Loop homology algebra of a closed manifold

Yves Félix, Jean-Claude Thomas and Micheline Vigué-Poirrier

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Abstract

The loop homology of a closed orientable manifold $M$ of dimension $d$ is the ordinary homology of the free loop space $LM = M^S$ with degrees shifted by $d$, i.e. $H_*(LM) = H_{*+d}(LM)$. M. Chas and D. Sullivan have defined a loop product on $H_*(LM)$ and an intersection morphism $I : H_*(LM) \to H_*(\Omega M)$. The algebra $H_*(LM)$ is commutative and $I$ is a morphism of algebras. In this paper we produce, for any field $\mathbb{k}$, a chain model that computes the algebra $H_*(LM; \mathbb{k})$ and the morphism $I$. We show that the kernel of $I$ is a nilpotent ideal and that the image of $I$ is contained in the center of $H_*(\Omega M; \mathbb{k})$, which is in general quite small.

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Let $M$ be a simply connected closed oriented $d$-dimensional (smooth) manifold and $\mathbb{k}$ be a principal ideal domain. In (2), Chas and Sullivan have constructed a product on the desuspension, $H_*(LM; \mathbb{k}) = H_{*+d}(LM; \mathbb{k})$, of the ordinary homology of the free loop space on $M$. This product, called the loop product, is defined at the chain level using both intersection product on the chains on $M$ and loop composition. (See 2.2 for more details). The loop homology of $M$ is the commutative graded algebra $H_*(LM; \mathbb{k})$. Our first result consists in producing an explicit model, which allows to compute the loop product (Theorem 4.4) for any field $\mathbb{k}$.

In their paper Chas and Sullivan construct also a morphism of graded algebras, called the intersection morphism,

$I : H_*(LM; \mathbb{k}) \to H_*(\Omega M; \mathbb{k}),$

where $H_*(\Omega M; \mathbb{k})$ is the usual (non commutative) Pontryagin algebra. Using a description at the chain level of the morphism $I$, we prove

**Theorem A.** (Theorem 5.2) For any field $\mathbb{k}$,

a) the kernel of $I$ is a nilpotent ideal of nilpotency index less than or equal to $d/2$,

b) the image of $I : H_*(LM; \mathbb{k}) \to H_*(\Omega M; \mathbb{k})$ lies in the center of $H_*(\Omega M; \mathbb{k})$.

Theorem A shows that the image of $I$ is in general very small comparatively to the expected growth of $H_*(LM; \mathbb{k})$. When $\mathbb{k}$ is a field of characteristic zero, Theorem A becomes more precise. Let us recall that an element $x \in \pi_q(M)$ is called a Gottlieb element ([S]-p.377), if the map $x \vee id_M : S^q \vee M \to M$ extends to the product $S^q \times M$. These elements generate a subgroup $G_*(M)$ of $\pi_*(\Omega M)$ via the isomorphism $\pi_*(\Omega M) \cong \pi_{*+1}(M)$. 

1
Finally, we denote by $\text{cat } M$ the Lusternik-Schnirelmann category of $M$ normalized so that $\text{cat } S^n = 1$.

**Theorem B.** (Theorem 6.3) If $k$ is a field of characteristic zero then

a) the kernel of $I$ is a nilpotent ideal of nilpotency index less than or equal to $\text{cat } M$.

b) $(\text{Im } I) \cap (\pi_*(\Omega M) \otimes k) = G_*(M) \otimes k$.

c) $\sum_{i=0}^n \dim (\text{Im } I \cap H_i(\Omega M; k)) \leq Cn^k$, some constant $C > 0$ and $k \leq \text{cat } M$.

Chas and Sullivan have proved that $I$ is surjective when $M$ is a Lie group, and in [4], Cohen, Jones and Yan have computed the algebra $H_*(LM; k)$ and the homomorphism $I$ when $M$ is a sphere or a complex projective space. Using our model we perform their results, proving in particular:

**Theorem C.** (Theorem 7.5) The intersection morphism $I : H_*(LM; \mathbb{Q}) \rightarrow H_*(\Omega M; \mathbb{Q})$ is surjective if and only if $M$ has the rational homotopy type of a product of odd dimensional spheres.

The starting point of our work is the Cohen-Jones isomorphism

$$f_* : \mathbb{H}_*(LM; k) \xrightarrow{\cong} HH^*(C^* M; C^* M)$$

which identifies the loop product with the Gerstenhaber product on $HH^*(C^* M; C^* M)$ (see 1.5) when $M$ is a 1-connected closed oriented manifold of dimension $d$ ([3]).

The Cohen-Jones ring isomorphism depends a priori on the smooth structure because its construction uses in an essential way the normal bundle of the embedding of the manifold into some $\mathbb{R}^n$. So even though $HH^*(C^* M, C^* M)$ is a homotopy invariant, there is no evidence that the ring isomorphism $f_*$ is independent of the smooth structure, and that, for a homotopy equivalence $g : M \rightarrow M'$, the induced map $g_* : \mathbb{H}_*(LM) \rightarrow \mathbb{H}_*(LM')$ is a ring isomorphism. Anyway this problem does not affect our results that only depend on the existence of a ring isomorphism and not on the isomorphism itself.

Denote by $\varepsilon : C^*(M) \rightarrow k$ the augmentation associated to the inclusion $\{m_0\} \hookrightarrow M$. Then the following result provides us a description of the intersection morphism in terms of Hochschild cohomology.

**Theorem D.** (Theorem 3.5) There is a commutative diagram of algebras

$$\begin{array}{ccc}
\mathbb{H}_*(LM; k) & \xrightarrow{\cong} & HH^*(C^* M; C^* M) \\
\uparrow f_* & & \uparrow \cong \\
H_*(\Omega M; k) & \xrightarrow{I} & H_*(\Omega M; k)
\end{array}$$

To prove Theorem A we also use the following algebraic result concerning the center of the enveloping algebra of a graded Lie algebra.

**Theorem E.** (Theorem 6.2) Let $L$ be a finite type graded Lie algebra defined on a field of characteristic zero, then the center of $UL$ is contained in the enveloping algebra on the radical of $L$.

The paper is organized as follows:

1 - Hochschild cohomology and Gerstenhaber product.
2 - Loop product and Hochschild cohomology.
3 - The intersection morphism $I : \mathbb{H}_*(LM) \rightarrow H_*(\Omega M)$. 

4 - A chain model for computing the loop product and $I$.
5 - The kernel and the image of $I$.
6 - Determination of $I$ when $k$ is a field of characteristic zero.
7 - Examples and applications.
8 - Hochschild cohomology and Poincaré duality.

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the paper.

1 - Hochschild cohomology and Gerstenhaber product
In this section we fix some notations and recall the standard definitions of Hochschild cohomology and of Gerstenhaber product.

1.1. Let $k$ be a principal ideal domain; modules, tensor product, linear homomorphism,... are defined over $k$. For notational simplicity, we avoid to mention $k$. If $V$ is a lower or upper graded module ($V_i = V^{-i}$) the suspension $s$ is defined by $(sV)_n = V_{n+1}$, $(sV)^n = V^{n-1}$.

1.2. Let $(A,d)$ be a differential graded augmented cochain algebra and $(N,d)$ be a differen-
tial graded $A$-bimodule, $A = \{ A^i \}_{i \geq 0}, N = \{ N^j \}_{j \in \mathbb{Z}}$ and $N = \ker(\epsilon : A \to k)$. The two-sided normalized bar construction,

$$
\mathcal{B}(N;A;N) = N \otimes T(sA) \otimes N,
$$

is defined as follows: For $k \geq 1$, a generic element $m[a_1|a_2|...|a_k]n \in \mathcal{B}_k(N;A;N)$ has degree $|m| + |n| + \sum_{i=1}^k (|a_i|)$. If $k = 0$, we write $m[\underline{n}] = m \otimes 1 \otimes n \in N \otimes T^0 sA \otimes N$.

The differential $d = d_0 + d_1$ is defined by:

$$
d_0 : \mathcal{B}_k(N;A;N) \to \mathcal{B}_{k-1}(N;A;N),
\qquad d_1 : \mathcal{B}_k(N;A;N) \to \mathcal{B}_{k-1}(N;A;N),
$$

with

$$
d_0(m[a_1|a_2|...|a_k]n) = d(m)[a_1|a_2|...|a_k]n - \sum_{i=1}^k (-1)^{\epsilon_i} m[a_1|a_2|...|d(a_i)|...|a_k]n
+ (-1)^{\epsilon_k+1} m[a_1|a_2|...|a_k]d(n)
$$

$$
d_1(m[a_1|a_2|...|a_k]n) = (-1)^{|m|} ma_1[a_2|...|a_k]n + \sum_{i=2}^k (-1)^{\epsilon_i} m[a_1|a_2|...|a_{i-1}a_i|...|a_k]n
- (-1)^{\epsilon_k} m[a_1|a_2|...|a_{k-1}]a_kn
$$

Here $\epsilon_i = |m| + \sum_{j<i} (|a_j|)$.

