Matrix model with manifest general coordinate invariance

Takehiro AZUMA\textsuperscript{†} and Hikaru KAWAI\textsuperscript{†‡}

\textsuperscript{†}Department of Physics, Kyoto University, Kyoto 606-8502, Japan

\textsuperscript{‡}Theoretical Physics Laboratory, RIKEN (The Institute of Physical and Chemical Research)
2-1 Hirosawa, Wako, Saitama 351-0198, Japan

Abstract

We present a formulation of a matrix model which manifestly possesses the general coordinate invariance when we identify the large $N$ matrices with differential operators. In order to build a matrix model which has the local Lorentz invariance, we investigate how the $so(9,1)$ Lorentz symmetry and the $u(N)$ gauge symmetry are mixed together. We first analyze the bosonic part of the model, and we find that the Einstein gravity is reproduced in the classical low-energy limit. We then present a proposal to build a matrix model which has $\mathcal{N} = 2$ SUSY and reduces to the type IIB supergravity in the classical low-energy limit.

\textsuperscript{1} e-mail address : azuma@gauge.scphys.kyoto-u.ac.jp
\textsuperscript{2} e-mail address : hkawai@gauge.scphys.kyoto-u.ac.jp
1 Introduction

A large $N$ reduced model has been proposed as a nonperturbative formulation of superstring theory\cite{2}\cite{3}. It is defined by the following action:

$$S = -\frac{1}{g^2} \text{Tr} \left( \frac{1}{4} [A_a, A_b][A^a, A^b] - \frac{1}{2} \bar{\psi} \Gamma^a [A_a, \psi] \right).$$

(1.1)

It is a large $N$ reduced model of 10-dimensional $\mathcal{N} = 1$ super Yang-Mills theory. Here, $\psi$ is a 10-dimensional Majorana-Weyl spinor field, and $A_a$ and $\psi$ are $N \times N$ hermitian matrices. This model is related to the type IIB superstring theory in that it is at the same time obtained by the matrix regularization of the Green-Schwarz action of the type IIB superstring theory (IIB matrix model is extensively reviewed in \cite{5}).

IIB matrix model has several evidences of describing gravitational interaction; such as the exchange of gravitons and dilatons which appears in the one-loop effective Lagrangian\cite{2}. However, it is an interesting issue to pursue a matrix model which describes the gravity more manifestly. Our goal is to build a matrix model with manifest general coordinate invariance, which is the fundamental principle of general relativity. In order to achieve this goal, we should consider a matrix model with local Lorentz invariance. This issue has been investigated in the context of supermatrix model in \cite{10}\cite{11}.

In these papers, the matrix model based on the super Lie algebra $u(1|16,16)$\cite{7}\cite{9} has been investigated as a generalization of IIB matrix model to analyze how the local Lorentz invariance is realized in a matrix model. This model has the following two features. One is the extended Lorentz symmetry, in which the infinitesimal parameters of $\text{so}(9,1)$ are extended to the elements of $u(N)$. In IIB matrix model, the information of the spacetime is embedded in the eigenvalues of the large $N$ matrices, and the general coordinate invariance is related to the $u(N)$ gauge symmetry\cite{4}. Therefore, in order to realize the coordinate-dependent Lorentz symmetry, it is natural to regard the parameters of $\text{so}(9,1)$ as the elements of $u(N)$. The other feature of the $u(1|16,16)$ supermatrix model is the existence of a higher rank tensor field. In order to formulate a matrix model in the curved spacetime, a spin connection term containing rank-3 gamma matrices must be included in the fermionic term. The $u(1|16,16)$ supermatrix model has been thus investigated as the model with higher rank tensor fields coupled to the fermions.

In this paper, we inherit this idea, and attempt another formulation of a matrix model which manifestly possesses the local Lorentz invariance. The starting point is to identify the large $N$ matrices with differential operators\cite{17}. This idea stems from the interpretation of the spacetime embedded in large $N$ matrices. In the twisted reduced model\cite{13}\cite{14}, the matrix $A_a$ represents the covariant derivative in the spacetime, whereas in IIB matrix model the matrix $A_a$ itself represents the coordinate of the spacetime. The relationship between these two viewpoints is elucidated by expanding the matrix $A_a$ around the classical solution $p_a$ as

$$A_a = p_a + a_a,$$

(1.2)

where $[p_a, p_b] = iB_{ab}$ and $B_{ab}$ is a real $c$-number. In \cite{6}, it has been pointed out that IIB matrix model is identified with the noncommutative Yang-Mills theory by this expansion. The $\text{so}(9,1)$ is a subalgebra of $u(1|16,16)$. For the detailed relationship between the super Lie algebra and the Lorentz symmetry, see \cite{8}.\footnote{\textit{so}(9,1) is a subalgebra of \textit{u}(1|16,16). For the detailed relationship between the super Lie algebra and the Lorentz symmetry, see \cite{8}.}
fermionic term of IIB matrix model \( \frac{1}{2g_s^2} Tr \bar{\psi} \Gamma^a [A_a, \psi] \) reduces to the action of the fermion in the flat space

\[
\int d^D x \bar{\psi}(x) i \Gamma^a (\partial_a \psi(x) + [a_a(x), \psi(x)]), \tag{1.3}
\]
in the classical low-energy limit.

With this idea in mind, we attempt to formulate a matrix model with the local Lorentz invariance, which in the classical low-energy limit reduces to the fermionic action on the curved spacetime

\[
S_F = \int d^{d+1} x \bar{\psi}(x) i \Gamma^a e_a^i(x) \left( \partial_i (e^{-\frac{i}{2}}(x) \bar{\psi}(x)) + [A_i(x), e^{-\frac{i}{2}}(x)\bar{\psi}(x)] \right) + \frac{1}{4} \Gamma^{ibc} \omega_{ibc}(x) \psi(x). \tag{1.4}
\]

