THE FELLER PROPERTY ON RIEMANNIAN MANIFOLDS

STEFANO PIGOLA AND ALBERTO G. SETTI

ABSTRACT. The asymptotic behavior of the heat kernel of a Riemannian manifold gives rise to the classical concepts of parabolicity, stochastic completeness (or conservative property) and Feller property (or $C^0$-diffusion property). Both parabolicity and stochastic completeness have been the subject of a systematic study which led to discovering not only sharp geometric conditions for their validity but also an incredible rich family of tools, techniques and equivalent concepts ranging from maximum principles at infinity, function theoretic tests (Khas’minkii criterion), comparison techniques etc... The present paper aims to move a number of steps forward in the development of a similar apparatus for the Feller property.

1. INTRODUCTION

This paper is a contribution to the theory of Riemannian manifolds satisfying the Feller (or $C_0$-diffusion) property for the Laplace-Beltrami operator. Since the appearance of the beautiful, fundamental paper by R. Azencott, [2], new insights into such a theory (for the Laplace operator) are mainly confined into some works by S.T. Yau, [25], J. Dodziuk, [8], P. Li and L. Karp, [17], E. Hsu, [13], [14], E.B. Davies, [7]. These papers, which have been extended to more general classes of diffusion operators (see e.g. [23], [16]) are devoted to the search of optimal geometric conditions for a manifold to enjoy the Feller property. In fact, with the only exception of Davies’, the geometric conditions are always subsumed to Ricci curvature lower bounds. The methods employed to reach their results range from estimates of solutions of parabolic equations (Dodziuk, Yau, Li-Karp) up to estimates of the probability of the Brownian motion on $M$ to be found in certain regions before a fixed time (Hsu). The probabilistic approach, which led to the best known condition on the Ricci tensor, relies on a result by Azencott (see also [14]) according to which $M$ is Feller if and only if, for every compact set $K$ and for every $t_0 > 0$, the Brownian motion $X_t$ of $M$ issuing from $x_0$ enters $K$ before the time $t_0$ with a probability that tends to zero as $x_0 \to \infty$.

Our point of view will be completely deterministic and, although parabolic equations will play a key role in a number of crucial sections, it will most often depend on elliptic equation theory.

2010 Mathematics Subject Classification. 58J05, 58J35, 58J65, 31B35.
Beside, and closely related, to the Feller property one has the notions of parabolicity and stochastic completeness. Recall that $M$ is said to be parabolic if every bounded above subharmonic function must be constant. Equivalently, the (negative definite) Laplace-Beltrami operator $\Delta$ of $M$ does not possess a minimal, positive Green kernel. From the probabilistic viewpoint, $M$ is parabolic if the Brownian motion $X_t$ enters infinitely many times a fixed compact set, with positive probability (recurrence). We also recall that $M$ is conservative or stochastically complete if, for some (hence any) constant $\lambda > 0$, every bounded, positive $\lambda$-subharmonic function on $M$ must be identically equal to 0. Here, $\lambda$-subharmonic means that $\Delta u \geq \lambda u$. Equivalently, $M$ has the conservative property if the heat kernel of $M$ has mass identically equal to 1. From the probabilistic viewpoint stochastic completeness means that, with probability 1, the Brownian paths do not explode to $\infty$ in a finite time. Clearly a parabolic manifold is stochastically complete.

Both parabolicity and stochastic completeness have been the subject of a systematic study which led to discovering not only sharp geometric conditions for their validity (in fact, volume growth conditions) but also an incredible rich family of tools, techniques and equivalent concepts ranging from maximum principles at infinity, function theoretic tests (Khas’minskii criterion), comparison techniques etc... The interested reader can consult e.g. the excellent survey paper by A. Grigor’yan, [11]. See also [20], [21] for the maximum principle perspective.

The present paper aims to move a number of steps forward in the development of a similar apparatus for the Feller property. Originally we also thought we would adopt an elliptic point of view. While, in many instances, this proves to be the most effective approach (for instance, in obtaining comparison results, or in the treatment of ends), in some cases it is not clear how to implement it, and we have to resort to the parabolic point of view (for instance studying minimal surfaces, or Riemannian coverings).

To make the treatment more readable, we decide to include the proofs of some of the basic results that are crucial in the development of the theory. Sometimes, we shall use a somewhat different perspective and more straightforward arguments. In fact, our attempt is to use a geometric slant from the beginning of the theory, most notably in interpreting the one dimensional case in terms of model manifolds.

2. Heat semigroup and the Feller property

Hereafter, $(M, \langle , \rangle)$ will always denote a connected, non-necessarily complete Riemannian manifold of dimension $\dim M = m$ and without boundary. Further requirements on $M$ will be specified when needed. The (negative definite) Laplace-Beltrami operator of $M$ is denoted by $\Delta$. With this sign convention, if $M = \mathbb{R}$, then $\Delta = \frac{d^2}{dx^2}$. Recall from the fundamental work by J. Dodziuk, [8], that $M$ possesses a positive, minimal heat kernel $p_t(x, y)$,
i.e., the minimal, positive solution of the problem

\[
\begin{cases}
\Delta p_t = \frac{\partial p_t}{\partial t}, \\
p_{0+}(x, y) = \delta_y(x).
\end{cases}
\]

Minimality means that if \( q_t(x, y) \) is a second, positive solution of \((2.1)\), then \( p_t(x, y) \leq q_t(x, y) \). According to Dodziuk construction, \( p_t(x, y) \) is obtained as

\[
p_t(x, y) = \lim_{n \to +\infty} p_{t}^{\Omega_n}(x, y),
\]

where, having fixed any smooth, relatively compact exhaustion \( \Omega_n \supset M \), \( p_{t}^{\Omega_n}(x, y) \) is the solution of the Dirichlet boundary value problem

\[
\begin{cases}
\Delta p_{t}^{\Omega_n} = \frac{\partial p_{t}^{\Omega_n}}{\partial t}, & \text{on } \Omega_n \\
p_{t}^{\Omega_n}(x, y) = 0, & \text{if } x \in \partial \Omega_n \text{ or } y \in \partial \Omega_n \\
p_{0+}^{\Omega_n}(x, y) = \delta_y(x).
\end{cases}
\]

The following properties hold:

(a) \( p_t(x, y) \geq 0 \) is a symmetric function of \( x \) and \( y \).
(b) \( \int_M p_t(x, z) p_s(z, y) \, dz = p_{t+s}(x, y), \forall t, s > 0, \) and \( \forall x, y \in M \)
(c) \( \int_M p_t(x, y) \, dy \leq 1, \forall t > 0 \) and \( \forall x \in M \).
(d) For every bounded continuous function \( u \) on \( M \) set

\[
P_t u(x) = \int_M p_t(x, y) \, u(y) \, dy,
\]

then \( P_t u(x) \) satisfies the heat equation on \( M \times (0, +\infty) \). Moreover, by (b) and (c), \( P_t \) extends to a contraction semigroup on every \( L^p \), called the heat semigroup of \( M \).

From the probabilistic viewpoint, \( p_t(x, y) \) represents the transition probability density of the Brownian motion \( X_t \) of \( M \). In this respect, property (c) stated above means that \( X_t \) is, in general, sub-Markovian. In case the equality sign holds for some (hence any) \( t > 0 \) and \( x \in M \) we say that \( M \) is stochastically complete.

Set \( C_0(M) = \{ u : M \to \mathbb{R} \text{ continuous : } u(x) \to 0, \text{ as } x \to \infty \} \).

**Definition 2.1.** The Riemannian manifold \((M, \langle, \rangle)\) is said to satisfies the \( C_0 \)-diffusion property or, equivalently, it satisfies the Feller property, if

\[(2.2)\]

\[
P_t u(x) \to 0, \text{ as } x \to +\infty
\]

for every \( u \in C_0(M) \).

Using property (c) of the heat kernel and a cut-off argument one can easily prove the following

**Lemma 2.2.** Assume that \( M \) is geodesically complete. Then \( M \) is Feller if and only if \((2.2)\) holds for every non-negative function \( u \in C_c(M) \).
**Proof.** Indeed, suppose that (2.2) holds for every $0 \leq u \in C_c(M)$. Let $v \in C_0(M)$ and define $v_\pm(x) = \max \{0, \pm v(x)\} \in C_0(M)$ so that $v(x) = v_+(x) - v_-(x)$. For every $R > 0$, fix a cut off function $0 \leq \xi_R \leq 1$ satisfying $\xi_R = 1$ on $B_R(0)$, and $\xi_R = 0$ on $M \setminus B_{2R}$. Then,

$$
(P_t v_\pm)(x) = (P_t (\xi_R v_\pm))(x) + (P_t ((1 - \xi_R)v_\pm))(x)
\leq (P_t (\xi_R v_\pm))(x) + \sup_{M \setminus B_R(0)} v_\pm.
$$

Since $0 \leq \xi_R v_\pm \in C_c(M)$, letting $x \to \infty$, we deduce

$$
\lim_{x \to \infty} (P_t v_\pm)(x) \leq \sup_{M \setminus B_R(0)} v_\pm.
$$

Since this inequality holds for every $R > 0$, letting $R \to +\infty$ we conclude

$$
\lim_{x \to \infty} (P_t v_\pm)(x) = 0.
$$

The converse is obvious. □

As a trivial consequence of Lemma 2.2, we point out the following interesting characterization.

**Corollary 2.3.** The geodesically complete manifold $M$ is Feller if and only if, for some $p \in M$ and for every $R > 0$,

$$
\int_{B_R(p)} p_t(x,y) \, dy \to 0, \text{ as } x \to +\infty.
$$

3. **Elliptic exterior boundary value problems and the Feller property**

This section is crucial for the development of most parts of the paper. As we will recall below, following Azencott, the Feller property can be characterized in terms of asymptotic properties of solutions of exterior boundary value problems. In this direction, a basic step is represented by the following

**Theorem 3.1.** Let $\Omega \subset\subset M$ be a smooth open set and let $q : M \to \mathbb{R}$ be a smooth function, $q(x) \geq 0$. Then, the problem

$$
\begin{cases}
\Delta h = q(x) h & \text{on } M \setminus \Omega \\
h = 1 & \text{on } \partial \Omega \\
h > 0 & \text{on } M \setminus \Omega,
\end{cases}
$$

has a (unique) minimal smooth solution $h : M \setminus \Omega \to \mathbb{R}$. Necessarily, $0 < h(x) \leq 1$. Furthermore, $h$ is given by

$$
h(x) = \lim_{n \to +\infty} h_n(x),
$$

where, for any chosen relatively compact, smooth exhaustion $\Omega_n \nearrow M$ with $\Omega \subset\subset \Omega_1$, $h_n$ is the solution of the boundary value problem

$$
\begin{cases}
\Delta h_n = q(x) h_n & \text{on } \Omega_n \setminus \overline{\Omega} \\
h_n = 1 & \text{on } \partial \Omega \\
h_n = 0 & \text{on } \partial \Omega_n.
\end{cases}
$$
Proof. Indeed, the sequence $h_n$ is increasing and $0 < h_n < 1$ in $\Omega_n \setminus \overline{\Omega}$ by the maximum principle. By standard arguments, $h_n$ together with all its derivatives converges, locally uniformly in $M \setminus \overline{\Omega}$, to a solution of (3.2). If $\tilde{h}$ is another solution of (3.2), then, again by the maximum principle $h_n \leq \tilde{h}$ on $\Omega_n \setminus \overline{\Omega}$, and passing to the limit $h \leq \tilde{h}$, showing that $h$ is minimal. □

The following result, originally due to Azencott, builds up the bridge between the parabolic and the elliptic viewpoints. For the sake of completeness we provide a direct proof.

