VOLUME PROPERTIES AND RIGIDITY ON SELF-EXPANDERS OF MEAN CURVATURE FLOW

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Abstract. In this paper, we mainly study immersed self-expander hypersurfaces in Euclidean space whose mean curvatures have some linear growth controls. We discuss the volume growths and the finiteness of the weighted volumes. We prove some theorems that characterize the hyperplanes through the origin as self-expanders. We estimate upper bound of the bottom of the spectrum of the drifted Laplacian. We also give the upper and lower bounds for the bottom of the spectrum of the $L$-stability operator and discuss the $L$-stability of some special self-expanders. Besides, we prove that the surfaces $\Gamma \times \mathbb{R}$ with the product metric are the only complete self-expander surfaces immersed in $\mathbb{R}^3$ with constant scalar curvature, where $\Gamma$ is a complete self-expander curve (properly) immersed in $\mathbb{R}^2$.

1. INTRODUCTION

In this article we study self-expanders that are self-expanding solutions for mean curvature flows. An $n$-dimensional smooth immersed submanifold $\Sigma$ in the Euclidean space $\mathbb{R}^m$ is called self-expander if its mean curvature vector $H$ satisfies the equation

\begin{equation}
H = \frac{1}{2} x^\perp,
\end{equation}

where $x^\perp$ denotes the normal component of the position vector $x$.

Equivalently, $\Sigma$ is a self-expander if and only if $\sqrt{t} \Sigma, t \in (0, \infty)$ is a mean curvature flow (MCF).

Self-expanders have a very important role in the study of MCF. They describe the asymptotic longtime behavior for MCF and the local structure of MCF after the singularities in the very short time. In [12], Ecker and Huisken studied MCF evolutions of entire graphical immersions. Under some assumptions on the initial hypersurface at infinity, they showed that the solution of MCF exists for all times $t > 0$ and converges to a self-expander. Stavrou [20] later proved the same result under weaker hypotheses that the initial hypersurface has a unique tangent cone at infinity.

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Self-expanders also appears in the mean curvature evolution of cones. In \cite{17}, Ilmanen studied the existence of E-minimizing self-expanding hypersurfaces which converge to prescribed closed cones at infinity. In \cite{11}, Ding studied self-expanders and their relationship to minimal cones in Euclidean space. Recently, Bernstein and Wang (\cite{3} and \cite{4}) obtained various results on asymptotically conical self-expanders. There are other works in self-expanders. See, for instance, \cite{1}, \cite{13}, \cite{21}, etc.

Recently in \cite{10}, the second author of the present paper and Zhou studied some properties of complete properly immersed self-expanders. Especially, they proved the discreteness of the spectrum of the drifted operator $\mathcal{L} = \Delta + \frac{1}{2} \langle x, \nabla \cdot \rangle$. In the case of self-expander hypersurfaces, they gave the lower bound estimate for the first eigenvalue $\lambda_1$ of the operator $\mathcal{L}$ and also proved that the bottom $\mu_1$ of the spectrum of the stability operator $L = \mathcal{L} + |A|^2 - \frac{1}{2}$ satisfies $\mu_1 \leq \lambda_1 + \frac{1}{2}$ (\cite{10} Theorems 1.3, 1.5). Besides, they proved the uniqueness of hyperplanes through the origin for mean convex self-expanders under some integrability conditions on the square of the norm of the second fundamental form (\cite{10} Theorems 1.4).

Motivated by the work in \cite{10}, in the present paper we study the topics discussed in \cite{10}. One of our strategies is to make use of the properties on the finiteness of weighted volumes and the volume growth upper estimate for self-expanders with some restriction on mean curvature. In order to prove these properties, we first prove Theorem 3.1 on a general Riemannian manifold which generalizes Theorem 5 in \cite{2} and is also of independent interest. Then we apply Theorem 3.1 to self-expanders in $\mathbb{R}^m$ and obtain the following result:

**Theorem 1.1.** Let $\Sigma$ be a complete $n$-dimensional properly immersed self-expander in $\mathbb{R}^m$, $n < m$. Assume that its mean curvature vector $H$ satisfies $|H|(x) \leq a|x| + b$, $x \in \Sigma$, for some constants $0 \leq a < \frac{1}{2}$ and $b > 0$. Then it holds that, for any $\alpha > \frac{4a^2}{1-4a}$,

1. $\int_{\Sigma} e^{-\frac{a}{4} |x|^2} d\sigma < \infty$.
2. The volume of $B_r(0) \cap \Sigma$ satisfies

\[
\text{Vol}(B_r(0) \cap \Sigma) \leq C(\alpha)e^{\frac{\alpha}{4} r^2},
\]

where $B_r(0)$ denotes the round ball in $\mathbb{R}^m$ of radius $r$ centered at the origin $0 \in \mathbb{R}^m$.

In particular, if $0 \leq a \leq \frac{1}{2\sqrt{2}}$, then the Gaussian weighted volume is finite, that is,

\[
\int_{\Sigma} e^{-\frac{a}{4} |x|^2} d\sigma < \infty.
\]
Remark 1.1. The partial conclusion in Theorem 1.1 ($\int_{\Sigma} e^{-\frac{|x|^2}{4a^2}} d\sigma < \infty$ if $0 \leq a < \frac{1}{2\sqrt{2}}$) can also be proved using Theorem 5 in [2]. It is noted that the restriction $a_2 < \frac{1}{4}$ should be added in the assumption of Theorem 5 in [2].

A natural question arises, whether (1.2) holds for any complete properly immersed self-expanders. In this direction, we have the following result:

Theorem 1.2. Let $\Sigma$ be a complete $n$-dimensional properly immersed self-expander in $\mathbb{R}^m$, $n < m$. If there are some constants $0 \leq a < \frac{1}{2}$, $b \geq 0$ and $r_0 > 0$ such that the mean curvature vector of $\Sigma$ satisfies that $|H|(x) \geq a|x| + b$ for $x \in \Sigma \setminus B_{r_0}(0)$, then

$$\int_{\Sigma} e^{-\frac{|x|^2}{4a^2}} d\sigma = \infty.$$  

In particular, if there are some constants $b \geq 0$ and $r_0 > 0$ such that the mean curvature vector of $\Sigma$ satisfies that $|H|(x) \geq \frac{1}{2\sqrt{2}} |x| + b$ for $x \in \Sigma \setminus B_{r_0}(0)$, then

$$\int_{\Sigma} e^{-\frac{|x|^2}{4}} d\sigma = \infty.$$  

In this paper, the notation $A$ denotes the second fundamental form of $\Sigma$. We obtain the following rigidity property of hyperplanes as self-expanders.

Theorem 1.3. Let $\Sigma$ be a complete properly immersed self-expander hypersurface in $\mathbb{R}^{n+1}$. Assume that its mean curvature $H$ satisfies $|H|(x) \leq a|x| + b$, $x \in \Sigma$, for some constants $0 \leq a < \frac{1}{2}$ and $b > 0$. If there exists $\beta > 0$, such that

$$|A|^2 H^2 + \frac{1}{2} H^2 + \beta A(x^T, x^T) H \leq 0,$$

then $\Sigma$ must be a hyperplane $\mathbb{R}^n$ through the origin, where $x^T$ denotes the tangent component of the position vector $x$.

Theorem 1.3 is a consequence of a more general result (see Theorem 4.1).

We also prove the following result:

Theorem 1.4. Let $\Sigma$ be a complete properly immersed self-expander hypersurface in $\mathbb{R}^{n+1}$. Assume that its mean curvature $H$ is bounded from below and satisfies $H(x) \leq a|x| + b$, $x \in \Sigma$, for some constants $0 \leq a < \frac{1}{2}$ and $b > 0$. If there exists $\alpha > \frac{4a^2}{1-4a^2}$ such that

$$|A|^2 H + \frac{H}{2} + \frac{\alpha + \frac{1}{4}}{4} A(x^T, x^T) \geq 0,$$

then $\Sigma$ must be a hyperplane $\mathbb{R}^n$ through the origin, where $x^T$ denotes the tangent component of the position vector $x$.  


In Section 5 of this paper, we study the problems related to the spectrum of the drifted Laplacian $\mathcal{L}$. In [10], the second author of the present paper and Zhou ([10, Theorems 1.1 and 1.3]) proved that the spectrum of the drifted Laplacian $\mathcal{L}$ on a properly immersed $n$-dimensional self-expander $\mathbb{R}^{n+k}$, $k \geq 1$, is discrete. In particular, the bottom $\lambda_1$ of the spectrum of $\mathcal{L}$ is the first weighted $L^2$ eigenvalue of $\mathcal{L}$. Further, for codimension 1 case, they proved that $\lambda_1 \geq \frac{n}{2} \inf_{x \in \Sigma} H^2$ and this lower bound is achieved if and only if the self-expander is the hyperplane through the origin. In this paper we give an upper bound for the bottom of the spectrum of the drifted Laplacian $\mathcal{L}$ and discuss the rigidity of the upper bound. More precisely, we prove that

**Theorem 1.5.** Let $\Sigma$ be a complete properly immersed self-expander hypersurface in $\mathbb{R}^{n+1}$. Assume that its mean curvature $H$ satisfies $|H|(x) \leq a|x| + b$, $x \in \Sigma$, for some constants $0 \leq a < \frac{1}{2\sqrt{2}}$ and $b > 0$. Then the bottom $\lambda_1$ of the spectrum of the drifted Laplacian $\mathcal{L} = \Delta + \frac{1}{2} \langle x, \nabla \cdot \rangle$ on $\Sigma$, i.e., the first weighted $L^2$ eigenvalue of $\mathcal{L}$ satisfies

$$\lambda_1 \leq \frac{n}{2} + \frac{\int_{\Sigma} H^2 e^{-\frac{|x|^2}{4}} d\sigma}{\int_{\Sigma} e^{-\frac{|x|^2}{4}} d\sigma},$$

with equality if and only if $\Sigma$ is the hyperplane $\mathbb{R}^n$ through the origin.

**Remark 1.2.** If $0 \leq a < \frac{1}{2}$, we have a general upper bound estimate for $\lambda_1$. See Theorem 5.1.

In this paper, we also study the $L$-stability operator for self-expanders:

$$L = \mathcal{L} + |A|^2 - \frac{1}{2}.$$

It is well known that a self-expander is noncompact (see, e.g., [3]). Thus the bottom $\mu_1$ of the spectrum of the operator $L$ may take $-\infty$ and for $\mu_1 > -\infty$, it may not be the lowest weighted $L^2$-eigenvalue for $L$. If $\mu_1 \geq 0$, $\Sigma$ is called $L$-stable. $L$-stability means that the second variation of its weighted volume is nonnegative for any compactly supported normal variation. In this paper, we obtain a lower bound for $\mu_1$ as follows:

**Theorem 1.6.** Let $\Sigma$ be a complete immersed self-expander hypersurface in $\mathbb{R}^{n+1}$. Then

$$\mu_1 \geq \frac{n + 1}{2} + \inf_{x \in \Sigma} \text{Scal}_\Sigma,$$

where $\inf_{x \in \Sigma} \text{Scal}_\Sigma$ denotes the infimum of the scalar curvature $\text{Scal}_\Sigma = H^2 - |A|^2$ of $\Sigma$.

