Productively Lindelöf and Indestructibly Lindelöf Spaces

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Abstract

There has recently been considerable interest in productively Lindelöf spaces, i.e. spaces such that their product with every Lindelöf space is Lindelöf. See e.g. [4], [29], [1], [26], and work in progress by Miller, Tsaban, and Zdomskyy, Repovs and Zdomskyy, and by Brendle and Raghavan. Here we make several related remarks about such spaces. Indestructible Lindelöf spaces, i.e. spaces that remain Lindelöf in every countably closed forcing extension, were introduced in [27]. Their connection with topological games and selection principles was explored in [25]. We find further connections here.

1 A sufficient condition for a space not to be productively Lindelöf

In [1], a set of four conditions was given for a regular Lindelöf space $X$ of countable type to be not productively Lindelöf. The conditions and proof

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were unnecessarily complicated because the authors wanted to produce a regular $Z$ such that $X \times Z$ was not Lindelöf. This extra effort was not necessary because we can prove the following result.

**Lemma 1.** Let $X$ be a Lindelöf space. If there is a Lindelöf space $Z$ such that $X \times Z$ is not Lindelöf, then there is such a Lindelöf $Z'$, which furthermore is 0-dimensional $T_1$ and hence regular.

**Proof.** Let $\{U_\alpha \times V_\alpha : \alpha < \kappa\}$ be an open cover of $X \times Z$ which does not have a countable subcover. Consider the following set-valued maps:

- $\Phi_X : X \to 2^\kappa$, $\Phi_X(x) = \{A \subseteq \kappa : \{\alpha : x \in U_\alpha\} \subseteq A\}$,
- $\Phi_Z : Z \to 2^\kappa$, $\Phi_Z(z) = \{C \subseteq \kappa : \{\alpha : z \in V_\alpha\} \subseteq C\}$,
- $\Phi_{X,Z} : X \times Z \to 2^\kappa$, $\Phi_{X,Z}(x, z) = \{B \subseteq \kappa : \{\alpha : \langle x, z \rangle \in U_\alpha \times V_\alpha\} \subseteq B\}$

By Lemma 2 of [33] each of these maps is compact-valued and upper semicontinuous. Lindelöfness is preserved by compact-valued upper semicontinuous maps, so $T = \Phi_Z(Z) \subseteq 2^\kappa$ is Lindelöf. To show $X \times T$ is not Lindelöf, it suffices to show $X' \times T$ is not Lindelöf, where $X' = \Phi_X(X)$. Consider the map $\Phi : X' \times T \to 2^\kappa$, $\Phi(A, C) = A \cap C$.

Notice that $A \cap C \neq \emptyset$ for any $A \in X'$ and $C \in T$. Indeed, find $\langle x, z \rangle \in X \times Z$ such that $\{\alpha : z \in V_\alpha\} \subseteq C$ and $\{\alpha : x \in U_\alpha\} \subseteq A$. Let $\alpha$ be such that $\langle x, z \rangle \in U_\alpha \times V_\alpha$. Then $\alpha \in A \cap C$. It follows from the above that $\mathcal{W} = \{W_\alpha : \alpha < \kappa\}$, where $W_\alpha = \{D \subseteq \kappa : \alpha \in D\}$ is an open cover of $X' \times T$. However $\mathcal{W}$ has no countable subcover, since if $\{W_\alpha : \alpha \in I\}$ covers $X' \times T$, $\{U_\alpha \times V_\alpha : \alpha \in I\}$ is a cover of $X \times Z$. Thus $X' \times T$ is not Lindelöf and hence neither is $X \times T$. \qed

**Definition.** $L(X)$, the Lindelöf number of $X$, is the least cardinal $\lambda$ such that every open cover of $X$ has a subcover of size $\leq \lambda$. The type of $X$, $T(X)$, is the least cardinal $\kappa$ such that for each compact $L \subseteq X$ there is a compact $K$ including $L$ such that there is a base of size $\leq \kappa$ for the open sets including $K$. (For $T_{3\frac{1}{2}} X$, this is equivalent to $L(\beta X - X) \leq \kappa$.) The weight of $X$, $w(X)$, is the least cardinal of a base.

Note that $T(X) \leq w(X)$. From Lemma 1, we obtain the following simplified version of the main theorem of [1]. The proof is also a simplified version of that in [1], so will be omitted.
Theorem 2. Let \( \langle X, \mathcal{T} \rangle \) be a Lindelöf space of countable type. Suppose there is a \( Y \subseteq X \) and a topology \( \rho \) on \( Y \) such that

\begin{enumerate}
  \item \( \mathcal{T}|Y \subseteq \rho \),
  \item \( \langle Y, \rho \rangle \) is not Lindelöf,
  \item any \( K \subseteq X \) that is \( \mathcal{T} \)-compact is such that \( K \cap Y \) is \( \rho \)-Lindelöf.
\end{enumerate}

Then \( X \) is not productively Lindelöf. Indeed there is a regular Lindelöf \( Z \) such that \( X \times Z \) is not Lindelöf.

