Fermi energies in fullerene compounds and cuprates are extremely small as consequence of the small number of charge carriers and are comparable to the phonon frequency scale. In this situation the conventional Migdal-Eliashberg theory does not hold anymore and nonadiabatic effects need to be taken into account. In previous studies, a generalization of Eliashberg theory in the nonadiabatic regime has been proposed to calculate normal state properties and the onset temperature $T_c$ of the superconductive phase. Here we extend the nonadiabatic theory below $T_c$ where the opening of the superconducting order parameter affects the nonadiabatic correction. The superconducting gap $\Delta$ is calculated in a self-consistent way. We find that large values of the ratio $2\Delta/T_c$ are obtained in the nonadiabatic theory by smaller electron-phonon coupling $\lambda$ than in Migdal-Eliashberg theory. This agrees with the picture that strong-coupling phenomenology can be achieved in nonadiabatic theory by “reasonable” values of $\lambda$. We apply our analysis to the case of the fullerene compounds.

1. Introduction

A common peculiar characteristic of many unconventional high-$T_c$ superconductors (cuprates, $A_3C_{60}$ compounds,...) is the narrowness of the electronic bands crossing the Fermi level, leading to Fermi energies $E_F$ of the same order of magnitude of the phonon energies $\omega_{ph}$ ($\omega_{ph}/E_F \approx 0.2 - 0.3$). In such a situation, the adiabatic assumption ($\omega_{ph}/E_F \ll 1$) on which Migdal’s theorem relies cannot be used anymore to justify the omission of vertex corrections in a diagrammatic theory. Motivated by this observation, a renewed interest has recently arose about the possible effects due to inclusion of the vertex corrections in the electron-phonon (or...
any kind of bosonic mediator) interaction. This task is even more relevant for the purely electronic interaction models (Hubbard, or $t - J$), where no justification at all to neglect such corrections exists.

In this perspective, in the past years, we have performed an intensive study of the vertex corrections in the normal state, and, afterwards, we have generalized the conventional Migdal-Eliashberg theory in order to include the first nonadiabatic corrections due to the breakdown of Migdal’s theorem \cite{2,3,4}. We have investigated the effect of the Migdal corrections on different quantities related to both the superconducting and the normal state properties, as for instance the superconducting critical temperature and its isotope effect $\alpha_{T_c}$, or the isotope effect on the effective electronic mass $\alpha_{m^*}$ \cite{5}. Significative nonadiabatic effects have been predicted on the electronic heat capacity, the spin susceptibility, the reduction rate of $T_c$ by impurity scattering, and they can be used as experimental tests of the nonadiabatic theory.

So far the nonadiabatic theory has focused on the analysis of normal state properties and of the superconducting transition ($T_c$, $\alpha_{T_c}$, ...) which can be calculated as instability of the normal state. However, a generalization of the nonadiabatic theory under $T_c$, in the superconducting state, is required in order to investigate nonadiabatic effects on superconductive quantities as the gap or the penetration depth. In particular, the finite and negative isotope effect on the electronic mass $m^*$ has been experimentally inferred by measurements of penetration depth in the superconducting state \cite{6}. It would be therefore of the highest interest to build up a nonadiabatic theory of the superconducting state and compare the isotope effect on the penetration depth at zero temperature with the predicted isotope effect on $m^*$ evaluated in the normal state. Self-consistent calculation of the superconducting gap $\Delta$ in nonadiabatic regime is also of primary relevance. A recent study of the experimental scenario of Rb$_3$C$_{60}$, based on the crossed analysis of $T_c$, $\alpha_{T_c}$ and $\Delta$, points out that the conventional Migdal-Eliashberg theory fails to describe superconductivity in the fullerides. On the other hand, preliminary investigations, restricted to $T_c$ and $\alpha_{T_c}$, suggest that the nonadiabatic theory could account much more properly for the superconducting properties \cite{7}. In order to complete such analysis and compare the results with the experimental ones, a self-consistent calculation of the gap in nonadiabatic theory is therefore needed.

In this contribution we outline the main ingredients of a nonadiabatic theory of the superconducting state. As first step we investigate the modification induced by the superconducting gap opening on the vertex corrections arising from the breakdown of Migdal’s theorem. Afterwards a generalization of the nonadiabatic equations in the superconducting state will be constructed.

