ON HOLOMORPHIC VECTOR FIELDS WITH MANY CLOSED ORBITS

LEONARDO CÂMARA & BRUNO SCARDUA

Abstract. We state some generalizations of a theorem due to G. Darboux, which originally states that a polynomial vector field in the complex plane exhibits a rational first integral and has all its orbits algebraic provided that it exhibits infinitely many algebraic orbits. In this paper, we give an interpretation of this result in terms of the classical Reeb stability theorems, for compact leaves of (non-singular) smooth foliations. Then we give versions of Darboux’s theorem, assuring, for a (non-singular) holomorphic foliation of any codimension, the existence of an open set of compact leaves provided that the measure of the set of compact leaves is not zero. As for the case of polynomial vector fields in the complex affine space of dimension $m \geq 2$, we prove suitable versions of the above results, based also on the very special geometry of the complex projective space of dimension $m$, and on the nature of the singularities of such vector fields we consider.

1. Introduction

One of the reasons for studying algebraic solutions of algebraic ordinary differential equations, is the fact that very basic examples of transcendental (non-algebraic) differentiable functions are given by solutions of such equations. These solutions however are not common. In fact, as it is well-known, for a generic choice on the coefficients of the coordinates, a polynomial complex vector field admits no algebraic solution. This is the subject of several works (cf. e.g. [26], [27], [35], [36], [37], [38], [39]).

On the other hand, an algebraic differential equation in dimension two with a sufficiently large number of algebraic solutions has all of its solutions in the algebraic class. This is stated in the following integrability theorem due to J. G. Darboux:

**Theorem 1.1** (Darboux [25]). Let $X$ be a polynomial vector field in $\mathbb{C}^2$. If $X$ exhibits infinitely many algebraic orbits, then all orbits are algebraic. In this case $X$ admits a rational first integral.

By an algebraic orbit we mean a non-singular orbit which is contained in an algebraic curve in $\mathbb{C}^2$. A rational function $R(x, y) = P(x, y)/Q(x, y)$, $P(x, y), Q(x, y) \in \mathbb{C}[x, y]$ is a first integral of $X$ if it is constant along the orbits of $X$. In this case, the orbits are algebraic and contained in the curves $\lambda P - \mu Q = 0$ for all $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{0\}$. Thus Theorem 1.1 is a complete result in dimension two.

Notice that, in the above theorem, the degree of the algebraic leaves has an upper bound (given by $\max\{\deg(P), \deg(Q)\}$). This somehow motivates the proof which is outlined as follows. The two basic facts are: (i) The vector field induces a natural dual polynomial 1-form $\omega$ of the same degree such that the integral curves of $\omega$ are the orbits of $X$. (ii) Given a reduced complex polynomial $f(x, y)$, the equation $\{f(x, y) = 0\}$ defines an algebraic orbit of $X$ if and only if $\frac{1}{f}(\omega \land df) =: \theta_f(x, y)dx \land dy$ is a polynomial two-form, and in this case we have $\deg(\theta_f) = \deg(\omega) - 1 = (X) - 1$. Since the space of complex polynomials of degree less than $\deg(X) - 1$ is a finite dimensional complex vector space, the hypothesis of infinitely many algebraic leaves ensures the existence of
two linearly independent polynomials \( f(x, y), g(x, y) \) such that \( \omega \wedge df = \omega \wedge dg = 0 \). Therefore \( R = f/g \) is a rational first integral for \( X \).

**Motivation for the foliation-geometrical approach.** The above algebraic argumentation is not available for the case of polynomial vector fields in dimension \( m \geq 3 \). A first problem involves the definition of algebraic curve in dimension \( m \geq 3 \) and the fact that not all algebraic curves are given by complete systems of polynomial equations. Another complication comes from the fact that it is not clear \textit{a priori} that the dual space of any vector field \( X \) is generated by \( m - 1 \) polynomial 1-forms satisfying the integrability conditions. Thus in order to understand the higher dimension situation one should deploy other more geometrical features. This is our main motivation for the introduction of the foliation framework and also for our current approach.

Theorem 1.1 is a sort of Reeb’s complete stability theorem for codimension one projective foliations, \textit{i.e.}, foliations in complex projective spaces. Let us first recall the classical result (see for instance [17, 30]): A smooth real oriented foliation of real codimension one in a compact connected manifold is a fibration by compact leaves if it exhibits a compact leaf with finite fundamental group. This result has many important consequences and motivates several questions in the theory of foliations. For instance: \textit{Is it true that a codimension one smooth foliation in a (connected) compact manifold with infinitely many compact leaves has all leaves compact?} The answer is clearly no, but this is true if the foliation is (transversely) real analytic.

There are versions of Reeb’s complete stability theorem for the class of holomorphic foliations (see [3]). In the holomorphic framework, it is proved in [5] that a non-singular transversely holomorphic codimension one foliation in a compact connected manifold admitting infinitely many compact leaves exhibits a transversely meromorphic first integral. All foliations mentioned so far are non-singular foliations. Similarly, in [18] it is proved that any (possibly singular) codimension one holomorphic foliation in a compact manifold having infinitely many closed leaves admits a meromorphic first integral. In particular, all leaves are closed off the singular set.

The problem of bounding the number of closed (off the singular set) leaves of a holomorphic foliation is known (at least in the complex algebraic framework) as Jouanolou’s problem, thanks to the pioneering results in [25], and has a wide range of contributions and applications in the algebraic-geometric setting.

From a more analytic-geometrical point of view, in [29] it is proved a global stability theorem for codimension \( k \geq 1 \) holomorphic foliations transverse to fibrations. In [10] the second author focus on the problem of existence of a stable compact leaf under the hypothesis of existence of a sufficiently large number of compact leaves. Recall that a leaf \( L \) of a compact foliation \( F \) is stable if it has a fundamental system of saturated neighborhoods (cf. [17], p. 376). The stability of a compact leaf \( L \in F \) is equivalent to the finiteness of its holonomy group \( \text{Hol}(F, L) \), and is also equivalent to the existence of a local bound for the volume of the leaves close to \( L \) ([17], Proposition 2.20, p. 103). As a partial converse, for smooth codimension one foliations Reeb proves that a compact leaf admitting a neighborhood consisting of compact leaves necessarily has finite holonomy. This is not true however in codimension greater or equal to 2. We also recall that a subset \( X \subset M \) of a differentiable \( m \)-manifold has zero measure if \( M \) admits an open cover by coordinate charts \( \varphi : U \subset M \to \varphi(U) \subset \mathbb{R}^m \) such that \( \varphi(U \cap X) \) has zero measure (with respect to the standard
Lebesgue measure in \(\mathbb{R}^m\)). Otherwise, we shall say that the set has positive measure. In \([10]\) it is proved the following measure stability theorem for non-singular holomorphic foliations:

**Theorem A** (Measure stability theorem) A holomorphic foliation \(F\) in a compact complex manifold \(M\) exhibits a compact stable leaf if and only if the set \(\Omega(F) \subset M\) of all compact leaves has positive measure.

In this paper we give a more general argumentation for the proof of Theorem A, based on the notion of measure concentration point of a subset of positive measure (cf. Definition \([4,11]\)) These ideas will be useful in the remaining part of the paper, in which we deal with the so called singular case. More precisely, we address the case of one-dimensional holomorphic foliations in projective spaces. Below we explain why.

The **singular foliations framework.** The structure of the orbits of a polynomial vector field \(X\) in \(\mathbb{C}^m\) is better understood by the study of their asymptotic behavior. For this sake, we introduce the complex projective space \(\mathbb{C}P^m\) as a natural compactification of the affine space \(\mathbb{C}^m\) and denote by \(E_{\infty}^{m-1}\) the hyperplane at infinity \(E_{\infty}^{m-1} = \mathbb{C}P^m \setminus \mathbb{C}^m\). Then we study the structure of the orbits of \(X\) in a neighborhood of the hyperplane at infinity. The best way to do that is by considering the notion of one-dimensional holomorphic foliation with singularities in complex manifolds. In few words, a one-dimensional holomorphic foliation with singularities in a complex manifold \(M\) consists of a non-singular one-dimensional holomorphic foliation in a open subset \(U = M \setminus \text{Sing}(F)\) and a discrete set of points (called singularities of the foliation) denoted by \(\text{Sing}(F)\). The notions of leaf, holonomy and so on, refer then to the underlying non-singular foliation and are defined as in the classical framework. It is well-known that any polynomial vector field defines a one-dimensional holomorphic foliation with singularities in the corresponding projective space. Conversely, any projective foliation is defined in an affine space by a polynomial vector field. Thus our approach will be based on this correspondence.

By an **algebraic leaf** of such a foliation, we mean a leaf contained in an algebraic curve. Such leaves are finitely punctured algebraic curves with such “ends” at singular points of the foliation. **Algebraic leaves of foliations in projective spaces play, in a certain sense, the role of compact leaves for (non-singular) smooth foliations.**

For singular foliations we need to introduce a notion of stability for algebraic leaves which at first sight seems to be stronger than one should expected. Let \(F\) be a holomorphic foliation with isolated singularities in a complex manifold \(M\) of dimension \(m \geq 2\). Given a non-singular point \(q \in M \setminus \text{Sing}(F)\) and a transverse disc \(\Sigma_q\) centered at \(q\), we shall denote by \(L_z\) the leaf of \(F\) through any \(z \in \Sigma_q\). The **virtual holonomy group** of the foliation \(F\) with respect to the transverse section \(\Sigma_q\) and base point \(q\) is defined as (cf. \([13]\)) the subgroup \(\text{Hol}^\text{virt}(F, \Sigma_q, q) \subset \text{Diff}(\Sigma_q, q)\) of germs of complex diffeomorphisms that preserve the leaves of the foliation, i.e.,

\[
\text{Hol}^\text{virt}(F, \Sigma_q, q) = \{ f \in \text{Diff}(\Sigma_q, q) : L_z = \hat{L}_{f(z)}, \forall z \in (\Sigma_q, q) \}
\]

Clearly, the virtual holonomy group contains the holonomy group, i.e.,

\[
\text{Hol}(F, L_q, \Sigma_q, q) \subset \text{Hol}^\text{virt}(F, \Sigma_q, q),
\]

where \(L_q \subset M\) is the leaf of \(F\) through \(q \in M\). If \(q_1\) and \(q_2\) belong to the same leaf of \(F\), then the corresponding virtual holonomy groups \(\text{Hol}^\text{virt}(F, \Sigma_{q_1}, q_1)\) and \(\text{Hol}^\text{virt}(F, \Sigma_{q_2}, q_2)\) are holomorphically conjugate by a germ of (holonomy) diffeomorphism \(h : (\Sigma_{q_1}, q_1) \to (\Sigma_{q_2}, q_2)\) for any transverse discs \(\Sigma_1 \ni q_1\) and \(\Sigma_2 \ni q_2\). Thus we shall refer to the virtual holonomy group corresponding to a leaf \(L \in F\) and denote it as \(\text{Hol}^\text{virt}(F, L)\), for general purposes.
Definition 1.2 (Finite virtual holonomy). Let $\Gamma \subset M$ be a connected invariant subset of $\mathcal{F}$. We shall say that each virtual holonomy group of $\Gamma$ is finite if for each point $q \in \Gamma \setminus (\Gamma \cap \text{Sing}(\mathcal{F}))$ the virtual holonomy group of the leaf $q \in L_q \subset \Gamma$ is finite.

We stress that $\Gamma$ is a union of leaves and singularities, and a priori the virtual holonomy groups of distinct leaves in $\Gamma$ are not related to each other. In order to relate such distinct groups we must introduce the notion of Dulac correspondence associated to suitable situations involving Siegel singularities (cf. Section 7).

