Swendsen-Wang dynamics for general graphs in the tree uniqueness region

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Abstract
The Swendsen-Wang (SW) dynamics is a popular Markov chain for sampling from the Gibbs distribution for the ferromagnetic Ising model on a graph $G = (V, E)$. The dynamics is conjectured to converge to equilibrium in $O(|V|^{1/4})$ steps at any (inverse) temperature $\beta$, yet there are few results providing $o(|V|)$ upper bounds. We prove fast convergence of the SW dynamics on general graphs in the tree uniqueness region. In particular, when $\beta < \beta_c(d)$ where $\beta_c(d)$ denotes the uniqueness/nonuniqueness threshold on infinite $d$-regular trees, we prove that the relaxation time (i.e., the inverse spectral gap) of the SW dynamics is $\Theta(1)$ on any graph of maximum degree $d \geq 3$. Our proof utilizes a monotone version of the SW dynamics which only updates isolated vertices. We establish that this variant of the SW dynamics has mixing time $O(\log |V|)$ and relaxation time $\Theta(1)$ on any graph of maximum degree $d$ for all $\beta < \beta_c(d)$. Our proof technology can be applied to general monotone Markov chains, including for example the heat-bath block dynamics, for which we obtain new tight mixing time bounds.

KEYWORDS
censoring, mixing time, relaxation time, spatial mixing, Swendsen-Wang dynamics

1 | INTRODUCTION

For spin systems, sampling from the associated Gibbs distribution is a key computational task with a variety of applications, notably including inference/learning [18] and approximate counting [27, 48].
In the study of spin systems, a model of prominent interest is the Ising model. This is a classical model in statistical physics, which was introduced in the 1920s to study the ferromagnet and its physical phase transition [24, 30]. More recently, the Ising model has found numerous applications in theoretical computer science, computer vision, social network analysis, game theory, biology, discrete probability and many other fields [6, 11, 12, 16, 37].

An instance of the (ferromagnetic) Ising model is given by an undirected graph $G = (V, E)$ on $n = |V|$ vertices and an (inverse) temperature $\beta > 0$. A configuration $\sigma \in \{+, -\}^V$ assigns a spin value (+ or −) to each vertex $v \in V$. The probability of a configuration $\sigma$ is proportional to

$$w(\sigma) = \exp \left( \beta \sum_{\{v, w\} \in E} \sigma(v)\sigma(w) \right),$$

where $\sigma(v)$ is the spin of $v$. The associated Gibbs distribution $\mu = \mu_{G, \beta}$ is given by $\mu(\sigma) = w(\sigma)/Z$, where the normalizing factor $Z$ is known as the partition function. Since $\beta > 0$ the system is ferromagnetic as neighboring vertices prefer to align their spins.

For general graphs Jerrum and Sinclair [25] presented an FPRAS for the partition function (which yields an efficient sampler); however, its running time is a large polynomial in $n$. Hence, there is significant interest in obtaining tight bounds on the convergence rate of Markov chains for the Ising model, namely, Markov chains on the space of Ising configurations $\{+, -\}^V$ that converge to Gibbs distribution $\mu$. A standard notion for measuring the speed of convergence to stationarity is the mixing time, which is defined as the number of steps until the Markov chain is close to its stationary distribution in total variation distance, starting from the worst possible initial configuration.

A simple, popular Markov chain for sampling from the Gibbs distribution is the Glauber dynamics, commonly referred to as the Gibbs sampler in some communities. This dynamics works by updating a randomly chosen vertex in each step in a reversible fashion. Significant progress has been made in understanding the mixing properties of the Glauber dynamics and its connections to the spatial mixing (i.e., decay of correlation) properties of the underlying spin system. In general, in the high-temperature region (small $\beta$) correlations typically decay exponentially fast, and one expects the Glauber dynamics to converge quickly to stationarity. For example, for the special case of the integer lattice $\mathbb{Z}^2$, in the high-temperature region it is well known that the Glauber dynamics has mixing time $\Theta(n \log n)$ [5, 9, 34]. For general graphs, Mossel and Sly [38] proved that the Glauber dynamics mixes in $O(n \log n)$ steps on any graph of maximum degree $d$ in the tree uniqueness region. Tree uniqueness is defined as follows: let $T_h$ denote a (finite) complete tree of height $h$ (by complete we mean all internal vertices have degree $d$). Fix the leaves to be all + spins, consider the resulting conditional Gibbs distribution on the internal vertices, and let $p^+_h$ denote the probability the root is assigned spin + in this conditional distribution; similarly, let $p^-_h$ denote the corresponding marginal probability with the leaves fixed to spin −. When $\beta < \beta_c(d)$, where $\beta_c(d)$ is such that

$$(d - 1) \tanh \beta_c(d) = 1,$$

then $\lim_{h \to \infty} p^+_h = \lim_{h \to \infty} p^-_h$ and we say tree uniqueness holds since there is a unique Gibbs measure on the infinite $d$-regular tree [41]. In the same setting, building upon the approach of Weitz [52] for the hard-core model, Li, Lu, and Yin [32] provide an FPTAS for the partition function, but the running time is a large polynomial in $n$.

In practice, it is appealing to utilize nonlocal (or global) chains which possibly update $\Omega(n)$ vertices in a step; these chains are more popular due to their presumed speed-up and for their ability to be naturally parallelized [29].
A notable example for the ferromagnetic Ising model is the Swendsen-Wang (SW) dynamics [49] which utilizes the random-cluster representation to derive an elegant Markov chain in which every vertex can change its spin in every step. The SW dynamics works in the following manner. From the current spin configuration $\sigma_t \in \{+,-\}^V$:

1. Consider the set of agreeing edges $E(\sigma_t) = \{(v, w) \in E : \sigma_t(v) = \sigma_t(w)\}$;
2. Independently for each edge $e \in E(\sigma_t)$, “percolate” by deleting $e$ with probability $\exp(-2\beta)$ and keeping $e$ with probability $1 - \exp(-2\beta)$; this yields $F_t \subseteq E(\sigma_t)$;
3. For each connected component $C$ in the subgraph $(V, F_t)$, choose a spin $s_C$ uniformly at random from $\{+,-\}$, and then assign spin $s_C$ to all vertices in $C$, yielding $\sigma_{t+1} \in \{+,-\}^V$.

The proof that the stationary distribution of the SW dynamics is the Gibbs distribution is nontrivial; see [10] for an elegant proof. The SW dynamics is also well-defined for the ferromagnetic Potts model, a natural generalization of the Ising model that allows vertices to be assigned $q$ different spins.

The SW dynamics for the Ising model is quite appealing as it is conjectured to mix quickly at all temperatures. Its behavior for the Potts model (which corresponds to $q > 2$ spins) is more subtle, as there are multiple examples of classes of graphs where the SW dynamics is torpidly mixing; that is, mixing time is exponential in the number of vertices of the graph; see, for example, [2–4, 14, 17, 19]. Despite the popularity [42, 43, 51] and rich mathematical structure [20] of the SW dynamics there are few results with tight bounds on its speed of convergence to equilibrium. In fact, there are few results proving the SW dynamics is faster than the Glauber dynamics (or the edge dynamics analog in the random-cluster representation). Most results derive as a consequence of analyses of these local dynamics. Recently, Guo and Jerrum [21] established that the mixing time of the SW dynamics on any graph and at any temperature is $O(|V|^{10})$. This bound, however, is far from the conjectured universal upper bound of $O(|V|^{1/4})$ [39], and once again their result derives from a bound on a local chain (the edge dynamics in the random-cluster representation).

In the special case of the mean-field Ising model, which corresponds to the underlying graph $G$ being the complete graph on $n$ vertices, Long, Nachmias, Ning, and Peres [33] provided a tight analysis of the mixing time of the SW dynamics. They prove that the mixing time of the mean-field SW dynamics is $\Theta(|V|^{1/4})$; this is expected to be the worst case and thus yields the aforementioned conjecture [39].

Another relevant case for which the speed of convergence is known is the two-dimensional integer lattice $\mathbb{Z}^2$ (more precisely, finite subsections of it). Blanca, Caputo, Sinclair, and Vigoda [1] recently established that the relaxation time of the SW dynamics is $O(1)$ in the high-temperature region. The relaxation time measures the speed of convergence to $\mu$ when the initial configuration is reasonably close to this distribution (a so-called “warm start”) [26, 28]. More formally, the relaxation time is equal to the inverse spectral gap of the transition matrix of the chain and is another well-studied notion of rate of convergence [31]. This result [1] applied a well-established proof approach [9, 34] which utilizes that $\mathbb{Z}^2$ is an amenable graph. Our goal in this paper is to establish results for general graphs of bounded degree.

Our inspiration is the result of Mossel and Sly [38] who proved $O(n \log n)$ mixing time of the Glauber dynamics for every graph of maximum degree $d$. When $\beta < \beta_c(d)$, in addition to uniqueness on the infinite $d$-regular tree, the ferromagnetic Ising model is also known to exhibit several key spatial mixing properties. For instance, Mossel and Sly [38] showed that when $\beta < \beta_c(d)$ a rather strong form of spatial mixing holds on graphs of maximum degree $d$; see Definition 9 and Lemma 10 in Section 3. Using this, together with the censoring result of Peres and Winkler [40] for the Glauber dynamics, they establish optimal bounds for the mixing and relaxation times of the Glauber dynamics. At a high-level,
the censoring result [40] says that extra updates by the Markov chain do not slow it down, and hence one can ignore transitions outside a local region of interest in the analysis of mixing times.

A Markov chain is monotone if it preserves the natural partial order on states; see Section 2 for a detailed definition. We generalize the proof approach of Mossel and Sly to apply to general (nonlocal) monotone Markov chains. This allows us to analyze a monotone variant of the SW dynamics, and a direct comparison of these two chains yields a new bound for the relaxation time of the SW dynamics.

**Theorem 1** Let $G$ be an arbitrary $n$-vertex graph of maximum degree $d$. If $\beta < \beta_c(d)$, then the relaxation time of the SW dynamics is $\Theta(1)$.

This tight bound for the relaxation time is a substantial improvement over the best previously known $O(n)$ bound which follows from Ullrich’s comparison theorem [50] combined with Mossel and Sly’s result [38] for the Glauber dynamics. We note that in Theorem 1, $d$ is assumed to be a constant independent of $n$ and thus the result holds for arbitrary graphs of bounded degree. We also mention that while spatial mixing properties are known to imply optimal mixing of local dynamics, only recently the effects of these properties on the rate of convergence of nonlocal dynamics have started to be investigated [1]. In general, spatial mixing properties have proved to have a number of powerful algorithmic applications in the design of efficient approximation algorithms for the partition function using the associated self-avoiding walk trees (see, e.g., [15, 32, 44–47, 52]).

There are three key components in our proof approach. First, we generalize the recursive/inductive argument of Mossel and Sly [38] from the Glauber dynamics to general (nonlocal) monotone dynamics. Since this approach relies crucially on the censoring result of Peres and Winkler [40] which only applies to the Glauber dynamics, we also need to establish a modest extension of the censoring result. For this, we use the framework of Fill and Kahn [13]. Finally, we require a monotone Markov chain that can be analyzed with these new tools and which is naturally comparable to the SW dynamics. To this end we utilize the Isolated-vertex dynamics which was previously used in [1].

The Isolated-vertex dynamics operates in the same manner as the SW dynamics, except in step 3 only components of size 1 choose a new random spin (other components keep the same spin as in $\sigma_t$). We prove that the Isolated-vertex dynamics is monotone. Combining these new tools we obtain the following result.

**Theorem 2** Let $G$ be an arbitrary $n$-vertex graph of maximum degree $d$. If $\beta < \beta_c(d)$, then the mixing time of the Isolated-vertex dynamics is $O(\log n)$, and its relaxation time is $\Theta(1)$.

Our result for censoring may be of independent interest, as it applies to a fairly general class of nonlocal monotone Markov chains. Indeed, combined with our generalization of Mossel and Sly’s results [38], it gives a general method for analyzing monotone Markov chains.

As the first application of this technology, we are able to establish tight bounds for the mixing and relaxation times of the block dynamics. Let $\{B_1, \ldots, B_r\}$ be a collection of sets (or blocks) such that $B_i \subseteq V$ and $V = \bigcup_i B_i$. The heat-bath block dynamics with blocks $\{B_1, \ldots, B_r\}$ is a Markov chain that in each step picks a block $B_i$ uniformly at random and updates the configuration in $B_i$ with a new configuration distributed according to the conditional measure in $B_i$ given the configuration in $V \setminus B_i$.

**Theorem 3** Let $G$ be an arbitrary $n$-vertex graph of maximum degree $d$ and let $\{B_1, \ldots, B_r\}$ be an arbitrary collection of blocks such that $V = \bigcup_{i=1}^r B_i$. If $\beta < \beta_c(d)$, then the mixing time of the block dynamics with blocks $\{B_1, \ldots, B_r\}$ is $O(r \log n)$, and its relaxation time is $O(r)$. 
We observe that there are no restrictions on the geometry of the blocks $B_i$ in the theorem other than $V = \cup_i B_i$. These optimal bounds were only known before for certain specific collections of blocks.