Throughout this paper, the indices \( a, b, c, \cdots \) and \( i, j, k, \cdots \) both run over \( 0, 1, \cdots, d-1 \). The former and the latter denote the indices of the Minkowskian and the curved spacetime, respectively \( \Box \). \( d \) is the spacetime dimension, and we focus on the case in which \( d = 10 \). In considering the correspondence with the matrix model, we need to absorb \( e(x) \) into the definition of the fermionic field, since it is not \( \int d^d x e(x) \) but \( \int d^d x \) that corresponds to the trace of the large \( N \) matrices. And we regard the "spinor root density"

\[
\Psi(x) = e^{\frac{i}{2}}(x) \psi(x) \tag{1.5}
\]
as the fundamental quantity. Then, the action is rewritten as

\[
S_F = \int d^d x \bar{\psi}(x) e^{\frac{i}{2}}(x) i \Gamma^a e_a^i(x) \left\{ \partial_i (e^{-\frac{i}{2}}(x) \bar{\psi}(x)) + [A_i(x), e^{-\frac{i}{2}}(x)\bar{\psi}(x)] \right\} + \frac{1}{4} \Gamma^{ibc} \omega_{ibc}(x) \psi(x)
\]

\[
= \int d^d x \left\{ \bar{\Psi}(x) i \Gamma^a \left[ e_a^i(x) \partial_i + \frac{1}{2} e_a^i(x) \omega_{ica}(x) + e_a^i(x) e_b^j(x) \partial_i e^{-\frac{i}{2}}(x) \right] \Psi(x) + i \bar{\Psi}(x) \Gamma^a e_a^i(x) [A_i(x), \Psi(x)] + \frac{i}{4} \bar{\Psi}(x) \Gamma^{ab} e_{a}^{i}(x) \omega_{ia23}(x) \Psi(x) \right\}
\]

\[
= \int d^d x \left\{ \bar{\Psi}(x) i \Gamma^a e_a^i(x) \left( \partial_i \Psi(x) + [A_i(x), \bar{\Psi}(x)] \right) + \frac{i}{4} \bar{\Psi}(x) \Gamma^{ab} e_{a}^{i}(x) \omega_{ia23}(x) \Psi(x) \right\},
\tag{1.6}
\]

where we have utilized in the last equality the fact that the fermionic field \( \Psi(x) \) is a Majorana one. The corresponding matrix model is formulated as

\[
S'_F = \frac{1}{2} Tr \bar{\psi} \Gamma^a [A_a, \psi] + \frac{i}{2} Tr \bar{\psi} \Gamma^{abc} \{ A_{abc}, \psi \}. \tag{1.7}
\]

\footnote{In this paper, we adopt the following notation of the gamma matrices:

\[
\{ \Gamma^a, \Gamma^b \} = 2 \eta^{ab}, \quad \text{where} \quad \eta^{ab} = \text{diag}(-1, +1, \cdots, +1).
\]

The gamma matrices are real and satisfy

\[
(\Gamma^a)^\dagger = (T \Gamma^a) = \begin{cases} 
-\Gamma^a & (a = 0) \\
+\Gamma^a & (a = 1, 2, \cdots, 9).
\end{cases}
\]
In promoting the action (1.7) to the matrix model, we have identified the covariant derivative with the commutator with the rank-1 matrix $A_a$. The rank-3 term is a naive product of the spin connection and the fermion, and it is natural to promote the product to the anticommutator of the large $N$ hermitian matrices.  

It is difficult to find an action invariant under this transformation, we instead promote this term to the anticommutator of the hermitian matrices as

$$\delta S = \frac{1}{4} \Gamma^{a_{1}a_{2}a_{3}} \varepsilon_{a_{1}a_{2}} \psi,$$

(1.9)

and we take the action (1.8). The hermiticity of the fermionic field is now sacrificed, and the actions (1.7) and (1.8) are no longer equivalent. The price we must pay for this alteration is that the product $A_a \psi$ does not directly correspond to the covariant derivative. The local Lorentz transformation of the action (1.8) is

$$\delta S' = \frac{1}{4} Tr \bar{\psi} \left[ \Gamma^a A_a + i \Gamma^{a_{1}a_{2}a_{3}} A_{a_{1}a_{2}a_{3}} \right] \psi.$$

(1.10)

Since we are now considering the local Lorentz transformation and their space-time dependent parameters are promoted to $u(N)$ matrices, the infinitesimal parameters $\varepsilon_{ab}$ are $u(N)$ matrices. Then, the commutator

$$[i \Gamma^{a_{1}a_{2}a_{3}} A_{a_{1}a_{2}a_{3}}, \Gamma^{b_1b_2} \varepsilon_{b_1b_2}] = \frac{i}{2} \left[ \Gamma^{a_{1}a_{2}a_{3}} A_{a_{1}a_{2}a_{3}}, \varepsilon_{b_1b_2} \right] + \frac{i}{2} \left[ \Gamma^{a_{1}a_{2}a_{3}} \varepsilon_{b_1b_2}, A_{a_{1}a_{2}a_{3}} \right]$$

(1.11)

must include the rank-5 gamma matrices in 10 dimensions, and the action (1.8) is not invariant under the local Lorentz transformation. Repeating this procedure, we find that it is necessary to introduce all the terms of odd-rank gamma matrices to make the action invariant.

The similar thing holds true of the generator of the local Lorentz transformation (1.9). In order for the algebra to close, only the rank-2 terms are not sufficient, because the commutator

$$[\Gamma^{a_{1}a_{2}} \varepsilon_{a_{1}a_{2}}, \Gamma^{b_1b_2} \varepsilon'_{b_1b_2}] = \frac{1}{2} \left[ \Gamma^{a_{1}a_{2}} \varepsilon_{a_{1}a_{2}}, \varepsilon'_{b_1b_2} \right] + \frac{1}{2} \left[ \Gamma^{a_{1}a_{2}} \varepsilon'_{b_1b_2}, \varepsilon_{a_{1}a_{2}} \right]$$

(1.12)

For the readers' convenience, we summarize the commutation relations of the hermitian and anti-hermitian operators. Let $H$ and $A$ be the set of the hermitian and anti-hermitian operators, respectively. When $h, h_1, h_2 \in H$ and $a, a_1, a_2 \in A$, their commutation relations are as follows:

$$[h_1, h_2] \in A, \ [h, a] \in H, \ [a_1, a_2] \in A, \ \{h_1, h_2\} \in H, \ \{h, a\} \in A, \ \{a_1, a_2\} \in H.$$
includes the rank-4 gamma matrices. Likewise, we find that the algebra of the local Lorentz transformation must include all the even-rank gamma matrices.

Our formulation of the matrix model is based on this observation, and we analyze a matrix model which includes all odd ranks of 10-dimensional gamma matrices coupled to the fermion from the beginning. We identify these matrices with differential operators, and investigate whether our matrix model includes the supergravity in the classical low-energy limit.

