Saturating the quantum Cramér–Rao bound using LOCC

Sisi Zhou\textsuperscript{1,2,3}*, Chang-Ling Zou\textsuperscript{4} and Liang Jiang\textsuperscript{1,2,3}

\textsuperscript{1} Departments of Applied Physics and Physics, Yale University, New Haven, CT 06511, United States of America
\textsuperscript{2} Yale Quantum Institute, Yale University, New Haven, CT 06520, United States of America
\textsuperscript{3} Pritzker School of Molecular Engineering, The University of Chicago, IL 60637, United States of America
\textsuperscript{4} Key Laboratory of Quantum Information, CAS, University of Science and Technology of China, Hefei, Anhui 230026, People’s Republic of China

*Author to whom any correspondence should be addressed.

E-mail: sisi.zhou@yale.edu

Keywords: quantum metrology, quantum Fisher information, quantum Cramér–Rao bound, local operations and classical communication

Abstract

The quantum Cramér–Rao bound (QCRB) provides an ultimate precision limit allowed by quantum mechanics in parameter estimation. Given any quantum state dependent on a single parameter, there is always a positive-operator valued measurement (POVM) saturating the QCRB. However, the QCRB-saturating POVM cannot always be implemented efficiently, especially in multipartite systems. In this paper, we show that the POVM based on local operations and classical communication is QCRB-saturating for arbitrary pure states or rank-two mixed states with varying probability distributions over fixed eigenbasis. Local measurements without classical communication, however, is not QCRB-saturating in general.

1. Introduction

Quantum metrology [1–6] is the study of designing high-precision quantum sensors to estimate physical parameters in quantum systems. It focuses on the ultimate precision achievable in parameter estimation, allowed by the theory of quantum mechanics. It has wide applications ranging from frequency spectroscopy and clocks [7–14] to gravitational-wave detectors and interferometry [15–18]. Lying in the center of quantum metrology is the quantum Cramér–Rao bound (QCRB) [19–22], which provides a lower bound of parameter estimation error:

\[
\delta \theta \geq \frac{1}{\sqrt{NJ(\rho_0)}}.
\]

Here \(\theta\) is the parameter to be estimated, e.g. magnetic field frequency, \(\delta \theta\) is the standard deviation of the \(\theta\)-estimator, \(\rho_0\) is the density matrix describing the quantum sensor as a function of \(\theta\), and \(N\) is the number of repeated experiments. \(J(\rho_0)\) is the so-called quantum Fisher information (QFI) [19–22] quantifying the sensitivity of a quantum sensor.

QFI can be viewed as the maximum Fisher information (FI) among all possible POVMs, where FI is the classical version of QFI as a measure of sensitivity [23–25]. It is a function of the probability distribution of measurement results. In a quantum system, the probability distribution is provided by the symmetric logarithmic derivative operator (SLD) usually saturates the QCRB [22]. However, in general, the...
eigenstates of SLD could be highly-entangled states over subsystems, and the optimal measurement requires global measurements (GM) (figure 1(a)) that might be challenging to implement experimentally [35].

Local measurements (LM) (figure 1(b)), performed separately on each subsystem, were shown to saturate the QCRB in many cases [1, 36–38]. For example, it is proven in [1] that for GHZ-type states evolving under local Hamiltonians with identical terms, LM can saturate the QCRB. However, by counting the number of degrees of freedom in LM and the QCRB-saturating condition, one can show that LM, in general, is not sufficient to saturate the QCRB in multipartite systems (see appendix E for proof). Compared to LM, local operations and classical communication (LOCC) (figure 1(c)) is a larger class of measurements which allows classical communication of measurement results so that the measurement basis performed on one subsystem could be determined by the measurement results from others [39–43], which has been demonstrated in many experimental platforms compatible with local measurement and adaptive control [44–47]. It is a restricted class of quantum operations [48–50] that cannot generate entanglement between subsystems. For example, it cannot fully distinguish the four Bell states [51]. Nevertheless, LOCC can distinguish any two orthogonal quantum states and, in particular, tell the quantum state itself from the state it evolves into, making it a potential candidate to saturate the QCRB. The power of LOCC protocols in achieving optimal performance has also been demonstrated in other contexts [53–57].

In this paper, we consider only quantum states $\rho_0$ in finite-dimensional Hilbert spaces. We prove that LOCC is QCRB-saturating for two types of quantum states: (i) arbitrary pure states $\rho_0 = |\psi_0\rangle \langle \psi_0|$ and (ii) rank-two mixed states $\rho_0 = p_0 |\psi_0\rangle \langle \psi_0| + (1 - p_0) |\psi_1\rangle \langle \psi_1| (0 < p_0 < 1)$, where $|\psi_0\rangle$ are fixed basis independent of $\theta$.

In the following, we first review the necessary and sufficient condition for QCRB-saturating measurements in finite-dimensional Hilbert spaces. Then we prove the existence of QCRB-saturating LOCC for states of type (i) and type (ii). Finally, we show that LM is not QCRB-saturating in general. For bipartite pure states, we found an interesting example where there is no QCRB-saturating projective LM but there is a QCRB-saturating LM.

2. QCRB-saturating POVM

To quantify the distinguishability of two neighboring probability distributions, the FI is defined by

$$F(\{P_x(\theta)\}) = \sum_x \frac{1}{P_x(\theta)} \left( \frac{\partial P_x(\theta)}{\partial \theta} \right)^2,$$

where $x$ is the label of measurement results, $P_x(\theta)$ is the probability of obtaining $x$ when the parameter is equal to $\theta$, satisfying $P_x(\theta) \geq 0$ and $\sum_x P_x(\theta) = 1$. For a quantum state $\rho_0$, $P_x(\theta) = \text{Tr}(\rho_0 E_x)$ for a POVM described by a set of non-negative operators $\{E_x\}$ satisfying $\sum_x E_x = I$, and the FI...
\[ F(P_\theta(\theta)) \leq \text{Tr}(\rho_0 L_\theta^2) = J(\rho_0). \]  

Here, \( L_\theta \) is the SLD, a Hermitian matrix defined by \( \partial_\theta \rho_0 = \frac{1}{\sqrt{2}}(L_\theta \rho_0 + \rho_0 L_\theta) \). The FI is equal to the QFI \( J(\rho_0) \) if and only if,

\[ E_{\lambda_1}^{1/2} \rho_{ij}^{1/2} = \lambda_1 E_{\lambda_1}^{1/2} L_\theta \rho_{ij}^{1/2}, \quad \forall x, \]

for some real \( \lambda_1 \), and for all \( x \) such that \( \text{Tr}(E_x \rho_0) = 0 \), \( \text{Tr}(E_x L_\theta \rho_0 L_\theta) = 0 \). We call any POVM \( \{ E_x \} \) satisfying equation (4) QCRB-saturating. Further simplifications of equation (4) leads to (see appendix A)

**Theorem 1** ([22]). \( \{ E_x \} \) is QCRB-saturating if and only if

\[ E_{\lambda_1}^{1/2} M_{ij} E_{\lambda_1}^{1/2} = 0, \quad \forall \ i, j, x, \]

and

\[ \forall x \text{ s.t.} \text{Tr}(E_x \rho_0) = 0, \quad E_{\lambda_1}^{1/2} L_\theta \rho_{ij}^{1/2} = 0, \quad \forall i. \]

Here we use the diagonalization of the density matrix \( \rho_0 = \sum_k p_{0k} \langle \psi_{0k} | \psi_{0k} \rangle \) (\( p_{0k} > 0 \)) and

\[ M_{ij} = \langle \psi_{0i} | \langle \psi_{0j} | L_\theta - L_\theta | \langle \psi_{0i} \rangle \rangle \langle \psi_{0j} \rangle \|. \]

