Reversible entanglement in a Kerr-like interaction Hamiltonian: an integrable model

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(Dated: March 31, 2022)

An exactly soluble non-linear interaction Hamiltonian is proposed to study fundamental properties of the entanglement dynamics for a coupled non-linear oscillators. The time-evolved state is obtained analytically for initial products of two coherent and two number states and relevant informations are extracted from the dynamics of various quantities like subsystem linear and Von Neumann entropies, quadrature mean values, variances and Q-functions. We determined the re-coherence time scales and found among the interaction terms present in the Hamiltonian the one responsible for the entanglement in both cases. We identify the existence of two regimens for the entanglement dynamics in the case of initially coherent states: the short time, phase spread regimen where the entropy rises monotonically and the self-interference regimen where the entropy oscillates and re-coherence phenomenon can be observed. We also found that the break time from the first regimen to the second one becomes longer, as well as the re-coherence and reversibility times, as the Planck’s constant becomes much smaller than a typical action in phase space.

PACS numbers: 42.65.-k,03.65.Ud,42.50.-p

I. INTRODUCTION

Nowadays it is widely accepted that quantum entanglement is an essential ingredient for the implementation of quantum information processing devices [1]. It is notable that until recently quantum information science was restricted over discrete, finite dimensional Hilbert space elements, however, interest on continuous variables has been developing in various contexts: from teleportation [2], cryptography [3] to cloning [4]. This is closely connected to the recent advances of enhancing nonlinear coupling via electromagnetically induced transparency (EIT) mechanism [5] using a Bose-Einstein condensate, which has opened possibilities of strong non-linear interaction of ultra slow light pulses [6] of tiny energies belonging to different modes of electromagnetic fields [7, 8]. Such advances ranges from building a quantum logic gates based on photonic qubits without resorting to photonic crystals [9], entangling continuous variable states [3, 10], and preparing entangled states of radiation from mixed thermal states [11] in conventional media. Facing these recent developments, it is important to understand the entanglement properties of Kerr-type interaction particularly for Gaussian states, which are one of the states that can be generated by simple experimental devices like beam splitters and phase shifters.

In the present work, we have studied the entanglement properties of a model describing a bipartite system of two degrees of freedom, an integrable version of two quartic oscillators [12, 13], coupled via a Kerr-type interaction. One could think of some coupled cavities scheme with a Kerr medium [4, 14] as a possible realization of these type of interactions. The nonlinearity of each of the subsystem oscillator keep some of the properties present in one degree of freedom such as collapses and revivals of

the quadrature mean values of each field. Also, this soluble model in which the nonlinearity is present can be used to understand the role of Kerr-type nonlinearity on the entanglement dynamics of initially coherent Gaussian states. Another case we have considered is the non-classical number states [16] for the initial disentangled state, which does not entangle via Kerr-type coupling, but via usual bilinear coupling in a rotating wave approximation (RWA), and show several differences in the entanglement dynamics.

The paper is organized as follows: In section II the model is introduced and the exact solutions for initially disentangled states, in the cases of number states and coherent states, are presented. Analytical expressions for the subsystem density operators and some mean values are calculated. Section III is reserved to present the exact subsystem linear entropies for both initial states, analyze the conditions for re-coherences and recurrences and discuss the differences in the entanglement dynamics in the proposed cases. Also, we examine the differences of the short time dynamics where no interference phenomenon is possible and the longer time dynamics where such effects do affect the evolution of the subsystems. Another issue treated here is the limit where Planck’s constant becomes much smaller than the typical action in phase space. As we shall see, the two cases present distinct semi-classical behaviors. Finally in section IV we present our conclusions.

II. THE MODEL

Our theoretical model will be inspired on two field modes in a cavity with a low-loss Kerr media, described by the creation and annihilation operators $\hat{a}_k$ and $\hat{a}_k^\dagger$ ($k = 1, 2$), such that each mode has the usual non-linear
interaction term of the form $\chi \hat{a}^{\dagger 2}_k \hat{a}^2_k$. Besides this non-linear self-interaction term we add two coupling interaction between the field modes: one is the usual RWA coupling and the other is the Kerr-type non-linear interaction,

$$\hat{H}_{eff} = \sum_{k=1}^{2} \left[ \hbar \omega_k \left( \hat{a}^{\dagger 2}_k \hat{a}_k + \frac{1}{2} \right) + \hbar \lambda \left( \hat{a}^\dagger_1 \hat{a}_2 + \hat{a}^\dagger_2 \hat{a}_1 \right) + \hbar^2 g \left( \hat{a}^{\dagger 2}_1 \hat{a}^2_1 + \hat{a}^{\dagger 2}_2 \hat{a}^2_2 \right) + \hbar^2 g' \hat{a}^\dagger_1 \hat{a}_1 \hat{a}^\dagger_2 \hat{a}_2. $$

