Quantum battery based on superabsorption

Yudai Ueki,1 Shunsuke Kamimura,1,2 Yuichiro Matsuzaki,2,3 Kyo Yoshida,1 and Yasuhiro Tokura1†

1Faculty of Pure and Applied Sciences, University of Tsukuba, Tsukuba 305-8571, Japan
2Research Center for Emerging Computing Technologies, National Institute of Advanced Industrial Science and Technology (AIST), 1-1-1 Umezono, Tsukuba, Ibaraki 305-8568, Japan.
3NEC-AIST Quantum Technology Cooperative Research Laboratory, National Institute of Advanced Industrial Science and Technology (AIST), Tsukuba, Ibaraki 305-8568, Japan

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A quantum battery is a device where an energy is charged by using a quantum effect. Here, we propose a quantum battery with a charger system composed of $N$ qubits by utilizing a collective effect called a superabsorption. Importantly, the coupling strength between the quantum battery and charger system can be enhanced due to an entanglement. While the charging time scales as $\Theta\left(N^{-1/2}\right)$ by applying a conventional scheme, we can achieve a charging time $\Theta\left(N^{-1}\right)$ in our scheme. Our results open the path to ultra-fast charging of a quantum battery.

I. INTRODUCTION

Quantum thermodynamics is an emerging field to extend the conventional thermodynamics to microscopic systems where not only thermal but also quantum fluctuations should be taken into account [1][7]. One of the aims of quantum thermodynamics is to investigate whether quantum devices provide enhancement of performance over classical devices.

A quantum heat engine is one of the promising devices with the enhancement over the classical ones by using the quantum property [8][16]. It is possible to obtain a quadratically scaling power $P = \Theta(N^2)$ by using the quantum enhanced heat engine with an entanglement while a conventional separable engine shows a power of $P = \Theta(N)$ where $N$ denotes the number of qubits [15]. Here, the key feature of this scheme is to adopt a collective effect called a superabsorption that was proposed in Ref. [17], and its proof-of-concept experiment was recently demonstrated using barium atoms in an optical cavity [18].

A quantum battery is also a prominent research subject in quantum thermodynamics to charge an energy of quantum systems. As is the case for conventional batteries (such as lithium-ion batteries) using electrochemical reactions to convert chemical energy into electrical energy, the main issue of the quantum battery is to increase a performance of charging and discharging processes [19][22]. Such a quantum battery was first proposed in Ref. [19]. In Ref. [20], it was found that the use of entangling operations can improve the performance of the quantum battery compared with the one using only separable operations. Here, the performance is defined as a storabe energy in a quantum battery per a unit time. Also, in Ref. [22], it has been found that Dicke-quantum batteries composed of collective $N$-qubit systems give us a scaling $\sqrt{N}$ times larger compared to independent $N$-qubit batteries. In these studies, external pulses are applied to charge the isolated quantum battery.

On the other hand, the battery can also be charged by using an interaction with an environment [23][24]. In Ref. [24], when $N$ quantum batteries interact with $N$ charger systems that are prepared in a steady state with a population inversion, the charging time scales as $\Theta(N)$ by using a collective charging process. Here, the charging time is defined as how long it takes for the quantum battery to be a steady state. In Ref. [25], the improvement of the quantum battery performance due to the collective effects has been experimentally confirmed, and the charging time scales as $\Theta\left(N^{-1/2}\right)$ where $N$ denotes the number of atoms.

Here, we propose a quantum battery using a charger system composed of an entangled $N$-qubit system. Our quantum battery provides a charging time scaling as $\Theta(N^{-1})$. This is stark contrast to the conventional scheme where the charging time scales as $\Theta(N^{-1/2})$ by using $N$ three-level systems as a charger. The key factor for the enhanced charging time is utilizing the superabsorption for the charging process. The charger systems are prepared in an entangled $N$-qubit state called Dicke state via an interaction with the environment, and the quantum battery can strongly interact with the charger system in a collective way. An energy exchange between the charger and the battery occurs, and the battery, initially prepared in a ground state, can be eventually raised in an excited state with a necessary time scaling as $\Theta(N^{-1})$.

Our paper is organized as follows. In Sec. II, we review a charging model with one three-level system and one qubit battery. Also, we explain a charging model using $N$-qubit charger initially prepared in a separable state, where the charging time scales as $\Theta(N^{-1/2})$. In Sec. III, we explain our charging scheme with a charging time scaling as $\Theta(N^{-1})$ by using an $N$-qubit charger initially prepared in an entangled state. In Sec. IV, we conclude our discussion.

