A Work-Efficient Parallel Algorithm for Longest Increasing Subsequence

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Abstract

This paper studies parallel algorithms for the longest increasing subsequence (LIS) problem. Let $n$ be the input size and $k$ be the LIS length of the input. Sequentially, LIS is a simple textbook problem that can be solved using dynamic programming (DP) in $O(n \log n)$ work. However, parallelizing LIS is a long-standing challenge. We are unaware of any parallel LIS algorithm that has optimal $O(n \log n)$ work and non-trivial parallelism (i.e., $\tilde{O}(k)$ or $o(n)$ span). Here, the work of a parallel algorithm is the total number of operations, and the span is the longest dependent instructions.

This paper proposes a parallel LIS algorithm that costs $O(n \log k)$ work, $\tilde{O}(k)$ span, and $O(n)$ space, and is much simpler than the previous parallel LIS algorithms. We also generalize the algorithm to a weighted version of LIS, which maximizes the weighted sum for all objects in an increasing subsequence. Our weighted LIS algorithm has $O(n \log^2 n)$ work and $\tilde{O}(k)$ span.

We also implemented our parallel LIS algorithms. Due to simplicity, our implementation is light-weighted, efficient, and scalable. On input size $10^9$, our LIS algorithm outperforms a highly-optimized sequential algorithm (with $O(n \log k)$ cost) on inputs with $k \leq 3 \times 10^5$. Our algorithm is also much faster than the best existing parallel implementation by Shen et al. on all input instances.

1 Introduction

Given a sequence $A_1..n$, the LIS of $A$ is the longest subsequence (not necessarily contiguous) in $A$ that is strictly increasing. In this paper, we use LIS to refer to both the longest increasing subsequence of a sequence, and the problem to find such an LIS. The LIS problem has extensive applications (e.g., [28, 43, 27, 59, 56]). In this paper, we use $n$ to denote the input size, and $k$ to denote the LIS length of the input. LIS can be solved by dynamic programming (DP) using the following DP recurrence [26] (more details in Section 2).

$$dp[i] = \max(1, \max_{j < i, A_j < A_i} dp[j] + 1)$$

Sequentially, LIS is a simple textbook problem. We can iteratively compute $dp[i]$ and maintain a search structure to find $\max_{j < i, A_j < A_i} dp[j]$, which gives $O(n \log n)$ work. However, in parallel, LIS becomes challenging both in theory and in practice. In theory, we are unaware of efficient parallel LIS algorithm that has $O(n \log n)$ work and non-trivial parallelism ($o(n)$ or $O(k)$ span). In practice, we are unaware of parallel LIS implementations that outperform the sequential algorithm on general input distributions. This paper studies efficient parallel LIS solutions, with the goal of achieving simplicity and efficiency both theoretically and practically.

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1 The work of a parallel algorithm is the total number of operations in the algorithm, and the span is the longest dependence of instructions. We formally define work and span in Section 2.
Our work follows some recent research [13, 16, 35, 44, 58, 63, 20, 19, 21, 41, 61, 15] that directly parallelizes sequential algorithms. Such algorithms are usually simple and practical, given their connections to existing sequential algorithms. To achieve high parallelism in a “sequential” algorithm, the key is to identify the dependences [63, 20, 21, 61] among the objects. In the DP recurrence of LIS, processing an object \( x \) depends on all objects \( y \) before it, but does not need to wait for objects before it but larger than it.

An “ideal” parallel algorithm should process all objects in the proper order based on the dependences — it should (1) process as many objects as possible in parallel (as long as they do not depend on each other), and (2) process an object only when it is ready (all objects it depends on are finished), to avoid redundant work. To formalize the two requirements, we say an algorithm is round-efficient [61] if its span is \( \tilde{O}(D) \) for a computation with longest logical dependence length \( D \). In LIS, the logical dependence length given by the DP recurrence is the LIS length \( k \). We say an algorithm is work-efficient if its work is asymptotically the same as the best sequential algorithm. Work-efficiency is important in practice, since nowadays, the number of processors in a multicore machine is roughly polylogarithmic to input sizes. Therefore, a parallel algorithm is less practical if its blows up the work of a sequential algorithm significantly.

Unfortunately, there exists no parallel LIS algorithm with both work-efficiency and round-efficiency. The difficulty is that the number of dependences can be \( \Theta(n^2) \) (Figure 1(a)), and a work-efficient solution cannot even afford to generate all dependences. Most existing parallel LIS algorithms introduced a polynomial overhead in work [38, 51, 60, 67, 54, 55], or have \( \Theta(n) \) span [4]. The only algorithm with \( \tilde{O}(n) \) work and \( o(n) \) span we know of is from Krusche and Tiskin (\( O(n \log^2 n) \) work and \( \tilde{O}(n^{2.5}) \) span), which relies on complicated techniques from [68], and has no implementation. We review more related work in Section 5.

Our algorithm is based on the phase-parallel framework by Shen et al. [61]. They also proposed a parallel LIS algorithm using this framework. We refer to it as the SWGS algorithm, and review it in Section 2. The phase-parallel framework defines a rank for each input object as the LIS length ending at it (the DP value in Eq. (1)). Note that an object only depends on lower-rank objects. Therefore, the phase-parallel LIS algorithm processes all objects based on the increasing order of rank. However, the SWGS algorithm [61] takes \( O(n \log^3 n) \) work whp, \( \tilde{O}(k) \) span, and \( O(n \log n) \) space, and is quite complicated. In the experiments, the overhead in work and space limits the performance. On input size \( 10^8 \), the SWGS algorithm becomes slower than a sequential algorithm when the LIS length grows to more than 100.

This paper proposes a deterministic parallel LIS algorithm that is work-efficient \( O(n \log k) \) work), round-efficient \( \tilde{O}(k) \) span) and space-efficient \( O(n) \) space), and is much simpler than previous parallel LIS algorithms [61, 52]. Note that the work of our algorithm is parameterized on the LIS length \( k \), so the work can be \( o(n \log n) \) for inputs with small LIS lengths. Our algorithm is also based on the phase-parallel framework, but uses a more efficient approach to identify all objects with a certain rank in parallel, based on a simple parallel tournament tree. Our main result is summarized in Theorem 1.1.

**Theorem 1.1** (Main Theorem). Given a sequence \( A \) of size \( n \) and LIS length \( k \), the longest increasing subsequence (LIS) of \( A \) can be computed in parallel with \( O(n \log k) \) work, \( O(k \log n) \) span, and \( O(n) \) space.

We also generalize our algorithm to the weighted LIS problem, which has a similar DP recurrence as LIS but maximizes the weighted sum for all objects in an increasing subsequence.

\[
dp[i] = w_i + \max(0, \max_{j<i} \forall A_j < A_i \ dp[j])
\]

(2)

where \( w_i \) is the weight of the \( i \)-th input object. By combining our new algorithm and the idea in the SWGS algorithm, we achieve a parallel algorithm that is round-efficient and has \( \tilde{O}(n) \) work.

**Theorem 1.2** (Weighted LIS). Given a sequence \( A \) of size \( n \) and LIS length \( k \), the weighted LIS of \( A \) can be computed in parallel with \( O(n \log^2 n) \) work, \( O(k \log n) \) span and \( O(n \log n) \) space.
Figure 1: Illustration of parallel LIS and the SWGS Parallel LIS algorithm. (a) An input instance where the number of dependences is $\Theta(n^2)$. (b) An example on an input instance, the rank of each object, and dependences between objects (grey arrows). The blue arrows show an example of object-pivot pairs used in the SWGS algorithm, which is a subset of the dependences. (c) An illustration that plots all objects in (b) as points $(i, A_i)$ on a 2D planar. Each object only depends on other objects in its lower-left corner.