**Theorem 3.2.** The following are equivalent:

1. $M$ is Feller.
2. For some (hence any) open set $\Omega \subset M$ with smooth boundary and for some (hence any) constant $\lambda > 0$, the minimal solution $h : M \setminus \overline{\Omega} \to \mathbb{R}$ of the problem

   \[
   \begin{cases}
   \Delta h = \lambda h & \text{on } M \setminus \overline{\Omega} \\
   h = 1 & \text{on } \partial \Omega \\
   h > 0 & \text{on } M \setminus \overline{\Omega},
   \end{cases}
   \]

   satisfies

   \[ h(x) \to 0, \text{ as } x \to \infty. \]

Proof. Assume that $M$ is Feller, so that the heat semigroup $P_t$ maps $C_0(M)$ into itself. Let $\Omega$ be a relatively compact open set with smooth boundary and let $\lambda > 0$. We choose a continuous function $u \geq 0$ with support contained in $\Omega$ and let $u_t = P_t u(x)$ be the solution of the heat equation with initial data $u$, so that $u_t > 0$ on $M$ by the parabolic maximum principle. Next set

   \[ w(x) = \int_0^\infty u_t(x) e^{-\lambda t} dt. \]

Note that since $P_t$ is contractive on $L^\infty$ the integral is well defined. Moreover, $w(x) \to 0$ as $x \to \infty$. Indeed, it suffices to show that $w(x_n) \to 0$ for every sequence $x_n \to \infty$. Since $M$ is assumed to be Feller, for every $t \geq 0$, $u_t(x_n) \to 0$ as $n \to \infty$, and the required conclusion follows by dominated convergence.

Differentiating under the integral we obtain

\[
\Delta w(x) = \int_0^{+\infty} e^{-\lambda t} \Delta u_t = \int_0^{+\infty} e^{-\lambda t} \partial_t u_t = -u(x) + \int_0^{+\infty} \lambda e^{-\lambda t} u_t = -u(x) + \lambda w(x),
\]

so that $w$ satisfies

\[ \Delta w \leq \lambda w \quad \text{in } M \setminus \Omega. \]

Since $C = \inf_{\partial \Omega} w > 0$, $v = C^{-1} w \geq 1$ on $\partial \Omega$, $v(x) \to 0$ as $x \to \infty$. It follows that if $h_n$ is a sequence as in Theorem 3.1, $1 = h_n \leq v$ on $\partial \Omega$, and
0 = h_n < v on \partial \Omega_n, so h_n \leq v on \Omega_n \setminus \Omega, and passing to the limit 0 < h \leq v on M \setminus \Omega. Since v tends to zero as x \to \infty.

For the converse, assume that for a given relatively compact open set \Omega and \lambda > 0, the minimal solution h of (3.3) satisfies h(x) \to 0 as x \to \infty. As noted above, in order to verify that M is Feller it suffices to show that \Pi_t maps non-negative compactly supported functions into \( C_0(M) \). So let \( u \) be such a function. We consider an exhaustion \( \Omega_n \) of \( M \) with relatively compact domains with smooth boundary such that \( \Omega \cup \text{supp} u \subset \subset \Omega_1 \), and let \( \Pi_t^{\Omega_n} \) be the Dirichlet heat kernel of \( \Omega_n \), \( \Pi_t^{\Omega_n}(x, y) \neq \Pi_t(x, y) \), and therefore \( u_{n,t} = \Pi_t^{\Omega_n} u \neq \Pi_t u \). Moreover, since \( \Pi_t^{\Omega_n} \) vanishes if either \( x \) or \( y \) lie on \( \partial \Omega_n \), \( u_{n,t} = 0 \) on \( \partial \Omega_n \times [0, +\infty) \), and since the initial datum vanishes outside \( \Omega_1 \), for every \( n > 1 \), \( u_{n,0} = 0 \in \Omega_n \setminus \Omega_1 \).

Now we fix \( t > 0 \). Since \( h \) is strictly positive on \( M \setminus \Omega \), there exists a constant \( C \) such that \( Ch(x) \geq u_s(x) \geq u_{n,s}(x) \) for every \( x \in \partial \Omega_1 \) and \( s \in [0, t] \). It follows that the function \( v_t = Ch(x)e^{\lambda t} \) is a solution of the heat equation on \( M \setminus \Omega \) which satisfies \( v_s \geq u_{n,s} \) on \( (\Omega_n \setminus \Omega_1) \times [0, +\infty) \). By the parabolic maximum principle \( v_t(x) \geq u_{n,t}(x) \) in \( \Omega_n \setminus \Omega_1 \), and, letting \( n \to \infty \), \( u_t(x) \leq Ce^{\lambda t}h(x) \). Since \( h(x) \to 0 \) as \( x \to \infty \) we conclude that so does \( u_t \), as required.

**Remark 3.3.** It is worth pointing out that the elliptic characterization of the Feller property involves the minimal solution to problem (3.4) and not a generic solution. In fact, even on Feller manifolds, there could exist infinitely many positive solutions which are asymptotically non-zero. This fact is easily verified on a model manifold, and will be seen in Section 6. On the other hand we will see in Section 6 that on a stochastically complete manifold the minimal solution is the only one bounded positive solution.

4. Feller property on rotationally symmetric manifolds

We recall the following

**Definition 4.1.** Let \( g : \mathbb{R} \to \mathbb{R} \) be a smooth, odd function satisfying \( g'(0) = 1 \), \( g(r) > 0 \) for \( r > 0 \). A (complete, non-compact) model manifold with warping function \( g \) is the \( m \)-dimensional Riemannian manifold

\[
M_g^m = \left( [0, +\infty) \times S^{m-1}, dr^2 + g(r)^2 d\theta^2 \right)
\]

where \( d\theta^2 \) stands for the standard metric on the \((m-1)\)-dimensional sphere \( S^{m-1} \). We refer to the origin \( o \in M_g \) as the pole of the model. The \( r \)-coordinate in the polar decomposition of the metric represents the distance from \( o \).

It is well known that necessary and sufficient conditions for \( M_g^m \) to be parabolic or stochastically complete are expressed in terms of the solely warped function \( g \); see e.g. [11] and references therein. More precisely, the
model manifold $M_g^m$ is parabolic if and only if
\[ \frac{1}{g^{m-1}} \notin L^1( + \infty) \]
whereas $M_g^m$ is stochastically complete if and only if
\[ \int_0^\infty \frac{g^{m-1}(t)}{g^{m-1}(r)} \, dt \notin L^1( + \infty). \]

This section aims to provide a similar characterization for the validity of the Feller property on $M_g^m$, thus completing the picture.

The next result will enable us to use model manifolds as test and comparison spaces for the validity of the Feller property.

**Theorem 4.2.** Let $M_g^m$ be an $m$-dimensional model manifold with warping function $g$. Let $q(r(x)) \geq 0$ be a smooth, rotationally symmetric function. Let $h$ be the minimal solution of the problem
\[ \begin{cases} \Delta h = q(r) h & \text{on } M_g \setminus B_{R_0} \\ h = 1 & \text{on } \partial B_{R_0} \\ h > 0 & \text{on } M_g \setminus B_{R_0}, \end{cases} \]
where $B_{R_0}$ is the metric ball of radius $R_0 > 0$, centered at the pole of $M_g$. Then $h$ is rotationally symmetric.

**Proof.** Let $R_n$ be an increasing sequence with $R_1 > R_0$, and for every $n$ let $h_n$ be the solution of the problem
\[ \begin{cases} \Delta h_n = q(r(x))h_n & \text{on } B_{R_n} \setminus \overline{B}_{R_0} \\ h_n = 1 & \text{on } \partial B_{R_0} \\ h_n = 0 & \text{on } \partial B_{R_n}, \end{cases} \]
so that $h = \lim_n h_n$. By the maximum principle, the solution to (4.3) is unique, and since coefficients and boundary data are rotationally symmetric, so is $h_n$. Passing to the limit we conclude that $h$ is rotationally symmetric. \[\square\]

Combining Theorem 3.2 with Theorem 4.2 and recalling that the Laplacian of a radial function $u(r(x))$ on $M_g^m$ is given by
\[ \Delta u = u'' + (m-1) \frac{g'}{g} u', \]
we immediately deduce the following important

**Corollary 4.3.** Let $M_g^m$ be an $m$-dimensional model manifold with warping function $g$. Then $M_g^m$ is Feller if and only if, for some (hence any) $R_0 > 0$, the minimal solution of the following o.d.e. problem
\[ \begin{cases} h'' + (m-1) \frac{g'}{g} h' = \lambda h & \text{on } [R_0, +\infty) \\ h(R_0) = 1 \\ h(r) > 0 & \text{on } [R_0, +\infty), \end{cases} \]
satisfies
\begin{equation}
\lim_{r \to +\infty} h(r) = 0.
\end{equation}

In his fundamental paper, [2], Azencott gave necessary and sufficient conditions for a 1-dimensional diffusion to satisfy the Feller property. These conditions concern with the coefficients of the corresponding diffusion operator. On the other hand, in Corollary 4.3, we showed that the Feller property on a model manifold $M^m_g$ can be reduced to that of a special 1-dimensional diffusion. Therefore, we are able to give the following geometric interpretation of Azencott result.

**Theorem 4.4.** Let $M^m_g$ be an $m$-dimensional model manifold with warping function $g$. Then $M^m_g$ is Feller if and only if either
\begin{equation}
\frac{1}{g^{m-1}(r)} \in L^1(+\infty)
\end{equation}
or
\begin{equation}
(i) \frac{1}{g^{m-1}(r)} \notin L^1(+\infty) \quad \text{and} \quad (ii) \int_{g^{m-1}(r)}^{+\infty} \frac{g^{m-1}(t)}{g^{m-1}(r)} dt \notin L^1(+\infty).
\end{equation}

**Remark 4.5.** In case $g^{m-1} \notin L^1(+\infty)$ condition (4.7) (ii) has to be understood as trivially satisfied.

For the sake of completeness, we include a proof of Theorem 4.4 which is clearly modeled on Azencott arguments but it is somewhat more direct.