We mention the relation between our new proposal and the original IIB matrix model. In the quantum field theory, some different models which share the same symmetry are equivalent in the continuum limit, known as universality. We expect that the similar mechanism may hold true of large \( N \) matrix models and hence that various matrix models may have the same large \( N \) limit. Thus, there is a possibility that our new model is equivalent to IIB matrix model. We can expect that both IIB matrix model and our new model are equally authentic constructive definition of superstring theory in the large \( N \) limit. Even if IIB matrix model is not an eligible framework, we expect that our model may be closer to the true theory than IIB matrix model, because our model has larger symmetry.

This paper is organized as follows. In Section 2, we analyze the bosonic term of the matrix model as the preliminary step, and we clarify that the bosonic part reduces to the Einstein gravity in the classical low-energy limit. In Section 3, we consider the model coupled to a fermionic field. We present a proposal to formulate a matrix model which has \( \mathcal{N} = 2 \) SUSY algebra and reduces to the type IIB supergravity. Section 4 is devoted to the discussion and the outlook.

## 2 Bosonic part of the matrix model

Before going to the analysis of the supersymmetric matrix model, we first investigate the bosonic part:

\[
S = Tr_{N \times N} \left[ tr_{32 \times 32} V(m^2) \right].
\]  

(2.1)

The uppercase \( Tr \) and the lowercase \( tr \) denote the trace for the \( N \times N \) and \( 32 \times 32 \) matrices, respectively. The large \( N \) matrix \( m \) consists of all odd-rank gamma matrices in 10 dimensions:

\[
m = m_a \Gamma^a + \frac{i}{3!} m_{a_1 a_2 a_3} \Gamma^{a_1 a_2 a_3} - \frac{1}{5!} m_{a_1 \cdots a_5} \Gamma^{a_1 \cdots a_5} - \frac{i}{7!} m_{a_1 \cdots a_7} \Gamma^{a_1 \cdots a_7} + \frac{1}{9!} m_{a_1 \cdots a_9} \Gamma^{a_1 \cdots a_9},
\]  

(2.2)

where \( m_{a_1 \cdots a_{2n-1}} \) are large \( N \) hermitian matrices. \( m \) satisfies

\[
\Gamma^0 m^\dagger \Gamma^0 = m,
\]  

(2.3)

and thus the action (2.1) is hermitian. Odd powers of \( m \) must not appear in the action, since they would transform the fermion to that of the opposite chirality. The function \( V(m^2) \) is discussed in Section 2.3.
2.1 Identification of large $N$ matrices with differential operators

We identify the space of large $N$ matrices with that of the differential operators\[1]. By this identification, we can describe the differential operators on an arbitrary spin bundle over an arbitrary manifold in the continuum limit simultaneously, because they are embedded in the space of large $N$ matrices (some simple examples are provided in Appendix\[A\]). In the following, from the big space of large $N$ matrices, we pick up a subspace consisting of the differential operators over one manifold. We regard $\mathcal{M}$ as the differential operators over this manifold. Then, we analyze the effective theory of the fields appearing in the expansion of the differential operators (the explicit form is given later in (2.6)).

The space of the differential operator is infinite-dimensional, and in general the trace $Tr$ for this space is divergent. But, as we elucidate later, we choose a function $V(m^2)$ which decreases exponentially, and thus the trace is finite. We want to identify the dimensionless matrix $m$ with something similar to the Dirac operator, and we need to introduce $\mathcal{D}$ as an extension of the Dirac operator in the curved space acting on the ”spinor root density” (1.5). $\mathcal{D}$ has the dimension $[(\text{length})^{-1}]$, and $m$ is expressed as

$$m = \tau^{\frac{1}{2}} \mathcal{D},$$

$$\mathcal{D} = A_a \Gamma^a + \frac{i}{3!} A_{a_1 a_2 a_3} \Gamma^{a_1 a_2 a_3} - \frac{1}{5!} A_{a_1 \ldots a_5} \Gamma^{a_1 \ldots a_5} - \frac{i}{7!} A_{a_1 \ldots a_7} \Gamma^{a_1 \ldots a_7} + \frac{1}{9!} A_{a_1 \ldots a_9} \Gamma^{a_1 \ldots a_9}$$

(2.5)

where $\tau$ has the dimension $[(\text{length})^2]$. $\tau$ is similar to the Regge slope $\alpha'$ in string theory, and is introduced as a reference scale. This parameter $\tau$ is not a cut-off parameter. $V(m^2)$ is an exponentially decreasing function, and the damping factor is supplied by the action itself. Since $V(m^2)$ is a function of the dimensionless quantity $m$, $\tau$ represents a damping scale. When we approximate the differential operators by finite $N$ matrices, an $N$-dependent ultraviolet cut-off naturally appears. When we take $N$ to infinity, this cut-off becomes infinitely small. But the scale $\tau$ is completely independent of this ultraviolet cut-off, and takes a constant value even in the large $N$ limit. Thus, we can fairly take $N$ to infinity and identify the large $N$ matrices with the differential operators, at least when we investigate the effective theory at tree level. In this sense, our model differs from the induced gravity.

$A_{a_1 \ldots a_{2n-1}} (n = 1, 2, \ldots, 5)$ are hermitian differential operators and are expanded by the number of the derivatives. Because of the hermiticity, they are expanded as the anticommutator of differential operators and real functions:

$$A_{a_1 \ldots a_{2n-1}} = a_{a_1 \ldots a_{2n-1}} (x) + \sum_{k=1}^{\infty} \frac{i^k}{2} (\partial_{i_1} \cdots \partial_{i_k}, a^{(i_1 \cdots i_k)}_{a_1 \cdots a_{2n-1}} (x)).$$

(2.6)

The indices in the parentheses are symmetric, while the other indices are antisymmetric. Since $\mathcal{D}$ is an extension of the Dirac operator in the curved space, we find it natural to identify the function $a^{(i)}_{a}(x)$ with the vielbein of the background metric. Now, the operator $\mathcal{D}$ is expanded as

$$\mathcal{D} = e^{\frac{1}{2}}(x) \left[ i e_a^{(i)}(x) \Gamma^a \left( \partial_i + \frac{1}{4} \Gamma^{bc} \omega_{ibc}(x) \right) \right] e^{-\frac{1}{2}}(x)$$

$$+ \quad \text{(higher-rank terms)} + \text{(higher-derivative terms)}.$$

(2.7)
The other coefficients $a^{(i_1\cdots i_k)}_{a_1\cdots a_{2n-1}}(x)$ are regarded as the matter fields, and a simple dimensional analysis reveals that they have the dimension

$$a^{(i_1\cdots i_k)}_{a_1\cdots a_{2n-1}}(x) \sim [(\text{length})^{-1+k}].$$ (2.8)

