Closed conformal Killing–Yano tensor and geodesic integrability

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Abstract
Assuming the existence of a single rank-2 closed conformal Killing–Yano tensor with a certain symmetry we show that there exists mutually commuting rank-2 Killing tensors and Killing vectors. We also discuss the condition of separation of variables for the geodesic Hamilton–Jacobi equations.

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1. Introduction
Recently, it has been shown that the geodesic motion in the Kerr–NUT–de Sitter spacetime is integrable for all dimensions \([1–6]\). Indeed, the constants of motion that are in involution can be explicitly constructed from a rank-2 closed conformal Killing–Yano (CKY) tensor. In this paper, we answer the question raised in [5] under which general assumptions a CKY tensor implies the complete integrability of geodesic equation. We assume the existence of a single rank-2 closed CKY tensor with a certain symmetry for the \(D\)-dimensional spacetime \(M\) with a metric \(g\). It turns out that such a spacetime admits mutually commuting \(k\) rank-2 Killing tensors and \(k\) Killing vectors. Here we put \(D = 2k\) for even \(D\), and \(D = 2k – 1\) for odd \(D\). Although the existence of the commuting Killing tensors was shown in [5, 6], we reproduce it more directly. We also discuss the condition of separation of variables for the geodesic Hamilton–Jacobi equations using the result given by Benenti–Francaviglia [7] and Kalnins–Miller [8] (see also [9]).

2. Assumptions and main results
A two-form
\[ h = \frac{1}{2} h_{ab} \, dx^a \wedge dx^b, \quad h_{ab} = -h_{ba} \quad (2.1) \]
is called a conformal Killing–Yano (CKY) tensor if it satisfies
\[ \nabla_a h_{bc} + \nabla_b h_{ac} = 2\xi_c g_{ab} - \xi_a g_{bc} - \xi_b g_{ac}. \] (2.2)

The vector field \( \xi_a \) is called the associated vector of \( h_{ab} \), which is given by
\[ \xi_a = \frac{1}{D-1} \nabla^b h_{ba}. \] (2.3)

In the following we assume
\( (a1)\) \( dh = 0 \), \( (a2) \) \( \mathcal{L}_\xi g = 0 \), \( (a3) \) \( \mathcal{L}_\xi h = 0 \). (2.4)

Assumption \((a1)\) means that \((D-2)\)-form \( f = *h \) is a Killing–Yano (KY) tensor,
\[ \nabla_{(a} f_{ab)k_3 \cdots k_{D-1}} = 0. \] (2.5)

Note that equation (2.2) together with \((a1)\) is equivalent to
\[ \nabla_a h_{bc} = \xi_c g_{ab} - \xi_b g_{ac}. \] (2.6)

It was shown in [10] that the associated vector \( \xi \) satisfies
\[ \nabla_a \xi_b + \nabla_b \xi_a = \frac{1}{D-2} (R_a^c h_{bc} + R_b^c h_{ac}), \] (2.7)

where \( R_{ab} \) is a Ricci tensor. If \( M \) is Einstein, i.e. \( R_{ab} = \Lambda g_{ab} \), then
\[ \nabla_a \xi_b + \nabla_b \xi_a = 0. \] (2.8)

Thus, any Einstein space satisfies assumption \((a2)\) [10]. According to [5], we define \(2j\)-forms \( h^{(j)} \) \((j = 0, \ldots, k - 1)\):
\[ h^{(j)} = h \wedge h \wedge \cdots \wedge h = \frac{1}{(2j)!} h^{(j)}_{a_1 \cdots a_{2j}} dx^{a_1} \wedge \cdots \wedge dx^{a_{2j}}, \] (2.9)
where the components are written as
\[ h^{(j)}_{a_1 \cdots a_{2j}} = \frac{(2j)!}{2^j} h_{[a_1 a_2} h_{a_3 a_4} \cdots h_{a_{2j-1} a_{2j}}]. \] (2.10)

Since the wedge product of the two CKY tensors is again a CKY tensor [5], \( h^{(j)} \) are closed CKY tensors, and so \( f^{(j)} = *h^{(j)} \) are KY tensors. Explicitly, we have
\[ f^{(j)} = *h^{(j)} = \frac{1}{(D-2j)!} f^{(j)}_{a_1 \cdots a_{D-2j}} dx^{a_1} \wedge \cdots \wedge dx^{a_{D-2j}}, \] (2.11)
where
\[ f^{(j)}_{a_1 \cdots a_{D-2j}} = \frac{1}{(2j)!} \epsilon^{b_1 \cdots b_{2j}} a_1 \cdots a_{D-2j} h^{(j)}_{b_1 \cdots b_{2j}}. \] (2.12)

