Kohn’s theorem and Galilean symmetry

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Abstract

The relation between the separability of a system of charged particles in a uniform magnetic field and Galilean symmetry is revisited using Duval’s “Bargmann framework”. If the charge-to-mass ratios of the particles are identical, $e_a/m_a = \epsilon$ for all particles, then the Bargmann space of the magnetic system is isometric to that of an anisotropic harmonic oscillator. Assuming that the particles interact through a potential which only depends on their relative distances, the system splits into one representing the center of mass plus a decoupled internal part, and can be mapped further into an isolated system using Niederer’s transformation. Conversely, the manifest Galilean boost symmetry of the isolated system can be “imported” to the oscillator and to the magnetic systems, respectively, to yield the symmetry used by Gibbons and Pope to prove the separability. For vanishing interaction potential the isolated system is free and our procedure endows all our systems with a hidden Schrödinger symmetry, augmented with independent internal rotations. All these properties follow from the cohomological structure of the Galilei group, as explained by Souriau’s “décomposition barycentrique”.

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I. INTRODUCTION

Kohn’s theorem [1], commonly but vaguely ascribed to Galilean invariance, says that a system of charged particles in a uniform magnetic field can be decomposed into center-of-mass and relative motion if the charge/mass ratios are identical,

\[ \frac{e_a}{m_a} = \epsilon = \text{const.} \]  

(1)

The term “Galilean invariance” has been recently been criticized by Gibbons and Pope [2], though, who argue that their symmetry transformation \( \vec{x} \rightarrow \vec{x} + \vec{a}(t) \) is not of the usual Galilean form \( \vec{x} \rightarrow \vec{x} + \vec{b}t \), and belongs rather to the Newton-Hooke group.

In this Note we show that the two, apparently contradictory, statements can be conciliated: \( \vec{a}(t) \) is a Galilean boost, — but it acts in a way which is different from the usual one. Separability does follow therefore from “abstract” Galilean invariance — as it does from Newton-Hooke symmetry also. In detail, we show that when (1) holds the Bargmann space of the magnetic-background system is conformally related to an isolated system with ordinary boost symmetry, and “importing” it guarantees the existence of a rest frame also for the magnetic-background. The “imported boost” coincides with the symmetry used by Gibbons and Pope [2].

In the absence of an interaction potential, the system carries, moreover, a “hidden” Schrödinger symmetry obtained by “importing” that of a free system, augmented with internal rotations. Our results shed new light on Kohn’s theorem and generalize Souriau’s “décomposition barycentrique” [3].

II. A “RELATIVISTIC” PROOF OF KOHN’S THEOREM

We demonstrate our statements in the Kaluza-Klein-type framework [4] which says that the null geodesics of a manifold in \( d + 2 \) dimensions with Lorentz metric,

\[ ds^2 = d\vec{x}^2 + 2dt ds - \frac{2U}{m}(\vec{x},t)dt^2 \]  

(2)

project, for a particle in \( d + 1 \) dimensional non-relativistic space-time with coordinates \( (\vec{x},t) \), according to Newton’s equations, \( m\ddot{\vec{x}} = -\nabla U \). The generalization of (2) to \( N \) particles in
$d$ dimensions in a potential $U$ is provided by the $Nd + 2$ dimensional metric \[4\],
\[
\sum_{a=1}^{N} \frac{m_a}{m} d\vec{x}_a^2 + 2dt ds - \frac{2U}{m} dt^2 \quad \text{where} \quad m = \sum_{a=1}^{N} m_a.
\]

A remarkable property of the metric \[2\] is that it defines a preferential Newton-Cartan structure \[5\] on non-relativistic spacetime obtained by projecting out the “vertical” direction generated by the lightlike vector $\partial_s$ \[4\]. In the quadratic case $U = \pm \frac{1}{2} \omega^2 \vec{x}^2$, \[2\] describes, from the mechanical point of view, an attractive of repulsive harmonic oscillator \[4\]. In a relativistic language \[2\] is a pp-wave, and the quotient is Newton-Hooke space-time \[2, 5\], which carries a Newton-Hooke symmetry, represented by the isometries of the metric \[2, 4\].

