Generalized Bloch Theorem and Chiral Transport Phenomena

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Bloch theorem states the impossibility of persistent electric currents in the ground state of nonrelativistic fermion systems. We extend this theorem to generic systems based on the gauged particle number symmetry and study its consequences on the example of chiral transport phenomena. We show that the chiral magnetic effect can be understood as a generalization of the Bloch theorem to a nonequilibrium steady state, similarly to the integer quantum Hall effect. On the other hand, persistent axial currents are not prohibited by the Bloch theorem and they can be regarded as Pauli paramagnetism of relativistic matter. An application of the generalized Bloch theorem to quantum time crystals is also discussed.

I. INTRODUCTION

During 1930’s Felix Bloch demonstrated the impossibility of persistent electric currents in the ground state of interacting nonrelativistic systems [1]. This Bloch theorem invalidated the idea proposed by Landau and others that superconductivity is characterized by persistent ground-state currents [2]; see also Ref. [3] for its extension to nonrelativistic systems at finite temperature.

Recently, the idea of spontaneous currents has revived in a completely different context: the chiral magnetic effect (CME) [4–7] and chiral vortical effect (CVE) [8–10]. As originally argued by Vilenkin [4, 8], the CME and CVE are considered the “ground-state (or equilibrium) currents” in relativistic systems with chirality imbalance in a magnetic field or in a rotation [see Eq. (26) below]. Remarkably, they are manifestations of the topological nature of chiral fermions, and have a close connection with the topological and quantum phenomenon known as the axial anomaly in field theory [11, 12] and with the Berry curvature [13–20]. These chiral transport phenomena are expected to appear in a wide area of physics from condensed matter physics [5, 6] and nuclear physics [7, 21] to cosmology [22, 23] and astrophysics [24–26], and were studied in the framework of gauge-gravity duality [27, 28].

One can ask whether the Bloch theorem can be generalized to apply to the CME and CVE and whether they are really possible in the ground state or in equilibrium [1]. This question is also important for possible technological applications of the CME and CVE; if electric currents could flow even in equilibrium, one could make best use of them without energy loss, in contrast to the Ohm’s current that dissipates energy via Joule heat; see also Ref. [31] for a similar question from a different perspective and Refs. [32–35] for related issues in the context of Weyl semimetals [36–38].

The purpose of this paper is to resolve this question as well as to discuss other possible applications of the Bloch theorem. To this end, we first extend the Bloch theorem to generic systems, including relativistic systems, based on the consequence of the gauged U(1) particle number symmetry. This indicates that total chiral magnetic currents should vanish in the ground state of any system. Moreover, we explicitly show that the CME can be understood as a generalization of the Bloch theorem to a nonequilibrium steady state, similarly to the integer quantum Hall effect (IQHE) [39, 40].

We emphasize that the essence of the argument for the generalized Bloch theorem is the U(1) (vector) gauge symmetry; it is independent of the details of systems and is applicable to any particle number currents, not limited to the CME and CVE. On the other hand, as there is no such thing as the U(1) axial gauge symmetry, spontaneous axial currents in the ground state are not forbidden by the Bloch-type no-go theorem. We indeed show that the spontaneous axial current can be understood as Pauli paramagnetism of relativistic matter (see also Ref. [41]).

This paper is organized as follows. In Sec. II, we review the original argument of the Bloch theorem and its extension by Bohm to circulating currents in nonrelativistic systems. In Sec. III, we extend the Bloch theorem to generic systems. We also comment on its application to the question of quantum time crystals proposed by Wilczek [42]. In Sec. IV, we provide a physical derivation of the CME as a nonequilibrium steady current. Section V is devoted to our conclusions.
Throughout the paper, we set $\hbar = c = e = 1$ for simplicity unless otherwise stated. We will concentrate on systems at zero temperature.