The authors of [1] observe the following corollary.

Corollary 3. Let \( X \) be a Lindelöf regular space of countable type. If there is an uncountable \( Y \subseteq X \) such that for each compact subset \( K \) of \( X \), \( K \cap Y \) is countable, then \( Y \) is not productively Lindelöf.

2 L-productive spaces

Definition. A space \( X \) is \( \leq \kappa \)-L-productive if \( L(X \times Y) \leq L(Y) \) whenever \( \aleph_0 \leq L(Y) \leq \kappa \). A space \( X \) is L-productive if \( L(X \times Y) \leq L(Y) \) for all \( Y \). A space \( X \) is powerfully Lindelöf if \( X^\omega \) is Lindelöf.

Despite much effort, the following problem of E. A. Michael remains unsolved.

Problem 1. If \( X \) is productively Lindelöf, is \( X \) powerfully Lindelöf?

The best result so far is:

Lemma 4 [2]. The Continuum Hypothesis (CH) implies that if \( X \) is productively Lindelöf and regular and \( w(X) \leq \aleph_1 \), then \( X^\omega \) is Lindelöf.

Note that L-productive spaces are productively Lindelöf. Thus a more modest problem is:

Problem 2. Is every L-productive space powerfully Lindelöf?

We shall make some small progress toward solving this problem. Since we occasionally will deal with spaces that are not necessarily Lindelöf, it is convenient to assume from now on that all spaces are Tychonoff.
Definition. $Y \subseteq X$ is sSkinny if $|Y \cap K| < |Y|$ for every compact $K \subseteq X$. A collection $\mathcal{G}$ of subsets of $X$ is a $k$-cover if every compact subset of $X$ is included in a member of $\mathcal{G}$. $A(X)$, the Alster degree of $X$, is the least cardinal $\kappa$ such that every $k$-cover of $X$ by $G_\delta$’s has a subcover of size $\leq \kappa$. If $A(X) \leq \aleph_0$, we say $X$ is Alster.

Definition. A space $X$ is $\aleph_1$-L-productive if $L(X \times Y) \leq \aleph_1$ whenever $L(Y) \leq \aleph_2$.

Note that this does not imply productively Lindelöf.

Theorem 5 [2]. Alster spaces are powerfully Lindelöf.

Lemma 6. $\aleph_2^{\aleph_0} = \aleph_2$ implies if $w(X) \leq \aleph_2$ and $A(X) = \aleph_2$, then $X$ has a sSkinny subspace of size $\aleph_2$.

Proof. Let $\mathcal{G}$ be a $k$-cover of $X$ by $G_\delta$’s which has no subcover of size $\leq \aleph_1$. By hypothesis we may assume that $\mathcal{G} = \{G_\alpha\}_{\alpha < \omega_2}$. Pick $x_\alpha \in X - \left(\bigcup_{\beta < \alpha} G_\beta \cup \{x_\beta : \beta < \alpha\}\right)$. This defines $A = \{x_\alpha : \alpha < \omega_2\}$, for if the construction stopped at $\gamma < \omega_2$, by taking $\{G_\beta : \beta < \gamma\}$ together with a member of $\mathcal{G}$ containing $x_\beta$, for each $\beta < \gamma$, we would obtain a subcover of $\mathcal{G}$ of size $\leq \aleph_1$, contradiction. $A$ is sSkinny since $\mathcal{G}$ is a $k$-cover.

Theorem 7. If $X$ is Lindelöf, and if $T(X) \leq \aleph_1$ and $X$ has a sSkinny subspace of size $\aleph_2$, then $X$ is not $\aleph_1$-L-productive.

Proof. This is accomplished by a straightforward generalization of Theorem [2] and Corollary [3]. See [1] for their proofs.

Theorem 8. If CH and $2^{\aleph_1} = \aleph_2$, then every Lindelöf $\leq \aleph_1$-L-productive space with $T \leq \aleph_1$ and $w \leq \aleph_2$ is powerfully Lindelöf.

Proof. If $A(X) = \aleph_0$, then $X^\omega$ is Lindelöf by Theorem [4]. If $A(X) = \aleph_1$, then $L(X^\omega) \leq \aleph_1$ by repeating the proof of Theorem [5] in [2]. But we have:

Lemma 9 [7]. CH implies that if $X$ is productively Lindelöf and $L(X^\omega) \leq \aleph_1$, then $X$ is powerfully Lindelöf.

