QUANTITATIVE BESICOVITCH PROJECTION THEOREM FOR IRREGULAR SETS OF DIRECTIONS

DAMIAN DĄBROWSKI

Abstract. The classical Besicovitch projection theorem states that if a planar set \( E \) with finite length is purely unrectifiable, then almost all orthogonal projections of \( E \) have zero length. We prove a quantitative version of this result: if \( E \subset \mathbb{R}^2 \) is AD-regular and there exists a set of direction \( G \subset S^1 \) with \( \mathcal{H}^1(G) \gtrsim 1 \) such that for every \( \theta \in G \) we have \( \|\pi_{\theta} \mathcal{H}^1|_E\|_{L^\infty} \lesssim 1 \), then a big piece of \( E \) can be covered by a Lipschitz graph \( \Gamma \) with \( \text{Lip}(\Gamma) \lesssim 1 \). The main novelty of our result is that the set of good directions \( G \) is assumed to be merely measurable and large in measure, while previous results of this kind required \( G \) to be an arc.

As a corollary, we obtain a result on AD-regular sets which avoid a large set of directions, in the sense that the set of directions they span has a large complement. It generalizes the following easy observation: a set \( E \) is contained in some Lipschitz graph if and only if the complement of the set of directions spanned by \( E \) contains an arc.

Contents

1. Introduction 1
2. Sketch of the proof 8
3. Preliminaries 10
4. Main proposition and proof of Theorem 1.7 11
5. Rectangles and generalized cubes 13
6. Conical energies 20
7. Estimating interior energy and obtaining good cones 26
8. Estimating exterior energy 31
9. Proof of the key geometric lemma 33
Appendix A. Proof of Corollary 3.2 43
References 44

1. Introduction

1.1. Besicovitch projection theorem. A Borel set \( E \subset \mathbb{R}^2 \) is said to be purely unrectifiable if for any (1-dimensional) Lipschitz graph \( \Gamma \subset \mathbb{R}^2 \) we have

\[ \mathcal{H}^1(E \cap \Gamma) = 0. \]
One of the fundamental results of geometric measure theory is the Besicovitch projection theorem, which states that if \( E \subset \mathbb{R}^2 \) is purely unrectifiable and \( \mathcal{H}^1(E) < \infty \), then almost all orthogonal projections of \( E \) have zero length. We reformulate this result below in a way that is more suitable for the purpose of this article.

Let \( T := \mathbb{R}/\mathbb{Z} \), and for \( \theta \in T \) we set \( e_\theta := (\cos(2\pi \theta), \sin(2\pi \theta)) \), and \( \pi_\theta(x) := e_\theta \cdot x \), so that \( \pi_\theta : \mathbb{R}^2 \to \mathbb{R} \) is the orthogonal projection map to the line \( \ell_\theta := \text{span}(e_\theta) \).

**Definition 1.1.** Given a Borel set \( E \subset \mathbb{R}^2 \), we define its *Favard length* (also known as its *Buffon’s needle probability*) as

\[
\text{Fav}(E) = \int_0^1 \mathcal{H}^1(\pi_\theta(E)) \, d\theta.
\]

**Theorem A (Bes39).** Let \( E \subset \mathbb{R}^2 \) be an \( \mathcal{H}^1 \)-measurable set with \( 0 < \mathcal{H}^1(E) < \infty \). Suppose that \( \text{Fav}(E) > 0 \). Then, there exists a Lipschitz graph \( \Gamma \) such that

\[
\mathcal{H}^1(\Gamma \cap E) > 0.
\]

The planar result stated above is due to Besicovitch [Bes39], see [Mat95, Theorem 18.1] for a modern reference. A higher dimensional counterpart of Theorem A dealing with \( n \)-dimensional subsets of \( \mathbb{R}^d \), was shown by Federer [Fed47], see also an alternative proof due to White [Whi98]. In this paper we will only be concerned with 1-dimensional subsets of \( \mathbb{R}^2 \).

Note that Theorem A is a purely qualitative result: it gives no estimate on the size of \( \mathcal{H}^1(\Gamma \cap E) \), nor on the Lipchitz constant of \( \Gamma \). In the last thirty years many classical definitions and results of geometric measure theory have been quantified (see e.g. [Jon90, DS91, DS93a, AT15, TT15, Tol17]), finding applications in PDEs and harmonic analysis (see e.g. [Dav98, Tol03, Tol05, NTV14, AHM+16, AHM+20]). However, obtaining a quantitative counterpart to Theorem A proved to be a notoriously difficult problem. Beyond its intrinsic appeal, this question is closely related to Vitushkin’s conjecture, which we briefly discuss in Subsection 1.5.

The problem of quantifying Theorem A has seen a number of breakthroughs in the last few years [MO18, CT20, Orp21], which we will discuss shortly. In this article we make further progress on this question.

### 1.2. Quantifying Besicovitch projection theorem

In order to state our result, we need to quantify the finite length assumption of Theorem A.

**Definition 1.2.** We say that a set \( E \subset \mathbb{R}^2 \) is Ahlfors-David-regular, or AD-regular, if \( E \) is closed and there exists a constant \( C \geq 1 \) such that for all \( x \in E \) and \( 0 < r < \text{diam}(E) \)

\[
C^{-1}r \leq \mathcal{H}^1(E \cap B(x, r)) \leq Cr.
\]

We will say that \( E \) is AD-regular with constant \( C_0 \) if the inequality above holds with \( C = C_0 \).

The following conjecture, if true, would be a very satisfactory quantitative version of the Besicovitch projection theorem.

**Conjecture 1.3.** Let \( s \in (0, 1) \), \( C_0 \in (1, \infty) \), and let \( E \subset \mathbb{R}^2 \) be a bounded AD-regular set with constant \( C_0 \). Suppose that

\[
\text{Fav}(E) \geq s \text{diam}(E). \tag{1.1}
\]
Then, there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\text{Lip}(\Gamma) \lesssim_{s,C_0} 1$ and

$$\mathcal{H}^1(\Gamma \cap E) \gtrsim_{s,C_0} \mathcal{H}^1(E).$$

**Remark 1.4.** A weaker version of Conjecture 1.3 was stated by David and Semmes in 1993 [DS93], and very recently proved by Orponen [Orp21]. This is Theorem C discussed below.

**Remark 1.5.** The AD-regularity assumption in Conjecture 1.3 cannot be dropped nor replaced by the weaker assumption $\mathcal{H}^1(E) \sim \text{diam}(E)$, see [CDOV22, Proposition 6.1].

**Remark 1.6.** Observe that the assumption (1.1) implies that there exists an $\mathcal{H}^1$-measurable set $G \subset \mathbb{T}$ with $\mathcal{H}^1(G) \gtrsim s$ such that

$$\mathcal{H}^1(\pi_\theta(E)) \gtrsim s \text{diam}(E) \quad \text{for all } \theta \in G.$$  

(1.2)

That is, $\text{Fav}(E) \geq s \text{diam}(E)$ implies that there exists a big set $G$ of “good directions” where $E$ has big projections.

On the other hand, the existence of a set $G$ as above implies that $\text{Fav}(E) \gtrsim s^2 \text{diam}(E)$. Hence, the two conditions are equivalent, up to a constant. We stress that, a priori, the set of good directions $G$ arising from (1.1) is only measurable and large in measure, possibly very scattered and irregular.

Significant progress towards proving Conjecture 1.3 has been recently achieved by Martikainen and Orponen [MO18] and in the aforementioned work of Orponen [Orp21]. We make further progress by proving the following result.

**Theorem 1.7.** Let $s \in (0,1), C_0, M \in (1,\infty)$, and let $E \subset \mathbb{R}^2$ be a bounded AD-regular set with constant $C_0$. Set $\mu = \mathcal{H}^1|_E$. Assume that there exists an $\mathcal{H}^1$-measurable set $G \subset \mathbb{T}$ with $\mathcal{H}^1(G) \gtrsim s$ and such that

$$\|\pi_\theta \mu\|_{L^\infty(\mathbb{R})} \leq M \quad \text{for all } \theta \in G,$$  

(1.3)

where $\pi_\theta \mu$ is the push-forward of $\mu$ by $\pi_\theta$.

Then, there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\text{Lip}(\Gamma) \lesssim_{C_0,M} 1$ and

$$\mathcal{H}^1(\Gamma \cap E) \gtrsim_{s,C_0,M} \mathcal{H}^1(E).$$

Note that the $L^\infty$-condition (1.3) implies the big projections condition (1.2):

$$\mathcal{H}^1(\pi_\theta(E)) \gtrsim M^{-1} \mu(E) \gtrsim M^{-1} C_0^{-1} \text{diam}(E),$$

but in general (1.3) is much stronger than (1.2).

**Remark 1.8.** The main novelty of Theorem 1.7 is that it allows us to work with a set of directions $G \subset \mathbb{T}$ which is merely $\mathcal{H}^1$-measurable and large in measure, just like the set of good directions arising from Conjecture 1.3 (see Remark 1.6). Previous results of this type, which we discuss below, needed to assume something about projections in a large interval of directions. Just how big of a difference this makes is discussed further in Remark 1.13.
1.3. **Comparison with results of Martikainen and Orponen.** Let us compare Theorem 1.7 with the results from [MO18] and [Orp21]. We only state their planar versions for simplicity, but both have higher-dimensional counterparts.

**Theorem B (MO18).** Let \( s \in (0, 1), C_0, M \in (1, \infty) \), and let \( E \subset \mathbb{R}^2 \) be an AD-regular set with constant \( C_0 \). Let \( E_1 \subset E \cap B(0, 1) \) be an \( H^1 \)-measurable subset with \( H^1(E_1) \geq s \).

Set \( \mu = H^1|_{E_1} \).

Assume there exists \( \theta_0 \in \mathbb{T} \) such that for \( G = (\theta_0, \theta_0 + s) \) we have

\[
\int_G \| \pi_{\theta_\mu} \|_{L^2(\mathbb{R})}^2 d\theta \leq M. \tag{1.4}
\]

Then, there exists a Lipschitz graph \( \Gamma \subset \mathbb{R}^2 \) with \( \text{Lip}(\Gamma) \lesssim s, C_0, M \) and \( H^1(\Gamma \cap E_1) \gtrsim s, C_0, M H^1(E_1) \).

The result below was conjectured in [DS93b], and it was proved very recently by Orponen.

**Theorem C (Orp21).** Let \( s \in (0, 1), C_0 \in (1, \infty) \), and let \( E \subset \mathbb{R}^2 \) be an AD-regular set with constant \( C_0 \). Suppose that for every \( x \in E \) and \( 0 < r < \text{diam}(E) \) there exists \( \theta_{x,r} \in \mathbb{T} \) such that for all \( \theta \in G_{x,r} = (\theta_{x,r}, \theta_{x,r} + s) \) we have

\[
H^1(\pi_{\theta}(E \cap B(x,r))) \geq sr. \tag{1.5}
\]

Then, for every \( x \in E \) and \( 0 < r < \text{diam}(E) \) there exists a Lipschitz graph \( \Gamma_{x,r} \subset \mathbb{R}^2 \) with \( \text{Lip}(\Gamma_{x,r}) \lesssim s, C_0 \) and \( H^1(\Gamma_{x,r} \cap E \cap B(x,r)) \gtrsim s, C_0 H^1(E \cap B(x,r)) \).

Observe that none of the three results above (Theorem 1.7, Theorem B, Theorem C) implies any other, at least not in an obvious way. We summarize the main differences between them below.

Firstly, as already mentioned in Remark 1.8 in all three results we assume that \( H^1(G) \geq s \), but in Theorem 1.7 we only assume that \( G \) is \( H^1 \)-measurable, whereas in the other two results we assume that \( G \) is an interval. We achieved this improvement at the cost of assuming better regularity of \( \pi_{\theta_\mu} \) for each \( \theta \in G \) than in either Theorem B or Theorem C, compare (1.3) with (1.4) and (1.5).

Secondly, observe that Theorem 1.7 and Theorem B are “single-scale results”, whereas Theorem C is a “multi-scale result”, in the sense that in Theorem C one needs to assume that \( E \) has big projections at all scales and locations in order to get Lipschitz graphs covering \( E \). Obtaining a single-scale version of Theorem C is an open problem stated in [Orp21] Question 1.

Finally, Theorem B holds for large subsets of AD-regular sets, whereas Theorem 1.7 and Theorem C have only been proven for AD-regular sets.

1.4. **Related results.** In [DS93b] David and Semmes proved that if \( E \subset \mathbb{R}^2 \) is AD-regular, it satisfies the weak geometric lemma (a multi-scale flatness property), and \( H^1(\pi_{\theta}(E)) \gtrsim 1 \) for some \( \theta \in \mathbb{T} \) (a single direction is enough!), then \( E \) contains a big piece of a Lipschitz graph.

In [JKV97] the authors proved a quantitative Besicovitch projection theorem for sets \( E \) which are boundaries of open sets. The structure of sets with nearly maximal Favard
length was studied in [CDOV22]. A version of Besicovitch projection theorem for Radon measures was recently shown in [Tas22]. A version of the Besicovitch projection theorem for metric spaces was proved in [Bat20].

See [CT20, Dąb22] for the study of conical energies, which we also use in the proof of Theorem 1.7. Closely related concepts of conical defect and measures carried by Lipschitz graphs were studied in [BN21].

An alternative approach to quantifying Besicovitch projection theorem is to estimate the rate of decay of Favard length of $\delta$-neighbourhoods of certain purely unrectifiable sets. See [Mat90, PS02, Tao09, LZ10, BV10a, BV10b, NPV11, BLV14, Lab14, Wil17, Bon19, LM22].

The Besicovitch projection theorem, and some of the results mentioned above, have been also proven for generalized projections in place of orthogonal projection. See [HJJL12, BV11, CDT20, BT21, DT22].

1.5. Vitushkin’s conjecture. One of the main motivations for the study of Conjecture 1.3 is to complete the solution to Vitushkin’s conjecture, which asks for the relation between Favard length and analytic capacity. Different parts of the conjecture have been verified or disproved in [Cal77, Dav98, Mat86, JM88], but one question remains: given a 1-dimensional compact set $E \subset \mathbb{R}^2$ with non-$\sigma$-finite length and $\text{Fav}(E) > 0$, is the analytic capacity of $E$ positive? It is beyond the scope of this introduction to discuss this in detail, but let us mention that recent progress on this problem made in [CT20] and [DV22] used the ideas and results obtained in [MO18] and [Orp21], respectively. Solving Conjecture 1.3 (or even it’s weaker, multi-scale version) would immediately mark substantial progress on this question, see [DV22, Remark 1.9]. We refer the interested reader to [DV22] for details.

1.6. Directions spanned by sets. We give an application of Theorem 1.7 to directions spanned by sets.

**Definition 1.9.** Given a Borel set $E \subset \mathbb{R}^2$ we define the set of directions spanned by $E$ as

$$D(E) := \left\{ \frac{x-y}{|x-y|} : x, y \in E, x \neq y \right\} \subset S^1,$$

or, using our preferred parametrization of the circle,

$$D_T(E) := \frac{1}{2\pi} \arg(D(E)) \subset \mathbb{T}.$$

We will denote the complement of $D_T(E)$ by $G_T(E)$, and we will say that the directions in $G_T(E)$ are avoided by $E$.

Sets of directions spanned by subsets of $\mathbb{R}^d$ have been studied in [OS11, IMS12]. They are closely related to radial projections due to the fact that

$$D(E) = \bigcup_{x \in E} \pi_x(E \setminus \{x\}),$$

where $\pi_x(y) = \frac{x-y}{|x-y|}$ is the radial projection map from $x$. The behaviour of purely unrectifiable sets under radial projections was studied in [Mar54, SS06, BLZ16]. See also [Mat81, Cső00, Cső01, VV22, DG22, OSW22].
Remark 1.10. Given $G \subset \mathbb{T}$ and $x \in \mathbb{R}^2$, consider the cone $X(x, G) := \bigcup_{\theta \in G} \ell_{x, \theta}$, where $\ell_{x, \theta} = x + \text{span}(e_\theta)$. Note that if $E \subset \mathbb{R}^2$ satisfies $G_T(E) \neq \emptyset$, then
\[ E \cap X(x, G_T(E)) = \{x\} \quad \text{for all } x \in E, \]
and $G_T(E)$ is the largest subset of $\mathbb{T}$ with this property.

The following is an easy observation used in many geometric measure theory proofs (for example, in the proof of Theorem A).

Observation 1.11. A set $E \subset \mathbb{R}^2$ is contained in some Lipschitz graph $\Gamma \subset \mathbb{R}^2$ if and only if there exists a (non-degenerate) interval $I \subset \mathbb{T}$ such that
\[ I \subset G_T(E). \]
Furthermore, we have $\text{Lip}(\Gamma) \lesssim \mathcal{H}^1(I)^{-1}$. Usually this result is stated in terms of the “empty cone condition”
\[ E \cap X(x, I) = \{x\} \quad \text{for all } x \in E, \]
but this is equivalent by Remark 1.10. See [Mat95, Lemma 15.13] or [MO18, Remark 1.11] for an easy proof.

It is natural to ask if the following generalization of the observation above is true:

Question 1.12. Let $s \in (0, 1)$, $C_0 \geq 1$. Suppose that $E \subset \mathbb{R}^2$ is a bounded AD-regular set with constant $C_0$, and that
\[ \mathcal{H}^1(G_T(E)) \geq s. \]
Is it possible to find a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\text{Lip}(\Gamma) \lesssim s, C_0$ and
\[ \mathcal{H}^1(\Gamma \cap E) \gtrsim s, C_0 \mathcal{H}^1(E)? \]

Remark 1.13. Note that in Question 1.12 we added many assumptions compared to Observation 1.11, we weakened the conclusion, and the only assumption that is weaker in Question 1.12 is that we assume no additional structure on $G_T(E)$ beyond large $\mathcal{H}^1$-measure. This makes all the difference: the case of a big interval, as in Observation 1.11, is very easy, whereas Question 1.12 appears to be non-trivial. Similarly, the fact that Theorem 1.7 does not assume much regularity about the set of good directions $G$ leads to genuinely new difficulties compared to Theorem B and Theorem C, and it is not merely a cosmetic difference.

Using Theorem 1.7 we are able to answer affirmatively the following special case of Question 1.12.

Corollary 1.14. Let $s \in (0, 1)$, $C_0 \geq 1$. Suppose that $E \subset \mathbb{R}^2$ is a bounded AD-regular set with constant $C_0$, and that
\[ \mathcal{H}^1(G_T(E)) \geq s. \]
Suppose further that $E$ is a union of parallel line segments. Then, there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\text{Lip}(\Gamma) \lesssim s, C_0$ and
\[ \mathcal{H}^1(\Gamma \cap E) \gtrsim s, C_0 \mathcal{H}^1(E). \]
Proof. Let $\theta_0 \in \mathbb{T}$ be such that the line segments comprising $E$ are parallel to $\ell_{\theta_0}$. Set $G := G_T(E) \setminus (\theta_0 - 0.1s, \theta_0 + 0.1s)$.

