SIMPLE PROLONGS OF THE NON-POSITIVE PARTS OF GRADED LIE ALGEBRAS WITH CARTAN MATRIX IN CHARACTERISTIC 2

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Abstract. Over an algebraically closed field, an alternative to the method due to Kostrikin and Shafarevich was recently suggested. It produces all known simple finite-dimensional Lie algebras in characteristic $p > 2$. For $p = 2$, we investigate one of the steps of this method, interpret several other simple Lie algebras, previously known only as sums of their components, as Lie algebras of vector fields. One new series of exceptional simple Lie algebras is discovered, together with its "hidden supersymmetries".

In characteristic 2, certain simple Lie algebras are "desuperizations" of simple Lie superalgebras. Several simple Lie algebras we describe as results of generalized Cartan prolongation of the non-positive parts, relative a simplest (by declaring degree of just one pair of root vectors corresponding to opposite simple roots nonzero) grading by integers, of Lie algebras with Cartan matrix are "desuperizations" of characteristic 2 versions of complex simple exceptional vectorial Lie superalgebras. We list the Lie superalgebras (some of them new) obtained from the Lie algebras considered by declaring certain generators odd.

One of the simple Lie algebras obtained is the prolong relative to a non-simplest grading, so the classification to be obtained might be more involved than we previously thought.

1. Introduction

Hereafter, $K$ is an algebraically closed field of characteristic 2 unless indicated otherwise. For background, see [BGL1, LeP] and §2 which has less examples but a few more clarifications.

1.1. Main results of this paper. For $p = 2$ and the non-positive part of each Lie algebra $g(A)$ with indecomposable Cartan matrix $A$ (classified in [WK, BGL1]) for the simplest $\mathbb{Z}$-gradings of $g(A)$ and $A$ of size $\leq 4$, we compute the CTS prolong $(g_-, g_0)_N^\ast$. If $g(A)$ is not simple, we consider prolongs of the non-positive part of both $g(A)/\mathfrak{c}$, where $\mathfrak{c}$ is the center, and the simple derived of $g(A)/\mathfrak{c}$. We denote by $\mathbf{F}$ the desuperization functor, the one that forgets parity of the Lie superalgebra turning it into a Lie algebra (recall that $p = 2$). Considering the CTS-prolongs of the non-positive or negative parts (relative certain particular $\mathbb{Z}$-gradings) of exceptional (discovered by Weisfeiler and Kac) Lie algebras $\mathfrak{w}(3; a)/\mathfrak{c}$, $\mathfrak{w}'(3; a)/\mathfrak{c}$ and $\mathfrak{w}(4; a)$, we obtain several simple Lie algebras as desuperizations of certain Lie superalgebras.

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1This limitation is imposed by in-built weaknesses of Mathematica on which the package SuperLie we use for computer-aided studies is based. To advance without computer aid is hopeless. However, we conjecture that there are no new simple Lie algebras obtained as CTS prolongs of the non-positive parts of algebras $g(A)$ for Cartan matrices of larger size.

2Brown did not give any interpretation of the three series of Lie algebras he described only in components. For $\mathfrak{w}'(3; a)/\mathfrak{c}$, Brown [Bro] was the first to consider prolongs of its non-positive part for one of gradings corresponding to one of several Cartan matrices of $\mathfrak{w}(3; a)$. We interpret another Brown’s series, $D_4(3; \mathbb{N})$, as a desuperization of the exceptional simple vectorial Lie superalgebra $\mathfrak{v}(3; \mathbb{N}|8)$ described for constrained values of the shearing vector $\mathbb{N}$. We interpret the third Brown’s series in [BGLLS].
These simple Lie algebras are new, but not to us: they are desuperizations of the characteristic 2 analogs of certain simple exceptional complex vectorial Lie superalgebras, cf. [BGLS]. Completely new are our interpretations of these desuperizations:

(a) $\mathbf{F}(\mathfrak{mb}(3; N|8))$ as independent of parameter $a$ prolong of $\mathfrak{w}\mathfrak{t}(4; a)$,
(b) the main deform of the anti-bracket or Buttin superalgebra as the prolong of $\mathfrak{w}\mathfrak{t}^e(3; a)/\mathbb{C}$ for a certain Cartan matrix and grading of $\mathfrak{w}\mathfrak{t}(3; a)$;
(c) $\mathbf{F}(\mathfrak{v}\mathfrak{c}(3; N|8))$ which under certain restrictions on $N$ turns into the Brown algebra $D_4(3; N)$ being more general otherwise.

Completely new are the exceptional simple Lie algebra $\mathfrak{ir}(9; N)$ and its superizations $\mathfrak{ir}(3; N|6)$ and $\mathfrak{ir}(5; N|4)$; for $N = (1, \ldots, 1)$, they turn into $\mathfrak{c}_H(8)/\mathbb{C}$, $\mathfrak{c}_H(2|6)/\mathbb{C}$ and $\mathfrak{c}_H(4|4)/\mathbb{C}$, respectively.

Hidden supersymmetries of the Lie algebras $(\mathfrak{g}_-, \mathfrak{g}_0)_*,N$ are, by definition, the Lie superalgebras one can obtain from $(\mathfrak{g}_-, \mathfrak{g}_0)_*,N$ by declaring some of the generators odd. Our description of the prolongs makes the description of “hidden supersymmetries” of the prolongs obvious and explicit.

1.2. The KSh method. Over the algebraically closed fields $\mathbb{K}$ of characteristic $p \geq 7$, the Kostrikin-Shafarevich procedure for obtaining all simple finite dimensional Lie algebras consists of the following steps:

1) for input, take the two types of simple complex Lie algebras:
   a) those of the form $\mathfrak{g}(A)$ for a Cartan matrix $A$,
   b) infinite dimensional vectorial Lie algebras with polynomial coefficients;

2) among bases allowing integer structure constants select certain ones with the “smallest” constants (Chevalley bases for the algebras of the form $\mathfrak{g}(A)$ and divided powers for vectorial Lie algebras) thus getting particular $\mathbb{Z}$-forms of these complex Lie algebras;

3) tensor the $\mathbb{Z}$-forms obtained at step 2) by $\mathbb{K}$ over $\mathbb{Z}$;

4) select a simple (and finite dimensional in the vectorial Lie algebra case) subquotient, called a simple “relative” in what follows;

5) deform the results obtained at step 4);

6) classify isomorphisms between Lie algebras obtained at earlier steps.

In [L], conjectures describing ways to obtain all simple finite dimensional Lie algebras and superalgebras over an algebraically closed field $\mathbb{K}$ of characteristic 2 were offered. These conjectures were sharpened lately, but not sufficiently. In this paper, a sequel to [BGL1, BGLS, BGL2, LeP], we perform a step towards these classifications along one of the ways indicated in [L, GL]; this helps to make the conjecture more precise.

1.3. A reformulation of the procedures leading to the list conjectured by Kostrikin and Shafarevich. Let $\mathfrak{g} = \oplus_{i \in \mathbb{Z}} \mathfrak{g}_i$, $\mathfrak{g}_- = \oplus_{i \leq 0} \mathfrak{g}_i$ and $\mathfrak{g}_0 \subset \text{der}_0 \mathfrak{g}_-$ a subalgebra preserving $\mathbb{Z}$-grading of $\mathfrak{g}_-$. For $p = 5$, the Melikyan algebras are obtained by means of a generalized Cartan prolongation of another type of pairs $(\mathfrak{g}_-, \mathfrak{g}_0)$ as compared with the pairs of the input for a generalized Cartan prolongations listed in [P]. Actually, Melikyan’s examples, especially their interpretation as generalized prolongs of the non-positive part of $\mathfrak{g}(2)$, and the Yamaguchi theorem [Y] (we will recall it in §2, see also a more accessible than [Y] paper [GL]), hint at

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3We denote the exceptional algebras $\mathfrak{g}(2)$, $\mathfrak{f}(4)$, etc. by analogy with $\mathfrak{g}(n)$, and in order not to confuse with the 2nd, 4th, ith component $\mathfrak{g}_i$ of the $\mathbb{Z}$-graded algebra $\mathfrak{g} = \oplus \mathfrak{g}_i$. 
another approach to the construction of simple modular Lie algebras, the one we recall and apply in what follows, see also the steps listed in [L].

The proof of the generalized KSh conjecture for \( p > 3 \), mainly due to Premet and Strade, is based on earlier results by Block and Wilson (restricted case for \( p > 5 \)), and several other results by other authors; for a final touch and history, see [BGP].

In [GL], we offered a procedure not only leading to the same list of simple modular Lie algebras for \( p > 3 \) as produced by the KSh procedure and Melikyan’s examples, but which for \( p = 3 \), clarified several previously inexplicable examples, corrected earlier findings, and produced several new simple Lie algebras. Namely, our main ingredients are Lie algebras of the form \( \mathfrak{g}(A) \) only, while the main procedure is a generalization of Cartan prolongation procedure, either complete or — this is important! — partial:

for \( p \geq 3 \), up to deformations and passage to the derived algebras and their quotients modulo center, the simple finite dimensional modular Lie algebras are the results of the Cartan-Tanaka-Shchepochkina (CTS) prolongations (either complete or partial) of the non-positive parts (relative certain \( \mathbb{Z} \)-gradings) of the Lie algebras of the form \( \mathfrak{g}(A) \) or their derived.

Thus, instead of the two types of Lie algebras required by the KSh procedure as the input, we need only one type of Lie algebras (of the form \( \mathfrak{g}(A) \)) subjected to one type of constructions (CTS prolongations). Subsequent passage to the derived (first or second), factorization modulo center, and deforming are common features of both approaches; however, selection of isomorphisms, especially among the deforms, although common to both approaches, becomes much more involved for \( p = 3 \) and, especially, \( p = 2 \), cf. [KuCh, BLW, BLLS].

1.3.1. The list of simple modular Lie algebras related to those of the form \( \mathfrak{g}(A) \) and their “hidden supersymmetries”. For the classification of finite dimensional Lie algebras \( \mathfrak{g}(A) \) with indecomposable Cartan matrix \( A \) over algebraically closed fields \( \mathbb{K} \) of characteristic \( p > 0 \), see [WK] with corrections in [SK1] and clarifications in [BGL1]. In [BGL1], we gave precise definitions of Cartan matrix and related notions (Dynkin diagrams, Chevalley generators, and more) specific to the super and characteristic \( p > 0 \) cases, and classified finite dimensional Lie superalgebras \( \mathfrak{g}(A) \) with indecomposable Cartan matrix \( A \) over algebraically closed fields \( \mathbb{K} \) of characteristic \( p > 0 \). Each finite dimensional Lie superalgebra \( \mathfrak{g}(A) \) with an indecomposable \( A \) is either simple itself or \( \mathfrak{g}(i)/\mathfrak{c} \), where \( \mathfrak{c} \) is the center and \( i = 1 \) or 2, is simple.

The answer in the case \( p = 2 \) turned out to be very interesting: the Lie algebras of the form \( \mathfrak{g}(A) \) possess a hidden supersymmetry. More precisely,

\[
\text{Each finite dimensional Lie superalgebra } \mathfrak{g}(A) \text{ with indecomposable Cartan matrix } A \text{ can be obtained from the Lie algebra of the form } \mathfrak{g}(A) \text{ with the same } A \text{ by declaring any number of pairs (positive and respective negative) of its Chevalley generators odd.}
\]

(3) The vectorial Lie algebras possess same property (hidden supersymmetry). Moreover, (a) in order to understand what are all the analogs of orthogonal and symplectic Lie algebras for \( p = 2 \) (being interested in their prolongs such as Lie algebras of Hamiltonian or contact vector fields), we have to take into account super versions of orthogonal and symplectic Lie algebras, namely, the periplectic Lie superalgebras;

\[4\text{Actually, for } p = 2, \text{ the situation is even more involved and resembles that of simple vectorial Lie superalgebras over } \mathbb{C}, \text{ see [LS]. We consider the cases with the input Lie algebras distinct from those considered in this paper in [BGLS] [BLLS].}\]

\[5\text{For the definition of partial prolongations, algorithm including, see [Shch].}\]
(b) several of seemingly new examples we obtained as CTS prolongs are desuperizations of certain characteristic 2 analogs of exceptional simple complex vectorial Lie superalgebras.

Therefore, we have to give a necessary background concerning not only Lie algebras but Lie superalgebras as well.

1.3.2. If \( p = 2 \), other inputs are needed for the CTS-procedure in (2). For \( p = 2 \), the procedure conjecturally leading to the complete description of simple Lie algebras becomes much more complicated than (2), see [L]. In addition to the step a) in the following list of steps (4) leading to all simple finite dimensional Lie algebras for \( p > 3 \) and — conjecturally — for \( p = 3 \), we need at least the ingredients listed in other steps.

If \( p = 2 \), in steps a)–d) we should consider not only simplest \( Z \)-gradings:

A) prolongs (complete and partial) of

Aa) the non-positive parts of the Lie algebras of the form \( g(A) \) or their derived,

where \( A \) is indecomposable;

Ab) the non-positive parts of the orthogonal Lie algebra without Cartan matrix (or its first or second derived, or a central extension thereof), see [LeP];

Ac) the non-positive parts of the Shen algebra and of certain exceptional pairs

\((g_{-1}, g_0)\), where \( g_{-1} \) is a \( g_0 \)-module;

B) the results of application of the functor forgetting superstructure to the \( p = 2 \) analogs of Shchepochkina’s simple exceptional Lie superalgebras (partly listed in [BGLS]);

C) deforming the results obtained at step A) and B);

D) classification of isomorphisms between Lie algebras obtained at earlier steps

(this becomes even more involved for \( p = 2 \), cf. [BLW, KuCh]).

1.4. Related open problems and conjectures. In this paper we tackle step a) of the main conjecture (4) for the Cartan matrices of size \( \leq 4 \) and for simplest \( Z \)-gradings only (with one exception). We conjecture that prolongs of the non-positive parts of the Lie algebras with Cartan matrices of larger size and more complicated gradings (bar the above exception) return the initial algebra (as in the generic cases of the Yamaguchi’s theorem). To investigate this conjecture is an important problem.

2. Notation and the background

2.1. What Lie superalgebra in characteristic 2 is. Let us give a naive definition of a Lie superalgebra for \( p = 2 \). (For a scientific one, as a Lie algebra in the category of supervarieties, needed, for example, for a rigorous study and interpretation of odd parameters of deformations, see [LSh].) We define a Lie superalgebra as a superspace \( g = g_0 \oplus g_1 \) such that the even part \( g_0 \) is a Lie algebra, the odd part \( g_1 \) is a \( g_0 \)-module (made into the two-sided one by symmetry; more exactly, by anti-symmetry, but if \( p = 2 \), it is the same) and on \( g_1 \) a squaring (roughly speaking, the halved bracket) is defined as a map

\[
\begin{align*}
    x \mapsto x^2 & \quad \text{such that } (ax)^2 = a^2 x^2 \text{ for any } x \in g_1 \text{ and } a \in K, \\
    (x + y)^2 - x^2 - y^2 & \quad \text{is a bilinear form on } g_1 \text{ with values in } g_0.
\end{align*}
\]

(We use a minus sign, so the definition also works for \( p \neq 2 \).) The origin of this operation is as follows: If \( \text{char } K \neq 2 \), then for any Lie superalgebra \( g \) and any odd element \( x \in g_1 \), the Lie superalgebra \( g \) contains the element \( x^2 \) which is equal to the even element \( \frac{1}{2}[x, x] \in g_0 \). It is desirable to keep this operation for the case of \( p = 2 \), but, since it can not be defined in the same way, we define it separately, and then define the bracket of odd elements to be (this equation is valid for \( p \neq 2 \) as well):

\[
[x, y] := (x + y)^2 - x^2 - y^2.
\]
We also assume, as usual, that

- if \( x, y \in \mathfrak{g}_0 \), then \([x, y] \) is the bracket on the Lie algebra;
- if \( x \in \mathfrak{g}_0 \) and \( y \in \mathfrak{g}_1 \), then \([x, y] := l_r(y) = -[y, x] = -r_x(y)\), where \( l \) and \( r \) are the left and right \( \mathfrak{g}_0 \)-actions on \( \mathfrak{g}_1 \), respectively.

The Jacobi identity involving odd elements now takes the following form:

\[
[x^2, y] = [x, [x, y]] \quad \text{for any } x \in \mathfrak{g}_1, y \in \mathfrak{g}.
\]

If \( \mathbb{K} \neq \mathbb{Z}/2\mathbb{Z} \), we can replace the condition \( \ref{eq:7} \) on two odd elements by a simpler one:

\[
[x, x^2] = 0 \quad \text{for any } x \in \mathfrak{g}_1.
\]

Because of the squaring, the definition of derived algebras should be modified. For any Lie superalgebra \( \mathfrak{g} \), set \( \mathfrak{g}^{(0)} := \mathfrak{g} \) and

\[
\mathfrak{g}':=\left[\mathfrak{g},\mathfrak{g}\right]+\text{Span}\{g^2\mid g \in \mathfrak{g}_1\}, \quad \mathfrak{g}^{(i+1)}:=\left[\mathfrak{g}^{(i)},\mathfrak{g}^{(i)}\right]+\text{Span}\{g^2\mid g \in \mathfrak{g}_1^{(i)}\}.
\]

An even linear map \( r: \mathfrak{g} \rightarrow \mathfrak{gl}(V) \) is said to be a representation of the Lie superalgebra \( \mathfrak{g} \) (and the superspace \( V \) is said to be a \( \mathfrak{g} \)-module) if

\[
\begin{aligned}
r([x, y]) &= [r(x), r(y)] \quad \text{for any } x, y \in \mathfrak{g}; \\
r(x^2) &= (r(x))^2 \quad \text{for any } x \in \mathfrak{g}_1.
\end{aligned}
\]

2.1.1. Examples: Lie superalgebras preserving non-degenerate (anti-)symmetric forms. We say that two bilinear forms \( B \) and \( B' \) on a superspace \( V \) are equivalent if there is an even invertible linear map \( M: V \rightarrow V \) such that

\[
B'(x, y) = B(Mx, My) \quad \text{for any } x, y \in V.
\]

We fix some basis in \( V \) and identify a given bilinear form with its Gram matrix in this basis; we also identify any linear operator on \( V \) with its supermatrix in a fixed basis.

Then two bilinear forms (rather supermatrices) are equivalent if and only if there is an even invertible matrix \( M \) such that

\[
B' = MBMT^T, \quad \text{where } T \text{ is for transposition.}
\]

A bilinear form \( B \) on \( V \) is said to be symmetric if \( B(v, w) = B(w, v) \) for any \( v, w \in V \); a bilinear form is said to be anti-symmetric if \( B(v, v) = 0 \) for any \( v \in V \).

