1 Introduction

There is a deep and well studied relation between the geometry of the classical hyperbolic space and the Möbius geometry of its boundary at infinity. This relation can be generalized in a nice way to CAT($-1$) spaces.

Let $X$ be a CAT($-1$) space with boundary $Z = \partial X$. For every basepoint $o \in X$ one can define the Bourdon metric $\rho_o(x,y) = e^{-\langle x\vert y \rangle_o}$ on $Z$, where $\langle \cdot \vert \cdot \rangle_o$ is the Gromov product with respect to $o$, compare [B1]. For different basepoints $o, o' \in X$ the metrics $\rho_o, \rho_{o'}$ are Möbius equivalent and thus define a Möbius structure on $Z$. By [FS1] this Möbius structure is ptolemaic.

On the other hand, examples show that not every ptolemaic Möbius structure arises as boundary of a CAT($-1$) space. In this paper we enlarge the class of CAT($-1$) spaces in a way that this larger class corresponds exactly to the spaces which have a ptolemaic Möbius structure at infinity.

Definition 1.1. A metric space is called asymptotically PT$-1$, if there exists some $\delta > 0$ such that for all quadruples $x_1, x_2, x_3, x_4 \in X$ we have

$$e^{\frac{1}{2}(\rho_{1,3}+\rho_{2,4})} \leq e^{\frac{1}{2}(\rho_{1,2}+\rho_{3,4})} + e^{\frac{1}{2}(\rho_{1,4}+\rho_{2,3})} + \delta e^\frac{1}{2}\rho,$$

where $\rho_{i,j} = d(x_i, x_j)$ and $\rho = \max_{i,j} \rho_{i,j}$.

We discuss this curvature condition in more detail later and compare it in section 3 with the asymptotically CAT($-1$) condition, which is formulated in more familiar comparison terms. It turns out that CAT($-1$) are asymptotically PT$-1$ and that the relation between these spaces and the Möbius geometry of their boundaries can be expressed in the following two results:
Theorem 1.2. Let $X$ be asymptotically $PT_{-1}$, then $X$ is a boundary continuous Gromov hyperbolic space. For every basepoint $o \in X$, $\rho_o(x, y) = e^{-(x,y)_o}$ defines a metric on $\partial_{\infty}X$. For different basepoints these metrics are Möbius equivalent and thus define a canonical Möbius structure $M$ on $\partial_{\infty}X$. The Möbius structure $M$ is complete and ptolemaic.

Theorem 1.3. Let $(Z, M)$ be a complete and ptolemaic Möbius space. Then there exists an asymptotically $PT_{-1}$ space $X$ such that $\partial_{\infty}X$ with its canonical Möbius structure is Möbius equivalent to $(Z, M)$.

The results can be viewed as a characterization of the class of Gromov hyperbolic spaces, whose boundary allows a canonical Möbius structure.

In the proof we use a hyperbolic cone construction due to Bonk and Schramm [BoS], which associates to a metric space $(Z, d)$ a cone $(\text{Con}(Z), \rho)$. We show in Proposition 4.1 that if $(Z, d)$ is ptolemaic, then the cone is asymptotically $PT_{-1}$. This method can also be used obtain a characterization of visual Gromov hyperbolic spaces in the spirit of the [BoS]. Recall that two metric spaces $X$ and $Y$ are roughly similar, if there are constants $K, \lambda > 0$ and a map $f : X \to Y$ such that for all $x, y \in X$ \[ |\lambda d_X(x, y) - d_Y(f(x), f(y))| \leq K \] and in additional $\sup_{y \in Y} d_Y(y, f(X)) \leq K$.

A theorem of Bonk and Schramm states that a visual Gromov hyperbolic space with doubling boundary is roughly similar to a convex subset of the real hyperbolic space $H^n$ for some integer $n$.

We have a version without conditions on the boundary:

Theorem 1.4. Every visual Gromov hyperbolic space is roughly similar to some asymptotically $PT_{-1}$ space.

We discuss some open questions in Remark 3.2. The structure of the paper is as follows. In section 2 we recall the basic facts about metric Möbius geometry and boundary continuous Gromov hyperbolic spaces. In section 3 we introduce the $PT_\kappa$ property, discuss asymptotically $PT_{-1}$ spaces and prove Theorem 1.2. In section 4 we introduce hyperbolic cones and prove Theorem 1.3. The proof of Theorem 1.4 is in section 5.

2 Preliminaries

2.1 Möbius Structures

Let $Z$ be a set which contains at least two points. An extended metric on $Z$ is a map $d : Z \times Z \to [0, \infty]$, such that there exists a set $\Omega(d) \subset Z$ with cardinality $#\Omega(d) \in \{0, 1\}$, such that $d$ restricted to the set $Z \setminus \Omega(d)$ is a metric (taking only values in $[0, \infty)$) and such that $d(z, \omega) = \infty$ for all
If $\Omega(d)$ is not empty, we call the unique $\omega \in \Omega(d)$ simply the point at infinity of $(Z,d)$. We write $Z_\omega$ for the set $Z \setminus \{\omega\}$.

The topology considered on $(Z,d)$ is the topology with the basis consisting of all open distance balls $B_r(z)$ around points in $z \in Z_\omega$ and the complements $D^C$ of all closed distance balls $D = \overline{B}_r(z)$.

We call an extended metric space $(Z,d)$ complete, if first every Cauchy sequence in $Z_\omega$ converges and secondly if the infinitely remote point $\omega$ exists in case that $Z_\omega$ is unbounded. For example the real line $(\mathbb{R},d)$, with its standard metric is not complete (as extended metric space), while $(\mathbb{R} \cup \{\infty\},d)$ is complete.

