TORSION OF INJECTIVE MODULES AND WEAKLY PRO-REGULAR SEQUENCES

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ABSTRACT. Let $R$ a commutative ring, $a \subset R$ an ideal, $I$ an injective $R$-module and $S \subset R$ a multiplicatively closed set. When $R$ is Noetherian it is well-known that the $a$-torsion sub-module $\Gamma_a(I)$, the factor module $I/\Gamma_a(I)$ and the localization $I_S$ are again injective $R$-modules. We investigate these properties in the case of a commutative ring $R$ by means of a notion of relatively-$a$-injective $R$-modules. In particular we get another characterization of weakly pro-regular sequences in terms of relatively injective modules. Also we present examples of non-Noetherian commutative rings $R$ and injective $R$-modules for which the previous properties do not hold. Moreover, under some weak pro-regularity conditions we obtain results of Mayer-Vietoris type.

INTRODUCTION

In this note $a$ denotes an ideal of a commutative ring $R$. The torsion functor with respect to $a$ is denoted by $\Gamma_a(\cdot)$, that is $\Gamma_a(M) = \{m \in M \mid a^t m = 0 \text{ for some } t > 0\}$. We say that the $R$-module $M$ is $a$-torsion if $M = \Gamma_a(M)$. The right-derived functors of $\Gamma_a(\cdot)$, the local cohomology functors, are denoted by $H^t_a(\cdot)$. When the ring is Noetherian it is well-known that for every injective $R$-module $I$, its $a$-torsion sub-module $\Gamma_a(I)$ is again injective. Moreover, $I/\Gamma_a(I)$ and $I_S$ are again injective $R$-modules and the natural map $I \to I_S$ is surjective. Here $S \subset R$ denotes a multiplicatively closed set. This is no longer the case in general. The main aim of the present paper is to analyse these conditions in the more general case of a commutative ring $R$.

When the ring $R$ is Noetherian it is known that the pair $(R, a)$ has the following property (referred to as property $B$): for all $a$-torsion $R$-module $M$ and all $i > 0$ we have that $H^t_a(M) = 0$. But the case of a Noetherian ring is not the only one with this property. For an ideal $a$ of a commutative ring $R$ we show in [1, 6] that the pair $(R, a)$ has the property $B$ if and only if for all relatively-$a$-injective $R$-module $J$ it holds that $H_a^t(J)$ is again relatively-$a$-injective. Here an $R$-module $M$ is called relatively-$a$-injective if $\text{Ext}^t_a(R/b, M) = 0$ for all ideal $b$ containing a power of $a$ and all $i \geq 1$. We also provide an example of an injective $R$-module $I$ such that $\Gamma_a(I)$ is not relatively-$a$-injective (hence also not injective). The more restricted question to know when $\Gamma_a(I)$ is injective has been investigated by Quy and Rohrer in [8].

When the ideal $a$ is finitely generated, say by a sequence $\underline{a} = x_1, \ldots, x_h$, we also show in [2, 3] that the pair $(R, a)$ has the property $B$ if and only if the sequence $\underline{a}$ is weakly pro-regular (see [2], [10] and [11] for more details on the notion of weakly pro-regular sequences). That is, we prove another characterization of weakly pro-regular sequences (see also [24]). Note that the notion of weakly pro-regular sequences plays also an essential rôle in the study of both completion and local cohomology (see [11] for more details).

Now let $a$ and $b$ be two finitely generated ideals of a commutative ring $R$. Under some weak pro-regularity conditions we obtain in [3, 4] a long exact Mayer-Vietoris sequence involving these ideals. Under the same conditions, we then prove in [3, 7] that the set of ideals $\{a^n \cap b^n \mid n \geq$
1} defines the \((a \cap b)\)-adic topology (for a Noetherian ring, this also follows by the Artin-Rees Lemma).

After some investigations about the natural map \(M \to M_S\) we provide in [15] another example of an injective \(R\)-module \(I\) and a multiplicatively closed set \(S \subset R\) such that the localization \(I_S\) is not injective. That this does not hold in general was first shown by Dade in [4]. Note that the problems to know when \(I_S\) or \(\Gamma_a(I)\) are injective for an injective module \(I\) seem to be related, at least when \(S\) consists of the powers of a single element \(x\). In that case, we also show in [14] that the natural map \(M \to M_S\) is surjective if and only if \(M/\Gamma_{xR}(M)\) is relatively-\(xR\)-injective.

In a final section we relate the \(a\)-transform \(\mathcal{D}_a(M)\) of a module \(M\) to some module of fractions. When the ideal \(a\) is finitely generated, say by a sequence \(a = x_1, \ldots, x_k\), we provide in [5,3] and [5,4] natural injections \(\mathcal{D}_a(M) \hookrightarrow H^0(\check{\mathcal{D}}_a^1 \otimes_R M)\) and \(H^0_\mathcal{D}(M) \hookrightarrow H^1(\check{C}^\mathcal{C}_\mathcal{D} \otimes_R M)\), where \(\check{\mathcal{D}}_a^1\) and \(\check{C}^\mathcal{C}_\mathcal{D}\) denote respectively the global Čech complex and the Čech complex. It follows that the deviation of the zero-th global Čech cohomology from the ideal transform is isomorphic to the deviation of the first local Čech cohomology from the first local cohomology. We provide an example for which these deviations do not vanish, and we note that both deviations vanish when the sequence \(a\) is weakly pro-regular.

For commutative algebra we refer to [7], for homological algebra we also refer to [14].

1. RELATIVELY \(a\)-INJECTIVE MODULES

Let \(I\) be an injective module over the commutative ring \(R\) and let \(a \subset R\) denote an ideal. To obtain some informations on the submodule \(\Gamma_a(I)\) a relative notion will be useful.

**Definition 1.1.** Let \(a\) be an ideal of a commutative ring \(R\). We say that an \(R\)-module \(M\) is relatively-\(a\)-injective if \(\text{Ext}^i_R(R/b, M) = 0\) for all ideal \(b\) containing a power of \(a\) and all \(i \geq 1\).

**Remarks 1.2.** (a) For a relatively-\(a\)-injective \(R\)-module \(M\) we note that \(H^i_a(M) = 0\) for all \(i \geq 1\). That is because in general we have \(H^i_a(M) = \varinjlim \text{Ext}^i_R(R/a^i, M)\).

(b) Let \(0 \to M' \to M \to M'' \to 0\) be a short exact sequence of \(R\)-modules. If both \(M'\) and \(M\) are relatively-\(a\)-injective, then so is \(M''\), as is easily seen by the long exact sequence of the \(\text{Ext}^i_R(R/b, \cdot)\), where \(b\) is any ideal containing a power of \(a\). In particular the short sequences

\[
0 \to \text{Hom}_R(R/b, M') \to \text{Hom}_R(R/b, M) \to \text{Hom}_R(R/b, M'') \to 0
\]

and

\[
0 \to \Gamma_a(M') \to \Gamma_a(M) \to \Gamma_a(M'') \to 0
\]

are again exact.

(c) By a standard cohomological argument it follows that the local cohomology of an \(R\)-module \(M\) may be computed with a right resolution \(I^*\) of \(M\) consisting of relatively-\(a\)-injective \(R\)-modules: for such a resolution we have \(H^i(\Gamma_a(I^*)) \cong H^i_a(M)\).

(To see this let \(I^*\) be an injective resolution of \(M\). Then there is a quasi-isomorphism \(f : I^* \to I^*\) and the cone \(C(f)\) is a left-bounded exact complex of relatively-\(a\)-injective \(R\)-modules. By view of (b) we have that \(\Gamma_a(C(f))\) is exact. Hence \(\Gamma_a(f) : \Gamma_a(I^*) \to \Gamma_a(I^*)\) is a quasi-isomorphism. Or see e.g. [12] for more details around cohomological arguments.)