The condition equation (6), though not explicitly spelled out in [22], is necessary in order to deal with measurements satisfying \( \text{Tr}(\rho_0 E_x) = 0 \) [58]. From theorem 1, it is clear that rank-one projection onto the eigenstates of \( L_\theta \) satisfies equation (5). As an example, we consider sensing with \( n \)-partite GHZ states

\[ |\psi_0\rangle = \frac{1}{\sqrt{2}}(|0^n\rangle + e^{i\theta}|1^n\rangle), \]

which can be viewed as the evolution of \( |\psi_{in}\rangle = \frac{1}{\sqrt{2}}(|0^n\rangle + |1^n\rangle) \) under the Hamiltonian \( H = \frac{\theta}{\sqrt{2}} \sum_{i=1}^{n} \sigma_i \cdot e^{i\theta} \) after unit time, where \( \sigma_i \) is the Pauli-Z matrix acting on the \( i \)-th qubit. The corresponding SLD is

\[ L_\theta = n(e^{i\theta}|1^n\rangle \langle 0^n| - e^{-i\theta}|0^n\rangle \langle 1^n|), \]

whose eigenstates \( \frac{1}{\sqrt{2}}(|0^n\rangle \pm e^{i\theta}|1^n\rangle) \) also satisfies equation (6) and therefore induce a QCRB-saturating measurement. Saturating the QCRB using projective measurements onto these maximally entangled states requires coupling gates between subsystems and might be challenging for practical experimental implementations. Alternatively, it is well known that projection onto \(|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)\) of individual qubits is also QCRB-saturating [1]. However, the systematic approach to identify experimental-friendly QCRB-saturating POVM have never been discussed before.

### 3. LOCC protocol

For arbitrary quantum states, LOCC is not sufficient to saturate the QCRB. Consider the following two-qubit quantum state

\[ \rho_0 = \theta \rho_1 + (1 - \theta) \rho_2, \]

where

\[ \rho_1 = \frac{2}{3} |\beta_1\rangle \langle \beta_1| + \frac{1}{3} |\beta_2\rangle \langle \beta_2|, \]

\[ \rho_2 = \frac{1}{3} |\beta_1\rangle \langle \beta_1| + \frac{2}{3} |\beta_3\rangle \langle \beta_3|, \]

\(|\beta_i\rangle\}_{i=1,2,3}\) are three of the four Bell states (we do not care about the order of the labels). The SLD operator is

\[ L_\theta = \frac{1}{1 + \theta} |\beta_1\rangle \langle \beta_1| + \frac{1}{\theta} |\beta_2\rangle \langle \beta_2| + \frac{1}{(1 - \theta)} |\beta_3\rangle \langle \beta_3|, \]

whose coefficients of \(|\beta_i\rangle \langle \beta_i|\) are all different. Therefore \(|M_{ij}|\) in equation (5) contains terms proportional to \(|\beta_i\rangle \langle \beta_j|\) for all \( i \neq j \). If there is an LOCC such that equation (5) is satisfied, then

\[ E_{\lambda_1}^{1/2} |\beta_i\rangle \langle \beta_j| E_{\lambda_1}^{1/2} = 0, \quad \forall i, j, x, \]

contradicting the fact that the any three Bell states cannot be distinguished from each other using LOCC [51]. Therefore, LOCC cannot saturate the QCRB for \( \rho_0 \).

Now we consider LOCC as potential candidates to saturate the QCRB for the following two types of quantum states: (i) arbitrary pure states \( \rho_0 = |\psi_0\rangle \langle \psi_0| \), which is one of the most commonly used states in quantum metrology [1, 2]; and (ii) rank-two mixed states \( \rho_0 = p_0 |\psi_0\rangle \langle \psi_0| + (1 - p_0) |\psi_d\rangle \langle \psi_d| \), where \(|\psi_{0d}\rangle\) are independent of \( \theta \), which might find applications in quantum thermometry [59–61]. These states only have one
distinct $M$ in equation (5). For type (i) states, equations (5) and (6) becomes
\[ E_x^{1/2}M E_x^{1/2} = 0, \quad \forall x, \tag{15} \]
and
\[ \forall x \text{ s.t.} E_x^{1/2}|\psi_x\rangle = 0, \quad E_x^{1/2}(|\psi_x\rangle)^* = 0, \tag{16} \]
where $|\psi_x\rangle = (1 - |\psi_0\rangle \langle \psi_0|)|\psi_x\rangle$ and
\[ M = |\psi_0\rangle \langle \psi_0| - |\psi_x\rangle \langle \psi_x| \tag{17} \]
is a traceless anti-Hermitian matrix. For type (ii) states, we have $M = |\psi_0\rangle \langle \psi_1|$ and equation (6) is always satisfied.

In particular, for rank-one projective measurements where $E_x = |E_x\rangle \langle E_x|$, equation (5) becomes
\[ \langle E_x|M|E_x\rangle = 0, \quad \forall x. \] Let us define
\[ \tilde{M} = \begin{cases} M + i \left(|\psi_0\rangle \langle \psi_0| - \frac{I}{D}\right) & \text{for type (i)}, \\ M & \text{for type (ii)}, \end{cases} \tag{18} \]
where $D$ is the dimension of the entire Hilbert space. For type (i) states, $(E_x|M|E_x) = 0$ implies $(E_x|M|E_x) = 0$ and $|E_x|\langle |\psi_x|\rangle^2 = 1/D$ for all $x$. Equations (15) and (16) are satisfied. Thus $(E_x|M|E_x) = 0, \forall x$ is a sufficient condition for $(|E_x\rangle)$ to be QCRB-saturating. By constructing a LOCC measurement basis satisfying $(E_x|M|E_x) = 0, \forall x$, we prove the following theorem:

**Theorem 2.** For any multipartite state belonging to type (i) and (ii), there exists a QCRB-saturating LOCC measurement protocol.

In fact, the QCRB is saturable for arbitrary $n$ when one-way classical communication is allowed, where the measurement result of subsystem $s_k$ is classically communicated to $s_{k+1}, \ldots, s_n$ to assist the choice of their measurement basis. The corresponding POVM (figure 1(c)) is
\[ E_{x_1, \ldots, x_n} = E_{x_1}^{s_1} \otimes E_{x_2}^{s_2} \otimes \cdots \otimes E_{x_n}^{s_n}, \tag{19} \]
where $E_{x_1}^{s_1}$ are non-negative operators in subsystem $s_1$ satisfying $\sum_x E_{x_1}^{s_1} = I^{s_1}$.

The procedure to construct a QCRB-saturating rank-one projective LOCC, where $E_{x_1}^{s_1} = |E_{x_1}^{s_1}\rangle \langle E_{x_1}^{s_1}|$, with the structure in equation (19) can be summarized as follows:

1. Calculate $\tilde{M}_{s_1} = Tr_{s_2 \ldots s_n}(\tilde{M})$ by tracing out subsystems $s_2, \ldots, s_n$ in matrix $\tilde{M}$;
2. Find an orthonormal basis $|E_{x_1}^{s_1}\rangle$ in $s_1$ such that $\langle E_{x_1}^{s_1}|\tilde{M}_{s_1}|E_{x_1}^{s_1}\rangle = 0$;
3. Calculate $\tilde{M}_{s_1}^{x_2} = Tr_{s_3 \ldots s_n}(\tilde{M}_{s_1}^{x_2}|E_{x_2}^{s_2})$;
4. Find an orthonormal basis $|E_{x_2}^{s_2}\rangle$ in $s_2$ such that $\langle E_{x_2}^{s_2}|\tilde{M}_{s_1}^{x_2}|E_{x_2}^{s_2}\rangle = 0$;
5. Repeat steps (3)–(4) for subsystems $s_3, \ldots, s_n$.