II. BLOCH THEOREM FOR NONRELATIVISTIC HAMILTONIAN

A. No-go theorem for total ground-state currents

Let us briefly review the original argument of the Bloch theorem for a nonrelativistic electron system [1]. The Hamiltonian is given by

$$H_{NR} = \int d^3x \, \psi^\dagger(x) \left(-\frac{\nabla^2}{2m} - \mu\right) \psi(x)$$

$$+ \int d^3x d^3x' \, \psi^\dagger(x) \psi^\dagger(x') V(x-x') \psi(x') \psi(x'),$$

where $\mu$ is the chemical potential and $V(x-x')$ is the isotropic and homogeneous electron-electron interaction. For simplicity of notation, we here omit the spin degrees of freedom, but it is straightforward to generalize the argument to electrons with spin [3]. For later purpose, we also introduce the Hamiltonian density $H_{NR}$, which is related to $H_{NR}$ by

$$H_{NR} = \int d^3x \, H_{NR}(x).$$

Let us first assume that the ground state $|\Omega\rangle$ carries a nonzero electric current, $\langle J_{NR} \rangle \neq 0$, exists. Here and below, the expectation value of an operator $\mathcal{O}$ with respect to the ground state $|\Omega\rangle$ is denoted as $\langle \mathcal{O} \rangle$. The total current is defined by

$$J_{NR} = \int d^3x \, J_{NR}(x),$$

$$J_{NR}(x) = \frac{1}{2im} (\psi^\dagger \nabla \psi - \psi \nabla \psi^\dagger).$$

By definition, the ground state $|\Omega\rangle$ minimizes the total energy, $\langle H_{NR} \rangle \equiv \langle \Omega | H_{NR} | \Omega \rangle = E_{NR}^{min}$. We now consider the trial state,

$$|\Omega'\rangle = e^{i\delta p \cdot x} |\Omega\rangle,$$

with the momentum $\delta p$ being arbitrary at this moment. Taking the expectation value of $H_{NR}$ for the trial state $|\Omega'\rangle$, one finds that the potential energy does not change while the kinetic energy does. The total energy is given by

$$E_{NR}' = E_{NR}^{min} + \delta p \cdot \langle J_{NR} \rangle + \frac{(\delta p)^2}{2m} \langle N \rangle,$$

$$N = \int d^3x \, n(x), \quad n(x) = \psi^\dagger \psi.$$

where $E_{NR}' \equiv \langle \Omega' | H_{NR} | \Omega' \rangle$.

As we assumed that $\langle J_{NR} \rangle \neq 0$, if we choose the magnitude of $\delta p$ infinitesimally small so that the third term on the right-hand side of Eq. (6) is negligible, and if we choose its direction opposite to $\langle J_{NR} \rangle$, we have $E_{NR}' < E_{NR}^{min}$. However, this contradicts the original assumption that the ground state has the lowest energy. Therefore, one concludes that $\langle J_{NR} \rangle \neq 0$ is forbidden in the ground state. This completes the proof of the Bloch theorem.

B. No-go theorem for circulating currents

The above result itself does not forbid the presence of a ground-state circulating current, since its integral over space is vanishing. As shown by Bohm for nonrelativistic systems [1], however, the Bloch theorem can also be extended to such circulating currents in the thermodynamic limit. For completeness of the paper, we recapitulate Bohm’s result in this subsection.

We consider a ring with the width $\Delta r$ at radius $r$ ($\Delta r \ll r$) in polar coordinates $(r, \phi)$, and we shall take the thermodynamic limit ($r \to \infty$ with $\Delta r$ fixed) in the end. We define the circulating current and the energy as

$$J_{NR} \equiv \oint_C \vec{j}_{NR} \cdot d\ell = 2\pi r j_{NR},$$

$$\mathcal{E}_{NR} \equiv \oint_C (\mathcal{H}_{NR}) \cdot d\ell = 2\pi r \langle \mathcal{H}_{NR} \rangle,$$

where the line integral is taken along the circle with the radius $r$, and

$$j_{NR}(x) = -\frac{i}{mr} \psi^\dagger(x) \frac{\partial}{\partial \phi} \psi(x)$$

