Mirror coupling of reflecting Brownian motion
and an application to Chavel’s conjecture

Mihai N. Pascu
Faculty of Mathematics and Computer Science
Transilvania University of Braşov
Braşov – 500091, Romania
mihai.pascu@unitbv.ro
http://cs.unitbv.ro/~pascu

Abstract

In a series of papers, Burdzy et al. introduced the mirror coupling of reflecting Brownian motions in a smooth bounded domain $D \subset \mathbb{R}^d$, and used it to prove certain properties of eigenvalues and eigenfunctions of the Neumann Laplaceian on $D$.

In the present paper we show that the construction of the mirror coupling can be extended to the case when the two Brownian motions live in different domains $D_1, D_2 \subset \mathbb{R}^d$.

As applications of the construction, we derive a unifying proof of the two main results concerning the validity of Chavel's conjecture on the domain monotonicity of the Neumann heat kernel, due to I. Chavel ([12]), respectively W. S. Kendall ([16]), and a new proof of Chavel's conjecture for domains satisfying the ball condition, such that the inner domain is star-shaped with respect to the center of the ball.

Key words: couplings, mirror coupling, reflecting Brownian motion, Chavel's conjecture.

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1 Introduction

The technique of coupling of reflecting Brownian motions is a useful tool, used by several authors in connection to the study of the Neumann heat kernel of the corresponding domain (see [2], [3], [6], [11], [16], [17], etc).

In a series of papers, Krzysztof Burdzy et al. ([11], [2], [3], [6], [10]) introduced the mirror coupling of reflecting Brownian motions in a smooth domain $D \subset \mathbb{R}^d$ and used it in order to derive properties of eigenvalues and eigenfunctions of the Neumann Laplaceian on $D$.

In the present paper, we show that the mirror coupling can be extended to the case when the two reflecting Brownian motions live in different domains $D_1, D_2 \subset \mathbb{R}^d$.

The main difficulty in the extending the construction of the mirror coupling comes from the fact that the stochastic differential equation(s) describing the mirror coupling has a singularity at the times when coupling occurs. In the case $D_1 = D_2 = D$ considered by Burdzy et al., this problem is not a major problem (although the technical details are quite involved, see [2]), since after the coupling time the processes move together. In the case $D_1 \neq D_2$ however, this is a major problem: after the processes have coupled, it is possible for them to decouple (for example in the case when the processes are coupled and they hit the boundary of one of the domains).

It is worth mentioning that the method used for proving the existence of the solution is new, and it relies on the additional hypothesis that the smaller domain $D_2$ (or more generally $D_1 \cap D_2$) is a convex domain. This hypothesis allows us to construct an explicit set of solutions in a sequence of approximating polygonal domains for $D_2$, which converge to the desired solution.

As applications of the construction, we derive a unifying proof of the two most important results on the challenging Chavel's conjecture on the domain monotonicity of the Neumann heat kernel ([12], [16]), and a new proof of Chavel's conjecture for domains satisfying the ball condition, such that the inner domain is star-shaped with respect to the center of the ball. This is also a possible new line of approach for Chavel's conjecture (note that by the results in [4], Chavel's conjecture does not hold in its full generality, but the additional hypotheses under which this conjecture holds are not known at the present moment).

The structure of the paper is as follows: in Section 2 we briefly describe the construction of Burdzy et al. of the mirror coupling in a smooth bounded domain $D \subset \mathbb{R}^d$.

In Section 3, in Theorem 3.1, we give the main result which shows that the mirror coupling can be extended to the case when $D_2 \subset D_1$ are smooth bounded domains in $\mathbb{R}^d$ and $D_2$ is convex (some extensions of the theorem are presented in Section 5).

Before proceeding with the proof of theorem, in Remark 3.4, we show that the proof can be reduced to the case when $D_1 = \mathbb{R}^d$. Next, in Section 3.1, we show that in the case $D_2 = (0, \infty) \subset D_1 = \mathbb{R}$ the solution is essentially given by Tanaka's formula (Remark 3.5), and then we give the proof of the main theorem in the 1-dimensional case (Proposition 3.6).

In Section 3.2, we first prove the existence of the mirror coupling in the case when $D_2$ is a half-space in $\mathbb{R}^d$ and $D_1 = \mathbb{R}^d$ (Lemma 3.8), and then we use this result in order to prove the existence of the mirror coupling in the case when $D_2$ is a polygonal domain in $\mathbb{R}^d$ and $D_1 = \mathbb{R}^d$ (Theorem 3.9).

In Proposition 3.10, we present some of the properties of the mirror coupling in the particular case when $D_2$ is a convex polygonal domain and $D_1 = \mathbb{R}^d$, which are essential for the construction of the general mirror coupling.
In Section 4 we give the proof of the main Theorem 3.1. The idea of the proof is to construct a sequence \( (Y^n_t, X_t) \) of mirror couplings in \((D_n, \mathbb{R}^d)\), where \( D_n \nearrow D_2 \) is a sequence of convex polygonal domains in \( \mathbb{R}^d \). Then, using the properties of the mirror coupling in convex polygonal domains (Proposition 3.10), we show that the sequence \( Y^n_t \) converges to a process \( Y_t \), which gives the desired solution to the problem.

The last section of the paper (Section 5) is devoted to the applications and the extensions of the mirror coupling constructed in Theorem 3.1.

First, in Theorem 5.3 we use the mirror coupling in order to give a simple, unifying proof of the results of I. Chavel and W. S. Kendall on the domain monotonicity of the Neumann heat kernel (Chavel’s Conjecture 5.1). The proof is probabilistic in spirit, relying on the geometric properties of the mirror coupling.

Next, in Theorem 5.4 we show that Chavel’s conjecture also holds in the more general case when one can interpose a ball between the two domains, and the inner domain is star-shaped with respect to the center of the ball (instead of being convex). The analytic proof given here is parallel to the geometric proof of the previous theorem, and it can also serve as an alternate proof of it.

Without giving all the technical details, we discuss the extension of the mirror coupling to the case of smooth bounded domains \( D_{1,2} \subset \mathbb{R}^d \) with non-tangential boundaries, such that \( D_1 \cap D_2 \) is a convex domain.

The paper concludes with a discussion of the non-uniqueness of the mirror coupling. The lack of uniqueness is due to the fact that after coupling the processes may decouple, not only on the boundary of the domain, but also when they are inside the domain.

The two basic solutions give rise to the sticky, respectively non-sticky mirror coupling, and there is a whole range of intermediate possibilities. The stickiness refers to the fact that after coupling the processes “stick” to each other as long as possible (“sticky” mirror coupling, constructed in Theorem 3.1), or they can immediately split apart after coupling (“non-sticky” mirror coupling), the general case (weak/mild mirror coupling) being a mixture of these two basic behaviors.

We developed the extension of the mirror coupling having in mind the application to Chavel’s conjecture, for which the sticky mirror coupling is the “right” tool, but perhaps the other mirror couplings (the non-sticky and the mild mirror couplings) might prove useful in other applications.

2 Mirror couplings of reflecting Brownian motions

Reflecting Brownian motion in a smooth domain \( D \subset \mathbb{R}^d \) can be defined as a solution of the stochastic differential equation

\[
X_t = x + B_t + \int_0^t \nu_D(X_s) \, dL^X_s, \tag{2.1}
\]

where \( B_t \) is a \( d \)-dimensional Brownian motion, \( \nu_D \) is the inward unit normal vector field on \( \partial D \) and \( L^X_t \) is the boundary local time of \( X_t \) (the continuous non-decreasing process which increases only when \( X_t \in \partial D \)).

In [1], the authors introduced the mirror coupling of reflecting Brownian motion in a smooth domain \( D \subset \mathbb{R}^d \) (piecewise \( C^2 \) domain in \( \mathbb{R}^2 \) with a finite number of convex corners or a \( C^2 \) domain in \( \mathbb{R}^d \), \( d \geq 3 \)).
They considered the following system of stochastic differential equations:

\[ X_t = x + W_t + \int_0^t v_D (X_s) \, dL_s^X \quad (2.2) \]

\[ Y_t = y + Z_t + \int_0^t v_D (X_s) \, dL_s^Y \quad (2.3) \]

\[ Z_t = W_t - 2 \int_0^t \frac{Y_s - X_s}{||Y_s - X_s||^2} (Y_s - X_s) \cdot dW_s \quad (2.4) \]

for \( t < \xi \), where \( \xi = \inf \{ s > 0 : X_s = Y_s \} \) is the coupling time of the processes, after which the processes \( X \) and \( Y \) evolve together, i.e. \( X_t = Y_t \) and \( Z_t = W_t + Z_\xi - W_\xi \) for \( t \geq \xi \).

In the notation of [1], considering the Skorokhod map

\[ \Gamma : C([0, \infty) : \mathbb{R}^d) \to C([0, \infty) : \overline{D}) , \]

we have \( X = \Gamma(x + W) \), \( Y = \Gamma(y + Z) \), and therefore the above system is equivalent to

\[ Z_t = \int_0^{t \wedge \xi} G \left( \Gamma (y + Z)_s - \Gamma (x + W)_s \right) \, dW_s + 1_{t \geq \xi} \left( W_t - W_\xi \right) , \quad (2.5) \]

where \( \xi = \inf \{ s > 0 : \Gamma(x + W)_s = \Gamma (y + Z)_s \} \). In [1] the authors proved the pathwise uniqueness and the strong existence of the process \( Z_t \) in (2.5) (given the Brownian motion \( W_t \)).

In the above \( G : \mathbb{R}^d \to \mathcal{M}_{d \times d} \) denotes the function defined by

\[ G(z) = \begin{cases} 
H \left( \frac{z}{||z||} \right), & \text{if } z \neq 0 \\
0, & \text{if } z = 0 
\end{cases} , \quad (2.6) \]

where for a unitary vector \( m \in \mathbb{R}^d \), \( H(m) \) represents the linear transformation given by the \( d \times d \) matrix

\[ H(m) = I - 2m m', \quad (2.7) \]

that is

\[ H(m) v = v - 2 (m \cdot v) m \quad (2.8) \]

is the mirror image of \( v \in \mathbb{R}^d \) with respect to the hyperplane through the origin perpendicular to \( m \) (\( m' \) denotes the transpose of the vector \( m \), vectors being considered as column vectors).

The pair \( (X_t, Y_t)_{t \geq 0} \) constructed above is called a mirror coupling of reflecting Brownian motions in \( D \) starting at \( (x, y) \in \overline{D} \times \overline{D} \).

Remark 2.1. The relation (2.4) can be written in the equivalent form

\[ dZ_t = G(X_t - Y_t) \, dW_t, \]

which shows that for \( t < \xi \) the increments of \( Z_t \) are mirror images of the increments of \( W_t \) with respect to the hyperplane \( M_t \) of symmetry between \( X_t \) and \( Y_t \), justifying the name of mirror coupling.
3 Extension of the mirror coupling

The main contribution of the author is the observation that the mirror coupling introduced above can be extended to the case when the two reflecting Brownian motion have different state spaces, that is when $X_t$ is a reflecting Brownian motion in a domain $D_1$ and $Y_t$ is a reflecting Brownian motion in a domain $D_2$. Although the construction can be carried out in a more general setup (see the concluding remarks in Section 5), in the present section we will consider the case when one of the domains is strictly contained in the other.

