2-ADIC STRATIFICATION OF TOTIENTS

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Abstract. In this paper we study the multiplicities and the asymptotic behaviour of the numbers of totients in the strata given by 2-adic valuation.

1. Introduction

One of the main multiplicative functions in number theory is the well known Euler’s totient function, whose value for a positive integer number $n$ is

$$\phi(n) := \#\{1 \leq a \leq n \mid \gcd(a, n) = 1\}.$$  

A particular subject of study is the set $\mathcal{V}$ of totients, i.e. the set of the values taken by Euler’s function.

$$\mathcal{V} := \{1, 2, 4, 6, 8, 10, 12, 16, 18, \ldots\}.$$  

The distribution of suitable totients is yet subject of investigation of many authors and from many perspectives. One of the main objects of study is the multiplicity of a given positive integer $m \in \mathbb{N}$, that is the number of elements in its pre-image under $\phi$, namely

$$A(m) := |\phi^{-1}(m)|.$$  

Equivalently, $A(m)$ is the number of solutions of the equation $\phi(x) = m$.

Since Euler’s $\phi$-function is multiplicative, and for every prime number $p$ one has $\phi(p^n) = p^{n-1}(p-1), \forall n \geq 1$, we easily see that totients bigger than 1 are even numbers. Hence the most naive stratification of $\mathcal{V}$ can be made by taking totients with a fixed 2-adic valuation. More precisely, for each $\ell \geq 1$ we pick up the following subset of $\mathcal{V}$,

$$\mathcal{V}^\ell := \{m \in \mathcal{V}; m \equiv 2^\ell \mod 2^{\ell+1}\}.$$  

As usual, for each real number $x \in \mathbb{R}$ we write $\mathcal{V}^\ell(x) := \{m \in \mathcal{V}^\ell; m \leq x\}$, analogously for $\mathcal{V}(x)$. We also write $V^\ell(x)$ and $V(x)$ for the number of elements in $\mathcal{V}^\ell(x)$ and in $\mathcal{V}(x)$, respectively. For a fixed positive integer $\ell > 1$ two very natural questions can be made.

**Question 1.** What is the asymptotic behaviour of $V^\ell(x)$?

**Question 2.** What multiplicities of elements in $\mathcal{V}^\ell$ are possible?

We did not find in the literature papers addressed to answer the above two questions. In regarding to the order of the set $V(x)$ of all totients not greater than $x$, K. Ford obtained its exact order, see [F98 Thm. 1]. Before Ford, H. Maier and C. Pomerance, cf. [MP88], obtained a nice order for $V(x)$ that will be useful here.
In [P29] S. Pillai proved that the set of multiplicity of totients is unbounded, he proved that
\[ \limsup_{x \to \infty} \{ A(m) \; ; \; m \in \mathcal{V}(x) \} = \infty. \]

Pillai’s theorem can be considered as an insight to Sierpiński’s conjecture that says that for each \( k > 1 \) there is \( m \in \mathbb{N} \) such that \( A(m) = k \). Sierpiński’s conjecture was completely solved by K. Ford in [F99].

We divide this short paper in 3 sections. The section 2 deals with the simplest (and very easy) case where \( \ell = 1 \). The multiplicity of totients in \( \mathcal{V}^1 \) are just 2 or 4, and the elements in the pre-images of such totients are just a power of an odd prime and twice this power of an odd prime. So the main idea is that totients in \( \mathcal{V}^1 \) are just image of prime numbers \( \equiv 3 \mod 4 \), except those too rare totients in \( \mathcal{V}^1 \) whose pre-image has a power of prime number with exponent bigger than one. Hence \( \mathcal{V}^1(x) \) has asymptotic order \( \pi(x)/2 \), where \( \pi(x) \) stands for the number of primes numbers not bigger than \( x \), see Corollary 2.5.