Moreover, the equality in (1.5) holds if $\Sigma$ is also properly immersed, has constant scalar curvature and satisfies $|H|(x) \leq a|x| + b$, $x \in \Sigma$, for some constants $0 \leq a < \frac{1}{2\sqrt{2}}$ and $b > 0$. 
In [10], the second author of the present paper and Zhou proved that the mean convex self-expanders are $L$-stable. Here Theorem 1.6 implies that

**Corollary 1.1.** Let $\Sigma$ be a complete immersed self-expander hypersurface in $\mathbb{R}^{n+1}$. If the scalar curvature of $\Sigma$ satisfies

$$\text{Scal}_\Sigma \geq -\frac{n+1}{2},$$

then $\Sigma$ is $L$-stable.

The authors in [10] also proved that the inequality $\mu_1 \leq \lambda_1 + \frac{1}{2}$ holds on complete properly immersed self-expander hypersurfaces. In this paper, we obtain another upper bound for $\mu_1$. More precisely,

**Theorem 1.7.** Let $\Sigma$ be a complete properly immersed self-expander hypersurface in $\mathbb{R}^{n+1}$. Assume that its mean curvature $H$ satisfies $|H(x)| \leq a|x| + b$, $x \in \Sigma$, for some constants $0 \leq a < \frac{1}{2\sqrt{2}}$ and $b > 0$. Then the bottom $\mu_1$ of the spectrum of the $L$-stability operator $L$ satisfies

$$\mu_1 \leq \frac{n+1}{2} + \frac{\int_{\Sigma} \text{Scal}_\Sigma e^{-\frac{|x|^2}{4}} d\sigma}{\int_{\Sigma} e^{-\frac{|x|^2}{4}} d\sigma}. \quad (1.6)$$

If $\mu_1 > -\infty$, then $\int_{\Sigma} \text{Scal}_\Sigma e^{-\frac{|x|^2}{4}} d\sigma < \infty$ and the equality $(1.6)$ holds if and only if the curvature $\text{Scal}_\Sigma$ of $\Sigma$ is constant.

**Remark 1.3.** In the more general case of $0 \leq a < \frac{1}{2}$, we also obtain a general upper bound estimate for $\mu_1$. See Theorem 7.1.

Theorems 1.6 and 1.7 have the following consequence:

**Corollary 1.2.** Let $\Gamma$ be a complete immersed self-expander curve in $\mathbb{R}^2$ and $\Sigma$ be the self-expander hypersurface $\Gamma \times \mathbb{R}^{n-1}$ with the product metric, where $n \geq 1$. Then the bottom $\mu_1$ of the spectrum of the $L$-stability operator on $\Sigma$ is $\frac{n+1}{2}$. In particular, the bottom of the spectrum of the $L$-stability operator of $\Gamma$ is 1.

**Remark 1.4.** Note that in Corollary 1.2 we do not assume any hypothesis on the mean curvature.

Noting that the equality in $(1.6)$ holds if and only if the scalar curvature $\text{Scal}_\Sigma$ is constant, we are interested in characterizing self-expander hypersurfaces with constant scalar curvature, which is also of independent interest. In this direction, we prove Theorem 6.1 which states that $\Gamma \times \mathbb{R}^{n-1}$ with the product metric are the only complete self-expander hypersurfaces immersed in $\mathbb{R}^{n+1}$ with nonnegative scalar curvature, where $\Gamma$ is a complete immersed self-expander curve in $\mathbb{R}^2$. Theorem 6.1 is a consequence of Proposition 6.1 which states that a complete self-expander hypersurface immersed $\mathbb{R}^{n+1}$ different from a hyperplane and with nonnegative scalar curvature is of the form $\Gamma \times \mathbb{R}^{n-1}$, where $\Gamma$ is a complete non-trivial self-expander curve.
immersed in \( \mathbb{R}^2 \), if and only if the scalar curvature attains a local minimum on the open set \( \{ x \in \Sigma; H(x) \neq 0 \} \). In its proof, we use a result by Smoczyk which gave the equivalent property of the self-expander hypersurfaces \( \Gamma \times \mathbb{R}^{n-1} \) (Theorem 5.1 in [21]). In general, it would be interesting to ask if the following is true.

**Problem:** Let \( \Sigma \) be a complete immersed self-expander hypersurface in \( \mathbb{R}^{n+1} \) with constant scalar curvature. Is it true that \( \Sigma = \Gamma \times \mathbb{R} \) with the product metric, where \( \Gamma \) is a complete self-expander curve immersed in \( \mathbb{R}^2 \)?

For self-expander surfaces in \( \mathbb{R}^3 \), we obtain the following result:

**Theorem 1.8.** Let \( \Sigma \) be a complete immersed self-expander surface in \( \mathbb{R}^3 \). If the scalar curvature of \( \Sigma \) is constant, then \( \Sigma = \Gamma \times \mathbb{R} \) with the product metric, where \( \Gamma \) is a complete self-expander curve (properly) immersed in \( \mathbb{R}^2 \).

Self-expander curves in \( \mathbb{R}^2 \) have been classified (see the work of Ishimura [18] and Halldorsson [16]). Theorem 6.1 in [16] states that each of the complete self-expander curves immersed in \( \mathbb{R}^2 \) is convex, properly embedded and asymptotic to the boundary of a cone with vertex at the origin. It is the graph of an even function. The curves form a one-dimensional family parametrized by their distance to the origin, which can take on any value in \( [0, \infty) \).

Theorems 1.7 and 1.8 have the following consequence.

**Corollary 1.3.** Let \( \Sigma \) be a complete properly immersed self-expander surface in \( \mathbb{R}^3 \). Assume that \( |H|(x) \leq a|x|+b, \ x \in \Sigma \), for some constants \( 0 \leq a < \frac{1}{2 \sqrt{2}} \) and \( b > 0 \). Then

\[
\mu_1 \leq \frac{3}{2} + \frac{\int_{\Sigma} \text{Scal}_{\Sigma} e^{-\frac{|x|^2}{4}} d\sigma}{\int_{\Sigma} e^{-\frac{|x|^2}{4}} d\sigma}.
\]

If \( \mu_1 > -\infty \), then the equality (1.7) holds if and only if \( \Sigma = \Gamma \times \mathbb{R} \) with the product metric, where \( \Gamma \) is a complete self-expander curve (properly) immersed in \( \mathbb{R}^2 \).

For self-expander surfaces in \( \mathbb{R}^3 \), we also obtain the following result:

**Theorem 1.9.** Let \( \Sigma \) be a complete properly immersed self-expander surface in \( \mathbb{R}^3 \). If \( \Sigma \) has nonpositive scalar curvature and the norm of its second fundamental form is constant, then \( \Sigma \) is a plane \( \mathbb{R}^2 \) through the origin.

The rest of the paper is organized as follows: In Section 2 we recall some notations and basic facts. In Section 3 we prove Theorem 3.1 and apply it to prove the finiteness of weighted volumes and volume growth estimate of self-expanders, that is, Theorem 1.1. We also prove Theorem 1.2. In Section
we prove rigidity Theorems. In Section 5 we obtain an upper bound for the bottom of the spectrum of the drifted Laplacian \( \mathcal{L} \). In Section 6 we discuss self-expanders with constant scalar curvature and prove Theorem 1.8. We also prove Theorem 1.9. In Section 7 we obtain the upper and lower bounds for the bottom of the spectrum of the \( L \)-stability operator \( L \) and also discuss the \( L \)-stability of self-expanders.

2. Preliminaries

In this section, we will recall some concepts and basic facts.

Assume that \((M, g)\) is a smooth \(m\)-dimensional Riemannian manifold. Let \( \Sigma \) be an \( n\)-dimensional immersed submanifold in \( M \) with the induced metric \( g \). We will denote by \( d\sigma \) the volume form of \( \Sigma \). In this paper, unless otherwise specified, the notations with a bar, for instance \( \bar{\nabla} \) and \( \bar{\nabla}^2 \), denote the quantities corresponding the metric \( \bar{g} \) on \( M \). On the other hand, the notations like \( \nabla, \Delta \) denote the quantities corresponding the intrinsic metric \( g \) on \( \Sigma \).

The isometric immersion \( i : (\Sigma^n, g) \to (M^{n+k}, \bar{g}) \) is said to be properly immersed if, for any compact subset \( \Omega \) in \( M \), the pre-image \( i^{-1}(\Omega) \) is compact in \( \Sigma \).

Let \( A \) denote the second fundamental form of \((\Sigma, g)\), that is, at \( p \in \Sigma \),
\[
A(X, Y) = (\nabla_X Y)^\perp,
\]
where \( X, Y \in T_p \Sigma \), \( \perp \) denotes the projection onto the normal bundle of \( \Sigma \). The mean curvature vector \( H \) of \( \Sigma \) is defined as the trace of \( A \). If \( \Sigma \) is a hypersurface, its mean curvature \( H \) is defined by \( H = -H\mathbf{n} \), where \( \mathbf{n} \) is the unit normal field on \( \Sigma \).

Given a smooth function \( f \) on \( M \), define the weighted mean curvature vector \( H_f \) of a submanifold \((\Sigma, g)\) by \( H_f := H + (\nabla f)^\perp \). \( \Sigma \) is called \( f \)-minimal if its weighted mean curvature vector \( H_f \) vanishes identically, or equivalently if it satisfies
\[
(2.1) \quad H = -(\nabla f)^\perp.
\]
If \( \Sigma \) is a hypersurface, its weighted mean curvature \( H_f \) is defined by \( H_f = -H_f\mathbf{n} \). In particular \( \Sigma \) is \( f \)-minimal if and only if the weighted mean curvature satisfies that \( H_f = H - \langle \nabla f, \mathbf{n} \rangle = 0 \) or equivalently
\[
(2.2) \quad H = \langle \nabla f, \mathbf{n} \rangle.
\]

The weighted volume of a measurable subset \( S \subset \Sigma \) with respect to the function \( f \) is defined by
\[
(2.3) \quad V_f(S) := \int_S e^{-f} d\sigma.
\]
It is known that an $f$-minimal submanifold is a critical point of the weighted volume functional defined in (2.3). On the other hand, it is also a minimal submanifold under the conformal metric $\tilde{g} = e^{-2n}f g$ on $M$ (see, e.g. [6], [7]). When $(M, g)$ is the Euclidean space $(\mathbb{R}^{n+k}, g_0)$, there are very interesting examples of $f$-minimal submanifolds:

**Example 2.1.** If $f = \frac{|x|^2}{4} - \frac{|x|^2}{4},$ and $-\langle x, w \rangle$ respectively, where $w \in \mathbb{R}^{n+k}$ is a constant vector, an $n$-dimensional $f$-minimal submanifold $\Sigma$ is a self-shrinker, self-expander and translator for MCF in the Euclidean space $\mathbb{R}^{n+k}$ respectively.