where the constants $g$ and $g'$ depend upon the third-order non-linear susceptibility $\chi$ and no losses will be considered. Since our purpose here is to have an exactly solvable model with the nonlinearities in such a way that we could trace out the effect of each interaction term on the entanglement dynamics, we will just set $g' = 2g$ and also choose the resonant case ($\omega_1 = \omega_2$) such that the Hamiltonian can be re-written in the following form:

$$\hat{H} = \hbar \omega \sum_{k=1}^{2} \left( \hat{a}^{\dagger 2}_k \hat{a}_k + \frac{1}{2} \right) + \hbar \lambda \left( \hat{a}^\dagger_1 \hat{a}_2 + \hat{a}^\dagger_2 \hat{a}_1 \right) + \hbar^2 g \left( \hat{a}^{\dagger 2}_1 \hat{a}^2_1 + \hat{a}^{\dagger 2}_2 \hat{a}^2_2 \right) + \hbar^2 g' \hat{a}^\dagger_1 \hat{a}_1 \hat{a}^\dagger_2 \hat{a}_2 \right. \quad \text{and} \quad \left. \hat{H}_0 = \hat{V}_x \right. \quad \text{and} \quad \hat{V}_y. $$

Notice that the entire Kerr type interaction is now in the form of the square of the free Hamiltonian $\hat{H}_0$, and hence the following commutation relations hold

$$\left[ \hat{H}_0, \hat{V}_x \right] = \left[ \hat{V}_y, \hat{V}_x \right] = 0. \quad \text{(2)}$$

This property is important for two reasons: first, it is associated to the presence of a constant of motion $\hat{N} = \hat{a}^\dagger_1 \hat{a}_1 + \hat{a}^\dagger_2 \hat{a}_2$; and second, it allows us to study separately the effect of each interaction term in the evolution operator on the initial state, and hence its consequences on the entanglement process. In what follows, we study the cases of initially uncorrelated number and coherent states.

### A. Analytical solution for product of number states

For the case of initially disentangled number states of the harmonic oscillators:

$$|\psi(0)\rangle = |n_1\rangle \otimes |n_2\rangle. \quad \text{(3)}$$

We can write $\hat{H}$ in Eq. (1) in terms of two new quartic oscillators diagonalizing the RWA-coupling, using the transformation proposed by Zoubi et al.\cite{19},

$$\hat{a}_1 = \frac{\hat{a}^\dagger + \hat{a}^\dagger_2}{\sqrt{2}}, \quad \hat{a}_2 = \frac{\hat{a}^\dagger_1 - \hat{a}^\dagger_1}{\sqrt{2}}. \quad \text{(4)}$$

The resulting Hamiltonian can be written in terms of the number operators $\hat{N}_k = \hat{A}^\dagger_k \hat{A}_k$, of the new oscillators as follows:

$$\hat{H} = \hbar \left( \omega_0 + \lambda \right) \left( \hat{N}_1 + \frac{1}{2} \right) + \hbar \left( \omega_0 - \lambda \right) \left( \hat{N}_2 + \frac{1}{2} \right) + \hbar^2 g \left( \hat{N}_1 + \hat{N}_2 + 1 \right)^2. \quad \text{(5)}$$

Using Eq. (4), we can connect the two different basis of the two oscillators Hilbert space by the relation

$$|n_1, n_2\rangle_a = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} c_{i,j}(n_1, n_2) |\mathcal{N} - (i + j), (i + j)\rangle_A, \quad \text{(6)}$$

where

$$c_{i,j}(n_1, n_2) = (-1)^j \binom{n_1}{i} \binom{n_2}{j} \sqrt{\binom{\mathcal{N} - (i + j)}{2^n n_1! n_2!}}. \quad \text{(7)}$$

The sub-index ‘$a$’ or ‘$A$’ indicates the bosonic representation the ket belongs to. It is to be noticed that at the right hand side of Eq. (6), only those states with the total number $\mathcal{N}$ fixed by the initial state are present. Using the Hamiltonian in the diagonal form, Eq. (5), and Eq. (4) the time-evolved state can be written as

$$|\psi(t)\rangle = e^{-i\Phi} \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} c_{i,j}(n_1, n_2) e^{2i(i+j)\lambda t} \times |\mathcal{N} - (i + j), (i + j)\rangle_A, \quad \text{(8)}$$

where the quantity $\Phi = \omega_0 t(\mathcal{N} + 1) + \lambda \mathcal{N} + h g \mathcal{N}(\mathcal{N} + 1)^2$ is a global phase. This phase factor will be relevant in Section \cite{11} where this result will be used to construct the solution for the case of coherent states. Hence, for the number states, Eq. (8) shows that the dynamics of the stationary states are completely determined by the RWA coupling and the effect of non-linear part of the Hamiltonian is only in the global phase.