is the current density operator in polar coordinates. The total current, energy, and number of fermions on the ring are given by $J_{NR} = J_{NR} \Delta r$, $E_{NR} = \mathcal{E}_{NR} \Delta r$, and $N = 2\pi r \Delta r \langle n \rangle$, respectively. We denote the ground state by $|\Omega\rangle$, which has the lowest energy, $E_{NR} = E_{NR}^{min}$, or $\mathcal{E}_{NR} = \mathcal{E}_{NR}^{min}$ when divided by $\Delta r$.

Let us consider the total energy of the trial state,

$$|\Omega'\rangle = e^{ik\phi} |\Omega\rangle,$$

where $k$ is required to be some nonzero integer to ensure the single valuedness of the state. Taking the expectation value of $\mathcal{H}_{NR}$ for the trial state $|\Omega'\rangle$, one finds that the energy is shifted as

$$\mathcal{E}_{NR}' = \mathcal{E}_{NR}^{min} + 2\pi k \langle j_{NR} \rangle + \frac{\pi k^2}{mr} \langle n \rangle.$$
Because $\mathcal{E}_{\text{NR}} \geq \mathcal{E}_{\text{NR}}^{\text{min}}$ by definition of $\mathcal{E}_{\text{NR}}^{\text{min}}$, one must have the following inequality for any integer $k$:

$$k\langle j_{\text{NR}} \rangle + \frac{k^2}{2mr} \langle n \rangle \geq 0. \quad (13)$$

The necessary and sufficient condition for this is

$$|\langle j_{\text{NR}} \rangle| \leq \frac{\langle n \rangle}{2mr}. \quad (14)$$

Integrating over the area of the ring, $S = 2\pi r \Delta r$, we get

$$\frac{|\langle J_{\text{NR}} \rangle|}{N} \leq \frac{1}{2mr}. \quad (15)$$

So $\langle J_{\text{NR}} \rangle / N \to 0$ in the thermodynamic limit ($r \to \infty$ with $\Delta r$ fixed), and the circulating current is thermodynamically negligible in the ground state. This is the no-go theorem for circulating currents [1].

We note that persistent currents in a mesoscopic normal-metal ring, driven by an external magnetic flux $\Phi$ do not constitute a contradiction with this theorem. Indeed, the order of the magnitude of total electric current in the ring with circumference $L = 2\pi r$ is estimated as

$$|\langle J_{\text{mes}} \rangle| = -L \frac{dE_{\text{NR}}}{d\Phi} \sim v_F, \quad (16)$$

where $v_F \sim N/(mL)$ is the Fermi velocity; this is the same order as the upper bound in Eq. (15) and $|\langle J_{\text{mes}} \rangle| / N \to 0$ in the thermodynamic limit $L \to \infty$. On the other hand, “persistent currents” in a macroscopic superconducting ring are not actually in the ground state, but in the metastable state [1]: it can in principle decay into the genuine ground state with no circulating current (which has a lower energy), but its lifetime is so long that it can be regarded as quasi-equilibrium.

### III. GAUGE SYMMETRY AND EXTENSION OF BLOCH THEOREM

One can ask how general the Bloch theorem is and if it is also applicable to relativistic systems, boson systems, systems in electromagnetic fields, and so on. In the above proofs, what we made use of is not actually the details of the Hamiltonian, but is just the gauge symmetry. Guided by the consequence of the gauge symmetry, one can extend it to generic systems.

To see it more clearly, we consider a general Hamiltonian density of (charged or neutral) fermions, $\mathcal{H}(\psi)$. We denote the corresponding Lagrangian density as $\mathcal{L}(\psi)$. Our argument can easily be generalized to multi-component fermions, $\psi_i$ ($i = 1, 2, \ldots, N$), and to charged scalar fields, $\phi$. For the sake of simplicity, we shall consider the single-component fermion, $\psi$.

