Higher Spin Currents with Manifest $SO(4)$ Symmetry
in the Large $\mathcal{N} = 4$ Holography

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Abstract

The large $\mathcal{N} = 4$ nonlinear superconformal algebra is generated by six spin-1 currents, four spin-$\frac{3}{2}$ currents and one spin-2 current. The simplest extension of these 11 currents is described by the 16 higher spin currents of spins $(1, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 2, 2, 2, 2, 2, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{3})$. In this paper, by using the defining operator product expansions (OPEs) between the 11 currents and 16 higher spin currents, we determine the 16 higher spin currents (the higher spin-1, $\frac{3}{2}$ currents were found previously) in terms of affine Kac-Moody spin-$\frac{1}{2}, 1$ currents in the Wolf space coset model completely. An antisymmetric second rank tensor, three antisymmetric almost complex structures or the structure constant are contracted with the multiple product of spin-$\frac{1}{2}, 1$ currents. The Wolf space coset contains the group $SU(N + 2)$ and the level $k$ is characterized by the affine Kac-Moody spin-1 currents. After calculating the eigenvalues of the zeromode of the higher spin-3 current acting on the higher representations up to three (or four) boxes of Young tableaux in $SU(N + 2)$ in the Wolf space coset, we obtain the corresponding three-point functions with two scalar operators at finite $(N, k)$. Furthermore, under the large $(N, k)$ 't Hooft like limit, the eigenvalues associated with any boxes of Young tableaux are obtained and the corresponding three-point functions are written in terms of the 't Hooft coupling constant in simple form in addition to the two-point functions of scalars and the number of boxes.
1 Introduction

The duality \cite{1,2} between the higher spin gauge theory on $AdS_3$ space \cite{3,4} and the large $N$ ’t Hooft-like limit of the $W_N$ minimal models \cite{5,6,7,8,9} implies that it is necessary to obtain the higher spin currents in the boundary theory in order to see some consistency with the bulk theory \cite{10}. For example, the three-point functions of the higher spin currents with two scalar operators can be determined from the explicit forms for the higher spin currents.

The Goddard-Kent-Olive (GKO) coset construction \cite{5,6} for the higher spin-3 Casimir operator has been found in \cite{7,8} some time ago. The role of completely symmetric $SU(N)$ invariant tensor of rank-3 is crucial in this construction. In \cite{11}, the eigenvalues for the higher spin-3 current were studied in the context of \cite{1,2,10}. The same coset construction for the higher spin-4 Casimir operator has been studied in \cite{12} where the completely symmetric $SU(N)$ invariant tensor of rank 4 can be written in terms of the above rank-3 tensor and rank-2 Kronecker delta tensor. The higher spin-4 Casimir operator occurs in the operator product expansion (OPE) between the higher spin-3 Casimir operator and itself. The rank-4 tensor is given by the quadratic rank-3 tensor and quadratic rank-2 tensor. The eigenvalues for the higher spin-4 current are simply the ones for the higher spin-3 current multiplied by a factor which depends on the ’t Hooft-like coupling constant under the large $N$ ’t Hooft-like limit.

For the higher spin-5 Casimir operator, the above coset construction has been described in \cite{13} by focusing on the particular pole in the OPE between the known higher spin-3 Casimir operator and the known higher spin-4 Casimir operator. The triple product of rank-3 tensor appears in this construction: one factor from the former and the two factors from the latter. As before, the eigenvalues for the higher spin-5 current can be written in terms of the ones for the higher spin-4 current with additional simple factor under the large $N$ ’t Hooft-like limit. The GKO coset construction for the higher spin-4 Casimir operator in the orthogonal coset model has been discussed in \cite{14}. The completely symmetric $SO(2N)$ invariant tensor of rank-4 played an important role. See also the relevant papers in \cite{15,16}.

The $\mathcal{N}=1$ supersymmetric (higher spin currents) extension of the GKO coset construction in \cite{7,8} has been studied in \cite{17,18} by considering some condition for one of the levels of the coset model, which allows us to construct the fermionic spin-$\frac{3}{2}$ current, the superpartner of spin-2 stress energy tensor current (and the superpartners of bosonic higher spin currents) using a fermion field, along the line of \cite{19,20}. Still one can take the large $N$ ’t Hooft-like limit. Similarly, the $\mathcal{N}=1$ supersymmetric higher spin extension of the GKO coset construction in \cite{14} (in the orthogonal coset model) was obtained in the same paper \cite{14} by applying the
above level condition to the bosonic coset theory. The other type of $\mathcal{N} = 1$ supersymmetric GKO coset construction for the higher spin Casimir operators in the orthogonal coset model was found in [21] by looking at the paper [22] on the $\mathcal{N} = 1$ higher spin holography closely.

The $\mathcal{N} = 2$ supersymmetric (higher spin currents) extension of the GKO coset construction in [7, 8] (or in [17, 18]) has been studied in [23] by restricting further condition on the the remaining level in the $\mathcal{N} = 1$ supersymmetric coset model, which allows us to construct the additional fermionic spin-$\frac{3}{2}$ current and bosonic spin-1 current (and the superpartners of $\mathcal{N} = 1$ higher spin currents) using an additional fermion field, along the line of [24]. There are two kinds of adjoint fermions. The other type of $\mathcal{N} = 2$ supersymmetric GKO coset construction for the higher spin Casimir operators was found in [25, 26] by considering the $\mathcal{N} = 2$ Kazama-Suzuki model [27, 28] in the context of the $\mathcal{N} = 2$ higher spin holography [29, 30, 31].

The $\mathcal{N} = 3$ supersymmetric (higher spin currents) extension of the GKO coset construction in [25, 26] has been studied in [32] by considering some condition for the level of the coset model, which allows us to construct the additional spin-$\frac{3}{2}$ current, two bosonic spin-1 current and fermionic spin-$\frac{1}{2}$ current (and the $\mathcal{N} = 3$ superpartners of $\mathcal{N} = 2$ higher spin currents) using the various fermion fields, along the line of the $\mathcal{N} = 3$ higher spin holography [33].

Note that compared to the bosonic cases studied in [12, 13, 14, 15, 16], one can use the supersymmetry (or the fermionic spin-$\frac{3}{2}$ current) to determine the superpartners corresponding to the (higher spin) currents (inside of given supermultiplet) for the supersymmetric cases in [17, 18, 21, 23, 25, 26, 32]. In other words, for example, once the lowest component higher spin current in the given supermultiplet is known explicitly, then in principle all the other components in the same supermultiplet can be determined successively by starting with the OPE between the fermionic spin-$\frac{3}{2}$ current and the above lowest higher spin current.

It is natural to ask whether one can construct the GKO coset construction for the higher spin currents in the $\mathcal{N} = 4$ superconformal Wolf space coset model in the context of the $\mathcal{N} = 4$ higher spin holography [34]. See also later works in [35, 36, 37]. In [38], the explicit higher spin-1 current was determined in terms of adjoint spin-1, $\frac{1}{2}$ currents (See also [39]). The antisymmetric rank-2 tensor which has the coset indices appears in the overall factor in the higher spin-1 current. The structure constant tensor also appears in the linear term of adjoint spin-1 current. Furthermore, the four higher spin-$\frac{3}{4}$ currents was obtained. One of them has the above antisymmetric rank-2 tensor and each three of them contains the symmetric (and traceless) rank-2 tensor. These three symmetric tensors are given by the contraction between the three almost complex structures (which are antisymmetric), above antisymmetric rank-2 tensor and the metric. Two of the six higher spin-2 currents were obtained explicitly and four
of them were determined but contain some composite field between the known (higher spin) currents. Similarly, four higher spin-\(\frac{5}{2}\) currents are given by the combination of the above adjoint spin-1, \(\frac{1}{2}\) currents and the composite fields between the known (higher spin) currents. Finally, the higher spin-3 current is written in mixed form. For orthogonal Wolf space coset, see also [40, 41].

In this paper, one would like to continue to the previous work [38] but in different basis. Although the six spin-1 currents of the large \(\mathcal{N} = 4\) ‘nonlinear’ superconformal algebra [42, 43, 44, 45] are manifest in \(SU(2) \times SU(2)\) basis (their linear combination provides the six independent spin-1 currents transforming as \(SO(4)\) vector representation), the spin-\(\frac{5}{2}\) currents are manifest in \(SO(4)\) basis. One can use the \(\mathcal{N} = 4\) primary condition for the 16 higher spin current as done in previous supersymmetric examples. This holds for the extension of the large \(\mathcal{N} = 4\) ‘linear’ superconformal algebra [46, 47, 48, 49, 50] generated by 16 currents. It is known that the \(\mathcal{N} = 4\) higher spin multiplet transforms as linearly under the \(\mathcal{N} = 4\) stress energy tensor in \(\mathcal{N} = 4\) superspace. In other words, the right hand side of the \(\mathcal{N} = 4\) super OPE between them contains only the linear term in the \(\mathcal{N} = 4\) higher spin multiplet [51]. Moreover, one can write down this \(\mathcal{N} = 4\) super OPE in component approach. By the work of Goddard-Schwimmer [42], one can write down the 11 currents in the nonlinear version using 16 currents in the linear version. Similarly, the 16 higher spin currents in the nonlinear version can be written in terms of 16 higher spin currents in the linear version [52]. Therefore, the OPEs between the 11 currents and the 16 higher spin currents in nonlinear version can be written explicitly in [53].

What is the usefulness of the description in the \(SO(4)\) basis rather than \(SU(2) \times SU(2)\) basis? For the higher spin-3 case, according to the observation of [38] in \(SU(2) \times SU(2)\) basis, one should consider both the first and the second order poles of the OPE between the spin-\(\frac{5}{2}\) current and the higher spin-\(\frac{5}{2}\) current. The second order pole of this OPE contains many composite fields. One should also subtract some composite fields appearing in the first order pole of this OPE. On the other hands, what one should calculate in the \(SO(4)\) basis is the first order pole of the OPE between the spin-\(\frac{5}{2}\) current and the higher spin-\(\frac{5}{2}\) current. Moreover, what one subtracts in the first order pole is simply the composite field between the higher spin-1 current and the spin-2 current.

In [54], the conformal dimension, two \(SU(2)\) quantum numbers, and \(U(1)\) charge in the minimal (and higher) representations up to three boxes (of Young tableaux) in the Wolf space coset have been studied. The eigenvalues associated with the higher spin currents in the (minimal and higher) representations up to two boxes are also obtained. This implies that the three-point functions [55] of the higher spin currents with two scalar operators are
determined. Although the explicit (closed) expressions for the higher spin currents in terms of adjoint spin-1, \( \frac{1}{2} \) currents are known for the higher spin-1, \( \frac{3}{2} \) currents and the remaining higher spin-2, \( \frac{5}{2} \), 3 currents are known for several \( N \) values, one could calculate the above eigenvalues explicitly. If the explicit realization for the higher spin currents with the help of adjoint spin-1, \( \frac{1}{2} \) currents is known completely, then one can analyze the various eigenvalues for the higher spin currents (and three-point functions) further because one can calculate each eigenvalue for each term. There are two \( SO(4) \) singlets, the higher spin-1 current and the higher spin-3 current. Maybe one can express the three-point functions for the higher spin-3 current with two scalar operators (associated with any boxes of Young tableaux) in terms of those for the higher spin-1 current with the same two scalar operators at least under the large \( (N, k) \) \( 't \) Hooft like limit as in [14, 13].

In section 2, the 11 currents, for the large \( \mathcal{N} = 4 \) nonlinear superconformal algebra, in terms of adjoint spin-1, \( \frac{1}{2} \) currents are reviewed. In section 3, the higher spin-2, \( \frac{5}{2} \), 3 currents in terms of the adjoint spin-1, \( \frac{1}{2} \) currents are determined explicitly. Together with the known higher spin-1, \( \frac{3}{2} \) currents, all these 16 higher spin currents are obtained explicitly. In section 4, the eigenvalues and three-point functions are analyzed based on the results in section 3. In section 5, the summary of this paper is given and some open problems are presented. In Appendices, some details, which are related to the contents in previous sections, are given.

In [58] (See also the works of [59, 60]), using the decomposition of the scalar four-point functions by Virasoro conformal blocks, the three-point functions including \( \frac{1}{N} \) corrections in the two dimensional (bosonic) \( W_N \) minimal model were obtained using the result of [61] (see also the works of [62, 63]). As observed in [58], it is an open problem to obtain the three-point functions from the decomposition of four-point functions in the large \( \mathcal{N} = 4 \) holography.

In [64], by analyzing the BPS spectrum of string theory and supergravity theory on \( AdS_3 \times S^3 \times S^3 \times S^1 \), it has been found that the BPS spectra of both descriptions agree (where the world sheet approach is used). The appearance of an infinite stringy tower of massless higher spin fields at the critical level of WZW model (in the bosonic string on \( AdS_3 \)) was found in [65]. The similar analysis for the superstring on \( AdS_3 \) has been found in [66]. See also the relevant works in [67, 68]. Very recently, some signal for the transition from the black holes to long strings in the superstring on \( AdS_3 \) at the critical level is interpreted as the infinite tower of modes that become massless in [69]. See also the work of [70]. It would be interesting to see how the large \( \mathcal{N} = 4 \) superconformal higher spin and CFT duality arises in the context of these world sheet approaches.

\(^2\) The Thielemans package in [66] with a mathematica [67] is useful to check the OPEs.
2 The 11 currents which generate the large $\mathcal{N} = 4$ non-linear superconformal algebra

Let us consider the Wolf space coset in the “supersymmetric” version with groups $G = SU(N + 2)$ and $H = SU(N) \times SU(2) \times U(1)$. The operator product expansion between the spin-1 current $V^a(z)$ and the spin-$\frac{1}{2}$ current $Q^a(z)$ is described as \cite{71}

$$ V^a(z) V^b(w) = \frac{1}{(z-w)^2} k g^{ab} - \frac{1}{(z-w)} f^{abc} V^c(w) + \cdots, $$

$$ Q^a(z) Q^b(w) = -\frac{1}{(z-w)} (k + N + 2) g^{ab} + \cdots, \quad V^a(z) Q^b(w) = + \cdots. \quad (2.1) $$

The level $k$ is the positive integer.

The metric can be obtained from $g_{ab} = \frac{1}{2 c_G} f^{abc} f_{bcd}$ where $c_G$ is the dual Coxeter number of the group $G = SU(N + 2)$. The metric $g_{ab}$ is given by the generators of $SU(N + 2)$ in the complex basis \cite{72} as follows

$$ g_{ab} = \text{Tr}(T_a T_b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = g^{ab}, \quad a, b = 1, 2, \cdots, (N + 2)^2 - 1. \quad (2.2) $$

The commutation relation for the $SU(N + 2)$ generators is given by $[T_a, T_b] = f_{ab}^c T_c$. Due to the regular behavior between the spin-1 current and the spin-$\frac{1}{2}$ current, one can move any current in the composite fields between them freely. Any $SU(N + 2)$ group indices, $a, b, \cdots$ can be raised or lowered by using the above metric $g_{ab}$.

The four supersymmetry currents of spin-$\frac{3}{2}$, $\hat{G}^0(z)$ and $\hat{G}^i(z)$, the six spin-1 currents of $SU(2)_k \times SU(2)_N$, $A^{\pm i}(z)$ and the spin-2 stress energy tensor $\hat{T}(z)$ can be described as follows \cite{43, 45, 38}

$$ \hat{G}^0(z) = \frac{i}{(k + N + 2)} g_{\bar{a}b} Q^a V^b(z), \quad \hat{G}^i(z) = \frac{i}{(k + N + 2)} h_{\bar{a}b}^i Q^a V^b(z), $$

$$ A^{\pm i}(z) = -\frac{1}{4N} f_{\bar{a}b}^c h_{\bar{a}b}^i V^c(z), \quad A^{-i}(z) = -\frac{1}{4(k + N + 2)} h_{\bar{a}b}^i Q^a Q^b(z), $$

$$ \hat{T}(z) = \frac{1}{2(k + N + 2)^2} \left[ (k + N + 2) g_{\bar{a}b} V^a V^b + k g_{\bar{a}b} Q^a \partial Q^b + f_{\bar{a}b}^c g_{abc} g_{\bar{a}d} Q^e \partial Q^d V^c \right](z) $$

$$ -\frac{1}{(k + N + 2)} (A^{+i} + A^{-i})^2(z), \quad i = 1, 2, 3. \quad (2.3) $$

The Wolf coset indices, $\bar{a}, \bar{b}, \cdots$, run over $\bar{a}, \bar{b}, \cdots = 1, 2, \cdots, 4N$. The quantity $4N$ can be obtained from the number of generators $(N + 2)^2 - 1$ in $G = SU(N + 2)$ and the number of generators $(N^2 - 1) + 4$ in $H = SU(N) \times SU(2) \times U(1)$. In the spin-2 stress energy tensor,
the terms $A^{-i} A^{-i}$ which contain $(Q^{\mu} Q^{\bar{\nu}})(Q^{\bar{\kappa}} Q^{\bar{\lambda}})(z)$ can be further simplified by using the rearrangement lemmas \[73, 74\].

The three almost complex structures are given by the following $4N \times 4N$ matrices \[72\]

$$
h^{1}_{ab} = \begin{pmatrix}
0 & 0 & 0 & -i \\
0 & -i & 0 & 0 \\
i & 0 & 0 & 0 \\
-i & 0 & 0 & 0
\end{pmatrix},
$$

$$
h^{2}_{ab} = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
-i & 0 & 0 & 0
\end{pmatrix},
$$

$$
h^{3}_{ab} = \begin{pmatrix}
0 & 0 & i & 0 \\
0 & 0 & 0 & -i \\
-i & 0 & 0 & 0 \\
0 & i & 0 & 0
\end{pmatrix}.
$$

(2.4)

Each entry is $N \times N$ matrix. The last almost complex structure can be obtained from the first two $h^{3}_{ab} = h^{1}_{ab} g^{\bar{c}d} h^{2}_{\bar{a}b}$. They are antisymmetric and satisfy the quaternionic algebra \[50\] where one sees the presence of the Wolf space coset metric $g_{ab} \equiv h^{0}_{ab}$. See also \(3.17\). Note that this coset metric and three almost complex structures appear in the above four supersymmetry currents of spin-$\frac{3}{2}$.

Then the large $\mathcal{N} = 4$ nonlinear superconformal algebra \[42\] can be obtained explicitly using the above 11 currents. The 10 currents of spin $\frac{3}{2}$ and 1 are primary under the stress energy tensor. The nonlinear structure appears in the OPE between the spin-$\frac{3}{2}$ currents. The two levels of $SU(2)$‘s are given by $k$ and $N$ respectively.

One can rewrite the large $\mathcal{N} = 4$ nonlinear superconformal algebra in $SO(4)$ manifest way. With the $SO(4)$ singlet

$$
T(z) = \hat{T}(z),
$$

(2.5)

the spin-$\frac{3}{2}$ currents, transforming as the $SO(4)$ vector representation, are given by \[53\]

$$
G^{1}(z) = \hat{G}^{3}(z),
$$

$$
G^{2}(z) = \hat{G}^{0}(z),
$$

$$
G^{3}(z) = \hat{G}^{1}(z),
$$

$$
G^{4}(z) = -\hat{G}^{2}(z).
$$

(2.6)

Furthermore, the six spin-1 currents, $T_{\mu \nu}(z)$ transforming as the $SO(4)$ adjoint representation, can be obtained from the corresponding two adjoint spin-1 currents $A^{\pm i}(z)$ as follows \[51, 53\]

$$
T^{12}(z) = -i(A^{+3} - A^{-3})(z),
$$

$$
T^{13}(z) = -i(A^{+2} + A^{-2})(z),
$$

$$
T^{14}(z) = -i(A^{+1} + A^{-1})(z),
$$

$$
T^{23}(z) = i(A^{+1} - A^{-1})(z),
$$

$$
T^{24}(z) = -i(A^{+2} - A^{-2})(z),
$$

$$
T^{34}(z) = i(A^{+3} + A^{-3})(z).
$$

(2.7)

The $SO(4)$ invariant Kronecker delta $\delta^{\mu \nu}$ and epsilon tensors $\varepsilon^{\mu \nu \rho \sigma}$ appear in the corresponding large $\mathcal{N} = 4$ nonlinear superconformal algebra \[42, 43, 44, 45\]. The six $4 \times 4$ matrices $\alpha^{\pm i}_{\mu \nu}$ \[47\] relate the spin-1 currents $A^{\pm i}(z)$ to the spin-1 currents $T^{\mu \nu}(z)$.

\[3\]Because the $SU(N + 2)$ generators one uses in this paper are given by the ones in \[72\] rather than the ones in \[38\], one cannot use some identities appeared in \[38\] directly.
3 Construction of the higher spin currents

In this section, we would like to obtain the higher spin-3 current together with the higher spin-2 currents and the higher spin-$\frac{5}{2}$ currents in terms of the Kac-Moody currents, $V^a(z)$ and $Q^a(z)$. Some of the expressions for these higher spin currents are given in [38] but they are mixed with the currents presented in the previous section from the large $\mathcal{N} = 4$ nonlinear superconformal algebra. The main things in this section are to write them in terms of $V^a(z)$ and $Q^a(z)$ completely.

3.1 Known higher spin-1 current

The higher spin-1 current, which transforms as the $SO(4)$ singlet representation, can be written in terms of the spin-1 currents $V^a(z)$ and the spin-$\frac{1}{2}$ currents $Q^a(z)$. The tensorial structure with exact coefficients are determined by [38]

$$\Phi_0^{(1)}(z) = -\frac{1}{2(k + N + 2)} d^{0}_{ab} f^{ab}_{c} V^c(z) + \frac{k}{2(k + N + 2)^2} d^{0}_{ab} Q^a Q^b(z).$$

The overall tensor is antisymmetric and is given by the following $4N \times 4N$ matrix

$$d^{0}_{ab} = \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}. \tag{3.2}$$

Each entry is $N \times N$ matrix. Note that there exists no coset spin-1 current $V^c(z)$ term in the first term of (3.1) because of the fact that $f^{ab}_{c} = 0$. On the other hand, the quadratic spin-$\frac{1}{2}$ current terms contain only the coset spin-$\frac{1}{2}$ current. Because this higher spin-1 current belongs to the lowest component of the $\mathcal{N} = 4$ higher spin multiplet, the other 15 higher spin currents can be determined by the help of the four spin-$\frac{3}{2}$ supersymmetry currents presented in the previous section.

3.2 Known higher spin-$\frac{3}{2}$ currents

From the defining equation for the OPE between the spin-$\frac{3}{2}$ current presented in previous section and the higher spin-1 current described in the previous subsection, one can obtain the higher spin-$\frac{3}{2}$ current. From Appendix (A.1), one has the OPE

$$\hat{G}^\mu(z) \Phi^{(1)}_0(w) = -\frac{1}{(z - w)} \hat{\Phi}^{(1)\mu}_{\frac{3}{2}}(w) + \cdots. \tag{3.3}$$

\[\text{We use the hat notation for the (higher spin) currents in the } SU(2) \times SU(2) \text{ basis. The OPEs with hat notation are the same as the ones in Appendix A. One has } \hat{\Phi}^{(1)\mu}_{\frac{3}{2}}(z) = \Phi_0^{(1)}(z).\]
By using the explicit expressions in (2.3) and (3.1), one can calculate the OPE \( \hat{G}^\mu(z) \Phi_0^{(1)}(w) \).

On the one hand, the following relation holds by using (2.1) with the help of the description in [73]

\[
Q^\bar{a} V^b(z) V^c(w) = \frac{1}{(z-w)^2} k^g Q^\bar{a}(w) + \frac{1}{(z-w)} \left[ k^g \partial Q^\bar{a} + f^{\bar{c}d} Q^\bar{a} V^d(w) \right] + \cdots. \tag{3.4}
\]

On the other hand, there exist other contributions from

\[
Q^\bar{a} V^b(z) Q^\bar{e} Q^d(w) = \frac{1}{(z-w)^2} (k + N + 2) \left[ - g^\bar{c}b V^b Q^d + g^{\bar{d}a} Q^\bar{e} V^b \right] (w) + \cdots. \tag{3.5}
\]

Then one obtains the final OPE by combining the two equations (3.4) and (3.5) with the appropriate metric, almost complex structures, \( d \) tensors and the structure constants.

### 3.2.1 The second order pole

The second order pole of \( \hat{G}^\mu(z) \Phi_0^{(1)}(w) \) (and the similar term in the first order pole) vanishes due to the fact that

\[
f^{\bar{c}d} e g^{\bar{c}d} = 0. \tag{3.6}
\]

Note that the structure constant \( f^{\bar{c}d} e = 0 \) and the nonzero metric components are given by the coset indices or the subgroup indices (2.2) separately.

### 3.2.2 The first order pole

The first order pole of \( \hat{G}^\mu(z) \Phi_0^{(1)}(w) \) contains three terms. One can use the following identity

\[
d^0_{\bar{c}d} f^{\bar{e}a} e f^{\bar{e}b} g = -2(N + 2) d^0_{\bar{a}b} g^{\bar{a}b}, \tag{3.7}
\]

to simplify the last term in (3.4). The \( N \) dependence can be seen from the several \( N \) values, \( N = 3, 5, 7, \cdots \). By multiplying the almost complex structure originating from the spin-\( \frac{3}{2} \) currents into (3.7), it is obvious that

\[
h^\mu_{\bar{a}b} d^0_{\bar{c}d} f^{\bar{e}a} e f^{\bar{e}b} g = -2(N + 2) d^\mu_{\bar{a}b}, \tag{3.8}
\]

where we, in the right hand side of (3.8), introduce the product of \( d^0_{\bar{a}b} \) tensor (3.2), which is antisymmetric, and complex structures (2.4) together with the metric tensor as follows [38]

\[
d^0_{\bar{a}b} g^{\bar{c}b} h^\mu_{\bar{c}d} \equiv d^\mu_{\bar{a}d}, \quad \mu = 0, 1, 2, 3. \tag{3.9}
\]
The $d^i_{ab}$ tensor where $i = 1, 2, 3$ is symmetric, compared to the antisymmetric $d^i_{ab}$ tensor. Note that we introduce the (symmetric) metric tensor

$$g_{ab} \equiv h^0_{ab}, \quad (3.10)$$

compared to the antisymmetric almost complex structures $h^i_{ab}$.

Therefore, the higher spin-$\frac{3}{2}$ currents from (3.3) in $SU(2) \times SU(2)$ basis are given by

$$\hat{\Phi}^{(1),\mu}_{\frac{i}{2}}(z) = -\frac{i}{(k + N + 2)} d^\mu_{ab} Q^a V^b(z), \quad \mu = 0, 1, 2, 3, \quad (3.11)$$

which corresponds to the higher spin-$\frac{3}{2}$ currents $G^\mu(z)$ with minus sign in [38].

As done in (2.6) (the last one has different sign), one obtains the higher spin-$\frac{3}{2}$ currents which are in the $SO(4)$ vector representation as follows [38]

$$\Phi^{(1),1}_{\frac{i}{2}}(z) = -\frac{i}{(k + N + 2)} d^3_{ab} Q^a V^b(z), \quad \Phi^{(1),2}_{\frac{i}{2}}(z) = -\frac{i}{(k + N + 2)} d^0_{ab} Q^a V^b(z),$$

$$\Phi^{(1),3}_{\frac{i}{2}}(z) = -\frac{i}{(k + N + 2)} d^1_{ab} Q^a V^b(z), \quad \Phi^{(1),4}_{\frac{i}{2}}(z) = \frac{i}{(k + N + 2)} d^2_{ab} Q^a V^b(z). \quad (3.12)$$

The $d^\mu_{ab}$ tensor appearing in these expressions, which are characteristic of the higher spin-$\frac{3}{2}$ currents, is given by (3.9). In other words, one also has the similar OPE relation given in (3.3), for given the spin-$\frac{3}{2}$ current $G^\mu(z)$ and the higher spin-1 current $\Phi^{(1)}_0(w)$ in Appendix A.

### 3.3 Higher spin-2 currents

Let us consider the higher spin-2 currents. It is known that the defining OPEs between the spin-$\frac{3}{2}$ currents and the higher spin-$\frac{3}{2}$ currents are given by (from Appendix (A.1))

$$\hat{G}^\mu(z) \hat{\Phi}^{(1),\nu}_{\frac{i}{2}}(w) = -\frac{1}{(z-w)^2} 2\delta^{\mu\nu} \Phi^{(1)}_0(w) + \frac{1}{(z-w)} \left[ -\delta^{\mu\nu} \partial \Phi^{(1)}_0 + \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \hat{\Phi}^{(1),\rho\sigma}_{\frac{i}{2}} \right](w) + \cdots . \quad (3.13)$$

Let us analyze each pole in (3.13). In particular, the first order pole provides the higher spin-2 currents.

#### 3.3.1 The third order pole

The third order pole of $\hat{G}^\mu(z) \hat{\Phi}^{(1),\nu}_{\frac{i}{2}}(w)$ contains

$$h^\mu_{ab} d^\nu_{cd} g^{ca} g^{db} \quad (3.14)$$

One can check that the 16 cases of this expression ($\mu, \nu = 0, 1, 2, 3$) vanish by using (2.4), (3.10), (3.2), (3.9) and (2.2). This is consistent with the structure of the OPE in (3.13).
3.3.2 The second order pole

The second order pole is given by

\[
1 \frac{1}{(k + N + 2)} h_{ab}^\mu g^{ca} d_{cd}^\nu f e V^c(w) - \frac{k}{(k + N + 2)^2} h_{ab}^{\mu b} g^{db} d_{cd}^c Q^e Q^d(w). \tag{3.15}
\]

One can check the above OPE by considering five cases \((\mu, \nu) = (0,0), (i,i), (0,i), (i,0)\) and \((i,j)\). One expects that the first two will contribute to the OPE while the last three will do not contribute at all due to the delta tensor \(\delta^{\mu\nu}\) in the second order pole of (3.13).

- \((\mu, \nu) = (0, i)\) case

One obtains \(h_{ab}^0 g^{ca} d_{cd}^i = d_{kd}^i\) for the first term of (3.15) from the relation (3.10). Note that this is symmetric in the indices. Then the first term of (3.15) vanishes because the structure constant is antisymmetric in the indices of \(\bar{b}\) and \(\bar{d}\). Similarly, one obtains \(h_{ab}^i g^{cb} d_{cd}^i = d_{ci}^i\), which is also symmetric, for the second term of (3.15). Then due to the fermionic property of spin-\(\frac{1}{2}\) current, the quantity \(Q^e Q^a(w)\) is also antisymmetric by interchanging the indices. There is no nontrivial contribution from the second term. The total contribution is zero.

- \((\mu, \nu) = (i, 0)\) case

One obtains \(-h_{ba}^i g^{be} d_{cd}^0 = -d_{bd}^i \) (for the first term) where the following identity is used

\[
h_{ab}^{\mu} g^{ce} d_{cd}^0 = d_{ed}^{\mu}. \tag{3.16}
\]

This can be obtained from (3.9). Similarly, one obtains \(-h_{ab}^i g^{be} d_{cd}^0 = -d_{bd}^i\) (for the second term) which is symmetric in the indices. Therefore, the final contribution from (3.15) vanishes from the same analysis before.

- \((\mu, \nu) = (i, j)\) case where \(i \neq j\)

What happens in this case? One can use the identity which will appear in (3.18) and show that there is no contribution.

- \((\mu, \nu) = (0, 0)\) case

For the first term of (3.15), one has \(h_{ba}^0 g^{ac} d_{cd}^0 = d_{bd}^0\) by using the relation (3.16) and for the second term, one has \(h_{ab}^0 g^{bd} d_{cd}^0 = d_{ca}^0\). Then one can check the second order pole of (3.13) together with (3.1) by combining the factors in (3.15).

\[
\begin{align*}
\hline
\text{We will treat the simplification on each pole of OPE for general indices without fixing them explicitly. But it is rather difficult to use some identities between the tensors when there are too many indices one should simplify. Later, we will fix them and describe each pole for fixed indices.} \\
\text{For simplicity, we will concentrate on the main part of the coefficient in front of composite field. Sometimes, we do not care about the signs, the numerical values, \(k\) factors or \((k+N+2)\) factors in their descriptions. We will consider them in the final results.} \\
\end{align*}
\]

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In this case, one should describe some identity from the relation between the three almost complex structures. By multiplying \( d^0_{\alpha \beta} g^{\bar{f} \bar{b}} \) into the following relation between the three almost complex structures

\[
\left( \begin{array}{c} h^i_{\alpha \bar{c}} g^{\bar{c} \bar{d}} h^j_{\bar{d} \bar{b}} = \varepsilon^{ijk} h^k_{\alpha \bar{b}}, \quad i, j, \ldots = 1, 2, 3, \end{array} \right)
\]  

one obtains the following identity, by using (3.18) together with the antisymmetric properties of \( h^i_{\alpha \bar{b}} \) and \( d^0_{\alpha \bar{b}} \) (symmetric properties of \( h^0_{\alpha \bar{b}} \) and \( d^i_{\alpha \bar{b}} \),

\[
\begin{aligned}
\left( h^i_{\alpha \bar{c}} g^{\bar{c} \bar{d}} d^{j}_{\bar{d} \bar{e}} = \varepsilon^{ijk} d^k_{\alpha \bar{e}} - \delta^{ij} d^0_{\alpha \bar{e}}, \quad i, j, \ldots = 1, 2, 3, \right)
\end{aligned}
\]  

For the \((\mu, \nu) = (i, i)\) case (there is no sum in the index \(i\)), one obtains \(- h^i_{\beta \bar{d}} g^{\bar{a} \bar{c}} d^i_{\bar{c} \bar{d}} = d^0_{\beta \bar{d}}\) by using (3.18) for the first term of (3.15). Similarly, one has \( h^i_{\alpha \bar{b}} g^{\bar{b} \bar{d}} d^{i}_{\bar{d} \bar{e}} = - d^0_{\alpha \bar{c}} = d^0_{\alpha \bar{a}}\) for the second term. Again, the identity (3.18) is used. It is straightforward to see the second order pole of (3.13) for this case also.

### 3.3.3 The first order pole

Let us consider the first order pole. The explicit form is as follows

\[
-\frac{1}{(k + N + 2)^2} h^\mu_{\alpha \bar{b}} d^\nu_{\bar{c} \bar{d}} \left[ (k + N + 2) g^{\bar{a} \bar{c}} V^b V^d + k g^{\bar{b} \bar{d}} Q^{\bar{c}} \partial Q^{\bar{a}} + f^{\bar{d} \bar{b}} \partial Q^{\bar{c}} Q^{\bar{a}} V^e \right](w). \tag{3.19}
\]

- **\(\mu = \nu\) case**

  When one differentiates the second term of (3.15) with respect to the coordinate \(w\), then one obtains the second term of (3.19) with an extra coefficient 2. Similarly, one obtains the first term of (3.19) with an extra coefficient 2 after one differentiates the first term of (3.15) with respect to the coordinate \(w\). This is because one can reexpress \( f^{\bar{d} \bar{b}} \partial V^{\bar{e}} \) in terms of \(-[V^b, V^d]\) and the factor \( h^\mu_{\alpha \bar{b}} g^{\bar{a} \bar{c}} d^\nu_{\bar{c} \bar{d}}\) under the condition \((\mu, \nu) = (0, 0)\) or \((i, i)\) is antisymmetric in the indices \(\bar{b}\) and \(\bar{d}\). In other words, the expression \(V^b V^d\) can be written in terms of \(\frac{1}{2}[V^b, V^d]\) under the particular choice of \((\mu, \nu)\). Furthermore, one has the following property

\[
h^\mu_{\alpha \bar{b}} d^\mu_{\bar{c} \bar{d}} f^{\bar{d} \bar{b}} = h^\mu_{\alpha \bar{c}} d^\mu_{\alpha \bar{d}} f^{\bar{b} \bar{d}}, \quad \text{no sum over} \ \mu = 0, 1, 2, 3, \tag{3.20}
\]

which leads to the fact that the \(hdf\) factor is symmetric in the indices \(\bar{a}\) and \(\bar{c}\). This implies that the third term of (3.19) vanishes under the condition \(\mu = \nu\). Therefore, one can prove the Kronecker delta term in the first order pole of (3.13).

- **\(\mu \neq \nu\) case**

\footnote{One can check the relation (3.14) using the identity (3.18). Note that \(d^i\) tensor is traceless.}
One expects that from (3.13), the \((\mu, \nu) = (0, 1)\) case (which is given by (3.19) explicitly) provides the higher spin current with \((2, 3)\) component. Similarly, one obtains the \((3, 0)\) component higher spin current for \((\mu, \nu) = (1, 2)\). The remaining \((0, 1), (1, 2), (3, 1)\) and \((0, 2)\) components of higher spin currents can be obtained from the cases \((\mu, \nu) = (2, 3), (3, 0), (0, 2)\) and \((3, 1)\) respectively.

It turns out that the higher spin-2 currents in \(SU(2) \times SU(2)\) basis can be written as

\[
\Phi_1^{(1), \rho \sigma}(z) = -\frac{1}{2(k + N + 2)^2} \varepsilon^\rho \varepsilon \nu h^\nu_{\alpha \beta} d^\alpha_{\epsilon \delta} f_{\epsilon \delta} g^\alpha_{\epsilon \delta} \left( (k + N + 2) g^{\epsilon \nu} V^\nu V^\chi + k g^{\epsilon \nu} Q^\nu \partial Q^\nu + f^{\epsilon \nu} g^{\nu \rho} Q^\nu Q^\rho V^\rho \right)(z).
\] (3.21)

One should make sure that the expression in (3.19) is antisymmetric in the indices of \(\mu\) and \(\nu\). For the \((\mu, \nu) = (0, i)\) case, the first term of (3.19) contains \(h^0_{\alpha \beta} d^\alpha_{\epsilon \delta} g^\alpha_{\epsilon \delta}\) which becomes \(d^0_{\alpha \beta}\). On the other hand, for the \((\mu, \nu) = (i, 0)\) case, one has \(h^i_{\alpha \beta} d^\alpha_{\epsilon \delta} g^\alpha_{\epsilon \delta}\) which is equal to \(-h^i_{\beta \alpha} g^{\alpha \beta} d^0_{\alpha \beta}\). Then this becomes \(-d^i_{\beta \alpha}\) from the defining relation (3.16). The second term of (3.19) contains \(h^i_{\alpha \beta} d^\alpha_{\epsilon \delta} g^\alpha_{\epsilon \delta}\) which is given by \(d^i_{\alpha \beta}\) for \((\mu, \nu) = (0, i)\). For the \((\mu, \nu) = (i, 0)\) case, one has \(h^i_{\alpha \beta} d^0_{\epsilon \delta} g^\alpha_{\epsilon \delta}\) which can be written as \(-h^i_{\alpha \beta} g^{\alpha \beta} d^0_{\alpha \beta}\). Then this is equal to \(-d^i_{\alpha \beta}\). For \((\mu, \nu) = (i, j)\) case where \(i \neq j\), the expressions \(h^i_{\alpha \beta} d^\alpha_{\epsilon \delta} g^\alpha_{\epsilon \delta}\) and \(h^i_{\alpha \beta} d^\alpha_{\epsilon \delta} g^\alpha_{\epsilon \delta}\) corresponding to the first and second terms of (3.19) are antisymmetric in the indices \(i\) and \(j\) from (3.18). For the third term of (3.19), one can show that the following identity holds

\[
(h^i_{\alpha \beta} d^\alpha_{\epsilon \delta} + h^i_{\alpha \beta} d^\epsilon_{\alpha \delta}) f^{\epsilon \delta} e Q^\epsilon Q^\delta V^\epsilon = 0,
\] (3.22)

which can be checked for several \(N\) values. In (3.22), the composite field is necessary. Therefore, the higher spin-2 current in (3.21) except epsilon tensor is antisymmetric under the indices \(\mu\) and \(\nu\). For fixed \(\rho\) and \(\sigma\) indices, the numerical factor \(1/2\) is cancelled.

By identifying the correct indices in \(SO(4)\) manifest way, one obtains the following six higher spin-2 currents which transform as \(SO(4)\) adjoint representation, from (3.21),

\[
\Phi_1^{(1), 12}(z) = \frac{h^1_{\alpha \beta} d^\alpha_{\epsilon \delta}}{(k + N + 2)^2} \left[ (k + N + 2) g^{\epsilon \nu} V^\nu V^\chi + k g^{\epsilon \nu} Q^\nu \partial Q^\nu + f^{\epsilon \nu} g^{\nu \rho} Q^\nu Q^\rho V^\rho \right](z),
\]

\[
\Phi_1^{(1), 13}(z) = \frac{h^2_{\alpha \beta} d^\alpha_{\epsilon \delta}}{(k + N + 2)^2} \left[ (k + N + 2) g^{\epsilon \nu} V^\nu V^\chi + k g^{\epsilon \nu} Q^\nu \partial Q^\nu + f^{\epsilon \nu} g^{\nu \rho} Q^\nu Q^\rho V^\rho \right](z),
\]

\[
\Phi_1^{(1), 14}(z) = \frac{h^1_{\alpha \beta} d^\alpha_{\epsilon \delta}}{(k + N + 2)^2} \left[ (k + N + 2) g^{\epsilon \nu} V^\nu V^\chi + k g^{\epsilon \nu} Q^\nu \partial Q^\nu + f^{\epsilon \nu} g^{\nu \rho} Q^\nu Q^\rho V^\rho \right](z),
\]

\[
\Phi_1^{(1), 23}(z) = -\frac{h^2_{\alpha \beta} d^\alpha_{\epsilon \delta}}{(k + N + 2)^2} \left[ (k + N + 2) g^{\epsilon \nu} V^\nu V^\chi + k g^{\epsilon \nu} Q^\nu \partial Q^\nu + f^{\epsilon \nu} g^{\nu \rho} Q^\nu Q^\rho V^\rho \right](z),
\]

\[\text{Note that } \varepsilon^{0123} = 1.\]
Let us consider the OPE between the spin-\(3\) currents \(\hat{\Phi}^{3}\). The last two terms in (3.23) using (3.18).

\[ \Phi^{(1),24}_{1}(z) = -\frac{\hbar_{3}^{3}d_{e}^{3}}{(k + N + 2)^2} \left[ (k + N + 2) g^{\bar{e}e} V^{\bar{d}} V^{\bar{f}} + k g^{\bar{d}d} Q^{\bar{e}} \partial Q^{e} + f^{\bar{d}g} Q^{\bar{e}} Q^{g} V^{g} \right](z), \]
\[ \Phi^{(1),34}_{1}(z) = -\frac{\hbar_{3}^{3}d_{e}^{3}}{(k + N + 2)^2} \left[ (k + N + 2) g^{\bar{e}e} V^{\bar{d}} V^{\bar{f}} + k g^{\bar{d}d} Q^{\bar{e}} \partial Q^{e} + f^{\bar{d}g} Q^{\bar{e}} Q^{g} V^{g} \right](z). \] (3.23)

Compared to the ones in [38], these are very simple forms in the sense that the \(SO(4)\) adjoint indices are coming from the indices of almost complex structures and \(d\) tensors. They have the same WZW current dependent terms inside the bracket. One can further simplify the first two terms in (3.23) using (3.18).

### 3.4 Higher spin-\(\frac{5}{2}\) currents

Let us consider the OPE between the spin-\(\frac{3}{2}\) currents \(\hat{G}^{\mu}(z)\) (2.3) and the higher spin-2 currents \(\Phi^{(1),\nu\rho}_{1}(w)\) (3.21). The defining OPE is given by

\[
\hat{G}^{\mu}(z) \Phi^{(1),\nu\rho}_{1}(w) = \frac{1}{(z-w)^2} \left[ \frac{(N-k)}{(k+N+2)} (\delta^{\mu\nu} \Phi^{(1),\rho}_{1} - \delta^{\mu\rho} \Phi^{(1),\nu}_{1}) + \frac{(2 + 3k + 3N)}{(k+N+2)} \epsilon^{\mu\nu\rho\sigma} \Phi^{(1),\sigma}_{1} \right](w) \\
+ \frac{1}{(z-w)^2} \left[ \frac{i}{(k+N+2)} (\epsilon^{\sigma\nu\rho\mu} T^{\sigma\alpha} \Phi^{(1),\nu}_{1} - \epsilon^{\sigma\mu\rho\nu} T^{\sigma\alpha} \Phi^{(1),\rho}_{1}) + \epsilon^{\mu\nu\rho\sigma} \partial \Phi^{(1),\sigma}_{1} \right] \\
- \delta^{\mu\nu} (\Phi^{(1),\rho}_{1} - \frac{k-N}{3(k+N+2)} \partial \Phi^{(1),\rho}_{1} + \frac{i}{(k+N+2)} \epsilon^{\alpha\beta\rho\sigma} T^{\alpha\beta} \Phi^{(1),\sigma}_{1}) \]
\[ + \delta^{\mu\nu} (\Phi^{(1),\rho}_{1} - \frac{k-N}{3(k+N+2)} \partial \Phi^{(1),\rho}_{1} + \frac{i}{(k+N+2)} \epsilon^{\alpha\beta\rho\sigma} T^{\alpha\beta} \Phi^{(1),\sigma}_{1}) \] (3.24)

One can see the antisymmetric property between the indices \(\nu\) and \(\rho\) from the right hand side of this OPE. It is obvious to see those property in the left hand side.

Let us check whether the realization of the higher spin currents in terms of Wolf space coset currents satisfies each singular term in (3.24).

#### 3.4.1 The third order pole

The third order pole of this OPE contains \(f^{\bar{d}e} g^{\bar{f}e}\) or \(f^{\bar{d}g} g^{\bar{b}g}\) which vanishes according to (3.6).