Given these KY tensors, we can construct the rank-2 Killing tensors \( K^{(j)} \) obeying the equation
\[ \nabla_a K^{(j)}_{bc} = 0: \]
\[ K^{(j)}_{ab} = \frac{1}{(D-2j-1)! (j)!^2} f^{(j)}_{a_1 \cdots a_{D-2j-1}} f^{(j)}_{b_1 \cdots b_{D-2j-1}}. \] (2.13)

From \((a2)\) we have \( \mathcal{L}_\xi h^{(j)} = *\mathcal{L}_\xi h^{(j)} \) and hence assumption \((a3)\) yields
\[ \mathcal{L}_\xi h^{(j)} = 0, \quad \mathcal{L}_\xi f^{(j)} = 0, \quad \mathcal{L}_\xi K^{(j)} = 0. \] (2.14)

We also immediately obtain from (2.6)
\[ \nabla_\xi h^{(j)} = 0, \quad \nabla_\xi f^{(j)} = 0, \quad \nabla_\xi K^{(j)} = 0. \] (2.15)
Let us define the vector fields \( \eta^{(j)} \) by [11, 12]
\[
\eta^{(j)}_a = K^{(j)}_a b \xi_b.
\] (2.16)

Then we have
\[
\nabla (a \eta^{(j)} b) = \frac{1}{2} \mathcal{L}_\xi K^{(j)}_a b - \nabla \xi K^{(j)}_a b,
\] (2.17)

which vanishes by (2.14) and (2.15), i.e. \( \eta^{(j)} \) are Killing vectors.

Theorem 1 was proved in [5, 6].

**Theorem 1.** Under (a1) Killing tensors \( K^{(i)} \) are mutually commuting,
\[
[K^{(i)}, K^{(j)}]_S = 0.
\]
The bracket \([ , ]_S\) represents a symmetric Schouten bracket. The equation can be written as
\[
K^{(i)}_d \nabla^d K^{(j)}_{bc} - K^{(j)}_d \nabla^d K^{(i)}_{bc} = 0.
\] (2.18)

Adding assumptions (a2) and (a3) we prove

**Theorem 2.**
\[
\mathcal{L}_\eta^{(j)} h = 0.
\]

**Corollary.** Killing vectors \( \eta^{(i)} \) and Killing tensors \( K^{(j)} \) are mutually commuting,
\[
[\eta^{(i)}, K^{(j)}]_S = 0, \quad [\eta^{(i)}, \eta^{(j)}] = 0.
\]

### 3. Proof of theorems 1, 2

Let \( H, Q := -H^2, K^{(j)} \) be matrices with elements
\[
H^a_b = h^a_b, \quad Q^a_b = -h^a_c h^c_b, \quad (K^{(j)})^a_b = K^{(j)a}_b. \] (3.1)

The generating function of \( K^{(j)} \) can be read off from [5]
\[
K_{ab}(\beta) = \sum_{j=0}^{k-1} K^{(j)}_{ab} \beta^j = \det^{1/2}(I + \beta Q)(I + \beta Q)^{-1} [I]_{ab}.
\] (3.2)

Here \( k = [(D + 1)/2] \). Note that
\[
2 \det^{1/2}(I + \beta Q)(I + \beta Q)^{-1} [I]_b = \det(I + \sqrt{\beta} H)(I + \sqrt{\beta} H)^{-1} [I]_b + \det(I - \sqrt{\beta} H)(I - \sqrt{\beta} H)^{-1} [I]_b.
\] (3.3)

Since \( \det(\pm \sqrt{\beta} H)(\pm \sqrt{\beta} H)^{-1} [I]_b \) is a cofactor of the matrix \( I \pm \sqrt{\beta} H, (3.2) \) is indeed a polynomial of \( \beta \) of degree \( [(D - 1)/2] \).

For simplicity, let us define a matrix \( S(\beta) \) by
\[
S(\beta) := (I + \beta Q)^{-1}.
\] (3.4)

Using (2.6), we have
\[
\nabla_a \det^{1/2}(I + \beta Q) = -2 \beta \xi_d [HS(\beta)]^d_a \det^{1/2}(I + \beta Q),
\] (3.5)

\[
\nabla_a S_{bc}(\beta) = \beta S_{bd}(\beta) \xi^d [HS(\beta)]_{dc} - \beta S_{bd}(\beta) \xi^d [HS(\beta)]_{ac}
+ \beta [HS(\beta)]_{bc} \xi^d S_{ad}(\beta) - \beta [HS(\beta)]_{bc} \xi^d S_{ad}(\beta).
\] (3.6)
Combining these relations, we have
\[ \nabla_a K_{bc}(\beta) = \det^{1/2}(I + \beta Q) \xi^d X_{abc,d}(\beta), \]  \hspace{1cm} (3.7)
where
\[ X_{abc,d}(\beta) = 2\beta [HS(\beta)]_{ad} S_{bc}(\beta) - \beta [HS(\beta)]_{bd} S_{ca}(\beta) - \beta [HS(\beta)]_{cd} S_{ab}(\beta) + \beta S_{bd}(\beta) [HS(\beta)]_{ca} + \beta S_{cd}(\beta) [HS(\beta)]_{ba}. \]  \hspace{1cm} (3.8)

