HEINTZE-KARCHER INEQUALITY AND CAPILLARY HYPERSURFACES IN A WEDGE

XIAOHAN JIA, GUOFANG WANG, CHAO XIA, AND XUWEN ZHANG

Abstract. In this paper, we utilize the method of Heintze-Karcher to prove a "best" version of Heintze-Karcher-type inequality for capillary hypersurfaces in the half-space or in a wedge. One of new crucial ingredients in the proof is modified parallel hypersurfaces which are very natural to be used to study capillary hypersurfaces. A more technical part is a subtle analysis along the edge of a wedge. As an application, we classify completely embedded capillary constant mean curvature hypersurfaces that hit the edge in a wedge, which is a subtler case.

MSC 2020: 53C24, 35J25, 53C21
Keywords: Heintze-Karcher’s inequality, capillary hypersurface, CMC hypersurface, Alexandrov’s theorem.

1. Introduction

The study of capillary surfaces goes back to Thomas Young, who studied in 1805 the equilibrium state of liquid fluids. It was he who first introduced the notion of mean curvature and the boundary contact angle condition of capillarity, the so-called Young’s law. This problem was reintroduced and reformulated by Laplace and by Gauss later. For the history of capillary surfaces see Finn’s survey [14].

A capillary hypersurface $\Sigma$ in $\mathbb{R}^{n+1}$ with boundary $\partial \Sigma$ on a support hypersurface $S \subset \mathbb{R}^{n+1}$ is a critical point of the following functional

$$|\Sigma| - \cos \theta |\tilde{\Omega} \cap S|$$

among all compact hypersurfaces with boundary $\partial \Sigma$ on $S$ under a volume constraint. Here $\Omega$ is the bounded domain enclosed by $\Sigma$ and $S$, and $|\Sigma|$ is the $n$-dimensional area of $\Sigma$, and $\theta \in (0, \pi)$. Equivalently, a capillary hypersurface is a constant mean curvature (CMC) hypersurface with boundary which intersects the support hypersurface $S$ at a constant angle $\theta$. There has been a lot of interdisciplinary investigations on the stationary solutions and local minimizers of the above energy. For the interested reader, we refer to Finn’s book [13], which is an excellent survey on capillary surfaces.

Inspired by the recent development of the min-max theory for minimal surfaces and CMC surfaces [5, 29, 30], there have been a lot of works on free boundary minimal surfaces and CMC surfaces, which are a special class of capillary surfaces.

This work is supported by the NSFC (Grant No. 11871406, 12271449, 12126102).
with $\theta = \frac{\pi}{2}$, see for example [9, 16, 27, 44]. Very recently the min-max theory for capillary surfaces was developed in [10, 24].

One important application of the capillary surfaces was recently obtained by Chao Li in [23], where he utilizes capillary surfaces in a polyhedron to study Gromov’s dihedral rigidity conjecture. His work on capillary surfaces is related to this paper, in which we will consider capillary surfaces supported on a wedge.

The main objective of this paper is to make a complete classification of embedded capillary hypersurfaces in a wedge. Such a hypersurface can be seen as a model for capillary hypersurfaces in Riemannian polyhedra. For the precise definition of a wedge, see below in the Introduction. Our starting point is the Heintze-Karcher inequality. Let us first recall Heintze-Karcher’s theorem and Heintze-Karcher’s inequality for closed hypersurfaces.

In a seminal paper [18], Heintze-Karcher proved a general tubular volume comparison theorem for embedded Riemannian submanifolds, which generalizes the celebrated Bishop-Gromov’s volume comparison theorem in Riemannian geometry. For an embedded closed hypersurface $\Sigma$, which encloses a bounded domain $\Omega$, in an $(n + 1)$-dimensional Riemannian manifold of nonnegative Ricci curvature, Heintze-Karcher’s theorem reads as follows,

$$|\Omega| \leq \int_\Sigma \int_0^{c(p)} \left(1 - \frac{H(p)}{n} t\right)^n \, dt \, dA.$$  \hspace{1cm} (1)

Here $H(p)$ is the mean curvature of $\Sigma$ at $p$ and $c(p)$ is the length to reach the first focal point of $\Sigma$ from $p$ by the normal exponential map. As a direct consequence of (1), one deduces that

$$|\Omega| \leq \frac{n}{n + 1} \int_\Sigma \frac{1}{H} \, dA,$$  \hspace{1cm} (2)

provided that $\Sigma$ is strictly mean convex, namely, $H > 0$ on $\Sigma$. Nowadays, (2) is literally referred to as Heintze-Karcher’s inequality in hypersurfaces theory. A well-known new proof via Reilly’s formula [35] has been given by Ros [36]. Moreover, Ros [36] utilized the Heintze-Karcher inequality (2) to reprove the celebrated Alexandrov’s soap bubble theorem, which states that any embedded closed constant mean curvature hypersurfaces in $\mathbb{R}^{n+1}$ must be a round sphere.

Since then various Heintze-Karcher-type inequalities have been established in various circumstance. For instance, Montiel-Ros [32] and Brendle [3] established Heintze-Karcher-type inequalities in space forms and in certain warped product manifolds respectively, see also [26, 34]. The Heintze-Karcher inequality in $\mathbb{R}^{n+1}$ has been also established for sets of finite perimeter, see e.g. [11, 37]. Like the Alexandrov-Fenchel inequalities, the Heintze-Karcher inequality becomes one of fundamental geometric inequalities in differential geometry.

Inspired by the method of Ros [36], and also by the work of Brendle [3], we have proved a Heintze-Karcher-type inequality for hypersurfaces with free boundary in a unit ball [41] by using a generalized Reilly’s formula proved by Qiu-Xia [34]. However this method leads to a slightly different inequality if we consider hypersurfaces with capillary boundary in the unit ball or in the half-space in the
very recent work, [21]. Precisely we have established in the previous work [21] a version of Heintze-Karcher-type inequality for hypersurfaces in the half-space with capillary boundary, by using the solution to a mixed boundary value problem in the classical Reilly’s formula. Let 
\[ R^{n+1}_{+} := \{ x \in \mathbb{R}^{n+1} : \langle x, E_{n+1} \rangle > 0 \} \]
where \( E_{n+1} = (0, \cdots, 0, 1) \), and for an embedded, compact, strictly mean-convex hypersurface \( \Sigma \subset \mathbb{R}^{n+1}_{+} \) with capillary boundary with a constant contact angle \( \theta_0 \in (0, \pi/2] \), there holds
\[
\int_{\Sigma} \frac{1}{H} \, dA \geq \frac{n+1}{n} |\Omega| + \cos \theta_0 \left( \frac{\int_{\Sigma} \langle v, E_{n+1} \rangle \, dA}{\int_{\Sigma} H \langle v, E_{n+1} \rangle \, dA} \right)^2,
\]
with equality if and only if \( \Sigma \) is a spherical cap.

Inequality (3) is optimal, in the sense that the spherical caps achieve equality in (3). However it is not in the best form. For example, while we are able to use (3) to reprove the Alexandrov theorem for constant mean curvature (CMC) hypersurfaces in [21], it is not very helpful to handle the case of higher order mean curvatures (see (17) for the definition). In view of the following Minkowski formula
\[
\int_{\Sigma} n(1 - \cos \theta_0 \langle v, E_{n+1} \rangle) - H \langle x, v \rangle \, dA = 0,
\]
a possible best form, which was conjectured in [21], is
\[
\int_{\Sigma} \frac{1 - \cos \theta_0 \langle v, E_{n+1} \rangle}{H} \, dA \geq \frac{n+1}{n} |\Omega|.
\]
It is clear by using the Cauchy-Schwarz inequality that inequality (5) implies (3), provided that \( \langle v, E_{n+1} \rangle \) is non-negative. But without the non-negativity one does not know which one is stronger.

The first part of this paper is to establish this “best” version of the Heintze-Karcher inequality in a little more general setting and for whole range \( \theta_0 \in (0, \pi) \). Let \( \Sigma \) be a hypersurface in \( \mathbb{R}^{n+1}_{+} \) with (possibly non-connected) boundary \( \partial \Sigma \subset \partial \mathbb{R}^{n+1}_{+} \). The hypersurface intersects with the supported hyperplane \( \partial \mathbb{R}^{n+1}_{+} \) transversely.

**Theorem 1.1.** Let \( \theta_0 \in (0, \pi) \) and let \( \Sigma \subset \mathbb{R}^{n+1}_{+} \) be a smooth, compact, embedded, strictly mean convex \( \theta \)-capillary hypersurface, with \( \theta(x) \leq \theta_0 \) for every \( x \in \partial \Sigma \). Let \( \Omega \) denote the enclosed domain by \( \Sigma \) and \( \partial \mathbb{R}^{n+1}_{+} \). Then it holds
\[
\int_{\Sigma} \frac{1 - \cos \theta_0 \langle v, E_{n+1} \rangle}{H} \, dA \geq \frac{n+1}{n} |\Omega|.
\]
**Equality in (6) holds if and only if \( \Sigma \) is a \( \theta_0 \)-capillary spherical cap.**

Note that we also have removed the restriction that \( \theta_0 \leq \pi/2 \), comparing with the previous work [21]. As a direct application, we get the Alexandrov-type theorem for embedded capillary hypersurfaces in \( \mathbb{R}^{n+1}_{+} \) with constant \( r \)-th mean curvature, for any \( r \in \{1, \cdots, n\} \).

---

1In the paper we abuse a little bit the terminology of capillarity. A capillary hypersurface mentioned at the beginning of the Introduction is called a capillary CMC hypersurface in the paper.
Corollary 1.2. Let $\theta_0 \in (0, \pi)$ and $r \in \{1, \ldots, n\}$. Let $\Sigma \subset \mathbb{R}^{n+1}$ be a smooth, embedded, compact, $\theta_0$-capillary hypersurface with constant $r$-th mean curvature. Then $\Sigma$ is a $\theta_0$-capillary spherical cap.

Our proof of Theorem 1.1 is inspired by the original idea of Heintze-Karcher [18] (see also Montiel-Ros [32]) which uses parallel hypersurfaces to estimate the enclosed volume. However the ordinary parallel hypersurfaces do not work for capillary hypersurfaces. The one of key ingredients of this paper is a correct form of parallel hypersurfaces $\zeta(\Sigma, t)$ defined in (33). To prove Theorem 1.1, we need to show the surjectivity of $\zeta$ onto the enclosed domain $\Omega$, for which we discover an appropriate foliation by round spheres with simultaneously varied center and radius.

It is interesting to see that our proof of Theorem 1.1 provides a refinement of the ordinary Heintze-Karcher inequality for closed hypersurfaces, since any closed hypersurface can be viewed as a capillary hypersurface with an empty boundary. Hence we have

Corollary 1.3. Let $\Sigma$ be a closed, strictly mean convex hypersurface in $\mathbb{R}^{n+1}$ with enclosed domain $\Omega$. Then it holds

$$\int_{\Sigma} \frac{1}{H} \, dA - \max_{e \in S^n} \left| \int_{\Sigma} \frac{\langle v, e \rangle}{H} \, dA \right| \geq \frac{n + 1}{n} |\Omega|.$$  

Equality in (1.1) holds if and only if $\Sigma$ is a round sphere.

In the second part of this paper, we study hypersurfaces with capillary boundary in a wedge domain. Here we simply call it a wedge. An ordinary wedge, we call it a classical wedge in this paper, is the unbounded closed region determined by two intersecting hyperplanes with dihedral angle $\alpha$, which is also called an opening angle, lying in $(0, \pi)$. There have been many works on the study of the stability of CMC capillary hypersurfaces (c.f. [4, 25, 39, 43]) and on embedded CMC capillary hypersurfaces in wedges (c.f. [28, 31, 33]). Comparing with the half-space case, a big difference is that the Alexandrov’s reflection method might fail in the case of wedges, though the authors in [28, 31, 33] managed to modify Alexandrov’s reflection to obtain their classification results in certain cases. It is interesting that our method to establish the Heintze-Karcher-type inequality works in the wedge case, and even works in a more general setting. See also the recent development of this method in the anisotropic setting [19, 20].

