The two-thirds law for pairwise velocity and origin of critical MOND acceleration from
distributions of density, velocity, and acceleration in dark matter flow

Zhijie (Jay) Xu¹,a

¹. Computational Mathematics Group, Physical and Computational Sciences Directorate, Pacific Northwest National Laboratory, Richland, WA 99352, USA

Abstract:

The statistics of density, velocity, and acceleration plays an essential role for probing large scale structure formation and direct/indirect detection experiments. In this paper, the scale and redshift dependence of density, velocity and acceleration distributions in dark matter flow are extensively studied. A halo-based non-projection approach is proposed to perform the statistical analysis: i) all halos are identified with particles divided into halo and out-of-halo particles such that distributions can be studied separately; ii) instead of projecting particle field onto structured grid that involves information loss, analysis is performed over all pairs on a given scale; iii) scale and redshift dependence of distributions can be studied by the variation of generalized kurtosis.

For density distribution, Delaunay tessellation is used to reconstruct the density field. The particle overdensity $\delta$ has an asymmetric distribution with a long tail $\propto \delta^{-3}$. The log-density $\eta$ evolves from Gaussian to a bimodal distribution at $z=0$, with two peaks corresponding to high density from halo particles and low density from out-of-halo particles. Without projecting onto grid, density correlation is obtained directly by counting all pairs on a given scale and cannot be positive on all scales due to normalization. Density spectrum and fluctuation functions are obtained

---

¹) Electronic mail: zhijie.xu@pnnl.gov; zhijiexu@hotmail.com
from correlations and compared with theory. Simple analytical models for all second order density
statistics are developed.

The scale dependence of velocity field is studied for longitudinal velocity $u_L$ or $u'_L$, velocity
difference $\Delta u_L = u'_L - u_L$ (or pairwise velocity) and sum $\Sigma u_L = u'_L + u_L$ of two longitudinal
velocities. Fully developed velocity fields are never Gaussian on any scale despite that they can
be initially Gaussian. $\Sigma u_L$ and $u_L$ have the same distribution on small scale and the correlation $\rho_L$
between $u_L$ and $u'_L$ approaches $1/2$. While on large scale, $\Sigma u_L$ and $\Delta u_L$ have the same
distribution with correlation $\rho_L \to 0$. The first moment of $\Delta u_L$ (mean pairwise velocity) is
analytically modelled on both small and large scales. The second moment of $\Delta u_L$ (structure
function) approaches $2u^2$ on small scale and a two-thirds law can be identified for a reduced
structure function that is $\propto \left(-\epsilon_u\right)^{2/3} r^{2/3}$, where $\epsilon_u$ is the constant rate of energy cascade. A
constant length scale is introduced $r_s = u_0^3/\epsilon_u$, below which the two-thirds law is valid. The two-
thirds law can be generalized to all even moments of pairwise velocity $\left\langle (\Delta u_L)^{2n} \right\rangle$, while odd
moments $\left\langle (\Delta u_L)^{2n+1} \right\rangle \propto r$ satisfy generalized stable clustering hypothesis (GSCH).

The distributions of velocities are analytically modeled. On small scale, both $u_L$ and $\Sigma u_L$ can
be modelled by a $X$ distribution to maximize system entropy. While explicit form for distribution
of $\Delta u_L$ is still unknown, the moments and kurtosis of $\Delta u_L$ can be analytically estimated based on
a halo-size dependent correlation coefficient $\rho_L$. On intermediate scale, distributions of $u_L$ and
$\Delta u_L$ becomes significantly non-symmetric with non-zero skewness, a necessary feature for inverse
energy cascade. On large scale, both $\Delta u_L$ and $\Sigma u_L$ approach the same distribution that can be
modelled by a logistic function. The redshift evolution of distributions of different velocities is studied via the evolution of generalized kurtosis. With time, all velocities become non-Gaussian and the evolution approximately follows the prediction of a $X$ distribution with a decreasing $\alpha$ to gradually maximize system entropy. However, velocities on large scale usually evolves at a much slower pace than velocities on small scale because of stronger gravity on small scale.

The redshift evolution of acceleration distributions is studied for halo and out-of-halo particles respectively, with a long tail $\propto a_{hp}^{-3}$ at large acceleration from halo particles in core region. The typical acceleration ($\sim 10^{-10} \text{m/s}^2$) in halos matches the critical acceleration scale $a_m$ in MOND, which can be determined by the constant rate of energy transfer $\varepsilon_u$, i.e. $a_m \propto \varepsilon_u /u$. For particles with acceleration below that scale, particle energy is proportional to its velocity. For particles subject to an external force and moving through a fluid involving both velocity and acceleration fluctuations, both deep-MOND and standard Newtonian behavior can be recovered. The fluctuation of acceleration in halos possibly plays the role of critical acceleration scale in MOND, while the fluctuation of halo acceleration is much smaller and on the order of $\sim 10^{-12} \text{m/s}^2$. In this regard, the MOND theory is intrinsically consistent with (or a direct result of) self-gravitating collisionless dark matter flow that involves both velocity and acceleration fluctuations.
## Contents

### Nomenclature

5

1. **Introduction** ......................................................................................................................................................... 6

2. **The N-body simulations and numerical data** .................................................................................................... 9

3. **Density distribution/correlation/dispersion functions and matter power spectrum** ........................................ 9

   3.1 One-point probability distributions of density field........................................................................................... 10

   3.2 Two-point statistical measures of density field.................................................................................................. 17

      3.2.1 Two-point density correlation functions from radial distribution function................................................ 17

      3.2.2 Specific potential/kinetic energy from density correlation function.......................................................... 18

      3.2.2 Density spectrum/dispersion functions and real space distribution of density fluctuation......................... 19

   3.3 Two-point second order statistical measures of density from N-body simulation............................................. 21

   3.4 Modeling second order statistical measures of density field............................................................................. 26

4. **Characterizing probability distributions of velocity fields** ............................................................................ 28

   4.1 Generalized kurtosis, moments, and structure functions of velocity fields......................................................... 30

   4.2 Generalized kurtosis of velocity fields from N-body simulation......................................................................... 31

   4.3 First moment of velocity fields and pair conservation equation ........................................................................ 34

   4.4 Second moments of velocities ............................................................................................................................ 39

   4.5 Two-thirds law for even order moments of pairwise velocity (structure functions) ........................................ 41

   4.6 Odd moments of pairwise velocity and Generalized stable clustering hypothesis (GSCH) .............................. 47

5. **Distributions of velocity on different scales and their redshift evolution** ..................................................... 49

   5.1 Modeling velocity distributions on small scale ................................................................................................. 49

   5.2 The limiting distribution and moments of pairwise velocity $\Delta \mu_L$ on small scale ...................................... 50

   5.3 Velocity distributions on the intermediate scale................................................................................................ 54

   5.4 Modeling the velocity distributions on large scale............................................................................................ 57

   5.5 The redshift evolution of velocity distributions ................................................................................................. 60

6. **Distributions of acceleration and critical acceleration scale in MOND** ............................................................. 62

   6.1 The redshift evolution of proper acceleration .................................................................................................. 62

   6.2 The critical acceleration scale in MOND ............................................................................................................ 67

7. **Conclusions** ................................................................................................................................................... 72
Nomenclature

See supplementary information
1. Introduction

The cosmic peculiar velocity and density fields contain rich information for the dynamics of self-gravitating collisionless dark matter flow (SG-CFD) from large scale to the highly non-linear small scales. Statistics of velocity and density fields is crucial for fundamental questions regarding structure formation and dynamics. The statistical analysis of velocity fields was previously applied to describe the evolution of a system of self-gravitating collisionless particles using BBGKY equations [1]. The pairwise velocity has been introduced to probe the cosmological density parameter [2, 3], and the two-point correlation function was introduced to quantify the cosmic velocity field from real dataset [4, 5]. In addition, velocity distribution has profound implications for detection experiments. For predicted DM-nucleon scattering in direct detection [6, 7], the detection rate of scattering is proportional to the inverse (or -1) moment of distribution. That rate is very sensitive to the high velocity tail of distribution. For indirect search [8, 9], the annihilation cross section is directly dependent on the distribution of relative velocity. Velocity distribution of dark matter particles is expected to be different from Maxwell-Boltzmann. This is confirmed by simulations [10, 11] and theory from maximum entropy principle [12]. The one-point distribution of matter density is another fundamental property for gravitational lensing and nonlinear clustering. The study of matter density has a long history dating back to 1930s when Hubble found the matter distribution is non-Gaussian and can be approximate by log-normal distribution [13]. The interests and efforts are still ongoing both theoretically and numerically [14, 15].

While directly measuring velocity and density fields from real samples is still challenging in practice, tremendous information can be obtained from N-body simulation, an invaluable tool to study the dynamics of collisionless flow in both linear and nonlinear regime [16-19]. However, it is not trivial to extract and characterize the statistics of velocity and density from N-body
simulations. There is a fundamental problem as velocity and density are only sampled at discrete locations of particle position in N-body simulations. That sampling has a poor quality at locations with low particle density [20]. The standard approach computes the power spectrum of velocity, density (and its gradients) in Fourier space [21-23], where cloud-in-cell (CIC) [24] or triangular-shaped-cloud (TSC) schemes are used to project density and velocity fields onto regular grids. This will unavoidably introduce sampling errors in density and velocity fields [25, 26]. Both real-space and Fourier-space data contain the same information, while directly working in real-space data avoids the information distortion due to field projection and the associated errors due to the conversion between Fourier- and real-space.

In this paper, a new halo-based non-projection approach is proposed for statistical analysis: i) instead of projecting particle fields onto structured grid, analysis is performed in real space by the statistics over all pairs on different scales. This will maximumly preserve and utilize the information from N-body simulation; ii) Based on the halo description of N-body system, distributions should evolve differently in halos and out-of-halo region. Therefore, all halos in N-body system are identified and all particles are divided into halo and out-of-halo particles such that distributions can be studied separately; iii) Scale and redshift dependence of distributions can be studied by the variation of generalized kurtosis for a given distribution.

From this practice, tremendous amount of knowledge was learned that can be compared with the statistical theory of isotropic, homogeneous, and incompressible turbulence [27-30]. One example is the distribution of pairwise velocity. For incompressible flow, there exist an inertial range in energy spectrum with a constant energy flux followed by a dissipation range dominant by viscosity. A universal form was established for longitudinal velocity structure function of \( m \)th order (\( m \)th moment of pairwise velocity) in inertial range [31],

\[
\ldots
\]
\[ S^{lp}_m(r) = \left\langle (u'_{L} - u_{L})^m \right\rangle = \beta_m (\varepsilon_u)^{m/3} r^{m/3}, \]  

where \( u'_{L} \) and \( u_{L} \) are longitudinal velocities, \( \beta_m \) is a universal constant, and \( \varepsilon_u \) is the energy dissipation rate. Specifically, the second order structure function (i.e. pairwise velocity dispersion for cosmic velocity) \( S^{lp}_2(r) = \beta_2 \varepsilon_u r^{2/3} \) with \( \beta_2 \approx 2 \) is known as the two-thirds law, while \( S^{lp}_2(r) \propto r^2 \) in the dissipation range where viscous force is dominant. The third order structure function \( S^{lp}_3(r) = -4/5 \varepsilon_u r \) with \( \beta_3 = -4/5 \) is known as the four-fifths law that can be exactly derived from Navier-stokes equation [29]. Similarly, \( S^{lp}_3(r) \propto r^3 \) in the dissipation range for incompressible flow. However, the dark matter flow exhibits completely different behavior due to collisionless nature and long-range interaction.

By studying the scale and redshift dependence of density, velocity, and acceleration distributions using a halo-based non-projection approach, we can i) show the redshift evolution of one-point density and log-density distributions for halo and out-of-halo particles with analytical models for second order density statistics; ii) demonstrate that velocity fields are non-Gaussian on all scales despite that they can be initially Gaussian; iii) model velocity distributions separately on small and large scales; iv) identify universal two-thirds law for (reduced) even order structure functions and liner scaling for odd order structure functions; v) determine the acceleration fluctuations in halos and of halos, where fluctuation in halos \( (\sim 10^{-10} \text{ m/s}^2) \) matches the critical acceleration in MOND \( a_m \) and fluctuation of halos is much smaller \( (\sim 10^{-12} \text{ m/s}^2) \); vi) determine the critical acceleration \( a_m \) by the rate of energy transfer \( \varepsilon_u \), i.e. \( a_m \propto \varepsilon_u / u \).

Finally, this paper is organized as follows: Section 2 introduces the N-body simulation data, followed by statistical measures for density in Section 3. The redshift and scale dependence of
velocity distributions are presented and modeled in Sections 4 and 5. The distribution of acceleration and relation to critical acceleration in MOND are discussed in Sections 6.

2. The N-body simulations and numerical data

The numerical data were public available and generated from the N-body simulations carried out by the Virgo consortium. A comprehensive description of the data can be found in [32, 33]. As the first step, the current study was carried out using the simulation runs with $\Omega = 1$ and the standard CDM power spectrum (SCDM) to focus on the matter-dominant gravitational flow of collisionless matter. Similar analysis can be extended to other simulations with different model assumptions in the future. The same set of data has been widely used in studies from clustering statistics [33] to formation of cluster halos in large scale environment [34], and test of models for halo abundances and mass functions [35]. More details are provided in supplementary information.

3. Density distribution/correlation/dispersion functions and matter power spectrum

Various statistical measures can be introduced to characterize the velocity field in self-gravitating collisionless flow [36, 37], i.e. the real-space correlation, dispersion and structure functions, and power spectrums in Fourier space. They are related to each other through kinematic relations for a given type of flow. The real-space correlation functions are the most fundamental quantity and building blocks of statistical theory for any stochastic field. The statistical measures of density field are also a primary focus of many existing literature. This section will introduce/review these fundamental statistical descriptions and the relations between different measures, along with results from N-body simulations. Analytical models for these statistical measures are also presented.
3.1 One-point probability distributions of density field

Projecting particle field onto structured grid usually involves information loss and numerical noise. Without projecting onto grid, Delaunay tessellation is used in this section to reconstruct the density field and maximumly preserve information in N-body data. For a particle at location $\mathbf{x}$, the overdensity $\delta(\mathbf{x})$ and log-density $\eta(\mathbf{x})$ are defined as

$$\delta(\mathbf{x}) = \frac{\rho(\mathbf{x})}{\rho_0} - 1 \quad \text{and} \quad \eta(\mathbf{x}) = \log \left( 1 + \delta(\mathbf{x}) \right) = \log \left( \frac{\rho(\mathbf{x})}{\rho_0} \right),$$

where $\rho(\mathbf{x}) = m_p/V_p$ is a local matter density at comoving coordinate $\mathbf{x}$, $m_p$ is the particle mass, $V_p$ is the volume occupied by that particle, and $\rho_0$ is the mean (comoving) density. In linear theory, $\eta(\mathbf{x}) \approx \delta(\mathbf{x})$ for small $\delta(\mathbf{x}) \ll 1$ on large scale. They are different on small scale in the nonlinear regime. Due to normalization that total volume should equal the sum of all particle volumes ($\sum V_p = V$), the redshift evolution of distributions of $\delta$ and $\eta$ should always satisfy

$$\left\langle \frac{1}{1 + \delta(\mathbf{x})} \right\rangle = 1 \quad \text{and} \quad \left\langle e^{-\eta(\mathbf{x})} \right\rangle = 1.$$

Unlike the velocity field, density is not a variable automatically carried by and computed for each particle in N-body simulation. The Delaunay tessellation can be applied to reconstruct the density field from a discrete set of particles [38, 39]. All particles in system are first connected by a set of non-overlapping tetrahedra (triangles in two-dimension). The volume $V_p$ that each particle occupies can be determined from the volume of its surrounding tetrahedral. The density $\rho(\mathbf{x})$ of each particle can be subsequently computed. This enables us to compute density distribution for particles in halos and out-of-halo particles separately.
By computing local density for each particle from simulation in Section 2, Fig. 1a) presents the redshift evolution of one-point density distribution $\delta(x)$ for all particles in N-body system. Due to the gravitational collapse on small scale, $\delta(x)$ evolves from an initial Gaussian (symmetric) at high redshift to a “double-power-law” distribution (asymmetric and highly skewed toward $\delta > 0$) at $z=0$ with a long tail $\propto \delta^{-3}$. The distribution is approximately $\propto \delta^{-1}$ for small $\delta$. For comparison, the distribution of density can also be obtained by projecting particles onto structure grid using Cloud-in-Cell (CIC) assignment scheme with a given size of grid ($\Delta x$). Results of (grid-based) density distributions with different $\Delta x$ are presented in Fig. 1b) with an approximate scaling of $\propto \delta^{-2}$. For grid-based density, $\langle \delta \rangle = 0$. Because of the limit of grid resolution, the grid-based density is much lower than particle density from Delaunay tessellation.
Figure 1. a) The redshift evolution of density distribution $\delta$ from $z=10$ to $z=0$. Density evolves from an initial Gaussian to an asymmetric distribution with a long tail $\propto \delta^{-3}$. Initial Gaussian at high redshift has two branches ($\delta > 0$ and $\delta < 0$); b) density distribution of $(1+\delta)$ obtained by projecting particles onto structured grid with a CIC scheme and different grid size $\Delta x$ for redshift $z=0$. The distribution shows an approximate scaling of $\propto \delta^{-2}$. The particle density from Delaunay tessellation is also plotted for comparison with greater density values than grid-based density.

Similarly, Fig. 2 plots the redshift evolution of log-density distribution $\eta(x)$ from $z=10$ to $z=0$. A bimodal distribution is gradually developed from an initial Gaussian distribution. The first peak corresponds to out-of-halo particles in the low-density region that do not belong to any halos with $\langle \eta \rangle < 0$. The second peak comes from all particles residing in halos with higher density and wider dispersion. A simple bimodal equation can be used to fit this distribution,

$$f(\eta) = \frac{c_1}{\sqrt{2\pi}\sigma_1} \exp \left(\frac{(\eta - \mu_1)^2}{2\sigma_1^2}\right) + \left(1 - c_1\right) \frac{1}{\sqrt{2\pi}\sigma_2} \exp \left(\frac{(\eta - \mu_2)^2}{2\sigma_2^2}\right)$$  \hspace{1cm} (4)
with best fitting parameters $c_1 = 0.404$, $\mu_1 = -0.30$, $\sigma_1 = 1.212$, $\mu_2 = 4.256$, $\sigma_2 = 2.979$ at $z=0$.

