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§1. Solitons and the Schrödinger operator

The modern theory of exactly solvable one-dimensional and two-dimensional Schrödinger operators

\[ L = -\partial_x^2 + u(x) \quad (n = 1), \]

\[ L = (\partial + B(x,y))(\partial + A(x,y)) + 2V(x,y), \quad \partial = \partial_x - i\partial_y, \quad z = x + iy \quad (n = 2) \]

cannot be separated from the theory of spatial one- and two-dimensional integrable nonlinear systems of soliton theory. This is especially so in the rapidly decaying and periodic case (when the coefficients of the operators are rapidly decaying and periodic functions of the space variables).

I. One-dimensional case \((n = 1)\). In the one-dimensional case, it follows from the papers [GGKM, L], where the famous KdV equation

\[ u_t = 6uu_x - u_{xxx} \]

was solved in the rapidly decaying case that it can be viewed as a kind of “symmetry” of the spectral theory of the class of operators \( L = -\partial_x^2 + u(x) \) acting in the Hilbert space \( L^2(\mathbb{R}) \). This means that the KdV equation can be written in the form

\[ L_t = AL - LA = [A, L], \]

where \( A \) is the linear differential operator \( A = -4\partial_x^3 + 3(u\partial_x + \partial_x u) \). Therefore, for example, the eigenvalues \( L\psi = \lambda_i\psi \) of the operator \( L \) on any translation-invariant class of functions of \( x \) on the line \( \mathbb{R} \) turn out to be integrals of the KdV equation: \( d\lambda_i/dt = 0 \). Using the previously solved inverse scattering problem, KdV was solved in [GGKM] for rapidly decaying functions. Beginning with S. P. Novikov’s
paper [N74], in the articles [DN74, D75, IM, L75, MV] the solution of the periodic problems for nonlinear KdV from soliton theory was obtained simultaneously with that of the inverse problems for the spectral theory of the linear operator \( L \), which had not been previously solved in the periodic case (see the surveys [DMN, Kr77, DKN2] and the book [Sol]). The higher analogs of KdV yield an infinite-dimensional commutative group of such “spectral symmetries” for the operator \( L \). The finite-dimensional orbits of this group generate the famous rapidly decaying and quasiperiodic (“multi-soliton and finite-zone”) solutions of KdV as well as the remarkable exactly solvable Schrödinger operators on the line in \( L^2(\mathbb{R}) \). Here the hyperelliptic \( \Theta \)-functions of Riemann surfaces make their appearance, together with a series of other now widely known formulas and results. Such are the consequences of continuous spectral symmetry.

A few years ago some interesting advances were made [W, SY, S1] in the application of discrete symmetries, such as sequences of the well-known Bäcklund–Darboux transformations, to the spectral theory of the operator \( L = -\partial_x^2 + u(x) \). Probably the most significant were obtained by A. B. Shabat and A. P. Veselov [VS1].

II. Two-dimensional topologically trivial case \((n = 2)\). Beginning with the papers [M, DKN1] in 1976, as the basis of the two-dimensional analog of the theory of integrable systems associated with the two-dimensional Schrödinger operator, it is customary to take a representation of the form

\[
L_t = L\hat{A} + \hat{B}L,
\]

where \( L \) is the two-dimensional Schrödinger operator (see above), \( \hat{A} \) and \( \hat{B} \) are differential operators on the plane \((x, y)\). Here the data of the inverse problem for the spectral theory of the linear operator \( L \) are taken from one level only, namely from the set of solutions of \( L\psi = 0 \) under some “boundary” condition or other on \( \psi \). For the operator \( L \) with doubly-periodic coefficients one takes the Bloch waves

\[
\hat{T}_\alpha \psi = \exp\{ip_\alpha T_\alpha \} \psi, \quad L\psi = 0, \quad \alpha = 1, 2.
\]

Here \( T_\alpha \) are the periods, \( \alpha = 1, 2 \), while \( \hat{T}_\alpha \) are the translation operators by the period \( T_\alpha \), and \( p_\alpha \) is the quasi-momentum. The set of all complex Bloch waves of zero level \( L\psi = 0 \) constitute a one-dimensional complex manifold \( \Gamma : \psi(P, z, \overline{z}) \), \( P \in \Gamma \) (the Fermi curve). Without going into the details of the analytic properties of the function \( \psi \), first clarified in [DKN1], let us note the following: exactly solvable topologically trivial two-dimensional Schrödinger operators correspond to the case in which one Fermi curve \( \Gamma \) is of finite genus. The selection of the inverse problem data corresponding to the purely potential operators

\[
L = \partial \overline{\zeta} + V(x, y)
\]

was carried out in [NV1] (see the survey [NV2]). The simplest two-dimensional system of soliton theory, generalizing KdV, is of the form

\[
\frac{\partial V}{\partial t} = \left( \partial^3 + \overline{\zeta} \right)V + \partial(uV) + \overline{\zeta}(\pi V), \quad \overline{\zeta}u = 3\partial V
\]

and can be represented as

\[
\frac{\partial L}{\partial t} = [L, \hat{B}] + fL,
\]

\[
\hat{B} = \left( \partial^3 + \overline{\zeta} \right) + (u\partial + \overline{\pi}), \quad f = \partial u + \overline{\pi}.
\]
There is an entire infinite-dimensional commutative group of systems of this form (the higher analogs of two-dimensional KdV or Novikov–Veselov hierarchy).

Such is the two-dimensional analog of the continuous spectral symmetry of the class of Schrödinger operators. In contrast to the one-dimensional case, it acts only on the eigenfunctions of one level $L\psi = 0$.

Already at the end of 18th century, Laplace (see [L]) proposed substitutions acting on the solutions of the equation $L\psi = 0$ and transforming them into solutions of a new Schrödinger equation $\tilde{L}\tilde{\psi} = 0$. We call them “Laplace transformations”. In the present paper we use the Laplace transformations as a kind of “discrete symmetry” for the spectral theory of the Schrödinger operator $L$ corresponding to one level $L\psi = 0$, i.e., being in good agreement with the ideology of two-dimensional soliton theory described above.

More precisely, we shall construct new classes of two-dimensional Schrödinger operators that are exactly solvable at one or several energy levels also in the topologically nontrivial case, a case about which little is known (practically nothing, except the papers [AC, DN80] and the survey of one of the authors [N83]). It is precisely this last part of the present paper that we regard as its main result. Part of the results of the present paper was already announced in [NV3].

An alternative approach, based not only on Laplace transformations, but also on additional symmetries, is being developed by A.B. Shabat and A.P. Veselov in [S2], [SV2] (in preparation). Its relation with spectral theory of topologically nontrivial operators (see below) is totally unclear.

§2. Two-dimensional Schrödinger operator in a magnetic field. Topologically nontrivial operators. The Pauli operator

The general two-dimensional Schrödinger operator on the Euclidean plane $\mathbb{R}^2$ has the form

$$L = (\partial + B)(\partial + A) + 2V = (\partial + A)(\partial + B) + 2U,$$

where $A$, $B$, $V$, and $U$ are functions of $(x, y)$, $z = x + iy$. It is characterized by the electric potential $V(x, y)$ and the magnetic field (directed along the third axis):

$$2H(x, y) = B_z - A_\tau, \quad U = V + H$$

up to gauge transformations

$$L \rightarrow e^f Le^{-f}, \quad \psi \rightarrow e^f \psi,$$

where $f(x, y)$ is a smooth function and $\psi \in L^2(\mathbb{R}^2)$.

We shall consider the periodic case, when the group of periods is given on the plane, this group being generated by the basis vectors $\vec{T}_1$ and $\vec{T}_2$, so that the magnetic field and the electric potential are doubly periodic:

$$H(\vec{x} + \vec{T}_1) = H(\vec{x}) = H(\vec{x} + \vec{T}_2), \quad \vec{x} = (x, y),$$

$$V(\vec{x} + \vec{T}_1) = V(\vec{x}) = 0V(\vec{x} + \vec{T}_2).$$

Denote by $K$ the elementary cell in the $(x, y)$-plane, i.e., the interior of the parallelogram with vertices

$$K = \{0, 0 + \vec{T}_1, 0 + \vec{T}_1 + \vec{T}_2\}.$$
By the magnetic flux we mean the expression:

\[ [H] = \int_K H(x, y) \, dx \, dy. \]

**Definitions.**

a) The case in which \([H] = 0\) is called topologically trivial. Here all the coefficients of the operator \(L\) are doubly-periodic.

b) The case in which \([H] \neq 0\) is called topologically nontrivial. The coefficients of the operator \(L\) are not doubly-periodic functions on the \((x, y)\)-plane. Special attention will be paid to the case in which \([H] = 2\pi n\), where \(n\) is an integer. We shall say that this is the integer case.