Our algorithms are also simple to program, and we expect it to be the algorithm of choice in implementations in the parallel setting. We tested our algorithms on a 96-core machine. Our implementation is light-weighted, efficient and scalable. Our LIS algorithm outperforms SWGS in all tests, and is faster than highly-optimized sequential algorithms on reasonable LIS length (e.g., up to $k = 3 \times 10^3$ for $n = 10^5$). To the best of our knowledge, this is the first parallel LIS implementation that can outperform the efficient sequential algorithm in a large input parameter space. On the weighted version, our algorithm is up to 2.5x faster than SWGS and 7x faster than a highly-optimized sequential algorithm. We believe the performance is enabled by the simplicity and theoretical-efficiency of our new algorithms.

2 Preliminaries

Notation. We use $\tilde{O}(f(n))$ to denote $O(f(n) \cdot \polylog(n))$. We use $O(f(n))$ with high probability (whp) (in $n$) to mean $O(cf(n))$ with probability at least $1 - n^{-c}$ for $c \geq 1$. We use $\log n$ as a short form for $1 + \log_2(n+1)$. We use $A_i$ and $A[i]$ interchangeably to denote the $i$-th object in an array or a sequence. We use $A[i..j]$ or $A_{i..j}$ to indicate the $i$-th to the $j$-th element in an array (or sequence) $A$.

Computational Model. We use the work-span model in the classic multithreaded model with binary-forking [22, 5, 15]. We assume a set of threads that share the memory. Each thread acts like a sequential RAM plus a fork instruction that forks two new child threads running in parallel. When both child threads finish, the parent thread continues. In our pseudocode, we use $s_1 || s_2$ to mean the statements $s_1$ and $s_2$ can run in parallel. A parallel-for is simulated by fork for a logarithmic number of levels. A computation can be viewed as a DAG (directed acyclic graph). The work $W$ of a parallel algorithm is the total number of operations in this DAG, and the span (depth) $S$ is the longest path in the DAG. An algorithm is work-efficient if its work is asymptotically the same as the best sequential algorithm. The randomized work-stealing scheduler can execute such a computation in $W/P + O(S)$ time whp in $W$ [22, 5, 40].

Different from the PRAM [62] model, this model assumes loose synchronization and is a practical model supported by most existing libraries [57, 53, 45, 47, 23]. It is used in recent papers for shared-memory parallel algorithms (a short list: [2, 7, 1, 17, 14, 25, 31, 6, 21, 30, 29, 39, 3]). Our algorithms can also be analyzed on PRAM and have the same work and span bounds.

Longest Increasing Subsequence (LIS). Given a sequence $A_{1..n}$ of $n$ input objects, $A'_{1..m}$ is a subsequence of $A$ if $A'_i = A_{s_i}$, where $1 \leq s_1 < s_2 < \ldots s_m \leq n$. The longest increasing subsequence (LIS) of $A$ is the longest subsequence $A^*$ of $A$ where $\forall i < n, A'_i < A^*_{i+1}$. Throughout the paper, we use $n$ to denote the input size, and $k$ to denote the LIS length of the input.
LIS can be solved using dynamic programming (DP) with the DP recurrence in Eq. (1). Here \( dp[i] \) (called the DP value of object \( i \)) is the LIS of \( A_{1:i} \) ending with \( A_i \). Following the terminology in DP algorithms, we call each \( dp[i] \) a state. When we attempt to compute state \( i \) from state \( j < i \), we say \( j \) is a decision for state \( i \). We say \( j \) is the best decision of the state \( i \) if \( j = \arg \max_{j < i, \langle A_j < A_i \rangle} dp[j] \).

The LIS problem generalizes to the weighted LIS problem with DP recurrence in Eq. (2). Sequentially, both LIS and weight LIS can be solved in \( O(n \log n) \) work. This is also the lower bound \cite{36} w.r.t. the number of comparisons. For (unweighted) LIS, there exists an \( O(n \log k) \) sequential algorithm \cite{50}.

**Data Structures.** We use two data structures: the tournament tree and the range tree. A tournament tree \( T \) on \( n \) records is a complete binary tree with \( 2n - 1 \) nodes (see Figure 3). It can be represented implicitly as an array \( T[1..(2n - 1)] \). The last \( n \) elements are the leaves, where \( T[i] \) stores the \((i - n + 1)\)-th record in the dataset. The first \( n - 1 \) elements are internal nodes, each storing the minimum value of its two children. The left and right children of \( T[i] \) are \( T[2i] \) and \( T[2i + 1] \), respectively. We will use a parallel tournament tree to maintain a dynamic prefix-min structure. We will use the following theorem about the tournament tree.

**Theorem 2.1.** (Parallel Tournament Trees \cite{32, 15}) A tournament tree can be constructed from \( n \) elements in \( O(n) \) work and \( O(\log n) \) span. Given a set \( S \) of \( m \) leaves in the tournament tree with size \( n \), the number of ancestors of all the nodes in \( S \) is \( O(m \log(n/m)) \).

Implementing a parallel tournament tree is simple. For instance, if we want to construct a tournament tree, we can simply recursively construct the left and right trees in parallel, and update the root value.

We use the parallel range tree \cite{8, 64} to answer 2D range-max queries. For a set of points \((x_i, y_i)\) with weight \( w_i \) on the 2D plane, the range-max query \((p, q)\) asks for the maximum weight among all points in its lower-left corner \((-\infty, p) \times (-\infty, q)\). A range tree \cite{8} is a two-level binary search tree where the outer tree is an index of the \( x \)-coordinates of the points. Each tree node maintains an inner tree storing the same set of points in its subtree, but keyed on the \( y \)-coordinates \cite{12, 11}. We let each tree node store the maximum weight in its subtree to answer range-max queries. We will use the following theorem about range trees.

**Theorem 2.2.** (Parallel Range Trees \cite{65, 64, 15}) Given a set of \( n \) points \((x_i, y_i)\) with weight \( w_i \) on a 2D plane, there exists a data structure that supports the following operations with \( O(n \log n) \) space:

- constructing the data structure in \( O(n \log n) \) work and \( O(\log^2 n) \) span,
- answering the 2D range-max query in \( O(\log^2 n) \) work and span, and
- updating the weights of a set of \( m \leq n \) points in \( O(m \log^2 n) \) work and \( O(\log^2 n) \) span.

**Dependence Graph \cite{63, 20, 21, 61}.** In a sequential iterative algorithm, we can analyze the logical dependencies between iterations (objects) to achieve parallelism. Such dependences can be represented in a directly acyclic graph (DAG), called a dependence graph (DG). In a DG, each vertex is an object in the algorithm. An edge from \( u \) to \( v \) means that \( v \) can be processed only when \( u \) has been finished. We say \( u \) depends on \( v \) in this case. Figure 1(b) illustrates the dependences in LIS. We say an object is ready when all its predecessors have finished. To execute a DG with depth \( D \), we say an algorithm is round-efficient\(^2\) if its span is \( \tilde{O}(D) \). In LIS, the dependence depth given by the DP recurrence is \( k \), the LIS length.

**Phase-Parallel Algorithms and the SWGS Algorithm.** The high-level idea of the phase-parallel algorithm is to assign each object \( x \) a rank, noted as \( \text{rank}(x) \), which indicates the earliest phase when the

\(^2\) We note that round-efficiency does not guarantee optimal span, since round-efficiency is with respect to a given DG. One can re-design a completely different algorithm with a shallower DG and get a better span.
object can be processed. For simplicity, here we only explain the concepts based on the LIS problem. In LIS, the rank of each object is the LIS length ending at it (the DP value computed by Eq. (1)). We also define the rank of a sequence \( A \) as the LIS length of \( A \). An object only depends on other objects with lower ranks. Therefore, the phase-parallel LIS algorithm (Algorithm 1) processes all objects with LIS length \( i \) (in parallel) in round \( i \). We call the objects processed in round \( i \) (\( F_i \) in Algorithm 1) the frontier of this round. Figure 1(b) and (c) present examples of the ranks in the LIS problem.

