CAUSALITY AND LEGENDRIAN LINKING FOR HIGHER DIMENSIONAL SPACETIMES

VLADIMIR CHERNOV

Abstract. Let \((X^{m+1}, g)\) be an \((m + 1)\)-dimensional globally hyperbolic spacetime with Cauchy surface \(M^m\), and let \(\tilde{M}^m\) be the universal cover of the Cauchy surface. Let \(\mathcal{N}_X\) be the contact manifold of all future directed unparameterized light rays in \(X\) that we identify with the spherical cotangent bundle \(ST^*M\). Jointly with Stefan Nemirovski we showed that when \(\tilde{M}^m\) is not a compact manifold, then two points \(x, y \in X\) are causally related if and only if the Legendrian spheres \(S_x, S_y\) of all light rays through \(x\) and \(y\) are linked in \(N_X\).

In this short note we use the contact Bott-Samelson theorem of Frauenfelder, Labrousse and Schlenk to show that the same statement is true for all \(X\) for which the integral cohomology ring of a closed \(\tilde{M}\) is not the one of the CROSS (compact rank one symmetric space).

If \(M\) admits a Riemann metric \(\mathbf{g}\), a point \(x\) and a number \(\ell > 0\) such that all unit speed geodesics starting from \(x\) return back to \(x\) in time \(\ell\), then \((M, \mathbf{g})\) is called a \(Y_x^\ell\)-manifold. Jointly with Stefan Nemirovski we observed that causality in \((M \times \mathbb{R}, \mathbf{g} \oplus -t^2)\) is not equivalent to Legendrian linking. Every \(Y_x^\ell\)-Riemann manifold has compact universal cover and its integral cohomology ring is the one of a CROSS. So we conjecture that Legendrian linking is equivalent to causality if and only if one can not put a \(Y_x^\ell\) Riemann metric on a Cauchy surface \(M\).

All manifolds, maps etc. are assumed to be smooth unless the opposite is explicitly stated, and the word smooth means \(C^\infty\).

1. Introduction

Let \(M\) be a not necessarily orientable, connected manifold of dimension \(m \geq 2\) and let \(\pi_M : ST^*M \to M\) be its spherical cotangent bundle. The manifold \(ST^*M\) carries a canonical co-oriented contact structure. An isotopy \(\{L_t\}_{t \in [0, 1]}\) of Legendrian submanifolds in a co-oriented contact manifold is called respectively positive, non-negative if it can be parameterised in such a way that the tangent vectors of all the trajectories of individual points lie in respectively positive, non-negative tangent half-spaces defined by the contact structure.

For the introduction of basic notions from Lorentz geometry we follow our paper [12].

Let \((X^{m+1}, g)\) be an \((m + 1)\)-dimensional Lorentz manifold and \(x \in X\). A nonzero \(v \in T_xX\) is called timelike, spacelike, non-spacelike (causal) or null (lightlike) if \(g(v, v)\) is respectively negative, positive, non-positive or zero. An piecewise smooth curve is timelike if all of its velocity vectors are timelike. Null and non-spacelike curves are defined similarly. The Lorentz manifold \((X, g)\) has a unique Levi-Cevita connection, see for example [3, page 22], so we can talk about timelike and null geodesics. A submanifold \(M \subset X\) is spacelike if \(g\) restricted to \(TM\) is a Riemann metric.

All non-spacelike vectors in \(T_xX\) form a cone consisting of two hemicones, and a continuous with respect to \(x \in X\) choice of one of the two hemicones (if it exists) is called the time...
orientation of \((X, g)\). The vectors from the chosen hemicones are called future directed. A time oriented Lorentz manifold is called a spacetime and its points are called events.

For \(x\) in a spacetime \((X, g)\) its causal future \(J^+(x) \subset X\) is the set of all \(y \in X\) that can be reached by a future pointing causal curve from \(x\). The causal past \(J^-(x)\) of the event \(x \in X\) is defined similarly.

Two events \(x, y\) are said to be causally related if \(x \in J^+(y)\) or \(y \in J^+(x)\).

A spacetime is said to be globally hyperbolic if \(J^+(x) \cap J^-(y)\) is compact for every \(x, y \in X\) and if it is causal, i.e. it has no closed non-spacelike curves. The classical definition of global hyperbolicity requires \((X, g)\) to be strongly causal rather than just causal, but these two definitions are equivalent, see Bernal and Sanchez [6, Theorem 3.2].