As mentioned above, algebraic leaves play the role of compact leaves in the framework of foliations in projective spaces. This idea can be extended in the sense of the following:

Definition 1.3 (stable and quasi-compact leaf). A leaf of a foliation $\mathcal{F}$ in a complex manifold $M$ will be called closed if it is closed off the singular set $\text{Sing}(\mathcal{F})$. Therefore, in the one dimensional case the classical Remmert-Stein extension theorem ([22]) ensures that the closure $L \subset L \cup \text{Sing}(\mathcal{F}) \subset M$ is a pure one-dimensional analytic subset of $M$. A closed leaf $L$ is called stable if its virtual holonomy group is finite. By a quasi-compact leaf of $\mathcal{F}$ we shall mean a closed leaf $L$ such that the closure $L \subset M$ is compact. This may occur even if the manifold is not compact, as in the case of product manifolds or fiber spaces.

Remark 1.4. If a compact leaf has finite holonomy, then by Reeb’s local stability lemma (Lemma 2.1) it has a fundamental system of invariant neighborhoods with all leaves compact. As we shall see (Section 2, Lemma 4.7), this implies the finiteness of the leaf’s virtual holonomy group. Thus the notion of stable closed leaf extends the notion of stable compact leaf.

The notion of foliation with singularities is detailed in Section 5. Recall that a non-degenerate singularity of a one-dimensional holomorphic foliation in a complex manifold is an isolated point in whose neighborhood the foliation is defined by a holomorphic vector field with a non-singular linear part. Our next result refers to quasi-compact leaves of foliations with non-degenerate singularities in surfaces.

Theorem B Let $\mathcal{F}$ be a holomorphic foliation of dimension one with non-degenerate singularities in the complex surface $M^2$. Assume that the set $\Omega(\mathcal{F}) \subset M$, union of all quasi-compact leaves of $\mathcal{F}$, has positive measure. Then $\Omega(\mathcal{F})$ contains some open nonempty set.

Indeed, in the above situation we prove the existence of a stable graph (cf. Definition 8.1) for the foliation. In case the surface is compact, this is just the result proved in [5] for two dimensional complex manifolds.

A non-degenerate singularity is classified as a Siegel type singularity in case the origin $0 \in \mathbb{R}^2$ belongs to the convex hull of the set of eigenvalues of the linear part of the vector field at the singular point. Otherwise, it is classified as a Poincaré type singularity (cf. Section 6). A Siegel type singularity in dimension $m$ will be called normal if, up to a local change of coordinates, the coordinate hyperplanes are invariant. In this case, we may write the vector field as $X = \sum_{j=1}^m \lambda_j x_j \frac{\partial}{\partial x_j} + (x_1 \cdots x_m) \cdot X_2(x_1, \cdots, x_m)$, where $X_2$ is holomorphic and vanishes at the origin $0 \in \mathbb{C}^m$. As it is well-known, each Siegel type singularity in dimension $m = 2$ is normal (cf. [15], [28]). In dimension $m \geq 3$ this occurs if the origin belongs to the interior of the convex hull of the eigenvalues. In this case the singularity is called a strict Siegel type singularity [16].
In terms of foliations in projective spaces, our main result is called Algebraic Stability theorem and reads as follows:

**Theorem C (Algebraic Stability theorem)** Let $\mathcal{F}$ be a holomorphic foliation of dimension one with non-degenerate singularities in the $m$-dimensional complex projective space $\mathbb{CP}^m$. Assume that the set $\Omega(\mathcal{F}) \subset \mathbb{CP}^m$, union of all algebraic leaves of $\mathcal{F}$, has positive measure. Then $\mathcal{F}$ has a stable algebraic leaf $L_0 \subset \Omega(\mathcal{F})$. If the dimension $m = 3$ and the (Siegel) singularities are normal, then all singularities in $L_0$ are analytically linearizable exhibiting local holomorphic first integrals.

We point out that in case $m = 3$ (and even in case $m \geq 3$) we actually prove the existence of (an algebraic leaf which is contained in) a stable graph (cf. Definition 8.4). In the situation of Theorem C above, we do not know whether there is an invariant nonempty open subset of $\mathbb{CP}^m$ consisting of algebraic leaves of $\mathcal{F}$ (see Conjecture 9.8).

**Outline of the proof of Theorem C.** A rough sketch of the argumentation in the paper is as follows (we focus on Theorem C): As a first step, we prove the existence of a measure concentration algebraic leaf $L_0$, which means an algebraic leaf such that every neighborhood has a positive measure set of algebraic leaves. Using this and the fact that the singularities are non-degenerate (and the well known analytic and topological descriptions of such singularities provided by the Poincaré linearization theorem and the Poincaré-Dulac normal form), we are able to prove that this leaf is contained in the interior of the set of algebraic leaves. This comes from the study of the holonomy groups of the leaf (via Burnside's theorem) and of the non-dicritical (i.e., isolated from the set of separatrices) adjacent leaves, i.e., leaves accumulating at singular points contained in the closure of $L_0$. An important fact is the passage, through the leaves, from a neighborhood of $L_0$ to a neighborhood of such a non-dicritical adjacent leaf, which is granted by the construction of a Dulac map associated to such a singularity corner. This technic allows us to spread the finiteness properties from the leaf $L_0$ to its adjacent leaves, which are proved to be also algebraic. Thus we construct a kind of “stable algebraic graph” for the foliation. This invariant set has finite holonomy groups in a more wide sense, which shall be introduced later. Also, an important fact is the notion of relative order of a given leaf, introduced in this paper. The very special geometry of the complex projective space, as well as the holomorphic character of the foliation, play a fundamental role in the definition of this notion. Using classical results from complex geometry, as Chow’s theorem on the algebraicity of analytic sets in complex projective spaces, and Remmert-Stein extension for analytic subsets of open subsets of complex spaces, we are able to prove that a leaf of the foliation is algebraic if and only if it has finite relative order. This characterization and the so called transverse uniformity lemma allow us to control the behavior of the leaves in the boundary of suitable sets of algebraic leaves, proving under which conditions these boundary leaves are also algebraic. In few words, a finite relative order leaf is algebraic, and a leaf in the boundary of a set of leaves of uniformly bounded ordered is also algebraic.

### 2. Finite holonomy and local stability

We begin by recalling one of the very basic tools we need in this paper. The classical local stability theorem of Reeb reads as follows ([8, 17]):

**Lemma 2.1 (Reeb local stability theorem).** Let $L_0$ be a compact leaf with finite holonomy of a smooth foliation $\mathcal{F}$ of real codimension $k \geq 1$ in a manifold $M$. Then there is a fundamental system of invariant neighborhoods $W$ of $L_0$ in $\mathcal{F}$ such that every leaf $L \subset W$ is compact, has a
finite holonomy group and admits a finite covering onto \( L_0 \). Moreover, for each neighborhood \( W \) of \( L_0 \) there is an \( F \)-invariant tubular neighborhood \( \pi : W' \subset W \rightarrow L_0 \) of \( L_0 \) with the following properties:

1. Every leaf \( L' \subset W' \) is compact with finite holonomy group;
2. If \( L' \subset W' \) is a leaf, then the restriction \( \pi|_{L'} : L' \rightarrow L_0 \) is a finite covering map;
3. If \( x \in L_0 \), then \( \pi^{-1}(x) \) is a transverse of \( F \);
4. There is an uniform bound \( k \in \mathbb{N} \) such that for each leaf \( L' \subset W' \) we have \( #(L' \cap \pi^{-1}(x)) \leq k \).

Let now \( F \) be a codimension \( k \) holomorphic foliation in a complex manifold \( M \). Given a point \( p \in M \), the leaf through \( p \) is denoted by \( L_p \). We denote by \( \text{Hol}(F, L_p) = \text{Hol}(L_p) \) the holonomy group of \( L_p \) (cf. [8, 17]). This is an equivalence class defined by conjugacy, and we shall denote by \( \text{Hol}(L_p, \Sigma_p, p) \) its representative evaluated at a local transverse disc \( \Sigma_p \) centered at the point \( p \in L_p \).

The group \( \text{Hol}(L_p, \Sigma_p, p) \) is therefore a subgroup of the group of germs \( \text{Diff}(\Sigma_p, p) \) which is identified with the group \( \text{Diff}(\mathbb{C}^k, 0) \) of germs at the origin \( 0 \in \mathbb{C}^k \) of complex diffeomorphisms.

One of the main tools in the proof of the local stability theorem and in our work is the following result.

**Lemma 2.2** (transverse fibration and transversal uniformity lemma, [8]). Let \( F \) be a \( C^r \) foliation in a manifold \( M \). Given a leaf \( L \in F \) and a compact connected subset \( K \subset L \), there are neighborhoods \( K \subset U \subset W \subset M \), with \( U \) open in \( L \) and \( W \) open in \( M \), and a \( C^r \) retraction \( \pi : W \rightarrow U \) such that the fiber \( \pi^{-1}(x) \) is transverse to the restriction \( F|_W \) for each \( x \in U \). Given two points \( q_1, q_2 \in L_0 \) in a same leaf \( L_0 \) of \( F \), there are transverse discs \( \Sigma_1 \) and \( \Sigma_2 \), centered at \( q_1, q_2 \) respectively, and a diffeomorphism \( h : \Sigma_1 \rightarrow \Sigma_2 \) such that for any leaf \( L \) of \( F \) we have \( h(L \cap \Sigma_1) = L \cap \Sigma_2 \).

### 3. Periodic Groups and Groups of Finite Exponent

Next we present Burnside’s and Schur’s results on periodic linear groups. Let \( G \) be a group with identity \( e_G \in G \). The group is periodic if each element of \( G \) has finite order. A periodic group \( G \) is periodic of bounded exponent if there is an uniform upper bound for the orders of its elements. This is equivalent to the existence of \( m \in \mathbb{N} \) with \( g^m = 1 \) for all \( g \in G \) (cf. [29]). Thus a group which is periodic of bounded exponent is also called a group of finite exponent. The following classical results are due to Burnside and Schur.

**Theorem 3.1** (Burnside, 1905 [6], Schur, 1911 [31]). Let \( G \subset \text{GL}(k, \mathbb{C}) \) be a complex linear group.

1. (Burnside) If \( G \) is of finite exponent \( \ell \) (but not necessarily finitely generated), then \( G \) is finite; actually we have \( |G| \leq \ell^k \).
2. (Schur) If \( G \) is finitely generated and periodic (not necessarily of bounded exponent), then \( G \) is finite.

Using this result we prove the following

**Lemma 3.2** ([29]). About periodic groups of germs of complex diffeomorphisms we have:

1. A finitely generated periodic subgroup \( G \subset \text{Diff}(\mathbb{C}^k, 0) \) is necessarily finite. A (non necessarily finitely generated) subgroup \( G \subset \text{Diff}(\mathbb{C}^k, 0) \) of finite exponent is necessarily finite.
2. Let \( G \subset \text{Diff}(\mathbb{C}^k, 0) \) be a finitely generated subgroup. Assume that there is an invariant connected neighborhood \( W \) of the origin in \( \mathbb{C}^k \) such that each point \( x \) is periodic for each element \( g \in G \). Then \( G \) is a finite group.
(3) Let \( G \subseteq \text{Diff}(\mathbb{C}^k, 0) \) be a (non necessarily finitely generated) subgroup such that for each point \( x \) close enough to the origin, the pseudo-orbit of \( x \) is periodic of (uniformly bounded) order \( \leq \ell \) for some \( \ell \in \mathbb{N} \), then \( G \) is finite.

The above lemma is partially extended in Lemma 4.2.

