Abstract

We give new upper and lower bounds on the minimax sample complexity of differentially private mean estimation of distributions with bounded $k$-th moments. Roughly speaking, in the univariate case, we show that

$$n = \Theta\left(\frac{1}{\alpha^2} + \frac{1}{\alpha^{k-1} \varepsilon}\right)$$

samples are necessary and sufficient to estimate the mean to $\alpha$-accuracy under $\varepsilon$-differential privacy, or any of its common relaxations. This result demonstrates a qualitatively different behavior compared to estimation absent privacy constraints, for which the sample complexity is identical for all $k \geq 2$. We also give algorithms for the multivariate setting whose sample complexity is a factor of $O(d)$ larger than the univariate case.

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## Contents

1 Introduction ................................................. 3  
   1.1 Results, Techniques, and Discussion .................. 3  
   1.2 Related Work .......................................... 6  

2 Preliminaries .................................................. 7  
   2.1 Privacy Preliminaries ................................... 7  
   2.2 Basic Differentially Private Mechanisms ............... 8  

3 Estimating in One Dimension ................................ 10  
   3.1 Technical Lemmata ..................................... 10  
   3.2 The Algorithm ......................................... 12  
       3.2.1 Private Range Estimation........................... 12  
       3.2.2 Private Mean Estimation........................... 14  
   3.3 Estimating with CDP and Approximate DP .............. 15  
   3.4 Lower Bound ........................................... 16  

4 Estimating in High Dimensions with CDP ................... 17  
   4.1 Technical Lemmata ..................................... 17  
   4.2 The Algorithm ......................................... 18  

5 Estimating in High Dimensions with Pure DP ............. 20  
   5.1 Technical Lemma ...................................... 20  
   5.2 The Algorithm ......................................... 22  

6 Lower Bounds for Estimating High-Dimensional Distributions 26  

References ....................................................... 28  

A Useful Inequalities .......................................... 32
1 Introduction

Given samples $X_1, \ldots, X_n$ from a distribution $D$, can we estimate the mean of $D$? This is the problem of mean estimation which, alongside hypothesis testing, one of the most fundamental questions in statistics. As a result, answers to this problem are known in fairly general settings. For instance, the empirical mean is known to be an optimal estimate of a distribution’s true mean under minimal assumptions.

That said, statistics like the empirical mean put aside any concerns related to the sensitivity, and might vary significantly based on the addition of a single datapoint in the dataset. While this is not an inherently negative feature, it becomes a problem when the dataset contains personal information, and large shifts based on a single datapoint could potentially violate the corresponding individual’s privacy. In order to assuage these concerns, we consider the problem of mean estimation under the constraint of differential privacy (DP) [DMNS06], considered by many to be the gold standard of data privacy. Informally, an algorithm is said to be differentially private if its distribution over outputs is insensitive to the addition or removal of a single datapoint from the dataset. Differential privacy has enjoyed widespread adoption, including deployment in by Apple [Dif17], Google [EPK14], Microsoft [DKY17], and the US Census Bureau for the 2020 Census [DLS+17].

In this vein, a recent line of work [KV18, KLSU19, BKSW19] gives nearly optimal differentially private algorithms for mean estimation of sub-Gaussian random variables. Roughly speaking, to achieve accuracy $\alpha$ under $\epsilon$-differential privacy in a $d$-dimensional setting, one requires $n = \tilde{O}(\frac{d}{\alpha^2} + \frac{d}{\epsilon \alpha})$ samples, a mild cost of privacy over the non-private sample complexity of $O(\frac{d}{\alpha^2})$, except when $\epsilon$ is very small (corresponding to a very high level of privacy). However, these results all depend on the strong assumption that the underlying distribution being sub-Gaussian. Indeed, many sources of data in the real world are known to be heavy-tailed in nature, and thus we require algorithms which are effective even under these looser restrictions. Thus, the core question of this work is

What is the cost of privacy when estimating the mean of heavy-tailed distributions?

We make progress on this question by giving both algorithms and lower bounds for differentially private mean estimation on distributions with bounded $k$-th moments, for $k \geq 2$. In particular, for univariate distributions, we show that the optimal worst-case sample complexity depends critically on the choice of $k$, which is qualitatively different from the non-private case.

1.1 Results, Techniques, and Discussion

In this section, we will assume familiarity with some of the most common notions of differential privacy: pure $\epsilon$-differential privacy, $\rho$-zero-concentrated differential privacy, and approximate $(\epsilon, \delta)$-differential privacy. In particular, one should know that these are in (strictly) decreasing order of strength, formal definitions appear in Section 2.

We first focus on the univariate setting, proving tight upper and lower bounds for estimation subject to bounds on every possible moment.
Theorem 1.1 (Theorems 3.5 and 3.8). For every \( k \geq 2, 0 < \epsilon, \alpha < 1, \) and \( R > 1, \) there is an \( \epsilon \)-DP algorithm that takes
\[
n = O \left( \frac{1}{\alpha^2} + \frac{1}{\epsilon \alpha^{k-1}} + \frac{\log(R)}{\epsilon} \right)
\]
samples from an arbitrary distribution \( D \) with mean \( \mu \) such that \( \mu \in (-R, R) \) and \( \mathbb{E} \left[ |D - \mu|^k \right] \leq 1 \) and returns \( \hat{\mu} \) such that, with high probability, \( |\hat{\mu} - \mu| \leq \alpha. \) Moreover, any such \( \epsilon \)-DP algorithm requires \( n = \Omega \left( \frac{1}{\alpha^2} + \frac{1}{\epsilon \alpha^{k-1}} + \frac{\log(R)}{\epsilon} \right) \) samples in the worst case.\(^*\)

Note that, absent privacy constraints, the sample complexity of mean estimation with bounded \( k \)-th moments is \( n = O(1/\alpha^2) \) samples, for any \( k \geq 2. \) However, if we require the algorithm to be differentially private, there is a qualitatively different picture in which the cost of privacy decays as we have stronger bounds on the moments of the distribution. Our upper bounds follow a noised and truncated-empirical-mean approach. While this is similar to prior work on private mean estimation [KV18, KLSU19, CWZ19, BS19], we must be more aggressive with our truncation than before. In particular, for the Gaussian case, strong tail bounds allow one to truncate in a rather loose window and not remove any points if the data was actually sampled from a Gaussian. Since we consider distributions with much heavier tails, trying to not discard any points would result in a very wide truncation window, necessitating excessive amounts of noise. Instead, we truncate in a way that balances the two sources of error: bias due to valid points being discarded, and the magnitude of the noise due to the width of the truncation window. To be a bit more precise, our setting of parameters for truncation can be viewed in two different ways: either we truncate so that (in expectation) \( 1/\epsilon \) points are removed, and we require \( n \) to be large enough to guarantee accuracy, or we truncate so that \( \alpha^{k/(k-1)} \) probability mass is removed, and we require \( n \) to be large enough to guarantee privacy. These two perspectives on truncation are equivalent when \( n \) is at the critical value that makes up our sample complexity.

Our lower bound is proved via hypothesis testing. We demonstrate that two distributions that satisfy the conditions and are indistinguishable with fewer than the prescribed number of samples. Due to an equivalence between pure and approximate differential privacy in this setting, our lower bounds hold for the most permissive privacy notion of \((\epsilon, \delta)\)-DP, even for rather large values of \( \delta. \)

Turning to the multivariate setting, we provide separate algorithms for concentrated and pure differential privacy, both of which come at a multiplicative cost of \( O(d) \) in comparison to the univariate setting. We state the concentrated DP result first.

Theorem 1.2 (Theorem 4.5). For every \( d, k \geq 2, \epsilon, \alpha > 0, \) and \( R > 1, \) there is a polynomial-time \( \frac{\epsilon^2}{2} \)-zCDP algorithm that takes
\[
n \geq O \left( \frac{d}{\alpha^2} + \frac{d}{\epsilon \alpha^{k-1}} + \frac{\sqrt{d \log(R) \log(d)}}{\epsilon} \right)
\]

\(^*\)Analogous tight bounds hold for zCDP and \((\epsilon, \delta)\)-DP, and these bounds differ only in the dependence on \( R \) in the final term. In particular, \( \Omega(1/\alpha^2 + 1/\epsilon \alpha^{k/(k-1)}) \) samples are necessary for any of the variants of differential privacy.
samples from an arbitrary distribution $D$ on $\mathbb{R}^d$ with mean vector $\mu$ such that $\|\mu\|_2 \leq R$ and bounded $k$-th moments $\sup_{v \in S^{d-1}} \mathbb{E}\left[|\langle v, D - \mu \rangle|^k\right] \leq 1$ and returns $\hat{\mu}$ such that, with high probability, $\|\hat{\mu} - \mu\|_2 \leq \alpha$.

Similar to the univariate case, we rely upon a noised and truncated empirical mean (with truncation to an $\ell_2$ ball). The computations required to bound the bias of the truncated estimator are somewhat more involved and technical than the univariate case.

Our pure-DP multivariate mean estimator has the following guarantees.

**Theorem 1.3** (Theorem 5.8). For every $d, k \geq 2, \epsilon, \alpha > 0$, and $R > 1$, there is a (possibly exponential time) pure $\epsilon$-DP algorithm that takes

$$n \geq O\left(\frac{d}{\alpha^2} + \frac{d}{\epsilon \alpha \frac{1}{\sqrt{R}}} + \frac{d \log(R) \log(d)}{\epsilon}\right)$$

samples from an arbitrary distribution $D$ on $\mathbb{R}^d$ with mean vector $\mu$ such that $\|\mu\|_2 \leq R$ and bounded $k$-th moments $\sup_{v \in S^{d-1}} \mathbb{E}\left[|\langle v, D - \mu \rangle|^k\right] \leq 1$ and returns $\hat{\mu}$ such that, with high probability, $\|\hat{\mu} - \mu\|_2 \leq \alpha$.

We discuss the similarities and differences between Theorems 1.2 and 1.3. First, we note that the first two terms in the sample complexity are identical, similar to the multivariate Gaussian case, where distribution estimation under pure and concentrated DP share the same sample complexity [KLSU19, BKSW19]. This is contrary to certain problems in private mean estimation, where an $O(\sqrt{d})$ factor often separates the two complexities [BUV14, SU15, DSS15]. It appears that these qualitative gaps may or may not arise depending on the choice of norm and the assumptions we put on the underlying distribution (see Section 1.1.4 of [KLSU19] and Remark 6.4 of [BKSW19]) for more discussion. We point out that the estimator of Theorem 1.3 is not computationally efficient, while the estimator of Theorem 1.2 is. However, even for the well structured Gaussian case, no computationally-efficient algorithm is known under pure DP [KLSU19, BKSW19].

Technically, our multivariate-pure-DP algorithm is quite different from our other algorithms. It bears significant resemblance to approaches based on applying the “Scheffé estimator” to a cover for the family of distributions [Yat85, DL96, DL97, DL01]. These approaches reduce an estimation problem to a series of pairwise comparisons (i.e., hypothesis tests) between elements of the cover. However, outside of density estimation, we are not aware of any other problems in this space which are solved by applying pairwise comparisons to elements of a cover, as they instead often appeal to uniform convergence arguments. In particular, we believe our algorithm is the first to use this approach for the problem of mean estimation. We cover the space of candidate means, and perform a series of tests of the form “Which of these two candidates is a better fit for the distribution’s mean?” As mentioned before, there are often gaps between our understanding of multivariate estimation under pure and concentrated DP, and the primary reason is that the Laplace and Gaussian mechanisms have sensitivities based on the $\ell_1$ and $\ell_2$ norms, respectively. We avoid paying the extra $O(\sqrt{d})$ which often arises in the multivariate setting by reducing to a series of univariate problems—given two candidate means, we can project the problem onto the line which connects the two. By choosing whichever candidate wins all of its comparisons, we can get an accurate estimate for the mean overall. Crucially, using techniques from [BKSW19], we only pay logarithmically in the size of the cover.

Finally, we prove some lower bounds for multivariate private mean estimation.
**Theorem 1.4** (Theorem 6.1). Any pure ε-DP algorithm that takes samples from an arbitrary distribution on \( \mathbb{R}^d \) with bounded 2nd moments and returns \( \hat{\mu} \) such that \( \| \hat{\mu} - \mu \|_2 \leq \alpha \) requires \( n = \Omega \left( \frac{d}{\epsilon \alpha^2} \right) \) samples from \( D \) in the worst case.

In addition to showing that Theorem 1.3 is optimal for the case of \( k = 2 \), it specifically shows a qualitative difference between distributions with bounded 2nd moment and bounded \( k \)-th moment for \( k > 2 \). In the latter case the additional sample complexity due to privacy can be of lower order than the sample complexity without privacy, whereas for \( k = 2 \) it cannot be unless \( \epsilon \) is a constant.