1.3. For any differential graded algebra $A$, let $A^{op}$ be the opposite graded algebra, $a^{op}b = (-1)^{|a||b|}b \cdot a$, and $A^e = A \otimes A^{op}$ be the enveloping algebra. Any differential graded $A$-bimodule $N$ is a differential graded $A^e$-module.

Let $A$ and $N$ as in 1.2. The Hochschild cochain complex $C^*(A;N)$ of $A$ with coefficients in $N$ is the differential graded module \cite{9,11}:

$$
C^*(A;N) = \text{Hom}_{A^e}(\mathcal{B}(A;A),N),
C^n(A,M) = \prod_{p-q=n} \text{Hom}_{A^e}(\mathcal{B}(A,A)^p,N^q),
$$

equipped with the standard differential $D$ defined by $Df = d \circ f - (-1)^{|f|} f \circ d$. The cohomology of the complex $C^*(A;N)$ is called the Hochschild cohomology of $A$ with values in $N$, and is denoted $\text{HH}^*(A;N)$.
This definition extends the classical one since:

1.4. Lemma (\[7\]-Lemma 4.3) If \( A \) is a differential graded algebra such that \( A \) is a \( \mathbb{k} \)-free graded module then the multiplication in \( A \) extends in a semi-free resolution of \( A^e \)-modules

\[
m : \underline{\mathcal{B}}(A, A, A) \to A.
\]

This means that \( m \) is a quasi-isomorphism of differential graded \( A \)-bimodules which well behaves with quasi-isomorphisms of differential graded \( A \)-bimodules. In particular, we have the following lifting lemma

Lifting Homotopy Lemma. For any quasi-isomorphism \( \varphi : A' \to A \) there exists a unique (up to homotopy in the category of differential graded bimodules) quasi-isomorphism \( \hat{m} : \underline{\mathcal{B}}(A, A, A) \to A' \) such that \( m \simeq \varphi \circ \hat{m} \).

1.5. Recall that \( \underline{\mathcal{B}}(A) = \underline{\mathcal{B}}(\mathbb{k}; A; \mathbb{k}) := (T(sA), d) \) is a differential graded coalgebra with

\[
d([a_1|a_2|\cdots|a_k]) = -\sum_{i=1}^{k}(-1)^{s_i}[a_1|a_2|\cdots|d(a_i)|\cdots|a_k] + \sum_{i=2}^{k}(-1)^{s_i}[a_1|a_2|\cdots|a_{i-1}|a_i|\cdots|a_k].
\]

The canonical isomorphism of graded modules \( \text{Hom}_{A^e}(\underline{\mathcal{B}}(A; A; A), N) = \text{Hom}(T(sA), N) \), carries on \( \text{Hom}(T(sA), N) \) a differential \( D' \). Observe that the differential \( D' \) is not the canonical differential \( D \) of \( \text{Hom}(\underline{\mathcal{B}}(A), N) \) except when \( N \) is the trivial bimodule \( \mathbb{k} \).

If \( N = A \), Gerstenhaber \[9\] has proved that the usual cup product on \( \text{Hom}(T(sA), A) \) makes \( \text{Hom}(T(sA), A), D' \) a differential graded algebra such that the induced product on \( HH^*(A; A) \), called the Gerstenhaber product, is commutative.

2 - Loop product and Hochschild cohomology

In this section, for the reader convenience and in order to precise notations, we recall the definition of the loop product defined by Chas and Sullivan, the interpretation given by Cohen and Jones and the relation between the loop product and the Gerstenhaber product.

Let \( X \) be topological space and \( C_*(X) \) be the singular chain coalgebra (coefficients in \( \mathbb{k} \)) with coproduct \( \Delta_X \):

\[
C_k(X) \xrightarrow{\text{diagonal}} C_k(X \times X) \xrightarrow{\text{AW}} \bigoplus_{k_1+k_2=k} C_{k_1}(X) \otimes C_{k_2}(X), \quad c \mapsto \sum_i c_i \otimes c_i',
\]

where \( \text{AW} \) denotes the usual Alexander-Whitney chain equivalence.

The singular cochain algebra on \( X \) is the differential graded algebra \( C^*(X) := (C_*(X))^\vee \) with product (cup product),

\[
\cup : C^p(X) \otimes C^q(X) \to C^{p+q}(X), \quad (f \cup g)(c) = \sum_i (-1)^{|c_i||g|} f(c_i)g(c'_i).
\]

Observe that the choice of a point \( x_0 \) in \( X \) determines an augmentation \( \varepsilon : C^*(X) \to \mathbb{k} \).

2.1. Let \( M \) be a 1-connected \( \mathbb{k} \)-Poincaré duality space of formal dimension \( d \), i.e., \( M \) is 1-connected with a class \([M] \in H_d(M)\) such that cap product with \([M] \) induces an isomorphism

\[- \cap [M] : H^*(M) \to H_{d-*}(M).
\]
(We restrict to 1-connected spaces in order to simplify the definition. In this case the formal dimension \( d \) does not depend on \( l_k \).) Every simply-connected closed oriented \( d \)-dimensional manifold is a \( k \)-Poincaré duality space.

Let us denote also by \([M]\) a cycle in \( C_d(X)\) representing the class \([M] \in H_d(M)\), and denote by \( P: H_*(M) \to H^{d-*}(M)\) the linear isomorphism, inverse of \(- \cap [M]\).

The intersection product of two homology classes of \( M \) is nowadays defined from the cup product in the following way:

\[
H_k(M) \otimes H_l(M) \xrightarrow{\cdot} H_{k+l-d}(M)
\]

\[
\mathcal{P} \otimes \mathcal{P} \downarrow
\]

\[
H^{d-k}(M) \otimes H^{d-l}(M) \xrightarrow{-\cup -} H^{2d-(k+l)}(M)
\]

The intersection product makes \( \mathbb{H}_*(M) = H_{*+d}(M) \) a commutative graded algebra with unity \([M] \in \mathbb{H}_0(M)\). Originally, the intersection product was defined for a polyhedron. In their paper [3] Cohen and Jones uses the following point of view. Let \( i: M \to \mathbb{R}^{n+d} \) be the embedding of \( M \) into a codimension \( n \) euclidean space. We denote by \( \nu^n \to M \) the normal bundle of this embedding, and by \( Th(\nu^n) \) the associated Thom space, which is Spanier-Whitehead dual to \( M_+ \), i.e. \( M \) with a disjoint base point. More precisely, denote by \( M^{-TM} \) the spectrum

\[
M^{-TM} = \Sigma^{-(n+d)} Th(\nu^n).
\]

Then the diagonal map \( \Delta: M \to M \times M \) induces a map of spectra \( \Delta^*: M^{-TM} \wedge M^{-TM} \to M^{-TM} \) that makes \( M^{-TM} \) into a ring spectrum. When \( M \) is orientable, the ring structure is compatible, via the Thom isomorphism, with the intersection product:

\[
H_q(M^{-TM}) \otimes H_r(M^{-TM}) \xrightarrow{\tau \otimes \tau} H_{q+r}(M^{-TM})
\]

\[
\Delta \downarrow \text{intersection}
\]

\[
H_{q+d}(M) \otimes H_{r+d}(M) \xrightarrow{\text{intersection}} H_{q+r+d}(M).
\]

2.2. The loop product has been defined geometrically by Chas and Sullivan [2] in the following way. Consider again the evaluation map \( ev: LM \to M \) and let \( \sigma: \Delta^n \to LM \) and \( \tau: \Delta^m \to LM \) be singular simplices. Then \( ev \circ \sigma \) and \( ev \circ \tau \) are singular simplices of \( M \). At each point \((s, t) \in \Delta^n \times \Delta^m\) where \( q \circ \sigma (s) = q \circ \tau (t) \), the composition of the loops \( \sigma(s) \) and \( \tau(t) \) can be defined. If we assume that the map \( (ev \circ \sigma, ev \circ \tau): \Delta^n \times \Delta^m \to M \times M \) is transverse to the diagonal map \( M \to M \times M \) then (as shown by Chas and Sullivan) this defines a chain \( \sigma \cdot \tau \in C_{n+m-d}(LM) \), and produces a commutative and associative multiplication

\[
\mathbb{H}_q(LM) \otimes \mathbb{H}_r(LM) \to \mathbb{H}_{q+r}(LM), \quad a \otimes b \mapsto a \bullet b,
\]

whose unity is the image of \([M]\) via the homomorphism induced by the canonical section.