In section 3 we deal with the case \( \ell \geq 2 \). While the multiplicities of totients in \( \mathcal{V}^1 \) are far from being exciting, the multiplicities of totients in \( \mathcal{V}^\ell \) with \( \ell \geq 2 \) seems to be unbounded, i.e. \( \limsup_{m \in \mathcal{V}^\ell} A(m) = \infty \). It is very simple to see that if there is some \( \ell_0 \) such that the set of multiplicities of totients in \( \mathcal{V}^{\ell_0} \) is unbounded, then the set of multiplicities of totients in \( \mathcal{V}^{\ell_0+n} \) is also unbounded for all \( n \geq 0 \), c.f. Proposition 3.1. We are able to show that Dickson’s \( k \)-tuples Conjecture implies that \( \ell_0 = 2 \), see Theorem 3.2. The remaining of Section 3 is devoted to provide a different approach to get information on \( \ell_0 \). The main idea lies in Theorem 3.3 and involves of finding a suitable lower bound for the number \( S^{\ell_0}(x) \) of the solutions in \( \mathcal{V}^\ell(x) \) of the equation \( \phi(z) \leq x \). To do so, first we provide the following upper bound addressing to Question 4 above
\[ V^\ell(x) = O_{\ell} \left( \frac{x}{\log x} \left( \log \log x \right)^\ell \right), \]
that derives from a classical result due to G. Hardy and S. Ramanujan, see Theorem 3.4. We also show the inequality
\[ S^{\ell_0+n}(2^n x) \geq \frac{S^\ell(x)}{2^n} \quad \forall \; n \geq 0, \]
c.f. Lemma 3.8. Now, for each real number \( x > 2 \), there is a positive integer \( \ell(x) \) such that \( S^{\ell(x)}(x) > \frac{1}{2\ell(x)} V(x) \). So the key is to show the existence of a positive integer \( \ell_0 \) that is given by
\[ \ell_0 := \liminf_{x \to \infty} \min\{ \ell(x) \}. \]
If we assume that such \( \ell_0 \) exists, through the equation
\[ S^{\ell_0}(2^n x) \geq \frac{1}{2^n} S^{\ell_0}(x) > \frac{1}{2\ell_0+m} \frac{V(x)}{\sqrt{V_0+m}(2^n x)}, \]
where \( (x_i) \) is a suitable sequence of positive real numbers that goes to infinity, and using a result involving the above order of \( V^\ell(x) \) and the order of \( V(x) \) due to H. Maier and C. Pomerance, c.f. Corollary 3.4, then we can show
\[ \limsup_{m \in \mathcal{V}^{\ell_0+n}} A(m) = \infty, \quad \forall n \geq 0, \]
that is precisely the contents of our Theorem 3.9.
The proof of the existence of the above lim inf seems to be workable, see some computations on Table 3 below. We strongly believe that \( \ell_0 \) is equal to 2, as suggested by Dickson’s \( k \)-tuples conjecture. We also note that proving that \( \ell_0 \) is a positive integer, then for each \( \ell \geq \ell_0 \), the multiplicities of \( \mathcal{V}^\ell \) are unbounded. So one should determine all the positive integers that are realized as the multiplicity of totients in \( \mathcal{V}^\ell \), equivalently, the number of the solutions of the equation \( \phi(x) = m \) where \( m \in \mathcal{V}^\ell \). Hence, to show that \( \ell_0 \) exists, is an insight to a Sierpiński-type problem on each stratum \( \mathcal{V}^\ell \).

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2. The simplest case \( \ell = 1 \)

The set of prime numbers is denoted by \( \mathcal{P} \) and \( \pi(x) \) stands for the number of prime numbers not greater than \( x \). We also use the big \( O \) and small \( o \) standard notations. The following notation will be useful throughout this paper: given any subset \( U \) of the positive integers and \( x \in \mathbb{R} \) a real number, \( U(x) \) stands for the elements of \( U \) not greater than \( x \),

\[
U(x) := \{ n \in U; n \leq x \},
\]

and it is clear that \( |U(x)| \) denotes the magnitude of \( U(x) \).