In fact we shall consider generalized wedges which are determined by finite many mutually intersecting hyperplanes. To be more precise, let $W$ be the unbounded closed region in $\mathbb{R}^{n+1}(n \geq 2)$, which are determined by finite many mutually intersecting hyperplanes $P_1, \ldots, P_L$, for some integer $1 \leq L \leq n + 1$, such that the dihedral angle between $P_i$ and $P_j$, $i \neq j$, lies in $(0, \pi)$. We call such $W$ a generalized wedge. $P_i \cap P_j, i \neq j$ is called an edge of the wedge $W$. If $L = 2$, we call $W$ a classical wedge. Let $\bar{N}_i$ be the outwards pointing unit normal to $P_i$ in $W$ for $i = 1, \ldots, L$. Thus $\{\bar{N}_1, \ldots, \bar{N}_L\}$ are linearly independent. Up to a translation,
HEINTZE-KARCHER INEQUALITY AND CAPILLARY HYPERSURFACES

we may assume that the origin $O \in \bigcap_{i=1}^{L} P_i$. Given $\vec{\theta}_0 = (\theta_0^1, \ldots, \theta_0^L) \in \prod_{i=1}^{L} (0, \pi)$. Given $\vec{\theta}_0 = (\theta_0^1, \ldots, \theta_0^L) \in \prod_{i=1}^{L} (0, \pi)$. Given $\vec{\theta}_0 = (\theta_0^1, \ldots, \theta_0^L) \in \prod_{i=1}^{L} (0, \pi)$. Given $\vec{\theta}_0 = (\theta_0^1, \ldots, \theta_0^L) \in \prod_{i=1}^{L} (0, \pi)$. Given $\vec{\theta}_0 = (\theta_0^1, \ldots, \theta_0^L) \in \prod_{i=1}^{L} (0, \pi)

Now we define an important vector $k_0$ associated with $W$ and $\vec{\theta}_0$ by

$$k_0 = \sum_{i=1}^{L} c_i \vec{N}_i,$$

where $c_i$ is such that $\langle k_0, \vec{N}_i \rangle = \cos \theta_0^i$. We say that $\Sigma$ is a $\vec{\theta}$-capillary hypersurface in $W$ with $\vec{\theta} = (\theta^1, \theta^2, \ldots, \theta^L)$ if it intersects $\partial W$ at contact angle $\theta^i(x)$ for $x \in \partial \Sigma \cap P_i$. It is easy to see that $k_0$ is the center of the $\vec{\theta}_0$-capillary spherical cap with radius 1. The following key assumption (8) has a clear geometric meaning that the unit sphere centered at $k_0$ intersects the edge of the wedge $W$.

We shall prove the following Heintze-Karcher-type inequality in a wedge.

**Theorem 1.4.** Let $W \subset \mathbb{R}^{n+1}$ be a generalized wedge whose boundary consists of $L$ mutually intersecting closed hyperplanes $\{P_i\}_{i=1}^{L}$ and $\vec{\theta}_0 \in \prod_{i=1}^{L} (0, \pi)$. Assume that $|k_0| \leq 1$. (8)

Let $\Sigma \subset W$ be a smooth, compact, embedded, strictly mean convex $\vec{\theta}$-capillary hypersurface with $\theta^i(x) \leq \theta_0^i$ for $x \in \partial \Sigma \cap P_i$, $i = 1, \ldots, L$. Let $\Omega$ be the enclosed domain by $\Sigma$ and $\partial W$. Assume in addition that $\Sigma$ does not hit the edges of $W$, i.e.,

$$\Sigma \cap (P_i \cap P_j) = \emptyset, \quad i \neq j.$$ (9)

Then

$$\int_{\Sigma} \frac{1 + \langle \nu, k_0 \rangle}{H} \, dA \geq \frac{n+1}{n} |\Omega|,$$ (10)

with equality if and only if $\Sigma$ is a $\vec{\theta}_0$-capillary spherical cap.

The idea of proof of Theorem 1.4 is similar to that of Theorem 1.1, by using a suitable family of parallel hypersurfaces $\xi(\Sigma, t)$ which relates $k_0$. To show the surjectivity of $\xi$, assumption (9) plays a crucial role. Actually, this condition was required in previous related papers, except [28]. See Remark 1.10. In this paper we are able to remove the additional assumption (9) in a classical wedge, i.e., $L = 2$. Precisely, we have the following

**Theorem 1.5.** When $L = 2$, Theorem 1.4 holds true without assumption (9).

The proof of Theorem 1.5 relies on a delicate analysis on the edge, which is the most technical part of this paper. A special case, $L = 2$ and $\vec{\theta}_0 = (\frac{\pi}{2}, \frac{\pi}{2})$, i.e., $\Sigma$ is a free boundary hypersurface, for which $k_0 = 0$, (10) was proved by Lopez in [28] via Reilly’s formula. It is a natural question to ask if Theorem 1.4 holds true for $L > 2$ without (9). Theorem 1.5 leads us to believe that assumption (9) is unnecessary.

Now we make some remarks on condition (8).

**Remark 1.6.**
(i) In view of the Heintze-Karcher inequality (10) we have established, (8) could not be removed.

(ii) When \( L = 1 \), it is nothing but the half-space case and (8) is satisfied automatically. When \( L = 2 \), (8) is equivalent to

\[
|\pi - (\theta^1_0 + \theta^2_0)| \leq \alpha \leq \pi - |\theta^1_0 - \theta^2_0|.
\]

where \( \alpha \in (0, \pi) \) is the opening angle of the wedge, or the dihedral angle between \( P_1 \) and \( P_2 \), see Lemma A.1. Similarly, \( |k_0| < 1 \) is equivalent to (11) with strict inequalities.

(iii) By virtue of Lemma A.2, Condition (8) is satisfied, provided there exists a \( \theta_0 \)-capillary hypersurface \( \Sigma \) in \( W \) which satisfies \( \Sigma \cap P_1 \cap P_2 \neq \emptyset \).

(iv) Condition (8) is also closely related to the existence of elliptic points of capillary hypersurfaces. See Section 5 below.

(v) Condition (11) appears also in the regularity of capillary surfaces at corner. See the last paragraph of the Introduction.

As applications of Theorem 1.4 and Theorem 1.5, we prove an Alexandrov-type theorem and a non-existence result for embedded CMC capillary hypersurfaces in a wedge.

**Theorem 1.7.** Let \( W \subset \mathbb{R}^{n+1} \) be a classical wedge whose boundary consists of \( P_1, P_2 \) with \( \theta_0 = \prod_{i=1}^{2} (0, \pi) \). Let \( \Sigma \subset W \) be a smooth, compact and embedded \( \theta_0 \)-capillary hypersurface with constant \( r \)-mean curvature, \( r \in \{1, \cdots, n\} \). Assume \( \Sigma \cap P_1 \cap P_2 \neq \emptyset \). Then \( \Sigma \) is a \( \theta_0 \)-capillary spherical cap.

**Theorem 1.8.** Let \( W \subset \mathbb{R}^{n+1} \) be a classical wedge whose boundary consists of \( P_1, P_2 \) with \( \theta_0 = \prod_{i=1}^{2} (0, \pi) \). Then there exists no smooth, compact and embedded \( \theta_0 \)-capillary, CMC hypersurface such that \( \Sigma \cap P_1 \cap P_2 = \emptyset \) and \( |k_0| \leq 1 \). Moreover, there exists no smooth, compact, embedded, \( \theta_0 \)-capillary hypersurface of constant \( r \)-mean curvature for some \( r \in \{2, \cdots, n\} \), such that \( \Sigma \cap P_1 \cap P_2 = \emptyset \) and \( |k_0| < 1 \).

As a consequence of Theorem 1.7, Theorem 1.8 and Remark 1.6 (iii), we have the following

**Theorem 1.9.** Let \( W \subset \mathbb{R}^{n+1} \) be a classical wedge whose boundary consists of \( P_1, P_2 \) with \( \theta_0 = \prod_{i=1}^{2} (0, \pi) \). Let \( \Sigma \subset W \) be a smooth, compact and embedded \( \theta_0 \)-capillary CMC hypersurface. Then \( \Sigma \) is a \( \theta_0 \)-capillary spherical cap which intersects with the edge \( P_1 \cap P_2 \) if and only if \( |k_0| \leq 1 \).

Several remarks and questions are in order.

**Remark 1.10.**

(i) McCuan [31, Theorem 2] proved that any \( \theta_0 \)-capillary spherical cap which is disjoint with the edge must satisfy \( \theta^1_0 + \theta^2_0 > \pi + \alpha \). This condition implies that \( |k_0| > 1 \), see Lemma A.1.
(ii) Lopez [28] proved an Alexandrov-type theorem for embedded CMC capillary surfaces with \( \tilde{\theta}_0 = \left( \frac{\pi}{2}, \frac{\pi}{2} \right) \), i.e., the free boundary case. Note that in this case, \( |k_0| \leq 1 \) is automatically satisfied. Hence Theorem 1.9 covers Lopez’s result.

In contrast to it a ring-type CMC free hypersurface in a wedge was constructed by Wente in [42], which is certainly not embedded. It is natural to ask whether there exist immersed ring-type CMC \( \tilde{\theta}_0 \)-hypersurfaces for a general \( \tilde{\theta}_0 \).

(iii) Park [33] classified the embedded CMC capillary ring-type spanners, which are topologically annuli and disjoint with the edge. Our Theorem 1.7 classified all embedded CMC capillary surfaces intersecting with the edge, without any topological condition.

(iv) McCuan [31] proved a non-existence result for the embedded CMC capillary ring-type spanners with

\[
\theta_0^1 + \theta_0^2 \leq \pi + \alpha, \tag{12}
\]

by developing spherical reflection technique, when \( n = 2 \). Note that the angle relation (12) is weaker than \( |k_0| \leq 1 \), see Lemma A.1. However, our Theorem 1.8 requires no topological assumption. Moreover it holds for any dimensions.

(v) For stable CMC capillary hypersurfaces in a classical wedge, Choe-Koiso [4] proved that such a surface is a part of a sphere without the angle condition (11), but with condition (9) and with the embeddness of \( \partial \Sigma \) for \( n = 2 \) or the convexity of \( \partial \Sigma \) for \( n \geq 3 \). It is an interesting question to ask if an immersed stable CMC capillary hypersurface in a wedge is a part of a sphere, without any further conditions, c.f., [17, 39, 41].

We end the Introduction with a few supplement on the study of capillary hypersurfaces in a wedge domain. A nonparametric capillary surface is a graph of a function \( f \) over a domain, say \( \Omega \), which satisfies the constant mean curvature equation with a corresponding capillary boundary condition. This is an equilibrium free surface of a fluid in a cylindrical container. When the domain \( \Omega \) has a corner, then this nonparametric surface can be viewed as a capillary surface in a wedge (or in a wedge domain). There have been a lot of research on such a problem, especially after Concus-Finn [6], where it was already observed that the opening angle of the wedge and both contact angles should satisfy certain conditions for the existence. See also [7]. Later in [8] Concus-Finn proved that \( f \) is continuous at a given corner if (11) holds, while if (11) does not hold there is no solution in one case and in the left case, namely \( \alpha > \pi - |\theta_0^1 - \theta_0^2| \), they conjectured that \( f \) has a jump discontinuity at the corner. See the Concus-Finn rectangle in [22], Figure 2. This conjecture was solved by Lancaster in [22] with the methods developed by Allard [2] and especially by Simon [38]. The latter was crucially used in a very recent work of Chao Li [23] mentioned at the beginning of the Introduction. See also his further work [12] with Edelen on surfaces in a polyhedral domain, which is also closely related to surfaces in a wedge domain.
The rest of the paper is organized as follows. In Section 2, we collect some basic facts about wedges and capillary hypersurfaces in wedges. In Section 3 and Section 4, we prove the main theorems on the Heintze-Karcher inequality in the half-space and a wedge. In Section 5, we prove the Alexandrov-type theorem and the non-existence result for CMC capillary hypersurfaces in a wedge.

