On quantum computation of Kloosterman sums

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Abstract. We give two quantum algorithms for computing (twisted) Kloosterman sums attached to a finite field \( F \) of \( q \) elements. The first algorithm computes a quantum state containing, as its coefficients with respect to the standard basis, all Kloosterman sums for \( F \) twisted by a given multiplicative character, and runs in time polynomial in \( \log q \). The second algorithm computes a single Kloosterman sum to a prescribed precision, and runs in time quasi-linear in \( \sqrt{q} \).

1. Introduction

Let \( p \) a prime number, and let \( r \) a positive integer. Let \( F \) be a finite field of \( q \) elements, where \( q = p^r \). An additive character of \( F \) is a group homomorphism

\[
\psi: F \to \mathbb{C}^*,
\]

and a multiplicative character of \( F \) is a group homomorphism

\[
\chi: F^* \to \mathbb{C}^*.
\]

There are many interesting number-theoretical constructions involving such characters.

Definition. Let \( \psi \) be a non-trivial additive character of \( F \). For all \( a \in F^* \) and all multiplicative characters \( \chi \) of \( F \), the (twisted) Kloosterman sum \( Kl_\psi(a, \chi) \) is defined as

\[
Kl_\psi(a, \chi) = \sum_{x, y \in F} \chi(x)\psi(x + y).
\]

If \( \chi \) is the trivial character, then \( Kl_\psi(a, \chi) \) is real. In general, a straightforward computation shows that \( Kl_\psi(a, \chi) \) satisfies

\[
Kl_\psi(a, \chi) = \chi(-a)\overline{Kl_\psi(a, \chi)},
\]

where the bar on the right-hand side denotes complex conjugation. The following (optimal) bound on absolute values of Kloosterman sums is a famous result of Weil [15]:

\[
|Kl_\psi(a, \chi)| \leq 2\sqrt{q}.
\]

No general closed formula for Kloosterman sums is known, except when \( q \) is odd and \( \chi \) is the unique character \( \chi_2: F^* \to \{\pm 1\} \) of order 2. The Kloosterman sums \( Kl_\psi(a, \chi_2) \) are also known as Salié sums in honour of the explicit formula

\[
Kl_\psi(a, \chi_2) = G_\psi(\chi_2) \sum_{y \in F^*_{\chi_2^{-1}a}} \psi(y)
\]

found by Salié [10]; here \( G_\psi(\chi_2) = \sum_{a \in F^*} \chi_2(a)\psi(a) \) is the Gauss sum of the character \( \chi_2 \).

The problem of efficiently computing or approximating Kloosterman sums does not seem to have a satisfactory solution as of this writing; see [12, Problem 3.45] (= [13, Problem 54]) and [3, §8.1]. In this article, we are especially interested in quantum algorithms. An efficient (classical or quantum) algorithm for approximating Kloosterman sums would have applications to quantum algorithms for finding certain hidden non-linear structures as in work of Childs, Schulman and Vazirani [4].

Although we have not been able to completely solve the problem of efficiently computing Kloosterman sums, we do have the following two results. The first of these might be particularly applicable as a building block in future quantum algorithms.
Theorem 1.1. There exists a quantum algorithm that, given a finite field $F$, a non-trivial additive character $\psi$ of $F$ and a multiplicative character $\chi$ of $F$, computes an $n$-qubit state whose coefficients on the standard basis are the values $Kl_\psi(a, \chi)/\sqrt{N_q}$ for $a \in F^\times$, and whose running time is polynomial in $\log q$ as $q \to \infty$. Here $n = \lceil \log_2(q-1) \rceil$, and $N_q$ is an explicit positive integer only depending on $\# F$ and on whether the character $\chi$ is trivial.

In our second result, we bound the quantum complexity of computing the normalised Kloosterman sums $Kl_\psi(a, \chi)/\sqrt{q}$ to within a prescribed absolute error.

Theorem 1.2. There exists a quantum algorithm that, given a finite field $F$, a non-trivial additive character $\psi$ of $F$, an element $a \in F^\times$, a multiplicative character $\chi$ of $F$ and real numbers $\delta, \epsilon \in (0, 1)$, computes an approximation to $Kl_\psi(a, \chi)/\sqrt{q}$ that has absolute error at most $\epsilon$ with probability at least $1 - \delta$, and whose running time is linear in $\sqrt{q}/(\delta \epsilon)$ times a power of $\log q$.

The main idea of the proof of Theorem 1.1 is to relate Kloosterman sums to Gauss sums. We recall that the Gauss sum of a multiplicative character $\chi$ of $F$ (with respect to a fixed non-trivial additive character $\psi$) is defined as

$$G_\psi(\chi) = \sum_{a \in F^\times} \chi(a)\psi(a).$$

Quantum computation of Gauss sums was studied by van Dam and Seroussi [14], who constructed a unitary operator having Gauss sums (normalised to have absolute value 1) as eigenvalues. In fact, one of our ingredients is the main algorithm of [14].

Our strategy, inspired by Katz [8, §4.0], is to use the multiplicative Fourier transform (as opposed to the additive Fourier transform used in [14]) to reduce the problem of computing the function $a \mapsto Kl_\psi(a, \chi)$ for a given multiplicative character $\chi$ of $F$ to the problem of computing the function mapping a multiplicative character $\chi'$ of $F$ to the complex number $G_\psi(\chi')G_\psi(\chi\chi')$. In order to deduce Theorem 1.2 from Theorem 1.1, we use amplitude amplification [2].