#### A. Generalized no-go theorem for total currents

Let us first prove the generalized Bloch-type no-go theorem for total currents. We assume the existence of the ground state $|\Omega\rangle$ which has the lowest ground-state energy, $\langle H \rangle = E_{\text{min}}$, and carries a nonvanishing total current, $\langle J \rangle \neq 0$. Here the total particle number current is defined by

$$J = \int d^3r \, j(x), \quad (17)$$

where

$$j = \frac{\partial \mathcal{L}}{\partial (\nabla \psi)} \frac{\delta \psi}{\delta \theta} + \text{h.c.} \quad (18)$$

is the Noether current associated with the global U(1) particle number symmetry, $\psi \to e^{i\theta} \psi$. The Noether theorem ensures that $\nabla \cdot j = 0$ in the static limit.

Let us consider the total energy for the trial state $|\Omega'\rangle$ defined by Eq. (5), $\langle \Omega' | H(\psi') | \Omega' \rangle$. This is equivalent to the total energy for the Hamiltonian in terms of the new field,

$$\psi'(x) = e^{i\theta(x)} \psi(x), \quad (19)$$

in the ground state, $\langle \Omega | H(\psi') | \Omega \rangle$. Here we assumed that the kinetic term is bilinear in $\psi$ and the interaction term is invariant under Eq. (19).

The point is that Eq. (19) is regarded as the “gauge transformation,”

$$\psi'(x) = e^{i\theta(x)} \psi(x), \quad (20)$$

with $\theta(x) = \delta p \cdot x$. By promoting $\theta(x)$ to a general scalar function of $x$, one can generally show, by following the standard procedure (see, e.g., Ref. [3]), that the corresponding variation of the Hamiltonian density is given by

$$\delta \mathcal{H} = \nabla \cdot (\delta j) = \nabla \theta \cdot j, \quad (21)$$

to first order in $\nabla \theta$. Here $j$ is the Noether current in Eq. (18). We stress that Eq. (21) takes the unique form dictated by the symmetry (although the expression of $j$ itself depends on the details of the Hamiltonian).

Setting $\theta(x) = \delta p \cdot x$, performing the integral over space, and taking the expectation value with respect to $|\Omega\rangle$, one finds the shift of the total energy as

$$\delta E = \delta p \cdot \langle J \rangle + O(\delta p^2). \quad (22)$$

This reproduces Eq. (6) to first order in $\delta p$ for the nonrelativistic Hamiltonian. The form of the first term on the right-hand side of Eq. (22) is determined solely by the
symmetry, while that of the second term may depend on the details of the Hamiltonian. As it is sufficient to consider an infinitesimally small $|\delta p|$ for our purpose, the second term at order $O(\delta p^2)$ is irrelevant. If $\langle J \rangle \neq 0$ in the ground state, the total energy is lowered by choosing $\delta p$ in the opposite direction as $\langle J \rangle$, which then contradicts the original assumption. Therefore, it follows that $\langle J \rangle = 0$ in the ground state of any system.

In essence, the (gauged) $U(1)$ particle number symmetry of a system prohibits spontaneous particle number currents in the ground state, independently of the form of the Hamiltonian. Note that, in the presence of external static electromagnetic fields, we need to consider the Hamiltonian that also depends on the gauge field, $H(\psi, A_\mu)$. Because $\langle \Omega' | H(\psi, A_\mu) | \Omega' \rangle = \langle \Omega | H(\psi', A_\mu) | \Omega \rangle$ with the gauge field being not transformed, our argument is directly applicable to this case as well.

