Hot $QED$ beyond ladder graphs

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Abstract

At finite temperature a breakdown of the hard thermal loop expansion arises whenever external momenta are light-like or tend to very soft scales. A resummation of ladder graphs is important in these cases where the effects of infrared or light-cone singularities are enhanced. We show that in hot $QED$ another class of diagrams is also relevant at leading order due to long range magnetic interactions and therefore recent studies about ladder expansions need to be corrected. A general cancellation of the hard modes damping effects still occurs near the light-cone or in the infrared region. The validity of an improved version of the hard thermal loop resummation scheme is discussed.

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1 Introduction

The loop expansion for gauge theories at high temperature suffers from a number of problems due to the extreme nature of the infrared divergences present. To address these difficulties the hard thermal loop expansion was devised [1–6]. Although successful in resolving many paradoxes, there still remain some fundamental problems with this resummation scheme in certain limits outside of its range of validity. One particular problem is that of the damping rate of a fast fermion, where a self-consistent calculational scheme outside of the hard thermal loop expansion has been used [7,8]. Another class of such problems involves processes sensitive to the behaviour near the light-cone, where the soft photon production rate estimates [9,10] already signal a breakdown of this expansion. In the same context, it was found that a resummation of asymptotic thermal masses for hard modes leaves the gauge invariance of the effective action intact [11,12].

But for such processes, a natural question is to know whether or not additional resum- mations beyond those of asymptotic thermal masses are required. A candidate is the anomalously strong damping of hard modes from interactions with soft ones. However if damping is to be taken into account, modifying only the propagators violates the Ward
identities and is therefore not sufficient. Furthermore the damping arising from interactions with soft modes cannot be incorporated into an effective action which summarizes the effects of integrating out the hard modes only. Vertex Corrections are necessary to restore gauge invariance and leads to a ladder resummation. Perturbation expansions which include ladder graphs have been considered in many works. Ladders arise for example in the context of the eikonal expansion of gauge theories [13,14]. In the infrared limit of the polarization tensor in hot QED [7], but within a simplified model using a constant damping, it was claimed that this damping term cancels out from all components of $\Pi^{\mu\nu}$. Subsequent studies [12,15] in scalar QED have led to the same conclusion for specific limits. More recently [16,17] a simple and general way to eliminate the damping terms has been put forward. A justification for such a compensation can be provided by an argument of gauge-independence. In the estimate of the damping term, keeping the external line on mass-shell while introducing a finite infrared cutoff $\mu$ gives a gauge-independent contribution [18]. But with propagators the integration is carried out over the real axis and usually leaves the momentum off mass-shell. Inserting the damping term therefore leads to gauge-dependent pieces. For energetic fermions these pieces are overwhelmed by a gauge-independent factor $\ln(1/e)$ but problems of gauge-dependence still remain when going beyond a simple logarithmic approximation at leading order. In that respect a general compensation of the damping terms in the expression of the photon polarization tensor is necessary. In short, specific calculations show that the usual hard thermal loop term seems to be recovered in the infrared region.

It is worth recalling that there are simple arguments based on kinetic equations analysis that suggest the same conclusion. In the framework of the scalar theory, the effect of the infinite set of ladders is shown to be generated by a collision term in the transport equations [19]. Extending this statement to QED, it can be shown that the effect of collisions is suppressed at length scales $(e^2T)^{-1}$ and starts to become important only at the order $(e^4T)^{-1}$ [20].

This study extends the diagrammatic approach of Refs. [7,12,16]. The main motivation is to determine all the relevant diagrams in the light-cone and in the infrared region. The main statement is that not only ladders contribute at leading order (at least beyond a simple logarithmic approximation), but also a full class of graphs with soft photon exchanges. Since the mechanism of cancellation is connected to Ward identities, this does not prevent to get a final answer in the infrared and in a 'weak' light-cone limit (as it will be explained later on). Although the result in these cases is expected to be the hard thermal loop, it is useful to study how the damping terms actually disappear. It will be seen that the demonstration is not only restricted to specific limits such as the static or the zero-momentum limits. It is finally a preparation to investigate the light-cone problem where the effects of an asymptotic mass show up. Already it demonstrates that the improved resummation scheme proposed in Ref. [11] seems to be justified but might not be complete. Finally it is worth mentioning that Refs. [7,12,16,17] left technical ambiguities that can only be overcome with an improved version of the vertex proposed in this paper, in a comparable way to what has been done at zero temperature in Ref. [21]. The next section presents a study of all the relevant graphs, using a precise power counting, the third part
is devoted to a resummation of these diagrams for the specific cases mentioned above, and the final demonstration concerns the mechanism of cancellation of the damping terms.

2 Leading diagrams

2.1 Ladder graphs

Several works have been partly devoted to the cancellation of ladder graphs in an effective expansion. That has been done either within a simplified model, namely with a constant damping, in $QED$ [7] or scalar $QED$ [12,15], or with a momentum-dependent damping using algebraic compensations [16,17]. But it would be interesting to see to which extent algebraic cancellations survive even when taking into account all the leading order diagrams, not only ladders. That was not the case in [16,17]. However, before going beyond a ladder resummation, it is worth recalling, even briefly, the power counting arguments put forward in these previous studies. Also it is important to insist on the equivalence between infrared limit and light-cone limit in this power counting, as the light-cone limit has just been superficially treated in Ref. [16].

Throughout this paper, the retarded/advanced formalism [22] is adopted, more specifically the conventions of Aurenche and Becherrawy. The structure of Green functions as 'tree-like diagrams' can easily be seen. Nevertheless all the calculations can be performed within different real time formalisms, for example the Schwinger-Keldysh technique [23]. The simple $ee\gamma$ one-loop vertex with $R/A$ prescriptions and the exchange of a soft photon reads

\[
\tilde{V}_{RAR}^\mu (P, Q, -R) = -e^2 \int \frac{d^4 L}{(2\pi)^4} P^\mu_{\rho\sigma}(L) (\gamma^\rho(R + L)\gamma^\mu(P + L)\gamma^\sigma) \\
\left\{ \left( \frac{1}{2} + n(l_0) \right) \left( *\Delta_R^i(L) - *\Delta_A^i(L) \right) \Delta_R(P + L)\Delta_A(R + L) \right. \\
+ \left( \frac{1}{2} - n_F(p_0 + l_0) \right) (\Delta_R(P + L) - \Delta_A(P + L)) \Delta_A(R + L) *\Delta_R^i(L) \\
+ \left( \frac{1}{2} - n_F(r_0 + l_0) \right) (\Delta_R(R + L) - \Delta_A(R + L)) \Delta_R(P + L) *\Delta_A^i(L) \right\} , \tag{2.1}
\]

where $*\Delta(L)$ is the effective propagator resummed within the hard thermal loop scheme. The particular example of the transverse component (in covariant gauges) is taken. Each statistical factor is associated with a cut internal line and the graph itself appears as a sum of tree-diagrams. A crucial point (unnoticed in Ref. [16,17]) is that the leading term is given by the Bose-Einstein distribution. Splitting the product of fermion propagators $\Delta(P)\Delta(R)$ leads to
\[
\tilde{V}_{RAR}^\mu(P, Q, -R) = -e^2 \int \frac{d^4 L}{(2\pi)^4} P^\mu_\rho L (\gamma^\rho (R + L) \gamma^\mu (P + L) \gamma^\sigma) 
\]
\[
\times \frac{1}{n(l_0) \rho T} (\Delta_A(R + L) - \Delta_R(P + L)).
\] (2.2)

A straightforward estimate can be done when the denominator \(2Q.(P + L) + Q^2\) is of the same order as \(2P.Q + Q^2\). Then the leading vertex has the same magnitude as

\[
\tilde{V}_{RAR}^\mu(P, Q, -R) \sim -e^2 \frac{2P^\mu + Q^\mu}{2Q.P + Q^2} \int \frac{d^4 L}{(2\pi)^4} P^\mu_\rho L (\gamma^\rho P \gamma^\sigma) n(l_0) \rho T(L) 
\]
\[
\times (\Delta_A(R + L) - \Delta_R(P + L)).
\] (2.3)

Under what circumstances the denominator can be extracted from the integral remains to be seen. The difference of self-energies written above has the same order as the damping rate \(i.e. e^2T\) (discarding the well-known problems of logarithmic infrared divergences which do no affect the order of magnitude). This is compensated by the term \(1/(2P.Q + Q^2)\) which brings an extra factor \(1/e^2\). The vertex is therefore of the same order as its tree-level counterpart. From this power counting it was already extensively explained in previous works how the generation of further ladder graphs leads to the same order. It can be easily verified from the expressions of the multi-loop graphs of Ref. [22] that the leading contributions involve only tree-diagrams with cut photon lines. This estimate enables to distinguish between three different limits:

- The infrared limit where the photon momentum lies in a very soft scale, \(q_0, q \sim O(e^2T)\).
- A weak light-cone limit for the vertex where the photon is soft \(q_0, q \sim O(eT)\) but real or almost real \(q_0 \sim q + O(e^2T)\). If \(P\) is one of the electron momenta, the angle of emission is \(\hat{p}.\hat{q} \sim \pm 1 + O(e)\).
- A strong light-cone limit with the same conditions for the photon momentum but \(\hat{p}.\hat{q} \sim \pm 1 + O(e^2)\). This is typically the limit of interest for multiple scatterings phenomena. Also if the fermion propagators are resummed, the terms involving an asymptotic mass contribute [11].

It is important to mention that diagrams with hard photons exchanges do not contribute, although the phase-space is larger. This is due to non-trivial compensations between the trace and some denominators. One of the propagators always gets suppressed and the graphs are not enhanced as in the previous case.

It will be seen that ladders with soft photon exchanges cancel against the corresponding leading order self-energies without vertex corrections (rainbow diagrams). This is due to a mechanism related to Ward identities. Here the status concerning the spectral density has not been precised. Unlike the damping rate problem the external momentum of the self-energy is a variable. Kinematics does not forbid the exchanges of time-like photons, even though it is the Landau damping part of the spectral density which should give the dominant contribution, the fermion is still close to its mass-shell. Now with the Landau damping part the photon momentum can reach the infrared scale \(e^2T\) due to the absence
of magnetic screening and this may change the estimate of multi-loop diagrams. If it turns out that self-energies with vertex corrections at leading order contribute too, then graphs other than ladders might be relevant for consistency and therefore it is necessary to go beyond what has been done in [7,12,16,17].

2.2 Two-loop self-energy

The two-loop retarded self-energy for an on-shell electron is considered. Since the exchange of transverse photons is characterized by the absence of (static) magnetic screening, a naive power counting suggests that the infrared scale of $O(e^2 T)$ could dramatically change the order of higher loop diagrams. This is due to power-like infrared divergences and, as a first example, the simple vertex correction of the self-energy might contribute at the same leading order as its bare counterpart. Simple estimates made in [7] within a simplified model led to the conclusion that such vertex corrections are subleading, while it has been pointed out in [24] that these corrections conspire to give a leading order evaluation of the damping rate. There is a need to clarify the situation, since the presence of leading vertex corrections in the self-energies would imply that ladders are not the only relevant diagrams in the infrared and light-cone limits of the polarization tensor.