### 1.2 Related Work

The most closely related works to ours are [BDMN05, BUV14, SU15, DSS+15, SU17, KV18, KLSU19, CWZ19, BS19, AMB19], which study differentially private estimation of the mean of a distribution. Some of these focus on restricted cases, such as product distributions or sub-Gaussians, which we generalize by making weaker moment-based assumptions. Some instead study more general cases, including unrestricted distributions over the hypercube – by making assumptions on the moments of the generating distributions, we are able to get better sample complexities. The work of Bun and Steinke [BS19] explicitly studies mean estimation of distributions with bounded second moment, but their sample complexity can be roughly stated as \( O(1/\alpha^2 \epsilon^2) \), whereas we prove a tight bound of \( \Theta(1/\alpha^2 \epsilon) \). Furthermore, we go beyond second-moment assumptions, and show a hierarchy of sample complexities based on the number of moments which are bounded.

Among other problems, Duchi, Jordan, and Wainwright [DJW13, DJW17] study univariate mean estimation with moment bounds under the stricter constraint of local differential privacy. Our univariate results and techniques are similar to theirs: morally the same algorithm and lower-bound construction works in both the local and central model. Translating their results to compare to ours, they show that the sample complexity of mean estimation in the local model is \( O(1/\alpha^{2k} \epsilon^2) \), the square of the “second term” in our sample complexity for the central model. However, their investigation is limited to the univariate setting, while we provide new algorithms and lower bounds for the multivariate setting as well. There has also been some work on locally private mean estimation in the Gaussian case [GRS19, JKMW19].

This is just a small sample of work in differentially private distribution estimation, and there has been much study into learning distributions beyond mean estimation. These are sometimes (but not always) equivalent problems – for instance, learning the mean of a Gaussian distribution with known covariance is equivalent to learning the distribution in total variation distance. Diakonikolas, Hardt, and Schmidt [DHS15] gave algorithms for learning structured univariate distributions. Privately learning mixtures of Gaussians was considered in [NRS07, KSSU19]. Bun, Nissim, Stemmer, and Vadhan [BNSV15] give an algorithm for learning distributions in Kolmogorov distance. Acharya, Kamath, Sun, and Zhang [AKSZ18] focus on estimating properties of a distribution, such as the entropy or support size. Smith [Smi11] gives an algorithm which allows one to estimate asymptotically normal statistics with optimal convergence rates, but no finite sample complexity guarantees. Bun, Kamath, Steinke, and Wu [BKSW19] give general tools for private hypothesis selection and apply this to learning many distribution
classes of interest.

In the non-private setting, there has recently been significant work in mean estimation of distributions with bounded second moments, in the “high probability” regime. That is, we wish to estimate the mean of a distribution with probability $1 - \beta$, where $\beta > 0$ might be very small. While the empirical mean is effective in the sub-Gaussian case, more advanced techniques are necessary to achieve the right dependence on $1/\beta$ when we only have a bound on the second moment. A recent series of papers has focused on identifying effective methods and making them computationally efficient [LM19b, Hop18, CFB19, DL19, LM19a]. This high-probability consideration is not the focus of the present work, though we note that, at worst, our estimators incur a multiplicative factor of $\log(1/\beta)$ in achieving this guarantee. We consider determining the correct dependence on the failure probability with privacy constraints an interesting direction for future study.

Our work bears a significant resemblance to a line on hypothesis selection, reducing to pairwise comparisons using the Scheffé estimator. This style of approach was pioneered by Yatracos [Yat85], and refined in subsequent work by Devroye and Lugosi [DL96, DL97, DL01]. After this, additional considerations have been taken into account, such as computation, approximation factor, robustness, and more [MS08, DDS12, DK14, SOAJ14, AJOS14, DKK+16, AFJ+18, BKM19, BKSW19]. As mentioned before, to the best of our knowledge, we are the first to apply this pairwise-comparison-based approach combined with a net-based argument for a problem besides density estimation.

2 Preliminaries

We formally state what it means for a distribution to have its $k^{th}$ moment bounded.

**Definition 2.1.** Let $D$ be a distribution over $\mathbb{R}^d$ with mean $\mu$. We say that for $k \geq 2$, the $k^{th}$ moment of $D$ is bounded by $M$, if for every unit vector $v \in S^{d-1}$,

$$\mathbb{E}\left[|\langle X - \mu, v \rangle|^k\right] \leq M.$$

Also, we define $B_r(p) \subset \mathbb{R}^d$ to be the ball of radius $r > 0$ centered at $p \in \mathbb{R}^d$.

### 2.1 Privacy Preliminaries

**Definition 2.2** (Differential Privacy (DP) [DMNS06]). A randomized algorithm $M : \mathcal{X}^n \to \mathcal{Y}$ satisfies $(\epsilon, \delta)$-differential privacy ((\epsilon, \delta))-DP if for every pair of neighboring datasets $X, X' \in \mathcal{X}^n$ (i.e., datasets that differ in exactly one entry),

$$\forall Y \subseteq \mathcal{Y} \quad \mathbb{P}[M(X) \in Y] \leq e^\epsilon \cdot \mathbb{P}[M(X') \in Y] + \delta.$$  

When $\delta = 0$, we say that $M$ satisfies $\epsilon$-differential privacy or pure differential privacy.

**Definition 2.3** (Concentrated Differential Privacy (zCDP) [BS16]). A randomized algorithm $M : \mathcal{X}^n \to \mathcal{Y}$ satisfies $\rho$-zCDP if for every pair of neighboring datasets $X, X' \in \mathcal{X}^n$,

$$\forall \alpha \in (1, \infty) \quad D_\alpha (M(X)||M(X')) \leq \rho \alpha,$$
where \( D_\alpha(M(X)\|M(X')) \) is the \( \alpha \)-Rényi divergence between \( M(X) \) and \( M(X') \).

Note that zCDP and DP are on different scales, but are otherwise can be ordered from most-to-least restrictive. Specifically, \((\varepsilon, 0)\)-DP implies \( \frac{\varepsilon}{2} \)-zCDP, which implies \((\varepsilon \sqrt{\log(1/\delta)}, \delta)\)-DP for every \( \delta > 0 \) [BS16].

Both these definitions are closed under post-processing and can be composed with graceful degradation of the privacy parameters.

**Lemma 2.4 (Post Processing [DMNS06, BS16]).** If \( M : X^n \rightarrow Y \) is \((\varepsilon, \delta)\)-DP, and \( P : Y \rightarrow Z \) is any randomized function, then the algorithm \( P \circ M \) is \((\varepsilon, \delta)\)-DP. Similarly if \( M \) is \( \rho \)-zCDP then the algorithm \( P \circ M \) is \( \rho \)-zCDP.

**Lemma 2.5 (Composition of DP [DMNS06, DRV10, BS16]).** If \( M \) is an adaptive composition of differentially private algorithms \( M_1, \ldots, M_T \), then the following all hold:

1. If \( M_1, \ldots, M_T \) are \((\varepsilon_1, \delta_1), \ldots, (\varepsilon_T, \delta_T)\)-DP then \( M \) is \((\varepsilon, \delta)\)-DP for
   \[
   \varepsilon = \sum_t \varepsilon_t \quad \text{and} \quad \delta = \sum_t \delta_t.
   \]

2. If \( M_1, \ldots, M_T \) are \((\varepsilon_0, \delta_1), \ldots, (\varepsilon_0, \delta_T)\)-DP for some \( \varepsilon_0 \leq 1 \), then for every \( \delta_0 > 0 \), \( M \) is \((\varepsilon, \delta)\)-DP for
   \[
   \varepsilon = \varepsilon_0 \sqrt{6T \log(1/\delta_0)} \quad \text{and} \quad \delta = \delta_0 + \sum_t \delta_t
   \]

3. If \( M_1, \ldots, M_T \) are \( \rho_1, \ldots, \rho_T \)-zCDP then \( M \) is \( \rho \)-zCDP for \( \rho = \sum_t \rho_t \).

### 2.2 Basic Differentially Private Mechanisms.

We first state standard results on achieving privacy via noise addition proportional to sensitivity [DMNS06].

**Definition 2.6 (Sensitivity).** Let \( f : X^n \rightarrow \mathbb{R}^d \) be a function, its \( \ell_1 \)-sensitivity and \( \ell_2 \)-sensitivity are

\[
\Delta f,1 = \max_{X,X' \in \mathcal{X}^n} \| f(X) - f(X') \|_1 \quad \text{and} \quad \Delta f,2 = \max_{X,X' \in \mathcal{X}^n} \| f(X) - f(X') \|_2,
\]

respectively. Here, \( X \sim X' \) denotes that \( X \) and \( X' \) are neighboring datasets (i.e., those that differ in exactly one entry).

For functions with bounded \( \ell_1 \)-sensitivity, we can achieve \( \varepsilon \)-DP by adding noise from a Laplace distribution proportional to \( \ell_1 \)-sensitivity. For functions taking values in \( \mathbb{R}^d \) for large \( d \), it is more useful to add noise from a Gaussian distribution proportional to the \( \ell_2 \)-sensitivity, to get \((\varepsilon, \delta)\)-DP and \( \rho \)-zCDP.

---

\[ \text{Given two probability distributions } P, Q \text{ over } \Omega, \] \[ D_\alpha(P \| Q) = \frac{1}{\alpha - 1} \log \left( \sum_x P(x)^\alpha Q(x)^{1-\alpha} \right). \]
Lemma 2.7 (Laplace Mechanism). Let $f : \mathcal{X}^n \rightarrow \mathbb{R}^d$ be a function with $\ell_1$-sensitivity $\Delta_{f,1}$. Then the Laplace mechanism

$$M(X) = f(X) + \text{Lap}\left(\frac{\Delta_{f,1}}{\varepsilon}\right)^{\otimes d}$$

satisfies $\varepsilon$-DP.

Lemma 2.8 (Gaussian Mechanism). Let $f : \mathcal{X}^n \rightarrow \mathbb{R}^d$ be a function with $\ell_2$-sensitivity $\Delta_{f,2}$. Then the Gaussian mechanism

$$M(X) = f(X) + \mathcal{N}\left(0, \left(\frac{\Delta_{f,2}}{\sqrt{2\ln(2/\delta)}}\right)^2 I_{d \times d}\right)$$

satisfies $(\varepsilon,\delta)$-DP. Similarly, the Gaussian mechanism

$$M_f(X) = f(X) + \mathcal{N}\left(0, \left(\frac{\Delta_{f,2}}{\sqrt{2\rho}}\right)^2 I_{d \times d}\right)$$

satisfies $\rho$-zCDP.

Lemma 2.9 (Private Histograms). Let $(X_1, \ldots, X_n)$ be samples in some data universe $U$, and let $\Omega = \{h_u\}_{u \subset U}$ be a collection of disjoint histogram buckets over $U$. Then we have $\varepsilon$-DP, $\rho$-zCDP, and $(\varepsilon,\delta)$-DP histogram algorithms with the following guarantees.

1. $\varepsilon$-DP: $\ell_\infty$ error - $O\left(\log(|U|/\varepsilon \beta)\right)$ with probability at least $1 - \beta$; run time - $\text{poly}(n, \log(|U|/\varepsilon \beta))$

2. $\rho$-zCDP: $\ell_\infty$ error - $O\left(\sqrt{\log(|U|/\varepsilon \beta)}\right)$ with probability at least $1 - \beta$; run time - $\text{poly}(n, \log(|U|/\rho \beta))$

3. $(\varepsilon,\delta)$-DP: $\ell_\infty$ error - $O\left(\log(|U|/\varepsilon \beta)\right)$ with probability at least $1 - \beta$; run time - $\text{poly}(n, \log(|U|/\varepsilon \beta))$

Part 1 follows from [BV19]. Part 2 follows trivially by using the Gaussian Mechanism (Lemma 2.8) instead of the Laplace Mechanism (Lemma 2.7) in Part 1. Part 3 holds due to [BNS16, Vad17].

Finally, we recall the widely used exponential mechanism.

Lemma 2.10 (Exponential Mechanism [MT07]). The exponential mechanism $M_{\varepsilon, S, \text{SCORE}}(X)$ takes a dataset $X \in \mathcal{X}^n$, computes a score ($\text{SCORE} : \mathcal{X}^n \times S \rightarrow \mathbb{R}$) for each $p \in S$ with respect to $X$, and outputs $p \in S$ with probability proportional to $\exp\left(\frac{\varepsilon \cdot \text{SCORE}(X,p)}{2 \cdot \Delta_{\text{SCORE},1}}\right)$, where

$$\Delta_{\text{SCORE},1} = \max_{p \in S} \max_{X \sim X' \in \mathcal{X}^n} |\text{SCORE}(X, p) - \text{SCORE}(X', p)|.$$

It satisfies the following.

1. $M$ is $\varepsilon$-differentially private.