The main result is the following:

**Theorem 3.1.** Let $D_{1,2} \subset \mathbb{R}^d$ be smooth bounded domains (piecewise $C^2$-smooth boundary with convex corners in $\mathbb{R}^2$, or $C^2$-smooth boundary in $\mathbb{R}^d$, $d \geq 3$ will suffice) with $D_2 \subset D_1$ and $D_2$ convex domain, and let $x \in \overline{D}_1$ and $y \in \overline{D}_2$ be arbitrarily fixed points. Given a $d$-dimensional Brownian motion $W_t, t \geq 0$ starting at 0 on a probability space $(\Omega, \mathcal{F}, P)$, there exists a strong solution of the following system of stochastic differential equations

\[
X_t = x + W_t + \int_0^t \nu_{D_1}(X_s) \, dL_s^X \tag{3.1}
\]

\[
Y_t = y + Z_t + \int_0^t \nu_{D_2}(Y_s) \, dL_s^Y \tag{3.2}
\]

\[
Z_t = \int_0^t G(Y_s - X_s) \, dW_s \tag{3.3}
\]

or equivalent

\[
Z_t = \int_0^t G(\tilde{\Gamma}(y + Z)_s - \Gamma(x + W)_s) \, dW_s, \tag{3.4}
\]

where $\Gamma$ and $\tilde{\Gamma}$ denote the corresponding Skorokhod maps which define the reflecting Brownian motion $X = \Gamma(x + W)$ in $D_1$, respectively $Y = \tilde{\Gamma}(y + Z)$ in $D_2$, and $G : \mathbb{R}^d \to M_{d \times d}$ denotes the following modification of the function $G$ defined in the previous section:

\[
G(z) = \begin{cases} 
H\left(\frac{z}{\|z\|}\right), & \text{if } z \neq 0 \\
I_z, & \text{if } z = 0
\end{cases} \tag{3.5}
\]

**Remark 3.2.** As it will follow from the proof of the theorem, with the choice of $G$ above, the solution of the equation (3.4) in the case $D_1 = D_2 = D$ is the same as the solution of the equation (2.5) considered by the authors in [1] (as also pointed out by the authors, the choice of $G(0)$ is irrelevant in this case).

Therefore, the above theorem is a natural generalization of the mirror coupling to the case when the two processes live in different spaces. We will refer to a solution $(X_t, Y_t)$ given by the above theorem as a mirror coupling of reflecting Brownian motions in $(D_1, D_2)$ starting from $(x, y) \in \overline{D}_1 \times \overline{D}_2$, with driving Brownian motion $W_t$.

As indicated in Section 5 the solution of (3.4) is not pathwise unique, due to the fact that the stochastic differential equation has a singularity at the times when coupling occurs. The general mirror coupling can be thought as depending on a parameter which is a measure of the stickiness of
the coupling: once the processes $X_t$ and $Y_t$ have coupled, they can either move together until one of them hits the boundary (sticky mirror coupling - this is in fact the solution constructed in the above theorem), or they can immediately split apart after coupling (non-sticky mirror coupling), and there is a whole range of intermediate possibilities (see the discussion at the end of Section 5).

As an application, in Section 5 we will use the former mirror coupling to give a unifying proof of Chavel's conjecture on the domain monotonicity of the Neumann heat kernel for domains $D_{1,2}$ satisfying the ball condition, although the other possible choices for the mirror coupling might prove useful in other contexts.

Before carrying out the proof, we begin with some preliminary remarks which will allow us to reduce the proof of the above theorem to the case $D_1 = \mathbb{R}^d$.

Remark 3.3. The main difference from the case when $D_1 = D_2 = D$ considered by the authors in [1] is that after the coupling time $\xi$ the processes $X_t$ and $Y_t$ may decouple. For example, if $t \geq \xi$ is a time when $X_t = Y_t \in \partial D_2$, the process $Y_t$ (reflecting Brownian motion in $D_2$) receives a push in the direction of the inward unit normal to the boundary of $D_2$, while the process $X_t$ behaves like a free Brownian motion near this point (we assumed that $D_2$ is strictly contained in $D_1$), and therefore the processes $X$ and $Y$ will drift apart, that is they will decouple. Also, as shown in Section 5 because the function $G$ has a discontinuity at the origin, it is possible that the solutions decouple even when they are inside the domain $D_2$. This shows that without additional assumptions, the mirror coupling is not uniquely determined (there is no pathwise uniqueness of (3.4)).

Remark 3.4. To fix ideas, for an arbitrarily fixed $\epsilon > 0$ chosen small enough such that $\epsilon < \text{dist}(\partial D_1, \partial D_2)$, we consider the sequence $(\xi_n)_{n \geq 1}$ of coupling times and the sequence $(\tau_n)_{n \geq 0}$ of times when the processes are $\epsilon$-decoupled ($\epsilon$-decoupling times, or simply decoupling times by an abuse of language) defined inductively by

\[
\begin{align*}
\xi_n &= \inf\{ t > \tau_{n-1} : X_t = Y_t \}, \quad n \geq 1, \\
\tau_n &= \inf\{ t > \xi_n : \|X_t - Y_t\| > \epsilon \}, \quad n \geq 1,
\end{align*}
\]

where $\tau_0 = 0$ and $\xi_1 = \xi$ is the first coupling time.

To construct the general mirror coupling (that is, to prove the existence of a solution to (3.1) – (3.3) above, or equivalent to (3.4)), we proceed as follows.

First note that on the time interval $[0, \xi]$, the arguments used in the proof of Theorem 2 in [1] (pathwise uniqueness and the existence of a strong solution $Z$ of (3.4)) do not rely on the fact that $D_1 = D_2$, hence the same arguments can be used to prove the existence of a strong solution of (3.4) on the time interval $[0, \xi_1] = [0, \xi]$. Indeed, given $W_t$, (3.1) has a strong solution which is pathwise unique (the reflecting Brownian motion $X_t$ in $D_1$), and therefore the proof of pathwise uniqueness and the existence of a strong solution of (3.4) is the same as in [1] considering $D = D_2$. Also note that as also pointed out by the authors, the value $G(0)$ is irrelevant in their proof, since the problem is constructing the processes until they meet, that is for $Y_t - X_t \neq 0$, for which their definition of $G$ is the same as in (3.5).

We obtain therefore the existence of a strong solution $Z_t$ to (3.4) on the time interval $[0, \xi_1]$. By this we understand that the process $Z$ verifies (3.4) for all $t \leq \xi_1$ and $Z_t$ is $\mathcal{F}_t$ measurable for $t \leq \xi_1$, where $(\mathcal{F}_t)_{t \geq 0}$ denotes the corresponding filtration of the driving Brownian motion $W_t$.

For an arbitrarily fixed $T > 0$, if $\xi_1 < T$, we can extend $Z$ to a solution of (3.4) on the time interval...
[0, T] as follows. Consider \( \xi_1^T = \xi_1 \land T \), and note that if \( Z \) solves (3.4), then

\[
Z_{\xi_1^T + t} - Z_{\xi_1^T} = \int_{\xi_1^T}^{\xi_1^T + t} G \left( \tilde{\Gamma} \left( y + Z \right)_s - \Gamma \left( x + W \right)_s \right) dW_s,
\]

where

\[
= \int_0^t G \left( \tilde{\Gamma} \left( y + Z \right)_{\xi_1^T + s} - \Gamma \left( x + W \right)_{\xi_1^T + s} \right) dW_{\xi_1^T + s}.
\]

By the uniqueness results on the Skorokhod map (in the deterministic sense), we have

\[
\tilde{\Gamma} \left( y + Z \right)_{\xi_1^T + s} = \tilde{\Gamma} \left( y + Z \right)_{\xi_1^T} - Z_{\xi_1^T} + Z_{\xi_1^T + s}.
\]

and

\[
\Gamma \left( x + W \right)_{\xi_1^T + s} = \Gamma \left( \Gamma \left( x + W \right)_{\xi_1^T} - W_{\xi_1^T} + W_{\xi_1^T + s} \right).
\]

for \( s \geq 0 \).

It is known that \( \tilde{W}_s = W_{\xi_1^T + s} - W_{\xi_1^T} \) is a Brownian motion starting at the origin, with corresponding filtration \( \mathcal{F}_s = \sigma \left( B_{\xi_1^T + u} - B_{\xi_1^T} : 0 \leq u \leq s \right) \) independent of \( \mathcal{F}_{\xi_1^T} \).

Setting \( Z_t = Z_{\xi_1^T + t} - Z_{\xi_1^T} \) and combining the above equations we obtain

\[
Z_t = \int_0^t G \left( \tilde{\Gamma} \left( y + Z \right)_{\xi_1^T} + \tilde{Z} \right)_s - \Gamma \left( \Gamma \left( x + W \right)_{\xi_1^T} + \tilde{W} \right)_s d\tilde{W}_s,
\]

which is the same as the equation (3.4) for \( \tilde{Z} \), with the initial points \( x, y \) of the coupling replaced by \( Y_{\xi_1^T} = \tilde{\Gamma} \left( y + Z \right)_{\xi_1^T} \), respectively \( X_{\xi_1^T} = \Gamma \left( x + W \right)_{\xi_1^T} \), and the Brownian motion \( W \) replaced by \( \tilde{W} \).

If we assume the existence of a strong solution \( \tilde{Z}_t \) of (3.6) until the first \( \epsilon \)-decoupling time, by patching \( Z \) and \( \tilde{Z} \) we obtain that

\[
Z_t \mathbb{1}_{t \leq \xi_1^T} + \tilde{Z}_{t - \xi_1^T} \mathbb{1}_{t - \xi_1^T \leq \tau_1^T}
\]

is a strong solution to (3.4) on the time interval \([0, \tau_1^T]\), where \( \tau_1^T = \tau_1 \land T \).

If \( \tau_1^T = T \), we are done. Otherwise, since at time \( \tau_1^T \) the processes \( X \) and \( Y \) are \( \epsilon \) units apart, we can apply again the results in [11] (with the Brownian motion \( W_{\tau_1^T + t} - W_{\tau_1^T} \) instead of \( W_t \), and the starting points of the coupling \( X_{\tau_1^T} \) and \( Y_{\tau_1^T} \) instead of \( x \) and \( y \)) in order to obtain a strong solution of (3.4) until the first coupling time. By patching we obtain the existence of a strong solution of (3.4) on the time interval \([0, \xi_2^T]\).

Proceeding inductively as indicated above, since only a finite number of coupling / decoupling times \( \xi_n \) and \( \tau_n \) can occur in the time interval \([0, T]\), we can construct a strong solution \( Z \) to (3.4) on the time interval \([0, T]\) for any \( T > 0 \) (and therefore on \([0, \infty)\)), provided we show the existence of strong solutions of equations of type (3.6) until the first \( \epsilon \)-decoupling time.

