Calculation of circular plates with assuming shear deformations

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Abstract. The problem of calculating circular plates by the finite element method taking into account shear deformations is considered. Transverse forces can be approximated by constant or piecewise constant functions. The necessary relations for triangular finite elements are obtained. It is shown that the proposed method can be used in combination with traditional finite elements for thin plates obtained by the finite element method in displacements. A comparison of the solutions obtained by the proposed method with other known solutions for circular plates regarding shear is given. It is shown that displacements from shear deformations are determined independently of displacements associated with bending. The obtained results demonstrate the convergence of the solution to the exact one when grinding the finite element mesh and good accuracy for considering shear deformations.

1. Introduction
The circular plates are widely used as common structures of various objects of civil and industrial construction. For calculations of building structures, the finite element method is currently used [1-5]. Often, in modern constructions thick and multi-layered slabs are used. The classical theory of Kirchhoff plate bending is based on assumption of the direct normals assumption and, therefore, does not allow for the shear deformations. The finite elements, which are developed based on the Kirchhoff theory, are can used only for the calculation of thin plates [1-2]. Therefore, Timoshenko–Mindlin’s theory of bending plates are widely application for calculating thick plates [6-16].

Thus, construction the models with considering shear deformations, which are alternatives by the finite element method in displacements, is actual for the bending plates. The purpose of this work is to develop the method for calculating the plates with accounting shear deformations based on the functional of additional energy and the principle of possible displacements [17-22], as well as comparing the solutions obtained for plates with different support conditions with solutions of the other methods.

2. Methods
Solving the problems of plate bending with considering the shear deformations due to transverse forces, we will obtain based on the functional of additional energy for an isotropic plate (for simplicity, we assume that there are no specified displacements):
\[ II^e = \frac{1}{2} \frac{12}{E \cdot t^3} \int \left( M_x^2 + M_y^2 - 2\nu M_x M_y + 2(1+\nu) M_{xy}^2 \right) d\Omega + \frac{k(1+\nu)}{E \cdot t} \int \left( Q_x^2 + Q_y^2 \right) d\Omega \rightarrow \text{min}. \tag{1} \]

\( E \) is the modulus of elasticity of the material; \( t \) is the plate thickness; \( \nu \) - Poisson's ratio; \( k \) - coefficient, which considering the parabolic law of change of the tangential stresses across the thickness of the plate. We write the functional (1) in matrix form that is more convenient for solving by the finite element method:

\[ II^e = \frac{1}{2} \{ M \}^T \{ E \}^{-1} \{ M \} d\Omega + \frac{1}{2} \{ Q \}^T \{ E_{sh} \}^{-1} \{ Q \} d\Omega \rightarrow \text{min}. \tag{2} \]

In expression (2) the following notation is entered:

\[ \{ M \} = \begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix}, \{ Q \} = \begin{bmatrix} Q_x \\ Q_y \end{bmatrix}, \{ E \}^{-1} = \frac{12}{E \cdot t^3} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} , \{ E_{sh} \}^{-1} = \frac{12(1+\nu)}{5E \cdot t} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{3} \]

In the functional (2), the first member is associated with the bending deformations of the plate, the second - with shear deformations by transverse forces. In accordance with the principle of minimum of additional energy, the functions of moments and shear forces must satisfy the corresponding differential equations of equilibrium and static boundary conditions. Since, in the general case, it is almost impossible to select such functions, we will operate as follows. Divide the circular plate into triangular finite elements. On the region of the finite element, we approximate the moment and shear force fields by piecewise constant functions (Figure 1a). Below we show that transverse forces can also be approximated by the functions, which are constant over the finite element region (Figure 1b).

**Figure 1.** Approximation of moments and shear forces in the region of finite elements: a) piecewise constant moments and shear forces; b) constant transverse forces.

