Discrete coherent states for $n$ qubits

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Discrete coherent states for a system of $n$ qubits are introduced in terms of eigenstates of the finite Fourier transform. The properties of these states are pictured in phase space by resorting to the discrete Wigner function.

Keywords: Discrete phase space; coherent states; Wigner function.

1. Introduction

Discrete quantum systems were studied originally by Weyl and Schwinger and later by many authors. However, many concepts that appear sharp for continuous systems become fuzzy when one tries to apply them to discrete ones. The reason is that the in the continuum we have only one harmonic oscillator, while for finite systems, there are a number of candidates for that role, each one with its own virtues and drawbacks.

Coherent states constitute an archetypical example of the situation: for the standard harmonic oscillator they are well understood, and sensible generalizations have been devised to deal with systems with more general dynamical groups. However, in spite of interesting advances, the discrete counterparts are still under heated discussion.

Number states are eigenstates of the Fourier transform, but they are not the only ones: coherent states, with their associated Gaussian wave functions, also are. In our opinion, this subtle, yet obvious, observation has not been taken in due consideration in this field.
From this perspective, a decisive step was given by Mehta, who obtained the eigenstates of the finite Fourier transform. The purpose of this paper is to further explore this path, showing how we can define physically feasible coherent states for qubits that are also Fourier eigenstates.

Since the natural arena of these discrete coherent states is the phase space, we use the discrete Wigner function to picture the corresponding results. The problem of generalizing the Wigner function to finite systems has a long story. Using the notions presented in the comprehensive review, we construct a Wigner function for these coherent states and discuss some of their properties.

2. Coherent states for qubits

We consider identical qubits (i.e., noninteracting spin 1/2 systems). We recall that the Dicke states, belonging to the symmetric subspace of the representation of SU(2)$^\otimes n$, are given by

$$|n, k\rangle = \sqrt{\frac{k!(n-k)!}{n!}} \sum_{k} P_k(|1_1, 1_2, \ldots, 1_k, 0_{k+1}, \ldots, 0_n\rangle),$$

where $\{P_k\}$ denotes the complete set of all the possible permutations of the qubits. These states can be expressed in terms of the elements of the Galois field $\text{GF}(2^n)$ using the standard decomposition in the self-dual basis (a short review of the concepts of finite fields needed in this paper is presented in the appendix)

$$|1_1, 1_2, \ldots, 1_k, 0_{k+1}, \ldots, 0_n\rangle \mapsto |\xi_1 + 1\xi_2 + \ldots + 1\xi_k + 0\xi_{k+1} + \ldots + 0\xi_n\rangle.$$ (2)

Now, consider a SU(2) coherent state

$$|\xi\rangle = \frac{1}{(1 + |\xi|^2)^{n/2}} \sum_{k=0}^{n} \sqrt{\frac{n!}{k!(n-k)!}} \xi^k |n, k\rangle,$$ (3)

where the complex number $\xi$ is related with the angular coordinates $(\theta, \varphi)$ on the Bloch sphere by

$$\xi = \cot(\theta/2) e^{-i\varphi}.$$ (4)

Using the previous correspondence, $|\xi\rangle$ can be recast as

$$|\xi\rangle = \frac{1}{(1 + |\xi|^2)^{n/2}} \sum_{\gamma \in \text{GF}(2^n)} \xi^{h(\gamma)} |\gamma\rangle,$$ (5)

where the function $h(\gamma)$, when applied to the field element $\gamma = \sum_{k=1}^{n} \gamma_k \sigma_k$ indicates the number of nonzero coefficients $\gamma_k$.

In this case, the Fourier operator is

$$F = \frac{1}{2^{n/2}} \sum_{\mu, \nu \in \text{GF}(2^n)} \chi(\mu\nu) |\mu\rangle \langle \nu|,$$ (6)
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\( \chi \) being an additive character defined in (A.4). As it is well known, \( F^2 = 1 \), so if we impose that the states \( |\xi\rangle \) are also eigenstates of \( F \) we are lead to

\[ F|\xi\rangle = \pm|\xi\rangle. \tag{7} \]

This immediately implies (all the spins are pointing in the same direction) that there are two SU(2) coherent states (with \( \xi_{\pm} = \pm \sqrt{2} - 1 \)) that simultaneously are eigenstates of the Fourier operator: they are precisely our candidates to be coherent states for \( n \) qubits. In particular, \( |\xi_{\pm}\rangle \) satisfy the following condition

\[ \frac{1}{2^{n/2}} \sum_{\mu, \gamma \in \text{GF}(2^n)} \xi^h(\gamma) \chi(\mu \gamma)|\mu\rangle = \pm \sum_{\mu \in \text{GF}(2^n)} \xi^h(\mu)|\mu\rangle, \tag{8} \]

or, equivalently,

\[ \frac{1}{2^{n/2}} \sum_{\gamma \in \text{GF}(2^n)} \xi^h(\gamma) \chi(\mu \gamma) = \pm \xi^h(\mu), \tag{9} \]

and the minus sign may appear only for odd number of qubits.