II. BATTERY CHARGING WITH SEPARABLE $N$ THREE-LEVEL CHARGERS

Let us review the conventional charging model with separable states. We consider the charger system with a three-level system and the quantum battery system with a qubit. The total
Hamiltonian $H_{\text{tot}}$ is given as follows.

$$H_{\text{tot}} = H_{3}\text{-level} + H_B^1 + H_1^1,$$

$$H_{3}\text{-level} = \sum_{i=0}^{2} E_i \left| i \right>_C \langle i |,$$  

$$H_B^1 = \frac{\Delta}{2} \sigma_z, \Delta \equiv E_1 - E_0,$$

$$H_1^1 = g \left( \left| 0 \right>_C \left( \left| 1 \right>_B \left| 0 \right> + h.c. \right) \right),$$

where $E_i (i = 0, 1, 2), E_0 < E_1 < E_2$ is the eigenenergy of the charger system, $\Delta$ is the energy of the quantum battery system, $g$ is a coupling strength between the charger and quantum battery. We prepare the initial system $\left| \psi^1(0) \right> = \left| 1 \right>_C \left| 0 \right>_B$ where the charger state is prepared as $\left| 1 \right>_C$ and the battery state is prepared as $\left| 0 \right>_B$. A steady state of the charger system after a coupling with two thermal baths can be $\left| 1 \right>_C$ by adjusting the parameters where one of the thermal baths induces a transition between $\left| 0 \right>_C$ and $\left| 2 \right>_C$, while the other thermal baths induces a transition between $\left| 2 \right>_C$ and $\left| 1 \right>_C$, and this is called a population inversion where the population of the first excited state becomes higher than that of the ground state $[14, 17, 24]$. The purpose of the charging is to obtain a battery state of $\left| 1 \right>_B$ from the initial state. The battery state $\rho_B^N(t)$ can be described as

$$\rho_B^N(t) = \text{Tr}_C \left[ e^{-iH_{\text{tot}}t} \left| \psi^1(0) \right> \left< \psi^1(0) \right| e^{iH_{\text{tot}}t} \right]$$

$$= \cos^2(gt) \left| 0 \right>_B \left< 0 \right| + \sin^2(gt) \left| 1 \right>_B \left< 1 \right|, \quad (5)$$

where Tr$_C$ denotes partial trace of the charger system. When we turn off the interaction at $gt = \pi/2$, we obtain a state of $\left| 1 \right>_B$, and this means that the charging is done. Next, let us explain a scheme to use $N$-three-level systems as a charger. We consider the charger system with $N$-three-level systems and the quantum battery system with a qubit. Strictly speaking, we need three-level systems for the charger systems, because this allows us to use a population inversion when the charger system becomes a steady state after the coupling with thermal baths, as we mentioned above. However, once we successfully obtain the population inversion for the charger system, the dynamics between the quantum battery and charger system is confined in a subspace spanned by $\left| 0 \right>_C$ and $\left| 1 \right>_C$, and so we consider only this subspace for simplicity. We define the collective operators $J_z$ and $J_{\pm} = \frac{1}{2} \sum_{i=1}^{N} \sigma_z^i$ and $J_{\pm} = \sum_{i=1}^{N} \sigma_z^i$. The total Hamiltonian $H_{\text{tot}}^{\text{sep}}$ is given by follows.

$$H_{\text{tot}}^{\text{sep}} = H_{\text{tot}} - H_B^1 + H_1^1,$$

$$H_N = \omega_A J_z + \Omega J_z^2,$$

$$H_B^1 = \frac{\omega_A + \delta}{2} \sigma_z^1,$$

$$H_1^1 = g \left( \sigma_+ \otimes \sigma_+ + \sigma_- \otimes \sigma_- \right), \quad (9)$$

where $\omega_A$ denotes a frequency of the qubits for the charger system and a quantum battery and $g$ is coupling strength between the charger and quantum battery. This Hamiltonian was experimentally realized by using a superconducting qubit and an electron-spin ensemble $[26, 29]$. We prepare the initial system $\left| \psi^N(0) \right> = \left| 11 \cdots 1 \right>_C \left| 0 \right>_B$ where the charger state is prepared as all excited states, $\left| 11 \cdots 1 \right>_C$, and the battery state is prepared as $\left| 0 \right>_B$. The purpose of the charging is to obtain a battery state of $\left| 1 \right>_B$ from the initial state. The battery state $\rho_B^N(t)$ can be described as

$$\rho_B^N(t) = \text{Tr}_C \left[ e^{-iH_{\text{tot}}^{\text{sep}}t} \left| \psi^N(0) \right> \left< \psi^N(0) \right| e^{iH_{\text{tot}}^{\text{sep}}t} \right]$$

$$= \cos^2 \left( \sqrt{N} gt \right) \left| 0 \right>_B \left< 0 \right| + \sin^2 \left( \sqrt{N} gt \right) \left| 1 \right>_B \left< 1 \right|,$$

$$\left| W \right>_C = \frac{1}{\sqrt{N}} \left( \left| 111 \cdots 0 \right>_C + \cdots + \left| 011 \cdots 1 \right>_C \right). \quad (11)$$

From this analysis the necessary time to obtain $\left| 1 \right>_B$ from $\left| 0 \right>_B$ is $t = \pi/2\sqrt{N} g$. This means that the charging time for the battery scale as $\Theta(N^{-1/2})$ in this model. Such a behavior was theoretically predicted in $[30, 31]$. 

