The fate of Hamilton’s Hodograph in Special and General Relativity

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November 12, 2018

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Abstract

The hodograph of a non-relativistic particle motion in Euclidean space is the curve described by its momentum vector. For a general central orbit problem the hodograph is the inverse of the pedal curve of the orbit, (i.e. its polar reciprocal), rotated through a right angle. Hamilton showed that for the Kepler/Coulomb problem, the hodograph is a circle whose centre is in the direction of a conserved eccentricity vector. The addition of an inverse cube law force induces the eccentricity vector to precess and with it the hodograph. The same effect is produced by a cosmic string. If one takes the relativistic momentum to define the hodograph, then for the Sommerfeld (i.e. the special relativistic Kepler/Coulomb problem) there is an effective inverse cube force which causes the hodograph to precess. If one uses Schwarzschild coordinates one may also define a a hodograph for timelike or null geodesics moving around a black hole. Their pedal equations are given. In special cases the hodograph may be found explicitly. For example the orbit of a photon which starts from the past singularity, grazes the horizon and returns to future singularity is a cardioid, its pedal equation is Cayley’s sextic the inverse of which is Tschirhausen’s cubic. It is also shown that that provided one uses Beltrami coordinates, the hodograph for the non-relativistic Kepler problem on hyperbolic space is also a circle. An analogous result holds for the the round 3-sphere. In an appendix the hodograph of a particle freely moving on a group manifold equipped with a left-invariant metric is defined.
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1 Introduction

The Kepler or Coulomb problem is well known to exhibit many remarkable features and these are often ascribed to a hidden $SO(3,1) \ E(3)$ or $SO(4)$ symmetry of phase space generated by a conserved, so-called Runge-Lenz, vector. Less well known is the relation to a fact discovered by Hamilton [1,2] that the hodograph of Kepler problem is a circle [3] and the associated conserved eccentricity vector [3].

It was Hamilton himself who both named and defined the hodograph associated with the non-relativistic motion of a particle as the curved described by its velocity vector $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ or up to a factor of its mass, its momentum vector $\mathbf{p} = m\mathbf{v}$. It is not immediately obvious how to generalise this concept for a relativistic particle moving in the flat spacetime of special relativity, and even less in the curved spacetime of general relativity. In the case of special relativity it is an obvious guess is to replace Newtonian time $t$ by propertime $\tau$ along the orbit and we shall shortly that this works in a particular case. In general relativity there is no natural coordinate system in which to define the velocity and even if one picks a particular coordinate system, the presence of curvature will prevent the necessary parallel transport of the velocity vector required to define a curve as the locus of its endpoint. Expressed in another way, in a general curved tangent or co-tangent space, there is no natural projection on to one of its fibres. [4]

In the years following Hamilton’s discovery there was a considerable interest in the hodograph and in particular to the hodographs of central orbit problems and a number of interesting results were obtained which in many cases allow a straightforward construction of the hodograph either geometrically, or analytically [4–9].

This suggests that while searching for a general extension of the hodograph concept to general relativity, it might be fruitful to look at those cases, typically spherically symmetric spacetimes, for which geodesic motion may be reduced to a central orbit problem. In particular, it seems worthwhile to look at the Schwarzschild solution from this point of view. There still remains however some ambiguity as what radial coordinate to use. It is clear from treatments in standard textbooks that by far the simplest for our purposes is the traditional Schwarzschild coordinate $r$ and the associated radial and tangential velocities $\frac{dr}{d\tau}$ and $r^2 \frac{d\phi}{d\tau}$, where $\tau$ is propertime for timelike geodesics and an affine parameter for null geodesics.

The organization of the paper is as follows. In section 1 the definition and some properties of the hodograph in non-relativistic mechanics are recalled and generalised to a relativistic particle moving in flat spacetime under the influence of Coulomb interaction, a problem studied by Sommerfeld. It is shown how relativistic effects cause the hodograph to precess. In section 3 the notion of a hodograph is extended to photon orbits in the background of a Schwarzschild black hole and its pedal equation given. For a particular case we find the hodograph curve to be Tschirhausen’s cubic. In section 4 massive particles are treated. In section 5 It is shown that in Beltrami coordinates, for the Kepler problem on hyperbolic space the hodograph is also a circle. In the appendix the hodograph of a particle moving freely on a group manifold equipped with a left-invariant metric is described in terms of generalised Euler equations. Section 6 contains a conclusion with some future prospects.