2. Vertex Corrections in the Superconducting State

Migdal’s theorem is a basic assumption in the conventional theory of superconductivity and can be considered the equivalent of Born-Oppenheimer approximation in quantum field theory. The electronic self-energy $\Sigma$ due to the phonon interaction
can be expressed in exact way as:

$$\Sigma(k) = -\int d^4q \, G(k)D(q)\Gamma(k,q),\quad (1)$$

where $k$ and $q$ are momentum-frequency quadrivectors and $G$, $D$, $\Gamma$ are respectively the electron and phonon propagators and the vertex function. Migdal’s theorem states the vertex corrections to the bare vertex are of the order of the adiabatic parameter $\omega_{ph}/E_F$:

$$\Gamma = 1 + P = 1 + O\left(\frac{\omega_{ph}}{E_F}\right),\quad (2)$$

where $\omega_{ph}$ is the characteristic phonon frequency scale and $E_F$ the Fermi energy.

In high-$T_c$ materials, where $\omega_{ph}/E_F$ is not negligible, the vertex correction $P$ cannot be anymore neglected and needs to be explicitly taken into account.

In the past years we have generalized the Eliashberg equations at $T_c$ to include nonadiabatic vertex corrections due to the violation of Migdal’s theorem. A crucial role is played by the momentum-frequency ($q-\omega$) structure of the vertex correction $P(q,\omega)$, schematized by the two representative limits, the static ($q \to 0, \omega = 0$) and dynamic ($q = 0, \omega \to 0$) one. Vertex corrections are mainly positive in the dynamic ($v_F|q|/\omega < 1$) and negative in the static ($v_F|q|/\omega > 1$) regime. It is thus clear that the resulting effect of the vertex corrections will depend on the global balance between positive and negative parts and can be affected by the physics of the system. For instance, strong correlation has been shown to select mainly positive parts leading to an enhanced effective electron-phonon coupling. On the contrary, non magnetic impurity scattering enlarges the negative parts yielding a reduction of $T_c$ even in s-wave superconductivity. A similar situation is recovered when the superconducting gap opens under $T_c$. In Fig. 1 the sign of the vertex function in the $(\omega,Q)$ space of the exchanged frequency and momentum ($Q = 2|q|/k_F$ with $k_F$ the Fermi vector) is plotted. Solid and dashed lines represent respectively the $P(Q,\omega) = 0$ curve in the superconducting and normal state. In the latter one the $P(Q,\omega) = 0$ curve crosses the point ($Q = 0, \omega = 0$) where the vertex correction is non-analytical, in agreement with the static and dynamic limits. As shown in the figure, the opening of the superconducting gap removes this non-analytical point and the dynamic and static limit are found to be identical and both negatives, for $T < T_c$.

### 3. Superconducting Gap and Strong Coupling Phenomenology

To calculate the superconducting gap in a selfconsistent way, including the nonadiabatic corrections, we need a set of generalized Eliashberg equations. The nonadiabatic corrections are constructed, in the superconducting as in the normal state, within a perturbative scheme based on $\lambda \omega_{ph}/E_F$ parameter ($\lambda$ being the electron-phonon coupling constant). In order to obtain explicit expressions of the vertex and cross scattering corrections, we make the following assumptions:

- a flat density of states in the half filling case
Figure 1: Plot in the $\omega, Q$ space of the curve on which vertex function takes zero value; the dashed line corresponds to the normal state, solid ones to the superconducting state for different values of the gap: (from top to the bottom) $\Delta/\omega_0 = 0.096$, $\Delta/\omega_0 = 0.168$, $\Delta/\omega_0 = 0.217$.

- an Einstein phonon spectrum with frequency $\omega_0$;
- the bare electron-phonon vertex is given by:

$$|g_{k,k+q}|^2 = g^2 \left( \frac{2k_F}{q_c} \right)^2 \theta(q_c - |q|)$$

where $q_c$ is a cut off on the exchanged momenta due to the effect of electronic correlations.

