Existence and upper semicontinuity of time-dependent attractors for the non-autonomous nonlocal diffusion equations

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Abstract

In this paper, under some appropriate assumptions, we prove the existence of the minimal time-dependent pullback $D_{t}^{H_{t}}$-attractors $A_{D_{t}^{H_{t}}}$ for the non-autonomous nonlocal diffusion equations in time-dependent space $H_{t}(\Omega)$. Next, in same phase space, using a priori estimate and energy methods we establish the existence of time-dependent pullback attractors $\{A_{\xi}(t)\}_{t\in\mathbb{R}}$ and the upper semicontinuity of $\{A_{\xi}(t)\}_{t\in\mathbb{R}}$ and the global attractor $A$ of equation (1.1) with $\xi = 0$, that is,

$$\lim_{\xi \to 0^{+}} \text{dist}_{H_{t}}(A_{\xi}(t), A) = 0.$$ 

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1 Introduction

In the past few decades, many scholars have devoted to obtaining the well-posedness of solutions by studying the attractors of partial differential equations (see [6, 13, 20, 28, 36]). It is worth mentioning that diffusion equations are also a prevalent research direction since they are applied in many disciplines such as physics, chemistry and biology (see [1, 33, 37, 38]).

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In this paper, we consider the following non-autonomous nonlocal diffusion equations

\[
\begin{cases}
    u_t - \varepsilon(t) \Delta u_t - a(l(u))\Delta u = f(u) + \xi h(x,t) & \text{in } \Omega \times (\tau, \infty), \\
    u = 0 & \text{on } \partial\Omega \times (\tau, \infty), \\
    u(x, \tau) = u_\tau & x \in \Omega,
\end{cases}
\]  

(1.1)

in time-dependent space \( \mathcal{H}_t(\Omega) \), where \( \Omega \subset \mathbb{R}^N (N \geq 3) \) is a bounded domain with smooth boundary \( \partial\Omega \), the initial time \( \tau \leq t \in \mathbb{R} \), \( \xi \) is a small positive parameter and the definition and properties of \( \mathcal{H}_t(\Omega) \) can be seen in §2.

Assume the function \( \varepsilon(t) \in C^1(\mathbb{R}) \) is a decreasing bounded function with respect to the parameter \( t \) satisfying

\[
\lim_{t \to +\infty} \varepsilon(t) = 0,
\]

(1.2)

and there exists a constant \( L > 0 \) such that

\[
\sup_{t \in \mathbb{R}} (|\varepsilon(t)| + |\varepsilon'(t)|) \leq L.
\]

(1.3)

Besides, the function \( a(l(u)) \in C(\mathbb{R}; \mathbb{R}^+) \) is the nonlocal diffusion term of equation (1.1) and satisfying

\[
0 < m \leq a(s) \leq M, \quad \forall s \in \mathbb{R},
\]

(1.4)

where \( m \) and \( M \) are constants. Furthermore, assume \( l(u) : L^2(\Omega) \to \mathbb{R} \) is a continuous linear functional acting on \( u \) that satisfies for some \( g \in L^2(\Omega), \)

\[
l(u) = l_g(u) = \int_{\Omega} g(x) u(x) dx.
\]

(1.5)

In addition, suppose the nonlinear term \( f \in C(\mathbb{R}, \mathbb{R}) \) and satisfies

\[
|f(u) - f(v)| \leq C (|u|^p + |v|^p + 1) |u - v|,
\]

(1.6)

and

\[
\lim \sup_{|s| \to \infty} \frac{f(s)}{s} < \lambda_1,
\]

(1.7)

where \( p \leq \frac{4}{N-2} \), \( u, v, s \in \mathbb{R} \) and \( \lambda_1 \) is the first eigenvalue of \(-\Delta\) in \( H^1_0(\Omega) \) with the homogeneous Dirichlet boundary conditions.

Throughout this paper, the inner product of \( L^2(\Omega) \) is denoted by \((\cdot, \cdot)\), and the corresponding norm is written as \( \| \cdot \| \). Then we also assume that the function \( f \) admits the following decomposition

\[
f = f_0 + f_1,
\]

(1.8)
for any \( u \) general diffusion equation seen in our paper \([27]\). The equation (1.1) is a nonlocal diffusion equation and the function \( a \) is a similar result of this problem in \( H \) with \( f \) and \( u \). Caraballo, Herrera-Cobos and Marín-Rubio \([6]\) studied the existence of minimal pullback attractors is a nonlocal diffusion term and many people considered analogous problems in the past decade. Caraballo, Herrera-Cobos and Marín-Rubio \([7]\) discussed the existence of pullback attractors of \( u_t - a(l(u))\Delta u = f(u) + h(t) \) in \( L^2(\Omega) \) and \( H^1(\Omega) \). Later on, Peng, Shang and Zheng \([23]\) proved a similar result of this problem in \( H^1_0(\Omega) \). In addition, Caraballo, Herrera-Cobos and Marín-Rubio \([7]\) discussed the existence of pullback attractors of \( u_t - (1 - \varepsilon)a(l(u))\Delta u = f(u) + \varepsilon h(t) \) with \( \varepsilon \in [0, 1] \) and the upper semicontinuity of attractors in \( L^2(\Omega) \), which means the family of pullback attractors corresponding global compact attractor associates with the autonomous nonlocal limit problem when \( \varepsilon \to 0 \). Then they also proved the existence of the minimal pullback attractors for \( p \)-Laplacian reaction-diffusion equation \( u_t - a(l(u))\Delta_p u = f(u) + h(t) \) in \( L^2(\Omega) \) and \( L^p(\Omega) \) with \( p \geq 2 \) in \([8]\). Besides, they established the existence of regular pullback attractors as well as their upper semicontinuous in \( H^1(\Omega) \) of \( u_t - g_1(\varepsilon)a(l(u))\Delta u = \tilde{g}_1(\varepsilon)f(u) + g_0(\varepsilon)h(t) \) in \([9]\), where \( g_1 \in C([0, 1]; (0, \infty)) \), \( \tilde{g}_1 \in C([0, 1]; [0, \infty)) \) and \( g_0 \in C([0, 1]). \) Additionally, in \([10]\) they proved the existence of pullback \( \mathcal{D} \)-attractors in \( L^2(\Omega) \) and \( H^1(\Omega) \).

Next we will introduce some results related to upper semicontinuity of diffusion equations. Guo and Wang \([18]\) proved the global attractor \( A \) of \( u_t - \nu \Delta u + f(u) + \lambda_0 u + g(x) = 0 \) is upper semicontinuity at 0 with respect to the global attractor \( \{A_L\} \), where \( A \) and \( \{A_L\} \) are obtained when \( \Omega = \mathbb{R} \) and \( \Omega = [-L, L] \), respectively. Later Carvalho, José and Robinson \([12]\) took into account the continuity of pullback attractors for evolution processes. Furthermore, Wang and Qin \([32]\) studied the upper semicontinuity of attractors of \( u_t - \Delta u_t - \Delta u = f(u) + \varepsilon g(x, t) \) in \( H^1_0(\Omega) \). Anh and Bao \([2]\) established the upper semicontinuity of pullback \( \mathcal{D} \)-attractors of \( u_t - \varepsilon \Delta u_t - \)
\[ \Delta u + f(u) = g(t) \text{ in } H_0^1(\Omega). \] Besides, Anh and Bao [3] and Wang [30] obtained similar results of \[ u_t - \varepsilon \Delta u_t - \Delta u + f(x, u) + \lambda u = g(x, t) \text{ in } L^2(\mathbb{R}^N) \] and \[ u_t - \varepsilon \Delta u_t - \Delta u = f(u) + g(x) + \varepsilon h(t) \text{ in } H_0^1(\Omega), \] respectively. Moreover, some authors also considered the upper semicontinuity between global attractors and uniform attractors (see [14, 15, 31, 34, 35]).