[4] However, as described in the appendix, this can be done if the base space is a group manifold.
2 Hodographs for Central Orbits

We begin by recalling some material on central orbits, and the theory of plane curves, not all of which is as familiar today as it was formerly. The pedal equation of a curve $\gamma$ in the plane, given say in polar coordinates $(r, \phi)$ with respect to an origin $S$ by an equation of the form $r = r(\phi)$, is a relation, $p = f(r)$, between the radial distance $r$ of a point $P$ on the curve from the origin and the perpendicular distance $p$ from the origin $S$ to the tangent to the curve at the point $(r, \phi)$. Concretely one eliminates $\phi$ from

$$p = \frac{r^2}{\sqrt{r^2 + r^2 (\frac{du}{d\phi})^2}} = \frac{1}{\sqrt{u^2 + (\frac{du}{d\phi})^2}},$$

(2.1)

where $u = \frac{1}{r}$.

Now a central orbit with acceleration $F(u)$ towards the centre $S$ and angular momentum per unit mass $h$ satisfies Binet’s equation

$$\frac{d^2 u}{d\phi^2} + u = \frac{F(u)}{h^2 u^2}$$

(2.2)

and has a first integral of the form

$$\frac{1}{2} \left( (\frac{du}{d\phi})^2 + u^2 \right) = \int_0^u \frac{F(u')}{h^2 u'^2} du' + C$$

(2.3)

where $C$ is a constant of integration. Thus the pedal equation of a central orbit is given by

$$\frac{1}{p^2} = 2 \left( \int_0^{1/h} \frac{F(u)}{h^2 u^2} du + C \right).$$

(2.4)

In fact we also have a converse: if the pedal equation of a particle orbit satisfying Kepler’s third law may be cast in the form (2.4) we may deduce the necessary central force.

The hodograph of a particle is the curve swept out by its velocity vector, the velocity vector being parallely transported to a fixed origin. For a central orbit the origin is the centre towards which the force is directed. It is then the case [5] that the hodograph is the inverse with respect to the centre of the pedal curve of the orbit turned through a right angle. This is illustrated in figure 1.

The point $P$ is the current point of the orbit $b$ subject to a central force directed towards $S$, then, since $SY$ is orthogonal to the tangent $PY$ of the orbit at the point $P$, $Y$ is the current point of the pedal curve $g$. Now since the force is central we have $pv = h$, where $p = |SY|$, $v$ is the velocity of the orbit at $P$ and $h$ is the conserved angular momentum per unit mass. Thus if $Q$ is chosen to lie on $SY$ so that $|SY||SQ| = h$, then $|SQ| = v$. It follows that, since $SQ$ is at right angles to the tangent $PY$, the curve $r$ is the hodograph of the orbit $b$ rotated through a right angle. More over $Q$ is the inverse of the point $Y$ in the circle $S$ of radius $\sqrt{h}$.

The inverse of the pedal of a curve is also called the reciprocal polar or just the reciprocal of the curve [5][7]. The name has its origin in the fact that the reciprocal polar of the reciprocal polar of a curve is the original curve (cf [10]).
Figure 1: An orbit $b$, its pedal curve $g$. The inverse of $g$ in the circle $S$ is its polar reciprocal $r$ which is the hodograph turned through a right angle. The curve $o$ is the pedal curve of the polar reciprocal and its inverse in the circle $S$ is the original orbit $b$.

This may be seen from the diagram. If $Z$ is chosen so that $Z$ is perpendicular to $SP$, then $Z$ is the current point of the pedal curve $o$ of the hodograph $r$. However, by construction the triangles $SZQ$ and $SYP$ are similar so that

$$\frac{|SQ|}{|SZ|} = \frac{|SP|}{|SY|}, \quad \Rightarrow \quad |SP||SZ| = |SQ||SY| = h,$$

and therefore $P$ is the inverse of the point $Z$ in the circle $s$. Thus the orbit $b$ is inverse of the pedal curve $o$ of the hodograph and hence it is the hodograph of the hodograph $r$.

In conclusion that The hodograph of a central orbit is the reciprocal polar turned through a right angle.

