QUANTIZATION VIA CLASSICAL ORBITS

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Abstract

A systematic method for calculating higher-order corrections of the relativistic semiclassical fixed-energy amplitude is given. The central scheme in computing corrections of all orders is related to a time ordering operation of an operator involving the Van Vleck determinant. This study provides us a new viewpoint for quantization.

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I. CORRECTION OF ALL ORDER FOR THE SEMICLASSICAL RELATIVISTIC FIXED-ENERGY AMPLITUDE

It was pointed out by Van Vleck [1] that the semiclassical approximation of the propagator in quantum mechanics can be expressed via the superposition of terms involving the classical action in the exponent, and allowing for more than one possible classical paths between two specified points in a given time interval \((t_b - t_a)\):

\[
K_{sc}(x_b, x_a; t_b - t_a) = \sum_{\text{classical orbit}} \left( \frac{M}{2\pi\hbar} \right)^{D/2} \sqrt{\det \left( -\frac{\partial^2 R}{\partial x_b \partial x_a} \right)} \exp \left\{ \frac{i}{\hbar} R(x_b, x_a; t_b - t_a) \right\},
\]

(1.1)

where \( R \) is the Hamilton principal function, and \( x_b, x_a \) is the terminal points of the orbits. This resulting formula are often analytically quite complicated, but they have the great merit of describing almost all the physics. Especially, they are often astonishingly accurate; this is important, because it is precisely in the semiclassical limit that many of the standard calculational methods of wave mechanics converge very slowly [2].

In this letter, we would like to present the semiclassical approximation of the relativistic fixed-energy amplitude (Green’s function). The corrections of all orders is given by a “time” like ordering operator. This study provides us a new viewpoint for quantization.

The starting point is Kleinert’s path integral representation of the fixed-energy amplitude of a relativistic particle in external static electromagnetic fields [3–5]

\[
G(x_b, x_a; E) = \frac{i\hbar}{2Mc} \int_0^\infty ds \int \mathcal{D} \rho(\lambda) \Phi [\rho(\lambda)] \int \mathcal{D}^3 x(\lambda) \exp \left\{ -\frac{A_E}{\hbar} \right\} \rho(0) \tag{1.2}
\]

with the action

\[
A_E = \int_{\lambda_a}^{\lambda_b} d\lambda \left[ \frac{M}{2\rho(\lambda)} \dot{x}^2(\lambda) - i(e/c) A(x) \cdot \dot{x}(\lambda) - \rho(\lambda) \frac{(E - V(x))^2}{2Mc^2} + \rho(\lambda) \frac{Mc^2}{2} \right],
\]

(1.3)

where \( s \) in Eq. (1.2) is defined as

\[
s_b - s_a = \int_{\lambda_a}^{\lambda_b} d\lambda \rho(\lambda), \tag{1.4}
\]
in which \( \rho(\lambda) \) is an arbitrary dimensionless fluctuating scale variable, \( \rho(0) \) is the terminal point of the function \( \rho(\lambda) \), and \( \Phi[\rho(\lambda)] \) is some convenient gauge-fixing functional \([3-6]\). The only condition on \( \Phi[\rho(\lambda)] \) is that

\[
\int \mathcal{D}\rho(\lambda) \Phi[\rho(\lambda)] = 1. \tag{1.5}
\]

\( \hbar/Mc \) is the well-known Compton wave length of a particle of mass \( M \), \( A(x) \) is the vector potential, \( V(x) \) is the scalar potential, \( E \) is the system energy, and \( x \) is the spatial part of the \((3 + 1)\) vector \( \vec{x} = (x, \tau) \).