When the problem (1.1) is compared with the general diffusion equations, it can be found that it adds the terms \( a(\cdot), \varepsilon(t), f(u) \) and \( \xi h(x, t) \), which undoubtedly increases challenges. Meanwhile, these terms make some conventional methods lose their effect. Therefore, we now introduce these difficulties and explain our methods, they should be creative fresh attempts.

(1) The results of this paper are all obtained on the time-dependent space \( \mathcal{H}_t(\Omega) \). Thus since the time-dependent function \( \varepsilon(t) \) exists in the norm of \( \mathcal{H}_t(\Omega) \), when \( u \) or \( u_t \) is used as the test function of the equation (1.1) in the energy estimation, the resulting equation cannot be directly estimated using the Gronwall inequality. To make it effective, we use the transformation

\[
\varepsilon(t) \frac{d}{dt} \| \nabla u \|^2 = \frac{d}{dt} \left( \varepsilon(t) \| \nabla u \|^2 \right) - \varepsilon(t) \| \nabla u \|^2,
\]

which combined with \( \varepsilon(t) \) is a decreasing function can prove the desired results.

(2) In this paper, (1.2) and (1.4) are weaker than the conditions in Qin and Yang [27]. In fact, the weakening of these conditions forces us to be more refined in the calculations of the pullback absorbing sets. In addition, we use two methods to discuss the existence of the minimal pullback \( \mathcal{D}_{\sigma}^{\mathcal{H}_t} \)-attractors \( \mathcal{A}_{\mathcal{D}_{\sigma}^{\mathcal{H}_t}} \) and pullback attractors \( \mathcal{A}_{\xi} \). On the one hand, in §4 we prove the compactness of \( \mathcal{A}_{\mathcal{D}_{\sigma}^{\mathcal{H}_t}} \) by introducing continuous functions. On the other hand, in §5 we use the theorem proved in Wang and Qin [32] to derive the desired results, which is also novel when compared with Qin and Yang [27].

(3) It is worth mentioning that the nonlinear term \( f(u) \) and the external force term \( \xi h(x, t) \) makes problem (1.1) be studied in a more general functional framework. To obtain the dissipative properties of the process, we assume that \( f \) satisfies (1.8) – (1.11), which are weaker than the conditions in Wang and Qin [32].

The structure of this paper is organized as follows. In §2, we introduce some useful abstract definitions, theorems and lemmas. Then in §3, we obtain the existence and uniqueness of the solutions of problem (1.1) by the standard Faedo-Galerkin approximations and compactness argument. Furthermore, we shall derive the existence of the minimal pullback \( \mathcal{D}_{\sigma}^{\mathcal{H}_t} \)-attractors \( \mathcal{A}_{\mathcal{D}_{\sigma}^{\mathcal{H}_t}} \),
the pullback attractors $A_\xi = \{A_\xi(t)\}_{t \in \mathbb{R}}$ and the upper semicontinuity of $\{A_\xi(t)\}_{t \in \mathbb{R}}$ and the global attractor $A$ of equation (1.1) with $\xi = 0$ in §4 and §5, respectively.

2 Preliminaries

Before proving the main results, we first introduce some necessary abstract concepts in this section, such as basic definitions and properties of function spaces and attractors.

Let $\{X_t\}_{t \in \mathbb{R}}$ be a family of normed spaces with norm $\| \cdot \|_{X_t}$ and metric $d_{X_t}(\cdot, \cdot)$. The closed $R$-ball with the origin as the center and $R$ as the radius in $\{X_t\}_{t \in \mathbb{R}}$ is denoted as $\bar{B}_{X_t}(0, R) = \{u \in X_t : \|u\|_{X_t} \leq R\}$.

The Hausdorff semidistance between two nonempty sets $S_1, S_2 \subset \{X_t\}_{t \in \mathbb{R}}$ is denoted by $\text{dist}_{X_t}(S_1, S_2) = \sup_{x \in S_1} \inf_{y \in S_2} \|x - y\|_{X_t}$.

Next, we recall some notations about the family of the Hilbert spaces $D(\mathcal{A}_s^2)$ with $\mathcal{A} = -\Delta$ and $s \in \mathbb{R}$. These spaces were widely used to study attractors (see [22, 29, 32]). To simplify the notations, let $H^s = D(\mathcal{A}_s^2)$ and its inner product and norm are denoted as $(\cdot, \cdot)_{H^s} = (\mathcal{A}_s^2 \cdot, \mathcal{A}_s^2 \cdot)$ and $\| \cdot \|_{H^s} = \| \mathcal{A}_s^2 \cdot \|$, respectively.

**Lemma 2.1** ([22]) The properties of the space $H^s = D(\mathcal{A}_s^2)$ are as follows:

(i) Assume that $s_1 > s_2$, then the embedding $D(\mathcal{A}_{s_1}^2) \hookrightarrow D(\mathcal{A}_{s_2}^2)$ is compact.

(ii) Assume that $s \in [0, \frac{n}{2})$, then the embedding $D(\mathcal{A}_s^2) \hookrightarrow L^{\frac{2n}{n-2s}}(\Omega)$ is continuous.

(iii) Assume that $s_0 > s_1 > s_2$, then for any $\delta > 0$, there exists constants $\epsilon$ and $C(\epsilon) = C_\epsilon(s_0, s_1, s_2)$ such that

$$\|\mathcal{A}_{s_1}^2 u\| \leq \epsilon \|\mathcal{A}_{s_0}^2 u\| + C(\epsilon)\|\mathcal{A}_{s_2}^2 u\|.$$ 

(iv) Assume that $s_1, s_2 \in (0, 1)$ and let $u \in H^{s_1}(\Omega) \cap H^{s_2}(\Omega)$, then for any constant $\theta \in (0, 1)$, there exists a constant $C(\theta) > 0$ such that

$$\|u\|_{H^{(1-\theta)s_1 + \theta s_2}} \leq C(\theta)\|u\|_{H^{s_1}}^{1-\theta}\|u\|_{H^{s_2}}^\theta.$$ 

In addition, for any $t \in \mathbb{R}$, the time-dependent space $H_t(\Omega)$ is endowed with the norm

$$\|u\|_{H_t}^2 = \|u\|_2^2 + \epsilon(t)\|\nabla u\|_2^2.$$
In particular, assume the space $H^\alpha_t(\Omega)$, more regular than $H_t(\Omega)$, is endowed with the norm

$$\|u\|_{H^\alpha_t}^2 = \|A_t^{\frac{\alpha}{2}}u\|^2 + \varepsilon(t)\|A_t^{\frac{1+\alpha}{2}}u\|^2.$$

**Definition 2.2** A process or a two-parameter semigroup on $H_t(\Omega)$ is a family $\{U(t, \tau) : t, \tau \in \mathbb{R}, t \geq \tau\}$ of mapping $U(t, \tau) : H_\tau \to H_t$ satisfies that $U(t, \tau)u = u$ for any $u \in H_\tau$ and $U(t, s)U(s, \tau) = U(t, \tau)$ for all $t \geq s \geq \tau$.

**Definition 2.3** The process $\{U(t, \tau)\}_{t \geq \tau}$ on $H_t(\Omega)$ is said to be continuous, if for any $t \geq \tau$ the mapping $U(t, \tau) : H_\tau \to H_t$ is continuous.

**Definition 2.4** The process $\{U(t, \tau)\}_{t \geq \tau}$ on $H_t(\Omega)$ is said to be closed, if for any sequence $\{x_n\} \subset H_t(\Omega)$ the equality $U(t, \tau)x = y$ can be concluded from $x_n \to x \in H_t(\Omega)$ and $U(t, \tau)x_n \to y \in H_t(\Omega)$.