2. Let $\text{OPT}_{\text{SCORE}}(X) = \max_{p \in S} |\text{SCORE}(X, p)|$. Then

$$\mathbb{P}\left[\text{SCORE}(X, M_{\varepsilon, S, \text{SCORE}}(X)) \leq \text{OPT}_{\text{SCORE}}(X) - \frac{2 \Delta_{\text{SCORE},1}}{\varepsilon}(\ln(|S| + t))\right] \leq e^{-t}.$$
3 Estimating in One Dimension

In this section, we discuss estimating the mean of a distribution whilst ensuring pure DP. Obtaining CDP and approximate DP algorithms for this is trivial, once we have the algorithm for pure DP, as shall be discussed towards the end of the upper bounds section. Finally, we show that our upper bounds are optimal.

3.1 Technical Lemmata

Here, we lay out the two main technical lemmata that we would use to prove our main results for the section. The first lemma says that if we truncate the distribution to within a large interval that is centered close to the mean, then the mean of this truncated distribution will be close to the original mean.

**Lemma 3.1.** Let $D$ be a distribution over $\mathbb{R}$ with mean $\mu$, and $k^{th}$ moment bounded by 1. Let $\rho \in \mathbb{R}$, $0 < \tau < \frac{1}{16}$, and $\xi = \frac{C \tau}{1 + \tau}$ for a constant $C \geq 6$. Let $X \sim D$, and $Z$ be the following random variable.

$$Z = \begin{cases} 
\rho - \xi & \text{if } X < \rho - \xi \\
X & \text{if } \rho - \xi \leq X \leq \rho + \xi \\
\rho + \xi & \text{if } X > \rho + \xi 
\end{cases}$$

If $|\mu - \rho| \leq \frac{\xi}{2}$, then $|\mu - \mathbb{E}[Z]| \leq \tau$. 

**Proof.** Without loss of generality, we assume that $\rho \geq \mu$, since the argument for the other case is symmetric. Let $a = \rho - \xi$ and $b = \rho + \xi$.

$$|\mu - \mathbb{E}[Z]| \leq |\mathbb{E}[(X - a)1_{X < a}]]| + |\mathbb{E}[(X - b)1_{X > b}]]|$$

Now, we compute the first term on the right hand side. The second term would follow by an identical argument.

$$|\mathbb{E}[(X - a)1_{X < a}]]| = |\mathbb{E}[(X - \mu - (a - \mu))1_{X < a}]]|$$

$$\leq \mathbb{E}[(|X - \mu|)1_{X < a}]] + (|a - \mu|)\mathbb{E}[1_{X < a}]]$$

$$\leq \left(\mathbb{E}[(|X - \mu|^k)]\right)^{\frac{1}{k}} (\mathbb{P}[X < a])^{\frac{k}{k-1}} + (|a - \mu|)\mathbb{P}[X < a]$$

$$\leq \left(\frac{2}{C}\right)^{k-1} \tau + C \left(\frac{2}{C}\right)^k \tau$$

$$= 3 \left(\frac{2}{C}\right)^{k-1} \tau$$

In the above, the first inequality follows from linearity of expectations, triangle inequality, and Lemma A.8. The second inequality follows from Lemma A.7, and the third inequality follows from the fact that

$$\mathbb{P}[X < a] \leq \mathbb{P}[X < \mu - \frac{\xi}{2}] \leq \left(\frac{2}{C}\right)^{k} \tau^{\frac{k}{k-1}}.$$
which holds due to Lemma A.1. Similarly, we can bound the second term in (1) as follows:

$$|E[(X - b)1_{X > b}]| \leq 4\left(\frac{2}{C}\right)^{k-1}\tau.$$

Substituting these two values in Inequality 1, we get that

$$|\mu - E[Z]| \leq 7\left(\frac{2}{C}\right)^{k-1}\tau \leq \tau.$$

The next lemma says that if we take a large number of samples from any distribution over \(\mathbb{R}\), whose \(k\)th moment is bounded by 1, then with high probability, the empirical mean of the samples lies close to the mean of the distribution.

**Lemma 3.2.** Let \(D\) be a distribution over \(\mathbb{R}\) with mean \(\mu\) and \(k\)th moment bounded by 1. Suppose \((X_1, \ldots, X_n)\) are samples from \(D\), where

\[n \geq O\left(\frac{1}{\alpha^2}\right)\]

Then with probability at least 0.9,

$$\left|\frac{1}{n} \sum_{i=1}^{n} X_i - \mu\right| \leq \alpha.$$

**Proof.** Using Lemma A.8, we know that

$$E\left[|X - \mu|^2\right] \leq E\left[|X - \mu|^k\right]^{\frac{2}{k}} \leq 1.$$

Let \(Z = \frac{1}{n} \sum_{i=1}^{n} X_i\). Then we have the following.

$$E\left[|Z - \mu|^2\right] = \frac{1}{n^2} E\left[\sum_{i=1}^{n} |X_i - \mu|^2\right] \leq \frac{1}{n^2} E\left[\sum_{i=1}^{n} |X_i|^2\right] = \frac{1}{n^2} \sum_{i=1}^{n} E\left[|X_i|^2\right] \leq \frac{1}{n}.$$

Then using Lemma A.1, we have

$$\mathbb{P}\left[|Z - \mu| > \alpha\right] \leq \frac{1}{\sqrt{n\alpha}} \leq 0.9.$$
3.2 The Algorithm

Here, we give an $\varepsilon$-DP algorithm to estimate the mean. The main algorithm consists of two parts: limiting the data to a reasonable range so as to achieve privacy, and to limit the amount of noise added for it to get optimal accuracy; and mean estimation in a differentially private way. We analyze the two separately.

3.2.1 Private Range Estimation

Here, we explore the first part of the algorithm, that is, limiting the range of the data privately. We do that in a way similar to that of [KV18]. To summarize, we use differentially private histograms (Lemma 2.9) to find the bucket with the largest number of points. Due to certain moments of the distribution being bounded, the points tend to concentrate around the mean. Therefore, the above bucket would be the one closest to the mean, and by extending the size of the bucket a little, we could get an interval that contains a large number of points along with the mean. We show that the range is large enough that the mean of the distribution truncated to that interval will not be too far from the original mean.

**Algorithm 1:** Pure DP Range Estimator $\text{PDPRE}_{\varepsilon,\alpha,R}(X)$

**Input:** Samples $X_1, \ldots, X_n \in \mathbb{R}$. Parameters $\varepsilon, \alpha, R > 1$.

**Output:** $[a, b] \in \mathbb{R}$.

Set parameters: $r \leftarrow 10/\alpha^{1/k}$

// Estimate range
Divide $[-R - 2r, R + 2r]$ into buckets: $[-R - 2r, -R), \ldots, [-2r, 0), [0, 2r), \ldots, [R, R + 2r]$

Run Pure DP Histogram for $X$ over the above buckets

Let $[a, b]$ be the bucket that has the maximum number of points

Let $I \leftarrow [a - 2r, b + 2r]$

Return $I$

**Theorem 3.3.** Let $\mathcal{D}$ be a distribution over $\mathbb{R}$ with mean $\mu \in [-R, R]$ and $k^{th}$ moment bounded by 1. Then for all $\varepsilon > 0$ and $0 < \alpha < \frac{1}{16}$, there exists an $\varepsilon$-DP algorithm that takes

$$n \geq O\left(\frac{1}{\alpha} + \frac{\log(R\alpha)}{\varepsilon}\right)$$

samples from $\mathcal{D}$, and outputs $I = [a, b] \subset \mathbb{R}$, such that with probability at least 0.9, the following conditions all hold:

1. $b - a \in \Theta\left(\frac{1}{\alpha k^{1/2}}\right)$.
2. At most $\alpha n$ samples lie outside $I$.
3. $\mu \in I$ and $b - \mu, \mu - a \geq \frac{10}{\alpha \varepsilon^{1/2}}$.

**Proof.** We separate the privacy and accuracy proofs for Algorithm 1 for clarity.
Privacy:
Privacy follows from Lemma 2.9 and post-processing of the private output of private histograms (Lemma 2.4).

Accuracy:
The first part follows because the intervals are deterministically constructed to have length $6r \in \Theta\left(\frac{1}{\alpha^{1/k}}\right)$.
Let $(X_1, \ldots, X_n)$ be independent samples from $D$. We know from Lemma A.1 that,

$$\Pr_{X \sim D} \left[ \left| X - \mu \right| > \frac{10}{\alpha^k} \right] \leq \frac{\alpha}{10^k}.$$  

Using Lemma A.3, we have, $\Pr \left[ |i| : X_i \not\in [\mu - r, \mu + r] > \alpha n \right] < 0.05$, because $n \geq O(1/\alpha)$. Therefore, there has to be a bucket that contains at least $0.5(1 - \alpha)n \geq \frac{n}{4}$ points from the dataset, which implies that the bucket containing the maximum number of points has to have at least $\frac{n}{4}$ points. Now, from Lemma 2.9, we know that the noise added to any bucket cannot exceed $\frac{n}{16}$. Therefore, the noisy value for the largest bucket has to be at least $\frac{3n}{16}$. Since, all these points lie in a single bucket, and include points that are not in the tail of the distribution, the mean lies in either the same bucket, or in an adjacent bucket because the distance from the mean is at most $r$. Hence, the constructed interval of length $6r$ contains the mean and at least $1 - \alpha$ fraction of the points. From the above, since the mean is at most $r$ far from at least one of $a$ and $b$, the end points of $I$ must be at least $r$ far from $\mu$.  

The CDP equivalent of Algorithm 1 (that we call, ”CDPRE“) could be created by using the CDP version of private histograms as mentioned in Lemma 2.9. Its approximate DP version (which we call, ”ADPRE“) could be obtained via approximate differentially private histograms as mentioned in the same lemma.

**Theorem 3.4.** Let $D$ be a distribution over $\mathbb{R}$ with mean $\mu \in [-R, R]$ and $k^{th}$ moment bounded by 1. Then for all $\epsilon, \delta, \rho > 0$ and $0 < \alpha < \frac{1}{16}$, there exist $(\epsilon, \delta)$-DP and $\rho$-zCDP algorithms that take

$$n_{(\epsilon, \delta)} \geq O\left(\frac{1}{\alpha} + \frac{\log(1/\delta)}{\epsilon}\right)$$

and

$$n_\rho \geq O\left(\frac{1}{\alpha} + \sqrt{\frac{\log(R\alpha)}{\rho}}\right)$$

samples from $D$ respectively, and output $I = [a, b] \subset \mathbb{R}$, such that with probability at least 0.9, the following hold.

1. $b - a \in \Theta\left(\frac{1}{\alpha^{1/k}}\right)$.
2. At most $\alpha n$ samples lie outside $I$.
3. $\mu \in I$ and $b - \mu, \mu - a \geq \frac{10}{\alpha^{1/k}}$.  

13
3.2.2 Private Mean Estimation

Now, we detail the second part of the algorithm, that is, private mean estimation. In the previous step, we ensured that the range of the data is large enough that the mean of the truncated distribution would not be too far from the original mean. Here, we show that this range is small enough, that the noise we add to guarantee privacy is not too large. This would imply that the whole algorithm, whilst being differentially private, would also be accurate without adding a large overhead in the sample complexity.

Algorithm 2: Pure DP 1-Dimensional Mean Estimator PDPODME_{\epsilon,\alpha,R}(X)

| Input: | Samples $X_1, \ldots, X_{2n} \in \mathbb{R}$. Parameters $\epsilon, \alpha, R > 0$. |
|--------|------------------------------------------------------------------|
| Output: | $\hat{\mu} \in \mathbb{R}$ |

Set parameters: $m \leftarrow 200 \log(2/\beta)$ $Z \leftarrow (X_1, \ldots, X_n)$ $W \leftarrow (X_{n+1}, \ldots, X_{2n})$

// Partition the dataset, and estimate mean on each subset
For $i \leftarrow 1, \ldots, m$
    Let $Y^i \leftarrow (W_{(i-1)\frac{n}{m}+1}, \ldots, W_{i\frac{n}{m}})$ and $Z^i \leftarrow (Z_{(i-1)\frac{n}{m}+1}, \ldots, Z_{i\frac{n}{m}})$
    // Find small interval containing the mean and large fraction of points
    $I_i \leftarrow \text{PDPRE}_{\epsilon,\alpha, R}(Z_i)$ and $r_i \leftarrow |I_i|$
    // Truncate to within the small interval above
    For $y \in Y^i$
        If $x < I_i$
            Set $x$ to be the nearest end-point of $I_i$
    // Estimate the mean
    $\bar{\mu}_i \leftarrow \frac{1}{n} \sum_{y \in Y^i} y + \text{Lap}\left(\frac{mr_i}{\epsilon n}\right)$
    // Median of means to select a good mean with high probability
    $\bar{\mu} \leftarrow \text{Median}(\mu_1, \ldots, \mu_m)$
Return $\bar{\mu}$

Theorem 3.5. Let $D$ be a distribution over $\mathbb{R}$ with mean $\mu \in [-R, R]$ and $k^{th}$ moment bounded by 1. Then for all $\epsilon, \alpha, \beta > 0$, there exists an $\epsilon$-DP algorithm that takes

$$n \geq O\left(\frac{\log(1/\beta)}{\alpha^2} + \frac{\log(1/\beta)}{\epsilon \alpha} + \frac{\log(R)\log(1/\beta)}{\epsilon}\right)$$

samples from $D$, and outputs $\bar{\mu} \in \mathbb{R}$, such that with probability at least $1 - \beta$,

$$|\mu - \bar{\mu}| \leq \alpha.$$ 

Proof. We first prove the privacy guarantee of Algorithm 2, then move on to accuracy.