Finally, if $A(X) = \aleph_2$, then $X$ has a sSkinny subspace of size $\aleph_2$ by Lemma [6]. Then by Theorem [7] $X$ is not $\aleph_1$-L-productive, a contradiction.

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Unfortunately, we do not know how to generalize Theorem 8 to higher weights, even for spaces of countable type, because of the dependence of the proof of Lemma 9 on Lemma 4. However, we do have a variation of Theorem 8:

**Theorem 10.** Suppose CH and $2^\aleph_1 = \aleph_2$. Then every Lindelöf $\leq \aleph_1$-productive space with $T \leq \aleph_1$ and size $\leq \aleph_2$ is powerfully Lindelöf.

**Proof.** Take a countably closed elementary submodel $M$ of $H_\theta$ of size $\aleph_2$, $\theta$ sufficiently large and regular, with $X$ and its topology in $M$. Without loss of generality, assume $X \subseteq M$. $X_M$ [17] is the topology on $X \cap M$ generated by $\{U \cap M : U \in M, U \text{ open in } X\}$. In the special case of $X \subseteq M$, $X_M$ is just a weaker topology on $X$.

**Lemma 11** [17]. For $X$ of countable type, $X_M$ is a perfect image of a subspace of $X$. Furthermore, each $x \in X_M$ is a member of its pre-image.

It follows that if $X$ is of countable type and $X \subseteq M$, then $X_M$ is a perfect image of $X$.

**Lemma 12.** Perfect maps preserve countable type.

This is probably due to Arhangel’ski˘ı; it is quoted without attribution in [6]. It follows that a Lindelöf $\leq \aleph_1$-productive space $X$ of countable type and size $\leq \aleph_2$ will map onto a Lindelöf $\leq \aleph_1$-productive $X_M$ of countable type and weight $\leq \aleph_2$, which is then powerfully Lindelöf. As in [29], we argue that $(X^\omega)_M = (X_M)^\omega$. If there were an open cover $U$ of $X^\omega$ without a countable subcover, there would be one in $M$. Then $\{U \cap M : U \in M\}$ would cover $(X^\omega)_M$. Take $U' = \{U_n : n < \omega\} \subseteq U \cap M$, a countable subcover. $M$ is countably closed so $U' \in M$. Then $M \models U'$ covers $X^\omega$, so $U'$ does cover $X^\omega$. □

### 3 Rothberger and indestructible spaces and the $\aleph_1$-Borel Conjecture

**Definition.** A space is **Rothberger** if for each sequence $\{U_n\}_{n<\omega}$ of open covers of $X$, there are $U_n \in U_n$ such that $\{U_n : n < \omega\}$ is an open cover. Equivalently [23], if Player ONE does not have a winning strategy in the game $G_1^\omega(\mathcal{O}, \mathcal{O})$ in which in inning $n$, ONE picks an open cover and Player
TWO picks an element of it, with ONE winning if the selections do not form an open cover. A space is indestructible if it generates a Lindelöf topology in any countably closed forcing extension. Equivalently [25] if ONE does not have a winning strategy in the $\omega_1$-length game $G_{\omega_1}(\mathcal{O},\mathcal{O})$ defined analogously to $G_1(\mathcal{O},\mathcal{O})$.

**Definition.** A space $X$ is projectively countable if whenever $f : X \to Y$, $Y$ separable metrizable (equivalently, $Y = \mathbb{R}$ or $Y = [0,1]^{\omega_2}$, or etc.), $f(X)$ is countable. Projectively $\sigma$-compact is defined similarly. $X$ is projectively $\aleph_1$ if whenever $f : X \to [0,1]^{\omega_1}$, then $|f(X)| \leq \aleph_1$.

Projectively countable Lindelöf spaces are Rothberger [19], [5], [26]; in fact

**Proposition 13** [22]. Borel’s Conjecture is equivalent to the assertion that a space is Rothberger if and only if it is projectively countable.

Surprisingly, productive Lindelöfness can substitute for Borel’s Conjecture:

**Theorem 14.** Suppose $X$ is a productively Lindelöf Rothberger space. Then $X$ is projectively countable.

**Proof.** By Corollary [3] it suffices to show that if $f : X \to \mathbb{R}$, then compact subspaces of $f(X)$ are countable. If such a subspace were uncountable, it would include a perfect subset and hence a copy of the Cantor set. But then a closed, hence Rothberger subset of $X$ would map onto the Cantor set, which is not Rothberger.