**Algorithm 1**: The framework of phase-parallel LIS algorithm

1. \( i \leftarrow 1 \)
2. \( \text{while } A \neq \emptyset \) do  
   \( \quad F_i \leftarrow \{x \in A : \text{the LIS ending at } x \text{ has length } i \} \)
   \( \quad \text{Process all } x \in F_i \text{ in parallel (compute the DP values) and set them as finished.} \)
   \( \quad A \leftarrow A \setminus F_i \)
   \( \quad i \leftarrow i + 1 \)

The SWGS algorithm uses a wake-up scheme to find the frontiers in LIS. Each object \( A_i \) is viewed as a 2D point \((i, A_i)\). An object only depends on objects in its lower-left corner (Figure 1(c)). Hence, the readiness of an object \( x \) can be checked using a range tree in \( O(\log^2 n) \) cost. To find the ready objects, each object \( x \) is attached to pivot \( p_x \), which is a random unfinished ancestor (Figure 1(b)). An object \( x \) is “waked up” (checking its readiness) only when \( p_x \) is finished. If \( x \) is not ready, we assign a new pivot to \( x \) and let it sleep again. The number of wake-ups per object can be bounded by \( O(\log n) \) whp. The algorithm has \( O(n \log^3 n) \) work whp, \( O(k \log^2 n) \) span, and \( O(n \log n) \) space (due to the use of the range tree). Our algorithm is also based on the phase-parallel framework but overcomes the overhead in SWGS, and is also much simpler.

### 3 Our Algorithms

#### 3.1 Longest Increasing Subsequence

We start with the (unweighted) LIS problem. Our algorithm is also based on the phase-parallel framework [61] but uses a much simpler idea to make it work-efficient. The work overhead in the SWGS algorithm comes from two aspects: range queries on the range tree, and the wake-up scheme. The \( O(\log n) \) space overhead comes from the range tree. Therefore, we want to 1) use a more efficient (and simpler) data structure than the range tree to reduce both work and space, and 2) wake up and process an object only when it is ready to avoid the wake-up scheme. We start with defining the prefix-min object in a sequence.

**Definition 3.1** (Prefix-min Objects). Given a sequence \( A_{1..n} \), we say \( A_i \) is a prefix-min object if for all \( j < i \), we have \( A_i \leq A_j \), i.e., \( A_i \) is (one of) the smallest object among \( A_{1..i} \).

Our algorithm is based on the observation in Lemma 3.1, which is simple and intuitive. Recall that the rank of an object \( A_i \) is exactly its DP value, which is the length of LIS ending at \( A_i \).

**Lemma 3.1**. In a sequence \( A \), an object \( A_i \) has rank 1 iff. \( A_i \) is a prefix-min object. An object \( A_i \) has rank \( r \) iff. after removing all objects with ranks smaller than \( r \), \( A_i \) is a prefix-min object.

**Proof**. We will prove the theorem inductively.

We first show that the base case is true. We start with the “if” direction. For a prefix-min object \( A_i \), \( A_i \) is the smallest object among \( A_{1..i} \). Thus, there exists no \( A_j \) such that \( a_j < a_i, j < i \). Based on Eq. (1), \( dp[i] = 1 \).
Algorithm 2: The parallel (unweighted) LIS algorithm

```
Input: A sequence $A_{1..n}$ with comparison function $<$
Output: All DP values (ranks) of $A_{1..n}$

1: int rank[1..n]  // rank[i]: the LIS length ending at $A_i$
2: int $r$  // $r$ is a global variable denoting the current round.
3: $r$ ← 0
4: Initialize the tree $T$
5: Function LIS(sequence $A_{1..n}$)
6:   while $T[1] ≠ +∞$ do  // The tournament tree is not empty.
7:      $r ← r + 1$
8:      ProcessFrontier()  // Find all $A_j$ s.t. it is the smallest object among $A_{1..j}$. Set rank[j] = $r$ and remove $A_j$.
9:   return rank[1..n]

10: Function ProcessFrontier()
11:   PrefixMin(1, +∞)

12: Function PrefixMin(int $i$, int LMin)
13:   if $T[i] > LMin$ then return  // Skip if the min of the subtree is larger than LMin.
14:      if $i ≥ n$ then  // Found a leaf node in the frontier.
15:         dp[i] ← $r$
16:         $T[i] ← +∞$  // Set its rank as $r$.
17:      else // Deal with an internal node. Recurse on both children. Pass the min value of the left subtree to the right subtree.
18:         PrefixMin(2i, LMin) || PrefixMin(2i + 1, min(LMin, $T[2i]$))  // The two recursive calls run in parallel.
19:         $T[i] ← \min(T[2i], T[2i + 1])$  // After the recursive calls, update the value.
```

For the “only-if” direction, note that if $dp[i] = 1$, the LIS ending at $A_i$ has length 1. Assume to the contrary that there exists $j < i$ such that $A_j < A_i$. Then the LIS ending at $A_i$ is at least 2, which contradicts the assumption. Therefore, $dp[i] = 1$ also indicates that $A_i$ is the smallest element among $A_{1..i}$.

Assume for all $r < t$, Lemma 3.1 is true. We will prove that the lemma is true for $r = t$. We first show the “if” direction. Based on the inductive hypothesis, after removing all objects with rank smaller than $t$, a (remaining) object $A_i$ must have $\text{rank}(A_i) ≥ t$. Since $A_i$ is the smallest object among all remaining objects in $A_{1..i}$, all objects in $A_{1..i}$ smaller than $A_i$ must have been removed and thus have rank at most $t - 1$. From Eq. (1), $\text{rank}(A_i) ≤ t - 1 + 1 = t$. Therefore, $\text{rank}(A_i) = t$.

For the “only-if” direction, note that if $dp[i] = t$, $A_i$ has rank $t$ and must be remaining after removing objects with ranks smaller than $t$. We will then prove that $A_i$ is a prefix-min object. Let $S = \{A_j : A_j < A_i, j < i\}$. We first show that $\text{rank}(x) < t$ for all $x ∈ S$. This is because $A_i$ depends on all objects in $S$—if any object $x ∈ S$ has $\text{rank}(x) ≥ t$, $A_i$ must have rank at least $t + 1$. This means that all objects in $S$ must have been removed. In this case, $A_i$ must be no larger than all remaining objects before it. Therefore, $dp[i] = t$ indicates that $A_i$ is a prefix-min object after removing all objects with rank smaller than $t$.

Based on Lemma 3.1, we can design a simple phase-parallel algorithm for LIS (Algorithm 2). For simplicity, we first focus on computing the DP values (ranks) of all objects in $A_{1..n}$, which also gives the LIS length of the input. Later we show how to output a specific LIS of the input sequence. The main loop of Algorithm 2 is in Lines 6–8. In round $r$, we identify the frontier $F_r$ as all the prefix-min objects. Their DP values will be set as $r$. We then remove all the objects in $F_r$ and repeat. Figure 2 illustrates this algorithm by showing a “prefix-min” value $pre_i$ for each object, which is the smallest value up to each object. Note that
Round 1
- **Input**: 52 31 45 26 61 10 39 44
- **Objects** $A_i$: 52 31 45 26 61 10 39 44
- **Prefix-Min $pre_i$**: 52 31 31 26 26 10 10 10

Round 2
- **Input**: 52 31 45 26 61 10 39 44
- **Objects** $A_i$: x x 45 x 61 x 39 44
- **Prefix-Min $pre_i$**: x x 45 x 45 x 39 39

Round 3
- **Input**: 52 31 45 26 61 10 39 44
- **Objects** $A_i$: x x x x 61 x x 44
- **Prefix Min**: x x x x 61 x x 44

**DP values (LIS length)**: 1 1 2 1 3 1 2 3

$pre_i$: smallest value up to this object (inclusive)

**Rank**:
- Rank=1 ($F_1$)
- Rank=2 ($F_2$)
- Rank=3 ($F_3$)

Figure 2: An illustration of Algorithm 2. The figure also shows $pre_i$ for each object, which is the smallest object up to this object (inclusive). If $A_i = pre_i$, it is a prefix-min object. In round $r$, Algorithm 2 finds all prefix-min objects, sets their DP values as $r$, removes them, and updates the $pre_i$ values. The objects identified in round $r$ all have rank $r$ (i.e., the LIS length ending at this object is $r$).