4. Measure and finiteness

Let us pave the way to the proof of Theorem A. For the sake of simplicity, we will adopt the following notation: if a subset \( X \subseteq M \) is not a zero measure subset, then we shall say that it has positive measure and write \( \text{med}(X) > 0 \). This may cause no confusion since we are not considering any specific measure on \( M \) and we shall be dealing only with the notion of zero measure subset stated in Section 4. Nevertheless, we notice that if \( X \subseteq M \) writes as a countable union \( X = \bigcup_{n \in \mathbb{N}} X_n \) of subsets \( X_n \subseteq M \), then \( X \) has zero measure in \( M \) if and only if \( X_n \) has zero measure in \( M \) for all \( n \in \mathbb{N} \). In terms of our notation, we have therefore \( \text{med}(X) > 0 \) if and only if \( \text{med}(X_n) > 0 \) for some \( n \in \mathbb{N} \).

4.1. Measure concentration points. Here we introduce a fundamental notion in our argumentation.

**Definition 4.1.** Given a subset \( X \subseteq M \) of a differentiable manifold, a point \( p \in M \) will be called a measure concentration point of \( X \) if the set \( V \cap X \) has positive measure in \( M \) for any open neighborhood \( p \in V \subseteq M \). The set of measure concentration points of \( X \) will be denoted by \( \mathcal{C}_\mu(X) \). Clearly, \( \mathcal{C}_\mu(X) \subseteq \overline{X} \). If we denote by \( \text{Int}(X) \subseteq M \) the set of interior points of \( X \), then \( \text{Int}(X) \subseteq \mathcal{C}_\mu(X) \).

**Lemma 4.2.** If \( X \subseteq M \) is a subset, then a boundary point \( p \in \partial \mathcal{C}_\mu(X) \) is a measure concentration point of \( X \). In other words, \( \partial \mathcal{C}_\mu(X) \subseteq \mathcal{C}_\mu(X) \).

**Proof.** Without loss of generality, we may assume \( M \) is an open subset of \( \mathbb{R}^n \). Given a neighborhood \( V \ni p \), there is a small \( R > 0 \) such that the ball centered at \( p \) with radius \( R \) is contained in \( V \), i.e., ball \( B(p; R) \subseteq V \subseteq \mathbb{R}^n \). Since \( p \in \partial \mathcal{C}_\mu(X) \), there is a point \( q \in \mathcal{C}_\mu(X) \) such that \( |q - p| < R/2 \). Given now the neighborhood \( B(q; R/3) \) of \( q \), since \( q \in \mathcal{C}_\mu(X) \), we have \( \text{med}(X \cap B(q; R/3)) > 0 \). On the other hand, we have \( X \cap V \supset X \cap B(p; R) \supset X \cap B(q; R/3) \). Therefore, \( \text{med}(X \cap V) > 0 \). This proves that \( p \in \mathcal{C}_\mu(X) \).

**Lemma 4.3.** Given a subset \( X \subseteq M \), the set \( \mathcal{C}_\mu(X) \) has no isolated points.

**Proof.** Suppose that \( p \in \mathcal{C}_\mu(X) \) is an isolated measure concentration point, then there is an open neighborhood \( W \) of \( p \) in \( M \) such that \( (W \setminus \{p\}) \cap \mathcal{C}_\mu(X) = \emptyset \). Therefore, given a point \( q \in W \setminus \{p\} \), there is an open neighborhood \( q \in V_q \subseteq W \setminus \{p\} \) of \( q \) such that \( X \cap W_q \) has zero measure. The open cover \( W \setminus \{p\} \cap \bigcup_{j \in \mathbb{N}} W_{q_j} \) admits a countable subcover \( W \setminus \{p\} \subseteq \bigcup_{j \in \mathbb{N}} W_{q_j} \). Since \( X \cap W_{q_j} \) has zero measure for each \( j \in \mathbb{N} \), we conclude that \( X \cap W \) has zero measure, therefore \( p \) cannot be a measure concentration point of \( X \). This leads to a contradiction.

**Lemma 4.4.** Let \( X \subseteq M \) be a subset such that \( X = \bigcup_{k \in \mathbb{N}} X_k \), where \( \mathcal{C}_\mu(X_k) = \emptyset \) for each \( k \in \mathbb{N} \). Then \( \text{med}(X) = 0 \).

**Proof.** Since \( \mathcal{C}_\mu(X_k) = \emptyset \), it follows that \( \text{med}(X_k) = 0 \) for each \( k \in \mathbb{N} \). Thus \( \text{med}(X) = 0 \).

Indeed, we can prove more:
Lemma 4.5. Let \( X \subset M \) be a subset such that \( X \cap C_\mu(X) = \emptyset \). Then \( \text{med}(X) = 0 \).

Proof. Since \( X \cap C_\mu(X) = \emptyset \), for each point \( x \in X \) there is a neighborhood \( x \in W_x \subset M \) such that \( \text{med}(W_x \cap X) = 0 \). Therefore \( X \subset \bigcup_{x \in X} W_x \). By choosing a countable subcover \( X \subset \bigcup_{j \in \mathbb{N}} W_{x_j} \) we obtain \( X \subset \bigcup_{j \in \mathbb{N}} (W_{x_j} \cap X) \) and since \( \text{med}(X \cap W_{x_j}) = 0, \forall j \in \mathbb{N} \), conclude that \( \text{med}(X) = 0 \). \( \square \)

The following result is very natural.

Lemma 4.6. Let \( \Omega \subset M \) be an \( \mathcal{F} \)-invariant subset. Then the set \( C_\mu(\Omega) \) is also invariant by the foliation \( \mathcal{F} \).

Proof. Consider a non-singular point \( q \in C_\mu(\Omega) \), using a local trivialization chart \((W, \varphi)\) for \( \mathcal{F} \), we conclude that the plaque \( q \in Q \subset W \) of the leaf \( L_q \) of \( \mathcal{F} \) through \( q \) is also contained in the set \( C_\mu(\Omega) \). This shows that \( C_\mu(\Omega) \) is (locally invariant and therefore) invariant. \( \square \)

4.2. Subgroups with uniformly bounded pseudo-orbits orders. Now we shall obtain a slight generalization of Lemma 3.2. First we need some notation. Let \( S \) be a complex manifold, \( S_p := (S, p) \) the germ of \( S \) at the point \( p \in S \), and \( V \) a neighborhood of \( p \) in \( S \). Then we say that \( q \in V \) is a periodic point with respect to \( G \) (or \( G \)-periodic point for short) if any germ \( g \in G \) has a representative map \( g : V \rightarrow S \) such that \( g^{(j)}(q) \in V \) for all \( j = 0, 1, \ldots, k_g \) and \( g^{(k_g)}(q) = q \). In particular, the minimum possible value \( k_g \in \mathbb{N} \) satisfying the previous property is called the order of the pseudo-orbit of \( z \). Further, we say that \( q \) is a periodic point with respect to \( G \) with uniformly bounded pseudo-orbits if there is \( \ell \in \mathbb{N} \) such that \( k_g \leq \ell \) for all \( g \in G \). For any subgroup \( G \subset \text{Diff}(S_p, p) \) we denote by \( O_G(S_p, \ell) \) the set of \( G \)-periodic points whose pseudo-orbits orders are uniformly bounded by \( \ell \in \mathbb{N} \).

Lemma 4.7. Let \( G \) be a (non necessarily finitely generated) subgroup of \( \text{Diff}(S_p, p) \) such that \( p \in C_\mu(O_G(S_p, \ell)) \) for some \( \ell \in \mathbb{N} \). Then \( G \) is finite.

Proof. Now consider a map germ \( g \in G \) and pick a neighborhood \( V \) of \( p \) in \( S \) such that the representative \( g : V \rightarrow S \) (of the germ \( g \)) and its iterates \( g, g^2, \ldots, g^\ell \) are defined in \( V \). Since the orders of the orbits in \( O_G(S_p, \ell) \) are uniformly bounded by \( \ell \), then \( O_G(S_p, \ell) \cap V \) is contained in the analytic subset \( X_\ell := \bigcup_{m=0}^{\ell} \{ z \in V : g^{(m)}(z) = z \} \). Therefore, since \( \text{med}(O_G(S_p, \ell)) > 0 \) for some \( \ell \in \mathbb{N} \), then \( O_G(S_p, \ell) \cap V = V \) (i.e., \( g^{(k)} = \text{id} \), for some \( k \leq \ell \)). This shows that each germ \( g \in G \) is periodic of order \( k_g \leq \ell \) for some uniform \( \ell \in \mathbb{N} \). This implies that \( G \) is finite by Lemma 3.2. \( \square \)

4.3. Proof of the measure stability theorem. Let us sketch the proof of Theorem A. First we show the following preliminary results for a compact complex manifold \( M \) endowed with a non-singular holomorphic foliation \( \mathcal{F} \).

Claim 4.8. There is a finite number of relatively compact open discs \( T_j \subset M \), \( j = 1, \ldots, r \), such that:

1. Each \( T_j \) is transverse to \( \mathcal{F} \) and the closure \( \overline{T_j} \) is contained in the interior of a transverse disc \( \Sigma_j \rightarrow \mathcal{F} \);
2. Each leaf of \( \mathcal{F} \) intersects at least one of the discs \( T_j \).

Proof. Since \( M \) is compact, it is enough to show that, for each point \( p \in M \), there is an open neighborhood \( U_p \subset M \) of \( p \), and a relatively compact open disc \( T_p \subset U_p \) whose closure \( \overline{T_p} \) is contained in the interior of a disc \( \Sigma_p \) transverse to \( \mathcal{F} \), and such that each leaf of \( \mathcal{F} \) intersecting \( U_p \) crosses the disc \( T_p \). But this is an immediate consequence of the local trivialization property of foliations and of the fact that \( M \) is a locally compact topological space. \( \square \)
Let \( \{T_1, \cdots, T_r\} \subset M \) be as in the above claim, then we call \( T = \bigcup_{j=1}^r T_j \) a complete transversal to \( \mathcal{F} \) in \( M \).

**Remark 4.9.** As it is well known, the set of leaves of a foliation is not necessarily a manifold (not even a Hausdorff topological space). Therefore, introducing the concept of a complete transversal to \( \mathcal{F} \), the above claim allows to use the notion of measure concentration point in the space of leaves of a foliation \( \mathcal{F} \) defined in a compact manifold \( M \).

Let

\[
\Omega(\mathcal{F}, T) = \{ L \in \mathcal{F} : \#(L \cap T) < \infty \},
\]

then a leaf \( L \in \Omega(\mathcal{F}, T) \) is called a finite order leaf with respect to the complete transversal \( T \). In particular, \( \Omega(\mathcal{F}, T) = \bigcup_{k=1}^{\infty} \Omega(\mathcal{F}, T, k) \), where

\[
\Omega(\mathcal{F}, T, k) = \{ L \in \mathcal{F} : \#(L \cap T) \leq k \}.
\]

Since \( M \) is compact we have:

**Claim 4.10.** A leaf of \( \mathcal{F} \) is compact if and only if it has finite order. In other words, \( \Omega(\mathcal{F}) = \Omega(\mathcal{F}, T) \) as collections of leaves.

