Strichartz and local smoothing estimates for stochastic dispersive equations with linear multiplicative noise

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Abstract. We study a quite general class of stochastic dispersive equations with linear multiplicative noise, including especially the Schrödinger and Airy equations. The pathwise Strichartz and local smoothing estimates are derived here in both the conservative and non-conservative case. In particular, we obtain the $\mathbb{P}$-integrability of constants in these estimates, where $\mathbb{P}$ is the underlying probability measure. Several applications are given to nonlinear problems, including local well-posedness of stochastic nonlinear Schrödinger equations with variable coefficients and lower order perturbations, integrability of global solutions to stochastic nonlinear Schrödinger equations with constant coefficients. As another consequence, we prove as well the large deviation principle for the small noise asymptotics.

Keywords: Local smoothing estimates; pseudo-differential operators, stochastic dispersive equation; Strichartz estimates.

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1 Introduction and main results

We are concerned with the stochastic dispersive equation with linear multiplicative noise

\[ dX(t) = iP(x,D)X(t)dt + F(t)dt - \mu X(t)dt + X(t)dW, \quad t \in (0,T), \quad (1.1) \]

\[ X(0) = X_0. \]

Here, \( X \) is a complex-valued function on \([0,T] \times \mathbb{R}^d\), \( T \in (0,\infty) \), \( P(x,D) \) is a pseudo-differential operator of order \( m \geq 2 \), \( D_j = -i\partial_{x_j}, \, D = (D_1, \cdots, D_d) \). The term \( W \) is a colored Wiener process of the form

\[ W(t,x) = \sum_{j=1}^{N} \mu_j e_j(x) \beta_j(t), \quad t \geq 0, \, x \in \mathbb{R}^d, \quad (1.2) \]

and \( \mu(x) = \frac{1}{2} \sum_{j=1}^{N} |\mu_j|^2 e_j^2(x) \), where \( \mu_j \in \mathbb{C}, e_j \) are real-valued functions, and \( \beta_j(t) \) are independent real Brownian motions on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with natural filtration \((\mathcal{F}_t)_{t \geq 0}\). For simplicity, we consider \( N < \infty \), but the arguments in this paper extend also to the case where \( N = \infty \). We assume that \( X_0 \) is \( \mathcal{F}_0 \) measurable and \( F \) is \( \{\mathcal{F}_t\} \)-adapted throughout this paper.

Stochastic dispersive equations arise in various fields of physics. An important model is the stochastic nonlinear Schrödinger equation where \( P(x,D) = -\Delta, \, F = -\lambda i|X|^{\alpha-1}X, \, \lambda = \pm 1 \) and \( \alpha \in (1,\infty) \). This equation is proposed as a model for the propagation of nonlinear dispersive waves in nonlinear or random media, the coefficient \( \lambda = 1 \) (resp. \(-1\)) corresponds to the focusing (resp. defocusing) case. See e.g. [2, 19, 20]. In particular, when \( \text{Re} \mu_j = 0, \, 1 \leq j \leq N \), \( -\mu X dt + X dW \) is indeed the Stratonovitch product \( X \circ dW \), and the mass of the homogeneous solution is pathwisely conserved. This case will be called the conservative case in this paper. In the non-conservative case, i.e. \( \text{Re} \mu_j \neq 0 \) for some \( 1 \leq j \leq N \), this equation plays an important role in the application to open quantum systems. See e.g. [11, 12, 29]. An important feature in this case is, that the mass of the homogeneous solution is no longer a constant, but a positive continuous martingale, which implies conservation in mean norm square which is crucial to define the so called “physical probability law” (see [11]). For other dispersive type equations, see e.g. [21] for the stochastic Korteweg-de Vries (KdV) equation where \( P(x,D) = D^3 \) and \( F = \frac{1}{2}\partial_x X^2 \), and [16, 17, 22, 23] for Schrödinger and KdV equations with modulated dispersion.
Unlike the usual parabolic case, the principle operator of a dispersive equation usually generates a unitary group in the standard $L^2$ space. Thus, a global smoothing effect is excluded in Sobolev spaces $H^s(\mathbb{R}^d)$, $s > 0$, which is the source of many difficulties to study nonlinear problems. Furthermore, although the principle operator is monotone, the variational approach (see e.g. [38]) is not applicable to stochastic dispersive equations, due to the lack of coercivity of the principle operator.

Here we shall study Strichartz and local smoothing estimates for stochastic dispersion equations, which are two most stable ways of measuring dispersion and play an important role to study nonlinear problems.

The Strichartz estimates give space-time integrability of solutions, while the local smoothing estimates allow to gain $(m-1)/2$ derivatives of solutions on every bounded domains. We refer to [31, 34, 39, 40, 41, 44] for Strichartz estimates and [15, 25, 26, 27, 32, 34, 35, 39] for local smoothing estimates in the deterministic case.

For stochastic nonlinear Schrödinger equations, the global well-posedness was first studied in [19, 20] for general multiplicative noises. The key Strichartz estimates for the stochastic convolution were proved there by using the theory of $\gamma$-radonifying operators, which, as the role of Hilbert-Schmidt operators on Hilbert spaces, allows to treat noises in Banach spaces. An improved stochastic Strichartz estimates was proved in [14], based on which global well-posedness was obtained on a two-dimensional compact manifold. See [30] for the global well-posedness in the full mass subcritical case via the stochastic Strichartz estimates in [14]. See also the recent work [13] for martingale solutions in the energy space on compact manifolds.

Recently, using a different approach based on the rescaling transformation (see (2.11) below), global well-posedness for stochastic nonlinear Schrödinger equations with linear multiplicative noise has been proved in the optimal mass and energy subcritical cases ([6, 7]), where the key role is played by the pathwise Strichartz and local smoothing estimates. It should also be mentioned that, the rescaling approach is quite robust and fits well with the theory of maximal operators. In particular, solutions continuously depend on the initial condition pathwisely and satisfy the strict cocycle property, so give rise to stochastic dynamical systems (see [11]). We refer to [8] for stochastic logarithmic Schrödinger equations, [9] for noise effect in the non-conservative case, and [10] for optimal control problems.

In this paper, we prove the pathwise Strichartz and local smoothing estimates for quite general stochastic dispersive equations with linear multi-
plicative noise in a uniform manner, including especially the Schrödinger and Airy equations.

Moreover, motivated by scattering and optimal control problems, we also obtain explicit upper bounds and $\mathbb{P}$-integrability of constants in these estimates. In particular, for the homogeneous Schrödinger and Airy equations, the constants in the local smoothing estimates are exponentially $\mathbb{P}$-integrable in the conservative case.

Several applications are given to nonlinear problems. Pathwise local well-posedness is proved for stochastic nonlinear Schrödinger equations with variable coefficients and also with lower order perturbations. Moreover, the $\mathbb{P}$-integrability of global solutions are obtained for the stochastic nonlinear Schrödinger equation mentioned above in both the mass and energy sub-critical cases, which can be viewed as a complement to [6, 7, 9] and is of importance for optimal control problems (see [10]).

Another application we obtain in this paper is that the large deviation principle for the small noise asymptotics for general linear stochastic dispersive equations, as well as stochastic nonlinear Schrödinger equations with variable coefficients, in the conservative case.

Notations. For any $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$ and any multi-index $\alpha = (\alpha_1, \cdots, \alpha_d)$, $\langle x \rangle = (1 + |x|^2)^{1/2}$, $|\alpha| = \sum_{j=1}^d \alpha_j$, $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}$, $\langle \partial_x \rangle = (I - \Delta_x)^{1/2}$. Let $D_{x_j} = -i\partial_{x_j}$, $D_x^\alpha$ is defined similarly. We will use the notation $\xi$ for the phase variable.

Given $1 \leq p \leq \infty$, $p'$ is the conjugate number, i.e., $1/p' + 1/p = 1$. $L^p = L^p(\mathbb{R}^d)$ is the space of $p$-integrable complex functions with the norm $| \cdot |_{L^p}$. In particular, $L^2$ is the Hilbert space with the inner product $\langle u, v \rangle_2 = \int u(x)v(x)dx$, and $| \cdot |_2 = | \cdot |_{L^2}$. As usual, $W^{s,p} = |D|^{-s}L^p(\mathbb{R}^d)$, $W^{s,p} = (\langle D \rangle)^{-s}L^p(\mathbb{R}^d)$, and $H^1 = W^{1,2}$. Let $\mathcal{S}$ denote the space of rapid decreasing functions and $\mathcal{S}'$ the dual space of $\mathcal{S}$. For any $f \in \mathcal{S}$, $\hat{f}$ is the Fourier transform of $f$, i.e. $\hat{f}(\xi) = \int e^{-ix\cdot\xi}f(x)dx$.

For any Banach space $\mathcal{X}$, $L^p(0,T;\mathcal{X})$ is the space of $p$-integrable $\mathcal{X}$-valued functions with the norm $\| \cdot \|_{L^p(0,T;\mathcal{X})}$, and $C([0,T];\mathcal{X})$ is the space of continuous $\mathcal{X}$-valued functions with the super norm in $t$. For two Banach spaces $\mathcal{X}, \mathcal{Y}$, $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is the space of linear continuous operators from $\mathcal{X}$ to $\mathcal{Y}$, and $\mathcal{L}(\mathcal{X}) = \mathcal{L}(\mathcal{X}, \mathcal{X})$.

Throughout this paper, we use $C(\cdots)$ for various constants that may change from line to line.
2 Formulations of main results

To begin with, let us introduce suitable spaces for Strichartz and local smoothing estimates.

Set \( H^s_b := \{ v \in \mathcal{S}' : \langle x \rangle^s (I - \Delta)^{s/2} v \in L^2 \} \) with the norm \( \| v \|_{H^s_b} = \| \langle x \rangle^s (I - \Delta)^{s/2} v \|_2 \). For any \( s \in [0, \infty) \), \( p,q \in [2, \infty] \), set

\[
\mathcal{X}_{T,s,p,q} := L^q(0, T; \dot{W}^{s,p}) \cap L^2(0, T; H^{m-1}_I),
\]
equipped with the norm \( \| u \|_{\mathcal{X}_{T,s,p,q}} = \| u \|_{L^q(0, T; \dot{W}^{s,p})} + \| u \|_{L^2(0, T; H^{m-1}_I)} \).

The dual space of \( \mathcal{X}_{T,s,p,q} \) is denoted by \( \mathcal{X}_{T,s,p,q}' \), and \( \| u \|_{\mathcal{X}_{T,s,p,q}'} := \inf \{ \| u_1 \|_{L^{q'}(0, T; \dot{W}^{-s,p'})} + \| u_2 \|_{L^2(0, T; H^{-(m-1)/2}_I)} : u = u_1 + u_2, u_1 \in L^{q'}(0, T; \dot{W}^{-s,p'}), u_2 \in L^2(0, T; H^{-(m-1)/2}_I) \} \).

We say that \( a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d) \) is a symbol of class \( S^m \), if for any multi-indices \( \alpha, \beta \in \mathbb{N}^d \), \( |\partial^\alpha_x \partial^\beta_\xi a(x, \xi)| \leq C_{\alpha, \beta} \langle \xi \rangle^{m-|\alpha|} \). The semi-norms \( |a|^{(l)}_{S^m} \), \( l \in \mathbb{N} \), are defined by

\[
|a|^{(l)}_{S^m} = \max_{|\alpha + \beta| \leq l} \sup_{\xi, \eta \in \mathbb{R}^{2d}} \{ |\partial^\alpha_x \partial^\beta_\xi a(x, \xi)| \langle \xi \rangle^{-(m-|\alpha|)} \}.
\]

Moreover, \( \Psi_a \) (or \( a(x, D) \)) denotes the pseudo-differential operator associated with the symbol \( a(x, \xi) \), i.e.,

\[
\Psi_a v(x) = a(x, D)v(x) = (2\pi)^{-d} \int e^{ix \cdot \xi} a(x, \xi) \hat{v}(\xi) d\xi, \quad v \in \mathcal{S}.
\]

In this case, we write \( \Psi_a \in S^m \) (or \( a(x, D) \in S^m \)) when no confusion arises.

We shall assume that the principle symbol and the spatial functions in the noise satisfy the following assumptions:

(A0) \( P(x, \xi) = P_1(x, \xi) + iP_2(x, \xi), \) \( P_1 \in S^m, \) \( P_2 \in S^{m-1}, \) \( P_j(x, \xi) \) are real polynomials of \( \xi \), and \( P(x, D) \) is self-adjoint.

(A1) Asymptotical flatness. For any multi-indices \( \alpha, \beta \),

\[
|\partial^\alpha_\xi \partial^\beta_x P_1(x, \xi)| \leq C_{\alpha \beta} \langle \xi \rangle^{-2} \langle \xi \rangle^{m-|\alpha|}, \quad \beta \neq 0,
\]

\[
|\partial^\alpha_\xi \partial^\beta_x P_2(x, \xi)| \leq C_{\alpha \beta} \langle \xi \rangle^{-2} \langle \xi \rangle^{m-1-|\alpha|}.
\]

Moreover, for \( 1 \leq j \leq N, e_j \in C^\infty(\mathbb{R}^d) \), and

\[
|\partial^\alpha_x e_j(x)| \leq C_{\alpha} \langle \xi \rangle^{-2}, \quad \alpha \neq 0.
\]
(A2) Non-trapping condition. The bicharacteristic flow associated with the principle symbol $P(x, \xi)$ is non-trapped. More precisely, let $(X, \Xi)(t, x, \xi)$ be a flow generated by the Hamiltonian vector field

$$H_P = \sum \partial_{\xi_j} P(x, \xi) \partial_{x_j} - \partial_{x_j} P(x, \xi) \partial_{\xi_j},$$

that is, $(X, \Xi)$ is a solution to

$$\frac{dX}{dt} = \nabla_\xi P(X, \Xi), \quad X(0) = x,$$

$$\frac{d\Xi}{dt} = (-1) \nabla_x P(X, \Xi), \quad \Xi(0) = \xi.$$

Then, for any $(x, \xi) \in T^*\mathbb{R}^d / \{0\}$, $|X(t)| \to \infty$, as $t \to \pm \infty$.

(A3) Strichartz-type estimate. There exists $(s, p, q) \in [0, \infty) \times [2, \infty) \times \left(2, \infty\right]$ such that

$$\|e^{itP}u\|_{L^q(0,\infty);\dot{W}^{s,p}} \leq C|u|_2. \quad (2.2)$$

The triple $(s, p, q)$ such that (2.2) holds is called admissible, and the set of all admissible triples is denoted by $A$. In particular, $(0, 2, \infty) \in A$. (Note that, we do not treat the endpoint case $p = \infty$, $q = 2$ here.)

Remark 2.1. Assumptions (A0)–(A2) are mainly required for local smoothing estimates. See Theorem 4.1 and Lemma 4.4 below. See also [15] for more general conditions for local smoothing estimates. We mention that, for $P$ elliptic with variable coefficients under some appropriate flatness conditions at infinity, it was proved in [27] that the local smoothing estimate is equivalent to the non-trapping condition of the Hamiltonian flow.

Remark 2.2. The smoothness of the spatial functions $e_j$ is assumed for technical reasons, particularly, to perform pseudo-differential calculus. One can also assume $e_j \in C^N_b(\mathbb{R}^d)$ as in [35] with $N$ large enough such that the pseudo-differential calculus can be carried out. One may also weaken the regularity of $e_j$ by using paradifferential calculus as in [44, 39].

Example 2.3. Schrödinger operator. Consider $P(x, D) = \sum_{j,k=1}^d D_j a^{jk}(x) D_k$, $x \in \mathbb{R}^d$, $d \geq 1$, $a^{jk}$ are real valued, symmetric, and satisfy some appropriate conditions (see Assumptions (B1) and (B2) below). In this case,
\[ P(x, \xi) = \sum_{j,k=1}^{d} a^{jk}(x)\xi_j \xi_k - i \partial_j a^{jk}(x) \xi_k. \]  
In particular, when \( a^{jk} = \delta_{jk} \), \( P(x, D) = -\Delta \). We have the admissible set (see [37])

\[ \mathcal{A} = \left\{ (s, p, q) : s = 0, \frac{2}{q} = d\left( \frac{1}{2} - \frac{1}{p} \right), (p, q) \in [2, \infty) \times (2, \infty) \right\}. \]  
(2.3)

The pairs \((p, q)\) in (2.3) are called Strichartz pairs below.

**Example 2.4.** Airy operator. Consider \( P(x, D) = D^3 \), \( d = 1 \), and so \( P(x, \xi) = \xi^3 \), \( \xi \in \mathbb{R} \). This operator mainly arises in the (generalized) KdV equations, for which we have (see [33, (1.5)])

\[ \mathcal{A} = \left\{ (s, p, q) : s = \theta \alpha, p = \frac{2}{1 - \theta}, q = \frac{6}{\theta(\alpha + 1)}, (\theta, \alpha) \in [0, 1) \times [0, 1/2] \right\}. \]

**Example 2.5.** Generalization of the Schrödinger operator \( P(x, D) = (-\Delta)^m \), \( m \in \mathbb{N} \), \( m \geq 1 \), \( d \geq 2 \). We have (see Remark (a) on p.49 in [34])

\[ \mathcal{A} = \left\{ (s, p, q) : s = \theta \frac{m}{2} (d(m - 1)), p = \frac{2}{1 - \theta}, q = \frac{4}{d\theta}, \theta \in [0, \frac{2}{d}) \right\}. \]

Moreover, for the generalization of the Airy operator \( P(x, D) = D^m \), \( m \in \mathbb{N} \), \( m \geq 3 \), \( d = 1 \), we have (see [34, Theorem 2.1])

\[ \mathcal{A} = \left\{ (s, p, q) : s = \frac{(m - 2)\theta}{4}, p = \frac{2}{1 - \theta}, q = \frac{4}{\theta}, \theta \in [0, 1) \right\}. \]

The main result of this paper is formulated below.