Similarly note also that \(\text{Ext}^i_R(R/b, M) \cong H^i(\text{Hom}_R(R/b, I))\), where \(b\) is any ideal containing a power of \(a\).

For relatively \(a\)-injective modules the \(\text{Ext}\)-depth and the \(\text{Tor}\)-codepth with respect to \(a\) come also into play.

**Definitions 1.3.** (see also [12] and [11]) Let \(a\) be an ideal of a commutative ring \(R\) and \(M\) an \(R\)-module. The \(\text{Ext}\)-depth and the \(\text{Tor}\)-codepth of \(M\) with respect to \(a\) are defined respectively
Ext  

Corollary 1.5. Let module. Assume that the ideal relatively-

\( \text{E-dp}(a, M) = \inf \{i \mid \Ext^i_R(R/a, M) \neq 0\} \)

\( \text{T-codp}(a, M) = \inf \{i \mid \Tor^i_R(R/a, M) \neq 0\} \).

where the infimum is taken over the ordered set \( \mathbb{N} \cup \{\infty\} \). Therefore \( \text{E-dp}(a, M) = \infty \) means that \( \Ext^i_R(R/a, M) = 0 \) for all \( i \geq 0 \).

Recall that \( \text{E-dp}(a, M) = \inf \{i \in \mathbb{N} \mid H^i_{\text{gp}}(M) \neq 0\} \) and that

\[ \text{E-dp}(a, M) = \text{E-dp}(a^n, M) \leq \text{E-dp}(b, M) \]

\[ \text{T-codp}(a, M) = \text{T-codp}(a^n, M) \leq \text{T-codp}(b, M) \]

for all \( n \geq 1 \) and any ideal \( b \) containing a power \( a^t \) of \( a \) (see [12] Propositions 5.3.15, 5.3 16, 5.3.11] and [11] Chapters 3 and 5).

When the ideal \( a \) is finitely generated recall also that

\( \text{E-dp}(a, M) = \infty \) if and only if \( \text{T-codp}(a, M) = \infty \),

see [12] 6.1.8 or [11] Chapter 5] for a complex version.

In this section we investigate the following question: Given an ideal \( a \) of a commutative ring \( R \) and a relatively-a-injective \( R \)-module \( J \), when do we have that \( \Gamma_a(J) \) also is relatively-a-injective? We may also wonder when the quotient \( J/\Gamma_a(J) \) is relatively-a-injective. It turns out that both of these properties are equivalent.

**Proposition 1.4.** Let \( a \) be an ideal of the commutative ring \( R \) and let \( J \) denote a relatively-a-injective \( R \)-module. The following conditions are equivalent:

(i) \( \Gamma_a(J) \) is relatively-a-injective.

(ii) \( J/\Gamma_a(J) \) is relatively-a-injective.

(iii) \( \text{E-dp}(a, J/\Gamma_a(J)) = \infty \).

**Proof.** We consider the exact sequence

\[ 0 \rightarrow \Gamma_a(J) \rightarrow J \rightarrow J/\Gamma_a(J) \rightarrow 0 \]

and note that the implication (i)\( \Rightarrow \) (ii) follows by the remark in [13] (b).

For the remaining part of the proof we first note that

\[ \text{Hom}_R(R/a, \Gamma_a(J)) \cong \text{Hom}_R(R/a, J) \quad \text{and} \quad \text{Hom}_R(R/a, J/\Gamma_a(J)) = 0. \]

Then we consider the long exact sequence of the \( \Ext^i_R(R/a, -) \) applied to the above short exact sequence. If \( J/\Gamma_a(J) \) is relatively-a-injective, it yields by the previous remark that \( \Ext^i_R(R/a, J/\Gamma_a(J)) = 0 \) for all \( i \geq 0 \), i.e. \( \text{E-dp}(a, J/\Gamma_a(J)) = \infty \) by the definition.

Now assume \( \text{E-dp}(a, J/\Gamma_a(J)) = \infty \), so that \( \Ext^i_R(R/b, J/\Gamma_a(J)) = 0 \) for any ideal \( b \) containing a power of \( a \) and all \( i \geq 0 \) (see [13]). By view of the long exact cohomology sequence of the \( \Ext^i_R(R/b, -) \) we get the isomorphisms \( \Ext^i_R(R/b, \Gamma_a(J)) \cong \Ext^i_R(R/b, J) \) for all \( i \geq 0 \). Whence \( \Ext^i_R(R/b, \Gamma_a(J)) = 0 \) for all \( i > 0 \) since \( J \) is relatively-a-injective. This finishes the proof. \( \square \)

**Corollary 1.5.** Let \( a \) be an ideal of the commutative ring \( R \) and let \( J \) be a relatively-a-injective \( R \)-module. Assume that the ideal \( a \) is finitely generated, say by a sequence \( x = x_1, \ldots, x_k \). If \( \Gamma_a(J) \) is relatively-a-injective, then

\[ J = (x_1^{n_1}, \ldots, x_k^{n_k})J + \Gamma_a(J) \]

for every \( k \)-tupel \( (n_1, \ldots, n_k) \in (\mathbb{N}_+)^k \).
Proof. Write $a^n$ for the ideal generated by $x_1^{n_1}, \ldots, x_k^{n_k}$ for the $k$-tuple $n = (n_1, \ldots, n_k)$. Note that $a^n$ contains a power of $a$. If $\Gamma_a(J)$ is relatively-$a$-injective, then $\text{E-dp}(a, J/\Gamma_a(J)) = \infty$ (see [1.4]). Whence

$$\text{T-codp}(a, J/\Gamma_a(J)) = \infty = \text{T-codp}(a^n, J/\Gamma_a(J))$$

(see [1.3]). In particular $J/\Gamma_a(J) = a^n \cdot (J/\Gamma_a(J))$ and the conclusion follows. \hfill \square

In the case of a singly generated ideal see also the more precise [4.4] Here is the main result of this section. It is a refinement of [11] Proposition 2.7.10.

**Proposition 1.6.** Let $a$ denote any ideal of the commutative ring $R$. The following conditions are equivalent:

(i) $H^i_a(M) = 0$ for all $i > 0$ and any $a$-torsion $R$-modules $M$.

(ii) $\Gamma_a(J)$ is relatively-$a$-injective for any relatively-$a$-injective $R$-module $J$.

(iii) $\Gamma_a(J)$ is relatively-$a$-injective for any injective $R$-modules $J$.

**Proof.** (i)$\Rightarrow$(ii): This is in [11] Proposition 2.7.10. Let us recall the proof. Let $J$ denote a relatively-$a$-injective $R$-module and put $N = J/\Gamma_a(J)$. We consider the short exact sequence

$$0 \rightarrow \Gamma_a(J) \rightarrow J \rightarrow N \rightarrow 0$$

and the associated long exact sequence in local cohomology. By the condition in (i) we have that $H^i_a(\Gamma_a(J)) = 0$ for all $i \geq 1$. Furthermore, by the hypothesis on $J$ we also have $H^i_a(J) = 0$ for all $i \geq 1$ (see [1.2]). By the local cohomology long exact sequence it follows that $H^i_a(N) = 0$ for all $i \geq 0$. Hence

$$\text{E-dp}(a, N) = \inf\{i \in \mathbb{N} \mid \text{Ext}_R^i(R/a, N) \neq 0\} = \infty,$$

see [1.3]. Therefore we get that $\text{Ext}_R^i(R/b, N) = 0$ for all $i \geq 0$ and all ideals $b$ containing a power of $a$ (see again [1.3]). Now the conclusion follows by the long exact sequence of the $\text{Ext}_R^i(R/b, \cdot)$ associated to the above short exact sequence.

(ii)$\Rightarrow$(iii): This is obvious.