We say that a quadruple $(x,y,z,w) \in Z^4$ is admissible, if no entry occurs three or four times in the quadruple. We denote with $Q \subset Z^4$ the set of admissible quadruples. We define the cross ratio triple as the map $\text{crt} : Q \to \Sigma \subset \mathbb{R}P^2$ which maps admissible quadruples to points in the real projective plane defined by

$$
\text{crt}(x,y,z,w) = (d(x,y)d(z,w) : d(x,z)d(y,w) : d(x,w)d(y,z)),
$$

here $\Sigma$ is the subset of points $(a : b : c) \in \mathbb{R}P^2$, where all entries $a,b,c$ are nonnegative or all entries are non-positive.

We use the standard conventions for the calculation with $\infty$. If $\infty$ occurs once in $Q$, say $w = \infty$, then $\text{crt}(x,y,z,\infty) = (d(x,y) : d(x,z) : d(y,z))$. If $\infty$ occurs twice , say $z = w = \infty$ then $\text{crt}(x,y,\infty,\infty) = (0 : 1 : 1)$.

Similar as for the classical cross ratio there are six possible definitions by permuting the entries and we choose the above one.

A map $f : Z \to Z'$ between two extended metric spaces is called M"{o}bius, if $f$ is injective and for all admissible quadruples $(x,y,z,w)$ of $X$,

$$
\text{crt}(f(x),f(y),f(z),f(w)) = \text{crt}(x,y,z,w).
$$

M"{o}bius maps are continuous.

Two extended metric spaces $(Z,d)$ and $(Z,d')$ are M"{o}bius equivalent, if there exists a bijective M"{o}bius map $f : Z \to Z$. In this case also $f^{-1}$ is a M"{o}bius map and $f$ is in particular a homeomorphism.

We say that two extended metrics $d$ and $d'$ on the same set $Z$ are M"{o}bius equivalent, if the identity map $\text{id} : (Z,d) \to (Z,d')$ is a M"{o}bius map. M"{o}bius equivalent metrics define the same topology on $Z$. It is also not difficult to check that for M"{o}bius equivalent metrics $d$ and $d'$ the space $(Z,d)$ is complete if and only if $(Z,d')$ is complete.

The M"{o}bius equivalence of metrics of metrics on a given set $Z$ is clearly an equivalence relation. A M"{o}bius structure $\mathcal{M}$ on $Z$ is an equivalence class of extended metrics on $Z$. 

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A pair \((Z, M)\) of a set \(Z\) together with a Möbius structure \(M\) on \(Z\) is called a Möbius space. A Möbius structure well defines a topology on \(Z\), thus a Möbius space is in particular a topological space. Since completeness is also a Möbius invariant we can speak about complete Möbius structures.

In general two metrics in \(M\) can look very different. However if two metrics have the same remote point at infinity, then they are homothetic (see \([FS2]\)):

**Lemma 2.1.** Let \(M\) be a Möbius structure on a set \(X\), and let \(d, d' \in M\), such that \(\omega \in X\) is the remote point of \(d\) and of \(d'\). Then there exists \(\lambda > 0\), such that \(d'(x, y) = \lambda d(x, y)\) for all \(x, y \in X\).

An extended metric space \((Z, d)\) is called a Ptolemy space, if for all quadruples of points \(\{x, y, z, w\} \in Z^4\) the Ptolemy inequality holds

\[
d(x, y) d(z, w) \leq d(x, z) d(y, w) + d(x, w) d(y, z)
\]

We can reformulate this condition in terms of the cross ratio triple. Let \(\Delta \subset \Sigma\) be the set of points \((a : b : c) \in \Sigma\), such that the entries \(a, b, c\) satisfy the triangle inequality. This is obviously well defined.

Then an extended space is Ptolemy, if \(\text{crt}(x, y, z, w) \in \Delta\) for all allowed quadruples \(Q\).

This description shows that the Ptolemy property is Möbius invariant and thus a property of the Möbius structure \(M\).

The importance of the Ptolemy property comes from the following fact (see e.g. \([FS2]\)):

**Theorem 2.2.** A Möbius structure \(M\) on a set \(Z\) is Ptolemy, if and only if for all \(\omega \in Z\) there exists \(d_\omega \in M\) with \(\Omega(d_\omega) = \{\omega\}\).

The metric \(d_\omega\) can be obtained by metric involution. If \(d\) is a metric on \(Z\) then

\[
d_\omega(z, z') = \frac{d(z, z')}{d(\omega, z)d(\omega, z')}
\]
gives the required metric.

### 2.2 Boundary continuous Gromov hyperbolic spaces

We recall some basic facts from the theory of Gromov hyperbolic spaces, compare e.g. \([BS]\).

A metric space \((X, d)\) is called Gromov hyperbolic if there exists some \(\delta > 0\) such that for all quadruples \(x_1, x_2, x_3, x_4 \in X\) we have

\[
\rho_{1,3} + \rho_{2,4} \leq \max\{\rho_{1,2} + \rho_{3,4}, \rho_{1,4} + \rho_{2,3}\} + \delta,
\]

where \(\rho_{i,j} = d(x_i, x_j)\).
For three points \(x, y, z \in X\) one defines the Gromov product

\[
(x|y)_z = \frac{1}{2}(|zx| + |zy| - |xy|),
\]

where we write \(|xy|\) as a short version of \(d(x, y)\).

A sequence \((x_i)\) converges at infinity, if for some (and hence every) basepoint \(o \in X\) we have \(\lim_{i<j \to \infty} (x_i|x_j)_o = \infty\). Two such sequences \((x_i), (y_i)\) are called equivalent, if \(\lim (x_i|y_i)_o = \infty\). The boundary \(\partial_\infty X\) consist of the equivalence classes of these sequences.

For two points \(\zeta, \xi \in \partial_\infty X\) and a base point \(o \in X\) one defines

\[
(\zeta|\xi)_o = \inf \liminf_{i \to \infty} (x_i|y_i)_o \tag{1}
\]

where the infimum is taken over all sequences \((x_i) \in \zeta\) and \((y_i) \in \xi\). In a similar way we also define \((x|\xi)_o\), where \(o, x \in X\) and \(\xi \in \partial_\infty X\).