2. Notations and Preliminaries

Let $\Omega$ be the bounded domain in $W$ with piecewise smooth boundary $\partial \Omega = \Sigma \cup (\bigcup_{i=1}^L T_i)$, where $\Sigma = \partial \Omega \cap W$ is a smooth compact embedded $\tilde{\theta}$-capillary hypersurface in $W$ and $T_i = \partial \Omega \cap P_i$. Denote the corners by $\Gamma_i = \Sigma \cap T_i$, which are smooth, co-dimension two submanifolds in $\mathbb{R}^{n+1}$. For the sake of simplicity, we denote by $\Gamma$ the union of $\Gamma_i$, i.e., $\Gamma = \bigcup_{i=1}^L \Gamma_i$. We use the following notation for normal vector fields. Let $\nu$ and $\tilde{N}_i$ be the outward unit normal to $\Sigma$ and $P_i$ (with respect to $\Omega$) respectively. Let $\mu_i$ be the outward unit co-normal to $\Gamma_i = \partial \Sigma \cap P_i \subset \Sigma$ and $\tilde{\nu}_i$ be the outward unit co-normal to $\Gamma_i \subset T_i$. Under this convention, along each $\Gamma_i \{\nu, \mu_i\}$ and $\{\tilde{\nu}_i, \tilde{N}_i\}$ span the same 2-dimensional plane and have the same orientation in the normal bundle of $\partial \Sigma \subset \mathbb{R}^{n+1}$. Hence one can define the contact angle function along each $\Gamma_i$, $\theta_i : \Gamma_i \rightarrow (0, \pi)$ by

$$\mu_i(x) = \sin \theta^i(x) \tilde{N}_i + \cos \theta^i(x) \tilde{\nu}_i(x), \quad (13)$$

$$\nu(x) = -\cos \theta^i(x) \tilde{N}_i + \sin \theta^i(x) \tilde{\nu}_i(x). \quad (14)$$

Let $\tilde{\theta}(x)$ denote the $L$-tuple $(\theta^1(x), \ldots, \theta^L(x))$. We call $\Sigma$ a $\tilde{\theta}$-capillary hypersurface, if we want to emphasize the contact angle function. We also use $\tilde{\theta}_0$-capillary hypersurface to denote such a hypersurface with $\tilde{\theta} \equiv \tilde{\theta}_0$, a vector $\tilde{\theta}_0 \in \prod_{i=1}^L (0, \pi)$.

We denote by $\tilde{\nabla}, \tilde{\Delta}, \tilde{\nabla}^2$ and $\text{div}$, the gradient, the Laplacian, the Hessian and the divergence on $\mathbb{R}^{n+1}$ respectively, while by $\nabla, \Delta, \nabla^2$ and $\text{div}$, the gradient, the Laplacian, the Hessian and the divergence on the smooth part of $\partial \Omega$, respectively. Let $g, h$ and $H$ be the first, second fundamental forms and the mean curvature of the smooth part of $\partial \Omega$ respectively. Precisely, $h(X, Y) = \langle \tilde{\nabla}_X \nu, Y \rangle$ and $H = \text{tr}_\nu(h)$. In particular, since $P_i$ is planar, the second fundamental form $h_i \equiv 0$, correspondingly, the mean curvature $H_i$ vanishes.

We need the following structural lemma for compact hypersurfaces in $\mathbb{R}^{n+1}$ with boundary, which is well-known and widely used, see [1, 21].

**Lemma 2.1.** Let $\Sigma \subset \mathbb{R}^{n+1}$ be a smooth compact hypersurface with boundary. Then it holds that

$$n \int_{\Sigma} \nu dA = \int_{\partial \Sigma} \{\langle x, \mu \rangle \nu - \langle x, \nu \rangle \mu \} \, ds. \quad (15)$$

**Proof.** Let $Z = \langle \nu, e \rangle x^T - \langle x, \nu \rangle e^T$ for any constant vector $e$, where $x^T$ and $e^T$ denote the tangential component of $x$ and $e$ respectively. One computes that

$$\text{div}(Z) = n \langle \nu, e \rangle.$$  

Integration by parts yields the assertion. \qed
The following lemma is well-known when the capillary hypersurfaces are bounded by containers with totally umbilical boundaries, in particular, a wedge in $\mathbb{R}^{n+1}$, see e.g., [1, 25, 41].

**Lemma 2.2.** Let $W \subset \mathbb{R}^{n+1}$ be a wedge and $\theta_i^0 \in (0, \pi)$ for $i = 1, \ldots, L$. If $\Sigma \subset W$ is a smooth $\theta_0$-capillary hypersurface, then along $\partial \Sigma$, $\mu_i$ is a principal direction of $\Sigma$.

**Proof.** For the completeness we provide a proof. It suffices to prove that $h(\mu_i, X) = 0$ for any vector $X$ tangent to $\partial \Sigma$. Indeed,

$$h(\mu_i, X) = \langle \nabla_X \mu_i, \nu \rangle = \langle \nabla_X (\sin \theta^i N_i + \cos \theta^i \tilde{v}_i), -\cos \theta^i N_i + \sin \theta^i \tilde{v}_i \rangle$$

$$= \langle \nabla_X \tilde{v}_i, N_i \rangle = -h_i(\tilde{v}_i, X) = 0,$$

where we have used (13), the constancy of $\theta_i$, the fact that $\tilde{v}_i, N_i$ are unit vector fields, and $h_i = 0$ since $P_i$ are totally geodesic. This completes the proof. □

The $r$-th mean curvature $H_r$ of $\Sigma$ is defined by the identity:

$$P_n(t) = \prod_{i=1}^n (1 + t \kappa_i) = \sum_{i=0}^n \binom{n}{i} H_i t^i$$

for all real number $t$. Thus $H_1 = \frac{H}{n}$ is the mean curvature of $\Sigma$ and $H_n$ is the Gaussian curvature, and we adopt the convention that $H_0 = 1$.

We have the following Minkowski-type formula for $\theta_0$-capillary hypersurfaces.

**Proposition 2.3.** Let $W \subset \mathbb{R}^{n+1}$ be a wedge and $\Sigma \subset W$ be a $\theta_0$-capillary hypersurface. Then it holds that for $r = 1, \ldots, n$,

$$\int_{\Sigma} (H_{r-1} (1 + \langle x, k_0 \rangle) - H_r \langle x, \nu \rangle) \, dA = 0. \tag{18}$$

In particular, if $L = 1$, i.e. $\Sigma \subset \mathbb{R}^{n+1}_+$, then

$$\int_{\Sigma} H_{r-1} (1 - \cos \theta_0 \langle x, E_{n+1} \rangle) - H_r \langle x, \nu \rangle \, dA = 0. \tag{19}$$

**Proof:** The case for $r = 1$ has been proved in [25, Lemma 5]. For the sake of completeness, we include the proof here.

Since

$$\text{div}(x^T) = n - H \langle x, \nu \rangle,$$

by using integration by parts in $\Sigma$, we get

$$\int_{\Sigma} (n - H \langle x, \nu \rangle) \, dA = \sum_{i=1}^L \int_{\Gamma_i} \langle x, \mu_i \rangle \, ds_i. \tag{20}$$

From the capillary boundary condition (13) it is easy to see that on each $\Gamma_i$

$$-\cos \theta_i^0 \langle x, \nu \rangle + \sin \theta_i^0 \langle x, \mu_i \rangle = \langle x, N_i \rangle = 0. \tag{21}$$
By (15), (13) and (21), we get
\[
\int_{\Sigma} n \langle \nu, k_0 \rangle \, dA = \sum_{i=1}^{L} \int_{\Gamma_i} \left( \langle x, \mu_i \rangle \langle \nu, k_0 \rangle - \langle x, \nu \rangle \langle \mu_i, k_0 \rangle \right) \, ds_i
\]
\[
= \sum_{i=1}^{L} \int_{\Gamma_i} \left( \langle x, \mu_i \rangle \langle \nu, k_0 \rangle + \langle x, \nu \rangle \frac{- \cos \theta_0^i}{\sin \theta_0^i} (1 + \langle \nu, k_0 \rangle) \right) \, ds_i
\]
\[
= \sum_{i=1}^{L} \int_{\Gamma_i} \left( \langle x, \mu_i \rangle \langle \nu, k_0 \rangle - \langle x, \mu_i \rangle (1 + \langle \nu, k_0 \rangle) \right) \, ds_i
\]
\[
= - \sum_{i=1}^{L} \int_{\Gamma_i} \langle x, \mu_i \rangle \, ds_i. \tag{22}
\]
It follows from (20) and (22) that
\[
\int_{\Sigma} n (1 + \langle \nu, k_0 \rangle) - H \langle x, \nu \rangle \, dA = 0. \tag{23}
\]

Now we prove (18) for general \( r \). For a small real number \( t > 0 \), consider a family of hypersurfaces with boundary \( \Sigma_t \), defined by
\[
y := \varphi_t(x) = x + t(\nu(x) + k_0) \quad x \in \Sigma.
\]
We claim that \( \Sigma_t \) is also a \( \theta_0 \)-capillary hypersurface in \( \mathbf{W} \). In fact, if \( e_1, \ldots, e_n \) are principal directions of a point of \( \Sigma \) and \( k_i \) are the corresponding principal curvatures, we have
\[
(\varphi_t)_* (e_i) = \tilde{\nabla}_{e_i} \varphi_t = (1 + t k_i) e_i, \quad i = 1, \ldots, n. \tag{24}
\]
From (24), we see that \( \nu_t(y) = \nu(x) \), where \( \nu_t(y) \) denotes the outward unit normal of \( \Sigma_t \) at \( y = \varphi_t(x) \). Moreover, the capillarity condition (14) implies: for any \( x \in \partial \Sigma \cap P_i \), we have
\[
\langle \nu(x) + k_0, \tilde{N}_i \rangle = - \cos \theta_0^i + \cos \theta_0^i = 0, \tag{25}
\]
in other words, \( \varphi_t(x) \in P_i \), and hence \( \partial \Sigma_t \subset \varphi_t(\partial \Sigma) \). In view of this, we have:
\[
\langle \nu_t(y), \tilde{N}_i \rangle = \langle \nu(x), \tilde{N}_i \rangle = - \cos \theta_0^i; \tag{26}
\]
that is, \( \Sigma_t \) is also a \( \theta_0 \)-capillary hypersurface in \( \mathbf{W} \).

Therefore, we can exploit (23) to find that
\[
\int_{\Sigma_t = \varphi_t(\Sigma)} n (1 + \langle \nu_t, k_0 \rangle) - H(t) \langle y, \nu_t \rangle \, dA_t(y) = 0. \tag{26}
\]
By (24), the tangential Jacobian of \( \varphi_t \) along \( \Sigma \) at \( x \) is just
\[
J^\Sigma \varphi_t(x) = \prod_{i=1}^{n} (1 + t k_i(x)) = \mathcal{P}_n(t), \tag{27}
\]
where \( \mathcal{P}_n(t) \) is the polynomial defined in (17). Moreover, using (24) again, we see that the corresponding principal curvatures are given by
\[
\kappa_i(\varphi_t(x)) = \frac{\kappa_i(x)}{1 + t k_i(x)}. \tag{28}
\]
Hence fix \( x \in \Sigma \), the mean curvature of \( \Sigma \) at \( \varphi_t(x) \), say \( H(t) \), is given by
\[
H(t) = \frac{\mathcal{P}'_n(t)}{\mathcal{P}_n(t)} = \frac{\sum_{i=0}^{n} i(i)_H H_i t_i^{-1}}{\mathcal{P}_n(t)},
\]
where \( H_i = H_i(x) \) is the \( i \)-th mean curvature of \( \Sigma \) at \( x \).

Using the area formula, (27) and (29), we find from (26) that
\[
\int_{\Sigma} (1 + \langle x, k_0 \rangle) \mathcal{P}_n(t) - t (1 + \langle x, k_0 \rangle) \mathcal{P}'_n(t) - \mathcal{P}'_n(t) \langle x, \nu \rangle \, dA_x = 0.
\]
As the left hand side in this equality is a polynomial in the time variable \( t \), this shows that all its coefficients vanish, and hence
\[
\int_{\Sigma} (1 + \langle x, k_0 \rangle) H_r - H_r \langle x, \nu \rangle \, dA, \quad r = 1, \ldots, n.
\]
This gives (18). \( \square \)

We have also
\[
(n + 1)|\Omega| = \int_{\Sigma} \langle x, \nu \rangle \, dA,
\]
which is easy to prove. If one views (32) as one of (18) with \( r = 0 \), the Heintze-Karcher inequality (6) that we want to prove could also be viewed one of them with \( r = -1 \). Certainly now it is an inequality, instead of an equality.

**Remark 2.4.** An alternative proof of Proposition 2.3 can be given as that of [40, Proposition 2.5], where the Minkowski-type formula for the half-space case has been proved.

In the sequel, \( \Sigma \) will be always referred to as a smooth, embedded capillary hypersurface.

3. **Heintze-Karcher Inequality in the Half-Space**

**Proof of Theorem 1.1.** Let \( \Sigma \subset \mathbb{R}^{n+1}_+ \) be a \( \theta \)-capillary hypersurface with \( \theta \leq \theta_0 \) along \( \partial \Sigma \). For any \( x \in \Sigma \), let \( \{ e_i = e_i(x) \} \) be the set of unit principal vectors of \( \Sigma \) at \( x \) and \( \{ \kappa_i(x) \} \) the set of corresponding principal curvatures. Since \( \Sigma \) is strictly mean convex,
\[
\max_i \kappa_i(x) \geq \frac{H(x)}{n} > 0, \text{ for } x \in \Sigma.
\]
We define
\[
Z = \left\{ (x, t) \in \Sigma \times \mathbb{R} : 0 < t \leq \frac{1}{\max_i \kappa_i(x)} \right\},
\]
and
\[
\zeta : Z \to \mathbb{R}^{n+1},
\]
\[
\zeta(x, t) = x - t (\nu(x) - \cos \theta_0 E_{n+1}).
\]
\( \zeta \) gives a family of hypersurfaces \( \zeta(\Sigma, t) \), which are the modified parallel hypersurfaces mentioned above.

