ON CYCLES AND COVERINGS ASSOCIATED TO A KNOT

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Abstract. Let $K$ be a knot, $G$ be the knot group, $K$ be its commutator subgroup, and $x$ be a distinguished meridian. Let $\Sigma$ be a finite abelian group. The dynamical system introduced by D. Silver and S. Williams in [S],[SW1] consisting of the set $\text{Hom}(K, \Sigma)$ of all representations $\rho : K \to \Sigma$ endowed with the weak topology, together with the homeomorphism

$$\sigma_x : \text{Hom}(K, \Sigma) \to \text{Hom}(K, \Sigma); \quad \sigma_x \rho(a) = \rho(xax^{-1}) \quad \forall a \in K, \rho \in \text{Hom}(K, \Sigma)$$

is finite, i.e. it consists of several cycles. In [L] we found the lengths of these cycles for $\Sigma = \mathbb{Z}/p$, $p$ is prime, in terms of the roots of the Alexander polynomial of the knot, mod $p$. In this paper we generalize this result to a general abelian group $\Sigma$. This gives a complete classification of depth 2 solvable coverings over $S^3 \setminus K$.

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1. Introduction

Let $K$ be a knot, $X$ be the knot complement in $S^3$, $X = S^3 \setminus K$, $X_{\infty}$ be the infinite cyclic cover of $X$, and $X_d$ be the cyclic cover of $X$ of degree $d$.

Let $G$ be the knot group, $K$ be its commutator subgroup, and $\Sigma$ be a finite group. Let $x$ be a distinguished meridian of the knot. The dynamical system introduced by D. Silver and S. Williams in [S] and [SW1] consisting of the set $\text{Hom}(K, \Sigma)$ of all representations $\rho : K \to \Sigma$ endowed with the weak topology, together with the homeomorphism $\sigma_x$ (the shift map):

$$\sigma_x : \text{Hom}(K, \Sigma) \to \text{Hom}(K, \Sigma); \quad \sigma_x \rho(a) = \rho(xax^{-1}) \quad \forall a \in K, \rho \in \text{Hom}(K, \Sigma).$$

is a shift of finite type ([SW1]). Moreover, if $\Sigma$ is abelian, this dynamical system is finite, i.e. it consists of several cycles ([SW2],[K]). In [L] we calculated the lengths of these cycles and their lcm (least common multiple) for $\Sigma = \mathbb{Z}/p$, $p$ prime,

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in terms of the roots of the Alexander polynomial of the knot, mod \( p \). Our goal is to
generalize these results to an arbitrary finite abelian group \( \Sigma \). This gives a complete
classification of solvable depth \( 2 \) coverings of \( S^3 \setminus K \). (By a solvable covering of
depth \( n \) we mean a composition of \( n \) regular coverings \( M_0 \to M_1 \to \ldots \to M_n \) with
Corresponding groups \( \Gamma_i \), such that \( \Gamma_0 \triangleleft \Gamma_1 \triangleleft \ldots \triangleleft \Gamma_n \) and \( \Gamma_{i+1}/\Gamma_i \) is abelian.)

Let \( \Delta(t) = c_0 + c_1 t + \ldots + c_n t^n \) be the Alexander polynomial of the knot \( K \), and
\( B - tA \) its Alexander matrix of size, say, \( m \times m \), corresponding to the Wirtinger
presentation. From \([L]\) we know that

\[
(1.1) \quad \text{Hom}(K, \mathbb{Z}/p) \cong (\mathbb{Z}/p)^n \quad \text{where} \quad n = \deg(\Delta(t) \mod p).
\]

It turns out that the same result is true for a target group \( \mathbb{Z}/p^r \):

\[
(1.2) \quad \text{Hom}(K, \mathbb{Z}/p^r) \cong (\mathbb{Z}/p^r)^n \quad \text{where} \quad n = \deg(\Delta(t) \mod p).
\]

In section 2 we give a proof of (1.2) for two-bridge knots. In section 3 we prove a
general result about solutions of the recurrence equation

\[
(1.3) \quad Bx_j - Ax_{j+1} = 0,
\]

where \( x_j \in \mathcal{X}, \mathcal{X} \) and \( \mathcal{Y} \) are finite modules, and \( A, B : \mathcal{X} \to \mathcal{Y} \) are module homomorphisms. We then use this result in section 4 to prove (1.2) for an arbitrary
knot. In section 5 we describe the set of periods and calculate their lcm for target
group \( \mathbb{Z}/p^r \), based on similar results for the target group \( \mathbb{Z}/p \), obtained in \([L]\).
We then generalize these results for any finite abelian group \( \Sigma \).

In section 6 we describe the relation between the shift \( \sigma_x \) on \( \text{Hom}(K, \Sigma) \) and
the pullback map \( \tau^* \) corresponding to the meridian \( x \), on the space of regular
coverings over \( X_\infty \). In section 7 we construct a regular covering \( p : N \to X_\delta \) with
the group of deck transformations \( \Sigma \), corresponding to a surjective homomorphism
\( \rho \in \text{Hom}(K, \Sigma) \) with \( \sigma^d_x \rho = \rho \), and prove that any regular covering of \( X_\delta \) with
the group of deck transformations \( \Sigma \) can be obtained in this way. We conclude
the paper by formulating our results in terms of \( p \)-adic representations of \( K \) and
associated solenoids and flat principal bundles.