In order to prove this claim, since \( \tilde{\Gamma} \left( y + Z \right)_{\xi_1^T} \) and \( \Gamma \left( x + W \right)_{\xi_1^T} \) are \( \mathcal{F}_{\xi_1^T} \) measurable and the \( \sigma \)-algebra \( \mathcal{F}_{\xi_1^T} \) is independent of the filtration \( \mathcal{F} = (\mathcal{F}_t)_{t \geq 0} \) of the driving Brownian motion \( \tilde{W}_t \), it suffices to show that for any starting points \( x = y \in \partial D_2 \) of the mirror coupling, there exists a strong solution of (3.4) until the first \( \epsilon \)-decoupling time \( \tau_1 \). Since \( \epsilon < \text{dist} (\partial D_1, \partial D_2) \), it follows that the
process \( X_t \) cannot reach the boundary \( \partial D_1 \) before the first \( \epsilon \)-decoupling time \( \tau_1 \), and therefore we can consider that \( X_t \) is a free Brownian motion in \( \mathbb{R}^d \), that is, we can reduce the proof of Theorem 3.1 to the case when \( D_1 = \mathbb{R}^d \).

We will give the proof of the Theorem 3.1 first in the 1-dimensional case, then we will extend it to the case of polygonal domains in \( \mathbb{R}^d \), and we will conclude with the proof in the general case.

### 3.1 The 1-dimensional case

From Remark 3.4 it follows that in order to construct the mirror coupling in the 1-dimensional case, it suffices to consider \( D_1 = \mathbb{R} \) and \( D_2 = (0, a) \), and to show that for an arbitrary choice \( x \in [0, a] \) of the starting point of the mirror coupling, \( \epsilon \in (0, a) \) sufficiently small and \( (W_t)_{t \geq 0} \) a 1-dimensional Brownian motion starting at \( W_0 = 0 \), we can construct a strong solution on \( [0, \tau_1] \) of the following system

\[
\begin{align*}
X_t &= x + W_t \\
Y_t &= x + Z_t + L_t^Y \\
Z_t &= \int_0^t G(Y_s - X_s) \, dW_s
\end{align*}
\]

where \( \tau_1 = \inf\{s > 0 : |X_s - Y_s| > \epsilon\} \) is the first \( \epsilon \)-decoupling time and the function \( G : \mathbb{R} \to \mathcal{M}_{1 \times 1} \equiv \mathbb{R} \) is given in this case by

\[
G(x) = \begin{cases} 
-1, & \text{if } x \neq 0 \\
+1, & \text{if } x = 0.
\end{cases}
\]

Remark 3.5. Before proceeding with the proof, it is worth mentioning that the heart of the construction is Tanaka’s formula. To see this, consider for the moment \( a = \infty \), and note that Tanaka’s formula

\[
|x + W_t| = x + \int_0^t \text{sgn}(x + W_s) \, dW_s + L_t^0(x + W)
\]

gives a representation of the reflecting Brownian motion \(|x + W_t|\) in which the increments of the martingale part of \(|x + W_t|\) are the increments of \( W_t \) when \( x + W_t \in [0, \infty) \), respectively the opposite (minus) of the increments of \( W_t \) in the other case \( (L_t^0(x + W)) \) denotes here the local time at 0 of \( x + W_t \).

Since \( x + W_t \in [0, \infty) \) is the same as \(|x + W_t| = x + W_t\), from the definition of the function \( G \) it follows that the above can be written in the form

\[
|x + W_t| = x + \int_0^t G(|x + W_s| - (x + W_s)) \, dW_s + L_t^{x+W},
\]

which shows that a strong solution to (3.7) – (3.9) above (in the case \( a = \infty \)) is given explicitly by \( X_t = x + W_t \), \( Y_t = |x + W_t| \) and \( Z_t = \int_0^t \text{sgn}(x + W_s) \, dW_s \).

We have the following:
Proposition 3.6. Given a 1-dimensional Brownian motion \((W_t)_{t\geq 0}\) starting at \(W_0 = 0\), a strong solution on \([0, \tau_1]\) of the system (3.7) – (3.9) is given by

\[
\begin{cases}
X_t = x + W_t \\
Y_t = |a - (x + W_t - a)| \\
Z_t = \int_0^t \text{sgn} (W_s) \text{sgn} (a - W_s) \, dW_s
\end{cases}
\]

where \(\tau_1 = \inf\{s > 0 : |X_s - Y_s| > \epsilon\}\) and

\[
\text{sgn} (x) = \begin{cases}
+1, & \text{if } x \geq 0 \\
-1, & \text{if } x < 0
\end{cases}
\]

Proof. Since \(\epsilon < a\), it follows that for \(t \leq \tau_1\) we have \(X_t = x + W_t \in (-a, 2a)\), and therefore

\[
Y_t = |a - (x + W_t - a)| = \begin{cases}
-(x + W_t), & x + W_t \in (-a, 0) \\
x + W_t, & x + W_t \in [0, a] \\
2a - x - W_t, & x + W_t \in (a, 2a)
\end{cases}
\]

(3.11)

Applying the Tanaka-Itô formula to the function \(f(z) = |a - |z - a||\) and to the Brownian motion \(X_t = x + W_t\), for \(t \leq \tau_1\) we obtain

\[
Y_t = x + \int_0^t \text{sgn} (x + W_s) \text{sgn} (a - x - W_s) \, d (x + W_s) + L_t^0 - L_t^a
\]

\[
= x + \int_0^t \text{sgn} (x + W_s) \text{sgn} (a - x - W_s) \, dW_s + \int_0^t \nu_{D_2} (Y_s) \, d \left( L_s^0 + L_s^a \right),
\]

where \(L_t^0 = \sup_{s \leq t} (x + W_s)^-\) and \(L_t^a = \sup_{s \leq t} (x + W_s - a)^+\) are the local times of \(x + W_t\) at 0, respectively at \(a\), and \(\nu_{D_2} (0) = +1, \nu_{D_2} (a) = -1\).

From (3.11) and the definition (3.10) of the function \(G\) we obtain

\[
\text{sgn} (x + W_s) \text{sgn} (a - x - W_s) = \begin{cases}
-1, & x + W_s \in (-a, 0) \\
+1, & x + W_s \in [0, a] \\
-1, & x + W_s \in (a, 2a)
\end{cases}
\]

\[
= \begin{cases}
+1, & X_s = Y_s \\
-1, & X_s \neq Y_s
\end{cases}
\]

\[
= G(Y_s - X_s),
\]

and therefore the previous formula can be written equivalently

\[
Y_t = x + Z_t + \int_0^t \nu_{D_2} (Y_s) \, dL_s^Y,
\]

where

\[
Z_t = \int_0^t G(Y_s - X_s) \, dW_s
\]

and \(L_t^Y = L_t^0 + L_t^a\) is a continuous nondecreasing process which increases only when \(x + W_t \in \{0, a\}\), that is only when \(Y_t \in \partial D_2\). 

\[
\square
\]

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3.2 The case of polygonal domains

In this section we will consider the case when $D_2 \subset D_1 \subset \mathbb{R}^d$ are polygonal domains (domains bounded by hyperplanes in $\mathbb{R}^d$). From Remark 3.4 it follows that we can consider $D_1 = \mathbb{R}^d$ and therefore it suffices to prove the existence of a strong solution of the following system

\begin{align*}
X_t &= X_0 + W_t \\
Y_t &= Y_0 + Z_t + \int_0^t \nu_{D_2}(Y_s) \, dL_s^Y \\
Z_t &= \int_0^t G(Y_t - X_s) \, dW_s
\end{align*}

or equivalently of the equation

\begin{equation}
Z_t = \int_0^t G \left( Y_0 + Z_s - X_0 - W_s \right) \, dW_s, \tag{3.15}
\end{equation}

where $W_t$ is a $d$-dimensional Brownian motion starting at $W_0 = 0$ and $X_0 = Y_0 \in D_2$.

The construction relies on the following skew product representation of Brownian motion in spherical coordinates:

\begin{equation}
X_t = R_t \Theta_t^d, \tag{3.16}
\end{equation}

where $R_t = \|X_t\| \in \text{BES}(d)$ is a Bessel process of order $d$ and $\Theta_t \in \text{BM}(S^{d-1})$ is an independent Brownian motion on the unit sphere $S^{d-1}$ in $\mathbb{R}^d$, run at speed

\begin{equation}
\sigma_t = \int_0^t \frac{1}{R_s^2} ds, \tag{3.17}
\end{equation}

which depends only on $R_t$.

Remark 3.7. One way to construct the Brownian motion $\Theta_t = \Theta_t^{d-1}$ on the unit sphere $S^{d-1} \subset \mathbb{R}^d$ is to proceed inductively on $d \geq 2$, using the following skew product representation of Brownian motion on the sphere $\Theta_t^{d-1} \in S^{d-1}$ (see [15]):

\begin{equation}
\Theta_t^{d-1} = \left( \cos \theta_t^1, \sin \theta_t^1 \Theta_t^{d-2} \right),
\end{equation}

where $\theta^1 \in \text{LEG}(d-1)$ is a Legendre process of order $d - 1$ on $[0, \pi]$, and $\Theta_t^{d-2} \in S^{d-2}$ is an independent Brownian motion on $S^{d-2}$, run at speed

\begin{equation}
\alpha_t = \int_0^t \frac{1}{\sin^2 \theta_s^1} ds.
\end{equation}

Therefore, if $\theta^1, \ldots, \theta_t^{d-1}$ are independent processes, with $\theta^i \in \text{LEG}(d - i)$ on $[0, \pi]$ for $i = 1, \ldots, d - 2$, and $\theta_t^{d-1}$ is a 1-dimensional Brownian (note that $\Theta_0^{d-1} = \left( \cos \theta_0^1, \sin \theta_0^1 \right) \in S^1$ is a Brownian motion on $S^1$), Brownian motion $\Theta_t^{d-1}$ on the unit sphere $S^{d-1} \subset \mathbb{R}^d$ is given by

\begin{equation}
\Theta_t^{d-1} = \left( \cos \theta_t^1, \sin \theta_t^1 \cos \theta_t^2, \sin \theta_t^1 \sin \theta_t^2 \cos \theta_t^3, \ldots, \sin \theta_t^1 \cdots \sin \theta_t^{d-1} \sin \theta_t^{d-1} \right),
\end{equation}
or by
\[ \Theta_t^{d-1} = (\theta_t^1, \ldots, \theta_t^{d-2}, \Theta_t^{d-1}) \]  \hspace{1cm} (3.18)
in spherical coordinates.

To construct the solution of (3.12) – (3.14), we first consider the case when \( D_2 \) is a half-space \( \mathcal{H}_d^+ = \{ (z^1, \ldots, z^d) \in \mathbb{R}^d : z^d > 0 \} \).