Privacy:
The step of finding a good interval $I_i$ is $\frac{\epsilon}{\epsilon}$-DP by Theorem 3.3. Then the step of estimating the mean by adding Laplace noise is $\frac{\epsilon}{\epsilon}$-DP by Lemma 2.7. Therefore, by Lemma 2.5, each iteration is $\epsilon$-DP. Since, we use each disjoint part of the dataset only once in the entire loop, $\epsilon$-DP still
holds. Finally, we operate on private outputs to find the median, therefore, by Lemma 2.4, the algorithm is \( \varepsilon \)-DP.

**Accuracy:**
We fix an iteration \( i \), and discuss the accuracy of that step. The accuracy of the rest of the iterations would be guaranteed in the same way, since all iterations are independent.

**Claim 3.6.** Fix \( 1 \leq i \leq m \). Then in iteration \( i \), if
\[
|Y_i| \geq O\left(\frac{1}{\alpha^2} + \frac{1}{\varepsilon \alpha \epsilon_i} + \frac{\log(R)}{\epsilon}\right),
\]
then with probability at least 0.7, \( |\mu_i - \mu| \leq \alpha \).

**Proof.** We know from Theorem 3.3 that with probability at least 0.9, \( \mu \in I_i \), such that if \( I_i = [a, b] \), then \( \mu - a, b - \mu \in \Omega\left(\frac{1}{\alpha^2}\right) \). Therefore, from Lemma 3.1, the mean of the truncated distribution (let’s call it \( \mu_i' \)) will be at most \( \frac{\alpha}{2} \) from \( \mu \). But from Lemma 3.2, we know that \( |\mu_i - \mu_i'| \leq \frac{\alpha}{2} \) with probability at least 0.9. Finally, from Lemma A.5, with probability at least 0.9, the Laplace noise added is at most \( \frac{\alpha}{2} \) because we have at least \( O\left(\frac{1}{\varepsilon \alpha \epsilon_i}\right) \) samples. Therefore, by triangle inequality, \( |\mu_i - \mu| \leq \alpha \), and by the union bound, this happens with probability at least 0.7.

Now, by the claim above, and using Lemma A.4, more than \( \frac{m}{2} \) iterations should yield \( \mu_i \) that are \( \alpha \) close to \( \mu \), which happens with probability at least \( 1 - \beta \) (because \( m \geq O(\log(1/\beta)) \)). Therefore, the median, that is, \( \hat{\mu} \) is at most \( \alpha \) far from \( \mu \) with probability at least \( 1 - \beta \). \( \square \)

### 3.3 Estimating with CDP and Approximate DP

The same algorithm could be used to get CDP guarantees by using CDPRE instead of Algorithm 1, and using the Gaussian Mechanism (Lemma 2.8) instead of the Laplace Mechanism (Lemma 2.7). To get approximate DP guarantees, Algorithm 2, with the exception of using ADPRE instead, could be used.

**Theorem 3.7.** Let \( \mathcal{D} \) be a distribution over \( \mathbb{R} \) with mean \( \mu \in [-R, R] \) and \( k \)th moment bounded by 1. Then for all \( \varepsilon, \delta, \rho, \alpha, \beta > 0 \), there exist \((\varepsilon, \delta)\)-DP and \( \rho \)-zCDP algorithms that take
\[
n_{(\varepsilon, \delta)} \Omega\left(\frac{\log(1/\beta)}{\alpha^2} + \frac{\log(1/\beta)}{\varepsilon \alpha \epsilon_i} + \frac{\log(1/\delta) \log(1/\beta)}{\epsilon}\right)
\]
and
\[
n_{\rho} \geq O\left(\frac{\log(1/\beta)}{\alpha^2} + \frac{\log(1/\beta)}{\sqrt{\rho \alpha \epsilon_i}} + \frac{\sqrt{\log(R) \log(1/\beta)}}{\sqrt{\rho}}\right)
\]
samples from \( \mathcal{D} \) respectively, and output \( \hat{\mu} \in \mathbb{R} \), such that with probability at least 1 - \( \beta \),
\[
|\mu - \hat{\mu}| \leq \alpha.
\]
3.4 Lower Bound

Now, we prove a lower bound that matches our upper bound, showing that our upper bounds are optimal.

**Theorem 3.8.** Let \( D \) be a distribution with mean \( \mu \in (-1, 1) \) and \( k \)th moment bounded by 1. Then given \( \epsilon, \delta, \alpha > 0 \), any \((\epsilon, \delta)\)-DP algorithm takes

\[
    n \geq \Omega \left( \frac{1}{\epsilon \alpha^{k-1}} \right)
\]

samples to estimate \( \mu \) to within \( \alpha \) absolute error with constant probability.

**Proof.** We construct two distributions that are “close”, and show that any \((\epsilon, \delta)\)-DP algorithm that distinguishes between them requires a large number of samples.

**Claim 3.9.** Let \( \epsilon, \delta, \alpha > 0 \), and \( D_1, D_2 \) be two distributions on \( \mathbb{R} \) defined as follows.

\[
    D_1 \equiv \mathbb{P}_{X \sim D_1} [X = 0] = 1
\]

\[
    D_2 \equiv \begin{cases} 
    X = 0 & \text{with probability } 1 - p \\
    X = \tau & \text{with probability } p 
\end{cases}
\]

Where in the above, \( \tau > 0 \), \( p \tau = \alpha \) and \( p \leq \frac{1}{\alpha^{k-1}} \). Then the following holds.

1. \( \mathbb{E}_{X \sim D_2} [|X - p \tau|^k] \leq 1 \)

2. Any \((\epsilon, \delta)\)-DP algorithm that can distinguish between \( D_1 \) and \( D_2 \) with constant probability requires at least \( \frac{1}{\epsilon \alpha^{k-1}} \) samples.

**Proof.** For the first part, note that \( \mathbb{E}_{X \sim D_2} [X] = p \tau \). Then we have the following.

\[
    \mathbb{E}_{X \sim D_2} [|X - p \tau|^k] = p |\tau - p \tau|^k + (1 - p) |p \tau|^k \\
    = p (1 - p) \tau^k (1 - p)^{k-1} + p^k \\
    \leq p \tau^k \\
    \leq 1. \quad \text{(Using our restrictions on } p, \tau \text{)}
\]

Now, for the second part, we know that

\[
    \left| \mathbb{E}_{X \sim D_1} [X] - \mathbb{E}_{X \sim D_2} [X] \right| = \alpha.
\]

Suppose we take \( n \) samples each from \( D_1 \) and \( D_2 \). Then by Theorem 11 of [ASZ18], we get that

\[
    pn \in \Omega \left( \frac{1}{\epsilon} \right) \\
    \Rightarrow n \in \Omega \left( \frac{1}{\epsilon \alpha^{k-1}} \right).
\]

We conclude by using the equivalence of pure and approximate DP for testing problems (e.g., Lemma 5 of [ASZ18]).
Finally, since being able to learn to within $\alpha$ absolute error implies distinguishing two distributions that are at least $2\alpha$ apart, from the above claim, the lemma holds.

4 Estimating in High Dimensions with CDP

In this section, we give a computationally efficient, $\rho$-zCDP algorithm for estimating the mean of a distribution with bounded $k$th moment. The analogous $(\varepsilon, \delta)$-DP algorithm would be the same, with the same analysis, and we state the theorem for it at the end.

Remark 4.1. Throughout the section, we assume that the dimension $d$ is greater than some absolute constant ($32\ln(4)$). If it is less than that, then we can just use our one-dimensional estimator from Theorem 3.7 multiple times to individually estimate each coordinate with constant multiplicative overhead in sample complexity.

4.1 Technical Lemmata

Similar to our one-dimensional distribution estimator, the idea is to aggressively truncate the distribution around a point, and compute the noisy empirical mean. We first have to define what truncation in high dimensions means. The definition essentially says that if a point lies outside the specified range (in this case, a sphere around a point), then we project the point on to the surface of the sphere in the direction towards the centre.

Definition 4.2. Let $\rho, x \in \mathbb{R}^d$, and $r > 0$. Then we define $\text{trunc}(\rho, r, x)$ as follows.

$$\text{trunc}(\rho, r, x) = \begin{cases} x & \text{if } \|\rho - x\|_2 \leq r \\ y \text{ st. } \|y - \rho\|_2 = r \text{ and } y = \rho + \gamma \cdot (\rho - x) \text{ for some } \gamma \in \mathbb{R} & \text{if } \|\rho - x\|_2 > r \end{cases}$$

Similarly, for a dataset $S = (X_1, \ldots, X_n) \in \mathbb{R}^{n \times d}$, we define $\text{trunc}(\rho, r, S)$ as the dataset $S' = (X'_1, \ldots, X'_n)$, where for each $1 \leq i \leq n, X'_i = \text{trunc}(\rho, r, X_i)$.

Now, we show that the mean of the distribution, truncated to within a large-enough ball centered close to the mean, does not move too far from the original mean. Our lemmas are somewhat similar to techniques in [DKK+17] (see Lemma A.18), but their results are specific to bounded second moments ($k = 2$), while we focus on the case of general $k$.

Lemma 4.3. Let $D$ be a distribution over $\mathbb{R}^d$ with mean $\mu$, and $k$th moment bounded by 1, where $k \geq 2$. Let $\rho \in \mathbb{R}^d$, $0 < \tau < \frac{1}{16}$, and $\xi = \frac{C\sqrt{d}}{\tau^{1/4}}$ for a constant $C > 2$. Let $X \sim D$, and $Z$ be the following random variable.

$$Z = \text{trunc}(\rho, \xi, X)$$

If $\|\mu - \rho\|_2 \leq \frac{\xi}{5}$, then $\|\mu - \mathbb{E}[Z]\|_2 \leq \tau$.

Proof. By self-duality of the Euclidean norm, it is sufficient to prove that for each unit vector $v \in \mathbb{R}^d$, $\langle \mu - \mathbb{E}[Z], v \rangle \leq \tau$. Let $\gamma = \mathbb{E}[Z]$. Then we have the following.

$$\langle \mu - \mathbb{E}[Z], v \rangle = \mathbb{E}\left[\langle X - \gamma, v \rangle 1_{X \in B_\varepsilon(\rho)}\right]$$
\[
\leq \left( \mathbb{E} \left[ \left( X - \gamma, v \right)^k \right] \right)^{\frac{1}{k}} \left( \mathbb{E} \left[ 1_{X \in B_{\xi}(\rho)} \right] \right)^{\frac{k-1}{k}} \tag{Lemmata A.8 and A.7}
\]
\[
\leq 1 \cdot \left( \mathbb{P} \left[ \|X - \mu\|_2 > \frac{\xi}{2} \right] \right)^{\frac{k-1}{k}} \leq \tau. \tag{Lemma A.2}
\]

Our last lemma shows that the empirical mean of a set of samples from a high-dimensional distribution with bounded $k^{th}$ moment is close to the mean of the distribution.

**Lemma 4.4.** Let $D$ be a distribution over $\mathbb{R}^d$ with mean $\mu$ and $k^{th}$ moment bounded by 1. Suppose for $\tau > 0$, $(X_1, \ldots, X_n)$ are samples from $D$, where

\[
n \geq O \left( \frac{d}{\tau^2} \right).
\]

Then with probability at least 0.9,

\[
\left\| \frac{1}{n} \sum_{i=1}^{n} X_i - \mu \right\|_2 \leq \tau.
\]

**Proof.** Suppose $\Sigma$ is the covariance matrix of $D$. We know that $\|\Sigma\|_2 = 1$. Let $\bar{\mu} = \sum_{i=1}^{n} X_i$. Then

\[
\mathbb{E} [\bar{\mu}] = \mu \quad \text{and} \quad \mathbb{E} [(\bar{\mu} - \mu)^T (\bar{\mu} - \mu)] = \frac{1}{n} \Sigma.
\]

Using Lemma A.2, we have the following.

\[
\mathbb{P} \left[ \|\bar{\mu} - \mu\|_2 > \tau \right] \leq \left( \sqrt{\frac{d}{n\tau^2}} \right)^{\frac{k-1}{k}} \leq 0.9.
\]

\[\square\]

### 4.2 The Algorithm

We finally state the main theorem of the section here. Algorithm 3 first computes a rough estimate of the mean using the one-dimensional mean estimator from Theorem 3.7 that lies at most $\sqrt{d}$ from $\mu$. Then it truncates the distribution to within a small ball around the estimate, and uses the Gaussian Mechanism (Lemma 2.8) to output a private empirical mean.

