THE WORK OF FEDERICO RODRIGUEZ HERTZ ON ERGODICITY OF DYNAMICAL SYSTEMS

DMITRY DOLGOPYAT
(Communicated by Giovanni Forni)

ABSTRACT. We review recent advances on ergodicity of partially and nonuniformly hyperbolic systems describing, in particular, important contributions of Federico Rodriguez Hertz and his collaborators.

1. HOPF ARGUMENT

One inspiring source for the development of the modern theory of dynamical systems is the work of Poincaré on the three body problem. Poincaré made a crucial observation that some systems are so complicated that it is impossible to obtain exact formulas for every trajectory and so one has to aim for a less precise description of the trajectories.

Ergodicity is a basic example of such an approach. Namely, if the system is ergodic then almost every trajectory is uniformly distributed in the phase space.

By now there are several methods to establish ergodicity, the most successful of which are harmonic analysis and the Hopf argument. The former method requires the system to have a high degree of symmetry while the latter does not have this constraint.

The basic idea of the Hopf argument is simple. Given a continuous map $f$ of a metric space $M$ the stable and unstable manifolds of a point $x \in M$ are defined by

$$W^s(x) = \{y : d(f^n x, f^n y) \to 0 \text{ as } n \to \infty\},$$
$$W^u(x) = \{y : d(f^{-n} x, f^{-n} y) \to 0 \text{ as } n \to \infty\}.$$

Call a point $x$ regular if

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \phi(f^n x) = \lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \phi(f^{-n} x) =: \bar{\phi}(x).$$

(1)

Birkhoff Ergodic Theorem shows that the set $\mathcal{R}$ of regular points has full measure with respect to each invariant probability measure.

We say that $x$ and $y$ in $\mathcal{R}$ are equivalent if for any continuous function $\phi$ we have $\bar{\phi}(x) = \bar{\phi}(y)$. Then $\mu$ is ergodic provided that $\mu$-almost all $x$ and $y$ are
where the Hopf argument was implemented is uniformly hyperbolic systems. In this case the problem of intermediate points was resolved by Hopf \[27\] under the assumption that stable and unstable foliations are \(\mathcal{H}\) invariant splitting
\[
T_\Lambda M = E_u \oplus E_s,
\]
where \(df\) expands \(E_u\) and contracts \(E_s\). Given an invariant measure \(\mu\), its basin is the set of points whose forward dynamics is described statistically by \(\mu\). That is
\[
\mathcal{B}(\mu) = \{x : \forall \phi \in C(M), \bar{\phi}(x) = \mu(\phi)\}.
\]
For uniformly hyperbolic systems the Hadamard-Perron Theorem [2] guarantees existence of unstable manifolds tangent to \(E_u\) and stable manifolds tangent to \(E_s\). Uniform transversality of stable and unstable manifolds makes uniformly hyperbolic systems a perfect ground for applying the Hopf argument. In this case the problem of intermediate points was resolved by Hopf [27] under the assumption that stable and unstable foliations are \(C^1\) by showing that \(x\) and \(y\) can be joined by many chains, and, by Fubini theorem, most chains are good.

The \(C^1\) assumption was later removed by Anosov and Sinai [2, 3] by showing that stable and unstable foliations are absolutely continuous. A foliation is called \textit{absolutely continuous} if any set of zero Lebesgue measure intersects almost every leaf on a set of zero leafwise measure. Absolute continuity allows to apply Fubini theorem which is needed to show that most chains on Figure 2 are good.

**Figure 1. A Hopf chain**

We say that \(x\) and \(y\) belong to the same \textit{accessibility class} if they can be connected by a Hopf chain (2).

The problem is that the existence of a Hopf chain connecting \(x\) and \(y\) does not imply that \(x\) and \(y\) are equivalent since the intermediate points may not belong to \(\mathcal{R}\). Therefore extra work is needed to establish ergodicity. The first case where the Hopf argument was implemented is \textit{uniformly hyperbolic systems}.

We need some terminology. Let \(\Lambda\) be a hyperbolic attractor, that is \(\Lambda = \bigcap_{n \geq 0} f^n U\), where \(U\) is an open forward invariant set. \(\Lambda\) is called (uniformly) \textit{hyperbolic} if there is an \(f\)-invariant splitting
\[
T_\Lambda M = E_u \oplus E_s,
\]
where \(df\) expands \(E_u\) and contracts \(E_s\). Given an invariant measure \(\mu\), its basin is the set of points whose forward dynamics is described statistically by \(\mu\). That is
\[
\mathcal{B}(\mu) = \{x : \forall \phi \in C(M), \bar{\phi}(x) = \mu(\phi)\}.
\]

For uniformly hyperbolic systems the Hadamard-Perron Theorem [2] guarantees existence of unstable manifolds tangent to \(E_u\) and stable manifolds tangent to \(E_s\). Uniform transversality of stable and unstable manifolds makes uniformly hyperbolic systems a perfect ground for applying the Hopf argument. In this case the problem of intermediate points was resolved by Hopf [27] under the assumption that stable and unstable foliations are \(C^1\) by showing that \(x\) and \(y\) can be joined by many chains, and, by Fubini theorem, most chains are good.

The \(C^1\) assumption was later removed by Anosov and Sinai [2, 3] by showing that stable and unstable foliations are absolutely continuous. A foliation is called \textit{absolutely continuous} if any set of zero Lebesgue measure intersects almost every leaf on a set of zero leafwise measure. Absolute continuity allows to apply Fubini theorem which is needed to show that most chains on Figure 2 are good.
Combining this idea with the work of Smale on structure of hyperbolic attractors [50], work of Sinai and Bowen on symbolic coding of hyperbolic systems [48, 10], and the work of Sinai and Ruelle on invariant measures for symbolic systems [49, 47], one obtains the following result.