Since indestructibility is the game version of Rothberger up one cardinal, it is reasonable to see whether Proposition [13] and Theorem [14] have generalizations to indestructibility. One difficulty we should first dispose of is the question of whether indestructibility is the right generalization of Rothberger for this context, or whether it is more appropriate to consider the selection principle variation:

**Definition.** A space is $\omega_1$-Rothberger if whenever $\{U_\alpha\}_{\alpha < \omega_1}$ are open covers, there is a selection $U_\alpha \in U_\alpha$, $\alpha < \omega_1$, such that $\bigcup\{U_\alpha : \alpha < \omega_1\}$ is a cover.
In [25], Scheepers and Tall ask whether \( \omega_1 \)-Rothberger is the same as indestructible. The latter easily implies the former but Dias and Tall [10] exhibit a destructible Lindelöf space which, under CH, is \( \omega_1 \)-Rothberger.

**Example 1.** The lexicographic order topology on \( 2^{\omega_1} \) is a compact destructible (see [10]) space of size \( 2^{\aleph_1} \) and weight \( 2^{\aleph_0} \), with no isolated points, which does not include a copy of \( 2^{\omega_1} \) and indeed does not even have a closed subset mapping onto \( 2^{\omega_1} \). CH implies the space is \( \omega_1 \)-Rothberger [10].

Under CH then, there is a space which is \( \omega_1 \)-Rothberger but also not projectively \( \aleph_1 \). Thus “indestructibility” is the appropriate generalization of “Rothberger” to use in attempting to generalize Proposition 13. Let us make the following definition:

**Definition** [30]. The \( \aleph_1 \)-Borel Conjecture is the assertion that a Lindelöf space is indestructible if and only if it is projectively \( \aleph_1 \).

There have been several quite different attempts to generalize Borel’s Conjecture - see [8], [12], [14].

**Proposition 15** [30]. Lévy-collapse an inaccessible cardinal to \( \aleph_2 \). Then CH and the \( \aleph_1 \)-Borel Conjecture hold.

In fact (see below), the \( \aleph_1 \)-Borel Conjecture implies CH. The inaccessible is necessary [10]; see below.

It should be straightforward to generalize Theorem [14] (possibly assuming CH) to obtain something like:

\[
\star \qquad \leq \aleph_1 \text{-productive indestructible spaces are projectively } \aleph_1.
\]

In fact, as we shall see below (Corollary [17]), this is consistently false.

It is instructive to see what happens when one naively tries to prove \( \star \) by stepping up the proof of Theorem [14] one cardinal, replacing the Cantor set by a copy of \( 2^{\omega_1} \). A crucial step in the proof fails: Example 1 is a space of size \( 2^{\aleph_1} \) without isolated points, which does not include a copy of \( 2^{\omega_1} \), yet under CH has weight \( \aleph_1 \). As an ordered space, this space is hereditarily normal, so by Šapirovskii’s mapping theorem (see e.g. [15]) cannot have a closed subspace mapping onto \( 2^{\omega_1} \).

Given that this attempt to generalize the proof of Theorem [14] in order to obtain \( \star \) fails, is there another way to get it? Well, of course the \( \aleph_1 \)-Borel Conjecture trivially implies \( \star \), but is that extra hypothesis necessary? It is:
Proposition 16. If $\aleph_2$ is not inaccessible in $L$, there is a compact indestructible space of weight $\aleph_1$ and cardinality greater than $\aleph_1$.

Corollary 17. If $\aleph_2$ is not inaccessible in $L$, there is an $L$-productive indestructible space which is not projectively $\aleph_1$.

Proof. The example of Proposition 16 is the compact line (which has weight $\aleph_1$) obtained from a Kurepa tree [31]. Compact spaces are obviously $L$-productive. Spaces of weight $\aleph_1$ are embeddable in $[0, 1]^{\omega_1}$. □

It follows that (17) is equiconsistent with the apparently stronger $\aleph_1$-Borel Conjecture, for if (17) holds, $\aleph_2$ is inaccessible in $L$ and so we can obtain that Conjecture.

Notice incidentally that one could generalize the proof of Theorem 14 if one knew that perfect subspaces of size $\geq \aleph_2$ of $[0, 1]^{\omega_1}$ included destructible compact subspaces. The $\aleph_1$-Borel Conjecture assures this. We can’t do better; the Kurepa line of Proposition 16 is compact indestructible, and hence has every compact subspace indestructible.

One might assume that projectively countable spaces are projectively $\aleph_1$; in fact, this is undecidable!

Example 2. If there is a Kurepa tree without an Aronszajn subtree (as there is in $L$ [9]), then there is a Lindelöf linearly ordered $P$-space ($G_\delta$‘s open) of weight $\aleph_1$ and size $> \aleph_1$ [16]. Such a space is obviously not projectively $\aleph_1$, yet every $P$-space is projectively countable.

On the other hand,

Theorem 18. The $\aleph_1$-Borel Conjecture implies that projectively countable Lindelöf spaces are projectively $\aleph_1$.