After a little algebra we can re-write the above state Eq. (8) in a more instructive form:

$$|\psi(t)\rangle = \hat{\Gamma}_{12}^{(n_1)}(t) \hat{\Gamma}_{21}^{(n_2)}(t) |0, 0\rangle, \quad \text{(9)}$$

where

$$\hat{\Gamma}_{kk}^{(n)}(t) = \frac{\hat{a}_k^\dagger \cos \lambda t - \hat{a}_k^\dagger |\sin \lambda t|^n}{\sqrt{n}}. \quad \text{(9)}$$

The sub-indexes “$12$” ($“21”$) of the operators $\hat{\Gamma}$ emphasize the entanglement features of the dynamics. Here, we omitted the remaining global phase, and used the fact that the vacuum state in both representations spaces must be the same.

Now, we can extract some informations of entanglement process of number states under the action solely of
the RWA interaction. At times $T_l = \frac{T}{l + \frac{1}{2}}$, with $l$ integer, Eq. \(\text{(B)}\) gives

$$|\psi(T_l)\rangle = (-i)^{n_1+n_2}(-1)^l(n_1+n_2) |n_2, n_1\rangle,$$

indicating that the system is in a disentangled state, but not the same as the initial one. However, at times $\tau_l = \frac{T}{l}$, we recover the exact initial state (modulo an overall phase):

$$|\psi(\tau_l)\rangle = (-1)^l(n_1+n_2) |n_1, n_2\rangle.$$  \(\text{(11)}\)

Hence, for the case of initially disentangled number states, we have found two characteristic times: (a) the re-coherence times $T_l$, for the which the subsystems recover the purity of the initial state, and (b) the recurrence times $\tau_l$, when the evolved state becomes equal to the initial state. This allows us to classify $|\psi(0)\rangle$ in Eq. \(\text{(B)}\) as a *reversible state* for this particular interaction. Consequently, we may define a period of reversibility for such states: $\tau_{\text{rev}} = \frac{T}{l}$. For the special case of equal initial number states ($n_1 = n_2$), the difference between the re-coherence and recurrence times disappears and the reversibility occurs earlier, as can be seen in Eq. \(\text{(12)}\).

**B. Analytical solution for coherent states**

Consider an initially disentangled product of two coherent states as follows:

$$|\psi(0)\rangle = |\alpha_1\rangle \otimes |\alpha_2\rangle = e^{-\frac{|\alpha_1|^2}{2}} e^{-\frac{|\alpha_2|^2}{2}} \sum_{n,m} \frac{\alpha_1^n \alpha_2^m}{\sqrt{n!m!}} |n, m\rangle.$$  \(\text{(12)}\)

The time-evolved state can be obtained using the previous result Eq. \(\text{(B)}\), including the $N$-dependent phase $\Phi$

$$|\psi(t)\rangle = e^{-\frac{|\alpha_1|^2 t - |\alpha_2|^2 t}{2}} \sum_{n,m} \frac{\alpha_1^n(t) \alpha_2^m(t)}{\sqrt{n!m!}} e^{-i g t (n+m+1)} t \times$$

$$\times \hat{r}_1^{(n)}(t) \hat{r}_2^{(m)}(t) |0, 0\rangle,$$

where $\alpha_k(t) = \alpha_k e^{-i (h\omega_k + 2h \beta_k t) t}$. In this result we have already omitted a global phase.

It is easy to see that for $g = 0$ the RWA-coupling does not entangle the oscillators, since the two summations of Eq. \(\text{(B)}\) can be factorized, and each oscillator remains in a coherent form

$$|\psi(t)\rangle|_g = 0 = \hat{D}_1 |\beta_1(t)\rangle \hat{D}_2 |\beta_2(t)\rangle |0, 0\rangle = |\beta_1(t), \beta_2(t)\rangle,$$  \(\text{(14)}\)

where

$$\beta_j(t) = (\alpha_j e^{-i h \omega_j t} \cos \lambda t - i \alpha_j e^{-i h \omega_j t} \sin \lambda t),$$  \(\text{(15)}\)

and $\hat{D}_k[\beta_k(t)] = e^{i \beta_k \hat{a}_k^\dagger - i \beta_k^* \hat{a}_k}$ is the displacement operator in the phase space of the $k$-th oscillator.