**Theorem 1** (see [11]). We can decompose \( \Lambda = \bigcup_{j=1}^{m} \Lambda_j \), where \( \Lambda_j \) are invariant and there are ergodic measures \( \mu_j \) supported on \( \Lambda_j \) such that \( \bigcup_{j=1}^{m} \mathcal{B}(\mu_j) \) has full measure in \( U \).

Moreover, \( \mathcal{B}(\mu_j) \) is equal up to a set of Lebesgue measure 0 to the homoclinic class

\[
\mathcal{H}(p_j) = \{ x \in U : W^s(x) \cap W^u(p_j) = 0 \text{ and } W^u(x) \cap W^s(p_j) = 0 \},
\]

where \( p_j \) is any periodic point in \( \Lambda_j \).

In addition, \( \Lambda_j \) can be further decomposed as \( \Lambda_j = \bigcup_{k=1}^{r_j} \Lambda_{jk} \) so that \( f \Lambda_{jk} = \Lambda_{j(k+1) \mod r_j} \) and \( f^r \big|_{\Lambda_{jk}} \) is mixing.

There are two natural ways to extend this result: partial hyperbolicity and nonuniform hyperbolicity. Federico Rodriguez Hertz made important contributions in both cases. Partially hyperbolic systems are discussed in Section 2 while nonuniformly hyperbolic systems are treated in Section 3.

## 2. Partial Hyperbolicity

### 2.1. Ergodicity of partially hyperbolic systems.**

A diffeomorphism \( f \) is called **partially hyperbolic** if there exists an \( f \)-invariant splitting \( TM = E_u \oplus E_c \oplus E_s \) and there are constants \( C > 0 \) and \( \theta < 1 \) such that, if \( v_u \in E_u, v_c \in E_c, v_s \in E_s \) are unit vectors, then

\[
\| df^n(v_u) \| \geq C\theta^{-n}, \quad \| df^n(v_c) \| \leq C\theta^n,
\]

\[
\| df^n(v_c) \| \leq C\theta^n\| df^n(v_u) \|, \quad \| df^n(v_s) \| \leq C\theta^n\| df^n(v_c) \|.
\]
For partially hyperbolic systems the distributions $E_u$ and $E_s$ are uniquely integrable, and they are tangent to foliations $W^u$ and $W^s$ ([26]). This sets the stage for the Hopf argument.

In fact, it was shown by Brin and Pesin [13] under a number of technical conditions, including the assumption that $W_u$ and $W_s$ are Lipschitz, then two regular points in the same accessibility class are equivalent. Unfortunately, the Lipschitz condition is quite rarely satisfied (unless there are additional symmetries). So a lot of non trivial work went into weakening the Lipschitz assumption. This was done by Grayson, Pugh, Shub, Burns, and Wilkinson [24, 37, 38, 16]. They succeeded to weaken the regularity assumptions on the center direction to center bunching. The diffeomorphism is called center bunched if $(df|E_c)$ is close to conformal. The weakest center bunching assumption is due to [16]. It says that if $v_u \in E_u$, $v'_c, v''_c \in E_c$, $v_s \in E_s$ are unit vectors, then

$$
\frac{||df^n v_s||}{||df(v'_c)||} \leq \frac{||df^n v_u||}{||df(v''_c)||}.
$$

For our purposes, it suffices to note that if dim$E_c = 1$, then the middle term becomes one and so (4) is always satisfied (for a suitable Riemannian metric) since $df$ expands $E_u$ and contracts $E_s$.

The upshot of the above cited works is that if $f$ is volume preserving and center bunched, then almost all points in the same accessibility class are equivalent. Therefore, it is convenient to make the following definition. We say that $f$ is essentially accessible if any measurable set consisting of accessibility classes has either 0 or full measure. If there is only one accessibility class, then we say that $f$ has accessibility property.

**Theorem 2 ([16]).** If $f$ is volume preserving, center bunched, and essentially accessible, then it is ergodic.

Let us call diffeomorphism $f$ stably ergodic if there exists $r > 0$ such that any diffeomorphism $g$ which is $C^r$-close to $f$ is ergodic. Theorem 2 allows to show that many classical examples of partially hyperbolic systems are stably ergodic ([15, 30, 39, 53]).

Let me give a sample result of this type. Let $f : M \to M$ be an Anosov diffeomorphism, $G$ be a compact Lie group and $\tau : M \to G$ be a smooth function. Consider a compact extension $F_{f,\tau}$ acting on $M \times G$ by the formula

$$
F_{f,\tau}(x, g) = (fx, \tau(x)g).
$$

**Theorem 3.**

(a) ([12]) Accessibility classes are orbits of subgroups of $G$.

(b) ([12]) For a typical $(f, \tau)$, $F_{f,\tau}$ has the accessibility property.

(c) ([15]) If $M$ is a nilmanifold and $F_{f,\tau}$ has the accessibility property, then it is stably ergodic.