A straightforward estimate of this two-loop graph can be obtained using the simplified transverse spectral density introduced in Ref. [24]

$$\rho_T(Q) = \frac{2\pi}{q^2} \delta(q_0). \quad (2.4)$$

The complete calculation, starting with the exact transverse density, will be presented in the appendix. The retarded self-energy in the $R/A$ formalism [22] can be written as

$$-i\Sigma_{RR}(P) = -e^2 \int \frac{d^4Q}{(2\pi)^4} n(q_0) \Delta_R(P + Q) P_{\mu\nu}^t(Q) \left[ \tilde{V}_{ARA}^\mu(-Q, P + Q, -P) \Delta_R(Q) 
- \tilde{V}_{RAA}^\mu(-Q, P + Q, -P) \Delta_A(Q) \right] (P + Q) \gamma^\nu, \quad (2.5)$$

with the expression

$$\tilde{V}_{ARA}^\mu(-Q, P + Q, -P) = -e^2 \int \frac{d^4K}{(2\pi)^4} n(k_0) \rho_T(K) P_{\rho\sigma}^t(K) \Delta_R(P + Q + K) \Delta_R(P + K) \gamma^\rho (P + Q + K) \gamma^\sigma. \quad (2.6)$$

In this specific case the $R/A$ indice of the photon does not modify the expression of the vertex

$$\tilde{V}_{ARA}^\mu(-Q, P + Q, -P) = \tilde{V}_{RRA}^\mu(-Q, P + Q, -P). \quad (2.7)$$
A direct evaluation of the order of magnitude can be obtained if the self-energy itself is replaced by a simpler quantity without modifying the order. Such a quantity may be the imaginary part of the trace \( \text{Tr}(P\Sigma(P)) \). In order to simply evaluate the order it always makes sense to take \( P^2 = 0 \). Next, the usual simplifications can be made, i.e. keeping only hard terms in the trace and using the approximate Bose-Einstein factors \( T/q_0 \) and \( T/k_0 \) for soft photons. Finally the damping of a particle with positive energy is chosen. Retaining only the positive energy parts of the propagators gives

\[
\frac{1}{4p_0} \text{Tr}(P\Sigma_{RR}(P)) = (e^2T)^2 \int \frac{d^3q}{(2\pi)^3} \int \frac{dq_0}{(2\pi)^3q_0} \rho_T(Q) \int \frac{d^3k}{(2\pi)^3k_0} \rho_T(K)
\]

\[
\text{Im} \frac{1}{q_0 - \hat{p} \cdot \hat{q} + i\epsilon} \frac{1}{k_0 - \hat{p} \cdot \hat{k} + i\epsilon} \frac{1}{q_0 + k_0 - \hat{p} \cdot (\hat{q} + \hat{k}) + i\epsilon} \left(1 - (\hat{p} \cdot \hat{q})^2\right) \left(1 - (\hat{p} \cdot \hat{k})^2\right).
\] (2.8)

where the spectral density depicting the exchange of static magnetic photons is given by the simplified form of Eq. [2.4]. Upper and lower cut-offs need to be introduced by hand. These can be respectively the plasmon frequency \( \omega_p \) and the parameter \( \mu \), the former being of order \( eT \) and the latter of order \( e^2T \). Taking into account the non trivial phase space due to kinematical constraints, it turns out that the expression of Eq. [2.8] can be decomposed into three parts

\[
A_1 = -2i\pi(e^2T)^2 \int_{\mu}^{\omega_p} \frac{dq}{(2\pi)^2q^2} \int_{-1}^{1} dx \left(\frac{P}{x^2} - 1\right) \int_{\mu}^{\omega_p} \frac{dk}{(2\pi)^2k}
\]

\[
= \frac{4i}{(2\pi)^3(e^2T)^2} \frac{1}{\mu} \ln \left(\frac{\omega_p}{\mu}\right)
\] (2.9)

\[
A_2 = i\pi(e^2T)^2 \int_{\mu}^{\omega_p} \frac{dq}{(2\pi)^2q^2} \int_{-1}^{1} dx \left(\frac{P}{x^2} - 1\right) \int_{q}^{\omega_p} \frac{dk}{(2\pi)^2k} \int_{-1}^{1} dy \delta \left(y + \frac{q}{k} x\right) (1 - y^2)
\]

\[
= -\frac{2i}{(2\pi)^3(e^2T)^2} \frac{1}{\mu} \left(\ln \left(\frac{\omega_p}{\mu}\right) - \frac{5}{6}\right)
\] (2.10)

\[
A_3 = i\pi(e^2T)^2 \int_{\mu}^{\omega_p} \frac{dq}{(2\pi)^2q^2} \int_{\mu}^{q} \frac{dk}{(2\pi)^2k} \int_{-\frac{k}{q}}^{\frac{k}{q}} dx \left(\frac{P}{x^2} - 1\right) \int_{-1}^{1} dy \delta \left(y + \frac{q}{k} x\right) (1 - y^2)
\]

\[
= -\frac{2i}{(2\pi)^3(e^2T)^2} \frac{1}{\mu} \left(\ln \left(\frac{\omega_p}{\mu}\right) - \frac{5}{6}\right)
\] (2.11)
A cancellation of the leading singularity \((1/\mu) \ln(\omega_p/\mu)\) occurs between these three parts. Still a term \(1/\mu\) does not get eliminated.

\[
\frac{1}{4p_0} \text{Tr} (P\Sigma_{RR}(P)) = A_1 + A_2 + A_3 = (e^2 T)^2 \frac{10i}{3(2\pi)^3} \frac{1}{\mu}.
\]

(2.12)

This means that, when dealing with complete fermion propagators instead of bare ones, dressed by full self-energies \(\Sigma(P + L_i)\), the diagram is sensitive to the scale \(e^2 T\) with \(\Sigma(P + L_i)\) providing possible cut-offs. In conclusion, the two-loop graph is at the same leading order \(e^2 T\) as the one-loop self-energy. Actually multi-loop vertex corrections must be taken into account whenever transverse photons are considered.

### 2.3 Non-planar diagram

The importance of a ladder resummation in the infrared limit is a well-known fact. Since the cancellation of the damping terms is related to Ward identities, ladders by themselves correspond to the elimination of self-energies without vertex corrections. However it has been noticed in the previous section that vertex corrections contribute at the same leading order. Therefore diagrams in the \(ee\gamma\) vertex strictly related to these corrections via Ward identities must also be taken into account. In particular, non-planar graphs are shown to be relevant contrary to what has been claimed in Ref. [16] and repeated in ref. [17].

In [16,17] the example of a higher loop crossed graph is mentioned. The case of a scalar \(\lambda \phi^3\) theory at zero temperature is considered but with a generalization for other theories at finite temperature in sight. Using the same notations as in [16,17] the two-loop non-planar diagram is written as

\[
-i\Pi(K) = (-i\lambda)^6 \int dR_1 \, dR_2 \, dP \, \Delta(P + R_1 + R_2) \Delta(P + R_1 + R_2 + K) \Delta(R_1) \\
\Delta(P + R_2) \Delta(P + R_1 + K) \Delta(R_2) \Delta(P) \Delta(P + K),
\]

(2.13)

where \(\Delta(K) = 1/(K^2 + i\epsilon)\). It is claimed that this kind of graphs does not contribute in the same way as the ladder graphs. While the product of propagators \(\Delta(P + R_1 + R_2) \Delta(P + R_1 + R_2 + Q)\) would be split in the infrared limit \(2Q \cdot R_1 + R_2 \ll Q^2 + 2Q \cdot P\) as

\[
\Delta(P + R_1 + R_2) \Delta(P + R_1 + R_2 + Q) \\
= i \frac{\Delta(P + R_1 + R_2) - \Delta(P + R_1 + R_2 + Q)}{Q^2 + 2Q \cdot P},
\]

(2.14)

splitting the product \(\Delta(P + R_2) \Delta(P + R_1 + Q)\) in the same limit \(2Q \cdot R_1 \ll Q^2 + 2Q \cdot P\)
then would produce
\[
\Delta(P + R_2) \Delta(P + R_1 + Q) \approx i \frac{\Delta(P + R_2) - \Delta(P + R_1 + Q)}{Q^2 + 2Q \cdot P + (P + R_1)^2 - (P + R_2)^2}.
\] (2.15)

By itself it would not lead to a cancellation of a factor of $\lambda^2$ in the numerator due to the presence of the $(P + R_1)^2 - (P + R_2)^2$ term. The argument put forward is that even by furthermore restricting the phase space so that $P \cdot R_i$ and $R_i^2$ ($i = 1, 2$) is sufficiently small, this introduces extra factors of $\lambda$ in the numerator coming from the momentum integral over $P$. The conclusion drawn is that in the infrared limit such crossed graphs are suppressed relative to the ladder graphs.

However the point is that, when considering transverse spectral densities in gauge theories there is no restriction of the phase-space (extra factors $\lambda$ in the numerator) because of the absence of magnetic screening at the order $\lambda T$. This is responsible for the well-known infrared sensitivity of the hard fermion damping rate. Therefore terms like $P \cdot R_i$ and $R_i^2$ ($i = 1, 2$) can be sufficiently small without rendering the corresponding regions of integration negligible. More precisely, a power counting can be provided for this crossed diagram. Only bare fermion propagators are considered at this stage. Soft photons propagators are resummed within the HTL scheme as they should, and the approximate spectral density of Eq. [2.4] may be used. The basic expression of the crossed graph in the $R/A$ formalism is already complicated. But it can be found that the leading pieces correspond to the tree-like terms with cut internal photon lines. The dominant contribution of the crossed diagram with the $RAR$ prescriptions as a particular example, is

\[
\tilde{V}_{RAR}^\mu(P, Q, -R) = e^4 \int [dL_1]^t_{\alpha \beta} [dL_2]^t_{\gamma \rho} \gamma^\alpha(R + L_1) \Delta_A(R + L_1) \gamma^\rho(R + L_1 + L_2) \\
\quad \quad \\quad \quad \Delta_A(R + L_1 + L_2) \gamma^\mu(P + L_1 + L_2) \Delta_R(P + L_1 + L_2) \\
\quad \quad \\quad \quad \gamma^\beta(P + L_2) \Delta_R(P + L_2) \gamma^\rho,
\] (2.16)

where the shorthand $[dL_i]$ stands for
\[
[dL_i]^t_{\alpha \beta} = \frac{d^4 L_i}{(2\pi)^4} n(l_{i0}) \rho_T(L_i) P^i_{\alpha \beta}(L_i).
\] (2.17)

The usual decomposition of the fermion propagator reads
\[
\Delta_R(P) = \frac{i}{2\Omega_P} \left( \frac{1}{p_0 - \Omega_P + i\epsilon} - \frac{1}{p_0 + \Omega_P + i\epsilon} \right).
\] (2.18)