As a second application of our technology, we consider another monotone variant of the SW dynamics, which we call the Monotone SW dynamics. This chain proceeds exactly like the SW dynamics, except that in step 3 each connected component $C$ is assigned a new random spin only with probability $1/2^{|C|-1}$ and is not updated otherwise; see Section 7 for a precise definition. We derive the following bounds.

**Theorem 4**  Let $G$ be an arbitrary $n$-vertex graph of maximum degree $d$. If $\beta < \beta_c(d)$, then the mixing time of the Monotone SW dynamics is $O(\log n)$, and its relaxation time is $\Theta(1)$.

The remainder of the paper is structured as follows. Section 2 contains some basic definitions and facts used throughout the paper. In Section 3 we study the Isolated-vertex dynamics and establish Theorem 2. Theorem 1 for the SW dynamics will follow as an easy corollary of these results. In Section 3 we also state our generalization of Mossel and Sly’s approach [38] for nonlocal dynamics (Theorem 11) and our censoring result (Theorem 7). The proofs of these theorems are included in Sections 4 and 5, respectively. Finally, the full proofs of Theorems 3 and 4 are provided in Sections 6 and 7, respectively.

## 2 | BACKGROUND

In this section we provide a number of standard definitions that we will refer to in our proofs. For more details see the book [31].

### 2.1 | Ferromagnetic Ising model

Given a graph $G = (V, E)$ and a real number $\beta > 0$, the ferromagnetic Ising model on $G$ consists of the probability distribution over $\Omega_G = \{+,-\}^V$ given by

$$
\mu_{G,\beta}(\sigma) = \frac{1}{Z(G, \beta)} \exp \left[ \beta \sum_{(u,v) \in E} \sigma(u)\sigma(v) \right],
$$

where $\sigma \in \Omega_G$ and

$$
Z(G, \beta) = \sum_{\sigma \in \Omega_G} \exp \left[ \beta \sum_{(u,v) \in E} \sigma(u)\sigma(v) \right]
$$

is called the partition function.

### 2.2 | Mixing and relaxation times

Let $P$ be the transition matrix of an ergodic (i.e., irreducible and aperiodic) Markov chain over $\Omega_G$ with stationary distribution $\mu = \mu_{G,\beta}$. Let $P^t(X_0, \cdot)$ denote the distribution of the chain after $t$ steps starting from $X_0 \in \Omega_G$, and let

$$
T_{\text{mix}}(P, \epsilon) = \max_{X_0 \in \Omega} \min \left\{ t \geq 0 : \|P^t(X_0, \cdot) - \mu(\cdot)\|_{TV} \leq \epsilon \right\}.
$$

The mixing time of $P$ is defined as $T_{\text{mix}}(P) = T_{\text{mix}}(P, 1/4)$. 


If $P$ is reversible with respect to (w.r.t.) $\mu$, the spectrum of $P$ is real. Let $1 = \lambda_1 > \lambda_2 \geq \ldots \geq \lambda_{|\Omega|} \geq -1$ denote its eigenvalues. The absolute spectral gap of $P$ is defined by $\lambda(P) = 1 - \lambda^*$, where $\lambda^* = \max\{|\lambda_2|, |\lambda_{|\Omega|}|\}$. $T_{\text{rel}}(P) = \lambda(P)^{-1}$ is called the relaxation time of $P$, and is another well-studied notion of rate of convergence to $\mu$ [26, 28].

### 2.3 Couplings and grand couplings

A (one step) coupling of a Markov chain $\mathcal{M}$ over $\Omega_G$ specifies, for every pair of states $(X_t, Y_t) \in \Omega_G \times \Omega_G$, a probability distribution over $(X_{t+1}, Y_{t+1})$ such that the processes $\{X_t\}$ and $\{Y_t\}$, viewed in isolation, are faithful copies of $\mathcal{M}$, and if $X_t = Y_t$ then $X_{t+1} = Y_{t+1}$. Let $\{X^\sigma_t\}_{t \geq 0}$ denote an instance of $\mathcal{M}$ started from $\sigma \in \Omega_G$. A grand coupling of $\mathcal{M}$ is a simultaneous coupling of $\{X^\sigma_t\}_{t \geq 0}$ for all $\sigma \in \Omega_G$.

### 2.4 Monotonicity

For two configurations $\sigma, \tau \in \Omega_G$, we say $\sigma \geq \tau$ if $\sigma(v) \geq \tau(v)$ for all $v \in V$ (assuming “+” > “-”). This induces a partial order on $\Omega_G$. The ferromagnetic Ising model is monotone w.r.t. this partial order, since for every $B \subseteq V$ and every pair of configurations $\tau_1, \tau_2$ on $B$ such that $\tau_1 \geq \tau_2$ we have $\mu(\cdot \mid \tau_1) \geq \mu(\cdot \mid \tau_2)$, where $\geq$ denotes stochastic domination. (For two distributions $\nu_1, \nu_2$ on $\Omega_G$, we say that $\nu_1$ stochastically dominates $\nu_2$ if for any increasing function $f \in \mathbb{R}[\Omega_G]$ we have $\sum_{\sigma \in \Omega_G} \nu_1(\sigma) f(\sigma) \geq \sum_{\sigma \in \Omega_G} \nu_2(\sigma) f(\sigma)$, where a vector or function $f \in \mathbb{R}[\Omega_G]$ is increasing if $f(\sigma) \geq f(\tau)$ for all $\sigma \geq \tau$.)

Suppose $\mathcal{M}$ is an ergodic Markov chain over $\Omega_G$ with stationary distribution $\mu$ and transition matrix $P$. A coupling of two instances $\{X_t\}, \{Y_t\}$ of $\mathcal{M}$ is a monotone coupling if $X_{t+1} \geq Y_{t+1}$ whenever $X_t \geq Y_t$. We say that $\mathcal{M}$ is a monotone Markov chain and $P$ is a monotone transition matrix if $\mathcal{M}$ has a monotone grand coupling.

### 2.5 Comparison inequalities

The Dirichlet form of a Markov chain with transition matrix $P$ reversible w.r.t. $\mu$ is defined for any $f, g \in \mathbb{R}[\Omega_G]$ as

$$\mathcal{E}_P(f, g) = \langle f, (I - P)g \rangle_\mu = \frac{1}{2} \sum_{\sigma, \tau \in \Omega_G} \mu(\sigma)P(\sigma, \tau)(f(\sigma) - f(\tau))(g(\sigma) - g(\tau)),$$

where $\langle f, g \rangle_\mu = \sum_{\sigma \in \Omega_G} \mu(\sigma)f(\sigma)g(\sigma)$ for all $f, g \in \mathbb{R}[\Omega_G]$.

If $P$ and $Q$ are the transition matrices of two monotone Markov chains reversible w.r.t. $\mu$, we say that $P \leq Q$ if $\langle Pf, g \rangle_\mu \leq \langle Qf, g \rangle_\mu$ for every increasing and positive $f, g \in \mathbb{R}[\Omega_G]$. Note that $P \leq Q$ is equivalent to $\mathcal{E}_P(f, g) \geq \mathcal{E}_Q(f, g)$ for every increasing and positive $f, g \in \mathbb{R}[\Omega_G]$.

### 3 ISOLATED-VERTEX DYNAMICS

In this section we consider a variant of the SW dynamics known as the Isolated-vertex dynamics which was first introduced in [1]. We shall use this dynamics to introduce a general framework for analyzing monotone Markov chains for the Ising model and to derive our bounds for the SW dynamics. Specifically, we will prove Theorems 1 and 2 from the introduction.
Throughout the section, let $G = (V, E)$ be an arbitrary $n$-vertex graph of maximum degree $d$, $\mu = \mu_G, \beta$ and $\Omega = \Omega_G$. Given an Ising model configuration $\sigma_t \in \Omega$, one step of the Isolated-vertex dynamics is given by:

1. Consider the set of agreeing edges $E(\sigma_t) = \{(v, w) \in E : \sigma_t(v) = \sigma_t(w)\}$;
2. Independently for each edge $e \in E(\sigma_t)$, delete $e$ with probability $\exp(-2\beta)$ and keep $e$ with probability $1 - \exp(-2\beta)$; this yields $F_t \subseteq E(\sigma_t)$;
3. For each isolated vertex $v$ in the subgraph $(V, F_t)$ (i.e., those vertices with no incident edges in $F_t$), choose a spin uniformly at random from $\{+, -\}$ and assign it to $v$ to obtain $\sigma_{t+1}$; all other (nonisolated) vertices keep the same spin as in $\sigma_t$.

We use $IV$ to denote the transition matrix of this chain. The reversibility of $IV$ with respect to $\mu$ was established in [1]. Observe also that in step 3, only isolated vertices are updated with new random spins, whereas in the SW dynamics all connected components are assigned new random spins. It is thus intuitive that the SW dynamics converges faster to stationarity than the Isolated-vertex dynamics. This intuition was partially captured in [1], where it was proved that

$$T_{rel}(SW) \leq T_{rel}(IV). \quad (4)$$

The Isolated-vertex dynamics exhibits various properties that vastly simplify its analysis. These properties allow us to deduce, for example, strong bounds for both its relaxation and mixing times. Specifically, we show (in Theorem 2) that when $\beta < \beta_c(d)$, $T_{mix}(IV) = O(\log n)$ and $T_{rel}(IV) = \Theta(1)$; see (2) for the definition of $\beta_c(d)$, Theorem 1 from the introduction then follows from (4).

A comparison inequality like (4) but for mixing times is not known, so Theorem 2 does not yield a $O(\log n)$ bound for the mixing time of the SW dynamics as one might hope. Direct comparison inequalities for mixing times are rare, since almost all known techniques involve the comparison of Dirichlet forms, and there are inherent penalties in using such inequalities to derive mixing times bounds.

The first key property of the Isolated-vertex dynamics is that, unlike the SW dynamics, this Markov chain is monotone. Monotonicity is known to play a key role in relating spatial mixing (i.e., decay of correlation) properties to fast convergence of the Glauber dynamics. For instance, for spin systems in lattice graphs, sophisticated functional analytic techniques are required to establish the equivalence between a spatial mixing property known as strong spatial mixing and optimal mixing of the Glauber dynamics [34–36]. For monotone spin systems such as the Ising model a simpler combinatorial argument yields the same sharp result [9]. This combinatorial argument is in fact more robust, since it can be used to analyze a larger class of Markov chains, including for example the systematic scan dynamics [1].

**Lemma 5** For all graphs $G$ and all $\beta > 0$, the Isolated-vertex dynamics for the Ising model is monotone.

The proof of Lemma 5 is given in Section 3.1. The second key property of the Isolated-vertex dynamics concerns whether moves (or partial moves) of the dynamics could be censored from the evolution of the chain without possibly speeding up its convergence. Censoring of Markov chains is a well-studied notion [13, 23, 40] that has found important applications [7, 8, 38].

We say that a stochastic $|\Omega| \times |\Omega|$ matrix $Q$ acts on a set $A \subseteq V$ if for all $\sigma, \sigma' \in \Omega$:

$$Q(\sigma, \sigma') \neq 0 \text{ iff } \sigma(V \setminus A) = \sigma'(V \setminus A).$$
Also recall that $P \leq P_A$ if $\langle Pf, g \rangle_\mu \leq \langle P_A f, g \rangle_\mu$ for any pair of increasing positive functions $f, g \in \mathbb{R}^\Omega$.

**Definition 6** Let $G$ be an arbitrary graph and let $\beta > 0$. Consider an ergodic and monotone Markov chain for the Ising model on $G$, reversible w.r.t. $\mu = \mu_{G,\beta}$ with transition matrix $P$. Let $\{P_A\}_{A \subseteq V}$ be a collection of monotone stochastic matrices reversible w.r.t. $\mu$ with the property that $P_A$ acts on $A$ for every $A \subseteq V$. We say that $\{P_A\}_{A \subseteq V}$ is a censoring for $P$ if $P \leq P_A$ for all $A \subseteq V$.

As an example, consider the *heat-bath Glauber dynamics* for the Ising model on the graph $G = (V, E)$. Recall that in this Markov chain a vertex $v \in V$ is chosen uniformly at random (u.a.r.) and a new spin is sampled for $v$ from the conditional distribution at $v$ given the configuration on $V \setminus v$. For every $A \subseteq V$, we may take $P_A$ to be the $|\Omega| \times |\Omega|$ transition matrix of the censored heat-bath Glauber dynamics that ignores all moves outside of $A$. That is, if the randomly chosen vertex $v \in V$ is not in $A$, then the move is ignored; otherwise the chain proceeds as the standard heat-bath Glauber dynamics.

It is easy to check that $P_A$ is monotone and reversible w.r.t. $\mu$. Moreover, it was established in [13, 40] that $P \leq P_A$ for every $A \subseteq V$, and thus the collection $\{P_A\}_{A \subseteq V}$ is a censoring for the heat-bath Glauber dynamics. This particular censoring has been used to analyze the speed of convergence of the Glauber dynamics in various settings (see [7, 8, 38, 40]), since it can be proved that censored variants of the Glauber dynamics—where moves of $P$ are replaced by moves of $P_A$—converge more slowly to the stationary distribution [13, 40]. Consequently, it suffices to analyze the speed of convergence of the censored chain, and this could be much simpler for suitably chosen censoring schemes.