#### 3.4.2 The second order pole

Let us calculate the second order pole of the above OPE. The OPE between \(Q^{\bar{a}} V^{\bar{b}}(z)\) and \(V^{\bar{d}} V^{\bar{f}}(w)\) provides the following second order pole

\[
(k g^{\bar{d}b} Q^{\bar{a}} V^{\bar{f}} - f^{\bar{d}e} f^{\bar{f}g} Q^{\bar{a}} V^{g} + k g^{\bar{f}b} Q^{\bar{a}} V^{d})(w). \] (3.25)
Similarly, the OPE between $Q^a V^b(z)$ and $Q^c \partial Q^e(w)$ leads to the following second order pole

$$(k + N + 2) g^{ca} Q^c V^b(w).$$

Finally, the OPE between $Q^a V^b(z)$ and $Q^e Q^g V^h(w)$ implies the following second order pole

$$(k + N + 2) (g^{ca} f^{bg} h Q^e V^h - g^{ba} f^{bg} h Q^e V^h)(w).$$

Although there exists the cubic term of spin-$\frac{1}{2}$ current, the factor $f^{\bar{f} \bar{g}} g$ with $g^{\bar{g} \bar{b}}$ does not contribute to the final second order pole. See also Appendix (B.1).

Then one obtains the final second order pole after multiplying all the factors correctly

$$\frac{-i}{2(k + N + 2)} \varepsilon^{\mu \nu \rho} h^{\alpha \beta} h^{\sigma \delta} d^{\epsilon \gamma} \left[ (k + N + 2) g^{\bar{c} \bar{a}} (3.25) + k g^{\bar{f} \bar{g}} (3.26) + f^{\bar{f} \bar{g}} (3.27) \right] (w).$$

As before, one can analyze this expression for five different cases in order to see (3.24).

- $(\mu, \nu) = (0, 0)$

Let us consider the case where the indices $\mu$ and $\nu$ are equal to each other. The first term of (3.28) contains $\varepsilon^{\mu \nu \rho} h^{\alpha \beta} h^{\sigma \gamma} d^{\epsilon \gamma} g^{\bar{c} \bar{a}}$. There is no sum over the index $\nu$. One can express this as $\pm \varepsilon^{\mu \nu \rho} h^{\alpha \beta} g^{\bar{a} \bar{d}} d^{\epsilon \gamma}$ with $h^{\sigma \gamma}$. For the index $\sigma$ is 0 (when the index $\nu$ is $i$), then one has $+$ sign while for the index $\sigma$ is equal to $i$ (when the index $\nu$ is 0), then one has $-$ sign. When $\mu = \nu = 0$, the factor $h^{\sigma \gamma} d^{\epsilon \gamma}$ can be reduced to $\varepsilon^{\sigma \gamma \epsilon} d^{\epsilon \gamma}$ according to (3.18). Then one obtains $-\varepsilon^{\mu \nu \rho} \varepsilon^{\sigma \gamma \epsilon} d^{\epsilon \gamma}$ which is equal to $-2 d^{\epsilon \gamma}$. The third term of (3.28) contains $\varepsilon^{\mu \nu \rho} h^{\alpha \beta} h^{\sigma \gamma} d^{\epsilon \gamma}$ which can be written as $\varepsilon^{\mu \nu \rho} h^{\alpha \beta} h^{\sigma \gamma} d^{\epsilon \gamma}$ with $(\mu, \nu) = (0, 0)$.

As before, the factor $h^{\nu \rho} g^{\bar{a} \bar{d}} d^{\epsilon \gamma}$ becomes $d^{\epsilon \gamma}$. Furthermore, the expression $d^{\epsilon \gamma} h^{\sigma \gamma} d^{\epsilon \gamma}$ gives $-\varepsilon^{\sigma \gamma \epsilon} d^{\epsilon \gamma}$. Then one obtains the final expression $-\varepsilon^{\mu \nu \rho} \varepsilon^{\sigma \gamma \epsilon} d^{\epsilon \gamma}$ again. This expression can be simplified as $-\varepsilon^{\sigma \gamma \epsilon} d^{\epsilon \gamma}$. The fourth term of (3.28) contains $\varepsilon^{\mu \nu \rho} h^{\alpha \beta} h^{\sigma \gamma} d^{\epsilon \gamma} f^{\bar{f} \bar{g}}$ which can be written as $\varepsilon^{\mu \nu \rho} d^{\epsilon \gamma} f^{\bar{f} \bar{g}}$. The factor $h^{\sigma \gamma} d^{\epsilon \gamma} f^{\bar{f} \bar{g}}$, which is $-h^{\sigma \gamma} d^{\epsilon \gamma} f^{\bar{f} \bar{g}}$, can be simplified as $-\varepsilon^{\sigma \gamma \epsilon} d^{\epsilon \gamma}$. Then one obtains $d^{\epsilon \gamma} f^{\bar{f} \bar{g}} e$. Here, one has the following identity

$$d^{\epsilon \gamma} f^{\bar{f} \bar{g}} e = N d^{\epsilon \gamma} f^{\bar{f} \bar{g}} e.$$ 

Then the final contribution is given by $-N \varepsilon^{\mu \nu \rho} \varepsilon^{\sigma \gamma \epsilon} d^{\epsilon \gamma} f^{\bar{f} \bar{g}} e = N d^{\epsilon \gamma} f^{\bar{f} \bar{g}} e$. By collecting the correct
signs, one obtains that the final contribution is given by

$$-rac{(k - N)}{(k + N + 2)} \frac{d^0}{(k + N + 2)} d^0 Q^3 V^b(w) = -\frac{(k - N)}{(k + N + 2)} \delta^{\mu\nu} \hat{\Phi}^{(1),\rho}(w),$$  \hspace{1cm} (3.30)

where at the final stage of (3.30), the previous expression for the higher spin-$\frac{3}{2}$ current (3.31) is inserted. Therefore, one observes that this is coincident with the second order pole of (3.24).

- $(\mu, \nu) = (i, i)$

So far, the $(\mu, \nu) = (0, 0)$ case is considered. Let us move on the $(\mu, \nu) = (i, i)$ case. One can see, for example, Appendices (C.11), (C.17), (C.20) or (C.23). One should have the corresponding expressions although the detailed calculations are not presented in this paper.

- $(\mu, \nu) = (0, i)$

Let us describe the case $\mu \neq \nu$. In particular, for $(\mu, \nu) = (0, i)$ case, The first term of (3.28) contains $\varepsilon^{\nu\rho\sigma\alpha} h_{ab}^\mu h_{cd}^\sigma d_{ef}^\rho g^{\bar{e}\bar{f}}$. One can further simplify this as $\varepsilon^{\nu\rho\sigma\alpha} h_{cd}^\rho d_{ef}^\rho g^{\bar{e}\bar{f}}$. When the index $\sigma = 0$, then this can be written as $\varepsilon^{\nu\rho\alpha} d_{af}^\rho$. When the index $\sigma = j$, one has $-\varepsilon^{\nu\rho\bar{j}} d_{af}^\rho$. Then one obtains $2\varepsilon^{\nu\rho\alpha} d_{af}^\rho (\equiv 2\varepsilon^{\nu\rho\alpha} d_{af}^\rho)$. Similarly, the third term of (3.28) is given by $\varepsilon^{\nu\rho\sigma\alpha} h_{ab}^\mu h_{cd}^\sigma d_{ef}^\rho g^{\bar{e}\bar{f}}$. This becomes $\varepsilon^{\nu\rho\sigma\alpha} h_{ed}^\rho d_{ef}^\rho g^{\bar{e}\bar{f}}$. One sees that this can be written in terms of $2\varepsilon^{\nu\rho\alpha} d_{af}^\rho$ by realizing the role of indices $\bar{d}$ and $\bar{a}$ (corresponding to $\bar{a}$ and $\bar{f}$ in the previous case). The fourth term of (3.28) is given by $\varepsilon^{\nu\rho\sigma\alpha} h_{ab}^\mu h_{cd}^\sigma d_{ef}^\rho g^{\bar{f}\bar{d}} g^{\bar{e}\alpha}$. This is equivalent to $\varepsilon^{\nu\rho\sigma\alpha} h_{ed}^\rho d_{ef}^\rho g^{\bar{f}\bar{d}}$. By interchanging the indices between $\bar{b}$ and $\bar{d}$ or between $\bar{c}$ and $\bar{f}$, there is no sign change because the nonzero contributions arise when the index $\sigma = 0$ and the index $\alpha = j$ or the index $\sigma = j$ and the index $\alpha = 0$. Then one obtains that the contribution from the fourth term is $2\varepsilon^{\nu\rho\alpha} d_{af}^\rho$.

From the second term of (3.28) $\varepsilon^{\nu\rho\sigma\alpha} h_{ab}^\mu h_{cd}^\sigma d_{ef}^\rho g^{\bar{e}\bar{f}} f^{\bar{d}g} e^{\bar{f}e}$, one can simplify this as follows. There are $\varepsilon^{ijk} h_{ab}^0 h_{cd}^0 d_{ef}^\rho g^{\bar{e}\bar{f}} f^{\bar{d}g} e^{\bar{f}e} f^{\bar{d}g}$ and $\varepsilon^{ijk} h_{ab}^0 h_{cd}^0 d_{ef}^\rho g^{\bar{e}\bar{f}} f^{\bar{d}g} e^{\bar{f}e} f^{\bar{d}g}$ for $\rho = j$ case. For the former, one can use the definition of (3.9). One obtains $-\varepsilon^{ijk} h_{ab}^0 d_{df}^k f^{\bar{d}g} e^{\bar{f}e} f^{\bar{d}g}$. Now one can use the identity (3.29) for the $df$ factor and obtains $-N \varepsilon^{ijk} d_{af}^k$. For the latter, one can use the Kronecker delta between the metric tensor and one has the same contribution. Therefore, the final contribution is given by $-2N \varepsilon^{ijk} d_{af}^k$. One has $\varepsilon^{\nu\rho\sigma\alpha} h_{ab}^\mu h_{cd}^\sigma d_{ef}^\rho g^{\bar{e}\bar{f}} f^{\bar{d}g} e^{\bar{f}e} f^{\bar{d}g} h$ for the fifth term of (3.28). Then one has $\varepsilon^{ijk} h_{ab}^0 h_{cd} h_{ef}^0 g^{\bar{e}\bar{f}} f^{\bar{d}g} e^{\bar{f}e} f^{\bar{d}g}$ for $\rho = j$ case. The first one can be written as $-\varepsilon^{ijk} h_{cd}^0 d_{ef}^0 f^{\bar{d}g} e^{\bar{f}e} f^{\bar{d}g} h$. The $df$ factor can be simplified as $-(N + 2) d_{af}^0 g^{\bar{e}\bar{f}}$ by using the following identity

$$d_{af}^0 f^{\bar{d}g} e^{\bar{f}e} f^{\bar{d}g} = -(N + 2) d_{af}^0 g^{\bar{e}\bar{f}}.$$  \hspace{1cm} (3.31)

Then this (3.31) leads to $(N + 2) \varepsilon^{ijk} d_{af}^k$ with the help of (3.9). Combining with the second contribution $N \varepsilon^{ijk} d_{af}^k$, one has the final contribution $(2N + 2) \varepsilon^{ijk} d_{af}^k$ \footnote{For the sixth term of (3.28), one has $\varepsilon^{\nu\rho\sigma\alpha} h_{ab}^\mu h_{cd}^\sigma d_{ef}^\rho g^{\bar{e}\bar{f}} f^{\bar{d}g} e^{\bar{f}e} f^{\bar{d}g} h$. This is equivalent to the sum}. Therefore, one

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has the final expression as follows
\[
\frac{(2 + 3k + 3N)}{(k + N + 2)} \varepsilon^{\mu \nu \rho \sigma} d_{ab}^\sigma Q^\alpha V^\beta (w) = \frac{(2 + 3k + 3N)}{(k + N + 2)} \varepsilon^{\mu \nu \rho \sigma} \hat{\Phi}_{\frac{3}{2}}^{(1), \sigma} (w), \quad (3.32)
\]
with (3.11). Therefore, this (3.32) coincides with the second order pole of (3.24).

One can analyze the following other cases also but the detailed descriptions are not given in this paper.
• \((\mu, \nu) = (i, 0)\)

For example, one sees Appendix (C.8) and the presence of \(\Phi_{\frac{3}{2}}^{(1), \sigma} (w)\) in \(SO(4)\) basis is expected.

• \((\mu, \nu) = (i, j)\) with \(i \neq j\)

Appendices (C.1), (C.3) or (C.14) correspond to this case. The higher spin-\(\frac{3}{2}\) current \(\Phi_{\frac{3}{2}}^{(1), j} (w)\) should appear.

### 3.4.3 The first order pole

In order to determine the higher spin-\(\frac{5}{2}\) currents, one should obtain the first order pole of the OPE between \(\hat{G}^\mu (z)\) and \(\Phi_{\frac{3}{2}}^{(1), \nu} (w)\). By collecting Appendix (F.1) of [38] or Appendix B with the appropriate factors, the first order pole is described as
\[
\frac{i}{2(k + N + 2)^3} \varepsilon^{\rho \sigma \alpha \beta} h^\mu_{ab} h^\alpha_{\rho \sigma} d_{\epsilon \bar{f}}^\beta \left[-(k + N + 2) g^{\epsilon \bar{e}} \right.
\]
\[
\times \left(k (g^{\bar{d} \bar{b}} \partial Q^\bar{a} V^\bar{f} + g^{\bar{f} \bar{b}} \partial Q^\bar{a} V^\bar{d}) + f^{\bar{d} \bar{g}} Q^\bar{a} V^\bar{f} V^g + f^{\bar{f} \bar{g}} Q^\bar{a} V^\bar{d} V^g + f^{\bar{d} \bar{g}} f^{\bar{f} \bar{g}} \partial (Q^\bar{a} V^\bar{h}) \right)
\]
\[
- k g^{\epsilon \bar{d}} (k + N + 2) (g^{\bar{c} \bar{a}} Q^\epsilon \partial V^\bar{b} - g^{\bar{c} \bar{a}} \partial Q^\epsilon V^\bar{b})
\]
\[
- f^{\bar{g} \bar{b}} (k + N + 2) (g^{\bar{c} \bar{a}} Q^\epsilon V^{\bar{b}} V^g - g^{\bar{c} \bar{a}} Q^\epsilon V^{\bar{b}} V^g) + k g^{\bar{d} \bar{h}} Q^\epsilon Q^\bar{e} \partial Q^\bar{a} + f^{\bar{g} \bar{h}} Q^\epsilon Q^\bar{e} Q^\bar{a} \partial V^h \left]. \right
\]

We would like to simplify this in order to extract the higher spin-\(\frac{5}{2}\) currents. In principle, the five cases can be considered.
• \((\mu, \nu) = (0, 0)\)

One can collect the quadratic terms with a derivative. Let us analyze the first term of (3.33). As described in previous subsection, the result is given by \(-2d_{ab}^\mu\). Similarly, the second term of (3.33) contributes to the similar quantity \(-2d_{cd}^\mu\) with different indices. The sixth term of (3.33) leads to the similar expression \(2d_{be}^\mu\). The seventh term has \(\varepsilon^{\rho \sigma \alpha \beta} h^\mu_{ab} h^\alpha_{\rho \sigma} d_{\epsilon \bar{f}}^\beta g^{\epsilon \bar{d}} g^{\epsilon \bar{a}}\) which can be simplified as \(\varepsilon^{\rho \sigma \alpha \beta} h^\mu_{ab} h^\alpha_{\rho \sigma} d_{\epsilon \bar{f}}^\beta g^{\epsilon \bar{d}} g^{\epsilon \bar{a}}\). Furthermore, one obtains \(\varepsilon^{\rho \sigma \alpha \beta} h^\mu_{\rho \sigma} g^{\epsilon \bar{d}} d_{\epsilon \bar{f}}^\beta\). Using the identity (3.15), one has \(\varepsilon^{\rho \sigma \alpha \beta} h^\mu_{ab} h^\alpha_{\rho \sigma} d_{\epsilon \bar{f}}^\beta = 2d_{ab}^\mu\) from previous analysis.
Let us look at the third term of (3.33). The expression $\varepsilon^{0\rho\sigma\alpha} h_{\alpha b}^0 h_{\epsilon d}^\sigma d_{\epsilon f}^\rho g^{\epsilon\epsilon} f^{\bar{d}g}$ can be expressed as $-\varepsilon^{0\rho\sigma\alpha} h_{\alpha b}^0 h_{\epsilon d}^\sigma d_{\epsilon f}^\rho g^{\epsilon\epsilon} f^{\bar{d}g}$. According to (3.18), one has $-\varepsilon^{0\rho\sigma\alpha} h_{\alpha b}^0 h_{\epsilon d}^\sigma \varepsilon^{\sigma\epsilon\epsilon} f^{\bar{d}g} g^{\epsilon\epsilon} f^{\bar{d}g}$. Therefore, one obtains $2 h_{\alpha b}^0 d_{\epsilon f}^\rho f^{\bar{d}g}$. The fourth term is given by $\varepsilon^{0\rho\sigma\alpha} h_{\alpha b}^0 h_{\epsilon d}^\sigma g^{\epsilon\epsilon} d_{\epsilon f}^\rho f^{\bar{d}g}$. One then has $2 h_{\alpha b}^0 d_{\epsilon f}^\rho f^{\bar{d}g}$. With the field $Q^\beta V^d V g$, the fourth term is equal to the third term. From the eighth term, one has $\varepsilon^{0\rho\sigma\alpha} h_{\alpha b}^0 h_{\epsilon d}^\sigma d_{\epsilon f}^\rho g^{\epsilon\epsilon} f^{\bar{d}g}$. So this can be written as $\varepsilon^{0\rho\sigma\alpha} h_{\alpha b}^0 h_{\epsilon d}^\sigma d_{\epsilon f}^\rho f^{\bar{d}g}$. Similarly, the ninth term can be written as $\varepsilon^{0\rho\sigma\alpha} h_{\alpha b}^0 h_{\epsilon d}^\sigma d_{\epsilon f}^\rho f^{\bar{d}g}$. Note that one can check the following identity

$$ (h_{bd}^{ij} d_{ef}^i - h_{ed}^{ij} d_{bf}^i) f^{fd} g Q^\beta V^b V g = 0. \quad (3.34) $$

Using this property (3.34), one can simplify the eighth and ninth terms (the summation over the indices $\sigma$ and $\alpha$ is taken) further. The tenth term vanishes identically due to (3.6)\footnote{Let us consider the fifth term of (3.33). One has $\varepsilon^{0\rho\sigma\alpha} h_{\alpha b}^0 h_{\epsilon d}^\sigma d_{\epsilon f}^\rho g^{\epsilon\epsilon} f^{\bar{d}g} f^{\bar{d}g}$. From the identity (3.18), one has $-\varepsilon^{0\rho\sigma\alpha} h_{\alpha b}^0 \varepsilon^{\sigma\epsilon\epsilon} f^{\bar{d}g} g^{\epsilon\epsilon} f^{\bar{d}g}$. From (3.29), the $dfg$ factor can be reduced to $N h_{\alpha b}^0 \varepsilon^{\sigma\epsilon\epsilon} f^{\bar{d}g}$. This leads to $-N \varepsilon^{0\rho\sigma\alpha} \varepsilon^{\sigma\epsilon\epsilon} f^{\bar{d}g} g^{\epsilon\epsilon} f^{\bar{d}g}$. The last term of (3.33) is $\varepsilon^{0\rho\sigma\alpha} h_{\alpha b}^0 h_{\epsilon d}^\sigma d_{\epsilon f}^\rho f^{fd} g^{\epsilon\epsilon} f^{\bar{d}g}$. One has the following identity $h_{\alpha b}^0 (h_{\epsilon d}^{ij} d_{ef}^i + h_{ed}^{ij} d_{bf}^i) f^{fd} g^{\epsilon\epsilon} f^{\bar{d}g} = 0$. This implies that the summation over the indices $\sigma$ and $\alpha$ can be simplified further.\textsuperscript{11}}

It turns out that the first order pole of (3.33) with free index $\rho$ and fixed indices $(\mu, \nu) = (0, 0)$ in $SU(2) \times SU(2)$ basis can be described as (the last term of (3.33) remains)

$$ \left[ (k + N + 2) \left[ (3k + N) d_{ab}^0 \partial Q^\beta V^b + (N - k) d_{ab}^0 Q^\beta \partial Q^\beta V^b + 2 h_{ab}^0 d_{df}^g f^{df} g Q^\beta V^f V g + \frac{1}{2} \varepsilon^{0\rho\sigma\alpha} h_{\alpha b}^0 h_{\epsilon d}^\sigma d_{\epsilon f}^\rho f^{fd} g Q^\beta V^b V g + \frac{1}{2} \varepsilon^{0\rho\sigma\alpha} h_{\alpha b}^0 h_{\epsilon d}^\sigma d_{\epsilon f}^\rho f^{fd} g Q^\beta V^b V g \right] \right]. \quad (3.35) $$

Therefore, this quantity will be crucial to obtain the higher spin-3 current eventually even in the fixed indices. In particular, the last quartic term in (3.35) has rather complicated tensor indices compared to other terms. We will fix the index $\rho$ later.

There are other cases one should consider as follows. The detailed calculations on these cases are ignored in this paper.

- $(\mu, \nu) = (i, i)$

For the same $(\mu, \nu) = (1, 1)$ but with $\rho = 2$, one sees Appendix (C.23) in $SO(4)$ basis. Appendix (C.11) corresponds to $(\mu, \nu) = (3, 3)$ with $\rho = 0$. For the same $(\mu, \nu) = (3, 3)$ but with $\rho = 1$, one sees Appendix (C.17). Moreover, for the same $(\mu, \nu) = (3, 3)$ but with $\rho = 2$, one sees Appendix (C.20). The higher spin-3\textsubscript{2} current $\Phi^{(1)\rho} (z)$ can be determined explicitly.
• \((\mu, \nu) = (0, i)\)

One observes this case in (3.51) in \(SO(4)\) basis when \(i = 3\). The index \(\rho\) is the same as the index \(\mu\).

• \((\mu, \nu) = (i, 0)\)

One sees the particular case with \(i = 2\) in Appendix (C.3) in \(SO(4)\) basis where the index \(\rho\) is the same as index \(\mu\). The higher spin-\(\frac{5}{2}\) current \(\Phi^{(1),\mu}(z)\) can be determined explicitly.

• \((\mu, \nu) = (i, j)\) with \(i \neq j\)

One can see Appendix (C.1) in \(SO(4)\) basis corresponding to \((i, j) = (1, 3)\). Appendix (C.3) corresponds to \((i, j) = (2, 3)\). Similarly, the \((\mu, \nu) = (2, 1)\) case can be seen from Appendix (C.14). All of these examples have the same index for \(\mu\) and \(\rho\). The higher spin-\(\frac{5}{2}\) current \(\Phi^{(1),\mu}(z)\) can be seen explicitly.

Let us find the four higher spin-\(\frac{5}{2}\) currents explicitly. There are three different ways for writing down each higher spin-\(\frac{5}{2}\) current from (3.21), the possibility of \(\rho\). We will present one way for each higher spin-\(\frac{5}{2}\) current and the remaining two ways will appear in Appendix C.

3.4.4 The first higher spin-\(\frac{5}{2}\) current

In order to proceed further, it is better to consider the fixed \(\rho\) case. Let us look at the final term of (3.35) when \(\rho = 1\) (Other cases are described in Appendix C). One has

\[
\varepsilon^{0123} h^0_{\bar{c} \bar{b}} h^2_{\bar{a} \bar{d}} d^3_{a f} f^f_{\bar{g} \bar{h}} f^{g h} = \\
-\frac{1}{2} d^2_{\bar{a} \bar{b}} h^3_{\bar{c} \bar{d}} - \frac{1}{4} d^2_{\bar{a} \bar{d}} h^3_{\bar{b} \bar{c}} - \frac{1}{4} d^2_{\bar{c} \bar{b}} h^3_{\bar{a} \bar{d}} + \frac{1}{4} d^2_{\bar{a} \bar{c}} h^3_{\bar{b} \bar{d}} + \frac{1}{4} d^2_{\bar{b} \bar{c}} h^3_{\bar{a} \bar{d}} - \frac{1}{2} d^3_{\bar{a} \bar{b} \bar{c}} + \frac{1}{4} d^3_{\bar{a} \bar{b} \bar{d}} h^2_{\bar{c} \bar{c}} \\
+ \frac{1}{4} d^3_{\bar{a} \bar{c} \bar{d}} h^2_{\bar{b} \bar{b}} - \frac{1}{4} d^3_{\bar{b} \bar{c} \bar{d}} h^2_{\bar{a} \bar{a}} - \frac{1}{4} d^0_{\bar{a} \bar{b}} h^1_{\bar{c} \bar{d}} + \frac{1}{2} d^0_{\bar{a} \bar{c}} h^1_{\bar{b} \bar{d}} - \frac{1}{2} d^0_{\bar{b} \bar{c}} h^1_{\bar{a} \bar{d}} - \frac{1}{2} d^0_{\bar{a} \bar{d}} h^1_{\bar{c} \bar{b}} + \frac{1}{4} d^0_{\bar{a} \bar{d}} h^1_{\bar{c} \bar{b}} - \frac{1}{4} d^0_{\bar{b} \bar{d}} h^1_{\bar{a} \bar{c}} + \frac{1}{4} d^0_{\bar{b} \bar{d}} h^1_{\bar{a} \bar{c}} - \frac{1}{4} d^0_{\bar{b} \bar{d}} h^1_{\bar{a} \bar{c}}. \tag{3.36}
\]

One can simplify the quartic term in (3.35) with (3.36). Among the first five terms of (3.36), the second and the fourth terms remain the nonzero contributions because the \(d^2\) tensor is symmetric while the quartic term is antisymmetric in the indices \(\bar{a}, \bar{b}\) and \(\bar{c}\). The second term can be written as \(d^2_{\bar{a} \bar{b}} h^3_{\bar{c} \bar{d}}\). With quartic term, one can interchange the indices \(\bar{a}\) and \(\bar{b}\) and obtains \(-d^2_{\bar{a} \bar{b}} h^3_{\bar{c} \bar{d}}\). Therefore, one has the same contribution as the one in fourth term. For the next five terms in (3.36), one is left with the seventh and ninth terms according to the similar analysis done before. The ninth term can be rewritten as \(d^3_{\bar{a} \bar{b} \bar{c}} h^2_{\bar{d} \bar{d}}\) with other factor. By interchanging the indices \(\bar{a}\) and \(\bar{b}\), one has \(-d^3_{\bar{a} \bar{b} \bar{c}} h^2_{\bar{d} \bar{d}}\). Then one has the same contribution as the one in seventh term. For the next six terms, the 11th, 13th and 16th can be combined together and the remaining terms vanish. Finally, there are no contributions from the last
four terms. Then one obtains for the quartic term in (3.35) as follows
\[\left(\frac{1}{2}d_{ab}^2 h_{ac}^b + \frac{1}{2}d_{ad}^3 h_{be}^2 - d_{ab}^0 h_{ed}^1\right) Q^\alpha Q^\beta V^d(w).\] (3.37)

Now one would like to determine the higher spin-5/2 current which is a primary. One can focus on the particular OPE of (3.24) with the indices \(\mu = 0, \nu = 0\) and \(\rho = 1\). By noting that \(G^\alpha(z) = G^2(z)\) from (2.3) and \(\Phi^{(1),01}_1(w) = \Phi^{(1),23}_1(w)\) from (3.21) and (3.23), one can have the following OPE in the \(SO(4)\) manifest basis
\[G^2(z) \Phi^{(1),23}_1(w) = + \cdots + \frac{1}{(z-w)} \left[-\Phi^{(1),3}_\frac{1}{2} \right.\]
\[\left. - \frac{2i}{(k + N + 2)} T^{24} \Phi^{(1),1}_\frac{1}{2} + \frac{4i}{(k + N + 2)} T^{14} \Phi^{(1),2}_\frac{1}{2} + \frac{(N-k)}{3(k + N + 2)} \partial \Phi^{(1),3}_\frac{1}{2}\right](w) + \cdots\] (3.38)
where the second order pole is ignored. All the hat notations appearing in the right hand side of the OPE (3.24) are gone. See also the corresponding OPE in Appendix (A.1).

Then it is obvious that the higher spin-5/2 current can be read off and is given by
\[\Phi^{(1),3}_\frac{1}{2}(z) = -\frac{2i}{(k + N + 2)} T^{12} \Phi^{(1),4}_\frac{1}{2} - \frac{2i}{(k + N + 2)} T^{24} \Phi^{(1),1}_\frac{1}{2} + \frac{4i}{(k + N + 2)} T^{14} \Phi^{(1),2}_\frac{1}{2}\]
\[+ \frac{(N-k)}{3(k + N + 2)} \partial \Phi^{(1),3}_\frac{1}{2} - (3.35)\] with \(\rho = 1\), (3.39)
where the relation (3.37) is used. Furthermore, one should rewrite the composite expressions appearing in (3.39) in terms of WZW adjoint currents. The spin-1 currents are presented in (2.7) and (2.3) while the higher spin-\(\frac{5}{2}\) currents are presented in (3.12). Because the spin-1 currents \(A^{-i(z)}\) are quadratic terms in (2.3), one should consider the following normal ordered product
\[(Q^\alpha Q^\beta)(Q^d V^\bar{\epsilon}) = Q^\alpha Q^\beta Q^d V^\bar{\epsilon} - [Q^d V^\bar{\epsilon}, Q^\beta Q^\beta] + (Q^d V^\bar{\epsilon}, Q^\alpha) Q^\beta - Q^\alpha \{Q^d V^\bar{\epsilon}, Q^\beta\}\]
\[= Q^\alpha Q^\beta Q^d V^\bar{\epsilon} + (k + N + 2) g^{\alpha\beta} \partial Q^\beta V^\bar{\epsilon} - (k + N + 2) g^{\alpha\beta} \partial Q^\alpha V^\bar{\epsilon}.\] (3.40)
Note that there is a minus sign in the last term of the first line of (3.40) due to the exchange of the two fermionic quantities. Each (anti)commutator can be obtained from each defining OPE.

It turns out that one obtains the following relations
\[T^{12} \Phi^{(1),4}_\frac{1}{2} = -\frac{1}{4N(k + N + 2)} f_{\alpha\beta\gamma} h_{ab}^3 d_{de}^2 Q^d V^e V^\bar{\epsilon} + \frac{1}{4(k + N + 2)} h_{ab}^3 d_{cd}^2 Q^\alpha Q^\beta Q^d V^\bar{\epsilon}\]
\[+ \frac{1}{2N(k + N + 2)} d_{ab}^1 \partial Q^\alpha V^\beta,\]
\[T^{24} \Phi^{(1),1}_\frac{1}{2} = \frac{1}{4N(k + N + 2)} f_{\alpha\beta\gamma} h_{ab}^2 d_{de}^3 Q^d V^e V^\bar{\epsilon} - \frac{1}{4(k + N + 2)} h_{ab}^2 d_{cd}^3 Q^\alpha Q^\beta Q^d V^\bar{\epsilon}\]
\[+ \frac{1}{2N(k + N + 2)} d_{ab}^1 \partial Q^\alpha V^\beta,\]
The relations (3.18) and (3.16) are used in the first two and the last one in (3.41) respectively. One can check that the quartic term of the first quantity in (3.41) corresponds to the first term of (3.37) and those in the second quantity in (3.41) does to the second term of (3.37). Furthermore, as one moves $V^g$ in (3.35) to the left at one step (in order to combine with the cubic terms in (3.41)), then the extra derivative terms can be reduced to the second term of (3.35) with different coefficients by using the identities (3.29) and (3.18).

Finally, one obtains the higher spin-\(\frac{5}{2}\) current which will used for the higher spin-3 current

\[
\Phi_{\frac{5}{2}}^{(1),3}(z) = \frac{i}{(k+N+2)^2} \left[ \frac{1}{2N} h_{ab}^3 d_{de}^3 - \frac{1}{2N} h_{ab}^2 d_{de} - \frac{1}{N} h_{ab}^1 d_{de}^0 \right]
- 2 h_{bd}^0 d_{ae}^1 - h_{bc}^2 d_{ad}^3 + h_{bd}^2 d_{ae}^3 \right] f^{abc} V^c V^e - \frac{4i(3 + 2k + N)}{3(k+N+2)^2} d_{ab}^1 Q^a V^b(z)
+ \frac{i}{(k+N+2)^2} \left[ d_{bd}^2 h_{ac}^3 + d_{bd}^3 h_{bc}^2 - d_{ab}^0 h_{cd}^1 + d_{cd}^0 h_{ab}^1 \right] Q^a Q^b Q^c V^d(z)
+ \frac{4i(3 + 2k + N)}{3(k+N+2)^2} d_{ab}^1 Q^a \partial V^b(z).
\]

(3.42)

Note that the last term of the quartic term in (3.42) originates from the one in (3.41). Compared to (3.35), the coefficients of all the derivative terms are changed as explained before. One presents this higher spin-\(\frac{5}{2}\) current in two different ways in Appendix C (The index \(\mu\) in \(SO(4)\) basis is 1 or 4). They will differ from (3.42) only by the cubic term.

### 3.4.5 The second higher spin-\(\frac{5}{2}\) current

Let us determine the second higher spin-\(\frac{5}{2}\) current. Let us consider the first order pole in (3.35) for the indices \(\mu = 0, \nu = 0\) and \(\rho = 2\). One has the following identity

\[
\varepsilon^{0231} h_{cb}^0 h_{bd}^3 d_{af}^1 f^{f'g'} g'^d f^d =
- \frac{1}{4} d_{ab}^1 h_{cd}^3 - \frac{1}{4} d_{ac}^1 h_{db}^3 + \frac{1}{4} d_{ad}^1 h_{cb}^3 - \frac{1}{2} d_{ab}^0 h_{cd}^3 + \frac{1}{2} d_{ac}^0 h_{db}^3 - \frac{1}{2} d_{ad}^0 h_{cb}^3 - \frac{1}{2} d_{ab}^2 h_{cd}^3 + \frac{1}{2} d_{ac}^2 h_{db}^3 - \frac{1}{2} d_{ad}^2 h_{cb}^3
- \frac{1}{4} d_{ac}^0 h_{db}^2 + \frac{1}{4} d_{ad}^0 h_{bc}^2 - \frac{1}{4} d_{ac}^2 h_{db}^2 + \frac{1}{4} d_{ad}^2 h_{bc}^2 - \frac{1}{2} d_{ac}^0 h_{db}^1 + \frac{1}{2} d_{ad}^0 h_{bc}^1 - \frac{1}{4} d_{ac}^3 h_{db}^1 - \frac{1}{4} d_{ad}^3 h_{bc}^1
+ \frac{1}{4} d_{db}^3 h_{ac}^1 + \frac{1}{2} d_{db}^3 h_{da}^1 - \frac{1}{4} d_{ab}^2 h_{cd}^0 + \frac{1}{4} d_{ac}^2 h_{db}^0 - \frac{1}{4} d_{ad}^2 h_{cb}^0 - \frac{1}{4} d_{db}^2 h_{ac}^0.
\]

(3.43)
There are no contributions for the terms having the \( d^i \) tensor with two indices among \( \tilde{a}, \tilde{b} \) or \( \tilde{c} \) because they are symmetric under the two indices while the composite field \(-Q^\tilde{a} Q^\tilde{b} Q^\tilde{c} V^\tilde{d} \) is antisymmetric. \(^{12}\) Finally, one obtains the resulting expression for the quartic term in (3.35)

\[
(-\frac{1}{2} d_{\tilde{e} \tilde{d}} h_{\tilde{a} \tilde{b}}^3 + \frac{1}{2} d_{\tilde{e} \tilde{d}} h_{\tilde{a} \tilde{b}}^1 + d_{\tilde{a} \tilde{b}} h_{\tilde{c} \tilde{d}}^2) Q^\tilde{a} Q^\tilde{b} Q^\tilde{c} V^\tilde{d}(w).
\] (3.44)

As before, now one would like to determine the second higher spin-\( \frac{5}{2} \) current which is a primary. One can focus on the particular OPE of (3.24) with the indices \( \mu = 0, \nu = 0 \) and \( \rho = 2 \). By noting that \( \hat{G}^\mu(z) = G^2(z) \) from (2.6) and \( \hat{\Phi}^{(1),02}(w) = -\Phi^{(1),24}(w) \) from (3.21) and (3.23), one can have the following OPE in the \( SO(4) \) manifest basis

\[
G^2(z) \Phi^{(1),24}_1(w) = + \cdots + \frac{1}{(z-w)} \left[ -\Phi^{(1),4}_\frac{1}{2} + \frac{2i}{(k+N+2)} T^{12} \Phi^{(1),3}_\frac{1}{2} \right. \\
+ \frac{2i}{(k+N+2)} T^{23} \Phi^{(1),1}_\frac{1}{2} - \frac{4i}{(k+N+2)} T^{13} \Phi^{(1),2}_\frac{1}{2} + \frac{(N-k)}{3(k+N+2)} \partial \Phi^{(1),4}_\frac{1}{2} + (3.35) \right] (w) + \cdots , \tag{3.45}
\]

where the second order pole is ignored. All the hat notations appearing in the right hand side of the OPE (3.24) are gone. See also the corresponding OPE in Appendix (A.1).

Then it is obvious that the second higher spin-\( \frac{5}{2} \) current can be read off and is given by

\[
\Phi^{(1),4}_\frac{1}{2}(z) = \frac{2i}{(k+N+2)} T^{12} \Phi^{(1),3}_\frac{1}{2} + \frac{2i}{(k+N+2)} T^{23} \Phi^{(1),1}_\frac{1}{2} - \frac{4i}{(k+N+2)} T^{13} \Phi^{(1),2}_\frac{1}{2} \\
+ \frac{(N-k)}{3(k+N+2)} \partial \Phi^{(1),4}_\frac{1}{2} + (3.35) \text{ with } \rho = 2, \tag{3.46}
\]

where the relation (3.44) is used. Furthermore, one should rewrite the composite expressions appearing in (3.46) in terms of WZW adjoint currents. One obtains

\[
T^{12} \Phi^{(1),3}_\frac{1}{2} = \frac{1}{4N(k+N+2)} f_{\tilde{a} \tilde{b} \tilde{c}} h_{\tilde{a} \tilde{b}}^3 d_{\tilde{c} \tilde{d}}^1 Q^\tilde{d} V^\tilde{c} V^\tilde{e} - \frac{1}{4(k+N+2)^2} h_{\tilde{a} \tilde{b}}^3 d_{\tilde{c} \tilde{d}}^1 Q^\tilde{a} Q^\tilde{b} Q^\tilde{c} V^\tilde{d} \\
+ \frac{1}{2(k+N+2)} d_{\tilde{a} \tilde{b}}^1 \partial Q^\tilde{a} V^\tilde{b},
\]

\[
T^{23} \Phi^{(1),1}_\frac{1}{2} = -\frac{1}{4N(k+N+2)} f_{\tilde{a} \tilde{b} \tilde{c}} h_{\tilde{a} \tilde{b}}^1 d_{\tilde{c} \tilde{d}}^3 Q^\tilde{d} V^\tilde{c} V^\tilde{e} + \frac{1}{4(k+N+2)^2} h_{\tilde{a} \tilde{b}}^1 d_{\tilde{c} \tilde{d}}^3 Q^\tilde{a} Q^\tilde{b} Q^\tilde{c} V^\tilde{d} \\
+ \frac{1}{2(k+N+2)} d_{\tilde{a} \tilde{b}}^2 \partial Q^\tilde{a} V^\tilde{b},
\]

\[
T^{13} \Phi^{(1),2}_\frac{1}{2} = \frac{1}{4N(k+N+2)} f_{\tilde{a} \tilde{b} \tilde{c}} h_{\tilde{a} \tilde{b}}^2 d_{\tilde{c} \tilde{d}}^0 Q^\tilde{d} V^\tilde{c} V^\tilde{e} + \frac{1}{4(k+N+2)^2} h_{\tilde{a} \tilde{b}}^2 d_{\tilde{c} \tilde{d}}^0 Q^\tilde{a} Q^\tilde{b} Q^\tilde{c} V^\tilde{d} \\
- \frac{1}{2(k+N+2)} d_{\tilde{a} \tilde{b}}^2 \partial Q^\tilde{a} V^\tilde{b}. \tag{3.47}
\]

\(^{12}\) That is, the first, the second, the fifth, the twelfth, the thirteenth, the sixteenth, the seventeenth, the eighteenth of (3.35), become zero. One can combine the third and the fourth terms. Similarly the fourteenth and the fifteenth terms can be combined each other. The last two terms vanish because the \( h^0 \) tensor is symmetric while the composite field is antisymmetric. The sixth and the eighth terms can be combined. The ninth and the tenth terms can contribute similarly.
The relations (3.18) and (3.16) are used in the first two and the last one in (3.47) respectively. One can check that the quartic term of the first quantity in (3.47) corresponds to the first term of (3.44) and those in the second quantity in (3.47) does to the second term of (3.44). Furthermore, as one moves $V^g$ in (3.35) to the left at one step (in order to combine with the cubic terms in (3.47)), then the extra derivative terms can be reduced to the second term of (3.35) with different coefficients by using the identities (3.29) and (3.18).

Therefore, the final second higher spin-$\frac{5}{2}$ current in $SO(4)$ basis can be described as

\[
\Phi^{(1,4)}_{\frac{5}{2}}(z) = \frac{i}{(k + N + 2)^2} \left[ \frac{1}{2N} h_{ab}^3 d_{de}^1 - \frac{1}{2N} h_{ab}^1 d_{de}^3 - \frac{1}{N} h_{ab}^2 d_{de}^0 \right]
+ 2 h_{bd}^0 d_{de}^2 - h_{be}^1 d_{ad}^3 + h_{bd}^1 d_{ae}^3 \right] f^{ab} c Q^d V^c V^e + \frac{4i(3 + 2k + N)}{3(k + N + 2)^2} \partial Q^a V^b(z)
+ \frac{i}{(k + N + 2)^3} \left[ - d_{cd}^1 h_{ab}^5 + d_{cd}^0 h_{cd}^2 - d_{ad}^0 h_{be}^3 + d_{cd}^3 \right] Q^a Q^b Q^c V^d(z)
- \frac{4i(k + 2N)}{3(k + N + 2)^2} d_{ab}^2 Q^a \partial V^b(z).
\]

(3.48)

Note that the third term of the quartic term in (3.48) originates from the one in (3.47). Compared to (3.35), the coefficients of all the derivative terms are changed as explained before. One presents this second higher spin-$\frac{5}{2}$ current in two different ways in Appendix C (The index $\mu$ in $SO(4)$ basis is 1 or 3). They will differ from (3.48) only by the cubic term.

### 3.4.6 The third higher spin-$\frac{5}{2}$ current

Let us determine the third higher spin-$\frac{5}{2}$ current. Let us consider the first order pole in (3.35) for the indices $\mu = 0$, $\nu = 0$ and $\rho = 3$. One has the following identity

\[
\varepsilon^{0312} h_{0}^{0} h_{1}^{0} h_{1}^{0} d_{a}^{2} f^{f} f^{g} f^{g} \partial d^{1} f^{f} g^{g} h_{0}^{0} h_{1}^{0} h_{1}^{0} d_{a}^{2} =
- \frac{1}{4} d_{ab}^{0} h_{cd}^{3} + \frac{1}{4} d_{ab}^{0} h_{bc}^{3} - \frac{1}{4} d_{ab}^{0} h_{cd}^{3} + \frac{1}{4} d_{ab}^{0} h_{bc}^{3} + \frac{1}{4} d_{ab}^{1} h_{cd}^{5} + \frac{1}{4} d_{ab}^{1} h_{bc}^{5} - \frac{1}{4} d_{ab}^{1} h_{cd}^{5} - \frac{1}{4} d_{ab}^{1} h_{bc}^{5}
+ \frac{1}{4} d_{ab}^{0} h_{ae}^{3} + \frac{1}{2} d_{ab}^{0} h_{da}^{3} - \frac{1}{4} d_{ab}^{0} h_{ae}^{3} - \frac{1}{4} d_{ab}^{0} h_{da}^{3} + \frac{1}{4} d_{ae}^{2} h_{be}^{1} - \frac{1}{4} d_{ae}^{2} h_{be}^{1} + \frac{1}{4} d_{ae}^{1} h_{be}^{1} - \frac{1}{4} d_{ae}^{1} h_{be}^{1}
- \frac{1}{4} d_{ab}^{0} h_{ce}^{3} + \frac{1}{4} d_{ab}^{0} h_{dc}^{3} - \frac{1}{4} d_{ab}^{0} h_{ce}^{3} + \frac{1}{4} d_{ab}^{0} h_{dc}^{3} - \frac{1}{4} d_{ab}^{0} h_{ce}^{3} + \frac{1}{4} d_{ab}^{0} h_{dc}^{3}.
\]

(3.49)

The fifth, sixth, ninth, tenth, eleventh, fourteenth, fifteenth, sixteenth, seventeenth and eighteenth terms in (3.49) vanish$^{13}$. Then, one obtains the simple form as follows

\[
(\frac{1}{2} d_{cd}^{1} h_{ab}^{3} - \frac{1}{2} d_{cd}^{1} h_{ab}^{3} + d_{ab}^{0} h_{cd}^{3}) Q^a Q^b Q^c V^d(w).
\]

(3.50)

$^{13}$ The sum of the second and fourth terms become zeroes. The seventh and the eighth terms can combine each other. Similarly, one can combine the twelfth and the thirteenth terms. Finally, the first, the second and the last terms can be combined.
As before, now one would like to determine the third higher spin-$\frac{5}{2}$ current which is a primary. One can focus on the particular OPE of \((3.24)\) with the indices $\mu = 0, \nu = 0$ and $\rho = 3$. By noting that $\hat{\Phi}^0(z) = G^2(z)$ from \((2.6)\) and $\hat{\Phi}^{1,03}_1(w) = -\Phi^{1,12}_1(w)$ from \((3.21)\) and \((3.23)\), one can have the following OPE in the $SO(4)$ manifest basis

$$G^2(z) \Phi^{1,12}_1(w) = + \cdots + \frac{1}{(z-w)} \left[ \Phi^{(1),1}_1 + \frac{2i}{(k+N+2)} T^{23} \Phi^{(1),4}_1 \right],$$

$$- \frac{2i}{(k+N+2)} T^{24} \Phi^{(1),3}_1 + \frac{4i}{(k+N+2)} T^{34} \Phi^{(1),2}_1 - \frac{(N-k)}{3(k+N+2)} \partial \Phi^{(1),1}_1 \right] (w) + \cdots, \quad (3.51)$$

where the second order pole is ignored. All the hat notations appearing in the right hand side of the OPE \((3.24)\) are gone. See also the corresponding OPE in Appendix A.1.