### 2.2 Local Lorentz invariance

We next investigate the local Lorentz symmetry of this action. The local Lorentz transformation is

$$\delta m = [m, \varepsilon], \quad \varepsilon = -i\varepsilon_0 + \frac{1}{2!}\Gamma^{a_1a_2}\varepsilon_{a_1a_2} + \frac{i}{4!}\Gamma^{a_1\cdots a_4}\varepsilon_{a_1\cdots a_4} - \frac{1}{6!}\Gamma^{a_1\cdots a_6}\varepsilon_{a_1\cdots a_6} - \frac{i}{8!}\Gamma^{a_1\cdots a_8}\varepsilon_{a_1\cdots a_8} + \frac{1}{10!}\Gamma^{a_1\cdots a_{10}}\varepsilon_{a_1\cdots a_{10}}.$$ (2.9)

The large $N$ matrix $\varepsilon$ satisfies

$$\Gamma^0 \varepsilon^\dagger \Gamma^0 = \varepsilon,$$ (2.10)

and represents the local Lorentz transformation, with $\varepsilon_{a_1\cdots a_{2n}}$ ($n = 0, \cdots, 5$) being hermitian. As we have observed in Section 1, all the even-rank gamma matrices are necessary in order for the algebra to close. The invariance of the action is verified as

$$\delta S = Tr\left[tr\left(2V'(m^2)\varepsilon [m, \varepsilon]\right)\right] = 0,$$ (2.11)

where $V'(x)$ denotes $V'(x) = \frac{\partial V(x)}{\partial x}$. The cyclic property of the trace still holds true of the space of the differential operators, if we assume that all the coefficients damp rapidly at infinity; i.e.

$$\lim_{|x|\to \infty} a^{(i_1\cdots i_k)}_{a_1\cdots a_{2n-1}}(x) = 0.$$ (2.12)

Under this assumption, the trace of the commutator between the function and the differential operator is

$$Tr\left([\partial_j, a^{(i_1\cdots i_k)}_{a_1\cdots a_{2n-1}}(x)]\right) = \int d^dx \langle x|\partial_j a^{(i_1\cdots i_k)}_{a_1\cdots a_{2n-1}}(x)|x\rangle = \int d^dx \langle x|\partial_j a^{(i_1\cdots i_k)}_{a_1\cdots a_{2n-1}}(x)|x\rangle.$$ (2.13)

The surface term of this integral vanishes due to the assumption (2.12), and the cyclic property of the trace is ensured even for the trace of differential operators.

### 2.3 Determination of $V(m^2)$

We next discuss the condition $V(m^2)$ should satisfy. In this section, we set the background metric to be flat for brevity. We require that the differential operator in the flat space $m_0 = i\tau^\frac{1}{2}\Gamma^a\partial_a$ should be a classical solution of this matrix model. To investigate this condition, we consider the Laplace transformation of the function $V(u)$

$$V(u) = \int_0^\infty ds g(s) e^{-su}.$$ (2.14)
The bosonic part of the action is thus expressed as
\[
\sum \text{Tr} V_k A \text{ scalar. Therefore, the coefficient is odd, the total number of the indices is also odd and this term cannot contract the indices to constitute a expansion. From (2.8), the dimension of the term } \prod \text{ the trace we discern that the function } V \text{ is linear term, we focus on the linear terms with the indices contracted as should vanish in the action. Since only the scalar fields can constitute a Lorentz invariant linear term, we should be canceled in the action.}
\]

A dimensional analysis shows that the linear terms of the fields \(0, 1, 2, \cdots\). Since \(a^{(i)}(x)\) is identified with the vielbein, the cancellation of the linear term \(a^{(a)}(x)\) means the absence of the cosmological constant in this matrix model. The linear terms of their derivatives \((\partial_{a_i} \cdots \partial_{a_m} a^{(a_{i_1 \cdots i_l})}(x))\) disappear after integrating in the action. A simple dimensional analysis indicates that the linear terms of the fields \(a^{(a_{i_1 \cdots i_l})}(x)\) are included in \(A_{-l}(x)\). Therefore, we demand that the Seeley de Witt coefficients \(A_{-l}(x)\) should be canceled in the action.

The condition for \(m_0\) to be a classical solution is thus translated into the following statement:
\[
\int_0^\infty ds g(s) s^{D-\frac{d}{2} - n} = 0, \text{ for } n = 0, -1, -2, \cdots.
\]

Now, let us rewrite the condition (2.16) in terms of the function \(V(u)\). Noting
\[
\int_0^\infty du V(u) u^{\alpha - 1} = \int_0^\infty du \int_0^\infty ds g(s) e^{-s u} u^{\alpha - 1} = \Gamma(\alpha) \int_0^\infty ds g(s) s^{-\alpha},
\]
we discern that the function \(V(u)\) must satisfy
\[
\int_0^\infty du u^{\frac{d}{2} + n} V(u) = 0, \text{ for } n = -1, 0, 1, 2, \cdots.
\]

6A dimensional analysis shows that the terms of the order \(O(\tau^{-\frac{d}{2} + l})\) for half integer \(l\) do not appear in this expansion. From (2.3), the dimension of the term \(\prod_{i=1}^n (\partial_{a_1} \cdots \partial_{a_m} a^{(a_{i_1 \cdots i_l})}(x))\) is \([\text{length}]^D\), where \(D = -n + \sum_{i=1}^n (-p_i + l_i)\). This term is included in the Seeley de Witt coefficient \(A_{-l}(x)\). On the other hand, the number of the indices in this term is \(\sum_{i=1}^n (2m_i + 1 - p_i + l_i)\), which is equal to \(D\) modulo 2. When \(D\) is odd, the total number of the indices is also odd and this term cannot contract the indices to constitute a scalar. Therefore, the coefficient \(A_l(x)\) for half integer \(l\) is prohibited due to the Lorentz invariance of the action. However, they may survive when \(d\) is odd, with the antisymmetric tensor \(\epsilon_{i_1 \cdots i_d}\) being accompanied.