III. BATTERY CHARGING WITH SUPERABSORPTION

Here, we introduce our scheme to charge the quantum battery with a charging time to scale as $\Theta(N^{-1})$ by using $N$ charger qubits.

A. Hamiltonian

We consider the charger system and the quantum battery system. The former one is composed of $N$-qubits while the latter one is composed of two-qubits. The total Hamiltonian $H_{\text{tot}}$ is given as follows.

$$H_{\text{tot}} = H_N + H_B + H_I,$$

$$H_N = \omega_A J_z + \Omega J_z^2,$$

$$H_B = \frac{\omega_A + \delta}{2} \sigma_z^1 + \frac{\omega_A + 2\Omega}{2} \delta \sigma_z^2,$$

$$H_I = 2g \left( \sigma_+^1 \sigma_+^2 \right) \otimes J_z,$$

where $H_N(H_B)$ denotes the Hamiltonian for $N$-qubit system (2-qubit system), $H_I$ denotes the interaction Hamiltonian between charger (N-qubit) and quantum battery (2-qubit), $\omega_A$ denotes a frequency of the each $N$-qubits, $\Omega$ denotes a total coupling constant between the $N$-qubits, $g$ denotes a coupling constant between charger and battery. Here, let us introduce $J_{\pm}^{1}$ and $J_{\pm}^{2}$, which are the part of the ladder operator. They are defined as follows.

$$J_+^{1} = J_+^{2} = J_+ = J_{-} = J_{-}^{2} = J_{-} = J_{+}^{1} = J_{+} = J_{+}^{2},$$

$$J_{-}^{1} = \sum_{M=0}^{N/2} \sqrt{\alpha_M} \left| M \right>_C \left< M - 1 \right|,$$

$$J_{+}^{1} = \sum_{M=0}^{N/2} \sqrt{\alpha_M} \left| M \right>_C \left< M + 1 \right|,$$

$$J_{+}^{2} = \sum_{M=0}^{N/2} \sqrt{\alpha_M} \left| M \right>_C \left< M + 1 \right|, \quad J_{-} = J_{-}^{2} = J_{-} = J_{+}^{1}, \quad (18)$$
Quantum battery  $g$  Charger

\begin{align*}
|1\rangle_B & \quad |1\rangle_C \\
|0\rangle_B & \quad |0\rangle_C \\
\end{align*}

\begin{align}
\Delta_{3/2} &= |\omega_A + 2\Omega| + \delta \text{ denotes the frequency of qubit 2, and this is detuned from } \Delta_{3/2} \text{ by } \delta.
\end{align}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Schematic of three-level charging model.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{Schematic of N-level charging model. Each frequency of qubits are $\omega_A$ and initial state of qubits are excited state $|1\rangle_C$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3.png}
\caption{Schematic of $A$-type Dicke states.}
\end{figure}

Here, we introduced the Dicke states, which are the simultaneous eigenstates of $J^2$ and $J_z$. These can be written as $|J, M\rangle$, and the corresponding eigenvalues are $J(J+1)$ and $M$. In this paper, we take Dicke states as $|M\rangle = |N/2, M\rangle$ in the subspace with total angular momentum $J = N/2$ and assume $N$ is odd. We assume conditions of strong coupling, i.e., $|\Omega| > \omega_A$. Also, we assume $\omega_A > 0$ and $\Omega < 0$. These conditions allow to construct a $A$-type structure for the Dicke states between $|3/2\rangle_C, |1/2\rangle_C$, and $|-1/2\rangle_C$ as shown in the FIG 3. In this case, $|1/2\rangle_C$ has the highest energy in the charger system. This means that $J_{+1}$ and $J_{-2}$ play a role in inducing a transition from a higher energy state to a lower energy state in the charger system. On the other hand, $J_{-1}$ and $J_{+2}$ induce a transition from a lower energy state to a higher energy state in the charger system. We are going to use a rotating wave approximation (RWA) for $gN \ll \omega_A$. In the RWA, we typically ignore terms that oscillate with a high frequency. In our case, terms such as $(\sigma_+^{(1)} + \sigma_-^{(2)})J_{+1}$, $(\sigma_+^{(1)} + \sigma_-^{(2)})J_{-2}$, $(\sigma_+^{(1)} + \sigma_-^{(2)})J_{-1}$, $(\sigma_+^{(1)} + \sigma_-^{(2)})J_{+2}$ will be dropped. So, by using the RWA, we obtain $H_I \simeq g(A + A^\dagger)$ where $A \equiv (\sigma_+^{(1)} + \sigma_-^{(2)})(J_{+1} + J_{-2})$.