Using the machinery from [13, 40], we can show that given a censoring (as defined in Definition 6), the strategy just mentioned for Glauber dynamics can be used for general monotone Markov chains.

**Theorem 7** Let $G$ be an arbitrary graph and let $\beta > 0$. Let $\{X_t\}$ be an ergodic monotone Markov chain for the Ising model on $G$, reversible w.r.t. $\mu = \mu_{G,\beta}$ with transition matrix $P$. Let $\{P_A\}_{A \subseteq V}$ be a censoring for $P$ and let $\{\hat{X}_t\}$ be a censored version of $\{X_t\}$ that sequentially applies $P_{A_1}, P_{A_2}, P_{A_3}, \ldots$ where $A_i \subseteq V$. If $X_0, Y_0$ are both sampled from a distribution $v$ over $\Omega$ such that $v/\mu$ is increasing, then the following hold:

1. $X_t \leq \hat{X}_t$ for all $t \geq 0$;
2. Let $\hat{P}^t = P_{A_1} \ldots P_{A_t}$. Then, for all $t \geq 0$

$$\|P^t(X_0, \cdot) - \mu(\cdot)\|_{TV} \leq \|\hat{P}^t(X_0, \cdot) - \mu(\cdot)\|_{TV}.$$  

If $v/\mu$ is decreasing, then $X_t \geq \hat{X}_t$ for all $t \geq 0$.

The proof of this theorem is provided in Section 5.

We define next a specific censoring for the Isolated-vertex dynamics. For $A \subseteq V$, let $IV_A$ be the transition matrix for the Markov chain that given an Ising model configuration $\sigma_t$ generates $\sigma_{t+1}$ as follows:

1. Consider the set of agreeing edges $E(\sigma_t) = \{(v, w) \in E : \sigma_t(v) = \sigma_t(w)\}$;
2. Independently for each edge $e \in E(\sigma_t)$, delete $e$ with probability $\exp(-2\beta)$ and keep $e$ with probability $1 - \exp(-2\beta)$; this yields $F_t \subseteq E(\sigma_t)$;
3. For each isolated vertex $v$ of the subgraph $(V, F_t)$ in the subset $A$, choose a spin uniformly at random from $\{+, -\}$ and assign it to $v$ to obtain $\sigma_{t+1}$; all other vertices keep the same spin as in $\sigma_t$.

**Lemma 8** The collection of matrices $\{IV_A\}_{A \subseteq V}$ is a censoring for the Isolated-vertex dynamics.
The proof of Lemma 8 is provided in Sec 3.2. To establish Theorem 2 we show that a strong form of spatial mixing, which is known to hold for all \( \beta < \beta_c(d) \) [38], implies the desired mixing and relaxation times bounds for the Isolated-vertex dynamics. We define this notion of spatial mixing next.

For \( v \in V \) and \( R \in \mathbb{N} \), let \( B(v, R) = \{ u \in V : \text{dist}(u, v) \leq R \} \) denote the ball of radius \( R \) around \( v \), where \( \text{dist}(\cdot, \cdot) \) denotes graph distance. Also, let \( S(v, R) = B(v, R + 1) \setminus B(v, R) \) be the external boundary of \( B(v, R) \). For any \( A \subseteq V \), let \( \Omega_A = \{ +, - \}^A \) be the set of all configurations on \( A \); hence \( \Omega = \Omega_G = \Omega_V \).

For \( v \in V \), \( u \in S(v, R) \) and \( \tau \in \Omega_{S(v,R)} \), let \( \tau_u^+ \) (resp., \( \tau_u^- \)) be the configuration obtained from \( \tau \) by changing the spin of \( u \) to + (resp., to −) and define

\[
a_u = \sup_{\tau \in \Omega_{S(v,R)}} \left| \mu\left( v = + \mid S(v, R) = \tau_u^+ \right) - \mu\left( v = + \mid S(v, R) = \tau_u^- \right) \right|,
\]

where “\( v = + \)” represents the event that the spin of \( v \) is + and “\( S(v, R) = \tau_u^+ \)” (resp., “\( S(v, R) = \tau_u^- \)”)
stands for the event that \( S(v, R) \) has configuration \( \tau_u^+ \) (resp., \( \tau_u^- \)).

**Definition 9** We say that aggregate strong spatial mixing (ASSM) holds for \( R \in \mathbb{N} \), if for all \( v \in V \)

\[
\sum_{u \in S(v, R)} a_u \leq \frac{1}{4}.
\]

**Lemma 10** (Lemma 3, [38]) For all graphs \( G \) of maximum degree \( d \) and all \( \beta < \beta_c(d) \), there exists an integer \( R = R(\beta, d) \in \mathbb{N} \) such that ASSM holds for \( R \).

Theorem 2 is then a direct corollary of the following more general theorem. The proof of this general theorem, which is provided in Section 4, follows closely the approach in [38] for the case of the Glauber dynamics, but key additional considerations are required to establish such result for general (nonlocal) monotone Markov chains. The main new innovation in our proof is the use of the more general Theorem 7, instead of the standard censoring result in [40].

**Theorem 11** Let \( \beta > 0 \) and \( G \) be an arbitrary \( n \)-vertex graph of maximum degree \( d \) where \( d \) is a constant independent of \( n \). Consider an ergodic monotone Markov chain for the Ising model on \( G \), reversible w.r.t. \( \mu = \mu_{G, \beta} \) with transition matrix \( P \). Suppose \( \{ P_A \}_{A \subseteq V} \) is a censoring for \( P \). If ASSM holds for a constant \( R > 0 \), and for any \( v \in V \) and any starting configuration \( \sigma \in \Omega \)

\[
T_{\text{mix}}(P_{B(v,R)}) \leq T,
\]

then \( T_{\text{mix}}(P) = O(T \log n) \) and \( T_{\text{el}}(P) = O(T) \).

We note that \( T_{\text{mix}}(P_{B(v,R)}) \) denotes the mixing time from the worst possible starting configuration, both in \( B(v, R) \) and in \( V \setminus B(v, R) \). (Since \( P_{B(v,R)} \) only acts in \( B(v, R) \), the configuration in \( V \setminus B(v, R) \) remains fixed throughout the evolution of the chain and determines its stationary distribution.)

We now use Theorem 11 to establish Theorem 2. In Sections 6 and 7, Theorem 11 is also used to establish Theorems 3 and 4 from the introduction, concerning the mixing time of the block dynamics and a monotone variant of the SW dynamics.

**Proof of Theorem 2** By Lemma 5 the Isolated-vertex dynamics is monotone, and by Lemma 8 the collection \( \{ TV_A \}_{A \subseteq V} \) is a censoring for \( TV \). Moreover, Lemma 10 implies that there exists a constant \( R \) such that ASSM. Thus, to apply Theorem 11 all that is needed is a bound for \( T_{\text{mix}}(TV_{B(v,R)}) \) for all
\(v \in V\). For this, we can use a crude coupling argument. Since \(|B(v, R)| \leq d^R\), the probability that every vertex in \(B(v, R)\) becomes isolated is at least
\[
e^{-2\beta d |B(v, R)|} \geq e^{-2\beta d^{R+1}}.
\]

Starting from two arbitrary configurations in \(B(v, R)\), if all vertices become isolated in both configurations, then we can couple them with probability 1. Hence, we can couple two arbitrary configurations in one step with probability at least \(\exp(-2\beta d^{R+1})\). Thus, \(T_{\text{mix}}(\mathcal{I}V_{B(v, R)}) = \exp(O(\beta d^{R+1})) = O(1)\), and the result then follows from Theorem 11.

**Proof of Theorem 1** Follows from Theorem 2 and the fact that \(T_{\text{rel}}(SW) \leq T_{\text{rel}}(IV)\), which was established in Lemma 4.1 from [1].

### 3.1 Monotonicity of the Isolated-vertex dynamics

In this section, we show that the Isolated-vertex dynamics is monotone by constructing a monotone grand coupling; see Section 2 for the definition of a grand coupling. In particular, we prove Lemma 5.

**Proof of Lemma 5** Let \(\{X^\sigma_t\}_{t \geq 0}\) be an instance of the Isolated-vertex dynamics starting from \(\sigma \in \Omega\); that is, \(X^\sigma_0 = \sigma\). We construct a grand coupling for the Isolated-vertex dynamics as follows. At time \(t\):

1. For every edge \(e \in E\) independently, pick a number \(r_t(e)\) uniformly at random from \([0, 1]\);
2. For every vertex \(v \in V\) independently, choose a uniform random spin \(s_t(v)\) from \(\{+, -\}\);
3. For every \(\sigma \in \Omega\):
   
   (i) Obtain \(F^\sigma_t \subseteq E\) by including the edge \(e = \{u, v\}\) in \(F^\sigma_t\) iff \(X^\sigma_t(u) = X^\sigma_t(v)\) and \(r_t(e) \leq 1 - e^{-2\beta}\).
   
   (ii) For every \(v \in V\), set \(X^\sigma_{t+1}(v) = s_t(v)\) if \(v\) is an isolated vertex in the subgraph \((V, F^\sigma_t)\); otherwise, set \(X^\sigma_{t+1}(v) = X^\sigma_t(v)\).

This is clearly a valid grand coupling for the Isolated-vertex dynamics. We show next that it is also monotone.

Suppose \(X^\sigma_t \geq X^\sigma_{t+1}\). We need to show that \(X^\sigma_{t+1} \geq X^\sigma_{t+1}\) after one step of the grand coupling. Let \(v \in V\). If \(v\) is not isolated in either \((V, F^\sigma_t)\) or \((V, F^\sigma_{t+1})\), then the spin of \(v\) remains unchanged in both \(X^\sigma_{t+1}\) and \(X^\sigma_{t+1}\), and \(X^\sigma_{t+1}(v) = X^\sigma_t(v) \geq X^\sigma_t(v) = X^\sigma_{t+1}(v)\). On the other hand, if \(v\) is isolated in both \((V, F^\sigma_t)\) and \((V, F^\sigma_{t+1})\), then the spin of \(v\) is set to \(s_t(v)\) in both instances of the chain; hence, \(X^\sigma_{t+1}(v) = s_t(v) = X^\sigma_{t+1}(v)\).

Suppose next that \(v\) is isolated in \((V, F^\sigma_t)\) but not in \((V, F^\sigma_{t+1})\). Then, \(X^\sigma_{t+1}(v) = s_t(v)\) and \(X^\sigma_{t+1}(v) = X^\sigma_t(v)\). The only possibility that would violate \(X^\sigma_{t+1}(v) \geq X^\sigma_{t+1}(v)\) is that \(X^\sigma_{t+1}(v) = -\), \(X^\sigma_t(v) = +\) and \(X^\sigma_{t+1}(v) = X^\sigma_t(v) = +\). If this is the case, then \(X^\sigma_t(v) = X^\sigma_{t+1}(v)\). Moreover, since \(X^\sigma_t \geq X^\sigma_{t+1}\), all neighbors of \(v\) assigned “+” in \(X^\sigma_t\) are also “+” in \(X^\sigma_{t+1}\); thus if \(v\) is isolated in \((V, F^\sigma_t)\) then \(v\) is also isolated in \((V, F^\sigma_{t+1})\). This leads to a contradiction, and so \(X^\sigma_{t+1}(v) \geq X^\sigma_{t+1}(v)\). The case in which \(v\) is isolated in \((V, F^\sigma_t)\) but not in \((V, F^\sigma_{t+1})\) follows from an analogous argument.

We can use the same grand coupling to show that \(\mathcal{I}V_A\) is also monotone for all \(A \subseteq V\). The only required modification in the construction is that if \(v \in V \setminus A\), then the spin of \(v\) is not updated in either copy. This gives the following corollary.

**Corollary 12** \(\mathcal{I}V_A\) is monotone for all \(A \subseteq V\).
3.2 Censoring for the Isolated-vertex dynamics

In this section we show that the collection \( \{ IV_A \}_{A \subseteq V} \) is a censoring for \( IV \). Specifically, we prove Lemma 8.

Proof of Lemma 8 For all \( A \subseteq V \), we need to establish that \( IV_A \) is reversible w.r.t. \( \mu = \mu_{G, \beta} \), monotone and that \( IV \leq IV_A \). Monotonicity follows from Corollary 12. To establish the other two facts we use an alternative representation of the matrices \( IV \) and \( IV_A \) that was already used in [1] and is inspired by the methods in [50].

Let \( \Omega_j = 2^E \times \Omega \) be the joint configuration space, where configurations consist of a spin assignment to the vertices together with a subset of the edges of \( G \). The joint Edwards-Sokal measure \( \nu \) on \( \Omega_j \) is given by

\[
\nu(F, \sigma) = \frac{1}{Z_j} p^{|F|} (1 - p)^{|E| \setminus |F|} \mathbb{1}(F \subseteq E(\sigma)),
\]

where \( p = 1 - e^{-2\beta} \), \( F \subseteq E, \sigma \in \Omega, E(\sigma) = \{ \{ u, v \} \in E : \sigma(u) = \sigma(v) \} \), and \( Z_j \) is the partition function [10].

