Exact solutions for the interacting tachyonic–dark matter system

Ramón Herrera*,1 Diego Pavón†,2 and Winfried Zimdahl‡3

1Instituto de Física, Pontificia Universidad Católica de Valparaíso,
Avenida Brasil 2950, Casilla 4059, Valparaíso, Chile
2Departamento de Física, Facultad de Ciencias,
Universidad Autónoma de Barcelona,
08193 Bellaterra (Barcelona), Spain
3Institut für Theoretische Physik, Universität zu Köln, 50937 Köln, Germany

Abstract

We find exact solutions leading to power law accelerated expansion for a homogeneous, isotropic and spatially flat universe, dominated by an interacting mixture of cold dark matter and a tachyonic field such that the ratio of the energy densities of both components at late times is constant and no coincidence problem arises.

* E-mail address: ramon.herrera.a@mail.ucv.cl
† E-mail address: diego.pavon@uab.es
‡ E-mail address: zimdahl@thp.Uni-Koeln.DE
I. INTRODUCTION

Recently the tachyon field introduced by Sen in a series of papers [1], has attracted some attention in cosmology [2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. As shown by Bagla et al. [12], the Lagrangian of the tachyon field \( L = -V(\varphi)\sqrt{1 - \partial^a \varphi \partial_a \varphi} \) arises as a straightforward generalization of the Lagrangian of a relativistic particle, \( L = -m\sqrt{1 - \dot{q}^2} \), much in the same way as the Lagrangian of the scalar field generalizes the Lagrangian of a non-relativistic particle. Likewise, its stress energy tensor has the structure of a perfect fluid and it can be seen as the sum of dust matter and a cosmological constant whereby it may play the role of both dark matter and dark energy [6, 12]. This latter feature together with the fact that the pressure associated to the tachyon field is negative - a key ingredient in Einstein’s relativity to produce accelerated expansion - may explain the interest in using it to account for the present state of the Universe [13]. Further, it has been argued that tachyonic fields may describe a larger variety of interesting physical situations than quintessence fields [11]. Besides, when the tachyon field potential reduces to a constant, its equation of state coincides with that of the Chaplygin gas - see, e.g., [14].

To study the cosmological dynamics in the presence of tachyonic matter, several authors have resorted to the potential \( V(\varphi) \propto \varphi^{-2} \) (where \( \varphi \) is the tachyon field), since it leads to a power–law solution for the scale factor of the Robertson–Walker metric. (Bear in mind that in the case of a minimally coupled scalar field a power law behavior requires an exponential potential.) Here we are interested in situations where the cosmic medium may at present be regarded as a mixture of two components, namely a tachyon field (acting as dark energy) and pressureless dust (i.e., cold dark matter) such that the ratio between their energy densities is a constant, thus indicating a solution of the coincidence problem which afflicts many approaches to late acceleration. (Obviously, this makes sense only, if the tachyon field is not yet near its long–time limit in which its equation of state approaches that for dust as well). This is accomplished by assuming an interaction between both components so that they do not conserve separately as the Universe expands. This approach parallels somewhat a previous study in which a generic quintessence scalar field interacted with cold dark matter in such a way that the resulting dynamics was compatible with late acceleration and solved the coincidence problem [15].

We find exact solutions to the system of dark matter and tachyon field equations for
different choices of \( V(\varphi) \) such that the ratio between the energy densities of both components is kept constant at late times and the scale factor of the Robertson–Walker metric obeys a power law.

The next section presents the relevant field equations of the tachyonic field. In section III the interaction between the latter and cold dark matter is considered and some solutions are found, first when \( \dot{\varphi}^2 \) is held constant and then when this constraint is relaxed. Finally, section IV summarizes our findings.

II. BASIC TACHYON FIELD EQUATIONS

We begin by succinctly recalling the basic equations of the tachyon field to be used in the next section where the interaction with dark matter is incorporated.