We remark that the sequence \((x_i|y_i)_o\) does not necessarily converge, therefore we need the complicated definition in (1).

A Gromov hyperbolic space is called boundary continuous, if the Gromov product extends continuously to the boundary in the following way: if \((x_i), (y_i)\) are sequences in \(X\) which converge to points \(x, y\) in \(X\) or \(\partial_\infty X\), then \((x_i|y_i)_o \to (x|y)_o\) for all base points \(o \in X\). For boundary continuous spaces one can define nicely Busemann functions. If \(\omega \in \partial_\infty X\) and \(o \in X\) a base point, then

\[
b_{\omega,o}(x) = \lim_{i \to \infty} (|xw_i| - |w_i|_o) \tag{2}
\]

where \(w_i \to \omega\) is the Busemannfunction of \(\omega\) normalized to have the value 0 at the point \(o \in X\). We have the formula:

\[
b_{\omega,o}(x) = (\omega|o)_x - (\omega|x)_o \tag{3}
\]

We also define form \(\omega \in \partial_\infty X\) a base point \(o \in X\) and \(x, y, z\) from \(X\) or \(\partial_\infty X \setminus \{\omega\}\)

\[
(x|y)_\omega,o = (x|y)_o - (\omega|x)_o - (\omega|y)_o.
\]

### 3 Asymptotic PT\(_\kappa\) spaces

In subsection 3.1 we define general PT\(_\kappa\) spaces. Then in section 3.2 we introduce the more general notion of asymptotic PT\(_\kappa\) spaces and compare it in section 3.3 with the notion of asymptotically CAT(\(\kappa\)) spaces. Finally in section 3.4 we prove Theorem 1.2.
3.1 The $PT_\kappa$ inequality

A metric space $(X, d)$ is called a $PT_\kappa$-space, if for points $x_1, x_2, x_3, x_4 \in X$, we have
\[
\text{sn}_\kappa\left(\frac{\rho_{1,3}}{2}\right) \cdot \text{sn}_\kappa\left(\frac{\rho_{2,4}}{2}\right) \leq \text{sn}_\kappa\left(\frac{\rho_{1,2}}{2}\right) \cdot \text{sn}_\kappa\left(\frac{\rho_{3,4}}{2}\right) + \text{sn}_\kappa\left(\frac{\rho_{1,4}}{2}\right) \cdot \text{sn}_\kappa\left(\frac{\rho_{2,3}}{2}\right)
\]
(4)

where $\rho_{i,j} = d(x_i, x_j)$ and $\text{sn}_\kappa$ is the function
\[
\text{sn}_\kappa(x) = \begin{cases} 
\frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa} x) & \text{if } \kappa > 0, \\
x & \text{if } \kappa = 0, \\
\frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa} x) & \text{if } \kappa < 0.
\end{cases}
\]

In the case that $\kappa > 0$ we assume in addition that the diameter is bounded by $\pi \sqrt{\kappa}$.

It is well known that the standard space forms $M_\kappa^n$ of constant curvature $\kappa$ satisfy the $PT_\kappa$ inequality. For the euclidean space this is the classical ptolemaic inequality and for the other spaces it is proved in [H]. By comparison we obtain the result also for $\text{CAT}(\kappa)$-spaces.

**Proposition 3.1.** Every $\text{CAT}(\kappa)$ space satisfies the $PT_\kappa$ inequality.

**Proof.** A $\text{CAT}(\kappa)$ spaces, $\kappa \in \mathbb{R}$, can be characterized by a 4-point condition, [BH, Proposition 1.11]. Suppose $x_i \in X$ for $0 \leq i \leq 4$, with $x_0 = x_4$, and $x_0 = x_4$, there exist four points $\bar{x}_i \in M_\kappa^2$ with $\bar{x}_0 = \bar{x}_4$ such that
\[
d(x_i, x_{i-1}) = |\bar{x}_i - \bar{x}_{i-1}|, \quad 1 \leq i \leq 4,
\]
\[
d(x_1, x_3) \leq |\bar{x}_1 - \bar{x}_3| \quad \text{and} \quad d(x_2, x_4) \leq |\bar{x}_2 - \bar{x}_4|.
\]
Since $M_\kappa^2$ satisfy the $PT_\kappa$ inequality the result follows. \qed

**Remark 3.2.** The following questions arises naturally: is a geodesic $PT_\kappa$ space $\text{CAT}(\kappa)$? A positive answer would imply a nice four point characterization of $\text{CAT}(\kappa)$ spaces. In the case $\kappa = 0$ this is not true (see [FLS]), but the counterexamples are not locally compact and there are partial positive results in the locally compact case (see e.g. [BuFW], [MS]). For $\kappa < 0$ the question is completely open.

3.2 Asymptotic $PT_\kappa$ inequality for $\kappa < 0$

One obtains the asymptotic $PT_\kappa$ property (for $\kappa < 0$) by weakening equation the $PT_\kappa$ inequality and allowing some error term. Instead of equation (4) we require that for some universal $\delta \geq 0$ we have
It implies e.g. that X as

\[ \text{asymptotic PT} \]

\[ \kappa \]

\[ \text{Remark 3.3.} \] The asymptotic \( \text{PT}_\kappa \) condition is a strong curvature condition. It implies e.g. that \( X \) does not contain flat strips: if a space contains a flat strip of width \( a > 0 \), then it contains quadruples with \( \rho_{1,3} = \rho_{2,4} = \sqrt{t^2 + a^2} \), \( \rho_{1,2} = \rho_{3,4} = t \) and \( \rho_{2,3} = \rho_{1,4} = a \). These quadruples do not satisfy the asymptotic \( \text{PT}_\kappa \) inequality for fixed \( \kappa < 0 \), \( \delta \geq 0 \) and \( t \to \infty \).

**Proposition 3.5.** Let \( 0 > \kappa' > \kappa \). If \( X \) is asymptotic \( \text{PT}_\kappa \), then \( X \) is asymptotic \( \text{PT}_{\kappa'} \).