Fitted curve is plotted in the same figure with about 60% particles in halos and 40% out-of-halo particles. This is consistent with the result of Eq. (38) in [40] for 60% of total mass in all halos, where there is a continuous exchange of mass from out-of-halo to halos due to inverse mass cascade. Particles in halos should have an average density close to $\delta = \Delta_c - 1$, where the critical density ratio $\Delta_c = 18\pi^2$ from a spherical collapse model or a two-body collapse model (TBCM, Eq. (89) in [41]) such that $\langle \eta \rangle \approx 5.17$ (close to $\mu_2$ as the mean density for all halo particles).

Figure 2 reflects the effect of inverse mass cascade on density distributions [40].

---

**Figure 2.** Distribution of log-density $\eta(x)$ at different redshifts $z$. The log-density $\eta(x)$ evolves from a relatively Gaussian to an approximately bimodal distribution at $z=0$ with two peaks corresponding to halo (60%) and out-of-halo (40%) particles. Inverse mass cascade leads to continuous mass exchange from out-of-halo to halos and the formation of two peaks in density distribution.
It is also natural to check the density distributions of particles in halos and out-of-halo particles separately. By identifying all halos in entire system and dividing all particles into halo and out-of-halo particles, Figure 3 presents the redshift evolution of distributions of \( \eta(x) \) for halo and out-of-halo particles, respectively. For out-of-halo particles, the distribution is relatively Gaussian (or \( \delta(x) \) is lognormal) with mean density decreasing with time. For halo particles, the log-density distribution evolves with increasing mean density due to the formation of halos.

![Figure 3. Redshift evolution of log-density distributions \( \eta(x) \) for two different types of particles. For out-of-halo particles, distribution is relatively Gaussian with a decreasing and negative mean density due to more out-of-halo particles absorbed into halos. For halo particles, the distribution evolves with an increasing mean log-density.](image)

To characterize the time evolution of distribution of any random variable \( \tau \), statistical quantities such as the skewness and kurtosis should be used. A generalized kurtosis \( K_n(\tau) \) reads
where the central moment of order $n$ for random variable $\tau$ reads

$$S_n^{cp}(\tau) = \langle (\tau - \langle \tau \rangle)^n \rangle.$$  \hfill (6)

The odd order generalized kurtosis should vanish for symmetric distributions. Specifically for Gaussian distribution, $K_3 = K_5 = 0$, $K_2 = 1$, $K_4 = 3$, $K_6 = 15$, and $K_8 = 105$.

Figure 4. The redshift evolution of generalized kurtosis of log-density $\eta$ for two different types of particles. The density distribution for out-of-halo particles is relatively Gaussian with generalized kurtosis $K_4 \approx 3$ and $K_6 \approx 15$ at $z=0$. The density distribution for halo particles becomes more symmetric with vanishing odd order generalized kurtosis, while even order kurtosis $K_4 \rightarrow 2$ and $K_6 \rightarrow 7$. 

\[ K_n(\tau) = \frac{\langle (\tau - \langle \tau \rangle)^n \rangle}{\langle (\tau - \langle \tau \rangle)^2 \rangle^{n/2}} = \frac{S_n^{cp}(\tau)}{S_2^{cp}(\tau)^{n/2}}, \]  \hfill (5)
Figure 4 presents the redshift evolution of the generalized kurtosis. For out-of-halo particles, the generalized kurtosis ($K_3(\eta)$ to $K_6(\eta)$) is relatively independent of time. The distribution is relatively Gaussian with $K_4 \approx 3$ and $K_6 \approx 15$ at $z=0$, such that the distribution of $\delta$ for out-of-halo particles is approximately log-normal. The distribution for halo particles approaching a symmetric distribution with vanishing odd order kurtosis and even order kurtosis $K_4 \to 2$ and $K_6 \to 7$. Figure 5 plots the variation of the mean and standard deviation of log-density with time. For out-of-halo particles, the mean log-density decreases with time and $\langle \eta \rangle < 0$ after $z=1$ (or $a=0.5$), while the mean log-density of all halo particles increase with time ($mean(\eta) \propto a^{1/2}$). The power-law scaling of $std(\eta) \propto a^{1/2}$ can also be found from the plot.

Figure 5. The variation of mean and standard deviation of log-density $\eta(x)$ with scale factor $a$. The mean log-density for out-of-halo particles decreases with time and $\langle \eta \rangle < 0$
after $z=1$. The mean log-density of all halo particles increases with time and the standard deviation $\text{std} (\eta) \propto a^{1/2}$ can be found from this plot.

3.2 Two-point statistical measures of density field

3.2.1 Two-point density correlation functions from radial distribution function

The gravitational interaction between collisionless particles leads to correlations in particle positions. Following the statistical mechanics of molecular liquid, we start from the radial distribution function $g(r)$ at a given scale factor $a$, a quantity to measure the averaged particle density from an arbitrary reference particle. The number of particles in spherical shell of thickness $dr$ at a distance $r$ from the reference particle can be written as:

$$dN_p = g(r) \frac{N_p}{V} 4\pi r^2 dr,$$ (7)

where $N_p/V$ is the mean density of particles, $N_p$ is the total number of particles in the system, and $V$ is the volume. The mean comoving density $\rho_0 = N_p m_p / V$. The normalization condition can be verified such that

$$\int_0^\infty g(r) 4\pi r^2 dr = \frac{N_p - 1}{N_p} V.$$ (8)

The two-point second order density correlation function at a given scale factor $a$ is given by $\xi(r)$ that can be related to the radial distribution function $g(r)$ as

$$\xi(r) = \langle \delta(x) \delta(x+r) \rangle = g(r) - 1,$$ (9)

with the normalization condition,

$$\int_0^\infty \xi(r, a) 4\pi r^2 dr = -V/N_p < 0,$$ (10)
such that $\xi(r,a)$ cannot be positive on all scales. Density must be negatively correlated on some scale $r$. Two length scales can be defined from density correlation (plotted in Fig. 10),

$$l_{\delta 0}(a) = \int_0^\infty \xi(r,a) \, dr \quad l_{\delta 1}(a) = \int_0^\infty \xi(r,a) \, rdr .$$

(11)

3.2.2 Specific potential/kinetic energy from density correlation function

In principle, the specific potential energy (per mass) of any system with particles interacting via a pairwise potential $V_g(r)$ can be related to the radial distribution function $g(r)$ as,

$$PE = \frac{2\pi \rho_0}{m_p} \int_0^\infty r^2 \left[ g(r) - 1 \right] V_g(r) \, dr ,$$

(12)

where $\rho_0$ is the mean density. With $V_g(r) = -Gm_p^2/r$ for gravitational interaction, the specific potential energy in physical coordinate should be

$$P_y(a) = -\frac{2\pi G \rho_0}{a} \int_0^\infty \xi(r,a) \, rdr = -\frac{2\pi G \rho_0}{a} l_{\delta 1}^2 = -\frac{3H_0^2 l_{\delta 1}^2}{4a} < 0 .$$

(13)

The specific kinetic energy of entire system can be related to the potential energy via a cosmic energy equation [42-44] (Eq. (1) in [45]), i.e.,

$$\frac{\partial (K_p + P_y)}{\partial t} + H (2K_p + P_y) = 0 ,$$

(14)

with an exact solution of

$$K_p = a^{-2} \int_0^a aP_y \, da - P_y .$$

(15)

The specific kinetic energy can be finally related to the density correlation as [46],

$$K_p = \frac{3}{4} H_0^2 a^{-1} \left[ \int_0^\infty \xi(r,a) \, rdr - a^{-1} \int_0^a \int_0^\infty \xi(r,a) \, rdr \, da \right] = \frac{3}{4} H_0^2 a^{-1} \left( l_{\delta 1}^2 - a^{-1} \int_0^a l_{\delta 1}^2 \, da \right) .$$

(16)
The evolution of kinetic and potential energy of N-body system in an expanding background may be approximated by a power law solution that is proportional to time \( t \) (Eq. (40) in [45]),

\[
K_p = -\varepsilon_u t, \quad P_p = \frac{7}{5} \varepsilon_u t,
\]

where the rate of energy production \( \varepsilon_u \) is negative and a constant of time,

\[
\varepsilon_u = -\frac{3}{2} \frac{\partial u^2}{\partial t} \approx -\frac{3}{2} \frac{u_0^2}{t_0}.
\]

The correlation length \( l_{\delta}^2 \) may be related to the energy production rate as (Eqs. (13) and (17)),

\[
l_{\delta}^2 (a) = \int_0^\infty \xi (r, a) r dr = -\frac{56}{45} \frac{\varepsilon_u}{H_0^3} a^{5/2}.
\]

3.2.2 Density spectrum/dispersion functions and real space distribution of density fluctuation

The density spectrum \( E_\delta (k, a) \) in Fourier-space and real-space correlation function \( \xi (r, a) \) are related through the Fourier transformation pair:

\[
E_\delta (k, a) = \frac{2}{\pi} \int_0^\infty \xi (r, a) kr \sin (kr) dr,
\]

\[
\xi (r, a) = \int_0^\infty E_\delta (k, a) \frac{\sin (kr)}{kr} dk.
\]

In Peebles’ convention [47], the usual matter power spectrum \( P_\delta (k, a) \) reads

\[
P_\delta (k, a) = 2\pi^2 E_\delta (k, a) / k^2.
\]

The dimensionless power spectrum \( \Delta_\delta^2 (k, a) \) (the power per logarithmic interval) reads

\[
\Delta_\delta^2 (k, a) = E_\delta (k, a) k.
\]

The variance of the density fluctuation (density dispersion function), i.e. the density fluctuation contained in all scales above \( r \) should be
$$\sigma^2_\delta (r, a) = \int_{-\infty}^{\infty} E_\delta (k, a) W(k r)^2 \, dk ,$$  \hspace{1cm} (24)

where $W(x = k r)$ is a window function when smoothed with a filter of size $r$. For a typical tophat spherical filter, where $r$ is the radius of filter, the window function is

$$W(x) = \frac{3}{x^3} \left[ \sin(x) - x \cos(x) \right] = 3 \frac{j_1(x)}{x},$$  \hspace{1cm} (25)

where $j_1(x) = \frac{\sin(x)}{x^2} - \frac{\cos(x)}{x}$

is the first order spherical Bessel function of the first kind. With $W(0) = 1$, the variance of density fluctuation $\sigma^2_\delta (0) \to \infty$, i.e. diverging with $r \to 0$. Relation between correlation $\xi (r)$ and dispersion $\sigma^2_\delta (r)$ for a tophat filter in Eq. (25) can be derived exactly from Eqs. (21) and (24),

$$\xi (2r) = \frac{1}{72 r^2} \frac{\partial}{\partial r} \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^3 \frac{\partial}{\partial r} \left( \sigma^2_\delta (r) r^4 \right) \right) \right].$$  \hspace{1cm} (27)

For a power-law density spectrum $E_\delta (k) \equiv b k^{-m}$, a power-law correlation is expected,

$$\xi (r) = -2b \Gamma(-m) \sin \left( \frac{m\pi}{2} \right) r^{m-1},$$  \hspace{1cm} (28)

along with a power-law density dispersion function

$$\sigma^2_\delta (r) = 72 \cdot 2^m b (1 + m)(4 + m) \Gamma(-5 - m) \sin \left( \frac{m\pi}{2} \right) r^{m-1}.$$  \hspace{1cm} (29)

It can be easily verified that Eqs. (28) and (29) satisfy the relation in Eq. (27). Finally, the real-space distribution of density fluctuation in scale $r$ can be written as,

$$E_{\delta r} (r) = -\frac{\partial \sigma^2_\delta (r)}{\partial r}.$$  \hspace{1cm} (30)

This distribution can be related to the density spectrum function as,
\[ E_{\delta r}(r) = \frac{\partial \sigma^2_{\delta}(r)}{\partial r} \] r^2 = -4 \int_0^{\infty} E_{\delta}(\frac{x}{r}) W(x) W'(x) \, dx. \]  

(31)

The fluctuation distribution in real space \( E_{\delta r}(r) \) is equivalent to and contain the same information as the density spectrum in Fourier space. For a power-law density spectrum, \( E_{\delta}(k) \equiv bk^{-m} \), the fluctuation distribution function \( E_{\delta r}(r) \) can be exactly related to \( E_{\delta}(k) \) as,

\[ E_{\delta r}(r) r^2 = E_{\delta}(\frac{x_0}{r}) \text{ and } x_0 = \frac{1}{2} \left[ -72(m^2 - 1)(4 + m) \Gamma(-5 - m) \sin \left( \frac{m \pi}{2} \right) \right]^{-\frac{1}{m}}. \]  

(32)

With density correlation \( \xi(r) \) fully determined from simulation data, we can easily translate it into the density dispersion function \( \sigma^2_{\delta}(r) \) via Eq. (27), the spectrum function using Eq. (20), and the real-space fluctuation distribution \( E_{\delta r}(r) \) via Eq. (30) (See Figs. 7, 8, 11, and 12).

3.3 Two-point second order statistical measures of density from N-body simulation

In this section, second order statistical measures for density are obtained from N-body simulation in Section 2. Algorithms are developed to find all pairs with a given separation \( r \) and computing the average of these statistical measures over all pairs. We first compute the radial distribution function \( g(r) \) (Eq. (7)) by counting all pairs at a given distance of \( r \). Density correlation can be obtained from Eq. (9). In this way, we avoid projecting particle field onto the structured grid and maximumly preserve information from N-body simulation data.

Figure 6 presents the density correlation function \( \xi(r) \) (using Eq. (9)) at \( z=0 \) (solid blue curve). The velocity dispersion \( \sigma^2_{\delta}(r) \) is obtained using Eq. (27) and plotted in solid purple with \( \sigma^2_{\delta}(r = 8 \text{ Mpc}/h) = \sigma^2_{\delta} \) matching the model input in Table S1. The density correlation \( \xi(r) < 0 \) at
scale \( r > 30 \, \text{Mpc}/h \), as required by normalization in Eq. (10). To validate the algorithm, we compared with other model predictions [33]. The linear (blue dash) and nonlinear theory prediction (red dash) are both obtained using the Fourier transform of the density spectrum model from [33]. Note that both predictions significantly underestimate the negative density correlation comparing to N-body results (blue solid). The models for \( \xi(r) \) and \( \sigma^2_\delta(r) \) in Section 3.4 (Eqs. (33) and (35)) are also plotted (dotted lines) that capture the negative density correlation.

![Figure 6](image)

Figure 6. Two-point second order density correlation function \( \xi(r) \) (solid blue) varying with scale \( r \) at \( z=0 \). The negative density correlation can be identified for scale larger than 30Mpc/h. The linear (blue dash) and nonlinear predictions (red dash) are also presented in the same plot and both underestimate the negative correlation. The dispersion function \( \sigma^2_\delta(r) \) (solid purple) is obtained from Eq. (27). Models for \( \xi(r) \) and \( \sigma^2_\delta(r) \) (Eqs. (33) and (35)) are also presented in the same figure that capture the negative correlation.

The power spectrum \( E_\delta(k) \) can be obtained by Fourier transform (Eq. (20)) of correlation function \( \xi(r) \) that is directly obtained from N-body simulation. Three spectrums (\( E_\delta(k) \), \( P_\delta(k) \)
from Eq. (22), and $\Delta_\delta^2(k)$ from Eq. (23) at $z=0$ are presented in Fig. 7 in solid lines and compared against the nonlinear theory prediction (dash) from [33].

Figure 7. Without projecting particles onto structured grid, density power spectrums $E_\delta(k)$, $P_\delta(k)$, and $\Delta_\delta^2(k)$ (solid curve) can be obtained from correlation $\xi(r)$. The nonlinear theory prediction (dash) is also presented for comparison. The power spectrum $E_\delta(k)$ was obtained by Fourier transform of correlation $\xi(r)$ in Fig. 6 using Eq. (20).

The density correlation $\xi(r,a)$ at different redshift $z$ can be similarly obtained for $z=5$ to $z=0$ and presented in Fig. 8. The variation of $\xi(r,a)$ at a given scale $r$ with scale factor $a$ is plotted in Fig. 9. The density correlation $\xi(r,a) \propto a^2$ on large scale $r$ that is still in the linear regime, while $\xi(r,a)$ is approximately $\propto a^{3/2}$ on small scale $r$ that is in the nonlinear regime.
Figure 8. Two-point second order density correlation function $\xi(r,a)$ varying with scale $r$ at different redshifts $z=0, 0.1, 0.3, 0.5, 1.0, 1.5, 2.0, 3.0$ and 5.0. Negative density correlation can be identified for $r>30\text{Mpc}/h$. 
Figure 9. Two-point second order density correlation $\xi(r,a)$ varying with scale factor $a$ on different scales $r = 0.1, 0.3, 0.5, 1.0, 3.0, 5.0$ and $10 \text{ Mpc}/h$. The correlation $\xi(r,a) \propto a^2$ on large scale $r$ that is in the linear regime, and approximately $\propto a^{5/2}$ on small scale.

The variation of two length scales (defined in Eq. (11)) with scale factor $a$ are plotted in Fig. 10. Two comoving correlation lengths show a limiting scaling of $l_{\delta 0}(a) \propto a^{5/2}$ and $l_{\delta 1}(a) \propto a^{5/4}$. The specific potential energy computed by Eq. (13) using $l_{\delta 1}$ is in good agreement with the potential energy directly obtained from simulation. Both have a limiting scaling of $P_y(a) \propto a^{3/2}$ (Eq. (17)).

