Kalman filtering applied to the calibration of a high-resolution analog-to-digital converter

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Abstract. The paper proposes the application of the Kalman filter in dc calibrations of analog-to-digital converters with a Zener or a Josephson dc voltage reference standard. Repeated measurements over the entire full-scale range of an analog-to-digital converter with posterior Kalman filtering allow optimal estimation of its deviations in respect to a dc reference to be obtained, yielding improved assessment of converter’s linearity.

1. Introduction

The Kalman filter is mainly used to tackle estimations problems from measurements of the state of nonstationary (and noisy) processes [1-3]. It has extensively been investigated since the 1960’s and finds wide application in many different areas such as space crafting, determination of ballistic trajectories and orbiting, radar, and adaptive adjustment of antenna arrays among others. It plays an important role in the military and defense applications. This paper presents the application of the Kalman filter in metrology to estimate with enhanced denoising properties and to a very high quality, the deviations of the readings of an analog-to-digital converter (ADC) or digital voltmeter (DVM) by calibrating it against a dc voltage reference of quantum or classical base.

2. Definition of the problem

Figure 1 depicts schematically the calibration problem of an ADC corrupted by noise in which only the initial state $x(0)$ of the reference (at the instant zero) is known with a variance $\sigma_0^2$ (i.e., assumed Gaussian distributed, $N(x(0), \sigma_0^2)$). The ADC readings $y(n)$ may eventually be biased (i.e., with an offset) and to vary randomly with a variance $\sigma_v^2$. Although this problem seems of trivial nature, normally solved by averaging readings up to a certain number to avoid the effect of flicker noise, a Wiener filter is, by concept, predestined to deliver best estimations under stationary conditions. However, aiming at tackling in the future more complex and nonstationary problems related to the traceability of alternating (ac) quantities, we propose the use of the Kalman filter as a general treatment. Only the dc calibration of an ADC is here of interest.

The treatment of the Kalman filter can be found in extensive literature (see references). Figure 2 shows the Kalman filter in the form of a block diagram with feedbacks, as originally devised from control theory. In this case $x(n)$ denotes the state vector of a dynamic system at time $n$ described by a transition state matrix $A$ based on previous (n-1) states. Best linear estimates of the state of the system at time $n$ given the observations or measurements $y(k)$ for index $k = 1, 2, \ldots, n$ are denoted with vector $\hat{x}(n|n)$. 

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Similarly, \( \hat{x}(n|n-1) \) stands for the best estimate at time \( n \) given the \((n-1)\) prior observations \( i.e., y(k) \) for \( k = 1 \) to \((n-1)\). Noise vectors of the dynamic system and of measurements are denoted by \( w(n) \) and \( v(n) \) respectively. The measurement vector \( y(n) \) is related to \( x(n) \) by an observation matrix \( C(n) \) with additive noise \( v(n) \) as shown in the insert in figure 2. Measurements at time \( n \), \( i.e., y(n) \) are compared with measurement estimates \( \hat{y}(n|n-1) \) employing the past \((n-1)\) observations, and the deviations are corrected with a Kalman gain \( K(n) \), which is continuously updated in a feedback loop (or tracking loop) so that the error vector \( e(n|n) \) of estimates of \( x(n) \) \( i.e., \hat{x}(n|n) \) with respect to \( x(n) \) tends to a minimum. The \( z^{-1} \) factor stands for the delay operator of one sample, as is usually done in digital signal processing. Feedback loops of the type shown in figure 2 are central in electronics and akin to the known feed-forward feedback, which gives the utmost performance in respect to the antagonic requirements of accuracy and speed.

Figure 1. Direct calibration of an ADC with a reference at state \( x(n) \) from measurements (or sampled values) \( y(n) \) at instants \( n \) corrupted by noise.

Figure 2. General block diagram of the discrete Kalman filter.
3. Kalman filtering

From figure 2, estimates at time $n$ employing all $n$ samples are expressed by

$$\hat{x}(n|n) = \hat{x}(n|n-1) + K(n) [y(n) - C(n) \hat{x}(n|n-1)]$$  \(1\)

Similarly, estimates at time $n$ employing the $(n-1)$ samples are recursively dependent on previous estimates as

$$\hat{x}(n|n-1) = A(n-1) \hat{x}(n-1|n-1)$$  \(2\)

The Kalman gain $K(n)$ is dependent on the noise covariances and on the Hermitian transpose of the observation matrix $C(n)$, i.e., $C(n)^H$ [2-3].