2. Case of a two-bridge knot

Let \( \Delta(t) \) be the Alexander polynomial of a two-bridge knot \( K \) and \( n \) be the degree of
\( \Delta(t) \) mod \( p \). Since the Alexander polynomial is defined up to multiplication by
\( t^k, k \in \mathbb{Z} \), and has symmetric coefficients, we can write

\[
\Delta(t) = pd_k t^{-k} + \ldots + pd_1 t^{-1} + c_0 + c_1 t + \ldots + c_n t^n + pd_1 t^{n+1} + \ldots + pd_k t^{n+k} = 0,
\]

where \( c_i, d_i \) are integers and \( c_0 = c_n \) is not divisible by \( p \). Similarly to the Theorem
9.1 in \([L]\), we can prove that \( \text{Hom}(K, \mathbb{Z}/p^r) \) is isomorphic to the space of bi-infinite
sequences \( \{ x_i \}_{i \in \mathbb{Z}}, x_i \in \mathbb{Z}/p^r \), satisfying the following recurrence equation mod \( p^r \):

\[
(2.1) \quad pd_k x_{-k+j} + \ldots + pd_1 x_{-1+j} + c_0 x_j + c_1 x_{j+1} + \ldots + c_n x_{n+j} + \ldots + pd_k x_{n+k+j} = 0
\]

From \([L]\) we know that \( \text{Hom}(K, \mathbb{Z}/p) \cong (\mathbb{Z}/p)^n \) where \( n = \deg(\Delta(t) \mod p) \). The
same is true for target groups \( \mathbb{Z}/p^r \).

**Theorem 2.1.** \( \text{Hom}(K, \mathbb{Z}/p^r) \cong (\mathbb{Z}/p^r)^n \) where \( n = \deg(\Delta(t) \mod p) \).
Proof. We will prove that $x_0, x_1, \ldots, x_{n-1} \in \mathbb{Z}/p^r$ uniquely determine the sequence \( \{x_i\}_{i \in \mathbb{Z}} \), \( x_i \in \mathbb{Z}/p^r \), satisfying equation (2.1). The proof is by induction. For \( r = 1 \), given \( x_0, x_1, \ldots, x_{n-1} \in \mathbb{Z}/p \), \( x_n \) is uniquely determined mod \( p \) by the equation
\[
0x_0 + c_1x_1 + \ldots + c_nx_n = 0 \mod p.
\]
So, \( x_0, x_1, \ldots, x_{n-1} \mod p \) uniquely determine the whole sequence \( \{x_i\}_{i \in \mathbb{Z}} \mod p \), satisfying (2.2). This proves the base of induction.

Suppose the statement is true for \( r \). Fix \( x_0, x_1, \ldots, x_{n-1} \mod p^{r+1} \) and let \( \{x_i\}_{i \in \mathbb{Z}} \) be the sequence satisfying equation:
\[
px_kx_{-k} + \ldots + pd_1x_{-1} + 0x_0 + c_1x_1 + \ldots + c_nx_n + \ldots + pd_kx_{n+k} = 0 \mod p^r.
\]
It is uniquely determined mod \( p^r \), by induction assumption. But then all the terms of (2.3) except \( c_nx_n \) are determined mod \( p^{r+1} \). So \( x_n \) and hence the whole sequence \( \{x_i\}_{i \in \mathbb{Z}} \) is uniquely determined mod \( p^{r+1} \) by \( x_0, x_1, \ldots, x_{n-1} \mod p^{r+1} \).

3. Linear matrix recurrence equations

**Theorem 3.1.** Let \( \mathcal{X}, \mathcal{Y} \) be two finite modules of the same order, over the same ring \( R \). Let \( A, B : \mathcal{X} \rightarrow \mathcal{Y} \) be modules homomorphisms such that \( \ker A \cap \ker B = 0 \).

Consider the following recurrence equation:
\[
Bx_j - Ax_{j+1} = 0
\]
Then \( \mathcal{X} = \mathcal{V} \oplus A \oplus B \), where \( \mathcal{V} = \{ v \in \mathcal{X} : \text{there exists a bi-infinite sequence } \ldots v_{-1}, v_0 = v, v_1, \ldots, \text{ satisfying equation (3.1)} \} \)
\( \mathcal{A} = \{ a \in \mathcal{X} : \text{there exists an infinite sequence } \ldots, a_{-1}, a_0 \text{ satisfying (3.1)} \} \) and \( a_{-1} = 0 \) for sufficiently large \( i \).
\( \mathcal{B} = \{ b \in \mathcal{X} : \text{there exists an infinite sequence } b_0 = b, b_1, b_2, \ldots, \text{ satisfying (3.1)} \} \) and \( b_1 = 0 \) for sufficiently large \( i \).

**Proof.** The proof is by induction in the order of \( \mathcal{X} \) and \( \mathcal{Y} \). Consider a diagram:

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{A} & \mathcal{Y} \\
\downarrow{\pi_1} & & \downarrow{\pi_2} \\
\mathcal{Y}/\ker A & \xrightarrow{A} & \mathcal{Y}/B(\ker A) \\
\end{array}
\]

where by definition, \( \pi_1 \) and \( \pi_2 \) are factorization maps; \( [x] = \pi_1(x) \); and
\[
\bar{A}([x]) = \pi_2 \circ A(x), \quad \bar{B}([x]) = \pi_2 \circ B(x).
\]
This diagram is not commutative, but its left- and right-hand triangles are commutative. Note that \( \mathcal{X}/\ker A \) and \( \mathcal{Y}/B(\ker A) \) are modules over \( R \) of the same order, since \( B \) is injective on \( \ker A \).

