Noether symmetries and conserved quantities of fractional nonconservative singular systems

Mingliang Zheng¹
Shcool of mechanical and electrical, Taihu University of Wuxi, Wuxi, 214064, China

¹Email: zhmlwxcstu@163.com

Abstract. The Noether theory of fractional nonconservative singular systems is studied based on fractional factor derivative method in form space. The Lagrange equations with fractional factor are established through the variational principle. The criterion equation and the conserved quantities are further studied according to the fractional order Hamilton action quantity maintain invariance under the infinitesimal transformation. Finally, an example is given to illustrate the application. The results show that comparing with the conservative systems, the nonconservative forces have impact on the Noether identity, but because of enhancing the invariance condition, it does not change the form of Noether type conserved quantities, at the same time, we use fractional factor method to study the nonconservative singular systems, some conclusions are highly natural consistent with the classical integer order singular systems, so the fractional factor can establish the connection between the fractional order systems and the integer order systems.

1. Introduction
Fractional order dynamics plays an important role in the study of complex classical dynamics theory and quantum mechanics theory, especially in the field of chaos and micro environment. At present, there are two main forms of research on fractional dynamics in the world. One is the fractional order dynamics of sequence form, represented by the physicist Fred Riewe [1-2]. The other is fractional order dynamics of the order of alpha form, represented by mathematicians Om.P.Agrawal [3] and Vasily E.Tarasov [4]. In the dynamic analysis, the study of symmetries and conserved quantities of fractional variational problems is an important aspect of fractional order dynamic system. In recent years, the fractional Lagrange system [5-6] and fractional Hamilton system [7-8], fractional order Birkhoff system [9], fractional nonconservative dynamical systems [10], fractional order generalized Hamilton system [11-12] and fractional order nonholonomic mechanical systems such as [13] have achieved certain results.Khalil [14] and Abdeljawad [15] proposed a new method of fractional calculus recently. The definition of this derivative is the limit form, The fractional order can be transformed into integer order by using polynomial function. Fu [16-17] respectively studied the Noether, Lie symmetries and conserved quantities of fractional order Lagrange and Hamilton systems based on the joint Caputo derivatives and the uniform fractional derivatives. Fu [18] obtains some new results on the equations of motion and integral factors of holonomic fractional Lagrange systems based on fractional factor.

Study on the symmetries of fractional nonsingular systems have been obtained some results, However, under the Legendre transformation, when the singular Lagrange system transits to the phase space and is described by the Hamilton system, there exists an inherent constraint between its
canonical variables, which is called the constrained Hamilton system [19]. Many important dynamical systems in reality are constrained Hamilton system model [20], such as supersymmetry, supergravity, electromagnetic field, relativistic motion of the particle, superstring and Yang-Mills field etc. However, research on the variational problem and the symmetries of fractional constraint Hamilton system is rarely reported, almost at the beginning stage. In this paper, a new definition of fractional derivative of fractional factor is given, and then the theory of Noether symmetry for fractional singular systems is established.

2. Fractional factor and fractional derivative
As everyone knows, Riemann-Liouville fractional derivative, Grunwald-Letnikov fractional derivative and Caputo fractional derivative are the integral form of the definition, it has only linear optimality, but its basic properties of calculus with integer order calculus is not a natural consistency. Recently, a novel fractional derivative whose definition and important properties follows [18].

The order derivative \( D_\alpha (f) = f^\alpha \) of function \( f(t) \), which is defined with fractional factor:

\[
D_\alpha (f) = f^\alpha (t) = \lim_{\Delta t \to 0} \frac{f(t + e^{-(1-\alpha)\Delta t}) - f(t)}{\Delta t} = \frac{df(t)}{d_\alpha t}
\]

(1)

Fractional integral based on fractional factor can be used as:

\[
I_a^b f(t) = \lim_{\max(\Delta t) \to 0} \sum_{r=1}^{n} f(\xi_r)\Delta t = \int_a^b f(t)dt = \int_a^b e^{-(1-\alpha)t} f(t)dt
\]

(2)

The exchange relations between isochronous variational and fractional order operators, and the fractional differential rule of composite functions are:

\[
\delta D_\alpha q = \delta (e^{-(1-\alpha)t}) = D_\alpha \delta q = e^{-(1-\alpha)t}\frac{d\delta q}{dt}
\]

(3)

\[
D_\alpha (f \cdot g) = D_\alpha (f) \cdot g + D_\alpha (g) \cdot f
\]

