RECOVERY OF THE $C^\infty$ JET FROM THE BOUNDARY DISTANCE FUNCTION

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Abstract. For a compact Riemannian manifold with boundary, we want to find the metric structure from knowledge of distances between boundary points. This is called the “boundary rigidity problem”. If the boundary is not concave, which means locally not all shortest paths lie entirely in the boundary, then we are able to find the Taylor series of the metric tensor ($C^\infty$ jet) at the boundary (see [3,8]). In this paper we give a new reconstruction procedure for the $C^\infty$ jet at non-concave points on the boundary using the localized boundary distance function.

A closely related problem is the “lens rigidity problem”, which asks whether the lens data determine metric structure uniquely. Lens data include information of boundary distance function, the lengths of all geodesics, and the locations and directions where geodesics hit the boundary. We give the first examples that show that lens data cannot uniquely determine the $C^\infty$ jet. The example include two manifolds with the same boundary and the same lens data, but different $C^\infty$ jets. With some additional careful work, we can find examples with different $C^1$ jets, which means the boundaries in the two lens-equivalent manifolds have different second fundamental forms.

1. Introduction

Let $(M,g)$ be a compact Riemannian manifold with smooth boundary, and let $\tau : M \times M \to \mathbb{R}$ be the distance function given by $g$. The boundary rigidity problem asks whether we can recover $g$ from the information of $\tau|_{\partial M}$. That is, whether we can uniquely determine the Riemannian metric of $M$, knowing the distances from boundary points to boundary points. Obviously if we pullback the metric $g$ by a diffeomorphism $f : M \to M$ that fixes every boundary point, the resulting metric $f^*g$ gives the same boundary distance function as before, but it is different from the original metric $g$. So the natural question is, whether this is the only obstruction to unique determination. If the answer is positive for $(M,g)$, then it is called boundary rigid.

One would like to know whether a given manifold with boundary is boundary rigid. If in some cases we have affirmative answers, we further want to have a
procedure to recover the interior metric structure from the information of boundary ("chordal") distances. The $C^\infty$ jet at a point of a Riemannian manifold is, roughly speaking, the Taylor series of the metric tensor at the point. Therefore to recover the $C^\infty$ jet at boundary points is the first step of the recovery of the entire interior metric structure.

In arguments about the boundary rigidity problem, often one needs to extend $(M, g)$ beyond its boundary, and here people care about the smoothness of the extension. The extension of $g$ is smooth if and only if the $C^\infty$ jets computed from both sides of the boundary agree.

There are results on the boundary determination of $C^\infty$ jets. Michel [4] proved that boundary distances uniquely determine the Taylor series of $g$ up to order 2, and in [5] he proved the same result without order limitations but with $\dim(M) = 2$, both with convex boundaries. Here convexity roughly means that the distance of two sufficiently close boundary points should be realized by a geodesic whose interior does not intersect the boundary. In [3] there is an elementary proof that the $C^\infty$ jet is uniquely determined by the boundary distance function if the boundary is convex. However the results above are not constructive. In [8], Uhlmann and Wang applied the result of [6] and used a suitably chosen reference metric, and gave a recovery procedure of $C^\infty$ jet on the boundary from localized boundary distance function. Here “localized” means we do not need to know $\tau$ for all pairs of points in $\partial M$, but we only should know $\tau$ restricted to an open neighborhood of the diagonal of $\partial M \times \partial M$, that is, the distance between close enough pairs. It should be noted that the arguments in [3] and [8] also apply to non-concave boundary (see Definition 2.1) without much modification.

Up to now, the only result for concave boundary is [7], Theorem 1. The statement is, if a geodesic segment $\gamma$ is tangent to the boundary at one end $p$, and the other end $q$ lies on the boundary, then under some generic no conjugate points condition, we can recover the $C^\infty$ jet at $p$ based on the lengths of geodesic segments in a neighborhood of $\gamma$. The argument in section 3 of this paper is similar to [7].

In the first part (section 2 and 3) of this paper, we give the same results as in [8], that is, a procedure to recover the $C^\infty$ jet at boundary points, but our argument is relatively elementary. We also directly adopt the weaker assumption that the boundary is non-concave, as opposed to “convex” as in previous results.

In the second part (section 4 and 5) of this paper, we give the first known example that shows the lens data do not always determine the boundary $C^\infty$ jet. Here “lens data” include the information of $\tau|_{\partial M}$ and the lengths of all maximal geodesics, together with the locations and the directions whenever they hit the
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boundary (see Definition 4.1 or [7] section 1 for detail). So the results in [3], [8], and the first part of this paper show that lens data uniquely determine $C^\infty$ jet near non-concave points. Meanwhile, the results in [7] should imply: We can uniquely recover $C^\infty$ jet near “generic” concave points, from the lens data of geodesics with bounded length, which are “almost” tangent to the boundary. In the example in section 4 the boundary is totally geodesic, so nearby geodesics have unbounded length, although each of them hits the boundary in finite time. In the example in section 5 the boundary is strictly concave, but every complete geodesic tangent to the boundary has infinite length. Therefore, the examples in this paper fall in the gap between non-concave results (Theorem 3.8 of this paper, [3], and [8]) and the concave result [7].