**Proof.** From the asymptotic \( \text{PT}_\kappa \) inequality, we obtain that

\[
e^{-\frac{\kappa}{2}(\rho_{1,3} + \rho_{2,4})} \leq e^{-\frac{\kappa}{2}(\rho_{1,2} + \rho_{3,4})} + e^{-\frac{\kappa}{2}(\rho_{1,4} + \rho_{2,3})} + \delta e^{-\frac{\kappa}{2} \rho}
\]

Here \( \rho = \max_{i,j} \rho_{i,j} \) for \( i, j = 1, 2, 3, 4 \).

Since we know that for \( 0 \leq x \leq 1 \)

\[(a + b)^x \leq a^x + b^x, a > 0, b > 0.\]

Hence

\[
e^{-\frac{\kappa}{2}(\rho_{1,3} + \rho_{2,4})} = (e^{-\frac{\kappa}{2}(\rho_{1,3} + \rho_{2,4})})^{\frac{\kappa}{\kappa'}} \leq (e^{-\frac{\kappa}{2}(\rho_{1,2} + \rho_{3,4})} + e^{-\frac{\kappa}{2}(\rho_{1,4} + \rho_{2,3})} + \delta e^{-\frac{\kappa}{2} \rho})^{\frac{\kappa}{\kappa'}} \leq e^{-\frac{\kappa}{2}(\rho_{1,2} + \rho_{3,4})} + e^{-\frac{\kappa}{2}(\rho_{1,4} + \rho_{2,3})} + \delta' e^{-\frac{\kappa}{2} \rho}
\]

It satisfies the asymptotic \( \text{PT}_{\kappa'} \) inequality. \( \square \)

By scaling an asymptotic \( \text{PT}_\kappa \) space with the factor \( \frac{1}{\sqrt{-\kappa}} \) we obtain an asymptotic \( \text{PT}_{-1} \) space. Therefore we will discuss in the sequel mainly \( \text{PT}_{-1} \) spaces.
3.3 Asymptotically CAT(\(\kappa\)) spaces

We relate the asymptotic PT(\(\kappa\)) property to some condition which is formulated in familiar comparison terms.

**Definition 3.6.** Let \(\kappa < 0\). A metric space \((X, d)\) is called asymptotically CAT(\(\kappa\)), if there exists some \(\delta \geq 0\) such that for every quadruple of points \(x_1, x_2, x_3, x_4\) in \(X\) there are comparison points \(\overline{x}_1, \overline{x}_2, \overline{x}_3, \overline{x}_4\) in \(M^2_\kappa\) such that

\[
\begin{align*}
    d(x_1, x_2) &= |x_1 \overline{x}_2|, & d(x_2, x_3) &= |x_2 \overline{x}_3|, \\
    d(x_3, x_4) &= |x_3 \overline{x}_4|, & d(x_4, x_1) &= |x_4 \overline{x}_1|, \\
    d(x_1, x_3) &\leq |x_1 \overline{x}_3|, & d(x_2, x_4) &\leq |x_2 \overline{x}_4| + \delta, \\
    \text{sn}_\kappa\left(\frac{d(x_2, x_4)}{2}\right) &\leq \text{sn}_\kappa\left(\frac{|x_2 \overline{x}_4|}{2}\right) + \delta.
\end{align*}
\]

**Remark 3.7.** This definition makes also sense for \(\kappa = 0\), then the last inequality is just \(d(x_2, x_4) \leq |x_2 \overline{x}_4| + 2\delta\). Then the condition is the rough CAT(0) condition of \([BuF]\). In general, if one replaces the last condition (also for \(\kappa < 0\)) simply by the condition \(d(x_2, x_4) \leq |x_2 \overline{x}_4| + \delta\), then one obtains, what is called rough CAT(\(\kappa\)) in \([BuF]\). For \(\kappa < 0\) this is equivalent to Gromov hyperbolicity and brings no new information.

**Lemma 3.8.** If \((X, d)\) is asymptotically CAT(\(\kappa\)), then it is also asymptotically PT\(_\kappa\).

**Proof.** Let \(X\) be asymptotically CAT(\(\kappa\)) with constant \(\delta\). Let \(x_1, x_2, x_3, x_4 \in X\) be given and let \(\overline{x}_1, \overline{x}_2, \overline{x}_3, \overline{x}_4 \in M^2_\kappa\) be comparison points according to the asymptotic CAT(\(\kappa\)) property. Let \(\rho_{i,j} = d(x_i, x_j), \overline{\rho}_{i,j} = |\overline{x}_i \overline{x}_j|\) and \(\rho = \max \rho_{i,j}\). Then

\[
\begin{align*}
    \text{sn}_\kappa\left(\frac{\rho_{1,3}}{2}\right) \text{sn}_\kappa\left(\frac{\rho_{2,4}}{2}\right) - \frac{\delta}{\sqrt{-\kappa}} e^{-\frac{\sqrt{-\kappa}}{\rho}} &
    \leq \text{sn}_\kappa\left(\frac{\rho_{1,3}}{2}\right) \left(\text{sn}_\kappa\left(\frac{\rho_{2,4}}{2}\right) - \delta\right) \\
    &\leq \text{sn}_\kappa\left(\frac{\overline{\rho}_{1,3}}{2}\right) \text{sn}_\kappa\left(\frac{\overline{\rho}_{2,4}}{2}\right) \\
    &\leq \text{sn}_\kappa\left(\frac{\rho_{1,2}}{2}\right) \text{sn}_\kappa\left(\frac{\rho_{3,4}}{2}\right) + \text{sn}_\kappa\left(\frac{\rho_{1,4}}{2}\right) \text{sn}_\kappa\left(\frac{\rho_{2,3}}{2}\right) \\
    &= \text{sn}_\kappa\left(\frac{\rho_{1,2}}{2}\right) \text{sn}_\kappa\left(\frac{\rho_{3,4}}{2}\right) + \text{sn}_\kappa\left(\frac{\rho_{1,4}}{2}\right) \text{sn}_\kappa\left(\frac{\rho_{2,3}}{2}\right).
\end{align*}
\]