**Remark 2.5** It is obvious that if a process is continuous, then it is closed.

**Definition 2.6** For any $\sigma > 0$, let $\mathcal{D}$ be a nonempty class of all families of parameterized sets $\hat{D} = \{D(t) : t \in \mathbb{R}\} \subset \Gamma(H_t)$ such that

$$\lim_{\tau \to -\infty} \left( e^{\sigma \tau} \sup_{u \in D(\tau)} \|u\|_{H_t}^2 \right) = 0,$$

where $\Gamma(H_t)$ denotes the family of all nonempty subsets of $H_t(\Omega)$, then $\mathcal{D}$ will be called a tempered universe in $\Gamma(H_t)$.

**Definition 2.7** A process $\{U(t, \tau)\}_{t \geq \tau}$ is pullback $\mathcal{D}$-asymptotically compact on $H_t(\Omega)$, if for any $t \in \mathbb{R}$, any $\hat{D} \in \mathcal{D}$, any sequence $\{\tau_n\}_{n \in \mathbb{N}^+} \subset (-\infty, t]$ and any sequence $\{x_n\}_{n \in \mathbb{N}^+} \subset D(\tau_n) \subset H_t(\Omega)$, the sequence $\{U(t, \tau)x_n\}_{n \in \mathbb{N}^+}$ is relatively compact in $H_t(\Omega)$ when $\tau_n \to -\infty$.

**Definition 2.8** A family $\tilde{D} = \{\tilde{D}(t) : t \in \mathbb{R}\}$ is pullback absorbing for the process $\{U(t, \tau)\}_{t \geq \tau}$ on $H_t(\Omega)$, if for any $t \in \mathbb{R}$ and any bounded subsets $B \subset H_t(\Omega)$, there exists some constants $T(t, B) > 0$ such that $U(t, t - \tau)B \subset \tilde{D}(t)$ for any $\tau \geq T(t, B)$.

**Definition 2.9** A family $\tilde{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \in \Gamma(H_t)$ is pullback $\mathcal{D}$-absorbing for the process $\{U(t, \tau)\}_{t \geq \tau}$ on $H_t(\Omega)$, if for any $t \in \mathbb{R}$ and $\hat{D} \in \mathcal{D}$, there exists a $\tau_0 = \tau_0(t, \hat{D}) < t$ such that $U(t, \tau)D(\tau) \subset D_0(t)$ for any $\tau \leq \tau_0(t, \hat{D})$. 

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Definition 2.10 (5) The set $A$ is called a global attractor for the process $\{U(t, \tau)\}_{t \geq \tau}$ on $\mathcal{H}_t(\Omega)$ if the following properties hold:

(i) $A$ is compact;
(ii) $A$ is invariant; and
(iii) $A$ attracts each bounded subset of $\mathcal{H}_t(\Omega)$.

Definition 2.11 (32) A family of compact sets $\mathcal{A}_\xi = \{A_\xi(t)\}_{t \in \mathbb{R}}$ is said to be a pullback attractor for the process $\{U(t, \tau)\}_{t \geq \tau}$ on $\mathcal{H}_t(\Omega)$ if the following properties hold:

(i) $\mathcal{A}_\xi$ is invariant, i.e., $U(t, \tau)A_\xi(\tau) = A_\xi(t)$ for any $\tau \leq t$; and
(ii) $\mathcal{A}_\xi$ is pullback attracting, i.e.,
\[
\lim_{\tau \to -\infty} \text{dist}_{\mathcal{H}_t}(U(t, t - \tau)B, A_\xi(t)) = 0,
\]
for any bounded subset $B \in \mathcal{H}_t(\Omega)$.

Definition 2.12 (23, 39) A family $\mathcal{A}_D = \{A_D(t) : t \in \mathbb{R}\} \subset \Gamma(\mathcal{H}_t)$ is said to be a minimal time-dependent pullback $\mathcal{D}$-attractor for the process $\{U(t, \tau)\}_{t \geq \tau}$ on $\mathcal{H}_t(\Omega)$ if the following properties hold:

(i) the set $A_D(t)$ is compact in $\mathcal{H}_t(\Omega)$ for any $t \in \mathbb{R}$;
(ii) $A_D$ is pullback $\mathcal{D}$-attracting in $\mathcal{H}_t(\Omega)$, i.e.,
\[
\lim_{\tau \to -\infty} \text{dist}_{\mathcal{H}_t}(U(t, \tau)D(\tau), A_D(t)) = 0,
\]
for any $\hat{D} \in \mathcal{D}$ and $t \in \mathbb{R}$;
(iii) $A_D$ is invariant, i.e., $U(t, \tau)A_D(\tau) = A_D(t)$, for any $\tau \leq t$; and
(iv) $A_D$ is minimal, i.e., if $\hat{C} = \{C(t) : t \in \mathbb{R}\} \subset \Gamma(\mathcal{H}_t)$ is a family of closed sets, which is pullback $\mathcal{D}$-attracting, then $A_D(t) \subset C(t)$ for any $t \in \mathbb{R}$.

Remark 2.13 The uniqueness of the minimal pullback attractor can be derived from (iv).

García-Luengo, Marín-Rubio and Real [17] proved the following lemma, which is a direct method to obtain the existence of the minimal time-dependent pullback $\mathcal{D}$-attractors.

Lemma 2.14 (17) Assume that $\{U(t, \tau)\}_{t \geq \tau}$ is closed on $\mathcal{H}_t(\Omega)$, $\mathcal{D}$ is a universe in $\Gamma(\mathcal{H}_t)$, $\hat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \Gamma(\mathcal{H}_t)$ is a pullback $\mathcal{D}$-absorbing family for $\{U(t, \tau)\}_{t \geq \tau}$ and $\{U(t, \tau)\}_{t \geq \tau}$
is pullback $\hat{D}_0$-asymptotically compact, then the family $A_D = \{A_D(t) : t \in \mathbb{R}\}$ with

$$A_D = \bigcup_{\hat{D} \in D} \Lambda(t, \hat{D})^{H_t} = \bigcup_{\hat{D} \in D} \bigcup_{s \leq t \leq s} U(t, \tau)D(\tau)^{H_t}$$

is the minimal pullback $D$-attractor for the process $\{U(t, \tau)\}_{t \geq \tau}$. In addition, if $\hat{D}_0 \in D$, then

$$A_D(t) = \bigcap_{s \leq t \leq s} U(t, \tau)D(\tau)^{H_t} \subset D_0(t)^{H_t}$$

for all $t \in \mathbb{R}$.

**Corollary 2.15** ([10 23]) Let $D^{H_t}_F$ be the universe of fixed nonempty bounded subsets of $H_t(\Omega)$. Namely, $D^{H_t}_F$ is the class of all families $\hat{D} = \{D(t) : t \in \mathbb{R}\}$, where $D(t)$ is a fixed nonempty bounded subsets of $H_t(\Omega)$.

**Corollary 2.16** ([10 23]) Under the assumptions of Lemma 2.14, if $D^{H_t}_F \subset D$, then the minimal time-dependent pullback attractors $A^{H_t}_F$ and $A_D$ exist and satisfy $A^{H_t}_F \subset A_D$ for all $t \in \mathbb{R}$. Besides, if for some $T \in \mathbb{R}$ the set $\bigcup_{t \leq T} D_0(t)$ is a bounded subset of $H_t(\Omega)$, then $A^{H_t}_F = A_D$ for all $t \leq T$.

In order to prove the upper semicontinuity of the time-dependent pullback attractors and the global attractor, it is necessary to introduce the following lemmas.