On the smooth metric measure space $(\Sigma, g, e^{-f}d\sigma)$, there is a very important second-order elliptic operator: the drifted Laplacian $\Delta_f = \Delta - \langle \nabla f, \nabla \cdot \rangle$. It is well known that $\Delta_f$ is a densely defined self-adjoint operator in $L^2(\Sigma, e^{-f}d\sigma)$, i.e. for $u$ and $v$ in $C_0^\infty(\Sigma)$, it holds that

\begin{equation}
\int_\Sigma (\Delta_f u) v e^{-f}d\sigma = -\int_\Sigma \langle \nabla u, \nabla v \rangle e^{-f}d\sigma.
\end{equation}

Since we will study the spectrum problems in this paper, we recall some facts in spectral theory (see more details in, e.g. [14], [19]). Consider the Schrödinger operator on $\Sigma$:

$$S = \Delta_f + q, \quad q \in L^\infty_{loc}(\Sigma).$$

The weighted $L^2$ spectrum of $S$ is called the spectrum of $S$ for short whenever there is no confusion. The bottom $s_1$ of the spectrum of $S$ can be characterized by

\begin{equation}
\begin{aligned}
s_1 &= \inf \left\{ \frac{\int_\Sigma (|\nabla \varphi|^2 - q \varphi^2) e^{-f}d\sigma}{\int_\Sigma \varphi^2 e^{-f}d\sigma}; \varphi \in C_0^\infty(\Sigma), \int_\Sigma \varphi^2 e^{-f}d\sigma \neq 0 \right\}.
\end{aligned}
\end{equation}

In general, if $\Sigma$ is noncompact, the bottom $s_1$ may not be the weighted $L^2$ eigenvalue and may take $-\infty$.

Now we give especial notations on self-expanders. In the following, unless otherwise specified, let $\Sigma$ be an $n$-dimensional self-expander in $\mathbb{R}^{n+k}, k \geq 1$, that is, $\Sigma$ satisfies the equation

\begin{equation}
H = \frac{x^\perp}{2}
\end{equation}

In the case of codimension 1, the mean curvature $H$ of a self-expander $\Sigma$ satisfies that

\begin{equation}
H = -\frac{1}{2} \langle x, n \rangle.
\end{equation}

Observe that taking $f = -\frac{|x|^2}{4}$, a self-expander $\Sigma$ can be viewed as an $f$-minimal submanifold in $\mathbb{R}^{n+k}$ since it satisfies the equation (2.1).
The weighted volume of a measurable subset \( S \subset \Sigma \) is given by
\[
V(S) := \int_S e^{\frac{|x|^2}{4}} d\sigma.
\]

The weighted \( L^2 \) inner product of functions \( u \) and \( v \) in \( L^2(\Sigma, e^{\frac{|x|^2}{4}} d\sigma) \) is defined by
\[
\langle u, v \rangle_{L^2(\Sigma, e^{\frac{|x|^2}{4}} d\sigma)} = \int_{\Sigma} u v e^{\frac{|x|^2}{4}} d\sigma.
\]

Denote by \( \mathcal{L} \) the drifted Laplacian on \( \Sigma \), i.e.
\[
\mathcal{L} = \Delta + \frac{1}{2} \langle x, \nabla \cdot \rangle.
\]

The bottom \( \lambda_1 \) of the spectrum of \( \mathcal{L} \) can be given by
\[
\lambda_1 = \inf \left\{ \frac{\int_{\Sigma} |\nabla \varphi|^2 e^{\frac{|x|^2}{4}} d\sigma}{\int_{\Sigma} \varphi^2 e^{\frac{|x|^2}{4}} d\sigma}; \varphi \in C^\infty_0(\Sigma), \int_{\Sigma} \varphi^2 e^{\frac{|x|^2}{4}} d\sigma \neq 0 \right\}.
\]

From (2.9), \( \lambda_1 \) is nonnegative. For self-expanders, the stability operator for \( \Sigma \) appeared in the second variation formula of the weighted volume is a Schrödinger operator:
\[
\mathcal{L} = \mathcal{L} + |A|^2 - \frac{1}{2} = \Delta + \frac{1}{2} \langle x, \nabla \cdot \rangle + |A|^2 - \frac{1}{2}.
\]

**Definition 2.1.** A self-expander \( \Sigma \) is said to be \( \mathcal{L} \)-stable if the following inequality holds for all \( \varphi \in C^\infty_0(\Sigma) \),
\[
- \int_{\Sigma} \varphi (L \varphi) e^{\frac{|x|^2}{4}} d\sigma = \int_{\Sigma} \left( |\nabla \varphi|^2 - (|A|^2 - \frac{1}{2}) \varphi^2 \right) e^{\frac{|x|^2}{4}} d\sigma \geq 0.
\]

\( \mathcal{L} \)-stability of \( \Sigma \) is equivalent to that the second variation of its weighted volume is nonnegative for any compactly supported normal variation. Denote the bottom of the spectrum of \( \mathcal{L} \) by \( \mu_1 \). \( \mathcal{L} \)-stability means \( \mu_1 \geq 0 \).

For self-expander hypersurfaces, the following equations are known (see, for instance, [10]).

**Proposition 2.1.** If \( \Sigma \subset \mathbb{R}^{n+1} \) is a self-expander hypersurface, then
\[
\mathcal{L} H = -(|A|^2 + \frac{1}{2}) H,
\]
\[
\mathcal{L} H^2 = -H^2(2|A|^2 + 1) + 2|\nabla H|^2,
\]
\[
\mathcal{L} |A|^2 = -|A|^2(2|A|^2 + 1) + 2|\nabla A|^2,
\]
\[
\mathcal{L}(\text{Scal}_\Sigma) = -\text{Scal}_\Sigma(2|A|^2 + 1) + 2|\nabla H|^2 - 2|\nabla A|^2.
\]

3. **Weighted volumes of properly immersed self-expanders**

In this section, motivated by Theorem 1.1 in [5], Theorem 1.1 in [9] and Theorem 5 in [2], we prove the following Theorem 3.1 which deals with the
growth of volumes of the level sets of adequate functions on a general Riemannian manifold. Next, we prove Theorem 1.1.

Recall that a function $h$ on a complete noncompact Riemannian manifold $M$ is said to be proper if, for any bounded closed subset $I \subset \mathbb{R}$, the inverse image $h^{-1}(I)$ is compact in $M$.

**Theorem 3.1.** Let $(X,g)$ be a complete noncompact Riemannian manifold. Assume that $\alpha > 0$, $\beta > 0$, $a_0$, $a_1$ and $a_2$ are constants satisfying $a_2 < \frac{\beta}{4}$. If $h$ is a proper nonnegative $C^2$ function on $X$ such that

$$\Delta h - \alpha|\nabla h|^2 + \beta h \leq a_2 r^2 + a_1 r + a_0 \quad (3.1)$$

and

$$\Delta h \leq a_2 r^2 + a_1 r + a_0 \quad (3.2)$$

hold on the sets $D_r = \{x \in X; 2\sqrt{h} \leq r\}$ for all $r > 0$, then

(i) The integral

$$\int_X e^{-\alpha h} dv < \infty. \quad (3.3)$$

(ii) For all $r > 0$ the volume of the set $D_r$ satisfies

$$V(r) \leq Ce^{\varepsilon(a_2 r^2 + a_1 r + a_0) + \frac{\beta r^2}{4\varepsilon}}, \text{ for any } \varepsilon > 0, \quad (3.4)$$

and

$$V(r) \leq Ce^{\frac{\alpha r^2}{2}}, \quad (3.5)$$

where $\gamma = \frac{\beta}{\alpha}$ and $C = C(\alpha) = \int_X e^{-\alpha h} dv < \infty.$

**Remark 3.1.** If $a_2 = 0$ and $a_1 = 0$, (3.3) was obtained in [8] (see (1.6) of Theorem 1.1 in [8]). However, in this case, the level sets $D_r$ have polynomial volume growth as proved in [8].

**Remark 3.2.** Taking $\alpha = \beta = 1$ in Theorem 3.1 we obtain Theorem 5 in [2]. It worth mentioning that the restriction $a_2 < \frac{1}{4}$ should be added in the assumption of Theorem 5 in [2].

**Proof of Theorem 3.1.** Let $\gamma = \frac{\beta}{\alpha}$ and $k(r) = a_2 r^2 + a_1 r + a_0$. Since $h$ is proper, the following integral $I(t)$ is well defined.

$$I(t) = \frac{1}{t^{k(r)}} \int_{D_r} e^{-\alpha h} dv, \quad t > 0.$$

$$I'(t) = t^{-k(r)-1} \int_{D_r} e^{-\alpha h} \left(\frac{\beta h}{t^\gamma} - k(r)\right) dv. \quad (3.6)$$
On the other hand, for \( t \geq 1, \gamma > 0 \),

\[
\int_{\mathcal{D}_r} \text{div} \left( e^{-\frac{\alpha}{t^\gamma} \nabla h} \right) dv = \int_{\mathcal{D}_r} e^{-\frac{\alpha}{t^\gamma} \left( \Delta h - \frac{\alpha}{t^\gamma} |\nabla h|^2 \right)} dv \\
= \int_{\mathcal{D}_r} e^{-\frac{\alpha}{t^\gamma}} \left( (1 - \frac{1}{t^\gamma}) \Delta h + \frac{1}{t^\gamma} \left( \Delta h - \alpha |\nabla h|^2 \right) \right) dv \\
\leq \int_{\mathcal{D}_r} e^{-\frac{\alpha}{t^\gamma}} \left( (1 - \frac{1}{t^\gamma}) k(r) + \frac{1}{t^\gamma} (\Delta h - \beta h + k(r)) \right) dv \\
= \int_{\mathcal{D}_r} e^{-\frac{\alpha}{t^\gamma}} \left( k(r) - \frac{\beta h}{t^\gamma} \right) dv.
\]

(3.7)

Substituting (3.7) into (3.6) gives

\[
I'(t) \leq -t^{-k(r)-1} \int_{\mathcal{D}_r} \text{div}(e^{-\frac{\alpha}{t^\gamma} \nabla h}) dv.
\]

At the regular value \( r \) of \( h \) and for \( t \geq 1 \), by Stokes' Theorem, we have

\[
I'(t) \leq -t^{-k(r)-1} \int_{\partial \mathcal{D}_r} \left\langle e^{-\frac{\alpha}{t^\gamma} \nabla h}, \frac{\nabla h}{|\nabla h|} \right\rangle dv \\
= -t^{-k(r)-1} \int_{\partial \mathcal{D}_r} e^{-\frac{\alpha}{t^\gamma}} |\nabla h| dv \leq 0.
\]

Integrating \( I'(t) \) over \( t \) from 1 to \( e^\varepsilon > 1 \), where \( \varepsilon > 0 \), we get

\[
\frac{1}{e^\varepsilon k(r)} \int_{\mathcal{D}_r} e^{-\frac{\alpha}{t^\gamma}} dv \leq \int_{\mathcal{D}_r} e^{-\alpha h} dv.
\]

(3.8)

Since the integral in (3.8) is right continuous in \( r \), (3.8) holds for all \( r > 0 \). Note \( 2\sqrt{\gamma} \leq r \) over \( \mathcal{D}_r \). (3.8) implies that, for all \( r > 0 \),

\[
\frac{1}{e^k(r) + \frac{\alpha}{2e^\gamma}} \int_{\mathcal{D}_r} dv \leq \frac{1}{e^\varepsilon k(r)} \int_{\mathcal{D}_r} e^{-\frac{\alpha}{t^\gamma}} dv \\
\leq \int_{\mathcal{D}_r} e^{-\alpha h} dv.
\]