Let \( T \) be the \(|\Omega| \times |\Omega_j| \) matrix given by:

\[
T(\sigma, (F, \tau)) = \mathbb{1}(\sigma = \tau) \mathbb{1}(F \subseteq E(\sigma)) p^{|F|} (1 - p)^{|E(\sigma) \setminus F|},
\]

where \( \sigma \in \Omega \) and \( (F, \tau) \in \Omega_j \). The matrix \( T \) corresponds to adding each edge \( \{ u, v \} \in E \) with \( \sigma(u) = \sigma(v) \) independently with probability \( p \), as in step 1 of the Isolated-vertex dynamics. Let \( L_2(\nu) \) and \( L_2(\mu) \) denote the Hilbert spaces \( (\mathbb{R}|\Omega|, \langle \cdot, \cdot \rangle_\nu) \) and \( (\mathbb{R}|\Omega|, \langle \cdot, \cdot \rangle_\mu) \) respectively. The matrix \( T \) defines an operator from \( L_2(\nu) \) to \( L_2(\mu) \) via vector-matrix multiplication. Specifically, for any \( f \in \mathbb{R}|\Omega| \) and \( \sigma \in \Omega \)

\[
Tf(\sigma) = \sum_{(F, \tau) \in \Omega_j} T(\sigma, (F, \tau)) f(F, \tau).
\]

It is easy to check that the adjoint operator \( T^* : L_2(\mu) \rightarrow L_2(\nu) \) of \( T \) is given by the \(|\Omega_j| \times |\Omega| \) matrix

\[
T^*((F, \tau), \sigma) = \mathbb{1}(\tau = \sigma),
\]

with \( (F, \tau) \in \Omega_j \) and \( \sigma \in \Omega \). Finally, for \( A \subseteq V \), \( F_1, F_2 \subseteq E \) and \( \sigma, \tau \in \Omega \) let

\[
Q_A((F_1, \sigma), (F_2, \tau)) = \mathbb{1}(F_1 = F_2) \mathbb{1}(F_1 \subseteq E(\sigma) \cap E(\tau)) \mathbb{1}(\sigma(I^c_A(F_1)) = \tau(I^c_A(F_1))) \cdot 2^{-|I_A(F_1)|}
\]

where \( I_A(F_1) \) is the set of isolated vertices of \( (V, F_1) \) in \( A \) and \( I^c_A(F_1) = V \setminus I_A(F_1) \), and similarily for \( F_2 \). For ease of notation we set \( Q = Q_V \). It follows straightforwardly from the definition of these matrices that \( IV = TQ^* \) and \( IV_A = TJ A T^* \) for all \( A \subseteq V \). It is also easy to verify that \( Q = Q^2 = Q^* \), \( Q_A = Q^2_A = Q^*_A \) and that \( Q = Q_A Q_Q_A \); see [1].

The reversibility of \( IV_A \) w.r.t. \( \mu \) follows from the fact that \( IV_A^* = (TQ_A T^*)^* = TQ_A T^* = IV_A \). This implies that \( IV_A \) is self-adjoint and thus reversible w.r.t. \( \mu \) [31].

To establish that \( IV \leq IV_A \), it is sufficient to show that for every pair of increasing and positive functions \( f_1, f_2 : \mathbb{R}|\Omega| \rightarrow \mathbb{R} \) on \( \Omega \), we have

\[
\langle f_1, IV f_2 \rangle_\mu \leq \langle f_1, IV_A f_2 \rangle_\mu.
\]
Now,
\[
\langle f_1, TV_{A f_2} \rangle_\mu = \langle f_1, TQ_A T^* f_2 \rangle_\mu = \langle f_1, TQ_A^2 T^* f_2 \rangle_\mu = \langle Q_A T^* f_1, Q_A T^* f_2 \rangle_\nu = \langle \hat{f}_1, \hat{f}_2 \rangle_\nu,
\]
where \( \hat{f}_1 = Q_A T^* f_1 \) and \( \hat{f}_2 = Q_A T^* f_2 \). Similarly,
\[
\langle f_1, TV_{f_2} \rangle_\mu = \langle f_1, TQ_A Q^2 A T^* f_2 \rangle_\mu = \langle Q_A T^* f_1, Q_A T^* f_2 \rangle_\nu = \langle Q \hat{f}_1, Q \hat{f}_2 \rangle_\nu.
\]
Thus, it is sufficient for us to show that \( \langle Q \hat{f}_1, Q \hat{f}_2 \rangle_\nu \leq \langle \hat{f}_1, \hat{f}_2 \rangle_\nu \).

Consider the partial order on \( \Omega_J \) where \((F, \sigma) \geq (F', \sigma')\) iff \( F = F' \) and \( \sigma \geq \sigma' \).

**Claim 13** Suppose \( f : \mathbb{R}^{[\Omega]} \to \mathbb{R} \) is an increasing positive function. Then, \( \hat{f} : \mathbb{R}^{[\Omega]} \to \mathbb{R} \) where \( \hat{f} = Q_A T^* f \) is also increasing and positive.

Given \( \omega \in \Omega_J \), let \( \rho_\omega(\cdot) = Q(\omega, \cdot) \); that is, \( \rho_\omega \) is the distribution over \( \Omega_J \) after applying \( Q \) from \( \omega \). We have
\[
\hat{Q} \hat{f}_1(\omega) = \sum_{\omega' \in \Omega_J} Q(\omega, \omega') \hat{f}_1(\omega') = \mathbb{E}_{\rho_\omega}[\hat{f}_1].
\]
Similarly, we get \( \hat{Q} \hat{f}_2(\omega) = \mathbb{E}_{\rho_\omega}[\hat{f}_2] \).

For a distribution \( \pi \) on a partially ordered set \( S \), we say \( \pi \) is positively correlated if for any increasing functions \( f, g \in \mathbb{R}^{[S]} \) we have \( \mathbb{E}_\pi[fg] \geq \mathbb{E}_\pi[f] \mathbb{E}_\pi[g] \). Since \( \rho_\omega \) is a product distribution over the isolated vertices in \( \omega \), \( \rho_\omega \) is positively correlated for any \( \omega \in \Omega_J \) by Harris inequality (see, e.g., Lemma 22.14 in [31]). By Claim 13, \( \hat{f}_1 \) and \( \hat{f}_2 \) are increasing. We then deduce that for any \( \omega \in \Omega_J \):
\[
\hat{Q} \hat{f}_1(\omega) \hat{Q} \hat{f}_2(\omega) = \mathbb{E}_{\rho_\omega}[\hat{f}_1] \mathbb{E}_{\rho_\omega}[\hat{f}_2] \leq \mathbb{E}_{\rho_\omega}[\hat{f}_1, \hat{f}_2].
\]
Putting all these facts together, we get
\[
\langle \hat{Q} \hat{f}_1, Q \hat{f}_2 \rangle_\nu = \sum_{\omega \in \Omega_J} \hat{Q} \hat{f}_1(\omega) \hat{Q} \hat{f}_2(\omega) \nu(\omega) \leq \sum_{\omega \in \Omega_J} \mathbb{E}_{\rho_\omega}[\hat{f}_1, \hat{f}_2] \nu(\omega)
\]
\[
= \sum_{\omega, \omega' \in \Omega_J} \hat{f}_1(\omega') \hat{f}_2(\omega') \rho_\omega(\omega') \nu(\omega) = \sum_{\omega, \omega' \in \Omega_J} \hat{f}_1(\omega') \hat{f}_2(\omega') \rho_\omega(\omega') \nu(\omega')
\]
\[
= \langle \hat{f}_1, \hat{f}_2 \rangle_\nu,
\]
where the second to last equality follows from the reversibility of \( Q \) w.r.t. \( \nu \); namely,
\[
\rho_\omega(\omega') \nu(\omega) = Q(\omega, \omega') \nu(\omega) = Q(\omega', \omega) \nu(\omega') = \rho_{\omega'}(\omega) \nu(\omega').
\]
This implies that (10) holds for every pair of increasing positive functions, and the theorem follows.

We conclude this section with the proof of Claim 13.

**Proof of Claim 13** From the definition of \( T^* \) we get \( T^* f(F, \sigma) = f(\sigma) \) for any \( (F, \sigma) \in \Omega_J \). Let \( (F, \sigma), (F', \tau) \in \Omega_J \) be such that \( \sigma \geq \tau \). Then,
\[
\hat{f}(F, \sigma) = Q_A T^* f(F, \sigma) = \sum_{(F', \sigma') \in \Omega_J} Q_A((F, \sigma), (F', \sigma')) f(\sigma').
\]
Recall that $Q_A((F, \sigma), (F', \sigma')) > 0$ iff $F = F'$ and $\sigma, \sigma'$ differ only in $I_A(F)$, the set of isolated vertices in $A$. If this is the case, then

$$Q_A((F, \sigma), (F', \sigma')) = \frac{1}{2|I_A(F)|}.$$ 

For $\xi \in \Omega_{I_A(F)}$, let $\sigma_\xi$ denote the configuration obtained from $\sigma$ by changing the spins of vertices in $I_A(F)$ to $\xi$; $\tau_\xi$ is defined similarly. (Recall that $\Omega_{I_A(F)}$ denotes the set of Ising configurations on the set $I_A(F)$.) Then, $\sigma_\xi \geq \tau_\xi$ for any $\xi \in \Omega_{I_A(F)}$ and

$$\hat{f}(F, \sigma) = \frac{1}{2|I_A(F)|} \sum_{\xi \in \Omega_{I_A(F)}} f(\sigma_\xi) \geq \frac{1}{2|I_A(F)|} \sum_{\xi \in \Omega_{I_A(F)}} f(\tau_\xi) = \hat{f}(F, \tau).$$

This shows that $\hat{f}$ is increasing.

\[\Box\]

4 | PROOF OF THEOREM 11

In [38], Mossel and Sly show that ASSM (see Definition 9) implies optimal $O(n \log n)$ mixing of the Glauber dynamics on any $n$-vertex graph of bounded degree [22]. Our proof of Theorem 11 follows the approach in [38]. The key new novelty is the use of Theorem 7.

**Proof of Theorem 11** Let $\{X^+_0\}, \{X^-_0\}$ be two instances of the chain such that $X^+_0$ is the “all plus” configuration and $X^-_0$ is the “all minus” one. Since the chain is monotone there exists a monotone grand coupling of $\{X^+_\}$ and $\{X^-\}$ such that $X^+_t \geq X^-_t$ for all $t \geq 0$. The existence of a monotone grand coupling implies that the extremal “all plus” and “all minus” are the worst possible starting configurations, and thus,

$$T_{\text{mix}}(P, \varepsilon) \leq T_{\text{coup}}(\varepsilon)$$

where $T_{\text{coup}}(\varepsilon)$ is the minimum $t$ such that $\Pr[X^+_t \neq X^-_t] \leq \varepsilon$, assuming $\{X^+_0\}$ and $\{X^-_0\}$ are coupled using the monotone coupling. Hence, it is sufficient to find $t$ such that for all $v \in V$

$$\Pr[X^+_t(v) \neq X^-_t(v)] \leq \frac{\varepsilon}{n},$$

since the result would follow from a union bound over the vertices.

Choose $R \in \mathbb{N}$ such that the ASSM property holds; see Lemma 10. Let $s \in \mathbb{N}$ be arbitrary and fixed. For each $v \in V$, we define two instances $\{Y^+_t\}$ and $\{Y^-_t\}$ of the censored chain that until time $s$ evolves as the chain $P$ and after time $s$ it evolves according to $P_{B(v, R)}$. By assumption $P_{B(v, R)}$ is also monotone, so the evolutions of $\{Y^+_t\}$ and $\{Y^-_t\}$ can be coupled as follows: up to time $s$, $\{Y^+_t\}$ and $\{Y^-_t\}$ are coupled by setting $Y^+_t = X^+_t$ and $Y^-_t = X^-_t$ for all $0 \leq t \leq s$; for $t > s$ the monotone coupling for $P_{B(v, R)}$ is used. Then, we have $X^+_t \geq X^-_t$ and $Y^+_t \geq Y^-_t$ for all $t \geq 0$.

Since $P \leq P_{B(v, R)}$ by assumption, and the distribution $\nu^+$ (resp., $\nu^-$) of $X^+_0$ (resp., $X^-_0$) is such that $\nu^+/\mu$ (resp., $\nu^-/\mu$) is trivially increasing (resp., decreasing), Theorem 7 implies $Y^+_t \geq X^+_t$ and $X^-_t \geq Y^-_t$ for all $t \geq 0$. Hence,

$$Y^+_t \geq X^+_t \geq X^-_t \geq Y^-_t.$$
Thus,

\[
\Pr[X^+_i(v) \neq X^-_i(v)] = \Pr[X^+_i(v) = +] - \Pr[X^-_i(v) = +] \\
\leq \Pr[Y^+_i(v) = +] - \Pr[Y^-_i(v) = +] \\
= \Pr[Y^+_i(v) \neq Y^-_i(v)],
\]

where the first and third equations follow from the monotonicity of \(\{X^+_i\}, \{X^-_i\}, \{Y^+_i\}\) and \(\{Y^-_i\}\) and the inequality from the fact that \(Y^+_i \geq X^+_i\) and \(Y^-_i \leq X^-_i\).