In view of Theorem 5 below it is interesting to note that, while essential accessibility is sufficient for ergodicity, it does not imply stable ergodicity. Indeed if $G = \mathbb{T}^2$ and $\tau(x) = (t(x)u_1, t(x)u_2)$, then $F_{f,\tau}$ will be ergodic for a typical function $t$ provided that $u_1/u_2 \not\in \mathbb{Q}$. However, by small perturbation one can
make $u_1/u_2$ rational, $u_1/u_2 = k_1/k_2$, in which case $F_{f,\tau}$ is not ergodic since
\[ \phi(x, g_1, g_2) = e^{2\pi i (k_1 g_1 + k_2 g_2)} \]
is an invariant function.

2.2. **Difficulties in proving ergodicity.** Theorem 3 and similar results provide an evidence for the conjecture of Pugh and Shub saying that stable ergodicity is dense among partially hyperbolic systems. While this conjecture is still open, Federico Rodriguez Hertz obtained several strong results in that direction. Before describing these results in detail let me mention the difficulty: the accessibility classes could be complicated.

To explain the problem let me compare the accessibility classes in dynamics with accessibility classes in control theory. In the setting of partially hyperbolic systems the accessibility class of $x$ denoted by $A(x)$ is the set of points which can be joined to $x$ by a piecewise smooth curve where each piece is tangent to either $E_u$ or $E_s$. More generally, fix two plane fields $E_1$ and $E_2$ and let $A(x)$ be the set of points which can be joined to $x$ by a piecewise smooth curve where each piece is tangent to either $E_1$ or $E_2$. If $E_1$ and $E_2$ are smooth, then the accessibility classes are nice.

Let $\text{Lie}(E_1, E_2)$ be the smallest subspace in the space of vector fields on $M$ which contains all vector fields tangent to either $E_u$ or $E_s$ and which is closed with respect to taking commutators. Let $\text{Lie}_p(E_1, E_2)$ be the set of all vectors $v \in T_p M$ such that $v = u(p)$ for some $u \in \text{Lie}(E_1, E_2)$.

**Theorem 4** (see e.g. [32, 37]).

(a) (Lorby) $A(x)$ is a manifold.
(b) (Chow) If $\text{Lie}_p(E_1, E_2) = T_p M$ for each $p \in M$, then $A(x) = M$ for each $x \in M$.
(c) (Nagano-Sussmann) If $A(x) = M$ for each $x \in M$ and $E_1, E_2$ are analytic, then $\text{Lie}_p(E_1, E_2) = T_p M$ for each $p \in M$.

So in the smooth case accessibility classes are manifolds and their tangent spaces could be obtained by local analysis. Unfortunately, for partially hyperbolic systems, the regularity of $E_u$ and $E_s$ is much lower than needed for applying Theorem 4. In fact, it was shown in [25] that even in the uniformly hyperbolic case $E_u$ and $E_s$ are generically not better than Hölder. Moreover, in many cases one can show that sufficient regularity of stable and unstable foliations implies that the system is algebraic (see e.g. [7]).

2.3. **Work of Rodriguez Hertz.** The first important contribution of Federico Rodriguez Hertz is his thesis [41] dealing with stable ergodicity of linear toral automorphisms with two dimensional center. Before that work there were no tools for proving stable ergodicity of non accessible systems. The precise statement of [41] is the following.

**Theorem 5** ([41]). Let $A : \mathbb{T}^d \to \mathbb{T}^d$ be a linear automorphism such that the spectrum of $A$ has two eigenvalues on the unit circle which are not the roots of unity and such that the characteristic polynomial of $A$ is irreducible over $\mathbb{Z}$ and can not be written in the form $q(t^m)$ for some $m > 1$. Then $A$ is stably ergodic.
Let me go briefly over the main steps of the proof. The key fact is that accessibility classes are reasonable. While they are not homogeneous spaces as in Theorem 3 they enjoy local homogeneity. Namely let $T$ be a submanifold transversal to $E_u \oplus E_s$. Given $x, y \in T$ which belong to the same accessibility class, there exists a continuous local map $\phi$ mapping a small neighborhood of $x$ in $T$ to a small neighborhood of $y$ in $T$ which preserves accessibility classes. The map $\phi$ is obtained by composing the holonomy maps along the Hopf chain joining $x$ and $y$. In particular, in general, $\phi$ has low regularity. If $\phi$ were smooth we could conclude that accessibility classes are manifolds ([40, 54]). In the present setting low dimensional topology arguments yield

**Lemma 6.** If $\dim(E_c) \leq 2$, then the accessibility classes are topological manifolds.

To present the idea of the argument consider the case $\dim(E_c) = 1$. Take a curve $T$ transversal to $E_u \oplus E_s$ (see Figure 3). Take $x \in T$ and consider all curves tangent to $E_u \cup E_s$ starting at $x$ and ending at $T$. There are two cases. Either all such curves end at $x$ in which case $A(x)$ is a codimension 1 surface or there is a curve $\gamma$ which does not end at $x$. Shortening all legs of $\gamma$ by a factor $t$ and adding two extra legs to land at $T$, if necessary, we obtain a family of curves $\gamma_t$ with $\gamma_0(1) = x$, $\gamma_1(1) = y$. Thus $A(x) \cap T$ contains a segment. Now local homogeneity implies that $A(x) \cap T$ is open proving the lemma.

![Figure 3](image_url)

**Figure 3.** Proof of Lemma 6 for one dimensional center. If there is a curve with nontrivial holonomy, then one can sweep an open set by shortening the legs of the original curve and adding two extra legs.