But the metric \[3\] is just one example of a “Bargmann” spacetime, whose characteristic feature is that it carries a covariantly constant lightlike vector \[4\]. More generally, the metric can also accommodate a vector potential \[6\] : the projections of the null geodesics of
\[
\begin{align*}
 ds^2 &= d\vec{x}^2 + 2dt (ds + \frac{e}{m} \vec{A}(\vec{x}) \cdot d\vec{x}) - \frac{2e}{m} U(\vec{x}) dt^2 \\
\end{align*}
\]
satisfy the usual [Lorentz] equations of motion of a non-relativistic particle in a (static) “electromagnetic” field $\vec{B} = \nabla \times \vec{A}$, $\vec{E} = -\nabla U$.

A remarkable feature is that, in the plane, the isotropic oscillator metric \[2\] with $U = \frac{1}{2} \omega^2 \vec{x}^2$ is indeed equivalent to the “magnetic” metric \[4\] with vector potential $A_i = -(B/2) \epsilon_{ij} x^j$, used to describe the motion in a uniform magnetic field $B$ perpendicular to the plane \[14\].

Switching to a rotating frame,
\[
\vec{x} = \begin{pmatrix} X^1 \\ X^2 \end{pmatrix} = R_B \vec{x} \equiv \begin{pmatrix} \cos \Omega t & \sin \Omega t \\ -\sin \Omega t & \cos \Omega t \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}; \quad \Omega = \frac{eB}{2m},
\]
completed with $T = t$ and $S = s$, carries the magnetic metric into that of the oscillator. (This is just the familiar Larmor trick in a new guise). $N$ particles in the plane with electric charges $e_a$ are described by adding to the metric \[3\] $2dt \sum_a (e_a/m) \vec{A}_a \cdot d\vec{x}_a$, where $\vec{A}_a = -(B/2) \epsilon_{ij} x^j$.

The generalization to $N$ particles being straightforward, we restrict ourselves henceforth to two charged planar particles in a constant magnetic field. With the same choice of gauge for $\vec{A}$ as above, we hence consider the $2 \times 2 + 1 + 1 = 6$-dimensional metric
\[
\sum_a \frac{m_a}{m} d\vec{x}_a^2 + 2dt ds - B \sum_a \frac{e_a}{m} (x^2_a dx^1 - x^1_a dx^2) dt - \frac{2V}{m} dt^2,
\]

\[6\]
where we have included an interaction potential \( V \equiv V(|\vec{x}_a - \vec{x}_b|) \) and dropped the external trapping potential \( U \) for simplicity. Then, applying (5) to each vector \( \vec{x}_a \) \( (a = 1, 2) \) yields
\[
\sum_a \frac{m_a}{m} d\vec{X}_a^2 + 2dTdS - \frac{2V}{m} dT^2 + \frac{\Omega}{m} \sum_a \left[ (m_a \Omega - e_a B) \vec{X}_a^2 \right] dT^2
\]
\[
+ \frac{1}{m} \sum_a \left[ (2m_a \Omega - e_a B) (X_a^1 dX_a^2 - X_a^2 dX_a^1) \right] dT. 
\]

