Josephson tunnel junctions drastically depend on their geometrical configurations. The shape of the junction determines the specific form of the magnetic-field dependence of the Josephson current. Here we address the magnetic diffraction patterns of specially shaped planar Josephson tunnel junctions in the presence of an in-plane magnetic field of arbitrary orientations. We focus on a wide ensemble of junctions whose shape is invariant under point reflection. We analyze the implications of this type of isometry and derive the threshold curves of junctions whose shape is the union or the relative complement of two point symmetric plane figures.
Identifying \( k_x \) and \( k_y \) with, respectively, \( \kappa H \cos \theta \) and \( \kappa H \sin \theta \), the critical current is achieved when \( 2\phi_0 = \pm \pi + k_y(y_2 + y_1) - k_x(x_2 + x_1) \) and we end out with the well-known double Fraunhofer diffraction pattern of a rectangular JTJ in an arbitrarily oriented magnetic field \[1\]:

\[
I_c^R(H, \theta) = I_c |A_R F_R(H, \theta) - A_v \mathcal{F}_v(H, \theta)|, \quad (9)
\]

where \( A_R = WL \), \( A_v = \omega l \) and the characteristic functions \( \mathcal{F}_v \) and \( \mathcal{F}_R \) are defined through the MDPs of the fictitious inner and outer rectangular junctions, respectively, \( \mathcal{F}_v(H, \theta) \equiv J_c |A_v \mathcal{F}_v(H, \theta)| \) and \( \mathcal{F}_R(H, \theta) \equiv J_c |A_R \mathcal{F}_R(H, \theta)| \). Interestingly, Eq.\[9\] still holds if the rectangles are not parallel \[10\]. Moreover, a similar expression also applies when one or both the rectangles are replaced by arbitrarily oriented concentric rhombuses or ellipses.

We remind that i) for a small diamond-like JTJ of diagonals \( P \) and \( Q \) parallel to Cartesian axes the characteristic function is \[11\]:

\[
\mathcal{F}_D(H, \theta) = 2 \frac{\cos[(\kappa H P \sin \theta)/2] - \cos[(\kappa H Q \cos \theta)/2]}{(\kappa H P \sin \theta/2)^2 - (\kappa H Q \cos \theta/2)^2}, \quad (10)
\]

and ii) for a planar JTJ delimited by an axis-aligned ellipse of principal semi-axes \( a \) and \( b \) it is \[12\]:

\[
\mathcal{F}_E(H, \theta) = 2 J_1 [\kappa H p_E(\theta)/2] / \kappa H p_E(\theta)/2^2, \quad (11)
\]

where \( J_1 \) the 1st order Bessel function of the first kind and \( p_E(\theta) \equiv 2 \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta} \) the length of the projection of the ellipse in the direction normal to the externally applied magnetic field. Eq.\[11\], first reported by Peterson et al. \[12\] in 1990, generalizes the so called Airy pattern of a circular junction \[1\] of radius \( r = a = b \).

Indeed, we found that the MDP of a complementary JTJ resulting from the difference (complement), \( s' = S - s \), of two concentric plane figures with two lines of symmetry (including unconventional shapes like, for example, crosses, bow-ties, s-shapes and figure-eights), can be expressed in terms of their areas, \( A_S \) and \( A_s \), and characteristic functions, \( \mathcal{F}_S \) and \( \mathcal{F}_s \), that is:

\[
I_c^S(H, \theta) = I_c |A_s \mathcal{F}_S(H, \theta) - A_v \mathcal{F}_v(H, \theta)|. \quad (12)
\]

A similar expression was proved for the sum (union), \( S \), of two disjoint figures, \( s \) and \( s' \), namely:

\[
I_c^S(H, \theta) = J_c |A_s \mathcal{F}_s(H, \theta) + A_v \mathcal{F}_v(H, \theta)|. \quad (13)
\]