A homogeneous \( \mathfrak{g} \)-linear map \( F \) is said to preserve a bilinear form \( B \), if \( F \)

\[
B(Fx, y) + (-1)^{p(x)p(F)}B(x, Fy) = 0 \quad \text{for any } x, y \in V.
\]

All linear maps preserving a given bilinear form constitute a Lie sub(super)algebra \( \text{aut}_B(V) \) of \( \mathfrak{gl}(V) \) denoted \( \text{aut}_B(n) \subset \mathfrak{gl}(n) \) in matrix realization and consisting of the supermatrices \( X \) such that

\[
BX + (-1)^{p(X)}X^\text{st}B = 0,
\]

where the supertransposition \( X^\text{st} \) acts as follows (in the standard format):

\[
\text{st}: \begin{pmatrix} A & B \\ C & D \end{pmatrix} \rightarrow \begin{pmatrix} A^t & -C^t \\ B^t & D^t \end{pmatrix}.
\]

A) The case of purely even space \( V \) of dimension \( n \) over a field of characteristic \( p \neq 2 \). Every non-zero form \( B \) can be uniquely represented as the sum of a symmetric and an anti-symmetric

\footnote{Hereafter, as always in Linear Algebra in superspaces, all formulas of linear algebra defined on homogeneous elements only are supposed to be extended to arbitrary ones by linearity.}

\footnote{Hereafter, \( p \) denotes both parity defining a superstructure and the characteristic of the ground field; the context is, however, always clear.}
form and it is possible to consider automorphisms and equivalence classes of each summand separately.

If the ground field $K$ of characteristic $p > 2$ satisfies $K^2 = K$, then there is just one equivalence class of non-degenerate symmetric even forms, and the corresponding Lie algebra $\text{aut}_B(V)$ is called orthogonal and denoted $\mathfrak{o}_B(n)$ (or just $\mathfrak{o}(n)$). Non-degenerate anti-symmetric forms over $V$ exist only if $n$ is even; in this case, there is also just one equivalence class of non-degenerate antisymmetric even forms; the corresponding Lie algebra $\text{aut}_B(n)$ is called symplectic and denoted $\mathfrak{sp}_B(2k)$ (or just $\mathfrak{sp}(2k)$). Both algebras $\mathfrak{o}_B(n)$ and $\mathfrak{sp}_B(2k)$ are simple.

If $p = 2$, the space of anti-symmetric bilinear forms is a subspace of symmetric bilinear forms. Also, instead of a unique representation of a given form as a sum of an anti-symmetric and symmetric form, we have a subspace of symmetric forms and the quotient space of non-symmetric forms; it is not immediately clear what to take for a representative of a given non-symmetric form. For an answer and classification, see Lebedev’s thesis [LeD] and [Le1]. There are no new simple Lie superalgebras associated with non-symmetric forms, so we confine ourselves to symmetric ones.

Instead of orthogonal and symplectic Lie algebras we have two different types of orthogonal Lie algebras (see Theorem 2.1.1a). Either the derived algebras of these algebras or their quotient modulo center are simple if $n$ is large enough, so the canonical expressions of the forms $B$ are needed as a step towards classification of simple Lie algebras in characteristic 2 which is an open problem, and as a step towards a version of this problem for Lie superalgebras, even less investigated.

In [Le1], Lebedev showed that, with respect to the above natural equivalence of forms, the following fact takes place:

2.1.1a. Theorem ([Le1]). Let $K$ be a perfect (i.e., such that every element of $K$ has a square root) field of characteristic 2. Let $V$ be an $n$-dimensional space over $K$.

1) For $n$ odd, there is only one equivalence class of non-degenerate symmetric bilinear forms on $V$.

2) For $n$ even, there are two equivalence classes of non-degenerate symmetric bilinear forms, one — with at least one non-zero element on the main diagonal of its Gram matrix — contains $1_n$ and the other one — all its Gram matrices are zero-diagonal — contains $S_n := \text{antidiag}(1, \ldots, 1)$ and $\Pi_n$, where

$$
\Pi_n = \begin{cases} 
\begin{pmatrix}
0 & 1_k \\
1_k & 0 
\end{pmatrix} & \text{if } n = 2k, \\
\begin{pmatrix}
0 & 0 & 1_k \\
1_k & 0 & 0 \\
0 & 1 & 0 
\end{pmatrix} & \text{if } n = 2k + 1.
\end{cases}
$$

Thus, every even symmetric non-degenerate form on a superspace of dimension $n_0|n_1$ over $K$ is equivalent to a form of the shape (here: $i = 0$ or $\bar{1}$ and each $n_i$ may equal to 0),

$$
B = \begin{pmatrix} B_0 & 0 \\ 0 & B_1 \end{pmatrix}, \quad \text{where } B_i = \begin{cases} 
1_{n_i} & \text{if } n_i \text{ is odd,} \\
either 1_{n_i} \text{ or } \Pi_{n_i} & \text{if } n_i \text{ is even.}
\end{cases}
$$

In other words, the bilinear forms with matrices $1_n$ and $\Pi_n$ are equivalent if $n$ is odd and non-equivalent if $n$ is even. The Lie superalgebra preserving the bilinear form $B$ is spanned by

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8In this paper, $K$ is algebraically closed; over fields algebraically non-closed, there are more types of symmetric forms.

9Since $a^2 - b^2 = (a - b)^2$ if $p = 2$, it follows that no element can have two distinct square roots.
the supermatrices which in the standard format are of the form
\[
\begin{pmatrix}
A_0 & B_0 C^T B_1^{-1} \\
C & A_1
\end{pmatrix},
\]
where \(A_0 \in \mathfrak{o}_B(n_0), A_1 \in \mathfrak{o}_B(n_1),\) and \(C\) is arbitrary \(n_1 \times n_0\) matrix.

By analogy with the orthosymplectic Lie superalgebra \(\mathfrak{oosp}\) in characteristic 0 we call the Lie superalgebra preserving the bilinear form \(B\) ortho-orthogonal and denote \(\mathfrak{oo}_B(n_0|n_1)\); usually, for clarity, we denote it \(\mathfrak{oo}_B(n_0|n_1)\), in particular, if \(B_0 = 1_{n_0}\) and \(B_1 = \Pi_{n_1}\) we write \(\mathfrak{oo}_{\Pi}(n_0|n_1)\).

Since, as is easy to see, \(\mathfrak{oo}_{\Pi}(n_0|n_1) \simeq \mathfrak{oo}_{\Pi}(n_1|n_0)\), we do not have to consider the Lie superalgebra \(\mathfrak{oo}_{\Pi}(n_0|n_1)\) separately in many questions, unless we study Cartan prolongations where the difference between these two incarnations of one algebra is vital: For the one, the prolong is finite dimensional (the automorphism algebra of the \(p = 2\) analog of the Riemann geometry), for the other one it is infinite dimensional (analog of the Lie superalgebra of Hamiltonian vector fields).

\(\text{B) For an odd symmetric form } B \text{ on a superspace of dimension } (n_0|n_1) \text{ to be non-degenerate, we need } n_0 = n_1, \text{ and every such form } B \text{ is equivalent to } \Pi_{k|k}, \text{ where } k = n_0 = n_1, \text{ and which is same as } \Pi_{2k} \text{ if the superstructure is forgotten. This form is preserved, over } \mathbb{K} \text{ for char } \mathbb{K} \neq 2, \text{ by linear transformations with supermatrices in the standard format of the shape}
\]
\[(13) \quad \begin{pmatrix} A & C \\ D & A^T \end{pmatrix}, \quad \text{where } A \in \mathfrak{gl}(k), C = C^T \text{ and } D = -D^T.\]

The Lie superalgebra of linear maps preserving \(B\) will be referred to as periplectic, as A. Weil suggested, and denoted \(\mathfrak{pe}_B(k)\) or just \(\mathfrak{pe}(k)\).

Note that even the superdimensions of the characteristic 2 versions of the Lie (super)algebras \(\mathfrak{aut}_B(k)\) differ from their analogs in other characteristics for both even and odd forms \(B\).

\(\text{C) Observe that}

\[
\text{The fact that two bilinear forms are inequivalent does not, generally, imply that the Lie (super)algebras that preserve them are not isomorphic.}
\]

In [Le1], Lebedev proved that for the non-degenerate symmetric forms, the implication spoken about in (14) is, however, true (bar a few exceptions), and therefore we have several types of non-isomorphic Lie (super) algebras (except for occasional isomorphisms intermixing the types, e.g., \(\mathfrak{oo}_{\Pi} \simeq \mathfrak{oo}_{\Pi}^\prime\) and \(\mathfrak{oo}_{\Pi}^\prime(6|2) \simeq \mathfrak{pe}(4)\)).

The problem of describing preserved bilinear forms has two levels: we can consider linear transformations (Linear Algebra) and arbitrary coordinate changes (Differential Geometry). In the literature, both levels are completely investigated, except for the case where \(p = 2\). More precisely, the fact that the non-split and split forms of the Lie algebras that preserve the symmetric bilinear forms are not always isomorphic was never mentioned. (Although known for the Chevalley groups preserving these forms, cf. [Sl], these facts do not follow from each other since there is no analog of Lie theorem on the correspondence between Lie groups and Lie algebras.) Here we consider the Linear Algebra aspect, for the Differential Geometry related to the objects considered here, see [LeP].

2.1.1b. Known facts: The case \(p = 2\). The following facts are given for clarity: lecturing on these results during the past several years we have encountered incredulity of the listeners based on several false premises intermixed with correct statements.

With any symmetric bilinear form \(B\) the quadratic form \(Q(x) := B(x, x)\) is associated. Arf has discovered the Arf invariant — an important invariant of non-degenerate quadratic forms
in characteristic 2. Two such forms are equivalent if and only if their Arf invariants are equal, see [Dye].

The other way round, given a quadratic form \( Q \), one defines a symmetric bilinear form, called the polar form of \( Q \), by setting

\[
B_Q(x, y) = Q(x + y) - Q(x) - Q(y).
\]

The Arf invariant can not, however, be used for classification of symmetric bilinear forms because one symmetric bilinear form can serve as the polar form for two non-equivalent (and having different Arf invariants) quadratic forms. Moreover, not every symmetric bilinear form can be represented as a polar form. If \( p = 2 \), the correspondence \( Q \leftrightarrow B_Q \) is not one-to-one.

In view of (14) the statement of the next Lemma (proved in [Le1]) is non-trivial.

2.1.1c. Lemma. 1) The Lie algebras \( \mathfrak{o}_I(2k) \) and \( \mathfrak{o}_I(2)(2k) \) are not isomorphic (though are of the same dimension); the same applies to their derived algebras:

2) \( \mathfrak{o}_I(2k) \not\cong \mathfrak{o}_I(2)(2k) \), though \( \dim \mathfrak{o}_I(2k) = \dim \mathfrak{o}_I(2)(2k) \);

3) \( \mathfrak{o}_I(2k) \not\cong \mathfrak{o}_I(2)(2k) \) unless \( k = 1 \).

Based on these results, Lebedev described all the (five) possible analogs of the Poisson bracket, and (there exists just one) contact bracket. Similar results for the odd bilinear form yield a description of the anti-bracket (a.k.a. Schouten or Buttin bracket), and the (peri)contact bracket, and (there exists just one) contact bracket. Similar results for the odd bilinear form can be represented as a polar form. If \( p = 2 \), the correspondence \( Q \leftrightarrow B_Q \) is not one-to-one.

In view of (14) the statement of the next Lemma (proved in [Le1]) is non-trivial.

2.1.1c. Lemma. 1) The Lie algebras \( \mathfrak{o}_I(2k) \) and \( \mathfrak{o}_I(2)(2k) \) are not isomorphic (though are of the same dimension); the same applies to their derived algebras:

2) \( \mathfrak{o}_I(2k) \not\cong \mathfrak{o}_I(2)(2k) \), though \( \dim \mathfrak{o}_I(2k) = \dim \mathfrak{o}_I(2)(2k) \);

3) \( \mathfrak{o}_I(2k) \not\cong \mathfrak{o}_I(2)(2k) \) unless \( k = 1 \).

Based on these results, Lebedev described all the (five) possible analogs of the Poisson bracket, and (there exists just one) contact bracket. Similar results for the odd bilinear form yield a description of the anti-bracket (a.k.a. Schouten or Buttin bracket), and the (peri)contact bracket, compare [LeP] with [LSH]. The quotients of the Poisson and Buttin Lie (super)algebras modulo center — analogs of Lie algebras of Hamiltonian vector fields, and their divergence-free subalgebras — are also described in [LeP].

2.2. Analogons of functions and vector fields for \( p > 0 \).

2.2.1. Divided powers. Let us consider the supercommutative superalgebra \( \mathbb{C}[x] \) of polynomials in \( a \) indeterminates \( x = (x_1, \ldots, x_a) \), for convenience ordered in a “standard format”, i.e., so that the first \( m \) indeterminates are even and the rest \( n \) ones are odd \( (m + n = a) \). Among the integer bases of \( \mathbb{C}[x] \) (i.e., the bases, in which the structure constants are integers), there are two canonical ones, — the usual, monomial, one and the basis of divided powers, which is constructed in the following way.

For any multi-index \( \underline{r} = (r_1, \ldots, r_a) \), where \( r_1, \ldots, r_m \) are non-negative integers, and \( r_{m+1}, \ldots, r_n \) are 0 or 1, we set

\[
u_i^{(r_i)} := \prod_{i=1}^{a} u_i^{r_i} \quad \text{and} \quad u^{\underline{r}} := \prod_{i=1}^{a} u_i^{(r_i)}.
\]

These \( u^{\underline{r}} \) form an integer basis of \( \mathbb{C}[x] \). Clearly, their multiplication relations are

\[
u^{\underline{r}} \cdot u^{\underline{s}} = \prod_{i=m+1}^{n} \min(1, 2 - r_i - s_i) \cdot (-1)^{m} \cdot \sum_{m<i<s \leq a} r_j s_i \cdot \left( \frac{r + s}{\underline{r}} \right) u^{(r + s)},
\]

(15)

where \( \left( \frac{r + s}{\underline{r}} \right) := \prod_{i=1}^{m} \left( \frac{r_i + s_i}{r_i} \right) \).

In what follows, for clarity, we will write exponents of divided powers in parentheses, as above, especially if the usual exponents might be encountered as well.

Now, for an arbitrary field \( K \) of characteristic \( p > 0 \), we may consider the supercommutative superalgebra \( \mathbb{K}[u] \) spanned by elements \( u^{\underline{r}} \) with multiplication relations [15]. For any \( m \)-tuple \( \underline{N} = (N_1, \ldots, N_m) \), where \( N_i \) are either positive integers or infinity, denote (we assume that
\[ p^n = \infty \]
\[ \mathcal{O}(m; \underline{N}) := \mathbb{K}[u; \underline{N}] := \text{Span}_\mathbb{K} \left( u(\omega) \mid r_i \begin{cases} < p^{N_i} & \text{for } i \leq m, \\ = 0 \text{ or } 1 & \text{for } i > m \end{cases} \right). \]

From (15) it is clear that \( \mathbb{K}[u; \underline{N}] \) is a subalgebra of \( \mathbb{K}[u] \). The algebra \( \mathbb{K}[u] \) and its subalgebras \( \mathbb{K}[u; \underline{N}] \) are called the \textit{algebras of divided powers}; they can be considered as analogs of the polynomial algebra. An important particular case: \( \mathcal{O}(m; \underline{N}_s) := \mathbb{K}[u; \underline{N}_s] \), where \( \underline{N}_s := (1, \ldots, 1) \), is the algebra of truncated polynomials.

Only one of these numerous algebras of divided powers \( \mathcal{O}(m; \underline{N}) \) are indeed generated by the indeterminates declared: If \( N_i = 1 \) for all \( i \). Otherwise, in addition to the \( u_i \), we have to add \( u_i^{\mathrm{div}} \) for all \( i \leq m \) and all \( k_i \) such that \( 1 < k_i < N_i \) to the list of generators. Since any derivation \( D \) of a given algebra is determined by the values of \( D \) on the generators, we see that \( \text{der}(\mathcal{O}[m; \underline{N}]) \) has more than \( m \) functional parameters (coefficients of the analogs of partial derivatives) if \( N_i \neq 1 \) for at least one \( i \). Define \textit{distinguished partial derivatives} by setting
\[ \partial_i(u_j^{(k)}) = \delta_{ij}u_j^{(k-1)} \quad \text{for any } k < p^{N_i}. \]

The simple vectorial Lie algebras over \( \mathbb{C} \) have only one parameter: the number of indeterminates. If \( \text{char } \mathbb{K} = p > 0 \), the vectorial Lie algebras acquire one more parameter: \( \underline{N} \). For Lie superalgebras, \( \underline{N} \) only concerns the even indeterminates.

The Lie (super)algebra of all derivations \( \text{der}(\mathcal{O}[m; \underline{N}]) \) turns out to be not so interesting as its \textit{Lie subsuperalgebra of distinguished derivations}: Let
\[ \text{vect}(m; \underline{N}) \oplus \text{a.k.a. } W(m; \underline{N}) \oplus \text{a.k.a. } \text{der}_{\text{dist}} \mathbb{K}[u; \underline{N}] = \text{Span}_\mathbb{K} \left( u(\omega) \partial_k \mid r_i \begin{cases} < p^{N_i} & \text{for } i \leq m, \\ = 0 \text{ or } 1 & \text{for } i > m, \end{cases} 1 \leq k \leq n \right) \]
be the general vectorial Lie algebra of distinguished derivations. The next notions are analogs of the polynomial algebra of the dual space.

2.2.2. Recapitulation: On vectorial Lie superalgebras, there are TWO analogs of \textbf{trace}. More precisely, there are \textit{traces} and their Cartan prolongs, called \textit{divergencies}. On any Lie (super)algebra \( \mathfrak{g} \) over a field \( \mathbb{K} \), a \textit{trace} is any map \( \text{tr} : \mathfrak{g} \rightarrow \mathbb{K} \) such that
\[ \text{tr}([\mathfrak{g}, \mathfrak{g}]) = 0. \]

The straightforward analogs of the trace are, therefore, the linear functionals that vanish on \( \mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}] \); the number of linearly independent traces is equal to \( \dim \mathfrak{g}/\mathfrak{g}' \); if \( \mathfrak{g} \) is a Lie superalgebra, these traces are called supertraces and they can be even or odd. Each trace is defined up to a non-zero scalar factor selected \textit{ad lib}.