**Claim:** \( \Omega \subset \zeta(Z) \).

Indeed, let us denote by \( B_r(x) \) the closed ball centered at \( x \) of radius \( r \), and \( S_r(x) = \partial B_r(x) \). For any \( y \in \Omega \), we consider a family of spheres \( \{ S_r(y - r \cos \theta_0 E_{n+1}) \}_{r \geq 0} \). Since \( y \in \Omega \) is an interior point, when \( r \) is small enough, we have \( B_r(y - r \cos \theta_0 E_{n+1}) \subset \subset \Omega \). Since \( |\cos \theta_0| < 1 \), it is easy to see that the spheres gives a foliation of \( \mathbb{R}^{n+1} \). Hence \( S_r(y - r \cos \theta_0 E_{n+1}) \) must touch \( \Sigma \) as we increase the radius \( r \). As a conclusion, for any \( y \in \Omega \), there exists \( x \in \Sigma \) and \( r_y > 0 \), such that \( S_r(y - r_y \cos \theta_0 E_{n+1}) \) touches \( \Sigma \) for the first time, at the point \( x \in \Sigma \). We have two cases.

**Case 1.** \( x \in \hat{\Sigma} \).

In this case, since \( x \in \hat{\Sigma} \), the sphere \( S_r(y - r_y \cos \theta_0 E_{n+1}) \) is tangent to \( \Sigma \) at \( x \) from the interior. It follows that \( r_y \leq \frac{1}{\max \kappa_i(x)} \). Invoking the definition of \( Z \) and \( \zeta \), we find that \( y \in \zeta(Z) \) in this case.

**Case 2.** \( x \in \partial \Sigma \).

We will rule out this case by the condition on the contact angle function \( \theta \) of \( \Sigma \). In this case, by the first touching property of \( x \), the contact angle \( \theta_y \) of \( S_r(y - r_y \cos \theta_0 E_{n+1}) \) with \( \partial \mathbb{R}^{n+1} \) is smaller than or equals to \( \theta(x) \), which is smaller than or equals to \( \theta_0 \), by assumption. (see Figure 1 for an illustration). However \( \theta_y \leq \theta_0 \) implies that \( \langle y, E_{n+1} \rangle < 0 \), a contradiction to \( y \in \Omega \subset \mathbb{R}^{n+1} \). The **Claim** is thus proved.

![Figure 1. Boundary touching.](image)

By a simple computation, we find

\[
\partial_t \zeta(x, t) = -(v(x) - \cos \theta_0 E_{n+1}),
\]

\[
\tilde{\nabla}_{e_i} \zeta(x, t) = (1 - t\kappa_i(x)) e_i.
\]
Hence the tangential Jacobian of $\zeta$ along $Z$, at $(x, t)$ is just

$$J^Z \zeta(x, t) = (1 - \cos \theta_0 \langle \nu, E_{n+1} \rangle) \prod_{i=1}^{n} (1 - t \kappa_i).$$

By virtue of the fact that $\Omega \subset \zeta(Z)$, the area formula yields

$$|\Omega| \leq |\zeta(Z)| = \int_{\zeta(Z)} H^0(\zeta^{-1}(y)) dy = \int_Z j^2 \zeta d\mathcal{H}^{n+1}$$

$$= \int_{\Sigma} dA \int_0^{\max\{\kappa_i(x)\}} (1 - \cos \theta_0 \langle \nu, E_{n+1} \rangle) \prod_{i=1}^{n} (1 - t \kappa_i(x)) dt.$$

By the AM-GM inequality, $1 - \cos \theta_0 \langle \nu, E_{n+1} \rangle > 0$ on $\Sigma$, and the fact that $\max\{\kappa_i(x)\} \geq H(x)/n$, we obtain

$$|\Omega| \leq \int_{\Sigma} dA \int_0^{\max\{\kappa_i(x)\}} (1 - \cos \theta_0 \langle \nu, E_{n+1} \rangle) \left( \frac{1}{n} \sum_{i=1}^{n} (1 - t \kappa_i(x)) \right)^n dt$$

$$\leq \int_{\Sigma} (1 - \cos \theta_0 \langle \nu, E_{n+1} \rangle) dA \int_0^{\frac{H(x)}{n}} \left( 1 - t \frac{H(x)}{n} \right)^n dt$$

$$= \frac{n}{n+1} \int_{\Sigma} \frac{(1 - \cos \theta_0 \langle \nu, E_{n+1} \rangle)}{H} dA,$$

which is (6).

The characterization of equality case in (6) follows from the classical one. Precisely, since the equality holds throughout the argument, the arithmetic mean-geometric mean (AM-GM) inequality assures the umbilicity of $\Sigma$, and it follows that $\Sigma$ is a spherical cap. Apparently, the contact angle of a spherical cap with a hyperplane is a constant, say $\theta$. It is easy to see that $\theta = \theta_0$, and hence $\Sigma$ must be a $\theta_0$-capillary spherical cap. Conversely, when $\Sigma$ is a $\theta_0$-capillary spherical cap, then $H$ is a positive constant. By virtue of the Minkowski formula (19) for $r = 1$, we see that equality in (6) holds.

Let us close this section with a remark. In the proof of $\Omega \subset \zeta(Z)$, our choice of the touching balls is enlightened by the following observation.

**Remark 3.1** (Foliation of $\theta_0$-capillary hypersurfaces). In [32], to prove the Heintze-Karcher inequality for closed hypersurfaces, one shall ‘sweepout’ the domain $\Omega$ by a foliation around any point $p \in \Omega$, whose leaves are level-sets of the distance function to $p$. The key point is that, such ‘sweep-outs’ coincides with the domain $\Omega$, if and only if $\Omega$ is a ball and $p$ is chosen to be the center.

In view of this, our choice of foliation in the capillary case is thus clear; we want to ‘sweepout’ the $\theta_0$-ball with the foliation, whose leaves are $\theta_0$-spherical caps (as illustrated in Figure 2).
Figure 2. Capillary foliation.

4. Heintze-Karcher Inequality in a Wedge

In this section, we prove Theorem 1.4 and Theorem 1.5 in a wedge $W$, following largely from the proof of the Heintze-Karcher-type inequality presented in the previous section.

Proof of Theorem 1.4. Let $\Sigma \subset W$ be a compact embedded hypersurface with $\tilde{\theta}(x)$-capillary boundary, where $\theta^i(x) \leq \theta^i_0$ for each $i$ and every $x \in \Gamma$. As above, for any $x \in \Sigma$, let $\{e_i\}$ be the set of principal unit vectors of $\Sigma$ at $x$ and $\{\kappa_i(x)\}$ the set of the corresponding principal curvatures. Now we define modified parallel hypersurfaces by

$$Z = \left\{ (x,t) \in \Sigma \times \mathbb{R} : 0 < t \leq \frac{1}{\max \kappa_i(x)} \right\},$$

$$\zeta(x,t) = x - t (\nu(x) + k_0), \ (x,t) \in Z.$$

As in Theorem 1.1, we shall show that $\Omega \subset \zeta(Z)$. Let $y \in \Omega$. We consider the sphere foliation $\{S_r(y + r \mathbf{k}_0)\}_{r \geq 0}$. By virtue of Lemma A.3, there exists some $r_y > 0$ such that $S_{r_y}(y + r_y \mathbf{k}_0)$ touches $\Sigma$ from the interior at a first touching point $x \in \Sigma$.

**Case 1.** $x \in \hat{\Sigma}$. We can get $y \in \zeta(Z)$, as argued in **Case 1, Theorem 1.1**.

**Case 2.** $x \in \partial \Sigma$. By assumption (9), $x \in \partial \Sigma \cap \tilde{P}_i$ for some $i$. We will rule out this case again by virtue of the capillarity of $\Sigma$. In this case, $S_{r_y}(y + r_y \mathbf{k}_0) \cap P_i$ is tangent to $\Sigma \cap P_i$, and the touching angle of $S_{r_y}(y + r_y \mathbf{k}_0)$ with $P_i$ at $x \in \Gamma_i$ must be smaller than $\theta^i(x)$, and it follows from the geometric relation that $y$ lies outside $W$. Precisely, up to a rotation, we may assume that the touching plane is $\{x_{n+1} = 0\}$, say $P_1$, and we denote by $\bar{\theta}^1(x)$ the touching angle, satisfying $\bar{\theta}^1(x) \leq \theta^1(x)$, due to the first touch. From the geometric relation (see Figure 3), we find, $y + r_y (\mathbf{k}_0 - \cos \bar{\theta}^1(x) \mathbf{N}_1) \in P_1$, the angle relation $\bar{\theta}^1(x) \leq \theta^1(x) \leq \theta^1_0 < \pi$ then implies that $y \in \mathbb{R}^{n+1} \setminus W$ \footnote{Notice that $\langle \mathbf{k}_0, \mathbf{N}_1 \rangle = \cos \theta^1_0$, which means moving along $\mathbf{k}_0$ with distance $r_y$ is indeed moving along $\mathbf{N}_1$ with distance $r_y \cos \theta^1_0$.}, which contradicts to the fact that $y \in \Omega \subset W$. Therefore, we complete the proof that $\Omega \subset \zeta(Z)$.
By a simple computation as Theorem 1.1, we see, the tangential Jacobian of $\zeta$ along $Z$ at $(x, t)$ is just

$$J^Z \zeta (x, t) = \left( 1 + (\nu, k_0) \right) n \prod_{i=1}^{n} \left( 1 - t k_i \right).$$

By a similar argument as Theorem 1.1, we conclude

$$|\Omega| \leq \frac{n}{n + 1} \int_{\Sigma} \frac{(1 + (\nu, k_0))}{H} dA.$$  \hspace{1cm} (34)

As proved in Theorem 1.1, if equality in (10) holds, then $\Sigma$ is umbilical, and hence spherical. To see that $\Sigma$ must be a $\theta_0$-capillary spherical cap, we need a different argument. As equalities hold throughout the argument, we have

$$|\Omega| = |\zeta(Z)| = \int_{\zeta(Z)} H^0(\zeta^{-1}(y)) dy.$$  \hspace{1cm} (35)

Moreover, (14) implies: for any $x \in \partial \Sigma \cap P_I$, there holds

$$\langle -(\nu(x) + k_0), \vec{N}_I \rangle = \cos \theta^I(x) - \cos \theta^I_0 \geq 0.$$  \hspace{1cm} (36)

Recall that $\vec{N}_I$ is the outwards pointing unit normal of $P_I$, and we have already showed in the previous proof that $\Omega \subset \zeta(Z)$. Thus, if $\theta^I(x) < \theta^I_0$ strictly at some $x \in \Sigma \cap P_I$, then it must be that $|\Omega| < |\zeta(Z)|$, which contradicts to (35). In other words, for any $x \in \partial \Sigma \cap P_I$, we must have $\theta^I(x) = \theta^I_0$, this shows that $\Sigma$ must be a $\theta_0$-capillary spherical cap. \hspace{1cm} \Box

Proof of Theorem 1.5. We note that the proof follows closely the one of Theorem 1.4. Precisely, thanks to Lemma A.3, we can use our foliation $\{S_r(y + r k_0)\}_{r \geq 0}$ to test the surjectivity of $\zeta$, i.e., $\Omega \subset \zeta(Z)$. One subtle point we have to be concerned with is that the first touching of $S_{r_0} \left( y + r_0 k_0 \right)$ with $\Sigma$ might occur at $\Sigma \cap P_1 \cap P_2$. 

Figure 3. Touching supporting hyperplanes in the interior.
Here we manage to rule this case out by a rather subtle analysis. In view of Remark 4.1 below, We only need to consider the 3-dimensional case, i.e., \( n + 1 = 3 \), in which case \( P_1 \cap P_2 \) is a line.

In the following we use \( v_{B_r}(x) \) to denote the outward unit normal of \( S_{r_y}(y + r_yk_0) \), \( T_x\Sigma \) to denote the tangent plane of \( \Sigma \) at \( x \), and \( l \) to denote a unit vector generating the line \( P_1 \cap P_2 \). Recall that \( \bar{N}_i \) is the outward unit normal of \( P_i \), \( i = 1, 2 \).