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2. Preliminaries

Throughout this paper we let $(M, g)$ be a compact $n$-dimensional Riemannian manifold with smooth boundary $\partial M$. Let $\tau$ be the distance function, and let $\rho = \tau^2$. We further introduce the notation $\tau_x(\cdot) = \tau(\cdot, x)$, and $\rho_x(\cdot) = \rho(\cdot, x)$. Notice that the distance might not be realized by a geodesic, and a curve realizing it can have non-geodesic parts in the boundary.

Let

\begin{equation}
\mu : \partial M \times \partial M \to \mathbb{R}
\end{equation}

be the distance function on the Riemannian manifold $(\partial M, g|_{\partial M})$. Note that $\mu$ is not $\tau|_{\partial M}$ ($\mu \geq \tau$ in general) although they may agree on some subset. Near “non-concave” points, $\tau|_{\partial M}$ contains more information than $\mu$.

Definition 2.1 (Concave and Non-Concave points). Let $x \in \partial M$. We say $\partial M$ is concave at $x$ if the second fundamental form of $\partial M$ is negative semi-definite at $x$, with respect to $\nu_x$ the inward-pointing unit normal vector. We call $\partial M$ non-concave at $x$ if it is not concave at $x$, that is, the second fundamental form has at least one positive eigenvalue.

In order to detect non-concave points from $\tau|_{\partial M}$, we state the following elementary proposition without proof.

Proposition 2.2. Let $x \in \partial M$. If $\partial M$ is concave in an open neighborhood of $x$, then there exists $\varepsilon > 0$ such that whenever $p, q \in \partial M$ satisfy $\mu(x, p) < \varepsilon$ and $\mu(x, q) < \varepsilon$, we have $\tau(p, q) = \mu(p, q)$. That means, for a pair of points close enough to $x$, the shortest path between them is along the boundary.
Since \((M, g)\) is extendable, we fix a compact \(n\)-dimensional Riemannian manifold without boundary \((\tilde{M}, \tilde{g})\), such that \(\partial M\) is an \((n - 1)\)-dimensional submanifold, the interior of \(M\) is a connected component of \(\tilde{M} - \partial M\), and \(g\) is the restriction of \(\tilde{g}\).

Next we define boundary normal coordinates of \(M\) near \(\partial M\). Let \((x_1, \ldots, x_{n-1})\) be a coordinate chart on the manifold \(\partial M\). For \(p \in M\) close enough to \(\partial M\), there is a unique point \(y = (y_1, \ldots, y_{n-1}) \in \partial M\) that is closest to \(p\). We then give \(p\) the coordinates \((y_1, \ldots, y_{n-1}, y_n)\) where \(y_n\) is defined to be the distance from \(p\) to \(y\). In such coordinates, \(g_{in} = \delta_{in}\), and the curves \(c(t) = (y_1, \ldots, y_{n-1}, t)\) are geodesics perpendicular to \(\partial M\) at \(t = 0\).

Knowing the \(C^\infty\) jet of \(g\) on \(\partial M\) is equivalent to, with respect to a given boundary normal coordinates, knowing the derivatives \(\frac{\partial^k}{\partial x_n^k} g_{ij}\) for all \(k \geq 0\) and indices \(i, j\), where \(g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\). Clearly if we know the jet with respect to one choice of boundary normal coordinates, we are able to find the jet with respect to every choice of boundary normal coordinates, knowing the coordinate change on the boundary. For each integer \(l \geq 0\), knowledge of \(C^l\) jet means knowledge of all the \(\frac{\partial^k}{\partial x_n^k} g_{ij}\) with \(k \leq l\). In this paper we find the jet only under boundary normal coordinates, and see \cite{3} Theorem 2.1 for the precise statement for general coordinates.

The key identity in the jet recovery procedure is the Eikonal equation,

\[|\nabla \tau_p| = 1, \quad p \in M,\]

wherever the function \(\tau_p\) is smooth. In coordinate charts the Eikonal equation is

\[g^{ij} \frac{\partial \tau_p}{\partial x_i} \frac{\partial \tau_p}{\partial x_j} = 1.\]

Here we adopt Einstein summation convention, where \(i, j\) ranges from 1 to \(n\), and matrix \((g^{ij})\) is the inverse of \((g_{ij})\). In boundary normal coordinates, this becomes

\[g^{\alpha\beta} \frac{\partial \tau_p}{\partial x_\alpha} \frac{\partial \tau_p}{\partial x_\beta} + \left(\frac{\partial \tau_p}{\partial x_n}\right)^2 = 1,\]

where \(\alpha\) and \(\beta\) range from 1 to \((n - 1)\).

