The solution to the strong CP problem at the BCS level of chiral rotations.

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We briefly review the cases of forced and spontaneous chiral symmetry breaking. In particular the chiral condensate of $q\bar{q}$ pairs is parametrized with two angles, $\phi$ which measures the chiral condensation, and $\theta$ which measures the chiral rotation. The strong CP problem arises when it is assumed that the current quark masses of the Standard Model, have a $\theta$ phase that differs from the phase induced by the instanton term which is originated in the QCD sector of the Standard Model. We show how chiral symmetry breaking may solve the strong CP problem at the BCS level. We show variationally that the physical vacuum is CP conserving and therefore the interplay of the instanton and the current mass terms cannot produce any strong CP violation. We also study the possible effect of $\theta$ in weak interactions, and conclude that it does not produce weak CP.
Within the strong interacting sector of the Standard Model, we have 3 different mechanisms for chiral symmetry breaking.

First, we have the the lagrangian mass term

\[ \mathcal{L}_m = m_u \bar{\Psi}_u \Psi_u + m_d \bar{\Psi}_d \Psi_d + m_s \bar{\Psi}_s \Psi_s + \ldots \]  

where the \( m_u, m_d, m_s \) current quark masses, explicitly break chiral symmetry.

Another well known mechanism for chiral symmetry breaking is furnished by the ’t Hooft coupling of the quark flavour determinant to the instanton,

\[ \mathcal{L}_g = \frac{C}{g^3} \exp \left( -\frac{8\pi^2}{g^2} \right) \text{det}_{sl} \left[ \bar{\Psi}_s (1 + \gamma_5) \Psi_t \right] + H.c, \]  

where \( s, t \) are flavour indices.

Finally the third cause of chiral symmetry breaking is due to the quark condensate \( \langle \bar{\Psi} \Psi \rangle \) which is generated dynamically .

In what follows we will make use of Valatin-Bogolubov transformations to construct explicit examples of general orthogonal Fock spaces consistent with quark condensation and chiral rotations. It is this residual freedom to self-consistently choose the Fock space that constitutes the cornerstone of the mechanism allowing for the removal of CP violating phases.

The condensed vacuum \( |\tilde{0}\rangle \) can be generated from the trivial vacuum \( |0\rangle \) with a Bogolubov-Valatin transformation, generated by the parity invariant creator \( ^3P_0 \) of quark antiquark pairs ,

\[
C_{ck}^\dagger = \sum_{s_1 s_2} b_{s_1 ck}^\dagger M_{s_1 s_2 - \hat{k}}^\dagger d_{s_2 ck}^\dagger
\]

\[
M_{s_1 s_2 \hat{k}}^\dagger = -\sqrt{6} \sum_{m \bar{s}_1} \left( \begin{array}{c} 110 \\ m \bar{l} \end{array} \right) \hat{k}_{1m} \left( \begin{array}{c} \frac{1}{2} \frac{1}{2} 1 \\ s_1 \bar{s}_2 \bar{l} \end{array} \right)
\]

\[
M^\dagger = \hat{\sigma} \hat{k}^i \sigma_2
\]

Then the \( |\tilde{0}\rangle \) is given by,

\[
|\tilde{0}\rangle = \sum_k \phi_k \left( C_{ck}^\dagger - C_{ck} \right) |0\rangle = \prod_{ck} \left( \cos^2 \phi_k + \sin \phi_k \cos \phi_k C_{ck}^\dagger + \frac{\sin^2 \phi_k}{2} C_{ck}^\dagger C_{ck}^\dagger + \frac{\sin^2 \phi_k}{2} C_{ck}^\dagger \right) |0\rangle,
\]

