{A,B^*}}, \lambda^{A,B^*}(T) \geq |\log \alpha| + \log(\eta)\right). \]

Due to the SLLN (6), we have

\[ P_A\left(\lim_{n \to \infty} \frac{\lambda^{A,B^*}(n)}{n} = I^{A,B^*}\right) = 1. \]
Therefore, by Lemma F.1, it follows that $\epsilon_{\alpha,B^*}(T) \to 0$ as $\alpha \to 0$ uniformly in $T$.

Putting everything together we have

$$P_A(D = B) \leq \eta + \epsilon_{\alpha,B^*}(T) + P_A(T \geq q_{\ell_\alpha}, D = B),$$

and summing over $B \in \mathcal{U}_k(A)$ we obtain

$$P_A(D \in \mathcal{U}_k(A)) \leq |\mathcal{U}_k(A)|\eta + \epsilon_{\alpha}(T) + P_A(T \geq q_{\ell_\alpha}),$$

where $\epsilon_{\alpha}(T) := \sum_{B \in \mathcal{U}_k(A)} \epsilon_{\alpha,B^*}(T) \to 0$ as $\alpha \to 0$ uniformly in $T$. Due to (39), we have

$$P_A(T \geq q_{\ell_\alpha}) \geq 1 - \alpha - \epsilon_{\alpha}(T) - |\mathcal{U}_k(A)|\eta.$$

Since $(T,D) \in \Delta_k(\alpha)$ is arbitrary and $\alpha \in (0,1)$ also arbitrary, taking the infimum over $(T,D)$ and letting $\alpha \to 0$ we obtain

$$\liminf_{\alpha \to 0} \inf_{(T,D) \in \Delta_k(\alpha)} P_A(T \geq q_{\ell_\alpha}) \geq 1 - |\mathcal{U}_k(A)|\eta.$$ 

Finally, letting $\eta \to 0$ we obtain (38), which completes the proof. \qed

**B.4. Proof of Theorem 3.3.** The following fact about set operations will be needed:

(41) Let $A, B \subset [J]$ and $C = A \triangle B$. Then $A \triangle C = B$.

**Proof.** Fix $A \subset [J]$ and consider the stopping time

$$T^A(b) := \inf \left\{ n \geq 1 : \lambda^{A,C}(n) \geq b \quad \forall C /\in \mathcal{U}_k(A) \right\}.$$

Under the conditions of the lemma, from Lemma F.2 in the Appendix it follows that as $b \to \infty$ we have

$$\mathbb{E}_A[T^A(b)] \leq \frac{b(1 + o(1))}{D_A(k)}.$$

Thus, it suffices to show that $T_S(b) \leq T^A(b)$ for any given $b > 0$. In what follows, we fix $b > 0$ and suppress the dependence on $b$. By the definition of the Sum-Intersection rule, it suffices to show that

$$\sum_{i \in B} |\lambda_i(T^A)| \geq b, \quad \forall B \subset [J] : |B| = k.$$
To this end, fix $B \subset [J]$ with $|B| = k$ and set $C = A \triangle B$. Then, from (41) we have that $B = A \triangle C$. Since $|B| \geq k$, it follows that $C \notin \mathcal{U}_k(A)$, and by the definition of $T^A$ we have $\lambda^{A,C}(T^A) \geq b$. As a result,

$$b \leq \lambda^{A,C}(T^A) = \sum_{i \in A \setminus C} \lambda^i(T^A) - \sum_{j \in C \setminus A} \lambda^j(T^A) \leq \sum_{i \in A \setminus C} |\lambda^i(T^A)| = \sum_{i \in B} |\lambda^i(T^A)|.$$

The proof is complete in view of (42).

**B.5. Proof of Corollary 3.4.**

**Proof.** Fix $A \subset [J]$. For (i) it suffices to show that for any $b > 0$

$$P_A(|A \triangle D_I(b, b)|) \leq C^I_k e^{-kb}.$$

The proof is identical to that of Theorem 3.1 as long as we replace the inequalities in (37) by

$$- \sum_{i \in B_1} |\lambda^i(T_I)| - \sum_{j \in B_2} |\lambda^j(T_I)| \leq -kb.$$

In order to prove (ii), setting $k = 1$ in Theorem 3.3 we have as $b \to \infty$

$$(43) \quad E_A[T_I(b, b)] \leq \frac{b (1 + o(1))}{\min_{C \neq A} I^{A,C}}.$$

If condition (11) is satisfied, then $\min_{C \neq A} I^{A,C} = I$. Therefore, if $b \sim |\log \alpha|/k$, from (43) we have that as $\alpha \to 0$

$$E_A[T_I] \leq \frac{|\log \alpha|}{kI} (1 + o(1)).$$

Further, this asymptotic upper bound agrees with the asymptotic lower bound in (20), since $D_A(k) = kI$ when condition (11) holds. Thus, the proof is complete.

**B.6. Proof of Theorem 3.5.**

**Proof.** Since $k \leq (J + 1)/2$ is fixed, we write $n^*(\alpha)$ (resp. $n_{NP}(\alpha)$) for $n^*(k, \alpha)$ (resp. $n_{NP}(k, \alpha)$) for simplicity. By Theorem 3.3, for any $A \subset [J]$ we have

$$N^*_A(k, \alpha) \sim \frac{|\log \alpha|}{D_A(k)} \quad \text{as} \quad \alpha \to 0.$$
Thus, it suffices to show that

\[
\liminf_{\alpha \to 0} \frac{n^*(\alpha)}{|\log(\alpha)|} \geq \frac{1}{\sum_{j=1}^{2k-1} C(j)} \quad \text{and} \quad n_{NP}(\alpha) \sim \frac{|\log(\alpha)|}{\sum_{j=1}^{k} C(j)}.
\]

(i) Let us first focus on \(n^*(\alpha)\). By its definition (13), there exists

\[
D^*(\alpha) \in \Delta_{fix}(n^*(\alpha)) \cap \Delta_k(\alpha).
\]

Denote \(P\) the probability measure for data in all streams. For any \(A \subset [J]\) with \(|A| = 2k - 1\), we consider the following simple versus simple problem:

\[
H'_0 : P = P_\emptyset \quad \text{vs.} \quad H'_1 : P = P_A,
\]

where \(P_A\) is defined in (2). Consider the following procedure for (45):

\[
\bar{D}^*(\alpha) = \begin{cases} 
0 & \text{if } |D^*(\alpha)| < k \\
1 & \text{if } |D^*(\alpha)| \geq k
\end{cases}
\]

Then by definition of \(D^*(\alpha)\), we have

\[
P_\emptyset(\bar{D}^*(\alpha) = 1) = P_\emptyset(|D^*(\alpha)| \geq k) \leq \alpha,
\]

\[
P_A(\bar{D}^*(\alpha) = 0) = P_A(|D^*(\alpha)| < k) \leq \alpha,
\]

where the second inequality uses the fact that \(|A| = 2k - 1\). Thus

\[
\frac{1}{n^*(\alpha)} \log(\alpha) \geq \frac{1}{n^*(\alpha)} \log \left( \frac{1}{2} P_\emptyset(\bar{D}^*(\alpha) = 1) + \frac{1}{2} P_A(\bar{D}^*(\alpha) = 0) \right).
\]

By Chernoff’s lemma G.1,

\[
\liminf_{\alpha \to 0} \frac{1}{n^*(\alpha)} \log \left( \frac{1}{2} P_\emptyset(\bar{D}^*(\alpha) = 1) + \frac{1}{2} P_A(\bar{D}^*(\alpha) = 0) \right) \geq -\Phi^A(0)
\]

where \(\Phi^A(0) := \sup_{\theta \in \mathbb{R}} \left\{ -\log \left( E_\emptyset \left[ e^{\theta A_{\emptyset}(1)} \right] \right) \right\} \). Due to independence,

\[
\Phi^A(0) = \sup_{\theta \in \mathbb{R}} \left\{ \sum_{j \in A} -\log \left( E_0 \left[ e^{\theta j(1)} \right] \right) \right\} \leq \sum_{j \in A} \Phi^j(0) = \sum_{j \in A} C_j,
\]

As a result,

\[
\liminf_{\alpha \to 0} \frac{1}{n^*(\alpha)} \log(\alpha) \geq -\sum_{j \in A} C_j.
\]
Maximizing the lower bound over $A \subset [J]$ with $|A| = 2k - 1$, we obtain the inequality in (44).

(ii) We now focus on $n_{NP}(\alpha)$. By definition, there exists some $\tilde{h} \in \mathbb{R}^J$ such that

$$(n_{NP}(\alpha), \tilde{D}(\alpha)) \in \Delta_k(\alpha),$$

where $\tilde{D}(\alpha) := D_{NP}(n_{NP}(\alpha), \tilde{h})$.

Denote

$$p_j := P_0^j(\tilde{D}_j(\alpha) = 1) = P_0^j\left(\lambda_j(n_{NP}(\alpha)) > \tilde{h}_j n_{NP}(\alpha)\right)$$

$$q_j := P_1^j(\tilde{D}_j(\alpha) = 0) = P_1^j\left(\lambda_j(n_{NP}(\alpha)) \leq \tilde{h}_j n_{NP}(\alpha)\right)$$

For any $A_1, A_2 \subset [J]$ such that $A_1 \cap A_2 = \emptyset$ and $|A_1 \cup A_2| = k$,

$$\alpha \geq P_{A_1}\left(\cap_{j \in A_1}\{\tilde{D}_j(\alpha) = 0\} \cap \cap_{i \in A_2}\{\tilde{D}_i(\alpha) = 1\}\right) = \prod_{j \in A_1} q_j \prod_{i \in A_2} p_i,$$

$$\alpha \geq P_{A_2}\left(\cap_{j \in A_1}\{\tilde{D}_j(\alpha) = 1\} \cap \cap_{i \in A_2}\{\tilde{D}_i(\alpha) = 0\}\right) = \prod_{j \in A_1} p_j \prod_{i \in A_2} q_i.$$

Since $A_1, A_2$ are arbitrary, we have for any $A \subset [J]$ with $|A| = k$

$$\alpha \geq \prod_{j \in A} \max\{p_j, q_j\},$$

which implies that

$$\log(\alpha) \geq \sum_{j \in A} \max\{\log(p_j), \log(q_j)\} \geq \sum_{j \in A} \log(p_j/2 + q_j/2).$$

Thus, again by Chernoff’s Lemma G.1,

$$\liminf_{\alpha \to 0} \frac{1}{n_{NP}(\alpha)} \log(\alpha) \geq -\sum_{j \in A} F_j(0).$$

Maximizing the lower bound over $A \subset [J]$ with $|A| = k$, we have

$$\liminf_{\alpha \to 0} \frac{n_{NP}(\alpha)}{\log(\alpha)} \geq \frac{1}{\sum_{j=1}^k C(j)}.$$

The lower bound is achieved when $\tilde{h} = 0$, and this proves the equivalence in (44).
B.7. Bernoulli example under the generalized mis-classification rate. Suppose that for each \( j \in \{1, \ldots, J\} \), \{\( X^j(n) : n \in \mathbb{N} \)\} are i.i.d. Bernoulli random variables and that there is a constant \( p \in (0, 1/2) \) such that
\[
H_0^j : \mathbb{P}(X^j(1) = 1) = p \quad \text{versus} \quad H_1^j : \mathbb{P}(X^j(1) = 1) = 1 - p := q.
\]
In this case, \( T_0^j = T_1^j = H(p) \), where
\[
H(x) := x \log \left( \frac{x}{1-x} \right) + (1-x) \log \left( \frac{1-x}{x} \right).
\]
Further,
\[
\Phi(0) = \sup_{\theta \in \mathbb{R}} \left\{ -\log(p^\theta q^{1-\theta} + p^{1-\theta} q^\theta) \right\} = \log \frac{1}{2\sqrt{p(1-p)}}.
\]
By Theorem 3.5, for any \( A \subset \{1, \ldots, J\} \) we have
\[
\liminf_{\alpha \to 0} \frac{n^*(k, \alpha)}{N_A^*(k, \alpha)} \geq \frac{kH(p)}{(2k-1)\Phi(0)}, \quad \lim_{\alpha \to 0} \frac{n_{NP}(k, \alpha)}{N_A^*(k, \alpha)} = H(p)/\Phi(0).
\]
In Figure 8, we plot \( H(p)/\Phi(0) \) as a function of \( p \).