**Lemma 2.1.** Given an integer number \( t > 0 \) and considering the set

\[
\mathcal{R}_t := \{ k \in \mathcal{V} \mid q^j \in \phi^{-1}(k) \text{ with } j \geq t \text{ and } q \in \mathcal{P} \},
\]

we get

\[
|\mathcal{R}_t(x)| = o(\sqrt{x}).
\]

**Proof.** For every \( k \in \mathcal{R}_t(x) \) there is a prime number \( q \) and an integer \( m \geq t \) such that

\[
x \geq k = q^m - q^{m-1} \geq q^t - q^{t-1} \geq q^{t-1}/2.
\]

Hence, from the above inequality we get the upper bound \( q \leq \sqrt{2x} \). Thus, by the Prime Number Theorem

\[
|\mathcal{R}_t(x)| \leq \frac{\pi(\sqrt{2x})}{2x},
\]

and so \( \mathcal{R}_t(x) = o(\sqrt{2x}) = o(\sqrt{x}) \). \( \square \)

For the sake of organization of the presentation we just state the next result, omitting its quite trivial proof.

**Lemma 2.2.** For each odd positive integer, \( A(2r) \in \{0, 2, 4\} \). If \( A(2r) = 2 \), then \( \phi^{-1}(2r) = \{ p^n, 2p^n \} \), with \( p \) an odd prime and \( n > 0 \). If \( A(2r) = 4 \), then \( 2r + 1 \) is a prime number and \( \phi^{-1}(2r) = \{ 2r + 1, q^m, 4r + 2, 2q^m \} \) with \( q \) a prime number and \( m > 1 \).

Next we study the distribution of totients 2 modulo 4. Let us start by taking the following useful sets

\[
\mathcal{V}^1_k = \{ 2r; r \text{ odd and } A(2r) = k \}, \text{ for each } k = 2, 4.
\]
Table 1. The number of totients $2 \mod 4 \leq x$ with a fixed multiplicity

| $x$   | $\pi(x)$ | $|\mathcal{Y}_2^1(x)|$ | $|\mathcal{Y}_2^3(x)|$ | $|\mathcal{Y}_2^4(x)|/\pi(x)$ |
|-------|----------|------------------------|------------------------|-------------------------------|
| $10^3 + 2$ | 168      | 87                     | 5                      | 0.517857...                  |
| $10^4 + 2$ | 1229     | 625                    | 8                      | 0.508543...                  |
| $10^5 + 2$ | 9592     | 4831                   | 14                     | 0.503648...                  |
| $10^6 + 2$ | 78498    | 39400                  | 20                     | 0.501923...                  |
| $10^7 + 2$ | 664579   | 332606                 | 34                     | 0.500476...                  |
| $10^8 + 2$ | 5761455  | 2881495                | 78                     | 0.500133...                  |

Corollary 2.3. $\lim_{x \to \infty} \frac{|\mathcal{Y}_2^4(x)|}{\sqrt{x}} = 0.$

Proof. Just note that $\mathcal{Y}_2^1(x) \subseteq \mathcal{R}_2(x)$.

Let us introduce some useful notation, $\mathcal{P}(k, j) = \{ p \in \mathcal{P} : p \equiv j \mod k \}$ and $\pi(x; k, j) = |\mathcal{P}(x; k, j)|.$ It is just easy to see that if $p \in \phi^{-1}(2r)$ with $r \equiv 1 \mod 2$, then $p \in \mathcal{P}(4, 3)$.

Theorem 2.4. For every real number $x > 0$,

$$|\mathcal{Y}_2^1(x)| \sim \frac{\pi(x)}{2}.$$

Proof. Let us consider the subset $\mathcal{A}$ of $\mathcal{Y}^1$ given by the elements $m \equiv 2 \mod 4$ such that $m + 1$ is prime, and $\mathcal{B} := \mathcal{Y}^1 \setminus \mathcal{A}.$ Thus

$$|\mathcal{Y}_2^1(x)| = |\mathcal{A}(x)| + |\mathcal{B}(x)|.$$

Moreover, from the above two Lemmas, $|\mathcal{B}(x)| = o(\sqrt{x}),$ since $\sqrt{x} = o(\pi(x)),$ the Prime Number Theorem implies that $|\mathcal{B}(x)| = o(\pi(x)).$ Hence

$$\liminf_{x \to \infty} \frac{|\mathcal{Y}_2^1(x)|}{\pi(x)} = \liminf_{x \to \infty} \frac{|\mathcal{A}(x)|}{\pi(x)} \quad \text{and} \quad \limsup_{x \to \infty} \frac{|\mathcal{Y}_2^1(x)|}{\pi(x)} = \limsup_{x \to \infty} \frac{|\mathcal{A}(x)|}{\pi(x)}.$$