**Proof.** A compact leaf intersects a complete transversal only a finite number of times, since the union of the intersection points is a discrete subset of a compact set. Conversely, assume a leaf \( L \in \mathcal{F} \) has finite order with respect to a complete transversal \( T \) as above. We claim that \( L \subset M \) is closed and therefore compact. In fact, if \( L \) is not closed, then it has an accumulation point \( p \in \overline{L} \setminus L \). Using the local trivializations of \( \mathcal{F} \) at \( p \), we conclude that there is an arbitrarily small transverse disc \( \Sigma_p \), centered at \( p \), such that \( L \cap \Sigma_p = +\infty \). The leaf \( L_p \ni p \) necessarily intersects the collection \( T \) at some interior point, say \( q \in L_p \cap T_j \), for a certain \( j \in \{1, \ldots, r\} \). Choose a simple path \( \gamma: [0, 1] \to L_p \) joining \( \gamma(0) = p \) to \( \gamma(1) = q \). Then the corresponding holonomy map gives a germ of diffeomorphism \( h_\gamma: (\Sigma_p, p) \to (T_j, q) \). Since for any arbitrarily small disc \( \Sigma_p \) we have \( \#(L \cap \Sigma_p) = +\infty \), choosing a representative for the above holonomy map, we have \( \#(h_\gamma(L \cap \Sigma_p) \cap T_j) = +\infty \). Since \( h_\gamma(L \cap \Sigma_p) \subset L \cap T_j \), we conclude that \( \#(L \cap T_j) = +\infty \). Therefore \( L \) cannot have finite order with respect to \( T \). This proves the claim. \( \square \)

**Remark 4.11.** It is important to notice that even if a priori the notion of order cannot be defined with respect to a given complete transversal, thanks to the above claim, the notion of finite order is intrinsic to the leaf.

Now let us deal with the proof of the theorem itself. Since the necessary part of the statement is immediate, we only have to verify the sufficient part. Thus, from now on, we assume \( \text{med}(\Omega(\mathcal{F})) > 0 \). But recall from **Claim 4.10** that \( \Omega(\mathcal{F}) = \Omega(\mathcal{F}, T) = \bigcup_{n \in \mathbb{N}} \Omega(\mathcal{F}, T, n) \), thus there is \( n \in \mathbb{N} \) such that \( \text{med}(\Omega(\mathcal{F}, T, n)) > 0 \).

Now recall that a leaf \( L \in \Omega(\mathcal{F}) \) is a measure concentration point of the set of compact leaves \( \Omega(\mathcal{F}) \) if for any open neighborhood \( W \) of \( L \) the intersection \( W \cap \Omega(\mathcal{F}) \) has positive measure.

**Claim 4.12.** There is a compact leaf \( L_0 \subset C(\Omega(\mathcal{F})) \).

\(^1\)Although \( \Omega(\mathcal{F}) \) and \( \Omega(\mathcal{F}, T) \) are considered as collections of leaves, when we refer to the measure of these sets, we are considering the measure of the union of the leaves in each of these sets. This should cause no misunderstanding.
Proof. Suppose that for each compact leaf \( L \in \Omega(F) \) and each neighborhood \( V_L \) of \( L \) in \( M \) there is a neighborhood \( W_L \subset V_L \) of \( L \) in \( M \) such that \( \text{med}(W_L \cap \Omega(F)) = 0 \). In particular, there is an open cover \( \Omega(F) \subset \bigcup_{L \in \Omega(F)} W_L \) such that \( \text{med}(W_L \cap \Omega(F, T, n)) = 0 \). Since this open cover admits a countable subcover \( \Omega(F) \subset \bigcup_{n \in \mathbb{N}} W_n \) with \( \text{med}(W_n) = 0 \) for all \( n \in \mathbb{N} \), then \( \text{med}(\Omega(F)) = 0 \); a contradiction.

Applying now Claim 4.10 and Lemma 4.4, we conclude that \( L_0 \subset \mathcal{C}_\mu(\Omega(F, T, n)) \) for some \( n \in \mathbb{N} \).

Claim 4.13. The holonomy group of \( L_0 \) is finite.

Proof. Since \( \text{med}(\Omega(F)) > 0 \), Claim 4.10 and the comments before it guarantee the existence of a positive integer \( n \in \mathbb{Z}_+ \) such that \( \text{med}(\Omega(F, T, n)) > 0 \). Now pick \( p \in L_0 \cap T \) and a disc \( \Sigma \subset \overline{\Sigma} \subset T \) transverse to \( F \) and centered at \( p \). For each \( z \in \Sigma \), denote the leaf through \( z \) by \( L_z \). If \( L_z \in \Omega(F, T, n) \), then \( \#(L_z \cap \Sigma) \leq n \). Let \( X := \{ z \in W : \#(L_z \cap \Sigma) \leq n \} \). Since \( \text{med}(\Omega(F, T, n)) > 0 \), then Claim 4.12 ensures that \( \text{med}(X) > 0 \).

Now consider a holonomy map germ \( h \in H := \text{Hol}(F, L_0, \Sigma, p) \) and choose a sufficiently small subdisc \( W \subset \Sigma \) such that the representative \( h : W \to \Sigma \) (of the germ \( h \)) and its iterates \( h, h^2, \ldots, h^{n+1} \) are defined in \( W \). Since \( X \) has positive measure, then \( \text{med}(\Omega(H, \Sigma, n)) > 0 \). The result then follows by Lemma 4.4.

In view of Claim 4.13 and Reeb local stability Lemma 2.1, the proof of Theorem A is finished.

5. Holomorphic foliations with singularities

Recall that a singular holomorphic foliation in a complex manifold \( M \) of dimension \( m \geq 2 \) is a pair \( \mathcal{F} = (\mathcal{F}', \text{Sing}(\mathcal{F})) \), where \( \text{Sing}(\mathcal{F}) \subset M \) is an analytic subset of \( M \) of dimension less or equal to \( (\mathcal{F}) - 1 \), and \( \mathcal{F}' \) is a holomorphic foliation in the usual sense (without singularities) in the open set \( M' = M \setminus \text{Sing}(\mathcal{F}) \subset M \). The leaves of \( \mathcal{F} \) are defined as the leaves of the foliation \( \mathcal{F}' \). The set \( \text{Sing}(\mathcal{F}) \) is called the singular set of \( \mathcal{F} \). The dimension of \( \mathcal{F} \) is defined as the dimension of \( \mathcal{F}' \). In the one-dimensional case there is an open cover \( \{ U_j \}_{j \in J} \) of \( M \) such that in each \( U_j \) the foliation \( \mathcal{F} \) is defined by a holomorphic vector field \( X_j \) satisfying the following property: if \( U_i \cap U_j \neq \emptyset \), then \( X_i \mid_{U_i \cap U_j} = g_{ij} \cdot X_j \mid_{U_i \cap U_j} \) for some non-vanishing holomorphic function \( g_{ij} \). The leaves of the restriction \( \mathcal{F} \mid_{U_j} \) are the nonsingular orbits of \( X_j \) in \( U_j \) while \( \text{Sing}(\mathcal{F}) \cap U_j = \text{Sing}(X_j) \).

An isolated singularity of a one-dimensional foliation \( \mathcal{F} \) in a manifold \( M \) is called non-degenerate if there is some open neighborhood \( p \in U \subset M \) where the foliation is induced by a holomorphic vector field \( X \) with non-singular linear part \( DX(p) \) at \( p \in \text{Sing}(\mathcal{F}) \).

Let \( p \in M \) be an isolated singularity of a one-dimensional singular foliation \( \mathcal{F} \) in \( M \). Given a neighborhood \( p \in U \subset M \) where \( \mathcal{F} \) has no other singularity than \( p \), we denote by \( \mathcal{F}(U) := \mathcal{F} \mid_U \) the restriction of \( \mathcal{F} \) to \( U \). A leaf of \( \mathcal{F}(U) \) accumulating only at \( p \) is closed off \( p \), thus by Remmert-Stein extension theorem \([22]\) it is contained in an irreducible analytic curve through \( p \). Such a curve is called a local separatrix of \( \mathcal{F} \) through \( p \). In the two dimensional case, the classical definition says that \( p \in \text{Sing}(\mathcal{F}) \) is a dicritical singularity if for some neighborhood \( p \in U \subset M \) the restriction \( \mathcal{F}(U) \) has infinitely many separatrices through \( p \). We extend this terminology in a natural way to higher dimensions: for \( m = (\mathcal{F}) \geq 2 \), the singularity shall be called dicritical if it admits infinitely many separatrices. This singularity is called absolutely dicritical if all the leaves close enough to it are contained in local separatrices. For our purposes, the important factor is whether the set of separatrices has positive measure or not. In dimension \( m \geq 2 \), we shall say that \( p \) is a \( \mu \)-dicritical
singularity if for arbitrarily small open neighborhoods $U \ni p$ the set $\text{Sep}(\mathcal{F}, U)$ of local separatrices of $\mathcal{F}(U)$ through $p$ has positive measure in $U$.

In dimension two, an isolated singularity is dicritical if and only if it is $\mu$-dicritical. This is a straightforward consequence of the theorem of resolution of singularities by blow-ups ([13, 34]). Nevertheless, in dimension three a linear vector field with eigenvalues $1, 1$ and $-1$ is not $\mu$-dicritical, but exhibits infinitely many separatrices (contained in the plane spanned by the positive eigenvalues).

By Newton-Puiseaux parametrization theorem, the topology of a separatrix is the one of a disc. Further, the separatrix minus the singularity is biholomorphic to a punctured disc. In particular, given a separatrix $S_p$ through a singularity $p \in \text{Sing}(\mathcal{F})$, we may choose a loop $\gamma \in S_p \setminus \{p\}$ generating the (local) fundamental group $\pi_1(S_p \setminus \{p\})$. The corresponding holonomy map $h_\gamma$ is defined in terms of a germ of complex diffeomorphism at the origin of a local disc $\Sigma$ transverse to $\mathcal{F}$ and centered at a non-singular point $q \in S_p \setminus \{p\}$. This map is well-defined up to conjugacy by germs of holomorphic diffeomorphisms, and is generically referred to as local holonomy of the separatrix $S_p$ with respect to the singularity $p$. Let us denote by $\Omega(\mathcal{F}, \Sigma, k)$ the union of the leaves $L$ of $\mathcal{F}$ such that $L$ meets the transverse disc $\Sigma$ at most in $k$ points, i.e., $\#(L \cap \Sigma) \leq k$.

**Lemma 5.1.** Let $\mathcal{F}$ be a one-dimensional holomorphic foliation with an isolated singularity at $p \in M$. Let $S$ be a local separatrix of $\mathcal{F}$ through $p$ and $\Sigma$ a local disc transversal to $\mathcal{F}$ centered at a non-singular point $q \in S_p \setminus \{p\}$ (close enough to $p$) such that $S \subset C_\mu(\Omega(\mathcal{F}, \Sigma, k))$ for some $k \in \mathbb{N}$. Then the local holonomy of $S$ with respect to $p$ is a periodic map.

**Proof.** Let $H := \text{Hol}(\mathcal{F}, L, \Sigma, q)$. Since $S \subset C_\mu(\Omega(\mathcal{F}, \Sigma, k))$, then $\text{med}(O_H(\Sigma_q, k)) > 0$. The result then follows by Lemma 4.7. □

### 6. Siegel Type and Poincaré Type Singularities

Let $\mathcal{F}$ be a one-dimensional holomorphic foliation with singularities in a manifold $M$. Assume $p \in \text{Sing}(\mathcal{F})$ is a nondegenerate singularity, i.e., for some neighborhood $p \in U \subset M$ the restriction $\mathcal{F}|_U$ is given by a holomorphic vector field $X$ with a non-singular linear part at $p$. There are two possibilities: If the convex hull in $\mathbb{R}^2$ of the set of eigenvalues of the linear part $DX(p)$ contains the origin, then we say that the singularity is in the Siegel domain, otherwise it is the Poincaré domain ([2, 15, 7, 40]).

**6.1. Singularities in the Poincaré domain.** A singularity in the Poincaré domain is either analytically linearizable or exhibits resonant eigenvalues and is analytically conjugate to a polynomial form with resonant monomials called Poincaré-Dulac normal form ([2, 15]). For such a non-linear normal form, the separatrices are contained in coordinate hyperplanes and not all hyperplanes are invariant. In particular, the set of separatrices is a zero measure subset.