B. Generalized no-go theorem for circulating currents

This Bloch-type no-go theorem can also be generalized to circulating currents in general systems. We consider a ring with the width $\Delta r$ at radius $r$ ($\Delta r \ll r$) as in Sec. [113] and consider the total energy for the trial state $|\Omega'\rangle$ defined by Eq. (11). This energy is equal to the one in terms of the new field,

$$\psi'(\mathbf{x}) = e^{ik\phi}\psi(\mathbf{x}), \quad (23)$$

in the ground state, $\langle \Omega | H(\psi') | \Omega \rangle$. We then regard Eq. (23) as the gauge transformation [20] with $\theta = k\phi$. We can concentrate on the kinetic term in the $\phi$ direction, since the other kinetic and interaction terms in the Hamiltonian remain unchanged under this transformation. For general scalar function $\theta(\phi)$, one can show that [see Eq. (21)]

$$\delta \mathcal{H} = \frac{1}{r} \frac{\partial \theta}{\partial \phi} j + O(r^{-2}). \quad (24)$$

Taking $\theta = k\phi$ and performing the line integral in the ground state, one finds that the new field in Eq. (23) shifts the energy $\mathcal{E}$ as

$$\delta \mathcal{E} = 2\pi k \langle j \rangle + O(r^{-1}). \quad (25)$$

The first term on the right-hand side above reproduces the term in Eq. (12) for the nonrelativistic Hamiltonian; again, the form of this term is determined only by the gauge symmetry and is universal, regardless of the details of the Hamiltonian. In the thermodynamic limit ($r \to \infty$ with $\Delta r$ fixed), the second term at order $O(r^{-1})$ in Eq. (25) is irrelevant. To satisfy $\delta \mathcal{E} \geq 0$ for any integer $k$, we must have $\langle j \rangle = 0$. This completes the proof of the generalized Bloch theorem.

C. Application to quantum time crystals

The generalized Bloch theorem for circulating currents can be directly applied to the question of (a class of) quantum time crystals (QTC) recently proposed by Wilczek [42] (see also Refs. [47, 48] for attempts of realization). The QTC is a hypothetical state of matter that spontaneously breaks the continuous translational symmetry in time, analogously to the usual crystals that spontaneously breaks the continuous translational symmetry in space.

As a concrete realization of the QTC, a system that allows for time-dependent persistent circulating currents in the ground state of a ring is proposed [42]. Recall here that one needs to take the thermodynamic limit to have any spontaneous symmetry breaking. However, according to the generalized Bloch theorem above, such a current-carrying ground state is prohibited in the thermodynamic limit (although it is possible in a finite volume). A similar result was obtained in the language of quantum mechanics in Ref. [19]. This seems also consistent with a more general argument for the absence of the QTC [50].

We remark that the Bloch theorem itself does not exclude a QTC characterized by something different from persistent circulating currents.

IV. GENERALIZED BLOCH THEOREM VS. CHIRAL TRANSPORT PHENOMENA

A. Chiral magnetic effect, gauge invariance, and boundary conditions

As mentioned in the introduction, for the Hamiltonian of chiral fermions with chirality imbalance in a magnetic field or in a rotation, “ground-state (or equilibrium) currents” are expected to develop. These chiral magnetic effect (CME) [4, 6, 7] and chiral vortical effect (CVE) [8, 10] are computed using the equilibrium field theory as

$$\langle j_{\text{CME}} \rangle = \frac{1}{2\pi^2} \mu_5 B, \quad \langle j_{\text{CVE}} \rangle = \frac{1}{\pi^2} \mu_5 \omega, \quad (26)$$

respectively. Here $j$ is the current density, $\mu = (\mu_R + \mu_L)/2$ and $\mu_5 = (\mu_R - \mu_L)/2$ are the vector and chiral chemical potentials, $B$ is the magnetic field, $\omega$ is the
vorticity, and the expectation value is taken in the ground state or in equilibrium.

When one considers the homogeneous system, the total chiral magnetic or chiral vortical current seems nonvanishing in the ground state. However, the generalized Bloch theorem above suggests that such a state is not the true ground state. We here provide an alternative explanation based on the gauge invariance that the total chiral magnetic current should vanish in the ground state. (See also Ref. [46] for a related discussion.)