Figure 10. The variation of two comoving correlation lengths $l_{\delta 0}$ and $l_{\delta 1}$ ($\text{Mpc}/h$) with scale factor $a$. Both correlation lengths are derived from density correlation $\xi(r,a)$ (Eq. (11)) with a limiting scaling $l_{\delta 1}(a) \propto a^{5/4}$ (Eq. (19)) and $l_{\delta 0}(a) \propto a^{5/2}$. The potential energy $P_y(a)$ (in unit of $(\text{Km})^2/\text{s}^2$) using Eq. (13) is in good agreement with $P_y(a)$ that is directly computed from simulation, both of which show a scaling of $P_y(a) \propto a^{3/2}$. 
3.4 Modeling second order statistical measures of density field

The density correlation on large scale can be analytically derived from velocity correlation functions (Section 5 in [36]). The exponential correlation for transverse velocity is a direct result of combined kinematics and dynamics on large scale (Section 6.3 in [37]), which leads to a simple form of density correlation (Eq. (121) in [36]),

\[
\bar{\xi}(r, a) = \frac{1}{(aHf(\Omega_0))^2} \cdot \frac{a_o u^2}{rr_2} \exp \left( -\frac{r}{r_2} \right) \left[ \left( \frac{r}{r_2} \right)^2 - 7 \left( \frac{r}{r_2} \right) + 8 \right],
\]

(33)

with parameter \( a_o u^2 = 0.45 u_o^2 a \) at \( z=0 \) and \( u^2(a) \) is the one-dimension velocity dispersion (Fig. 18 in [36]). The only comoving length scale in this model \( r_2 = 23.14 \, \text{Mpc}/h \) is independent of time and might be related to the size of sound horizon. Values of \( a_0 \) and \( u^2 \) are listed in Table 1 for different redshift \( z \). Obviously \( a_o u^2 \propto a \) is consistent with the scaling \( \bar{\xi}(r, a) \propto a^2 \) in the linear theory. The correlation turns to negative at \( \sqrt{0.5} \approx 0.71 \), \( r_2 \approx 33 \, \text{Mpc}/h \) from Eq. (33). The model of Eq. (33) is also plotted in Fig. 6 that matches the N-body simulation on large scale. The average correlation \( \bar{\xi}(r, a) \) on large scale should read,

\[
\bar{\xi}(r, a) = \frac{3}{r^3} \int_0^r \bar{\xi}(y, a) y^2 dy = \frac{a_o u^2}{(aHf(\Omega_0))^2} \frac{3}{rr_2} \exp \left( -\frac{r}{r_2} \right) \left( 4 - \frac{r}{r_2} \right),
\]

(34)

that can be related to the mean pairwise velocity via pair conservation equation (Eq. (41)).

Table 1. Values for parameter \( a_0(a) \) and one-dimensional velocity dispersion \( u(a) \) (km/s)

| \( z \) | 0  | 0.1 | 0.3 | 0.5 | 1.0 | 1.5 | 2.0 | 3.0 |
|--------|----|-----|-----|-----|-----|-----|-----|-----|
| \( a_0(a) \) | 0.451 | 0.463 | 0.486 | 0.509 | 0.559 | 0.604 | 0.643 | 0.694 |
| \( u(a) \) | 354.61 | 335.42 | 303.37 | 277.67 | 231.29 | 199.76 | 177.15 | 148.61 |
Figure 11. Density dispersion function $\sigma_\delta^2 (r)$ at different redshift $z$ obtained from density correlation $\xi(r)$ using Eq. (27). Model from Eq. (35) is plotted for comparison with good agreement with simulation data on large scale.

The density dispersion function $\sigma_\delta^2 (r)$ can be obtained using Eqs. (27) and (33),

$$
\sigma_\delta^2 (r) = \frac{1}{(aH_f(\Omega_0))^2} \frac{9a_0u^2}{2r^2} \left\{ 3 \left( \frac{r_2}{r} \right)^4 + \left( \frac{r_2}{r} \right)^2 \right\}
- \exp \left\{ -\frac{2r}{r_2} \right\} \left[ 1 + \left( \frac{r_2}{r} \right)^2 \right] \left[ 3 \left( \frac{r_2}{r} \right)^4 + 6 \left( \frac{r_2}{r} \right)^2 + 4 \right], \quad (35)
$$

where $\sigma_\delta^2 (r) \propto a^2 r^{-4}$ for large $r \to \infty$ and $\sigma_\delta^2 (r) \propto a^2 r^{-1}$ for $r \to 0$. Figure 11 plots the variation of $\sigma_\delta^2 (r)$ obtained from $\xi(r)$ at different redshifts $z$, along with the model in Eq. (35).
The real-space distribution of density fluctuation, i.e. function $E_{\delta r}(r)$ can be subsequently obtained from $\sigma_\delta^2(r)$ (Eq. (30)) and presented in Fig. 12. The density fluctuation increases with time on all scales, compared to the real-space distribution of kinetic energy (Fig.10 in [36]).

![Figure 12. Real-space distribution of density fluctuation $E_{\delta r}(r)$ on scale $r$ obtained from density dispersion function $\sigma_\delta^2(r)$ using Eq. (30). Density fluctuation increases with time on all scales, while fluctuation on small scale increases faster than large scale.]

4. **Characterizing probability distributions of velocity fields**

The nature of velocity distributions is studied in this section. To understand how velocity distributions vary with scale $r$ and redshift $z$, we are interested in three types of velocities on different scale $r$, i.e. the longitudinal velocity $u_L$ or $u'_L$, velocity difference (or pairwise velocity)
\[ \Delta u_L = u'_L - u_L, \] and velocity sum \[ \Sigma u_L = u_L + u'_L. \]

For a given pair of particles with velocities \( u, u' \), and separation vector \( r = x' - x \), the longitudinal velocity and transverse velocity read (Fig. 13),

\[
\begin{align*}
  u_L &= u \cdot \hat{r} \quad \text{and} \quad u'_L = -(u \times \hat{r} \times \hat{r}) = u - (u \cdot \hat{r}) \hat{r}, \\
  u'_T &= u' \cdot \hat{r} \quad \text{and} \quad u'_T = -(u' \times \hat{r} \times \hat{r}) = u' - (u' \cdot \hat{r}) \hat{r},
\end{align*}
\]

(36)

where \( \hat{r} = r/r \) is the normalized unit vector.

Fig 13. Schematic plot of longitudinal and transverse velocities on scale \( r \).

For any given scale \( r \), all particle pairs with a separation between \( r \) and \( r + dr \) (\( dr = 0.001 \text{Mpc}/h \) in this work) are identified and the particle position and velocity are recorded to compute the velocity distribution and associated statistical quantities by averaging that quantity over all pairs on the same scale \( r \). By this way, information is maximumly preserved without projecting particle velocity onto structured grid.

In addition to three velocities that are scale dependent, we are also interested in the distribution of four particle velocities, i.e. the velocity of all particles in entire system \( (u_p) \), the velocity of all halo particles \( (u_{hp}) \), the velocity of all out-of-halo particles \( (u_{op}) \), and the velocity of all halos identified in the system \( (u_h, \text{the mean velocity of all particles in the same halo}) \). The distributions
of these velocities are not scale dependent, but redshift dependent. The redshift evolution of these distributions will significantly improve our understanding of velocity field for dark matter flow.

4.1 Generalized kurtosis, moments, and structure functions of velocity fields

Just like the density distributions in Section 3.1, velocity distributions can be best characterized by generalized kurtosis in Eqs. (5) and (6). The first example is the generalized kurtosis for velocity difference $\Delta u_L$ (pairwise velocity),

$$K_n(\Delta u_L, r) = \frac{\langle (\Delta u_L - \langle \Delta u_L \rangle)^n \rangle}{\langle (\Delta u_L - \langle \Delta u_L \rangle)^2 \rangle^{n/2}} = \frac{S_n^{cp}(\Delta u_L, r)}{S_2^{cp}(\Delta u_L, r)^{n/2}}, \quad (37)$$

where the central moment of order $n$ for $\Delta u_L$ reads

$$S_n^{cp}(\Delta u_L, r) = \left\langle \left( \Delta u_L - \langle \Delta u_L \rangle \right)^n \right\rangle. \quad (38)$$

The $n$th order longitudinal structure function of $\Delta u_L$ is defined as,

$$S_n^{lp}(r) = \left\langle (\Delta u_L)^n \right\rangle = \left\langle (u_L - u_L)^n \right\rangle, \quad (39)$$

Remarks: For incompressible hydrodynamics, $\langle u_L \rangle = \langle \Delta u_L \rangle = \langle \Sigma u_L \rangle = 0$ on all scales of $r$ and $S_n^{cp}(\Delta u_L, r) = S_n^{lp}(r)$, namely the central moment of $\Delta u_L$ equals the structure function defined in Eq. (39). For self-gravitating collisionless dark matter flow (SG-CFD), two particles tend to approach each other under gravity that leads to a non-zero longitudinal velocity $\langle u_L \rangle = -\langle \Delta u_L \rangle/2 > 0$, and central moment $S_n^{cp}(\Delta u_L, r) \neq S_n^{lp}(r)$ for SG-CFD. Due to symmetry, $\langle \Sigma u_L \rangle = 0$ and the distribution of $\Sigma u_L$ is always symmetric on all scale $r$ with vanishing odd moments.
4.2 Generalized kurtosis of velocity fields from N-body simulation

Figure 14. Redshift evolution of generalized kurtosis for velocity of all particles (blue), all halo particles (red), and all out-of-halo particles (black). Gaussian distribution is presented as green dash lines. All velocities quickly become non-Gaussian with time to maximize system entropy, while the evolution of out-of-halo particle velocity is at a much slower pace than that of halo particles due to weak gravity on large scale.

Figure 14 presents the time variation of general kurtosis for velocity $u_p$ (for all particles), $u_{hp}$ (for halo particles), and $u_{op}$ (for out-of-halo particles). Different order of kurtosis for Gaussian distribution is plotted as green dash lines for comparison. All velocities are initially Gaussian. The velocity distribution of halo particles $u_{hp}$ deviates from Gaussian much faster than the distribution of out-of-halo particles $u_{op}$ due to the stronger gravitational interaction in halos than between halos. All velocities become non-Gaussian with time to maximize system entropy [12].
Figure 15 plots the even order generalized kurtosis (4th order -- bottom, 6th order – middle, and 8th order – top) of three velocities \( (u_L, \Delta u_L, \text{ and } \Sigma u_L) \) at \( z=0 \). The 4th, 6th and 8th kurtosis of Gaussian distribution are also plotted in the same figure with \( K_4 = 3 \), \( K_6 = 15 \), and \( K_8 = 105 \). Clearly, distributions of three velocities are non-Gaussian on all scales due to the long-range nature of gravity. This is important as it poses challenges on any theory that assumes the Gaussianity of velocity fields. Velocity field of fully developed self-gravitating collisionless dark matter flow (SG-CFD) is not Gaussian on any scale despite that they can be initially Gaussian. By contrast, for incompressible hydrodynamics with short range interactions, the distribution of \( u_L \) is nearly Gaussian on large scale and \( \Delta u_L \) is also Gaussian on large scale and only becomes non-Gaussian on small scale due to strong viscous force.

Distribution of \( \Sigma u_L \) approaches the distribution of \( u_L \) on small scale with a limiting correlation (between \( u_L \) and \( \dot{u}_L \) ) \( \rho_L = 0.5 \) between two velocities (Fig. 15 in [36]). This is expected as \( r \to 0 \), the sum velocity \( \lim_{r \to 0} \Sigma u_L = \lim_{r \to 0} \left( u_L(\mathbf{x}) + u_L(\mathbf{x}) \right) \) will become the total velocity \( \mathbf{u}(\mathbf{x}) \) at location \( \mathbf{x} \). Longitudinal velocities \( u_L \) and \( \dot{u}_L \) along many different directions will simply collapse into \( \mathbf{u}(\mathbf{x}) \) and this also requires \( \rho_L = 0.5 \), i.e.

\[
\lim_{r \to 0} \left( \left( u_L + \dot{u}_L \right)^2 \right) = \lim_{r \to 0} |\mathbf{u}(\mathbf{x})|^2 = 3 \lim_{r \to 0} \langle u_L^2 \rangle. \tag{40}
\]

On large scale, the distribution of \( \Sigma u_L \) approaches the distribution of \( \Delta u_L \) with correlation \( \rho_L = 0 \) between \( u_L \) and \( \dot{u}_L \). This is also expected as the sum and difference of two independent random variables with symmetric distribution should follow the same distribution. Finally, on both small and large scales, generalized kurtosis approaches constant such that there exist unique (limiting) probability distributions that are independent of scale \( r \) when \( r \to 0 \) or \( r \to \infty \). While
on the intermediate scale around 1Mpc/h, all three velocity distributions exhibit the greatest value of generalized kurtosis of different order.

Figure 15. The generalized kurtosis (4th, 6th, and 8th order) of three velocities varying with scale $r$ at $z=0$. The generalized kurtosis of Gaussian distribution is plotted in the same figure (magenta) for comparison. All velocity distributions are non-Gaussian on all scales due to the long-range gravitational interaction, despite that they can be initially Gaussian. The distribution of $\Sigma u_L$ approaches that of $u_L$ on small scale, while the distribution of $\Delta u_L$ approaches that of $\Delta u_L$ on large scale. There exist limiting probability distributions for all velocities on both ends of small and large scales.

Figure 16 plots the variation of odd order generalized kurtosis ($K_3(\Delta u_L, r)$ and $K_5(\Delta u_L, r)$) with scale $r$ at $z=0$. The third order kurtosis $K_3(\Delta u_L, r)$ (skewness) vanishes on both small and large scales, where the distribution of $\Delta u_L$ is symmetric. The skewness $K_3(\Delta u_L, r)<0$ on the
intermediate scale (the distribution of $\Delta u_L$ skews toward positive side). The negative skewness can be an important signature of inverse cascade of kinetic energy across halo mass scales [48].

Figure 16. The odd order generalized kurtosis of pairwise velocity $\Delta u_L$ varying with scale $r$ at $z=0$. The third order kurtosis $K_3(\Delta u_L, r)$ (skewness) vanishes on both small and large scales, where distribution of $\Delta u_L$ is symmetric. The skewness $K_3(\Delta u_L, r)<0$ on the intermediate scale (distribution skews toward positive side). This negative skewness on the intermediate scale should be a result of inverse cascade of kinetic energy.

4.3 First moment of velocity fields and pair conservation equation

While the generalized kurtosis can be used to characterize distributions of different velocities, the moments of velocity distributions can be studied in detail to provide more insights. To validate the algorithm identifying pairs from N-body simulation, the mean pairwise velocity (first moment)
on all scales can be compared against pair conservation equation that relates the pairwise velocity with density correlation [47],

\[
\frac{\langle \Delta u_L \rangle}{Har} = -\frac{(1 + \bar{\xi}(r,a)) \partial \ln (1 + \bar{\xi}(r,a))}{3(1 + \bar{\xi}(r,a)) \partial \ln a},
\]

(41)

where \( \bar{\xi}(r,a) = 3r^{-3} \int_0^r \xi(y,a) y^2 dy \) is the volume averaged correlation function (model provided in Eq. (34) for large scale). For linear regime, \( \bar{\xi} \ll 1 \) and \( \partial \ln \bar{\xi} / \partial \ln a = 2 \), we have

\[
\frac{\langle \Delta u_L \rangle}{Har} = -\frac{2\bar{\xi}(r,a)(1 + \bar{\xi}(r,a))}{3(1 + \bar{\xi}(r,a))} \approx -\frac{2}{3} \bar{\xi}(r,a).
\]

(42)

For nonlinear regime with \( \bar{\xi} \gg 1 \) and assuming the scaling with scale factor as \( \bar{\xi}(r,a) \propto a^\alpha \) (Fig. 9) and \( \partial \ln \bar{\xi} / \partial \ln a = \alpha \), Eq. (41) reduces to,

\[
\frac{\langle \Delta u_L \rangle}{Har} = -\frac{\alpha(1 + \bar{\xi}(r,a))}{3(1 + \bar{\xi}(r,a))}.
\]

(43)

On small scale, if stable clustering hypothesis (\( \langle \Delta u_L \rangle = -Har \) demonstrated in [41]) is assumed combined with a self-similar gravitational clustering with \( \bar{\xi}(r,a) \propto a^\alpha r^\gamma \), then we should have

\[
\frac{\langle \Delta u_L \rangle}{Har} = -1 \text{ and } \alpha = \gamma + 3.
\]

(44)

Figure 17 plots the variation of the first moment \( S^1_L(r) = \langle \Delta u_L \rangle \) (mean pairwise velocity) with scale \( r \) (normalized by Hubble constant) at \( z=0 \). Results are compared with the prediction from pair conservation equation for both linear (black dash from Eq. (42)) and nonlinear regime (red dash from Eq. (43) with \( \alpha = 5/2 \) from Fig. 9) using the density correlation \( \bar{\xi}(r,a) \) obtained from N-body simulation. Blue line is the normalized pairwise velocity computed directly from
simulation by identifying all particle pairs and associated velocities. Good match with pair conservation equation validates our numerical implementation.

Figure 17. The variation of first moment of longitudinal velocity $S_{iL}(r) = \langle \Delta u_L \rangle$ (mean pairwise velocity) with scale $r$ at $z=0$ (normalized by Hubble constant $H$) from N-body simulation (blue solid). Results are compared with predictions using pair conservation equation for both linear (black dash from Eq. (42)) and nonlinear regime (red dash from Eq. (43)). Predictions are made with density correlation $\xi(r)$ from N-body simulation.

On small scale, an exact expression can be identified from stable clustering hypothesis (also analytically demonstrated by a two-body collapse model in [41]),

$$\langle \Delta u_L \rangle = -Har \quad \text{and} \quad \langle u_L \rangle = Har/2.$$  \hspace{0.5cm} \text{(45)}

For nonlinear regime below a characteristic scale $r_c = 1.3a^{1/2} \text{Mpc}/h$ where the longitudinal velocity correlation equals the transverse velocity correlation (Fig. 3 in [36]), a better relation to fit the simulation data reads
\[ \langle \Delta u_L \rangle = -Har - ua^{-5/3} \left( \frac{r}{r_a} \right)^{5/2}. \]  

(46)

On large scale, from pair conservation Eq. (42), the mean pairwise velocity is,

\[ \langle \Delta u_L \rangle \approx -\frac{2}{3} Har \bar{\xi} (r, a) = -\frac{2Ha}{r^2} \int_0^r \bar{\xi} (y) y^2 dy. \]  

(47)

With Eq. (34) for mean correlation \( \bar{\xi} (r, a) \), the mean pairwise velocity is simply the derivative of velocity correlation \( R_2 = \langle u \cdot u' \rangle \) (see Eq. (120) in [36]), we should have

\[ \langle \Delta u_L \rangle = \frac{2}{aH f (\Omega_0)} \frac{\partial R_2}{\partial r} = \frac{2a u^2}{aH r_2} \exp \left( -\frac{r}{r_2} \right) \left( \frac{r}{r_2} - 4 \right). \]  

(48)

Figure 18. Mean (first moment) longitudinal velocity \( \langle u_L \rangle \) and velocity difference \( \langle \Delta u_L \rangle \) varying with scale \( r \) at different redshift \( z \), normalized by velocity dispersion \( u(a) \). Note that \( \langle \Delta u_L \rangle = -2 \langle u_L \rangle \) is not zero, while \( \langle u_L \rangle = -\langle u_L \rangle \) and velocity sum \( \langle \Sigma u_L \rangle = 0 \) on all
scales. For SG-CFD, the velocity field $\mathbf{u}$ and vector $\mathbf{r}$ between two particles are correlated that leads to a nonzero longitudinal velocity $\langle u_L \rangle = \langle \mathbf{u} \cdot \mathbf{r} \rangle > 0$.