In steps (2) and (4), we use the following lemma:

**Lemma 1.** Given any traceless matrix $\tilde{M} \in \mathbb{C}^{d \times d}$, there exists an complete orthonormal basis $\{|u_i\rangle\}_{i=1}^d$ in $\mathbb{C}^d$ such that $\langle u_i|\tilde{M}|u_i\rangle = 0$ for all $i$.

A constructive proof can be found in appendix B. Our construction is mathematically reminiscent of the one provided in [32] where LOCC is used to distinguish two multiparticle orthogonal quantum states, but our construction does not require extending the dimension of each subsystem to be a power of two. In fact, parameter estimation is closely related to state discrimination. Projective measurements $(|E_x\rangle \langle E_x|)$ distinguishes two orthogonal quantum states $|\psi_0\rangle$ as long as $\langle E_x|\psi_0\rangle \langle \psi_0|E_x\rangle = 0$ appendix C. It is then clear that a measurement distinguishing an orthonormal basis $\{|\psi_i\rangle\}$ is also QCRB-saturating when estimating $\theta$ in the probability coefficients for any mixed quantum states $\sum_k p_{0,k}|\psi_k\rangle \langle \psi_k| (p_{0,k} > 0)$.

In appendix D, we prove an example of a four-qubit system with a nearest neighbour interaction Hamiltonian, where the parameter to estimated is the strength of the Hamiltonian. We use the algorithm described above to calculate the LOCC measurement basis and plot them in the Bloch spheres.
4. Local measurements

LM in a \( n \)-partite system \( \{ s_1, \ldots, s_n \} \) (figure 1(b)) has the following structure

\[
E_{s_1, \ldots, s_n} = E_{s_1}^a \otimes E_{s_2}^a \otimes \cdots \otimes E_{s_n}^a,
\]

where \( s_k \) is the \( k \)th measurement result and \( \{ E_{s_k}^a \} \) is a POVM in subsystem \( s_k \). One may wonder whether LM would be sufficient to saturate the QCRB, as for GHZ states. It is not possible in general for sufficiently large \( n \), because the number of the degrees of freedom in LM grows linearly as the number of qubits increases but that in the quantum states grows exponentially (see the detailed proof in appendix F).

For bipartite pure states, however, the argument above does not hold and the problem should be treated carefully. Consider \( |\psi_\theta \rangle \in \mathcal{H}_i \otimes \mathcal{H}_2 \) where \( \dim \mathcal{H}_i = d_i \) and \( \dim \mathcal{H}_2 = d_2 \). Let

\[
|\psi_\theta \rangle = \sum_y A_y |i \rangle |j \rangle, \quad |\psi_\theta^* \rangle = \sum_y B_y |i \rangle |j \rangle.
\]

The orthogonality condition \( \langle \psi_\theta | \psi_\theta^* \rangle = 0 \) implies \( \text{Tr}(A^\dagger B) = 0 \). Then we have the following lemma:

**Lemma 2.** There exists a QCRB-saturating LM for \( |\psi_\theta \rangle \) if and only if there are isometries \( U \) and \( V \) satisfying \( UU^\dagger = I \in \mathbb{C}^{d_i \times d_i} \) and \( VV^\dagger = I \in \mathbb{C}^{d_2 \times d_2} \) such that \( C = U^\dagger AV, \; D = U^\dagger BV \) satisfying

\[
C_{ij}D_{ij}^* = C_{ij}D_{ij}, \quad \forall i, j,
\]

and

\[
\forall i, j, \quad \text{if} \; C_{ij} = 0, \; D_{ij} = 0.
\]

**Proof.** On one hand, given \( U \) and \( V \) satisfying equations (22) and (23), let \( U = \sum_i |E_{s_1}^a \rangle \langle i|, \; V = \sum_j |E_{s_2}^a \rangle \langle j| \) ( \( \leftrightarrow \) means complex conjugate). Then \( |E_{s_1}^a \rangle \otimes |E_{s_2}^a \rangle = |E_{s_1}^a \rangle \otimes |E_{s_2}^a \rangle \langle E_{s_1}^a | \otimes |E_{s_2}^a | \), \( \forall i, j \) is a LM and the QCRB-saturating conditions equations (5) and (6) become

\[
\langle \langle E_{s_1}^a | \otimes \langle E_{s_2}^a | \rangle \langle \psi_\theta | \rangle \rangle \langle \psi_\theta^* | \rangle \rangle = \langle \langle E_{s_1}^a | \otimes \langle E_{s_2}^a | \rangle \langle \psi_\theta^* | \rangle \rangle \langle \psi_\theta | \rangle \rangle, \quad \forall i, j,
\]

and

\[
\langle \langle E_{s_1}^a | \otimes \langle E_{s_2}^a | \rangle \rangle |\psi_\theta \rangle = 0, \quad \text{if} \; \langle \langle E_{s_1}^a | \otimes \langle E_{s_2}^a | \rangle \rangle |\psi_\theta \rangle = 0,
\]

which are equivalent to equations (22) and (23). Here \( \{ |E_{s_1}^a \rangle \} \) and \( \{ |E_{s_2}^a \rangle \} \) are not unit vectors in general. When they are, \( U \) and \( V \) are unitary operators and give rise to a QCRB-saturating rank-one projective LM.

On the other hand, given a QCRB-saturating LM \( \{ E_{s_1}^a \otimes E_{s_2}^a \} \), let \( E_{s_1}^a = \sum_k |E_{s_1}^a \rangle \langle E_{s_1}^a | \rangle \) and \( E_{s_2}^a = \sum_k |E_{s_2}^a \rangle \langle E_{s_2}^a | \rangle \) where \( |E_{s_1}^a \rangle \langle E_{s_2}^a | \rangle = 0 \) if \( k = k' \) and positive if \( k = k' \). Then \( U = \sum_{i,k} |E_{s_1}^a \rangle \langle E_{s_2}^a | \rangle \langle i, k \rangle \) and \( V = \sum_{j,k} |E_{s_2}^a \rangle \langle E_{s_1}^a | \rangle \langle j, k \rangle \) satisfy equations (22) and (23).

Using lemma 2, we make the following observations on the QCRB-saturating LM for bipartite pure states:

(a) Parameter estimation is not equivalent to orthogonal state discrimination—there exists \( |\psi_\theta \rangle \) and \( |\psi_\theta^* \rangle \) such that they cannot be distinguished using LM, but there exists a QCRB-saturating LM for them. We show in appendix F that

\[
|\psi_\theta \rangle = \frac{1}{\sqrt{2}}(|00 \rangle + |1+ \rangle), \quad |\psi_\theta^* \rangle = \frac{1}{\sqrt{2}}(|01 \rangle + |1- \rangle)
\]

is the desired example. Note that this also implies LM is not always QCRB-saturating for type (ii) bipartite states.

(b) LM is not QCRB-saturating for all bipartite pure states—equations (5) and (6) cannot always be satisfied simultaneously. Consider a two-qubit system where \( A = I/2 \) and \( B = \sqrt{2} \sigma_x / 2 \). Suppose equations (22) and (23) are both satisfied for some \( U \) and \( V \). Then \( U^\dagger (AB^\dagger - BA^\dagger) U \) and \( V^\dagger (A^\dagger B - B^\dagger A) V \) are zero-diagonal (i.e. all diagonal elements are zero), which implies \( \text{Re}[U_{ij}^\dagger U_{jk}] = \text{Re}[V_{ij}^\dagger V_{jk}] = 0, \; \forall i, j, \) without loss of generality, assume \( U_{ii} \), \( V_{ii} \in \mathbb{R}, \) \( U_{ij}, V_{ij} \in i \mathbb{R} \). Then \( C_{ij} \in \mathbb{R}, \; D_{ij} \in \mathbb{C}^{2 \times 2} \). Therefore \( C_{ij} \) and \( D_{ij} \) cannot be simultaneously non-zero. According to equation (25), we must have \( D_{ij} = 0 \) for all \( i, j \) which is not possible.

