On the Dimensional Regularization of QCD Helicity Amplitudes With Quarks

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Abstract: We study QCD helicity amplitudes with an arbitrary number of (massive) quarks, keeping unobserved (loop) particles in fixed integer $D_s$ dimensions. We find a suitable embedding of external four-dimensional fermion states into higher dimensional spaces. This allows to identify the $D_s$ dependence of amplitudes with external quarks at one and two loops, permitting an analytic continuation in $D_s$. Explicitly we focus on ’t Hooft-Veltman and four-dimensional helicity schemes for which we provide a compact prescription for the computation of one- and two-loop amplitudes amenable for numerical implementation.
1 Introduction

Dimensional regularization is a well established framework, which is used extensively in perturbative computations in quantum field theories. It is a key ingredient for precise predictions for collisions of high energy particles at modern colliders. Various dimensional regularization schemes exist, which differ in their placement of observed (or external) particles and unobserved (or loop) particles in spaces with a dimensionality other than 4. For a recent review see [1]. A particular regularization scheme is typically chosen to provide the most convenient and efficient description of the problem at hand. While analytic approaches are more flexible when treating spin states in dimensional regularization, numerical approaches to the computation of loop amplitudes, such as numerical unitarity [2–5], are more limited and two conditions have to be considered:

i The spaces on which the scheme is defined have to allow for the existence of well-defined finite-dimensional representations for all external and loop particles;

ii The dimensionality of these spaces should be as small as possible to allow for an efficient implementation.
The first condition is in contradiction with the analytic continuation in the dimension parameter of loop-particle spin spaces $D_s$ away from integer values used in dimensional regularization. We discuss this feature in the context of the ‘t Hooft-Veltman (HV) scheme [6] and the four-dimensional helicity (FDH) scheme [7, 8], which keep external particles in four dimensions. These two schemes differ in the dimensionality $D_s$. In HV it is set to $D_s = 4 - 2\epsilon$ and in FDH to $D_s = 4$. We treat the two schemes uniformly by keeping $D_s$ unspecified. While the extraction of the analytic dependence on $D_s$ has been solved in the literature for gauge fields, the treatment of external fermionic fields is incomplete and we provide a complete solution.

There exist two known approaches in numerical computations to get around the problem raised by condition i. Either some parts of the computation have to be performed with additional analytic input as in refs. [5, 9–12], or the (polynomial) dependence on $D_s$ has to be reconstructed through sampling over integer $D_s$, which we refer to as dimensional reconstruction [4, 13, 14]. In both cases the scheme has to be complemented by a precise meaning of external particles being four-dimensional. This is straightforward in pure Yang-Mills theory by embedding four-dimensional polarization states into e.g. the upper components of $D_s$ dimensional ones, but requires some care when external fermions are present since the embedding of four-dimensional states into $D_s$-dimensional Dirac spinors is ambiguous. We propose a systematic prescription for the treatment of external fermions in the HV/FDH dimensional regularization schemes based on the ideas of [15], which extends the developments of [10, 11, 13, 16, 17] in what regards the embedding of external fermion states. We average over embeddings corresponding to a partial trace over the higher-dimensional components of spinor chains. Our prescription is essential for any numerical multi-loop computation of dimensionally regularized helicity amplitudes with external fermions. Using only a single embedding of external fermions leads to spurious integrand terms such as square roots of loop momenta contractions. At one loop such terms are encountered for massive external fermions (see e.g. [10, 17]), and beyond one loop also for massless external fermions. In proper analytic approaches to computations in the HV/FDH schemes, such as [18], these spurious integrand terms are not present.

In this work, we present a method to address the conditions i and ii simultaneously, thereby allowing for an efficient numerical computation of dimensionally regularized one- and two-loop amplitudes with external quarks. We use explicit representations of the Clifford algebra in integer $D_s$ dimensions, our embedding prescription for external fermions, and the tensor product structure of Clifford algebras in $D_s$ dimensions to analytically identify the full $D_s$ dependence of one- and two-loop QCD helicity amplitudes with an arbitrary number of external quarks. The coefficients of polynomials in $D_s$ are expressed in terms of diagrams with modified particle content. We then extrapolate $D_s$ to a desired value to obtain an amplitude in HV or FDH schemes. We thereby extend the known decompositions by particle content for pure Yang-Mills theories at one and two loops [19–21] to the full QCD spectrum including massive external quarks. A direct application of dimensional reconstruction techniques [4, 13, 14] for n-loop amplitudes requires computations in $n + 1$ different (even) integer $D_s > 4$ dimensions. Contrary to that, our approach uses only a single minimal dimension, which is determined by the number of non-trivial directions of loop momenta. This allows to alleviate the exponential growth\(^1\) of the dimensionality of corresponding representations for amplitudes with external quarks.

We validated our approach by implementing it for the computation of dimensionally regulated one-loop amplitudes in a new version of the BlackHat library, which we used to provide phenomenologically relevant predictions for processes involving massive external quarks [22]. In this work we consider bare color-ordered QCD amplitudes. However since the argument presented relies mostly on the Lorentz index structure of the particles involved, an extension to a broader class of

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\(^1\)Representations of the Clifford algebra generated from $D_s$-dimensional space have the dimensionality of $2^{D_s/2}$. 

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theories is possible.

This paper is organized as follows. In sec. 2 we review the considered regularization schemes. In sec. 3 we establish the notation used in this work and state the properties of Clifford algebras and spinor states in $D_s$ dimensions. In sec. 4 we formulate the embedding prescription for external fermions. We then work out the $D_s$ dependence of helicity amplitudes at one and two loops and use it to obtain amplitudes in HV and FDH schemes. In sec. 5 we discuss the connection of our approach to the four-dimensional formulation (FDF) of FDH [10] at one loop. In sec. 6 we give our conclusions. We provide implementation details of the computational prescription at one loop in Appendix D.

2 FDH and HV Schemes

We follow the notation for different regularization schemes introduced in [1]. It is based on distinguishing a four-dimensional Minkowski space $S[4]$, an infinite-dimensional space of loop momentum integration $QS[4]$ with $D = 4 - 2\epsilon$ and an enlarged space ($D_s \geq D$) of internal spin states $QS[D_s]$, such that

$$QS[D_s] = QS[4] \oplus QS[n_s] = S[4] \oplus QS[n_s-2\epsilon], \quad n_s - 2\epsilon = D_s - 4. \quad (2.1)$$

We restrict our discussion to amplitudes computed in the FDH and HV schemes, that is all external particles are kept in $S[4]$, while loop particles are extended to $QS[D_s]$. These two schemes differ in the value of $D_s$, which we choose to keep general postponing the specialization. A split of tensor and Clifford algebras is induced by

$$g_{[D_s]}^\mu^\nu = g_{[4]}^\mu^\nu + g_{[D_s-4]}^\mu^\nu, \quad (2.2a)$$

leading to the properties

$$(g_{(\text{dim})})^\mu_\mu = \text{dim,} \quad (g_{[4]})^{\mu\nu}(g_{[4]}[D_s-4])_{\mu\nu} = 0, \quad (2.2b)$$

with $\text{dim} \in \{D_s, 4, D_s - 4\}$ and

$$A_{[4]}^\mu \equiv g_{[4]}^\mu A_{[D_s]}, \quad A_{[D_s-4]}^\mu \equiv g_{[D_s-4]}^\mu A_{[D_s]}, \quad (2.2c)$$

where $A_{[D_s]}^\mu$ is any object carrying a vector index, like momenta or gamma matrices. We use a left superscript in brackets to indicate the dimensionality $D_s$ of the object explicitly, as in $(D_s)A$, whenever there is a potential ambiguity.