Thus \(X\) is asymptotically PT\(_\kappa\) with constant \(\frac{\delta}{\sqrt{-\kappa}}\).

\[\square\]

3.4 Properties of asymptotically PT\(_{-1}\) spaces

**Proposition 3.9.** An asymptotic PT\(_{-1}\) metric space is a Gromov hyperbolic space.
Proof. The asymptotic $PT_{-1}$ inequality is
\[
e^{\frac{1}{2} (\rho_1 + \rho_2)} \leq e^{\frac{1}{2} (\rho_1 + \rho_3)} + e^{\frac{1}{2} (\rho_1 + \rho_4)} + \delta e^{\frac{1}{2} \rho}
\]
Using the triangle inequality, we see
\[
\rho \leq \max\{\rho_1, 2, \rho_3, \rho_4, \rho_1 + \rho_2, \rho_3 + \rho_4, \rho_1 + \rho_2, \rho_3 + \rho_4\}
\]
which then implies
\[
e^{\frac{1}{2} (\rho_1 + \rho_2)} \leq (\delta + 1)(e^{\frac{1}{2} (\rho_1 + \rho_3)} + e^{\frac{1}{2} (\rho_1 + \rho_4)})
\]
and hence
\[
\rho_1, 2, \rho_3, \rho_4, \rho_1 + \rho_2, \rho_3 + \rho_4, \rho_1 + \rho_2, \rho_3 + \rho_4 \leq \delta e^{\frac{1}{2} \rho}
\]
Thus $X$ is a Gromov hyperbolic space.

Lemma 3.10. Let $X$ be an asymptotic $PT_{-1}$ space. Let $(x_i)$, $(x'_i)$ and $(y_i)$
be sequences in $X$ satisfying
\[
\lim_{i \to \infty} (x_i, x'_i)_o = \infty, \quad \lim_{i \to \infty} (x_i, y_i)_o = a, \quad o \in X.
\]
Then
\[
\lim_{i \to \infty} (x'_i, y_i)_o = a
\]
Proof. From the asymptotic $PT_{-1}$ inequality, we obtain
\[
e^{\frac{1}{2} (|x'_i| + |x_i|)} - e^{\frac{1}{2} (|y_i| + |x_i|)} - \delta e^{\frac{1}{2} \rho_i} \leq e^{\frac{1}{2} (|o y_i| + |x_i|)}
\]
\[
\leq e^{\frac{1}{2} (|o y_i| + |x_i|)} + e^{\frac{1}{2} (|x'_i| + |o y_i|)} + \delta e^{\frac{1}{2} \rho_i},
\]
where $\rho_i = \max\{|o x_i|, |o x'_i|, |o y_i|, |x_i|, |y_i|, |x'_i|, |y'_i|\}$.
Dividing both sides by $e^{\frac{1}{2} (|o x_i| + |o x'_i| + |o y_i|)}$, we obtain
\[
e^{-(x'_i|y_i|)} - e^{-(x_i|y_i|)} - E_i \leq e^{-(x_i|y_i|)} \leq e^{-(x'_i|y_i|)} + E_i + E_i,
\]
where $E_i = \delta e^{\frac{1}{2} (\rho_i - |o x_i| - |o x'_i| - |o y_i|)}$. Note that by triangle inequalities
\[
|o x_i| + |o x'_i| + |o y_i| - \rho_i \geq \min\{|o x_i|, |o x'_i|, 2(x_i|x'_i|)\},
\]
and hence $E_i \to 0$ by our assumptions. Taking the limit, we obtain
\[
\lim_{i \to \infty} (x'_i|y_i|) = \lim_{i \to \infty} (x_i|y_i|) = a
\]
As an immediate consequence we get

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Corollary 3.11. An asymptotic PT\(_{-1}\) space is boundary continuous.

**Theorem 3.12.** Let \(X\) be an asymptotic PT\(_{-1}\) metric space and \(o \in X\), then

\[
\rho_o(x, y) = e^{-(x|y)_o}, \quad x, y \in \partial\infty X
\]

is a metric on \(\partial\infty X\) which is PT\(_0\).

**Proof.** First, we show that \(\rho_o\) is a metric on \(\partial\infty X\). For given three points \(x, y, z \in \partial\infty X\), choose sequences \((x_i) \in x, (y_i) \in y, (z_i) \in z\). By boundary continuity we have \((x|z)_o = \lim_{i \to \infty}(x_i|z_i)_o\). Then

\[
e^{-(x|z)_o} = \lim_{i \to \infty} e^{\frac{1}{2}[|x_i|z_i|]-|x_i|o|z_i|o]} = \lim_{i \to \infty} e^{\frac{1}{2}[|x_i|o|y_i|+|z_i|o]} e^{\frac{1}{2}[|x_i|z_i|+|o|y_i|]}
\]

From the asymptotic PT\(_{-1}\) inequality, we have

\[
e^{\frac{1}{2}[|x_i|z_i|+|o|y_i|]} \leq e^{\frac{1}{2}[|y_i|z_i|+|o|z_i|]} + e^{\frac{1}{2}[|x_i|y_i|+|o|z_i|]} + \delta e^{\frac{1}{2}\rho_i}
\]

where \(\rho_i = \max\{|x_i|z_i|, |o|y_i|, |o|z_i|, |x_i|y_i|, |x_i|z_i|, |y_i|z_i|\}\). Thus

\[
e^{-(x|z)_o} \leq \lim_{i \to \infty} e^{\frac{1}{2}[|x_i|y_i|]-|o|z_i|]} + \lim_{i \to \infty} e^{\frac{1}{2}[|y_i|z_i|]-|o|z_i|]} + \lim_{i \to \infty} E_i,
\]

where \(E_i = \delta e^{\frac{1}{2}(\rho_i-|x_i|z_i|)-|o|y_i|}-|o|z_i|)]\). Again we easily check that \(E_i \to 0\) and we obtain in the limit the desired ptolemaic inequality i.e.