We claim that

$$|\mathcal{A}(x)| \sim \frac{\pi(x)}{2}.$$

Since $m + 1$ must be a prime equivalent to 3 modulo 4, we get

$$\{m + 1 \in \mathcal{P}(x + 1; 4, 3)\} = \mathcal{A}(x) \cup \{m + 1 \in \mathcal{P}(x + 1; 4, 3); A(m) = 4\}.$$

Thus

$$\pi(x; 4, 3) = |\mathcal{A}(x)| + |\{m + 1 \in \mathcal{P}(x + 1; 4, 3); A(m) = 4\}|.$$

We know that $|\{m + 1 \in \mathcal{P}(x + 1; 4, 3); A(m) = 4\}| = o(\pi(x)).$ Now, the Prime Number Theorem in Arithmetic Progression assure that

$$\pi(x; 4, 3) \sim \frac{\pi(x)}{2}.$$

Hence

$$\lim_{x \to \infty} \frac{|\mathcal{A}(x)|}{\pi(x)} = \frac{1}{2}.$$

This finishes the claim and proves the theorem.
From the above Theorem and the Corollary 2.3 we have
\[
\lim_{x \to \infty} \frac{V_1(x)}{\pi(x)} = \lim_{x \to \infty} \frac{|\mathcal{Y}_1^1(x)| + |\mathcal{Y}_1^2(x)|}{\pi(x)} = \lim_{x \to \infty} \frac{|\mathcal{Y}_2^1(x)|}{\pi(x)},
\]
and so get the main result of this section.

**Corollary 2.5.** \(|\mathcal{Y}_1^1(x)| \sim \frac{\pi(x)}{2}\)

**Remark 2.6.** It follows from the above Theorem that for each \(k > 1\)
\[
|\{m \in \mathcal{Y}^k(x) : A(m) \geq 2\}| \gg \frac{\pi(x)}{2^k}.
\]

**Corollary 2.7.** \(|\mathcal{Y}_1^4(x)| = o(|\mathcal{Y}_1^2(x)|)\).

**Proof.** Since \(|\mathcal{Y}_1^2(x)| \sim \frac{x}{2 \log x}\) and \(|\mathcal{Y}_1^4(x)| = o(\sqrt{x})\), we get
\[
\frac{|\mathcal{Y}_1^4(x)|}{|\mathcal{Y}_1^2(x)|} = O\left(\frac{\sqrt{x} \log x}{x}\right) = O\left(\frac{\log x}{\sqrt{x}}\right).
\]
Therefore,
\[
\lim_{x \to \infty} \frac{|\mathcal{Y}_1^4(x)|}{|\mathcal{Y}_1^2(x)|} = 0.
\]

\[\square\]

### 3. When \(\ell \geq 2\)

The next natural step is to try to answer the Questions 1 and 2 when \(\ell = 2\), i.e. considering totients that are 4 mod 8. So are allowed until two odd prime divisors in the pre-image of such totients, hence the computations become more involved. The first examples suggest that the multiplicities of totient in \(\mathcal{Y}^2\) can assume very large values. For example, using a simple computer one can see in a few minutes that the multiplicities of totients in \(\mathcal{Y}^2(10^7)\) assume all values between 2 and 35 and the biggest one is 42, there is some gaps between 35 and 42 for such \(x\). In a couple of hours one can see that all numbers between 2 and 72 are realized as multiplicities of totients in \(\mathcal{Y}^2(3 \cdot 10^3 + 2 \cdot 10^6)\), and the maximal multiplicity attained is 94. Doing the same computations for \(\ell = 3, 4, 5, 6, 7\) very large multiplicities also appear, but as \(\ell\) increases, large multiplicities are attained by even smaller totients. For example, all numbers between 2 and 82 are realized as the multiplicities of totients in \(\mathcal{Y}^3(10^6)\) and the biggest multiplicity is 169, see Table 2.