As it is only a matter of determining the order of magnitude of a graph, various simplifications can be made. First the first term of the decomposition above corresponds to an electron propagating forward in time, and the second term to an antiparticle also propagating forward in time. All the quantities involve soft photons and are related to the
picture of an energetic electron (or positron) close to its mass-shell undergoing multiple interactions with soft photons. The anihilation process between a particle and an antiparticle is subleading compared to simple scattering processes of the same particle. In the expression of the vertex above only the positive energy parts of the propagators may be retained, in order to find out just the order of magnitude. This strictly follows the lines of the previous estimate concerning the two-loop self-energy diagram. Splitting the product $\Delta(P + L_1 + L_2)\Delta(\mathbb{R} + L_1 + L_2)$ and keeping only the positive energy components give the term

$$
\tilde{V}_{RAR}^\mu(P, Q, -R) = e^A \int [dL_1]_{\alpha\beta}^I [dL_2]^I_{\sigma\rho} \gamma^\alpha(R + L_1)\Delta^\mu_{R}(R + L_1)\gamma^\sigma(R + L_1 + L_2)
$$

where the $i\epsilon$ prescription is absorbed into a re-definition of $q_0$. It will become clear in the next sections that the same procedure is employed when dealing with resummed propagators. The common denominator $1/(q_0 + \Omega_P + L_1 + L_2 - \Omega_{R + L_1 + L_2})$ has the same order as $1/(q_0 - \Omega_P + \Omega_R)$. Under what circumstances this replacement can actually be done will be seen in details later on. The last step of the power counting consists in taking particular terms easily calculable under specific conditions from expression [2.19]. Such a procedure should not change the order of magnitude. This is the case when taking the imaginary part $(1/4p_0)\text{ImTr}(P\tilde{V})$ evaluated with both $P$ and $R$ on-shell (internal momenta are close to their mass-shell). It turns out that

$$
\tilde{V}_{RAR}^\mu(P, Q, -R) \sim \frac{\hat{P}^\mu}{q_0 + \Omega_P - \Omega_R} \left[ \frac{1}{4p_0} \text{ImTr}(P\Sigma_1(P)) - \frac{1}{4p_0} \text{ImTr}(P\Sigma_2(P)) \right],
$$

where

$$
\frac{1}{4p_0} \text{ImTr}(P\Sigma_1(P)) = -(e^2 T)^2 \int \frac{d^3l_1}{(2\pi)^3} \int \frac{d^3l_2}{(2\pi)^3} \frac{(1 - (\hat{p} \cdot \hat{l}_1)^2)}{(1 - (\hat{p} \cdot \hat{l}_2)^2)} \frac{1}{\hat{p} \cdot \hat{l}_1 + i\epsilon} \frac{1}{\hat{p} \cdot \hat{l}_2 - i\epsilon} \frac{1}{\hat{p} \cdot (\hat{l}_1 + \hat{l}_2) - i\epsilon}.
$$

(2.21)

Thereby $(1/4p_0)\text{ImTr}(P\Sigma_2(P))$ has the same expression except a $+ i\epsilon$ prescription in the last denominator, and the sign $\sim$ means ‘of the same order as’. In the same way as with the two-loop self-energy, the infrared cut-off $\mu \sim O(e^2 T)$ must be introduced. The term $1/\mu$ does not get canceled

$$
\tilde{V}_{RAR}^\mu(P, Q, -R) \sim \frac{\hat{P}^\mu}{q_0 + \Omega_P - \Omega_R} (e^2 T)^2 \frac{1}{(2\pi)^3} \frac{1}{\mu} \ln \left( \frac{\omega_P}{\mu} \right),
$$

(2.22)
and shows the sensitivity to the infrared regime (quantities $\sim O(e^2T)$). With resummed fermion propagators the crossed diagram will be of the same order as the ladders. The inclusion of higher order crossed graphs is necessary. This is consistent with the fact that vertex corrections in the self-energies are relevant at leading order.

### 2.4 Resummed fermion propagator

It is important at this stage to review all the relevant diagrams involved in the self-energy. All these graphs have to be resummed to form the propagator of a hard fermion close to its mass-shell. First, as advocated in Ref. [11], the hard thermal loop self-energy must be considered. The well-known two-structure functions of the HTL term read

$$\Sigma_{HTL}(P) = a p_0 \gamma_0 + b \vec{p} \cdot \vec{\gamma},$$

(2.23)

where

$$a = -\frac{e^2 T^2}{2 p p_0} \ln (\frac{p_0 + P}{p_0 - p}), \quad b = \frac{e^2 T^2}{p^2} \left[ 1 - \frac{p_0}{2 p} \ln (\frac{p_0 + P}{p_0 - p}) \right].$$

(2.24)

Also the self-energy $\Sigma$ involving soft longitudinal and transverse photon exchanges must be resummed. The dressed (for example retarded) propagator is given by

$$S_R(P) = \frac{i}{P - \Sigma_{HTL}(P) - \Sigma_R(P)} = \frac{i P + O(e^2)}{P^2 - 2 a p_0^2 - 2 b p^2 - P \Sigma_R(P) - \Sigma_R(P) P}.$$  

(2.25)

The combination of the HTL structure functions gives the asymptotic mass [11]

$$m^2_\infty = 2(p_0^2 a + p^2 b) = \frac{1}{4} e^2 T^2.$$  

(2.26)

It is also necessary to introduce (with implicitly the same definition for the advanced counterpart)

$$\sigma_R(P) = -\frac{1}{4 p_0} \text{Tr} (P \Sigma_R(P)),$$

(2.27)

such that the imaginary part of this quantity just corresponds to the damping rate. Although the real part might be subleading, it is better to keep this real term in the following.
It will get eliminated along with the damping in a general mechanism of cancellation. The fermion propagator finally reads

$$S_R(P) = \frac{iP + O(e^2)}{P^2 - m_\infty^2 + 2p_0\sigma_R(P)}, \quad (2.28)$$

where it is worth noting that without $\sigma$ (and damping resummation) the propagator introduced in Ref. [11] is obviously recovered. The 'one loop' contribution to $\Sigma$ is

$$\Sigma_1^R(P) = \sum_{i=t,l} (-ie^2) \int [dL]_{\alpha\beta}\gamma^\alpha S_R(P + L)\gamma^\beta, \quad (2.29)$$

where in fact the resummed propagator is not necessary in the longitudinal part. There is no sensitivity to the infrared scale due to Debye screening. The 'two-loop' self-energy involves just the simple vertex correction with transverse exchanges

$$\Sigma_2^R(P) = -ie^2 \int [dL]_{\alpha\beta}\tilde{V}^\alpha_{ARR}(-P, L, P + L)S_R(P + L)\gamma^\beta. \quad (2.30)$$

It is therefore the Landau damping part of the spectral density which gives the dominant piece here. The time-like part leads to subleading terms. That will be the same for the multi-loop vertices. However for the ladders and the 'one-loop' self-energy kinematics does not forbid completely processes with (time-like) quasi-particles. In order to simplify the notations the entire expression of the transverse spectral density may be kept for all the diagrams, thus including negligible terms in most of the cases. This will be implicit for higher-order vertices. The following 'N-loop' self-energies correspond to all the possible vertex corrections (with the same resummed fermion propagator). There are four 'three-loop' graphs, and by the multiple insertions of a photon leg along the fermion line, each of these graphs is related to a subset of 'N-loop' $ee\gamma$ vertices (always with the same fermion propagator) via Ward identities. The complete self-energy introduced above may be written as

$$\Sigma_R(P) = \Sigma_1^R(P) + \Sigma_2^R(P) + \sum_{K=1}^{4} \Sigma_K^R(P) + ... \quad (2.31)$$

3 Ladders and diagram resummation

3.1 Preliminaries

In this section a method is described for including all the diagrams discussed previously in an effective expansion. Unlike what has been done before [7,16,17], this does not only involve ladders but all the loop graphs of the four-fermion amplitude which cannot be
disconnected by cutting two fermion lines. These contain resummed fermion propagators as well as soft photon propagators (longitudinal and transverse for the ladders, transverse for all the other cases). A first non-trivial example was the subset of 'two-loop' graphs, namely the crossed diagram with the two symetric vertex corrections. The next subset of 'three-loop' diagrams for example is determined from the four 'three-loop' self-energies via Ward identities (i.e. via the insertions of an external photon leg). This can be summarized in the Bethe-Salpeter equation with an infinite kernel

\[ \tilde{V}^\mu(P, Q, -R) = \gamma^\mu + \int \frac{d^4 P'}{(2\pi)^4} K(P, -P', R', -R) K_R R' \Delta(R') \left( \gamma^\mu + \tilde{V}^\mu(P', Q, -R') \right) P' \Delta(P') K_P, \]  

(3.1)

where

\[ K(P, -P', R', -R) K_P \cdot K_R = K^1(P, -P', R', -R) K^1_P \cdot K^1_R + K^2(P, -P', R', -R) K^2_P \cdot K^2_R + \ldots \]

\[ R = P + Q; \quad R' = P' + Q. \]

In this equation, the kernel \( K \) represents the infinite series of four-electron amplitudes. \( K^1 \) corresponds to all the simple ladders, \( K^2 \) to the crossed and vertex correction graph resummation, etc...

Previous works on the ladders [7,12,16] have led to the following formula for the vertex

\[ \tilde{V}_\mu(P, Q, -R) \approx \frac{2 P_\mu + Q_\mu}{Q^2 + 2Q \cdot P} \left[ \Sigma(P) - \Sigma(R) \right], \]

(3.2)

where the \( R/A \) prescriptions have been omitted, and the self-energies do not include vertex corrections. Although this expression corresponds to a ladder summation, it is not well-defined due to the absence of an \( i \epsilon \) prescription or a cut-off in the denominator. In particular sticking this form into the polarization tensor does not solve the problem and leaves the same ambiguity [12,16]. In order to guess the correct formula for the vertex a good starting point is to consider the one loop graph with bare propagators in a \( \lambda \phi^3 \) theory

\[ \tilde{V}(P, Q, -R) = -\lambda^2 \int [dL] \Delta(R + L) \Delta(P + L). \]

(3.3)

The propagator can be split into a positive energy part and a negative one. Thereby the product \( \Delta(P + L)\Delta(R + L) \) leads to subleading denominators for mixed (positive/negative energies) terms. The vertex now reads

\[ \tilde{V}(P, Q, -R) \approx \lambda^2 \int [dL] \left[ \frac{1}{q_0 + \Omega_{P+L} - \Omega_{R+L}} \left( \frac{1}{p_0 + l_0 - \Omega_{P+L}} - \frac{1}{r_0 + l_0 - \Omega_{R+L}} \right) \right] \]
Anticipating the fact that under specific circumstances, especially the infrared limit, denominators like \( q_0 + \Omega_P + \Omega_R - \Omega_{P+L} \) can be approximated by \( q_0 + \Omega_P - \Omega_R \) and therefore can be extracted from the integral, finally yields

\[
\tilde{V}(P, Q, -R) = \frac{1}{q_0 + \Omega_P - \Omega_R} \left( \Sigma^+(P) - \Sigma^+(R) \right) + \frac{1}{q_0 + \Omega_R - \Omega_P} \left( \Sigma^-(P) - \Sigma^-(R) \right),
\]

where

\[
\Sigma^\pm(P) = \pm \lambda^2 \int [dL] \frac{1}{4p^2} \frac{1}{p_0 + l_0 \mp \Omega_{P+L}}.
\]
be understood that this addition is unimportant as long as calculations are performed at leading order.

### 3.2 Ladder resummation

In previous works [7,12,16] a resummation of ladders has been considered, discarding the possible inclusion of more complicated diagrams (see Fig. 1). Deferring the discussion about these diagrams to the following sections, the first step consists in including the expression of Eq. [3.8] in the first term of the Bethe-Salpeter equation corresponding to a ladder summation. The complete form of the vertex [3.8] should lift any ambiguity concerning the absence of the \( i \epsilon \) prescriptions in denominators such as \( 1/P.Q \) and enables to get general expressions in the infrared, not just specific limits like \( \Pi^{00}(q_0,0) \) and \( \Pi^{0i}(0,q \to 0) \).

