Fusion rules for the continuum sectors of the Virasoro algebra with $c = 1$

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Abstract

The Virasoro algebra with $c = 1$ has a continuum of superselection sectors characterized by the ground state energy $h \geq 0$. Only the discrete subset of sectors with $h = s^2$, $s \in \frac{1}{2} \mathbb{N}_0$, arises by restriction of representations of the $SU(2)$ current algebra at level $k = 1$. The remaining continuum of sectors is obtained with the help of (localized) homomorphisms into the current algebra. The fusion product of continuum sectors with discrete sectors is computed. A new method of determining the sector of a state is used.

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1 Introduction

“Fusion rules” describe the product of two superselection charges and the decomposition of the product into irreducible charges. They thus constitute an important characteristics for the charge structure of a quantum field theory.

The general definition of the composition of charges (“DHR product”) was first given in \cite{DHR}. In two-dimensional conformal quantum field theory, other notions of fusion \cite{Fuchs, Runkel} became more popular, but every evidence shows \cite{Fuchs, Runkel} that these describe the same abstract charge structure.

The actual computation of the fusion rules in concrete models is in general a difficult task, and almost always relies on some specific apriori knowledge. If the QFT at hand is the fixpoint subalgebra of another QFT with respect to a compact gauge group, then

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harmonic analysis determines the composition law for those sectors which appear in the decomposition of the vacuum sector of the larger algebra \[2\]. The fusion rules then follow the composition of the representations of the gauge group. In low-dimensional theories, a gauge group is in general not present, but in favorable cases, modular transformation properties \[6\] or “null vectors” \[3, 4\] can be exploited.

In the present letter we treat a model where the standard strategies are not applicable: the chiral stress-energy tensor of a 1+1-dimensional conformal quantum field theory with \(c = 1\). (A chiral field can be treated like a “one-dimensional QFT”.) Its algebra \(A\) is the fixpoint algebra of the chiral \(SU(2)\), level \(k = 1\), current algebra \(B\) with respect to its global \(SU(2)\) symmetry \[8, 9\], and the positive-energy representations of the current algebra contain a discrete series of superselection sectors of \(A\). But besides the discrete series there is a continuum of further sectors which do not arise by restriction from \(B\). These sectors have no “null vectors” and hence infinite asymptotic dimension \[9\], so that the Verlinde formula or Nahm’s prescription are not applicable.

We adopt a method due to Fredenhagen \[10\] for the computation of the fusion rules: A charged state \(\omega\) is described by a positive map \(\chi\) of the algebra into itself such that

\[
\omega = \omega_0 \circ \chi
\]

where \(\omega_0\) is the vacuum state. The correspondence between states and positive maps is 1:1 provided the charge is strictly localized. This yields a product of states defined by

\[
\omega_1 \times \omega_2 := \omega_0 \circ \chi_1 \circ \chi_2.
\]

The GNS representation \(\pi_{\omega_1 \times \omega_2}\) is always a subrepresentation of the DHR product of GNS representations \(\pi_{\omega_1} \times \pi_{\omega_2}\) \[10\], and is expected to exhaust it as the positive maps vary within their equivalence class.

For two states \(\omega_1\) and \(\omega_2\) belonging to the discrete and continuous sectors, respectively, we shall determine (by a new method) the sectors to which the product states belong.

## 2 Fusion rules for \(c = 1\)

The superselection sectors of the stress-energy tensor with \(c = 1\) are uniquely determined by their ground state energy \(h \geq 0\) for the conformal Hamiltonian \(L_0\). The sectors \([h = s^2]\) with \(s \in \mathbb{N}_0\), arise as subrepresentations of the vacuum representation of the \(SU(2)\) current algebra \(B\), and those with \(s \in \mathbb{N}_0 + \frac{1}{2}\) arise in the spin-\(\frac{1}{2}\) representation of \(B\). Those with \(h \notin (\frac{1}{2}\mathbb{Z})^2\) constitute the continuum. For each of these representations, the partition function is well known \[11\]:

\[
\text{Tr} \exp(-\beta \pi_h(L_0)) = \begin{cases} 
    t^h p(t) & \text{if } h \notin (\frac{1}{2}\mathbb{Z})^2, \\
    (t^{s^2} - t^{(s+1)^2}) p(t) & \text{if } h = s^2, \quad s \in \frac{1}{2}\mathbb{N}_0,
\end{cases}
\]
where \( t = e^{-\beta} \) and \( p(t) = \prod_n (1 - t^n)^{-1} \).