Let $\theta \in G$ and $y \in \pi_\theta(E)$. Since $E$ avoids the direction $\theta$, we get that $E$ is a graph over $\ell_{\theta}$, and it consists of segments forming angle $\angle(\ell_{\theta_0}, \ell_{\theta}) \sim |\theta - \theta_0|$ with $\ell_{\theta} = (\ell_{\theta_0})^\perp$. It follows that

$$\pi_\theta \mathcal{H}^1|_E(y) = \lim_{h \to 0} \frac{\mathcal{H}^1(E \cap (\pi_\theta)^{-1}((y - h, y + h)))}{h} \lesssim \lim_{h \to 0} \frac{|\theta - \theta_0|^{-1}h}{h} \lesssim s^{-1}.$$ 

Hence, $\|\pi_\theta \mathcal{H}^1|_E\|_\infty \lesssim s^{-1}$. Since

$$\mathcal{H}^1(G) \geq \mathcal{H}^1(G_T(E)) - 0.2s \geq \frac{s}{2},$$

we may apply Theorem 1.7 (with $G^\perp$ instead of $G$) to find the desired Lipschitz graph $\Gamma$ with $\text{Lip}(\Gamma) \lesssim_{s, C_0} s$ and $\mathcal{H}^1(\Gamma \setminus E) \gtrsim_{s, C_0} \mathcal{H}^1(E)$.

We mention another interesting question in the same vein, which is essentially a qualitative version of Question 1.12.

It follows from the definition of purely unrectifiable sets and Observation 1.11 that if $E$ is purely unrectifiable and $\mathcal{H}^1(E) > 0$, then $D_T(E)$ is dense in $\mathbb{T}$. What can be said about $\mathcal{H}^1(D_T(E))$?

**Question 1.15.** Suppose that $E \subset \mathbb{R}^2$ is purely unrectifiable, and $0 < \mathcal{H}^1(E) < \infty$. Do we have

$$\mathcal{H}^1(D_T(E)) = \mathcal{H}^1(\mathbb{T})?$$

The answer is yes for *homogeneous sets* (examples of which include self-similar sets satisfying the strong separation condition for which the linear parts of the similarities contain no rotations) by [RS19, Proposition 3.1]; in fact, for such sets Rossi and Shmerkin proved that $D_T(E) = \mathbb{T}$. To the best of our knowledge, the question is open for general purely unrectifiable sets. Up until recently it wasn’t even clear if $\dim_H(D_T(E)) = 1$, but this follows from a recent paper of Orponen, Shmerkin, and Wang [OSW22].

1.7. **Plan of the article.** In Section 2 we sketch the proof of Theorem 1.7. In Section 3 we introduce some notation, list all the parameters appearing in the proof, and remind some useful results from [CT20] and [Dąb22]. In Section 4 we state our main proposition, Proposition 4.1, and we show how it can be used to prove Theorem 1.7. We prove the main proposition in Sections 5–9. In Section 6 we introduce a “dyadic grid of rectangles” adapted to Proposition 4.1 and we prove some basic measure estimates on these rectangles. Section 6 contains a stopping time argument and a corona decomposition involving conical energies. In Sections 7–9 we estimate these energies. Finally, in Appendix A we prove one of the results from Section 3.

**Acknowledgments.** I am grateful to Alan Chang, Tuomas Orponen, Xavier Tolsa, and Michele Villa for inspiring discussions.

I was supported by the Academy of Finland via the projects *Incidences on Fractals*, grant No. 321896, and *Quantitative rectifiability and harmonic measure beyond the Ahlfors-David-regular setting*, grant No. 347123.
2. Sketch of the proof

Suppose that \( E \subset \mathbb{R}^2 \) is bounded and AD-regular, \( \mu = \mathcal{H}^1|_E, G \subset \mathbb{T} \) satisfies \( \mathcal{H}^1(G) \gtrsim 1 \), and for all \( \theta \in G \) we have \( \| \pi_\theta \mu \|_\infty \lesssim 1 \). Using Proposition 3.1, which is a result from [CT20], it is easy to show that this implies

\[
\int_{\mathbb{R}^2} \mu(X(x,G_{\perp},r)) \frac{dr}{r} d\mu(x) \lesssim \mu(E),
\]

where \( X(x,G_{\perp},r) = X(x,G_{\perp}) \cap B(x,r) \), and \( X(x,G_{\perp}) \) is the union of lines passing through \( x \) with directions perpendicular to those from \( G \). See [3,1] for the precise definition.

Estimate (2.1) is reminiscent of Proposition 3.3, which was observed in [Dąb22] but is essentially due to [MO18]. This result says that if the estimate (2.1) holds with \( G \) which is a large interval, then one can find a big piece of a Lipschitz graph inside \( E \). The problem is, the set \( G \) given by Theorem 1.7 may be a very complicated set, possibly consisting of many tiny intervals, or not containing any intervals at all.

This issue is addressed by our main proposition, Proposition 4.1. Roughly speaking, it says that if we start with a set of “good directions” \( G \) which almost fills an interval \( J \), then the goodness of \( G_J \) propagates to all of \( J \), and even to the enlarged interval \( 3J \). More precisely, given an interval \( J \subset \mathbb{T} \), possibly very short, and a set \( G_J \subset J \) with \( \mathcal{H}^1(J \setminus G_J) \leq \varepsilon \mathcal{H}^1(J) \), where \( \varepsilon > 0 \) is very small, and under some additional technical assumptions involving \( \| \pi_\theta \mu \|_\infty \), one has

\[
\int_E \int_0^{\text{diam}(E)} \frac{\mu(X(x,3J,r))}{r} \frac{dr}{r} d\mu(x) \leq C_{\text{Prop}} \left( \int_E \int_0^{\text{diam}(E)} \frac{\mu(X(x,G_J,r))}{r} \frac{dr}{r} d\mu(x) + \mathcal{H}^1(J) \mu(E) \right). \tag{2.2}
\]

Crucially, the constants \( \varepsilon \) and \( C_{\text{Prop}} \) do not depend on \( \mathcal{H}^1(J) \).

Using the idea of the good set \( G \) propagating and becoming larger, we are able to apply Proposition 4.1 iteratively, so that after a bounded number of iterations we end up with an estimate (2.1) with the set \( G \) replaced by some interval \( J_0 \) with \( \mathcal{H}^1(J_0) \sim 1 \). This allows us to use Proposition 3.3 to obtain a big piece of Lipschitz graph inside \( E \). All of this is done in Section 4 assuming that Proposition 4.1 is true. The remainder of the paper is dedicated to the proof of Proposition 4.1.

In Section 5 we consider a “dyadic lattice of rectangles” \( D = \bigcup_k D_k \), where each \( D_k \) is a partition of \( E \). The rectangles we work with have a very large, but fixed, aspect ratio equal to \( \mathcal{H}^1(J)^{-1} \), and they all point in the same direction, corresponding to the mid-point of \( J \). A priori, the fact that \( \mu \) is AD-regular only tells us that a rectangle \( Q \in D \) satisfies

\[
\ell(Q) \lesssim \mu(Q) \lesssim \mathcal{H}^1(J)^{-1} \ell(Q),
\]

where \( \ell(Q) \) denotes the length of the shorter side of \( Q \). This is no good: it is crucial that our estimates do not explode as \( \mathcal{H}^1(J) \to 0 \). Luckily, due to one of the assumptions on \( \| \pi_\theta \mu \|_\infty \), we show in Lemma 5.1 that \( \mu(Q) \sim \ell(Q) \). So in a sense, we need the \( L^\infty \)-norm in (1.3), and not just the \( L^2 \)-norm as in Theorem 1.3, to ensure that our rectangles are “AD-regular”.
In Section 6 we introduce conical energies $\mathcal{E}_G(Q)$ and $\mathcal{E}_J(Q)$, associated to $G_J$ and $3J$, respectively. They are essentially local versions of double intergals from (2.2), so that
\[
\int_{\mathbb{R}^2} \int_0^{\text{diam}(E)} \frac{\mu(X(x,G_J,r))}{r} \, dr \, d\mu(x) \sim \sum_{Q \in D} \mathcal{E}_G(Q) \mu(Q),
\]
and an analogous estimate holds for $3J$ and $\mathcal{E}_J(Q)$. Inspired by [CT20], we conduct a stopping time argument and a corona decomposition of $D$ into a family of trees $\text{Tree}(R)$, $R \in \text{Top}$. What we gain is that for any $R \in \text{Top}$ and most $x \in R$ the cone $X(x,G_J)$ does not intersect $E$ at the scales associated to $\text{Tree}(R)$.

In Sections 7 and 8 we prove that for any $R \in \text{Top}$
\[
\sum_{Q \in \text{Tree}(R)} \mathcal{E}_J(Q) \mu(Q) \lesssim \sum_{Q \in \text{Tree}(R)} \mathcal{E}_G(Q) \mu(Q) + \mathcal{H}^1(J) \mu(R),
\]
which is enough to obtain (2.2). To prove the estimate above, we divide $\mathcal{E}_J(Q)$ into an “interior” conical energy $\mathcal{E}_J^{\text{int}}(Q)$ associated to $0.5J$, and an “exterior” conical energy $\mathcal{E}_J^{\text{ext}}(Q)$ associated to $3J \setminus 0.5J$. In Section 7 we deal with the interior part. This is another important point where we use the technical assumptions related to $\lVert \pi_0 \mu \rVert_\infty$: together with AD-regularity of $E$ they allow us to get a strong, pointwise estimate $\mathcal{E}_J^{\text{int}}(Q) \lesssim \mathcal{E}_G(Q)$. As a corollary, we get that for $R \in \text{Top}$ and all $x \in R$ the cone $X(x,0.5J)$ does not intersect $E$ at the scales associated to $\text{Tree}(R)$.

Finally, in Section 8 we estimate the exterior energy $\mathcal{E}_J^{\text{ext}}(Q)$. The argument uses the key geometric lemma of this article, Lemma 8.4, which we prove in Section 9. The proof is purely geometric, and we believe it is the true heart of this article.

A simplified version of Lemma 8.4 says the following:

**Key Geometric Lemma (simplified).** Let $A \subset B(0,1) \subset \mathbb{R}^2$ be an AD-regular sets consisting of horizontal segments. Let $J \subset \mathbb{T}$ be an interval such that $\mathcal{H}^1(J) \leq c$ for a small absolute constant $c > 0$, and such that $X(0,J)$ contains the vertical axis. Assume that
\[
A \cap X(x,J) = \{x\} \quad \text{for every } x \in A.
\]
Suppose that there is a point $y \in A$ and a scale $r \in (0,1)$ such that
\[
A \cap X(y,3J,2r) \setminus B(y,r) \neq \emptyset.
\]
Then, there exists an interval $K \subset \mathbb{R}$, which is a connected component of $\mathbb{R} \setminus \pi_0(A)$ (where $\pi_0$ is the projection to the horizontal axis), such that $\mathcal{H}^1(K) \sim \mathcal{H}^1(J)r$ and $\pi_0(y) \in CK$ for some absolute $C \geq 1$.

It is not too difficult to show using this lemma that a set $A$ as above satisfies
\[
\int_A \int_0^{\text{diam}(A)} \frac{\mathcal{H}^1(A \cap X(x,3J,r))}{r} \, dr \, d\mathcal{H}^1(x) \lesssim \mathcal{H}^1(J) \mathcal{H}^1(A).
\]
This is essentially where the last term in (2.2) comes from.
3. PRELIMINARIES

3.1. Notation. Given $x \in \mathbb{R}^2$ and $\theta \in \mathbb{T}$ we set

$$e_{\theta} := (\cos(2\pi \theta), \sin(2\pi \theta)) \in \mathbb{S}^1,$$

$$\pi_{\theta}(x) := e_{\theta} \cdot x,$$

$$\ell_{x,\theta} := x + \text{span}(e_{\theta}),$$

$$\ell_{\theta} := \ell_{0,\theta}.$$

For $x \in \mathbb{R}^2$ and a measurable set $I \subset \mathbb{T}$ we define the cone centered at $x$ with directions in $I$ as

$$X(x, I) = \bigcup_{\theta \in I} \ell_{x,\theta}.$$

Note that we do not require $I$ to be an interval. We also set $I^\perp = I + 1/4$.

For $0 < r < R$ we define truncated cones as

$$X(x, I, r) = X(x, I) \cap B(x, r),$$

$$X(x, I, r, R) = X(x, I, R) \setminus B(x, r).$$

In case $I = [\theta - a, \theta + a]$, we have an algebraic characterization of $X(x, I)$: $y \in X(x, I)$ if and only if

$$|\pi^1_{\theta} (y) - \pi^1_{\theta} (x)| \leq \sin (2\pi a)|x - y|. \tag{3.1}$$

We will denote by $\Delta$ the usual family of half-open dyadic intervals on $[0, 1) \simeq \mathbb{T}$. If $J \in \Delta$, then $\Delta(J)$ denotes the collection of dyadic intervals contained in $J$. For $I \in \Delta \setminus \{[0, 1)\}$, the notation $I^\perp$ will be used for the dyadic parent of $I$.

Given an interval $I \subset \mathbb{T}$ and $C > 0$, we will write $CI$ to denote the interval with the same midpoint as $I$ and length $C \mathcal{H}^1(I)$.

The closure of a set $A$ will be denoted by $\overline{A}$, and its interior by $\text{int}(A)$.

3.2. Constants and parameters. Whenever we write $f \lesssim g$, this should be understood as “there exists an absolute constant $C > 0$ such that $f \leq C g$.” We will write $f \lesssim_A g$ if we allow the constant $C$ to depend on some parameter $A$. We also write $f \sim g$ to denote $g \lesssim f \lesssim g$, and similarly $f \sim_A g$ stands for $g \lesssim_A f \lesssim_A g$.

Throughout the proof we use many constants and parameters. We list the most important ones here for reader’s convenience. The notation $C_1 = C_1(C_2)$ means “$C_1$ is a parameter whose value depends on the value of parameter $C_2$”.

- $C_0 \geq 1$ is the AD-regularity constant of the set $E$.
- $M \geq 1$ is the constant bounding the $L^\infty$-norm of projections in the assumptions of Theorem 1.7 and Proposition 4.1.
- $s \in (0, 1)$ is the constant from the assumption $\mathcal{H}^1(G) \geq s$ in Theorem 1.7.
- $\varepsilon = \varepsilon(C_0, M) \in (0, 1)$ is a constant appearing in Proposition 4.1; see [4.1]. It is chosen in Lemma 7.2. One could take $\varepsilon = c C_0^{-1} M^{-1}$ for some small absolute $c \in (0, 1)$.
- $C_{\text{prop}} = C_{\text{prop}}(C_0, M) > 1$ is a big constant appearing in the conclusion of Proposition 4.1.
- $c_1 \in (0, 1)$ is a small absolute constant appearing in the assumption $\mathcal{H}^1(J) \leq c_1 C_0^{-1} M^{-1}$ of Proposition 4.1. It is fixed above 0.4.
• $\rho = 1/1000$ is the constant from Theorem 5.3 so that for $Q \in D_k$ we have $\ell(Q) = 4\rho^k$.
• $A = A(C_0, M) \geq 1000$ is a large constant appearing in the definition of $E_G(Q)$ (6.1). It is fixed in Lemma 9.8, one could take $A = CC_0M$ for some absolute $C \geq 1000$.
• $\delta = \delta(A, M, C_0) \in (0, 1)$ is the BCE-parameter, appearing in (5.3). It is fixed in Lemma 7.3.
• $N \sim C_0M$ is a parameter appearing in the definition of rectangles $G_i$, below (9.2). It’s exact value is chosen in Lemma 9.5.

3.3. Useful results on cones and projections. We recall some results that will be useful in our proof. The proposition below is a simplified version of Corollary 3.3 from [CT20].

Proposition 3.1. Let $\mu$ be a finite, compactly supported Borel measure on $\mathbb{R}^2$, and $I \subset \mathbb{T}$ an open set. Then,
\[
\int_{\mathbb{R}^2} \int_0^\infty \frac{\mu(X(x, I, r))dr}{r}d\mu(x) \lesssim \int_I \|\pi_\theta^r\mu\|_2^2 d\theta.
\]
We get the following corollary.

Corollary 3.2. Let $E \subset \mathbb{R}^2$ and $G \subset \mathbb{T}$ be as in Theorem 1.7, and let $\mu = H^1|_E$. Then,
\[
\int_{\mathbb{R}^2} \int_0^\infty \frac{\mu(X(x, G^\perp, r))dr}{r}d\mu(x) \lesssim MH^1(G)\mu(E),
\]
where $G^\perp = G + 1/4$.

If $G$ is open, then this follows almost immediately from Proposition 3.1. The case of a general measurable set $G$ is a long and uninspiring exercise in measure theory, so we postpone it to the appendix.

The following result is a simplified version of Proposition 10.1 from [Dąb22], which in turn is a consequence of Proposition 1.12 from [MO18].

Proposition 3.3. Let $E \subset \mathbb{R}^2$ be a bounded AD-regular set with constant $C_0$. Let $F \subset E$ be such that $H^1(F) \geq \kappa H^1(E)$. Assume there exists an interval $J \subset \mathbb{T}$ with $H^1(J) = s$ such that for $H^1$-a.e. $x \in F$
\[
\int_0^1 \frac{H^1(X(x, J, r) \cap F)}{r} dr \leq M.
\]
Then, there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\text{Lip}(\Gamma) \lesssim s$ and $H^1(F \cap \Gamma) \gtrsim_{s, C_0, M, \kappa} H^1(F)$.

4. Main proposition and proof of Theorem 1.7

The following is our main proposition.

Proposition 4.1. Let $1 \leq C_0, M < \infty$. There exist constants $0 < \varepsilon < 1 < C_{\text{Prop}} < \infty$, which depend on $M, C_0$, such that the following holds. Assume that:
(a) $E \subset \mathbb{R}^2$ is a bounded AD-regular set with constant $C_0$, and set $\mu = H^1|_E$,
(b) $J \subset \mathbb{T}$ is an interval with $\mathcal{H}^1(J) \leq c_1 C_0^{-1} M^{-1}$, where $c_1 > 0$ is a small absolute constant.

c) there exists $\theta_0 \in 3J$ such that $\|\pi_{\theta_0}^\perp \mu\|_{\infty} \leq M$.

d) $G \subset J$ is a closed set which satisfies

$$\mathcal{H}^1(G) \geq (1 - \varepsilon)\mathcal{H}^1(J), \quad (4.1)$$

(e) for every interval $I$ comprising $J \setminus G$ there exists $\theta_I \in 3I$ such that $\|\pi_{\theta_I}^\perp \mu\|_{\infty} \leq M$.