**Theorem 4.5.** Let $D$ be a distribution over $\mathbb{R}^d$ with mean $\mu \in B_R(\bar{0})$ and $k^{th}$ moment bounded by 1. Then for all $\rho, \alpha > 0$, there exists a polynomial-time, $\rho$-zCDP algorithm that takes

\[
n \geq O \left( \frac{d}{\alpha^2} + \frac{d}{\sqrt{\rho} \alpha^{k-1}} + \frac{\sqrt{d} \log(R) \log(d)}{\sqrt{\rho}} \right)
\]

samples from $D$, and outputs $\hat{\mu} \in \mathbb{R}^d$, such that with probability at least 0.7,

\[
\|\mu - \hat{\mu}\|_2 \leq \alpha.
\]

18
Algorithm 3: CDP High-Dimensional Mean Estimator CDPHDE_{\rho,\alpha,R}(X)

| Input: | Samples $X_1, \ldots, X_{2n} \in \mathbb{R}^d$. Parameters $\rho, \alpha, R > 0$. |
|--------|----------------------------------------------------------------------------------|
| Output: | $\hat{\mu} \in \mathbb{R}^d$. |

Set parameters: $Y \leftarrow (X_1, \ldots, X_n)$ $Z \leftarrow (X_{n+1}, \ldots, X_{2n})$ $r \leftarrow \frac{4 \sqrt{d}}{\alpha \frac{1}{\epsilon R}}$

// Obtain a rough estimate of the mean via coordinate-wise estimation

For $i \leftarrow 1, \ldots, d$

$c_i \leftarrow \text{CDPODME}_{\frac{\rho}{d^2}, 0.1, \frac{1}{d}, R}(Y^i)$

Let $\vec{c} \leftarrow (c_1, \ldots, c_d)$

// Truncate to within a small ball around the mean

Let $Z' \leftarrow \text{trunc}(\vec{c}, r, Z)$

// Estimate the mean

$\hat{\mu} \leftarrow \frac{1}{n} \sum_{z \in Z'} z + \mathcal{N}\left(\vec{c}, \frac{2r^2}{\rho n^2} \mathbf{1}_{d \times d}\right)$

Return $\hat{\mu}$

Proof. The proofs of privacy and accuracy (for Algorithm 3) are separated again as follows.

Privacy:
Privacy follows from Lemmata 3.7, 2.8, and 2.5 (since the $\ell_2$-sensitivity of the estimation step is $\frac{2r}{n}$).

Accuracy:
The first step finds a centre $c_i$ for each coordinate $i$, such that the $i$th coordinate of the mean is at most 1 far from $c_i$ in absolute distance. Therefore, $\vec{c}$ is at most $\sqrt{d}$ away from $\mu$. By Theorem 3.7, this happens with probability at least 0.9.

Now, by Lemma 4.3, the mean of the truncated distribution around $\vec{c}$ (that we call $\mu'$) is at most $\alpha$ far from $\mu$ is $\ell_2$ distance. Therefore, by Lemma 4.4, the empirical mean of the truncated distribution ($\bar{\mu}$) will be at most $\alpha$ far from $\mu'$ with probability at least 0.9.

Finally, let $z = (z_1, \ldots, z_d)$ be the noise vector added in the estimation step. and let $S_z = \sum_{i \in [d]} z_i$.

Then since $d \geq 32 \ln(4)$, by Lemma A.6, we have the following.

\[
P\left[\left|\sum_{i=1}^{d} z_i^2 - \frac{2dr^2}{\rho n^2}\right| \geq 0.5 \times \frac{2dr^2}{\rho n^2}\right] \leq 0.1.
\]

Therefore, it is enough to have the following.

\[
\frac{3dr^2}{\rho n^2} \leq \alpha^2
\]

\[
\iff n \geq \frac{\sqrt{48d^2}}{\rho \alpha \frac{1}{\epsilon R}} = \frac{4\sqrt{3}d}{\sqrt{\rho \alpha \frac{1}{\epsilon R}}}
\]

19
This is what we required in our sample complexity. Hence, by the union bound and rescaling $\alpha$ by a constant, we get the required result.

To get the analogous $(\epsilon, \delta)$-DP algorithm, we just use ADPODME instead of CDPODME in the first step, and keep the rest the same.

**Theorem 4.6.** Let $D$ be a distribution over $\mathbb{R}^d$ with mean $\mu \in B_{\mathbb{R}}(\bar{0})$ and $k^{th}$ moment bounded by 1. Then for all $\epsilon, \delta, \alpha > 0$, there exists a polynomial-time, $(\epsilon, \delta)$-DP algorithm that takes

$$n \geq O\left( \frac{d}{\alpha^2} + \frac{d \sqrt{\log(1/\delta)}}{\epsilon \alpha^{k-1}} + \frac{\sqrt{d \log(1/\delta)} \log(d)}{\epsilon} \right)$$

samples from $D$, and outputs $\widehat{\mu} \in \mathbb{R}^d$, such that with probability at least 0.7,

$$\|\mu - \widehat{\mu}\|_2 \leq \alpha.$$

5 Estimating in High Dimensions with Pure DP

We prove an upper bound for mean estimation in case of high-dimensional distributions with bounded $k^{th}$ moment, whilst having pure DP guarantee for our algorithm. It involves creating a cover over $[R, R]^d$, and using the Exponential Mechanism (Lemma 2.10) for selecting a point that would be a good estimate for the mean with high probability. Note that while this algorithm achieves sample complexity that is linear in the dimension, it is computationally inefficient.

5.1 Technical Lemma

We start by stating two lemmata that would be used in the proof of accuracy of our proposed algorithm. Unlike Lemma 3.1, the first lemma says that if we truncate a one-dimensional distribution with bounded $k^{th}$ moment around a point that is far from the mean, then the mean of this truncated distribution would be far from the said point.

**Lemma 5.1.** Let $D$ be a distribution over $\mathbb{R}$ with mean $\mu$, and $k^{th}$ moment bounded by 1. Let $\rho \in \mathbb{R}$, $0 < \tau < \frac{1}{16}$, and $\xi = \frac{C}{\tau^{k-1}}$ for a universal constant $C$. Let $X \sim D$, and $Z$ be the following random variable.

$$Z = \begin{cases} 
\rho - \xi & \text{if } X < \rho - \xi \\
X & \text{if } \rho - \xi \leq X \leq \rho + \xi \\
\rho + \xi & \text{if } X > \rho + \xi
\end{cases}$$

If $\rho > \mu + \frac{\xi}{2}$, then the following holds.

$$\mathbb{E}[Z] \in \begin{cases} 
\left[ \mu - \frac{\xi}{8}, \mu + \frac{\xi}{8} \right] & \text{if } \frac{\xi}{2} < |\rho - \mu| \leq \frac{17\xi}{16} \\
\left[ \rho - \xi, \rho - \frac{15\xi}{16} \right] & \text{if } |\rho - \mu| > \frac{17\xi}{16}
\end{cases}$$

If $\rho < \mu - \frac{\xi}{2}$, then the following holds.

$$\mathbb{E}[Z] \in \begin{cases} 
\left[ \mu - \frac{\xi}{8}, \mu + \frac{\xi}{8} \right] & \text{if } \frac{\xi}{2} < |\rho - \mu| \leq \frac{17\xi}{16} \\
\left[ \rho + \frac{15\xi}{16}, \rho + \xi \right] & \text{if } |\rho - \mu| > \frac{17\xi}{16}
\end{cases}$$
Proof. Without loss of generality, we assume that \( \rho \geq \mu \), since the argument for the other case is symmetric. Let \( a = \max \{ \mu + \xi / 16, \rho - \xi \} \), \( b = \rho + \xi \), and \( q = P[X - \mu > |a - \mu|] \). Then the highest value \( \mathbb{E}[Z] \) can achieve is when \( 1 - q \) probability mass is at \( a \) and \( q \) of the mass is at \( b \). This is because the mass that lies beyond \( a \) is \( q \), and \( b \) is the highest value that \( X \) can take because of truncation. We get the following:

\[
\mathbb{E}[Z] \leq (1 - q)a + qb \\
= a + q(b - a). \tag{2}
\]

Now, there are two cases. First, when \( a = \mu + \xi / 16 \), and second, when \( a = \rho - \xi \). In the first case, since \( \mu + \xi / 16 \geq \rho - \xi \), it must be the case that \( \rho - \mu \leq 15\xi / 16 \). So, we have the following from (2).

\[
\mathbb{E}[Z] \leq \mu + \xi / 16 + q\left( \rho + \xi - \mu - \frac{\xi}{16} \right) \\
= \mu + \frac{\xi}{16} + q(\rho - \mu) + \frac{15q\xi}{16} \\
\leq \mu + \frac{\xi}{16} + 2q\xi.
\]

From Lemma A.1, we know that

\[
q = P[|X - \mu| > \xi / 16] \leq \left( \frac{16}{\xi} \right)^k \left( \frac{16}{C} \right)^k \tau^{k-1}.
\]

This, along with our restrictions on \( C \) and \( \tau \), gives us,

\[
\mathbb{E}[Z] \leq \mu + \frac{\xi}{8}.
\]

We now have to show that \( \mathbb{E}[Z] \geq \mu - \xi / 8 \). We have the following:

\[
\mathbb{E}[Z] \geq (1 - q)(\mu - \xi / 16) + q(\rho - \xi) \\
= \mu - \frac{\xi}{16} + q\left( \frac{\xi}{16} - \xi + \rho - \mu \right) \\
= \mu - \frac{\xi}{16} - \frac{15q\xi}{16} + q(\rho - \mu) \\
\geq \mu - \frac{\xi}{8}.
\]

For the second case, we have the following from (2):

\[
\mathbb{E}[Z] \leq \rho - \xi + q(\rho + \xi - \rho + \xi) \\
= \rho - \xi + 2q\xi.
\]

From Lemma A.1, we know that

\[
q = P[|X - \mu| > |\rho - \xi - \mu|] \leq P[|X - \mu| > \xi / 16] \leq \left( \frac{16}{\xi} \right)^k \left( \frac{16}{C} \right)^k \tau^{k-1}.
\]

This gives us,

\[
\mathbb{E}[Z] \leq \rho - \frac{15\xi}{16}.
\]

Now it is trivial to see that \( \mathbb{E}[Z] \geq \rho - \xi \) because the minimum value \( Z \) can take is \( \rho - \xi \). \( \square \)
The guarantees of the next lemma are similar to Lemma 3.2, except that this one promises a much higher correctness probability. It is just the well-known median of means estimator, which is adapted for the case of distributions with bounded $k^{th}$ moment.

**Lemma 5.2 (Median of Means).** Let $D$ be a distribution over $\mathbb{R}$ with mean $\mu$ and $k^{th}$ moment bounded by 1. For $0 < \beta < 1$, and $\alpha > 0$, suppose $(X_1, \ldots, X_n)$ are independent samples from $D$, such that

$$n \geq O\left(\frac{\log(1/\beta)}{\alpha^2}\right).$$

Let $m \geq 200 \log(2/\beta)$. For $i = 1, \ldots, m$, suppose $Y^i = (X_{(i-1)m+1}, \ldots, X_{(i-1)m+n/n})$ and $\mu_i$ is the empirical mean of $Y^i$. Define $\bar{\mu} = \text{Median}(\mu_1, \ldots, \mu_m)$. Then with probability at least $1 - \beta$,

$$|\bar{\mu} - \mu| \leq \alpha.$$

**Proof.** Since $\frac{n}{m} \geq n_k$, by Lemma 3.2, with probability at least 0.9, for a fixed $i$, $|\mu_i - \mu| \leq \alpha$. Now, because $m \geq 200 \log(2/\beta)$, using Lemma A.4, we have that with probability at least $1 - \beta$, at least $0.8m$ of the $\mu_i$'s are at most $\alpha$ far from $\mu$. Therefore, their median has to be at most $\alpha$ far from $\mu$. \hfill \Box

### 5.2 The Algorithm

Our algorithm for estimating the mean with pure differential privacy is a so-called, “cover-based algorithm”. It creates a net of points, some of which could be used to get a good approximation for the mean, then privately chooses one of those points with high probability. This is done by assigning a “score” to each point in the net, which depends on the dataset, such that it ensures that the points which are good, have significantly higher scores than the bad points. Privacy comes in for the part of selecting a point because the score is directly linked with the dataset. So we use the Exponential Mechanism (Lemma 2.10) for this purpose.