Our clue is now that if the particles have the same charge to mass ratios, (1), then, choosing the rotation frequency as \( \Omega = \epsilon B/2 \) carries the constant-magnetic-field-metric, (6), into
\[
\sum_a \frac{m_a}{m} d\vec{X}_a^2 + 2dTdS - \frac{2}{m} \left( \frac{\omega^2}{2} \sum_a m_a \vec{X}_a^2 + V \right) dT^2, \quad \omega^2 = \epsilon^2 \frac{B^2}{4}, \quad (7)
\]
which is the metric for an anisotropic oscillator in \( d = 2 + 2 \) dimensions, augmented with the potential \( V \) [15]. The two-particle metric (7) plainly decomposes into center-of-mass and relative coordinates. Putting
\[
\vec{X}_0 = \frac{m_1 \vec{X}_1 + m_2 \vec{X}_2}{m}, \quad \vec{Y} = \sqrt{\frac{m_1 m_2}{m^2}} (\vec{X}_1 - \vec{X}_2)
\]
and calling \( V(|\vec{Y}| m^2 (m_1 m_2)^{-1/2}) \) again \( V(|\vec{Y}|) \) with some abuse of notations (7) is indeed written as
\[
\left\{ d\vec{X}_0^2 - \omega^2 \vec{X}_0^2 dT^2 \right\} + \left\{ d\vec{Y}^2 - (\omega^2 \vec{Y}^2 + 2V(|\vec{Y}|)/m) dT^2 \right\} + 2dTdS, \quad (9)
\]
The first curly bracket here clearly describes the center-of-mass which behaves as a planar particle of mass \( m \) in an attractive oscillator field, to which the “internal vector” \( \vec{Y} \) adds two more dimensions, interpreted as an “internal oscillator” with an interaction potential. Note that the “external” and “internal” oscillators have identical frequencies \( \omega \) and also that the anisotropic oscillator became isotropic when expressed in the new coordinates. The null geodesics of the metric (9) project to the decoupled system of planar oscillators
\[
\frac{d^2 \vec{X}_0}{dT^2} + \omega^2 \vec{X}_0 = 0, \quad \frac{d^2 \vec{Y}}{dT^2} + \omega^2 \vec{Y} + \frac{1}{m} \vec{\nabla}_Y V = 0, \quad (10)
\]
The center-of-mass, \( \vec{X}_0 \), performs an elliptic “deferent” motion around the origin, to which \( \vec{Y} \) adds an “epicycle” with the same oscillator frequency, plus some internal interaction. Transforming back to the magnetic background, we have
\[
\left\{ dx_0^2 + \epsilon B(x_0 \times dx_0) dt \right\} + \left\{ dy^2 + \epsilon B(y \times dy) dt - 2 \frac{V(|y|)}{m} dT^2 \right\} + 2dTdS, \quad (11)
\]
\[
\dot{x}_0^i = -2\omega \epsilon^ij \dot{x}_0^j, \quad \dot{y}^j = -2\omega \epsilon^ij \dot{y}^j - \partial_{y^j} V, \quad (12)
\]
where \( \vec{x}_0 = R_B^{-1} \vec{X} \) is the magnetic center-of-mass and \( \vec{y} = R_B^{-1} \vec{Y} \) is the internal coordinate.
The decomposition (9) [or (10)] allows us to infer that the system admits two independent and separately conserved angular momenta, since one can consider independent external and internal rotations,

\[\begin{align*}
X_0 \to R_{\text{ext}} X_0, & \quad Y \to Y, \quad L_0 = m \vec{X} \times \dot{\vec{X}}, \\
X_0 \to \hat{X}_0, & \quad \vec{Y} \to R_{\text{int}} \vec{Y}, \quad L_{\text{int}} = m \vec{Y} \times \dot{\vec{Y}},
\end{align*}\]

(13)

where \(R_{\text{int}}\) and \(R_{\text{ext}}\) are planar rotation matrices. The first rotation corresponds to rotating the center of mass alone, and the second corresponds to rotating it around its center of mass. The (separate) conservations of the two angular momenta can be checked directly using the equations of motion (10) or (12).

### III. MAPPING TO AN ISOLATED SYSTEM AND HIDDEN SCHröDINGER SYMMETRY

Another remarkable feature of the metric (2) [with \(x \sim X, t \sim T\)] is that, in the quadratic case \(2U = \pm \omega^2 \vec{X}^2\) and for uniform \(B = B(t)\), it is conformally flat [4, 6]. For \(B = \text{const.}\), lifting Niederer’s transformation [9] to Bargmann space according to

\[\Xi_a = \frac{X_a}{\cos \omega T}, \quad \tau = \tan \frac{\omega T}{\omega}, \quad \Sigma = s - \frac{\omega}{2} \vec{X}^2 \tan \omega T\]

(14)

maps in fact each half-period of the oscillator conformally into the free metric [9],

\[d\vec{\Xi}^2 + 2d\tau d\Sigma = \cos^{-2} \omega T (d\vec{X}^2 + 2dT dS - 2U dT^2)\]