Suppose that the statement of the theorem is true for \( \mathcal{X}/\ker A \) and operators \( \bar{A} \) and \( \bar{B} \):
\[
\mathcal{X}/\ker A = \bar{\mathcal{V}} \oplus \bar{A} \oplus \bar{B},
\]
where all the sequences in definition of \( \bar{\mathcal{V}} \), \( \bar{A} \), \( \bar{B} \) satisfy the equation:
\[
\bar{B}[x] - \bar{A}[x] = 0.
\]
Then we will prove that
\begin{equation}
\mathcal{X} = \mathcal{V} \oplus \mathcal{A} \oplus \mathcal{B},
\end{equation}

Take any \( u \in \mathcal{X} \). By induction assumption \([u] = [v] + [a] + [b] \), where \([v] \in \mathcal{V} \), \([a] \in \mathcal{A} \), \([b] \in \mathcal{B} \). We find lifts \( v, a, b \) of \([v], [a], [b] \) to \( \mathcal{V}, \mathcal{A}, \mathcal{B} \) respectively. Let \( \ldots, [v_{-1}], [v_0] = [v], [v_1], \ldots \) satisfy \( B[v_i] - A[v_{i+1}] = [0], i \in \mathbb{Z} \). Take any lift \( \ldots, y_{-1}, y_0, y_1, \ldots \) Then \( By_i - Ay_{i+1} = x_i \in B(\ker A) \). So \( x_i = Bw_i \) for some \( w_i \in \ker A \). Then
\[ B(y_i - w_i) - A(y_{i+1} - w_{i+1}) = 0. \]

So \( v_i = y_i - w_i \) satisfy \( (3.3) \) and \( v = v_0 \in \mathcal{V} \) is a desired lift of \([v] \).

Similarly, for \([a] \in \mathcal{A} \) there exists a sequence \( \ldots, [a]_{-1}, [a_0] = [a] \), satisfying \( (3.3) \) with \([a]_{-1} = [0] \) for \( i \geq N \). As before, we can find a lift \( \{a_{-i}\}_{i \geq 0} \), satisfying \( Ba_{-i} - Aa_{-(i-1)} = 0 \). Note that \( a_{-i} \in \ker A \) for \( i \geq N \). We have
\[ B \cdot 0 = Aa_{-N}. \]

But then the sequence \( \ldots, 0, 0, a_{-N}, a_{-(N-1)}, \ldots, a_0 \) also satisfies \( (3.3) \), so \( a = a_0 \in \mathcal{A} \) is a desired lifting.

We repeat the same argument to prove that \([b] \) has a lift \( b \in \mathcal{B} \). If \( \{b_i\}_{i \geq 0} \) satisfies \( (3.3) \) and \([b]_i = 0 \) for \( i \geq N \), we find a lift \( \{b_i\}_{i \geq 0} \) satisfying \( (3.3) \). Since \( b_i \in \ker A \) for \( i \geq N \), and \( Bb_i - Ab_{i+1} = 0 \), we have also \( b_i \in \ker B \) for \( i \geq N-1 \), hence \( b_i = 0 \) for \( i \geq N-1 \), since by assumption \( \ker A \cap \ker B = 0 \). So \( b = b_0 \in \mathcal{B} \) is a desired lift. Since \( \pi_1(u) = \pi_1(v + a + b), u = v + a + b + \tilde{a} \), where \( \tilde{a} \in \ker A \) and so \( \tilde{a} \in \mathcal{A} \). The step of induction is done.

Since we can interchange the roles of \( A \) and \( B \), it remains to prove the statement of the theorem in the case when \( A \) and \( B \) are monomorphisms and hence are isomorphisms, since \( |\mathcal{X}| = |\mathcal{Y}| \). In this case any element \( x \in \mathcal{X} \) has a bi-infinite continuation \( x_i = (A^{-1}B)^i x \), satisfying \( (3.3) \). The theorem is proven. \( \square \)

4. MAIN RESULT FOR A GENERAL KNOT

In this section we prove that the Theorem \( \ref{thm:main} \) holds for any knot. Let \( B - tA \) be the Alexander matrix of a general knot \( K \) arising from the Wirtinger presentation of the knot group \( G \). Here \( A, B \) are \( m \times m \) matrices with elements \( 0, \pm 1 \).

**Theorem 4.1.** Dynamical system \((\text{Hom}(K, \Sigma), \sigma_x)\) is conjugate to the left shift in the space of bi-infinite sequences \( \{y_j\}_{j \in \mathbb{Z}} \), \( y_j \in (\Sigma)^m \) satisfying recurrence equation
\begin{equation}
B y_j - A y_{j+1} = 0.
\end{equation}

For the target group \( \mathbb{Z}/p \) this result is proven in \( \cite{L}, \) Theorem 4.2. For a general abelian group \( \Sigma \) the proof is identical.