This framework is reminiscent of the classic approach of density estimation via Scheffé estimators (see, e.g., [DL01]), which has recently been privatized in [BKSW19]. In particular, in a similar way, we choose from the cover by setting up several pairwise comparisons, and privatize using the exponential mechanism in the same way as [BKSW19]. This is where the similarities end: the method for performing a comparison between two elements is quite different, and the application is novel (mean estimation versus density estimation). We adopt this style of pairwise comparisons to reduce the problem from $d$ dimensions to 2 dimensions: indeed, certain aspects of pure differential privacy are not well understood in high-dimensional settings, so this provides a new tool to get around this roadblock. To the best of our knowledge, we are the first to use pairwise comparisons for a statistical estimation task besides general density estimation. Most cover-based arguments for other tasks instead appeal to uniform convergence, which is not clear how to apply in this setting when we must preserve privacy.

The first task is to come up with a good SCORE function. It means that it should satisfy two properties. First: the points $O(\alpha)$ close to $\mu$ should have very high scores, but the ones further than that must have very low scores. Second: the function should have low sensitivity so that we don’t end up selecting a point with low utility (SCORE). We create a required function, which is based on games or “matches” between pairs of points, as defined below.
**Definition 5.3 (Match between Two Points).** Let $X^1, \ldots, X^m \in \mathbb{R}^{n \times d}$ be datasets, $X$ be their concatenation, and $p, q$ be points in $\mathbb{R}^d$, and $\xi > 0$. Suppose $Y^1, \ldots, Y^m$ are the respective datasets after projecting their points on to the line $p - q$ and truncating to within $B_\xi(p)$, and $\mu_1, \ldots, \mu_m$ are the respective empirical means of $Y_i$'s. Let $\mu' = \text{Median}(\mu_1, \ldots, \mu_m)$. Then we define the function $\text{Match}_{X, \xi} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ as follows.

$$
\text{Match}_{X, \xi}(p, q) = \begin{cases} 
\text{Tie} & \text{if } \|p - q\|_2 \leq 20\alpha \\
\text{Win} & \text{if } \|p - \mu'\|_2 < \|q - \mu'\|_2 \\
\text{Lose} & \text{if } \|p - \mu'\|_2 \geq \|q - \mu'\|_2 
\end{cases}
$$

Note that the above definition is not symmetric.

**Definition 5.4 (Score of a Point).** Let $X^1, \ldots, X^m \in \mathbb{R}^{n \times d}$ be datasets, $X$ be their concatenation, and $p$ be a point in $\mathbb{R}^d$, and $\alpha, \xi > 0$. We define $\text{SCORE}_{X, \xi, D, \alpha}(p)$ with respect to a domain $D \subset \mathbb{R}^d$ of a point $p$ to be the minimum number of points of $X$ that need to be changed to get a dataset $\overline{X}$ so that there exists $q \in D$, such that $\text{Match}_{\overline{X}, \xi}(p, q) = \text{Lose}$. If for all $q \in D \setminus \{p\}$ and all $Y \in \mathbb{R}^{m \times d}$, $\text{Match}_{Y, \xi}(p, q) \neq \text{Lose}$, then we define $\text{SCORE}_{Y, \xi, D, \alpha}(p) = n\alpha$.

If the context is clear, we abbreviate the quantity to $\text{SCORE}_{X}(p)$.

By the above definition, if there already exists a $q \in D$, such that $\text{Match}_{X, \xi}(p, q) = \text{Lose}$, then $\text{SCORE}_{X}(p) = 0$. Let $S$ be the set as defined in Algorithm 2. We start by showing that the points in $S$ that are close to $\mu$ have a high score with high probability.

---

**Algorithm 4:** Pure DP High-Dimensional Mean Estimator PDPHDME$_{\epsilon, \alpha, R}(X)$

```
Input: Samples $X_1, \ldots, X_{2n} \in \mathbb{R}^d$. Parameters $\epsilon, \alpha, R > 0$.
Output: $\widehat{\mu} \in \mathbb{R}^d$.

Set parameters: $\xi \leftarrow \frac{60}{\alpha^2} \quad Y \leftarrow (X_1, \ldots, X_n) \quad Z \leftarrow (X_{n+1}, \ldots, X_{2n})$

// Reduce search space
For $i \leftarrow 1, \ldots, d$
    $c_i \leftarrow \text{PDPODME}_{\frac{\epsilon}{\sqrt{d}}, \alpha, R}(Y^i)$
    $I_i \leftarrow [c_i - \alpha, c_i + \alpha]$
    $J_i \leftarrow [c_i - \alpha, c_i - \alpha + \frac{\alpha}{\sqrt{d}}, \ldots, c_i + \alpha - \frac{\alpha}{\sqrt{d}}, c_i + \alpha]$

// Find a good estimate
Let $S \leftarrow J_1 \times \cdots \times J_d$
Compute $\text{SCORE}_{X, \xi, S, \alpha}(p)$ with respect to $Z$ for every $p \in S$
Run Exponential Mechanism w.r.t. $\text{SCORE}$ (sensitivity 1, privacy budget $\epsilon$) to output $\widehat{\mu} \in S$

Return $\widehat{\mu}$
```

**Lemma 5.5.** Let $m \geq 400d \log(8 \sqrt{d}/\beta)$ be the same quantity as in Definition 5.3, and let $S_\leq \subset S$, such that for all $p \in S_\leq$, $\|p - \mu\|_2 \leq 5\alpha$. If

$$
n \geq O\left(\frac{m}{\alpha^2}\right),
$$

then

$$
\mathbb{P}\left[\exists p \in S_\leq, \text{ st. } \text{SCORE}_{X, \xi, S, \alpha}(p) < \frac{4n\alpha}{\sqrt{d}}\right] \leq \beta.
$$
Proof. Fix a \( p \in S \). First, note that \( p \) cannot lose to any other point that is at most \( 5\alpha \) far from \( \mu \). So, it can only lose or win against points that are at least \( 15\alpha \) far from \( \mu \).

Now, let \( q \in S \) be any point that is at least \( 20\alpha \) away from \( p \), and let \( \ell_q \) be the line \( p - q \). If we project \( \mu \) on to \( \ell_q \), the projected mean \( \mu_0 \) will be at most \( 5\alpha \) away from \( p \) as projection cannot increase the distance between the projected point any of the points on the line. This implies that \( \mu_0 \) will be at least \( 15\alpha \) far from \( q \).

From Lemma 3.1, we know that the mean of the distribution truncated around \( p \) for a sufficiently large \( \xi \) (which we call \( \mu_p \)) is at most \( \alpha \) far from \( \mu_0 \). So, by Lemma 5.2, we know that with probability at least \( 1 - \frac{\beta}{|S|} \), the median of means (\( \mu' \)) is at most \( \alpha \) far from \( \mu_p \). This implies that \( \mu' \) is more than \( 14\alpha \) far from \( q \), and at most \( 6\alpha \) far from \( p \). Therefore, to lose to \( q \), \( \mu' \) will have to be moved by at least \( 4\alpha \) towards \( q \).

Since moving a point in \( X \) can move \( \mu' \) by at most \( \frac{2m\xi}{n} \), it means that we need to change at least \( \frac{2m\xi}{n} \) points each from at least \( 0.4m \) of the sub-datasets in \( X \) to make \( p \) lose to \( q \) (from the proof of Lemma 5.2), since we want to have the means of at least half of the sub-datasets to be closer to \( q \). Taking the union bound over all pairs of points in \( S \), we get the desired error probability bound. This proves the claim. \( \square \)

Now, we prove that the points in \( S \) that are far from \( \mu \) have a very low score.

Lemma 5.6. Let \( m \geq 400d \log(8\sqrt{d}/\beta) \) be as in Definition 5.3, and let \( S_\alpha \subset S \), such that for all \( p \in S_\alpha \), \( \|p - \mu\|_2 > 20\alpha \). If

\[
n \geq O\left(\frac{m}{\alpha^2}\right),\]

then

\[
P[\exists p \in S_\alpha, \text{ st. } \text{SCORE}_{X,\xi,S,\alpha}(p) > 0] \leq \beta.
\]

Proof. Fix a \( p \in S_\alpha \). We have to deal with two cases here. First, when \( \|\mu - p\|_2 \leq \frac{\xi}{2} \), and when \( \|\mu - p\|_2 > \frac{\xi}{2} \).

For the first case, let \( z \) be the point in \( S \) that is nearest to \( \mu \), and let \( \ell_z \) be the line \( p - z \). Suppose \( \mu_1 \) is the projection of \( \mu \) on to \( \ell_z \). Then \( \mu_1 \) will be at most \( \alpha \) from \( z \). By Lemma 3.1, the mean of the distribution truncated around \( p \) (which we call \( \mu_p \)) will be at most \( \alpha \) far from \( \mu_1 \). Then by Lemma 5.2, with probability at least \( 1 - \frac{\beta}{|S|} \), the median of means (\( \mu' \)) will be at most \( \alpha \) far from \( \mu_p \), hence, at most \( 3\alpha \) far from \( z \). This implies that \( \mu' \) will be at least \( 17\alpha \) far from \( p \). Therefore, the score of \( p \) will be 0, since it has already lost to \( z \).

In the second case, let \( \ell_p \) be the line \( p - \mu \). Suppose \( \mu_1 \) is the mean of the distribution projected on to \( \ell_p \) and truncated around \( p \), and let \( z \) be the point in \( S \) that is closest to \( \mu_1 \). If \( \mu_1 = z \), then we’re done because by Lemma 5.1, \( \mu_1 \) is at least \( \frac{15\xi}{16} \) far from \( p \), and then by Lemma 5.2, with probability at least \( 1 - \frac{\beta}{|S|} \), the median of means lies \( \alpha \) close to \( \mu_1 \), and is closer to \( \mu_1 \) than it is to \( p \).

If not, then we have to do some more work. Now, let \( \ell_z \) be the line \( p - z \), let \( \mu_z \) be the projection of \( \mu \) on to \( \ell_z \), and let \( \mu_2 \) be the projection of \( \mu_1 \) on to \( \ell_z \). Using basic geometry, we have the following,

\[
\frac{\|p - \mu\|_2}{\|p - \mu_z\|_2} = \frac{\|p - \mu_1\|_2}{\|p - \mu_2\|_2}
\]
\[\iff \|p - \mu_z\|^2 = \frac{\|p - \mu\|^2 \|p - \mu_z\|^2}{\|p - \mu_z\|^2} = \frac{\|p - \mu\|^2 (\|p - \mu_1\|^2 - \alpha)}{\|p - i\mu_1\|^2} = \|p - \mu\|^2 \left(1 - \frac{\alpha}{\|p - \mu_1\|^2}\right) \geq \|p - \mu\|^2 \left(1 - \frac{16\alpha}{15\xi}\right) = \frac{15\|p - \mu\|^2}{16}\] (Triangle Inequality)

If \(\|p - \mu_z\| \leq \frac{\xi}{2}\), then by a similar argument as in the first case, \(z\) wins against \(p\) because the mean of the truncated distribution is close to \(\mu_z\), and the empirical median of means is close to that mean with probability at least \(1 - \frac{\beta}{|S|^2}\), hence, closer to \(z\) than to \(p\). If not, then by Lemma 5.1, the mean of the distribution projected on to \(\ell_z\), and truncated around \(p\) will be at most \(\frac{\xi}{16}\) far from \(p - \xi\), so by Lemma 5.2, the median of means (\(\mu'\)) will be at most \(\alpha\) far from that mean with probability at least \(1 - \frac{\beta}{|S|^2}\). This implies that it will be at most \(\frac{\xi}{16} + 2\alpha\) far from \(z\), but will be at least \(\frac{15\xi}{16} - 2\alpha\) from \(p\), which means that \(p\) will lose to \(z\) by default.

Taking the union bound over all sources of error, and all pairs of points in \(S\), we get the error probability of \(4\beta\), which we can rescale to get the required bounds.

Finally, we prove that the \(\text{SCORE}\) function has low sensitivity.

**Lemma 5.7.** The \(\text{SCORE}\) function satisfies the following:

\[\Delta_{\text{SCORE}, 1} \leq 1.\]

**Proof.** Let \(X\) be any dataset, and \(p \in \mathbb{R}^d\) be a point in the domain in question, and let \(\text{SCORE}_X(p)\) be the score of \(p\). Suppose \(X'\) is a neighbouring dataset of \(X\). Then by changing a point \(x\) in \(X\) to \(x'\) (to get \(X'\)), we can only change the score of \(p\) by 1. Let the median of means of projected, truncated \(X\) be \(\mu_1\), and that of \(X'\) be \(\mu_2\).

Suppose \(p\) was already losing to a point \(q\), that is, its score was 0. Then switching from \(X\) to \(X'\) can either imply that \(\mu_2\) is further from \(p\) than \(\mu_1\) was from \(p\), or it could go further. In the first case, \(p\) would still lose to \(q\). In the second case, if \(\mu_2\) is closer to \(p\) than it is to \(q\), then the score of \(p\) would increase at most by 1 because we can switch back to \(X\) from \(X'\) by switching one point; or \(p\) could still be losing to \(q\), in which case, the score wouldn’t change at all.