**Theorem 2.6.** Assume (A0)-(A3). Let \((s_1, p_1, q_1), (s_2, p_2, q_2) \in \mathcal{A}\) be any admissible triples. Then, we have

(i). For any \( \mathcal{F}_0 \)-measurable \( X_0 \in L^2 \), \( \{\mathcal{F}_t\} \)-adapted \( F \in X'_{T,-s_2,p_2,q_2}; \mathbb{P}\)-a.s., the solution \( X \) to (1.1) satisfies that \( \mathbb{P}\)-a.s.

\[ \|e^{-\Phi(W)} X\|_{X_{T,s_1,p_1,q_1}} \leq D(e, e^*, T)(\|X_0\|_2 + \|e^{-\Phi(W)} F\|_{X'_{T,-s_2,p_2,q_2}}). \]  
(2.4)

Here,

\[ \Phi(W) = W - \frac{1}{2} t \sum_{j=1}^{N} (|\mu_j|^2 + \mu_j^2) e_j^2, \]  
(2.5)
Remark 2.8. The process $e$ is actually related to the martingale property of homogeneous solutions in the non-conservative case, while $e^*$ arises in the duality case and involves the semi-martingale $M^*$ in (4.8) below.

Remark 2.9. The upper bound in (2.4) (and also (2.7)) can be improved in the homogeneous case (see Theorem 1.1 and Lemma 5.2 below), but we will not seek the optimal upper bound here. However, we have the $\mathbb{P}$-integrability of constants and solutions which are important for optimal control problems (see [10]).

More precisely, similarly to [7, Lemma 3.6], we have that for any $1 \leq \rho < \infty$, \( \mathbb{E} \sup_{t \in [0,T]} (M(t) + M^*(t))^\rho \leq C(\rho) \), where $M$ and $M^*$ are as in (4.6) and (1.8) below respectively. Thus, it follows from (2.6) that

\[
D(e, e^*, T) \in \bigcap_{1 \leq \rho < \infty} L^\rho(\Omega).
\]
Moreover, when $X_0 \in L^p(\Omega; L^2)$ and $F \in L^p(\Omega; \mathcal{X}_{T,-s_0,p_0,q_0}^{T,s,p,q})$ for some admissible triple $(s_0, p_0, q_0) \in \mathcal{A}$ and for all $1 \leq \rho < \infty$, we have that

$$X \in \bigcap_{1 \leq \rho < \infty} \bigcap_{(s,p,q) \in \mathcal{A}} L^\rho(\Omega; \mathcal{X}_{T,s,p,q}).$$

In particular, by the mild reformulation of (1.1), this implies the $\mathbb{P}$-integrability of the stochastic convolution, i.e.,

$$\int_0^\cdot e^{i(-s)P(x,D)}X(s)\,dW(s) \in \bigcap_{1 \leq \rho < \infty} \bigcap_{(s,p,q) \in \mathcal{A}} L^\rho(\Omega; \mathcal{X}_{T,s,p,q}).$$

**Remark 2.10.** The $\mathbb{P}$-integrability of solutions to (1.1) can be proved by the stochastic Strichartz estimate in [14]. Here, we obtain quantitative information for general dispersive equations with linear multiplicative noise. As a matter of fact, we obtain (2.4) in the pathwise way. In particular, for the Schrödinger and Airy operators, the constants in the homogeneous local smoothing estimates are exponentially $\mathbb{P}$-integrable in the conservative case. See Theorem 4.1 and Remark 4.3 below. Moreover, stochastic dispersive equations with lower order perturbations can be treated here, and the pathwise estimates are applicable as well to the large deviation principle for the small noise asymptotics in the conservative case. See Theorems 6.3 and 6.6 below.

Below we present the Strichartz and local smoothing estimates for stochastic dispersive equations with lower order perturbations.

**Theorem 2.11.** Consider the equation

$$dX = iP(x,D)Xdt + b(t,x,D)Xdt + Fdt - \mu X(t)dt + X(t)dW, \quad (2.8)$$

$$X(0) = X_0.$$  

Here, $P(x,D)$, $\mu$ and $W$ are as in (1.1). For each $x, \xi \in \mathbb{R}^d$, $t \mapsto b(t,x,\xi)$ is an adapted continuous process satisfying that, for any multi-indices $\alpha, \beta$,

$$\sup_{t \in [0,T]} |\partial_x^\alpha \partial_x^\beta b(t,x,\xi)| \leq g(T)C_{\alpha,\beta} \langle x \rangle^{-2} \langle \xi \rangle^{m-1-|\alpha|}, \quad \mathbb{P} - a.s., \quad (2.9)$$

where $t \mapsto g(t)$ is an $\{\mathcal{F}_t\}$-adapted continuous process.
Then, under Assumptions (A0)–(A3), for any \((s_i, p_i, q_i) \in \mathcal{A}, \ i = 1,2,\) and for any \(\mathcal{F}_0\)-measurable \(X_0 \in L^2, \{\mathcal{F}_t\}\)-adapted \(F \in \mathcal{X}'_{T,-s_2,p_2',q_2'}, \mathbb{P}\)-a.s., the solution \(X\) to (2.8) satisfies that \(\mathbb{P}\)-a.s.,
\[
\|e^{-\Phi(W)}X\|_{\mathcal{X}_{T,s_1,p_1,q_1}} \leq C_T(|X_0|^2 + \|e^{-\Phi(W)}F\|_{\mathcal{X}'_{T,-s_2,p_2',q_2'}}),
\]
(2.10)
where \(\Phi(W)\) is as in (2.5) and \(t \mapsto C_t\) is adapted, increasing and continuous.

Remark 2.12. The constant \(C_T\) in (2.10) may not be \(\mathbb{P}\)-integrable, due to the lack of martingale property of homogeneous solutions to (2.8) with lower order perturbations in general. In fact, in the derivation of (2.10) the Gronwall inequality will produce a non-integrable double exponential of Brownian motions (see e.g. (4.40) below).

As mentioned above, several applications are given to nonlinear problems, which are actually main motivations for the estimates in Theorems 2.6 and 2.11. We first show the pathwise local well-posedness for stochastic nonlinear Schrödinger equations with variable coefficients and lower order perturbations, in the full mass (sub)critical range of the exponents of nonlinearity. See Theorem 6.1 below. For the typical stochastic nonlinear Schrödinger equation as studied in [6, 7, 19, 20, 30], we also prove the \(\mathbb{P}\)-integrability of global solutions which is important for optimal control problems. See Theorem 6.2.

Moreover, these estimates apply also to the large deviation principle for the small noise asymptotics for linear stochastic dispersive equations, as well as stochastic nonlinear Schrödinger equations with variable coefficients, in the conservative case. See Theorems 6.3 and 6.6 below.

The proof of Theorem 2.6 relies on the rescaling approach as in [6, 7, 9]. The rescaling transformation is in fact a Doss-Sussman type transformation in infinite dimension, which reduces the stochastic dispersive equation (1.1) to a random equation (2.12) below with lower order perturbations.

This point of view allows pathwise analysis of stochastic partial differential equations, including sharp pathwise estimates of stochastic solutions, path-by-path uniqueness and random attractors. See e.g. [3, 4] for stochastic porous media equations and the total variations flow. Moreover, the rescaling approach fits quite well with the theory of maximal monotone operators and indeed reveals the structure of stochastic equations. See e.g. [5, 8]. We would also like to mention that, the damped effect of the noise in the non-conservative case, completely different from that in the conservative case, can be revealed by the rescaling approach (see [9]).
Below we shall see that the rescaling transformation leaves the principle symbol unchanged but produces lower order perturbations in the resulting random equation. In the light of this structural feature, we shall prove Strichartz estimates via the control of lower order perturbations.

More precisely, let

$$u := e^{-\Phi(W)}X,$$  \hspace{1cm} (2.11)

where $\Phi(W)$ is as in (2.5). By (1.1) we have

$$\partial_t u(t) = iP_t(x, D)u(t) + f(t)$$  \hspace{1cm} (2.12)

with the initial datum $u(0) = X_0$, $f(t) = e^{-\Phi(W(t))}F(t)$, and

$$P_t(x, D) = e^{-\Phi(W(t,x))}P(x, D)e^{\Phi(W(t,x))}, \quad t \in [0, T].$$

In particular, $P_0(x, \xi) = P(x, \xi)$. The equivalence of solutions to (1.1) and (2.12) can be proved similarly as in [6, Lemma 6.1]. Note that

$$P_t = e^{-\Phi(W(t))}Pe^{\Phi(W(t))} = P + e^{-\Phi(W(t))}[P, e^{\Phi(W(t))}].$$  \hspace{1cm} (2.13)

It follows that

$$\partial_t u(t) = iP_t u(t) + ie^{-\Phi(W(t))}[P, e^{\Phi(W(t))}]u(t) + f(t),$$  \hspace{1cm} (2.14)

where $e^{-\Phi(W)}[P, e^{\Phi(W)}]$ is of lower order $m-1$. Therefore, the original problem is now reduced to this random equation.

**Theorem 2.13.** Assume (A0)-(A3). (i). For any $(s_i, p_i, q_i) \in \mathcal{A}$, $i = 1, 2$, and for any $\mathcal{F}_0$-measurable $u_0 \in L^2$ and $\{\mathcal{F}_t\}$-adapted $f \in \mathcal{X}'_{T, -s_2, p_2', q_2'}$, $\mathbb{P}$-a.s., the solution $u$ to (2.12) satisfies $\mathbb{P}$-a.s. that

$$\|u\|_{\mathcal{X}_{T, s_1, p_1, q_1}} \leq D(e, e^*, T)(|u_0|_2 + \|f\|_{\mathcal{X}'_{T, -s_2, p_2', q_2'}}),$$  \hspace{1cm} (2.15)

where $D(e, e^*, T)$ is as in (2.6).

(ii). Assume in addition that $P(x, \xi) = P(\xi)$, $u_0 \in H^1$, and $\partial_{x_j} f \in \mathcal{X}'_{T, -s_3, p_3', q_3'}$ for some $(s_3, p_3, q_3) \in \mathcal{A}$, $1 \leq j \leq d$, $\mathbb{P}$-a.s. Then,

$$\|u\|_{\mathcal{X}_{T, s_1, p_1, q_1}} \leq (1 + \beta_T^* + T)^m D(e, e^*, T)(|u_0|_{H^1} + \|f\|_{\mathcal{X}'_{T, -s_2, p_2', q_2'}}$$

$$+ \|\partial_{x_j} f\|_{\mathcal{X}'_{T, -s_3, p_3', q_3'}}).$$  \hspace{1cm} (2.16)
Remark 2.14. The proof presented below applies as well to the stochastic dispersive equation of more general form

\[ dX = iP(x, D)X dt + F dt - \mu X dt + \lambda X dt + \sum_{j=1}^{N} F_k X d\beta_k(t), \quad (2.17) \]

where \( X(0) = X_0, \lambda \in \mathbb{C}, \mu = \frac{1}{2} \sum_{j=1}^{N} |F_k|^2 \), and \( F_k \) are adapted complex valued functions on \( \mathbb{R}^+ \times \mathbb{R}^d \). Under appropriate conditions of \( F_k \), we can perform the rescaling transformation \( u = e^{-\Phi(\beta) - \lambda t} X \) with

\[ \Phi(\beta)(t, x) = \sum_{j=1}^{N} \int_{0}^{t} F_k(s, x)\beta_k(s) - \frac{1}{2} \sum_{j=1}^{N} \int_{0}^{t} (|F_k(s, x)|^2 + F_k^2(s, x)) ds \]

to reduce (2.17) to the random equation below

\[ \partial_t u = i\tilde{P}_i(x, D) u + e^{-\Phi(\beta) - \lambda t} F, \]

where \( \tilde{P}_i(x, D) = e^{-i\tilde{\Phi}(\beta)(t)} P(x, D) e^{i\Phi(\beta)(t)} \). Thus, using similar arguments below one can prove Strichartz and local smoothing estimates for (2.17).

Below we are mainly concerned with the estimates in Theorems 2.6, 2.11 and 2.13. The global well-posedness for (1.1), (2.8) and (2.12) can be proved via approximation procedure with smooth initial data and smooth inhomogeneous parts.

There is an extensive literature on Strichartz estimates for the free group \( \{e^{itP(x, D)}\}_{t \in \mathbb{R}} \), of which the standard proof is based on dispersive estimates, e.g., \( |e^{-it\Delta} u_0|_{L^\infty} \leq Ct^{-d/2} |u_0|_{L^1} \) for the Schrödinger operator, and \( |e^{-it\partial_x^3} u_0|_{L^\infty} \leq Ct^{-\frac{1}{2}} |u_0|_{L^1} \) for the Airy operator. See [31, 37, 43].

However, it is much more difficult to prove Strichartz estimates for operators with lower order perturbations and, as a matter of fact, dispersive estimates do not hold in general (see e.g. [44]).

Inspired by the work [39, 40, 41], we shall prove Strichartz estimates by using local smoothing estimates under appropriate asymptotically flat conditions, which allow to control lower order perturbations. For this purpose, we first prove the local smoothing estimates for (2.14) in the homogeneous case (see Theorem 4.1 below), which actually plays the key role in the proof of Theorem 2.13.
It should be mentioned that, local smoothing estimates for more general operators with lower order perturbations were studied in [15], where, however, the dependence on time of the constants is implicit.

In order to obtain the $\mathbb{P}$-integrability of constants, we prove precisely upper bounds of constants, by estimating the remainders in the expansion of compositions of pseudo-differential operators and also the semi-norms of pseudo-differential operators (see Lemma 3.2 and Corollary 3.3 below). We first treat the easier conservative case in the spirit of [18], and then, for the non-conservative case, we perform the energy method to a new equation as in [35], combined with the Gårding inequality (3.6) and the interpolation estimate (3.10) below. Moreover, the Gronwall inequality used in [15] produces a non-integrable double exponential boundedness of Brownian motions. Instead, we shall use the martingale property of homogeneous solutions to obtain the $\mathbb{P}$-integrability of constants.

Once the homogeneous local smoothing estimates are obtained, by virtue of Assumption (A3) and the Christ-Kiselev lemma, we obtain homogeneous Strichartz estimates and prove Theorem 2.13 by duality arguments, thereby proving Theorem 2.6 via the rescaling transformation.

The remainder of this paper is structured as follows. In Section 3 we review basic results of pseudo-differential operators and present necessary estimates used in subsequent sections. Section 4 is mainly concerned with the homogeneous local smoothing estimates, and Section 5 is devoted to the proof of the main results. In Section 6 we present several applications concerning stochastic nonlinear Schrödinger equation as well as large deviation principle for small noise asymptotics. Finally, for simplicity of exposition, some technical computations are postponed to the Appendix, i.e., Section 7.

3 Preliminary

We first review some basic results of pseudo-differential operators. For more details we refer to [35, 36, 37, 45].

Lemma 3.1. ([36, Theorem 2.6, Theorem 3.1]) Let $a_i \in S^{m_i}, i = 1, 2$. Then, $\Psi_{a_1} \circ \Psi_{a_2} = \Psi_a$ with

$$a(x, \xi) = (2\pi)^{-d} \int \int e^{-iy \cdot \eta} a_1(x, \xi + \eta) a_2(x + y, \xi) dyd\eta \in S^{m_1 + m_2}. $$
Moreover, we have the expansion
\[ a(x, \xi) = \sum_{|\alpha| < n} \frac{1}{\alpha!} \partial_\xi^\alpha a_1(x, \xi) D_x^\alpha a_2(x, \xi) + n \sum_{|\gamma| = n} \int_0^1 \frac{(1 - \theta)^{n-1}}{\gamma!} r_{\gamma, \theta}(x, \xi) d\theta, \]
for any \( n \geq 1 \), where
\[ r_{\gamma, \theta}(x, \xi) = (2\pi)^{-d} \int e^{-i y \cdot \eta} a_1(x, \xi + \theta \eta) D_x^\gamma a_2(x + y, \xi) dy d\eta, \quad (3.1) \]
and \( \{r_{\gamma, \theta}(x, \xi)\}_{|\theta| \leq 1} \) is a bounded symbol of \( S^{m_1 + m_2 - |\gamma|} \).

Note that, the commutator \( i[\Psi_a, \Psi_b] := i(\Psi_a \Psi_b - \Psi_b \Psi_a) \) is an operator with symbol in \( S^{m_1 + m_2 - 1} \), and the principle symbol is the Poisson bracket
\[ H_{ab} := \{a,b\} = \sum_{j=1}^d \partial_{\xi_j} a \partial_{x_j} b - \partial_{\xi_j} b \partial_{x_j} a. \]

The lemma below allows to estimate the remainder term (3.1) in the expansion of composition of pseudo-differential operators.

**Lemma 3.2.** Let \( a, b \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d) \) be such that for any multi-indices \( \alpha, \beta \) with \( |\alpha + \beta| = l \), \( |\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \leq C_1(l) \langle x \rangle^{\rho_1(\beta)} \langle \xi \rangle^{m_1 - |\alpha|}, \) \( |\partial_\xi^\beta \partial_x^\alpha b(x, \xi)| \leq C_2(l) \langle x \rangle^{\rho_2(\alpha)} \langle \xi \rangle^{m_2 - |\alpha|}, \) where \( \rho_i(\beta) = \rho_i(|\beta|) \) are decreasing on \( |\beta| \). Set
\[ c_\theta(x, \xi) = \int \int e^{-i y \cdot \eta} a(x, \xi + \theta \eta) b(x + z, \xi) dy d\eta dz, \]
where \( \theta \in [0, 1] \). Then,
\[ |c_\theta(x, \xi)| \leq CC_1(l)C_2(k) \langle x \rangle^{\rho_1(0) + \rho_2(0)} \langle \xi \rangle^{m_1 + m_2} \quad (3.2) \]
for \( l, k \) such that \( l > |\rho_2(0)| + d, k > |m_1| + d, \) where \( C \) is independent of \( \theta \).