A homological presentation of the loop product in the smooth case goes as follows. Let \( N \to M \) be the normal bundle to the inclusion \( \Delta: M \to M \times M \) and \( (ev)^*(N) \) the pull back bundle on \( LM \times_M LM \). We write \( N_D, N_S, (ev)^*(N)_D, (ev)^*(N)_S \) for the corresponding disk and sphere bundles. We have homotopy equivalences

\[
\varphi_1: (N_D, N_S) \to (V, \partial V)
\]

\[
\varphi_2: ((ev)^*(N)_D, (ev)^*(N)_S) \to ((ev)^{-1}(V), (ev)^{-1}(\partial V)),
\]
where $V$ is a tubular neighborhood of $\Delta(M)$ into $M \times M$. The map $\varphi_1$ is the usual exponential map and $\varphi_2$ is defined by

$$\varphi_2(x, v, c) = \gamma(x, v)^{-1} \circ c \circ \gamma(x, v),$$

with $x \in \Delta(M), v \in N_x, c \in LM \times_M LM$ and where $\gamma(x, v)$ denotes the geodesic ray of length $\|v\|$ starting from $x$ with tangent vector $v$. Then the Chas-Sullivan loop product can be defined as the composite

$$H_q(LM) \otimes H_r(LM) \to H_{q+r}(LM \times LM,(ev)^{-1}(M \times M \setminus \Delta(M)))$$

$$\cong (1) H_{q+r}((ev)^{-1}(V),(ev)^{-1}(\partial V)) \cong H_{q+r}((ev)^*(N)_D,(ev)^*(N)_S)$$

$$\cong (2) H_{q+r-d}(LM \times LM) \xrightarrow{\text{composition}} H_{q+r-d}(LM),$$

where (1) is usual excision and (2) the Thom isomorphism for an oriented fiber bundle.

2.3. In ([3]), Cohen and Jones give another description of the Chas-Sullivan product. Since we need relative versions of their construction we recall here the main steps of their construction. Using the notations of 2.1, let $\text{Th}(ev^*(\nu^n))$ be the Thom space of the pullback bundle $ev^*(\nu^n) \to LM$ and define the Thom spectrum

$$LM^{-TM} = \Sigma^{-(n+d)}\text{Th}(ev^*(\nu^n)).$$

2.4. Theorem. ([3], Theorem 1.3) $LM^{-TM}$ is a ring spectrum whose multiplication

$$\mu : LM^{-TM} \wedge LM^{-TM} \to LM^{-TM}$$

is compatible with the Chas-Sullivan product in the sense that the following diagram commutes

$$\begin{array}{ccc}
H_q(LM^{-TM}) \otimes H_r(LM^{-TM}) & \xrightarrow{\mu^*} & H_{q+r}(LM^{-TM}) \\
\downarrow \tau & & \downarrow \tau \\
H_{q+d}(LM) \otimes H_{r+d}(LM) & \xrightarrow{\tau} & H_{q+r+d}(LM),
\end{array}$$

where $\tau$ denotes the Thom isomorphism.

2.5. Theorem. ([3], Theorem 3) $LM^{-TM}$ is the geometric realization of a cosimplicial spectrum $(\mathbb{L}_M)_*$. The $k$-simplices are given by

$$(\mathbb{L}_M)_k = (M^k)_+ \wedge M^{-TM}$$

and the ring structure is realized on the cosimplicial level by pairings

$$\mu_k : ((M^k)_+ \wedge M^{-TM}) \wedge ((M^*)_+ \wedge M^{-TM}) \to (M^{k+r})_+ \wedge M^{-TM}$$

defined by

$$\mu_k((x_1, \ldots, x_k;a) \wedge (y_1, \ldots, y_r;b)) = (x_1, \ldots, x_k, y_1, \ldots, y_r; \Delta^*(a \wedge b))$$

where $\Delta^*$ is the ring structure defined on $M^{-TM}$ in 2.1.

2.6. Let $\Delta^k$ be the standard $k$-simplex

$$\Delta^k = \{(x_1, \ldots, x_k) | 0 \leq x_1 \leq x_2 \leq \ldots \leq x_k \leq 1 \}$$
and consider the maps
\[ f_k : \Delta^k \times LM \to M^{k+1}, \quad f_k(x_1, \ldots, x_k, c) = (c(0), c(x_1), \ldots, c(x_k)). \]
Denote by \( \tilde{f}_k : LM \to \text{Map}(\Delta^k, M^{k+1}) \) the adjoint of \( f_k \). The product of the \( \tilde{f}_k \) induces an homeomorphism
\[ f : LM \to \text{Map}_\Delta(M^*, M^{*+1}) \]
which induces, when \( M \) is simply connected, a linear isomorphism in homology
\[ \mathbb{H}_*(LM) \xrightarrow{\cong} HH^*(C^*(M), C^*(M)) \]
Pulling back the normal bundle \( \nu^n \to M \) along the maps \( ev \) and \( p_1 \) in the diagram
\[ \begin{array}{ccc}
\Delta^k \times LM & \xrightarrow{f_k} & M^{k+1} \\
\downarrow_{ev} & \downarrow_{p_1} & \downarrow = \\
M & = & M,
\end{array} \]
we get maps of Thom spectra
\[ f_k : (\Delta^k)_+ \wedge LM^{-TM} \to M^{-TM} \wedge (M^k)_+. \]
By gluing together the adjoint maps, we get a map of spectra
\[ f : LM^{-TM} \to \prod_k \text{Map}((\Delta^k)_+, M^{-TM} \wedge (M^k)_+). \]
Now, using the homotopy equivalence \( C_*(M^{-TM}) \simeq C^{-*}(M_+) \), and taking singular chains we get a morphism connecting \( C_*(LM^{-TM}) \) to the Hochschild cochains on \( C^*(M) \),
\[ f_* : C_{*-k}(LM^{-TM}) \to C^k(C^*(M), C^*(M)). \]
The commutativity of \( f_* \) with the differentials is not a trivial point. One reason is that, on the spectrum level, the Atiyah duality map \( M^{-TM} \to \text{Map}(M, S^0) \) is a map of ring spectra and of bimodules of spectra (\text{[5]}).

2.7. Theorem. (\text{[3]}, Theorem 12) Let \( M \) be a compact simply connected manifold. Then the morphism \( f_* \) is a quasi-isomorphism and the following diagram of chain complexes is commutative
\[ \begin{array}{ccc}
C_*(LM^{-TM}) \otimes C_*(LM^{-TM}) & \xrightarrow{f_* \otimes f_*} & C_*(LM^{-TM}) \\
\downarrow_{f_* \otimes f_*} & \downarrow_{f_*} & \downarrow = \\
C^*(C^*(M), C^*(M)) \otimes C^*(C^*(M), C^*(M)) & \xrightarrow{\cup} & C^*(C^*(M), C^*(M)).
\end{array} \]

3. The intersection morphism \( I : \mathbb{H}_*(LM) \to H_*(\Omega M) \)

3.1. Let \( i : N \hookrightarrow M \) be the injection of an open set in \( M \). We define the mapping space \( L_N M \) as the pullback
\[ \begin{array}{ccc}
L_N M & \xrightarrow{i'} & LM \\
\downarrow & \downarrow_{ev} & \downarrow = \\
N & \xrightarrow{i} & M
\end{array} \]
The space \( L_N M \) is the space of loops that originate in \( N \). By restriction, the loop product induces a product on \( \mathbb{H}_*(L_N M) \) so that the morphism \( H_*(i') : \mathbb{H}_*(L_N M) \to \mathbb{H}_*(LM) \) becomes a multiplicative morphism.
We will now follow verbatim the lines of the Cohen-Jones construction in order to compute $\mathbb{H}_*(L_N M)$ in terms of Hochschild cohomology. With the notation of section 2.3, we define the Thom ring spectra

$$L_N M^{-TM} = \Sigma^{-(n+d)} Th(i^* ev^*(v^n)) .$$

The Thom map $\tau$ is then a multiplicative isomorphism

$$H_*(\tau) : H_*(L_N M^{-TM}) \xrightarrow{\cong} H_{*+d}(L_N M) .$$

The morphisms $f_k$ restrict naturally to morphisms

$$f_k : L_N M^{-TM} \land \Delta^k \to N^{-TM} \land (M^k)_+$$

that induce, exactly in the same way as in the original case $N = M$, a multiplicative isomorphism

$$H_*(L_N M^{-TM}) \xrightarrow{\cong} \text{HH}_*(C^*(M), C_*(N^{-TM})) .$$

Now by the Spanier-Whitehead duality, there is a homotopy equivalence

$$C_*(N^{-TM}) \simeq C_*(M, M \setminus N) ,$$

and the map $f_*$ restricts naturally to a quasi-isomorphism

$$f_*^N : C_{*+k}(L_N M^{-TM}) \to C^k(C_*(M), C^*(M, M \setminus N)) .$$

We therefore have

**3.2. Theorem.** When $M$ and $N$ are simply connected, there is a commutative diagram of algebras in which horizontal lines are isomorphisms

$$\varphi_N : \mathbb{H}_*(L_N M) \xleftarrow{\cong} H_*(L_N M^{-TM}) \xrightarrow{\cong} \text{HH}_*(C^*(M), C_*(M, M \setminus N)) \xrightarrow{\cong} H_*(i^* ev^*(v^n))$$

$$\varphi_M : \mathbb{H}_*(LM) \xleftarrow{\cong} H_*(LM^{-TM}) \xrightarrow{\cong} \text{HH}_*(C^*(M), C^*(M)) ,$$

where, as usual, $\mathbb{H}_p = H_{p+d}$.