Substituting the CME in Eq. (26) into the interaction term between the gauge field and the current,

$$H_{\text{int}} = \int d^3 x \mathbf{A} \cdot \mathbf{j}, \quad (27)$$

we have

$$H_{\text{CS}} = \frac{\mu_5}{2\pi^2} \int d^3 x \mathbf{A} \cdot \mathbf{B}. \quad (28)$$

This is the effective Chern-Simons term induced at finite $\mu_5$. Note that this is gauge invariant up to surface terms. By the gauge transformation, $\mathbf{A} \rightarrow \mathbf{A} - \nabla \Lambda$ with $\Lambda(x)$ being any scalar function, this energy is shifted as

$$\Delta H_{\text{CS}} = \frac{\mu_5}{2\pi^2} \int_S \Lambda(x) \mathbf{B} \cdot d\mathbf{S}, \quad (29)$$

where $S$ is the boundary of the region under consideration. To maintain the gauge invariance (i.e., $\Delta H_{\text{CS}} = 0$) for any $\Lambda$, one can take the following boundary condition at $S$: (i) $\langle \mathbf{j} \rangle \cdot d\mathbf{S} = 0$, or (ii) the periodic boundary condition for $\langle \mathbf{j} \rangle$.

In fact, this requirement is related to the conservation of the particle number, and is not limited to the CME. We consider $N$ fermions in a finite (but sufficiently large) volume region $V$ with the boundary $S = \partial V$. We assume the local current conservation, $\partial_t J^\mu = 0$ with $J^\mu$ being the particle number current. However, the local current conservation does not necessarily mean the global charge conservation. Indeed, using the local current conservation, one has

$$\partial_t N = - \int_S \langle \mathbf{j} \rangle \cdot d\mathbf{S}, \quad (30)$$

which can be nonzero unless one chooses the boundary condition at $S$ appropriately. In order for $N$ to be conserved in the region $V$, one needs to choose the boundary condition (i) or (ii) above.

For the boundary condition (i), one can show that

$$\langle J^i \rangle = \int d^3 x \partial_k (x^i j^k) = \int_S x^i \langle j^k \rangle dS^k = 0, \quad (31)$$

which is the same conclusion as the generalized Bloch theorem. In other words, if $\langle J \rangle \neq 0$ in the region $V$, it means that $\partial_t N \neq 0$, and then the system under consideration would not be static. Note that this argument is not limited to the CME or CVE and is applicable to any system with the boundary condition (i). This argument, however, cannot simply be carried over to the case of the boundary condition (ii) and to circulating currents; in those cases, one needs to resort to the Bloch-type argument to show vanishing total currents in the ground state, as we have shown above.

### B. Chiral magnetic effect as a nonequilibrium steady current

We now explicitly show that the circulating chiral magnetic current can be understood as a generalization of the Bloch theorem to a nonequilibrium steady state. Our argument is similar to the one by Thouless [40], which reformulates Laughlin’s argument for the integer quantum Hall effect (IQHE) [39] as an extension of the argument for the Bloch theorem. To make our discussion clear, we restore the units $\hbar$, $c$, and $e$ in this subsection.

We consider noninteracting massless Dirac fermions (right- and left-handed massless chiral fermions) in a torus with the cross section $S$ whose inside is pierced by a homogeneous magnetic field $B$. This is illustrated in Fig. 1. We assume to maintain different chemical potentials $\mu_R$ for right-handed fermions, and $\mu_L$ for left-handed fermions in a torus. We also introduce the magnetic flux $\Phi$ threading the hole of the torus.