Then with the help of (3.7), it is easy to check that the following relations hold:
\[ \nabla_a K_{bc}(\beta) = 0. \]  \hspace{1cm} (3.9)

Therefore we have
\[ \nabla_a K_{(j)bc}(\beta) = 0. \]  \hspace{1cm} (3.10)

**Proof of theorem 1.** In terms of generating function, theorem 1 (2.18) can be written as follows:
\[ K_{e(a}(\beta_1) \nabla^e K_{bc)}(\beta_2) - K_{e(a}(\beta_2) \nabla^e K_{bc)}(\beta_1) = 0. \]  \hspace{1cm} (3.11)

Let
\[ F_{abc}(\beta_1, \beta_2) := \frac{K_{e(a}(\beta_1) \nabla^e K_{bc)}(\beta_2)}{\det^{1/2}(I + \beta_1 Q) \det^{1/2}(I + \beta_2 Q)}. \]  \hspace{1cm} (3.12)

(3.11) is equivalent to
\[ F_{abc}(\beta_1, \beta_2) - F_{abc}(\beta_2, \beta_1) = 0. \]  \hspace{1cm} (3.13)

Using the explicit form of \( \nabla^e K_{bc}(\beta_2) \), we have
\[ F_{abc}(\beta_1, \beta_2) = \beta_2 \xi^d S_{ea}(\beta_1) \times 2[H S(\beta_2)]_{ad} S_{bc}(\beta_2) - [H S(\beta_2)]_{bd} S_{ca}(\beta_2) - [H S(\beta_2)]_{cd} S_{ab}(\beta_2) + \beta S_{bd}(\beta_2) [H S(\beta_2)]_{ca} + \beta S_{cd}(\beta_2) [H S(\beta_2)]_{ba}. \]  \hspace{1cm} (3.14)

Then
\[ F_{abc}(\beta_1, \beta_2) = 2\beta_2 \xi^d [S_{bc}(\beta_2) [H S(\beta_1) S(\beta_2)]_{ad} - [S(\beta_1) S(\beta_2)]_{bc} [H S(\beta_2)]_{ad}]. \]  \hspace{1cm} (3.15)

Note that
\[ \beta_2 S(\beta_2) - \beta_1 S(\beta_1) = (\beta_2 - \beta_1) S(\beta_1) S(\beta_2). \]  \hspace{1cm} (3.16)

Then
\[ F_{abc}(\beta_1, \beta_2) - F_{abc}(\beta_2, \beta_1) = 2(\beta_2 - \beta_1) \xi^d ([S(\beta_1) S(\beta_2)]_{bc} [H S(\beta_1) S(\beta_2)]_{ad} - [S(\beta_1) S(\beta_2)]_{bd} [H S(\beta_1) S(\beta_2)]_{ad}) = 0. \]  \hspace{1cm} (3.17)

This completes the proof of theorem 1. \( \square \)

Let \( \eta_a(\beta) \) be the generating function of \( \eta_a^{(j)} \):
\[ \eta_a(\beta) = \sum_{j=0}^{k-1} \eta_a^{(j)} \beta^j = K_{ab}(\beta) \xi^b. \]  \hspace{1cm} (3.17)
Proof of theorem 2. In terms of the generating function (3.17), the theorem 2 is equivalent to

$$L_{\eta(\beta)} h_{ab} = 0. \quad (3.18)$$

The left-hand side is

$$L_{\eta(\beta)} h_{ab} = \eta^c(\beta) \nabla_c h_{ab} + h_{cb} \nabla_a \eta^c(\beta) + h_{ac} \nabla_b \eta^c(\beta). \quad (3.19)$$

Using (2.6), the first term on the right-hand side of (3.19) becomes

$$\eta^c(\beta) \nabla_c h_{ab} = \xi_b \eta_a(\beta) - \xi_a \eta_b(\beta). \quad (3.20)$$

Let us examine the second and third terms.

$$U_{ab}(\beta) := h_{cb} \nabla_a \eta^c(\beta) + h_{ac} \nabla_b \eta^c(\beta)$$

$$= h_{cb} \nabla_a (K^c_d(\beta) \xi^d) + h_{ac} \nabla_b (K^c_d(\beta) \xi^d)$$

$$= [K(\beta) H]_{ab} \xi_d + [K(\beta) H]_{ad} \xi_b \xi^d + \xi^d (h_{cb} \nabla_a K^c_d(\beta) + h_{ac} \nabla_b K^c_d(\beta)). \quad (3.21)$$

Note that

$$[K(\beta) H]_{ab} \nabla_a \xi^d + [K(\beta) H]_{ad} \nabla_b \xi^d = L_{\xi} [K(\beta) H]_{ab} - \nabla_c [K(\beta) H]_{ab} = 0. \quad (3.22)$$

Here we have used (2.14) and (2.15).