**Proof.** Assume first the validity of (4.6). For every $n \in \mathbb{N}$, consider the function
\begin{equation}
G_n(r) = \int_r^n \frac{1}{g^{m-1}(t)} dt
\end{equation}
and note that $u_n(r) = G_n(r) / G_n(1)$ solves the Dirichlet problem
\begin{equation}
\begin{cases}
\Delta u_n = 0 & \text{on } B_n(0) \setminus B_1(0) \\
u_n = 1 & \text{on } \partial B_1(0) \\
u_n = 0 & \text{on } \partial B_n(0).
\end{cases}
\end{equation}
Let $\lambda > 0$ be fixed and let $h_n(r)$ be the (rotationally symmetric) solution of
\begin{equation}
\begin{cases}
\Delta h_n = \lambda h \quad & \text{on } B_n(0) \setminus B_1(0) \\
h = 1 & \text{on } \partial B_1(0) \\
h = 0 & \text{on } \partial B_n.
\end{cases}
\end{equation}
By the maximum principle
\begin{equation}
h_n(r) \leq u_n(r), \text{ on } B_n(0) \setminus B_1(0).
\end{equation}
Letting $n \to +\infty$ we deduce that the minimal, positive solution $h(r)$ of
\begin{equation}
\begin{cases}
\Delta h = \lambda h \quad & \text{on } M^m_g \setminus B_1(0) \\
h = 1 & \text{on } \partial B_1(0)
\end{cases}
\end{equation}
THE FELLER PROPERTY ON RIEMANNIAN MANIFOLDS

satisfies

\[(4.13) \quad h(r) \leq c \int_{r}^{+\infty} \frac{1}{g^{m-1}(t)} dt,\]

with \(c = \int_{1}^{+\infty} g^{1-m}(t) dt > 0\). It follows that \(h(r) \to 0\) as \(r \to +\infty\) proving that \(M^{m}_g\) is Feller.

Suppose now that conditions (4.7) (i) and (ii) are met and, as above, let \(h\) be the minimal positive solution of (4.12). Explicitly, this means that \(h(r) \geq 0\) satisfies

\[(4.14) \begin{cases} (g^{m-1}h')' = \lambda g^{m-1}h \text{ on } (1, +\infty) \\ h(1) = 1. \end{cases}\]

Note that, in particular, \(g^{m-1}h'\) is increasing. On the other hand, by Lemma 5.2 of the previous section, \(h'(r) < 0\) on \((1, +\infty)\) and, therefore,

\[(4.15) \quad g^{m-1}(r) h'(r) \to b \leq 0, \text{ as } r \to +\infty.\]

We claim that, in fact, \(b = 0\). Indeed, suppose the contrary. Then, having fixed \(\varepsilon > 0\) satisfying \(b + \varepsilon < 0\), we can choose \(r_0 \gg 1\) such that

\[−g^{m-1}(r) h'(r) \geq -(b + \varepsilon) > 0, \text{ on } [r_0, +\infty).\]

Whence, integrating on \([r_0, +\infty]\) yields

\[(4.16) \quad h(r_0) \geq -\lim_{r \to +\infty} h(r) + h(r_0) \geq -(b + \varepsilon) \int_{r_0}^{+\infty} \frac{1}{g^{m-1}(t)} dt,\]

which contradicts (4.7) (i). This proves the claim. Keeping in mind this fact, we now integrate (4.14) on \([r, +\infty)\) and we get

\[(4.17) \quad -g^{m-1}(r) h'(r) = \lambda \int_{r}^{+\infty} g^{m-1}(t) h(t) dt \geq \lambda \lim_{t \to +\infty} h(t) \int_{r}^{+\infty} g^{m-1}(t) dt.\]

Accordingly, if \(g^{m-1} \notin L^1(+\infty)\) we necessarily have \(\lim_{t \to +\infty} h(t) = 0\) and \(M^{m}_g\) is Feller. On the other hand, if \(g^{m-1} \in L^1(+\infty)\), integrating (4.17) once more we deduce

\[(4.18) \quad h(1) \geq -\lim_{t \to +\infty} h(t) + h(1) \geq \lambda \lim_{t \to +\infty} h(t) \int_{1}^{+\infty} \frac{\int_{r}^{+\infty} g^{m-1}(t) dt}{g^{m-1}(r)} dr.\]

Because of (4.7) (ii), this latter forces \(\lim_{t \to +\infty} h(t) = 0\) and \(M^{m}_g\) is again Feller.

Conversely, we now suppose that the model \(M^{m}_g\) is Feller, we assume that condition (4.6) is not satisfied and we prove the validity of (4.7) (ii). If \(g^{m-1} \notin L^1(+\infty)\) then there is nothing to prove. Otherwise, we note that,
as above, $g^{m-1}(r) h'(r) \to 0$, as $r \to +\infty$. Therefore, according to the first line in (4.17), we have

$$-g^{m-1}(r) h'(r) = \lambda \int_r^{+\infty} g^{m-1}(t) h(t) \, dt$$

and integrating this latter on $[1, +\infty]$ finally gives

$$+\infty = - \lim_{r \to +\infty} \log h(r) \leq \lambda \int_1^{+\infty} \frac{\int_r^{+\infty} g^{m-1}(t) \, dt}{g^{m-1}(r)} \, dr,$$

as desired. \hfill \Box

Recall that, on $M^m_g$,

(4.19) \quad \text{vol} (\partial B_r) = c_m g^{m-1}(r)

where $c_m$ is the volume of the Euclidean unit sphere $S^{m-1}$. In particular, by the co-area formula,

(4.20) \quad \text{vol} (M^m_g) - \text{vol} (B_r) = c_m \int_r^{+\infty} g^{m-1}(t) \, dt.

Therefore Theorem 4.4 can be restated more geometrically by saying that $M^m_g$ is Feller if either

(4.21) \quad \frac{1}{\text{vol} (\partial B_r)} \in L^1 (+\infty)

or

(a) \quad \frac{1}{\text{vol} (\partial B_r)} \notin L^1 (+\infty) \quad \text{and}

(b) \quad \frac{\text{vol} (M_g)}{\text{vol} (\partial B_r)} - \frac{\text{vol} (B_r)}{\text{vol} (\partial B_r)} \notin L^1 (+\infty).

From these considerations we deduce, in particular, the validity of the next

**Corollary 4.6.** Every model manifold $M^m_g$ with infinite volume has the Feller property.

**Remark 4.7.** We have already recalled that a necessary and sufficient condition for $M_g$ to be non-parabolic is that $g^{1-m} \in L^1 (+\infty)$. In fact, if $o$ denotes the pole of $M_g$, then the function

$$G(x, o) := \int_{r(x)}^{+\infty} \frac{dt}{g^{m-1}(t)}$$

is the Green kernel with pole $o$ of the Laplace-Beltrami operator of $M_g$. In passing, note also that $G(x, o) \to 0$ as $x \to \infty$. According to Theorem 4.4, a non-parabolic model $M_g$ is Feller. Since parabolicity implies stochastic completeness, we immediately deduce that a stochastically incomplete model
is Feller. On the other hand, neither parabolicity, nor, a fortiori, stochastic completeness imply the Feller property. This is shown in the next example.

**Example 4.8.** Having fixed \( \beta > 2 \) and \( \alpha > 0 \), let \( g(t) : \mathbb{R} \to \mathbb{R} \) be any smooth, positive, odd function satisfying \( g'(0) = 1 \) and \( g(r) = \exp(-\alpha r^\beta) \) for \( r \geq 10 \). Then,

\[
1 \leq g(r) = \exp(\alpha r^\beta) \notin L^1(+\infty).
\]

Moreover,

\[
\int_{r}^{+\infty} \frac{g(t)}{g(r)} dt \asymp r^{1-\beta} \in L^1(+\infty).
\]

With this preparation, consider the 2-dimensional model \( M^2_g \). As observed above, by \( (4.23) \) \( M^2_g \) is parabolic. On the other hand, according to Theorem 4.4, condition \( (4.24) \) implies that \( M^2_g \) is not Feller.

5. **Monotonicity properties and non-uniqueness of bounded solutions of the exterior problem**

In Theorem 4.4 we were able to characterize the validity of the Feller property on a model manifold in terms of minimal solutions \( h \) of the o.d.e. problem \( (4.4) \), namely

\[
\begin{aligned}
&h'' + (m-1) \frac{g'}{g} h' = \lambda h, \text{ on } [R_0, +\infty) \\
&h(R_0) = 1 \\
&h(r) > 0 \text{ on } [R_0, +\infty),
\end{aligned}
\]

It should be noted that \( h \) enjoys interesting monotonicity properties. First of all, we point out the following

**Lemma 5.1.** Let \( R_0, \lambda > 0, m \in \mathbb{N} \) and let \( g : [R_0, +\infty) \to (0, +\infty) \) be a smooth function. Assume that \( u : [R_0, +\infty) \to [0, +\infty) \) is a non-negative solution of the inequality

\[
(5.1) \quad u'' + (m-1) \frac{g'}{g} u' \geq \lambda u.
\]

Suppose \( u'(R_1) \geq 0 \) for some \( R_1 \geq R_0 \). Then \( u'(r) \geq 0 \) for every \( r \geq R_1 \).

**Proof.** Write inequality \( (5.1) \) in the form

\[
(5.2) \quad \frac{(g^{m-1}u')'}{g^{m-1}} \geq \lambda u.
\]

Therefore, integrating on \([R_1, r]\) gives

\[
(5.3) \quad u'(r) \geq \frac{\lambda \int_{R_1}^{r} u(t) g^{m-1}(t) dt + g^{m-1}(R_1) u'(R_1)}{g^{m-1}(r)} \geq 0,
\]

as claimed. \( \square \)
Actually, much more can be said if we impose some further condition on the coefficient $g$. Namely, we have the following

**Lemma 5.2.** Assume that $1/g^{m-1} \notin L^1(+\infty)$. Let $h$ be the minimal (bounded is enough) solution of (4.4). Then $h(r)$ is a strictly decreasing function.

We are going to prove (a more general version of) this result by using the point of view of potential theory on model manifolds. We need to recall the following characterization of parabolicity due to L.V. Ahlfors (see [1] Theorem 6.C. See also [22] Theorem 4).

**Theorem 5.3.** A Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$ of dimension $m \geq 2$ is parabolic if and only if given a open set $G \subset M$ and a bounded above solution $f$ of $\Delta f \geq 0$ on $G$ it holds

$$\sup_G f \leq \sup_{\partial G} f.$$  \hspace{1cm} (5.4)

In particular, if we assume that $G = M \setminus \Omega$ for some $\Omega \subset \subset M$, it turns out that the function

$$r \mapsto \sup_{\partial B_r} f, \quad \forall r >> 1,$$  \hspace{1cm} (5.5)

is decreasing. Since the minimal solution $h$ of the problem

$$\begin{cases}
\Delta h = \lambda h & \text{on } M \setminus \bar{\Omega} \\
h = 1 & \text{on } \partial \Omega \\
h > 0 & \text{on } M \setminus \Omega,
\end{cases}$$  \hspace{1cm} (3.4)

must satisfy $0 < h \leq 1$, we obtain the following conclusion.

**Lemma 5.4.** Let $(M, \langle \cdot, \cdot \rangle)$ be a complete, parabolic manifold and $\Omega \subset \subset M$. Let $h : M \setminus \Omega \to \mathbb{R}$ be the minimal solution of problem (3.4). Then, $\sup_{\partial B_r} h$ is a decreasing function of $r >> 1$.

Since for the model manifold $M_g^m$ the condition $g^{1-m} \notin L^1(+\infty)$ is equivalent to parabolicity and since the minimal solution $h$ of (4.4) is nothing but the the minimal solution of (3.4) on $M_g^m$, the (weak) monotonicity property asserted in Lemma 5.2 immediately follows from Lemma 5.4. In order to conclude that, in fact, $h$ is strictly decreasing, suppose by contradiction that $h'(R_1) \geq 0$ for some $R_1 \geq R_0$. Then, by Lemma 5.4, $h'(r) \geq 0$ for every $r \geq R_1$. On the other hand, we have just proved that $h' \leq 0$. Therefore $h(r) \equiv h(R_1)$ for every $r \geq R_1$ which is clearly impossible.