Using this particular result, we can solve the general case of initially disentangled coherent states in a more intuitive way. The commutation relations \(\text{(B)}\) allow us to apply separately the piece of the evolution operator associated to the non-linear term of $\hat{H}$, Eq. \(\text{(A)}\), and use the previously derived results [Eqs. \(\text{(13, 14)}\)]. The final exact solution for the temporal evolution of two initially coherent states is given by

$$|\psi(t)\rangle = e^{-\frac{\hat{H}}{\hbar} t} e^{\frac{i}{\hbar} \hat{H} t} |\alpha_1, \alpha_2\rangle = e^{-\frac{\hat{H}^2}{2 \hbar} t} |\beta_1(t), \beta_2(t)\rangle$$

$$= e^{-\frac{\hat{H}^2}{2 \hbar} t} [\beta_1(t) e^{-\frac{i 2 g t^2}{2 \hbar}} n \sum_{n,m} e^{-i g t (n+m+1)} t \times$$

$$\times [\beta_1(t) e^{-\frac{i 2 g t^2}{2 \hbar}} n \sum_{n,m} e^{-i g t (n+m+1)} t \beta_2(t) e^{-\frac{i 2 g t^2}{2 \hbar}} m] \sqrt{n!} \sqrt{m!} |n, m\rangle.$$  \(\text{(16)}\)

Here, the expressions of $\beta_k(t)$ are the ones depicted in Eq. \(\text{(13)}\).

In order to calculate other quantities like the mean values and variances of the quadrature operators, and discuss the phenomenon of collapses and revivals, let us calculate the density operator of the system. Since we are interested in the case where the global system is isolated, the total density operator is a projector onto the state $|\psi(t)\rangle$.

$$\hat{\rho}(t) = \langle \psi(t) | \psi(t) \rangle = e^{-\frac{\hat{H}^2}{\hbar} t} \sum_{n,m,n',m'} e^{-i g t (n+m+1) (n'+m'+1)} t \times$$

$$\times \beta_1^n \beta_1^m \beta_2^{n'} \beta_2^{m'} \langle n, m | n', m' \rangle,$$

where we have defined $\Lambda = |\beta_1|^2 + |\beta_2|^2$, a constant of motion associated to the mean value of $\hat{N}$. We shall denote from here on $\beta_k(t)$ simply as $\beta_k$. We calculated the reduced density operators $\hat{\rho}_k(t)$ ($k = 1, 2$) by tracing over the undesired degree of freedom, corresponding to one of the original oscillators:

$$\hat{\rho}_1(t) = e^{-\frac{\hat{H}^2}{\hbar} t} \sum_{n,m,n',m'} \frac{\beta_1^n \beta_1^m \beta_2^{n'} \beta_2^{m'}}{\sqrt{n!} \sqrt{m!} \sqrt{n'!} \sqrt{m'!}} \langle n, m | n', m' \rangle,$$

with a similar expression for $\hat{\rho}_2(t)$. The field quadrature operators $\hat{Q}_k$ and $\hat{P}_k$ are given in terms of the creation and annihilation operators as follows:

$$\langle \hat{Q}_k \rangle = \sqrt{\frac{\hbar}{2}} \left[ \begin{array} {cc} 1 & 1 \\ -i & i \end{array} \right] \langle \hat{a}_k \rangle,$$

and analytical expressions for the mean values are given by

$$\langle \hat{Q}_k \rangle(t) = Tr_k[\hat{Q}_k \hat{\rho}_k(t)] = \sqrt{2 \hbar} Re \left[ \beta_k(t) e^{-3 \omega_k t} e^{-\frac{i}{\hbar} (1 - e^{-2 i \omega_k t})} \right],$$  \(\text{(20)}\)

where $\beta_k(t)$ is the displacement operator in the phase space of the $k$-th oscillator.
\[ \langle \hat{P}_k \rangle (t) = \text{Tr}_k [\hat{P}_k \hat{\rho}_k (t)] \]
\[ = \sqrt{2\hbar} \text{Im} \left[ \beta_k (t) e^{-3i\omega_g t} e^{-\frac{\hbar}{2} (1 - e^{-2i\omega_g t})} \right], \]

where we defined the frequency associated to the non-linear term as \( \omega_g = \hbar g \). The above expressions contain low integer fractions of the re-coherence times. But, in contrast to the case coupled to a reservoir \[21\], we will see in what follows that at the revival time the subsystem entropy goes to zero (re-coherence), and the system do come back to the initial state (recurrence) under appropriate conditions.

Another initially disentangled state that can be solved is the product of the form \( |\alpha_1 \rangle \otimes |n_2 \rangle \), where this type of analysis can be done and find the times at re-coherence allows superposition states \[22\].