Sticking the expression [3.8] into the first term of Eq. [3.1] yields

\[
\tilde{V}_{RAR}^\mu(P, Q, -R) = \sum_{i=t,l} (-e^2) \int [dL]_\rho \Delta_R(P + L)\Delta_A(R + L)\gamma^\rho(R + L) \left[ \gamma^\mu + \tilde{V}_{RAR}^\mu(P + L, Q, -R - L) \right] (P + L)\gamma^\sigma,
\]

where the \( RAR \) indices have been chosen as a particular example, the generalization to any kind of \( R/A \) vertex being straightforward. The indice 1 is a reminder that the vertex is a 'one-loop' graph with resummed fermion propagators and dressed transverse and longitudinal photons. First it is useful to define

\[
\sigma^{\pm}_{R/A}(P) = -\frac{1}{4p_0} \text{Tr} \left( P \Sigma^{\pm}_{R/A}(P) \right), \quad \Sigma(P) = \Sigma^+(P) + \Sigma^-(P).
\]

Contracting the expression of the vertex with the spinors leads to

\[
(R + L)\gamma^\mu(P + L) = 2(P^\mu + L^\mu)(R + L) - (R + L)(P + L)\gamma^\mu \approx 2P^\mu P,
\]

\[
(R + L)v^\mu\Sigma^+(P + L)(P + L) = \frac{1}{2} v^\mu \text{Tr} \left[ (P + L)\Sigma^+(P + L) \right] (R + L)
- v^\mu(R + L)(P + L)\Sigma^+(P + L) \approx -2P^\mu\sigma^+(P)P,
\]

where soft terms may be discarded. Furthermore the phase-space is limited to the regions where the electrons are close to their mass-shell \( (P^2 \sim O(e) \text{ at most}) \) which justifies some of the approximations made above. This can be seen on complex analysis grounds. Replacing the damping terms of the propagators by simple \( i \epsilon \) prescriptions, the discontinuities are simple poles. Picking up these poles involves the constraints \( p_0 + l_{i_0} \approx \Omega_{P+L_i} \) or \( p_0 + l_{i_0} \approx -\Omega_{P+L_i} \) for a specific fermion momentum. This remains true for the other
momenta whose differences are given by the soft terms \( L_i \). These approximate conditions should not be modified when coming back to the resummed propagators. This argument will be repeated throughout this paper. The expression written above now reads (with still a complex energy \( q_0 = \text{Re} q_0 + i\epsilon \))

\[
\tilde{V}_{RAR}^{1\mu} (P, Q, -R) = \sum_{i=t, l} (-e^2) \int [dL]_i \rho^\sigma \Delta_R(P + L) \Delta_A(R + L) 2P^\mu \gamma^\rho P^\gamma^\sigma \\
\left[ 1 - \frac{\sigma_R^I(P + L) - \sigma_A^+(R + L)}{q_0 + \Omega_{P+L} - \Omega_{R+L}} - \frac{\sigma_R^L(P + L) - \sigma_A^-(R + L)}{q_0 + \Omega_{R+L} - \Omega_{P+L}} \right].
\] (3.13)

The fermion propagator may be decomposed into a positive energy and a negative energy part. As explained above the contribution of the resummed vertex in the infrared and on the light-cone becomes relevant only when the momenta \( P + L_i \) approach their mass-shell, either with a positive energy or a negative one, namely \( p_0 \simeq \pm \Omega_P \). That enables to split the entire expression of the vertex into a positive and a negative part. Under these conditions, both the self-energy \( \sigma^+ \) inside the positive component of the propagator and \( \sigma^- \) inside the negative one become negligible. It turns out that

\[
\Delta_{R/A}(P) = \frac{i}{P^2 - m_\infty^2 + 2p_0\sigma_{R/A}(P)} \\
\simeq \frac{i}{2p} \left( \frac{1}{p_0 - \Omega_P + \sigma_{R/A}^+(P)} - \frac{1}{p_0 + \Omega_P + \sigma_{R/A}^-(P)} \right) \\
= \Delta_{R/A}^+(P) + \Delta_{R/A}^-(P).
\] (3.14)

Considering the product of propagators \( \Delta_R(P)\Delta_A(R) \) the previous decomposition leads to

\[
\Delta_R(P)\Delta_A(R) \simeq \frac{i}{2p} \left( \frac{1}{q_0 + \Omega_P - \sigma_R^+(P) + \sigma_A^+(R)} \right) \left[ \Delta_R^+(P) - \Delta_A^+(R) \right]
\]
\[ \frac{1}{q_0 + \Omega_R - \Omega_P + \sigma_\Lambda(R) - \sigma_\Lambda^{-1}(R)} \left[ \Delta_R^{-1}(P) - \Delta_\Lambda^{-1}(R) \right], \tag{3.15} \]

where crossed (positive times negative) terms exhibit subleading common denominators. For the same reason as above when the momenta \( P + L_i \) are almost on-shell, with a positive energy (resp. negative) the self-energies \( \sigma^- \) (resp. \( \sigma^+ \)) are subleading. This gives for instance

\[ \left[ \Delta_R^+(P) - \Delta_\Lambda^+(R) \right] \left[ 1 - \frac{\sigma_R^+(P) - \sigma_\Lambda^+(R)}{q_0 + \Omega_P - \Omega_R} \right] \]
\[ \approx \left[ \Delta_R^+(P) - \Delta_\Lambda^+(R) \right] \left[ 1 - \frac{\sigma_R^+(P) - \sigma_\Lambda^+(R)}{q_0 + \Omega_P - \Omega_R} \right], \tag{3.16} \]

along with the analogous approximation for the negative energy counterpart. The vector \( P^\mu/p \) can finally be approximated by the unit vector \((1, \hat{p}^i)\) within the positive part and by \((-1, \hat{p}^i)\) within the negative one. This estimate is valid only at leading order and leaves soft subleading terms unconsidered. Since again soft corrections are unimportant, these vectors can be replaced respectively by \( u^\mu \) and \(-v^\mu\) in order to satisfy strictly Ward identities for the vertex. The mechanism of algebraic cancellation between denominators and numerators with damping then occurs

\[ V_{RAR}^{1\mu}(P, Q, -R) = \sum_{i=l, t} \left( -ie^2 \right) \int [dL] \gamma_\mu \gamma^\sigma \left( \frac{u^\mu}{q_0 + \Omega_P + L - \Omega_R + L} \right) \left[ \Delta_R^+(P + L) - \Delta_\Lambda^+(R + L) \right] \]
\[ - \left[ \Delta_R^+(R + L) \right] + \left[ \frac{v^\mu}{q_0 + \Omega_R + L - \Omega_P + L} \right] \left[ \Delta_R^{-1}(P + L) - \Delta_\Lambda^{-1}(R + L) \right]. \tag{3.17} \]

A basic point is to see to which extent the common denominators without damping can be extracted from the integral. Taking explicitly the expressions \( \Omega_{P+L} \) and \( \Omega_{R+L} \), namely \( \sqrt{(\hat{p} + \hat{l})^2} \) and \( \sqrt{(-\hat{r} + \hat{l})^2} \), with the soft terms from \( L \) and \( Q \), an expansion of these denominators can be written as

\[ q_0 + \Omega_{P+L} - \Omega_{R+L} = q_0 - \hat{p} \cdot \hat{q} \left( 1 - \frac{m^2}{2\hat{p}^2} \right) \]
\[ \frac{q^2}{2\hat{p}} \left( 1 - \hat{p} \cdot \hat{q} \right) - \frac{\hat{q}}{\hat{p}} \left[ \hat{l} - \hat{p}(\hat{p} \cdot \hat{l}) \right] \]
\[ + q F(\hat{p}, \hat{q}, \hat{l}) + O(e^4) \]
\[ = q_0 + \Omega_P - \Omega_R - \frac{\hat{q}}{\hat{p}} \left[ \hat{l} - \hat{p}(\hat{p} \cdot \hat{l}) \right] + O(e^3) \tag{3.18} \]

where \( F \) is a function of the three vectors and of order \( e^2 \) at most. In the infrared limit \( (q_0, q \sim O(e^2 T)) \) and below the terms following \( q_0 + \Omega_P - \Omega_R \) can always be neglected. The difference \( q_0 + \Omega_P - \Omega_R \) is equal to \( q_0 - \hat{p} \hat{q} \) and the asymptotic mass gives a subleading contribution (except on the light-cone). On the light-cone if \( \hat{p} \cdot \hat{q} \sim \pm 1 + O(e) \) the subsequent terms remain of order \( e^3 T \) (\( \hat{q} \sim \hat{p} \) and \( (\hat{q}/p).((\hat{l} - \hat{p}(\hat{p} \cdot \hat{l})) \sim O(e^3 T) \)) and can
be also discarded. However when the presence of the asymptotic mass becomes necessary
($\hat{p}\cdot\hat{q} \sim \pm 1 + O(e^2)$), $q_0 + \Omega_P - \Omega_R$ is of the same order as the other contributions
and cannot be extracted. Therefore, except when $q_0 + \Omega_P - \Omega_R \sim O(e^{2T})$ on the light-cone,
the replacements $1/(q_0 + \Omega_{P+L} - \Omega_{R+L})$ by $1/(q_0 + \Omega_P - \Omega_R)$ and
$1/(q_0 + \Omega_{R+L} - \Omega_{P+L})$ by $1/(q_0 + \Omega_R - \Omega_P)$ are always justified
and the denominators above may be extracted from the integral. The result for the 'one-loop' vertex related to the simple ladder resummation
is

$$V_{RAR}^{1\mu}(P, Q, -R) = \frac{\bar{u}^\mu}{q_0 + \Omega_P - \Omega_R} \left( \Sigma_{R}^{1+}(P) - \Sigma_{A}^{1+}(R) \right)$$

with the 'one-loop' self-energies with resummed fermion and photon propagators and the
simplest vertex correction. Thus this first step clearly indicates that it is necessary to
go beyond ladder resummation in order to recover the complete expression [3.8] of the
vertex.

3.3 Two-loop diagrams resummation

In the previous subsection, only the simple ladder resummation has been considered. It is
now important to go beyond this and to take into account vertex corrections and crossed
diagrams since these graphs have been shown to contribute also at leading order. Therefore
the 'two-loop' graph of the vertex has to be investigated (see Fig. 2). Still fermion
propagators are resummed with the 'complete' leading order self-energy. The procedure
leading to an algebraic cancellation of self-energies is much similar to the previous one
with ladders.