Given an angle \( \varphi \in \mathbb{R} \), we introduce the rotation matrix \( R(\varphi) \in M_{d \times d} \) which leaves invariant the first \( d-2 \) coordinates and rotates clockwise by the angle \( \alpha \) the remaining 2 coordinates, that is
\[
R(\alpha) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & 1 & 0 & 0 \\
0 & \cdots & \cos \varphi & -\sin \varphi \\
0 & \cdots & \sin \varphi & \cos \varphi
\end{pmatrix}.
\]  \hspace{1cm} (3.19)

We have the following:

Lemma 3.8. Let \( D_2 = \mathcal{H}_d^+ = \{ (z^1, \ldots, z^d) \in \mathbb{R}^d : z^d > 0 \} \) and assume that
\[ Y_0 = R(\varphi_0) X_0 \]  \hspace{1cm} (3.20)
for some \( \varphi_0 \in \mathbb{R} \).

Consider the reflecting Brownian motion \( \tilde{\theta}_t^{d-1} \) on \([0, \pi]\) with driving Brownian motion \( \theta_t^{d-1} \), where \( \theta_t^{d-1} \) is the \((d-1)\) spherical coordinate of \( G(Y_0 - X_0) X_t \), given by (3.16) – (3.18) above, that is:
\[ \tilde{\theta}_t^{d-1} = \theta_t^{d-1} + L_0^\varphi (\tilde{\theta}_t^{d-1}) - L_\pi (\tilde{\theta}_t^{d-1}), \quad t \geq 0, \]
and \( L_0^\varphi (\tilde{\theta}_t^{d-1}), L_\pi (\tilde{\theta}_t^{d-1}) \) represent the local times of \( \tilde{\theta}_t^{d-1} \) at 0, respectively at \( \pi \).

A strong solution of the system (3.12) – (3.14) is explicitly given by
\[
Y_t = \begin{cases}
R(\varphi_t) G(Y_0 - X_0) X_t, & t < \xi \\
X_{t\mid \xi}, & t \geq \xi
\end{cases}
\]  \hspace{1cm} (3.21)
where \( \xi = \inf \{ t > 0 : X_t = Y_t \} \) is the coupling time, the rotation angle \( \varphi_t \) is given by
\[ \varphi_t = L_0^\varphi (\tilde{\theta}_t^{d-1}) - L_\pi (\tilde{\theta}_t^{d-1}), \quad t \geq 0, \]
and for \( z = (z^1, z^2, \ldots, z^d) \in \mathbb{R}^d \) we denoted by \( |z|_d = (z^1, z^2, \ldots, |z|^d) \).

Proof. Recall that for \( m \in \mathbb{R}^d - \{0\} \), \( G(m) \nu \) denotes the mirror image of \( \nu \in \mathbb{R}^d \) with respect to the hyperplane through the origin perpendicular to \( m \).

By Itô formula, we have
\[ Y_{t\wedge \xi} = Y_0 + \int_0^{t\wedge \xi} R(\varphi_s) G(Y_0 - X_0) dX_s + \int_0^{t\wedge \xi} R(\varphi_s + \frac{\pi}{2}) G(Y_0 - X_0) dL_s. \]  \hspace{1cm} (3.22)
Figure 1: The mirror coupling of a free Brownian motion $X_t$ and a reflecting Brownian motion $Y_t$ in the half-space $\mathcal{H}_d^+$. 

Note that the composition $R \circ G$ (a symmetry followed by a rotation) is a symmetry, and since $\|Y_t\| = \|X_t\|$ for all $t \geq 0$, it follows that $X_t$ and $Y_t$ are symmetric with respect to a hyperplane passing through the origin for all $t \leq \xi$. Therefore, from the definition (3.5) of the function $G$ it follows that we have $Y_t = G(Y_t - X_t)X_t$ for all $t \leq \xi$.

Also note that when $L^0_t(\bar{\theta}^{-1})$ increases, $Y_s \in \partial D_2$ and we have

$$R\left(\varphi + \frac{\pi}{2}\right)G(Y_0 - X_0)X_s = R\left(\pi \right)Y_s = \nu_{D_2}(Y_s),$$

and if $L^\pi_t(\bar{\theta}^{-1})$ increases, $Y_s \in \partial D_2$ and we have

$$R\left(\varphi + \frac{\pi}{2}\right)G(Y_0 - X_0)X_s = R\left(\pi \right)Y_s = -\nu_{D_2}(Y_s).$$

It follows that the relation (3.22) can be written in the equivalent form

$$Y_t \wedge \xi = Y_0 + \int_0^{t \wedge \xi} G(Y_s - X_s) \, dX_s + \int_0^{t \wedge \xi} \nu_{D_2}(Y_s) \, dL^Y_s,$$

where $L^Y_t = L^0_t(\bar{\theta}^{-1}) + L^\pi_t(\bar{\theta}^{-1})$ is a continuous non-decreasing process which increases only when $Y_t \in \partial D_2$, and therefore $Y_t$ given by (3.21) is a strong solution of the system (3.12) – (3.14) for $t \leq \xi$.

For $t \geq \xi$, we have $Y_t = |X_t| = (X^1_t, X^2_t, \ldots, X^d_t)$, and proceeding similarly to the 1-dimensional
case, by Tanaka formula we obtain:

\[
Y_{t \wedge \xi} = Y_{\xi} + \int_{\xi}^{t \wedge \xi} (1, \ldots, 1, \text{sgn} (X^d_s)) \, dX_s + \int_{\xi}^{t \wedge \xi} (0, \ldots, 0, 1) L^0_t (X^d) \tag{3.23}
\]

\[
= Y_{\xi} + \int_{\xi}^{t \wedge \xi} G (Y_s - X_s) \, dX_s + \int_{\xi}^{t \wedge \xi} \nu_{D_2} (Y_s) L^Y_t,
\]

since in this case

\[
G (Y_s - X_s) = \begin{cases}
(1, \ldots, 1, +1), & X_s = Y_s \\
(1, \ldots, 1, -1), & X_s \neq Y_s
\end{cases}
\]

\[
= \begin{cases}
(1, \ldots, 1, +1), & X^d_s \geq 0 \\
(1, \ldots, 1, -1), & X^d_s < 0
\end{cases}
\]

\[
= (1, \ldots, 1, \text{sgn} (X^d_s)).
\]

The process \(L^Y_t = L^0_t (X^d)\) in (3.23) is a continuous non-decreasing process which increases only when \(Y_t \in \partial D_2\) \((L^0_t (X^d))\) represents the local time at 0 of the last cartesian coordinate \(X^d\) of \(X\), which shows that \(Y_t\) also solves (3.12) – (3.14) for \(t \geq \xi\), and therefore \(Y_t\) is a strong solution of (3.12) – (3.14) for \(t \geq 0\), concluding the proof.

Consider now the case of a general polygonal domain \(D_2 \subset \mathbb{R}^d\). We will show that a strong solution of the system (3.12) – (3.14) can be constructed from the previous lemma by choosing the appropriate coordinate system.

Consider the times \((\sigma_n)_{n \geq 0}\) at which the solution \(Y_t\) hits different bounding hyperplanes of \(\partial D_2\), that is \(\sigma_0 = \inf \{s \geq 0 : Y_s \in \partial D_2\}\) and inductively

\[
\sigma_{n+1} = \inf \left\{t \geq \sigma_n : Y_t \in \partial D_2 \text{ and } Y_t \text{ belongs to hyperplanes of } \partial D_2 \text{ different from } \partial D_2 \right\}, \quad n \geq 0.
\]  

If \(X_0 = Y_0 \in \partial D_2\) belong to a certain bounding hyperplane of \(D_2\), we can chose the coordinate system so that this hyperplane is \(\mathcal{H}_d = \{(z^1, \ldots, z^d) \in \mathbb{R}^d : z^d = 0\}\) and \(D_2 \subset \mathcal{H}_d^+\), and we let \(\mathcal{H}_d\) be any bounding hyperplane of \(D_2\) otherwise.

By Lemma 3.8 it follows that on the time interval \([\sigma_0, \sigma_1)\), the strong solution of (3.12) – (3.14) is given explicitly by (3.21).

If \(\sigma_1 < \infty\), we distinguish two cases: \(X_{\sigma_1} = Y_{\sigma_1}\), and \(X_{\sigma_1} \neq Y_{\sigma_1}\). Let \(\mathcal{H}\) denote the bounding hyperplane of \(D\) which contains \(Y_{\sigma_1}\), and let \(\nu_{\mathcal{H}}\) denote the unit normal to \(\mathcal{H}\) pointing inside \(D_2\).

If \(X_{\sigma_1} = Y_{\sigma_1} \in \mathcal{H}\), choosing again the coordinate system conveniently, we may assume that \(\mathcal{H}\) is the hyperplane \(\mathcal{H}_d = \{(z^1, \ldots, z^d) \in \mathbb{R}^d : z^d = 0\}\), and on the time interval \([\sigma_1, \sigma_2)\) the coupling \((X_{\sigma_1 + t}, Y_{\sigma_1 + t})_{t \in [0, \sigma_2 - \sigma_1]}\) is given again by Lemma 3.8.

If \(X_{\sigma_1} \neq Y_{\sigma_1} \in \mathcal{H}\), in order to apply Lemma 3.8 we have to show that we can choose the coordinate system so that the condition (3.20) holds. If \(Y_{\sigma_1} - X_{\sigma_1}\) is a vector perpendicular to \(\mathcal{H}\), by choosing

\[1\text{Since 2-dimensional Brownian motion does not hit points a.s., the } d\text{-dimensional Brownian motion } Y_t\text{ does not hit the edges of } D_2 ((d - 2)\text{-dimensional hyperplanes in } \mathbb{R}^d)\text{ a.s., thus there is no ambiguity in the definition.}
the coordinate system so that $\mathcal{H} = \mathcal{H}_d = \{(z^1, \ldots, z^d) \in \mathbb{R}^d : z^d = 0\}$, the problem reduces to the 1-dimensional case (the first $d-1$ coordinates of $X$ and $Y$ are the same), and it can be handled as in Proposition 3.6 by the Tanaka formula. The proof being similar, we omit it.

If $X_{\sigma_1} \neq Y_{\sigma_1} \in \mathcal{H}$ and $Y_{\sigma_1} - X_{\sigma_1}$ is not orthogonal to $\mathcal{H}$, consider $\tilde{X}_{\sigma_1} = \text{pr}_\mathcal{H} X_{\sigma_1}$ the projection of $X_{\sigma_1}$ onto $\mathcal{H}$, and therefore $\tilde{X}_{\sigma_1} \neq Y_{\sigma_1}$. The plane of symmetry of $X_{\sigma_1}$ and $Y_{\sigma_1}$ intersects the line determined by $\tilde{X}_{\sigma_1}$ and $Y_{\sigma_1}$ at a point, and we consider this point as the origin of the coordinate system (note that the intersection cannot be empty, for otherwise the vectors $Y_{\sigma_1} - X_{\sigma_1}$ and $\tilde{X}_{\sigma_1} - \tilde{X}_{\sigma_1}$ were parallel, which is impossible since then $Y_{\sigma_1} - X_{\sigma_1}, Y_{\sigma_1} - \tilde{X}_{\sigma_1}$ and $Y_{\sigma_1} - X_{\sigma_1}, \tilde{X}_{\sigma_1} - \tilde{X}_{\sigma_1}$ were perpendicular pairs of vectors, contradicting $\tilde{X}_{\sigma_1} \neq Y_{\sigma_1}$ – see Figure 2).