Let

$$V_{ab}(\beta) := \frac{\xi^d h_{ad} \nabla_b K^c_d(\beta)}{\det^{1/2}(I + \beta Q)}. \quad (3.23)$$

Then

$$U_{ab}(\beta) = \det^{1/2} (I + \beta Q) (V_{ab}(\beta) - V_{ba}(\beta)) = 2 \det^{1/2} (I + \beta Q) V_{\{ab\}}(\beta). \quad (3.24)$$

Using (3.7), we have

$$V_{ab}(\beta) = \beta \xi^d \xi^f \{[H S(\beta)]_{ad} [H S(\beta)]_{bf} - S_{df} [Q S(\beta)]_{ad} + [Q S(\beta)]_{ad} S_{bf}(\beta)\}, \quad (3.25)$$

$$2V_{\{ab\}}(\beta) = \beta \xi^d \xi^f \{[Q S(\beta)]_{ad} S_{bf}(\beta) - S_{ad}(\beta) [Q S(\beta)]_{bf}\}. \quad (3.26)$$

Note that

$$\beta Q S(\beta) = I - S(\beta). \quad (3.27)$$

Then

$$2V_{\{ab\}}(\beta) = \beta \xi^d \xi^f \{g_{ad} S_{bf}(\beta) - S_{ad}(\beta) g_{bf}\} = \xi_a S_{bf}(\beta) \xi^f - \xi_b S_{ad}(\beta) \xi^d. \quad (3.28)$$

Therefore

$$U_{ab}(\beta) = \xi_a \eta_b(\beta) - \xi_b \eta_a(\beta). \quad (3.29)$$

Adding (3.20) and (3.29), we have

$$L_{\eta(\beta)} h_{ab} = 0. \quad (3.30)$$

This completes the proof of theorem 2. \(\square\)

The first relation of corollary is equivalent to

$$L_{\eta^{(i)}} K^{(j)} = 0, \quad (3.31)$$

which immediately follows from theorem 2.

The second relation of corollary is equivalent to

$$L_{\eta^{(i)}} \eta^{(j)} = 0. \quad (3.32)$$
Note that
\[ \mathcal{L}_\xi \xi = [\xi, \xi] = 0, \tag{3.33} \]
\[ \mathcal{L}_\xi \eta^{(j)a} = \mathcal{L}_\xi (K^{(j)a}_b \xi^b) \]
\[ = (\mathcal{L}_\xi K^{(j)a}_b) \xi^b + K^{(j)a}_b (\mathcal{L}_\xi \xi^b) \]
\[ = 0. \tag{3.34} \]

Here we have used (2.14) and (3.33). Then
\[ \mathcal{L}_{\eta^{(i)}} \xi = [\eta^{(j)}, \xi] = -\mathcal{L}_\xi \eta^{(j)} = 0. \tag{3.35} \]

Now, using this relation and (3.31), we easily see that
\[ \mathcal{L}_{\eta^{(i)}} \eta^{(j)a} = \mathcal{L}_{\eta^{(i)}} (K^{(j)a}_b \xi^b) \]
\[ = (\mathcal{L}_{\eta^{(i)}} K^{(j)a}_b) \xi^b + K^{(j)a}_b (\mathcal{L}_{\eta^{(i)}} \xi^b) \]
\[ = 0. \tag{3.36} \]

This completes the proof of corollary.

4. Separation of variables in the Hamilton–Jacobi equation

A geometric characterization of the separation of variables in the geodesic Hamilton–Jacobi equation was given by Benenti–Francaviglia [7] and Kalnins–Miller [8]. Here, we use the following result in [8].

**Theorem.** Suppose there exists a \( N \)-dimensional vector space \( A \) of rank-2 Killing tensors on \( D \)-dimensional space \((M, g)\). Then the geodesic Hamilton–Jacobi equation has a separable coordinate system if and only if the following conditions hold1:

(i) \( [A, B]_S = 0 \) for each \( A, B \in A \).
(ii) There exist \((D - n)\)-independent simultaneous eigenvectors \( X^{(a)} \) for every \( A \in A \).
(iii) There exist \( n \)-independent commuting Killing vectors \( Y^{(\alpha)} \).
(iv) \( [A, Y^{(\alpha)}]_S = 0 \) for each \( A \in A \).
(v) \( N = (2D + n^2 - n)/2 \).
(vi) \( g(X^{(a)}, X^{(b)}) = 0 \) if \( 1 \leq a < b \leq D - n \), and \( g(X^{(a)}, Y^{(\alpha)}) = 0 \) for \( 1 \leq a \leq D - n \), \( D - n + 1 \leq \alpha \leq D \).