\[
e^{-(x|y)_o-(z|w)_o} \leq e^{-(x|y)_o-(z|w)_o} + e^{-(y|z)_o-(x|w)_o}.
\]

Choose sequences \((x_i) \in x, (y_i) \in y, (z_i) \in z, (w_i) \in w\). Since we have

\[
e^{-(x_i|z_i)_o-(y_i|w_i)_o} = e^{\frac{1}{2}[|x_i|o|y_i|+|z_i|o|+|w_i|o]} e^{\frac{1}{2}[|x_i|z_i|+|y_i|w_i|]}
\]

\[
\leq e^{\frac{1}{2}[|x_i|o|y_i|+|z_i|o|+|w_i|o]} e^{\frac{1}{2}[|x_i|y_i|+|z_i|w_i|]}
\]

\[
+ e^{\frac{1}{2}[|y_i|z_i|+|w_i|]} + \delta e^{\frac{1}{2}\rho_i}
\]

\[
= e^{-(x_i|y_i)_o-(z_i|w_i)_o} + e^{-(y_i|z_i)_o-(x_i|w_i)_o} + E_i,
\]

where \(\rho_i = \max\{|x_i|y_i|, |x_i|z_i|, |x_i|w_i|, |y_i|z_i|, |y_i|w_i|, |z_i|w_i|\}\) and

\[
E_i = \delta e^{\frac{1}{2}(\rho_i-|x_i|z_i|)-|o|y_i|}-|o|z_i|)]\). Again we see that \(E_i \to 0\) and we obtain in the limit the desired ptolemaic inequality.

\[\square\]

**Remark 3.13.** The above result implies in particular that the asymptotic upper curvature bound (see [BF]) of an asymptotic PT\(_{\kappa}\) space is bounded above by \(\kappa\).
4 Hyperbolic cones over Möbius spaces

In this chapter we prove Theorem 1.3. Therefore we give (bases on [BoS]) a construction, how to associate to a ptolemaic Möbius space \((Z, \mathcal{M})\) a hyperbolic space \(X\) (which turns out to be asymptotically \(PT_{-1}\)), such that \((Z, \mathcal{M})\) is the canonical Möbius structure of \(\partial_\infty X\).

Let \((Z, \mathcal{M})\) be a complete ptolemaic Möbius space. We choose a point \(\omega \in Z\) and an extended metric \(d \in \mathcal{M}\) from the Möbius structure, such that \(\{\omega\} = \Omega(d)\) is the point at infinity. Such a metric exists by Theorem 2.2 and this metric is unique (up to homothety) by Lemma 2.1.

We take now the metric space \((Z_\omega, d)\), where \(Z_\omega = Z \setminus \{\omega\}\) and apply the cone construction of [BoS] to it. The space \(\text{Con}(Z_\omega)\) has properties analogous to the hyperbolic convex hull of a set in the boundary of a real hyperbolic space. Set

\[
\text{Con}(Z_\omega) = Z_\omega \times (0, \infty).
\]

Define \(\rho : \text{Con}(Z_\omega) \times \text{Con}(Z_\omega) \to [0, \infty)\) by

\[
\rho((z, h), (z', h')) = 2 \log \left( \frac{d(z, z') + h \lor h'}{\sqrt{hh'}} \right).
\]

(5)

It turns out that \(\rho\) satisfies the triangle inequality and is thus a metric, see [BoS]. We write \(|zz'| = d(z, z')\) for \(z, z' \in Z_\omega\).

**Proposition 4.1.** \((\text{Con}(Z_\omega), \rho)\) is asymptotically \(PT_{-1}\).

**Proof.** Given arbitrary four points \(x_i = (z_i, h_i) \in \text{Con}((Z_\omega, d)), z_i \in (Z_\omega, d), i = 1, 2, 3, 4,\) we have

\[
e^{-\frac{\rho(x_i, x_j)}{2}} = \frac{|z_iz_j| + h_i \lor h_j}{\sqrt{h_ih_j}}, \quad i \neq j.
\]

i.e.

\[
|z_iz_j| = \sqrt{h_ih_j} e^{\frac{\rho(x_i, x_j)}{2}} - h_i \lor h_j, \quad i \neq j.
\]

(6)

Since \((Z, \mathcal{M})\) is a complete ptolemaic Möbius space, \((Z_\omega, d)\) is a complete metric space which satisfies the \(PT_0\) inequality, hence we obtain

\[
|z_1z_2| |z_3z_4| + |z_1z_4| |z_2z_3| \geq |z_1z_3| |z_2z_4|.
\]

Replacing \(|z_iz_j|\) by (6), we have the following inequality

\[
\begin{align*}
&\left( \sqrt{h_1h_2} e^{\frac{\rho(x_1, x_2)}{2}} - h_1 \lor h_2 \right) \left( \sqrt{h_3h_4} e^{\frac{\rho(x_3, x_4)}{2}} - h_3 \lor h_4 \right) \\
&\quad + \left( \sqrt{h_1h_4} e^{\frac{\rho(x_1, x_4)}{2}} - h_1 \lor h_4 \right) \left( \sqrt{h_2h_3} e^{\frac{\rho(x_2, x_3)}{2}} - h_2 \lor h_3 \right) \\
&\quad \geq \left( \sqrt{h_1h_3} e^{\frac{\rho(x_1, x_3)}{2}} - h_1 \lor h_3 \right) \left( \sqrt{h_2h_4} e^{\frac{\rho(x_2, x_4)}{2}} - h_2 \lor h_4 \right).
\end{align*}
\]
This can be written as
\[
\sqrt{h_1h_2h_3h_4}(e^{\frac{\rho(x_1,x_2)}{2}} + e^{\frac{\rho(x_3,x_4)}{2}} - e^{\frac{\rho(x_1,x_3)}{2}} + e^{\frac{\rho(x_2,x_4)}{2}}) - \sqrt{h_1h_2(h_3 \vee h_4)e^{\frac{\rho(x_1,x_2)}{2}} - \sqrt{h_3h_4(h_1 \vee h_2)e^{\frac{\rho(x_1,x_3)}{2}}} - \sqrt{h_1h_4(h_2 \vee h_3)e^{\frac{\rho(x_2,x_4)}{2}}} - \sqrt{h_2h_3(h_1 \vee h_4)e^{\frac{\rho(x_2,x_3)}{2}}} + (h_1 \vee h_2)(h_3 \vee h_4) + (h_1 \vee h_4)(h_2 \vee h_3) - (h_1 \vee h_3)(h_2 \vee h_4) \geq 0.
\]