The space $QS[D_s]$ is formally infinite-dimensional for non-integer $D_s$. However, $D_s$ can be chosen to be integer and then dim $\left(QS[D_s]\right) = D_s$. In this case an explicit representation of tensor and spinor algebras is possible (see e.g. [23]). After all sources of $D_s$ dependence are identified, an FDH or an HV amplitude is obtained by setting $D_s = 4$ or $D_s = 4 - 2\epsilon$ respectively.

If four-dimensional external momenta are embedded trivially in $QS[D_s]$, an integrand of a loop amplitude can only depend on $(D - 4)$ directions of loop momenta through contractions between themselves. Rotational symmetry of the loop integration thus allows, without loss of generality, to write the loop momentum as

$$\ell_{[D]}^\alpha = \ell_{[4]}^\alpha + \ell_{[D-4]}^\alpha \equiv \ell_{[4]}^\alpha + i \sum_j \mu_j n_j^\alpha, \quad n_j^2 = 1, \quad (2.3a)$$

$$\ell_{[D]}^2 = \ell_{[4]}^2 + \ell_{[D-4]}^2 \equiv \ell_{[4]}^2 + \sum_j \mu_j^2, \quad (2.3b)$$

where $\{n_j^\alpha\}$ span non-trivial directions in $QS[D-4]$, with dim$(\{n_j^\alpha\})$ given by the number of loops.
3 Clifford Algebra and Spinors in Integer $D_s$

In this section, we review the definition of the objects of different dimensionality that we handle in this work and the notation that we adopt for them. In particular, we require representations of the Clifford algebra and associated spinor states in arbitrary integer dimensions.

3.1 Representations of the Clifford Algebra

A Clifford algebra $\mathcal{C}_1, D_s - 1(\mathbb{R})$ generated from an even integer $D_s$-dimensional Minkowski spacetime with a signature $\{+, -, \ldots, -\}$ has a defining property

$$\{\Gamma^\mu, \Gamma^\nu\} = 2g^{\mu\nu},$$  \hspace{1cm} (3.1)

where $\Gamma^\mu$ are the generating elements (which we refer to as gamma matrices) and $g^{\mu\nu}$ are the metric tensor components. Its $D_t = 2^{D_s/2}$-dimensional representation can be constructed recursively [23]:

$$(D_t)^\Gamma^\mu = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes (D_t^{D_s})^\Gamma^\mu, \quad \mu = 0, \ldots, D_s - 3,$$  \hspace{1cm} (3.2a)

$$(0 \ i \ i \ 0) \otimes (D_t^{D_s})^\Gamma^\nu, \quad \mu = D_s - 2,$$  \hspace{1cm} (3.2b)

$$(0 \ 1 \ -1 \ 0) \otimes (D_t^{D_s})^\Gamma^\nu, \quad \mu = D_s - 1$$  \hspace{1cm} (3.2c)

with

$$(D_t)^\Gamma^\star = i^{D_s/2-1} \Gamma^0 \cdots \Gamma^{D_s-1}, \quad \Gamma^\star \Gamma^\star = 1,$$  \hspace{1cm} (3.2d)

which can also be represented as

$$(D_t)^\Gamma^\star = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes (D_t^{D_s})^\Gamma^\star.$$  \hspace{1cm} (3.2e)

We choose the four-dimensional Dirac $\gamma$-matrices as the starting point of the iteration:

$$^{(4)}\Gamma^\mu \equiv \gamma^\mu,$$  \hspace{1cm} \hspace{1cm} (4.4)

$$^{(4)}\Gamma^\star \equiv \gamma_5.$$  \hspace{1cm} (3.3)

It is useful to expose the tensor product structure of the Clifford algebra $\mathcal{C}_{1, D_s - 1} \equiv \mathcal{C}_{0, D_s - 4} \otimes \mathcal{C}_{1, 3}$. The first four matrices in $D_s \geq 4$ obtained with the iteration in eq. (3.2a), can be written as

$$g^{\mu\nu^{(D_t)}} \Gamma_{\nu[D_s]}^{(D_s)} \equiv (D_t)^{\Gamma_{\nu[D_s]}^{(D_s)}} = \begin{cases} (D_t^{D_s})^\Gamma^\nu \otimes \gamma^\mu, & \mu \leq 3, \\ 0, & \mu \geq 4, \end{cases}$$  \hspace{1cm} (3.4)

that is they are $D_t \times D_t$ dimensional matrices with $D_t/4$ copies of the four dimensional gamma matrices on the diagonal. The remaining matrices can be written as

$$g^{\mu\nu^{(D_s-4)}} \Gamma_{\nu[D_s]}^{(D_s-4)} \equiv (D_t)^{\Gamma_{\nu[D_s]}^{(D_s-4)}} = \begin{cases} 0, & \mu \leq 3, \\ (D_t^{D_s})^\gamma^\mu \otimes \gamma_5, & \mu \geq 4, \end{cases}$$  \hspace{1cm} (3.5)

where $g^\mu$ are $(D_t/4)$-dimensional matrices obtained from iteration of eqs. (3.2b) and (3.2c). $g^\mu$ are generating elements of an Euclidean Clifford algebra $\mathcal{C}_{0, D_s - 4}$. We denote the dimensionality of its representation as $D_t \equiv D_t/4$. 

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\[ D_{s} \text{ dimensions the } D_{s}/2 \text{ independent fermionic states } u_{k} \text{ must satisfy the polarization sum} \]
\[ \sum_{k=1}^{D_{s}/2} u_{k} \bar{u}_{k} = \ell + m, \quad (3.6) \]
as well as the Dirac equation
\[ (\ell - m)u_{k} = 0, \quad \bar{u}_{k}(\ell - m) = 0. \quad (3.7) \]

The states constructed by the action of the inverse Dirac operator on arbitrary reference spinors \[ q_{k} \] in \[ D_{s} \] dimensions
\[ u_{k}(\ell, q_{k}) = \frac{(\ell + m)}{\sqrt{N}} q_{k}, \quad \bar{u}_{k}(\ell, q_{k}) = \frac{(\ell + m)}{\sqrt{N}}, \quad (3.8) \]
with a suitable normalization \[ N \] satisfy the requirements of eqs. (3.6) and (3.7) by construction. A \[ D_{s}-\text{dimensional basis for the reference spinors } q_{k} \] can be constructed recursively:
\[ (D_{s})q_{k} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \left( \frac{\sqrt{t}}{q} \right) q_{k}, \quad (D_{s})q_{k+D_{s}/4} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \left( \frac{\sqrt{t}}{q} \right) q_{k}, \quad (3.9a) \]
\[ (D_{s})\bar{q}_{k} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \left( \frac{\sqrt{t}}{q} \right) \bar{q}_{k}, \quad (D_{s})\bar{q}_{k+D_{s}/4} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \left( \frac{\sqrt{t}}{q} \right) \bar{q}_{k}, \quad (3.9b) \]
with \( k \in \{0, \ldots, D_{s}/4 - 1\} \). We choose as the starting point of this recursive definition in \( D_{s} = 4 \) the Weyl spinors
\[ (4)q_{1} \equiv |q, +\rangle, \quad (4)q_{2} \equiv |q, -\rangle, \quad (3.10a) \]
\[ (4)\bar{q}_{1} \equiv \langle q, +\rangle, \quad (4)\bar{q}_{2} \equiv \langle q, -\rangle, \quad (3.10b) \]
representing states with definite helicity.