3. The motion equations of fractional nonconservative singular systems
The form of fractional order mechanical systems is determine by generalized coordinates \( q_s(t) \), the Lagrange function is \( L(t, q_s, D_\alpha q_s) \), the non-potential and non-conservative force is \( Q^\alpha(t, q_s, D_\alpha q_s) \). The Hamilton variational principle of fractional order systems with nonconservative forces is:

\[
\delta S = \int_a^b [\delta L(t, q_s, D_\alpha q_s) + Q^\alpha(t) \delta q_s] dt = 0
\]

(4)

According to the formula (3), we have:

\[
\delta S = \int_a^b \left[ \frac{\partial L}{\partial q_s} + Q^\alpha \right] \delta q_s + \frac{\partial L}{\partial D_\alpha q_s} D_\alpha \delta q_s \right] dt = 0
\]

(5)

And because:

\[
\frac{\partial L}{\partial D_\alpha q_s} D_\alpha q_s = D_\alpha \left( \frac{\partial L}{\partial D_\alpha q_s} q_s \right) - D_\alpha \left( \frac{\partial L}{\partial D_\alpha q_s} \right) \delta q_s
\]

(6)

The formula (6) is substituted into the formula (5), we have:
\[ \delta S = \int_{a}^{b} \left[ \left( \frac{\partial L}{\partial q_s} - D_{a} \left( \frac{\partial L}{\partial D_{a} q_s} \right) + Q_{s}^* \right) \delta q_s \right] d \alpha t + \int_{a}^{b} \left[ D_{a} \left( \frac{\partial L}{\partial D_{a} q_s} \right) \right] d \alpha t \]
\[ = \int_{a}^{b} \left[ \left( \frac{\partial L}{\partial q_s} - D_{a} \left( \frac{\partial L}{\partial D_{a} q_s} \right) + Q_{s}^* \right) \delta q_s \right] d \alpha t + \int_{a}^{b} \left[ \left( \frac{\partial L}{\partial D_{a} q_s} \right) \right] d \alpha t \]
\[ = \int_{a}^{b} \left[ \left( \frac{\partial L}{\partial q_s} - D_{a} \left( \frac{\partial L}{\partial D_{a} q_s} \right) + Q_{s}^* \right) \delta q_s \right] d \alpha t + \int_{a}^{b} \left[ \left( \frac{\partial L}{\partial D_{a} q_s} \right) \right] d \alpha t \]
\[ = \int_{a}^{b} \left[ \left( \frac{\partial L}{\partial q_s} - D_{a} \left( \frac{\partial L}{\partial D_{a} q_s} \right) + Q_{s}^* \right) \delta q_s \right] d \alpha t \]

Due to the \( \delta q_s \) is arbitrary, so:

\[ D_{a} \left( \frac{\partial L}{\partial D_{a} q_s} \right) = 0 \]  

(8)

It is completely equivalent to the motion equations of fractional order nonconservative systems are obtained through the D’Alembert-Lagrange principle.

Spreading the formula (8), and the fractional order generalized acceleration are obtained:

\[ \left[ \frac{\partial^2 L}{\partial D_{a} q_s \partial D_{a} q_{k}} \right] \frac{d}{dt} D_{a} q_{k} = e^{(1-\alpha)y} \frac{\partial L}{\partial q_s} + e^{(1-\alpha)\gamma} Q_{s}^* - \frac{\partial^2 L}{\partial D_{a} q_{s} \partial q_{k}} \dot{q}_{k} - \frac{\partial^2 L}{\partial D_{a} q_{s} \partial t} \]  

(9)

Because of the singularity, the generalized accelerations are not all expressed, but a part of them can be solved, so it can be recorded:

\[ \frac{d}{dt} D_{a} q_{k} = A_{k}(t, q, \dot{q}, D_{a} q) (k = 1,2,...,r) \]  

(10)

And the (n- r) relationships are:

\[ \varphi_{j}(t, q, \dot{q}, D_{a} q) = 0 (j = 1,2,...,n-r) \]  

(11)

4. Noether symmetry of fractional nonconservative singular systems

Set \( \varepsilon \) as an infinitesimal parameter, \( \xi_{0}, \xi_{s} \) are the generating elements of infinitesimal transformation, the infinitesimal transformations contain time and generalized coordinates:

\[ t^{*} = t + \varepsilon \xi_{0} (t, q, \dot{q}, D_{a} q_{s}), q_{s}^{*} = q_{s} + \varepsilon \xi_{s} (t, q, \dot{q}, D_{a} q_{s}) \]  