Now, suppose \(p\) was winning against all points that are more than \(30\alpha\) away from \(p\) with respect to \(X\). Let \(q\) be a point that determined the score of \(p\). If switching to \(X'\) made \(\mu_2\) closer to \(p\) than \(\mu_1\) was, then the score can only increase by 1 because we can always switch back to \(X\), and get the original score. If it moved \(\mu_2\) closer to \(q\) than \(\mu_1\) was, then the score can only
decrease by 1. This is because \( q \) determined \( \text{SCORE}_X(p) \) via some optimal strategy, and changing a point of \( X \) cannot do better than that. Therefore, the sensitivity is 1, as required.

We can now move on to the main theorem of the section.

**Theorem 5.8.** Let \( D \) be a distribution over \( \mathbb{R}^d \) with mean \( \mu \in B_R(\vec{0}) \) and \( k \)th moment bounded by 1. Then for all \( \varepsilon, \alpha, \beta > 0 \), there exists an \( \varepsilon \)-DP algorithm that takes

\[
  n \geq O \left( \frac{d \log(d/\beta)}{\alpha^2} + \frac{d \log(d/\beta)}{\varepsilon \alpha^{\frac{1}{k-1}}} + \frac{d \log(R) \log(d/\beta)}{\varepsilon} \right)
\]

samples from \( D \), and outputs \( \widehat{\mu} \in \mathbb{R}^d \), such that with probability at least \( 1 - \beta \),

\[
  \| \mu - \widehat{\mu} \|_2 \leq \alpha.
\]

**Proof.** We again separate the proofs of privacy and accuracy of Algorithm 4.

**Privacy:**
The first step is \( \varepsilon \)-DP from Lemma 3.5, and from Lemma 2.5. The second step is \( \varepsilon \)-DP from Lemmata 5.7 and 2.10. Therefore, the algorithm is \( 2\varepsilon \)-DP.

**Accuracy:**
The first step is meant to reduce the size of the search space. From Lemma 3.5, we have that for each \( i \), the distance between \( c_i \) and the mean along the \( i \)th axis is at most \( \alpha \). So, \( I_i \) contains the mean with high probability, and is of length \( 2\alpha \) by construction.

We know from Lemma 2.10 that with high probability, the point returned by Exponential Mechanism has a high \( \text{SCORE} \). So, it will be enough to argue that with high probability, only the points in \( S \), which are \( O(\alpha) \) close to \( \mu \), have a high quality score, while the rest have \( \text{SCORE} \) close to 0. This exactly what we have from Lemmata 5.5 and 5.6. Let \( \text{OPT}_{\text{SCORE}}(\cdot) \) be the maximum score of any point in \( S \). Then we know that \( \text{OPT}_{\text{SCORE}}(\cdot) \geq \frac{4\pi\alpha}{5\xi} \), and that the points that have this score have to be at most 20\( \alpha \) far from \( \mu \). From Lemma 2.10, we know that with probability at least \( 1 - \beta \),

\[
  \text{SCORE}(X, \widehat{\mu}) \geq \text{OPT}_{\text{SCORE}}(\cdot) - \frac{2\Delta_{\text{SCORE},1}}{\varepsilon} (\log(|S|) + \log(1/\beta))
\]

\[
  \geq \frac{4\pi\alpha}{5\xi} - \frac{2}{\varepsilon} \left( d \log \left( 4\sqrt{d} \right) + \log(1/\beta) \right)
\]

\[
  \geq O \left( n\alpha^{\frac{k}{k-1}} \right). \quad \text{(Because of our bounds on } n \text{ and } \xi \text{)}
\]

Therefore, we get a point that is at most 20\( \alpha \) far from \( \mu \). Rescaling \( \alpha \) and \( \beta \) by constants, we get the required result.

\[ \square \]

### 6 Lower Bounds for Estimating High-Dimensional Distributions

**Theorem 6.1.** Suppose \( A \) is an \((\varepsilon, 0)\)-DP algorithm and \( n \in \mathbb{N} \) is a number is such that, for every product distribution \( P \) on \( \mathbb{R}^d \) such that \( \mathbb{E}[P] = \mu \) and \( \sup_{v: \|v\|_1 = 1} \mathbb{E}\left[ \langle v, P - \mu \rangle^2 \right] \leq 1 \),

\[
  \mathbb{E}_{X_1, \ldots, X_n \sim P \cdot A} \left[ \| A(X) - \mu \|^2 \right] \leq \alpha^2.
\]

Then \( n = \Omega \left( \frac{d}{\alpha^2 \varepsilon} \right) \).
The proof uses a standard packing argument, which we encapsulate in the following lemma.

**Lemma 6.2.** Let \( \mathcal{P} = \{P_1, P_2, \ldots\} \) be a family of distributions such that, for every \( P_i, P_j \in \mathcal{P}, \) \( d_{TV}(P_i, P_j) \leq \tau. \) Suppose \( A \) is an \( (\epsilon, 0) \)-DP algorithm and \( n \in \mathbb{N} \) is a number such that, for every \( P_i \in \mathcal{P}, \)

\[
\mathbb{P}_{X_1, \ldots, X_n \sim P_i, A} [A(X) = i] \geq 2/3,
\]

then \( n = \Omega\left( \frac{\log |\mathcal{P}|}{\epsilon \tau} \right). \)

**Proof.** We will define a packing as follows. As a shorthand, define

\[
\begin{cases}
0 & \text{w.p. } 1 - \frac{a^2}{d} \\
\frac{\sqrt{d}}{a} & \text{w.p. } \frac{a^2}{d}
\end{cases}
\]

For \( c \in \{0, 1\}^d, \) let

\[
P_c = \bigotimes_{j=1}^d Q_{c_j}
\]

be the product of the distributions \( Q_0 \) and \( Q_1 \) where we choose each coordinate of the product based on the corresponding coordinate of \( c. \)

Note that \( d_{TV}(Q_0, Q_1) \leq a^2/d, \) and therefore, for every \( c, c' \in \{0, 1\}^d, \) \( d_{TV}(P_c, P_{c'}) \leq a^2. \) Let \( C \subseteq \{0, 1\}^d \) be a code of relative distance \( 1/4. \) That is, every distinct \( c, c' \in C \) differ on at least \( d/4 \) coordinates. By standard information-theoretic arguments, there exists such a code such that \( |C| = 2^{\Omega(d)}. \) We will define the packing to be \( \mathcal{P} = \{P_c\}_{c \in C}. \) By Lemma 6.2, if there is an \( (\epsilon, 0) \)-DP algorithm \( A \) that takes \( n \) samples from an arbitrary one of the distribution \( P_c \in \mathcal{P} \) and correctly identifies \( P_c \) with probability at least \( 2/3, \) then \( n = \Omega\left( \frac{d}{\alpha^2 \epsilon} \right). \)

We make two more observation about the distributions in \( \mathcal{P}. \) First, since this is a product distribution, its 2nd moment is bounded by the maximum 2nd moment of any coordinate, so

\[
\sup_{v: \|v\|_2 = 1} \mathbb{E}((v, P - \mu)^2) \leq \max\{\text{Var}[Q_0], \text{Var}[Q_1]\} \leq 1.
\]

Second, since any distinct \( c, c' \) differ on \( d/4 \) coordinates, and \( \mathbb{E}[Q_1 - Q_0] = \alpha/\sqrt{d}, \) we have that for every distinct \( c, c', \)

\[
\|\mathbb{E}[P_c - P_{c'}]\|_2 \geq \sqrt{\frac{d}{4}} \cdot \frac{\alpha}{\sqrt{d}} = \frac{\alpha}{2}.
\]

By a standard packing argument, any \( (\epsilon, 0) \)-DP algorithm that takes \( n \) samples from \( P_c \) for an arbitrary \( c \in C, \) and correctly identifies \( c, \) must satisfy \( n = \Omega\left( \frac{d}{\alpha^2 \epsilon} \right). \) Therefore, if we can estimate the mean to within \( \ell_2^2 \) error \( < \alpha^2/64, \) we can identify \( c \) uniquely. Moreover, if \( A \) satisfies

\[
\mathbb{E}_{X_1, \ldots, X_n \sim P, A} \left[ \|A(X) - \mathbb{E}[P]\|_2^2 \right] < \alpha^2/192
\]

for every distribution \( P \) with bounded 2nd moment, then by Markov’s inequality, we have

\[
\mathbb{P}_{X_1, \ldots, X_n \sim P, A} \left[ \|A(X) - \mathbb{E}[P]\|_2^2 < \alpha^2/64 \right] \geq 2/3.
\]

Therefore, any \( (\epsilon, 0) \)-DP algorithm \( A \) with low expected \( \ell_2^2 \) error must have \( n = \Omega\left( \frac{d}{\alpha^2 \epsilon} \right). \) The theorem now follows by a change-of-variables for \( \alpha. \) \( \square \)
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References

[AFJ+18] Jayadev Acharya, Moein Falahatgar, Ashkan Jafarpour, Alon Orlitsky, and
Ananda Theertha Suresh. Maximum selection and sorting with adversarial com-
parators. *Journal of Machine Learning Research*, 19(1):2427–2457, 2018.

[AJOS14] Jayadev Acharya, Ashkan Jafarpour, Alon Orlitsky, and Ananda Theertha Suresh.
Sorting with adversarial comparators and application to density estimation. In *Proceedings of the 2014 IEEE International Symposium on Information Theory*, ISIT ’14,
pages 1682–1686, Washington, DC, USA, 2014. IEEE Computer Society.

[AKSZ18] Jayadev Acharya, Gautam Kamath, Ziteng Sun, and Huanyu Zhang. Inspectre:
Privately estimating the unseen. In *Proceedings of the 35th International Conference on
Machine Learning*, ICML ’18, pages 30–39. JMLR, Inc., 2018.

[AMB19] Marco Avella-Medina and Victor-Emmanuel Brunel. Differentially private sub-
Gaussian location estimators. *arXiv preprint arXiv:1906.11923*, 2019.

[ASZ18] Jayadev Acharya, Ziteng Sun, and Huanyu Zhang. Differentially private testing of
identity and closeness of discrete distributions. In *Advances in Neural Information
Processing Systems 31*, NeurIPS ’18, pages 6878–6891. Curran Associates, Inc., 2018.

[BDMN05] Avrim Blum, Cynthia Dwork, Frank McSherry, and Kobbi Nissim. Practical privacy:
The SuLQ framework. In *Proceedings of the 24th ACM SIGMOD-SIGACT-SIGART
Symposium on Principles of Database Systems*, PODS ’05, pages 128–138, New York,
NY, USA, 2005. ACM.

[BKM19] Olivier Bousquet, Daniel M. Kane, and Shay Moran. The optimal approximation
factor in density estimation. In *Proceedings of the 32nd Annual Conference on Learning
Theory*, COLT ’19, pages 318–341, 2019.

[BKSW19] Mark Bun, Gautam Kamath, Thomas Steinke, and Zhiwei Steven Wu. Private
hypothesis selection. In *Advances in Neural Information Processing Systems 32*, NeurIPS
’19, pages 156–167. Curran Associates, Inc., 2019.

[BNS16] Mark Bun, Kobbi Nissim, and Uri Stemmer. Simultaneous private learning of
multiple concepts. In *Proceedings of the 7th Conference on Innovations in Theoretical
Computer Science*, ITCS ’16, pages 369–380, New York, NY, USA, 2016. ACM.
[BNSV15] Mark Bun, Kobbi Nissim, Uri Stemmer, and Salil Vadhan. Differentially private release and learning of threshold functions. In Proceedings of the 56th Annual IEEE Symposium on Foundations of Computer Science, FOCS ’15, pages 634–649, Washington, DC, USA, 2015. IEEE Computer Society.

[BS16] Mark Bun and Thomas Steinke. Concentrated differential privacy: Simplifications, extensions, and lower bounds. In Proceedings of the 14th Conference on Theory of Cryptography, TCC ’16-B, pages 635–658, Berlin, Heidelberg, 2016. Springer.

[BS19] Mark Bun and Thomas Steinke. Average-case averages: Private algorithms for smooth sensitivity and mean estimation. In Advances in Neural Information Processing Systems 32, NeurIPS ’19, pages 181–191. Curran Associates, Inc., 2019.

[BUV14] Mark Bun, Jonathan Ullman, and Salil Vadhan. Fingerprinting codes and the price of approximate differential privacy. In Proceedings of the 46th Annual ACM Symposium on the Theory of Computing, STOC ’14, pages 1–10, New York, NY, USA, 2014. ACM.

[BV19] Victor Balcer and Salil Vadhan. Differential privacy on finite computers. Journal of Privacy and Confidentiality, 9(2), Sep. 2019.

[CFB19] Yeshwanth Cherapanamjeri, Nicolas Flammarion, and Peter L. Bartlett. Fast mean estimation with sub-Gaussian rates. In Proceedings of the 32nd Annual Conference on Learning Theory, COLT ’19, pages 786–806, 2019.