**Case 1.** \( \bar{N}_i \parallel v_{B_r}(x) \) for some \( i \).

Without loss of generality, we assume \( \bar{N}_2 \) is parallel to \( v_{B_r}(x) \). Hence the sphere \( S_{r_y}(y + r_yk_0) \) touches the plane \( P_2 \) only at the point \( x \). Since \( \bar{N}_1 \) and \( \bar{N}_2 \) are not parallel, then \( \bar{N}_1 \) is not parallel to \( v_{B_r} \), thus the intersection of \( S_{r_y}(y + r_yk_0) \) with \( P_1 \) must be a circle. Since \( S_{r_y}(y + r_yk_0) \cap P_2 = \{x\} \), we know that \( S_{r_y}(y + r_yk_0) \cap P_1 \) touches \( P_1 \cap P_2 \) only at \( x \). Hence \( l \) is the tangential vector of \( S_{r_y}(y + r_yk_0) \cap P_1 \) at \( x \). Since \( x \) is the first touching point of \( B_r(y + r_yk_0) \) with \( \Sigma \) from the interior, we see that \( l \) is also the tangential vector of \( \partial \Sigma^1 = \Sigma \cap P_1 \) at \( x \) (see Figure 4).

![Figure 4](image)

It follows that \( S_{r_y}(y + r_yk_0) \cap P_1 \) is tangent to \( \partial \Sigma^1 \) at \( x \). We are now in the same situation as in **Case 2.** Theorem 1.1 or Theorem 1.4. Hence the contact angle of \( S_{r_y}(y + r_yk_0) \) with \( P_1 \) at \( x \) must be smaller than \( \theta^1(x) \), which is a contradiction, since the contact angle of \( S_{r_y}(y + r_yk_0) \) with \( P_1 \) is \( \theta^1_0 \geq \theta^1(x) \) by assumption.

**Case 2.** Neither \( \bar{N}_i \) is parallel to \( v_{B_r}(x) \), i.e., \( v_{B_r}(x) \wedge \bar{N}_i \neq 0 \) for \( i = 1, 2 \).

In this case, \( S_{r_y}(y + r_yk_0) \cap P_i(i = 1, 2) \) must be circles.

**Case 2.1.** \( P_1 \cap P_2 \subset T_x\Sigma \).

Since \( P_1 \) is not parallel to \( P_2 \), there exists exactly one of \( P_i(i = 1, 2) \), say \( P_1 \), which does not coincide with \( T_x\Sigma \). Then \( \Sigma \) is not tangent to \( P_1 \) at \( x \) and \( \partial \Sigma^1 + \Sigma \cap P_1 \).
is a curve near $x$. Since $l \in T_x \Sigma$ and $l \perp \vec{N}_1$, we see that $l$ is the tangential vector of $\partial \Sigma^1$. Since $x$ is the first touching point of $S_{r_i} (y + r_y k_0)$ with $\Sigma$, $l$ is also the tangential vector of $S_{r_i} (y + r_y k_0) \cap P_1$. Hence $S_{r_i} (y + r_y k_0) \cap P_1$ is tangent to $\Sigma \cap P_1$ at $x$. We are again in the position as in Case 2, Theorem 1.1 or Theorem 1.4, and consequently we get a contradiction.

**Case 2.2.** $P_1 \cap P_2 \notin T_x \Sigma$.

In this case, $\vec{N}_i, i = 1, 2$, cannot be parallel to $\nu(x)$. For simplicity of notation, in the following we use $\nu, \nu_{B_r}, \theta^i$ to indicate $\nu(x), \nu_{B_r}(x), \theta^i(x)$, respectively, and we adopt the notation

$$\partial \Sigma^i = \Gamma_i = \Sigma \cap P_i, \quad \partial B^i_r = S_{r_i} (y + r_y k_0) \cap P_i.$$  

Let $T_{\partial \Sigma^i}$ be the unit tangent vector of $\partial \Sigma^i$ at $x$ such that $\langle T_{\partial \Sigma^i}, \vec{N}_j \rangle > 0$ for $i \neq j$ and $T_{\partial B^i_r}$ be the unit tangent vector of $\partial B^i_r$ at $x$ such that $\langle T_{\partial B^i_r}, \vec{N}_j \rangle > 0$ for $i \neq j$. Since $\nu$ and $\vec{N}_i$ are perpendicular to $T_{\partial \Sigma^i}$, we know that $T_{\partial \Sigma^i}$ is parallel to $\nu \wedge \vec{N}_i$. Similarly, since $\nu_{B_r}$ and $\vec{N}_i$ are perpendicular to $T_{\partial B^i_r}$, we have that $T_{\partial B^i_r}$ is parallel to $\nu_{B_r} \wedge \vec{N}_i$.

Without loss of generality, we may assume that the origin $0 \in \partial \Omega$ and $x \neq 0$. Let $l$ be the unit tangent vector $\frac{\gamma_i}{|\gamma_i|}$. With such choice of $l$, we claim that $\langle l, \nu \rangle > 0$ and $\langle l, \nu_{B_r} \rangle > 0$. Indeed, recall that by capillarity, $\nu$ is expressed as

$$\nu = -\cos \theta^i \vec{N}_i + \sin \theta^i \vec{v}_i, \quad i = 1, 2.$$  

Here $\vec{v}_i$ is the outward unit normal vector of $\partial \Sigma^i$ in $P_i$ at $x$. Since $\partial \Sigma^i$ lies on the same side of the line $P_1 \cap P_2$ in $P_i$, it yields that $\langle l, \vec{v}_i \rangle > 0$, thus we get

$$\langle l, \nu \rangle = -\cos \theta^i \langle l, \vec{N}_i \rangle + \sin \theta^i \langle l, \vec{v}_i \rangle = \sin \theta^i \langle l, \vec{v}_i \rangle > 0.$$  

We will show that $\langle l, \nu_{B_r} \rangle > 0$. Since $S_{r_i} (y + r_y k_0)$ touches $\Sigma$ from the interior at $x$, we have $B_{r_i} (y + r_y k_0) \cap P_i \in \tilde{\Omega} \cap P_i$ for $i = 1, 2$. Combining with $0 \in \tilde{\Omega} \cap P_1 \cap P_2$, it follows that $l$ is pointing outward of $B_r$ at $x$, see Figure 5, thus we obtain $\langle l, \nu_{B_r} \rangle > 0$.  

Note that $\vec{N}_1$ and $\vec{N}_2$ are perpendicular to $l$. Without loss of generality, we may assume

$$l = -\frac{\vec{N}_1 \wedge \vec{N}_2}{|\vec{N}_1 \wedge \vec{N}_2|}.$$

A direct computation then yields

$$\langle \nu \wedge \vec{N}_1, \vec{N}_2 \rangle = \langle \vec{N}_1 \wedge \vec{N}_2, \nu \rangle = -|\vec{N}_1 \wedge \vec{N}_2| \langle l, \nu \rangle < 0,$$

which implies the following fact: the two vectors $T_{\partial \Sigma^1}$ and $\nu \wedge \vec{N}_1$ are in the opposite direction. Therefore,

$$T_{\partial \Sigma^1} = -\frac{\nu \wedge \vec{N}_1}{|\nu \wedge \vec{N}_1|}.$$

By a similar argument, we obtain

$$T_{\partial \Sigma^2} = \frac{\nu \wedge \vec{N}_2}{|\nu \wedge \vec{N}_2|}, \quad T_{\partial B^i} = -\frac{\nu_{B^i} \wedge \vec{N}_1}{|\nu_{B^i} \wedge \vec{N}_1|}, \quad T_{\partial B^i} = \frac{\nu_{B^i} \wedge \vec{N}_2}{|\nu_{B^i} \wedge \vec{N}_2|}.$$

Thanks to the fact that $x$ is the first touching point, we must have (see Figure 6)

$$\langle T_{\partial B^i}, l \rangle \geq \langle T_{\partial \Sigma^i}, l \rangle, \quad i = 1, 2. \quad (37)$$
Let \( \eta^i \in (0, \pi) \) be such that \( \langle \nu_{B_i}, \vec{N}_i \rangle = -\cos \eta^i \) for each \( i = 1, 2 \). We can carry out the following computations.

\[
\langle T_{\partial B_i}, l \rangle = \langle -\frac{\nu_{B_i} \wedge \vec{N}_1}{|\nu_{B_i} \wedge \vec{N}_1|} - \frac{\vec{N}_1 \wedge \vec{N}_2}{|\vec{N}_1 \wedge \vec{N}_2|}, \vec{N}_2 \rangle = -\frac{\langle \nu_{B_i}, \vec{N}_2 \rangle - \langle \nu_{B_i}, \vec{N}_1 \rangle \langle \vec{N}_1, \vec{N}_2 \rangle}{|\nu_{B_i} \wedge \vec{N}_1| \cdot |\vec{N}_1 \wedge \vec{N}_2|} = \frac{\cos \eta^2 + \cos \eta^1 \cos \alpha}{\sin \eta^1 \sin \alpha}.
\]

Similarly,

\[
\langle T_{\partial B_2}, l \rangle = \frac{\cos \eta^1 + \cos \eta^2 \cos \alpha}{\sin \eta^2 \sin \alpha},
\]

\[
\langle T_{\partial \Sigma_1}, l \rangle = \frac{\cos \theta^2 + \cos \theta^1 \cos \alpha}{\sin \theta^1 \sin \alpha},
\]

\[
\langle T_{\partial \Sigma_2}, l \rangle = \frac{\cos \theta^1 + \cos \theta^2 \cos \alpha}{\sin \theta^2 \sin \alpha}.
\]

Plugging into (37), we thus obtain

\[
\frac{\cos \eta^2 + \cos \eta^1 \cos \alpha}{\sin \eta^1 \sin \alpha} \geq \frac{\cos \theta^2 + \cos \theta^1 \cos \alpha}{\sin \theta^1 \sin \alpha}, \tag{38}
\]

\[
\frac{\cos \eta^1 + \cos \eta^2 \cos \alpha}{\sin \eta^2 \sin \alpha} \geq \frac{\cos \theta^1 + \cos \theta^2 \cos \alpha}{\sin \theta^2 \sin \alpha}. \tag{39}
\]
A crucial observation is that (38) is equivalent to (see the end of Appendix A)
\[
\sin \theta^1 \left( \cos \eta^2 - \cos (\theta^2 - \theta^1 + \eta^1) \right) \\
+ \left( \cos (\theta^2 - \theta^1) + \cos \alpha \right) \sin (\theta^1 - \eta^1) \geq 0.
\] (40)

On the one hand, we have
\[
\langle x, \vec{N}_1 \rangle \triangleq \langle x - (y + r_y k_0), \vec{N}_1 \rangle + \langle y + r_y k_0, \vec{N}_1 \rangle \\
= r_y \langle \nu_{B_r}, \vec{N}_1 \rangle + \langle y, \vec{N}_1 \rangle + r_y \langle k_0, \vec{N}_1 \rangle.
\] (41)

Since \( x \in P_1 \) and \( y \in \tilde{W} \), we have: \( \langle x, \vec{N}_1 \rangle = 0 \) and \( \langle y, \vec{N}_1 \rangle < 0 \). It follows from (41) that
\[
\langle \nu_{B_r}, \vec{N}_1 \rangle > -\langle k_0, \vec{N}_1 \rangle = -\cos \theta^1_0.
\]

By our angle assumption \( \theta^1 \leq \theta^1_0 \), we get
\[
-\cos \eta^1 = \langle \nu_{B_r}, \vec{N}_1 \rangle > -\cos \theta^1_0 \geq -\cos \theta^1,
\]
which implies
\[
\eta^1 > \theta^1.
\] (42)

On the other hand, since \( P_1 \cap P_2 \not\subset T_x \Sigma \), by Lemma A.2, we see that \( |k(x)| < 1 \), where \( k(x) := \sum_{i=1}^2 c_i(x) \tilde{N}_i \), satisfying \( \langle k(x), \tilde{N}_i \rangle = \cos \theta^i \). In view of Lemma A.1, it implies that
\[
\cos (\theta^2 - \theta^1) + \cos \alpha > 0.
\] (43)

From (42) and (43), we know that
\[
(\cos (\theta^2 - \theta^1) + \cos \alpha) \sin (\theta^1 - \eta^1) < 0.
\]

Combining with (40), we find
\[
\cos \eta^2 - \cos (\theta^2 - \theta^1 + \eta^1) > 0.
\] (44)

By a similar argument we obtain from (39) that
\[
\eta^2 > \theta^2,
\] (45)

and
\[
\cos \eta^1 - \cos (\theta^1 - \theta^2 + \eta^2) > 0.
\] (46)

We may assume \( \theta^2 \geq \theta^1 \) without loss of generality, then it follows from (45) that
\[
0 < \theta^1 < \theta^1 - \theta^2 + \eta^2 \leq \eta^2 < \pi.
\]

Therefore, back to (44), we deduce that
\[
\eta^1 < \theta^1 - \theta^2 + \eta^2,
\] (47)

In the meanwhile, from (46), we deduce
\[
\eta^2 < \theta^2 - \theta^1 + \eta^1.
\] (48)

Apparently, (47) and (48) lead to a contradiction.
In conclusion, we have showed that the first touching point cannot occur at any \( x \in \Sigma \cap P_1 \cap P_2 \). The rest of the proof follows similarly from Theorem 1.4. 