The positive maps describing the charged states are of the form (cf. Lemma 2.1)
\[
\chi = \mu \circ \alpha_g|_A.
\]
Here \( g \) is a smooth \( SU(2) \) valued function, and \( \alpha_g \) the automorphism of the current algebra \( B \) induced from the local gauge transformation (Bogolyubov automorphism) of the underlying chiral fermion doublet,
\[
\psi_i(x) \mapsto \sum_j \psi_j(x) g_{ji}(x).
\]
\( \mu = \int d\mu(k) \gamma_k \) is the average over the global gauge group \( SU(2) \) acting by automorphisms \( \gamma_k \). Since \( \mu \) is a positive map of \( B \) onto \( A \), \( \chi_g \) is a positive map of \( A \) onto \( A \).

The induced action of \( \alpha_g \) on the currents \( j(f) \equiv \sum j^a(f_a) = \int :\psi(x)f(x)\psi(x)^* : dx \) (with an \( su(2) \) valued test function \( f(x) = \sum f_a(x) T^a \)) is explicitly computed as
\[
\alpha_g(j(f)) = j(gfg^{-1}) - \frac{i}{2\pi} \int Tr(fg^{-1}\partial g)1,
\]
and its restriction to the Sugawara stress-energy tensor \( T = \frac{\pi}{3} \sum g_{ab} : j^a j^b : \) is
\[
\alpha_g(T(f)) = T(f) - ij(f\partial g^{-1}) - \frac{1}{4\pi} \int f Tr(\partial gg^{-1}\partial gg^{-1})1.
\]
The central terms arise, of course, from normal ordering. To be specific, we choose the functions
\[
g_q(x) = \begin{pmatrix}
\exp(iq\lambda(x)) & 0 \\
0 & \exp(-iq\lambda(x))
\end{pmatrix}
\]
where \( \lambda(x) = -i \log \frac{1 + ix}{1 - ix} \) interpolates between \( \lambda(-\infty) = -\pi \) and \( \lambda(+\infty) = +\pi \), and \( q \) is a real parameter whose role as a charge will be exhibited in Lemma 2.1.

At this point, we have to distinguish the quasilocal algebras \( A_{\text{local}} \) and \( B_{\text{local}} \) generated by field operators smeared with test functions, and the global algebras \( A_{\text{global}} \) and \( B_{\text{global}} \) generated by field operators smeared with “admissible” functions which are test functions up to polynomials of order \( 2(d - 1) \) where \( d \) is the scaling dimension. It is well known\(^1\) that the fields as distributions extend to these enlarged test function spaces, so that
\[
L_n = \frac{1}{2} \int (1 - ix)^{1-n}(1 + ix)^{1+n} T(x) dx \quad \text{and} \quad Q_n^a = \int (1 - ix)^{-n}(1 + ix)^n j^a(x) dx
\]
\(^1\)It appears that one could also use the embedding \( T = \pi : jj : \) of \( A \) into a \( U(1) \) current algebra \( C \). The problem would be that the conditional expectation \( \mu \) which takes the homomorphisms \( \alpha_g : A \to C \) back onto \( A \) in order to obtain \( \chi = \mu \circ \alpha_g|_A \) is not explicitly known in that case.
are defined as closed unbounded operators. The specific automorphisms \( \alpha_q \equiv \alpha_{q'} \) extend to the operators \( Q^3_n \in B_{\text{global}} \) and \( L_n \in A_{\text{global}} \):
\[
\alpha_q(Q^3_n) = Q^3_n + q \delta_{n,0}, \quad \alpha_q(L_n) = L_n + 2q Q^3_n + q^2 \delta_{n,0} \quad (q \in \mathbb{R}),
\]
but they extend to \( Q^\pm_n \in B_{\text{global}} \) only if \( q \in \frac{1}{2} \mathbb{Z} \):
\[
\alpha_q(Q^\pm_n) = Q^\pm_{n \pm 2q} \quad (q \in \frac{1}{2} \mathbb{Z}).
\]