Then,

$$\int_E \int_0^{\text{diam}(E)} \frac{\mu(X(x,3J,r))}{r} \, dr \, d\mu(x) \leq C_{\text{Prop}} \left( \int_E \int_0^{\text{diam}(E)} \frac{\mu(X(x,G,r))}{r} \, dr \, d\mu(x) + \mathcal{H}^1(J) \mu(E) \right).$$

Remark 4.2. In the proposition above, the interval $J$ may be open, closed, or half-open, it doesn’t make a difference. In the conclusion we may take $3J$ to be a closed interval (in fact, the same proof gives the conclusion also with $CJ$ replacing 3J, if we let $C_{\text{Prop}}$ depend on $C$ as well, and as long as $\mathcal{H}^1(CJ) \leq c_1 C_0^{-1} M^{-1}$).

We prove Proposition 4.1 in Sections 5–9. Now let us show how it can be used to prove Theorem 1.7. We begin by proving a corollary of Proposition 4.1, which looks quite similar to Proposition 4.1 itself; the crucial difference is that it deals with sets $G \subset J$ with $\mathcal{H}^1(G) < (1 - \varepsilon)\mathcal{H}^1(J)$. Recall that for a dyadic interval $I \in \Delta$ we denote by $I_1$ the dyadic parent of $I$.

**Corollary 4.3.** Let $1 \leq C_0, M < \infty$. Let $\varepsilon = \varepsilon(M, C_0), C_{\text{Prop}} = C_{\text{Prop}}(M, C_0)$ be as in Proposition 4.1. Assume that:

(a) $E \subset \mathbb{R}^2$ is a bounded AD-regular set with constant $C_0$, and $\mu = \mathcal{H}^1|_E$,

(b) $J \subset \mathbb{T}$ is a dyadic interval with $\mathcal{H}^1(J) \leq c_1 C_0^{-1} M^{-1}$, where $c_1 > 0$ is as in Proposition 4.1,

(c) $G \subset J$ is a finite union of closed dyadic intervals, which satisfies

$$0 < \mathcal{H}^1(G) < (1 - \varepsilon)\mathcal{H}^1(J), \quad (4.2)$$

(d) denoting the collection of maximal dyadic intervals contained in $J \setminus G$ by $\mathcal{B}_\Delta$, for every $I \in \mathcal{B}_\Delta$ there exists $\theta_I \in I^1$ such that $\|\pi_{\theta_I}^\perp \mu\|_{\infty} \leq M$.

Then, there exists a closed set $G^* \subset J$ such that

$$\mathcal{H}^1(G^*) \geq (1 + \varepsilon)\mathcal{H}^1(G), \quad (4.4)$$

which is a finite union of closed dyadic intervals, such that

$$\mathcal{H}^1(G^*) \geq (1 + \varepsilon)\mathcal{H}^1(G), \quad (4.4)$$

and

$$\int_E \int_0^{\text{diam}(E)} \frac{\mu(X(x,G^*,r))}{r} \, dr \, d\mu(x) \leq C_{\text{Prop}} \left( \int_E \int_0^{\text{diam}(E)} \frac{\mu(X(x,G,r))}{r} \, dr \, d\mu(x) + \mathcal{H}^1(J) \mu(E) \right). \quad (4.5)$$
Moreover, denoting by \( B_{\Delta,*} \) the collection of maximal dyadic intervals contained in \( J \setminus G_\ast \), we have
\[
B_{\Delta,*} \subset B_\Delta. \tag{4.6}
\]

The statement above is quite involved, but it is very well-suited for its iterative application later on: note that the resulting set \( G_\ast \) satisfies all the same assumptions as the set \( G \) we started with, except perhaps for the measure assumption (4.2).

We divide the proof of Corollary 4.3 into several steps.

**Definition of \( G_\ast \).** Let \( I \subset \Delta(J) \) be the family of maximal dyadic intervals such that for every \( I \in I \)
\[
\mathcal{H}^1(I \cap G) \geq (1 - \varepsilon)\mathcal{H}^1(I). \tag{4.7}
\]
Since \( G \) is a finite union of closed dyadic intervals, we get immediately that
\[
G \subset \bigcup_{I \in I} \overline{I},
\]
and that \( I \) is a finite family. Observe that the intervals in \( I \) are pairwise disjoint by maximality. Moreover, we have \( J \notin I \) due to (4.2), so that all \( I \in I \) are strictly contained in \( J \).

Consider the family \( I^1 = \{I^1\}_{I \in I} \subset \Delta(J) \), where \( I^1 \) denotes the dyadic parent of \( I \), and let \( I_\ast \) be the family of maximal dyadic intervals from \( I^1 \). The intervals in \( I_\ast \) are pairwise disjoint by maximality, and the family \( I_\ast \) is finite because \( I \) is finite. We set
\[
G_\ast := \bigcup_{l \in I_\ast} \overline{I}.
\]
It remains to show that \( G_\ast \) satisfies (4.3), (4.4), (4.5), and (4.6).

**Proof of (4.3).** Note that
\[
G \subset \bigcup_{l \in I} \overline{I} \subset \bigcup_{l \in I_\ast} \overline{I} = \bigcup_{l \in I_\ast} \overline{I} = G_\ast.
\]
Since \( I_\ast \subset \Delta(J) \), we also have \( G_\ast \subset J \). \( \square \)

**Proof of (4.4).** Recall that \( I \) was defined as the collection of maximal dyadic intervals where (4.7) holds. Let \( I \in I_\ast \). We know that \( I \) is a parent of some \( I' \in I \), and \( I' \) is a maximal interval where (4.7) holds. It follows that \( I \) does not satisfy (4.7), which means that
\[
\mathcal{H}^1(I \cap G) < (1 - \varepsilon)\mathcal{H}^1(I),
\]
or equivalently,
\[
\mathcal{H}^1(I \setminus G) \geq \varepsilon\mathcal{H}^1(I).
\]
Using this estimate we compute
\[
\mathcal{H}^1(G_\ast) = \sum_{l \in I_\ast} \mathcal{H}^1(I) = \sum_{l \in I_\ast} \mathcal{H}^1(I \cap G) + \sum_{l \in I_\ast} \mathcal{H}^1(I \setminus G)
\]
\[
= \mathcal{H}^1(G) + \sum_{l \in I_\ast} \mathcal{H}^1(I \setminus G) \geq \mathcal{H}^1(G) + \varepsilon \sum_{l \in I_\ast} \mathcal{H}^1(I)
\]
\[
= \mathcal{H}^1(G) + \varepsilon \mathcal{H}^1(G_\ast) \geq (1 + \varepsilon)\mathcal{H}^1(G).
\]
This shows (4.4). \( \square \)
We checked all the assumptions of Proposition 4.1, and so we may conclude that Corollary 4.3, and:

Proof of (4.5). Without loss of generality, we may assume that \( \text{diam}(E) = 1 \). Fix \( I \in \mathcal{I}_s \), and let \( J_I \in \mathcal{I} \) be an interval such that \( (J_I)^1 = I \). We claim that we may apply Proposition 4.1 with \( J = J_I \) and \( G = G \cap J_I \). Indeed, assumption (a) is the same as in Corollary 4.3 and:

- assumption (b) holds since \( H^1(J_I) \leq H^1(J) \leq c_1 C_0^{-1} M^{-1} \).
- assumption (c) holds because \( (J_I)^1 = I \) has non-empty intersection with both \( G \) and \( J \setminus G \), so in particular \( I \) strictly contains some \( K \in \mathcal{B}_\Delta \). We assumed that there exists \( \theta_K \in K^1 \subset I \) such that \( ||\pi_{\theta_K}^{1} \mu||_{\infty} \leq M \). Since \( I \subset 3J_I \), we may take \( \theta_0 = \theta_K \).
- assumption (d) follows from the definition of \( \mathcal{I} \) (4.7).
- assumption (e) holds because any interval \( K \) comprising \( J_I \setminus G \) contains some dyadic interval \( K' \in \mathcal{B}_\Delta \), and since \( (K')^1 \subset 3K \), we may take \( \theta_K := \theta_{K'} \).

We checked all the assumptions of Proposition 4.1 and so we may conclude that

\[
\int_E \int_0^1 \frac{\mu(X(x, 3J_I, r))}{r} \frac{dr}{r} d\mu(x) \leq C_{\text{Prop}} \int_E \int_0^1 \frac{\mu(X(x, G \cap J_I, r))}{r} \frac{dr}{r} d\mu(x) + C_{\text{Prop}} H^1(J_I) \mu(E).
\]

Summing over \( I \in \mathcal{I}_s \) yields

\[
\int_E \int_0^1 \frac{\mu(X(x, G_s, r))}{r} \frac{dr}{r} dH^1(x) = \sum_{I \in \mathcal{I}_s} \int_E \int_0^1 \frac{\mu(X(x, I, r))}{r} \frac{dr}{r} d\mu(x) \leq \sum_{I \in \mathcal{I}_s} \int_E \int_0^1 \frac{\mu(X(x, 3J_I, r))}{r} \frac{dr}{r} d\mu(x) \leq \sum_{I \in \mathcal{I}_s} C_{\text{Prop}} \int_E \int_0^1 \frac{\mu(X(x, G \cap J_I, r))}{r} \frac{dr}{r} d\mu(x) + \sum_{I \in \mathcal{I}_s} C_{\text{Prop}} H^1(J_I) \mu(E) \leq C_{\text{Prop}} \int_E \int_0^1 \frac{\mu(X(x, G, r))}{r} \frac{dr}{r} d\mu(x) + C_{\text{Prop}} H^1(J) \mu(E).
\]

This shows (4.5). \( \square \)

Proof of (4.6). Let \( I \in \mathcal{B}_{\Delta,*} \), so that

\[
I \cap G_s = \emptyset \quad \text{and} \quad I^1 \cap G_s \neq \emptyset.
\]  

We want to prove that \( I \in \mathcal{B}_\Delta \). Since \( G \subset G_s \), it is clear that \( I \cap G = \emptyset \), so we only need to show that

\[
I^1 \cap G \neq \emptyset.
\]  

Let \( I' \) be the dyadic sibling of \( I \), that is, the unique interval \( I' \in \Delta(J) \) such that \( I \cup I' = I^1 \). It follows from (4.8) that \( I' \cap G_s \neq \emptyset \). By the definition of \( G_s \), there exists \( P \in \mathcal{I}_s \) such that \( P \cap I' \neq \emptyset \). Hence, we have either \( P \subset I' \) or \( I' \subset P \). The latter would imply \( I^1 \subset P \), which is not possible because \( I \cap P \subset I \cap G_s = \emptyset \). Thus, we have \( P \subset I' \).
Let $J_P \in \mathcal{I}$ be such that $P = (J_P)^1$. By the definition of $\mathcal{I}$, we have
\[ \mathcal{H}^1(J_P \cap G) \geq (1 - \varepsilon) \mathcal{H}^1(J_P). \]
Since $J_P \subset P \subset I'$, it follows that $I' \cap G \neq \emptyset$. In particular, the parent $(I')^1 = I^1$ satisfies $I^1 \cap G \neq \emptyset$. This gives (4.9), and concludes the proof of (4.6).

This finishes the proof of Corollary 4.3.

4.1. Proof of Theorem 1.7

Preliminaries. Recall that $G^\perp = G + 1/4$. Let $J_0 \subset T$ be a dyadic interval with
\[ 2^{-1} c_1 C_0^{-1} M^{-1} \leq \mathcal{H}^1(J_0) \leq c_1 C_0^{-1} M^{-1} \]
and such that
\[ \mathcal{H}^1(J_0 \cap G^\perp) \geq s \mathcal{H}^1(J_0). \]
It is clear that such interval exists since $\mathcal{H}^1(G^\perp) = \mathcal{H}^1(G) \geq s$. Using inner regularity of Lebesgue measure, we may find a closed subset $G' \subset G^\perp \setminus J_0$ such that
\[ \mathcal{H}^1(G') \geq \frac{1}{2} \mathcal{H}^1(G^\perp \setminus J_0) \geq \frac{s}{2} \mathcal{H}^1(J_0). \]

Let $\varepsilon = \varepsilon(C_0, M)$ be as in Proposition 4.1. We define $\mathcal{G} \subset \Delta(J_0)$ as the family of maximal dyadic intervals such that for every $I \in \mathcal{G}$
\[ \mathcal{H}^1(I \cap G') \geq (1 - \varepsilon) \mathcal{H}^1(I). \]
It follows from Lebesgue differentiation theorem that
\[ \mathcal{H}^1\left( G' \setminus \bigcup_{I \in \mathcal{G}} I \right) = 0. \]
In particular,
\[ \mathcal{H}^1\left( \bigcup_{I \in \mathcal{G}} I \right) \geq \mathcal{H}^1(G') \geq \frac{s}{2} \mathcal{H}^1(J_0). \]
Let $G_0 \subset \mathcal{G}$ be a finite sub-collection such that
\[ \mathcal{H}^1\left( \bigcup_{I \in G_0} I \right) \geq \frac{1}{2} \mathcal{H}^1\left( \bigcup_{I \in \mathcal{G}} I \right) \geq \frac{s}{4} \mathcal{H}^1(J_0). \]
Set
\[ G_0 = \bigcup_{I \in G_0} I, \]
so that $G_0$ is a finite union of closed dyadic intervals.

Without loss of generality, we may assume that $\text{diam}(E) = 1$. For each $I \in G_0$ we apply Proposition 4.1 (with $J = I$ and $G = G' \cap I$; it is straightforward to see that all the assumptions are satisfied) to conclude that
\[
\int_E \int_0^1 \frac{\mu(X(x, I, r))}{r} \frac{dr}{r} d\mu(x) \leq C_{\text{Prop}} \int_E \int_0^1 \frac{\mu(X(x, G' \cap I, r))}{r} \frac{dr}{r} d\mu(x) + C_{\text{Prop}} \mathcal{H}^1(I) \mu(E). \tag{4.11}
\]
Summing (4.11) over $I \in \mathcal{G}_0$ we get
\[
\int_E \int_0^1 \frac{\mu(X(x, G_0, r))}{r} \, dr \, d\mu(x) \leq C_{\text{Prop}} \int_E \int_0^1 \frac{\mu(X(x, G', r))}{r} \, dr \, d\mu(x) + C_{\text{Prop}} \mathcal{H}^1(G_0) \mu(E). \tag{4.12}
\]

Notice also that if $B_{\Delta,0}$ are maximal dyadic intervals contained in $J_0 \setminus G_0$, and $I \in B_{\Delta,0}$, then $I^1$ contains some interval from $\mathcal{G}_0$, and in particular $I^1 \cap G' \neq \emptyset$. Since $G' \subseteq G^\perp$, we get from (1.3) that there exists $\theta_I \in I^1$ such that $\|\pi_{\theta_I, \mu}\| \leq M$. Hence, $G_0$ satisfies all the assumptions of Corollary 4.3, except perhaps for the measure assumption (4.4).

**Iteration.** We are in position to start the iteration. Assume for a moment that $\mathcal{H}^1(G_0) < (1 - \varepsilon)\mathcal{H}^1(J_0)$ so that $G_0$ satisfies all the assumptions of Corollary 4.3. We apply Corollary 4.3 and we define $G_1 := (G_0)_*$, so that
\[
\mathcal{H}^1(G_1) \geq (1 + \varepsilon)\mathcal{H}^1(G_0) \geq \frac{s(1 + \varepsilon)}{4} \mathcal{H}^1(J_0),
\]
and all the other conclusions of Corollary 4.3 hold for $G_1$. If $\mathcal{H}^1(G_1) < (1 - \varepsilon)\mathcal{H}^1(J_0)$, then we may apply Corollary 4.3 yet again to get a set $G_2 := (G_1)_*$.

In general, if after $k$-applications of Corollary 4.3 we get a set $G_k := (G_{k-1})_*$ satisfying $\mathcal{H}^1(G_k) < (1 - \varepsilon)\mathcal{H}^1(J_0)$, then we may continue applying Corollary 4.3. If for some $k = k_0$ we get $\mathcal{H}^1(G_{k_0}) \geq (1 - \varepsilon)\mathcal{H}^1(J_0)$, then we may apply Proposition 4.1 instead (with $G = G_{k_0}, J = J_0$), so that
\[
\int_E \int_0^1 \frac{\mu(X(x, 3J_0, r))}{r} \, dr \, d\mu(x) \leq C_{\text{Prop}} \int_E \int_0^1 \frac{\mu(X(x, G_{k_0}, r))}{r} \, dr \, d\mu(x) + C_{\text{Prop}} \mathcal{H}^1(J_0) \mu(E).
\]
Recall that for each $k$ we had $G_{k+1} = (G_k)_*$, so that by (4.5)
\[
\int_E \int_0^1 \frac{\mu(X(x, G_{k+1}, r))}{r} \, dr \, d\mu(x) \leq C_{\text{Prop}} \int_E \int_0^1 \frac{\mu(X(x, G_k, r))}{r} \, dr \, d\mu(x) + C_{\text{Prop}} \mathcal{H}^1(J_0) \mu(E).
\]
Putting the two estimates above together (the second one used $k_0$ times), and also recalling (4.12), we get
\[
\int_E \int_0^1 \frac{\mu(X(x, 3J_0, r))}{r} \, dr \, d\mu(x) \leq C_{\text{Prop}}^{k_0 + 1} \int_E \int_0^1 \frac{\mu(X(x, G_0, r))}{r} \, dr \, d\mu(x) + (k_0 + 1)C_{\text{Prop}}^{k_0 + 1} \mathcal{H}^1(J_0) \mu(E)
\]
\[
\leq C_{\text{Prop}}^{k_0 + 2} \int_E \int_0^1 \frac{\mu(X(x, G', r))}{r} \, dr \, d\mu(x) + (k_0 + 2)C_{\text{Prop}}^{k_0 + 2} \mathcal{H}^1(J_0) \mu(E). \tag{4.13}
\]
Bounding the number of iterations. We claim that the iteration ends (i.e. we obtain a set $G_{k_0}$ with $\mathcal{H}^1(G_{k_0}) \geq (1 - \varepsilon)\mathcal{H}^1(J_0)$) after at most

$$k_0 \lesssim s, \varepsilon$$

steps. Indeed, we had

$$\mathcal{H}^1(G_0) = \mathcal{H}^1\left( \bigcup_{I \in G_0} I \right) \geq \frac{s}{4} \mathcal{H}^1(J_0),$$

and so by (4.4) for each $G_k$ we have a lower bound

$$\mathcal{H}^1(G_k) \geq (1 + \varepsilon)\mathcal{H}^1(G_{k-1}) \geq (1 + \varepsilon)^k \mathcal{H}^1(G_0) \geq \frac{s(1 + \varepsilon)^k}{4} \mathcal{H}^1(J_0).$$

Taking $k_0 = k_0(s, \varepsilon)$ so large that $s(1 + \varepsilon)^{k_0}/4 \geq (1 - \varepsilon)$, we see that the iterative procedure described above ends after at most $k_0$ applications of Corollary 4.3.