**Remark 4.1.** We make a remark for the case of higher dimensions \( \mathbb{R}^{n+1} \). In this case, \( P_1 \cap P_2 \) is a \((n-1)\)-plane and \( \partial \Sigma \cap P_1 \cap P_2 \) is a closed \((n-2)\)-submanifold, playing the role as \( \partial (\partial \Sigma \cap P_1) \) or \( \partial (\partial \Sigma \cap P_2) \). By letting \( l \) be the unique unit vector in \( P_1 \cap P_2 \) which is orthogonal to \( \partial \Sigma \cap P_1 \cap P_2 \), and \( T_{\partial \Sigma} \) be the tangent vector of \( \partial \Sigma \cap P_i \) at \( x \) which is orthogonal to \( \partial \Sigma \cap P_1 \cap P_2 \), and \( T_{\partial B_i} \) be the tangent vector of \( \partial B_r \cap P_i \) at \( x \) which is orthogonal to \( \partial \Sigma \cap P_1 \cap P_2 = \partial B_r \cap P_1 \cap P_2 \) (because \( x \) is the first touching point), we may reduce the problem to the 3-dimensional case. In other words, all the vectors at \( x \) we considered in the proof lie in the orthogonal 3-subspace of \( T_x (\partial \Sigma \cap P_1 \cap P_2) \).

## 5. Alexandrov-Type Theorem

It is well-known that there exists at least one elliptic point for a closed embedded hypersurface in \( \mathbb{R}^{n+1} \). Recall that an elliptic point is a point at which the principal curvatures are all positive with respect to the outward unit normal. We shall show that this fact is true for capillary hypersurfaces in the half-space, or in a wedge whenever condition (8) holds. Recall that for a wedge case, the existence of an elliptic point is no longer true if without any restriction. An example is easy to be found among ring type capillary surfaces a wedge. See a figure in [31].

We first need the following lemma (compare to Remark 3.1).

**Lemma 5.1.** Let \( W \) be a generalized wedge and \( y \in \bigcap_{i=1}^{L} P_i \). If \( |k_0| < 1 \), then for any \( r > 0 \), the sphere \( S_r (y + rk_0) \) intersects \( P_i \) at angle \( \theta_i^0 \). In particular, in \( \mathbb{R}^{n+1}_+ \), the sphere \( S_r (y - r \cos \theta_0 E_{n+1}) \) intersects \( \partial \mathbb{R}^{n+1}_+ \) at angle \( \theta_0 \).

**Proof.** First, if \( |k_0| < 1 \), it is easy to see that for any \( r > 0 \), \( S_r (y + rk_0) \) intersects \( P_i \). The outward unit normal to \( S_r (y + rk_0) \) at \( x \in \bar{P}_i \) is given by

\[
v_{B_r}(x) = \frac{x - (y + rk_0)}{r}.
\]

Since \( x - y \in P_i \), we see that

\[
\langle v_{B_r}(x), \tilde{N}_i \rangle = -\langle k_0, \tilde{N}_i \rangle = -\cos \theta_i^0.
\]

**Proposition 5.2.** Let \( \Sigma \subset \mathbb{R}^{n+1}_+ \) be a smooth, compact, embedded \( \theta_0 \)-capillary hypersurface. Then there exists at least one elliptic point on \( \Sigma \).

**Proof.** Let \( \Omega \) be the enclosed region of \( \Sigma \) and \( \partial \mathbb{R}^{n+1}_+, T = \partial \Omega \cap \partial \mathbb{R}^{n+1}_+ \), we fix a point \( y \in \bar{T} \). Consider the family of the open balls \( B_r (y - r \cos \theta_0 E_{n+1}) \). By Lemma 5.1, \( S_r (y - r \cos \theta_0 E_{n+1}) \) intersects \( \partial \mathbb{R}^{n+1}_+ \) at the angle \( \theta_0 \). Since \( \Sigma \) is compact, for \( r \) large enough, \( \Sigma \subset B_r (y - r \cos \theta_0 E_{n+1}) \). Hence we can find the smallest \( r \), say \( r_0 > 0 \), such that \( S_{r_0} (y - r_0 \cos \theta_0 E_{n+1}) \) touches \( \Sigma \) at a
first time at some \( x \in \Sigma \). For simplicity, we abbreviate \( S_{r_0}(y - r_0 \cos \theta_0 E_{n+1}) \) by \( S_{r_0} \).

If \( x \in \hat{\Sigma} \), then \( \Sigma \) and \( S_{r_0} \) are tangent at \( x \). If \( x \in \partial \Sigma \), from the fact that both \( \Sigma \) and \( S_{r_0} \) intersect with \( \partial \mathbb{R}^{n+1} \) at the angle \( \theta_0 \), we conclude again that \( \Sigma \) and \( S_{r_0} \) are tangent at \( x \).

In both cases, we have that the principal curvatures of \( \Sigma \) at \( x \) are bigger than or equal to \( 1/r_0 > 0 \), which implies that \( x \) is an elliptic point.

\[ \square \]

**Proposition 5.3.** Let \( W \subset \mathbb{R}^{n+1} \) be a classical wedge. Let \( \Sigma \subset W \) be a smooth, compact, embedded \( \hat{\theta}_0 \)-capillary surface. If \( \Sigma \cap P_1 \cap P_2 \neq \emptyset \), then there exists at least one elliptic point on \( \Sigma \).

**Proof.** In view of Remark 4.1, we need only consider 3-dimensional case.

Since the corners \( \Gamma_i \) (\( i = 1, 2 \)) are smooth co-dimension two submanifolds in \( \partial \mathbb{R}^{n+1} \), \( \partial \Sigma \) is embedded in \( \mathbb{R}^{n+1} \) (see Section 2), when \( \Sigma \cap P_1 \cap P_2 \neq \emptyset \), we must have that \( \partial \Omega \cap P_1 \cap P_2 \) is also a co-dimension two submanifold in \( \mathbb{R}^{n+1} \). This means it has non-trivial interior relative to the topology of \( P_1 \cap P_2 \), therefore we are free to choose any \( y \) in the interior of \( \partial \Omega \cap P_1 \cap P_2 \) as the initial center of the sphere foliation \( \{ S_r(y + r k_0) \}_{r \geq 0} \). Thanks to Lemma A.4, any such foliation must have a first touching point with \( \Sigma \) from outside as \( r \) decreases from \( \infty \).

We consider the sphere \( S_{r_0}(y + r_0 k_0) \) which touches \( \Sigma \) for the first time at some \( x \in \Sigma \). Following the argument in Proposition 5.2, we see that, if \( x \in \hat{\Sigma} \) or \( x \in \partial \Sigma \setminus (P_1 \cap P_2) \), then \( x \) must be an elliptic point of \( \Sigma \). Therefore, it remains to consider the case \( x \in \partial \Sigma \cap (P_1 \cap P_2) \).

Since \( x, y \in P_1 \cap P_2 \), we find

\[
\begin{align*}
\langle v_B, (x), -\vec{N}_1 \rangle &= \cos \theta_0^1 = \langle v(x), -\vec{N}_1 \rangle, \\
\langle v_B, (x), -\vec{N}_2 \rangle &= \cos \theta_0^2 = \langle v(x), -\vec{N}_2 \rangle.
\end{align*}
\]

This implies that either (a) \( v(x) = v_B, (x) \), or (b) \( v(x) = v_B, (x) + a l \), where \( l \) is a unit vector perpendicular to both \( \vec{N}_1 \) and \( \vec{N}_2 \), and \( a \) is a non-zero constant.

We claim that the situation (b) does not occur. Otherwise, by the fact that \( v(x) \) and \( v_B, (x) \) are unit vectors, we can deduce that \( a = -2 \langle v_B, (x), l \rangle \neq 0 \). Thus we have

\[
\langle v(x), l \rangle = \langle v_B, (x) + a l, l \rangle = -\langle v_B, (x), l \rangle \neq 0.
\]

Note that \( x \neq y \) and \( x - y \in \bar{\Omega} \cap B_{r_0}(y + r_0 k_0) \), by the fact that \( v(x) \) and \( v_B, (x) \) are outward normal vectors, we have

\[
\langle v(x), x - y \rangle \geq 0, \quad \langle v_B, (x), x - y \rangle \geq 0.
\]

Since \( x, y \in P_1 \cap P_2 \), \( l \) is parallel to \( x - y \). It follows that \( \langle v(x), l \rangle \) and \( \langle v_B, (x), l \rangle \) have the same sign, which contradicts (49) and concludes the claim.

From the claim, we see that only the situation (a) happens. This means, \( S_r \) and \( \Sigma \) are tangent at \( x \). Thus the principal curvatures of \( \Sigma \) at \( x \) are bigger than or equal to \( 1/r_0 > 0 \), which implies \( x \) is an elliptic point.

\[ \square \]
In view of the proof of Proposition 5.3, the only place where we used the condition \( \Sigma \cap P_1 \cap P_2 \neq \emptyset \) is that we have to use it in Lemma A.4, as \( |k_0| = 1 \), to conclude that the sphere foliation \( \{ S_r(y + r k_0) \}_{r \geq 0} \) must touch \( \Sigma \). In this regard, we can remove the extra assumption \( \Sigma \cap P_1 \cap P_2 \neq \emptyset \) by strengthening the angle condition to be \( |k_0| < 1 \). Indeed, we have the following

**Proposition 5.4.** Let \( W \subset \mathbb{R}^{n+1} \) be a classical wedge. Let \( \Sigma \subset W \) be a smooth, compact, embedded \( \tilde{\theta}_0 \)-capillary hypersurface with \( |k_0| < 1 \). Then there exists at least one elliptic point on \( \Sigma \).

**Proposition 5.5.** Let \( W \subset \mathbb{R}^{n+1} \) be a classical wedge. Let \( \Sigma \subset W \) be a smooth, compact, embedded \( \tilde{\theta}_0 \)-capillary hypersurface. If \( \Sigma \) is of constant mean curvature \( H \), then \( H > 0 \), provided \( \pi - \alpha \leq \theta_1^0 + \theta_2^0 \).

**Proof.** We remark that the free boundary case, that is \( \tilde{\theta}_0 = \left( \frac{\pi}{2}, \frac{\pi}{2} \right) \) has been proved by Lopez [28]. We consider here the general case.

To begin, we take \( p_0 \in \partial \Sigma \) to be the point of maximal distance of \( \partial \Sigma \) from the edge \( P_1 \cap P_2 \). Assume without loss of generality that \( p_0 \in P_2 \). Let \( P \) be the family of planes parallel to the edge of \( W \) and having contact angle \( \theta_2^0 \) with \( P_2 \). Starting from one of such a plane near infinite and moving it among this family until the first time that one of plane \( P \) in \( P \) touches \( \Sigma \). By definition of \( p_0 \) and \( P \), it is only possible that the first touching occurs at certain interior point of \( \Sigma \), at \( p_0 \in P_2 \) or at some \( p_1 \in \Sigma \cap P_1 \).

For the former two cases, one can use the strong maximum principle for the interior point and the Hopf-lemma for the boundary point, and obtain \( H > 0 \). For the last case, we shall use our angle assumption \( \pi - \alpha \leq \theta_1 + \theta_2 \).

Indeed, if \( \pi - \alpha < \theta_1 + \theta_2 \), then the first touching point must not occur at any points of \( \Sigma \cap P_1 \): since \( P \) is a plane having contact angle \( \theta_2^0 \) with \( P_2 \), we know that the contact angle of \( \Sigma \) with \( P_1 \), say \( \theta_1^1 \), is \( \pi - \alpha - \theta_2^0 \). By the angle assumption, we find

\[
\theta_1^1 = \pi - \alpha - \theta_2^0 < \theta_0^1,
\]  

which is not possible if \( P \) touches \( \Sigma \cap P_1 \) from outside for the first time.

If \( \pi - \alpha = \theta_1^1 + \theta_2^0 \), then \( \theta_1^1 = \theta_0^1 \), \( \theta_2^0 = \theta_0^1 \) from the above discussion, and we can use the Hopf’s boundary point lemma again to find that \( H > 0 \). This completes the proof. \( \square \)

Now, we are in the position to prove the Alexandrov-type theorem and the non-existence theorem.