This sequence $pre_i$ is not maintained in our algorithm, but is just used here for illustration. In each round, we find all objects $A_i$ such that $A_i = pre_i$. Then we remove these values, update the prefix-min values $pre_i$, and repeat. In round $r$, all identified prefix-min objects have rank $r$.

To achieve work-efficiency, we cannot re-compute the prefix-min values of the entire subsequence after each round. In Algorithm 2, we use a tournament tree $T$ to efficiently identify the frontier and dynamically remove objects. The tournament tree stores all input objects in the leaves. We always round up the number of leaves to a power of 2 to make it a full binary search tree. Each internal node stores the minimum value among all values in its subtree. When we traverse the tree $T[i]$, note that if the smallest object on its left is smaller than the value at $T[i]$, we can skip the entire subtree. Using the internal nodes, we can maintain the minimum value of the prefix before any subtree and skip irrelevant subtrees to save work.

In particular, we can identify the frontier of each round in parallel using a divide-and-conquer algorithm. The function ProcessFrontier traverses the tournament tree $T$ by calling PrefixMin starting at the root. The function PrefixMin$(i, LMin)$ traverses the subtree at node $i$, and finds all leaves $v$ in this subtree s.t. 1) $v$ is no more than any leaves before $v$ in this subtree, and 2) $v$ is no more than $LMin$. The argument $LMin$ records the smallest value in $T$ before the subtree at $T[i]$. If the smallest value in subtree $T[i]$ is larger than $LMin$, we can skip the entire subtree (Line 13), because no object in this subtree can be a prefix-min object (they are all larger than $LMin$). Otherwise, there are two cases. The first case is when $T[i]$ is a leaf (Lines 14–16). Since its value is no more than $LMin$, it must be a prefix-min object. Therefore, we set its DP value as the current round number $r$ (Line 15) and remove it by setting its value as $+\infty$ (Line 16). The second case is when this node is an internal node (Lines 17–19), we can recurse on both subtrees to find the desired objects (Line 18). For the left subtree, we directly use the $LMin$ value from the current function invocation. For the right subtree, we need to further consider the minimum value in the left subtree. Therefore, we take the minimum of the current $LMin$ and the smallest value in the left subtree ($T[2i]$), and then set it as the $LMin$ value of the right recursive call. These two recursive calls can be executed in parallel. After both recursive calls return, we update the value at $T[i]$ (Line 19) because some values in the subtree may have been removed (set to $+\infty$). We present an example in Figure 3, which illustrates finding the first frontier for the input in Figure 2.

The correctness of Algorithm 2 is straightforward based on Lemma 3.1. We now prove the cost of Algorithm 2 in Theorem 3.1.

**Theorem 3.1.** Algorithm 2 computes the LIS of the input sequence $A$ in $O(n \log k)$ work and $O(k \log n)$ span, where $n$ is the length of the input sequence $A$, and $k$ is the LIS length of $A$. 

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Figure 3: The parallel tournament tree used in Algorithm 2. The leaf nodes store the input sequence \((A_1, n)\). Each internal node stores the minimum value in its subtree. The figure illustrates finding the first frontier in the example in Figure 2. The algorithm recursively traverses the tree from the root (two subtrees are visited in parallel), and maintains a \(LMin\) value for each subtree, which is the smallest value before this subtree. If the \(LMin\) value is smaller than the value stored at the subtree root, the entire subtree is skipped. For example, at the yellow node with value 39, the smallest value \(LMin\) before it is 10. This means that no leaves in this subtree can be a prefix-min object, so this subtree is skipped. Blue nodes are relevant nodes, which contain at least one leaf in the frontier. Yellow nodes are not relevant but still visited, which are the children of some relevant nodes. Grey nodes are not visited.

Proof. Constructing the tournament tree takes \(O(n)\) work and \(O(\log n)\) span. We then focus on the main loop (Lines 6-8) of the algorithm. For the span, note that the algorithm runs in \(k\) rounds. In each round, executing \textsc{ProcessFrontier} means to recurse \(O(\log n)\) steps. Therefore, the span of the algorithm is \(O(k \log n)\). Next we show that the function \textsc{ProcessFrontier} in round \(r\) takes \(O(m_r \log (n/m_r))\) work, where \(m_r\) is frontier size of this round, i.e., the number of prefix-min objects identified in this round.

First note that visiting a tournament tree node has a constant cost. Therefore, the work is asymptotically the number of nodes visited in the algorithm. We say a node is relevant if at least one object in its subtree is in the frontier. Based on Theorem 2.1, there are \(O(m_r \log (n/m_r))\) relevant nodes.

If Line 14 is executed (i.e., Line 13 does not return), the smallest object in this subtree is no more than \(LMin\) and must be a prefix-min object, and this node is relevant. There are also other nodes visited but skipped by Line 13. Executing Line 13 for subtree \(i\) means that \(i\)'s parent executed Line 14 and went to Line 17, so \(i\)'s parent is relevant. This indicates that a node is visited either because it is relevant, or its parent is relevant. Since every node has at most two children, the number of visited nodes is asymptotically the same as all relevant nodes. which is \(O(m_r \log (n/m_r))\). Hence, the total number of visited nodes is:

\[
\sum_{r=1}^{k} m_r \log(n/m_r)
\]

\[
\leq \sum_{i=1}^{k} \frac{n/k}{\log(n/k)} \quad \text{(the function } f(x) = 1 + x \log_2(1 + n/x) \text{ is concave)}
\]

\[= n \log k
\]

This proves the work bound of the algorithm. \(\square\)

Note that the work bound of Theorem 3.1 is parameterized on the LIS length \(k\). For small \(k\), the work can be \(o(n \log n)\). For example, if the input sequence is strictly decreasing, Algorithm 2 only needs \(O(n)\) work because the algorithm will find all objects in the first round in \(O(n)\) work and finishes.
Reporting an LIS of the Input Sequence

To report a specific LIS of the input sequence, we will slightly modify Algorithm 2 to compute the best decision \(d[i]\) (see Section 2) for each object \(i\). Namely, \(A_{d[i]}\) is \(A_i\)'s previous object in the LIS ending at \(A_i\). Then starting from an object with the largest rank, we can iteratively find an LIS in \(O(k)\) work and span. Our idea is based on a simple observation:

Lemma 3.2. For an object \(A_i\) with rank \(r\), let \(A_{d[i]}\) be the smallest object with rank \(r - 1\) before \(A_i\), then \(d[i]\) is the best decision for state \(i\), i.e., \(d[i] = \arg \max_{j : j < L A_j < A_i} \text{dp}[j]\).

Proof. Since \(\text{rank}(A_i) = r\) and \(\text{rank}(A_{d[i]}) = r - 1\), clearly \(\text{dp}[d[i]] = \max_{j : j < L A_j < A_i} \text{dp}[j]\). We just need to show that \(A_{d[i]} < A_i\), such that it is a candidate of \(A_i\)'s previous object in the LIS. Assume to the contrary that \(A_{d[i]} \geq A_i\). Note that \(A_{d[i]}\) is the smallest object with rank \(r - 1\) before \(A_i\). Therefore, for any \(A_j\) where \(\text{rank}(A_j) = r - 1\) and \(j < i\), we have \(A_j \geq A_{d[i]} \geq A_i\). Hence, \(A_i\) cannot obtain a DP value of \(r\), which leads to a contradiction.