End of the proof. Taking into account estimates (4.13) and (4.14), the fact that $\varepsilon = \varepsilon(M, C_0)$, $C_{\text{Prop}} = C_{\text{Prop}}(M, C_0)$, $\mathcal{H}^1(J) \leq 1$, and that $G' \subset G^\perp$, we get

$$\int_E \int_0^1 \frac{\mu(X(x, 3J_0, r))}{r} \frac{dr}{r} d\mu(x) \leq C(M, C_0, s) \int_E \int_0^1 \frac{\mu(X(x, G^\perp, r))}{r} \frac{dr}{r} d\mu(x) + C(M, C_0, s) \mu(E).$$

Hence, by Corollary 3.2

$$\int_E \int_0^1 \frac{\mu(X(x, 3J_0, r))}{r} \frac{dr}{r} d\mu(x) \lesssim_{M, C_0, s} \mu(E).$$

Let $M_0 = M_0(M, C_0, s)$ be a big constant. We define

$$E_* := \left\{ x \in E : \int_0^1 \frac{\mu(X(x, 3J_0, r))}{r} \frac{dr}{r} \leq M_0 \right\}.$$

By Chebyshev’s inequality, if $M_0$ is chosen big enough, we have

$$\mu(E_*) \geq \frac{\mu(E)}{2}.$$

Applying Proposition 3.3 to $E_*$ and $3J_0$, and recalling that $\mathcal{H}^1(J_0) \sim C_0^{-1}M^{-1}$, we obtain a Lipschitz graph $\Gamma$ with $\text{Lip}(\Gamma) \lesssim_{M, C_0} 1$ and

$$\mathcal{H}^1(\Gamma \cap E_*) \gtrsim_{C_0, M, M_0} \mu(E).$$

This finishes the proof of Theorem 1.7.

The remainder of the paper is dedicated to the proof of Proposition 4.1.
5. Rectangles and Generalized Cubes

Suppose that $E \subset \mathbb{R}^2$ is a bounded AD-regular set with constant $C_0$, and set $\mu = \mathcal{H}^1|_E$. Since Proposition 4.1 is scale-invariant, we may assume without loss of generality that $\text{diam}(E) = 1$.

Let $J, G \subset \mathbb{T}$ be as in Proposition 4.1. By rotating $E$, we may assume that $J$ is centered at $1/4$, so that the cone $X(0, J)$ is centered on the vertical axis. Note that that $\pi_0 = \pi_{1/4}$ is the projection to the horizontal axis, i.e., $\pi_0(x, y) = x$. Recall that there exists $\theta_0 \in 3J$ such that

$$\|\pi_0\mu\|_{\infty} \leq M. \quad (5.1)$$

5.1. Rectangles. Throughout the article we will be working with many rectangles, typically with one side much longer than the other. Let us fix some notation.

Given a rectangle $\mathcal{R} \subset \mathbb{R}^2$, we will denote the length of its shorter side by $\ell(\mathcal{R})$, and the length of its longer side by $\mathcal{L}(\mathcal{R})$. We will also write $\theta(\mathcal{R}) \in [0, 1/2) \subset \mathbb{T}$ to denote the “direction” of $\mathcal{R}$, so that $\ell_{\theta(\mathcal{R})}$ is parallel to the longer sides of $\mathcal{R}$ (for squares, it doesn’t matter which of the two directions we choose).

Given a constant $C > 0$ and a rectangle $\mathcal{R}$, we will sometimes write $C\mathcal{R}$ to denote the (unique) rectangle with the same center as $\mathcal{R}$, $\ell(C\mathcal{R}) = C\ell(\mathcal{R})$, $\mathcal{L}(C\mathcal{R}) = C\mathcal{L}(\mathcal{R})$, and such that their longer sides are parallel to each other.

Most of the rectangles $\mathcal{R}$ we will be working with will have a fixed direction $\theta(\mathcal{R}) = 1/4$, and a fixed aspect ratio $\mathcal{L}(\mathcal{R})/\ell(\mathcal{R}) = \mathcal{H}^1(J)^{-1}$. In other words, they will be very tall, vertically aligned rectangles. We fix notation specific to these rectangles. Given $x \in \mathbb{R}^2$ and $r > 0$ we set

$$\mathcal{R}(x, r) = x + \left[ -\frac{r}{2}, \frac{r}{2} \right] \times \left[ -\frac{r}{2\mathcal{H}^1(J)}, \frac{r}{2\mathcal{H}^1(J)} \right],$$

so that $\ell(\mathcal{R}(x, r)) = r$ and $\mathcal{L}(\mathcal{R}(x, r)) = \mathcal{H}^1(J)^{-1}r$. Note that $\pi_0(\mathcal{R}(x, r)) = \pi_0(x) + [-r/2, r/2]$.

Lemma 5.1. Let $\mathcal{R}$ be a rectangle, and suppose that for some $\theta \in \mathbb{T}$ with

$$|\theta - \theta(\mathcal{R})| \lesssim \frac{\ell(\mathcal{R})}{\mathcal{L}(\mathcal{R})} \quad (5.2)$$

we have $\|\pi_0\mu\|_{L^\infty} \leq M$. Then,

$$\mu(\mathcal{R}) \lesssim M\ell(\mathcal{R}). \quad (5.3)$$

Proof. Let $\mathcal{R}$ and $\theta$ be as above, and set $\alpha = |\theta - \theta(\mathcal{R})| \cdot 2\pi$. It follows from elementary trigonometry that

$$\mathcal{H}^1(\pi_0(\mathcal{R})) = \ell(\mathcal{R}) \left( \cos(\alpha) + \frac{\mathcal{L}(\mathcal{R})}{\ell(\mathcal{R})} \sin(\alpha) \right).$$

From (5.2) we have $\alpha \lesssim \frac{\ell(\mathcal{R})}{\mathcal{L}(\mathcal{R})}$, and so

$$\mathcal{H}^1(\pi_0(\mathcal{R})) \lesssim \ell(\mathcal{R}).$$

Since $\|\pi_0\mu\|_{L^\infty} \leq M$, we get

$$\mu(\mathcal{R}) \leq \mu((\pi_0)^{-1}(\pi_0(\mathcal{R}))) \leq M\mathcal{H}^1(\pi_0(\mathcal{R})) \lesssim M\ell(\mathcal{R}).$$
5.2. Generalized dyadic cubes. We say that a metric space \((X,d)\) has a finite doubling property if any ball \(B_X(x,2r) \subset X\) can be covered by finitely many balls of the form \(B_X(x_i,r)\). The following is a special case of Theorem 2.1 from [KRS12].

**Theorem 5.3** ([KRS12]). Let \(\rho = 1/1000\). Suppose that \((X,d)\) is a metric space with the finite doubling property. Then, for every \(k \in \mathbb{Z}\) there exists a collection \(D_k\) of generalized cubes on \(X\) such that the following hold:

1. For each \(k \in \mathbb{Z}\), \(X = \bigcup_{Q \in D_k} Q\), and the union is disjoint.
2. If \(Q_1, Q_2 \in \bigcup_k D_k\) satisfy \(Q_1 \cap Q_2 \neq \emptyset\), then either \(Q_1 \subset Q_2\) or \(Q_2 \subset Q_1\).
3. For every \(Q \in D_k\) there exists \(x_Q \in Q\) such that
   \[B_X(x_Q,0.4\rho^k) \subset Q \subset B_X(x_Q,2\rho^k).\]

Consider \(X = E\) endowed with the metric
\[
d((x_1,y_1),(x_2,y_2)) = \max \left( |x_1 - x_2|, \mathcal{H}^1(J) |y_1 - y_2| \right).
\]
Note that for \(x \in E\) and \(r > 0\), the ball with respect to \(d\) is of the form \(B_E(x,r) = \mathcal{R}(x,2r) \cap E\).

It is clear that \((E,d)\) has the finite doubling property, and so we may use Theorem 5.3 to obtain a lattice of generalized cubes \(D = \bigcup_{k \in \mathbb{Z}} D_k\) associated to \((E,d)\).

Given \(Q \in D_k\), we will write
\[
\ell(Q) := 4\rho^k,
\]
\[
\mathcal{C}(Q) := \{ P \in D_{k+1} : P \subset Q \},
\]
\[
D(Q) := \{ P \in D : P \subset Q, \ell(P) \leq \ell(Q) \}.
\]
Observe that \(Q \subset \mathcal{R}(x_Q,\ell(Q)) \cap E\). We set
\[
\mathcal{R}_Q := \mathcal{R}(x_Q,\ell(Q)),
\]
\[
\mathcal{L}(Q) := \mathcal{H}^1(J)^{-1} \ell(Q),
\]
so that \(\ell(\mathcal{R}_Q) = \ell(Q)\) and \(\mathcal{L}(\mathcal{R}_Q) = \mathcal{L}(Q)\). Note that if \(P,Q \in D\) satisfy \(P \cap Q = \emptyset\) and \(\ell(P) \geq \ell(Q)\), then by (3) in Theorem 5.3 we have \(d(x_P,x_Q) \geq 0.1\ell(P) \geq 0.05\ell(P) + 0.05\ell(Q)\), so in particular \(0.1\mathcal{R}_P \cap 0.1\mathcal{R}_Q = \emptyset\). We set
\[
\mathcal{R}(Q) := 0.1\mathcal{R}_Q.
\]
We record for future reference that
\[ \mathcal{R}(Q) \cap E \subset Q \subset \mathcal{R}_Q \cap E, \]
\[ 2\mathcal{R}_Q \subset 2\mathcal{R}_P \quad \text{if } P \subset Q = \emptyset, \]
\[ \mathcal{R}(Q) \cap \mathcal{R}(P) = \emptyset \quad \text{if } P \cap Q = \emptyset. \]

Observe also that for any \( C > 0 \) such that \( C\ell(Q) \lesssim \text{diam}(E) = 1 \) we have
\[ CC_0 \ell(Q) \lesssim \mu(C\mathcal{R}_Q) \lesssim CM\ell(Q). \] (5.7)

In particular,
\[ C_0 \ell(Q) \lesssim \mu(Q) \lesssim M\ell(Q). \]

6. Conical energies

Let \( A = A(C_0, M) \geq 1000 \) be a large constant which we will fix later on. Inspired by \cite{CT20} and \cite{Dąb22}, we introduce the following conical energy associated to the set of directions \( G \subset J \). For any \( Q \in D \) we set
\[ \mathcal{E}_G(Q) := \frac{1}{\mu(Q)} \int_{2\mathcal{R}_Q} \int_{A^{-1}\mathcal{L}(Q)} \frac{\mu(X(x,G,r))}{r} \frac{dr}{r} d\mu(x). \] (6.1)

We have the following easy upper bound for \( \mathcal{E}_G(Q) \).

\textbf{Lemma 6.1.} \textit{For any } \( Q \in D \text{ we have}
\[ \mathcal{E}_G(Q) \lesssim_{A,M,C_0} \mathcal{H}^1(J). \] (6.2)

\textbf{Proof.} Observe that for any \( x \in 2A\mathcal{R}_Q \) and \( r \in (A^{-1}\mathcal{L}(Q), A^3\mathcal{L}(Q)) \) we have
\[ X(x,G,r) \subset X(x,J,A^3\mathcal{L}(Q)) \subset \mathcal{R}(x,A^4\ell(Q)), \]
so that
\[ \mu(X(x,G,r)) \leq \mu(\mathcal{R}(x,A^4\ell(Q))) \lesssim A^4M\ell(Q). \]

Hence,
\[ \mathcal{E}_G(Q) = \frac{1}{\mu(Q)} \int_{2\mathcal{R}_Q} \int_{A^{-1}\mathcal{L}(Q)} \frac{\mu(X(x,G,r))}{r} \frac{dr}{r} d\mu(x) \]
\[ \lesssim_{A,M} \frac{1}{\mu(Q)} \int_{2\mathcal{R}_Q} \int_{A^{-1}\mathcal{L}(Q)} \frac{\ell(Q)}{r} \frac{d\mu(x)}{r} \]
\[ \sim_A \mathcal{H}^1(J) \frac{\mu(2A\mathcal{R}_Q)}{\mu(Q)} \lesssim_{A,M,C_0} \mathcal{H}^1(J). \]
\[ \square \]
6.1. **Stopping time argument.** Given a small constant $\delta = \delta(A, M, C_0) > 0$, we consider the following stopping time condition. For $R \in \mathcal{D}$, we define the family $\text{BCE}(R)$ as the family of maximal cubes $Q \in \mathcal{D}(R)$ such that

$$
\sum_{S \in \mathcal{D}, Q \subset S \subset R} \mathcal{E}_G(S) \geq \delta \mathcal{H}^1(J).
$$

(6.3)

We define also $\text{Tree}(R)$ as the subfamily of $\mathcal{D}(R)$ consisting of cubes that are not strictly contained in any cube from $\text{BCE}(R)$. Note that it may happen that $R \in \text{BCE}(R)$, in which case $\text{Tree}(R) = \{R\}$.

**Lemma 6.2.** For any $R \in \mathcal{D}$ we have

$$
\sum_{Q \in \text{Tree}(R) \setminus \text{BCE}(R)} \mathcal{E}_G(Q) \mu(Q) \leq \delta \mathcal{H}^1(J) \mu(R),
$$

(6.4)

and

$$
\delta \mathcal{H}^1(J) \sum_{P \in \text{BCE}(R)} \mu(P) \leq \sum_{Q \in \text{Tree}(R)} \mathcal{E}_G(Q) \mu(Q) \lesssim_{A, M, C_0} \mathcal{H}^1(J) \mu(R).
$$

(6.5)

**Proof.** We start by proving (6.4). Observe that

$$
\sum_{Q \in \text{Tree}(R) \setminus \text{BCE}(R)} \mathcal{E}_G(Q) \mu(Q) = \sum_{Q \in \text{Tree}(R) \setminus \text{BCE}(R)} \int \mathcal{E}_G(Q) \mathbb{1}_Q(x) \, d\mu(x)
$$

$$
= \int \sum_{Q \in \text{Tree}(R) \setminus \text{BCE}(R)} \mathcal{E}_G(Q) \mathbb{1}_Q(x) \, d\mu(x).
$$

Let $x \in \bigcup_{Q \in \text{Tree}(R) \setminus \text{BCE}(R)} Q$, and let $P \in \text{Tree}(R) \setminus \text{BCE}(R)$ be a cube with $x \in P$. Recalling that $P \notin \text{BCE}(R)$ and the definition of $\text{BCE}(R)$ (6.3), we get

$$
\sum_{P \subset Q \subset R} \mathcal{E}_G(Q) < \delta \mathcal{H}^1(J).
$$

Since $P$ was an arbitrary cube with $P \in \text{Tree}(R) \setminus \text{BCE}(R)$ and $x \in P$, this gives

$$
\sum_{Q \in \text{Tree}(R) \setminus \text{BCE}(R)} \mathcal{E}_G(Q) \mathbb{1}_Q(x) \leq \delta \mathcal{H}^1(J).
$$

Integrating over $x \in \bigcup_{Q \in \text{Tree}(R) \setminus \text{BCE}(R)} Q \subset R$ yields

$$
\sum_{Q \in \text{Tree}(R) \setminus \text{BCE}(R)} \mathcal{E}_G(Q) \mu(Q) \leq \delta \mathcal{H}^1(J) \mu(R).
$$

This proves (6.4).

The upper bound in (6.5) follows from (6.4) and the trivial bound (6.2) applied to $Q \in \text{BCE}(R)$:

$$
\sum_{Q \in \text{BCE}(R)} \mathcal{E}_G(Q) \mu(Q) \lesssim_{A, M, C_0} \mathcal{H}^1(J) \sum_{Q \in \text{BCE}(R)} \mu(Q) \leq \mathcal{H}^1(J) \mu(R).
$$
Now we prove the lower bound in (6.5). We have
\[
\sum_{Q \in \text{Tree}(R)} \mathcal{E}_G(Q) \mu(Q) = \int \sum_{Q \in \text{Tree}(R)} \mathcal{E}_G(Q) \mathbb{1}_Q(x) \, d\mu(x) \\
\geq \int \sum_{P \in \text{BCE}(R)} \sum_{Q \in \text{Tree}(R), P \subset Q} \mathcal{E}_G(Q) \mathbb{1}_P(x) \, d\mu(x). \quad (6.6)
\]
By (6.3) we have for every \( P \in \text{BCE}(R) \)
\[
\sum_{Q \in \text{Tree}(R), P \subset Q} \mathcal{E}_G(Q) \geq \delta \mathcal{H}^1(J).
\]
Hence,
\[
\int \sum_{P \in \text{BCE}(R)} \sum_{Q \in \text{Tree}(R), P \subset Q} \mathcal{E}_G(Q) \mathbb{1}_P(x) \, d\mu(x) \geq \delta \mathcal{H}^1(J) \sum_{P \in \text{BCE}(R)} \mu(P).
\]
Together with (6.6), this gives the desired estimate. \(\square\)

6.2. Corona decomposition. We are ready to perform the corona decomposition. Let \( k(J) \in \mathbb{Z} \) be the largest integer such that for \( Q \in \mathcal{D}_{k(J)} \) we have
\[
\mathcal{L}(Q) = 4 \mathcal{H}^1(J)^{-1} \rho^{k(J)} \geq 1.
\]
Set \( \mathcal{D}_s = \bigcup_{k \geq k(J)} \mathcal{D}_k \), and
\[
\text{Top}_0 = \{ \mathcal{D}_{k(J)} \}.
\]
If \( \text{Top}_k \) has already been defined, we set
\[
\text{Top}_{k+1} = \bigcup_{R \in \text{Top}_k} \bigcup_{Q \in \text{BCE}(R)} \text{Ch}(Q).
\]
Finally,
\[
\text{Top} = \bigcup_{k \geq 0} \text{Top}_k.
\]
Observe that
\[
\bigcup_{R \in \text{Top}} \text{Tree}(R) = \mathcal{D}_s.
\]
The following is a fairly standard computation.