In this case, the Hamiltonian of the $N$-qubit system can be diagonalized as follows.

\begin{align}
H_N = \sum_{M=-N/2}^{N/2} E_M |M\rangle_C \langle M|, \quad E_M = \omega_A M + \Omega M^2, \\
M \in \left\{ -\frac{N}{2}, -\frac{N}{2} + 1, \ldots, \frac{N}{2} \right\}.
\end{align}

Also, we describe the energy eigenstates of the $N$-qubit system as the Dicke states. The energy differences between the Dicke states are written as $\Delta_M = E_M - E_{M-1} = \omega_A + \Omega(2M - 1)$. Let us denote the frequency of the qubit 1 by $\omega_A + \delta$, which is detuned from $\Delta_{1/2}$ by $\delta$. On the other hand, $|\omega_A + 2\Omega| + \delta$ denotes the frequency of qubit 2, and this is detuned from $\Delta_{3/2}$ by $\delta$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4.png}
\caption{Schematic of three-level charging model.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig5.png}
\caption{Schematic of $A$-type Dicke states.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig6.png}
\caption{Schematic of $A$-type Dicke states.}
\end{figure}

\begin{align}
U(t, 0) &= I - i \int_0^t H_I(t')dt' \\
&\quad - \int_0^t H_I(t')dt' \int_0^t H_I(t'')dt'' + O(g^3). \tag{22}
\end{align}

Under the assumptions of $gN \ll \delta \ll \omega_A < |\Omega|$ and $1 \ll N\gamma$, the first order of the interaction term induces transitions between states with a large energy difference, and these have terms to oscillate with a high frequency. So the second order term, which includes a resonant transition, becomes the relevant term. Then we obtain

\begin{align}
U(t, 0) &= I - itH_{\text{eff}} \simeq e^{-itH_{\text{eff}}}. \tag{23}
\end{align}
where we use the interaction Hamiltonian $H_I$ without the rotating wave approximation. We consider the battery state $\rho(0)$ of the battery as different from the exact Hamiltonian for $\omega_A = 1$ where the condition of $\delta \ll \omega_A \ll \Omega$ is violated. On the other hand, in Fig. 6, we observe a deviation of the dynamics with the effective Hamiltonian from that of the exact Hamiltonian for $\delta \ll \omega_A \ll \Omega$ is violated.

Next, we analyze the population of $(|0\rangle_B, |1\rangle_B)$ when the charging is done when the ergotropy of the battery state $\rho_B(t)$ described as

$$
\rho_B(t) = \text{Tr}_N \left[ e^{-iH_{\text{tot}}t} |\psi(0)\rangle \langle \psi(0)| e^{iH_{\text{tot}}t} \right].
$$

Firstly, we compare the dynamics with the effective Hamiltonian $H_{\text{eff}}$ with that of the exact Hamiltonian $H_{\text{tot}}$. Actually, as shown in Fig. 5, the dynamics with the effective Hamiltonian is different from the exact Hamiltonian for $\omega_A = 1$ where the condition of $\delta \ll \omega_A \ll \Omega$ is violated. On the other hand, in Fig. 6, we observe a deviation of the dynamics with the effective Hamiltonian from that of the exact Hamiltonian for $\delta \ll \omega_A \ll \Omega$ is violated.

C. Numerical analysis

The total Hamiltonian for numerical simulations is given by

$$
H_{\text{tot}} = H_0 + H_I
$$

where we use the interaction Hamiltonian $H_I$ without the rotating wave approximation. We consider the battery state $\rho_B(t)$ described as

$$
\rho_B(t) = \text{Tr}_N \left[ e^{-iH_{\text{tot}}t} |\psi(0)\rangle \langle \psi(0)| e^{iH_{\text{tot}}t} \right].
$$

Firstly, we compare the dynamics with the effective Hamiltonian $H_{\text{eff}}$ with that of the exact Hamiltonian $H_{\text{tot}}$. Actually, as shown in Fig. 5, the dynamics with the effective Hamiltonian is different from the exact Hamiltonian for $\omega_A = 1$ where the condition of $\delta \ll \omega_A \ll \Omega$ is violated. On the other hand, in Fig. 6, we observe a deviation of the dynamics with the effective Hamiltonian from that of the exact Hamiltonian for $\delta \ll \omega_A \ll \Omega$ is violated.
Here, we define $N$ fixed when the number of the qubits becomes more than 50, $\omega_A = 10$, $\Omega = -2.3\omega_A$ and $N = 101$.