In the same way, working in the relative case, we get

**3.3. Theorem.** There is a sequence of isomorphisms of algebras

$$\varphi_{M,N} : \mathbb{H}_*(LM, L_N M) \xleftarrow{\cong} H_*(LM^{-TM}, L_N M^{-TM}) \xrightarrow{\cong} \text{HH}_*(C_*(M), C^*_*(M, M \setminus N)) ,$$

making commutative the diagram

$$\begin{array}{ccc}
\mathbb{H}_*(L_N M) & \xrightarrow{\varphi_N} & \text{HH}_*(C^*(M), C^*_*(M, M \setminus N)) \\
\downarrow & & \downarrow \\
\mathbb{H}_*(LM) & \xrightarrow{\varphi_M} & \text{HH}_*(C^*(M), C^*(M)) \\
\downarrow & & \downarrow \\
\mathbb{H}_*(LM, L_N M) & \xrightarrow{\varphi_{M,N}} & \text{HH}_*(C^*(M), C^*_*(M, M \setminus N)) \\
\end{array}$$
3.4. In [2], Chas and Sullivan define the intersection morphism $I : \mathbb{H}_\ast(LM) \to H_\ast(\Omega M)$ by associating to an $q$-cycle in $LM$ its intersection with the space of based loops at $m_0$, $\Omega(M, m_0) = \Omega M$, viewed as a codimension $d$-submanifold. It follows directly from the definition that $I$ transforms the loop product into the Pontryagin product.

A slightly different exposition of the intersection morphism works as follows. Fix a Riemannian metric on $M$, choose a geodesic disc $D^d$ centered at the base point $m_0$ and consider the map

$$\mu : D^d \times \Omega M \to LM,$$

where $\gamma_x$ denotes the geodesic ray from $m_0$ to $x$, and $\ast$ denotes the composition of paths. Let us denote by $E$ the subspace $L_{M\setminus\{m_0\}}M$. We consider the commutative diagram of complexes

$$
\begin{array}{ccc}
\mathcal{C}_\ast(D^d) \otimes \mathcal{C}_\ast(\Omega M) & \xrightarrow{EZ} & \mathcal{C}_\ast(D^d \times \Omega M) \\
\alpha \otimes \text{id} & \downarrow & \mathcal{C}_\ast(LM) \\
\mathcal{C}_\ast(D^d, \partial D^d) \otimes \mathcal{C}_\ast(\Omega M) & \xrightarrow{\psi} & \mathcal{C}_\ast(LM, E)
\end{array}
$$

where $EZ$ means the Eilenberg-Zilber map, $\alpha$ the canonical surjections, and $\psi$ is the quotient map. Clearly $H_\ast(\psi) : H_\ast(\Omega M) \to \mathbb{H}_\ast(LM, E)$ is an isomorphism of algebras. The intersection morphism $I$ coincides with the composition $H(\psi)^{-1} \circ H(\alpha)$:

$$H_\ast(\Omega M) \xrightarrow{H(\alpha)} H_\ast(LM, E) \xrightarrow{H(\psi)} \left( H_\ast(D^d, \partial D^d) \otimes H_\ast(\Omega M) \right)_q \cong H_{q-d}(\Omega M).$$

Our next result describes the homomorphism $I$ in terms of Hochschild cohomology. Denote by $\varepsilon : \mathcal{C}_\ast(M) \to \mathcal{k}$ the augmentation induced by the inclusion $\{m_0\} \hookrightarrow M$. We then have:

3.5 Theorem. Let $M$ be a simply connected closed oriented $d$-dimensional manifold. There exists an isomorphism of graded algebras $\Theta$ that makes commutative the following diagram

$$
\begin{array}{ccc}
HH^\ast(\mathcal{C}_\ast(M); \mathcal{C}_\ast(M)) & \xrightarrow{HH^\ast(\mathcal{C}_\ast(M); \varepsilon)} & HH^\ast(\mathcal{C}_\ast(M); \mathcal{k}) \\
\cong & & \Theta \cong \\
\mathbb{H}_\ast(LM) & \xrightarrow{I} & H_\ast(\Omega M).
\end{array}
$$

Proof. We take $N = M\setminus\{m_0\}$ in Theorem 3.3. We remark that the map $HH^\ast(\mathcal{C}_\ast(M); \varepsilon)$ is the composite

$$HH^\ast(\mathcal{C}_\ast(M), \mathcal{C}_\ast(M)) \to HH^\ast(\mathcal{C}_\ast(M), \mathcal{C}_\ast(M\setminus N)) \xrightarrow{\tilde{\varepsilon}} HH^\ast(\mathcal{C}_\ast(M), \mathcal{k}),$$

where $\tilde{\varepsilon}$ is an isomorphism. We define the isomorphism $\Theta$ to be the composition

$$H_\ast(\Omega M) \xrightarrow{H_\ast(\psi)} \mathbb{H}_\ast(LM, E) \xrightarrow{\varphi_{M, M\setminus\{m_0\}}} HH^\ast(\mathcal{C}_\ast(M), \mathcal{C}_\ast(M\setminus N)) \xrightarrow{\tilde{\varepsilon}} HH^\ast(\mathcal{C}_\ast(M), \mathcal{k}).$$

With these definitions, the above diagram commutes trivially. \qed

4 - A chain model for computing the loop product and $I$.

In this section we construct, for any field of coefficients $\mathcal{k}$, an explicit model for the loop product at the chain level.
4.1. Recall the Adams Cobar construction $\Omega C$ on a coaugmented differential graded coalgebra $C = \mathbb{k} \oplus \bar{C}$. This is the differential graded algebra $(T(s^{-1}C), d)$, where $d = d_1 + d_2$ is the unique derivation determined by:

$$d_1 s^{-1}c = -s^{-1}dc, \quad \text{and} \quad d_2 s^{-1}c = \sum_i (-1)^{|c_i|} s^{-1}c_i \otimes s^{-1}c_i', \quad c \in \bar{C},$$

where the reduced coproduct of $c \in \bar{C}$ is written $\bar{\Delta}c = \sum_i c_i \otimes c_i'$. For sake of simplicity we put $\langle x_1|x_2|\cdots|x_n \rangle := s^{-1}x_1 \otimes \cdots \otimes s^{-1}x_n$.

4.2. Assume $\mathbb{k}$ is a field, and $M$ is a 1-connected compact $d$-dimensional manifold. Denote by $f : (T(V), d) \to C^*(M)$ a free minimal model for the singular cochain algebra on $M$ (\cite{A}), i.e. $(T(V), d)$ is a differential graded algebra, $f$ is a quasi-isomorphism of differential graded algebras, and $d(V) \subset T^{\geq 2}(V)$. The differential graded algebra $(T(V), d)$ is uniquely defined, up to isomorphism, by the above properties. Moreover, $V^p \cong H^{p-1}(\Omega M)$, (\cite{A}). Denote by $S$ a complement of the vector space generated by the cocycles of degree $d$. The differential graded ideal $J = (T(V))^\geq d \oplus S$ is acyclic and the quotient algebra $A = T(V)/J$ is a finite dimensional graded differential algebra.

4.3. Since $A$ is finite dimensional, the dual algebra $A^\vee$ is a differential graded coalgebra and we consider the differential graded algebra

$$\Omega A^\vee = (T(W), d),$$

with in particular, $W \cong \text{Hom}(s\bar{A}, \mathbb{k})$, and $\Omega A^\vee = \text{Hom}(\Omega A, \mathbb{k}) = (\text{Hom}(T(s\bar{A}), \mathbb{k}), D)$ (see 1.5).