An immediate consequence of the above discussion and of Poincaré-Dulac normal form theorem ([15, 2]) for singularities in the Poincaré domain is the following result.

**Lemma 6.1.** A $\mu$-dicritical non-degenerate singularity is necessarily in the Poincaré domain and is analytically linearizable: in suitable local coordinates, it is of the form $\dot{x} = Ax$ for some diagonal linear map $A \in \text{GL}(m, \mathbb{Z}_+)$ with positive integer coefficients. The same holds if the set of leaves which are closed off the singular set is a positive measure set.

A singularity as above will be called radial type singularity.
6.2. Singularities in the Siegel domain. We consider the following situation: $\mathcal{F}$ is a one-dimensional foliation defined in a neighborhood of the origin $0 \in \mathbb{C}^m$, which is assumed to be a (non-degenerate) Siegel type singularity. We recall that the singularity is of normal type if we can choose small polydiscs $U \ni 0$ centered at the origin endowed with local coordinates $x = (x_1, \ldots, x_m) \in U$ such that:

(i) The coordinate hyperplanes $H_j = \{x_j = 0\}$ are invariant by the foliation.

(ii) The coordinate axes $O_{x_j}$ contain separatrices whose holonomy maps, denoted by $h_j$, are related to the analytic classification of the germ of the foliation at the origin (see section 5.3).

Coordinates as above will be called adapted to the singularity. The existence of such adapted coordinates (in dimension $m \geq 3$) for any Siegel type singularity is discussed in [11]. Fix a holomorphic vector field $X$ in $U$ with an isolated singularity at $p$ defining the restriction $\mathcal{F}(U) := \mathcal{F}|_U$. Let $\lambda_j \in \mathbb{C}$ be the eigenvalue of $DX(0)$ corresponding to the eigenvector tangent to the $O_{x_j}$ axis. For a suitable choice of $X$ and of the local adapted coordinates we can write

\[
X = \sum_{j=1}^{m} \lambda_j x_j \frac{\partial}{\partial x_j} + (x_1 \cdots x_m) \cdot X_2(x_1, \ldots, x_m)
\]

where $\lambda_j \in \mathbb{C}, \forall j$, and $X_2$ is a vector field defined in a neighborhood of the origin of the coordinate system $(x_1, \ldots, x_m)$.

As we shall see, under suitable conditions we can assure that the eigenvalues are integral numbers (cf. Lemma 4.7). Indeed, we may choose transverse discs $\Sigma_j$ centered at points $q_j : (x_j = a_j) \in O_{x_j}$. Then the holonomy maps $h_j$ have representatives given by local diffeomorphisms $h_j : (\Sigma_j, q_j) \to (\Sigma_j, q_j)$ of the form

\[
h_j(x_1, \ldots, \hat{x}_j, \ldots, x_n) = (x_1a_1^j(x_1, \ldots, \hat{x}_j, \ldots, x_m), \ldots, \hat{x}_j, \ldots, x_m a_m^j(x_1, \ldots, \hat{x}_j, \ldots, x_m)),
\]

where $a_i^j$ is a holomorphic function in a neighborhood of the origin $0 \in \mathbb{C}^{m-1}$, for all $i = 1, \ldots, j, \ldots, m$ (the hat $\hat{x}_j$ stands for omitting that coordinate). The map $h_j$ has linear part given by $Dh_j(0) \cdot (x_1, \ldots, \hat{x}_j, \ldots, x_n) = (\exp(\frac{\lambda_j}{a_j}) \cdot x_1, \ldots, \exp(\frac{\lambda_j}{a_j}) \cdot x_n)$. If the map $h_j$ has set of periodic orbits with positive measure, then it is periodic as we have already seen above (cf. Lemmas 4.7 and 5.1). This implies that the quotients $\frac{\lambda_j}{a_j}$ are rational numbers for every $i \neq j$. Therefore, up to dividing the generating vector field $X$ by a suitable complex number, we may assume that each eigenvalue $\lambda_j$ is a rational number. Such a Siegel singularity will be called resonant. Resonant Siegel type singularities exhibit at most one non-dicritical separatrix. In dimension $m = 3$, a resonant Siegel type singularity exhibits exactly one non-dicritical separatrix, i.e., one corresponding to an eigenvalue of signal different from the signal of other two eigenvalues. That is the key point in the discussion that follows.

6.3. Analytic linearization of Siegel singularities. We end this paragraph dealing with the analytic linearization of Siegel singularities. Consider a germ of a Siegel singularity at the origin $0 \in \mathbb{C}^m$. In the 2-dimensional case (i.e., $m = 2$), the local holonomy of a separatrix gives the full analytic classification of the singularity (cf. e.g. [40]). Nevertheless, this is more delicate in case $m \geq 3$. Indeed, a Siegel type singularity may look like a dicritical singularity when restricted to suitable invariant planes.

In order to better describe generic isolated singularities in the $m$-dimensional case ($m \geq 3$) we need some notation.
Definition 6.2 (Condition (⋆)). Let $X$ be a germ of a holomorphic vector field at the origin such that the origin $0 \in \mathbb{C}^m$ is a singularity in the Siegel domain. We say that $X$ satisfies condition (⋆) if there is a real line $L \subset \mathbb{C}$ through the origin avoiding all the eigenvalues of $X$ such that one of the connected components of $\mathbb{C} \setminus L$ contains just one eigenvalue of $X$.

The above condition holds for $X$ if and only if holds for any vector field $Y$ such that $X$ and $Y$ are tangent. Condition (⋆) implies that $X$ is in the Siegel domain, but it is stronger than this last. Denote by $\lambda(X)$ the isolated eigenvalue of $X$ and by $S_X$ its corresponding invariant manifold (the existence is granted by the classical invariant manifold theorem). We call $S_X$ the distinguished axis or distinguished separatrix of $X$. The singularity will be called holonomy-linearizable if the holonomy map associated to the distinguished separatrix is analytically conjugate to its linear part. The notions of analytically linearizable and holonomy-linearizable are strongly related as we will see in what follows.

In [16] it is proved the following result:

Theorem 6.3 ([16]). Let $X$ and $Y$ be two normal Siegel type germs of holomorphic vector fields with an isolated singularity at the origin $0 \in \mathbb{C}^n$ satisfying condition (⋆). Let $h_X$ and $h_Y$ be the holonomies of $X$ and $Y$ relatively to $S_X$ and $S_Y$, respectively. Then $X$ and $Y$ are analytically equivalent if and only if the holonomies $h_X$ and $h_Y$ are analytically conjugate.

For a fibered version of the above result we refer to [24], p. 1656. Notice that a resonant (i.e., a rational eigenvalues) germ of a holomorphic vector field in the Siegel domain necessarily satisfies condition (⋆), otherwise it would have all eigenvalues with the same signal (positive or negative) and would be in the Poincaré domain. Therefore, an immediate consequence of the above results is the following:

Lemma 6.4. Let $X$ be a resonant normal Siegel type holomorphic vector field germ at the origin $0 \in \mathbb{C}^3$ satisfying condition (⋆). Then the following conditions are equivalent:

(i) The germ of foliation $\mathcal{F}(X)$ induced by $X$ is analytically linearizable.

(ii) $\mathcal{F}(X)$ is holonomy-linearizable, i.e., the holonomy map of $\mathcal{F}(X)$ relatively to the separatrix $S_X$ tangent to the eigenspace associated to the distinguished axis of $X$ is analytically conjugate to its linear part.

In terms of our notion of measure concentration point we have:

Lemma 6.5. Let $\mathcal{F}$ be a one-dimensional holomorphic foliation with non-degenerate singularities in a three-dimensional complex manifold $M^3$. Denote by $\Omega^0(\mathcal{F})$ the set of all leaves which are closed off the singular set $\text{Sing}(\mathcal{F}) \subset M^3$. Given a singularity $p \in M^3$, suppose that $p \in \mathcal{C}_\mu(\Omega^0(\mathcal{F}))$. Then we have two possibilities:

(i) The singularity is in the Poincaré domain and it is analytically linearizable, indeed it is of radial type;

(ii) The singularity is in the Siegel domain and resonant. If $p$ is a normal singularity, then $\mathcal{F}$ is analytically linearizable with rational eigenvalues at this singular point.

Proof. From what we have seen in Lemma 6.1, for any singularity in the Poincaré domain, the leaves not contained in separatrices are not closed off the singular set. Therefore, since $p \in \mathcal{C}_\mu(\Omega^0(\mathcal{F}))$, the singularity is $\mu$-dicritical and the result follows.

Now assume the singularity is in the Siegel domain. We may then assume that the coordinate hyperplanes are invariant as well as the coordinate axes, which are supposed to be tangent to the
eigenvectors of (the linear part at the origin of) the generating vector field. Since $p \in C_\mu(\Omega^0(\mathcal{F}))$, for any arbitrarily small neighborhood $U$ of $p$ in $M$, the set of leaves of $\mathcal{F}|_U$, closed off $p$ has positive measure. Therefore, this set is not contained in the coordinate hyperplanes or any countable set of hypersurfaces. Thus, the holonomy maps $h_j$, $j = 1, \ldots, m$, associated to the coordinate axes, have positive measure sets of periodic orbits. From Lemma 4.7, this implies that the maps $h_j$ are periodic (and therefore, analytically linearizable) and the eigenvalues of the generating vector field can be assumed to be rational numbers. Hence, the singularity is resonant and holonomy-analytically linearizable.

7. Dulac correspondence at a Siegel type corner

The framework in this section is motivated by Lemma 6.5. Let $\mathcal{F}$ be a one-dimensional foliation defined in a neighborhood of the origin $0 \in \mathbb{C}^m$, which is assumed to be a normal Siegel type singularity. We can choose small polydiscs $U \ni 0$ centered at the origin endowed with adapted local coordinates $x = (x_1, \ldots, x_m) \in U$ as in [6.2]. Again we fix disc type transverse sections $\Sigma_j = \{x_j = a_j\} \subset U$ for some sufficiently close to the origin $a_j \in \mathbb{C}$, $j = 1, \ldots, n$ and denote the intersection points by $q_j = \Sigma_j \cap \mathcal{O}_{x_j}$.

Fix a holomorphic (generator) vector field $X$ in $U$ with an isolated singularity at $p$ defining the restriction $\mathcal{F}(U)$. Denote by $\lambda_j \in \mathbb{C}$ the eigenvalue of $DX(0)$ corresponding to the eigenvector tangent to the $\mathcal{O}_{x_j}$ axis and write $X$ as in equation (1). From now on, we assume that $0$ is an holonomy-analytically linearizable singularity with rational eigenvalues.

The Dulac correspondence will be defined as a correspondence $D_{ij}$ from certain subsets of $\Sigma_i$ onto certain subsets of $\Sigma_j$ as follows (cf. [9, 33]).

The general idea is motivated by the following two dimensional picture: Since $\lambda_1 \cdot \lambda_2 < 0$, the topological analytic description of Siegel plane singularities says that any leaf $L_z \in \mathcal{F}$ passing through $z \in \Sigma_i$ must intersect $\Sigma_j$ provided that $z$ is close enough to the origin of $\Sigma_i$. Therefore, we associate to the intersection points $L_z \cap \Sigma_i$, the intersection points $L_z \cap \Sigma_j$. We shall write $D_{ij}(z)$ to denote this subset $L_z \cap \Sigma_j$ just for simplicity. The Dulac correspondence is a multivalued correspondence $D_{ij} : \Sigma_i \rightarrow \Sigma_j$, which is obtained by following the local leaves of $\mathcal{F}|_U$ from $\Sigma_i$ to $\Sigma_j$.