![Figure 8](image)

\( \Phi(0) \) is plotted as a function of \( p \in (0, 1/2) \).

APPENDIX C: PROOFS REGARDING THE GENERALIZED FAMILYWISE ERROR RATES

C.1. Proof of Theorem 4.1. The goal in this subsection is to show that for any \( a, b > 0 \) and \( A \subset \{1, \ldots, J\} \) we have
\[
P_A(|D_L \setminus A| \geq k_1) \leq Q(k_1) e^{-b}, \quad P_A(|A \setminus D_L| \geq k_2) \leq Q(k_2) e^{-a},
\]
where \( Q(k) := 2^k C_k^j \). We start with a lemma that shows how to select the thresholds for procedures \( \delta_\ell \), \( 0 \leq \ell < k_1 \) and \( \tilde{\delta}_\ell \), \( 0 \leq \ell < k_2 \).
Lemma C.1. Assume that (5) holds. Fix $A \subset [J]$. Let $B_1 \subset A^c$ with $|B_1| = k_1$, and $B_2 \subset A$ with $|B_2| = k_2$.

(i) Fix any $0 \leq \ell < k_1$. For any event $\Gamma \in F_{\hat{\tau}_\ell}$, we have
\[
\mathbb{P}_A(B_1 \subset \hat{D}_\ell) \leq C_{k_1}^\ell e^{-b}, \quad \mathbb{P}_A(B_2 \subset \hat{D}_\ell \setminus \Gamma) \leq e^{-a} \mathbb{P}_{A \setminus B_2}(\Gamma).
\]

(ii) Fix any $0 \leq \ell < k_2$. For any event $\Gamma \in F_{\hat{\tau}_\ell}$, we have
\[
\mathbb{P}_A(B_1 \subset \hat{D}_\ell \setminus \Gamma) \leq C_{k_2}^\ell e^{-b} \mathbb{P}_{A \cup B_1}(\Gamma), \quad \mathbb{P}_A(B_2 \subset \hat{D}_\ell^c) \leq C_{k_2}^\ell e^{-a}.
\]

Proof. We will only prove (i), since (ii) can be shown in a similar way. Fix $0 \leq \ell < k_1$. By definition, $\hat{D}_\ell$ rejects the nulls in the $\ell$ streams with the least significant non-positive LLR, in addition to the nulls in the streams with positive LLR. Thus,
\[
\{B_1 \subset \hat{D}_\ell\} \subset \bigcup_{M \subset B_1, |M| = k_1 - \ell} \Pi_M,
\]
where $\Pi_M := \{\lambda^j(\hat{\tau}_\ell) > 0 \ \forall j \in M\}$.

With a change of measure from $\mathbb{P}_A \to \mathbb{P}_C$, where $C = A \cup M$, we have
\[
\mathbb{P}_A(\Pi_M) = \mathbb{E}_C \left[ \exp\{\lambda^A,C(\hat{\tau}_\ell)\}; \Pi_M \right] = \mathbb{E}_C \left[ \exp \left\{- \sum_{j \in M} \lambda^j(\hat{\tau}_\ell) \right\}; \Pi_M \right].
\]

By the definition of $\hat{\tau}_\ell$, on the event $\Pi_M$ we have $\sum_{j \in M} \lambda^j(\hat{\tau}_\ell) \geq b$. Thus $\mathbb{P}_A(\Pi_M) \leq e^{-b}$. Since the number of such $M$ is no more than $C_{k_1}^\ell$, the first inequality in (i) follows from Boole's inequality.

On the other hand, we observe that on the event $\{B_2 \subset \hat{D}_\ell^c\}$ we have
\[
\sum_{j \in B_2} \lambda^j(\hat{\tau}_\ell) \leq -a.
\]

Thus, with a change of measure from $\mathbb{P}_A \to \mathbb{P}_{A \setminus B_2}$ we have
\[
\mathbb{P}_A(B_2 \subset \hat{D}_\ell^c, \Gamma) \leq \mathbb{E}_{A \setminus B_2} \left[ \exp \left\{ - \sum_{j \in B_2} \lambda^j(\hat{\tau}_\ell) \right\}; \Gamma \right] \leq e^{-a} \mathbb{P}_{A \setminus B_2}(\Gamma),
\]
which completes the proof.

Proof of Theorem 4.1. We will only establish the upper bound for $\mathbb{P}_A(|A \setminus D_L| \geq k_2)$, since the other inequality can be established similarly. Observe that
\[
\{|A \setminus D_L| \geq k_2\} \subset \bigcup_{B \subset A: |B| = k_2} \{B \subset D_L^c\}.
\]
Since the union consists of at most $C_{k_2}^J$ events, by Boole’s inequality it suffices to show that the probability of each event is upper bounded by $2^{k_2}e^{-a}$. Fix an arbitrary $B \subset A$ with $|B| = k_2$. Further observe that

$$\{B \subset D_L^c\} \subset \bigcup_{\ell=0}^{k_1-1} \tilde{\Gamma}_{B,\ell} \bigcup \bigcup_{\ell=1}^{k_2-1} \tilde{\Gamma}_{B,\ell},$$

where

$$\tilde{\Gamma}_{B,\ell} := \{B \subset \hat{D}_{\ell}^c\} \cap \{D_L = \hat{D}_{\ell}\}, \quad \check{\Gamma}_{B,\ell} := \{B \subset \hat{D}_{\ell}^c\}.$$

By Boole’s inequality it follows that $P_A(B \subset D_L^c)$ is upper bounded by

$$\sum_{\ell=0}^{k_1-1} P_A(\tilde{\Gamma}_{B,\ell}) + \sum_{\ell=1}^{k_2-1} P_A(\check{\Gamma}_{B,\ell}) \leq \sum_{\ell=0}^{k_1-1} e^{-a}P_A(B \setminus D_L = \hat{D}_{\ell}) + \sum_{\ell=1}^{k_2-1} C_{k_2}^\ell e^{-a} \leq e^{-a} + e^{-a} \left( \sum_{\ell=1}^{k_2-1} C_{k_2}^\ell \right) \leq 2^{k_2}e^{-a},$$

where the first inequality follows from Lemma C.1, and the second from the fact that $\{D_L = \hat{D}_{\ell}\}$ are disjoint events. Thus, the proof is complete.

### C.2. Proof of Lemma 4.2.

**Proof.** We will only prove the inequality for $\hat{\tau}_\ell$, as the proof of the inequality for $\check{\tau}_\ell$ is similar. Fix $A$ and $0 \leq \ell < k_1$. We introduce the following classes of subsets

$$\mathcal{M}_1 = \{B \subset A : |B| = k_1 - \ell\},$$

$$\mathcal{M}_0 = \left\{B \subset A^c : |B| = k_2, \max_{i \in A} \lambda_i(n) > 0 \text{ and } \min_{j \notin A} \lambda_j(n) < 0 \right\}.$$

Clearly, we have $\hat{\tau}_\ell \leq \tau'$, where

$$\tau' := \inf\{n \geq 1 : \min_{B \in \mathcal{M}_1} \sum_{i \in B} \lambda_i(n) \geq b \text{ and } \min_{B \in \mathcal{M}_0} \sum_{j \in B} \lambda_j(n) \leq -a, \min_{i \in A} \lambda_i(n) > 0 \text{ and } \max_{j \notin A} \lambda_j(n) < 0\}.$$

Thus, by an application of Lemma F.2, we have

$$E_A[\tau'] \leq \max \left\{ \frac{b}{\min_{B \in \mathcal{M}_1} \sum_{j \in B} I_j^1}, \frac{a}{\min_{B \in \mathcal{M}_0} \sum_{j \in B} I_j^2} \right\} (1 + o(1)).$$

By definition, for any $B_1 \in \mathcal{M}_1$ and $B_0 \in \mathcal{M}_0$ we have

$$\sum_{j \in B_1} I_j^1 \geq D_1(A; 1, k_1 - \ell), \quad \sum_{j \in B_0} I_j^2 \geq D_0(A^c; 1 + \ell, k_2 + \ell).$$
therefore we conclude that

\[ E_A[\tau'] \leq \max \left\{ \frac{b}{D_1(A; 1, k_1 - \ell)}, \frac{a}{D_0(A^c; 1 + \ell, k_2 + \ell)} \right\} (1 + o(1)), \]

which proves the inequality for \( \hat{\tau}_\ell \).

C.3. An important lemma. In this subsection we establish a lemma that is critical in establishing the lower bound in Theorem 4.3. To state the result, let us denote by

\[ U_{k_1,k_2}(A) = \{ C \subset [J] : |C \setminus A| < k_1 \text{ and } |A \setminus C| < k_2 \}, \]

the collection of sets that are “close” to \( A \), according to the generalized familywise error rates. Since \( k_1, k_2 \) are fixed integers, in order to lighten the notation, in this subsection we write

\[ L(A; \alpha, \beta) \text{ for } L_{A}(k_1, k_2, \alpha, \beta). \]

Lemma C.2. Let \( A \subset [J], B \in U_{k_1,k_2}(A), \alpha, \beta \in (0,1) \).

1. If \( |B| \geq k_1 \) and \( |B^c| \geq k_2 \), then there exists \( B_1^*, B_2^* \subset [J] \) such that

\[ (i) \ |B \setminus B_1^*| = k_1, \ |B_2^* \setminus B| = k_2, \quad (ii) \ \frac{|\log(\alpha)|}{ TA,B_1^* } \lor \frac{|\log(\beta)|}{ TA,B_2^* } \geq L(A; \alpha, \beta) \]

2. If \( |B| < k_1 \), then there exists \( B_2^* \subset [J] \) such that

\[ (i) \ |B_2^* \setminus B| = k_2, \quad (ii) \ \frac{|\log(\beta)|}{ TA,B_2^* } \geq L(A; \alpha, \beta). \]

3. If \( |B^c| < k_2 \), there exists \( B_1^* \subset [J] \) such that

\[ (i) \ |B \setminus B_1^*| = k_1, \quad (ii) \ \frac{|\log(\alpha)|}{ TA,B_1^* } \geq L(A; \alpha, \beta). \]

The proof relies on the following two lemmas.

Lemma C.3. Let \( G \subset A \subset F \subset [J] \). Denote \( s_1 = |A \setminus G| \) and \( s_2 = |F^c| \).

Then, for any positive integer \( n \) we have

\[ D_1(G, 1,n) \leq D_1(A, 1 + s_1, n + s_1), \]

\[ D_0(F \setminus A, 1,n) \leq D_0(A^c, 1 + s_2, n + s_2). \]
Proof. We start with the first inequality. We can assume \( n \leq |G| \), since otherwise both sides are equal to \( \infty \).