We will use the convention that \(i, j\) range from 1 to \(n\), and \(\alpha, \beta\) range from 1 to \((n - 1)\). We assume we are always in boundary normal coordinates near \(\partial M\). We will write \(\partial_{x_i}\) for \(\frac{\partial}{\partial x_i}\), and \(\partial_{x_i, x_j}\) for \(\frac{\partial^2}{\partial x_i \partial x_j}\), and so on. The reader should view the function \(\tau\) as \(\tau(x, y)\) and

\[\tau(x_1, \ldots, x_n, y_1, \ldots, y_n),\]
so that formulas like $\partial_x \tau(p, q)$ and $\partial_y \tau(p, q)$ will make sense. We treat $\rho = \tau^2$
similarly.

3. RECOVERY OF $C^\infty$ JET

We have the following lemmas.

Lemma 3.1. Let $V_\varepsilon \subset M \times M$ be the set of pairs $(x, y)$ satisfying the following
properties: $\tau(x, y)$ is realized by a geodesic in $M$, and $\tau(x, y) \leq \varepsilon$.
Then there exists an $\varepsilon > 0$ such that $\rho$ is a smooth function on $V_\varepsilon$.

The lemma is easy to prove if $M$ has no boundary. But when $M$ has a boundary,
we may first prove the property for the extension $(\tilde{M}, \tilde{g})$ with its corresponding $\tilde{\rho}$, and use the fact that $\rho|_{V_\varepsilon}$ is the same as $\tilde{\rho}|_{V_\varepsilon}$. Recall that a function being smooth in
(a subset of) a manifold with boundary (and possibly corner) means the manifold
together with the function can be extended into a bigger one without boundary
such that the function is still smooth.

Notice that we cannot replace $\rho$ in the last lemma with $\tau$, because $\tau$ is not
smooth where $x = y$. Smoothness is the reason why we use distance squared rather
than distance itself.

Lemma 3.2. Let $c : (-\varepsilon, +\varepsilon) \to M$ be a smooth curve in $M$, which may intersect
$\partial M$. If for each $t$ the distance between $c(t)$ and $c(0)$ is realized by a minimizing
geodesic of $M$, then we have

\[ 2|c'(0)|^2 = \frac{\partial^2}{\partial t^2} \bigg|_{t=0} \rho(c(t), c(0)). \]

Proof. If $c$ is a geodesic the statement is clearly true. If $c'(0) = 0$ the statement is
also easy to prove.

Otherwise, we may think of $c'(t)$ as coming from a vector field $X$ in a neighborhood
of $c(0) \in M$. This will give rise to a vector field $\tilde{X} = (X, 0)$ in $M \times M$. Then
we look at the right side,

\[ \frac{\partial^2}{\partial t^2} \bigg|_{t=0} \rho(c(t), c(0)) = \tilde{X}(\tilde{X}\rho) \]

\[ = \text{Hess}(\tilde{X}, \tilde{X}) + (\nabla_{\tilde{X}} \tilde{X}) \rho \]

\[ = \text{Hess}(\tilde{X}, \tilde{X}), \]

where all expressions are evaluated at $(c(0), c(0))$, a critical point of $\rho$. However,
$\text{Hess}(\tilde{X}, \tilde{X})$ only depends on $\tilde{X}$ at the point, which is $(c'(0), 0)$, so the right side
of equation (3) only depends on $c'(0)$. This means we might as well assume $c$ is a
godesic. \[ \Box \]
Now we are ready to recover the jet from the localized boundary distance function. We present the recovery procedure in four steps, i.e. Proposition 3.3, 3.4, 3.6, and 3.7.

**Proposition 3.3.** We can recover the $C^0$ jet from the localized boundary distance function.

This is easy because $C^0$ jet is simply $g_{ij}|_{\partial M}$ the Riemannian metric tensor. From the localized boundary distance function, we are able to compute the length of any smooth curve in $\partial M$. The curve lengths will tell us the metric tensor.

We start the recovery procedure for higher order jets. The idea underlying the proofs of the following propositions (Proposition 3.4, 3.6, and 3.7) is derived from [7], section 3.

**Proposition 3.4.** If $\partial M$ is non-concave at $y$, then we can recover the $C^1$ jet near $y \in \partial M$ from the localized boundary distance function, with respect to a given boundary normal coordinates.

We need a definition for the proof of this proposition.

**Definition 3.5** (Convex direction). Let $\xi$ be a vector tangent to $\partial M$. We can find a geodesic $\gamma : (-\varepsilon, +\varepsilon) \to \partial M$ with $\gamma'(0) = \xi$. (Here $\gamma$ may not be a geodesic in $M$.) Let $\nabla$ be the covariant derivative in $M$, and $\nu$ the inward-pointing unit normal at appropriate points in $\partial M$. We call $\xi$ a convex direction if $\langle \nabla_{\gamma'(0)} \gamma', \nu \rangle > 0$.

Certainly the set of convex directions compose an open subset of $T(\partial M)$. By definition, $\partial M$ is non-concave at $y$ if and only if there is at least one, and hence a nonempty open set of convex directions based at $y$.