Figure 18 plots the mean longitudinal velocity $\langle u_L \rangle$ and velocity difference $-\langle \Delta u_L \rangle$ at different redshift $z$. Note that $\langle \Delta u_L \rangle = -2 \langle u_L \rangle$ vanishes on both small and large scales. Since $\langle u_L \rangle = -\langle u_L \rangle$, mean velocity sum $\langle \Sigma u_L \rangle = 0$ on all scales. $\langle u_L \rangle > 0$ and $\langle \Delta u_L \rangle < 0$ indicate that two particles are moving toward each other due to gravity. By contrast, $\langle u_L \rangle = \langle \Delta u_L \rangle = \langle \Sigma u_L \rangle = 0$ on all scales for incompressible collisional hydrodynamics, where $\mathbf{u}$ and $\mathbf{r}$ are independent of each other. Models for pairwise velocity on small and large scales (Eqs. (46) and (48)) are also presented for comparison. The entire range of $\langle \Delta u_L \rangle$ can be modeled by smoothly connecting two models on small and large scales (see Eq. (147) in [36]).
Figure 19. The variation of second moment of $\langle u_L^2 \rangle$, $\langle \Delta u_L^2 \rangle$ and $\langle \Sigma u_L^2 \rangle$ at $z=0$, normalized by velocity dispersion $u_0^2$ of entire system. On small scale, $\langle \Delta u_L^2 \rangle = \langle u_L^2 \rangle = \langle \Sigma u_L^2 \rangle / 3 = 2u^2$ and on large scale $\langle \Delta u_L^2 \rangle = \langle \Sigma u_L^2 \rangle = 2 \langle u_L^2 \rangle = 2u^2$. The difference between the second order longitudinal structure function $S^{lp}_2(r) = \langle \Delta u_L^2 \rangle$ and central moment $S^{cp}_2(\Delta u_L, r)$ is due to the nonzero $\langle \Delta u_L \rangle$ on intermediate scale. By contrast, $S^{lp}_2 = S^{cp}_2$ and $\langle u_L^2 \rangle = u^2$ on all scales for incompressible hydrodynamics. Model for longitudinal dispersion $\langle u_L^2 \rangle$ in Eq. (59) is also plotted (dotted line) for comparison.

4.4 Second moments of velocities

Figure 19 plots the second moments and central moments (normalized by $u^2$ ) of three velocities $u_L$, $\Delta u_L$, and $\Sigma u_L$ on all scales at $z=0$. The longitudinal velocities ($u_L$ and $u_L^\prime$) must be strongly correlated on small scale due to gravitational interaction and uncorrelated on large scale. The correlation between longitudinal velocities $u_L$ and $u_L^\prime$ leads to

$$\langle \Delta u_L^2 \rangle = 2 \langle u_L^2 \rangle (1 - \rho_L)$$

and

$$\langle \Sigma u_L^2 \rangle = 2 \langle u_L^2 \rangle (1 + \rho_L).$$

(49)

where $\rho_L$ is the correlation coefficient between $u_L$ and $u_L^\prime$. On small scale with $\rho_L = 1/2$ for $r \to 0$ (pairs in small halos are fully correlated, while pairs in large halos are uncorrelated, such that the average correlation is around $1/2$, Eq. (40) and Section 3.3.1 in [36]),

$$\langle \Delta u_L^2 \rangle = \langle u_L^2 \rangle = \langle \Sigma u_L^2 \rangle / 3 = 2u^2.$$  

(50)

On large scale with $\rho_L = 0$ for $r \to \infty$,

$$\langle \Delta u_L^2 \rangle = \langle \Sigma u_L^2 \rangle = 2 \langle u_L^2 \rangle = 2u^2.$$  

(51)

By contrast, the incompressible collisional hydrodynamics should have $\langle \Delta u_L^2 \rangle = 0$ and $\langle \Sigma u_L^2 \rangle = 4$ on small scale with $\rho_L = 1$ when $r \to 0$, and $\langle u_L^2 \rangle = u^2$ on all scales. The difference between the
second moments and the central moments of $u_L$ and $\Delta u_L$ on intermediate scale is due to the non-zero first moment $\langle u_L \rangle$ and $\langle \Delta u_L \rangle$, as shown in Fig. 19. All second moments increase with $r$ initially and decrease for $r > r_\epsilon$. Model for $\langle u_L^2 \rangle$ on small scale is proposed in Eq. (59), while model for $\langle u_L^2 \rangle$ on large scale is proposed in our previous work (Eq. (136) in [36]).

By identifying all pairs of particles with different separation $r$, we can compute the variance of velocity on different scales $r$, namely the total variance $\langle u^2 \rangle = \langle u \cdot u \rangle$, the longitudinal variance $\langle u_L^2 \rangle$, and the transverse variance $\langle u_T^2 \rangle = \langle u_r \cdot u_r \rangle$, where

$$\langle u^2 \rangle = \langle u \cdot u \rangle = \langle u_L^2 \rangle + \langle u_r \cdot u_r \rangle,$$

(52)

Figure 20. The variation of velocity dispersions $\langle u^2 \rangle = \langle u \cdot u \rangle$, $\langle u_L^2 \rangle$, and $\langle u_T^2 \rangle = \langle u_r \cdot u_r \rangle$ with scale $r$ at $z=0$ (normalized by $u_0^2$). The initial increase of all dispersions with $r$ for $r < r_\epsilon$ is mostly due to the increasing velocity dispersion with halo size on small scale. With
more pairs of particles from different halos on larger scale \( r > r_i \), dispersion starts to decrease with \( r \). With all pairs of particles from different halos, velocity dispersion reaches a plateau with \( \langle u^2 \rangle = 3 \langle u_L^2 \rangle = 3u^2 \).

Figure 20 plots three velocity dispersions \( \langle u^2 \rangle \), \( \langle u_L^2 \rangle \), and \( \langle u_T^2 \rangle \) on different scale \( r \) at \( z=0 \). The initial increase of all three dispersions with \( r \) for \( r < r_i \) (pair of particles are more likely from same halos) is mostly due to the increase of velocity dispersion with halo size. With more pairs of particles from different halos for scale \( r > r_i \), the velocity dispersions sharply decrease with \( r \). At some large-scale \( r \), almost all pair of particles are from different halos, where velocity dispersions reach a plateau with \( \langle u^2 \rangle = 3 \langle u_T^2 \rangle = 3u^2 \).

For particle pairs separated by scale \( r \), the longitudinal and transverse velocities are comparable on both small and large scales. However, \( \langle u_L^2 \rangle > \langle u_T^2 \rangle / 2 \) on intermediate scales with \( \langle u_L^2 \rangle > \langle u^2 \rangle / 3 \), i.e. energy is not equipartitioned on intermediate scale. The velocity dispersion on small scale \( \langle u_L^2 \rangle \big|_{r=0} \approx 2 \langle u_L^2 \rangle \big|_{r=\infty} \), i.e. the kinetic energy on small scale is twice of that on large scale.

The variation of pairwise velocity dispersion (or the second order longitudinal structure function)
\[
S^p_2 = \langle (\Delta u_L)^2 \rangle = \langle (u_L - u_L)^2 \rangle
\]
and velocity sum \( \langle (\Sigma u_L)^2 \rangle = \langle (u_L + u_L)^2 \rangle \) are also plotted in the same figure for comparison.

4.5 Two-thirds law for even order moments of pairwise velocity (structure functions)

Now we focus on the second order structure function \( S^p_2 \) (pairwise velocity dispersion in Eq. (39)) that is defined as
\[
S^p_2 (r) = \langle (\Delta u_L)^2 \rangle = 2 \langle u_L^2 \rangle - L_2 (r),
\]
(53)
and a modified version of longitudinal structure function $S_{2}^{l}(r)$

$$S_{2}^{l}(r) = 2(u^2 - L_{z}(r)).$$  \hspace{1cm} (54)

With equations for $\langle u_{L}^{2} \rangle$ (Eq. (136) in [36]) and longitudinal correlation function $L_{z} = \langle u_{L} u_{L}^{'} \rangle$ (see Eq. (111) in [36]), $S_{2}^{lp}(r)$ on large scale can be easily calculated. The structure function $S_{2}^{l}(r)$ on small scale is also identified to follow a one-fourth law $\propto r^{1/4}$ (see Eq. (137) in [36]). However, the knowledge of structure function $S_{2}^{lp}(r)$ on small scale is still missing.

Figure 21. The variation of second order longitudinal structure functions $S_{2}^{lp}(r)/u^2$ (pairwise velocity dispersion) with scale $r$ and redshifts $z$. The two limits $S_{2}^{lp}(r \to 0) = S_{2}^{lp}(r \to \infty) = 2u^2$ due to correlation coefficients $\rho_L = 1/2$ and $\rho_L = 0$ between longitudinal velocities $u_{L}$ and $u_{L}^{'}$ on small and large scales. Two second order structure functions $S_{2}^{lp}(r) \approx S_{2}^{l}(r)$ for high redshift $z$ when velocity is still Gaussian and small scale structures are not fully developed.
Figure 21 presents the variation of pairwise velocity dispersion $S_{2}^{lp}(r)$ with scale $r$ and redshift $z$ with limits $S_{2}^{lp}(r \rightarrow 0) = S_{2}^{lp}(r \rightarrow \infty) = 2u^2$ due to correlation coefficient $\rho_L = 1/2$ and $\rho_L = 0$ on small and large scales. In addition, $S_{2}^{lp}(r) \approx S_{2}^{lp}(r)$ for high redshift $z$, when velocity distribution is nearly Gaussian and $\langle u_L^2 \rangle \approx u^2$ on all scales (Eqs. (53) and (54) or Fig. 22 in [36]).

Both two-thirds and four-fifths laws (Eq. (1)) in incompressible hydrodynamics are no longer valid for SG-CFD due to the collisionless nature of flow. However, since the peculiar velocity field is of constant divergence on small scale, second order structure and correlation functions for peculiar velocity should satisfy the same kinematic relations as if the peculiar velocity field is incompressible [36]. In addition, just like the direct energy cascade in 3D incompressible turbulence, there also exists a constant energy flux $\varepsilon_u < 0$ in the mass propagation range for inverse kinetic energy cascade from small to large mass scales (see Eqs. (27) and (48) in [48]) in SG-CFD. Therefore, we expect the second order structure function $S_{2}^{lp}(r)$ on the small scale should be determined by the constant energy flux $\varepsilon_u$ in some way.

Since the viscous effect is not present, a reduced structure function $S_{2r}^{lp} = S_{2}^{lp} - 2u^2$ can be introduced with limit $\lim_{r \rightarrow 0} S_{2r}^{lp} = 0$. The limiting pairwise velocity dispersion is inherent to all particle pairs with $r \rightarrow 0$ and equals the kinetic energy on small scale, $\lim_{r \rightarrow 0} S_{2}^{lp} = \lim_{r \rightarrow 0} \langle u_L^2 \rangle = 2u^2$. The reduced structure function $S_{2r}^{lp}$ is the extra pairwise velocity dispersion purely due to the inverse energy cascade. It should be determined by and only by the constant energy flux $\varepsilon_u$ ($m^2/s^3$) and scale $r$. By a simple dimensional analysis, $S_{2r}^{lp}$ must follow a two-thirds law, i.e. $S_{2r}^{lp} \propto (-\varepsilon_u)^{2/3} r^{2/3}$. Here to test this idea, Figure 22 plots the variation of reduced second order
structure function $S_{2r}^{lp}$ with scale $r$ at different redshifts $z$. The range with $S_{2r}^{lp} \propto r^{2/3}$ can be clearly identified that is formed along with the formation of halo structures. This range gradually extends to smaller and smaller length scales with time. This is a very interesting finding that the constant energy flux $\varepsilon_u$ determines the new two-thirds law for a reduced second order structure function $S_{2r}^{lp}$ in self-gravitating collisionless flow.

$$S_{2r}^{lp} = \left( S_2^{lp} - 2 u^2 \right)$$

Figure 22. The variation of reduced longitudinal structure functions $S_{2r}^{lp} = \left( S_2^{lp} - 2 u^2 \right)$ with scale $r$ at different redshifts $z$, normalized by velocity dispersion $u^2 (a)$. A scaling of $S_{2r}^{lp} \propto \left( -\varepsilon_u \right)^{2/3} r^{2/3}$ can be clearly identified in a range that is gradually expanding with time, where $\varepsilon_u < 0$ is the constant rate of energy production. The model from Eq. (55) is also presented for comparison.

As expected, the reduced structure function quickly converges to $S_{2r}^{lp} \propto \left( -\varepsilon_u \right)^{2/3} r^{2/3}$ with halo structures developed. The length scale at which $S_2^{lp}$ is at its maximum is about $r_d \approx 0.7 a Mpc/h$, 


same as the length scale for \( u_{i}^{2} \) (see Fig. 20 in [36]). Therefore, second order longitudinal structure function (pairwise velocity dispersion) on small scale can be finally modelled as,

\[
S_{2}^{lp}(r) = u^{2}\left[2 + \beta_{2}^{*}\left(\frac{r}{r_{s}}\right)^{2/3}\right] = 2u^{2} + a^{3/2}\beta_{2}^{*}\left(-\varepsilon_{u}\right)^{2/3}r^{2/3},
\]

(55)

where the length scale \( r_{s} \) is purely determined by \( u_{0} \) and \( \varepsilon_{u} \) with

\[
r_{s} = \frac{u_{0}^{3}}{\varepsilon_{u}} = \frac{4}{9} \frac{u_{0}}{H_{0}} = \frac{2}{3} u_{0} t_{0} \approx 1.58\text{Mpc}/h,
\]

(56)

which is roughly the scale below which two-thirds law is valid. Energy flux \( \varepsilon_{u} \) is estimated as,

\[
-\varepsilon_{u} = \frac{3}{2} \frac{u_{0}^{3}}{t_{0}} = \frac{9}{4} u_{0}^{3} H_{0} \approx 0.6345 \frac{u_{0}^{3}}{\text{Mpc}/h} = 4.6 \times 10^{-7}\text{m}^{2}/\text{s}^{3}.
\]

(57)

Constant \( \beta_{2}^{*} \approx 9.5 \) can be found from Fig. 22, where the model (55) is also presented for comparison. With model for \( S_{2}^{lp}(r) \) in Eq. (55), Eq. (53), and model for longitudinal correlation \( L_{2}(r) \) (see Eq. (138) in [36]),

\[
L_{2}(r) = u^{2}\left[1 - \left(\frac{r}{r_{1}}\right)^{n}\right],
\]

(58)

the dispersion \( \langle u_{i}^{2} \rangle \) of longitudinal velocity (in Fig. 20) on small scale can be finally modeled as,

\[
\langle u_{i}^{2} \rangle = u^{2}\left[2 - \left(\frac{r}{r_{1}}\right)^{n} + \frac{1}{2}\beta_{2}^{*}\left(\frac{r}{r_{s}}\right)^{2/3}\right],
\]

(59)

where \( n \approx 1/4 \) and \( r_{1}(a) \approx 19.4a^{-3}\text{Mpc}/h \).

Next, the higher order structure functions can be similarly studied. Figure 23 plots the variation of even and odd order structure functions \( S_{2n+1}^{lp}(r) \) with scale \( r \) at \( z=0 \). It is now clear that the original Kolomogrov’s scaling (Eq. (1)) for incompressible flow does not apply for self-gravitating
collisionless dark matter flow. On small scale, all even order reduced structure functions follow
the same scaling of $S_{2n}^{(p)} \propto \beta_{2n}^* r^{2/3}$, while all odd order structure functions follow $S_{2n+1}^{(p)} \propto r$.

Figure 23. The variation of even and odd order structure functions with scale $r$ at $z=0$. The
plot demonstrates that even order reduced structure functions scales as $S_{2n}^{(p)} \propto \beta_{2n}^* r^{2/3}$ on
small scale (Eq. (60), while odd order structure functions scales as $S_{2n+1}^{(p)} \propto r$. The numbers
2, 30, 1280... are related to the generalized kurtosis $K_{2n}(\Delta u_L, r)$ for the limiting
distribution of pairwise velocity $\Delta u_L$ when $r \to 0$ (Eq. (60)).