We assume that the Killing tensors \( K^{(j)} \) and \( K^{(ij)} = \eta^{(i)} \otimes \eta^{(j)} + \eta^{(j)} \otimes \eta^{(i)} \) given in section 2 form a basis for \( A \). Note that in the odd-dimensional case the last Killing Yano tensor \( f^{(k-1)} \) is a Killing vector, and hence the corresponding Killing tensor \( K^{(k-1)} \propto f^{(k-1)} f^{(k-1)} \) is reducible [5]. Then, it is easy to see that conditions (1)–(6) hold. Indeed, the relation \( K^{(ij)} K^{(k)} = K^{(j)} K^{(ik)} \) implies that there exist simultaneous eigenvectors \( X^{(a)} \) for \( K^{(ij)} \) satisfying conditions (2) and (6). Other conditions are direct consequences of theorem 1 and corollary.

5. Example

Finally, we describe the Kerr–NUT–de Sitter metric as an example, which was fully studied in [1–6, 13, 14]. The \( D \)-dimensional metric takes the form [13]:

1 We put \( n_2 = 0 \) for theorem 4 in [8]. This condition is satisfied in the case of a positive definite metric \( g \).
(a) $D = 2n$

$$g = \sum_{\mu=1}^{n} \frac{dx_{\mu}^2}{Q_{\mu}} + \sum_{\mu=1}^{n} Q_{\mu} \left( \sum_{k=0}^{n-1} A_{\mu}^{(k)} d \psi_k \right)^2. \tag{5.1}$$

(b) $D = 2n + 1$

$$g = \sum_{\mu=1}^{n} \frac{dx_{\mu}^2}{Q_{\mu}} + \sum_{\mu=1}^{n} Q_{\mu} \left( \sum_{k=0}^{n-1} A_{\mu}^{(k)} d \psi_k \right)^2 + S \left( \sum_{k=0}^{n} A_{\mu}^{(k)} d \psi_k \right)^2. \tag{5.2}$$

The functions $Q_{\mu}$ are given by

$$Q_{\mu} = \frac{X_{\mu}}{U_{\mu}}, \quad U_{\mu} = \prod_{1 \leq \nu < \mu \leq n} (x_{\mu}^2 - x_{\nu}^2), \tag{5.3}$$

where $X_{\mu}$ is a function depending only on $x_{\mu}$ and

$$A_{\mu}^{(k)} = \sum_{1 \leq v_1 < \cdots < v_k \leq n} \prod_{(i \neq \mu)}^{v_k} (x_{\mu}^2 - x_{v_1}^2) \quad A^{(k)} = \sum_{1 \leq v_1 < \cdots < v_k \leq n} x_{v_1}^2 x_{v_2}^2 \cdots x_{v_k}^2, \quad S = \frac{c}{A^{(n)}} \tag{5.4}$$

with a constant $c$. The CKY tensor is written as [2]

$$h = \frac{1}{2} \sum_{k=0}^{n-1} dA^{(k+1)} \wedge d \psi_k \tag{5.5}$$

with the associated vector $\xi = \partial / \partial \psi_0$. Assumptions (a1), (a2) and (a3) are clearly satisfied.

The commuting Killing tensors $K^{(j)}$ and Killing vectors $\eta^{(j)}$ are calculated as [2, 3]

$$K^{(j)} = \sum_{\mu=1}^{n} A_{\mu}^{(j)} (e^{\mu} e^{\mu} + e^{\mu+n} e^{\mu+n}) + \epsilon A^{(j)} e^{2n+1} e^{2n+1}, \tag{5.6}$$

$$\eta^{(j)} = \frac{\partial}{\partial \psi_j} \tag{5.7}$$

where $\epsilon = 0$ for $D = 2n$ and $1$ for $D = 2n + 1$. The 1-forms $\{e^{\mu}, e^{\mu+n}, e^{2n+1}\}$ are orthonormal bases defined by

$$e^{\mu} = \frac{dx_{\mu}}{\sqrt{Q_{\mu}}}, \quad e^{\mu+n} = \sqrt{Q_{\mu}} \left( \sum_{k=0}^{n-1} A_{\mu}^{(k)} d \psi_k \right), \quad e^{2n+1} = \sqrt{S} \left( \sum_{k=0}^{n} A_{\mu}^{(k)} d \psi_k \right). \tag{5.8}$$

Note added. In the successive paper [15], we found that a single CKY tensor satisfying assumptions (a1), (a2) and (a3) leads inevitably to the Kerr–NUT–de Sitter spacetime.

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Appendix A. Generating function of $K_{ab}^{(j)}$

In this appendix, we rederive the expression of the generating function of $K^{(j)}$ directly from the definition (2.13).