Generalizing (14), we observe that

\[\Xi_a = \frac{\vec{X}_a}{\cos \omega T}, \quad \tau = \tan \frac{\omega T}{\omega}, \quad \Sigma = s - \frac{\omega}{2} \left(\sum_a \frac{m_a}{m} \vec{X}_a^2\right) \tan \omega T\]

(15)

carries the two-particle oscillator metric (7) into

\[\frac{1}{1 + \omega^2 \tau^2} \left(\sum_a \frac{m_a}{m} d\vec{\Xi}_a^2 + 2d\tau d\Sigma - 2V^* d\tau^2\right)\]

(16)

where \(V^* = V(|\vec{\Xi}_1 - \vec{\Xi}_2|/\sqrt{(1 + \omega^2 \tau^2)})\), which is conformal to an isolated system with some time-dependent interaction. The latter is plainly decomposed into center-of-mass, \(\vec{\Xi}_0 = \sum_a m_a \vec{X}_a/m\), and relative coordinate, \(\vec{\Upsilon} = \sqrt{1 + \omega^2 \tau^2} (\vec{\Xi}_1 - \vec{\Xi}_2)\), as confirmed by writing (16) as,

\[\frac{1}{1 + \omega^2 \tau^2} \left(d\vec{\Xi}_0^2 + d\vec{\Upsilon}^2 + 2d\tau d\Sigma - \frac{2V^*}{m(1 + \omega^2 \tau^2)} d\tau^2\right)\]

(17)
Proceeding backwards, the c-o-m decompositions (9) and (11) are recovered.

Note that the potential became, in general, time-dependent. A notable exception is the Calogero case when the interaction potential is a sum of inverse-squares, which is conformally invariant [10].

Now the isolated system (16) is clearly invariant under Galilei boosts acting in the ordinary way, \( \Xi_a \rightarrow \Xi_a + \beta \tau, \tau \rightarrow \tau \). Boosts only act on the center-of-mass, \( \Xi_0 \), and leave the internal coordinate, \( \Upsilon \), invariant. Then the inverse of (15) “exports” these boosts to the oscillator background, and combining with the Larmor rotation (5) backwards transports everything to our original space while respecting the c-o-m decomposition, \( \vec{X}_a \rightarrow \vec{X}_a + (\omega^{-1} \sin \omega t) \vec{\beta} \) and \( \vec{x}_a \rightarrow \vec{x}_a + \vec{a}(t) \), respectively, where

\[
\vec{a}(t) = (\omega^{-1} \sin \omega t) R_B^{-1}(t) \vec{\beta}
\]  

(18)

The new, “hidden” boosts look quite different from the usual Galilean expression; being isometries, they also belong to the Newton-Hooke group, and are in fact identical to those in [2], as solution of \( \ddot{\vec{a}} = \epsilon \vec{a} \times B \).

All boosts only act on the center-of-mass and leave the relative coordinate invariant. Their role can be understood as follows. The c-o-m of the isolated system, \( \Xi_0 \), moves as a free particle, i.e., with constant velocity. It can be brought therefore to a halt by a suitable (ordinary) Galilei boost: the massive system has a rest-frame. Transforming backwards through (15) to the oscillator context provides us with the (elliptical) motion of the oscillator c-o-m \( \vec{X}_0 \) — whereas it also “exports” the boost symmetry to the oscillator problem, yielding “hidden boosts”. Applied to \( \vec{X}_0 \), the latter still brings the c-o-m motion to a rest. Acting with (5) backwards provides us, at last, with the c-o-m motion of the original charge system in the uniform magnetic background. (It is amusing to check that the rotation (5) backwards converts the oscillator-ellipses into circles in the magnetic problem, as it should.) All this is decoupled from the internal motions.

For \( V \equiv 0 \) we have, in particular, an isolated system of free particles for which the boost symmetry plainly extends to (centrally extended) Schrödinger symmetry, identified with \( \partial_\Sigma \)-preserving conformal transformations of the metric [4, 6]. Then this 9-parameter Schrödinger symmetry can be “exported” backwards: the full system becomes “hiddenly” Schrödinger symmetric [6, 11].