The above argument also shows that $\dim A(x)$ is upper semicontinuous. Next, using algebraic topology the author shows that the partition into accessibility classes is minimal, that is, there are no nontrivial open sets consisting of accessibility classes. Now Lemma 6 shows that there are only three possibilities:

(i) $A(x)$ is open for some $x$. Then $f$ is accessible by minimality.
(ii) $A(x) \cap T$ is a curve for each $x$. This is shown to be incompatible with the fact that $A(0) \cap T$ is invariant by a map which is close to rotation. (Recall that for the unperturbed linear map $0$ is a fixed point and the restriction of
A to a center space is a rotation. For a small perturbation, there is a fixed point close to 0 which possesses an invariant manifold where the dynamics is close to a rotation.)

(iii) $A(x) \cap T$ is discrete for each $x$. A standard argument then shows that the accessibility partition of the perturbed map $f$ is close to the accessibility partition of the linear model. The accessibility classes of the linear map are Diophantine planes. One then invokes a KAM result of Moser to promote the topological conjugacy to the smooth conjugacy which in turn implies that $f$ is essentially accessible.

Apart from playing an important role in the proof of Theorem 5, Lemma 6 has several spectacular applications in case $\dim(E_c) = 1$ which are described below.

**Theorem 7 ([44]).** If $f$ is a partially hyperbolic volume preserving diffeomorphism of a three dimensional nilmanifold, then either $f$ has accessibility property or the manifold is $T^3$.

In particular, there are manifolds where all partially hyperbolic diffeomorphisms are ergodic.

**Theorem 8 ([45]).** For partially hyperbolic systems with one dimensional center accessibility is open and dense.

(The openness of accessibility was established in [20] while density is due to [45]).

To prove both theorems the authors consider the set $\Gamma(f)$ of points whose accessibility class has codimension 1. It is a closed invariant set saturated by stable and unstable manifolds. Thus there are three possibilities. Either

(i) $\Gamma(f) = \emptyset$, or
(ii) $\Gamma(f) = M$ and it is foliated by leaves tangent to $E_u \oplus E_s$, or
(iii) $\Gamma(f)$ is a closed lamination.

In case (iii), the fact that $f$ is area preserving and there only finitely many gaps in $\Gamma$ of a given size implies that the boundary leaves of $\Gamma(f)$ have Anosov dynamics and, hence, they contain many periodic points.

Now to prove Theorem 8 it suffices to prove that generically there is at least one open accessibility class (which rules out case (ii)) and all periodic points belong to open classes (which rules out case (iii)). Both of those genericity statements could be achieved by local perturbations.

In the proof of Theorem 7 cases (ii) and (iii) are ruled out by using deep results about two dimensional foliations of three dimensional manifolds.

3. **Nonuniform Hyperbolicity**

3.1. **Nonuniformly hyperbolic attractors.** The second direction of weakening uniform hyperbolicity assumption is nonuniform hyperbolicity. Namely, we say that a measure is hyperbolic if all its Lyapunov exponents are non-zero. Then there exists a measurable $f$ invariant splitting $TM = E^+(x) \oplus E^-(x)$ and there are
measurable functions $c(x) > 0$ and $\lambda(x) > 1$ such that if $\nu_+ \in E_+$, $\nu_- \in E_-$ are unit vectors then

\begin{equation}
||df^n(x)\nu_+|| \geq c(x)\lambda(x)^n, \quad ||df^n(x)\nu_-|| \leq c(x)\lambda(x)^{-n}.
\end{equation}

Pesin theory (see [5]) guarantees that for almost every $x$ there exist an unstable manifold $W^u(x)$ tangent to $E^+$ and a stable manifold $W^s$ tangent to $E^-$. This makes it possible to speak about Hopf chains and accessibility classes. In particular, given a point $x$ its *Hopf brush*

$$
\Lambda(x) = \bigcup_{y \in W^u(x)} W^s(y)
$$

is a set of points which are connected to $x$ by a two-leg chain. It is proven in [34] that Hopf brushes have positive Lebesgue measure.

We say that an invariant measure $\mu$ has the *SRB property*\(^1\) if it has conditional densities on unstable manifolds, that is, $\mu$ admits a disintegration

$$
\mu = \int_T \mu_\gamma(t) d\lambda(t)
$$

where $\gamma(t)$ is the unstable manifold passing through $t$ and $\mu_\gamma$ has a density on $\gamma$.

**Theorem 9** (Pesin, Pugh-Shub, see [5, 34, 36]). *If $\mu$ has the SRB property, then for $\mu$ almost any $x$, $\Lambda(x) \in \mathcal{B}(\mu)$. In particular, there are at most countably many SRB measures.*

In general, $\Lambda(x)$ can be a complicated fractal set. In particular, there indeed may be countably many SRB measures ([21]) and basins of different measures may be intermingled ([18, 29]).

While in many specific examples (such as Hénon attractors, Lorenz equation, dispersive billiards, etc.) the uniqueness of SRB measures has been established ([6, 17, 35, 55]), the general results on ergodicity were lacking due to the complicated geometry of Hopf brushes.