Suppose, for the sake of simplicity, that the lattice is rectangular,

\[ \vec{T}_1 = (T_1, 0), \quad \vec{T}_2 = (0, T_2). \]

**Definition.** The magnetic translations are the following transformations:

\[ \hat{T}_1 \psi(x, y) = \exp \{if_1\} \psi(x + T_1, y), \]
\[ \hat{T}_2 \psi(x, y) = \exp \{if_2\} \psi(x + T_2, y), \]

where

\[ A(x + T_1, y) - A(x, y) = i\partial f_1, \quad B(x + T_1, y) - B(x, y) = i\vec{\partial} f_1, \]
\[ A(x, y + T_2) - A(x, y) = i\partial f_2, \quad B(x, y + T_2) - B(x, y) = i\vec{\partial} f_2. \]

It is known that magnetic translations commute with the operators \(L\), i.e., \(L\hat{T}_i = \hat{T}_i L\). They also satisfy the following commutation relation:

\[ \hat{T}_2 \hat{T}_1 = e^{i[H]} \hat{T}_1 \hat{T}_2, \]

where \([H]\) is the magnetic flux through the elementary cell \(K\). These properties can be verified by an elementary substitution.

Thus relation (4) implies that only in the integer case \([H] = 2\pi n\) can we define the magnetic Bloch functions:

\[ \hat{T}_k \psi(\vec{x}, \vec{p}) = \exp (ip_k T_k) \psi(\vec{x}, \vec{p}), \quad k = 1, 2. \]

Here the magnetic quasi-momenta \((p_1, p_2)\) range over the points of the two-dimensional torus \(\vec{p} \in T^2\) corresponding to the reciprocal lattice \(K^*\). For fixed quasi-momentum \((p_1, p_2)\), the operator \(L\) acts as an elliptic operator on the sections of the complex linear bundle over the torus \(T^2\) with the connection whose curvature \(H(x, y)\) is determined by formula (5). If the magnetic field, the electric potential, and the quasi-momenta \((p_1, p_2)\) are real, then this operator is semibounded, selfadjoint, and has a discrete spectrum of finite multiplicity

\[ L\psi_j = -\varepsilon_j \psi_j, \quad \varepsilon_j = \varepsilon_j(p_1, p_2), \quad j = 0, 1, 2, \ldots, \varepsilon_0 \leq \varepsilon_1 \leq \varepsilon_2 \leq \ldots, \]

for given \((p_1, p_2)\). If the energy is large \((\varepsilon \to \infty)\), then the sizes of the magnetic zones

\[ \Delta_j = \max_{p \in T^2} \varepsilon_j(p) - \min_{p \in T^2} \varepsilon_j(p) \]

tend to zero: \(\Delta_j \to 0\) as \(j \to \infty\). It is assumed that the magnetic flux is nonzero: \([H] \neq 0\).

It is of interest to study the topology of generic operators, initiated in [N81] (also see the survey [N83]).
**Definition.** A selfadjoint operator \( L \) with smooth doubly-periodic \((V,H), [H] = 2\pi n\) is called *generic* if for all \( j \) and all \((p_1,p_2) \in T^2_*\) we have

\[
\varepsilon_j(p_1,p_2) \neq \varepsilon_k(p_1,p_2), \quad j \neq k.
\]

In this case the eigenfunctions \( L\psi_j = \varepsilon_j \psi_j \) in the Hilbert space \( \mathcal{H} \) of sections of the bundle over the \((x,y)\)-torus \( T^2 \) depend on the quasi-momenta \((p_1,p_2) \in T^2_*\) as on parameters. This generates a linear complex bundle \( \eta_j \) over the torus \( T^2_* \), depending on the number \( j \). The corresponding Chern classes \( c_1(\eta_j) \), whose study was initiated in [N81, N83], as was later established by physicists [T], play a fundamental role in the theory of the integer quantum Hall effect, discovered in 1981. These classes may assume different random series of values.

**Problem.** Study the asymptotic properties of the integers \( c_1(\eta_j) \) as \( j \to \infty \) for generic operators \( L \) when \([H] = 2\pi n\).

The eigenfunction \( \psi_j(x,y,p_1,p_2) \), where \( L\psi_j = \varepsilon_j(p_1,p_2) \psi_j \), in the generic case possesses a manifold of zeros

\[
\psi_j(x,y,p_1,p_2) = 0.
\]

This equation expresses two real conditions. Hence, generally speaking, the manifold of zeros \( M_j \) is a two-dimensional real submanifold in the Cartesian product of tori

\[
M_j \subset T^2 \times T^2_* \quad \text{and} \quad M_j = \tilde{M}_j \cup \tilde{M}_j,
\]

where \( \tilde{M}_j \) is projected into a finite number of points of the torus \( T^2_* \), while \( \tilde{M}_j \) at a generic point is given locally in the form of a graph

\[
p_1 = F_{j_1}(x,y), \quad p_2 = F_{j_2}(x,y).
\]

The problem of studying the analytic properties of the collection of vector functions \( \vec{p} = \vec{F}_j(x,y) \) was posed in [N83]. It seems probable that the problem of the spectral theory of the operator \( L \) can be effectively solved on the basis of these notions.

Of special interest is the nonrelativistic Pauli operator for spin \( 1/2 \) on the plane with an electric field parallel to the plane and a magnetic field perpendicular to it. This is a vector operator. It has the form (the charge, mass, and Plank constant are assumed equal to 1)

\[
P = \sum_{\alpha=1}^{2} (\partial_\alpha - iA_\alpha)^2 + (W + \sigma_3 H), \quad (6)
\]

where \( x_1 = x, \ x_2 = y, \ \partial_\alpha = \partial/\partial_\alpha, \) while \((A_1, A_2)\) is the vector potential, \( W \) is the electric potential, \( H = \partial_1 A_2 - \partial_2 A_1 \) is the magnetic field, and

\[
\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]
The operator $P$ has the form of a direct sum of two scalar operators
\[ P = L_+ \oplus L_- , \]
\[ L_+ = \sum_{\alpha=1}^{2} (\partial_\alpha - iA_\alpha)^2 + (W + H) , \quad L_- = \sum_{\alpha=1}^{2} (\partial_\alpha - iA_\alpha)^2 + (W - H) . \] (7)

For a zero electric field $W = 0$, the Pauli operator may be factorized as
\[ L_+ = (\partial + A)(\partial + B) = (\partial + B)(\partial + A) + 2H , \]
\[ L_- = (\partial + B)(\partial + A) = (\partial + A)(\partial + B) - 2H . \] (8)

It is convenient to choose a “real” gauge $B = -A$, $A = A_2 + iA_1$. We shall also impose the usual Lorentz condition $\partial_1 A_1 + \partial_2 A_2 = 0$ or $\text{Im}(\partial A) = 0$. (9)

According to the results of [AC, DN80], in the absence of any electric field, the Pauli operator $(-P)$, which is always nonnegative under these conditions, has the point $\varepsilon = 0$ as its ground state for rapidly decaying fields $H$ (see [AC]) and doubly-periodic $H$ when $[H] \neq 0$ (see [DN80]). Here if $[H] > 0$, then the ground state corresponds to the + sector and satisfies the first order equation
\[ (\partial + B)\psi = 0 . \]

The equation $L_- \psi = 0$ in this case has no solutions with the necessary boundary conditions. If $[H] = 2\pi n$, $n > 0$, then the basis of the magnetic Bloch states may be written in the form
\[ \psi = e^{\varphi} \prod_{j=1}^{n} \sigma(z - a_j)e^{az} , \quad a_j \in K , \]
where
\[ \Delta\varphi = -H , \quad \varphi = \frac{1}{\pi} \iint_{K} \ln|\sigma(z - z')| H(z') \ dx \ dy , \quad z = x + iy ; \]
here $a_1, \ldots, a_n$ are arbitrary constants, while $a$ is determined by them from the condition requiring the quasi-momenta to be real (see [DN80]).

The degree of degeneracy for the level $\varepsilon = 0$ is the same as that for the Landau level for $H = \text{const}$. The magnetic field enters into the formula only via $\varphi$ and $a$.