Let $S(t) : X_t \to X_t$ be a $C_0$-semigroup on $\{X_t\}_{t \in \mathbb{R}}$ and assume there exists a global attractor $A$ for $S(t)$. Then we use a non-autonomous term depending on a small parameter $\xi \in (0, \xi_0]$ to perturb the semigroup $S(t)$, so as to obtain a non-autonomous dynamical system driven by the process $\{U_\xi(t, \tau)\}_{t \geq \tau}$. Moreover, for any $t \in \mathbb{R}$, $\tau \in \mathbb{R}^+$ and $x \in X_t$, assume that

$$\lim_{\xi \to 0} d_{X_t} (U_\xi(t, t-\tau)x, S(\tau)x) = 0 \quad (2.1)$$

uniformly on any bounded set of $\{X_t\}_{t \in \mathbb{R}}$.

**Lemma 2.17** ([11]) Assume that (2.1) holds and for any small $\xi \in (0, \xi_0]$, there exists a pullback attractor $A_\xi = \{A_\xi(t) : t \in \mathbb{R}\}$ and a compact set $K \subset \{X_t\}_{t \in \mathbb{R}}$ such that

$$\lim_{\xi \to 0} d_{X_t} (A_\xi(t), K) = 0, \quad (2.2)$$

then the upper semicontinuity of the attractors holds, that is,

$$\lim_{\xi \to 0} d_{X_t} (A_\xi(t), A) = 0. \quad (2.3)$$
Lemma 2.18 (32) Assume the family \( D_\xi = \{ D_\xi(t) \}_{t \in \mathbb{R}} \) is pullback absorbing for process \( \{ U(t, \tau) \}_{t \geq \tau} \) on \( \{ X_t \}_{t \in \mathbb{R}} \), and for any \( \xi \in (0, \xi_0] \), the set \( K_\xi = \{ K_\xi(t) \}_{t \in \mathbb{R}} \) is a family of compact sets in \( \{ X_t \}_{t \in \mathbb{R}} \).

Suppose \( U_\xi(\cdot, \cdot) = U_{1, \xi}(\cdot, \cdot) + U_{2, \xi}(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \times X_t \to X_t \) such that

(i) for any \( t \in \mathbb{R}, \xi \in (0, \xi_0] \), \( x_{t-\tau} \in D_\xi(t-\tau) \) and \( \tau > 0 \) the following inequality holds

\[
\| U_{1, \xi}(t, t-\tau)x_{t-\tau} \|_{X_t} \leq \Phi(t, \tau),
\]

where \( \Phi(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+ \) and satisfies \( \lim_{\tau \to \infty} \Phi(t, \tau) = 0 \); and

(ii) for any \( t \in \mathbb{R}, \xi \in (0, \xi_0] \) and \( T \geq 0 \), the set \( \bigcup_{0 \leq \tau \leq T} U_{2, \xi}(t, t-\tau)D_\xi(t-\tau) \) is bounded and there exists a \( T_{D_\xi}(t) > 0 \) such that

\[
U_{2, \xi}(t, t-\tau)D_\xi(t-\tau) \subset K_\xi(t)
\]

for all \( \tau \geq T_{D_\xi}(t) \) and there exists a compact set \( K \subset X_t \) such that

\[
\lim_{\xi \to 0} \text{dist}_{X_t}(K_\xi(t), K) = 0.
\]

Then for each \( \xi \in (0, \xi_0] \) there exists a pullback attractor and (2.2) holds.

3 Existence and uniqueness of solutions

Studying the attractors of an equation generally requires proving the existence and uniqueness of solutions. Thus in this section we shall first discuss the existence and uniqueness of weak solutions to problem (1.1).

Definition 3.1 A weak solution to problem (1.1) is a function \( u \in C([\tau, T], \mathcal{H}_t(\Omega)) \) for any \( \tau < T \), with \( u(x, \tau) = u_\tau, \) and such that

\[
\frac{d}{dt} [(u(t), \varphi) + \varepsilon(t)(\nabla u(t), \nabla \varphi)] + (2a(l(u)) - \varepsilon'(t)) (\nabla u(t), \nabla \varphi) = 2(f(u(t)), \varphi) + 2\xi(h(t), \varphi)
\]

for all test functions \( \varphi \in H_0^1(\Omega) \).

Remark 3.2 The equation (3.1) should be understood in the sense of the generalized function space \( \mathcal{D}'(\tau, +\infty) \).
**Corollary 3.3** If \(u(x,t)\) is a weak solution of problem (1.1), then the following energy equality holds

\[
\|u(t)\|^2 + \varepsilon(t)\|\nabla u(t)\|^2 + \int_t^s (2a(l(u)) - \varepsilon'(r)) \|\nabla u(r)\|^2 \, dr \\
= \|u(s)\|^2 + \varepsilon(s)\|\nabla u(s)\|^2 + 2\int_s^t (f(u(r)), u(r)) \, dr + 2\xi \int_s^t (h(r), u(r)) \, dr,
\]

for all \(\tau \leq s \leq t\).

We first prove the following lemma.

**Lemma 3.4** If the function \(u\) is bounded in \(L^\infty(\tau, T; H^1_0(\Omega))\), then \(f(u)\) is bounded in \(L^{\gamma q}(\tau, T; L^{\gamma q}(\Omega))\), where \(q = \frac{2(N+2)}{N\gamma}\) with \(0 < \gamma < \min\left\{\frac{N+2}{N-2}, 2+\frac{4}{N}\right\}\).

**Proof.** From the assumptions (1.8), (1.9), (1.11) and \(p \leq \frac{4}{\frac{N-2}{N}}\), it follows that there exists a constant satisfying \(0 < \gamma < \min\left\{\frac{N+2}{N-2}, 2+\frac{4}{N}\right\}\) such that

\[
|f(u)| \leq |f_0(u)| + |f_1(u)| \leq C(|u|^{\gamma} + 1).
\]

Let \(\gamma q = \frac{2N+4}{N}\), then we can conclude that \(q > 1\) and \(2 < \gamma q < \frac{2N}{N-2}\). Meanwhile, take \(\theta = \frac{N-2}{N}\), then \(\theta \in (0,1)\). Letting \(\gamma q = \frac{2N\theta}{N-2} + 2(1-\theta)\) and using (3.3) and the Hölder inequality, we obtain

\[
\|f(u)\|_{L^{\gamma q}(\Omega)}^q \leq C \int_\Omega |u|^{\gamma q} \, dx \\
\leq C \int_\Omega |u|^{\frac{2N\theta}{N-2}} \, dx \\
= C \int_\Omega |u|^{\frac{2N\theta}{N-2} + 2(1-\theta)} \, dx \\
\leq C \int_\Omega |u|^{\frac{2N\theta}{N-2}} \, dx (\int_\Omega |u|^{2} \, dx)^{1-\theta}.
\]

Then from (3.4) and since when \(N \geq 3\) the embedding \(H^1_0(\Omega) \subset L^{\frac{2N}{N-2}}(\Omega)\) is continuous (see [16]), the following inequalities hold

\[
\|f(u)\|_{L^{\gamma q}(\Omega)}^q \leq C + C\|u\|_{L^{\frac{2N\theta}{N-2}}(\Omega)}^{\frac{2N\theta}{N-2}} \|u\|_{L^{\gamma q}(\Omega)}^{2(1-\theta)} \\
\leq C + C\|u\|_{L^{\frac{2N}{N-2}}(\Omega)}^{\frac{2N\theta}{N-2}} \|u\|_{L^{\gamma q}(\Omega)}^{2(1-\theta)} \\
= C + C\|u\|_{L^{\frac{2N}{N-2}}(\Omega)}^{\frac{2N\theta}{N-2}} \\
\leq C + C\|\nabla u\|_{L^{\frac{2N}{N-2}}(\Omega)}^{2}.
\]

The proof is complete by (3.5). □

Now we prove the existence and uniqueness of the solutions to problem (1.1).
Theorem 3.5 Assume that $a(\cdot)$ is a local Lipschitz continuous function and satisfies $1.4$, $l(\cdot)$ is given in $1.5$, $f \in C(\mathbb{R}, \mathbb{R})$ and satisfies $1.6 - 1.11$, $h \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ and the initial value $u_\tau \in H_t(\Omega)$, then for any $\tau \in \mathbb{R}$ and $t \geq \tau$, there exists a weak solution to problem $1.11$. Moreover, the solution $u$ depends continuously on its initial value.