Then the \( |\tilde{0}\rangle \) is given by,

\[
|\tilde{0}\rangle = \sum_{\phi_k} \left( C_{ck}^\dagger - C_{ck} \right) |0\rangle = \prod_{ck} \left( \cos^2 \phi_k + \sin \phi_k \cos \phi_k C_{ck}^\dagger + \frac{\sin^2 \phi_k}{2} C_{ck}^\dagger C_{ck}^\dagger + \frac{\sin^2 \phi_k}{2} C_{ck}^\dagger \right) |0\rangle,
\]
Now using the Dirac fermion fields,

\[ \Psi = \frac{1}{\sqrt{\text{V}}} \sum_{sk} (u_{sk} b_{sk} + v_{sk} d_{s-k}^\dagger) e^{ik.x} \]  

with the chiral spinors normalized with \( u^\dagger u = 1 \), \( v^\dagger v = 1 \) in order to be able to use massless spinors, we get the vacuum expectation value for the quark condensate,

\[ \langle \bar{\Psi} \Psi \rangle = \langle \tilde{0} | \bar{\Psi} \Psi | \tilde{0} \rangle = -2 \int \frac{d^3 k}{(2\pi)^3} \sin \left[ -2\phi(k) + \text{atan} \left( m/k \right) \right] \]  

where \([ -2\phi(k) ] \) is positive in such a way that this expectation value is always negative and never vanishes. For current quark masses smaller than the scale of the strong confining interaction, this VEV practically has a constant value.

This vacuum is an example of a wider class of vacua which will be denoted in this paper by \(| \tilde{\theta} \rangle \). From the point of view of chiral physics, the extent of chiral symmetry breaking can be associated to the norm of a vector in a 2 dimensional space. A cartesian basis of this space can be defined by the "x-direction" of the scalar condensate \( \bar{\Psi} \Psi \) and by the orthogonal "y-direction" of the pseudoscalar condensate \( i \bar{\Psi} \gamma_5 \Psi \).

Therefore it can be said with all generality that the fermion part of the strong interaction vacuum, must lie somewhere in a circle in this x-y plane and can be indexed by a polar angle which we denote by \( \theta \). This polar angle parametrizes the chiral rotations generated by the \( Q_5 \) operator \( \bar{\Psi} \gamma_0 \gamma_5 \Psi \).

To arrive at the vacuum \(| \tilde{\theta} \rangle \) from the trivial unstable vacuum \(|0\rangle \) we have to define an extended condensate generator in order to include any combination of scalar \( ^3P_0 \) and pseudoscalar \( i \, ^1S_0 \) quark-antiquark pair creations, which we do by replacing in the equations \[ M \] by \( M_\theta \),

\[ M_\theta^\dagger = \left[ \cos(\theta) \vec{\sigma} \cdot \hat{k} + i \sin(\theta) \right] i\sigma_2 \]  

which in turn defines \( C_\theta^\dagger \) as,

\[ C_\theta^\dagger = \cos(\theta) C_s^\dagger + \sin(\theta) C_{ip}^\dagger. \]
In Eq. \[8\] \(C_s\) stands for the \(^3P_0\) creator \(C^\dagger_{ck}\). \(C_{ip}\) represents the pseudoscalar quark-antiquark creator, where the pseudoscalar spin wave function is \(\uparrow\downarrow - \downarrow\uparrow = i\,\sigma_2\). It is also possible to rotate directly from the \(|\tilde{0}\rangle\) to the \(|\tilde{\theta}\rangle\) with the generator \(Q_5\). We can show that we get the algebra,

\[
[Q_{ip}, Q_5] = -2iQ_s, \quad [Q_s, Q_5] = 2iQ_{ip}
\]  

(9)

\[
Q_5 = \int d^3x \bar{\Psi}\gamma_5 \Psi = \sum b^\dagger \hat{\sigma}.\hat{k}b - d^\dagger \hat{\sigma}.\hat{k}d
\]

(10)

\(Q_5\) generates the chiral rotation of the vacua,

\[
|\tilde{\theta}\rangle = e^{i\frac{\theta}{2}Q_5}|\tilde{0}\rangle
\]

(11)

and the effect of this generator is depicted in Fig.\[1\].