Thus if $(r_o, p_0), (r_p, p_p), (r_h, p_h)$ are pedal relations for the orbit, pedal of the orbit, and hodograph respectively, we have

$$r_p = p_o \quad p_p = \frac{p_o^2}{r_o}, \quad \iff \quad p_o = r_p, \quad r_o = \frac{r_p^2}{p_p}$$

$$r_h = \frac{h}{p_o} \quad p_h = \frac{h}{r_o}, \quad \iff \quad p_o = \frac{h}{r_h}, \quad r_o = \frac{h}{p_h}$$

$$r_h = \frac{h}{p_p} \quad p_h = \frac{h}{p_p}, \quad \iff \quad r_p = \frac{h}{r_h}, \quad p_p = \frac{hp_h}{r_h^3}.$$  

As an example the pedal equation of an ellipse with semi-latus rectum $l$ and semi-major axis $a$
with respect to its focus is
\[
\frac{l}{p_0^2} = \frac{2}{r_o} - \frac{1}{a}.
\]  
(2.7)

The pedal equation of the pedal curve of the ellipse is therefore
\[
2pPa = r_p^2 + la.
\]  
(2.8)

Now the pedal equation of a circle of radius \(A\) with respect to a point distance \(B\) from the centre of circle is easily seen to be
\[
2pA = r^2 + A^2 - B^2
\]  
(2.9)

and therefore the pedal of an ellipse with respect to its centre is a circle. But the inverse of a circle is a circle and hence the hodograph is a circle, as claimed by Hamilton. One may readily check that if one starts either from (2.7) or (2.8) and uses (2.6) one obtains
\[
\frac{2p_h}{h} = \frac{1}{a} + \frac{lr^2}{h^2}.
\]  
(2.10)

which is indeed a circle, as found by Hamilton.

More succintly, \([6]\) if the pedal equation of the orbit is \(p = f(r)\), that of the hodograph is \(\frac{b}{r} = f(\frac{b}{r})\), and the pedal curve of the orbit has pedal equation \(r^2 = pf(p)\).

### 2.1 Geometrical and Physical Interpretation of the Hodograph

Tait and Steele \([5]\) draw attention to some properties of the hodograph. The luminous flux incident on an orbiting planet (assuming the inverse square law and ignoring aberration) is proportional to
\[
\frac{1}{r^2} = \frac{1}{Gm} |\dot{p}| = \frac{1}{Gmn} \frac{ds}{dt}.
\]  
(2.11)

*Thus the total luminous flux received in a time interval \(t\) is proportional to the length described by the hodograph in that time.*

Hamilton \([2]\) followed by Tait and Steele \([5]\) also assert that

It is evident that the path apparently described by a fixed star, in consequence of the *Aberration* of light, is the Hodograph of the Earth’s orbit, and is therefore a circle in a plane parallel to the ecliptic, and of the same dimensions for all stars.

For more recent discussions see \([13,14]\)

### 2.2 Revolving Orbits, Global Monopoles and Cosmic Strings

A useful result in the theory of central orbits, due originally to Newton and much discussed of late \([15,19]\), is that *Given a central orbit problem with central acceleration \(F(u)\) and orbits \(u = u(\phi, h)\), the orbits of the associated central orbit problem with central acceleration \(F(u) + Au^3\) are given by \(u = u(B\phi, Bh)\), where \(B = \sqrt{1 - \frac{A}{n^2}}\)*
To see why, recall that Binet’s equation for the original orbit reads

\[ \frac{d^2 u}{d\phi^2} + u = \frac{F(u)}{u^2 h^2}, \]  

(2.12)

and for the modified orbit

\[ \frac{d^2 u}{d\bar{\phi}^2} + \left(1 - \frac{A}{\bar{h}}\right)u = \frac{F(u)}{u^2 \bar{h}^2}. \]  

(2.13)

If \( \bar{\phi} = B\phi \) and \( \bar{h} = Bh \) this becomes

\[ \frac{d^2 u}{d\bar{\phi}^2} + u = \frac{F(u)}{u^2 \bar{h}^2}, \]  

(2.14)

which is of the same form as (2.12).

If successive the apses of the original orbit are separated by an amount \( \Delta\phi \) then for the modified orbit they will be separated by an amount \( \Delta(B\phi) \). In other words the modified orbit will appear to precess relative to the original orbit at a rate of \( \left(\frac{1}{B} - 1\right)2\pi \) per revolution.

From a geometrical point of view it is as if the original problem on the equatorial plane with polar coordinates \( (r = \frac{1}{u}, \phi) \), with \( 0 \leq \phi \leq 2\pi \) to a locally flat cone with coordinates \( (r = \frac{1}{u}, \bar{\phi}) \) with \( 0 \leq \bar{\phi} \leq 2\pi B \) with deficit angle \( \delta = (1 - B)2\pi \). If \( A > 0 \) the deficit angle is a true deficit, \( \delta > 0 \) and the precession gives rise to an advance of the apses.