7Since \(\frac{d}{2} + n > 0\) for \(n = -1, 0, 1, 2, \cdots\), we need not worry about the possibility that \(\Gamma\left(\frac{d}{2} + n + 1\right)\) vanishes.
There are various choices for such functions, and one example is the following function:\footnote{\textsuperscript{8}}

\[
V_0(u) = \frac{\partial^{\frac{d}{2} - 1} \left( \exp(-u^\frac{d}{2}) \sin(u^\frac{d}{2}) \right)}{\partial u^{\frac{d}{2} - 1}}. \tag{2.19}
\]

Going back to the general case, when \( V(m^2) \) is chosen in order that \( m_0 = i \tau_\Sigma^a \partial_a \) becomes a classical solution, we understand that the action reduces to the Einstein gravity in the classical low-energy limit. The absence of the linear term \( a^a(x) \) ensures that the graviton is massless, since the mass terms (including the cross-terms with the other fields) vanish due to the general coordinate invariance. Then, the Einstein gravity is derived from the Seeley de Witt expansion of the trace \( Tr[tr(e^{-\tau D^2})] \) when we bequeath only the curved space Dirac operator in the expansion (2.7):\footnote{\textsuperscript{9}}

\[
Tr[tr(e^{-\tau D^2})] = \int d^4x \frac{32}{(2\pi \tau)^{\frac{d}{2}}} e(x) \left( \frac{R(x)}{\tau} + \cdots \right), \tag{2.20}
\]

where 32 comes from the trace \( tr \). It is obvious from the dimensional analysis that this term is included in \( A_1(x) \). When \( A_1(x) \) survives in the action, i.e. \( \int_{0}^{\infty} ds g(s) s^{-\frac{d}{2} + 1} \neq 0 \) or equivalently \( \int_{0}^{\infty} du V(u) u^{\frac{d}{2} - 2} \neq 0 \), the trace \( Tr[tr V(m^2)] \) includes the Einstein gravity in the Seeley de Witt expansion.

We next investigate the mass terms and the kinetic terms of the matter fields. A dimensional analysis shows that the mass terms and the kinetic terms (including the cross terms) are included in the following Seeley de Witt coefficients:

\[
\begin{align*}
\text{mass terms: } a^{(i_1 \cdots i_k)}_{a_1 \cdots a_{2n-1}}(x) a^{(j_1 \cdots j_l)}_{a_1 \cdots a_{2n-1}}(x) & \in A_{1 - \frac{d}{2}(k + l)}(x), \tag{2.21} \\
\text{kinetic terms: } (\partial_k a^{(i_1 \cdots i_k)}_{a_1 \cdots a_{2n-1}}(x)) (\partial_l a^{(j_1 \cdots j_l)}_{a_1 \cdots a_{2n-1}}(x)) & \in A_{2 - \frac{d}{2}(k + l)}(x). \tag{2.22}
\end{align*}
\]

where \( k + l \) is an even number. In particular, the mass terms and the kinetic terms of the odd-rank anti-symmetric fields \( a_{a_1 \cdots a_{2n-1}}(x) \) are included in \( A_1(x) \) and \( A_2(x) \), respectively. Generically\footnote{\textsuperscript{10}}, these fields are massive and are decoupled in the classical low-energy limit.

Next, we consider the fields \( a^{(i)}_{a_1 \cdots a_{2n-1}}(x) \). As we will explain in Section 3, especially the fields \( a^{(i)}_{a_1 \cdots a_{2n}}(x) \) (with \( n = 1, 2, 3, 4 \)) are identified with the anti-symmetric tensor fields in the type IIB supergravity. Their mass terms are included in \( A_0(x) \), whereas the kinetic terms reside in \( A_1(x) \). These fields are therefore massless. This observation provides a positive evidence in constructing a matrix model which reduces to the type IIB supergravity, which is the goal of the discussion of the next section.

\footnote{\textsuperscript{8}We can verify that this function satisfies (2.18) as follows\footnote{\textsuperscript{10}}. We note that \( \int_{0}^{\infty} dy g^n e^{-ay} = m! a^{-m-1} \) for \( a = \exp(\frac{d}{2}) = \frac{1}{\sqrt{2}} \) and \( m = 0, 1, 2, \cdots \). This is a real number when \( m - 3 \) is a multiple of 4. Taking the imaginary part of the both hand sides, we obtain \( \int_{0}^{\infty} dy y^{4n+3} \sin(\sqrt{\frac{y}{2}}) \exp(-\sqrt{\frac{y}{2}}) = 0 \), for \( n = 0, 1, 2, \cdots \). We make a substitution \( u = \frac{y}{2} \) and integrate by part to verify that (2.19) satisfies (2.18). Another choice is the function \( V(u) = \frac{\partial^{\frac{d}{2} - 1} \left( \sin u^{\frac{d}{2}} \right)}{\partial u^{\frac{d}{2} - 1}} \), which corresponds to the limit \( a = \exp(\frac{d}{4}) \). In this case, we need to introduce a proper damping factor.}

\footnote{\textsuperscript{9}The square density \( e^{rac{d}{2}}(x) \) can be treated by using the cyclic rule of the trace.}

\footnote{\textsuperscript{10}The computation of the trace \( Tr[tr(e^{-\tau D^2})] \) indicates that the mass term \( a^a(x) a_a(x) \) vanishes, but \( a_a(x) \) is presumably massive because there is no reason that the cross-terms \( a^a(x) a^{j_1 \cdots j_2} a(x) \) also vanish.}
However, it is not clear whether the higher-spin fields $a^{(i_1\cdots i_k)}_{a_1\cdots a_{2n-1}}(x)\ (k = 2, 3, \cdots)$ are massive, since the mass terms and the kinetic terms belong to the coefficients $A_{1-k}(x)$ and $A_{2-k}(x)$ respectively and both of them vanish in the action.

3 Supersymmetric Action

In the previous section, we have investigated the bosonic part of the matrix model, and we have elucidated that the Einstein gravity is derived from the bosonic action in the classical low-energy limit.

We next add the fermionic term and present a proposal to construct a matrix model which has $\mathcal{N} = 2$ SUSY and reduces to the type IIB supergravity in the classical low-energy limit. If we succeed in building such a matrix model, this will be an extension of IIB matrix model with the gravity encoded more manifestly and provide a strong evidence that a matrix model is an eligible framework to describe the gravitational interaction. The suggestion in this section is rather conjectural. We start with the following action:

$$S_S = \text{Tr}[trV_S(m^2)] + \text{Tr}\bar{\psi}m\psi. \quad (3.1)$$

Here, $m$ is a large $N$ matrix already defined in (2.2). $m$ is identified with the differential operators and expanded by the number of the derivatives in the same way as is explained in the previous section.