Proof. Consider the approximate solution $u_k(t, \tau; u_\tau) = \sum_{j=1}^{k} r_{k,j}(t)\omega_j(x)$, where $j, k \in \mathbb{N}^+$, $\{\omega_j\}_{j=1}^{\infty}$ is a basis of $H^2(\Omega) \cap H^1_0(\Omega)$ and orthonormal in $L^2(\Omega)$. The Faedo-Galerkin method needs to find an approximate sequence $\{u_k\}$ so that the following approximate system holds:

$$
\begin{aligned}
\frac{d}{dt}[u_k(t, \omega_j) + \varepsilon(t)(\nabla u_k(t), \nabla \omega_j)] + (2a(l(u_k)) - \varepsilon'(t))(\nabla u_k(t), \nabla \omega_j) \\
= 2(f(u_k(t)), \omega_j) + 2\xi(h(t), \omega_j), \quad \forall t \in [\tau, +\infty),
\end{aligned}
$$

(3.6)

Step 1: (A priori estimate for $u_k$) Multiplying (3.6) by the test function $\gamma_{k,j}(t)$ and then summing $j$ from 1 to $k$, we obtain

$$
\frac{d}{dt}(\|u_k(t)\|^2 + \varepsilon(t)\|\nabla u_k(t)\|^2) + (2a(l(u_k)) - \varepsilon'(t))\|\nabla u_k(t)\|^2
= 2(f(u_k(t)), u_k(t)) + 2\xi(h(t), u_k(t)).
$$

(3.7)

From (1.7) there exists some constants $0 < \tilde{\eta} < \frac{3}{2}m\lambda_1$ and $C_0 > 0$ such that

$$
2(f(u_k(t)), u_k(t)) \leq \left(\frac{3}{2}m\lambda_1 - 2\tilde{\eta}\right)\|u_k(t)\|^2 + 2C_0.
$$

(3.8)

Using the Young and the Poincaré inequalities, it follows that

$$
2\xi(h(x,t), u_k(t)) \leq \frac{2\varepsilon^2}{m\lambda_1}\|h(x,t)\|^2 + \frac{m}{2}\|\nabla u_k(t)\|^2.
$$

(3.9)

Substituting (3.8) and (3.9) into (3.7), and then by (1.4) and the Poincaré inequality, we can derive

$$
\frac{d}{dt}(\|u_k(t)\|^2 + \varepsilon(t)\|u_k(t)\|^2) + (2\tilde{\eta} - \varepsilon'(t))\|\nabla u_k(t)\|^2 \leq \frac{2\varepsilon^2}{m\lambda_1}\|h(x,t)\|^2 + 2C_0.
$$

(3.10)

Integrating (3.10) from $\tau$ to $t$, we deduce

$$
\|u_k(t)\|^2 + \varepsilon(t)\|\nabla u_k(t)\|^2 + \int_{\tau}^{t}(2\tilde{\eta} - \varepsilon'(s))\|\nabla u_k(s)\|^2 ds
\leq \|u_k(\tau)\|^2 + \varepsilon(\tau)\|\nabla u_k(\tau)\|^2 + \frac{2\varepsilon^2}{m\lambda_1}\int_{\tau}^{t}\|h(x,s)\|^2 ds + 2C_0(t - \tau).
$$

(3.11)
From (3.11) and notice that $\varepsilon(t)$ is a decreasing bounded function, we obtain that for any $T > t$, 
\{u_k\} is bounded in $L^\infty(\tau, T; L^2(\Omega)) \cap L^\infty(\tau, T; H^1_0(\Omega)) \cap L^2(\tau, T; H^1_0(\Omega))$, then we deduce
\[
\{u_k\} \text{ is bounded in } L^\infty(\tau, T; H^t_t(\Omega)). \tag{3.12}
\]

Multiplying the approximate system (3.16) by $\gamma'_{k,j}(t)$ and summing $j$ from 1 to $k$, then by (1.4), we obtain
\[
\|(u_k(t))_t\|^2 + \varepsilon(t) \|\nabla (u_k(t))_t\|^2 + \dfrac{m}{2} \dfrac{d}{dt} \|\nabla u_k(t)\|^2 \leq (f(u_k(t)), u_k(t)) + \varepsilon(h(x,t), u_k(t)). \tag{3.13}
\]

Using the Young inequality, we obtain
\[
\dfrac{1}{2} \|(u_k(t))_t\|^2 + \varepsilon(t) \|\nabla (u_k(t))_t\|^2 + \dfrac{m}{2} \dfrac{d}{dt} \|\nabla u_k(t)\|^2 \leq \dfrac{1}{4} \|f(u_k(t))\|^2 + \dfrac{\varepsilon^2}{4} \|h(x,t)\|^2. \tag{3.14}
\]

Integrating (3.14) from $\tau$ to $t$, we deduce
\[
\int_\tau^t \left(\dfrac{1}{2} \|(u_k(s))_s\|^2 + \varepsilon(s) \|\nabla (u_k(s))_s\|^2\right) ds + \dfrac{m}{2} \|\nabla u_k(t)\|^2 \\
\leq \dfrac{m}{2} \|\nabla u_k(\tau)\|^2 + \dfrac{1}{4} \int_\tau^t \|f(u_k(s))\|^2 ds + \dfrac{\varepsilon^2}{4} \int_\tau^t \|h(x,s)\|^2 ds. \tag{3.15}
\]

By Lemma 3.4, it follows that
\[
\{f(u_k)\} \text{ is bounded in } L^q(\tau, T; H^1_0(\Omega)). \tag{3.16}
\]

Then from (3.15) and (3.16), through similar calculations and estimations to (3.12), we deduce
\[
\{\partial_t u_k\} \text{ is bounded in } L^\infty(\tau, T; H^t_t(\Omega)). \tag{3.17}
\]

From (3.12), (3.16), (3.17), the compactness arguments and the Aubin-Lions lemma (see [19]), we derive that there exists a subset of \{u_k\} (still marked as \{u_k\}), $u \in L^\infty(\tau, T; H^t_t(\Omega)) \cap L^2(\tau, T; H^1_0(\Omega))$ and $\partial_t u \in L^\infty(\tau, T; H^t_t(\Omega))$ such that
\[
u_k \rightharpoonup u \quad \text{weakly-star in } L^\infty(\tau, T; H^t_t(\Omega)); \tag{3.18}
\]
\[
u_k \rightharpoonup u \quad \text{weakly in } L^2(\tau, T; H^1_0(\Omega)); \tag{3.19}
\]
\[
f(u_k) \rightharpoonup f(u) \quad \text{weakly in } L^q(\tau, T; L^q(\Omega)); \tag{3.20}
\]
\[
a(l(u_k)) u_k \rightharpoonup a(l(u)) u \quad \text{weakly in } L^2(\tau, T; H^1_0(\Omega)); \tag{3.21}
\]
\[
\partial_t u_k \rightharpoonup \partial_t u \quad \text{weakly in } L^2(\tau, T; H^t_t(\Omega)); \tag{3.22}
\]
\[ u_k \to u \quad \text{a.e.} \quad (x, t) \in \Omega \times [\tau, +\infty). \tag{3.23} \]