FIG. 7. Plot of the population of the battery state against time. $P_{ij}$ denotes the population of $|ij\rangle_B$, $i, j \in \{0, 1\}$. We choose the parameters as $g = 10^{-3}$, $\omega_A = 10$, $\delta = 10gN$, $\omega = -2.3\omega_A$ and $N = 31$.

In conclusion, we propose a quantum battery with a charger system composed of $N$-entangled qubits. We utilize a super-absorption for an entanglement-enhanced energy exchange between the charger system and quantum battery. Our scheme provides a scaling of a charging time $\Theta(N)$ while a conventional scheme provides a scaling of a charging time $\Theta(N^{-1/2})$. Our results pave the way for the realization of a ultra-fast quantum battery.

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Appendix A: Derivation of the effective Hamiltonian

In this section, we derive the effective Hamiltonian

$$H_{\text{eff}} = \frac{g^2}{\delta} \sqrt{a_{1/2}} \sqrt{a_{3/2}} (\tilde{L}_+ + \tilde{L}_-),$$

$$\tilde{L}_+ = \sigma_+^{(2)} \sigma_{-}^{(1)} (3/2)_{C} (-1/2), \quad \tilde{L}_- = (\tilde{L}_+)^\dagger$$

from the unitary operator (A9). We define $\mu_1 = \frac{1}{2} \left( \frac{2\omega_A + \delta}{\Omega} + 1 \right)$, $\mu_2 = \frac{1}{2} \left( \frac{\delta}{\Omega} + 1 \right)$, $\mu_3 = \frac{1}{2} \left( \frac{\delta}{\Omega} + 3 \right)$, and $\mu_4 = \frac{1}{2} \left( \frac{3\omega_A - \delta}{\Omega} - 1 \right)$. We adjust parameters to satisfy the following

$$\mu_i \neq M, \quad M \in \{-N/2, -N/2 + 1, \cdots, N/2\}$$

for all $\mu_i$, $i \in \{1, 2, 3, 4\}$. First, we expand the first order of interaction term $\int_0^t H_{\text{I}}(t') dt'$.
\[
\sum_{M=3/2}^{N/2} g \left\{ \sum_{M=3/2}^{N/2} e^{i(\omega_A + \delta + \Delta_M)t} \sqrt{a_M \sigma_M^1} |M\rangle \langle M - 1| + 2/2 e^{i(\omega_A + \delta - \Delta_M)t} \sqrt{a_M \sigma_M^+} |M\rangle \langle M - 1| + \sum_{M=-3/2}^{N/2} e^{i(\omega_A + 2\Omega + \delta - \Delta_M)t} \sqrt{a_M \sigma_M^2} |M\rangle \langle M - 1| + \sum_{M=-3/2}^{N/2} e^{i(\omega_A + 2\Omega + \delta - \Delta_M)t} \sqrt{a_M \sigma_M^2} |M\rangle \langle M - 1| + h.c. \right\}
\]

(A2)

It is worth mentioning that the first order of the interaction term induces transitions between states with a large energy difference. In this case, we have terms to oscillate with a high frequency, and these tend to be small. On the other hand, these terms also have a collective enhancement factor of \( \sqrt{a_M} \). We are going to evaluate whether these can be negligible or not as a total.

\[
N/2 \sum_{M=3/2}^{N/2} \frac{g}{i} (\omega_A + \delta + \Delta_M) \left( e^{i(\omega_A + \delta + \Delta_M)t} - 1 \right) \sqrt{a_M \sigma_M^1} |M\rangle \langle M - 1| + 2/2 \frac{g}{i} (\omega_A + \delta - \Delta_M) \left( e^{i(\omega_A + \delta - \Delta_M)t} - 1 \right) \sqrt{a_M \sigma_M^+} |M\rangle \langle M - 1| + \sum_{M=3/2}^{N/2} \frac{g}{i} (\omega_A + 2\Omega + \delta - \Delta_M) \left( e^{i(\omega_A + 2\Omega + \delta - \Delta_M)t} - 1 \right) \sqrt{a_M \sigma_M^2} |M\rangle \langle M - 1| + \sum_{M=-3/2}^{N/2} \frac{g}{i} (\omega_A + 2\Omega + \delta - \Delta_M) \left( e^{i(\omega_A + 2\Omega + \delta - \Delta_M)t} - 1 \right) \sqrt{a_M \sigma_M^2} |M\rangle \langle M - 1| + h.c.
\]

By choosing suitable parameters, these terms are negligible as we show below.