$\psi$ is a Weyl fermion, but is no longer hermitian. As we have observed in Section 1, it is difficult to construct a matrix model which is invariant under the local Lorentz transformation of the type $\delta \psi = \frac{1}{4}\Gamma^{ab}\{\varepsilon_{ab}, \psi\}$. Then, we define the local Lorentz transformation of the matrix model as $\delta \psi = \varepsilon \psi$ and sacrifice the hermiticity of the fermionic field. This fermionic field is also identified with differential operators and expanded as

$$\psi = \left(\chi(x) + \sum_{l=1}^{\infty} \chi^{(i_1\cdots i_l)}(x)\partial_{i_1}\cdots\partial_{i_l}\right)e^{-(\tau D^2)^\alpha}, \quad (3.2)$$

where the fermionic fields $\chi^{(i_1\cdots i_l)}(x)$ possess the dimension $[(\text{length})^l]$. We temporarily introduce the damping factor $e^{-(\tau D^2)^\alpha}$ so that the trace $\text{Tr}\bar{\psi}m\psi$ should be finite. The power $\alpha$ is chosen in accordance with the function $V_S(m^2)$ and, for example, when we choose the function (2.19), the power is $\alpha = \frac{1}{4}$.

This action is invariant under the local Lorentz transformation

$$\delta m = [m, \varepsilon], \quad \delta \psi = \varepsilon \psi, \quad (3.3)$$

where $\varepsilon$ is already defined in (2.9). The transformation of $\bar{\psi}$ is readily seen to be $\delta \bar{\psi} = -\bar{\psi}\varepsilon$. The local Lorentz invariance is verified as

$$\delta S_S = 2\text{Tr}[tr(V'_S(m^2)m[m, \varepsilon])] + \text{Tr}[tr(\bar{\psi}[m, \varepsilon]\psi)] = 0, \quad (3.4)$$

when we assume that the coefficients $a^{(i_1\cdots i_k)}_{a_1\cdots a_{2n-1}}(x)$ and $\chi^{(i_1\cdots i_k)}(x)$ damp rapidly at infinity.

The function $V_S(m^2)$ is to be determined so that it satisfies the following conditions. First, the differential operator in the flat space $m_0 = i\tau^\frac{1}{2}\Gamma^a\partial_a$ must be a classical solution, and
$V_S(m^2)$ must satisfy the criterion (2.18). Second, in order for the model to reduce to the type IIB supergravity in the classical low-energy limit, only the fields $a^{(i)}_{ia_1a_2...a_{2n}}(x)$ ($n = 1, 2, 3, 4$), which are interpreted as the anti-symmetric even-rank fields in the type IIB supergravity, must be massless and the other fields must be massive. It is not clear whether (2.19) is an eligible function, because we have yet to understand whether the higher-spin fields are massive. This problem will be reported in more detail elsewhere.

We next consider the structure of the supersymmetry of this matrix model. The SUSY transformation is given by

$$\delta_\epsilon \psi = \epsilon \bar{\psi} + \bar{\psi} \epsilon, \quad \delta_\epsilon \bar{\psi} = 2V''_S(m^2)\epsilon.$$  \hspace{1cm} (3.5)

It readily follows that the SUSY transformation of $\bar{\psi}$ is

$$\delta_\epsilon \bar{\psi} = 2\bar{\epsilon}V'_S(m^2).$$  \hspace{1cm} (3.6)

The invariance of the action under this SUSY transformation is verified as

$$\delta_\epsilon S_S = Tr \left[ tr \left( (2V'_S(m^2)m(\epsilon \bar{\psi} + \bar{\psi}\epsilon)) + \psi(\epsilon \bar{\psi} + \bar{\psi}\epsilon)\psi 
+ 2\bar{\psi}mV'_S(m^2)\epsilon + 2\bar{\epsilon}mV'_S(m^2)\psi \right) \right] = 0.$$  \hspace{1cm} (3.7)

We analyze the commutator of this SUSY transformation especially when it is possible to perform the Taylor expansion of $V_S(u)$ around $u = 0$ as

$$V_S(u) = \sum_{k=1}^\infty \frac{a_{2k}}{2^k} u^k,$$  \hspace{1cm} (3.8)

in order to simplify the analysis using the equations of motion

$$\frac{\partial S_S}{\partial \bar{\psi}} = 2m\psi = 0, \quad \frac{\partial S_S}{\partial \psi} = 2\bar{\psi}m = 0.$$  \hspace{1cm} (3.9)

The following analysis is not applied to the function (2.19), because the Taylor expansion around $u = 0$ is impossible\footnote{The invariance of the action under the local Lorentz transformation (3.4) and the SUSY transformation (3.7) holds true without assuming the Taylor expansion around $u = 0$.}. However, in the case of (3.8), the commutator can be simplified as

$$[\delta_\epsilon, \delta_\xi]m = 2[\xi \bar{\epsilon} - \epsilon \bar{\xi}, V'_S(m^2)],$$  \hspace{1cm} (3.10)

$$[\delta_\epsilon, \delta_\xi]\psi = 2\psi \left( \epsilon m \frac{V'_S(m^2) - V'(0)}{m^2} \xi - \bar{\epsilon} m \frac{V'_S(m^2) - V'(0)}{m^2} \bar{\xi} \right).$$  \hspace{1cm} (3.11)

The explicit derivation is presented in Appendix. B.

In order to see the structure of the $\mathcal{N} = 2$ SUSY, we separate the SUSY parameters into the hermitian and the antihermitian parts as

$$\epsilon = \epsilon_1 + i\epsilon_2, \quad \xi = \xi_1 + i\xi_2,$$  \hspace{1cm} (3.12)
where $\epsilon_1$, $\epsilon_2$, $\xi_1$ and $\xi_2$ are Majorana-Weyl fermions. For simplicity, we assume that $\epsilon$ and $\xi$ are c-numbers and that the background metric is flat. We now clarify that the translation of $A_a$ comes from the quartic term of $m$ in the Taylor expansion (3.8):

$[\delta_\epsilon, \delta_\xi]A_a = \frac{1}{16} tr ([\delta_\epsilon, \delta_\xi]m \Gamma^a)$

$= \frac{1}{16} \sum_{k=2}^{\infty} a_{2k} tr (\bar{\epsilon} m^{2k-2} \Gamma^a - \epsilon \bar{\xi} m^{2k-2} \Gamma^a - m^{2k-2} \xi \bar{\epsilon} \Gamma^a + m^{2k-2} \epsilon \bar{\xi} \Gamma^a)$