Remark. The problem of computing Kloosterman sums is a “natural problem” in the sense that it does not involve a black box. Other natural problems for which efficient quantum algorithms have been developed are Shor’s algorithms for integer factorisation and discrete logarithms [11], van Dam and Seroussi’s algorithm for Gauss sums mentioned above, and algorithms for computing unit groups and ideal class groups of number fields; see [6], [5] and [1].

To put the above theorems in a broader perspective, we briefly discuss equidistribution properties of Kloosterman sums. Although these results are not used in the remainder of this article, they could be used as a statistical check on possible future implementations.

We first consider the case where $\chi:F^\times \to C^\times$ is the trivial character 1, so we consider the Kloosterman sums

$$Kl_\psi(a, 1) = \sum_{x, y \in F \atop xy = a} \psi(x + y).$$

Theorem 1.3 (Katz [8, Chapter 13]). For each prime power $q$, let $F_q$ be a finite field of $q$ elements, and let $\psi_q$ be a non-trivial additive character of $F_q$. As $q \to \infty$, the normalised Kloosterman sums $Kl_\psi_q(a, 1)/\sqrt{q}$ for $a \in F^\times$ become equidistributed with respect to the probability measure $\frac{1}{2\pi} \sqrt{4-x^2} \, dx$ on $[-2, 2]$.

The measure in Theorem 1.3 is the Sato–Tate measure for the compact Lie group SU(2); this is the distribution of traces of elements of SU(2) that are randomly distributed with respect to the Haar measure. In view of Weil’s bound (1.2) and the fact that $Kl_\psi(a, 1)$ is real, we can write

$$Kl_\psi(a, 1) = 2\sqrt{q} \cos(\theta_\psi(a))$$

for a unique angle $\theta_\psi(a) \in [0, \pi]$. Theorem 1.3 can equivalently be stated by saying that as $q \to \infty$, the angles $\theta_\psi(a)$ for $a \in F^\times$ become equidistributed with respect to the probability measure $\frac{1}{\pi} \sin(\theta) \, d\theta$ on $[0, \pi]$.

A similar equidistribution result holds when the character $\chi$ is allowed to vary; this was kindly explained to the author by Nicholas M. Katz.
Theorem 1.4 (Katz, personal communication). For each prime power \( q \), let \( F_q \) be a finite field of \( q \) elements, and let \( \psi_q \) be a non-trivial additive character of \( F_q \). As \( q \to \infty \), the normalised Kloosterman sums \( Kl_{\psi_q}(a, \chi)/\sqrt{q} \), for \( a \in F_q^\times \) and \( \chi \) a multiplicative character of \( F_q \), become equidistributed with respect to the probability measure on the complex disc of radius 2 given in polar coordinates \((r, \theta)\) by \( \frac{1}{\pi} \sqrt{4 - r^2} \, dr \, d\theta \).

The measure in Theorem 1.4 is the Sato–Tate measure for the compact Lie group \( U(2) \). It follows from (1.1) and (1.2) that the two roots of the polynomial

\[
f_\psi(a, \chi) = t^2 - \frac{Kl_{\psi}(a, \chi)}{\sqrt{q}} t + \chi(-a) \in \mathbb{C}[t]
\]

have absolute value 1, so there are two angles \( \theta_\psi(a, \chi), \theta'_\psi(a, \chi) \) (unique up to ordering) satisfying

\[
Kl_{\psi}(a, \chi) = \sqrt{q} (\exp(i\theta_\psi(a, \chi)) + \exp(i\theta'_\psi(a, \chi))),
\]

\[
\chi(-a) = \exp(i(\theta_\psi(a, \chi) + \theta'_\psi(a, \chi))).
\]

(For \( \chi = 1 \), these angles are \( \pm \theta_\psi(a) \).) In other words, there is a unique conjugacy class in \( U(2) \) having characteristic polynomial \( f_\psi(a, \chi) \). Katz’s proof of Theorem 1.4 shows that as \( q \to \infty \), these conjugacy classes are equidistributed for the probability measure on the space of conjugacy classes induced by the Haar measure on \( U(2) \).

We end this introduction with two questions that appear natural to ask in the light of the above results.

**Question.** Can one find a family of curves or higher-dimensional varieties over finite fields for which knowing the quantum state from Theorem 1.1 leads to an efficient algorithm for counting points?

**Question.** Can one construct a unitary operator that has (a subset of) the numbers \( \exp(i\theta_\psi(a, \chi)) \) and \( \exp(i\theta'_\psi(a, \chi)) \) as eigenvalues and that can be efficiently implemented on a quantum computer? The construction of such an operator will necessarily be closely connected to a proof of Weil’s bound and to making the Kloosterman sheaves of Katz [8] computationally accessible.