(iii)$\Rightarrow$(i): Let $M$ denote an $a$-torsion $R$-module and let $I^0$ denotes its injective hull. There is a short exact sequence

$$0 \rightarrow M \rightarrow \Gamma_a(I^0) \rightarrow M_1 \rightarrow 0$$

where $\Gamma_a(I^0)$ is relatively-$a$-injective by the condition in (iii). Note that the module $M_1$ also is $a$-torsion. We iterate the process and obtain a right resolution $J^\bullet$ of $M$ by means of $a$-torsion relatively-$a$-injective modules. Then, by view of [1.2] we have that

$$H^i_a(M) \cong H^i(\Gamma_a(J^\bullet)) = H^i(J^\bullet) = 0$$

for all $i > 0$. \hfill \square

Here is an explicit example of an injective $R$-module $I$ such that $\Gamma_a(I)$ is not relatively-$a$-injective.

**Example 1.7.** Let $R = \mathbb{k}[[x]]$ denote the power series ring in one variable over the field $\mathbb{k}$. Let $E = E_k(k)$ denote the injective hull of the residue field. Then define $S = R \otimes E$, the idealization of $R$ by its $R$-module $E$. That is, $S = R \oplus E$ as an $R$-module with a multiplication on $S$ defined by $(r, r') \cdot (e, e') = (r'r, re + r'e)$ for all $r, r' \in R$ and $e, e' \in E$. By a result of Faith [3] we have that the commutative ring $S$ is self-injective. More precisely, there is an isomorphism of $S$-modules $\text{Hom}_R(S, E) \cong S$ (see [11] Theorem A.4.6). We consider the ideal $a := (x, 0)S$ of $S$ and note $\Gamma_a(S) = 0 \otimes E$. Then $\Gamma_a(S)$ is not injective as an $S$-module (see [11] 2.8.8).

Here we claim that moreover $\Gamma_a(S)$ is not relatively-$a$-injective as an $S$-module. First note that $a = xR \otimes E$ because $E$ is $x$-divisible (mulplication by $x$ is surjective on $E$). Hence $S/a \cong \mathbb{k}$. 


Since \( S/0 \cong E \cong R \) it follows that the short exact sequence \( 0 \to R \xrightarrow{\alpha} R \to k \to 0 \) is also a short exact sequence of \( S \)-modules. As it is not split exact it yields that \( \text{Ext}_{2}^{1}(k, R) \neq 0 \). Then consider the short exact sequence of \( S \)-modules

\[
0 \to 0 \times E \to S \to R \to 0
\]

and the associated long exact sequence of the \( \text{Ext}_{2}^{1}(k, \cdot) \). It induces the exact sequence

\[
0 = \text{Ext}_{1}^{1}(k, S) \to \text{Ext}_{1}^{1}(k, R) \to \text{Ext}_{2}^{1}(k, 0 \times E).
\]

Hence \( \text{Ext}_{2}^{1}(S/a, 0 \times E) = \text{Ext}_{2}^{1}(k, 0 \times E) \neq 0 \) and \( \Gamma(S) = 0 \times E \) is not relatively-a-injective.

In this example note that the ascending sequence of ideals \( 0 :_{S}(x, 0)^{t} = 0 \times 0 :_{E} x^{t}, t > 0 \) does not stabilizes.

We end the section with other properties of relatively-a-injective \( R \)-modules, though we do not really need these in this paper.

**Proposition 1.8.** Let \( a \) be an ideal of a commutative ring \( R \) and \( J \) a relatively-a-injective \( R \)-module. Then the following holds:

(a) \( \text{Hom}_{R}(R/b, J) \) is \( R/b \)-injective for all ideals \( b \) containing a power of \( a \).

(b) \( \text{Ext}_{i}^{1}(N, J) = 0 \) for all a-torsion module \( N \) and all \( i \geq 1 \).

**Proof.** Let \( I^{\bullet} \) denote a injective resolution of \( J \).

The statement in (a) was already in [11, 2.7.2]. Let us recall the proof. We fix an ideal \( b \) containing some power of \( a \). For any ideal \( c \) containing \( b \) the complex \( \text{Hom}_{R}(R/c, I^{\bullet}) \) is exact in degree \( i > 0 \) by the assumption on \( J \). In particular, the complex \( \text{Hom}_{R}(R/b, I^{\bullet}) \) provides an injective \( R/b \)-resolution of the \( R/b \)-module \( \text{Hom}_{R}(R/b, J) \). But \( \text{Hom}_{R}(R/c, I^{\bullet}) = \text{Hom}_{R/b}(R/c, \text{Hom}_{R}(R/b, I^{\bullet})) \) as is easily seen. It follows that \( \text{Ext}_{1}^{1}(R/b, J) = 0 \). Hence \( \text{Hom}_{R}(R/b, J) \) is \( R/b \)-injective by Baer’s injectivity criterion.

(b) There is a direct system of augmented complexes

\[
0 \to \text{Hom}_{R}(R/a^{i}, J) \to \text{Hom}_{R}(R/a^{i}, I^{0}) \to \text{Hom}_{R}(R/a^{i}, I^{1}) \to \cdots.
\]

These complexes are exact by the assumption on \( J \). They are even split-exact by view of (a). By passing to the direct limit it follows that the complex

\[
0 \to \Gamma_{a}(J) \to \Gamma_{a}(I^{0}) \to \Gamma_{a}(I^{1}) \to \cdots.
\]

also is split-exact and so is the complex

\[
0 \to \text{Hom}_{R}(N, \Gamma_{a}(J)) \to \text{Hom}_{R}(N, \Gamma_{a}(I^{0})) \to \text{Hom}_{R}(N, \Gamma_{a}(I^{1})) \to \cdots.
\]

Because \( \text{Hom}_{R}(N, I^{\bullet}) \cong \text{Hom}_{R}(N, \Gamma_{a}(I^{\bullet})) \) when \( N \) is a-torsion it follows \( \text{Ext}_{i}^{1}(N, J) = 0 \) for all \( i \geq 1 \).

For other examples on relatively-a-injective modules, see [11, Chapter 2, Section 7]. For other examples, see also the interesting paper of Quy and Rohrer [8].

2. **Weakly pro-regular sequences and injectivity**

We shall obtain another characterization of weakly pro-regular sequences. Let \( x = x_{1}, \ldots, x_{k} \) denote a sequence of elements of a commutative ring \( R \). For a natural number \( n \) let \( x_{m}^{n} = x_{m}^{1}, \ldots, x_{m}^{n} \). For natural numbers \( m \geq n \) there is a natural map of the Koszul homology modules \( H_{i}(\{x_{m}^{1}, \ldots, x_{m}^{n}\}; R) \rightarrow H_{i}(\{x_{m}^{n}\}; R) \) for all \( i \geq 0 \). The sequence \( x \) is called weakly pro-regular provided the inverse system \( \{H_{i}(\{x_{m}^{n}\}; R) \mid n \geq 1 \} \) is pro-zero for all \( i \geq 1 \), that is, for all \( i > 0 \) and any \( n \) there is a natural number \( m \geq n \) such that the natural map \( H_{i}(\{x_{m}^{n}\}; R) \rightarrow H_{i}(\{x_{m}^{m}\}; R) \) is zero. Weakly pro-regular sequences were introduced in [2] and [10] (see also [11] Chapter 7, Section 3).
Recalls 2.1. Let \( \underline{x} = x_1, \ldots, x_k \) denote a sequence of elements in the commutative ring, write \( a \) for the ideal generated by this sequence and \( \mathcal{C}_{\underline{x}} \) for the Čech complex constructed on this sequence. Then for all \( R \)-modules (resp. complexes) \( M \) there are natural homomorphisms \( H^i_a(M) \to H^i(\mathcal{C}_{\underline{x}} \otimes_R M) \) and it is known that these are isomorphisms when the sequence \( \underline{x} \) is weakly pro-regular (see \[2\] or \[10\] or \[11\] 7.4.1) applied to an injective resolution of \( M \). The main reason for this is given by the following result.