The general form for even order structure function $S_{2n}^{(p)}(r)$ can be precisely modeled as,

$$S_{2n}^{(p)}(r) = u^{2n} \left[ 2^n K_{2n}(\Delta u_L, 0) + \beta_{2n}^* \left( \frac{r}{r_s} \right)^{2/3} \right], \quad (60)$$

where $K_{2n}(\Delta u_L, r = 0)$ is the generalized kurtosis on the smallest scale that we can find from Fig.
15 (given in Table 3 and modeled by Eq. (80)). The universal constants $\beta_{2n}^*$ are determined as
\[ \beta_2^\ast = 9.5, \quad \beta_4^\ast = 300, \quad \beta_6^\ast = 2.25 \times 10^4, \quad \text{and} \quad \beta_8^\ast = 2.75 \times 10^6 \] (61)

or using an approximate relation of

\[ \beta_{2n}^\ast \approx 10^{1.826n-1.003}. \] (62)

4.6 Odd moments of pairwise velocity and Generalized stable clustering hypothesis (GSCH)

![Graph showing the variation of ratio \( \left( \frac{S_{21}^{lp}}{S_{22}^{lp}S_{11}^{lp}} \right) \) for \( n = 1, 2, \) and 3 with scale \( r \) at \( z=0 \). For \( n=1 \), this ratio is around 3 as predicted by the generalized stable clustering hypothesis (GSCH) [41].](image)

The mean pairwise velocity (first moment) \( S_1^{lp} (r) = -Har \) on small scale (Fig. 18) is from the stable clustering hypothesis, which can be precisely demonstrated by a two-body collapse model in expanding background (see Eq. (117) in [41]). The same model can be extended to higher order moments, i.e. the generalized stable clustering hypothesis (see Eq. (123) in [41]) where
\[ S_{2n+1}^{lp}(r) = (2n+1) S_t^{lp}(r) S_{2n}^{lp}(r), \quad (63) \]

or
\[ S_{2n+1}^{lp}(r) = - (2n+1) Har S_{2n}^{lp}(r) = -2^n (2n+1) K_{2n} (\Delta u_L, r = 0) Har u^2. \quad (64) \]

where the generalized kurtosis on the smallest scale \( K_{2n} (\Delta u_L, r = 0) \) is presented in next section (Table 3 and Eq. (80)). With odd moments from Fig. 23, Fig. 24 presents the ratio \( S_{2n+1}^{lp}(r)/[ S_t^{lp}(r) S_{2n}^{lp}(r) ] \) for \( n = 1, 2 \) and \( 3 \) at \( z = 0 \). For \( n = 1 \), this ratio is around 3 on small scale. For \( n = 2 \) and \( 3 \), this ratio slightly deviates from predicted value of \( (2n+1) \) with much greater noises. Finally, Table 2 presents a complete comparison of velocity field between incompressible hydrodynamics and self-gravitating collisionless dark matter flow (SG-CFD).

Table 2. The comparison of velocity fields between incompressible and SG-CFD

| Quantity          | Incompressible flow | SG-CFD                          |
|-------------------|---------------------|---------------------------------|
| \( \langle \hat{u}_L \rangle = \langle \mathbf{u} \cdot \mathbf{r} \rangle \) | 0 for all scale \( r \) | \( \lim_{r \to 0, \infty} \langle \hat{u}_L \rangle = 0 \), varying with \( r \) |
| \( \langle u_L^2 \rangle \) | \( u_0^2 \) for all scale \( r \) | \( \lim_{r \to 0} \langle u_L^2 \rangle = 2u_0^2 \), \( \lim_{r \to \infty} \langle u_L^2 \rangle = u_0^2 \) |
| \( \langle u_L^3 \rangle \) | 0 for all scale \( r \) | \( \lim_{r \to 0, \infty} \langle u_L^3 \rangle = 0 \), varying with \( r \) |
| PDF of \( u_L \) | Gaussian            | Non-gaussian on all scales      |
| \( \langle \Delta u_L \rangle \) | 0 for all scale \( r \) | \( \lim_{r \to 0, \infty} \langle \Delta u_L \rangle = 0 \), varying with \( r \) |
| \( \langle \Delta u_L^2 \rangle \) | \( \lim_{r \to 0} \langle \Delta u_L^2 \rangle = 0 \), \( \lim_{r \to \infty} \langle \Delta u_L^2 \rangle = u_0^2 \) | \( \lim_{r \to 0} \langle \Delta u_L^2 \rangle = 2u_0^2 \), \( \lim_{r \to \infty} \langle \Delta u_L^2 \rangle = 2u_0^2 \) |
| \( K_3(\Delta u_L) \) | \( \lim_{r \to 0} K_3(\Delta u_L) = -0.4 \), \( \lim_{r \to \infty} K_3(\Delta u_L) = 0 \) | \( \lim_{r \to 0} K_3(\Delta u_L) = 0 \), varying with \( r \) |
| \( K_4(\Delta u_L) \) | \( \lim_{r \to 0} K_4(\Delta u_L) \approx 4 \) (depends on Re), \( \lim_{r \to \infty} K_4(\Delta u_L) = 3 \) (Gaussian) | \( \lim_{r \to 0} K_4(\Delta u_L) = 7.5 \), \( \lim_{r \to \infty} K_4(\Delta u_L) = 4.2 \) |
| \( \langle \sum u_L \rangle \) | 0 on all scales      | 0 on all scales                  |
| \( \langle \sum u_L^2 \rangle \) | \( \lim_{r \to 0} \langle \sum u_L^2 \rangle = 4u_0^2 \), \( \lim_{r \to \infty} \langle \sum u_L^2 \rangle = 2u_0^2 \) | \( \lim_{r \to 0} \langle \Delta u_L^2 \rangle = 6u_0^2 \), \( \lim_{r \to \infty} \langle \Delta u_L^2 \rangle = 2u_0^2 \) |
5. Distributions of velocity on different scales and their redshift evolution

5.1 Modeling velocity distributions on small scale

On small scale, velocities \( u_L \) and \( \Sigma u_L \) should have the same limiting distribution with \( r \to 0 \) (Fig. 15). Velocity distribution that maximizes the system entropy was studied in our previous work [12]. Based on the halo description of self-gravitating system, \( u_L \) on small scale should follow a \( X \) distribution to maximize system entropy. The distribution reads (see Eq. (32) in [12]),

\[
X(v) = \frac{1}{2\alpha v_0} \frac{e^{-\frac{1}{\alpha^2 v_0^2}}}{K_1(\alpha)},
\]

where \( \alpha \) is a shape parameter and \( K_\alpha(x) \) is the modified Bessel function of the second kind. The velocity scale \( v_0 \) satisfies

\[
\frac{K_\alpha(\alpha)}{K_1(\alpha)} v_0^2 = \langle u_L^2 \rangle,
\]

where \( \langle u_L^2 \rangle \) is the dispersion of velocity \( u_L \) in Fig. 19. It can be estimated that \( v_0^2 \approx 0.84 u_0^2 \) with \( \langle u_L^3 \rangle(r = 0.1 \text{Mpc}/h) = 2.5 u_0^2 \) (from Fig. 19) at \( z=0 \). With shape parameter \( \alpha \approx 1.33 \) and \( v_0^2 = 0.84 u_0^2 \), the \( X \) distribution is plotted in Fig. 25 for comparison with the distribution of velocity \( u_L \). The velocity sum \( \Sigma u_L \) on a small scale should also follow the same \( X \) distribution with a different variance, i.e. \( \langle (\Sigma u_L)^2 \rangle \approx 3 \langle (u_L)^2 \rangle \). All distributions are symmetric on small scale.

The longitudinal velocity has a finite limiting correlation \( \rho_L = 1/2 \) at \( r = 0 \) (Eq. (58) and Fig. 15 in [36]) such that the limiting distribution of velocity difference (pairwise velocity) \( \Delta u_L \) must be different from the distribution of \( u_L \) (different Kurtosis in Fig. 15). This effect, i.e. the correlation between longitudinal velocities \( u_L \) and \( u'_L \) decreases with increasing halo size, was not
considered in previous work when finding the distribution of $\Delta u_L$ [49]. The distribution of $\Delta u_L$ on small scale cannot be Gaussian because of strong gravitational interaction (also see Fig. 15), whose moment and kurtosis is estimated in next Section.

![Distributions of $u_L$, $\Delta u_L$, and $\Sigma u_L$ on scale of $r=0.1\text{Mpc/h}$ at $z=0$, i.e. $\log_{10} P_{u_L}$ vs. $u_L/u_0$. All distributions are symmetric with a vanishing skewness. The $X$ distribution that maximizes the system entropy (Eq. (65)) is also plotted and matches the distribution of longitudinal velocity $u_L$. Distributions have a Gaussian core for small velocity and exponential wings for large velocity.](image)

Figure 25. Distributions of $u_L$, $\Delta u_L$, and $\Sigma u_L$ on scale of $r=0.1\text{Mpc/h}$ at $z=0$, i.e. $\log_{10} P_{u_L}$ vs. $u_L/u_0$. All distributions are symmetric with a vanishing skewness. The $X$ distribution that maximizes the system entropy (Eq. (65)) is also plotted and matches the distribution of longitudinal velocity $u_L$. Distributions have a Gaussian core for small velocity and exponential wings for large velocity.

5.2 The limiting distribution and moments of pairwise velocity $\Delta u_L$ on small scale

The limiting distribution of $\Delta u_L$ when $r \to 0$ is different from the distribution of $u_L$ (Figs. 15 and 25). The explicit form of that distribution is still unknown and should be explored in the future. However, the moments of $\Delta u_L$ when $r \to 0$ can be estimated as follows. This is required to compute the generalized kurtosis in Eq. (60) for modeling pairwise velocity on small scale. Let’s
start from the $H$ distribution for the fraction of particles with a velocity dispersion $\sigma_v^2$ (i.e. the virial dispersion of a halo that particle belongs to) [12, 50]. For system with total $N$ collisionless particles, total number of particles in a halo group with all halos of the same size $n_p$ should be

$$n_p N_h = N H\left(\sigma_v^2\right) d\sigma_v^2,$$

(67)

where $N_h$ is the number of halos of size $n_p$ in a halo group. Let’s assume the number of particle pairs $n_{\text{pair}}$ with a small separation $r$ from halos of size $n_p$ is proportional to the halo size $n_p$ with a power law, $n_{\text{pair}} = \mu_p \left(n_p\right)^{\alpha_p}$, where $\mu_p$ is a proportional constant. The maximum number of pairs for a given halo size $n_p$ is $n_{\text{pair}} = n_p \left(n_p - 1\right)/2$ if all $n_p$ particles in that halo collapse to a single point, where we should have $\alpha_p = 2$. In reality, $\alpha_p$ should satisfy $1 < \alpha_p < 2$. The number of pairs in a halo group ($N_h$ halos with same size $n_p$) reads,

$$N_h n_{\text{pair}} = N \mu_p \left(n_p\right)^{\alpha_p - 1} H\left(\sigma_v^2\right) d\sigma_v^2,$$

or the total number of pairs with a given separation $r$ in entire N-body system should be,

$$N_{\text{pair}} = \int_0^\infty N \mu_p \left(n_p\right)^{\alpha_p - 1} H\left(\sigma_v^2\right) d\sigma_v^2.$$

(68)

From virial theorem, the halo size $\sigma_v^2 \propto \left(n_p\right)^{2/3}$ and we can write $n_p = \mu_v \left(\sigma_v^2 / \sigma_h^2\right)^{3/2}$, where $\sigma_h^2$ is the halo velocity dispersion (the dispersion of halo velocity) that is independent of halo mass (see Fig. 2 in [48]) and $\mu_v$ is a proportional constant. The $H$ distribution is naturally related to the dimensionless halo mass function $f\left(\nu\right)$ (see Eq. (59) in [50]),

$$H\left(\sigma_v^2\right) d\sigma_v^2 = f\left(\nu\right) d\nu,$$

(69)

where the dimensionless variable $\nu = \sigma_v^2 / \sigma_h^2$. Therefore, Eq. (68) can be transformed to
\[
\frac{N_{\text{pair}}}{N \mu_p (\mu_t)^{\alpha_t^{-1}}} = \int_0^\infty f(\nu) \nu^{2(\alpha_t^{-1})} d\nu \quad \text{or} \quad \int_0^\infty \beta_p f(\nu) \nu^p d\nu = 1,
\]  
(70)

\text{i.e. } \beta_p f(\nu) \nu^p d\nu \text{ is the fraction of pairs in halo group with a given size where the exponent } \quad p = 3(\alpha_p - 1)/2.

(71)

Since velocity \( u_L \) for all particles from the same halo group is nearly Gaussian (see Fig. 3 in [12]), the distribution of \( \Delta u_L = u_L' - u_L \) can be obtained from the joint Gaussian distribution of \( u_L \) and \( u_L' \) with a halo size-dependent correlation \( \rho_{\text{cor}}(n_p) \),

\[
P_{\Delta u_L}(x) = \int_0^\infty \frac{1}{\sqrt{2\pi \sigma}} e^{-x^2/[2(1-\rho_{\text{cor}})]} \beta_p f(\nu) \nu^p d\nu.
\]  
(72)

The correlation \( \rho_{\text{cor}} \) (see Eq. (58) in [36]) and the total particle velocity dispersion \( \sigma^2 \) read

\[
\rho_{\text{cor}}(n_p) = \sigma_h^2/\sigma^2 \quad \text{and} \quad \sigma^2(n_p) = \sigma_v^2(n_p) + \sigma_h^2,
\]  
(73)

where \( \sigma_h^2 \) and \( \sigma_v^2 \) are the halo velocity dispersion and halo virial dispersion, respectively. The moment generating function and the corresponding moments can be obtained from Eq. (72),

\[
\int_{-\infty}^\infty P_{\Delta u_L}(x) e^{xt} dx = \int_0^\infty \beta_p f(\nu) \nu^p e^{(1-\rho_{\text{cor}}) \sigma_v^2} d\nu = \int_0^\infty \beta_p f(\nu) \nu^p e^{\sigma_h^2} d\nu,
\]  
(74)

\[
M_m(\Delta u_L) = \frac{m!}{(m/2)!} \int_0^\infty \beta_p f(\nu) \nu^{m/2} d\nu \sigma_h^m.
\]  
(75)

We can use the double-\( \lambda \) mass function (see Eq. (98) in [40]) that is proposed based on the inverse mass cascade theory, where the dimensionless mass function \( f(\nu) \) reads,

\[
f(\nu) = f_{\text{d} \lambda}(\nu) = \left( \frac{2\sqrt{\eta_0}}{\Gamma(q/2)} \right)^{-q} \nu^{q/2-1} \exp \left( -\frac{\nu}{4\eta_0} \right),
\]  
(76)
where $\eta_0 = 0.76$ and $q = 0.556$ for the best fit of mass function to simulation data. The normalization factor in Eq. (70) should be

$$\beta_p = \frac{N\mu_\mu (\mu_\nu)^{\alpha-1}}{N_{\text{pair}}} = \frac{\Gamma(q/2)}{\left(2\sqrt{\eta_0}\right)^2 \Gamma(p+q/2)}. \tag{77}$$

Finally, the distribution of pairwise velocity $P_{\text{aul}}$ should satisfy (from Eq. (74))

$$\int_{-\infty}^{\infty} P_{\text{aul}}(x)e^{xt} \, dx = \frac{1}{\left(1-4\eta_0\sigma_h^2 t^2\right)^{p+q/2}}, \tag{78}$$

such that the moments of any order $m$ can be obtained as,

$$M_m(\Delta u_L) = \frac{m!(2\sqrt{\eta_0})^m}{(m/2)!\Gamma(p+q/2)} \Gamma\left(\frac{1}{2}(m+2p+q)\right)\sigma_h^m. \tag{79}$$

The generalized kurtosis for pairwise velocity $\Delta u_L = u_L^* - u_L$ on small scale is,

$$K_{2n}(\Delta u_L) = \frac{(2n)!\Gamma(n+p+q/2)\Gamma(p+q/2)^{n-1}}{n!2^n \Gamma(1+p+q/2)^n}. \tag{80}$$

With $\eta_0 = 0.76$ and $q = 0.556$ for double-$\lambda$ mass function, $\beta_p \approx 1.5426$ from Eq. (77). Using the Kurtosis values for $\Delta u_L$ on small scale from simulation (Table 3), the parameter $p \approx 0.36$ or exponent $\alpha_p \approx 1.24$ (from Eq. (71)) can be obtained, and the total number of pairs $N_{\text{pair}}$ with $r \to 0$ should be (from Eq. (77))

$$\frac{N_{\text{pair}}}{N} = \frac{\mu_\mu (\mu_\nu)^{\alpha-1}}{\beta_p}, \tag{81}$$

where both constants $\mu_\mu$ and $\mu_\nu$ can be obtained from simulation ($\mu_\mu \approx 0.21$ and $\mu_\nu \approx 14$ from N-body simulation in Section 2).
The general kurtosis for $\Delta u_L$ computed from Eq. (80) are listed in table 3 and agrees well with N-body simulation. Table 3 lists the generalized kurtosis of three types of velocities on small and large scales, both from simulations and corresponding models. Again, the pairwise velocity $\Delta u_L$ is usually approximated by an exponential (Laplace) distribution [49]. This seems not accurate as the generalized kurtosis of distribution of $\Delta u_L$ from N-body simulations does not agree with that of exponential distribution on both small and large scales.

Table 3. The limiting distributions of velocity fields on small and large scales at $z=0$

| Velocity fields | Distribution | $4^{th}$ Kurtosis | $6^{th}$ Kurtosis | $8^{th}$ Kurtosis |
|-----------------|--------------|-------------------|-------------------|-------------------|
| $r \to 0 \quad u_L, \Sigma u_L$ | N-body, $z=0$, Fig. 14 | 4.8 | 57 | 1200 |
| $r \to 0 \quad \Delta u_L$ | N-body, $z=0$, Fig. 14 | 7.5 | 160 | 6000 |
| $r \to 0 \quad u_L, \Sigma u_L$ | $X(x)$ | 4.6 | 48.9 | 944.8 |
| $r \to 0 \quad \Delta u_L$ | Eq. (80) | 7.7 | 159.24 | 6356 |
| $r \to \infty \quad \Delta u_L, \Sigma u_L$ | N-body, $z=0$, Fig. 14 | 4.181 | 41.46 | 670.8 |
| $r \to \infty \quad u_L$ | N-body, $z=0$ Fig. 14 | 5.39 | 85.78 | 2800 |
| $r \to \infty \quad \Delta u_L, \Sigma u_L$ | Logistic (Eq. (82)) | 4.2 | 279/7 | 685.8 |
| $r \to \infty \quad u_L$ | $P_{ul}(x)$ (Eq. (85)) | 5.4 | 78.4 | 2269.8 |

Laplace distribution | 6 | 90 | 2520 |
Gaussian distribution | 3 | 15 | 105 |

5.3 Velocity distributions on the intermediate scale

Figure 26 presents velocity distributions on the intermediate scale $r=1.3\text{Mpc}/h$. Distributions of $\Delta u_L$ and $u_L$ are non-symmetric with nonzero skewness that is necessary as the kinetic energy cascaded from small scale needs to be consumed (dissipated) to grow halos on the intermediate scale. While the velocity sum $\Sigma u_L$ is symmetric on all scales.
Figure 26. Distributions of $u_L$, $\Delta u_L$, and $\Sigma u_L$ on intermediate scale of $r = 1.3$ Mpc/h at $z=0$, i.e. $\log_{10} P_{ul}$ vs. $u_L / u_0$. Distribution of $\Sigma u_L$ is symmetric, while the distribution of $\Delta u_L$ is non-symmetric with non-zero (negative) skewness (Fig. 28) and skew toward positive side. This is a necessary feature of inverse energy cascade. The distribution of $u_L$ is also non-symmetric with a non-zero mean $\langle u_L \rangle$. 
Figure 27. The redshift evolution of generalized kurtosis for pairwise velocity $\Delta u_L$ at redshift $z=2.0, 1.0, 0.3,$ and $0$. Kurtosis for Gaussian distribution are also plotted for reference (straight purple lines). The distribution of $\Delta u_L$ is non-Gaussian on both small and large scales, while the distribution evolution is much faster on small scale due to strong gravitational interaction in halos (also see Fig. 30).