**Table 2. maximal multiplicity in \(\mathcal{Y}^\ell(x)\)**

| \(\mathcal{Y}^\ell(x)\) | \(x = 10^6\) | \(x = 5 \cdot 10^6\) | \(x = 10^7\) | \(x = 5 \cdot 10^7\) |
|------------------------|-------------|----------------|-------------|----------------|
| \(\max A(\mathcal{Y}^2(x))\) | 32 | 34 | 42 | 57 |
| \(\max A(\mathcal{Y}^3(x))\) | 169 | 250 | 277 | 427 |
| \(\max A(\mathcal{Y}^4(x))\) | 463 | 745 | 860 | 1427 |
| \(\max A(\mathcal{Y}^5(x))\) | 998 | 1804 | 1961 | 3732 |
| \(\max A(\mathcal{Y}^6(x))\) | 1401 | 2222 | 3887 | 6239 |
| \(\max A(\mathcal{Y}^7(x))\) | 1375 | 3258 | 4076 | 7807 |
Now, it is only natural to make a weak version of Question 2 in the following way.

**Question 3.** Is \( \limsup_{m \in \mathcal{V}} A(m) = \infty \) for each \( \ell \geq 2 \)?

Next we see an elementary result concerning the above Question 3.

**Proposition 3.1.** If there exists \( \ell_0 \) such that \( \limsup_{m \in \mathcal{V}^\ell_0} A(m) = \infty \), then for each \( n \geq 0 \), \( \limsup_{m \in \mathcal{V}^{\ell_0+n}} A(m) = \infty \).

**Proof.** Take \( m \in \mathcal{V}^{\ell_0} \) such that \( A(m) = k \). For each prime number \( p \equiv 3 \mod 4 \) and \( 3 \mod 5 \) (2

The above Proposition shows that we should prove the existence of a suitable \( \ell_0 \), ideally the smallest one, where \( \limsup_{m \in \mathcal{V}^{\ell_0}} A(m) = \infty \). Now we give a conditional proof that such \( \ell_0 = 2 \) using Dickson’s \( k \)-tuples Conjecture, c.f. \[Rivera96\].

**Theorem 3.2.** Dickson’s \( k \)-tuples Conjecture implies that \( \ell_0 = 2 \).

**Proof.** We start by noting that each \( A(m) = k \), then for all \( 1 \leq i, j \leq k \) we have

\[
m := \phi(p_i \cdot q_i) = (p_i - 1)(q_i - 1) = (p_j - 1)(q_j - 1) = \phi(p_j \cdot q_j)
\]

and so \( A(m) \geq k \) with \( m \in \mathcal{V}^2 \).

The remaining of this paper is devoted to give a sufficient condition on the existence of a suitable \( \ell_0 \) using a completely different approach. We start by considering a family of suitable functions \( S^k \), with \( k \in \mathbb{N} \), such that for each \( x \in \mathbb{R} \) its value by \( S^k \) is the sum of all multiplicities of the totients in \( \mathcal{V}^k(x) \), namely \( S^k(x) := \sum_{m \in \mathcal{V}^k(x)} A(m) \).

**Theorem 3.3.** If there is a function \( f : \mathbb{R} \to \mathbb{R} \) such that

i. \( \liminf_{x \to \infty} f(x) = \infty \), and

ii. \( S^f(x) \geq V^f(x) \cdot f(x) \), for some increasing sequence \( (x_i)_{i \in \mathbb{N}}, \)

then \( \limsup_{m \in \mathcal{V}^\ell} A(m) = \infty \).