A sequence \( \underline{x} = x_1, \ldots, x_k \) of elements in a commutative ring \( R \) is weakly pro-regular if and only if

\[
H^i(\mathcal{C}_{\underline{x}} \otimes_R I) = 0 \text{ for all injective } R\text{-modules } I \text{ and all } i > 0 \text{ (see \[11\] Lemma 7.3.3)}.
\]

Note that a one length sequence \( x \) is weakly pro-regular if and only if the ascending sequence of ideals \( 0 :_R x^t, t > 0 \), stabilizes. Note also that any finite sequence of elements in a Noetherian ring is weakly pro-regular. (This may be proved with the Artin-Rees Lemma, see \[12\] 4.3.3 or \[11\] A.2.3.)

For more informations on the notion of weakly pro-regular sequences, we refer to \[10\] or \[11\] Chapter 7, Section 3.

Let \( I \) denote an injective \( R \)-module. In the preceding section we investigated when \( \Gamma_a(I) \) is relatively-\( a \)-injective. The more general question to know when \( \Gamma_a(I) \) is injective has been investigated by Quy and Rohrer (see \[8\]). In this interesting paper Quy and Rohrer proved the following proposition.

Proposition 2.2. (see \[8\] Proposition 3.2) Let \( a \) be an ideal of the commutative ring \( R \) generated by the sequence \( \underline{x} = x_1, \ldots, x_k \). If for all injective \( R \)-modules \( I \) one has that \( \Gamma_a(I) \) is again injective, then the sequence \( \underline{x} \) is weakly pro-regular,

We note that \[2.2\] is a particular case of the following more precise theorem, which also provide us with a new characterization of weakly pro-regular sequences.

Theorem 2.3. Let \( a \) be a finitely generated ideal of a commutative ring \( R \) generated by the sequence \( \underline{x} = x_1, \ldots, x_k \). Then the following are equivalent:

(i) \( H^i_a(M) = 0 \) for all \( i > 0 \) and every \( a \)-torsion \( R \)-module \( M \).

(ii) \( \underline{x} \) is a weakly pro-regular sequence.

(iii) \( \Gamma_a(I) \) is relatively-\( a \)-injective for every injective \( R \)-modules \( I \).

Proof. (i)⇒(ii): Let \( I \) be any injective \( R \)-modules and consider the short exact sequence

\[
0 \to \Gamma_a(I) \to I \to I/\Gamma_a(I) \to 0.
\]

By view of the condition (i) and by Proposition \[1.6\] we have that \( \Gamma_a(I) \) is relatively-\( a \)-injective. By view of Proposition \[1.4\] we then have that \( \text{E-dp}(a, I/\Gamma_a(I)) = \infty \).

Now let \( K^*(\underline{x}^t) \) denote the ascending Koszul complex constructed on the sequence \( \underline{x}^t = x_1^t, \ldots, x_k^t \) (see \[12\] or \[11\] Chapter 5 Section 2). By the Ext-depth sensitivity of the Koszul complex we also have that \( H^i(K^*(\underline{x}^t) \otimes_R I/\Gamma_a(I)) = 0 \) for all \( i \geq 0 \) (see \[12\] Theorem 6.1.6) or \[11\] Chapter 5. It follows that the complex \( \mathcal{C}_{\underline{x}} \otimes_R I/\Gamma_a(I) \) is exact since \( H^i(\mathcal{C}_{\underline{x}} \otimes_R I/\Gamma_a(I)) \cong \lim H^i(K^*(\underline{x}^t) \otimes_R I/\Gamma_a(I)) \). But the complex \( \mathcal{C}_{\underline{x}} \) is a complex of flat \( R \)-modules and induces a short exact sequence of complexes

\[
0 \to \mathcal{C}_{\underline{x}} \otimes_R \Gamma_a(I) \to \mathcal{C}_{\underline{x}} \otimes_R I \to \mathcal{C}_{\underline{x}} \otimes_R I/\Gamma_a(I) \to 0.
\]

Because \( \mathcal{C}_{\underline{x}} \otimes_R \Gamma_a(I) \cong \Gamma_a(I) \) as is easily seen we now obtain that \( H^i(\mathcal{C}_{\underline{x}} \otimes_R I) = 0 \) for all \( i > 0 \). Hence the sequence \( \underline{x} \) is weakly pro-regular by view of the recalls in \[2.1\].

(ii)⇒(i): This is rather obvious. Let \( M \) be any \( a \)-torsion modules. Then \( \mathcal{C}_{\underline{x}} \otimes_R M \cong M \) as is easily seen and the conclusion follows since \( H^i_a(M) \cong H^i(\mathcal{C}_{\underline{x}} \otimes_R M) \) when the sequence \( \underline{x} \) is weakly pro-regular (see \[2.1\]).
Proposition 3.2. Let \( \mathfrak{m} \) be defined by \( \mathfrak{m} \mapsto (a, b) \) (relations \( \mathfrak{m} \cdot a = 0 \)). Then the short exact sequence 

\[ 0 \rightarrow \mathfrak{m} \cap \mathfrak{n} \rightarrow \mathfrak{m} + \mathfrak{n} \rightarrow \mathfrak{m} / \mathfrak{m} \cap \mathfrak{n} \rightarrow 0 \]

for each \( \mathfrak{m}, \mathfrak{n} \), induces short exact sequences 

\[ 0 \rightarrow \mathfrak{m} / \mathfrak{n} \rightarrow \mathfrak{m} / (\mathfrak{m} \cap \mathfrak{n}) \rightarrow \mathfrak{m} / \mathfrak{m} \cap \mathfrak{n} \rightarrow 0 \]

and

\[ 0 \rightarrow \mathfrak{m} / (\mathfrak{m} \cap \mathfrak{n}) \rightarrow \mathfrak{m} / \mathfrak{m} \cap \mathfrak{n} \rightarrow \mathfrak{m} / \mathfrak{m} \cap \mathfrak{n} \rightarrow 0 \]

These are first results for the torsion of injective modules.

Proposition 3.2. Let \( \mathfrak{a}, \mathfrak{b} \subset R \) two ideals of the commutative ring \( R \) and let \( M \) be any \( R \)-module.

(a) We have \( \Gamma_{\mathfrak{a}+\mathfrak{b}}(M) = \Gamma_{\mathfrak{a}}(M) \cap \Gamma_{\mathfrak{b}}(M) \) and a short exact sequence 

\[ 0 \rightarrow \Gamma_{\mathfrak{a}+\mathfrak{b}}(M) \rightarrow \Gamma_{\mathfrak{a}}(M) \oplus \Gamma_{\mathfrak{b}}(M) \rightarrow \Gamma_{\mathfrak{a}}(M) + \Gamma_{\mathfrak{b}}(M) \rightarrow 0. \]

(b) Moreover, if \( M = I \) is injective, then 

\[ \Gamma_{\mathfrak{a}}(I) + \Gamma_{\mathfrak{b}}(I) \cong \lim \Hom_R(R/\mathfrak{a}^n \cap \mathfrak{b}^n, I). \]

Corollary 2.4. Let \( \mathfrak{a} \) be an ideal of the commutative ring \( R \) generated by the sequence \( x = x_1, \ldots, x_k. \) The following conditions are equivalent.

(i) \( J/\Gamma_{\mathfrak{a}}(J) \) is relatively-\( \mathfrak{a} \)-injective for every relatively-\( \mathfrak{a} \)-injective \( R \)-module \( J. \)

(ii) \( \Gamma_{\mathfrak{a}}(J) \) is relatively-\( \mathfrak{a} \)-injective for every relatively-\( \mathfrak{a} \)-injective \( R \)-module \( J. \)

(iii) \( x \) is a weakly pro-regular sequence.