(See the Appendix for the proof.)

**Corollary 3.3.** Let \( a, b \) be as in Lemma 3.2. For any multi-indices \( \alpha, \alpha', \beta, \beta' \) and \( \theta \in [0, 1] \), let \( c_\theta(x, \xi) \) be as in Lemma 3.2 and
\[ \tilde{c}_\theta(x, \xi) := \int \int e^{-i y \cdot \eta} \partial_\xi^\alpha \partial_x^\beta a(x, \xi + \theta \eta) \partial_\xi^{\alpha'} \partial_x^{\beta'} b(x + z, \xi) dy d\eta dz. \]
Then, there exists $C$ independent of $\theta$ such that
\[
|\partial_\xi^\beta \partial_x^\gamma c_\theta(x,\xi)| \leq CC_1(|\alpha + \beta| + l)C_2(|\alpha + \beta| + k) \langle x \rangle^{\rho_1(0) + \rho_2(0)} \langle \xi \rangle^{m_1 + m_2 - |\alpha|},
\]
and
\[
|\tilde{c}_\theta(x,\xi)| \leq CC_1(|\alpha + \beta| + l)C_2(|\alpha' + \beta'| + k) \langle x \rangle^{\rho_1(\beta) + \rho_2(\beta')} \langle \xi \rangle^{m_1 + m_2 - |\alpha + \alpha'|},
\]
where $C_i(\cdot), \rho_i(\cdot), i = 1, 2, l, k$ are as in Lemma 3.2.

**Lemma 3.4.** Let $a \in S^0$, $p \in (1, \infty)$. There exist a constant $C$ and $l \in \mathbb{N}$ such that
\[
\|\Psi_a\|_{L^p(L^p)} \leq C|a|_{S^0}^{(l)}. \tag{3.3}
\]

See [36, Theorem 4.1] for the case where $p = 2$ and [45, Chapter 1.2] for the general case where $p \in (1, \infty)$.

**Lemma 3.5.** (i) (Gårding inequality) Let $a \in S^1$ with $\text{Re}(x,\xi) \geq 0$ for $|\xi| \geq R$, $R \geq 1$. Then, there exist $j_0 = j_0(d) \in \mathbb{N}$ and $c(d,R)$, such that
\[
\text{Re} \langle \Psi_a f, f \rangle \geq -c(d,R)|a|_{S^1}^{(j_0)}|f|^2, \quad \forall f \in L^2. \tag{3.4}
\]
Moreover, there exists a constant $C(d) > 0$ such that
\[
c(d,R) \leq C(d) \langle R \rangle. \tag{3.5}
\]

(ii) Let $a \in S^{m-1}$, $m \geq 2$, $\text{Re}(x,\xi) \geq 0$ for any $|\xi| \geq R$, $R > 1$, and $|\partial_\xi^\alpha \partial_x^\beta a| \leq C_{\alpha,\beta} (x) x^{-2} \langle \xi \rangle^{m-1-|\alpha|}$ for any multi-indices $\alpha, \beta$. Then,
\[
\text{Re} \langle \Psi_a f, f \rangle \geq -CR\|f\|_{H_{-1}^{m-2}}^2, \quad \forall f \in H_{-1}^{m-2}. \tag{3.6}
\]
where $C$ is independent of $R$.

**Proof.** (i) (3.4) is the standard Garding estimate, see, e.g., [37, Lemma 10.3]. As regards (3.5), let $\varphi$ be a positive smooth function such that $\varphi(\xi) = 1$ if $|\xi| \leq 1$ and $\varphi(\xi) = 0$ if $|\xi| \geq 2$. Set $\varphi_R(\xi) := \tilde{C}(R)\varphi(\frac{\xi}{R})$, where $\tilde{C}(R) := \sup_{x \in \mathbb{R}^d, |x| \leq R} |a(x,\xi)| \leq |a|_{S^1}^{(j_0)} \langle R \rangle$. Then, $\text{Re}(a(x,\xi) + \varphi_R(\xi)) \geq 0$, $\forall x, \xi \in \mathbb{R}^d$. By (3.4), there exist $j_0 = j_0(d) \in \mathbb{N}$ and a constant $c(d)$, such that
\[
\text{Re} \langle \Psi_{a + \varphi_R} f, f \rangle_2 \geq -c(d)(|a + \varphi_R|_{S^1}^{(j_0)})|f|^2_2 \\
\geq -c(d)(|a|_{S^1}^{(j_0)} + |\varphi_R|_{S^1}^{(j_0)})|f|^2_2, \quad \forall f \in L^2. \tag{3.7}
\]
Note that, \( \varphi_R \in S^0 \), and for any \( l \geq 1 \), \( |\varphi_R|^{(l)}_{S^0} = \tilde{C}(R)|\varphi(\frac{1}{R})|^{(l)}_{S^0} \leq |a|^{(l)}_{S^1}|\varphi|^{(l)}_{S^0} \langle R \rangle \). Then, by Lemma 3.4, for some \( l \in \mathbb{N} \),

\[
|\Re \langle \Psi_{\varphi} f, f \rangle_2 | \leq C |a|^{(l)}_{S^1}|\varphi|^{(l)}_{S^0} \langle R \rangle |f|_2^2, \quad \forall f \in L^2.
\] (3.8)

Moreover, using the facts that \( \langle \xi \rangle^{-1} \leq \langle \xi / R \rangle^{-1} \) and \( \langle \xi \rangle^{-1+l} \leq R^{-l} \langle \xi / R \rangle^{-1+l} \) for any \( l \geq 1 \), we have \( |\varphi(\frac{1}{R})|^{(jo)}_{S^1} \leq |\varphi|^{(jo)}_{S^1} \). Hence,

\[
|\varphi_R|^{(jo)}_{S^1} = \tilde{C}(R)|\varphi(\frac{1}{R})|^{(jo)}_{S^1} \leq |a|^{(jo)}_{S^1}|\varphi|^{(jo)}_{S^0} \langle R \rangle .
\] (3.9)

Plugging (3.8) and (3.9) into (3.7), we have

\[
\Re \langle \Psi_a f, f \rangle_2 \geq -c(d)(|a|^{(j0)}_{S^1} + |a|^{(0)}_{S^1}|\varphi|^{(j0)}_{S^1} \langle R \rangle) |f|_2^2 - C |a|^{(j0)}_{S^1}|\varphi|^{(j0)}_{S^0} \langle R \rangle |f|_2^2
\]

for some \( l \in \mathbb{N} \), which implies (3.5).

(ii) Let \( g = \langle x \rangle^{-1} \langle D \rangle^{\frac{m-2}{2}} f \). By Lemma 3.1 and Corollary 3.3,

\[
\langle \Psi_a f, f \rangle = \left\langle \langle x \rangle \langle D \rangle^{\frac{m-2}{2}} \Psi_a \langle D \rangle^{\frac{m-2}{2}} \langle x \rangle g, g \right\rangle = \left\langle \left( \Psi_{\langle x \rangle^2 a(x, \xi)}^{(j0)} + \Psi_{\varphi} \right) g, g \right\rangle,
\]

where \( r \in S^0 \). Note that \( \langle \xi \rangle^{-(m-2)} \langle x \rangle^2 a(x, \xi) \in S^1 \), and \( \Re(\langle \xi \rangle^{-(m-2)} \langle x \rangle^2 a(x, \xi)) \geq 0 \) for any \( |\xi| \geq R \). Then, using (3.4) we obtain

\[
\Re \left\langle \Psi_{\langle x \rangle^2 a(x, \xi)}^{(j0)} g, g \right\rangle \geq -CR|g|_2^2 = -CR\|f\|_{H_{-1}^{m-2}}^2.
\]

Moreover, since \( r \in S^0 \), by Lemma 3.1, \( |\Re \langle \Psi_{\varphi} g, g \rangle | \leq C |g|_2^2 = C \|f\|_{H_{-1}^{(m-2)/2}}^2 \). Therefore, combining the estimates above we obtain (3.6).

**Lemma 3.6.** Fix \( m > 2 \). For any \( u \in \mathcal{S} \) and any \( \varepsilon \in (0, 1) \),

\[
\|u\|_{H_{-1}^{\frac{m-2}{2}}} \leq C \varepsilon^\frac{3}{4} \|u\|_{H_{-1}^{\frac{m-1}{2}}} + C \varepsilon^{-\frac{m-2}{4}} \|u\|_{2},
\] (3.10)

where \( C \) is independent of \( \varepsilon \). In particular,

\[
\|u\|_{H_{-1}^{\frac{m-2}{2}}} \leq C \left( \|u\|_{H_{-1}^{\frac{m-1}{2}}} |u|_2 \right)^{\frac{1}{2}},
\] (3.11)
Lemma 3.7. Let $\theta$ be a smooth nondecreasing function such that $\theta(\xi) = 0$ for $|\xi| \leq 1$, and $\theta(\xi) = 1$ for $|\xi| \geq 2$. Set $\theta_\varepsilon(\xi) := \theta(\varepsilon \xi)$. Then,
\[
(x)^{-1} \langle D \rangle^{\frac{m-2}{2}} = (x)^{-1} \langle D \rangle^{\frac{m-2}{2}} \theta_\varepsilon(D) + (x)^{-1} \langle D \rangle^{\frac{m-2}{2}} (1 - \theta_\varepsilon(D)) =: K_1 + K_2.
\]

Note that, $K_1 = a_\varepsilon(x, D) (x)^{-1} \langle D \rangle^{(m-1)/2}$, where $a_\varepsilon(x, D) := \langle x \rangle^{-1} \theta_\varepsilon(D) \langle D \rangle^{-1/2} \langle x \rangle \in S^0$, due to Corollary 3.3. Since $\langle \xi \rangle^{-1/2} \leq \varepsilon^{1/2}$ on the support of $\theta_\varepsilon$, $|a_\varepsilon(x, \xi)|^{(l)}_{S^0} \leq \varepsilon^{1/2} C(l)$, $\forall l \geq 1$. Hence,
\[
|K_1 u|_2 \leq \varepsilon^{\frac{1}{2}} C(l) \|x\|_{H^{-l-1}}. \tag{3.12}
\]

Moreover, since $\langle \xi \rangle \leq 4\varepsilon^{-1}$ on the support of $1 - \theta_\varepsilon(\xi)$, $|\langle \xi \rangle^{(m-2)/2} (1 - \theta_\varepsilon(\xi))|_{S^0} \leq C\varepsilon^{-(m-2)/2}$, we have
\[
|K_2 u|_2 \leq |\langle D \rangle^{\frac{m-2}{2}} (1 - \theta_\varepsilon(D)) u|_2 \leq C \varepsilon^{-\frac{m-2}{2}} |u|_2. \tag{3.13}
\]

Thus, (3.12) and (3.13) yield (3.10). (3.11) follows by optimization in $\varepsilon$. □

Lemma 3.7. Let $a \in S^0$ and $\theta$ be a smooth nondecreasing function such that $\theta(\xi) = 0$ if $|\xi| \leq 1$ and $\theta(\xi) = 1$ if $|\xi| \geq 2$. Set $\theta_R(\xi) := \theta(\frac{\xi}{R})$, $c_R(x, \xi) := e^{Ma(x, \xi) \theta_R(\xi)} \in S^0$, where $R, M \geq 1$. Let $\|a\|_\infty := |a|_{L^\infty(\mathbb{R}^d)}$.

(i). There exist $l \in \mathbb{N}$ and $C(l) > 0$ such that for any $R \geq 1$,
\[
\|\Psi_{c_R}\|_{\mathcal{L}(L^2)} + \|\Psi_{c_R}^{-1}\|_{\mathcal{L}(L^2)} \leq C(l) M^{l} e^{M\|a\|_\infty}. \tag{3.14}
\]

(ii) For $R = CM^{l} e^{2M\|a\|_\infty}$ with $C, l$ large enough, $\Psi_{c_R}$ is invertible, and
\[
\|\Psi_{c_R}^{-1}\|_{\mathcal{L}(L^2)} \leq C(l) M^{l} e^{M\|a\|_\infty}, \tag{3.15}
\]

(iii) For $R = CM^{l} e^{2M\|a\|_\infty}$ with $C$ and $l$ large enough, we have
\[
\|\Psi_{c_R}\|_{\mathcal{L}(H^{-l-1})} + \|\Psi_{c_R}^{-1}\|_{\mathcal{L}(H^{-l-1})} \leq C(l) M^{l} e^{2M\|a\|_\infty}. \tag{3.16}
\]

The proof of Lemma 3.7 is postponed to the Appendix.
4 Homogeneous local smoothing estimates

This section is mainly concerned with the local smoothing estimates for homogeneous solutions to \((2.12)\).

**Theorem 4.1.** Consider \((2.12)\) in the homogeneous case \(f \equiv 0\), i.e.,
\[
\partial_t u = iP_t(x, D)u, \quad u(0) = u_0. \tag{4.1}
\]
Assume \((A0)-(A2)\). Then, for any \(\mathcal{F}_0\)-measurable \(u_0 \in L^2\), \(\mathbb{P}\)-a.s., the solution \(u\) to \((2.12)\) satisfies \(\mathbb{P}\)-a.s. that
\[
\|u\|_{C([0,T];L^2)} + \|u\|_{L^2(0,T;H_{-1}^{m-1})} \leq \tilde{D}(e, T)|u_0|_2 \tag{4.2}
\]
Here,
\[
\tilde{D}(e, T) = C_1(1 + e_T + |e|_{L^1})^{\frac{l}{2}}(1 + \beta_T^* + T)^{\frac{l}{2}((m-2)^2+m)}e^{C_2(1+\beta_T^*+T)}, \tag{4.3}
\]
for some \(l \geq 1\), where \(C_1 > 0\), \(C_2 \geq 0\), \(\beta_T^* = \sup_{t \in [0,T]} |\beta(t)|\), and \(e\) is as in \((4.5)\) below. In particular, one can take \(l = e = 1\) and \(C_2 = 0\) in the conservative case.

**Remark 4.2.** Similarly to Remark 2.9, we have
\[
\tilde{D}(e, T) \in \bigcap_{1 \leq \rho < \infty} L^\rho(\Omega), \tag{4.4}
\]
and when \(u_0 \in L^\rho(\Omega; L^2)\) for all \(\rho \geq 1\),
\[
u \in \bigcap_{1 \leq \rho < \infty} L^\rho(\Omega; C([0,T];L^2) \cap L^2(0,T;H_{-1}^{m-1})).
\]

**Remark 4.3.** In the conservative case, for the Schrödinger operator \((m = 2)\) or the Airy operator \((m = 3)\), the constant \(\tilde{D}(e, T)\) is even exponentially \(\mathbb{P}\)-integrable. Moreover, if in addition \(u_0 \in L^\infty(\Omega; L^2)\), then the solution is also exponentially \(\mathbb{P}\)-integrable, that is, there exists \(\delta > 0\) such that
\[
\mathbb{E}\exp \left(\delta(\|u\|_{C([0,T];L^2)} + \|u\|_{L^2(0,T;H_{-1}^{m-1})})\right) < \infty.
\]
The key role in the proof of Theorem 4.1 is played by the pseudo-differential operator of order zero constructed in [15]. See also [25, 26, 32, 35] in the Schrödinger case.

**Lemma 4.4.** ([15, Lemma 7.1]) Assume (A0)–(A2). There exist \( \tilde{h}(x, \xi) \in S^0 \) and \( c_1, c_2 > 0 \), such that

\[
H_{\tilde{h}} P \leq -c_1 \frac{|\xi|^{m-1}}{\langle x \rangle^2} + c_2,
\]

and \( |\partial_x^\alpha \partial_{x}^\beta \tilde{h}(x, \xi)| \leq C_{\alpha \beta} \langle x \rangle^{-\rho(\beta)} \langle \xi \rangle^{-|\alpha|} \) for \( (x, \xi) \in T^* \mathbb{R}^d \), where \( \rho(\beta) \) is equal to 0 for \( \beta = 0 \), 1 for \( |\beta| = 1 \), 2 for \( |\beta| \geq 2 \).

As mentioned in Section 2, the Gronwall inequality as in [15] will produce a double exponential bound involving Brownian motions, which, however, is not \( \mathbb{P} \)-integrable in the non-conservative case. Instead, we use the martingale property of homogeneous solutions to control the \( L^2 \)-norm of solutions. Similar semi-martingales in the dual case will also be used in the next section.

As in [6, 7], we use the notations \( U(t,s), s,t \in [0, \infty) \), for evolution operators corresponding to (4.1). Their dual operators are denoted by \( U^*(t,s) \).

**Lemma 4.5.** (i) For any \( \mathcal{F}_0 \)-measurable \( u_0 \in L^2 \), \( \mathbb{P} \)-a.s., we have

\[
|U(t,0)u_0|^2 \leq e(t)|u_0|^2,
\]

where \( e(t) = |e^{-\Phi(W(t))}|^2_{L^\infty} M(t) \),

\[
M(t) = \exp \left\{ \sum_{j=1}^N \left[ \int_0^t v_j(s)d\beta_j(s) - \frac{1}{2} \int_0^t v_j^2(s)ds \right] \right\},
\]

and \( v_j = 2\text{Re} \left\langle \tilde{X}, \mu_j e_j \tilde{X} \right\rangle \frac{|\tilde{X}|^2}{2} \) with \( \tilde{X}(t) = e^{\Phi(W(t))} U(t,0)u_0, 1 \leq j \leq N \).