A Cauchy surface in \((X, g)\) is a subset such that every inextendible nonspace-like curve \(\gamma(t)\) intersects it at exactly one value of \(t\). A classical result is that \((X, g)\) is globally hyperbolic if and only if it has a Cauchy surface, see [19, pages 211-212]. Geroch [18] proved that every globally hyperbolic \((X, g)\) is homeomorphic to a product of \(\mathbb{R}\) and a Cauchy surface. Bernal and Sanchez [7, Theorem 1], [8, Theorem 1.1], [9, Theorem 1.2] proved that every globally hyperbolic \((X^{m+1}, g)\) has a smooth spacelike Cauchy surface \(M^m\) and that moreover for every smooth spacelike Cauchy surface \(M\) there is a diffeomorphism \(h: M \times \mathbb{R} \to X\) such that

- \(a\): \(h(M \times t)\) is a smooth spacelike Cauchy surface for all \(t\),
- \(b\): \(h(x \times \mathbb{R})\) is a future directed timelike curve for all \(x \in M\), and finally
- \(c\): \(h(M \times 0) = M\) with \(h|_{M \times 0}: M \to M\) being the identity map.

For a spacetime \(X\) we consider its space of light rays \(\mathcal{R} = \mathcal{N}_X\). By definition, a point \(\gamma \in \mathcal{R}\) is an equivalent class of inextendible future-directed null geodesics up to an orientation preserving affine reparametrization.

A seminal observation of Penrose and Low [21,22] is that the space \(\mathcal{R}\) has a canonical structure of a contact manifold (see also [23, 20, 10, 15]). A contact form \(\alpha_M\) on \(\mathcal{R}\) defining that contact structure can be associated to any smooth spacelike Cauchy surface \(M \subset X\).

Namely, consider the map

\[ i_M : \mathcal{N}_X \hookrightarrow T^*M \]

taking \(\gamma \in \mathcal{R}\) represented by a null geodesic \(\gamma \subset X\) to the 1-form on \(M\) at the point \(x = \gamma \cap M\) collinear to \(\langle \dot{\gamma}(x), \cdot \rangle_M\) and having unit length with respect to the induced Riemann metric on \(M\). This map identifies \(\mathcal{R}\) with the unit cosphere bundle \(ST^*M\) of the Riemannian manifold \(M\). Then

\[ \alpha_M := i_M^* \lambda_{\text{can}}, \]

where \(\lambda_{\text{can}} = \sum p_k dq^k\) is the canonical Liouville 1-form on \(T^*M\).

**Remark 1.1** (Bott-Samelson type result of Frauenfelder, Labrousse and Schlenk and its strengthening). The contact Bott-Samelson [11] type result of Frauenfelder, Labrousse and Schlenk [17, Theorem 1.13] says that if there is a positive Legendrian isotopy of a fiber \(S_x\) of \(ST^*M\) to itself, then the universal cover \(\widetilde{M}\) of \(M\) is compact and has the integral cohomology ring of a CROSS. (This result answers our question with Nemirovski [13, Example 8.3] and compactness of \(\widetilde{M}\) was first aproved in [13, Corollary 8.1].)

Our result with Nemirovski [14, Proposition 4.5] says that if there is a non-constant non-negative Legendrian isotopy of \(S_x\) to itself, then there is a positive Legendrian isotopy of \(S_x\) to itself. (Note that this positive Legendrian isotopy generally is not a perturbation of the non-negative constant Legendrian isotopy that was assumed.)

So we can somewhat strengthen [17, Theorem 1.13] to say that if there is a non-constant non-negative Legendrian isotopy of \(S_x\) to itself, then the universal cover \(\widetilde{M}\) of \(M\) is compact.
and has the integral cohomology ring of a CROSS, and in this work we will have to use the strengthened version of the contact Bott-Samelson theorem.

2. Main Results

Let \((X, g)\) be a globally hyperbolic spacetime with Cauchy surface \(M\). For a point \(x \in X\) we denote by \(S_x \subset N\) the Legendrian sphere of all (unparameterized, future directed) light rays passing through \(x\).

For two causally unrelated points \(x, y \in X\) the Legendrian link \((S_x, S_y)\) in \(N\) does not depend on the choice of the causally unrelated points. Under the identification \(N = ST^*M\) this link is Legendrian isotopic to the link of sphere-fibers over two points of some (and then any) spacelike Cauchy surface \(M\), see [16, Theorem 8], [12, Lemma 4.3] and [23]. We call a Legendrian link trivial if it is isotopic to such a link.