**Proof of Theorem 1.7.** Before we proceed the proof, we emphasize that the condition \( |k_0| \leq 1 \) ensures the validity of the inequality

\[ 1 + \langle \nu, k_0 \rangle \geq 0 \]

pointwisely on \( \Sigma \). In particular, as \( L = 1 \), we have

\[ 1 - \cos \theta_0 \langle \nu, E_{n+1} \rangle \geq 0 \]
along $\Sigma$.

On one hand, by virtue of Proposition 5.3 and Gärding’s argument [15] (see also [36, Section 3]), we know that $H_j$ are positive, for $j \leq r$. Applying Theorem 1.5 and using the Maclaurin inequality

$$H_1 \geq \frac{H_1}{r},$$

we find

$$(n + 1)H_1^{1/r} |\Omega| \leq H_r^{1/r} \int_{\Sigma} \frac{1 + \langle v, k_0 \rangle}{H/n} \, dA \leq \int_{\Sigma} (1 + \langle v, k_0 \rangle) \, dA,$$ (51)

and equality holds if and only if $\Sigma$ is a $\theta_0$-capillary spherical cap.

On the other hand, using the Minkowski formula (23) and the Maclaurin inequality again, we have

$$0 = \int_{\Sigma} (1 + \langle v, k_0 \rangle) H_{r-1} - H_r \langle x, v \rangle \, dA$$

$$\geq \int_{\Sigma} (1 + \langle v, k_0 \rangle) \frac{H_r}{r} - H_r \langle x, v \rangle \, dA$$

$$= H_r \frac{r}{r-1} \int_{\Sigma} 1 + \langle v, k_0 \rangle - H_r^{1/r} \langle x, v \rangle \, dA$$

$$= H_r \frac{r}{r-1} \left\{ \int_{\Sigma} 1 + \langle v, k_0 \rangle \, dA - (n + 1)H_r^{1/r} |\Omega| \right\},$$

where in the last equality we have used (32). Thus equality in (51) holds, and hence $\Sigma$ is a $\theta_0$-capillary spherical cap. This completes the proof. □

**Proof of Theorem 1.8.** For the CMC case, we notice that our condition (8) implies $\pi - \alpha \leq \theta_0^1 + \theta_0^2$ automatically, thanks to Lemma A.1. Therefore, we can use Proposition 5.5 to see that $\Sigma$ is of positive constant mean curvature.

For the constant $H_r$ case, since we assume $|k_0| < 1$, we may use Proposition 5.4 to conclude that $\Sigma$ has an elliptic point, from which we see that $H_r$ is a positive constant.

In view of this, arguing as in the proof of Theorem 1.7, we can use Proposition 2.3 together with Theorem 1.4 to show that the equality case happens in the Heintze-Karcher inequality (10), and hence $\Sigma$ must be a $\theta_0$-spherical cap. However, if this is the case, a simple geometric relation (see Figure 7) then implies that $\alpha + (\pi - \theta_0^1) + (\pi - \theta_0^2) < \pi$, i.e., $\pi + \alpha < \theta_0^1 + \theta_0^2$, which is incompatible with (11). The proof is complete. □
Appendix A. Miscellaneous Results in Wedge

Lemma A.1. For the case $L = 2$, namely, $W \subset \mathbb{R}^{n+1}$ is a classical wedge, (8) is equivalent to (11), i.e.,

$$|\pi - (\theta_0^1 + \theta_0^2)| \leq \alpha \leq |\theta_0^1 - \theta_0^2|.$$  

Similarly, (8) with strict inequality is equivalent to (11) with strict inequality.

Proof. First we note that (11) can be viewed of somewhat a compatible condition for admitting spherical caps in a wedge (a wedge is determined by its dihedral angle $\alpha$.) with prescribed angles $\tilde{\theta}_0 = (\theta_0^1, \theta_0^2)$.

By definition of $k_0$ and the fact that $\langle \tilde{N}_1, \tilde{N}_2 \rangle = \cos(\pi - \alpha) = -\cos \alpha$, we find

$$\begin{cases} c_1 - c_2 \cos \alpha = \cos \theta_0^1, \\ -c_1 \cos \alpha + c_2 = \cos \theta_0^2. \end{cases}$$

A simple computation then yields

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{\sin^2 \alpha} \begin{pmatrix} 1 & \cos \alpha \\ \cos \alpha & 1 \end{pmatrix} \begin{pmatrix} \cos \theta_0^1 \\ \cos \theta_0^2 \end{pmatrix}.$$
and it follows that
\[
|k_0|^2 = \langle k_0, c_1 \bar{N}_1 + c_2 \bar{N}_2 \rangle = c_1 \cos \theta_0^1 + c_2 \cos \theta_0^2
\]
\[
= (\cos \theta_0^1 \cos \theta_0^2) \frac{c_1}{c_2} \frac{c_2}{c_2}
\]
\[
= \cos^2 \theta_0^1 + \cos^2 \theta_0^2 + 2 \cos \theta_0^1 \cos \theta_0^2 \cos \alpha \sin^2 \alpha.
\]
Thus (8) can be rewritten as
\[
\cos^2 \theta_0^1 + \cos^2 \theta_0^2 + 2 \cos \theta_0^1 \cos \theta_0^2 \cos \alpha \leq \sin^2 \alpha.
\]
which is
\[
(\cos \alpha + \cos(\theta_0^1 + \theta_0^2))(\cos \alpha + \cos(\theta_0^1 - \theta_0^2)) \leq 0.
\]
Thus we see that (8) is equivalent to
\[
|\theta_0^1 - \theta_0^2| \leq \pi - \alpha \leq \theta_0^1 + \theta_0^2 \leq \pi + \alpha,
\]
which is just (11).

The following observation is important for our analysis in Theorem 1.5.

**Lemma A.2.** Let \( W \subset \mathbb{R}^{n+1} \) be a classical wedge. If \( \Sigma \subset W \) is a smooth, compact, embedded \( \bar{\theta} \)-capillary hypersurface. Let \( k : \partial \Sigma \to \mathbb{R}^{n+1} \) be given by \( k(x) = \sum_{i=1}^{2} c_i(x) \bar{N}_i \) such that \( \langle k(x), \bar{N} \rangle = \cos \theta^i(x) \). If \( \Sigma \cap P_1 \cap P_2 \neq \emptyset \), then we have \( |k| \leq 1 \) on \( \Sigma \cap P_1 \cap P_2 \). Moreover, for any \( x \in \Sigma \cap P_1 \cap P_2 \), \( P_1 \cap P_2 \subset T_x \Sigma \) if and only if \( |k(x)| = 1 \).

**Proof.** In view of Remark 4.1, we need only consider the 3-dimensional case.

Let \( x \in \Sigma \cap P_1 \cap P_2 \). In the following, we compute at \( x \). We have
\[
\langle \bar{N}_1, \bar{N}_2 \rangle = -\cos \alpha, \quad \langle v, \bar{N}_i \rangle = -\cos \theta^i, \quad i = 1, 2.
\]
Thus
\[
\langle \bar{N}_1 \wedge v, \bar{N}_2 \wedge v \rangle = \langle \bar{N}_1, \bar{N}_2 \rangle - \langle \bar{N}_1, v \rangle \langle \bar{N}_2, v \rangle
\]
\[
= -\cos \alpha - \cos \theta^1 \cos \theta^2.
\]
Since \( |\langle \bar{N}_1 \wedge v, \bar{N}_2 \wedge v \rangle| \leq \sin \theta^1 \sin \theta^2 \), we deduce
\[
|\cos \alpha + \cos \theta^1 \cos \theta^2| \leq \sin \theta^1 \sin \theta^2,
\]
which implies \( |k| \leq 1 \).

If \( P_1 \cap P_2 \subset T_x \Sigma \), we have \( |\langle \bar{N}_1 \wedge v, \bar{N}_2 \wedge v \rangle| = \sin \theta^1 \sin \theta^2 \). Then we get
\[
|\cos \alpha + \cos \theta^1 \cos \theta^2| = \sin \theta^1 \sin \theta^2,
\]
which is equivalent to \( |k(x)| = 1 \).

We point out that, in the discussion above, if \( \Sigma \) intersects \( P_1 \) and \( P_2 \) transversally at \( x \in P_1 \cap P_2 \), then Lemma A.2 is included in [23, Lemma 2.5].
Lemma A.3. Let $W \subset \mathbb{R}^{n+1}$ be a classical wedge. If $\Sigma \subset W$ is a smooth, compact, embedded $\bar{\theta}_0$-capillary with $|k_0| \leq 1$, then for any $y \in \Omega$, the family of spheres $\{S_r(y + r k_0)\}_{r \geq 0}$ must touch $\Sigma$.

Proof. The case $|k_0| < 1$ follows trivially, since $\{S_r(y + r k_0)\}_{r \geq 0}$ foliates the whole $\mathbb{R}^{n+1}$. As for $|k_0| = 1$, we proceed by the following observation.

Observation. Since $y \in S_r(y + r k_0)$ for any $r \geq 0$, with $\nu_{\Sigma}(y) = -k_0$. Moreover, 

$$B_r(y + r k_0) \to H_y^{-} := \{z \in \mathbb{R}^{n+1} : \langle z - y, k_0 \rangle > 0\} \quad \text{as} \ r \to \infty. \quad (52)$$

In other words, the family of spheres $S_r(y + r k_0)$ foliates the half-space $H_y^{-}$.

We claim that for any $y \in \Omega$, there holds $H_y^{-} \cap \Sigma \neq 0$. To see this, we consider the following situations separately.

Case 1. $\Sigma \cap P_1 \cap P_2 = \emptyset$.

By definition of $\Sigma$, we see that $\partial \Omega = \Sigma \cup T_1 \cup T_2$ with $T_1, T_2$ away from the edge $P_1 \cap P_2$ (see Figure 7 for illustration). Therefore for any $k_0 \in \mathbb{S}^n$, the open half-space determined by $y, k_0$ must intersect $\Sigma$.

Case 2. $\Sigma \cap P_1 \cap P_2 \neq \emptyset$.

In this case, since $\Sigma \cap P_1 \cap P_2 \neq \emptyset$, we must have $P_1 \cap P_2 \subset T_2$ for any $x \in \Sigma \cap P_1 \cap P_2$, according to Lemma A.2 (with $\bar{\theta} = \bar{\theta}_0$ chosen therein). However, this contradicts to our assumption on $\Sigma$, precisely, that $\Gamma_1, \Gamma_2$ are smooth codimension two submanifolds in $\mathbb{R}^{n+1}$. This proves the claim and hence completes the proof.

In the proof of Proposition 5.3, we use the family of spheres $S_r(y + r k_0)$ for some $y$ in the interior of $\Omega \cap P_1 \cap P_2$, the following observation is needed.

Lemma A.4. Let $W \subset \mathbb{R}^{n+1}$ be a classical wedge. If $\Sigma \subset W$ is a smooth, compact, embedded $\bar{\theta}_0$-capillary such that $\Sigma \cap P_1 \cap P_2 \neq \emptyset$, then for any $y$ in the interior of $\partial \Omega \cap P_1 \cap P_2$, the family of spheres $\{S_r(y + r k_0)\}_{r \geq 0}$ must touch $\Sigma$.

Proof. In view of Remark 4.1, we need only consider the 3-dimensional case.

By virtue of Lemma A.2, we have $|k_0| \leq 1$. The case $|k_0| < 1$ follows trivially from Lemma A.3. As for $|k_0| = 1$, by Observation above and the fact that $\langle y, k_0 \rangle = 0$ due to $y \in P_1 \cap P_2$, it suffices to show that $P_1, P_2 \subset H_0^{-}$.

Claim. In (7), we have $c_1, c_2 < 0$, provided that $|k_0| = 1$ and $\Sigma \cap P_1 \cap P_2 \neq \emptyset$.

If the claim holds, a direct computation yields: for any $z \in \bar{P}_1$, 

$$\langle z, k_0 \rangle = \langle z, c_1 \bar{N}_1 + c_2 \bar{N}_2 \rangle = c_2 \langle z, \bar{N}_2 \rangle.$$ 

Notice that $\{l, l_1, \bar{N}_1\}$ forms an orthonormal basis of $\mathbb{R}^3$, where $l$ is a fixed unit vector, parallel to $P_1 \cap P_2$, $l_1$ is the unit inwards pointing conormal of $\partial P_1 \cap P_1$. In this coordinate, since $z \in \bar{P}_1$, it can be expressed as $z = a_0 l + a_1 l_1 + 0 \bar{N}_1$ with $a_1 > 0$. It follows that $\langle z, k_0 \rangle = a_1 c_2 (\sin \alpha) > 0$. Similarly, we have: for any $z \in \bar{P}_2$, there holds $\langle z, k_0 \rangle > 0$, which implies that $P_1, P_2 \subset H_0^{-}$ and proves the Lemma.