(3.9)

Note that for \( r \geq 1 \)

\[
\int_{\mathcal{D}_r} e^{-\alpha h} dv - \int_{\mathcal{D}_{r-1}} e^{-\alpha h} dv = \int_{\mathcal{D}_r \setminus \mathcal{D}_{r-1}} e^{-\alpha h} dv \\
\leq e^{-\frac{\alpha r^2}{2e^\gamma}} \int_{\mathcal{D}_r} dv.
\]

(3.10)
By (3.9), (3.10)

\[
\int_{\mathcal{D}_r} e^{-\alpha h} dv - \int_{\mathcal{D}_{r-1}} e^{-\alpha h} dv \\
\leq e^{\varepsilon k(r) + \frac{a^2}{4\varepsilon} - \frac{\alpha (r-1)^2}{4}} \int_{\mathcal{D}_r} e^{-\alpha h} dv \\
= e^{(\varepsilon a_2 + \frac{\alpha}{4\varepsilon} - \frac{\alpha}{4})r^2 + (\varepsilon a_1 + \frac{\alpha}{2})r + (\varepsilon a_0 - \frac{\alpha}{4})} \int_{\mathcal{D}_r} e^{-\alpha h} dv.
\]

(3.11)

Since \(a_2 < \frac{\beta}{4}\), there exists a very small \(\varepsilon_0 = \varepsilon_0(\alpha, \beta, a_2)\) such that

\[
(\varepsilon_0 a_2 + \frac{\alpha}{4\varepsilon} - \frac{\alpha}{4}) < 0.
\]

Then there exists \(r_0 \geq 1\) such that for \(r \geq r_0\)

\[
e^{(\varepsilon a_2 + \frac{\alpha}{4\varepsilon} - \frac{\alpha}{4})r^2 + (\varepsilon a_1 + \frac{\alpha}{2})r + (\varepsilon a_0 - \frac{\alpha}{4})} \leq e^{-r}.
\]

Substituting into (5.11) gives that

\[
\int_{\mathcal{D}_r} e^{-\alpha h} dv \leq \frac{1}{1 - e^{-r}} \int_{\mathcal{D}_{r-1}} e^{-\alpha h} dv.
\]

(3.12)

Then for any positive integer \(N\), we have

\[
\int_{\mathcal{D}_{r+N}} e^{-\alpha h} dv \leq \left( \prod_{i=0}^{N} \frac{1}{1 - e^{-(r+i)}} \right) \int_{\mathcal{D}_{r-1}} e^{-\alpha h} dv.
\]

(3.13)

Noting that the infinite product \(\prod_{i=0}^{\infty} (1 - e^{-(r+i)})\) converges to a positive number and letting \(N\) tend to infinity, we get that \(\int_{X} e^{-\alpha h} dv < +\infty\).

Moreover by (3.9), for all \(r > 0\) and \(\varepsilon > 0\),

\[
\frac{1}{e^{\varepsilon k(r) + \frac{a^2}{4\varepsilon r}}} \int_{\mathcal{D}_r} dv \leq \int_{\mathcal{D}_r} e^{-\alpha h} dv \leq \int_{X} e^{-\alpha h} dv.
\]

Hence

\[
V(r) \leq C(\alpha)e^{\varepsilon k(r) + \frac{a^2}{4\varepsilon r}} = C(\alpha)e^{\varepsilon (a_2 r^2 + a_1 r + a_0) + \frac{\alpha^2}{4\varepsilon r}},
\]

where \(C(\alpha) = \int_{X} e^{-\alpha h} dv\).

Since \(\varepsilon > 0\) is arbitrary, letting \(\varepsilon \to 0\) yields that

\[
V(r) \leq C(\alpha)e^{\frac{\alpha^2}{4\varepsilon r^2}}.
\]

□

**Proof of Theorem 1.1.** Take \(h = \frac{|x|^2}{4}\), \(x \in \Sigma\). Since \(\Sigma\) is properly immersed in \(\mathbb{R}^n\), \(h(x)\) is proper on \(\Sigma\). We have that, on \(B_r(0) \cap \Sigma\),

\[
\Delta h = \frac{n}{2} + \langle \nabla h, H \rangle = \frac{n}{2} + |H|^2 \\
\leq a_2 r^2 + 2abr + (b^2 + \frac{n}{2}).
\]
In the above, we used the hypothesis: $|H|(x) \leq a|x| + b$, $x \in \Sigma$. We also have that, on $B_r(0) \cap \Sigma$,

$$\Delta h - a|\nabla h|^2 + ah = \frac{n}{2} + |H|^2 + \alpha|\nabla h|^2$$

$$= \frac{n}{2} + (1 + \alpha)|H|^2$$

$$\leq (1 + \alpha)a^2r^2 + 2(1 + \alpha)abr + (1 + \alpha)b^2 + \frac{n}{2}.$$  

Let $a_2 = (1 + \alpha)a^2$, $a_1 = 2(1 + \alpha)ab$, $a_0 = (1 + \alpha)b^2 + \frac{n}{2}$ and $\beta = \alpha$. For $\alpha > \frac{4a_2}{1 - 4a_2}$, where $0 \leq a < \frac{1}{2}$, it holds that $a_2 < \frac{\beta^2}{4}$. By applying Theorem 3.1 we obtain that $\int_{\Sigma} e^{-\frac{\alpha}{4}\frac{|x|^2}{r^2}} d\sigma < \infty$ for all $\alpha > \frac{4a_2}{1 - 4a_2}$ and the volume of $B_r(0) \cap \Sigma$ satisfies that, for all $r > 0$,

$$\text{Vol}(B_r(0) \cap \Sigma) \leq C(\alpha)e^{\frac{\alpha}{4}\frac{|x|^2}{r^2}}.$$  

In the particular case of $a < \frac{1}{2\sqrt{2}}$, since $a < \frac{1}{2\sqrt{2}}$ implies that $\frac{4a_2}{1 - 4a_2} < 1$, we may take $\alpha = 1$.  

By an argument analogous to the ones used in the proofs of Theorem 4.1 in [9] and Theorem 4 in [2], we may prove the following result: Let $\Sigma$ be a complete $n$-dimensional immersed self-expander in $\mathbb{R}^m$, $n < m$. If there exists $\alpha > 0$ such that $\int_{\Sigma} e^{-\frac{\alpha}{4}\frac{|x|^2}{r^2}} d\sigma < \infty$, then $\Sigma$ is properly immersed on $\mathbb{R}^m$. Hence Theorem 4.4 has the following consequence.

**Corollary 3.1.** Let $\Sigma$ be a complete $n$-dimensional immersed self-expander in $\mathbb{R}^m$, $n < m$. Assume that its mean curvature vector $H$ satisfies $|H|(x) \leq a|x| + b$, $x \in \Sigma$, for some constants $0 \leq a < \frac{1}{2}$ and $b > 0$. Then for $\alpha > \frac{4a_2}{1 - 4a_2}$ the following statements are equivalent:

(i) $\Sigma$ is properly immersed on $\mathbb{R}^m$.

(ii) There exist constants $\bar{C} = C(\alpha)$, $\bar{r}_0$, $\bar{r}_1$, $\bar{r}_2$, $\bar{r}_3 < \frac{\alpha}{4}$, such that

$$V(B_r(0) \cap \Sigma) \leq C(\alpha)e^{\frac{\alpha}{4}\frac{|x|^2}{r^2} + \frac{\alpha}{4}\frac{r^2}{r^2} + \frac{\alpha}{4}\frac{r^2}{r^2}}.$$  

(iii) $\int_{\Sigma} e^{-\frac{\alpha}{4}\frac{|x|^2}{r^2}} d\sigma < \infty$.

**Proof of Theorem 4.4.** Let $u = e^{-\frac{\alpha}{4}\frac{|x|^2}{r^2}}$. Since $\Sigma$ is properly immersed, using $\Delta|x|^2 = 2n + 4|H|^2$ and Stokes’ Theorem, we have

$$\int_{(B_r(0) \setminus B_{r_0}(0)) \cap \Sigma} u\Delta|x|^2 = \frac{1}{2n} \int_{(B_r(0) \setminus B_{r_0}(0)) \cap \Sigma} u|\nabla u|^2$$

$$- \frac{2}{n} \int_{(B_r(0) \setminus B_{r_0}(0)) \cap \Sigma} u|H|^2$$

$$= -\frac{1}{n} \int_{(B_r(0) \setminus B_{r_0}(0)) \cap \Sigma} \langle \nabla u, u^T \rangle + \frac{1}{n} \int_{\partial B_r(0) \cap \Sigma} u|x^T|$$

$$= -\frac{1}{n} \int_{\partial B_{r_0}(0) \cap \Sigma} u|x^T| - \frac{2}{n} \int_{(B_r(0) \setminus B_{r_0}(0)) \cap \Sigma} u|H|^2.$$  

(3.14)
On the other hand, using co-area formula we obtain

\[
\frac{d}{dt} \left( \frac{1}{t^n} \int_{(B_t(0) \setminus B_{r_0}(0)) \cap \Sigma} u \right) = -\frac{n}{t^{n+1}} \int_{(B_t(0) \setminus B_{r_0}(0)) \cap \Sigma} u + \frac{1}{t^n} \int_{\partial B_t(0) \cap \Sigma} \frac{u|x|}{|x^T|}. \tag{3.15}
\]

Substituting (3.14) into (3.15) and noting that \( |x| = t \) on \( \partial B_t(0) \cap \Sigma \), we have

\[
\frac{d}{dt} \left( \frac{1}{t^n} \int_{(B_t(0) \setminus B_{r_0}(0)) \cap \Sigma} u \right) = 1 \frac{1}{t^{n+1}} \int_{(B_t(0) \setminus B_{r_0}(0)) \cap \Sigma} \left( 2u|\mathbf{H}|^2 + \langle \nabla u, x^T \rangle \right)
\]

\[
+ \frac{1}{t^{n+1}} \left( \int_{\partial B_t(0) \cap \Sigma} u \frac{|x|^2 - |x^T|^2}{|x^T|} + \int_{\partial B_{r_0}(0) \cap \Sigma} u|x^T| \right)
\]

\[
\geq 1 \frac{1}{t^{n+1}} \int_{(B_t(0) \setminus B_{r_0}(0)) \cap \Sigma} \left( 2u|\mathbf{H}|^2 + \langle \nabla u, x^T \rangle \right)
\]

\[
- \frac{1}{t^{n+1}} \left( 2|\mathbf{H}|^2 - \frac{\alpha}{2}|x^T|^2 \right) e^{-\frac{\alpha}{4}|x|^2}
\]

\[
(3.16)
\]

Choose \( \alpha = \frac{4a^2}{1 - 4a^2} \), where \( 0 \leq a < \frac{1}{2} \). By the hypothesis \( |\mathbf{H}|(x) \geq a|x| + b \) for \( |x| \geq r_0 \), it follows that

\[
\frac{\partial}{\partial t} \left( \frac{1}{t^n} \int_{(B_t(0) \setminus B_{r_0}(0)) \cap \Sigma} e^{-\frac{\alpha}{4}|x|^2} \right) \geq 0 \quad \text{for} \quad t \geq r_0. \tag{3.17}
\]