![Figure 2: Construction of the appropriate coordinate system.](image)

Choose an orthonormal basis $\{e_1, \ldots, e_d\}$ in $\mathbb{R}^d$ such that $e_d = \nu_\mathcal{H}$ is the normal vector to $\mathcal{H}$ pointing inside $D_2$, $e_{d-1} = \frac{1}{\|Y_{\sigma_1} - X_{\sigma_1}\|} (Y_{\sigma_1} - X_{\sigma_1})$ is a unit vector lying in the 2-dimensional plane determined by the origin and the vectors $e_d$ and $Y_{\sigma_1} - X_{\sigma_1}$, and $\{e_1, \ldots, e_{d-2}\}$ is a completion of $\{e_{d-1}, e_d\}$ to an orthonormal basis in $\mathbb{R}^d$ (see Figure 2).

Note that by the construction, the vectors $e_1, \ldots, e_{d-2}$ are orthogonal to the 2-dimensional hyper-plane containing the origin and the points $X_{\sigma_1}$ and $Y_{\sigma_1}$, and therefore $X_{\sigma_1}$ and $Y_{\sigma_1}$ have the same (zero) first $d-2$ coordinates; also, since $X_{\sigma_1}$ and $Y_{\sigma_1}$ are at the same distance from the origin, it follows that $Y_{\sigma_1}$ can be obtained from $X_{\sigma_1}$ by a rotation which leaves invariant the first $d-2$ coordinates, which shows that the condition (3.20) of Lemma 3.8 is satisfied.

Since by construction the bounding hyperplane $\mathcal{H}$ of $D_2$ at $Y_{\sigma_1}$ is given by $\mathcal{H}_d = \{(z^1, \ldots, z^d) \in \mathbb{R}^d : z^d = 0\}$ and $D_2 \subset \mathcal{H}_d^+ = \{(z^1, \ldots, z^d) \in \mathbb{R}^d : z^d > 0\}$, we can apply Lemma 3.8 to deduce that on the time interval $[\sigma_1, \sigma_2)$ a solution of (3.12) – (3.14) is given by $(X_{\sigma_1+t}, Y_{\sigma_1+t})_{t \in [0, \sigma_2 - \sigma_1]}$.

Repeating the above argument we can construct inductively (in the appropriate coordinate systems) the solution of (3.12) – (3.14) on any time interval $[\sigma_n, \sigma_{n+1})$, $n \geq 1$, and therefore we obtain a strong solution of (3.12) – (3.14) defined for $t \geq 0$.

We summarize the above discussion in the following:

**Theorem 3.9.** If $D_2 \subset \mathbb{R}^d$ is a polygonal domain, for any $X_0 = Y_0 \in \overline{D_2}$, there exists a strong solution of the system (3.12) – (3.14).
Moreover, between successive hits of different bounding hyperplanes of $D_2$ (i.e. on each time interval $[\sigma_n, \sigma_{n+1})$ in the notation above), the solution is given by Lemma 3.8 in the appropriately chosen coordinate system.

We will refer to the solution $(X_t, Y_t)_{t \geq 0}$ constructed in the previous theorem as a mirror coupling of reflecting Brownian motions in $(\mathbb{R}^d, D_2)$ with starting point $X_0 = Y_0 \in \overline{D_2}$.

If $X_t \neq Y_t$, the hyperplane $M_t$ of symmetry between $X_t$ and $Y_t$ (the hyperplane passing through $\frac{X_t + Y_t}{2}$ with normal $m_t = \frac{1}{\| Y_t - X_t \|} (Y_t - X_t)$) will be referred to as the mirror of the coupling. For definiteness, when $X_t = Y_t$ we let $M_t$ denote any hyperplane passing through $X_t = Y_t$, for example we can choose $M_t$ such that it is a left continuous function with respect to $t$.

In the particular case of a convex polygonal domain $D_2$, some of the properties of the mirror coupling are contained in the following:

**Proposition 3.10.** If $D_2 \subset \mathbb{R}^d$ is a convex polygonal domain, for any $X_0 = Y_0 \in \overline{D_2}$, the mirror coupling given by the previous theorem has the following properties:

i) If the reflection takes place in the bounding hyperplane $\mathcal{H}$ of $D_2$ with inward unitary normal $\nu_{\mathcal{H}}$, then the angle $\angle (m_t; \nu_{\mathcal{H}})$ decreases monotonically to zero.

ii) When processes are not coupled, the mirror $M_t$ lies outside $D_2$.

iii) Coupling can take place precisely when $X_t \in \partial D_2$. Moreover, if $X_t \in D_2$, then $X_t = Y_t$.

iv) If $D_\alpha \subset D_\beta$ are two polygonal domains and $(Y_t^\alpha, X_t)$, $(Y_t^\beta, X_t)$ are the corresponding mirror coupling starting from $x \in \overline{D_\alpha}$, for any $t > 0$ we have

$$
\sup_{s \leq t} \| Y_s^\alpha - Y_s^\beta \| \leq \text{Dist} \left( D^\alpha, D^\beta \right) := \max_{x_\alpha \in \partial D_\alpha, x_\beta \in \partial D_\beta} \| x_\alpha - x_\beta \|.
$$

(3.25)

**Proof.**

i) In the notation of Theorem 3.9 on the time interval $[\sigma_0, \sigma_1)$ we have $Y_t = X_t$, so $\angle (m_t, \nu_{\mathcal{H}}) = 0$ and therefore the claim is verified in this case.

On an arbitrary time interval $[\sigma_n, \sigma_{n+1})$, in the appropriately chosen coordinate system, $Y_t$ is given by Lemma 3.8. For $t < \xi$, $Y_t$ is given by the rotation $R(\varphi_t)$ of $G(Y_0 - X_0)X_t$ which leaves invariant the first $(d-2)$ coordinates, and therefore

$$
\angle (m_t, \nu_{\mathcal{H}}) = \angle (m_0, \nu_{\mathcal{H}}) + \frac{L^0_t - L^\pi_t}{2},
$$

which proves the claim in this case (note that before the coupling time $\xi$ only one of the non-decreasing processes $L^0_t$ and $L^\pi_t$ is not identically zero).

Since for $t \geq \xi$ we have $Y_t = \left( X_t^1, \ldots, |X_t^d | \right)$, we have $\angle (m_t, \nu_{\mathcal{H}}) = 0$ which concludes the proof of the claim.

ii) On the time interval $[\sigma_0, \sigma_1)$ the processes are coupled, so there is nothing to prove in this case.

On the time interval $[\sigma_1, \sigma_2)$, in the appropriately chosen coordinate system we have $Y_t = \left( X_t^1, \ldots, |X_t^d | \right)$, thus the mirror $M_t$ coincides with the boundary hyperplane $\mathcal{H}_d = \partial D_2$.
\( \{ (z^1, \ldots, z^d) \in \mathbb{R}^d : z^d = 0 \} \) of \( D_2 \) where the reflection takes place, and therefore \( M_t \cap D_2 = \emptyset \) in this case.

Inductively, assume the claim is true for \( t < \sigma_n \). By continuity, \( M_{\sigma_n} \cap D_2 = \emptyset \), thus \( D_2 \) lies on one side of \( M_{\sigma_n} \). By the previous proof, the angle \( \angle (m_t, v_{\mathcal{K}}) \) between \( m_t \) and the inward unit normal \( v_{\mathcal{K}} \) to bounding hyperplane \( \mathcal{K} \) of \( D_2 \) where the reflection takes place decreases to zero. Since \( D_2 \) is a convex domain, it follows that on the time interval \( [\sigma_n, \sigma_{n+1}) \) we have \( M_t \cap D_2 = \emptyset \), concluding the proof.

iii) The first part of the claim follows from the previous proof (when the processes are not coupled, the mirror (hence \( X_t \)) lies outside \( D_2 \); by continuity, it follows that at the coupling time \( \xi \) we must have \( X_\xi = Y_\xi \in \partial D_2 \).

To prove the second part of the claim, consider an arbitrary time interval \( [\sigma_n, \sigma_{n+1}) \) between two successive hits of \( Y_t \) to different bounding hyperplanes of \( D_2 \). In the appropriately chosen coordinate system, \( Y_t \) is given by Lemma 3.8. After the coupling time \( \xi \), \( Y_t \) is given by \( Y_t = (X^1_t, \ldots, X^d_t) \), and therefore if \( X_t \in D_2 \) (thus \( X_t^d \geq 0 \)) we have \( Y_t = (X^1_t, \ldots, X^d_t) = X_t \), concluding the proof.

iv) Let \( M_t^\alpha \) and \( M_t^\beta \) denote the mirrors of the coupling in \( D^\alpha \), respectively \( D^\beta \), with the same driving Brownian motion \( X_t \).

Since \( Y_t^\alpha \) and \( X_t \) are symmetric with respect to \( M_t^\alpha \), and \( Y_t^\beta \) and \( X_t \) are symmetric with respect to \( M_t^\beta \), it follows that \( Y_t^\beta \) is obtained from \( Y_t^\beta \) by a rotation which leaves invariant the hyperplane \( M_t^\alpha \cap M_t^\beta \), or by a translation by a vector orthogonal to \( M_t^\alpha \) (in the case when \( M_t^\alpha \) and \( M_t^\beta \) are parallel).

The angle of rotation (respectively the vector of translation) is altered only when either \( Y_t^\alpha \) or \( Y_t^\beta \) are on the boundary of \( D^\alpha \), respectively \( D^\beta \). Since \( D^\alpha \subset D^\beta \) are convex domains, the angle of rotation (respectively the vector of translation) decreases when \( Y_t^\beta \in D^\beta \) or when \( Y_t^\alpha \in \partial D^\alpha \) and \( \left( Y_t^\beta - Y_t^\alpha \right) \cdot v_{D^\alpha} (Y_t^\alpha) > 0 \) (in these cases \( Y_t^\beta \) and \( Y_t^\alpha \) receive a push such that the distance \( \|Y_t^\alpha - Y_t^\beta\| \) is decreased), thus the maximum distance \( \|Y_t^\alpha - Y_t^\beta\| \) is attained when \( Y_t^\alpha \in \partial D^\alpha \) and \( \left( Y_t^\beta - Y_t^\alpha \right) \cdot v_{D^\alpha} (Y_t^\alpha) \leq 0 \), and the formula follows.

\[ \Box \]

4 The proof of Theorem 3.1

By Remark 3.4, it suffices to consider the case when \( D_1 = \mathbb{R}^d \) and \( D_2 \subset \mathbb{R}^d \) is a convex bounded domain with smooth boundary. To simplify the notation, we will drop the index and write \( D \) for \( D_2 \) in the sequel.