We choose a homogeneous linear basis $e_i$ for $\bar{A}$, and its dual basis $w_i$ for $W$. This determines the constants of structure $\alpha^i_{jk}$ and $\rho^i_j$:

$$\langle w_i, se_k \rangle = -(-1)^{|w_i|}\delta_{ik}, \quad e_i \cdot e_j = \sum_k \alpha^k_{ij} e_k, \quad d(e_i) = \sum_j \rho^j_i e_j$$

$$d(w_i) = \sum_{jk} a^{jk}_i w_j w_k + \sum_j \beta^j_i w_j, \quad a^{jk}_i = (-1)^{|e_j|+|e_j|} \alpha^j_{ik}, \quad \beta^j_i = (-1)^{|w_j|} \rho^j_i.$$

4.4 Theorem. Let $\mathbb{k}$ be a field and $M$ be a 1-connected closed oriented manifold of dimension $d$. With notation introduced above

a) the derivation $D$ uniquely defined on the tensor product of graded algebras $A \otimes T(W)$ by

$$D(a \otimes 1) = d(a) \otimes 1 + \sum_j (-1)^{|a|+|e_j|} [a, e_j] \otimes w_j, \quad a \in A,$$

$$D(1 \otimes b) = 1 \otimes d(b) - \sum_j (-1)^{|e_j|} e_j \otimes [w_j, b], \quad b \in TW,$$

is a differential. Here $[\ ,\ ]$ denotes the Lie bracket in the graded algebras $A$ and $T(W)$.

b) the graded algebra $H_*(A \otimes T(W), D)$ is isomorphic to the loop algebra $\mathbb{H}_*(LM)$.

Proof.

a) is proved by a direct but laborious computation.\hfill \Box

b) is a direct consequence of theorem 2.7.

Observe that this model is dual to those constructed by one of us, \cite{A}.

4.5 Proposition. Let $\mathbb{k}$ be a field and $M$ be a 1-connected closed oriented manifold of dimension $d$. There is a cohomology spectral sequence of graded algebras such that

$$E_2 = HH^*(H^*(M), H^*(M)) \Rightarrow \mathbb{H}_*(LM).$$
4.6 Example. If $M$ is a formal space, (for instance $M$ is a simply connected compact Kähler manifold for $k = \mathbb{Q}$, (II), one can choose $A = H^*(M)$ and thus the algebras $HH^*(H^*(M); H^*(M))$ and $H_* (LM)$ are isomorphic graded vector spaces. If we put $H^* = H^*(M)$ and $H_* = H_* (M)$ the loop algebra $H_* (LM)$ is isomorphic to the graded algebra $H (H^* \otimes T (sH_*), D)$ with $D (a \otimes 1) = 0$, $a \in H^*$ and $D (1 \otimes b) = - \sum s (-1)^s e_j \otimes [w_j, b]$, $b \in H_*$.

4.7. The commutative case. Suppose that the algebra $C^* (M)$ is connected by a sequence of quasi-isomorphisms to a commutative differential graded algebra $(A, d)$. This is the case if either $k$ is of characteristic zero, or else if $k$ is a field of characteristic $p > d$ (II, Proposition 8.7). We can also suppose that $A$ is finite dimensional, $A^0 = k$, $A^1 = 0$, $A^{> d} = 0$ and $\overline{A^d} = k \omega$. Then formulas of Theorem 4.4 simplify as:

\[
\begin{align*}
D (a \otimes 1) &= d(a) \otimes 1, \\
D (1 \otimes b) &= 1 \otimes d(b) - \sum s (-1)^s e_j \otimes [w_j, b].
\end{align*}
\]

We can now interpret the intersection morphism in terms of models.

4.8 Theorem. Let $k$ be a field and $M$ be a 1-connected closed oriented manifold of dimension $d$. There is a commutative diagram of algebras

\[
\begin{align*}
H_* (LM) &\xrightarrow{\cong} H_* (A \otimes T (W), D) \\
\downarrow \quad i \downarrow &\quad \downarrow H (\varepsilon_A \otimes 1) \\
H_* (\Omega M) &\xrightarrow{\cong} H_* (T (W), d).
\end{align*}
\]

Proof. Recall that Hochschild cohomology $HH^*(A; M)$ is covariant in $M$ and contravariant in $A$. Moreover, if $f : A \rightarrow B$ is a quasi-isomorphism of differential graded algebras and $g : M \rightarrow M'$ is a quasi-isomorphism of $A$-bimodules, we have isomorphisms

\[
HH^*(B; M) \xrightarrow{\cong} HH^*(A; M) \xrightarrow{\cong} HH^*(A; M').
\]

Starting with Theorem 3.5, we obtain the following commutative diagram

\[
\begin{align*}
H_* (LM) &\xrightarrow{\cong} HH^*(C^* (M); C^* (M)) \xrightarrow{\cong} HH^*(A; A) \xrightarrow{\cong} H_* (A \otimes T (W), D) \\
\downarrow \quad i \downarrow &\quad \downarrow HH^*(C^* (M), \varepsilon) \quad \downarrow HH^*(A, \varepsilon_A) \quad \downarrow H (\varepsilon_A \otimes 1) \\
H_* (\Omega M) &\xrightarrow{\cong} HH^*(C^* (M); k) \xrightarrow{\cong} HH^*(A; k) \xrightarrow{\cong} H_* (T (W), d)
\end{align*}
\]

\]

5 - The kernel and the image of $I$

5.1. If $J$ is an ideal of an algebra $A$, we put $J^1 = J$ and $J^{n+1} = JJ^n$, $n \geq 1$ and, in the case $J$ is nilpotent, we define $\text{Nil} (J) = \sup \{ n \mid J^n \neq 0 \}$.

5.2 Theorem. Let $k$ be a field and $M$ be a simply connected closed oriented $d$-dimensional manifold.
Proof. a) By Theorem 4.8, the kernel of $I$ is generated by the classes of cocycles in $\hat{A} \otimes T(W)$. Since $A^1 = 0$ and $A^{>d} = 0$, the nilpotency of the kernel of $I$ is less than or equal to $d/2$.

b) Let $e_i$ and $w_i$ be the elements defined in 4.3 and $[\alpha]$ be an element in the image of $H(e_A \otimes id)$. Then $\alpha$ is a cocycle in $T(W)$ and there exist elements $\alpha_i$ in $T(W)$ such that $\alpha = \sum_i e_i \otimes \alpha_i$ is a cycle in $A \otimes T(W)$. A short calculation shows that the component of $e_i$ in $d(\alpha)$ is

$$(-1)^{|w_i|} \left( d(\alpha_i) - [w_i, \alpha] + \sum_j \beta^j_i \alpha_j + \sum_{j,k} a^{jk}_i (-1)^{|u||w_k|} \alpha_j w_k + \sum_{j,k} a^{kj}_i (-1)^{|w_k|} w_k \alpha_j \right).$$

Since this component must be 0, by Lemma 5.3 below there exists a surjective morphism $H(T(W), d) \otimes k[u] \to H(T(W), d)$ that maps $u$ to $[\alpha]$. This implies that $[\alpha]$ is in the center of $H(T(W), d) \cong H_*(\Omega M)$.

5.3 Lemma Assume $k$ is a field. Let $\alpha$ be a cycle in $(T(W), d)$ and let $u$ be a variable in the same degree. Then with the notations of 4.3 and 4.4, we have:

1. There exists a surjective quasi-isomorphism

$$\varphi : (T(w_i, u, w_i'), D) \to (T(W), d) \otimes (k[u], 0), \quad |w_i'| = |u| + |w_i| + 1,$

such that $\varphi(u) = u$, $\varphi(w_i) = w_i$ and $\varphi(w_i') = 0$, and with $D$ defined by

$$D(w_i') = [w_i, u] - \sum_j \beta^j_i w_i' - \sum_{j,k} a^{jk}_i (-1)^{|u||w_k|} w_j w_k - \sum_{j,k} a^{kj}_i (-1)^{|w_k|} w_k w_i'.$$

2. There exists a morphism of differential graded algebras $\rho : (T(w_i, u, w_i'), D) \to (T(W), d)$ such that $\rho(u) = \alpha$ and $\rho(w_i) = w_i$ if and only if there are elements $\alpha_i \in T(W)$ satisfying

$$d(\alpha_i) = [w_i, \alpha] - \sum_j \beta^j_i \alpha_j - \sum_{j,k} a^{jk}_i (-1)^{|u||w_k|} \alpha_j w_k - \sum_{j,k} a^{kj}_i (-1)^{|w_k|} w_k \alpha_j.$$