We then show how to identify the best decision \(d[i]\) for each \(A_i\). First of all, when executing PROCESS-FRONTIER in round \(r\), we can also output all objects of rank \(r\) into an array in parallel. This can be performed by traversing \(T\) twice, similar to the function PREFIXMIN. In the first traversal, we mark all prefix-min objects to be extracted in the frontier. On the way back of the recursive calls, we also compute the number of such prefix-min objects in each subtree. We call this the effective size of this subtree. The effective size at the root is exactly the frontier size \(m_r\), and we can allocate an array \(F_r[1..m_r]\) for the frontier. Then we traverse the tree once again. We recursively put objects in the left and right subtrees into \(F_r[\cdot]\) in parallel. Let the effective size of the left subtree be \(s\). Then the right tree can be processed in parallel to put objects in \(F_r\) from the \(s + 1\)-th slot. We then show that the objects in each frontier \(F_r\) are non-increasing.

Lemma 3.3. Given a sequence \(A\) and any integer \(r\), let \(F_r\) be the subsequence of \(A\) with all objects with rank \(r\). Then \(F_r\) is non-increasing for all \(r\).

Proof. Assume to the contrary that there exist \(A_i\) and \(A_j\), s.t. \(\text{rank}(A_i) = \text{rank}(A_j), i < j, A_i < A_j\). This means that we can add \(A_j\) after \(A_i\) in an LIS, so \(\text{dp}[j]\) is at least \(\text{dp}[i] + 1\). This leads to a contradiction since \(A_i\) and \(A_j\) have the same rank (DP values). Therefore, each frontier \(F_r\) is non-increasing.

Based on Lemma 3.3, the smallest object with rank \(r - 1\) before \(A_i\) is also the last object with rank \(r - 1\) before \(A_i\). Therefore, after we find the frontier \(F_r\), we can merge \(F_r\) with \(F_{r-1}\) based on the index, such that each object in \(F_r\) can find the last object before it with rank \(r - 1\). Using a parallel merge algorithm [46], this part takes \(O(\log n)\) span in each round and \(O(n)\) total work in the entire algorithm.

3.2 Weighted Longest Increasing Subsequence

The nice property of the (unweighted) LIS problem is that the DP value is the same as its rank. Hence in round \(r\), we just set the DP values of all objects in the frontier as \(r\). This is not true for the weighted LIS problem, so we need some additional techniques. It is not too hard to get an algorithm by combining our Algorithm 2 with the SWGS algorithm using range trees. We present the pseudocode in Algorithm 3. The high-level idea is the same as Algorithm 2. We also build a range tree (see Section 2) as in SWGS, which views each object in the input as a point \((i, A_i)\). In round \(r\), we also use the tournament tree \(T\) to find all prefix-min objects. Once such an object \(T[i]\) is identified (Line 21–24), we compute its DP value by using the range-max query on the range tree \(T_{range}\). After that, we mark this object \(T[i]\) as “updated” in the range tree (along all relevant paths) and removed it from the tournament tree. After traversing and processing
Algorithm 3: The parallel weighted LIS algorithm

Input: A sequence $A_{1..n}$ with weight comparison function $\prec$. Object $i$ has weight $w[i]$
Output: The DP values $dp[1..n]$ for each object $A_i$.

```
1 Struct Point
2   int x, y  // $x =$ index, $y = A_x$.
3   int dp    // The DP value of $A_x$ is the weight of this point.
4 Point p[1..n]
5 int dp[1..n] // dp[i]: the DP value of $A_i$.
6 Struct RangeTree(Point)
7   Contains 2D Points $(x_i, y_i)$ with weights
8   Supports RangeMax($p, q$) query: returns the maximum weight among all points $(x_i, y_i)$ where $x_i < p$ and $y_i < q$
9 RangeTree $T_{\text{range}}$

// $T$: the (implicit) tournament tree. The same as Algorithm 2.
10 int $T[1..(2n-1)]$
11 parallel_for_each $A_i \in A$ do $p[i] = (i, A_i, 0)$
12 Initialize the tournament tree $T$
13 Construct $T_{\text{range}}$ from $p[\cdot]$
14 while $T[1] \neq +\infty$ do // The tournament tree is not empty.
15   // Find all $A_j$ s.t. $A_j$ is a prefix-min object. Compute $dp[j]$ from the range tree $T_{\text{range}}$, and remove $A_j$.
16   ProcessFrontier()
17   Update all DP values by $dp[\cdot]$ in the range tree $T_{\text{range}}$ marked as “updated”, then clear the mark
18 Function ProcessFrontier()
19   // Deal with subtree rooted at $T[i]$. Find objects s.t.: 1) it is no more than any object before it, and 2) it is no more than LMin.
20   // Then compute the DP values of such objects from the range tree.
21 Function PrefixMin(int i, int LMin)
22   if $T[i] > LMin$ then return // Skip if the min of the subtree is larger than LMin
23   if $i \geq n$ then // Found a leaf node in the frontier.
24       $dp[i] \leftarrow T_{\text{range}}.\text{RangeMax}(i-n+1, A_{i-n+1}) + w[i-n+1]$ // Compute the DP value from a 2D range-max.
25       Mark object $i-n+1$ as “updated” in the range tree
26       $T[i] \leftarrow +\infty$ // Remove the object from the tournament tree.
27   else // Deal with an internal node. Recurse on both children. Pass the min value of the left subtree to the right subtree.
28       PrefixMin(2i, LMin) || PrefixMin(2i + 1, min(LMin, $T[2i]$)) // The two recursive calls run in parallel.
29       $T[i] \leftarrow \min(T[2i], T[2i+1])$ // After the recursive calls, update the value.
```

the frontier, we traverse the range tree and follow the “update” marks to update the newly-computed DP values for all objects in the current frontier. It is also easy to output a specific increasing subsequence with the highest weight. The best decision at any object $i$ can be returned by the range tree along with the maximum DP value in its lower-left corner. Then, using the best decisions of each object, we can iteratively output the optimal subsequence.

A range-max query in a range tree costs $O(log^2 n)$ work and span. Updating $m$ elements in the range tree in parallel takes $O(m log^2 n)$ work and $O(log^2 n)$ span. This gives the following theorem.

**Theorem 3.2.** Algorithm 3 computes the weighted LIS of an input sequence $A$ in $O(n log^2 n)$ work and $O(k log^2 n)$ span, where $n$ is the length of the input sequence $A$, and $k$ is the LIS length of $A$. 


4 Experiments

In addition to the new theoretical bounds, we also implement the parallel algorithms for both the LIS and weighted LIS problems. Our code is very simple and light-weighted. We release our code on GitHub [42]. We use the experimental results to examine our theoretical bounds, and show how the simplicity and theoretical efficiency of our new algorithms improve the performance over the existing results.

Experimental Setup. We run all experiments on a 96-core (192-hyperthread) machine equipped with four-way Intel Xeon Gold 6252 CPUs and 1.5 TiB of main memory. Our implementation is in C++ with ParlayLib [10]. All reported numbers are the averages of the last three runs among four repeated tests.

Input Data. We tested input size $n = 10^8$ and $n = 10^9$ with different ranks (LIS length $k$). We generate inputs with a certain pattern to vary the rank. In the input, we set $A_i = t \cdot i + s_i$, where $s_i$ is an independent random variable choosing from a uniform distribution. By changing $t$ and the distribution of $s_i$, we can control the input ranks (LIS length). We use random weights for the weighted LIS problem.