The \( D_{s}\)-dimensional spinors for four-dimensional momenta have a particularly simple form. To reflect the diagonal structure of the \( D_{s}\)-dimensional part of the first four gamma matrices in eq. (3.4) we split the index \( k \) labeling \( D_{s}/2 \) states into an index \( i \in \{0, 1\} \) denoting the helicities \{+ , -\} and an index \( j \in \{0, \ldots, D_{s}/2 - 1\} \) such that \( k = i + 2j \) and obtain
\[ (D_{s})u_{i,j}(p_{(4)}) = |j\rangle \otimes |p, \pm\rangle \equiv |p, \pm, j\rangle, \quad (D_{s})\bar{u}_{i,j}(p_{(4)}) = \langle j\rangle \otimes \langle p, \pm\rangle \equiv \langle p, \pm, j\rangle, \quad j \in \{0, \ldots, D_{s}/2 - 1\}, \quad (3.11) \]
with \( \{|j\rangle\} \) forming an orthonormal basis in the \( D_{s}\)-dimensional space corresponding to the representation of an Euclidian Clifford algebra, such that
\[ \langle p_{1}, \pm, j_{1}|p_{2}, \pm, j_{2}\rangle = \langle p_{1}, \pm|p_{2}, \pm\rangle \delta_{j_{1}j_{2}}. \quad (3.12) \]
We refer to the index \( j \) of the fermion states as the \( D_{s}\)-index. Note that there are in general \( D_{s}\) states for each helicity choice in \( D_{s} \) dimensions.

### 3.3 Normalized Partial Traces

In this work we make use of normalized partial traces over the subspace \( \tilde{D}_{s} \) of products of matrices \( \Gamma'_{D_{s}} \), which we define as a map
\[ \tilde{\text{Tr}}[:C\ell_{0,D_{s}/4} \otimes C\ell_{1,3}] \mapsto C\ell_{1,3}, \quad (3.13) \]
such that for all $\tilde{A} \in C_{\ell_0,D_s-4}$ and $B \in C_{\ell_1,3}$

$$\tilde{\text{Tr}}[\tilde{A} \otimes B] = \frac{1}{D_t} \text{Tr}[\tilde{A}] \cdot B,$$

(3.14)

where $\text{Tr}[\tilde{A}]$ is a usual trace in $C_{\ell_0,D_s-4}$. Given the recursive construction by eqs. (3.2) and (3.9), it can be explicitly evaluated as

$$(4)A_{ij} \equiv \left(\tilde{\text{Tr}} \left[(D_t)A\right]\right)_{ij} = \frac{1}{D_t} \sum_{k=1}^{D_s-4} (D_t)A_{i+4(k-1),j+4(k-1)},$$

(3.15)

where $(4)A_{ij}$ are the components of the image of $(D_t)A_{ij}$. This can be interpreted as an averaging over all possible embeddings of the four-dimensional states into a $D_t$-dimensional representation. We present some properties of normalized partial traces in the Appendix A.

The definition given by eq. (3.13) can be generalized for any choice of a tensor product split $C_{\ell_0,D_s-1} \otimes C_{\ell_1,x-1}$ with $4 \leq x \leq D_s - 2$. In particular we use the case with $x = 6$ for two-loop computations in section 4.3.

## 4 Helicity Amplitudes

We propose to compute integrands of dimensionally regularized loop helicity amplitudes in HV or FDH schemes using finite-dimensional representations of the Clifford algebra. The building blocks of our approach are amplitudes with loop particles in $QS_{[D_s]}$ and four-dimensional momenta and polarization states of external particles embedded in the enlarged $D_s$-dimensional space.

In the first part of this section, we work out the details of a correct treatment of external particles in the considered regularization schemes. In the second part, we then analyze the $D_s$ dependence of amplitudes at one and two loops. The resulting decomposition by particle content allows to compute in spaces of reduced dimensionality and HV or FDH amplitudes are obtained by extrapolating $D_s$ to the corresponding value.

### 4.1 Embedding Prescription

Given a representation of a Clifford algebra $C_{\ell_1,D_s-1}(\mathbb{R})$ in $D_1 = 2^{D_s}/2$ dimensions, one can construct a basis of $2^{D_s}$ elements

$$\left\{ \mathbb{1}, \Gamma^{[\nu_1]}, \Gamma^{[\nu_1\nu_2]}, \ldots, \Gamma^{[\nu_1\ldots\nu_{D_s}]} \right\}, \quad \nu_i \in \{0, \ldots, D_s - 1\},$$

(4.1)

from antisymmetric combinations of $\Gamma^\nu$ as follows:

$$\Gamma^{[\nu_1\ldots\nu_m]} \equiv \frac{1}{m!} \sum_{\sigma \in S_m} \text{sgn}(\sigma) \Gamma^{\sigma_1} \ldots \Gamma^{\sigma_m},$$

(4.2)

where the summation runs over all permutations and the parity $\text{sgn}(\sigma)$ of each permutation is given by the number of inversions in it and evaluates to $+1$ or $-1$ for even or odd permutations respectively.

Any product of gamma matrices

$$S^{\mu_1\ldots\mu_n} = \Gamma^{\mu_1} \ldots \Gamma^{\mu_n}$$

(4.3)

can be decomposed using the basis (4.1) as

$$S^{\mu_1\ldots\mu_n} = \alpha^{\mu_1\ldots\mu_n} \mathbb{1} + \alpha^{\mu_1\ldots\mu_n}_{\nu_1} \Gamma^{[\nu_1]} + \alpha^{\mu_1\ldots\mu_n}_{\nu_1\nu_2} \Gamma^{[\nu_1\nu_2]} + \ldots + \alpha^{\mu_1\ldots\mu_n}_{\nu_1\ldots\nu_{D_s}} \Gamma^{[\nu_1\ldots\nu_{D_s}]}.$$

(4.4)
For the special case of \( D_s = 4 \) we have:
\[
S_{[4]}^{\mu_1 \ldots \mu_n} = \alpha^{\mu_1 \ldots \mu_n} \mathbb{1} + \alpha^{\mu_1 \ldots \mu_n}_{\nu_1} \Gamma^{[\nu_1]} + \alpha^{\mu_1 \ldots \mu_n}_{\nu_1 \nu_2} \Gamma^{[\nu_1 \nu_2]} + \alpha^{\mu_1 \ldots \mu_n}_{\nu_1 \nu_2 \nu_3} \Gamma^{[\nu_1 \nu_2 \nu_3]} + \alpha^{\mu_1 \ldots \mu_n}_{\nu_1 \nu_2 \nu_3 \nu_4} \Gamma^{[\nu_1 \nu_2 \nu_3 \nu_4]}.
\] (4.5)

Using the property \((D_i)\Gamma_{[4]}^{\mu} = (\mathcal{D}_i)\mathbb{1} \otimes \gamma^\mu\) (see eq. (3.4)) and the normalized partial trace defined in section 3.3 we can identify the basis elements of the four-dimensional basis embedded trivially in \( D_t \) dimensions by
\[
(4) \Gamma^{[\nu_1 \ldots \nu_m]} = \overline{\text{Tr}} \left[ (D_t) \Gamma_{[4]}^{[\nu_1 \ldots \nu_m]} \right].
\] (4.6)

An important observation is that the basis elements \( \Gamma_{[4]}^{[\nu_1 \ldots \nu_m]} \) form a subset of the \( D_t \)-dimensional basis and we can thus split the basis decomposition of eq. (4.3) in \( D_t \) dimensions into a direct sum
\[
S_{[4]}^{\mu_1 \ldots \mu_n} = S_{[4]}^{\mu_1 \ldots \mu_n} \oplus S_{[D_s-4]}^{\mu_1 \ldots \mu_n}.
\] (4.7)