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2. Preliminaries

Let \( X_F \) denote the group of multiplicative characters of \( F \), i.e. group homomorphisms \( \chi: F^\times \to \mathbb{C}^\times \). For any function \( f: F^\times \to \mathbb{C} \), the *multiplicative Fourier transform* of \( f \) is the function

\[
\mathcal{M}f: X_F \to \mathbb{C}
\]

\[
\chi \mapsto \sum_{a \in F^\times} \chi(a) f(a).
\]

This defines a map

\[
\mathcal{M}: \mathbb{C}^{F^\times} \to \mathbb{C}^{X_F}.
\]

This is an isomorphism of \( \mathbb{C} \)-vector spaces; the inverse \( \mathcal{M}^{-1} \) maps a function \( \phi: X_F \to \mathbb{C} \) to the function

\[
\mathcal{M}^{-1}\phi: F^\times \to \mathbb{C}
\]

\[
a \mapsto \frac{1}{q-1} \sum_{\chi \in X_F} \chi(a)^{-1} \phi(\chi).
\]

In particular, for all functions \( f: F^\times \to \mathbb{C} \) we have the identity

\[
f = \mathcal{M}^{-1}\mathcal{M} f = \frac{1}{q-1} \sum_{\chi \in X_F} \mathcal{M} f(\chi) \chi^{-1} \quad \text{in} \quad \mathbb{C}^{F^\times},
\]
and for all functions $\phi : X_F \to C$ we have the identity

$$\phi = \mathcal{M}^{-1} \phi = \sum_{a \in F^\times} \mathcal{M}^{-1} \phi(a) \hat{a} \quad \text{in } C^{X_F},$$

where $\hat{a} \in C^{X_F}$ is the function $X_F \to C$ sending $\chi$ to $\chi(a)$.

For two functions $f, g : F^\times \to C$, the convolution of $f$ and $g$ is

$$f \ast g : F^\times \to C$$

$$a \mapsto \sum_{x, y \in F^\times \atop xy = a} f(x) g(y).$$

An elementary computation shows that

$$\mathcal{M} f(\chi) \mathcal{M} g(\chi) = \mathcal{M} (f \ast g)(\chi) \quad \text{for all } \chi \in X_F.$$

If $\psi$ is a non-trivial additive character of $F$ and $\chi$ is a (possibly trivial) multiplicative character of $F$, then the definition of the Gauss sum $G_\psi(\chi)$ immediately implies

$$\mathcal{M} \psi(\chi) = G_\psi(\chi). \quad (2.1)$$

3. Encodings into quantum states

For all $n \geq 0$, let $V_n$ denote the $2^n$-dimensional Hilbert space of $n$-qubit states; this can be identified with $(C^2)^\otimes n$ equipped with the standard Hermitian metric. The standard basis vectors are labelled $|0\rangle, |1\rangle, \ldots, |2^n - 1\rangle$. When we are dealing with states inside a tensor product of the form $V_{n_1} \otimes \cdots \otimes V_{n_k}$, we will refer to the $i$-th tensor factor as the $i$-th register.

The quantum Fourier transform modulo $q - 1$ is the unitary transformation $F_{q-1}$ on $n$-qubit states defined on the subspace $C|0\rangle + \cdots + C|q - 2\rangle$ by

$$F_{q-1}|m\rangle = \frac{1}{\sqrt{q - 1}} \sum_{d=0}^{q-2} \exp(2\pi i dm/(q - 1)) |d\rangle$$

and extended in an unspecified way to the orthogonal complement. This can be computed exactly using an efficient quantum algorithm [9], and we will use this for simplicity. In practice, an approximate Fourier transform would probably be more useful.

We fix generators $a_1$ and $\chi_1$ of the finite cyclic groups $F^\times$ and $X_F$, respectively, that are assumed to satisfy

$$\chi_1(a_1) = \exp(2\pi i/(q - 1)).$$

This choice determines isomorphisms

$$\mathbb{Z}/(q - 1)\mathbb{Z} \xrightarrow{\sim} F^\times$$

$$d \mapsto a_1^d$$

and

$$\mathbb{Z}/(q - 1)\mathbb{Z} \xrightarrow{\sim} X_F$$

$$m \mapsto \chi_1^m.$$
For all $a \in \mathbb{F}^*$, we write $d_a$ for the discrete logarithm of $a$ with respect to $a_1$, i.e. the unique $d \in \{0, 1, \ldots, q - 2\}$ such that $a_1^d = a$. Similarly, for all $\chi \in X_{\mathbb{F}}$, we write $m_\chi$ for the unique $m \in \{0, 1, \ldots, q - 2\}$ such that $\chi_1^m = \chi$. Then for all $a \in \mathbb{F}^*$ and $\chi \in X_{\mathbb{F}}$, we have

$$\chi(a) = \exp(2\pi im_\chi d_a/(q - 1)).$$

Let $n$ be the least integer such that $2^n \geq q - 1$. For all $a \in \mathbb{F}^*$, we write $|a\rangle$ for the $n$-qubit state $|d_a\rangle$, and for all $\chi \in X_{\mathbb{F}}$ we write $|\chi\rangle$ for the $n$-qubit state $|m_\chi\rangle$. Furthermore, let $l$ be the least integer such that $2^l \geq p$. We assume that we are given a basis $(b_0, \ldots, b_{r-1})$ for $\mathbb{F}$ as an $\mathbb{F}_p$-vector space. We encode elements of $\mathbb{F}_p$ as strings of $l$ bits, and elements of $\mathbb{F}$ by their coefficients with respect to $(b_0, \ldots, b_{r-1})$. In this way we get an encoding of elements of $\mathbb{F}$ as strings of $lr$ bits, and hence as $lr$-qubit states. We write $|x\rangle_+$ for the state defined by $x \in \mathbb{F}$ in this way. For $a \in \mathbb{F}^*$ we will use both representations $|a\rangle_+$ and $|a\rangle_-$.