In general relativity, his situation would arise in a an asymptotically flat spherically symmetric static metric if it were pierced by a cosmic string along the an axis of rotational symmetry and one is considering motion in the orthogonal equatorial plane (20). It would also arise or for an asymptotically conical spherically symmetric static metric such as that of a global monopole 21 with spacetime metric

\[ ds^2 = -dt^2 + dr^2 + B^2 r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]  

(2.15)

In 23 the non-relativistic Coulomb problem on such a cone was investigated. The orbits turned out to be precessing ellipses. Moreover there was a conserved Runge Lenz vector and the hodograph was of constant curvature, however because of the deficit angle neither the orbit nor its hodographs were closed curves because of the precession caused by the deficit angle. In the next section we will see that the relativistic Coulomb problem in globally flat Minkowski spacetime exhibits the same features.

### 2.3 Newton’s revolving orbit applied to the Sommerfeld problem

The relativistic Coulomb problem was studied both classically (and semi-classically) by Sommerfeld shortly after Bohr’s model of the atom in order to explain the fine structure of the spectral lines of hydrogen. The degeneracy found in the non-relativistic case is partially broken by relativistic effects. The complete treatment requires the Dirac equation. Here we deal with the the classical motion which, remarkably, may be reduced to a central orbit problem with proper time playing the role of Newtonian time and solved exactly. We start with energy and momentum conservation

\[ \sqrt{\frac{m^2 c^4}{4 \pi \epsilon_0 r} + c^2 p^2} - \frac{Ze^2}{4 \pi \epsilon_0 r} = E \]  

(2.16)
\[ c^2 p^2 = (W + \frac{Ze^2}{4\pi\epsilon_0 r})^2 - m^2 c^4 \]  \hspace{1cm}(2.17)

\[ \mathbf{x} \times \mathbf{p} = \mathbf{J} \]  \hspace{1cm}(2.18)

where

\[ \mathbf{p} = m \frac{d\mathbf{x}}{d\tau} \]  \hspace{1cm}(2.19)

and \( \tau \) is proper time. Thus

\[ p^2 = m^2 \left( \left( \frac{dr}{d\tau} \right)^2 + r^2 \left( \frac{d\phi}{d\tau} \right)^2 \right) \]  \hspace{1cm}(2.20)

\[ mr^2 \frac{d\phi}{d\tau} = J, \hspace{0.5cm} u = \frac{1}{r} \]  \hspace{1cm}(2.21)

\[ \left( \frac{du}{d\phi} \right)^2 + u^2 = \left( \frac{E}{cJ} + \frac{Ze^2}{4\pi\epsilon_0 cJ} u \right)^2 - \frac{m^2 c^2}{J^2} \]  \hspace{1cm}(2.22)

where

\[ \alpha^2 = \frac{e^2}{4\pi\epsilon_0 cJ} \]  \hspace{1cm}(2.23)

Note that semi-classically \( J = n\hbar, \ n \in \mathbb{Z} \). If \( n = 1 \) then by (2.23) \( \alpha \) is of course the usual fine structure constant. Classically we have

\[ \left( \frac{du}{d\phi} \right)^2 + (1 - Z^2 \alpha^4) \left( u - \frac{E}{cJ(1 - Z^2 \alpha^4)} \right)^2 = \frac{E^2 - m^2 c^4}{c^2 J^2} + \frac{E^2}{c^2 J^2(1 - Z^2 \alpha^4)} \]  \hspace{1cm}(2.24)

\[ \tilde{\phi} = \sqrt{1 - Z^2 \alpha^4} \phi \]  \hspace{1cm}(2.25)

We find

\[ \left( \frac{du}{d\phi} \right)^2 + \left( u - \frac{E}{cJ(1 - Z^2 \alpha^4)} \right)^2 = \frac{E^2 - m^2 c^4}{c^2 J^2(1 - Z^2 \alpha^4)} + \frac{E^2}{c^2 J^2(1 - Z^2 \alpha^4)^2} \]  \hspace{1cm}(2.26)

Clearly in the \( r, \tilde{\phi} \) plane the orbits are precisely conic sections with circular hodographs. In the physical \( r, \phi \) plane both the orbit and the hodograph will precess at a steady rate.

### 2.4 Non-Separability of Relativistic Euler Problem

Euler showed that the non-relativistic Kepler problem with two fixed centres is integrable by virtue of an extra constant of the motion. In view of the results of [24] who relates the constant to Hamilton’s eccentricity vector, it seems worthwhile asking integrability persists in the relativistic case. In the non-relativistic case, The extra constant of the motion is readily found by showing that the Hamilton-Jacobi
equation separates in prolate spheroidal coordinates. In fact using prolate spheroidal coordinates it is easy to see that the relativistic extension is not separable.

