ON FINITE TEMPERATURE CASIMIR EFFECT FOR DIRAC LATTICES

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We consider polarizable sheets modeled by a lattice of delta function potentials. The Casimir interaction of two such lattices is calculated at nonzero temperature. The heat kernel expansion for periodic singular background is discussed in relation with the high temperature asymptote of the free energy.

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1. The System

In recent years, we have witnessed the discoveries of ever new unusual properties of two-dimensional materials, such as 2d electron gas, graphene, or other monoatomic layers. In the present note we deal with a system which mimics two parallel monoatomic layers and discuss van der Waals and Casimir forces between them. Two-dimensional rectangular lattices of delta function potentials separated by a distance $b$ are considered. Our objective is to evaluate the vacuum energy of a scalar field in the background of these lattices and compute the finite temperature corrections.

A periodic delta potentials are a well studied in quantum mechanics, the simplest case being the Kronig-Penney model (‘Dirac comb’).\(^1\) In more than one dimension a Hamilton operator with a delta function potential is not self adjoint, however a self-adjoint extension may be defined. For a Laplace operator with three-dimensional $\delta$ function, $\Delta_a = \Delta + a\delta^3(x)$, the self-adjoint extension was analyzed\(^2\) in the QFT context. Close relation of self-adjoint extension and zero range potential approach with regularization and renormalization was traced.\(^3,4\)

The scattering on a single two-dimensional lattice 3D delta functions was considered in Ref. 4 for scalar and electromagnetic field. In Ref. 6 the scattering approach was used to obtain the separation dependent part of the vacuum energy for a scalar...
field in the presence of two parallel lattices. In the notations of this paper, the lattice sites of two rectangular 2D lattices, \( (A) \) and \( (B) \), are given by 3D vectors

\[
\vec{a}_n^A = \left( \frac{a_n + c}{b} \right), \quad \vec{a}_n^B = \left( \frac{a_n}{0} \right), \quad a_n = a \left( \frac{n_1}{n_2} \right),
\]

\( a_n \) are 2D vectors in \((x,y)\)-plane, \( n_1 \) and \( n_2 \) are integers, \( a \) is the lattice spacing, \( b \) is the separation, \( c \) is the displacement vector in \((x,y)\)-plane.

The wave equation for a scalar field \( \phi(\vec{x}) \),

\[
\left( -\omega^2 - \Delta + g \sum_n \left( \delta^{(3)}(\vec{x} - \vec{a}_n^A) + \delta^{(3)}(\vec{x} - \vec{a}_n^B) \right) \right) \phi(\vec{x}) = 0,
\]

endowed with two lattices of three dimensional delta functions is not well defined, and \( g \), having the dimension of length, should be regarded as a bare coupling.\(^4\)

In the scattering approach the separation dependent part of the vacuum energy is given by so called 'TGTG'-formula, see for example,\(^5\)

\[
E_0 = \frac{1}{2\pi} \int_0^\infty d\xi \text{Tr} \ln(1 - \mathcal{M}(i\xi, \vec{x}, \vec{x}')), \quad \mathcal{M} = T_A G_0 T_B G_0,
\]

where \( \xi \) is the imaginary frequency, \( \omega = i\xi \), and the trace is taken with respect to \( \vec{x} \). The free Green’s function is denoted by

\[
G_0(\vec{x} - \vec{x}') = \int \frac{d^2 k}{(2\pi)^2} \frac{e^{i\vec{k}(\vec{x} - \vec{x}') + i\Gamma|\vec{k}_3 - \vec{k}'_3|}}{2i\Gamma(k)}, \quad \Gamma(k) = \sqrt{\omega^2 - k^2 + i0},
\]

and the \( T \) operators describe scattering on the lattices \( A \) and \( B \)

\[
T_{A,B}(\vec{x}, \vec{x}') = \sum_{n,n'} \delta(\vec{x} - \vec{a}_n^{A,B}) \Phi^{-1}_{n,n'} \delta(\vec{a}_n^{A,B} - \vec{x}').
\]

Here \( \Phi^{-1}_{n,n'} \) is the inverse matrix to \( \Phi_{n,n'} = \frac{1}{\xi} \delta_{n,n'} - G_0(\vec{a}_n - \vec{a}_{n'}) \) with diagonal elements defined after the renormalization\(^4\) of the coupling \( g \) so that \( \Phi_{n,n} = 1/g \).