[CWZ19] T. Tony Cai, Yichen Wang, and Linjun Zhang. The cost of privacy: Optimal rates of convergence for parameter estimation with differential privacy. arXiv preprint arXiv:1902.04495, 2019.

[DDS12] Constantinos Daskalakis, Ilias Diakonikolas, and Rocco A. Servedio. Learning Poisson binomial distributions. In Proceedings of the 44th Annual ACM Symposium on the Theory of Computing, STOC ’12, pages 709–728, New York, NY, USA, 2012. ACM.

[DHS15] Ilias Diakonikolas, Moritz Hardt, and Ludwig Schmidt. Differentially private learning of structured discrete distributions. In Advances in Neural Information Processing Systems 28, NIPS ’15, pages 2566–2574. Curran Associates, Inc., 2015.

[Dif17] Differential Privacy Team, Apple. Learning with privacy at scale. https://machinelearning.apple.com/docs/learning-with-privacy-at-scale/appledifferentialprivacysystem.pdf, December 2017.

[DJW13] John C. Duchi, Michael I. Jordan, and Martin J. Wainwright. Local privacy and statistical minimax rates. In Proceedings of the 54th Annual IEEE Symposium on Foundations of Computer Science, FOCS ’13, pages 429–438, Washington, DC, USA, 2013. IEEE Computer Society.

[DJW17] John C. Duchi, Michael I. Jordan, and Martin J. Wainwright. Minimax optimal procedures for locally private estimation. Journal of the American Statistical Association, 2017.
Constantinos Daskalakis and Gautam Kamath. Faster and sample near-optimal algorithms for proper learning mixtures of Gaussians. In *Proceedings of the 27th Annual Conference on Learning Theory*, COLT ’14, pages 1183–1213, 2014.

Ilias Diakonikolas, Gautam Kamath, Daniel M. Kane, Jerry Li, Ankur Moitra, and Alistair Stewart. Robust estimators in high dimensions without the computational intractability. In *Proceedings of the 57th Annual IEEE Symposium on Foundations of Computer Science*, FOCS ’16, pages 655–664, Washington, DC, USA, 2016. IEEE Computer Society.

Ilias Diakonikolas, Gautam Kamath, Daniel M. Kane, Jerry Li, Ankur Moitra, and Alistair Stewart. Being robust (in high dimensions) can be practical. In *Proceedings of the 34th International Conference on Machine Learning*, ICML ’17, pages 999–1008. JMLR, Inc., 2017.

Bolin Ding, Janardhan Kulkarni, and Sergei Yekhanin. Collecting telemetry data privately. In *Advances in Neural Information Processing Systems 30*, NIPS ’17, pages 3571–3580. Curran Associates, Inc., 2017.

Luc Devroye and Gábor Lugosi. A universally acceptable smoothing factor for kernel density estimation. *The Annals of Statistics*, 24(6):2499–2512, 1996.

Luc Devroye and Gábor Lugosi. Nonasymptotic universal smoothing factors, kernel complexity and Yatracos classes. *The Annals of Statistics*, 25(6):2626–2637, 1997.

Luc Devroye and Gábor Lugosi. *Combinatorial methods in density estimation*. Springer, 2001.

Jules Depersin and Guillaume Lecué. Robust subgaussian estimation of a mean vector in nearly linear time. *arXiv preprint arXiv:1906.03058*, 2019.

Aref N. Dajani, Amy D. Lauger, Phyllis E. Singer, Daniel Kifer, Jerome P. Reiter, Ashwin Machanavajjhala, Simson L. Garfinkel, Scot A. Dahl, Matthew Graham, Vihesh Karwa, Hang Kim, Philip Lelecr, Ian M. Schmutte, William N. Sexton, Lars Vilhuber, and John M. Abowd. The modernization of statistical disclosure limitation at the U.S. census bureau, 2017. Presented at the September 2017 meeting of the Census Scientific Advisory Committee.

Cynthia Dwork, Frank McSherry, Kobbi Nissim, and Adam Smith. Calibrating noise to sensitivity in private data analysis. In *Proceedings of the 3rd Conference on Theory of Cryptography*, TCC ’06, pages 265–284, Berlin, Heidelberg, 2006. Springer.

Cynthia Dwork and Aaron Roth. The algorithmic foundations of differential privacy. *Foundations and Trends® in Machine Learning*, 9(3–4):211–407, 2014.

Cynthia Dwork, Guy N. Rothblum, and Salil Vadhan. Boosting and differential privacy. In *Proceedings of the 51st Annual IEEE Symposium on Foundations of Computer Science*, FOCS ’10, pages 51–60, Washington, DC, USA, 2010. IEEE Computer Society.
[DSS'15] Cynthia Dwork, Adam Smith, Thomas Steinke, Jonathan Ullman, and Salil Vadhan. Robust traceability from trace amounts. In Proceedings of the 56th Annual IEEE Symposium on Foundations of Computer Science, FOCS ’15, pages 650–669, Washington, DC, USA, 2015. IEEE Computer Society.

[EPK14] Úlfar Erlingsson, Vasyl Pihur, and Aleksandra Korolova. RAPPOR: Randomized aggregatable privacy-preserving ordinal response. In Proceedings of the 2014 ACM Conference on Computer and Communications Security, CCS ’14, pages 1054–1067, New York, NY, USA, 2014. ACM.

[GRS19] Marco Gaboardi, Ryan Rogers, and Or Sheffet. Locally private confidence intervals: Z-test and tight confidence intervals. In Proceedings of the 22nd International Conference on Artificial Intelligence and Statistics, AISTATS ’19, pages 2545–2554. JMLR, Inc., 2019.

[Hop18] Samuel B. Hopkins. Sub-Gaussian mean estimation in polynomial time. arXiv preprint arXiv:1809.07425, 2018.

[JKMW19] Matthew Joseph, Janardhan Kulkarni, Jieming Mao, and Zhiwei Steven Wu. Locally private Gaussian estimation. In Advances in Neural Information Processing Systems 32, NeurIPS ’19, pages 2980–2989. Curran Associates, Inc., 2019.

[KLSU19] Gautam Kamath, Jerry Li, Vikrant Singhal, and Jonathan Ullman. Privately learning high-dimensional distributions. In Proceedings of the 32nd Annual Conference on Learning Theory, COLT ’19, pages 1853–1902, 2019.

[KSSU19] Gautam Kamath, Or Sheffet, Vikrant Singhal, and Jonathan Ullman. Differentially private algorithms for learning mixtures of separated Gaussians. In Advances in Neural Information Processing Systems 32, NeurIPS ’19, pages 168–180. Curran Associates, Inc., 2019.

[KV18] Vishesh Karwa and Salil Vadhan. Finite sample differentially private confidence intervals. In Proceedings of the 9th Conference on Innovations in Theoretical Computer Science, ITCS ’18, pages 44:1–44:9, Dagstuhl, Germany, 2018. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

[LM19a] Gábor Lugosi and Shahar Mendelson. Mean estimation and regression under heavy-tailed distributions: A survey. Foundations of Computational Mathematics, 19(5):1145–1190, 2019.

[LM19b] Gábor Lugosi and Shahar Mendelson. Sub-Gaussian estimators of the mean of a random vector. The Annals of Statistics, 47(2):783–794, 2019.

[MS08] Satyaki Mahalanabis and Daniel Stefankovic. Density estimation in linear time. In Proceedings of the 21st Annual Conference on Learning Theory, COLT ’08, pages 503–512, 2008.

[MT07] Frank McSherry and Kunal Talwar. Mechanism design via differential privacy. In Proceedings of the 48th Annual IEEE Symposium on Foundations of Computer Science, FOCS ’07, pages 94–103, Washington, DC, USA, 2007. IEEE Computer Society.
[NRS07] Kobbi Nissim, Sofya Raskhodnikova, and Adam Smith. Smooth sensitivity and sampling in private data analysis. In Proceedings of the 39th Annual ACM Symposium on the Theory of Computing, STOC ’07, pages 75–84, New York, NY, USA, 2007. ACM.

[Smi11] Adam Smith. Privacy-preserving statistical estimation with optimal convergence rates. In Proceedings of the 43rd Annual ACM Symposium on the Theory of Computing, STOC ’11, pages 813–822, New York, NY, USA, 2011. ACM.

[SOAJ14] Ananda Theertha Suresh, Alon Orlitsky, Jayadev Acharya, and Ashkan Jafarpour. Near-optimal-sample estimators for spherical Gaussian mixtures. In Advances in Neural Information Processing Systems 27, NIPS ’14, pages 1395–1403. Curran Associates, Inc., 2014.

[SU15] Thomas Steinke and Jonathan Ullman. Interactive fingerprinting codes and the hardness of preventing false discovery. In Proceedings of the 28th Annual Conference on Learning Theory, COLT ’15, pages 1588–1628, 2015.

[SU17] Thomas Steinke and Jonathan Ullman. Between pure and approximate differential privacy. The Journal of Privacy and Confidentiality, 7(2):3–22, 2017.

[Vad17] Salil Vadhan. The complexity of differential privacy. In Yehuda Lindell, editor, Tutorials on the Foundations of Cryptography: Dedicated to Oded Goldreich, chapter 7, pages 347–450. Springer International Publishing AG, Cham, Switzerland, 2017.

[Yat85] Yannis G. Yatracos. Rates of convergence of minimum distance estimators and Kolmogorov’s entropy. The Annals of Statistics, 13(2):768–774, 1985.

[ZJS19] Banghua Zhu, Jiantao Jiao, and Jacob Steinhardt. Generalized resilience and robust statistics. arXiv preprint arXiv:1909.08755, 2019.

A Useful Inequalities

The following standard concentration inequalities are used frequently in this document.

**Lemma A.1** (Chebyshev’s Inequality). Let \( D \) be a distribution over \( \mathbb{R} \) with mean \( \mu \), and \( k \)th moment bounded by \( M \). Then the following holds for any \( a > 1 \).

\[
P_{X \sim D} \left[ |X - \mu| > aM^{\frac{1}{k}} \right] \leq \frac{1}{a^k}
\]

**Lemma A.2** (Concentration in High Dimensions [ZJS19]). Let \( D \) be a distribution over \( \mathbb{R}^d \) with mean \( \bar{0} \), and \( k \)th moment bounded by \( M \). Then the following holds for any \( t > 0 \).

\[
P_{X \sim D} \left[ \|X\|_2 > t \right] \leq M \left( \frac{\sqrt{d}}{t} \right)^k
\]
Lemma A.3 (Multiplicative Chernoff). Let $X_1,\ldots,X_m$ be independent Bernoulli random variables taking values in $\{0,1\}$. Let $X$ denote their sum and let $p = \mathbb{E}[X_i]$. Then for $m \geq \frac{12}{p} \ln(2/\beta)$,

$$
\mathbb{P}\left[X \notin \left[\frac{mp}{2}, \frac{3mp}{2}\right]\right] \leq 2e^{-mp/12} \leq \beta.
$$

Lemma A.4 (Bernstein’s Inequality). Let $X_1,\ldots,X_m$ be independent Bernoulli random variables taking values in $\{0,1\}$. Let $p = \mathbb{E}[X_i]$. Then for $m \geq \frac{5p}{\varepsilon^2} \ln(2/\beta)$ and $\varepsilon \leq p/4$,

$$
\mathbb{P}\left[\left|\frac{1}{m} \sum_{i=1}^{m} X_i - p\right| \geq \varepsilon \right] \leq 2e^{-\varepsilon^2 m/2(p+\varepsilon)} \leq \beta.
$$

Lemma A.5 (Laplace Concentration). Let $Z \sim \text{Lap}(t)$. Then $\mathbb{P}[|Z| > t \cdot \ln(1/\beta)] \leq \beta$.

Lemma A.6 (Gaussian Empirical Variance Concentration). Let $(X_1,\ldots,X_m) \sim \mathcal{N}(0,\sigma^2)$ be independent. If $m \geq \frac{8}{\tau^2} \ln(2/\beta)$, for $\tau \in (0,1)$, then

$$
\mathbb{P}\left[\left|\frac{1}{m} \sum_{i=1}^{m} X_i^2 - \sigma^2\right| > \tau \sigma^2\right] \leq \beta.
$$

We also mention two well-known and useful inequalities.

Lemma A.7 (Hölder’s Inequality). Let $X, Y$ be random variables over $\mathbb{R}$, and let $k > 1$. Then,

$$
\mathbb{E}[|XY|] \leq \left(\mathbb{E}[|X|^k]\right)^{\frac{1}{k}} \left(\mathbb{E}[|Y|^{\frac{k}{k-1}}]\right)^{\frac{k-1}{k}}.
$$

Lemma A.8 (Jensen’s Inequality). Let $X$ be an integrable, real-valued random variable, and $\psi$ be a convex function. Then

$$
\psi(\mathbb{E}[X]) \leq \mathbb{E}[\psi(X)].
$$