Turning to the internal part, the second line in (13) is obviously an “internal” symmetry
which can be added to the “external” (i.e., center-of-mass) one to yield a 10 parameter symmetry group, isomorphic to

\[(\text{centrally extended-Schrödinger}) \times \text{SO}(2)\]  

(19)
as symmetry of the oscillator [16], and combining with (5) extends the statement to the magnetic problem: as long as the charge-to-mass condition (1) holds, the 2-charge system carries the same “hidden” symmetry (19) – but realized in an “even more hidden way” [6].

IV. SOURIAU’S “DÉCOMPOSITION BARYCENTRIQUE”

Skipping technical details, we would like to mention that our results here fit perfectly into Souriau’s “décomposition barycentrique” [3]. Souriau argues in fact that having a rest frame and a corresponding center-of-mass decomposition only depend on the cohomological properties of the Galilei group, \(G\), and are independent of the concrete way Galilean symmetry is implemented.

Consider in fact an arbitrary Galilei-invariant mechanical system in \(d\) dimensions. Its “space of motions” \(M\) [Souriau’s abstract substitute for the phase space] is even dimensional. If its dimension is the lowest possible one, namely \(2d\), then the Galilei group acts on it transitively, and the space of motions is a coadjoint orbit endowed with the canonical symplectic structure of the centrally extended Galilei group: it describes a free spinless particle. Souriau calls it an elementary system.

If the dimension of \(M\) is at least \(2d + 2\), then the action of the Galilei group is not more transitive; the system is not more elementary. Then Souriau proves that, for non-vanishing mass, \(m \neq 0\), \(M\) is split into the direct product of the \(2d\) dimensional Galilei coadjoint orbit \(M_0\) with another symplectic manifold, \(M_{\text{int}}\),

\[M = M_0 \times M_{\text{int}}.\]  

(20)

\(M_0\) describes the center of mass, and \(M_{\text{int}}\) is characterized by the vanishing of all external Galilean conserved quantities: it describes the “internal” motions in the rest frame.

Moreover, while the Galilei group acts transitively and symplectically on \(M_0\), its subgroup composed of rotations and time translations acts independently and also symplectically on \(M_{\text{int}}\) which carries hence an internal angular momentum and internal energy.
V. THE QUANTUM PICTURE

Our investigations, presented so far classically, also apply in the quantum context. Remember first that a wave function, \( \psi \), is an equivariant function on Bargmann space, \( \partial_s \psi = \text{im} \psi \), and, for a scalar particle, the quantum counterpart of motion along null-geodesics is the massless Klein-Gordon equation \[ \psi = 0, \] (21)

where \( \Box \) is the Laplace-Beltrami operator associated with the Bargmann metric. Equivariance then implies that \( \Psi(\vec{x}, t) = e^{im_s \psi}(\vec{x}, t, s) \) is a well defined “ordinary” wave function, for which (21) reduces to a Schrödinger equation. In the oscillator and magnetic cases we get,

\[
\begin{align*}
    i \partial_T \Psi &= \left[ -\sum_a \left( \frac{1}{2m_a} \vec{\nabla}_a^2 - \frac{1}{2} \omega^2 m_a \vec{X}_a^2 \right) + V(|\vec{X}_1 - \vec{X}_2|) \right] \Psi, \quad (22) \\
    i \partial_t \Psi &= \left[ -\sum_a \frac{1}{2m_a} \left( \vec{\nabla}_a - ie_a \vec{A}_a \right)^2 + V(|\vec{x}_1 - \vec{x}_2|) \right] \Psi, \quad (23)
\end{align*}
\]

respectively, as expected. In c-o-m coordinates \[ \| \text{ and resp. } \| \text{, one readily finds instead,} \]