**Lemma 6.3.** We have
\[
\mathcal{H}^1(J) \mu(E) + \int_E \int_0^1 \frac{\mu(X(x, G, r))}{r} \, dr \, d\mu(x) \\
\sim_{A,M} \mathcal{H}^1(J) \mu(E) + \sum_{Q \in \mathcal{D}_s} \mathcal{E}_G(Q) \mu(Q). \quad (6.7)
\]
Recalling that, for any \( Q \in \mathcal{D}_k \) the rectangles \( 2A R_Q \) have only bounded overlaps (with bound depending on \( A \)), we have

\[
\sum_{Q \in \mathcal{D}_k} E_G(Q) \mu(Q) \sim \int_E \int_{A^{-1} H^1(J)^{-1} \rho^k} \frac{\mu(X(x, G, r))}{r} \, dr \, d\mu(x).
\]

Summing over \( k \geq k(J) \) we get

\[
\sum_{Q \in \mathcal{D}_k} E_G(Q) \mu(Q) \sim \int_E \int_{A^{-1} H^1(J)^{-1} \rho^k(J)} \frac{\mu(X(x, G, r))}{r} \, dr \, d\mu(x).
\]

Recalling that \( 1 \leq 4 H^1(J)^{-1} \rho^{k(J)} \lesssim 1 \), we get that

\[
\sum_{Q \in \mathcal{D}_k} E_G(Q) \mu(Q) \sim \int_E \int_{A^{-1} H^1(J)^{-1} \rho^{k(J)}} \frac{\mu(X(x, G, r))}{r} \, dr \, d\mu(x)
\]

for some constant \( 1 \leq C \lesssim 1 \). This is obviously no-smaller than the integral on the left hand side of (6.7).

To see the converse estimate, note that for \( r > 1 \) we have \( X(x, G, r) \cap E \subset R(x, 2H^1(J)) \), so that

\[
\int_E \int_{A^{-1} H^1(J)^{-1} \rho^{k(J)}} \frac{\mu(X(x, G, r))}{r} \, dr \, d\mu(x) \lesssim \int_E \int_{A^{-1} H^1(J)^{-1} \rho^{k(J)}} \frac{\mu(R(x, 2H^1(J)))}{r} \, dr \, d\mu(x)
\]

\[
\lesssim \frac{M H^1(J)}{E} \int_E \int_{A^{-1} H^1(J)^{-1} \rho^{k(J)}} \frac{1}{r^2} \, dr \, d\mu(x) \lesssim MH^1(J) \mu(E).
\]

The family \( \text{Top} \) satisfies the following packing condition.

**Lemma 6.4.** We have

\[
\sum_{R \in \text{Top}} \mu(R) \lesssim_{\delta, A} (H^1(J))^{-1} \int_E \int_{A^{-1} H^1(J)^{-1}} \frac{\mu(X(x, G, r))}{r} \, dr \, d\mu(x) + \mu(E). \tag{6.8}
\]

**Proof.** First, we use the fact that the cubes \( R \in \text{Top}_0 \) are pairwise disjoint to estimate

\[
\sum_{R \in \text{Top}_0} \mu(R) \leq \mu(E).
\]

This gives the second term on the right hand side of (6.8).

Moving on to \( \text{Top} \setminus \text{Top}_0 \), we compute

\[
\sum_{R \in \text{Top} \setminus \text{Top}_0} \mu(R) = \sum_{k \geq 0} \sum_{R \in \text{Top}_{k+1}} \mu(R) = \sum_{k \geq 0} \sum_{R \in \text{Top}_k} \sum_{Q \in \text{BCE}(R)} \sum_{P \in \text{Ch}(Q)} \mu(P)
\]

\[
= \sum_{k \geq 0} \sum_{R \in \text{Top}_k} \sum_{Q \in \text{BCE}(R)} \mu(Q) \lesssim \sum_{k \geq 0} \sum_{R \in \text{Top}_k} \sum_{Q \in \text{Tree}(R)} E_G(Q) \mu(Q)
\]

\[
= (\delta H^1(J))^{-1} \sum_{Q \in \mathcal{D}_k} E_G(Q) \mu(Q)
\]

\[
\lesssim_{A, M} (\delta H^1(J))^{-1} \int_E \int_{A^{-1} H^1(J)^{-1}} \frac{\mu(X(x, G, r))}{r} \, dr \, d\mu(x) + \delta^{-1} \mu(E).
\]
Consider the following conical energy associated to $3J$:

$$
\mathcal{E}_J(Q) := \frac{1}{\mu(Q)} \int_Q \int_{\rho \mathcal{L}(Q)} \frac{\mu(X(x, 3J, r))}{r} \, dr \, d\mu(x).
$$

Arguing as in (6.7), it is easy to show that

$$
\int_{\mathcal{E}} \int_0^1 \frac{\mu(X(x, 3J, r))}{r} \, dr \, d\mu(x) \lesssim \sum_{Q \in \mathcal{D}^*} \mathcal{E}_J(Q) \mu(Q). \quad (6.9)
$$

We divide the conical energy $\mathcal{E}_J(Q)$ into an “interior” and “exterior” part, which will be dealt with separately:

$$
\mathcal{E}^{\text{int}}_J(Q) := \frac{1}{\mu(Q)} \int_Q \int_{\rho \mathcal{L}(Q)} \frac{\mu(X(x, 0.5J, r))}{r} \, dr \, d\mu(x),
$$

$$
\mathcal{E}^{\text{ext}}_J(Q) := \frac{1}{\mu(Q)} \int_Q \int_{\rho \mathcal{L}(Q)} \frac{\mu(X(x, 3J \setminus 0.5J, r))}{r} \, dr \, d\mu(x).
$$

We define also the following modification of $\mathcal{E}^{\text{ext}}_J(Q)$

$$
\tilde{\mathcal{E}}^{\text{ext}}_J(Q) := \frac{1}{\mu(Q)} \int_Q \frac{\mu(X(x, 3J \setminus 0.5J, \rho \mathcal{L}(Q), \mathcal{L}(Q)))}{\mathcal{L}(Q)} \, d\mu(x).
$$

Lemma 6.5. We have

$$
\sum_{Q \in \mathcal{D}^*} \mathcal{E}^{\text{ext}}_J(Q) \mu(Q) \lesssim \sum_{Q \in \mathcal{D}^*} \tilde{\mathcal{E}}^{\text{ext}}_J(Q) \mu(Q). \quad (6.10)
$$

Proof. Given $x \in Q$, we set

$$
X(x, Q) = X(x, 3J \setminus 0.5J, \rho \mathcal{L}(Q), \mathcal{L}(Q)).
$$

If $Q = Q_0(x) \supset Q_1(x) \supset Q_2(x) \supset \ldots$ is a sequence of cubes such that for all $i \in \mathbb{N}$ we have $Q_{i+1}(x) \in \text{Ch}(Q_i(x))$ and $x \in Q_i(x)$, then

$$
\mu(X(x, 3J \setminus 0.5J, \mathcal{L}(Q))) = \sum_{i \in \mathbb{N}} \mu(X(x, Q_i(x))).
$$

Thus, for $x \in Q$ and $\rho \mathcal{L}(Q) < r < \mathcal{L}(Q)$

$$
\frac{\mu(X(x, 3J \setminus 0.5J, r))}{r} \lesssim \sum_{i \in \mathbb{N}} \frac{\mu(X(x, Q_i(x)))}{\mathcal{L}(Q)} = \sum_{i \in \mathbb{N}} \frac{\mu(X(x, Q_i(x)))}{\mathcal{L}(Q_i(x))} \cdot \frac{\ell(Q_i(x))}{\ell(Q)}.
$$

Integrating over $x \in Q$ and $\rho \mathcal{L}(Q) < r < \mathcal{L}(Q)$ yields

$$
\mathcal{E}^{\text{ext}}_J(Q) \mu(Q) \lesssim \sum_{P \in \mathcal{D}(Q)} \tilde{\mathcal{E}}^{\text{ext}}_J(P) \mu(P) \frac{\ell(P)}{\ell(Q)}.
$$
We sum over \( Q \in D_* \) and conclude that
\[
\sum_{Q \in D_*} \mathcal{E}^\text{ext}_J(Q)\mu(Q) \lesssim \sum_{Q \in D_*} \sum_{P \in \mathcal{D}(Q)} \hat{\mathcal{E}}^\text{ext}_J(P)\mu(P) \frac{\ell(P)}{\ell(Q)}
\]
\[
= \sum_{P \in D_*} \hat{\mathcal{E}}^\text{ext}_J(P)\mu(P) \sum_{Q \in D_*, Q \supset P} \frac{\ell(P)}{\ell(Q)} \lesssim \sum_{P \in D_*} \hat{\mathcal{E}}^\text{ext}_J(P)\mu(P),
\]
where in the last inequality we used the fact that the inner sum was a geometric series.

\[
\square
\]

We will prove the following estimates for the interior and exterior energies.

**Lemma 6.6.** If \( \varepsilon = \varepsilon(M, C_0) \) is chosen small enough, then for any \( R \in \text{Top} \) we have
\[
\sum_{Q \in \text{Tree}(R)} \mathcal{E}^\text{int}_J(Q)\mu(Q) \lesssim_{C_0} \sum_{Q \in \text{Tree}(R)} \mathcal{E}_G(Q)\mu(Q). \tag{6.11}
\]
Furthermore, if \( A = A(C_0, M) \) is chosen big enough, and \( \delta = \delta(A, M, C_0) \) is chosen small enough, then
\[
\sum_{Q \in \text{Tree}(R)} \hat{\mathcal{E}}^\text{ext}_J(Q)\mu(Q) \lesssim_{C_0, M} \mathcal{H}^1(J)\mu(R). \tag{6.12}
\]

We prove (6.11) in Section 7 and (6.12) in Section 8. Now we show how Proposition 4.1 follows from the estimates above.

**Proof of Proposition 4.1.** Recall that our goal is to prove
\[
\int_E \int_0^1 \frac{\mu(X(x, 3J, r))}{r} \frac{dr}{d\mu(x)} = C_0, M \int_E \int_0^1 \frac{\mu(X(x, G, r))}{r} \frac{dr}{d\mu(x)} + \mathcal{H}^1(J)\mu(E). \tag{6.13}
\]
By (6.9), the left hand side is bounded by
\[
\sum_{Q \in \mathcal{D}_*} \mathcal{E}_J(Q)\mu(Q) = \sum_{Q \in \mathcal{D}_*} \mathcal{E}^\text{int}_J(Q)\mu(Q) + \sum_{Q \in \mathcal{D}_*} \mathcal{E}^\text{ext}_J(Q)\mu(Q)
\]
\[
\lesssim \sum_{Q \in \mathcal{D}_*} \mathcal{E}^\text{int}_J(Q)\mu(Q) + \sum_{Q \in \mathcal{D}_*} \hat{\mathcal{E}}^\text{ext}_J(Q)\mu(Q)
\]
\[
= \sum_{R \in \text{Top}} \sum_{Q \in \text{Tree}(R)} \mathcal{E}^\text{int}_J(Q)\mu(Q) + \sum_{R \in \text{Top}} \sum_{Q \in \text{Tree}(R)} \hat{\mathcal{E}}^\text{ext}_J(Q)\mu(Q) =: S_1 + S_2.
\]
To estimate \( S_1 \), we apply (6.11) and (6.7) to conclude
\[
S_1 \lesssim \sum_{R \in \text{Top}} \sum_{Q \in \text{Tree}(R)} \mathcal{E}_G(Q)\mu(Q) \lesssim_{A, M} \int_E \int_0^1 \frac{\mu(X(x, G, r))}{r} \frac{dr}{d\mu(x)} + \mathcal{H}^1(J)\mu(E).
\]
Regarding \( S_2 \), using (6.12) and (6.8) yields
\[
S_2 \lesssim_M \int_E \int_0^1 \frac{\mu(X(x, G, r))}{r} \frac{dr}{d\mu(x)} + \mathcal{H}^1(J)\mu(E).
\]
Recalling that $\delta = \delta(A, M, C_0)$ and $A = A(C_0, M)$, this gives (6.13).

7. Estimating interior energy and obtaining good cones

7.1. Interior energy estimates. Recall that in Proposition 4.1 assumption (e), we assumed that $G$ is closed, and that for every interval $I$ comprising $J \setminus G$ there exists $\theta_I \in 3I$ such that $\|\pi_{\theta_I} \mu\|_{\infty} \leq M$. We use this property in the following lemma, which is the first step in estimating $\mathcal{E}_j^{int}(Q)$.

Lemma 7.1. For any $x \in \mathbb{R}^2$ and $0 < r < \infty$ we have

$$\mu(X(x, J \setminus G, r)) \lesssim M \mathcal{H}^1(J \setminus G) r.$$  

In particular, since $\mathcal{H}^1(J \setminus G) \leq \mathcal{H}^1(J)$, we have

$$\mu(X(x, J \setminus G, r)) \lesssim M \mathcal{H}^1(J) r.$$  

Proof. Let $B$ denote the intervals comprising $J \setminus G$, so that for every $I \in B$ there exists $\theta_I \in 3I$ such that $\|\pi_{\theta_I} \mu\|_{\infty} \leq M$. Clearly,

$$X(x, J \setminus G, r) = \bigcup_{I \in B} X(x, I, r).$$

Observe that each truncated cone $X(x, I, r)$ is contained in some rectangle $R/I$ which is centered at $x$, its direction $\theta(R/I) \in T$ coincides with the midpoint of $I$, and it satisfies $\ell(R/I) \sim \mathcal{H}^1(I) r$, $\mathcal{L}(R/I) \sim r$. Since

$$|\theta(R/I) - \theta_I| \leq 2 \mathcal{H}^1(I) \sim \frac{\ell(R/I)}{\mathcal{L}(R/I)},$$

we may use Lemma 5.1 (recall that $\|\pi_{\theta_I} \mu\|_{\infty} \leq M$) to conclude that

$$\mu(R/I) \lesssim M \ell(R/I) \sim M \mathcal{H}^1(I) r.$$ 

It follows that

$$\mu(X(x, J \setminus G, r)) \lesssim \sum_{I \in B} \mu(X(x, I, r)) \lesssim \sum_{I \in B} \mu(R/I) \lesssim M r \sum_{I \in B} \mathcal{H}^1(I) = M \mathcal{H}^1(J \setminus G) r.$$ 

Lemma 7.2. If $\varepsilon = \varepsilon(M, C_0)$ is small enough, then for any $x \in E$ and $0 < r < \infty$ we have

$$\mu(X(x, 0.9J, r)) \lesssim_{C_0} \mu(X(x, G, 2r)).$$  

In particular, $\mathcal{E}_j^{int}(Q) \lesssim_{C_0} \mathcal{E}_G(Q)$, and so (6.11) holds.

Proof. If $X(x, 0.9J, r) \cap E = \{x\}$, then there is nothing to prove, so suppose that $X(x, 0.9J, r) \cap E \neq \{x\}$.

Let $y \in X(x, 0.9J, r) \cap E \setminus \{x\}$, and let $0 < r_0 \leq r/2$ be such that $y \in E \cap X(x, 0.9J, r_0, 2r_0)$. Set $r_y = \varepsilon \mathcal{H}^1(J) r_0$ for some small absolute constant $c > 0$, and observe that if $c$ is chosen small enough, then $B(y, r_y) \subset X(x, J, r_0/2, 4r_0)$. 

We use Lemma 7.1 to estimate
\[ \mu(B(y, r_y) \cap X(x, J \setminus G, r_0/2, 4r_0)) \leq \mu(X(x, J \setminus G, r_0/2, 4r_0)) \]
\[ \overset{\text{7.1}}{\approx} M \varepsilon \mathcal{H}^1(J)r_0 \sim M \varepsilon r_y. \]

On the other hand, since \( y \in E, r_y < r_0 < \text{diam}(E) = 1 \), and \( B(y, r_y) \subset X(x, J, r_0/2, 4r_0) \), we get from AD-regularity of \( E \) that
\[ \mu(B(y, r_y) \cap X(x, J, r_0/2, 4r_0)) = \mu(B(y, r_y)) \gtrsim C_0^{-1} r_y. \]
The two estimates together give
\[ C_0^{-1} r_y \lesssim \mu(B(y, r_y) \cap X(x, J, r_0/2, 4r_0)) = \mu(B(y, r_y)) \approx \mu(X(x, J \setminus G, r_0/2, 4r_0)) \]
\[ \leq \mu(B(y, r_y) \cap X(x, J, r_0/2, 4r_0)) + C M \varepsilon r_y. \]
Hence, assuming \( \varepsilon = \varepsilon(M, C_0) \) small enough, we may absorb the second term on the right hand side to the left hand side, which gives
\[ \mu(B(y, r_y) \cap X(x, G, 2r)) \geq \mu(B(y, r_y) \cap X(x, G, r_0/2, 4r_0)) \]
\[ \gtrsim C_0^{-1} r_y \sim C_0 \mu(B(y, r_y)). \quad (7.3) \]

Now consider the family of balls
\[ \mathcal{B} = \{ B(y, r_y) : y \in X(x, 0.9J, r) \cap E \setminus \{ x \} \}. \]
By the 5r-covering lemma, we may find a countable sub-collection \( \mathcal{B}' = \{ B(y_i, r_{y_i}) \}_{i \in \mathcal{I}} \) of pairwise disjoint balls such that \( \{ B(y_i, 5r_{y_i}) \}_{i \in \mathcal{I}} \) covers all of \( X(x, 0.9J, r) \cap E \setminus \{ x \} \). Then,
\[ \mu(X(x, 0.9J, r) \cap E) \leq \mu \left( \bigcup_{i \in \mathcal{I}} B(y_i, 5r_{y_i}) \right) \leq \sum_{i \in \mathcal{I}} \mu(B(y_i, 5r_{y_i})) \]
\[ \sim C_0 \sum_{i \in \mathcal{I}} \mu(B(y_i, r_{y_i})) \overset{(7.3)}{\gtrsim} C_0 \sum_{i \in \mathcal{I}} \mu(B(y_i, r_{y_i}) \cap X(x, G, 2r)) \]
\[ = \mu \left( \bigcup_{i \in \mathcal{I}} B(y_i, r_{y_i}) \cap X(x, G, 2r) \right) \leq \mu(X(x, G, 2r)). \]

\[ \square \]

7.2. Obtaining good cones. We will say that a (possibly truncated) cone \( X \) is good if it satisfies
\[ X \cap E = \emptyset. \]
Similarly, we will say that a rectangle \( \mathcal{R} \) is good if \( \mathcal{R} \cap E = \emptyset \).