We can apply theorem \( \ref{thm:main} \) for modules \((\mathbb{Z}/p^r)^m \) and linear operators \( A, B : (\mathbb{Z}/p^r)^m \to (\mathbb{Z}/p^r)^m \) given by matrices \( A \) and \( B \) to get
\begin{equation}
(\mathbb{Z}/p^r)^m = \mathcal{V}_r \oplus \mathcal{A}_r \oplus \mathcal{B}_r,
\end{equation}

where \( \mathcal{V}_r = \{y \in (\mathbb{Z}/p^r)^m : \text{there exists a bi-infinite sequence } \ldots, y_{-1}, y_0 = y, y_1, \ldots, \text{ satisfying equation (4.1)}\}, \)
\( \mathcal{A}_r = \{a \in (\mathbb{Z}/p^r)^m : \text{there exists an infinite sequence } \ldots, a_{-1}, a_0 = a, \text{ satisfying (4.1) and } a_{-i} = 0 \text{ for sufficiently large } i \}, \)
\( \mathcal{B}_r = \{b \in (\mathbb{Z}/p^r)^m : \text{there exists an infinite sequence } b = b_0, b_1, b_2, \ldots, \text{ satisfying (4.1) and } b_i = 0 \text{ for sufficiently large } i \} \).
We will use the uniqueness of continuation that follows from the finiteness of \( \text{Hom}(K, \Sigma) \) for a finite abelian group \( \Sigma \) (see Proposition 3.7 [SW2] and Theorem 1 (ii) [K]). If \( \{x_i\}_{i \in \mathbb{Z}} \) and \( \{y_i\}_{i \in \mathbb{Z}} \) satisfy (1.1), then \( x_0 = y_0 \) implies \( x_i = y_i \) \( \forall i. \) In particular, for \( a \in A_r, \ a \neq 0, \) there is no infinite continuation to the right, satisfying (1.1), and for \( b \in B_r, \ b \neq 0, \) there is no infinite continuation to the left, satisfying (1.1). (Otherwise we would have two bi-infinite sequences: \( \ldots, 0, 0, \ldots, a_0, a_1, \ldots \) and \( \ldots, 0, 0, \ldots. \) So \( \text{Hom}(K, \mathbb{Z}/p^r) \) being isomorphic to the space of be-infinite sequences satisfying (1.1), is isomorphic to \( \mathcal{V}_r. \)

Since the only decomposition of \( (\mathbb{Z}/p^r)^m \) as a direct sum of three groups is:

\[
(\mathbb{Z}/p^r)^m \cong (\mathbb{Z}/p^r)^{nr} \oplus (\mathbb{Z}/p^r)^{l_r} \oplus (\mathbb{Z}/p^r)^{m_r}
\]

with \( n_r + l + m_r = m, \) it follows from (1.2) that \( \mathcal{V}_r \cong (\mathbb{Z}/p^r)^{n_r}. \) Consider the projection:

\[
\pi : (\mathbb{Z}/p^r)^m = \mathcal{V}_{r+1} \oplus A_{r+1} \oplus B_{r+1}
\]

\[
\pi : (\mathbb{Z}/p^r)^m = \mathcal{V}_r \oplus A_r \oplus B_r
\]

Clearly \( \pi(\mathcal{V}_{r+1}) \subset \mathcal{V}_r, \pi(A_{r+1}) \subset A_r, \pi(B_{r+1}) \subset B_r. \) It follows that \( n_r \) is the same for all \( r. \) Since from Theorem 5.5 [L] it immediately follows that \( n_1 = \deg(\Delta(t) \mod p), \) we have proven the following theorem:

**Theorem 4.2.** For any knot, \( \text{Hom}(K, \mathbb{Z}/p^r) \cong (\mathbb{Z}/p^r)^n, \) where \( n = \deg(\Delta(t) \mod p). \)

### 5. Least Common Multiple

**Proposition 5.1.** The dynamical system \( \text{Hom}(K, \mathbb{Z}/p^r), \sigma_x) \) is isomorphic to \( (\mathcal{V}_r, T_r), \) where \( T_r = (A|\mathcal{V}_r)^{-1}(B|\mathcal{V}_r). \)

**Proof.** Restrictions \( A|\mathcal{V}_k \) and \( B|\mathcal{V}_k \) are isomorphisms, since \( \ker A \in A_k \) and \( \ker B \in B_k. \) Also \( A\mathcal{V}_k = B\mathcal{V}_k \) since every element \( v \in \mathcal{V}_k \) has continuation to the right and to the left: there exist \( v_{-1} \) and \( v_1 \) such that \( Bv_{-1} = Av, \) \( Bv = Av_1. \) So \( T_r : \mathcal{V}_r \to \mathcal{V}_r \) is well defined, and since \( T_r \) is conjugate to the left shift in the space of sequences satisfying equation (1.3), the formula \( T_r = (A|\mathcal{V}_r)^{-1}(B|\mathcal{V}_r) \) is obvious. \( \square \)

In [L] we calculated the set of periods of orbits and their lcm for dynamical system \( (\text{Hom}(K, \Sigma), \sigma_x) \) with \( \Sigma = \mathbb{Z}/p \) in terms of orders and multiplicities of the roots of \( \Delta(t) \) \( \mod p. \) Now we find the lcm and the set of periods for \( \Sigma = \mathbb{Z}/p^r. \)

**Theorem 5.2.** Let \( d_r = \text{lcm of periods of orbits of } (\text{Hom}(K, \mathbb{Z}/p^r), \sigma_x). \) Then either \( d_i = d_1 \forall i, \) or \( \exists s \geq 1 \) such that \( d_1 = \ldots = d_s, \) and \( d_{s+i} = d_1 p^i. \)

**Proof.** The following diagram commutes:

\[
\begin{array}{ccccccccc}
\ldots & \xrightarrow{T_{k+1}} & \mathcal{V}_{k+1} & \xrightarrow{T_k} & \mathcal{V}_k & \xrightarrow{T_1} & \mathcal{V}_1 \\
\mathcal{V}_{k+1} & \xrightarrow{T_{k+1}} & \mathcal{V}_k & \xrightarrow{T_k} & \mathcal{V}_1 \\
\ldots & \xrightarrow{T_{k+1}} & \mathcal{V}_{k+1} & \xrightarrow{T_k} & \mathcal{V}_k & \xrightarrow{T_1} & \mathcal{V}_1 \\
\end{array}
\]