A well-defined treatment of loop helicity amplitudes in integer dimensions in the FDH or HV scheme requires external particle states to be in four-dimensions, cf. discussion in Sec. 2. Conversely, gamma matrices originating in interactions of loop particles are forced to be \( D_t \)-dimensional. Therefore, the spinors of external particles need to be embedded in the \( D_t \)-dimensional space. To this end, taking inspiration from the ideas of [15], we would have to require that the states \( (D_t)u_{\pm} \) and \( (D_t)\bar{u}_{\pm} \) representing external particles with definite helicity satisfy

\[
(D_t)\bar{u}_{\pm} (D_t)\Gamma_{[D_s-4]}^{[\mu_1 \ldots \mu_n]} (D_t)u_{\pm} = 0,
\] (4.8a)

\[
(D_t)\bar{u}_{\pm} (D_t)\Gamma_{[4]}^{[\mu_1 \ldots \mu_n]} (D_t)u_{\pm} = \overline{\text{Tr}} \left[ (D_t) \Gamma_{[4]}^{[\mu_1 \ldots \mu_n]} \right] u_{\pm},
\] (4.8b)

that is all matrix elements of four-dimensional states trivially embedded in a higher-dimensional space annihilate all \([D_s-4]\) basis elements in (4.7) while leaving all four-dimensional basis elements unchanged. However one can show that no such states can be found in finite-dimensional representations. To overcome this we propose a consistent prescription in which the conditions of eq. (4.8) are fulfilled by a normalized partial trace over \( \mathcal{D}_t \),

\[
(D_t)\bar{u}_{\pm} (D_t)S_{[4]}^{\mu_1 \ldots \mu_n} (D_t)u_{\pm} \iff \sum_{j=1}^{D_t} \langle p_1, \pm, j \rangle (D_t)S_{[4]}^{\mu_1 \ldots \mu_n} | p_2, \pm, j \rangle = \langle p_1, \pm \rangle \overline{\text{Tr}} \left[ (D_t) S_{[4]}^{\mu_1 \ldots \mu_n} \right] | p_2, \pm \rangle.
\] (4.9)

Indeed, we have
\[
\overline{\text{Tr}} \left[ (D_t) S_{[D_s-4]}^{\mu_1 \ldots \mu_n} \right] = 0
\] (4.10a)

\[
\overline{\text{Tr}} \left[ (D_t) S_{[4]}^{\mu_1 \ldots \mu_n} \right] = (4) S_{[4]}^{\mu_1 \ldots \mu_n},
\] (4.10b)

which correspond to eqs. (4.8a) and (4.8b) respectively. Eq. (4.10a) follows from the fact that basis elements are either antisymmetric in the \( \mathcal{D}_t \) space with respect to at least a single pair of indices or contain a single gamma matrix with \( \mu \geq 4 \). In both cases, the basis elements are eliminated by the partial trace. The relation in eq. (4.10b) follows from the defining property of the partial trace, see eq. (3.14).

\footnote{The basis in four dimensions is typically simplified making use of \( \gamma_5 \), however this is not advantageous for our argument.}
Thus the normalized partial trace over the $\tilde{D}_t$ space allows a consistent embedding of four-dimensional external fermion states into a $D_s$-dimensional space. This can be combined with an embedding of four-dimensional vector-boson states into a $D_s$-dimensional space, which does not pose any difficulty. An **integrand** of a loop helicity amplitude with all external particles embedded trivially is then schematically given by

$$A^{[D_s]}_{\text{external} \in S[4]} = \tilde{\text{Tr}}_{l_1} \cdots \tilde{\text{Tr}}_{l_n} \left[ A^{[D_s]}(j_{l_1}, \ldots, j_{l_n}) \right],$$

(4.11)

where $\tilde{\text{Tr}}_{l_i}$ denotes the substitution of eq. (4.9) applied to a pair of external fermions $i$. Note that $A^{[D_s]}(j_{l_1}, \ldots, j_{l_n})$ is an amplitude in $D_s$ dimensions with external fermions having open $\tilde{D}_t$-indices $j_{l_i}$, which means that eq. (4.11) defines a map from $D_t/2$ degrees of freedom per fermion to two degrees of freedom (two helicities) per fermion.

Amplitudes with multiple identical external quark-antiquark pairs can be decomposed into a properly antisymmetrized sums of amplitudes with distinct quark-antiquark pairs. Our embedding prescription is then applied to each of the amplitudes with distinct quark-antiquark pairs separately.

### 4.2 $D_s$ Dependence at One Loop

We now proceed to establish the explicit $D_s$ dependence of one-loop amplitudes by analyzing all possible non-trivial contributions to traces over $\tilde{D}_t$ in eq. (4.11). They can be grouped in four different classes depending on the type of particles in the loop: closed gluon loops, one fermion line entering the loop, two or more fermion lines entering the loop, and closed fermion loops.

In the following, we use the split of an axial gauge$^3$ gluon propagator in $D_s$ dimensions induced by a specific choice of the reference momentum (see Appendix C). Since also gluonic vertices respect this split-up, a $D_s$-dimensional gluon line in the loop decomposes into a four dimensional, “massive” gluon with a mass $\mu^2$ and $(D_s - 5)$ scalar components. The decomposition of pure gluon amplitudes by particle content, with the split-up into a “massive” gluon amplitude in four-dimensions$^4$ and a scalar amplitude, was previously established in [19]. In the remainder of this section, we establish a decomposition valid for the full QCD spectrum including several (massive) quark lines.

![Figure 1](image)

**Figure 1.** A schematic subdiagram for a contribution from a generic spinor line entering the loop. Bold lines and vertices represent objects in $D_s$ dimensions. Blobs represent parts of the diagram which explicit form is not important here.

When fermion lines enter (and leave) the loop, they contribute generic spinor chains to diagrams (Fig. 1). The source of gamma matrices contributing to the trace over $\tilde{D}_t$ is twofold. Vertices of quarks coupled to gluons carrying loop momentum are proportional to $\Gamma_{[D_s]}^\alpha$. Propagators of quarks with mass $m$ carrying loop momentum $\ell$ can be written, using relations (2.3), as

$$S_q(\ell_{[D_s]}) = i \frac{\ell_{[D_s]}^\alpha \Gamma^\alpha + m \cdot \mathbb{1}}{\ell_{[D_s]}^2 - m^2} = \frac{i \left( \ell_{[4]} + m \cdot \mathbb{1} + \mu \Gamma^4 \right)}{\ell_{[4]}^2 - \mu^2 - m^2},$$

(4.12)

$^3$A similar, but more involved argument can be carried out with the Feynman gauge propagator.

$^4$Or equivalently a massless gluon in five-dimensions.
where the only non-trivial in $\tilde{D}_t$ contribution comes from $\mu \Gamma^4$, since the first two terms inside the brackets are proportional to a unit matrix in $\tilde{D}_t$. A generic spinor string $T_{[D_s]}^{\mu\nu}$, with indices coupled to $D_s$-dimensional gluons in the loop left uncontracted, is then of the form

$$T_{[D_s]}^{\mu\nu} = (p_1, \pm) \frac{1}{\tilde{T}r} \left[ \prod_{k=0}^{n-1} \Gamma^4 \ldots \right] \left[ \prod_{k=0}^{n-1} \Gamma^\mu \ldots \right] |p_2, \pm\rangle, \quad (4.13)$$

where the dots $\cdots$ represent any number of gamma matrices which contribute only a unit matrix to the trace. The partial trace can be evaluated explicitly using eqs. (3.4) and (3.5). It is convenient to absorb the sign stemming from the square of even powers of $q^4$ insertions from quark propagators into the remaining four-dimensional $\gamma_5$ by

$$\ell_{[D_{s-4}]}^{\alpha} = \mu \left( q^4 \otimes \gamma_5 \right) \equiv \mu (-i q^4) \otimes (i \gamma_5), \quad (4.14)$$