Thus if \( r_1 \) and \( r_2 \) are the distances from two points a distance \( 2a \) apart on the axis of symmetry of a system \( u, v, \phi \) of prolate spheroidal coordinates the spacetime metric is given by

\[
d s^2 = -c^2 dt^2 + a^2 \left( \sinh^2 u + \sin^2 v \right) \left( du^2 + dv^2 \right) + a^2 \sinh^2 u \sin^2 v d\phi^2 ,
\]

(2.27)

and

\[
a r_1 = \frac{1}{ \cosh u + \cos v }, \quad a r_2 = \frac{1}{ \cosh u - \cos v } .
\]

(2.28)

The Hamilton-Jacobi equation for a particle of charge \( e \) moving in an electrostatic potential \( V \) is

\[
\frac{1}{a^2} (\partial_u S)^2 + (\partial_v S) + \frac{1}{a^2} \left( \frac{1}{\sinh^2 u} + \frac{1}{\sin^2 v} \right) (\partial_{\phi} S)^2 = \frac{1}{c^2} (\sinh^2 u + \sin^2 v) \left\{ (E + eV)^2 - m^2 c^4 \right\} .
\]

(2.29)

If

\[
V = \frac{q_1}{r_1} + \frac{q_2}{r_s} = \frac{1}{a} \frac{(q_1 + q_2) \cosh u + (q_1 - q_2) \cos v}{\sinh^2 u + \sin^2 v} ,
\]

(2.30)

the only term on the r.h.s. of (2.29) which is not the sum of a function of \( u \) only and a function of \( v \) only is \( \frac{e^2 V^2}{c^2} \) which vanishes in the non-relativistic limit.

Further discussion of the relativistic Euler problem may be found in [25,26].

3 Photons in the Schwarzschild metric

For photons in the Schwarzschild metric, i.e. for null geodesics one may reduce the problem to a central orbit problem for which

\[
F(u) = \frac{3M u^2}{\hbar^2 u^2} .
\]

(3.31)

Thus the force is an attraction inversely as the inverse fourth power of the distance. It is a striking fact that (3.31) is unaffected by the addition of a cosmological term to the metric [27,28].

The pedal equation of a photon orbit is therefore

\[
\frac{1}{p^2} = 2 \left( C + \frac{M}{r^3} \right)
\]

(3.32)

Note that for those orbits which reach infinity, we have

\[
\frac{1}{p^2} \approx 2C = \frac{1}{b^2}
\]

(3.33)

where \( b \) is the impact parameter.

A special case is given by \( C = 0 \), in which case

\[
\frac{1}{p^2} = 2M u^3 = \frac{2M}{r^3}
\]

(3.34)
which is the pedal equation of the cardioid \[11\] p. 118.

\[ r = M(1 + \cos \phi). \tag{3.35} \]

This has parametric equation

\[ x = M \cos \lambda(1 + \cos \lambda), \quad y = M \sin \lambda(1 + \cos \lambda) \tag{3.36} \]

and Cartesian equation

\[ (x^2 + y^2 - Mx)^2 = M^2(x^2 + y^2). \tag{3.37} \]

Thus the photon starts on the past singularity at \( \phi = -\pi \) moves outwards and grazes the horizon at \( \phi = 0 \) and then moves back inwards to the future singularity at \( \phi = \pi \).

The pedal equation of the pedal curve of the photon orbit is in general

\[ 2Mp^3 = r^4 - 2Cr^6, \tag{3.38} \]

which, if \( C = 0 \), is the pedal equation of Cayley’s sextic \[12\] p. 155.

\[ r = 2M \cos^3\left(\frac{\phi}{3}\right) \tag{3.39} \]

whose Cartesian equation is

\[ (x^2 + y^2 - 2Mx)^3 = 27M^2(x^2 + y^2)^2. \tag{3.40} \]

The inverse of Cayley’s sextic is Tschirhausen’s cubic whose pedal equation in units in which \( h = 1 \) is \[11\]

\[ 2Mr^2 = p^3. \tag{3.41} \]

Its Cartesian equation is

\[ 54My^2 = (2M - x)(x + 16M), \tag{3.42} \]

and its parametric equation

\[ x = 2M(1 - 3\lambda^2), \quad y = 2M\lambda(3 - t^2) \tag{3.43} \]

and whose polar equation is

\[ r = \frac{2M}{\cos^3\left(\frac{\phi}{\pi}\right)}. \tag{3.44} \]

Thus Tschirhausen’s cubic \[3.44\] turned through a right angle is the hodograph of the cardioidal photon orbit \[3.35\].