(ii) For any \( \mathcal{F}_0 \)-measurable \( u_0 \in L^2 \), \( \mathbb{P} \)-a.s., we have

\[
|U^*(0,t)u_0|^2 \leq e^*(t)|u_0|^2.
\]

Here \( e^*(t) = |e^{\Phi(W(t))}|^2_{L^\infty} M^*(t) \),

\[
M^*(t) := \exp \left\{ \sum_{j=1}^N \left[ - \int_0^t v_j^*(s)d\beta_j(s) - \int_0^t \frac{1}{2} v_j^2(s)ds + \int_0^t \tilde{v}_j^*(s)ds \right] \right\},
\]

\[
v_j^* = 2\text{Re} \left\langle X^*, \mu_j e_j X^* \right\rangle \frac{|X^*|^2}{2}, \text{ and } \tilde{v}_j^* = 2|X^*|^2 \text{Re} \left\langle X^*, (|\mu_j|^2 + \mu_j^2) e_j X^* \right\rangle
\]

with \( X^*(t) = e^{-\Phi(W(t))} U^*(0,t)u_0, 1 \leq j \leq N \).
Proof (i). Note that \( \tilde{X} \) is the homogeneous solution to (1.1). Using similar arguments as in the proof of [6, (1.4)], we have that \( |\tilde{X}|^2 \) is a continuous martingale with the representation

\[
|\tilde{X}(t)|^2 = |u_0|^2 + 2 \sum_{j=1}^{N} \int_0^t \text{Re} \mu_j \left\langle \tilde{X}(s), \tilde{X}(s)e_j \right\rangle^2 d\beta_j(s), \quad t \in [0, T]. \quad (4.9)
\]

This yields that \( |\tilde{X}(t)|^2 = |u_0|^2 M(t), \quad t \in [0, T], \) which implies (4.5).

(ii). Since \( \text{Id} = U(0,t) U(t,0), \partial_t U(0,t) = -iU(0,t) \Psi_{P_t}, \) we have

\[
\partial_t \langle U^*(0,t) u_0, z \rangle_2 = \langle u_0, -iU(0,t) \Psi_{P_t} z \rangle_2 = \langle i \Psi^*_{P_t} U^*(0,t) u_0, z \rangle, \quad \forall z \in H^m,
\]

which implies that \( v^*(t) := U^*(0,t) u_0 \) satisfies the equation

\[
\partial_t v^* = i \Psi^*_{P_t} v^*, \quad v^*(0) = u_0. \quad (4.10)
\]

Then, by Itô’s formula,

\[
dX^* = iP(x,D)X^* dt + \sum_{j=1}^{N} \left( \frac{1}{2} |\mu_j|^2 + \frac{1}{2} \right) e_j^2 X^* dt - X^* dW, \quad X^*(0) = u_0.
\]

This yields that

\[
|X^*(t)|^2 = |u_0|^2 + 4 \sum_{j=1}^{N} \left( \text{Re} \mu_j \right)^2 e_j^2 |X^*(t)|^2 dx dt
\]

\[
- 2 \sum_{j=1}^{N} \int_0^t \text{Re} \mu_j \left\langle X^*(s), X^*(s)e_j \right\rangle^2 d\beta_j(s), \quad t \in [0, T]. \quad (4.11)
\]

Thus, \( |X^*(t)|^2 = |u|^2 M^*(t), \quad t \in [0, T], \) which implies (4.7). \( \square \)

Below we first treat the easier conservative case in the spirit of [18].

Proof of Theorem 4.1 (Conservative case). Let \( \tilde{h} \in S^0 \) and \( \theta \) be as in Lemmas 4.4 and 3.7 respectively. Set \( h(x, \xi) := \tilde{h}(x, \xi) \theta(\xi) \in S^0. \) Note that, \( \Psi_{P_t} = \Psi_{P_t} = e^{-W(t)} \Psi_{P_t} e^{W(t)} \) and

\[
|u^*(t)|^2 = |u|^2 M^*(t), \quad t \in [0, T]. \quad (4.12)
\]

Using (2.12) we have

\[
\partial_t \text{Re} \langle u, \Psi_{P_t} u \rangle_2 = \text{Re} \langle i \Psi_{P_t} u, \Psi_{P_t} u \rangle_2 + \text{Re} \langle u, i \Psi_{P_t} \Psi_{P_t} u \rangle_2
\]

\[
= \text{Re} \langle u, i [\Psi_{P_t}, \Psi_{P_t}] u \rangle_2. \quad (4.13)
\]

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Since by (2.13),
\[ i[\Psi_h, \Psi_P] = i[\Psi_h, \Psi_P] + e^{-W(t)}[\Psi_P, e^{W(t)}] =: \Psi_a + \Psi_{b_t}, \] (4.14)
where \(a(x, \xi) \in S^{m-1}\) and \(b_t(x, \xi) \in S^{m-2}\). It follows that
\[ \partial_t \text{Re} \langle u, \Psi_h u \rangle_2 = \text{Re} \langle u, \Psi_a u \rangle_2 + \text{Re} \langle u, \Psi_{b_t} u \rangle_2. \] (4.15)

Below we perform separate estimates of the symbols \(a\) and \(b_t\) by expanding them up to zero order via Lemma 3.1. The key point here is that the commutator \(i[\Phi_h, \Phi_P]\) is an elliptic operator of order \(m-1\), which raises the local regularity of homogeneous solutions, while the lower order perturbations will then be controlled by the interpolation estimate (3.10). We mainly consider the case \(m > 2\), the case \(m = 2\) can be proved similarly.

First, by Lemma 3.1
\[ a(x, \xi) = H_h P + \sum_{2 \leq |\alpha| \leq m-1} \frac{i}{\alpha!} (\partial^\alpha_{\xi} h D^\alpha_x P - \partial^\alpha_{\xi} P D^\alpha_x h) + r_0(x, \xi) \]
\[ := H_h P + a_0(x, \xi) + r_0(x, \xi), \] (4.16)
where \(a_0 \in S^{m-2}\) and \(r_0 \in S^0\). Since by Lemma 3.4, \(H_h P \leq -c_1 \langle x \rangle^{-2} \langle \xi \rangle^{m-1} + c_2\) for \(|\xi| \geq 2\), using Lemma 3.5 (ii) we get
\[ \text{Re} \left\langle (-H_h P - \frac{c_1}{2} \Psi_{\langle \xi \rangle^{m-1}}) u, u \right\rangle \geq -c_1 \|u\|_{H^{-1}}^{m-2}, \]
which implies that
\[ \text{Re} \langle H_h P u, u \rangle \leq -\frac{c_1}{2} \text{Re} \left\langle \Psi_{\langle \xi \rangle^{m-1}} u, u \right\rangle + C \|u\|_{H^{-1}}^{m-2} + C |u|^2 \]
\[ \leq -\frac{c_1}{2} \|u\|_{H^{-1}}^{m-2} + C \|u\|_{H^{-1}}^{m-2} + C |u|^2. \] (4.17)
Moreover, by Assumption (A1) and Lemma 4.4, \(|\partial^\gamma_{\xi} \partial^\beta_x a_0(x, \xi)| \leq C \langle x \rangle^{-2} \langle \xi \rangle^{m-2-|\gamma|}\), implying that \(\Psi_{a_0} := \langle x \rangle^{D_{m-2}} \Psi_{a_0} \langle x \rangle^{-D_{m-2}} \in S^0\). Hence,
\[ |\langle u, \Psi_{a_0} u \rangle| = \left| \left\langle \Psi_{\langle \xi \rangle^{m-2}} u, \Psi_{a_0} \Psi_{\langle \xi \rangle^{m-2}} u \right\rangle \right| \leq C \|u\|_{H^{-1}}^{m-2}. \] (4.18)
Thus, taking together (4.16), (4.17) and (4.18) we obtain
\begin{equation}
\Re \langle u, \Psi_a u \rangle \leq -\frac{c_1}{2} \|u\|_{H^{-1}}^{\frac{m+1}{2}} + C\|u\|_{H^{-1}}^{\frac{m}{2}} + C|u|_2^2.
\end{equation}

As regards the symbol $b_t(x, \xi)$, by Lemma 3.1,
\begin{equation}
e^{-W(t)}[\Psi_P, e^{W(t)}] = \sum_{1 \leq |\alpha| \leq m} \frac{1}{\alpha!} \partial_\xi^\alpha \Psi_a(W),
\end{equation}
where $\psi_a(W) = e^{-W}D_x^\alpha e^W$, satisfying that $|\psi_a(W)| \leq C|\beta(t)|^{\alpha}$. Then, applying Lemma 3.1 again we get
\begin{equation}
\Psi_{bt} = i[\Psi_h, \sum_{1 \leq |\alpha| \leq m} \frac{1}{\alpha!} \partial_\xi^\alpha \Psi_a(W)]
= i \sum_{1 \leq |\alpha| \leq m-2} \frac{1}{\alpha!} \left[ \sum_{1 \leq |\beta| \leq m-|\alpha|-1} \frac{1}{\beta!} \left( \partial_\xi^\beta hD_x^\beta (\partial_\xi^\alpha \Psi_a(W)) 
- D_x^\beta h\partial_\xi^\beta (\partial_\xi^\alpha \Psi_a(W)) \right) + \Psi_{r_{2, \alpha}} \right] + i \sum_{|\alpha| = m-1, m} \frac{1}{\alpha!} [\Psi_h, \partial_\xi^\alpha \Psi_a(W)]
= : \Psi_{b_{t, 1}} + \Psi_{r_{t, 1}},
\end{equation}
where $\Psi_{r_{t, 1}} = i \sum_{1 \leq |\alpha| \leq m-2} \frac{1}{\alpha!} \Psi_{r_{2, \alpha}} + i \sum_{|\alpha| = m-1, m} \frac{1}{\alpha!} [\Psi_h, \partial_\xi^\alpha \Psi_a(W)] \in S^0$, $b_{t, 1} = b_t - r_{t, 1}$. Note that, by Assumption (A1), for any $l \in \mathbb{N}$ and any multi-indices $\gamma_1, \gamma_2$,
\begin{equation}
|r_{t, 1}|_{S^0}^{(l)} \leq C(l)(1 + |\beta_t|^m), \quad |\partial_\xi^{\gamma_1} \partial_\xi^{\gamma_2} b_{t, 1}(x, \xi)| \leq C(1 + |\beta_t|^{m-2}) \frac{(\xi^{m-2-|\gamma_1|})^{(l)}}{(x)^2}.
\end{equation}

Then, similarly to (4.18), we have
\begin{equation}
|\langle u, \Psi_{b_t} u \rangle| \leq C(1 + |\beta_t|^{m-2}) \|u\|_{H^{-1}}^{\frac{m+1}{2}} + C(1 + |\beta_t|^{m}) |u|_2^2.
\end{equation}

Now, applying (4.19) and (4.23) into (4.15) and using (3.10) we get
\begin{equation}
\partial_t \Re \langle u, \Psi_h u \rangle \leq -\frac{c_1}{2} \|u\|_{H^{-1}}^{\frac{m+1}{2}} + C(1 + |\beta_t|^{m-2}) \|u\|_{H^{-1}}^{\frac{m}{2}} + C(1 + |\beta_t|^{m}) |u|_2^2
\leq (\frac{c_1}{2} + \varepsilon C(1 + |\beta_t|^{m-2})) \|u\|_{H^{-1}}^{\frac{m+1}{2}} + \varepsilon(1 + |\beta_t|^{m}) |u|_2^2.
\end{equation}
Taking $\varepsilon = c_1(4C(1 + |\beta|^{m-2}))^{-1}$ we obtain
\[
\partial_t \text{Re} \langle u, \Psi_h u \rangle \leq -\frac{c_1}{4} \|u\|^2_{H^{\frac{m-1}{2}}} + C(1 + |\beta|^{(m-2)^2+m})|u|^2_2.
\] (4.24)

Thus, integrating over $[0, T]$ and using (4.12) we get
\[
\text{Re} \langle u(T), \Psi_h u(T) \rangle \leq \text{Re} \langle u_0, \Psi_h u_0 \rangle - \frac{c_1}{4} \|u\|^2_{L^2(0,T;H^{\frac{m-1}{2}})} + CT(1 + (\beta_T^*)^{(m-2)^2+m})|u_0|^2_2,
\]
which implies immediately that
\[
\|u\|^2_{L^2(0,T;H^{\frac{m-1}{2}})} \leq C(1 + T(1 + (\beta_T^*)^{(m-2)^2+m}))+ u_0|^2_2,
\] (4.25)
thereby proving (4.2) in the case $m > 2$.

The case $m = 2$ is easier. In this case, we do not have the lower order terms $a_0$ and $b_{t,1}$ in (4.16) and (4.21) respectively. Hence, instead of (4.19) and (4.23) we have
\[
\text{Re} \langle u, \Psi_a u \rangle \leq -\frac{c_1}{2} \|u\|^2_{H^{\frac{1}{2}}} + C|u|^2_2, \quad |\langle u, \Psi_b u \rangle| \leq C(1 + |\beta|^2)|u|^2_2.
\]
Therefore, arguing as those below (4.23) we obtain (4.25) with $m = 2$. □

Next we treat the non-conservative case, for which we will use the transformation as in [35] and perform the energy method to a new equation.

**Proof of Theorem 4.1 (Non-conservative case)** Let $\tilde{h}, \theta$ be as in previous the proof of the conservative case. Set $\theta_R(\xi) := \theta(\frac{\xi}{R})$, $h_R(x, \xi) := M\tilde{h}(x, \xi)\theta_R(\xi)$, and $c_R(x, \xi) := \exp\{h_R(x, \xi)\} \in S^0$, where $M \geq 1$ is to be chosen later, and
\[
R = CM^l e^{2M\|\tilde{h}\|_{\infty}}
\] (4.26)
for some $l \geq 1$ and $C$ large enough such that Lemma 3.7 holds.

By (2.12), $v := \Psi_{c_R} u$ satisfies the equation
\[
\partial_t v = i\Psi_{c_R} \Psi_{P_1} \Psi^{-1}_{c_R} v.
\]
In view of (2.13) we have
\[
\Psi_{c_R} \Psi_{P_1} \Psi^{-1}_{c_R} = \Psi_P + [\Psi_{c_R}, \Psi_P] \Psi^{-1}_{c_R} + \Psi_{c_R} e^{-\Phi(W)} [\Psi_P, e^{\Phi(W)}] \Psi^{-1}_{c_R}.
\]
Thus,
\[
\frac{1}{2} \partial_t |v|^2 = \Re \langle v, i \Psi_{c_R}, \Psi_P \rangle \Psi_{c_R}^{-1} v + \Re \langle v, i \Psi_{c_R} e^{-\Phi(W)} \Psi_P, e^{\Phi(W)} \rangle \Psi_{c_R}^{-1} v
\]
\[
= : \Re \langle v, \Psi_a v \rangle + \Re \langle v, \Psi_b v \rangle ,
\]
(4.27)
where \( \Psi_a = i [\Psi_{c_R}, \Psi_P] \Psi_{c_R}^{-1} \), \( \Psi_b = i \Psi_{c_R} e^{-\Phi(W)} [\Psi_P, e^{\Phi(W)}] \Psi_{c_R}^{-1} \).

Note that, unlike the conservative case, the symbols \( a, b_t \) have the same order \( m - 1 \). Below we separate, via Lemma 3.1, the principle symbols and the lower order symbols of \( a \) and \( b_t \). As in the conservative case, we will mainly consider the case \( m > 2 \), the case \( m = 2 \) can be proved similarly.