A.1. Auxiliary operators

It is convenient to introduce auxiliary fermionic creation/annihilation operators:

$$\bar{\psi}^a, \psi_a, \quad a = 1, 2, \ldots, D$$  \hspace{1cm} (A.1)

such that

$$\{\psi_a, \psi_b\} = 0, \quad \{\bar{\psi}^a, \bar{\psi}^b\} = 0, \quad \{\psi_a, \bar{\psi}^b\} = \delta^b_a.$$  \hspace{1cm} (A.2)

Also let

$$\bar{\psi}^a := g_{ab} \bar{\psi}^b, \quad \psi_a := g^{ab} \psi_b.$$  \hspace{1cm} (A.3)

The Fock vacuum is defined by

$$\psi_a |0\rangle = 0, \quad \langle 0 | \bar{\psi}^a = 0, \quad a = 1, 2, \ldots, D.$$  \hspace{1cm} (A.5)

with a normalization

$$\langle 0 | 0 \rangle = 1.$$  \hspace{1cm} (A.6)

With a 2-form $h$

$$h = \frac{1}{2} h_{ab} \, dx^a \wedge dx^b.$$  \hspace{1cm} (A.7)

let us associate the following operators:

$$h\bar{\psi} := \frac{1}{2} h_{ab} \bar{\psi}^a \bar{\psi}^b,$$  \hspace{1cm} (A.8)

$$h\psi := \frac{1}{2} h^{ab} \psi_a \psi_b.$$  \hspace{1cm} (A.9)

Note that

$$\langle h\bar{\psi} \rangle^j = \frac{1}{(2j)!} h^{(j)a_1\ldots a_{2j}} \bar{\psi}^{a_1} \ldots \bar{\psi}^{a_{2j}}.$$  \hspace{1cm} (A.10)

$$h^{(j)}_{a_1\ldots a_{2j}} = \langle 0 | \psi_{a_{2j}} \ldots \psi_{a_1} (h\bar{\psi})^j |0\rangle = (-1)^j \langle 0 | \psi_{a_1} \ldots \psi_{a_{2j}} (h\bar{\psi})^j |0\rangle.$$  \hspace{1cm} (A.11)

A.2. The generating function of $A^{(j)}$

Let

$$A^{(j)} := \frac{1}{(2j)! (j)!^2} \langle h^{(j)c_1\ldots c_{2j}} \rangle = \frac{(2j)!}{(2j)!^2} h^{[a_1b_1} \ldots h^{a_jb_j]} h_{[a_1b_1} \ldots h_{a_jb_j]}.$$  \hspace{1cm} (A.12)

$A^{(j)}$ is nontrivial for $j = 0, 1, \ldots, [D/2]$.

Note that

$$A^{(j)} = \frac{1}{(2j)! (j)!^2} \langle h^{(j)c_1\ldots c_{2j}} \rangle = \frac{1}{(2j)! (j)!^2} h^{(j)c_1\ldots c_{2j}} \times (-1)^j \langle 0 | \psi_{c_1} \ldots \psi_{c_{2j}} (h\bar{\psi})^j |0\rangle$$

$$= (-1)^j \langle 0 | \frac{(h\bar{\psi})^j (h\bar{\psi})^j}{j!} |0\rangle.$$  \hspace{1cm} (A.13)
Then we have

\[
\sum_{j=0}^{[D/2]} A^{(j)} \beta^j = \langle 0 | e^{-\sqrt{\beta} \psi} e^{\sqrt{\beta} \bar{\psi}} | 0 \rangle.
\]  \hspace{1cm} (A.14)

Let us introduce the vielbein

\[ g_{ab} = \delta_{ij} e^i_a e^j_b. \]  \hspace{1cm} (A.15)

(We assume the Euclidean signature.)

Let \( E \) be the matrix with elements

\[ E^i_a = e^i_a. \]  \hspace{1cm} (A.16)

Then

\[ H^{a b} = (E^{-1})^a_i \tilde{H}_{ij} E^j_b, \quad \tilde{H}_{ij} = -\tilde{H}_{ji}. \]  \hspace{1cm} (A.17)

Also let

\[ \theta^i = e^i_a \psi^a, \quad \bar{\theta}^i = e^i_a \bar{\psi}^a, \quad i = 1, 2, \ldots, D. \]  \hspace{1cm} (A.18)