**Proof of Proposition 3.4.** After possibly changing coordinates, we assume $\partial x_1$ is a convex direction at $y$ (and in a neighborhood too). Let $c(t)$ be a curve in $M$ such that $c'(t) = \partial x_1$, which means its coordinates representation is $(x_1 + t, x_2, \ldots, x_n)$. Applying lemma 3.2 we know

\begin{equation}
2g_{11}(p) = \partial_{x_1} \rho(p, p),
\end{equation}

where $p$ is not assumed to be on the boundary. Clearly both sides of the equation are smooth functions of $p$. We now let the point $p$ move in the direction $\partial x_n$ and take the derivative of equation (1),

\begin{align}
2\partial_{x_n} g_{11} &= \partial_{x_n} \partial_{x_1} \rho + \partial_{y_n} \partial_{x_1} \rho \\
&= \partial_{x_1} \partial_{x_n} \rho + \partial_{y_n} \rho.
\end{align}
We let $c : (−ε, +ε) → ∂M$ be the curve in $∂M$ with $c(0) = y$ and $c′ ≡ ∂x_1$. Since $∂x_1$ is a convex direction, we may assume for any point $x$ on $c$ which is not the same point as $y$, the distance between $y$ and $x$ is realized by a geodesic segment whose interior does not intersect $∂M$, and the geodesic is transversal to $∂M$ at both endpoints. So we know the value of $(∂_{x_n}τ)(x, y)$ from first variation of arclength, and similarly $(∂_{y_n}τ)(x, y)$. The values $(∂_{x_n}ρ)(x, y)$ and $(∂_{y_n}ρ)(x, y)$ are then easily recovered from localized $τ|_{∂M}$.

Since $∂_{x_1x_1}(∂_{x_n}ρ + ∂_{y_n}ρ)|_{(y, y)}$ only depends on the value of $(∂_{x_n}ρ + ∂_{y_n}ρ)(x, y)$ where $x$ is along the curve $c$, from equation (15) we find $∂_{x_n}g_{11}$|$_y$.

Now we use the fact that a symmetric $n × n$ tensor $(f_{ij})$ can be recovered by knowledge of $f_{ij}v_k^iv_j^k$ for $N = n(n + 1)/2$ “generic” vectors $v_k$, $k = 1, \ldots, N$, and we can find such $N$ vectors in any open set on the unit sphere.

We may choose appropriate $N$ perturbations of $∂x_1$, say $v_k$, which are all convex directions at $y$. Letting $(∂_{x_n}g_{ij})$ be the tensor described above, We find the values of $∂_{x_n}g_{ij}v_k^iv_j^k$ using the same method as above (change $∂x_1$ into $v_k$). They will tell us the values of $∂_{x_n}g_{ij}|_y$. □

Next we give the recovery procedure of $C^2$ jet, which applies Eikonal equation. The cases of higher order jets are essentially the same as $C^2$ jet.

**Proposition 3.6.** If $∂M$ is non-concave at $y$, then we can recover the $C^2$ jet near $y ∈ ∂M$ from the localized boundary distance function, with respect to a given boundary normal coordinates.

*Proof.* Clearly being non-concave is an open property for points in $∂M$, so we have already recovered $C^1$ jet near $y$ by Proposition 3.5. Again we assume without loss of generality that $∂x_1$ is a convex direction at $y$.

Now look at equation (1) again, and let $p$ move in the direction $∂x_n$, but this time we look at the second derivative:

$$2∂_{x_nx_n}g_{11} = (∂_{x_n} + ∂_{y_n})^2(∂_{x_1x_1}ρ)$$

(6)

$$= ∂_{x_1x_1}(∂_{x_nx_n}ρ + 2∂_{x_ny_n}ρ + ∂_{y_ny_n}ρ).$$

Again we let $c$ be a short enough curve in $∂M$ with $c(0) = y$ and $c′ ≡ ∂x_1$. For the same reason as in the proof of Theorem 3.4 to compute the value of right side of equation (6) we only need to know $(∂_{x_nx_n}ρ + 2∂_{x_ny_n}ρ + ∂_{y_ny_n}ρ)$ at $(x, y)$ where $x$ lies on $c$.

If $x = y$, it is easy to see the value of $(∂_{x_nx_n}ρ + 2∂_{x_ny_n}ρ + ∂_{y_ny_n}ρ)$ at $(x, y)$ is 0.

If $x ≠ y$, then we look at the Eikonal equation as in (2), in the following form,

$$g^{αβ}(q_1)(∂_{x_n}τ_y(x))(∂_{x_β}τ_y(x)) + (∂_{x_n}τ_y(x))^2 = 1.$$
Taking $\partial_{x_n}$ we get (with all terms evaluated at $x$)

\begin{equation}
\partial_{x_n} g^{\alpha\beta} (\partial_{x_\alpha \tau})(\partial_{x_\beta \tau}) + 2 g^{\alpha\beta} (\partial_{x_\alpha x_n \tau})(\partial_{x_\beta \tau}) + 2 (\partial_{x_n \tau})(\partial_{x_n x_n \tau}) = 0.
\end{equation}

In equation (8), the term $\partial_{x_n} g^{\alpha\beta}$ we already know because $(g^{\alpha\beta})$ is the inverse of $(g_{\alpha\beta})$ and we know $g_{\alpha\beta}$ and $\partial_{x_n} g_{\alpha\beta}$. Also we know $\partial_{x_n \tau}$ from the localized boundary distance function. We know $\partial_{x_n x_n \tau}$ because from the first variation formula we know $\partial_{x_n \tau}$ in a neighborhood of $x$ along the boundary.