Fix some \( 1 \leq i \leq n \). Then clearly the \( i^{th} \) smallest element in \( \{I^j : j \in G\} \) is not larger than the \( (i + |A \setminus G|)^{th} \) element in \( \{I^j : j \in A\} \). Thus, the first inequality follows from the definition of the \( D_1 \) function.

For the second inequality, it follows from the previous argument by replacing \( G \) by \( F \setminus A \), \( A \) by \( A^c \), and \( I^j \) by \( I^j_0 \).

Lemma C.4. Let \( \ell_1, \ell_2 \) be two non-negative integers such that \( \ell_1 < k_1 \) and \( \ell_2 < k_2 \). Then for any \( A \subset [K] \), and \( \alpha, \beta > 0 \), we have

\[
\frac{\log(\alpha)}{D_1(A, 1 + \ell_2, k_1 - \ell_1 + \ell_2)} + \frac{\log(\beta)}{D_0(A^c, 1 + \ell_1, k_2 - \ell_2 + \ell_1)} \geq L(A; \alpha, \beta).
\]

Proof. Let's consider the case that \( \ell_1 \geq \ell_2 \). When \( \ell_1 \leq \ell_2 \), the result can be proved in a similar way. Thus, denote \( \ell = \ell_1 - \ell_2 \). Then

\[
\frac{\log(\alpha)}{D_1(A, 1 + \ell_2, k_1 - \ell_1 + \ell_2)} + \frac{\log(\beta)}{D_0(A^c, 1 + \ell_1, k_2 - \ell_2 + \ell_1)}
\geq \frac{\log(\alpha)}{D_1(A, 1, k_1 - \ell)} + \frac{\log(\beta)}{D_0(A^c, 1 + \ell, k_2 + \ell)} = \hat{L}_A(\ell; \alpha, \beta) \geq L(A; \alpha, \beta)
\]

where the last line used the definition of \( \hat{L}_A \) and \( L \).

With above two lemmas, we are ready to present the proof of Lemma C.2. We illustrate the intuition of this proof in Figure 9.

Proof. Fix \( A \) and \( B \in U_{k_1, k_2}(A) \). By definition of the class \( U_{k_1, k_2}(A) \),

\[
\ell_1 := |B \setminus A| < k_1, \quad \ell_2 := |A \setminus B| < k_2.
\]

First, consider the case that \( |B| \geq k_1 \), which implies \( |A \cap B| \geq k_1 - \ell_1 \). Thus, we can find \( \Gamma_1 \subset A \cap B \) such that

\[
\Gamma_1 = k_1 - \ell_1, \quad \sum_{i \in \Gamma_1} T_i^1 = D_1(A \cap B, 1, k_1 - \ell_1)
\]

Set \( B^*_1 := A \setminus \Gamma_1 \). It is easy to see that

\[
A \setminus B^*_1 = \Gamma_1, \quad B \setminus B^*_1 = \Gamma_1 \cup (B \setminus A).
\]
Thus, \(|B \setminus B^*_1| = k_1\); further, viewing \(A \cap B\) as \(G\) in the Lemma C.3, and since \(\ell_2 = |A \setminus B|\), we have

\[ I^{A, B^*_1} = \sum_{i \in \Gamma_1} I^i_1 = D_1(1, k_1 - \ell_1) \leq D_1(1 + \ell_2, k_1 - \ell_1 + \ell_2). \]

Second, consider the case that \(|B^c| \geq k_2\), which implies \(|A^c \cap B^c| \geq k_2 - \ell_2\).

Thus, there exists \(\Gamma_2 \subset A^c \cap B^c\) such that

\[ \Gamma_2 = k_2 - \ell_2, \quad \sum_{j \in \Gamma_2} I^c_0 = D_0(A^c \cap B^c, 1, k_2 - \ell_2) \]

Set \(B^*_2 := A \cup \Gamma_2\). It is easy to see

\[ B^*_2 \setminus A = \Gamma_2, \quad B^*_2 \setminus B = \Gamma_2 \cup (A \setminus B). \]

Then \(|B^*_2 \setminus B| = k_2\). Further, viewing \(A \cup (A^c \cap B^c)\) as \(F\) in the Lemma C.3, and since \(\ell_1 = |B \setminus A| = |F^c|\), we have

\[ I^{A, B^*_2} = \sum_{j \in \Gamma_2} I^c_0 = D_0(A^c \cap B^c, 1 + \ell_1, k_2 - \ell_2 + 1) \]

It remains to show \(B^*_1\) and \(B^*_2\) satisfy the property (ii) in each case.

**Case 1:** \(|B| \geq k_1\) and \(|B^c| \geq k_2\). By the construction of \(B^*_1\) and \(B^*_2\) we have

\[
\frac{|\log(\alpha)|}{I^{A, B^*_1}} \lor \frac{|\log(\beta)|}{I^{A, B^*_2}} \\
\geq \frac{|\log(\alpha)|}{D_1(1, \ell_2 + 1, \ell_2 + k_1 - \ell_1)} \lor \frac{|\log(\beta)|}{D_0(A^c, \ell_1 + 1, \ell_1 + k_2 - \ell_2)} \\
\geq L(A; \alpha, \beta)
\]

where the last inequality is due to Lemma C.4.

**Case 2:** \(|B| < k_1\), which implies the following:

\[ |A| = |A \setminus B| + |A \cap B| = \ell_2 + |B| - \ell_1 < \ell_2 + k_1 - \ell_1, \]

thus, \(D_1(A, \ell_2 + 1, \ell_2 + k_1 - \ell_1) = \infty\). As a result,

\[
\frac{|\log(\beta)|}{I^{A, B^*_2}} \geq \frac{|\log(\alpha)|}{D_1(A, \ell_2 + 1, \ell_2 + k_1 - \ell_1)} \lor \frac{|\log(\beta)|}{D_0(A^c, \ell_1 + 1, \ell_1 + k_2 - \ell_2)} \\
\geq L(A; \alpha, \beta)
\]

where the last inequality is again due to Lemma C.4.

**Case 3:** \(|B^c| < k_2\). It can be proved in the same way as in case 2. \(\Box\)
Fig 9: The solid area are the streams with signal. The whole set \([J]\) is partitioned into four disjoint sets: \(A \setminus B\), \(A \cap B\), \(B \setminus A\), \(A^c \cap B^c\). If \(B \in \mathcal{U}_{k_1, k_2}(A)\), then \(\ell_1 < k_1\) and \(\ell_2 < k_2\).

### C.4. Proof of Theorem 4.3

As explained in the discussion following Theorem 4.3, it suffices to show that for any \(A \subset [J]\), as \(\alpha, \beta \to 0\),

\[ N^*_A(k_1, k_2, \alpha, \beta) \geq L_A(k_1, k_2, \alpha, \beta) (1 - o(1)). \]

Since \(k_1, k_2\) are fixed integers, in order to simplify the notation in this subsection we write

\[ L(A; \alpha, \beta) \text{ for } L_A(k_1, k_2, \alpha, \beta). \]

**Proof.** Fix \(A \subset [J]\). By the same argument as in the proof of Theorem 3.2, it suffices to show for every \(q \in (0, 1)\) we have:

\[ \liminf_{\alpha, \beta \to 0} \inf_{(T, D) \in \Delta_{k_1, k_2}(\alpha, \beta)} P_A(T \geq qL(A; \alpha, \beta)) \geq 1. \]

Fix \(q \in (0, 1)\) and let \((T, D)\) be any procedure in \(\Delta_{k_1, k_2}(\alpha, \beta)\). Then, by the definition of the class \(\mathcal{U}_{k_1, k_2}(A)\) in \((46)\) we have

\[ 1 - (\alpha + \beta) \leq P_A(D \in \mathcal{U}_{k_1, k_2}(\alpha, \beta)) = \sum_{B \in \mathcal{U}_{k_1, k_2}(\alpha, \beta)} P_A(D = B). \]

Fix \(B \in \mathcal{U}_{k_1, k_2}(\alpha, \beta)\), and let \(\eta > 0\). First, we assume that \(|B| \geq k_1\) and \(|B^c| \geq k_2\). Then \(P_A(D = B)\) is upper bounded by I + II, where

\[
I = P_A\left(\lambda^{A, B^c_1}(T) < \log\left(\frac{\eta}{\alpha}\right), D = B\right) + P_A\left(\lambda^{A, B^c_2}(T) < \log\left(\frac{\eta}{\beta}\right), D = B\right),
\]

\[
II = P_A\left(\lambda^{A, B^c_1}(T) \geq \log\left(\frac{\eta}{\alpha}\right), \lambda^{A, B^c_2}(T) \geq \log\left(\frac{\eta}{\beta}\right), D = B\right).
\]
where the sets $B_1^*$ and $B_2^*$ are selected to satisfy the conditions in Case 1 of Lemma C.2. Then, $|B \setminus B_1^*| \geq k_1$ and $|B_2^* \setminus B| \geq k_2$, and consequently

$$P_{B_1^*}(D = B) \leq \alpha \quad \text{and} \quad P_{B_2^*}(D = B) \leq \beta.$$ 

Thus, by change of measure $P_A \to P_{B_1^*}$ and $P_A \to P_{B_2^*}$, we have

$$P_A \left( \lambda_{A, B_i^*}^A(T) < \log \left( \frac{\eta}{\alpha} \right), D = B \right) \leq \eta, \quad \text{for} \ i = 1, 2$$

which shows that $I \leq 2\eta$. Moreover, it is obvious that

$$\Pi \leq \epsilon_{\alpha, \beta}^B(T) + P_A(T \geq qL(A; \alpha, \beta), D = B), \quad \text{where} \epsilon_{\alpha, \beta}^B(T) := P_A \left( T < qL(A; \alpha, \beta), \lambda_{A, B_i^*}^A(T) \geq \log \left( \frac{\eta}{\alpha} \right), \lambda_{A, B_i^*}^A(T) \geq \log \left( \frac{\eta}{\beta} \right) \right).$$

But by the construction of $B_1^*$ and $B_2^*$ we have

$$L(A; \alpha, \beta) \leq \ell_{\alpha, \beta} := \frac{\log(\alpha)}{L_{A, B_1^*}} \sqrt{\frac{\log(\beta)}{L_{A, B_2^*}}},$$

consequently

$$\epsilon_{\alpha, \beta}^B(T) \leq P_A \left( T < q\ell_{\alpha, \beta}, \lambda_{A, B_1^*}^A(T) \geq \log \left( \frac{\eta}{\alpha} \right), \lambda_{A, B_2^*}^A(T) \geq \log \left( \frac{\eta}{\beta} \right) \right),$$

and from Lemma F.1 it follows that $\epsilon_{\alpha, \beta}^B(T) \to 0$ as $\alpha, \beta \to 0$ uniformly in $T$.