We conclude this section showing that even on Feller manifolds, there could exist infinitely many positive solutions which are asymptotically non-zero.

**Example 5.5.** Let $M_g^m$ be an $m$-dimensional non-stochastically complete model manifold. According to Remark 4.7 $M_g^m$ is Feller. By an equivalent characterization of stochastic completeness (see, e.g., [11], Theorem 6.2) there exists a positive bounded function $u$ satisfying $\Delta u = \lambda u$. By a radialization argument if necessary, we may assume that $u$ is radial, and by
scaling, we may also suppose that, given \( R_0 > 0 \), we have \( u(R_0) = 1 \), so that \( u \) solves the problem

\[
\begin{cases}
    u'' + (m - 1) \frac{g'}{g} u' = \lambda u, \text{ on } (R_0, +\infty) \\
    u(R_0) = 1.
\end{cases}
\]

Note that by the maximum principle, the subharmonic function \( u \) cannot tend to zero at infinity. Next, let \( h(r(x)) \) be the (rotationally invariant) minimal, positive solution of (5.6). Since \( M^m_g \) is Feller, \( h(r) \to 0 \) as \( r \to \infty \), and in particular \( h \neq u \). For any fixed \( \alpha \) such that \( h'(R_0) < \alpha < u'(R_0) \), let \( v_{\alpha}(t) \) be the solution of the Cauchy problem

\[
\begin{cases}
    v_{\alpha}'' + (m - 1) \frac{g'}{g} v_{\alpha}' = \lambda v_{\alpha}, \text{ on } (R_0, +\infty) \\
    v_{\alpha}(R_0) = 1, \quad v_{\alpha}'(R_0) = \alpha.
\end{cases}
\]

Since \( v_{\alpha} - h \) and \( u - v_{\alpha} \) are solutions of the Cauchy problem

\[
\begin{cases}
    w'' + (m - 1) \frac{g'}{g} w' = \lambda w, \text{ on } (R_0, +\infty) \\
    w(R_0) = 0, \quad w'(R_0) > 0,
\end{cases}
\]

according to Lemma [5.1], they are both non-constant, increasing, hence positive, functions on \((R_0, +\infty)\). This means that \( h < v_{\alpha} < u \) on \((R_0, +\infty)\). Moreover, since by assumption \( h(t) \to 0 \), as \( t \to +\infty \), then, necessarily, \( v_{\alpha}(t) \neq 0 \). It follows that, for every such \( \alpha \), the radial function \( v_{\alpha}(r(x)) \) is a bounded positive solution of (3.7) which does not tend to zero at infinity.

6. Comparison with Model Manifolds

It is by now standard that parabolicity and stochastic completeness of a general manifold can be deduced from those of a model manifold via curvature comparisons. Such a result was obtained by Grigor’yan, [11]. In view of Section 4 we can now extend the use of this comparison technique to cover also the Feller property.

We begin with two comparison results for solutions of the exterior Dirichlet problem which, in some sense, can be considered as “Khas’minskii-type tests” for the Feller property. By comparison, recall that the original Khas’minskii test for parabolicity and stochastic completeness states that \( M \) is parabolic (resp. stochastically complete) if, for some \( \Omega \subset \subset M \), there exists a superharmonic function \( u > 0 \) on \( M \setminus \Omega \) (resp. a \( \lambda \)-superharmonic function \( u > 0 \) on \( M \setminus \Omega \)) such that \( u(x) \to +\infty \) as \( x \to \infty \). For a proof based on maximum principle techniques, we refer the reader to [20], [21].

Recall that, by a supersolution of the exterior problem

\[
\begin{cases}
    \Delta v = \lambda v \quad \text{on } M \setminus \Omega \\
    v = 1 \quad \text{on } \partial \Omega,
\end{cases}
\]
we mean a function $u$ satisfying
\[
\begin{cases}
\Delta u \leq \lambda u & \text{on } M \setminus \Omega \\
u \geq 1 & \text{on } \partial \Omega.
\end{cases}
\]
A subsolution is defined similarly by reversing all the inequalities.

**Proposition 6.1.** Let $\Omega$ be relatively compact open set with smooth boundary in the Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$ and let $\lambda > 0$. Let $u$ be a positive supersolution of (6.1) and let $h$ be the minimal, positive solution of (6.1). Then
\[
h \leq u, \text{ on } M \setminus \Omega.
\]
In particular, if $u(x) \to 0$ as $x \to \infty$ then $M$ is Feller.

**Proof.** Let $\{\Omega_n\}$ be a smooth exhaustion of $M$ and let $\{h_n\}$ be the corresponding sequence of functions defined in Theorem 3.1 with $q(x) = \lambda$. Thus $h_n \to h$ the minimal positive solution of (3.4) i.e.
\[
\begin{cases}
\Delta h = \lambda h & \text{on } M \setminus \bar{\Omega} \\
h = 1 & \text{on } \partial \Omega \\
h > 0 & \text{on } M \setminus \Omega.
\end{cases}
\]
Since $h_n \leq u$ on $\partial \Omega_n \cup \partial \Omega$, by the maximum principle we have
\[
h_n \leq u \text{ on } \Omega_n \setminus \Omega,
\]
and, letting $n \to +\infty$, we deduce
\[
h \leq u, \text{ on } M \setminus \Omega.
\]

As an application we get the following result which was first observed in [2].

**Corollary 6.2.** Let $M$ be non-parabolic with positive Green kernel $G(x, y)$. Suppose that, for some (hence any) $y \in M$, $G(x, y) \to 0$ as $x \to \infty$. Then $M$ is Feller.

Recall that the Green kernel is related to the heat kernel of $M$ by
\[
G(x, y) = \int_0^{+\infty} p_t(x, y) \, dt. \tag{6.2}
\]
In view of (6.2), the assumptions of Corollary 6.2 are satisfied whenever we are able to provide a suitable decay estimate on the heat kernel. However, heat kernel estimates may be used to obtain sharper results by using directly the definition of the Feller property. This will be exemplified in Section 8.

In spirit similar to that of Proposition 6.1, in order to deduce that the manifold at hand is non-Feller, one can compare with positive subsolutions of (6.1). Note that in this case, the conclusion holds under the additional assumption that the manifold is stochastically complete.
Theorem 6.3. Let \((M, \langle \cdot, \cdot \rangle)\) be a stochastically complete manifold, and let \(v\) be a bounded, positive subsolution of (6.1) in \(M \setminus \Omega\). Then
\[
h(x) \geq v(x).
\]
In particular, if
\[
v(x) \not\to 0, \text{ as } x \to \infty,
\]
then \(M\) is not Feller.

Proof. For every \(\epsilon > 0\), let \(v_\epsilon = v - h - \epsilon\), so that \(\Delta v_\epsilon \geq \lambda v_\epsilon - \lambda v_\epsilon^+\) on \(M \setminus \Omega\) and \(v_\epsilon = -\epsilon\) on \(\partial \Omega\). But then \(v_\epsilon^+ = \max\{0, v_\epsilon\}\) is bounded and satisfies \(\Delta v_\epsilon^+ \geq \lambda v_\epsilon^+\) on \(M\). The assumption that \(M\) is stochastically complete (see, e.g. [11], Theorem 6.2, or [19]) forces \(v_\epsilon^+ \equiv 0\), that is, \(u \leq h + \epsilon\). Letting \(\epsilon \to 0^+\) we deduce that \(v \leq h\). \(\square\)

In particular, if \(v\) is a bounded positive solution of (3.4) then, by minimality, \(v = h\) and we deduce the uniqueness property noted at the end of Section 3.

Corollary 6.4. Let \((M, \langle \cdot, \cdot \rangle)\) be a stochastically complete Riemannian manifold. Then, for every smooth open set \(\Lambda \subset \subset M\), problem (3.4) has a unique, bounded solution, namely, the minimal solution \(h\) constructed in Theorem 3.1.

Clearly, in order to deduce from Theorem 6.3 that \(M\) is not Feller, it is vital that the bounded subsolution \(v(x)\) does not converge to zero at infinity. In view of applications, we observe that such condition can be avoided up to replacing condition \(\Delta v \geq \lambda v\) with a suitable (and in some sense more restrictive) differential inequality. This is the content of the next

Corollary 6.5. Let \((M, \langle \cdot, \cdot \rangle)\) be a stochastically complete manifold. Assume that, for some smooth open set \(\Lambda \subset \subset M\), there exists a bounded solution \(u_\ast \leq u(x) \leq u^\ast\) of the differential inequality
\[
\Delta u \geq f(u), \text{ on } M \setminus \Omega,
\]
where \(f(t)\) is a \(C^1\) function on \(u_\ast \leq t \leq u^\ast\) such that
\[
(a) \ f(t) > 0, \ (b) \ f'(t) \leq \lambda,
\]
for some \(\lambda > 0\). Then \(M\) is not Feller.

Proof. Let
\[
F(t) = \int_{u_\ast}^{t} \frac{1}{f(s)} \, ds
\]
and define a new function \(v(x)\) on \(M \setminus \Omega\) by setting
\[
v(x) = e^{\lambda(F(u(x)) - F(u^\ast))}.
\]
Clearly,
\[
e^{-\lambda F(u^\ast)} \leq v(x) \leq 1,
\]
and by direct computations we deduce
\[ \Delta v \geq \lambda v. \]
The result now follows from Theorem 6.3. \( \square \)

As we have already noted above, the triviality of bounded positive \( \lambda \)-subharmonic functions is equivalent to stochastic completeness of the underlying manifold. As shown by Grigor’yan (see, e.g., [11]), the validity of a similar Liouville property when \( \lambda \) is replaced by a non-negative function is related to parabolicity.

This suggests that a comparison result similar to Theorem 6.3 holds for minimal solutions to the exterior problem (3.1)
\[
\begin{cases} 
\Delta h = q(x)h \text{ on } M \setminus \overline{\Omega} \\
h = 1 \text{ on } \partial \Omega \\
h > 0 \text{ on } M \setminus \overline{\Omega} ,
\end{cases}
\]
with \( q(x) \geq 0 \) on \( M \).

**Theorem 6.6.** Let \( (M, \langle, \rangle) \) be a parabolic manifold and let \( h \) be the minimal positive solution of (3.1). Assume that, for some smooth open set \( \Omega \subset \subset M \), and for some \( \lambda > 0 \), \( v \) is a positive subsolution of (3.1) Then
\[ h(x) \geq v(x). \]

Clearly, Theorem 6.6 yields a uniqueness result for bounded positive solutions to the exterior problem (3.1) companion to Corollary 6.4.