Let \( \mathcal{V} = \varprojlim \mathcal{V}_k, \ \mathcal{V}_k \subset (\mathbb{Z}_p)^m, \) where \( \mathbb{Z}_p \) is the set of \( p \)-adic numbers, and \( T : \mathcal{V} \to \mathcal{V}, \)

\( T = \varprojlim T_k. \) We will use the same notations for module homomorphisms and their matrices in the standard basis. Let \( E_r, \ E \) denote the identity isomorphisms of
(\mathbb{Z}/p^r)^n \text{ and } (\mathbb{Z}_p)^n \text{ respectively. We have } T_{d_1}^{d_1} = E_1, \text{ so either } T_{d_1}^{d_1} = E, \text{ and then } T_{d_1}^{d_1} = E, \forall r, \text{ or } T_{d_1}^{d_1} = E + p^r A \text{ for some } s \in \mathbb{Z}, s \geq 1, \text{ and not all elements of matrix } A \text{ are divisible by } p. \text{ In the later case } T_{d_1}^{d_1} = E_i, i = 1, \ldots, s. \text{ Since }
\begin{align*}
T_{d_1}^{d_1,k} = (E + p^s A)^k = E + kp^s A + C_k^2 p^{2s} A^2 + \ldots + p^{s-k} A^k,
\end{align*}
we have } T_{d_1}^{d_1} = E + p^{s+1} A_1, \text{ where not all elements of } A_1 \text{ are divisible by } p, \text{ and, by induction, } T_{d_1}^{d_1} = E + p^{s+i} A_i, \forall i \geq 1, \text{ where not all elements of } A_i \text{ are divisible by } p. \text{ Then } T_{d_1}^{d_1} = E_{s+i} \text{ and the statement of the theorem follows.} \tag*{□}

**Proposition 5.3.** Let } Q \subseteq \mathbb{N} \text{ be the set of all periods of } (\text{Hom}(K, \mathbb{Z}/p^r), \sigma_x). \text{ Then } Q_r \subseteq Q_{r+1}.

**Proof.** If } \{x_j\}_{j \in \mathbb{Z}}, x_j \in \mathbb{Z}/p^r \text{ is a sequence satisfying recurrence equation } \text{(4.1)} \mod p^r \text{ with period } d, \text{ then } \{px_j\}_{j \in \mathbb{Z}}, px_j \in \mathbb{Z}/p^{r+1} \text{ satisfies } \text{(4.1)} \mod p^{r+1} \text{ and has the same period.} \tag*{□}

Now we turn to a general finite abelian group } \Sigma, \text{ which is isomorphic to a direct sum of cyclic groups:
\begin{align*}
\Sigma = \bigoplus_{i \in I} \mathbb{Z}/p_i^{r_i}, \quad I \subseteq \mathbb{N}.
\end{align*}
Then
\begin{align*}
\text{Hom}(K, \Sigma) = \bigoplus_{i \in I} \text{Hom}(K, \mathbb{Z}/p_i^{r_i}) = \bigoplus_{i \in I} (\mathbb{Z}/p_i^{r_i})^{n_i}, \quad \text{where } n_i = \deg(\Delta(t) \mod p_i),
\end{align*}
and the original dynamical system is the product of dynamical systems:
\begin{align*}
(\text{Hom}(K, \Sigma), \sigma_x) = \bigoplus_{i \in I} (\text{Hom}(K, \mathbb{Z}/p_i^{r_i}), \sigma_x).
\end{align*}
Taking sums of orbits with different periods, we obtain the following proposition:

**Proposition 5.4.** (i) Let } d_i \text{ be lcm of periods of orbits of } (\text{Hom}(K, \mathbb{Z}/p_i^{r_i}), \sigma_x). \text{ Then lcm of periods of orbits of } (\text{Hom}(K, \Sigma), \sigma_x) \text{ is lcm}\{d_i, i \in I\}. \text{ (ii) Let } Q_i \text{ be the set of periods of orbits of } (\text{Hom}(K, \mathbb{Z}/p_i^{r_i}), \sigma_x). \text{ Then the set of periods for } (\text{Hom}(K, \Sigma), \sigma_x) \text{ is }
\begin{align*}
Q = \{\text{lcm}\{q_i, i \in I\}, q_i \in Q_i\}.
\end{align*}

6. **Pullback } \tau^* \text{ ON THE SPACE OF COVERINGS OVER } X_\infty

Let } p_\infty : X_\infty \longrightarrow \ X \text{ be the infinite cyclic covering over the complement of the knot, and let } \tau : X_\infty \longrightarrow X_\infty \text{ be the deck transformation corresponding to the loop } x. \text{ We will now give a geometric description of the transformation } \sigma_x \text{ earlier defined algebraically.

Let us remind the pullback construction. Let } P : E \rightarrow B \text{ and } f : Y \rightarrow B \text{ be two continuous maps. } \Gamma_P = \{(e, b) : e \in E, b \in B, P(e) = b\} \subseteq E \times B \text{ is the graph of } P. \text{ We have } \text{id} \times f : E \times Y \rightarrow E \times B. \text{ Then, by definition, the pullback of } P \text{ by } f, f^*(P) : (\text{id} \times f)^{-1} \Gamma_P \rightarrow Y \text{ is the projection onto the second coordinate. We have } (\text{id} \times f)^{-1} \Gamma_P = \{(e, y) : e \in E, y \in Y, P(e) = f(y)\}. \text{ The projection of this set}
onto the first coordinate, \( \tilde{f} \), is the lift of \( f \), since the following diagram commutes:

\[
\begin{array}{ccc}
(e, y) & \xrightarrow{\tilde{f}} & e \\
\downarrow{f^*(P)} & & \downarrow{p} \\
y & \xrightarrow{f} & f(y) = P(e)
\end{array}
\]

Note that if \( P \) is a (regular) covering then so is \( f^*(P) \).