Having plenty of good cones and rectangles will be crucial for estimating the exterior energy \( \tilde{E}^{ext}(Q) \) in Section 8. In the lemma below we use Lemma 7.2 and the BCE-stopping condition to find many good cones.
Lemma 7.3. If the BCE-parameter $\delta = \delta(A, M, C_0) \in (0, 1)$ is chosen small enough, then for all $R \in \text{Top}$, $Q \in \text{Tree}(R) \setminus \text{BCE}(R)$, and $x \in AR_Q \cap E$ we have

$$X(x, 0.5J, A^{-1}L(Q), A^{2}L(R)) \cap E = \emptyset.$$  

Proof. Assume the contrary: let $Q \in \text{Tree}(R) \setminus \text{BCE}(R)$, $x \in AR_Q \cap E$, and $y \in X(x, 0.5J, A^{-1}L(Q), A^{2}L(R)) \cap E$.

Let $P \in \text{Tree}(R) \setminus \text{BCE}(R)$ be such that $Q \subset P$ and $y \in X(x, 0.5J, A^{-1}L(P), A^{2}L(P))$, so that in particular

$$A^{-1}L(P) \leq |x - y| \leq A^{2}L(P).$$

Set

$$r_0 := A^{-2}\ell(P) = A^{-2}H^1(J)\mathcal{L}(P) \leq A^{-1}H^1(J)|x - y|.$$  \hfill (7.4)

We claim that if $A$ is chosen big enough, then for all $x' \in B(x, r_0)$ we have

$$B(y, r_0) \subset X(x', 0.9J, 2A^{2}\mathcal{L}(P)).$$  \hfill (7.5)

This is a simple geometric observation, see Figure 7.1. The rigorous computation goes as follows: first, observe that if $x' \in B(x, r_0)$, $y' \in B(y, r_0)$, then

$$|x' - y'| \geq |x - y| - 2r_0 \geq A\mathcal{H}^1(J)^{-1}r_0 - 2r_0 \geq \frac{A}{2\mathcal{H}^1(J)}r_0.$$  \hfill (7.3)

Thus, using the fact that $y \in X(x, 0.5J)$,

$$|\pi_0(x') - \pi_0(y')| \leq |\pi_0(x) - \pi_0(y)| + 2r_0 \leq \sin \left( \frac{H^1(J)}{2} \pi \right) |x - y| + 2r_0$$

$$\leq \sin \left( \frac{H^1(J)}{2} \pi \right) |x' - y'| + 4r_0 \leq \left( \sin \left( \frac{H^1(J)}{2} \pi \right) + \frac{8H^1(J)}{A} \right) |x' - y'|$$

$$\leq \sin \left( 0.9H^1(J)\pi \right) |x' - y'|,$$

assuming $A$ large enough. This shows $y' \in X(x', 0.9J)$. We also have $y' \in B(x', 2A^{2}\mathcal{L}(P))$ because

$$|x' - y'| \leq |x - y| + 2r_0 \leq A^{2}\mathcal{L}(P) + 2A^{-2}\ell(P) \leq 2A^{2}\mathcal{L}(P).$$

This gives the claim (7.5).

Since $x \in AR_P$ and $B(x, r_0) \subset 2AR_P$, we get from Lemma 7.2 that

$$E_G(P)\mu(P) = \int_{2AR_P} \int_{A^{-1}L(P)}^{A^{2}L(P)} \frac{\mu(X(x', G, r))}{r} \, d\mu(x')$$

$$\overset{\text{(7.2)}}{\geq} C_0 \int_{2AR_P} \int_{A^{-1}L(P)}^{A^{2}L(P)} \frac{\mu(X(x', 0.9J, r))}{r} \, d\mu(x')$$

$$\geq \int_{B(x, r_0)} \int_{2A^{2}L(P)}^{4A^{2}L(P)} \frac{\mu(X(x', 0.9J, r))}{r} \, d\mu(x')$$

$$\geq A \int_{B(x, r_0)} \int_{2A^{2}L(P)}^{4A^{2}L(P)} \frac{\mu(B(y, r_0))}{\mathcal{L}(P)} \, d\mu(x')$$

$$\geq \frac{\mu(B(x, r_0))\mu(B(y, r_0))}{\mathcal{L}(P)} \geq \frac{C_0^{-2}r_0^2}{\mathcal{L}(P)} \sim C_0 A \frac{\ell(P)^2}{\mathcal{L}(P)} = H^1(J)\ell(P).$$
Hence, 
\[ \mathcal{E}_G(P) \gtrsim_{C_0, A} H^1(J) \ell(P) \gtrsim_M H^1(J). \]
Recall that \( \mathcal{E}_G(P) \leq \delta H^1(J) \) because \( P \not\in \text{BCE}(R) \) (see the BCE stopping condition \((6.3))\). Assuming \( \delta = \delta(A, M, C_0) \) small enough, we arrive at a contradiction. \( \square \)

For brevity of notation, for \( R \in \text{Top} \) we define 
\[ T(R) = \text{Tree}(R) \setminus \text{BCE}(R). \]
In the next two lemmas we show that for any integer \( k \in \mathbb{Z} \), the family of intervals 
\[ \{ \pi_0(R_P) : P \in T_k(R) \} \]
has bounded overlaps. In other words, if we fix generation \( D_k \), then the rectangles associated to cubes in \( T_k(R) \) resemble a graph over the horizontal line \( \ell_0 \). This will be useful in Section 8.

**Lemma 7.4.** There exists an absolute constant \( C > 1 \) such that the following holds. Suppose that \( R \in D_* \) and \( Q \neq P \in D(R) \) are such that \( \ell(Q) = \ell(P) \), and
\[ X(x, 0.5J, \rho \mathcal{L}(Q), \mathcal{L}(R)) \cap E = \emptyset \quad \text{for all} \quad z \in E \cap 2R_Q. \] (7.6)
If \( \pi_0(R_Q) \cap \pi_0(R_P) \neq \emptyset \), then \( R_P \subset C R_Q \).

Note that the assumptions above are in particular satisfied for any \( Q, P \in D_k \cap \text{Tree}(R) \setminus \text{BCE}(R) \), by Lemma 7.3.
Taking $\pi_0(x) = \pi_0(z)$ and $z_p = z_p(z)$. Then, we have

$$|\pi_0(y_Q) - \pi_0(y_P)| = |\pi_0(y_Q - z_Q) - \pi_0(y_P - z_P)|$$

$$\leq |\pi_0(y_Q - z_Q)| + |\pi_0(y_P - z_P)| + |\pi_0(z_p - z_Q)| \leq \ell(Q) + \ell(P) + 0 = 2\ell(Q).$$

We claim that $|\pi_0^+(y_Q) - \pi_0^+(y_P)| \leq C'\mathcal{L}(Q)$ for some big absolute $C' > 1$. Indeed, if that was not the case, then the previous computation gives

$$|\pi_0(y_Q) - \pi_0(y_P)| \leq 2\ell(Q) = 2\mathcal{H}^1(J,\mathcal{L}(Q)) \leq \frac{2\mathcal{H}^1(J)}{C'}|y_Q - y_P|.$$

Taking $C' > 1$ large enough, we arrive at

$$y_P \in X(y_Q, 0.5J, \rho\mathcal{L}(Q), \mathcal{L}(R)),$$

which is a contradiction with (7.6). Hence, $|\pi_0^+(y_Q) - \pi_0^+(y_P)| \leq C'\mathcal{L}(Q)$.

Recall that $x_Q$ is the center of $R_Q$. It follows easily from the estimates above that for any $x \in R_P$

$$|\pi_0(x) - \pi_0(x_Q)| \leq |\pi_0(x) - \pi_0(y_P)| + |\pi_0(y_P) - \pi_0(y_Q)| + |\pi_0(y_Q) - \pi_0(x_Q)|$$

$$\leq \ell(P) + 2\ell(Q) + \ell(Q) = 4\ell(Q),$$

and

$$|\pi_0^+(x) - \pi_0^+(x_Q)| \leq |\pi_0^+(x) - \pi_0^+(y_P)| + |\pi_0^+(y_P) - \pi_0^+(y_Q)| + |\pi_0^+(y_Q) - \pi_0^+(x_Q)|$$

$$\leq \mathcal{L}(P) + C'\mathcal{L}(Q) + \mathcal{L}(Q) \lesssim \mathcal{L}(Q).$$

Thus, $R_P \subset CR_Q$ for some absolute $C > 1$. \hfill $\square$

Recall that that for $Q \in D_k$ we have $\ell(Q) = 4\rho^k$.

**Lemma 7.5.** Let $R \in T_k$ and $k \geq 0$. Then, the family of intervals $\{\pi_0(R_P)\}_{P \in T_k(R)}$ has bounded overlaps, i.e.

$$\sum_{P \in T_k(R)} \mathds{1}_{\pi_0(R_P)}(x) \lesssim 1 \quad \text{for all } x \in \mathbb{R}. \quad (7.7)$$

In particular, for any interval $K \subset \mathbb{R}$ we have

$$\# \left\{ P \in T_k(R) : \pi_0(R_P) \subset K \right\} \lesssim \frac{\mathcal{H}^1(K)}{\rho^k}. \quad (7.8)$$

**Proof.** Fix $Q \in T_k(R)$. Suppose that $P \in T_k(R)$ satisfies $\pi_0(R_Q) \cap \pi_0(R_P) \neq \emptyset$. We know from Lemma 7.3 that $Q$ and $P$ satisfy (7.6), and so it follows Lemma 7.4 that $R_P \subset CR_Q$. It remains to observe that

$$\# \left\{ P \in T(R) \cap D_k : R_P \subset CR_Q \right\} \lesssim C 1.$$

This gives (7.7).
To see (7.8), we compute

\[
\# \left\{ P \in T_k(R) : \pi_0(R_P) \subset K \right\} \leq \sum_{P \in T_k(R)} \frac{1}{\rho_k} \int_K 1_{\pi_0(R_P)}(x) \, dx
\]

\[
= \frac{1}{\rho_k} \int_K \sum_{P \in T_k(R)} 1_{\pi_0(R_P)}(x) \, dx \lesssim \frac{\mathcal{H}^1(K)}{\rho_k}.
\]

□

8. Estimating exterior energy

Recall that

\[
\tilde{E}^{\text{ext}}_J(Q) = \frac{1}{\mu(Q)} \int_Q \mu(X(x, 3J \setminus 0.5J, \rho \mathcal{L}(Q), \mathcal{L}(Q))) \, d\mu(x).
\]

Our goal is to prove the following.

**Lemma 8.1.** If \(A = A(C_0, M)\) is chosen large enough, then for any \(R \in \text{Top}\) we have

\[
\sum_{\mathcal{Q} \in \text{Tree}(R)} \tilde{E}^{\text{ext}}_J(Q) \mu(Q) \lesssim_{C_0, M} \mathcal{H}^1(J) \mu(R).
\]

This estimate will follow from the key geometric lemma below. In order to state it, we introduce some notation.

**Definition 8.2.** For \(R \in D_+\) we define \(U(R) \subset \mathbb{R}\) as

\[
U(R) := \pi_0(AR_R) \setminus \pi_0(AR_R \cap E) = \left[ \pi_0(x_R) - \frac{A \ell(R)}{2}, \pi_0(x_R) + \frac{A \ell(R)}{2} \right] \setminus \pi_0(AR_R \cap E).
\]

Denote by \(\text{Gap}(R)\) the family of connected components of \(U(R)\). Since \(E\) is closed, the elements of \(\text{Gap}(R)\) are intervals. We will call them gaps in \(\pi_0(AR_R \cap E)\).

Since the gaps are disjoint, and they have positive length, we get that \(\text{Gap}(R)\) is at most countable, and also

\[
\sum_{K \in \text{Gap}(R)} \mathcal{H}^1(K) \leq \mathcal{H}^1(U(R)) \leq \mathcal{H}^1(\pi_0(AR_R)) = A\ell(R). \quad (8.1)
\]

Given \(0 < r < \ell(R)\) we define the collection of gaps with length comparable to \(r\) as

\[
\text{Gap}(R, r) = \{ K \in \text{Gap}(R) : A^{-1}r \leq \mathcal{H}^1(K) \leq Ar \}.
\]

**Definition 8.3.** For \(R \in D_+\), we define the family \(\text{Bad}(R) \subset D(R)\) as the family of cubes \(Q \in D(R)\) for which there exists \(x \in Q\) such that

\[
X(x, 3J \setminus 0.5J, \rho \mathcal{L}(Q), \mathcal{L}(Q)) \cap E \neq \emptyset.
\]

Observe that if \(Q \notin \text{Bad}(R)\), then

\[
\tilde{E}^{\text{ext}}_J(Q) = \frac{1}{\mu(Q)} \int_Q \mu(X(x, 3J \setminus 0.5J, \rho \mathcal{L}(Q), \mathcal{L}(Q))) \, d\mu(x) = 0.
\]

The following is the key geometric lemma of this article.
Lemma 8.4. If $A = A(C_0, M)$ is chosen large enough, then the following holds. Suppose that $R \in D_*$ and $Q \in D(R)$ are such that

$$X(z, 0.5J, A^{-1}L(Q), A^2L(R)) \cap E = \emptyset \quad \text{for all } z \in AR_Q \cap E. \quad (8.2)$$

If $Q \in \text{Bad}(R)$, then there is a gap $K \in \text{Gap}(R, \ell(Q))$ such that

$$\pi_0(R_Q) \subset A^3K.$$

We defer the proof to the next section. Let us show how Lemma 8.1 follows from Lemma 8.4.

Proof of Lemma 8.1. Let $R \in \text{Top}$. Our goal is to prove

$$\sum_{Q \in \text{Tree}(R)} \tilde{E}^\text{ext}_J(Q) \mu(Q) \lesssim_{C_0, M} \mathcal{H}^1(J) \mu(R).$$

Recall that $\mathcal{T}(R) = \text{Tree}(R) \setminus \text{BCE}(R)$, $\mathcal{T}_k(R) = \mathcal{T}(R) \cap D_k$. If $Q \notin \text{Bad}(R)$, then $\tilde{E}^\text{ext}_J(Q) = 0$ trivially, and so it suffices to show

$$\sum_{Q \in \mathcal{T}(R) \cap \text{Bad}(R)} \tilde{E}^\text{ext}_J(Q) \mu(Q) + \sum_{Q \in \text{BCE}(R)} \tilde{E}^\text{ext}_J(Q) \mu(Q) \lesssim_{C_0, M} \mathcal{H}^1(J) \mu(R). \quad (8.3)$$

Observe that for any $x \in E$ we have

$$\mu(X(x, 3J \setminus 0.5J, L(Q), L(Q))) \leq \mu(R(x, 3\ell(Q))) \overset{(5.4)}{\lesssim} M \ell(Q),$$

and so for any $Q \in D_*$

$$\tilde{E}^\text{ext}_J(Q) \mu(Q) = \int_Q \mu(X(x, 3J \setminus 0.5J, L(Q), L(Q))) \frac{d\mu(x)}{L(Q)} \lesssim \frac{M \ell(Q)}{L(Q)} \mu(Q) = M \mathcal{H}^1(J) \mu(Q).$$

It follows that

$$\sum_{Q \in \mathcal{T}(R) \cap \text{Bad}(R)} \tilde{E}^\text{ext}_J(Q) \mu(Q) + \sum_{Q \in \text{BCE}(R)} \tilde{E}^\text{ext}_J(Q) \mu(Q) \lesssim M \mathcal{H}^1(J) \left( \sum_{Q \in \mathcal{T}(R) \cap \text{Bad}(R)} \mu(Q) + \sum_{Q \in \text{BCE}(R)} \mu(Q) \right).$$

Thus, to reach $(8.3)$, it suffices to show that the two sums on the right hand side above are bounded by $C(C_0, M) \mu(R)$. This is immediate for the second sum:

$$\sum_{Q \in \text{BCE}(R)} \mu(Q) \leq \mu(R).$$

What remains to show is that

$$\sum_{Q \in \mathcal{T}(R) \cap \text{Bad}(R)} \mu(Q) \lesssim_{C_0, M} \mu(R). \quad (8.4)$$
Let $Q \in \mathcal{T}(R) \cap \text{Bad}(R) \subset \text{Tree}(R) \setminus \text{BCE}(R)$. By Lemma 7.3, $R$ and $Q$ satisfy the empty cone assumption (8.2), and so we may use Lemma 8.4 to conclude that there is a gap $K \in \text{Gap}(R, \ell(Q))$ such that $\pi_0(\mathcal{R}_Q) \subset A^3K$. Hence,

$$\sum_{Q \in \mathcal{T}(R) \cap \text{Bad}(R)} \mu(Q) = \sum_{k \geq 0} \sum_{Q \in \mathcal{T}_k(R) \cap \text{Bad}(R)} \mu(Q) \leq \sum_{k \geq 0} K \in \text{Gap}(R, A^k) \sum_{Q \in \mathcal{T}_k(R), \pi_0(\mathcal{R}_Q) \subset A^3K} \mu(Q) \lesssim \sum_{k \geq 0} K \in \text{Gap}(R, A^k) \sum_{Q \in \mathcal{T}_k(R), \pi_0(\mathcal{R}_Q) \subset A^3K} M \ell(Q) \overset{(7.3)}{\lesssim} \sum_{k \geq 0} K \in \text{Gap}(R, A^k) M \rho^k \frac{\mathcal{H}^1(A^3K)}{\rho^k} \sim_{A,M} \sum_{k \geq 0} K \in \text{Gap}(R, A^k) K \overset{(8.1)}{\sim} A \ell(R).$$

Since $A = A(C_0, M)$ and $\mu(R) \gtrsim C_0^{-1} \ell(R)$, this gives the desired estimate (8.4). □

## 9. Proof of the Key Geometric Lemma

In this section we prove Lemma 8.4.