Let now \( \mathfrak{g} \) be a \( \mathbb{Z} \)-graded vectorial Lie superalgebra with \( \mathfrak{g}_- := \bigoplus_{i<0} \mathfrak{g}_i \) generated by \( \mathfrak{g}_{-1} \), and let \( \text{tr} \) be a (super)trace on \( \mathfrak{g}_0 \). The \textit{divergence} \( \text{div} : \mathfrak{g} \rightarrow \mathcal{F} \), where \( \mathcal{F} \) is the space of functions-coefficients, is an \( \text{ad}_{\mathfrak{g}_{-1}} \)-invariant prolongation of the trace satisfying the following conditions:
\[ \text{div} : \mathfrak{g} \rightarrow \mathcal{F} \text{ preserves the degree, i.e., } \deg \text{div} = 0; \]
\[ X_i(\text{div} D) = \text{div}[X_i, D] \text{ for all elements } X_i \text{ that span } \mathfrak{g}_{-1}; \]
\[ \text{div}|_{\mathfrak{g}_0} = \text{tr}; \]
\[ \text{div}|_{\mathfrak{g}_-} = 0. \]

By construction, the Lie (super)algebra \( \mathfrak{s}\mathfrak{g} := \text{Ker } \text{div}|_\mathfrak{g} \) of divergence-free elements of \( \mathfrak{g} \) is the complete prolong of \( (\mathfrak{g}_-, \text{Ker } \text{tr}|_{\mathfrak{g}_0}) \). This fact explains why we say that \text{div} is the prolongation of the trace.
Strictly speaking, divergences are not traces (they do not satisfy (18)) but for vectorial Lie (super)algebras they embody the idea of the trace (understood as property (18)) better than the traces. We denote the special (divergence free) subalgebra of a vectorial algebra $g$ by $sg$, e.g., $sgc(n|m)$. If there are several traces on $g_0$, there are several types of special subalgebras of $g$ and we need a different name for each.

2.3. Weisfeiler filtrations and gradings. Recall, see [LSH], that the Weisfeiler filtrations were initially used for description of simple (or primitive) transitive infinite dimensional Lie (super)algebras $L$ by selecting a maximal subalgebra $L_0$ of finite codimension. For the same reason we need these filtrations and associated gradings dealing with infinite dimensional algebras (if $N_i = \infty$ for at least one $i$).

Dealing with finite dimensional algebras, we can confine ourselves to maximal subalgebras of least codimension, or almost least, etc. Let $L_{-i}$ be a minimal $L_0$-invariant subspace strictly containing $L_0$, and $L_0$-invariant; for $i \geq 1$, set:

$$L_{-i} = [L_{-i}, L_{-i}] + L_{-i} \text{ and } L_i = \{ D \in L_{i-1} | [D, L_{-1}] \subset L_{i-1} \}.$$  

We thus get a filtration:

$$L = L_{-d} \supset L_{-d+1} \supset \cdots \supset L_0 \supset L_1 \supset \cdots$$

The $d$ in (20) is called the depth of $L$ and of the associated graded (the Weisfeiler graded) Lie superalgebra $g = \bigoplus_{-d \leq i} g_i$, where $g_i = L_i/L_{i+1}$.

2.4. What $g(A)$ is.

2.4.1. Warning: $\text{psl}$ has no Cartan matrix. The relatives of $sl$ and $\text{psl}$ that have Cartan matrices. For the most reasonable definition of Lie algebra with Cartan matrix over $\mathbb{C}$, see [K]. The same definition applies, practically literally, to Lie superalgebras and to modular Lie algebras and to modular Lie superalgebras. However, the usual sloppy practice is to attribute Cartan matrices to (usually simple) Lie (super)algebras none of which, strictly speaking, has a Cartan matrix!

Although it may look strange for those with non-super experience over $\mathbb{C}$, neither the simple modular Lie algebra $\text{psl}(pk)$, nor the simple modular Lie superalgebra $\text{psl}(a|pk + a)$, nor — in characteristic 0 — the simple Lie superalgebra $\text{psl}(a|a)$ possesses a Cartan matrix. Their central extensions, $\text{sl}(pk)$, the modular Lie superalgebra $\text{sl}(a|pk + a)$ for characteristic $p > 0$, and the Lie superalgebra $\text{sl}(a|a)$ for characteristic 0 — do not have Cartan matrix, either.

Their relatives possessing a Cartan matrix are, respectively, $\text{gl}(pk)$, $\text{gl}(a|pk + a)$, and $\text{gl}(a|a)$, and for the grading operator we take the matrix unit $E_{1,1}$.

Since all the Lie (super)algebras involved (the simple one, its central extension, the derivation algebras thereof) are often needed simultaneously (and only representatives of one of these types of Lie (super)algebras are of the form $g(A)$), it is important to have (preferably short and easy to remember) notation for each of them. For example, in addition to $\text{psl}$, $sl$, $\text{pgl}$ and $\text{gl}$, we have:

for $p = 2$: $\epsilon(7)$ is of dimension 134, then $\dim \epsilon(7)' = 133$, whereas the “simple core” is $\epsilon(7)/\epsilon$ of dimension 132;

for an analog of $\mathfrak{g}(2)$, having no Cartan matrix, see [BGLLS], where Shen’s and Brown’s descriptions are sharpened;

the orthogonal Lie algebras and their super analogs are considered in detail later.

---

\(^{10}\)If $p = 2$, the simple Lie (super)algebra may have more central extensions; these centrally extended algebras are even further, so to say, from possessing Cartan matrix.
In our main examples, \( \text{sdim} \mathfrak{g}(A)^{(i)}/\mathfrak{c} = d|\delta \) for a simple Lie (super)algebra \( \mathfrak{g}(A)^{(i)}/\mathfrak{c} \) whereas the notation \( D/d|\delta \) means that \( \text{sdim} \mathfrak{g}(A) = D|\delta \). The general formula is
\[
d = D - 2(\text{size}(A) - \text{rk}(A)) \quad \text{and} \quad i = \text{size}(A) - \text{rk}(A).
\]

2.4.2. What Cartan matrix is. Let \( A = (A_{ij}) \) be an \( n \times n \)-matrix with elements in \( \mathbb{K} \) with \( \text{rk} A = n - l \). Complete \( A \) to an \( (n+l) \times n \)-matrix \( \begin{pmatrix} A & B \\ \hline C & D \end{pmatrix} \) of rank \( n \). (Thus, \( B \) is an \( l \times n \)-matrix.)

Let the elements \( e_i^{\pm}, h_i, \) where \( i = 1, \ldots, n \), and \( d_k, \) where \( k = 1, \ldots, l \), generate a Lie superalgebra denoted \( \widetilde{\mathfrak{g}}(A, I) \), where \( I = (p_1, \ldots, p_n) \in (\mathbb{Z}/2)^n \) is a collection of parities \( (p(e_i^{\pm}) = p_i, \) the parities of the \( d_k \)'s being 0), free except for the relations
\[
[e_i^{\pm}, e_j^{\pm}] = \delta_{ij}h_i; \quad [h_i, e_j^{\pm}] = \pm A_{ij}e_j^{\pm}; \quad [d_k, e_j^{\pm}] = \pm B_{kj}e_j^{\pm}; \\
[h_i, h_j] = [h_i, d_k] = [d_k, d_m] = 0 \quad \text{for any} \ i, j, k, m.
\]

The Lie superalgebra \( \widetilde{\mathfrak{g}}(A, I) \) is \( \mathbb{Z}^n \)-graded with
\[
\text{deg} e_i^{\pm} = (0, \ldots, 0, \pm 1, 0, \ldots, 0) \\
\text{deg} h_i = \text{deg} d_k = (0, \ldots, 0) \quad \text{for any} \ i, k.
\]

Let \( \mathfrak{h} \) denote the linear span of the \( h_i \)'s and \( d_k \)'s. Let \( \widetilde{\mathfrak{g}}(A, I)^{\pm} \) denote the Lie subsuperalgebras in \( \widetilde{\mathfrak{g}}(A, I) \) generated by \( e_1^{\pm}, \ldots, e_n^{\pm} \). Then
\[
\widetilde{\mathfrak{g}}(A, I) = \mathfrak{g}(A, I)^- \oplus \mathfrak{h} \oplus \mathfrak{g}(A, I)^+,
\]
where the homogeneous component of degree \((0, \ldots, 0)\) is just \( \mathfrak{h} \).

The Lie subsuperalgebras \( \mathfrak{g}(A, I)^{\pm} \) are homogeneous in this \( \mathbb{Z}^n \)-grading, and there is a maximal homogeneous (in this \( \mathbb{Z}^n \)-grading) ideal \( \mathfrak{r} \) such that \( \mathfrak{r} \cap \mathfrak{h} = 0 \).

The ideal \( \mathfrak{r} \) is just the sum of homogeneous ideals whose homogeneous components of degree \((0, \ldots, 0)\) is trivial. As \( \text{rk} A = n - l \), there exists an \( l \times n \)-matrix \( T = (T_{ij}) \) of rank \( l \) such that
\[
TA = 0.
\]
Let
\[
c_i = \sum_{1 \leq j \leq n} T_{ij}h_j, \quad \text{where} \ i = 1, \ldots, l.
\]
Then, from the properties of the matrix \( T \), we deduce that
\begin{itemize}
  \item a) the elements \( c_i \) are linearly independent; let \( \mathfrak{c} \) be the space they span;
  \item b) the elements \( c_i \) are central, because
\end{itemize}
\[
[c_i, e_j^{\pm}] = \pm \left( \sum_{1 \leq k \leq n} T_{ik}A_{kj} \right) e_j^{\pm} = \pm (TA)_{ij}e_j^{\pm} \quad \text{and} \quad 0.
\]

The Lie (super)algebra \( \mathfrak{g}(A, I) \) is defined as the quotient \( \mathfrak{g}(A, I)/\mathfrak{r} \) and is called the Lie (super)algebra with Cartan matrix \( A \) (and parities \( I \)). Note that this coincides with the definition in \( \text{CE} \) of the contragredient Lie superalgebras, although written in a slightly different way. Condition (Z4) modified as
\[
\text{maximal homogeneous (in this Z^n-grading) ideal} \ \mathfrak{s} \ \text{such that} \ \mathfrak{s} \cap \mathfrak{h} = \mathfrak{c}
\]
leads to what in \( \text{CE} \) is called the centerless contragredient Lie superalgebra, cf. \( \text{[B3]} \).

\[\footnotesize\text{This word does not seem to mean anything in this context, and therefore this term, though often used, is ill chosen.}\]
By abuse of notation we denote by $e_i^+, h_i, d_k$ and $c$ their images in $\mathfrak{g}(A, I)$ and $\mathfrak{g}(A, I)'$.

The Lie superalgebra $\mathfrak{g}(A, I)$ inherits, clearly, the $\mathbb{Z}^n$-grading of $\mathfrak{g}(A, I)$. The non-zero elements $\alpha \in \mathbb{Z}^n \subset \mathbb{R}^n$ such that the homogeneous component $\mathfrak{g}(A, I)_\alpha$ is non-zero are called roots. The set $R$ of all roots is called the root system of $\mathfrak{g}$. Clearly, the subspaces $\mathfrak{g}_\alpha$ are purely even or purely odd, and the corresponding roots are said to be even or odd.

The additional to (22) relations that turn $\mathfrak{g}(A, I)^\pm$ into $\mathfrak{g}(A, I)^\pm$ are of the form $R_i = 0$ whose left sides are implicitly described as follows:

$$\sum_{\alpha \in \mathbb{Z}^n} \alpha = 0 \quad \text{and} \quad \sum_{\alpha \in \mathbb{Z}^n} \alpha = 0$$

(29)

For the explicit description of these additional relations, see [BGLL1].

2.4.3. Roots and weights. In this subsection, $\mathfrak{g}$ denotes one of the algebras $\mathfrak{g}(A, I)$ or $\tilde{\mathfrak{g}}(A, I)$.

The elements of $\mathfrak{h}^*$ are called weights. For a given weight $\alpha$, the weight subspace of a given $\mathfrak{g}$-module $V$ is defined as

$$V_\alpha = \{x \in V \mid \text{an integer } N > 0 \text{ exists such that } (\alpha(h) - \text{ad}_h)^N x = 0 \text{ for any } h \in \mathfrak{h}\}.$$  

Any non-zero element $x \in V$ is said to be of weight $\alpha$. For the roots, which are particular cases of weights if $p = 0$, the above definition is inconvenient: In the modular analog of the following useful statement summation should be over roots defined in the previous subsection.

2.4.3a. Statement ([K]). Over $\mathbb{C}$, the space $\mathfrak{g}$ can be represented as a direct sum of subspaces

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha.$$  

Note that $\mathfrak{h} \subsetneq \mathfrak{g}_0$ over $\mathbb{K}$, e.g., all weights of the form $p\alpha$ over $\mathbb{C}$ become 0.

2.4.4. Systems of simple and positive roots. In this subsection, $\mathfrak{g} = \mathfrak{g}(A, I)$, and $R$ is the root system of $\mathfrak{g}$.

For any subset $B = \{\sigma_1, \ldots, \sigma_m\} \subset R$, we set (we denote by $\mathbb{Z}_+$ the set of non-negative integers):

$$R_B^\pm = \{\alpha \in R \mid \alpha = \pm \sum n_i \sigma_i, \ n_i \in \mathbb{Z}_+\}.$$  

The set $B$ is called a system of simple roots of $R$ (or $\mathfrak{g}$) if $\sigma_1, \ldots, \sigma_m$ are linearly independent and $R = R_B^+ \cup R_B^-$. Note that $R$ contains basis coordinate vectors, and therefore spans $\mathbb{R}^n$; thus, any system of simple roots contains exactly $n$ elements.

Let $(\cdot, \cdot)$ be the standard Euclidean inner product in $\mathbb{R}^n$. A subset $R^+ \subset R$ is called a system of positive roots of $R$ (or $\mathfrak{g}$) if there exists $x \in \mathbb{R}^n$ such that

$$(\alpha, x) \in \mathbb{R} \setminus \{0\} \quad \text{for any } \alpha \in R,$$

(30)

$$R^+ = \{\alpha \in R \mid (\alpha, x) > 0\}.$$  

Since $R$ is a finite (or, at least, countable if $\dim \mathfrak{g}(A, I) = \infty$) set, so the set

$$\{y \in \mathbb{R}^n \mid \text{there exists } \alpha \in R \text{ such that } (\alpha, y) = 0\}$$

is a finite/countable union of $(n - 1)$-dimensional subspaces in $\mathbb{R}^n$, so it has zero measure. So for almost every $x$, condition (30) holds.

By construction, any system $B$ of simple roots is contained in exactly one system of positive roots, which is precisely $R_B^+$.  

2.4.4a. Statement. Any finite system $R^+$ of positive roots of $\mathfrak{g}$ contains exactly one system of simple roots. This system consists of all the positive roots (i.e., elements of $R^+$) that can not be represented as a sum of two positive roots.
We can not give an \textit{a priori} proof of the fact that each set of all positive roots each of which is not a sum of two other positive roots consists of linearly independent elements. This is, however, true for finite dimensional Lie algebras and superalgebras $\mathfrak{g}(A, I)$ if $p \neq 2$.

2.4.5. Normalization convention. Clearly,

$$\text{(31)} \quad \text{the rescaling } e_i^+ \mapsto \sqrt{\lambda_i}e_i^+, \text{ sends } A \rightarrow A' := \text{diag}(\lambda_1, \ldots, \lambda_n) \cdot A.$$  

Two pairs $(A, I)$ and $(A', I')$ are said to be \textit{equivalent} (and we write $(A, I) \sim (A', I')$) if $(A', I')$ is obtained from $(A, I)$ by a composition of a permutation of parities and a rescaling $A' = \text{diag}(\lambda_1, \ldots, \lambda_n) \cdot A$, where $\lambda_1 \ldots \lambda_n \neq 0$. Clearly, equivalent pairs determine isomorphic Lie superalgebras.

The rescaling affects only the matrix $A_B$, not the set of parities $I_B$. The Cartan matrix $A$ is said to be \textit{normalized} if

$$\text{(32)} \quad A_{jj} = 0 \text{ or } 1, \text{ or } 2,$$

where we let $A_{jj} = 2$ only if $p_j = 0$; in order to distinguish between the cases where $p_j = 0$ and $p_j = 1$, we write $A_{jj} = 0$ or $1$, instead of 0 or 1, if $p_j = 0$. \textbf{We will only consider normalized Cartan matrices; for them, we do not have to describe $I$.}

The row with a 0 or 0 on the main diagonal can be multiplied by any nonzero factor; usually (not only in this paper) we multiply the rows so as to make $A_B$ symmetric, if possible.

\textit{A posteriori}, for each \textbf{finite dimensional} Lie (super)algebra of the form $\mathfrak{g}(A)$ with indecomposable Cartan matrix $A$, the matrix $A$ is symmetrizable (i.e., it can be made symmetric by operation (31)) for any $p$. For affine and almost affine Lie (super)algebra of the form $\mathfrak{g}(A)$ this is not so, cf. [CCLL].

2.4.6. Equivalent systems of simple roots. Let $B = \{\alpha_1, \ldots, \alpha_n\}$ be a system of simple roots. Choose non-zero elements $e_i^\pm$ in the 1-dimensional (by definition) superspaces $\mathfrak{g}_{\pm\alpha_i}$; set $h_i = [e_i^\pm, e_i^-]$, let $A_B = (A_{ij})$, where the entries $A_{ij}$ are recovered from relations (22), and let $I_B = \{p(e_1), \ldots, p(e_n)\}$. Lemma 2.6.3b claims that all the pairs $(A_B, I_B)$ are equivalent to each other.

Two systems of simple roots $B_1$ and $B_2$ are said to be \textit{equivalent} if the pairs $(A_{B_1}, I_{B_1}) \sim (A_{B_2}, I_{B_2})$.

For the role of the “best” (first among equals) order of indices we propose the one that minimizes the value

$$\text{(33)} \quad \max_{i,j \in \{1, \ldots, n\} \text{ such that } (A_{B})_{ij} \neq 0} |i - j|$$

(i.e., gather the non-zero entries of $A$ as close to the main diagonal as possible).

2.4.7. Chevalley generators and Chevalley bases. We often denote the set of generators corresponding to a normalized matrix by $X_1^\pm, \ldots, X_n^\pm$ instead of $e_1^\pm, \ldots, e_n^\pm$; and call them, together with the elements $H_i := [X_i^+, X_i^-]$, and the derivatives $d_j$ added for convenience for all $i$ and $j$, the \textit{Chevalley generators}.

For $p = 0$ and the \textit{normalized} Cartan matrices of simple finite dimensional Lie algebras, there exists only one (up to signs) basis containing $X_i^\pm$ and $H_i$ in which $A_{ii} = 2$ for all $i$ and all structure constants are integer, cf. [ST]. Such a basis is called the \textit{Chevalley basis}.

Observe that, having normalized the Cartan matrix of $\mathfrak{o}(2n + 1)$ so that $A_{ii} = 2$ for all $i \neq n$ but $A_{nn} = 1$, we get another basis with integer structure constants. Clearly, this basis also qualifies to be called \textit{Chevalley basis}; for the Lie superalgebras, the basis normalized as in (32) is more appropriate than the one with $A_{ii} = 2$; for $p = 2$, the normalization (32) seems at the moment the only reasonable one:
2.4.8. **Conjecture.** If \( p > 2 \), then for finite dimensional Lie (super)algebras with indecomposable Cartan matrices normalized as in (32), there also exists only one (up to signs) analog of the Chevalley basis.