At this stage we just have to bear in mind that all the vacua \(|0\rangle, |\tilde{0}\rangle\) and \(|\tilde{\theta}\rangle\) are orthogonal and they belong to independent Fock spaces. For instance,

\[
\langle\tilde{0}|\tilde{\theta}\rangle = \prod_c k \left[ 1 - \sin^2 \left(\frac{\theta}{2}\right) \sin^2 (2\phi) \right] = \delta_{0,\theta}
\]

(12)

Thus all these vacua are independent and cannot be reached pertubatively. Each vacuum has its own field creators \(b^\dagger_{\theta \text{ck}}, d^\dagger_{\theta \text{ck}}\) and field annihilators \(b_{\theta \text{ck}}, d_{\theta \text{ck}}\) that can be used to generate the corresponding Fock spaces \(\mathcal{F}_\theta\).

When there is only one source of chiral symmetry breaking, like \(\mathcal{L}_m\), then we can trivially choose this vacuum to define the x-axis (\(\theta = 0\)) : \(|\tilde{0}\rangle\)

When a second source for chiral symmetry breaking exists, like for instance \(\mathcal{L}_g\), then we have an extra direction and a \(P\) violating phase could be thouht to appear. To quote 't Hooft, "if other mass terms or interaction terms occur in the Lagrangian that also violate chiral \(U(1)\), then they may have a phase factor different from these. We then find that our effective Lagrangian may violate \(P\), whereas \(C\) invariance is maintained."

This is the origin of the strong CP problem, in the sense that strong CP violations have not yet been observed experimentally. In other words, because weak and strong interactions
are independent, and in a sense one is skewed in relation to the other [4], the angles $\theta_{Lm}$ (which we had set to be the origin) and $\theta_{Lg}$ are naturally expected to be different. An arbitrary example of this effect is shown in Fig. [1].

The purpose of this paper is to show that a difference in the angles $\theta_m$ and $\theta_g$ may not cause strong CP violation.

If there were no explicit chiral symmetry breaking Lagrangians $L_m$ and $L_g$ then a continuous degenerate set of orthogonal vacua would exist, and the whole hadronic physics could suffer a chiral rotation from the vacuum $|\tilde{0}\rangle$ to another vacuum $|\tilde{\theta}\rangle$. For each $\theta$ we would find an identical replica of the whole hadronic world. Nothing in strong interaction physics will distinguish one vacuum from another.

This is easy to see in a simplified version of $L_g$ provided by the Lagrangian of Baluni [5] with only 1 flavor. Let us suppose that we have the following Lagrangian,

$$L_m + L_g + L_i = m \bar{\Psi} \Psi + \bar{\Psi}(A + iB\gamma_5)\Psi + L_i = m^* \bar{\Psi}e^{i\theta\gamma_5}\Psi + L_i$$

$$m^* = \sqrt{(m + A)^2 + B^2}, \quad \theta = arctan\frac{B}{m + A}$$

(13)

where $L_i$ contains the remainder of the Q.C.D. Lagrangian which is chiral invariant and should not concern us for the purpose of this paper. We can use the Valatin-Bogolubov transformation,

$$\Psi_\theta = e^{iQ\frac{\theta}{2}} \Psi e^{-iQ\frac{\theta}{2}} = e^{i\gamma_5\frac{\theta}{2}} \Psi, \quad \Psi_\theta = \frac{1}{\sqrt{V}} \sum u \ b_\theta + v \ d_\theta^\dagger$$

(14)

to get read of the complex CP violating masses. Thus if instead of the fields $\Psi$, we use the chiral rotated fields $exp(i\theta\gamma_5)\Psi$ then CP violation explicitly disappears from the Lagrangian for the vacuum $|\tilde{\theta}\rangle = e^{iQ\frac{\theta}{2}} |\tilde{0}\rangle$. It is not difficult to show that this choice minimizes the vacuum energy,

$$\mathcal{E} = \langle \bar{\Psi}\Psi \rangle [(m + A) \cos(\theta) + B \sin(\theta)] + \mathcal{E}_i.$$  

(15)
$|\tilde{\theta}\rangle$ is therefore the physical vacuum of the system, and for this physical vacuum there remains no evidence of strong CP.

