We propose an order parameter for a general one-dimensional gapped system with an open boundary condition. The order parameter can be computed from the ground state entanglement entropy of some regions near one of the boundaries. Hence, it is well-defined even in the presence of arbitrary interaction and disorder. We also show that it is invariant under a finite-depth local quantum circuit, suggesting its stability against an arbitrary local perturbation that does not close the energy gap. Further, it can unambiguously distinguish Majorana chain from a trivial chain under a global fermion parity conservation. We argue that the order parameter can be in principle measured in an optical lattice system.

Topological order is a new kind of order that cannot be described by Landau’s symmetry breaking paradigm. Quantum many-body systems with such an order are known to exhibit a number of interesting properties which have no counterpart in classical physics. Important examples include a quantized Hall conductance and a chiral gapless edge mode that propagates along the edge of a finite sample. For certain two-dimensional systems, topological ground state degeneracy may arise, which is stable against an arbitrary perturbation that is sufficiently weak. In the bulk of these systems, exotic particles obeying anyonic statistics may arise. Further, remnants of the information about these particles can be sometimes obtained from the ground state wavefunction alone.

In some sense, Kitaev’s Majorana chain also has some of these properties. It can encode a qubit in its ground state that is robust against a weak perturbation that respects the fermion parity, and one can construct networks of such chains to braid Majorana fermions. However, topological entanglement entropy - defined as a constant subcorrection term of the entanglement entropy - has not been discussed in this system to the best of author’s knowledge.

There are several reasons to believe why such a quantity might be ill-defined, or even completely nonexistent. For a one-dimensional system, entanglement entropy satisfies an area law. Therefore, the subleading term must vanish asymptotically, which is unlikely to contain any useful information about the phase. Furthermore, there are systems in two spatial dimensions that cannot be adiabatically connected to a trivial state yet do not have any topological entanglement entropy, e.g. integer quantum Hall state. These examples show that the nontriviality of the phase does not necessarily imply a nonzero value of topological entanglement entropy.

However, on the other hand, the author has recently proved an inequality between topological entanglement entropy and the topological ground state degeneracy. Since Majorana chain has two topologically protected ground states, one may expect some form of topological entanglement entropy to be present in the system.

We explicitly show that such a speculation is indeed correct. Furthermore, we present a stability argument based on a widely held belief: that adiabatic evolution can be simulated by a finite-depth local quantum circuit.

We note in passing that our framework is quite general, in that the only assumption we make about the Hamiltonian is its geometrically local structure and the existence of an energy gap that is independent of the system size. Nevertheless, we focus on the application to the Majorana chain for a number of reasons. An important open problem in the classification of gapped phases is to determine a full set of topological invariants that characterize the phase. In the case of a one-dimensional fermionic system, Kitaev has originally proposed a Pfaffian formula which can be used for a clean, translationally invariant system. For a disordered chain, there is a prescription proposed by Akhmerov et al. Alternatively, one may study the entanglement spectrum along a real-space cut. The advantage of this approach over the others is that it is applicable to interacting systems as well.

Both the degeneracy of the entanglement spectrum and our invariant can reliably distinguish the two different phases in the presence of interaction. However, there are important differences as well. For one thing, the degeneracy of the entanglement spectrum is a bulk invariant: it can be inferred from the reduced density matrix of some local region in the bulk. On the other hand, our invariant can be computed only from the reduced density matrix of some local region near the boundary. We emphasize, however, that the new invariant is a single number that can be easily computed from the entanglement entropies of some local regions. This is in contrast to the degeneracy of the entanglement spectrum, which needs an access to the entire spectrum.

Definition of the invariant

In order to motivate the definition of the invariant, we briefly mention the main result of Ref. There the author showed that the topological ground state degeneracy gives a rigorous lower bound for the topological entanglement entropy. We show that an analogous relationship exists in one-dimensional systems as well. For a bosonic chain without any symmetry, all gapped phases can be
adiabatically connected to a trivial state.\cite{20,21} Therefore, we do not expect anything interesting to happen. However, fermionic chain is known to have two different phases even in the absence of any symmetry.