Baseline Algorithms. We compare our algorithm to standard sequential LIS algorithms, and the existing parallel LIS implementation from SWGS [61]. We also show the running time of our algorithm on one core to indicate the work of the algorithm. SWGS works on both weighted and unweighted LIS problems with $O(n \log^2 n)$ work and $\tilde{O}(k)$ span, and we compare both of our algorithms (Algorithms 2 and 3) with it.

For the LIS problem, we compare our algorithm to a highly-optimized sequential implementation of the classic algorithm from [50], and call it Seq-BS. Seq-BS maintains an array $B$, where $B[r]$ is the smallest value of $A_i$ with rank $r$. Note that $B$ is monotonically increasing. Iterating $i$ from 1 to $n$, we binary search $A_i$ in $B$, and if $B[r] < A_i \leq B[r+1]$, we set $dp[i]$ as $r+1$. By the definition of $B[\cdot]$, we then update the value $B[r+1]$ to $A_i$ if $A_i$ is smaller than the current value in $B[r+1]$. The size of $B$ is at most $k$, and thus this algorithm has work $O(n \log k)$. This algorithm only works on the unweighted LIS problem.

For weighted LIS, we implement a tree-based algorithm, and call it Seq-AVL. This algorithm maintains an augmented search tree, which stores all input objects ordered by their values, and supports querying the range-max of DP values. Iterating $i$ from 1 to $n$, we simply query the maximum DP value in the tree among all objects with values less than $A_i$, and update $dp[i]$. We then insert $A_i$ (with $dp[i]$) to the tree and continue to the next object. This algorithm takes $O(n \log n)$ work, and we implement it with an AVL tree.

Our algorithms are always faster than the existing parallel implementation SWGS due to better work and span bounds. Our algorithms also outperform highly-optimized sequential algorithms up to reasonably large ranks (e.g., up to $k = 3 \times 10^5$ for $n = 10^9$). We note that the expected LIS length of a random sequence is $2\sqrt{n}$ (when $n \to \infty$) [48]. For our tests on $10^8$ and $10^9$ input sizes, our algorithm outperforms the sequential algorithm on ranks from 1 to large than $2\sqrt{n}$. We believe this is the first parallel LIS implementation that can outperform the efficient sequential algorithm in a large input parameter space.

4.1 Longest Increasing Subsequence (LIS)

Figure 4(a) shows the results on input size $n = 10^8$ with ranks (LIS lengths) from 1 to $10^6$. For our algorithm and Seq-BS, the running time first increases with $k$ gets larger because both algorithms have work $O(n \log k)$. When $k$ is sufficiently large, the running time then slightly goes down. This is because a large rank brings up better cache locality as each object is likely to extend its LIS from an object close to it. Our parallel algorithm is faster than the sequential algorithm for $k \leq 3 \times 10^4$ and gets slower afterwards. The slowdown comes from the lack of parallelism (the span bound is proportional to $k$). We note that our algorithm running on one core is only 3-10x slower than Seq-BS because of work-efficiency. With sufficient parallelism (e.g., on low-rank inputs), our performance is better than Seq-BS by up to 7x.
Figure 4: Experimental results on the LIS and weighted LIS. We vary the output size for each test. “Ours”= our LIS algorithm in Algorithm 2 running on all 96 cores. “Ours (seq)”= our LIS algorithm in Algorithm 2 running on one core. “Ours-W”=our weighted LIS algorithm in Algorithm 3 running on all 96 cores. “Seq-BS”= the sequential Seq-BS algorithm based on binary search. “Seq-AVL”= the sequential Seq-AVL algorithm based on the AVL tree. “SWGS”= the parallel algorithm SWGS from [61]. See more details in Section 4.

We only test SWGS on ranks up to $10^4$ because it costs too much time for larger ranks. On the existing experiment results, our new algorithm is always faster than SWGS (up to 188x) because of better work and span. We believe the simplicity in code also contributes to the improvement.

We also evaluate our algorithm on input size $n = 10^9$ and vary the ranks from 1 to $10^8$. We show the performance in Figure 4(b). We exclude SWGS in the comparison due to its space-inefficiency, since it ran out of memory to construct the range tree on $10^9$ elements. For $k \leq 3 \times 10^5$, our algorithm is consistently faster than Seq-BS (up to 9.1x). When the rank is large, the work in each round is not sufficient to get good parallelism, and the algorithm behaves as if it runs sequentially. Because of work-efficiency, even with large ranks, the performance of algorithm is comparable to Seq-BS (at most 3.4x slower).

Overall, our LIS algorithm performs well with reasonable ranks. Our algorithm obtains up to 41x self-speedup with $n = 10^8$ and up to 70x self-speedup with $n = 10^9$. Due to work-efficiency, our algorithm is scalable, and should perform especially well on large data because larger input sizes result in more work to better utilization parallelism.

4.2 Weighted LIS

We compare our weighted LIS algorithm (Algorithm 3) with SWGS and Seq-AVL on input size $n = 10^8$. We vary the rank from 1 to 3000, and show the results in Figure 4(c). Our algorithm is always faster than SWGS (up to 2.5x). The improvement comes from avoiding waking-up each object multiple times, because our algorithm directly finds each frontier efficiently and does not need the wake-up scheme.

Our algorithm also outperforms the sequential algorithm Seq-AVL with ranks up to 100. We note that the running time of the sequential algorithm decreases with increasing the LIS length $k$ because of better locality, while our algorithm performs worse with increasing $k$ because of larger span.

The results also imply the importance of work-efficiency in practice. To get better performance, we believe an interesting direction is to design a work-efficient parallel algorithm for the weighted LIS problem.
5 Related Work

LIS is widely-studied both sequentially and in parallel. LIS is in the category of sparse dynamic programming [33, 34, 37] and various sequential algorithms have been proposed [71, 9, 36, 50, 59, 27]. Sequentially, LIS can be solved in $O(n \log n)$ work. This is also the lower bound [36] w.r.t. the number of comparisons. In the parallel setting, LIS is studied both as general dynamic programming [18, 24, 66, 38] or on its own [51, 60, 67, 54, 55, 4, 68]. However, we are unaware of any work-efficient LIS algorithm with non-trivial parallelism ($o(n)$ or $O(k)$ span). Most existing parallel LIS algorithms introduced a polynomial overhead in work [38, 51, 60, 67, 54, 55], and/or have $\tilde{O}(n)$ span [4, 18, 24, 66] (many of them [18, 24, 66] focused on improving the I/O bound of LIS and related problems). The algorithm in [52] translates to $O(n \log^2 n)$ work and $\tilde{O}(n^{2/3})$ span, but it relies on complicated techniques of Monge Matrices [68]. Most of the parallel LIS algorithms are quite complicated and have no implementations. In fact, we are unaware of any parallel LIS implementation with competitive performance to the sequential $O(n \log k)$ or $O(n \log n)$ algorithm.

Many previous papers also proposed general methodologies to study dependences in sequential iterative algorithms to achieve parallelism [15, 20, 13, 61]. Their common idea is to (implicitly or explicitly) traverse the dependence graph. There are two major approaches, and both have led to many efficient algorithms. The first one is edge-centric [20, 19, 15, 21, 49, 44, 16, 35, 61], which identifies the ready objects by processing the successors of the newly-finished objects. The second approach is vertex-centric [63, 13, 58, 69, 61], which checks all unfinished objects in each round to process the ready ones. However, none of these frameworks directly guarantees work-efficiency for parallel LIS. The edge-centric algorithms need to evaluate all edges in the dependence graph, which has $\Theta(n^2)$ cost in the worst case for LIS. The vertex-centric algorithms need to check the readiness of all remaining objects in each round and require $k$ rounds, which means $\Omega(nk)$ work for LIS. The SWGS algorithm [61] combines the ideas in edge-centric and vertex-centric algorithms. This algorithm has $O(n \log^2 n)$ work whp and is round-efficient ($\tilde{O}(k)$ span) using $O(n \log n)$ space. It is sub-optimal in both work and space. Our algorithm improves the work and space bounds of the SWGS LIS algorithm, and also improves the work bound in weighted LIS. Our algorithm is also simpler and performs much better than SWGS in practice.