§3. LAPLACE TRANSFORMATIONS. MAIN RESULTS
(THE TOPOLOGICALLY TRIVIAL CASE)

For the Schrödinger operator $L = (\partial + B)(\partial + A) + 2V$, we start with an arbitrary solution of $L\psi = 0$. Then the function $\tilde{\psi} = (\partial + A)\psi$ satisfies the equation
\[ |V(\partial + A)V^{-1}(\partial + B) + 2V| \tilde{\psi} = 0 . \]
Thus we see that if $L\psi = 0$, then $\tilde{L}\tilde{\psi} = 0$, where
\[ \tilde{L} = V(\partial + A)V^{-1}(\partial + B) + 2V , \quad \tilde{\psi} = (\partial + A)\psi . \]
Since $V(\partial + A)V^{-1} = \partial + \tilde{A}$, where $\tilde{A} = A - (\ln V)_z$, we have $\tilde{L} = (\partial + \tilde{B})(\partial + \tilde{A}) + 2\tilde{V}$, where $\tilde{B} = B$, $\tilde{A} = A - (\ln V)_z$,
\[ \begin{align*}
\tilde{H} &= H + \frac{1}{2} \partial \tilde{A}(\ln V) , \\
\tilde{V} &= V + \tilde{H} .
\end{align*} \] (10)

It is essential that the final formulas (10) provide the transformations only in terms of physical quantities.
Lemma 1. Suppose that the potential \( V(x, y) \) and the magnetic field \( H(x, y) \) are real and doubly-periodic, and \( V \neq 0 \). In this case the magnetic fluxes \( [H] \) and \( \widetilde{[H]} \) coincide, while for the potentials we have the relation
\[
[\widetilde{V}] = [V] + [H].
\]

The proof follows from formula (10) for the Laplace transformations because \( \left[ \frac{\partial}{\partial \ell} \ln V \right] = 0 \).

Lemma 2. Let the potential \( V \) and the magnetic field \( H \) be real, while the operator \( L \) is given in the real Lorentz gauge
\[
L = (\vec{\sigma} + B)(\partial + A) + 2V, \quad B = -\vec{\sigma}, \quad \text{Im} A = 0.
\]
Then for any solution of \( L\psi = 0 \) the function \( \widetilde{\psi} = e^{-Q}(\partial + A)\psi \) satisfies the equation \( \widetilde{L}\widetilde{\psi} = 0 \), where \( \widetilde{L} = (\vec{\sigma} + \vec{B})(\partial + \vec{A}) + 2\vec{V} \), and the operator \( \widetilde{L} \) is gauge equivalent to the image of the Laplace transformation
\[
\widetilde{L} = e^{-Q}Le^{+Q}, \quad \widetilde{V} = \vec{V}, \quad \widetilde{B} = \vec{B} + Q\vec{\sigma}, \quad \widetilde{A} = \vec{A} + Qz.
\]

Now if \( Q = \ln \sqrt{V} \), then the new operator \( \widetilde{L} \) can be expressed in the real Lorentz gauge as follows
\[
\widetilde{B} = B + \frac{1}{2}(\ln V)\vec{\sigma}, \quad \widetilde{A} = A - \frac{1}{2}(\ln V)z, \quad -\widetilde{B} = \vec{A}, \quad \text{Im} \, \vec{A} = 0. \quad (11)
\]

The proof of this lemma also follows from the formulas specifying the Laplace transformations.

Speaking of the Laplace transformation for magnetic fields and potentials, we shall not distinguish, as a rule, operators from the same gauge equivalence class. But if we are concerned specifically with real selfadjoint operators, then by Laplace transformations we mean the transformations \( L \to \widetilde{L} \) effected in accordance to formulas (11).

Now let us consider infinite sequences of Laplace transformations: suppose that \( V_j = \exp f_j \), then
\[
e^{f_{j+1}} = e^{f_j} + H_{j+1}, \quad H_{j+1} = H_j + \frac{1}{2}\partial \ell f_j,
\]
whence
\[
\frac{1}{2}\Delta f_j = e^{f_{j+1}} - 2e^{f_j} + e^{f_{j-1}}. \quad (12)
\]

Lemma 3. After the substitution \( f_j = \varphi_j - \varphi_{j-1} \), the infinite sequence of Laplace transformations reduces to the well-known two-dimensional Toda lattice
\[
\frac{1}{2}\Delta \varphi_j = e^{\varphi_{j+1} - \varphi_j} - e^{\varphi_j - \varphi_{j-1}}. \quad (13)
\]

Indeed, the substitution \( f_j = \varphi_j - \varphi_{j-1} \) in (12) yields
\[
\frac{1}{2}\Delta \varphi_j = e^{\varphi_{j+1} - \varphi_j} - e^{\varphi_j - \varphi_{j-1}} + h(z, \vec{\sigma}),
\]
where \( h(z, \overline{z}) \) is a function that does not depend on the number \( j \). It is easy to get rid of it by using the change \( \varphi_j \rightarrow \varphi_j + \alpha(z, \overline{z}) \), where \( (1/2)\Delta \alpha = h \).

The two-dimensional Toda lattice together with the representation in the form of an (L-A)-pair was found by A. Mikha˘ ılov in [Mik] and, independently, in [LS, BUL] for finite chains with free ends, as a generalization of the Liouville equation related to Lie algebras. It was known to classical geometers (see [Dar, Tzi]) as a sequence of Laplace transforms. The connection of classical geometric studies with systems from soliton theory has been observed by A. M. Vassiliev (unpublished), but no attempts to use this observation were made. A consequence of this comparison is the following

**Proposition 1.** The algebro-geometric solutions (expressed in terms of \( \Theta \)-functions) found in [Kr81] for the two-dimensional Toda lattice generate for any \( j \) a two-dimensional Schrödinger operator \( L_j \) with quasiperiodic coefficients, which possess a Bloch solution \( L_j \psi = 0 \) with appropriate analytic properties on the same Riemann surface \( \Gamma \) of finite genus. Conversely, suppose that we are given a nonsingular algebraic curve \( \Gamma \) on which two distinguished points \( P_+ \), \( P_- \) with local parameters \( w_+, w_- \) are fixed, together with the divisor \( D = P_1 + \cdots + P_g \) of degree \( g \). If the operator \( L = L_0 \) corresponds to the data of the inverse problem \( (\Gamma, P_\pm, w_\pm, P_{1j}, \ldots, P_{gj}) \), then the operators \( L_j \) from the chain of Laplace transformations are obtained from the data \( (\Gamma, P_\pm, w_\pm, P_{1j}, \ldots, P_{gj}) \), and we have the following linear equivalence of divisors

\[
P_1^j + \cdots + P_g^j \sim j(P_+ - P_-) + P_1 + \cdots + P_g.
\]  

Recall (see [DKN]) that in the generic case \( P_1, \ldots, P_g \) are poles of order one of the Bloch solution of \( L \psi = 0 \) that do not depend on \( x, y \). Near the points \( P_\pm \) the function \( \psi \) has the asymptotics

\[
\psi \sim c_1(x, y) e^{z/w_+}(1 + \sigma(w_+)), \quad P \sim P_+, \quad z = x + iy.
\]

The function \( \psi \) is meromorphic on \( \Gamma \setminus (P_+ \cup P_-) \) and depends on \( x, y \) as on parameters. Such a function always satisfies an equation of the form \( L \psi = 0 \), where

\[
L = (\overline{\partial} + B)(\partial + A) + 2V, \quad A = -\partial \ln c_2, \quad B = -\overline{\partial} \ln c_1.
\]

Indeed, since \((\partial + A)c_2 = (\overline{\partial} + B)c_1 \equiv 0\), the function \((\overline{\partial} + B)(\partial + A) \psi \) possesses similar analytic properties and, therefore, differs from \( \psi \) only by a constant factor. It is easy to see that the function \( \widetilde{\psi} = (\partial + A) \psi \) has the same poles \( P_1, \ldots, P_g \) and asymptotics in \( P_\pm \) of the form

\[
\widetilde{\psi} = \widetilde{c}_1(x, y) w_+^{-1} e^{z/w_+}(1 + \sigma(w_+)), \quad P \sim P_+,
\]

\[
\widetilde{\psi} = \widetilde{c}_2(x, y) w_- e^{z/w_-}(1 + \sigma(w_-)), \quad P \sim P_-.
\]  

More generally, we have the following
Lemma 4. The function $\psi^{(j)}$, being the result of $j$ applications of the Laplace transformation to the operator $L$, besides its poles $P_1, \ldots, P_g$, possesses singularities at the points $P_{\pm}$ of the form
\begin{align*}
\psi^{(j)} &= c_1^{(j)}(x, y) w_j^j e^{z/w} (1 + \sigma(w^+)), \quad P \sim P_+,
\psi^{(j)} &= c_2^{(j)}(x, y) w_j^j e^{z/w} (1 + \sigma(w^-)), \quad P \sim P_-.
\end{align*}
(17)

The proof of Proposition 1 now follows if we compare the result of Lemma 4 with the construction of the algebraic solutions of the two-dimensional Toda lattice proposed by I. M. Krichever [Kr81]. Explicit formulas involving $\Theta$-functions for the coefficients of the corresponding operators $L_j$ can be found in [D81, p. 50] and [Kr81, p. 77]. They imply, in particular, that the coefficients are quasi-periodic functions of $x$ and $y$, so that the operators $L_j$ are topologically trivial.