III. ENTANGLEMENT PROPERTIES AND ITS SEMI-CLASSICAL BEHAVIOR.

In this section we are going to discuss the entanglement dynamics of time-evolved states found in Section II, namely, the product of number states and coherent states. We analyze the subsystem entropy, one important tool that gives a measure of entanglement for globally pure bipartite systems, and also the Husimi distribution or Q-function, in order to follow the evolution of the partial distribution in phase space. The Q-function for the global system is usually defined by:

\[ Q (\vec{q}, \vec{p}) = \frac{1}{\pi} \langle \gamma_1 \gamma_2 | \hat{\rho} (t) | \gamma_1 \gamma_2 \rangle, \]

where we choose \( \gamma_k = \frac{q_k + i p_k}{\sqrt{2\hbar}} \) \((k = 1, 2)\). Particularly, we investigate the role played by the non-linear interaction term in the Hamiltonian \[1\] which appears associated to the frequency \( \omega_g \) in the various analytical expressions derived. Moreover, we would like to find the behavior of the subsystem linear entropies in the semi-classical limit.

A. Entanglement properties for number state: linear and Von Neumann entropies.

We are interested in the calculation of both, the subsystem linear entropy (SLE) and the Von Neumann entropy, usually defined by

\[ \delta_k = 1 - \text{Tr}_k (\hat{\rho}_k^2) \]
\[ S_k = -\text{Tr}_k (\hat{\rho}_k \ln \hat{\rho}_k), \]

where the label \( k \) is associated to one of the degrees of freedom. For the initially disentangled number states Eq.(6), the reduced density operator is diagonal in the number state basis; therefore, it can be directly expressed in terms of its eigenvalues as follows

\[ \rho_k (t) = \sum_{l=0}^{n_1 + n_2} \lambda_l (t) |l\rangle \langle l|, \]

where the trace condition \( \sum \lambda_l (t) = 1 \) must be satisfied. The eigenvalues \( \lambda_l (t) \) are given by:

\[ \lambda_l (t) = \sum_{i,j=0}^{n_1} \sum_{m,m'=0}^{n_2} c_{i,m}(t) c^*_{i',m'}(t) \delta_{i=m,i'=m'} \delta_{n_1 - l, i - m} \]

FIG. 1: (a) Collapse and revival phenomenon in the evolution of the expectation value of the quadrature \( \hat{Q}_k \) and (b) the respective variance. Analytical results for \( \frac{\omega}{\omega_g} = 20, \frac{1}{\omega_g} = 2, q_{ko} = p_{ko} = 1.0, \Lambda = 4.0 \) and \( \hbar = 1 \). The same qualitative behaviors are observed for \( \langle \hat{P}_k \rangle (t) \) and \( \langle \Delta \hat{P}_k \rangle (t) \).
\[ c_{i,m}(t) = \binom{n_1}{i} \binom{n_2}{m} \sqrt{(n_1 - i + m)!(n_2 - i + m)! \over n_1!n_2!} \times (\cos \lambda t)^{n_1+n_2-(i+m)} (-\sin \lambda t)^{i+m}. \]  

(26)

Then, in terms of the reduced density operator eigenvalues, the subsystem entropies assume a familiar form:

\[ \delta_k(t) = 1 - \sum_{l=0}^{n_1+n_2} \lambda_l(t)^2 \]

\[ S_k(t) = -\sum_{l=0}^{n_1+n_2} \lambda_l(t) \ln \lambda_l(t). \]

(27)

The dependence of the eigenvalues on the periodic functions confirms the rule for the re-coherence times, Eqs.(10) and Eq.(11). To illustrate what has been discussed above, we present some simple examples:

- For \(|\psi(0)\rangle = |1, 0\rangle\) we have:
  \[ \delta_k(t) = {1 \over 2} \sin^2 2\lambda t, \]
  \[ S_k(t) = -\cos^2 \lambda t \ln (\cos^2 \lambda t) - \sin^2 \lambda t \ln (\sin^2 \lambda t). \]

(28)

- For \(|\psi(0)\rangle = |1, 1\rangle\) we have:
  \[ \delta_k(t) = \frac{\sin^2 2\lambda t}{4} (5 + 3 \cos 4\lambda t), \]
  \[ S_k(t) = -\cos^2 2\lambda t \ln (\cos^2 2\lambda t) - \sin^2 2\lambda t \ln (\sin^2 2\lambda t). \]

(29)