\[
\begin{align*}
    i \partial_T \Psi &= \left[ \left\{ -\frac{\vec{\nabla}_x^2}{2m} + \frac{m \omega^2}{2} \vec{X}_0^2 \right\} + \left\{ -\frac{\vec{\nabla}_y^2}{2m} + \frac{m \omega^2}{2} \vec{Y}^2 + V(|\vec{Y}|) \right\} \right] \Psi, \quad (24) \\
    i \partial_t \Psi &= \left[ \left\{ -\frac{1}{2m} \left( \vec{\nabla}_{x_0} - iq \vec{A}_{x_0} \right)^2 \right\} + \left\{ -\frac{1}{2m} \left( \vec{\nabla}_y - iq \vec{A}_y \right)^2 + V(|\vec{y}|) \right\} \right] \Psi, \quad (25)
\end{align*}
\]

where \( q = \sum_a e_a \) is the total electric charge. Both equations are plainly separable in c-o-m and internal coordinates.

The implementation of symmetries on wave functions can also be deduced from the Bargmann picture \[ \| \text{ and resp. } \| : \] a conformal transformation \( f \) of Bargmann space with conformal factor \( \Omega^2_f \), acts as \( \hat{f} \psi = \Omega_f f^* \psi \). If \( f \) also preserves the lightlike vector \( \partial_s \), then it projects into a transformation \( F(\vec{x}, t) \) of non-relativistic spacetime, and locally \( f(\vec{x}, t, s) = (F(\vec{x}, t), s + \sigma(\vec{x}, t)) \). Our symmetry is implemented therefore on an ordinary wave function according to

\[
\hat{F} \Psi(\vec{x}, t) = \Omega_f(\vec{x}, t) e^{i\sigma(\vec{x}, t)} \Psi(F(\vec{x}, t)). \quad (26)
\]
For our three types of boosts with parameter $\vec{\beta}$ we get, in particular,

$$e^{-i\vec{\beta} \cdot \sum a_m \vec{z}_a - m \vec{\beta} \cdot \vec{r}/2} \Psi(\vec{z}_a', \tau), \quad \vec{z}_a' = \vec{z}_a + \vec{\beta} \tau$$  \hspace{1cm} (27)

$$e^{-i \cos \omega T \vec{\beta} \cdot \sum a_m \vec{X}_a - m \vec{\beta} \cdot \vec{r} \sin \omega T} \Psi(\vec{X}_a', T), \quad \vec{X}_a' = \vec{X}_a + (\omega^{-1} \sin \omega T) \vec{\beta},$$  \hspace{1cm} (28)

$$e^{-i \cos \omega t R_B^{-1}(t) \vec{\beta} \cdot \sum a_m \vec{x}_a - m \vec{\beta} \cdot \vec{r} \sin \omega t} \Psi(\vec{x}_a', t), \quad \vec{x}_a' = \vec{x}_a + (\omega^{-1} \sin \omega t) R_B^{-1}(t) \vec{\beta}$$  \hspace{1cm} (29)

Note that (28) reduces to (27) as $\omega \to 0$.

VI. CONCLUSION

We have proved that the decomposition into center-of-mass and internal motion is indeed a consequence of Galilean symmetry as it is popularly said — but boosts act in a “hidden”, and not in the conventional way. As the Newton-Hooke group has a similar of cohomological structure as the Galilei group \cite{12}, this implies the separability, (9) and (10), of a particle system in an oscillator background directly from the group theory, applied to the Newton-Hooke group. Details will be presented elsewhere \cite{13}.

Souriau’s theorem also explains why we do not have similar properties in the relativistic case: the Poincaré group has trivial cohomology.

Acknowledgments

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[14] For simplicity, we took \( B(t) = B = \text{const} \) and work in the plane.

[15] The relation of the (non-commutative) Landau problem with an anisotropic harmonic oscillator has also been studied [7].

[16] The system has in fact yet one more conserved quantity namely the “internal energy” \( E_{\text{int}} \), related to the independent external and internal time translations [3]. This can be understood in a multiple-time framework [8], which allows for independent internal and external time translations. Here we only consider \( E_{\text{int}} = 0 \).

[17] These same cohomological properties determine central extensions [3].

[18] If the Bargmann manifold is curved, conformal invariance requires adding a curvature term to [21] [4, 6].