Considering the 'two-loop' crossed vertex when inserting the vertex [3.8] gives
\[ \tilde{V}_{CRAR}^{2\nu}(P, Q, -R) = e^4 \int [dL_1]_{\alpha \beta}[dL_2]_{\sigma \rho} \Delta_A(R + L_1) \gamma^\alpha (R + L_1) \Delta_A(R + L_1 + L_2) \gamma^\sigma (R + L_1 + L_2) \]  
\[ \gamma^\sigma (R + L_1 + L_2) \Delta_R(P + L_1 + L_2) \left( \gamma^\mu + \tilde{V}_{RAR}^{\mu}(P + L_1 + L_2, Q, -R - L_1 - L_2) \right) \]  
\[ (P + L_1 + L_2) \gamma^\beta \Delta_R(P + L_2)(P + L_2) \gamma^\rho. \]  

(3.20)

The product of propagators \( \Delta(P + L_1 + L_2) \Delta(R + L_1 + L_2) \) attached to the internal vertex must be split in the same way as for the ladders. Also the canonical decomposition of all the fermion propagators into positive and negative energy parts enables to separate the positive energy contribution of the expression above from its negative counterpart. As it was pointed out before, mixed terms (positive/negative energy terms) are shown to be subleading. The algebraic cancellation of the self-energies still occurs and is expressed through the replacement of \((q_0 \pm \Omega_{R+L_1+L_2} \mp \sigma_{R+L_1+L_2} \pm \sigma_{A+L_1+L_2})\) by the denominator \((q_0 \pm \Omega_{P+L_1+L_2} \mp \Omega_{R+L_1+L_2})\). This yields

\[ \tilde{V}_{CRAR}^{2\nu}(P, Q, -R) = i e^4 \int [dL_1]_{\alpha \beta}[dL_2]_{\sigma \rho} \Delta_A^+(R + L_1) \Delta_R^+(P + L_2) \gamma^\alpha (R + L_1) \gamma^\sigma \]  
\[ (R + L_1 + L_2)(P + L_1 + L_2) \gamma^\beta (P + L_2) \gamma^\rho \]  
\[ \times \left( \Delta_R^+(P + L_1 + L_2) - \Delta_A^+(R + L_1 + L_2) \right) + n.e. \]  

(3.21)

where the negative energy part has the same expression as its positive counterpart except for the propagators \( \Delta^- \) and the denominator \( \tilde{v}^\mu / (q_0 + \Omega_{R+L_1+L_2} - \Omega_{P+L_1+L_2}) \). In the infrared limit and on the light-cone \((\hat{p}, \hat{q} \sim \pm 1 + O(c))\), these denominators can be replaced by \((q_0 + \Omega_{P} - \Omega_{R})\) and \((q_0 + \Omega_{R} - \Omega_{P})\), since again the \( L_i \)-dependent terms may be neglected. It turns out

\[ \tilde{V}_{CRAR}^{2\nu}(P, Q, -R) = \frac{i e^4 \int [dL_1]_{\alpha \beta}[dL_2]_{\sigma \rho} \Delta_A^+(R + L_1) \Delta_R^+(P + L_2) \gamma^\alpha (R + L_1) \gamma^\sigma}{(R + L_1 + L_2)(P + L_1 + L_2) \gamma^\beta (P + L_2) \gamma^\rho \left( \Delta_R^+(P + L_1 + L_2) - \Delta_A^+(R + L_1 + L_2) \right) + n.e.} \]  

(3.22)

If now the ‘two-loop’ diagram with a vertex correction connected to the line of momentum \( P \) is considered, its expression reads

\[ \tilde{V}_{VRAR}^{2\nu}(P, Q, -R) = e^4 \int [dL_1]_{\alpha \beta}[dL_2]_{\sigma \rho} \Delta_A(R + L_1) \Delta_R(P + L_1) \Delta_R(P + L_1 + L_2) \]  
\[ \Delta_R(P + L_2) \gamma^\alpha (R + L_1) \left( \gamma^\mu + \tilde{V}_{RAR}^{\mu}(P + L_1, Q, -R - L_1) \right) (P + L_1) \gamma^\sigma \]  
\[ (P + L_1 + L_2) \gamma^\beta (P + L_2) \gamma^\rho. \]  

(3.23)

Again the product of propagators attached to the internal vertex, in this case \( \Delta(P + L_1) \Delta(R + L_1) \), has to be split. In the very same way as before, the ‘complete’ self-energies
get eliminated in the common denominators. The infrared limit and the `weak' light-cone limit ($\hat{p}.\hat{q} \sim \pm 1 + O(e)$) is the angle measuring the collinearity between the emitted photon and the fermion) allow to neglect the $L_i$-dependent terms in these denominators. Therefore the latter can be approximated by $q_0 + \Omega_\mu - \Omega_R$ and $q_0 + \Omega_R - \Omega_\mu$, and finally extracted from the integral. The decomposition between positive energy and negative energy parts of the propagators has to be used and by discarding subleading pieces mixing positive and negative energy terms leads to two contributions. This gives

$$\tilde{V}_{V_{1RAR}}^{2\mu}(P, Q, -R) = \frac{v^\mu}{q_0 + \Omega_\mu - \Omega_R}i e^4 \int [dL_1]^4 [dL_2]^4 \Delta_\Lambda^+(P + L_1 + L_2)$$

$$\Delta_R^+(P + L_2)\gamma^\alpha(P + L_1)\gamma^\sigma(P + L_1 + L_2)(P + L_1 + L_2)\gamma^\beta(P + L_2)\gamma^\rho$$

$$\left(\Delta_R^+(P + L_1) - \Delta_\Lambda^+(R + L_1)\right) + n.e. \quad (3.24)$$

Finally the last `two-loop' graph is the symmetric counterpart of the previous diagram with a vertex correction along the line carried by the momentum $R$. Repeating the same procedure and splitting this time the product $\Delta(P + L_2)\Delta(R + L_2)$, its expression in the infrared limit and on the light-cone ($\hat{p}.\hat{q} \sim \pm 1 + O(e)$) becomes

$$\tilde{V}_{V_{2RAR}}^{2\mu}(P, Q, -R) = \frac{v^\mu}{q_0 + \Omega_\mu - \Omega_R}i e^4 \int [dL_1]^4 [dL_2]^4 \Delta_\Lambda^+(R + L_1)$$

$$\Delta_R^+(R + L_1 + L_2)\gamma^\alpha(R + L_1)\gamma^\sigma(R + L_1 + L_2)(P + L_1 + L_2)\gamma^\beta(P + L_2)\gamma^\rho$$

$$\left(\Delta_R^+(P + L_2) - \Delta_\Lambda^+(R + L_2)\right) + n.e. \quad (3.25)$$

Therefore both, in the infrared limit and on the light-cone ($\hat{p}.\hat{q} \sim \pm 1 + O(e)$) the procedure is equivalent to a simple contraction with $Q^\mu$ times the vector $v^\mu/(q_0 + \Omega_\mu - \Omega_R)$ (and its negative energy counterpart). Adding the three previous `two-loop' diagrams leads to the difference of the self-energies at `two-loop' order (with resummed fermion propagator and the simplest vertex correction).

$$\tilde{V}_{RAR}^{2\mu}(P, Q, -R) = \frac{v^\mu}{q_0 + \Omega_\mu - \Omega_R} \left(\Sigma_R^+(P) - \Sigma_\Lambda^+(R)\right)$$

$$+ \frac{\bar{v}^\mu}{q_0 + \Omega_R - \Omega_\mu} \left(\Sigma_R^-(P) - \Sigma_\Lambda^-(R)\right). \quad (3.26)$$

Since this corresponds to Ward identities applied to resummed diagrams it is natural to repeat the same operation at any 'N-loop' order, i.e. with vertices which cannot be disconnected by cutting two fermion lines.
In the following 'N-loop vertices' are considered. The 'N-loop photon-electron-electron vertices' are the generalization of the previous ladder and 'two-loop graphs'. They involve resummed fermion propagators and the exchange of soft transverse photons. They cannot be disconnected by cutting two fermion lines. It is shown that for any of such 'N-loop diagrams' the action of the complete vertex of Eq. [3.8] is equivalent to the contraction with the photon momentum \( Q^\mu \). It is therefore natural to rely on Ward identities at any order to prove that the afore-mentioned vertex solves the Bethe-Salpeter equation in the infrared limit.

The 'N-loop graphs' are all related to a specific diagram of the self-energy with vertex corrections via Ward identities and form a subset of 'N-loop vertices'. Each 'N-loop diagram' can be written as

\[
\bar{V}_{M K r a r}^{N \mu}(P, Q, -R) = e^{2N} \int [dL_1]^t_{\alpha_1 \beta_1} [dL_2]^t_{\alpha_2 \beta_2} \ldots [dL_N]^t_{\alpha_N \beta_N} \gamma^{\alpha_1}(R + L_1) \\
\Delta^+(R + L_1) \ldots (R + L_i + \ldots + L_j) \Delta^+(R + L_i + \ldots + L_j) \Delta_R(P + L_i + \ldots + L_j) \\
\left( \gamma^\mu + \bar{V}_{R a r}^{\mu}(P + L_i + \ldots + L_j, Q, -R - L_i - \ldots - L_j) \right) (P + L_i + \ldots + L_j) \\
(3.27)
\]

where \( P + L_i + \ldots + L_j \) and \( R + L_i + \ldots + L_j \) correspond to the internal legs attached to the external photon line. From the arguments of the previous sections the leading contribution comes from the tree-like terms involving cut internal photon lines with the Bose-Einstein factors. Each fermion propagator can be split along the ways described before. That enables to split the vertex into a positive energy and a negative energy part. Also sticking the complete vertex of Eq. [3.8] into \( \bar{V}_{K r a r}^{N \mu} \) leads to the usual replacement of the common denominator \((q_0 + \Omega_{P + L_i + \ldots + L_j} - \Omega_{R + L_i + \ldots + L_j}) \) by \((q_0 + \Omega_{P + L_i + \ldots - L_j}) \). This follows strictly the lines of the previous sections. Then in the infrared region and near the light-cone without a too strong collinearity \( \hat{p}, \hat{q} \sim \pm 1 + O(\epsilon) \) the \( L_i \)-dependent terms may be neglected. The same reasoning can then be applied here as for the simple ladder resummation. It is enough to replace \( \bar{l} \) in the expansion of Eq. [3.18] by a more general vector like \( \bar{l}_i + \ldots + \bar{l}_j \). Discarding negligible contributions involving terms such as \( P^2 \) gives

\[
\bar{V}_{M K r a r}^{N \mu}(P, Q, -R) = i e^{2N} \int [dL_1]^t_{\alpha_1 \beta_1} [dL_2]^t_{\alpha_2 \beta_2} \ldots [dL_N]^t_{\alpha_N \beta_N} \gamma^{\alpha_1}(R + L_1) \\
\Delta^+(R + L_1) \ldots \frac{v^\mu}{q_0 + \Omega_{P + L_i + \ldots + L_j} - \Omega_{R + L_i + \ldots + L_j}} \left( \Delta^+_R(P + L_i + \ldots + L_j) \\
- \Delta^+_R(R + L_i + \ldots + L_j) \right) (P + L_i + \ldots + L_j) + n.e. \\
(3.28)
\]

As explained above the common denominator can be extracted from the integral in the same specific cases.
\[ \tilde{V}_{MKRAR}^{N\mu}(P, Q, -R) = \frac{v^\mu}{q_0 + \Omega_P - \Omega_R} i e^{2N} \int [dL_1]\alpha_1\beta_1 [dL_2]\alpha_2\beta_2 \cdots [dL_N]\alpha_N\beta_N \gamma^{\alpha_1} 
\]
\[ (R + L_1)\Delta^+_A(R + L_1) \cdots (\Delta^+_A(P + L_i + \ldots + L_j) - \Delta^+_A(R + L_i + \ldots + L_j)) \]
\[ (P + L_i + \ldots + L_j) \ldots + n.e. \] (3.29)

Therefore this is completely equivalent to a contraction of each graph with \( Q^\mu \) times the vector \( v^\mu/(q_0 + \Omega_P - \Omega_R) \) (and its negative energy counterpart \( \bar{v}^\mu/(q_0 + \Omega_R - \Omega_P) \)). The expressions obtained correspond to specific parts, the sum of these parts giving after cancellations a particular self-energy diagram.

\[ \tilde{V}_{MKRAR}^{N\mu}(P, Q, -R) = \frac{v^\mu}{q_0 + \Omega_P - \Omega_R} \left( \Sigma^{N^+}_{MKR}(P) - \Sigma^{N^+}_{MKA}(R) \right) \]
\[ + \frac{\bar{v}^\mu}{q_0 + \Omega_R - \Omega_P} \left( \Sigma^{N^-}_{MKR}(P) - \Sigma^{N^-}_{MKA}(R) \right). \] (3.30)

The above procedure may be repeated for each graph. Adding all the subset of diagrams (indice \( K \)) corresponding to a self-energy with a specific vertex correction

\[ \Sigma^{N^\pm}_{MKR}(P) = \sum_K \Sigma^{\pm}_{MKR}(P) \] (3.31)

and adding each self-energy with a particular vertex correction (indice \( M \)) gives the 'N-loop' self-energy

\[ \Sigma^N_{MR}(P) = \sum_M \Sigma^\pm_{MR}(P). \] (3.32)

Finally the sum of all the 'N-loop' self-energies, starting from the previous 'one-loop' (transverse and longitudinal photon exchanges) and 'two-loop' diagrams (only transverse) leads to the 'complete' self-energy at leading order. With the vector \( v^\mu/(q_0 + \Omega_P - \Omega_R) \) (and \( \bar{v}^\mu/(q_0 + \Omega_R - \Omega_P) \) in front, the vertex of Eq. [3.8] is recovered and therefore shown to be the solution of the Bethe-Salpeter equation.