Therefore, so far the only term in equation (8) we do not know is $\partial_{x_n x_n \tau}(x) = \partial_{x_n x_n \tau}(x, y)$, whose coefficient is $2 \partial_{x_n \tau}$, a nonzero number because of the transversality of the segment between $x$ and $y$ to $\partial M$. We can now immediately find value of $\partial_{x_n x_n \tau}(x, y)$ from the other terms. Then, we can find $\partial_{x_n x_n \rho}(x, y)$. If we interchange the roles of $x$ and $y$, we can find $\partial_{y_n y_n}(x, y)$. As for $\partial_{x_n y_n}(x, y)$, we simply take derivative of equation (7) with respect to $y_n$, that is, let $y$ move away from the boundary, and get (assuming all are taken at $(x, y)$)

\begin{equation}
2(\partial_{x_n y_n \tau})(\partial_{x_n \tau}) + 2 \partial_{x_n y_n \tau}(\partial_{x_n x_n \tau}) = 0,
\end{equation}

where we know all but $\partial_{x_n y_n \tau}(x, y)$. So we can find the value of $\partial_{x_n y_n \tau}(x, y)$ and hence $\partial_{x_n y_n \rho}(x, y)$.

Up to now, we have computed $(\partial_{x_n x_n \rho} + 2 \partial_{x_n y_n \rho} + \partial_{y_n y_n \rho})$ at $(x, y)$ with $x \in c$, so by equation (8), we can find $\partial_{x_n x_n g_{11}}(y)$.

Once again, we perturb $\partial_{x_1}$ a little to get sufficiently many vectors $v_k$ with convex directions. Carry out the procedure for every $v_k$ to know $(\partial_{x_n x_n g_{ij}}) v_k^i v_j^k$, and combine the values all together to find out all the $\partial_{x_n x_n g_{ij}}(y)$.

We may now proceed by induction.

**Proposition 3.7.** Let $k \geq 3$. If we have recovered the $C^{k-1}$ jet in an open neighborhood of $y \in \partial M$, then with respect to a given boundary normal coordinates, we can recover the $C^k$ jet of the same neighborhood from localized boundary distance function.

**Proof.** We let $p$ in equation (4) move towards the $n$th direction and take the $k$th derivative, and get the following equation,

\begin{equation}
2 \partial_{x_n}^k g_{11} |_p = (\partial_{x_n} + \partial_{y_n}^k (\partial_{x_1 z_1} \rho))_{(p,p)}
\end{equation}

\begin{equation}
= \partial_{x_1 z_1} \sum_{i=0}^{k} \binom{k}{i} \partial_{x_n}^i \partial_{y_n}^{k-i} \rho_{(p,p)}.
\end{equation}

Here we borrow notation from Theorem 3.6. The right side of equation (10) evaluated at $(y, y)$ only depends on

\begin{equation}
\partial_{x_n}^k \partial_{y_n}^{k-i} \rho(x, y),
\end{equation}
where \( x \) lies on the curve \( c \), and \( i = 0, 1, \ldots, k \).

If \( x = y \) all are simple to compute, and the values do not even depend on the manifold.

If \( x \neq y \), we first solve the problem when \( i = k \). We apply the operator \( \partial^{k-1}_x \) to equation (7), recalling the Eikonal equation holds wherever the gradient is smooth. The resulting equation has terms involving \( g^{\alpha\beta}, \partial_\alpha \tau_y, \partial_\beta \tau_y, \) and \( \tau_{q2} \), and each of them may carry the operator \( \partial_x \) at most \((k-1)\) times, except the last term

\[
2(\partial_x \tau_y)(\partial_x \tau_y),
\]

where \( \partial_x \tau_y \) is nonzero at \( x \) by transversality. Since \( x \neq y \) (which means \( \tau \neq 0 \)), knowing the derivatives of \( \rho \) up to order \((k-1)\) is equivalent to knowing the derivatives of \( \tau \) up to order \((k-1)\). It is also okay to move \( x \) along the boundary (i.e., take \( \partial_x \tau \)) because all procedures work in some open neighborhoods, with a change of coordinates if necessary. So by the inductive hypothesis, we know \( g^{\alpha\beta}, \partial_\alpha \tau, \partial_\beta \tau, \) and \( \tau \), along with their derivatives involving \( \partial_x \) up to \((k-1)\) times. Therefore, we can compute \( \partial^k \tau \), and hence \( \partial^k \rho \). Now we have finished the case \( i = k \).

If \( i = 0 \), we do the same procedure after interchanging \( x \) and \( y \).