We further assume that $\psi$ is given to us as the list of values $\psi(b_0), \ldots, \psi(b_{r-1})$. Then we can efficiently compute the unitary operator

$$U_\psi: V_{lr} \rightarrow V_{lr}$$

such that

$$U_\psi |x\rangle_+ = \psi(x) |x\rangle_+ \quad \text{for all } x \in \mathbb{F}. \quad (3.1)$$

We can also efficiently compute the Fourier transform

$$\mathcal{F}_{\mathbb{F}, \psi}: V_{lr} \rightarrow V_{lr}$$

defined on states of the form $|x\rangle_+$ by

$$\mathcal{F}_{\mathbb{F}, \psi} |x\rangle_+ = \frac{1}{\sqrt{q}} \sum_{y \in \mathbb{F}} \psi(xy) |y\rangle_+$$

and extended in an unspecified way to the orthogonal complement. Furthermore, we can efficiently compute the unitary operator

$$W: V_n \otimes V_n \rightarrow V_n \otimes V_n$$

mapping any state $|\chi\rangle_+ |a\rangle_-$ to the state $\exp(2\pi m_\chi d_a/(q - 1)) |\chi\rangle |a\rangle$ for all $\chi \in X_{\mathbb{F}}$ and $a \in \mathbb{F}^*$.

Using classical operations in the finite field $\mathbb{F}$, we can efficiently compute the unitary operators $W$. We assume that we are given a basis $\{e_0, e_1, \ldots, e_{n-1}\}$ for $\mathbb{F}$ as an $\mathbb{F}_p$-vector space. We encode elements of $\mathbb{F}_p$ as strings of $l$ bits, and elements of $\mathbb{F}$ by their coefficients with respect to $(e_0, e_1, \ldots, e_{n-1})$. In this way we get an encoding of elements of $\mathbb{F}$ as strings of $lr$ bits, and hence as $lr$-qubit states. We write $|x\rangle_+$ for the state defined by $x \in \mathbb{F}$ in this way. For $a \in \mathbb{F}^*$ we will use both representations $|a\rangle_+$ and $|a\rangle_-$.

We further assume that $\psi$ is given to us as the list of values $\psi(b_0), \ldots, \psi(b_{r-1})$. Then we can efficiently compute the unitary operator

$$U_\psi: V_{lr} \rightarrow V_{lr}$$

such that

$$U_\psi |x\rangle_+ = \psi(x) |x\rangle_+ \quad \text{for all } x \in \mathbb{F}. \quad (3.1)$$

We can also efficiently compute the Fourier transform

$$\mathcal{F}_{\mathbb{F}, \psi}: V_{lr} \rightarrow V_{lr}$$

defined on states of the form $|x\rangle_+$ by

$$\mathcal{F}_{\mathbb{F}, \psi} |x\rangle_+ = \frac{1}{\sqrt{q}} \sum_{y \in \mathbb{F}} \psi(xy) |y\rangle_+$$

and extended in an unspecified way to the orthogonal complement. Furthermore, we can efficiently compute the unitary operator

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mapping any state $|\chi\rangle_+ |a\rangle_-$ to the state $\exp(2\pi m_\chi d_a/(q - 1)) |\chi\rangle |a\rangle$ for all $\chi \in X_{\mathbb{F}}$ and $a \in \mathbb{F}^*$.

Using classical operations in the finite field $\mathbb{F}$, we can efficiently compute the unitary operators $W$. We assume that we are given a basis $\{e_0, e_1, \ldots, e_{n-1}\}$ for $\mathbb{F}$ as an $\mathbb{F}_p$-vector space. We encode elements of $\mathbb{F}_p$ as strings of $l$ bits, and elements of $\mathbb{F}$ by their coefficients with respect to $(e_0, e_1, \ldots, e_{n-1})$. In this way we get an encoding of elements of $\mathbb{F}$ as strings of $lr$ bits, and hence as $lr$-qubit states. We write $|x\rangle_+$ for the state defined by $x \in \mathbb{F}$ in this way. For $a \in \mathbb{F}^*$ we will use both representations $|a\rangle_+$ and $|a\rangle_-$.