Proof. Projectively countable Lindelöf spaces are Rothberger [5], [19], [26] and hence indestructible. By the $\aleph_1$-Borel Conjecture, they are then projectively $\aleph_1$. □

4 The $\aleph_1$-Hurewicz Property

This section was motivated by the idea that, just as Borel’s Conjecture implies that Rothberger spaces are Hurewicz [26], we should be able to prove
Theorem 19. The $\aleph_1$-Borel Conjecture implies that indestructible Lindel"of spaces are $\aleph_1$-Hurewicz.

where $\aleph_1$-Hurewicz is some natural generalization of the usual Hurewicz property. We should also be able to generalize the classic theorem that Hurewicz Čech-complete spaces are $\sigma$-compact so as to have $\aleph_1$-Hurewicz in the hypothesis and $\aleph_1$-compact (the union of $\aleph_1$ compact sets) in the conclusion. We could then prove

Theorem 20. The $\aleph_1$-Borel Conjecture implies that indestructible Lindel"of $\aleph_1$-Čech-complete spaces are $\aleph_1$-compact.

where “$\aleph_1$-Čech-complete” is a natural generalization defined below of “Čech-complete”.

There are several equivalent definitions of the Hurewicz property. See e.g. [28], [20]. We will use the following generalization of one such equivalent as our definition of $\aleph_1$-Hurewicz, because it enables us to prove Theorems 19 and 20.

Definition. A Lindel"of space is $\aleph_1$-Hurewicz if whenever \( \{U_\alpha : \alpha < \omega_1\} \) are open sets in $\beta X$ including $X$, there are closed sets \( \{F_\alpha : \alpha < \omega_1\} \) in $\beta X$ such that $X \subseteq \bigcup \{F_\alpha : \alpha < \omega_1\} \subseteq \bigcap \{U_\alpha : \alpha < \omega_1\}$.

Definition. A space is $\aleph_1$-compact if it is the union of $\aleph_1$ compact sets.

Note: “$\aleph_1$-compact” used to mean what is now called “countable extent”. It seems appropriate to repurpose the term.

Definition. A space $X$ is $\aleph_1$-Čech-complete if there are open covers \( \{U_\alpha\}_{\alpha<\omega_1} \) of $X$ such that any centered family of closed sets which, for each $\alpha$, contains a closed set included in some member of $U_\alpha$ has non-empty intersection.

Theorem 21. $\aleph_1$-Hurewicz, $\aleph_1$-Čech-complete spaces are $\aleph_1$-compact.

Proof. A routine generalization of the standard proof (see e.g. [11]) that Čech-complete spaces are $G_\delta$'s in their Stone-Čech compactifications establishes that an $\aleph_1$-Čech-complete space is a $G_{\aleph_1}$, i.e., is the intersection of $\aleph_1$ open sets in its Stone-Čech compactification. The theorem follows immediately. \qed
Definition. A space is **projectively \( \aleph_1 \)-compact** (**projectively \( \aleph_1 \)-Hurewicz**) if its continuous image in \([0,1]^{\omega_1}\) is always \( \aleph_1 \)-compact (\( \aleph_1 \)-Hurewicz).

The following standard fact follows, e.g., from Lemma 1.0 in [20].

**Lemma 22.** Let \( U \) be an open cover of a regular Lindelöf space \( X \). Then there exists a continuous function \( f : X \to \mathbb{R} \) such that \( f^{-1}([-n,n]) \) is included in a finite union of elements of \( U \) for every \( n \in \omega \).

Let us note that in the definition of the \( \aleph_1 \)-Hurewicz property the Stone-Čech compactification \( \beta X \) may be replaced by any other one.

**Theorem 23.** Every Lindelöf projectively \( \aleph_1 \)-Hurewicz space is \( \aleph_1 \)-Hurewicz.

*Proof.* Let \( X \) be a Lindelöf projectively \( \aleph_1 \)-Hurewicz space and \( \{W_\alpha : \alpha < \omega_1\} \) be a collection of open subsets of \( \beta X \) including \( X \). For every \( \alpha \) fix a cover \( U_\alpha \) of \( X \) by open subsets of \( \beta X \) whose closures are subsets of \( W_\alpha \). Set \( U'_\alpha = \{U \cap X : U \in U_\alpha\} \). By Lemma 22 for every \( \alpha \) there exists a continuous function \( f_\alpha : X \to \mathbb{R} \) such that \( f_\alpha^{-1}([-n,n]) \) is included in a union of finitely many elements of \( U'_\alpha \) for all \( n \). Now set \( f : X \to \mathbb{R}^{\omega_1}, f(x)(\alpha) = f_\alpha(x) \). Since \( \mathbb{R}^{\omega_1} \) is homeomorphic to a \( G_{\aleph_1} \)-subset of \([0,1]^{\omega_1}\) and \( X \) is projectively \( \aleph_1 \)-Hurewicz, there exists a collection \( K \) of compact subsets of \( \mathbb{R}^{\omega_1} \) such that \( |K| \leq \aleph_1 \) and \( f(X) \subseteq \bigcup K \). Therefore \( X \subseteq \bigcup_{K \in K} f^{-1}(K) \). It also follows from the above that for every \( K \in K \) and \( \alpha \in \omega_1 \) the preimage \( f^{-1}(K) \) is included in a finite union of elements of \( U'_\alpha \), and hence its closure in \( \beta X \) is included in \( \bigcap_{\alpha \in \omega_1} W_\alpha \). Thus