\[
\frac{g}{i} (\omega_A + \delta + \Delta_M) \sqrt{a_M} \approx \frac{2\omega_A + \delta + (2M - 1)\Omega}{2\omega_A + \delta + 2M\Omega} \approx \frac{1}{1 + (2M - 1)\frac{\Omega}{2\omega_A}} \approx 1
\]
\[
\frac{g}{i} (\omega_A + \delta - \Delta_M) \sqrt{a_M} \approx \frac{2\omega_A + \delta - (2M + 1)\Omega}{2\omega_A + \delta - 2M\Omega} \approx \frac{1}{1 - (2M + 1)\frac{\Omega}{2\omega_A}} \approx 1
\]
\[
\frac{g}{i} (\omega_A + 2\Omega + \delta - \Delta_M) \sqrt{a_M} \approx \frac{2\omega_A + 2\Omega + \delta - (2M - 1)\Omega}{2\omega_A + 2\Omega + \delta + 2M\Omega} \approx \frac{1}{1 + (2M - 1)\frac{\Omega}{2\omega_A}} \approx 1
\]
\[
\frac{g}{i} (\omega_A + 2\Omega + \delta - \Delta_M) \sqrt{a_M} \approx \frac{2\omega_A + 2\Omega + \delta - (2M + 1)\Omega}{2\omega_A + 2\Omega + \delta - 2M\Omega} \approx \frac{1}{1 - (2M + 1)\frac{\Omega}{2\omega_A}} \approx 1
\]

Therefore, we can drop the term of \( \int_0^t H_i(t')dt' \) for \( gN \ll \delta \ll \omega_A \ll |\Omega| \). Next, we expand the second order of the interaction term \( \sum_{i=0}^{N/2} H_i(t')dt' \). Since \( \mu_i \neq M \) for all \( i \), we obtain

\[
\int_0^t H_i(t')dt' \approx \sum_{i=1, M=3/2}^{N/2} \frac{g^2}{i} (\omega_A + \delta + \Delta_M) \left( e^{i(\omega_A + \delta + \Delta_M)t} - 1 \right) \sqrt{a_M \sigma_M^1} |M\rangle \langle M - 1| + \sum_{i=1, M=3/2}^{N/2} \frac{g^2}{i} (\omega_A + \delta - \Delta_M) \left( e^{i(\omega_A + \delta - \Delta_M)t} - 1 \right) \sqrt{a_M \sigma_M^+} |M\rangle \langle M - 1| + \sum_{i=1, M=3/2}^{N/2} \frac{g^2}{i} (\omega_A + 2\Omega + \delta - \Delta_M) \left( e^{i(\omega_A + 2\Omega + \delta - \Delta_M)t} - 1 \right) \sqrt{a_M \sigma_M^2} |M\rangle \langle M - 1|
\]
\[
\int_0^t dt' A^j(t') \left( e^{i(\omega_A + 2\Omega) + \delta + \Delta_M)t'} - 1 \right) \sqrt{\sigma_{M}^{2}} |M \rangle \langle M - 1| \frac{1}{\sqrt{\sigma_{M}^{2}} |M - 1\rangle \langle M|} \]

\[
= - \left( \sum_{1, M = 3/2}^{N/2} \frac{g^2}{i (\omega_A + \delta + \Delta_M)} \right) \int_0^t dt' A^j(t') \left( e^{-i(\omega_A + \delta + \Delta_M)t'} - 1 \right) \sqrt{\sigma_{M}^{2}} |M \rangle \langle M - 1| \]

\[
+ \sum_{1, M = -N/2}^{N/2} \frac{g^2}{i (\omega_A + \delta + \Delta_M)} \int_0^t dt' A^j(t') \left( e^{-i(\omega_A + 2\Omega + \delta + \Delta_M)t'} - 1 \right) \sqrt{\sigma_{M}^{2}} |M \rangle \langle M - 1| \]

\[
+ \sum_{1, M = -N/2}^{N/2} \frac{g^2}{i (\omega_A + 2\Omega + \delta + \Delta_M)} \int_0^t dt' A^j(t') \left( e^{-i(\omega_A + 2\Omega + \delta + \Delta_M)t'} - 1 \right) \sqrt{\sigma_{M}^{2}} |M \rangle \langle M - 1| \]

\[
+ \sum_{1, M = -N/2}^{N/2} \frac{g^2}{i (\omega_A + 2\Omega - \delta - \Delta_M)} \int_0^t dt' A^j(t') \left( e^{-i(\omega_A + 2\Omega - \delta - \Delta_M)t'} - 1 \right) \sqrt{\sigma_{M}^{2}} |M \rangle \langle M - 1| \]

\[
+ \sum_{M = 3/2}^{N/2} \frac{g^2}{i (\omega_A + \delta + \Delta_M)} \int_0^t dt' e^{i(\omega_A + 2\Omega + \delta + \Delta_M)t'} \left( e^{i(\omega_A + \delta + \Delta_M)t'} - 1 \right) e^{i\Delta_{M+1}t} \sqrt{\sigma_{M+1}^{2}} |M + 1\rangle \langle M - 1| \]