Let us vary the magnetic flux threading the torus adiabatically by one quantum unit,

$$\delta \Phi = \oint \delta \mathbf{A} \cdot d\ell = \frac{2\pi \hbar}{e}, \quad (32)$$

where $\delta \mathbf{A}$ is the change of the gauge field inside the torus. This leads to the trivial Aharonov-Bohm phase
for the fermions, \( \exp(-i e \delta \Phi / \hbar) = 1 \), and the system does not change from the original state. The only change that can happen is the transfer of \( N_B \) massless fermions from the Fermi surface of left-handed fermions to that of right-handed fermions (meaning that the system is in the nonequilibrium steady state). In the presence of the chemical potential difference between two Fermi surfaces, this transfer requires the energy \( N_B (\mu_R - \mu_L) \). Hence, by the gauge transformation in Eq. (32) together with the \( N_B \) fermions, the change of the total energy is given by

\[
\delta E = \int d^3 x \, j \cdot \delta A - N_B (\mu_R - \mu_L) = I \left( \frac{2\pi \hbar}{e} \right) - N_B (\mu_R - \mu_L),
\]

(33)

where \( I \) is the current flowing around the torus. As the system comes back to the original state, the total energy shift is zero in this process, \( \delta E = 0 \). We thus get

\[
\delta E = \frac{N_B e}{2\pi \hbar} (\mu_R - \mu_L).
\]

(34)

This can be viewed as a bulk version of Landauer-type formula \[^{[44]}\] with perfect transmission.

We now determine \( N_B \). The magnetic field inside the torus gives rise to the quantization of energy levels (Landau levels) for Dirac fermions. The fermions in the lowest Landau level are massless, and the degeneracy per unit transverse area is \( g_n = eB/(2\pi \hbar c) \); the number of gapless modes in the area \( S \) is given by \( N_B = g_n S \).

Substituting it into Eq. (34), we obtain

\[
j = \frac{e^2 \mu_5}{2\pi \hbar^2 c} B,
\]

(35)

where \( j = I/S \) is the current density and \( \mu_5 = (\mu_R - \mu_L)/2 \). This is exactly the expression of the CME in Eq. (26) in the units \( \hbar = c = e = 1 \). In this way, the CME can be seen as the current of \( N_B \) bulk states.

This argument clarifies not only the similarity between the CME and IQHE via Eq. (34) \[^{[3]}\] but also the difference that the current is carried by bulk (edge) modes for the CME (IQHE). Equation (35) evades the Bloch theorem, because the system is in the nonequilibrium steady state with keeping \( \delta E = 0 \), similarly to the IQHE; the current is driven by the “voltage” \( \mu_R - \mu_L \).

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\[^{3}\] In the original Laughlin’s argument for the IQHE \[^{[39]}\], the number of edge modes moved by the gauge transformation on a ribbon is some integer \( N_g \) (related to the Chern number), and \( \mu_R - \mu_L \) is replaced by the voltage between the two edges multiplied by the electric charge, \( eV \). Then Eq. (34) reduces to the familiar expression of the IQHE, \( I = N_g e^2 V / \hbar \).

C. Chiral separation effect as Pauli paramagnetism

So far we have concentrated on the CME at finite \( \mu_5 \). In the presence of \( \mu \), the spontaneous axial current,

\[
\langle j_5 \rangle = \frac{1}{2\pi^2} \mu B,
\]

(36)

is also considered to appear. This is called the chiral separation effect (CSE) \[^{[21]}\]. Contrary to the vector currents, such as the electric current, there is no conservation law for the axial charge nor the gauge symmetry corresponding to Eq. (19). This means that the Bloch-type no-go theorem is not applicable to the CSE and that the total axial current can appear even in the ground state. In the following, we shall indeed show that the CSE is purely a ground-state property of relativistic matter—Pauli paramagnetism. For simplicity and convenience, we consider a noninteracting relativistic Fermi gas at finite \( \mu \).