### 3.2. Work of Rodriguez Hertz.

Jointly with Jana Rodriguez Hertz, Ali Tahzibi and Raul Ures, Federico Rodriguez Hertz has obtained deep results about geometry of Hopf brushes, especially in the low dimensional setting. Let $\mu$ be a measure with nonzero Lyapunov exponents and the SRB property. Given a saddle $p$ let

$$
\Lambda^u(p) = \{x : W^u(x) \text{ intersects transversally } W^s(p)\},
\Lambda^s(p) = \{x : W^s(x) \text{ intersects transversally } W^u(p)\},
\Lambda(p) = \Lambda^s(p) \cap \Lambda^u(p).
$$

**Theorem 10** ([43])**.** (a) *If $\mu(\Lambda^s(p)) > 0$ and $\mu(\Lambda^u(p)) > 0$, then

$$
\mu(\Lambda^s(p)\Delta \Lambda^u(p)) = 1
$$

and $\Lambda(p)$ is an ergodic component.*

---

\(^1\)The definition of the *SRB property* is different in different papers. Here we use the terminology of Federico Rodriguez Hertz and his collaborators.
properties (see Figure 4).

hyperbolic ergodic measures with the SRB property.

\[ \theta \]

expense of allowing a worse bound on the functions in (6).

on sizes of stable and unstable manifolds. On the other hand, given a hyperbolic measure, the absolute continuity of the functions implies that, for \( \mu \) almost every point \( x \), Lebesgue almost every point inside \( W^u(x) \) is \( \mu \) typical. If this happens we say that \( W^u(x) \) is \( \text{supp}(\mu_j) \) saturated by unstable manifolds.

To explain the idea of the proof we need the notion of Pesin set. We refer the reader to [5] for the precise definition. Roughly speaking, a Pesin set is a set of points where the functions

\[ c(x), \quad \lambda(x) - 1, \quad \text{and} \quad \angle(E^+(x), E^-(x)) \]

are not too small and they do not deteriorate too quickly along the orbit of \( x \). In particular, the points from a given Pesin set have a uniform lower bounds on sizes of stable and unstable manifolds. On the other hand, given a hyperbolic measure \( \mu \) one can find a Pesin set of measure arbitrary close to 1 (at the expense of allowing a worse bound on the functions in (6)).

The proof of Corollary 11 relies on an elegant analytic trick. Let \( \mu_1, \mu_2 \) be two hyperbolic ergodic measures with the SRB property.

(b) If \( \nu \) is an ergodic component of \( \mu \), then there exists a saddle \( p \) with \( \nu(\Lambda(p)) = 1 \).

Theorem 10 provides an extension of Theorem 1 to the nonuniformly hyperbolic setting.

The intuition behind this theorem is that if \( \mu(\Lambda(p)) > 0 \), then most points on \( W^u(p) \) are forward typical for \( \mu \) and this ensures that \( W^u(p) \) and hence \( \bigcup_{x \in W^u(p)} W^s(x) \) belong to one component (see Figure 4).

Corollary 11 ([42]). Let \( f \) be a topologically transitive map of a surface. Then every hyperbolic measure with the SRB property is ergodic. Hence such a measure is unique.

To explain the idea of the proof we need the notion of Pesin set. We refer the reader to [5] for the precise definition. Roughly speaking, a Pesin set is a set of points where the functions

\[ c(x), \quad \lambda(x) - 1, \quad \text{and} \quad \angle(E^+(x), E^-(x)) \]

are not too small and they do not deteriorate too quickly along the orbit of \( x \). In particular, the points from a given Pesin set have a uniform lower bounds on sizes of stable and unstable manifolds. On the other hand, given a hyperbolic measure \( \mu \) one can find a Pesin set of measure arbitrary close to 1 (at the expense of allowing a worse bound on the functions in (6)).

The proof of Corollary 11 relies on an elegant analytic trick. Let \( \mu_1, \mu_2 \) be two hyperbolic ergodic measures with the SRB property.

Recall from Section 1 that for \( \mathcal{B}(\mu_j) \) is saturated by stable manifolds. Since \( \mu_j \) have the SRB property, the absolute continuity of the \( W^u \) implies that, for \( \mu_j \) almost every point \( x \), Lebesgue almost every point inside \( W^u(x) \) is \( \mu_j \) typical. If this happens we say that \( W^u(x) \) is \( \text{supp}(\mu_j) \) saturated by unstable manifolds.

\[ \text{FIGURE 4. Ergodicity of homoclinic classes} \]
Let $R_1$ be a rectangle containing many stable manifolds having the following properties (see Figure 5):

(a) they belong to $\mathcal{B}(\mu_1)$;
(b) they belong to a given Pesin set;
(c) they fully cross $R_1$.

This can be achieved taking a density point of some Pesin set of $\mu_1$ and letting $R_1$ to be a sufficiently small rectangle around it. We denote by $W^s$ the union of the stable manifolds having properties (a)–(c).

Let $W$ be a piece of an unstable manifold which is typical for $\mu_2$, is contained inside $R_1$ and fully crosses $R_1$. The existence of such manifold follows from the topological transitivity. The authors show, using uniform absolute continuity of $W^s$, that $W^s$ can be extended to a $C^1$ foliation of $R_1$. Then Sard Theorem guarantees that almost every leaf of that foliation intersects $W$ transversely. Therefore $W$ contains many points from $\mathcal{B}(\mu_1)$ and since it is typical for $\mu_2$ we can conclude that $\mu_2 = \mu_1$ proving the uniqueness.

4. ERGODICITY AND RIGIDITY

Federico Rodriguez Hertz obtained significant geometric information on the structure of ergodic decomposition in both partially hyperbolic and nonuniformly hyperbolic settings, which constitutes a significant advance in this classical subject. Perhaps equally important is that his work provides a new point of view on ergodicity of hyperbolic systems. Namely, he treated ergodicity as a rigidity problem. If the majority of systems are believed to be ergodic, then it makes sense to classify the nonergodic examples and to see if the system at hand belongs to a small list of exceptions. This new point of view allows one to bring to the spotlight new powerful tools of rigidity theory, in particular topological and geometric methods.
Currently, the interplay between hyperbolicity and rigidity is an active research topic\(^2\). Below I present a selection of works in this area making emphasis on the papers related to the results discussed in the two previous sections.