Let \( (D_n)_{n \geq 1} \) be an increasing sequence of convex polygonal domains in \( \mathbb{R}^d \) with \( \overline{D_n} \subset D_{n+1} \) and \( \bigcup_{n \geq 1} D_n = D \).

Consider \( (Y_t^m, X_t)_{t \geq 0} \), a sequence of mirror couplings in \( (D_n, \mathbb{R}^d) \) with starting point \( x \in D_1 \) and driving Brownian motion \( (W_t)_{t \geq 0} \) with \( W_0 = 0 \), given by Theorem 3.9.

By Proposition 3.10, for any \( t > 0 \) we have

\[
\sup_{s \leq t} |Y^m_s - Y^n_s| \leq \text{Dist} (D_n, D_m) = \max_{x_n \in \partial D_n, x_m \in \partial D_m} |x_n - x_m| \xrightarrow{n,m \to \infty} 0,
\]

where

\[
(x_m - x_n) \cdot v_{\partial D_n} (x_n) \leq 0
\]

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hence $Y_t^n$ converges a.s. in the uniform topology to a continuous process $Y_t$.

Since $(Y^n)_{n \geq 1}$ are reflecting Brownian motions in $(D_n)_{n \geq 1}$ and $D_n \nearrow D$, the law of $Y_t$ is that of a reflecting Brownian motion in $D$, that is $Y_t$ is a reflecting Brownian motion in $D$ starting at $x \in D$ (see [8]). Also note that since $Y_t^n$ are adapted to the filtration $\mathcal{F}_t = (\mathcal{F}_t)_{t \geq 0}$ generated by the Brownian motion $W_t$, so is $Y_t$.

By construction, the driving Brownian motion $Z_t^n$ of $Y_t^n$ satisfies

$$Z_t^n = \int_0^t G \left( Y_t^n - X_t \right) \, dW_t, \quad t \geq 0.$$ 

Consider the process

$$Z_t = \int_0^t G \left( Y_s - X_s \right) \, dW_s,$$

and note that since $Y$ is $\mathcal{F}_W$-adapted and $||G|| = 1$, by Lévy’s characterization of Brownian motion, $Z_t$ is a free $d$-dimensional Brownian motion starting at $Z_0 = 0$, also adapted to the filtration $\mathcal{F}_W$.

We will show that $Z_t$ is the driving process of the reflecting Brownian motion $Y_t$, that is, we will show that

$$Y_t = x + Z_t + L^Y_t = x + \int_0^t G \left( Y_s - X_s \right) \, dW_s + L^Y_t, \quad t \geq 0.$$ 

Note that the mapping $z \mapsto G(z)$ is continuous with respect to the norm $||A|| = \left\| a_{ij} \right\| = \sum_{i,j=1}^d a_{ij}^2$ of $d \times d$ matrices at all points $z \in \mathbb{R}^d - \{0\}$, hence $G \left( Y^n_s - X_s \right) \to G \left( Y_s - X_s \right)$ if $Y^n_s - X_s \neq 0$. If $Y_s - X_s = 0$, then either $Y_s = X_s \in D$ or $Y_s = X_s \in \partial D$.

If $Y_s = X_s \in D$, since $D_n \nearrow D$, there exists $N \geq 1$ such that $X_s \in D_N$, hence $X_s \in D_n$ for all $n \geq N$. By Proposition 3.10 it follows that $Y^n_s = X_s$ for all $n \geq N$, hence in this case we also have $G \left( Y^n_s - X_s \right) = G(0) \to G(0) = G \left( Y_s - X_s \right)$.

If $Y_s = X_s \in \partial D$, since $\overline{D}_n \subset D$ we have $Y^n_s - X_s \neq 0$, and therefore by the definition (3.5) of $G$ we
have:

\[
\int_0^t \left\| G \left( Y^n_s - X_s \right) - G \left( Y_s - X_s \right) \right\|^2 1_{Y^n_s \in \partial D} ds
\]

\[
= \int_0^t \left\| H \left( \frac{Y^n_s - X_s}{\left\| Y^n_s - X_s \right\|} \right) - I \right\|^2 1_{Y^n_s \in \partial D} ds
\]

\[
= \int_0^t \left\| I - 2 \frac{Y^n_s - X_s}{\left\| Y^n_s - X_s \right\|} \left( \frac{Y^n_s - X_s}{\left\| Y^n_s - X_s \right\|} \right)' - I \right\|^2 1_{Y^n_s \in \partial D} ds
\]

\[
= \int_0^t \left\| 2 \frac{Y^n_s - X_s}{\left\| Y^n_s - X_s \right\|} \left( \frac{Y^n_s - X_s}{\left\| Y^n_s - X_s \right\|} \right)' \right\|^2 1_{Y^n_s \in \partial D} ds
\]

\[
= 4 \int_0^t 1_{Y^n_s \in \partial D} ds
\]

\[
\leq 4 \int_0^t 1_{\partial D} (Y^n_s) ds
\]

\[
= 0,
\]

since $Y_t$ is a reflecting Brownian motion in $D$, and therefore it spends zero Lebesgue time on the boundary of $D$.

Since $\|G\| = 1$, using the above and the bounded convergence theorem we obtain

\[
\lim_{n \to \infty} \int_0^t \left\| G \left( Y^n_s - X_s \right) - G \left( Y_s - X_s \right) \right\|^2 ds = 0,
\]

and therefore by Doob's inequality it follows that

\[
E \sup_{t \leq t} \|Z^n_t - Z_t\|^2 \leq cE \|Z^n_t - Z_t\|^2 \leq cE \int_0^t \left\| G \left( Y^n_s - X_s \right) - G \left( Y_s - X_s \right) \right\|^2 ds \to 0,
\]

for any $t \geq 0$, which shows that $Z^n_t$ converges uniformly on compact sets to $Z_t = \int_0^t G \left( Y_s - X_s \right) dW_s$.

By construction, $Z^n_t$ is the driving Brownian motion for $Y^n_t$, that is

\[
Y^n_t = x + Z^n_t + \int_0^t v_{D^n} \left( Y^n_s \right) dL^n_s,
\]

and passing to the limit with $n \to \infty$ we obtain

\[
Y_t = x + Z_t + A_t = x + \int_0^t G \left( Y_s - X_s \right) dW_s + A_t, \quad t \geq 0,
\]

where $A_t = \lim_{n \to \infty} \int_0^t v_{D^n} \left( Y^n_s \right) dL^n_s$. 

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It remains to show that $A_t$ is a process of bounded variation. For an arbitrary partition $0 = t_0 < t_1 < \ldots t_l = t$ of $[0, t]$ we have

$$
E \sum_{i=1}^l \|A_{t_i} - A_{t_{i-1}}\| = \lim_{n \to \infty} E \sum_{i=1}^l \left\| \int_{t_{i-1}}^{t_i} \nu_{D_n} (Y^n_s) \, dL^n_s \right\|
$$

$$
\leq \lim \sup \left( E \int_{t_0}^t \int_{\partial D_n} p_{D_n} (s, x, y) \sigma_n (dy) \, ds \right)
$$

$$
\leq c \sqrt{t},
$$

where $\sigma_n$ is the surface measure on $\partial D_n$, and the last inequality above follows from the estimates in [5] on the Neumann heat kernels $p_{D_n} (t, x, y)$ (see the remarks preceding Theorem 2.1 and the proof of Theorem 2.4 in [7]).

From the above it follows that $A_t = Y_t - x - Z_t$ is a continuous, $\mathcal{F}^W$-adapted process (since $Y_t$, $Z_t$ are continuous, $\mathcal{F}^W$-adapted processes) of bounded variation.

By the uniqueness in the Doob-Meyer semimartingale decomposition of the reflecting Brownian motion $Y_t$ in $D$, it follows that

$$
A_t = \int_0^t \nu_D (Y_s) \, dL^Y_s, \quad t \geq 0,
$$

where $L^Y$ is the local time of $Y$ on the boundary $\partial D$. It follows that the reflecting Brownian motion $Y_t$ in $D$ constructed above is a strong solution to

$$
Y_t = x + \int_0^t G (Y_s - X_s) \, dW_s + \int_0^t \nu_D (Y_s) \, dL^Y_s, \quad t \geq 0,
$$

or equivalent, the driving Brownian motion $Z_t = \int_0^t G (Y_s - X_s) \, dW_s$ of $Y_t$ is a strong solution to

$$
Z_t = \int_0^t G (\tilde{Y} (y + Z)_s - X_s) \, dW_s, \quad t \geq 0,
$$

concluding the proof of Theorem 3.1.

### 5 Extensions and applications

As an application of the construction of mirror coupling, we will present a unifying proof of the two most important results on Chavel’s conjecture.

It is not difficult to prove that the Dirichlet heat kernel is an increasing function with respect to the domain. Since for the Neumann heat kernel $p_D (t, x, y)$ of a smooth bounded domain $D \subset \mathbb{R}^d$ we have

$$
\lim_{t \to \infty} p_D (t, x, y) = \frac{1}{\text{vol}(D)},
$$

the monotonicity in the case of the Neumann heat kernel should be reversed.

The above observation was conjectured by Isaac Chavel ([12]), as follows:
Conjecture 5.1 (Chavel's conjecture, [12]). Let $D_{1,2} \subset \mathbb{R}^d$ be smooth bounded convex domains in $\mathbb{R}^d$, $d \geq 1$, and let $p_{D_1}(t,x,y)$, $p_{D_2}(t,x,y)$ denote the Neumann heat kernels in $D_1$, respectively $D_2$. If $D_2 \subset D_1$, then
\begin{equation}
    p_{D_1}(t,x,y) \leq p_{D_2}(t,x,y),
\end{equation}
for any $t \geq 0$ and $x, y \in D_1$.

Remark 5.2. The smoothness assumption in the above conjecture is meant to insure the a.e. existence of the inward unit normal to the boundaries of $D_1$ and $D_2$, that is the boundaries should have a locally differentiable parametrization. Requiring that the boundary of the domain is of class $C^{1,\alpha}$ ($0 < \alpha < 1$) is a convenient hypothesis on the smoothness of the domains $D_{1,2}$.

In order to simplify the proof, we will assume that $D_{1,2}$ are smooth $C^2$ domains (the proof can be extended to a more general setup, by approximating $D_{1,2}$ by less smooth domains).

Among the positive results on Chavel conjecture, the most general known results (and perhaps the easiest to use in practice) are due to I. Chavel ([12]) and W. Kendall ([16]), and they show that if there exists a ball $B$ centered at either $x$ or $y$ such that $D_2 \subset B \subset D_1$, then the inequality (5.1) in Chavel's conjecture holds for any $t > 0$.

While there are also other positive results which suggest that Chavel's conjecture is true for certain classes of domains (see for example [11], [14]), in [4] R. Bass and K. Burdzy showed that Chavel's conjecture does not hold in its full generality (i.e. without additional hypotheses).