The proof of Theorem 6.6 uses some potential theory for diffusion operators on weighted manifolds. Given a smooth function \( w \) on \( M \), the \( w \)-Laplacian is defined as the diffusion operator
\[ \Delta_w f = e^w \text{div} ( e^{-w} \nabla f ) . \]
The corresponding weighted manifold \( (M, \langle, \rangle), e^{-w}d\text{vol} \) is said to be \( w \)-parabolic if every bounded above solution of \( \Delta_w u \geq 0 \) must be constant. As in the usual Riemannian case \( w = 0 \), one has that \( w \)-parabolicity of a geodesically complete manifold \( (M, \langle, \rangle) \) is implied by the volume growth condition
\[
\frac{1}{\text{vol}_w (\partial B_r (o))} \notin L^1 (+\infty) ,
\]
where we have set
\[
\text{vol}_w (\partial B_r (o)) = \int_{\partial B_r (o)} e^{-w} d\mathcal{H}^{m-1} ,
\]
and \( \mathcal{H}^{m-1} \) is the Riemannian \( (m - 1) \)-dimensional Hausdorff measure. Furthermore, one can relate the \( w \)-parabolicity to the vanishing of a suitable (weighted) capacity of compact subsets. More precisely, for any fixed closed set \( C \subset M \), define
\[
\text{cap}_w (C) = \inf \left\{ \int_M |\nabla u|^2 e^{-w} d\text{vol} : u \in C^\infty (M) \text{ s.t. } u \geq 1 \text{ on } C \right\} .
\]
Then, we have

**Lemma 6.7.** The weighted manifold \((M, \langle , \rangle, e^{-w}dv)\) is \(w\)-parabolic if and only if, for every compact set \(K \subset M\), \(\text{cap}_w(K) = 0\).

Finally, one has a weighted version of the Ahlfors-type characterization:

**Lemma 6.8.** The weighted manifold \((M, \langle , \rangle, e^{-w}dv)\) is \(w\)-parabolic if and only if given an open set \(G \subset M\) and a bounded above solution \(f\) of \(\Delta_w f \geq 0\) on \(G\) it holds

\[
\sup_G f \leq \sup_{\partial G} f.
\]

We are now ready to give the

**Proof (of Theorem 6.6).** Let \(h\) be the minimal solution of problem \((3.1)\). Then, the new function

\[
f = \frac{h}{v}
\]

satisfies

\[
\Delta_w f \leq 0,
\]

with

\[
w = -\log v^2.
\]

Furthermore, having set

\[
\sup_{M \setminus \Omega} v = v^* < +\infty,
\]

we have

\[
0 \leq \text{cap}_w (K) \leq (v^*)^2 \text{cap} (K) = 0,
\]

for every compact set \(K \subset M\), proving that \((M, \langle , \rangle, e^{-w}dv)\) is \(w\)-parabolic. By the global minimum principle on \(M \setminus \Omega\) we deduce

\[
f = \frac{h}{v} \geq \inf_{\partial \Omega} f \geq 1,
\]

as desired. \(
\)

We now apply Proposition 6.1 and Theorem 6.3 to obtain comparison results with model manifolds mentioned at the beginning of the section.

**Theorem 6.9.** Let \((M, \langle , \rangle)\) be a complete, \(m\)-dimensional Riemannian manifold

\(a)\) Assume that \(M\) has a pole \(o\). Suppose that the radial sectional curvature with respect to \(o\) satisfies

\[
\text{M Sec}_{\text{rad}} \leq G (r(x)) \quad \text{on } M,
\]

for some smooth even function \(G : \mathbb{R} \to \mathbb{R}\). Let \(g : [0, +\infty) \to [0 + \infty)\) be the unique solution of the Cauchy problem

\[
\begin{cases}
g'' + Gg = 0 \\
g(0) = 0, \quad g'(0) = 1.
\end{cases}
\]
If the \( m \)-dimensional model \( M_g \) has the Feller property then also \( M \) is Feller.

(b) Assume that the radial Ricci curvature of \( M \) satisfies
\[
M \text{Ric}(\nabla r, \nabla r) \geq (m - 1) G(r(x)),
\]
where \( r(x) = \text{dist}(x, o) \), for some \( o \in M \), and \( G : \mathbb{R} \to \mathbb{R} \) is a smooth, even function. Let \( g : [0, +\infty) \to [0, +\infty) \) be the unique solution of the problem
\[
\begin{aligned}
g'' + Gg &= 0 \\
g(0) &= 0, \quad g'(0) = 1.
\end{aligned}
\]
If the \( m \)-dimensional model \( M_g \) has finite volume and it does not satisfy the Feller property then also \( M \) is not Feller.

Proof. Let \( u \) be the minimal solution of
\[
\begin{aligned}
\frac{d^2}{dr^2} u + (m - 1) \frac{d}{dr} u' &= \lambda u \quad \text{on } [1, +\infty) \\
u(1) &= 1 \\
u(r) > 0 \quad \text{on } [1, +\infty).
\end{aligned}
\]
By Corollary 4.3 \( u(r) \to 0 \) as \( r \to +\infty \). In particular, according to Lemma 5.1 \( u' < 0 \) on \([1, +\infty)\).

Consider now the radial smooth function \( v(x) = u(r(x)) \) on \( M \setminus B(1) \). Note that
\[
\Delta v = u'' + u' \Delta r.
\]
Since \( u' < 0 \) and, by Hessian comparison,
\[
\Delta r \geq (m - 1) \frac{d'}{d'},
\]
we deduce
\[
\Delta v \leq u'' + (m - 1) \frac{d'}{d'} u' = \Delta_{M_g} u = \lambda u = \lambda v.
\]
Summarying,
\[
\begin{aligned}
\Delta v &\leq \lambda v \quad \text{on } M \setminus B(1) \\
v &= 1 \quad \text{on } \partial B(1) \\
v > 0 \quad \text{on } M \setminus B(1) \\
\lim_{r(x) \to +\infty} v(x) &= 0.
\end{aligned}
\]
To conclude, we apply the comparison principle stated in Proposition 6.1 above.

(b) By assumption, \( g^{m-1} \in L^1(+) \) so that \( 1/g^{m-1} \notin L^1(+) \). Since \( M_g \) is not Feller, by Theorem 4.4 we must have
\[
\int_0^{+\infty} g^{m-1}(t) \frac{dt}{g^{m-1}(r)} \in L^1(+) .
\]
Define
\[
\alpha(r) = \int_r^{+\infty} \int_s^{+\infty} g^{m-1}(t) \frac{dt}{g^{m-1}(s)} ds .
\]
A direct computation shows that
\[ M_g \Delta \alpha = 1. \]
Now consider
\[ v(x) = \alpha(r(x)) \text{ on } M \setminus B_1. \]
Clearly, \( v \) is a positive bounded function. Since \( \alpha' \leq 0 \), by Laplacian comparison we have
\[ \Delta v \geq 1. \]
On the other hand, by the Bishop volume comparison theorem it holds
\[ \text{vol}(M) \leq \text{vol}(M_g) < +\infty. \]
In particular \( M \) is parabolic. Applying Corollary 6.5 with the choice \( f(t) = 1 \) we conclude that \( M \) is not Feller. \( \square \)

7. Feller property on manifolds with many ends

It is a trivial consequence of Theorem 3.2 that Riemannian manifolds which are isometric outside a compact set have the same behavior with respect to the Feller property. The choice of a smooth compact set \( \Omega \) in the complete manifold \((M, \langle \cdot, \cdot \rangle)\) gives rise to a finite number of unbounded connected components, say \( E_1, \ldots, E_k \). They are called the ends of \( M \) with respect to \( \Omega \). Thus, the minimal solution \( h \) of (3.4) restricts to the minimal solution \( h_j \) of the same Dirichlet problem on \( E_j \) with respect to the compact boundary \( \partial E_j \). Furthermore, \( h \) tends to zero at infinity in \( M \) if and only if each function \( h_j(x) \to 0 \) as \( E_j \ni x \to \infty \).

This situation suggests to localize the definition of the Feller property to a given end by saying that \( E \) is Feller if, for some \( \lambda > 0 \), the minimal solution \( g : E \to (0, 1] \) of the Dirichlet problem
\[
\begin{cases}
\Delta g = \lambda g & \text{on int}(E) \\
g = 1 & \text{on } \partial E,
\end{cases}
\]
satisfies \( g(x) \to 0 \) as \( x \to \infty \). The usual exhausting procedure shows that \( g \) actually exists.

Now, let \( E_1, \ldots, E_k \) be the ends of \( M \) with respect to the compact set \( \Omega \). Then, we can enlarge slightly \( \Omega \) to a new compact \( \Omega' \) which encloses a small collar neighborhood \( W_j \) of each \( \partial E_j \subset E_j \). Since the validity of the Feller property on \( M \) is not sensitive of the chosen compact, we deduce that \( M \) is Feller if and only if each \( E'_j = E_j \setminus W_j \) is Feller. This implies that, in case we have isometries \( f_j : \partial E_j \to \partial D_j \) onto the boundaries of compact Riemannian manifolds \((D_j, \langle \cdot, \cdot \rangle_{D_j})\), then \( M \) is Feller if and only if so is each Riemannian gluing (without boundary) \( E_j \cup f_j D_j \). Recall that, by definition, \( E_j \cup f_j D_j \) has the original metrics outside a small bicollar neighborhood of the glued boundaries. Along the same lines we can easily obtain that \( M \) is Feller if and only if the Riemannian double \( D(E_j) \) of each end \( E_j \) has the same property. We have thus obtained the following
Proposition 7.1. Let \((M, \langle , \rangle)\) be a complete Riemannian manifold and let \(E_1, \ldots, E_k\) be the ends of \(M\) with respect to the smooth compact domain \(\Omega\). Then, the following are equivalent:

(1) \(M\) is Feller
(2) Each end \(E_j\) has the Feller property.
(3) Each end with a cap \(E_j \cup \Omega_j\) (if possible) has the Feller property.
(4) The double \(D(E_j)\) of each end has the Feller property.

Using this observation, one can easily construct new Feller or non-Feller manifolds from old ones by adding suitable ends. For instance, consider the equidimensional, complete Riemannian manifolds \(M\) and \(N\) and form their connected sum \(M \# N\). This latter is Feller if and only if both \(M\) and \(N\) has the Feller property.

In the special case of warped products with rotational symmetry, combining Theorem 4.4 with Proposition 7.1, we are able to obtain the following characterization.

Example 7.2. Consider the warped product of the form

\[ \mathbb{R} \times fS^{m-1} = \left( \mathbb{R} \times S^{m-1}, dr^2 + f(r)^2 d\theta^2 \right) \]

where \(f(r) > 0\) is a smooth function on \(\mathbb{R}\). This is a complete manifold (without boundary) with two ends. Let \(E_1 = (1, +\infty) \times fS^{m-1}\) and \(E_2 = (-\infty, 1) \times fS^{m-1}\) be the ends of \(\mathbb{R} \times fS^{m-1}\) with respect to the compact domain \(\Omega = [-1, 1] \times S^{m-1}\). Using the closed unit disc \(D^m\) as a cap, starting from \(E_1\) and \(E_2\) we can construct complete manifolds without boundary each isometric to a model manifold. Precisely, \(E_1\) gives rise to

\[ E_{g_1} = \left( [0, 1) \times S^{m-1}, dr^2 + g_1(r)^2 d\theta^2 \right) \]

where \(g_1 : [0, +\infty) \rightarrow [0, +\infty)\) satisfies \(g_1(r) = r\) if \(0 \leq r < 1 - \varepsilon\) and \(g_1(r) = f(r)\) if \(r > 1 + \varepsilon\). Similarly,

\[ E_{g_2} = \left( [0, +\infty) \times S^{m-1}, dr^2 + g_2(r)^2 d\theta^2 \right) \]

where \(g_2 : [0, +\infty) \rightarrow [0, +\infty)\) satisfies \(g_2(r) = r\) if \(0 \leq r < 1 - \varepsilon\) and \(g_2(r) = f(-r)\) if \(r > 1 + \varepsilon\). By Proposition 7.1, \(\mathbb{R} \times fS^{m-1}\) is Feller if and only if both \(E_{g_1}\) and \(E_{g_2}\) are Feller. Since, according to Theorem 4.4, the Feller property on model manifolds is completely characterized by the asymptotic behavior of the warping functions, we obtain the next

Corollary 7.3. The warped product \(\mathbb{R} \times fS^{m-1}\) has the Feller property if and only if both \(g(t) = f(t), t >> 1,\) and \(g(t) = f(-t), t << 1,\) satisfy either of the conditions (4.6) or (4.7) of Theorem 4.4.