The solutions of a periodic chain of period $N$, $\varphi_{n+N} \equiv \varphi_n$, correspond to curves with points $P_+, P_-$ such that $NP_+ - NP_- \sim 0$, i.e., curves on which there exists a meromorphic function with a unique pole of $N$th order at $P_+$ and a zero of order $N$ at $P_-$. To such solutions correspond cyclic chains of Laplace transformations. It turns out that under certain analytical assumptions all such chains are algebro-geometric, namely, we have the following result.

Proposition 2. Suppose that a chain of Laplace transformations is cyclic, $V_{i+N} \equiv V_i$, and all the potentials $V_i = \exp f_i$ are doubly periodic, smooth, real, and positive functions. Then for all $n$ the Schrödinger operators $L_n$ are algebro-geometric with respect to the zero energy level $L_n\psi = 0$.

For $N = 1$ from the relation (12) we get $\Delta f_1 = 0$, which in the doubly periodic case implies $f_1 = \text{const}$. The case $N = 2$ reduces to a similar question for the sinh-Gordon equation
$$\Delta \varphi + \sinh \varphi = 0.$$ 

The corresponding result in this case was first obtained by Pinkall and Sterling [PS]. A. Bobenko considerably simplified the proof [B]. The proof of Proposition 1 presented below is a direct generalization of Bobenko’s arguments.

It follows from Lemma 3 that cyclic chains of Laplace transformations correspond to doubly-periodic solutions of the periodic Toda lattices (13). It is known (see [Mik]) that such a chain may be represented as an $(L-A)$-pair of the form
$$[\partial + P, \overline{\partial} + Q] = 0,$$ 
where $P$ and $Q$ are the following matrices depending on the “spectral” parameter $\lambda$:
\begin{align*}
P &= \begin{pmatrix}
\partial \varphi_1 & 1 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 1 \\
\lambda & 0 & \ldots & \partial \varphi_n
\end{pmatrix},
Q &= \begin{pmatrix}
0 & \ldots & 0 & \lambda^{-1} e^{\varphi_n - \varphi_1} \\
\varphi_1 & \ldots & 0 & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ldots & \varphi_{n-1} - \varphi_n & 0
\end{pmatrix}.
\end{align*}

This allows to determine, in the standard way, an infinite series of commuting flows
$$[\partial + P, \partial_i + P_i] = 0 = [\overline{\partial} + Q, \partial_i + P_i],$$
(19)
where $\partial = \partial / \partial t_i$ and $P_i$ is a matrix polynomially depending on $\lambda$ (see, for example, [DS]). The derivatives $\xi^{(i)} = (\partial / \partial t_i)(\varphi)$ satisfy for all $i$ the linearized system

$$
\frac{1}{2} \Delta \xi_k^{(i)} = \xi_{k+1}^{(i)} e^{\varphi_{k+1} - \varphi_k} - \xi_k^{(i)} (e^{\varphi_{k+1} - \varphi_k} + e^{\varphi_k - \varphi_{k-1}}) + \xi_{k-1}^{(i)} e^{\varphi_k - \varphi_{k-1}}.
$$

(20)

This system is an elliptic linear system on a compact manifold (the torus $T^2$), hence there exists only a finite number of linearly independent $\xi^{(i)}$. This in its turn implies that any doubly-periodic solutions of the Toda lattice is stationary for an appropriate higher flow (19) and by a standard argument (see [DKN2]) is algebro-geometric. This concludes the proof of Proposition 2.

§4. Laplace transformations and topologically nontrivial periodic operators

As indicated above, in the nonsingular periodic case, the Laplace transformation

$$
H_{k+1} = H_k + \frac{1}{2} \Delta f_k, \quad e^{f_k} = V_k, \quad e^{f_{k+1}} = e^{f_k} + H_{k+1}
$$

preserves the magnetic flow $[H_{k+1}] = [H_k]$ and changes the mean value of the potential

$$
[e^{f_{k+1}}] = [e^{f_k}] + [H_{k+1}].
$$

Thus (nonsingular) cyclic chains with nonzero magnetic flux do not exist.

Definition. A semi-cyclic chain of length $n$ is a chain such that

$$
H_n = H_0, \quad V_n = V_0 + C_n.
$$

(21)

In the nonsingular case we obviously have

$$
C_n = n[H_0].
$$

(22)

Here we assume that all the operators in the chain $L_0, L_1, \ldots, L_n$ have nonsingular coefficients and the invariant expressions are doubly-periodic

$$
H_n(\vec{x} + T_1) = H_n(\vec{x} + T_2) = H_n(\vec{x}), \quad V_n(\vec{x} + T_1) = V_n(\vec{x} + T_2) = V_n(\vec{x}).
$$

Here $T_1, T_2$ are the basis vectors of the lattice of periods in the plane, and $\vec{x} = (x_1, x_2)$.

Example 1. Let $n = 1$. From the formulas for the Laplace transformations, we immediately obtain

$$
H_1 = H_0, \quad e^{f_1} = e^{f_0} + H_1, \quad \Delta f_0 = 0.
$$

(23)

In the nonsingular doubly-periodic case, we have

$$
f_1 = \text{const}, \quad f_0 = \text{const}, \quad H_0 = \text{const}.
$$
Thus the operator $L_1$ is of the form

$$L_1 = L_0 + C_1.$$  (24)

Here $L_1$ is the Landau operator in a homogeneous magnetic field and trivial (constant) potential. If we know solutions of $L_0 \psi_0 = 0$ in the Hilbert space $L_2(\mathbb{R}^2)$, then the function $\psi_1 = Q_0 \psi_0 = e^{\rho_0/2}(\partial + A_0)\psi_0$ gives the solution $\psi_1 \in L_2(\mathbb{R}^2)$. Iterating this procedure, we get

$$L_n = L_0 + nC_1, \quad \psi_n = Q_{n-1} \cdots Q_0(\psi_0) \in L_2(\mathbb{R}^2).$$  (25)

In this case we obtain the eigenfunctions of the operator $L_0$ with eigenvalues $(-nC_1)$ in the space $L_2(\mathbb{R}^2)$

$$L_0 \psi_n = (-nC_1) \psi_n, \quad n = 0, 1, 2, \ldots.$$  (26)

It is easy to see that

$$C_1 = H_0 = |H_0|/K = \text{const},$$  (27)

where $K$ is the volume of the elementary cell. We know that the Schrödinger operator $(-L_0)$ is semi-bounded. There exists a constant $E_0$ such that

$$\langle -L_0 \psi, \psi \rangle \geq -E_0 \langle \psi, \psi \rangle, \quad \psi \in L_2(\mathbb{R}^2).$$

Hence the solution $L_0 \psi_0 = 0$ belonging to $L_2(\mathbb{R}^2)$ is possible only in the case $|H_0| > 0$. If $W_0 = 0$ (or $V = H_0$), i.e., if

$$L_0 = (\partial + A)(\overline{\partial} + B) = (\partial_x + iA_1)^2 + (\partial_y + iA_2)^2,$$

then the solutions serving $L_2(\mathbb{R}^2)$ can be found in the form $(\overline{\partial} + B)\psi_0 = 0$ for the real Lorentz gauge $B = -\overline{A}$, $A_{1x} + A_{2y} = 0$. (This will be established below in connection with the theory of quasi-cyclic chains.) In this case the spectrum of the operator $(-L_0)$ is discrete and has the levels

$$\lambda_n = nC_1 = nH_0, \quad (-L_0)\psi_n = \lambda_n \psi_n.$$

This is precisely the spectrum of the Landau operator (up to a translation by the constant $H_0/2$). However, if $|H_0| < 0$, then we must begin with the operator

$$L'_0 = (\overline{\partial} + B)(\partial + A), \quad V = 0$$

and search for the solution in the form $(\partial + A)\psi' = 0$. The spectrum is obtained by applying the inverse Laplace transformation (and its iterations):

$$\psi'_1 = Q'_0 \psi'_0 = e^{g_0/2}(\partial + A)\psi'_0, \quad g_0 = \ln W_0,$$

$$L'_1 = L'_0 + C'_1, \quad C'_1 = -H_0 \quad \lambda_n = -nH_0, \quad (-L'_0)\psi_n = \lambda_n \psi_n.$$

This concludes Example 1.