$= \frac{1}{16} \sum_{k=2}^{\infty} a_{2k} (\bar{\epsilon} [m^{2k-2}, \Gamma^a] - \epsilon [m^{2k-2}, \Gamma^a])$

$= \frac{a_4}{16} (\bar{\xi} [\Gamma^{b_1} \Gamma^{b_2}, \Gamma^a] \epsilon - \bar{\epsilon} [\Gamma^{b_1} \Gamma^{b_2}, \Gamma^a] \xi) A_{b_1} A_{b_2} + \cdots = \frac{a_4}{16} (\bar{\xi} \Gamma^i \epsilon - \bar{\epsilon} \Gamma^i \xi) [A_i, A_a] + \cdots$  

(3.13)

where we have utilized the fact that the SUSY parameters $\epsilon$ and $\xi$ are c-numbers and thus that the matrix $m$ commutes with the SUSY parameters. Let us concentrate on the SUSY transformation of the field $a_a(x)$. The commutator $[A_i, A_a]$ represents the translation of $a_a(x)$ and its gauge transformation:

$[A_i, A_a] = [i \partial_i + a_i(x), i \partial_a + a_a(x)] + \cdots$

$= i (\partial_a a_a(x)) - i (\partial_a a_i(x)) + [a_i(x), a_a(x)] + \cdots$  

(3.14)

Therefore, we discern that the commutator of the SUSY transformations constitutes a bona fide translation for the vector fields.

However, this is not the case with the fermionic fields. The commutator is computed as

$[\delta_\epsilon, \delta_\xi] \psi = - \sum_{k=2}^{n} a_{2k} \psi (\bar{\epsilon} m^{2k-3} \epsilon - \epsilon \bar{\xi} m^{2k-3} \xi) + \cdots = -a_4 (\bar{\xi} \Gamma^j \epsilon - \bar{\epsilon} \Gamma^j \xi) \psi A_j + \cdots$

$= -2a_4 (\bar{\xi}_1 \Gamma^j \epsilon_1 + \bar{\xi}_2 \Gamma^j \epsilon_2) \psi A_j + \cdots$  

(3.15)

We explore the term $\psi A_j$ more carefully:

$\psi A_j = i \psi \partial_j + \cdots = \left( \chi(x) \partial_j + \sum_{l=1}^{\infty} \chi^{(i_1 \cdots i_l)}(x) \partial_{i_1} \cdots \partial_{i_l} \partial_j \right) e^{-(\tau D^2)^a} + \cdots$  

(3.16)

where $\cdots$ denotes the omission of the non-linear terms of the fields. Therefore, each fermionic field is transformed as

$[\delta_\epsilon, \delta_\xi] \chi(x) = 0 + \cdots$

$[\delta_\epsilon, \delta_\xi] \chi^{(i_1 \cdots i_{l+1})}(x) = -2a_4 (\bar{\xi}_1 \Gamma^j \epsilon_1 + \bar{\xi}_2 \Gamma^j \epsilon_2) \chi^{(i_1 \cdots i_l)}(x) \delta^{i_{l+1}j} + \cdots$  

(3.17)

The commutator of the SUSY transformation constitutes a translation for the vector fields, while the same does not hold true of the fermionic fields. However, the analysis (3.10) – (3.17) is applicable only to the case in which we can perform a Taylor expansion of $V_S(u)$ around
$u = 0$. We speculate that we may be able to construct a well-defined model with $\mathcal{N} = 2$ SUSY for the functions like (2.19), which do not have the Taylor expansion around $u = 0$.

We surmise that the fermionic fields $\chi(x)$ and $\chi^{(i)}(x)$, which are identified with the dilatino and the gravitino respectively, become massless due to the supersymmetry, when the even-rank fields $a^{(i)_{a_1\ldots a_n}}(x)$ are massless. If we choose the function $V_S(m^2)$ properly, the classical low-energy limit of this matrix model would reduce to the type IIB supergravity, and it is an interesting future problem to investigate this conjecture more deeply.

4 Conclusion

We have hitherto investigated the possibility to realize the general coordinate invariance manifestly in a matrix model. And in order to build a matrix model which describes the gravity, we should consider the local Lorentz invariance of the model. In realizing the local Lorentz invariance in a matrix model, the parameters of the Lorentz symmetry must be promoted to $u(N)$ matrices, because the information of the spacetime is encoded in the eigenvalues of the large $N$ matrices. The model thus has to include all the odd-rank gamma matrices in 10 dimensions. The starting point to construct such a matrix model is to identify the large $N$ matrices with differential operators. By this identification, it is possible to describe the differential operators on an arbitrary spin bundle over an arbitrary manifold simultaneously in the space of large $N$ matrices.

We have first considered the bosonic part of the model, and clarified that it reduces to the Einstein gravity in the classical low-energy limit. And we have extended the observation to the supersymmetric action, and have presented a proposal to construct a matrix model which has $\mathcal{N} = 2$ SUSY and reduces to the type IIB supergravity. It is a future problem to seek the explicit form of the function $V_S(m^2)$. If we find a proper model, this would be a nonperturbative formulation of the type IIB superstring theory, and become an extension of IIB matrix model with the information of the gravity encoded more manifestly. It is also interesting to elucidate the relation between our model and IIB matrix model. More analysis of these issues will be reported elsewhere.

Acknowledgment

The authors would like to express their gratitude to Keisuke Ohashi for valuable discussion. The works of T.A. were supported in part by Grant-in-Aid for Scientific Research from Ministry of Education, Culture, Sports, Science and Technology of Japan (#01282).

A Examples of the differential operators in the space of large $N$ matrices

This appendix is devoted to some simple examples of the differential operators embedded in the space of large $N$ matrices. The space of the large $N$ matrices includes the differential operators on an arbitrary spin bundle over an arbitrary manifold simultaneously. We now consider the differential operators of the scalar field on two different bundles over $S_1$. 