4.1. **Lyapunov exponents.** Since hyperbolicity provides a powerful tool for studying statistical properties of dynamical systems, it is desirable to obtain methods for proving that Lyapunov exponents are non-zero. For products of independent random matrices this was done by Furstenberg [23]. The dependent case remained much less understood even though significant progress was achieved in the work of Ledrappier [31]. The situation changed in the last decade due to the work Bonatti, Gomez-Mont, Viana, Avila and their collaborators ([9, 52, 4]). For cocycles over nonuniformly hyperbolic systems it was proven in [52] that the top Lyapunov exponent is positive on a open and dense set of smooth cocycles. To explain the idea of this work let \( A \) be a matrix valued cocycle over a diffeomorphism \( f \).

Consider an induced action on the projective bundle

\[
F(x, v) = \left( f x, \frac{A(x) v}{||A(x) v||} \right).
\]

The key observation of [52] is that if Lyapunov exponents of \( A \) are small (compared to the exponents of \( f \)), then \( F \) is *non-uniformly partially hyperbolic*. This allows one to employ the method of [31] to show that if the Lyapunov exponents vanish, then the accessibility classes of \( F \) are small, which does not occur generically.

Avila and Viana extend the techniques used for hyperbolic systems to partially hyperbolic setting. Here is a sample of their results.

**Theorem 12** ([4]). Let \( g \) be a symplectic diffeomorphism of \( \mathbb{T}^4 \) which is close to \( f(x) = Ax \ mod \mathbb{Z}^4 \), where \( A \) has two dimensional center. Then either \( g \) has non-zero Lyapunov exponents or \( g \) is conjugated to \( f \). In particular, \( g \) is Bernoulli.

Note that the last statement strengthens Theorem 5 in the symplectic setting.

4.2. **Random diffeomorphisms.** The results presented above deal with ergodicity of single diffeomorphism. In this section we discuss ergodicity of several diffeomorphisms. Let \( (f_1, f_2, \ldots, f_k) \) be a finite collection of transformations of a space \( M \) preserving the same measure \( \mu \). We say that this collection is *ergodic with respect to \( \mu \)* if every measurable set invariant under each transformation in our collection has either zero or full measure. Equivalently, one can consider a skew product \( \Sigma \times M \) on \( \Sigma = \{1, 2, \ldots, k\}^\mathbb{Z} \) given by

\[
F(\omega, x) = (\sigma \omega, f_{\omega_0}(x)),
\]

\(^2\)The present survey concentrates on using rigidity methods to study ergodic properties of hyperbolic systems. It is also very fruitful to use ergodic properties of hyperbolic systems to obtain new results in rigidity theory. The work of Federico Rodriguez Hertz in that direction is described in the companion survey [51].
where $\sigma$ is the shift. $F$ preserves the measures which are products of a Bernoulli measure on $\Sigma$ with $\mu$. (The meaning of $F$ is that the transformations applied at different times are chosen independently from our collection.) According to Kakutani Theorem [28] the ergodicity of the collection $(f_1, f_2, \ldots, f_k)$ is equivalent to the ergodicity $F$ with respect to any product measure described above.

Since the dynamics of hyperbolic systems has many common features with the shift $\sigma$, one can expect that the problem of ergodicity of several diffeomorphisms is similar to the problem of ergodicity of skew products over hyperbolic systems. The later skew products are often partially hyperbolic and the results of Section 2 apply. However, the geometry of partially hyperbolic systems is richer than the geometry of Bernoulli shift, so the problem of ergodicity of several maps is less understood.

The next result presents an instructive example of a stably ergodic system, where each map in the system has poor ergodic properties.

**Theorem 13.** [22] Let $f_1$ and $f_2$ be volume preserving maps of $\mathbb{S}^{2d}$ such that $f_j$ are close to rotations $R_j$. If $(R_1, R_2)$ generate a dense subgroup of $SO_{d+1}$, then the collection $(f_1, f_2)$ is ergodic.

Similarly to [41], a key step is to develop a KAM scheme and to show that a lack of stochasticity (in this case the presence of zero Lyapunov exponents) gives sufficient information to conclude vanishing of the obstructions to conjugacy to the linear model.

I would like also to mention that, recently, KAM machinery allowed to obtain several advances in the rigidity theory (see e.g. [19] and references therein) which go beyond the present survey. On the other hand, a recent work of Aaron Brown and Federico Rodriguez Hertz [14] provides a significant generalization of Theorem 13, at least, in the two dimensional case.

### 4.3. Homological equation.

The homological equation

$$\phi(x) = \Phi(x) - \Phi(fx), \quad (7)$$

where $\phi$ is a given function and $\Phi$ is an unknown plays a key role in several branches of dynamics, including rigidity theory, limit theorems and time changes. In particular, in many cases one would like to understand the regularity of the transfer function $\Phi$. If $f$ is volume preserving, then integrating both sides of (7) we obtain that, if (7) has a solution, then $\phi$ has zero mean. For functions of zero mean the study of (7) is closely related to the ergodicity of the skew product on $M \times \mathbb{R}$ given by

$$F(x, t) = (fx, t + \phi(x)) \quad (8)$$

(see e.g. [1, 12, 33]).

For partially hyperbolic systems the regularity of solutions to (7) is given by the following result.