Application of this result will be given in Section 9.
8. Isoperimetry and the Feller property

Using a general result by A. Grigor’yan, [10], we are going to show that a Riemannian manifold is Feller provided it satisfies a suitable isoperimetric inequality. As a consequence we will deduce that minimal submanifolds in Cartan-Hadamard manifolds (i.e., complete, simply connected manifolds with non-positive sectional curvature), and in particular Cartan-Hadamard manifolds themselves, are Feller. The latter result was proved by Azencott, [2], using different methods based on comparison arguments. Actually, in Section 6 above we developed comparison techniques which allowed us to prove the validity of the Feller property for manifolds with a pole which are not necessarily Cartan-Hadamard.

If Ω is a bounded domain with smooth boundary, we denote by $\lambda_1(\Omega)$ the smallest Dirichlet eigenvalue of $-\Delta$ in Ω. Note that by domain monotonicity $\lambda_1(\Omega)$ is a decreasing function of Ω, and since a Riemannian manifold is locally Euclidean, $\lambda_1(B_r(x_0)) \sim c_n r^{-2}$ as $r \to 0$.

**Theorem 8.1.** Let $(M, \langle \cdot, \cdot \rangle)$ be a complete manifold satisfying the Faber-Krahn isoperimetric inequality

\[(8.1) \quad \lambda_1(\Omega) \geq \Lambda(\text{vol}\Omega),\]

for every bounded domain $\Omega \subset M$, where $\Lambda$ is a positive decreasing function such that $1/(s\Lambda(s)) \in L^1(0+)$. Let $V(t)$ be the function defined by the formula

\[(8.2) \quad t = \int_0^{V(t)} \frac{ds}{s\Lambda(s)} ds,\]

and assume that there exists $T \in (0, +\infty]$ such that

\[(8.3) \quad \frac{tV'(t)}{V(t)} \text{ is bounded for } t \leq 2T \text{ and non-decreasing for } t > T.\]

Then $M$ is Feller.

**Proof.** Indeed, it follows from [10] Theorem 5.1 that the heat kernel $p_t(x, y)$ of $M$ satisfies the Gaussian estimate

\[(8.4) \quad p_t(x, y) \leq \frac{C}{V(\text{ct})} \exp \left( -\frac{d(x, y)^2}{D t} \right),\]

for some constants $C, c > 0$ and $D > 4$, and where $V(t)$ is the function defined in (8.2). A straightforward application of the dominated convergence theorem shows that for every continuous function of compact support and for every $t > 0$,

\[P_t u(x) = \int_M p_t(x, y) u(y) d\text{vol}(y) \to 0 \text{ as } x \to \infty,\]

and therefore $M$ is Feller. \(\Box\)
It follows from Cheeger’s inequality, see [10] Proposition 2.4, that if the isoperimetric inequality
\[ \text{vol}(\partial \Omega) \geq g(\text{vol}(\Omega)) \]
is valid for some \( g \) such that \( g(s)/s \) is non-increasing, then (8.1) holds with \( \Lambda(s) = \frac{1}{4} \left( \frac{g(s)}{s} \right)^2 \). Thus, for instance, if \( M \) supports an isoperimetric inequality of the type
\[ \text{vol}(\partial \Omega) \geq C(\text{vol}(\Omega))^{1-1/p} \]
for some \( p \geq m \), then (8.1) holds with \( \Lambda(s) = Cs^{-2/p} \) and the associated function \( V(t) = Ct^{p/2} \) satisfies condition (8.4). Note that (8.3) is equivalent to the validity of the \( L^1 \)-Sobolev inequality
\[ ||\nabla u||_{L^1} \geq S_{1,p} ||u||_{L^{\frac{2p}{p-2}}}, \forall u \in C^1_c(M). \]
On the other hand, it is known from work of G. Carron, [5], that the Faber-Krahn inequality
\[ \lambda_1(\Omega) \geq C(\text{vol}(\Omega))^{-2/p}, \quad p \geq m, p \neq 2, \]
is equivalent to the \( L^2 \)-Sobolev inequality
\[ ||\nabla u||_{L^2} \geq S_{2,p} ||u||_{L^{\frac{2p}{p-2}}}, \forall u \in C^1_c(M). \]
From these considerations we obtain the following

**Corollary 8.2.** Let \( M \) be isometrically immersed into a Cartan-Hadamard manifold. If its mean curvature vector field \( H \) satisfies
\[ ||H||_{L^m(M)} < +\infty, \]
then \( M \) is Feller. In particular,

(a) Every Cartan-Hadamard manifold is Feller.
(b) Every complete, minimal submanifold in a Cartan-Hadamard manifold is Feller.

**Proof.** Indeed, according to [15], there exists a constant \( c_m \) depending only on \( m \) such that, for every bounded domain \( C^1 \) function with compact support
\[ \left( \int_M |u|^{m/(m-1)} \text{dvol} \right)^{(m-1)/m} \leq c_m \int_M [||\nabla u| + |H||u||] \text{dvol}. \]
Since \( |H| \in L^m(M) \), there exists a compact set \( K \) such that
\[ \left( \int_{M \setminus K} |H|^m \text{dvol} \right)^{1/m} < \frac{1}{2c_m}, \]
and applying Hölder inequality to the second summand on the RHS, we deduce that the standard $L^1$ isoperimetric inequality

\[ \left( \int_M |u|^{m/(m-1)} \, d\text{vol} \right)^{(m-1)/m} \leq 2c_m \int_M |\nabla u| \, d\text{vol} \]

holds for every $C^1$ function $u$ with compact support in $M \setminus K$. A variation of a result of Carron, [4] (see [22], Theorem 13) implies that the $L^1$-Sobolev inequality (8.8) holds, possibly with a larger constant, for every compactly supported function on $M$. By the arguments preceding the statement of the corollary $M$ is Feller.

**Remark 8.3.** We note, in particular, that if $M$ is isometrically immersed in a Cartan-Hadamard manifold and its mean curvature is in $L^m$ then $M$ has infinite volume.

We also observe that the same arguments show that if the $L^2$-isoperimetric inequality (8.7) holds off a compact set then it holds everywhere and $M$ is Feller.

Finally, the above arguments show that, if $m \geq 3$, then a minimal submanifold in a Cartan-Hadamard space is non-parabolic, and its Green kernel decays at infinity. Of course this fails in dimension two, as the example of $\mathbb{R}^2$ shows.

We end this section by noting that one of the most important category of minimal surfaces is represented by those properly immersed in the ambient space. Recall that a map between topological spaces is proper if the preimage of a compact set is compact. Thus, intrinsically divergent sequences cannot accumulate at a finite point in the ambient space. In case $f : M \to N$ is a proper, minimal immersion of the complete $m$-dimensional manifold $M$ into the Cartan-Hadamard manifold $N$, the validity of the Feller property can be also obtained using direct heat kernel comparisons. More precisely, we can use the following result, [6], [18].

**Theorem 8.4.** Let $f : M \to N$ be an $m$-dimensional, complete, minimally immersed submanifold of the $n$-dimensional Cartan-Hadamard manifold $N$. Let $D$ be a compact domain in $M$ with Dirichlet heat kernel $p^D_{t}$. For any fixed $x \in D$, let $B_R(0)$ be the ball in $\mathbb{R}^m$ of radius

\[ R = \max_{y \in D} d_N(f(x), f(y)), \]

and let $p^B_D(0, v)$ be the corresponding Dirichlet heat kernel. Recalling that $p^B_D(0, v) = p^B_{R(0)}(|v|)$ depends only on $|v|$, then for every $t \geq 0$ and for every $y \in D$,

\[ p^{D}_{t} (x, y) \leq p^{B}_{t} (d_N(f(x), f(y))). \]

Now, let $x, y \in M$ and let $\{D_n\}$ be a smooth exhaustion of $M$ satisfying $x, y \in D_0$. Then, by Theorem 8.4 and by the parabolic comparison principle

\[ p^{D_{n}}_{t} (x, y) \leq p^{B_{n}}_{t} (d_N(f(x), f(y))) \leq p^{R^m}_{t} ((d_N(f(x), f(y)))) , \]
for every $t \geq 0$. On the other hand,

$$p_t^{D_n}(x, y) \to p_t^M(x, y), \text{ as } n \to +\infty.$$ 

Therefore, taking limits in (8.9) we obtain that, for every $x, y \in M$ and for every $t \geq 0$,

$$p_t^M(x, y) \leq p_t^{R_m}((d_N(f(x), f(y))).$$

It follows that, for any $R > 0$,

$$0 \leq \int_{B_R} p_t^M(x, y) \, dy \leq \int_{B_R} p_t^{R_m}((d_N(f(x), f(y))) \, dy.$$

Since, by assumption, $f$ is proper we have that $f(x) \to \infty$ as $x \to \infty$. Whence, taking limits in (8.11) and using the dominated convergence theorem on the RHS we conclude

$$\lim_{x \to \infty} \int_{B_R} p_t^M(x, y) \, dy = 0.$$ 

According to Corollary 2.3, this proves that $M$ is Feller.

9. Feller property and covering spaces

In this section we address the following

**Problem 9.1.** Suppose we are given a Riemannian covering $\pi : (\tilde{M}, \langle, \rangle) \to (M, \langle, \rangle)$. Is there any relation between the validity of the Feller property on the covering space $\tilde{M}$ and on the base manifold $M$?

By comparison, recall that $M$ is stochastically complete if and only if so is $\tilde{M}$. Passing from the covering to the base is easy via the use of bounded, $\lambda$-subharmonic functions. The converse seems to be non-trivial. A proof using stochastic differential equations can be found in the book by D. Elworthy, [9], but it would be nice to have a deterministic proof of this fact. Intuitively, Elworthy proof relies on the fact that (similarly to what happens for geodesics) Brownian paths in $M$ lift to Brownian paths in $\tilde{M}$ and, conversely, Brownian paths in $\tilde{M}$ project to Brownian paths in $M$.

As for parabolicity, the situation is quite different. Using subharmonic functions it is easy to see that if the covering manifold $\tilde{M}$ is parabolic then the base manifold $M$ is also parabolic. In general, the converse is not true, as shown e.g. by the twice punctured complex plane. This latter is a parabolic manifold which is universally covered by the (non-parabolic) Poincaré disk.