\[
= \sum_{M = 3/2}^{N/2} \frac{g^2}{i (\omega_A + \delta + \Delta_M)} \int_0^t dt' e^{i(\omega_A + 2\Omega + \delta + \Delta_M)t'} \left( e^{i(\omega_A + \delta + \Delta_M)t'} - 1 \right) \sqrt{\sigma_{M+1}^{2}} |M + 1\rangle \langle M - 1| \]

\[
+ \sum_{M = 3/2}^{N/2} \frac{g^2}{i (\omega_A + \delta + \Delta_M)} \left( e^{2i(\Delta_M + \delta)t} - 1 \right) \sqrt{\sigma_{M+1}^{2}} |M + 1\rangle \langle M - 1| \]

\[
+ \sum_{M = 3/2}^{N/2} \frac{g^2}{i (\omega_A + \delta + \Delta_M)} \left( e^{i(-\omega_A + \Delta_M + \delta)t} - 1 \right) \sqrt{\sigma_{M+1}^{2}} |M + 1\rangle \langle M - 1| \]
the other terms are negligible as we show below.

\[ \frac{g^2}{(\omega_A + \delta + \Delta_M)} \left( \frac{1}{(\Delta_M + \delta)^2} \right)^{\sqrt{a_{M+1}/a_M}} \approx \frac{g^2 O(N^2)}{\delta^2} \left( \frac{1}{1 + \frac{3\omega_A + 2(2M-1)\Omega}{\delta} + \frac{(2M-1)^2(2M-1)\omega_A + 2\omega_N}{\delta^2}} \right) \]

\[ \leq 1 \left\{ \begin{array}{l}
M \neq 1/2, g^2 N^2 / |\Omega|^2 \ll 1, \frac{|\Omega|^2}{\delta^2} \gg 1 \\
M = 1/2, g^2 N^2 / \omega_A^2 \ll 1, \frac{\omega_A}{\delta^2} \gg 1
\end{array} \right. \]

\[ \frac{g^2}{(\omega_A + \delta + \Delta_M)} \left( \frac{1}{(\Delta_M + \delta)^2} \right)^{\sqrt{a_{M+1}/a_M}} \approx \frac{g^2 O(N^2)}{\delta^2} \left( \frac{1}{1 + \frac{3\omega_A + 2(2M-1)\Omega}{\delta} + \frac{(2M-1)^2(2M-1)\omega_A + 2\omega_N}{\delta^2}} \right) \]

\[ \leq 1 \left\{ \begin{array}{l}
M \neq 1/2, g^2 N^2 / |\Omega|^2 \ll 1, \frac{|\Omega|^2}{\delta^2} \gg 1 \\
M = 1/2, g^2 N^2 / \omega_A^2 \ll 1, \frac{\omega_A}{\delta^2} \gg 1
\end{array} \right. \]

The other three terms of (A3) can be small for \( gN \ll \delta \ll \omega_A < |\Omega| \). Therefore, we can ignore the contribution from (A3). When an initial state is \( |\psi(0)\rangle = \{|-1/2\rangle_c |10\rangle_B \rangle \), the dominant term in (A4) is

\[ \sum_{2, M = \pm |N|/2}^{1/2} \frac{g^2}{i (\omega_A + \delta - \Delta_M)} \int_0^t dt'A_i(t') \left( e^{-i(\omega_A + \delta - \Delta_M)t'} - 1 \right) \sqrt{a_M} \sigma_{-1} |M\rangle \langle M - 1| \] \quad (A7)

and

\[ \sum_{3, M = \pm |N|/2}^{N/2} \frac{g^2}{i (\omega_A + 2\Omega) + \delta + \Delta_M)} \int_0^t dt'A_i(t') \left( e^{-i(\omega_A + 2\Omega) + \delta + \Delta_M)t'} - 1 \right) \sqrt{a_M} \sigma_{-1}^2 |M - 1\rangle \langle M| \] \quad (A8)