such that we have $(-i q^4)^2 = 1$. Using the fact that $T_{[D_s]}^{\mu\nu}$ is only non-vanishing for an even number of $\Gamma_{[D_{s-4}]}^{\mu\nu}$ insertions and trace properties from Appendix A we can evaluate it to

$$T_{[D_s]}^{\mu\nu} = \begin{cases} \left( \cdots \gamma^\mu \cdots \right), & \mu, \nu \leq 3, \ n \text{ even}, \\ \left( \cdots \gamma^5 \cdots \right) \Gamma_{[D_{s-4}]}^{\mu\nu}, & \mu, \nu \geq 4, \ n \text{ even}, \\ -i \left( \cdots \gamma^\mu \cdots \right) \Gamma_{[D_{s-4}]}^{\mu\nu}, & \mu \geq 4, \nu \leq 3, \ n \text{ odd}, \\ -i \left( \cdots \gamma^5 \cdots \right) \Gamma_{[D_{s-4}]}^{\mu\nu}, & \mu \leq 3, \nu \geq 4, \ n \text{ odd}, \end{cases} \quad (4.15)$$

where only the remainders of $D_s$-dimensional gamma matrices with open indices from eq. (4.13) are explicitly shown, and the contraction with external states is implicit.

![Figure 2](image_url)  

**Figure 2.** Examples of diagrams with quarks in the loop. Bold lines and vertices represent objects in $D_s$ dimensions.

The result of eq. (4.15) then needs to be contracted with the rest of the loop diagram $D_{[D_s]}^{\mu\nu}$. If only a single fermion line enters the loop (e.g. as in fig. 2a), we can use the decoupling of “massive” gluons and scalars and write

$$D_{[D_s]}^{\mu\nu} = \{ \cdots \} [4]_{\mu\nu} + \{ \cdots \} [4]_S \cdot g_{[D_{s-5}]}^{\mu\nu}, \quad (4.16)$$

where $\{ \cdots \}_{[4]}$ is a tensor with all $D_s$-dimensional gluons replaced by “massive” gluons and $\{ \cdots \}_S$ is a contribution with all $D_s$-dimensional gluons replaced by scalars. Thus the contraction takes the form

$$T_{[D_s]}^{\mu\nu} D_{[D_s]}^{\mu\nu} = T_{[D_s]}^{\mu\nu} \left( \{ \cdots \} [4]_{\mu\nu} + \{ \cdots \}_S \cdot g_{[D_{s-5}]}^{\mu\nu} \right) = d_G + (D_s - 5) d_S, \quad (4.17)$$

where $d_G$ is a diagram with “massive” gluons and $d_S$ is a diagram with scalars. In the second equality we used (4.15) and the property

$$g_{[D_{s-4}]}^{\mu\nu} |g_{[D_{s-5}]}^{\nu\mu} = g_{[D_{s-5}]}^{\mu\nu} = (D_s - 5). \quad (4.18)$$
For loop diagrams with multiple quark lines entering the loop (e.g. as in fig. 2b), the split-up of $D_s$-dimensional gluons leads to mixed diagrams with both “massive” gluons and scalars coupling to the same spinor string which have to be considered in general. However the contraction of a fermion line with a “massive” gluon on one side and with a scalar on the other side turns out to vanish. Indeed, in this case the last two cases from eq. (4.15) are chosen, which leads to

\[ g_{[D_s-5]}^{\mu\nu} g_{[D_s-4]\nu\delta} = g_{[D_s-5]}^{\mu\delta} = 0. \]  

(4.19)

Therefore, diagrams with “massive” gluons in the loop and those with scalars in the loop decouple in the case of multiple quark lines as well. The diagrams with scalars in the loop pick up a factor of $(D_s - 5)$ from contractions across multiple lines as in eq. (4.18).

Summing up, we found that all diagrams exhibit the same dimensional dependence and respect the decoupling of “massive” gluons and scalars. Hence a partially traced amplitude with external particles in $S[4]$ and any number of external quark pairs can be written in arbitrary $D_s$ as

\[ A_{\text{external} \in S[4]}^{[D_s]} = A_G + (D_s - 5) A_S, \]  

(4.20)

where $A_G$ is a sum of diagrams with “massive” gluons in the loop and $A_S$ is a sum of diagrams with scalars in the loop. Both $A_G$ and $A_S$ are defined in terms of four-dimensional objects with the additional requirement of only even number of $(D_s - 4)$-dimensional loop momentum insertions from the quark propagator (4.12) on each fermion line. The latter is in general impossible to ensure numerically only with four-dimensional representations of the Clifford algebra, hence it is the ultimate obstruction to such an attempt. Further discussion and details for the computation of $A_G$ and $A_S$ are provided in Appendix D.

Once the full dimensional dependence is established by eq. (4.20), we can use it to define an amplitude for continuous $D_s$. An FDH or HV amplitude can be then obtained by setting $D_s = 4$ or to $D_s = 4 - 2\epsilon$ correspondingly:

\[ A^{\text{FDH}} = A_G - A_S, \]  

\[ A^{\text{HV}} = A_G - (1 + 2\epsilon) A_S = A^{\text{FDH}} - 2\epsilon A_S. \]  

(4.21)

Finally, diagrams with a closed fermion loop (e.g. as in fig. 2c) do not have any $D_s$-dimensional vector index contractions, hence the only possible dimensional dependence is an additional total factor of $\tilde{D}_t$ from the loop trace, which we can cancel explicitly and get

\[ A_{\text{nf}}^{\text{FDH}} = A_{\text{nf}}^{\text{HV}} = \frac{1}{\tilde{D}_t} A_{\text{nf}}^{[D_s]}. \]  

(4.22)

We validated our approach by comparing NLO virtual matrix elements of most QCD processes with up to 7 partons including massive top and bottom quarks as well as those with an associated emission of a $W^\pm$ boson\(^5\) with publicly available tools.

### 4.3 $D_s$ Dependence at Two Loops

The results of sections 2, 3, and 4.1, in particular eq. (4.11), are valid at any loop order. However the analytical evaluation of partial traces that we accomplished in the previous section is not possible beyond one loop. Considering a two-loop computation we perform a two-step process: partial traces over dimensions higher than six can be performed analytically once and for all, and the remaining traces have to be computed explicitly.

At two loops we need at least $D_s = 6$ to embed two non-trivial directions of loop momenta (see eq. (2.3)). We split $D_s$-dimensional gluons in six-dimensional (6d) gluons and scalars. Similar to the

\(^5\) Chiral couplings can be incorporated using the t’Hooft-Veltman prescription, see Appendix B.
one-loop case, scalar lines start and end only on quarks lines, or they form a closed loop otherwise (see Appendix C). At one loop we chose an axial gauge to effectively reduce five-dimensional gluon to a four-dimensional “massive” gluon and keep quark propagators in four dimensions. Since the dimensionality of the loop momentum is even at two loops, such a decomposition is not beneficial.