The phase that is different from the trivial phase is Kitaev’s Majorana chain. Kitaev’s original model can be described by the following Hamiltonian:

\[ H = \sum_j -w a_j^\dagger a_{j+1} - \frac{1}{2} \mu (a_j^\dagger a_j - \frac{1}{2}) + \Delta a_j a_{j+1} + h.c., \quad (1) \]

where \( w \) is the hopping amplitude, \( \mu \) is the chemical potential, and \( \Delta \) is the superconducting gap. The operators \( a_j^\dagger \) and \( a_j \) are fermion creation and annihilation operators, and \( h.c. \) is the hermitian conjugate. If \( 2w > |\mu| \) and \( \Delta \neq 0 \), a gapless boundary mode arises.\cite{11} Two states \( |0\rangle \) and \( |1\rangle \) corresponding to each of the boundary modes form a set of degenerate ground states in the thermodynamic limit.

Each of these states have a fermion parity of +1 and −1, hence belonging to a different superselection sector. It is worth noting that a physical state cannot be in a superposition of the two sectors. Nevertheless, one can devise the following \textit{unphysical} state for the purpose of obtaining an inequality analogous to the one discussed in Ref.\cite{15}:

\[ \rho = \frac{1}{2} (|0\rangle \langle 0| + |1\rangle \langle 1|). \]

An important ingredient for deriving the inequality is the local indistinguishability of the two states, which asserts that they cannot be distinguished nor mapped into each other by applying any local operation.\cite{22}

Here we have an interesting (weaker) variant of the local indistinguishability condition. The two states can be mapped into each other by applying a parity-violating local operation near the boundary, but the expectation values of all the local observables are identical.\cite{23} Therefore, the reduced density matrix of \( |0\rangle \) and \( |1\rangle \) over any region that does not contain both boundaries must be close to each other in trace norm.

Now we apply the protocol introduced in Ref.\cite{13}. We partition the system into three contiguous subsystems, see FIG.\cite{4} Applying the strong subadditivity of entropy (SSA),\cite{24} one can arrive at the following inequality:

\[ S(AB) + S(BC) - S(B) \geq S(ABC) = 1. \]

\( AB, BC, \) and \( B \) are \textit{local} in a sense that they have access to only half of the unpaired Majorana fermion. Therefore, all the correlation functions supported on these local subsystems must be identical for both \( |0\rangle \) and \( |1\rangle \). In particular, the reduced density matrix of the two states must be identical over these local subsystems. Therefore, if the system is in one of the superselection sectors, \( i.e., \) \( |0\rangle \) or \( |1\rangle \) but not their superposition, \( S(AB) + S(BC) - S(B) \) must be at least larger or equal to 1. Further, by using the purity of the global state, we obtain the following inequality:

\[ \gamma \geq 1, \]

where the \textit{entropic invariant} for a one-dimensional system is defined as follows:

\[ \gamma := S(AB) + S(A) - S(B). \quad (2) \]

We emphasize again: that implicit in the definition of \( \gamma \) we assume the global state to be in one of the superselection sectors. This observation is related to the fact that there are two subtleties in the preceding discussion. First, for a pure state \( |\psi\rangle_{AB} \) over fermion modes \( A \) and \( B \), the entanglement entropy of \( A \) is in general \textit{not} equal to the entanglement entropy of \( B \). However, one can sidestep this problem if the global state is guaranteed to be in one of the superselection sectors.\cite{24} Secondly, SSA in its original form cannot be used: the underlying Hilbert space does not have a simple tensor product structure. However, Araki and Moriya has proved a variant of SSA over fermion modes, see Theorem 10.1 of Ref.\cite{25}. Therefore, the preceding argument is indeed valid for our system.

The nontrivial lower bound for \( \gamma \) strongly suggests that it is capable of distinguishing a trivial chain from a Majorana chain. If the chain is a renormalization-group fixed point of a trivial phase, \( i.e., \) \( w = \Delta = 0 \), \( \gamma \) is trivially 0. On the other hand, the aforementioned weak form of local indistinguishability condition should hold as long as the boundary mode can be described by the Majorana fermion. Therefore, if \( \gamma \) is measured to be some value that is appreciably larger than 0, one may conclude that the system is in a topologically nontrivial phase. There is, of course, a remote possibility that \( \gamma \) can attain a value close to 1 for a state that can be adiabatically connected from a trivial state. We show that such a possibility is strictly forbidden. In fact, we explicitly show that \( \gamma \) remains stable against any finite depth local quantum circuit, so long as \( \gamma \) is \textit{deformation invariant}.