Then it is obvious that the third higher spin-$\frac{5}{2}$ current can be read off and is given by

$$\Phi^{(1),1}_1(z) = - \frac{2i}{(k+N+2)} T^{23} \Phi^{(1),4}_1 + \frac{2i}{(k+N+2)} T^{24} \Phi^{(1),3}_1 - \frac{4i}{(k+N+2)} T^{34} \Phi^{(1),2}_1 + \frac{(N-k)}{3(k+N+2)} \partial \Phi^{(1),1}_1 - (3.35) \quad \text{with } \rho = 3, \quad (3.52)$$

where the relation \((3.50)\) is used. Furthermore, one should rewrite the composite expressions appearing in \((3.52)\) in terms of WZW adjoint currents. One obtains

\begin{align*}
T^{23} \Phi^{(1),4}_1 &= \frac{1}{4N(k+N+2)} f^{abc} h^{1}_{ab} d_{de}^{\alpha} Q^{\beta} V^{c} V^{\epsilon} - \frac{1}{4(k+N+2)^2} h^{1}_{ab} d_{cd}^{\alpha} Q^{\beta} Q^{\delta} V^{d} \\
&+ \frac{1}{2(k+N+2)} d_{ab}^{\beta} \partial Q^{a} V^{b},
\end{align*}

\begin{align*}
T^{24} \Phi^{(1),3}_1 &= \frac{1}{4N(k+N+2)} f^{abc} h^{2}_{ab} d_{de}^{\alpha} Q^{\beta} V^{c} V^{\epsilon} - \frac{1}{4(k+N+2)^2} h^{2}_{ab} d_{cd}^{\alpha} Q^{\beta} Q^{\delta} V^{d} \\
&- \frac{1}{2(k+N+2)} d_{ab}^{\beta} \partial Q^{a} V^{b},
\end{align*}

\begin{align*}
T^{34} \Phi^{(1),2}_1 &= - \frac{1}{4N(k+N+2)} f^{abc} h^{3}_{ab} d_{de}^{\alpha} Q^{\beta} V^{c} V^{\epsilon} - \frac{1}{4(k+N+2)^2} h^{3}_{ab} d_{cd}^{\alpha} Q^{\beta} Q^{\delta} V^{d} \\
&+ \frac{1}{2(k+N+2)} d_{ab}^{\beta} \partial Q^{a} V^{b}. \quad (3.53)
\end{align*}

The relations \((3.18)\) and \((3.16)\) are used in the first two and the last one in \((3.53)\) respectively.

\footnote{One can check that the quartic term of the first quantity in \((3.51)\) corresponds to the second term of \((3.50)\) and those in the second quantity in \((3.53)\) does to the first term of \((3.50)\). Furthermore, as one moves $V^g$ in \((3.35)\) to the left at one step (in order to combine with the cubic terms in \((3.53)\), then the extra derivative terms can be reduced to the second term of \((3.35)\) with different coefficients by using the identities \((3.29)\) and \((3.18)\).}
Therefore, the third higher spin-$\frac{5}{2}$ current in $SO(4)$ basis is given by

$$
\Phi^{(1,1)}_{\frac{5}{2}}(z) = \frac{i}{(k + N + 2)^2} \left[ \frac{1}{2N} h^{1}_{ab} d^{2}_{de} + \frac{1}{2N} h^{2}_{cd} d^{1}_{de} + \frac{1}{N} h^{3}_{ab} d^{0}_{cd} \right]
- \frac{2h^{0}_{bd} d^{3}_{ac} + h^{2}_{be} d^{1}_{ad} - h^{2}_{bd} d^{1}_{ae}}{f^{ab} c Q^{d} V^{c} V^{e}} \cdot 4i(3 + 2k + N) \left( 3(k + N + 2)^2 d^{3}_{ab} \partial Q^{a} Q^{b} \cdot (z) \right) + \frac{i}{(k + N + 2)^3} \left[ - d^{0}_{ab} h^{3}_{cd} - d^{1}_{cd} h^{2}_{ab} + d^{2}_{cd} h^{1}_{ab} + d^{0}_{cd} h^{3}_{ab} \right] Q^{a} Q^{b} Q^{c} V^{d}(z) + \frac{4i(k + 2N)}{3(k + N + 2)^2} d^{3}_{ab} Q^{a} \partial V^{b}(z).
$$

(3.54)

Note that the last term of the quartic term in (3.54) originates from the one in (3.53). Compared to (3.55), the coefficients of all the derivative terms are changed as explained before. One presents this third higher spin-$\frac{5}{2}$ current in two different ways in Appendix C \textcolor{red}{(The index $\mu$ in $SO(4)$ basis is 3 or 4). They will differ from (3.54) only by the cubic term.}

Let us describe the case of $(\mu, \nu) = (1, 1)$ with $\rho = 0$ in the following subsection.

\section*{3.4.7 The fourth higher spin-$\frac{5}{2}$ current}

Let us determine the fourth higher spin-$\frac{5}{2}$ current. Let us consider the first order pole in (3.55) for the indices $\mu = 1, \nu = 1$ and $\rho = 0$. One has the following identity

$$
\varepsilon^{0123} h^{1}_{cb} h^{3}_{cd} d^{3}_{ab} f^{fd}_{g} f^{g^{d} y_{d} =}
- \frac{1}{4} d^{3}_{ab} h^{3}_{cd} - \frac{1}{4} d^{3}_{ac} h^{3}_{db} + \frac{1}{4} d^{3}_{dc} h^{3}_{ab} + \frac{1}{4} d^{3}_{db} h^{3}_{ac} + \frac{1}{2} d^{3}_{cd} h^{3}_{da} - \frac{1}{4} d^{2}_{ab} h^{2}_{cd} - \frac{1}{4} d^{2}_{cd} h^{2}_{ab} + \frac{1}{4} d^{2}_{db} h^{2}_{ac} + \frac{1}{4} d^{2}_{ac} h^{2}_{db} + \frac{1}{2} d^{2}_{cd} h^{2}_{da} + \frac{1}{4} d^{1}_{cd} h^{1}_{ab} + \frac{1}{2} d^{1}_{cd} h^{1}_{da} - \frac{1}{2} d^{1}_{cd} h^{1}_{da} + \frac{1}{4} d^{1}_{cd} h^{1}_{da} + \frac{1}{4} d^{1}_{cd} h^{1}_{da}.
$$

(3.55)

Among the first five terms of (3.55), the nonzero contribution arises from the third and fourth terms. Therefore, the simple form for the quartic term one obtains is given by

$$
(\frac{1}{2} d^{3}_{cd} h^{3}_{ab} + \frac{1}{2} d^{2}_{cd} h^{2}_{ab} + d^{0}_{ab} h^{0}_{cd}) Q^{a} Q^{b} Q^{c} V^{d}(w).
$$

(3.56)

As before, now one would like to determine the fourth higher spin-$\frac{5}{2}$ current which is a primary. One can focus on the particular OPE of (3.24) with the indices $\mu = 1, \nu = 1$ and

\textcolor{red}{Similarly among the next five terms of (3.55), the nonzero contribution arises from the second, fourth and fifth terms. However, this leads to zero. For the last five terms, one sees that the third and fourth terms do not contribute. The remaining three terms can contribute.}
\( \rho = 0 \). By noting that \( \hat{C}^{1}(z) = G^{3}(z) \) from (2.6) and \( \hat{\Phi}^{(1),10}(w) = -\Phi^{(1),23}(w) \) from (3.21) and (3.23), one can have the following OPE in the \( SO(4) \) manifest basis

\[
G^{3}(z) \Phi^{(1),23}(w) = + \cdots + \frac{1}{(z - w)} \left[ \Phi^{(1),2} - \frac{2i}{(k + N + 2)} T^{13} \Phi^{(1),4} \right] + \frac{2i}{(k + N + 2)} T^{34} \Phi^{(1),1} + \frac{4i}{(k + N + 2)} T^{14} \Phi^{(1),3} - \frac{(N - k)}{3(k + N + 2)} \partial \Phi^{(1),2} \right](w) + \cdots, (3.57)
\]

where the second order pole is ignored. All the hat notations appearing in the right hand side of the OPE (3.21) are gone. See also the corresponding OPE in Appendix (A.1).

Then it is obvious that the fourth higher spin-\( \frac{5}{2} \) current can be read off and is given by

\[
\Phi^{(1),2}(z) = \frac{2i}{(k + N + 2)} T^{13} \Phi^{(1),4} + \frac{2i}{(k + N + 2)} T^{34} \Phi^{(1),1} - \frac{4i}{(k + N + 2)} T^{14} \Phi^{(1),3} + \frac{(N - k)}{3(k + N + 2)} \partial \Phi^{(1),2} - (3.35) \text{ with } \mu = \nu = 1, \rho = 0, \tag{3.58}
\]

where the relation (3.56) is used. Furthermore, one should rewrite the composite expressions appearing in (3.58) in terms of WZW adjoint currents. One obtains

\[
T^{13} \Phi^{(1),4} = -\frac{1}{4N(k + N + 2)} f^{\bar{a}\bar{b}} h_{\bar{a}\bar{b}}^{2} d_{\bar{c}\bar{d}}^{2} Q^{\bar{c}} V^{\bar{d}} - \frac{1}{4(k + N + 2)^{2}} h_{\bar{a}\bar{b}}^{2} d_{\bar{c}\bar{d}}^{2} Q^{\bar{a}} Q^{\bar{b}} Q^{\bar{c}} V^{\bar{d}}
\]

\[
- \frac{1}{2(k + N + 2)} d_{\bar{a}\bar{b}}^{0} \partial Q^{\bar{a}} V^{\bar{b}},
\]

\[
T^{34} \Phi^{(1),1} = -\frac{1}{4N(k + N + 2)} f^{\bar{a}\bar{b}} h_{\bar{a}\bar{b}}^{3} d_{\bar{c}\bar{d}}^{3} Q^{\bar{c}} V^{\bar{d}} - \frac{1}{4(k + N + 2)^{2}} h_{\bar{a}\bar{b}}^{3} d_{\bar{c}\bar{d}}^{3} Q^{\bar{a}} Q^{\bar{b}} Q^{\bar{c}} V^{\bar{d}}
\]

\[
- \frac{1}{2(k + N + 2)} d_{\bar{a}\bar{b}}^{0} \partial Q^{\bar{a}} V^{\bar{b}},
\]

\[
T^{14} \Phi^{(1),3} = \frac{1}{4N(k + N + 2)} f^{\bar{a}\bar{b}} h_{\bar{a}\bar{b}}^{1} d_{\bar{c}\bar{d}}^{1} Q^{\bar{c}} V^{\bar{d}} + \frac{1}{4(k + N + 2)^{2}} h_{\bar{a}\bar{b}}^{1} d_{\bar{c}\bar{d}}^{1} Q^{\bar{a}} Q^{\bar{b}} Q^{\bar{c}} V^{\bar{d}}
\]

\[
+ \frac{1}{2(k + N + 2)} d_{\bar{a}\bar{b}}^{0} \partial Q^{\bar{a}} V^{\bar{b}}. \tag{3.59}
\]

The relations (3.18) and (3.16) are used in the first two and the last one in (3.59) respectively.

Therefore, the fourth higher spin-\( \frac{5}{2} \) current in \( SO(4) \) basis is given by

\[
\Phi^{(1),2}(z) = \frac{i}{(k + N + 2)^{2}} \left[ - \frac{1}{2N} h_{\bar{a}\bar{b}}^{2} d_{\bar{c}\bar{d}}^{2} - \frac{1}{2N} h_{\bar{a}\bar{b}}^{3} d_{\bar{c}\bar{d}}^{3} - \frac{1}{N} h_{\bar{a}\bar{b}}^{1} d_{\bar{c}\bar{d}}^{1} \right]
\]

\( \text{[16]} \) One can check that the quartic term of the first quantity in (3.59) corresponds to the second term of (3.56) and those in the second quantity in (3.59) does to the first term of (3.56). Furthermore, as one moves \( V^{\bar{g}} \) in (3.35) to the left at one step (in order to combine with the cubic terms in (3.59), then the extra derivative terms can be reduced to the second term of (3.36) with different coefficients by using the identities (3.20) and (3.18).
obtains the higher spin-3 current $\Phi^{(1)}_3$. The index $i$ of the higher spin-$i$ can focus on the last term of this expression. Let us calculate the OPE between the spin-$1$ basis. There are no quintic, cubic or linear terms in spin-$5$ before. One presents this fourth higher spin-3 current in two different ways in Appendix C (The index $\mu$ in $SO(4)$ basis is $1$ or $4$). They will differ from (3.60) only by the cubic term.

Then, one has the four higher spin-$\frac{5}{2}$ currents in (3.42), (3.48), (3.54) and (3.60) in $SO(4)$ basis. There are no quintic, cubic or linear terms in spin-$\frac{3}{2}$ current. The next step is to obtain the higher spin-3 current using one of these higher spin-$\frac{5}{2}$ currents.

3.5 Higher spin-3 current

Let us describe how one can obtain the final higher spin-3 current. The following OPE (the first order pole) determines the explicit higher spin-3 current in $SO(4)$ basis

$$G^\mu(z)\Phi^{(1)\nu}_2(w) = \frac{1}{(z-w)^3} \left[ \frac{16(k-N)}{3(2+k+N)} \right] \delta^{\mu\nu} \Phi^{(1)}_0(w)$$

$$+ \frac{1}{(z-w)^2} \left[ -4 \Phi^{(1)\mu\nu}_1 - \frac{2(k-N)}{3(2+k+N)} \epsilon^{\mu\nu\rho\sigma} \Phi^{(1)\rho\sigma}_1 \right](w)$$

$$+ \frac{1}{(z-w)} \left[ \delta^{\mu\nu} ( -\Phi^{(1)}_2 + \frac{16(k-N)}{6(kN+5k+5N+4)} T_0 + \frac{2}{(2+k+N)} \epsilon^{\mu\nu\rho\sigma} G^{\rho} \Phi^{(1)\sigma}_2 \right)$$

$$- \partial \Phi^{(1)\mu\nu}_1 - \frac{(k-N)}{6(2+k+N)} \epsilon^{\mu\nu\rho\sigma} \partial \Phi^{(1)\rho\sigma}_1 + \frac{2i}{(2+k+N)} \epsilon^{\mu\nu\rho\sigma} T^{\rho\sigma} \Phi^{(1)}_0$$

$$- \frac{2i}{(2+k+N)} \epsilon^{\mu\nu\rho\sigma} T^{\rho\sigma} \partial \Phi^{(1)}_0 + \frac{2i}{(2+k+N)} (T^{\mu\nu} \Phi^{(1)\nu\rho}_1 - T^{\nu\rho} \Phi^{(1)\mu\rho}_1) \right](w) + \ldots \quad (3.61)$$

That is, when one looks at the first order pole of this OPE with the condition $\mu = \nu$, one obtains the higher spin-3 current $\Phi^{(1)}_2(w)$ from the explicit calculation of the left hand side of this OPE.

Instead of calculating the left hand side of this OPE directly, one considers some parts of the higher spin-$\frac{5}{2}$ current. For example, the higher spin-$\frac{5}{2}$ current is given by (3.39). One can focus on the last term of this expression. Let us calculate the OPE between the spin-$\frac{3}{2}$ current and the expression from the first order pole in (3.35) using the WZW currents.
3.5.1 The fourth order pole

According to Appendix (B.2), there is no fourth order pole.

3.5.2 The third order pole

Let us calculate the third order pole of the OPE between $\hat{G}^{\mu=i}(z)$ and the expression appearing the first order pole of (3.35) with $\rho = i$ (no sum over the index $i$). It turns out that

$$\frac{i}{(k + N + 2)^2} (3k + N) (k + N + 2) h^i_{ab} d^i_{cd} g^{\bar{c}\bar{a}} f^\bar{b} \bar{d} e V^c + (N - k) h^i_{ab} d^i_{cd} ((k + N + 2) g^{\bar{c}\bar{a}} f^\bar{b} \bar{d} e V^c - 2k g^{\bar{c}\bar{d}} Q^\bar{e} Q^\bar{a})$$

$$+ (2h^0_{ab} h^0_{cf} d^0_{dg} f^\bar{g} f - \frac{1}{2} h^i_{ab} \varepsilon^{0i\alpha\sigma} h^\sigma_{df} d^{\alpha \bar{e}} f^\bar{g} \bar{f} e + \frac{1}{2} h^i_{ab} \varepsilon^{0i\alpha\sigma} h^\sigma_{cf} d^{\alpha \bar{e}} f^\bar{g} \bar{f} e)$$

$$\times (-k + N + 2) g^{\bar{c}\bar{a}} (k g^{\bar{b} \bar{d}} V^e + f^\bar{f} h f^{\bar{e} \bar{g}} i_v + k g^{\bar{b} \bar{e}} V^d) - k f^\bar{d} h g^{\bar{e} \bar{h}} Q^\bar{e} Q^\bar{h} - k g^{\bar{c}\bar{a}} g^{\bar{b} \bar{f}} Q^\bar{e} Q^\bar{f} - k g^{\bar{c}\bar{a}} g^{\bar{b} \bar{f}} Q^\bar{e} Q^\bar{f}) \right) (3.62)$$

The first term of (3.62) has $h^i_{ab} g^{\bar{a}\bar{c}} d^i_{cd}$ which is written as $d^0_{bd}$ using (3.18). The second term can be reduced to $d^0_{bd}$ similarly. Let us consider $V^e(w)$ term appearing in the first term in the fourth line of (3.62). The first contribution is related to $-d^i_{gd} g^{\bar{d} \bar{h}} h^i_{ba} g^{\bar{a}\bar{c}} h^0_{cf}$ which is equal to $-d^i_{gd} g^{\bar{d} \bar{h}} h^i_{bf}$. Again this can be written as $h^i_{bf} g^{\bar{b} \bar{d}} d^i_{dg}$. With (3.18), this becomes $-d^0_{fd}$. The second contribution contains $\varepsilon^{0i\alpha\sigma} h^i_{ab} g^{\bar{a}\bar{c}} h^\sigma_{cf} d^{\alpha \bar{e}} d^0_{dg}$. Then the $hgh$ factor is given by $\varepsilon^{i\alpha\sigma} h^0_{bf}$. Then one has $-\varepsilon^{0i\alpha\sigma} \varepsilon^{i\alpha\sigma} h^i_{bf} g^{\bar{b} \bar{e}} d^0_{dg}$ which leads to $2d^0_{fd}$ from (3.18). The third contribution has $-\varepsilon^{0i\alpha\sigma} h^i_{ba} g^{\bar{a}\bar{c}} h^\sigma_{cf} g^{\bar{b} \bar{d}} d^0_{dg}$. From the $hgh$ factor, one can simplify this as $-\varepsilon^{0i\alpha\sigma} \varepsilon^{i\alpha\sigma} h^i_{bf} g^{\bar{b} \bar{d}} d^0_{dg}$. This leads to the final result $2d^0_{fd}$ as before. Let us look at $V^i(w)$ term which is the second term of the fourth line of (3.62). The first contribution has $-h^i_{ab} g^{\bar{a}\bar{c}} h^0_{cf}$ which is given by $-h^i_{bf}$. Together with the factor $d^0_{dg} f^{\bar{b} \bar{d}}$, one obtains $-h^i_{bf} d^i_{dg} f^{\bar{b} \bar{d}}$ which is symmetric under the indices $\bar{f}$ and $\bar{g}$ according to the identity in (3.20). On the other hand, there exists a factor $f^\bar{f} \bar{e}$ which is antisymmetric in the indices $\bar{g}$ and $\bar{f}$. Therefore, there is no contribution at all.

Let us collect the spin-1 dependent terms as follows:

$$-\frac{1}{(k + N + 2)^2} (N + 3k) + (N - k) - 2k - k - k) d^0_{ab} f^{\bar{a} \bar{b}} e V^c = \frac{-2(N - k)}{(k + N + 2)^2} d^0_{ab} f^{\bar{a} \bar{b}} e V^c. \quad (3.63)$$

The second contribution has the factor $-h^i_{ba} g^{\bar{a}\bar{c}} d^i_{cd}$ which is equal to $\varepsilon^{i\alpha\sigma} d^k_{bd}$. Furthermore, one has $\varepsilon^{0i\alpha\sigma} \varepsilon^{i\alpha\sigma} f^{\bar{b} \bar{d}} h^\sigma_{df} d^k_{bd}$ by adding the extra factor. The nonzero contribution arises when the two indices are equal to each other $\sigma = k$. One sees that this is also symmetric in the indices $\bar{f}$ and $\bar{g}$ in the presence of $f^{\bar{f} \bar{e}}$. In this case also there is no contribution. The third contribution has the factor $-h^i_{ba} g^{\bar{a}\bar{c}} h^0_{cf}$ which becomes $-\varepsilon^{i\alpha\sigma} h^k_{bf}$. Furthermore, with other factors one has $-\varepsilon^{0i\alpha\sigma} \varepsilon^{i\alpha\sigma} f^{\bar{b} \bar{d}} h^k_{bf} d^k_{cg}$. Again, the two indices are equal to each other $\sigma = k$ for nonzero contribution. Due to the symmetric property between the indices $\bar{f}$ and $\bar{g}$, there is no contribution. Let us focus on the $V^i(w)$ term which contains $f^{\bar{f} \bar{e}} g^{\bar{g} \bar{e}}$. According to the previous identity in (3.6), there is no contribution.
Now let us consider the spin-$\frac{1}{2}$ dependent terms. The second term of the second line of (3.62) has $h_{ab}^{0} g_{bd}^{0} d_{dc}^{0}$ which is equivalent to $-d_{ac}^{0}$.

The last term of the fourth line of (3.62) contains $d_{d_{g}}^{i} f_{a}^{i j} f_{d_{h}}^{j} g^{e h}$. One has the following identity for the first contribution

$$d_{d_{g}}^{i} f_{a}^{i j} f_{d_{h}}^{j} g^{e h} = -N d_{g h}^{i} g_{d_{g}}^{j} g^{e h}. \quad (3.64)$$

By combining the remaining factor, one obtains $h_{ab}^{0} h_{c}^{j} d_{h}^{0} g_{d_{g}}^{j} g^{e h}$ which becomes $h_{ab}^{0} g_{d_{g}}^{j} d_{h}^{0}$.

By using the identity (3.62), this can be simplified as $-d_{a}^{0}$. For the second contribution, one has the factor $h_{d_{f}}^{j} f_{a}^{i j} f_{d_{h}}^{j} g^{e h}$. By using the identity

$$h_{d_{f}}^{j} f_{a}^{i j} f_{d_{h}}^{j} g^{e h} = -N h_{d_{f}}^{i} g_{d_{f}}^{j} g^{e h}, \quad (3.65)$$

one can simplify further by combining the other factors. Effectively, two $f$ is gone in (3.65).

From the relation (3.17), one sees the factor $-h_{a}^{i} h_{c}^{j} g_{d_{g}}^{j} h_{d}^{0}$ which becomes $-\varepsilon_{ijk} h_{a}^{k}$. Finally one can combine this with the factor $d_{c}^{a}$. Using the relation (3.18), one has $-\varepsilon_{ijk} \varepsilon_{ikj} h_{a}^{k} g_{d_{g}}^{j} d_{c}^{a}$ which leads to the final result $2 d_{a}^{0}$. The third contribution can be calculated as done in the first contribution and it turns out that $-2 d_{a}^{0}$.

For the first term of the last line of (3.62), it is better to consider the particular indices. One can use the relation (3.37) for the factor $\varepsilon h d f f$ when the index \[^{18}\]

$$\rho = i = 1, \quad \text{in } SU(2) \times SU(2) \text{ basis} \quad \text{(or } \rho = i = 3, \quad \text{in } SO(4) \text{ basis}). \quad (3.66)$$

That is, the coefficient tensor of (3.37) is given by

$$\left( \frac{1}{2} d_{f i}^{2} h_{i}^{2} + \frac{1}{2} d_{e f}^{2} h_{e}^{2} - d_{e d}^{0} h_{e f}^{1} \right). \quad (3.67)$$

Then it is obvious to see that with the factor $h_{a}^{i} = 1$ and the factor $g_{ab}^{i} g_{d_{g}}^{j}$, one can simplify the corresponding terms. It turns out that the first term and the third term of (3.67) will contribute to $-\frac{1}{2} d_{d e}^{0}$ and $d_{d e}^{0}$ respectively while the second term doesn’t contribute at all. For the second term of the last line of (3.62), one can analyze similarly. It turns out that the second term and the third term of (3.67) will contribute to $\frac{1}{2} d_{e e}^{0}$ and $d_{e e}^{0}$ respectively while the first term doesn’t contribute at all. For the last term of the last line of (3.62), it turns out that the first term and the second term of (3.67) will contribute to $-\frac{1}{2} d_{c d}^{0}$ and $-\frac{1}{2} d_{c d}^{0}$ respectively. The third term will contribute to $4 N d_{c d}^{0}$ because of $\delta_{a}^{a} = 4 N$.

\[^{18}\text{We will consider this particular indices from now on.}\]
By collecting the all contributions (from spin-$\frac{1}{2}$ dependent parts) with the correct coefficients, one obtains

$$\left[ -2k(N-k) + 2kN + kN + \frac{k}{2} - k(4N+1) \right] d^{0}_{ab} Q^{a} \tilde{Q}^{b} = 2k(N-k) \frac{1}{(k+N+2)} d^{0}_{ab} Q^{a} \tilde{Q}^{b},$$

(3.68)

with the overall factor $\frac{1}{(k+N+2)}$. Then, one obtains the expression from the third order pole as the sum of (3.63) and (3.68). This leads to, by using the expression of (3.1),

$$\frac{4(N-k)}{(k+N+2)} \Phi^{(1)}_{0}(w).$$

(3.69)

Furthermore, the derivative term in (3.39) has the following OPE in $SO(4)$ basis with $G^{3}(z)$

$$G^{3}(z) \partial \Phi^{(1)}_{\frac{3}{2}}(w) = -\frac{1}{(z-w)^{3}} \Phi^{(1)}_{0}(w) - \frac{1}{(z-w)^{2}} \partial \Phi^{(1)}_{0}(w) - \frac{1}{(z-w)} \partial^{2} \Phi^{(1)}_{0}(w) + \cdots.$$  

(3.70)

Therefore, one can check the coefficient of the third order pole of (3.61) by realizing that, together with (3.69) (there is a minus sign in (3.39)),

$$\left[ -\frac{4(N-k)}{(k+N+2)} + \frac{(N-k)}{3(k+N+2)} \times (-4) \right] \Phi^{(1)}_{0}(w) = -\frac{16(N-k)}{3(k+N+2)} \Phi^{(1)}_{0}(w).$$

(3.71)

Then the total contribution in (3.71), from the third order pole (3.62) and the third order pole (3.70), coincides with the one in (3.61). One can easily check that there is no contribution from the OPE between the first three terms from (3.39) and $G^{3}(z)$ current.

So far, the $\mu = \nu = 1$ case in $SU(2) \times SU(2)$ basis (or the $\mu = \nu = 3$ case in $SO(4)$ basis) is considered. What happens for $\mu \neq \nu$ case? For example, when $\mu = 4$ and $\nu = 3$ in $SO(4)$ basis of (3.61) (or $\mu = 2$ and $\nu = 1$ in $SU(2) \times SU(2)$ basis), one expects that there is no contribution in the third order pole. For the quadratic terms in the spin-$\frac{1}{2}$ current, one obtains a symmetric $d^{3}$ tensor in the second and fourth lines of (3.62) where the $h$ tensor has the index 2 and the $d$ tensor or $\varepsilon$ tensor has the index 1. On the other hand, the composite field is antisymmetric. Therefore, there is no contribution. In the last line of (3.62) with above index assignment, each term has three contributions. For the first term, one can use the traceless condition for the $d^{4}$ tensor ($d^{4}_{ab} \epsilon^{ab} = 0$) in the second contribution and there is a symmetric $d^{3}$ tensor in the first and last ones. For the second term, one can use the traceless condition for the $d^{4}$ tensor in the first contribution and there is a symmetric $d^{3}$ tensor in the remaining ones. For the last term, one can use the traceless condition for the $d^{4}$ tensor in the third contribution and there is a symmetric $d^{3}$ tensor in the remaining ones.

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19 One can analyze the linear terms in the spin-1 current. It turns out that there is a symmetric $d^{3}$ tensor which shares the two indices of the structure constant. This leads to zero. Or there exists the factor $hdff$ which becomes identically zero. Furthermore, one can check that there will be no contributions from the first four terms of (3.39).

29
3.5.3 The second order pole

The second order pole (of the OPE between $\hat{G}^{\mu=i}(z)$ and the expression appearing the first order pole of the OPE (3.35) with $\rho = i$ (no sum over the index $i$)) can be described as

\[
\begin{align*}
&\frac{i}{(k+N+2)} \left[ (3k+N) (k+N+2) h_{\alpha \beta}^i d_{\alpha \beta}^j \right. (-(k+N+2) g^{\alpha \bar{a}} V^b V^d - k g^{\bar{a} \bar{b}} \partial Q^c Q^\bar{a}) \\
&+ (N-k) (k+N+2) h_{\alpha \beta}^i d_{\alpha \beta}^j ((k+N+2) g^{\alpha \bar{a}} f_{\bar{d} \bar{e}} \partial V^e - 2 k g^{\bar{d} \bar{b}} Q^\alpha \partial Q^\bar{a} - f_{\bar{d} \bar{e}} Q^c Q^\bar{a} V^e) \\
&+ (2 h_{\alpha \beta}^i h_{\sigma \tau}^0 d_{\sigma \tau}^j f^{\bar{a} \bar{f}} e - \frac{1}{2} h_{\alpha \beta}^i e^{0 \alpha \sigma} h_{\sigma \tau}^0 d_{\sigma \tau}^j f^{\bar{a} \bar{f}} e + \frac{1}{2} h_{\alpha \beta}^i e^{0 \bar{a} \sigma} h_{\sigma \tau}^0 d_{\sigma \tau}^j f^{\bar{a} \bar{f}} e) \\
&\left. \times (g^{\bar{a} \bar{c}} f^{\bar{d} \bar{f}} V^f V^e + (k+N+2) g^{\bar{a} \bar{c}} f_{\bar{e} f} V^d V^f - k g^{\bar{d} \bar{b}} Q^\alpha Q^\bar{a} V^e - f^{\bar{d} \bar{g}} (k g^{\bar{g} \bar{q}} Q^c \partial Q^\bar{a} + f^{\bar{e} f} Q^\bar{c} Q^\bar{d} V^g) - k g^{\bar{d} \bar{b}} Q^\alpha Q^\bar{a} V^e) \right)
\end{align*}
\]

(3.72)

Let us simplify this expression. First of all, let us look at the cubic terms. The last term of the second line in (3.72) has the $hdf$ factor which is symmetric in the indices of $\bar{a}$ and $\bar{c}$ according to the identity (3.20) while the $Q^\bar{a} Q^c (w)$ term is antisymmetric in these indices. Then there is no contribution from this term. Let us consider the last term of fourth line. The factor $h_{\alpha \beta}^i g^{\bar{d} \bar{g}} d_{\alpha \beta}^j$ can be simplified as $-d_{\alpha \beta}^0$. With other factor, this will lead to $-h_{\alpha \beta}^i d_{\alpha \beta}^j f^{\bar{a} \bar{f}} e$ which is symmetric in the indices of $\bar{a}$ and $\bar{c}$ according to the identity (3.20). One has zero contribution from this term also. The second contribution of this term has the factor $h_{\alpha \beta}^i g^{\bar{d} \bar{g}} h_{\sigma \tau}^0$ which is equal to $\varepsilon^{i \sigma k} h_{\alpha \beta}^k$. With other factor, this implies that one has $\varepsilon^{0 \alpha \sigma} \varepsilon^{i \sigma k} h_{\alpha \beta}^k d_{\sigma \tau}^j f^{\bar{a} \bar{f}} e$ which is also symmetric in the indices $\bar{a}$ and $\bar{c}$ and then there is no contribution. Similarly, the third contribution contains the factor $h_{\alpha \beta}^i g^{\bar{d} \bar{g}} d_{\alpha \beta}^j$ which is given by $\varepsilon^{i \sigma k} d_{\alpha \beta}^k$. This gives no contribution by considering other factors also. The second term of fifth line has the following property

\[
h_{\alpha \beta}^i h_{\sigma \tau}^0 d_{\sigma \tau}^j f^{\bar{a} \bar{f}} e f^{\bar{d} \bar{g}} h f^{\bar{c} \bar{h}} j, \quad \text{symmetric in the indices } \bar{a}, \bar{c}.
\]

(3.73)

Moreover, the composite field $Q^c Q^\bar{a} (w)$ is antisymmetric in their indices. The second contribution of this term contains $hhdf f f$ factor which is also symmetric under the indices $\bar{a}$ and $\bar{c}$. There is no contribution from the third contribution with same reason: the $hhdf f f$ factor is also symmetric under the indices $\bar{a}$ and $\bar{c}$. The last term of fifth line vanishes due to the previous relation (3.6).

Let us look at the sixth line. Here the relation (3.37) for the factor $\varepsilon hhdf f$, when the index $\rho = i = 1$ (3.66), can be used as before in (3.67). It is easy to see that by using the
previous property in (3.20), there is no contribution because the composite field \( Q^\alpha \bar{Q}^\beta(w) \) is antisymmetric in their indices. There are first two terms in the seventh line. One can check that there is no contribution from the same analysis.

Let us consider the quadratic terms in the spin-1/2 current. The last term of the first line contains \( d^0_{\bar{a}b} \). Similarly, the second term of the second line contains \( d^0_{\bar{a}b} \). The first term in the fifth line has three contributions. The identity in (3.64) can be used here. The coefficients of the contributions are given by \( 2kN \), \( kN \) and \( kN \) respectively. Then one obtains

\[
-\left[ -k(3k + N) - 2k(N - k) + 2kN + kN + kN \right] d^0_{\bar{a}b} Q^\alpha \partial Q^\beta = -\frac{k(N - k)}{2} \partial (d^0_{\bar{a}b} Q^\alpha Q^\beta), \tag{3.74}
\]

with \( \frac{1}{(k+N+2)^i} \) factor. One observes that there is no contribution from the last term in (3.72).

Let us describe the quadratic terms in the spin-1 current. Note that one can rewrite them in terms of the linear term with derivative by using the following property

\[
d^0_{\bar{a}b} V^\alpha V^\beta = \frac{1}{2} d^0_{\bar{a}b} [V^\alpha, V^\beta] = -\frac{1}{2} d^0_{\bar{a}b} f^{\bar{a}c}_{\bar{b}c} \partial V^c. \tag{3.75}
\]

The first term of (3.72) contains \( d^0_{\bar{a}b} \). Similarly, the first term of the second line has \( d^0_{\bar{a}b} f^{\bar{a}c}_{\bar{b}c} \). The first term of fourth line has the three contributions. There are no contributions in the first one and the last one. For the second contribution, the \( hdf \) factor leads to \( d^0_{\bar{a}b} \) with \( N \). The second term of fourth line has the three contributions. There is no contribution in the second one. For the first and last contributions, the \( hdf \) factor leads to \( d^0_{\bar{a}b} \) with \( N \). By collecting all the contributions, one obtains the final result, with (3.75), as follows:

\[
-\left[ \frac{1}{2} (3k + N) + (N - k) - N - \frac{N}{2} - \frac{N}{2} \right] d^0_{\bar{a}b} f^{\bar{a}c}_{\bar{b}c} \partial V^c = \frac{(N - k)}{2} d^0_{\bar{a}b} f^{\bar{a}c}_{\bar{b}c} \partial V^c, \tag{3.76}
\]

with the \( \frac{1}{(k+N+2)^i} \) factor.

Therefore, the total contribution from the second order pole, the sum of (3.74) and (3.76), is given by the derivative of the higher spin-1 current

\[
-\frac{(N - k)}{(k + N + 2)} \partial \Phi^{(1)}_0(w). \tag{3.77}
\]

Again the explicit form of (3.1) is used here (3.77). One can check that the coefficient of the second order pole of (3.61) vanishes by realizing that

\[
\left[ \frac{(N - k)}{(k + N + 2)} + \frac{(N - k)}{3(k + N + 2)} \times (-3) \right] = 0, \tag{3.78}
\]

where the second term comes from (3.70). Therefore, as we expect, this (3.78) is consistent with (3.61). As before, there is no contribution from the second order pole of the OPE between the first three terms from (3.39) and \( G^3(z) \) current.
So far, the \( \mu = \nu = 3 \) case in \( SO(4) \) basis is considered. What happens for \( \mu \neq \nu \) case? For example, when \( \mu = 4 \) and \( \nu = 3 \) in (3.61), one expects that there is a nontrivial contribution in the second order pole.

First of all, one can calculate the following OPE

\[
-G^4(z) \tilde{\Phi}^{(1),3}_{\frac{3}{2}}(w) = \frac{1}{(z-w)^2} \left[ -\frac{(k-N)}{3(k+N+2)} \Phi^{(1),12}_1 - \frac{6}{(k+N+2)} \Phi^{(1),34}_1 \right. \\
- \frac{4i}{(k+N+2)} T^{12} \Phi^{(1)}_0 \left] (w) + \mathcal{O}\left(\frac{1}{(z-w)}\right) . \right.
\]

(3.79)

Here \( \tilde{\Phi}^{(1),3}_{\frac{3}{2}}(w) \) is introduced by the first four terms in (3.39). The quartic term appearing in the last term of (3.72) corresponds to the one in the last term in the second order pole of (3.79). With the help of the tensor structure appearing in the coefficient of (3.37), one can check that the second term does not contribute and the first and the last terms can contribute. It turns out that one obtains, by multiplying the quartic terms of spin-\( \frac{1}{2} \) current,

\[
-\frac{k}{2(k+N+2)^4} d_{ab} h_{cd} Q^a Q^b Q^c Q^d,
\]

(3.80)

which is exactly same as the one in the last term appearing in the second order pole of (3.79) after some calculation. There is also the derivative term in \( T^{12} \Phi^{(1)}_0(w) \) which contains (3.80).

For the quadratic terms in the spin-\( \frac{1}{2} \) current, one can collect all the contributions and obtain

\[
- \left[ - (3k+N)k + 2k(N-k) - 2kN - k N - k N \right] d_{ab} Q^a \partial Q^b = k(5k+3N) d_{ab} Q^a \partial Q^b,
\]

together with \( \frac{1}{(k+N+2)^2} \) which is exactly same as the corresponding terms in

\[
\frac{(k-N)}{(k+N+2)} \Phi^{(1),12}_1 + \frac{2(1+2k+2N)}{(k+N+2)} \Phi^{(1),34}_1 - \frac{4i}{(k+N+2)} T^{12} \Phi^{(1)}_0 \right]. \]

(3.81)

Note that the factor \( h d g \) in the higher spin-2 currents in (3.81) reduces to the \( d^3 \) tensor \( ^{20} \).

For the quadratic terms in spin-1 current, one has the final expression

\[
\frac{1}{(k+N+2)^2} \left[ 3k + N + 2N + N + 2 \right] d_{ab} V^a V^b = \frac{3(5k+3N)}{(k+N+2)^2} d_{ab} V^a V^b,
\]

(3.82)

which corresponds to the ones for the first two terms in (3.81). There are also some contributions from the first term in the fourth line of (3.72) in addition to (3.82). The first

\[
\left[ \frac{4(k-N)}{3(k+N+2)} \Phi^{(1),12}_1 + 4 \Phi^{(1),34}_1 \right](w).
\]

By combining this with (3.79), one obtains the relation (3.81).
contribution can be simplified as $hdf$ factor which does not vanish but when one multiples the composite field $V^h V^e$, then this becomes zero. Similarly, the second contribution can be simplified as $hdf$ factor which does not vanish but when the composite field $V^h V^e$ is added, then this becomes zero. Finally, the second contribution has nontrivial nonzero expression. In this case, one arrives at the factor $hdf$.

$$d_{gd}^h h_{bf}^i f^{gf} e f^{bd}_h = \frac{1}{2N} h_{bd}^h f_{gf}^i f^{bd}_h - d_{gd}^h h_{bf}^i f^{gf} e f^{bd}_h.$$  \hspace{1cm} (3.83)

By acting on the composite field $V^h V^e$ into (3.83), then the second term vanishes and the remaining one is exactly same as the corresponding one in the last term in (3.81). \footnote{One can also analyze the mixed cubic terms. There is nontrivial factor $hdf$ in front of the composite field $Q_i Q_j V^9(w)$ in the fifth line of (3.72). One expects that this will be written in terms of the factor $hdf$. Although the explicit details for this calculation are ignored in this paper, the similar features will arise at the end of this section.}

Let us consider the final first order pole terms which will generate the higher spin-3 current in terms of WZW currents.

### 3.5.4 The first order pole

The first order pole (of the OPE between $\hat{G}^{\mu=i}(z)$ and the expression appearing the first order pole of the OPE (3.35) with $\rho = i$) can be written as

$$\frac{i}{(k + N + 2)} \left[ \frac{i}{(k + N + 2)^3} \right] (3k + N) \left( (k + N + 2) h_{ab}^i d_{cd}^i ((k + N + 2) (-g^{ca} V^d \partial V^b + \frac{1}{2} g^{ca} f^{bd}_e \partial^2 V^e) - k g^{db} \partial Q^e \partial Q^a - f^{bd}_e Q^e \partial Q^a \partial Q^c V^e) + (N - k) (k + N + 2) h_{ab}^i d_{cd}^i (-k (k + N + 2) g^{ca} V^b \partial V^d - k g^{db} Q^e \partial Q^a - f^{db}_e Q^e \partial (Q^a V^e)) + (2 h_{ab}^i \bar{d}_{ij}^t f^{bi}_k - \frac{1}{2} h_{ab}^i \varepsilon^{0i} f^{bi}_j k + \frac{1}{2} h_{ab}^i \varepsilon^{0i} h_{ij}^t f^{bi}_k) \right) \times (-k (k + N + 2) g^{ca} Q^b V^i k - k g^{db} Q^i \partial Q^a V^k - f^{db}_j Q^i \partial Q^a V^k V^l - f^{db}_j Q^i \partial Q^a V^j V^l - k g^{db} Q^i \partial Q^a V^k V^l - k g^{db} Q^i \partial Q^a V^j V^l) \right] (3.84)

$$h_{ab}^i \varepsilon^{0i} h_{cd}^j \bar{d}_{ij}^t f^{bi}_j (k + N + 2) g^{db} Q^i Q^j \partial Q^a V^m + (k + N + 2) g^{db} Q^i Q^j \partial Q^a V^m + f^{db}_j Q^i \partial Q^a V^m + k g^{db} Q^i Q^j \partial Q^a V^m) \right] (w), \hspace{1cm} \text{no sum over the index } i.$$

Let us simplify this first order pole of (3.84). The first term can be written as $-h_{bd}^i g^{ad} f^{bd}_i$ with other factors. This is equal to $d_{gd}^h$ from (3.18). Similarly, the second term has $d_{bd}^h$. The third term also has $-d_{ac}^i$ from the $hgd$ factor. The fourth term cannot be simplified further.
The fifth term contains $d_{\delta}^{k}$. The sixth term contains $-d_{\delta}^{0}$. The seventh term cannot be simplified further. Note that the term with derivative acting on $V^{\epsilon}$ becomes zero.