Let \( a \in X_\infty \), \( p_\infty(a) = x(0) \) and let \( p : (M, y) \to (X_\infty, a) \) be the covering corresponding to a group \( \Gamma \subset \pi_1(X_\infty, a) \), so that \( p_*\pi_1(M, y) = \Gamma \). Let \( p' : (M', y') \to (X_\infty, \tau^{-1}a) \) be the pull back of \( p \) by \( \tau \). It is a covering corresponding to the group \( \tau_\infty^{-1}\Gamma \subset \pi_1(X_\infty, \tau^{-1}a) \). Then \( \tau : X_\infty \to X_\infty \) lifts to a homeomorphism \( \hat{\tau} : M' \to M \) such that \( p \circ \hat{\tau} = \tau \circ p' \).

Let \( \hat{x} \) be the lift of \( x \) to \( X_\infty \) connecting \( \tau^{-1}a \) to \( a \). If \( \hat{x} \) is the lift of \( \hat{x} \) to \( M' \) beginning at \( y' \) and ending at \( y'' \), then \( p' : (M', y'') \to (X_\infty, a) \) is the covering corresponding to the group \( \hat{x}^{-1}(\tau_\infty^{-1}\Gamma)\hat{x} \subset \pi_1(X_\infty, a) \).

Let \( C \) denote the space of all coverings of \( X_\infty \) up to the usual equivalence. Let \( \mathcal{G} \) be the space of conjugacy classes of subgroups of \( \pi_1(X_\infty, a) \equiv K \). There is one-to-one correspondence between \( C \) and \( \mathcal{G} \). In what follows we will not distinguish notationally between a covering and its equivalence class, and between a subgroup and its conjugacy class.

The pullback transformation \( \tau^* : C \to C \), corresponds to the map \( \hat{\gamma} : \mathcal{G} \to \mathcal{G} \), \( \hat{\gamma} : \Gamma \mapsto \hat{x}^{-1}(\tau_\infty^{-1}\Gamma)\hat{x} \subset \pi_1(X_\infty, a) \), \( \forall \Gamma \subset \pi_1(X_\infty, a) \), which turns into the map \( \gamma \) acting on the subgroups of \( K \subset \pi_1(X, x(0)) \): \( \gamma(\Gamma) = x^{-1}\Gamma x, \forall \Gamma \subset K \).

Regular coverings of \( X_\infty \) correspond to normal subgroups \( \Gamma \subset K \), which in turn correspond to representations \( \rho \in \text{Hom}(K, \Sigma) \) such that \( \ker \rho = \Gamma \), in various groups \( \Sigma \). The corresponding map on the space \( \text{Hom}(K, \Sigma) \) is \( \sigma_x \), where \( \sigma_x \rho(\alpha) = \rho(x\alpha x^{-1}) \). Indeed, if \( \Gamma = \ker \rho \), then \( x^{-1}\Gamma x = \ker \sigma_x \rho \). In summary we can say that the shift \( \sigma_x \) in the space \( \text{Hom}(K, \Sigma) \) defined algebraically corresponds to the pullback action of the deck transformation \( \tau \) in the space of regular coverings over \( X_\infty \).

7. Coverings of finite degree

**Theorem 7.1.** There is one-to-one correspondence between the surjective elements \( \rho \in \text{Hom}(K, \Sigma) \) such that \( \sigma_x^d = \rho \) and regular coverings \( p : N \to X_d \) with the group of deck transformations \( \Sigma \).

**Proof.** Let \( \rho \) satisfy the condition of the theorem. Take a covering \( p_\rho : M \to X_\infty \) corresponding to \( \ker \rho \). Since \( \sigma_x^d \rho = \rho \), this covering coincides with its \( d \)-time pullback: \( \tau^d p_\rho = p_{\sigma_x^d \rho} = p_\rho \). We can lift \( \tau^d \) to \( \zeta : M \to M \) so that the following
diagram commutes:

\[
\begin{array}{ccc}
(M, y') & \xrightarrow{\zeta} & (M, y) \\
\downarrow{p_\rho} & & \downarrow{p_\rho} \\
(X_\infty, \tau^{-d} a) & \xrightarrow{\tau^d} & (X_\infty, a) \\
\downarrow{p\infty} & & \downarrow{p\infty} \\
(X, x(0)) & & (X, x(0))
\end{array}
\]

If \( \rho : K \to \Sigma \) is onto then \( \Sigma \cong K/ \ker \rho \) acts on \( M \) in the standard way: if \( \alpha \in \pi_1(X_\infty, a) \) is a loop and \( \bar{\alpha} \) is its lift to \( M \) starting at \( y \), it ends at \( \rho(\bar{\alpha})(y) \). Clearly the action of \( \Sigma \) commutes with \( \zeta \). So \( \Sigma \) acts on the space of orbits of \( \zeta \), \( N = M/\zeta \). These orbits project onto orbits of \( \tau^d \). Since \( X_\infty/\tau^d = X_d \), we obtained a regular covering \( p : N \to X_d \).