In the following, we will demonstrate that the broadest geometrical requirement for the validity of Eqs.\[12\] and \[13\] is the point-symmetry of the plane figures.
Let us consider a small JTJ whose shape has a second order point or central-inversion symmetry, that is to say, is invariant upon a 180° rotation around one point called center of symmetry, namely upon reflections in two perpendicular lines. If we pick any Cartesian system with origin in the center of symmetry, then the figure contour in the first (second) quadrant is reproduced specularly in the third (fourth) quadrant. One example of point-symmetric figure is illustrated by the gray shape in Figure 2. Any line drawn through the center of symmetry crosses the figure in two equidistant points. If we rotate the figure of any arbitrary angle, \( \gamma \), around its center of symmetry, we still obtain a point symmetric figure (see the dashed contour in Figure 2). Furthermore, if the \( S \) and \( s \subset S \) are two concentric point symmetric figures, the relative complement of \( s \) in \( S \) is also point symmetric. The category of point symmetric plane figures is wider that the set of figures with two perpendicular lines of symmetry; more generally, it includes regular digons (degenerate polygons with two edges and two vertices), parallelograms and other polygons having an even number of sides with opposite sides equal in length and parallel, to mention a few non-trivial ones.

By way of example of the use of Eq. (13) applied to point symmetric JTJs, a \(+\)-shape can be considered as the union of two point symmetric figure: a rectangle and the union of two disjoint squares. (Generalizing, an \( X \)-shaped surface is a point symmetric figure regardless of the angle between its legs; it results from the union of a parallelogram with the union of two disjoint trapezoids.)

For a point symmetric JTJ, in force of the sine function oddity, the surface integral over its surface \( S \) of \( \sin(k_x x - k_y y) \) is automatically zero; therefore, the \( \phi_0 \)-dependence of the Josephson current in Eq. (4) is simply sinusoidal, for any value of the field angle \( \theta \):

\[
I_J(H, \theta, \phi_0) = J_c A_s \mathcal{F}_S(H, \theta) \sin \phi_0,
\]

where we defined:

\[
\mathcal{F}_S(H, \theta) = \frac{1}{A_s} \int_S \cos [\kappa H (x \cos \theta - y \sin \theta)] \, dS.
\]

If we decompose the junction surface, \( S \), in two non-overlapping surfaces, \( s \) and \( s' \), then the total current, \( I_J \), is given by the sum of the currents in each surface:

\[
I_J^s(H, \theta, \phi_0) + I_J^{s'}(H, \theta, \phi_0).
\]

If and only if \( s \) and \( s' \) are point-symmetric concentric figures, \( I_J^s \) and \( I_J^{s'} \) have the same dependence on \( \phi_0 \) as \( I_J \) in Eq. (14). Therefore, Eq. (16) can be rewritten as:

\[
I_J^s(H, \theta, \phi_0) = J_c A_s [\mathcal{F}_S(H, \theta) + \mathcal{F}_{s'}(H, \theta)] \sin \phi_0,
\]

with \( \mathcal{F}_s \) and \( \mathcal{F}_{s'} \) defined as in Eq. (15). We observe here that upon a rotation around the center of symmetry of an angle \( \gamma \) relative to the \( Y \)-axis, \( \mathcal{F}(H, \theta) \) transforms to \( \mathcal{F}(H, \theta - \gamma) \), i.e., rotations do not affect the \( \phi_0 \)-dependence of \( I_J \). Inserting Eq. (17) into Eq. (1), we readily get Eq. (13). Eq. (12) can be derived in a similar fashion, if a surface \( s' \) is the relative complement of \( s \) in \( S \), with \( s \) and \( S \) point-symmetric and concentric plane figures.