Then the functional (2) can be written in the following form:

\[ II^e = \frac{1}{2} \{ M \}^T [ D ] \{ M \} + \frac{1}{2} \{ Q \}^T [ D_{sh} ] \{ Q \} \rightarrow \text{min}. \tag{4} \]

\( \{ M \} \) - vector of unknown nodal moments for the whole system; \( [ D ] \) - flexibility matrix for the whole system under bending; \( [ D_{sh} ] \) - flexibility matrix for the whole system under shear. Then, using the principle of possible displacements (Figure 2), we are construct algebraic equilibrium equations of the nodes of the grid of finite elements.

**Figure 2.** Possible displacements of node 1: a) bending state; b) shear state.
Under bending, the equilibrium equations are expressed through the nodal moments and can be represented in the following matrix form:

$$\{C_i\}^T\{M_i\} + \bar{P}_i = 0, \quad i \in \Xi_z.$$  \hspace{2cm} (5)

Under shear, the equilibrium equations are expressed only through the transverse forces and are represented in the follow form:

$$\{C_{sh,i}\}^T\{Q_i\} + \bar{P}_i = 0, \quad i \in \Xi_z.$$  \hspace{2cm} (6)

Thus, we have obtained the problem of minimizing quadratic function of several variables (4) with constraints in the form of system of linear algebraic equations. Unknown parameters are nodal moments and shear forces. To solve this problem, we use the well-known Lagrange multipliers method for account the equilibrium equations, and the penalty function method, for account static boundary conditions. Then, we get the following expression of the extended functional:

$$IT^e = \frac{1}{2}\{M\}^T\left[D\right]\{M\} + \frac{1}{2}\{Q\}^T\left[D_{sh}\right]\{Q\} + \sum_{i \in \Xi_f} w_i \left(\{C_i\}^T\{M_i\} + \bar{P}_i\right) + \sum_{i \in \Xi_f} w_{sh,i} \left(\{C_{sh,i}\}^T\{Q_i\} + \bar{P}_i\right) + \sum_{i \in \Xi_f} \alpha \left(M_{x,i} \cos^2 \phi + M_{y,i} \sin^2 \phi - 2M_{sy,i} \sin \phi \cos \phi - \tilde{M}_{x,i}\right)^2 + \sum_{i \in \Xi_f} \alpha \left((M_{y,j} - M_{x,j}) \sin \phi \cos \phi + M_{sy,j} (\cos^2 \phi - \sin^2 \phi) - \tilde{M}_{y,j}\right)^2 + \sum_{i \in \Xi_f} \alpha \left(Q_{x,j} \cos \phi + Q_{y,j} \sin \phi - \bar{Q}_j\right)^2 \rightarrow \text{min}. \quad (7)$$

$w_i$ - vertical displacement of the node $i$, associated with the bend of the plate; $w_{sh,i}$ - vertical displacement of the node $i$, associated with the shear of the plate cross sections; $\alpha$ - penalty parameter (large number). The expression of the functional (7) can be represented in a more compact matrix form:

$$IT^e = \frac{1}{2}\{M\}^T\left[D^*\right]\{M\} + \frac{1}{2}\{Q\}^T\left[D_{sh}^*\right]\{Q\} + \{M\}^T\{F^M\} + \{Q\}^T\{F^0\} + \{w\}^T\left((F) - [L]\{M\}\right) + \{w_{sh}\}^T\left((F) - [L_{sh}]\{Q\}\right) \rightarrow \text{min}. \quad \hspace{1cm} (8)$$

Equating to zero the derivatives of the functional $IT^e$ along the vectors $\{M\}$ and $\{w\}$, we obtain the system of equations, consisting of the equations of the compatibility of deformations and the equilibrium equations for bending:

$$\begin{bmatrix} \left[D^*\right] & -[L]^T \\ -[L] & 0 \end{bmatrix} \begin{bmatrix} \{M\} \\ \{w\} \end{bmatrix} = \begin{bmatrix} \{-F^M\} \\ \{-F\} \end{bmatrix}. \quad \hspace{2cm} (9)$$