Since \( \Sigma \) is properly immersed, there exists some \( t_0 > r_0 \) such that

\[
\int_{(B_{t_0}(0) \setminus B_{r_0}(0)) \cap \Sigma} e^{-\frac{\alpha}{4}|x|^2} > 0.
\]

Integrating (3.17) from \( t_0 \) to \( t > t_0 \), we get

\[
\int_{(B_t(0) \setminus B_{r_0}(0)) \cap \Sigma} e^{-\frac{\alpha}{4}|x|^2} d\sigma \geq \frac{t^n}{t_0^n} \int_{(B_{t_0}(0) \setminus B_{r_0}(0)) \cap \Sigma} e^{-\frac{\alpha}{4}|x|^2} d\sigma.
\]

Letting \( t \to \infty \) in the above inequality implies

\[
\int_{\Sigma} e^{-\frac{\alpha}{4}|x|^2} d\sigma = \infty.
\]

The particular case stated in the theorem holds by taking \( a = \frac{1}{2\sqrt{2}} \) and \( \alpha = 1 \).
4. Rigidity of hyperplanes

In this section we study the rigidity property of hyperplanes as self-expanders. First we state the following equations:

**Lemma 4.1.** Let \( \Sigma \) be an immersed self-expander hypersurface in \( \mathbb{R}^{n+1} \). Then for \( \alpha \in \mathbb{R} \) it holds that

\[
\mathcal{L}_\alpha H = -\frac{1}{2}H - |A|^2 H - \frac{\alpha + 1}{2}(x, \nabla H),
\]

\[
\mathcal{L}_\alpha H = -\frac{1}{2}H - |A|^2 H - \frac{\alpha + 1}{4} A(x^T, x^T),
\]

where the operator \( \mathcal{L}_\alpha = \Delta - \frac{\alpha}{2} \langle x, \nabla \cdot \rangle \) and \( x^T \) denotes the tangent component of \( x \).

**Proof.** Since \( \mathcal{L} H = -\frac{1}{2}H - |A|^2 H \),

\[
\mathcal{L}_\alpha H = -\frac{1}{2}H - |A|^2 H - \frac{\alpha + 1}{2} \langle x, \nabla H \rangle.
\]

Take a local orthonormal frame \( \{e_i\}, i = 1, \ldots, n \) for \( \Sigma \). From \( H = -\frac{1}{2} \langle x, n \rangle \),

\[
2\nabla_{e_i} H = - \langle \nabla_{e_i} x, n \rangle - \langle x, \nabla_{e_i} n \rangle
\]

\[
= h_{ij} \langle x, e_j \rangle
\]

and hence

\[
\langle x, \nabla H \rangle = \langle x, e_i \rangle \nabla_{e_i} H = \frac{1}{2} h_{ij} \langle x, e_i \rangle \langle x, e_j \rangle = \frac{1}{2} A(x^T, x^T).
\]

By this and Equation (4.1), we have that

\[
\mathcal{L}_\alpha H = -\frac{1}{2}H - |A|^2 H - \frac{\alpha + 1}{2} \langle x, \nabla H \rangle
\]

\[
= -\frac{1}{2}H - |A|^2 H - \frac{\alpha + 1}{4} A(x^T, x^T).
\]

\( \square \)

Now, we prove the following result:

**Theorem 4.1.** Let \( \Sigma \) be a complete immersed self-expander hypersurface in \( \mathbb{R}^{n+1} \). Assume that \( \delta \in \{1, 3, 5, \ldots\} \) and \( \alpha > 0 \). If \( \Sigma \) satisfies the following properties:

(i) \( |A|^2 H^2 + \frac{1}{2} H^2 + \frac{\alpha + 1}{4} A(x^T, x^T) H \leq 0, \)

(ii) \( \frac{1}{j} \int_{B_{2j}(p) \setminus B_j(p)} H^{\delta+1} e^{-\alpha \frac{|x|^2}{4}} \sigma \to 0 \) when \( j \to \infty \), for a fixed point \( p \in \Sigma \),

then \( \Sigma \) must be a hyperplane \( \mathbb{R}^n \) through the origin, where \( x^T \) denotes the tangent component of the position vector \( x \).
Proof. Let $\varphi \in C_0^\infty(\Sigma)$. From (4.2), hypothesis (i) and the value of $\delta$, we have

$$0 \leq \int_{\Sigma} \left( -\frac{1}{2} H - |A|^2 H - \frac{\alpha + 1}{4} A(x^T, x^T) \right) H^\delta \varphi^2 e^{-\alpha |x|^2}$$

$$= \int_{\Sigma} H^\delta \varphi^2 (\mathcal{L}_\alpha H) e^{-\alpha |x|^2}.$$

Further,

$$\int_{\Sigma} H^\delta \varphi^2 (\mathcal{L}_\alpha H) e^{-\alpha |x|^2} = -2 \int_{\Sigma} H^\delta \varphi \langle \nabla \varphi, \nabla H \rangle e^{-\alpha |x|^2}$$

$$- \delta \int_{\Sigma} \varphi^2 H^{\delta - 1} |\nabla H|^2 e^{-\alpha |x|^2}$$

$$\leq -\frac{\delta}{2} \int_{\Sigma} \varphi^2 H^{\delta - 1} |\nabla H|^2 e^{-\alpha |x|^2}$$

$$+ \frac{2}{\delta} \int_{\Sigma} |\nabla \varphi|^2 H^{\delta + 1} e^{-\alpha |x|^2}.$$

where $\delta > 0$. Therefore

$$\frac{\delta}{2} \int_{\Sigma} \varphi^2 H^{\delta - 1} |\nabla H|^2 e^{-\alpha |x|^2} \leq \frac{2}{\delta} \int_{\Sigma} |\nabla \varphi|^2 H^{\delta + 1} e^{-\alpha |x|^2}.$$

Choose $\varphi = \varphi_j$, where $\varphi_j$ are the nonnegative cut-off functions satisfying that $\varphi_j = 1$ on $B^\Sigma_j(p)$, $\varphi_j = 0$ on $\Sigma \setminus B^\Sigma_{2j}(p)$ and $|\nabla \varphi_j| \leq \frac{1}{2}$. By the monotone convergence theorem and hypothesis (ii), it follows that, on $\Sigma$,

$$H^{\delta - 1} |\nabla H|^2 = 0,$$

We claim that $H = 0$ on $\Sigma$. In fact, if $H(p) \neq 0$ for some $p \in \Sigma$, then there exists a neighborhood $B_\epsilon(p)$ such that $H \neq 0$ on $B_\epsilon(p)$. So $\nabla H = 0$ on $B_\epsilon(p)$ and hence $H = C$ on $B_\epsilon(p)$. By (2.12) we conclude that $H \equiv 0$ on $B_\epsilon(p)$ which contradicts with $H(p) = 0$. The claim implies that $\Sigma$ is the hyperplane $\mathbb{R}^n$ through the origin. \(\Box\)

We give the integrable property of the powers of the norm of mean curvature vector $H$ which will be used later.

**Lemma 4.2.** Let $\Sigma$ be a complete $n$-dimensional properly immersed self-expander in $\mathbb{R}^m$. Assume that $|H| \leq a|x| + b$, $x \in \Sigma$, for some constants $0 \leq a < \frac{1}{2}$ and $b > 0$. Then for $\delta \geq 0$ and $\alpha > \frac{4a^2}{1-4a}$,

$$\int_{\Sigma} |H|^\delta e^{-\alpha |x|^2} d\sigma < \infty.$$

**Proof.** Note that $|H| = |\frac{1}{2} x^+| \leq \frac{1}{2} |x|$. Hence it is easily to see that Theorem 1.1 implies the desired conclusion. \(\Box\)

A consequence of Theorem 4.1 is Theorem 1.3.
Proof of Theorem 1.3. The hypothesis \(|A|^2H^2 + \frac{1}{2}H^2 + \beta A(x^T, x^T)H \leq 0\), implies that \(A(x^T, x^T)H \leq 0\). Then for any \(\alpha > 4\beta - 1\),

\[
|A|^2H^2 + \frac{1}{2}H^2 + \frac{\alpha + 1}{4}A(x^T, x^T)H \leq 0,
\]

which is just the condition (i) of Theorem 4.1. Choose \(\alpha > \max\{\frac{\beta}{2} - \alpha, 4\beta - 1\}\). By Lemma 4.2, the condition (ii) of Theorem 4.1 is also satisfied. \(\square\)

In order to prove Theorem 1.4, we prove the following result.

**Theorem 4.2.** Let \(\Sigma\) be a complete immersed self-expander hypersurface in \(\mathbb{R}^{n+1}\). Assume that its mean curvature \(H\) is bounded from below. If there exists \(\alpha > 0\) such that the following conditions hold:

(i) \(|A|^2H^2 + \frac{1}{2}H^2 + \frac{\alpha + 1}{4}A(x^T, x^T)H \geq 0\),

(ii) \(\frac{1}{j^2} \int_{B_j^c(p) \setminus B_{2j}^c(p)} e^{-\alpha |x|^2} d\sigma \to 0\), when \(j \to \infty\), for a fixed point \(p \in \Sigma\),

then \(\Sigma\) must be a hyperplane \(\mathbb{R}^n\) through the origin, where \(x^T\) denotes the tangent component of the position vector \(x\).

**Proof.** Let us fix \(C = \inf_{x \in \Sigma} H\). From hypothesis (i) and (4.2) it follows that

\[
\mathcal{L}_\alpha(H - C) \leq 0.
\]

By the maximum principle, either \(H \equiv C\) or \(H > C\). If \(H \equiv C\), then \(\Sigma\) is the hyperplane through the origin. If \(H > C\), let us consider \(u := \log(H - C)\). A computing yields

\[
\Delta u = -|\nabla u|^2 + \frac{\Delta H}{H - C}.
\]

Combining (4.5) and (4.6), we get

\[
\mathcal{L}_\alpha u \leq -|\nabla u|^2.
\]

Let us consider the sequence \(\varphi_j\) of nonnegative cut-off function satisfying that \(\varphi_j = 1\) on \(B_j^2(p)\), \(\varphi_j = 0\) on \(\Sigma \setminus B_{2j}^2(p)\) and \(|\nabla \varphi_j| \leq \frac{1}{j}\).

Multiplying (4.7) by \(\varphi_j^2\) and integrating by parts we obtain

\[
\int_\Sigma \varphi_j^2 |\nabla u|^2 e^{-\alpha |x|^2} \leq - \int_\Sigma \varphi_j^2 (\mathcal{L}_\alpha u) e^{-\alpha |x|^2}
\]

\[
= \int_\Sigma 2\varphi_j \langle \nabla \varphi_j, \nabla u \rangle e^{-\alpha |x|^2}
\]

\[
\leq \frac{1}{2} \int_\Sigma \varphi_j^2 |\nabla u|^2 e^{-\alpha |x|^2} + 2 \int_\Sigma |\nabla \varphi_j|^2 e^{-\alpha |x|^2}.
\]

Therefore

\[
\int_\Sigma \varphi_j^2 |\nabla u|^2 e^{-\alpha |x|^2} \leq 4 \int_\Sigma |\nabla \varphi_j|^2 e^{-\alpha |x|^2}.
\]
By hypothesis (ii) and the dominated convergence theorem, we obtain
\[
\int_{\Sigma} |\nabla u|^2 e^{-\frac{\alpha |x|^2}{4}} = 0.
\]
In particular \(H\) must be a constant, but this contradicts the assumption that \(H > \inf_{x \in \Sigma} H\).
\(\square\)

As a consequence, Theorem 4.2 implies Theorem 1.4.