Let us look at the cubic in spin-1 currents. The first contribution is given by $h_{i\delta}^{i}$. The second contribution has the factor $-h_{i\delta}^{i} g_{k\delta}^{\epsilon} d_{\delta}^{0}$ which can be written as $-\varepsilon_{iak}^{\epsilon} d_{\delta}^{k}$. Combining with other piece, one obtains $-\varepsilon_{i\alpha \delta}^{\epsilon} \varepsilon_{iak}^{\epsilon} h_{i\delta}^{i} f_{\delta}^{g} f_{k}^{g}$. The third contribution has the factor $-h_{i\delta}^{i} g_{k\delta}^{\epsilon} h_{i\delta}^{k}$ which leads to $-\varepsilon_{iak}^{\epsilon} h_{i\delta}^{k}$. Then with other factor, one has $\varepsilon_{i\alpha \delta}^{\epsilon} \varepsilon_{iak}^{\epsilon} h_{i\delta}^{k} f_{\delta}^{g} f_{k}^{g}$. One can easily see that by considering the additional signs, the final contributions from the second and third terms are equal to each other. Let us move on the next term $Q_{i}^{a} \partial Q^{a} V^{k}(w)$. The first contribution has the factor $h_{i\delta}^{i} g_{k\delta}^{\epsilon} d_{\delta}^{k}$ which is equal to $-d_{\delta}^{0}$. Then one obtains the final form $h_{i\delta}^{i} f_{\delta}^{0} f_{k}^{g} f_{k}^{g}$. One can consider the second contribution. The $h_{i\delta}^{i} g_{k\delta}^{\epsilon} h_{k\delta}^{f}$ which is given by $\varepsilon_{iak}^{\epsilon} h_{i\delta}^{k}$. With other factor, one has $-\varepsilon_{i\alpha \delta}^{\epsilon} \varepsilon_{iak}^{\epsilon} h_{i\delta}^{k} f_{\delta}^{g} f_{k}^{g}$. Similarly, the third contribution can be obtained and one has the factor $h_{i\delta}^{i} g_{k\delta}^{\epsilon} d_{\delta}^{k} f_{\delta}^{g} f_{k}^{g}$. This becomes $\varepsilon_{iak}^{\epsilon} d_{\delta}^{k}$. Then one obtains $\varepsilon_{i\alpha \delta}^{\epsilon} \varepsilon_{iak}^{\epsilon} h_{i\delta}^{k} d_{\delta}^{k} f_{\delta}^{g} f_{k}^{g}$. One realizes that the final contributions from the second and third terms are equal to each other. Let us consider the term $Q_{i}^{a} \partial^{2} Q^{a}(w)$. One can see the factor $d_{\delta}^{k} f_{\delta}^{g} f_{\delta}^{g}$. From the identity

$$d_{\delta}^{k} f_{\delta}^{g} f_{\delta}^{g} = -N d_{\delta}^{k} g_{\delta}^{g} g_{\delta}^{g}, \quad (3.85)$$

one can simplify further. With other factor, one obtains $-N h_{i\delta}^{i} f_{\delta}^{0} f_{\delta}^{g} g_{\delta}^{g}$. Then one has $-N h_{i\delta}^{i} g_{k\delta}^{\epsilon} d_{\delta}^{k}$. The second contribution can be obtained similarly. By realizing that the factor $h_{i\delta}^{i} f_{\delta}^{g} f_{\delta}^{g} g_{\delta}^{k}$ can be written as further simple form, by using the identity

$$h_{i\delta}^{i} f_{\delta}^{g} f_{\delta}^{g} g_{\delta}^{k} = -N h_{i\delta}^{i} g_{\delta}^{g} g_{\delta}^{g} g_{\delta}^{g}, \quad (3.86)$$

with other factor, one has $N \varepsilon_{i\alpha \delta}^{\epsilon} h_{i\delta}^{i} d_{\delta}^{k} h_{i\delta}^{g} g_{\delta}^{g}$. Then from the relation (3.17), one can replace the $h_{i\delta}^{i} g_{k\delta}^{\epsilon} h_{i\delta}^{g} g_{\delta}^{g}$ by $h_{i\delta}^{i} g_{k\delta}^{\epsilon} h_{i\delta}^{g} g_{\delta}^{g}$. The above expression leads to $N \varepsilon_{i\alpha \delta}^{\epsilon} \varepsilon_{iak}^{\epsilon} h_{i\delta}^{k} d_{\delta}^{g} f_{\delta}^{g} f_{k}^{g}$. This is equal to $-2N d_{\delta}^{0}$ from (3.18). One can analyze the third contribution similarly. One can consider the factor $d_{\delta}^{0} f_{\delta}^{g} f_{\delta}^{g} = -N d_{\delta}^{0} g_{\delta}^{g} g_{\delta}^{g}$, with the help of the following identity

$$d_{\delta}^{0} f_{\delta}^{g} f_{\delta}^{g} = -N d_{\delta}^{0} g_{\delta}^{g} g_{\delta}^{g}, \quad (3.87)$$

one obtains $-N \varepsilon_{i\alpha \delta}^{\epsilon} h_{i\delta}^{i} h_{i\delta}^{g} d_{\delta}^{g} g_{\delta}^{g} g_{\delta}^{g}$. Then the $h_{i\delta}^{i} g_{k\delta}^{\epsilon} h_{i\delta}^{g} g_{\delta}^{g}$ which leads to $2N d_{\delta}^{0}$ according to (3.18). The contribution from the term $Q^{a}_{i} \partial Q^{a} V^{j}(w)$ vanishes because of (3.9). It would be interesting to observe the above identities (3.85), (3.86), and (3.87) in the general context.

Let us simplify the first term of the seventh line of (3.84). Again by using the coefficient tensor appearing in (3.31), one obtains the following factor

$$(k + N + 2) \left[ -\frac{1}{2} h_{i}^{2} d_{i}^{2} + \frac{1}{2} h_{i}^{3} d_{i}^{3} + h_{i}^{0} d_{i}^{0} \right] Q^{a}_{i} Q^{b} V^{c} V^{d}. \quad (3.88)$$
It turns out that the first term of the eighth line of (3.84) contains the factor \[ (k + N + 2) \left[ -\frac{1}{2} h_{ba}^3 d_{de}^3 - \frac{1}{2} h_{cd}^3 d_{bd}^3 + h_{ad}^1 d_{bc}^1 \right] Q^a Q^b V^c V^d. \] (3.89)

This (3.89) looks similar to (3.88).

Let us consider the third term of the fifth line. As before, it has three contributions. One cannot simplify them further. Let us look at the first term of the sixth line. According to the property of (3.73), the first contribution occurs when the derivative acts on $Q^a$. The corresponding tensor structure has the following form

\[
\left[ N h_{ca}^1 d_{db}^1 + \left( \frac{1}{4} + \frac{1}{2N} \right) h_{ab}^0 d_{cd}^1 - \frac{1}{4} h_{cd}^0 d_{ab}^0 - \frac{1}{4} h_{ab}^2 d_{cd}^2 - \frac{1}{4} h_{ab}^3 d_{cd}^3 \right] f^{\tilde{a} \tilde{b} \tilde{c} \tilde{d}}, \]

which acts on the composite field $Q^d \partial Q^e V^c(w)$. Of course, the $N$-dependence in here is nontrivial and one can observe by trying to determine from the several $N$ values cases. We will see that the above terms (3.90) contribute to the final higher spin-3 current which will appear in next subsection. Similarly, the second contribution contains

\[
\left[ -N h_{ca}^1 d_{db}^1 - \frac{1}{4} h_{ab}^0 d_{cd}^1 + \frac{1}{4} h_{cd}^0 d_{ab}^0 + \left( \frac{1}{4} + \frac{1}{2N} \right) h_{ab}^2 d_{cd}^2 + \frac{1}{4} h_{ab}^3 d_{cd}^3 \right] f^{\tilde{a} \tilde{b} \tilde{c} \tilde{d}}. \]

This looks like (3.90) but the tensorial structure is different from each other. One can see the $N$-dependence from the explicit expressions for the several $N$ cases. The corresponding quantity for the third contribution is given by

\[
\left[ N h_{ca}^1 d_{db}^1 + \frac{1}{4} h_{ab}^0 d_{cd}^1 - \frac{1}{4} h_{cd}^0 d_{ab}^0 - \frac{1}{4} h_{ab}^2 d_{cd}^2 - \left( \frac{1}{4} + \frac{1}{2N} \right) h_{ab}^3 d_{cd}^3 \right] f^{\tilde{a} \tilde{b} \tilde{c} \tilde{d}}. \]

As in (3.90) and (3.91), the tensorial structure of (3.92) looks similar but is different from those expressions.

Let us move the next term. The first contribution can have the following tensor structure

\[
\begin{align*}
h_{ca}^1 h_{bc}^0 d_{cd}^3 f^{\tilde{c} \tilde{d}} f^{\tilde{b} \tilde{a}} k & = -\frac{1}{4} d_{ab}^1 h_{cd}^3 - \frac{1}{4} d_{ac}^1 h_{db}^3 - \frac{1}{4} d_{bc}^1 h_{ad}^3 + \frac{1}{4} d_{db}^1 h_{ac}^3 + \frac{1}{2} d_{ab}^3 h_{cd}^3 + \frac{1}{2} d_{ac}^3 h_{db}^3 - \frac{1}{2} d_{ab}^2 h_{cd}^2 - \frac{1}{2} d_{ac}^2 h_{db}^2 - \frac{1}{2} d_{bc}^2 h_{ad}^2 - \frac{1}{2} d_{bd}^2 h_{ac}^2 + \frac{1}{4} d_{ab}^3 h_{cd}^3 + \frac{1}{4} d_{ac}^3 h_{db}^3 + \frac{1}{4} d_{bc}^3 h_{ad}^3 - \frac{1}{4} d_{bd}^3 h_{ac}^3 + \frac{1}{4} d_{ab}^0 h_{cd}^0 + \frac{1}{4} d_{ac}^0 h_{db}^0 + \frac{1}{4} d_{bc}^0 h_{ad}^0 + \frac{1}{4} d_{bd}^0 h_{ac}^0.
\end{align*}
\]

\[22\] Similarly, it is easy to see that the second term of the eighth line has the factor $(k + N + 2) \left[ -\frac{1}{2} h_{ba}^3 d_{de}^3 - \frac{1}{2} h_{bd}^3 d_{ce}^3 - h_{ad}^1 d_{bc}^1 \right] Q^a Q^b V^c V^d$. One cannot simplify the third term of the eighth line further. The last term of (3.84) contains the factor \[ -\frac{1}{2} h_{ba}^3 d_{de}^3 + \frac{1}{2} h_{bd}^3 d_{ce}^3 + h_{ad}^1 d_{bc}^1 \] $Q^a Q^b Q^c V^d$. 

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which acts on the composite field \( \bar{Q}^{\bar{b}} Q^a V^c V^d(w) \). One can further simplify this (3.93) using the property of the indices \( \bar{i} \) and \( \bar{a} \).

Therefore, one can summarize the field contents appearing in the first order pole in (3.84). The spin-\( \frac{1}{2} \) dependent terms of (3.84) is given by

\[
\frac{k}{(k + N + 2)^2} (d_{ab}^0 h_{cd}^0 + \frac{1}{2} h_{ab}^0 d_{cd}^1 + \frac{1}{2} h_{ab}^3 d_{cd}^3) Q^{\bar{b}} Q^a Q^c \partial Q^d(z) - \frac{k(3 + N)}{(k + N + 2)^3} d_{ab}^0 Q^{\bar{b}} \partial^2 Q^{\bar{b}}(z) + \frac{k}{(k + N + 2)^3} d_{ab}^0 \partial Q^a \partial Q^{\bar{b}}(z). \tag{3.94}
\]

The expression which depends on the spin-1 current of (3.84) only is given by

\[
\frac{1}{(k + N + 2)^2} (2h_{ab}^1 d_{cd}^1 + h_{ab}^2 d_{cd}^2 + h_{ab}^3 d_{cd}^3) f^{\bar{b} d}_{k} V^a V^c V^d(z) - \frac{4k}{(k + N + 2)^2} d_{ab}^0 V^{\bar{b}} \partial V^{\bar{b}}(z) - \frac{(3 + N)}{2(k + N + 2)^2} d_{ab}^0 f^{\bar{a} c}_{k} \partial^2 V^c(z). \tag{3.95}
\]

The first two terms appearing only in this particular first order pole are new in the sense that they cannot be obtained from the known (higher spin) currents. It is obvious that the last term can be seen from the second derivative of the higher spin-1 current. The remaining mixed terms of (3.84) are given by

\[
- \frac{1}{(k + N + 2)^2} \left( -\frac{1}{2} d_{\bar{a} b}^2 h_{\bar{c} d}^3 + \frac{1}{2} d_{\bar{a} b}^3 h_{\bar{c} d}^3 - \frac{1}{2} d_{\bar{a} b}^0 h_{\bar{c} d}^0 \right) h_{a f}^1 f_{k i}^g Q^{\bar{a}} Q^b Q^c \partial Q^d V^e(z) - \frac{1}{(k + N + 2)^3} \left( -2d_{\bar{a} c}^2 h_{\bar{d} b}^3 + 2d_{\bar{a} d}^3 h_{\bar{c} b}^3 - 2d_{\bar{a} c}^2 h_{\bar{d} b}^2 + 2d_{\bar{a} d}^1 h_{\bar{c} b}^1 \right) d_{a b}^1 \partial h_{\bar{d} c}^1 - 2d_{\bar{a} d}^0 h_{\bar{b} c}^1 - 2d_{\bar{a} d}^0 h_{\bar{b} c}^1 - 2d_{\bar{a} d}^0 h_{\bar{b} c}^1 - 2d_{\bar{a} d}^0 h_{\bar{b} c}^1.
\]

Similarly, one obtains the following structure for the second contribution

\[
\begin{align*}
& h_{\bar{a} b}^l, h_{\bar{a} c}^l, d_{\bar{a} d}^3 f^{\bar{c} a}_{k} f^{\bar{c} b}_{d} = \\
& \quad \frac{1}{4} d_{\bar{a} b}^1 h_{\bar{c} d}^3 + \frac{1}{4} d_{\bar{a} b}^3 h_{\bar{c} d}^3 + \frac{1}{2} d_{\bar{a} b}^2 h_{\bar{c} d}^3 + \frac{1}{2} d_{\bar{a} b}^3 h_{\bar{c} d}^3 + \frac{1}{4} d_{\bar{a} b}^0 h_{\bar{c} d}^0 + \frac{1}{4} d_{\bar{a} b}^3 h_{\bar{c} d}^0 + \frac{1}{2} d_{\bar{a} b}^0 h_{\bar{c} d}^0 + \frac{1}{2} d_{\bar{a} b}^0 h_{\bar{c} d}^0 \\
& - \frac{1}{4} d_{\bar{a} b}^1 h_{\bar{c} d}^3 + \frac{1}{4} d_{\bar{a} b}^3 h_{\bar{c} d}^3 + \frac{1}{2} d_{\bar{a} b}^2 h_{\bar{c} d}^3 + \frac{1}{2} d_{\bar{a} b}^3 h_{\bar{c} d}^3 + \frac{1}{4} d_{\bar{a} b}^0 h_{\bar{c} d}^0 + \frac{1}{4} d_{\bar{a} b}^3 h_{\bar{c} d}^0 + \frac{1}{2} d_{\bar{a} b}^0 h_{\bar{c} d}^0 + \frac{1}{2} d_{\bar{a} b}^0 h_{\bar{c} d}^0.
\end{align*}
\]

which acts on the composite field \( \bar{Q}^{\bar{b}} Q^a V^c V^d(w) \). Finally, the third contribution contains the following tensor structure

\[
\begin{align*}
h_{\bar{a} b}^l, h_{\bar{a} c}^l, d_{\bar{a} d}^3 f^{\bar{c} a}_{k} f^{\bar{c} b}_{d} = & - \frac{1}{4} d_{\bar{a} b}^1 h_{\bar{c} d}^3 - \frac{1}{4} d_{\bar{a} b}^3 h_{\bar{c} d}^3 - \frac{1}{4} d_{\bar{a} b}^0 h_{\bar{c} d}^0 + \frac{1}{4} d_{\bar{a} b}^3 h_{\bar{c} d}^0 + \frac{1}{4} d_{\bar{a} b}^3 h_{\bar{c} d}^0 + \frac{1}{2} d_{\bar{a} b}^2 h_{\bar{c} d}^2 - \frac{1}{2} d_{\bar{a} b}^2 h_{\bar{c} d}^2 - \frac{1}{4} d_{\bar{a} b}^1 h_{\bar{c} d}^1 - \frac{1}{2} d_{\bar{a} b}^1 h_{\bar{c} d}^1 - \frac{1}{4} d_{\bar{a} b}^1 h_{\bar{c} d}^1 - \frac{1}{2} d_{\bar{a} b}^1 h_{\bar{c} d}^1.
\end{align*}
\]

which acts on the composite field \( \bar{Q}^{\bar{b}} Q^a V^c V^d(w) \). The \( h f f \) factor is gone.
Recall that the expression appearing in (3.84) was obtained from the OPE between the spin-3 current and this quantity (3.98). By realizing the defining OPEs in Appendix 2, although the higher spin-3 current is not involved.

\[
\begin{align*}
\Phi^{(1),3}_{\frac{3}{2}} &\equiv -\frac{2i}{(k+N+2)} T^{12} \Phi^{(1),4}_{\frac{3}{2}} - \frac{2i}{(k+N+2)} T^{24} \Phi^{(1),1}_{\frac{3}{2}} + \frac{4i}{(k+N+2)} T^{14} \Phi^{(1),2}_{\frac{3}{2}} \\
&\quad + \frac{(N-k)}{3(k+N+2)} \partial \Phi^{(1),3}_{\frac{3}{2}}.
\end{align*}
\] (3.98)

In order to determine the higher spin-3 current, one should also calculate the OPE between the spin-3 current and this quantity (3.98). By realizing the defining OPEs in Appendix 2, one can calculate the following OPE

\[
\begin{align*}
\Phi^{(1),3}_{\frac{3}{2}}(w) &\equiv \frac{1}{(z-w)^3} \left[ \frac{4(k-N)}{3(k+N+2)} \right] \Phi^{(1)}_0(w) + \frac{1}{(z-w)^2} \left[ \frac{(k-N)}{(k+N+2)} \right] \partial \Phi^{(1)}_0(w) \\
&\quad + \frac{1}{(z-w)} \left[ -\frac{2i}{(k+N+2)} T^{12} \Phi^{(1),12} + \frac{2i}{(k+N+2)} T^{24} \Phi^{(1),24} \\
&\quad - \frac{4i}{(k+N+2)} T^{14} \Phi^{(1),14} + \frac{(k-N)}{3(k+N+2)} \partial^2 \Phi^{(1)}_0(w) \right] + \cdots.
\end{align*}
\] (3.99)
Let us focus on the first order pole of (3.99). On the other hand, according to the first order pole of (3.61) when $\mu = \nu = 3$ in $SO(4)$ basis, one obtains the very simple terms
\[
-\Phi_2^{(1)} + \frac{16(k - N)}{(4 + 5k + 5N + 6kN)} \Phi_0^{(1)} T. \tag{3.100}
\]
Compared to the one [38] in $SU(2) \times SU(2)$ basis, there are not too many composite fields except the higher spin-3 current itself in this pole. This implies that it turns out the higher spin-3 current can be described as
\[
\Phi_2^{(1)}(z) = \Phi_2^{(1)}(z) + \frac{16(k - N)}{(4 + 5k + 5N + 6kN)} \Phi_0^{(1)} T(z) + \frac{2i}{(k + N + 2)} T^{12} \Phi_1^{(1), 12}(z) \tag{3.101}
\]
where the last four terms come from the first order pole of (3.99). Therefore, the remaining things are to write down and simplify the last five terms appearing in (3.101) using the WZW currents. One should use the explicit expressions in (2.3), (3.1) and (3.23).

Let us consider the third term of (3.101). One should write down the known (higher spin) currents in terms of WZW currents. The nontrivial parts arise from the quadratic term in $T^{12}(z)$ and the spin-$\frac{1}{2}$ dependent parts in $\Phi_1^{(1), 12}(z)$. One should use the following identity from the normal ordering product [7], [7], [7], [24]
\[
(Q^a Q^b)(Q^f \partial Q^d) = Q^a Q^b Q^f \partial Q^d - [Q^a Q^b] Q^f, Q^d + [Q^f \partial Q^d, Q^a] Q^b + Q^a [Q^f \partial Q^d, Q^b] = Q^a Q^b Q^f \partial Q^d + (k + N + 2) \left( \frac{1}{2} g^{ad} Q^f \partial^2 Q^b - g^{af} Q^d \partial Q^b \right) + \frac{1}{2} g^{bd} \partial^2 Q^a Q^f - g^{bf} Q^a \partial Q^d. \tag{3.102}
\]
In this case, there are no anticommutators and the signs are coming from the usual bosonic case. In the second line, one substitutes the various commutators and obtains the final result It turns out that the following composite field can be expressed in terms of WZW currents
\[
T^{12} \Phi_1^{(1), 12}(z) = -i \frac{4N(k + N + 2)}{4} h_{ab} d_{cd}^{3} f_{a b c}^{3} e V^c V^d(z) + i \frac{k}{4N(k + N + 2)} h_{ab} d_{cd}^{3} f_{a b c}^{3} e V^c \partial Q^d V^e (z) \nonumber
\]
\[
+ i \frac{4N(k + N + 2)}{4(k + N + 2)^2} h_{ab} d_{cd}^{3} f_{a b c}^{3} e f_{b e h}^{3} Q^f Q^d V^c V^h(z) + \frac{i}{4} \frac{4N(k + N + 2)}{4(k + N + 2)^2} h_{ab} d_{cd}^{3} Q^a Q^b V^c V^d(z) \nonumber
\]

24 Similarly, the following identity is also used
\[
(Q^a Q^b)(Q^f Q^d V^h) = Q^a Q^b Q^f Q^d V^h - [Q^a Q^b] Q^f Q^d V^h, Q^a Q^b + [Q^f Q^d V^h, Q^a] Q^b + Q^a [Q^f Q^d V^h, Q^b] = Q^a Q^b Q^f Q^d V^h + (k + N + 2) \left( g^{bd} \partial Q^a Q^f V^h + g^{bf} Q^d \partial Q^b Q^f V^h - g^{ad} \partial Q^b Q^f V^h - g^{bf} Q^a \partial Q^d Q^f V^h \right), \tag{3.103}
\]
for the other nontrivial part of the composite field. Again, there are no anticommutators in the first line of this normal ordered product (3.103). One should calculate the various rather complicated OPEs.
In doing this calculation, the identities (3.16) and (3.17) are used. There are also three kinds of composite fields: spin-$\frac{1}{2}$ dependent part, spin-1 dependent part and mixed ones.

Similarly, one can obtain the following result for the fourth term of (3.101) as follows:

$$T^{14} \Phi_{1}^{(1),14}(z) = -\frac{i}{4N(k + N + 2)} h_{ab}^{1} d_{cd}^{1} f_{i}^{ab} e V^{i} V^{d}(z) - \frac{i}{4N(k + N + 2)} h_{ab}^{1} d_{cd}^{1} f_{i}^{ab} e Q^{i} \partial Q^{d} V^{c}(z)$$

$$+ \frac{i}{4N(k + N + 2)} h_{ab}^{1} d_{cd}^{1} f_{i}^{ab} e V^{i} V^{d}(z) - \frac{i}{4N(k + N + 2)} h_{ab}^{1} d_{cd}^{1} f_{i}^{ab} e Q^{i} \partial Q^{d} V^{c}(z)$$

$$- \frac{i}{2(k + N + 2)^{2}} h_{de}^{1} f_{i}^{de} h Q^{i} Q^{d} Q^{h}(z) + \frac{i}{2(k + N + 2)^{2}} h_{de}^{1} f_{i}^{de} h \partial Q^{i} Q^{d} V^{h}(z).$$

(3.104)

where the identities (3.16) and (3.17) can be used. Finally, one obtains the following expression for the fifth term of (3.101):

$$T^{24} \Phi_{1}^{(1),24}(z) = -\frac{i}{4N(k + N + 2)} h_{ab}^{1} d_{cd}^{1} f_{i}^{ab} e V^{i} V^{d}(z) - \frac{i}{4N(k + N + 2)} h_{ab}^{1} d_{cd}^{1} f_{i}^{ab} e Q^{i} \partial Q^{d} V^{c}(z)$$

$$- \frac{i}{4N(k + N + 2)} h_{ab}^{1} d_{cd}^{1} f_{i}^{ab} e V^{i} V^{d}(z) - \frac{i}{4N(k + N + 2)} h_{ab}^{1} d_{cd}^{1} f_{i}^{ab} e Q^{i} \partial Q^{d} V^{c}(z)$$

$$- \frac{i}{2(k + N + 2)^{2}} h_{de}^{1} f_{i}^{de} h Q^{i} Q^{d} Q^{h}(z) + \frac{i}{2(k + N + 2)^{2}} h_{de}^{1} f_{i}^{de} h \partial Q^{i} Q^{d} V^{h}(z).$$

(3.105)

Here the identities (3.16) and (3.17) can be used to simplify the various expressions. Let us move on the second term of (3.101). In order to determine the composite field $\Phi_{0}^{(1)} T$ (from (3.1), (2.3) and (2.5)) in terms of WZW currents, one should use the following identity

$$(Q^{a} Q^{b})(Q^{c} Q^{d} Q^{e} Q^{f}) = Q^{a} Q^{b} Q^{c} Q^{d} Q^{e} Q^{f} - [Q^{c} Q^{d} Q^{e} Q^{f}, Q^{a} Q^{b}]$$

$$+ [Q^{c} Q^{d} Q^{e} Q^{f}, Q^{a}] Q^{b} + Q^{a} [Q^{c} Q^{d} Q^{e} Q^{f}, Q^{b}]$$

$$= Q^{a} Q^{b} Q^{c} Q^{d} Q^{e} Q^{f} + (k + N + 2) (g^{ab} \partial Q^{c} Q^{d} Q^{e} Q^{f} - g^{ab} \partial Q^{a} Q^{b} Q^{d} Q^{e} Q^{f})$$

$$- g^{ab} \partial Q^{a} Q^{b} Q^{c} Q^{d} Q^{f} + g^{ab} \partial Q^{a} Q^{b} Q^{c} Q^{d} Q^{f} + g^{ab} \partial Q^{a} Q^{b} Q^{c} Q^{d} Q^{f}$$

$$- g^{ab} \partial Q^{a} Q^{b} Q^{c} Q^{d} Q^{f} + g^{ab} \partial Q^{a} Q^{b} Q^{c} Q^{d} Q^{f} + g^{ab} \partial Q^{a} Q^{b} Q^{c} Q^{d} Q^{f}).$$

(3.107)

as well as (3.102), and (3.103). In obtaining this expression (3.107), one should calculate the various complicated OPEs from which the commutators can be determined [73]. It turns out
that the composite field is

\[
\Phi_0^{(1)}(T(z) = -\frac{k}{32(k + N + 2)^3} \partial_{\alpha} h^i_{\alpha \beta} h^j_{\alpha \beta} Q^a Q^b Q^\alpha Q^\beta Q^\alpha Q^\beta \partial Q^\alpha Q^\beta
\]

\[
- \frac{k}{4(k + N + 2)^4} \partial_{\alpha} h^i_{\alpha \beta} \partial \partial Q^\alpha Q^\beta
\]

\[
- \frac{1}{4(k + N + 2)^2} \partial_{\alpha} h^i_{\alpha \beta} \partial Q^\alpha Q^\beta Q^\alpha Q^\beta \partial Q^\alpha Q^\beta V^\alpha V^\beta
\]

\[
+ \frac{1}{32(k + N + 2)^3} \partial_{\alpha} h^i_{\beta \alpha} h^j_{\beta \alpha} \partial Q^\alpha Q^\beta Q^\alpha Q^\beta \partial Q^\alpha Q^\beta V^V^\alpha V^\alpha + \frac{k}{4(k + N + 2)^3} \partial_{\alpha} h^i_{\beta \alpha} h^j_{\beta \alpha} \partial Q^\alpha Q^\beta Q^\alpha Q^\beta \partial Q^\alpha Q^\beta V^V^\alpha V^\alpha.
\)

Because the composite field \( \Phi_0^{(1)}(T(z) \) is written in terms of adjoint spin-1, \( \frac{1}{2} \) currents, one obtains some eigenvalues for the higher spin-1 current in the higher representations indirectly (the conformal dimension, which is the zeromode eigenvalue of \( T(z) \), in any representations are known explicitly). In next section, we will use this property very frequently.

From the previous results, \((3.101), (3.104), (3.105), (3.106), (3.108), (3.97)\) and second derivative term of the higher spin-1 current \((3.1)\), one obtains the final higher spin-3 current which consists of the three independent pieces as follows:

\[
\Phi_2^{(1)}(z) = \Phi_2^{(1),Q}(z) + \Phi_2^{(1),V}(z) + \Phi_2^{(1),Q,V}(z).
\)

Here the higher spin-3 current which depends on the spin-1, \( \frac{1}{2} \) current is given by

\[
\Phi_2^{(1),Q}(z) = -\frac{k(k - N)}{2(4 + 5k + 5N + 6kN)(k + N + 2)^5} \partial_{\alpha} h^i_{\alpha \beta} h^j_{\alpha \beta} Q^a Q^b Q^\alpha Q^\beta \partial Q^\alpha Q^\beta
\]

\[
+ \frac{k(4 + 11k + 4k^2 - N + 2kN)}{(4 + 5k + 5N + 6kN)(k + N + 2)^4} \partial_{\alpha} h^i_{\alpha \beta} \partial Q^\alpha Q^\beta \partial Q^\alpha Q^\beta
\]

\[
+ \frac{k(4 + k + 9N + 6kN)}{(4 + 5k + 5N + 6kN)(k + N + 2)^4} \partial_{\alpha} h^i_{\alpha \beta} h^j_{\alpha \beta} \partial Q^\alpha Q^\beta \partial Q^\alpha Q^\beta
\]

\[
- \frac{2k(12 + 14k + 4k^2 + 28N + 39kN + 12k^2N + 5N^2 + 6kN^2)}{3(4 + 5k + 5N + 6kN)(k + N + 2)^3} \partial_{\alpha} Q^a \partial Q^b
\]

\[
+ \frac{4k(12 + 14k + 4k^2 + 28N + 39kN + 12k^2N + 5N^2 + 6kN^2)}{3(4 + 5k + 5N + 6kN)(k + N + 2)^3} \partial_{\alpha} Q^a \partial Q^a Q^b
\].
The first term of this expression originates from (3.108). The second and third terms come from the one in (3.94), the corresponding term in (3.108), those in (3.104), (3.105) and (3.106). The remaining derivative terms originate from the ones in (3.94), (3.108), (3.104), (3.105), (3.106) and the second derivative of the higher spin-1 current. All the coefficients which depend on $N$ and $k$ appearing in (3.110) are very important to obtain the eigenvalues in the Wolf space coset representations in next section. Furthermore, the fourth term of (3.110) will play an important role for the eigenvalues under the large ($N, k$) ’t Hooft-like limit.

The higher spin-3 current which depends on the spin-1 current is given by

$$
\Phi^{(1)}_{2,V}(z) \equiv \frac{16(k - N)}{(4 + 5k + 5N + 6kN)} \left[ -\frac{1}{4(k + N + 2)^2} d_{ab}^0 f_{abc} h_{de}^i V^c V^d V^\bar{e} 
+ \frac{1}{32N^2(k + N + 2)^2} d_{ab}^0 f_{abc}^i h_{de}^f h_j^i f_{jde} h_{\bar{h}h} f_{\bar{h}h} V^c V^f V^j \right](z) 
+ \frac{1}{2N(k + N + 2)^2} (2h_{ab}^1 d_{\bar{c}d}^2 + h_{ab}^2 d_{\bar{c}d}^2 + h_{ab}^3 d_{\bar{c}d}^2) f_{\bar{a}e} V^e V^\bar{c} V^d(z) 
+ \frac{1}{(k + N + 2)^2} (2h_{ab}^1 d_{\bar{c}d}^2 + h_{ab}^2 d_{\bar{c}d}^2 + h_{ab}^3 d_{\bar{c}d}^2) f_{\bar{a}e}^i k V^\bar{a} V^e V^k(z) 
+ \frac{4k}{(k + N + 2)^2} d_{ab}^0 V^\bar{a} \partial V^b(z) - \frac{2(2k + N)}{3(k + N + 2)^2} d_{ab}^0 f_{\bar{a}e} c \partial^2 V^e(z). (3.111)
$$

The first two terms of this expression come from (3.108). The third term comes from the corresponding terms in (3.104), (3.105) and (3.106). The fourth term can be seen from (3.95). The fifth term originates from the one in (3.95). The last term was combined from the one in (3.95) and the derivative term of the higher spin-1 current. Note that the composite fields in the fourth and the fifth terms occur only in the first order pole (3.84). The other ones can be seen from the known composite fields made of the known (higher spin) currents. All the coefficients depending on $N$ and $k$ in (3.111) are crucial to obtain the eigenvalues in the Wolf space coset representations in next section. Furthermore, the fourth, the fifth and the sixth terms of (3.111) will play an important role for the eigenvalues under the large ($N, k$) ’t Hooft-like limit.

The higher spin-3 current which contain the remaining mixed terms is given by

$$
\Phi^{(3)}_{2,Q,V}(z) \equiv \frac{16(k - N)}{(4 + 5k + 5N + 6kN)} \left[ -\frac{1}{32(k + N + 2)^4} d_{ab}^0 h_{de}^i h_{a\bar{g}}^j f_{\bar{a}e}^i Q^d Q^f Q^\bar{g} V^c 
+ \frac{k}{4(k + N + 2)^4} d_{ab}^0 h_{de}^i h_{a\bar{g}}^j f_{\bar{a}e}^i Q^d Q^f Q^\bar{g} V^c 
- \frac{k}{16N(k + N + 2)^4} d_{ab}^0 h_{de}^i h_{a\bar{g}}^j f_{\bar{a}e}^i Q^d Q^f Q^\bar{g} Q^d V^c \right](z) 
+ \frac{1}{2(2k + N + 2)^4} (h_{ab}^1 d_{\bar{c}d}^2 f_{\bar{a}e}^i - 2h_{ab}^1 h_{de}^i d_{\bar{c}d}^2 - h_{ab}^2 h_{de}^i d_{\bar{c}d}^2 f_{\bar{a}e}^i) f_{\bar{a}e}^i Q^d Q^f Q^\bar{g} V^c(z)
$$

41
\[- \frac{1}{(k + N + 2)^4} \left( - \frac{1}{2} d_{bc}^2 h_{cd}^3 + \frac{1}{2} d_{ab}^3 h_{cd}^2 - h_{bc}^1 d_{cd}^0 h_{ab}^1 f_{\hat{g} e} Q^a Q^b Q^c V^e(z) \right) \]
\[+ \frac{16(k - N)}{(4 + 5k + 3N + 6kN)^3} \left[ - \frac{1}{4(k + N + 2)^3} d_{ab}^0 h_{ef}^0 h_{gh} f_{\hat{g} e} f_{\hat{d} f} Q^3 Q^b V^c V^d h + \frac{k}{4(k + N + 2)^3} d_{ab}^0 h_{ef} Q^a Q^b V^e V^d \right] \]
\[- \frac{k}{32N^2(k + N + 2)^3} d_{ab}^0 h_{ef}^1 f_{\hat{g} e} f_{\hat{d} f} h Q^a Q^b V^c V^d h \left( z \right) \]
\[+ \frac{1}{2N(k + N + 2)^3} \left( - \frac{1}{2} h_{bc}^1 d_{cd}^0 + h_{ab}^1 d_{cd}^3 - h_{bc}^1 h_{cd}^3 f_{\hat{g} e} f_{\hat{d} f} Q^3 Q^b V^c V^d h \left( z \right) \right) \]
\[- \frac{1}{(k + N + 2)^3} \left( - \frac{1}{2} d_{bc}^3 h_{cd}^0 h_{bc}^1 d_{cd}^3 + h_{bc}^1 h_{cd}^3 f_{\hat{g} e} f_{\hat{d} f} h Q^a Q^b V^c V^d h \left( z \right) \right) \]
\[+ \frac{1}{2N(k + N + 2)^3} h_{ab}^0 h_{ef}^0 d_{ef} f_{\hat{d} f} h Q^a Q^b V^c V^d h \left( z \right) \]
\[+ \frac{k}{(k + N + 2)^3} h_{ab}^0 h_{ef}^0 f_{\hat{d} f} h Q^a Q^b V^c V^d \left( z \right) \]
\[- \frac{1}{(k + N + 2)^3} \left( - \frac{1}{2} h_{bc}^1 d_{cd}^0 + h_{ab}^1 d_{cd}^3 - h_{bc}^1 h_{cd}^3 f_{\hat{g} e} f_{\hat{d} f} h Q^a Q^b V^c V^d h \left( z \right) \right) \]
\[+ \frac{k}{4N(k + N + 2)^3} d_{ab}^0 h_{ef}^1 f_{\hat{g} e} Q^a Q^b \partial Q^c V^e \left( z \right) \]
\[- \frac{k}{(k + N + 2)^3} d_{ab}^0 h_{ef}^1 f_{\hat{g} e} Q^a \partial Q^b V^e \left( z \right) \]
\[- \frac{k}{(k + N + 2)^3} \left( - h_{bc}^1 d_{cd}^0 + h_{ab}^1 d_{cd}^3 - h_{bc}^1 h_{cd}^3 f_{\hat{g} e} Q^a \partial Q^b V^e \left( z \right) \right) \]
\[+ \frac{k}{(k + N + 2)^3} \left( - h_{bc}^1 d_{cd}^0 + h_{ab}^1 d_{cd}^3 - h_{bc}^1 h_{cd}^3 f_{\hat{g} e} Q^a \partial Q^b V^e \left( z \right) \right) \]
\[- \frac{k}{(k + N + 2)^3} \left( - \frac{1}{2} \right) h_{ab}^0 h_{ef}^1 f_{\hat{g} e} + \frac{1}{N} h_{ab}^1 d_{cd}^3 f_{\hat{g} e} + h_{cd}^1 h_{ab}^0 f_{\hat{d} f} \left( z \right) \]
\[+ \left( 1 + \frac{1}{2N} \right) h_{ab}^0 d_{cd}^3 f_{\hat{g} e} + \left( 1 + \frac{1}{2N} \right) h_{ab}^0 d_{cd}^3 f_{\hat{d} f} \right] Q^a \partial Q^b V^e \left( z \right). \] (3.112)

First of all, the expressions having the factor 16(k – N) are coming from the ones in (3.108). The fourth and the ninth lines of (3.112) can be seen from the ones in (3.104), (3.105) and (3.106). The fifth line comes from the first term of (3.96). The tenth and eleventh lines are coming from the ones in (3.104), (3.105) and (3.106) and the corresponding terms in (3.96). The twelfth to fourteenth lines are the same as the ones in (3.96). The fourth and fifth lines from the below of (3.112) are the same as the ones in (3.104), (3.105) and (3.106). Finally, the last three lines come from the ones in (3.96). There are terms which are obtained from the first order pole in (3.84). For the eigenvalue calculation in next section, this mixed terms
do not contribute to the eigenvalues for the minimal (and higher) representations.

It would be interesting to determine the higher spin-3 current in the basis of $38$ and see any differences with the above higher spin-3 current.

### 3.5.6 Other way of obtaining the higher spin-3 current using the higher spin-\(\frac{5}{2}\) current (3.42)

One can determine the above higher spin-3 current by considering the higher spin-$\frac{5}{2}$ current (3.42) directly. In order to use previous relations, one can move $V^e$ appearing in the cubic terms of (3.42) to the right. Then one sees that the extra terms are given by $-f^{e}{}_{g} Q^d \partial V^g$. One can check the following identity, by multiplying the structure constant to the cubic term of (3.42),

$$
\begin{align*}
& - \left[ \frac{1}{2N} h^3_{ab} d^2_{de} - \frac{1}{2N} h^2_{ab} d^3_{de} + \frac{1}{N} h^1_{ab} d^0_{de} - 2 h^0_{bd} d^1_{ae} - 2 h^2_{bd} d^3_{ae} + h^0_{bd} d^3_{ae} \right] f^{\alpha \beta}_{c} f^{e}{}_{g} \\
& = (1 + 1 + 2 - 2N - N - N) d_{i j}^{1} = 4(1 - N) d_{i j}^{1}.
\end{align*}
$$

(3.113)

Then one can write down the (intermediate) first order pole in the OPE between $G^a(z)$ and $\Phi^{(1),3} (w)$ which should be equal to (3.100). By adding the contribution from (3.113), the first order pole of the OPE between $G^a(z)$ and $\Phi^{(1),3} (w)$ is given by

$$
\begin{align*}
& - \frac{1}{(k + N + 2)^2} \left[ \frac{1}{2N} h^3_{ab} d^2_{de} - \frac{1}{2N} h^2_{ab} d^3_{de} + \frac{1}{N} h^1_{ab} d^0_{de} - 2 h^0_{bd} d^1_{ae} - 2 h^2_{bd} d^3_{ae} + h^0_{bd} d^3_{ae} \right] \\
& \times f^{\alpha}{}_{k} f_{ij}^{1} \left[ - (k + N + 2) g^{\tilde{a}} V^{\tilde{b}} \partial V^{\tilde{c}} V^{k} - k g^{\tilde{a}} Q^{\tilde{c}} \partial Q^{\tilde{b}} V^{k} - f^{\tilde{b} \tilde{c}} Q^{\tilde{b}} \partial Q^{\tilde{c}} V^{k} V^{l} \\
& - f^{\tilde{b} \tilde{c}} f_{ij}^{kl} Q^{\tilde{b}} \partial (Q^{\tilde{a}} V^{m}) - f^{\tilde{b} \tilde{c}} f_{ij}^{kl} Q^{\tilde{b}} \partial Q^{\tilde{a}} V^{k} V^{l} - \frac{k}{2} f^{\tilde{b} \tilde{c}} f_{ij}^{kl} Q^{\tilde{b}} \partial^{2} Q^{\tilde{a}} - k g^{\tilde{a}} Q^{\tilde{b}} \partial Q^{\tilde{a}} V^{j} \right] \\
& + \frac{1}{(k + N + 2)^2} 4(1 - N) h_{i j}^{1} d_{cd}^{i} \\
& \times \left[ - (k + N + 2) g^{\tilde{a}} V^{\tilde{b}} \partial V^{\tilde{c}} - k g^{\tilde{a}} Q^{\tilde{c}} \partial^{2} Q^{\tilde{a}} - f^{\tilde{b} \tilde{c}} Q^{\tilde{c}} \partial (Q^{\tilde{a}} V^{e}) \right] \\
& + \frac{4(3 + 2k + N)}{3(k + N + 2)^2} h_{i j}^{1} d_{cd}^{i} \\
& \times \left[ (k + N + 2) (g^{\tilde{a}} V^{\tilde{d}} \partial V^{\tilde{b}} + \frac{1}{2} g^{\tilde{a}} f^{b e} \partial \partial^{2} V^{e}) - k g^{\tilde{a}} Q^{\tilde{c}} \partial Q^{\tilde{a}} - f^{b e} Q^{b} \partial Q^{e} \partial V^{e} \right] \\
& - \frac{1}{(k + N + 2)^2} \left[ d_{ij}^{1} h_{ik}^{3} + d_{ij}^{3} h_{jk}^{3} - d_{ij}^{0} h_{ik}^{1} + h_{ij}^{0} d_{ik}^{0} \right] h_{ab}^{1} \\
& \times \left[ (k + N + 2) (g^{\tilde{a}} Q^{\tilde{b}} V^{\tilde{d}} V^{l} - g^{\tilde{a}} Q^{\tilde{b}} Q^{\tilde{d}} V^{b} V^{l} + g^{\tilde{a}} Q^{\tilde{b}} Q^{\tilde{d}} V^{b} V^{l}) \right]
\end{align*}
$$
\[ + f_{\bar{m}} f_m Q^k Q^\bar{j} Q^\bar{a} V^m + k g_{\bar{k}} g^k Q^\bar{j} Q^\bar{a} \partial Q^\bar{a} \]  
(3.114)

\[- \frac{4(k + 2N)}{3(k + N + 2)^3} h_{\bar{a} b} d_{c \bar{d}} \left[ - (k + N + 2) g^{\bar{a} \bar{b}} \partial V^\bar{d} - k g^{\bar{a} \bar{b}} Q^{\bar{c}} \partial^2 Q^{\bar{a}} - f^{\bar{a} \bar{b} \bar{c}} Q^{\bar{c}} \partial (Q^{\bar{a}} V^\bar{c}) \right]. \]

One would like to see that (3.114) should be equal to (3.100) together with (3.109), (3.110), (3.111), (3.112) and (3.108).

For the quartic terms in the spin-\(\frac{1}{2}\) currents, the identities (3.16) and (3.17) can be used and it turns out that

\[- \frac{k}{(k + N + 2)^4} \left[ h_{\bar{a} b}^3 d_{c \bar{d}}^2 + h_{\bar{a} b}^2 d_{c \bar{d}} d_{c \bar{d}} + h_{\bar{a} b}^0 d_{c \bar{d}}^0 + h_{\bar{a} b}^1 d_{c \bar{d}}^1 \right] Q^{\bar{a}} Q^{\bar{b}} Q^{\bar{c}} \partial Q^{\bar{d}}(z), \]
(3.115)

which is exactly the same as the corresponding terms in (3.100) by using (3.110) and (3.108). Note that according to (3.100), there is a minus sign in front of higher spin-3 current and can be seen in (3.115).

For the quadratic term \(Q^{\bar{a}} \partial^2 Q^{\bar{b}}\), the contribution from the third line of (3.114) can be written in terms of

\[- \frac{1}{(k + N + 2)^2} \frac{k}{2} \left[ 1 + 1 + 2 - 2N - N - N \right] d_{\bar{a} b} Q^{\bar{a}} \partial^2 Q^{\bar{b}} = - \frac{2k(1 - N)}{(k + N + 2)^3} d_{\bar{a} b} Q^{\bar{a}} \partial^2 Q^{\bar{b}}, \]
(3.116)

by identifying the \(hdhfhg\) factors for generic \(N\) correctly. By collecting other contributions from (3.114), it turns out that one obtains, together with (3.116), the final result

\[- \frac{2k(3k + 2k + N)}{3(k + N + 2)^3} d_{\bar{a} b} Q^{\bar{a}} \partial^2 Q^{\bar{b}}, \]
(3.117)

which is exactly the same as the corresponding terms in (3.100) together with (3.110) and (3.108). The other quadratic terms can be simplified as

\[- \frac{4k(3k + 2k + N)}{3(k + N + 2)^3} d_{\bar{a} b} \partial Q^{\bar{a}} \partial Q^{\bar{b}}, \]
(3.118)

where the identity (3.16) is used. One sees that this is exactly the same as the corresponding terms in (3.100) together with (3.110) and (3.108). Then, one obtains the total expression in (3.117) and (3.118).

For the spin-1 dependent terms, one obtains

\[- \frac{4(k + 2)}{(k + N + 2)^2} d_{\bar{a} b} V^{\bar{a}} \partial V^{\bar{b}}(z) + \frac{2(3 + 2k + N)}{3(k + N + 2)^2} d_{\bar{a} b} f^{\bar{a} \bar{b} \bar{c}} \partial^2 V^\bar{e}(z), \]
(3.119)

by using the identity (3.16). Furthermore, the cubic terms are described as

\[- \frac{1}{2N(k + N + 2)^2} \left( 2h_{\bar{a} b}^1 d_{c \bar{d}} + h_{\bar{a} b}^2 d_{c \bar{d}} + h_{\bar{a} b}^3 d_{c \bar{d}} \right) f^{\bar{a} \bar{b} \bar{c}} V^{\bar{e}} V^{\bar{d}} V^\bar{e}(z) \]

\[- \frac{1}{(k + N + 2)^2} \left( 2h_{\bar{a} b}^1 d_{c \bar{d}} + h_{\bar{a} b}^2 d_{c \bar{d}} + h_{\bar{a} b}^3 d_{c \bar{d}} \right) f^{\bar{a} \bar{b} \bar{c}} V^{\bar{e}} V^{\bar{d}} V^\bar{e}(z), \]
(3.120)
by using the identities \((3.16)\) and \((3.17)\). In order to compare the previous expressions \((3.100)\) with \((3.119)\) and \((3.120)\), one should move \(V^e\) in the first term of \((3.120)\) to the left. This will give the additional contributions to \((3.119)\) and it turns out that they are exactly the same as the corresponding terms in \((3.100)\) together with \((3.111)\) and \((3.108)\).