Our main results are related to the following class of operators.
**Definition.** A *quasi-cyclic chain* of Laplace transformations of length $n$ is a chain $L_0, L_1, \ldots, L_n$ such that

$$C_n + H_n = V_n = e^{f_n}, \quad H_0 = V_0 = e^{f_0}. \quad (28)$$

Suppose further that the magnetic flux is positive and integer-valued, i.e., $[H_0] = 2\pi m > 0$.

**Example 2.** Let $n = 2$. Let us write out the equations that follow from the conditions of semi-cyclicity and quasi-cyclicity:

$$\frac{1}{2} \Delta g = -C_2 - 4e^{a/2} \sinh g, \quad g = f_0 - a/2 \quad \text{(semi-cyclic case)}, \quad (29)$$

$$\frac{1}{2} \Delta f_0 = C_2 - 2e^{f_0}, \quad e^{f_2} = 2C_2 - e^{f_0} \quad \text{(quasi-cyclic case)}. \quad (30)$$

In both cases we have $C_2 = 2[H_0] > 0$. This is especially important in the case of quasi-cyclic chains. Consider the solutions of these equations ($n = 2$) that do not depend on $y$. We obviously have the following solutions

$$x = \int \frac{dg}{\sqrt{C_2g + 4e^{a/2} \cosh g + C}}, \quad \text{(semi-cyclic case)},$$

$$x = \int \frac{dg}{\sqrt{2e^g - C_2g + C}}, \quad g = f_0, \quad \text{(quasi-cyclic case)}.$$

It is clear that for the corresponding values of the constant $C$ in both cases we obtain wide classes of smooth nonsingular periodic solutions. Being independent of $y$, the expressions above determine doubly-periodic operators with arbitrary period in $y$. For the magnetic field we have

$$H_0 = C_2 - 2e^{a/2} \sinh(f_0 - a/2) \quad \text{(semi-cyclic case)}, \quad (31)$$

$$V_2 = 2C_2 - e^{f_0} = e^{f_2} = H_2 + C_2,$$

$$H_2 = C_2 - e^{f_0} = H_1 \quad \text{(quasi-cyclic case)}. \quad (32)$$

To ensure smoothness and nonsingularity of the operator $L_2$ in the quasi-cyclic case, we must have $e^{f_0} < C_2$. Solutions satisfying this condition certainly exist, as can be seen from the formulas. Since the function $f_0(x)$, as well as the potential and magnetic field, does not depend on $y$, it follows that we can choose such a (real) gauge so that

$$L = (\partial_x + iA_1)^2 + (\partial_y + iA_2)^2 + 2V, \quad A_1 = 0, \quad A_2(x) = A_2, \quad -A'_2 = H(x),$$

where $L = L_0$ (semi-cyclic chains) and $L = L_2$ (quasi-cyclic chains). In both cases, after the substitution $\psi = e^{iky}\varphi(x, k)$, we obtain the following equation in $x$:

$$\Lambda_k \varphi_k = 0,$$

$$\Lambda_k = \partial_x^2 - (k + A_2(x))^2 + (2V(x) - A'_2(x)). \quad (33)$$
In the Landau case, we have $A_2 = Hx$, $V = H/2$. After that the substitution $x' = x - kH^{-1}$ the operator reduces to the harmonic oscillator $A_{k0} = \partial_x^2 - H^2 x^2$.

In the case of a nonhomogeneous magnetic field we have $H(A_2) = Hx + A_0^2(x)$, where $A_0^2(x)$ is periodic with period $T_1$. The substitution $x' = x + kH^{-1}$ reduces this equation to the family of equations

$$\Lambda_k \varphi_k = 0,$$

$$\Lambda_k = \partial_x^2 - x'^2 H^2 + U(x' - kH^{-1}) - 2kHA_0^2(x),$$

$$U = 2V(x) - H(x) - (A_0^2(x))^2.$$

The potential $V_2$ for the operator $L_2$ in both cases has the form

$$V_2 = V_0 + C_2 \quad (\text{semi-cyclic case}), \quad V_2 = H_2 + C_2 \quad (\text{quasi-cyclic case}).$$

The quasi-cyclic operators reduce to a one-dimensional family of oscillator-like operators:

$$\Lambda_k \varphi_k = 0,$$

$$\Lambda_k = \partial_x^2 - \overline{H}^2 x'^2 - 2\overline{H} x' A_0^2(x) + U(x),$$

$$U = 2V(x) - H(x) - (A_0^2(x))^2.$$ (34)

The proof of Proposition 3 is based on the fact that the Laplace transformation takes magnetic Bloch functions to magnetic Bloch functions. This fact is readily verified by substitution and direct calculation. If $\psi_n \neq 0$, then the proposition is proved. Under its assumptions, if $\psi_n \equiv 0$, then it follows that $Q_{n-1}\psi_{n-1} = 0$. Suppose that $j$ satisfies $\psi_j = 0$ and $\psi_{j-1} \neq 0$. We have $Q_{j-1}\psi_{j-1} = 0$. Thus the operator $L_{j-1}$ coincides with half of the Pauli operator

$$L_- = L_{j-1} = (\partial + B_{j-1})(\partial + A_{j-1}).$$

However, we know that the eigenfunctions of the principal state are possible for $L_-$ here only for negative magnetic flux $[H_{j-1}] < 0$, whereas by assumption we have $[H_{j-1}] = [H_0] > 0$. This contradiction proves Proposition 3.
Proposition 4. Equation (30) of the quasi-cyclic chain of length $n = 2$ possesses smooth solutions, essentially depending on both variables $(x, y)$, doubly-periodic with respect to the square lattice with any period $T > 2\pi C_2^{-1/2}$ sufficiently close to $2\pi C_2^{-1/2}$. These solutions differ little from $f_0 = \ln(C_2/2)$.

The proof of this proposition (necessary to establish that equation (30) does possess a large class of nontrivial doubly periodic solutions) was communicated to the authors by S. Kuksin.

Conjecture. For $n \geq 5$, the equation of the quasi-cyclic chain does not have any nonconstant periodic solutions close to a constant. The constant is isolated.

It seems probable that these equations have no solutions already in the linear approximation. For $n = 2, 3, 4$ solutions in the linear approximation can be found.

Theorem. Suppose we are given a quasi-cyclic chain of length $n$ of operators $L_0, \ldots, L_n$ with smooth nonsingular positive potentials $V_0, \ldots, V_{n-1}$ and nonsingular magnetic fields $H_0, \ldots, H_n$ with positive flow $[H_0] = \cdots = [H_n]$, the potentials and fields being doubly-periodic with period lattice $\mathbb{T}_1, \mathbb{T}_2$. In this case the Schrödinger operator $L_n$ possesses two nondegenerate levels $0, +C_n$ each of which is isomorphic to the Landau level in a homogeneous magnetic field and zero potential. The eigenfunctions of the levels $0, +C_n$ can be computed explicitly.

Proof. Let us construct the eigenfunctions of the levels $0, +C_n$. By the quasi-cyclicity condition, we have $V_n = e^{f_n} = H_n + C_n$. Hence the operator $L = L_n - C_n$ has the form

$$L = L_n - C_n = (\partial + A_n)(\bar{\partial} + B_n).$$

Since the flow $[H_n]$ is positive and integer, we can extract the eigenfunctions of the ground state from the paper [DN80]

$$(L_n - C_n)\psi = 0, \quad \psi = e^{\varphi} \prod \sigma(z - a_1), \ldots, \sigma(z - a_m)e^{az},$$

$$\Delta \varphi_n = -\frac{1}{\pi} \iint_K \ln|\sigma(z - z')|H_n(z') d^2z', \quad z' = (x', y'), \quad z = (x, y).$$

Here $K$ is the elementary cell, we do not specify the class $a$ and constants $a_1, \ldots, a_m$ here. The eigenfunctions of the level $L_n\psi = 0$ are obtained by Laplace transformations from the functions $L_0\psi_0 = 0$. Indeed, we have

$$\psi_n = Q_{n-1} \cdots Q_0(\psi_0), \quad \text{where} \quad Q_j = e^{f_j/2}(\partial + A_j).$$

Let us put

$$\psi_0 = e^{\varphi_0} \prod_{j=0}^m \sigma(z - a_j)e^{a_jz}, \quad \text{where} \quad \Delta \varphi_0 = -\frac{1}{\pi} \iint_K \ln|\sigma(z - z')|H_0(z') d^2z'.$$

The constant $a$ is related to $(a_1, \ldots, a_m)$ by formulas presented in [DN80]. The functions $\psi_0, \psi_1, \ldots, \psi_n$ are magnetic Bloch by construction. We already know that the level $(L_n - C_n)\psi = 0$ is isolated, discrete, and isomorphic to the Landau level [DN80, N83]. Let us prove this for the level $C_n$. The subspace of functions (36) obtained is isomorphic to the Landau level (by construction) and determines
a subspace in the Hilbert space \( L_2(\mathbb{R}^2) \). Let us prove that there is nothing more at this level. Let us apply the inverse Laplace transformation to the operator \( L_n \).