\textit{Deformation invariance and stability}

At least on the perturbative level, the stability of topological entanglement entropy can be attributed to the existence of conditionally independent subsystems.\cite{26} The existence of such subsystems imply that a certain linear combination of entanglement entropy is deformation invariant. Colloquially, we say that a linear combination of entanglement entropy to be deformation invariant if it is invariant under a small deformation of the subsystems that preserves the topology.

For example, the mutual information between \( A \) and \( C \) is deformation invariant for a trivial state: entanglement entropies over any subsystems is 0. A more interesting example is the Majorana chain. More precisely, one can set the parameters of Eq.\cite{14} as \( \Delta = w > 0 \) and \( \mu = 0 \). In this case, Eq.\cite{14} reduces to the simplest Hamiltonian that
has Majorana zero modes near the boundary. By using Fidkowski’s formula for the entanglement spectrum, one can easily show that $I(A : C) = 1$ for a physical state as long as $B$ is not an empty set.\[27\]

We shall exploit the deformation invariance by approximating the adiabatic evolution by a finite-depth local quantum circuit. Such a circuit is defined as follows:

$$U_{lqc} = \Pi_{i=1}^{n} U_{i},$$

where $n$ is a constant that is independent of the system size, and $U_{i}$ is a product of unitary operators which have disjoint and geometrically local supports. We note in passing that currently there is no definitive proof that adiabatic evolution can be approximated by such a circuit. Here we nevertheless assume such an approximation exists, and study its consequences. As was the case for the SSA, there is a subtlety in using the notion of local unitary transformation in fermionic systems. The solution is to replace a product of unitary operators to a product of local fermionic unitary operators. The local fermionic unitary operator is defined as $U = e^{iH}$, where $H$ is some parity-preserving Hermitian operator made up of fermion creation and annihilation operators that act on a finite ball.\[28\]

We say that the quantum circuit $U_{lqc}$ has depth $n$ and width $w$ if (i) $U_{lqc}$ can be expressed as in Eq.\[3\] and (ii) $U_{i} = \otimes_{j} U_{i,j}$, where $U_{i,j}$ are unitary operators on disjoint supports with the support size bounded by $w$. We further assume that each of the supports are local, in a sense that they can be contained in a ball of finite radius. After a sufficient amount of coarse-graining, one can set $w$ to be 2. Also, we define

$$\gamma_{m}^{l} := I(A_{l} : C_{l})_{m},$$

where $A_{l}$ is the first $l$ sites of the lattice, $C_{l}$ the last $l$ sites of the lattice, and $I(A_{l} : C_{l})_{m}$ is the mutual information between $A_{l}$ and $C_{l}$ for a global state $|\psi_{m}\rangle := \Pi_{i=1}^{n} U_{m} |\psi_{0}\rangle$.

By using SSA and the invariance of the entanglement entropy under local unitary transformation, we obtain the following inequality:

$$\gamma_{m-1}^{l-1} \leq \gamma_{m}^{l} \leq \gamma_{m-1}^{l+1}.$$  \hspace{1cm} (4)

Recursively applying Eq.\[4\] the main result of this paper is obtained:

$$\gamma_{0-m}^{l-m} \leq \gamma_{m}^{l} \leq \gamma_{0+m}^{l+m}.$$  \hspace{1cm} (5)

Applied to a trivial state, Eq.\[5\] implies that $\gamma_{n}^{l} = 0$ for $n \leq \min(\frac{l}{2} - l, l)$. Similarly, we conclude that $\gamma_{n}^{l} = 1$ for the Majorana chain. Therefore, if the size of the subsystems are sufficiently large, $\gamma$ should remain invariant up to a small error that decays sufficiently fast with the subsystem size. For a realistic system with a finite correlation length $\xi$, we expect the correction term to be $O(e^{-l/\xi})$, where $l$ is the size of the subsystem.

**Renyi entropy analogue and its numerical benchmark**

Owing in part to the strong subadditivity of entropy, we have argued that $\gamma$ is an invariant that characterizes the phase. On one hand, this result is encouraging. One can compute the invariant by simply looking at the reduced density matrix of some local region that contains one of the boundaries. Therefore, if two systems have different values of $\gamma$, one can unambiguously tell that they are in a different phase. On the other hand, our result has its shortcomings. Entanglement entropy is an extremely hard quantity to measure experimentally. To the best of author’s knowledge, there is no simple known way of computing the von Neumann entropy without explicitly calculating all the eigenvalues of the density matrix.