We are thus left to prove the Claim. Indeed, by Lemma A.2, let $x \in \Sigma \cap P_1 \cap P_2$, then we have $P_1 \cap P_2 \in T_x \Sigma$. In view of this, we obtain: $\nu(x) \in \text{span}\{\bar{N}_1, \bar{N}_2\}$,
where $\nu(x)$ is the unit outward normal of $\Sigma$ at $x$. Due to the contact angle condition, we must have
\begin{equation}
\langle \nu(x), \bar{N}_i \rangle = -\cos \theta_0^i, \tag{53}
\end{equation}
comparing with the definition of $k_0$ (7), we thus find:
\[\nu(x) = -k_0 = -c_1\bar{N}_1 - c_2\bar{N}_2.\]
Since $\nu(x)$ is the outward unit normal of $\Sigma$ at $x$, for any $y_i \in \partial\Omega \cap \bar{P}_i, i = 1, 2$, we have
\[\langle x - y_i, \nu(x) \rangle > 0.\]
Indeed, it follows from $\nu(x) = -\cos \theta_0^i \bar{N}_i + \sin \theta_0^i \bar{v}_i(x)$ and the fact $\bar{v}_i(x)$ is orthogonal to $l$ and is pointing outward $P_i$ at $x$.
Meanwhile, since $y_i \in \bar{P}_i$, we definitely have
\[\langle -y_i, \bar{N}_i \rangle = 0, \quad \langle -y_i, \bar{N}_j \rangle > 0 \quad \text{for} \ j \neq i.\]

Recall that $x \in P_1 \cap P_2$ and hence $\langle x, \bar{N}_i \rangle = 0$ for each $i$, we thus obtain
\[\langle x - y_i, \bar{N}_i \rangle = 0, \quad \langle x - y_i, \bar{N}_j \rangle > 0 \quad \text{for} \ j \neq i.\]
Combining all above and invoking that $\nu(x) = -c_1\bar{N}_1 - c_2\bar{N}_2$, we thus find: $c_1 < 0, c_2 < 0$. This proves the Claim and completes the proof. 
\[\Box\]

**Proof of (38)$\iff$(40):** Using
\[\cos \theta^2 = \cos(\theta^2 - \theta^1) \cos \theta^1 - \sin(\theta^2 - \theta^1) \sin \theta^1,\]
we see that (38) is equivalent to
\[\left(\cos \eta^2 + \cos \eta^1 \cos \alpha\right) \sin \theta^1 \geq \left(\cos(\theta^2 - \theta^1) \cos \theta^1 - \sin(\theta^2 - \theta^1) \sin \theta^1 + \cos \theta^1 \cos \alpha\right) \sin \eta^1.\]
After rearranging, we get
\[\sin \theta^1 \cos \eta^2 + \cos \alpha \sin(\theta^1 - \eta^1) = \sin \theta^1 \cos \eta^2 + \cos \alpha (\sin \theta^1 \cos \eta^1 - \sin \eta^1 \cos \theta^1) \geq \cos(\theta^2 - \theta^1) (\cos \theta^1 \sin \eta^1 - \sin \theta^1 \cos \eta^1) + \sin \theta^1 \left(\cos (\theta^2 - \theta^1) \cos \eta^1 - \sin(\theta^2 - \theta^1) \sin \eta^1\right) = -\cos(\theta^2 - \theta^1) \sin(\theta^1 - \eta^1) + \sin \theta^1 \cos(\theta^2 - \theta^1 + \eta^1).\]
which is just (40). 
\[\Box\]
HEINTZE-KARCHER INEQUALITY AND CAPILLARY HYPERSURFACES

References

[1] Abdelhamid Ainouz and Rabah Souam, *Stable capillary hypersurfaces in a half-space or a slab*, Indiana Univ. Math. J. 65 (2016), no. 3, 813–831. MR3528820

[2] William K. Allard, *On the first variation of a varifold*, Ann. of Math. (2) 95 (1972), 417–491. MR307015

[3] Simon Brendle, *Constant mean curvature surfaces in warped product manifolds*, Publ. Math., Inst. Hautes Étud. Sci. 117 (2013), 247–269. MR3090261

[4] Jaigyoung Choe and Miyuki Koiso, *Stable capillary hypersurfaces in a wedge*, Pac. J. Math. 280 (2016), no. 1, 1–15. MR3441213

[5] Tobias H. Colding and Camillo De Lellis, *The min-max construction of minimal surfaces*, Surveys in differential geometry, Vol. VIII (Boston, MA, 2002), 2003, pp. 75–107. MR2039986

[6] Paul Concus and Robert Finn, *On the Behavior of a Capillary Surface in a Wedge*, Proc NAS 63 (1969), no. 2, 292–299. MR0207984

[7] , *On capillary free surfaces in a gravitational field*, Acta Math. 132 (1974), 207–223. MR670443

[8] , *Capillary wedges revisited*, SIAM J. Math. Anal. 27 (1996), no. 1, 56–69. MR1373146

[9] Camillo De Lellis and Jusuf Ramic, *Min-max theory for minimal hypersurfaces with boundary*, Ann. Inst. Fourier (Grenoble) 68 (2018), no. 5, 1909–1986. MR3893761

[10] Luigi De Masi and Guido De Philippis, *Min-max construction of minimal surfaces with a fixed angle at the boundary*, 2021. arXiv: 2111.09913.

[11] Matias Gonzalo Delgadino and Francesco Maggi, *Alexandrov’s theorem revisited*, Anal. PDE 12 (2019), no. 6, 1613–1642. MR3921314

[12] Nicholas Edelen and Chao Li, *Regularity of free boundary minimal surfaces in locally polyhedral domains*, Comm. Pure Appl. Math. 75 (2022), no. 5, 970–1031. MR4400905

[13] Robert Finn, *Equilibrium capillary surfaces*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 284, Springer-Verlag, New York, 1986. MR816345

[14] , *Capillary surface interfaces*, Notices Amer. Math. Soc. 46 (1999), no. 7, 770–781. MR1697840

[15] Lars Gårding, *An inequality for hyperbolic polynomials*, J. Math. Mech. 8 (1959), 957–965. MR0113978

[16] Qiang Guang, Martin Man-chun Li, Zichao Wang, and Xin Zhou, *Min-max theory for free boundary minimal hypersurfaces. II: General Morse index bounds and applications*, Math. Ann. 379 (2021), no. 3–4, 1395–1424. MR4238268

[17] Jinyu Guo, Guofang Wang, and Chao Xia, *Stable capillary hypersurfaces supported on a horosphere in the hyperbolic space*, Adv. Math. 409 (2022), Paper No. 108641. MR473639

[18] Ernst Heintze and Hermann Karcher, *A general comparison theorem with applications to volume estimates for submanifolds*, Ann. Sci. Éc. Norm. Supér. (4) 11 (1978), no. 4, 451–470. MR533065

[19] Xiaohan Jia, Guofang Wang, Chao Xia, and Xuwen Zhang, "Heintze-Karcher inequality for anisotropic free boundary hypersurfaces in convex domains", 2023. arXiv:2311.01162.

[20] Xiaohan Jia, Guofang Wang, Chao Xia, and Xuwen Zhang, *Alexandrov’s theorem for anisotropic capillary hypersurfaces in the half-space*, Arch. Ration. Mech. Anal. 247 (2023), no. 2, Paper No. 25, 19. MR4562813

[21] Xiaohan Jia, Chao Xia, and Xuwen Zhang, *A Heintze-Karcher-type inequality for hypersurfaces with capillary boundary*, J. Geom. Anal. 33 (2023), no. 6, Paper No. 177, 19. MR4567578

[22] Kirk E. Lancaster, *A proof of the Concus-Finn conjecture*, Pacific J. Math. 247 (2010), no. 1, 75–108. MR2718208

[23] Chao Li, *A polyhedron comparison theorem for 3-manifolds with positive scalar curvature*, Invent. Math. 219 (2020), no. 1, 1–37. MR4050100

[24] Chao Li, Xin Zhou, and Jonathan J. Zhu, *Min-max theory for capillary surfaces*, 2021. arXiv: 2111.09924.
[25] Haizhong Li and Changwei Xiong, *Stability of capillary hypersurfaces with planar boundaries*, J. Geom. Anal. 27 (2017), no. 1, 79–94. MR3606545
[26] Junfang Li and Chao Xia, *An integral formula and its applications on sub-static manifolds*, J. Differ. Geom. 113 (2019), no. 3, 493–518. MR4031740
[27] Martin Man-Chun Li and Xin Zhou, *Min-max theory for free boundary minimal hypersurfaces I—Regularity theory*, J. Differential Geom. 118 (2021), no. 3, 487–553. MR4285846
[28] Rafael López, *Capillary surfaces with free boundary in a wedge*, Adv. Math. 262 (2014), 476–483 (English). MR3228434
[29] Fernando C. Marques and André Neves, *Min-max theory of minimal surfaces and applications*, Mathematical Congress of the Americas, 2016, pp. 13–25. MR3457593
[30] _____, *Min-max theory and a proof of the Willmore conjecture*, Geometric analysis, 2016, pp. 277–300. MR3524219
[31] John McCuan, *Symmetry via spherical reflection and spanning drops in a wedge*, Pac. J. Math. 180 (1997), no. 2, 291–323. MR1487566
[32] Sebastián Montiel and Antonio Ros, *Compact hypersurfaces: the Alexandrov theorem for higher order mean curvatures*, Differential geometry, 1991, pp. 279–296. MR1173047
[33] Sung-ho Park, *Every ring type spanner in a wedge is spherical*, Math. Ann. 332 (2005), no. 3, 475–482. MR2181758
[34] Guohuan Qiu and Chao Xia, *A generalization of Reilly's formula and its applications to a new Heintze-Karcher type inequality*, Int. Math. Res. Not. 2015 (2015), no. 17, 7608–7619. MR3403995
[35] Robert C. Reilly, *Applications of the Hessian operator in a Riemannian manifold*, Indiana Univ. Math. J. 26 (1977), 459–472. MR0474149
[36] Antonio Ros, *Compact hypersurfaces with constant higher order mean curvatures*, Rev. Mat. Iberoam. 3 (1987), no. 3-4, 447–453. MR996826
[37] Mario Santilli, *Uniqueness of singular convex hypersurfaces with lower bounded k-th mean curvature*, 2020. arXiv: 1908.05952v4.
[38] Leon Simon, *Regularity of capillary surfaces over domains with corners*, Pacific J. Math. 88 (1980), no. 2, 363–377. MR607984
[39] Rabah Souam, *On stable capillary hypersurfaces with planar boundaries*, J. Geom. Anal. 33 (2023), no. 6, Paper No. 196, 10. MR4576390
[40] Guofang Wang, Liangjun Weng, and Chao Xia, *Alexandrov–Fenchel inequalities for convex hypersurfaces in the half-space with capillary boundary*, Math. Ann. (2023).
[41] Guofang Wang and Chao Xia, *Uniqueness of stable capillary hypersurfaces in a ball*, Math. Ann. 374 (2019), no. 3-4, 1845–1882. MR3985125
[42] Henry C. Wente, *Tubular capillary surfaces in a convex body*, Advances in geometric analysis and continuum mechanics (Stanford, CA, 1993), 1995, pp. 288–298. MR1356751
[43] Chao Xia and Xiuwen Zhang, *Uniqueness for volume-constraint local energy-minimizing sets in a half-space or a ball*, Advances in Calculus of Variations (2023).
[44] Xin Zhou and Jonathan J. Zhu, *Min-max theory for constant mean curvature hypersurfaces*, Invent. Math. 218 (2019), no. 2, 441–490. MR4011704

(J.X) School of Mathematics, Southeast University, 211189, Nanjing, P.R. China
*Email address*: jia92@mail.ustc.edu.cn

(G.W) Mathematisches Institut, Universit"at Freiburg, Ernst-Zermelo-Str. 1, 79104, Freiburg, Germany
*Email address*: guofang.wang@math.uni-freiburg.de

(C.X) School of Mathematical Sciences, Xiamen University, 361005, Xiamen, P.R. China
*Email address*: chaoxia@xmu.edu.cn

(X.Z) School of Mathematical Sciences, Xiamen University, 361005, Xiamen, P.R. China
*Email address*: xuwenzhang@stu.xmu.edu.cn