First, for the symbol \( a \), note that
\[
i [\Psi_{c_R}, \Psi_P] = \Psi_{H_{cR}P} + \Psi_{r_0'}, \Psi_{r_0''} + \Psi_{r_0},
\]
where \( \Psi_{r_0} = (\Psi_{r_0'} + \Psi_{r_0''}) \Psi_{c_R}^{-1} \). By Assumption (A1), Corollary 3.3 and straightforward computations, we have that for any multi-indices \( \alpha, \beta \), \( |\alpha + \beta| = \ell \), there exists \( \ell' \geq 1 \) such that
\[
|\partial^{\ell'}_{\xi^\beta} \partial_{\xi^\alpha} r_0| \leq C(l') M^\ell e^{2M\|\tilde{\eta}\|_\infty} \langle \xi \rangle^{m-2-l} \langle x \rangle^2.
\]
(4.29)
As regards the symbol \( b_t(x, \xi) \), we compute
\[
\Psi_{b_t} = i e^{-\Phi(W)} [\Psi_P, e^{\Phi(W)}] \Psi_{c_R}^{-1} + i [\Psi_{c_R}, e^{-\Phi(W)} [\Psi_P, e^{\Phi(W)}]] \Psi_{c_R}^{-1}
\]
\[
= \sum_{|\alpha| = 1} i \partial_{\xi}^\alpha P \psi_\alpha (\Phi(W))
\]
\[
+ \left( \sum_{2 \leq |\alpha| \leq m} \frac{i}{\alpha!} \partial_{\xi}^\alpha P \psi_\alpha (\Phi(W)) + i [\Psi_{c_R}, e^{-\Phi(W)} [\Psi_P, e^{\Phi(W)}]] \Psi_{c_R}^{-1} \right)
\]
\[
= : \Psi_{b_{t,1}} + \Psi_{b_{t,2}},
\]
(4.30)
where \( \psi_\alpha (\Phi(W)) = e^{-\Phi(W)} D_x^\alpha e^{\Phi(W)} \). Note that, by Assumption (A1),
\[
|b_{t,1}(x, \xi)| \leq C(1 + |\beta_t| + t) \langle \xi \rangle^{-2} \langle x \rangle^{m-1},
\]
(4.31)
and for any multi-indices $\gamma, \delta, |\gamma + \delta| = l$, there exists some $l'$ such that

$$|\partial_x^\gamma \partial_t^\delta b_{t,2}| \leq C(l')(1 + |\beta(t)| + t)^m M' e^{2M||\tilde{h}||_{\infty}} \langle \xi \rangle^{m-2-|\gamma|} \langle x \rangle^2. \quad (4.32)$$

Thus, it follows from (4.27), (4.28) and (4.30) that

$$\frac{1}{2} \partial_t |v|^2 = \text{Re} \left< v, (\Psi_{H_{b,t}} P + \Psi_{b_{t,1}}) v \right> + \text{Re} \left< v, (\Psi_{r_0} + \Psi_{b_{t,2}}) v \right>. \quad (4.33)$$

For the first term in the right hand side of (4.33), taking into account Lemma 4.4 and the boundedness of $b_{t,1}$, we take

$$M = \frac{4C}{c_1} (1 + \beta_t + T) \quad (4.34)$$

and get

$$\frac{1}{M} \text{Re}(H_{b,t} P + b_{t,1})(x, \xi) \leq -\frac{c_1}{4} \langle \xi \rangle^m + \frac{c_2}{M}, \quad |\xi| \geq 2R. \quad (4.35)$$

Then, arguing as in the proof of (4.17) we obtain

$$\text{Re} \left< v, (\Psi_{H_{b,t}} P + \Psi_{b_{t,1}}) v \right> \leq -\frac{c_1}{4} M \|v\|^2_{H^{-1}_{-1}} + CRM \|v\|^2_{H^{-m-2}_{-1}} + C|v|^2. \quad (4.36)$$

Regarding the second term in the right hand side of (4.33), similarly to (4.18), using (4.29), (4.32) we get

$$\text{Re} \left< v, (\Psi_{r_0} + \Psi_{b_{t,2}}) v \right> \leq C(l')(1 + |\beta_t| + t)^m M' e^{2M||\tilde{h}||_{\infty}} \|v\|^2_{H^{-m-2}_{-1}}. \quad (4.37)$$

Now, plugging (4.36) and (4.37) into (4.33) and using (3.10) we get

$$\frac{1}{2} \partial_t |v|^2 \leq (-\frac{c_1}{4} M + C(1 + |\beta_t| + t)^m M'' e^{2M||\tilde{h}||_{\infty}} \langle \xi \rangle^m \langle x \rangle^2$$

$$+ C(1 + |\beta_t| + t)^m M'' e^{2M||\tilde{h}||_{\infty}} \varepsilon^{-(m-2)} |v|^2_{H^{-m-1}_{-1}}$$

for some $l'' \geq 1$. Choosing $\varepsilon = c_1 (8C)^{-1} (1 + |\beta_t| + t)^{-m} M^{1-l''} e^{-2M||\tilde{h}||_{\infty}}$ yields

$$\frac{1}{2} \partial_t |v|^2 \leq -\frac{c_1}{8} M \|v\|^2_{H^{-m-1}_{-1}} + C(1 + |\beta_t| + t)^{(m-1)m} M''(m-1) e^{CM} |v|^2, \quad (4.38)$$
which implies that
\[
\frac{c_1}{8} M \|v\|^2_{L^2(0,T;H^{-m/2})} \leq \frac{1}{2} (|v_0|^2 + |v(T)|^2)
\]
\[+ C(1 + \beta_T + T)^{(m-1)m} M^{m(m-1)} e^{CM} \|v\|^2_{L^2(0,T;L^2)}.
\] (4.39)

Note that, applying Gronwall's inequality to (4.38) implies that
\[
\|v(t)\|^2_{L^\infty(0,T;L^2)} \leq C |v(0)|^2 \exp\left\{C(1 + \beta_T + T)^{(m-1)m} M^{m(m-1)} e^{CM}\right\},
\] (4.40)
which includes a non-integrable double exponential of Brownian motions and so cannot yield, via (4.39), the integrability of the \(L^2(0,T;H^{(m-1)/2})\)-norm of \(v\).

Instead, we use the martingale property of homogeneous solutions in Lemma 4.5 (i). Then, taking into account the boundedness of \(\Psi_{cR}, \Psi_{cR}^{-1}\) in \(H^{(m-2)/2}\) and \(L^2\), we obtain (4.22) for \(m > 2\) in the nonconservative case.

The case \(m = 2\) is easier. In this case, similarly to (4.36), applying Lemma 3.5 (i) we have
\[
\text{Re} \langle (H_{hR} P + b_t, 1)v, v \rangle \leq -\frac{c_1}{4} M \|v\|^2_{H^{1/2}} + CRM |v|^2.
\]
Moreover, we have that \(r_0 + b_{t_2} \in S^0\), which implies (4.37) with \(H^{(m-2)/2}\) replaced by \(L^2\). Then, similar arguments as above yield (4.39) with \(m = 2\), thereby proving (4.2). The proof is complete. \(\square\)

5 Proof of main results

We start with the Strichartz and local smoothing estimates of the free group \(\{e^{-itP(x,D)}\}\). For simplicity, we use the abbreviations \(X_{s,p,q}, X'_{s,p,q}\) for \(X_{T,s,p,q}\) and \(X'_{T,-s,p,q}\) respectively.

Lemma 5.1. Assume (A0)–(A3). For any \(u_0 \in L^2\) and admissible triple \((s,p,q) \in A\),
\[
\|e^{iP(x,D)}u_0\|_{X_{s,p,q}} \leq C(1 + T)^{\frac{1}{2}} |u_0|_2.
\] (5.1)

Moreover, for any \((s_1,p_1,q_1), (s_2,p_2,q_2) \in A\) and any \(f \in X'_{s_2,p_2,q_2}\),
\[
\left\| \int_0^t e^{i(s-P(x,D))} f(s) ds \right\|_{X_{s_1,p_1,q_1}} \leq C(1 + T) \|f\|_{X'_{s_2,p_2,q_2}}.
\] (5.2)

where \(C\) is independent of \(T\).
\textbf{Proof.} In order to prove \((5.1)\), in view of Assumption \((A3)\), we only need to prove that
\[
\|e^{iP(x,D)}u_0\|_{L^2(0,T;H^{-m\alpha}_{-1})}^2 \leq C(1 + T)|u_0|^2, \tag{5.3}
\]
For this purpose, let \(u(t) = e^{itP(x,D)}u_0\) and \(h\) be as in the proof of Theorem \(4.1\). Similarly to \((4.19)\),
\[
\partial_t \text{Re} \langle u, \Psi_h u \rangle \leq -\frac{c_1}{2} \|u\|_{H^{m\alpha}_{-1}}^2 + C\|u\|_{H^{m\alpha}_{-1}}^2 + C|u|^2,
\]
which along with \((3.10)\) implies that
\[
\partial_t \text{Re} \langle u, \Psi_h u \rangle \leq \left(-\frac{c_1}{2} + C\varepsilon\right)\|u\|_{H^{m\alpha}_{-1}}^2 + C(1 + \varepsilon^{-(m-2)})|u|^2.
\]
Thus, taking \(\varepsilon\) small enough we obtain \((5.3)\), thereby proving \((5.1)\).

Regarding \((5.2)\), we note that, for any \((s, p, q) \in A\) and \(z \in L^2\), by \((5.1)\),
\[
\left\langle \int_0^T e^{isP(x,D)}f(s)ds, z \right\rangle = \int_0^T \left\langle f(s), e^{-isP(x,D)}z \right\rangle_2 ds \leq C(1 + T)^{\frac{1}{2}} \|f\|_{\mathcal{X}_{s',p',q'}},
\]
which implies that
\[
\left\| \int_0^T e^{-isP(x,D)}f(s)ds \right\|_2 \leq C(1 + T)^{\frac{1}{2}} \|f\|_{\mathcal{X}_{s',p',q'}}. \tag{5.4}
\]

Now, let \(f = f_1 + f_2\), \(f_1 \in L^2(0,T;H^{-(m-1)/2}_{-1})\) and \(f_2 \in L^2(0,T;\dot{W}^{-s_2,p_2}_{-1})\). Since \(\int_0^T e^{i(t-s)P(x,D)}f_1(s)ds = e^{itP(x,D)}\int_0^T e^{-isP(x,D)}f_1(s)ds\), by \((5.1)\) and \((5.4)\),
\[
\left\| \int_0^T e^{i(t-s)P(x,D)}f_1(s)ds \right\|_{L^q(0,T;W^{s_1,p_1}_{-1})} \leq C(1 + T)^{\frac{1}{2}} \int_0^T \|e^{-isP(x,D)}f_1(s)ds\|_2 \leq C(1 + T)\|f_1\|_{L^2(0,T;H^{m\alpha}_{-1})},
\]
Then, since \(q_1 > 2\), by Christ-Kiselev’s lemma (see e.g. [42] Lemma 3.1)\(^2\),
\[
\left\| \int_0^T e^{i(t-s)P(x,D)}f_1(s)ds \right\|_{L^q(0,T;W^{s_1,p_1}_{-1})} \leq C(1 + T)\|f_1\|_{L^2(0,T;H^{m\alpha}_{-1})}. \tag{5.5}
\]

\(^2\)The proof of [42] Lemma 3.1\ also works for homogeneous Sobolev spaces.
Moreover, arguing as in the proof of (4.25) we have
\[ \partial v_t \in L^1(0,T;\dot{W}^{s_1,p_1}) \leq C(1 + T)\| f_2 \|_{L^q(0,T;\dot{W}^{-s_2,p_2'})}. \] (5.6)
Thus, setting \( v(t) := \int_0^t e^{-i(t-s)P(x,D)} f ds \) we obtain
\[ \| v \|_{L^q(0,T;\dot{W}^{s_1,p_1})} \leq C(1 + T)\| f \|_{\mathcal{X}'_{-s_2,p_2',q_2'}}. \] (5.7)

As regards the estimate for the \( L^2(0,T;H^{(m-1)/2}) \)-norm of \( v \), we first note that \( \partial_t |v(t)|^2 = 2Re \int v(t,x)\dot{f}(t,x) dx \), which implies that
\[ \| v \|_{L^2(0,T;L^2)}^2 \leq \| v \|_{\mathcal{X}_{s_2,p_2,q_2}} \| f \|_{\mathcal{X}'_{-s_2,p_2',q_2'}}. \] (5.8)
Moreover, arguing as in the proof of (4.25) we have
\[ \| v \|_{L^2(0,T;H^{m-1})}^2 \leq C|v(T)|^2 + C\| v \|_{L^2(0,T;L^2)}^2 + C|\langle f, \Psi_h v \rangle| + |\langle v, \Psi_h f \rangle|, \]
where \( \langle , \rangle \) is the pairing between \( \mathcal{X}_{s_2,p_2,q_2} \) and \( \mathcal{X}'_{-s_2,p_2',q_2'} \). Thus, by (5.8),
\[ \| v \|_{L^2(0,T;H^{m-1})}^2 \leq C(1 + T)\| f \|_{\mathcal{X}'_{-s_2,p_2',q_2'}} \| v \|_{\mathcal{X}_{s_2,p_2,q_2}} \]
\[ \leq \frac{1}{2} \| v \|_{\mathcal{X}_{s_2,p_2,q_2}}^2 + C_2^2 (1 + T)^2 \| f \|_{\mathcal{X}'_{-s_2,p_2',q_2'}}^2, \]
which along with (5.7) implies (5.2). The proof is complete. \( \Box \)

Below we prove Strichartz and local smoothing estimates for the homogeneous solutions to (1.1).

**Lemma 5.2.** Assume (A0)–(A3). For each \( \mathcal{F}_0 \)-measurable \( u_0 \in L^2, \mathbb{P}\text{-a.s.} \), and \( (s,p,q) \in \mathcal{A} \), we have \( \mathbb{P}\text{-a.s.} \)
\[ \| U(\cdot,0)u_0 \|_{\mathcal{X}_{s,p,q}} \leq C(e,T)|u_0|_2, \] (5.9)
where \( C(e,T) := C(1 + T)(1 + \beta_T^s + T)^m \tilde{D}(e,T) \) with \( \tilde{D}(e,T) \) as in (1.2).

**Proof.** Set \( u(t) := U(t,0)u_0, b(t,x,D) := ie^{-\Phi(W)(t,x)}[\Psi_P, e^{\Phi(W)(t,x)}], t \in [0,T] \). By (1.1),
\[ u(t) = e^{itP(x,D)}u_0 + \int_0^t e^{i(t-s)P(x,D)}b(s,x,D)u(s)ds. \]

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Then, by Lemma 5.1,
\[ \|u\|_{X_{s,p,q}} \leq C(1 + T)(|u_0|_2 + \|b(\cdot, \cdot, D)u\|_{L^2(0,T;H^\frac{m-1}{2})}). \]

Note that, by Corollary 3.3, the \( S(l) \) semi-norm of \( \langle x \rangle \langle D \rangle^{-(m-1)/2} b(t, x, D) \langle D \rangle^{-(m-1)/2} \langle x \rangle \) is bounded by \( C(l')(1 + |\beta_t| + t)^m \) for some \( l' \geq 1 \). Then,
\[ \|b(\cdot, \cdot, D)u\|_{L^2(0,T;H^\frac{m-1}{2})} \leq C(l')(1 + |\beta_t| + t)^m \|u\|_{L^2(0,T;H^\frac{m-1}{2})}. \]  
(5.10)

Therefore, in view of Theorem 4.1, we obtain (5.9). \( \Box \)

Similarly, for the dual operator \( U^*(0,t) \) we have

Lemma 5.3. Assume (A0)–(A3). For each \( \mathcal{F}_0 \)-measurable \( u_0 \in L^2, \mathbb{P}\text{-a.s.} \), and for any \( (s,p,q) \in \mathcal{A} \),
\[ \|U^*(0,\cdot)u_0\|_{X_{s,p,q}} \leq C(e^*,T)|u_0|_2, \ \mathbb{P} - \text{a.s.}, \]  
(5.11)

where \( C(e^*,T) \) is as in (5.9) with \( e^* \) replacing \( e \).

Proof. We note that,
\[ P_t^* = e^{\Phi(W)}P_0 e^{-\Phi(W)} = P + e^{\Phi(W)}[P, e^{-\Phi(W)}] = P + \sum_{1 \leq |\alpha| \leq m} \frac{1}{\alpha!} \partial^\alpha_t \psi_\alpha(-\Phi(W)) \]
with \( \psi_\alpha(-\Phi(W)) = e^{\Phi(W)}D_\alpha^2 e^{-\Phi(W)} \), which has similar structure as \( P_t \). Hence, arguing as in the proof of Theorem 4.1 and using (4.7), we also have the homogeneous local smoothing estimates for \( U^*(0,t)u_0 \), which consequently yields (5.11) by similar arguments as in the previous proof of Lemma 5.2 \( \Box \)

Corollary 5.4. Assume (A0)–(A3). For any \( (s,p,q) \in \mathcal{A} \) and \( \{\mathcal{F}_t\} \)-adapted \( f \in X'_{s,p',q'} \), \( \mathbb{P}\text{-a.s.}, \)
\[ \left| \int_0^T U(0,s)f(s)ds \right|_2 \leq C(e^*,T)\|f\|_{X'_{-s,p',q'}}, \ \mathbb{P} - \text{a.s.}, \]  
(5.12)

where \( C(e^*,T) \) is as in (5.11).
Since \(q\langle \int \) which along with Lemma 5.2 and Corollary 5.4 implies that
which implies (5.12) by duality. □

Proof. By Lemma \[\text{Lemma 5.3}\] for any \(z \in L^2\),

\[
\left\langle \int_0^T U(0, s) f(s) ds, z \right\rangle = \int_0^T \langle f(s), U^*(0, s) z \rangle ds \leq \|f\|_2 \|x'_{s,p,q} \| U^*(0, \cdot) z \|_2 \leq C(e^*, T) \|f\|_2 \|x'_{s,p,q} \|_2,
\]

which implies (5.12) by duality.

Proof of Theorem 2.13. (i). We reformulate (2.12) in the mild form

\[u(t) = U(t, 0) u_0 + \int_0^t U(t, s) f(s) ds\]

with \(f = e^{-\Phi(W)} F\). By Lemma \[\text{Lemma 5.2}\] we only need to prove (2.15) when \(u_0 \equiv 0\).

First we prove that for any \((s_i, p_i, q_i) \in \mathcal{A}, i = 1, 2,\)

\[
\|u\|_{L^{q_1}(0,T;W^{s_1,p_1})} \leq C(e, T) C(e^*, T) \|f\|_2 \|x'_{s_2,p_2,q_2} \|.
\]

(5.13)

In particular, taking \((s_1, p_1, q_1) = (0, 2, \infty)\) we have that for any \((s, p, q) \in \mathcal{A},\)

\[
\|u\|_{L^{\infty}(0,T;L^2)} \leq C(e, T) C(e^*, T) \|f\|_2 \|x'_{s,p,q} \|.
\]

(5.14)

The arguments is similar to that of (5.7). In fact, let \(f = f_1 + f_2, f_1 \in L^{q_1}(0,T;W^{s_2,p_2})\) and \(f_2 \in L^2(0,T;H_1^{m-\frac{1}{2}})\). Note that

\[
\int_0^T U(t, s) f_1(s) ds = \int_0^T U(t, 0) U(0, s) f_1(s) ds = U(t, 0) \int_0^T U(0, s) f_1(s) ds,
\]

which along with Lemma \[\text{Lemma 5.2}\] and Corollary \[\text{Corollary 5.4}\] implies that

\[
\left\| \int_0^T U(\cdot, s) f_1(s) ds \right\|_{L^{q_1}(0,T;W^{s_1,p_1})} \leq C(e, T) \left\| \int_0^T U(0, s) f_1(s) ds \right\|_2 \leq C(e, T) C(e^*, T) \|f_1\|_{L^{2}(0,T;H_1^{m-\frac{1}{2}})}.
\]

Since \(q_1 > 2\), using the Christ-Kiselev lemma we obtain

\[
\left\| \int_0^T U(\cdot, s) f_1(s) ds \right\|_{L^{q_1}(0,T;W^{s_1,p_1})} \leq C(e, T) C(e^*, T) \|f_1\|_{L^{2}(0,T;H_1^{m-\frac{1}{2}})}.
\]
Similarly, since \( q_1 > q_2' \), similar arguments as above yield that
\[
\left\| \int_0^T U(\cdot, s) f_2(s) ds \right\|_{L^q(0,T; W^{1,q_1})} \leq C(e, T) C(e^*, T) \|f_2\|_{L^{q_2'}(0,T; W^{-s_2,s_2'})}.
\]
Thus, combining these estimates together we obtain (5.13), as claimed.