Proof. The equivalence (i)\( \Leftrightarrow \) (ii) follows by Proposition 1.4. The equivalence (ii)\( \Leftrightarrow \) (iii) follows by Theorem 2.3 together with Proposition 1.6.

3. WEAKLY PRO-REGULAR SEQUENCES AND MAYER-VIEtorIS SEQUENCES

Local cohomology is a matter of adic topology, Mayer-Vietoris sequence concerns local cohomology with respect to two distinct ideals. Note that the local homology with respect to an ideal \( \mathfrak{a} \) only depends on the topological equivalence class of \( \mathfrak{a}. \) We say that two ideals \( \mathfrak{a} \) and \( \mathfrak{a}' \) are topologically equivalent if they define the same adic topology, that is if each power of \( \mathfrak{a} \) contains a power of \( \mathfrak{a}' \) and vice versa. More generally let \( \mathfrak{a} \) be an ideal of a commutative ring \( R. \) We say that a set of ideals \( \{ \mathfrak{c}_n \mid n \in \mathbb{N} \} \) defines the \( \mathfrak{a} \)-adic topology if these \( \mathfrak{c}_n \) form a basis of open neighbourhood of the \( \mathfrak{a} \)-adic topology, that is if each power of \( \mathfrak{a} \) contains one the \( \mathfrak{c}_n \) and if each \( \mathfrak{c}_n \) contains a power of \( \mathfrak{a}. \)

For example, let \( \mathfrak{a}, \mathfrak{b} \subset R \) be two ideals of the commutative ring \( R. \) Then the set \( \{ a^n + b^n \mid n \geq 1 \} \) defines the \( (\mathfrak{a} + \mathfrak{b}) \)-adic topology: for each integer \( n \) we have the following containment relations \( (\mathfrak{a} + \mathfrak{b})^{2n} \subset a^n + b^n \subset (\mathfrak{a} + \mathfrak{b})^n. \)

Suppose that the ring \( R \) is Noetherian and let \( \mathfrak{a}, \mathfrak{b} \subset R \) two ideals. Then the ideals \( \mathfrak{a} \cap \mathfrak{b} \) and \( \mathfrak{a} \cdot \mathfrak{b} \) define the same adic-topology (that is because they are finitely generated with the same radical). Recall also that the set of ideals \( \{ a^n \cap \mathfrak{b}^n \mid n \geq 1 \} \) defines the \( \mathfrak{a} \cap \mathfrak{b} \)-adic topology, this follows by the Artin-Rees Lemma (see [3]). We shall see that these facts also hold for a commutative ring under some weak pro-regularity conditions. To this end we need Mayer-Vietoris type results.

The following obvious lemma will be useful, its rôle in Mayer-Vietoris sequences is central.

Lemma 3.1. Let \( R \) be a commutative ring. Let \( M \) denote an \( R \)-module with \( M_1, M_2 \subset M \) two sub-modules. Then the short exact sequence 

\[ 0 \rightarrow M \rightarrow M \oplus M \rightarrow M \rightarrow 0 \]

defined by \( m \mapsto (m, m) \) and \( (m', m'') \mapsto m' - m'', m, m', m'' \in M, \) induces short exact sequences 

\[ 0 \rightarrow M_1 \cap M_2 \rightarrow M_1 \oplus M_2 \rightarrow M_1 + M_2 \rightarrow 0 \]

and 

\[ 0 \rightarrow M/(M_1 \cap M_2) \xrightarrow{i} M/M_1 \oplus M/M_2 \xrightarrow{p} M/(M_1 + M_2) \rightarrow 0 \]

Here are first results for the torsion of injective modules.
Proof. (a) The inclusion \( \Gamma_{a+b}(M) \subseteq \Gamma_a(M) \cap \Gamma_b(M) \) is trivial. Let \( m \in \Gamma_a(M) \cap \Gamma_b(M) \) and therefore \( a^n m = 0 = b^n m \) for some \( n \geq 1 \). Then we have \( (a^n + b^n) m = 0 \) and \( (a + b)^{2n} m = 0 \), that is \( m \in \Gamma_{a+b}(M) \). The short exact sequence follows (see [3,1] applied to the sub-modules \( \Gamma_a(M) \) and \( \Gamma_b(M) \) of \( M \)).

(b) By view of [3,1] there is the short exact sequences

\[
0 \to \text{Hom}_R(R/a^n + b^n, I) \to \text{Hom}_R(R/a^n, I) \oplus \text{Hom}_R(R/b^n, I) \to \text{Hom}_R(R/a^n \cap b^n, I) \to 0.
\]

for all \( n \geq 1 \). With the natural homomorphisms these form a short exact sequence of direct systems. By passing to the direct limit there is a short exact sequence

\[
0 \to \Gamma_{a+b}(I) \to \Gamma_a(I) \oplus \Gamma_b(I) \to \lim \text{Hom}_R(R/a^n \cap b^n, I) \to 0.
\]

The last claim follows by comparing this short exact sequence with the one in (a).

Then recall that there is also a Mayer-Vietoris long exact sequence for Čech cohomology.

**Theorem 3.3.** (see [11, Theorem 9.4.3] or [13]) Let \( \underline{x} = x_1, \ldots, x_k \) and \( y = y_1, \ldots, y_l \) be two sequences in a commutative ring \( R \). We form the sequence \( \underline{x} \cdot y \) and denote by \( \underline{z} \) the sequence formed by the \( z_{ij} = x_i y_j \) in any order. Let \( X \) denote an \( R \)-complex. Then there is a long exact sequence

\[
\ldots \to H^i(\check{\mathcal{C}}_{\underline{x} \cdot y} \otimes_R X) \to H^i(\check{\mathcal{C}}_{\underline{x}} \otimes_R X) \oplus H^i(\check{\mathcal{C}}_{\underline{y}} \otimes_R X) \to H^i(\check{\mathcal{C}}_{\underline{z}} \otimes_R X) \to H^{i+1}(\check{\mathcal{C}}_{\underline{x} \cdot y} \otimes_R X) \to \ldots
\]

Proof. The proof makes use of the change of rings homomorphism

\[
\underline{z}[X_1, \ldots, X_k, Y_1, \ldots, Y_l] \to \underline{R} : X_i \mapsto x_i, Y_i \mapsto y_i,
\]

where \( X_1, \ldots, X_k, Y_1, \ldots, Y_l \) are indeterminates, and the Mayer-Vietoris long exact sequence for Noetherian rings (see e.g. [11, 9.4.2]) together with the recalls in [2,1] For more details see [11, 9.4.3].

We now obtain a Mayer-Vietoris long exact sequence in a rather large generality.

**Theorem 3.4.** With the notations of [3,3] put \( a = \underline{x} \underline{R} \) and \( b = \underline{y} \underline{R} \), so that \( a \cdot b = \underline{z} \underline{R} \). Suppose that the three sequences \( \underline{x}, \underline{y}, \underline{xy} \) are weakly pro-regular.

(a) The sequence \( \underline{z} \) is also weakly pro-regular.

(b) There is a long exact sequence

\[
\ldots \to H^i_{a+b}(X) \to H^i_a(X) \oplus H^i_b(X) \to H^i_{a+b}(X) \to H^{i+1}_{a+b}(X) \to \ldots
\]

for any \( R \)-complex \( X \).

Proof. The first statement was already in [11, Corollary 9.4.4]. It follows by the above theorem together with the characterization of weakly pro regular sequences recalled in [2,1] Then the second follows by the first together with Theorem [3,3] Note with [2,1] that the Čech cohomology with respect to a weakly pro-regular sequence coincides with the local homology with respect to the ideal generated by this sequence.

A second Mayer-Vietoris type result concerns the torsion of injective modules.