It is without lost the generally that the functional \( \Phi[\rho(\lambda)] \) is taken as the \( \delta \)-functional \( \delta[\rho - 1] \) to fix the value of \( \rho(\lambda) \) to unity \([3–5,7]\). Then the lowest order’s approximation of the fixed-energy amplitude for a relativistic system is given by \([3,4,8]\)

\[
G_{\text{sc}}(x_b, x_a; E) = \frac{\hbar}{2Mc (2\pi\hbar i)^{D/2}} \int_{s_a}^{s_b} ds_b e^{[\frac{i\varepsilon(s_b-s_a)}{\hbar}]} K_{\text{sc}}(x_b, s_b | x_a, s_a)
\]

\[
= \frac{\hbar}{2Mc (2\pi\hbar i)^{D/2}} \int_{s_a}^{s_b} ds_b e^{[\frac{i\varepsilon(s_b-s_a)}{\hbar}]} D_E^{1/2}(x_b, s_b | x_a, s_a)e^{[\frac{iA_E(x_b, s_b | x_a, s_a)]}{\hbar}} \tag{1.6}
\]

with the pseudoaction

\[
A_E = \int_{s_a}^{s_b} ds \left[ \frac{1}{2} \dot{x}^2(s) + \frac{e}{c} A(x, s) \cdot \dot{x}(s) + \frac{1}{2Mc^2} \left(V^2(x, s) - 2EV(x, s)\right) \right], \tag{1.7}
\]

where \( D_E \) is the second derivative with respect to \( A_E \) and is given by

\[
\det \left[ -\partial_{x^i} \partial_{s^a} A_E(x_b, s_b | x_a, s_a) \right], \tag{1.8}
\]

and the pseudoenergy \( \varepsilon \) is defined as \((E^2 - M^2c^4)/2Mc^2\). We have assumed that \( A(x, s) \) and \( V(x, s) \) are functions of coordinate \( x \) and the timelike parameter \( s \). This will be useful for reducing to non-relativistic semi-classical approximation.

To get the higher order corrections, we make a reasonable conjecture \([9]\)

\[
G_{\text{all}}(x_b, x_a; E) = \frac{\hbar}{2Mc (2\pi\hbar i)^{D/2}} \int_{s_a}^{s_b} ds_b e^{[\frac{i\varepsilon(s_b-s_a)}{\hbar}]} K_{\text{all}}(x_b, s_b | x_a, s_a)
\]

\[
= \frac{\hbar}{2Mc (2\pi\hbar i)^{D/2}} \int_{s_a}^{s_b} ds_b e^{[\frac{i\varepsilon(s_b-s_a)}{\hbar}]} \tag{1.6}
\]
\[
\times \left\{ D_E^{1/2}(x_b, s_b \mid x_a, s_a) e^{\hat{A}_E(x_b, s_b \mid x_a, s_a)} \right\} \sum_{k=0}^{\infty} \hbar^k g^{(k)}(x_b, s_b \mid x_a, s_a) \right\}. \tag{1.9}
\]

The subscript “all” stand for the all order corrections. An important observation in evaluating the unknown functions \( g^{(k)}(x_b, x_a; s) \) is that the curly bracket above satisfies the Schrödinger-like equation
\[
\left[ \hat{H}_E (-i\hbar \partial_{x_b}, x_b; s) - i\hbar \partial_s \right] K_{\text{all}}(x_b, s_b \mid x_a, s_a) = -i\hbar \delta (s_b - s_a) \delta^3 (x - x_a), \tag{1.10}
\]
where \( \hat{H}_E \) is the Hamilton operator
\[
\hat{H}_E(\hat{p}, x; s) = (\hat{p} - e/c A(x, s))^2/2M + [2E V(x, s) - V^2(x, s)]/2Mc^2 \tag{1.11}
\]
with \( \hat{p} = -i\hbar \partial_x \). Since the boundary condition given in Eq. (1.6) and the limiting property is \( K_{\text{all}}(x_b, s_a \mid x_a, s_a) = \delta^3 (x_b - x_a) \) for the pseudopropagator \( K_{\text{all}} \), we obtain the relations
\[
\begin{cases}
g^{(0)}(x_b, s_b \mid x_a, s_a) = 1 \\
g^{(k)}(x_b, s_a \mid x_a, s_a) = 0, \quad k \geq 1.
\end{cases} \tag{1.12}
\]
To go further, let us insert the pseudopropagator \( K_{\text{all}} \) in Eq. (1.9) into Eq. (1.10). Three equalities arise from this operation. They are
\[
\frac{1}{2M} \left[ \partial_x A_E(x, s \mid x_a, s_a) - e/c A^i(x, s) \right]^2 + V_E(x, s) = \partial_s A_E(x, s \mid x_a, s_a), \tag{1.13}
\]
where \( V_E(x, s) \) is defined as \([2E V(x, s) - V^2(x, s)]/2Mc^2 \),
\[
\partial_s D_E^{1/2}(x, s \mid x_a, s_a) = -\frac{1}{2M} \left[ 2(\partial_x D_E^{1/2}) \left( \partial_x A_E - e/c A^i, A_E \right) + D_E^{1/2} \partial_x \left( \partial_x A_E - e/c A^i \right) \right], \tag{1.14}
\]
and the iterative equation
\[
\begin{cases}
\partial_s + \frac{1}{M} \left[ \partial_x A_E(x, s \mid x_a, s_a) - e/c A^i(x, s) \right] \partial_x \right) g^{(k)}(x, s \mid x_a, s_a) \\
= \hat{O}_E(x, s \mid x_a, s_a) g^{(k-1)}(x, s \mid x_a, s_a). \tag{1.15}
\end{cases}
\]
Here the action of the operator $\hat{O}_E$ is defined as