(c) Equation (5) itself can be satisfied for all \( A, B \in \mathbb{C}^{2 \times d}, \; d \geq 2 \). According to lemma 1, there exists a unitary matrix \( U = \sum_{i,j} |E_{s_1}^a \rangle \langle i| \) such that \( U^\dagger (AB^\dagger - BA^\dagger) U \) is zero-diagonal. Furthermore, according to lemma 1, there exists a unitary matrix \( V = \sum_{j,k} |E_{s_2}^a \rangle \langle j| \) such that both \( B^\dagger |E_{s_1}^a \rangle \langle A - A^\dagger |E_{s_1}^a \rangle \langle E_{s_1}^a | B \) and
B^\dagger E_2^\dagger |A - A^\dagger E_2^\dagger B are zero-diagonal. Now we have the desired U and V. In practice, one could first find U and V using this procedure and check whether equation (23) is also satisfied. If so, we have a QCRB-saturating LM. The U and V constructed here are both unitary and the corresponding LM is projective.

(d) Projective LM is distinct from general LM—When \( A, B \in \mathbb{C}^{3 \times 3} \), an example exists where there is no QCRB-saturating projective LM, but there is a QCRB-saturating LM. So far, we have shown that projective LOCC is sufficient to saturate the QCRB for type (i) and (ii) states and projective LM is sufficient to satisfy equation (5) for bipartite pure states when \( A, B \in \mathbb{C}^{2 \times d} \). However, when \( A, B \in \mathbb{C}^{3 \times 3} \), projective LM is not as powerful as general LM. We show in appendix G that LOCC is sufficient condition for bipartite pure states when \( A, B \in \mathbb{C}^{d \times d} \).

To sum up, we have found a bipartite pure state where there is no QCRB-saturating LM. We also show that the QCRB saturability problem for pure bipartite states distinguishes projective LM from general LM. However, it is not clear whether our examples could be generalized. It is an interesting open question to classify bipartite pure states by the existence of the QCRB-saturating LM.

5. Conclusion and outlook

We have investigated the QCRB-saturating measurement to maximize the sensitivity of quantum sensors. For arbitrary pure states or rank-two mixed states with fixed eigenbasis, we have developed the QCRB-saturating LOCC protocol, feasible with many physical platforms by local measurement and adaptive control [44–47]. Our LOCC protocol may have applications in extensive parameter estimation and calibration scenarios, including criticality-based quantum metrology [62, 38], quantum thermometry [59–61] and various other cases in many body physics [63–65].

Our LOCC sensing protocol crucially relies on the fact that two orthogonal states can be distinguished using LOCC, so that it can be QCRB-saturating for pure states or rank-two mixed states with fixed eigenbasis. In practice, the quantum states could suffer various decoherences and our protocol might not be able to saturate the QCRB for general mixed states or for multi-parameter sensing [66–73]. To tackle the decoherence, we may apply dynamical decoupling to suppress time-correlated noises [74–76], or introduce quantum error correction to restore unitary evolution in logical subspace even in the presence of Markovian noises [77–82]. Therefore, it will be intriguing to further investigate LOCC sensing protocol combined with quantum error correction.

Acknowledgments

We thank the anonymous reviewer for pointing out the loophole in theorem 1 which is now fixed by adding equation (6). We thank Steven Flamia, Arpit Dua, Wen-long Ma, Shengjun Wu, Zi-wen Liu and Yau Wing Li for helpful discussions. We acknowledge support from the ARL-CDQI (W911NF-15-2-0067), ARO (W911NF-14-1-0011, W911NF-14-1-0563), ARO MURI (W911NF-16-1-0349), AFOSR MURI (FA9550-14-1-0052, FA9550-15-1-0015), NSF (EFMA-1640959), Alfred P Sloan Foundation (BR2013-049), and Packard Foundation (2013-39273).

Appendix A. The necessary and sufficient condition for QCRB-saturating POVM

In this appendix, we prove theorem 1 in the main text. The classical Fisher information \( F(\rho_q) \) satisfies

\[
F(\rho_q) = \sum_{x : \text{Tr}(E_x \rho_q) = 0} (\text{Tr}(E_x \partial_q \rho_q))^2 \frac{\text{Tr} (\text{Re}[\text{Tr}(E_x L_q \rho_q)])^2}{\text{Tr}(E_x \rho_q)} \leq \sum_{x : \text{Tr}(E_x \rho_q) = 0} \frac{(\text{Tr}(E_x L_q \rho_q))^2}{\text{Tr}(E_x \rho_q)} \leq \sum_{x : \text{Tr}(E_x \rho_q) = 0} \text{Tr}(E_x L_q \rho_q) \leq \text{Tr}(L^2_q \rho_q) \equiv f(\rho_q),
\]

(A1)
where the first equality holds true when
\[ \text{Im} [\text{Tr}(E_x L_\theta \rho_0)] = 0, \quad \text{for all } x, \]
(A2)

the second equality holds true when
\[ E_x^{1/2} \rho_0^{1/2} = \lambda_x E_x^{1/2} L_\theta \rho_0^{1/2}, \quad \lambda_x \in \mathbb{C}, \quad \text{for all } x, \]
(A3)

based on the use of the Cauchy–Schwarz inequality, and the third equality holds true when
\[ \forall x \text{ s.t. } \text{Tr}(E_x \rho_0) = 0, \quad \text{Tr}(E_x L_\theta \rho_0 L_\theta) = 0 \iff E_x^{1/2} L_\theta |\psi_i\rangle = 0, \quad \forall i. \]
(A4)

For pure states,
\[ L_\theta = 2(|\partial_\theta \psi\rangle \langle \psi| + |\psi\rangle \langle \partial_\theta \psi|), \]
\[ J(\rho_0) = 4 |\langle \partial_\theta \psi| \psi\rangle|^2, \]
(A5)\hspace{1cm} (A6)

and in general when \( \rho_0 = \sum_k p_{0,k} |\psi_{0,k}\rangle \langle \psi_{0,k}|, \)
\[ L_\theta = \sum_{j_k} \frac{2}{p_{0,j} + p_{0,k}} \langle \psi_{0,j} \partial_\theta \rho_0 | \psi_{0,k}\rangle |\psi_{0,j}\rangle \langle \psi_{0,k}|, \]
\[ J(\rho_0) = \sum_{j_k} \frac{2}{p_{0,j} + p_{0,k}} |\langle \psi_{0,j} \partial_\theta \rho_0 | \psi_{0,k}\rangle|^2. \]
(A7)\hspace{1cm} (A8)

Combining equations (A2) and (A3), we get
\[ E_x^{1/2} \rho_0^{1/2} = \lambda_x E_x^{1/2} L_\theta \rho_0^{1/2}, \quad \lambda_x \in \mathbb{R}, \quad \text{for all } x. \]
(A9)

Therefore equation (A9) is a necessary and sufficient condition for a POVM \( \{E_x\} \) to be QCRB-saturating.