Then we have \( \theta_i = \theta^i, \bar{\theta}_i = \bar{\theta}^i \), and

\[ \{\theta_i, \theta_j\} = 0, \quad \{\bar{\theta}_i, \bar{\theta}_j\} = 0, \quad \{\theta_i, \bar{\theta}_j\} = \delta_{ij}, \]  \hspace{1cm} (A.19)

for \( i, j = 1, 2, \ldots, D \). It is well known that any real antisymmetric matrix can be block diagonalized by some orthogonal matrix. Therefore, we can choose the vielbein such that \( \tilde{H} \) has a block diagonal form and

\[
h_\psi = \sum_{\mu=1}^n \lambda_\mu \theta_\mu \bar{\theta}_{n+\mu}, \quad h_{\bar{\psi}} = \sum_{\mu=1}^n \lambda_\mu \bar{\theta}_\mu \bar{\theta}_{n+\mu},
\]  \hspace{1cm} (A.20)

for \( n = [D/2] \). Here we assume that \( \lambda_\mu \neq 0 \). Note that

\[ EQE^{-1} = \text{diag}(\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2, \lambda_1^2, \lambda_2^2, \ldots). \]  \hspace{1cm} (A.21)

For odd \( D \), the last diagonal entry equals zero.

Then

\[
\langle 0 | e^{-\sqrt{\beta} \psi} e^{\sqrt{\beta} \bar{\psi}} | 0 \rangle = \prod_{\mu=1}^n (1 - \sqrt{\beta} \lambda_\mu \theta_\mu \bar{\theta}_{n+\mu})(1 + \sqrt{\beta} \lambda_\mu \bar{\theta}_\mu \bar{\theta}_{n+\mu})
\]

\[
= \prod_{\mu=1}^n (1 + \beta \lambda_\mu^2)
\]

\[
= \det^{1/2}(I + \beta Q).
\]  \hspace{1cm} (A.22)

Here \( I \) is the \( D \times D \) identity matrix.

We have the generating function of \( A^{(j)} \):

\[
\sum_{j=0}^{[D/2]} A^{(j)} \beta^j = \det^{1/2}(I + \beta Q) = \det(I + \sqrt{\beta} H) = \det(I - \sqrt{\beta} H). \]  \hspace{1cm} (A.23)
A.3. Recursion relations for $K^{(j)}$

The Levi-Civita tensor satisfies

$$\varepsilon^{a_1 \ldots a_r c_1 \ldots c_{D-r}} g_{b_1 \ldots b_r c_1 \ldots c_{D-r}} = r! (D-r)! \delta_{b_1 \ldots b_r}^{a_1 \ldots a_r}.$$  \hspace{1cm} (A.24)

Using (A.24), we can check that $K^{(j)}_{ab}$ has the following form:

$$K^{(j)}_{ab} = A^{(j)}_{ab} + \frac{1}{(2j-1)!(j-1)!} h^{(j)}_{a r_1 \ldots r_j} h^{(j)}_{b r_1 \ldots r_j}.$$  \hspace{1cm} (A.25)

Here $A^{(j)}$ is defined by (A.12).

It is possible to show that

$$\frac{1}{(2j-1)!(j-1)!} h^{(j)}_{a r_1 \ldots r_j} h^{(j)}_{b r_1 \ldots r_j} = h^{(j)}_{e r_1 \ldots r_j} K^{(j-1)}_{e b}.$$  \hspace{1cm} (A.26)

In the matrix notation, $K^{(j)}$ satisfies the following recursion relation:

$$K^{(j)} = A^{(j)} I + H K^{(j-1)} H.$$  \hspace{1cm} (A.27)

Therefore, we can see that $K^{(j)}$ commutes with $H$. Thus

$$K^{(j)} = A^{(j)} I - Q K^{(j-1)}.$$  \hspace{1cm} (A.28)

With the initial condition

$$K^{(0)} = I, \quad K^{(0)}_{ab} = g_{ab},$$

we easily find that

$$K^{(j)} = \sum_{l=0}^{j} (-1)^{j-l} A^{(j-l)} Q^l,$$  \hspace{1cm} (A.30)

or

$$K^{(j)}_{ab} = \sum_{l=0}^{j} (-1)^l A^{(j-l)} (Q^l)^{a} b.$$  \hspace{1cm} (A.31)

We immediately see that

$$K^{(j)} K^{(j)} = K^{(j)} K^{(j)}.$$  \hspace{1cm} (A.32)

Using (A.23), we can see that $K^{(k)} = 0$ for $k = [(D+1)/2]$. Indeed, by setting $\beta = -x^{-1}$,

$$\sum_{j=0}^{\lfloor D/2 \rfloor} (-1)^j A^{(j)} x^{-j} = \text{det}^{1/2}(I - x^{-1} Q) = x^{-D/2} \text{det}^{1/2}(x I - Q).$$  \hspace{1cm} (A.33)

For $D = 2k$,

$$\sum_{j=0}^{k} (-1)^{k-j} A^{(j)} x^{k-j} = (-1)^k \text{det}^{1/2}(x I - Q).$$  \hspace{1cm} (A.34)