**Theorem 14 ([54]).** Let $f$ be a partially hyperbolic volume preserving diffeomorphism with the accessibility property. Let $\phi \in C^\infty(M)$. If (7) admits a measurable solution, then it admits a smooth solution.
The starting point of [54] is that if (7) has a solution, then the accessibility classes of the skew product (8) are measurably isomorphic to $M$. A difficult problem is to show that this isomorphism is smooth. Important ingredients include regularity theory of the sets which are saturated by stable and unstable manifolds developed in the study of stable ergodicity [16] and the analysis of locally homogenous sets similar to [40, 41].

5. CONCLUSION

The last two decades saw significant advances in understanding ergodic and other statistical properties of a large number of classical dynamical systems. Federico Rodríguez Hertz has made a significant contributions to this subject. The present survey contains merest sketches of his work and related developments. More detailed reviews are contained in [8, 46]. However, I urge the readers who want to get a better appreciation of this subject to go to the original papers for a wealth of beautiful ideas and powerful techniques.

REFERENCES

[1] J. Aaronson, An Introduction to Infinite Ergodic Theory, Math. Surv. & Monographs, 50, AMS, Providence, RI, 1997.
[2] D. V. Anosov, Geodesic flows on closed Riemannian manifolds of negative curvature, Trudy Mat. Inst. Steklov., 90 (1967), 209 pp.
[3] D. V. Anosov and Ya. G. Sinai, Some smooth ergodic systems, Russian Math. Surveys, 22 (1967), 103–167.
[4] A. Avila and M. Viana, Extremal Lyapunov exponents: An invariance principle and applications, Invent. Math., 181 (2010), 115–189.
[5] L. Barreira and Ya. B Pesin, Nonuniform Hyperbolicity. Dynamics of Systems with Nonzero Lyapunov Exponents, Encyclopedia Math., Appl., 115, Cambridge Univ. Press, Cambridge, 2007.
[6] M. Benedicks and M. Viana, Solution of the basin problem for Hénon-like attractors, Invent. Math., 143 (2001), 375–434.
[7] Y. Benoist, P. Foulon and F. Labourie, Flots d’Anosov a distributions stable et instable differentiables, J. AMS, 5 (1992), 33–74.
[8] C. Bonatti, L. J. Díaz and M. Viana, Dynamics Beyond Uniform Hyperbolicity. A Global Geometric and Probabilistic Perspective, Encycl. Math. Sci., 102, Mathematical Physics, III, Springer-Verlag, Berlin, 2005.
[9] C. Bonatti, X. Gómez-Mont and M. Viana, Généricité d’exposants de Lyapunov non-nuls pour des produits déterministes de matrices, Ann. Inst. H. Poincaré Anal. Non Linéaire, 20 (2003), 579–624.
[10] R. Bowen, Markov partitions for Axiom A diffeomorphisms, Amer. J. Math., 92 (1970), 725–747.
[11] R. Bowen, Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms, 2nd revised edition, Lecture Notes Math., 470, Springer-Verlag, Berlin, 2008.
[12] M. I. Brin, Topological transitivity of a certain class of dynamical systems, and flows of frames on manifolds of negative curvature, Funkcional. Anal. i Priložen., 9 (1975), 9–19.
[13] M. I. Brin and Ya. B. Pesin, Partially hyperbolic dynamical systems, Izv. Akad. Nauk SSSR Ser. Mat., 38 (1974), 170–212.
[14] A. W. Brown and F. Rodríguez Hertz, Measure rigidity for random dynamics on surfaces and related skew products, arXiv:1506.06826.
[15] K. Burns and A. Wilkinson, Stable ergodicity of skew products, *Ann. ENS.*, 32 (1999), 859–889.

[16] K. Burns and A. Wilkinson, On the ergodicity of partially hyperbolic systems, *Ann. of Math. (2)*, 171 (2010), 451–489.

[17] N. Chernov and R. Markarian, *Chaotic Billiards*, Math. Surveys & Monographs, 127, AMS, Providence, RI, 2006.

[18] E. Colli, Infinitely many coexisting strange attractors, *Ann. Inst. H. Poincaré*, 15 (1998), 539–579.

[19] D. Damjanović, Hamilton’s theorem for smooth Lie group actions, in *Ergodic Theory and Dynamical Systems* (ed. Idris Assani), De Gruyter Proc. in Math., De Gruyter, Berlin, 2014, 117–127.

[20] P. Didier, Stability of accessibility, *Ergodic Th. Dyn. Syst.*, 23 (2003), 171–1731.

[21] D. Dolgopyat, H. Hu and Ya. B. Pesin, An example of a smooth hyperbolic measure with countably many ergodic components, *Proc. Symp. Pure Math.*, 69 (2001), 95–106.

[22] D. Dolgopyat and R. Krikorian, On simultaneous linearization of diffeomorphisms of the sphere, *Duke Math. J.*, 136 (2007), 475–505.

[23] H. Furstenberg, Noncommuting random products, *Trans. AMS*, 108 (1963), 377–428.

[24] M. Grayson, C. Pugh and M. Shub, Stably ergodic diffeomorphisms, *Ann. of Math. (2)*, 140 (1994), 295–329.

[25] B. Hasselblatt and A. Wilkinson, Prevalence of non-Lipschitz Anosov foliations, *Erg. Th. Dynam. Sys.*, 19 (1999), 643–656.

[26] M. W. Hirsch, C. C. Pugh and M. Shub, *Invariant Manifolds*, Lecture Notes Math., 583, Springer-Verlag, Berlin-New York, 1977.