For the particular case of a dc calibration with a “constant” dc voltage reference (i.e., $x(n) = x(n-1)$) in which $w(n) = 0$ and only the initial conditions are previously known (as stated in figure 1), it follows that

$$\hat{x}(0|0) = E\{x(0)\} = x(0)$$  \(3\)

and the initial covariance

$$P(0|0) = E\{x(0)x^H(0)\} = \sigma_0^2$$  \(4\)

where $E\{\cdot\}$ stands for the expectancy of a random quantity. Further, according to figure 3, other matrices simplify to $A = 1$, $C = 1$, and $x(n|n)$ to a scalar quantity, whose Kalman estimates for $n \geq 1$ are:

$$\hat{x}(n) = \hat{x}(n-1) + \frac{\sigma_0^2}{n\sigma_0^2 + \sigma_v^2} [y(n) - \hat{x}(n-1)]$$  \(5\)

In calibrations, the ADC deviations at time $n$, i.e., $\delta(n)$ from the reference (or nominal) are of interest and are expressed as

$$\delta(n) = \hat{x}(n) - x(0)$$  \(6\)

To illustrate the application of the Kalman filter, suppose that a 28-bit integrating ADC with a noise spectral density of 200 nV/Hz$^{1/2}$ has to be calibrated against a 10 V dc-reference with a known
“uncertainty” $\sigma_0$ of 1 $\mu$V (root-mean-square). Acquiring equally spaced samples $y(n)$ with an aperture time of nearly 1 s at a 1 s sampling rate leads to a $\sigma_v$ of approximately 141 nV. Figure 4 displays the ADC $\delta(n)$ deviations for simulated observations $y(n)$ with added Gaussian distributed noise. Application of equation (5) demonstrates the enhanced noise suppression of the Kalman filter as $\delta(n) \rightarrow 0$ for large $n$. Although the ADC readings spread considerably, the estimated values $\hat{x}(n)$ barely vary. Some small systematic deviations though persist, which are dependent on the start conditions.

Figure 4. Simulated Kalman filtering for an ADC calibration at its full-scale of 10 V.

Figure 5. Simulated outcome of five repetitions of the Kalman filter on five different data records of 330 values each.

Figure 5 shows the outcome of five independent repetitions of the Kalman filter on five different records. For $n > 300$ and the case in question, the $\delta$ deviations are smaller than 10 nV or 1 into $10^9$ parts of full-scale (10 V).

This procedure can be repeated for many x(0) start values within the whole full-scale range of the ADC to access its linearity.

Figure 6 shows an experimental result of the Kalman filtering for an 8 ½ digit 3458A digital multimeter (DMM) at 10 V. Zener x(0) start value was obtained from a conventional Josephson Voltage Standard calibration performed right before the DMM calibration. In this case, the voltage offset was not compensated and an estimation of its value can be made from figure 6 (around 1 $\mu$V).
4. Uncertainty of the Kalman estimates

The Kalman filter has strong ties to Bayesian inference, where prior information is used for estimation purposes. An approach which takes into account only the moments of the posterior probability distribution to determine the dynamic state of a system [4] demonstrates that the optimal solution for this problem is the Kalman’s filtering algorithm.

For the steady model considered in this paper, information about the state at time n is described by a Gaussian distribution whose mean and variance are respectively

\[ \hat{x}(n) = \frac{\sigma_v^2 \hat{x}(n-1) + \sigma_{n-1}^2 y(n)}{\sigma_v^2 + \sigma_{n-1}^2} \quad (7) \]

\[ \sigma_n^2 = \frac{\sigma_0^2 \sigma_v^2}{n \sigma_0^2 + \sigma_v^2} \quad (8) \]

Equation (7) is equivalent to equation (5). The moments of the distribution will approach limits as n gets large or approaches infinity, i.e., \( \sigma_n^2 \) approaches \( \sigma_v^2 / n \) and \( \hat{x}(n) \) the average of the observations, becoming increasingly insensitive to additional observations y.
It is interesting to compare these findings with results obtained from the Bayesian framework used in
traditional ‘stationary’ metrology. In this case, information about the state at time \( n \) is described by a
Gaussian distribution whose mean is [5]

\[
\hat{x}(n) = \frac{\sigma^2 \hat{x}(0) + \sigma_v^2 \sum^n y(n)}{n\sigma_0^2 + \sigma_v^2}
\]

(9)

The variance is also given by equation (8).

Equations (5), (7) and (9) are all equivalent leading to the same numerical result.

Figure 7 shows the differences of the mean of observations from 1 to \( n \) and the Kalman estimates at time
\( n \). For large \( n \) and neglecting 1/f - noise effects, these differences get close to zero on the limit as
expected. Although these differences appear negligible in this example, the Kalman filter delivers the
optimum estimates when nonstationarity is rampant.

Figure 8 shows a simulated outcome of Kalman estimates with the evaluated Kalman uncertainties as
given in equation (8).

![Figure 7](image7.png)

**Figure 7.** Deviations of the mean of measurements in respect to Kalman estimates in
dependence of \( n \).

![Figure 8](image8.png)

**Figure 8.** A simulated outcome of Kalman estimates with the evaluated Kalman uncertainties as given in equation (8).

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