Due to translational invariance with respect to the lattice step the momentum \( k \) may be split into quasi momentum and integer part, \( k = q + \frac{2\pi}{a} N \). The infinite momentum integration is replaced by \( \int d^2 k = \int d^2 q \sum_N \) with the components of \( q = (q_1, q_2) \) restricted to \(-\pi/a \leq q_{1,2} < \pi/a\). One can derive \( \text{Tr} \ln(1 - \mathcal{M}) \) in (3) expanding the logarithm in powers of \( \mathcal{M}(i\xi) \), which after some transformations\(^5\) appear to be diagonal with respect to \( q \). Consequently the expression (3) can be written in a Lifshitz-like form. For the vacuum energy per lattice cell we get

\[
E_0 = \frac{1}{2} \int_0^\infty \frac{d\xi}{\pi} a^2 \int \frac{d^2 q}{(2\pi)^2} \ln \left( 1 - |h(i\xi, q)|^2 \right),
\]

with

\[
h(\omega, q) = \frac{1}{a^2} \sum_N e^{i\Gamma(k)k + i\frac{2\pi}{a} N e} 2\Gamma(k) \tilde{\phi}(k), \quad \tilde{\phi}(k) = \frac{1}{g} - \frac{1}{4\pi} \sum_n \frac{1}{|a_n|} e^{i\omega|a_n| + ik a_n}
\]
(in the primed sum the term with \( n = 0 \) is dropped). For weak coupling \( g \) it corresponds to the vacuum energy of parallel plates with “reflection coefficient”

\[
r(\omega, k) = g/(2a^2)(\omega^2 - k^2)^{-1/2}.
\]

The general formula (6) obtained in Ref. 6 for the vacuum energy of two lattices at zero temperature will be used to find finite temperature corrections.

2. Finite Temperature

To find the finite temperature corrections the Matsubara formalism is used. The free energy is given by the sum over Matsubara frequencies \( i\xi \rightarrow \xi_n = 2\pi T n \),

\[
F = \frac{T}{2} \sum_{n=-\infty}^{\infty} \text{Tr} \ln (1 - \mathcal{M}(\xi_n)) = \frac{T a^2}{2} \sum_{n=-\infty}^{\infty} \int \frac{d^2q}{(2\pi)^2} \ln (1 - |h(\xi_n, q)|^2). \tag{8}
\]

We define three temperature regions with respect to the parameters of the model, namely the lattice spacing \( a \) and the separation of two lattices \( b \). Low temperature corresponds to \( Ta, Tb < 1 \). Medium temperature meets the inequality \( Ta < 1 < Tb \). This area requires numerical studies. High temperature obeys the condition \( 1 < Ta < Tb \). Here it is worth mentioning that when \( b \ll a \) the problem reduces to the interaction of two \( \delta \) sources.

For low temperature, \( Ta, Tb < 1 \), the Abel-Plana formula may be used

\[
F = E_0 + F_T, \quad F_T = \frac{i}{2} \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} n_T(\xi) \text{Tr} [\ln(1 - \mathcal{M}(i\xi)) - \ln(1 - \mathcal{M}(-i\xi))]. \tag{9}
\]