Next we describe the Dulac correspondence on each specific case we need. The main hypothesis being that the singularity is Siegel resonant, normal and holonomy-linearizable.

7.1. Dimension two. Suppose $m = 2$. In this case holonomy-analytically linearizable foliations and analytically linearizable foliations are equivalent notions. Thus, we are actually assuming that the origin is a linearizable singularity in the Siegel domain. We may then choose local holomorphic coordinates $(x, y) \in U$ such that the local separatrices $D_i$ and $D_j$ through the singularity are given by $D_i : (x = 0)$, $D_j : (y = 0)$, and such that $\mathcal{F}|_U$ is given by $\lambda xy - \mu ydx = 0$, $q_0 : x = y = 0$, where $\lambda, \mu \in \mathbb{Q}$ and $\frac{\lambda}{\mu} \in \mathbb{Q}_-$. We fix the local transverse sections as $\Sigma_j = (x = 1)$ and $\Sigma_i = (y = 1)$, such that $\Sigma_i \cap D_i = q_i \neq q_0$ and $\Sigma_j \cap D_j = q_j \neq q_0$. Let us denote by $h_o \in \text{Diff}(\Sigma_i, q_i)$ the local holonomy map of the separatrix $D_i$ corresponding to the corner $q_0$. Then we have $h_o(x) = \exp(2\pi i \frac{\lambda}{\mu} \sqrt{-1}) \cdot x$.

The Dulac correspondence is therefore given by

$$D_{ij} : (\Sigma_i, q_i) \rightarrow (\Sigma_j, q_j), D_{ij}(x_o) = \frac{\mu}{\lambda} x_o.$$

7.2. Dimension three. We may assume that the eigenvalues of the linear part of $X$ are $\lambda_1 \in \mathbb{Q}_-$ and $\lambda_2, \lambda_3 \in \mathbb{Q}_+$. The coordinate plane $H_1 = E(x_2 x_3) = \{x_1 = 0\}$ is invariant by $\mathcal{F}(U)$ and we denote this foliation by $\mathcal{F}(U)_{x_2 x_3}$. Furthermore, this planar foliation is analytically linearizable
and absolutely dicritical, i.e., all leaves are contained in separatrices and accumulate at the origin. In particular, the closure of each leaf $L \in \mathcal{F}(U)$ is a (unique) separatrix $\Gamma = L \cup \{0\} \subset E(x_1x_3)$ through the origin. By the analytic-linearization (or, more generally, by the topological analytic description of Siegel singularities), we know that there is a germ at the origin of an analytic surface $H(x_1, \Gamma)$ that is invariant by the vector field $X$, contains the axis $O_{x_1}$, and the curve $\Gamma$. The surface $H(x_1, \Gamma)$ meets the disc $\Sigma_1$ transversely at a 1-disc $\Sigma_1(\Gamma)$ centered at $q_1$. Also, given any point $q_2 \in \Gamma \setminus \{0\}$ and a transverse 2-disc $\Sigma_2$ centered at $q_2$, the surface $H(x_1, \Gamma)$ meets $\Sigma_2$ transversely at a 1-disc $\Sigma_2(\Gamma)$. Reasoning as in the dimension two case above, we obtain a Dulac correspondence $D_{1,\Gamma}: \Sigma_1(\Gamma) \to \Sigma_2(\Gamma)$ by following the local leaves of the foliation on the invariant variety $H(x_1, \Gamma)$.

A pair of separatrices of a non-degenerate Siegel type singularity is called Siegel pair if the quotient of the eigenvalues corresponding to these separatrices is a negative real number. As a consequence of the above considerations, we obtain the following complement to Lemma \ref{lem:existence}:

**Lemma 7.1.** Let $\mathcal{F}$ be a one-dimensional holomorphic foliation with non-degenerate singularities in a complex manifold $M$. Let $\mathcal{X}(\mathcal{F}) \subset M \setminus \text{Sing}(\mathcal{F})$ be any invariant set of leaves. Let $p \in \text{Sing}(\mathcal{F}) \subset M$ be a normal Siegel type singularity. Given a Siegel type pair of separatrices $S_p^p, S_p^q$ we have $S_p \subset C_\gamma(\mathcal{X}(\mathcal{F}))$ if and only if $S_p^q \subset C_\gamma(\mathcal{X}(\mathcal{F}))$, where $C_\gamma(\mathcal{X}(\mathcal{F}))$ denotes the set of measure concentration points of the set $\mathcal{X}(\mathcal{F})$.

8. Stable graphs

In order to motivate our notion below, we recall the classical real framework. Let $X$ be a smooth vector field in a real surface $N^2$ with isolated singularities. By a graph for $X$, we mean an invariant compact subset $\Gamma \subset N$ consisting of singularities and orbits such that the $\alpha$-limit and the $\omega$-limit of each orbit of $X$ contain some singularity in the graph. This notion admits a natural extension to the complex two-dimensional case as follows:

Let $\mathcal{F}$ be a one-dimensional foliation with non-degenerate singularities in a complex surface $M^2$.

**Definition 8.1** (graph in complex surfaces). A graph of $\mathcal{F}$ is an invariant compact connected analytic subset $\Gamma \subset M$ of pure-dimension one such that:

1. The singularities of $\mathcal{F}$ in $\Gamma$ are all of Siegel type;
2. Each leaf $L \subset \Gamma$ is contained in an analytic curve and accumulates at some singularity $p \in \text{Sing}(\mathcal{F}) \cap \Gamma$;
3. Any local separatrix through $p \in \Gamma \cap \text{Sing}(\mathcal{F})$ is contained in $\Gamma$.

Next we obtain the following stability theorem for graphs:

**Proposition 8.2.** Let $\mathcal{F}$ be a holomorphic foliation of dimension one with non-degenerate singularities in a compact surface $M^2$. Suppose $\mathcal{F}$ has a stable graph $\Gamma \subset M$, then there is a fundamental system of invariant neighborhoods $W$ of $\Gamma$ in $M$ such that each leaf intersecting $W$ is quasi-compact. In such a neighborhood the foliation admits a holomorphic first integral.

**Proof.** The proof is somehow similar to the proof of the main result in \cite{28}, where it is proved the existence of a holomorphic first integral for a germ of a non-dicritical foliation in dimension two, provided that the leaves are closed off the singular point. In \cite{28} it is used an induction argument, since the singularity is not necessarily non-degenerate. This is not the case here. Nevertheless, their construction of invariant neighborhoods and holomorphic first integrals, from the finiteness of the combined holonomy groups of the projective lines in the exceptional divisor of the resolution of singularities, can be repeated here with minor changes. Let us give the main steps. First we
remark that all singularities in $\Gamma$ are linearizable of the local form $nxdy - mydx = 0$, $n, m \in \mathbb{Z}_+$. Therefore, each such a singularity exhibits a local holomorphic first integral of the form $f = x^my^n$. Now, the finiteness of the holonomy groups allow the holonomy extension of these first integrals to a neighborhood of each irreducible component $\Gamma_j \subset \Gamma$ of the graph $\Gamma$. Here the procedure is the same as in the case of a single blow-up in $[28]$. This gives a holomorphic first integral $f_j$ defined in a neighborhood $W_j$ of $\Gamma_j$ in $M$, in such a way that the union $W = \bigcup_j W_j$ is an invariant neighborhood of $\Gamma$. Moreover, thanks to the finiteness of the virtual holonomy groups, i.e., of the combined holonomy groups of the components of $\Gamma$, the first integrals $f_j$ are such that each corner $q \in \Gamma_i \cap \Gamma_j \neq \emptyset$ is a singularity and in some neighborhood of this singularity we have $f_i^{n_{ij}} = f_j^{m_{ij}}$ for $n_{ij}, m_{ij} \in \mathbb{N}$. This shows the existence of a holomorphic first integral $f$ in $W$, which is defined on each $W_j$ by an expression like $f|_{W_j} = f_j^{\nu_j}$ for a suitable $\nu_j \in \mathbb{N}$.

Proof of Theorem B. Let $F$ be as in Theorem B and denote by $\Omega(F) \subset M$ the union of quasi-compact leaves of $F$. Since $\Omega(F)$ has positive measure, then arguing as in Lemma 4.5, we can assure the existence of a leaf $L_0 \subset C_\nu(\Omega(F))$. On the other hand, we conclude from Lemma 4.7 the finiteness of the holonomy group of $L_0$. Analogously, the virtual holonomy group of $L_0$ is finite. Finally, applying induction and the existence of first integrals nearby each separatrix meeting $L_0$ ($[28]$, we construct a stable graph $\Gamma$ for $F$. □

Example 8.3. Let $X$ be the polynomial vector field $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \lambda z\frac{\partial}{\partial z}$ on $\mathbb{C}^3$, where $\lambda \in \mathbb{R} \setminus \mathbb{Q}$. Then the hyperplane $H \subset \mathbb{CP}^3$ given by $\Gamma \cap \mathbb{C}^3 = \{z = 0\}$ is invariant by the foliation $F$ induced by $X$ in $\mathbb{CP}^3$. Moreover: (1) the restriction $F|_H$ is biholomorphically equivalent to the “radial foliation” induced in $\mathbb{CP}^2$ by the vector field $\tilde{F}(x, y) = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$. (2) On the other hand, since $\lambda$ is irrational, the holonomy groups of leaves in $H$ are not finite. Thus, though algebraic, the leaves in $H$ do not fit into the above framework.

Now we extend the above notions to dimension $m \geq 2$. For this sake, we shall introduce some notation: Let $F$ be a one-dimensional foliation with a finite number of singularities in a complex manifold $M^m$ of dimension $m \geq 2$. Given a leaf $L$ of $F$, we say that a leaf $L_1$ is in Siegel pairing with $L$ if there is a singularity $p \in L \cap L_1$ such that $L$ and $L_1$ induce distinct separatrices $S(L_p)$ and $S(L_1)_p$ at $p$, which are of Siegel pair type. Then we denote by $\Gamma_1(L)$ the union of $L$ and all such leaves $L_1$ which are in Siegel pairing with $L$. In the same way, we consider all leaves which are in Siegel pairing with the leaves in $\Gamma_1(L)$ and its union will be denoted by $\Gamma_2(L)$. Since the singular set of $F$ is finite, we may proceed this way and by induction we obtain a subset $\Gamma(L) \subset M$, which shall be called the Siegel component of $L$.

Definition 8.4 (graph in complex manifolds). Suppose all the singularities of $F$ are non-degenerate. A graph of $F$ is an invariant compact connected analytic subset $\Gamma \subset M$ of pure-dimension one such that $\Gamma = \Gamma(L)$, the Siegel component of some leaf $L$ of $F$. The graph $\Gamma \subset M$ of $F$ is called stable if each virtual holonomy group of $\Gamma$ is finite in the sense of Definition 1.2.

In particular, all the Siegel type singularities of $F$ in $\Gamma$ are resonant and holonomy-linearizable.