6 Conclusion

In this paper, we present the first work-efficient parallel algorithm for the longest-increasing subsequence (LIS) problem that has non-trivial parallelism ($\tilde{O}(k)$ span for an input sequence with LIS length $k$). Our algorithm is much simpler than existing parallel LIS algorithms. Theoretical efficiency and simplicity also enable a practical implementation with good performance. Some interesting future directions include achieving work-efficiency and good performance for the weighted LIS problem in parallel, and designing a work-efficient parallel LIS algorithm with $o(n)$ or even polylogarithmic span.

References

[1] Umut A. Acar, Guy E. Blelloch, and Robert D. Blumofe. The data locality of work stealing. *Theoretical Computer Science (TCS)*, 35(3), 2002.

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3Note that if the input only contains integers in a small range, the work can be optimized to $O(n \log \log n)$ using van Emde Boas trees [70]. In this paper, we focus on general inputs with no assumptions on the input type.
[2] Kunal Agrawal, Jeremy T. Fineman, Kefu Lu, Brendan Sheridan, Jim Sukha, and Robert Utterback. Provably good scheduling for parallel programs that use data structures through implicit batching. In *ACM Symposium on Parallelism in Algorithms and Architectures (SPAA)*, 2014.

[3] Zafar Ahmad, Rezaul Chowdhury, Rathish Das, Pramod Ganapathi, Aaron Gregory, and Mohammad Mahdi Javanmard. Low-span parallel algorithms for the binary-forking model. In *ACM Symposium on Parallelism in Algorithms and Architectures (SPAA)*, pages 22–34, 2021.

[4] Muhammad Rashed Alam and M Sohel Rahman. A divide and conquer approach and a work-optimal parallel algorithm for the lis problem. *Information Processing Letters*, 113(13):470–476, 2013.

[5] Nimar S Arora, Robert D Blumofe, and C Greg Plaxton. Thread scheduling for multiprogrammed multiprocessors. *Theory of Computing Systems (TOCS)*, 34(2):115–144, 2001.

[6] Naama Ben-David, Guy E. Blelloch, Jeremy T Fineman, Phillip B Gibbons, Yan Gu, Charles McGuffey, and Julian Shun. Implicit decomposition for write-efficient connectivity algorithms. In *IEEE International Parallel and Distributed Processing Symposium (IPDPS)*, 2018.

[7] Michael A Bender, Jeremy T Fineman, Seth Gilbert, and Charles E Leiserson. On-the-fly maintenance of series-parallel relationships in fork-join multithreaded programs. In *ACM Symposium on Parallelism in Algorithms and Architectures (SPAA)*, pages 133–144, 2004.

[8] Jon Louis Bentley and Jerome H Friedman. Data structures for range searching. *ACM Computing Surveys*, 11(4):397–409, 1979.

[9] Sergei Bespamyatnikh and Michael Segal. Enumerating longest increasing subsequences and patience sorting. *Information Processing Letters*, 76(1-2):7–11, 2000.

[10] Guy E. Blelloch, Daniel Anderson, and Laxman Dhulipala. Parlaylib—a toolkit for parallel algorithms on shared-memory multicore machines. In *ACM Symposium on Parallelism in Algorithms and Architectures (SPAA)*, pages 507–509, 2020.

[11] Guy E. Blelloch, Daniel Ferizovic, and Yihan Sun. Just join for parallel ordered sets. In *ACM Symposium on Parallelism in Algorithms and Architectures (SPAA)*, 2016.

[12] Guy E. Blelloch, Daniel Ferizovic, and Yihan Sun. Joinable parallel balanced binary trees. *ACM Transactions on Parallel Computing*, 9(2):1–41, 2022.

[13] Guy E. Blelloch, Jeremy T. Fineman, Phillip B. Gibbons, and Julian Shun. Internally deterministic parallel algorithms can be fast. In *ACM Symposium on Principles and Practice of Parallel Programming (PPOPP)*, 2012.

[14] Guy E. Blelloch, Jeremy T. Fineman, Phillip B. Gibbons, and Harsha Vardhan Simhadri. Scheduling irregular parallel computations on hierarchical caches. In *ACM Symposium on Parallelism in Algorithms and Architectures (SPAA)*, 2011.

[15] Guy E. Blelloch, Jeremy T. Fineman, Yan Gu, and Yihan Sun. Optimal parallel algorithms in the binary-forking model. In *ACM Symposium on Parallelism in Algorithms and Architectures (SPAA)*, 2020.

[16] Guy E. Blelloch, Jeremy T Fineman, and Julian Shun. Greedy sequential maximal independent set and matching are parallel on average. In *ACM Symposium on Parallelism in Algorithms and Architectures (SPAA)*, 2012.

[17] Guy E. Blelloch, Phillip B. Gibbons, and Harsha Vardhan Simhadri. Low depth cache-oblivious algorithms. In *ACM Symposium on Parallelism in Algorithms and Architectures (SPAA)*, 2010.

[18] Guy E. Blelloch and Yan Gu. Improved parallel cache-oblivious algorithms for dynamic programming. In *SIAM Symposium on Algorithmic Principles of Computer Systems (APOCS)*, 2020.

[19] Guy E. Blelloch, Yan Gu, Julian Shun, and Yihan Sun. Parallel write-efficient algorithms and data structures for computational geometry. In *ACM Symposium on Parallelism in Algorithms and Architectures (SPAA)*, 2018.
[20] Guy E. Blelloch, Yan Gu, Julian Shun, and Yihan Sun. Parallelism in randomized incremental algorithms. *J. ACM*, 2020.

[21] Guy E. Blelloch, Yan Gu, Julian Shun, and Yihan Sun. Randomized incremental convex hull is highly parallel. In *ACM Symposium on Parallelism in Algorithms and Architectures (SPAA)*, 2020.

[22] Robert D. Blumofe and Charles E. Leiserson. Space-efficient scheduling of multithreaded computations. *SIAM J. on Computing*, 27(1), 1998.

[23] Philippe Charles, Christian Grothoff, Vijay A. Saraswat, Christopher Donawa, Allan Kielstra, Kemal Ebcioglu, Christoph von Praun, and Vivek Sarkar. X10: an object-oriented approach to non-uniform cluster computing. In *Symposium on Object-oriented Programming, Systems, Languages and Applications (OOPSLA)*, pages 519–538, 2005.

[24] Rezaul A. Chowdhury and Vijaya Ramachandran. Cache-efficient dynamic programming algorithms for multicore. In *ACM Symposium on Parallelism in Algorithms and Architectures (SPAA)*. ACM, 2008.

[25] Richard Cole and Vijaya Ramachandran. Resource oblivious sorting on multicore. ACM *Transactions on Parallel Computing (TOPC)*, 3(4), 2017.

[26] Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein. *Introduction to Algorithms (3rd edition)*. MIT Press, 2009.

[27] Maxime Crochemore and Ely Porat. Fast computation of a longest increasing subsequence and application. *Information and Computation*, 208(9):1054–1059, 2010.

[28] Arthur L Delcher, Simon Kasif, Robert D Fleischmann, Jeremy Peterson, Owen White, and Steven L Salzberg. Alignment of whole genomes. *Nucleic acids research*, 27(11):2369–2376, 1999.

[29] Laxman Dhulipala, Guy E Blelloch, Yan Gu, and Yihan Sun. Pac-trees: Supporting parallel and compressed purely-functional collections. In *ACM Conference on Programming Language Design and Implementation (PLDI)*, 2022.

[30] Laxman Dhulipala, Guy E Blelloch, and Julian Shun. Low-latency graph streaming using compressed purely-functional trees. In *ACM Conference on Programming Language Design and Implementation (PLDI)*, pages 918–934, 2019.