Now we prove that any regular covering over \( X_d \) with the group of deck transformations \( \Sigma \) can be obtained in this way: namely, for any covering (that is convenient to denote by) \( p_2 : N \to X_d \) with \( \Sigma \) as the group of deck transformations, \( \exists \rho \in \text{Hom}(K, \Sigma) \) such that \( \sigma_2^d(\rho) = \rho \) and the covering \( \varepsilon_2 : M \to X_\infty \) corresponding to the subgroup \( \ker \rho \), such that \( N = M/\zeta \), \( \zeta \) being a lift of \( \tau^d \).

Consider a diagram

\[
\begin{array}{ccc}
N & \xrightarrow{p_2} & X_d \\
\downarrow{p_1} & & \downarrow{p_1} \\
X_\infty & \xrightarrow{p_1} & X_d
\end{array}
\]

where \( p_2 \) is a regular covering with a group of deck transformations \( \Sigma \), and \( p_1 \) is an infinite cyclic covering with the generator \( \tau^d \). Let us consider the pullback of \( p_2 \) by \( p_1 \). Let \( M \subset N \times X_\infty \), \( M = \{(a, x) \mid p_2a = p_1x\} \). Then we have two covering maps \( \varepsilon_1 \) and \( \varepsilon_2 \), \( \varepsilon_1(a, x) = a \), \( \varepsilon_2(a, x) = x \), such that the following diagram commutes:

\[
\begin{array}{ccc}
M & \xrightarrow{\varepsilon_1} & N \\
\downarrow{\varepsilon_2} & & \downarrow{p_2} \\
X_\infty & \xrightarrow{p_1} & X_d
\end{array}
\]

For \( y \in X_\infty \), \( (a_1, y), (a_2, y), \ldots, (a_s, y) \) are all preimages of \( y \) under \( \varepsilon_2 \), where \( a_1, a_2, \ldots, a_s \) are all preimages of \( x = p_1(y) \) under \( p_2 \), and \( (a, y_1), (a, y_2), \ldots \), are all preimages of \( a \in N \) under \( \varepsilon_1 \), where \( y_1, y_2, \ldots \) are all preimages of \( p_2(a) \) under \( p_1 \).

Since \( \tau^d \) is a generator of the group of deck transformations of \( p_1 \), \( \zeta = (\text{id}, \tau^d) \) is a generator of the group of deck transformations of \( \varepsilon_1 \), while \( \{(\sigma, \text{id}) \mid \sigma \in \Sigma\} \cong \Sigma \) is the group of deck transformations of \( \varepsilon_2 \).

For any \( \beta \in K \) let \( \beta \) be its lift to \( M \) starting at \( (y_0, \beta(0)) \) and ending at \( (y_1, \beta(0)) \), where \( y_0, y_1 \in N \). There exists a unique \( \sigma \in \Sigma \) such that \( \sigma y_0 = y_1 \). Take \( \rho(\beta) = \sigma \). It is easy to see that \( \beta \in \ker \rho \) iff \( x^d(\tau^d \circ \beta)x^{-d} \in \ker \rho \). So, \( \ker \rho = \ker \sigma^d_x(\rho) \).

Since we can think of \( \rho \) as the homomorphism \( \rho : K \to K/ \ker \rho \cong \Sigma \), we have \( \sigma^d_x(\rho) = \rho \). \( \square \)
8. *p*-adic solenoids

The above results can be summarized in terms of solenoids fibered over manifolds $X$ and $X_\infty$.

Let us have a family of coverings $p_n : S_n \to B$, $n = 0, 1, 2, \ldots$, over the same $m$-dimensional manifold $B$. We say that they form a tower if there is a family of coverings $g_n : S_n \to S_{n-1}$ such that $p_n = p_{n-1} \circ g_n$. In this case we can form the inverse limit $S = \varprojlim S_n$ by taking the space of sequences $z = \{z_n\}_{n=0}^\infty$, $z_n \in S_n$ such that $g_n(z_n) = z_{n-1}$. Endow $S$ with the weak topology. It makes the natural projection $p_\infty : S \to B$, $z \mapsto z_0$, a locally trivial fibration with Cantor fibers (as long as $\deg p_n \to \infty$). Moreover, $S$ has a “horizontal” structure of $m$-dimensional lamination. If it is minimal (i.e., if all the leaves are dense in $S$), it is called a solenoid over $B$.

If all the coverings $p_n$ are regular with the group of deck transformations $\Sigma_n$, then $S$ is a flat principal $\Sigma$-bundle over $B$ with $\Sigma = \varinjlim \Sigma_n$. This means that

1. $p_\infty : S \to B$ is a locally trivial fibration with fiber $\Sigma$; $\forall b \in B$, $\exists U \subset B$, $U \ni b$ and a homeomorphism $\phi_U$ such that the following diagram commutes:

$$
\begin{array}{ccc}
p^{-1}(U) & \xrightarrow{\phi_U} & U \times \Sigma \\
p_\infty \downarrow & & \downarrow \\
U & & U
\end{array}
$$

2. If $U \cap V \neq \emptyset$ and $h_{U \cap V}$ is defined by commutative diagram

$$
\begin{array}{ccc}
p^{-1}(U \cap V) & \xrightarrow{\phi_U} & (U \cap V) \times \Sigma \\
h_{U \cap V} & & \xrightarrow{h_{U \cap V}} (U \cap V) \times \Sigma \\
(U \cap V) \times \Sigma & \xrightarrow{\phi_V} & \phi_V
\end{array}
$$

then $\exists a = a_{U \cap V} \in \Sigma$, such that $h_{U \cap V}(b, \sigma) = (b, \sigma + a)$.