To eliminate \( x \) in equation (A9), one may first rewrite it via vectorization:
\[ (E_x^{1/2} \otimes I)(\rho_0^{1/2}) = \lambda_x (E_x^{1/2} \otimes I)(L_\theta \rho_0^{1/2}), \]
(A10)

where \( |A\rangle = \sum_j |i\langle A|j\rangle |j\rangle \). Note that
\[ |v\rangle = \lambda |w\rangle, \quad \lambda \in \mathbb{R} \iff |v\rangle \langle w| - |w\rangle \langle v| = 0. \]
(A11)

It means that equation (A10) is equivalent to
\[ (E_x^{1/2} \otimes I)(\rho_0^{1/2}) \langle L_\theta \rho_0^{1/2} | - L_\theta \rho_0^{1/2} \rangle \langle L_\theta \rho_0^{1/2} |) (E_x^{1/2} \otimes I) = 0. \]
(A12)

Assuming \( \rho_0 = \sum_k p_{0,k} |\psi_{0,k}\rangle \langle \psi_{0,k}| (p_{0,k} > 0), \) equation (A9) is simplified to
\[ E_x^{1/2} M_{ij} E_x^{1/2} = 0, \quad \forall i, j, x. \]
(A13)

where \( M_{ij} = |\psi_{ij}\rangle \langle \psi_{ij}| L_\theta - L_\theta |\psi_{ij}\rangle \langle \psi_{ij}|. \) In particular, for rank-one projective measurements \( \{E_x = |E_x\rangle \langle E_x|\} \), equation (A9) becomes
\[ \langle E_x | M_{ij} | E_x \rangle = 0, \quad \forall i, j, x. \]
(A14)

When \( \rho_0 = |\psi\rangle \langle \psi| \) is pure and \( p_0 = 1, \) the necessary and sufficient condition becomes
\[ E_x^{1/2} M_{00} E_x^{1/2} = 0, \quad \forall x, \]
(A15)

where \( M_{00} = |\psi\rangle \langle \psi| L_\theta - L_\theta |\psi\rangle \langle \psi|. \) When \( \rho_0 = p_0 |\psi\rangle \langle \psi| + (1 - p_0) |\psi_i\rangle \langle \psi_i| \) where \( |\psi_{ij}\rangle \) is independent of \( \theta, \) the necessary and sufficient condition becomes
\[ E_x^{1/2} |\psi\rangle \langle \psi| E_x^{1/2} = 0, \quad \forall x, \]
(A16)

because \( M_{00} = M_{11} = 0 \) and \( M_{01} = -M_{10} = -\frac{\partial_\theta p_0}{p_0 (1 - p_0)} |\psi\rangle \langle \psi_i|. \)

**Appendix B. QCRB-saturating LOCC**

We first prove lemma 1 which will become quite useful in constructing QCRB-saturating LOCC:

**Proof.** We only need to prove any two traceless Hermitian matrices \( M_1 \) and \( M_2 \) can be simultaneously zero-diagonalized. We first consider the case where \( d = 2, \) i.e. \( M_1 \) and \( M_2 \) are 2-by-2 traceless Hermitian matrices. Let

\[ \begin{bmatrix} a & b \\ -b & c \end{bmatrix} \]
\[
M_k = \begin{pmatrix}
  a_k & b_k e^{i\theta_k} \\
  b_k e^{-i\theta_k} & -a_k
\end{pmatrix}, 
\]

\( k = 1, 2, \) and

\[
U = \begin{pmatrix}
  \cos \beta & -\sin \beta e^{i\alpha} \\
  \sin \beta e^{-i\alpha} & \cos \beta
\end{pmatrix}. 
\]

Then \( U^\dagger M_k U \) has zero diagonal elements equivalent to

\[
a_k (\cos^2 \beta - \sin^2 \beta) = -2b_k \cos \beta \sin \beta \cos (\alpha - \phi_k) 
\]

\[
\iff \cot 2\beta = -\frac{b_k}{a_k} \cos (\alpha - \phi_k), \quad k = 1, 2. 
\]

It can be solved by first finding \( \alpha \) satisfying \( b_k \alpha \cos (\alpha - \phi_k) = b_k \alpha \cos (\alpha - \phi_k) \) and then solving \( \beta \) using the equation above. For higher dimension, lemma 1 can be proven by induction. Suppose lemma 1 holds for \( d < \bar{d} \). Then when \( d = \bar{d} + 1 \), we only need to find some \(|\psi\rangle\) such that \( \langle \psi | M_1 | \psi \rangle = \langle \psi | M_2 | \psi \rangle = 0 \). The rest follows by the induction assumption by simultaneously diagonalizing \( M_1 \) and \( M_2 \) in the \( \bar{d} \) dimensional orthogonal subspace perpendicular to \(|\psi\rangle\). Now we prove the existence of \(|\psi\rangle\). Without loss of generality (WLOG), we assume \( M_1 \neq 0 \) is diagonal,

\[
M_1 = \begin{pmatrix}
  A_1 & 0 \\
  0 & A_2
\end{pmatrix}, 
\]

where we divide the Hilbert space into the direct sum of two subspaces and put \( M_1 \) in a block-diagonal form such that \( A_1 > 0 \) and \( A_2 < 0 \). Meanwhile,

\[
M_2 = \begin{pmatrix}
  \Sigma_1 & B \\
  B^T & \Sigma_2
\end{pmatrix}. 
\]

We can always rescale \( M_2 \) such that one of the following situations occurs:

(a) \( \text{Tr}(A_1) = \text{Tr}(\Sigma_1) > 0 \) and \( \text{Tr}(A_2) = \text{Tr}(\Sigma_2) < 0 \). Then by the induction assumption, there are \(|\psi_1\rangle\) and \(|\psi_2\rangle\), s.t.

\[
\langle \psi_1 | A_1 | \psi_1 \rangle > 0, \quad \langle \psi_2 | A_2 | \psi_2 \rangle < 0. 
\]

Let \(|\psi\rangle = \cos \beta |\psi_1 \rangle \oplus \sin \beta e^{i\alpha} |\psi_2 \rangle \), we have

\[
\langle \psi | M_1 | \psi \rangle = \cos^2 \beta \langle \psi_1 | A_1 | \psi_1 \rangle + \sin^2 \beta \langle \psi_2 | A_2 | \psi_2 \rangle, 
\]

\[
\langle \psi | M_2 | \psi \rangle = \cos^2 \beta \langle \psi_1 | \Sigma_1 | \psi_1 \rangle + \sin^2 \beta \langle \psi_2 | \Sigma_2 | \psi_2 \rangle + 2 \cos \beta \sin \beta \text{Re}[e^{-i\alpha} \langle \psi_1 | B | \psi_2 \rangle]. 
\]

Clearly, there is a solution \((\alpha, \beta)\), s.t. \( \langle \psi | M_1 | \psi \rangle = \langle \psi | M_2 | \psi \rangle = 0 \).

(b) \( \text{Tr}(\Sigma_2) = \text{Tr}(\Sigma_2) = 0 \). Then by the induction assumption, there are \(|\psi_1\rangle\) and \(|\psi_2\rangle\), s.t.

\[
\langle \psi_1 | A_1 | \psi_1 \rangle > 0, \quad \langle \psi_1 | \Sigma_1 | \psi_1 \rangle = 0, 
\]

\[
\langle \psi_2 | A_2 | \psi_2 \rangle < 0, \quad \langle \psi_2 | \Sigma_2 | \psi_2 \rangle = 0. 
\]

Let \(|\psi\rangle = \cos \beta |\psi_1 \rangle \oplus \sin \beta e^{i\alpha} |\psi_2 \rangle \), we have

\[
\langle \psi | M_1 | \psi \rangle = \cos^2 \beta \langle \psi_1 | A_1 | \psi_1 \rangle + \sin^2 \beta \langle \psi_2 | A_2 | \psi_2 \rangle, 
\]

\[
\langle \psi | M_2 | \psi \rangle = 2 \cos \beta \sin \beta \text{Re}[e^{-i\alpha} \langle \psi_1 | B | \psi_2 \rangle]. 
\]

Clearly, there is a solution \((\alpha, \beta)\), s.t. \( \langle \psi | M_1 | \psi \rangle = \langle \psi | M_2 | \psi \rangle = 0 \).