Equation (5) is the abstract form of the SU(2) coherent state. It factorizes in a product of single-qubit states when represented in the self-dual basis, i.e.,

\[ |\xi\rangle = \frac{1}{(1 + |\xi|^2)^{n/2}} \sum_{c_1, \ldots, c_n \in \mathbb{Z}_2} \xi^h(\sum_{k=1}^{n} c_k \sigma_k)|c_1\rangle \ldots |c_n\rangle = \prod_{j=1}^{n} \frac{|\langle 0 \rangle + \xi|1\rangle\rangle}{(1 + |\xi|^2)^{1/2}}, \tag{10} \]

\( c_k \) being the expansion coefficients of \( \gamma \) in that basis. The operator transforming from the arbitrary basis \( \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\} \) into a factorized form is always a permutation given by

\[ P = \sum_{\mu \in \text{GF}(2^n)} (|\mu'_1\rangle \ldots |\mu'_n\rangle) (|\mu_1\rangle \ldots |\mu_n\rangle), \tag{11} \]

where

\[ \mu = \sum_{i=1}^{n} \mu_i \varepsilon_i = \sum_{i=1}^{n} \mu'_i \sigma_i, \quad \mu \in \text{GF}(2^n), \quad \mu_i, \mu'_i \in \mathbb{Z}_2. \tag{12} \]

Let us examine the simple yet illustrative example of a two-qubit coherent state. In its abstract form it reads as

\[ |\xi\rangle = \frac{1}{1 + \xi^2}(|0\rangle + \xi|\sigma\rangle + \xi|\sigma^2\rangle + \xi^2|\sigma^3\rangle). \tag{13} \]

In the self-dual basis (\( \sigma, \sigma^2 \)) we have the representation

\[ |0\rangle = |00\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |\sigma\rangle = |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |\sigma^2\rangle = |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |\sigma^3\rangle = |11\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{14} \]
in such a way that
\[
|\xi\rangle = \frac{1}{\sqrt{1 + \xi^2}} \begin{pmatrix} \xi^2 \\ \xi \\ 1 \end{pmatrix} = \frac{1}{\sqrt{1 + \xi^2}} |\xi\rangle \otimes \frac{1}{\sqrt{1 + \xi^2}} |1\rangle.
\] (15)

In a non self-dual basis \((\sigma, \sigma^3)\) we have
\[
|0\rangle = |00\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, |\sigma\rangle = |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, |\sigma^3\rangle = |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, |\sigma^2\rangle = |11\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},
\] (16)

and
\[
|\xi\rangle = \frac{1}{\sqrt{1 + \xi^2}} \begin{pmatrix} \xi^2 \\ \xi \\ 1 \end{pmatrix},
\] (17)

which cannot be factorized. The transition operator for this case is
\[
P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\] (18)

and it is nothing but a CNOT gate performing the operation
\[
|00\rangle + |01\rangle \rightarrow |00\rangle + |11\rangle.
\] (19)

3. Discrete Wigner function

To gain further insights into the coherent states \(|\xi_\pm\rangle\) we proceed to picture them in phase space. To this end, we first note that, while in the continuous case it is possible to translate a state by an infinite distance, this is clearly not possible if the space is finite. To “prevent” a state from “escaping” the finite phase space it is natural and convenient to use the field \(GF(2^n)\) in the representation of the states.

In consequence, we denote by \(|\alpha\rangle\), with \(\alpha \in GF(2^n)\), an orthonormal basis in the Hilbert space of the system. Operationally, the elements of the basis can be labeled by powers of a primitive element, and the basis reads
\[
\{|0\rangle, |\sigma\rangle, \ldots, |\sigma^{2^n-1} = 1\rangle\}. \tag{20}
\]

These vectors are eigenvectors of the operators \(Z_\beta\) belonging to the generalized Pauli group, whose generators are now defined as
\[
Z_\beta = \sum_{\alpha \in GF(2^n)} \chi(\alpha \beta) |\alpha\rangle \langle \alpha|, \quad X_\beta = \sum_{\alpha \in GF(2^n)} |\alpha + \beta\rangle \langle \alpha|, \tag{21}
\]
so that
\[ Z_\alpha X_\beta = \chi(\alpha \beta) X_\beta Z_\alpha. \] (22)