Below we estimate the \( L^2(0,T; H_{\frac{m-1}{2}}) \)-norm of \( u \). We shall consider the conservative and non-conservative cases respectively.

**Conservative case.** Similarly to (4.25), we have
\[
\|u\|_{L^2(0,T; H_{\frac{m-1}{2}})} \leq C(1 + T(1 + (\beta_T^*)^{(m-2)^2 + m})) \|u\|_{L^\infty(0,T; L^2)} + C(\|f, \Psi_h u\| + \|f, \Psi^*_h u\|),
\]
where \( \langle \cdot, \cdot \rangle \) is the pairing between \( \mathcal{X}_{s_2,p_2,q_2} \) and \( \mathcal{X}'_{s_2,p_2,q_2} \); and \( h \in S^0 \) is the symbol as in the proof of Theorem 4.1. Since \( \Psi_h, \Psi^*_h \in \mathcal{L} \) \( (\mathcal{X}_{s_2,p_2,q_2}) \), the last term of the right hand side above is bounded by
\[
C\|f\|_{\mathcal{X}_{s_2,p_2,q_2}} \|u\|_{\mathcal{X}_{s_2,p_2,q_2}} \leq \frac{1}{4} \|u\|_{\mathcal{X}_{s_2,p_2,q_2}}^2 + 4C\|f\|_{\mathcal{X}'_{s_2,p_2,q_2}}^2.
\]
Plugging this into (5.13) and using (5.14) we get
\[
\|u\|_{L^2(0,T; H_{\frac{m-1}{2}})} \leq \frac{1}{4} \|u\|_{\mathcal{X}_{s_2,p_2,q_2}}^2 + C(1 + T(1 + |\beta_T^*|^{(m-2)^2 + m})) (C(e, T) C(e^*, T))^2 \|f\|_{\mathcal{X}'_{s_2,p_2,q_2}}^2,
\]
which along with (5.13) implies that
\[
\|u\|_{\mathcal{X}_{s_1,p_1,q_1}} \leq C(1 + T(1 + |\beta_T^*|^{(m-2)^2 + m})) (C(e, T) C(e^*, T))^2 \|f\|_{\mathcal{X}'_{s_2,p_2,q_2}}^2,
\]
thereby proving (2.15) in the conservative case.

**Non-conservative case.** Let \( v := \Psi_{eR} u \), where \( c_R \) is as in the proof of Theorem 4.1. Then, \( v \) satisfies the equation
\[
\partial_t v = i\Psi_{eR} \Psi_{eR}^{-1} v + \Psi_{eR} f, \quad v(0) = 0.
\]
Arguing as in the proof of (4.39) and using (5.14) we get
\[
\|v\|_{L^2(0,T; H_{\frac{m-1}{2}})} \leq C|v_T|^2 + C(1 + |\beta_T^* + T|^{(m-2)^2 + m}) M^{(m-1)|e|} C \|v\|_{L^2(0,T; L^2)}^2
\]
\[
\quad + C\|v, \Psi_{eR} f\|,
\]
(5.16)
for some \( l \geq 1 \), where \( M = 4c_1^{-1}C(1 + \beta_T^* + T) \). Since \( \Psi_{cR}^{-1} \in \mathcal{L}(H_{-1}^{m-1}) \), \( \Psi_{cR} \in \mathcal{L}(L^2) \cap \mathcal{L}(X_{s,p,q}), \Psi_{eR}^* \in \mathcal{L}(X_{s,p,q}) \), and the norms are bounded by the semi-norms of \( c_{R1}^1 \) and \( c_R \), we get

\[
\|u\|_{L^2(0,T;H_{-1}^{m-1})}^2 \\
\leq CM' e^{CM}(1 + T)(1 + \beta_T^* + T)^{m-1}m(C(e,T)C(e^*,T))^2\|f\|_{X_{-\gamma_3,\gamma_3',\gamma_3'^2}}^2 \\
+ CM' e^{CM}\|u\|_{X_{s,p_2,q_2}}\|f\|_{X_{-\gamma_3,\gamma_3'^2}}^2 \\
\leq CM' e^{CM}(1 + T)(1 + \beta_T^* + T)^{m-1}m(C(e,T)C(e^*,T))^2\|f\|_{X_{-\gamma_3,\gamma_3',\gamma_3'^2}}^2 + \frac{1}{2}\|u\|_{X_{s,p_2,q_2}}^2,
\]

which along with (5.13) implies (2.15). The statement (i) is proved.

(ii). For each \( 1 \leq j \leq d \), let \( w_j := \partial_{x_j} u \). By (2.12),

\[
\partial_t w_j = iP_t(x,D)w_j + iP_t(x,D)\partial_{x_j} u + \partial_{x_j} f.
\]

Let \( g(t,x,D) = iP_t(x,D) \). Similarly to (1.21),

\[
P_t = P + e^{-\Phi(W)}[P,e^{\Phi(W)}] = P + \sum_{1 \leq |\alpha| \leq m} \frac{1}{\alpha!} \Psi_{\alpha}^{\Phi}(W).
\]

Then, since \( P \) is independent of \( x \), we have

\[
g(t,x,\xi) = (-1)^m \sum_{1 \leq |\alpha| \leq m} \frac{1}{\alpha!} \partial_{\xi}^\alpha PD_{x_j}\psi_{\alpha}(\Phi(W)),
\]

which implies that

\[
|\partial_{\xi}^\beta \partial_{\xi}^\gamma g(t,x,\xi)| \leq C_{\beta\gamma}(1 + \beta_T^* + T)^m \frac{\langle \xi \rangle^{m-1-|\beta|}}{\langle x \rangle^2}
\]

for any multi-indices \( \beta, \gamma \). Thus, applying (2.4) we obtain

\[
\|w_j\|_{X_{s_1,1,p_1,q_1}} \\
\leq D(e,e^*,T)(|\partial_{x_j} u(0)|_2 + \|g(t,x,D)u\|_{L^2(0,T;H_{-1}^{m-1})}) + \|\partial_{x_j} f\|_{X_{-\gamma_3,\gamma_3',\gamma_3'^2}} \\
\leq CD(e,e^*,T)(|u(0)|_{H^1} + (1 + \beta_T^* + T)^m \|u\|_{L^2(0,T;H_{-1}^{m-1})}) + \|\partial_{x_j} f\|_{X_{-\gamma_3,\gamma_3',\gamma_3'^2}}.
\]

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which implies (2.10). The proof of Theorem 2.13 is complete.

Proof of Theorem 2.11. The proof follows the lines as those in the proof of Theorems 4.1 and 2.13. In fact, we can derive similar equations as (4.13) and (4.27) in the conservative and non-conservative cases respectively. In each case, the lower perturbation only contributes the $H^{(m-2)/2}$-norm of $u$, which can be controlled by the interpolation estimate (3.10). Then, similarly to (4.40), instead of using Lemma 4.5 we apply the Gronwall inequality to control the $L^2$-norm of the solution and then obtain the homogeneous local smoothing estimates, which consequently implies the inhomogeneous estimates (2.10) by analogous arguments as in the proof of Theorem 2.13.

We conclude this section with a simplified proof without duality argument for (2.4) in the conservative case, but with $(s_1, p_1, q_1) = (s_2, p_2, q_2)$. In fact, applying Lemma 5.1 and (5.10) to (2.14) we get

\[
\|u\|_{X_T, s, p, q} \leq C(1 + T)(|X_0|_2 + \|b(\cdot, \cdot, D)u\|_{L^2(0, T; H^{m-2})}) + \|f\|_{X'_T, s, p, q},
\]

where $b(t, x, D) := ie^{-W(t, x)}[\Psi_p, e^{W(t, x)}]$, $f = e^{-W}F$. Then, by (5.15),

\[
\|u\|_{X_T, s, p, q} \leq C(1 + T)^{\frac{3}{2}}(1 + \beta^* + T)^{\frac{(m-2)^2 + m}{2}} + \|u\|_{L^\infty(0, T; L^2)} + (1 + T)(1 + \beta^* + T)^m \|f\|_{X'_T, s, p, q},
\]

Therefore, similarly to (5.8), since $\|u\|_{L^\infty(0, T; L^2)} \leq 2\|u\|_{X_T, s, p, q}^1 \|f\|_{X'_T, s, p, q}^{1/2}$, using Cauchy’s inequality $ab \leq \varepsilon a^2 + \varepsilon^{-1}b^2$ we obtain similar estimate as (2.4) but with $(s_1, p_1, q_1) = (s_2, p_2, q_2)$.

6 Applications

This section contains several applications to nonlinear problems, including well-posedness, integrability of global solutions and large deviation principle.
### 6.1 Stochastic nonlinear Schrödinger equation with variable coefficients

Consider the stochastic nonlinear Schrödinger equation with variable coefficients and lower order perturbations

\[
\frac{dX(t)}{dt} = i \sum_{j,k=1}^{d} D_j a^{jk}(x) D_k X(t) dt + b(t,x) \cdot D X(t) dt + c(t,x) X(t) dt
\]

\[-\lambda i |X|^{\alpha-1} X(t) dt + \mu X(t) dt + X(t) dW(t),
\]

\[X(0) = X_0 \in L^2, \quad (6.1)\]

where \(a^{jk}\) are real valued and symmetric, \(1 \leq j, k \leq d\), \(W\) and \(\mu\) are as in (1.1), \(b(t,x), c(t,x), t \geq 0\) are continuous \(\{F_t\}\)-adapted processes in \(C^d\) and \(C\) respectively; the coefficient \(\lambda = 1\) (resp. \(\lambda = -1\)) corresponds to the focusing (resp. defocusing) case and \(\alpha \in (1, \infty)\). We assume that

(B1) **Ellipticity.** There exists \(C > 0\) such that

\[C^{-1} |\xi|^2 \leq \langle a^{jk} \xi, \xi \rangle \leq C |\xi|^2.\]

(B2) **Asymptotic flatness.** For any multi-index \(\beta \neq 0\),

\[|\partial^\beta_x a^{jk}(x)| \leq C_{\beta} \langle x \rangle^{-2}\]

and for \(1 \leq k, l \leq d\),

\[\sum_{j \in \mathbb{Z}} \sup_{2^j \leq |x| \leq 2^{j+1}} |x|^2 (|\partial_{x_k x_l} a(x)| + |x||\partial_{x_k} a(x)| + |a(x) - I_n|) \leq \varepsilon \ll 1.\]

Moreover, for any multi-index \(\alpha\),

\[\sup_{t \in [0,T]} (|\partial_x^\alpha b(t,x)| + |\partial_x^\alpha c(t,x)|) \leq g(T) C_\alpha \langle x \rangle^{-2}, \quad \mathbb{P} - a.s.,\]

where \(g(t), t \geq 0\), is an \(\{F_t\}\)-adapted continuous process.

**Theorem 6.1.** Assume (B1), (B2) and the asymptotic flatness of \(a^{jk}\) in (2.1). Let \(\alpha \in [1, 1 + 4/d]\) and \(X_0\) be \(\mathcal{F}_0\)-measurable, and \(X_0 \in L^2\), \(\mathbb{P}\)-a.s.. Then, there exist a stopping time \(\tau(\leq T)\) and a unique solution \(X\) to (6.1) on \([0, \tau]\), such that \(X \in C([0, \tau]; L^2) \cap \mathcal{X}_{\tau, p,q}\) for any Strichartz pair \((p,q)\), \(\mathbb{P}\)-a.s. Moreover, \(\tau = T\), \(\mathbb{P}\)-a.s., if \(\alpha \in [1, 1 + 4/d]\) and \(b, c\) vanish.
Proof of Theorem 6.1. It follows from [44] that the smallness condition \(6.2\) precludes the existence of trapped bicharacteristics and Assumptions (A2) and (A3) hold for the operator \(P(x,D) = D_j a^{jk}(x) D_k\). Thus, Theorem 2.11 yields pathwise Strichartz and local smoothing estimates for the linear part of equation \(6.1\). Consequently, similar arguments as in the proof of [6, Lemma 4.2] yield the local well-posedness. In the mass subcritical case where \(\alpha \in [1, 1 + 4/d)\) and \(b,c\) vanish, similarly to (4.9), we have the martingale property of \(|X(t)|^2\). Thus, arguing as in the proof of [6, Proposition 3.2] we obtain the global existence of solutions to \(6.1\), i.e. \(\tau = T, \mathbb{P}\text{-a.s.} \square\)

6.2 Integrability of global solutions

Below we consider the typical stochastic nonlinear Schrödinger equation with power nonlinearity as in [6]–[10], namely,

\[
id X = \Delta X dt + \lambda |X|^{\alpha-1} X dt - i\mu X dt + iX dW, \tag{6.3}
\]

with \(X(0) = X_0\) being \(\mathcal{F}_0\)-measurable. As mentioned in Section 1, global well-posedness of (6.3) was first studied in [19, 20]. Pathwise global well-posedness in the full mass and energy subcritical cases has been recently obtained in [6, 7]. See also [30] for the full mass subcritical case.

Motivated by optimal control problems (see [10]), we shall prove \(L^\rho(\Omega)\)-integrability of global solutions in both mass and energy subcritical cases, which can be viewed as a complement to [6, 7].

**Theorem 6.2.** Assume the asymptotically flat condition of \(e_j, 1 \leq j \leq N\), as in (2.1). Let \(\alpha \in (1, 1 + 4/d)\) (resp. \(\alpha \in (1, 1 + 4/d)\) if \(\lambda = 1\), and \(\alpha \in (1, 1 + 4/(d - 2)_+)\) if \(\lambda = -1\)) and \(X_0 \in L^\rho(\Omega; L^2)\) (resp. \(L^\rho(\Omega; H^1)\)) for any \(1 \leq \rho < \infty\).

Then, for each \(0 < T < \infty\), there exists a unique global solution \(X\) to (6.3) on \([0, T]\), such that \(X \in C([0, T]; L^2)\) (resp. \(X \in C([0, T]; H^1)\)), \(\mathbb{P}\text{-a.s.}\), and for any Strichartz pair \((p,q)\) and \(1 \leq \rho < \infty\),

\[
\mathbb{E}||X||^\rho_{L^p_x, p,q} < \infty \quad (\text{resp. } \mathbb{E}(||X||^\rho_{L^p_{xT}, p,q} + ||\nabla X||^\rho_{L^p_{xT}, p,q}) < \infty). \tag{6.4}
\]

**Proof.** The global well-posedness follows from similar arguments as in [6, 7]. For the integrability of solutions, let us first consider the \(L^2\) case.

Choose the Strichartz pair \((p,q) = (\alpha + 1, \frac{4(\alpha + 1)}{d(\alpha - 1)})\) and set \(u = e^{-\Phi(W)} X\). As in the proof of [6] (4.9), (4.10), applying Theorem 2.6 and Hölder’s
inequality we have that for any Strichartz pair \((p,q)\),
\[
\|u\|_{X_{t,p,q}} \leq C_T(|X_0|_2 + t^\theta \gamma_T \|u\|_{L^p(0,T;L^p)}),
\]  
(6.5)
where \(\gamma_T = e^{(\alpha-1)|\Phi(W)|_{L^\infty(0,T;L^\infty)}}, \theta = 1-d(\alpha-1)/4 \in (0,1)\), and \(C_T \in L^p(\Omega)\) for any \(1 \leq p < \infty\). Then, taking
\[
t = (\alpha - 1)^{\frac{\alpha-1}{\alpha}} \gamma_T^{-\frac{\alpha}{\alpha-1}} (|X_0|_2 + 1)^{-\frac{\alpha-1}{\alpha}} C_T^{-\frac{\alpha}{\alpha-1}} \gamma_T^{-\frac{\alpha}{\alpha-1}}(\leq T),
\]
and using \([10, \text{Lemma 6.1}]\) we get
\[
\|u\|_{X_{t,p,q}} \leq \frac{\alpha}{\alpha - 1} C_T |X_0|_2 \leq \frac{\alpha}{\alpha - 1} C_T \|u\|_{C([0,T];L^2)}.
\]
Iterating similar arguments on \([jt, (j+1)t \wedge T], 1 \leq j \leq [T/t]\), since \(1/p < 1/2\), we obtain
\[
\|u\|_{X_{t,p,q}} \leq \frac{2\alpha}{\alpha - 1} C_T \left(\frac{T}{t}\right) + 1 + \frac{\alpha}{\alpha - 1} C_T \|u\|_{C([0,T];L^2)} \leq \frac{2\alpha}{\alpha - 1} C_T \left(\frac{T}{t}\right) + 1 + \frac{\alpha}{\alpha - 1} C_T \|u\|_{C([0,T];L^2)}.
\]
(6.6)
Therefore, since \(\|X\|_{X_{t,p,q}} \leq C(l) \sup_{t \in [0,T]} \|e^{(\Phi(W))_{t}}\|_{L^\infty(0,T;L^\infty)}\) for some \(l \geq 1\), using the \(L^p(\Omega)\)-integrability of \(C_T, e^{(\Phi(W))_{t}}\) \(\|L^\infty(0,T;L^\infty)\) and \(\|X\|_{C([0,T];L^2)}\) (see \([7, \text{Lemma 3.6}]\) we obtain \((6.4)\) in the \(L^2\) case.