We further assume that $\psi$ is given to us as the list of values $\psi(b_0), \ldots, \psi(b_{r-1})$. Then we can efficiently compute the unitary operator

$$U_\psi: V_{lr} \rightarrow V_{lr}$$

such that

$$U_\psi |x\rangle_+ = \psi(x) |x\rangle_+ \quad \text{for all } x \in \mathbb{F}. \quad (3.1)$$

We can also efficiently compute the Fourier transform

$$\mathcal{F}_{\mathbb{F}, \psi}: V_{lr} \rightarrow V_{lr}$$

defined on states of the form $|x\rangle_+$ by

$$\mathcal{F}_{\mathbb{F}, \psi} |x\rangle_+ = \frac{1}{\sqrt{q}} \sum_{y \in \mathbb{F}} \psi(xy) |y\rangle_+$$

and extended in an unspecified way to the orthogonal complement. Furthermore, we can efficiently compute the unitary operator

$$W: V_n \otimes V_n \rightarrow V_n \otimes V_n$$

mapping any state $|\chi\rangle_+ |a\rangle_-$ to the state $\exp(2\pi m_\chi d_a/(q - 1)) |\chi\rangle |a\rangle$ for all $\chi \in X_{\mathbb{F}}$ and $a \in \mathbb{F}^*$.
4. Computing Gauss sums

Let \( 1 \in X_F \) be the trivial multiplicative character of \( F \). It is well known that for all multiplicative characters \( \chi \in X_F \), the Gauss sum \( G_{\psi}(\chi) \) has absolute value

\[
|G_{\psi}(\chi)| = \sqrt{L_{\chi}},
\]

where

\[
L_{\chi} = \begin{cases} 
  q & \text{if } \chi \neq 1, \\
  1 & \text{if } \chi = 1.
\end{cases}
\]  

Furthermore, we have

\[
G_{\psi}(1) = -1.
\]

In particular, (4.1) implies

\[
\sum_{\chi \in X_F} |G_{\psi}(\chi)|^2 = (q - 2)q + 1 = (q - 1)^2. \tag{4.2}
\]

We sketch a variant of the algorithm of van Dam and Seroussi [14] that realises the Gauss sum \( G_{\psi}(\chi) \) for a non-trivial element \( \chi \in X_F \) as a phase shift. This algorithm is based on the fact that for all \( y \in F \) we have the identity

\[
\sum_{a \in F^*} \chi(a)\psi(ay) = \begin{cases} 
(\chi(y)^{-1}G_{\psi}(\chi)) & \text{if } y \in F^*, \\
0 & \text{if } y = 0.
\end{cases} \tag{4.3}
\]

**Algorithm 4.1** (Compute a single Gauss sum). Given a finite field \( F \), a non-trivial additive character \( \psi \) of \( F \) and a quantum state \( |\chi\rangle_+ \) encoding a non-trivial multiplicative character \( \chi \) of \( F \), this algorithm outputs the state \( q^{-1/2}G_{\psi}(\chi)|\chi\rangle_+ \).

1. Start with the state \( |\chi\rangle_+|1\rangle_x = |\chi\rangle_+|0\rangle \).
2. Apply \( F_{q^{-1}} \) to the second register to obtain the state \( \frac{1}{\sqrt{q-1}}\sum_{a \in F^*} |\chi\rangle_+|a\rangle_x \).
3. Apply \( W \) to obtain the state \( \frac{1}{\sqrt{q-1}}\sum_{a \in F^*} \chi(a)|\chi\rangle_+|a\rangle_x \).
4. Apply \( \exp_{a_1} \) to the second register to obtain the state \( \frac{1}{\sqrt{q-1}}\sum_{a \in F^*} \chi(a)|\chi\rangle_+|a\rangle_x \).
5. Apply the Fourier transform \( F_{F,\psi} \) to obtain the state

\[
\frac{1}{\sqrt{q(q-1)}}\sum_{y \in F} \left( \frac{1}{\sqrt{q-1}}\sum_{a \in F^*} \chi(a)\psi(ay) \right) |\chi\rangle_+|y\rangle_x = \frac{1}{\sqrt{q(q-1)}}G_{\psi}(\chi) \sum_{y \in F^*} \chi(y^{-1})|\chi\rangle_+|y\rangle_x,
\]

where the equality follows from (4.3).
6. Apply \( \log_{a_1} \) to the second register to obtain the state \( \frac{1}{\sqrt{q(q-1)}}G_{\psi}(\chi) \sum_{y \in F^*} \chi(y^{-1})|\chi\rangle_+|y\rangle_x \).
7. Apply \( W \) to obtain the state \( \frac{1}{\sqrt{q(q-1)}}G_{\psi}(\chi) \sum_{y \in F^*} |\chi\rangle_+|y\rangle_x \).
8. Apply \( F_{q^{-1}} \) to the second register to obtain the state \( q^{-1/2}G_{\psi}(\chi)|\chi\rangle_+|1\rangle_x \).

In fact, it turns out to be slightly easier to give an algorithm that computes all Gauss sums \( G_{\psi}(\chi) \) for \( \chi \in X_F \) simultaneously, using the multiplicative rather than the additive Fourier transform to make the Gauss sums appear. In our algorithm for computing Kloosterman sums, we will use both this algorithm and the algorithm of van Dam and Seroussi. We define the \( n \)-qubit state

\[
|G_{\psi}\rangle = \frac{1}{q-1} \sum_{\chi \in X_F} G_{\psi}(\chi)|\chi\rangle_+; \tag{4.4}
\]

note that this state is normalised because of (4.2).

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Algorithm 4.2 (Compute the vector of Gauss sums). Given a finite field $F$ and a non-trivial additive character $\psi$ of $F$, this algorithm outputs the state $|G_\psi\rangle$.