(12)

Under the (12), the Hamilton action quantity of the system is changed to:

\[ \Delta S = I_{a}^{ab} \{ \Delta L + L \frac{d \Delta t}{dt} \} \]
\[ = I_{a}^{ab} \left[ \frac{\partial L}{\partial t} \Delta t + \frac{\partial L}{\partial q_s} \Delta q_s + \frac{\partial L}{\partial D_{a} q_s} \Delta D_{a} q_s + L \frac{d \Delta t}{dt} \right] \]

(13)

Be aware:

\[ \Delta t = \varepsilon \xi_{0} \]
\[ \Delta q_s = \Delta q_s - \dot{q}_s \Delta t = \varepsilon (\xi_{s} - \dot{q}_s \xi_{0}) = \varepsilon \xi_{s} \]
\[ \Delta D_{a} q_s = \Delta D_{a} q_s + D_{a+1} q_s \Delta t = D_{a} \Delta q_s + D_{a+1} q_s \Delta t \]

(14)

Below, we introduce the definition and criterion equations (Noether identities) of the Noether symmetric transformation for fractional order singular systems.
**Definition:** If the Hamilton action quantity of the fractional order singular systems is the invariant under the infinitesimal transformation (12), that is, for each generating element $\xi_0, \xi_s$, it is always established:

$$\Delta S = -I_a^{ab} \left[ \frac{d}{da^t}(eG_N) + Q_s^a \frac{\partial q_s}{\partial D_a q_s} \right]$$ (15)

The $G_N(t_s, q_s, D_a q_s)$ is a gauge function, then, the infinitesimal transformation is the Noether generalized quasi symmetric transformation.

Simultaneous the formula (13), (14) and (15), we can obtain the criterion equation (Noether identity) of the Noether quasi symmetric transformation:

$$\frac{\partial L}{\partial t_0} \xi_0 + \frac{\partial L}{\partial q_s} \xi_s + \frac{\partial L}{\partial D_a q_s} (D_a \xi_s - \dot{q}_s D_a \xi_0) + L \dot{\xi}_0 + Q_s^a (\xi_s - \dot{q}_s \xi_0) + D_a G_N = 0$$ (16)

The invariance of the intrinsic constraint equations (11) under the infinitesimal transformation (12) are the following limit equations:

$$\frac{\partial \phi_j}{\partial t_0} + \frac{\partial \phi_j}{\partial q_s} + (\xi_s - \dot{q}_s \xi_0) \frac{\partial \phi_j}{\partial D_a q_s} + [(D_a \xi_s - \dot{q}_s \xi_0) + \frac{d}{da} D_a q_s] \frac{\partial \phi_j}{\partial D_a q_s} = 0$$ (17)

5. **The conservation of Noether symmetry**

There is a close and profound relationship between the symmetry and the conserved quantity of the mechanical systems, and the Noether symmetry can directly lead to the conservation.

According to the relation between total variation and isochronous variation, the formula (13) can also be expressed as:

$$\Delta S = \delta S + \delta \Delta t$$

$$= I_a^{ab} \left\{ \frac{\partial L}{\partial q_s} \frac{\partial \phi_j}{\partial q_s} + \frac{\partial L}{\partial D_a q_s} \frac{\partial \phi_j}{\partial D_a q_s} \right\} + e^{(1-a)t} L \Delta t$$

$$= I_a^{ab} \left\{ \frac{d}{da^t} [e^{(1-a)t} L \xi_0 + \frac{\partial L}{\partial D_a q_s} \xi_s - \dot{q}_s \xi_0 + D_a \frac{\partial L}{\partial D_a q_s} \xi_s] \right\}$$

Simultaneous the formula (8), (14) and (15), the formula (18) can be reduced to:

$$I_a^{ab} e^{(1-a)t} \frac{d}{da^t} \left\{ e^{(1-a)t} L \xi_0 + \frac{\partial L}{\partial D_a q_s} (\xi_s - \dot{q}_s \xi_0) + G_N \right\} = 0$$ (19)

Sum deduced from above, we can easily obtain the Noether theorem for fractional order singular systems:

**Theorem:** If the generating elements $\xi_0, \xi_s, D_a q_s$, and the gauge function, $G_N(t_s, q_s, D_a q_s)$ satisfy the criterion equation (16) and the limit equations (17) of Noether generalized quasi symmetric transformation for fractional singular Lagrange systems, then the system have the Noether type conserved quantities:

$$I = e^{(1-a)t} L \xi_0 + \frac{\partial L}{\partial D_a q_s} (\xi_s - \dot{q}_s \xi_0) + G_N = \text{const}$$ (20)

6. **Illustrated example**

The Lagrange function of fractional order systems with three degrees of freedom is:

$$L = \frac{1}{2} [(D_a q_1)^2 + (D_a q_2)^2] - (q_1 - q_2) D_a q_3$$ (21)
$0 < \alpha < 1$. The nonconservative generalized forces are:

$$Q_i^\alpha = -D_\alpha q_2, \quad Q_i^\alpha = D_\alpha q_1, \quad Q_i^\alpha = 0$$  \hspace{1cm} (22)

Please study the Noether symmetry and conserved quantity. According to the equations (10), the system’s differential equations are:

$$\frac{d}{dt} D_\alpha q_1 = -e^{(1-\alpha)t}D_\alpha q_2 - e^{(1-\alpha)t}D_\alpha q_3$$
$$\frac{d}{dt} D_\alpha q_2 = e^{(1-\alpha)t}D_\alpha q_1 + e^{(1-\alpha)t}D_\alpha q_3$$  \hspace{1cm} (23)

The rank of hessian matrix of $L$ is $r = 2$, thus, there is one inherent constraint:

$$\dot{q}_1 - \dot{q}_2 = 0$$  \hspace{1cm} (24)

According to the Noether generalized quasi symmetric criterion equation (16), we have:

$$-D_\alpha q_1 \ddot{\xi}_1 + D_\alpha q_3 \ddot{\xi}_2 + D_\alpha q_1 (D_\alpha \ddot{\xi}_1 - \dot{q}_1 D_\alpha \ddot{\xi}_0) + D_\alpha q_2 (D_\alpha \ddot{\xi}_2 - \dot{q}_2 D_\alpha \ddot{\xi}_0) -$$
$$\left( q_1 - \dot{q}_2 \right) (D_\alpha \ddot{\xi}_3 - \dot{q}_3 D_\alpha \ddot{\xi}_0) + L \ddot{\xi}_0 - D_\alpha q_2 (\ddot{\xi}_1 - \dot{q}_1 \ddot{\xi}_0) + D_\alpha q_1 (\ddot{\xi}_2 - \dot{q}_2 \ddot{\xi}_0) + D_\alpha G_N = 0$$  \hspace{1cm} (25)

According to the limit equations (17), we have:

$$\left( \ddot{\xi}_1 - \dot{q}_1 \ddot{\xi}_0 \right) - \left( \ddot{\xi}_2 - \dot{q}_2 \ddot{\xi}_0 \right) = 0$$  \hspace{1cm} (26)

The infinitesimal generating element and the gauge function in (25) and (26) may be considered as the following solution:

$$\ddot{\xi}_0 = 0, \ddot{\xi}_1 = 1, \ddot{\xi}_2 = 1, \ddot{\xi}_3 = 0, G_N = 0$$  \hspace{1cm} (27)

According to (20), the Noether generalized quasi symmetry leads to the conserved quantity is:

$I = D_\alpha q_1 + D_\alpha q_2 = const$  \hspace{1cm} (28)

7. Conclusions

In order to solve the large number of singular Lagrange function problems in modern physics and mechanics, in this paper, the Noether symmetry theory has been extended to fractional order nonconservative singular systems based on a fractional factor, a series of new results have been given. The fractional order motion differential equations are add to a mixture of the first derivative and the fractional partial derivative in comparing with the integer order singular systems. A part of the generalized accelerations can be solved, the remaining part can expressed as constraints, so the singularity have impact on the Noether symmetry, the generating elements need to maintain the constraints for invariant. The generating elements function and gauge function are related to the nonconservative forces, but the nonconservative forces don’t appear in the Noether type conserved quantities, the form between the classical Noether and the fractional Noether conserved quantities is coincident, it only adds to a fractional factor, it shows that the biggest feature of the fractional factor derivative method is that the fractional order systems can be transformed into an integer order systems. The method in this paper can be applied to other fractional order dynamics problems, such as stability, symmetric perturbation and adiabatic invariants, etc.

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