### 4 Cancellation of damping terms

In this section the polarization tensor \( \Pi^{\mu\nu}(Q) \) at leading order (\( i.e. \) the order \( e^2T^2 \)) is considered. It is worth emphasizing again that this is no longer valid for subleading quantities, especially \( \Pi^{\mu\nu}_\mu(Q) \). This is due to the approximations made when deriving the vertex of Eq. [3.8]. In the infrared limit the latter is shown to be the solution of the Bethe-Salpeter equation. In the case of the light-cone limit, the afore-mentioned vertex is not the complete solution (in particular when the emission angle \( \hat{p}, \hat{q} \) approaches \( \pm 1 \)). It is nevertheless always interesting to look for the expression of \( \Pi^{\mu\nu}(Q) \) provided by this vertex even in
that case. The main result is that with a resummation of the fermion propagators with a damping, the latter drops out in the final expression as it was pointed out in Ref. [7,12]. But unlike [7,12] the assumption of a constant damping is not required. Also here the general infrared limit (outside the light-cone) has been investigated, not only specific terms such as \( \Pi^{00}(q_0,0) \) and \( \Pi^{ii}(0,q \to 0) \). Finally there are no ambiguities any longer in denominators like \( 1/P.Q \) due to the absence of \( i \epsilon \) prescriptions as it was the case with the vertex advocated in Ref. [7].

In the \( R/A \) formalism, the retarded part of the tensor can be written as

\[
\begin{align*}
  i\Pi^{\mu\nu}_{RR}(Q) &= -e^2 \text{Tr} \int \frac{d^4p}{(2\pi)^4} \left( P\gamma^\mu R \right) \left\{ \left( \frac{1}{2} - n_F(p_0) \right) \Delta_R(R) \left[ \gamma^\nu \right. \\
  &\quad + \hat{V}_{RAA}^\nu(P,Q,-R) \Delta_R(P) - \left( \gamma^\nu + \hat{V}_{AAR}^\nu(P,Q,-R) \right) \Delta_A(P) \\
  &\quad + \left( \frac{1}{2} - n_F(r_0) \right) \Delta_A(P) \left[ \left( \gamma^\nu + \hat{V}_{ARR}^\nu(P,Q,-R) \right) \Delta_R(R) \right. \\
  &\quad - \left( \gamma^\nu + \hat{V}_{ARR}^\nu(P,Q,-R) \right) \Delta_A(R) \left. \right\} .
\end{align*}
\] (4.1)

Contracting the spinors gives a part sensitive to the infrared or light-cone region plus possible tadpole terms. These tadpoles are not concerned by vertex and damping corrections. Taking \( \Pi^{00} \) as a particular example, the relevant expression is

\[
\begin{align*}
  i\Pi^{00}_{RR}(Q) &= -e^2 \int \frac{d^4p}{(2\pi)^4} 8p_0^2 \left\{ \left( \frac{1}{2} - n_F(p_0) \right) \left[ \left( 1 - \frac{\sigma_R^+(P) - \sigma_R^-(R)}{q_0 + \Omega_P - \Omega_R} \right) \\
  &\quad \cdot \frac{\sigma_R^-(P) - \sigma_R^+(R)}{q_0 + \Omega_R - \Omega_P} \Delta_R(R) \Delta_R(P) - \Delta_R(R) \Delta_A(P) \left( 1 - \frac{\sigma_A^+(P) - \sigma_A^+(R)}{q_0 + \Omega_P - \Omega_R} \right) \\
  &\quad \cdot \frac{\sigma_A^+(P) - \sigma_A^-(R)}{q_0 + \Omega_R - \Omega_P} \right] + \left( \frac{1}{2} - n_F(r_0) \right) \left[ \Delta_A(R) \Delta_A(P) \left( 1 - \frac{\sigma_A^+(P) - \sigma_A^+(R)}{q_0 + \Omega_P - \Omega_R} \right) \\
  &\quad \cdot \frac{\sigma_A^+(P) - \sigma_A^+(R)}{q_0 + \Omega_R - \Omega_P} \right) \right\} .
\end{align*}
\] (4.2)

Splitting the products of propagators give denominators containing the retarded and advanced \( \sigma \)’s. These denominators cancel against the numerators of the internal vertices written above. What remains as usual are the differences between \( \Delta^\pm(P) \) and \( \Delta^\pm(R) \). More explicitly

\[
\begin{align*}
  i\Pi^{00}_{RR}(Q) &= -4ie^2 \int \frac{d^4p}{(2\pi)^4} p \left\{ \left( \frac{1}{2} - n_F(p_0) \right) \left[ \frac{1}{q_0 + \Omega_P - \Omega_R} \left( \Delta_R^+(P) - \Delta_R^+(R) \right) \\
  &\quad - \frac{1}{q_0 + \Omega_R - \Omega_P} \left( \Delta_R^-(R) - \Delta_R^-(P) \right) \right] \right\}.
\end{align*}
\]
\[ + \frac{1}{q_0 + \Omega_R - \Omega_P} \left( \Delta_A^+(P) - \Delta_A^+(R) \right) \right) \left( \frac{1}{2} - n_F(r_0) \right) \left( \frac{1}{q_0 + \Omega_P - \Omega_R} \left( \Delta_A^+(P) - \Delta_A^+(R) \right) \right) \]

\[ - \Delta_A^+(R) - \frac{1}{q_0 + \Omega_R - \Omega_P} \left( \Delta_A^+(P) - \Delta_A^-(P) \right) \right) \left( \frac{1}{2} - n_F(r_0) \right) \left( \frac{1}{q_0 + \Omega_P - \Omega_R} \left( \Delta_A^+(P) - \Delta_A^-(R) \right) \right) \right) \right]. \quad (4.3) \]

It is then possible to get the cancellation of propagators, the \( R/A \) prescriptions of which remain unchanged for the same statistical factor. This gives just cut propagators for the variable associated to the statistical factor

\[ i \Pi_{00}^{RR}(Q) = -4ie^2 \int \frac{d^4p}{(2\pi)^4} \left\{ \left( \frac{1}{2} - n_F(p_0) \right) \left[ \frac{1}{q_0 + \Omega_P - \Omega_R} \left( \Delta_A^+(P) - \Delta_A^+(P) \right) \right) \right) \left( \frac{1}{2} - n_F(r_0) \right) \left( \frac{1}{q_0 + \Omega_P - \Omega_R} \left( \Delta_A^+(P) - \Delta_A^-(R) \right) \right) \right) \right). \quad (4.4) \]

It can be noticed at this stage that the improved hard thermal loop is recovered if the dampings (imaginary parts of the \( \sigma \)'s) inside the propagators are replaced by the usual \( i\epsilon \) prescriptions. In this case the differences of propagators just give Dirac functions. At first glance this approximation by \( \delta \) functions seems justified if the scale \( e^2T \) of the damping compared to the hard scale is taken into account. However the expression above contains Breit-Wigner functions but with energy dependent widths. Thus it seems more appropriate to look for a rigorous treatment of these terms. The main point is that the cancellation of retarded-advanced products replaced by simple propagators allows to convert the integral over \( p_0 \) into a contour integral and to choose the complex half-plane without the discontinuities from the damping terms. For the retarded propagators of the expression above, a closed contour in the upper half-plane avoiding the fermion Matsubara frequencies \( \omega_n = 2i\pi(n+\frac{1}{2})T \) on the imaginary axis gives no contribution. This contour can be composed of the real axis, the sum \( C_1 \) of two quarter-circles expanding at infinity and a part \( C_2 \) encircling clockwise the upper poles of the statistical factor. The integration over the real axis can be replaced by an integration over \(-C_1-C_2\). Relabeling the variables \( p_0 \) and \( r_0 \) as \( z \), the expression of the polarization tensor coming from the retarded propagators is

\[ i \Pi_{00}^{00\text{mp}}(Q) = 2e^2 \int_{-C_1-C_2} \frac{dz}{2\pi} \int \frac{d^3p}{(2\pi)^3} \left\{ \left( \frac{1}{2} - n_F(z) \right) \left[ \frac{1}{q_0 + \Omega_P - \Omega_R} \times \right. \right. \]

\[ \times \left( \frac{1}{z - \Omega_P + \sigma^+(z, p)} - \frac{1}{z - \Omega_R + \sigma^+(z, r)} \right) - \frac{1}{q_0 + \Omega_R - \Omega_P} \times \]

\[ \times \left( \frac{1}{z + \Omega_P + \sigma^-(z, p)} - \frac{1}{z + \Omega_R + \sigma^-(z, r)} \right) \right\} \]. \]
Each propagator taken individually yields a finite contribution over $-C_1$. In order to get denominators falling off as $1/z^2$ when $|z|$ tends to infinity, the function

$$f_\pm(z) = \left(\frac{1}{2} - n_F(z)\right) \left(\frac{1}{z - \Omega_P + \sigma^+(z, p)} - \frac{1}{z - \Omega_R + \sigma^+(z, r)}\right),$$

must be considered. It has the property $zf_\pm(z) \to 0$ when $|z| \to \infty$, wherever it is analytical, which is the case on $C_1$. Therefore it is a specific sum of propagators which gives no contribution over $-C_1$. The next step consists in writing the contribution of $-C_2$, namely the sum of the residues of the Fermi-Dirac factor for the poles located in the upper half-plane. The same reasoning can be applied for the advanced propagators with a contour in the lower half-plane. The contribution is finally reduced to the sum of the residues corresponding to the Matsubara frequencies in the lower half-plane. Adding the parts from the retarded and advanced propagators gives

$$i\Pi^0_{RR}(Q) = 4i\pi e^2 \int \frac{d^3p}{(2\pi)^3} \sum_n \left[ \frac{1}{q_0 + \Omega_P - \Omega_R} \frac{1}{\omega_n - \Omega_P + \sigma^+(\omega_n, p)} - \frac{1}{q_0 + \Omega_P - \Omega_R} \frac{1}{\omega_n - \Omega_R + \sigma^+(\omega_n, r)} \right] \frac{1}{q_0 + \Omega_R - \Omega_P \omega_n + \Omega_P + \sigma^-(\omega_n, p)} + \frac{1}{q_0 + \Omega_R - \Omega_P \omega_n + \Omega_R + \sigma^-(\omega_n, r)}.$$ (4.7)