The equations for the wave function renormalization factor $Z_n$ and the gap function $\Delta_n$, containing the first order corrections beyond Migdal's limit, for $T < T_c$, can be written as follows:

$$Z_n = 1 + \frac{2T}{\omega_n} \sum_m \lambda_Z(Q_c; i\alpha_m, i\alpha_n) \frac{Z_m \omega_m}{\alpha_m} \arctan \left( \frac{E_F}{\alpha_m} \right)$$

$$Z_n \Delta_n = 2T \sum_m \lambda_{\Delta}(Q_c; i\alpha_m, i\alpha_n) \frac{Z_m \Delta_m}{\alpha_m} \arctan \left( \frac{E_F}{\alpha_m} \right)$$

where $\alpha_n = Z_n \sqrt{\omega_n^2 + \Delta_n^2}$ and $\omega_n$ are fermionic Matsubara frequencies. The terms $\lambda_Z$ and $\lambda_{\Delta}$ represent the electron-phonon kernel functions calculated at first order in $\omega_0/E_F$ for the diagonal and off-diagonal interactions. It is important to stress that $\lambda_Z$ and $\lambda_{\Delta}$ contain different nonadiabatic corrections and have therefore different functional forms. In the $T \to T_c$ limit, equations (3) and (4) reduce to the linearized ones, derived in Ref. 3. In Fig. 2 we show the diagrammatic expression.
of the generalized Eliashberg equations for normal and anomalous self-energy, including the first nonadiabatic correction, for \( T < T_c \). By using this closed set of selfconsistent equations, one can calculate the wave function renormalization factor \( Z_n \) and the superconducting gap \( \Delta_n \), in Matsubara frequency space.

By solving numerically the generalized Eliashberg equations, we have investigated the effects of the nonadiabatic corrections on the gap function and on the \( 2\Delta/T_c \) ratio. We find that, for a fixed value of \( T_c \), nonadiabatic corrections suppress the gap function because of the enhancement of the negative vertex region due to the opening of the superconducting gap. In this case, however, adiabatic and nonadiabatic solutions correspond to two different values of \( \lambda \), of which the larger one is required in Migdal-Eliashberg theory. For a given \( \lambda \), instead, nonadiabatic corrections increase the value of the gap and the \( 2\Delta/T_c \) ratio as well (see Fig. 3).

To clarify this result we note that for a given value of \( \lambda \) nonadiabatic corrections produce a large enhancement of \( T_c/\omega_0 \) and of \( \Delta/\omega_0 \), namely give arise to a “strong coupling” phenomenology defined by a non negligible ratio between the superconducting energy scale and the phononin one. For not too large values of \( \omega_0/E_F \), this effect is more relevant than the gap suppression due to the enhancement of negative contributions in the vertex corrections.

An interesting result of the nonadiabatic theory in superconducting state is that the electron-phonon coupling constant \( \lambda \) required to obtain a given \( 2\Delta/T_c \) ratio, whithin nonadiabatic framework, is found to be significantly smaller than the one needed in Migdal-Eliashberg theory.

Migdal-Eliashberg theory is often used to describe the superconducting properties of the alkali-doped \( \text{C}_{60} \) compounds. In order to account for the experimental critical temperature \( T_c = 30 \text{ K} \) a very large microscopical electron-phonon coupling constant (\( \lambda > 3 \)) is needed. On the other hand, the comparison of phononic and
electronic energies gives compelling evidence of the nonadiabaticity in these materials. Recent studies show that the experimental scenario in Rb$_3$C$_{60}$ ($T_c = 30$ K, $\alpha_{T_c} = 0.21$) can be naturally reproduced in the nonadiabatic theory of superconductivity with reasonable values of $\lambda$ and $\omega_0$. This analysis should be completed by taking into account also the size of the superconducting gap as estimated by experimental measurements. To this aim the present results can be used to determine the values of the superconducting gap $\Delta$ and of the ratio $2\Delta/T_c$. We have shown that the measured $2\Delta/T_c = 4.2$ corresponds to a value of the microscopical parameter $\lambda$ which is significantly smaller in nonadiabatic theory than the one derived by using conventional Migdal-Eliashberg framework, in agreement with the above discussion.

Figure 3: Plot of the ratio $2\Delta/T_c$ as function of $\lambda$; solid line corresponds to the nonadiabatic case and dashed line to the Migdal-Eliashberg result.

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