1. Start with the state $|1\rangle_x = |0\rangle$.
2. Apply $F_{q-1}$ to obtain the state $\frac{1}{\sqrt{q-1}} \sum_{a \in F^\times} |a\rangle_x$.
3. Apply $\exp_{a_1}$ to obtain the state $\frac{1}{\sqrt{q-1}} \sum_{a \in F^\times} |a\rangle_+$.
4. Apply the operator $U_\psi$ defined in (3.1) to obtain the state $\frac{1}{\sqrt{q-1}} \sum_{a \in F^\times} \psi(a)|a\rangle_+$.
5. Apply $\exp_{a_1}$ to obtain the state $\frac{1}{\sqrt{q-1}} \sum_{a \in F^\times} \psi(a)|a\rangle_x$.
6. Apply $F_{q-1}$ to obtain the state $\frac{1}{q-1} \sum_{\chi \in X_d} M_\psi(\chi)|\chi\rangle_\star = \frac{1}{q-1} \sum_{\chi \in X_d} G_\psi(\chi)|\chi\rangle_\star = |G_\psi\rangle$, where the equalities follow from (2.1) and (4.4).

5. Computing Kloosterman sums

In this section, we prove Theorem 1.1 by exhibiting an algorithm for computing Kloosterman sums as in the theorem. Our algorithm is based on the following observation.

**Proposition 5.1.** Let $F$ be a finite field, let $\psi$ be a non-trivial additive character of $F$, and let $\chi$ be a multiplicative character of $F$. The multiplicative Fourier transform of the function $F^\times \rightarrow \mathbb{C}$ mapping $a$ to $K_l(\psi(a, \chi))$ is the function

$$\Gamma_\psi^\chi : X_n \rightarrow \mathbb{C}$$

$$\chi' \mapsto \sum_{a \in F^\times} K_l(\psi(a, \chi)\chi'(a)) = G_\psi(\chi\chi')G_\psi(\chi').$$

**Proof.** This is a special case of [8, Scholium 4.0.1].

We write

$$N_\chi = \sum_{a \in F^\times} |K_l(\psi(a, \chi))|^2.$$

A straightforward computation using Proposition 5.1, Parseval’s identity (i.e. the fact that the operator $(q-1)^{-1/2} M$ is unitary) and (4.1) yields

$$N_\chi = \begin{cases} q^2 - q - 1 & \text{if } \chi = 1, \\ q^2 - 2q & \text{if } \chi \neq 1. \end{cases}$$

Note that the last case cannot occur if $q = 2$, so $N_\chi$ is strictly positive.

Given a non-trivial additive character $\psi$ of $F$ and a (possibly trivial) multiplicative character $\chi$ of $F$, we define the $n$-qubit state

$$|K_l(\psi)\rangle = \frac{1}{\sqrt{N_\chi}} \sum_{a \in F^\times} K_l(\psi(a, \chi))|a\rangle. \quad (5.1)$$

We also define two $n$-qubit states

$$|G_l(\psi)\rangle = \frac{1}{\sqrt{(q-1)N_\chi}} \sum_{\chi' \in X_d} \Gamma_l(\psi(\chi')|\chi\rangle_\star,\langle \chi'|),$$

$$|\tilde{G}_l(\psi)\rangle = \frac{1}{q-1} \left( |G_l(\chi\psi)\rangle_\star + \frac{1}{\sqrt{q}} \sum_{\chi' \in X_d \setminus \{1\}} \Gamma_l(\psi(\chi')|\chi\rangle_\star) \right).$$

By Proposition 5.1, we have

$$F_{q-1} |K_l(\psi)\rangle = |G_l(\psi)\rangle,$$

or equivalently

$$F_{q-1}^\dagger |\tilde{G}_l(\psi)\rangle = |K_l(\psi)\rangle,$$

which shows that computing $|K_l(\psi)\rangle$ is equivalent to computing $|G_l(\psi)\rangle$. However, the state $|\tilde{G}_l(\psi)\rangle$ appears to be easier to obtain (starting from just $\psi$ and $\chi$) than $|G_l(\psi)\rangle$; see Algorithm 5.3 below. That having been said, the states $|G_l(\psi)\rangle$ and $|\tilde{G}_l(\psi)\rangle$ are very close for large $q$, as we will now show.
Lemma 5.2. The asymptotic behaviour of the inner product $\langle \tilde{\Gamma}^X_\psi | \Gamma^X_\psi \rangle$ as $q \to \infty$ is given by
\[
\langle \tilde{\Gamma}^X_\psi | \Gamma^X_\psi \rangle = \begin{cases} 
1 - \frac{q^2}{2} + O(q^{-5/2}) & \text{if } \chi = 1, \\
1 - \frac{2q}{p} + O(q^{-3/2}) & \text{if } \chi \neq 1.
\end{cases}
\]