Lemma 1 is then proved. \( \square \)

To find the QCRB-saturating LOCC in theorem 2, we only need to find an orthonormal basis which has the structure \( E_{\psi_1, \ldots, \psi_n} = E_{\psi_1}^{\otimes n} \otimes E_{\psi_2}^{\otimes n} \otimes \cdots \otimes E_{\psi_n}^{\otimes n} \) and satisfy \( \langle E_{\psi_1, \ldots, \psi_n} | \tilde{M} | E_{\psi_1, \ldots, \psi_n} \rangle = 0 \) as well. It can be constructed by the following procedure:

(1) Find an orthonormal basis \( \{|E_{\psi_1}^{\otimes n}\rangle\}_{\psi_1=1}^{\dim E_{\psi_1}} \) which zero-diagonalizes \( \tilde{M}^{\psi_1} = \text{Tr}_{\psi_2, \ldots, \psi_n} (\tilde{M} E_{\psi_1}^{\otimes n}) \) for all \( \psi_1 \).

(2) Find an orthonormal basis \( \{|E_{\psi_1}^{\otimes n}\rangle\}_{\psi_1=1}^{\dim E_{\psi_1}} \) which zero-diagonalizes \( \tilde{M}^{\psi_1} = \text{Tr}_{\psi_2, \ldots, \psi_n} (\tilde{M} E_{\psi_1}^{\otimes n}) \).
(3) Find an orthonormal basis \( \{ |E_{xi_1...x_i} \rangle \}^{\dim s_i}_{x_i=1} \) which zero-diagonalizes
\[
M_{x_i=1...x_i-1}^{s_i} = \text{Tr}_{x_i+1...s_i} \left( E_{xi_1...x_i-1} \cdots (E^{s_i}_{xi} |M| E^{s_i}_{xi}) \cdots |E_{xi_1...x_i-1} \rangle \langle E_{xi_1...x_i-1} | \right) 
\] (B15)
till \( k = n \).

Then one can easily verify
\[
E_{x_i...x_n} = \sum_{x_1...x_{i-1}} E^{s_i}_{x_i} \cdots \sum_{x_n} E^{s_n}_{x_n} 
\] (B16)
is QCRB-saturating, where
\[
E^{s_i}_{x_i} = |E^{s_i}_{x_i} \rangle \langle E^{s_i}_{x_i}|. 
\] (B17)

Note that the proof of lemma 1 is constructive. It means the QCRB-saturating LOCC can be calculated directly from matrix \( M \).

Appendix C. Distinguishing two orthogonal quantum states

In [52], the distinguishability of two multipartite orthogonal states \( \{ |\psi_{0,1} \rangle \} \) via LOCC is shown by writing them as
\[
|\psi_0 \rangle = \sum_{s_0 \in s_0} \alpha_{s_0...s_n} |x_0 \rangle_{s_0...s_{n-1}} 
\] (D1)
\[
|\psi_1 \rangle = \sum_{s_1 \in s_1} \alpha_{s_0...s_n} |x_1 \rangle_{s_0...s_{n-1}} 
\] (D2)
where \( s_i \in \{ \text{dim}(s_i) \} \), \( s_0 \cap s_1 = \emptyset \), \( \alpha_{s_0...s_n} \) are probability amplitudes and \( \langle x_k | x_k' \rangle_{s_0...s_{n-1}} = \delta_{x_kx_k'} \). As we can see, it is equivalent to the QCRB-saturating condition for rank-two mixed states with fixed eigenbasis \( |\psi_{0,1} \rangle \),
\[
\langle E_x | \psi_0 \rangle \langle \psi_1 | E_x \rangle = 0 \quad \forall |E_x \rangle. 
\] (D3)
The LOCC measurement basis \( |E_x \rangle \) corresponds to \( |x_0 \rangle |x_1 \rangle \cdots |x_n \rangle_{s_0...s_{n-1}} \).

Appendix D. An example of the LOCC protocol

Here we demonstrate our LOCC protocol by considering an open boundary Hamiltonian in a four-qubit system
\[
H = \theta \sum_{i=1}^3 \sigma^{x,i} \sigma^{x-i,-1}, 
\] (E1)
where \( \sigma^{x,i} \) is the Pauli-\( X \) matrix acting on the \( i \)th qubit and a Dicke state input
\[
|\psi_{in} \rangle = \frac{1}{2} \left( |1000 \rangle + |0100 \rangle + |0010 \rangle + |0001 \rangle \right). 
\] (E2)
The parameter we want to estimate is the Hamiltonian strength \( \theta \) which is encoded in \( |\psi_0 \rangle \) through \( |\psi_0 \rangle = e^{-iHt} |\psi_{in} \rangle \) after unit time evolution. Numerical search suggests that there is no QCRB-saturating LM and LOCC is necessary. As shown in figure 2, we demonstrate our LOCC protocol by directly calculating a set of QCRB-saturating LOCCs for \( \theta \in [0, \frac{\pi}{4}] \). The tree structure illustrates the choice of measurement basis for qubit \( s_i \) dependent on the results from \( \{ s_1, \ldots, s_{i-1} \} \) via classical communication. Note that the LOCC protocol illustrated in figure 2 is not unique and there could be other LOCCs that are also QCRB-saturating due to remaining degrees of freedom.

Appendix E. LM is not sufficient to saturate the QCRB for multipartite systems

Here we want to show LM is not sufficient to saturate the QCRB for sufficiently large \( n \) in generalized sensing scenarios, where \( n \) is the number of subsystems. Consider Hilbert space \( \mathcal{H}^\otimes n \) where \( \dim \mathcal{H} = d \). For type (i) or (ii) states, let
\[
E_{x_i...x_n} = E^{s_i}_{x_i} \otimes \cdots \otimes E^{s_n}_{x_n} 
\] (F1)
be a QCRB-saturating local measurement satisfying
\[
(E^{s_i}_{xx} \otimes \cdots \otimes E^{s_n}_{xx}) M (E^{s_i}_{xx} \otimes \cdots \otimes E^{s_n}_{xx}) = 0, \quad \forall (x_i, \ldots, x_n) \in [n_1] \times \cdots \times [n]. 
\] (F2)
where \( r = \{1, 2, \ldots, r\} \). Let \( \{|E_{x_i}^a\rangle\}_{a} \) be a basis of the support of \( E_{x_i}^a \). Then equation (F2) implies
\[
\text{Tr} (M (F_1^b \otimes \cdots \otimes F_n^b)) = 0, \tag{F3}
\]
for all Hermitian \( F_i^b \in \text{span} \{|E_{x_i}^a\rangle \langle E_{x_i}^a|, \forall x_i, j\rangle \} \).