Proof. We define $D(w_i')$ by the above formula. Proving that $D^2 = 0$ is an easy and standard computation. The morphism

$$\varphi : (T(w_i, u, w_i'), D) \to (T(w_i), d) \otimes (k[u], 0)$$

defined by $\varphi(w_i) = w_i$, $\varphi(u) = u$ and $\varphi(w_i') = 0$ is a surjective homomorphism of differential graded algebras. To prove that $\varphi$ is a quasi-isomorphism, we filter each differential graded algebra by putting $u$ in filtration degree 0 and the other variables in filtration degree one. We are then reduced to prove that $\varphi : (T(w_i, u, w_i'), D) \to (T(w_i), 0) \otimes (k[u], 0)$, $d(w_i) = 0, d(w_i') = [w_i, u]$, is a quasi-isomorphism. Denote by $K$ the kernel of $\tilde{\varphi}$ and consider the short exact sequence of complexes

$$0 \to (K \otimes E, D) \to (T(w_i, u, w_i') \otimes E, D) \xrightarrow{\tilde{\varphi}\otimes 1} ((T(w_i) \otimes k[u]) \otimes E, \tilde{D}) \to 0,$$
where $E$ is the linear span of the elements $1, sw_i, su$ and $sw'_i$, and where $D$ is defined by 

$$D(sw_i) = w_i \otimes 1, D(su) = u \otimes 1, D(sw'_i) = w'_i - (-1)^{|w'_i|} w_i \otimes su + (-1)^{|u||w'_i|+|u|} u \otimes sw_i.$$ 

By construction, $(T(sw_i, u, sw'_i) \otimes E, D)$ and $(T(w_i) \otimes k[u] \otimes E, \overline{D})$ are contractible and therefore quasi-isomorphic. Now a non-zero cocycle of lowest degree in $K$ is a non-trivial cocycle in the complex $(K \otimes E, D)$. Therefore $H_\ast(K) = 0$ and $\varphi$ is a quasi-isomorphism.

Part (2) of the Lemma follows directly from the expression of $D$. \hfill $\square$

**6 - Determination of $I$ when $k$ is a field of characteristic zero**

In this section $k$ is a field of characteristic zero.

6.1. By Theorem 5.2, the image of $I$ is contained in the center of $H_\ast(\Omega M)$. On the other hand, by the Milnor-Moore theorem (e.g \cite{8}-Theorem 21.5), $H_\ast(\Omega M)$ is the universal enveloping algebra of the homotopy Lie algebra $L_M = \pi_\ast(\Omega M) \otimes k$ (\cite{8}-p.294).

Let $L$ be any graded algebra. The center, $Z(U L)$, of the universal enveloping algebra $U L$ contains the universal enveloping algebra of the center of the Lie algebra, $U Z(L)$. However the inclusion can be strict. Consider for instance the Lie algebra $L = \mathbb{L}(a, b)/([b, b], [a, a, b])$, with $|a| = 2$ and $|b| = 1$. The element $(ab - ba)b$ is in the center of $U L$, but not in $U Z(L)$. We denote by $R(L)$ the sum of all solvable ideals in $L$, (\cite{8}-p.495).

6.2 Theorem. If $L = \{L_i\}_{i \geq 1}$ is a graded Lie algebra over a field of characteristic zero satisfying $\dim L_i < \infty$ then $Z(U L) \subset U R(L)$.

Proof. It is well known that in characteristic zero, $U L$ decomposes into a direct sum

$$U L = \bigoplus_{k \geq 0} \Gamma^k(L)$$

where the $\Gamma^k(L)$ are sub-vector spaces that are stable for the adjoint representation of $L$ on $U L$: $\Gamma^0(L) = k$, $\Gamma^1(L) = L$, and $\Gamma^n(L)$ is the sub-vector space generated by the elements $\varphi(x_1, \ldots, x_n) = \sum_{\sigma \in \Sigma_n} \varepsilon_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}$, $x_i \in L$. The coproduct $\Delta$ of $U L$ respects the decomposition, i.e.

$$\Delta : \Gamma^n(L) \to \bigoplus_{p+q=n} \Gamma^p(L) \otimes \Gamma^q(L).$$

If we denote by $\Delta_p$ the component of $\Delta$ in $\Gamma^p(L) \otimes \Gamma^{n-p}(L)$ then

$$\Delta_p(\varphi(x_1, \ldots, x_n)) = \sum_{\tau \in S_h p} \varepsilon_\tau \varphi(x_{\tau(1)}, \ldots, x_{\tau(p)}) \otimes \varphi(x_{\tau(p+1)}, \ldots, x_{\tau(n)}),$$

where $S_h p$ denotes the set of $p$-shuffles of the set $\{1, 2, \ldots, n\}$. This implies that the composition $\Gamma^n(L) \xrightarrow{\Delta_n} \Gamma^p(L) \otimes \Gamma^{n-p}(L) \xrightarrow{\text{multiplication}} U L$ is the multiplication by $\binom{n}{p}$.

We then consider the composite $c$

$$c : \Gamma^n(L) \xrightarrow{\Delta_n} L \otimes \Gamma^{-1}(L) \xrightarrow{1 \otimes \Delta^1} L \otimes L \otimes \Gamma^{-2}(L) \to \ldots \to L \otimes^n.$$

Let $\alpha \in U L$ be an element in the center of $U L$, $\alpha = \sum_{i=1}^n \alpha_i$ with $\alpha_i \in \Gamma^i(L)$. Since $\Gamma^i(L)$ is stable by adjunction, each $\alpha_i$ is in the center of $U L$. Therefore we can assume that $\alpha \in \Gamma^n(L)$. We write $c(\alpha)$ as a sum of monomials $x_i \otimes \ldots \otimes x_{i_n}$. Since $\text{mult} \circ c : \Gamma^n(L) \to U L$ is the multiplication by $n!$, the element $\alpha$ belongs to the Lie algebra generated by the $x_{i_j}$. Suppose that in the decomposition of $c(\alpha)$ the number of monomials
is minimal, then for each \( r, 1 \leq r \leq n \), the elements \( x_{i_1} \otimes \ldots \otimes x_{i_{r-1}} \otimes x_{i_r+1} \ldots \otimes x_{i_n} \) are linearly independent. Since \([\alpha, x] = 0, x \in L\), we obtain the equation:

\[
0 = \sum_{k=1}^{n} \left( \sum_{i} (-1)^{\left| (x_{i_1} + \ldots + x_{i_{k-1}}) \right|} x_{i_1} \otimes \ldots \otimes [x, x_{i_k}] \otimes \ldots \otimes x_{i_n} \right)
\]

Let us assume that the \( x_{i_k} \) are ordered by increasing degrees then the elements \( x_{i_k} \) with maximal degree belong to \( Z(L) \). The above equation shows also that \([x_{i_k}, x]\) belongs to the subvector space generated by the elements \( x_{i_k} \) with higher degree. A decreasing induction on the degree shows that all the \( x_{i_k} \) belong to \( R(L) \).

6.3. Denote by \( X_0 \) the 0-localization of a simply connected space \( X \). The Lusternik-Schnirelmann category of \( X_0 \), \( \text{cat} X_0 \), is less than or equal to the Lusternik-Schnirelmann of \( X \), \( \text{cat} X \). Moreover the invariant \( \text{cat} X_0 \) is easier to compute than \( \text{cat} X \), \((\mathbb{S})\text{-§-27})\).

**Theorem.** Let \( M \) be a simply connected oriented closed manifold and \( \mathfrak{L} \) is a field of characteristic zero then

a) The kernel of \( I \) is a nilpotent ideal and \( \text{Nil} (\text{Ker} (I)) \leq \text{cat} M_0 \).

b) \( (\text{Im} I) \cap (\pi_*(\Omega M) \otimes \mathfrak{L}) = G_s(M) \otimes \mathfrak{L} \).

c) \( \sum_{i=0}^{n} \dim (\text{Im} I \cap H_i(\Omega M; \mathfrak{L})) \leq C n^k \), some constant \( C > 0 \) and \( k \leq \text{cat} M_0 \).