Proof. First, we note that
\[
|\Gamma^X_\psi(1)|^2 = |G_\psi(\chi)G_\psi(1)|^2 = L_\chi,
\]
with $L_\chi$ as in (4.1). Furthermore, we have
\[
\sum_{\chi' \in X_F \setminus \{1\}} |\Gamma^X_\psi(\chi')|^2 = qM_\chi,
\]
where
\[
M_\chi = \begin{cases} 
q^2 - 2q & \text{if } \chi = 1, \\
q^2 - 3q + 1 & \text{if } \chi \neq 1.
\end{cases}
\]
In view of this, we define two $n$-qubit states
\[
|\Gamma^X_\psi\rangle = \frac{1}{\sqrt{L_\chi}} \Gamma^X_\psi(1)|\psi\rangle,
\]
\[
|\Gamma^X_\psi\rangle = \frac{1}{\sqrt{qM_\chi}} \sum_{\chi' \in X_F \setminus \{1\}} \Gamma^X_\psi(\chi')|\chi\rangle.
\]
The state $|\tilde{\Gamma}^X_\psi\rangle$ can be decomposed as
\[
|\tilde{\Gamma}^X_\psi\rangle = \alpha_+^X|\Gamma^X_\psi\rangle_+ + \alpha_-^X|\Gamma^X_\psi\rangle_-,
\]
where
\[
\alpha_+^X = \frac{L_\chi}{q - 1} \quad \text{and} \quad \alpha_-^X = \frac{M_\chi}{q - 1}.
\] (5.2)
Similary, the state $|\Gamma^X_\psi\rangle$ can be decomposed as
\[
|\Gamma^X_\psi\rangle = \beta_+^X|\Gamma^X_\psi\rangle_+ + \beta_-^X|\Gamma^X_\psi\rangle_-,
\]
where
\[
\beta_+^X = \sqrt{L_\chi} \quad \text{and} \quad \beta_-^X = \sqrt{qM_\chi}.
\] (5.3)
Expanding $\alpha_\pm^X$ and $\beta_\pm^X$ in power series in $q^{-1/2}$ and substituting these in the equality
\[
\langle \tilde{\Gamma}^X_\psi | \Gamma^X_\psi \rangle = \alpha_+^X \beta_+^X + \alpha_-^X \beta_-^X,
\]
we obtain the claim. \hfill \Box

We will now give an algorithm for computing $|\tilde{\Gamma}^X_\psi\rangle$ and then show how one can use amplitude amplification to transform this into an algorithm for computing $|\Gamma^X_\psi\rangle$.

**Algorithm 5.3 (Compute the state $|\tilde{\Gamma}^X_\psi\rangle$).** Given a finite field $F$, a non-trivial additive character $\psi$ of $F$ and a quantum state $|\chi\rangle_*$ encoding a (possibly trivial) multiplicative character $\chi$ of $F$, this algorithm outputs the state $|\chi\rangle_*|\tilde{\Gamma}^X_\psi\rangle$.

1. Apply Algorithm 4.2 to an auxiliary $n$-qubit register to compute the state
\[
|\chi\rangle_*|G_\psi\rangle = \frac{1}{q - 1} \sum_{\chi' \in X_F} G_\psi(\chi')|\chi\rangle_*|\chi\rangle_*.
\]
2. Apply the operator $|\chi\rangle_*|\chi\rangle_* \mapsto |\chi\rangle_*|\chi^{-1}\rangle_*$ to obtain the state
\[
\frac{1}{q - 1} \sum_{\chi' \in X_F} G_\psi(\chi')|\chi\rangle_*|\chi^{-1}\rangle_* = \frac{1}{q - 1} \sum_{\chi' \in X_F} G_\psi(\chi|\chi\rangle_*|\chi\rangle_*.
\]
3. Apply either a phase change by $-1$ (for $\chi' = 1$) or Algorithm 4.1 (for $\chi' = 1$) or Algorithm 4.1 (for $\chi' = 1$) or Algorithm 4.1 (for $\chi' = 1$) to the second register to obtain the state $|\chi\rangle_*|\Gamma^X_\psi\rangle$.
For all $\chi \in X_F$, let $Z_\psi^X$ denote the unitary operator on the second $n$-qubit register defined by the above algorithm, so we have in particular

$$Z_\psi^X |0\rangle = |\tilde{\Gamma}_\psi^X\rangle = \alpha_\chi^X |\Gamma_\psi^X\rangle_+ + \alpha_\chi^X |\Gamma_\psi^X\rangle_-.$$  

We define three angles $\theta_\chi, \omega_\chi, \rho_\chi$ by

$$\sin \theta_\chi = \alpha_\chi^X, \quad \cos \theta_\chi = \alpha_\chi^X,$$

$$\sin \omega_\chi = \beta_\chi^X, \quad \cos \omega_\chi = \beta_\chi^X,$$

$$\rho_\chi = \theta_\chi - \omega_\chi.$$

Then we have

$$|\Gamma_\psi^X\rangle = \beta_\chi^X |\Gamma_\psi^X\rangle_+ + \beta_\chi^X |\Gamma_\psi^X\rangle_- = R_\chi |\tilde{\Gamma}_\psi^X\rangle,$$

where $R_\chi$ is the unitary operator on $C|\Gamma_\psi^X\rangle_+ + C|\Gamma_\psi^X\rangle_-$ defined by the matrix $(\frac{\cos \rho_\chi}{\sin \rho_\chi} - \frac{\sin \rho_\chi}{\cos \rho_\chi})$ with respect to the basis $(|\Gamma_\psi^X\rangle_+, |\Gamma_\psi^X\rangle_-)$.