\[
X \subseteq \bigcup_{K \in K} \text{cl}_{\beta X} f^{-1}(K) \subseteq \bigcap_{\alpha < \omega_1} W_\alpha,
\]

which completes our proof.

**Corollary 24.** Lindelöf projectively \( \aleph_1 \) spaces are \( \aleph_1 \)-Hurewicz.

Theorems 19 and 20 follow immediately.

The reason we are interested in \( \aleph_1 \)-compactness is because by Lemma 9 we have:

**Lemma 25** [7]. **CH implies productively Lindelöf \( \aleph_1 \)-compact spaces are powerfully Lindelöf.**
This result could be used to establish that \( CH \) implies productively Lindelöf Čech-complete spaces are powerfully Lindelöf, since Lindelöf Čech-complete spaces are perfect preimages of separable metrizable spaces, which latter have cardinality \( \leq 2^{\aleph_0} \), but we shall not do so because it is known without \( CH \) that Lindelöf Čech-complete (indeed \( p \)-)spaces are powerfully Lindelöf.

Incidentally, let us mention:

**Theorem 26.** The \( \aleph_1 \)-Borel Conjecture implies Lindelöf Čech-complete spaces are \( \aleph_1 \)-compact.

**Proof.** The \( \aleph_1 \)-Borel Conjecture implies \( CH \), since \([0,1]^{\omega_1}\) is indestructible and has weight \( \leq \aleph_1 \). A Lindelöf Čech-complete space is a perfect preimage of a separable metric space, and hence is the union of \( \leq 2^{\aleph_0} \) compact sets. \( \square \)

A straightforward generalization of known results is:

**Theorem 27.** \( 2^{\aleph_1} = \aleph_2 \) implies \( \aleph_1 \)-productive spaces are projectively \( \aleph_1 \)-compact.

**Proof.** Let \( X \) be the continuous image in \([0,1]^{\omega_1}\) of an \( \aleph_1 \)-L-productive space. Then \( X \) is \( \aleph_1 \)-L-productive. If \( X \) is not \( \aleph_1 \)-compact, \( Y = [0,1]^{\omega_1} - X \) is not the intersection of \( \aleph_1 \) open sets. Since \( w(Y) \leq \aleph_1 \) by assumption there are \( \leq \aleph_2 \) open sets about \( Y \) such that every open set about \( Y \) includes one. We may therefore form a strictly decreasing \( \omega_2 \)-sequence \( \{G_\alpha : \alpha < \omega_2\} \) of intersections of \( \aleph_1 \) open subsets of \([0,1]^{\omega_1}\) about \( Y \). Pick \( z_\alpha \in (G_\alpha - G_{\alpha+1}) \cap X \). Take \( Z = Y \cup \{z_\beta : \beta < \omega_2\} \) and make each \( z_\beta \) isolated. Then \( L(Z) \leq \aleph_1 \), but \( L(X \times Z) > \aleph_1 \), contradiction. \( \square \)

Clearly projectively \( \aleph_1 \) implies projectively \( \aleph_1 \)-compact implies projectively \( \aleph_1 \)-Hurewicz.

Recall the space obtained from a Kurepa tree with no Aronszajn subtree (Example 2). Since \( P \)-spaces are projectively countable, this is an example of a projectively countable Lindelöf space which is not projectively \( \aleph_1 \). In fact, it is not projectively \( \aleph_1 \)-compact. To see this, note that its weight is \( \aleph_1 \), so it is embedded in \([0,1]^{\omega_1}\). But compact \( P \)-spaces are finite. \( \square \)

Lindelöf \( P \)-spaces are Rothberger and hence indestructible 25, but the \( \aleph_1 \)-Borel Conjecture is unavailable, so it is not immediately obvious whether or not this space \( Y \) is (projectively) \( \aleph_1 \)-Hurewicz. It is Hurewicz, since Lindelöf \( P \)-spaces are Hurewicz 25.
We could use Theorems 19 and 21 and Lemma 25 to prove that indestructible, productively Lindelöf, $\aleph_1$-Čech-complete spaces are powerfully Lindelöf, assuming the $\aleph_1$-Borel Conjecture, but we can do better:

**Theorem 28.** Assume CH. Suppose that $X$ is a regular $\aleph_1$-Čech-complete space which is productively Lindelöf. Then $X$ is powerfully Lindelöf.