The analogs of Chevalley bases for \( p = 2 \) are not described yet; we conjecture that the methods of a recent paper [CR] should solve the problem.

2.5. **Ortho-orthogonal and periplectic Lie superalgebras.** In this section, \( p = 2 \) and \( \mathbb{K} \) is perfect. We also assume that \( n_0, n_1 > 0 \).

2.5.1. **Non-degenerate even supersymmetric bilinear forms and ortho-orthogonal Lie superalgebras.** For \( p = 2 \), there are, in general, four equivalence classes of inequivalent non-degenerate even supersymmetric bilinear forms on a given superspace. Any such form \( B \) on a superspace \( V \) of superdimension \( n_0|n_1 \) can be decomposed as follows:

\[
B = B_0 \oplus B_1,
\]

where \( B_0, B_1 \) are symmetric non-degenerate forms on \( V_0 \) and \( V_1 \), respectively. For \( i = 0, 1 \), the form \( B_i \) is equivalent to \( 1_{n_i} \) if \( n_i \) is odd, and equivalent to \( 1_{n_i} \circ \Pi_{n_i} \) if \( n_i \) is even. So every non-degenerate even symmetric bilinear form is equivalent to one of the following forms (some of them are defined not for all dimensions):

\[
\begin{align*}
B_{II} &= 1_{n_0} \oplus 1_{n_1}; \\
B_{III} &= \Pi_{n_0} \oplus 1_{n_1} \text{ if } n_0 \text{ is even}; \\
B_{IV} &= \Pi_{n_0} \oplus \Pi_{n_1} \text{ if } n_0, n_1 \text{ are even}.
\end{align*}
\]

We denote the Lie superalgebras that preserve the respective forms by \( \mathfrak{o}_I(n_0|n_1) \), \( \mathfrak{o}_{II}(n_0|n_1) \), \( \mathfrak{o}_{III}(n_0|n_1) \), \( \mathfrak{o}_{IV}(n_0|n_1) \), respectively. Now let us describe these algebras.

2.5.1a. **\( \mathfrak{o}_{II}(n_0|n_1) \).** If \( n \geq 3 \), then the Lie superalgebra \( \mathfrak{o}'_{II}(n_0|n_1) \) is simple. This Lie superalgebra has no Cartan matrix.

2.5.1b. **\( \mathfrak{o}_{III}(n_0|n_1) \) (\( n_1 = 2k_1 \)).** The Lie superalgebra \( \mathfrak{o}'_{III}(n_0|n_1) \) is simple, it has Cartan matrix if and only if \( n_0 \) is odd; this matrix has the following form (up to a format; all possible formats — corresponding to \( * = 0 \) or \( * = \bar{0} \) — are described in Table 7 below):

\[
\begin{pmatrix}
  \ddots & \ddots & \cdots & \ddots \\
  \ddots & * & 1 & 0 \\
  \ddots & 1 & * & 1 \\
  \cdots & 0 & 1 & 1
\end{pmatrix}
\]

(34)

In particular, the Lie algebra \( \mathfrak{g} = \mathfrak{o}(1)(2n+1) \) with Cartan matrix (33) with \( * = \bar{0} \) can be considered as the Lie algebra of matrices of the form (recall that \( ZD(n) \) is the space of symmetric matrices with zeros on the main diagonal)

\[
\begin{pmatrix}
  A & X & B \\
  Y^T & 0 & X^T \\
  C & Y & A^T
\end{pmatrix}, \quad \text{where} \quad A \in \mathfrak{gl}(n); \quad B, C \in ZD(n); \quad X, Y \text{ are column } n\text{-vectors.}
\]

(35)

2.5.1c. **\( \mathfrak{o}_{III}(n_0|n_1) \) (\( n_0 = 2k_0, n_1 = 2k_1 \)).** If \( n = n_0 + n_1 \geq 6 \), then

\[
\begin{align*}
\text{if } & k_0 + k_1 \text{ is odd, then the Lie superalgebra } \mathfrak{o}_{II}(2k_0|2k_1) \text{ is simple;} \\
\text{if } & k_0 + k_1 \text{ is even, then the Lie superalgebra } \mathfrak{o}_{II}(2k_0|2k_1)/\mathbb{K}1_{n_0|n_1} \text{ is simple.}
\end{align*}
\]

(36)

Each of these simple Lie superalgebras is also close to a Lie superalgebra with Cartan matrix. To describe this Cartan matrix Lie superalgebra in most simple terms, we will choose a slightly different realization of \( \mathfrak{o}_{III}(2k_0|2k_1) \): Let us consider it as the algebra of linear transformations
that preserve the bilinear form $\Pi_{2k_0+2k_1}$ in the supermatrix format $k_0|k_1|k_0|k_1$. Then the algebra $\mathfrak{o}(i)(2k_0|2k_1)$ is spanned by supermatrices of format $k_0|k_1|k_0|k_1$ and of the form

$$A \in \begin{cases} \mathfrak{gl}(k_0|k_1) & \text{if } i \leq 1, \\ \mathfrak{sl}(k_0|k_1) & \text{if } i \geq 2, \\ \text{symmetric matrices} & \text{if } i = 0; \\ \text{symmetric zero-diagonal matrices} & \text{if } i \geq 1. \end{cases}$$

(37) $\begin{pmatrix} A & C \\ D & A^T \end{pmatrix}$

where $C, D$ are $\mathfrak{g}(i)$.

If $i \geq 1$, these derived algebras have a non-trivial central extension given by the following cocycle:

$$F\left(\begin{pmatrix} A & C \\ D & A^T \end{pmatrix}, \begin{pmatrix} A' & C' \\ D' & A'^T \end{pmatrix}\right) = \sum_{1 \leq i < j \leq k_0 + k_1} (C_{ij}D'_{ij} + C'_{ij}D_{ij})$$

(note that this expression resembles $\frac{1}{2} \text{tr}(CD' + C'D)$). We will denote this central extension of $\mathfrak{o}(i)(2k_0|2k_1)$ by $\mathfrak{ooc}(i, 2k_0|2k_1)$.

Let

$$I_0 := \text{diag}(1_{k_0|k_1}, 0_{k_0|k_1}).$$

Then the corresponding Cartan matrix Lie superalgebra is

$$\mathfrak{ooc}(2, 2k_0|2k_1) \cong K I_0 \quad \text{if } k_0 + k_1 \text{ is odd};$$

$$\mathfrak{ooc}(1, 2k_0|2k_1) \cong K I_0 \quad \text{if } k_0 + k_1 \text{ is even}.$$ (40)

2.5.2. The non-degenerate odd supersymmetric bilinear forms. Periplectic Lie superalgebras. In this subsection, $m \geq 3$.

If $m$ is odd, then the Lie superalgebra $\mathfrak{pc}(2)_B(m)$ is simple;

$$\mathfrak{pc}(2)_B(m)/\mathfrak{k}1_{m|m}$$

If $m$ is even, then the Lie superalgebra $\mathfrak{pc}(2)_B(m)/\mathfrak{k}1_{m|m}$ is simple.

If we choose the form $B$ to be $\Pi_{m|m}$, then the algebras $\mathfrak{pc}(i)_B(m)$ consist of matrices of the form (37); the only difference from $\mathfrak{o}(i)(2m|2m)$ is the format which in this case is $m|m$.

Each of these simple Lie superalgebras has a 2-structure. Note that if $p \neq 2$, then the Lie superalgebra $\mathfrak{pc}_B(m)$ and its derived algebras are not close to Cartan matrix Lie superalgebras (because, for example, their root system is not symmetric). If $p = 2$ and $m \geq 3$, then they are close to Cartan matrix Lie superalgebras; here we describe them.

The algebras $\mathfrak{pc}(i)_B(m)$, where $i > 0$, have non-trivial central extensions with cocycles (38); we denote these central extensions by $\mathfrak{pec}(i, m)$. Let us introduce another matrix

$$I_0 := \text{diag}(1_m, 0_m).$$

(43)
Then the Cartan matrix Lie superalgebras are

\[
\begin{align*}
\text{pec}(2, m) \ltimes \mathbb{K}I_0 & \text{ if } m \text{ is odd; } \\
\text{pec}(1, m) \ltimes \mathbb{K}I_0 & \text{ if } m \text{ is even. }
\end{align*}
\]

The corresponding Cartan matrix has the form (41); the only condition on its format is that the last two simple roots must have distinct parities. The corresponding Dynkin diagram is shown in Table 7, all its nodes, except for the “horns”, may be both \(\otimes\) or \(\odot\), see (51).

2.5.3. **Superdimensions.** The following expressions (with a + sign) are the superdimensions of the relatives of the ortho-orthogonal and periplectic Lie superalgebras that possess Cartan matrices. To get the superdimensions of the simple relatives, one should replace +2 and +1 by −2 and −1, respectively, in the two first lines and the four last ones:

\[
\begin{align*}
\text{dim } \mathfrak{o}(1; 2k) \ltimes \mathbb{K}I_0 & = 2k^2 - k \pm 2 & \text{ if } k \text{ is even; } \\
\text{dim } \mathfrak{o}(2; 2k) \ltimes \mathbb{K}I_0 & = 2k^2 - k \pm 1 & \text{ if } k \text{ is odd; } \\
\text{dim } \mathfrak{o}'(2k + 1) & = 2k^2 + k \\
\text{sdim } \mathfrak{o}'(2k_0 + 1|2k_1) & = 2k_0^2 + k_0 + 2k_1^2 + k_1 = 2k_0^2 + (2k_0 + 1) \\
\text{sdim } \mathfrak{ooc}(1; 2k_0|2k_1) \ltimes \mathbb{K}I_0 & = 2k_0^2 - k_0 + 2k_1^2 - k_1 \pm 2 | 4k_0k_1 & \text{ if } k_0 + k_1 \text{ is even; } \\
\text{sdim } \mathfrak{ooc}(2; 2k_0|2k_1) \ltimes \mathbb{K}I_0 & = 2k_0^2 - k_0 + 2k_1^2 - k_1 \pm 1 | 4k_0k_1 & \text{ if } k_0 + k_1 \text{ is odd; } \\
\text{sdim } \text{pec}(1; m) \ltimes \mathbb{K}I_0 & = m^2 \pm 2 | m^2 - m & \text{ if } m \text{ is even; } \\
\text{sdim } \text{pec}(2; m) \ltimes \mathbb{K}I_0 & = m^2 \pm 1 | m^2 - m & \text{ if } m \text{ is odd. }
\end{align*}
\]

2.5.4. **An example.** Let us explain why the simple Lie algebras like \(\mathfrak{psl}(np)\) over \(\mathbb{K}\) of characteristic \(p > 0\) does not have Cartan matrix and how its “too small” toral subalgebra (i.e., a subalgebra of diagonal matrices) should be fixed (enlarged so that the enlarged algebra would possess a Cartan matrix.

Consider the case of orthogonal Lie algebra as most complicated one. Let size \(A = k\), i.e., consider orthogonal \(2k \times 2k\)-matrices. The Chevalley generators are:

\[
\begin{align*}
e^+_i & = E^{i,i+1} + E^{k+i+1,k+i}, & \text{for } i = 1, \ldots, k - 1; \\
e^+_k & = E^{k-1,2k} + E^{k,2k-1} \\
e^-_i & = (e^+_i)^T, & \text{for } i = 1, \ldots, k.
\end{align*}
\]

Let us start with \(k = 2n + 1\) and the algebra \(\mathfrak{o}^{(2)}_n(4n + 2)\). The Cartan matrix has rank \(k - 1\) in this case; the degeneration is caused by the fact that two last rows are the same. This means that the element \(h_{k-1} - h_k\) is central in \(\mathfrak{g}(A)\); also, this element belongs to \(\mathfrak{g}'(A)\) since \(h_i = [e^+_i, e^-_i]\). But in the orthogonal algebra we have

\[
[e^+_{k-1}, e^-_{k-1}] = [e^+_k, e^-_k] = E^{k-1,k-1} + E^{k,k} + E^{2k-1,2k-1} + E^{2k,2k}.
\]

So we essentially have \(h_{k-1} = h_k\) in the “non-fixed” algebra. To fix this, we need to construct a non-trivial central extension of \(\mathfrak{o}^{(2)}_n(4n + 2)\) such that \([e^+_{k-1}, e^-_{k-1}] - [e^+_k, e^-_k]\) is the central element. The extension \(\mathfrak{o}(2; 4n + 2)\) satisfies this property.

The Lie algebra \(\mathfrak{g}(A)\) also contains an additional grading element \(d_1\) such that its action is determined by a row we add to \(A\) for it to have rank \(k\). We can choose \((0, \ldots, 0, 1)\) as such a row, i.e., we have

\[
[d_1, e^+_i] = 0 \text{ for } i = 1, \ldots, k - 1; \quad [d_1, e^+_k] = e^+_k.
\]

It is easy to check that the matrix \(I_0 = \text{diag}(1,k,0_k)\) acts in exactly this way. So \(\mathfrak{g}(A)\) is isomorphic to \(\mathfrak{o}(2; 4n + 2) \ltimes \mathbb{K}I_0\).

Now let us consider the case \(k = 2n\). Let us start with \(\mathfrak{o}^{(2)}_n(4n)\) again (not \(\mathfrak{o}'_n(4n)\)). In this case the matrix \(A\) has rank \(k - 2\). One degeneration is again two last rows being equal; the other one is that the sum of all odd-numbered rows is equal to 0. So again, first we move from
$o_{II}^{(2)}(4n)$ to its central extension $o(2,4n)$. Fortunately, we do not need to add another central element, the corresponding sum of $h_i$ is already central in the algebra: $\sum_{1 \leq i \leq n} [\epsilon_{2i-1}^{+}, \epsilon_{2i-1}^{-}] = 1_{4n}$.

Now we need to add two grading elements determined by two rows we add to $A$ to make the rank of the enlarged matrix equal to $k$. We can choose the first row to be $(0, \ldots, 0, 1)$ again, $d_1$ is $I_0$ again. We can choose $(1, 0, \ldots, 0)$ as the second row, and the needed action coincides with the action of the matrix $E_{1,1}^{1,k} + E_{k+1,k+1}^{k+1,k+1}$. This is one of the matrices present in $s_{II}^{(2)}(4n)$ but absent in $s_{II}^{(2)}(4n)$ (since its trace is non-zero), so by adding it to the algebra we just get $o(1;4n) \ltimes \mathbb{K}I_0$ from $o(2;4n) \ltimes \mathbb{K}I_0$.

2.5.5. Summary: The types of Lie superalgebras preserving non-degenerate symmetric forms. In addition to the isomorphisms $o_{II}(a|b) \simeq o_{II}(b|a)$, there is the only “occasional” isomorphism intermixing the types of Lie superalgebras preserving non-degenerate symmetric forms: $o_{II}(6|2) \simeq \mathfrak{p}c(4)$.

Let $\hat{g} := g \ltimes \mathbb{K}I_0$. We have the following types of non-isomorphic Lie (super)algebras:

| No Relative Has Cartan Matrix | With Cartan Matrix |
|-------------------------------|-------------------|
| $o_{II}(2n + 1|2m + 1)$, $o_{II}(2n + 1|2m)$ | $o(\hat{i};2n)$, $o'(2n + 1)$; $\mathfrak{p}c(\hat{i};k)$ |
| $o_{II}(2n|2m)$, $o_{II}(2n|2m)$; $o(2n)$ | $o(\hat{i};2n|2m)$, $o_{II}(2n + 1|2m)$ |

2.5.5a. On various versions of the orthogonal Lie algebra, and its prolong, for $p = 2$. Let us begin with $g = \mathfrak{h}(2n)$ and $\mathfrak{h}(0|m)$, and for $p = 0$ for simplicity. Both these algebras can be realized on generating functions (in even and odd indeterminates, respectively) with the well-known brackets. The component of Lie-degree 0 (in the standard $\mathbb{Z}$-grading of $g$) is spanned by monomials of degree 2 and is isomorphic to $\mathfrak{sp}(2n)$ for $\mathfrak{h}(2n)$ and $\mathfrak{o}(m)$ for $\mathfrak{h}(0|m)$. If we forget the parity of the indeterminates for a moment and look at the basis of $g_0$, the only difference between $\mathfrak{sp}(2n)$ and $\mathfrak{o}(m)$ is in the fact that the generating functions of the basis elements of $\mathfrak{sp}(2n)$ contain squares of the indeterminates, whereas the generating functions of the basis elements of $\mathfrak{o}(m)$ do not contain squares.

Revenons à nos moutons, i.e., to $p = 2$ and Lie algebras (no super!). In this case, as A. Lebedev explained in $[\text{LeL}] [\text{LeP}]$, the what he denoted by $\mathfrak{o}$ with various sub- and superscripts looks more like the good old $\mathfrak{sp}$, whereas both $\mathfrak{o}_{I}'$ and $\mathfrak{o}_{II}'$ are the true analogs of the usual $\mathfrak{o}$. Indeed: as modules over themselves and for $p \neq 2$, we have $\mathfrak{sp}(W) = S^2(W)$, whereas $\mathfrak{o}(V) = E^2(V)$; while for $p = 2$, we have $S^2(V) \supset E^2(V)$.

Consider the Cartan prolongs of the pairs $(V, \mathfrak{o}(V))$ and $(V, \mathfrak{o}'(V))$ and realize these prolongs by generating functions. We see that $(V, \mathfrak{o}(V))_{*,N}$ and $(V, \mathfrak{o}'(V))_{*,N}$ resemble $\mathfrak{h}(2n)$ and $\mathfrak{h}(0|m)$, respectively.

But the second case can be also interpreted as follows: we declared $N_i = 1$ for all coordinates of the shearing vector $N$. Observe that here we are talking about the shearing parameter for generating functions! The shearing parameter in the realization of the elements of the algebra by vector fields does not demonstrate this effect, cf. $[\text{LeP}]$.

One can also take an intermediate road: set $N_i = 1$ for SOME $i$, setting $N_i > 1$ for the remaining values of $i$. Then $g_0$ becomes isomorphic to something in-between $\mathfrak{o}$ and $\mathfrak{o}'$: In terms of generating functions, we add (divided) squares of those indeterminates $x_i$ for which $N_i > 1$. In particular, if such an indeterminate is unique, then $\mathfrak{o}'$ is augmented by ONE element only.