**Corollary 3.5.** Let \( a \) and \( b \) two finitely generated ideals of the commutative ring \( R \), generated respectively by the sequences \( \underline{x} = x_1, \ldots, x_k \) and \( y = y_1, \ldots, y_l \). Assume that the three sequences \( \underline{x}, \underline{y}, \underline{xy} \) are weakly pro-regular. For any injective \( R \)-module \( I \) we then have a short exact sequence

\[
0 \to \Gamma_{a+b}(I) \to \Gamma_a(I) \oplus \Gamma_b(I) \to \Gamma_{a+b}(I) \to 0.
\]

Moreover \( \Gamma_{a+b}(I) = \Gamma_{a-b}(I) = \Gamma_a(I) + \Gamma_b(I) \).
Proof. The short exact sequence is a particular case of the long exact sequence in [3.4]. This sequence together with the one in [3.2](a) implies that $\Gamma_{a\cap b}(I) = \Gamma_a(I) + \Gamma_b(I)$. Then we have the following obvious containment relations

$\Gamma_{a\cap b}(I) \subseteq \Gamma_a(I) \subseteq \Gamma_{a\cap b}(I),$

they finish the proof. \hfill $\square$

Now we are prepared for our last purpose in this section. We need a technical lemma.

**Lemma 3.6.** Let $a$ be an ideal of a commutative ring $R$ and let $\{c_n\}_{n \geq 1}$ denote a descending sequence of ideals such that $a^n \subset c_n$ for all $n \geq 1$. Suppose that the natural map

$$\lim_n \text{Hom}_R(R/c_n, I) \to \lim_n \text{Hom}_R(R/a^n, I)$$

is an isomorphism for any injective $R$-module $I$. Then the set of ideals $\{c_n\}_{n \geq 1}$ defines the $a$-adic topology.

**Proof.** The short exact sequences $0 \to c_n/a^n \to R/a^n \to c_n \to 0$ induce a direct system of short exact sequences

$$0 \to \text{Hom}_R(R/c_n, I) \to \text{Hom}_R(R/a^n, I) \to \text{Hom}_R(c_n/a^n, I) \to 0$$

for any injective $R$-module $I$. By passing to the limit there is a short exact sequence

$$0 \to \lim_n \text{Hom}_R(R/c_n, I) \to \lim_n \text{Hom}_R(R/a^n, I) \to \lim_n \text{Hom}_R(c_n/a^n, I) \to 0.$$

Because of our assumption it follows that $\lim_n \text{Hom}_R(c_n/a^n, I) = 0$ for any injective $R$-module $I$. Now let us fix an $n \in \mathbb{N}_+$ and let $f : c_n/a^n \to l^0$ denote an injection into some injective $R$-module $l^0$. Note that $f \in \text{Hom}_R(c_n/a^n, l^0)$. Because of the vanishing $\lim_n \text{Hom}_R(c_n/a^n, l^0) = 0$ there must be an integer $m \geq n$ such that the image of $f$ in $\text{Hom}_R(c_m/a^m, l^0)$ is zero. That is, the composite of the maps

$$c_m/a^m \to c_n/a^n \xrightarrow{f} l^0$$

is zero. Since $f$ is an injection it follows that the first map is zero, and that $c_m \subset a^n$. \hfill $\square$

The following emphasizes again the ubiquity of the weak pro-regularity conditions.

**Theorem 3.7.** Let $a$ and $b$ two finitely generated ideals of the commutative ring $R$, generated respectively by the sequences $x = x_1, \ldots, x_k$ and $y = y_1, \ldots, y_l$. Assume that the three sequences $x, y, xy$ are weakly pro-regular.

Then the set of ideals $\{a^n \cap b^n\}_{n \geq 1}$ defines the $a \cap b$-adic topology as well as the $a \cdot b$-adic topology. In particular the ideals $a \cdot b$ and $a \cap b$ define the same adic topology.

**Proof.** Note first that $(a \cdot b)^n \subset (a \cap b)^n \subset a^n \cap b^n$ for all $n \geq 1$. By view of Corollary [3.5] we get the natural isomorphisms

$$\lim_n \text{Hom}_R(R/(a \cdot b)^n, I) \cong \lim_n \text{Hom}_R(R/(a \cap b)^n, I) \cong \Gamma_a(I) + \Gamma_b(I).$$

By Proposition [3.2] we have the natural isomorphism

$$\Gamma_a(I) + \Gamma_b(I) \cong \lim_n \text{Hom}_R(R/a^n \cap b^n, I).$$

Putting this together it yields natural isomorphisms

$$\lim_n \text{Hom}_R(R/(a \cdot b)^n, I) \cong \lim_n \text{Hom}_R(R/(a \cap b)^n, I) \cong \lim_n \text{Hom}_R(R/a^n \cap b^n, I)$$

and the conclusion follows by Lemma [3.6]. \hfill $\square$
Remark 3.8. Let us look again at the long exact sequence in 3.4. By view of the above 3.7 we may now replace in it the product ideal \(a \cdot b\) by the intersection \(a \cap b\). Note that \(H^i_{a \cap b}(X) \cong H^i_{a \cdot b}(X)\) for all \(i\) and all complexes \(X\) because the ideals \(a \cdot b\) and \(a \cap b\) define the same adic topology.

4. Modules of Fractions and Injectivity

Now let \(S\) denote a multiplicatively closed subset in the ring \(R\). For an \(R\)-module \(M\) we denote by \(\iota_{S,M}\) the natural map \(M \to M_S\), and by \(K_S(M)\) its kernel. There is the question to know when the localization \(I_S\) of an injective \(R\)-module \(I\) is again injective. This was claimed to be true by Rotman (see [11, 3.76]) and shown to be incorrect by Dade (see his interesting paper [4]), though Dade did not provide an explicit example (a first concrete example of an injective module that does not localize can be found in [11, A.5.4]). This problem seems to be related to the question when \(\Gamma_S(I)\) is an injective \(R\)-module. Note that \(K_S(M) = \sum_{s \in S} \Gamma_{sR}(M)\). We now investigate a little bit in this direction and provide a further example.

The following elementary lemma will be used repeatedly.

Lemma 4.1. Let \(S \subset R\) denote a multiplicatively closed set in the commutative ring \(R\). For an \(R\)-module \(M\) the natural homomorphism

\[
\iota_{S,M} : M \to M_S, \quad m \mapsto m/1,
\]

is surjective if and only if \(M/K_S(M) = s \cdot M/K_S(M)\) for all \(s \in S\). The last condition is equivalent to \(M = K_S(M) + s \cdot M\) for all \(s \in S\).

Proof. First note that the last equivalence is easily seen.

If the map \(\iota_{S,M}\) is surjective, then \(M_S \cong M/K_S(M)\), in particular \(M/K_S(M) = s \cdot M/K_S(M)\) for all \(s \in S\).

Conversely, assume that \(M/K_S(M) = s \cdot M/K_S(M)\) for all \(s \in S\). This means that the multiplications by any \(s \in S\) are surjective on \(M/K_S(M)\). As they are also injective on \(M/K_S(M)\) it follows that \(M/K_S(M) \cong M_S\). Whence \(\iota_{S,M}\) is surjective.

\(\square\)

Proposition 4.2. Let \(S\) denote a multiplicatively closed set in the commutative ring \(R\) and let \(M\) be an \(R\)-module. If the \(R\)-module \(M/K_S(M)\) is injective, then \(M/K_S(M) \cong M_S\).

Proof. For all \(s \in S\) the multiplication by \(s\) on \(M/K_S(M)\) is always injective. When the \(R\)-module \(M/K_S(M)\) is injective it follows that these multiplications are also surjective. Whence \(\iota_{S,M}\) is surjective (see 4.1) and \(M/K_S(M) \cong M_S\).

\(\square\)

Corollary 4.3. Let \(S\) denote a multiplicatively closed set in the commutative ring \(R\) and let \(I\) denote an injective \(R\)-module. Assume that \(K_S(I)\) is injective. Then the localized module \(I_S\) is injective and \(I_S \cong I/K_S(I)\).