Using again (6) we obtain
\[
e^{\frac{\rho(x_1,x_2)}{2}} + e^{\frac{\rho(x_3,x_4)}{2}} - e^{\frac{\rho(x_1,x_3)}{2}} + e^{\frac{\rho(x_2,x_4)}{2}} \geq \frac{(h_3 \vee h_4)|z_1z_2| + (h_1 \vee h_2)|z_3z_4| + (h_2 \vee h_3)|z_1z_4| + (h_1 \vee h_4)|z_2z_3|}{\sqrt{h_1h_2h_3h_4}} - \frac{(h_2 \vee h_4)|z_1z_3| + (h_1 \vee h_3)|z_2z_4|}{\sqrt{h_1h_2h_3h_4}} + (h_1 \vee h_2)(h_3 \vee h_4) + (h_1 \vee h_4)(h_2 \vee h_3) - (h_1 \vee h_3)(h_2 \vee h_4) \quad (7)
\]

Since \((a \vee b)(c \vee d) = ac \vee ad \vee bc \vee bd, a, b, c, d \in \mathbb{R}\), we easily obtain that
\[(h_1 \vee h_2)(h_3 \vee h_4) + (h_1 \vee h_4)(h_2 \vee h_3) \geq (h_1 \vee h_3)(h_2 \vee h_4)\]

which shows that the last term in (7) is nonnegative and can be omitted.

We use below that
\[(h_i \vee h_j) + \sqrt{h_ih_j} \geq h_i + h_j\]
for all \(h_i, h_j \geq 0\).

Let \(\rho = \max_{i \neq j} \rho_{i,j}\). Then again by (6) \(|z_iz_j| \leq \sqrt{h_1h_2} e^{\frac{1}{2}\rho}\) and thus
\[
e^{\frac{\rho(x_1,x_2)}{2}} + e^{\frac{\rho(x_3,x_4)}{2}} - e^{\frac{\rho(x_1,x_3)}{2}} + e^{\frac{\rho(x_2,x_4)}{2}} \geq \frac{(h_3 \vee h_4)|z_1z_2| + (h_1 \vee h_2)|z_3z_4| + (h_2 \vee h_3)|z_1z_4| + (h_1 \vee h_4)|z_2z_3|}{\sqrt{h_1h_2h_3h_4}} - \frac{(h_2 \vee h_4)|z_1z_3| + (h_1 \vee h_3)|z_2z_4|}{\sqrt{h_1h_2h_3h_4}} + \frac{|z_1z_2|}{\sqrt{h_1h_2}} + \frac{|z_3z_4|}{\sqrt{h_3h_4}} + \frac{|z_1z_4|}{\sqrt{h_1h_4}} + \frac{|z_2z_3|}{\sqrt{h_2h_3}} \geq \frac{(h_3 + h_4)|z_1z_2| + (h_1 + h_2)|z_3z_4| + (h_2 + h_3)|z_1z_4| + (h_1 + h_4)|z_2z_3|}{\sqrt{h_1h_2h_3h_4}} - \frac{(h_2 \vee h_4)|z_1z_3| + (h_1 \vee h_3)|z_2z_4|}{\sqrt{h_1h_2h_3h_4}} \geq 0
\]

Therefore \(X\) is asymptotic \(PT_{-1}\).

To finish the proof of Theorem \(\text{IX}\) we have to show that \(\partial_{\infty} X\) can be canonically identified with \(Z\).
We chose a base point $z_0 \in Z_\omega$ and then $o := (z_0, 1)$ as base point of $X$. We define for simplicity $|z| := |zz_0|$. For $x = (z, h)$ and $x' = (z', h')$ in $X$ we compute
\[
(x|x')_o = \log\left(\frac{|z| + h \lor 1)(|z'| + h' \lor 1)}{|zz'| + h \land h'}\right).
\]

Lemma 4.2. A sequence $x_i = (z_i, h_i)$ in $X$ converges at infinity, if and only if one of the following holds

1. $(z_i)$ is a Cauchy sequence in $Z_\omega$ and $h_i \to 0$.
2. $(|z_i| + h_i) \to \infty$.

Proof. We show first the if implication:

Assume 1. that $(z_i)$ is a Cauchy sequence and $h_i \to 0$. Then equation (8) immediately implies that $\lim_{i,j \to \infty} (z_i|z_j)_o = \infty$.

Assume 2. that $(|z_i| + h_i) \to \infty$. For given $i, j$ let
\[
M_{i,j} = \max\{(|z_i| + h_i \lor 1), (|z_j| + h_j \lor 1)\},
\]
\[
m_{i,j} = \min\{(|z_i| + h_i \lor 1), (|z_j| + h_j \lor 1)\}.
\]

One easily sees
\[
M_{i,j} \geq \frac{1}{4}(|z_i z_j| + h_i \lor h_j)
\]
thus
\[
(x_i|x_j)_o = \log\left(\frac{m_{i,j} M_{i,j}}{|z_i z_j| + h_i \lor h_j}\right) \geq \log\left(\frac{1}{4} m_{i,j}\right)
\]
and hence $\lim_{i,j \to \infty}(x_i|x_j)_o = \infty$.