Figure 28. The redshift evolution of skewness $K_3$ (third order generalized kurtosis) of $\Delta u_L$ on intermediate scale. The skewness $K_3 \approx 0$ on small scale and $K_3 < 0$ on intermediate scale. A non-zero skewness is a necessary feature of inverse energy cascade.

Figure 27 plots the redshift variation of generalized kurtosis $K_4$, $K_6$ and $K_8$ of pairwise velocity $\Delta u_L$ for $z=0, 0.3, 1,$ and $2.0$. Kurtosis of Gaussian distribution is also plotted for reference. All velocities are initially Gaussian. With most pairs of particles from the same halo, the distribution of pairwise velocity $\Delta u_L$ on small scale converges to limiting distribution (Eq. (80)) much faster due to strong intra-halo gravitational interaction. While the distribution of $\Delta u_L$ for particle pairs from different halos on large scale converges much slower due to weaker inter-halo gravitational
interaction from greater distance. We will revisit this in Fig. 30. Kurtosis on the intermediate scales is much greater than that on both small and large scales. Figure 28 plots the variation of $K_3$ (or skewness) of pairwise velocity $\Delta u_L$ for $z=0, 0.3, 1, \text{ and } 2.0$ on small and intermediate scales. The skewness $K_3 \approx 0$ on small scale and $K_3 < 0$ on intermediate scale. A non-zero skewness is a necessary feature of inverse energy cascade.

5.4 Modeling the velocity distributions on large scale

On large scale, velocities $\Delta u_L$ and $\Sigma u_L$ have the same distribution with $r \to \infty$ (Fig. 15 and Table 3). The distribution of $u_L$ at $r \to \infty$ has greater kurtosis than $\Delta u_L$ and $\Sigma u_L$. The non-Gaussian feature on large scale could be a manifestation of the long-range nature of gravitational interaction. By contrast, velocity is always Gaussian on large scale for incompressible hydrodynamic turbulence with short range interaction.

The distribution of pairwise velocity $\Delta u_L$ on large scale is usually assumed to be exponential in literature that is not smooth (non-differentiable) at zero velocity (Fig. 29). An improvement can be a Logistic distribution for both $\Delta u_L$ and $\Sigma u_L$ with a variance of $(s \pi)^2 / 3 = 2u^2$, where $u^2$ is the one dimensional velocity dispersion of the entire N-body system (or the variance of $u_L$ on large scale),

$$P_{\Delta u_L}(x) = \frac{1}{4s} \text{sech}^2 \left( \frac{x}{2s} \right),$$

and for large $x$, the Logistic distribution reduces to an exponential distribution,

$$P_{\Delta u_L}(x \to \infty) \approx \frac{1}{s} \exp \left( -\frac{x}{s} \right). \quad (82)$$
Figure 29. a) Distributions of \( u_L, \Delta u_L, \) and \( \Sigma u_L \) on scale of \( r = 100 \) Mpc/h at \( z=0 \), i.e. \( \log_{10} P_{stL} \) vs. \( u_L/u_0 \) (normalized by \( u_0 \)). On large scale \( r \), all distributions are symmetric. A logistic distribution can be used to model the distribution of \( \Delta u_L \) and \( \Sigma u_L \). At large velocity, all distributions approach exponential function; b) Comparison of proposed distributions with simulation data for small velocities. Logistic distribution for \( \Delta u_L \) shows better agreement for small velocities than exponential approximation.

Let’s assume \( P_{ul} \) being the limiting distribution of \( u_L \) when \( r \to \infty \). With \( \rho_L = 0 \) at \( r \to \infty \), two distributions should satisfy the convolution

\[
P_{\Delta u_L} (z) = \int_{-\infty}^{\infty} P_{ul} (x) P_{ul} (z-x) dx.
\]  

Using the characteristic function, the Fourier transform of two distributions should satisfy,

\[
\hat{P}_{\Delta u_L} (t) = \left[ \hat{P}_{ul} (t) \right]^2 = \frac{\pi st}{\sinh(\pi st)}.
\]  

The moment-generating function of \( u_L \) can be found (Eq. (84)) with a variance of \( (\pi s)^2 / 6 = u^2 \),

\[
MGF_{P_{ul}} (t) = \sqrt{\frac{\pi st}{\sin(\pi st)}}.
\]  

The explicit form of distribution \( P_{ul} (x) \) from Eq. (84) is not available but can be obtained numerically from Eq. (85), via inverse Fourier transform. All distributions (Eqs. (82) and (85)) are plotted in Fig. 29 and compared with the numerical data showing good agreement. All distributions are approximately exponential at large velocity. On large scale, pair of particles are not likely residing in the same halo, where the assumption of pairs being from the same halo is invalid. With pair of particles from different halos, velocity on large scale should reflect the velocity of halos that particles reside in. Kurtosis of velocity distributions on large scale is presented in Table 3.
5.5 The redshift evolution of velocity distributions

Finally, the redshift evolution of distributions of all different types of velocities is summarized in this section. This includes velocity $u_p$ of all particles, velocity $u_{hp}$ of all halo particles, velocity $u_{op}$ of all out-of-halo particles, velocity $u_h$ of all halos, and three longitudinal velocities $u_L$, $\Delta u_L$, and $\sum u_L$ on small and large scales, respectively. If any velocity always follows a family of $X$ distribution with a varying shape parameter $\alpha$ at different time (Eq. (65)), the redshift evolution of that velocity distribution can be reduced to the redshift dependence of shape parameter $\alpha \equiv \alpha(z)$. Therefore, the redshift evolution of velocity distributions can be presented as the variation of generalized kurtosis. The $m$th order generalized kurtosis of $X$ distribution can be found from its moments (see Eq. (41) in [12]),

$$K_m(X) = \left( \frac{2K_1(\alpha)}{K_2(\alpha)} \right)^{m/2} \frac{\Gamma\left((1+m)/2\right)}{\sqrt{\pi}} \frac{K_{1+m/2}(\alpha)}{K_1(\alpha)}. \quad (86)$$

Figure 30 presents a summary of the redshift evolution of velocity distributions in terms of the generalized kurtosis of different order ($4^{th}$, $6^{th}$, $8^{th}$, and $10^{th}$ kurtosis from both simulations and Eq. (86)). With increasing time, all velocities become non-Gaussian and the evolution approximately follows the prediction of $X$ distribution with a decreasing $\alpha$. In principle, the halo velocity ($u_h$), the out-of-halo particle velocity ($u_{op}$), and the halo particle velocity ($u_{hp}$) should all follow a $X$ distribution to maximize system entropy, just like the longitudinal velocity on small scale (Eq. (65)). The distributions of halo velocity ($u_h$) and out-of-halo particle velocity ($u_{op}$) have similar distributions and evolve at a much slower pace than the distributions of halo particle velocity ($u_{hp}$) because gravity is much stronger on small scale. This is also consistent with the finding that virial
equilibrium is established much faster for particles in halos (due to stronger gravity) than for halos themselves (see Fig. 9 in [48]).

Figure 30. The redshift evolution of generalized kurtosis $K_6$, $K_8$, and $K_{10}$ with $K_4$ for different types of velocities, i.e. velocity $u_p$ of all particles in system, velocity $u_{hp}$ of all halo particles, velocity $u_{op}$ of all out-of-halo particles, velocity $u_h$ of all halos, and longitudinal velocity $u_L$, pairwise velocity $\Delta u_L$, and velocity sum $\sum u_L$ on small and large scales. All velocities are initially Gaussian with shape parameter $\alpha = \infty$ for $X$ distribution and gradually evolving toward non-Gaussian with a decreasing $\alpha$ with time. The evolution (approximately) follows the prediction (gray lines) of $X$ distribution. The distributions of out-of-halo particles $u_{op}$ and halo velocity $u_h$ matches each other and evolves at a much slower pace compared to halo particles $u_p$. Halos can be treated as macro-particle with similar velocity distribution as that of out-of-halo particles.
6. Distributions of acceleration and critical acceleration scale in MOND

6.1 The redshift evolution of proper acceleration

This section presents the distribution of proper acceleration $a_p$ by computing the total force of every particle in N-body simulation, i.e. the proper acceleration for particle $i$ is

$$a_p = \frac{Gm_i}{a^2} \sum_{j \neq i} \frac{x_j - x_i}{|x_j - x_i|^3},$$

where $x_i$ and $x_j$ are comoving spatial coordinates of particles $i$ and $j$, $a$ is the scale factor, and summation is running over all other particles except $i$. Periodic boundary is applied for force calculation with a total of 26 repeats of simulation domain in three-dimension. Figure 31 plots the redshift variation of distribution of acceleration $a_p$, i.e. the distribution of Cartesian component $[a_{px}, a_{py}, a_{pz}]$ of acceleration vector $a_p$ for all particles. The particle acceleration evolves from an initial (relatively) Gaussian distribution at high redshift to a distribution with a long tail $\propto a_p^{-3}$ for large acceleration in halo core region. Tail starts to form at $z=5$ due to the formation of halos.

Just like the density/velocity distributions, we divide all particles into halo particles and out-of-halo particles since distributions are evolving differently for two different types of particles. Figure 32 plots the redshift evolution of distributions of $a_{hp}$ for halo particles (solid lines) and $a_{op}$ for out-of-halo particles (dash lines). The long tail $\propto a_p^{-3}$ at large acceleration comes from halo core region with higher density. The maximum particle acceleration is determined by the highest density at halo core and independent of redshift. With inverse mass cascade from small to large mass scale [40] and more particles reside in halo outer region, the distribution gradually extends to smaller acceleration. The distribution of $a_{op}$ for out-of-halo particles is relatively Gaussian for all redshifts. Acceleration decreases with time for both types of particles due to expanding space.
Figure 31. The redshift evolution of the distribution of particle acceleration $a_p$. A long tail $\propto a_p^{-3}$ is gradually formed due to the formation of halo structures.
Figure 32. The redshift evolution of halo particle acceleration $a_{hp}$ (solid lines) and out-of-halo particle acceleration $a_{op}$ (dash lines). A long tail $\propto a_{hp}^{-3}$ at large acceleration is a typical feature from halo particles in core region. With time, distribution of $a_{hp}$ gradually extends to smaller acceleration with more particles in the outer region of halos. The distribution of $a_{op}$ for out-of-halo particles is relatively Gaussian. The out-of-halo particles with the greatest acceleration should be close to the surface of halos to merge with (red arrow). For both types of particles, acceleration decreases with time (Fig. 33). Critical acceleration scale in MOND ($a_m \approx 10^{-10}\, m/s^2$) is also indicated in the plot (black arrow).

Figure 33 plots the time variation of typical accelerations ($\sqrt{3} \times$ standard deviation of distributions in Figs. 31 and 32, i.e. the root-mean-square) for all particles (blue), halo particles (black), out-of-halo particles (red), and halos (green), where the factor $\sqrt{3}$ is for the magnitude of acceleration vector in 3D space. Typical accelerations decrease with time (approximately $\propto a^{-3/4}$ for halo particles and $\propto a^{-1/2}$ for out-of-halo particles and halos). The only exception is the halo particle acceleration at $z=0.3$ (red circle) that is greater than $z=0.1$. On large scale, the motion of halos and out-of-halo particles have similar acceleration and velocity (see Figs. 30 and 33).
Figure 33. The variation of typical (root-mean-square) accelerations with scale factor \( a \) for all particles (\( a_p : \) blue), halo particles (\( a_{hp} : \) black), out-of-halo particles (\( a_{op} : \) red), and halos (\( a_h : \) green), respectively. All accelerations decrease with time. At \( z=0 \), the typical acceleration of halo particles \( a_{hp} \) matches the critical acceleration \( a_c = 1.2 \times 10^{-10} \text{ m/s}^2 \) in modified Newtonian dynamics (MOND). The acceleration of halos \( a_h \) matches the out-of-halo particle acceleration \( a_{op} \) and is much smaller (\( \sim 10^{-12} \text{ m/s}^2 \)) due to weaker gravity.

By identifying all halos in N-body system and grouping halos according to their size \( n_p \) (the number of particles in halo) or mass \( m_h = n_p m_p \), the acceleration \( a_{hp} \) and velocity \( v_{hp} \) of halo particles can be further decomposed into the acceleration and velocity due to intra-halo interaction,

\[
\mathbf{a}_i = \mathbf{a}_{hp} - \langle \mathbf{a}_{hp} \rangle_h = \mathbf{a}_{hp} - \mathbf{a}_h \quad \text{and} \quad \mathbf{v}_i = \mathbf{v}_{hp} - \langle \mathbf{v}_{hp} \rangle_h = \mathbf{v}_{hp} - \mathbf{v}_h ,
\]

and due to inter-halo interaction (the acceleration and velocity of halos),

\[
\mathbf{a}_h = \langle \mathbf{a}_{hp} \rangle_h = {1 \over n_p} \sum_{k=1}^{n_p} \mathbf{a}_{hp} \quad \text{and} \quad \mathbf{v}_h = \langle \mathbf{v}_{hp} \rangle_h = {1 \over n_p} \sum_{k=1}^{n_p} \mathbf{v}_{hp} ,
\]

where \( \langle \cdot \rangle_h \) stands for the average over all halo particles for a given halo.

On halo level, the typical acceleration in each halo \( a_{h}^i \) can be computed as the root-mean-square of intra-halo particle acceleration \( a_{hp}^i \) for all particles in the same halo, i.e. \( a_{h}^i = \langle a_{hp}^i \rangle_h^{1/2} \).

The acceleration \( \mathbf{a}_{h} \) of a given halo can be computed as the mean acceleration of all particles in the same halo, i.e. \( \mathbf{a}_{h} = \langle \mathbf{a}_{hp} \rangle_h \) in Eq. (89).

On halo group level (a group of all halos with same size \( n_p \)), the typical acceleration in halo (\( a_{hg}^i \)) can be computed as the mean of intra-halo acceleration \( a_{h}^i \) of all halos in the same group, i.e. \( a_{hg}^i = \langle a_{h}^i \rangle_g \), where \( \langle \cdot \rangle_g \) stands for average over all halos in the same group. The typical acceleration of halo (\( a_{hg} \)) can be computed as the root-mean-square of halo acceleration \( \mathbf{a}_{h} \) for all
halos in the same group, i.e. \( a_{hg} = \left( \langle |\mathbf{a}_p|^2 \rangle_g \right)^{1/2} \). Similar statistics were also applied to particle velocity \( \mathbf{v}_{hp} \) (Section 4.1 in [48]) to obtain the halo virial dispersion \( \sigma_v^2 \) and halo velocity dispersion \( \sigma_h^2 \).

Figure 34. The variation of typical acceleration in halos (\( a_{hg}^i \): solid lines) and acceleration of halos (\( a_{hg} \): dash lines) with halo size \( n_p \) at different redshifts \( z \). Halo mass \( m_h = n_p m_p \). Both accelerations decrease with time at a given halo size, while acceleration in halos increases with halo size (roughly follows \( a_{hg}^i \propto (m_h)^{2/3} a^{-1} \) for small \( n_p \) and \( a_{hg} \propto (m_h)^{1/3} a^{-1} \) for large \( n_p \)) and reaches about \( 10^{-10} \text{m/s}^2 \) for large halos. Acceleration of halos \( a_{hg} \) is independent of halo size \( n_p \), much smaller and on the order of \( 10^{-12} \text{m/s}^2 \).

Figure 34 plots the variation of typical accelerations in halos (\( a_{hg}^i \): solid lines) and acceleration of halos (\( a_{hg} \): dash lines) with halo size \( n_p \) at different redshifts \( z \). Halo mass \( m_h = n_p m_p \) with \( m_p = 2.27 \times 10^{11} M_\odot /h \). For same halo size \( n_p \), both accelerations decrease with time.
Acceleration in halos $a_{hg}^i$ increases with halo size and reaches about $10^{-10} \text{ m/s}^2$ for large halos. Acceleration of halos $a_{hg}$ is relatively independent of halo size, much smaller than $a_{hg}^i$ and roughly on the order of $10^{-12} \text{ m/s}^2$ due to weaker gravity between halos on large scale.

6.2 The critical acceleration scale in MOND

Note that the typical acceleration in halos ($a_{hp}$ in Fig. 33) at $z=0$ matches the critical acceleration scale $a_m = 1.2 \times 10^{-10} \text{ m/s}^2$ in modified Newtonian dynamics (symbol $a_0$ in MOND) [51]. The critical acceleration $a_m$ might be related to the fluctuation of acceleration in self-gravitating collisionless dark matter flow. The value of $a_m$ ($\sim 1.2 \times 10^{-10} \text{ m/s}^2$) is still empirical and phenomenological without a good theory. A brief estimation of the critical acceleration $a_m$ can be made here based on the inverse mass/energy cascade in dark matter flow [40, 48].

In a finite time interval $\Delta t$, the hierarchical structure merging might involve multiple substructures merging into a single large structure. For an infinitesimal time interval $dt$, that process should involve the merging of two and only two substructures such that the two-body collapse is the most elementary process for halo mass accretion and inverse mass cascade [41]. As shown in Fig. 35, the elementary merging between a halo and a single merger facilitates the mass and energy cascade [40, 48]. Let’s consider a two-body merging, where a single merger has a mass $dm$, typical velocity $u(a)$, and a typical acceleration $a_m(a)$ right before merging with a halo of mass $m_h$. The velocity and acceleration of single merger are likely aligned on large scale (velocity $u$ and acceleration $a_m$ point to the same direction). Due to the gravitational interaction with halo to be merged, the single merger right on the boundary (dash line) has a relative motion toward the
center of halo \( u_r = u \cot(\theta_{ur}) \) (radial velocity) and an acceleration toward halo center with \( a_r = a_m \cot(\theta_{ur}) \) (radial acceleration).