The cases with restrictions on the coordinates of the shearing vector of the form $N_i = 1$ for some $i$ can also be interpreted as certain analogs of divergence. We will need several of them. There are two types of Cartan prolongs of the derived orthogonal Lie algebras $\mathfrak{o}_B^{(1)}$. These prolongs — ‘little” Hamiltonian Lie algebras, $\mathfrak{h}_I(n; \overline{N}^2)$ and $\mathfrak{h}_II(2k; \overline{N}^2)$ — consist of vector
fields $A = \sum_{1 \leq i \leq n} A_i \partial_i$, elements of the “full” Lie algebras $\mathfrak{h}_f(n; \underline{N})$ and $\mathfrak{h}_\Pi(2k; \underline{N})$, satisfying the following conditions:

\begin{align}
\text{for } \mathfrak{o}_f^{(1)}(n): & \quad \partial_i A_i = 0 \text{ for all } i = 1, \ldots, n; \\
\text{for } \mathfrak{o}_\Pi^{(1)}(2k): & \quad \partial_i A_{k+i} = \partial_{k+i} A_i = 0 \text{ for all } i = 1, \ldots, k.
\end{align}

(48)

There is also $\mathfrak{sl}_\Pi(2k)$, the Cartan prolong of the second derived Lie algebra $\mathfrak{o}_\Pi^{(2)}(2k)$ consisting of divergence-free elements of $\mathfrak{h}_\Pi(2k; \underline{N})$. In [ILL], we set:

\begin{align}
\mathfrak{h}_f(n; \underline{N}) := \langle \text{id, } \mathfrak{o}_f(n) \rangle_{\underline{N}}; & \quad \mathfrak{h}_S(n; \underline{N}) := \langle \text{id, } \mathfrak{o}_S(n) \rangle_{\underline{N}}; \\
\mathfrak{h}_f(n) := \langle \text{id, } \mathfrak{c}(\mathfrak{o}_f(1)(n)) \rangle_\star; & \quad \mathfrak{h}_S(n) := \langle \text{id, } \mathfrak{c}(\mathfrak{o}_S(1)(n)) \rangle_\star,
\end{align}

(49)

where $\mathfrak{h}$ from [ILL] is the same as $\mathfrak{sl}_\Pi$ in [LeP]. Now, denote by $\mathbf{F}(\mathfrak{l}(n; \underline{N}|n))$ the subalgebra of “half-divergence”-free Hamiltonian vector fields, see subsec. 3.6 of [ILL] and eq. (2.15) of [LeP]:

\begin{align}
\langle \text{id, } \mathfrak{o}_\Pi^{(1)}(2n) \rangle_\star := \{ H_f \mid \sum \frac{\partial^2 f}{\partial \alpha \partial \beta} = 0, \quad \underline{N} = \underline{N}_s \}.
\end{align}

(50)

2.6. Dynkin diagrams. A usual way to represent simple Lie algebras over $\mathbb{C}$ with integer Cartan matrices is via graphs called, in the finite dimensional case, Dynkin diagrams. The Cartan matrices of certain interesting infinite dimensional simple Lie superalgebras $\mathfrak{g}$ (even over $\mathbb{C}$) can be non-symmetrizable or have entries belonging to the ground field $\mathbb{K}$. Still, it is always possible to assign an analog of the Dynkin diagram to each (modular) Lie (super)algebra with Cartan matrix, provided the edges and nodes of the graph (Dynkin diagram) are rigged with an extra information. Although these analogs of the Dynkin graphs are not uniquely recovered from the Cartan matrix (and the other way round), they give a graphic presentation of the Cartan matrices and help to observe some hidden symmetries.

Namely, the Dynkin diagram of a normalized $n \times n$ Cartan matrix $A$ is a set of $n$ nodes connected by multiple edges, perhaps endowed with an arrow, according to the usual rules ([K]) or their modification, most naturally and unambiguously formulated by Serganova: compare [ELS] with vague definitions in [WK, FSS]. In what follows, we recall these rules, and further improve them to fit the modular case.

2.6.1. Nodes. To every simple root there corresponds

\begin{align}
\text{a node } \circ & \quad \text{if } p(\alpha_i) = 0 \text{ and } A_{ii} = 2; \\
\text{a node } \ast & \quad \text{if } p(\alpha_i) = 0 \text{ and } A_{ii} = 1; \\
\text{a node } \bullet & \quad \text{if } p(\alpha_i) = 1 \text{ and } A_{ii} = 1; \\
\text{a node } \bigcirc & \quad \text{if } p(\alpha_i) = 1 \text{ and } A_{ii} = 0; \\
\text{a node } \odot & \quad \text{if } p(\alpha_i) = 0 \text{ and } A_{ii} = 0.
\end{align}

(51)

The Lie algebras $\mathfrak{sl}(2)$ and $\mathfrak{o}(3)'$ with Cartan matrices $(2)$ and $(\overline{1})$, respectively, and the Lie superalgebra $\mathfrak{osp}(1|2)$ with Cartan matrix $(1)$ are simple.

The Lie algebra $\mathfrak{gl}(2)$ with Cartan matrix $(0)$ and the Lie superalgebra $\mathfrak{gl}(2|2)$ with Cartan matrix $(0)$ are solvable of dim $4$ and $\text{sdim } 2|2$, respectively. Their derived algebras are the Heisenberg algebra $\mathfrak{hei}(2) := \mathfrak{hei}(2|0) \simeq \mathfrak{sl}(2)$ and the Heisenberg superalgebra $\mathfrak{hei}(0|2) \simeq \mathfrak{sl}(1|1)$ of (super)dimension $3$ and $1|2$, respectively.

2.6.1a. Remark. A posteriori (from the classification of simple Lie superalgebras with Cartan matrix and of polynomial growth) we find out that for $p = 0$, the simple root $\circ$ can only occur if $\mathfrak{g}(A, I)$ grows faster than polynomially. Thanks to classification again, if $\text{dim } \mathfrak{g} < \infty$, the
simple root $\odot$ can not occur if $p > 3$; whereas for $p = 3$, the Brown Lie algebras are examples of $\mathfrak{g}(A)$ with a simple root of type $\odot$; for $p = 2$, such roots are routine.

2.6.2. Edges. If $p = 2$ and $\dim \mathfrak{g}(A) < \infty$, the Cartan matrices considered are symmetric. If $A_{ij} = a$, where $a \neq 0$ or 1, then we rig the edge connecting the $i$th and $j$th nodes by a label $a$.

If $p > 2$ and $\dim \mathfrak{g}(A) < \infty$, then $A$ is symmetrizable, so let us symmetrize it, i.e., consider $DA$ for an invertible diagonal matrix $D$. Then, if $(DA)_{ij} = a$, where $a \neq 0$ or $-1$, we rig the edge connecting the $i$th and $j$th nodes by a label $a$.

If all off-diagonal entries of $A$ belong to $\mathbb{Z}/p$ and their representatives are selected to be non-positive integers, we can draw the Dynkin diagram as for $p = 0$, i.e., connect the $i$th node with the $j$th one by max$(|A_{ij}|, |A_{ji}|)$ edges rigged with an arrow $>$ pointing from the $i$th node to the $j$th if $|A_{ij}| > |A_{ji}|$ or in the opposite direction if $|A_{ij}| < |A_{ji}|$.

2.6.3. Reflections. Let $R^+$ be a system of positive roots of Lie superalgebra $\mathfrak{g}$, and let $B = \{\sigma_1, \ldots, \sigma_n\}$ be the corresponding system of simple roots with some corresponding pair $(A = A_B, I = I_B)$. Then the set $(R^+ \setminus \{\sigma_k\}) \bigsqcup \{-\sigma_k\}$ is a system of positive roots for any $k \in \{1, \ldots, n\}$. This operation is called the reflection in $\sigma_k$; it changes the system of simple roots by the formulas

$$r_{\sigma_k}(\sigma_j) = \begin{cases} -\sigma_j & \text{if } k = j, \\ \sigma_j + B_{kj}\sigma_k & \text{if } k \neq j, \end{cases}$$

where

$$B_{kj} = \begin{cases} -\frac{2A_{kj}}{A_{kk}} & \text{if } p_k = 0, \ A_{kk} \neq 0, \text{ and } -\frac{2A_{kj}}{A_{kk}} \in \mathbb{Z}/p\mathbb{Z}, \\ p - 1 & \text{if } p_k = 0, \ A_{kk} \neq 0 \text{ and } -\frac{2A_{kj}}{A_{kk}} \notin \mathbb{Z}/p\mathbb{Z}, \\ -\frac{A_{kj}}{A_{kk}} & \text{if } p_k = 1, \ A_{kk} \neq 0, \text{ and } -\frac{A_{kj}}{A_{kk}} \in \mathbb{Z}/p\mathbb{Z}, \\ 1 & \text{if } p_k = 1, \ A_{kk} = 0, \ A_{kj} \neq 0, \\ 0 & \text{if } p_k = 1, \ A_{kk} = A_{kj} = 0, \\ p - 1 & \text{if } p_k = 0, \ A_{kk} = 0, \ A_{kj} \neq 0, \\ 0 & \text{if } p_k = 0, \ A_{kk} = 0, \ A_{kj} = 0, \end{cases}$$

where we consider $\mathbb{Z}/p\mathbb{Z}$ as a subfield of $\mathbb{K}$.

The name “reflection” is used because in the case of (semi)simple finite-dimensional Lie algebras this action extended on the whole $R$ by linearity is a map from $R$ to $R$, and it does not depend on $R^+$, only on $\sigma_k$. This map is usually denoted by $r_{\sigma_k}$ or just $r_k$. The map $r_{\sigma_i}$ extended to the $\mathbb{R}$-span of $R$ is reflection in the hyperplane orthogonal to $\sigma_i$ relative the bilinear form dual to the Killing form.

The reflections in the even (odd) roots are said to be even (odd). A simple root, and reflection in it, is called isotropic, if the corresponding row of the Cartan matrix has zero on the diagonal, and non-isotropic otherwise.

If there are isotropic simple roots, the reflections $r_\alpha$ do not, as a rule, generate a version of the Weyl group because the product of two reflections in nodes not connected by one (perhaps, multiple) edge is not defined\textsuperscript{12}. These reflections just connect pair of “neighboring” systems of simple roots and there is no reason to expect that we can multiply two distinct such reflections.

\textsuperscript{12}The ideas of the paper \cite{SV} might be helpful here.
In the general case (of Lie superalgebras and \( p > 0 \)), the action of a given isotropic reflections \([52]\) cannot, generally, be extended to a linear map \( R \rightarrow R \). For Lie superalgebras over \( \mathbb{C} \), one can extend the action of reflections by linearity to the root lattice but this extension preserves the root system only for \( \mathfrak{sl}(m|n) \) and \( \mathfrak{osp}(2m+1|2n) \), cf. \([Sc]\).

If \( \sigma_i \) is an odd isotropic root, then the corresponding reflection sends one set of Chevalley generators into a new one:

\[
\bar{X}_i^\pm = X_i^\pm; \quad \bar{X}_j^\pm = \begin{cases} [X_i^\pm, X_j^\pm] & \text{if } A_{ij} \neq 0, 0, \\
X_j^\pm & \text{otherwise.}
\end{cases}
\]

2.6.3a. Remark. The description of the numbers \( B_{ik} \) is empirical and based on classification \([BGL1]\): For infinite-dimensional Lie (super)algebras these numbers might be different. In principle, in the second, fourth and penultimate cases, the matrix \([53]\) can be equal to \( kp - 1 \) for any \( k \in \mathbb{N} \), and in the last case any element of \( \mathbb{K} \) may occur. For \( \dim \mathfrak{g} < \infty \), this does not happen (and it is of interest to investigate at least the simplest infinite dimensional case — the modular analog of \([CCLL]\)).

The values \( -\frac{2A_{kj}}{A_{kk}} \) and \( -\frac{A_{kj}}{A_{kk}} \) are elements of \( \mathbb{K} \), while the roots are elements of a vector space over \( \mathbb{R} \). Therefore the expressions in the first and third cases in \([53]\) should be understood as “the minimal non-negative integer congruent to \( -\frac{2A_{kj}}{A_{kk}} \) or \( -\frac{A_{kj}}{A_{kk}} \), respectively”. (If \( \dim \mathfrak{g} < \infty \), these expressions are always congruent to integers.)

There is known just one exception: If \( p = 2 \) and \( A_{kk} = A_{jk} \), then \( -\frac{2A_{jk}}{A_{kk}} \) should be understood as 2, not 0.

2.6.3b. On neighboring root systems. Serganova \([Sc]\) proved (for \( p = 0 \)) that there is always a chain of reflections connecting \( B_1 \) with some system of simple roots \( B'_2 \) equivalent to \( B_2 \) in the sense of definition \([2.4.6]\). Here is the modular version of Serganova’s Lemma. Observe that Serganova’s statement is not weaker: Serganova used only odd reflections.

Lemma (\([LeD]\)). For any two systems of simple roots \( B_1 \) and \( B_2 \) of any simple finite dimensional Lie superalgebra with Cartan matrix, there is always a chain of reflections connecting \( B_1 \) with \( B_2 \).

2.7. The Lie (super)algebras of the form \( \mathfrak{g}(A) \). Their simple subquotients \( \mathfrak{g}'(A)/\mathfrak{c} \).

2.7.1. Over \( \mathbb{C} \). Kaplansky was the first (see his newsletters in \([Kapp]\)) to discover the exceptional algebras \( \mathfrak{aq}(2) \) and \( \mathfrak{ab}(3) \) (he dubbed them \( \Gamma_2 \) and \( \Gamma_3 \), respectively) and a parametric family \( \mathfrak{osp}(4|2; \alpha) \) (he dubbed it \( \Gamma(A, B, C) \)); our notation reflect the fact that \( \mathfrak{aq}(2)_0 = \mathfrak{sl}(2) \oplus \mathfrak{g}(2) \) and \( \mathfrak{ab}(3)_0 = \mathfrak{sl}(2) \oplus \mathfrak{o}(7) \) (\( \mathfrak{o}(7) \) is \( B_3 \) in Cartan’s nomenclature). Kaplansky’s description (irrelevant to us at the moment except for the fact that \( A, B \) and \( C \) are on equal footing) of what we now identify as \( \mathfrak{osp}(4|2; \alpha) \), a parametric family of deformations of \( \mathfrak{osp}(4|2) \), made an \( S_3 \)-symmetry of the parameter manifest (to A. A. Kirillov, and he informed us, in 1976). Indeed, since \( A + B + C = 0 \), and \( \alpha \in \mathbb{C} \cup \infty \) is the ratio of the two parameters remaining after \( A, B \) and \( C \) were constrained, we get an \( S_3 \)-action on the plane \( A + B + C = 0 \) which in terms of \( \alpha \) is generated by the transformations:

\[
a \mapsto -1 - a, \quad a \mapsto \frac{1}{a}.
\]

The other transformations generated by \([55]\) are

\[
a \mapsto -\frac{1 + a}{a}, \quad a \mapsto -\frac{1}{a + 1}, \quad a \mapsto -\frac{a}{a + 1}.
\]
This symmetry should have immediately sprang to mind since \( \mathfrak{osp}(4|2; a) \) is strikingly similar to \( \mathfrak{wt}(3; a) \) found 5 years earlier, cf. \[56\], and since \( S_3 \simeq \text{SL}(2; \mathbb{Z}/2) \).

2.7.2. Modular Lie algebras and Lie superalgebras.

2.7.2a. \( p = 2 \), Lie algebras. Weisfeiler and Kac \[WK\] discovered two new parametric families that we denote \( \mathfrak{wt}(3; a) \) and \( \mathfrak{wt}(4; a) \) for Weisfeiler and Kac algebras.

\( \mathfrak{wt}(3; a) \), where \( a \neq 0, 1 \), of dim 18 is a non-super version of \( \mathfrak{osp}(4|2; a) \) (although no \( \mathfrak{osp} \) exists for \( p = 2 \)); the dimension of its simple subquotient \( \mathfrak{wt}(3; a)/\mathfrak{c} \) is equal to 16; the inequivalent Cartan matrices are:

1) \[
\begin{pmatrix}
\bar{0} & a & 0 \\
 a & \bar{0} & 1 \\
 0 & 1 & 0 \\
\end{pmatrix},
\]
2) \[
\begin{pmatrix}
\bar{0} & 1 & a + 1 & 0 \\
 1 & \bar{0} & a & 0 \\
 a + 1 & a & \bar{0} & a \\
 0 & a & 0 & \bar{0} \\
\end{pmatrix},
\]
3) \[
\begin{pmatrix}
\bar{0} & a & 0 & 0 \\
 a & \bar{0} & a + 1 & 0 \\
 0 & a + 1 & \bar{0} & 1 \\
 0 & 0 & 1 & \bar{0} \\
\end{pmatrix}.
\]

Weisfeiler and Kac investigated also which of these algebras are isomorphic and the answer is as follows:

\[
\mathfrak{wt}(3; a) \simeq \mathfrak{wt}(3; a') \iff a' = \frac{\alpha a + \beta}{\gamma a + \delta}, \text{where } \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}(2; \mathbb{Z}/2)
\]
\[ (56) \]

\[
\mathfrak{wt}(4; a) \simeq \mathfrak{wt}(4; a') \iff a' = \frac{1}{a}.
\]

2.7.2b. \( p = 2 \), Lie superalgebras. The same Cartan matrices as for \( \mathfrak{wt} \) algebras but with arbitrary distribution of 0’s on the main diagonal correspond to Lie superalgebras \( b\mathfrak{gl}(3; a) \) and \( b\mathfrak{gl}(4; a) \) discovered in \[BGL1\]. The conditions when they are isomorphic are the same as in \[56\], they have the same inequivalent Cartan matrices, and are considered also only if \( a \neq 0, 1 \) (since otherwise they are not simple). We have \( \text{sdim } b\mathfrak{gl}(3; a) = 10/8/8 \) and \( \text{sdim } b\mathfrak{gl}(4; a) = 18/16 \).

2.8. Yamaguchi’s theorem. Let \( s := \bigoplus_{i \geq -d} s_i \) be a simple finite dimensional Lie algebra over \( \mathbb{C} \). Let \((s_\ast)_\ast = (s_\ast, g_{0\ast})_\ast \) be the Cartan prolong with the maximal possible \( g_0 := \text{der}_0(s_\ast) \). As is now well-known \[K\], any \( \mathbb{Z} \)-grading of the finite dimensional Lie algebra \( g \) with Cartan matrix

\[
\begin{pmatrix}
g_{ij} & \\
\end{pmatrix}
\]

is given by a vector \( r = (r_1, \ldots, r_{\text{rk}g}) \), where \( r_i \in \mathbb{Z} \) for all \( i \), by setting

\[
\deg X^\pm_i = \pm r_i.
\]

We say that a grading is simplest if \( r_i = \delta_{i0} \) for some \( i_0 \). The indices \( i \) for which \( r_i = 1 \) will be called “selected” (assuming \( r_j = 0 \) for all non-selected indices \( j \)).