For the only if part assume that we have given a sequence $x_i = (z_i, h_i)$ with $\lim_{i,j \to \infty}(x_i|x_j)_o = \infty$.

We first show that there cannot exist two subsequences $(x_{i_k})$ and $(x_{i_l})$ of $(x_i)$, such that $|z_{i_k}| + h_{i_k} \to \infty$ for $k \to \infty$ and $|z_{i_l}| + h_{i_l} \leq M$ for all $l$. If to the contrary such sequences would exist, then we easily obtain using triangle inequalities that
\[
|z_{i_k}| + h_{i_k} \lor 1 - 2M - 1 \leq |z_{i_k} z_{i_l}| + h_{i_k} \lor h_{i_l} \leq |z_{i_k}| + h_{i_k} \lor 1 + 2M + 1
\]
and hence $\limsup(x_{i_k}|x_{i_l})_o$ is finite, a contradiction.

Thus either $|z_i| + h_i \to \infty$ and we are in case 2 or $|z_i| + h_i$ is bounded. The boundedness and $(x_i|x_j)_o \to \infty$ implies $\log(|z_i z_j| + h_i \lor h_j) \to \infty$ and hence $(z_i)$ is a Cauchy sequence and $h_i \to 0$.

Lemma 4.3. One can identify $Z$ with $\partial_\infty X$ in a canonical way.

Proof. We define a map $\chi : Z \to \partial_\infty X$ by $z \mapsto [(z, \frac{1}{2})]$ for $z \in Z_\omega$ and $\omega \mapsto [(z_0, i)]$; here $[\ ]$ denotes the equivalence class of the corresponding sequences. Formula (N) shows that this map is injective. Let now $\xi \in \partial_\infty X$ be given and be represented by a sequence $x_i = (z_i, h_i)$. If $|z_i| + h_i \to \infty$ then
\((x_i(z_0, i))_i \to \infty\) and \(\xi = \chi(\omega)\). If \(h_i \to 0\) and \((z_i)\) a Cauchy sequence in \(Z_\omega\), then the \(z := \lim z_i\) exists, since \((Z, \mathcal{M})\) is a complete Möbius structure.

One easily checks \(\xi = \chi(z)\).

**Lemma 4.4.** The canonical Möbius structure of \(\partial_\infty X\) equals to \(\mathcal{M}\).

**Proof.** We consider on \(\partial_\infty X\) the canonical Möbius structure which is given by the metric \(\rho_o(z, z') = e^{-(z|z')o}\). Using metric involution we consider the extended metric in the same Möbius class with \(\omega\) as infinitely remote point. This metric is given for \(z, z' \in Z_\omega\) by

\[
\rho_{\omega,o}(z, z') = \rho_o(z, z') \rho_o(\omega, z) \rho_o(\omega, z') = e^{-(z|z')_\omega,o}.
\]

Now

\[
(z|z')_{\omega,o} = (z|z')_o - (\omega|z)_o - (\omega|z').
\]

By formula (8) we have

\[
(\omega|z)_o = \lim_{i \to \infty} \log(i(|z_i| + 1) = \log(|z| + 1)
\]

and in the same way \((\omega|z')_o = \log(|z'| + 1)\). Using formula (8) we see that for \(z, z' \in Z_\omega\)

\[
(z|z')_o = \log(\frac{|z| + 1)(|z'| + 1)}{|zz'|}.
\]

Now we easily compute

\[
(z|z')_{\omega,o} = -\log(|zz'|),
\]

and hence

\[
\rho_{\omega,o}(z, z') = |zz'|.
\]

\[\Box\]

### 5 Proof of Theorem 1.4

Our proof relies on the following result, which is a combination of Proposition 4.1 and Theorem [BoS, Theorem 8.2]. For the notion of a visual Gromov hyperbolic space we also refer to that paper or [BS]. Two space \((X, d_X)\) and \((Y, d_Y)\) are rough isometric, if there exists \(f : X \to Y\) and a constant \(K \geq 0\) such that for all \(x, y \in X\)

\[
|d_X(x, y) - d_Y(f(x), f(y))| \leq K
\]

and in additional \(\sup_{y \in Y} d_Y(y, f(X)) \leq K\).
Proposition 5.1. Assume that $X$ is a visual Gromov hyperbolic space such that $e^{-\epsilon(\cdot)}_o$ is bilipschitz to a ptolemaic metric $d$ on $\partial_\infty X$, the $X$ is rough isometric to an asymptotically $\text{PT}_{-1}$ space.

Proof. Consider the truncated Cone $\text{Con}^T(\partial_\infty X, d)$, which is defined as $\text{Con}^T(\partial_\infty X) = \partial_\infty X \times (0, D]$, where $D = \text{diam}(\partial_\infty X, d)$ again with the metric defined by (5). This is the cone considered in [BoS], where it is shown that $X$ is rough isometric to $\text{Con}^T(\partial_\infty X, d)$. Since by Proposition 4.1 the cone is $\text{PT}_{-1}$, the result follows.

Now we can finish the proof of Theorem 1.4. We start with some visual Gromov hyperbolic space $(X, d_X)$ with some base point $o \in X$. There exists some $\epsilon > 0$, such that the function $e^{-\epsilon(\cdot)}_o$ is bilipschitz to a metric $\rho(\cdot,\cdot)$ on $\partial_\infty X$ (see e.g. [BS, Theorem 2.2.7]). By a result of Lytchak (see [FSI, Proposition 8]) $\rho^{\frac{1}{2}}$ is a ptolemaic. Clearly $e^{-\epsilon\frac{1}{2}(\cdot)}_o$ is bilipschitz to the metric $\rho^{\frac{1}{2}}(\cdot,\cdot)$. Thus the visual Gromov hyperbolic space $(X, \rho^{\frac{1}{2}}d_X)$ satisfies the assumptions of Proposition 5.1 and is rough isometric to an asymptotically $\text{PT}_{-1}$ space. Hence $(X, d_X)$ is rough similar to this space.

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