For the purpose of this letter it suffices to calculate the vacuum expectation value of the hamiltonian density, which can be performed with the help of the Wick theorem.

Let us define a class of 't Hooft determinants for n flavours as,

$$D_n = \frac{1}{2} \left[ \det s(1 + \gamma_5)t + \det s(1 - \gamma_5)t \right]$$

where we drop the $\Psi$ and from now on $s$, $t$ represent the Dirac fields of flavour $s$, $t$. In Eq. (16) only the even terms in $\gamma_5$ which are CP conserving survive. In fact this is simply equivalent to insert an even number of $\gamma_5$ matrices in the flavour determinant $det_s\bar{s}t$ in all possible different ways. For instance we get the mass term for 1 flavour, whereas for 2 flavours we already get 4 terms, while for 3 flavors we would have 24 terms and so on,

$$D_1 = \bar{u}u$$

$$D_2 = \bar{u}u \, \bar{d}d - \bar{u}d \bar{d}u + \bar{u}\gamma_5u \, \bar{d}\gamma_5d - \bar{u}\gamma_5d \, \bar{d}\gamma_5u$$

$$D_3 = \bar{u}u \, \bar{d}d \, \bar{s}s + \ldots$$

(17)

Let us see how $D_n$ transforms with a general chiral rotation of angles $\theta_u$, $\theta_d$, $\ldots$. We find,

$$D_n \rightarrow D_{n}^{\theta_u, \theta_d, \ldots}$$

$$= \frac{1}{2} e^{i(\theta_u + \theta_d + \ldots)} \det s(1 + \gamma_5)t$$

$$+ \frac{1}{2} e^{-i(\theta_u + \theta_d + \ldots)} \det s(1 - \gamma_5)t$$

$$= \cos(\theta_u + \theta_d + \ldots)D_n + i \sin(\theta_u + \theta_d + \ldots)D_5^n$$

(18)

Where $D_5^n$ has the same definition as $D_n$ except that it has an odd number of insertions of $\gamma_5$. While $D_n$ is CP invariant, $D_n^5$ is a pseudoscalar. Because all the angles come in a sum, we see that from the point of view of chiral rotations $D_n$ has a single dimension, and thus is equivalent to a mass term in the flavour U(1) direction.

In order to study the vacuum energy we have to calculate the vacuum expectation values of the operators $D_n$. This can be performed with the Wick contraction technique. As an
illustration we give $\langle D_n \rangle$ for the simpler case where all the flavours have the same condensate. This is not a bad approximation for the u, d quarks and of course this can also be calculated in the general case of unequal quark condensates.

$$\langle D_n \rangle = \langle \bar{0}|D_n|\bar{0} \rangle = G \langle \bar{\Psi} \Psi \rangle^n$$

(19)

where $G$ is a geometrical factor,

$$G_1 = 1, \quad G_2 = \frac{3}{2}, \quad G_3 = 3, \ldots$$

(20)

In the general case we get for the vacuum energy density

$$\langle \bar{\theta}_u \bar{\theta}_d \ldots |D_n|\bar{\theta}_u \bar{\theta}_d \ldots \rangle = \cos(\theta_u + \theta_d + \ldots)\langle D_n \rangle$$

$$\mathcal{E} = \mathcal{E}_i + m_u \langle \bar{u}u \rangle \cos(\theta_u) + m_d \langle \bar{d}d \rangle \cos(\theta_d)$$

$$+ \ldots + K \langle D_n \rangle \cos(\theta_u + \theta_d + \ldots - \theta_g)$$

(21)

where $\langle D_n \rangle$ is a given number and $K$ is given by

$$K = 2 \frac{C}{g^8} \exp\left(\frac{-8\pi^2}{g^2}\right)$$

(22)

In the litterature the $\theta_g$ is regarded as fixed. However for completness we will also consider the scenario where $\theta_g$ is a variational parameter. This is an easy scenario insofar that the evident solution is that $\theta_g$ will be "aligned" with the current mass chiral angles, $\theta_f = \theta_g = 0$. This is the combination that yields the lowest vacuum energy. In this case the instanton Lagrangian does not produce any CP violation either in strong or in weak interactions.