- For \(|\psi(0)\rangle = |2, 0\rangle\) we have:
  \[ \delta_k(t) = \frac{\sin^2 2\lambda t}{4} (13 + 3 \cos 4\lambda t), \]
  \[ S_k(t) = -\cos^4 \lambda t \ln (\cos^4 \lambda t) - \sin^4 \lambda t \ln (\sin^4 \lambda t) + \]
  \[ -\sin^2 2\lambda t \ln (\sin^2 2\lambda t). \]

(30)

These analytical results for the SLE are shown in Fig.2(a). An interesting fact is the appearance of the oscillatory structures between two successive re-coherences for particular values of \(n_k\). This is illustrated in Fig.2(b) where the SLE is plotted for \(n_1 = 3\) and several successive values of \(n_2\). We can see that both, the number of oscillations and the maximum value of SLE, increases as \(n_2\) is increased. This feature can be understood from Eq.(24), where the number of accessible states is given by \(n_1 + n_2 + 1\) which increases with increasing \(n_2\). This accounts for the increase in the maximum value, and also the subsystem is allowed to pass through many more possible mixed states. Notice that no dependence on \(\hbar\) or \(g\) is essentially left in the expression of the time-evolved state \([8]\), and the same happens to the calculated SLE’s \([Eqs. (28, 29)]\). This is the case where the non-linear interaction plays no role, and the semi-classical limit is related to the presence of many photons (large \(n_k\)).

**B. Entanglement properties for coherent states and semi-classical behavior.**

In order to study the entanglement dynamics and reversibility properties for the initial product of coherent states, we calculate the SLE defined in Eq. (23). Using previous result Eq. (16), an exact expression for this quantity can be calculated:

\[ \delta_1(t) = 1 - Tr_1 (\hat{\rho}_1^2(t)) = 1 - e^{-2|\beta_1(t)|^2} \sum_{n,m} |\beta_1(t)|^{2n} |\beta_1(t)|^{2m} \over n!m! \times e^{-4|\beta_2(t)|^2 \sin^2 [\hbar g T_1/2\pi]}. \]

(31)

In a similar fashion as in the case of the number states, we can find the re-coherence times studying the conditions under which \(\delta_1(t)\) is equal to zero, indicating that the subsystems are disentangled. A simple analysis of Eq.(31) shows that this happens in the following situations:

1. For any initial conditions excluding the vacuum state \((0, 0)\), the re-coherence times associated to the non-linear interaction are \(\hbar g T_1 = l\pi\), when the
2. For those initial conditions such that one of the arguments $\beta_k(t) = 0$, one can find other instants when SLE is zero. These conditions are associated exclusively with RWA coupling and special initial conditions. In terms of the quadratures of the initial values $\alpha_k$, using the expression (16) one can write these conditions as follows:

$$q_1 q_2 + p_1 p_2 = 0,$$

and we have disentangled states for times $t$: $t = \frac{1}{\hbar} \arctan(-\frac{q_2}{p_2})$, or $t = \frac{1}{\hbar} \arctan(\frac{p_2}{q_2})$.

Among these re-coherence times, we can identify those at which we also have recurrences by examining Eq. (16). Only the first class of re-coherences produces recurrences, and this happens when the following condition is satisfied: for a given value of $T_1$, when the ratio $\frac{a_1}{a_2}$ and $\frac{b_1}{b_2}$ are integer numbers.

Now we will choose a particular parameter sets to illustrate the semi-classical limit of the SLE, since in contrast to the case of the number states, we have an explicit dependence on $\hbar$, actually on the frequency $\omega_g = \hbar g$. This allows us to study the semi-classical behavior of the SLE. The natural parameter to measure the ‘quantumness’ of the system here is the ratio $R = \frac{\Lambda}{\Lambda}$, where $\Lambda$ (defined in Section 3) is a characteristic action in phase space. In Fig. 3 we plotted the time evolution of the SLE for several values of $R$. Here, we adopted the convention to fix the value of $\Lambda$ and vary $\hbar$ instead of the opposite way. The first thing to be noticed is the fact that all curves coincide at the short time scale, where the SLE increases monotonically until it reaches the maximum value. The maximum value of SLE depends on $R$, which increases as we let $R \ll 1$. This has to do with the increasing number of accessible states as we let the spectrum become denser, and also implies in loss of information. We will call this first regimen the ‘phase spread regimen’ in connection to the behavior of the $Q$-function of each subsystem that we shall see in what follows. After reaching the maximum value, oscillations start to happen in the SLE, which can be seen as a partial recovering of coherence, until the first re-coherence time $T_1 = \frac{\pi}{\hbar R}$ (see Eq. (12)) at which $\delta_k = 0$ and the subsystem recovers purity. This second regimen will be called ‘self-interference regimen’ since the time-evolved subsystem $Q$-function shows the phenomenon of self-interference typical of the Kerr-type nonlinearity. This is consistent with the fact that in the limit $R \to 0$ the re-coherence time goes to infinity and the initial purity will never be recovered. We will call break time $t_b$, the time at which the transition from the phase spread regimen (rising) to the self-interference (oscillating) regimen occurs. It is clear from Fig. 3 that this time increases with the inverse power of $R$, and we also expect it to go to infinity in the classical limit $R = 0$.