Recall our earlier definitions of \(B(v, R)\) as the ball of radius \(R\) and \(S(v, R)\) as the external boundary of \(B(v, R)\); that is, \(B(v, R) = \{u \in V : \text{dist}(u, v) \leq R\}\) and let \(S(v, R) = B(v, R + 1) \setminus B(v, R)\). For ease of notation let \(A = B(v, R + 1) = B(v, R) \cup S(v, R)\) and for \(\sigma^+, \sigma^- \in \Omega_A\) let \(\mathcal{F}_3(\sigma^+, \sigma^-)\) be the event \(\{X^+_i(A) = \sigma^+, X^-_i(A) = \sigma^-\}\). Then, for \(t > s\) we have

\[
\Pr[Y^+_i(v) \neq Y^-_i(v) \mid \mathcal{F}_3(\sigma^+, \sigma^-)] \leq \|\Pr[Y^+_i(v) = + \mid \mathcal{F}_3(\sigma^+, \sigma^-)] - \mu(v = + \mid \tau^+)\| + \|\mu(v = + \mid \tau^+) - \mu(v = + \mid \tau^-)\| + \|\Pr[Y^-_i(v) = + \mid \mathcal{F}_3(\sigma^+, \sigma^-)] - \mu(v = + \mid \tau^-)\|,
\]

where \(\mu = \mu_{G, \beta}, \tau^+ = \sigma^+(S(v, R))\) and \(\tau^- = \sigma^-(S(v, R))\).

Observe that \(\mu(\cdot \mid \tau^+)\) and \(\mu(\cdot \mid \tau^-)\) are the stationary measures of \(\{Y^+_i\}\) and \(\{Y^-_i\}\) respectively, and recall that by assumption

\[
\max_{\sigma \in \Omega} T_{\max}(P_{B(v, R)}, \sigma) \leq T.
\]

Hence, for \(t = s + T \log_4[8|A|]\), we have

\[
\|\Pr[Y^+_i(v) = + \mid \mathcal{F}_3(\sigma^+, \sigma^-)] - \mu(v = + \mid \tau^+)\| \leq \frac{1}{8|A|},
\]

and similarly

\[
\|\Pr[Y^-_i(v) = + \mid \mathcal{F}_3(\sigma^+, \sigma^-)] - \mu(v = + \mid \tau^-)\| \leq \frac{1}{8|A|}.
\]

We bound next \(\|\mu(v = + \mid \tau^+) - \mu(v = + \mid \tau^-)\|\). For \(u \in S(v, R)\), let \(a_u\) be defined as in (5) and let \(S(v, R) = \{u_1, u_2, \ldots, u_l\}\) with \(l = |S(v, R)|\). Let \(\tau_0, \tau_1, \ldots, \tau_l\) be a sequence of configurations on \(S(v, R)\) such that \(\tau_j(u_k) = \tau^+(u_k)\) for \(j < k \leq l\) and \(\tau_j(u_k) = \tau^-(u_k)\) for \(1 \leq k \leq j\). That is, \(\tau_0 = \tau^+, \tau_l = \tau^-\) and \(\tau_j\) is obtained from \(\tau_{j-1}\) by changing the spin of \(u_j\) from \(\tau^+(u_j)\) to \(\tau^-(u_j)\). The triangle inequality then implies that

\[
\|\mu(v = + \mid \tau^+) - \mu(v = + \mid \tau^-)\| \leq \sum_{j=1}^{l} \|\mu(v = + \mid \tau_{j-1}) - \mu(v = + \mid \tau_j)\| \\
\leq \sum_{j=1}^{l} \mathbb{I}\{\tau^+(u_j) \neq \tau^-(u_j)\} \cdot a_{u_j} \\
= \sum_{u \in S(v, R)} \mathbb{I}\{\sigma^+(u) \neq \sigma^-(u)\} \cdot a_u.
\]

(14)
Hence, plugging (12) to (14) into (11), we get

\[ \Pr[Y^+_t(v) \neq Y^-_t(v) \mid F^+_s(\sigma^+, \sigma^-)] \leq \frac{1}{4|A|} + \sum_{u \in S(v, R)} \mathbb{1}\{\sigma^+(u) \neq \sigma^-(u)\} \cdot a_u. \]

Now, if \( X^+_t(A) = X^-_t(A) \), then \( Y^+_t(A) = Y^-_t(A) \) for all \( t \geq s \). Therefore,

\[ \Pr[Y^+_t(v) \neq Y^-_t(v)] = \sum_{\sigma^+ \neq \sigma^- \in \Omega_4} \Pr[Y^+_t(v) \neq Y^-_t(v) \mid F^+_s(\sigma^+, \sigma^-)] \Pr[F^+_s(\sigma^+, \sigma^-)] \]

\[ \leq \frac{\Pr[X^+_t(A) \neq X^-_t(A)]}{4|A|} + \sum_{\sigma^+ \neq \sigma^- \in \Omega_4} \sum_{u \in S(v, R)} \mathbb{1}\{\sigma^+(u) \neq \sigma^-(u)\} \cdot a_u \cdot \Pr[F^+_s(\sigma^+, \sigma^-)] \]

\[ = \frac{\Pr[X^+_t(A) \neq X^-_t(A)]}{4|A|} + \sum_{u \in S(v, R)} \Pr[X^+_u(u) \neq X^-_u(u)] \cdot a_u. \]

By union bound,

\[ \frac{\Pr[X^+_t(A) \neq X^-_t(A)]}{4|A|} \leq \frac{1}{4|A|} \sum_{u \in A} \Pr[X^+_u(u) \neq X^-_u(u)] \leq \frac{1}{4} \max_{u \in V} \Pr[X^+_u(u) \neq X^-_u(u)]. \]

Moreover, the ASSM property (see Lemma 10) implies that

\[ \sum_{u \in S(v, R)} \Pr[X^+_u(u) \neq X^-_u(u)] \cdot a_u \leq \max_{u \in V} \Pr[X^+_u(u) \neq X^-_u(u)] \sum_{u \in S(v, R)} a_u \]

\[ \leq \frac{1}{4} \max_{u \in V} \Pr[X^+_u(u) \neq X^-_u(u)]. \]

Thus, we conclude that for every \( v \in V \)

\[ \Pr[X^+_t(v) \neq X^-_t(v)] \leq \Pr[Y^+_t(v) \neq Y^-_t(v)] \leq \frac{1}{2} \max_{v \in V} \Pr[X^+_u(u) \neq X^-_u(u)] \]

for \( t = s + T \log_4 |8|A| \). Taking the maximum over \( v \)

\[ \max_{v \in V} \Pr[X^+_t(v) \neq X^-_t(v)] \leq \frac{1}{2} \max_{v \in V} \Pr[X^+_u(v) \neq X^-_u(v)]. \]

Iteratively, we get that for \( \hat{T} = T \log_4 |8|A| \) \( \log_2 \left[ \frac{\hat{T}}{\epsilon} \right] \)

\[ \max_{v \in V} \Pr[X^+_\hat{T}(v) \neq X^-_\hat{T}(v)] \leq \frac{\epsilon}{n}. \]

This implies that \( T_{\text{mix}}(P, \epsilon) \leq T \log_4 |8|A| \) \( \log_2 \left[ \frac{\hat{T}}{\epsilon} \right] \), so taking \( \epsilon = 1/4 \) it follows that \( T_{\text{mix}}(P) = O(T \log n) \) as desired. Moreover, since for \( \epsilon > 0 \)

\[ (T_{\text{rel}}(P) - 1) \log(2\epsilon)^{-1} \leq T_{\text{mix}}(P, \epsilon), \]

taking \( \epsilon = n^{-1} \) yields that \( T_{\text{rel}}(P) = O(T) \); see Theorem 12.5 in [31].
5 | PROOF OF THEOREM 7

Proof of Theorem 7  By assumption, $X_t$ has distribution $vP^t$ while $\hat{X}_t$ has distribution $v\hat{P}^t$ where $\hat{P}^t = P_{A_1} \ldots P_{A_t}$. Since $\{P_A\}_{A \subseteq V}$ is a censoring for $P$, we have $P \leq P_A$ for all $A \subseteq V$. We show first that this implies $P^t \leq \hat{P}^t$.

Recall that $P_{A_i}$ may be viewed as an operator from $L_2(\mu)$ to $L_2(\mu)$. The reversibility of $P_{A_i}$ w.r.t. $\mu$ implies that $P_{A_i}$ is self-adjoint; that is, $P^*_{A_i} = P_{A_i}$. Also, since $P$ is monotone, $P^k f$ is increasing for any integer $k > 0$ and any increasing function $f$; see Proposition 22.7 in [31]. Combining these facts, we have that for any pair of increasing positive functions $f, g : \mathbb{R}^{[\Omega]} \to \mathbb{R}$

$$\langle f, P^t g \rangle_\mu = \langle f, P(P^{t-1} g) \rangle_\mu \leq \langle f, P_{A_t}(P^{t-1} g) \rangle_\mu = \langle P_{A_t} f, P^{t-1} g \rangle_\mu.$$  

Note also that $P_{A_t}$ is monotone, so $P_{A_t} f$ is increasing. Iterating this argument, we obtain

$$\langle f, P^t g \rangle_\mu \leq \langle P_{A_t} f, P^{t-1} g \rangle_\mu \leq \ldots \leq \langle P_{A_t} \ldots P_{A_1} f, g \rangle_\mu = \langle f, \hat{P}^t g \rangle_\mu.$$  

This shows that $P^t \leq \hat{P}^t$.

To prove $X_t \leq \hat{X}_t$, we need to show that for any increasing function $g$

$$\sum_{\sigma \in \Omega} \nu P^t(\sigma)g(\sigma) \leq \sum_{\sigma \in \Omega} \nu \hat{P}^t(\sigma)g(\sigma). \quad (15)$$

Let $h : \mathbb{R}^{[\Omega]} \to \mathbb{R}$ be the function given by $h(\tau) = \nu(\tau)/\mu(\tau)$ for $\tau \in \Omega$. Then we have

$$\sum_{\sigma \in \Omega} \nu P^t(\sigma)g(\sigma) = \sum_{\sigma \in \Omega} \left( \sum_{\tau \in \Omega} \nu(\tau)P^t(\tau, \sigma) \right)g(\sigma) = \sum_{\sigma, \tau \in \Omega} \nu(\tau)P^t(\tau, \sigma)g(\sigma)$$  

$$= \sum_{\sigma, \tau \in \Omega} \mu(\tau)P^t(\tau, \sigma)g(\sigma) h(\tau) = \langle h, P^t g \rangle_\mu.$$

Similarly,

$$\sum_{\sigma \in \Omega} \nu \hat{P}^t(\sigma)g(\sigma) = \langle h, \hat{P}^t g \rangle_\mu.$$

The function $h$ is increasing by assumption, and thus (15) follows immediately from the fact that $P^t \leq \hat{P}^t$. This establishes part 1 of the theorem. Part 2 of the theorem follows from part 1 and Lemma 2.4 in [40].

6 | BLOCK DYNAMICS

As an application of the technology introduced in Section 3, in this section we study the mixing and relaxation times of the block dynamics. Let $G = (V, E)$ be a graph of maximum degree at most $d$. Let $D = \{B_1, \ldots, B_r\}$ be a family of $r$ subsets of $V$ such that $\cup_{i=1}^r B_i = V$. Given a configuration $\sigma_t \in \Omega$ at time $t$, one step of the block dynamics is given by:

1. Pick $k \in \{1, 2, \ldots, r\}$ uniformly at random;
2. Sample $\sigma_{t+1}(B_k)$ from $\mu(\cdot \mid \sigma_t(V \setminus B_k))$ and set $\sigma_{t+1}(v) = \sigma_t(v)$ for all $v \notin B_k$. 

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[References]

[31] [32] [33] [34] [35] [36] [37] [38] [39] [40]
Let $B^D$ be the transition matrix of the block dynamics with respect to $D$. For ease of notion, we write $B = B^D$ and consider the collection $D$ of blocks to be fixed. We show that when $\beta < \beta_c(d)$ the block dynamics has mixing time $O(r \log n)$ and relaxation time $O(r)$, which proves Theorem 3 from the introduction. This is done using the general framework introduced in Section 3, which requires showing that the block dynamics is monotone, and that it has a censoring.

The following is a standard fact about the block dynamics.

**Lemma 14** For all graphs $G = (V, E)$, all $\beta > 0$ and any collection of blocks $D = \{B_1, \ldots, B_r\}$ such that $\cup_{i=1}^{r} B_i = V$, the block dynamics for the Ising model is monotone.

**Proof** A grand coupling is constructed as follows: let $\{X^\sigma_t\}_{t \geq 0}$ denote the chain that starts from $\sigma \in \Omega$; that is, $X^\sigma_0 = \sigma$. At time $t$, for all chains $\{X^\sigma_t : \sigma \in \Omega\}$ we choose the same uniform random block $B_t$ and then fix some order $\{v_1, \ldots, v_r\}$ of the vertices in $B$. For each $j = 1, \ldots, r'$ and $\sigma \in \Omega$, the spin of $X^\sigma_{t+1}(v_j)$ is sampled from the conditional distribution given $X^\sigma_t(v_1), \ldots, X^\sigma_{t+1}(v_{j-1})$ and $X^\sigma_t(V \setminus B)$. To update each $v_j$, we can use the standard grand coupling for the single-site Glauber dynamics; that is, for all $\sigma \in \Omega$ choose the same uniform random number $r_t(v_j)$ from $[0, 1]$, and set $X^\sigma_{t+1}(v_j)$ to be “+” if and only if

$$r_t(v_j) \leq \mu(v_j = + | X^\sigma_{t+1}(v_1), \ldots, X^\sigma_{t+1}(v_{j-1}), X^\sigma_t(V \setminus B)).$$

It is straightforward to check that this gives a monotone grand coupling.