Proof. By the assumption on \(K_S(I)\) the short exact sequence

\[
0 \to K_S(I) \to I \to I/K_S(I) \to 0
\]

is split exact and \(I/K_S(I)\) is injective. We conclude by 4.2.

\(\square\)

In the case when the multiplicatively closed subset \(S\) of \(R\) consists of the powers of a single element we have a refinement of Proposition 4.2.

Proposition 4.4. Let \(x \in R\) be an element. For an \(R\)-module \(M\) the following conditions are equivalent:

\(\text{(i)}\) The \(R\)-module \(M/\Gamma_{xR}(M)\) is relatively-\(xR\)-injective.

\(\text{(ii)}\) \(E\text{-dp}(xR, M/\Gamma_{xR}(M)) = \infty\).

\(\text{(iii)}\) \(M/\Gamma_{xR}(M) = x \cdot M/\Gamma_{xR}(M)\).

\(\text{(iv)}\) The natural map \(M \to M_x\) is surjective.
Proof. (i)⇒(ii): This is clear, note that always \( \text{Hom}_R(R/ xR, M/ \Gamma_\Delta R(M)) = 0. \)

(ii)⇒(iii): With the condition in (ii) we also have \( \text{Ext}_R^i(xR, M/ \Gamma_\Delta R(M)) = \infty, \) see 4.1. This implies condition (iii).

(iii)⇒(iv): This follows by 4.1.

(iv)⇒(i): With the condition in (iv) we have \( M/ \Gamma_\Delta R(M) \cong M_\Delta. \) Then note that always \( \text{Ext}_R^i(xR, M_\Delta) = 0 \) for all \( i \geq 0 \) because multiplication by \( x \) acts as an automorphism on \( M_\Delta. \)

We are ready for the discussion of the following example. Note first that an \( R_S \)-module is \( R \)-injective if and only if it is \( R_S \)-injective (see [4] or [11, A.5.1]). That is because \( \text{Hom}_R(M, N) \cong \text{Hom}_{R_S}(M, N) \) for two \( R_S \)-modules \( M, N \) and because \( R_S \) is \( R \)-flat.

**Example 4.5.** (A) Let \( \mathbb{k} \) denote a field and \( x, y \) two variables over \( \mathbb{k}. \) Let \( R = \mathbb{k}[|x, y|] \) denote the formal power series ring over \( \mathbb{k}. \) Let \( E = E_R(R/ m) \) denote the injective hull of the residue field. We define \( S = R \times E \) as the idealization of \( R \) by \( E. \) Then \( \text{Hom}_R(S, E) \) is an injective \( S \)-module and \( \text{Ext}_R^i(S, E) \cong S \) as follows by a Theorem of Faith (see [5] or [11, A.4.6]).

Let \( (x, 0) \in S \). We claim that the localization \( S_{(x, 0)} \) of \( S \) is not injective \( S \)-module. This is equivalent to the fact that \( S_{(x, 0)} \) is not self-injective (see above). But \( S_{(x, 0)} \cong R_\Delta \) as is easily seen and \( R_\Delta \) is not self-injective.

(B) Moreover, let \( \check{C}_{(x, 0)} : 0 \to S \to S_{(x, 0)} \to 0 \) denote the \( \check{C}ech \) complex of \( S \) with respect to the one length sequence \((x, 0)\). Then it follows that

\[
\Gamma_{(x, 0)}(S) = H^0(\check{C}_{(x, 0)}) = 0 \times E, \quad \text{and} \quad H^1(\check{C}_{(x, 0)}) = (R_\Delta / R) \times 0,
\]

so that \( H^1(\check{C}_{(x, 0)}(S)) \neq 0 \) for the injective \( S \)-module \( S \).

Finally, because the natural map \( S \to S_{(x, 0)} \) is not surjective, the module \( S/ \Gamma_{(x, 0)} S \) is not isomorphic to \( S_{(x, 0)} \). Whence it follows by 4.2 that \( S/ \Gamma_{(x, 0)}(S) \) and also \( \Gamma_{(x, 0)}(S) \) are not \( S \)-injective modules. Moreover \( S/ \Gamma_{(x, 0)}(S) \) and \( \Gamma_{(x, 0)}(S) \) are not relatively-\((x, 0)S\)-injective modules by 4.4 and 4.3.

(C) Another feature of the example in (A) is the following. Let \( S^N \) denote the direct product of copies of \( S \) over the index set \( \mathbb{N}. \) Then \( S^N \) is an injective \( S \)-module (as a direct product of injective \( S \)-modules). Now let \( S^{(\mathbb{N})} \) be the direct sum of copies of \( S \) over the index set \( \mathbb{N}. \) We claim that \( S^{(\mathbb{N})} \) is not an injective \( S \)-module. To this end look at the short exact sequence

\[
0 \to S^{(\mathbb{N})} \to S^{(\mathbb{N})} \to S_{(x, 0)} \to 0
\]

as it is derived from the isomorphism \( S_{(x, 0)} \cong \varprojlim \{S_n, (x, 0)\} \) with the direct system \( S_n = S \) and \( S_n \to S_{n+1} \) multiplication by \( (x, 0) \) for all \( n \geq 1. \) Assuming that \( S^{(\mathbb{N})} \) is \( S \)-injective it implies that the above sequence is split exact and therefore \( S_{(x, 0)} \) is \( S \)-injective. This is not true by (A).

There is the more general question to know when the natural homomorphism \( I \to I_S \) is surjective for an injective module \( I. \) Note that this is not always the case, as shown by example 4.5 (B). In the case when the multiplicatively closed subset \( S \) of \( R \) consists of the powers of a single element the answer is rather simple.

**Proposition 4.6.** Let \( x \) denote an element of the commutative ring \( R. \) The following conditions are equivalent:

(i) The natural map \( I \to I_x \) is surjective for any relatively-\( xR \)-injective module \( I. \)

(ii) The natural map \( I \to I_x \) is surjective for any injective module \( I. \)

(iii) The one length sequence \( x \) is weakly pro-regular.

Proof. (i)⇒(ii): This is obvious.
(ii)⇒(iii): If $I \to I_x$ is onto, then $H^1(\check C_x \otimes R) = 0$. The claim follows by the recalls in 2.1.

(iii)⇒(i): Assume that the one length sequence $x$ is weakly pro-regular and let $J$ be any relatively-$xR$-injective $R$-module. On one hand we have $H^1(\check C_x \otimes_R J) = H^1_{xR}(J)$ (see 2.4). On the other hand we also have $H^1_{xR}(J) = 0$ (see 2.2(a)). Whence the natural map $J \to I_x$ is onto. 

\[
\square
\]

5. Ideal Transforms

In the final section we relate our results to the ideal transforms.

Definition 5.1. For an ideal $a \subset R$ of a commutative ring $R$ and an $R$-module $M$ the $a$-transform of $M$ is defined by $\mathcal{D}_a(M) = \lim_i \text{Hom}_R(a^n, M)$.

For this and related results we refer to [8] and [11, Chapter 12, section 5].

First we shall discuss in detail the $a$-transform of an $R$-module $M$.

Proposition 5.2. Let $a \subset R$ denote an ideal of the commutative ring $R$. Let $M$ be an $R$-module.

(a) There is a natural homomorphism $\tau_M : M \to \mathcal{D}_a(M)$ and a short exact sequence

\[
0 \to \Gamma_a(M) \to M \xrightarrow{\tau_M} \mathcal{D}_a(M) \to H^1_a(M) \to 0.
\]

(b) Let $S$ denote a multiplicatively closed subset of $R$ such that $a \cap S \neq \emptyset$. There is a natural homomorphism $\xi_M : \mathcal{D}_a(M) \to M_S$ such that $\xi_M \circ \tau_M = i_{S,a} : M \to M_S$.