The starting point is the Dirac Hamiltonian density,

\[
\mathcal{H}_{\text{Dirac}} = \psi^\dagger (-i \alpha \cdot D - \mu) \psi,
\]

(37)

where \( \alpha = \gamma^0 \gamma, \ D = \nabla + i A \), and \( \psi \) is the four-component Dirac field. Using the equation of motion, one can rewrite the interaction term between the gauge field and the current into the form of the “Pauli term,”

\[
\mathcal{H}_{\text{int}} = i \frac{2\mu}{\hbar} A \cdot (\psi^\dagger \nabla \psi) - \frac{1}{2\mu} \psi^\dagger B \cdot \sigma \psi,
\]

(38)

up to total derivatives. This shows that free massless Dirac fermions at finite \( \mu \) has the magnetic moment \( \gamma = e/(2\mu) \) at the tree level (see also Refs. \[^{[16]}\] \[^{[19]}\] \[^{[20]}\]). This is similar to the magnetic moment for massive Dirac fermions at \( \mu = 0 \).

Below we take the magnetic field in the \( z \) direction, \( B = (0,0,B) \). The “Zeeman splitting” in the second term changes the particle energy depending on the spins,

\[
\delta \epsilon_{\sigma z} = -\gamma \sigma_z B.
\]

(39)

This in turn leads to the change of the distribution functions of fermions,

\[
\delta n_{\sigma z} = \frac{\partial n_{\sigma z}}{\partial \epsilon_{\sigma z}} (\delta \epsilon_{\sigma z} - \delta \mu),
\]

(40)

where \( n_{\sigma z} = \theta(\mu - |p|) \). Because \( \delta \mu \) is a scalar quantity, \( \delta \mu \) must be an even function of \( B \), and \( \delta \mu \propto B^2 \) at the leading order. At first order in \( B \) (for sufficiently small \( B \)), the variation of the chemical potential \( \delta \mu \) is thus negligible. Then the total number of particle with spin \( \sigma \) is given by

\[
\delta n_{\sigma} = \int \frac{d^3 p}{(2\pi)^3} \delta n_{\sigma z} = \frac{1}{2} N(\mu) \gamma \sigma_z B,
\]

(41)
where \( N(\mu) = \mu^2/\pi^2 \) is the density of states at the Fermi surface including spin degrees of freedom.

The axial current is expressed as the net spin polarization,

\[
\langle j_5^z \rangle = \langle \psi^\dagger \Sigma^z \psi \rangle = \delta n_\uparrow - \delta n_\downarrow
\]

where \( \Sigma^i = \gamma^5 \gamma^0 \gamma^i \) is the spin operator. From Eq. (41), this current can be computed as

\[
\langle j_5^z \rangle = N(\mu) \gamma B = \frac{1}{2\pi^2} \mu B,
\]

which is nothing but the CSE in Eq. (36).

Although the CME and CSE look superficially similar in expressions (26) and (36), they are different in that the Bloch theorem is applicable to the former, but not to the latter. This is intimately related to the presence (absence) of the U(1) vector (axial) gauge symmetry.

V. CONCLUSION

In this paper, we have extended the Bloch theorem to generic systems based on the consequence of the gauged U(1) particle number symmetry. The generalized Bloch theorem excludes the possibility of the chiral magnetic and chiral vortical effects and quantum time crystals as persistent currents in the thermodynamic limit. We have also shown that the chiral magnetic effect can be understood as the nonequilibrium steady current, similarly to the integer quantum effect.

The crux of the proof of the generalized Bloch theorem for vector currents is the U(1) (vector) gauge symmetry. As there is no such thing as the U(1) axial gauge symmetry, the Bloch-type no-go theorem is not applicable to the axial current. We have indeed shown that the chiral separation effect is the spontaneous axial current in the ground state. It would be interesting to apply our arguments to other currents, such as heat currents and spin currents.

Finally, it should be possible to extend the Bloch-type no-go theorem considered in this paper to generic systems at finite temperature, in a way similar to Ref. [3].

ACKNOWLEDGMENTS

We thank Yoji Ohashi, Keiji Saito, Atsuo Shitade, and Ryo Yokokura for useful discussions. We especially thank Yoji Ohashi for drawing their attention to Refs. [1, 3] and critical reading of the manuscript. This work was supported by JSPS KAKENHI Grant Number 26887032.

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