The proof in the \(H^1\) case is similar. Indeed, as in the proof of \([7, (2.25)]\), using Theorem 2.6 and the Sobolev imbedding \(|u|_{L^p} \leq D|u|_{H^1}\) we have
\[
\|u\|_{X_{t,p,q}} + \|\nabla u\|_{X_{t,p,q}} \leq C_T(|X_0|_{H^1} + t^\theta D(T)\|u\|_{C([0,T];H^1)}^{\alpha-1})\|L^p(0,t;W^{1,p})\),
\]
where \(\theta = 1 - 2/q \in (0,1)\), and \(D(T) = \alpha D^{\alpha-1}(\|\nabla \Phi(W)\|_{L^\infty(0,T;L^\infty)} + 2)\gamma_T\). Taking \(t = (2C_T D(T)\|y\|_{C([0,T];H^1)}^{\alpha-1})^{-1/\theta}\) and using iterated arguments we get
\[
\|u\|_{X_{t,p,q}} + \|\nabla u\|_{X_{t,p,q}} \leq 8C_T \left(\frac{T}{t}\right) + 1 + \frac{\alpha}{\alpha - 1} C_T \|u\|_{C([0,T];H^1)}.
\]
(6.7)
Since \(\|X\|_{C([0,T];H^1)}\) is \(L^p(\Omega)\)-integrable (see \([10, (2.3)]\)), and so are the coefficients \(C_T\) and \(\gamma_T\), we obtain \((6.4)\) in the \(H^1\) case. \(\Box\)
6.3 Large deviation principle

We first consider the large deviation principle (LDP) for the small noise asymptotics for (1.1) in the conservative case. Consider

\[ dX^\varepsilon(t) = iP(x,D)X^\varepsilon(t)dt + F(t)dt - \varepsilon\mu X^\varepsilon(t)dt + \sqrt{\varepsilon}X^\varepsilon(t)dW, \]

(6.8)

where \( X^\varepsilon(0) = X_0 \in L^2, \mathbb{P}\text{-a.s.} \), \( X_0 \) is \( \mathcal{F}_0 \)-measurable, \( \mu \) and \( W \) are as in (1.2), \( \text{Re}\mu_j = 0 \), \( 1 \leq j \leq N \), \( \varepsilon \in (0,1) \), and \( F \) is \( \{\mathcal{F}_t\} \)-adapted, \( F \in \mathcal{X}_{T,-s,p,q}' \) for some \( (s,p,q) \in A \), \( \mathbb{P}\text{-a.s.} \).

Let \( C_0([0,T];\mathbb{R}^N) = \{u \in C([0,T];\mathbb{R}^N) : u(0) = 0\} \). Introduce the map \( G : C_0([0,T];\mathbb{R}^N) \rightarrow \mathcal{X}_{T,s,p,q} ' \) such that for any \( g = (g_1, \cdots, g_N) \in C_0([0,T];\mathbb{R}^N) \), \( u^g := G(g) \) solves the equation

\[ \partial_t u^g(t) = ie^{-\tilde{\Phi}(g)}P(x,D)e^{\tilde{\Phi}(g)}u^g(t) + e^{-\tilde{\Phi}(g)}F(t), \]

(6.9)

where \( u^g(0) = X_0 \), and \( \tilde{\Phi}(g) = \sum_{j=1}^N \mu_j e_j g_j \).

Moreover, define the map \( S : C_0([0,T];\mathbb{R}^N) \rightarrow \mathcal{X}_{T,s,p,q} \) by

\[ S(g) = e^{\tilde{\Phi}(g)}G(g), \quad \forall g \in C_0([0,T];\mathbb{R}^N). \]

(6.10)

Thus, by the rescaling (2.11),

\[ X^\varepsilon = S(\sqrt{\varepsilon} \beta) = e^{\tilde{\Phi}(\sqrt{\varepsilon} \beta)}G(\sqrt{\varepsilon} \beta), \]

(6.11)

where \( \beta = (\beta_1, \cdots, \beta_N) \) are \( N \) dimensional real valued Brownian motions.

**Theorem 6.3.** The family \( \{X^\varepsilon\} \) satisfies a LDP on \( \mathcal{X}_{T,s,p,q} \) of speed \( \varepsilon \) and a good rate function

\[ I(w) = \frac{1}{2} \inf_{g \in H^1(0,T;\mathbb{R}^N):w=S(g)} \|\dot{g}\|^2_{L^2(0,T;\mathbb{R}^N)}, \]

(6.12)

where \( \dot{g} \) denotes the derivative of \( g \).

The key observation here is, that the solution map \( G \) of the reduced equation (6.9) is continuous from \( C([0,T];\mathbb{R}^N) \) to \( \mathcal{X}_{T,s,p,q} \), i.e., the solution to (6.9) depends continuously on lower order perturbations. This fact implies, via the representation formula (6.11) of the stochastic solution to (6.8), the large deviation principle for \( S \) by virtue of Varadhan’s contraction principle.
Lemma 6.4. The map \( G : C_0([0, T]; \mathbb{R}^N) \mapsto \mathcal{X}_{T,s,p,q} \) is continuous.

Proof. Let \( g_n, g \in C_0([0, T]; \mathbb{R}^N) \), \( g_n \rightarrow g \) in \( C_0([0, T]; \mathbb{R}^N) \), as \( n \rightarrow \infty \). Set \( u_n = G(g_n) \). Then, by (6.9), if \( u(x, D, g_n) := ie^{-\tilde{\phi}(g_n)}[P(x, D), e^{\tilde{\phi}(g_n)}] \),
\[
\partial_t u_n(t) = iP(x, D)u_n + b(x, D, g_n)u_n + e^{-\tilde{\phi}(g_n)}F(t).
\]
Define \( u, b(x, D, g) \) similarly as above. Then,
\[
\partial_t (u_n - u) = iP(u_n - u) + (b(x, D, g_n)u_n - b(x, D, g)u) + (e^{-\tilde{\phi}(g_n)} - e^{-\tilde{\phi}(g)})F(t)
\]
\[
= ie^{-\tilde{\phi}(g_n)}P(x, D)e^{\tilde{\phi}(g_n)}(u_n - u) + (b(x, D, g_n) - b(x, D, g))u
\]
\[
+ (e^{-\tilde{\phi}(g_n)} - e^{-\tilde{\phi}(g)})F(t).
\]

Note that, Strichartz and local smoothing estimates as in (2.10) also holds for \( e^{-\tilde{\phi}(g_n)}P(x, D)e^{\tilde{\phi}(g_n)} \) with \( e^{-\Phi(W)}X \) and \( e^{-\Phi(W)}F \) replaced by \( u_n \) and \( e^{-\tilde{\phi}(g_n)}F \) respectively, and the constant \( C_T \) is independent of \( n \), due to the boundedness of \( \sup_n \|g_n(t)\|_{C([0,T];\mathbb{R}^N)} \). Then, taking into account \( \langle x \rangle \langle D \rangle^{-\frac{m-1}{2}} \langle b(x, D, g_n) - b(x, D, g) \rangle \langle D \rangle^{-\frac{m-1}{2}} \langle x \rangle \in S^0 \), and using Lemma 3.4, we obtain
\[
\|u_n - u\|_{X_{T,s,p,q}} \leq C_T \|b(x, D, g_n) - b(x, D, g)\|_{L^2([0,T];H^\frac{m-1}{2})} + C_T \|e^{-\tilde{\phi}(g_n)} - e^{-\tilde{\phi}(g)}\|_{X_{T,s,p,q}}
\]
\[
\leq C_T \sup_{t \in [0,T]} \|b(x, \xi, g_n(t)) - b(x, \xi, g(t))\|_{C^{l-1}(\mathbb{R}^m)} + C_T \|e^{-\tilde{\phi}(g_n(t))} - e^{-\tilde{\phi}(g(t))}\|_{S^0_{1,0}}
\]
for some \( l \geq 1 \).

Thus, since
\[
b(x, \xi, g_n) - b(x, \xi, g) = i \sum_{1 \leq |\alpha| \leq m} \frac{1}{\alpha!} \partial_\alpha^2 P(\psi_\alpha(\tilde{\Phi}(g_n)) - \psi_\alpha(\tilde{\Phi}(g)) ),
\]
where \( \psi_\alpha(\tilde{\Phi}(g_n)) = e^{-\tilde{\phi}(g_n)}D_x^\alpha e^{\tilde{\phi}(g_n)} \) and \( \psi_\alpha(\tilde{\Phi}(g)) \) is defined similarly, using the convergence of \( \{g_n\} \) we get that \( \sup_{t \in [0,T]} \|b(x, \xi, g_n(t)) - b(x, \xi, g(t))\|_{C^{l-1}} \rightarrow 0 \), and \( \sup_{t \in [0,T]} \|e^{-\tilde{\phi}(g_n(t))} - e^{-\tilde{\phi}(g(t))}\|_{S^0_{1,0}} \rightarrow 0 \), as \( n \rightarrow \infty \), which implies that \( \|u_n - u\|_{X_{T,s,p,q}} \rightarrow 0 \), thereby completing the proof. \( \square \)
Corollary 6.5. The map \( S : C([0, T]; \mathbb{R}^N) \to X_{T,s,p,q} \) is continuous.

Proof. Let \( g_n, g, u_n, u \) be as in the proof of Lemma 6.4 and set \( X_n = S(g_n), X = S(g) \). Then, by (6.10),
\[
\|X_n - X\|_{X_{T,s,p,q}} \leq \|e^{\tilde{\Phi}(g_n)(t)}(u_n - u)\|_{X_{T,s,p,q}} + \|e^{\tilde{\Phi}(g_n)} - e^{\tilde{\Phi}(g)}\|_{X_{T,s,p,q}}.
\]
Similarly to Lemma 3.7, \( \|e^{\tilde{\Phi}(g_n)}\|_{L^2(0, T; \mathbb{R}^N)} \leq C|e^{\tilde{\Phi}(g_n)}|_{S^0} \) for some \( l \in \mathbb{N} \). Thus, using Lemma 6.4 and the convergence of \( g_n \) we obtain
\[
\|X_n - X\|_{X_{T,s,p,q}} \leq C \sup_{n \in \mathbb{N}} \sup_{t \in [0, T]} |e^{\tilde{\Phi}(g_n)(t)}|_{S^0} \|u_n - u\|_{X_{T,s,p,q}} + C \sup_{t \in [0, T]} |e^{\tilde{\Phi}(g_n)(t)} - e^{\tilde{\Phi}(g)(t)}|_{S^0} \|u\|_{X_{T,s,p,q}} \to 0,
\]
which finishes the proof. \( \square \)

Proof of Theorem 6.3. By Schilder’s theorem (see e.g. [24, Theorem 5.2.3]), \( \{\sqrt{\varepsilon} \beta\} \) satisfies the LDP of speed \( \varepsilon \) and the good rate function
\[
I^\beta = \frac{1}{2} \inf_{\dot{g} \in H^1(0, T; \mathbb{R}^N)} \|\dot{g}\|^2_{L^2(0, T; \mathbb{R}^N)}.
\]
Then, by virtue of the continuity of \( S \) in Corollary 6.5 and Varadhan’s contraction principle (see [24, Theorem 4.2.1]) we prove (6.12). \( \square \)

We conclude this section with the large deviation principle for the nonlinear Schrödinger equation (6.1) with variable coefficients in the case where \( b \) and \( c \) vanish. See also [28] for the case of constant coefficients.

As above, for any Strichartz pair \((p, q)\), introduce the map \( \tilde{G} : C_0([0, T]; \mathbb{R}^N) \to X_{T,p,q} \), such that for any \( g \in C_0([0, T]; \mathbb{R}^N) \), \( u^g = \tilde{G}(g) \) solves the equation
\[
\partial_t u^g = i e^{-\tilde{\Phi}(g)} P(x, D) e^{\tilde{\Phi}(g)} u^g - \lambda|u^g|^\alpha u^g,
\]
and \( u^g(0) = X_0 \), where \( P(x, D) = \sum_{j,k=1}^d D_j a^{jk}(x) D_k \).

Set \( \tilde{S}(g) := e^{\tilde{\Phi}(g)} \tilde{G}(g) \), \( g \in C_0([0, T]; \mathbb{R}^N) \). Then, \( X^\varepsilon = \tilde{S}(\sqrt{\varepsilon} \beta) \) solves (6.1) with \( b, c = 0 \) and \( W, \mu \) replaced by \( \sqrt{\varepsilon} W \) and \( \varepsilon \mu \) respectively.

Theorem 6.6. Assume the conditions of Theorem 6.1 to hold. Assume in addition that \( \alpha < 1, 1 + 4/d \), \( b \) and \( c \) vanish, and \( \text{Re} \mu_j = 0, 1 \leq j \leq N \).
Then, for any Strichartz pair \((p, q)\), the family \(\{X^\varepsilon\}\) satisfies the LDP on \(\mathcal{X}_{T,p,q}\) of speed \(\varepsilon\) and a good rate function

\[
\tilde{I}(w) = \frac{1}{2} \inf_{g \in H^1(0,T;\mathbb{R}^N), w = \tilde{S}(g)} \|\dot{g}\|_{L^2([0,T] ; \mathbb{R}^N)}^2,
\]

where \(\dot{g}\) denotes the derivative of \(g\).

**Lemma 6.7.** Assume the conditions of Theorem 6.6 to hold. Then, for any Strichartz pair \((p, q)\), the map \(\tilde{G} : C_0([0, T]; \mathbb{R}^N) \to \mathcal{X}_{T,p,q}\) is continuous.

**Proof.** Let \(g_n, g\) be as in the proof of Lemma 6.4. Set \(u_n = \tilde{G}(g_n)\), \(u = \tilde{G}(g)\) and choose the Strichartz pair \((p_0, q_0) = (\alpha + 1, \frac{4(\alpha+1)}{d(\alpha-1)})\).

We first claim that,

\[
\sup_{n \geq 1} \|u_n\|_{\mathcal{X}_{T,p_0,q_0}} < \infty.
\]

In fact, similarly to (6.13),

\[
\|u_n\|_{\mathcal{X}_{T,p_0,q_0}} \leq C_T (|X_0| + t^\theta \|u_n\|_{L^q(0,T;L^p)}), \quad \forall t \in (0, T),
\]

where \(\theta = 1 - d(\alpha - 1)/4 \in (0, 1)\), \(C_T\) is independent of \(n\). As in the proof of [10, (3.16)], taking \(t = \alpha - \alpha/\theta (\alpha - 1)^{(\alpha-1)/\theta} (|X_0| + 1)^{-\alpha/\theta} C_T^{-\alpha/\theta}(\leq T)\), we have

\[
\|u_n\|_{\mathcal{X}_{T,p_0,q_0}} \leq \frac{\alpha}{\alpha - 1} C_T |X_0|.
\]

Since \(|u_n(t)| \leq |X_0|, \forall t \in [0, T]\), we can iterate similar arguments on \([jt, (j+1)t \wedge T], 1 \leq j \leq \lfloor T/t \rfloor\), and obtain

\[
\|u_n\|_{\mathcal{X}_{T,p_0,q_0}} \leq \frac{2\alpha}{\alpha - 1} C_T (\frac{T}{t} + 1)^\frac{1}{\theta} |X_0|, \quad \forall n \geq 1,
\]

which implies (6.15), as claimed.

Now, note that, similarly to (6.13),

\[
\partial_t(u_n - u) = ie^{-\Phi(g_n)} P(x, D)e^{\Phi(g_n)} (u_n - u) + (b(x, D, g_n) - b(x, D, g))u - \lambda i(|u_n|^{\alpha-1}u_n - |u|^{\alpha-1}u),
\]

where \(b(x, D, g_n)\) and \(b(x, D, g)\) are defined as in (6.13).
Applying Strichartz estimates and Hölder’s inequality we obtain
\[
\|u_n - u\|_{x_{t,p,q}} \leq C_T(w_n(u) + \tilde{C}t^\theta \|u_n - u\|_{L^\infty(0,t;L^p)}) ,
\]
where \( \tilde{C} = \alpha(\sup_{n \geq 1} \|u_n\|_{L^\infty(0,T;L^p)}^{-1} + \|u\|_{L^\infty(0,T;L^p)}^{-1}) < \infty \), and
\[
w_n(u) = \|(b(x,D,g_n) - b(x,D,g))u\|_{L^2(0,T;H^{-\frac{1}{2}})} \\
\leq C \sup_{t \in [0,T]} \|b(x,\xi,g_n(t)) - b(x,\xi,g(t))\|_{S^1} \|u\|_{L^2(0,T;H^{-\frac{1}{2}})} \to 0.
\]

Thus, taking \( t = (2C_T\tilde{C})^{-1/\theta} \) we obtain \( \|u_n - u\|_{x_{t,p,q}} \leq 2C_Tw_n(u) \to 0 \).

Thus, as \( t \) is independent of \( n \), iterating similar estimates on \( [jt, (j+1)t \wedge T] \), \( 1 \leq j \leq [T/t] \), we obtain \( \|u_n - u\|_{x_{t,p,q}} \to 0 \) and complete the proof. \( \square \)

Now, using similar arguments as in the proof of Theorem 6.3 we prove Theorem 6.6

7 Appendix

Proof of Lemma 3.2 First note that, for any \( l, k \geq 1 \),
\[
|c_0(x,\xi)| = \left| \int \int e^{-iz \cdot \eta} \langle \eta \rangle^{-k} (\partial_x)^k (\partial_z)^{-l} \langle \partial_\eta \rangle^l (a(x,\xi + \theta \eta)b(x + z,\xi))d\eta dz \right|
\leq CC_1(l)C_2(k) \langle \xi \rangle^{\rho_1(0)} \langle \xi \rangle^{m_2} \int K_1(\xi,\eta)d\eta \int K_2(x,z)dz , \tag{7.1}
\]
where \( K_1(\xi,\eta) = \langle \eta \rangle^{-k} \langle \xi + \theta \eta \rangle^{m_1-l} \) and \( K_2(x,z) = \langle z \rangle^{-l} \langle x + z \rangle^{\rho_2(0)} \).