(Our basis of \( SU(2) \) and hence of the fields \( j^a \) is such that \([Q^3_n, Q^3_m] = 2Q^3_{n+m} + n \delta_{n+m,0}, \]
\([Q^\pm_n, Q^\pm_m] = Q^\pm_{n+m}, [Q^3_n, Q^3_m] = \frac{1}{2} n \delta_{n+m,0} \).)

Our first Lemma establishes the relation between the parameters \( q \) and \( h \):

2.1. Lemma: The state \( \omega_q \equiv \omega_0 \circ \chi_q \equiv \omega_0 \circ \mu \circ \alpha_q |_A = \omega_0 \circ \alpha_q |_A \) is a ground state in the irreducible sector \([h = q^2] \).

Proof: Since the operators \( L_n \) and \( Q^3_n \) \((n \geq 0)\) annihilate the vacuum, \( \alpha_q(L_n) \) annihilate the vacuum for \( n > 0 \) and \( \alpha_q(L_0) \) has eigenvalue \( q^2 \). It follows that \( \omega_q \) is a ground state for \( L_0 \) with ground state energy \( q^2 \) Q.E.D.

Thus, in order to compute the fusion rules \([h_1] \times [h_2] \) (where \( h_i = q_i^2 \)) one has to determine the GNS representation for the product state
\[
\omega_{q_1} \times \omega_{q_2} = \omega_0 \circ \alpha_{q_1} \circ \mu \circ \alpha_{q_2} |_A = \int_{SU(2)} d\mu(k) \omega_0 \circ \alpha_{q_1} \circ \gamma_k \circ \alpha_{q_2} |_A.
\]

This state is a continuous mixture of states \( \omega_k \equiv \omega_0 \circ \alpha_k \) induced by the homomorphisms
\[
\alpha_k \equiv \alpha_{q_1} \circ \gamma_k \circ \alpha_{q_2} |_A
\]
of \( A_{\text{local}} \) into \( B_{\text{local}} \). (We suppress the explicit reference to the involved charges \( q_1 \) and \( q_2 \).)

These homomorphisms extend to \( A_{\text{global}} \) for generic \( k \in SU(2) \) only if \( q_1 \in \frac{1}{2} \mathbb{Z} \), as can be seen from the above transformation formulae. The following argument is more physical: If one evaluates \( \omega_k(T(f)^2) \) for test functions \( f \), then one finds that the contributions from the current two-point functions diverge for generic \( q \) as \( f \) is replaced by the function \( \frac{1}{2}(1 + x^2) \). Hence the operator \( L_0 = \frac{1}{2} \int (1 + x^2) T(x) dx \) has a finite expectation value but infinite variance in these states.

This is why we shall restrict ourselves to the case \( q_1 \in \frac{1}{2} \mathbb{Z} \). Since \( q \) and \( -q \) give rise to the same sector \([h = q^2] \), we shall even assume \( q_1 \in \frac{1}{2} \mathbb{N}_0 \).