For pure Yang-Mills theory the dimensional dependence is given \[21\] by

\[
A^{[D_s]}_{\text{external} \in S(4)} = A^{[6]}_G + (D_s - 6) A^{[6]}_{GS} + (D_s - 6)^2 A^{[6]}_S ,
\]

where \(A^{[6]}_G\) receives contributions from diagrams with only 6d gluons; \(A^{[6]}_{GS}\) from diagrams with one closed scalar loop; \(A^{[6]}_S\) from diagrams with two closed scalar loops. The latter is only possible for factorized (or “one-loop-squared”) topologies, i.e. when two loops do not share a propagator (see e.g. fig. 3b). Compared to the one-loop case there is no decoupling of 6d gluons and scalars at the level of full amplitudes since the \((D_s - 6)\) coefficient contains diagrams with both 6d gluons and scalars. Another subtlety is that one is forced to consider two different “flavors” of scalars for the evaluation of \(A^{[6]}_S\) since the four-gluon vertex couples scalars coming from different \(D_s\).

\[\text{Figure 3.} \quad \text{Diagrams contributing to the } (D_s - 6)^2 \text{ coefficient. Quark-scalar vertices originating from equal indices are represented by the same shape.}\]

In what follows we extend eq. (4.23) to amplitudes with external fermions. In this case we have to compute partial traces as prescribed by eq. (4.11). The tensor product structure of the Clifford algebra allows us to accomplish this in two steps:

\[
\widetilde{\text{Tr}}_{D_s \to 4} [ \cdot ] = \widetilde{\text{Tr}}_{6 \to 4} \left[ \widetilde{\text{Tr}}_{D_s \to 6} [ \cdot ] \right] ,
\]

where the traces \(\widetilde{\text{Tr}}_{D_s \to 4} [ \cdot ]\) are defined in the section 3.3; and \(\widetilde{\text{Tr}}_{D_s \to 6} [ \cdot ]\) can be defined similarly with the target \(D_s = 4\) space replaced by the \(D_s = 6\) one. First we trace over all degrees of freedom beyond \(D_s = 6\), which can be done once and for all. The remaining trace from \(D_s = 6\) to \(D_s = 4\) in general has to be evaluated explicitly (as in eq. (3.15)). In order to evaluate the inner trace we split gamma matrices as

\[
(D_s) \Gamma^\mu_{[6]} = (\tilde{D}_s^\mu) \mathbb{1} \otimes (8) \Gamma^\mu_{[6]} ,
\]

\[
(D_s) \Gamma^\mu_{[D_s - 6]} = (\tilde{D}_s^\mu) \psi^\mu \otimes (8) \Gamma^* ,
\]

where \(\tilde{D}_s^\mu = D_s^\mu / 8\) (cf. eqs. (3.4) and (3.5)). The quark-scalar vertex is thus proportional to \((8) \Gamma^*\) since it arises from gluon polarizations beyond six dimensions.

The only source of matrices \(\Gamma^\mu_{[D_s - 6]}\) that are non-trivial in the \((D_s - 6)\) space are interactions of \(D_s\)-dimensional gluons and quarks. Since traces over odd numbers of gamma matrices vanish, there are three possibilities:

1. Quarks only couple to 6d gluons, which means that no \(\psi^\mu\) insertions are present and the \(\widetilde{\text{Tr}}_{D_s \to 6} [ \cdot ]\) is trivial.
2. Two quark-scalar vertices give \( \tilde{\text{Tr}}_{D_s \to 6} \left[ h^\mu h^\nu \right] = g^{'\mu\nu}_{[D_s - 6]} \), which contracted with a scalar line contributes a \((D_s - 6)\) factor (see fig. 4).

3. Four quark-scalar vertices give

\[
T^\mu_\nu\delta_\sigma = \tilde{\text{Tr}}_{D_s \to 6} \left[ h^\mu h^\nu h^\delta h^\sigma \right] = g^\mu_\nu g^\delta_\sigma - g^\mu_\sigma g^\nu_\delta + g^\mu_\delta g^\nu_\sigma + g^\mu_\delta g^\nu_\sigma,
\]

which can be contracted with scalars in two non-equivalent ways:

\[
T^\mu_\nu\delta_\sigma g^\mu_\sigma = (D_s - 6)^2, \quad (4.27a)
\]

\[
T^\mu_\nu\delta_\sigma g^\mu_\delta = 2(D_s - 6) - (D_s - 6)^2. \quad (4.27b)
\]

The above possibilities are illustrated by fig. 3a and fig. 5 respectively.

In cases 1 and 2 an additional power of \((D_s - 6)\) from a closed scalar loop is possible for factorized topologies.

Overall we see that the \(D_s\) dependence matches the general structure of eq. (4.23) and can be expressed as

\[
A^{[D_s]}_{\text{external } S_4} = \tilde{A}^{[6]}_G + (D_s - 6)\tilde{A}^{[6]}_{6S} + (D_s - 6)^2 \tilde{A}^{[6]}_{SS},
\]

where tildes on the RHS remind that the trace \( \tilde{\text{Tr}}_{6 \to 4} \left[ \cdot \right] \) is to be evaluated. Here \( \tilde{A}^{[6]}_G \) is computed by replacing all \(D_s\)-dimensional gluons with 6d gluons. All diagrams with both 6d gluons and scalars, as well as doubled diagrams with alternating scalar lines (as in fig. 5) contribute to \( \tilde{A}^{[6]}_{6S} \). The latter follows from eq. (4.27b). The \((D_s - 6)^2\) coefficient \( \tilde{A}^{[6]}_{SS} \), compared to the pure Yang-Mills case, receives contributions from generic diagrams with scalars (as in fig. 3a). The alternating scalar diagrams contribute with a minus sign which arises from eq. (4.27b).

Once the \(D_s\) dependence is established an FDH or HV amplitude can be obtained by insertion of appropriate values of \(D_s\). The non-trivial difference between the two is given by

\[
A^{HV} - A^{FDH} = 2 \left( 4\tilde{A}^{[6]}_S - \tilde{A}^{[6]}_{6S} \right) \epsilon + 4\tilde{A}^{[6]}_S \epsilon^2,
\]

\[
(4.29)
\]
which underlines the known problems with consistency of the “naive” FDH scheme beyond one loop [1, 24–26].

Figure 6. Examples of diagrams with closed quark loops. The diagram (a) is proportional to the number of flavors and contributes to the \((D_s - 6)\) coefficient. The diagram (b) is proportional to the number of flavors squared.

In a similar way we consider amplitudes with closed quark loops. When there is one closed quark loop at most two quark-scalar vertices are possible (see e.g. fig. 6a), which gives at most a linear dependence on \((D_s - 6)\):

\[
A_{nf, \text{external} \in S^{(4)}} = \tilde{D}_t' \left( \tilde{A}_{nf}^{[6]} \varphi + (D_s - 6) \tilde{A}_{nf}^{[6]} s \right),
\]

(4.30a)

where a factor \(\tilde{D}_t'\) comes from an additional partial trace over the quark loop. In case of two closed quark loops (see e.g. fig. 6b) no \((D_s - 6)\)-dimensional index contractions are possible and again we have a factor \(\tilde{D}_t'\) from each quark loop:

\[
A_{nf^2, \text{external} \in S^{(4)}} = \left( \tilde{D}_t' \right)^2 \tilde{A}_{nf}^{[6]}.
\]

(4.30b)

Summing up, eqs. (4.28) and (4.30) together with eq. (3.15) provide an efficient algorithm for evaluation of dimensionally regularized two-loop amplitudes with massless or massive quarks employing representations of the minimal dimensionality.

5 Connection to the FDF

The recent four-dimensional (re)formulation (FDH) of FDH [10] provides a computational prescription for one-loop amplitudes in four dimensions together with selection rules that have to be evaluated diagrammatically. The FDF scheme is defined with the additional requirement to remove odd powers of \(\mu\) from the integrand. We find that for amplitudes with gluons and up to one quark pair, the application of the FDF prescription is equivalent to our decomposition by particle content for one-loop amplitudes in eq. (4.21a). This can be seen by explicitly comparing selection rule factors and the Feynman rules presented in Appendix D. However, for amplitudes involving multiple (massive) quark pairs the FDF prescription generates integrands which differ from those of FDH. In what follows we indicate the source of these spurious integrands, which are not present in our approach.