Putting everything together, we have

(47) $P_A(D = B) \leq 2\eta + \epsilon_{\alpha, \beta}^B(T) + P_A(T \geq qL(A; \alpha, \beta), D = B).$

In a similar way we can show that equation (47) remains valid when $|B| < k_1$ or $|B^c| < k_2$. Thus summing over $B \in \mathcal{U}_{k_1, k_2}(A)$ we have

$$P_A(D \in \mathcal{U}_{k_1, k_2}(A)) \leq 2Q\eta + \epsilon_{\alpha, \beta}(T) + P_A(T \geq qL(A; \alpha, \beta), D \in \mathcal{U}_{k_1, k_2}(A)),$$

where $Q = |\mathcal{U}_{k_1, k_2}(A)|$ is a constant, and $\epsilon_{\alpha, \beta}(T) = \sum_{B \in \mathcal{U}_{k_1, k_2}(A)} \epsilon_{\alpha, \beta}^B(T)$. Since each summand goes to 0, we have $\epsilon_{\alpha, \beta}(T) \to 0$ as $\alpha, \beta \to 0$ uniformly in $T$. Therefore,

$$P_A(T \geq qL(A; \alpha, \beta)) \geq 1 - (\alpha + \beta) - 2Q\eta - \epsilon_{\alpha, \beta}(T)$$

The proof is complete after taking the infimum over the class $\Delta_{k_1, k_2}(\alpha, \beta)$, letting $\alpha, \beta \to 0$ and letting $\eta \to 0$. \qed
C.5. Proof of Corollary 4.4.

Proof. The error control for \( \delta_0 \) follows by setting \( \ell = 0 \) in Lemma C.1. The error control for the Intersection rule \( \delta_I \) can be established by a simple modification of the proof of Lemma C.1. If assumptions (11) and (12) hold, then from (26) it follows that for every \( A \subset [J] \) we have

\[
L_A(k_1, k_1, \alpha, \alpha) = \frac{\log(\alpha)}{k_1 I}.
\]

Further, setting \( \ell = 0 \) for \( \tau_0 \), and \( k = 1 \) for \( T_I \) in the first inequality of Lemma 4.2, we have as \( b \to \infty \)

\[
E_A[\tau_0(b, b)] \leq \frac{b}{k_1 I}(1 + o(1)), \quad E_A[\tau_I(b, b)] \leq \frac{b}{I}(1 + o(1)).
\]

Thus, if \( b \) is selected as in the statement of the corollary, then the quantity \( L_A(k_1, k_1, \alpha, \alpha) \) provides an asymptotic power bound for both \( E_A[\tau_0] \) and \( E_A[\tau_I] \). Thus, the proof is complete.

C.6. Proof of Theorem 4.5.

Proof. Since \( k_1, d \) are fixed, for simplicity we write \( n^*(\beta) \) and \( \tilde{n}(\beta) \) for \( n^*(k_1, k_1, \beta^d, \beta) \) and \( \bar{n}_{NP}(k_1, k_1, \beta^d, \beta) \), respectively. (i) Let us first focus on \( n^*(\beta) \). By its definition (13), there exists some

\[
D^*(\beta) \in \Delta_{fix}(n^*(\beta)) \cap \Delta_{k_1,k_1}(\beta^d, \beta).
\]

Fix any \( A \subset [J] \) such that \( |A| = 2k_1 - 1 \). Denote \( P \) the probability measure for data in all streams, and consider the simple versus simple testing problem (45) and the following procedure \( \bar{D}^*(\beta) := \begin{cases} 0 & \text{if } |D^*(\beta)| < k_1 \\ 1 & \text{if } |D^*(\beta)| \geq k_1. \end{cases} \)

Then, by definition of \( D^*(\beta) \) we have

\[
P_0(\bar{D}^*(\beta) = 1) = P_0(|D^*(\beta)| \geq k_1) \leq \alpha = \beta^d,
\]

\[
P_A(\bar{D}^*(\beta) = 0) = P_A(|D^*(\beta)| < k_1) \leq \beta,
\]

and by the generalized Chernoff’s Lemma G.1,

\[
\liminf_{\beta \to 0} \frac{\log(\beta)}{n^*(\beta)} \geq \liminf_{\beta \to 0} \frac{1}{n^*(\beta)} \log \left( \frac{1}{2} P_0^{1/d}(\bar{D}^*(\beta) = 1) + \frac{1}{2} P_A(\bar{D}^*(\beta) = 0) \right) \geq -\frac{\Phi^A(\tilde{h}_d^A)}{d}.
\]
where \( \tilde{h}_d^A \) is a solution to \( \Phi^A(z)/d = \Phi^A(z) - z \), and for any \( z \in \mathbb{R} \)

\[
\Phi^A(z) := \sup_{\theta \in \mathbb{R}} \left\{ z\theta - \sum_{j \in A} \log \left( \mathbb{E}_0^j \left[ e^{\theta \lambda^j(1)} \right] \right) \right\} \\
= \sup_{\theta \in \mathbb{R}} \left\{ z\theta - |A| \log \left( \mathbb{E}_0^1 \left[ e^{\theta \lambda^1(1)} \right] \right) \right\} = |A| \Phi(z/|A|).
\]

Here, the second equality is due to homogeneity (27). By definition (29), \( \Phi(h_d)/d = \Phi(h_d) - h_d \), which implies

\[
\Phi^A(|A|h_d)/d = \Phi^A(|A|h_d) - (|A|h_d).
\]

Thus, \( \tilde{h}_d^A = |A|h_d \), and

\[
\Phi^A(\tilde{h}_d^A)/d = |A|\Phi(h_d)/d = \frac{2k_1 - 1}{d} \Phi(h_d),
\]

which completes the proof of (i).

(ii) We now focus on \( \hat{n}(\beta) \). By definition, there exists \( h_\beta \in \mathbb{R} \) such that

\[
(\hat{n}(\beta), \hat{D}(\beta)) \in \Delta_{k_1,k_1}(\beta^d, \beta), \quad \text{where } \hat{D}(\beta) := D_{NP}(\hat{n}(\beta), h_\beta \mathbf{1}_J),
\]

where \( \mathbf{1}_J \in \mathbb{R}^J \) is a vector of all ones. Due to homogeneity (27), set

\[
\begin{align*}
p_\beta & := P_0^1(\hat{D}^1(\beta) = 1) = P_0^1 \left( \lambda^1(\hat{n}(\beta)) > h_\beta \hat{n}(\beta) \right), \\
q_\beta & := P_1^1(\hat{D}^1(\beta) = 0) = P_1^1 \left( \lambda^1(\hat{n}(\beta)) \leq h_\beta \hat{n}(\beta) \right).
\end{align*}
\]

For any \( A \subset [J] \) such that \( |A| = k_1(= k_2) \),

\[
\begin{align*}
\beta^d & \geq P_\emptyset \left( \bigcap_{j \in A} \{ \hat{D}(\alpha)^j = 1 \} \right) = (p_\beta)^{k_1}, \\
\beta & \geq P_{[J]} \left( \bigcap_{j \in A} \{ \hat{D}(\alpha)^j = 0 \} \right) = (q_\beta)^{k_1},
\end{align*}
\]

which implies that

\[
\frac{1}{\hat{n}(\beta)} \frac{\log(\beta)}{k_1} \geq \frac{1}{\hat{n}(\beta)} \log \left( \frac{1}{2} P_{1/d}^{1/d} + \frac{1}{2} q_\beta \right).
\]

Then, again by the generalized Chernoff’s lemma G.1 we have

\[
\lim \inf_{\beta \to 0} \frac{\hat{n}(\beta)}{|\log(\beta)|} = \frac{d}{k_1 \Phi(h_d)}.
\]

Further, the same argument shows that the equality is obtained with \( h = h_d \), which completes the proof of (ii). \( \square \)
In this section, we prove Theorem 6.1 in Section 6. We first establish a universal asymptotic lower bound on the expected sample size of procedures that control generalized familywise error rates under composite hypotheses (Subsec. D.1). Then, we show that this lower bound is achieved by the Leap rule with the adaptive log-likelihood statistics in (31) (Subsec. D.2 and D.3). Further, we demonstrate numerically that the Intersection rule (10) and the asymmetric Sum-Intersection rule (23) with the adaptive statistics fail to achieve asymptotic optimality in the composite case (Subsec. D.4). We conclude this section with a discussion on the adaptive statistics and alternative local test statistics (Subsec. D.5).

D.1. Lower bound on the expected sample size. Fix any \( A \subset [J] \) and \( \theta = (\theta^1, \ldots, \theta^J) \in \Theta_A \).

Case 1: Assume for now that the infima in (33) are attained, i.e., there exists \( \tilde{\theta} = (\tilde{\theta}^1, \ldots, \tilde{\theta}^J) \in \Theta_{A^c} \) such that
\[
\mathcal{I}_0^j(\theta^j) = I^j(\theta^j, \tilde{\theta}^j) \text{ for every } j \in A^c,
\]
\[
\mathcal{I}_1^j(\theta^j) = I^j(\theta^j, \tilde{\theta}^j) \text{ for every } j \in A.
\]

Any procedure \((T, D) \in \Delta_{k_1, k_2}^{\text{comp}}(\alpha, \beta)\) controls the generalized familywise error rates below \( \alpha \) and \( \beta \) when applied to the multiple testing problem with the following simple hypotheses for each stream:
\[
H_0^j : \gamma^j = \theta^j \text{ versus } H_1^j : \gamma^j = \tilde{\theta}^j, \quad j \in A^c,
\]
\[
H_0^j : \gamma^j = \tilde{\theta}^j \text{ versus } H_1^j : \gamma^j = \theta^j, \quad j \in A,
\]
where we write \( \gamma^j \) for the generic local parameter in j-th stream to distinguish it from the j-th component of \( \theta \).

Then, under assumptions (32) and (34), by Theorem 4.3 we have
\[
\liminf_{\alpha, \beta \to 0} N_{A, \theta}^x(k_1, k_2, \alpha, \beta)/L_{A, \theta}(k_1, k_2, \alpha, \beta) \geq 1,
\]
where

\[
L_{A, \theta}(k_1, k_2; \alpha, \beta) := \min \left\{ \min_{0 \leq \ell < k_1} \tilde{L}_{A, \theta}(\ell; \alpha, \beta), \min_{0 \leq \ell < k_2} \tilde{L}_{A, \theta}(\ell; \alpha, \beta) \right\},
\]

\[
\tilde{L}_{A, \theta}(\ell; \alpha, \beta) := \max \left\{ \frac{|\log(\alpha)|}{D_1(A, \theta; \ell, 1) + \log(\beta)}, \frac{|\log(\beta)|}{D_0(A^c, \theta; \ell + 1, \ell + k_2)} \right\},
\]

\[(49)\]

\[
\tilde{L}_{A, \theta}(\ell; \alpha, \beta) := \max \left\{ \frac{|\log(\alpha)|}{D_1(A, \theta; \ell + 1, \ell + k_1) + \log(\beta)}, \frac{|\log(\beta)|}{D_0(A^c, \theta; 1, k_2 - \ell)} \right\},
\]

\[
D_1(A, \theta; \ell, u) = \sum_{j=\ell}^{u} \mathcal{I}_{1}^{(j)}(A, \theta), \quad D_0(A^c, \theta; \ell, u) = \sum_{j=\ell}^{u} \mathcal{I}_{0}^{(j)}(A^c, \theta),
\]

and

\[
\mathcal{I}_{1}^{(1)}(A, \theta) \leq \ldots \leq \mathcal{I}_{1}^{(|A|)}(A, \theta)
\]

is the increasingly ordered sequence of \(\{\mathcal{I}_{1}^{(j)}(\theta^j), j \in A\}\), and

\[
\mathcal{I}_{0}^{(1)}(A^c, \theta) \leq \ldots \leq \mathcal{I}_{0}^{(|A^c|)}(A^c, \theta)
\]

is the increasingly ordered sequence of \(\{\mathcal{I}_{0}^{(j)}(\theta^j), j \in A^c\}\). As before, the convention is that

\[
\mathcal{I}_{0}^{(j)}(A, \theta) = \infty \text{ if } j > |A|, \quad \mathcal{I}_{0}^{(j)}(A^c, \theta) = \infty \text{ if } j > |A^c|.
\]

**Case 2:** In general, the infima in (33) are not attained. However, under the separability assumption (34), for any \(\epsilon > 0\) there exists \(\tilde{\theta}_\epsilon = (\tilde{\theta}_1^\epsilon, \ldots, \tilde{\theta}_t^\epsilon) \in \Theta_{A^c}\) such that

\[
I^1(\theta^j, \tilde{\theta}_\epsilon^j) \leq (1 + \epsilon) I^1_0(\theta^j) \text{ for any } j \in A^c, \quad I^1(\theta^j, \tilde{\theta}_\epsilon^j) \leq (1 + \epsilon) I^1_0(\theta^j) \text{ for any } j \in A.
\]

Applying again Theorem 4.3 to the following multiple testing problem with simple hypotheses:

\[
H_0^j : \gamma^j = \theta^j \text{ versus } H_1^j : \gamma^j = \tilde{\theta}_\epsilon^j, \quad j \in A^c, \quad H_0^j : \gamma^j = \tilde{\theta}_\epsilon^j \text{ versus } H_1^j : \gamma^j = \theta^j, \quad j \in A,
\]

we have

\[
\liminf N_{A, \theta}^*(k_1, k_2; \alpha, \beta) / L_{A, \theta}(k_1, k_2; \alpha, \beta) \geq 1/(1 + \epsilon).
\]

Since \(\epsilon\) is arbitrary, (48) still holds.