**Step 2: (Verify the continuity of \( u \))** Let \( \bar{u} = u_k - u \), then \( \bar{u} \) satisfies
\[
\frac{d}{dt} \| \bar{u} \|^2_{H_t} = \frac{d}{dt} \left( \| \bar{u} \|^2 + \varepsilon(t) \| \nabla \bar{u} \|^2 \right) + 2(a(l(u_k)) - \varepsilon'(t)) \| \nabla \bar{u} \|^2 \\
= 2(a(l(u)) - a(l(u_k))) (\nabla u, \nabla \bar{u}) + 2(f(u_k) - f(u), \bar{u}). \tag{3.24} 
\]

Noting that the function \( a(\cdot) \) is local Lipschitz continuous and by the Young and the Cauchy inequalities, we can obtain
\[
2(a(l(u)) - a(l(u_k))) (\nabla u, \nabla \bar{u}) \leq 2m \| \nabla u \|^2 + \frac{(La(R))^2 \| I \|^2 \| \nabla u \|^2 \| u_k - u \|^2}{2m}, \tag{3.25} 
\]
where \( L_a(R) \) is the Lipschitz constant of the function \( a(\cdot) \) in \([-R, R]\).

By (1.6), the Hölder, the Poincaré inequalities, definition of the space \( H_t(\Omega) \) and notice that \( p \leq \frac{4}{N-2} \), then for any \( 0 < \alpha < \min \left\{ 1, \frac{4-(N-2)p}{2} \right\} \), we obtain
\[
2(f(u_k) - f(u), \bar{u}) \leq C \int_\Omega (1 + |u_k|^p + |u|^p) |\bar{u}|^2 \, dx \\
\leq C \left( \int_\Omega (1 + |u_k|^p + |u|^p)^{\frac{2N}{N-2(p+1)}} dx \right)^{\frac{N-2(p+1)}{2N}} \left( \int_\Omega |\bar{u}|^{\frac{2N}{N-2}} dx \right)^{\frac{N-2(p+1)}{2N}} \\
\leq C(1 + \| A^\frac{1}{p} u_k \|^p + \| A^\frac{1}{p} u \|^p) \| A^\frac{1}{p} \bar{u} \| \| A^\frac{1}{p} u \| \\
\leq C(1 + \| A^\frac{1}{p} u_k \|^p + \| A^\frac{1}{p} u \|^p)((\| \bar{u} \|^2 + \varepsilon(t) \| \nabla \bar{u} \|^2) \\
\leq C(1 + \| A^\frac{1}{p} u_k \|^\frac{4}{N-2} + \| A^\frac{1}{p} u \|^\frac{4}{N-2}) \| \bar{u} \|^2_{H_t}. \tag{3.26} 
\]

Substituting (3.25) and (3.26) into (3.24), from (1.4), \( \varepsilon(t) \) is a decreasing function and using the Sobolev embedding theorem (see [20]), we can derive
\[
\frac{d}{dt} \| \bar{u} \|^2_{H_t} = \frac{d}{dt} \left( \| \bar{u} \|^2 + \varepsilon(t) \| \nabla \bar{u} \|^2 \right) \leq C \left( \| \bar{u} \|^2 + \varepsilon(t) \| \nabla \bar{u} \|^2 \right). \tag{3.27} 
\]

Applying the generalized Gronwall lemma (see [24, 25, 26]) to (3.27), we can conclude
\[
\| \bar{u} \|^2_{H_t} = \| \bar{u}(t) \|^2 + \varepsilon(t) \| \nabla \bar{u}(t) \|^2 \leq e^{C(t-\tau)}(\| \bar{u}_\tau \|^2 + \varepsilon(\tau) \| \nabla \bar{u}_\tau \|^2). \tag{3.28} 
\]

**Step 3: (Verify the initial value \( u_\tau \))** Choosing a suitable test function \( \varphi \in C^1([\tau, T]; H^1(\Omega)) \) with \( \varphi(T) = 0 \), then it follows that
\[
\int^T_\tau - (u, \varphi) \, ds + \int^T_\tau \int_\Omega \varepsilon(s) \nabla u_s \nabla \varphi \, dx \, ds = \int^T_\tau \int_\Omega a(l(u))(\Delta u)\varphi \, dx \, ds \\
- \int^T_\tau \int_\Omega (f(u) + \xi h(x, s)) \varphi \, dx \, ds = (u(\tau), \varphi(\tau)). \tag{3.29} 
\]
In the same way as in the Faedo-Galerkin approximations, we conclude
\[
\int_\tau^T -(u_k, \varphi')ds + \int_\tau^T \int_\Omega \varepsilon(s) \nabla(u_k)_s \nabla \varphi dxds - \int_\tau^T \int_\Omega a(l(u_k))(\Delta u_k)\varphi dxds - \int_\tau^T (f(u_k) + \xi h(x, s))\varphi ds = (u_k(\tau), \varphi(\tau)).
\]

(3.30)

Taking limits as \(k \to \infty\) in (3.30) and since \(u_k(\tau) \to u_\tau\), we obtain
\[
\int_\tau^T -(u, \varphi')ds + \int_\tau^T \int_\Omega \varepsilon(s) \nabla u_s \nabla \varphi dxds - \int_\tau^T \int_\Omega a(l(u))(\Delta u)\varphi dxds - \int_\tau^T (f(u) + \xi h(x, s))\varphi ds = (u_\tau(\tau), \varphi(\tau)).
\]

(3.31)

Then \(u(\tau) = u_\tau\) directly holds.

From the above estimations, we can obtain that \(u\) is a weak solution of problem (1.1). □

**Theorem 3.6** Under the assumptions of Theorem 3.5, if the weak solution of problem (1.1) exists, then it is a unique solution.

**Proof.** Assuming that \(u^1\) and \(u^2\) are two solutions corresponding to the initial values \(u^1_\tau\) and \(u^2_\tau\), respectively, and satisfying
\[
\begin{cases}
  u^1_t - \varepsilon(t) \Delta u^1_t - a(l(u^1)) \Delta u^1 = f(u^1) + \xi h(x, t) & \text{in } \Omega \times (\tau, \infty), \\
  u^1 = 0 & \text{on } \partial \Omega \times (\tau, \infty), \\
  u^1(x, \tau) = u^1_\tau(x), & x \in \Omega,
\end{cases}
\]

(3.32)

and
\[
\begin{cases}
  u^2_t - \varepsilon(t) \Delta u^2_t - a(l(u^2)) \Delta u^2 = f(u^2) + \xi h(x, t) & \text{in } \Omega \times (\tau, \infty), \\
  u^2 = 0 & \text{on } \partial \Omega \times (\tau, \infty), \\
  u^2(x, \tau) = u^2_\tau(x), & x \in \Omega.
\end{cases}
\]

(3.33)

Subtracting (3.33) from (3.32) and taking \(u = u^1 - u^2\) as the test function of the resulting equation, we derive
\[
\frac{d}{dt} \left[ ||u||^2 + \varepsilon(t)||\nabla u||^2 \right] + 2(a(l(u^1)) - \varepsilon'(t))||\nabla u||^2 = 2(a(l(u^2)) - a(l(u^1)))(\nabla u^2, \nabla u) + 2(f(u^1) - f(u^2), u).
\]

(3.34)

Performing similar calculations to the proof of Theorem 3.5 can obtain the following inequality
\[
||u||^2_{H_\tau} = ||u(t)||^2 + \varepsilon(t)||\nabla u(t)||^2 \leq C(t-\tau)(||u_\tau||^2 + \varepsilon(\tau)||u_\tau||^2).
\]

(3.35)

Consequently, the uniqueness of the solution follows readily. □
Corollary 3.7 Thanks to Theorems 3.5 and 3.6, problem (1.1) has a continuous process

\[ U(t, \tau) : \mathcal{H}_t(\Omega) \to \mathcal{H}_t(\Omega) \]

with \( U(t, \tau)u_\tau \) being the unique weak solution of (1.1) respects to initial datum \( u_\tau \).