As an example, we consider the diagram in Fig. 7, that is the exchange of two “massive” gluons (cf. Appendix C) between two quark lines. Each quark propagator contributes a term \(\tilde{I}_{[D - 4]} = i\mu\gamma_5\) (c.f. eq. (4.14)). If both quark lines are massive, there is a non-vanishing contribution of the form

\[
\langle p_2 \gamma^\alpha (i\mu\gamma_5) \gamma^\beta | p_1 \rangle \langle p_4 \gamma^\rho (i\mu\gamma_5) \gamma^\sigma | p_3 \rangle D_{\alpha\sigma} D_{\beta\rho}^g \sim \mu^2 \cdots.
\]

(5.1)

Terms with odd numbers of \(\mu\) insertions that combine to even powers of \(\mu\) across multiple lines generically appear in FDF with two or more massive quark lines. These terms constitute a non-vanishing difference between integrands in the FDF and FDH schemes. In our approach these terms vanish due to the trace over \(\tilde{D}_t\) on each fermion line.
Figure 7. A diagram for the exchange of two “massive” gluons between two quark lines in the FDF. It contains terms that are proportional to $\mu^2$, which vanish in our approach.

We propose a modification of the FDF prescription to correctly reproduce one-loop FDH integrands with multiple (massive) quark lines. Following the notation of ref. [10], one performs the substitutions

\[ g^{\alpha\beta}_{[D_s-4]} \rightarrow G^{AB}, \quad \gamma^\alpha_{[D_s-4]} \rightarrow \gamma^5 \Gamma^A, \quad \ell^\alpha_{[D_s-4]} \rightarrow i\mu Q^A, \]  

(5.2)

where symbols $\Gamma^A$ do not have a finite-dimensional representation. The FDF is then based on the formulation of contraction rules for the objects $G^{AB}, \Gamma^A$ and $Q^A$. Amongst them, the rule

\[ Q^A \Gamma^A = 1 \]  

(5.3)

is derived by applying the substitutions (5.2) to

\[ \ell^\dagger_{[D_s-4]} \ell_{[D_s-4]} = -\mu^2 Q^A \Gamma^A Q^B \Gamma^B = -\mu^2. \]  

(5.4)

However, the symbols $\Gamma^A$ have not been equipped with a proper algebraic structure and in general $Q^A \Gamma^A Q^B \Gamma^B = 1$ does not imply $Q^A \Gamma^A = 1$. The corrected selection rule drawn from eq. (5.4) is thus

\[ Q^A \Gamma^A Q^B \Gamma^B = 1. \]  

(5.5)

A further addition to the basic rules of the FDF approach is to give a proper prescription for contracting indices across different quark lines equivalent to eq. (4.15). Then the FDF becomes a valid (re)formulation of FDH applicable to amplitudes with multiple quark lines.

6 Conclusions

The main result of the paper is a decomposition by particle content of one- and two-loop QCD helicity amplitudes regularized in HV or FDH schemes given by eqs. (4.20), (4.22), and eqs. (4.28), (4.30) respectively. It is valid for any number of massless or massive fermions. It is formulated using only finite integer-dimensional representations of tensor and spinor algebras and extends the previous results obtained for pure Yang-Mills theories [19–21]. Compared to a direct application of dimensional reconstruction [4, 13], our method requires the computation in a single $D_s$ dimension, which is determined by the number of non-trivial directions of loop momenta. This is in particular beneficial for numerical unitarity applications [2–5, 12, 27, 28] and can be exploited in the future for computation of two-loop amplitudes with external quarks.

To achieve this we formulated a precise notion of external four-dimensional spinor states embedded into higher dimensional spaces in eqs. (4.9) and (4.11). An important consequence is that no spurious integrands such as square roots of loop momenta contractions are encountered. This embedding of states of external particles is required for any numerical approach for computation of multi-loop amplitudes with external quarks.
We found that the FDF prescription [10], applied to one-loop amplitudes with gluons and those with a single quark pair, is equivalent to a special case of the formula (4.21a). In case of multiple (massive) quark pairs however, the FDF scheme contains spurious terms which are not present in our approach. We provided an improvement of FDF selection rules to make it fully compatible with our approach, therefore with FDH, also in the case of amplitudes involving multiple quark lines.

Finally, we provided some details of the algorithms implemented in a new version of the Black-Hat library, which has been used for the computation of phenomenologically relevant NLO QCD corrections [22].

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A Evaluation of Normalized Partial Traces

Here we list some properties of normalized partial traces defined in the section 3.3 which we use throughout the paper.

From eqs. (3.4) and (3.14) one deduces that the matrices $\Gamma^\mu_{[4]}$ only contribute a unit matrix to the trace over $\tilde{D}_t$, 

$$\tilde{\text{Tr}} \left[ \Gamma^\mu_{[4]} \cdots \right] = \gamma^\mu \tilde{\text{Tr}} \left[ \cdots \right]. \quad (A.1)$$

The only non-trivial contribution thus comes from partial traces over products of $\Gamma^\mu_{[D_s-4]}$. This can be demonstrated by using eqs. (3.4) and (3.5), the fact that \{ $\Gamma^\mu_{[4]}; \Gamma^\nu_{[D_s-4]}$ \} = 0, and repeatedly applying eq. (A.1).

It follows from eq. (3.14) that evaluation of partial traces of products of $\Gamma^\mu_{[D_s-4]}$ is equivalent to evaluation of normal traces of products of matrices $g^\mu$, which themselves are elements of a Clifford algebra. Thus all the usual trace properties apply. In particular we have

$$\text{Tr} \left[ g^{\mu_1} \cdots g^{\mu_n} \right] = 0, \quad n \text{ odd}, \quad (A.2a)$$

$$\text{Tr} \left[ g^\mu g^\nu \right] = g^{\mu\nu}_{[D_s-4]}, \quad (A.2b)$$

$$\text{Tr} \left[ g^\mu q^\nu q^\delta \sigma \right] = g^{\mu\nu}_{[D_s-4]} g^{\delta\sigma}_{[D_s-4]} - g^{\mu\delta}_{[D_s-4]} g^{\nu\sigma}_{[D_s-4]} + g^{\mu\sigma}_{[D_s-4]} g^{\nu\delta}_{[D_s-4]} \quad (A.2c)$$

We note that our rules for computation of normalized partial traces are consistent with the implementation found in [18], where similar objects appear and no connection to finite-dimensional representations is considered.

B $\gamma_5$ in $D_s$ Dimensions

The t’Hooft-Veltman/Breitenlohner-Maison [6, 29] prescription for $\gamma_5$ can be realized in integer $D_s$ by

$$(D_s) \gamma_5 \equiv (\bar{D}_t) \mathbb{1} \otimes \gamma_5. \quad (B.1)$$
such that the following relations hold
\[
\left\{ (D_\lambda)^\gamma_5, (D_\lambda)^\mu [4] \right\} = 0, \quad (D_\lambda)^\gamma_5, (D_\lambda)^\mu [D_\gamma - 4] = 0. \quad (B.2)
\]
As a consequence we have
\[
\text{Tr} [\Gamma_5 \Gamma^\alpha \Gamma^\nu \Gamma^\sigma] \neq 0. \quad (B.3)
\]
A proper treatment of the spurious anomalies associated to the fact that $\Gamma_5$ does not anticommute with other Dirac matrices has to be ensured.