Above discussions leads to the following theorem.

**Theorem D.1.** If (32) and (34) hold, then (48) holds for every \(A \subset [J]\) and \(\theta \in \Theta_A\).
D.2. Error control of the Leap rule with adaptive log-likelihood ratios. We start with the following observation.

**Lemma D.2.** Fix $A \subset [J]$, $\Theta = (\theta^1, \ldots, \theta^J) \in \Theta_A$. For each $j \in [J],$

$$L^j_n := \exp \left( \ell^*_j(n) - \ell^j(n, \theta^j) \right), \quad n \in \mathbb{N}$$

is an $\{\mathcal{F}_n\}$-martingale under $\mathbb{P}_{A, \theta}$ with expectation 1.

**Proof.** By definition,

$$L^j_n = L^j_{n-1} \cdot \frac{p^j_{\hat{\theta}^j_{n-1}}(X^j(n)|\mathcal{F}^j_{n-1})}{p^j_{\theta^j}(X^j(n)|\mathcal{F}^j_{n-1})}.$$

Clearly, $L^j_n \in \mathcal{F}_n$ for any $n \in \mathbb{N}$. Further, since $\hat{\theta}^j_{n-1} \in \mathcal{F}_{n-1}$,

$$E_{A, \theta} \left[ \frac{p^j_{\hat{\theta}^j_{n-1}}(X^j(n)|\mathcal{F}^j_{n-1})}{p^j_{\theta^j}(X^j(n)|\mathcal{F}^j_{n-1})} \bigg| \mathcal{F}_{n-1} \right] = \int \frac{p^j_{\hat{\theta}^j_{n-1}}(z|\mathcal{F}^j_{n-1})}{p^j_{\theta^j}(z|\mathcal{F}^j_{n-1})} p^j_{\theta^j}(z|\mathcal{F}^j_{n-1}) = 1,$$

which implies $E_{A, \theta}[L^j_n|\mathcal{F}_{n-1}] = L^j_{n-1}$. Further, since $\hat{\theta}_0$ is deterministic, $E_{A, \theta}[L^j_1] = 1$, which completes the proof.

By Lemma D.2 and due to independence across streams, for any subset $M \subset [J]$, there exists a probability measure $Q_{A, \theta, M}$ such that for any $n \in \mathbb{N},$

$$dQ_{A, \theta, M}(\mathcal{F}_n) = \prod_{j \in M} \exp \left( \ell^*_j(n) - \ell^j(n, \theta^j) \right).$$

Next, we establish the error control of the Leap rule with adaptive log-likelihood ratios. The proof is almost identical to Theorem 4.1.

**Theorem D.3.** Assume (33) and (34) hold. For any $\alpha, \beta \in (0, 1)$ we have that the Leap rule $\delta^*_L(a, b) \in \Delta_{k_1, k_2}^{\text{comp}}(\alpha, \beta)$ when the thresholds are selected as follows:

$$a = |\log(\beta)| + \log(2^{k_2}C^{L}_{k_2}), \quad b = |\log(\alpha)| + \log(2^{k_1}C^{L}_{k_1}).$$

**Proof.** Just as Theorem 4.1 follows from Lemma C.1 (see the proof in Appendix C.1), in the same way Theorem D.3 follows by the next Lemma.
Lemma D.4. Assume (33), (34) hold. Fix $A \subset [J]$, $\Theta = (\theta^1, \ldots, \theta^J) \in \Theta_A$. Let $B_1 \subset A^c$ with $|B_1| = k_1$, and $B_2 \subset A$ with $|B_2| = k_2$.

(i) Fix any $0 \leq \ell < k_1$. For any event $\Gamma \in \mathcal{F}_{\hat{\tau}_\ell}$, we have

$$P_{A, \theta}(B_1 \subset \hat{D}_{\ell}^c) \leq C_{k_1} e^{-b}, \quad P_{A, \theta}(B_2 \subset (\hat{D}_{\ell}^*)^c, \Gamma) \leq e^{-a} Q_{A, \theta, B_2}(\Gamma).$$

(ii) Fix any $0 \leq \ell < k_2$. For any event $\Gamma \in \mathcal{F}_{\hat{\tau}_\ell}$, we have

$$P_{A, \theta}(B_1 \subset \hat{D}_{\ell}, \Gamma) \leq e^{-b} Q_{A, \theta, B_1}(\Gamma), \quad P_{A, \theta}(B_2 \subset (\hat{D}_{\ell}^*)^c) \leq C_{k_2} e^{-a}.$$

Proof. The proof is similar to that of Lemma C.1. We only indicate the differences by working out the first inequality in (i).

As in the proof of Lemma C.1, by definition, $\hat{D}_{\ell}^*$ rejects the nulls in the $\ell$ streams with the least significant non-positive LLR, in addition to the nulls in the streams with positive LLR. Thus,

$$\{B_1 \subset \hat{D}_{\ell}^*\} \subset \bigcup_{M \subset B_1, |M| = k_1 - \ell} \Pi_M,$$

where $\Pi_M := \{\lambda^j_\ell(\hat{\tau}_\ell) > 0 \ \forall j \in M\}$,

and by Boole’s inequality it suffices to show that $P_{A, \theta}(\Pi_M) \leq e^{-b}$ for every $M \subset B_1$ with $|M| = k_1 - \ell$.

By definition, for any $j \in M \subset B_1 \subset A^c$, since $\theta^j \in \Theta^j_0$,

$$\ell^j_0(n) \geq \ell^j(n, \theta^j) \quad \text{for any } n \in \mathbb{N}.$$

Then, by the definition of the adaptive log-likelihood ratio statistics (31), we have

$$\Pi_M \subset \left\{ \sum_{j \in M} \left( \ell^j_*(\hat{\tau}_\ell) - \ell^j_0(\hat{\tau}_\ell) \right) \geq b \right\} \subset \left\{ \sum_{j \in M} \left( \ell^j_*(\hat{\tau}_\ell) - \ell^j(\hat{\tau}_\ell, \theta^j) \right) \geq b \right\}.$$

By the above observation, the definition of $Q_{A, \theta, M}$ (50), and likelihood ratio identity, on the event $\Pi_M$,

$$\frac{dQ_{A, \theta, M}}{dP_{A, \theta}}(\mathcal{F}_{\hat{\tau}_\ell}) \geq e^b,$$

and the proof is complete by changing the measure from $P_{A, \theta}$ to $Q_{A, \theta, M}$.
D.3. Asymptotic optimality of the Leap rule with adaptive log-likelihood ratios. The asymptotic optimality follows after we establish an asymptotic upper bound on the expected sample size of the Leap rule. The following result is similar to Lemma 4.2.

**Lemma D.5.** Assume (34) and (35) hold. For any $A \subset [J]$ and $\theta \in \Theta_A$, as $a, b \to \infty$,

\[
E_{A, \theta}[\hat{\tau}_{LE}^L] \leq \max \left\{ \frac{b(1 + o(1))}{D_1(A, \theta; 1, k_1 - 1)}, \frac{a(1 + o(1))}{D_0(A^c, \theta; \ell + 1, \ell + k_2)} \right\}, 0 \leq \ell < k_1,
\]

\[
E_{A, \theta}[\hat{\tau}_{LE}^L] \leq \max \left\{ \frac{b(1 + o(1))}{D_1(A, \theta; 1, k_1 - 1)}, \frac{a(1 + o(1))}{D_0(A^c, \theta; 1, k_2 - 2)} \right\}, 0 \leq \ell < k_2,
\]

where the denominators are defined in (49).

**Proof.** Under assumption (35), the proof uses the same argument as in that for Lemma 4.2 in Subsection C.2. □

Now Theorem 6.1 follows from Theorem D.1, Lemma D.3 and Lemma D.5.

D.4. Simulations for composite case. Here we consider a “homogeneous” multiple testing problem on the normal means with known variance. Specifically, we assume that for each $j \in [J]$, the observations in the $j$-th stream, $\{X_j(n) : n \in \mathbb{N}\}$, are i.i.d. with common distribution $\mathcal{N}(\theta_j, 1)$, and for a given constant $\mu > 0$, that does not depend on $j$, we want to test

\[
H_0^j : \theta_j \leq 0 \quad \text{versus} \quad H_1^j : \theta_j \geq \mu.
\]

(51) Using $\mathcal{N}(0, 1)$ as our reference measure, for each $j \in [J]$ we have

\[
\ell^j(n, \theta_j) = n \left( \theta_j X^j(n) - \frac{1}{2} \theta_j^2 \right), \quad \text{where} \quad X^j(n) := \frac{1}{n} \sum_{i=1}^{n} X^j(i).
\]

Further, for any $\theta_j, \bar{\theta}_j$, we have $I^j(\theta_j, \bar{\theta}_j) = \frac{1}{2}(\theta_j - \bar{\theta}_j)^2$, and

\[
I_0^j(\theta_j) = \frac{1}{2}(\theta_j - \mu)^2 \quad \text{for} \quad \theta_j \leq 0, \quad I_1^j(\theta_j) = \frac{1}{2}(\theta_j)^2 \quad \text{for} \quad \theta_j \geq \mu.
\]

Clearly, the null and the alternative hypotheses are separated in the sense of (34). Further, condition (32) is satisfied due to [19].
The adaptive log-likelihood process \( (30) \) for the \( j \)-th stream in this context takes the following form: \( \ell_0^j = 0 \), and for \( n \geq 1 \),

\[
\ell^j_n(n) = \sum_{i=1}^{n} \left( X^j(i) \hat{\theta}^j_{i-1} - \frac{1}{2} (\hat{\theta}^j_{i-1})^2 \right),
\]

If we choose to use the maximum likelihood estimators \( \{\hat{\theta}_n\} \) in above definition, i.e., \( \hat{\theta}_n = \overline{X}^j(n) \), the one-sided complete convergence condition \( (35) \) is established in \( [39] \) (Page 278-279). Thus, by Theorem 6.1, the Leap rule is asymptotically optimal in this setup.