Now we will suppose that $\theta_g = \theta$ is fixed and show that even in this case strong CP violation does not occur. The minimum condition is simply obtained when we differentiate Eq.[21] with respect to the $\theta_f$, and we get the set of coupled equations,

$$m_u \langle \bar{u}u \rangle \sin(\theta_u) = -\sin(\theta_u + \theta_d + \ldots - \theta)K\langle D_n \rangle$$

$$m_d \langle \bar{d}d \rangle \sin(\theta_d) = -\sin(\theta_u + \theta_d + \ldots - \theta)K\langle D_n \rangle$$

$$\ldots$$

(23)
that can be easily solved numerically. For small chiral angles and small current masses the system becomes linear and we get the simple solutions,

$$\theta_f = \theta \frac{m_f^{-1}}{\sum_s m_s^{-1}}.$$  

(24)

Thus only the lightest flavours are actually rotated because this corresponds to an average weighted by the inverse masses. If we inspect the light $\pi, K, \eta, \eta'$ pseudoscalar meson masses with the help of the Gell-Mann, Oakes and Renner relations, together with the heavy meson masses, we get the approximate mass ratios,

$$m_d \simeq 2m_u, \ m_s \simeq 30m_d < K\langle D_n \rangle / \langle \bar{\Psi}\Psi \rangle$$

$$\ll \ m_c \simeq 6m_s, \ m_b \simeq 3m_c, \ m_t \simeq 15m_b$$

(25)

which in turn yield the chiral rotations,

$$\theta_u \simeq \frac{2}{3}\theta, \ \theta_d \simeq \frac{1}{3}\theta, \ \theta_s \simeq \frac{1}{90}\theta, \ \theta_c \simeq \theta_b \simeq \theta_t \simeq 0$$

(26)

We now verify that within the minimum energy vacuum $|\tilde{\theta}\rangle$ the Lagrangian becomes CP invariant. We have,

$$D_n = \langle D_n \rangle + \bar{u}u\langle D_n^u \rangle + \bar{d}d\langle D_n^d \rangle + \ldots$$

$$+\text{higher order terms}$$

(27)

Where $\langle D_n^f \rangle = \frac{\langle D_n \rangle}{\langle ff \rangle}$ is simply equal to $\langle D_n \rangle$ except for a contraction of $\bar{f}f$ or $f\bar{f}$. In the same way we get,

$$D_n^5 = 0 + \bar{u}\gamma_5u\langle D_n^u \rangle + \bar{d}\gamma_5d\langle D_n^d \rangle + \ldots$$

$$+\text{higher order terms}$$

(28)

thus we verify, with the help of equation [23] that in the normal ordered Hamiltonian all the CP violating terms cancel up to quadratic terms in field operators.
\[ \mathcal{H} = \mathcal{E} + \mathcal{H}_{i2} + \\
+ [m_u \cos(\theta_u) + \cos(\theta_u + \theta_d + \ldots - \theta)K\langle D_n^u \rangle] \bar{u} u \\
+ [m_d \cos(\theta_d) + \cos(\theta_u + \theta_d + \ldots - \theta)K\langle D_n^d \rangle] \bar{d} d \\
+ \ldots \\
+i [m_u \sin(\theta_u) + \sin(\theta_u + \theta_d + \ldots - \theta)K\langle D_n^u \rangle] \bar{u} \gamma_5 u \\
+i [m_d \sin(\theta_d) + \sin(\theta_u + \theta_d + \ldots - \theta)K\langle D_n^d \rangle] \bar{d} \gamma_5 d \\
+ \ldots \\
+ \text{higher order terms} \\
= \mathcal{E} + \mathcal{H}_{i2} + m_u^* \bar{u} u + m_d^* \bar{d} d + \ldots \\
+ i [0] \bar{u} \gamma_5 u + i [0] \bar{d} \gamma_5 d + \ldots + \text{higher order terms} \quad (29) \]

A word of caution is in order. It is well known that the BCS formalism does not exactly minimize the vacuum energy. To solve this problem one has to include the effects of coupled channels in the vacuum and this will be not pursued here. However the BCS approach is already enough to show the cancellation of the CP violating quadratic terms considered in the litterature [3].