4 Existence of the minimal time-dependent pullback \( \mathcal{D} \)-attractors

In this section, we will verify the existence of the minimal time-dependent pullback \( \mathcal{D} \)-attractors for the process \( \{U(t, \tau)\}_{t \geq \tau} \) in \( \mathcal{H}_t(\Omega) \). To prove it, we will check the four properties mentioned in Definition 2.12, therefore we first estimate the following lemma.

Lemma 4.1 Under the assumptions of Theorems 3.5 and 3.6, then for any \( t \geq \tau \), the solution of problem (1.1) satisfies

\[ \|u(t)\|^2_{\mathcal{H}_t} = \|u(t)\|^2 + \varepsilon(t)\|\nabla u(t)\|^2 \leq e^{-\sigma \tau} \|u_{t-\tau}\|^2_{\mathcal{H}_t} + \frac{\xi e^{-\sigma t}}{\eta} \int_{-\infty}^{t} e^{\sigma s}\|h(x,s)\|^2 ds + \frac{2C_1}{\sigma}, \]

where \( 0 < \sigma < \delta_1 < \min \left\{ \eta, \frac{-\varepsilon'(t)}{\varepsilon(t)} \right\} \) with \( 0 < \eta < m\lambda_1 \).

Proof. From (3.2), it easily follows that the weak solution \( u \) satisfies

\[
\frac{d}{dt} \left[ \|u(t)\|^2 + \varepsilon(t)\|\nabla u(t)\|^2 \right] + (2a(l(u(t))) - \varepsilon'(t)) \|\nabla u(t)\|^2 \\
= 2(f(u(t)), u(t)) + 2\xi(h(x,t), u(t)).
\] (4.2)

By (1.4), we obtain

\[ 2m\|\nabla u(t)\|^2 \leq 2a(l(u))\|\nabla u(t)\|^2. \] (4.3)

Besides, we conclude from (1.7) there exists \( 0 < \eta < m\lambda_1 \) such that

\[ (f(u(t)), u(t)) \leq (m\lambda_1 - \eta) \|u(t)\|^2 + C_1. \] (4.4)

Using the Young and the Cauchy inequalities, we can derive

\[ 2\xi(h(x,t), u(t)) \leq \frac{\xi}{\eta} \|h(x,t)\|^2 + \xi\eta\|u(t)\|^2. \] (4.5)

Inserting (4.3) - (4.5) into (4.2), we obtain

\[
\frac{d}{dt} \left[ \|u(t)\|^2 + \varepsilon(t)\|\nabla u(t)\|^2 \right] + (2m - \varepsilon'(t))\|\nabla u(t)\|^2 \\
\leq 2(m\lambda_1 - \eta) \|u(t)\|^2 + 2C_1 + \frac{\xi}{\eta} \|h(x,t)\|^2 + \xi\eta\|u(t)\|^2.
\] (4.6)
Then by the Poincaré inequality, we can derive
\[ \frac{d}{dt} \left[ \|u(t)\|^2 + \varepsilon(t)\|\nabla u(t)\|^2 \right] + (2\eta - \xi \eta)\|u(t)\|^2 - \varepsilon'(t)\|\nabla u(t)\|^2 \leq 2C_1 + \frac{\xi}{\eta}\|h(x,t)\|^2. \tag{4.7} \]

Taking \( \xi \) is so small that \( (2\eta - \xi \eta)\|u(t)\|^2 > \eta\|u(t)\|^2 \), which substituted into (4.7) leads to
\[ \frac{d}{dt} \left[ \|u(t)\|^2 + \varepsilon(t)\|\nabla u(t)\|^2 \right] + \eta\|u(t)\|^2 - \varepsilon'(t)\|\nabla u(t)\|^2 \leq 2C_1 + \frac{\xi}{\eta}\|h(x,t)\|^2. \]

Taking \( 0 < \sigma < \delta_1 < \min \left\{ \eta, \frac{\varepsilon'(t)}{\sigma \varepsilon(t)} \right\} \) with \( 0 < \eta < m\lambda_1 \), then we can derive
\[ \frac{d}{dt} \left[ \|u(t)\|^2 + \varepsilon(t)\|\nabla u(t)\|^2 \right] + \delta_1(\|u(t)\|^2 + \varepsilon(t)\|\nabla u(t)\|^2) \leq 2C_1 + \frac{\xi}{\eta}\|h(x,t)\|^2. \tag{4.8} \]

Then by a simple calculation, we can conclude
\[ \frac{d}{dt} (e^{\sigma t}(\|u(t)\|^2 + \varepsilon(t)\|\nabla u(t)\|^2)) + (\delta_1 - \sigma)e^{\sigma t}(\|u(t)\|^2 + \varepsilon(t)\|\nabla u(t)\|^2) \leq 2C_1 e^{\sigma t} + \frac{\xi}{\eta} e^{\sigma t}\|h(x,t)\|^2. \tag{4.9} \]

Integrating (4.9) from \( t - \tau \) to \( t \), we deduce
\[
\begin{align*}
\|u(t)\|^2 + \varepsilon(t)\|\nabla u(t)\|^2 &+ (\delta_1 - \sigma)e^{-\sigma \tau} \int_{t-\tau}^{t} e^{\sigma s} (\|u(s)\|^2 + \varepsilon(s)\|\nabla u(s)\|^2) \, ds \\
&\leq e^{-\sigma \tau}(\|u_{t-\tau}\|^2 + \varepsilon(t - \tau)\|\nabla u_{t-\tau}\|^2) + \frac{\xi}{\eta} e^{-\sigma \tau} \int_{-\infty}^{t} e^{\sigma s}\|h(x,s)\|^2 \, ds + \frac{2C_1}{\sigma}. \tag{4.10}
\end{align*}
\]

From the definition of \( \mathcal{H}_t(\Omega) \) and (1.23), (1.11) follows directly. \( \square \)

**Definition 4.2 (Tempered universe)** For each \( \sigma > 0 \), let \( \mathcal{D}^\mathcal{H}_t \) be the class of all families of nonempty subsets \( \hat{D} = \{ D(t) : t \in \mathbb{R} \} \subset \Gamma(\mathcal{H}_t) \) such that
\[
\lim_{\tau \to -\infty} \left( e^{\sigma \tau} \sup_{v \in D(\tau)} \|v\|_{\mathcal{H}_t}^2 \right) = 0.
\]

**Remark 4.3** The universe \( \mathcal{D}^L_\mathcal{F} \subset \mathcal{D}^\mathcal{H}_ \alpha \) and \( \mathcal{D}^H_\alpha \) is inclusion-closed, which means that if \( \hat{D} \in \mathcal{D}^\mathcal{H}_ \alpha \) and \( \hat{D}' = \{ D'(t) : t \in \mathbb{R} \} \subset \Gamma(\mathcal{H}_t) \) satisfies that \( D'(t) \subset D(t) \) for all \( t \in \mathbb{R} \), then \( \hat{D}' \in \mathcal{D}^\mathcal{H}_ \alpha \).

Based on the above results, adding some suitable growth conditions to the function \( h \) of problem (1.1), then we obtain the existence of the \( \mathcal{D}^\mathcal{H}_t \)-absorbing family of process \( \{ U(t,\tau) \}_{t \geq \tau} \) on \( \mathcal{H}_t(\Omega) \).