To distinguish from the simulations in the simple versus simple setup, we refer to the Leap rule with adaptive statistics as “Leap*” rule. We will compare the Leap* rule with the following procedures:

1. **Asymmetric Sum-Intersection* rule**: replace the log-likelihood ratio statistics \( \lambda^j(n) \), in the definition of the asymmetric Sum-Intersection rule \( (23) \), by the adaptive version \( \lambda^j_n(n) \) \( (30) \).

2. **Intersection* rule**: replace the log-likelihood ratio statistics \( \lambda^j(n) \), in the definition of the Intersection rule \( (10) \), by the adaptive version \( \lambda^j_n(n) \) \( (30) \).

3. **MNP rule**: for a fixed-sample size \( n \), in each stream, we run the Neyman-Pearson rule with the same threshold \( h > 0 \), which is the most powerful test for each stream due to the monotone likelihood ratio property. Formally,

\[
\delta_{NP}(n, h) := (n, D_{NP}(n, h)), \quad D_{NP}(n, h) := \{j \in [J] : \overline{X}^j(n) > h\},
\]

For simulation purposes, we assume that the tolerance on the two types of mistakes is the same, in the sense that \( (12) \) holds. As in Section 6, we denote the true parameter as \( (A, \theta) \), where \( \theta = (\theta^1, \ldots, \theta^J) \) \( \in \Theta_A \).

**D.4.1. Thresholds selection via simulation.** For each \( j \in [J] \) and \( \theta^j \leq 0 \) the distribution of \( \{\lambda^j(n) : n \in \mathbb{N}\} \) under \( P^j_{\mu - \theta^j} \) is the same as the distribution of \( \{-\lambda^j(n) : n \in \mathbb{N}\} \) under \( P^j_{\theta^j} \). Since \( (12) \) holds, we should equate the thresholds \( a \) and \( b \) in the Leap* rule. Further, we only need to focus on the generalized familywise error rate of Type I.

For a fixed parameter \( a (= b) \), we use simulation to find out the maximal probability of the Leap* rule committing \( k_1 \) false positive mistakes, i.e.

\[
\max_{(A, \theta) : A \subset [J], \theta \in \Theta_A} P_{A, \theta}(|D^*_L \setminus A| \geq k_1).
\]
Then we try different values for \( a \) and select the one for which the above quantity is equal to \( \alpha \). Note that the maximum is over \( \theta \in \Theta \). However, for \( \theta^j \leq \tilde{\theta} \), \( \{ \lambda^j(n) : n \in \mathbb{N} \} \) under \( P_{\theta^j} \) is stochastically larger than \( \{ \lambda^j(n) : n \in \mathbb{N} \} \) under \( P_{\tilde{\theta}} \), in the sense that for any \( n \in \mathbb{N} \) and \( x \in \mathbb{R} \),

\[
P_{\theta^j}(\lambda^j(n) \leq x) \geq P_{\tilde{\theta}}(\lambda^j(n) \leq x).
\]

As a result, the maximal probability is achieved by the boundary cases, i.e., \( \theta \in \{0, \mu\}^J \).

The same discussion applies to the other two sequential procedures. For the MNP rule, (12) implies that \( h = \frac{1}{2} \mu \), and for a fixed \( n \), the maximal probability of making \( k_1 \) false positives is also achieved by \( \theta \in \{0, \mu\}^J \).

**D.4.2. Practical considerations.** The first few estimators of \( \theta \) will typically be quite noisy, since they are estimated based on only a few observations. However, from (30) or (53) we observe that their effect will persist. Thus, in practice it is preferable to take an initial sample of fixed size, say \( n_0 \), and use these observations only to obtain good initial estimates of the unknown parameter.

Specifically, we assume that for each \( j \in J \), \( X^j(-n_0), \ldots, X^j(-1) \) are i.i.d. with distribution \( \mathcal{N}(\theta^j, 1) \), and we define for \( n \geq 0 \) the following maximum likelihood estimator

\[
\tilde{\theta}^j_n := \frac{\sum_{i=-n_0}^{-1} X^j(i) + \sum_{i=1}^{n} X^j(i)}{n_0 + n},
\]

which includes the initial samples. The definitions of the log-likelihood process (52) and the adaptive log-likelihood process (53) remain unchanged. By taking an initial sample of fixed size, the asymptotic expected sample size of the Leap* rule is not affected. Further, if we enlarge the \( \sigma \)-field by including the initial samples, i.e.,

\[
\tilde{\mathcal{F}}_n := \mathcal{F}_n \vee \sigma \left( X^j(i) : j \in [J], i \in \{-n_0, \ldots, -1\} \right),
\]

then the key Lemma D.2, used to establish the error control of Leap* rule, still holds. Thus, taking an initial sample does not affect the asymptotic optimality of the Leap* rule.

**D.4.3. Simulation results.** We consider the problem (51) with \( J = 20 \), \( \mu = 0.2 \), \( k_1 = k_2 = 2 \) and the initial sample size \( n_0 = 10 \). Based on the previous discussion, we set \( a = b \) for the sequential methods. For a fixed threshold \( a \), we use simulation to find out the maximal probability (over
\( \theta \in \Theta \) of committing \( k_1 \) false positives (Err), and the expected sample size (ESS) under a particular \( P_{A, \theta} \), where \( A = \{1, \ldots, 10\} \) and

\[
(54) \quad \theta = (\theta^1, \ldots, \theta^J), \quad \theta^j = \begin{cases} 
0.7 & \text{if } j = 1, \ldots, 10 \\
-0.3 & \text{if } j = 11, \ldots, 19 \\
0 & \text{if } j = 20.
\end{cases}
\]

For the MNP rule, we set \( h = \frac{1}{2} \mu \), and use simulation to find out the maximal probability of committing \( k_1 \) false positives for each fixed \( n \in \mathbb{N} \). The results are shown in Figure 10.

From Figure 10, we observe that the other procedures have a different “slope” compared to the asymptotically optimal Leap* rule, which indicates that they fail to be asymptotically optimal. Further, since sequential methods are adaptive to the true \( \theta \), the gains over fixed-sample size procedures increase as \( \theta \) is farther from the boundary cases.

D.5. Discussion on the local test statistics. When there is only one stream (i.e. \( J = 1 \)), the adaptive log-likelihood ratio statistic (31) was first proposed in [31] in the context of power one tests, and later extended by [28] to sequential multi-hypothesis testing. There are two other popular choices for the local test statistics in the case of composite hypotheses.

The first one is to follow the approach suggested by Wald [42] and replace \( \lambda^j(n) \) in the Leap rule (24) by the following mixture log-likelihood ratio
statistic:
\[ \log \left( \frac{\int_{\Theta_j} \exp \left( \ell(n, \theta) \right) \omega^j_1(d\theta)}{\int_{\Theta_j} \exp \left( \ell(n, \theta) \right) \omega^j_0(d\theta)} \right) , \]
where \( \omega^j_0, \omega^j_1 \) are two probability measures on \( \Theta_j^0 \) and \( \Theta_j^1 \) respectively. The second is to replace \( \lambda^j(n) \) in the Leap rule (24) by the \textit{generalized} log-likelihood ratio (GLR) statistic \( \ell^j_1(n) - \ell^j_0(n) \). When there is only one stream (i.e. \( J = 1 \)), the corresponding sequential test has been studied in [25] for one-parameter exponential family, in [7] for multi-parameter exponential family, and in [23] for separate families of hypotheses.

We have chosen the adaptive log-likelihood ratio statistics (31) in this paper mainly because they allow for explicit and universal error control. Indeed, with this choice of statistics, the upper bounds on the error probabilities rely on a change-of-measure argument, in view of Lemma D.2, whereas this argument breaks down when we use GLR or mixture statistics.

APPENDIX E: SEQUENTIAL TESTING OF TWO COMPOSITE HYPOTHESES IN EXPONENTIAL FAMILY

In this section, we show that (35) holds if each stream has i.i.d. observations from an exponential family distribution, both the null and alternative parameter spaces are compact, and the maximal likelihood estimator is used in the adaptive log-likelihood statistics (31). Note that (35) is a condition on each \textit{individual stream}, thus in this section we drop the superscript \( j \).

Let \( \{X_n : n \in \mathbb{N}\} \) be a sequence of i.i.d. random vectors in \( \mathbb{R}^d \) with common density
\[ p_\theta(x) = \exp \left( \theta^T x - b(\theta) \right) \]
with respect to some measure \( \nu \), where superscript \( T \) means transpose. We assume that the natural parameter space
\[ \Theta := \{ \theta \in \mathbb{R}^d : \int p_\theta(x) \nu(dx) < \infty \} \]
is an open subset of \( \mathbb{R}^d \). For any \( \theta, \tilde{\theta} \in \Theta \), the Kullback-Leibler divergence between \( p_\theta \) and \( p_{\tilde{\theta}} \) is denoted by
\[ I(\theta, \tilde{\theta}) := \mathbb{E}_\theta \left[ \log \frac{p_\theta(X_1)}{p_{\tilde{\theta}}(X_1)} \right] = (\theta - \tilde{\theta})^T \nabla b(\theta) - (b(\theta) - b(\tilde{\theta})) , \]
where \( \nabla \) stands for the gradient. We denote by \( \{\ell(n, \theta) : n \in \mathbb{N}\} \) the log-likelihood process:
\[ \ell(n, \theta) := \sum_{i=1}^{n} \log p_\theta(X_i) = \sum_{i=1}^{n} \left( \theta^T X_i - b(\theta) \right) \quad \text{for } n \in \mathbb{N} . \]
We assume that $\Theta_0, \Theta_1$ are two disjoint, compact subsets of $\Theta$, and denote by $$\hat{\theta}_n := \arg \max_{\theta \in \Theta_0 \cup \Theta_1} \ell(n, \theta)$$ the maximum likelihood estimator based on the data up to time $n$ over the set $\Theta_0 \cup \Theta_1$. Picking any deterministic $\tilde{\theta}_0 \in \Theta$, we define $$\ell^*(n) := \sum_{i=1}^{n} \log p_{\tilde{\theta}_{i-1}}(X_i) = \sum_{i=1}^{n} (\tilde{\theta}_i^T X_i - b(\tilde{\theta}_{i-1}))$$ for $n \in \mathbb{N}$.

The main result of this subsection is summarized in the following theorem.

**Theorem E.1.** Let $\theta \in \Theta_1$ and set $I(\theta) := \inf_{\tilde{\theta}_0 \in \Theta_0} I(\theta, \tilde{\theta}_0)$. Then, for any $\epsilon > 0$,

$$\sum_{n=1}^{\infty} P_{\theta} \left( \frac{\ell^*(n) - \ell_0(n)}{n} - I(\theta) < \epsilon \right) < \infty,$$

where $\ell_0(n) := \sup_{\tilde{\theta}_0 \in \Theta_0} \ell(n, \tilde{\theta}_0)$.