The operators (21) can be factorized into tensor products of powers of single-particle Pauli operators \( \sigma_z \) and \( \sigma_x \), whose expression in the standard basis of the two-dimensional Hilbert space is
\[ \hat{\sigma}_z = |1\rangle \langle 1| - |0\rangle \langle 0|, \quad \hat{\sigma}_x = |0\rangle \langle 1| + |1\rangle \langle 0|. \] (23)

This factorization can be carried out by mapping each element of GF\( (2^n) \) onto an ordered set of natural numbers. As we have already seen, a convenient choice for this is the self-dual basis, since the finite Fourier transform factorizes then into a product of single-particle Fourier operators, which leads to
\[ Z_\alpha = \hat{\sigma}^{a_1}_z \otimes \cdots \otimes \hat{\sigma}^{a_n}_z, \quad X_\beta = \hat{\sigma}^{b_1}_x \otimes \cdots \otimes \hat{\sigma}^{b_n}_x, \] (24)
where \((a_1, \ldots, a_n)\) and \((b_1, \ldots, b_n)\) are the expansion coefficients of \( \alpha \) and \( \beta \), respectively, in the self-dual basis.

It was shown that the operators
\[ D(\alpha, \beta) = \phi(\alpha, \beta) Z_\alpha X_\beta, \] (25)
where \( \phi(\alpha, \beta) \) is a phase, form an operational basis in the discrete phase space. The unitarity condition imposes the condition \( \phi^2(\alpha, \beta) = \chi(-\alpha \beta) \). These displacement operators (or phase-point operators in the notation of Wootters) allows us to introduce a Hermitian kernel
\[ \Delta(\alpha, \beta) = \frac{1}{2^n} \sum_{\mu, \nu \in \text{GF}(2^n)} \chi(\alpha \nu - \beta \mu) D(\mu, \nu), \] (26)
in terms of which we can define a well-behaved Wigner function as
\[ W_\rho(\alpha, \beta) = \text{Tr}[\rho \Delta(\alpha, \beta)], \] (27)
where \( \rho \) is the density matrix of the system.

After some calculations, the Wigner function for our coherent states turns out to be
\[ W_{\xi^+}(\alpha, \beta) = \frac{1}{2^n} \frac{1}{(1 + |\xi|^2)^n} \sum_{\mu, \nu, \gamma \in \text{GF}(2^n)} \xi^{h(\gamma)} \xi^{h(\gamma + \nu)} \chi(\alpha \nu + \beta \mu + \mu \nu + \mu \gamma) \phi(\mu, \nu). \] (28)

A plot of this function for the case of three qubits is shown in Fig. 1. We also note that the marginal distributions take a very simple form:
\[ \sum_{\alpha} W_{\xi^+}(\alpha, \beta) = \frac{|\xi^{h(\beta)}|^2}{(1 + |\xi|^2)^n}, \quad \sum_{\beta} W_{\xi^+}(\alpha, \beta) = \frac{|\xi^{h(\alpha)}|^2}{(1 + |\xi|^2)^n}. \] (29)

To conclude we wish to mention that it is also possible to introduce the notion of squeezing for these states. In the basis of the eigenstates of \( Z_\alpha \), such an operator has the following form
\[ S_\lambda = \sum_{\kappa \in \text{GF}(2^n)} |\kappa\rangle \langle \lambda \kappa|. \] (30)
The following relations hold
\[ S_\lambda^\dagger Z_\alpha S_\lambda = Z_{\alpha \lambda^{-1}}, \quad S_\lambda^\dagger X_\alpha S_\lambda = X_{\alpha \lambda}, \] (31)
so that
\[ \langle \xi_\pm | S_\lambda^\dagger X_\alpha S_\lambda | \xi_\pm \rangle = \langle \xi_\pm | S_\lambda^\dagger Z_{\alpha \lambda^{-1}} S_\lambda | \xi_\pm \rangle. \] (32)

4. Conclusions

In summary, we have formulated a new sensible approach to deal with coherent states for a system of \( n \) qubits. The associated discrete Wigner function has also been worked out. Some related problems, as the behavior under time evolution or the extension to systems of qudits, will be addressed elsewhere.