Let us illustrate the differences between the two regimens by means of the subsystem $Q$-function which will show the proper signatures in each regimen. In Fig. 4 we plotted $Q$-function in various instants of the phase spread regimen for the same Hamiltonian parameters used in Fig. 3 and the quantumness parameter $R = 0.025$. The sequence of plots is in the quadrature plane of the oscillator-1, beginning at $t = 0$ until $t \approx t_b$. Where the center of the initially coherent Gaussian wave-packet follows essentially the classical trajectory, as predicted by the Ehrenfest theorem, circulating around the origin and, due to the nature of the self-interaction term, the packet itself spreads in phase angle in the phase space. During the interval of time before the front of the packet reaches its tail, no re-coherences can happen.

In Fig. 5 another sequence of contour plots of $Q$-functions of the subsystem-1 shows a time evolution during the self-interference regimen in the interval of time after the break time $t_b$, until the first re-coherence time $T_1$. We also add a plot at the recurrence time ($\tau_1 = 2T_1$ for this particular set of parameters). It is remarkable the appearance of several peaks along the annular region, a signature of self-interference phenomenon where a kind of standing waves with $M$-peaks forms at times $t = T_1/M$ (for $t > t_b$), when the SLE assumes the value $\delta_k \approx 1 - \frac{1}{2} M^2$ at a local minimum. This kind of behavior is a hallmark of the self-interference regimen, an essentially quantum phenomena which is at the core of re-coherence and reversibility in this system. This sequence of phenomena phase spread and self-interference is reproduced many times latter since the entanglement process in this case is reversible. We call the attention to the fact that
IV. CONCLUSIONS

We have solved analytically the problem of two resonant RWA-interacting fields in the presence of non-linear Kerr-like interactions. The time-evolved state was exactly determined and this allowed us to identify analytically how the collapses and revivals are produced in the quadrature mean values for the initially coherent state. All properties of the entanglement dynamics have been studied for initial states which are products of both coherent and number states.

In particular, we presented all the necessary conditions to the re-coherence of the initially non-entangled number and coherent states. We have calculated the exact expressions for the subsystem entropies in both cases. Also, some conditions for the recurrence has been established for the coherent state case, related to the commensurability of the physical frequencies of the model.

We also identified, in the case of coherent initial states, two distinct regimes of entanglement: the first one (phase spread regimen) happens during the time where the initial coherent state spreads in phase angle in the phase space; whereas the second one (self-interference regimen) occurs when the phase spread state starts to self-interfere. The time at which self-interference becomes important for the evolution of this type of initial states, we call it break time ($t_b$), and it delimits the beginning of the essentially quantum processes responsible for the re-coherence. We also have shown that the self-interference of each oscillator produces the standing waves, where no single Schrödinger cat state is allowed but a mixture of Schrödinger cat states, consistent with the entanglement. Finally, we show how in the semiclassical limit ($\hbar \to 0$) both the re-coherence times and the recurrence times go away and the entanglement process becomes irreversible for all practical purposes in this type of model for Gaussian initial states.

Acknowledgments

The authors acknowledge K. F. Romero for helpful discussions and Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) (Contracts No.300651/85-6, No.146010/99-0) and Fundação de Amparo à Pesquisa de São Paulo (FAPESP) (Contract No.98/13617-4) for financial support.

APPENDIX A: A DEMONSTRATION OF MIXTURE OF SCHRÖDINGER CAT STATES

We first re-write Eq. (16) in a more compact notation

$$|\psi(t)\rangle = \sum_{n,m} c_n (\gamma_1) c_m (\gamma_2) e^{-i\gamma t (n+m)z^2} |n,m\rangle (A1)$$
where $\gamma_k = \beta_k e^{i2qth}$ and $\beta_k$ are functions of time. We are interested in the instants $t_{r,s} = \frac{r+s}{g}T$, where $r$ and $s$ are mutually prime with $r < s$. Using the discrete Fourier transform

$$e^{-i\pi n^2 \frac{r+s}{g}} = \sum_{q=0}^{l-1} a_q^{(r,s)} e^{-i2\pi n \frac{r+s}{g}}$$