A censoring for the block dynamics is constructed as follows. For any $A \subseteq V$, let $B_A$ denote the transition matrix of the censored chain that given a configuration $\sigma_t \in \Omega$ generates $\sigma_{t+1}$ as follows:

1. Pick $k \in \{1, 2, \ldots, r\}$ uniformly at random;
2. Sample $\sigma_{t+1}(A \cap B_k)$ from $\mu(\cdot | \sigma_t(V \setminus (A \cap B_k)))$ and set $\sigma_{t+1}(v) = \sigma_t(v)$ for all $v \notin A \cap B_k$.

Note that if the chosen block $B_k$ has no intersection with $A$, then the censored chain $B_A$ will not update any spin in this step.

**Lemma 15** The collection of matrices $\{B_A\}_{A \subseteq V}$ is a censoring for the block dynamics.

Theorem 3 then follows straightforwardly from Lemma 14, Lemma 15 and Theorem 11.

**Proof of Theorem 3** By Lemma 14 the block dynamics is monotone and by Lemma 15 the collection $\{B_A\}_{A \subseteq V}$ is a censoring for $B$. Furthermore, Lemma 10 implies that ASSM holds for some constant $R > 0$. Hence, to apply Theorem 11 it is sufficient to bound $T_{\text{mix}}(B_{B(v, R)})$ for all $v \in V$. For this, we shall use a crude coupling argument. For each vertex $w$ in $B(v, R)$, the probability that $w$ is updated in one step is at least $1/r$ since at least one of the $r$ blocks contains it. Thus, $w$ is not updated after $T = \lceil r \ln(2|B(v, R)|) \rceil$ steps with probability at most

$$\left(1 - \frac{1}{r}\right)^T \leq e^{-T/r} \leq \frac{1}{2|B(v, R)|}.$$  

Consequently, the probability that all vertices in $B(v, R)$ are updated at least once after $T$ steps is at least $1/2$ by union bound.
Now, consider two instances of the block dynamics chain with arbitrary starting configurations in $B(v, R)$ coupled as in the proof of Lemma 14. Suppose a vertex $w \in B(v, R)$ is updated at time $t$. Observe that $w$ is set to “+” with probability at least 
\[
\frac{e^{-\beta d}}{e^{\beta d} + e^{-\beta d}} > \frac{1}{2}e^{-2\beta d}
\]
and to “−” with at least the same probability. Hence, if the uniform random number $r_t(w)$ we pick in the coupling satisfies $r_t(w) \leq \frac{1}{2}e^{-2\beta d}$, then $w$ is set to “+” in both configurations. Similarly, if $r_t(w) \geq 1 - \frac{1}{2}e^{-2\beta d}$, then $w$ is always set to “−”. This implies that the spins at $w$ couple with probability at least $e^{-2\beta d}$ each time $w$ is updated. Moreover, the event $r_t(w) \in [0, \frac{1}{2}e^{-2\beta d}] \cup [1 - \frac{1}{2}e^{-2\beta d}, 1]$ is independent for distinct $w$ and $t$. Therefore, after all vertices in $B(v, R)$ are updated, the probability that the two configurations couple is at least $e^{-2\beta d |B(v, R)|}$.

Combining this with the fact that with probability at least $1/2$ all vertices in $B(v, R)$ are updated after $T$ steps, we get that the probability that two configurations couple at time $T$ is at least
\[
\frac{1}{2} \cdot e^{-2\beta d |B(v, R)|} = \Omega(1).
\]
This shows that for any $v \in V$
\[
T_{\text{mix}}(B_{B(v, R)}) = O(T) = O(r),
\]
and the result then follows from Theorem 11.

We conclude this section with the proof of Lemma 15.

**Proof of Lemma 15** The reversibility of $B_A$ follows by definition for all $A \subseteq V$. Moreover, the argument in the proof of Lemma 14 also shows that $B_A$ is monotone for all $A \subseteq V$. Thus, it suffices to show that $B \leq B_A$ for all $A \subseteq V$.

For any $D \subseteq V$, let $R_D$ be the transition matrix corresponding to the heat-bath update in $D$; namely, sampling vertices in $D$ from the (conditional) Ising distribution conditioned on the configuration of the vertices outside $D$. Thus,
\[
B = \frac{1}{r} \sum_{i=1}^{r} R_{B_i},
\]
and for any $f_1, f_2 \in \mathbb{R}^{[2]}$ we have
\[
\langle f_1, B f_2 \rangle_\mu = \frac{1}{r} \sum_{i=1}^{r} \langle f_1, R_{B_i} f_2 \rangle_\mu.
\]
The corresponding equality holds for $B_A$ and $R_{A \cap B}$. Hence, it suffices to show that for any $i \in \{1, \ldots, r\}$ and any increasing positive functions $f_1, f_2 \in \mathbb{R}^{[2]}$ we have
\[
\langle f_1, R_{B_i} f_2 \rangle_\mu \leq \langle f_1, R_{A \cap B} f_2 \rangle_\mu. \tag{16}
\]
For $D \subseteq V$, the matrix $R_D$ can be viewed as an operator from $L_2(\mu)$ to $L_2(\mu)$. Recall that $R_D^*$ denotes the adjoint operator of $R_D$. The following properties of the block dynamics follow immediately from its
definition: \( R_{B_i} = R_{B_i}^2 = R_{B_i}^* = R_{A \cap B_i}^* = R_{A \cap B_i}^2 = R_{B_i}^* \) and \( R_{B_i} = R_{A \cap B_i} R_{B_i} R_{A \cap B_i} ; \) see [1]. Let \( \hat{f}_1 = R_{A \cap B_i} f_1 \) and \( \hat{f}_2 = R_{A \cap B_i} f_2 \). Then

\[
\langle f_1, R_{A \cap B_i} f_2 \rangle_\mu = \langle f_1, R_{A \cap B_i}^2 f_2 \rangle_\mu = \langle R_{A \cap B_i}, f_1, R_{A \cap B_i}, f_2 \rangle_\mu = \langle \hat{f}_1, \hat{f}_2 \rangle_\mu,
\]

and

\[
\langle f_1, R_{B_i} f_2 \rangle_\mu = \langle f_1, R_{A \cap B_i} R_{B_i} R_{A \cap B_i} f_2 \rangle_\mu = \langle R_{B_i}, R_{A \cap B_i}, f_1, R_{B_i}, R_{A \cap B_i}, f_2 \rangle_\mu = \langle R_{B_i} \hat{f}_1, R_{B_i} \hat{f}_2 \rangle_\mu.
\]

Given \( \sigma \in \Omega \), let \( \rho_\sigma(\cdot) = R_{B_i}(\sigma, \cdot) \); that is, \( \rho_\sigma \) is a distribution over \( \Omega_{B_i} \) that results from applying the transition \( R_{B_i} \) to \( \sigma \). It follows that

\[
R_{B_i} \hat{f}_1(\sigma) = \sum_{\tau \in \Omega} R_{B_i}(\sigma, \tau) \hat{f}_1(\tau) = \mathbb{E}_{\rho_{\sigma}}[\hat{f}_1].
\]

Similarly, we have \( R_{B_i} \hat{f}_2(\sigma) = \mathbb{E}_{\rho_{\sigma}}[\hat{f}_2] \). Notice that the distribution \( \rho_\sigma \) is a (conditional) Ising model distribution on \( B_i \), and thus it is positively correlated for any \( \sigma \in \Omega \); see Theorem 22.16 in [31]. Also, since \( R_{A \cap B_i} \) is monotone, \( \hat{f}_1 \) and \( \hat{f}_2 \) are increasing functions; see, for example, Proposition 22.7 in [31]. This implies that for any \( \sigma \in \Omega \)

\[
R_{B_i} \hat{f}_1(\sigma) R_{B_i} \hat{f}_2(\sigma) = \mathbb{E}_{\rho_{\sigma}}[\hat{f}_1] \mathbb{E}_{\rho_{\sigma}}[\hat{f}_2] \leq \mathbb{E}_{\rho_{\sigma}}[\hat{f}_1 \hat{f}_2].
\]

We then deduce that

\[
\langle R_{B_i} \hat{f}_1, R_{B_i} \hat{f}_2 \rangle_\mu = \sum_{\sigma \in \Omega} R_{B_i} \hat{f}_1(\sigma) R_{B_i} \hat{f}_2(\sigma) \mu(\sigma) \leq \sum_{\sigma \in \Omega} \mathbb{E}_{\rho_{\sigma}}[\hat{f}_1 \hat{f}_2] \mu(\sigma)
\]

\[
= \sum_{\sigma, \tau \in \Omega} \hat{f}_1(\tau) \hat{f}_2(\tau) \rho_{\sigma}(\tau) \mu(\sigma) = \sum_{\sigma, \tau \in \Omega} \hat{f}_1(\tau) \hat{f}_2(\tau) \rho_{\tau}(\sigma) \mu(\tau)
\]

\[
= \langle \hat{f}_1, \hat{f}_2 \rangle_\mu
\]

where in the second to last equality we use the reversibility of \( R_{B_i} \) w.r.t. \( \mu \); namely,

\[
\rho_\sigma(\tau) \mu(\sigma) = R_{B_i}(\sigma, \tau) \mu(\sigma) = R_{B_i}(\tau, \sigma) \mu(\tau) = \rho_\tau(\sigma) \mu(\tau)
\]

for all \( \sigma, \tau \in \Omega \). This shows that (16) holds for any two increasing positive functions \( f_1, f_2 \) and the theorem follows.

\[\square\]

7 | MONOTONE SW DYNAMICS

As a second application of our technology for analyzing monotone Markov chains, in this section we consider a monotone variant of the SW dynamics, which we call the Monotone SW dynamics. Let \( G = (V, E) \) be an arbitrary graph of maximum degree \( d = O(1) \). Given an Ising configuration \( \sigma_t \in \Omega \) at time \( t \), one step of the Monotone SW dynamics is given by:

1. Consider the set of agreeing edges \( E(\sigma_t) = \{(v, w) \in E : \sigma_t(v) = \sigma_t(w)\} \);
2. Independently for each edge \( e \in E(\sigma_t) \), delete \( e \) with probability \( \exp(-2\beta) \) and keep \( e \) with probability \( 1 - \exp(-2\beta) \); this yields \( F_t \subseteq E(\sigma_t) \);
3. For each connected component $C \subseteq V$ in the subgraph $(V, F_t)$ do the following:
   (i) With probability $1/2^{|C|-1}$, choose a spin $s_C$ uniformly at random from $\{+, -\}$ and assign spin $s_C$ to all vertices in $C$;
   (ii) Otherwise, with probability $1 - 1/2^{|C|-1}$, every vertex in $C$ keeps the same spin as in $\sigma_t$.

We denote the transition matrix of this chain by $\mathcal{MSW}$. Notice that in step 3, all isolated vertices (components of size 1) are updated with new random spins, and that the probability of a connected component $C$ being updated decays exponentially with the size of $C$. For comparison, recall that in the SW dynamics all connected components are updated in each step, while in the Isolated-vertex dynamics only the isolated vertices are. Using the machinery from [50], we can show that $\mathcal{MSW}$ is ergodic and reversible with respect to $\mu$.

**Claim 16** For all graphs $G$ and all $\beta > 0$, the Monotone SW dynamics for the Ising model is ergodic and reversible with respect to $\mu$.

The proof of Claim 16 is postponed to Section 7.2. We show next that, just like the Isolated-vertex dynamics, the Monotone SW dynamics also has mixing time $O(\log n)$ and relaxation time $\Theta(1)$ when $\beta < \beta_c(d)$. That is, we establish Theorem 4 from the introduction. For this, we use the framework from Section 3 for monotone Markov chains. The first step is then to establish that the Monotone SW dynamics is indeed monotone.

**Lemma 17** For all graphs $G$ and all $\beta > 0$, the Monotone SW dynamics for the Ising model is monotone.

The proof of Lemma 17 is provided in Section 7.1. Next, we construct a censoring for the Monotone SW dynamics as follows. For $A \subseteq V$, let $\mathcal{MSW}_A$ be the transition matrix for the Markov chain that given a configuration $\sigma_t \in \Omega$ generates $\sigma_{t+1}$ as follows:

1. Consider the set of agreeing edges $E(\sigma_t) = \{(v, w) \in E : \sigma_t(v) = \sigma_t(w)\}$;
2. Independently for each edge $e \in E(\sigma_t)$, delete $e$ with probability $\exp(-2\beta)$ and keep $e$ with probability $1 - \exp(-2\beta)$; this yields $F_t \subseteq E(\sigma_t)$;
3. For each connected component $C \subseteq V$ of the subgraph $(V, F_t)$ that is completely contained in $A$ (i.e., $C \subseteq A$), do the following:
   (i) With probability $1/2^{|C|-1}$, choose a spin $s_C$ uniformly at random from $\{+, -\}$ and assign spin $s_C$ to all vertices in $C$;
   (ii) Otherwise, with probability $1 - 1/2^{|C|-1}$, every vertex in $C$ keeps the same spin as in $\sigma_t$;

   All other vertices keep the same spin as in $\sigma_t$.