Now we exploit the fact that \( \gamma_k \) is implemented by a unitary operator in \( B_{\text{global}} \) of the form \( U(k) = \exp(i \sum \kappa_a Q^3_a) \) on which \( \alpha_{q_1} \) is well defined. Hence
\[
\alpha_k = \text{Ad}(V(k)) \circ \alpha_{q_1} \circ \alpha_{q_2} |_A = \text{Ad}(V(k)) \circ \alpha_{q_1+q_2} |_A
\]
with \( V(k) = \alpha_{q_1}(U(k)) = \exp(i \sum \kappa_\alpha \alpha_{q_1}(Q_0^\alpha)) \). It is more convenient to express \( U(k) \) in the form

\[
U(k) = \exp(i \frac{k^*_2}{k_1} Q^-_0 k_1^2 Q^+_0) \exp(i \frac{k^*_2}{k_1} Q^+_0) \quad \text{for} \quad k = \left( \begin{array}{c} k_1 \\ ik_2^* \\ k_1^* \end{array} \right).
\]

\( (k_1^2 Q^0_0 \) is well defined since \( 2Q^3_0 \) has integer spectrum.) Application of \( \alpha_{q_1} \) yields

\[
V(k) = k_1^{q_1} \exp(i \frac{k^*_2}{k_1} Q^-_{2q_1}) k_1^{2Q^3_1} \exp(i \frac{k^*_2}{k_1} Q^+_0) k_1^{2Q^3_1}.
\]

2.2. Lemma: The product state \( \omega_{q_1} \times \omega_{q_2} \) is a convex integral over states \( \omega_0 \circ \alpha_k \), \( k \in SU(2) \). Each state \( \omega_0 \circ \alpha_k \) on \( A \) is a finite convex sum

\[
\omega_0 \circ \alpha_k = \sum_{\nu=0}^{2q_1} \frac{(2q_1 - \nu)!}{(2q_1)! \nu!} |k_1|^{2(2q_1 - \nu)} |k_2|^{2\nu} \omega^{(\nu)}_{q_1,q_2}
\]

of states

\[
\omega^{(\nu)}_{q_1,q_2}(\cdot) = \frac{(2q_1 - \nu)!}{(2q_1)! \nu!} (|Q^-_{2q_1}\rangle^\nu \Omega, \alpha_{q_1+q_2}(\cdot)(Q^-_{2q_1}\rangle^\nu \Omega).
\]

Since only the weights depend on the group element \( k \in SU(2) \), the product state \( \omega_{q_1} \times \omega_{q_2} \) is a finite convex sum of the same states \( \omega^{(\nu)}_{q_1,q_2} \).

Proof: The first statement just summarizes the precedent discussion. We have \( \omega_0 \circ \alpha_k = (V(k)^* \Omega, \alpha_{q_1+q_2}(\cdot)V(k)^* \Omega) \), and \( V(k)^* \Omega = (k_1^*)^{2q_1} \exp(-i \frac{k^*_2}{k_1} Q^-_{2q_1}) \Omega \) because \( Q^a_0 \) annihilate the vacuum for \( n \geq 0 \) (remember our choice \( q_1 \in \frac{1}{2} N_0 \)). The power series expansion of the exponential yields vectors \( (Q^-_{2q_1}\rangle^\nu \Omega \) with energy \( 2q_1 \nu \) and Cartan charge (the eigenvalue of \( Q^3_0 \)) \( C = -\nu \). These vectors vanish for \( \nu > 2q_1 \) because the vacuum Hilbert space \( H \) of \( B \) does not contain vectors with energy less than \( C^2 \). This fact is read off the following expression for the partition function for the vacuum representation:

\[
\text{Tr} \exp(-\beta L_0 - \eta Q^3_0) = \sum_{j \in N_0} \sum_{m=-j}^{j} z^m (t^j - t^{(j+1)^2})^p(t) \quad (z = e^{-\eta}, t = e^{-\beta})
\]

in which the power of \( t \) is always at least the square of the power of \( z \). Since \( \alpha_{q_1+q_2}(L_n) \) does not change the Cartan charge \( C \), the vectors \( (Q^-_{2q_1}\rangle^\nu \Omega \) have only diagonal matrix elements for \( \alpha_{q_1+q_2}(A) \), showing the convex decomposition. The proper normalization of the states \( \omega^{(\nu)}_{q_1,q_2}(1) = 1 \) can be checked recursively in \( \nu \).

Q.E.D.