While strong CP violation disappears, a chiral rotation of the quark fields might affect weak interactions. The Cabibo-Kobayashi-Maskawa [3] matrix, responsible for the weak decay of quarks is not invariant for chiral transformations that are flavour dependent. In the usual Standard Model formalism, it appears in the form,

\[ \mathcal{L}_{C.K.M.} = \frac{g}{2} \left( W_\mu^+ \sum_{s,f} u_{Ls} \gamma^\mu V_{sf} d_{Lf} + H.c. \right) \quad (30) \]

where \( u_s \) corresponds to the Dirac field for the flavours \( u, c, t \) and \( d_f \) corresponds to the the Dirac fields for the flavours \( d, s, b \). In the CKM matrix a flavour dependent chiral rotation would produce the phase \( (\theta_s - \theta_f)/2 \) for the terms with \( \bar{u}_{Ls} \gamma^\mu d_{Lf} \) and the opposite phase for the hermitean conjugate terms \( \bar{d}_{rf} \gamma^\mu u_{rs} \). This \( \theta \) phase is independent of the single CP violating phase of the standard model. For instance in the case where only the light flavours suffer a chiral rotation to their strong CP invariant values \( \theta_f \), \( V \) will be transformed into
\[ V \rightarrow V' = \begin{pmatrix} V_{ud}e^{i(\theta_u - \theta_d)/2} & V_{us}e^{i(\theta_u - \theta_s)/2} & V_{ub}e^{i\theta_u/2} \\ V_{cd}e^{-i\theta_d/2} & V_{cs}e^{-i\theta_s/2} & V_{cb} \\ V_{td}e^{-i\theta_d/2} & V_{ts}e^{-i\theta_s/2} & V_{tb} \end{pmatrix}. \] (31)

It is easy to verify that if \( V \) was originally unitary then \( V' \) is also unitary. The new phases \( \theta_f/2 \) cancel, and we get,

\[ V'^\dagger V' = V'^\dagger V' = 1 \] (32)

Moreover the new phases can be rotated out of the CKM matrix if we rotate each flavour \( f \) with a simple phase \( e^{-i\theta_f/2} \), which is not a chiral phase and will vanish in any other terms in the Lagrangian where Dirac fields always come in pairs \( \bar{f} \ldots f \). To verify that this mechanism is not a candidate to explain weak CP, then we see that the products

\[ Im[V'_{ud} V'^*_{us} V'^*_{cd} V_{cs}] = Im[V_{ud} V^*_{us} V^*_{cd} V_{cs}] \] (33)

that measure weak CP violation are invariant with regard to the phases \( \theta_f \). We thus conclude that the mechanism that preserves strong CP invariance does not affect the CP of weak interactions.

An interesting conclusion of this work is that the \( \pi_0 \) valley can be seen in the direction \( \theta_d = \theta - \theta_u \) between the \( \eta \) twin peaks as we show in Fig.[2]. The \( \eta \) direction is defined with \( \theta_d = \theta_u \), and the curvature is essentially forced by the instanton term. We thus can extend the Gell-Mann Oakes Renner relation to the \( \eta \) meson providing we replace \( m_u \langle \bar{u}u \rangle \) by \( K \langle D_n \rangle \) in order to get, in the limit of small \( \theta \),

\[ K \langle D_n \rangle \simeq m^2_\eta f^2_\pi \] (34)

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FIGURES

FIG. 1. We show the chiral of $\mathcal{L}_m, \mathcal{L}_g$. For historical reasons we choose a vanishing $\theta_m$. The physical $\tilde{\theta}$ is also visible.

FIG. 2. Here we show how the vacuum energy (in arbitrary units) changes when 2 different flavours are rotated. In this example we have $\theta = \pi/3$. The $\pi$ valley is in the middle of the $\eta$ hills, with a visible saddle point.