[31] Laxman Dhulipala, Charlie McGuffey, Hongbo Kang, Yan Gu, Guy E Blelloch, Phillip B Gibbons, and Julian Shun. Semi-asymmetric parallel graph algorithms for NVRAMs. *Proceedings of the VLDB Endowment (PVLDB)*, 13(9), 2020.

[32] Xiaojun Dong, Yan Gu, Yihan Sun, and Yunming Zhang. Efficient stepping algorithms and implementations for parallel shortest paths. In *ACM Symposium on Parallelism in Algorithms and Architectures (SPAA)*, 2021.

[33] David Eppstein, Zvi Galil, Raffaele Giancarlo, and Giuseppe F Italiano. Sparse dynamic programming i: linear cost functions. *J. ACM*, 39(3):519–545, 1992.

[34] David Eppstein, Zvi Galil, Raffaele Giancarlo, and Giuseppe F Italiano. Sparse dynamic programming ii: convex and concave cost functions. *J. ACM*, 39(3):546–567, 1992.

[35] Manuela Fischer and Andreas Noever. Tight analysis of parallel randomized greedy mis. In *ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 2152–2160, 2018.

[36] Michael L Fredman. On computing the length of longest increasing subsequences. *Discrete Mathematics*, 11(1):29–35, 1975.

[37] Zvi Galil and Kunsoo Park. Dynamic programming with convexity, concavity and sparsity. *Theoretical Computer Science (TCS)*, 92(1), 1992.

[38] Zvi Galil and Kunsoo Park. Parallel algorithms for dynamic programming recurrences with more than O(1) dependency. *J. Parallel Distrib. Comput.*, 21(2), 1994.
[39] Michael Goodrich, Riko Jacob, and Nodari Sitchinava. Atomic power in forks: A super-logarithmic lower bound for implementing butterfly networks in the nonatomic binary fork-join model. In *ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 2141–2153. SIAM, 2021.

[40] Yan Gu, Zachary Napier, and Yihan Sun. Analysis of work-stealing and parallel cache complexity. In *SIAM Symposium on Algorithmic Principles of Computer Systems (APOCS)*, pages 46–60. SIAM, 2022.

[41] Yan Gu, Zachary Napier, Yihan Sun, and Letong Wang. Parallel cover trees and their applications. In *ACM Symposium on Parallelism in Algorithms and Architectures (SPAA)*, pages 259–272, 2022.

[42] Yan Gu, Zheqi Shen, Yihan Sun, and Zijin Wan. Work-efficient parallel implementations on longest increasing subsequence. https://github.com/ucrparlay/nLogn-LIS, 2022.

[43] Dan Gusfield. Algorithms on strings, trees, and sequences: Computer science and computational biology. *ACM Sigact News*, 28(4):41–60, 1997.

[44] William Hasenplaugh, Tim Kaler, Tao B. Schardl, and Charles E. Leiserson. Ordering heuristics for parallel graph coloring. In *ACM Symposium on Parallelism in Algorithms and Architectures (SPAA)*, 2014.

[45] https://www.threadingbuildingblocks.org.

[46] Joseph JáJá. *Introduction to Parallel Algorithms*. Addison-Wesley Professional, 1992.

[47] http://docs.oracle.com/javase/tutorial/essential/concurrency/forkjoin.html.

[48] Kurt Johansson. The longest increasing subsequence in a random permutation and a unitary random matrix model. *Mathematical Research Letters*, 5(1):68–82, 1998.

[49] Mark T Jones and Paul E Plassmann. A parallel graph coloring heuristic. 14(3):654–669, 1993.

[50] Donald E. Knuth. *The Art of Computer Programming, Volume III: Sorting and Searching*. Addison-Wesley, 1973.

[51] Peter Krusche and Alexander Tiskin. Parallel longest increasing subsequences in scalable time and memory. In *International Conference on Parallel Processing and Applied Mathematics*, pages 176–185. Springer, 2009.

[52] Peter Krusche and Alexander Tiskin. New algorithms for efficient parallel string comparison. In *ACM Symposium on Parallelism in Algorithms and Architectures (SPAA)*, pages 209–216, 2010.

[53] Charles E. Leiserson. Cilk. In David A. Padua, editor, *Encyclopedia of Parallel Computing*. Springer, 2011.

[54] Takaaki Nakashima and Akihiro Fujiwara. Parallel algorithms for patience sorting and longest increasing subsequence. In *International Conference in Networks, Parallel and Distributed Processing and Applications*, pages 7–12, 2002.

[55] Takaaki Nakashima and Akihiro Fujiwara. A cost optimal parallel algorithm for patience sorting. *Parallel processing letters*, 16(01):39–51, 2006.

[56] Ryan O’Donnell and John Wright. A primer on the statistics of longest increasing subsequences and quantum states. *SIGACT News*.

[57] http://www.openmp.org.

[58] Xinghao Pan, Dimitris Papailiopoulos, Samet Oymak, Benjamin Recht, Kannan Ramchandran, and Michael I Jordan. Parallel correlation clustering on big graphs. In *Advances in Neural Information Processing Systems (NIPS)*, pages 82–90, 2015.

[59] Craige Schensted. Longest increasing and decreasing subsequences. *Canadian Journal of Mathematics*, 13:179–191, 1961.

[60] David Semé. A cgm algorithm solving the longest increasing subsequence problem. In *International Conference on Computational Science and Its Applications*, pages 10–21. Springer, 2006.
Many sequential iterative algorithms can be parallel and (nearly) work-efficient. In ACM Symposium on Parallelism in Algorithms and Architectures (SPAA), 2022.

Yossi Shiloach and Uzi Vishkin. Finding the maximum, merging, and sorting in a parallel computation model. J. Algorithms, 2(1), 1981.

Julian Shun, Yan Gu, Guy E. Blelloch, Jeremy T Fineman, and Phillip B Gibbons. Sequential random permutation, list contraction and tree contraction are highly parallel. In ACM-SIAM Symposium on Discrete Algorithms (SODA), pages 431–448, 2015.

Yihan Sun and Guy E Blelloch. Parallel range, segment and rectangle queries with augmented maps. In SIAM Symposium on Algorithm Engineering and Experiments (ALENEX), pages 159–173, 2019.

Yihan Sun, Daniel Ferizovic, and Guy E Blelloch. Pam: Parallel augmented maps. In ACM Symposium on Principles and Practice of Parallel Programming (ACM), 2018.

Yuan Tang, Ronghui You, Haibin Kan, Jesmin Jahan Tithi, Pramod Ganapathi, and Rezaul A Chowdhury. Cache-oblivious wavefront: improving parallelism of recursive dynamic programming algorithms without losing cache-efficiency. In ACM Symposium on Principles and Practice of Parallel Programming (PPOPP), 2015.

Garcia Thierry, Myoupo Jean-Frédéric, and Semé David. A work-optimal cgm algorithm for the lis problem. In ACM Symposium on Parallelism in Algorithms and Architectures (SPAA), pages 330–331, 2001.

Alexander Tiskin. Fast distance multiplication of unit-monge matrices. Algorithmica, 71(4):859–888, 2015.

Daniel Tomkins, Timmie Smith, Nancy M Amato, and Lawrence Rauchwerger. Sccmulti: an improved parallel strongly connected components algorithm. ACM Symposium on Principles and Practice of Parallel Programming (PPOPP), 49(8):393–394, 2014.

Peter van Emde Boas. Preserving order in a forest in less than logarithmic time and linear space. Information Processing Letters, 6(3):80–82, 1977.

I-Hsuan Yang, Chien-Pin Huang, and Kun-Mao Chao. A fast algorithm for computing a longest common increasing subsequence. Information Processing Letters, 93(5):249–253, 2005.