The stress–energy tensor of the tachyon field

\[
T_{ab}^{(\varphi)} = \frac{V(\varphi)}{\sqrt{1 + \varphi^a \varphi_a}} \left[-g_{ab} (1 + \varphi^c \varphi_c) + \varphi_a \varphi_b \right],
\]

 admits to be cast into the form of a perfect fluid

\[
T_{ab}^\varphi = \rho_\varphi u_a u_b + p_\varphi (g_{ab} + u_a u_b),
\]

where the energy density and pressure are given by

\[
\rho_\varphi = \frac{V(\varphi)}{\sqrt{1 - \dot{\varphi}^2}} \quad \text{and} \quad p_\varphi = -V(\phi) \sqrt{1 - \dot{\varphi}^2},
\]

respectively, with

\[
\dot{\varphi} \equiv \varphi_a u^a = \sqrt{-g^{ab} \varphi_a \varphi_b} \quad \text{and} \quad u_a = -\frac{\varphi_a}{\dot{\varphi}}, \quad \text{with} \quad u^a u_a = -1.
\]

Furthermore if the tachyon field interacts only gravitationally, its evolution equation reads

\[
\frac{\ddot{\varphi}}{1 - \dot{\varphi}^2} + 3H \dot{\varphi} + \frac{V'}{V} = 0,
\]
where \( H \equiv \dot{a}/a \) is the Hubble factor and \( a \) the scale factor of the FLRW metric; the prime indicates derivation respect to \( \varphi \). The latter expression can be recast as

\[
\dot{\rho}_\varphi = -3H \varphi^2 \rho_\varphi ,
\]

(6)

this implies that for any \( \dot{\varphi}^2 < 1 \) the energy density of tachyon field decays at a lower rate than that for dust. It approaches the behavior of dust for \( \dot{\varphi}^2 \to 1 \). In this limit the tachyon may be considered a pressureless dark matter component.

As mentioned above, we shall assume that in addition to the tachyon field (with \( \dot{\varphi}^2 < 1 \)) a cold dark matter fluid, of energy density \( \rho_m \), enters the cosmic medium. Thus, the Friedmann equation for this two component system in a spatially flat FLRW universe can be written as

\[
H^2 = \frac{8\pi G}{3} \left[ \frac{V(\varphi)}{\sqrt{1 - \dot{\varphi}^2}} + \rho_m \right] .
\]

(7)

III. THE \( \varphi \)CDM INTERACTING MODEL

Henceforward we shall assume that both components -the tachyon field and the cold dark matter- do not conserve separately but that they interact through a term \( Q \) (to be specified later) according to

\[
\dot{\rho}_m + 3H \rho_m = Q ,
\]

(8)

\[
\dot{\rho}_\varphi + 3H \varphi^2 \rho_\varphi = -Q .
\]

(9)

The interaction term \( Q \) is to be determined by the condition that the ratio between the energy densities \( r \equiv \rho_m/\rho_\varphi \) remains constant at late times. One easily realizes that a suitable interaction between both components,

\[
Q = 3H \frac{r}{(r + 1)^2} (1 - \dot{\varphi}^2) \rho ,
\]

(10)
where $\rho = \rho_m + \rho_\phi$ is the total energy density of the cosmic substratum, leads to the desired result. Since $\dot{\varphi}^2 < 1$ we have $Q > 0$. Therefore, for a stationary ratio $r$ to exist, a transfer of energy from the tachyon field to the matter component is required. A stability analysis of the stationary solution parallel to that in [15] reveals that when $Q/3H \propto \rho$ in the vicinity of the stationary solution, then $r$ is stable for any $r < 1$ (and $\dot{\varphi}^2 < 1$). In particular, the stability is compatible with accelerated expansion (see below). In the presence of the above interaction the evolution equation for $\varphi$ (Eq. (5)) generalizes to

$$\frac{\ddot{\varphi}}{1 - \dot{\varphi}^2} + 3H \dot{\varphi} + 3H \frac{1 - \dot{\varphi}^2}{\dot{\varphi}} \frac{r}{r + 1} + \frac{V'}{V} = 0 \, .$$

(11)

A. $\dot{\varphi}^2 = \text{const}$

When $\dot{\varphi}^2 = \text{const}$, the solution

$$\rho_m, \rho_\phi, V \propto a^{-\frac{2r + \dot{\varphi}^2}{3 + r + \dot{\varphi}^2}} \, .$$

(12)

readily follows. It corresponds to a power law expansion

$$a(t) \propto t^n \, , \quad \text{with} \quad n = \frac{2}{3} \frac{r + 1}{3 + r + \dot{\varphi}^2} = \text{constant} \, .$$

(13)

Thus, the temporal evolution is given by $\rho_m, \rho_\phi \propto t^{-2}$. Thereby, one has accelerated expansion for

$$n > 1 \quad \iff \quad \dot{\varphi}^2 < \frac{2 - r}{3} \, ,$$

(14)

i.e., $\dot{\varphi}^2$ has to be sufficiently small. For $\rho_m = r = 0$ one recovers the result of the single-component case [2].
The solution just found is a very particular one as, generally speaking, one should expect that $\dot{\varphi}$ does not vanish. Therefore, we next focus on finding analytical solutions such that $\dot{\varphi} \neq 0$ but, as before, keeping the ratio $r$ between the energy densities constant.