Theorem \([Y]\). Over \( \mathbb{C} \), equality \((s_\ast)_\ast = s \) holds almost always. The exceptions (cases where \( s = \bigoplus_{i \geq -d} s_i \) is a partial prolong in \((s_\ast)_\ast = (s_\ast, g_{0\ast})_\ast \) are

1) \( s \) with the grading of depth \( d = 1 \) (in which case \((s_\ast)_\ast = \text{vect}(s^\ast) \));
2) \( s \) with the grading of depth \( d = 2 \) and \( \text{dim } s_{-2} = 1 \), i.e., with the “contact” grading, in which case \((s_\ast)_\ast = \mathfrak{t}(s^\ast) \) (these cases correspond to “selecting” the nodes on the Dynkin graph connected with the node representing the maximal root on the extended graph);
3) \( s \) is either \( \mathfrak{sl}(n+1) \) or \( \mathfrak{sp}(2n) \) with the grading determined by “selecting” the first and the \( i \)th of simple coroots, where \( 1 < i < n \) for \( \mathfrak{sl}(n+1) \) and \( i = n \) for \( \mathfrak{sp}(2n) \). (Observe that \( d = 2 \) with \( \text{dim } s_{-2} > 1 \) for \( \mathfrak{sl}(n+1) \) and \( d = 3 \) for \( \mathfrak{sp}(2n) \)).
Moreover, the equality $(s - s_0)_* = s$ also holds almost always. The cases where the equality fails (the ones where a projective action is possible) are $\mathfrak{sl}(n+1)$ or $\mathfrak{sp}(2n+2)$ with the grading determined by “selecting” only one (the first) simple coroot; $s = \mathfrak{vect}(n)$ or $\mathfrak{t}(2n+1)$, respectively.

2.8.1. **Remark.** First, Yamaguchi’s cases (for $p = 0$) where the CTS prolongs return the initial algebra are precisely the cases where restrictions on $N$ are imposed if we pass to $p > 0$. More exactly, to describe the complete prolong, NO restrictions should be imposed on $N$.

It so happens that (even if $p = 0$) certain indeterminates, in terms of which the CTS prolong is described, can not enter in degrees greater than something. For $p > 0$, this imposes certain restrictions on $N$ dictated by the very structure of the Lie algebra whose non-positive part we are prolonging.

For example, if we write $N = (1, n, 1)$, this does not mean that WE have imposed any constraints on the first and third coordinates of $N$; it is the non-positive (or negative, depending on the problem) part of the algebra to be prolonged imposes these constraints on these coordinates.

Therefore, not only in the case where there are no restrictions on $N$ but also in all cases where at least one of the coordinates of $N$ is not restricted, the COMPLETE prolong is of infinite dimension. The space $\mathfrak{g}_1$ can not generate the complete prolong, or any part of it with sufficiently great value of at least one of coordinates of $N$. The algebra generated by $\mathfrak{g}_1$ gives us restrictions on coordinates of $N$ imposed by $\mathfrak{g}_1$.

3. **Simple Lie algebras as CTS-prolongs of the non-positive parts of $\mathfrak{g}(A)$**

For the definition of the grading vector $r$ in tables below, see (57); for a $\mathbb{Z}$-graded Lie algebra $\mathfrak{g} = \bigoplus \mathfrak{g}_i$, we set $\mathfrak{g}_{\leq 0} := \bigoplus_{i \leq 0} \mathfrak{g}_i$. Consider Cartan matrices as their size grows. In tables (60) and (70), we provisionally (until we identify the algebra $\mathfrak{g}_*,N$ with a known algebra) denote the prolongs $\mathfrak{g}_*,N$ with $\dim \mathfrak{g}_{\leq 0} = D$ by $D(D; \mathbb{N})$.

### 3.1. Size=1.

| $\mathfrak{g}$ | Cartan matrix | $r$ | Prolong of $\mathfrak{g}_{\leq 0}$ for this $r$ |
|----------------|---------------|-----|----------------------------------|
| $\mathfrak{sl}(3)$ | \( \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \) | \( (1) \) | $\mathfrak{vect}(1; \mathbb{N})$ |

### 3.2. Size=2.

| $\mathfrak{g}$ | Cartan matrix | $r$ | Prolong of $\mathfrak{g}_{\leq 0}$ for this $r$ |
|----------------|---------------|-----|----------------------------------|
| $\mathfrak{sl}(3)$ | \( \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \) | \( (10) \) | $\mathfrak{vect}(2; \mathbb{N})$ |
| | | \( (01) \) | $\mathfrak{vect}(2; \mathbb{N})$ |
| $\mathfrak{so}(5)$ | \( \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \) | \( (10) \) | $\mathfrak{so}(3; \mathbb{N}) = \mathfrak{h}_1(3; \mathbb{N})$ |
| | | \( (01) \) | $\mathfrak{so}(3; \mathbb{N}) = \mathfrak{t}(3; \mathbb{N})$ |

3.2.0a. **What $\mathfrak{B}(2m - 1; \mathbb{N})$ and $\mathfrak{C}(2m - 1; \mathbb{N})$ are.** For $m = 2$, consider the generating functions in $x_1, x_2, x_3$ with $N = (1, n, 1)$ and the Poisson bracket:

\[
\{ f, g \} = \frac{\partial f}{\partial x_1} \frac{\partial g}{\partial x_3} + \frac{\partial f}{\partial x_3} \frac{\partial g}{\partial x_1} + \frac{\partial f}{\partial x_2} \frac{\partial g}{\partial x_2}.
\]

Having factorized modulo center (generated by constants) we get $\mathfrak{h}_1(3; \mathbb{N})$, and its elements can be represented as

\[
H_f = \frac{\partial f}{\partial x_1} \partial_{x_3} + \frac{\partial f}{\partial x_3} \partial_{x_1} + \frac{\partial f}{\partial x_2} \partial_{x_2}.
\]
New simple Lie algebras in characteristic 2

Setting $\deg_{\text{Lie}} = \deg - 2$, where $\deg x_i = 1$ for all $i$, we see that the height of $\frak{h}_{II}(3; \frak{N})$ relative the grading $\deg_{\text{Lie}}$ is equal to $2^n - 1$, and

$$\frak{g}_k = \text{Span}(x_2^{(k+2)}, x_2^{(k+1)}x_1, x_2^{(k+1)}x_3, x_2^{(k)}x_1x_3) \text{ for } 0 \leq k < 2^n - 1,$$

whereas $\dim \frak{g}_{2^n-1} = 1$. Since the same arguments hold for $m > 2$ as well, we arrive at the following verdict:

$$\frak{B}(2m - 1; \frak{N}) \cong \frak{h}_{II}(2m - 1; \frak{N}).$$

By comparing non-positive parts of the $\mathbb{Z}$-graded Lie algebras $\frak{C}(2m - 1; \frak{N})$ we deduce:

$$\frak{C}(3; \frak{N}) \cong \frak{t}(3; \frak{N}),$$

$$\frak{C}(2m - 1; \frak{N}) = \frak{o}_{II}(2m + 1) \text{ for } m > 2.$$

3.3. Size=3.

3.3.1. Derivarions and central extensions. When the size of Cartan matrix is $\geq 3$ it might be non-invertible, and hence the algebras $\frak{g}(A)$ have to be replaced, see the left column of table (60), with their quotients modulo center; the same applies to the derived of $\frak{g}(A)$. These latter algebras may have, for $A$ of small rank, non-trivial outer derivations and in order to identify the “extra” elements of CTS-prolongs with some of these derivations we have to know all of them. For example, we already know that $\frak{psl}(4)$ is a desuperization of $\frak{psl}(2|2) \cong \frak{h}'(0|4)$, and hence has at least three outer derivations of degrees $\pm 4$ and 0 with respect to the grading defined on Chevalley generators by setting $\deg X_i^\pm = \pm 1$ (these cocycles turn $\frak{h}'(0|4)$ into, respectively, $\frak{h}(0|4)$ — twice, and $\frak{pgl}(2|2) \cong \frak{h}'(0|4) \in KE$, where $E = \sum \theta_i \partial_i$ is the Euler operator). In reality there are 7 central extensions, see [BGLL2].

For reasons explained in [BGLL2], there should be no less than 7 outer derivations of $\frak{psl}(4)$; there are precisely 7 of them, see [BGLL2].

3.3.2. The table. Explanations of isomorphisms in the right-most column of table (60) are given under it. For a description of the Buttin algebras $\frak{b}_\lambda(m)$ and their various Weisfeiler regradings $\frak{b}_\lambda(m; r)$ over $\mathbb{C}$, see [LSh]; their analogs for $p = 2$ are described in [LeP]. In particular, $\frak{b}_\lambda(2; 2; \frak{N})$ is the nonstandard regrading of $\frak{b}_\lambda(2; \frak{N})$ corresponding to $\deg \xi_i = 0$ for both odd indeterminates $\xi_i$, see [LSh]. For $p = 2$, the odd indeterminates correspond to those of the desuperization, whose degree can not exceed 1.
| \( \mathfrak{g} \) | Matrix \( A \) | \( r \) | Prolong of \( \mathfrak{g}^{\leq 0} \) for this \( r \) |
|---|---|---|---|
| \( \mathfrak{pgl}(4) \) | \[
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\] | \( (100) \) | \( \text{vect}(3; N) \) |
| \( \mathfrak{psl}(4) \) | \[
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\] | \( (010) \) | \( \text{F}(\mathfrak{h}(04)) \) |
| | | \( (001) \) | \( \text{vect}(3; N) \) |
| \( \mathfrak{so}_3'(7) \) | \[
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 1
\end{pmatrix}
\] | \( (100) \) | \( 5(5; N) = h_7(5; 1, 1, 1, 1) \) |
| | | \( (010) \) | \( \mathfrak{so}_3'(7) \) in 2 outer derivations |
| | | \( (001) \) | \( \mathfrak{so}_3'(7) \) in 3 outer derivations |
| \( \mathfrak{mt}'(3; \lambda)/\mathfrak{c} \) | \[
\begin{pmatrix}
0 & \lambda & 0 \\
\lambda & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\] | \( (100) \) | \( \mathfrak{S}(5; \lambda; N) = \mathfrak{F}(b_1(2; 2; N)) \) |
| | | \( (010) \) | \( \mathfrak{mt}'(3; \lambda)/\mathfrak{c} \) |
| | | \( (001) \) | \( \tilde{\mathfrak{S}}(5; \lambda; N) = \mathfrak{F}(b_1(2; 2; N)) \) |
| | | \( (001) \) | \( \mathfrak{S}((5; \lambda; N) = \mathfrak{F}(b_1(2; 2; N)) \) |

For \( \mathfrak{mt}(3; \lambda)/\mathfrak{c} \), the prolong of the non-positive part is the same as for the respective line for \( \mathfrak{mt}(3; \lambda)/\mathfrak{c} \) plus one outer derivation, except for the second line of the first Cartan matrix in which case it is isomorphic to \( \mathfrak{F}(\mathfrak{t}(1; 1; 1)) \) independently of parameter \( \lambda \).

3.3.3. Elucidating table \([60]\): realization of \( 5(5; N) \) by vector fields. We have \( \mathfrak{g}_0 \cong \mathfrak{o}'(5) \oplus \mathfrak{c} \), where \( \mathfrak{c} = \mathbb{R} \mathfrak{c} \) and \( \mathfrak{g}_0 \) is the tautological \( \mathfrak{g}_0 \)-module. This case was studied in \([\text{IL}_1]\).

Recall that the shearing parameter is of the form \( N \in \mathbb{N} \).

3.3.3a. A description of \( \mathfrak{G}(5; N; \lambda) := (\mathfrak{g}_0, \mathfrak{g}_0)_* \mathfrak{N} \) for \( \mathfrak{g} = \mathfrak{mt}(3; \lambda) \) with the first Cartan matrix. The grading \( r = (100) \) gives \( \text{dim}(\mathfrak{mt}(3; \lambda)_-) = 4 \). The realization by vector fields is as follows:

| \( \mathfrak{g}_0 \) | the generators |
|---|---|
| \( \mathfrak{g}_0 \cong \mathfrak{sl}(3) \) | \( x_1, x_2, x_3, x_4 \), \( Z_2 = \lambda x_2 x_1 + (1 + \lambda) x_4 x_3, H_2 = [Z_2, Y_2], \) \ Y_3 = x_2 x_1, \ Z_3 = x_3 x_2, \ H_3 = [Z_3, Y_3], \ Y_4 = [Y_2, Y_3], \ Z_4 = [Z_2, Z_3] |

Our computation shows that \( N = (1, n, m, 1) \).

3.3.3b. A description of \( \mathfrak{G}(5; N; \lambda) := (\mathfrak{g}_-, \mathfrak{g}_0)_* \mathfrak{N} \) for \( \mathfrak{g} = \mathfrak{mt}(3; \lambda) \) with the second Cartan matrix. The grading \( r = (100) \) gives \( \text{dim}(\mathfrak{mt}(3; \lambda)_-) = 4 \). The realization by vector fields is as follows:

| \( \mathfrak{g}_0 \) | the generators |
|---|---|
| \( \mathfrak{g}_0 \cong \mathfrak{sl}(3) \) | \( x_1, x_2, x_3, x_4 \), \( Z_2 = (\lambda + 1) x_2 x_1 + \lambda x_4 x_3, H_2 = [Z_2, Y_2], \) \ Y_3 = x_2 x_1, \ Z_3 = \lambda (x_3 x_2 + x_4 x_3), \ Y_4 = [Y_2, Z_4], \ Z_4 = [Z_2, Z_3] |

Our computation shows that \( N = (n, 1, 1, m) \).

3.3.3c. Further elucidating Table \([60]\). There is no mistake in the description of prolongs for \( \mathfrak{o}'(7) \). The gradings \( (010) \) and \( (001) \) give the algebra plus different number of linearly independent outer derivations. There is no reason why prolongs of negative parts relative derivations must yield the same number of outer derivations.

Let us explain where 2 or 3 outer derivations in line 2 (concerning \( \mathfrak{o}'(7) \)) come from. The heart of the matter lies in generating functions \( \rho_1^2 \) and \( \rho_1^2 \) one disregards when considering \( \mathfrak{o}' \) instead of \( \mathfrak{o} \). The generating functions which, in the regraded, as \( r \) varies, algebra might appear
in the components of non-positive degree can not, of course, appear. But the ones that should appear in components of positive degree may appear and do so.

This is most clear if you consider the weights for \( p = 0 \). Obviously, for \( p = 2 \) several distinct weights coincide (modulo 2), but for us it is only important that we can realize the initial algebra to be regraded on generating functions (homogeneous degree 2 polynomials in 7 indeterminates).

The weights (roots) of the elements in the initial algebra are well-known: \( \pm \varepsilon_i \pm \varepsilon_j \) and \( \pm \varepsilon_i \). Accordingly, the simple roots are

\[
\alpha_1 = \varepsilon_1 - \varepsilon_2, \quad \alpha_2 = \varepsilon_2 - \varepsilon_3, \quad \alpha_3 = \varepsilon_3.
\]

Setting \( \deg X_{\alpha_1} = 1 \) whereas \( \deg X_{\alpha_i} = 0 \) for \( i = 2, 3 \), we get \( g_0 = \mathfrak{g}_\varepsilon(5) \oplus \mathbb{K} \), the depth is equal to 1, and the prolong is isomorphic to \( h \).

Setting \( \deg X_{\alpha_2} = 1 \) whereas \( \deg X_{\alpha_i} = 0 \) for \( i \neq 2 \), we get the algebra of depth 2, and the component \( g_1 \) contains elements of weight \( \varepsilon_1 \) and \( \varepsilon_2 \). If now we set

\[
\text{weight}(p_i) = 1 \quad \text{and hence weight}(q_i) = -1,
\]

then \( p_1^2 \) and \( p_2^2 \) belong to \( g_2 \) and personify 2 outer derivations of \( g \) (since, e.g., \( [p_2^2, q_i] = p_1 \theta \in g_1 \)). In this grading the element of weight \( \varepsilon_3 \) lies in \( g_0 \), and so \( p_3^2 \) should also lie in \( g_0 \), which is impossible by hypothesis.

Now, setting \( \deg X_{\alpha_3} = 1 \) whereas \( \deg X_{\alpha_i} = 0 \) for \( i \neq 3 \), the component \( g_1 \) contains all the three elements of weight \( \varepsilon_i \). Accordingly, all the three \( p_i^2 \) lie in \( g_2 \) (and serve as outer derivations).

3.3.3d. \((\mathfrak{w}(\lambda; 3)/c)_{\leq 0} \cong \mathbf{F}(\mathfrak{t}(1; 1|4)))_{\leq 0} \) for \( r = (010) \) and the first Cartan matrix. This case is an exceptional one and does not fit the pattern of the other gradings because in this case the isomorphism of non-positive parts of two algebras occurs and therefore their prolongs coincide. We have

\[
\begin{array}{c|c}
\mathfrak{g}_i & \text{the generators} \\
\hline
\mathfrak{g}_{-2} & w_1 = \partial_1 \\
\mathfrak{g}_{-1} & w_2 = \partial_2, w_3 = \partial_3, w_4 = x_3 \partial_1 + \partial_4, w_5 = x_2 \partial_1 + \partial_5 \\
\mathfrak{g}_0 & \cong \mathfrak{g}_\varepsilon(4) \oplus \varepsilon \cong X_1^+ = x_2 \partial_3 + x_4 \partial_5, X_2^+ = \lambda x_3 \partial_1 + x_5 \partial_4, X_3^+ = x_4 x_5 \partial_1 + x_4 \partial_2 + x_5 \partial_3 \\
\mathfrak{h}(4) \times 2 \text{ outer} & H_1 = [X_1^+, X_3^+] = \lambda H_2, H_3 = (\lambda + 1) x_1 \partial_1 + \lambda x_3 \partial_1 + x_4 \partial_4 + (\lambda + 1) x_5 \partial_5, \\
\text{derivatives} & H_2 = [X_2^+, X_1^+] = x_2 \partial_2 + x_3 \partial_1 + x_4 \partial_4 + x_5 \partial_5. \\
\end{array}
\]

An isomorphism between the non-positive parts of \( \mathbf{F}(\mathfrak{t}(1; 1|4)) \), see \( \mathbb{L} \mathbb{E} \mathbb{P} \), and \( \mathfrak{w}(\lambda; 3)/c \) goes as follows, where we briefly write \( f \) instead of contact vector field \( K_f \):

\[
\begin{array}{cccc}
w_2 & \longleftrightarrow \xi_1 & X_2 & \longleftrightarrow \xi_2 \eta_1 & H_1 & \longleftrightarrow \lambda (\xi_1 \eta_1 + \xi_2 \eta_2) \\
w_3 & \longleftrightarrow \xi_2 & X_2^+ & \longleftrightarrow \lambda \xi_1 \eta_2 & H_3 & \longleftrightarrow \lambda \xi_2 \eta_2 + (1 + \lambda) t \\
w_4 & \longleftrightarrow \eta_2 & X_1^- & \longleftrightarrow \eta_1 \eta_2 & H_2 & \longleftrightarrow \xi_1 \eta_1 + \xi_2 \eta_2 \\
w_5 & \longleftrightarrow \eta_1 & X_1^+ & \longleftrightarrow \xi_1 \xi_2 & d & \longleftrightarrow \xi_2 \eta_2 + t \\
\end{array}
\]

Recall that the contact bracket corresponding to the bracket \([K_f, K_g] \) is

\[
\{f, g\}_{k.b.} = \frac{\partial f}{\partial t} (1 - E')(g) + (1 - E')(f) \frac{\partial g}{\partial t} + \{f, g\}_{p.b.},
\]

where \( E' = \sum \xi_i \partial_{\xi_i} \) and the Poisson bracket is:

\[
\{f, g\}_{p.b.} = \frac{\partial f}{\partial \xi_1} \frac{\partial g}{\partial \eta_1} + \frac{\partial g}{\partial \xi_1} \frac{\partial f}{\partial \eta_1} + \frac{\partial f}{\partial \xi_2} \frac{\partial g}{\partial \eta_2} + \frac{\partial g}{\partial \xi_2} \frac{\partial f}{\partial \eta_2}.
\]

The non-positive parts determine the isomorphic prolongs that do not depend on \( \lambda \); the algebra \( \mathfrak{w}(3; \lambda) \) is a subalgebra of \( \mathbf{F}(\mathfrak{t}(1; 1|4)) \), a partial prolong.
3.3.3e. **What \( \mathfrak{g}(5; \underline{N}; \lambda) \) and \( \widetilde{\mathfrak{g}}(5; \underline{N}; \lambda) \) are isomorphic to.** For various Cartan matrices and various simplest regradings \( r \), we have \( \dim(\mathfrak{wt}(3; \lambda)_-) = 4 \), see (60). The realization by vector fields is given in eqs. (61) and (62). Let us compare the representations of \( g_0 \) in \( g_{-1} \) in these two cases.