For the quintic terms, one obtains

\[
- \frac{1}{(k + N + 2)^4} \left[ d_{bc}^2 h_{cd}^2 + d_{bd}^3 h_{cd}^2 - d_{bc}^0 h_{cd}^1 + h_{bc}^1 d_{cd}^0 \right] h_{ab}^1 f^l_{ij} e Q^a Q^b Q^c Q^d V^e, \tag{3.121}
\]

which is equal to the previous expressions \((3.100)\) corresponding to the quintic terms together with \((3.112)\) and \((3.108)\). One can simplify this \((3.121)\) further by using the indices \(a, \ldots, d\).

For the quartic terms appearing in the third line from the below of \((3.114)\), one can use the identities \((3.16)\) and \((3.17)\). For the quartic terms appearing in the third line of \((3.114)\), one can use some identity between the \(hhff\) factors and the \(h\) factors with some \(N\) dependent coefficients for the first three terms. For the last three terms one has the following identity

\[
\left[ -2 h_{\tilde{a}i}^0 d_{\tilde{a}i}^1 - h_{\tilde{a}i}^2 d_{\tilde{a}i}^3 + h_{\tilde{a}i}^3 d_{\tilde{a}i}^1 \right] f^{\tilde{a}j \tilde{b} l}_{k} h_{\tilde{a}b}^1 f^{kl}_{ji} Q^i Q^\tilde{a} Q^\tilde{j} V^l =
\]
\[
- \left[ d_{ab}^3 h_{cd}^3 + d_{ac}^3 h_{db}^3 + d_{bc}^3 h_{da}^3 - d_{ab}^3 h_{ac}^3 - d_{cb}^3 h_{da}^3 + d_{ab}^2 h_{cd}^2 + d_{ac}^2 h_{db}^2 
+ d_{de}^2 h_{ab}^2 - d_{db}^2 h_{ac}^2 - d_{ca}^2 h_{da}^2 - d_{ab}^1 h_{cd}^1 + d_{bc}^1 h_{da}^1 - d_{ac} h_{db}^1 - d_{ac} h_{db}^1 + d_{ac} h_{db}^1 
+ d_{db}^1 h_{ac}^1 + d_{ab}^0 h_{cd}^0 - d_{ab}^0 h_{db}^0 - d_{bc}^0 h_{da}^0 + d_{ac}^0 h_{de}^0 \right] Q^i Q^b V^e V^d. \tag{3.122}
\]

It turns out, together with \((3.122)\), that by collecting with all the contributions from \(Q^a Q^b V^e V^d\), one can check the coincidence of the corresponding terms in \((3.100)\) together with \((3.112)\) and \((3.108)\).

Let us consider the quartic terms appearing in the second line of \((3.114)\). Then the first three contributions are equal to the ninth line of \((3.112)\) with minus sign. Moreover, the contribution from the last three terms is equal to the ones in the twelfth, thirteenth, and the fourteenth line of \((3.112)\) with minus sign respectively.

Let us describe the final terms, the cubic terms with one derivative. The previous identities \((3.16)\) and \((3.17)\) can be used also. There are nontrivial identities corresponding to the first terms in the third line of \((3.114)\). That is,

\[
\begin{align*}
 h_{\tilde{a}i}^3 d_{\tilde{j}i}^2 f^{\tilde{a}l}_{k} h_{\tilde{a}b}^1 f^{kl}_{ji} m & = - (h_{\tilde{a}b}^1 d_{\tilde{c}d}^1 + \bar{h}_{\tilde{a}b}^1 d_{\tilde{c}d}^2) f^{\tilde{a}b}_{e}, \\
 h_{\tilde{a}i}^2 d_{\tilde{j}i}^3 f^{\tilde{a}l}_{k} h_{\tilde{a}b}^1 f^{kl}_{ji} m & = (h_{\tilde{a}b}^1 d_{\tilde{c}d}^1 + \bar{h}_{\tilde{a}b}^1 d_{\tilde{c}d}^3) f^{\tilde{a}b}_{e}, \\
 h_{\tilde{a}i}^1 d_{\tilde{j}i}^0 f^{\tilde{a}l}_{k} h_{\tilde{a}b}^1 f^{kl}_{ji} m & = (\bar{h}_{\tilde{a}b}^1 d_{\tilde{c}d}^2 + \bar{h}_{\tilde{a}b}^1 d_{\tilde{c}d}^3) f^{\tilde{a}b}_{e}. \tag{3.123}
\end{align*}
\]
Then one can check that these coincide with the ones in (3.100) together with (3.112) and (3.108).

The remaining two subsections maybe be skipped without any discontinuity and the readers can go to the next section 4 directly.

### 3.5.7 Other way of obtaining the higher spin-3 current using the higher spin-\( \frac{5}{2} \) current in Appendix (C.15)

In Appendix C, there are other two different expressions Appendix (C.15) and Appendix (C.18), corresponding to the above higher spin-\( \frac{5}{2} \) current in (3.124). Let us see how the higher spin-\( \frac{3}{2} \) current (3.42) and the same quantity given in Appendix (C.15) differ from each other. The difference appears in the cubic terms (the first six terms). Then after one calculates the OPE between the spin-\( \frac{3}{2} \) current and the higher spin-\( \frac{5}{2} \) current in (C.18), corresponding to the above higher spin-\( \frac{5}{2} \) current in (C.15), one obtains

\[
\begin{align*}
-\frac{1}{(k+N+2)^2} & \left[ -\frac{1}{2N} h_{ab}^3 d_{cd}^3 \bar{d}_{ij}^3 + \frac{1}{2N} h_{ab}^1 d_{cd}^1 \bar{d}_{ij}^1 + \frac{1}{N} h_{ab}^3 d_{ij}^3 + 2 h_{ab}^2 \bar{d}_{ij}^2 - h_{ab}^0 d_{ij}^0 - h_{ab}^2 d_{ij}^2 \right] \\
\times f^a_{\bar{k} b} \left[ -(k+N+2)g^{\bar{i}a} V^b V^j V^k - k g^{\bar{j}b} Q^i V^j V^k - k \right. \\
\left. - \frac{1}{2} f_{\bar{i} b} e_{\bar{i} b} Q^i \partial^2 Q^j - k g^{\bar{i}b} Q^i \partial Q^j V^k \right]
\end{align*}
\]

+the remaining terms starting from the fourth line to the last (3.114).

One should check whether the three lines of (3.124) are equal to those (the first three lines) in (3.114) or not.

One can simplify the cubic terms in the spin-1 current as before. It turns out that

\[
\begin{align*}
-\frac{1}{2N(k+N+2)^2} (h_{ab}^1 d_{cd}^1 + h_{ab}^2 d_{cd}^2 + 2 h_{ab}^3 d_{cd}^3) f^{\bar{a} b \bar{c} d} e V^c V^d V^e(z) \\
-\frac{1}{(k+N+2)^2} (h_{ca}^1 d_{db}^1 + h_{ca}^2 d_{db}^2 + 2 h_{ca}^3 d_{db}^3) f^{\bar{c} a b d} e V^\bar{a} V^c V^e(z).
\end{align*}
\]

By identifying (3.120) with (3.123), one obtains the nontrivial relation between the various tensors as follows:

\[
\frac{1}{2N} (h_{ab}^1 d_{cd}^1 - h_{ab}^3 d_{cd}^3) f^{\bar{a} b \bar{c} d} e = -(h_{ca}^1 d_{db}^1 - h_{ca}^3 d_{db}^3) f^{\bar{c} a b d} e. \quad (3.126)
\]

This identity can be also observed in the different context of Appendix C.

From the contributions of the cubic terms in the second line of (3.124) and the corresponding terms in (3.114), one obtains the following nontrivial relation between the tensors

\[
\frac{1}{2N} (-h_{ab}^1 d_{cd}^1 - h_{ab}^3 d_{cd}^3) f^{\bar{a} b \bar{c} d} e = (h_{bc}^0 d_{ad}^0 - h_{bc}^2 d_{ad}^2) f^{\bar{a} b \bar{c} d} e. \quad (3.127)
\]
Due to the relative sign change in the left hand sides of (3.126) and (3.127), one can obtain that each $hdf$ factor can be written in terms of other quantities in the right hand side by adding or subtracting them.

One can check the coincidence of the quadratic terms. For the cubic terms with one derivative, one can use the previous relations in (3.123) and the other remaining terms can be checked explicitly.

3.5.8 Other way of obtaining the higher spin-3 current using the higher spin-$\frac{5}{2}$ current in Appendix (C.18)

After one calculates the OPE between the spin-$\frac{3}{2}$ current $G^3(z)$ and the higher spin-$\frac{5}{2}$ current in Appendix (C.18), one obtains the first order pole of this OPE as follows:

$$\frac{1}{(k + N + 2)^3} \left[ \frac{1}{2N} h_{\nu \bar{\nu}}^2 d_{ij}^2 + \frac{1}{2N} h_{\nu \bar{\nu}}^2 d_{ij}^0 - \frac{1}{N} h_{\nu \bar{\nu}}^2 d_{ij}^0 - 2 h_{\nu \bar{\nu}} d_{ij}^0 - h_{\nu \bar{\nu}} d_{ij}^0 - h_{\nu \bar{\nu}} d_{ij}^0 \right]$$

$$\times f^{\bar{\nu} \bar{\nu}} \frac{1}{k} h_{\bar{a} \bar{b}} \left[ - (k + N + 2) g^{\bar{a} \bar{b}} V^j V^j V^k - k g^{\bar{a} \bar{b}} Q^i \partial Q^i V^k - f^{\bar{b} i} Q^i Q^i V^k - f^{j l} Q^i Q^i V^k \right]$$

$$- f^{k l} Q^i \partial (Q^i V^m) - f^{k l} Q^i Q^i V^l \varepsilon - \frac{k}{2} f^{\bar{b} i} g^{k l} Q^i \partial 2 Q^i - k g^{k l} Q^i \partial Q^i V^j$$

+ the remaining terms of starting from the fourth line to the last (3.114). (3.128)

One should check that the three lines of (3.128) are equal to those (the first three lines) in (3.114) as before.

One can simplify the cubic terms in the spin-1 current as before. It turns out that

$$- \frac{1}{2N(k + N + 2)^2} (h_{\bar{a} \bar{b}} d_{\bar{c} \bar{d}}^1 + 2 h_{\bar{a} \bar{b}} d_{\bar{c} \bar{d}}^2 + h_{\bar{a} \bar{b}} d_{\bar{c} \bar{d}}^3) f^{\bar{a} \bar{b}} e V^e V^d V^e(z)$$

$$- \frac{1}{(k + N + 2)^2} (h_{\bar{a} \bar{b}} d_{\bar{c} \bar{d}}^1 + 2 h_{\bar{a} \bar{b}} d_{\bar{c} \bar{d}}^2 + h_{\bar{a} \bar{b}} d_{\bar{c} \bar{d}}^3) f^{\bar{a} \bar{b}} e V^a V^b V^e(z).$$ (3.129)

By identifying (3.120) with (3.129), one obtains the nontrivial relation between the tensors

$$\frac{1}{2N} (- h_{\bar{a} \bar{b}} d_{\bar{c} \bar{d}}^1 + h_{\bar{a} \bar{b}} d_{\bar{c} \bar{d}}^2) f^{\bar{a} \bar{b}} e = (h_{\bar{c} \bar{d}} d_{\bar{a} \bar{b}}^1 - h_{\bar{c} \bar{d}} d_{\bar{a} \bar{b}}^2) f^{\bar{a} \bar{b}} e.$$ (3.130)

This identity can be also observed in the different context of Appendix C.

From the contributions of the cubic terms in the second line of (3.128) and the corresponding terms in (3.114), one obtains the following nontrivial relation

$$\frac{1}{2N} (- h_{\bar{a} \bar{b}} d_{\bar{c} \bar{d}}^2 - h_{\bar{a} \bar{b}} d_{\bar{c} \bar{d}}^1) f^{\bar{a} \bar{b}} e = (h_{\bar{c} \bar{d}} d_{\bar{a} \bar{b}}^0 - h_{\bar{c} \bar{d}} d_{\bar{a} \bar{b}}^3) f^{\bar{a} \bar{b}} e.$$ (3.131)

Due to the relative sign change in the left hand sides of (3.130) and (3.131), each $hdf$ factor can be written in terms of other quantities in the right hand side by adding or subtracting
them. One can check that all the other remaining terms in (3.128) are equal to those in (3.114) each other.

In Appendix D, there are also other possibilities to express the higher spin-3 current using the different \((\mu, \nu)\) cases. It would be interesting to check whether the other cases also lead to the previous higher spin-3 current with nontrivial relations between the various tensors.

4 Eigenvalues and Three-point functions

We reinterpret the previous results in [54] by examining the structure of (3.110) and (3.111) carefully, calculate the eigenvalues of the higher spin-3 current acting on the other higher representations and extract the corresponding three-point functions. We also analyze its large \((N, k)\) ’t Hooft-like limit in terms of two-point functions of two scalar operators with ’t Hooft coupling constant-dependent coefficients (and the number of boxes of Young tableaux).

4.1 The \((0; \Lambda)\) representations up to three boxes

The relevant part of the higher spin-3 current acting on this representation is given by (3.110) where there are four kinds of independent terms. It turns out that there is no third order pole between the six multiple product of spin-\(\frac{1}{2}\) current in the higher spin-3 current (corresponding to the first term in (3.110)) and the spin-\(\frac{1}{2}\) current. Similarly, there is no third order pole between the six multiple product of spin-\(\frac{1}{2}\) current in the higher spin-3 current and the quadratic term in the spin-\(\frac{1}{2}\) current.

However, there exist third order poles between the six multiple product of spin-\(\frac{1}{2}\) current in the higher spin-3 current (3.110) and the cubic term in the spin-\(\frac{1}{2}\) current as follows:

\[
d_{ab} h_{c\ell}^i h_{ef}^i Q^a Q^b Q^c Q^d Q^e Q^f(z) Q^g Q^h Q^i(w) =
-\frac{1}{(z - w)^3} \left[ g^{g\bar{a}} g^{h\bar{b}} Q^d Q^e Q^f(z) Q^g Q^h Q^i(w)
+ g^{g\bar{a}} g^{h\bar{b}} g^{i\bar{d}} Q^b Q^e Q^f(z) Q^g Q^h Q^i(w)
+ g^{g\bar{a}} g^{h\bar{b}} g^{i\bar{d}} Q^a Q^e Q^f(z) Q^g Q^h Q^i(w)
+ g^{g\bar{a}} g^{h\bar{b}} g^{i\bar{d}} Q^a Q^d Q^f(z) Q^g Q^h Q^i(w)
+ g^{g\bar{a}} g^{h\bar{b}} g^{i\bar{d}} Q^a Q^d Q^e Q^f(z) Q^g Q^h Q^i(w)
+ g^{g\bar{a}} g^{h\bar{b}} g^{i\bar{d}} Q^a Q^d Q^e Q^f(z) Q^g Q^h Q^i(w)
- g^{g\bar{a}} g^{h\bar{b}} g^{i\bar{d}} Q^a Q^d Q^e Q^f(z) Q^g Q^h Q^i(w)
+ g^{g\bar{a}} g^{h\bar{b}} g^{i\bar{d}} Q^a Q^d Q^e Q^f(z) Q^g Q^h Q^i(w)
\right] + O(1/(z - w)^2).
\] (4.1)

We focus on the third order pole only for the calculation of the eigenvalues. Only after one specifies the indices \(\bar{g}, \bar{h}\), and \(\bar{i}\), one can determine the coefficient of the composite field
$Q^g Q^h Q^i(w)$ in the right hand side of (4.1).

It turns out that there is no third order pole between the quartic product of spin-$\frac{1}{2}$ current (corresponding to the second term of (3.110)) in the higher spin-3 current and the spin-$\frac{1}{2}$ current. There are third order poles between the quartic product of spin-$\frac{1}{2}$ current in the higher spin-3 current (3.110) and the quadratic term in the spin-$\frac{1}{2}$ current, by considering the second term of (3.110), as follows:

$$d_{ab}^0 h_{cd}^0 Q^a Q^b Q^c \partial Q^d(z) Q^e Q^f(w) =$$

$$\frac{1}{(z-w)^3} (k + N + 2)^2 d_{ab}^0 h_{cd}^0 \left[ -g^{ea} g^{fd} Q^b Q^e + g^{ed} g^{fb} Q^a Q^e - g^{ed} g^{fc} Q^a Q^b \right. + g^{ea} g^{fd} Q^b Q^e - g^{eb} g^{fd} Q^a Q^e + g^{ec} g^{fd} Q^a Q^b \right] (w) + O\left(\frac{1}{(z-w)^2}\right). \tag{4.2}$$

As before, the third order pole is the relevant terms. Similarly, there are third order poles between the quartic product of spin-$\frac{1}{2}$ current in the higher spin-3 current (3.110) and the cubic term in the spin-$\frac{1}{2}$ current as follows:

$$d_{ab}^0 h_{cd}^0 Q^a Q^b Q^c \partial Q^d(z) Q^e Q^f Q^g(w) =$$

$$\frac{1}{(z-w)^3} (k + N + 2)^2 d_{ab}^0 h_{cd}^0 \left[ -2g^{ea} g^{fd} (Q^b Q^c) Q^g + 2g^{ed} g^{fb} (Q^b Q^e) Q^g + 2g^{ed} g^{fc} Q^a Q^f Q^g \right. + 2g^{ea} g^{fd} (Q^b Q^c) Q^g + 2g^{ea} g^{fd} (Q^b Q^c) Q^g + 2g^{ea} g^{fd} (Q^b Q^c) Q^g \left. -2g^{ea} g^{fd} (Q^b Q^c) Q^g + 2g^{ea} g^{fd} (Q^b Q^c) Q^g + 2g^{ea} g^{fd} (Q^b Q^c) Q^g \right] (w) + O\left(\frac{1}{(z-w)^2}\right). \tag{4.3}$$

Moreover, one has, for the third term of (3.110) with the sum over the index $i$,

$$h_{ab}^i d_{cd}^i Q^a Q^b Q^c \partial Q^d(z) Q^e Q^f Q^g(w) =$$

$$\frac{1}{(z-w)^3} (k + N + 2)^2 h_{ab}^i d_{cd}^i \left[ -g^{ea} g^{fd} Q^b Q^c + g^{ed} g^{fb} Q^a Q^e - g^{ed} g^{fc} Q^a Q^b \right. + g^{ea} g^{fd} Q^b Q^c - g^{eb} g^{fd} Q^a Q^e + g^{ec} g^{fd} Q^a Q^b \right] (w) + O\left(\frac{1}{(z-w)^2}\right). \tag{4.3}$$

Similarly, one obtains (with the sum over the index $i$)

$$h_{ab}^i d_{cd}^i Q^a Q^b Q^c \partial Q^d(z) Q^e Q^f Q^g(w) =$$

$$\frac{1}{(z-w)^3} (k + N + 2)^2 h_{ab}^i d_{cd}^i \left[ -2g^{ea} g^{fd} (Q^b Q^c) Q^g + 2g^{ea} g^{fd} (Q^b Q^c) Q^g + 2g^{ea} g^{fd} (Q^b Q^c) Q^g \right. + 2g^{ea} g^{fd} (Q^b Q^c) Q^g + 2g^{ea} g^{fd} (Q^b Q^c) Q^g + 2g^{ea} g^{fd} (Q^b Q^c) Q^g \left. -2g^{ea} g^{fd} (Q^b Q^c) Q^g + 2g^{ea} g^{fd} (Q^b Q^c) Q^g + 2g^{ea} g^{fd} (Q^b Q^c) Q^g \right] (w) + O\left(\frac{1}{(z-w)^2}\right). \tag{4.4}$$
\[-2g^{\bar{f}d}g^{\bar{g}a}Q^\bar{a}Q^\bar{b}Q^\bar{c} + 2g^{f\bar{a}}g^{\bar{g}d}Q^\bar{d}Q^\bar{b}Q^\bar{c}\right](w) + \mathcal{O}\left(\frac{1}{(z-w)^2}\right). \quad \text{(4.5)}\]

It turns out that there are third order poles between the quadratic product of spin-$\frac{1}{2}$ current (corresponding to the fourth term of (3.110)) in the higher spin-3 current (3.110) and the spin-$\frac{1}{2}$ current as follows:

\[d_{ab}^{0} Q^{\bar{a}} \partial^{2} Q^{\bar{b}}(z) Q^{\bar{c}}(w) = -\frac{1}{(z-w)^3} 2(k + N + 2) d_{ab}^{0} Q^{\bar{a}}(w) + \mathcal{O}\left(\frac{1}{(z-w)^2}\right). \quad \text{(4.6)}\]

Moreover, there are third order poles between the quadratic product of spin-$\frac{1}{2}$ current in the higher spin-3 current (3.110) and the quadratic terms in the spin-$\frac{1}{2}$ current as follows:

\[d_{ab}^{0} Q^{\bar{a}} \partial^{2} Q^{\bar{b}}(z) Q^{\bar{d}} Q^{\bar{d}}(w) = -\frac{1}{(z-w)^3} 2(k + N + 2) d_{ab}^{0} Q^{\bar{a}} Q^{\bar{d}} + g^{\bar{d}\bar{b}} Q^{\bar{e}} Q^{\bar{a}}(w) + \mathcal{O}\left(\frac{1}{(z-w)^2}\right). \quad \text{(4.7)}\]

There are third order poles between the quadratic product of spin-$\frac{1}{2}$ current in the higher spin-3 current (3.110) and the cubic terms in the spin-$\frac{1}{2}$ current as follows:

\[d_{ab}^{0} Q^{\bar{a}} \partial^{2} Q^{\bar{b}}(z) Q^{\bar{e}} Q^{\bar{d}} Q^{\bar{d}}(w) = -\frac{1}{(z-w)^3} 2(k + N + 2) d_{ab}^{0} Q^{\bar{a}} Q^{\bar{d}} Q^{\bar{e}} + g^{\bar{d}\bar{b}} Q^{\bar{e}} Q^{\bar{a}} + g^{\bar{e}\bar{b}} Q^{\bar{c}} Q^{\bar{d}} Q^{\bar{d}}(w) + \mathcal{O}\left(\frac{1}{(z-w)^2}\right). \quad \text{(4.8)}\]

One can easily check that there are no contributions from the last term of higher spin-3 current in (3.110) with cubic, quadratic or linear terms in the spin-$\frac{1}{2}$ currents.

From the third order poles of the above OPEs, one can determine the various eigenvalues in the representations as follows.

### 4.1.1 The eigenvalue in $(0; f)$ representation

Because the nontrivial third order pole in the OPE between the higher spin-3 current and the spin-$\frac{1}{2}$ current arises in (4.6), by combining the coefficient of the fourth line of (3.110) and the eigenvalue in (4.6), one can determine the final eigenvalue in this representation as follows:

\[
\phi_{2}^{(1)}(0; \square) = \frac{-2k(12k^2 N + 4k^2 + 6kN^2 + 39kN + 14k + 5N^2 + 28N + 12)}{3(k + N + 2)^3(6kN + 5k + 5N + 4)} \times \left[-2(k + N + 2)\right] = \frac{4k(12k^2 N + 4k^2 + 6kN^2 + 39kN + 14k + 5N^2 + 28N + 12)}{3(k + N + 2)^2(6kN + 5k + 5N + 4)} \rightarrow \frac{4}{3}(2 - \lambda)(1 - \lambda). \quad \text{(4.9)}
\]
Here the large \((N, k)\) 't Hooft-like limit is defined by \[\lambda \equiv \frac{(N + 1)}{(N + k + 2)}\] \[\text{fixed.} \quad (4.10)\]

The leading behavior term is denoted by the boldface notation.

Note that the nontrivial contribution from the composite field \(\Phi_{0}^{(1)} T\) occurs in the fourth term of \((3.108)\) according to \((4.6)\). Then the eigenvalue, by combining the coefficient with the eigenvalue, is given by

\[
\frac{k(2k + 3)}{8(k + N + 2)^3} \times \left[ -2(k + N + 2) \right] = -\frac{k(2k + 3)}{4(k + N + 2)^2}. \quad (4.11)
\]

Then the zeromode \([7]\) of the composite field \(\Phi_{0}^{(1)} T\) is given by

\[
\sum_{p \leq -1} (\Phi_{0}^{(1)})_{p} T_{-p} + \sum_{p > 0} T_{-p} (\Phi_{0}^{(1)})_{p}
\]

which do not contribute to the eigenvalue because the positive mode \(T_{-p}\) or \((\Phi_{0}^{(1)})_{p}\) acting on the state corresponding to this representation vanishes. Once one knows one of the eigenvalue, then the other one can be determined from \((4.11)\). For example, the conformal dimension for this representation can be obtained from the formula \[\[34\]

\[
h(\Lambda_{+}; \Lambda_{-}) = \frac{C^{(N+2)}(\Lambda_{+})}{(k + N + 2)} - \frac{C^{(N)}(\Lambda_{-})}{(k + N + 2)} - \frac{\hat{u}^2}{N(N + 2)(k + N + 2)} + n. \quad (4.12)
\]

It turns out that \(h(0; \square) = \frac{(2k + 3)}{4(k + N + 2)} \) \[34\]. Then from \((4.11)\), one can split as follows

\[
\phi_{0}^{(1)} (0; \square) h(0; \square) = \left[ -\frac{k}{(k + N + 2)} \right] \times \left[ \frac{(2k + 3)}{4(k + N + 2)} \right]. \quad (4.13)
\]

In this way, one can conclude indirectly that the eigenvalue of the higher spin-1 current is given by \(
\phi_{0}^{(1)} (0; \square) = -\frac{k}{(k + N + 2)} \)
which was found in \[54\] by calculating the OPE between the second term of \((3.1)\) and the spin-\(\frac{1}{2}\) current \(Q^{4*}\) where \(\bar{A}^{*} = 1^{*}, \cdots, (2N)^{*}\) and reading off the first order pole.

It is straightforward to write down the three-point function of the higher spin-3 current with the scalar operator corresponding to \((0; f)\) and the scalar operator corresponding to \((0; \bar{f})\) from \((4.9)\).

### 4.1.2 The eigenvalue in \((0; \text{symm})\) representation with two boxes

From the nontrivial results for the OPEs between the quartic terms and the quadratic term in \((4.2)\) and \((4.3)\) and the OPE between the quadratic terms and the quadratic term in \((4.7)\),
the corresponding eigenvalue can be obtained by considering the coefficients in (3.110) (and fixing the indices $\bar{e}, \bar{f}$ and $\bar{c}, \bar{d}$ for the symmetric representation) as follows:

\[
\phi^{(1)}_2(0; \square) = \frac{k(4k^2 + 2kN + 11k - N + 4)}{(k + N + 2)^4(6kN + 5k + 5N + 4)} \times \left[-4(k + N + 2)^2\right]
\]

\[
+ \frac{k(6kN + k + 9N + 4)}{(k + N + 2)^4(6kN + 5k + 5N + 4)} \times \left[-12(k + N + 2)^2\right]
\]

\[
- \frac{2k(12k^2N + 4k^2 + 6kN^2 + 39kN + 14k + 5N^2 + 28N + 12)}{3(k + N + 2)^4(6kN + 5k + 5N + 4)} \times \left[-4(k + N + 2)\right] = \frac{8}{3}(2 - \lambda)(1 - \lambda), \quad (4.14)
\]

where the limit in (4.10) is taken and one sees that the leading contribution in this limit occurs in the last term denoted by the boldface notation. The corresponding eigenvalue is the twice of the one in previous subsection and this leads to the fact that the large $(N, k)$ 't Hooft like behavior in this representation is the twice of the one in (4.9).

Again, by collecting the second, the third and the fourth terms in (3.108) together with their eigenvalues, (4.2), (4.3) and (4.7) where one of the quadratic terms is given by $Q^1 \cdot Q^{(N+1)^*}$, one obtains the following eigenvalue corresponding to the composite field $\Phi^{(1)}_0 T$

\[
\frac{k(2k + 3)}{8(k + N + 2)^4} \times \left[-4(k + N + 2)^2\right] - \frac{k}{4(k + N + 2)^4} \times \left[-12(k + N + 2)^2\right]
\]

\[
+ \frac{k(2k + 3)}{8(k + N + 2)^3} \times \left[-4(k + N + 2)\right] = -\frac{2k^2}{(k + N + 2)^2}. \quad (4.15)
\]

By substituting the conformal dimension in this representation using the formula (4.12) or the equation (2.17) of [54], the following decomposition is, from (4.15), valid

\[
\phi^{(1)}_0(0; \square) h(0; \square) = \left[-\frac{2k}{(k + N + 2)}\right] \times \left[\frac{k}{(k + N + 2)}\right]. \quad (4.16)
\]

One can see that the eigenvalue for the higher spin-1 current in (4.16) is equal to the one in (5.20) of [54].

The three-point function of the higher spin-3 current with the scalar operator corresponding to (0; $\square$) and the scalar operator corresponding to (0; $\square$) can be read off from (4.14).

### 4.1.3 The eigenvalue in (0; antisymm) representation with two boxes

As done in previous subsection, by using the relations (4.2), (4.3) and (4.4), the following eigenvalue is obtained

\[
\phi^{(1)}_2(0; \square) = \frac{k(4k^2 + 2kN + 11k - N + 4)}{(k + N + 2)^4(6kN + 5k + 5N + 4)} \times \left[-4(k + N + 2)^2\right]
\]
\[
\frac{k(6kN + k + 9N + 4)}{(k + N + 2)^4(6kN + 5k + 5N + 4)} \times \left[ 4(k + N + 2)^2 \right] \\
- \frac{2k(12k^2N + 4k^2 + 6kN^2 + 39kN + 14k + 5N^2 + 28N + 12)}{3(k + N + 2)^3(6kN + 5k + 5N + 4)} \times \left[ -4(k + N + 2) \right] = \\
\frac{8k(12k^2N - 2k^2 + 6kN^2 + 45kN - k + 5N^2 + 43N + 12)}{3(k + N + 2)^2(6kN + 5k + 5N + 4)} \rightarrow \frac{8}{3}(2 - \lambda)(1 - \lambda). \quad (4.17)
\]

where the limit in (4.10) is taken and the leading contribution in this limit occurs in the last term denoted by the boldface notation. The eigenvalues from (4.7) in this subsection and previous subsection are the same as each other and the large \((N, k)\) ’t Hooft like behavior in (4.17) and (4.14) is the same.

By combining the second, the third and the fourth terms in (3.108) together with their eigenvalues, (4.2), (4.3) and (4.7) correctly where one of the quadratic terms is given by \(Q^1, Q^2\), one obtains the following eigenvalue corresponding to the composite field \(\Phi^{(1)}_0 T\)

\[
\frac{k(2k + 3)}{8(k + N + 2)^4} \times \left[ -4(k + N + 2)^2 \right] - \frac{k}{4(k + N + 2)^4} \times \left[ 4(k + N + 2)^2 \right] \\
+ \frac{k(2k + 3)}{8(k + N + 2)^3} \times \left[ -4(k + N + 2) \right] = -\frac{2k(k + 2)}{(k + N + 2)^2}. \quad (4.18)
\]

After substituting the conformal dimension in this representation using the formula (4.12) or the equation (2.19) of [54], it is obvious see that the following decomposition from (4.18) is valid

\[
\phi^{(1)}_0 (0; \begin{array}{c} \text{ } \end{array}) h(0; \begin{array}{c} \text{ } \end{array}) = \left[ -\frac{2k}{(k + N + 2)} \right] \times \left[ \frac{(k + 2)}{(k + N + 2)} \right]. \quad (4.19)
\]

The eigenvalue for the higher spin-1 current is equal to the one in (5.21) of [54].

For the three-point function of the higher spin-3 current with the scalar operator corresponding to \((0; \begin{array}{c} \text{ } \end{array})\) and the scalar operator corresponding to \((0; \begin{array}{c} \text{ } \end{array})\) one can use the relation (4.17).

4.1.4 The eigenvalue in \((0; \text{antisymm})\) representation with three boxes

Because the nontrivial third order pole in the OPE between the higher spin-3 current and the cubic terms in the spin-\(\frac{1}{2}\) current arises in (4.1), (4.5), (4.4) and (4.8), by combining the coefficients of (3.110) and the eigenvalues, one can determine the final eigenvalue in this representation as follows:

\[
\phi^{(1)}_2 (0; \begin{array}{c} \text{ } \end{array}) = -\frac{k(k - N)}{2(6kN + 5k + 5N + 4)(k + N + 2)^5} \times \left[ 48(k + N + 2)^3 \right]
\]
where the limit in (4.10) is taken and the leading contribution in this limit occurs in the last term denoted by the boldface notation. Note that the large \((N, k)\) 't Hooft like behavior in this representation is the three times of the one in (4.9).

By combining the first, second, the third and the fourth terms in (3.108) together with their eigenvalues, (4.1), (4.5), (4.4) and (4.8) correctly where one of the quadratic terms is given by \(Q_1^* Q_2^* Q_3^*\), one obtains the following eigenvalue corresponding to the composite field \(\Phi(1)\)

\[
\begin{align*}
&- \frac{k}{32(k + N + 2)^5} \times \left[ 48(k + N + 2)^3 \right] + \frac{k(2k + 3)}{8(k + N + 2)^4} \times \left[ -12(k + N + 2)^2 \right] \\
&- \frac{k}{4(k + N + 2)^4} \times \left[ 12(k + N + 2)^2 \right] + \frac{k(2k + 3)}{8(k + N + 2)^3} \times \left[ -6(k + N + 2) \right] = \\
&- \frac{9k(2k + 5)}{4(k + N + 2)^2}.
\end{align*}
\]

(4.21)

After substituting the conformal dimension in this representation using the formula (4.12) or the equation (2.20) of [54], it is obvious see that the following decomposition from (4.21) satisfies

\[
\phi_0^{(1)}(0; \, \square \, \square \, \square \, h(0; \, \square \, \square \, \square) = \left[ - \frac{3k}{(k + N + 2)} \right] \times \left[ \frac{3(2k + 5)}{4(k + N + 2)} \right].
\]

(4.22)

The eigenvalue for the higher spin-1 current is equal to the one in (9.1) of [54].

For the three-point function of the higher spin-3 current with the scalar operator corresponding to \((0; \, \square \, \square \, \square \, \square \) and the scalar operator corresponding to \((0; \, \square \, \square \, \square) one can use the relation (4.20) explicitly.

### 4.1.5 The eigenvalue in \((0; \text{mixed})\) representation

By using the relations (4.1), (4.5), (4.4) and (4.8), one obtains the following eigenvalue

\[
\phi_2^{(1)}(0; \, \square \, \square) = - \frac{k(k - N)}{2(6kN + 5k + 5N + 4)(k + N + 2)^5} \times \left[ -48(k + N + 2)^3 \right]
\]

\[
\phi_2^{(1)}(0; \, \square \, \square) = - \frac{k(k - N)}{2(6kN + 5k + 5N + 4)(k + N + 2)^5} \times \left[ -48(k + N + 2)^3 \right]
\]
\[ + \frac{k(4k^2 + 2kN + 11k - N + 4)}{(k + N + 2)^4(6kN + 5k + 5N + 4)} \times \left[ -12(k + N + 2)^2 \right] \\
+ \frac{k(6kN + k + 9N + 4)}{(k + N + 2)^4(6kN + 5k + 5N + 4)} \times \left[ -12(k + N + 2)^2 \right] \\
- \frac{2k(12k^2N + 4k^2 + 6kN^2 + 39kN + 14k + 5N^2 + 28N + 12)}{3(k + N + 2)^3(6kN + 5k + 5N + 4)} \times \left[ -6(k + N + 2) \right] = \\
\frac{4k(12k^2N - 8k^2 + 6kN^2 + 15kN - 16k + 5N^2 - 2N - 12)}{(k + N + 2)^2(6kN + 5k + 5N + 4)} \rightarrow 4(2 - \lambda)(1 - \lambda), \quad (4.23) \]

where the limit in (4.10) is taken and the leading contribution in this limit occurs in the last term denoted by the boldface notation. Because the eigenvalue for the last term is the same as the one in previous subsection, the large \((N, k)\) ’t Hooft like behavior is the same as the one in (4.20).

By combining the first, second, the third and the fourth terms in (3.108) together with their eigenvalues, (4.1), (4.5), (4.4) and (4.8) correctly where one of the quadratic terms is given by \(Q^2 \cdot Q^{(N+1)} \cdot Q^1\), one obtains the following eigenvalue corresponding to the composite field \(\Phi_0^{(1)} T\)

\[ - \frac{k}{32(k + N + 2)^5} \times \left[ -48(k + N + 2)^3 \right] + \frac{k(2k + 3)}{8(k + N + 2)^4} \times \left[ -12(k + N + 2)^2 \right] \\
- \frac{k}{4(k + N + 2)^4} \times \left[ -12(k + N + 2)^2 \right] + \frac{k(2k + 3)}{8(k + N + 2)^3} \times \left[ -6(k + N + 2) \right] = \frac{9k(2k + 1)}{4(k + N + 2)^2}. \quad (4.24) \]

After substituting the conformal dimension in this representation using the formula (4.12) or the equation (2.23) of [54], it is obvious see that the following decomposition from (4.24) satisfies

\[ \phi_2^{(1)}(0; \begin{array}{c} \hline \hline \end{array}) h(0; \begin{array}{c} \hline \hline \end{array}) = \left[ - \frac{3k}{(k + N + 2)} \right] \times \left[ \frac{3(2k + 1)}{4(k + N + 2)} \right]. \quad (4.25) \]

The eigenvalue for the higher spin-1 current in (4.25) is equal to the one in section 9 of [54].

The three-point function of the higher spin-3 current with the scalar operator corresponding to \((0; \begin{array}{c} \hline \hline \end{array})\) and the scalar operator corresponding to \((0; \begin{array}{c} \hline \hline \end{array})\) can be obtained from the relation (4.23) explicitly.

### 4.2 The \((0; \Lambda_{-})\) representations with more than four boxes

So far, the eigenvalues are obtained in the representations up to three boxes. One can consider the higher representations with more than four boxes. The simplest cases are the antisymmetric representations with more than four boxes.
4.2.1 The eigenvalue in \((0;\text{antisymm})\) representation with more than four boxes

For the four boxes in the antisymmetric representation, the minimum value for \(N\) is given by \(N = 5\). One can visualize the spin-\(\frac{3}{2}\) currents in the following \((N+2) \times (N+2)\) matrix

\[
\begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & Q_1^* & Q^{(N+1)*} \\
0 & 0 & 0 & \cdots & 0 & Q_2^* & Q^{(N+2)*} \\
0 & 0 & 0 & \cdots & 0 & Q_3^* & Q^{(N+3)*} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & Q_N^* & Q^{(2N)*} \\
Q^{N+1} & Q^{N+2} & Q^{N+3} & \cdots & Q^{2N} & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0
\end{pmatrix}
\]  
(4.26)

Then one of the four boxes in the antisymmetric representations is given by \(Q_1^* \ Q_2^* \ Q_3^* \ Q_4^*\) in the view of (4.26).

It turns out (the explicit form for the OPE between the higher spin-3 current (3.110) and the quartic term in the spin-\(\frac{3}{2}\) current is not given) that the eigenvalue is given by

\[
\phi_2^{(1)}(0;\text{antisymm}) = -\frac{k(k - N)}{2(6kN + 5k + 5N + 4)(k + N + 2)^5} \times \left[ 192(k + N + 2)^3 \right] \\
+ \frac{k(4k^2 + 2kN + 11k - N + 4)}{(k + N + 2)^4(6kN + 5k + 5N + 4)} \times \left[ -24(k + N + 2)^2 \right] \\
+ \frac{k(6kN + k + 9N + 4)}{(k + N + 2)^4(6kN + 5k + 5N + 4)} \times \left[ 24(k + N + 2)^2 \right] \\
- \frac{2k(12k^2N + 4k^2 + 6kN^2 + 39kN + 14k + 5N^2 + 28N + 12)}{3(k + N + 2)^3(6kN + 5k + 5N + 4)} \times \left[ -8(k + N + 2) \right] \\
\frac{16(12k^2N - 14k^2 + 6kn^2 + 57kN - 49k + 5N^2 + 91N + 12)}{3(k + N + 2)^2(6kN + 5k + 5N + 4)} \rightarrow \frac{16}{3}(2 - \lambda)(1 - \lambda),
\]

where the limit in (4.10) is taken and the leading contribution in this limit occurs in the last term denoted by the boldface notation. Because the eigenvalue for the last term is four times of the one in previous subsection 4.1.1, the large \((N, k)\) 't Hooft like behavior is four times of the one in (4.9).

By combining the first, second, the third and the fourth terms in (3.108) together with their eigenvalues correctly where one of the quadratic terms is given by \(Q_1^* \ Q_2^* \ Q_3^* \ Q_4^*\), one obtains the following eigenvalue corresponding to the composite field \(\Phi_0^{(1)}T\)

\[
-\frac{k}{32(k + N + 2)^5} \times \left[ 192(k + N + 2)^3 \right] + \frac{k(2k + 3)}{8(k + N + 2)^4} \times \left[ -24(k + N + 2)^2 \right] \\
-\frac{k}{4(k + N + 2)^4} \times \left[ 24(k + N + 2)^2 \right] + \frac{k(2k + 3)}{8(k + N + 2)^3} \times \left[ -8(k + N + 2) \right] = \]

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One can calculate the conformal dimension in this representation by using the formula (4.12).

\[ h(0; \square) = -\frac{4(N - 4)(1 + \frac{1}{N}) - \frac{4(N + 2)^2}{N(N + 2)(k + N + 2)} + 2 = \frac{2(k + 3)}{(k + N + 2)}. \]  

(4.29)

This calculation was not done in [54]. Then one has the following decomposition from (4.28) and (4.29)

\[ \phi_0^{(1)} (0; \square) h(0; \square) = \left[ -\frac{4k}{(k + N + 2)} \right] \times \left[ \frac{2(k + 3)}{(k + N + 2)} \right]. \]  

(4.30)

Therefore, one obtains the eigenvalue for the higher spin-1 current in this representation indirectly. See also (5.50) of [54]. On the other hand, one can check by calculating the OPE between the higher spin-1 current and \( Q^1 \ast Q^2 \ast Q^3 \ast Q^4 \ast \) and reading off the first order pole.

The three-point function of the higher spin-3 current with the scalar operator corresponding to \( (0; \square) \) and the scalar operator corresponding to \( (0; \square) \) can be obtained from the relation (4.27) explicitly as follows:

\[
\langle \mathcal{O}(0; \square) \mathcal{O}(0; \square) \Phi^{(1)} \rangle = \left\langle \frac{16k(12k^2N - 14k^2 + 6kN^2 + 57kN - 49k + 5N^2 + 91N + 12)}{3(k + N + 2)^2(6kN + 5k + 5N + 4)} \right\rangle \langle \mathcal{O}(0; \square) \mathcal{O}(0; \square) \rangle
\]

\[ \rightarrow \frac{16}{3} (2 - \lambda)(1 - \lambda) \langle \mathcal{O}(0; \square) \mathcal{O}(0; \square) \rangle = -\frac{4}{3} (2 - \lambda) \langle \mathcal{O}(0; \square) \mathcal{O}(0; \square) \Phi^{(1)} \rangle, \]  

(4.31)

where the limit in (4.10) is taken at the final stage. Furthermore, one can use the relation of the three-point function for the higher spin-1 current (containing the factor \( -4(1 - \lambda) \)) and obtain the explicit relation between two three-point functions, which is simple linear expression of the 't Hooft coupling constant \( \lambda \).

### 4.2.2 The eigenvalue in \( (0; \text{antisymm}) \) representation with \( p \equiv |\Lambda_-| \) boxes

As in previous subsection, one can consider the \( p \) multiple product \( Q^1 \ast Q^2 \ast Q^3 \ast \cdots Q^p \ast \) (see the matrix (4.26)) as one of the antisymmetric representation with \( p \) boxes. One should calculate
the corresponding eigenvalues as done in (4.27). In particular, the eigenvalue for the operator 
\( d_{ab}^0 Q^a \partial^2 Q^b \) is the \( p \) times of the one for the fundamental representation from the observations in (4.6), (4.7) and (4.8). One expects that the generalization of three-point function of the higher spin-3 current in (4.31) with the scalar operator corresponding to \((0;\text{antisymm})\) and the scalar operator corresponding to \((0;\text{antisymm})\) can be written in terms of \( \frac{4p}{3} (2 - \lambda)(1 - \lambda) \) multiplied by the two-point function between the two scalar operators under the large \((N, k)\) \('t Hooft like limit (4.10). That is, one has

\[
\langle \mathcal{O}(0; \Lambda_-) \mathcal{O}(0; \Lambda_-) \Phi^{(1)}_2 \rangle \rightarrow \frac{4|\Lambda_-|}{3} (2 - \lambda)(1 - \lambda) \langle \mathcal{O}(0; \Lambda_-) \mathcal{O}(0; \Lambda_-) \rangle = -\frac{4}{3} (2 - \lambda) \langle \mathcal{O}(0; \Lambda_-) \mathcal{O}(0; \Lambda_-) \Phi^{(1)}_0 \rangle. \tag{4.32}
\]

The \(|\Lambda_-|\) is the number of boxes of Young tableaux. Here the relation for the three-point function of the higher spin-1 current with two scalar operators under the large \((N, k)\) \('t Hooft like limit (from its finite result) is used. In other words, the eigenvalue is given by

\[
-\frac{|\Lambda_-|}{k+N+2} \rightarrow -|\Lambda_-|(1 - \lambda) \text{ which generalizes the cases in (4.13), (4.19), (4.22) or (4.30).}
\]

Note that although the above analysis is for the antisymmetric representations, this result holds (under the large \((N, k)\) \('t Hooft-like limit) for any representations which have the number of total boxes \(|\Lambda_-|\). One can check that the OPE between the quadratic term in the fourth line of (3.110) corresponding to the leading behavior and the \(|\Lambda_-|\) multiple product of spin-\(\frac{1}{2}\) current implies the coefficient, \(-2(k+N+2) \times |\Lambda_-|\), in the third order pole.