(c) If $M = J$ is relatively-$a$-injective, then $\mathcal{D}_a(J) \cong J/\Gamma_a(J)$.

Proof. The proof of (a) is well-known. We only need to take the direct limit of the direct system of exact sequences with the natural maps

\[
0 \to \text{Hom}_R(R/a^n, M) \to M \to \text{Hom}_R(a^n, M) \to \text{Ext}^1_R(R/a^n, M) \to 0.
\]

For the proof of (b) we choose first an element $x \in a \cap S$. Then we define homomorphisms

\[
\text{Hom}_R(a^n, M) \to M_S \text{ by } g_n \to g_n(x^n)/x^n.
\]

They do not depend on the particular choice of $x$ as easily seen. Moreover they provide a direct system of sequences

\[
M \to \text{Hom}_R(a^n, M) \to M_S.
\]

We take direct limits and obtain the wanted homomorphism $\xi_M : \mathcal{D}_a(M) \to M_S$ such that $\xi_M \circ \tau_M = i_{S,a} : M \to M_S$.

The proof of (c) follows by (a) since $H^1_a(J) \cong \lim \text{Ext}^1_R(R/a^n, J) = 0$ for a relatively-$a$-injective $R$-module $J$. 

\[
\square
\]

Assume now that the ideal $a$ of $R$ is finitely generated, say by the sequence $x = x_1, \ldots, x_k$. We define the $R$-complex $D_\Sigma$ as the kernel of the natural surjective map $C_\Sigma \to R$ (see [11, 6.1.6] for more details). For an $R$-module $M$ we have a natural injection $M/\Gamma_a(M) \hookrightarrow H^1(D_\Sigma \otimes_R M)$. This follows because the complex $C_\Sigma \otimes_R M$ has the form

\[
C_\Sigma \otimes_R M : 0 \to M \xrightarrow{d^0} \oplus_{i=1}^r M_{x_i} \xrightarrow{d^1} \oplus_{1 \leq i < j \leq k} M_{x_i x_j} \xrightarrow{d^2} \cdots \to M_{x_1 \cdots x_k} \to 0
\]

and because $M/\Gamma_a(M) \cong \text{Im}(d^0) \subseteq \text{Ker}(d^1) = H^1(D_\Sigma \otimes_R M)$.

Proposition 5.3. Let $x = x_1, \ldots, x_k$ denote a sequence of elements and $a = xR$. For any $R$-module $M$ there is an injective homomorphism $\rho_M : D_a(M) \to \oplus_{i=1}^k M_{x_i}$ such that $\rho_M \circ \tau_M$ is the natural map $M \to \oplus_{i=1}^k M_{x_i}$. Moreover it induces an injection

\[
D_a(M) \hookrightarrow H^1(D_\Sigma \otimes_R M).
\]
Proof. By view of [5, 2] there are homomorphisms $\xi^i_M : D_a(M) \to M_i$ for $i = 1, \ldots, k$ such that the composite $M \xrightarrow{ri} D_a(M) \to M_i$ is the natural map $M \to M_i$. Then the homomorphism $\rho_M$ is defined by $D_a(M) \to \oplus_{i=1}^k M_i$, $f \mapsto (\xi^i_M(f))_{i=1}^k$.

In order to show that $\rho_M$ is injective suppose that $f \in D_a(M)$ maps to zero, i.e. $(\xi^i_M(f))_{i=1}^k = 0$. Let $g_n \in \text{Hom}_R(a^n, M)$ denote a representative of $f$. Then $g_a(x^n_i)/x^n_i = 0$ for $i = 1, \ldots, k$. Therefore, there is an integer $m \geq n$ such that $x^m_1 g_n(x^m_i) = 0$ for all $i = 1, \ldots, k$. Because of $a^{n+m+k} \subseteq (x^{n+m}_1, \ldots, x^{n+m}_k)R$ this implies $g_a(a^{n+m+k}) = 0$. Whence the restriction of $g_a$ to $a^{n+m+k}$ is zero and therefore $f = 0$.

For the final claim we have that $H^1(\check{D}_\bar{\chi} \otimes_R M) = \text{Ker}(\oplus_i M_i \xrightarrow{d^i} \oplus_{i<j} M_{i,j})$ and note that $D_a(M) \subseteq \text{Ker} d^i$. Indeed let $f \in D_a(M)$ and $g_n \in \text{Hom}_R(a^n, M)$ a representative of $f$. This $f$ is mapped to $(g_n(x^n_i)/x^n_i)^{i=1}_k \in \oplus_{i=1}^k M_i$ which is well defined and belongs to $\text{Ker} d^i$ since in $M_i$ we have the equalities $g_n(x^n_i)/x^n_i = g_n(x^n_j)/x^n_j$ for all $i, j \in \{1, \ldots, k\}$.

□

Corollary 5.4. Let $\chi$ and $a$ be as in [5, 3]. Let $M$ denote an $R$-module.

(a) The natural map $H^1_\chi(M) \to H^1(\check{C}_\chi \otimes_R M)$ is injective and there is a commutative diagram with exact rows

$$
\begin{array}{cccccc}
0 & \to & \Gamma_a(M) & \to & M & \to & D_a(M) \\
 & & \parallel & & \downarrow & & \parallel \\
0 & \to & H^0(\check{C}_\chi \otimes_R M) & \to & M & \to & H^1(\check{D}_\chi \otimes_R M) \\
& & & & \downarrow & \uparrow & \\
& & & & H^1(\check{C}_\chi \otimes_R M) & \to & 0.
\end{array}
$$

In particular there is an isomorphism $H^1(\check{D}_\chi \otimes_R M)/D_a(M) \cong H^1(\check{C}_\chi \otimes_R M)/H^1_\chi(M)$.

(b) The natural injection $M/\Gamma_a(M) \hookrightarrow H^1(\check{D}_\chi \otimes_R M)$ factors through the injection $D_a(M) \to H^1(\check{D}_\chi \otimes_R M)$.

(c) Suppose that the sequence $\chi$ is weakly pro-regular. Then we have the isomorphism $D_a(M) \cong H^1(\check{D}_\chi \otimes_R M)$.

Proof. (a) There is a short exact sequence $0 \to \check{D}_\chi \otimes_R M \to \check{C}_\chi \otimes_R M \to M \to 0$ of complexes (see [11, 6.1.6]). The short exact sequence at the bottom of the diagram follows by the associated long exact sequence in cohomology, while the one at the top is shown in [5, 2](a). The commutativity of the diagram is rather obvious. Because the third vertical map $D_a(M) \to H^1(\check{D}_\chi \otimes_R M)$ is injective (see [5, 3]), so is the fourth one. The last statement in (a) follows now.

(b) The statement follows by the diagram.

(c) If the sequence $\chi$ is weakly pro-regular the fourth map in the diagram is an isomorphism (see [2, 1]). Whence the third map also is an isomorphism.

The isomorphism in [5, 4](c) originally proved for a Noetherian ring is known as Deligne’s formula (see [6, Exercise 3.7] or [3, 20.1.14]). It does not hold necessarily either for non-Noetherian rings nor for injective modules.

Example 5.5. Take the ring $S = R \times E$ of Example 1.7 and take $a = (x, 0)S$. We noted that $S$ is self-injective and with $\Gamma_a(S) = 0 \times E$. Hence $D_a(S) = R \subsetneq \oplus_{x \in S} H^1(\check{D}_{(x, 0)} \otimes_S S) = S_{(x, 0)} = R_X$.

Note added in proof. Theorem 2.3 has been shown independently by R. Vyas and A. Yekutieli (see: Weak Stability, and the Noncommutative MGM Equivalence, J. Algebra 513 (2018), 265-325), where the notion of “flasque module” is used instead of “relative injective”. Thanks to A. Yekutieli for drawing our attention to their paper.
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