9. One-dimensional foliations in complex projective spaces

Since the Cousin multiplicative problem always admits a solution in $\mathbb{C}^{m+1} \setminus \{0\}$, $m \geq 2$, a one-dimensional holomorphic foliation with singularities $F$ in $\mathbb{CP}^m$ is always defined in any affine space $\mathbb{C}^m \subset \mathbb{CP}^m$ by a polynomial vector field with isolated singularities. From now on, by foliation we shall mean a one-dimensional holomorphic foliation with singularities. In this paper, by an
algebraic leaf we mean a leaf $L \subset \mathbb{CP}^m$ of the (non-singular) foliation, such that the closure $\overline{L} \subset \mathbb{CP}^m$ is algebraic of dimension one. Equivalently, $L$ is contained in an algebraic curve. In this case, $\Lambda(L) := \overline{L}$ is an algebraic invariant curve and we have $\Lambda(L) \setminus L = \overline{L} \cap \text{Sing}(F) \subset \text{Sing}(F)$. Given a foliation $\mathcal{F}$ in $\mathbb{CP}^m$, we denote by $\Omega(\mathcal{F})$ the collection of all algebraic leaves of $\mathcal{F}$. This is therefore a collection of leaves of $\mathcal{F}$, not a subset of $\mathbb{CP}^m$. Given a leaf $L \subset \text{Sep}(\mathcal{F})$, we denote by $\Delta = \Lambda(L)$ the algebraic curve $\overline{L} \subset \overline{L} \cup \text{Sing}(\mathcal{F}) \subset \mathbb{CP}^m$, i.e., the corresponding algebraic curve containing the leaf $L$. Then we shall denote by $\Omega(\mathcal{F}) \subset \mathbb{CP}^m$ the union of all such algebraic curves $\Lambda(L)$ with $L \in \text{Sep}(\mathcal{F})$.

Recall that a Leaf $L$ of $\mathcal{F}$ is quasi-compact if it is closed off the singular set of the $\mathcal{F}$, i.e., $\overline{L} \setminus L \subset \text{Sing}(\mathcal{F})$. The following simple criterion will be useful for our intents:

**Lemma 9.1.** A leaf $L$ of a one-dimensional holomorphic foliation $\mathcal{F}$ in $\mathbb{CP}^m$ is algebraic if and only if it is quasi-compact.

**Proof.** According to a theorem of Chow ([22]), complex analytic subsets of projective spaces are indeed algebraic subsets. Therefore, a leaf $L \subset \mathbb{CP}^m$ is algebraic if and only if its closure $\overline{L} \subset \mathbb{CP}^m$ is an analytic subset of dimension one. In addition, by the classical extension theorem of Remmert and Stein ([22]), the closure $\overline{L} \subset \mathbb{CP}^m$ is analytic of dimension one if and only if $\overline{L} \setminus L$ is contained in a analytic subset of dimension zero. Therefore, a leaf $L$ of $\mathcal{F}$ is algebraic if and only if $\overline{L} \setminus L \subset \text{Sing}(\mathcal{F})$. □

### 9.1. Algebraic leaves and finite order.

**Definition 9.2 ([32]).** Let $\mathcal{F}$ be a foliation of codimension $k$ in a manifold $M$ (perhaps non-compact). A compact total transverse section of $\mathcal{F}$ is a compact $k$-manifold $T \subset M$ (possibly with boundary) such that every leaf of $\mathcal{F}$ intersects the interior of $T$.

**Lemma 9.3 ([22]).** Let $\mathcal{F}$ be a holomorphic foliation of dimension 1 in $\mathbb{CP}^m$ with (finite) singular set $\text{Sing}(\mathcal{F}) \subset \mathbb{CP}^m$. There exists a finite collection of immersed closed discs $D_j \subset \mathbb{CP}^m$, $j = 1, \ldots, r$ pairwise disjoint such that: (i) $D_j \approx \{ z \in \mathbb{C}^{n-1} : |z| \leq 1 \}$ and $D_j = D_j \setminus \partial D_j$ is transverse to $\mathcal{F}$. (ii) Each leaf of $\mathcal{F}$ intersects at least one of the open discs $D_j$. In other words, the manifold with boundary $T = D_1 \cup \cdots \cup D_r$ is a compact total transverse section to $\mathcal{F}$.

**Proof.** Denote by $\{ p_1, \ldots, p_r \}$ the singular set of $\mathcal{F}$. Choose small neighborhoods $U_j \ni p_j$ diffeomorphic to polydiscs $\{ z \in \mathbb{C}^n : |z| \leq 1 \}$ centered at $p_j$. Then $X := \mathbb{CP}^m \setminus \bigcup_{j=1}^r U_j$ is compact. Therefore, there exists an open cover $\bigcup_{\alpha \in A} U_{\alpha} = \mathbb{CP}^m \setminus \text{Sing}(\mathcal{F})$ by distinguished neighborhoods $U_{\alpha}$, such that in each $U_{\alpha}$ we have an embedded disc $\Sigma_{\alpha} \approx \mathbb{D}$ transverse to $\mathcal{F}|_{U_{\alpha}}$ satisfying the following property: if $p \in U_{\alpha}$, then $L_p \cap \Sigma_{\alpha} \neq \emptyset$. Since $X$ is compact, there exists a finite subcovering $X \subset U_{\alpha_1} \cup \cdots \cup U_{\alpha_t}$. In particular, for any $p \in X$, the leaf $L_{p}$ intersects some $\Sigma_{\alpha_j}, j \in \{1, \ldots, \ell\}$. On the other hand, a leaf cannot remain in a polydisc $U_j$ for it cannot be bounded in the affine space. Therefore, any leaf intersects $X$ and thus the interior of some $\Sigma_j$. □

Therefore, Lemmas 2.2 and 9.3 motivate the following:

**Definition 9.4** (relative order of a leaf). Let $\mathcal{F}$ be a one-dimensional foliation in $\mathbb{CP}^m$ with finite singular set $\text{Sing}(\mathcal{F}) \subset \mathbb{CP}^m$. We say that a leaf $L \in \mathcal{F}$ has finite order if for some compact total transverse section $T \subset \mathbb{CP}^m$ to $\mathcal{F}$ the intersection $L \cap T$ is a finite set. Fixed such section $T$, the number of intersection points between $L$ and $T$ is called the relative order of $L$ with respect to $T$ and denoted by $\text{ord}(L,T)$, i.e., $\text{ord}(L,T) = \#(L \cap T)$. 

According to Lemma 9.1, a leaf $L$ of $\mathcal{F}$. Nevertheless, for our purposes it is enough to observe the following: from Lemma 2.2, given a leaf $L$ of $\mathcal{F}$, a point $q \in L$ and a transverse disc $\Sigma$ to $\mathcal{F}$ centered at a point $q \in L$ we have (see the proof of Claim 4.10)

$$\#(L \cap \Sigma) \leq \text{ord}(L, \mathcal{F}).$$

**Lemma 9.6.** Given a holomorphic foliation $\mathcal{F}$ of dimension $k$ in $\mathbb{C}P^m$ and a compact total transverse section $T \subset \mathbb{C}P^m$ to $\mathcal{F}$, a leaf $L \in \mathcal{F}$ is algebraic if and only if it has finite (relative) order (i.e., $\text{ord}(L, T) < \infty$). More precisely, we have $\Omega(\mathcal{F}) = \bigcup_{n \in \mathbb{N}} \Omega(\mathcal{F}, T, n)$, where

$$\Omega(\mathcal{F}, T, n) := \{L \in \mathcal{F} : \#(L \cap T) \leq n\}.$$

**Proof.** According to Lemma 9.1, a leaf $L \subset \mathbb{C}P^m$ is algebraic if and only if $\overline{L} \setminus L \subset \text{Sing}(\mathcal{F})$. If $L$ is algebraic, then $\overline{L}$ is a compact algebraic curve in $\mathbb{C}P^m$. Since $\overline{L} \cap T = L \cap T$ and $\overline{L}$ is compact, then $\#(L \cap T) < \infty$. Therefore, every algebraic leaf has finite relative order. Conversely, let $L \in \mathcal{F}$ be a leaf with finite relative order with respect to $T$. In particular, by the same arguments used in the proof of Claim 4.10, the leaf $L$ is closed off the singular set. Therefore, by the initial remark, $L$ is an algebraic leaf.

Another simple but very useful remark is the following:

**Lemma 9.7.** Let a leaf $L \in \mathcal{F}$ be such that $L \subset \mathcal{C}_\mu(\Omega(\mathcal{F}, T, k))$ for some $k \in \mathbb{N}$ and some total transverse section $T \subset \mathbb{C}P^m$. Then $L$ is algebraic.

**Proof.** Suppose by contradiction that $L \subset \mathcal{C}_\mu(\Omega(\mathcal{F}, T, k))$ is not algebraic. Then Lemma 9.1 says that $L$ is not closed off $\text{Sing}(\mathcal{F})$. Therefore, there is a non-singular accumulation point $q_\infty \in \overline{L} \setminus L$, $q_\infty \notin \text{Sing}(\mathcal{F})$. Given an arbitrarily small transverse disc $\Sigma_{q_\infty}$ to $\mathcal{F}$ centered at $q_\infty$, we have $\#(L \cap \Sigma_{q_\infty}) = \infty$. Then there is a disc $T_j \subset T$ such that $L_{q_\infty}$ meets $T_j$ at an interior point, say $q \in T_j$. Choose now a point $p \in L$ and a transverse disc $\Sigma_p$ centered at $p$. By the Transverse uniformity lemma (Lemma 2.2), there is a map from the disc $\Sigma_p$ to $\Sigma_{q_\infty}$ and thus to the disc $T_j$. We conclude that for any $w \in \Sigma_p$ close enough to $p$, we have $\#(L_w \cap T_j) \geq k!$. In particular, $L_w$ has order greater than $k$. This shows that $\text{med}(\Sigma_p \cap \Omega(\mathcal{F}, T, k)) = 0$, yielding a contradiction to $L \subset \mathcal{C}_\mu(\Omega(\mathcal{F}, T, k))$.

We propose the following conjecture:

**Conjecture 9.8.** Let $\mathcal{F}$ be a holomorphic foliation of dimension one with non-degenerate singularities in the complex projective space $\mathbb{C}P^3$ admitting a stable graph $\Gamma(L)$, where $L \in \mathcal{F}$ is a stable algebraic leaf of $\mathcal{F}$. Then all leaves are algebraic.

### 10. Proof of the Algebraic Stability theorem

In what follows, we consider the following situation: $\mathcal{F}$ is a holomorphic foliation of dimension one with non-degenerate singularities in the complex projective space $\mathbb{C}P^m$ such that the set $\Omega(\mathcal{F})$, union of all algebraic leaves of $\mathcal{F}$, has positive measure. From Lemmas 1.3 and 1.4, the set $\mathcal{C}_\mu(\Omega(\mathcal{F}))$ is invariant and contains some leaf, say $L_0 \subset \mathcal{C}_\mu(\Omega(\mathcal{F}))$. However, this is not enough for our purposes, since we cannot control the degree (or the order) of the algebraic leaves accumulating at $L_0$. Actually, we may say more: from Lemma 9.3, we may choose a compact total transverse section $T \subset \mathbb{C}P^m$ to $\mathcal{F}$. Using then Lemmas 4.4 and 9.6 we immediately conclude that:
Lemma 10.1. Under the hypothesis of Theorem C, there is a leaf \( L_0 \in \mathcal{F} \) such that \( L_0 \subset C_\mu(\Omega(\mathcal{F}, T, k)) \neq \emptyset \) for some \( k \in \mathbb{N} \) and some total transverse section \( T \subset \mathbb{CP}^m \).

From now on, we assume the dimension is \( m = 3 \). In this case, a Siegel singularity \( p \in \text{Sing}(\mathcal{F}) \) exhibits a distinguished separatrix, also called non-dicritical separatrix, and denote by \( S_p \). Using then Lemmas 6.5 and 7.1 we obtain:

Lemma 10.2. Assume the dimension is \( m = 3 \). Let \( p \in C_\mu(\Omega(\mathcal{F}, T, k)) \cap \text{Sing}(\mathcal{F}) \) be a Siegel type singularity, which is a measure concentration point of \( \Omega(\mathcal{F}, T, k) \). Then the singularity of \( \mathcal{F} \) at \( p \), say \( \mathcal{F}_p \), is resonant. Moreover, the non-dicritical (i.e., the distinguished) separatrix \( S_p \) of \( \mathcal{F} \) through \( p \) is also contained in the set of measure concentration points of \( \Omega(\mathcal{F}, T, k) \) and has finite holonomy map with respect to the singularity \( \mathcal{F}_p \). In particular, if \( \mathcal{F}_p \) is normal, then it is a resonant analytically linearizable singularity admitting holomorphic first integrals.