It seems unclear if this can be accomplished at all by a perfect (i.e., non–dissipative) fluid model of the dark matter. But as we shall see, this can be achieved by assuming that the dark matter component is dissipative, i.e., endowed with a dissipative pressure $\pi_m$ whose origin may lie either in the interactions between the particles that make up the dark matter fluid or in their decay and/or mutual annihilation -for a recent short review on models of self–interacting CDM see [16]. As a consequence, the energy balance equation for the matter component now reads

$$\dot{\rho}_m + 3H(\rho_m + \pi_m) = Q ,$$

(15)

where the strength of the interaction $Q$ will differ from that in Eq. (10). The presence of $\pi_m$ (which ought to be negative for expanding fluids as required by the second law of thermodynamics -see, e.g., [17]) in the dark matter fluid is only natural since (barring superfluids) all matter fluids found in Nature are dissipative -see e.g. [18]. Further, this quantity is crucial to solve the coincidence problem of late acceleration in models where the dark matter and dark energy conserve separately [19]. Here we shall use the imperfect fluid degree of freedom to obtain a power-law solution for $\dot{\varphi} \neq$ constant under the condition $r = \text{const}$.

The energy balance equation for the tachyon field can be written as

$$\dot{\rho}_\varphi + 3H\varphi^2\rho_\varphi = -Q ,$$

(16)

or equivalently,

$$\frac{\ddot{\varphi}}{1 - \dot{\varphi}^2} + 3H\dot{\varphi} + \frac{V'(\varphi)}{V(\varphi)} = -\sqrt{1 - \dot{\varphi}^2} \frac{Q}{V'\varphi} Q .$$

(17)

Again, we have left the interaction term unspecified. To determine it we impose (as we did above) that the ratio of both energy densities remains constant (i.e., we demand that
the coincidence problem should be solved). Thus, in addition to equations (15) and (16) we require that \( \dot{r} = (\rho_m/\rho_\varphi)' = 0 \). As a consequence, the interaction term is now given by

\[
Q = 3H \frac{r}{(r+1)^2} \left( \frac{\pi_m}{\rho_m} - \frac{p_\varphi}{\rho_\varphi} \right) \rho .
\]  

(18)

This corresponds to

\[
\frac{\dot{\rho}_m}{\rho_m} = \frac{\dot{\rho}_\varphi}{\rho_\varphi} = 3H \left[ 1 + \frac{p_\varphi + \pi_m}{\rho} \right] = -3H \left[ 1 - \frac{1 - \dot{\varphi}^2}{1 + r + \frac{r}{b_0^2}} + \frac{\pi_m}{\rho} \right].
\]  

(19)

For the expression in the brackets on the right hand side to be a constant when \( \dot{\varphi}^2 \) is admitted to vary, the last term must cancel the \( \dot{\varphi}^2 \) term. This suggests an ansatz

\[
\pi = -b^2 \rho ,
\]  

(20)

where

\[
b^2 = b_0^2 + \frac{\dot{\varphi}^2}{r+1},
\]  

(21)

with \( b_0^2 = \text{const} \), which implies

\[
\frac{\dot{\rho}_m}{\rho_m} = \frac{\dot{\rho}_\varphi}{\rho_\varphi} = -3H \left[ \frac{r}{r+1} - b_0^2 \right].
\]  

(22)

Physically, this means that the ratio of the total pressure, which is \( p_\varphi + \pi_m \), to the total energy density \( \rho \) is required not to depend on \( \dot{\varphi}^2 \). In other words, \( \pi_m \) has to be chosen such that the total equation of state of the cosmic medium is constant, while at the same time \( \dot{\varphi}^2 \) is not. Thus, the dependences of \( \rho_m \) and \( \rho_\varphi \) on the scale factor are found to be:

\[
\rho_m \propto a^{-\nu}, \quad \rho_\varphi \propto a^{-\nu}, \quad \nu = 3 \left[ \frac{r}{r+1} - b_0^2 \right].
\]  

(23)

Under this condition the interaction term is now given by

\[
Q = 3H \frac{r}{(r+1)^2} \left[ 1 - \frac{1+r}{r} \left( b_0^2 + \dot{\varphi}^2 \right) \right] \rho .
\]  

(24)
By virtue of the relationship $\rho \propto a^{-\nu}$ the Friedmann equation (7) leads to $a(t) \propto t^{2/\nu}$. It readily follows that $\rho \propto \rho_m \propto \rho_\varphi \propto t^{-2}$ for the energy densities. Again, for a power-law solution to exist, a transfer of energy from the tachyon field to the matter component is required (i.e., one must have $Q > 0$), as well as $\nu < 2$ (tantamount to $r/(r+1) < (2/3) + b_0^2$), to have accelerated expansion.