One may continue the chain described above. The pedal equation of Tschirhausen’s cubic is a parabola with focus at the origin

\[ \frac{4M}{r} = 1 + \cos \phi \tag{3.45} \]
and the inverse of this parabola with respect to the origin

\[
\frac{r}{M} = 1 + \cos \phi
\]  

(3.46)

is a cardioid. All four curves are sinusoidal spirals of the form \((\frac{r}{M})^a = \sin(a\phi)\) with \(a = \frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, -\frac{1}{2}\) for the cardioid, Cayley sextic, Tschirhausen’s cubic, and parabola respectively.

In general the hodograph of the photon orbits has pedal equation

\[
2Ch^2 + \frac{2Mp^3}{h} = 0
\]

(3.47)

The null geodesics are given in general in terms of Weierstrass’s elliptic function

\[
\frac{1}{r} = \frac{1}{6M} + \frac{2}{M} p(\phi + c),
\]

(3.48)

where \(c\) is a constant of integration.(see e.g. [29]) and the cardioid is one of three cases where the Weierstrass function reduces to a trigonometric or hyperbolic trigonometric function. The other two have \(C = \frac{1}{34M^2}\) and take the form

\[
\frac{1}{r} = \frac{1}{3M} - \frac{1}{1 + \pm \cosh \phi}.
\]

(3.49)

These start from infinity or the singularity and endlessly encircle the circular photon orbit at \(r = 3M\).

### 4 Massive particles moving in the Schwarzschild metric

For a massive particle

\[
\frac{F(u)}{h^2u^2} = 3Mu^2 + \frac{M}{h^2},
\]

(4.50)

and we have a sum of an inverse fourth and inverse square law attraction. and the pedal equation for both cases is given by

\[
r^4 - 2Cr^6 = 2Mp^3 + \frac{2\epsilon Mpr^4}{h^2}
\]

(4.51)

where \(\epsilon = 0\) in the massless case and \(\epsilon = 1\) in the massive case.

Both the massless and massive orbits may be solved in terms of Weierstrass functions [29] and in some cases are equivalent problems. If \(v = u + a\) and \(\tilde{\phi} = \sqrt{1 - 6Ma}\phi\), and

\[
a^2 - \frac{a}{3M} + \frac{1}{3h^2} = 0,
\]

(4.52)

one finds

\[
\frac{d^2v}{d\tilde{\phi}^2} + v = \frac{3M}{1 - 6aM}v^2.
\]

(4.53)
One may regard (4.52) either as an equation for 
\[ a = \frac{1}{6M} \pm \frac{1}{3} \sqrt{\frac{1}{4M^2} - \frac{1}{h^2}} \] (4.54)
or an equation for 
\[ h^2 = \frac{1}{3} \left(1 - \frac{1}{6M^2} - \frac{1}{30M^2} \right) \] (4.55)
In either case given a photon orbit 
\[ r = r_p(\phi, M) \]
that is a solution of (3.31), then
\[ \frac{1}{r} = \frac{1}{r_p(\sqrt{1 - 6Ma\phi}, \frac{M}{1-6Ma})} + a \] (4.56)
is a solution of (4.50).

5 Central Orbits in Hyperbolic space

The Kepler problem in hyperbolic space has been studied since the nineteenth century [30][31]. More recently Higgs [33] and independently and later Chernikov [34] discussed its remarkable integrability problems. A recent extensive review is given in [32], see also [35].