**Proof.** Let $\langle U_\alpha : \alpha < \omega_1 \rangle$ be a sequence of open covers of $X$ witnessing its $\aleph_1$-Čech-completeness. Without loss of generality, each $U_\alpha$ is locally finite and countable. Let us write $U_\alpha$ in the form $\{ U_\alpha n : n \in \omega \}$ and consider the relation

$$R = \{ (r, x) \in \omega^{\omega_1} \times X : x \in \bigcap_{\alpha \in R_1} U_\alpha r(\alpha) \}.$$

**Claim 29.** The set-valued map $R_1 : X \to \omega^{\omega_1}$ assigning to $x \in X$ the set $\{ r \in \omega^{\omega_1} : (r, x) \in R \}$ is compact-valued and upper semicontinuous.

**Proof.** It is clear that $R_1(x)$ is closed in $\omega^{\omega_1}$ for all $x \in X$. Moreover, since every $U_\alpha$ is locally finite, we conclude that the set $\{ r(\alpha) : r \in R_1(x) \}$ is finite for every $x \in X$ and $\alpha \in \omega_1$. Thus $R_1$ is compact-valued.

Now let $O \subseteq \omega^{\omega_1}$ be an open set including $R_1(x)$ for some $x \in X$. Passing to a subset of $O$ including $R_1(x)$, if needed, we may additionally assume that $O = \bigcup \{ [s] : s \in \text{pr}_A(R_1(x)) \}$ for some $F \in [\omega_1]^{<\omega}$, where $\text{pr}_A : \omega^{\omega_1} \to \omega^A$ is the natural projection map for every $A \subseteq \omega_1$ and $[s] = \{ r \in R_1(X) : r \upharpoonright F = s \}$ for all $s \in \omega^F$. Set

$$U = X \setminus \bigcup_{\alpha \in F} \{ \bigcap_{\alpha \in F} U_\alpha t(\alpha) : t \in \omega^F \setminus \text{pr}_F(R_1(x)) \}.$$

Then $x \in U$. Moreover, since all $U_\alpha$’s are locally finite, so is the family $\{ \bigcap_{\alpha \in F} U_\alpha t(\alpha) : t \in \omega^F \}$, and hence $U$ is open. A direct verification shows that $R_1(y) \subseteq O$ for all $z \in U$, which completes our proof.

**Claim 30.** The set-valued map $R_t : R_1(X) = \bigcup_{x \in X} R_t(x) \to X$ assigning to $r$ the set $\bigcap_{\alpha \in \omega_1} U_\alpha r(\alpha) = \{ x \in X : (r, x) \in R \}$ is compact-valued and upper semicontinuous.

**Proof.** Given any $r \in R_t(X)$ let us observe that $R_t(r)$ is closed in $X$ and therefore Lindelöf. If $R_t(r)$ is not compact then there exists a decreasing sequence $\langle Z_n : n \in \omega \rangle$ of closed subsets of $R_t(r)$ with empty intersection. Set
\[ C_n = \overline{U_r(n)} \cap Z_n \text{ for } n \in \omega \text{ and } C_\alpha = \overline{U_r(\alpha)} \text{ for } \alpha \in \omega_1 \setminus \omega. \] Then the sequence \( \langle C_\alpha : \alpha \in \omega_1 \rangle \) is centered, \( C_\alpha \) is included in the closure of some element of \( U_\alpha \) for all \( \alpha \), and \( \bigcap_{\alpha \in \omega_1} C_\alpha = \emptyset \), a contradiction.

Let us fix an open neighborhood \( U \) of \( R_r(r) \). Then there exists a finite \( F \subseteq \omega_1 \) such that \( \bigcap_{\alpha \in F} \overline{U_r(\alpha)} \subseteq U \), as otherwise the family \( \{ \bigcap_{\alpha \in F} U_r(\alpha) \setminus U : F \in [\omega_1]^{<\omega} \} \) would be centered and have empty intersection, thus contradicting the choice of the sequence \( \langle U_\alpha : \alpha < \omega_1 \rangle \). It follows from the above that \( R_r([r \upharpoonright F]) \subseteq U \), which completes the proof. \( \square \)

We are now in a position to finish the proof of Theorem 28. Since \( X \) is productively Lindelöf and \( R_l \) is compact-valued and upper semicontinuous, so is its image \( R_l(X) \subseteq \omega^{\omega_1} \). By CH and Lemma 4 all productively Lindelöf spaces of weight \( \aleph_1 \) are powerfully Lindelöf, and hence so is \( R_l(X) \). Since \( R_r \) is compact-valued and upper semicontinuous and \( X = R_r(R_l(X)) \), we conclude that \( X \) is powerfully Lindelöf as well. \( \square \)

5 Projective \( \sigma \)-compactness in finite powers does not imply productive Lindelöfness

E. A. Michael [21] proved under CH that productively Lindelöf spaces are projectively \( \sigma \)-compact. In fact, since finite powers of productively Lindelöf spaces are productively Lindelöf, under CH they are projectively \( \sigma \)-compact. It is natural to wonder whether having finite powers projectively \( \sigma \)-compact is sufficient, perhaps assuming CH, to conclude productive Lindelöfness. It isn’t; an example of Todorcevic [32] will establish this.