### 9.1. Preliminaries

Suppose that $R \in \mathcal{D}_*$ and $Q \in \mathcal{D}(R)$ are as in the assumptions of Lemma 8.4, so that they satisfy

$$X(z, 0.5J, A^{-1}L(Q), A^2L(R)) \cap E = \emptyset \text{ for all } z \in AR_Q \cap E, \quad (9.1)$$

and assume that $Q \in \text{Bad}(R)$, which means that there exists $x \in Q$ such that

$$X(x, 3J \setminus 0.5J, \rho L(Q), L(R)) \cap E \neq \emptyset.$$

Let $y \in X(x, 3J \setminus 0.5J, \rho L(Q), L(R)) \cap E$. See Figure 9.1 for an overview of our setup. The plan is as follows. We want to find a gap $K \in \text{Gap}(R, \ell(Q))$ such that

$$\pi_0(\mathcal{R}_Q) \subset A^3K.$$

To achieve this, we will find a rectangle $\mathcal{Y}$ satisfying $\mathcal{Y} \cap E = \emptyset$ (in our terminology: “$\mathcal{Y}$ is a good rectangle”) of size roughly $\ell(Q) \times L(R)$, such that $\pi_0(\mathcal{Y}) \supset \pi_0(AR_R)$, and such that $\mathcal{Y}$ lies between $x$ and $y$, in the sense that $\pi_0(x)$ and $\pi_0(y)$ lie on different sides of the interval $\pi_0(\mathcal{Y})$. See the yellow rectangle in Figure 9.1. The properties above tell us that

$$\pi_0(\mathcal{Y}) \cap \pi_0(AR \cap E) = \emptyset,$$

so that $\pi_0(\mathcal{Y})$ is contained in some interval $K \in \text{Gap}(R)$. One can also see that $K$ necessarily satisfies $\mathcal{H}^1(K) \sim A \ell(Q)$, so that $K \in \text{Gap}(R, \ell(Q))$. This will be our desired gap.
Figure 9.1. The big white rectangle is $AR_R$, the small white rectangle is $R_Q$, the orange double-truncated cone is $X(x, 3J \setminus 0.5J, \rho \mathcal{L}(Q), \mathcal{L}(Q))$, the yellow rectangle is the desired good rectangle $Y$.

The double truncated cone $X(x, 3J \setminus 0.5J, \rho \mathcal{L}(Q), \mathcal{L}(Q))$ has 4 connected components (see the orange cone in Figure 9.1 or Figure 9.2). Without loss of generality, we may assume that $y$ lies in the lower right connected component, so that $\pi_0(x) < \pi_0(y)$ and $\pi_0^+(x) > \pi_0^+(y)$ (the proof for other cases is completely analogous). Note that, since $y \in X(x, 3J \setminus 0.5J, \rho \mathcal{L}(Q), \mathcal{L}(Q))$, we have

$$\pi_0(y) - \pi_0(x) \sim \ell(Q),$$

and

$$\pi_0^+(x) - \pi_0^+(y) \sim \mathcal{L}(Q).$$
9.2. Finding a leftist rectangle. Recall that our desired good rectangle \( Y \) will be of size roughly \( \ell(Q) \times L(R) \) and will satisfy \( \pi_{0k}(Y) \supset \pi_{0k}(AR_R) \). Note that any good cone arising from (9.1) already almost contains a rectangle with these properties, except for a missing \( \ell(Q) \times L(Q) \) rectangle close to the center of the cone (see the red cone in Figure 9.6). Our goal is to find an auxiliary good rectangle \( B \) of size roughly \( \ell(Q) \times L(Q) \), which will fill the missing piece of the good cone. See the blue rectangle in Figure 9.6.

The good rectangle \( B \) will be contained in something we called “a leftist rectangle”. In order to define it, we first consider the rectangle

\[
\mathcal{G} := \left\{ z \in \mathbb{R}^2 : \pi_0(x) \leq \pi_0(z) \leq \pi_0(y), |\pi_0(z) - \pi_0(y)| \leq \frac{|\pi^1_0(x) - \pi^1_0(y)|}{2}\right\},
\]

see the gray rectangle in Figure 9.2. Note that \( \ell(\mathcal{G}) = |\pi_0(x) - \pi_0(y)| \sim \ell(Q), L(\mathcal{G}) = |\pi^1_0(x) - \pi^1_0(y)| \sim L(Q) \), and the mid-point of its right edge is \( y \).

Let \( N > 1 \) be a large integer satisfying

\[
N \sim MC_0,
\]

whose precise value will be fixed later on.

We divide \( \mathcal{G} \) into \( 2N + 1 \) sub-rectangles \( \mathcal{G}_{-N}, \ldots, \mathcal{G}_0, \ldots, \mathcal{G}_N \) such that \( \ell(\mathcal{G}_i) = \ell(\mathcal{G}) = |\pi_0(x) - \pi_0(y)| \) and \( L(\mathcal{G}_i) = L(\mathcal{G})/(2N + 1) = |\pi^1_0(x) - \pi^1_0(y)|/(2N + 1) \). We enumerate...
Lemma 9.1. We have check this in the lemma below.

\[
G_i := \left\{ z \in \mathbb{R}^2 : \pi_0(x) \leq \pi_0(z) \leq \pi_0(y), \quad \frac{(2i-1)\mathcal{L}(G)}{2(2N+1)} \leq \pi_0^+(z) - \pi_0^+(y) \leq \frac{(2i+1)\mathcal{L}(G)}{2(2N+1)} \right\}
\]

It is not immediately clear that \(\ell(G_i)\) and \(\mathcal{L}(G_i)\) as we defined them satisfy \(\ell(G_i) \leq \mathcal{L}(G_i)\), and that \(G_i\)'s look as portrayed in Figure 9.3 as opposed to being very flat. We check this in the lemma below.

Lemma 9.1. We have \(\ell(G_i) \leq \mathcal{L}(G_i)\).

Proof. Recall that \(\ell(G_i) = \ell(G) \sim \ell(Q)\), and

\[
\mathcal{L}(G_i) = \frac{\mathcal{L}(G)}{2N+1} \sim \frac{\mathcal{L}(Q)}{2N} = \frac{\mathcal{H}^1(J)^{-1}(Q)}{N} = \frac{\ell(G_i)}{\mathcal{H}^1(J)N} \leq \frac{\ell(G_i)}{\mathcal{H}^1(J)MC_0}.
\]

(9.3)

Assumption (b) of Proposition 4.1 stated that \(\mathcal{H}^1(J) \leq c_1C_0^{-1}M^{-1}\), where \(c_1 > 0\) is a small absolute constant. Assuming \(c_1\) to be small enough, the above estimates give

\[
\mathcal{L}(G_i) \geq \ell(G_i).
\]

(9.4)

\(\square\)

The following three definitions are easier to digest together with the right hand side of Figure 9.3

Definition 9.2. For each \(G_i\) with \(G_i \cap E \neq \emptyset\), let \(z_i \in G_i \cap E\) be a point such that

\[
\pi_0(z_i) = \inf_{z \in G_i \cap E} \pi_0(z).
\]

We will call \(z_i\) the leftmost point of \(G_i \cap E\). Note that the left-most point is well-defined because \(G_i\) and \(E\) are closed. It might be non-unique, but we do not care.

Definition 9.3. If \(-N \leq i, j \leq N\) and \(G_i \cap E \neq \emptyset\), then we will write \(G_i \prec G_j\) if either \(G_j \cap E = \emptyset\) or \(\pi_0(z_i) \leq \pi_0(z_j)\). In other words, \(G_i \prec G_j\) means that there is no point of \(G_j \cap E\) to the left of \(z_i\).

Definition 9.4. For \(-N + 1 \leq i \leq N - 1\), we will say that \(G_i\) is a leftist rectangle if \(G_i \cap E \neq \emptyset\) and we have \(G_i \prec G_{i-1}\) and \(G_i \prec G_{i+1}\). That is, the point \(z_i\) is the leftmost point of \((G_{i-1} \cup G_i \cup G_{i+1}) \cap E\).

Lemma 9.5. There exists \(-N + 1 \leq i \leq N - 1\) such that \(G_i\) is a leftist rectangle.

Proof. Suppose the opposite, so that none of the rectangles is leftist. In particular, \(G_0\) is not leftist. This means that either \(G_0 \cap E = \emptyset\), or for some \(i \in \{-1, 1\}\) we have \(G_i \prec G_0\). Since \(y \in G_0 \cap E\), the second alternative holds. Without loss of generality assume that \(G_1 \prec G_0\).

Since \(G_1\) is not leftist, but \(G_1 \prec G_0\), we get that \(G_2 \prec G_1\). In particular, \(G_2 \cap E \neq \emptyset\). Continuing in this way, we get for \(1 \leq j \leq N - 1\) that \(G_{j+1} \prec G_j \prec G_{j-1}\). In particular, for all \(1 \leq j \leq N\) we have \(z_j \in G_j \cap E \neq \emptyset\).
Let \( 1 \leq j \leq N \). By (9.4), we have \( B(z_j, \ell(G_j)) \subset 3G_j \), and so
\[
\mu(3G_j) \geq \mu(B(z_j, \ell(G_j))) \geq C_0^{-1}\ell(G_j).
\]
Since the rectangles \( \{3G_j\}_{j=1}^N \) have bounded overlap, and they are all contained in \( 3G \), we get that
\[
\mu(3G) \geq \sum_{j=1}^N \mu(3G_j) \geq \sum_{j=1}^N C_0^{-1}\ell(G_j) = NC_0^{-1}\ell(G).
\]  
(9.5)
Recall that \( \ell(G) = |x - y| \) and \( \mathcal{L}(G) = |\pi_0(x) - \pi_0(y)| \sim \mathcal{H}^1(J)^{-1}\ell(G) \), so that \( 3G \subset \mathcal{R}(y, C\ell(G)) \) for some absolute constant \( C > 1 \). Hence, we get from (5.4) that
\[
\mu(3G) \leq \mu(\mathcal{R}(y, C\ell(G))) \lesssim M\ell(G).
\]  
(9.6)
In the definition of \( N \) (9.2), we assumed \( N \sim MC_0 \). Let \( N = \lceil C' M C_0 \rceil \), where \( C' > 1 \) is a big absolute constant. Pitting (9.5) against (9.6) and choosing \( C' > 1 \) large enough, we reach a contradiction. \( \square \)

The combination of Lemma 9.5 and the following lemma will complete the proof of the key geometric lemma.

**Lemma 9.6.** If \( G_i \) is a leftist rectangle, then \( \pi_0(z_i) \) is the right endpoint of some gap \( K \in \text{Gap}(R, \ell(Q)) \) with \( \pi_0(\mathcal{R}_Q) \subset A^3K \).
Figure 9.4. The blue rectangle is $B$. In Lemma 9.7 we show that $\ell(B) \sim \ell(G) \sim \ell(Q)$.

We divide the proof of Lemma 9.6 into several steps.

9.3. Small good rectangle $B$. Assume that $G_i$ is a leftist rectangle. We define

$$B := \{ z \in G_{i-1} \cup G_i \cup G_{i+1} : \pi_0(z) \leq \pi_0(z_i) \}, \quad (9.7)$$

$$= \{ z \in G_{i-1} \cup G_i \cup G_{i+1} : \pi_0(x) \leq \pi_0(z) \leq \pi_0(z_i) \}, \quad (9.8)$$

see the blue rectangle in Figure 9.4. A priori it might happen that $\pi_0(z_i) = \pi_0(x)$, in which case $B$ would be a degenerate rectangle (a segment). We show in Lemma 9.7 below that this is not the case.

Note that

$$L(B) = L(G_{i-1}) + L(G_i) + L(G_{i+1}) = 3L(G) \frac{2N+1}{N} \sim \frac{L(Q)}{N},$$

and also $\ell(B) = |\pi_0(z_i) - \pi_0(x)|$.

Since $G_i$ is a leftist rectangle, it follows immediately from the definitions of leftist rectangles and leftmost points that

$$\text{int}(B) \cap E = \emptyset, \quad (9.9)$$

so that $\text{int}(B)$ is a good (open) rectangle.

**Lemma 9.7.** We have $|\pi_0(z_i) - \pi_0(x)| = \ell(B) \sim \ell(Q)$.

**Proof.** Since $B \subset G$, it is clear that

$$\ell(B) \leq \ell(G) \sim \ell(Q),$$

so we only need to prove $\ell(B) \gtrsim \ell(G) \sim \ell(Q)$. See Figure 9.5 to get some intuition on why this is true. We give a formal argument below.

Assume the contrary, so that $\ell(B) \leq c \ell(G)$ for some small absolute constant $0 < c < 1$. We claim that if $0 < c < 1$ is chosen small enough, then

$$B \subset X(x, 0.5J, A^{-1}L(Q), A\mathcal{L}(Q)). \quad (9.10)$$
Figure 9.5. On the left we see the full picture, on the right we zoom in on the dashed-border rectangle. The white rectangle is $R_Q$, the gray rectangle is $G$, the blue rectangle is $B$, the red double-truncated cone is $X(x, 0.5J, A^{-1}\mathcal{L}(Q), A\mathcal{L}(Q))$. The red cone has an empty intersection with $E$ by (9.1), whereas $B$ contains the point $z_i \in E$. Thus, $B$ cannot be fully contained in the red cone, which gives $\ell(B) \gtrsim \ell(Q)$.

To see that, observe that if $z \in B$, then

$$|\pi_0(z) - \pi_0(x)| \leq \ell(B) \leq c\ell(G) \sim c\ell(Q),$$

and also, since $B \subset G$,

$$\frac{\mathcal{L}(G)}{2} \leq |\pi_0^+(z) - \pi_0^+(x)| \leq \frac{3\mathcal{L}(G)}{2}.$$  

In particular, $|\pi_0^+(z) - \pi_0^+(x)| \sim \mathcal{L}(G) \sim \mathcal{L}(Q) = \mathcal{H}^1(J)^{-1}\ell(Q)$. It follows that

$$|\pi_0(z) - \pi_0(x)| \lesssim c\mathcal{H}^1(J)|\pi_0^+(z) - \pi_0^+(x)|.$$  

If $0 < c < 1$ is chosen small enough, we get that $z \in X(x, 0.5J)$. 

Since

$$|x - z| \sim |\pi_0(z) - \pi_0(x)| + |\pi_0^+(z) - \pi_0^+(x)| \sim \mathcal{L}(Q),$$

we also have $z \in X(x, 0.5J, A^{-1}\mathcal{L}(Q), A\mathcal{L}(Q))$ if $A$ is chosen large enough. This shows (9.10).
Recall that $X(x, 0.5L, A^{-1}L(Q), A\mathcal{L}(Q)) \cap E = \emptyset$ by the assumption (9.1). At the same time, $B$ contains $z_i \in E$. This contradicts (9.10). Hence,

$$\ell(B) \geq c\ell(G) \sim \ell(Q).$$

\[\square\]

9.4. **Big good rectangle $\mathcal{Y}$.** Consider the rectangle $\mathcal{Y}$ defined as

$$\mathcal{Y} := \{ z \in \mathbb{R}^2 : \pi_0(z_i) - A^{-1}\ell(Q) \leq \pi_0(z) \leq \pi_0(z_i), \ |\pi_0(z) - \pi_0(z_i)| \leq 2A\mathcal{L}(R) \},$$
see the yellow rectangle in Figure 9.6. Note that $\ell(Y) = A^{-1} \ell(Q)$, $\mathcal{L}(Y) = 4A\mathcal{L}(R)$, and the mid-point of its right edge is $z_i$.

Our plan is the following. First, we will show that $Y$ is contained in the union of the good cone $X(z_i, 0.5J, A^{-1}\mathcal{L}(Q), A^2\mathcal{L}(R))$ (the red cone in the figure) and the good rectangle $B$ (the blue rectangle in the figure). Since the interiors of these two have empty intersections with $E$, we will conclude that $\text{int}(Y) \cap E = \emptyset$. This will give us $K \in \text{Gap}(R, \ell(Q))$ with $\pi_0(R_Q) \subset A^iK$, the desired gap in $\pi_0(AR_R \cap E)$.

**Lemma 9.8.** If $A = A(C_0, M)$ is chosen big enough, then

$$\text{int}(Y) \subset \text{int}(B) \cup X(z_i, 0.5J, A^{-1}\mathcal{L}(Q), A^2\mathcal{L}(R)).$$

**Proof.** This is easy to believe in after looking at Figure 9.6 for a minute or two, but for the sake of completeness, we provide the computations below. They are easier to follow keeping Figure 9.6 in mind.

Let

$$\mathcal{Y}_1 := \{z \in \mathbb{R}^2 : \pi_0(z_i) - A^{-1}\ell(Q) < \pi_0(z) < \pi_0(z_i), |\pi_0^+(z) - \pi_0^+(z_i)| < \mathcal{L}(G_i)\},$$

$$\mathcal{Y}_2 := \text{int}(Y) \setminus \mathcal{Y}_1,$$

so that $\text{int}(Y) = \mathcal{Y}_1 \cup \mathcal{Y}_2$. We claim that

$$\mathcal{Y}_1 \subset \text{int}(B),$$

and

$$\mathcal{Y}_2 \subset X(z_i, 0.5J, A^{-1}\mathcal{L}(Q), A^2\mathcal{L}(R)).$$

First we prove (9.12). By Lemma 9.7 we have $\ell(\mathcal{Y}_1) = A^{-1}\ell(Q) \leq \ell(B)$, assuming $A$ big enough. Since $z_1$ lies on the right edges of both $\mathcal{Y}_1$ and $B$, this immediately gives $\pi_0(\mathcal{Y}_1) \subset \pi_0(\text{int}(B))$. On the other hand, recall that $z_i \in G_i$ and

$$\pi_0^+(B) = \pi_0^+(G_{i-1}) \cup \pi_0^+(G_i) \cup \pi_0^+(G_{i+1}),$$

see Figure 9.4. It follows that

$$\pi_0^+(\mathcal{Y}_1) = (\pi_0^+(z_i) - \mathcal{L}(G_i), \pi_0^+(z_i) + \mathcal{L}(G_i)) \subset \pi_0^+(\text{int}(B)).$$

Since both $\mathcal{Y}_1$ and $\text{int}(B)$ are open rectangles with sides parallel to the axes, we conclude that $\mathcal{Y}_1 \subset \text{int}(B)$.

We move on to (9.13). First, observe that for $z \in \mathcal{Y}_2$ we have, by the definition of $\mathcal{Y}$,

$$|z - z_i| \leq (A^{-2}\ell(Q)^2 + 4A^2\mathcal{L}(R)^2)^{1/2} \leq 3A\mathcal{L}(R),$$

and also, since $z \notin \mathcal{Y}_1$,

$$|z - z_i| \geq |\pi_0^+(z) - \pi_0^+(z_i)| \geq \mathcal{L}(G_i) = \frac{\mathcal{L}(G)}{2N + 1} \geq \frac{\mathcal{L}(Q)}{N} \geq \frac{\mathcal{L}(Q)}{MC_0} \mathcal{L}(Q)/MC_0.$$

Thus, assuming $A = A(M, C_0)$ large enough, we have

$$z \in B(z_i, A^2\mathcal{L}(R)) \setminus B(z_i, A^{-1}\mathcal{L}(Q)).$$

It remains to show $z \in X(z_i, 0.5J)$. Note that

$$|\pi_0(z) - \pi_0(z_i)| \leq A^{-1}\ell(Q) = A^{-1}\mathcal{H}^1(J, \mathcal{L}(Q))$$

$$= MC_0A^{-1}\mathcal{H}^1(J)\frac{\mathcal{L}(Q)}{MC_0} \leq MC_0A^{-1}\mathcal{H}^1(J) |\pi_0^+(z) - \pi_0^+(z_i)|.$$
Assuming $A = A(M, C_0)$ large enough, this gives $z \in X(z_i, 0.5J)$. □

**Lemma 9.9.** We have $\text{int}(Y) \cap E = \emptyset$.

**Proof.** Recall that $z_i \in \mathcal{G} \cap E$, and $\mathcal{G} \subset A\mathcal{R}_Q$. Thus, $z_i \in A\mathcal{R}_Q \cap E$, and so we get from (9.1) that

$$X(z_i, 0.5J, A^{-1}\mathcal{L}(Q), A^2\mathcal{L}(R)) \cap E = \emptyset.$$  

We also have $\text{int}(B) \cap E = \emptyset$ by (9.9). Hence, it follows from (9.11) that $\text{int}(Y) \cap E = \emptyset$. □

9.5. **Mind the gap.** We are finally ready to find the gap $K \in \text{Gap}(R, \ell(Q))$ with $\pi_0(R_Q) \subset A^4K$.