12
We first consider the trivial bundle with the periodic condition $f(1) = f(0)$. We discretize the region $0 \leq x \leq 1$ into small slices of spacing $\epsilon = \frac{1}{N}$. Then, the differential operator is approximated by the finite difference as

$$\partial_x f \left(\frac{k}{N}\right) \rightarrow \frac{1}{2} \left( \frac{f \left(\frac{k+1}{N}\right) - f \left(\frac{k}{N}\right)}{\epsilon} + \frac{f \left(\frac{k}{N}\right) - f \left(\frac{k-1}{N}\right)}{\epsilon} \right) = \frac{N}{2} \left( f \left(\frac{k+1}{N}\right) - f \left(\frac{k-1}{N}\right) \right). \quad (A.1)$$

Due to the periodic condition, this finite difference is expressed by the large $N$ matrix as

$$\partial_x \rightarrow A = \frac{N}{2} \begin{pmatrix} 0 & 1 & \cdots & -1 \\ -1 & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 1 & -1 & 0 & \cdots \end{pmatrix}. \quad (A.2)$$

(2) $\mathbb{Z}_2$-twisted bundle (Möbius strip)

We consider the similar problem with respect to the $\mathbb{Z}_2$-twisted bundle, in which the antiperiodic condition $f(1) = -f(0)$ is imposed. Paying attention to this antiperiodicity, we understand
that the finite difference in the discretized space is expressed as

\[
\partial_x \rightarrow A = \frac{N}{2} \begin{pmatrix}
0 & 1 & 1 \\
-1 & 0 & 1 \\
-1 & 0 & 1 \\
\ddots & \ddots & \ddots \\
-1 & -1 & 0
\end{pmatrix}.
\] (A.3)

**B  Proof of (3.10) and (3.11)**

This appendix is devoted to the explicit computation of the commutation relations of the SUSY transformation. First, (3.10) is verified by considering the differences of the following two SUSY transformations:

\[
m \delta \rightarrow m + \xi \bar{\psi} + \psi \xi \delta \rightarrow m + (\epsilon + \xi) \bar{\psi} + \psi (\bar{\epsilon} + \bar{\xi}) + 2 \xi \bar{\epsilon} V'_S(m^2) + 2 V'_S(m^2) \epsilon \bar{\xi},
\]

\[
m \delta \rightarrow m + \epsilon \bar{\psi} + \psi \epsilon \delta \rightarrow m + (\epsilon + \xi) \bar{\psi} + \psi (\bar{\epsilon} + \bar{\xi}) + 2 \epsilon \bar{\xi} V'_S(m^2) + 2 V'_S(m^2) \xi \bar{\epsilon}.
\]

(3.11) is verified likewise, but we need to use the Taylor expansion of \(V_S(u)\) around \(u = 0\):

\[
\psi \xrightarrow{\delta \epsilon} \psi + 2 V'_S(m^2) \xi \\
\delta \xi \psi \xrightarrow{\delta \epsilon} \psi + 2 V'_S(m^2) (\epsilon + \psi) + \sum_{k=2}^{\infty} a_{2k} \left[ (\epsilon \bar{\psi} + \psi \epsilon) m^{2k-3} + m (\epsilon \bar{\psi} + \psi \epsilon) m^{2k-4} \cdots + m^{2k-3} (\epsilon \bar{\psi} + \psi \epsilon) \right] \xi,
\]

\[
\psi \delta \epsilon \psi \xrightarrow{\delta \xi} \psi + 2 V'_S(m^2) \epsilon \\
\delta \xi \psi \xrightarrow{\delta \epsilon} \psi + 2 V'_S(m^2) (\epsilon + \psi) + \sum_{k=2}^{\infty} a_{2k} \left[ (\xi \bar{\psi} + \psi \xi) m^{2k-3} + m (\xi \bar{\psi} + \psi \xi) m^{2k-4} \cdots + m^{2k-3} (\xi \bar{\psi} + \psi \xi) \right] \epsilon.
\]

Utilizing the equations of motion (3.3) and the fact that the combinations of the fermionic fields \(\bar{\psi} \epsilon\) and \(\bar{\psi} \xi\) vanish because the fermions are Weyl ones, we find that the commutator of the SUSY transformation is

\[
[\delta \epsilon, \delta \xi] \psi = \sum_{k=2}^{\infty} a_{2k} \left[ \psi \bar{\epsilon} m^{2k-3} \xi - \psi \bar{\xi} m^{2k-3} \epsilon \right] = 2 \psi \left( \bar{\epsilon} m \left( \frac{V'_S(m^2) - V'_S(0)}{m^2} \right) \xi - \bar{\xi} m \left( \frac{V'_S(m^2) - V'_S(0)}{m^2} \right) \epsilon \right).
\]

**References**

[1] A. H. Chamseddine and A. Connes, Commun. Math. Phys. **186**, 731 (1997) [hep-th/9606001]

[2] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, Nucl. Phys. B **498**, 467 (1997) [hep-th/9612115]

[3] M. Fukuma, H. Kawai, Y. Kitazawa and A. Tsuchiya, Nucl. Phys. B **510**, 158 (1998) [hep-th/9705128]

[4] S. Iso and H. Kawai, Int. J. Mod. Phys. A **15**, 651 (2000) [hep-th/9903217]
[5] H. Aoki, S. Iso, H. Kawai, Y. Kitazawa, A. Tsuchiya and T. Tada, Prog. Theor. Phys. Suppl. 134 (1999) 47 [hep-th/9908038].

[6] H. Aoki, N. Ishibashi, S. Iso, H. Kawai, Y. Kitazawa and T. Tada, Nucl. Phys. B 565, 176 (2000) [hep-th/9908141].

[7] L. Smolin, Nucl. Phys. B 591, 227 (2000) [hep-th/0002009].

[8] E. Bergshoeff and A. Van Proeyen, Class. Quant. Grav. 17, 3277 (2000) [hep-th/0003261].

[9] L. Smolin, [hep-th/0006137].

[10] T. Azuma, S. Iso, H. Kawai and Y. Ohwashi, Nucl. Phys. B 610, 251 (2001) [hep-th/0102168].

[11] T. Azuma, [hep-th/0103003].

[12] T. Eguchi and H. Kawai, Phys. Rev. Lett. 48, 1063 (1982).

[13] A. Gonzalez-Arroyo and M. Okawa, Phys. Rev. D 27, 2397 (1983).

[14] A. Gonzalez-Arroyo and C. P. Korthals Altes, Phys. Lett. B 131, 396 (1983).

[15] P. Gilkey, "Invariance Theory, The Heat Equation, And the Atiyah-Singer Index Theorem" Publish or Perish (1984)

[16] T. W. Körner, "Fourier Analysis" Cambridge University Press (1988) Chap. 6

[17] H. Kawai, "Constructing New Types of Matrix Models" Talks in Tohwa International Symposium in String Theory (2001), Fukuoka, Japan. 
http://f-16.tohwa-u.ac.jp/symposium/32.htm