The problem has thus been reduced to the determination of the GNS representations \( \pi^{(\nu)}_{q_1,q_2} \) for the states \( \omega^{(\nu)}_{q_1,q_2} \). One can easily compute that these states are eigenstates of \( L_0 \).
with energy \((q_1 + q_2)^2 - 2\nu q_2\), but they are not ground states in general. It is therefore not possible to determine the sectors directly via their ground state energies. Instead, it turns out to be possible to compute the partition function for the representations induced by these states. This is our main result.

2.3. Proposition: Let \(q_1 \in \frac{1}{2}\mathbb{N}_0\). If \(q_2 \notin \frac{1}{2}\mathbb{Z}\), then \(\pi_{q_1,q_2}^{(\nu)}\) is irreducible and belongs to the sector \(| h = (q_1 + q_2 - \nu)^2 \rangle\). If \(q_2 \in \frac{1}{2}\mathbb{Z}\), then \(\pi_{q_1,q_2}^{(\nu)}\) is a direct sum of sectors from the set \(\{ | h = s^2 \rangle : s \in | q_1 + q_2 - \nu | + \mathbb{N}_0 \}\).

Proof: The vector \((Q_{-2q_1})^\nu\Omega\) has Cartan charge \(C = -\nu\). This value is not changed by application of \(\alpha_{q_1+q_2}(L_n)\), hence \(\pi_{q_1,q_2}^{(\nu)}\) is a subrepresentation of the representation \(\alpha_{q_1+q_2}\) on the subspace \(H_{C=-\nu} = P_{-\nu}H\) of Cartan charge \(-\nu\) in the vacuum representation of \(B\). The partition function for the latter representation is

\[
\text{Tr} P_{-\nu} \exp(-\beta \alpha_{q_1+q_2}(L_0)) = e^{-(q_1+q_2)^2} \beta \cdot \text{Tr} P_{-\nu} \exp(-\beta L_0 - 2(q_1 + q_2)\beta Q_0^3).
\]

From the previous expression for the vacuum partition function, we obtain

\[
\text{Tr} P_{-\nu} \exp(-\beta L_0 - \eta Q_0^3) = z^{-\nu} t^{\nu^2} p(t) \quad (z = e^{-\eta}, t = e^{-\beta})
\]

by collecting the terms \(z^{-\nu}\), and hence

\[
\text{Tr} P_{-\nu} \exp(-\beta \alpha_{q_1+q_2}(L_0)) = t^{(q_1+q_2-\nu)^2} p(t).
\]

If \(q_1 + q_2 - \nu \notin \frac{1}{2}\mathbb{Z}\), then this is the partition function of the irreducible sector \(| h = (q_1 + q_2 - \nu)^2 \rangle\). Hence \(\alpha_{q_1+q_2}(A)\) acts irreducibly on \(H_{C=-\nu}\), and must coincide with its subrepresentation \(\pi_{q_1,q_2}^{(\nu)}\). If on the other hand \(q_1 + q_2 - \nu \in \frac{1}{2}\mathbb{Z}\), then the above equals the sum of the partition functions \((t^{s^2} - t^{(s+1)^2})p(t)\) of the sectors \(| h = s^2 \rangle\) with \(s \in | q_1 + q_2 - \nu | + \mathbb{N}_0\). Thus \(\pi_{q_1,q_2}^{(\nu)}\) is the direct sum of a subset of these sectors. Q.E.D.

As mentioned in the introduction, the product of states, computed here, might accidently not exhaust the DHR product. But this degeneracy disappears if the positive map \(\chi_2\) is perturbed by the adjoint action of some isometry \(a \in A\). We note that the argument leading to Prop. 2.3 is in fact stable if \(\chi_{q_2}\) is replaced by \(\text{Ad}(a^*)\circ \chi_{q_2}\). Namely, because \(a\) is \(SU(2)\)-invariant, one has \(\text{Ad}(a^*)\circ \gamma_k \circ \alpha_{q_2} = \text{Ad}(U(k)a^*) \circ \alpha_{q_2}\), so it is sufficient to replace in the above argument the vectors \((Q_{-2q_1})^\nu\Omega\) by the perturbed vectors \(\alpha_{q_1}(a) (Q_{-2q_1})^\nu\Omega\) which still belong to \(H_{C=-\nu}\). In the case \(q_2 \notin \frac{1}{2}\mathbb{Z}\), the perturbed GNS representation \(\pi_{q_1(q_2)}^{(\nu)}\) will still belong to the irreducible sector \(| h = (q_1 + q_2 - \nu)^2 \rangle\).