We note that both the proof of Chavel (the case when $D_1$ is a ball centered at either $x$ or $y$) and Kendall (the case when $D_2$ is a ball centered at either $x$ or $y$) relies in an essential way that one of the domains is a ball: the first uses an integration by parts technique, while the later uses a coupling argument of the radial parts of Brownian motion, and none of these proofs seem to be easily applicable to the other case.

Using the mirror coupling, we can derive a simple, unifying proof of these two important results, as follows:

Theorem 5.3. Let $D_2 \subset D_1 \subset \mathbb{R}^d$ be smooth bounded domains and assume that $D_2$ is convex. If for $x, y \in D_2$ there exists a ball $B$ centered at either $x$ or $y$ such that $D_2 \subset B \subset D_1$, then for all $t \geq 0$ we have
\begin{equation}
    p_{D_1}(t,x,y) \leq p_{D_2}(t,x,y).
\end{equation}

Proof. Consider $x, y \in D_2$ arbitrarily fixed and assume that $D_2 \subset B = B(y,R) \subset D_1$ for some $R > 0$. By eventually approximating the convex domain $D_2$ by convex polygonal domains, it suffices to prove the claim in the case when $D_2$ is a convex polygonal domain.

Let $(X_t, Y_t)$ be a mirror coupling of reflecting Brownian motions in $(D_1, D_2)$ starting at $x \in D_2$. The idea of the proof is to show that for all times $t \geq 0$, $Y_t$ is at a distance from $y$ no greater than the distance from $X_t$ to $y$.

Let $t_0 \geq 0$ be a time when the processes are at the same distance from $y$, and let $t_1 \geq t_0$ be the first time after $t_0$ when the process $X_t$ hits the boundary of $D_1$.

Note that by the ball condition we have $\|X_t - y\| = R > \|Y_t - y\|$ for any $t \geq 0$, and in particular this holds for $t = t_1$. Since the processes $X_t$ and $Y_t$ are continuous, the distances from $X_t$ and $Y_t$ to $y$ are continuous functions of $t$, and therefore in order to prove the claim it suffices to show that
\[\|Y_t - y\| \leq \|X_t - y\| \text{ for all } t \in [t_0, t_1].\] Also note that on the time interval \([t_0, t_1]\) the process \(X_t\) behaves like a free Brownian motion.

We distinguish the following cases:

i) The processes are coupled at time \(t_0\) (i.e. \(X_{t_0} = Y_{t_0}\));

In this case, the distances from \(X_t\) and \(Y_t\) to \(y\) will remain equal until the first time when the processes hit the boundary of \(D_2\). Since on the time interval \([t_0, t_1]\) the process \(X_t\) behaves like a free Brownian motion, by Proposition 3.10 ii) it follows that when processes are not coupled, the mirror \(M_t\) of the coupling lies outside the domain \(D_2\). Since the domain \(D_2\) is assumed convex, this shows in particular that the mirror \(M_t\) of the coupling cannot separate the points \(Y_t\) and \(y\), and therefore the distance from \(Y_t\) to \(y\) is smaller than or equal to the distance from \(X_t\) to \(y\), for all \(t \in [t_0, t_1]\).

ii) The processes are decoupled at time \(t_0\);

In this case, since \(|Y_{t_0} - y| = |X_{t_0} - y|\) and \(X_{t_0} \neq Y_{t_0}\), the hyperplane \(M_{t_0}\) of symmetry between \(X_{t_0}\) and \(Y_{t_0}\) passes through the point \(y\), so \(M_{t_0}\) does not separate the points \(Y_{t_0}\) and \(y\).

The processes \(X_t\) and \(Y_t\) will remain at the same distance from \(y\) until the first time when \(Y_t \in \partial D_2\). Since on the time interval \([t_0, t_1]\) the process \(X_t\) behaves like a free Brownian motion, by Theorem 3.9 it follows that between successive hits of different boundary hyperplanes of \(D_2\), the mirror \(M_t\) of the coupling describes a rotation which leaves invariant \(d - 2\) coordinate axes. Moreover, by Proposition 3.10 the rotation is directed in such a way that the angle \(\angle(m_t, v)\) between the normal \(m_t = \frac{1}{\|Y_t - X_t\|} (Y_t - X_t)\) of \(M_t\) and the inner normal \(v\) of the bounding hyperplane \(\mathcal{H}\) of \(D_2\) where the reflection takes place decreases monotonically to zero (see Figure 1).

Since the hyperplane \(M_{t_0}\) does not separate the points \(Y_{t_0}\) and \(y\), simple geometric considerations show that \(M_t\) will not separate the points \(Y_t\) and \(y\) for all \(t \in [t_0, t_1]\), and therefore \(|Y_t - y| \leq |X_t - y|\) for all \(t \in [t_0, t_1]\), concluding the proof of the claim.

We showed that for any \(t \geq 0\) we have \(|Y_t - y| \leq |X_t - y|\), and therefore

\[P^x(\|X_t - y\| < \epsilon) \leq P^x(\|Y_t - y\| < \epsilon),\]

for any \(\epsilon > 0\) and \(t \geq 0\).

Dividing the above inequality by the volume of the ball \(B(y, \epsilon)\) and passing to the limit with \(\epsilon \searrow 0\), from the continuity of the transition density of the reflecting Brownian motion in the space variable we obtain

\[p_{D_1}(t, x, y) \leq p_{D_2}(t, x, y), \quad t \geq 0,\]

concluding the proof of the theorem.

As also pointed out by Kendall in [16], we note that in the above theorem the convexity of the larger domain \(D_1\) is not needed in order to derive the validity of condition (5.1) in Chavel’s conjecture. We can also replace the hypothesis on the convexity of the smaller domain \(D_2\) by the weaker hypothesis that \(D_2\) is a star-shaped domain with respect to either \(x\) or \(y\), as follows:

**Theorem 5.4.** Let \(D_2 \subset D_1 \subset \mathbb{R}^d\) be smooth bounded domains. If for \(x, y \in D_2\) there exists a ball \(B\) centered at either \(x\) or \(y\) such that \(D_2 \subset B \subset D_1\) and \(D_2\) is star-shaped with respect to the center of the ball, then for all \(t \geq 0\) we have

\[p_{D_1}(t, x, y) \leq p_{D_2}(t, x, y).\] (5.3)
Proof. We will present an analytic proof which parallels the geometric proof of the previous theorem. Consider \(x, y \in D_2\) arbitrarily fixed and assume that \(D_2 \subset B = B(y, R) \subset D_1\) for some \(R > 0\) and \(D_2\) is a star-shaped domain with respect to \(y\).

By eventually approximating \(D_2\) with star-shaped polygonal domains, it suffices to prove the claim in the case when \(D_2\) is a polygonal star-shaped domain.

Let \((X_t, Y_t)\) be a mirror coupling of reflecting Brownian motions in \((D_1, D_2)\) starting at \(x \in D_2\). The idea of the proof is to show that for all times \(t \geq 0\), \(Y_t\) is at a distance from \(y\) no greater than the distance from \(X_t\) to \(y\).

We can reduce the proof to the case when \(D_1 = \mathbb{R}^d\) as follows. Consider the sequences of stopping times \((\xi_n)_{n \geq 1}\) and \((\tau_n)_{n \geq 1}\) defined inductively by

\[
\begin{align*}
\tau_0 &= 0, \\
\xi_n &= \inf \{ t > \tau_{n-1} : X_t \in \partial D_1 \}, \quad n \geq 1, \\
\tau_n &= \inf \{ t > \xi_n : \|X_t - y\| = \|Y_t - y\| \}, \quad n \geq 1.
\end{align*}
\]

Note that by the ball condition we have \(\|X_{\xi_n} - y\| > \|Y_{\xi_n} - y\|\) for any \(n \geq 1\), and therefore \(\|X_t - y\| \geq \|Y_t - y\|\) for any \(n \geq 1\) and any \(t \in [\xi_n, \tau_n]\). In order to prove that the same inequality holds on the intervals \([\tau_n, \xi_{n+1}]\) for \(n \geq 0\), we proceed as follows.

On the set \(\{\tau_n < \infty\}\), the pair \((\tilde{X}_t, \tilde{Y}_t) = (X_{\tau_n+t}, Y_{\tau_n+t})\) defined for \(t \leq \xi_{n+1} - \tau_n\) is a mirror coupling in \((\mathbb{R}^d, D_2)\) with driving Brownian motion \(\tilde{W}_t = W_{\tau_n+t} - W_{\tau_n}\) (and \(\tilde{Z}_t = Z_{\tau_n+t} - Z_{\tau_n}\)), and starting points \((\tilde{X}_0, \tilde{Y}_0) = (X_{\tau_n}, Y_{\tau_n})\) independent of the filtration of \(B_t\) (see Remark 3.4). In order to prove the claim it suffices to show that for any points \(u \in \mathbb{R}^d\) and \(v \in D_2\) with \(\|u - y\| = \|v - y\|\), the mirror coupling \((X_t, Y_t)\) in \((\mathbb{R}^d, D_2)\) with starting points \((X_0, Y_0) = (u, v)\) verifies

\[
\|X_t - y\| \geq \|Y_t - y\|, \quad t \geq 0. \tag{5.4}
\]

Consider therefore a mirror coupling \((X_t, Y_t)\) in \((\mathbb{R}^d, D_2)\) with starting points \((X_0, Y_0) = (u, v)\) in \(\mathbb{R}^d \times D_2\) satisfying \(\|u - y\| = \|v - y\|\).

If \(u = v\), from the construction of the mirror coupling it follows that \(X_t = Y_t\) until the process \(Y_t\) hits the boundary of \(D_2\), and therefore the inequality in (5.4) holds for these values of \(t\). After the process \(Y_t\) hits a bounding hyperplane of \(D_2\), by Lemma 3.8 it follows that in an appropriate coordinate system \(Y_t\) is given by \(Y_t = (X_t^1, \ldots, X_t^{d-1}, |X_t^d|)\), until the time \(\sigma\) when the process \(Y\) hits a different bounding hyperplane of \(D_2\), and therefore the inequality in (5.4) is again verified for the corresponding values of \(t\) (in the chosen coordinate system we must have \(y = (y^1, \ldots, y^d)\) with \(y^d > 0\), and therefore \(\|X_t - y\|^2 - \|Y_t - y\|^2 = 2y^d (|X_t^d| - X_t^d) \geq 0\)). If at time \(\sigma\) the processes are coupled (i.e. \(X_\sigma = Y_\sigma \in \partial D_2\)), we can apply the above argument inductively, and find a time \(\sigma_1\) when the processes are decoupled and \(\|X_t - y\| \geq \|Y_t - y\|\) for all \(t \leq \sigma_1\).