The separation between the different scales can now be used to prove the irrelevance of the damping contribution at leading order. The frequencies $\omega_n$ all belong to the hard scale, along with $\Omega_P$ or $\Omega_R$. The imaginary parts given by $\omega_n$ like the real part given by $\Omega_P$ or $\Omega_R$ overwhelm the real and imaginary parts of the $\sigma$’s. But the soft term $\Omega_P - \Omega_R \sim \hat{p}.\hat{q}$ should intervene in the differences written above. In the light-cone limit with $q \sim O(eT)$, the vertex and the damping resummations are only relevant when the emission angle is close to $\pm 1$. In that case $\hat{p}.\hat{q}$ remains soft and much larger than the $\sigma$’s. In the infrared case $q \sim O(e^2T)$, and $\hat{p}.\hat{q}$ has therefore the same order (for the main part of the phase-space the angle is not too small and does not alter this estimate). But the order of magnitude of the $\sigma$’s when going away from the real axis should not exceed the scale of order $e^3T$ contrary to what happens when $p_0 \simeq \Omega_P$. A simple power counting shows that a simplified expression corresponding to the 'one-loop' self-energy is $e$ times the order of $\hat{p}.\hat{q}$. Therefore in all these cases, provided that the value of $q$ is not too small (below $O(e^2T)$), the self-energies contributions should be negligible. What is left is nothing else than the term obtained with the simple $i\epsilon$ prescriptions and delta functions in real time. Neglecting the $\sigma$’s, the usual procedure of deforming the contours when going from the imaginary time to the real time formalism can be used. The contribution given by simple poles is re-established.
\begin{align}
\Pi_{RR}^{00}(Q) &= -4i\pi e^2 \int \frac{d^4P}{(2\pi)^4} \left\{ \left( \frac{1}{2} - n_F(p_0) \right) \left[ \frac{1}{q_0 + \Omega_P - \Omega_R} \delta(p_0 - \Omega_P) \right. \\
&\quad - \frac{1}{q_0 + \Omega_R - \Omega_P} \delta(p_0 + \Omega_P) \right. \\
&\quad - \left( \frac{1}{2} - n_F(r_0) \right) \left[ \frac{1}{q_0 + \Omega_P - \Omega_R} \delta(r_0 - \Omega_R) \right. \\
&\quad - \frac{1}{q_0 + \Omega_R - \Omega_P} \delta(r_0 + \Omega_R) \right\} ,
\end{align}

and the \textit{improved hard thermal loop} is recovered. In the form written above, it is a complex \( q_0 = \text{Re} q_0 + i\epsilon \) \((-i\epsilon \text{ for the advanced Green function})\) which is considered. This is explicitly mentioned in the previous sections. This allows to obtain straightforwardly the imaginary part or Landau damping contribution. In the estimate above, it can be seen from Eq. \([4.7]\) that the limits \( \Pi_{00}(q_0, 0) \) or \( \Pi_{ii}(0, q \to 0) \) have a particular status since \( \Omega_R \to \Omega_P \) and \( \sigma(\omega_n, r) \to \sigma(\omega_n, p) \). They both involve terms such as \((\Omega_P - \Omega_R + \sigma(\omega_n, p) - \sigma(\omega_n, r))/((\omega_n - \Omega_P + \sigma(\omega_n, p))^2 \) which can be approximated by \((\Omega_P - \Omega_R)/(\omega_n - \Omega_P)^2 \) for any value of \( q \) below the order \( e^2T \). This leads to the same answers as without damping contributions.

In short, sticking the expression of the complete vertex into the polarization tensor at leading order gives the \textit{improved hard thermal loop} introduced in Ref. \([11]\). However this does not give the complete solution in the light-cone limit. In the infrared limit, where the vertex satisfying the Bethe-Salpeter equation can be found, the tensor just corresponds to the usual \textit{hard thermal loop} as expected.

## 5 Conclusion

It has been shown that in the infrared and in the light-cone region in \textit{QED} ladders are not the only leading diagrams for the resummed vertex when going beyond a simplified model. Due to long range magnetic interactions, specific non-planar graphs and graphs with vertex corrections have to be considered at leading order. They all involve exchanges of soft photons and resummed fermion propagators. Both, in the infrared limit and in a weak light-cone limit an improved vertex could be derived which solves the Bethe-Salpeter equation. The resummation of all these vertex diagrams cancels against self-energy insertions containing soft photons. This compensation is algebraic for the most part. Technical ambiguities are lifted when the products of propagators with different \( R/A \) prescriptions cancel out. This does not require any particular input on the self-energies, but only very general properties. The result is found to be the \textit{improved hard thermal loop} expression, which in the infrared or weak light-cone limits just corresponds to the usual \textit{HTL} term. In the strong light-cone limit where the effects of the asymptotic mass become important, the improved vertex is no longer a solution of the Bethe-Salpeter equation. Basic approximations valid in the former cases are no longer legitimate. Other methods for solving this problem are necessary and will be the subject of future work \([26]\).
APPENDIX

A  Two-loop self-energy

In this appendix the two-loop diagram of the self-energy is calculated using the correct transverse spectral density $\rho_T$ (more exactly its Landau damping part). The purpose is clearly to consider all the relevant scales for the internal momenta, and not only the very soft regime $O(e^2 T)$. It is also interesting to see how a complete calculation which includes dynamical screening reaches the conclusion already established with the approximate spectral density given in Eq. [2.4].

The starting point is given by the the form of Eq. [2.8], where $p_0 = p$. After recombining the denominators by change of variables, this yields

$$
\zeta(P) = \frac{1}{4p_0} \text{Tr} (P \Sigma_{RR}(P)) \\
= (e^2 T)^2 \int \frac{d^3 q}{(2\pi)^3} \int \frac{dq_0}{2\pi q_0} \rho_T(Q) \int \frac{d^3 k}{(2\pi)^3} \int \frac{dk_0}{2\pi k_0} \rho_T(K) \left(1 - (\hat{p} \cdot \hat{q})^2\right) \\
(1 - (\hat{p} \cdot \hat{k})^2) \left(i\pi \delta(q_0 + k_0 - \hat{p} \cdot (\hat{q} + \hat{k})) \frac{P}{(q_0 - \hat{p} \cdot \hat{q})^2} - 2i\pi \delta(k_0 - \hat{p} \cdot \hat{k}) \frac{P}{(q_0 - \hat{p} \cdot \hat{q})^2}\right). 
$$

(A.1)

This expression may be split into two parts, associated respectively to $\delta(k_0 - \hat{p} \cdot \hat{k})$ and $\delta(q_0 + k_0 - \hat{p} \cdot (\hat{q} + \hat{k}))$. The first part can easily be computed with the correct spectral densities, due to a complete separation (or factorization) between the terms involving $Q$ and $K$. First the integration over the angles gives

$$
\zeta_1(P) = 2i\pi (e^2 T)^2 \int_{-k}^k \frac{dk}{(2\pi)^2} \int_{-k}^k \frac{dk_0}{2\pi k_0} \rho_T(K) \left(1 - \frac{k_0^2}{k^2}\right) \\
\int_{-q}^q \frac{dq}{(2\pi)^2} \int \frac{dq_0}{2\pi} \rho_T(Q) \left(\frac{4}{q_0} + \frac{2}{q} \ln \left|\frac{q_0 - q}{q_0 + q}\right|\right). 
$$

(A.2)

The well-known sum rules [25,8] read:

$$
\int_{-k}^k \frac{dk_0}{2\pi k_0} \rho_T(K) = \frac{1}{k^2} - \frac{Z_T(k)}{\omega_T^2(k)}, \quad \int_{-k}^k \frac{dk_0}{2\pi k_0} \rho_T(K) = 1 - Z_T(k). 
$$

(A.3)
where $Z_T(k)$ and $\omega_T(k)$ are the residue and the transverse dispersion relation, respectively. These sum rules can be used for carrying out the integration over $q_0$. However, the two-loop graph is only at leading order, when $q$ can reach the infrared limit, i.e. the order $e^2T$. Thus the logarithmic term is shown to be negligible. Since there is clearly a separation between two scales, the soft one of order $eT$ and the infrared or very soft scale of order $e^2T$, an intermediate scale parameter $k^* \sim O(e\sqrt{eT})$ can be introduced. The first part may then be written as

$$\zeta_1(P) = \frac{4i}{(2\pi)^3}(e^2T)^2 \frac{1}{\mu} \int_0^\infty \frac{dk}{k} Z_T(k) \left(1 - \frac{k^2}{\omega_T^2(k)}\right)$$

$$= \frac{4i}{(2\pi)^3}(e^2T)^2 \frac{1}{\mu} \ln \left(\frac{k^*}{\mu}\right) + \frac{4i}{(2\pi)^3}(e^2T)^2 \frac{1}{\mu} \int_0^\infty \frac{dk}{k} Z_T(k) \left(1 - \frac{k^2}{\omega_T^2(k)}\right), \quad (A.4)$$

where the integral has been decomposed into an infrared and a soft scale contribution.

The calculation of the second part is by far more complicated, due to the non trivial phase-space. The kinematical constraints give, for $k > q$:

$$1 > \cos \theta > -1,$$

$$k - q > q_0 + k_0 > q - k, \quad (1)$$

$$q + k > q_0 + k_0 > k - q, \quad (2)$$

$$\frac{q_0 + k_0 + k}{q} > \cos \theta > -1,$$

$$q - k > q_0 + k_0 > -q - k, \quad (3)$$

and for $k < q$:

$$\frac{q_0 + k_0 + k}{q} > \cos \theta > \frac{q_0 + k_0 - k}{q},$$

$$q - k > q_0 + k_0 > k - q, \quad (1')$$

$$q + k > q_0 + k_0 > q - k, \quad (2')$$

$$\frac{q_0 + k_0 + k}{q} > \cos \theta > -1,$$

$$k - q > q_0 + k_0 > -q - k, \quad (3')$$

Using the sum rules of Eq. [A.3] seems useless with the structure of the phase space. However under specific conditions further simplifications can still be made. First the case where either $k$ or $q$ is not in the infrared limit, but still remains of order $eT$ can be treated. The other variable must be necessarily of order $e^2T$ in order to keep the two loop diagram dominant. The case with both variables in the infrared region will be considered afterwards. In order to keep $k$ or $q$ of order $eT$, the scale parameter $k^*$ previously introduced may be used. With $k$ larger than $k^*$ for instance, $q$ lies in the infrared. For very soft momenta, the whole density of states is concentrated around zero energy, $q_0 \ll q$ and for the dominant part $q \ll k^* \ll k$. With these conditions, the regions
(2) and (3) give negligible contributions and region (1) is reduced to \(-k < k_0 < k\). The variable \(q_0\) is still limited by positive and negative values. The expression corresponding to this part can be written as:

\[
\zeta_2^{(1)}(P) = i\pi (e^2 T)^2 \int_{k^*}^{\infty} \frac{dk}{(2\pi)^2} k \int_{-k}^{k} \frac{dk_0}{2\pi k_0} \rho_T(K) \int_{k^*}^{\infty} \frac{dq}{(2\pi)^2} q^2 \int_{-q}^{q} \frac{dq_0}{2\pi q_0} \rho_T(Q) \\
\int_{-1}^{1} d\cos \theta \frac{1 - \cos^2 \theta}{(q_0 - q \cos \theta)^2} \left(1 - \frac{1}{k^2} (q_0 + k_0 - q \cos \theta)^2\right).
\]