Finally, if \( 0 < i < k \), we have at least one \( \partial_x \) and one \( \partial_y \) applied to \( \rho \) in formula (11). To proceed, we can apply \( \partial^{i-1}_x \partial^k_y \) to Eikonal equation (7). We then use the same method as in the case \( i = k \). Note that \( \partial_y g^{\alpha\beta}(x) \equiv 0 \), because \( \partial_y \) does not move point \( x \).

So far we have found \( \partial^k_{x}g_{11}|_y \).

We perturb \( \partial_x \) a little to get sufficiently many vectors \( v \) with convex directions. Carry out the procedure for every such \( v \) to know \( (\partial^k_{x}g_{ij})v^iv^j \), and put the results all together to determine all the \( \partial^k_{x}g_{ij}|_y \). \( \square \)

If we combine the results of Proposition 3.4, Proposition 3.6, and Proposition 3.7, we have the following

**Theorem 3.8.** Suppose \( \partial M \) is non-concave at \( y \), and \( D \subset \partial M \times \partial M \) is an open neighborhood of \((y, y)\). Then we can recover the \( C^\infty \) jet of \( g \) at \( y \) based on the information of \( \tau|_D \).

If we want to weaken the assumption in the theorem, we can try to detect non-concave points of \( \partial M \) by information about \( \tau|_{\partial M} \) only. The contrapositive statement of Proposition 2.2 is, if in any open neighborhood of \( y \) in \( \partial M \), we can find \( x_1, x_2 \) with \( \tau(x_1, x_2) < \mu(x_1, x_2) \), then \( y \) is not in the interior of the (closed) set of concave points, i.e., \( y \) is in the closure of non-concave points. But we can recover
$C^\infty$ jets near non-concave points, and jets are continuous (because $g$ is extendable), so we know the jet at $y$.

**Theorem 3.9.** Suppose $y \in \partial M$. If for every neighborhood $D$ of $(y, y) \in \partial M \times \partial M$, we have $\tau|_D$ and $\mu|_D$ do not entirely agree, then we can recover $C^\infty$ jet of $g$ at $y$.

This can help us know the interior metric structure if we a priori assume the manifold, metric, and boundary are analytic. Observe that the set of non-concave points is open, and we have the following

**Theorem 3.10.** Suppose $(M, \partial M, g)$ is analytic. If for any connected component of $\partial M$, we have a point $y$ satisfying the hypothesis of Theorem 3.8, then we can recover the $C^\infty$ jet of $g$ at all points of $\partial M$.

This can lead to lens rigidity results in the category of analytic metrics, with some assumptions such as “every unit speed geodesic hits the boundary in finite time”, see [9].

In Theorems 3.9 and 3.10, the hypothesis is simply “the localized chordal distance function at the boundary does not agree with the localized in-boundary distance function”. One is tempted to remove the words “localized”, which means we now have the question: for an analytic Riemannian manifold with boundary, if $\tau$ does not entirely agree with $\mu$, can we compute the $C^\infty$ jet? The answer is negative, because of the examples described in the next section.

### 4. Examples of different $C^2$ Jets

In this section we are going to give an example of two manifolds which have the same boundary and the same lens data but different $C^\infty$ jets. The idea of the example is borrowed from [1] section 2, and [2] section 6. The idea in [1] and [2] is, if we have a surface of revolution with two circles as boundary, then in some sense, the lens data only depends on the measures of the sublevel sets of radius function along a meridian. We can find distinct smooth functions $f_1, f_2$ both with domain $[a, b]$, such that they have the same measure for every sublevel set.

Before giving the example, we give the definition of lens data and lens equivalence.

**Definition 4.1.** Let $(M, \partial M, g)$ be a Riemannian manifold with boundary, and let $\partial(\text{SM})$ be the set of unit vectors with base point at boundary. Define set $\Omega \subset \partial(\text{SM}) \times \partial(\text{SM}) \times \mathbb{R}^+$ to be the set of 3-tuples $(\gamma'(0), \gamma'(T), T)$ that satisfies: (1) $\gamma$ is a unit speed geodesic, (2) $\langle \gamma'(0), \nu \rangle > 0$ i.e. $\gamma'(0)$ points inwards, and (3) $T$ is the first moment at which $\gamma$ hits $\partial M$ again. The description above depends
on the interior structure, so we orthogonally project \( \partial(SM) \) to \( \overline{B(\partial M)} \) the closed ball bundle on \( \partial M \). This projection maps \( \Omega \) to \( \Omega' \subset B(\partial M) \times B(\partial M) \times \mathbb{R}^+ \).

We define lens data to be the information of \( \Omega' \) and \( \tau|_{\partial M} \). We say two Riemannian manifolds with boundary are lens equivalent if they have the same boundary and the same lens data i.e. the same \( \Omega' \) and \( \tau|_{\partial M} \).

Consider the strip \( S \) defined as \( \mathbb{R} \times [0, L] \), with standard coordinates \((x, y)\) where \( 0 \leq y \leq L \). Obviously \( S \) has a natural structure of manifold with boundary. Define a Riemannian metric \( g \) on \( S \) as

\[
\begin{align*}
g_{yy} &= 1, \\
g_{xy} &= 0, \\
g_{xx} &= f(y).
\end{align*}
\]

Here \( f : [0, L] \to \mathbb{R} \) is a smooth function, such that \( f(0) = f(L) = 1 \) and \( f(y) \geq 1 \) for all \( y \in (0, L) \). Under certain circumstances, this manifold can be viewed as the universal cover of a surface of revolution in \( \mathbb{R}^3 \).