In this case $\Sigma$ acts on $S$ preserving fibers, so that for all $\alpha \in \Sigma$ the following diagram commutes:

$$
\begin{array}{ccc}
p^{-1}(U) & \xrightarrow{\phi_U} & U \times \Sigma \\
T_\alpha \downarrow & & \downarrow (b, \sigma) \mapsto (b, \sigma + \alpha) \\
p^{-1}(U) & \xrightarrow{\phi_U} & U \times \Sigma
\end{array}
$$

(we consider the case of an abelian $\Sigma$).

Given a principal flat $\Sigma$-bundle and a point $b \in B$, we can consider the monodromy action of $K = \pi_1(B, b)$ on the fiber $p_{-1}^{-1}(b)$. Each element $\gamma \in K$ acts as a translation by some $\rho(\gamma) \in \Sigma$. (Let us cover the image of $\gamma$ by neighborhoods $U_0, U_1, \ldots, U_n$ from the definition of flat principal $\Sigma$-bundle, such that $U_i \cap U_{i+1} \neq \emptyset$, $U_n = U_0$. The monodromy action of $\gamma$ on $p^{-1}(b)$ is the translation by $\rho(\gamma) = \sum_{i=0}^{n-1} \alpha_{U_i, U_{i+1}}$.) This action gives us a representation $\rho : K \to \Sigma$.

Vice versa, given a representation $\rho : K \to \Sigma$, we can construct a flat principal $\Sigma$-bundle over $B$ by taking the suspension of the $K$-action. The suspension space $S$ is defined as the quotient of $\Sigma \times B$, where $B$ is the universal covering of $B$, by the diagonal action of $K : (\sigma, y) \sim (\sigma + \rho(\alpha), \alpha(y)) \forall \sigma \in \Sigma, y \in \hat{B}$ and $\alpha(y)$ being the application of $\alpha \in K \cong \pi_1(B, b)$ to $y$. Indeed, it is easy to see that if we choose...
a base point \( y \in \pi^{-1}b \subset \hat{B} \), then the elements of \( p^{-1}_b \subset S \) can be “enumerated” by elements of \( \Sigma \), and that conditions (i) and (ii) in the definition of a flat principal \( \Sigma \)-bundle are satisfied.

Thus, the space \( \mathcal{C}(\Sigma) \) of principal flat \( \Sigma \)-bundles over \( B \) (mod a natural equivalence) is identified with the space of representations \( \rho : K \rightarrow \Sigma \).

In the case of \( B = X_\infty \) and \( \Sigma = \mathbb{Z}_p \), where \( \mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n \) is the group of \( p \)-adic numbers, the space \( \mathcal{C}(\mathbb{Z}_p) \) of flat principal \( \mathbb{Z}_p \)-bundles (mod natural equivalence) is identified with the space of \( p \)-adic representations \( \text{Hom}(K, \mathbb{Z}_p) \). To the bundle

\[
\begin{array}{c}
\mathbb{Z}_p \\
\downarrow p_{\infty}
\end{array} \quad \begin{array}{c}
S \\
\downarrow X_\infty
\end{array}
\]

corresponding to a representation \( \rho \), there are associated \( \mathbb{Z}/p^n \)-bundles

\[
\begin{array}{c}
\mathbb{Z}/p^n \\
\downarrow p_{\infty}
\end{array} \quad \begin{array}{c}
S_r \\
\downarrow X_\infty
\end{array}
\]

corresponding to homomorphisms \( \rho_r : K \rightarrow \mathbb{Z}/p^n \), where \( \rho_r \) is the composition

\[
\begin{array}{c}
K \\
\rho \downarrow \pi
\end{array} \quad \begin{array}{c}
\mathbb{Z}_p \\
\downarrow \mathbb{Z}/p^n
\end{array}
\]

\( \pi \) being the natural projection. Clearly, \( S_r \) form a tower of coverings and \( S = \varprojlim S_r \).

Note that \( S_r \) is connected iff \( \rho_r : K \rightarrow \mathbb{Z}/p^n \) is onto. In the case when all \( \rho_r \) are onto, \( S \) is a solenoid over \( X_\infty \). If for some \( r \), \( \rho_r \) is not onto, \( S_r \) is disconnected.

The pullback action of the deck transformation \( \tau \) on \( \mathcal{C}(\mathbb{Z}_p) \) corresponds to the \( \sigma_x \)-action in \( \text{Hom}(K, \mathbb{Z}_p) \).

The latter space is a finite dimensional \( \mathbb{Z}_p \)-module. Let us endow it with the sup-norm. Then any invertible operator \( A : \text{Hom}(K, \mathbb{Z}_p) \rightarrow \text{Hom}(K, \mathbb{Z}_p) \) becomes an isometry. Since \( \text{Hom}(K, \mathbb{Z}_p) \) is compact, \( A \) is almost periodic in the sense that the cyclic operator group \( \{ A^n \}_{n \in \mathbb{Z}} \) is precompact. The closure of this group is called the Bohr compactification of \( A \) (see [Lyu]). Theorem 5.2 provides us with a description of this group for \( \sigma_x \):

**Theorem 8.1.** The Bohr compactification of the operator

\[
\sigma_x : \text{Hom}(K, \mathbb{Z}_p) \rightarrow \text{Hom}(K, \mathbb{Z}_p)
\]

is the inverse limit of the cyclic groups \( \mathbb{Z}/d_n \) where the \( d_n \) are the least common multiples described by Theorem 5.2.

We can also consider solvable coverings over the knot complement \( X \) described in §7. Taking their inverse limits, we obtain various solenoids over \( X \).

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