The angle $\rho_\chi$ satisfies the inequality

$$|\sin \rho_\chi| \leq \sin(2\theta_\chi)$$

or equivalently

$$|\alpha_\chi^X \beta_\chi^X - \alpha_\chi^X \beta_\chi^X| \leq 2\alpha_\chi^X \alpha_\chi^X,$$

which is not hard to verify using (5.2) and (5.3). Using the techniques of Høyer [7, §III], we therefore see that $R_\chi$ can be implemented using one application of $Z_\psi^X$, one application of $(Z_\psi^X)^\dagger$ and a number of (conditional) phase shifts.

**Algorithm 5.4** (Compute the vector of Kloosterman sums). Given a finite field $F$, a non-trivial additive character $\psi$ of $F$ and a quantum state $|\chi\rangle_+$ encoding a (possibly trivial) multiplicative character $\chi$ of $F$, this algorithm outputs the state $|\chi\rangle_+|\text{Klo}_\psi^X\rangle$.

1. Apply Algorithm 5.3 to obtain the state $|\chi\rangle_+|\tilde{\Gamma}_\psi^X\rangle$.
2. Apply the operator $R_\chi$ to the second register to obtain the state $|\chi\rangle_+|\Gamma_\psi^X\rangle$.
3. Apply $\mathcal{F}_q$ to the second register to obtain the state $|\chi\rangle_+|\text{Klo}_\psi^X\rangle$.

**6. Computing single Kloosterman sums**

In this section, we prove Theorem 1.2. As before, let $F$ be finite field, let $\psi$ be a non-trivial additive character of $F$, and let $\chi$ be a multiplicative character of $F$.

We can summarise Theorem 1.1 by saying that there exists a unitary operator $A$ acting on an $n$-qubit register such that

$$A |0\rangle = |\text{Klo}_\psi^X\rangle.$$  

Consider any $a \in F^\times$. By (5.1), the quantum amplitude of the state $|a\rangle_\chi$ in $|\text{Klo}_\psi^X\rangle$ is

$$\kappa(a) = \frac{1}{\sqrt{N_\chi}} \text{Klo}_\psi(a, \chi),$$

and our goal is to estimate $\kappa(a)$. To do this, we first initialise an auxiliary qubit $|u\rangle$ to the state $\frac{1}{\sqrt{2}}(|0\rangle + \rho|1\rangle)$, where $\rho$ is a complex number of absolute value 1 that will be chosen below. We initialise an $n$-qubit register conditionally on the auxiliary qubit $|u\rangle$ by either setting it to $|a\rangle_\chi$ (if $u = 0$) or applying the operator $A$ (if $u = 1$). This gives the state

$$\frac{1}{\sqrt{2}}(|0\rangle |a\rangle_\chi + \rho|1\rangle |\text{Klo}_\psi^X\rangle).$$
We then apply a Hadamard transform to the auxiliary qubit. In total, we obtain a unitary operator $B_{a,\rho}$ acting on a 1-qubit register and an $n$-qubit register and satisfying

$$B_{a,\rho}(|0\rangle|0\rangle) = \frac{1}{2} \left( |0\rangle|a\rangle_X + \rho(|0\rangle|K\langle\chi| + |1\rangle|a\rangle_X - \rho|1\rangle|K\langle\chi| \right).$$

The quantum amplitude of $|0\rangle|a\rangle_X$ in this state equals $(1 + \rho\kappa)/2$.

Let real numbers $d, e \in (0, 1)$ be given. Using amplitude estimation [2, Theorem 12] with parameters

$$k = \left\lceil 1 + \frac{1}{2d} \right\rceil \quad \text{and} \quad M = \left\lceil \frac{k\pi}{\sqrt{1 + e - 1}} \right\rceil,$$

we can compute an approximation to the probability amplitude

$$P_{a,\rho} = |1 + \rho\kappa(a)|^2/4$$

that has absolute error at most $e$ with probability at least $1 - d$. Here $M$ is the number of evaluations of $B_{a,\rho}$ performed by the amplitude estimation algorithm; from the choice of $M$ in (6.1), we deduce that this satisfies $M = O(1/(de))$.

We now take $\rho \in \{1, \zeta, \zeta^2\}$, where $\zeta \in \mathbb{C}$ is a primitive cube root of unity. A straightforward computation shows

$$\kappa(a) = \frac{4}{3}(P_{a,1} + \zeta^2P_{a,\zeta} + \zeta P_{a,\zeta^2}).$$

We can therefore compute an approximation to $\kappa(a)$ that has absolute error at most $4e$ with probability at least $(1 - d)^3$ from approximations of $P_{a,1}$, $P_{a,\zeta}$, $P_{a,\zeta^2}$ as above. Multiplying by $\sqrt{N}/\sqrt{q} < \sqrt{q}$, we obtain an approximation of $K\kappa(a, \chi)/\sqrt{q}$ that has absolute error at most $4e\sqrt{q}$ with probability at least $(1 - d)^3$.

For any $\delta, e \in (0, 1)$, taking $e = \epsilon/(4\sqrt{q})$ and $d = 1 - (1 - \delta)^{1/3}$, and taking $M$ as in (6.1), we obtain an approximation of $K\kappa(a, \chi)/\sqrt{q}$ that has absolute error at most $e$ with probability at least $1 - \delta$, using $O(\sqrt{q}/(d\epsilon))$ applications of the operators $B_{a,\rho}$ for $\rho \in \{1, \zeta, \zeta^2\}$. This implies Theorem 1.2.

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