Proof of Theorem 1.4. From the assumption on the mean curvature of \(\Sigma\), it follows that
\[
|H| = \left|\frac{\alpha + 1}{4} A(x^T, x^T)\right| \leq a |x| + b,
\]
for some constants \(0 < a < \frac{1}{2}\) and \(b > 0\). Therefore, by Lemma 4.2, the condition (ii) of Theorem 4.2 is satisfied.
\(\square\)

Theorem 1.4 also has the following consequence:

Corollary 4.1. Let \(\Sigma\) be a complete properly immersed self-expander hypersurface in \(\mathbb{R}^{n+1}\). Assume that its mean curvature \(H\) satisfies \(H(x) \leq a |x| + b\), \(x \in \Sigma\), for some constants \(0 < a < \frac{1}{2}\) and \(b > 0\). If \(A(x^T, x^T)\) is bounded from above and there exists \(\alpha > \frac{4a^2}{1-2a}\) such that
\[
|A|^2 H + H^2 + \frac{\alpha + 1}{4} A(x^T, x^T) \geq 0,
\]
then \(\Sigma\) must be a hyperplane \(\mathbb{R}^n\) through the origin, where \(x^T\) denotes the tangent component of the position vector \(x\).

Proof. We claim that \(\inf_{x \in \Sigma} H > -\infty\). In fact, if \(\inf_{x \in \Sigma} H = -\infty\), then there exists a sequence \(\{p_k\}\) in \(\Sigma\) such that \(H(p_k) \to -\infty\) when \(k \to \infty\). By (4.9), we have
\[
|A|^2(p_k) \leq -\frac{1}{2} - \frac{(\alpha + 1) A(p_k^T, p_k^T)}{4H(p_k)}.
\]
By the hypothesis that \(A(x^T, x^T)\) is bounded from above, (4.10) implies that \(|A|^2(p_k) < 0\) for \(k\) large enough, that is a contradiction. Therefore \(\inf_{x \in \Sigma} H > -\infty\).

Now applying Theorem 1.4 we complete the proof.
\(\square\)

5. Upper bound of \(\lambda_1\)

In this section, we prove the upper bound estimate of the first eigenvalue \(\lambda_1\) for the drifted Laplacian \(\mathcal{L}\).

Let \(\Sigma\) be a complete \(n\)-dimensional immersed self-expander in \(\mathbb{R}^m\) (not necessarily hypersurface). Then, the functions \(v = e^{-\frac{\alpha |x|^2}{8}}\), \(\alpha \in \mathbb{R}\), satisfy
\[
\mathcal{L} v + \left(\frac{(\alpha + 1)n}{4} + \frac{\alpha + 1}{2} |H|^2 - \frac{(\alpha + 1)(\alpha - 1)}{16} |x^T|^2\right) v = 0 \text{ on } \Sigma.
\]
In fact, it was proved in [10, Lemma 3.1] that for any smooth functions \( u, f \) and \( h \), it holds that
\[
\Delta f(ue^h) = e^h \{ \Delta f - 2h u + \langle \nabla h - f, \nabla h \rangle u \}.
\]
Substituting \( u = 1 \), \( f = -\frac{|x|^2}{4} \) and \( h = -\frac{\alpha + 1}{8}|x|^2 \) into the above equality yields (5.1).

The following integrability properties on \( v \) hold.

**Lemma 5.1.** Let \( \Sigma \) be a complete properly \( n \)-dimensional immersed self-expander in \( \mathbb{R}^n \). Assume that its mean curvature vector \( H \) satisfies \( |H|(x) \leq a|x| + b \), \( x \in \Sigma \), for some constants \( 0 \leq a < \frac{1}{2} \) and \( b > 0 \). Then for \( \alpha > \frac{4a^2}{1 - 4a^2} \), \( v = e^{-\frac{\alpha + 1}{8}|x|^2} \) satisfies
\[
(5.2) \quad \int_{\Sigma} v^2 e^{\frac{|x|^2}{4}} d\sigma < \infty \quad \text{and} \quad \int_{\Sigma} |\nabla v|^2 e^{\frac{|x|^2}{4}} d\sigma < \infty.
\]

**Proof.** Note that
\[
v^2 e^{\frac{|x|^2}{4}} = e^{-\frac{\alpha - 1}{4}|x|^2}
\]
and
\[
|\nabla v|^2 e^{\frac{|x|^2}{4}} = \frac{(\alpha + 1)^2}{16} |x|^2 e^{-\frac{\alpha}{4}|x|^2}.
\]
It is easily to see that Theorem (1.1) implies the desired conclusion. \( \Box \)

Now we give an inequality on the first eigenvalue \( \lambda_1 \).

**Theorem 5.1.** Let \( \Sigma \) be a complete properly immersed self-expander hypersurface in \( \mathbb{R}^{n+1} \). Assume that its mean curvature \( H \) satisfies \( |H|(x) \leq a|x| + b \), \( x \in \Sigma \), for some constants \( 0 \leq a < \frac{1}{2} \) and \( b > 0 \). Then the bottom \( \lambda_1 \) of the spectrum of the drifted Laplacian \( \mathcal{L} = \Delta + \frac{1}{2} \langle x, \nabla \cdot \rangle \) on \( \Sigma \), i.e. the first weighted \( L^2 \) eigenvalue of \( \mathcal{L} \) satisfies
\[
(5.3) \quad \lambda_1 \leq \frac{(\alpha + 1)\pi}{4} + \frac{\int_{\Sigma} \left( \frac{1}{2} H^2 - \frac{(\alpha - 1)}{16} |x|^2 \right) e^{-\frac{\alpha}{4}|x|^2} d\sigma}{\int_{\Sigma} e^{-\frac{\alpha}{4}|x|^2} d\sigma},
\]
for all \( \alpha > \frac{a^2}{\frac{1}{2} - a^2} \).

**Proof.** By Theorem 1.1 in [10], the spectrum of the operator \( \mathcal{L} \) is discrete and \( \lambda_1 \) is the first weighted \( L^2 \) eigenvalue of \( \mathcal{L} \). Then for any \( \phi \in C_0^\infty(\Sigma) \)
\[
(5.4) \quad \lambda_1 \int_{\Sigma} \phi^2 e^{\frac{|x|^2}{4}} \leq \int_{\Sigma} |\nabla \phi|^2 e^{\frac{|x|^2}{4}}.
\]
Take \( v = e^{-\frac{\alpha + 1}{8}|x|^2} \), where \( \alpha \) is any constant satisfying \( \alpha > \frac{a^2}{\frac{1}{2} - a^2} \).
Choose \( \phi = \varphi_j v \), where \( \varphi_j \) are the nonnegative cut-off functions satisfying that \( \varphi_j \) is 1 on \( B_j(0) \), \( |\nabla \varphi_j| \leq 1 \) on \( B_{j+1}(0) \setminus B_j(0) \), and \( \varphi_j = 0 \) on \( \Sigma \setminus B_{j+1}(0) \). Substitute \( \phi \) in (5.4):

\[
\lambda_1 \int \varphi_j^2 v^2 e^{\frac{|x|^2}{4}} \leq \int |\nabla (\varphi_j v)|^2 e^{\frac{|x|^2}{4}}.
\]

(5.5)

Note

\[
\int \varphi_j^2 \nabla v^2 e^{\frac{|x|^2}{4}} \leq 2 \int \varphi_j^2 |\nabla v|^2 e^{\frac{|x|^2}{4}} + 2 \int |\nabla \varphi_j|^2 v^2 e^{\frac{|x|^2}{4}},
\]

and \( v \in W^{1,2}(\Sigma, e^{\frac{|x|^2}{4}} d\sigma) \). Letting \( j \to \infty \) in (5.6) and using the monotone convergence theorem,

\[
\int |\nabla (\varphi_j v)|^2 e^{\frac{|x|^2}{4}} \to \int |\nabla v|^2 e^{\frac{|x|^2}{4}}.
\]

Besides, since

\[
\int \varphi_j v (-\mathcal{L} v) e^{\frac{|x|^2}{4}} = \int \langle \nabla v, \nabla (\varphi_j v) \rangle e^{\frac{|x|^2}{4}} \leq \frac{1}{2} \int |\nabla v|^2 e^{\frac{|x|^2}{4}} + \frac{1}{2} \int |\nabla (\varphi_j v)|^2 e^{\frac{|x|^2}{4}},
\]

letting \( j \to \infty \) in (5.7), Lemma 5.1 and the monotone convergence theorem yield

\[
\int \varphi_j v (-\mathcal{L} v) e^{\frac{|x|^2}{4}} = \int |\nabla v|^2 e^{\frac{|x|^2}{4}}.
\]

Then let \( j \to \infty \) in (5.5) and use the monotone convergence theorem again. We have

\[
\lambda_1 \int v^2 e^{\frac{|x|^2}{4}} \leq \int |\nabla v|^2 e^{\frac{|x|^2}{4}}
\]

\[
= \int \varphi_j v (-\mathcal{L} v) e^{\frac{|x|^2}{4}}
\]

\[
= \int \left( \frac{(\alpha + 1)n}{4} + \frac{\alpha + 1}{2} H^2 - \frac{(\alpha + 1)(\alpha - 1)}{16} |x|^2 \right) v^2 e^{\frac{|x|^2}{4}}.
\]

(5.8)

Therefore

\[
\lambda_1 \leq \frac{(\alpha + 1)n}{4} + \frac{(\alpha + 1)}{16} \int \left( \frac{1}{2} H^2 - \frac{(\alpha - 1)}{16} |x|^2 \right) e^{-\frac{|x|^2}{4}}.
\]

(5.9)

Now we prove Theorem 1.5, which is a corollary of Theorem 5.1.

Proof of Theorem 1.5. Noting the assumption \( a < \frac{1}{2 \sqrt{2}} \), we can take \( \alpha = 1 \) in Theorem 5.1. Then
If the equality in (5.10) holds, (5.8) with $\alpha = 1$ becomes the equality and hence $v = e^{-\frac{1}{4}|x|^2} \in W^{1,2}(\Sigma, e^{\frac{|x|^2}{4}}d\sigma)$ is the first eigenfunction of the operator $\mathcal{L}$ satisfying
\[
\mathcal{L}v + \lambda_1 v = 0.
\]
Taking $\alpha = 1$ in (5.1), we have
\[
\mathcal{L}v + \left(\frac{n}{2} + H^2\right)v = 0 \text{ on } \Sigma.
\]
(5.11) and (5.12) imply $H = \text{constant}$. By
\[
\mathcal{L}H + (|A|^2 + \frac{1}{2})H = 0,
\]
we conclude that $H \equiv 0$. Thus $\Sigma$ is a hyperplane passing through the origin. \hfill \Box

6. SELF-EXPANDER HYPERSURFACES WITH CONSTANT SCALAR CURVATURE

In this section we prove Theorem 1.8, which characterizes the complete self-expander surfaces immersed in $\mathbb{R}^3$ with constant scalar curvature. We also prove Theorem 1.9 which characterizes the complete self-expander surfaces properly immersed in $\mathbb{R}^3$ with the second fundamental form constant in norm and nonpositive scalar curvature.