C Gluon Propagator in Axial Gauge

The color-stripped gluon propagator in $D_s$-dimensional ghost-free axial gauge is given by [30]
\[
D^{\alpha \beta}_{[D_s]} = \frac{i}{\ell^2_{[D_s]}} \left[ -g^{\alpha \beta}_{[D_s]} + \frac{\ell^2_{[D]} \eta^\alpha_{[D]} + \ell^2_{[D]} \eta^\beta_{[D]}}{\ell^2_{[D]} \cdot \eta_{[D]}} - \frac{(\eta^2_{[D]} + \lambda \ell^2_{[D]} i \ell_{[D]}^\alpha / \ell_{[D]}^\beta)}{(\ell^2_{[D]} \cdot \eta_{[D]})^2} \right], \quad (C.1)
\]
with the parameter $\lambda$ and the $D$-dimensional reference vector $\eta^\alpha_{[D]}$. We choose the reference vector $\eta^\alpha_{[D]} = i \mu n^\alpha_5$ with $\eta^2_{[D]} = -\mu^2$, proportional to the unit vector in the fifth dimension $n^\alpha_5$. We set the parameter $\lambda \rightarrow 0$ and split the metric and loop momentum into four- and higher-dimensional parts. The above expression then simplifies to
\[
D^{\alpha \beta}_{[D_s]} = \frac{i}{\ell^2_{[4]}} - \frac{\ell^2_{[D]} \eta^\alpha_{[4]} \eta^\beta_{[4]}}{\mu^2} - g^{\alpha \beta}_{[D_s - 5]} \equiv D^{\alpha \beta}_{G_{[4]}} + D^{\alpha \beta}_{S_{[D_s - 5]}}, \quad (C.2)
\]
where we used the fact that the $(D_s - 4)$-dimensional part of the metric can be written as
\[
g^{\alpha \beta}_{[D_s - 4]} = \sum_{k=4}^{D_s - 1} n^\alpha_k n^\beta_k. \quad (C.3)
\]
The $D_s$-dimensional gluon propagator in axial gauge thus decomposes into the propagator of a “massive” vector particle in four dimensions and scalar propagators of higher-dimensional components. For on-shell loop momenta with $\ell^2_{[D]} = 0$, the reference vector $\eta^\alpha_{[D]} = \ell^\alpha_{[4]} - \ell^\alpha_{[D_s - 5]}$ leads to the same split-up into a “massive” four-dimensional gluon and scalars.

The only non-vanishing contractions of $n_{k \geq 5}$ are with itself via metric tensors, since external polarizations and momenta as well as the “massive” gluon propagator are four-dimensional and the loop momentum is five-dimensional. As a consequence, scalar lines can start and end only on quark lines, or otherwise they form a closed loop. The three-gluon vertex in $D_s$ dimensions, with two gluons polarized along $n_k$, corresponds to a scalar-scalar-gluon-gluon vertex [4]
\[
V^{ggg}_{\alpha \beta \gamma}(p_1, p_2, p_3) \epsilon^\alpha_i n^\beta_k n^\gamma_l \sim \epsilon^\alpha_i (p_2 - p_3)_\alpha, \quad (C.4)
\]
where $p_i$ are momenta and $\epsilon^\alpha_i$ is the trivially embedded four-dimensional polarization. And a similar argument applies to the scalar-scalar-gluon-gluon vertex. Thus “massive” four-dimensional gluons and each scalar degree of freedom decouple.

D Implementation Details

In this appendix, we provide some details required for a numerical implementation of the prescription we found in section 4.2 for one-loop amplitudes, see eq. (4.21a). It allows to compute one-loop
QCD amplitudes involving multiple (massive) quark lines regularized in the FDH scheme in a compact way. An FDH amplitude is given by the difference of an amplitude with all $D_s$ dimensional gluons replaced by a four-dimensional “massive” gluon in the loop and one with a scalar particle in the loop, see eq. (4.21a). Furthermore, these amplitudes are defined in terms of four-dimensional objects with the additional requirement of only even numbers of $(D_s-4)$-dimensional loop momenta insertions from quark propagators on each fermion line enforced.

We propose to keep track of the number of $(D_s-4)$ insertions from quark propagators to enforce this requirement. One way to achieve this numerically is to use a double copy of fermionic states and vertices and to adapt the $\mu$-dependent piece of the propagator to switch between upper and lower components, see Feynman rules in Table 1. For external and internal fermionic particles, we construct states as described in the section 3.2 with $D_s = 6$. For external particles, only the states with $\tilde{D}_s$-index $j = 0$ are considered, which projects out terms with even numbers of insertions of $(D_s-4)$-dimensional loop momenta. One can equally think of this as keeping track of even powers of $\mu$. Note, that this is not the same as computing an amplitude in $D_s = 6$ dimensions.

An additional simplification can be exploited in case when quarks are massless. $\gamma_5$ matrices acting on Weyl spinors preserve representations as opposed to normal slashed matrices. Hence, spinor products with an odd number of $\gamma_5$ matrices, inserted in place of slashed matrices, produce a mismatch between representations and have to vanish. The requirement to have only even numbers of $(D_s-4)$-dimensional loop momentum insertions is thus automatically fulfilled. In this situation, it suffices to compute with a four-dimensional representation of gamma matrices and the quark propagator

$$
^{(4)}S_Q(\ell_{[D]}^{(4)}) = \frac{i^{(4)}\ell_{[4]}^{(4)} + i\mu\gamma_5 + m}{\ell_{[4]}^{2} - \mu^2 - m^2},
$$

with $m = 0$.

A similar simplification applies in case of closed fermion loops. Odd numbers of $(D_s-4)$-dimensional loop momenta insertions lead to vanishing traces over odd numbers of $\gamma_5$. Consequently, the requirement of having even numbers of insertions is always fulfilled and it suffices to compute in four-dimensions with the quark propagator of eq. (D.1), even with $m \neq 0$.

In the two cases discussed above the simplified quark propagator given by eq. (D.1) is identical to the one of the FDF approach [10].
Table 1. Color-ordered QCD Feynman rules for the direct computation of FDH one-loop amplitudes. Shown are the rules for “massive” gluons and scalars in the loop. The requirement to have even numbers of insertions of higher-dimensional loop momentum along each fermion line is enforced by construction.

**Propagators**

\[
\ell_{\mu} \ell^{\nu} = \frac{i}{\ell^4 - \mu^2} \left[ -g_{\mu\nu} + \frac{P_{\mu}[4] P_{\nu}[4]}{\mu^2} \right]
\]

\[
\ell_{\mu} = \frac{i}{\ell^4 - \mu^2 - m^2} \left[ (2) \mathbb{1} \otimes \left( (4) f_{[4]} + m (4) \mathbb{1} \right) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes i\gamma_5 \right]
\]

\[
\ell_{\mu} = -\frac{i}{\ell^4 - \mu^2}
\]

**Vertices**

\[
\mu, \nu = \frac{i}{\sqrt{2}} \left[ g_{\mu\nu} (p_1[4] - p_2[4])^\rho + g_{\nu\rho} (p_2[4] - p_3[4])^\mu + g_{\rho\sigma} (p_3[4] - p_1[4])^\nu \right]
\]

\[
\mu, \nu, \rho, \sigma = i g_{\mu\nu} g_{\rho\sigma} - \frac{i}{2} \left[ g_{\mu\nu} g_{\rho\sigma} + g_{\mu\rho} g_{\nu\sigma} \right]
\]

\[
\mu, 1, 2, \rho, 3 = \frac{i}{\sqrt{2}} \left( (2) \mathbb{1} \otimes \mathbb{1} \right) \gamma^\mu
\]

\[
\mu, 1, 2, \rho, 3 = -\frac{i}{\sqrt{2}} \left( (2) \mathbb{1} \otimes \mathbb{1} \right) \gamma^5
\]

\[
\mu, 1, 2, \rho, 3 = -\frac{i}{\sqrt{2}} \left( (2) \mathbb{1} \otimes \mathbb{1} \right) \gamma^5
\]
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