4.3 The \((\Lambda_+; \Lambda_-)\) representations up to two boxes for \(\Lambda_+\)

Let us consider the case where the representation \(\Lambda_-\) appears in the branching of \(\Lambda_+\) under the \(SU(N)_k \times SU(2)_k \times U(1)\). There is a trivial \(l^- = 0\) quantum number. The \(l^+\) quantum number and \(\hat{u}\) charge can be read off from the multiple product of \((\Box, 1)_1 + (1, 2)_{-\frac{N}{2}}\) in \[34]. It is known that the branching rules for the symmetric and antisymmetric representations satisfy \[34, 75, 76\]

\[
\Box \rightarrow (\Box, 1)_2 + (\Box, 2)_{1-\frac{N}{2}} + (1, 3)_{-N},
\]

\[
\boxed{1} \rightarrow (\Box, 1)_2 + (\Box, 2)_{1-\frac{N}{2}} + (1, 1)_{-N}. \tag{4.33}
\]

In this subsection, the previous known results are reinterpreted and some new features are presented based on the results of section 3.

4.3.1 The eigenvalue in \((f;0)\) representation

The eigenvalue of the zeromode for the higher spin current appearing in the first term of (3.111) in this representation is given by \((18, 50, 98)\) for \(N = 3, 5, 7\) respectively. Then the general \(N\)
dependence is given by $2N^2$. Similarly, From the eigenvalues $(-648, -3000, -8232, -17496)$ of the zeromode for the higher spin current appearing in the second term of (3.111) for $N = 3, 5, 7, 9$, the general $N$ behavior is given by $-24N^3$. The eigenvalue of the zeromode for the higher spin current appearing in the third term of (3.111) in this representation is given by $(-72, -200, -392)$ for $N = 3, 5, 7$ respectively. Then the general $N$ dependence can be read off and it is given by $-8N^2$. From the eigenvalues $(0, 0, 0)$ of the zeromode for the higher spin current appearing in the fourth term of (3.111) for $N = 3, 5, 7$, the general $N$ behavior is given by 0. The eigenvalue of the zeromode for the higher spin current appearing in the fifth term of (3.111) in this representation is given by $(-3, -5, -7)$ for $N = 3, 5, 7$ respectively. Then the general $N$ dependence can be read off and it is given by $-N$. Finally, from the eigenvalues $(12, 20, 28)$ of the zeromode for the higher spin current appearing in the last term of (3.111) for $N = 3, 5, 7$, the general $N$ behavior is given by $4N$. Then one obtains the following eigenvalue by collecting all the contributions with correct coefficients

$$
\phi_2^{(1)}(\Box; 0) = \frac{16(k - N)}{(6k + 5k + 5N + 4)} \left[ -\frac{1}{4(k + N + 2)^2} \times 2N^2 + \frac{1}{32N^2(k + N + 2)^2} \times (-24N^3) \right]
+ \frac{1}{2N(k + N + 2)^2} \times (-8N^2) + \frac{1}{(k + N + 2)^2} \times 0 - \frac{4k}{(k + N + 2)^2} \times (-N)
- \frac{2(2k + N)}{3(k + N + 2)^2} \times 4N = -\frac{4N(6k^2N + 5k^2 + 12kN^2 + 39kN + 28k + 4N^2 + 14N + 12)}{3(k + N + 2)^2(6kN + 5k + 5N + 4)}
\rightarrow -\frac{4}{3} \lambda (\lambda + 1),
$$

(4.34)

where the limit in (4.10) is taken at the final stage. It is obvious to see that the leading contributions in this limit come from the fifth and the sixth terms of (4.34).

By noting that the eigenvalue for the zeromode of $\Phi_0^{(1)} T$ can be read off from the first two terms in (4.34), it is given by

$$
\left[ -\frac{1}{4(k + N + 2)^2} \times 2N^2 + \frac{1}{32N^2(k + N + 2)^2} \times (-24N^3) \right] = -\frac{N(2N + 3)}{4(k + N + 2)^2}.
$$

(4.35)

Then one can decompose this (4.35) as the product of two eigenvalues

$$
\phi_0^{(1)}(\Box; 0) h(\Box; 0) = \left[ -\frac{N}{(k + N + 2)} \right] \times \left[ \frac{(2N + 3)}{4(k + N + 2)} \right],
$$

(4.36)

where the conformal dimension is substituted. For example, the equation (2.9) of [54] can be used or one can use the formula in (4.12). Now one can see the equation (4.4) of [54] for the eigenvalue of the higher spin-1 current in this representation.

The three-point function of the higher spin-3 current with the scalar operator corresponding to $(\Box; 0)$ and the scalar operator corresponding to $(\Box; 0)$ can be obtained from the relation (4.34) explicitly.
4.3.2 The eigenvalue in (f;f) representation

The eigenvalue of the zeromode for the higher spin current appearing in the first term of (3.111) in this representation is given by \((-8, -8, -8)\) for \(N = 3, 5, 7\) respectively. Similarly, the eigenvalues are given by \((0, 0, 0)\) of the zeromode for the higher spin current appearing in the second or third term for \(N = 3, 5, 7\). From the eigenvalues \((-24, -40, -56)\) of the zeromode for the higher spin current appearing in the fourth term for \(N = 3, 5, 7\), the general \(N\) behavior is given by \(-8N\). The eigenvalues are \((2, 2, 2)\) of the zeromode for the higher spin current appearing in the fifth term for \(N = 3, 5, 7\). The eigenvalues of the zeromode for the higher spin current appearing in the last term are \((-8, -8, -8)\) for \(N = 3, 5, 7\). By collecting all the contributions with correct coefficients, one determines the final eigenvalue as follows:

\[
\phi^{(1)}_2(\square \Box) = \frac{16(k - N)}{(6kN + 5k + 5N + 4)} - \frac{1}{4(k + N + 2)^2} \times (-8) + \frac{1}{32N^2(k + N + 2)^2} \times 0
\]

where the limit in (4.10) is taken at the final stage. It is obvious to see that the leading contributions \((\frac{1}{N}\text{ term})\) in this limit come from the fourth, the fifth and the sixth terms of (4.37).

One can decompose the zeromode of \(\Phi^{(1)}_0 T\) appearing in the first and second terms of (4.37) as

\[
\phi^{(1)}_0(\square \Box) h(\square \Box) = \left(\frac{2}{(k + N + 2)}\right) \times \left[\frac{1}{(k + N + 2)}\right],
\]

where the equation (2.10) of [54] for the conformal dimension is substituted. Then one sees the coincidence with the equation (5.1) of [54] for the eigenvalue of the higher spin-1 current.

It is straightforward to write down the three-point function of the higher spin-3 current with the scalar operator corresponding to \((f; 0)\) and the scalar operator corresponding to \((\bar{f}; 0)\) from (4.37) as before.

4.3.3 The eigenvalue in (symm;0) representation with two boxes

The eigenvalue of the zeromode for the higher spin current appearing in the first term of (3.111) in this representation is given by \((72, 200, 392)\) for \(N = 3, 5, 7\) respectively. Then the general \(N\) dependence is given by \(8N^2\). By collecting all the contributions with correct coefficients, one determines the final eigenvalue as follows:

\[
\phi^{(1)}_2(\Box \square) = \frac{16(k - N)}{(6kN + 5k + 5N + 4)} - \frac{1}{4(k + N + 2)^2} \times (-8) + \frac{1}{32N^2(k + N + 2)^2} \times 0
\]

where the limit in (4.10) is taken at the final stage. It is obvious to see that the leading contributions \((\frac{1}{N}\text{ term})\) in this limit come from the fourth, the fifth and the sixth terms of (4.37).

One can decompose the zeromode of \(\Phi^{(1)}_0 T\) appearing in the first and second terms of (4.37) as

\[
\phi^{(1)}_0(\square \Box) h(\square \Box) = \left(\frac{2}{(k + N + 2)}\right) \times \left[\frac{1}{(k + N + 2)}\right],
\]

where the equation (2.10) of [54] for the conformal dimension is substituted. Then one sees the coincidence with the equation (5.1) of [54] for the eigenvalue of the higher spin-1 current.

It is straightforward to write down the three-point function of the higher spin-3 current with the scalar operator corresponding to \((f; 0)\) and the scalar operator corresponding to \((\bar{f}; 0)\) from (4.37) as before.

From the eigenvalues \((-3456, -16000, -43904, -93312)\) of the zeromode for the higher spin current appearing in the second term for \(N = 3, 5, 7, 9\), the general \(N\) behavior is given by \(-128N^3\). From the eigenvalues \((-144, -400, -784)\) of the zeromode for the higher spin current appearing in the third term for \(N = 3, 5, 7\), the general \(N\) behavior is given by \(-16N^2\). From the eigenvalues \((0, 0, 0)\) of the zeromode for the higher spin current appearing in the fourth term for \(N = 3, 5, 7\), the general \(N\) behavior is given by 0.
coefficients, one determines the final eigenvalue as follows:
\[
\phi^{(1)}_{2}(\square\square; 0) = \frac{16(k - N)}{(6kN + 5k + 5N + 4)} \left[ -\frac{1}{4(k + N + 2)^2} \times 8N^2 + \frac{1}{32N^2(k + N + 2)^2} \times (-128N^3) \right]
+ \frac{1}{2N(k + N + 2)^2} \times (-16N^2) \times 0 - \frac{4k}{(k + N + 2)^2} \times (-2N) - \frac{2(2k + N)}{3(k + N + 2)^2} \times 8N = -\frac{8N(6k^2N + 5k^2 + 12kN^2 + 45kN + 43k - 2N^2 - N + 12)}{3(k + N + 2)^2(6kN + 5k + 5N + 4)}
\rightarrow -\frac{8}{3} \lambda(\lambda + 1),
\]
(4.39)

where the limit in (4.10) is taken at the final stage. It is obvious that the leading contributions in this limit come from the fifth and the sixth terms of (4.39) as before. The eigenvalue in this limit is the twice of the one in (4.34).

In particular, the first and second terms of (4.39) contain
\[
\left[ -\frac{1}{4(k + N + 2)^2} \times 8N^2 + \frac{1}{32N^2(k + N + 2)^2} \times (-128N^3) \right] = -\frac{2N(N + 2)}{(k + N + 2)^2}.
\]
(4.40)

One can also decompose this (4.40) as follows:
\[
\phi^{(1)}_{0}(\square; 0) \phi^{(1)}(\square; 0) = \left[ -\frac{2N}{(k + N + 2)} \right] \times \left[ \frac{(N + 2)}{(k + N + 2)} \right],
\]
(4.41)

where one uses the equation (2.13) of [54]. This implies that one sees the eigenvalue for the higher spin-1 current given in the equation (5.18) of [54].

The third and fourth terms of (4.39) are given by
\[
\left[ \frac{1}{2N(k + N + 2)^2} \times (-16N^2) + \frac{1}{(k + N + 2)^2} \times 0 \right] = -\frac{8N}{(k + N + 2)^2}.
\]
(4.42)

Moreover, there are also other eigenvalues corresponding to these two terms on this representation
\[
\left[ \frac{1}{2N(k + N + 2)^2} \times (-24N^2) + \frac{1}{(k + N + 2)^2} \times 4N \right] = -\frac{8N}{(k + N + 2)^2},
\]
(4.43)

which is equal to (4.42). According to the first equation of (4.33), there are triplet states under the SU(2)_k. This leads to the same eigenvalue, (4.42) or (4.43), for the higher spin-3 current in this representation although the eigenvalues corresponding to the third and fourth terms are different from each other.

The three-point function of the higher spin-3 current with the scalar operator corresponding to $\square\square\square; 0$ and the scalar operator corresponding to $\square\square; 0$ can be read off from (4.39).

From the eigenvalues $(-6, -10, -14)$ of the zeromode for the higher spin current appearing in the fifth term for $N = 3, 5, 7$, the general $N$ behavior is given by $-2N$. From the eigenvalues $(24, 40, 56)$ of the zeromode for the higher spin current appearing in the last term for $N = 3, 5, 7$, the general $N$ behavior is given by $8N$.  

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4.3.4 The eigenvalue in (symm;symm) representation with two, and two boxes

The eigenvalue of the zeromode for the higher spin current appearing in the first term of (3.111) in this representation is given by \((-32, -32, -32)\) for \(N = 3, 5, 7\) respectively.\(^{28}\)

Then one obtains the following eigenvalue

\[
\phi_2^{(1)}(\begin{array}{cc}
\mathcal{M} & \mathcal{M} \\
\mathcal{M} & \mathcal{M}
\end{array}) = \frac{16(k-N)}{(6kN + 5k + 5N + 4)} \left[ -\frac{1}{4(k+N+2)^2} \times (-32) + \frac{1}{32N^2(k+N+2)^2} \times 0 \right] + \frac{1}{2N(k+N+2)^2} \times 0 + \frac{1}{(k+N+2)^2} \times (-16N - 16) - \frac{4k}{(k+N+2)^2} \times 4 \\
- \frac{2(2k+N)}{3(k+N+2)^2} \times (-16) = \frac{16(6k^2N + 5k^2 - 6kN^2 - 18kN + 13k - 5N^2 - 43N - 12)}{3(k+N+2)^2(6kN + 5k + 5N + 4)} \\
\rightarrow -\frac{16}{3N} \lambda(2\lambda - 1),
\]

\(^{(4.44)}\)

where the limit in \((4.10)\) is taken at the final stage. It is obvious that the leading contributions in this limit come from the fourth, fifth and the sixth terms of \((4.44)\) as before. The eigenvalue in this limit is the twice of the one in \((4.37)\). Note that the number of boxes is increased by two respectively.

From the first and the second terms of \((4.44)\), one can decompose the corresponding eigenvalue as

\[
\phi_0^{(1)}(\begin{array}{cc}
\mathcal{M} & \mathcal{M} \\
\mathcal{M} & \mathcal{M}
\end{array}) h(\begin{array}{cc}
\mathcal{M} & \mathcal{M} \\
\mathcal{M} & \mathcal{M}
\end{array}) = \left[ \frac{4}{(k+N+2)} \right] \times \left[ \frac{2}{(k+N+2)} \right],
\]

\(^{(4.45)}\)

where the equation (3.5) of [54] can be used. Then one sees the coincidence with the equation (5.22) of [54] for the eigenvalue of the higher spin-1 current.

The three-point function of the higher spin-3 current with the scalar operator corresponding to \(\begin{array}{cc}
\mathcal{M} & \mathcal{M} \\
\mathcal{M} & \mathcal{M}
\end{array}\) and the scalar operator corresponding to \(\begin{array}{cc}
\mathcal{M} & \mathcal{M} \\
\mathcal{M} & \mathcal{M}
\end{array}\) can be read off from \((4.44)\) as before.

4.3.5 The eigenvalue in (symm;f) representation with two boxes

The eigenvalue of the zeromode for the higher spin current appearing in the first term of (3.111) in this representation is given by \((14, 54, 110)\) for \(N = 3, 5, 7\) respectively. Then the general \(N\) dependence is given by \((2N^2 + 4N - 16)\)\(^{29}\). Then one obtains the following eigenvalue

\[\cdots\]

\[^{28}\] Similarly, from the eigenvalues \((0, 0, 0)\) appearing in the second or third term for \(N = 3, 5, 7\), the general \(N\) behavior can be read off. From the eigenvalues \((-64, -96, -128)\) appearing in the fourth term for \(N = 3, 5, 7\), the general \(N\) behavior is given by \((-16N - 16)\). The eigenvalue appearing in the fifth term in this representation is given by \((4, 4, 4)\) for \(N = 3, 5, 7\) respectively. Finally, from the eigenvalues \((-16, -16, -16)\) appearing in the last term for \(N = 3, 5, 7\), the general \(N\) behavior can be obtained.

\[^{29}\] Similarly, from the eigenvalues \((-216, -1800, -5880, -13608)\) appearing in the second term for \(N = 3, 5, 7, 9\), the general \(N\) behavior is given by \((-24N^3 + 48N^2)\). From the eigenvalues \((-120, -280, -504)\) appearing in the third term for \(N = 3, 5, 7\), the general \(N\) behavior is given by \((-8N^2 - 16N)\). From the
\[
\phi_2^{(1)}(\square\square) = \frac{16(k - N)}{(6kN + 5k + 5N + 4)} \left[ -\frac{1}{4(k + N + 2)^2} \times (2N^2 + 4N - 16) + \frac{1}{32N^2(k + N + 2)^2} \times (-24N^3 + 48N^2) \right]
+ \frac{1}{2N(k + N + 2)^2} \times (-8N^2 - 16N) + \frac{1}{(k + N + 2)^2} \times (-16N) - \frac{4k}{(k + N + 2)^2} \times (-N + 2)
- \frac{2(2k + N)}{3(k + N + 2)^2} \times (4N - 8) =
\frac{-4(6k^2N^2 - 7k^2N - 10k^2 + 12kN^3 + 87kN^2 + 106kN - 44k + 4N^3 + 42N^2 + 140N + 24)}{3(k + N + 2)^2(6kN + 5k + 5N + 4)}
\rightarrow -\frac{4}{3} \lambda(\lambda + 1),
\]

(4.46)

where the limit in (4.10) is taken at the final stage. It is obvious that the leading contributions in this limit come from the fifth and the sixth terms of (4.46) as before. The eigenvalue in this limit is the same as the one in (1.34). In other words, the boldface parts of two expressions are the same as each other.

The first and second terms of (4.46) contain
\[
\left[ -\frac{1}{4(k + N + 2)^2} \times (2N^2 + 4N - 16) + \frac{1}{32N^2(k + N + 2)^2} \times (-24N^3 + 48N^2) \right] =
\frac{-(N - 2)(2N + 11)}{4(k + N + 2)^2}.
\]

(4.47)

One can decompose this (4.47) as
\[
\phi_0^{(1)}(\square\square) h(\square\square) = \left[ -\frac{(N - 2)}{(k + N + 2)} \right] \times \left[ \frac{(2N + 11)}{4(k + N + 2)} \right],
\]

(4.48)

where the equation (3.6) of [54] is used. Then one can see the equation (5.23) of [54] in this expression.

The three-point function of the higher spin-3 current with the scalar operator corresponding to (\square\square) and the scalar operator corresponding to (\square\square) can be read off from (4.46).

4.3.6 The eigenvalue in (antisymm;0) representation with two boxes

The eigenvalue of the zeromode for the higher spin current appearing in the first term of (3.111) in this representation is given by (72, 200, 392) for \(N = 3, 5, 7\) respectively. Then the eigenvalues \((-48, -80, -112)\) appearing in the fourth term for \(N = 3, 5, 7\), the general \(N\) behavior is given by \((-16N)\). The eigenvalue appearing in the fifth term in this representation is given by \((-1, -3, -5)\) for \(N = 3, 5, 7\) respectively. Then the general \(N\) dependence is given by \((-N + 2)\). Finally, from the eigenvalues \((4, 12, 20)\) appearing in the last term for \(N = 3, 5, 7\), the general \(N\) behavior is given by \((4N - 8)\).
The general $N$ dependence is given by $8N^2$ \[^{30}\]. By collecting all the contributions with correct coefficients, one determines the final eigenvalue as follows:

$$
\phi_2^{(1)}(\overline{\mathbf{B}}^0, 0) = \frac{16(k - N)}{(6kN + 5k + 5N + 4)} \left[ -\frac{1}{4(k + N + 2)^2} \times (8N^2) + \frac{1}{32N^2(k + N + 2)^2} \times 0 \right] \\
+ \frac{1}{2N(k + N + 2)^2} \times 0 + \frac{1}{(k + N + 2)^2} \times 8N - \frac{4k}{(k + N + 2)^2} \times (-2N) \\
- \frac{2(2k + N)}{3(k + N + 2)^2} \times 8N = \\
- \frac{8N(6k^2N + 5k^2 + 12kN^2 + 9kN - 11k - 2N^2 - 7N - 12)}{3(k + N + 2)^2(6kN + 5k + 5N + 4)} \rightarrow -\frac{8}{3} \lambda (\lambda + 1), \tag{4.49}
$$

where the limit in (4.10) is taken at the final stage. It is obvious that the leading contributions in this limit come from the fifth and the sixth terms of (4.49) as before. The eigenvalue in this limit is the twice of the one in (4.34). The number of boxes is increased by two.

The following decomposition corresponding to the first and second terms of (4.49) can be seen from the previous results in the equations (2.14) and (5.19) of \[^{54}\]

$$
\phi_0^{(1)}(\overline{\mathbf{B}}^0, 0) h(\overline{\mathbf{B}}^0, 0) = \left[ -\frac{2N}{(k + N + 2)} \right] \times \left[ \frac{N}{(k + N + 2)} \right]. \tag{4.50}
$$

The three-point function of the higher spin-3 current with the scalar operator corresponding to $(\overline{\mathbf{B}}^0, 0)$ and the scalar operator corresponding to $(\overline{\mathbf{B}}^0, 0)$ can be read off from (4.49).

### 4.3.7 The eigenvalue in (antisymmm;antisymmm) representation with two, and two boxes

The eigenvalue of the zeromode for the higher spin current appearing in the first term of \[^{31}\] in this representation is given by \((-32, -32, -32)\) for $N = 3, 5, 7$ respectively. Then one obtains the following eigenvalue

$$
\phi_2^{(1)}(\overline{\mathbf{E}}^0, 0) = \frac{16(k - N)}{(6kN + 5k + 5N + 4)} \left[ -\frac{1}{4(k + N + 2)^2} \times (32) + \frac{1}{32N^2(k + N + 2)^2} \times 0 \right] \\
+ \frac{1}{2N(k + N + 2)^2} \times 0 + \frac{1}{(k + N + 2)^2} \times (-16N + 16) - \frac{4k}{(k + N + 2)^2} \times 4
$$

\[^{30}\] Similarly, there are the eigenvalues $(0, 0, 0)$ appearing in the second or third term for $N = 3, 5, 7$. From the eigenvalues $(24, 40, 56)$ appearing in the fourth term for $N = 3, 5, 7$, the general $N$ behavior is given by $8N$. From the eigenvalues $(-6, -10, -14)$ appearing in the fifth term for $N = 3, 5, 7$, the general $N$ behavior is given by $-2N$. From the eigenvalues $(24, 40, 56)$ appearing in the last term for $N = 3, 5, 7$, the general $N$ behavior is given by $8N$.

\[^{31}\] Similarly, there exist the eigenvalues $(0, 0, 0)$ appearing in the second or third term for $N = 3, 5, 7$. From the eigenvalues $(-32, -64, -96)$ appearing in the fourth term for $N = 3, 5, 7$, the general $N$ behavior is given by $-16N + 16$. The eigenvalue appearing in the fifth term in this representation is given by $(4, 4, 4)$ for $N = 3, 5, 7$ respectively. Finally, the eigenvalues are given by $(-16, -16, -16)$ appearing in the last term for $N = 3, 5, 7$. 

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The eigenvalue in (antisymm;f) representation with two boxes

The three-point function of the higher spin-3 current with the scalar operator corresponding to \((\square \square)\) and the scalar operator corresponding to \((\square \square)\) can be obtained from the relation \((4.51)\) explicitly.

### 4.3.8 The eigenvalue in (antisymm;f) representation with two boxes

The eigenvalue of the zeromode for the higher spin current appearing in the first term of \((3.111)\) in this representation is given by \((6,30,70)\) for \(N = 3, 5, 7\) respectively. By collecting all the contributions with correct coefficients, one determines the final eigenvalue as follows:

\[
\phi_2^{(1)}[\square \square] = \phi_0^{(1)}[\square \square] h[\square \square] = \frac{16}{3}\left(\frac{1}{(k + N + 2)^2}\times (2N^2 - 4N) + \frac{1}{32N^2(k + N + 2)^2}\times (-24N^3 + 48N^2)\right) + \frac{2}{N(k + N + 2)^2}\times (-8N^2 + 16N) + \frac{1}{(k + N + 2)^2}\times 0 - \frac{4k}{(k + N + 2)^2}\times (-N + 2) - \frac{2(2k + N)}{3(k + N + 2)^2}\times (4N - 8) - \frac{4(N - 2)(6k^2N + 5k^2 + 12kN^2 + 39kN + 28k + 4N^2 + 14N + 12)}{3(k + N + 2)^2(6kN + 5k + 5N + 4)} \rightarrow -\frac{4}{3}\lambda(\lambda + 1),
\]

where the limit in \((4.10)\) is taken at the final stage. It is obvious that the leading contributions in this limit come from the fifth and the sixth terms of \((4.53)\) as before. The eigenvalue in this limit is the same as the one in \((4.37)\). In other words, the boldface parts are increased by two.

One decomposes the expression appearing in the first two terms of \((4.51)\) as

\[
\phi_0^{(1)}[\square \square] h[\square \square] = \frac{4}{(k + N + 2)^2} \times \frac{2}{(k + N + 2)},
\]

by using the previous results in the equations \((3.16)\) and \((5.36)\) of \([54]\).

32 Then the general \(N\) dependence is given by \((2N^2 - 4N)\). Similarly, from the eigenvalues \((-216, -1800, -5880, -13608)\) appearing in the second term for \(N = 3, 5, 7, 9\), the general \(N\) behavior is given by \((-24N^3 + 48N^2)\). From the eigenvalues \((-24, -120, -280)\) appearing in the third term for \(N = 3, 5, 7\), the general \(N\) behavior is given by \((-8N^2 + 16N)\). From the eigenvalues \((0, 0, 0)\) appearing in the fourth term for \(N = 3, 5, 7\), the general \(N\) behavior is given by \(0\). From the eigenvalues \((-1, -3, -5)\) appearing in the fifth term for \(N = 3, 5, 7\), the general \(N\) behavior is given by \((-N + 2)\). From the eigenvalues \((4, 12, 20)\) appearing in the last term for \(N = 3, 5, 7\), the general \(N\) behavior is given by \((4N - 8)\).
From the expression coming from the first two terms of (4.53)
\[
\left[ -\frac{1}{4(k + N + 2)^2} \times (2N^2 - 4N) + \frac{1}{32N^2(k + N + 2)^2} \times (-24N^3 + 48N^2) \right] = \\
-\frac{(N - 2)(2N + 3)}{4(k + N + 2)^2},
\]
one decomposes this (4.54) as
\[
\phi_0^{(1)}(□□) \ h(□□) = \left[ -\frac{(N - 2)}{(k + N + 2)} \right] \times \left[ \frac{(2N + 3)}{4(k + N + 2)} \right],
\]
by using the previous results in the equations (3.17) and (5.37) of [54].

The three-point function of the higher spin-3 current with the scalar operator corresponding to (□□) and the scalar operator corresponding to (□□) can be obtained from the relation (4.53) explicitly.

4.4 The \((\Lambda_+; \Lambda_-)\) representations with three boxes for \(\Lambda_+\)

It is known that the following branching rules for symmetric, mixed and antisymmetric representations hold [54]
\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} & \rightarrow (□□□\ 1)_3 + (□□\ 2)_{2-N} + (□\ 3)_{1-N} + (1, 4)_{2-N} , \\
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} & \rightarrow (□\ 1)_{3-N} + (□\ 2)_{2-N} + (□\ 3)_{1-N} + (1, 2)_{2-N}, \\
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} & \rightarrow (□\ 1)_{3-N} + (□\ 2)_{2-N} + (1\ 1)_{1-N} .
\end{align*}
\]

In this subsection, some new features in the representations with more than three boxes are presented.

4.4.1 The eigenvalue in (symm;0) representation with three boxes

The eigenvalue of the zero mode for the higher spin current appearing in the first term of (3.111) in this representation is given by \((162, 450, 882)\) for \(N = 3, 5, 7\) respectively. Then the general \(N\) dependence can be read off and it is given by \(18N^2\) \(^{33}\). One obtains the final

\(^{33}\) Similarly, from the eigenvalues \((-9720, -45000, -123480, -262440)\) appearing in the second term for \(N = 3, 5, 7, 9\), the general \(N\) behavior is given by \(-360N^3\). The eigenvalue appearing in the third term in this representation is given by \((-432, -1200, -2352)\) for \(N = 3, 5, 7\) respectively. Then the general \(N\) dependence can be read off and it is given by \(-48N^2\). From the eigenvalues \((36, 60, 84)\) appearing in the fourth term for \(N = 3, 5, 7\), the general \(N\) behavior is given by \(12N\). The eigenvalue appearing in the fifth term in this representation is given by \((-9, -15, -21)\) for \(N = 3, 5, 7\) respectively. Then the general \(N\) dependence can be read off and it is given by \(-3N\). Finally, from the eigenvalues \((36, 60, 84)\) appearing in the last term for \(N = 3, 5, 7\), the general \(N\) behavior is given by \(12N\).
eigenvalue as follows:

$$\phi_{2}^{(1)(\square; 0)} = \frac{16(k - N)}{(6kN + 5k + 5N + 4)} \left[ -\frac{1}{4(k + N + 2)^2} \times 18N^2 + \frac{1}{32N^2(k + N + 2)^2} \times (-360N^3) \right]$$

$$+ \frac{1}{2N(k + N + 2)^2} \times (-48N^2) + \frac{1}{(k + N + 2)^2} \times 12N - \frac{4k}{(k + N + 2)^2} \times (-3N)$$

$$- \frac{2(2k + N)}{3(k + N + 2)^2} \times 12N = -\frac{4N(6k^2N + 5k^2 + 12kN^2 + 51kN + 64k - 8N^2 - 22N + 12)}{(k + N + 2)^2(6kN + 5k + 5N + 4)}$$

$$\rightarrow -4\lambda (\lambda + 1), \quad (4.57)$$

where the limit in (4.10) is taken at the final stage. It is obvious that the leading contributions in this limit come from the fifth and the sixth terms of (4.57). The eigenvalue in this limit is the third times of the one in (4.34). The boldface parts are increased by three.

On the other hand, the other eigenvalue of the zeromode for the higher spin current appearing in the third term of (3.111) in this representation is given by $(-288, -800, -1568)$ for $N = 3, 5, 7$ respectively. Then the general $N$ dependence can be read off and it is given by $-32N^2$. From the other eigenvalue $(12, 20, 28)$ of the zeromode for the higher spin current appearing in the fourth term of (3.111) for $N = 3, 5, 7$, the general $N$ behavior is given by $4N$. From the previous results of third and fourth terms of (4.57), one has the following value

$$\left[ \frac{1}{2N(k + N + 2)^2} \times (-48N^2) + \frac{1}{(k + N + 2)^2} \times 12N \right] = -\frac{12N}{(k + N + 2)^2}. \quad (4.58)$$

Moreover, one has the same total eigenvalue for each different eigenvalue (according to the first equation of (4.56) there are quartic states under SU(2)$_k$ in this representation) as follows:

$$\left[ \frac{1}{2N(k + N + 2)^2} \times (-32N^2) + \frac{1}{(k + N + 2)^2} \times 4N \right] = -\frac{12N}{(k + N + 2)^2}. \quad (4.59)$$

Therefore, one obtains the same eigenvalue in (4.57) because of the coincidence between (4.58) and (4.59).

By realizing that the eigenvalue of the zeromode of $\Phi_{0}^{(1)} T$ is given by

$$\left[ -\frac{1}{4(k + N + 2)^2} \times 18N^2 + \frac{1}{32N^2(k + N + 2)^2} \times (-360N^3) \right] = -\frac{9N(2N + 5)}{4(k + N + 2)^2}, \quad (4.60)$$

from the first two terms in (4.57), one can decompose this eigenvalue (4.60) as the following form

$$\phi_{0}^{(1)(\square; 0)} h(\square; 0) = \left[ -\frac{3N}{(k + N + 2)} \right] \times \left[ \frac{3(2N + 5)}{4(k + N + 2)} \right], \quad (4.61)$$
where the conformal dimension for this representation appearing in the equation (3.29) of [54] is substituted. Then one observes that the eigenvalue for the zeromode of the higher spin-1 current in this representation can be read off and it is given by

\[
\phi^{(1)}_{0}(\text{symm}; 0) = -\frac{3N}{(k + N + 2)} = 3\phi^{(1)}_{0}(\Box; 0),
\]

where the relation in (4.36) is used. On the other hand, the other eigenvalue of the zeromode for the higher spin-1 current appearing in the first term of (3.1) in this representation is given by \((-\frac{9}{k+5}, -\frac{15}{k+7}, -\frac{21}{k+9})\) for \(N = 3, 5, 7\) respectively. The result of (4.62) implies that the multiplicity 3 on the eigenvalue \(\phi^{(1)}_{0}(\text{symm}; 0)\) is valid at finite \((N, k)\) (as well as under the large \((N, k)\) ’t Hooft like limit). The three-point function of the higher spin-1 current with two scalars is

\[
< \mathcal{O}(\text{symm}; 0) \mathcal{O}(\text{symm}; 0) \Phi^{(1)}_{0} > \equiv -\frac{3N}{(k + N + 2)} < \mathcal{O}(\text{symm}; 0) \mathcal{O}(\text{symm}; 0) > \rightarrow -3\lambda < \mathcal{O}(\text{symm}; 0) \mathcal{O}(\text{symm}; 0) >, \tag{4.63}
\]

where the limit in (4.10) is taken at the final stage.

Note that one observes the relation between two eigenvalues

\[
\phi^{(1)}_{2}(\text{symm}; 0) = 3\phi^{(1)}_{2}(\Box; 0) = 3 \times \left[ -\frac{4}{3} \lambda (\lambda + 1) \right], \tag{4.64}
\]

under the large \((N, k)\) ’t Hooft like limit, as anticipated in [54]. In (4.64), the previous result (4.34) is used.

The three-point function of the higher spin-3 current with the scalar operator corresponding to \((\text{symm}; 0)\) and the scalar operator corresponding to \((\text{symm}; 0)\) can be obtained from the relation (4.57) explicitly

\[
< \mathcal{O}(\text{symm}; 0) \mathcal{O}(\text{symm}; 0) \Phi^{(1)}_{2} > \equiv \left[ -\frac{4N(6k^2N + 5k^2 + 12kN^2 + 51kN + 64k - 8N^2 - 22N + 12)}{(k + N + 2)^2(6kN + 5k + 5N + 4)} \right] < \mathcal{O}(\text{symm}; 0) \mathcal{O}(\text{symm}; 0) > \rightarrow -4\lambda(\lambda + 1) < \mathcal{O}(\text{symm}; 0) \mathcal{O}(\text{symm}; 0) > = \frac{4}{3} (\lambda + 1) < \mathcal{O}(\text{symm}; 0) \mathcal{O}(\text{symm}; 0) \Phi^{(1)}_{0} >, \tag{4.65}
\]

where the limit in (4.10) is taken at the final stage and the relation (4.63) is used. One can see that the three-point function (4.65) is a multiple of the one in the subsection 4.3.1 because of (4.64).

### 4.4.2 The eigenvalue in (symm; symm) representation with three, and three boxes

The eigenvalue of the zeromode for the higher spin current appearing in the first term of (3.111) in this representation is given by \((-72, -72, -72)\) for \(N = 3, 5, 7\) respectively [54]. By

\[34\) Similarly, the eigenvalues appearing in the second or third term are given by \((0, 0, 0)\) for \(N = 3, 5, 7, 9\). The eigenvalue appearing in the fourth term in this representation is given by \((-120, -168, -216)\) for \(N = 3, 5, 7\).\]
collecting all the contributions with correct coefficients, one determines the final eigenvalue as follows:

\[
\phi^{(1)}_2(\square; \square) = \frac{16(k - N)}{6kN + 5k + 5N + 4} \left[ -\frac{1}{4(k + N + 2)^2} \times (-72) + \frac{1}{32N^2(k + N + 2)^2} \times 0 \right] \\
+ \frac{1}{2N(k + N + 2)^2} \times 0 + \frac{1}{(k + N + 2)^2} \times \left( -\frac{24N - 48}{4k} \right) - \frac{2(2k + N)}{3(k + N + 2)^2} \times (-24) = \frac{8(6k^2N + 5k - 6kN^2 - 36kN + 10k - 5N^2 - 70N - 24)}{(k + N + 2)^2(6kN + 5k + 5N + 4)} \\
\rightarrow -\frac{8}{N} \lambda (2\lambda - 1),
\]

(4.66)

where the limit in (4.10) is taken at the final stage. It is obvious that the leading contributions in this limit come from the fourth, the fifth and the sixth terms of (4.66). The eigenvalue in this limit is the third times of the one in (4.37). The number of boxes is increased by three.

By realizing that the eigenvalue of the zeromode of \( \Phi^{(1)}_0 T \) is given by

\[
\phi^{(1)}_0(\square; \square) = \frac{6}{(k + N + 2)} \times \left[ \frac{6}{(k + N + 2)} \right] \times \left[ \frac{3}{(k + N + 2)} \right],
\]

(4.68)

where the conformal dimension for this representation appearing in the subsection 3.3.1 of [54] is substituted. Then one sees that the eigenvalue for the zeromode of the higher spin-1 current in this representation can be read off and it is given by, together with (4.38),

\[
\phi^{(1)}_0(\square; \square) = \frac{6}{(k + N + 2)} = 3\phi^{(1)}_0(\square; \square),
\]

(4.69)

which implies that the multiplicity 3 on the eigenvalue \( \phi^{(1)}_0(\square; \square) \) in (4.69) is valid at finite \((N, k)\) (as well as under the large \((N, k)\) ’t Hooft like limit). On the other hand, the other eigenvalue of the zeromode for the higher spin-1 current appearing in the first term of (3.1) in this representation is given by \((\frac{6}{(k+5)}, \frac{6}{(k+7)}, \frac{6}{(k+9)})\) for \( N = 3, 5, 7 \) respectively. The three-point function of the higher spin-1 current with scalars is given by

\[
< \mathcal{O}(\square; \square) \mathcal{O}(\square; \square) \mathcal{O}^{(1)}_0(\square; \square) > = \left[ \frac{6}{(k + N + 2)} \right] < \mathcal{O}(\square; \square) \mathcal{O}(\square; \square) > \\
\rightarrow \frac{6\lambda}{N} < \mathcal{O}(\square; \square) \mathcal{O}(\square; \square) >,
\]

(4.70)

respectively. Then the general \( N \) dependence can be read off and it is given by \((-24N - 48)\). The eigenvalues are \((6, 6, 6)\) appearing in the fifth term for \( N = 3, 5, 7 \). The eigenvalues appearing in the last term are given by \((-24, -24, -24)\) for \( N = 3, 5, 7 \).
where the limit in (4.10) is taken at the final stage.

Note that one observes the relation between two eigenvalues

\[ \phi^{(1)}_{2}(\begin{array}{c}
\bullet
\bullet
\bullet
\end{array}) = 3\phi^{(1)}_{2}(\begin{array}{c}
\bullet
\bullet
\end{array}) = 3 \times \left[ -\frac{8}{3N} \lambda (2\lambda - 1) \right], \tag{4.71} \]

under the large \((N, k)\) 't Hooft like limit, as anticipated in [54]. In (4.71), the previous relation (3.37) is used.

The three-point function of the higher spin-3 current with the scalar operator corresponding to \((\begin{array}{c}
\bullet
\bullet
\bullet
\end{array})\) and the scalar operator corresponding to \((\begin{array}{c}
\bullet
\bullet
\end{array})\) can be obtained from the relation (4.66) explicitly

\[ < \mathcal{O}(\begin{array}{c}
\bullet
\bullet
\bullet
\end{array}) \mathcal{O}(\begin{array}{c}
\bullet
\bullet
\end{array}) \Phi^{(1)}_{2} > \\
= \left[ \frac{8(6k^{2}N + 5k - 6kN^{2} - 36kN + 10k - 5N^{2} - 70N - 24)}{(k + N + 2)(6kN + 5k + 5N + 4)} \right] < \mathcal{O}(\begin{array}{c}
\bullet
\bullet
\bullet
\end{array}) \mathcal{O}(\begin{array}{c}
\bullet
\bullet
\end{array}) > \\
\rightarrow -\frac{8}{N} \lambda (2\lambda - 1) < \mathcal{O}(\begin{array}{c}
\bullet
\bullet
\bullet
\end{array}) \mathcal{O}(\begin{array}{c}
\bullet
\bullet
\end{array}) \Phi^{(1)}_{2} > = \\
\frac{4}{3} (2\lambda - 1) < \mathcal{O}(\begin{array}{c}
\bullet
\bullet
\bullet
\end{array}) \mathcal{O}(\begin{array}{c}
\bullet
\bullet
\end{array}) \Phi^{(1)}_{2} >, \tag{4.72} \]

where the limit in (4.10) is taken at the final stage. Moreover, the relation in (4.70) is used.

The three-point function (4.72) is a multiple of the one in the subsection 4.3.2 according to (4.71).

### 4.4.3 The eigenvalue in \((\text{symm; symm})\) representation with three, and two boxes

The eigenvalue of the zeromode for the higher spin current appearing in the first term of (3.111) in this representation is given by \((-22, 26, 90)\) for \(N = 3, 5, 7\) respectively. Then the general \(N\) dependence can be read off and it is given by \((2N^{2} + 8N - 64)\)\(^{35}\). One obtains the final eigenvalue as follows:

\[ \phi^{(1)}_{2}(\begin{array}{c}
\bullet
\bullet
\bullet
\end{array}) = \frac{16(k - N)}{(6kN + 5k + 5N + 4)} \left[ -\frac{1}{4(k + N + 2)^{2}} \times (2N^{2} + 8N - 64) + \frac{1}{32N^{2}(k + N + 2)^{2}} \times (96N^{2} - 24N^{3}) \right] \\
+ \frac{1}{2N(k + N + 2)^{2}} \times (-8N^{2} - 32N) + \frac{1}{(k + N + 2)^{2}} \times (-32N - 32) - \frac{4k}{(k + N + 2)^{2}} \times (-N + 4) \\
- \frac{2(2k + N)}{3(k + N + 2)^{2}} \times (4N - 16) \]

\(^{35}\) Similarly, from the eigenvalues \((216, -600, -3528, -9720)\) appearing in the second term for \(N = 3, 5, 7, 9\), the general \(N\) behavior is given by \((-24N^{2} + 96N^{2})\). The eigenvalue appearing in the third term in this representation is given by \((-168, -360, -616)\) for \(N = 3, 5, 7\) respectively. Then the general \(N\) dependence can be read off and it is given by \((-8N^{2} - 32N)\). From the eigenvalues \((1, -1, -3)\) appearing in the fourth term for \(N = 3, 5, 7\), the general \(N\) behavior is given by \((-N + 4)\). The eigenvalue appearing in the fifth term in this representation is given by \((-9, -15, -21)\) for \(N = 3, 5, 7\) respectively. Then the general \(N\) dependence can be read off and it is given by \(-3N\). Finally, from the eigenvalues \((-4, 4, 12)\) appearing in the last term for \(N = 3, 5, 7\), the general \(N\) behavior is given by \((4N - 16)\).
\[-\frac{4(6k^2N^2 - 19k^2N - 20k^2 + 12kN^2 + 135kN^2 + 328kN - 64k + 4N^2 + 70N^2 + 484N + 144)}{3(k + N + 2)^2(6kN + 5k + 5N + 4)} \rightarrow -\frac{4}{3} \lambda (\lambda + 1), \tag{4.73}\]

where the limit in (4.10) is taken at the final stage. It is obvious that the leading contributions in this limit come from the fifth and the sixth terms of (4.73). The eigenvalue in this limit is the same as the one in (4.34). The boldface parts are the same as each other.

By realizing that the eigenvalue of the zeromode of \(\Phi_0^{(1)} T\) is given by

\[
\left[ \frac{1}{2N(k + N + 2)} \times (2N^2 + 8N - 64) + \frac{1}{(k + N + 2)^2} \times (96N^2 - 24N^3) \right] = -\frac{(N - 4)(2N + 19)}{4kN + 2N + 2}, \tag{4.74}\]

from the first two terms of (4.73), one can decompose this eigenvalue (4.74) as the following form

\[
\phi_0^{(1)}(\bullet; \bullet) h(\bullet; \bullet) = \left[ \frac{(4 - N)}{(k + N + 2)} \right] \times \left[ \frac{(2N + 19)}{4(k + N + 2)} \right], \tag{4.75}\]

where the conformal dimension for this representation appearing in the subsection 3.3.2 of 54 is substituted into (4.75). Then one sees that the eigenvalue for the zeromode of the higher spin-1 current in this representation can be read off and it is given by

\[
\phi_0^{(1)}(\bullet; \bullet) = \frac{(4 - N)}{(k + N + 2)} \rightarrow -\lambda = \phi_0^{(1)}(\square; 0), \tag{4.76}\]

under the large \((N, k)\) 't Hooft like limit. On the other hand, the other eigenvalue of the zeromode for the higher spin current appearing in the first term of (3.1) in this representation is given by \((\frac{1}{(k+5)}, -\frac{1}{(k+7)}, -\frac{3}{(k+9)})\) for \(N = 3, 5, 7\) respectively. The three-point function of the higher spin-1 current with scalars is given by

\[
< O(\bullet; \bullet) O(\bullet; \bullet) \Phi_0^{(1)} >= \left[ \frac{(4 - N)}{(k + N + 2)} \right] < O(\square; \square) O(\square; \square) > \rightarrow -\lambda < O(\square; \square) O(\square; \square) >, \tag{4.77}\]

where the limit in (4.10) is taken at the final stage.