In order to prove Theorem 1.8 we need the following result on the complete self-expander hypersurfaces immersed in $\mathbb{R}^{n+1}$ with nonnegative scalar curvature.

Proposition 6.1. Let $\Sigma$ be a complete immersed self-expander hypersurface in $\mathbb{R}^{n+1}$. Assume that $\Sigma$ is different from a hyperplane and has nonnegative scalar curvature. Then $\Sigma = \Gamma \times \mathbb{R}^{n-1}$, where $\Gamma$ is a complete non-trivial self-expander curve immersed in $\mathbb{R}^2$, if and only if the scalar curvature attains a local minimum on the open set $\{x \in \Sigma; H(x) \neq 0\}$.

Proof. Since $\text{Scal}_\Sigma = H^2 - |A|^2$ it follows that
\[
\nabla\text{Scal}_\Sigma = 2H\nabla H - \nabla|A|^2.
\]
Therefore
\[
4H^2|\nabla H|^2 = \langle \nabla\text{Scal}_\Sigma, \nabla\text{Scal}_\Sigma + 2\nabla|A|^2 \rangle + 4|A|^2|\nabla|A||^2.
\]
This together with (2.15) imply that on the set $\{x \in \Sigma; H(x) \neq 0\}$ the following holds
\[
\Delta \text{Scal}_\Sigma + \left\langle \nabla \text{Scal}_\Sigma, \frac{x}{2} - \frac{\nabla \text{Scal}_\Sigma + 2\nabla |A|^2}{2H^2} \right\rangle \\
= -\text{Scal}_\Sigma (2|A|^2 + 1) + 2\frac{|A|^2}{H^2} |\nabla |A||^2 - 2|\nabla A|^2 \\
\leq -\text{Scal}_\Sigma (2|A|^2 + 1) + 2|\nabla |A||^2 - 2|\nabla A|^2 \\
\leq 0.
\]

In the above we also use the hypothesis \(\text{Scal}_\Sigma \geq 0\) and the inequality \(|\nabla |A||^2 - 2|\nabla A|^2 \geq 0\).

If \(\text{Scal}_\Sigma\) attains a local minimum on the set \(\{x \in \Sigma; H(x) \neq 0\}\), the maximum principle implies that there exists an open set \(U \subset \{x \in \Sigma; H(x) \neq 0\}\) such that \(\text{Scal}_\Sigma\) is constant on \(U\). Further (6.3) implies that \(\text{Scal}_\Sigma = 0\) on \(U\). This implies that \(\frac{|A|^2}{H^2}\) attains a local maximum on the open set \(\{x \in \Sigma; H(x) \neq 0\}\). Smoczyk ([21, Theorem 5.1]) proved that for a self-expander hypersurface immersed in \(\mathbb{R}^{n+1}\) different from a linear subspace, it is of the form \(\Gamma \times \mathbb{R}^{n-1}\), where \(\Gamma\) is a nontrivial self-expander curve in \(\mathbb{R}^2\), if and only if, the function \(\frac{|A|^2}{H^2}\) attains a local maximum on the open set \(\{x \in \Sigma; H(x) \neq 0\}\). Hence we conclude that \(\Sigma = \Gamma \times \mathbb{R}^{n-1}\) where \(\Gamma\) is a nontrivial self-expander curve in \(\mathbb{R}^2\). This completes the proof. \(\square\)

Proposition 6.1 have the following consequence.

**Theorem 6.1.** Let \(\Sigma\) be a complete immersed self-expander hypersurface in \(\mathbb{R}^{n+1}\) with nonnegative constant scalar curvature. Then \(\Sigma = \Gamma \times \mathbb{R}^{n-1}\) with the product metric, where \(\Gamma\) is a complete self-expander curve immersed in \(\mathbb{R}^2\).

**Proof.** We consider two cases:
(i) Case of \(H \equiv 0\). Obviously, \(\Sigma\) is a hyperplane through the origin.
(ii) Case of \(H \neq 0\). By Proposition 6.1 we conclude that \(\Sigma = \Gamma \times \mathbb{R}^{n-1}\) where \(\Gamma\) is a nontrivial self-expander curve in \(\mathbb{R}^2\). Combining these two cases completes the proof. \(\square\)

Now we prove Theorem 1.8.

**Proof of Theorem 1.8.** The classical Hilbert’s theorem says that there is no complete surface immersed in \(\mathbb{R}^3\) with negative constant scalar curvature. Hence Theorem 6.1 implies Theorem 1.8.

In the case that \(\Sigma\) is properly immersed, we may give an alternative direct proof without using Hilbert’s theorem as follows.

First, we prove that \(\text{Scal}_\Sigma \geq 0\). Since \(\text{Scal}_\Sigma = H^2 - |A|^2\) is constant, we have
\[
0 = \nabla \text{Scal}_\Sigma = 2H \nabla H - 2|A|^2
\]
and, from (2.13),
\[ \text{Scal}_\Sigma(2|A|^2 + 1) = 2|\nabla H|^2 - 2|\nabla A|^2. \]

Since \( \Sigma \) is properly immersed, there exists \( p \in \Sigma \) which minimizes \( |x| \). At the point \( p \), we have
\[ \nabla H(p) = 0. \]

Choose a local orthonormal frame \( \{e_1, e_2\} \) such that the coefficients of the second fundamental form are \( h_{ij}(p) = \lambda_i \delta_{ij} \), for \( i, j = 1, 2 \). By the definition,
\[ |\nabla H|^2 = (h_{111} + h_{221})^2 + (h_{112} + h_{222})^2. \]
This together with (6.6) implies
\[ h_{111} = -h_{221} \quad \text{and} \quad h_{222} = -h_{112}. \]

By (6.5), we conclude that \( \text{Scal}_\Sigma = 0 \) at \( p \).

Using an idea similar to the proof of the Theorem 1.8, we prove Theorem 1.9.

**Proof of Theorem 1.9.** Since \( \Sigma \) is properly immersed, there exists \( p \in \Sigma \) which minimizes \( |x| \). At the point \( p \), we have
\[ \nabla H(p) = 0. \]

Choose a local orthonormal frame \( \{e_1, e_2\} \) such that the coefficients of the second fundamental form are \( h_{ij}(p) = \lambda_i \delta_{ij} \), for \( i, j = 1, 2 \). By the definition,
\[ |\nabla H|^2 = (h_{111} + h_{221})^2 + (h_{112} + h_{222})^2. \]
This together with (6.6) implies
\[ h_{111} = -h_{221} \quad \text{and} \quad h_{222} = -h_{112}. \]

If \( h_{111} = h_{222} = 0 \), then
\[ |\nabla A|^2 = h_{111}^2 + h_{222}^2 + 3h_{112}^2 + 3h_{221}^2 = 0. \]

By (6.5), we conclude that \( \text{Scal}_\Sigma = 0 \) at \( p \). Therefore, we have that the constant \( \text{Scal}_\Sigma \geq 0 \). By Theorem 6.1, we conclude that \( \Sigma = \Gamma \times \mathbb{R} \) with the product metric, where \( \Gamma \) is a complete self-expander curve immersed in \( \mathbb{R}^2 \). \( \square \)
and, by (2.14),

\[(6.11) \quad |\nabla A|^2 = |A|^2 (|A|^2 + \frac{1}{2}).\]

Combining (6.10) with (6.9) yields

\[(6.11) \quad (h_{11} - h_{22}) h_{111} = (h_{11} - h_{22}) h_{222} = 0.\]

If \(h_{11} = h_{22}\), then \(\text{Scal}_\Sigma = H^2 - |A|^2 = |A|^2 \geq 0\). This together with hypothesis \(\text{Scal}_\Sigma \leq 0\) implies that \(|A| = 0\) at \(p\). Since \(|A|\) is constant, \(|A| = 0\) on \(\Sigma\).

If \(h_{111} = h_{222} = 0\), then

\[|\nabla A|^2 = h_{111}^2 + h_{222}^2 + 3h_{112}^2 + 3h_{221}^2 = 0.\]

By (6.11), we conclude that \(|A| = 0\) at \(p\). Therefore, \(|A| = 0\) on \(\Sigma\). Thus \(\Sigma\) is a plane passing through the origin. □

7. Properties of stability operator \(L\)

In this section, we discuss the \(L\)-stability of self-expanders and estimate the bottom spectrum of the \(L\)-stability operator \(L = \mathcal{L} + |A|^2 - \frac{1}{2}\).

Proof of Theorem 7.6. Let \(v = e^{-\frac{|x|^2}{4}}\). From (5.1) it follows that

\[(7.1) \quad \mathcal{L} v + \left(\frac{n}{2} + H^2\right) v = 0.\]

Denote \(w = \log(v)\). From (7.1), we have

\[\mathcal{L} w + |\nabla w|^2 = -\frac{n}{2} - H^2.\]

For any \(\psi \in C_0^\infty(\Sigma)\),

\[
\frac{n}{2} \int_\Sigma \psi^2 e^{\frac{|x|^2}{4}} + \int_\Sigma H^2 \psi^2 e^{\frac{|x|^2}{4}} = -\int_\Sigma |\nabla w|^2 \psi^2 e^{\frac{|x|^2}{4}} - \int_\Sigma \psi^2 (\mathcal{L} w)e^{\frac{|x|^2}{4}} \\
= -\int_\Sigma |\nabla w|^2 \psi^2 e^{\frac{|x|^2}{4}} + 2 \int_\Sigma \psi \langle \nabla \psi, \nabla w \rangle e^{\frac{|x|^2}{4}} \\
\leq \int_\Sigma |\nabla \psi|^2 e^{\frac{|x|^2}{4}}.
\]

Therefore

\[
\int_\Sigma \left( |\nabla \psi|^2 - \left( |A|^2 - \frac{1}{2}\right) \psi^2 \right) e^{\frac{|x|^2}{4}} \geq \int_\Sigma \left( H^2 - |A|^2 + \frac{n + 1}{2} \right) \psi^2 e^{\frac{|x|^2}{4}} \\
= \int_\Sigma \left( \text{Scal}_\Sigma + \frac{n + 1}{2} \right) \psi^2 e^{\frac{|x|^2}{4}} \\
\geq \left( \frac{n + 1}{2} + \inf_{x \in \Sigma} \text{Scal}_\Sigma \right) \int_\Sigma \psi^2 e^{\frac{|x|^2}{4}}.
\]

(7.2)
Hence

\[(7.3) \quad \mu_1 \geq \frac{n+1}{2} + \inf_{x \in \Sigma} \text{Scal}_\Sigma.\]

Now we have assumption that $\Sigma$ is proper, satisfies $|H|(x) \leq a|x| + b$, $x \in \Sigma$, for some constants $0 \leq a < \frac{1}{2\sqrt{2}}$ and $b > 0$, and has constant scalar curvature $\text{Scal}_\Sigma = \inf_{x \in \Sigma} \text{Scal}_\Sigma$.