In the studies of two-dimensional systems however, it has been known for a while that Renyi entanglement entropy can be used as an alternative to the von Neumann entanglement entropy.\[29\] Further, several proposals have been recently made to measure the Renyi-2 entanglement entropy of a generic quantum many-body system.\[30\]\[32\] Therefore, it is natural to ask if the Renyi entropy variant of our order parameter shows a similar behavior.

Here we attempt to make a similar approach by defining a Renyi-2 entropy variant of $\gamma$:

$$\gamma_{2} := S_{2}(A) + S_{2}(AB) - S_{2}(B),$$

FIG. 1: A schematic representation of the sequence of inequalities leading to Eq.\[4\]. The shaded region represents a local unitary transformation $U_{m}$, and the vertical line represents the partition of the system into $A, B$, and $C$. For the first and the last equality, we have used the fact that entanglement entropy is invariant under a local unitary transformation. The inequalities are simple consequences of SSA.
where $S_2(\rho) = -\log \text{Tr}(\rho^2)$. The result is plotted in FIG. 2, together with the plot for $\gamma$. While the stability argument presented in this paper cannot be applied to $\gamma_2$, its behavior under the change of the parameters shows an excellent agreement with that of $\gamma$. Our numerical result indicates that $\gamma_2$ might serve as a viable candidate for unambiguously distinguishing Majorana chain from a trivial chain.

![FIG. 2: We have plotted the value of $|1 - \gamma|$ and $|1 - \gamma_2|$ over a range of parameters. The length of the chain is 60, and the size of the subsystem $A,B$, and $C$ were all set to 20. We have allowed the chemical potential $\mu = \mu_0(1 + \eta x)$ to have a disorder, where $\eta$ is the disorder parameter and $x \in [-1, 1]$ is a uniform probability distribution. The value of $\gamma$ and $\gamma_2$ was obtained without any averaging over the disorder probability distribution. Without disorder, e.g., (a) and (b), the value of $\gamma$ and $\gamma_2$ both faithfully represent the phase as expected. Even in the presence of disorder, e.g., (c) and (d), the order parameters are highly reliable except near the phase boundary.](image)

- **Outlook**

In this paper, we have proposed a new order parameter for a generic one-dimensional system and argued that it is stable under an adiabatic evolution. Our conclusion was based on the assumption that the adiabatic evolution can be approximated by a finite-depth local quantum circuit. The actual (rigorous) stability proof shall be presented elsewhere.

While $\gamma$ is likely to be difficult to measure in a realistic experimental setting, one might be able to measure $\gamma_2$ by employing a recent proposal by Pichler et al. [32], which is particularly well-suited for measuring the Renyi-2 entropy of optically trapped fermionic atoms. In light of their proposal, the optical realization of the Majorana chain [33] will be an interesting testbed for detecting a smoking-gun signature of the long-range entanglement. It is worth noting that the Majorana chain sidesteps the key drawback that plagues essentially all the current proposals for measuring entanglement entropy: that the accuracy of the measurement grows exponentially with the entanglement entropy. Since Majorana chain is gapped, all of its entanglement entropy must be bounded by a constant. However, it should be also noted that the stability argument given in this paper does not hold for $\gamma_2$. Therefore, more work is needed to understand its stability.

We also note that the origin of the $\Omega(1)$ deviation of the order parameter from 0 can be traced back to (i) the existence of the states that have the same local reduced density matrices and (ii) the global superselection rule. A linear combination of the two states may lead to a different local reduced density matrix, but such a state does not follow the superselection rule. Nevertheless, for a physical state that satisfies the superselection rule, the invariant $\gamma$ is a well-defined quantity. In the case of the Majorana chain, the superselection rule was dictated by the global parity conservation law. It will be interesting to apply the invariant in the context of different symmetry, which is left for the future work.

We close with a remark that one must be careful about the possibility that the global fermion parity is not exact. Such a scenario may occur if the environment can couple to the gapless boundary mode. However, even under such condition, the stability argument for $\gamma$ remains to be valid. Therefore, if one can observe a significant deviation of $\gamma$ from 0 under the change of the parameters describing the chain, one may interpret it as an evidence that the chain is in a topologically nontrivial phase.

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