Example 3. There is a \( \gamma \)-space which has all finite powers projectively countable but is not productively Lindelöf.

Recall the definition of a \( \gamma \)-space:

**Definition** [13]. A cover of \( X \) is an \( \omega \)-**cover** if each finite subset of \( X \) is included in some member of the cover. A space is a \( \gamma \)-**space** if for every open \( \omega \)-cover \( U \), there is a sequence of elements of \( U \), \( U_n \), \( n < \omega \), such that every member of \( X \) is in all but finitely many \( U_n \)'s.

\( \gamma \)-spaces are Lindelöf; in fact, finite powers of \( \gamma \)-spaces are \( \gamma \) [18]. It is easy to see that every continuous image of a \( \gamma \)-space is a \( \gamma \)-space.
Lemma 31. All metrizable $\gamma$-spaces are zero-dimensional.

Proof. By II.3.3 and II.3.6 respectively of [3], $\gamma$-spaces are $\phi$-spaces, and $\phi$-spaces have small inductive dimension 0. But for Lindelöf metrizable spaces, that is the same as being 0-dimensional.

In [32], Todorcevic constructs a stationary Aronszajn line which is $\gamma$, projectively countable, and not productively Lindelöf. We claim that his space actually has all finite powers Lindelöf and projectively countable. It will suffice to prove the following claim, for then all even powers of $X$ – and hence all finite powers of $X$ – are projectively countable.

Claim 32. If $X$ is projectively countable, $X^2$ is Lindelöf, and all continuous metrizable images of $X^2$ are zero-dimensional, then $X^2$ is projectively countable.

Proof. Let $f : X^2 \rightarrow Y$ be a continuous map for some metrizable $Y$. We need to show that $f[X^2]$ is countable. Without loss of generality, $f$ is surjective, and hence $Y$ is zero-dimensional and has countable weight. Let $\mathcal{B}$ be a countable base of $Y$ consisting of clopen sets and $\mathcal{C} = \{f^{-1}(B) : B \in \mathcal{B}\}$. Then every $C \in \mathcal{C}$ is a clopen subset of $X^2$, and thus it may be written as a union $\bigcup_{i \in I_C} U_i \times V_i$ for some clopen subsets $U_i, V_i$ of $X$. Since $X^2$ is Lindelöf, we can assume that each $I_C$ is countable. It will be also convenient for us to assume that $I_{C_0} \cap I_{C_1} = \emptyset$ if $C_0 \neq C_1$. Set $I = \bigcup_{C \in \mathcal{C}} I_C$ and consider maps $g, h : X \rightarrow 2^I$ defined as follows: $g(x)(i) = 1$ (respectively $h(x)(i) = 1$) if and only if $x \in U_i$ (respectively $x \in V_i$). It follows from the above that $g$ and $h$ are continuous. Since $I$ is countable, $g[X]$ and $h[X]$ are countable as well. We claim that if $g(x_0) = g(x_1)$ and $h(y_0) = h(y_1)$ then $f(x_0, y_0) = f(x_1, y_1)$. (This easily implies that $f[X^2]$ is countable.) Suppose that $f(x_0, y_0) \neq f(x_1, y_1)$. Then there exists $B \in \mathcal{B}$ such that $f(x_0, y_0) \in B$ but $f(x_1, y_1) \notin B$. Then $(x_0, y_0) \in C$ but $(x_1, y_1) \notin C$, where $C = f^{-1}(B) \in \mathcal{C}$. Let $i \in I_C$ be such that $(x_0, y_0) \in U_i \times V_i$ and notice that $(x_1, y_1) \notin U_i \times V_i$. This means that either $x_1 \notin U_i$ or $y_1 \notin V_i$. In the first case we have $g(x_0)(i) \neq g(x_1)(i)$, while in the second case $h(y_0)(i) \neq h(y_1)(i)$. In any case, $(g(x_0), h(y_0)) \neq (g(x_1), h(y_1))$, which completes our proof.

We do not know whether Todorcevic’s space is powerfully Lindelöf. If it is, it would show that even the addition of “powerfully Lindelöf” to “projectively $\sigma$-compact in finite powers” would fail to characterize productive Lindelöfness.

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