Figure 35. The schematic plot for inverse mass/energy cascade from a series of merging between halo and single mergers. For an infinitesimal interval \( dt \), that process should involve the merging of a halo (mass \( m_h \)) with a single merger (mass \( dm \)). The single merger has a typical velocity \( u(a) \) and a typical acceleration \( a_m(a) \). Dash line represents the boundary of that halo.

The angle \( \theta_{ur} \) can be related to the critical value \( \beta_{s2} \) for an equilibrium two-body collapse (Eq. (104) in [41]), i.e.

\[
\cot(\theta_{ur}) = \frac{u_r}{v_{cir}} = \beta_{s2} = \frac{1}{3\pi},
\]

where \( \beta_{s2} \) is a constant that is related to the critical halo density as \( \Delta_c = 2/(\beta_{s2})^2 = 18\pi^2 \) (Eq. (89) in [41]), and \( v_{cir} \) is the circular velocity (\( u \approx v_{cir} \)). This is also true for isothermal halos where the ratio between circular velocity and radial flow is \( v_{cir}/u_r = 3\pi \) (Eq. (29) in [52]).
Now let’s try to compute the constant rate of energy cascade in collisionless dark matter flow. Firstly, that rate \( \varepsilon_u \) in \( m^2/s^3 \) represents the energy flux/transfer across halos of different mass scales. It can be determined by the change of kinetic energy for all halo particles due to intra-halo motion defined in Eqs. (88) and (89), i.e. a dot product between \( \mathbf{a}_{hp}^i \) and \( \mathbf{v}_{hp}^i \),

\[
\varepsilon_u = -\langle \mathbf{a}_{hp}^i \cdot \mathbf{v}_{hp}^i \rangle = -\langle \mathbf{a}_{hp} \cdot \mathbf{v}_{hp} \rangle + \langle \mathbf{a}_h \cdot \mathbf{v}_h \rangle ,
\]

(91)

where \( \mathbf{a}_{hp} \) and \( \mathbf{v}_{hp} \) are vectors of halo particle acceleration and velocity, \( \mathbf{a}_{hp}^i \) and \( \mathbf{v}_{hp}^i \) are vectors of intra-halo acceleration and velocity relative to the motion of halos. Here the average is taken over all halo particles. Terms \( \mathbf{a}_h \) and \( \mathbf{v}_h \) are the acceleration and velocity of halos, i.e. the mean acceleration and velocity of all particles in a given halo. The change of kinetic energy due to the motion of halos (term 1 in Eq. (91)) does not contribute to energy cascade and should be excluded.

Secondly, the inverse mass/energy cascade is facilitated by a series of merging with single mergers [40, 48]. The rate of kinetic energy cascade might be directly determined from a typical merging process, i.e. a two-body collapse model (TBCM) [41]. During each merging, the kinetic energy transferred from small to large mass scale (from \( m_h \) to \( m_h + dm \) ) comes from the change of kinetic energy of that single merger at the instant of merging, mostly via the relative motion along radial direction. For a single merger with typical velocity \( u(a) \) and acceleration \( a_m \), the rate of energy transfer \( \varepsilon_u \) can be approximated by

\[
\varepsilon_u = -a_m u_r = -a_m(a) \cot(\theta_{ar}) u(a) \cot(\theta_{ar}) ,
\]

(92)

where \( \varepsilon_u < 0 \) for inverse cascade from small to large mass scale. This expression resembles the first expression in Eq. (91), but is obtained from the elementary two-body collapse.
Thirdly, by considering the energy evolution in entire N-body system, the rate of energy transfer is approximately the change of specific kinetic energy for entire system, or rate of energy production. It is a constant of time (Fig. 1 in [45] or Fig. 10 in [48]) and should read

$$\varepsilon_u \approx \frac{3 u^2}{2 t} = -\frac{9}{4} H_0 u_0^2 = 4.6 \times 10^{-7} \frac{m^2}{s^3}. \quad (93)$$

Combining Eqs. (90)-(93) together, the typical acceleration scale can be related to $\varepsilon_u$ as

$$a_m(a) = \frac{-\Delta \varepsilon_u}{u} = (3\pi)^2 \frac{\varepsilon_u}{u} = \frac{81}{4} \pi^2 H_0 \frac{u_0^2}{u}, \quad (94)$$

where with $u_0 = 354.61 \text{ km/s}$ and $H_0 = 100h \text{ km/(s \cdot Mpc)} \approx 1.6202 \times 10^{-18} \text{ km/s}$ (with $h = 0.5$),

$$a_m(a = 1) \approx 200H_0u_0 \approx 1.2 \times 10^{-10} \text{ km/s}^2. \quad (95)$$

Finally, the typical acceleration scale $a_m$ in MOND can be determined by $\varepsilon_u$ (Eq. (94)), the constant rate of kinetic energy cascade. Just like the peculiar velocity, there exists a fluctuation of acceleration in self-gravitating collisionless dark matter flow. The typical scale of acceleration fluctuation possibly plays the role of the critical acceleration scale in MOND, i.e. the MOND theory might be an intrinsic property of and fully consistent with the theory of dark matter flow.

More insights can be briefly outlined here for the so-called deep-MOND behavior, where particle acceleration is much smaller than the fluctuation of acceleration ($|\Delta a_p| \ll a_m$). According to maximum entropy distributions for particles in self-gravitating collisionless flow (Eqs. (48) and (49) in [12]), the specific particle energy $\varepsilon(v)$ of particles has a linear scaling with $|v|$ for large velocity, and a parabolic scaling for small velocity,

$$\varepsilon(v) \approx \frac{3}{2}\left(1 + \frac{2}{n}\right)v_0 |v| \quad \text{for} \quad |v| \gg v_0,$$
Here $v_0$ is a typical velocity scale in maximum entropy distribution $X$ (Eq. (65)), $\alpha$ is a shape parameter of $X$ distribution, and $n \approx 1$ is the effective potential exponent for virial theorem. Specific particle energy $\varepsilon(v)$ includes both kinetic and potential energy. Using virial theorem, the specific kinetic energy for particles with a given speed $|v|$ should also follow $\varepsilon_K(v) \propto |v|^2$ for low-speed ($|v| \ll v_0$), which is the standard Newtonian behavior. However, $\varepsilon_K(v) \propto v_0 |v|$ for high-speed particles ($|v| \gg v_0$) from maximum entropy can also be a direct manifestation of the external field effect that is often discussed in literature. These high-speed particles are usually in the outer region of halos with extremely small acceleration. Their dynamics is much easier to be affected by the presence of external gravitational field due to the long-range nature of gravity.

To simplify the calculation, let’s assume a one-dimensional self-gravitating collisionless system with a typical velocity scale $v_0$ and an acceleration scale $a_m$ due to the fluctuation of velocity and acceleration. A particle with mass $m_p$, velocity $v_p$, and acceleration $a_p = \frac{dv_p}{dt}$, moves through this one-dimensional collisionless fluid and subjects to an external force $F_p$. This is an analogue of the standard Brownian motion with particle moving through a viscous liquid. To respect the nature of self-gravitating collisionless flow, the velocity dispersion $v_p^2$ of that particle should increase linearly with time (Eq. (93)) with a proportional constant $\varepsilon_u$ ($\theta_{ur}$ in Eq. (92) does not present for 1D system),

$$\frac{1}{2} \frac{dv_p^2}{dt} = v_p \frac{dv_p}{dt} = a_p v_p = a_m v_0 = -\varepsilon_u.$$  

(97)
The external force $F_p$ can be computed from the change of specific kinetic energy,

$$F_p v_p = m_p \frac{d\varepsilon_K}{dt}.$$  \hspace{1cm} (98)

For deep-MOND regime where $a_p \ll a_m$ or $v_p \gg v_0$, the specific kinetic energy

$$\varepsilon_K(v) = v_0 v_p$$  \hspace{1cm} (99)

is proportional to speed $v_p$ (from Eq. (96)). Force in deep-MOND regime can be obtained from Eqs. (97) and (98) such that external force is proportional to square of acceleration, i.e.

$$F_p = m_p v_0 a_p = m_p \frac{a_p^2}{a_m}.$$  \hspace{1cm} (100)

On the other hand, for particles with low speed or high acceleration $a_p \gg a_m$ or $v_p \ll v_0$, the specific kinetic energy $\varepsilon_K(v) = v_p^2 / 2$ such that the standard Newton’s law $F_p = m_p a_p$ can be fully recovered from Eq. (98). Two key ingredients are necessary for this simplistic view of deep-MOND behavior: i) the constant rate of energy cascade for dark matter flow (Eq. (97)); and ii) the kinetic energy proportional to particle speed for low acceleration (Eqs. (96) and (99)). If $a_m$ and $v_0$ are the standard deviation of acceleration and velocity, Eq. (97) presents a “uncertainty” principle such that the more precisely velocity is determined, the less precisely its acceleration can be determined, and vice versa. Future study is required to refine and develop this simple idea.

7. Conclusions

By identifying all halos in entire N-body system and dividing all particles into halo particles and out-of-halo particles that do not belong to any halos, the redshift and scale dependence of density, velocity and acceleration distributions are extensively investigated. Instead of projecting
particle field onto structured grid that usually involves information loss and unnecessary noise, Delaunay tessellation is used to reconstruct the comoving density field and maximally preserve information in simulation data. The particle overdensity $\delta$ evolves from an initial Gaussian to an asymmetric distribution with a long tail $\propto \delta^{-3}$ (Fig. 1). The log-density $\eta$ evolves from Gaussian to a bimodal distribution at $z=0$, with two peaks corresponding to the high density for halo particles and low density for out-of-halo particles (Fig. 2). The log-density distribution of out-of-halo particles has a negative mean decreasing with time, while that of halo particles has an increasing mean, both due to the continuous mass transfer from out-of-halo to halos (Fig. 5).

Without projecting the density field onto grid, we first compute the radial distribution function $g(r)$ for all scale $r$ from N-body simulation. The second order correlation $\xi(r)$ can be obtained from $g(r)$ (Eq. (9)) and plotted in Figs. 6, 8 and 9. The density correlation cannot be positive on all scales due to the normalization (Eq. (10)). The density spectrum $E_\delta$ and fluctuation function $\sigma_\delta^2$ can be obtained from $\xi(r)$ using Eqs. (20) and (27), and presented in Figs. 6, 7, 11. Function $E_\delta$ reflects the real-space distribution of density fluctuation on different scales (Eq. (30) and Fig. 12) and contains the same information as density spectrum $E_\delta$ (Eq. (31)). Analytical models for correlation and dispersion functions on large scale are also presented in Eqs. (33) and (35).

The scale dependence of velocity field is studied for the longitudinal velocity $u_L$ or $u'_L$, velocity difference $\Delta u_L = u'_L - u_L$ (or pairwise velocity), and velocity sum $\Sigma u_L = u_L + u'_L$ (see Fig. 13). Fully developed velocity field is never Gaussian on any scale despite that they can be initially Gaussian (Fig. 14 and 15). By contrast, velocity distribution is nearly Gaussian on large scale for incompressible flow. Distribution of $\Sigma u_L$ approaches that of $u_L$ on small scale with the correlation
(between $u_L$ and $u'_L$) $\rho_L \to 0.5$. While on large scale, the distribution of $\Sigma u_L$ approaches that of $\Delta u_L$ with correlation $\rho_L \to 0$. Combining pair conservation equation and density correlation, the first moment of $\Delta u_L$ (pairwise velocity) can be analytically modelled on small and large scales (Eqs. (46), (48) and Fig. 18). The second moment of three types of velocities is presented in Figs. 19 and 20, with an initial increase followed by a sharp decrease on the intermediate scale.

The second moment of $\Delta u_L$, i.e. the pairwise velocity dispersion $S_2^{lp}(r) = \langle (\Delta u_L)^2 \rangle$, approaches $2u^2$ on small scale (Fig. 21). A two-thirds law can be identified for a reduced structure function such that $S_2^{lp} = (S_2^{lp} - 2u^2) \propto (-\varepsilon_u)^{2/3} r^{2/3}$ (Eq. (55) and Fig. 22), where $\varepsilon_u$ is the constant rate of energy production. A constant length scale can be introduced as $r_s = u_0^3/\varepsilon_u$, below which the two-thirds law is valid. Model for longitudinal velocity dispersion $\langle u_L^2 \rangle$ on small scale can be derived (Eq. (59) and Fig. 19). The two-thirds law can be generalized to all even order structure functions $\langle (\Delta u_L)^{2n} \rangle$ (Eq. (60) and Fig. 23), while odd order structure functions $\langle (\Delta u_L)^{2n+1} \rangle$ should satisfy generalized stable clustering hypothesis (GSCH in Eq. (63) and Fig. 24). A complete comparison between incompressible flow and SG-CFD is listed in Table 2.

The distributions of three types of different velocities can be analytically modeled on small and large scales, respectively. On small scale, both velocities $u_L$ and $\Sigma u_L$ can be modelled by a $X$ distribution to maximize system entropy (Fig. 25 and Eq. (65)). Explicit form for distribution of pairwise velocity $\Delta u_L$ on small scale is still unknown. However, the moments and kurtosis of $\Delta u_L$ can be analytically estimated (Eqs. (79) and (80)) using joint Gaussian distribution with a halo-size dependent correlation coefficient $\rho_L$. On intermediate scale, distributions of $u_L$ and $\Delta u_L$ becomes significantly non-symmetric with non-zero skewness, a necessary feature of inverse
energy cascade. On large scale, both $\Delta u_L$ and $\Sigma u_L$ approach the same distribution and can be modelled by a logistic function (Eq. (82) and Fig. 29). The distribution of $u_L$ can also be analytically obtained in Eq. (84). The limiting distributions of different velocities on small and large scales are summarized in Table 3. The redshift evolution of velocity distributions is summarized in Fig. 30. With increasing time, all velocities become non-Gaussian and the evolution approximately follows the prediction of a $X$ distribution with a decreasing $\alpha$ to maximize system entropy. However, the distribution of velocities on large scale usually evolves at a much slower pace than the distribution of velocities on small scale because of stronger gravity on small scale.

Finally, the redshift evolution of acceleration distribution for halo and out-of-halo particles are studied (Figs. 31 and 32). A long tail $\propto a_{hp}^{-3}$ at large acceleration is a typical feature from halo particles in core region. With increasing time, acceleration gradually extends to smaller range due to halo mass accretion and more particles residing in the outer region of halos with smaller acceleration. The root-mean-square acceleration decreases with time (Fig. 33) due to the expansion of space. We note that the typical acceleration in halos matches the critical acceleration scale $a_m$ in MOND, i.e. the critical acceleration scale might be related to the fluctuation of acceleration. Based on the theory of inverse mass/energy cascade, the typical acceleration can be related to the rate of energy production $\varepsilon_u$, i.e. $a_m \propto \varepsilon_u / u$ in Eq. (94). Combining the particle energy from maximum entropy principle and constant rate of energy cascade, both deep-MOND and standard Newtonian behavior can be recovered (Eq. (100)). The typical scale of acceleration fluctuation possibly plays the role of the critical acceleration scale in MOND.
Reference

1. Davis, M. and P.J.E. Peebles, *Integration of Bbgky Equations for Development of Strongly Nonlinear Clustering in an Expanding Universe*. Astrophysical Journal Supplement Series, 1977. 34(4): p. 425-450.

2. Ferreira, P.G., et al., *Streaming velocities as a dynamical estimator of Omega*. Astrophysical Journal, 1999. 515(1): p. L1-L4.

3. Juszkiewicz, R., et al., *Evidence for a low-density universe from the relative velocities of galaxies*. Science, 2000. 287(5450): p. 109-112.

4. Gorski, K., *On the Pattern of Perturbations of the Hubble Flow*. Astrophysical Journal, 1988. 332(1): p. L7-L11.

5. Gorski, K.M., et al., *Cosmological Velocity Correlations - Observations and Model Predictions*. Astrophysical Journal, 1989. 344(1): p. 1-19.

6. Kuhlen, M., et al., *Dark matter direct detection with non-Maxwellian velocity structure*. Journal of Cosmology and Astroparticle Physics, 2010(2).

7. Ullio, P. and M. Kamionkowski, *Velocity distributions and annual-modulation signatures of weakly-interacting massive particles*. Journal of High Energy Physics, 2001(3).

8. Zhao, Y., et al., *Constraint on the velocity dependent dark matter annihilation cross section from gamma-ray and kinematic observations of ultrafaint dwarf galaxies*. Physical Review D, 2018. 97(6).

9. Petac, M., P. Ullio, and M. Valli, *On velocity-dependent dark matter annihilations in dwarf satellites*. Journal of Cosmology and Astroparticle Physics, 2018(12).

10. Kazantzidis, S., J. Magorrian, and B. Moore, *Generating equilibrium dark matter halos: Inadequacies of the local Maxwellian approximation*. Astrophysical Journal, 2004. 601(1): p. 37-46.

11. Wojtak, R., et al., *The distribution function of dark matter in massive haloes*. Monthly Notices of the Royal Astronomical Society, 2008. 388(2): p. 815-828.

12. Xu, Z., *The maximum entropy distributions of collisionless particle velocity, speed, and energy for statistical mechanics of self-gravitating collisionless flow (SG-CFD)*. arXiv:2110.03126v1 [astro-ph.CO], 2021.

13. Hubble, E., *The distribution of extra-galactic nebulae*. Astrophysical Journal, 1934. 79(1): p. 8-76.

14. Bernardeau, F. and L. Kofman, *Properties of the Cosmological Density Distribution Function*. Astrophysical Journal, 1995. 443(2): p. 479-498.

15. Klypin, A., et al., *Density distribution of the cosmological matter field*. Monthly Notices of the Royal Astronomical Society, 2018. 481(4): p. 4588-4601.

16. Angulo, R.E., et al., *Scaling relations for galaxy clusters in the Millennium-XXL simulation*. Monthly Notices of the Royal Astronomical Society, 2012. 426(3): p. 2046-2062.