We emphasize that in step 3 only the components completely contained in $A$ may be assigned new random spins.

**Lemma 18** The collection of matrices $\{\mathcal{MSW}_A\}_{A \subseteq V}$ is a censoring for the Monotone SW dynamics.

The proof of Lemma 18 is provided in Section 7.2.

Theorem 4 then follows immediately from Lemma 17, Lemma 18, and Theorem 11.

**Proof of Theorem 4** By Lemma 17 the Monotone SW dynamics is monotone, and by Lemma 18 the collection $\{\mathcal{MSW}_A\}_{A \subseteq V}$ is a censoring for $\mathcal{MSW}$. Lemma 10 shows that there exists a constant
Step 3 of the Monotone SW dynamics is equivalent to the following two steps:

Proof of Lemma 17

We use the grand coupling from the proof of Lemma 17 with a slight modification. Namely, in step 3(ii) a component $C$ is updated under the additional condition that $C \subseteq A$. Suppose \{$X_t^\sigma\}$, \{$X_t^\tau\}$

7.1 Monotonicity of the Monotone SW dynamics

In this section, we prove Lemma 17. The proof is similar to that of Lemma 5 for the Isolated-vertex dynamics.

Proof of Lemma 17  Step 3 of the Monotone SW dynamics is equivalent to the following two steps:

3’. For each vertex $v \in V$, choose a spin $s_v$ uniformly at random from $\{+, -\}$;

4’. For each connected component $C = \{v_1, \ldots, v_k\}$ in the subgraph $(V, F_i)$ where $k = |C|:

(a) If $s_{v_1} = s_{v_2} = \cdots = s_{v_i}$, then assign spin $s_v$ to $v_i$ for all $i$;

(b) Otherwise, each $v_i$ keeps the same spin as in $\sigma$.

In step 4’, the probability that a connected component $C$ is updated with the new spin is exactly $1/2^{|C|-1}$; thus, step 3’ and 4’ are equivalent to step 3 of the Monotone SW dynamics.

Let $(X_t^\sigma)_{t \geq 0}$ be an instance of the Monotone SW dynamics starting from $\sigma \in \Omega$; that is, $X_0^\sigma = \sigma$. We construct a grand coupling for the Monotone SW dynamics as follows. At time $t$:

1. For every edge $e \in E$, pick a number $r(e)$ uniformly at random from $[0, 1]$;
2. For every vertex $v \in V$, choose a uniform random spin $s_v(v)$ from $\{+, -\}$;
3. For every $\sigma \in \Omega$:

(a) Obtain $F_t^\sigma \subseteq E$ by including the edge $e = \{u, v\}$ in $F_t^\sigma$ iff $X_t^\sigma(u) = X_t^\sigma(v)$ and $r(e) \leq 1 - e^{-2\beta}$;

(b) For every connected component $C = \{v_1, \ldots, v_k\}$ in the subgraph $(V, F_t^\sigma)$ and every $i = 1, \ldots, k$, set $X_{t+1}^\sigma(v_i) = s_i(v)$ if $s_i(v_1) = \cdots = s_i(v_k)$; otherwise, set $X_{t+1}^\sigma(v_i) = X_t^\sigma(v_i)$.

We show next that this grand coupling is monotone.

Suppose $X_t^\sigma \geq X_t^\tau$. We need to show that $X_{t+1}^\sigma \geq X_{t+1}^\tau$. Let $v \in V$ be arbitrary. If the spin of $v$ is updated in step 3(ii) in both $X_{t+1}^\sigma$ and $X_{t+1}^\tau$, then $X_{t+1}^\sigma(v) = s_t(v) = X_{t+1}^\tau(v)$. Similarly, if $v$ keeps its original spin in step 3(ii) in both $X_{t+1}^\sigma$ and $X_{t+1}^\tau$, then $X_{t+1}^\sigma(v) = X_t^\sigma(v) \geq X_t^\tau(v) = X_{t+1}^\tau(v)$.

It remains to consider the cases where $v$ is updated in exactly one of $X_{t+1}^\sigma$ and $X_{t+1}^\tau$. Assume first that $v$ is updated in $X_{t+1}^\sigma$ but keeps its original spin in $X_{t+1}^\tau$. Suppose for sake of contradiction that $X_{t+1}^\sigma(v) < X_{t+1}^\tau(v)$; that is, $X_t^\sigma(v) = +, X_t^\tau(v) = +$. Then, $X_t^\sigma(v) = X_t^\tau(v) = +$ by assumption. Let $C_t^\sigma(v)$ (resp., $C_t^\tau(v)$) be the connected component in the subgraph $(V, F_t^\sigma)$ (resp., $(V, F_t^\tau)$) that contains $v$. Since $X_t^\sigma \geq X_t^\tau$, all vertices assigned “+” in $X_t^\tau$ are also “+” in $X_t^\tau$; thus, by the way edges are coupled we have $C_t^\sigma(v) \subseteq C_t^\tau(v)$. The fact that $v$ is updated in $X_{t+1}^\sigma$ implies that $s_t(u) = s_t(v)$ for all $u \in C_t^\sigma(v)$, and in particular, for all $u \in C_t^\tau(v)$. Then, $v$ should also be updated in $X_{t+1}^\tau$, which contradicts our assumption. The case in which $v$ is updated in $X_{t+1}^\tau$ but not in $X_{t+1}^\sigma$ follows by an analogous argument.

In similar manner, we can show that $\mathcal{M}SW_A$ is also monotone for all $A \subseteq V$.

Corollary 19  $\mathcal{M}SW_A$ is monotone for all $A \subseteq V$.

Proof  We use the grand coupling from the proof of Lemma 17 with a slight modification. Namely, in step 3(ii) a component $C$ is updated under the additional condition that $C \subseteq A$. Suppose \{$X_t^\sigma\}$, \{$X_t^\tau\}$
are two instances of $\mathcal{MSW}_A$ starting from $\sigma$ and $\tau$, respectively, and that $X^\sigma_t \geq X^\tau_t$. Let $v \in V$. If $v$ is updated in both $X^\sigma_{t+1}$ and $X^\tau_{t+1}$, or it is updated in neither of the two chains, then $X^\sigma_{t+1}(v) \geq X^\tau_{t+1}(v)$. Now, suppose $v$ is updated in $X^\sigma_{t+1}$ but not in $X^\tau_{t+1}$, and for the sake of contradiction that $X^\sigma_{t+1}(v) < X^\tau_{t+1}(v)$. Then, $X^\sigma_{t+1}(v) = -X^\tau_{t+1}(v) = +$ and $X^\tau_{t+1}(v) = X^\tau_t(v) = +$. This implies that all vertices in $C^\tau_t(v)$ (the connected component in $(V,F^\tau)$ containing $v$) receive the same uniform random spin and that $C^\tau_t(v) \subseteq A$. Since $C^\tau_t(v) \subseteq C^\sigma_t(v)$, the same property holds for $C^\sigma_t(v)$. Thus, $v$ is also updated in $X^\tau_{t+1}$, leading to a contradiction. The case when $v$ is updated in $X^\tau_{t+1}$ but not in $X^\sigma_{t+1}$ follows by an analogous argument.

7.2 | Censoring for the Monotone SW dynamics

In this section we prove Lemma 18. The ideas in this proof are similar to those in the proof of Lemma 8. Namely, we introduce a “joint” configuration space denoted by $\Omega^m$. Configurations in $\Omega^m$ are triples $(F, \sigma, C)$ where $F$ is a subset of the edges, $\sigma$ a spin assignment to the vertices and $C$ a set of “marked” connected components of the subgraph $(V,F)$. We show that the transition matrix $\mathcal{MSW}$ of the Monotone SW dynamics can then be decomposed as the product of five matrices, four of which correspond to projections or liftings between the spaces $\Omega$, $\Omega_j$ and $\Omega^m_j$ and one that corresponds to a trivial resampling in $\Omega^m_j$.

**Proof of Lemma 18** For all $A \subseteq V$, we need to show that $\mathcal{MSW}_A$ is reversible with respect to $\mu$, monotone and that $\mathcal{MSW} \leq \mathcal{MSW}_A$. Monotonicity of $\mathcal{MSW}_A$ follows from Corollary 19. To prove the other two facts, we establish a decomposition of the matrices $\mathcal{MSW}_A$ and $\mathcal{MSW}$ as a product of simpler matrices, in similar fashion to what was done for the matrices $I V_A$ and $I V$ in Section 3.2.

Recall that $\Omega_j = 2^E \times \Omega$ is the joint configuration space. For $F \subseteq E$, let $(F)\Omega_j$ denote the set of all connected components of the subgraph $(V,F)$. We define the marked joint configuration space $\Omega^m_j \subseteq 2^E \times \Omega \times 2^J$ by

$$\Omega^m_j = \{(F, \sigma, C) : (F, \sigma) \in \Omega_j, C \subseteq (F)\Omega_j\}.$$  

Connected components in $C$ are said to be marked. Observe that both $\mathcal{MSW}$ and $\mathcal{MSW}_A$ “lift” a configuration from $\Omega$ to one in $\Omega_j$ (by adding the edges in step 2), which is then lifted to a configuration in $\Omega^m_j$ (by marking the components that will be updated in step 3).

Let $\nu_m$ be the marked joint measure on $\Omega^m_j$ where each $(F, \sigma, C) \in \Omega^m_j$ is assigned probability

$$\nu_m(F, \sigma, C) = \nu(F, \sigma) \prod_{C \in C} \frac{1}{2^{|C|-1}} \prod_{\substack{C \in (F)\Omega_j \setminus C \in C}} \left(1 - \frac{1}{2^{|C|-1}}\right).$$

Recall that $\nu$ is the joint Edwards-Sokal measure defined in (7). Drawing a sample from the marked joint measure $\nu_m$ can be achieved in the following way: first draw a sample $(F, \sigma)$ from the joint measure $\nu$, and then for each connected component $C \in (F)\Omega_j$ independently mark $C$ (i.e., include $C$ in $C$) with probability $1/2^{|C|-1}$.

Let $S$ be the $|\Omega_j| \times |\Omega^m_j|$ matrix given by

$$S((F_1, \sigma), (F_2, \tau, C)) = \mathbb{1}\left((F_1, \sigma) = (F_2, \tau) \right) \prod_{C \in C} \frac{1}{2^{|C|-1}} \prod_{C \in (F_2)\Omega_j} \left(1 - \frac{1}{2^{|C|-1}}\right),$$

where $(F_1, \sigma) \in \Omega_j$ and $(F_2, \tau, C) \in \Omega^m_j$. Hence, the matrix $S$ corresponds to the process of marking the connected components of a joint configuration as described above. Let $L_2(\nu_m)$ denote the Hilbert space $(\mathbb{R}^{[\Omega^m_j]}, \langle \cdot, \cdot \rangle_{\nu_m})$. We can view $S$ as an operator from $L_2(\nu_m)$ to $L_2(\nu)$. The adjoint operator of $S$ can be obtained straightforwardly.
Claim 20  The adjoint operator $S^* : L_2(\nu) \rightarrow L_2(\nu_m)$ of $S$ is given by the $|\Omega^m_j| \times |\Omega_j|$ matrix
\[
S^* ((F_2, \tau, C), (F_1, \sigma)) = \mathbb{1} ((F_2, \tau) = (F_1, \sigma)) .
\]

Note that $S^*$ corresponds to dropping all the marks from the components to obtain a configuration in $\Omega_j$.

For $A \subseteq V$ and $F \subseteq E$, let $C_A(F)$ be the subset of $C(F)$ that contains all the connected components completely contained in $A$; let $C_A^c(F) = C(F) \setminus C_A(F)$. For $F \subseteq E$ and $C \subseteq C(F)$, let $C' = C(F) \setminus C$ be the set of all unmarked connected components. Also recall that for $\sigma \in \Omega$, $E(\sigma) = \{\{u, v\} \in E : \sigma(u) = \sigma(v)\}$. Given $A \subseteq V$, we define an $|\Omega^m_j| \times |\Omega^m_j|$ matrix $K_A$ indexed by the configurations of $\Omega^m_j$, which corresponds to resampling all marked connected components that are completely contained in $A$. That is, for $(F_1, \sigma, C_1), (F_2, \tau, C_2) \in \Omega^m_j$:
\[
K_A((F_1, \sigma, C_1), (F_2, \tau, C_2)) = \mathbb{1}(F_1 = F_2) \mathbb{1}(F_1 \subseteq E(\sigma) \cap E(\tau)) \mathbb{1}(C_1 = C_2) \mathbb{1}(\sigma(C_1^c(F_1)) = \tau(C_1^c(F_1))) \mathbb{1}(\sigma(C_1^c) = \tau(C_1^c)) \cdot 2^{-|C_A(F)|} .
\]

where for a collection $U$ of subsets of vertices, we write $\sigma(U') = \tau(U')$ if $\sigma(U) = \tau(U)$ for all $U \in U'$.