Note that there is a relation between the two eigenvalues

\[
\phi_2^{(1)}(\bullet; \bullet) = \phi_2^{(1)}(\square; 0), \tag{4.78}\]

under the large \((N, k)\) 't Hooft like limit, as anticipated in 54. The previous relation (4.34) is used in (4.78).

It is straightforward to write down the three-point function of the higher spin-3 current with the scalar operator corresponding to \((\bullet; \bullet; \bullet)\) and the scalar operator corresponding to
Obtains the final eigenvalue (3.111) in this representation is given by (96, 256, 480) for \( N = 3, 5, 7 \) respectively. One obtains the final eigenvalue

\[
\phi_2^{(1)}(\square \square \square) = \frac{16(k - N)}{(6kN + 5k + 5N + 4)} \left[ -\frac{1}{4(k + N + 2)^2} \times (8N^2 + 16N - 24) + \frac{1}{32N^2(k + N + 2)^2} \times (128N^2 - 128N^3) \right] + \frac{1}{2N(k + N + 2)^2} \times (-24N^2 - 48N) + \frac{1}{(k + N + 2)^2} \times (-24N - \frac{4k}{(k + N + 2)^2} \times (-2N + 2) - \frac{2(2k + N)}{3(k + N + 2)^2} \times (8N - 8) - \frac{4(12N^2 - 2k^2N - 10k^2 + 24kN^3 + 192kN^2 + 317kN - 38k - 4N^3 + 35N^2 + 302N + 72)}{3(k + 3N^2 + 128N^2) \times (6kN + 5k + 5N + 4)} \right]
\]

where the limit in (4.10) is taken at the final stage. It is obvious that the leading contributions in this limit come from the fifth and the sixth terms of (4.80). The eigenvalue in this limit is the twice of the one in (4.34). The boldface parts are increased by two.

By realizing that the eigenvalue of the zeromode of \( \Phi_0^{(1)} T \) is given by

\[
\left[ \frac{1}{2N(k + N + 2)^2} \times (8N^2 + 16N - 24) + \frac{1}{(k + N + 2)^2} \times (128N^2 - 128N^3) \right] = -\frac{2(N - 1)(N + 5)}{(k + N + 2)^2}, \quad (4.81)
\]

The general \( N \) dependence is given by (8N^2 + 16N - 24). Similarly, from the eigenvalues (−2304, −12800, −37632, −82944) appearing in the second term for \( N = 3, 5, 7, 9 \), the general \( N \) behavior is given by (−128 + 128N^2). The eigenvalue appearing in the third term in this representation is given by (−260, −840, −1512) for \( N = 3, 5, 7 \) respectively. Then the general \( N \) dependence can be read off and it is given by (−24N^2 − 48N). From the eigenvalues (−72, −120, −168) appearing in the fourth term for \( N = 3, 5, 7 \), the general \( N \) behavior is given by −24N. The eigenvalue appearing in the fifth term in this representation is given by (−4, −8, −12) for \( N = 3, 5, 7 \) respectively. Then the general \( N \) dependence can be read off and it is given by (−2N + 2). Finally, from the eigenvalues (16, 32, 48) appearing in the last term for \( N = 3, 5, 7 \), the general \( N \) behavior is given by (8N − 8).
from the first two terms of (4.80), one can decompose this eigenvalue (4.81) as the following form
\[
\phi^{(1)}(\begin{array}{c}
\boxdot \\
\boxbullet
\end{array}) = \frac{2(1 - N)}{(k + N + 2)} \times \frac{(N + 5)}{(k + N + 2)},
\]
where the conformal dimension for this representation appearing in the subsection 3.3.3 of [54] is substituted into (4.82). Then one sees that the eigenvalue for the zeromode of the higher spin-1 current in this representation can be read off and it is given by
\[
\phi^{(1)}_0(\begin{array}{c}
\boxbullet \\
\boxbullet
\end{array}) = 2(1 - N)(k + N + 2) - 2\lambda,
\]
under the large \((N, k)\) 't Hooft like limit. On the other hand, the other eigenvalue of the zeromode for the higher spin current appearing in the first term of (3.1) in this representation is given by \((-\frac{4}{k+5}, -\frac{8}{k+7}, -\frac{12}{k+9})\) for \(N = 3, 5, 7\) respectively. The three-point function is given by
\[
\langle \mathcal{O}(\begin{array}{c}
\boxbullet \\
\boxbullet
\end{array}) | \Phi^{(1)}_0(\begin{array}{c}
\boxbullet \\
\boxbullet
\end{array}) | \mathcal{O}(\begin{array}{c}
\boxbullet \\
\boxbullet
\end{array}) \rangle = 2(1 - N)(k + N + 2) - 2\lambda \langle \mathcal{O}(\begin{array}{c}
\boxbullet \\
\boxbullet
\end{array}) | \mathcal{O}(\begin{array}{c}
\boxbullet \\
\boxbullet
\end{array}) \rangle \rightarrow -2\lambda \langle \mathcal{O}(\begin{array}{c}
\boxbullet \\
\boxbullet
\end{array}) | \mathcal{O}(\begin{array}{c}
\boxbullet \\
\boxbullet
\end{array}) \rangle,
\]
where the limit in (4.10) is taken at the final stage.

Note that one observes the relation between the eigenvalues
\[
\phi^{(1)}_2(\begin{array}{c}
\boxbullet \\
\boxbullet
\end{array}) = 2\phi^{(1)}_2(\emptyset, 0) = 2 \times \left[ -\frac{4}{3} \lambda (\lambda + 1) \right],
\]
under the large \((N, k)\) 't Hooft like limit, as anticipated in [54]. In (4.85), one uses (4.34).

The other eigenvalue of the zeromode for the higher spin current appearing in the third term of (3.111) in this representation is given by \((-240, -560, -1008)\) for \(N = 3, 5, 7\) respectively. Note that there are triplet states in this representation from (4.56). Then the general \(N\) dependence can be read off and it is given by \((-16N^2 - 32N)\). From the other eigenvalues \((-52, -92, -132)\) of the zeromode for the higher spin current appearing in the fourth term of (3.111) for \(N = 3, 5, 7\), the general \(N\) behavior is given by \((-20N + 8)\). It turns out that
\[
\phi^{(1)}_2(\begin{array}{c}
\boxbullet \\
\boxbullet
\end{array}) = \frac{16(k - N)}{(6kN + k + 5N + 4)} \left[ -\frac{1}{4(k + N + 2)^2} \times (8N^2 + 16N - 24) + \frac{1}{32N^2(k + N + 2)^2} \times (128N^2 - 128N^3) \right] + \frac{1}{2N(k + N + 2)^2} \times (-16N^2 - 32N) + \frac{1}{(k + N + 2)^2} \times (-20N + 8) - \frac{4k}{(k + N + 2)^2} \times (2 - 2N) - \frac{2(2k + N)}{3(k + N + 2)^2} \times (8N - 8)
\]
\[
= -\frac{4(12k^2N^2 - 2k^2N - 10k^2 + 24kN^3 + 156kN^2 + 215kN - 98k - 4N^3 + 5N^2 + 218N + 24)}{3(k + N + 2)^2(6kN + 5k + 5N + 4)} \rightarrow -\frac{8}{3} \lambda (\lambda + 1).
\]
Although the eigenvalue (4.86) is different from the one in (4.80) at finite \((N, k)\) 't Hooft like limit is the same.

It is straightforward to write down the three-point function of the higher spin-3 current with the scalar operator corresponding to \((\begin{array}{c}3 \\ N \end{array})\) and the scalar operator corresponding to \((\begin{array}{c}3 \\ 3 \end{array})\) from (4.80)

\[
\left< \mathcal{O}(\begin{array}{c}3 \\ N \end{array}) \mathcal{O}(\begin{array}{c}3 \\ 3 \end{array}) \Phi_1^{(1)} \right> = \left[ \frac{4(12k^2N^2 - 2k^2N - 10k^2 + 24kN^3 + 192kN^2 + 317kN - 38k - 4N^3 + 35N^2 + 302N + 72)}{3(k + N + 2)^2(6kN + 5k + 5N + 4)} \right] 
\times \left< \mathcal{O}(\begin{array}{c}3 \\ 3 \end{array}) \mathcal{O}(\begin{array}{c}3 \\ N \end{array}) \Phi_0^{(1)} \right> 
= \frac{4}{3} \lambda (\lambda + 1) \left< \mathcal{O}(\begin{array}{c}3 \\ N \end{array}) \mathcal{O}(\begin{array}{c}3 \\ 3 \end{array}) \Phi_0^{(1)} \right>,
\]

where the limit in (4.10) is taken at the final stage and one uses (4.84). In (4.87), the three-point function is a multiple of the one in the subsection 4.3.1.

4.4.5 The eigenvalue in (antisymm; antisymm) representation with three, and three boxes

The eigenvalue of the zeromode for the higher spin current appearing in the first term of (3.111) in this representation is given by \((-72, -72, -72)\) for \(N = 3, 5, 7\) respectively \(^{37}\). One obtains the final eigenvalue as follows:

\[
\phi_2^{(1)} = \frac{16(k - N)}{(6kN + 5k + 5N + 4)} \left[ -\frac{1}{4(k + N + 2)^2} \times (-72) + \frac{1}{32N^2(k + N + 2)^2} \times 0 \right] 
+ \frac{1}{2N(k + N + 2)^2} \times 0 + \frac{1}{(k + N + 2)^2} \times (-24N + 48) - \frac{4k}{(k + N + 2)^2} \times 6 
- \frac{2(2k + N)}{3(k + N + 2)^2} \times (-24) = \frac{8(6k^2N + 5k^2 - 6kN^2 + 36kN + 70k - 5N^2 - 10N + 24)}{(k + N + 2)^2(6kN + 5k + 5N + 4)} 
\rightarrow \frac{-8}{N} \lambda (2\lambda - 1),
\]

where the limit in (4.10) is taken at the final stage. It is obvious that the leading contributions in this limit come from the fifth and the sixth terms of (4.88). The eigenvalue in this limit is the third times of the one in (4.37). The boldface parts are increased by three.

By realizing that the eigenvalue of the zeromode of \(\Phi_0^{(1)} T\) is given by

\[
\left[ -\frac{1}{4(k + N + 2)^2} \times (-72) + \frac{1}{32N^2(k + N + 2)^2} \times 0 \right] = \frac{18}{(k + N + 2)^2},
\]

\(^{37}\) Similarly, the eigenvalues appearing in the second term are \((0, 0, 0)\) for \(N = 3, 5, 7\). The eigenvalue appearing in the third term in this representation is given by \((0, 0, 0)\) for \(N = 3, 5, 7\) respectively. From the eigenvalues \((-24, -72, -120)\) appearing in the fourth term for \(N = 3, 5, 7\), the general \(N\) behavior is given by \((-24N + 48)\). The eigenvalue appearing in the fifth term in this representation is given by \((6, 6, 6)\) for \(N = 3, 5, 7\) respectively. Finally, from the eigenvalues \((-24, -24, -24)\) appearing in the last term for \(N = 3, 5, 7\), the general \(N\) behavior can be seen.
from the first two terms of (4.88), one can decompose this eigenvalue (4.89) as the following form

\[ \phi^{(1)}_0 (\square \square) h(\square \square) = \left[ \frac{6}{(k + N + 2)} \right] \times \left[ \frac{3}{(k + N + 2)} \right], \]  

(4.90)

where the conformal dimension for this representation appearing in the subsection 3.4.1 of [54] is substituted into (4.90). Then one sees that the eigenvalue for the zeromode of the higher spin-1 current in this representation can be read off and it is given by, together with (4.38),

\[ \phi^{(1)}_0 (\square \square) = \frac{6}{k + N + 2} = 3 \phi^{(1)}_0 (\square \square), \]  

(4.91)

which implies that the multiplicity 3 on the eigenvalue \( \phi^{(1)}_0 (\square \square) \) is valid at finite \( (N, k) \) (as well as under the large \( (N, k) \) 't Hooft like limit). On the other hand, the other eigenvalue of the zeromode for the higher spin-1 current appearing in the first term of (3.1) in this representation is given by \( \left( \frac{6}{(k+5)}, \frac{6}{(k+7)}, \frac{6}{(k+9)} \right) \) for \( N = 3, 5, 7 \) respectively. The three-point function is given by

\[ \langle \mathcal{O}(\square \square) \mathcal{O}(\square \square) \Phi^{(1)}_0 \rangle = \left[ \frac{6}{(k + N + 2)} \right] \langle \mathcal{O}(\square \square) \mathcal{O}(\square \square) \rangle \rightarrow \frac{6 \lambda}{N} < \mathcal{O}(\square \square) \mathcal{O}(\square \square) >. \]  

(4.92)

where the limit in (4.10) is taken at the final stage.

Note that one observes the eigenvalue

\[ \phi^{(2)}_0 (\square \square) = 3 \phi^{(1)}_0 (\square \square) = 3 \times \left[ - \frac{8}{3N} \lambda (2 \lambda - 1) \right], \]  

(4.93)

under the large \( (N, k) \) 't Hooft like limit, as anticipated in [54]. In (4.93), the previous result (4.37) is used.

It is straightforward to write down the three-point function of the higher spin-3 current with the scalar operator corresponding to \( (\square \square) \) and the scalar operator corresponding to \( (\square \square) \) from (4.88)

\[ \langle \mathcal{O}(\square \square) \mathcal{O}(\square \square) \Phi^{(1)}_2 \rangle \]

\[ = \left[ \frac{8(6k^2 N + 5k^2 - 6kN^2 + 36kN + 70k - 5N^2 - 10N + 24)}{(k + N + 2)(6kN + 5k + 5N + 4)} \right] \langle \mathcal{O}(\square \square) \mathcal{O}(\square \square) \rangle \]

\[ \rightarrow - \frac{8}{N} \lambda (2 \lambda - 1) < \mathcal{O}(\square \square) \mathcal{O}(\square \square) \Phi^{(1)}_0 \rangle = - \frac{4}{3} (2 \lambda - 1) \langle \mathcal{O}(\square \square) \mathcal{O}(\square \square) \Phi^{(1)}_0 \rangle, \]  

(4.94)

where the limit in (4.10) is taken at the final stage and the relation (4.92) is used. In (4.94), the three-point function is a multiple of the one in the subsection 4.3.2.
4.4.6 The eigenvalue in (antisymm; antisymm) representation with three, and two boxes

The eigenvalue of the zeromode for the higher spin current appearing in the first term of (3.111) in this representation is given by \((-6, 10, 42)\) for \(N = 3, 5, 7\) respectively. Then the general \(N\) dependence can be read off and it is given by \((2N^2 - 8N)\) \(^{38}\). By collecting all the contributions with correct coefficients, one determines the final eigenvalue as follows:

\[
\phi_2^{(1)}(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}) = \frac{16(k - N)}{(6kN + 5k + 5N + 4)} \left[ -\frac{1}{4(k + N + 2)^2} \times (2N^2 - 8N) + \frac{1}{32N^2(k + N + 2)^2} \times (96N^2 - 24N^3) \right] + \frac{1}{2N(k + N + 2)^3} \times (32N - 8N^2) + \frac{1}{(k + N + 2)^2} \times 0 - \frac{4k}{(k + N + 2)^2} \times (-N + 4) - \frac{2(2k + N)}{3(k + N + 2)^2} \times (4N - 16) = -\frac{4}{3} \lambda (\lambda + 1),
\]

(4.95)

where the limit in (4.10) is taken at the final stage. It is obvious that the leading contributions in this limit come from the fifth and the sixth terms of (4.95). The eigenvalue in this limit is the same as the one in (4.31). The boldface parts are the same as each other.

By realizing that the eigenvalue of the zeromode of \(\Phi_0^{(1)} T\) is given by

\[
\left[ -\frac{1}{4(k + N + 2)^2} \times (2N^2 - 8N) + \frac{1}{32N^2(k + N + 2)^2} \times (96N^2 - 24N^3) \right] = -\frac{(N - 4)(2N + 3)}{4(k + N + 2)^2},
\]

(4.96)

from the first two terms of (4.95), one can decompose this eigenvalue (4.96) as the following form

\[
\phi_0^{(1)}(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}) h(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}) = \left[ \frac{(4 - N)}{(k + N + 2)} \right] \times \left[ \frac{(2N + 3)}{4(k + N + 2)} \right],
\]

(4.97)

where the conformal dimension for this representation appearing in the subsection 3.4.2 of \[54\] is substituted into (4.97). Then one sees that the eigenvalue for the zeromode of the higher spin-1 current in this representation can be read off and it is given by

\[
\phi_0^{(1)}(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}) = \frac{(4 - N)}{(k + N + 2)} \rightarrow -\lambda = \phi_0^{(1)}(\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}),
\]

(4.98)

\(^{38}\) Similarly, from the eigenvalues \((216, -600, -3528, -9720)\) appearing in the second term for \(N = 3, 5, 7, 9\), the general \(N\) behavior is given by \((-24N^3 + 96N^2)\). The eigenvalue appearing in the third term in this representation is given by \((24, -40, -108)\) for \(N = 3, 5, 7\) respectively. Then the general \(N\) dependence can be read off and it is given by \((32N - 8N^2)\). There are eigenvalues \((0, 0, 0)\) appearing in the fourth term for \(N = 3, 5, 7\). From the eigenvalues \((1, -1, -3)\) appearing in the fifth term for \(N = 3, 5, 7\), the general \(N\) behavior is given by \((-N + 4)\). From the eigenvalues \((-4, 4, 12)\) appearing in the last term for \(N = 3, 5, 7\), the general \(N\) behavior is given by \((4N - 16)\).
under the large \((N,k)\) 't Hooft like limit. On the other hand, the other eigenvalue of the zeromode for the higher spin-1 current appearing in the first term of (3.11) in this representation is given by \(\left(\frac{1}{k+5}, -\frac{1}{k+7}, -\frac{3}{k+9}\right)\) for \(N = 3, 5, 7\) respectively.

The three-point function of the higher spin-1 current with two scalars is given by

\[
< \mathcal{O}(\boxtimes) \mathcal{O}(\boxtimes) \Phi_0^{(1)} > = \left[ \frac{(4-N)}{(k+N+2)} \right] < \mathcal{O}(\boxtimes) \mathcal{O}(\boxtimes) > \rightarrow -\lambda < \mathcal{O}(\boxtimes) \mathcal{O}(\boxtimes) >, \tag{4.99}
\]

where the limit in (4.10) is taken at the final stage. Note that one observes the eigenvalue

\[
\phi_2^{(1)}(\boxtimes) = \phi_2^{(1)}(\square 0), \tag{4.100}
\]

under the large \((N,k)\) 't Hooft like limit, as anticipated in \([54]\). Again the relation (4.34) is used in (4.100).

It is straightforward to write down the three-point function of the higher spin-3 current with the scalar operator corresponding to \((\boxtimes, \boxtimes)\) and the scalar operator corresponding to \((\boxtimes, \boxtimes)\) from (4.95)

\[
< \mathcal{O}(\boxtimes) \mathcal{O}(\boxtimes) \Phi_2^{(1)} >
= \left[ -\frac{4(N-4)(6k^2N + 5k^2 + 12kN^2 + 39kN + 28k + 4N^2 + 14N + 12)}{3(k+N+2)^2(6kN + 5k + 5N + 4)} \right] < \mathcal{O}(\boxtimes) \mathcal{O}(\boxtimes) >
\rightarrow -\frac{4}{3}(\lambda + 1) < \mathcal{O}(\boxtimes) \mathcal{O}(\boxtimes) > = \frac{4}{3}(\lambda + 1) -\lambda < \mathcal{O}(\boxtimes) \mathcal{O}(\boxtimes) \Phi_0^{(1)} >, \tag{4.101}
\]

where the limit in (4.10) is taken at the final stage and the relation (4.99) is used. In (4.101), the three-point function is the same as the one in the subsection 4.3.1 under the limit.

### 4.4.7 The eigenvalue in \((\text{antisymm; } f)\) representation with three boxes

The eigenvalue of the zeromode for the higher spin current appearing in the first term of (3.111) in this representation is given by \((32, 128, 288)\) for \(N = 3, 5, 7\) respectively. Then the general \(N\) dependence can be read off and it is given by \((8N^2 - 16N + 8)\). By collecting all the contributions with correct coefficients, one determines the final eigenvalue as follows:

\[
\frac{16(k-N)}{6kN + 5k + 5N + 4} \left[ -\frac{1}{4(k+N+2)^2} \times (8N^2 - 16N + 8) + \frac{1}{32N^2(k+N+2)^2} \times 0 \right]
\]

Similarly, there are eigenvalues \((0,0,0,0)\) appearing in the second or third term for \(N = 3, 5, 7, 9\). From the eigenvalues \((32, 64, 96)\) appearing in the fourth term for \(N = 3, 5, 7\), the general \(N\) behavior is given by \((16N - 16)\). From the eigenvalues \((-4, -8, -12)\) appearing in the fifth term for \(N = 3, 5, 7\), the general \(N\) behavior is given by \((-2N + 2)\). From the eigenvalues \((16, 32, 48)\) appearing in the last term for \(N = 3, 5, 7\), the general \(N\) behavior is given by \((8N - 8)\).
\[ + \frac{1}{2N(k + N + 2)^2} \times 0 + \frac{1}{(k + N + 2)^2} \times (16N - 16) - \frac{4k}{(k + N + 2)^2} \times (-2N + 2) \]
\[ - \frac{2(2k + N)}{3(k + N + 2)^2} \times (8N - 8) \]
\[ = - \frac{8(N - 1)(6k^2N + 5k^2 + 12kN^2 - 9kN - 38k - 2N^2 - 10N - 24)}{3(k + N + 2)^2(6kN + 5k + 5N + 4)} \rightarrow - \frac{8}{3} \lambda(\lambda + 1), \]

where the limit in (4.10) is taken at the final stage. It is obvious that the leading contributions in this limit come from the fifth and the sixth terms of (4.102). The eigenvalue in this limit is the twice of the one in (4.34). The boldface parts are increased by two.

By realizing that the eigenvalue of the zero-mode of \( \Phi_1^{(1)} \) is given by

\[ \phi_0^{(1)}(\square) = \left[ -\frac{2(N - 1)}{(k + N + 2)} \right] \times \left[ \frac{(N - 1)}{(k + N + 2)} \right], \]

where the conformal dimension for this representation is substituted into (4.104). Then one sees that the eigenvalue for the zero-mode of the higher spin-1 current in this representation can be read off and it is given by

\[ \phi_0^{(1)}(\square) = \frac{-2(N - 1)}{(k + N + 2)} \rightarrow -2\lambda, \]

under the large \((N, k)\) 't Hooft like limit. On the other hand, the other eigenvalue of the zero-mode for the higher spin-1 current appearing in the first term of (3.1) in this representation is given by \((-\frac{1}{(k+5)}, -\frac{8}{(k+7)}, -\frac{12}{(k+9)})\) for \(N = 3, 5, 7\) respectively.

The three-point function of the higher spin-1 current with two scalars is given by

\[ < \square \square \square \square \phi_0^{(1)}(\square) > = \left[ -\frac{2(N - 1)}{(k + N + 2)} \right] < \square \square \square \square \square > \rightarrow -2\lambda < \square \square \square \square \square >, \]

where the limit in (4.10) is taken at the final stage. Note that one observes the eigenvalue

\[ \phi_2^{(1)}(\square) = 2\phi_2^{(1)}(\square, 0) = 2 \times \left[ -\frac{4}{3} \lambda(\lambda + 1) \right], \]

under the large \((N, k)\) 't Hooft like limit, as anticipated in [54]. In (4.107), one uses the previous relation (4.34).
It is straightforward to write down the three-point function of the higher spin-3 current with the scalar operator corresponding to \( (\square, \square) \) and the scalar operator corresponding to \( (\square, \square) \) from (4.102)

\[
< \mathcal{O}(\square, \square) \mathcal{O}(\square, \square) \Phi_2^{(1)} >
= \left[ -\frac{8(N-1)(6k^2N + 5k^2 + 12kN^2 - 9kN - 38k - 2N^2 - 10N - 24)}{3(k + N + 2)^2(6kN + 5k + 5N + 4)} \right] < \mathcal{O}(\square, \square) \mathcal{O}(\square, \square) >
\rightarrow -\frac{8}{3} \lambda (\lambda + 1) < \mathcal{O}(\square, \square) \mathcal{O}(\square, \square) >= \frac{4}{3} (\lambda + 1) < \mathcal{O}(\square, \square) \mathcal{O}(\square, \square) \Phi_0^{(1)} >,
\tag{4.108}
\]

where the limit in (4.10) is taken at the final stage and the relation (4.106) is used. There is a simple relation the three-point function in (4.108) and the one in the subsection 4.3.1.

4.4.8 The eigenvalue in \((\Lambda_+, \Lambda_-)\) representation with more than four boxes

One can obtain the following branching rules for the symmetric and antisymmetric representations with four boxes \([75, 76]\)

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\square
\end{array}
\end{array} & \rightarrow (\square, 1)_4 + (\square, 2)_3 + (\square, 2)_3 - N + (\square, 4)_{1 - 2N} + (1, 5)_{1 - 2N}, \\
\begin{array}{c}
\begin{array}{c}
\square
\end{array}
\end{array} & \rightarrow (\square, 1)_4 + (\square, 2)_{3 - 2N} + (\square, 1)_{2 - N}.
\end{align*}
\]

Then four-index symmetric (or antisymmetric) parts of the SU\((N + 2)\) representation can be obtained from the generators of the fundamental representation of SU\((N + 2)\) by using the projection operator. By acting on the space

\[
T_a \otimes 1 \otimes 1 \otimes 1 + T_a \otimes 1 \otimes 1 + 1 \otimes 1 \otimes T_a + 1 \otimes 1 \otimes 1 \otimes T_a,
\]

where 1 is \((N + 2) \times (N + 2)\) unit matrix, the generators for the symmetric (or antisymmetric) representation for the SU\((N + 2)\) can be determined.

For more than five boxes, one can analyze similarly. One observes that for the higher representations \((\Lambda_+, \Lambda_+)\), the eigenvalues (corresponding to the higher spin-3 current) of the fourth, fifth and sixth terms of (3.111) (under the large \((N, k)\) ’t Hooft limit) behave as \((-8N, 2, -8)\) for one box, \((-16N, 4, -16)\) for two boxes and \((-24N, 6, -24)\) for three boxes. This implies that the basic quantity is the one for one box. The remaining one is a multiple of this quantity.

For the higher representations \((\Lambda_+, \Lambda_-)\), the eigenvalues of the fifth and sixth terms of (3.111) (under the large \((N, k)\) ’t Hooft limit) behave as \((-N, 4N)\) for one box which is equal
to \((|\Lambda_+| - |\Lambda_-|), (-2N, 8N)\) for two boxes and \((-3N, 12N)\) for three boxes. In this case also the basic quantity is the one for one box. The remaining one is a multiple of this quantity.

Similarly, for the higher representations \((\Lambda_+, \Lambda_-)\), the eigenvalues (corresponding to the higher spin-1 current) under the large \((N, k)\) 't Hooft limit behave as \(\frac{2}{(k+N+2)}\) for one box, \(\frac{4}{(k+N+2)}\) for two boxes and \(\frac{6}{(k+N+2)}\) for three boxes. This implies that the basic quantity is the one for one box. See also (4.38), (4.45), (4.52), (4.68), and (4.91). The remaining one is a multiple of this quantity.

For the higher representations \((\Lambda_+, \Lambda_-)\), the eigenvalues (of the higher spin-1 current) under the large \((N, k)\) 't Hooft limit behave as \(2(k+N+2)\) for one box, \(4(k+N+2)\) for two boxes and \(6(k+N+2)\) for three boxes. This implies that the basic quantity is the one for one box. See also the equations (4.36), (4.41), (4.48), (4.55), (4.61), (4.76), (4.83), (4.98) and (4.105).

For the three-point functions of the higher spin current with two scalars, one expects that the following relations satisfy

\[
\begin{align*}
<\mathcal{O}(\Lambda_+;\Lambda_+\Lambda_+^\dagger)\Phi_2^{(1)}>& \quad \rightarrow \quad -\frac{8|\Lambda_+|}{3N} \lambda (2\lambda - 1) <\mathcal{O}(\Lambda_+;\Lambda_+\Lambda_+^\dagger)\Phi_2^{(1)}> \\
&= \quad -\frac{4}{3} (2\lambda - 1) <\mathcal{O}(\Lambda_+;\Lambda_+\Lambda_+^\dagger)\Phi_2^{(1)}>, \\
<\mathcal{O}(\Lambda_+;\Lambda_-\Lambda_-^\dagger)\Phi_2^{(1)}>& \quad \rightarrow \quad -\frac{4(|\Lambda_+| - |\Lambda_-|)}{3} \lambda (\lambda + 1) <\mathcal{O}(\Lambda_+;\Lambda_-\Lambda_-^\dagger)\Phi_2^{(1)}> \\
&= \quad \frac{4}{3} (\lambda + 1) <\mathcal{O}(\Lambda_+;\Lambda_-\Lambda_-^\dagger)\Phi_2^{(1)}>.
\end{align*}
\]

In the first case of (4.109), the case of \(\Lambda_- = \Lambda_+\) is considered and the leading behavior is given by \(\frac{1}{N}\) for finite \(|\Lambda_+|\). The coefficient of the two-point function is a multiple of the quantity appearing in (4.37). The three-point function of higher spin-3 current is written in terms of the one of higher spin-1 current. One observes the simple factor \((2\lambda - 1)\) appears in their ratios. In the second case of (4.109) where \(|\Lambda_-| \leq |\Lambda_+|\), the coefficient of the two-point function is a multiple of the quantity appearing in (4.34). The simple factor \((\lambda + 1)\) appears in the ratio of the three-point functions of higher spin-3, 1 currents.

5 Conclusions and outlook

We have found the 16 higher spin currents where the higher spin-2 currents are given by (3.23), the higher spin-\(\frac{5}{2}\) currents are given by (3.42), (3.48), (3.54), and (3.60), the higher spin-3 current is given by (3.109) with (3.110), (3.111) and (3.112), together with the known higher spin-1 current (3.1) and the known higher spin-\(\frac{3}{2}\) currents (3.12). Based on the explicit
forms in (3.110) and (3.111) for the higher spin-3 current, the eigenvalues and the three-point functions are analyzed for various higher representations. Under the large $(N, k)$ 't Hooft limit, one has very simple expressions for the three-point functions of the higher spin-1, 3 currents with two scalar operators in (4.32) and (4.109).

Let us present some open problems in some related directions of this paper.

- Checking of the remaining OPEs of Appendix (A.1)
  So far, some of the OPEs in Appendix (A.1) using the adjoint spin-1, $\frac{1}{2}$ currents has been not analyzed fully. It would be interesting to observe whether there exist any nontrivial identities between the various tensors.

- The eigenvalues and three-point functions in other higher representations
  We did not consider the eigenvalues for the higher representations with mixed three boxes for the $\Lambda_+$. One should obtain the corresponding $SU(N + 2)$ generators for several $N$ values. For $N = 3$, they were given in [54].

- The next lowest 16 higher spin currents in terms of adjoint spin-$\frac{1}{2}, 1$ currents
  We have considered the lowest 16 higher spin currents. It would be interesting to observe the three-point functions for the higher spin-2, 4 current (in higher representations) living in the next $\mathcal{N} = 4$ multiplet.

- The lowest 16 higher spin currents in terms of adjoint spin-$\frac{1}{2}, 1$ currents in orthogonal Wolf space coset
  The defining OPE relations between the 11 currents and the lowest 16 higher spin currents ($s = 2$) in [53] are valid in this orthogonal Wolf space coset. The lowest higher spin-2 current has an explicit form in [40]. It is straightforward to continue to the present calculation and obtain these 16 higher spin currents using the adjoint spin-1, $\frac{1}{2}$ currents.

- The bulk theory
  It would be interesting to construct the bulk dual theory and observe whether one can see the corresponding three-point functions which should be equal to the ones in this paper.

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A The OPEs between the 11 currents and the lowest 16 higher spin currents

For convenience, the explicit OPEs [53] (by putting $s = 1$) between the 11 currents and the 16 higher spin currents are given by

\[ T(z) \left( \begin{array}{c}
\Phi^{(1)}_0 \\
\Phi^{(1),\mu}_1 \\
\Phi^{(1),\mu\nu}_2
\end{array} \right) (w) = \frac{1}{(z-w)^2} \left( \begin{array}{c}
\Phi^{(1)}_0 \\
\partial \Phi^{(1),\mu}_1 \\
\partial \Phi^{(1),\mu\nu}_2
\end{array} \right) (w) + \frac{1}{(z-w)^3} \left[ \Phi^{(1),\mu}_1 (w) \right] + \cdots, \]

\[ G^\mu(z) \Phi^{(1),\nu}_2 (w) = \frac{1}{(z-w)^3} \left[ \frac{8(k-N) (12kN + 19k + 19N + 26)}{3(k + N + 2) (6kN + 5k + 5N + 4)} \right] \Phi^{(1),\mu}_1 (w) \]

\[ + \frac{1}{(z-w)^2} \left[ - 5 \Phi^{(1),\mu}_1 + \frac{24(k-N)}{(6kN + 5k + 5N + 4)} (G^\mu \Phi^{(1)}_0 + \frac{2}{3} \partial \Phi^{(1),\mu}_1) - \frac{8}{(2k + N + 4)} \varepsilon^{\mu\nu\sigma T^\nu T^\sigma \Phi^{(1),\sigma}_2} \right] \]

\[ + \frac{1}{(z-w)} \left[ - \partial \Phi^{(1),\mu}_1 - \frac{4}{(2k + N + 4)} \varepsilon^{\mu\nu\sigma \partial T^\nu \Phi^{(1),\sigma}_2} + \frac{8(k-N) (12kN + 19k + 19N + 26)}{3(k + N + 2) (6kN + 5k + 5N + 4)} \right] \cdots, \]

\[ G^\mu(z) \Phi^{(1),\nu}_1 (w) = \frac{1}{(z-w)^3} \left[ \frac{16(k-N)}{3(2k + N)} \right] \delta^{\mu\nu} \Phi^{(1)}_0 (w) + \frac{1}{(z-w)^2} \left[ - 4 \Phi^{(1),\mu\nu}_1 \right] \]

\[ - \frac{2k-N}{3(2k + N)} \varepsilon^{\mu\nu\rho\sigma} \partial \Phi^{(1),\rho\sigma}_1 (w) + \frac{1}{(z-w)} \left[ \delta^{\mu\nu} (- \Phi^{(1)}_2) + \frac{16(k-N)}{(6kN + 5k + 5N + 4)} \Phi^{(1)}_0 T \right] \]

\[ - \partial \Phi^{(1),\mu\nu}_1 - \frac{(k-N)}{6(2k + N)} \varepsilon^{\mu\nu\rho\sigma} \partial \Phi^{(1),\rho\sigma}_1 + \frac{2i}{(2k + N)} \varepsilon^{\mu\nu\rho\sigma \partial T^\nu \Phi^{(1),\sigma}_0} \]

\[ - \frac{2}{(2k + N)} \varepsilon^{\mu\nu\rho\sigma \partial T^\nu \partial \Phi^{(1)}_0} + \frac{2i}{(2k + N)} (T^{\mu\rho} \Phi^{(1),\rho\nu}_1 - T^{\nu\rho} \Phi^{(1),\mu\rho}_1) - \frac{2}{(2k + N)} \varepsilon^{\mu\nu\rho\sigma} G^{\rho} \Phi^{(1),\sigma}_2 \]

\[ G^\mu(z) \Phi^{(1),\nu}_1 (w) = \frac{1}{(z-w)^2} \left[ \frac{(N-k)}{(k-N+2)} \right] \left( \delta^{\mu\nu} \Phi^{(1)}_2 - \delta^{\mu\rho} \Phi^{(1),\nu}_2 + \frac{2(2k + 3k + 3N)}{(k+N+2)} \varepsilon^{\mu\nu\rho\sigma} \Phi^{(1),\sigma}_2 \right) \]

\[ + \frac{1}{(z-w)} \left[ \frac{i}{(k+N+2)} \varepsilon^{\sigma\rho\mu T^{\sigma\alpha} \Phi^{(1),\nu}_2} - \varepsilon^{\sigma\alpha\rho T^{\sigma\alpha} \Phi^{(1),\rho}_2} + \varepsilon^{\mu\nu\rho\sigma} \partial \Phi^{(1),\sigma}_2 \right] \]

\[ - \delta^{\mu\nu} \Phi^{(1),\rho}_2 + \frac{(k-N)}{3(k+N+2)} \partial \Phi^{(1),\rho}_1 + \frac{i}{(k+N+2)} \varepsilon^{\sigma\rho\mu T^{\sigma\alpha} \Phi^{(1),\sigma}_2} \]

\[ + \delta^{\mu\nu} \Phi^{(1),\nu}_2 + \frac{(k-N)}{3(k+N+2)} \partial \Phi^{(1),\nu}_1 + \frac{i}{(k+N+2)} \varepsilon^{\sigma\rho\mu T^{\sigma\alpha} \Phi^{(1),\sigma}_2} \right] (w) + \cdots, \]

\[ G^\mu(z) \Phi^{(1)}_0 (w) = - \frac{1}{(z-w)^2} \left[ 2 \delta^{\mu\nu} \Phi^{(1)}_0 + \frac{1}{(z-w)} \left[ - \delta^{\mu\nu} \delta \Phi^{(1)}_0 + \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \Phi^{(1),\sigma}_2 \right] \right] (w) + \cdots, \]

\[ G^\mu(z) \Phi^{(1),\nu}_0 (w) = - \frac{1}{(z-w) \Phi^{(1),\mu}_1 (w) + \cdots, \]
\[ T^{\mu\nu}(z) \Phi^{(1)}_2(w) = \frac{1}{(z-w)^2} \left[ 4i \Phi_1^{(1),\mu} + \frac{16(k-N)}{(6kN+5k+5N+4)} T^{\mu\nu} \Phi^{(1)}_0 \right] (w) + \cdots, \]

\[ T^{\mu\nu}(z) \Phi^{(1),\rho}_2(w) = \frac{1}{(z-w)^2} \left[ \frac{4i(k-N)}{3(2+k+N)} (\delta^{\mu\rho} \Phi^{(1),\nu}_2 - \delta^{\nu\rho} \Phi^{(1),\mu}_2) - 4i \varepsilon^{\mu\nu\rho\sigma} \Phi^{(1),\sigma}_2 \right] (w) \]

\[ - \frac{1}{(z-w)} i \left[ \delta^{\mu\rho} \Phi^{(1),\nu}_2 - \delta^{\nu\rho} \Phi^{(1),\mu}_2 \right] (w) + \cdots, \]

\[ T^{\mu\nu}(z) \Phi^{(1),\rho\sigma}_2(w) = \frac{1}{(z-w)^2} 2i \varepsilon^{\mu\nu\rho\sigma} \Phi^{(1),\rho\sigma}_0 (w) \]

These OPEs are obtained from the $\mathcal{N} = 4$ primary condition in the $\mathcal{N} = 4$ superspace in the linear version \[51\].

**B The OPEs between the spin-$\frac{3}{2}$ composite fields and the spin-$2, \frac{5}{2}$ composite fields**

Let us present the OPEs between the spin-$\frac{3}{2}$ composite fields appearing in the spin-$\frac{3}{2}$ currents (2.3) and the spin-2 composite fields appearing in the higher spin-2 currents (3.23) as follows:

\[ Q^a V^b(z) V^c V^d(w) = \frac{1}{(z-w)^2} \left[ k g^{\dot{c}b} Q^a V^d - f^{\dot{c}de} Q^a V^g + k g^{\dot{d}b} Q^a V^c \right] (w) \]

\[ + \frac{1}{(z-w)} \left[ k (g^{\dot{c}b} \partial Q^a V^d + g^{\dot{d}b} \partial Q^a V^c) + f^{\dot{c}de} Q^a V^d V^e + f^{\dot{d}b} e f^{de} \partial (Q^a V^f) \right] (w), \]

\[ Q^a V^b(z) Q^c \partial Q^d(w) = \frac{1}{(z-w)^2} (k+N+2) \left[ g^{\dot{a}c} Q^d V^b \right] (w) + \frac{(k+N+2)}{(z-w)} \left[ g^{\dot{a}c} Q^c \partial V^b - g^{\dot{c}a} \partial Q^a V^b \right] (w), \]

\[ Q^a V^b(z) Q^c \partial Q^d V^e(w) = \frac{1}{(z-w)^2} (k+N+2) \left[ g^{\dot{a}c} f^{be} Q^d V^h - g^{\dot{b}a} f^{be} Q^e V^h \right] (w) \]

\[ + \frac{1}{(z-w)} \left[ (k+N+2) (g^{\dot{a}c} Q^d V^e - g^{\dot{c}a} Q^d V^b) + k g^{\dot{b}a} Q^c \partial Q^a + f^{\dot{c}e} f^{\dot{d}b} Q^a Q^d V^f \right] (w) + \cdots. \]

In particular, the explicit form for the first order pole is very crucial to determine the higher spin-$\frac{5}{2}$ current.

The OPEs between the spin-$\frac{3}{2}$ composite fields appearing in the spin-$\frac{3}{2}$ currents (2.3) and the spin-$\frac{5}{2}$ composite fields appearing in the higher spin-$\frac{5}{2}$ currents (3.42) are presented as follows:

\[ Q^a V^b(z) \partial Q^c V^d(w) = \frac{1}{(z-w)^2} (k+N+2) \left[ g^{\dot{c}a} f^{bd} V^e \right] (w) \]

\[ + \frac{1}{(z-w)^2} \left[ (k+N+2) g^{\dot{b}a} V^b V^d - k g^{\dot{d}b} \partial Q^c Q^a \right] (w). \]
The higher spin pole is very crucial to determine the higher spin-3 current. When one looks at the equation (3.24), there are three cases of the higher spin-3 current using the different tensor components

\[ Q^a V^b(z) Q^c \partial V^d(w) = \left( \frac{1}{(z-w)} \right)^3 \left[ (k + N + 2) g^{a\alpha} f^{b\beta} e \partial V^c + 2 k g^{d\delta} \partial Q^e \partial Q^a - f^{b\delta} e Q^a \partial Q^e V^c \right] (w) + \cdots, \]

\[ \gamma^a V^b(z) Q^c \partial V^d(w) = \left( \frac{1}{(z-w)} \right)^3 \left[ (k + N + 2) g^{a\alpha} f^{b\beta} e \partial V^c - 2 k g^{d\delta} Q^e \partial Q^a - f^{b\delta} e Q^a \partial Q^e V^c \right] (w) + \cdots, \]

Some of these expressions are given in [38]. In particular, the explicit form for the first order pole is very crucial to determine the higher spin-3 current.

### C The higher spin-5/2 currents using the different tensor components

When one looks at the equation (3.24), there are three cases of the higher spin-5/2 current for each \( \rho \) index for given indices \( \mu = \nu \) (or for each \( \nu \) index for given indices \( \mu = \rho \)). Note that for nontrivial case, the index \( \rho \) is not equal to \( \mu = \nu \) (or the index \( \nu \) is not equal to \( \mu = \rho \)).
For three different $\rho, \mu$ and $\nu$ cases, there is no higher spin-$\frac{5}{2}$ current in the first order pole.

### C.1 The higher spin-$\frac{5}{2}$ current $\Phi^{(1),1}_{\frac{5}{2}}$

In (3.54), one of the expression for this higher spin-$\frac{5}{2}$ current is described. Let us consider two other expressions for the same higher spin-$\frac{5}{2}$ current as follows. For example, one has the following OPE from Appendix A, with $\mu = \rho = 3$ and $\nu = 1$ in $SO(4)$ basis,

$$G^3(z) \Phi^{(1),13}_{\frac{5}{2}}(w) = \frac{1}{(z-w)^2} \frac{(k-N)}{(k+N+2)} \Phi^{(1),1}_{\frac{5}{2}}(w) + \frac{1}{(z-w)} \left[ \Phi^{(1),1}_{\frac{5}{2}} + \frac{2i}{(k+N+2)} T^{23} \Phi^{(1),4}_{\frac{5}{2}} \right]$$

$$+ \frac{2i}{(k+N+2)} T^{34} \Phi^{(1),2}_{\frac{5}{2}} - \frac{4i}{(k+N+2)} T^{24} \Phi^{(1),3}_{\frac{5}{2}} - \frac{(N-k)}{3(k+N+2)} \partial \Phi^{(1),1}_{\frac{5}{2}} \right](w). \quad (C.1)$$

From this expression of the first order pole, one can write down the higher spin-$\frac{5}{2}$ current with the help of (3.53) as follows:

\[
\Phi^{(1),1}_{\frac{5}{2}}(z) = \frac{i}{(k+N+2)^2} \left[ -\frac{1}{2N} h^1_{ab} d^3_{cd} + \frac{1}{2N} h^3_{ab} d^0_{cd} + \frac{1}{N} h^2_{ab} d^1_{cd} \\
+ 2h^3_{cd} d^2_{ae} - h^3_{be} d^0_{ad} - h^2_{bd} d^1_{ae} \right] f^{ab}_{c} Q^d V^e V^\bar{e} - \frac{4i(3+2k+N)}{3(k+N+2)^2} d^3_{ab} \partial Q^a V^b(z) \quad (C.2)
\]

\[
+ \frac{i}{(k+N+2)^2} \left[ -d^0_{ab} h^3_{cd} - d^1_{cd} h^2_{ab} + d^2_{cd} h^1_{ab} + d^3_{cd} h^0_{ab} \right] Q^a Q^b V^d(z) + \frac{4i(k+2N)}{3(k+N+2)^2} d^3_{ab} Q^a \partial V^b(z).
\]

Compared to the previous result in (3.54), the difference arises in the cubic terms.