Obviously the curves \( \gamma_0(t) = (x_0, t) \) and \( \gamma_L(t) = (x_0, L - t) \) are unit speed geodesics, for any \( x_0 \in \mathbb{R} \). So the normal (i.e. “n-th” in previous sections) direction is simply the \( y \) direction, and the \( C^\infty \) jet of \( S \) is determined by

\[
\frac{\partial^k f}{\partial y^k}, \quad k = 1, 2, \ldots
\]

If we let index 1 stand for \( x \) and 2 for \( y \), a straightforward computation of Christoffel symbols shows

\[
\begin{align*}
\Gamma^2_{22} &= \Gamma^2_{12} = \Gamma^1_{22} = \Gamma^1_{11} = 0, \\
\Gamma^2_{11} &= -\frac{1}{2} f'(y), \\
\Gamma^1_{12} &= \frac{1}{2} \frac{f'(y)}{f(y)}.
\end{align*}
\]

Let \((x(t), y(t))\) be a geodesic of \( S \) parametrized by arclength. Then it will satisfy the second order system

\[
\frac{d^2 x_k}{dt^2} + \sum_{i,j} \Gamma^k_{ij} \frac{dx_i}{dt} \frac{dx_j}{dt}, \quad k = 1, 2,
\]

where \( i, j \) ranges over 1, 2, and \( x_1 = x, \ x_2 = y \).

**Lemma 4.2.** Along a geodesic, \( \frac{dx}{dt} \cdot f(y) \) is constant.
Proof.

\[
\frac{d}{dt} \left( \frac{dx}{dt} \cdot f(y) \right) = \frac{dx}{dt} \frac{d}{dt} f(y) + \frac{d^2 x}{dt^2} \cdot f(y) \\
= \frac{dx}{dt} \frac{dy}{dt} f'(y) - f(y) \sum_{i,j} \Gamma_{ij}^1 \frac{dx_i}{dt} \frac{dx_j}{dt} \\
= \frac{dx}{dt} \frac{dy}{dt} f'(y) - f(y) \cdot 2 \Gamma_{12}^1 \frac{dx}{dt} \frac{dy}{dt} \\
= \frac{dx}{dt} \frac{dy}{dt} f'(y) - f(y) \cdot \frac{f'(y)}{f(y)} \frac{dx}{dt} \frac{dy}{dt} \\
= 0.
\]

□

The Lemma above is Clairaut’s relation when \( S \) is a surface of revolution. The Lemma does not require that the geodesic is unit speed, but from now on we assume all geodesics in discussion are of unit speed.

Since \( f(y) \) is never 0, we know either \( \frac{dx}{dt} \) is constant zero or never changes sign. Since \( g_{ij} = \delta_{ij} \) at the boundary, we know each geodesic leaves \( S \) at the same angle as when it enters \( S \). Also, in each geodesic, \( \left| \frac{dx}{dt} \right| \) assumes its maximum on the boundary because \( f(y) \) is minimal there. Therefore, since

\[
\left( \frac{dy}{dt} \right)^2 + \left( \frac{dx}{dt} \right)^2 g_{xx} = 1,
\]

we know \( \frac{dy}{dt} \) never changes sign in the interior. This means each entering geodesic transversal to the boundary goes all the way to the other component of the boundary, and hits the boundary with the “same” direction as it entered.

Suppose \( (x(t), y(t)), t \in [0, T] \) is such a maximal geodesic, and without loss of generality we assume \( \frac{dy}{dt} > 0 \) which is equivalent to the geodesic entering \( S \) at \( y = 0 \) and leaving \( S \) at \( y = L \). Since \( \frac{dy}{dt} \) is positive and smooth, we have

\[
T = \int_0^L \frac{dt}{dy} dy \\
= \int_0^L \left( \frac{dy}{dt} \right)^{-1} dy \\
= \int_0^L (1 - x'(t)^2 \cdot f(y))^{-\frac{1}{2}} dy \\
= \int_0^L \left( 1 - \frac{x'(0)^2}{f(y)} \right)^{-\frac{1}{2}} dy,
\]
and

\[ x(T) - x(0) = \int_0^L \frac{dx}{dy} dy = \int_0^L \frac{dx}{dt} \left( \frac{dy}{dt} \right)^{-1} dy = \int_0^L \frac{x'(0)}{f(y)} \left( 1 - \frac{x'(0)^2}{f(y)^2} \right)^{-\frac{1}{2}} dy, \]

Now let’s consider two different strips of this kind, \( S_1, S_2 \) with \( L = 2\pi \) and

\[
\begin{align*}
    f_1(y) &= 2 - \cos(y), \\
    f_2(y) &= 2 - \cos(2y).
\end{align*}
\]

Consider a geodesic in \( S_1 \) and one in \( S_2 \) entering them at the same location and same direction, i.e., \( x_1(0) = x_2(0) \) and \( x_1'(0) = x_2'(0) \). Then obviously \( T_1 = T_2 \) and \( x_1(T_1) = x_2(T_2) \), because for each real number \( r \), the sublevel sets \( \{ f_1 \leq r \} \) and \( \{ f_2 \leq r \} \) have the same measure.