$$\hat{O}_E(x, s \mid x_a, s_a)g^{(k-1)}(x, s \mid x_a, s_a) = D_E^{-1/2}(x, s \mid x_a, s_a)\partial_i^2 \left[D_E^{1/2}g^{(k-1)}\right].$$

(1.16)

Eqs. (1.13) and (1.14) is just the Hamilton-Jacobi equation and continuity equation, respectively. Eq. (1.15) will provide us the information of each order. To solve it, we note that the middle bracket in Eq. (1.15) precisely satisfies the first of the equation:

$$\frac{d}{ds}x^i(s) = \frac{1}{M} \left[ \partial_{x^i}A_E(x, s \mid x_a, s_a) - \frac{e}{c}A^i(x, s) \right] \bigg|_{x=x(s)}$$

(1.17)

if we identify $g^{(k)}(x, s \mid x_a, s_a) = g^{(k)}(x(s; x_b, s_b \mid x_a, s_a), s \mid x_a, s_a)$. The left-hand side of Eq. (1.13) now equals the total derivative

$$\frac{d}{ds}g^{(k)}(x(s; x_b, s_b \mid x_a, s_a), s \mid x_a, s_a) = \text{left hand side of Eq. (1.15)}.$$  

(1.18)

From this, it is easy to find the explicitly solution, subject the condition in Eq. (1.12),

$$g^{(k)}(x_b, s_b \mid x_a, s_a)$$

$$= \frac{i}{2M} \int_{s_a}^{s_b} ds \hat{O}_E(x, s \mid x_a, s_a)g^{(k-1)}(x, s \mid x_a, s_a) \bigg|_{x=x(s; x_b, s_b \mid x_a, s_a)}.$$  

(1.19)

It is useful to introduce a pseudotime-ordering operator which orders the pseudotimes successively

$$\hat{T}_s \left( \hat{O}_E(s_n)\hat{O}_E(s_{n-1})\cdots\hat{O}_E(s_1) \right) \equiv \hat{O}_E(s_{i_n})\hat{O}_E(s_{i_{n-1}})\cdots\hat{O}_E(s_{i_1}),$$  

(1.20)

where $s_{i_n}, \cdots, s_{i_1}$ are the pseudotimes $s_n, \cdots, s_1$ relabeled in the causal order, so that

$$s_{i_n} \geq s_{i_{n-1}} \geq \cdots \geq s_{i_1}.$$  

(1.21)

With this formal operator, the expansion can be rewritten in a more compact form and is given by

$$G_{all}(x_b, x_a; E) = \frac{\hbar}{2Mc} \frac{1}{(2\pi\hbar)^{D/2}} \int_{s_a}^{s_b} ds_b e^{\left[\frac{\hbar}{2\epsilon(s_b-s_a)}\right]} D_E^{1/2}(x_b, s_b \mid x_a, s_a).$$
\[ \times e^{iA_E(x_b,s_b|x_{a},s_{a})} \hat{T}_s \exp \left\{ \frac{i\hbar}{2M} \int_{s_{a}}^{s_{b}} ds \hat{O}_E(x, s \mid x_{a}, s_{a}) \right\} . \] (1.22)