We first consider type (i) states, then if we define \( |\psi_i^b\rangle = (1 - |\psi_0\rangle \langle \psi_0|) \hat{\partial}_i \psi \),
\[
M = |\psi_0\rangle \langle \psi_0^b| - |\psi_0^b\rangle \langle \psi_0|, \tag{F4}
\]
Suppose \( \mathcal{H}^{\otimes n} = \mathcal{H}_0 \otimes \mathcal{H}_2 \) where \( \mathcal{H}_0 = \mathcal{H}^{\otimes m} \) with \( m \leq n/2 \). Then the reduce matrix \( M_i = \text{Tr}_{\mathcal{H}_2}(M) \) after tracing out \( \mathcal{H}_2 \) could be an arbitrary traceless anti-Hermitian matrix (up a real factor) by choosing
\[
|\psi_0\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} \langle i|_{\mathcal{H}_0}| i\rangle_{\mathcal{H}_0}, \tag{F5}
\]
\[
|\psi_0^b\rangle = \frac{1}{\sqrt{\text{Tr}(-M_i^2)}} \sum_{i,j=1}^{d} M_{i,j}|i\rangle_{\mathcal{H}_0}| j\rangle_{\mathcal{H}_0}. \tag{F6}
\]
According to equation (F3), \( M_i \) to satisfy
\[
\text{Tr} (M_i (F_1^b \otimes \cdots \otimes F_m^b)) = 0, \tag{F7}
\]
for all Hermitian \( F_i^b \in \text{span} \{|E_{x_i}^a\rangle \langle E_{x_i}^a|, \forall x_i, j\rangle \}. \) There must exists a (not necessarily orthonormal) basis \( \{|e_{i_k}^a\rangle\}_{k=1}^d \) for each \( k \) such that
\[
\left( \bigotimes_{k=1}^{m} |e_{i_k}^a\rangle \right) M_i \left( \bigotimes_{k=1}^{m} |e_{i_k}^a\rangle \right) = 0, \quad \forall i_k = 1, 2, \ldots, d. \tag{F8}
\]

Note that the degree of freedom of an arbitrary traceless anti-Hermitian matrix \( M_i \) is \( d^{2m} - 1 \). But the degree of freedom for matrices satisfying equation (F8) is at most \( d^{2m} - d^m + md^4 \) which is smaller than \( d^{2m} \) for large enough \( m \). Here \( d^m \) is the minimum number of equations of constraints in equation (F8) and \( md^4 \) is the local freedoms in choosing the basis. Therefore equation (F8) could not be satisfied for arbitrary \( M_i \), implying that equation (F3) could not be satisfied for all possible \( M_i \). For type (ii) states, the same argument holds if we replace \( |\psi_0\rangle \) and \( |\psi_0^b\rangle \) above with \( |\psi_0\rangle \) and \( |\psi_1\rangle \).

---

**Figure 2.** Plotting the LOCC measurement basis for each qubit on a Bloch sphere in a four-qubit system described by equations (E1) and (E2). The eigenstates of Pauli matrices \( \{0, 1\}, \{\pm\} \) and \( \{L, R\} = \{\frac{0 \pm \pi}{2}\} \) are labeled on the Bloch sphere. Columns from top to bottom each represent measurement on qubit \( s_1, s_2, s_3, s_4 \) and arrows represent how the measurement basis should be chosen based on previous measurement results \( x = 0, 1 \). The measurement basis is represented by a point on the surface of the Bloch sphere which corresponds to \( x = 0 \). The color indicates the value of Hamiltonian strength \( \theta \in [0, \frac{\pi}{2}] \). Note that the measurement on first qubit does not change with time because \( M^{x_1} = \text{Tr}_{x_2 \ldots x_4}(M) \) and \( \text{Tr}_{x_2 \ldots x_4}(|\psi\rangle\langle \psi| - \frac{1}{d}I) \) are always proportional to \( s_1 \).
Appendix F. Equation (26) cannot be distinguished using LM

Consider a two-qubit system. Here we show that
\[ |\psi_h\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \quad |\psi_v\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \]  
(G1)
cannot be distinguished using LM, i.e. there is no LM \( \{ E_{x_1}^h \otimes E_{x_2}^h \} \) such that
\[ \langle \psi_h | E_{x_1}^h \otimes E_{x_2}^h | \psi_h \rangle \cdot \langle \psi_v | E_{x_1}^h \otimes E_{x_2}^h | \psi_v \rangle = 0, \quad \forall x_1, x_2. \]  
(G2)
Choose \( |\phi_1\rangle \) and \( |\phi_2\rangle \) in the support of some \( E_{x_1}^h \) and \( E_{x_2}^h \) respectively. Then
\[ \langle \psi | \phi_1 \otimes \phi_2 \rangle \langle \phi_1 \otimes \phi_2 | \psi \rangle = 0. \]  
(G3)
Since the identity operator \( I \) is in the span of all \( |\phi_2\rangle \langle \phi_2| \), we must have
\[ \langle \psi | \phi_1 \rangle \langle \phi_1 | \psi \rangle = 0. \]  
(G4)
Note that both \( \langle \phi_1 | \psi_h \rangle \) and \( \langle \phi_1 | \psi_v \rangle \) are not zero because \( |\psi_h\rangle \) and \( |\psi_v\rangle \) are entangled. Let \( |e^{(1)}_i\rangle \propto \langle \phi_1 | \psi_h \rangle \) and \( |e^{(2)}_i\rangle \propto \langle \phi_1 | \psi_v \rangle \) be unit vectors. We must have either \( |\phi_2\rangle = |e^{(1)}_i\rangle \) or \( |\phi_2\rangle = |e^{(2)}_i\rangle \) because
\[ \langle e^{(2)}_i | \phi_2 \rangle \langle \phi_2 | e^{(2)}_i \rangle = 0. \]  
Therefore, there are two orthonormal basis \( |e^{(1)}_i\rangle \propto |e^{(2)}_i\rangle \) such that
\[ \langle e^{(1)}_i | \otimes e^{(2)}_j | \psi_h \rangle \langle e^{(1)}_i | \otimes e^{(2)}_j | \psi_v \rangle = 0, \quad \forall i, j = 1, 2. \]  
(G5)
Since \( |\psi_h\rangle \) and \( |\psi_v\rangle \) are both entangled, we must have
\[ |\psi_h\rangle = \lambda_1 |e^{(1)}_1\rangle \otimes |e^{(2)}_1\rangle, \quad |\psi_v\rangle = \mu_1 |e^{(1)}_1\rangle \otimes |e^{(2)}_1\rangle + \mu_2 |e^{(1)}_2\rangle \otimes |e^{(2)}_2\rangle, \]  
for some non-zero \( \lambda_{1,2} \) and \( \mu_{1,2} \). Clearly, equation (26) does not have this form. Therefore, they cannot be distinguished using LM.

On the other hand, there exists an LM such that the QCRB is saturated. For example,
\[ |E^h_{x_1}\rangle = |0\rangle, \quad |E^h_{x_2}\rangle = |1\rangle, \]  
(G7)
\[ |E^v_{x_1}\rangle = \cos \frac{\pi}{8} |0\rangle + \sin \frac{\pi}{8} |1\rangle, \quad |E^v_{x_2}\rangle = -\sin \frac{\pi}{8} |0\rangle + \cos \frac{\pi}{8} |1\rangle. \]  
(G8)

Appendix G. There is no QCRB-saturating projective LM for equation (27)

Here we prove there is no QCRB-saturating projective LM when
\[ A = \begin{pmatrix} \sqrt{2}/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}, \quad B = \begin{pmatrix} \sqrt{2}i/2 & 0 & 0 \\ 0 & -i/2 & 0 \\ 0 & 0 & -i/2 \end{pmatrix} \]  
(H1)
but there is a QCRB-saturating general LM. From the proof of lemma 2, we can see that it is equivalent to the statement that there are no unitaries \( U \) and \( V \) such that equations (22) and (23) are both satisfied, but there are isometries \( U \) and \( V \) such that these conditions are satisfied.