**Proof.** We start by noting that

\[
S^f(x) = \sum_{m \in \mathcal{V}^f(x)} A(m) \leq V^f(x) \cdot \max\{A(m) ; m \in \mathcal{V}^f(x)\}.
\]

If we assume that \( A(m) = o(f(m)) \) with \( m \in \mathcal{V}^\ell(x) \), then

\[
\frac{S^f(x)}{f(x) \cdot V^f(x)} = o(1).
\]
Now, condition (ii) ensures that
\[
\liminf_{x \to \infty} \frac{S_\ell(x)}{f(x) \cdot V^\ell(x)} \geq 1,
\]
which contradicts equation (2). Hence \( A(n) \neq o(f(n)) \), and from condition (1) we are done. \( \square \)

Of course that we seek for some function \( f \) satisfying the hypothesis of the above Theorem 3.3. The first step is try to measure the magnitude of \( V^\ell(x) \), addressing to Question 4. This is the content of the following theorem.

**Theorem 3.4.** For each \( \ell \geq 1 \), \( V^\ell(x) = O_{\ell} \left( \frac{x}{\log x} (\log \log x)^{\ell} \right) \).

**Proof.** Since the number of prime divisors of an element in the pre-image of a totient in \( V^\ell \) is at most \( \ell + 1 \), it follows that
\[
V^\ell(x) < \sum_{i=1}^{\ell+1} \pi_i(x),
\]
where \( \pi_i(x) = \# \{ n \leq x ; \omega(n) = i \} \) is the number of elements not bigger than \( x \) such that in their prime factorization appear exactly \( i \) different prime numbers. In [HR17] Hardy and Ramanujan proved that
\[
\pi_i(x) < M \left( \frac{x}{\log x} \frac{(\log \log x + c)^{i-1}}{(i-1)!} \right).
\]
From the above two inequalities the results follows easily as follows.
\[
V^\ell(x) = O_{\ell} \left( \sum_{i=1}^{\ell+1} \frac{x}{\log x} \frac{(\log \log x)^i}{(i-1)!} \right) = O_{\ell} \left( \frac{x}{\log x} (\log \log x)^\ell \right).
\]
\( \square \)

We strongly believe that the above upper bound for \( V^\ell(x) \) can be improved using better sieve methods. Compare, for example, the above bound in the particular case \( \ell = 1 \) to Corollary 2.5 of the previous section. On the other hand, for the purposes of this paper the above upper bound fits nicely. The next theorem is due to Maier and Pomerance and it can be found in [MP88].

**Theorem 3.5 (Maier–Pomerance).**
\[
V(x) = \frac{x}{\log x} \exp \left( (C + o(1))(\log \log x)^2 \right).
\]

The proof of the following useful result follows easily from Theorem 3.4 and from Theorem 3.5.

**Corollary 3.6.** For any \( \ell \in \mathbb{N} \) and any real number \( M > 0 \) we have
\[
\lim_{x \to \infty} \frac{V(x)}{V^\ell(Mx)} = \infty.
\]

For each real number \( x > 0 \), let \( k_0 := k_0(x) \) be the smallest natural number such that \( S^\ell(x) = 0 \) for every \( \ell > k_0 \).

**Lemma 3.7.** \( k_0(x) = \left\lfloor \frac{\log x}{\log 2} \right\rfloor \).
Proof. Let $x > 0$ be a real number. Since $\phi(2^{\ell+1}) = 2^\ell$ is the smallest totient in $\nu^\ell(x)$, $S^\ell(x) \neq 0$ if and only if $x \geq 2^\ell$. Hence $k_0(x) = \lfloor \log x / \log 2 \rfloor$.

\[\ell \leq 15 299 = 13 325 = 12 326 = 11 289 = 10 264 = 9 215 = 8 169 = 7 125 = 6 29 = 5 13 = 4 4 \]

We tried to give a proof of the Hypothesis, but unfortunately we were not able to.

\[\phi(2n_i) = 2\phi(n_i) \in \nu^{\ell+1}(2x)\]

and so $S^{\ell+1}(2x) \geq S^\ell(x) \cdot 2^{-1}$.