Considering the phase change coming from the conjugate points of classical orbits \([10,11]\) and the various classical trajectories joining the \((x_{a}, s_{a})\) and \((x_{b}, s_{b})\) in the generic space-time, we have

\[
G_{\text{all}}(x_b, x_a; E) = \frac{\hbar/2Mc}{(2\pi\hbar)^{D/2}} \sum_{\text{classical orbit}} \int_{s_{a}}^{\infty} ds_{b} e^{i\varepsilon(s_{b} - s_{a})} D_{E}^{1/2}(x_b, s_b \mid x_{a}, s_{a})
\]

\[
\times e^{iA_E(x_a,s_a|x_{b},s_{b})} \hat{T}_s \exp \left\{ \frac{i\hbar}{2M} \int_{s_{a}}^{s_{b}} ds \hat{O}_E(x, s \mid x_{a}, s_{a}) \right\} . \] (1.23)

This result relates to the operation of Van Vleck determinant. Since the summation of the perturbation series should converges to the fixed-energy amplitude, we have a new point of view for quantization which just relates to the topology and the summation of classical orbits. Contrary to the Feynman’s path integral, where many nonclassical paths are summed, we just need to consider the physical paths here. We believe that classical orbits are the most fundamental factor for quantization. This idea was exploited by Bohr’s in 1913 by postulating the famous rule that only a countable number of orbits are allowed by the quantum condition \(\oint pdq = nh (n \in \mathbb{N})\). However, this “old” quantum mechanics is not sufficient, because it does not consider the topology of the classical orbits \([12]\).

The idea of summing the classical orbits for quantization can be found in the trace formula \([11]\). The energy spectra produced by taking the trace of the non-relativistic version of Eq. (1.23) even in the lowest order approximation are highly accurate. Another evidence for quantization via classical orbits comes from gauge transformation. The well-known space-time technique in path integral is a gauge transformation \([3]\). It stands for the over-summation of the classical orbits, think for instance of gauge fixing techniques in gauge field theory where divergence arises for over counting the gauge fields when we perform the sum over gauge fields using path integral.

Eq. (1.23) is also suitable for the non-relativistic systems. This is achieved by replacing \(V_E(x,s)\) and \(\varepsilon\) with \(V(x,s)\) and \(E\), respectively. Numerous physical problems such as
tunneling effect, quantization of classical chaotic systems, and the systems related to the semiclassical quantization require higher order corrections. We hope that our result may offer a useful tool for performing these calculations. Particularly, the formula given in Eq. (1.23) may provide systems failing to be quantized using Feynman’s method an alternate way of quantization via classical orbits.

Details about the calculations discussed in this letter and illustrative examples will be presented elsewhere. We expect that the formula in Eq. (1.23) provide detailed microscopic information via the classical orbits.

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REFERENCES

[1] J. H. Van Vleck, Proc. Natn. Acad. Sci. 14, 178 (1928).

[2] M. V. Berry and K. E. Mount, Rep. Prog. Phys. 35, 315 (1972).

[3] H. Kleinert, Phys. Lett. A 212, 15 (1996).

[4] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics and Polymer Physics*, World Scientific, Singapore, 1995.

[5] D. H. Lin, J. Phys. A 30 3201 (1997); A 30 4365 (1997); A 31 4785 (1998); A 31 7577 (1998); hep-th/9708144 to appear in J. Math. Phys.; hep-th/9709152.

[6] K. Fujikawa, Prog. Theor. Phys. 96, 863 (1996) (hep-th/9609029); (hep-th/9608052).

[7] D. H. Lin, *Path Integral on Relativistic Spinless Potential Problems*, talk given at the sixth International Conference on Path-Integrals from peV to TeV 50 Years from Feynman’s Paper, Florence, Italy, 25-29 August 1998, to appear in the Proceedings.

[8] H. Kleinert and D. H. Lin, *Relativistic Trace Formula for Bound States in Terms of Classical Periodic Orbits*, quant-ph/9807068.

[9] M. Roncadelli, Phys. Rev. Lett. 72, 1145 (1994).

[10] J. B. Keller, Ann. Phys. 4, 180 (1958).

[11] M. Gutzwiller, *Chaos in Classical and Quantum Mechanics*, Springer, Berlin, 1990.

[12] D. Wintgen, K. Richter, and G. Tanner, CHAOS 2, 19 (1992).