The trick is to use Beltrami coordinates \( r \), with \( r = |r| = \tanh \chi \) in which the Lobachevsky metric is
\[ ds^2 = \frac{dr^2}{(1 - r^2)} + \frac{(r \cdot dr)^2}{(1 - r^2)^2} \] (5.57)
and in which free particles move on straight lines. The canonical momenta are
\[ p = \frac{\dot{r} - r^3 \dot{\phi}}{(1 - r^2)^2} \] (5.58)
Consider any spherically symmetric potential \( V(r) \). The conserved orbital angular momenta are
\[ L = r \times p = \frac{r \times \dot{r}}{(1 - r^2)} \] (5.59)
The motion lies in a plane and angular momentum conservation and energy conservation lead to the constancy of the angular momentum per unit mass \( h \) and the energy \( E \)
\[ h = \frac{r^2 \dot{\phi}}{(1 - r^2)}, \] (5.60)
\[ E = \frac{1}{2} \left[ \frac{r^2}{(1 - r^2)^2} + \frac{r^2 \dot{\phi}^2}{(1 - r^2)} \right] + V(r). \] (5.61)
Elimination of the time gives
\[ E = \frac{1}{2} h^2 \left[ \frac{1}{r^4} \left( \frac{dr}{d\phi} \right)^2 + \frac{1}{r^2} \right] + V(r) - \frac{h^2}{2}. \] (5.62)
This is exactly of the same form, for any potential \( V(r) \) as a central orbit problem in Euclidean space \( \mathbb{E}^3 \) with flat metric \( ds^2 = dr^2 \). Indeed if we set \( u = \frac{1}{r} \) we have, if \( p \) is given by (2.1),
\[ \frac{1}{p^2} = \frac{1}{h^2} \left( 2E - V(r) + h^2 \right) \] (5.63)
which is of the same form as (2.4).
5.1 The Kepler Problem in Hyperbolic space

In Lobachevsky space, translations do not commute but they continue to give conserved quantities if the potential vanishes. Thus if

\[ \pi = p - (r.p)r = \frac{\dot{r}}{(1-r^2)}, \]  
\[ \dot{\pi} = -\nabla V + (r.\nabla V)r. \]  

(5.64) (5.65)

In particular, if we chose for \( V \) a spherically symmetric harmonic function

\[ V = \Phi = \frac{q}{4\pi r} \]  

(5.66)

we find that

\[ \dot{\pi} = \frac{q}{4\pi}(1-r^2)\frac{r}{r^3} \]  

(5.67)

whence we obtain the constant Runge-Lenz vector,

\[ K = L \times \pi + \frac{q}{4\pi}\frac{\ddot{r}}{r}, \quad \dot{K} = 0. \]  

(5.68)

5.2 The Hodograph is a Circle

We define this to be the curve swept out by the vector \( \pi \). Since

\[ \dot{\pi} = \frac{q}{4\pi}r(1-r^2), \]  

(5.69)

so that the the tangent vector of the hodograph is in the radial direction and the angle \( \tilde{\psi} \) the tangent makes with a fixed direction is \( \phi \). Moreover if \( \tilde{s} \) is the arc-length along the hodograph

\[ \frac{d\tilde{s}}{dt} = |\dot{\pi}| = \frac{q}{4\pi r^2}(1-r^2). \]  

(5.70)

Now the radius of curvature \( \tilde{\rho} \) of the hodograph is given by

\[ \tilde{\rho} = \frac{d\tilde{s}}{d\tilde{\psi}} = \frac{d\tilde{s}}{d\phi} = \frac{d\tilde{s}}{dt} \frac{dt}{d\phi} = \frac{q}{4\pi \tilde{h}}. \]  

(5.71)

Thus the hodograph is a plane curve with a constant radius of curvature, i.e. a circle.

6 Conclusion

In this paper, The extension of of Hamilton’s notion of a hodograph to cover a particle moving in a curved background, possibly relativistically, has been studied. In flat space time the extension to include relativistic effects appears to present no great problems, even though relativistic effects may lead quantities which are conserved non-relativistically no longer being conserved. If in a curved spacetime the problem reduces, on choosing suitable coordinates, to a central orbit problem one may still define the hodograph in straightforward way. The case of geodesics in the Schwarzschild solution
has been treated in detail but the procedure adopted would work for any spherically symmetric static metric. Of course in that case, there is some freedom in the choice of radial coordinate and the example of hyperbolic space shows that an appropriate choice can lead to dramatic simplifications.

Les obvious is how to proceed if the metric is not spherically symmetric. In the case of free motion on a group manifold one may regard the analogue of Euler equations for a top as giving the hodograph. The case of hyperbolic space, which is a coset rather than a group manifold suggests a possible route to explore in the future. Another question for future study would be the Sommerfeld problem on hyperbolic space.

7 Acknowledgement

I would like to thank Peter Horvathy for his interest in this work and also Thanu Padmanabhan who suggested to me some years ago that the hodograph for Coulomb motion on hyperbolic space might be a circle.

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8 Appendix: The Hodograph on a Group Manifold

We start by giving general treatment of Hamiltonian mechanics on a group manifold, obtaining the Euler equations and the equations for the time dependence of the coordinates on the group manifold.