After \( n \) steps we obtain the operator \( L_0 \), while from the functions \( \psi_n \) we get \( \psi_0 \). For the level \( L_0 \psi_0 = 0 \) we already know that there is nothing more at this level, except the functions found in [DN80]. If the \( n \)-fold Laplace transformation \((\psi_n) \to (\psi_0)\) is an isomorphism, then the Theorem is proved. If it is not an isomorphism, the inverse Laplace transformation has a nontrivial kernel. This means, for \( n > 1 \), that for some number \( j \) to the operator \( L_j \) corresponds a function \( \phi_j \) such that \( L_j \phi_j = 0 \) and \((\partial + B_j) \phi_j = 0\). Arguing as in the proof of Proposition 3, we see that \( L_j \) is quasi-cyclic of length \( j < n \), where the constant \( C_j \) is zero. Having in mind the fact that the magnetic fields are nonsingular and the relation \( f_j = \ln V_j \), we see that \( C_j = j[H_0] \neq 0 \); this contradiction proves the theorem.

**Remark.** As the authors were told by Hector de Vega, the equation \( \Delta g = 1 - e^g \) was obtained as the “Bogomolny reduction” of the Ginzburg–Landau equations for a special value of their parameter that once divides superconductors of types I and II (see an explicit quotation to the work of de Vega and Shaposhnik in [NV3]). The applicability of the Ginzburg–Landau equations at this point is not clear (possibly it necessitates some corrections, as it was explained to the authors by experts from the Landau Institute). However, in this situation the “physical magnetic field” differs from ours by a constant (in such a way that the magnetic flux in the case of smooth nonsingular solutions is zero). The physical magnetic flux in the Ginzburg–Landau case entirely consists of singularities. The study of solutions with isolated singularities is meaningful in our case also. In this case nontrivial decreasing solutions as \( |z| \to \infty \) are possible in the case \( C_2 \neq 0 \) as well, where we have an elliptic Liouville equation, \( \Delta g = -e^g \). Its general solution is given by the following

**Proposition.** For an arbitrary analytic function \( f(z) \) the function \( \varphi \) defined as

\[
e^\varphi = \frac{|f(z)|^2 |f'(z)|^2}{(1 + |f(z)|^4)^2}
\] (38)

satisfies the following elliptic version of Liouville equation

\[
\Delta \varphi + 32 e^\varphi = 0.
\] (39)

**Proof.** It can be checked that the function \( \varphi \) of the form

\[
e^\varphi = \frac{r^2}{(1 + r^4)^2},
\]

where \( r = |z|^2 \) satisfies this equation. Now the statement follows from the following symmetry of Liouville equation: if \( \varphi(z, \overline{z}) \) is a solution of (39) then \( \varphi(f(z), \overline{f(z)}) + 2 \log |f'(z)| ) \), where \( f(z) \) is analytic, is a solution too.

Taking a rational or elliptic function for \( f(z) \) we obtain solutions with singularities at the poles. The search for solutions with singularities also makes sense in the case \( C_2 \neq 0 \).
Appendix I (S. P. Novikov)

Difference analogs of Laplace transformations
and two-dimensional Toda lattices

In the continuous case, the Laplace transformation $L \to \tilde{L}$, $\psi \to \tilde{\psi}$ was formally the same in the elliptic and the hyperbolic case: one could be obtained from the other by the formal substitution

$$\partial \to \partial_x, \quad \overline{\partial} \to \partial_y,$$

$$(\overline{\partial} + B)(\partial + A) + 2V \to (\partial_x + B)(\partial_u + A) + 2V$$

(here, of course, the global statements of the problem are entirely different).

For difference operators, the hyperbolic and selfadjoint elliptic case will already differ on the formal level. The separating out of classes of difference operators that possess Laplace transformations seems important, since this would indicate the most natural classes of operators with maximal “hidden symmetry” (in some sense) and therefore with the nicest properties.

1. Hyperbolic case. Let the operator $L$ be of the form [Kr85]

$$L \psi_n = \psi_n + a_n T_1 \psi_n + b_n T_2 \psi_n + c_n T_1 T_2 \psi_n = 0,$$  

(39)

where $n = (n_1, n_2)$, $N_j \in \mathbb{Z}$, and $T_1, T_2$ are the translations determining the lattice,

$$T_1 \psi_n = \psi_{n+T_1}, \quad T_2 \psi_n = \psi_{n+T_2}, \quad n + T_1 = (n_1 + 1, n_2), \quad n + T_2 = (n_1, n_2 + 1).$$

Lemma 5. The operator $L$ has a factorization of the form

$$L = f_n [(1 + u_n T_1)(1 + v_n T_2) + w_n].$$

(40)

The proof of the lemma is elementary. We come to the following formulas

$$f_n = (1 + w_n)^{-1} = a_{n-T_1} b_n c_{n-T_1}, \quad u_n = a_n / f_n, \quad v_n = b_n / f_n.$$  

(41)

Note that formulas (41) are local, i.e., to obtain the factorization we need not solve any difference equations.

Definitions. The Laplace transformation is the assignment $L \to \tilde{L}$, $\psi \to \tilde{\psi}$, where

$$\tilde{L} \psi = 0, \quad L \psi = 0,$$

$$\tilde{L} = \frac{w_n}{1 + w_n} [(1 + v_n T_2) w_n^{-1} (1 + u_n T_1) + 1], \quad \tilde{\psi} = (1 + v_n T_2) \psi_n.$$  

(42)

The inverse Laplace transformation is the assignment $L \to \bar{L}$, $\psi \to \bar{\psi}$, where

$$\bar{L} \bar{\psi} = 0, \quad \bar{\psi} = (1 + g_n T_1) \psi_n, \quad L = g_n [(1 + p_n T_2) (1 + q_n T_1) + s_n].$$

Remark 1. Just as in the continuous case, the Laplace transformation and its inverse are indeed inverse to each other. This is verified in an obvious way.

Remark 2. It is easy to see that to each pair of basis periods $(T_1^{\pm 1}, T_2^{\pm 1})$ and $(T_2^{\pm 1}, T_1^{\pm 1})$ there corresponds a Laplace transformation quite similar to (40)–(42), where we have the pair $(T_1, T_2)$ and $(T_2, T_1)$ for the inverse transformation. The algebra generated by all Laplace transformations is extremely interesting in this case.

In the gauge equivalence classes $L \to g L g^{-1}$, the potential $w_n$ and the nonphysical magnetic field $H_n$ are invariants,

$$w_n + 1 = C_n^{-1} a_n^{-1} b_n^{-1}, \quad e^{H_n} = a_n b_n T_2 b_n^{-1} a_n T_2.$$  

(43)
Lemma 6. The Laplace transformation depends only on the invariants of the gauge transformation and can be expressed as

\[ e^{\tilde{H}_n} = e^{H_n}w_{n+T_2}w_{n+T_1}w_{n}^{-1}w_{n+T_1+T_2}^{-1}, \]
\[ 1 + \tilde{w}_{n+T_1} = e^{-\tilde{H}_n}(1 + w_{n+T_2}). \] (44)

The proof is a direct verification of the formula.

The infinite chain of Laplace transformations obtained according to formulas (44) leads to a completely discrete analog of the two-dimensional Toda lattice for the magnitudes \( w_n^{(k)} \), where the values of \( H_n^{(k)} \) are expressed via (44)

\[ e^{H_n^{(k+1)}} = \frac{1 + w_n^{(k+2)}}{1 + w_{n+T_1}}. \]

Here \( (w_n^{(k)}, H_n^{(k)}) \) is obtained by the Laplace transformation from \( (w_n^{(k-1)}, H_n^{(k-1)}) \), \( k \in \mathbb{Z}, \ n = (n_1, n_2) \). This system should be compared with [KLWZ].

Example 3. Let us consider a cyclic chain of Laplace transformations

\[ H_n^{(k+m)} = H_n^{(k)}, \quad w_n^{(k+m)} = w_n^{(k)}. \]

The case \( m = 1 \) is trivial. For \( m = 2 \) we obtain a discrete analog of the sinh-Gordon equation, and after reduction, we get

\[ w_n^{(1)} = C(w_n^{(0)})^{-1}, \]
\[ w_{n+T_1+T_2} = w_n^{-1}(C + w_{n+T_1})(C + w_{n+T_2})(1 + w_{n+T_1})^{-1}(1 + w_{n+T_2})^{-1}. \]

For \( C = 1 \) the equation degenerates.