(A3)

we re-write Eq. (A1) at the mentioned instants as follows:

\[
|\psi(t_{r,s})\rangle = \sum_{n,m} c_n (\gamma_1) c_m (\gamma_2) e^{-i\pi (n+m)^2} |n,m\rangle
\]

\[
= \sum_{q,p=0}^{l-1} a_q a_p \sum_{n,m} \left( \eta_{q} e^{-i2\pi m \frac{r+s}{g}} \right) c_m (\eta_{2p}) |n,m\rangle
\]

(A6)

Here $\eta_{q} = \gamma_k e^{-2\pi \frac{r+s}{g}} = \beta_k e^{-2\pi \left( \frac{r+s}{g} \right)}$, and we have defined the following cat-like states for the oscillator-1

$$|C_m\rangle = \sum_{q=0}^{l-1} a_q |\eta_{q} e^{-i2\pi m \frac{r+s}{g}}\rangle.$$  

(A8)

Then, constructing the global density operator and tracing over the oscillator-2, i.e., summing over the left over number states, we finally get the following mixture of cat states:

$$\rho_1(t_{r,s}) = \sum_{m} \xi_m |C_m\rangle \langle C_m|$$  

(A9)

$$\xi_m = \sum_{p,p'=0}^{l-1} a_p a_p' c_m (\eta_{2p}) c_m^* (\eta_{2p'}).$$  

(A10)

References:

[1] R.F. Werner, in *Quantum Information*, Springer Tracts in Modern Physics vol. 173 (Springer, Heidelberg, 2001).

[2] S.L. Braunstein and H.J. Kimble, Phys. Rev. Lett. **80**, 869 (1998); A. Furusawa et al., Science **282**, 706 (1998).

[3] T.C. Ralph, Phys. Rev. A **61**, 030303 (R) (2000); F. Grosshans and P. Grangier, Phys. Rev. Lett. **88**, 057902 (2002); Ch. Silberhorn et al., *ibid.* **88**, 167902 (2002).

[4] S.L. Braunstein et al., Phys. Rev. Lett. **86**, 4938 (2001); J. Fiurásek, *ibid.* **86**, 4942 (2001).

[5] H. Schmidt and A. Imamoğlu, Opt. Lett. **21**, 1936 (1996); *ibid.* **23**, 1936 (1998); S.E. Harris, Phys. Today **50** (7), 36 (1997); S.E. Harris, J.E. Field, and A. Imamoğlu, *ibid.*, **64**, 1107 (1990); J.P. Marangos, J. Mod. Opt. **45**, 471 (1998).

[6] L.V. Hau et al. Nature **397**, 594 (1999).

[7] M.D. Lukin and A. Imamoğlu, Phys. Rev. Lett. **84**, 1419 (2000).

[8] L. Deng, *et al.*, Phys. Rev. Lett. **88**, 143902 (2002).

[9] D. Petrosyan and G. Kurizki, Phys. Rev. A **64**, 023810 (2001).

[10] Ch. Silberhorn et al., Phys. Rev. Lett. **86**, 4267 (2001).

[11] R. Filip, *et al.*, Phys. Rev. A **65**, 043802 (2002).

[12] G. J. Milburn, *Phys. Rev. A* **33** (1), 674 (1986).

[13] G.S. Agarwal and J. Banerji, *Phys. Rev. A* **57** (5), 674 (1998).

[14] T. Opatrný and D.-G. Welsch, Phys. Rev. A **64**, 023805 (2001).

[15] M. O. Scully and M.S. Zubairy, *Quantum Optics*, Cambridge University Press, Cambridge, 1997.

[16] M. Brune, *et al.*, Phys. Rev. Lett. **65**, 976 (1990).

[17] S.G. Mokarzel, A.N. Salgueiro and M.C. Nemes, *Phys. Rev. A* **65**, 044101 (2002).

[18] This choice of parameters is important when one seeks a well defined classical counterpart for the quantum Hamiltonian whose dynamics will be associated in the limit $\hbar \to 0$.

[19] H. Zoubi, M. Orenstien, A. Ron, Phys. Rev. A **62**, 033801 (2000).

[20] G. S. Agarwal and R. R. Puri, *Phys. Rev. A* **39** (6), 2969 (1989); and references therein.

[21] G. J. Milburn and C.A. Holmes, Phys. Rev. Lett. **56**, 2237 (1986).

[22] R.M. Angelo, L. Sanz and K. Furuya, work in preparation.

[23] A more detailed study about this case where Shrödinger cat states can be generated will be given in [24].

[24] The connection of the break time with the Ehrenfest time will be given elsewhere.

[25] J. Banerji, *PRAMANA - J. Phys.* **56** (2 & 3), 267-280 (2001).

[26] P. Ehrenfest, *Z. Phys.* **45**, 455 (1927).