**Proof.** a) By \((\mathbb{S})\text{-Theorems 29.1 and 28.5})\), \( C^*(M; \mathbb{Q}) \) is connected by a sequence of quasi-isomorphisms to a connected finite dimensional commutative differential graded algebra \((A, d)\) satisfying \( \text{Nil} (\overline{\alpha}) \leq n \) for \( n > \text{cat} M_0 \). Thus we conclude as in the proof of theorem 5.2).

b) The differential graded algebra \( \Omega(A^\prime) = (T(W), d) \) is the universal enveloping algebra on the graded Lie algebra \( \mathcal{L}_M = (\mathcal{L}(W), d) \), and the differential graded algebra \( (T(W \oplus \mathfrak{L}u \oplus W'), D) \) is the universal enveloping algebra of the differential graded Lie algebra \( \mathcal{L}_M^1 = (\mathcal{L}(W \oplus k u \oplus W'), D) \), \((\mathbb{S})\text{-p.289})\), with

\[
\begin{align*}
d(w_j) &= \sum_j \beta_j^i w_j + \sum_{j,k} \frac{\partial_j^k}{2} a^{ij}_k [w_j, w_k], \\
D(w'_j) &= [w_i, u] - \sum \beta_j^i w'_j - \sum_{j,k} a^{ij}_k (-1)^{\left| w_k \right|} [w_k, w'_j].
\end{align*}
\]

By construction \( \mathcal{L}_M \) is a free Lie model for \( M \) and \( \mathcal{L}_M^1 \) is a free Lie model for \( M \times S^n \) with \( n = |u| + 1 \), \((\mathbb{S})\text{-§24})\). Moreover there exists a bijection between homotopy classes of maps:

\[
[X \times S^n, X] \cong \left( [\mathcal{L}(W \oplus \mathfrak{L}u \oplus W'), D], (\mathcal{L}(W), d) \right).
\]

Therefore a homomorphism \( \varphi : (\mathcal{L}(W \oplus \mathfrak{L}u \oplus W'), D) \rightarrow (\mathcal{L}(W), d) \) such that \( \varphi(u) = \alpha \) and \( \varphi(w) = w, w \in W \), corresponds to a map \( f : M \times S^n \rightarrow M \) which extends \( id_M \cup g : M \times S^n \rightarrow M \), such that \( |g| = \alpha \) modulo the identifications \( \pi_n(M) \otimes \mathfrak{L} \cong \pi_{n-1}(\Omega M) \otimes \mathfrak{L} \cong H_{n-1}(\mathcal{L}(W), d) \). This means exactly that \( \text{Image} I \cap (\pi_*(\Omega M) \otimes \mathfrak{L}) = G_s(M) \otimes \mathfrak{L} \).

c) By, Theorems 36.4, 36.5 and 35.10 of \((\mathbb{S})\) we know that if \( L = \pi_*(\Omega M) \otimes \mathfrak{L} \) then \( R(L) \) is finite dimensional and \( \dim R(L)_{\text{even}} \leq \text{cat} M_0 \). We conclude using the graded Poincaré-Birkhoff-Witt theorem \((\mathbb{S})\text{-Theorem 21.1})\): \( \mathcal{Z}(UL) \subset UR(L) \cong \Lambda(R(L)_{\text{odd}}) \otimes \mathfrak{L} \{(R(L)_{\text{even}})\}. \)

\[\square\]

**7 - Examples and applications**

In this section we assume that \( \mathfrak{L} \) is a field.
7.1 The spheres $S^n$. The loop homology of spheres has been computed by Cohen, Jones and Yan using a spectral sequence similar to those obtained in 4.5. Our model leads to a direct computation of the loop homology and the intersection homomorphism $I$.

Since the differential graded algebra $C^*(S^n)$ is quasi-isomorphic to $(H^*(S^n),0) = (\wedge u/u^2,0), |u| = n$, by Example 4.6, $H_*(LS^n)$ is isomorphic as an algebra to

$$H^*(\wedge u \otimes T(v), D), |v| = n - 1, |u| = -n, D(u) = 0, D(v) = u \otimes [v,v].$$

When $n$ is odd, $D = 0$, $H_*(LS^n) \cong \wedge u \otimes T(v)$ and $I = \varepsilon \otimes 1 : \wedge u \otimes T(v) \to T(v)$. When $n$ is even, $D(v^{2n}) = 0$, $D(v^{2n+1}) = 2u \otimes v^{2n+2}$. Therefore a set of generators of $H_*(LS^n)$ is given by the elements $c = 1 \otimes v^2, b = u \otimes v, a = u \otimes 1$ and,

$$H_*(LS^n) \cong \wedge (b) \otimes \mathbb{k}[a,c]/(2ac, a^2, ab), |a| = -n, |b| = -1, |c| = 2n - 2.$$

The homomorphism $I : H_*(LS^n) \to H_*(\Omega S^n) = T(v)$ is given by: $I(c) = v^2, I(a) = I(b) = 0$.

7.2 An example where $I$ is the trivial homomorphism. Let $M$ be the connected sum $M = (S^3 \times S^3 \times S^3) \# (-S^3 \times S^3 \times S^3)$. The wedge $N = (S^3 \times S^3 \times S^3) \vee (S^3 \times S^3 \times S^3)$ is then obtained by attaching a 9-dimensional cell to $M$ along the homotopy class determined by the collar between the two components of $M$. Recall that

$$\pi_*(\Omega N) \otimes \mathbb{Q} \cong Ab(a,b,c) \coprod Ab(e,f,g),$$

where $Ab(u,v,w)$ means the abelian Lie algebra generated by $u, v$ and $w$ considered in degree 2. The inclusion $i : M \to N$ induces a surjective map $\pi_* (\Omega M) \otimes \mathbb{Q} \to \pi_* (\Omega N) \otimes \mathbb{Q}$, This means that the attachment of the cell is inert in the sense of [S]-p.503. Therefore, ([S]-Theorem 38.5),

$$\pi_* (\Omega M) \otimes \mathbb{Q} \cong Ab(a,b,c) \coprod Ab(e,f,g) \coprod \mathbb{L}(x)$$

with $|x| = 7$. In particular $R(L)$ is zero, and by Theorems 5.2 and 6.2, when $\mathbb{k}$ is of characteristic zero, the homomorphism $I$ is trivial.

7.3 Product of two manifolds. If $A$ and $B$ are differential graded algebras, we have a natural isomorphism of algebras

$$HH^*(A \otimes B, A \otimes B) \cong HH^*(A,A) \otimes HH^*(B,B).$$

Therefore, from Theorem 2.7, for closed oriented simply connected manifolds $M$ and $N$, the isomorphism

$$\mathbb{H}_*(L(M \times N)) \cong \mathbb{H}_*(LM) \otimes \mathbb{H}_*(LN)$$

respects the Chas-Sullivan product. Moreover the intersection homomorphism $I_{M \times N}$ identifies to $I_M \otimes I_N$.

7.4. Lie groups. Let $\mathbb{k}$ be a field of characteristic zero and $G$ be a connected Lie group. Since $G$ has the rational homotopy type of a product of odd dimensional spheres, we obtain

$$\mathbb{H}_*(LG; \mathbb{Q}) \cong \wedge (u_1, \ldots, u_n) \otimes \mathbb{T}(v_1, \ldots, v_n),$$

and $I_G$ is onto. This example generalizes in:

7.5 Theorem. Let $\mathbb{k}$ be a field of characteristic zero and $M$ be a simply connected closed oriented d-dimensional manifold. The intersection homomorphism $I : \mathbb{H}_*(LM) \to$
$H_*(\Omega M)$ is surjective if and only if $M$ has the rational homotopy type of a product of odd dimensional spheres.

**Proof.** When $M$ has the rational homotopy type of the product of odd dimensional spheres, then $I$ is clearly surjective. Conversely, if $I$ is surjective, then $\pi_*(\Omega M) \otimes \mathbb{Q} = G_*(M) \otimes \mathbb{Q}$. Thus, $\pi_*(M) \otimes \mathbb{Q} = G_{\text{odd}} \otimes \mathbb{Q}$. Let $\{f_i : S^{n_i} \to M, i = 1, \ldots, r\}$ represent a given linear basis of $\pi_*(M) \otimes \mathbb{Q}$, and let $\varphi_i : S^{n_i} \times M \to M$ be maps that restrict to $f_i \vee \text{id}_M$ on $S^{n_i} \vee M$. Then the composition

$$S^{n_1} \times \ldots \times S^{n_r} \to S^{n_1} \times \ldots \times S^{n_r} \times M \xrightarrow{1 \times \varphi_1} S^{n_1} \times \ldots \times S^{n_r} \times M \xrightarrow{1 \times \varphi_{r-1}} \ldots \xrightarrow{1 \times \varphi_1} M$$

induces an isomorphism on the homotopy groups. Therefore, $M$ has the rational homotopy type of a product of odd dimensional spheres. □

8 - Hochschild cohomology and Poincaré duality

When two $A$-bimodules $M$ and $N$ are quasi-isomorphic as bimodules, then the Hochschild cohomologies $HH^*(A; M)$ and $HH^*(A; N)$ are isomorphic. In this section we relate the Hochschild cohomology of the singular cochains algebra on $X$ with coefficients in itself and with coefficients in the singular chains on $X$ when $X$ is a Poincaré duality space. The usual cap product with the fundamental class is not a bimodule morphism. However the cohomology $HH^n(C^*(M); C_*(M))$ and $HH^{n-d}(C^*(M); C^*(M))$ are shown to be isomorphic. This point is not directly related to the constructions built in sections 4 to 8, but has its own importance. For this reason we have added this point in the last section of this paper.

8.1. Let $V$ be a graded module, then $V^\vee$ denotes the graded dual, $V^\vee = \text{Hom}_k(V, k)$, and $(\langle - , - \rangle : V^\vee \otimes V \to k$ denotes the duality pairing. We denote by $\lambda_V : V \to V^{\vee \vee}$ the natural inclusion defined by $\langle \lambda_V(v), \xi \rangle = (-1)^{|v||\xi|} \langle \xi, v \rangle$.