Given a real number $x$, there is $2 \leq \ell(x) \leq k_0(x)$ such that $S^{\ell(x)}(x) > \frac{1}{2^{\ell(x)}}V(x)$. Otherwise $x < S(x) = \sum_{i=1}^{k_0(x)} S^\ell(x) < V(x)$, getting a contradiction. Hence, for each positive real number $x$, set

$$\ell(x) := \min \{ \ell(x) \}.$$  

**Hypothesis.** There is $\ell_0 := \liminf_{x \to \infty} \ell(x) < \infty$.

We tried to give a proof of the Hypothesis, but unfortunately we were not able until now. Computations suggest that $\ell_0$ does exist, see Table 3. It is also clear that Dickson’s $k$-tuples Conjecture is stronger that our Hypothesis.

**Table 3.** Collecting $(2^\ell \cdot S^\ell(x))/V(x)$

| $\ell$  | $x = 10^6$ | $x = 5 \cdot 10^6$ | $x = 10^7$ | $x = 5 \cdot 10^7$ | $x = 10^8$ |
|--------|-----------|-------------------|-----------|-------------------|---------|
| 2      | 4,6044... | 4,4593...        | 4,764...  | 4,2755...         | 4,2223... |
| 3      | 13,3598... | 13,3172...      | 13,2956... | 13,2309...        | 13,1984... |
| 4      | 29,8629... | 30,5975...      | 30,8712... | 31,4799...        | 31,7125... |
| 5      | 54,3445... | 57,7380...      | 59,0381... | 61,9642...        | 63,1584... |
| 6      | 86,5368... | 95,3994...      | 98,8584... | 107,0031...       | 110,4274... |
| 7      | 125,1458... | 143,0769...    | 150,4968... | 167,9442...       | 175,5206... |
| 8      | 169,4072... | 200,5425...    | 213,6786... | 245,7202...       | 260,0693... |
| 9      | 215,2255... | 265,4653...    | 285,8896... | 339,0014...       | 363,3946... |
| 10     | 264,8755... | 333,3547...    | 364,4399... | 445,1494...       | 482,6826... |
| 11     | 289,2330... | 392,9665...    | 441,5500... | 557,9906...       | 613,1423... |
| 12     | 326,8434... | 452,8810...    | 504,6976... | 666,9787...       | 748,3791... |
| 13     | 325,5704... | 488,7035...    | 567,5796... | 768,3839...       | 869,4729... |
| 14     | 290,2446... | 497,8492...    | 598,7011... | 855,6433...       | 984,6031... |
| 15     | 299,1556... | 492,8571...    | 598,0896... | 908,2700...       | 1076,7877... |

**Theorem 3.9.** Assuming the Hypothesis,

$$\limsup_{m \in \nu_{\ell_0+n}} A(m) = \infty, \forall n \geq 0.$$
Proof. We first prove in the case that \( n = 0 \). Take \( \ell_0 \) as in Hypothesis. Thus there is a sequence \((x_i)\) with \( x_i \to \infty \) such that
\[
S^{\ell_0}(x_i) > \frac{1}{2^{\ell_0}} V(x_i) = \frac{1}{2^{\ell_0}} \frac{V(x_i)}{V_{\ell_0}(x_i)} V^{\ell_0}(x_i).
\]
From Corollary 3.6 we know that \( 2^{-\ell_0} \frac{V(x_i)}{V_{\ell_0}(x_i)} \) goes to infinity when \( i \to \infty \). Hence the Theorem 3.3 assures that \( \limsup_{m \in V_{\ell_0}} A(m) = \infty \).

Now take any integer \( n \geq 1 \). From Lemma 3.8 and the Hypothesis there is a sequence \((x_i)\) that goes to infinity such that
\[
S^{\ell_0+n}(2^n x_i) \geq \frac{1}{2^n} S^{\ell_0}(x_i) > \frac{V(x_i)}{2^{\ell_0+n} V_{\ell_0+m}(2^n x_i)} V^{\ell_0+m}(2^n x_i)
\]
and again by the Corollary 3.6 and Theorem 3.3 we are done. \qed

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