If piecewise constant approximations are used for the moment fields, then the we easily get inverse matrix $\left[D^*\right]^{-1}$ analytically. Therefore, we can easily express the vector $\{M\}$ from the first matrix equation:

$$\{M\} = \left[D^*\right]^{-1}[L]^T\{w\} + \left[D^*\right]^{-1}\{F^M\}. \quad \hspace{2cm} (10)$$

Substituting expression (10) into the second matrix equation of (9), we obtain the system of linear algebraic equations for determining the vector $\{w\}$:
\[
[K]\{w\} = \{F\} + [L][D^c]^{-1}\{F^0\}, \quad [K] = [L][D^c]^{-1}[L]^T.
\]

The expressions for the elements of the vector \(\{F\}\), matrices \([D]\), \([L]\), the algorithm of their formation and examples of the calculation of bent plates corresponding to the Kirchhoff theory, are given in [21,22].

To determine the displacement vector \(\{w_{sh}\}\) and the vector of transverse forces \(\{Q\}\) associated with shear deformations, we equate to zero the derivatives of the functional \(J^T\) along the vectors \(\{Q\}\) and \(\{w_{sh}\}\). Then we get the following system of equations, like the system of equations (9):

\[
\begin{bmatrix}
[D_{sh}^T] - [L_{sh}]^T \\
-L_{sh} \\
0
\end{bmatrix}
\begin{bmatrix}
\{Q\} \\
\{w_{sh}\}
\end{bmatrix}
= \begin{bmatrix}
-\{F^0\} \\
-\{F\}
\end{bmatrix}.
\]

The solution of the system of equations (12) can be performed in the following sequence:

\[
[K_{sh}] = [L_{sh}][D_{sh}^T]^{-1}[L_{sh}], \quad [K_{sh}]\{w_{sh}\} = \{F\} + [L_{sh}][D_{sh}^T]^{-1}\{F^0\},
\]

\[
\{Q\} = [D_{sh}^T]^{-1}[L_{sh}]\{w_{sh}\} + [D_{sh}^T]^{-1}\{F^0\}.
\]

We obtain the equilibrium equations for triangular finite elements. The equilibrium equation for possible displacement of node of the triangular finite element can be obtained directly in the global coordinate system. For this, the possible displacements of the points of the finite element \(k\) on shearing will express using the triangular coordinates:

\[
\delta w_{sh,i}(x,y) = T_i, \quad i = 1,2,3 \quad T_i = \frac{1}{2A^k}(a_i + b_i x + c_i y).
\]

\[
a_i = x_{i1}y_{i2} - x_{i2}y_{i1}, \quad b_i = y_{i1} - y_{i2}, \quad c_i = x_{i2} - x_{i1}.
\]

\(A^k\) - area of the triangular element; \(x_i, y_i\) - coordinates of the node \(i\). Triangular coordinates are natural coordinates of triangular area. The function \(T_i\) takes the value 1 at node \(i\) and the value zero at other two nodes. With the possible displacement of node, constant shear deformations arise in cross sections:

\[
\delta y_{x,i} = \frac{\partial (\delta w_{sh,i})}{\partial x} = \frac{b_i}{2A^k}, \quad \delta y_{y,i} = \frac{\partial (\delta w_{sh,i})}{\partial y} = \frac{c_i}{2A^k}.
\]

Consider the case of piecewise constant approximations of transverse forces in finite element region. The vector of nodal forces for triangular finite element, expressed in the global coordinate system, will have the following form:

\[
\{Q^k\}^T = \begin{bmatrix}
Q_{i,1} & Q_{j,1} & Q_{i,2} & Q_{j,2} & Q_{i,3} & Q_{j,3}
\end{bmatrix}.
\]