Thus, combining Lemma 2.2 with the Proposition, we obtain

2.4. Corollary: Let \(q_1 \in \frac{1}{2}\mathbb{N}_0\) and \(q_2 \in \mathbb{R} \setminus \frac{1}{2}\mathbb{Z}\). The fusion rules for the sectors \(| h_i = q_i^2 \rangle\) are

\[
[h_1] \times [h_2] = \bigoplus_{\nu=0}^{2q_1} [h(\nu)] \quad \text{with} \quad h(\nu) = (q_1 + q_2 - \nu)^2.
\]
3 Comments

We have studied the decomposition into irreducibles of the product of sectors (“fusion rules”) for the chiral stress-energy tensor with \( c = 1 \). We succeeded to compute the fusion rules for two sectors with ground state energies \( h_i \) where \( [h_1] \) is a special sector, \( h_1 \in (\frac{1}{2}N_0)^2 \), and \( [h_2] \) belongs to the continuum of sectors, \( h_2 \in \mathbb{R}_+ \setminus (\frac{1}{2}N_0)^2 \), Cor. 2.4. This result was not accessible by the prevailing methods for the computation of fusion rules. The case where both sectors belong to the continuum should in principle also be studied with the present method, but becomes technically very intricate.

When both sectors \([h_i]\) are special, we would have expected \( SU(2) \)-like fusion rules \(^4\) since the special sectors \([h = s^2]\), \( s \in \frac{1}{2}N_0 \), arise by restriction of the vacuum and spin-\( \frac{1}{2} \) representations of \( B \) to the fixpoint algebra \( A \) on the subspaces of \( SU(2) \) charge \( s \). This is, however, not reproduced by Prop. 2.3 and Lemma 2.2: Although the unperturbed states \( \omega_{q_1,q_2}^{(o)} \) have finite energy and hence only finitely many of the possible sectors according to Prop. 2.3 really contribute to them, this limitation will disappear if \( \chi_{q_2} \) is perturbed as described above. Moreover, if \( h_i = s_i^2 \) with \( 0 < s_2 < s_1 \), the sectors \([h = s^2]\) with \( 0 \leq s < |s_1 - s_2| \) should not occur according to \( SU(2) \), while they are not excluded by Prop. 2.3, and are really found to be present by more explicit computations.

This state of affairs has a simple explanation: For \( q \in \frac{1}{2}\mathbb{Z} \), the positive maps \( \chi_q \) transfer not only the \( SU(2) \) charge \( s = |q| \) but in fact, as explained below, a mixture of all charges \( s \in |q| + \mathbb{N}_0 \). These admixtures are not seen if evaluated in the vacuum state (Lemma 2.1), but become visible if evaluated in a generic state of \( A \), e.g., upon perturbation of \( \chi_q \). The product states \( \omega_0 \circ \chi_q \circ \chi_{q_2} \), too, are sensitive to admixtures to \( \chi_{q_2} \), which accounts for the presence of “too many” sectors contributing to the fusion rules as inferred from Lemma 2.2 and Prop. 2.3.