Let us begin with \( \mathfrak{g}(5; \underline{N}; \lambda) \). The highest weight vector in \( g_{-1} \) is the one killed by all the \( Y_i \). From eq. (61) we see that this is \( \partial_4 \).

Now look how the \( Z_i \) act on it:

\[
\begin{align*}
\partial_4 & \rightarrow (1 + \lambda) \partial_3 \rightarrow (1 + \lambda) \partial_2 \rightarrow \lambda(\lambda + 1) \partial_1 \rightarrow z_2, z_3 \rightarrow 0 \\
0 & \rightarrow 0 \rightarrow 0 
\end{align*}
\]

To compute the weights, we need an explicit form of the \( H_i \):

\[
H_2 = (\lambda + 1)(x_3 \partial_3 + x_4 \partial_4) + \lambda(x_1 \partial_1 + x_2 \partial_2); \quad H_3 = x_2 \partial_2 + x_3 \partial_3.
\]

Thus, the weight diagram of our representation is as follows:

\[
\begin{array}{cccc}
\partial_4 & \partial_3 & \partial_2 & \partial_1 \\
(\lambda + 1, 0) & (\lambda + 1, 1) & (\lambda, 1) & (\lambda, 0)
\end{array}
\]

Now, pass to \( \widetilde{\mathfrak{g}}(5; \underline{N}; \lambda) \). In order to avoid a confusion with the previous discussion, let us denote the basis elements of \( g_0 \) by small letters: \( y_i, z_i, h_i \). This \( y_3 = \lambda x_1 \partial_3 + (\lambda + 1)x_2 \partial_4 \) differs from that of (62) by a factor \( \lambda \) and \( z_3 = x_3 \partial_1 + x_4 \partial_2 \) differs from that of (62) by a factor \( \lambda^{-1} \). Neither \( h_3 \), nor Cartan matrix are affected.

We have:

\[
h_2 = (\lambda + 1)(x_1 \partial_1 + x_2 \partial_2) + \lambda(x_3 \partial_3 + x_4 \partial_4), \quad h_3 = \lambda(x_1 \partial_1 + x_3 \partial_3) + (\lambda + 1)(x_2 \partial_2 + x_4 \partial_4).
\]

Assuming the \( y_i \) to be positive root vector we again see that \( \partial_4 \) is a highest weight vector. The \( z_i \) act on it as follows:

\[
\begin{align*}
\partial_4 & \rightarrow z_2 \rightarrow \lambda \partial_3 \rightarrow z_3 \rightarrow \lambda \partial_1 \rightarrow z_2, z_3 \rightarrow 0 \\
\partial_2 & \rightarrow z_2 \rightarrow (\lambda + 1) \partial_1 \rightarrow z_2, z_3 \rightarrow 0 
\end{align*}
\]

and the weight diagram of the representation is of the following form:

\[
\begin{array}{cccc}
\partial_4 & \partial_3 & \partial_2 & \partial_1 \\
(\lambda, \lambda + 1) & (\lambda, \lambda) & (\lambda + 1, \lambda + 1) & (\lambda + 1, \lambda)
\end{array}
\]

Now, let us change the basis of \( g_{-1} \). First, let us interchange \( x_2 \) with \( x_3 \). We see that

\[
h_3 = \lambda(x_1 \partial_1 + x_2 \partial_2) + (\lambda + 1)(x_3 \partial_3 + x_4 \partial_4) = H_2
\]

whereas

\[
h_2 = (\lambda + 1)(x_1 \partial_1 + x_3 \partial_3) + \lambda(x_2 \partial_2 + x_4 \partial_4).
\]

Then \( h = h_2 + h_3 = x_1 \partial_1 + x_4 \partial_4 \).

Now, let us interchange \( x_1 \) with \( x_2 \), and \( x_3 \) with \( x_4 \). Then \( h_2 \) does not vary, whereas \( h \) turns into \( x_2 \partial_2 + x_3 \partial_3 = H_3 \).

As a result, we have performed the permutation of indeterminates

\[
\begin{pmatrix}
1 & 2 & 3 & 4 \\
2 & 4 & 1 & 3
\end{pmatrix}
\]
which sends the other vectors of $\mathfrak{G}(5; N; \lambda)_0$ as follows:

\[
\begin{align*}
y_3 &\mapsto \lambda x_2 \partial_1 + (\lambda + 1) x_4 \partial_3 = Z_2, \\
y_6 &\mapsto x_2 \partial_3 = Y_3, \\
y_2 &\mapsto x_2 \partial_4 + x_1 \partial_3 = Y_5, \\
z_3 &\mapsto x_1 \partial_2 + x_3 \partial_4 = Y_2, \\
z_6 &\mapsto x_3 \partial_2 = Z_3, \\
z_2 &\mapsto (\lambda + 1) x_4 \partial_2 + \lambda x_3 \partial_1 = Z_5.
\end{align*}
\]

In order to see how vector diagram (65) turns into (67), pass to another “Borel” subalgebra (with positive generators $Z_2$ and $Y_5$ and respective negative ones $Y_2$ and $Z_5$) and change basis according to eq. (69). Now $\partial_3$ becomes a highest weight vector and the images of $g_0$ in $\mathfrak{g}(\mathfrak{g}_{-1})$ coincide. The Cartan prolongs of these two pairs $(\mathfrak{g}_{-1}, g_0)$ are isomorphic for $\mathfrak{G}$ and $\tilde{\mathfrak{G}}$.

For illustration, let us express the elements of $g_0$ (for $\mathfrak{G}$, not for $\tilde{\mathfrak{G}}$) by matrices. The most simple form is obtained in the basis $\partial_2, \partial_3, \partial_1, \partial_4$:

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \text{where } A \text{ and } C \text{ are arbitrary,}
\]

\[
D = \begin{pmatrix} \lambda \cdot \text{tr} A & 0 \\ 0 & (\lambda + 1) \cdot \text{tr} A \end{pmatrix}, \quad \text{and if } C = \begin{pmatrix} \alpha \\ \gamma \end{pmatrix}, \quad \text{then } B = \begin{pmatrix} \delta & \frac{\lambda + 1}{\lambda} \cdot \beta \\ \gamma & \frac{\lambda}{\lambda + 1} \cdot \alpha \end{pmatrix}.
\]

It is not difficult to verify that for $g := \tilde{\mathfrak{G}}(5; N; \lambda)$, the $g_0$-action on $g_{-1}$ is precisely the $\text{vect}(2; N_s)$-action on the space of $\lambda$-densities; observe that $\mathfrak{sl}(3) = F(\mathfrak{sl}(1|2)) \cong F(\text{vect}(0|2))$.

In other words, the prolong is a desuperization of $\mathfrak{b}_{\lambda}(2; 2; N)$, the nonstandard regrading of $\mathfrak{b}_{\lambda}(2; N)$ corresponding to $\deg \xi_i = 0$ for both odd indeterminates $\xi_i$, see [LSN]. In the version of the prolong we consider, both free shearing parameters (corresponding to the even indeterminates) are taken equal to 2. Here is the place where the difference between $N$ for the generating functions, which correctly describes the algebra, and $N$ for coefficients of vector fields, which yields a wrong description since some information becomes lost, see “non-existent generating functions” $\theta^2$ in [LeP].

3.3.3f. Remarks. 1) This simple Lie algebra — prolong of $\mathfrak{w} \mathfrak{t}(3; \lambda)$ — first appeared (without interpretation, in components) in [Bro]. It is difficult not to admire the computational skill of Brown and compare the difficult calculation made by bare hands in [Bro] with easiness brought to us by code SuperLie.

2) Although the Lie algebra $\mathfrak{w} \mathfrak{t}(3; \lambda)$ is defined for $\lambda \neq 0, 1$, its prolong $\mathfrak{b}_{\lambda}(2; 2; N)$ is well-defined for all values of $\lambda$. For $\lambda = 0, 1$, these prolongs are not simple: for $\lambda = 0$, it has a center the quotient modulo which is simple, for $\lambda = 1$, it has a simple ideal of codimension 1.
3.4. Size=4.

| $\mathfrak{g}$ | Cartan matrix | $r$ | Prolong of $\mathfrak{g}_{\leq 0}$ for this $r$ |
|----------------|----------------|----------------|--------------------------------------------------|
| $\mathfrak{oc}(1;8)/\xi \ltimes \mathbb{K}J_0$ | $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ | (1000) | $6(6) = \mathfrak{h}_{11}(6) \times 2$ outer derivations |
| $\mathfrak{oc}(1;8)/\xi = \mathfrak{o}_{11}(2;8)/\xi$ | $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ | (1000) | $\tilde{\mathfrak{h}}_{11}(6)$, see eq. (2.6) |
| $\mathfrak{st}(5)$ | $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ | (1000) | $\mathfrak{vect}(4;\mathcal{N})$ |
| $\mathfrak{so}'_1(9)$ | $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ | (1000) | $\mathfrak{h}_1(7;1,1,1,n,1,1,1)$, see eq. (2.6) |
| $\mathfrak{mt}(4,a)$ | $\begin{pmatrix} 0 & a & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ | (1000) | $\mathfrak{mt}(4;\mathcal{N})$ |

3.4.1. The Lie algebra $11(11;\mathcal{N};a)$ is a desuperization of $\mathfrak{mb}(3;\mathcal{N}|8)$. The grading $r = (0100)$ gives $\dim(\mathfrak{mt}(3;a)_-) = 11$. The CTS prolong $(\mathfrak{g}_-;\mathfrak{g}_0)_+$ gives a Lie algebra that we denote by $11(11;\mathcal{N};a)$. Its non-positive part is precisely as that of $\mathfrak{mb}(3;\mathcal{N}|8)$ in which we consider the odd indeterminates even. Our computation shows that the coordinates of the shearing vector corresponding to the odd indeterminates can only be equal to 1, other being arbitrary.

In the $\mathbb{Z}$-grading considered, the negative part of $11(11;\mathcal{N};a)$ is independent of $a$. It first appears in the 0th component, and only in the three vectors $Z_1$, $H_1 = [Z_1,Y_1]$ and $H_2$. But $Z_1 = a \cdot Z'_1$, where $Z'_1$ does not depend on $a$. So we can replace $Z_1$ by $Z'_1$, and $H_1$ by $[Z'_1,Y_1]$ and the new basis elements do not depend on $a$.

Finally, replacing $H_2$ by $H_2 + H_4$, we get

$$H_2 + H_4 = a(x_1\partial_1 + x_3\partial_3 + x_4\partial_4 + x_5\partial_5 + x_7\partial_7 + x_9\partial_9 + x_{11}\partial_{11}) = a \cdot H.$$

Now, replace $H_2$ by $H$; we get a basis of $\mathfrak{g}_0$ independent of $a$.

The prolong — desuperization of $\mathfrak{mb}(3;\mathcal{N}|8)$ — is described more explicitly in [BGLS]. The realization by vector fields is as follows:
3.4.2. Prolong of the non-positive parts of $\mathfrak{o}_\Pi(2n)/c$ and its derived.

3.4.2a. For $r = (10 \ldots 0)$. We consider the Lie algebras obtained from the algebras with Cartan matrix by factorizing modulo center and without $\mathbb{K}I_0$, i.e., $\mathfrak{o}_\Pi^{(2)}(4k + 2)$ and $\mathfrak{o}_\Pi'(4k)/c$, and in both cases the 0-th part is isomorphic to $\mathfrak{o}_\Pi'(\dim - 2)$, where $\dim = 4k + 2$ or $4k$, respectively.

Note that the $4k + 2$-dimensional case can be described just like the $4k$-dimensional one: $\mathfrak{o}_\Pi^{(2)}(4k + 2) = \mathfrak{o}_\Pi'(4k + 2)/c$. (It is just that it is usually more convenient to use a simpler description in terms of derived algebra instead of quotient algebra.) So we can talk about $\mathfrak{o}_\Pi'(\dim)/c$ in both cases.

If we consider the 0th part of $\mathfrak{o}_\Pi'(\dim)$ (before factorization by center) in that grading, it is isomorphic to $\mathfrak{o}_\Pi'(\dim - 2) \oplus \mathbb{K}c$, where $c$ is a central element acting by identity on the $(-1)$st part (take $d = E^{1,1} + E^{k+1,k+1}$). The algebra $\mathfrak{o}_\Pi'(\dim - 2)$ contains the identity matrix $1_{\dim - 2}$ which is also central and also acts on the $(-1)$st part by identity. The center of $\mathfrak{o}_\Pi'(\dim)$ modulo which we factorize the algebra consists of elements that act on the $(-1)$st part by 0, that is, $\mathbb{K}(1_{\dim - 2} + c)$. The resulting quotient algebra and its action on the $(-1)$st component are the same as if we factorized modulo $\mathbb{K}c$, i.e., it is just $\mathfrak{o}_\Pi'(\dim - 2)$.

In table (70) we write $\mathfrak{o}_\Pi'(1; 8)/c$ because we consider prolongs of (non-positive parts relative a certain $\mathbb{Z}$-grading of) Lie algebras with Cartan matrix. In reality we do not have to first centrally extend an algebra just to factorize it modulo this center the next moment.

3.4.2b. For $2n > 8$ and $r = (0 \ldots 01)$. The prolongation returns $\mathfrak{o}_\Pi'(2n) \ltimes n$ outer derivations.

3.4.2c. For $2n > 8$ and $r = (0 \ldots 0100)$. The prolongation returns $\mathfrak{o}_\Pi'(2n) \ltimes (n - 2)$ outer derivations.

3.4.2d. For $\mathfrak{o}_\Pi'(8)$, the Dynkin diagram is most symmetric. For $\mathfrak{o}_\Pi'(8)$, there are only the two non-equivalent cases:
(a) For the grading \( r = (1000) \) (“selected” is any of the end-points of the Dynkin diagram), the components \( g_i \) for \( i \leq 0 \) are as follows:

| \( g_i \) | the generators |
|----------|---------------|
| \( g_{-1} \) | \( \partial_1, \partial_2, \partial_3, \partial_4, \partial_5, \partial_6 \) |

\[
(71)
\]

The prolong, denoted by \( 6(6) \), is of dimension 64 whatever \( \mathcal{N} \). We see that \( \dim 6'(6) = 62 \). The lowest weight vectors of the \( 6'(6)_0 \)-module \( 6'(6)_1 \) are as follows:

\[
\begin{align*}
v_1 &= x_1 x_2 \partial_1 + x_1 x_3 \partial_5 + x_2 x_3 \partial_6 \\
v_2 &= x_1 x_2 \partial_1 x_1 x_4 \partial_5 + x_2 x_4 \partial_6 \\
v_3 &= x_1 x_2 \partial_2 + x_1 x_5 \partial_5 + x_2 x_5 \partial_6 + x_1 x_3 \partial_3 + x_1 x_4 \partial_4 + x_3 x_4 \partial_6
\end{align*}
\]

The modules generated by \( v_1 \) and \( v_2 \) are of dimension 7, that generated by \( v_3 \) is of dimension 6. Since \( \dim g_1 = 20 \), these modules constitute a direct sum (to be sure: there are three highest weight vectors).

The two vectors that are missing in \( 6'(6) \) as compared with \( 6(6) \) act on \( 6'(6) \) as outer derivatives; one lies in the component \( 6(6)_0 \), the other one in \( 6(6)_4 \); just as in \( \mathbb{L} \).

(b) For the grading \( r = (0100) \) (“selected” is the branching node), we have the following Chevalley basis (the \( X_i^\pm \) are Chevalley generators; of the 4 elements of the maximal torus only \( H_1 \) and \( H_2 \) survive after factorization modulo center):

| \( g_i \) | the generators |
|----------|---------------|
| \( g_{-2} \) | \( w_1 = \partial_1 \) |
| \( g_{-1} \) | \( X_i^- := w_2 = \partial_2, w_3 = \partial_3, w_4 = \partial_4, w_5 = \partial_5, w_6 = x_5 \partial_1 + \partial_6, w_7 = x_4 \partial_1 + \partial_7, \\
w_8 = x_3 \partial_1 + \partial_8; w_9 = x_2 \partial_1 + \partial_9 \) |

\[
(72)
\]

We have \( \tilde{g}_0 := [g_{-1}, g_1] \simeq \mathfrak{h}(6) \ltimes 1 \) outer derivative (which is \( H_2 \)). The \( \tilde{g}_0 \)-module \( g_1 \) is irreducible of dimension 8, with the lowest weight vector

\[
v := X^+_2 = x_1 x_2 \partial_1 + x_3 x_4 x_5 \partial_1 + x_1 x_9 \partial_0 + x_2 x_3 \partial_3 + x_2 x_4 \partial_4 + x_2 x_5 \partial_5 + x_2 x_9 \partial_0 + \]

\[
+ x_3 x_4 \partial_6 + x_3 x_5 \partial_7 + x_3 x_8 \partial_0 + x_4 x_5 \partial_8 + x_4 x_7 \partial_0 + x_5 x_6 \partial_9
\]

Further, set \( 9(9; \mathcal{N}) := (g_{-1}, \tilde{g}_0)_* \mathcal{N} \); by standard criteria this is a simple Lie algebra; we have \( (g_{-1}, \tilde{g}_0)_* \mathcal{N} = 9(9; \mathcal{N}) \ltimes 6 \) outer derivatives in the highest component.