The work of the internal transverse forces of the \(k\)-th finite element on the possible displacement of the node \(i\) is expressed as an integral:

\[
\delta U^k_{i,j} = \int_{\Omega_{i,j}} (\delta y_{x,i} Q_x + \delta y_{y,i} Q_y) dA = \frac{b_i}{2A^k} \sum_{j=1}^{3} Q_{x,j}A^k_{ij} + \frac{c_i}{2A} \sum_{j=1}^{3} Q_{y,j}A^k_{ij}.
\]
area of part of $k$-th triangular finite element is adjoined to the node $j$. We introduce the vector, that combines the values of the works of transverse forces on possible displacements of finite element nodes:

$$\{\delta U^k_i\}^T = (\delta U_{i,j}^k, \delta U_{2,j}^k, \delta U_{3,j}^k).$$

(19)

Then we get

$$\{\delta U^k_i\} = [L_{ik}]{[Q]^k},$$

(20)

$$[L_{ik}] = \frac{1}{2A^k} \begin{bmatrix}
    b_1A^k_1 & c_1A^k_1 & b_1A^k_2 & c_1A^k_2 & b_1A^k_3 & c_1A^k_3 \\
    b_2A^k_1 & c_2A^k_1 & b_2A^k_2 & c_2A^k_2 & b_2A^k_3 & c_2A^k_3 \\
    b_3A^k_1 & c_3A^k_1 & b_3A^k_2 & c_3A^k_2 & b_3A^k_3 & c_3A^k_3
\end{bmatrix}.$$  

(21)

In the case of using constant approximations of transverse forces, the vector of unknowns for the triangular element will have the following form:

$$\{Q_i^k\}^T = (Q_{x,k}, Q_{z,k}).$$

(22)

Calculating the integral (18), we get:

$$[L_{ik}] = \frac{1}{2} \begin{bmatrix}
    b_1 & c_1 \\
    b_2 & c_2 \\
    b_3 & c_3
\end{bmatrix}.$$  

(23)

From matrices $[L_{ik}]$ for triangular finite elements, in accordance with the numbering of the nodes and the elements, the global matrix $[L_{ik}]$ for the whole system is formed.

3. Result and discussion

According to the proposed method, calculations were made for hinged supported circular plate on the effect of uniformly distributed load with different finite element grids (Figure 3). The following parameters of the plate were taken: $E = 10^7$ kN/m$^2$, $\mu = 0.3$, $q = 10$ kN/m$^2$, $R = 3$ m. The exact solution for the displacement of the center of the slab according to the Timoshenko-Mindlin theory

$$w = \frac{qR^4(5 + \mu)}{64D(1 + \mu)} + \frac{qR^2}{4kGt}, \quad k = 5/6.$$  

(24)

Figure 3. Finite element grids for circular plates.

Table 1 presents the results of calculations by the proposed method and the exact values of the displacement of the center of the plate for various ratios of plate thickness to radius.
Table 1. Center's displacement of the plate \( w \cdot 10^3 \ m \).

| \( t/R \) | Grid a | Grid b | Grid c | Exact   |
|--------|--------|--------|--------|---------|
| 0.01   | 2245.18| 2119.67| 2090.06| 2086.88 |
| 0.1    | 2.2424 | 2.1419 | 2.1140 | 2.1104  |
| 0.2    | 0.28038| 0.27615| 0.27298| 0.27256 |
| 0.4    | 0.03509| 0.03872| 0.03849| 0.03845 |

The results presented in the table demonstrate the convergence of the solution to the exact one when grinding the finite element mesh and good accuracy for considering shear deformations. With the smallest grid, the numerical solution almost equal with the exact one.

4. Conclusion
The proposed method of solving in stresses is based on the fundamental principles of structural mechanics - the principle of minimum of additional energy and the principle of possible displacements. Using the principle of possible displacements, we are construct algebraic equilibrium equations of the nodes of the grid of finite elements. The moment and shear force fields are approximated by piecewise constant functions. Lagrange multipliers method is using for account the equilibrium equations, and the penalty function method, for account static boundary conditions. The obtained results demonstrate the convergence of the solution to the exact one when grinding the finite element mesh and good accuracy for considering shear deformations.

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