Given a Lie group $G$, coordinates $x^\mu$, i.e. group elements $G \ni g = g(x^\mu)$, and left and right invariant Cartan-Maurer forms

$$g^{-1}dg = \lambda^a e_a, \quad dgg^{-1} = \rho^a e_a$$

(8.72)

with $e_a$ a basis for the Lie algebra $\mathfrak{g}$ such that

$$[e_a, e_b] = C^c_{ab} e_c$$

(8.73)

$$d\lambda^c = \frac{1}{2} C^c_{ab} \lambda^a \wedge \lambda^b, \quad d\rho^c = \frac{1}{2} C^c_{ab} \rho^a \wedge \rho^b$$

(8.74)

we pass to the co-tangents space $T^*G = G \times \mathfrak{g}^*$ with Darboux coordinates $(x^\mu, p_\mu)$.

The left and right invariant vector field $L^\mu_a$ and $R^\mu_a$ dual to $\lambda^a_\mu, \rho^a_\mu$ repectively,

$$\lambda^a_\mu L^\mu_b = \delta^a_b, \quad \rho^a_\mu R^\mu_b = \delta^a_b$$

(8.75)

satisfy

$$[L_a, L_b] = C^c_{ab} L_c \quad [R_a, L_b] = 0, \quad [R_a, R_b] = -C^c_{ab} R_c$$

(8.76)

and generate right and left translations on $G$. Quantum mechanically one often inserts $i$’s so that if $\hat{R}_a = \frac{i}{2} R_a, \hat{L}_a = \frac{i}{2} L_a$ then

$$[\hat{R}_a, \hat{R}_b] = iC^c_{ab} \hat{R}_c$$

(8.77)
\[ [\hat{L}_a, \hat{L}_b] = -iC_{ac}^b \hat{L}_c. \] (8.78)

We may define moment maps into \( g^* \), the dual of the Lie algebra,
\[ M_a = p_\mu L^\mu_a, \quad N_a = p_\mu R^\mu_a, \] (8.79)
with Poisson brackets
\[ \{M_a, M_b\} = -C_{ac}^b M_b \quad \{M_a, N_b\} = 0, \quad \{N_a, N_b\} = C_{ac}^b M_b \] (8.80)
which generate the lifts of right and left translation to \( T^*G \). A Hamiltonian \( H = H(x^\mu, p_\mu) \) which is left-invariant satisfies
\[ \dot{N}_a = \{N_a, H\} = 0, \] (8.81)
and so the moment maps \( N_a \) are constants of the motion. By contrast, the moment maps generating right actions, \( M_a \), are time-dependent
\[ \dot{M}_a = \{M_a, H\} \neq 0, \] (8.82)

A left-invariant Lagrangian may be constructed from combinations of left-invariant velocities or angular velocities
\[ \omega^a = \lambda^a_\mu \dot{x}^\mu \] (8.83)
Thus the Hamiltonian is a combination of the momenta maps \( M_a \),
\[ H = H(M_a) \] (8.84)

Thus (8.82) provide an autonomous 1st order system of ODE’s on \( g^* \) for the moment maps \( M_a \) called the Euler equations. To obtain the motion on the group, one uses the equation
\[ \dot{x}^\mu = \frac{\partial H}{\partial p_\mu} \] (8.85)

Now
\[ p_\mu = M_a \lambda^a_\mu \] (8.86)
and so
\[ \dot{x}^\mu = L^a_\mu \frac{\partial H}{\partial M_a} \] (8.87)

The method described above can reasonably be called hodographic. Hamilton [1, 2] defined the hodograph of a particle motion \( x = x(t) \) in \( \mathbb{E}^3 \) as the the curve described by velocity vector \( v(t) = \frac{dx}{dt} \), a construction very similar to the Gauss map for surfaces in \( \mathbb{E}^3 \). Hamilton then discovered [1, 2] the elegant result that the hodograph for Keplerian motion is a circle.

Since velocity space and momentum space are naturally identified in this case we may think about the motion in phase space \( T^*\mathbb{E}^3 = (x, p) \), and then observed that because \( \mathbb{E}^3 \) is flat, there is, in addition to the standard vertical projection \( (x, p) \rightarrow (x, 0) \), a well defined horizontal map or hodographic projection \( (x, p) \rightarrow (0, p) \). For a general configuration space \( Q \), the co-tangent manifold \( T^*Q \), will not admit a well-defined horizontal projection. However if \( Q = G \), a group manifold, then it does, and the Euler equations govern the motion of the hodograph.