In [12, Theorem A] and [13, Theorem 10.4] we proved the following result. Assume that the universal cover \(\tilde{M}\) of a Cauchy surface of \(M\) of \(X\) is not compact and events \(x, y \in X\) are causally related. Then the Legendrian link \((\mathcal{G}_x, \mathcal{G}_y)\) is nontrivial.

In the case where \(M = \mathbb{R}^3\) this proved the Legendrian Low conjecture of Natario and Tod [23]. The question to explore relations between causality and linking was motivated by the observations of Low [21, 22] and appeared on Arnold’s problem lists as a problem communicated by Penrose [1, Problem 8], [2, Problem 1998-21].

In this work we prove the following Theorem.

**Theorem 2.1.** Assume that two events \(x, y\) in a globally hyperbolic spacetime \(X\) are causally related and the universal cover \(\tilde{M}\) of a Cauchy surface \(M\) of \(X\) is compact but does not have integral cohomology ring of a compact rank one symmetric space (CROSS). Then the Legendrian link \((\mathcal{G}_x, \mathcal{G}_y)\) is nontrivial.

**Proof.** The beginning of the proof follows the one of our [12, Theorem A].

If \(x, y\) are on the same null geodesic then the Legendrian link \((\mathcal{G}_x, \mathcal{G}_y)\) has a double point and hence is singular. So we do not have to consider this case.

Without the loss of generality we can assume that \(y \in J^+(x)\) and hence \(\mathcal{G}_y\) is connected to \(\mathcal{G}_x\) by a non-negative Legendrian isotopy, see [12, Proposition 4.2].

Suppose that \(\mathcal{G}_x\) and \(\mathcal{G}_y\) are Legendrian unlinked, i.e., that the link \(\mathcal{G}_x, \mathcal{G}_y\) is Legendrian isotopic to the link \(F \sqcup F'\) formed by two different fibers of \(ST^*M\). By the Legendrian isotopy extension theorem, there exists a contactomorphism \(\Psi : ST^*M \rightarrow ST^*M\) such that \(\Psi(\mathcal{G}_x \sqcup \mathcal{G}_y) = F \sqcup F'\). Hence, we get a non-negative Legendrian isotopy connecting two different fibers \(F\) and \(F'\) of \(ST^*M\). But the Legendrian link \(F \sqcup F'\) is symmetric, i.e., it is Legendrian isotopic to \(F' \sqcup F\). Hence there exists a contactomorphism of \(ST^*M\) exchanging the two link components and we also have a non-negative Legendrian isotopy connecting \(F'\) to \(F\).

Composing the non-negative Legendrian isotopy from \(F\) to \(F'\) and from \(F'\) to \(F\) we get a non-constant non-negative Legendrian isotopy from a fiber of \(ST^*M\) to itself.

Finally Remark [11] says that \(\tilde{M}\) has the integral cohomology ring of a CROSS. This contradicts to our assumptions.

\[\Box\]

**Remark 2.2.** Let \((M, g)\) be a Riemann manifold having a point \(x\) and a positive number \(\ell\) such that all the unit speed geodesics from \(x\) return back to \(x\) in time \(\ell\), then \((M, g)\) is called a \(Y^*\) Riemann manifold. The result of Bérdard-Bergery [5, Theorem 7.37], [4] says that the universal cover \(\tilde{M}\) in this case is compact and the rational cohomology ring of \(\tilde{M}\) has exactly one generator. Clearly the co-geodesic flow on \(M\) gives a positive Legendrian
constant non-negative Legendrian isotopy connecting the fiber to itself. So the Bott-Samelson [11] type result of Frauenfelder, Labrousse and Schlenk [17, Theorem 1.13] says more, namely that \( \tilde{M} \) has the integral cohomology ring of a CROSS.

In [13, Example 10.5] we observed that when \((M, \mathcal{F})\) is a \(Y^x_\ell\) Riemann manifold, then causality in the globally hyperbolic \((M \times \mathbb{R}, \mathcal{F} \oplus -dt^2)\) is not equivalent to Legendrian linking of sphere of light rays. In particular, causality is not equivalent to Legendrian linking in the case where \((M, \mathcal{F})\) is a metric quotient of the standard sphere. The Thurston Elliptization Conjecture (solved as the part of the Thurston Geometrization Conjecture) by Perelman [25, 26, 27] says that every 3-dimensional \(M\) whose universal cover is compact (and hence is the sphere by the Poincare conjecture) is a metric quotient of the sphere. So the results of Theorem 2.1 are new and interesting only for spacetimes of dimension greater than four.