First, note that $z_i \in A\mathcal{R}_Q \subset A\mathcal{R}_R$. Since $\mathcal{L}(Y) = 4A\mathcal{L}(R)$ and $z_i$ is the mid-point of the right edge of $Y$, it follows that

$$\{z \in A\mathcal{R}_R : \pi_0(z) \in \pi_0(\text{int}(Y))\} \subset \text{int}(Y)$$

Together with Lemma 9.9, this gives

$$\{z \in A\mathcal{R}_R \cap E : \pi_0(z) \in \pi_0(\text{int}(Y))\} \subset \text{int}(Y) \cap E = \emptyset.$$ 

Hence,

$$\pi_0(A\mathcal{R}_R \cap E) \cap \pi_0(\text{int}(Y)) = \emptyset.$$ 

This means that the open interval $\pi_0(\text{int}(Y)) = (\pi_0(z_i) - A^{-1}\ell(Q), \pi_0(z_i))$ is contained in some gap $K \in \text{Gap}(R)$. We have

$$\mathcal{H}^1(K) \geq \mathcal{H}^1(\pi_0(\text{int}(Y))) = A^{-1}\ell(Q).$$

Note that $x, z_i \in A\mathcal{R}_R \cap E$. Thus, $\pi_0(x), \pi_0(z_i) \notin K$, and also $\pi_0(z_i)$ lies on the right end-point of $K$. By Lemma 9.7

$$\pi_0(z_i) - \pi_0(x) = \ell(B) > A^{-1}\ell(Q) = \mathcal{H}^1(\pi_0(\text{int}(Y))),$$

so that

$$\pi_0(x) \leq \pi_0(z_i) - \mathcal{H}^1(\pi_0(\text{int}(Y))).$$

This means that $\pi_0(x)$ lies “to the left” of the interval $\pi_0(\text{int}(Y))$, and in consequence, “to the left” of the gap $K$. Since $\pi_0(z_i)$ is the right end-point of $K$, it follows from Lemma 9.7 that

$$\mathcal{H}^1(K) \leq |\pi_0(x) - \pi_0(z_i)| = \ell(B) \sim \ell(Q).$$

So we have $A^{-1}\ell(Q) \leq \mathcal{H}^1(K) \lesssim \ell(Q)$. In particular, $K \in \text{Gap}(R, \ell(Q))$.

Finally, we have

$$\text{dist}(\pi_0(R_Q), K) \leq \text{dist}(\pi_0(x), K) \leq |\pi_0(x) - \pi_0(z_i)| \lesssim \ell(Q) \leq A\mathcal{H}^1(K),$$

and so $\pi_0(R_Q) \subset A^3K$. This finishes the proof of Lemma 9.6 and of the key geometric lemma.
Appendix A. Proof of Corollary 3.2

In this section we prove Corollary 3.2, which we repeat below for reader’s convenience.

**Corollary A.1.** Let $E \subset \mathbb{R}^2$ and $G \subset \mathbb{T}$ be as in Theorem 1.7 and let $\mu = \mathcal{H}^1|_E$. Then,

$$\int_{\mathbb{R}^2} \int_0^\infty \frac{\mu(X(x, G^+, r))}{r} \frac{dr}{r} d\mu(x) \lesssim M \mathcal{H}^1(G) \mu(E),$$

where $G^+ = G + 1/4$.

**Proof.** If the set $G$ is open, then we can immediately apply Proposition 3.1 to estimate

$$\int_{\mathbb{R}^2} \int_0^\infty \frac{\mu(X(x, G^+, r))}{r} \frac{dr}{r} d\mu(x) \lesssim \int_G \|\pi_\theta \mu\|_2^2 d\theta = \int_G \int_{\mathbb{R}} |\pi_\theta \mu(x)|^2 dx d\theta \lesssim M \int_G \int_{\mathbb{R}} \pi_\theta \mu(x) dx d\theta = M \mathcal{H}^1(G) \mu(E), \quad (A.1)$$

which is the desired inequality.

The general case will follow from the classical Besicovitch projection theorem and approximation. Suppose that $G$ is not open. Note that the assumption (1.3) implies that $\mathcal{H}^1(\pi_\theta(E)) > 0$ for all $\theta \in G$, and even $\mathcal{H}^1(\pi_\theta(F)) > 0$ for all $F \subset E$ with $\mathcal{H}^1(F) > 0$. Since $\mathcal{H}^1(G) > 0$, we get from the classical Besicovitch projection theorem, Theorem A, that $E$ is rectifiable, so that

$$E = \bigcup_{i=1}^\infty \Gamma_i \cup Z,$$

where $\Gamma_i$ is a measurable subset of a graph of a $C^1$-function, and $\mathcal{H}^1(Z) = 0$. For $N \geq 1$ set

$$E_N := \bigcup_{i=1}^N \Gamma_i,$$

and $\mu_N = \mathcal{H}^1|_{E_N}$.

Fix $\theta \in G$. Since $\|\pi_\theta \mu\|_\infty \leq M$, we have that for each $i \in \mathbb{N}$ and $\mathcal{H}^1$-a.e. point $x \in \Gamma_i$, the line tangent to $\Gamma_i$ at $x$ cannot be perpendicular to $\ell_\theta$, and even

$$\angle(T_x \Gamma_i, \ell_\theta) \leq \frac{\pi}{2} - CM^{-1}$$

for some absolute constant $0 < C < 1$. Hence, if $|\theta' - \theta| \leq cM^{-1}$ for some small absolute constant $0 < c < 1$, then we have

$$\angle(T_x \Gamma_i, \ell_{\theta'}) \leq \frac{\pi}{2} - C'M^{-1}.$$  

It follows that if $|\theta' - \theta| \leq cM^{-1}$, then for any $i \in \mathbb{N}$ we have $\|\pi_{\theta'} \mathcal{H}^1|_{\Gamma_i}\|_\infty \lesssim M$. Thus,

$$\|\pi_{\theta'} \mu_N\|_\infty \leq \sum_{i=1}^N \|\pi_{\theta'} \mathcal{H}^1|_{\Gamma_i}\|_\infty \lesssim NM.$$

By the outer regularity of Lebesgue measure, there exists a sequence of open sets $G_k \supset G$ such that

$$\mathcal{H}^1(G_k \setminus G) \leq \frac{1}{k}.$$
Without loss of generality we may assume that each $G_k$ is contained in a $cM^{-1}$-neighbourhood of $G$, so that for all $\theta \in G$ we have $\|\pi_{\theta} Mu\|_{\infty} \leq \|\pi_{\theta} \mu\|_{\infty} \leq M$ and for all $\theta \in G_k \setminus G$ we have $\|\pi_{\theta} Mu\|_{\infty} \lesssim NM$. Then, repeating the computation from (A.1) yields
\[
\int_{\mathbb{R}^2} \int_0^\infty \frac{\mu_N(\pi(x, G_k, r))}{r} dr d\mu_N(x) \lesssim \int_{G_k} \|\pi_{\theta} \mu\|_{L^2}^2 d\theta \leq M\mathcal{H}^1(G)\mu_N(E) + MN\mathcal{H}^1(G_k \setminus G)\mu_N(E). \tag{A.2}
\]

Note that $\mu_N(\pi(x, G, r)) \leq \liminf_{k \to \infty} \mu_N(\pi(x, G_k, r))$, and so by Fatou’s lemma
\[
\int_{\mathbb{R}^2} \int_0^\infty \frac{\mu_N(\pi(x, G, r))}{r} dr d\mu_N(x) \leq \int_{\mathbb{R}^2} \int_0^\infty \liminf_{k \to \infty} \frac{\mu_N(\pi(x, G_k, r))}{r} dr d\mu_N(x) \leq \liminf_{k \to \infty} \left( M\mathcal{H}^1(G)\mu_N(E) + MN\mathcal{H}^1(G_k \setminus G)\mu(E) \right) = M\mathcal{H}^1(G)\mu_N(E) \leq M\mathcal{H}^1(G)\mu(E). \tag{A.3}
\]

Now, fix $0 < r < \infty$. We claim that
\[
f_N(r) := \int_{\mathbb{R}^2} \mu_N(\pi(x, G, r)) dr \xrightarrow{N \to \infty} \int_{\mathbb{R}^2} \mu(\pi(x, G, r)) dr =: f(r).
\]
Indeed, we have
\[
|f(r) - f_N(r)| = \int_{\mathbb{R}^2} \mu(\pi(x, G, r)) dr - \int_{\mathbb{R}^2} \mu_N(\pi(x, G, r)) dr = \int_{E \setminus E_N} \mu(\pi(x, G, r)) dr - \mu_N(\pi(x, G, r)) dr \leq \mu(E) \cdot \mu(E \setminus E_N) + \mu(E_N) \cdot \mu(E \setminus E_N) \xrightarrow{N \to \infty} 0.
\]

Hence, by Fatou’s lemma and Fubini’s theorem
\[
\int_{\mathbb{R}^2} \int_0^\infty \frac{\mu_N(\pi(x, G, r))}{r} dr d\mu_N(x) = \int_0^\infty f(r) \frac{dr}{r^2} = \liminf_{N \to \infty} \int_0^\infty f_N(r) \frac{dr}{r^2} \leq \liminf_{N \to \infty} \int_0^\infty f_N(r) \frac{dr}{r^2} \leq \liminf_{N \to \infty} \int_{\mathbb{R}^2} \int_0^\infty \frac{\mu_N(\pi(x, G, r))}{r} dr d\mu_N(x) \lesssim \liminf_{N \to \infty} M\mathcal{H}^1(G)\mu(E) = M\mathcal{H}^1(G)\mu(E).
\]

\[\square\]

References

[AHM+16] J. Azzam, S. Hofmann, J. M. Martell, S. Mayboroda, M. Mourgoglou, X. Tolsa, and A. Volberg. Rectifiability of harmonic measure. \textit{Geom. Funct. Anal.}, 26(3):703–728, 2016. doi:10.1007/s00039-016-0371-x.

[AHM+20] J. Azzam, S. Hofmann, J. M. Martell, M. Mourgoglou, and X. Tolsa. Harmonic measure and quantitative connectivity: geometric characterization of the $L^p$-solvability of the Dirichlet problem. \textit{Invent. Math.}, 222(3):881–993, 2020. doi:10.1007/s00222-020-00984-5.

[AT15] J. Azzam and X. Tolsa. Characterization of $n$-rectifiability in terms of Jones’ square function: Part II. \textit{Geom. Funct. Anal.}, 25(5):1371–1412, 2015. doi:10.1007/s00039-015-0334-7.
[HJJL12] R. Hovila, E. Järvenpää, M. Järvenpää, and F. Ledrappier. Besicovitch-Federer projection theorem and geodesic flows on Riemann surfaces. Geom. Dedicata, 161(1):51–61, 2012. doi:10.1007/s10711-012-9693-5

[IMS12] A. Iosevich, M. Moungoglu, and S. Senger. On sets of directions determined by subsets of $\mathbb{R}^d$. J. Anal. Math., 116(1):355–369, 2012. doi:10.1007/s11854-012-0010-x

[JKV97] P. W. Jones, N. H. Katz, and A. Vargas. Checkerboards, Lipschitz functions and uniform rectifiability. Rev. Mat. Iberoam., 13(1):189–210, 1997. doi:10.4171/rmi/219

[JM88] P. W. Jones and T. Murai. Positive analytic capacity but zero Buffon needle probability. Pacific J. Math., 133(1):99–114, 1988. doi:10.2140/pjm.1988.133.99

[Jon90] P. W. Jones. Rectifiable sets and the traveling salesman problem. Invent. Math., 102(1):1–15, 1990. doi:10.1007/BF01233415

[KRS12] A. Käenmäki, T. Rajala, and V. Suomala. Existence of doubling measures via generalised nested cubes. Proc. Amer. Math. Soc., 140(9):3275–3281, 2012. doi:10.1090/S0002-9939-2012-11161-X

[Łab14] I. Łaba. Recent Progress on Favard Length Estimates for Planar Cantor Sets. In Operator-Related Function Theory and Time-Frequency Analysis, volume 9 of Abel Symposia, pages 117–145. Springer, Cham, 2014. doi:10.1007/978-3-319-08557-9_5

[ŁM22] I. Łaba and C. Marshall. Vanishing sums of roots of unity and the Favard length of self-similar product sets. Preprint, 2022. doi:10.48550/arXiv.2202.07555

[ŁZ10] I. Łaba and K. Zhai. The Favard length of product Cantor sets. Bull. London Math. Soc., 42(6):997–1009, 2010. doi:10.1112/blms/bdq059

[Mar54] J. M. Marstrand. Some Fundamental Geometrical Properties of Plane Sets of Fractional Dimensions.Proc. London Math. Soc., s3-4(1):257–302, 1954. doi:10.1112/plms/s3-4.1.257

[Mat81] P. Mattila. Integralgeometric properties of capacities. Trans. Amer. Math. Soc., 266(2):539–554, 1981. doi:10.1090/S0002-9947-1981-0617550-8

[Mat86] P. Mattila. Smooth Maps, Null-Sets for Integralgeometric Measure and Analytic Capacity. Ann. Of Math., 123(2):303–309, 1986. doi:10.2307/1971273

[Mat90] P. Mattila. Orthogonal Projections, Riesz Capacities, and Minkowski Content. Ann. Of Math., 132(2):303–309, 1990. doi:10.1512/iumj.1990.39.39011

[Mat95] P. Mattila. Geometry of sets and measures in Euclidean spaces: fractals and rectifiability, volume 44 of Cambridge Stud. Adv. Math. Cambridge Univ. Press, Cambridge, UK, 1995. doi:10.1017/CBO9780511623813

[MO18] H. Martikainen and T. Orponen. Characterising the big pieces of Lipschitz graphs property using projections. J. Eur. Math. Soc. (JEMS), 20(5):1055–1073, 2018. doi:10.4171/JEMS/782

[NPV11] F. Nazarov, Y. Peres, and A. Volberg. The power law for the Buffon needle probability of the four-corner Cantor set. St. Petersburg Math. J., 22(1):61–72, 2011. doi:10.1090/S1061-0022-2010-01133-6

[NTV14] F. Nazarov, X. Tolsa, and A. Volberg. On the uniform rectifiability of AD-regular measures with bounded Riesz transform operator: the case of codimension 1. Acta Math., 213(2):237–321, 2014. doi:10.1007/s11511-014-0120-7

[Orp21] T. Orponen. Plenty of big projections imply big pieces of Lipschitz graphs. Invent. Math., 226(2):653–709, 2021. doi:10.1007/s00222-021-01055-2

[OS11] T. Orponen and T. Sahbtab. Radial projections of rectifiable sets. Ann. Acad. Sci. Fenn. Math., 36:677–681, 2011. doi:10.5186/aasfm.2011.3634

[OSW22] T. Orponen, P. Shmerkin, and H. Wang. Kaufman and Falconer estimates for radial projections and a continuum version of Beck’s Theorem. Preprint, 2022. doi:10.48550/arXiv.2209.00348

[PS02] Y. Peres and B. Solomyak. How likely is Buffon’s needle to fall near a planar Cantor set? Pacific J. Math., 204(2):473–496, 2002. doi:10.2140/pjm.2002.204.473

[RS19] E. Rossi and P. Shmerkin. Hölder coverings of sets of small dimension. J. Fractal Geom., 6(3):285–299, 2019. doi:10.1471/jfg/78
[SS06] K. Simon and B. Solomyak. Visibility for self-similar sets of dimension one in the plane. *Real Anal. Exchange*, 32(1):67–78, 2006. doi:10.14321/realanalexch.32.1.0067

[Tao09] T. Tao. A quantitative version of the Besicovitch projection theorem via multiscale analysis. *Proc. London Math. Soc.*, 98(3):559–584, 2009. doi:10.1112/plms/pdn037.

[Tas22] E. Tasso. Rectifiability of a class of integralgeometric measures and applications. *Preprint*, 2022. doi:10.48550/arXiv.2206.14044

[Tol03] X. Tolsa. Painlevé’s problem and the semiadditivity of analytic capacity. *Acta Math.*, 190(1):105–149, 2003. doi:10.1007/BF02393237.

[Tol05] X. Tolsa. Bilipschitz maps, analytic capacity, and the Cauchy integral. *Ann. of Math. (2)*, 162(3):1243–1304, 2005. doi:10.4007/annals.2005.162.1241

[Tol17] X. Tolsa. Rectifiable measures, square functions involving densities, and the Cauchy transform. *Mem. Amer. Math. Soc.*, 245(1158), 2017. doi:10.1090/memo/1158.

[TT15] X. Tolsa and T. Toro. Rectifiability via a square function and Preiss’ theorem. *Int. Math. Res. Not. IMRN*, 2015(13):4638–4662, 2015. doi:10.1093/imrn/rnu082.

[VV22] D. Vardakis and A. Volberg. Geometry of planar curves intersecting many lines at a few points. *St. Petersburg Math. J.*, 33(6):1047–1062, 2022. doi:10.1090/spmj/1712

[Whi98] B. White. A new proof of Federer’s structure theorem for \( k \)-dimensional subsets of \( \mathbb{R}^N \). *J. Amer. Math. Soc.*, 11(3):693–701, 1998. doi:10.1090/S0894-0347-98-00267-7

[Wil17] B. Wilson. Sets with Arbitrarily Slow Favard Length Decay. *Preprint*, 2017. doi:10.48550/arXiv.1707.08137.

**Department of Mathematics and Statistics, University of Jyväskylä, P.O. Box 35 (MaD), FI-40014 University of Jyväskylä, Finland**

**Email address:** damian.m.dabrowski@jyu.fi