The matrix $K_A$ defines an operator from $L_2(\nu_m)$ to $L_2(\nu_m)$. In the following claim, which is proved later, several key properties of the matrix $K_A$ are established.

Claim 21  For all $A \subseteq V$, $K_A = K_A^2 = K_A^*$. For ease of notation we set $K = K_V$. Recall that the matrix $T$ defined in (8) is an operator from $L_2(\nu)$ to $L_2(\mu)$ and $T^*$ defined in (9) is its adjoint operator. The following claim is an analogue of Fact 4.5 in [1] for the Monotone SW dynamics.

Claim 22  For all $A \subseteq V$, $\mathcal{M}SW_A = TSK_A S^* T^*$. The reversibility of $\mathcal{M}SW_A$ with respect to $\mu$ follows immediately from Claims 21 and 22 for all $A \subseteq V$:
\[
\mathcal{M}SW_A^* = (TSK_A S^* T^*)^* = TSK_A S^* T^* = \mathcal{M}SW_A ,
\]
so it is self-adjoint and thus reversible.

To establish that $\mathcal{M}SW \leq \mathcal{M}SW_A$, it is sufficient to show that for every pair of increasing and positive functions $f_1, f_2 : \mathbb{R}^{[\Omega]} \rightarrow \mathbb{R}$ on $\Omega$, we have
\[
\langle f_1, \mathcal{M}SW f_2 \rangle_\mu \leq \langle f_1, \mathcal{M}SW_A f_2 \rangle_\mu .
\]

Let $\hat{f}_1 = K_A S^* T^* f_1$ and $\hat{f}_2 = K_A S^* T^* f_2$. Then, using Claims 21 and 22,
\[
\langle f_1, \mathcal{M}SW f_2 \rangle_\mu = \langle f_1, TSK_A S^* T^* f_2 \rangle_\mu = \langle K_A S^* T^* f_1, K_A S^* T^* f_2 \rangle_{\nu_m} = \langle \hat{f}_1, \hat{f}_2 \rangle_{\nu_m} .
\]

Similarly, since $K = K_A K^2 K_A$ by Claim 21, we have
\[
\langle f_1, \mathcal{M}SW f_2 \rangle_\mu = \langle f_1, TSK_A K^2 K_A S^* T^* f_2 \rangle_\mu = \langle K_A S^* T^* f_1, K_A S^* T^* f_2 \rangle_{\nu_m} = \langle \hat{f}_1, \hat{f}_2 \rangle_{\nu_m} .
\]

Thus, it is sufficient for us to show that $\langle K_A \hat{f}_1, \hat{f}_2 \rangle_{\nu_m} \leq \langle \hat{f}_1, \hat{f}_2 \rangle_{\nu_m}$. Consider the partial order on $\Omega^m_j$ where $(F_1, \sigma, C_1) \geq (F_2, \tau, C_2)$ if and only if $F_1 = F_2, C_1 = C_2$ and $\sigma \geq \tau$. The following property of the matrix $K_A S^* T^*$ will be useful.
Claim 23 Suppose \( f : \mathbb{R}^{\Omega \setminus m} \rightarrow \mathbb{R} \) is an increasing positive function. Then, \( \hat{f} : \mathbb{R}^{\Omega \setminus m} \rightarrow \mathbb{R} \) where \( \hat{f} = K_{\Lambda}S^{*}T^{*}f \) is also increasing and positive.

Given \( \omega \in \Omega \setminus m \), let \( \rho_{\omega} = K(\omega, \cdot) \) be the distribution over \( \Omega \setminus m \) that results after applying \( K \) (i.e., assigning uniform random spins to all marked connected components) from \( \omega \). We get

\[
K_{\Lambda}f_{1}(\omega) = \sum_{\omega' \in \Omega \setminus m} K(\omega, \omega')f_{1}(\omega') = \mathbb{E}_{\rho_{\omega}}[f_{1}]
\]

and similarly \( K_{\Lambda}f_{2}(\omega) = \mathbb{E}_{\rho_{\omega}}[f_{2}] \). Given \( \omega \), the distribution \( \rho_{\omega} \) is a product distribution over the spin assignments of all the marked connected components in \( \omega \). Therefore, \( \rho_{\omega} \) is positive correlated for any \( \omega \in \Omega \setminus m \) by Harris inequality (see, e.g., Lemma 22.14 in [31]). Since \( f_{1} \) and \( f_{2} \) are increasing by Claim 23, we deduce that for any \( \omega \in \Omega \setminus m \)

\[
K_{\Lambda}f_{1}(\omega)K_{\Lambda}f_{2}(\omega) = \mathbb{E}_{\rho_{\omega}}[f_{1}] \mathbb{E}_{\rho_{\omega}}[f_{2}] \leq \mathbb{E}_{\rho_{\omega}}[\hat{f}_{1}\hat{f}_{2}].
\]

Hence,

\[
\langle K_{\Lambda}f_{1}, K_{\Lambda}f_{2} \rangle_{v_{m}} = \sum_{\omega \in \Omega \setminus m} K_{\Lambda}f_{1}(\omega)K_{\Lambda}f_{2}(\omega)v_{m}(\omega) \leq \sum_{\omega \in \Omega \setminus m} \mathbb{E}_{\rho_{\omega}}[\hat{f}_{1}\hat{f}_{2}]v_{m}(\omega)
\]

\[
= \sum_{\omega, \omega' \in \Omega \setminus m} \hat{f}_{1}(\omega')\hat{f}_{2}(\omega')\rho_{\omega}(\omega')v_{m}(\omega) = \sum_{\omega, \omega' \in \Omega \setminus m} \hat{f}_{1}(\omega')\hat{f}_{2}(\omega')\rho_{\omega}(\omega')v_{m}(\omega')
\]

\[
\leq \langle \hat{f}_{1}, \hat{f}_{2} \rangle_{v_{m}}
\]

where the second to last equality follows from the fact that \( K \) is reversible with respect to \( v_{m} \); namely,

\[
\rho_{\omega}(\omega')v_{m}(\omega') = K(\omega, \omega')v_{m}(\omega') = K(\omega', \omega)v_{m}(\omega') = \rho_{\omega'}(\omega)v_{m}(\omega').
\]

Hence, (17) holds for every pair of increasing positive functions and the theorem follows.

\[ \blacksquare \]

7.3 Proof of auxiliary facts

In this section we give proofs to Claims 16, 20, 21, 22, and 23.

Proof of Claim 16 In one step of the Monotone SW dynamics, there is a positive probability that all vertices are isolated, in which case each vertex receives a uniform random spin. Thus, for any \( \sigma, \tau \in \Omega \) we have \( MSW(\sigma, \tau) > 0 \). This implies that the chain is ergodic (i.e., irreducible and aperiodic). The reversibility of \( MSW \) with respect to \( \mu \) follows from Claims 21 and 22: \( MSW^{*} = (T_{\Lambda}S^{*}T^{*})^{*} = T_{\Lambda}S^{*}T^{*} = MSW \), so it is self-adjoint and thus reversible.

\[ \blacksquare \]

Proof of Claim 20 We need to show that for any \( f \in \mathbb{R}^{\Omega \setminus m} \) and \( g \in \mathbb{R}^{\Omega \setminus m} \) we have \( \langle f, Sg \rangle_{v} = \langle S^{*}f, g \rangle_{v_{m}} \). Since

\[
\langle f, Sg \rangle_{v} = \sum_{(F_{1}, \sigma) \in \Omega \setminus m} v(F_{1}, \sigma)f(F_{1}, \sigma)Sg(F_{1}, \sigma)
\]

\[
= \sum_{(F_{1}, \sigma) \in \Omega \setminus m} v(F_{1}, \sigma)S((F_{1}, \sigma), (F_{2}, \tau, C))f(F_{1}, \sigma)g(F_{2}, \tau, C)
\]
Proof of Claim 21

The matrix $\mathcal{A}$ contains in $\mathcal{A}$ this shows that

Moreover, for $F, \tau, C \in \Omega_1^m$ we have

This follows immediately from the definition of the matrices $S$ and $S^*$:

Hence, $S^*$ is the adjoint operator of $S$.

Proof of Claim 22

The matrix $K_A$ is symmetric for any $A \subseteq V$, and for $(F, \sigma, C), (F, \tau, C) \in \Omega_1^m$ we have

Moreover, for $(F_1, \sigma, C_1), (F_2, \tau, C_2) \in \Omega_1^m$ such that $K_A(\{(F_1, \sigma, C_1), (F_2, \tau, C_2)\}) \neq 0$, we have $F_1 = F_2$ and $C_1 = C_2$. Combining these facts, we get

This shows that $K_A$ is reversible with respect to $\nu_m$ and so $K_A^* = K_A$ for all $A \subseteq V$.

Since the matrix $K_A$ assigns an independent uniform random spin to each marked connected component contained in $A$, doing this process twice is equivalent to doing it once. This gives $K_A^2 = K_A = K_A^*$ for all $A \subseteq V$ as claimed.

Proof of Claim 22

We will prove the special case where $A = V$. The same argument works for arbitrary $A \subseteq V$. Recall that for $\sigma \in \Omega$, $E(\sigma) = \{(u, v) \in E : \sigma(u) = \sigma(v)\}$. For any $\sigma, \tau \in \Omega$, we have

where

and

and

and

and
Moreover, direct calculations show that for any \( \sigma \in \Omega \) and any \( (F, \tau, C) \in \Omega^m_J \) we have

\[
\text{TSK}(\sigma, (F, \tau, C)) = \mathbb{1}(F \subseteq E(\sigma) \cap E(\tau)) \mathbb{1}(C \subseteq C(F)) \cdot p^{F(1 - p)^{|E(\sigma) \setminus F|}} \cdot \prod_{C \in \mathcal{C}} \left( \frac{1}{2|C|-1} \cdot \frac{1}{2} \right) \prod_{C \in \mathcal{C}(F) \setminus C} \left( 1 - \frac{1}{2|C|-1} \right) \mathbb{1}(\sigma(C) = \tau(C)),
\]

and for any \( (F, \xi, C) \in \Omega^m_J, \tau \in \Omega \)

\[
S^*T^*((F, \xi, C), \tau) = \mathbb{1}(\xi = \tau).
\]

Therefore, we deduce that for any \( \sigma, \tau \in \Omega \),

\[
\text{TSKS}^*T^*(\sigma, \tau) = \sum_{(F, \xi, C) \in \Omega^m_J} \text{TSK}(\sigma, (F, \xi, C))S^*T^*((F, \xi, C), \tau)
\]

\[
= \sum_{(F, \tau, C) \in \Omega^m_J} \text{TSK}(\sigma, (F, \tau, C))
\]

\[
= \mathcal{MSW}(\sigma, \tau).
\]

This implies that \( \mathcal{MSW} = \text{TSKS}^*T^* \) as claimed. \( \blacksquare \)

**Proof of Claim 23** By the definition of the matrices \( T^* \) and \( S^* \), we have \( S^*T^*f(F, \sigma, C) = f(\sigma) \) for any \( (F, \sigma, C) \in \Omega^m_J \). Suppose \( (F, \sigma, C), (F, \tau, C) \in \Omega^m_J \) and \( \sigma \geq \tau \). Then,

\[
\hat{f}(F, \sigma, C) = K_A S^*T^*f(F, \sigma, C) = \sum_{(F', \sigma', C') \in \Omega^m_J} K_A((F, \sigma, C),(F', \sigma', C')) f(\sigma').
\]

Recall that \( C_A(F) \) is the set of all connected components in \((V, F)\) that are completely contained in \( A \). Let

\[
U_A = U_A(F, C) = \bigcup_{C \in C_A(F) \cap C} C
\]

be the subset of vertices in the marked components completely contained in \( A \). Let \( \Phi_{U_A} \subseteq \{+,-\}^{U_A} \) be the set of all spin configurations on \( U_A \) such that vertices from the same component receive the same spin. For \( \xi \in \Phi_{U_A} \), we use \( \sigma_\xi \) (resp., \( \tau_\xi \)) to denote the configuration obtained from \( \sigma \) (resp., \( \tau \)) by replacing the spins of \( U_A \) with \( \xi \). Then, \( \sigma_\xi \geq \tau_\xi \) for any \( \xi \in \Phi_{U_A} \). By definition of the matrix \( K_A \), for any \( (F, \sigma, C), (F', \sigma', C') \in \Omega^m_J \), \( K_A((F, \sigma, C),(F', \sigma', C')) = 2^{-|C_A(F) \cap C|} \) if and only if \( F = F', C = C' \) and \( \sigma, \sigma' \) differ only in \( U_A \); otherwise, it equals 0. Thus, we deduce that

\[
\hat{f}(F, \sigma, C) = 2^{-|C_A(F) \cap C|} \sum_{\xi \in \Phi_{U_A}} f(\sigma_\xi) \geq 2^{-|C_A(F) \cap C|} \sum_{\xi \in \Phi_{U_A}} f(\tau_\xi) = \hat{f}(F, \tau, C).
\]

Hence, \( \hat{f} \) is increasing for any \( A \subseteq V \). \( \blacksquare \)

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