(A.5)

Again keeping only dominant terms, with the strict condition \(q_0 \ll q\), a simplified term for the trace in the numerator is obtained

\[
\zeta_2^{(1)}(P) = \frac{-2i}{(2\pi)^3} (e^2 T)^2 \int_{k^*}^{\infty} \frac{dk}{k} Z_T(k) \left(1 - \frac{k^2}{\omega_T^2(k)}\right) \int_{\mu}^{\infty} dq \left(\frac{1}{q^2} + \frac{1}{3k^2}\right).
\]

(A.6)

The remaining leading terms are the divergent part \(1/\mu\) and \(1/k^*\). This last contribution is expected to get canceled afterwards. Thus

\[
\zeta_2^{(1)}(P) = \frac{-2i}{(2\pi)^3} (e^2 T)^2 \int_{k^*}^{\infty} \frac{dk}{k} Z_T(k) \left(1 - \frac{k^2}{\omega_T^2(k)}\right) + \frac{4i}{(2\pi)^3} (e^2 T)^2 \frac{1}{3k^*}.
\]

(A.7)

With \(q\) larger than \(k^*\), \(k\) is forced to lie in the infrared and the case \(k_0 \ll k\) has to be considered with the divergent contribution given by \(k \ll k^* \ll q\). Now regions (2') and (3') are negligible and region (1') simplifies to \(-q < q_0 < q\) with the same restrictions for the angle. The starting formula is then

\[
\zeta_2^{(1')}(P) = i\pi (e^2 T)^2 \int_{\mu}^{k^*} \frac{dk}{(2\pi)^2} k \int_{k^*}^{\infty} \frac{dk_0}{2\pi k_0} \rho_T(K) \int_{\mu}^{\infty} \frac{dq}{(2\pi)^2} q^2 \int_{-q}^{q} \frac{dq_0}{2\pi q_0} \rho_T(Q) \\
\int_{\frac{q_0 + k_0 - k}{q}}^{\frac{q_0 + k_0 + k}{q}} d\cos \theta \frac{1 - \cos^2 \theta}{(q_0 - q \cos \theta)^2} \left(1 - \frac{1}{k^2} (q_0 + k_0 - q \cos \theta)^2\right).
\]

(A.8)

The use of similar simplifications gives

\[
\zeta_2^{(1')}(P) = \frac{-2i}{(2\pi)^3} (e^2 T)^2 \int_{k^*}^{\infty} \frac{dq}{q} Z_T(q) \left(1 - \frac{q^2}{\omega_T^2(q)}\right) \int_{\mu}^{\infty} dk \left(\frac{1}{k^2} + \frac{1}{3q^2}\right).
\]

(A.9)

Inverting the names of the variables \(K\) and \(Q\) gives exactly the same contribution as
Eq. [A.7]. The sum of the two divergent parts Eq. [A.7] and Eq. [A.9] exactly cancel against the term of Eq. [A.4] where $k$ is restricted to the simple soft scale $k^* \ll k$.

Finally the contributions where both variables $k$ and $q$ lies in the infrared remain to be inspected. In these cases, simplifications concerning all the regions of the phase space can no longer be made. For the region (1) of the phase space, the integration over the angles leads to

\[
\zeta_2^{(1)}(P) = -4i\pi(e^2 T)^2 \int_{\mu}^{k^*} \frac{dq}{(2\pi)^2 q^2} \int_{\frac{q}{2\pi q_0}}^{k^*} \frac{dq_0}{2\pi q_0} \rho_T(Q) \int_{q}^{k} \frac{dk}{(2\pi)^2 k} \int_{q-k-q_0}^{k-q-q_0} \frac{dk_0}{4k} \frac{3\omega_p^2}{k^4 + \left(\frac{3\pi\omega_p^2 k_0}{4k}\right)^2} \left(\frac{1}{q^2} + \frac{1}{3k^2}\right). \tag{A.10}
\]

The Landau damping part of the spectral density is taken for a very soft momentum $k \ll eT$. With the dominant contribution, $k_0/k$ remains of order $e^2$. The same argument concerns also $q_0/q$ since $q$ is in the infrared, too. Therefore all the terms involving $q_0$ and $k_0$ in the expression of the trace in the numerator can reasonably be neglected. But the point is to keep nevertheless $q_0$ in the bounds of integration over $k_0$, since it is not yet known under what circumstances $k - q$ becomes very small and comparable to $q_0$. Carrying out the integration over $k_0$ gives

\[
\zeta_2^{(1)}(P) = -4i\pi(e^2 T)^2 \int_{\mu}^{k^*} \frac{dq}{(2\pi)^2 q^2} \int_{q}^{k^*} \frac{dk}{(2\pi)^2 k} \left(\frac{1}{q^2} + \frac{1}{3k^2}\right) \frac{dq_0}{2\pi} \frac{3\omega_p^2}{2q} \frac{1}{q^4 + \left(\frac{3\pi\omega_p^2 q_0}{4q}\right)^2} \left(\arctan\left(\frac{3\pi\omega_p^2 (k - q - q_0)}{4k^3}\right) + \arctan\left(\frac{3\pi\omega_p^2 (k - q + q_0)}{4k^3}\right)\right), \tag{A.11}
\]

where the bounds for the integration over $q_0$ have been replaced by $(-\infty, \infty)$ in order to make the next step of the calculation tractable. This is allowed since, although the function is not correct when $q_0/q$ is no longer of order $e^2$, the whole contribution becomes subleading. At this stage the Parseval relation between functions and their Fourier transforms may be used to perform the integration over $q_0$. Thus, a two-dimensional integral is obtained

\[
\zeta_2^{(1)}(P) = -4i\pi(e^2 T)^2 \int_{\mu}^{k^*} \frac{dq}{(2\pi)^2 q^2} \int_{q}^{k^*} \frac{dk}{(2\pi)^2 k} \left(\frac{1}{q^2} + \frac{1}{3k^2}\right) \frac{2}{\pi} \arctan\left(\frac{3\pi\omega_p^2 (k - q)}{4(k^3 + q^3)}\right). \tag{A.12}
\]

The integrations over $q$ and $k$ can be performed. Two cases we have to be considered when integrating over $q$ for instance. First the region where $q$ is close enough to $k$ to make the
arctan not equal to $\pi/2$. Second the region where the arctan is actually equal to $\pi/2$ plus a negligible correction of $O(e)$. In both cases the replacement

$$\arctan \left( \frac{3\pi \omega^2_p (k-q)}{4(k^3 + q^3)} \right) \rightarrow \arctan \left( \frac{3\pi \omega^2_p (k-q)}{8k^3} \right),$$

(A.13)

renders the integration tractable. Without entering into details, it can be shown that the leading term gives the same result as when directly replacing the arctan by $\pi/2$ in Eq. [A.12]. This result could be anticipated at the level of this equation, since the region where the difference $k-q$ is very small gives a negligible contribution, since no such terms appear in the denominator. The result for this contribution may be written as

$$\zeta_{(1)}^{(1)}(P) = \frac{-2i}{(2\pi)^3} \left( e^2 T \right)^2 \frac{1}{\mu} \left( \ln \left( \frac{k^*}{\mu} \right) - \frac{5}{6} \right) - \frac{4i}{(2\pi)^3} \left( e^2 T \right)^2 \frac{1}{3k^*}.$$ 

(A.14)

As for the region (1'), the specific bounds of integration over the angle yields a different trace in the numerator compared to the previous case. Once again terms proportional to the energies $k_0$ and $q_0$ can be neglected. The analogous formula of Eq. [A.10] is therefore

$$\zeta_{(1')}^{(1)}(P) = -4i\pi (e^2 T)^2 \int_{\mu}^{k^*} \frac{dk}{(2\pi)^2} k^2 \int \frac{dk_0}{2\pi k_0} \rho_T(K) \int \frac{dq}{(2\pi)^2} q \left( \frac{1}{k^2} + \frac{1}{3q^2} \right) \frac{3\omega^2_p}{4q} \frac{1}{q^4 + \left( \frac{3\pi \omega^2_p q_0}{4q} \right)^2},$$

(A.15)

Finally exchanging $Q$ and $K$ gives straightforwardly the same final answer as Eq. [A.14]

$$\zeta_{(1')}^{(1)}(P) = \frac{-2i}{(2\pi)^3} \left( e^2 T \right)^2 \frac{1}{\mu} \left( \ln \left( \frac{k^*}{\mu} \right) - \frac{5}{6} \right) - \frac{4i}{(2\pi)^3} \left( e^2 T \right)^2 \frac{1}{3k^*}.$$ 

(A.16)

The final step concerns the contributions corresponding to the regions (2), (3), (2') and (3'). Focusing only on $\zeta_{(2)}^{(2)}$ and $\zeta_{(2)}^{(3)}$ new constraints must be taken into account (see the inequalities of regions (2) and (3)) for the trace in the numerator. With these constraints, after neglecting subleading terms and carrying out integrations over the energies in the same way as before, it turns out

$$| \zeta_{(2)}^{(2)}(P) + \zeta_{(2)}^{(3)}(P) | < \frac{\pi}{(2\pi)^4} \left( e^2 T \right)^2 \int_{\mu}^{k^*} \frac{dk}{q} \left( \frac{3}{q^2} + \frac{2}{k^2} + \frac{1}{kq} \right) \frac{2}{\pi} \left( \arctan \left( \frac{3\pi \omega^2_p (k+q)}{4(k^3 + q^3)} \right) - \arctan \left( \frac{3\pi \omega^2_p (k-q)}{4(k^3 + q^3)} \right) \right).$$

(A.17)
The first arctan can obviously be replaced by $\pi/2$, since the two momenta are in the infrared. The correction is always smaller by at least a factor $O(e)$. For the second arctan, the calculation can be carried out in the same way as for the parts corresponding to (1) and (1'). Again the region where $k - q$ is small gives a negligible contribution. Discarding corrections smaller by a factor $O(e)$, and keeping only the terms $(1/\mu) \ln(k^*/\mu)$, $1/\mu$ and $1/k^*$, this part cancels against its counterpart from the first arctan. Therefore $\zeta^{(2)}_2(P)$ and $\zeta^{(3)}_2(P)$ can safely be neglected. It can be shown that the calculation for the regions (2') and (3') leads straightforwardly to the same conclusion.

Adding the terms of Eq. [A.14] and Eq. [A.16] where both momenta lies in the infrared, with the parts of Eq. [A.7] and Eq. [A.9] with one of the momenta in the simple soft regime, the scale parameter $k^*$ gets canceled, as it should be. The divergent terms with 'soft factors' $k > k^*$ in Eq. [A.7] and $q > k^*$ in Eq. [A.9] compensate their counterpart of Eq. [A.4]. The logarithmic divergences in Eq. [A.14] and Eq. [A.16] get suppressed by the analogous term in the same Eq. [A.4]. The only remaining terms are the simple power-like divergences of Eq. [A.14] and Eq. [A.16]. The result is finally the same as with the simplified spectral densities

$$\zeta(P) = (e^2 T)^2 \frac{10i}{3(2\pi)^3} \frac{1}{\mu}, \quad \text{(A.18)}$$

but the compensation of soft scale parts ($q$ or $k$ of order $eT$) had to be proven and the effect of the dynamical screening in the infrared shown to be negligible. Finally, it can be mentioned that the same cancellation of the strongest divergence occurs for three-loop diagrams, with remaining leading order terms, although the calculations will not be reported here.

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