Here \( n_T(\xi) = (\exp(|\xi|/T) - 1)^{-1} \) is the Boltzmann factor. At low temperatures the integral in (9) is determined by small \( \xi \). After the development around \( \xi = 0 \), each power of \( \xi \) adds a power of \( T \),

\[
F_T = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\xi}{\exp(\frac{\xi}{T}) - 1} [\mathcal{M}_1 \xi + \mathcal{M}_3 \xi^3 + \ldots] = \mathcal{M}_1 \pi T^2 \frac{\pi^2}{6} + \mathcal{M}_3 \pi^3 T^4 \frac{1}{15} + \ldots, \tag{10}
\]

\[
\mathcal{M}_1 = -\text{Tr} \frac{\mathcal{M}'}{1 - \mathcal{M}}, \quad \mathcal{M}_3 = -\text{Tr} \left\{ \frac{\mathcal{M}^3}{3(1 - \mathcal{M})^3} + \frac{\mathcal{M}' \mathcal{M}''}{2(1 - \mathcal{M})^2} + \frac{\mathcal{M}'''}{6(1 - \mathcal{M})} \right\} \ldots
\]

At high temperature the leading contribution is given by the zeroth term of the Matsubara sum (8) which is proportional to the derivative of the spectral zeta function, \( \zeta'(0) \), of considered system. The subleading terms are expressed through the coefficients of the heat kernel expansion \( K(t) = \sum_j e^{-\lambda_j^2} t \sim (4\pi t)^{-3/2} \sum_{n=0}^{\infty} t^{n/2} a_{n/2}, \quad t \to 0 \). Thus the entire high temperature expansion acquires the form,

\[
F(T) \sim -\frac{T}{2} \zeta'(0) + a_0 T^4 \frac{\pi^2}{90} - \frac{a_{1/2}}{4\pi^{3/2}} T^3 \frac{\pi^2}{h^2} \zeta_0(3) - \frac{a_{1/2}}{24} T^2 \frac{\pi^2}{h^3} + \ldots \tag{11}
\]

The heat kernel in the spectral problem (2) with two parallel delta lattices \( A \) and \( B \) is given by an integral equation,

\[
K(x, y; t) = K_0(x, y; t) + g \int_0^t ds \sum_{i=A,B} \sum_n K_0(x, a_n^i; t - s) K(a_n^i, y; s), \tag{12}
\]

\[
(t \to 0)
\]
with the free heat kernel $K_0(x, y; t) = (4\pi t)^{-3/2} e^{-\frac{(x-y)^2}{4t}}$. The integral equation (12) can be iterated.\cite{7,8} After taking the trace one arrives at $K(t) = K_0(t) + K(1)(t) + K(2)(t) + K(3)(t) + \ldots$, where

$$
K_0(t) = \frac{1}{(4\pi t)^{3/2}} V, \quad K(1)(t) = \frac{2N^2 g}{(4\pi)^{3/2} t^{1/2}}, \ldots \quad (13)
$$

Here $N$ is the number of the lattice points. Unfortunately, starting from $g^2$ term we come across divergent integrals which may be regularized by point-splitting. Similar problem was discussed in Ref. 8. It is worth comparing (13) with the exact solution for the heat kernel trace of the operator with a single three-dimensional $\delta$-function, obtained in Ref. 2

$$
K(t) = \frac{1}{(4\pi t)^{3/2}} + \frac{1}{2} e^{\frac{-4z^2}{g^2}} \left[ 1 - \Phi \left( \frac{4\pi}{g} \sqrt{t} \right) \right], \quad \Phi(z) = \frac{2}{\sqrt{\pi}} \int_0^z dx e^{-x^2}. \quad (14)
$$

The small $g$ expansion of (14) coincides in its leading terms with $K_0(t)$ and $K(1)(t)$, up to volume and area factors. Thus, from (13) the leading heat kernel coefficients $a_0$ and $a_{1/2}$ may be extracted.

### 3. Conclusion

We considered the Casimir effect for two-dimensional lattices of delta functions at zero and finite temperature. Our approach is based on the $TGTG$ formula, with $T$-operators derived in terms of lattice sums. The generalization to finite temperature is tricky but straightforward. Here scaling properties of the system and relations between various limiting cases proved to be useful at low and high temperatures. The heat kernel of the Laplace operator with double delta lattice potential was analyzed in relation with the high temperature asymptote of the free energy. Medium temperature region is left for numerical study.

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