[27] E. Hopf, Statistik der geodätischen Linien in Mannigfaltigkeiten negativer Krümmung, *Ber. Verh. Sächs. Akad. Wiss. Leipzig*, 91 (1939), 261–304.

[28] S. Kakutani, Random ergodic theorems and Markoff processes with a stable distribution, in *Proc. 2nd Berkeley Symposium on Math. Stat. & Prob.*, Univ. of California Press, Berkeley-Los Angeles, 1951, 247–261.

[29] I. Kan, Open sets of diffeomorphisms having two attractors, each with an everywhere dense basin, *Bull. AMS (N.S.)*, 31 (1994), 68–74.

[30] A. Katok and A. Kononenko, Cocycles’ stability for partially hyperbolic systems, *Math. Res. Lett.*, 3 (1996), 191–210.

[31] F. Ledrappier, Quelques propriétés des exposants caractéristiques, Lecture Notes Math., 1097, Springer-Verlag, Berlin, 1984, 305–396.

[32] R. Montgomery, *A Tour of Subriemannian Geometries, Their Geodesics and Applications*, Math. Surveys & Monographs, 91, AMS, Providence, RI, 2002.

[33] W. Parry and M. Pollicott, Skew products and Livsic theory, in *Representation Theory, Dynamical Systems, and Asymptotic Combinatorics*, AMS Transl. Ser. 2, 217, AMS, Providence, RI, 2006, 139–165.

[34] Ya. B. Pesin, Characteristic Ljapunov exponents, and smooth ergodic theory, *Russian Math. Surveys*, 32 (1977), 55–114.

[35] Ya. B. Pesin, Dynamical systems with generalized hyperbolic attractors: Hyperbolic, ergodic and topological properties, *Erg. Th. Dynam. Sys.*, 12 (1992), 123–151.

[36] C. C. Pugh and M. Shub, Ergodic attractors, *Trans. AMS*, 312 (1989), 1–54.

[37] C. C. Pugh and M. Shub, Stable ergodicity and stable accessibility, in *Differential Equations and Applications (Hangzhou, 1996)*, Int. Press, Cambridge, MA, 258–268.

[38] C. C. Pugh and M. Shub, Stably ergodic dynamical systems and partial hyperbolicity, *J. Complexity*, 13 (1997), 125–179.

[39] C. Pugh, M. Shub and A. Starkov, Unique ergodicity, stable ergodicity, and the Mautner phenomenon for diffeomorphisms, *Discrete Contin. Dyn. Syst.*, 14 (2006), 845–855.

[40] D. Repovš, A. B. Skopenkov and E. V. Ščepin, C1-homogeneous compacta in Rn are C1-submanifolds of Rn, *Proc. AMS*, 124 (1996), 1219–1226.
[41] F. Rodriguez Hertz, Stable ergodicity of certain linear automorphisms of the torus, *Ann. of Math.* (2), 162 (2005), 65–107.

[42] F. Rodriguez Hertz, M. A. Rodriguez Hertz, A. Tahzibi and R. Ures, Uniqueness of SRB measures for transitive diffeomorphisms on surfaces, *Comm. Math. Phys.*, 306 (2011), 35–49.

[43] F. Rodriguez Hertz, M. A. Rodriguez Hertz, A. Tahzibi and R. Ures, New criteria for ergodicity and nonuniform hyperbolicity, *Duke Math. J.*, 160 (2011), 599–629.

[44] F. Rodriguez Hertz, M. A. Rodriguez Hertz and R. Ures, Partial hyperbolicity and ergodicity in dimension three, *J. Mod. Dyn.*, 2 (2008), 187–208.

[45] F. Rodriguez Hertz, M. A. Rodriguez Hertz and R. Ures, Accessibility and stable ergodicity for partially hyperbolic diffeomorphisms with 1D-center bundle, *Invent. Math.*, 172 (2008), 353–381.

[46] F. Rodriguez Hertz, M. A. Rodriguez Hertz and R. Ures, Accessibility and abundance of ergodicity in dimension three: A survey, *Publ. Mat. Urug.*, 12 (2011), 177–198.

[47] D. Ruelle, A measure associated with axiom-A attractors, *Amer. J. Math.*, 98 (1976), 619–654.

[48] Ya. G. Sinai, Construction of Markov partitionings, *Funct. An., Appl.*, 2 (1968), 64–89, 70–80.

[49] Ya. G. Sinai, Gibbs measures in ergodic theory, *Uspehi Mat. Nauk*, 27 (1972), 37–105.

[50] S. Smale, Differentiable dynamical systems, *Bull. AMS*, 73 (1967), 747–817.

[51] R. Spatzier, On the work of Rodriguez Hertz on rigidity in dynamics, *J. Mod. Dyn.*, 10 (2016), 191–207.

[52] M. Viana, Almost all cocycles over any hyperbolic system have nonvanishing Lyapunov exponents, *Ann. of Math.* (2), 167 (2008), 643–680.

[53] A. Wilkinson, Stable ergodicity of the time-one map of a geodesic flow, *Erg. Th. Dynam. Sys.*, 18 (1998), 1545–1587.

[54] A. Wilkinson, The cohomological equation for partially hyperbolic diffeomorphisms, *Astérisque*, 358 (2013), 75–165.

[55] L.-S. Young, What are SRB measures, and which dynamical systems have them?, *J. Stat. Phys.*, 108 (2002), 733–754.

DMITRY DOLGOYAT <dolgopyat@math.umd.edu>: Department of Mathematics, University of Maryland, College Park, MD 20742, USA