Let us now consider the Feller property. To begin with, consider the easiest case of coverings with a finite number of sheets. As expected, we have the following

**Proposition 9.2.** Let $\pi : (\tilde{M}, \langle, \rangle) \to (M, \langle, \rangle)$ be a $k$-fold Riemannian covering, $k < +\infty$. Then, $\tilde{M}$ is Feller if and only if $M$ is Feller.
Proof. Let $\Omega \subset M$ be fixed and set $\Omega = \pi^{-1}(\Omega) \subset \hat{M}$. Let $\lambda > 0$ be a chosen number. We are going to show that the minimal, positive $\lambda$-harmonic function $h$ on $M \setminus \Omega$ with boundary data $h|_{\partial \Omega} = 1$ is related to the minimal, positive $\lambda$-harmonic function $\hat{h}$ on $\hat{M} \setminus \hat{\Omega}$ with boundary data $\hat{h}|_{\partial \hat{\Omega}} = 1$ by

$$
\hat{h} = h \circ \pi.
$$

Indeed, let $\Omega_n \not\supset M$ be a compact exhaustion and, for each $n$, let $h_n$ be the solution of the Dirichlet problem

$$
\begin{cases}
\Delta h_n = \lambda h_n & \text{on } \Omega_n \setminus \Omega \\
h_n = 1 & \text{on } \partial \Omega \\
h_n = 0 & \text{on } \partial \Omega_n.
\end{cases}
$$

Then, $\hat{\Omega}_n = \pi^{-1}(\Omega_n) \not\supset \hat{M}$ is a compact exhaustion and, since $\pi$ is a local isometry,

$$
\hat{h}_n = h_n \circ \pi
$$
solves the analogous Dirichlet problem on $\hat{\Omega}_n \setminus \hat{\Omega}$. The desired relation between $h$ and $\hat{h}$ follows by letting $n \to +\infty$.

Now, suppose $\hat{M}$ is Feller. We show that $M$ must be Feller, i.e., for every $\varepsilon > 0$ there exists a compact set $K \subset M$ such that, for every $x \in M \setminus K$, $h(x) < \varepsilon$. Let $\varepsilon > 0$ be fixed. Then, by assumption, there is a compact $\hat{K} \subset \hat{M}$ such that $\hat{h}(\hat{x}) < \varepsilon$ for every $\hat{x} \in \hat{M} \setminus \hat{K}$. Define $K = \pi(\hat{K}) \subset M$ and the further compact subset $\hat{K}_1 = \pi^{-1}(K) \subset \hat{M}$. Clearly, $\hat{K} \subset \hat{K}_1$ so that $\hat{h}(\hat{x}) < \varepsilon$ whenever $\hat{x} \in \hat{M} \setminus \hat{K}_1$. It follows from (9.1) that, for every $x \in M \setminus K$, $h(x) = \hat{h}(\hat{x}) < \varepsilon$, where $\hat{x} \in \pi^{-1}(x)$ is chosen arbitrarily.

Assume now that $M$ is Feller. We show that $\hat{M}$ is Feller. To this end, having fixed $\varepsilon > 0$, let $K \subset M$ be a large compact set such that $h(x) < \varepsilon$ for every $x \in M \setminus K$. Let us consider the compact set $\hat{K} = \pi^{-1}(K) \subset \hat{M}$. If $\hat{x} \in \hat{M} \setminus \hat{K}$ then $x = \pi(\hat{x}) \in M \setminus K$ and, according to (9.1) we deduce $\hat{h}(\hat{x}) = h(x) < \varepsilon$, completing the proof.

Observe that there are two key points in the above proof:

(A) a $k$-fold covering map is proper;

(B) for a $k$-fold covering, conditions $\hat{x} \to \hat{\infty}$ and $x = \pi(\hat{x}) \to \infty$ are essentially the same.

Obviously, the situation changes drastically if we consider an $\infty$-fold Riemannian covering $\pi : \hat{M} \to M$. Violating (A) yields that the Feller property does not descend on the base manifold. In the next example we show that

$$
\hat{M} \text{ Feller} \not\implies M \text{ Feller}.
$$

Example 9.3. Consider the 2-dimensional warped product $M = \mathbb{R} \times_f S^1$ where $f(t) = e^t$. According to Corollary 7.3, we can use Example 4.1 to
deduce that $M$ is not Feller. Note that the Gaussian curvature of $M$ is given by
\[ K(t, \theta) = -\frac{f''(t)}{f(t)} \leq 0. \]

Therefore the universal covering $\hat{M}$ is Cartan-Hadamard, hence Feller by Theorem 6.9.

By contrast, the reverse implication

\[ M \text{ is Feller} \implies \hat{M} \text{ is Feller} \]

holds, and it is the content of the main theorem of this section.

It is not obvious how to achieve the proof using the elliptic point of view, so we adopt the heat kernel point of view. We begin with a simple lemma that will be used in the proof. Recall that since manifolds are second countable, $\pi_1(M)$ is necessarily countable. Moreover the compact open topology induced by its action on $\hat{M}$ coincides with the discrete topology. To say that the sequence $\gamma_k \to \infty$ in the compact open topology means that $\gamma_k$ is eventually in the complement of any finite set.

**Lemma 9.4.** Let $\hat{B}$ be a ball in $\hat{M}$. Then for every $\hat{x} \in \hat{M}$ and every sequence $\{\gamma_k\} \to \infty$ in $\pi_1(M)$

\[ \lim_{k} \int_{\hat{B}} \hat{p}_t(\gamma_k \hat{x}, \hat{y}) d\hat{y} = 0. \]  

**Proof.** Assume first that $\hat{B} = \hat{B}_r(\hat{z})$ is contained in a fundamental domain, so that $\pi: \hat{B}_r(\hat{z}) \to B_r(\pi(\hat{z}))$ is an isometry and

\[ \pi^{-1}(B_r(\pi(\hat{z}))) = \bigcup_{\gamma \in \pi_1(M)} \gamma \hat{B}_r(\hat{z}) \]

is a disjoint union. It follows that for every $\hat{x}$

\[ \sum_{\gamma \in \pi_1(M)} \int_{\gamma \hat{B}_r(\hat{z})} \hat{p}_t(\hat{x}, \hat{y}) d\hat{y} \leq \int_{\hat{M}} \hat{p}_t(\hat{x}, \hat{y}) d\hat{y} \leq 1. \]

In particular,

\[ \int_{\gamma^{-1} \hat{B}_r(\hat{z})} \hat{p}_t(\hat{x}, \hat{y}) d\hat{y} \to 0 \text{ as } \gamma \to \infty, \]

and therefore

\[ \int_{\gamma_k^{-1} \hat{B}_r(\hat{z})} \hat{p}_t(\hat{x}, \hat{y}) d\hat{y} \to 0 \text{ as } k \to \infty. \]

On the other hand, since $\pi_1(M)$ acts isometrically on $\hat{M}$, for every $\gamma$, $\hat{p}_t(\hat{x}, \gamma^{-1} \hat{y})$ solves the heat equation and converges to $\delta_{\gamma \hat{z}}$ as $t \to 0+$, so, by minimality,

\[ \hat{p}_t(\gamma \hat{x}, \hat{y}) = \hat{p}_t(\hat{x}, \gamma^{-1} \hat{y}). \]

This, together with a change of variables gives

\[ \int_{\hat{B}_r(\hat{z})} \hat{p}_t(\gamma_k \hat{x}, \hat{y}) d\hat{y} = \int_{\hat{B}_r(\hat{z})} \hat{p}_t(\hat{x}, \gamma_k^{-1} \hat{y}) d\hat{y} = \int_{\gamma_k^{-1} \hat{B}_r(\hat{z})} \hat{p}_t(\hat{x}, \hat{y}) d\hat{y}. \]
and \( [1, 2] \) is proved in this case. The case where \( \hat{B} \) is not contained in a fundamental domain is dealt with using a standard covering argument. □

**Theorem 9.5.** If \( M \) is Feller then so is \( \hat{M} \).

**Proof.** We need to show that if \( \hat{B}_o \) is a ball is \( \hat{M} \), then, for every sequence \( \hat{x}_k \to \infty \) in \( \hat{M} \),

\[
\int_{\hat{B}_o} \hat{p}_t(\hat{x}_k, \hat{y}) d\hat{y} \to 0 \quad \text{as} \quad k \to \infty.
\]

Fix \( \epsilon > 0 \). According to a result of M. Bordoni, \([3]\), for every \( 0 \leq \varphi \in C_c(M) \), we have

\[
(9.3) \quad \int_{\hat{M}} \hat{p}_t(\hat{x}, \hat{y}) (\varphi \circ \pi(\hat{y})) d\hat{y} \leq \int_{\hat{M}} p_t(\pi(\hat{x}), y) \varphi(y) dy
\]

(and equality holds if \( M \), or equivalently \( \hat{M} \), is stochastically complete), so, if \( \varphi \) is a cut-off function such that \( \pi(\hat{B}_o) \subset \{ \varphi = 1 \} \),

\[
\int_{\hat{B}_o} \hat{p}_t(\hat{x}, \hat{y}) d\hat{y} \leq \int_{\hat{M}} p_t(\pi(\hat{x}), y) \varphi(y) dy.
\]

Since \( M \) is Feller, there exists a ball \( D \subset M \) such that, if \( \pi(\hat{x}_k) \notin \overline{D} \) then the integral on the right hand side is less than \( \epsilon \), and therefore

\[
\int_{\hat{B}_o} \hat{p}_t(\hat{x}_k, \hat{y}) d\hat{y} < \epsilon.
\]

Next let \( \hat{B} \) be a ball in \( \hat{M} \) such that \( \pi(\hat{B}) \supset \overline{D} \), so that \( \pi^{-1}(\overline{D}) \subset \bigcup_{\gamma \in \pi_1(M)} \gamma \hat{B} \).

Without loss of generality, we may therefore assume that \( \pi(\hat{x}_k) \in \overline{D} \) for every \( k \), so that there exist \( \hat{z}_k \in B \) and a sequence \( \gamma_k \in \pi_1(M) \) such that \( \hat{x}_k = \gamma_k \hat{z}_k \). Since \( \hat{x}_k \to \infty \) in \( \hat{M} \), \( \gamma_k \to \infty \) in \( \pi_1(M) \). Moreover, since \( \gamma_k \) is an isometry of \( \hat{M} \),

\[
(\hat{z}, t) \to \int_{\hat{B}_o} \hat{p}_t(\gamma_k \hat{z}, \hat{y}) d\hat{y}
\]

is a solution of the heat equation on \( M \), by the parabolic mean value inequality (see, e.g., [24]), there exists a constant \( C(\hat{B}, t) \) depending only on the geometry of \( 2\hat{B} \) and \( t \) such that, for every \( \hat{z}_k \in B \)

\[
\int_{\hat{B}_o} \hat{p}_t(\gamma_k \hat{z}_k, \hat{y}) d\hat{y} \leq C(\hat{B}, t) \int_{t/2}^t ds \int_{2\hat{B}} d\hat{z} \int_{\hat{B}_o} \hat{p}_s(\gamma_k \hat{z}, \hat{y}) d\hat{y},
\]

and the right hand side tends to zero as \( k \to \infty \) by Lemma 9.4 and the dominated convergence theorem. □

We remark that the proof actually shows that if \( \hat{x}_k \to \infty \) but \( \pi(\hat{x}_k) \) is contained in a compact set for every \( k \) then

\[
\int_{\hat{B}_o} \hat{p}_t(\gamma_k \hat{z}_k, \hat{y}) d\hat{y} \to 0 \quad \text{as} \quad k \to \infty
\]

without having to assume that \( M \) be Feller.
We also note that the proof of Proposition 2.4 in [3] can be easily adapted to show that (9.3) always holds with equality in the case of \( k \)-fold coverings. This could be used to give an alternative proof of Proposition 9.2.