Appendix A. Galois fields

We briefly recall the minimum background of finite fields needed to proceed through this paper. The reader interested in more mathematical details is referred, e.g., to the excellent monograph by Lidl and Niederreiter.\(^{13}\)

A commutative ring is a set \( R \) equipped with two binary operations, called addition and multiplication, such that it is an Abelian group with respect the addition, and the multiplication is associative. Perhaps, the motivating example is the ring of integers \( \mathbb{Z} \) with the standard sum and multiplication. On the other hand, the simplest example of a finite ring is the set \( \mathbb{Z}_n \) of integers modulo \( n \), which has exactly \( n \) elements.

A field \( F \) is a commutative ring with division, that is, such that 0 does not equal 1 and all elements of \( F \) except 0 have a multiplicative inverse (note that 0 and 1 here stand for the
identity elements for the addition and multiplication, respectively, which may differ from the familiar real numbers 0 and 1. Elements of a field form Abelian groups with respect to addition and multiplication (in this latter case, the zero element is excluded).

The characteristic of a finite field is the smallest integer\( p \) such that
\[
\sum_{i=0}^{p-1} 1 = 0
\]
and it is always a prime number. Any finite field contains a prime subfield \( \mathbb{Z}_p \) and has \( d = p^n \) elements, where \( n \) is a natural number. Moreover, the finite field containing \( p^n \) elements is unique and is called the Galois field \( \text{GF}(p^n) \).

Let us denote as \( \mathbb{Z}_p[x] \) the ring of polynomials with coefficients in \( \mathbb{Z}_p \). Let \( P(x) \) be an irreducible polynomial of degree \( n \) (i.e., one that cannot be factorized over \( \mathbb{Z}_p \)). Then, the quotient space \( \mathbb{Z}_p[x]/P(x) \) provides an adequate representation of \( \text{GF}(p^n) \). Its elements can be written as polynomials that are defined modulo the irreducible polynomial \( P(x) \).

The multiplicative group of \( \text{GF}(p^n) \) is cyclic and its generator is called a primitive element of the field.

As a simple example of a nonprime field, we consider the polynomial \( x^2 + x + 1 = 0 \), which is irreducible in \( \mathbb{Z}_2 \). If \( \sigma \) is a root of this polynomial, the elements \( \{0, 1, \sigma, \sigma^2 = \sigma + 1 = \sigma^{-1}\} \) form the finite field \( \text{GF}(2^2) \) and \( \sigma \) is a primitive element.

A basic map is the trace
\[
\text{tr}(\alpha) = \alpha + \alpha^2 + \ldots + \alpha^{p^n-1},
\]
which satisfies
\[
\text{tr}(\alpha + \beta) = \text{tr}(\alpha) + \text{tr}(\beta),
\]
and leaves the prime field invariant. In terms of it we define the additive characters as
\[
\chi(\alpha) = \exp\left[\frac{2\pi i}{p} \text{tr}(\alpha)\right],
\]
and posses two important properties:
\[
\chi(\alpha + \beta) = \chi(\alpha)\chi(\beta), \quad \sum_{\alpha \in \text{GF}(p^n)} \chi(\alpha\beta) = p^n\delta_{0,\beta}.
\]

Any finite field \( \text{GF}(p^n) \) can be also considered as an \( n \)-dimensional linear vector space. Given a basis \( \{\theta_k\}, \) \( (k = 1, \ldots, n) \) in this vector space, any field element can be represented as
\[
\alpha = \sum_{k=1}^{n} a_k \theta_k,
\]
with \( a_k \in \mathbb{Z}_p \). In this way, we map each element of \( \text{GF}(p^n) \) onto an ordered set of natural numbers \( \alpha \leftrightarrow (a_1, \ldots, a_n) \).

Two bases \( \{\theta_1, \ldots, \theta_n\} \) and \( \{\theta'_1, \ldots, \theta'_n\} \) are dual when
\[
\text{tr}(\theta_k\theta'_l) = \delta_{k,l}.
\]
A basis that is dual to itself is called self-dual.

There are several natural bases in $\mathbb{GF}(p^n)$. One is the polynomial basis, defined as

$$\{1, \sigma, \sigma^2, \ldots, \sigma^{n-1}\},$$

where $\sigma$ is a primitive element. An alternative is the normal basis, constituted of

$$\{\sigma, \sigma^p, \ldots, \sigma^{p^{n-1}}\}.$$

The choice of the appropriate basis depends on the specific problem at hand. For example, in $\mathbb{GF}(2^2)$ the elements $\{\sigma, \sigma^2\}$ are both roots of the irreducible polynomial. The polynomial basis is $\{1, \sigma\}$ and its dual is $\{\sigma^2, 1\}$, while the normal basis $\{\sigma, \sigma^2\}$ is self-dual.

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