As an application of the above theory, let us consider a JTJ shaped in a rectangular annulus, as that shown in Figure 1, upon the assumption that its widths are much smaller than its mean dimensions \( \bar{w} \) and \( \bar{l} \), i.e.,

\[
2\Delta w \equiv W - w < < (W + w)/2 \equiv \bar{w} \quad \text{and} \quad 2\Delta l = L - l < < (L + l)/2 \equiv \bar{l}.
\]

Under such small widths approximation, we can readily compute the MDP of the annular junction, inserting Eq. (8) in Eq. (12):

\[
I_{cR}^\Delta(H, \theta) = J_c A_{\Delta R} \mathcal{F}_{\Delta R}(H, \theta) \sin \phi_0,
\]

where \( A_{\Delta R} = 2(\bar{w}\Delta l + \bar{l}\Delta w) \) is the annulus area and

\[
\mathcal{F}_{\Delta R}(H, \theta) = \frac{\Delta l \cos[(\kappa H \bar{l} \cos \theta)/2] \sin[(\kappa H \bar{w} \sin \theta)/2]}{[\kappa H (\bar{w}\Delta l + \bar{l}\Delta w) \sin \theta]/2} + \frac{\Delta w \cos[(\kappa H \bar{w} \sin \theta)/2] \sin[(\kappa H \bar{l} \cos \theta)/2]}{[\kappa H (\bar{w}\Delta l + \bar{l}\Delta w) \cos \theta]/2}.
\]

If the field is applied along the annulus diagonal, then \( \bar{w} \sin \theta = \bar{l} \cos \theta = \bar{p}_R/2 \equiv \bar{p}_R(\theta)/2 \) and the last expression, independently of \( \Delta l \) and \( \Delta w \), reduces to a (single) Fraunhofer pattern:

\[
\mathcal{F}_{\Delta R}(H) = \frac{\sin \kappa H \bar{p}_R/2}{\kappa H \bar{p}_R/2}.
\]
and resonances which also involves surface integrals.

the magnetic dependence of the amplitudes of the Fiske
ties can also be invoked to simplify the computation of
given in Eq.(12) [13]. Furthermore, the additive proper-
vious works dealing with point symmetric JTJs and in
centers of symmetry aligned with the magnetic field.
several inner boundary of point symmetric plane figures
whose surface is determined by an outer boundary and
applied to multiply connected Josephson tunnel junctions
does not alter the its MDP. A similar reasoning can be
annular junction along the direction of the applied field
other words, the translation of the inner boundary of an
Eq.(15) transforms to (\(\kappa H p R/4\)) = (\(\kappa H p R/4\)) ×

Therefore, the MDP of a narrow rectangular annular
JTJ, when compared to that of a full rectangular junc-
tion, although qualitatively similar, has an halved peri-
odicity and a slower sidelobes suppression.

If \(s\) and \(s'\) are not concentric, Eqs. (12) and (13) only
apply when the magnetic field is aligned with the two
centers of symmetry, that is, when the field angle is \(\theta =
\arctan x_c/y_c\), where \((x_c, y_c)\) are the coordinates of one
center relative to the other. In fact, upon a translation
\((x, y) = (x' + x_c, y' + y_c)\), the quantity \((x \cos \theta - y \sin \theta)\) in
Eq.(15) transforms to \((x' \cos \theta - y' \sin \theta + x_c \cos \theta - y_c \sin \theta)\)
and \(x_c \cos \theta - y_c \sin \theta = 0\) only when \(\tan \theta = x_c/y_c\). In
other words, the translation of the inner boundary of an
annular junction along the direction of the applied field
does not alter the its MDP. A similar reasoning can be
applied to multiply connected Josephson tunnel junctions
whose surface is determined by an outer boundary and
several inner boundary of point symmetric plane figures
with centers of symmetry aligned with the magnetic field.

The symmetry properties were not envisaged in pre-
vious works dealing with point symmetric JTJs and in
which the calculated MDPs were not cast in the form
given in Eq.(12) [13]. Furthermore, the additive proper-
ties can also be invoked to simplify the computation of
the magnetic dependence of the amplitudes of the Fiske-
resonances which also involves surface integrals.

\[
F_R(H) = \left( \frac{\sin \kappa H p R/4}{\kappa H p R/4} \right)^2.
\]

\[I_c^R(H, \theta) = J_c A_R \sin \left( \frac{\kappa H W (\sin(\theta - \gamma)/2)}{\kappa H W (\sin(\theta - \gamma)/2)} \times
\sin \left( \frac{\kappa H L (\cos(\theta - \gamma)/2)}{\kappa H L (\cos(\theta - \gamma)/2)} \right) \right).
\]

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