2. Laplace transformations for selfadjoint real difference operators. The ordinary discretization of the Schrödinger operator, say, on a square lattice, where at the given point \( n = (n_1, n_2) \) one takes the (weighted) sum of the values of \( \psi \) at the same point and four its nearest neighbors \( (n_1 \pm T_1, n_2), (n_1, n_2 \pm T_2) \), does not possess the factorization necessary for the construction of the Laplace transformation. The latter can be obtained for the operator \( L \) if we use the equilateral triangular lattice, where each vertex has 6 nearest neighbors,

\[ |T_1^\pm1| = |T_2^\pm1| = |(T_1T_2^{-1})^\pm1|. \]

The operator \( L \) has the form

\[ L = a_n + b_nT_1 + c_nT_2 + d_n + T_1T_2^{-1} + b_n-T_1^{-1} + c_n-T_2^{-1} + d_{T_2} + nT_2T_1^{-1}. \] (45)

Lemma 7. An operator of the form (45) with real coefficients possesses the following factorization

\[ L = (x_n + y_nT_1 + z_nT_2)(x_n + y_n-T_1^{-1} + z_n-T_2^{-1}) + w_n. \]
The proof is a simple calculation. Note that here, just as in the hyperbolic case (40), (41), the factorization is given by local algebraic formulas, which do not require the solutions of difference equations. We also have the inverse factorization

\[ L = (x'_n + y'_n T_1^{-1} + z_n T_2^{-1})(x'_n + y'_n - T_1 T_2^{-1} + z_n - T_2 T_1^{-1}) + w'_n. \]

To any solution of \( L \psi = 0 \) a new solution is assigned:

\[ \widetilde{L} \widetilde{\psi} = 0, \quad \widetilde{L} = (x_n + y_n - T_1 T_2^{-1} + y_n - T_2^{-1} T_1) w_n^{-1} (x_n + y_n T_1 + z_n T_2) + 1, \]

\[ \widetilde{\psi}_n = (x_n + y_n - T_1 T_2^{-1} + z_n - T_2 T_1^{-1}) \psi_n. \]

We also have an inverse Laplace transformation induced by the inverse factorization. The two transformations are mutually inverse up to a gauge transformation. Here the discrete analog of the two-dimensional Toda lattice, as well as those of the cyclic, semi-cyclic, and quasicyclic chains arise. They will be studied in a forthcoming paper.

Remark. To each pair of neighboring basis periods of the same length \( (T_1, T_2) \), \( (T_1^{-1} T_2, T_1^{-1}) \), \( (T_2^{-1} T_1, T_2^{-1}) \), \( (T_1, T_2^{-1}) \), and \( (T_1^{-1} T_2, T_1^{-1}) \) there corresponds one Laplace transformation. The algebra of these transformations is being studied.

Introducing complex values for the fields, we also obtain the discretization of the Schrödinger operator in a magnetic field (what we considered above is a discretization of real operators only, where physical magnetic field is equal to zero.)

APPENDIX II (S. P. Novikov, I. A. Taimanov)

DIFFERENCE ANALOGS OF THE HARMONIC OSCILLATOR

Let us consider the real selfadjoint second order difference operator acting in the space \( L^2(\mathbb{Z}) \), \( n \in \mathbb{Z} \), and given by

\[ L \psi_n = v_n \psi_n + c_{n-1} \psi_{n-1} + c_n \psi_{n+1}, \]

\[ L = c_{n-1} T^{-1} + c_n T + v \cdot 1, \]

where \( T \colon n \to n + 1 \) is the translation operator along the lattice. Here \( c_n, v_n \in \mathbb{R} \), \( T^+ = T^{-1} \), and \( L^+ = L \).

As in the continuous case, one defines the factorization of the operator \( L \):

(I) \[ L + \alpha = (a_n + b_{n-1} T^{-1})(a_n + b_n T) \quad \text{or} \]

(II) \[ L + \alpha = (p_n + q_n T)(p_n + q_{n-1} T^{-1}). \]

Even without requiring the coefficients to be real, such a factorization will exist for any \( \alpha \). In order to find the values of \( a_n, b_n, p_n, \) and \( q_n \) from \( v_n \) and \( c_n \), we arrive at the difference analog of the Riccati equation, which appears when we factorize the continuous Schrödinger operator of the form

\[ -\partial_x^2 + u(x) + \alpha = -(\partial_x + v)(\partial_x - v), \quad v^2 + v_x = \alpha + u(x). \]
In the discrete case we have

(I) \[ v_n + \alpha = a_n^2 + b_{n-1}^2, \quad c_n = a_nb_n, \]

(II) \[ v_n + \alpha = p_n^2 + q_{n-1}^2, \quad c_n = q_np_n. \]

If all the coefficients \( a_n \) and \( b_n \) are real, then

(I) \[ L + \alpha = QQ^+, \quad Q^+ = a_n + b_nT. \]

Similarly, if all the coefficients \( p_n \) and \( q_n \) are real,

(II) \[ L + \alpha = RR^+, \quad R^+ = p_n + q_nT^{-1}. \]

The Darboux–Bäcklund transformations \( B_\alpha \), by definition, are given by

(I) \[ L \rightarrow Q^+Q = \tilde{L} = B^{(I)}_\alpha L, \]

(II) \[ L \rightarrow R^+R = \tilde{L} = B^{(II)}_\alpha L. \]

It is obvious that these transformations can be taken to be inverse to each other after an appropriate factorization:

\[ B^{(II)}_0(B^{(I)}_\alpha L) = L + \alpha, \quad B^{(I)}_0(B^{(II)}_\alpha L) = L + \alpha. \]

(Recall that in contrast with the two-dimensional case, the factorization here requires solving a Riccati type equation and is therefore nonunique.) The cyclic chains, just as in the continuous case, are determined from the condition:

\[ L_N = B_{\alpha_N} \cdots B_{\alpha_0} L, \quad L_j = B_{\alpha_j}L_{j-1}, \quad L_0 = L_N. \]

It is not difficult, by following [VS1], to prove the following

**Proposition 1.** Cyclic chains satisfying the condition \( \sum_{j=1}^N \alpha_j = 0 \) consist of finite-zone difference operators \( L_j \) corresponding to one and the same Riemann surface of genus \( g \leq \lceil N/2 \rceil \) (see [DMN, DKN2]).

More interesting are the chains for which \( \sum_{j=1}^N \alpha_j = h > 0 \).

**The first difference analog of the harmonic oscillator.** Already for \( N = 1 \) nontrivial phenomena arise. Consider the cyclicity conditions

\[ L = QQ^+ - h, \quad \tilde{L} = Q^+Q = B_hL, \quad Q^+Q = QQ^+ + h, \quad Q^+ = a_n + b_nT. \quad (1) \]

From (1) we obtain an equation of Riccati type, which implies

\[ a_n = a = \text{const}, \quad b_n^2 - b_{n-1}^2 = h \quad (2) \]

or \( b_n = \sqrt{nh + b_0^2} \). Suppose that further

\[ a = 1. \quad (3) \]

We come to the following conclusion: the operator \( L = QQ^+ \) is not defined in \( L_2(\mathbb{Z}) \) as a real operator.
Proposition 2. 1 The operator \( L = QQ^+ \) described by formulas (1)–(3) determines a real selfadjoint operator \( L \) in the space \( \mathcal{H}_l \subset L^2(\mathbb{Z}) \) if and only if \( l \in \mathbb{Z} \) (the quantization condition).

(Here \( \psi_n \in \mathcal{H}_l \) if and only if \( \psi_n = 0 \) for \( n \leq -b_0^2/h = l \)). The space \( \mathcal{H}_l \) is isomorphic to \( L^2(\mathbb{Z}^+) \) with zero boundary condition for \( k = 0 \).

Proof. If the number \( l \) is an integer, then

\[
Q^\pm_l = (1 \pm \sqrt{(n-l)hT}).
\]

Setting \( n = l + k \), we come to \( \mathbb{Z}^+ \). It is readily verified by substitution that the operators \( L^\pm = Q^\pm Q^\pm_l \) take the space \( \mathcal{H}_l \) to itself if and only if \( l \) is an integer.