The above discussion shows that without loss of generality we may further reduce the proof of the claim to the case when \((u, v) \in \mathbb{R}^d \times D_2\) with \(u \neq v\) and \(\|u - y\| \geq \|v - y\|\). Also, the above discussion shows that it is enough to prove (5.4) for all values of \(t \leq \zeta\), where \(\zeta = \inf \{ s > 0 : X_s = Y_s \}\) is the first coupling time.
The mirror coupling defined by (3.1) – (3.3) becomes in the case
\[
X_t = u + W_t \quad \text{(5.5)}
\]
\[
Y_t = v + Z_t + \int_0^t v_{D_2}(Y_s) \, dL_s^Y \quad \text{(5.6)}
\]
\[
Z_t = \int_0^t G(Y_s - x_s) \, dW_s \quad \text{(5.7)}
\]
where \(G\) is given by (3.5). In order to prove the claim we will show that
\[
R_t = \|X_t - y\|^2 - \|Y_t - y\|^2 \geq 0, \quad t \leq \zeta, \quad \text{(5.8)}
\]
where \(\zeta\) is the first coupling time.

Using the Itô formula it can be shown that the process \(R_t\) verifies the stochastic differential equation
\[
R_t = R_0 - 2 \int_0^t \frac{Y_s - x_s}{\|Y_s - x_s\|^2} \cdot dW_s - 2 \int_0^t (Y_s - y) \cdot v_{D_2}(Y_s) \, dL_s^Y, \quad t \leq \zeta. \quad \text{(5.9)}
\]
The process \(B_t = -2 \int_0^t \frac{Y_s - x_s}{\|Y_s - x_s\|^2} \cdot dW_s\) is a continuous local martingale on \([0, \zeta]\), with quadratic variation
\[
A_t = 4 \sum_{i=1}^d \int_0^{t \land \zeta} \frac{(Y_{s}^i - x_{s}^i)^2}{\|Y_s - x_s\|^2} \, ds = \int_0^{t \land \zeta} \frac{4}{\|Y_s - x_s\|^2} \, ds, \quad t \geq 0, \quad \text{(5.10)}
\]
and therefore by Lévy’s characterization of Brownian motion it follows that \(\bar{B}_t = B_{\alpha_t \land \zeta}\) is a 1-dimensional Brownian motion (possibly stopped at time \(\zeta\), if \(A_\zeta < \infty\)), where the time change \(\alpha_t = \inf\{s \geq 0 : A_s > t\}\) is the inverse of the nondecreasing process \(A_t\).

Setting \(\bar{X}_t = X_{\alpha_t \land \zeta}, \bar{Y}_t = Y_{\alpha_t \land \zeta}, \bar{R}_t = R_{\alpha_t \land \zeta}\) and \(\bar{L}_t^Y = L_{\alpha_t \land \zeta}^Y\), from (5.9) we obtain
\[
\bar{R}_t = \bar{R}_0 + \int_0^t \bar{R}_s \, d\bar{B}_s - \int_0^t (\bar{Y}_s - y) \cdot v_{D_2}(\bar{Y}_s) \, d\bar{L}_s^Y, \quad t \geq 0. \quad \text{(5.11)}
\]
Since the polygonal domain \(D_2\) is assumed star-shaped with respect to the point \(y\), geometric considerations show that
\[
(z - y) \cdot v_{D_2}(z) \leq 0, \quad \text{(5.12)}
\]
for all the points \(z \in \partial D_2\) for which the inside pointing normal \(v_{D_2}(z)\) at the boundary point \(z\) of \(D_2\) is defined, that is for all points \(z \in \partial D_2\) not lying on the intersection of two bounding hyperplanes of \(D_2\). Since the reflecting Brownian motion \(Y_t\) does not hit the set of these exceptional points with positive probability, we may assume that the above condition is satisfied for all points, and therefore
\[
(\bar{Y}_s - y) \cdot v_{D_2}(\bar{Y}_s) \leq 0 \quad \text{a.s}, \quad \text{(5.13)}
\]
for all times \(s \geq 0\) when \(\bar{Y}_s \in \partial D_2\).

Since \(\bar{L}_s^Y\) is a nondecreasing process of \(t \geq 0\), a standard comparison argument for solutions of stochastic differential equations shows that the solution \(\bar{R}_t\) of (5.11) satisfies \(\bar{R}_t \geq \rho_t\) for all \(t \geq 0\), where \(\rho_t\) is the solution of the stochastic differential equation
\[
\rho_t = \bar{R}_0 + \int_0^t \rho_s \, d\bar{B}_s, \quad t \geq 0. \quad \text{(5.14)}
\]
The last equation has the explicit solution $\rho_t = R_0 e^\tilde{B}_t - \frac{1}{2} t$, and since by hypothesis $R_0 = \|u - y\|^2 - \|v - y\|^2 \geq 0$, we obtain

$$R_{t, \wedge \zeta} = \tilde{R}_t \geq \rho_t = R_0 e^\tilde{B}_t - \frac{1}{2} t \geq 0, \quad t \geq 0,$$

(5.15)

and therefore $R_t = \|X_t - y\|^2 - \|Y_t - y\|^2 \geq 0$ for all $t \leq \zeta$, concluding the proof of the claim.

By the initial remarks, it follows that if $(X_t, Y_t)$ is a mirror coupling in $(D_1, D_2)$ with starting point $X_0 = Y_0 = x$, then

$$\|X_t - y\| \geq \|Y_t - y\|, \quad t \geq 0.$$  

(5.16)

As in the proof of the last theorem, this shows that $p_{D_1}(t, x, y) \leq p_{D_2}(t, x, y)$ for all $t \geq 0$, concluding the proof.

We have chosen to carry out the construction of the mirror coupling in the case of smooth domains with $\overline{D_2} \subset D_1$ and $D_2$ convex, having in mind the application to Chavel’s conjecture. However, although the technical details can be considerably longer, it is possible to construct the mirror coupling in a more general setup.

For example, in the case when $D_1$ and $D_2$ are disjoint domains, none of the difficulties encountered in the construction of the mirror coupling occur (the possibility of coupling/decoupling), so the constructions extends immediately to this case.

The two key ingredients in our construction of the mirror coupling were the hypothesis $\overline{D_2} \subset D_1$ (needed in order to reduce by a localization argument the construction to the case $D_1 = \mathbb{R}^d$) and the hypothesis on the convexity of the inner domain $D_2$ (which allowed us to construct a solution of the equation of the mirror coupling in the case $D_1 = \mathbb{R}^d$).

Replacing the first hypothesis by the condition that the boundaries $\partial D_1$ and $\partial D_2$ are not tangential (needed for the localization of the construction of the mirror coupling) and the second one by condition that $D_1 \cap D_2$ is a convex domain, the arguments in the present construction can be modified in order to give rise to a mirror coupling of reflecting Brownian motion in $(D_1, D_2)$ (see Figure 3).

Figure 3: Generic smooth domains $D_{1,2} \subset \mathbb{R}^d$ for the mirror coupling: $D_1, D_2$ have non-tangential boundaries and $D_1 \cap D_2$ is a convex domain.

Remark 5.5. Even though the construction of the mirror coupling was carried out without the additional assumption on the convexity of the inner domain $D_2$ in the case when $D_2$ is a polygonal
domain (see Theorem 3.9), we cannot extend the construction of the mirror coupling to the general case of smooth domains $D_2 \subset D_1$.

This is due to the fact that the stochastic differential equation which defines the mirror coupling has a singularity (discontinuity) when the processes couple, and we cannot prove the convergence of solutions in the approximating domains (as in the proof of Theorem 3.1). The convexity of the inner domain is an essential argument for this proof, which allowed us to handle the discontinuity of the stochastic differential equation which defines the mirror coupling: considering an increasing sequence of approximating domains $D_n \nearrow D_2$, the convexity of $D_2$ was used to show that if the coupling occurred in the case of the mirror coupling in $(\mathbb{R}^d, D_N)$, then coupling also occurred in the case of the mirror coupling in $(\mathbb{R}^d, D_n)$, for all $n \geq N$.

It is easy to construct an example of a non-convex domain $D_2$ and a sequence of approximating domains $D_n \nearrow D_2$ such that the mirror coupling $(X_t, Y^n_t)$ in $(\mathbb{R}^d, D_n)$ does not have the above-mentioned property, and therefore we cannot prove the existence of the mirror coupling using the same ideas as in Theorem 3.1. However, this does not imply that the mirror coupling cannot be constructed by other methods in a more general setup.

We conclude with some remarks on the non-uniqueness of the mirror coupling in general domains. To simplify the ideas, we will restrict to the 1-dimensional case when $D_2 = (0, \infty) \subset D_1 = \mathbb{R}$.

Fixing $x \in (0, \infty)$ as starting point of the mirror coupling $(X_t, Y_t)$ in $(D_1, D_2)$, the equations of the mirror coupling are

$$X_t = x + W_t \quad (5.17)$$

$$Y_t = x + Z_t + L^Y_t \quad (5.18)$$

$$Z_t = \int_0^t G (Y_s - X_s) \, dW_s \quad (5.19)$$

where in this case

$$G(z) = \begin{cases} -1, & \text{if } z \neq 0 \\ +1, & \text{if } z = 0 \end{cases}.$$  

Until the hitting time $\tau = \{s > 0 : Y_s \in \partial D_2\}$ of the boundary of $\partial D_2$ we have $L^Y_t \equiv 0$, and with the substitution $U_t = -\frac{1}{2} (Y_t - X_t)$, the stochastic differential for $Y_t$ becomes

$$U_t = \int_0^t \frac{1 - G(Y_s - X_s)}{2} \, dW_s = \int_0^t \sigma (U_s) \, dW_s, \quad (5.20)$$

where

$$\sigma(z) = \frac{1 - G(z)}{2} = \begin{cases} 1, & \text{if } z \neq 0 \\ 0, & \text{if } z = 0 \end{cases}.$$  

By a result of Engelbert and Schmidt ([13]) the solution of the above problem is not even weakly unique, for in this case the set of zeroes of the function $\sigma$ is $N = \{0\}$ and $\sigma^{-2}$ is locally integrable on $\mathbb{R}$.

In fact, more can be said about the solutions of (5.20) in this case. It is immediate that both $U_t \equiv 0$ and $U_t = W_t$ are solutions to (5.20), and it can be shown that an arbitrary solution can be obtained...
from $W_t$ by delaying it when it reaches the origin (sticky Brownian motion with sticky point the origin).

Therefore, until the hitting time $\tau$ of the boundary, we obtain as solutions

$$Y_t = X_t = x + W_t$$  \hspace{1cm} (5.21)

and

$$Y_t = X_t - 2W_t = x - W_t,$$  \hspace{1cm} (5.22)

and an intermediate range of solutions, which agree with (5.21) for some time, then switch to (5.22) (see [18]).

Correspondingly, this gives rise to mirror couplings of reflecting Brownian motions for which the solutions stick to each other after they have coupled (as in (5.21)), or they immediately split apart after coupling (as in (5.22)), and there is a whole range of intermediate possibilities. The first case can be referred to as sticky mirror coupling, the second as non-sticky mirror coupling, and the intermediate possibilities as weak/mild sticky mirror coupling.

The same situation occurs in the general setup in $\mathbb{R}^d$, and it is the cause of lack uniqueness of the stochastic differential equations which defines the mirror coupling. In the present paper we detailed the construction of the sticky mirror coupling, which we considered to be the most interesting, both from the point of view of the construction and of the applications, although the other types of mirror coupling might prove useful in other applications.

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