Computer-aided experiments show that \( \mathcal{N} = (n, 1, \ldots, 1) \). The Lie algebra \( 9(9; \mathcal{N}) \) is, clearly, an exceptional subalgebra of \( \mathfrak{g}(9; \mathcal{N}) \), a partial prolong.

The Lie algebra \( \mathfrak{o}_0^4(8)/\mathfrak{c} \) is the result of desuperization of an exceptional (in the sense described in eq. (92) in \[\mathbb{BGLN}\]) simple Lie superalgebra \( \mathfrak{o}_m^4(4|4)/\mathfrak{c} \). One can superize \( \mathfrak{o}_0^4(8)/\mathfrak{c} \) by assuming that any one of the 4 pairs of Chevalley generators is odd; in addition to \( \mathfrak{o}_m^4(4|4)/\mathfrak{c} \) this yields \( \mathfrak{o}_m^4(2|6)/\mathfrak{c} \simeq \mathfrak{pe}(4)/\mathfrak{c} \). Equivalently, declaring parities of Chevalley generators imposes certain restriction on parities of the indeterminates, see the elements of \( g_{-1} \) in eq. (72).

So in addition to prolongs of the non-positive part of \( \mathfrak{o}_m^4(4|4)/\mathfrak{c} \) which does not differ from that of \( \mathfrak{o}_m^4(8)/\mathfrak{c} \), except parities of its elements: we declare \( x_3, x_5, x_6, x_8 \) odd for \( \mathfrak{o}_m^4(2|6)/\mathfrak{c} \).
and all, except $x_1$, odd for $o''_{III}(4|4)/c$; in this case is $g_0 = h(2|4) \times 3$ outer derivatives (or $h(6) \times 3$ outer derivatives, respectively).

The four of the authors suggest to designate this exceptional simple Lie algebra

$$(73) \quad \mathfrak{ir}(9; N) := 9(9; N)$$

and its superizations $\mathfrak{ir}(3; N|6)$ and $\mathfrak{ir}(5; N|4)$.

3.4.2e. Brown’s algebra $D_4(3; N)$, see [Bro], as a desuperization of $\mathfrak{ulc}(3; N|8)$. Set $\mathcal{L} = \mathcal{L}(3; N) = \mathcal{L}_0 \oplus \mathcal{L}_1$, where $\mathcal{L}_0 = \mathfrak{svect}(3; N)$ while $\mathcal{L}_1 = \mathcal{O}(3; N) \oplus \mathcal{O}(3; N)_2$ is the direct sum of two copies of $\mathcal{O}(3; N)$ indexed for convenience, the elements of $\mathcal{O}(3; N)_2$ will be barred. Let the action of $\mathcal{L}_0$ on $\mathcal{L}_1$ be the natural one, for any $f, g \in \mathcal{O}(3; N)_1$ and $\bar{f}, \bar{g} \in \mathcal{O}(3; N)_2$, set

$$[f, g] = [\bar{f}, \bar{g}] = 0 \text{ and } [f, \bar{g}] = \nabla f \times \nabla g,$$

where $\nabla f = \sum (\partial_i f) \partial_i$ and $D \times E$ is determined by bi-linearity over $\mathcal{O}(3; N)$ and the rules

$$\partial_1 \times \partial_i = 0 \quad \text{for } i = 1, 2, 3$$
$$\partial_i \times \partial_j = \partial_k \quad \text{for } (i, j, k) \text{ a permutation of } (1, 2, 3).$$

Brown showed that $\mathcal{L}$ is a Lie algebra and endowed it with a $\mathbb{Z}$-grading as follows: Recall that the standard $\mathbb{Z}$-grading of the algebra of functions $\mathcal{O}(3; N)$ (degree of each indeterminate equals to 1) induces a $\mathbb{Z}$-grading (also called standard) of $\mathfrak{svect}(3; N)$ and its homogeneous subalgebras, such as $\mathfrak{svect}(3; N)$.

Now, let $(\mathcal{L}_0)_{2i} := \mathfrak{svect}(3; N)_i$ and $(\mathcal{L}_1)_{2i-3} := (\mathcal{O}(3; N)_{1i}) \oplus (\mathcal{O}(3; N)_2)_i$ with respect to the standard $\mathbb{Z}$-grading of the algebras in the right hand sides.

Brown showed that $\mathcal{L}_{-3}/\mathcal{L}_{-2}$ is the center in $\mathcal{L}$, and $D_4(3; N) := \mathcal{L}/\mathcal{L}_{-3}$ is a simple Lie algebra; in particular, we have $D_4(3; N)_0 := \mathfrak{o}(8)^{(2)}/c$.

Brown’s description of $D_4(3; N)$ reproduced above can be formulated in a very simple way as the complete CTS prolong of its non-positive part, where

$$(74) \quad g_0 = \mathfrak{sl}(V), \text{ where } \dim V = 3, \ g_{-1} = V \oplus \nabla, \ g_{-2} = V,$$

and where $\nabla$ is another copy of the tautological representation of $\mathfrak{sl}(3)$ whereas the bracket

$$(75) \quad E^2(g_{-1}) \longrightarrow g_{-2}$$

is the cross product of vectors of any 3-dimensional space (in other words, we identify $V$ and $\nabla$ with $\mathfrak{o}'(3)$ and the map $(75)$ is just the bracket in $\mathfrak{o}'(3)$.

To obtain the Lie algebra $\mathfrak{sl}(3)$ as the 0th part, we have to consider not the simplest grading, but the one of the form $r = (0011)$. This grading is not, however, a Weisfeiler one (since the component $g_{-1}$ is not an irreducible $g_0$-module) and the only Weisfeiler regradings of this algebra are the ones from table (70). Apart from giving an interpretation of the Brown’s algebra, our answer shows the true number of independent parameters the vector $N$ depends on.

Now observe a remarkable likeness of the negative parts of the following pairs: Brown’s $\mathcal{L}$ and the Lie superalgebra $m\mathfrak{b}$, see subsec. [3.4.1] as well as those of $\mathcal{L}$ and $\mathfrak{ulc}(3|6)$, see also [ShP]. To prove that $D_4(3; N)$ is a desuperization of $\mathfrak{ulc}(3|6)$ and $\mathfrak{ulc}(3|6)_0$ is contained in $D_4(3; N)_0$ due to the constraints imposed on $N$, consider the complete prolong of the negative part of $D_4(3; N)$ without any restrictions on $N$. We see that indeed $D_4(3; N)_0 = \mathfrak{sl}(3) \oplus \mathfrak{gl}(2)$ which turns $D_4(3; N)_{-1}$ into an irreducible $D_4(3; N)_0$-module and the grading $r = (0011)$ into a Weisfeiler one. Our computations confirm (Brown’s description) that $N = (n_1, n_2, n_3, 1, 1, 1, 1, 1, 1)$. The component $g_1$ is of dimension 12, the sum of 4 irreducible 3-dimensional $D_4(3; N)_0$ modules for $g = D_4(3; N)$ with the following lowest weight vectors (resp. dim $g_1 = 18$ for $g = \mathfrak{ulc}(3; N|6)$,
only the first 2 vectors are generators of the irreducible $\mathfrak{utc}(3; \mathbb{N}[6])_0$-modules of dimension 8 and 6, respectively:

$$
v_1 = x_1 x_4 \partial_2 + x_1 x_6 \partial_3 + x_4 x_6 x_7 \partial_3 + x_1 \partial_9 + x_4 x_6 \partial_8 + x_4 x_7 \partial_9$$

$$
v_2 = x_1 x_4 \partial_1 + x_2 x_4 \partial_2 + x_2 x_6 \partial_3 + x_1 \partial_7 + x_2 \partial_9 + x_4 x_6 \partial_6 + x_4 x_7 \partial_7 + x_4 x_8 \partial_8 + x_4 x_9 \partial_9$$

$$
v_3 = x_1 x_5 \partial_2 + x_1 x_7 \partial_3 + x_5 x_6 \partial_8 + x_5 x_7 \partial_9$$

$$
v_4 = x_1 x_5 \partial_1 + x_2 x_5 \partial_2 + x_2 x_7 \partial_3 + x_1 \partial_6 + x_2 \partial_8 + x_5 x_6 \partial_6 + x_5 x_7 \partial_7 + x_5 x_8 \partial_8 + x_5 x_9 \partial_9$$

Clearly, Brown’s $D_4(3; \mathbb{N})$ is a partial prolong of the non-positive part of the above algebra with $g_0 = \mathfrak{sl}(3)$. The partial prolong of the non-positive part and just one of the two irreducible $g_0$-modules in $D_4(3; \mathbb{N})_1$ is $\mathfrak{vs}c(3; \mathbb{N}_s)$. We consider partial prolongs with $g_0 = \mathfrak{sl}(3) \oplus \mathfrak{gl}(2)$ in [BGLS].

We have:

| $g_i$    | the generators                      |
|----------|-------------------------------------|
| $g_{-2}$ | $u_1 = \partial_1, u_2 = \partial_2, u_3 = \partial_3$ |
| $g_{-1}$ | $u_2 = \partial_2, u_3 = \partial_3, u_4 = \partial_1 + \partial_6, u_5 = \partial_5 + \partial_9, u_6 = \partial_6 + \partial_1 + \partial_9, u_7 = \partial_7 + \partial_9$ |

$D_4(3; \mathbb{Z})_0 = \mathfrak{sl}(3)$

$\mathfrak{utc}(3; \mathbb{N}[6])_0 = \mathfrak{sl}(3) \oplus \mathfrak{gl}(2)$

with the above with:

| $\mathfrak{V}$ | $\mathfrak{sl}(3) \oplus \mathfrak{gl}(2)$ |
|-----------------|------------------------------------------|
| $\mathfrak{V}_1$ | $x_1 x_6 \partial_1 + x_6 x_9 \partial_2 + x_8 x_9 \partial_3 + x_7 \partial_8 + x_8 \partial_9 + x_9 \partial_7, Z_1 = x_2 x_9 + x_8 x_9 \partial_1 + x_8 \partial_8 + x_9 \partial_7, Z_2 = x_3 x_2 + x_4 x_7 \partial_1 + x_4 \partial_8 + x_7 \partial_9, Z_3 = x_3 x_2 + x_4 x_7 \partial_1 + x_4 \partial_8 + x_7 \partial_9, H_1 = [Z_1, Z_2], H_2 = [Z_2, Z_3]$ |

3.4.2f. $\mathfrak{utc}(4|3; K)$: recapitulation. If $p = 0$, we have $\dim g_1 = 18$, same as above for $p = 2$. If $p = 0$, then the $g_0$-module $g_1$ possesses a 12-dimensional submodule $V = S^2(\mathfrak{id}_{4|3}(3)) \otimes S^2(\mathfrak{id}_{4|3}(2))$ (this is what comes from $\mathfrak{sl}(3)$, the common parts of the two glued superalgebras $\mathfrak{le}(3)$ and $\mathfrak{le}(3; 3)$, see [ShP]); the quotient of $g_1$ modulo this submodule is of dim 6 and isomorphic to the tensor product $S^2(\mathfrak{id}_{4|3}(3)) \otimes S^2(\mathfrak{id}_{4|3}(2))$ of tautological modules over $\mathfrak{sl}(3)$ and $\mathfrak{sl}(2)$. Clearly, for $p \neq 0$ (except, perhaps, for $p = 3$; we have to check), both $V$ and the quotient are irreducible. The quotient module is not a direct summand: $g_1$ is an indecomposable $g_0$-module.

Let us denote the indeterminates of functions that generate the two copies of $\mathfrak{le}(3)$ with common part $\mathfrak{sl}(3)$ by $u_1, u_2, u_3, \xi_1, \xi_2, \xi_3$ and $u_1', u_2', u_3', \xi_1', \xi_2', \xi_3'$, respectively, assuming that if

$\deg \xi f(u, \xi) = 1$ and $\Delta f = 0$,

harmonic functions being singled out by the “odd Laplacian” $\Delta = \sum \frac{\partial^2}{\partial u_i \partial \xi_i}$;

then we identify

$$f(u, \xi) \text{ with } f(u', \xi'),$$

$$f(u) \text{ with } \sum \frac{\partial f(u')}{\partial \xi_i'} \xi_j' \xi_k' \text{ for any even permutation } (i, j, k) \text{ of } (1, 2, 3).$$

Now, set $\deg \xi_i = \deg \xi_i' = 1$ and $\deg u_i = \deg u_i' = 2$ for all indices, whereas the degree of the element of $\mathfrak{utc}$ with generating function $f$ is $\deg \mathfrak{Lie}(f) = \deg(f) - 3$. Since we factorize both copies of the spaces of generation functions modulo constants, the depth of the resulting Lie superalgebra is equal to 2.

The component $g_{-2}$ is spanned by functions of degree 1, i.e., by $\xi_i = \xi_i'$ (3 elements).

The component $g_{-1}$ is spanned by functions of degree 2, i.e., $u_i = \xi_i' \xi_i' \text{ and } \xi_i \xi_j = u_i'$ for any even permutation $(i, j, k)$ of $(1, 2, 3)$.

The component $g_0$ is spanned by functions of degree 3. These are

$$\mathfrak{sl}(3) = \{ f \in \operatorname{Span}(u_i \xi_j) \mid \Delta f = 0 \} = \{ f \in \operatorname{Span}(u_i' \xi_j') \mid \Delta f = 0 \},$$
and also
\[ \mathfrak{gl}(2) = \text{Span}(\sum u_i \xi_i, \sum u'_i \xi'_i, \xi_1 \xi_2 \xi_3, \xi'_1 \xi'_2 \xi'_3). \]

The component \( \mathfrak{g}_1 \) is spanned by degree 4 functions: 6 monomials of degree 2 in \( u \) span \( V \), and 9 monomials of the form \( u_i \xi_\alpha \xi_\beta \) span a subspace \( W \); analogous monomials in primed indeterminates span \( V' \) and \( W' \). How to glue these spaces?

We have \( W \supset W_0 \) (and \( W' \supset W'_0 \)), where subspaces \( W_0 \) and \( W'_0 \) consist of harmonic functions. We see (from \[\text{ShP}\]) that \( V \) is glued with \( W'_0 \), while \( W_0 \) with \( V' \). The subspace \( V \oplus W_0 = V' \oplus W'_0 \) obtained is precisely \( S^2(\text{id}_{\mathfrak{sl}(3)}) \otimes \text{id}_{\mathfrak{sl}(2)} \) as \( \mathfrak{g}_0 \)-module; it is exactly component of degree 1 of the subalgebra \( \mathfrak{sl}' \) in the grading considered.

The quotient of \( \mathfrak{g}_1 \) modulo this submodule is a \( \mathfrak{g}_0 \)-module of dimension 6 isomorphic to \( \text{id}_{\mathfrak{sl}(3)} \otimes \text{id}_{\mathfrak{sl}(2)} \); the \( \mathfrak{g}_0 \)-module \( \mathfrak{g}_1 \) is indecomposable.

3.4.2g. About \( \mathfrak{vle}(4|3; 1) \). Let \( \mathfrak{g} \) be a Lie algebra, \( \mathcal{O}(x) \) an associative algebra of “functions” in indeterminates \( x \) (polynomials, divided powers, etc.), and \( d \) a derivation of \( \mathfrak{g} \). The expression

\[ X \mapsto 1 \otimes X + d \otimes \text{div}(X), \text{ where } X \in \mathfrak{vect}(x), \]

determines a \( \mathfrak{vect}(x) \)-action on \( \mathfrak{h} = \mathfrak{g} \otimes \mathcal{O}(x) \) commuting with the operator \( d \otimes 1 \).

Now, if we identify the \((-1)\)st component of \( \mathfrak{vle}(4|3; 1) \) with \( V \otimes \Lambda(2) \), then the 0th component would contain an ideal \( \mathfrak{sl}(2) \otimes \Lambda(2) \), a subalgebra isomorphic to \( \mathfrak{vect}(2) \), acting on the \((-1)\)st component as \( 1 \otimes X + E_{11} \otimes \text{div}(X) \), where \( X \in \mathfrak{vect}(2) \), and commuting with the ideal, and instead of the center we have to add the derivation \( E_{11} \otimes 1 \), where \( E_{11} \) is a matrix unit.
Table 1. Dynkin diagrams for Lie superalgebras: \( p = 2 \)

| Diagrams | \( \mathfrak{g} \) | \( v \) | \( ev \) | \( od \) | \( png \) | \( ng \leq \min(\ast, \ast) \) |
|----------|----------------|-------|-------|-------|-------|-----------------|
| 1) \( \cdots \) | \( ooc(2; 2k_0|2k_1) \times \mathbb{K}I_0 \) if \( k_0 + k_1 \) is odd; \( ooc(1; 2k_0|2k_1) \times \mathbb{K}I_0 \) if \( k_0 + k_1 \) is even. | \( k_0 + k_1 \) | \( k_0 - 2 \) | \( k_1 \) | \( \bar{0} \) | \( 2k_0 - 4, 2k_1 \) |
| 2) \( \cdots \) | \( k_0 - 1 \) | \( k_1 - 1 \) | \( 2k_0 - 3, 2k_1 - 1 \) |
| 3) \( \cdots \) | \( ooc'_{III}(2k_0 + 1|2k_1) \) | \( k_0 + k_1 \) | \( k_0 - 1 \) | \( k_1 \) | \( \bar{0} \) | \( 2k_0 - 2, 2k_1 - 1 \) |
| 4) \( \cdots \) | \( k_1 - 1 \) | \( k_0 - 1 \) | \( 2k_0 - 1, 2k_1 - 2 \) |
| 5) \( \cdots \) | \( pec(2; m) \times \mathbb{K}I_0 \) for \( m \) odd; \( pec(1; m) \times \mathbb{K}I_0 \) for \( m \) even. | \( m \) | \( \) |

**Notation** The Dynkin diagrams in Table ?? correspond to Cartan matrix Lie superalgebras close to ortho-orthogonal and periplectic Lie superalgebras. Each thin black dot may be \( \otimes \) or \( \odot \); the last five columns show conditions on the diagrams; in the last four columns, it suffices to satisfy conditions in any one row. Horizontal lines in the last four columns separate the cases corresponding to different Dynkin diagrams. The notation are: \( v \) is the total number of nodes in the diagram; \( ng \) is the number of “grey” nodes \( \otimes \)’s among the thin black dots; \( png \) is the parity of this number; \( ev \) and \( od \) are the number of thin black dots such that the number of \( \otimes \)’s to the left from them is even and odd, respectively.
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