We come to the following operators in \( L^2(\mathbb{Z}^+) \):

\[
L^\pm = Q^\pm Q^\pm_l, \quad Q^\pm_l = (1 \pm \sqrt{h}kT).
\]

The spectrum of the operator \( L^\pm = Q^\pm Q^\pm_l \) in the space \( L^2(\mathbb{Z}^+) \) is the same as that of the ordinary harmonic oscillator. The ground state will be

\[
L^\pm \psi_0^\pm = 0, \quad Q^\pm_0 \psi_0^\pm = 0, \quad \psi_{0k}^\pm = \begin{cases} (\pm 1)^{k-1}/\sqrt{h^{k-1}(k-1)!}, & k \geq 1, \\ 0, & k \leq 0. \end{cases}
\]

The higher eigenfunctions have the form

\[
\psi_m^\pm = Q^m_\pm \psi_0^\pm, \quad L^\pm \psi_m^\pm = mh\psi_m^\pm, \quad m \in \mathbb{Z}^+.
\]

Lemma 1. The eigenfunctions \( \psi_m^\pm \) of the operator \( L^\pm \) have the form

\[
\psi_{m\pm}^\pm = \frac{(\pm 1)^{k-1}}{\sqrt{h^{k-1}(k-1)!}} \cdot P_m(k) \cdot \Theta(k).
\]

Here the \( P_j(k) \) are polynomials such that

\[
P_j(k) = (1 - h(k-1)T^{-1})P_{j-1}(k), \quad P_0 \equiv 1, \quad \text{and} \quad \Theta(k) = \begin{cases} 1, & k \geq 1, \\ 0, & k \leq 0. \end{cases}
\]

The expression \( \psi_0^\pm \) is a Poisson distribution, while the polynomials \( P_j(k) \) are orthogonal with Poisson weight \( \psi_0^2 \) on \( \mathbb{Z}^+ \). Undoubtedly, they are known, although their relationship with harmonic oscillators and bosonic commutation relations, most probably, was never discussed.2

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1 Similar operators were constructed in [AS90]. However, the analysis of the corresponding formulas does not reveal if their authors have the same thing in mind as we do. Apparently our results are new, at least methodically.

2 These polynomials are known as the Charlet polynomials.
The second difference analog of the harmonic oscillator. Let \( N = 1 \). Put \( L = QQ^+ \) and \( \tilde{L} = Q^+Q \). We shall introduce a family of operators \( L \) depending on two constants \( c, a \in \mathbb{R} \):

\[
L(c, a) = Q(c, a)Q^+(c, a), \quad \tilde{L}(c, a) = Q^+(c, a)Q(c, a)
\]

so as to have

\[
a^2Q^+(c, a)Q(c, a) = Q(ca^2, a)Q^+(ca^2, a) + D, \quad D = a^2 - 1. \tag{4}
\]

Let us put

\[
Q^+ = 1 + ca^nT, \quad Q = 1 + ca^{n-1}T^{-1}, \quad a \neq 0, c \neq 0.
\]

Consider the transformation \( \tau: n \to 1 - n \). We have the formula

\[
\tau Q(c, a) = Q^+(c, a^{-1}) \tau. \tag{5}
\]

The relations (4)–(5) are extremely interesting. Apparently, they have not appeared previously.

**Theorem.** The spectrum of the operator \( L(c, a) \) for \( \lambda \) in the semi-interval \([0, 1)\) in the Hilbert space \( L^2(\mathbb{Z}) \) has the form

1. \( a > 1 \), \( \lambda_n = 1 - a^{-2n}, \quad n \geq 0 \).
2. \( a < 1 \), \( \lambda_n = 1 - a^{2n}, \quad n \geq 1 \).

The eigenfunctions are the following:

1. \( a > 1 \),
   \[
   \psi_{0k}(c, a) = (-1)^k c^{-k} a^{-(k-1)k/2}, \quad k \in \mathbb{Z},
   \]
   \[
   \psi_n(c, a) = Q(c, a)Q(ca^2, a) \cdots Q(ca^{2n-2}, a)\psi_0(ca^{2n}, a),
   \]
   \[
   \psi_{nk}(c, a) = P_n(k, c, a)\psi_{0k}(c, a),
   \]
   \[
   P_n(k, c, a) = (a^{-2k} - ca^{2n-2})^n, \quad n \geq 1.
   \]
2. \( a < 1 \),
   \[
   \psi_1(c, a) = \tau \psi_0 \left( \frac{1}{a^2}, \frac{1}{a} \right),
   \]
   \[
   \psi_n(c, a) = Q^+ \left( \frac{1}{a^2}, a \right) \cdots Q^+ \left( \frac{1}{a^2n^2-2}, a \right) \psi_1 \left( \frac{1}{a^2n-2}, a \right), \quad n \geq 2.
   \]

**Remarks.**

1. For \( \lambda \geq 1 \) the spectrum of the operators \( L(c, a) \) is not known to the authors. We conjecture that it is continuous.
2. If \( a > 1 \) and \( c < 0 \), then the operator \( L \) may be considered in \( L^2(\mathbb{R}) \) so that its restriction to the family of lattices \( \delta + \mathbb{Z} \subset \mathbb{R} \) yields our family \( L(c, a) \):

\[
L_-(m, \gamma) = (1 - \gamma^{x-m-1}T^{-1})(1 - \gamma^{x-m}T), \quad \gamma > 0, \ m \in \mathbb{Z}.
\]

The ground state \( \Phi_0(x) \ (L\Phi_0(x) = 0) \) is a function satisfying

\[
(1 - \gamma^{x-m}T)\Phi_0(x) = 0 \quad \text{or} \quad \Phi_0(x) = \gamma^{x-m}\Phi_0(x + 1).
\]
It has the form
\[ \Phi_0(x) = g(x)\gamma^{x(2m+1-x)/2}, \]
where \( g(x) = g(x+1) \) is any function of period 1.

Let us normalize the ground state by defining \( g(x) \) from a continuum of normalizing conditions so that the expression \( \Phi_0^2(x) \) is a normalized probability distribution on any lattice of the form \( \delta + \mathbb{Z} \subset \mathbb{R} \):

\[
g^{-1}(x) = \exp \left( \frac{ax(m+1)}{2} - \frac{ax^2}{2} \right) \Theta \left[ \frac{a(m+1)}{2} - ax \mid a \right],
\]
where \( a = 2\ln\gamma > 0 \) and \( \Theta[u \mid a] \) is a theta function:

\[
\Theta[u \mid a] = \sum_{n \in \mathbb{Z}} \exp \left( -\frac{an^2}{2} + nu \right).
\]

It follows from the analytical properties of theta functions ([BE]) that \( g(x) \) is a smooth function with period 1. The normed ground state is given by the formula

\[
\Phi_0(x) = \frac{\gamma^{x(x-1)/2}}{\Theta[(m+1-2x)\ln\gamma \mid 2\ln\gamma]}.
\]

Thus the cyclic chains are of two types:

1) \( L = B_{\alpha N} \cdots B_{\alpha} L = L' \), where \( L \) acts in \( \mathcal{L}_2(\mathbb{Z}) \) for an appropriate “quantization” of the parameters;

2) the operator \( L' \) coincides with \( L \) after multiplication by a constant and a translation along \( x \) in the natural realization in \( \mathcal{L}_2(\mathbb{R}) \), where the restriction to the lattices \( (\delta + \mathbb{Z}) \subset \mathbb{R} \) generates a family of discrete operators participating in the definition of the cyclic chain similarly to the case \( N = 1 \) considered above.

\textit{Added in proof.} In the paper [SVZ], the problem was essentially posed already. However, that paper contains unmotivated restrictions. For example, the constant \( \delta = \ln |c| / \ln a \) is assumed rational in [SVZ]. Further, the assertion in [SVZ] according to which the relation (4) above or the corresponding relation (14) in [SVZ] and its consequences (15)–(17) “clearly define a spectrum generating algebra” is incorrect. This assertion is certainly false, for the case in which \( \psi_0 \) satisfies the equation \( Q^+ \psi_0 = 0 \) and growth exponentially. This is indeed the case when \( q > 1 \), a situation omitted in the papers [AS91, SVZ]. In this situation the authors have found a different ground state with eigenvalue \( 1 - q \) \((q^{-1} = a^2)\) by using the additional symmetry \( \tau \) (see Theorem 2).

Moreover, the assertion that the relations mentioned above “clearly define a spectrum generating algebra” is incorrect for another reason: it gives no information on the spectrum of the Schrödinger operator \( L = QQ^+ \) with \( \lambda \geq 1 \) in the Hilbert space \( \mathcal{L}_2(\mathbb{Z}) \). According to our conjecture (see Remark 1 above) this spectrum is continuous and occupies the entire strip \( \lambda \geq 1 \).

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