If we take quotients of \( S_1 \) and \( S_2 \), both by \( x \)-axis slides of multiples of 100, we have two cylinders with identical lens data but different \( C^\infty \) jets. Furthermore, both are compact and analytic. If we want the boundary to be connected, we can take the quotients of the cylinders by an orientation-reversing involution, which gives us two Möbius bands.

**Theorem 4.3.** There is an example of two analytic Riemannian manifolds with isometric boundaries and identical lens data, but different \( C^\infty \) jets at the boundaries.

The examples \( S_1 \) and \( S_2 \) have same lens data, same \( C^1 \) jet, but different \( C^2 \) jet. If one wants a pair of examples of different \( C^1 \) jets, then the idea still works, but to construct an example we need to care about the smoothness at the peaks and the smooth extendability at boundary, as in the following section.

### 5. Examples of different \( C^1 \) Jets

In this section we give an example of two manifolds which have the same boundary and the same lens data but different \( C^1 \) jets. Knowledge of the \( C^1 \) jet is equivalent to knowledge of the second fundamental form of the boundary as a submanifold, so different \( C^1 \) jet means different “shape” of the embedding.

The setup is the same as in the previous section. The only modifications are the functions \( f_1 \) and \( f_2 \). Let \( L = 14 \).
Let \( f_1 : [0, 14] \to \mathbb{R} \) be a smooth function that satisfies the following properties:

\[
\begin{align*}
  f_1(x) &= x + 1, \quad \text{if } 0 \leq x \leq 1; \\
  f_1'(x) &= 0, \quad \text{if } x \in [1, 3), \quad \text{and } f_1'(3) = 0; \\
  f_1(3 + t) &= f_1(3 - t) \quad \text{if } t \in [0, 1]; \\
  f_1'(x) &= 0, \quad \text{if } x \in (3, 6); \\
  f_1(x) &= 1, \quad \text{if } 6 \leq x \leq 7; \\
  f_1(7 + t) &= f_1(7 - t) \quad \text{if } t \in [0, 7].
\end{align*}
\]

We have some freedom of choice here, but it is crucial and possible to make \( f_1 \) smooth. Intuitively, \( f_1 \) starts at 1 and increases linearly in the first time period, near the peak \( f_1 \) is symmetric, then it smoothly decreases to the constant function 1, and later it copies its own mirror image.

In order to define \( f_2 \), we define the non-increasing function \( H : [1, +\infty) \to [0, 6] \),

\[
H(y) = m(\{ x \in [0, 6] \mid f_1(x) \geq y \}),
\]

where \( m(\cdot) \) is the Lebesgue measure.

We think of \( f_2 \) as the “horizontal central lineup” of \( f_1 \). That is, \( f_2 : [0, 14] \to L \) should satisfy the following:

\[
\begin{align*}
  \text{if } x \in [0, 3), \quad \text{then } f_2(x) &= y \quad \text{if and only if } x = 3 - \frac{H(y)}{2}; \\
  f_2(3) &= f_1(3); \\
  f_2(3 + t) &= f_2(3 - t), \quad \text{if } t \in [0, 3]; \\
  f_2(x) &= 1, \quad \text{if } x \in [6, 7]; \\
  f_2(7 + t) &= f_2(7 - t) \quad \text{if } t \in [0, 7].
\end{align*}
\]

Obviously \( f_2 \) is uniquely determined by \( f_1 \), and they have the same measure for every sublevel set. The only possibilities of non-smoothness of \( f_2 \) are at 0, 3, 6, 8, 11, 14. We have \( f_2 = f_1 \) near 3 and 11 from the symmetry of \( f_1 \) near the peaks. It is not hard to see \( f_2 \) is smooth at 6, because the graph of \( f_2 \) near (6, 1) is a linear transformation of that of \( f_1 \) near the same point. The smoothness near 8 is guaranteed for the same reason. Finally, from the symmetry of \( f_2 \), the smooth extendability of \( f_2 \) at 0 and 14 immediately follows.

Observe that \( f_1'(0) = 1 = -f_1'(14) \) but \( f'_2(0) = f'_2(14) = 0 \), i.e., the boundary is concave in \( S_1 \) but totally geodesic in \( S_2 \). We now use the same argument as in last section. The strips \( S_1, S_2 \) defined by \( f_1, f_2 \) are lens equivalent, but have different \( C^1 \) jets. If we want, we can take quotients to make the strips compact, and to make the boundaries connected, as in the last section.
**Theorem 5.1.** There is an example of two Riemannian manifolds with isometric boundaries and identical lens data, but different $C^1$ jets at the boundaries.

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