We first consider \( K_1(\xi,\eta) \). Note that, for \( \eta \in \Lambda := \{ \eta : |\eta| < \langle \xi \rangle /2 \} \), \( \langle \xi \rangle /2 \leq \langle \xi + \theta \eta \rangle \leq 3 \langle \xi \rangle /2 \). Then,
\[
\int_\Lambda K_1(\xi,\eta)d\eta \leq C \int_\Lambda \langle \eta \rangle^{-k} \langle \xi \rangle^{m_1-l} d\eta \leq C \langle \xi \rangle^{m_1-l+d},
\]
which implies that for \( l > d \), \( \int_\Lambda K_1(\xi,\eta)d\eta \leq C \langle \xi \rangle^{m_1} \). Moreover, for \( \eta \in \Lambda^c \), \( \langle \xi + \theta \eta \rangle \leq \langle \xi \rangle + |\eta| \leq 3|\eta| \). Since \( \langle \eta \rangle \geq |\eta| \), we obtain
\[
\int_{\Lambda^c} K_1(\xi,\eta)d\eta \leq C \int_{\Lambda^c} |\eta|^{-k+(m_1)+} d\eta \leq C \langle \xi \rangle^{-k+(m_1)+d} \leq C \langle \xi \rangle^{m_1},
\]

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where we chose $k$ such that $-k + (m_1)_+ + d < -(m_1)_-$, where $(m_1)_+ = \max\{m_1, 0\}$, $(m_1)_- = -(m_1)_+$. Thus, for $l > d$, $k > |m_1| + d$, we have

$$
\int K_1(\xi, \eta) d\eta \leq C \langle \xi \rangle^{m_1}. 
$$

(7.2)

The estimate for $K_2(x, z)$ is similar. Set $\Omega := \{z : |z| \leq \langle x \rangle / 2\}$. For $z \in \Omega$, $\langle x \rangle / 2 \leq \langle x + z \rangle \leq 3 \langle x \rangle / 2$, and so

$$
\int_\Omega K_2(x, z) dz \leq C \langle x \rangle^{\rho_2(0)} \int_\Omega \langle z \rangle^d dz \leq C \langle x \rangle^{\rho_2(0)},
$$

if $l > d$. Moreover, for $z \in \Omega^c$, $\langle x + z \rangle \leq 3|z|$. Then,

$$
\int_{\Omega^c} K_2(x, z) dz \leq C \int_{\Omega^c} |z|^{-(l + \rho_2(0))} dz \leq C \langle x \rangle^{-(l + \rho_2(0))} \leq C \langle x \rangle^{\rho_2(0)},
$$

if $l$ is large enough such that $-l + (\rho_2(0))_+ + d < -(\rho_2(0))_-$. Thus, for $l > (\rho_2(0))_+ + d$, we have

$$
\int K_2(x, z) dz \leq C \langle x \rangle^{\rho_2(0)}. 
$$

(7.3)

Therefore, plugging (7.2) and (7.3) into (7.1) we obtain (3.2).

\[ \square \]

Proof of Lemma 3.7. For simplicity, we set $a_R(x, \xi) := a(x, \xi) \theta_R(\xi)$. (i). By straightforward computations, for any $l \in \mathbb{N}$,

$$
\|c_{R_{S_0}}^{(l)} + c_{R_{S_0}}^{-1}^{(l)}\|_S \leq C(l) M^l e^{M\|a\|_\infty},
$$

(7.4)

which along with Lemma 3.4 implies (3.14).

(ii). By Lemma 3.1, $\Psi_{c_R} \Psi_{c_R^{-1}} = I + \Psi_{e_R}$ with

$$
e_R = \frac{1}{(2\pi)^d} \int_0^1 \sum_{|\gamma|=1} \int e^{iy \cdot \eta} \partial_\xi^\gamma c_R(x, \xi + \theta \eta) D_x^\gamma c_R^{-1}(x + y, \xi) dyd\eta d\theta \in S^0.
$$

Using Corollary 3.3 we have that for any $l \in \mathbb{N}$, $|e_R|_S^{(l)} \leq C(l') \sum_{|\alpha|=1} |\partial_\xi^\alpha c_R|_S^{(l')} |\partial_\xi^\alpha c_R^{-1}|_S^{(l')}$ for some $l' \in \mathbb{N}$. Below we shall prove that

$$
|\partial_\xi^\alpha c_R|_S^{(l')} |\partial_\xi^\alpha c_R^{-1}|_S^{(l')} \leq C(l') R^{-1} M^{2M^l} e^{2M\|a\|_\infty}
$$

(7.5)
Then, in view of Lemma 3.4 for $R = C(l)M^{2l}e^{2M\|a\|_{\infty}}$ with $l$ and $C(l)$ large enough, $\|\Psi_{e_R}\|_{L^2(L^2)}$ is less than $1/2$, which yields that $I + \Psi_{e_R}$ is invertible and $\Psi_{e_R}^{-1} = \Psi_{e_R}^{-1}(I + \Psi_{e_R})^{-1}$, thereby implying (3.15) by (7.4).

It remains to prove (7.5). Note that

$$|\partial^\alpha_x c_R(\xi)|_{S^0} = M|c_R\partial^\alpha_x a_R(\xi)|_{S^0} \leq M|c_R(\xi)|_{S^0} \partial^\alpha_x a_R(\xi)_{S^0} \leq C(l)M^{-1}|c_R(\xi)|_{S^0}. \quad (7.6)$$

Similarly, since $|\partial^\alpha_x a_R(\xi)|_{S^0} = |\theta_R \partial^\alpha_x a(\xi)|_{S^0} \leq C(l)$, we have

$$|\partial^\alpha_x c_R(\xi)|_{S^0} = |Mc^{-1}_R\partial^\alpha_x a_R(\xi)|_{S^0} \leq C(l)M^{-1}|c_R(\xi)|_{S^0}. \quad (7.7)$$

Thus, (7.5) follows from (7.4), (7.6) and (7.7), and (3.15) is proved.

(iii) First note that for any $f \in H_{-\frac{1}{2}}^m$,

$$\|\Psi_{e_R}f\|_{H_{-\frac{1}{2}}^m} = |b(x, D)\langle x\rangle^{-1}\langle D\rangle^{m-\frac{1}{2}}f|_{2} \leq C|c_R(\xi)|_{S^0}\|f\|_{H_{-\frac{1}{2}}^{m-\frac{1}{2}}}$$

for some $l \in \mathbb{N}$, where $b(x, D) := \langle x\rangle^{-1}\langle D\rangle^{(m-1)/2}\Psi_{e_R} \langle D\rangle^{-(m-1)/2}\langle x\rangle \in S^0$ due to Corollary 3.3 and we used Lemma 3.4.

Similarly, since $\Psi_{e_R}^{-1} = \Psi_{e_R}^{-1}(I + \Psi_{e_R})^{-1}$, we have that

$$\|\Psi_{e_R}^{-1}f\|_{H_{\frac{1}{2}}^{m-1}} \leq C|c_R^{-1}(\xi)|_{S^0}\|r_R(\xi)|_{S^0}\|f\|_{H_{-\frac{1}{2}}^{m-1}}$$

for some $l_0 \geq 1$, where $r_R \in S^0$ is the symbol of $(I + \Psi_{e_R})^{-1}$.

Below we claim that, for $R = C(l)M^{l}e^{2M\|a\|_{\infty}}$ with $C(l)$ and $l$ large enough, there exists $C$ independent of $M$ and $R$, such that

$$|r_R(\xi)|_{S^0} \leq C. \quad (7.8)$$

For this purpose, let $e_k \in S^0$ be the symbol of $\Psi_{e_R}^k$. From the proof of (ii) we see that

$$\Psi_{e_R} = (I + \Psi_{e_R})^{-1} = \sum_{k=0}^{\infty}(-1)^k\Psi_{e_R}^k = \sum_{k=0}^{\infty}(-1)^k\Psi_{e_k},$$

which implies that $r_R = \sum_{k=0}^{\infty}(-1)^k e_k$. Note that, for any multi-indices $\alpha, \beta$, $|\alpha + \beta| \leq l_0$, by Corollary 3.3 (7.6) and (7.7), there exits $l \geq 1$ such that

$$|\partial^\alpha_x \partial^\beta_s e_R(x, \xi)| \leq C(l)|c_R(\xi)|_{S^0}|c_R^{-1}(\xi)|_{S^0}M^{l-1}R^{-1}\langle \xi\rangle^{-|\alpha|} = \varepsilon(R, l, \langle \xi\rangle^{-|\alpha|}.\)
where $\varepsilon(R,l) = C(l)|c^\theta_{R|S^0}|c^{-\theta}_{R|S^0}M^2R^{-1} \rightarrow 0$, as $R \rightarrow \infty$. Applying Corollary 3.3 again we have $|\partial^\alpha_\xi \partial^\beta_x e_k(x,\xi)| \leq C^k(l_0)\varepsilon^k(R,l) \langle \xi \rangle^{-|\alpha|}$.

Therefore, for $R \geq 2C(l_0)C(l)|c^\theta_{R|S^0}|c^{-\theta}_{R|S^0}M^2$, we obtain

$$|\partial^\alpha_\xi \partial^\beta_x r_R(x,\xi)| = \left| \sum_{k=0}^\infty (-1)^k \partial^\alpha_\xi \partial^\beta_x e_k(x,\xi) \right| \leq 2 \langle \xi \rangle^{-|\alpha|},$$

which implies (7.8) as claimed and so proves (3.16). The proof is complete. □

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**References**

[1] L. Arnold, Random dynamical systems. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.

[2] O. Bang, P.L. Christiansen, F. If, O. K.O. Rasmussen, Y.B. Gaididei, Temperature effects in a nonlinear model of monolayer Scheibe aggregates, Phys. Rev. E 49, (1994), 4627-4636.

[3] V. Barbu, G. Da Prato, M. Röckner, Stochastic porous media equations and self-organized criticality. Comm. Math. Phys. 285 (2009), no.3, 901-923.

[4] V. Barbu, M. Röckner, Stochastic variational inequalities and applications to the total variations flow perturbed by linear multiplicative noise. Arch. Ration. Mech. Anal. 209 (2013), no. 3, 797-834.

[5] V. Barbu, M. Röckner, An operatorial approach to stochastic partial differential equations driven by linear multiplicative noise, J. Eur. Math. Soc. (JEMS) 17 (2015), no. 7, 1789-1815.

[6] V. Barbu, M. Röckner, D. Zhang, The stochastic nonlinear Schrödinger equation with multiplicative noise: the rescaling approach, *J. Nonlinear Sciences*, 24 (2014), 383-409.
[7] V. Barbu, M. Röckner, D. Zhang, Stochastic nonlinear Schrödinger equations. Nonlinear Anal. 136 (2016), 168-194.

[8] V. Barbu, M. Röckner, D. Zhang, The stochastic logarithmic Schrödinger equation. J. Math. Pures Appl. (9) 107 (2017), no. 2, 123-149.

[9] V. Barbu, M. Röckner, D. Zhang, Stochastic nonlinear Schrödinger equation: no blow-up in the non-conservative case, J. Differential Equations, http://dx.doi.org/10.1016/j.jde.2017.08.030.

[10] V. Barbu, M. Röckner, D. Zhang, Optimal bilinear control of nonlinear stochastic Schrödinger equations driven by linear multiplicative noise, to appear in Ann. Probab.

[11] A. Barchielli, M. Gregoratti, Quantum Trajectories and Measurements in Continuous Case. The Diffusive Case, Lecture Notes Physics, 782, Springer Verlag, Berlin, 2009.

[12] A. Barchielli, A.S. Holevo, Constructing quantum measurement processes via classical stochastic calculus. Stochastic Process. Appl. 58 (1995), no. 2, 293-317.

[13] Z. Brzeźniak, F. Hornung, L. Weis, Martingale solutions for the stochastic nonlinear Schrödinger equation in the energy space, arXiv:1707.05610v1.

[14] Z. Brzeźniak, A. Millet, On the stochastic Strichartz estimates and the stochastic nonlinear Schrödinger equation on a compact Riemannian manifold. Potential Anal. 41 (2014), no. 2, 269-315.

[15] H. Chihara, Smoothing effects of dispersive pseudodifferential equations. Comm. Partial Differential Equations 27 (2002), no. 9-10, 1953-2005.

[16] K. Chouk, M. Gubinelli, Nonlinear PDEs with modulated dispersion II: KdV equations, 2014. ArXiv Preprint arXiv:1406.7675.

[17] K. Chouk, M. Gubinelli, Nonlinear PDEs with modulated dispersion I: Nonlinear Schrödinger equations. (English summary) Comm. Partial Differential Equations 40 (2015), no. 11, 2047-2081.
[18] W. Craig, T. Kappeler, W. Strauss, Microlocal dispersive smoothing for
the Schrödinger equation. Comm. Pure Appl. Math. 48 (1995), no. 8,
769-860.

[19] A. de Bouard, A. Debussche, A stochastic nonlinear Schrödinger equa-
tion with multiplicative noise, Comm. Math. Phys., 205 (1999), 161-181.

[20] A. de Bouard, A. Debussche, The stochastic nonlinear Schrödinger equa-
tion in $H^1$, Stoch. Anal. Appl., 21 (2003), 97-126.

[21] A. de Bouard, A. Debussche, The Korteweg-de Vries equation with mul-
tiplicative homogeneous noise, Stochastic Differential Equations : The-
ory and Applications, P.H. Baxendale and S.V. Lototsky Ed., Interdisci-
plinary Math. Sciences, vol. 2, World Scientific, 2007.

[22] A. de Bouard, A. Debussche, The nonlinear Schrödinger equation with
white noise dispersion. J. Funct. Anal. 259 (2010), no. 5, 1300-1321.

[23] A. Debussche, Y. Tsutsumi, 1D quintic nonlinear Schrödinger equa-
tion with white noise dispersion. J. Math. Pures Appl. (9) 96 (2011), no. 4,
363-376.

[24] A. Dembo, O. Zeitouni, Large deviations techniques and applicat-
ions. Corrected reprint of the second (1998) edition. Stochastic Modelling and
Applied Probability, 38. Springer-Verlag, Berlin, 2010.

[25] S. Doi, On the Cauchy problem for Schrödinger type equation and the
regularity of solutions, J. Math. Kyoto Univ., 1994, 34 (2), 319-328.

[26] S. Doi, Remarks on the Cauchy problem for Schrödinger-type equations,
Comm. PDE, 21 (1996), 163-178.

[27] S. Doi, Smoothing effects for Schrödinger evolution equation and global
behavior of geodesic flow. Math. Ann., 318 (2000), 355-389.

[28] E. Gautier, Uniform large deviations for the nonlinear Schrödinger equa-
tion with multiplicative noise. Stochastic Process. Appl. 115 (2005), no.
12, 1904-1927.

[29] A.S. Holevo, On dissipative stochastic equations in a Hilbert space.
Probab. Theory Related Fields 104 (1996), no. 4, 483-500.
[30] F. Hornung, The nonlinear stochastic Schrödinger equation via stochastic Strichartz estimates, arXiv: 1611.07325.

[31] M. Keel, T. Tao. Endpoint Strichartz estimates. Amer. J. Math., 120(5): 955-980, 1998.

[32] C.E. Kenig, The Cauchy Problem for Quasilinear Schrödinger Equation (Following Kenig-Ponce-Vega), arXiv: 1309.3291v1.

[33] C.E. Kenig, G. Ponce, L. Vega, On the (generalized) Korteweg-de Vries equation. Duke Math. J. 59 (1989), 585-610.

[34] C.E. Kenig, G. Ponce, L. Vega, Oscillatory Integrals and Regularity of Dispersive Equations. Indiana Univ. Math. J. (1991) 40, 33-69.

[35] C.E. Kenig, G. Ponce, L. Vega, The Cauchy problem for quasi-linear Schrödinger equations. Invent. Math. 158 (2004), no. 2, 343-388.

[36] H. Kumano-go, Pseudodifferential operators. Translated from the Japanese by the author, Rémi Vaillancourt and Michihiro Nagase. MIT Press, Cambridge, Mass.-London, 1981.

[37] F. Linares, G. Ponce, Introduction to Nonlinear Dispersive Equations, Springer, 2009.

[38] W. Liu, M. Röckner, Stochastic partial differential equations: an introduction. Universitext. Springer, Cham, 2015.

[39] J. Marzuola, J. Metcalfe, D. Tataru, Strichartz estimates and local smoothing estimates for asymptotically flat Schrödinger equations, J. Funct. Anal., 255 (6) (2008), 1479-1553.

[40] L. Robbiano, C. Zuily, Strichartz estimates for the Schrödinger equation with variable coefficients, Mém.Soc.Math.Fr.(N.S.) 101-102 (2005), 208PP.

[41] I. Rodnianski, W. Schlag, Time decay for solutions of Schrödinger equations with rough and time-dependent potentials. Invent. Math. 155 (2004), no. 3, 451-513.
[42] H. Smith, C. Sogge, Global Strichartz estimates for nontrapping perturbations of the Laplacian. Comm. Partial Differential Equations 25 (2000), no. 11-12, 2171-2183.

[43] T. Tao. Nonlinear Dispersive Equations. Local and Global Analysis, CBMS Regional Conference Series in Mathematics, 106, AMS (2006).

[44] D. Tataru, Parametrices and dispersive estimates for Schrödinger operators with variable coefficients. Amer. J. Math. 130 (2008), no. 3, 571-634.

[45] M. E. Taylor, Tools for PDE. Pseudodifferential operators, paradifferential operators, and layer potentials. Mathematical Surveys and Monographs, 81. American Mathematical Society, Providence, RI, 2000.