**Proof.** Observe that for any $\theta_0 \in \Theta_0$,

$$\ell^*(n) - \ell(n, \tilde{\theta}_0) = \ell^*(n) - \ell(n, \theta) + \ell(n, \theta) - \ell(n, \theta_0) - nI(\theta, \theta_0) + nI(\theta, \theta_0),$$

which implies that

$$\ell^*(n) - \ell_0(n) = \ell^*(n) - \ell(n, \theta) + \inf_{\tilde{\theta}_0 \in \Theta_0} (\ell(n, \theta) - \ell(n, \theta_0) - nI(\theta, \theta_0) + nI(\theta, \theta_0))$$

$$\geq \ell^*(n) - \ell(n, \theta) + \inf_{\tilde{\theta}_0 \in \Theta_0} (\ell(n, \theta) - \ell(n, \theta_0) - nI(\theta, \theta_0)) + nI(\theta).$$

As a result, it suffices to show that

$$\frac{1}{n}(\ell^*(n) - \ell(n, \theta)) \xrightarrow{P_{\theta}, \text{completely}} 0,$$

$$\frac{1}{n} \inf_{\tilde{\theta}_0 \in \Theta_0} (\ell(n, \theta) - \ell(n, \theta_0) - nI(\theta, \theta_0)) \xrightarrow{P_{\theta}, \text{completely}} 0,$$

which are the content of the next two lemmas.

**Remark E.1.** The sequence in (55) concerns the behavior of the maximal likelihood estimator for the exponential family distribution, while the sequence in (56) concerns the uniform behavior over $\Theta_0$. \qed
LEMMA E.2. For any \( \theta \in \Theta \), as \( n \to \infty \), \( \frac{1}{n}(\ell_s(n) - \ell(n, \theta)) \) converges completely to zero under \( P_\theta \).

PROOF. Since \( \Theta_0 \) and \( \Theta_1 \) are compact, there exists \( K > 0 \) such that
\[
\max\{\|\tilde{\theta}\|, I(\theta, \tilde{\theta})\} < K \quad \text{for any } \tilde{\theta} \in \Theta_0 \cup \Theta_1,
\]
where we use \( \|\cdot\| \) to denote the Euclidean distance.

Observe that \( \frac{1}{n}(\ell_s(n) - \ell(n, \theta)) = \frac{1}{n}M_n - \frac{1}{n}R_n \), where
\[
M_n := \ell_s(n) - \ell(n, \theta) + \sum_{i=1}^{n} I(\theta, \hat{\theta}_{i-1}) = \sum_{i=1}^{n} (\tilde{\theta}_{i-1} - \theta)^T (X_i - \nabla b(\theta)),
\]
\[
R_n := \sum_{i=1}^{n} I(\theta, \hat{\theta}_{i-1})
\]

Denote \( \mathcal{F}_n := \sigma(X_1, \ldots, X_n) \) the \( \sigma \)-field generated by the first \( n \) observations. Then \( \{M_n : n \in \mathbb{N}\} \) is an \( \{\mathcal{F}_n\} \)-martingale, since \( E[X_i] = \nabla b(\theta) \) due to the property of the exponential family and \( \tilde{\theta}_{n-1} \in \mathcal{F}_{n-1} \). Further, the martingale difference sequence \( \{(\tilde{\theta}_{i-1} - \theta)^T (X_i - \nabla b(\theta)) : i \in \mathbb{N}\} \) is bounded in \( L^p \) for any \( p > 2 \). Indeed, by Cauchy-Schwarz inequality,
\[
\sup_{i \in \mathbb{N}} E[(\tilde{\theta}_{i-1} - \theta)^T (X_i - \nabla b(\theta))]^p \leq (2K)^p E\|X_1 - \nabla b(\theta)\|^p < \infty.
\]

Then by [36], we conclude \( \frac{1}{n}M_n \) converges completely to zero under \( P_\theta \).

It remains to show that \( \frac{1}{n}R_n \) converges completely to zero under \( P_\theta \). Fix any \( \epsilon > 0 \). Since \( I(\theta, \tilde{\theta}) \) is continuous in \( \tilde{\theta} \), there exists \( \delta > 0 \) such that if \( \|\tilde{\theta} - \theta\| \leq \delta \), \( I(\theta, \tilde{\theta}) \leq \epsilon/2 \). Define three random times
\[
\eta_1 := \sup\{n \in \mathbb{N} : |R_n| > n \epsilon\},
\]
\[
\eta_2 := \sup\{n \in \mathbb{N} : |I(\theta, \tilde{\theta}_n)| > \epsilon/2\}, \quad \eta_3 := \sup\{n \in \mathbb{N} : \|\tilde{\theta}_n - \theta\| > \delta\}
\]

By Theorem 5.1 in [28], there exist constant \( c_1 \) and \( c_2 \) such that \( P_\theta(\eta_3 > n) \leq c_1 \exp(-c_2 n) \) for any \( n \in \mathbb{N} \). In particular,
\[
E_\theta[\eta_3] < \infty.
\]

Clearly, \( \eta_2 \leq \eta_3 \), which implies that \( E_\theta[\eta_2] < \infty \). We next show that \( \eta_1 \leq 2\delta K \eta_2 / \epsilon \). Indeed, for \( n \geq 2K \eta_2 / \epsilon \),
\[
\frac{1}{n} |R_n| \leq \frac{1}{n} \left( \sum_{i=1}^{\eta_2} I(\theta, \tilde{\theta}_{i-1}) + \sum_{i=\eta_2+1}^{n} I(\theta, \tilde{\theta}_{i-1}) \right) \leq \frac{K \eta_2 + n \epsilon/2}{n} \leq \epsilon.
\]
Thus $E_\theta[\eta_1] < \infty$, which implies $\frac{1}{n} R_n$ converges to zero quickly. (See Chapter 2.4.3 in [39] for formal definition of quick convergence.) Due to Lemma 2.4.1 in [39], quick convergence implies complete convergence, and thus $\frac{1}{n} R_n$ converges to zero completely.

Lemma E.3. Assume the conditions in Theorem E.1 hold. Then

$$\frac{1}{n} \inf_{\theta_0 \in \Theta_0} (\ell(n, \theta) - \ell(n, \theta_0) - n I(\theta, \theta_0)) \xrightarrow{P_{\theta} \text{ completely} \quad n \to \infty} 0.$$ 

Proof. By definition, we have

$$\frac{1}{n} \inf_{\theta_0 \in \Theta_0} (\ell(n, \theta) - \ell(n, \theta_0) - n I(\theta, \theta_0)) = \frac{1}{n} \inf_{\theta_0 \in \Theta_0} \sum_{i=1}^{n} (\theta - \theta_0)^T (X_i - \nabla b(\theta))$$

$$= \inf_{\theta_0 \in \Theta_0} (\theta - \theta_0)^T \left( \frac{1}{n} \sum_{i=1}^{n} (X_i - \nabla b(\theta)) \right).$$

Denote $\theta_j, \theta_{0,j}, X_{i,j}$ and $\nabla_j b(\theta)$ the $j^{th}$ dimension of the $\mathbb{R}^d$ vectors $\theta, \theta_0, X_i$ and $\nabla b(\theta)$. Since $\Theta_0, \Theta_1$ is compact, there exists $K > 0$ such that

$$|\theta_j|, |\theta_{0,j}| \leq K, \text{ for any } 1 \leq j \leq d, \theta_0 \in \Theta_0.$$ 

By triangle inequality,

$$\left| \frac{1}{n} \inf_{\theta_0 \in \Theta_0} (\ell(n, \theta) - \ell(n, \theta_0) - n I(\theta, \theta_0)) \right| \leq 2K \sum_{j=1}^{d} \left| \frac{1}{n} \sum_{i=1}^{n} (X_{i,j} - \nabla_j b(\theta)) \right|.$$ 

But for each $1 \leq j \leq d$, since $E[\xi_{i,j}^2] < \infty$, by [19],

$$\frac{1}{n} \sum_{i=1}^{d} (X_{i,j} - \nabla_j b(\theta)) \xrightarrow{P_{\theta} \text{ completely} \quad n \to \infty} 0,$$

which completes the proof. 

Appendix F: Two Renewal-Type Lemmas

In this section, we present two renewal-type lemmas about general discrete stochastic process, which may be of independent interest.
Lemma F.1. Let \( \{ \xi_i(n) : n \in \mathbb{N} \} \ (i = 1, 2) \) be two stochastic processes on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Suppose that for some positive \( \mu_1, \mu_2 \),

\[
\mathbb{P}\left( \lim_{n \to \infty} \frac{1}{n} \xi_i(n) = \mu_i \right) = 1 \quad \text{for} \quad i = 1, 2.
\]

Let \( c \) be a fixed constant. Then for any \( q \in (0, 1) \),

\[
\lim_{b \to \infty} \sup_T \mathbb{P}\left( T \leq q \frac{b}{\mu_1}, \xi_1(T) \geq b + c \right) = 0, \tag{57}
\]

\[
\lim_{a, b \to \infty} \sup_T \mathbb{P}\left( T \leq q \left( \frac{a}{\mu_1} \vee \frac{b}{\mu_2} \right), \xi_1(T) \geq a + c, \xi_2(T) \geq b + c \right) = 0. \tag{58}
\]

where the supremum is taken over all random time \( T \).

Proof. Since \( c \) is fixed, we assume \( c = 0 \) without loss of generality. Denote \( N_b = \lfloor q \frac{b}{\mu_1} \rfloor \), and \( \epsilon_q = \frac{1}{q} - 1 > 0 \). Notice that \( \mathbb{P}(T \leq q \frac{b}{\mu_1}, \xi_1(T) \geq b) \) is upper bounded by

\[
\mathbb{P}\left( \max_{1 \leq n \leq N_b} \xi_1(n) \geq b \right) \leq \mathbb{P}\left( \frac{1}{N_b} \max_{1 \leq n \leq N_b} \xi_1(n) \geq (1 + \epsilon_q)\mu_1 \right) \to 0
\]

where the convergence follows directly from [13, Lemma A.1]. Thus, the proof of (57) is complete.

For the second part, assume (58) does not hold. Then, there exists some \( \epsilon > 0 \) and a sequence \((a_n, b_n)\) with \( a_n, b_n \to \infty \) such that \( p_n \geq \epsilon \) for large \( n \in \mathbb{N} \), where

\[
p_n := \sup_T \mathbb{P}\left( T \leq q \left( \frac{a_n}{\mu_1} \vee \frac{b_n}{\mu_2} \right), \xi_1(T) \geq a_n, \xi_2(T) \geq b_n \right).
\]

We can assume \( a_n / \mu_1 \geq b_n / \mu_2 \) for every \( n \in \mathbb{N} \), since otherwise we can take a subsequence and the following argument will still go through. Thus,

\[
\epsilon \leq p_n \leq \sup_T \mathbb{P}\left( T \leq q \frac{a_n}{\mu_1}, \xi_1(T) \geq a_n \right),
\]

which contradicts with (57). Thus the proof is complete. \( \square \)

Remark F.1. Note that in (58) there is no restriction on the way \( a, b \) approach infinity, and that \( T \) is not required to be a stopping time.