By Inequality (1.6) in Theorem 1.7, which will be proved later in this paper, we have

\[(7.4) \quad \mu_1 \leq \frac{n+1}{2} + \int_{\Sigma} \text{Scal}_\Sigma e^{-\frac{|x|^2}{4}} = \frac{n+1}{2} + \inf_{x \in \Sigma} \text{Scal}_\Sigma.\]

Inequalities (7.3) and (7.4) induce $\mu_1 = \frac{n+1}{2} + \text{Scal}_\Sigma$ and hence the equality in (7.3) holds $\Box$

Theorem 1.6 induces Corollary 1.1 directly. Now we prove the following Theorem 7.1, whose proof is similar to that of Theorem 5.1.

**Theorem 7.1.** Let $\Sigma$ be a complete properly immersed self-expander hypersurface in $\mathbb{R}^{n+1}$. Assume that its mean curvature $H$ satisfies $|H|(x) \leq a|x| + b$, $x \in \Sigma$, for some constants $0 \leq a < \frac{1}{2\sqrt{2}}$ and $b > 0$. Then the bottom $\mu_1$ of the spectrum of the $L$-stability operator $L$ satisfies

\[(7.5) \quad \mu_1 \leq \frac{(\alpha + 1)n + 2}{4} + \int_{\Sigma} \left[ \text{Scal}_\Sigma + (\alpha - 1) \left( \frac{1}{2} H^2 - \frac{(\alpha + 1)(\alpha - 1)}{16} |x|^2 \right) \right] e^{-\frac{\alpha}{4} |x|^2} d\sigma\]

for all $\alpha > \frac{a^2}{\frac{a^2}{4} - a^2}$.

**Proof.** $\mu_1$ may be finite or $-\infty$. If $\mu_1$ is $-\infty$, the inequality (7.3) holds. Suppose $\mu_1 > -\infty$. From the variational characterization of $\mu_1$, we have that for any $\psi \in C_0^\infty(\Sigma)$,

\[(7.6) \quad \mu_1 \int_{\Sigma} \psi^2 e^{-\frac{|x|^2}{4}} \leq \int_{\Sigma} |\nabla \psi|^2 e^{-\frac{|x|^2}{4}} - \int_{\Sigma} (|A|^2 - \frac{1}{2}) \psi^2 e^{-\frac{|x|^2}{4}},\]

Let $v = e^{-\frac{\alpha + 1}{8} |x|^2}$, where $\alpha$ is any constant satisfying $\alpha > \frac{a^2}{\frac{a^2}{4} - a^2}$. Recall (5.1), i.e.,

\[(7.7) \quad \mathcal{L} v + \left( \frac{(\alpha + 1)n}{4} + \frac{\alpha + 1}{2} H^2 - \frac{(\alpha + 1)(\alpha - 1)}{16} |x|^2 \right) v = 0 \text{ on } \Sigma.\]

Lemmas 5.1 and 4.2 state that $v \in W^{1,2}(\Sigma, e^{-\frac{|x|^2}{4}} d\sigma)$ and $\int_{\Sigma} H^2 e^{-\frac{\alpha}{4} |x|^2} < \infty$. Choose $\psi = \varphi_j v$, where $\varphi_j$ are the nonnegative cut-off functions satisfying
that \( \varphi_j \) is 1 on \( B_j(0) \), \( |\nabla \varphi_j| \leq 1 \) on \( B_{j+1}(0) \setminus B_j(0) \), and \( \varphi = 0 \) on \( \Sigma \setminus B_{j+1}(0) \). Substitute \( \psi \) in (7.6):

\[
(7.8) \quad \mu_1 \int_{\Sigma} \varphi_j^2 v^2 e^{\frac{|x|^2}{4}} \leq \int_{\Sigma} |\nabla (\varphi_j v)|^2 e^{\frac{|x|^2}{4}} - \int_{\Sigma} (|A|^2 - \frac{1}{2}) \varphi_j^2 v^2 e^{\frac{|x|^2}{4}}.
\]

Note \( \mu_1 > -\infty \). Letting \( j \to \infty \) in (7.8) and using the monotone convergence theorem, (7.8) implies that

\[
\mu_1 \int_{\Sigma} v^2 e^{\frac{|x|^2}{4}} \leq \int_{\Sigma} |\nabla v|^2 e^{\frac{|x|^2}{4}} - \int_{\Sigma} (|A|^2 - \frac{1}{2}) v^2 e^{\frac{|x|^2}{4}}
\]

\[
= \int_{\Sigma} v(-\mathcal{L} v) e^{\frac{|x|^2}{4}} - \int_{\Sigma} (|A|^2 - \frac{1}{2}) v^2 e^{\frac{|x|^2}{4}}
\]

\[
= \int_{\Sigma} \left( \frac{\alpha + 1}{4} n + \frac{\alpha + 1}{2} H^2 - \frac{(\alpha + 1)(\alpha - 1)}{16} |x_T|^2 \right) v^2 e^{\frac{|x|^2}{4}}
\]

\[
- \int_{\Sigma} (|A|^2 - \frac{1}{2}) v^2 e^{\frac{|x|^2}{4}}
\]

\[
(7.9) \quad = \int_{\Sigma} \left[ \frac{\alpha + 1}{4} n + 2 + \text{Scal}_\Sigma + (\alpha - 1) \left( \frac{H^2}{2} - \frac{\alpha + 1}{16} |x_T|^2 \right) \right] v^2 e^{\frac{|x|^2}{4}}.
\]

Therefore

\[
(7.10) \quad \mu_1 \leq \frac{(\alpha + 1)n + 2}{4} + \frac{\int_{\Sigma} \left[ \text{Scal}_\Sigma + (\alpha - 1) \left( \frac{H^2}{2} - \frac{\alpha + 1}{16} |x_T|^2 \right) \right] e^{\frac{|x|^2}{4}}}{\int_{\Sigma} e^{\frac{|x|^2}{4}}}.
\]

\[\square\]

Theorem 7.1 implies Theorem 1.7 as follows:

**Proof of Theorem 1.7.** The proof is similar to that of Theorem 1.5. Since \( a < \frac{1}{2\sqrt{2}} \), we can take \( \alpha = 1 \) in Theorem 7.1. Then

\[
(7.11) \quad \mu_1 \leq \frac{n + 1}{2} + \frac{\int_{\Sigma} \text{Scal}_\Sigma e^{\frac{|x|^2}{4}}}{\int_{\Sigma} e^{\frac{|x|^2}{4}}}.
\]

If the equality in (7.11) holds, (7.9) with \( \alpha = 1 \) becomes the equality. Namely,

\[
(7.12) \quad \mu_1 \int_{\Sigma} v^2 e^{\frac{|x|^2}{4}} = \int_{\Sigma} |\nabla v|^2 e^{\frac{|x|^2}{4}} - \int_{\Sigma} (|A|^2 - \frac{1}{2}) v^2 e^{\frac{|x|^2}{4}}.
\]

Thus \( \mu_1 \) is attained on the function \( v = e^{-\frac{1}{4} |x|^2} \in W^{1,2}(\Sigma, e^{\frac{|x|^2}{4}} d\sigma) \). One property of the bottom of spectrum (that can be proved similar to [12, Theorem 10.10]) induces that \( v = e^{-\frac{1}{4} |x|^2} \) must satisfies the equation

\[
(7.13) \quad \mathcal{L} v + (|A|^2 - \frac{1}{2}) v + \mu_1 v = 0.
\]
Taking $\alpha = 1$ in (7.7), we have
\[ \mathcal{L}v + \left( \frac{n}{2} + H^2 \right) v = 0 \text{ on } \Sigma. \] (7.14)

By (7.13) and (7.14), it follows that
\[ \text{Scal}_\Sigma = H^2 - |A|^2 = \mu_1 - \frac{n + 1}{2} = \text{constant}. \]
Conversely, if $\text{Scal}_\Sigma = \text{constant}$, by (7.11) and Theorem 1.6 it follows that
\[ \mu_1 = \text{Scal}_\Sigma + \frac{n + 1}{2}. \]

In order to prove Corollary 1.2, we need the following known fact (see Theorem 3.20 in [15]). For the convenience of the readers, we include its proof here.

**Lemma 7.1.** Let $\gamma$ be a complete immersed self-expander curve in $\mathbb{R}^2$ whose geodesic curvature $H$ satisfies $H = -\frac{1}{2} \langle x, n \rangle$. Then either $\gamma$ is a straight line through the origin or $H^2$ is positive and bounded.

**Proof.** Without loss of generality, we assume that $\gamma$ is parametrized by its arc-length. Differentiating $H = -\frac{1}{2} \langle x, n \rangle$ and noting that $\nabla_{\gamma'} n = H \gamma'$ yield
\[ 2H' = -H \langle x, \gamma' \rangle. \] (7.15)

On the other hand, differentiating $|x|^2$, we get
\[ (|x|^2)' = 2 \langle \gamma', x \rangle. \] (7.16)

Combining (7.15) and (7.16) it follows that
\[ e^{-\frac{|x|^2}{2}} \left( H^2 e^{-\frac{|x|^2}{2}} \right)' = 2HH' + \frac{1}{2} H^2 (|x|^2)' = -H^2 \langle x, \gamma' \rangle + H^2 \langle x, \gamma' \rangle = 0. \]
Therefore $H^2 = Ce^{-\frac{|x|^2}{2}}$, for some constant $C$. This completes the proof. \( \square \)

With Lemma 7.1 we prove Corollary 1.2.

**Proof of Corollary 1.2.** Since $\Gamma$ is a complete immersed self-expander in $\mathbb{R}^2$, it is properly embedded by Theorem 6.1 in [16]. Since $\Sigma = \Gamma \times \mathbb{R}^{n-1}$, $\text{Scal}_\Sigma \equiv 0$. By Lemma 7.1, the mean curvature of $\Sigma$ is bounded. Hence, Theorems 1.6 and 1.7 imply that $\mu_1 = \frac{n + 1}{2}$. \( \square \)

Theorems 1.7 and 6.1 imply

**Corollary 7.1.** Let $\Sigma$ be a complete properly immersed self-expander in $\mathbb{R}^{n+1}$. Assume that $\text{Scal}_\Sigma \geq 0$ and $|H|(x) \leq a|x| + b$, $x \in \Sigma$, for some constants $0 \leq a < \frac{1}{2\sqrt{2}}$ and $b > 0$. Then
\[ \mu_1 \leq \frac{n + 1}{2} + \frac{\int_{\Sigma} \text{Scal}_\Sigma e^{-\frac{|x|^2}{4}} d\sigma}{\int_{\Sigma} e^{-\frac{|x|^2}{4}} d\sigma}. \] (7.17)
The equality holds if and only if $\Sigma = \Gamma \times \mathbb{R}^{n-1}$ with the product metric, where $\Gamma$ is a complete self-expander curve (properly) immersed in $\mathbb{R}^2$.

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