Proof. Since each algebraic leaf is closed off the singular set, we have \( \Omega(\mathcal{F}, T, k) \subset \Omega(\mathcal{F}) \subset \Omega^0(\mathcal{F}) \) in the notation of Lemma 6.5. The result follows from Lemmas 6.5 and 7.1 as well as the considerations in §§ 6.1 and 6.2. In fact, since it is (non-degenerate and) non-\( \mu \)-dicritical, \( p \) is a singularity necessarily in the Siegel domain. Then, for instance by the topological description of Siegel type singularities ([7], [4]) we conclude that any positive measure invariant set \( X \) that accumulates at the singularity \( p \), also accumulates at any non-dicritical separatrix of the foliation through \( p \); indeed, there are two possibilities; either \( X \) is contained in some proper invariant analytic variety \( \Gamma(X) \), that contains the singularity, or \( X \) accumulates at some separatrix of \( \mathcal{F} \) through \( p \). In the first case \( X \) has zero measure and, a fortiori, we have \( p \notin C_\mu(X) \). These considerations prove some separatrix \( S_p \) of \( \mathcal{F} \) through \( p \) is also contained in \( C_\mu(\Omega(\mathcal{F}, T, k)) \). Lemma 6.1 then assures the finiteness of the holonomy maps of the separatrices. This already says that \( p \) is a resonant holonomy-linearizable singularity.

The next step is the linearization of all (normal) singularities, as follows:

Lemma 10.3. Let \( p \in C_\mu(\Omega(\mathcal{F}, T, k)) \cap \text{Sing}(\mathcal{F}) \) be a non-degenerate singularity which is a measure concentration point of \( \Omega(\mathcal{F}, T, k) \). Then \( p \in \text{Sing}(\mathcal{F}) \) is analytically linearizable of radial type or it is a holonomy-linearizable singularity of resonant Siegel type, which is analytically linearizable in case it is normal. Given a leaf \( L_0 \subset C_\mu(\Omega(\mathcal{F}, T, k)) \) all the singularities \( q \in \text{Sing}(\mathcal{F}) \cap \Gamma(L_0) \) are analytically linearizable with rational eigenvalues.

Proof. If the singularity \( q \in \text{Sing}(\mathcal{F}) \cap \Gamma(L_0) \) is in the Poincaré domain, then the analytic linearization follows from Lemma 6.5. On the other hand, if \( q \) is a resonant normal singularity in the Siegel domain, then we apply Lemma 10.2 in order to ensure the linearization. Finally, the rationality of the eigenvalues comes from \( L_0 \subset C_\mu(\Omega(\mathcal{F}, T, k)) \).

Proposition 10.4. Let \( \mathcal{F} \) be as in Theorem C, then there exists a finite sequence of quadratic blow-ups \( \pi : \tilde{M} \to \mathbb{CP}^m \) at the \( \mu \)-dicritical singularities in \( C_\mu(\Omega(\mathcal{F})) \) such that \( \tilde{M} \) is a compact complex manifold and the induced pull-back foliation \( \tilde{\mathcal{F}} = \pi^*(\mathcal{F}) \) is a foliation with the following properties:

(i) The singularities of \( \tilde{\mathcal{F}} \) are of non-degenerate type;
(ii) The set \( \Omega(\tilde{\mathcal{F}}) \subset \tilde{M} \), union of algebraic leaves of \( \tilde{\mathcal{F}} \), has positive measure;
(iii) The singularities of \( \tilde{\mathcal{F}} \) are non-\( \mu \)-dicritical singularities or \( \mu \)-dicritical singularities which are not measure concentration points of the set \( \Omega(\tilde{\mathcal{F}}) \subset \tilde{M} \) of algebraic leaves of \( \tilde{\mathcal{F}} \).
Proof of Proposition 10.4. Let $\mathcal{F}$ be as in the statement. By Lemma 6.1 a singularity $p \in C_\mu(\Omega(\mathcal{F}))$ is either non-$\mu$-dicritical or of radial type. For a singularity of radial type $p_1 \in \text{Sing}(\mathcal{F})$, a single quadratic (punctual) blow-up at this singular point, say $\pi_1: M_1 \to \mathbb{CP}^m$, produces a pull-back foliation $\mathcal{F}_1 = \pi_1^{-1}(\mathcal{F})$ with the following properties:

1. $\mathcal{F}_1|_{M_1\setminus\pi_1^{-1}(p_1)}$ is equivalent to $\mathcal{F}|_{\mathbb{CP}^m\setminus\{p_1\}}$;
2. The exceptional divisor $\pi_1^{-1}(p_1) \cong \mathbb{CP}^{m-1}$ contains no singularity of $\mathcal{F}_1$ and is everywhere transverse to $\mathcal{F}_1$;
3. The set of algebraic leaves of $\mathcal{F}_1$ is birationally equivalent, by the map $\pi: M_1 \to \mathbb{CP}^m$, to the set of algebraic leaves of $\mathcal{F}$. In particular, $\Omega(\mathcal{F}_1)$ has positive measure if and only if $\Omega(\mathcal{F})$ has positive measure.

By the proper mapping theorem [22], a leaf $L$ of $\mathcal{F}$ is algebraic if and only if the inverse image $\pi_1^{-1}(L) \subset M_1$ is a finite union of algebraic leaves of $\mathcal{F}_1$ and (possibly) some singular points. In particular, we have $\text{med}(\Omega(\mathcal{F}_1)) = 0$ if and only if $\text{med}(\Omega(\mathcal{F})) = 0$. Thus, a finite iteration of this process at the $\mu$-dicritical and measure concentration singular points of $\mathcal{F}$ gives the desired result.

\[ \square \]

Notation and convention 10.5. Consider $\mathcal{F}$ and $\tilde{\mathcal{F}}$ as in the above proposition. For the sake of simplicity, from now on we shall write $\mathbb{CP}^m$ for $\tilde{\mathbb{M}}$ and $\mathcal{F}$ as for $\tilde{\mathcal{F}}$. This corresponds to say that, after the blowing-up procedure, the foliation $\mathcal{F}$ has no $\mu$-dicritical singularity in $C_\mu(\Omega(\mathcal{F})) \subset \mathbb{CP}^m$.

Given an algebraic leaf $L \in \Omega(\mathcal{F})$ of $\mathcal{F}$, denote by $\Lambda(L) = L \cup (L \cap \text{Sing}(\mathcal{F}))$ the irreducible algebraic curve containing $L$. In particular, $\Lambda(L)$ is the union of $L$ and all the local separatrices tangent to $L$. If we denote by $\Omega(\mathcal{F}) \subset \mathbb{CP}^m$ the closure of $\Omega(\mathcal{F})$ in $\mathbb{CP}^m$, then

\[ \Omega(\mathcal{F}) = \bigcup_{L \in \Omega(\mathcal{F})} \Lambda(L) = \Omega(\mathcal{F}) \cup (\text{Sing}(\mathcal{F}) \cap \Omega(\mathcal{F})) \subset \Omega(\mathcal{F}) \cup \text{Sing}(\mathcal{F}). \]

The set $\Omega(\mathcal{F})$ is the union of all algebraic invariant curves of $\mathcal{F}$.

Proposition 10.6. Let $L \subset C_\mu(\Omega(\mathcal{F}, T, k))$, then $\Gamma(L)$ is a stable graph of $\mathcal{F}$.

Summing up, we have the following result.

Proposition 10.7. Let $\mathcal{F}$ be a foliation with non-degenerate singularities in $\mathbb{CP}^m$. Then:

1. Each leaf $L \subset C_\mu(\Omega(\mathcal{F}, T, k))$ is algebraic;
2. Each leaf $L \in \Omega(\mathcal{F})$ accumulates at some singularity of $\mathcal{F}$;
3. Let $L_0 \subset C_\mu(\Omega(\mathcal{F}, T, k))$ and $p \in \text{Sing}(\mathcal{F}) \cap \overline{L_0}$ be a non-$\mu$-dicritical singularity and denote by $S(L_0)_p$ the local separatrix induced by $L_0$ at $p$. Then $p$ is a Siegel type singularity and $S(L_0)_p \subset C_\mu(\Omega(\mathcal{F}, T, k))$;
4. Let $L \subset C_\mu(\Omega(\mathcal{F}, T, k))$. Given a singularity $p \in \text{Sing}(\mathcal{F}) \cap \overline{L} = \text{Sing}(\mathcal{F}) \cap \Lambda(L)$, for each separatrix $S_p$ forming a Siegel pair with a local branch $S_p$ of $\Lambda(L)$ at $p$, we also have $S_p \subset C_\mu(\Omega(\mathcal{F}, T, k))$;
5. Given $L \subset C_\mu(\Omega(\mathcal{F}, T, k))$, the corresponding Siegel component also satisfies $\Gamma(L) \subset C_\mu(\Omega(\mathcal{F}, T, k))$.

Proof. Item (1) is proved in Lemma 3.7. For item (2), notice that the closure of $L \in \Omega(\mathcal{F})$ must be algebraic, thus the result follows immediately. Item (3) is a straightforward consequence of Lemma 10.3. Item (4) comes directly from Lemma 7.1, yielding immediately item (5). \[ \square \]
Let be given a leaf \( L_0 \subset C_\mu(\Omega(F, T, k)) \). By Proposition 10.7 above, \( \Gamma(L_0) \) is an invariant set consisting of a union of algebraic curves containing \( \Lambda(L_0) \) and such that \( \Gamma(L_0) \subset C_\mu(\Omega(F, T, k)) \).

**Lemma 10.8.** Given a leaf \( L_0 \subset C_\mu(\Omega(F, T, k)) \), all the virtual holonomy groups of \( \Gamma(L_0) \) are finite and the normal singularities \( q \in \text{Sing}(F) \cap \Gamma(L_0) \) are all analytically linearizable with rational eigenvalues. In case the Siegel component \( \Gamma(L_0) \) of \( L_0 \) has just normal singularities, it is a stable graph admitting a holomorphic first integral nearby it.

**Proof.** By hypothesis, all the singularities in \( \Gamma(L_0) \) are in the Siegel domain, thus \( \Gamma(L_0) \) is a connected invariant subset contained in \( \Omega(F) \). Applying Lemma 10.3 we conclude that each singularity in \( \Gamma(L_0) \) is holonomy-analytically linearizable. Furthermore, all the normal Siegel singularities are indeed linearizable. In case \( \Gamma(L_0) \) has just normal singularities, then reasoning as in the proof of Proposition 10.2 we can construct a holomorphic first integral in a neighborhood of \( \Gamma(L_0) \).

End of the proof of Algebraic Stability theorem. Theorem C is proved as follows. By hypothesis \( \text{med}(\Omega(F)) > 0 \), thus Lemma 10.1 ensures the existence of a leaf \( L_0 \subset C_\mu(\Omega(F, T, k)) \neq \emptyset \). By Proposition 10.7(1) and Lemma 10.8, \( L_0 \) is algebraic and stable. Finally, in the three dimensional case, if all the Siegel singularities are normal, then Lemma 10.8 assures that the Siegel component \( \Gamma(L_0) \) of \( L_0 \) is a stable graph admitting a holomorphic first integral.

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