If we set $x$ to be an eigenvalue of $Q$, the RHS becomes zero. Therefore, we can see that

$$K^{(k)} = \sum_{l=0}^{k} (-1)^l A^{(k-l)} Q^l = 0, \quad \text{for } D = 2k.$$  \hspace{1cm} (A.35)

Similarly, for $D = 2k - 1$,

$$\sum_{j=0}^{k-1} (-1)^{k-j} A^{(j)} x^{k-j} = (-1)^k x^{1/2} \text{det}^{1/2}(x I - Q).$$  \hspace{1cm} (A.36)
Thus
\[ K^{(k)} = \sum_{j=1}^{k} (-1)^j A^{(k-j)} Q^j = 0, \quad \text{for} \quad D = 2k - 1. \] (A.37)

Also note that \( A^{(j)} = 0 \) for \( j \geq \lfloor D/2 \rfloor + 1 \). Therefore the recursion relations (A.28) become trivial for \( j \geq k + 1 \) and \( K^{(j)} = 0 \) for \( j \geq k \). \( K^{(j)} \) can be written as (A.30) for all \( j \geq 0 \) but are nontrivial only for \( j = 0, 1, \ldots, k - 1 \).

Using (A.30) and (A.23), we can see that the generating function of \( K^{(j)} \) is
\[ K(\beta) := \sum_{j=0}^{k-1} K^{(j)} \beta^j = \det^{1/2}(I + \beta Q)(I + \beta Q)^{-1}. \] (A.38)

\subsection*{A.4. Proof of (A.26)}

The LHS of (A.26) is
\[ \frac{1}{(2j-1)!(j!)^2} h_{a_1 \ldots c_{2j-1}}^{(j)} h_{b_1 \ldots c_{2j-1}}^{(j)} \]
\[ = \frac{1}{(2j-1)!(j!)^2} h_{b_1 \ldots c_{2j-1}}^{(j)} \times (-1)^j (0|\psi_a \psi_{c_1} \ldots \psi_{c_{2j-1}} (h\psi)^j |0) \]
\[ = \frac{(-1)^{j-1}}{(2j-1)!(j!)^2} h_{b_1 \ldots c_{2j-1}}^{(j)} (0|\psi_{c_1} \ldots \psi_{c_{2j-1}} (h\psi)^j |0) \]
\[ = \frac{(-1)^{j-1}}{(2j)!(j!)^2} h_{b_1 \ldots c_{2j}}^{(j)} (0|\psi_{c_1} \ldots \psi_{c_{2j}} \psi_b \psi_a (h\psi)^j |0) \]
\[ = (-1)^{j-1} (0 \frac{(h\psi)^j}{j!} \psi_b \psi_a (h\psi)^j |0). \] (A.39)

Then
\[ K_{ab}^{(j)} = (-1)^j g_{ab} (0 \frac{(h\psi)^j}{j!} |0) - (-1)^j \frac{(h\psi)^j}{j!} \psi_b \psi_a (h\psi)^j |0) \]
\[ = (-1)^j (0 \frac{(h\psi)^j}{j!} [(\psi_a, \bar{\psi}_b) - \bar{\psi}_b \psi_a] (h\psi)^j |0) \]
\[ = (-1)^j (0 \frac{(h\psi)^j}{j!} \psi_a \bar{\psi}_b (h\psi)^j |0). \] (A.40)

Thus
\[ K_{ab}^{(j)} = (-1)^j (0 \frac{(h\psi)^j}{j!} \psi_a \bar{\psi}_b (h\psi)^j |0). \] (A.41)

Note that
\[ [\psi_a, h\psi] = h_{a\bar{a}} \bar{\psi}^\bar{a}, \] (A.42)
\[ \psi_a (h\psi)^j |0) = j h_a \bar{\psi}^\bar{a} (h\psi)^j |0), \] (A.43)
\[ [h\psi, \psi_b] = \psi_b h^b, \] (A.44)
\[ (0)(h\psi)^j \bar{\psi}_b = j (0)(h\psi)^j \psi_b h^b. \] (A.45)
Then

\[
\text{(LHS of (A.26))} = \frac{1}{(2j - 1)! (j!)^2} h_a^{(j)} \bar{h}_{ac_1 \ldots c_{2j-1}} h_b^{(j)c_1 \ldots c_{2j-1}} \bigr/
\]

\[
= (-1)^{j-1} (0) \frac{(h \psi)^j}{j!} \bar{\psi}_a (h \psi)^j j! (0)
\]

\[
= h_a^\mu (-1)^{j-1} (0) \frac{(h \psi)^j}{(j-1)!} \bar{\psi}_a \frac{(h \psi)^j}{(j-1)!} (0) h^\mu b
\]

\[
= h_a^\mu (j^{(j-1)}) h^\mu b
\]

\[
= \text{(RHS of (A.26))} \quad \text{(A.46)}
\]

This completes the proof of (A.26).

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