It is not currently known whether every compact simply connected manifold whose integer cohomology ring is the one of a CROSS admits a \(Y^x_\ell\) Riemann metric. So we make the following Conjecture.

**Conjecture 2.3.** Assume that a globally hyperbolic spacetime \(X\) is such that one can not put a \(Y^x_\ell\) Riemann metric on its Cauchy surface \(M\). Then two events \(x, y \in X\) are causally related if and only if the Legendrian link \((\mathcal{S}_x, \mathcal{S}_y)\) is non-trivial.

In [12, Theorem C] we proved the following result. Assume that a Cauchy surface \(M\) has a cover diffeomorphic to an open domain in \(\mathbb{R}^m\) and two events \(x, y \in X\) are such that \(y \in J^+(x)\) but \(x, y\) do not belong to a common light geodesic. Then the Legendrian links \(\mathcal{S}_x \sqcup \mathcal{S}_y\) and \(\mathcal{S}_y \sqcup \mathcal{S}_x\) are not Legendrian isotopic. (In this case there is a non-negative Legendrian isotopy from \(\mathcal{S}_y\) to \(\mathcal{S}_x\), but there is no such non-negative Legendrian isotopy from \(\mathcal{S}_x\) to \(\mathcal{S}_y\).) One of the key ingredients in the proof is that for such \(M\) there is no non-constant non-negative Legendrian isotopy of a fiber of \(ST^*M\) to itself. In [13, Corollary 8.1, Remark 8.2] we showed that for the case where the universal cover \(\tilde{M}\) of \(M\) is not a compact manifold there is no non-negative Legendrian isotopy of a fiber of \(ST^*M\) to itself. So the result and the proof of [12, Theorem C] immediately generalize to this case.

In this work we use the Bott-Samelson [11] type Theorem of Frauenfelder, Labrousse and Schlenk [17, Theorem 1.13] to get the following result.

**Theorem 2.4.** Assume that two events \(x, y\) in a globally hyperbolic spacetime \(X\) are causally related and the universal cover \(\tilde{M}\) of a Cauchy surface \(M\) of \(X\) is compact but does not have integral cohomology ring of a compact rank one symmetric space (CROSS). Then the Legendrian link \((\mathcal{S}_x, \mathcal{S}_y)\) is not Legendrian isotopic to \((\mathcal{S}_y, \mathcal{S}_x)\).

**Proof.** The beginning of the proof follows the one of our [12, Theorem C].

Suppose that the links \(\mathcal{S}_x, \mathcal{S}_y\) and \(\mathcal{S}_y, \mathcal{S}_x\) are Legendrian isotopic. By the Legendrian isotopy extension theorem, there exists an auto contactomorphism \(\Psi\) such that \(\Psi(\mathcal{S}_x, \mathcal{S}_y) = (\mathcal{S}_y, \mathcal{S}_x)\). Without loss of generality we assume that \(y\) is in the causal past of \(x\). Let \(\{\Lambda_t\}_{t \in [0, 1]}\) be a non-negative Legendrian isotopy in \(ST^*M\) connecting \(\mathcal{S}_x\) to \(\mathcal{S}_y\) provided by [12, Proposition 4.2]. Then \(\{\Psi(\Lambda_t)\}_{t \in [0, 1]}\) is a non-negative Legendrian isotopy connecting \(\mathcal{S}_y\) to \(\mathcal{S}_x\). Composing these two isotopies, we obtain a non-constant non-negative Legendrian isotopy connecting \(\mathcal{S}_x\) to itself.

Recall now that \(\mathcal{S}_x\) is Legendrian isotopic to a fiber of \(ST^*M\). By the Legendrian isotopy extension theorem, there exists a contactomorphism \(\Phi\) taking \(\mathcal{S}_x\) to that fiber. The Legendrian isotopy connecting \(\mathcal{S}_x\) to itself constructed above is taken by \(\Phi\) to a non-constant non-negative Legendrian isotopy connecting the fiber to itself.
Finally Remark \[1\] says that \( \tilde{M} \) has the integral cohomology ring of a CROSS. This contradicts to our assumptions.

\[\square\]

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**Department of Mathematics, 6188 Kemeny Hall, Dartmouth College, Hanover, NH 03755, USA**

*E-mail address*: Vladimir.Chernov@dartmouth.edu