10. On the curvature condition by E. Hsu and some remarks on the role of volumes

As we mentioned in the Introduction, a complete Riemannian manifold is Feller if the Ricci tensor is assumed. In this direction, the best known result in the literature is the following theorem by Hsu, [13], [14], which extends previous work by Yau, Dodziuk and Li-Karp.

**Theorem 10.1.** Let \((M, \langle , \rangle)\) be a complete, non compact Riemannian manifold of dimension \( \dim M = m \). Assume that

\[
M \text{Ric} \geq - (m - 1) G^2 (r (x)),
\]

where \( r (x) = \text{dist} (x, o) \) is the distance function from a fixed reference point \( o \in M \) and \( G \) is a positive, increasing function on \([0, +\infty)\) satisfying

\[
\frac{1}{G} \notin L^1 (+\infty).
\]

Then \( M \) is Feller.

**Remark 10.2.** Unlike a similar result for the validity of the stochastic completeness, this theorem is not a comparison-type theorem. As indicated in Section 9, curvature comparisons should go exactly in the opposite direction (the same direction of heat kernel comparisons). Hsu theorem, like its “predecessors”, is a genuine estimating result. The proof supplied by Hsu is very probabilistic in nature. To the best of our knowledge there is no deterministic proof (neither for general manifolds nor for the easiest case of models) and we feel its discovery would be very interesting.

According to Theorem 10.1, \( M \) is Feller provided its Ricci curvature does not decay to \(-\infty\) too much quickly. To fix ideas one may think of \( G (t) \) as the function \( t \Pi_{j=1}^{n} \log^{(j)} (t) \), where \( \log^{(j)} (t) \) denotes the \( j \)-th iterated logarithm and \( n \in \mathbb{N} \) is arbitrarily large. Using the results of Section 4 we are able to prove that such curvature condition is, in some sense, sharp.

**Example 10.3.** Let \( G : \mathbb{R} \to \mathbb{R} \) be a smooth, increasing function satisfying

\[
(a) \ G (r) > 0, \ (b) \ \limsup_{r \to +\infty} \frac{G' (r)}{G (r)^2} = \alpha < +\infty, \ (c) \ \frac{1}{G (r)} \in L^1 (+\infty).
\]

Fix \( \beta > \alpha \) and let \( g (t) : \mathbb{R} \to \mathbb{R} \) be any smooth, positive, odd function such that \( g' (0) = 1 \) and \( g (r) = \exp \left( -\beta \int_{0}^{r} G (t) \ dt \right) \) for \( r \geq 10 \). Let us consider the \( m \)-dimensional model manifold \( M^m_g \) with warping function \( g \). Direct
computations show that the radial sectional curvature of $M^m_g$ satisfies

$$K_{rad}(r) = -\frac{g''(r)}{g(r)} \leq -\beta (\beta - \alpha) G(r)^2, \quad r \gg 1.$$  

We claim the validity of the following conditions

$$g(r)^m \in L^1(\mathbb{R}^+), \quad \frac{\int_r^{+\infty} g(t)^{m-1} dt}{g(r)^{m-1}} \in L^1(\mathbb{R}^+).$$  

Indeed, since $G$ is positive and increasing, 

$$g(r)^m \leq \exp\left(- (m-1) \beta G(0) r\right), \quad r \gg 1,$$

thus proving (10.5) (a). On the other hand, note that, for $r \gg 1$,

$$G(r) \int_r^{+\infty} g(t)^{m-1} dt \leq \int_r^{+\infty} G(t) \exp\left(- (m-1) \beta \int_0^r G(t) dt\right) = \frac{1}{(m-1) \beta} \exp\left(- (m-1) \beta \int_0^r G(t) dt\right),$$

so that

$$G(r) \int_r^{+\infty} g(t)^{m-1} dt \to 0, \quad \text{as } r \to +\infty.$$  

Therefore, de l’Hospital rule applies and gives

$$\limsup_{r \to +\infty} \frac{G(r) \int_r^{+\infty} g(t)^{m-1} dt}{g(r)^{m-1}} \leq \limsup_{r \to +\infty} \frac{G'(r) \int_r^{+\infty} g(t)^{m-1} dt - G(r) g(r)^{m-1}}{(m-1) g(r)^{m-2} g'(r)} \leq \limsup_{r \to +\infty} \frac{-G(r) g(r)^{m-1}}{(m-1) g(r)^{m-2} g'(r)} \leq \frac{1}{\beta (m-1)},$$

which implies

$$\frac{\int_r^{+\infty} g(t)^{m-1} dt}{g(r)^{m-1}} \leq \frac{1}{\beta (m-1) G(r)} \in L^1(\mathbb{R}^+).$$

This shows the validity of (10.5) (b).

Now, condition (10.5) (a) implies $1/g^{m-1} \notin L^1(\mathbb{R}^+)$. Therefore, from (10.5) (b) and applying Theorem 10.2, we conclude that $M^m_g$ is not Feller.

According to Theorem 10.1, and in view of the above example, the search of more general (or even new) conditions ensuring the validity of the Feller property should not involve pointwise curvature lower bounds. For instance, variations on the theme could be obtained using integral curvature bounds. More importantly, one is naturally led to ask whether a solely volume growth condition suffices. In this respect, we quote the following intriguing question addressed by Li and Karp, [17].
Problem 10.4. Let $(M, \langle , \rangle)$ be a complete Riemannian manifold. Assume that, for some reference origin $o \in M$,

$$\text{vol} \ (B_{R+1} (o) \setminus B_{R} (o)) \geq e^{-AR^2},$$

for every $R >> 1$ and for some constant $A > 0$. Does $M$ satisfy the Feller property?

Actually, as a consequence of Corollary 7.3 it seems that the volume growth (decay) of a general complete manifold is not so tightly related to the validity of the Feller property. Indeed, one can always take $\mathbb{R} \times f \mathbb{S}^{m-1}$ and prescribe the asymptotic behavior of $f \ (t)$ at $-\infty$ and $+\infty$ in such a way that the volume growth is slow (even finite) or fast but at least one of the ends is not Feller. This suggests that possible conditions on volumes, such as those specified in Problem 10.4 should be localized on each of the ends of the manifold.

Problem 10.5. Let $(M, \langle , \rangle)$ be a complete Riemannian manifold which is connected at infinity and satisfies (10.6). Is $M$ Feller?

To the best of our knowledge, only specific examples are used to conjecture a positive answer to this question, $\cite{17}$. For instance, it is reasonable to approach the problem by first assuming that $M$ has only one end which is a cylindrical end, namely $E = (0, +\infty) \times f \Sigma$ for some compact manifold $\Sigma$. However, so far, even in the easiest case $\Sigma = S^{m-1}$ it is unknown whether condition (10.6) implies the validity of the Feller property.

References

[1] L.V. Ahlfors, L. Sario, Riemann Surfaces. Princeton University Press. Princeton N.J. 1960.
[2] R. Azencott, Behavior of diffusion semi-groups at infinity. Bull. Soc. Math. France 102 (1974), 193–240.
[3] M. Bordoni, Comparing heat operators through local isometries and fibrations. Bull. Soc. math. France. 128 (2000), 151–178.
[4] G. Carron, Une suite exacte en $L^2$-cohomologie. Duke Math. J. 95 (1998), no. 2, 343–372.
[5] G. Carron, Inégalités isopérimétriques et inégalités de Faber-Krahn. Séminaire de Théorie Spectrale et Géométrie, No. 13, Année 1994–1995, 63–66.
[6] S.-Y. Cheng, P. Li, S.-T. Yau, Heat Equations on Minimal Submanifolds and Their Applications. Amer. J. Math., 106 (1984), 1033-1065.
[7] E.B. Davies, Heat kernel bounds, conservation of probability and the Feller property. Festschrift on the occasion of the 70th birthday of Shmuel Agmon. J. Anal. Math. 58 (1992), 99–119.
[8] J. Dodziuk, Maximum principle for parabolic inequalities and the heat flow on open manifolds. Indiana Univ. Math. J. 32 (1983), no. 5, 703–716.
[9] K.D. Elworthy, Stochastic differential equations on manifolds. London Mathematical Society Lecture Note Series, 70. Cambridge University Press, Cambridge-New York, 1982. iii+326 pp.
[10] A. Grigor’yan, Heat kernel upper bounds on a complete non-compact manifold, Revista Math. Iberoam. 10 (1994), 395-452.
[11] A. Grigor’yan, Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds, Bulletin of Amer. Math. Soc. 36 (1999) 135-249.
[12] R. Greene and H.H. Wu, Function Theory on Manifolds Which Possess a Pole, Lecture Notes in Math. n. 699, Springer Verlag, Berlin, 1979
[13] E.-P. Hsu, Heat semigroup on a complete Riemannian manifold. Ann. Probab. 17 (1989), no. 3, 1248–1254.
[14] E.-P. Hsu, Stochastic Analysis on Manifolds. Graduated Studies in Mathematics, 38, American Mathematical Society.
[15] D. Hoffman, J. Spruck, Sobolev and isoperimetric inequalities for Riemannian submanifolds. Comm. Pure and Appl. Math. 27 (1974), 715-727.
[16] X.-D. Li, Liouville theorems for symmetric diffusion operators on complete Riemannian manifolds. J. Math. Pures Appl. 84 (2005), 1295–1361.
[17] P. Li, L. Karp, The heat equation on complete Riemannian manifolds. Unpublished. [http://www.math.uci.edu/~pli/]
[18] S. Markvorsen, On the Heat Kernel Comparison Theorems for Minimal Submanifolds. Proc. Amer. Math. Soc., 97 (1986), 479-482.
[19] S. Pigola, M. Rigoli, A.G. Setti, A remark on the maximum principle and stochastic completeness. Proc. Amer. Math. Soc. 131 (2002), 1283–1288.
[20] S. Pigola, M. Rigoli, A.G. Setti, Maximum principles on Riemannian manifolds and applications. Mem. Amer. Math. Soc. 174 (2005), no. 822, x+99 pp.
[21] S. Pigola, M. Rigoli, A.G. Setti, Some non-linear function theoretic properties of Riemannian manifolds. Revista Mat. Iberoam. 22 (2006), 801–831.
[22] S. Pigola, A.G. Setti, M. Troyanov, The topology at infinity of a manifold supporting an $L^{q,p}$-Sobolev inequality. [arXiv:1007.1761]
[23] Z. Qian, On conservation of probability and the Feller property. Ann. Probab. 24 (1996), 280–292.
[24] L. Saloff-Coste, Aspects of Sobolev-type Inequalities. London Mathematical Society Lecture Notes Series 289, Cambridge University Press, 2002.
[25] S.T. Yau, Some function-theoretic properties of complete Riemannian manifold and their applications to geometry. Indiana Univ. Math. J. 25 (1976), no. 7, 659–670.

Dipartimento di Fisica e Matematica, Università dell’Insubria - Como, via Valleggio 11, I-22100 Como, ITALY
E-mail address: stefano.pigola@uninsubria.it

Dipartimento di Fisica e Matematica, Università dell’Insubria - Como, via Valleggio 11, I-22100 Como, ITALY
E-mail address: alberto.setti@uninsubria.it