We conclude that by a suitable choice of the imperfect fluid degree of freedom $\pi_m$ it is indeed possible to obtain a power law solution with $r = \text{constant}$ and $\dot{\varphi} \neq \text{const}$. The magnitude of this pressure is largely dictated by the dynamics of the tachyon field. Since from the outset $\pi_m$ was introduced to account for interactions within the matter component, this may seem surprising at a first glance. The point here is that the requirement $r = \text{constant}$ establishes a strong coupling of the dynamics of the components which are no longer independent of each other. Only if $\pi_m$ is such that $\pi_m/\rho$ follows the (not yet known) dynamics of $\dot{\varphi}^2$, the described power-law solution is possible. As we shall see, such a configuration may serve as a toy model which admits exact solutions for the cosmological dynamics.

A stability analysis of the stationary solution may be performed as in [15] where the relationship $Q = 3Hc^2\rho$, with $c^2$ a constant, was hypothesized. By introducing the ansatz

$$\frac{\rho_m}{\rho_\varphi} = \left(\frac{\rho_m}{\rho_\varphi}\right)_{st} + \epsilon \quad \left(\epsilon \ll \left(\frac{\rho_m}{\rho_\varphi}\right)_{st}\right),$$

(25)

where the subscript $st$ denotes ‘stationary’, in

$$\left(\frac{\rho_m}{\rho_\varphi}\right) = \frac{\rho_m}{\rho_\varphi} \left[\frac{\dot{\rho}_m}{\rho_m} - \frac{\dot{\rho}_\varphi}{\rho_\varphi}\right],$$

and retaining terms up to first order in the perturbation we get

$$\dot{\epsilon} = 3H \left[c^2(r+1) - \frac{1}{r+1}\right] \epsilon.$$

(26)

Hence, the stationary solution will be stable for $c < 1/(r + 1)$. Given the currently favored observational data $\rho_m \approx 0.3$ and $\rho_\varphi \approx 0.7$ [13] we get $c < 0.7$.

Next we seek an exact solution to Eq. (17) with the interaction term given by Eq. (24) for different potentials.
(i) For $V(\varphi(t)) = \beta t^{-m}$, where $\beta$ and $m$ are positive–definite constants (bearing in mind that $\rho_m = r\rho_\varphi$ and $\rho_\varphi = \gamma t^{-2}$, with $\gamma$ a constant), the solution is

$$\varphi(t) = t \ 2F_1 \left( \left[ \frac{-1}{2}, \frac{-1}{2} \right], \left[ \frac{-5 + 2m}{2(-2 + m)} \right]; \frac{(\beta/\gamma)^2}{t^{4-2m}} \right),$$

(27)

where $2F_1$ is the hypergeometric function [20]. When $m = 2$ the function $2F_1$ collapses to a constant, thereby

$$\varphi(t) \propto t \implies V(\varphi) \propto \varphi^{-2}.$$ 

This recovers a particular case considered in Ref.[12].

(ii) For $V(\varphi(t)) = \alpha t^2 \left(1 - \frac{\beta}{t^2}\right)^{1/2}$ with $\alpha$ and $\beta$ positive–definite constants, one finds that

$$\varphi(t) = \beta^{1/2} \ln t,$$

(28)

where the integration constant has been set to zero. This potential may look a bit contrived, however, it becomes nearly exponential, $V(\varphi) \sim e^{-2\varphi}$ (which is another case considered in [12]), for $\beta < 1$.