Let us explain why \( \chi_q \) is capable of transferring the “wrong” charges if \( q \in \frac{1}{2}\mathbb{Z} \), but not if \( q \notin \frac{1}{2}\mathbb{Z} \), and why this is not in conflict with the statement in \(^{12}\) that the correspondence between states and positive maps is 1:1. The argument is very similar to the one in the proof of Prop. 2.3. If \( \chi_q \) is evaluated in some perturbed state \( \omega = (a\Omega, \cdot a\Omega) \) with \( a \in A \), we have \( \omega \circ \chi_q = \omega \circ \alpha_q \) since \( a \) and \( \omega_0 \) are \( SU(2) \) invariant. Thus the GNS representation \( \pi_\omega \) for \( \omega \) is a subrepresentation of the representation \( \alpha_q \) on the subspace \( H_{C=0} = P_0H \) of Cartan charge \( C = 0 \) in the vacuum representation of \( B \) (to which \( a\Omega \) belongs). The partition function for this representation has been computed above (putting \( q_1 = 0, \nu = 0, q_2 = q \)):

\[
\text{Tr} \ P_0 \exp(-\beta \alpha_q(L_0)) = t^{q^2} p(t).
\]

This is the character of the irreducible representation \([h = q^2]\) if \( q \notin \frac{1}{2}\mathbb{Z} \), but is the sum of infinitely many irreducible characters for \([h = s^2], s \in |q| + \mathbb{N}_0 \), if \( q \in \frac{1}{2}\mathbb{Z} \).

By testing with suitable operators \( a \in A_{\text{global}} \), one finds that the “wrong” sectors are indeed present. Remember that the 1:1 correspondence between states and positive maps
requires that the charge is strictly localized, while the automorphisms $\alpha_q$ in our analysis are only asymptotically localized (the derivative $\partial g_q(x)$ vanishes asymptotically). Of course our choice for $\alpha_q$ was dictated by the simplicity of the transformation formulae for $L_n$ and $Q^a_n$. The unpleasant feature of the wrong sectors is the price for that simplification.

The fusion rules in Cor. 2.4 are not affected by this complication.

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References

[1] S. Doplicher, R. Haag, J.E. Roberts: Local observables and particle statistics I, Commun. Math. Phys. 23, 199-230 (1971), and II, Commun. Math. Phys. 35, 49-85 (1974).
[2] S. Doplicher, R. Haag, J.E. Roberts: Fields, observables and gauge transformations I, Commun. Math. Phys. 13, 1-23 (1969).
[3] A.A. Belavin, A.M. Polyakov, A.B. Zamolodchikov: Infinite conformal symmetry in two-dimensional quantum field theory, Nucl. Phys. B 241, 333-380 (1984).
[4] W. Nahm: Quasi-rational fusion products, Int. J. Mod. Phys. 8, 3693-3702 (1994).
[5] K. Fredenhagen, K.-H. Rehren, B. Schroer: Superselection sectors with braid group statistics and exchange algebras I, Commun. Math. Phys. 125, 201-226 (1989), and II, Rev. Math. Phys. SI1 (special issue), 113-157 (1992).
[6] A. Wassermann: Operator algebras and conformal field theory III, Invent. Math. 133, 467-538 (1998).
[7] E. Verlinde: Fusion rules and modular transformations in 2D conformal field theory, Nucl. Phys. B 330, 360-376 (1988).
[8] I.B. Frenkel, Representations of Kac-Moody algebras and dual resonance models, in: Lect. Notes Appl. Math. 21, eds. M. Flato et al., AMS, Providence, RI, 1985, pp. 325-353.
[9] K.-H. Rehren: A new view of the Virasoro algebra, Lett. Math. Phys. 30, 125-130 (1994).
[10] K. Fredenhagen: Product of states, in: Groups and Related Topics, eds. R. Gielerrak et al., Kluwer Academic Press, Dordrecht, 1992, pp. 199-209.
[11] V. Kac: Infinite Dimensional Lie Algebras, Birkhäuser Verlag, Basel, 1983.
[12] M. Lüscher, G. Mack: Global conformal invariance in quantum field theory, Commun. Math. Phys. 41, 203-234 (1975).
[13] H.R. Tuneke: Produkt von Superauswahlsektoren des chiralnen Energie-Impuls-Tensors mit $c = 1$, Diploma thesis, Göttingen, 2000 (in German).