We evaluate these two terms. First, for (A7), we obtain

\[ \sum_{2, M = \pm |N|/2}^{1/2} \frac{g^2}{i (\omega_A + \delta - \Delta_M)} \int_0^t dt'A_i(t') \left( e^{-i(\omega_A + \delta - \Delta_M)t'} - 1 \right) \sqrt{a_M} \sigma_{-1}^2 |M\rangle \langle M - 1| \]

\[ = \sum_{M = \pm |N|/2}^{1/2} \frac{1}{i (\omega_A + \delta - \Delta_M)} \int_0^t dt' e^{i(\omega_A + 2\Omega) + \delta)t'} \left( e^{-i(\omega_A + \delta - \Delta_M)t'} - 1 \right) \]

\[ J_{\pm 1}(t) \sqrt{a_M} \sigma_{-1}^2 |M\rangle \langle M - 1| \]

\[ = \sum_{M = \pm |N|/2}^{1/2} \frac{g^2}{i (\omega_A + \delta - \Delta_M)} \int_0^t dt' e^{i(\omega_A + 2\Omega) + \delta)t'} \left( e^{-i(\omega_A + \delta - \Delta_M)t'} - 1 \right) e^{i\Delta_M t'} \sqrt{a_M} \sigma_{+1} \delta_M |M\rangle \langle M - 1| \]

\[ = \sum_{M = \pm |N|/2}^{1/2} \frac{N/2}{i (\omega_A + \delta - \Delta_M)} \int_0^t dt' e^{i(\omega_A + 2\Omega) + \delta)t'} \left( e^{-i(\omega_A + \delta - \Delta_M)t'} - 1 \right) e^{i\Delta_M/2 t'} \]

\[ = \sum_{M = \pm |N|/2}^{1/2} \frac{g^2}{i (\omega_A + \delta - \Delta_M)} \int_0^t dt' e^{i(\omega_A + 2\Omega) + \delta)t'} \left( e^{-i(\omega_A + \delta - \Delta_M)t'} - 1 \right) e^{i\Delta_M/2 t'} \]

\[ = g^2 \sum_{M = \pm |N|/2}^{1/2} \frac{g^2}{i (\omega_A + 2\Omega) + \delta + \Delta_M)} \int_0^t dt' \left( 1 - e^{i\delta t} \right) \sqrt{a_M} \sigma_{-1}^2 |3/2\rangle \langle -1/2| \]

\[ = g^2 \sum_{M = \pm |N|/2}^{1/2} \frac{1}{i (\omega_A + 2\Omega) + \delta + \Delta_M)} \left( t - \frac{1}{i \delta} \right) \sqrt{a_M} \sigma_{-1}^2 |3/2\rangle \langle -1/2| \]

\[ \approx -ig^2 t \sqrt{a_M} \sigma_{-1}^2 |3/2\rangle \langle -1/2| \]

where we use \( 1 \ll \delta t \) in the last line. Also, we impose a condition of \( 1 \ll gNt \) so that the second order \( \frac{g^2 N^2}{\delta^2} \) of the interaction should be larger than the first order \( \frac{g^2 N^2}{\delta^2} \). Since we assume \( 1 \ll gNt \) and \( gN \ll \delta \), the necessary condition is written as \( 1 \ll gNt \ll \delta t \). Second, for (A8), we obtain

\[ \sum_{3, M = \pm |N|/2}^{N/2} \frac{g^2}{i (\omega_A + 2\Omega) + \delta + \Delta_M)} \int_0^t dt' A_i(t') \left( e^{-i(\omega_A + 2\Omega) + \delta + \Delta_M)t'} - 1 \right) \sqrt{a_M} \sigma_{-1}^2 |M - 1\rangle \langle M| \]

\[ \approx -ig^2 t \sqrt{a_M} \sigma_{-1}^2 |3/2\rangle \langle -1/2| \]

From these calculations, the second order of the interaction can be described as

\[ \int_0^t H_j^I(t')dt' \int_0^t H_j^I(t'')dt'' \approx \frac{ig^2 t}{\sqrt{a_{3/2}^2}} \left( \vec{L}_+ + \vec{L}_- \right) \]
\[
\hat{L}_+ \equiv \sigma^+_1 \sigma^+_2 |3/2\rangle \langle -1/2|.
\]

Therefore, we obtain the following.

\[
|\psi(t)\rangle \simeq |\psi(0)\rangle - \frac{ig^2 t}{\delta} \sqrt{a_{1/2} a_{3/2}} \left( \hat{L}_+ + \hat{L}_- \right). \quad (A9)
\]

We can this rewrite as

\[
\frac{|\psi(t)\rangle - |\psi(0)\rangle}{t} = \frac{ig^2 t}{\delta} \sqrt{a_{1/2} a_{3/2}} \left( \hat{L}_+ + \hat{L}_- \right)
\]

and, we obtain

\[
\frac{d|\psi(t)\rangle}{dt} \simeq -iH_{\text{eff}} |\psi(t)\rangle \quad (A10)
\]

where we define the effective Hamiltonian as

\[
H_{\text{eff}} \equiv \frac{ig^2 t}{\delta} \sqrt{a_{1/2} a_{3/2}} \left( \hat{L}_+ + \hat{L}_- \right).
\]

By solving the (A10), we obtain a state at a time \( t \) with the effective Hamiltonian. This kind of approximation has been used in quantum optics \([32]\), but we firstly apply this technique to the model of the quantum battery with a superabsorption.

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