IV. CONCLUDING REMARKS

In this paper we have considered that the present accelerated expansion of our flat FLRW Universe is driven by an interacting mixture of cold dark matter and a tachyonic field. The interaction was not fixed from the outset but derived from the requirement that the ratio between the energy densities of both components remains constant such that there is no coincidence problem. We have found an exact solution when $\ddot{\varphi}^2 = \text{constant}$ and two exact solutions when $\ddot{\varphi} \neq 0$ for specific potentials -Eqs. (27) and (28). All these solutions imply power law expansions. For solutions of this type to exist when $\ddot{\varphi} \neq 0$, a negative scalar pressure in the matter component is required, in order to keep the overall equation of state parameter of the cosmic medium constant.
One should be aware that our model is not complete in the sense that (i) it is unable to provide a dynamical approach towards a stationary energy density ratio and (ii) it does not include a radiation dominated era at early times. To achieve this one should resort to a more general approach, maybe similar to the one by Chimento et al. [21] that extended the scenario of Ref. [15] (a scalar field interacting with cold dark matter) in such a way that both a stationary ratio is dynamically approached and the radiation era is retrieved for early times. This, as well as to find solutions other than power law expansions will be the subject of future research.

Acknowledgments

This work has been partially supported by the “Ministerio de Educación de Chile” through MECESUP Project FSM 9901, the US A 0108 grant, the Spanish Ministry of Science and Technology under grant BFM–2003–06033, the “Direcció General de Recerca de Catalunya” under grant 2001 SGR–00186, and the Deutsche Forschungsgemeinschaft.

[1] A. Sen, JHEP04(2002)048; JHEP07(2002)065; Mod. Phys. Lett. A 17, 1797 (2002).
[2] G.W. Gibbons, Phys. Lett. B 537, 1 (2002).
[3] T. Padmanabhan, Phys. Rev. D 66, 021301 (2002); T. Padmanabhan, and T.R. Choudhury, Phys. Rev. D 66, 081301 (2002).
[4] M. Fairbairn, and M.H.G. Tytgat, Phys. Lett. B 546, 1 (2002).
[5] A. Feinstein, Phys. Rev. D 66, 063511 (2002).
[6] D. Choudhury, D. Ghoshal, D.P. Jatkar, and S. Panda, Phys. Lett. B 544, 231 (2002).
[7] A. Frolov, L. Kofman, and A.A. Starobinsky, Phys. Lett. B 545, 8 (2002).
[8] L. Kofman, and A. Linde, JHEP07(2002)065.
[9] G. Shiu, and I. Wasserman, Phys. Lett. B 541, 6 (2002).
[10] L.R.W. Abramo, and F. Finelli, Phys. Lett. B 575, 165 (2003).
[11] V. Gorini et al., LANL preprint hep-th/0311111.
[12] J.S. Bagla, H.K. Jassal, and T. Padmanabhan, Phys. Rev. D 67, 063504 (2003).

[13] S. Perlmutter, Int. J. Theor. Phys. A 15, Suppl. 1B, 715 (2000);
    J.L. Tonry et al., Astrophys. J. 594, 1 (2003);
    B.J. Barris et al., LANL preprint astro-ph/0310843 Astrophys J. (in the press);
    Proceedings of the I.A.P. Conference “On the Nature of Dark Energy”, held in Paris, edit. P. Brax et al. (Frontiere Group, Paris, 2002);
    Proceedings of the Conference “Where Cosmology and Fundamental Physics Meet” held in Marseille, edit. S. Basa et al. (in the press).

[14] A.Yu Kamenshchik, U. Moschella, and V. Pasquier, Phys. Lett. B 511, 265 (2001).

[15] W. Zimdahl, D. Pavón, and L.P. Chimento, Phys. Lett. B 521, 133 (2001).

[16] J.P. Ostriker, and P. Steinhardt, Science 300, 1909 (2003), LANL preprint astro-ph/0306402

[17] L. Landau, and E.M. Lifshitz, “Mécanique des Fluides” (MIR, Moscou, 1971);
    A.L. Fetter, and J.D. Walecka, “Theoretical Mechanics of Particles and Continua” (McGraw-Hill, New York, 1980).

[18] G.K. Batchelor, “An Introduction to Fluid Dynamics” (Cambridge University Press, Cambridge, 1967).

[19] L.P. Chimento, A.S. Jakubi, and D. Pavón, Phys. Rev. D 67, 087302 (2003).

[20] A. Prudnikov, Y. Brychkov, and O. Marichev, “More Special Functions”, Gordon and Breach Science Publisher, (1990);
    K. Roach, “Hypergeometric Function Representation”, in Proceedings of ISSA 96’, ACM, 301-308 (New York, 1996).

[21] L.P., Chimento, S.A. Jakubi, D. Pavón, and W. Zimdahl, Phys. Rev. D 67, 083513 (2003).