8.2. Let $X$ be topological space. The $C^*(X)$-bimodule structures on $C_*(X)$ and $C^*(X)^\vee$ are explicitly defined by:

\[
f \cdot c \cdot g := (-1)^{|c||f|+|g|+|f|+|g|}|g \otimes \text{id} \otimes f)(\Delta_X \otimes \text{id}) \circ \Delta_X(c), \quad c \in C_*(X),
\]

\[
\langle f \cdot \alpha \cdot g, h \rangle := (-1)^{|f|}|\alpha; g \cup h \cup f|, \quad f, g, h \in C^*(X), \alpha \in C^*(X)^\vee.
\]

Remark that the associativity properties of $AW$ and of $\Delta_X$ imply directly that $C_*(X)$ is a graded $C^*(X)$-bimodule.

Let $1 \in C^0(X)$ be the 0-cochain which value is 1 on the points of $X$. The usual cap product is then defined by

\[
C^p(X) \otimes C_k(X) \longrightarrow C_{k-p}(X), \quad f \otimes c \mapsto f \cap c = f \cdot c \cdot 1 = \sum_i (-1)^{|c_i|+|f|} c_i f(c'_i).
\]

The cap product with a cycle $x \in C_k(X)$ is a well defined homomorphism of differential graded modules, but is not a “degree $k$ homomorphism” of $C^*(X)$-bimodules. However,

8.3. **Theorem.** Let $X$ be a path connected space and $c \in C_*(X)$ be a cycle of degree $k > 0$. Then there exists a (degree $k$) morphism of $C^*(X)$-bimodules

\[
\gamma_c : B(C^*(X), C^*(X), C^*(X)) \to C_*(X)
\]

such that

- $\gamma_c(1[]1) = c$,
\[
\bullet H(\gamma_c) \circ H(m)^{-1} : H^*(X) \to H_*(X) \text{ is the cap product by } [c], \ m \text{ is the quasi-isomorphism of } C^*(X)\text{-module defined in 1.4.}
\]

Recall that \( \gamma_c \) is a degree \( k \) morphism of \( C^*(X) \)-bimodules means that the following two properties are satisfied:

a) \( d \circ \gamma_c = (-1)^k \gamma_c \circ d \),

b) \( \gamma_c(f \cdot \alpha \cdot g) = (-1)^{|f|k} f \cdot \gamma_c(\alpha) \cdot g \),

for \( f, g \in C^*(X) \) and \( \alpha \in \mathbb{B}(C^*(X), C^*(X), C^*(X)) \).

**Proof.** For simplicity we denote by \( A^e \) the enveloping algebra of \( A = C^*(X) \) and by \( B \) the differential graded \( C^*(X) \)-bimodule \( \mathbb{B}(C^*(X), C^*(X), C^*(X)) \).

Recall the loop space fibration \( \text{ev} : X^{S^1} \to X \), \( \gamma \mapsto \gamma(0) = \gamma(1) \) with the canonical section \( \sigma : X \to X^{S^1} \), \( x \mapsto \) the constant loop at \( x \). Jones defined a quasi-isomorphism of differential graded modules ([23] Theorem 8),

\[
J_* : B \otimes A^e \to C^*(X^{S^1})
\]

making commutative the following diagram of differential graded modules

\[
\begin{array}{ccc}
B \otimes A^e & \xrightarrow{J_*} & C^*(X^{S^1}) \\
\downarrow i & & \nearrow \mathbb{C}^*(X) \\
C^*(X) & &
\end{array}
\]

where \( i : C^*(X) \to B \otimes A^e C^*(X) \), \( f \mapsto 1[[1 \otimes f] \), denotes the canonical inclusion. Let \( \rho \) be the composite \( C^*(\sigma) \circ J_* \) then \( \rho \) is a retraction of \( i : \rho \circ i = id \).

Let \( u \in C^k(X)^v \), \( k > 0 \), be a cycle. Using the canonical isomorphism of differential graded modules

\[
\Psi : \text{Hom}(B \otimes A^e, A_k) \to \text{Hom}_{A^e}(B, A^v), \quad (\Psi(\theta)(\alpha))(f) = \theta(\alpha \otimes f),
\]

we define the map

\[
\theta_u : \mathbb{B}(C^*(X), C^*(X), C^*(X)) \to (C^*(X))^v, \quad \theta_u = \Psi(\circ \rho).
\]

The element \( \theta_u \) is a \( k \)-cycle in \( \text{Hom}_{A^e}(B, A^v) \) and for any \( f \in A \), \( \theta_u(1[[1 \otimes f) = u \circ \rho \circ i(f) = u(f) \).

Since the linear map

\[
\lambda : C^*(X) \to C^*(X)^v
\]

is a morphism of differential graded \( C^*(X) \)-bimodules, for a cycle \( c \in C_k(X) \), we have a morphism

\[
\theta_{\lambda(c)} : \mathbb{B}(C^*(X), C^*(X), C^*(X)) \to (C^*(X))^v
\]

with \( \theta_{\lambda(c)}(1[[1) = \lambda(c) \).

Since \( \mathbb{B}(C^*(X), C^*(X), C^*(X)) \) is semifree, we deduce from the lifting homotopy property (cf. 1.4) a morphism of \( C^*(X) \)-bimodules

\[
\gamma_c : \mathbb{B}(C^*(X), C^*(X), C^*(X)) \to C^*(X)
\]

making commutative, up to homotopy, the diagram

\[
\begin{array}{ccc}
\mathbb{B}(C^*(X), C^*(X), C^*(X)) & \xrightarrow{\theta_{\lambda(c)}} & C^*(X)^v \\
\| & & \uparrow \lambda \\
\mathbb{B}(C^*(X), C^*(X), C^*(X)) & \xrightarrow{\gamma_c} & C^*(X)
\end{array}
\]
and such that $\gamma_c(1[1]) = c$.

The equality $H(\gamma_c) \circ H(m)^{-1} = \cap[c]$ comes from the commutativity of the diagram

$$\begin{array}{ccc}
\mathfrak{M}_0(C^*(X), C^*(X), C^*(X)) & \xrightarrow{\theta_{\lambda(c)}} & C^*(X)^{\vee} \\
\downarrow m & & \uparrow \lambda \\
C^*(X) & \xrightarrow{-\cap[c]} & C_*(X)
\end{array}$$

i.e., for any $f, g, h \in C^*(X)$, we have $\langle \theta_{\lambda(c)}(f[g]), h \rangle = \langle \lambda \circ (-\cap[c]) \circ m(f[g]), h \rangle$.

As a special case, we deduce:

**8.4. Theorem.** Let $M$ be a 1-connected $k$-Poincaré duality space of formal dimension $d$. Then there are quasi-isomorphisms of $C^*(M)$-bimodules

$$C^*(M) \xrightarrow{\mathfrak{M}} \mathfrak{B}(C^*(M), C^*(M), C^*(M)) \xrightarrow{\gamma} C_*(M)$$

where $m$ is defined in lemma 1.3 and $\gamma = \gamma_{[M]}$ with $[M] \in H_d(M)$ a fundamental class of $M$. In particular, the composite, $H(m) \circ H(\gamma)^{-1}$ is the Poincaré isomorphism $\mathcal{P} : H_*(M) \to H^{d-*}(M)$.

Applying Hochschild cohomology, we obtain

**8.5 Theorem.** Let $M$ be a 1-connected $k$-Poincaré duality space of formal dimension $d$ then there exist natural linear isomorphisms

$$D : HH^n(C^*(M); C_*(M)) \xrightarrow{\cong} HH^{n-d}(C^*(M); C^*(M)).$$

**Proof.** Let $\varphi : N \to N'$ be a homomorphism of differential graded $A$-bimodules and assume that $A$ is a $k$-module. Then we deduce from 1.4 (see [7] for more details) that $\varphi$ induces an isomorphism of graded modules

$$HH^*(A; N) \to HH^*(A; N').$$

Theorem 8.5 follows directly from Theorem 8.4 when one observes that the suspended map $s^d\gamma$ is a quasi-isomorphism of differential graded $C^*(X)$-bimodules.

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felix@math.ucl.ac.be  
Jean-claude.thomas@univ-angers.fr  
vigue@math.univ-paris13.fr

Département de mathématique  
Département de mathématique  
Département de mathématique

Université Catholique de Louvain  
Faculté des Sciences  
Institut Galilée

2, Chemin du Cyclotron  
2, Boulevard Lavoisier  
93430 Villetaneuse, France

1348 Louvain-La-Neuve, Belgium  
49045 Angers, France