17. Springel, V., *The cosmological simulation code GADGET-2*. Monthly Notices of the Royal Astronomical Society, 2005. 364(4): p. 1105-1134.

18. Peebles, P.J.E., et al., *A Model for the Formation of the Local Group*. Astrophysical Journal, 1989. 345(1): p. 108-121.

19. Efstathiou, G., et al., *Numerical Techniques for Large Cosmological N-Body Simulations*. Astrophysical Journal Supplement Series, 1985. 57(2): p. 241-260.

20. Jennings, E., C.M. Baugh, and S. Pascoli, *Modelling redshift space distortions in hierarchical cosmologies*. Monthly Notices of the Royal Astronomical Society, 2011. 410(3): p. 2081-2094.

21. Hahn, O., R.E. Angulo, and T. Abel, *The properties of cosmic velocity fields*. Monthly Notices of the Royal Astronomical Society, 2015. 454(4): p. 3920-3937.
22. Pueblas, S. and R. Scoccimarro, *Generation of vorticity and velocity dispersion by orbit crossing*. Physical Review D, 2009. 80(4).
23. Jelic-Cizmek, G., et al., *The generation of vorticity in cosmological N-body simulations*. Journal of Cosmology and Astroparticle Physics, 2018(9).
24. Hockney, R.W. and J.W. Eastwood, *Computer Simulation Using Particles*. 1988, Bristol, PA, USA: Taylor & Francis.
25. Baugh, C.M. and G. Efstathiou, *A Comparison of the Evolution of Density Fields in Perturbation-Theory and Numerical Simulations*. 1. Nonlinear Evolution of the Power Spectrum. Monthly Notices of the Royal Astronomical Society, 1994. 270(1): p. 183-198.
26. Baugh, C.M., E. Gaztanaga, and G. Efstathiou, *A Comparison of the Evolution of Density Fields in Perturbation-Theory and Numerical Simulations*. 2. Counts-in-Cells Analysis. Monthly Notices of the Royal Astronomical Society, 1995. 274(4): p. 1049-1070.
27. Taylor, G.I., *Statistical theory of turbulence Part 1-4*. Proceedings of the royal society A, 1935. 151: p. 421.
28. Taylor, G.I., *Production and dissipation of vorticity in a turbulent fluid*. Proceedings of the Royal Society of London Series a-Mathematical and Physical Sciences, 1938. 164(A916): p. 0015-0023.
29. de Karman, T. and L. Howarth, *On the statistical theory of isotropic turbulence*. Proceedings of the Royal Society of London Series a-Mathematical and Physical Sciences, 1938. 164(A917): p. 0192-0215.
30. Batchelor, G.K., *The Theory of Homogeneous Turbulence*. 1953, Cambridge, UK: Cambridge University Press.
31. Kolmogorov, A.N., *A Refinement of Previous Hypotheses Concerning the Local Structure of Turbulence in a Viscous Incompressible Fluid at High Reynolds Number*. Journal of Fluid Mechanics, 1962. 13(1): p. 82-85.
32. C. S. Frenk, et al., *Public Release of N-body simulation and related data by the Virgo consortium*. arXiv:astro-ph/0007362v1 2000.
33. Jenkins, A., et al., *Evolution of structure in cold dark matter universes*. Astrophysical Journal, 1998. 499(1): p. 20.
34. Colberg, J.M., et al., *Linking cluster formation to large-scale structure*. Monthly Notices of the Royal Astronomical Society, 1999. 308(3): p. 593-598.
35. Sheth, R.K., H.J. Mo, and G. Tormen, *Ellipsoidal collapse and an improved model for the number and spatial distribution of dark matter haloes*. Monthly Notices of the Royal Astronomical Society, 2001. 323(1): p. 1-12.
36. Xu, Z., *The statistical theory of self-gravitating collisionless dark matter flow and the correlation, structure, and dispersion functions for velocity, density, and potential fields*. arXiv:2202.00910 [astro-ph.CO], 2022.
37. Xu, Z., *The statistical theory of self-gravitating collisionless dark matter flow and high order kinematic and dynamic relations for velocity correlations on small and large scales*. arXiv:2202.02991 [astro-ph.CO], 2022.
38. Romano-Diaz, E. and R. van de Weygaert, *DeLaunay Tessellation Field Estimator analysis of the PSCz local Universe: density field and cosmic flow*. Monthly Notices of the Royal Astronomical Society, 2007. 382(1): p. 2-28.
39. Bernardeau, F. and R. van de Weygaert, *A new method for accurate estimation of velocity field statistics*. Monthly Notices of the Royal Astronomical Society, 1996. 279(3): p. 693-711.
40. Xu, Z., *Inverse mass cascade of self-gravitating collisionless flow and effects on halo mass functions*. arXiv:2109.09985v1 [astro-ph.CO], 2021.
41. Xu, Z., A non-radial two-body collapse model (TBCM) for gravitational collapse of dark matter in expanding background and generalized stable clustering hypothesis (GSCP). arXiv:2110.05784v1 [astro-ph.CO], 2021.

42. Irvine, W.M., Local Irregularities in a Universe Satisfying the Cosmological Principle. 1961, HARVARD UNIVERSITY.

43. Layzer, D., A Preface to Cosmogony. I. The Energy Equation and the Virial Theorem for Cosmic Distributions. Astrophysical Journal, 1963. 138: p. 174.

44. Mo, H.J., Y.P. Jing, and G. Borner, Analytical approximations to the low-order statistics of dark matter distributions. Monthly Notices of the Royal Astronomical Society, 1997. 286(4): p. 979-993.

45. Xu, Z., The evolution of energy, momentum, and spin parameter in self-gravitating collisionless dark matter flow and integral constants on large and small scales. arXiv:2202.04054 [astro-ph.CO], 2022.

46. Sheth, R.K., et al., Linear and non-linear contributions to pairwise peculiar velocities. Monthly Notices of the Royal Astronomical Society, 2001. 325(4): p. 1288-1302.

47. Peebles, P.J.E., The Large-Scale Structure of the Universe. 1980, Princeton, NJ: Princeton University Press.

48. Xu, Z., Inverse and direct cascade of kinetic and potential energy for self-gravitating collisionless dark matter flow and effects of halo shape on energy cascade. arXiv:2110.13885v1 [astro-ph.GA], 2021.

49. Sheth, R.K., The distribution of pairwise peculiar velocities in the non-linear regime. Monthly Notices of the Royal Astronomical Society, 1996. 279(4): p. 1310-1324.

50. Xu, Z., Mass functions of dark matter halos from maximum entropy distributions for self-gravitating collisionless flow. arXiv:2110.09676v1 [astro-ph.CO], 2021.

51. Milgrom, M., A Modification of the Newtonian Dynamics as a Possible Alternative to the Hidden Mass Hypothesis. Astrophysical Journal, 1983. 270(2): p. 365-370.

52. Xu, Z., Inverse mass cascade of self-gravitating collisionless flow and effects on halo deformation, energy, size, and density profiles. arXiv:2109.12244v1 [astro-ph.CO], 2021.
Supplementary information for

The two-thirds law for pairwise velocity and origin of critical MOND acceleration from
distributions of density, velocity, and acceleration in dark matter flow

Zhijie (Jay) Xu\textsuperscript{1,a}

1. Computational Mathematics Group, Physical and Computational Sciences Directorate, Pacific
Northwest National Laboratory, Richland, WA 99352, USA

\textsuperscript{a) Electronic mail: zhijie.xu@pnnl.gov; zhijiexu@hotmail.com}
Contents

Nomenclature ................................................................................................................................................ 3

The N-body simulations and numerical data ................................................................................................ 6
### Nomenclature

| Symbol | S.I. Unit | Physical Meaning |
|--------|-----------|------------------|
| $\Omega_0$ | Dimensionless | Matter content |
| $\Lambda$ | $1/m^2$ | Universe constant |
| $h$ | Dimensionless | Dimensionless Hubble constant |
| $\Gamma$ | Dimensionless | $\Gamma = \Omega_0 h$ Shape parameter for the density power spectrum |
| $\sigma_8$ | Dimensionless | Density fluctuation at $r = 8 Mpc/h$ |
| $L$ | $m$ | Size of simulation box |
| $N_p$ | Dimensionless | |
| $m_p$ | Dimensionless | Mass of collisionless particle |
| $l_{\text{soft}}$ | $m$ | Gravitational soften length |
| $V$ | $m^3$ | Volume of simulation box |
| $PE$ | $m^2/s^2$ | The specific potential energy of the N-body system |
| $P_p$ | $m^2/s^2$ | The specific potential energy in physical coordinate |
| $K_p$ | $m^2/s^2$ | The specific peculiar kinetic energy |
| $H_0, H$ | $1/s$ | Hubble constants (‘0’ stands for the current epoch) |
| $t_{0, t}$ | $s$ | Time (‘0’ stands for the current epoch) |
| $\rho_0$ | $kg/m^3$ | Mean matter density (‘0’ stands for the current epoch) |
| $\varepsilon_\alpha$ | $m^2/s^3$ | Constant rate of energy transfer |
| $a$ | Dimensionless | Scale factor |
| $z$ | Dimensionless | Redshift |
| $\phi$ | $m^2/s^2$ | Gravitational potential field |
| $m_h^*$ | $kg$ | Characteristic mass scale of halos |
| $\delta(x)$ | Dimensionless | Overdensity field |
| $\rho(x)$ | $kg/m^3$ | Matter density |
| $\eta(x)$ | Dimensionless | Log-density field |
| $g(r)$ | Dimensionless | Radial distribution function |
| $\xi(r)$ | Dimensionless | Two-point density correlation function |
| $\bar{\xi}(r)$ | Dimensionless | Volume averaged density correlation |
| $l_{\delta 0}, l_{\delta 1}$ | $m$ | Correlation length from density field |
| $E_\delta(k)$ | $m$ | The power spectrum of particle density field |
| $P_\delta(k)$ | $m^3$ | The matter power spectrum in cosmology |
| Symbol | Dimensionless | Description |
|--------|---------------|-------------|
| \( \sigma^2_d(r) \) | The variance of density fluctuation smoothed at scale of \( r \) |
| \( \Delta^2_d(k) \) | The dimensionless power spectrum of particle density field |
| \( W(x) \) | The window function when smoothed with a filter of size \( r \) |
| \( j_0(x), j_1(x) \) | Zeroth/first order spherical Bessel function of the first kind |
| \( E_{sr}(r) \) | The real-space distribution of density fluctuation in scale \( r \) |

| Symbol | Dimensionless | Description |
|--------|---------------|-------------|
| \( u, u_0 \) | The root-mean-square particle velocity |
| \( a_0 \) | Parameter for velocity correlation |
| \( r_z \) | Constant length scale for velocity correlation |
| \( r_t \) | Length scale where longitudinal and transverse velocity correlations are comparable |
| \( u_L, u_T \) | Longitudinal velocity on scale \( r \) |
| \( u_r, u_T \) | Transverse velocity on scale \( r \) |
| \( \Delta u_L(r) \) | Longitudinal velocity difference (pairwise) on scale \( r \) |
| \( \Sigma u_L(r) \) | Longitudinal velocity sum on scale \( r \) |
| \( u_p \) | The velocity of any particles in entire system |
| \( u_{hp} \) | The velocity of halo particles |
| \( u_{op} \) | The velocity of out-of-halo particles |
| \( u_h \) | The velocity of halos identified in the system |
| \( K_n(\tau, r) \) | The generalized kurtosis for random variable \( \tau \) on scale \( r \) |
| \( S^p_0(\tau, r) \) | The nth order central moment of random variable \( \tau \) |
| \( S^p_0(r) \) | The nth order longitudinal structure function |
| \( S^p_{aw}(r) \) | The nth order reduced longitudinal structure function |
| \( \rho_L \) | Correlation coefficient between \( u_L \) and \( u_T \) |
| \( R_z(r) \) | The total velocity correlation functions |
| \( L_z(r) \) | The longitudinal velocity correlation function |
| \( T_z(r) \) | The lateral (transverse) velocity correlation function |
| \( S^p_z(r) \) | Original second order longitudinal structure function |
| \( S^p_{2z}(r) \) | Reduced second order longitudinal structure function |
| \( S^p_3(r) \) | Modified second order longitudinal structure function |
| \( \beta^*_{2n} \) | Constant of two-third law for even order structure functions |
| Symbol          | Definition                                                                 |
|-----------------|-----------------------------------------------------------------------------|
| $X(v)$          | $s/m$ The X distribution for particle velocity                              |
| $H\left(\sigma_v^2\right)$ | $s^2/m^2$ The H distribution for particles with a virial dispersion $\sigma_v^2$ |
| $\alpha$       | Dimensionless Shape parameter in X distribution                            |
| $v_0$           | $m/s$ Velocity scale in X distribution                                      |
| $K_n(x)$        | Dimensionless Modified Bessel function of the second kind                   |
| $N_h$           | Dimensionless Total number of halos in the system                           |
| $m_h$           | $kg$ Mass of a given halo                                                   |
| $n_h$           | Dimensionless Number of halos in a halo group                              |
| $n_p$           | Dimensionless Number of particles in a halo                                |
| $n_{pair}$      | Dimensionless Number of pairs for two particles with separation $r$ in a halo |
| $N_{pair}$      | Dimensionless Total number of pairs with a separation $r$ in entire system  |
| $\sigma_v^2(n_p)$ | $m^2/s^2$ Size-dependent halo virial dispersion                           |
| $\sigma_h^2$   | $m^2/s^2$ Halo velocity dispersion                                           |
| $\sigma_v^2(n_p)$ | $m^2/s^2$ Size-dependent total dispersion of velocity                      |
| $\rho_{cor}(n_p)$ | Dimensionless Correlation coefficient for velocity of pair of particles    |
| $a_p$           | $m/s^2$ Proper acceleration of particles                                    |
| $a_{hp}$        | $m/s^2$ Acceleration of halo particles                                       |
| $a_{op}$        | $m/s^2$ Acceleration of out-of-halo particles                               |
| $a_h$           | $m/s^2$ Acceleration of halos                                               |
| $a_m$           | $m/s^2$ Critical acceleration scale in MOND                                 |
| $\theta_{sr}$   | Dimensionless Angle between vector $\mathbf{r}$ and velocity $\mathbf{u}$ for single merger |
| $\Delta_c$      | Dimensionless Critical halo mass density                                    |
The N-body simulations and numerical data

The numerical data were public available and generated from the N-body simulations carried out by the Virgo consortium, an international collaboration aims to perform large N-body simulations for the formation of structure. A comprehensive description of the data can be found in [1, 2]. As the first step, the current study was carried out using the simulation runs with $\Omega = 1$ and the standard CDM power spectrum (SCDM) to focus on the matter-dominant gravitational flow of collisionless particles. Similar analysis can be extended to other simulation runs with different model assumptions and parameter in the future.

The current simulation includes about 17 million particles with particle mass about $2.27 \times 10^{11} M_\odot/h$. The simulation box sizes around 240 Mpc/h, where $h$ is the Hubble constant in units of $100\, km/Mpc\cdot s$. The same set of data has been widely used in a number of studies from clustering statistics to formation of cluster halos in large scale environment [3], and test of models for halo abundances and mass functions [4]. Some key numerical parameters of the N-body simulations were listed in Table S1. In N-body simulations, it is common to adopt a unit system different from the S.I. units. The current units used in N-body simulations are listed in Table S2.

The friends-of-friends algorithm (FOF) was used to identify all halos from the simulation data that depends only on a dimensionless parameter $b$, which defines the linking length $b(\frac{N}{V})^{-1/3}$, where $V$ is the volume of the simulation box. Halos were identified with a linking length parameter of $b = 0.2$ in this work. All halos identified from the simulation data were first grouped into halo groups of different sizes according to halo mass $m_h$ (or in terms of $n_p$, the number of particles in the halos), where $m_h = n_p m_p$. The total mass for a halo group of mass $m_g$ is $m_g = m_h n_h$, where $n_h$ is the number of halos in each group.
Table S1. Numerical parameters of N-body simulation for SCDM1

| Run   | $\Omega_0$ | $\Lambda$ | $h$ | $\Gamma$ | $\sigma_v$ | $L(Mpc/h)$ | $N_p$ | $m_p(M_\odot/h)$ | $l_{soft}(Kpc/h)$ |
|-------|------------|-----------|-----|----------|------------|------------|-------|-----------------|------------------|
| SCDM1 | 1.0        | 0.0       | 0.5 | 0.5      | 0.51       | 239.5      | 256\(^3\) | $2.27 \times 10^{11}$ | 36               |

Table S2. Unit system used in N-body simulation for SCDM1

| Time       | Mass       | Length | Velocity | Specific Energy | $H_0$     | $t_0$     |
|------------|------------|--------|----------|----------------|-----------|-----------|
| $s \cdot Mpc$ | $M_\odot$ | $Mpc$ | $Km$ | $\left(\frac{Km}{s}\right)^2$ | $100 \frac{h \cdot Km}{s \cdot Mpc}$ | $\frac{2}{3} \times 10^{-2} \frac{s \cdot Mpc}{h \cdot Km}$ |

For matter dominant Eds model, we have the relations between those parameters,

$$H_0^2 = \frac{8 \pi G \rho_0}{3}, \quad Ht = H_0 t_0 = \frac{2}{3},$$

(1)

where $H$, $H_0$, and $\rho_0$ are the Hubble constants and matter density at time $t$ and time $t_0$ of current epoch. The gravitational constant in the simulation units can be expressed as,

$$G = 4.3016 \times 10^{-6} \frac{Mpc \cdot Km^2}{M_\odot \cdot s^2}.$$  

(2)
Reference

1. C. S. Frenk, et al., Public Release of N-body simulation and related data by the Virgo consortium. arXiv:astro-ph/0007362v1 2000.
2. Jenkins, A., et al., Evolution of structure in cold dark matter universes. Astrophysical Journal, 1998. 499(1): p. 20.
3. Colberg, J.M., et al., Linking cluster formation to large-scale structure. Monthly Notices of the Royal Astronomical Society, 1999. 308(3): p. 593-598.
4. Sheth, R.K., H.J. Mo, and G. Tormen, Ellipsoidal collapse and an improved model for the number and spatial distribution of dark matter haloes. Monthly Notices of the Royal Astronomical Society, 2001. 323(1): p. 1-12.