Furthermore, one can also consider the following OPE from Appendix A with $\mu = \rho = 4$ and $\nu = 1$ in $SO(4)$ basis,

$$G^4(z) \Phi^{(1),14}_{\frac{5}{2}}(w) = \frac{1}{(z-w)^2} \frac{(k-N)}{(k+N+2)} \Phi^{(1),1}_{\frac{5}{2}}(w) + \frac{1}{(z-w)} \left[ \Phi^{(1),1}_{\frac{5}{2}} - \frac{2i}{(k+N+2)} T^{24} \Phi^{(1),3}_{\frac{5}{2}} \right]$$

$$+ \frac{2i}{(k+N+2)} T^{34} \Phi^{(1),2}_{\frac{5}{2}} - \frac{4i}{(k+N+2)} T^{23} \Phi^{(1),4}_{\frac{5}{2}} - \frac{(N-k)}{3(k+N+2)} \partial \Phi^{(1),1}_{\frac{5}{2}} \right](w). \quad (C.3)$$

Similarly, the same higher spin-$\frac{5}{2}$ current with the help of (3.53) can be written as

\[
\Phi^{(1),1}_{\frac{5}{2}}(z) = \frac{i}{(k+N+2)^2} \left[ -\frac{1}{2N} h^2_{ab} d^1_{cd} + \frac{1}{2N} h^1_{ab} d^3_{cd} - \frac{1}{N} h^0_{ab} d^2_{cd} \\
-2h^3_{cd} d^1_{ae} - h^3_{be} d^0_{ad} + h^2_{bd} d^1_{ae} \right] f^{ab}_{c} Q^d V^e V^\bar{e} - \frac{4i(3+2k+N)}{3(k+N+2)^2} d^3_{ab} \partial Q^a V^b(z) \quad (C.4)
\]

\[
+ \frac{i}{(k+N+2)^2} \left[ -d^0_{ab} h^1_{cd} - d^1_{cd} h^2_{ab} + d^2_{cd} h^1_{ab} + d^3_{cd} h^0_{ab} \right] Q^a Q^b V^d(z) + \frac{4i(k+2N)}{3(k+N+2)^2} d^3_{ab} Q^a \partial V^b(z).
\]

Compared to the previous results in (3.54) and Appendix (C.2), the difference arises in the cubic terms.
Therefore, one has two relations between three identical results. From (3.54) and Appendix (C.2), one has

\[
\left[ -\frac{1}{2N} h_{\bar{a}b}^1 d_{de}^2 + \frac{1}{2N} h_{\bar{a}b}^2 d_{de}^1 + \frac{1}{N} h_{\bar{a}b}^3 d_{de}^0 - 2 h_{\bar{b}d}^0 d_{\bar{a}e} - h_{\bar{b}e}^2 d_{\bar{a}d}^1 - h_{b\bar{d}}^2 d_{\bar{a}e} \right] f_{\bar{a}b}^c = 0,
\]

This can be simplified further as

\[
\left[ -\frac{1}{2N} h_{\bar{a}b}^2 d_{de}^1 + \frac{1}{2N} h_{\bar{a}b}^3 d_{de}^0 - h_{\bar{b}d}^0 d_{\bar{a}e} - h_{\bar{b}e}^1 d_{\bar{a}d}^2 - h_{b\bar{d}}^1 d_{\bar{a}e} \right] f_{\bar{a}b}^c = 0, \tag{C.5}
\]

by using the identity

\[
\left[ h_{\bar{b}d}^\mu d_{\bar{a}e}^\nu - h_{\bar{b}e}^\mu d_{\bar{a}d}^\nu \right] f_{\bar{a}b}^c = 0. \tag{C.6}
\]

This can be checked for several $N$ values.

From (3.54) and Appendix (C.4), one has

\[
\left[ -\frac{1}{2N} h_{\bar{a}b}^2 d_{de}^1 + \frac{1}{2N} h_{\bar{a}b}^3 d_{de}^0 - 2 h_{\bar{b}d}^0 d_{\bar{a}e} - h_{\bar{b}e}^2 d_{\bar{a}d}^1 - h_{b\bar{d}}^2 d_{\bar{a}e} \right] f_{\bar{a}b}^c = 0,
\]

Moreover, one obtains with the help of Appendix (C.6)

\[
\left[ \frac{1}{2N} h_{\bar{a}b}^1 d_{de}^2 + \frac{1}{2N} h_{\bar{a}b}^3 d_{de}^0 - 2 h_{\bar{b}d}^0 d_{\bar{a}e} - h_{\bar{b}e}^1 d_{\bar{a}d}^2 + h_{b\bar{d}}^1 d_{\bar{a}e} \right] f_{\bar{a}b}^c = 0. \tag{C.7}
\]

The identities Appendices (C.5) and (C.7) are nontrivial relations which will be useful to check other relations appearing in Appendix A. It would be interesting to prove these identities in general without using the above defining relations.

### C.2 The higher spin-$\frac{5}{2}$ current $\Phi_{\frac{5}{2}}^{(1),2}$

In (3.60), one of the expression for this higher spin-$\frac{5}{2}$ current is found. Let us consider two other expressions for the same higher spin-$\frac{5}{2}$ current as follows. Let us consider $\mu = 4 = \rho$ and $\nu = 2$ in $SO(4)$ basis and the OPE from Appendix A can be written as

\[
G^4(z) \Phi_{\frac{5}{2}}^{(1),24}(w) = \frac{1}{(z-w)^2} \frac{(k-N)}{(k+N+2)} \Phi_{\frac{5}{2}}^{(1),2}(w) + \frac{1}{(z-w)} \left\{ \Phi_{\frac{5}{2}}^{(1),2} + \frac{2i}{(k+N+2)} T^{14} \Phi_{\frac{5}{2}}^{(1),3} \right\} - \frac{2i}{(k+N+2)} T^{34} \Phi_{\frac{5}{2}}^{(1),1} - \frac{4i}{(k+N+2)} T^{13} \Phi_{\frac{5}{2}}^{(1),4} - \frac{(N-k)}{3(k+N+2)} \partial \Phi_{\frac{5}{2}}^{(1),2}(w). \tag{C.8}
\]
From the first order pole, after substituting the relations in (3.59), the higher spin-$\frac{5}{2}$ current can be described as

$$ \Phi_{(1/2)}^{(1,2)}(z) = \frac{i}{(k + N + 2)^2} \left[ -\frac{1}{2N} h_{ab}^1 d_{de}^1 - \frac{1}{2N} h_{ab}^3 d_{de}^3 - \frac{1}{N} h_{ab}^2 d_{de}^2 \right]$$

$$ -2 h_{bd}^2 d_{ae}^2 - h_{bd}^3 d_{ae}^3 - h_{bd}^1 d_{ae}^1 \] f^{ab}_{c d} Q^c V^e V^f(z)$$

$$+ \frac{i}{(k + N + 2)^2} \left[ -d_{cd}^3 h_{ab}^3 - d_{cd}^2 h_{ab}^2 - d_{ab}^0 h_{cd}^0 - d_{ab}^1 h_{cd}^1 \right] Q^a Q^b Q^c V^d(z) + \frac{4i(k + 2N)}{3(k + N + 2)^2} d_{ab}^0 Q^a \partial V^b(z).$$

(C.9)

Compared to the previous results in (3.60) and Appendix (C.9), the difference arises in the cubic terms as before. Therefore, one has a relation between two identical results. From (3.60) and Appendix (C.9), one has

$$ \left[ -\frac{1}{2N} h_{ab}^2 d_{de}^2 - \frac{1}{2N} h_{ab}^3 d_{de}^3 - \frac{1}{N} h_{ab}^1 d_{de}^1 - 2 h_{bd}^1 d_{ae}^1 - h_{bd}^3 d_{ae}^3 - h_{bd}^2 d_{ae}^2 \right] f^{ab}_{c d} = 0,$$

(C.10)

which was observed in (3.130) in different context.

Furthermore, one can also consider the following OPE with $\mu = \nu = 1$ and $\rho = 2$ in $SO(4)$ basis

$$ G^1(z) \Phi_{(1/2),12}^{(1,12)}(w) = -\frac{1}{(z-w)^2} \frac{(k-N)}{(k+N+2)} \Phi_{(1/2)}^{(1,2)}(w) + \frac{1}{(z-w)} \left[ -\Phi_{(1/2)}^{(1,2)} + \frac{2i}{(k + N + 2)} T_{13}^{T^{13}} \Phi_{(1/2)}^{(1/4)} \right]$$

$$ - \frac{2i}{(k + N + 2)^2} T_{14}^{T^{14}} \Phi_{(1/2)}^{(1/3)} + \frac{4i}{(k + N + 2)^3} T_{15}^{T^{15}} \Phi_{(1/2)}^{(1/1)} + \frac{(N-k)}{(3(k+N+2)^2)} \partial \Phi_{(1/2)}^{(1/2)}(w).$$

(C.11)

Similarly, from the first order pole, the higher spin-$\frac{5}{2}$ current with the help of (3.59) can be written as

$$ \Phi_{(1/2)}^{(1,2)}(z) = \frac{i}{(k + N + 2)^2} \left[ -\frac{1}{2N} h_{ab}^2 d_{de}^2 - \frac{1}{2N} h_{ab}^3 d_{de}^3 - \frac{1}{N} h_{ab}^1 d_{de}^1 \right]$$

$$ -2 h_{bd}^2 d_{ae}^2 - h_{bd}^3 d_{ae}^3 - h_{bd}^1 d_{ae}^1 \] f^{ab}_{c d} Q^c V^e V^f(z)$$

$$+ \frac{i}{(k + N + 2)^2} \left[ -d_{cd}^3 h_{ab}^3 - d_{cd}^2 h_{ab}^2 - d_{ab}^0 h_{cd}^0 - d_{ab}^1 h_{cd}^1 \right] Q^a Q^b Q^c V^d(z) + \frac{4i(k + 2N)}{3(k + N + 2)^2} d_{ab}^0 Q^a \partial V^b(z).$$

(C.12)

Compared to the previous results in (3.60) and Appendix (C.12), the difference arises in the cubic terms. From (3.60) and Appendix (C.12), one has

$$ \left[ -\frac{1}{2N} h_{ab}^2 d_{de}^2 - \frac{1}{2N} h_{ab}^3 d_{de}^3 - \frac{1}{N} h_{ab}^1 d_{de}^1 - 2 h_{bd}^1 d_{ae}^1 - h_{bd}^3 d_{ae}^3 - h_{bd}^2 d_{ae}^2 \right] f^{ab}_{c d} = 87
cubic terms as before. From (3.42) and Appendix (C.15), one has

\[
\left[ -\frac{1}{2N} h^{2}_{ab} d_{de}^{2} - \frac{1}{2N} h^{1}_{ab} d_{de}^{1} - \frac{1}{N} h^{3}_{ab} d_{de}^{3} - 2 h^{0}_{bd} d_{ae}^{0} - h^{3}_{bd} d_{ae}^{3} - h^{1}_{bd} d_{ae}^{1} \right] f^{\bar{a}b}_{c} = 0.
\]

Then one has

\[
\left[ \frac{1}{2N} h^{3}_{ab} d_{de}^{3} - \frac{1}{2N} h^{1}_{ab} d_{de}^{1} - h^{3}_{bd} d_{ae}^{3} + h^{3}_{bd} d_{ae}^{3} \right] f^{\bar{a}b}_{c} = 0,
\]

which was observed in (3.126) in section 3. It would be interesting to prove these identities Appendices (C.10) and (C.13) from the group theoretical context.

### C.3 The higher spin-$\frac{5}{2}$ current $\Phi^{(1),3}_{\frac{5}{2}}$

In (3.42), one of the expression for this higher spin-$\frac{5}{2}$ current is obtained. Let us consider two other expressions for the same higher spin-$\frac{5}{2}$ current as follows. Let us consider $\mu = 4 = \rho$ and $\nu = 3$ in $SO(4)$ basis and the OPE from Appendix A can be written as

\[
G^i(z) \Phi^{(1),3}_{\frac{5}{2}}(w) = \frac{1}{(z-w)^2} (k-N) \Phi^{(1),3}_{\frac{5}{2}}(w) + \frac{1}{(z-w)} \left[ \Phi^{(1),3}_{\frac{5}{2}} - \frac{2i}{(k+N+2)} T^{14} \Phi^{(1),1}_{\frac{5}{2}} \right] (w).
\]

After substituting the relations in (3.41), the higher spin-$\frac{5}{2}$ current can be described as

\[
\Phi^{(1),3}_{\frac{5}{2}}(z) = \frac{i}{(k+N+2)^2} \left[ -\frac{1}{2N} h^{2}_{ab} d_{de}^{2} + \frac{1}{2N} h^{1}_{ab} d_{de}^{1} + \frac{1}{N} h^{3}_{ab} d_{de}^{3} \right]
\]

\[
+ 2 h^{2}_{bd} d_{ae}^{0} - h^{2}_{bd} d_{ae}^{0} - h^{0}_{bd} d_{ae}^{0} + \frac{4i(3+2k+N)}{3(k+N+2)^2} d^{0}_{ab} \partial Q^a V^b(z)
\]

\[
+ \frac{i}{(k+N+2)^3} \left[ d^{0}_{bd} h^{3}_{ae} + d^{0}_{bd} h^{2}_{ae} + d^{0}_{bd} h^{1}_{ae} + d^{0}_{bd} h^{1}_{ae} \right] Q^a Q^b \partial V^d(z) + \frac{4i(k+N+2)}{3(k+N+2)^2} d^{0}_{ab} Q^a \partial V^b(z).
\]

Compared to the previous results in (3.42) and Appendix (C.15), the difference arises in the cubic terms as before. From (3.42) and Appendix (C.15), one has

\[
\left[ -\frac{1}{2N} h^{3}_{ab} d_{de}^{2} + \frac{1}{2N} h^{1}_{ab} d_{de}^{1} - h^{3}_{bd} d_{ae}^{3} + h^{3}_{bd} d_{ae}^{3} \right] f^{\bar{a}b}_{c} = 0.
\]

Then it is easy to see that

\[
\left[ -\frac{1}{2N} h^{3}_{ab} d_{de}^{2} + \frac{1}{2N} h^{1}_{ab} d_{de}^{1} - h^{0}_{bd} d_{ae}^{0} - h^{2}_{bd} d_{ae}^{2} \right] f^{\bar{a}b}_{c} = 0.
\]
Let us consider $\mu = 1 = \nu$ and $\rho = 3$ in $SO(4)$ basis and the OPE from Appendix A can be written as

$$G^1(z) \Phi_i^{(1),13}(w) = \frac{1}{(z-w)^2} \frac{(k-N)}{(k+N+2)} \Phi_i^{(1),3}(w) + \frac{1}{(z-w)} \left[ - \Phi_i^{(1),3} - \frac{2i}{(k+N+2)} T^{12} \Phi_i^{(1),4} \right].$$

Again, after substituting the relations in (3.41), the higher spin-$\frac{5}{2}$ current can be described as

$$\Phi_i^{(1),3}(z) = \frac{i}{(k+N+2)^2} \left[ \frac{1}{2N} h_{ab}^2 d_{bc}^2 + \frac{1}{2N} h_{ab} d_{bc}^3 - \frac{1}{N} h_{ab}^2 d_{bc}^3 - \frac{1}{2N} h_{ab}^3 d_{bc}^2 + \frac{1}{2N} h_{ab}^2 d_{bc}^3 - \frac{1}{N} h_{ab}^3 d_{bc}^2 \right] f^{abc} Q^a V^c - \frac{4i(3 + 2k + N)}{3(k+N+2)^2} d_{ab} \partial Q^a V^b(z) - 2h_{bd}^2 d_{ae}^2 + h_{be}^3 d_{ad}^2 + d_{ab}^0 h_{ed}^1 + d_{cd}^0 h_{ab}^1 \right] Q^a Q^b V^d(z) + \frac{4i(k + 2N)}{3(k+N+2)^2} d_{ab} Q^a \partial V^b(z).$$

Compared to the previous results in (3.42) and Appendix (C.18), the difference arises in the cubic terms. From (3.42) and Appendix (C.18), one has

$$\left[ \frac{1}{2N} h_{ab}^3 d_{de}^2 - \frac{1}{2N} h_{ab}^2 d_{de}^3 + \frac{1}{N} h_{ab}^2 d_{de}^3 - 2h_{bd}^0 d_{ae}^1 - h_{be}^3 d_{ad}^2 + h_{bd}^3 d_{ae}^2 \right] f^{abc} = 0.$$ 

Moreover, one has from Appendix (C.6)

$$\left[ \frac{1}{2N} h_{ab}^2 d_{de}^2 + \frac{1}{2N} h_{ab}^3 d_{de}^2 - \frac{1}{2N} h_{ab}^2 d_{de}^3 - 2h_{bd}^3 d_{ae}^2 + h_{be}^3 d_{ad}^2 + h_{bd}^3 d_{ae}^2 \right] f^{abc} = 0.$$

It would be interesting to see where one can use these two identities Appendices (C.16) and (C.19) in the various OPEs.

### C.4 The higher spin-$\frac{5}{2}$ current $\Phi_i^{(1),4}$

In (3.48), one of the expression for this higher spin-$\frac{5}{2}$ current is determined. Let us consider two other expressions for the same higher spin-$\frac{5}{2}$ current as follows. Let us consider $\mu = 1 = \nu$ and $\rho = 4$ in $SO(4)$ basis and the OPE from Appendix A can be written as

$$G^1(z) \Phi_i^{(1),14}(w) = \frac{1}{(z-w)^2} \frac{(k-N)}{(k+N+2)} \Phi_i^{(1),4}(w) + \frac{1}{(z-w)} \left[ - \Phi_i^{(1),4} + \frac{2i}{(k+N+2)} T^{12} \Phi_i^{(1),3} \right].$$

$$- \frac{2i}{(k+N+2)} T^{13} \Phi_i^{(1),2} + \frac{4i}{(k+N+2)} T^{23} \Phi_i^{(1),1} + \frac{(N-k)}{3(k+N+2)} \partial \Phi_i^{(1),4} \right] (w).$$
Again, after substituting the relations in (3.47), the higher spin-$\frac{5}{2}$ current can be described as

$$
\Phi^{(1),4}_{\frac{5}{2}}(z) = \frac{i}{(k + N + 2)^2} \left[ \frac{1}{2N} h_{ab}^3 d_{de}^1 - \frac{1}{2N} h_{ab}^2 d_{de}^0 - \frac{1}{N} h_{ab}^1 d_{de}^3 \right] f_{\tilde{a} \tilde{b} \tilde{c}}^\tilde{d} Q^\tilde{d} V^\tilde{c} V^\tilde{e} + \frac{4i(3 + 2k + N)}{3(k + N + 2)^2} d_{ab}^2 \partial Q^a \partial V^b(z)
$$

(C.21)

Compared to the previous results in (3.48) and Appendix (C.21), the difference arises in the cubic terms as before. From (3.48) and Appendix (C.21), one has

$$
\left[ \frac{1}{2N} h_{ab}^3 d_{de}^1 - \frac{1}{2N} h_{ab}^2 d_{de}^0 - \frac{1}{N} h_{ab}^1 d_{de}^3 \right] f_{\tilde{a} \tilde{b} \tilde{c}}^\tilde{d} = 0.
$$

(C.22)

Let us consider $\mu = 3 = \nu$ and $\rho = 4$ in $SO(4)$ basis and the OPE from Appendix A can be written as

$$
G^3(z) \Phi^{(1),34}_{1}(w) = -\frac{1}{(z-w)^2} \frac{(k-N)}{(k + N + 2)} \Phi^{(1),4}_{\frac{5}{2}}(w) + \frac{1}{(z-w)} \left[ -\Phi^{(1),4}_{\frac{5}{2}} - \frac{2i}{(k + N + 2)} T^{13} \Phi^{(1),2}_{\frac{5}{2}} \right] + \frac{4i}{(k + N + 2)} T^{12} \Phi^{(1),3}_{\frac{5}{2}} + \frac{(N-k)}{3(k + N + 2)} \partial \Phi^{(1),4}_{\frac{5}{2}}(w).
$$

(C.23)

Again, after substituting the relations in (3.47), the higher spin-$\frac{5}{2}$ current can be described as

$$
\Phi^{(1),4}_{\frac{5}{2}}(z) = \frac{i}{(k + N + 2)^2} \left[ -\frac{1}{2N} h_{ab}^2 d_{de}^0 - \frac{1}{2N} h_{ab}^1 d_{de}^3 + \frac{1}{N} h_{ab}^3 d_{de}^1 \right] f_{\tilde{a} \tilde{b} \tilde{c}}^\tilde{d} Q^\tilde{d} V^\tilde{c} V^\tilde{e} + \frac{4i(3 + 2k + N)}{3(k + N + 2)^2} d_{ab}^2 \partial Q^a \partial V^b(z)
$$

(C.24)

Compared to the previous results in (3.48) and Appendix (C.24), the difference arises in the cubic terms. From (3.48) and Appendix (C.24), one has

$$
\left[ \frac{1}{2N} h_{ab}^3 d_{de}^1 - \frac{1}{2N} h_{ab}^2 d_{de}^0 - \frac{1}{N} h_{ab}^1 d_{de}^3 \right] f_{\tilde{a} \tilde{b} \tilde{c}}^\tilde{d} = 0.
$$

(C.22)
Then by using the identity Appendix (C.6), one has
\[
- \frac{1}{2N} h_{ab}^3 d_1^{de} - \frac{1}{2N} h_{ab}^2 d_0^{de} + h_{bd}^1 d_{ae} - h_{bd}^1 d_{ae} \right] f^{abc} = 0. \quad \text{(C.25)}
\]
As before, it is an open problem to observe the identities Appendices (C.22) and (C.25) in the general context.

D Different routes for obtaining the higher spin-3 current

Recall that the higher spin-3 current arises in the OPE between \( G^\mu(z) \) and the higher spin-\( \frac{5}{2} \) current \( \Phi^{(1)_\nu}(w) \) appearing in Appendix (A.1) with the condition \( \mu = \nu \). In section 3, the \( \mu = \nu = 3 \) case in \( SO(4) \) basis is considered. In this Appendix, we describe the other cases, \( \mu = \nu = 1, 2 \) and 4.

D.1 From the second higher spin-\( \frac{5}{2} \) current \( \Phi^{(1)_\nu}_2 \)

Let us introduce some part of the first order pole in (3.45) as follows:
\[
\bar{\Phi}^{(1),4}_2 = \frac{2i}{(k + N + 2)} T^{12} \Phi^{(1),3}_2 + \frac{2i}{(k + N + 2)} T^{23} \Phi^{(1),1}_2 - \frac{4i}{(k + N + 2)} T^{13} \Phi^{(1),2}_2 + \frac{(N - k)}{3(k + N + 2)} \partial \Phi^{(1),4}_2.
\]

In order to obtain the higher spin-3 current using the higher spin-\( \frac{5}{2} \) current (3.48), one should also calculate the OPE between \( G^4(z) \) and \( \Phi^{(1),4}_2(w) \) \( \mu = \nu = 4 \) in \( SO(4) \) basis. From the defining OPEs in Appendix (A.1), one can calculate this OPE as follows:
\[
G^4(z) \Phi^{(1),4}_2(w) = \frac{1}{(z - w)^3} \left\{ \frac{4(k - N)}{3(k + N + 2)} \Phi^{(1)}_0(w) + \frac{1}{(z - w)^2} \left\{ \frac{(k - N)}{(k + N + 2)} \partial \Phi^{(1)}_0(w) \right\} \right.
\]
\[
+ \frac{1}{(z - w)} \left\{ - \frac{2i}{(k + N + 2)} T^{12} \Phi^{(1),12}_1 - \frac{2i}{(k + N + 2)} T^{23} \Phi^{(1),23}_1 \right. \right.
\]
\[
- \frac{4i}{(k + N + 2)} T^{13} \Phi^{(1),13}_1 + \frac{(k - N)}{3(k + N + 2)} \partial^2 \Phi^{(1)}_0 \right\} (w) + \cdots. \quad \text{(D.1)}
\]

Then one can write down the higher spin-3 current as in (3.101). In other words, one has 1) the first order pole in (3.84) with \( \mu = \nu = 4 \) in \( SO(4) \) basis, which comes from the OPE between \( G^4(z) \) and the first order pole of (3.35), 2) the second term of (3.101) and 3) the first order pole of Appendix (D.1). It is an open problem to observe any nontrivial identities between the various tensors from the two expressions for the same higher spin-3 current.

There exists other expression for the same higher spin-\( \frac{5}{2} \) current in Appendix (C.21). From its defining equation in Appendix (C.20), one introduces the following quantity
\[
\Phi^{(1),4}_2 = \frac{2i}{(k + N + 2)} T^{12} \Phi^{(1),3}_2 - \frac{2i}{(k + N + 2)} T^{13} \Phi^{(1),2}_2 + \frac{4i}{(k + N + 2)} T^{23} \Phi^{(1),1}_2 + \frac{(N - k)}{3(k + N + 2)} \partial \Phi^{(1),4}_2.
\]

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Let us introduce some part of the first order pole in (3.51) as follows:

$$\tilde{\varphi}^{(1),4}_\frac{1}{2}(w) = \frac{2i}{(k+N+2)} T^{23} \varphi^{(1),4}_\frac{1}{2} - \frac{2i}{(k+N+2)} T^{24} \varphi^{(1),3}_\frac{1}{2} + \frac{4i}{(k+N+2)} T^{34} \varphi^{(1),2}_\frac{1}{2} - \frac{(N-k)}{3(k+N+2)} \partial \varphi^{(1),1}_\frac{1}{2}. $$

The following OPE can be calculated from Appendix (A.1)

$$G^4(z) \tilde{\varphi}^{(1),4}_\frac{1}{2}(w) = \frac{1}{(z-w)^3} \left[ \frac{4(k-N)}{3(k+N+2)} \right] \Phi^{(1)}_0(w) + \frac{1}{(z-w)^2} \left[ \frac{(k-N)}{(k+N+2)} \right] \partial \Phi^{(1)}_0(w)$$

$$+ \frac{1}{(z-w)} \left[ - \frac{2i}{(k+N+2)} T^{12} \Phi^{(1),12}_1 - \frac{2i}{(k+N+2)} T^{13} \Phi^{(1),13}_1 \right]$$

$$- \frac{4i}{(k+N+2)} T^{23} \Phi^{(1),23}_1 + \frac{(k-N)}{3(k+N+2)} \Phi^{(1)}_0(w) \right] (w) + \cdots. \quad (D.2)$$

Then as before, one can write down the higher spin-3 current as in (3.101). In other words, one has 1) the first order pole in (3.84) with $\mu = \nu = 4$ in $SO(4)$ basis, which comes from the OPE between $G^4(z)$ and the first order pole of (3.35), 2) the second term of (3.101) and 3) the first order pole of Appendix (D.2). One expects that there will be nontrivial identities by identifying two expressions for the same higher spin-3 current.

There exists other expression for the same higher spin-$\frac{5}{2}$ current in Appendix (C.24). From its defining equation in Appendix (C.20), one introduces the following quantity

$$\tilde{\varphi}^{(1),4}_\frac{1}{2} = - \frac{2i}{(k+N+2)} T^{13} \varphi^{(1),2}_\frac{1}{2} + \frac{2i}{(k+N+2)} T^{23} \varphi^{(1),1}_\frac{1}{2} + \frac{4i}{(k+N+2)} T^{12} \varphi^{(1),3}_\frac{1}{2} + \frac{(N-k)}{3(k+N+2)} \partial \varphi^{(1),4}_\frac{1}{2}. $$

The following OPE can be calculated from Appendix (A.1)

$$G^4(z) \tilde{\varphi}^{(1),4}_\frac{1}{2}(w) = \frac{1}{(z-w)^3} \left[ \frac{4(k-N)}{3(k+N+2)} \right] \Phi^{(1)}_0(w) + \frac{1}{(z-w)^2} \left[ \frac{(k-N)}{(k+N+2)} \right] \partial \Phi^{(1)}_0(w)$$

$$+ \frac{1}{(z-w)} \left[ - \frac{2i}{(k+N+2)} T^{13} \Phi^{(1),12}_1 - \frac{2i}{(k+N+2)} T^{23} \Phi^{(1),13}_1 \right]$$

$$- \frac{4i}{(k+N+2)} T^{12} \Phi^{(1),23}_1 + \frac{(k-N)}{3(k+N+2)} \Phi^{(1)}_0(w) \right] (w) + \cdots. \quad (D.3)$$

The higher spin-3 current can be determined as in (3.101). In other words, one has 1) the first order pole in (3.84) with $\mu = \nu = 4$, which comes from the OPE between $G^4(z)$ and the first order pole of (3.35), 2) the second term of (3.101) and 3) the first order pole of Appendix (D.2). There should be nontrivial identities by identifying two expressions for the same higher spin-3 current.

**D.2 From the third higher spin-$\frac{5}{2}$ current $\Phi^{(1),1}_\frac{1}{2}$**

Let us introduce some part of the first order pole in (3.51) as follows:

$$\tilde{\varphi}^{(1),1}_\frac{1}{2} = \frac{2i}{(k+N+2)} T^{23} \varphi^{(1),4}_\frac{1}{2} - \frac{2i}{(k+N+2)} T^{24} \varphi^{(1),3}_\frac{1}{2} + \frac{4i}{(k+N+2)} T^{34} \varphi^{(1),2}_\frac{1}{2} - \frac{(N-k)}{3(k+N+2)} \partial \varphi^{(1),1}_\frac{1}{2}. $$

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One should also calculate the OPE between $G^1(z)$ and $\tilde{\Phi}^{(1),1}_{\frac{5}{2}}(w)$ ($\mu = \nu = 1$). From the defining OPEs in Appendix (A.1), one can calculate this OPE as follows:

$$G^1(z) \tilde{\Phi}^{(1),1}_{\frac{5}{2}}(w) = \frac{1}{(z-w)^3} \left[ -\frac{4(k-N)}{3(k+N+2)} \right] \Phi_0^{(1)}(w) + \frac{1}{(z-w)^2} \left[ -\frac{(k-N)}{3(k+N+2)} \right] \partial \Phi_0^{(1)}(w)$$

$$+ \frac{1}{(z-w)} \left[ \frac{2i}{(k+N+2)} T^{23} \Phi_1^{(1),23} + \frac{2i}{(k+N+2)} T^{24} \Phi_1^{(1),24} \right]$$

$$+ \frac{4i}{(k+N+2)} T^{34} \Phi_1^{(1),34} - \frac{(k-N)}{3(k+N+2)} \partial^2 \Phi_0^{(1)}(w) + \cdots. \quad (D.4)$$

Then one can write down the higher spin-3 current as in (3.101). In other words, one has 1) the first order pole in (3.84) with $\mu = \nu = 4$ in $SO(4)$ basis, which comes from the OPE between $G^4(z)$ and the first order pole of (3.35), 2) the second term of (3.101) and 3) the first order pole of Appendix (D.4).

There exists other expression for the same higher spin-$\frac{5}{2}$ current in Appendix (C.2). From its defining equation in Appendix (C.1), one introduces the following quantity

$$\tilde{\Phi}^{(1),1}_{\frac{5}{2}} = \frac{2i}{(k+N+2)} T^{23} \Phi_1^{(1),4} + \frac{2i}{(k+N+2)} T^{34} \Phi_1^{(1),2} - \frac{4i}{(k+N+2)} T^{24} \Phi_1^{(1),3} - \frac{(N-k)}{3(k+N+2)} \partial \Phi_0^{(1),1}. \quad (D.5)$$

The following OPE can be calculated from Appendix (A.1)

$$G^1(z) \tilde{\Phi}^{(1),1}_{\frac{5}{2}}(w) = \frac{1}{(z-w)^3} \left[ -\frac{4(k-N)}{3(k+N+2)} \right] \Phi_0^{(1)}(w) + \frac{1}{(z-w)^2} \left[ -\frac{(k-N)}{3(k+N+2)} \right] \partial \Phi_0^{(1)}(w)$$

$$+ \frac{1}{(z-w)} \left[ \frac{2i}{(k+N+2)} T^{23} \Phi_1^{(1),23} + \frac{2i}{(k+N+2)} T^{34} \Phi_1^{(1),34} \right]$$

$$+ \frac{4i}{(k+N+2)} T^{24} \Phi_1^{(1),24} - \frac{(k-N)}{3(k+N+2)} \partial^2 \Phi_0^{(1)}(w) + \cdots. \quad (D.5)$$

The higher spin-3 current can be determined as in (3.101). In other words, one has the first order pole in (3.84) with $\mu = \nu = 1$ in $SO(4)$ basis, which comes from the OPE between $G^1(z)$ and the first order pole of (3.35), the second term of (3.101) and the first order pole of Appendix (D.5).

There exists other expression for the same higher spin-$\frac{5}{2}$ current in Appendix (C.4). From its defining equation in Appendix (C.3), one introduces the following quantity

$$\tilde{\Phi}^{(1),1}_{\frac{5}{2}} = -\frac{2i}{(k+N+2)} T^{24} \Phi_1^{(1),3} + \frac{2i}{(k+N+2)} T^{34} \Phi_1^{(1),2} + \frac{4i}{(k+N+2)} T^{23} \Phi_1^{(1),4} - \frac{(N-k)}{3(k+N+2)} \partial \Phi_0^{(1),1}. \quad (D.5)$$

The following OPE can be calculated from Appendix (A.1)

$$G^1(z) \tilde{\Phi}^{(1),1}_{\frac{5}{2}}(w) = \frac{1}{(z-w)^3} \left[ -\frac{4(k-N)}{3(k+N+2)} \right] \Phi_0^{(1)}(w) + \frac{1}{(z-w)^2} \left[ -\frac{(k-N)}{3(k+N+2)} \right] \partial \Phi_0^{(1)}(w)$$

$$+ \frac{1}{(z-w)} \left[ \frac{2i}{(k+N+2)} T^{23} \Phi_1^{(1),23} + \frac{2i}{(k+N+2)} T^{24} \Phi_1^{(1),24} \right]$$

$$+ \frac{4i}{(k+N+2)} T^{34} \Phi_1^{(1),34} - \frac{(k-N)}{3(k+N+2)} \partial^2 \Phi_0^{(1)}(w) + \cdots. \quad (D.5)$$
Then one can write down the higher spin-3 current as in (3.101). In other words, one has
1) the first order pole in (3.35),
2) the second term of (3.101) and
3) the first order pole of Appendix (D.6).

D.3 From the fourth higher spin-\(\frac{5}{2}\) current \(\Phi^{(1),2}_{\frac{5}{2}}\)

Let us introduce some part of the first order pole in (3.57) as follows:

\[
\Phi_{\frac{5}{2}}^{(1),2} = \frac{2i}{(k + N + 2)} T^{13} \Phi_{\frac{5}{2}}^{(1),3} \equiv - \frac{2i}{(k + N + 2)} T^{13} \Phi_{\frac{5}{2}}^{(1),1} + \frac{4i}{(k + N + 2)} T^{14} \Phi_{\frac{5}{2}}^{(1),3} - \frac{(N - k)}{3(k + N + 2)} \partial \Phi_{\frac{5}{2}}^{(1),2}.
\]

One should also calculate the OPE between \(G^2(z)\) and \(\Phi_{\frac{5}{2}}^{(1),2}(w)\) (\(\mu = \nu = 2\)). From the defining OPEs in Appendix (A.1), one can calculate this OPE as follows:

\[
G^2(z) \Phi_{\frac{5}{2}}^{(1),2}(w) = \frac{1}{(z - w)^3} \left[ \frac{-4(k - N)}{3(k + N + 2)} \right] \Phi_{0}^{(1)}(w) + \frac{1}{(z - w)^2} \left[ \frac{-(k - N)}{(k + N + 2)} \right] \partial \Phi_{0}^{(1)}(w)
+ \frac{1}{(z - w)} \left[ \frac{2i}{(k + N + 2)} T^{13} \Phi_{1}^{(1),13} + \frac{2i}{(k + N + 2)} T^{34} \Phi_{1}^{(1),34} \right]
+ \frac{4i}{(k + N + 2)} T^{14} \Phi_{1}^{(1),14} - \frac{(k - N)}{3(k + N + 2)} \partial \Phi_{0}^{(1)}(w) + \cdots.
\]

Then one can write down the higher spin-3 current as in (3.101). In other words, one has
1) the first order pole in (3.84) with \(\mu = \nu = 4\) in \(SO(4)\) basis, which comes from the OPE between \(G^4(z)\) and the first order pole of (3.35),
2) the second term of (3.101) and
3) the first order pole of Appendix (D.7).
Then one can write down the higher spin-3 current as in (3.101). In other words, one has the first order pole in (3.84) with \( \mu = \nu = 2 \) in \( SO(4) \) basis, which comes from the OPE between \( G^2(z) \) and the first order pole of (3.35), the second term of (3.101) and the first order pole of Appendix (D.8).

There exists other expression for the same higher spin-\( \frac{5}{2} \) current in Appendix (C.12). From its defining equation in Appendix (C.11), one introduces the following quantity

\[
\tilde{\Phi}^{(1),2} = -\frac{2i}{(k+N+2)} T_{13}^{(1),4} - \frac{2i}{(k+N+2)} T_{14}^{(1),3} - \frac{4i}{(k+N+2)} T_{34}^{(1),1} - \frac{(N-k)}{3(k+N+2)} \partial \Phi^{(1),2}.
\]

The following OPE can be calculated from Appendix (A.1)

\[
G^2(z) \tilde{\Phi}^{(1),2}(w) = \frac{1}{(z-w)^3} \left[ \frac{-4(k-N)}{3(k+N+2)} \right] \Phi_0^{(1)}(w) + \frac{1}{(z-w)^2} \left[ \frac{-(k-N)}{(k+N+2)} \right] \partial \Phi_0^{(1)}(w) + \frac{1}{(z-w)} \left[ \frac{2i}{(k+N+2)} T_{13}^{(1),13} + \frac{2i}{(k+N+2)} T_{14}^{(1),14} \right] \Phi_0^{(1)}(w) + \frac{4i}{(k+N+2)} T_{34}^{(1),34} - \frac{(k-N)}{3(k+N+2)} \partial^2 \Phi_0^{(1)}(w) + \cdots. \tag{D.9}
\]

Then one can write down the higher spin-3 current as in (3.101). In other words, one has the first order pole in (3.84) with \( \mu = \nu = 4 \) in \( SO(4) \) basis, which comes from the OPE between \( G^4(z) \) and the first order pole of (3.35), the second term of (3.101) and the first order pole of Appendix (D.9).

References

[1] M. R. Gaberdiel and R. Gopakumar, Phys. Rev. D 83, 066007 (2011).
[2] M. R. Gaberdiel and R. Gopakumar, JHEP 1207, 127 (2012).
[3] S. F. Prokushkin and M. A. Vasiliev, Nucl. Phys. B 545, 385 (1999).
[4] S. Prokushkin and M. A. Vasiliev, hep-th/9812242.
[5] P. Goddard, A. Kent and D. I. Olive, Phys. Lett. 152B, 88 (1985).
[6] P. Goddard, A. Kent and D. I. Olive, Commun. Math. Phys. 103, 105 (1986).
[7] F. A. Bais, P. Bouwknegt, M. Surridge and K. Schoutens, Nucl. Phys. B 304, 348 (1988).
[8] F. A. Bais, P. Bouwknegt, M. Surridge and K. Schoutens, Nucl. Phys. B 304, 371 (1988).
[9] V. A. Fateev and S. L. Lukyanov, Int. J. Mod. Phys. A 3, 507 (1988).
[10] M. R. Gaberdiel and R. Gopakumar, J. Phys. A 46, 214002 (2013).
[11] M. R. Gaberdiel and T. Hartman, JHEP 1105, 031 (2011).
[12] C. Ahn, JHEP 1202, 027 (2012).
[13] C. Ahn and H. Kim, JHEP 1401, 012 (2014) Erratum: [JHEP 1401, 174 (2014)].
[14] C. Ahn, Eur. Phys. J. C 77, no. 6, 394 (2017).
[15] C. Ahn, JHEP 1205, 040 (2012).
[16] C. Ahn and J. Paeng, Class. Quant. Grav. 30, 175004 (2013).
[17] C. Ahn, JHEP 1307, 141 (2013).
[18] C. Ahn, JHEP 1304, 033 (2013).
[19] C. Ahn, K. Schoutens and A. Sevrin, Int. J. Mod. Phys. A 6, 3467 (1991).
[20] K. Hornfeck and E. Ragoucy, Nucl. Phys. B 340, 225 (1990).
[21] C. Ahn and J. Paeng, JHEP 1401, 007 (2014).
[22] T. Creutzig, Y. Hikida and P. B. Ronne, JHEP 1302, 019 (2013).
[23] C. Ahn, Phys. Rev. D 94, no. 12, 126014 (2016).
[24] R. Gopakumar, A. Hashimoto, I. R. Klebanov, S. Sachdev and K. Schoutens, Phys. Rev. D 86, 066003 (2012).
[25] C. Ahn, JHEP 1301, 041 (2013).
[26] C. Ahn, JHEP 1208, 047 (2012).
[27] Y. Kazama and H. Suzuki, Nucl. Phys. B 321, 232 (1989).
[28] Y. Kazama and H. Suzuki, Phys. Lett. B 216, 112 (1989).
[29] T. Creutzig, Y. Hikida and P. B. Ronne, JHEP 1202, 109 (2012).
[30] C. Candu and M. R. Gaberdiel, JHEP 1309, 071 (2013).
[31] K. Hanaki and C. Peng, JHEP 1308, 030 (2013).
[32] C. Ahn and H. Kim, JHEP 1612, 001 (2016).
[33] T. Creutzig, Y. Hikida and P. B. Ronne, JHEP 1410, 163 (2014).
[34] M. R. Gaberdiel and R. Gopakumar, JHEP 1309, 036 (2013).
[35] M. R. Gaberdiel and R. Gopakumar, JHEP 1411, 044 (2014).
[36] M. R. Gaberdiel and R. Gopakumar, J. Phys. A 48, no. 18, 185402 (2015).
[37] M. R. Gaberdiel and R. Gopakumar, JHEP 1609, 085 (2016).
[38] C. Ahn and H. Kim, JHEP 1412, 109 (2014).
[39] C. Ahn, JHEP 1403, 091 (2014).
[40] C. Ahn, H. Kim and J. Paeng, Int. J. Mod. Phys. A 31, no. 16, 1650090 (2016).
[41] C. Ahn and J. Paeng, Class. Quant. Grav. 32, no. 4, 045011 (2015).
[42] P. Goddard and A. Schwimmer, Phys. Lett. B 214, 209 (1988).
[43] A. Van Proeyen, Class. Quant. Grav. 6, 1501 (1989).
[44] M. Gunaydin, J. L. Petersen, A. Taormina and A. Van Proeyen, Nucl. Phys. B 322, 402 (1989).
[45] S. J. Gates, Jr. and S. V. Ketov, Phys. Rev. D 52, 2278 (1995).
[46] A. Sevrin, W. Troost and A. Van Proeyen, Phys. Lett. B 208, 447 (1988).
[47] A. Sevrin, W. Troost, A. Van Proeyen and P. Spindel, Nucl. Phys. B 311, 465 (1988).
[48] K. Schoutens, Nucl. Phys. B 295, 634 (1988).
[49] A. Sevrin and G. Theodoridis, Nucl. Phys. B 332, 380 (1990).
[50] N. Saulina, Nucl. Phys. B 706, 491 (2005).
[51] C. Ahn and M. H. Kim, Eur. Phys. J. C 76, no. 7, 389 (2016).
[52] M. Beccaria, C. Candu and M. R. Gaberdiel, JHEP 1406, 117 (2014).
[53] C. Ahn, D. g. Kim and M. H. Kim, Eur. Phys. J. C 77, no. 8, 523 (2017).
[54] C. Ahn, arXiv:1711.07599 [hep-th].

[55] C. M. Chang and X. Yin, JHEP 1210, 024 (2012).

[56] K. Thielemans, Int. J. Mod. Phys. C 2, 787 (1991).

[57] Wolfram Research, Inc., Mathematica, Version 11.0, Champaign, IL (2016).

[58] Y. Hikida and T. Uetoko, Universe 3, no. 4, 70 (2017).

[59] Y. Hikida and T. Uetoko, Prog. Theor. Exp. Phys. (2017) 113B03.

[60] Y. Hikida and T. Uetoko, Phys. Rev. D 97, no. 8, 086014 (2018).

[61] K. Papadodimas and S. Raju, Nucl. Phys. B 856, 607 (2012).

[62] C. M. Chang and X. Yin, JHEP 1210, 050 (2012).

[63] C. M. Chang and X. Yin, Phys. Rev. D 88, no. 10, 106002 (2013).

[64] L. Eberhardt, M. R. Gaberdiel, R. Gopakumar and W. Li, JHEP 1703, 124 (2017).

[65] M. R. Gaberdiel, R. Gopakumar and C. Hull, JHEP 1707, 090 (2017).

[66] K. Ferreira, M. R. Gaberdiel and J. I. Jottar, JHEP 1707, 131 (2017).

[67] L. Eberhardt, M. R. Gaberdiel and W. Li, JHEP 1708, 111 (2017).

[68] L. Eberhardt, M. R. Gaberdiel and I. Rienacker, JHEP 1803, 097 (2018).

[69] G. Giribet, C. Hull, M. Kleban, M. Porrati and E. Rabinovici, arXiv:1803.04420 [hep-th].

[70] M. R. Gaberdiel and R. Gopakumar, arXiv:1803.04423 [hep-th].

[71] V. G. Kac and I. T. Todorov, Commun. Math. Phys. 102, 337 (1985).

[72] C. Ahn and H. Kim, JHEP 1510, 111 (2015).

[73] P. Bouwknegt and K. Schoutens, Phys. Rept. 223, 183 (1993).

[74] J. Fuchs, Nucl. Phys. B 318, 631 (1989).

[75] R. Slansky, Phys. Rept. 79, 1 (1981).

[76] R. Feger and T. W. Kephart, Comput. Phys. Commun. 192, 166 (2015).