Our goal is to prove there are no unitaries \( U \) and \( V \) such that equation (27) is satisfied. Now suppose \( U \) and \( V \) are unitary. According to the definitions of \( C \) and \( D \), we have
\[ C_{ij} = (U^\dagger AV)_{ij} = \frac{1}{4} (\sqrt{2} U^*_{ij} V_{ij} + U^*_{ij} V_{ij} + U^*_{ij} V_{ij} + U^*_{ij} V_{ij}), \]  
(H2)
\[ D_{ij} = (U^\dagger BV)_{ij} = \frac{1}{4} (\sqrt{2} i U^*_{ij} V_{ij} - U^*_{ij} V_{ij} - i U^*_{ij} V_{ij} + i U^*_{ij} V_{ij}), \]  
and
\[ C_i D^*_j - D_j C^*_i = \frac{1}{4} (\sqrt{2} U^*_{ij} V_{ij} + U^*_{ij} V_{ij} + U^*_{ij} V_{ij} + U^*_{ij} V_{ij})(-\sqrt{2} i U_{ij} V_{ij} + i U_{ij} V_{ij} + i U_{ij} V_{ij} + U_{ij} V_{ij}) \]  
\[ \quad - (\sqrt{2} U_{ij} V^*_{ij} + U_{ij} V^*_{ij} + U_{ij} V^*_{ij} + U_{ij} V^*_{ij})(\sqrt{2} i U_{ij} V_{ij} - i U_{ij} V_{ij} - i U_{ij} V_{ij} + i U_{ij} V_{ij}) \]  
\[ \quad = -i |U^*_{ij} V_{ij}|^2 + \frac{i}{2} |U^*_{ij} V_{ij} + U^*_{ij} V_{ij}|^2 = 0, \quad \forall i, j. \]  
(H4)
First note that the following two transformations do not change equation (22):

1. \( U \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} \), \( V \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} \)
2. \( U \rightarrow UD_1, \quad V \rightarrow VD_2 \), where \( D_1 \) and \( D_2 \) are arbitrary diagonal unitary matrices.
Therefore, WLOG, we assume \((U_{11}, U_{21}, U_{31}) = (\mathbb{R}_+ , \mathbb{R}_+, 0)\).

\[
2 |U_{11}|^2 |V_j|^2 = |U_{21}|^2 |V_j|^2, \quad \forall j,
\]

(H5)

\[
\Rightarrow 2 |U_{11}|^3 \sum_{j=1}^3 |V_j|^2 = |U_{21}|^3 \sum_{j=1}^3 |V_j|^2,
\]

(H6)

\[
\Rightarrow U_{11} = 1/\sqrt{3}, \quad U_{21} = \sqrt{2}/\sqrt{3}, \quad \text{and} \quad |V_j|^2 = |V_j|^2 = s_j, \quad \forall j.
\]

(H7)

WLOG, assume \(V_j = \sqrt{s_j} \sqrt{2} \sqrt{3}, \forall j\).

Consider the following two situations:

Situation (1): \(s_j = 0\) for some \(j\). Then \(|U_{11}|^2 V_j|^2 = 0\) and \(U_{3j} = 0\) for all \(i\), which is not possible.

Situation (2): \(s_j \neq 0, \forall j\). Then for \(i \neq 1\),

\[
2 |U_{11}|^3 |V_j|^2 = |U_{21}|^3 |V_j|^2 + |U_{31}|^3 |V_j|^2, \quad \forall j,
\]

(H8)

\[
\Rightarrow 2 |U_{11}|^3 = |U_{21}|^3 + |U_{31}|^3 \frac{1}{s_j} |V_j|^2 + \frac{2}{s_j} \cdot \text{Re}[U_{21}^* U_{31}^* V_j],
\]

(H9)

\[
\Rightarrow 2 \sum_{i \neq j} |U_{ii}|^2 = \sum_{i \neq j} |U_{2i}|^2 + \sum_{i \neq j} |U_{3i}|^2 \frac{1}{s_j} |V_j|^2,
\]

(H10)

\[
\Rightarrow |V_j|^2 = |V_j|^2 = 1/3.
\]

(H11)

According to equation (H9), for some \(i \neq 1\) such that \(U_{2i} \neq 0\),

\[
\text{Re}[U_{2i} U_{31}^* V_j] = \text{Re}[U_{2i}^* U_{31}^* V_j] = \text{Re}[U_{2i}^* U_{31}^* V_j].
\]

(H12)

We must have \(V_j \in \{e^{\theta j} \sqrt{3}, e^{\theta j} \sqrt{3} \} \) for all \(i\). This is not possible because in that case

\[
\sum_{i = 1}^3 V_{3i}^* V_{2i} = 0,
\]

(H13)

contradicting with the requirement that \(V\) is unitary.

On the other hand, let

\[
U = \begin{pmatrix}
1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\
\sqrt{2}/\sqrt{3} & -1/\sqrt{6} & -1/\sqrt{6} \\
0 & -i/\sqrt{2} & i/\sqrt{2}
\end{pmatrix}, \quad V = \begin{pmatrix}
e^{i\pi/4}/2 & e^{i3\pi/4}/2 & -e^{i\pi/4}/2 & -e^{i3\pi/4}/2 \\
1/2 & 1/2 & 1/2 & 1/2 \\
1/2 & -1/2 & 1/2 & -1/2
\end{pmatrix},
\]

(H14)

one can verify in this case both equations (22) and (23) are satisfied.

**ORCID iDs**

Sisi Zhou  
https://orcid.org/0000-0003-4618-8590

Chang-Ling Zou  
https://orcid.org/0000-0003-2484-7292

**References**

[1] Giovannetti V, Lloyd S and Maccone L 2006 *Phys. Rev. Lett.* 96 010401
[2] Giovannetti V, Lloyd S and Maccone L 2011 *Nat. Photon.* 5 222
[3] Degen C L, Reinhard F and Cappellaro P 2017 *Rev. Mod. Phys.* 89 035002
[4] Braun D, Adesso G, Benatti F, Floreanini R, Marzolino U, Mitchell M W and Pirandola S 2018 *Rev. Mod. Phys.* 90 035006
[5] Pezzè L, Smerzi A, Oberthaler M K, Schmied R and Treutlein P 2018 *Rev. Mod. Phys.* 90 035005
[6] Pirandola S, Bardhan B R, Gehring T, Weedbrook C and Lloyd S 2018 *Nat. Photon.* 12 724
[7] Sanders B and Milburn G 1995 *Phys. Rev. Lett.* 75 2944
[8] Bollinger J, Itano W M, Wineland D and Heinzen D 1996 *Phys. Rev. A* 54 R4649
[9] Hudja S F, Macchiavello C, Pellizzari T, Ekert A K, Plenio M B and Cirac J J 1997 *Phys. Rev. Lett.* 79 3865
[10] Leibfried D, Barrett M, Schautz T, Britton J, Chiaverini J, Itano W, Jost L, Langer C and Wineland D 2004 *Science* 304 1476
[11] Giovannetti V, Lloyd S and Maccone L 2004 *Science* 306 1330
[12] Bužek V, Derka R and Massar S 1999 *Phys. Rev. Lett.* 82 2207
[13] Valencia A, Scarcelli G and Shih Y 2004 *Appl. Phys. Lett.* 85 2655
[14] de Burgh M and Bartlett S D 2005 *Phys. Rev. A* 72 042301
[15] Caves C M 1981 *Phys. Rev. D* 23 1693
[16] Yurke B, McCall S L and Klauder J R 1986 *Phys. Rev. A* 33 4033
[17] Berry D and Wiseman H 2008 *Phys. Rev. Lett.* 85 5096
[18] Higgins B L, Berry D W, Bartlett S D, Wiseman H M and Pryde G J 2007 *Nature* 450 393
[19] Helstrom C W 1976 *Quantum Detection and Estimation Theory* (New York: Academic)
[20] Helstrom C 1968 *IEEE Trans. Inf. Theory* 14 234
[21] Paris M G 2009 *Int. J. Quantum Inf.* 7 125