The next lemma provides an upper bound on the expectation of the first time when multiple processes simultaneous cross given thresholds.
Lemma F.2. Let \( L \geq 2 \) and \( \{ \xi_\ell(n) : n \in \mathbb{N}\}_{\ell \in [L]} \) be \( L \) stochastic processes on some probability space \((\Omega, \mathcal{F}, P)\). Define the stopping time
\[
\nu(\vec{b}) := \inf\{ n \geq 1 : \xi_\ell(n) \geq b_\ell \text{ for every } \ell \in [L]\}
\]
where \( \vec{b} = \{b_1, \ldots, b_L\} \). Then for some positive \( \mu_1, \ldots, \mu_L \), we have
\[
E[\nu(\vec{b})] \leq \max_{\ell \in [L]} \left\{ \frac{b_\ell}{\mu_\ell} \right\} (1 + o(1)) \text{ as } \min_{\ell \in [L]} \{b_\ell\} \to \infty \tag{59}
\]
if one of the following conditions holds: (i). For each \( \ell \in [L] \) and any \( \epsilon > 0 \),
\[
\sum_{n=1}^{\infty} P\left( \left| \frac{1}{n} \xi_\ell(n) - \mu_\ell \right| \geq \epsilon \right) < \infty.
\]
(ii). For each \( \ell \in [L] \), \( \{\xi_\ell(n) : n \in \mathbb{N}\} \) has independent and identically distributed increment, and
\[
P\left( \lim_{n \to \infty} \frac{1}{n} \xi_\ell(n) = \mu_\ell \right) = 1.
\]

Proof. Denote \( N(\vec{b}) = \max_{\ell \in [L]} \{b_\ell/\mu_\ell\} \), and \( \vec{b}_{\min} = \min\{b_1, \ldots, b_L\} \).

First, assume condition (i) holds. Fix \( \epsilon \in (0, 1) \), and denote \( N_\epsilon(\vec{b}) = [N(\vec{b})/(1 - \epsilon)] \).

By definition of \( \nu(\vec{b}) \), we have
\[
\{\nu(\vec{b}) > n\} \subset \bigcup_{\ell \in [L]} \{\xi_\ell(n) < b_\ell\}
\]
By Boole’s inequality, for \( n > N_\epsilon(\vec{b}) \),
\[
P(\nu(\vec{b}) > n) \leq \sum_{\ell \in [L]} P(\xi_\ell(n) < b_\ell) \leq \sum_{\ell \in [L]} P\left( \frac{1}{n} \xi_\ell(n) < \frac{b_\ell}{N_\epsilon(\vec{b}) + 1} \right)
\leq \sum_{\ell \in [L]} P\left( \frac{1}{n} \xi_\ell(n) < (1 - \epsilon)\mu_\ell \right)
\leq \sum_{\ell \in [L]} P\left( \left| \frac{1}{n} \xi_\ell(n) - \mu_\ell \right| > \epsilon\mu_\ell \right),
\]
where we used the fact that \( n \geq N_\epsilon(\vec{b}) + 1 > \frac{N_\epsilon(\vec{b})}{1 - \epsilon} \geq \frac{b_\ell}{(1 - \epsilon)\mu_\ell} \). Thus
\[
E[\nu(\vec{b})] = \int_{0}^{\infty} P(\nu(\vec{b}) > t) \, dt \leq N_\epsilon(\vec{b}) + 1 + \sum_{n > N_\epsilon(\vec{b})} P(\nu(\vec{b}) > n)
\leq N_\epsilon(\vec{b}) + 1 + \sum_{\ell \in [L]} \sum_{n > N_\epsilon(\vec{b})} P\left( \left| \frac{1}{n} \xi_\ell(n) - \mu_\ell \right| > \epsilon\mu_\ell \right)
\]
Due to condition (i), we have
\[
\limsup_{\vec{b}_{\min} \to \infty} \frac{E[\nu(\vec{b})]}{N(\vec{b})} = \limsup_{\vec{b}_{\min} \to \infty} (1 - \epsilon) \frac{E[\nu(\vec{b})]}{N_\epsilon(\vec{b})} \leq 1 - \epsilon
\]

Since \( \epsilon \in (0, 1) \) is arbitrary, (59) holds.

Now assume that condition (ii) holds. Clearly, \( \nu(\vec{b}) \geq \nu_\ell(b_\ell) \), where
\[
\nu_\ell(b_\ell) := \inf\{n \geq 1 : \xi_\ell(n) \geq b_\ell\}
\]
for \( \ell \in [L] \).

Due to condition (ii), we have
\[
\liminf_{b_\ell \to \infty} \frac{\nu(\vec{b})}{b_\ell/\mu_\ell} \geq \lim_{b_\ell \to \infty} \frac{\nu_\ell(b_\ell)}{b_\ell/\mu_\ell} = 1 \text{ for } \ell \in [L],
\]
which implies \( \liminf_{b_\ell \to \infty} \nu(\vec{b})/N(\vec{b}) \geq 1 \). On the other hand, by the definition of \( \nu(\vec{b}) \), there exists \( \ell' \in [L] \) such that
\[
\xi_{\ell'}(\nu(\vec{b}) - 1) < b_{\ell'} \iff \frac{\xi_{\ell'}(\nu(\vec{b})) - b_{\ell'}}{\nu(\vec{b})/\mu_{\ell'}} \leq \frac{\xi_{\ell'}(\nu(\vec{b}) - 1)}{\nu(\vec{b})/\mu_{\ell'}}.
\]
Taking the minimum on the l.h.s., and maximum on the right, we have
\[
\min_{\ell \in [L]} \frac{\xi_{\ell}(\nu(\vec{b})) - b_\ell}{\nu(\vec{b})/\mu_\ell} \leq \max_{\ell \in [L]} \frac{\xi_{\ell}(\nu(\vec{b})) - \xi_{\ell}(\nu(\vec{b}) - 1)}{\nu(\vec{b})/\mu_\ell},
\]
which implies
\[
\frac{N(\vec{b})}{\nu(\vec{b})} = \max_{\ell \in [L]} \frac{b_\ell}{\nu(\vec{b})/\mu_\ell} \geq \min_{\ell \in [L]} \frac{\xi_{\ell}(\nu(\vec{b})) - \xi_{\ell}(\nu(\vec{b}) - 1)}{\nu(\vec{b})/\mu_\ell}
\]
where the last term will go to 1 as \( \vec{b}_{\min} \to \infty \) due to condition (ii). Thus, \( \liminf N(\vec{b})/\nu(\vec{b}) \geq 1 \) as \( \vec{b}_{\min} \to \infty \), which together with previous reverse inequality, shows that \( \nu(\vec{b})/N(\vec{b}) \to 1 \) almost surely as \( \vec{b}_{\min} \to \infty \). Thus, the proof would be complete if we can show the following:
\[
(\ast) \quad C_1 = \left\{ \frac{\nu(\vec{b})}{N(\vec{b})} : b_1, \ldots, b_L > 0 \right\} \text{ is uniformly integrable}
\]
Define \( \mu_{\max} = \max\{\mu_1, \ldots, \mu_L\} > 0, b_{\max} = \max\{b_1, \ldots, b_L\} \) and
\[
\nu'(c) = \inf\{n \geq 1 : \xi_\ell \geq c \text{ for every } \ell \in [L]\} \text{ for } c > 0.
\]
By Theorem 3 of [12], \( C_2 = \{ \nu'(c)/c : c > 0 \} \) is uniformly integrable. Observe that

\[
\nu(\bar{b}) \leq \nu'(b_{\text{max}}), \quad N(\bar{b}) \geq \frac{b_{\text{max}}}{\mu_{\text{max}}} \Rightarrow \frac{\nu(\bar{b})}{N(\bar{b})} \leq \mu_{\text{max}} \frac{\nu'(b_{\text{max}})}{b_{\text{max}}} \in \mu_{\text{max}} C_2.
\]

Since \( \mu_{\text{max}} \) is a constant, \( C_1 \) is dominated by a uniformly integrable family. Thus condition (*) holds, and the proof is complete.

**APPENDIX G: GENERALIZED CHERNOFF’S LEMMA**

In this section we present a generalization of Chernoff’s lemma [11, Corollary 3.4.6] that allows for different requirements on the type I and type II errors. Let \( \{X_n, n \in \mathbb{N} \} \) be a sequence of independent random variables with common density \( f \) relative to some \( \sigma \)-finite measure \( \nu \) and consider the following simple versus simple testing problem:

\[
H_0 : f = f_0 \quad \text{vs.} \quad H_1 : f = f_1.
\]

Let \( S_n \) be the class of \( F_n \)-measurable random variables taking value in \( \{0, 1\} \), where \( F_n = \sigma(X_1, \ldots, X_n) \). For any procedure \( D_n \in S_n \), denote

\[
p_n(D_n) := P_0(D_n = 1), \quad q_n(D_n) := P_1(D_n = 0),
\]

where \( P_i \) is the probability measure under \( H_i \) for \( i = 0, 1 \). Further, denoting \( Y := f_1(X_1)/f_0(X_1) \), we define

\[
\Phi(z) := \sup_{\theta \in \mathbb{R}} \left\{ z\theta - \log \left( E_0[Y^\theta] \right) \right\}, \quad I_0 := E_0[-\log(Y)], \quad I_1 := E_1[\log(Y)],
\]

with the possibility that either \( I_0 \) or \( I_1 \) is equal to \( \infty \). We assume that there exists \( h_d \in (-I_0, I_1) \) such that

\[
(60) \quad \Phi(h_d)/d = \Phi(h_d) - h_d.
\]

In particular, if \( d = 1 \), we can set \( h_d = 0 \).

**LEMMA G.1.** (Generalized Chernoff’s Lemma) For any \( d > 0 \),

\[
\lim_{n \to \infty} \inf_{D_n \in S_n} \frac{1}{n} \log \left( p_n^{1/d}(D_n) + q_n(D_n) \right) = -\frac{\Phi(h_d)}{d}.
\]

**REMARK G.1.** When \( d = 1 \), since we can select \( h_d = 0 \), it reduces to Chernoff’s Lemma [11, Corollary 3.4.6]. For \( d \neq 1 \), the proof is essentially the same, and we present it here for completeness.
Proof of Lemma G.1. For fixed $n \in \mathbb{N}$, due to the Neyman-Pearson Lemma, it suffices to consider the tests of the following form:

$$\delta_n(h) := 1 \Leftrightarrow \frac{1}{n} \lambda(n) \geq h,$$

where $\lambda(n) := \sum_{i=1}^{n} \log \frac{f_1(X_i)}{f_0(X_i)}$.

Then, we have

$$\inf_{D_n \in S_n} \log \left( p_n^{1/d}(D_n) + q_n(D_n) \right) \geq \inf_{h \in \mathbb{R}} \log \left( p_n^{1/d}(\delta_n(h)) + q_n(\delta_n(h)) \right).$$

Since $p_n(\delta_n(h))$ is decreasing in $h$ and $q_n(\delta_n(h))$ increasing in $h$, for any $h \in \mathbb{R}$, either $p_n(\delta_n(h)) \geq p_n(\delta_n(h_d))$ or $q_n(\delta_n(h)) \geq q_n(\delta_n(h_d))$. Thus

$$\inf_{D_n \in S_n} \log \left( p_n^{1/d}(D_n) + q_n(D_n) \right) \geq \log \min \left\{ p_n^{1/d}(\delta_n(h_d)), q_n(\delta_n(h_d)) \right\}.$$ 

By [11, Theorem 3.4.3], as $n \to \infty$,

$$\frac{1}{n} \log(p_n^{1/d}(\delta_n(h_d))) \to -\frac{\Phi(h_d)}{d}, \quad \frac{1}{n} \log(q_n(\delta_n(h_d))) \to -(\Phi(h_d) - h_d).$$

Thus, by the definition of $h_d$ in (60) and letting $n \to \infty$ we obtain

$$\liminf_{n \to \infty} \inf_{D_n \in S_n} \frac{1}{n} \log(p_n^{1/d}(D_n) + q_n(D_n)) \geq -\frac{\Phi(h_d)}{d}.$$

Clearly, the lower bound is attained by the Neyman-Pearson rule with threshold $h_d$, $\delta_n(h_d)$, which completes the proof.

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