**Lemma 4.4** Under the assumptions of Theorems 3.5 and 3.6, if \( h(x,t) \) also satisfies
\[
\int_{-\infty}^{0} e^{\sigma s}\|h(x,s)\|^2 \, ds < +\infty, \tag{4.11}
\]
for some $0 < \sigma < \delta_1 < \min \left\{ \eta, \frac{\varepsilon'(t)}{\varepsilon(t)} \right\}$ with $0 < \eta < m\lambda_1$. Then the family $\hat{D}_0 = \{D_0(t) : t \in \mathbb{R} \}$ with $D_0(t) = \bar{B}_{\mathcal{H}_t} (0, \rho \varepsilon(t))$, the closed ball in $\mathcal{H}_t(\Omega)$ of centre zero and radius $\rho \varepsilon(t)$, where

$$
\rho \varepsilon(t) = C_2 \left( \xi e^{-\sigma t} \int_{-\infty}^{t} e^{\sigma s} \| h(x,s) \|^2 ds + 1 \right)
$$

(4.12)

and $C_2 = \max \left\{ \frac{2}{\eta}, \frac{4C_1}{\sigma} \right\}$ is pullback $\mathcal{D}_{\sigma}^{\mathcal{H}_t}$-absorbing family of process $\{U(t, \tau)\}_{t \geq \tau}$ on $\mathcal{H}_t(\Omega)$. Moreover, $\hat{D}_0 \in \mathcal{D}_{\sigma}^{\mathcal{H}_t}$.

\textbf{Proof.} From the prove of Lemma 4.1 and Definition 4.2 we can derive that Lemma 4.4 follows directly.

To prove the existence of the minimal time-dependent pullback $\mathcal{D}_{\sigma}^{\mathcal{H}_t}$-attractors for the process $\{U(t, \tau)\}_{t \geq \tau}$, we shall check the compactness of process $\{U(t, \tau)\}_{t \geq \tau}$ on $\mathcal{H}_t(\Omega)$.

\textbf{Lemma 4.5} Under the assumptions of Lemma 4.3, for any $t \in \mathbb{R}$ and $\hat{D} \in \mathcal{D}_{\sigma}^{\mathcal{H}_t}$, there exists $\tau_1(t, \hat{D}) < t - 2$ such that for any $\tau \leq \tau_1(t, \hat{D})$ and any $u_\tau \in D(\tau)$, the following inequalities hold:

$$
\| u(r; \tau, u_\tau) \|^2_{\mathcal{H}_t} \leq \rho_1(t), \quad \forall r \in [t-2, t],
$$

(4.13)

and

$$
\int_{r-1}^{r} \| \nabla u(s; \tau, u_\tau) \|^2 ds \leq \rho_2(t), \quad \forall r \in [t-1, t],
$$

(4.14)

where

$$
\rho_1(t) = C_2 \left( 1 + \xi e^{-(\delta_1 t - \tau)} \int_{-\infty}^{t} e^{\sigma s} \| h(x,s) \|^2 ds \right),
$$

$$
\rho_2(t) = \frac{1}{2\eta + L} \left( \rho_1(t) + \frac{2\xi^2}{m\lambda_1} \max_{r \in [t-1, t]} \int_{r-1}^{r} \| h(x,s) \|^2 ds + 2C_0 \right),
$$

$\eta$, $C_0$ and $C_2$ are the same as they are in Theorem 3.5 and Lemma 4.1, respectively.

\textbf{Proof.} Let $\tau \leq \tau_1(t, \hat{D}) < t - 2$, then from Definition 4.2 and similar calculations to Lemma 4.1 we can derive that (4.13) holds. Then through similar calculations to (3.11), we can obtain

$$
\frac{d}{ds} (\| u_k(s) \|^2 + \varepsilon(s) \| \nabla u_k(s) \|^2) + (2\eta - \varepsilon'(t)) \| \nabla u_k(s) \|^2 \leq \frac{2\xi^2}{m\lambda_1} \| h(x,s) \|^2 + 2C_0.
$$

(4.15)

Integrating (4.15) from $r - 1$ to $r$, we deduce

$$
\| u_k(r) \|^2 + \varepsilon(r) \| \nabla u_k(r) \|^2 + \int_{r-1}^{r} (2\eta - \varepsilon'(s)) \| \nabla u_k(s) \|^2 ds.
$$

(4.16)

$$
\leq \| u_k(r-1) \|^2 + \varepsilon(r-1) \| \nabla u_k(r-1) \|^2 + \frac{2\xi^2}{m\lambda_1} \int_{r-1}^{r} \| h(x,s) \|^2 ds + 2C_0.
$$

By (1.3), we obtain

$$
\int_{r-1}^{r} (2\eta - \varepsilon'(s)) \| \nabla u_k(s) \|^2 ds \leq (2\eta + L) \int_{r-1}^{r} \| \nabla u_k(s) \|^2 ds.
$$

(4.17)
Then from (4.13), (4.16) and (4.17), we derive that for any $k \geq 1$

$$
\int_{r-1}^{r} \|\nabla u_k(s)\|^2 ds \leq \rho_2(t) \quad \forall r \in [t-1, t], \tau \leq \tau_1(t, \hat{D}), u_\tau \in D(\tau).
$$

(4.18)

From the proof of Theorem 3.5, we can conclude that $u_k \to u(t; \tau, u_\tau)$ weakly in $L^\infty(\tau, T; \mathcal{H}_t(\Omega)) \cap L^2(\tau, T; H^1_0(\Omega))$ for all $r \in [t-1, t]$. Then according to (4.13) and (4.18), the inequality (4.14) follows directly.

Now we will use the energy method to prove that the process $\{U(t, \tau)\}_{t \geq \tau}$ is pullback $\mathcal{D}^{H_t}$-asymptotically compact.

Lemma 4.6 Under the assumptions of Lemma 4.4, the process $\{U(t, \tau)\}_{t \geq \tau}$ on $\mathcal{H}_t(\Omega)$ is pullback $\mathcal{D}^{H_t}$-asymptotically compact.

Proof. Let $t \in \mathbb{R}$, $\hat{D} \in \mathcal{D}^{H_t}$, $\{\tau_k\} \subset (-\infty, t-2]$ with $\tau_k \to -\infty$, and $u_{\tau_k} \in D(\tau_k)$ for all $k \in \mathbb{N}^+$. According to Definition 2.7 if it is proved that the sequence $\{u(t; \tau_k, u_{\tau_k})\}$ is relatively compact in $\mathcal{H}_t$, then $u^k(t) = \{u(t; \tau_k, u_{\tau_k})\}$ is pullback $\mathcal{D}^{H_t}$-asymptotically compact.

By similar calculations in the proof of Theorems 3.5 and 3.6 and Lemma 4.5, it follows that there exists $\tau_1(t, \hat{D}) < t - 2$ such that $\{u^k\}_{k \geq k_1 \geq 1}$ is bounded in $L^\infty(t-2, t; \mathcal{H}_t(\Omega)) \cap L^2(t-2, t; H^1_0(\Omega))$, $\{f(u^k)\}_{k \geq k_1 \geq 1}$ is bounded in $L^q(t-2, t; L^q(\Omega))$ and $\{\partial_t u^k\}_{k \geq k_1 \geq 1}$ is bounded in $L^2(t-2, t; \mathcal{H}_t(\Omega))$. Then, by the Aubin-Lions compactness lemma, we derive that there exists $u \in L^2(t-2, t; \mathcal{H}_t(\Omega))$ with $\partial_t u \in L^2(t-2, t; \mathcal{H}_t(\Omega))$ such that

$$
u^k \to u \quad \text{weakly-star in } L^\infty(t-2, t; \mathcal{H}_t(\Omega));
$$

(4.19)

$$
u^k \to u \quad \text{weakly in } L^2(t-2, t; H^1_0(\Omega));
$$

(4.20)

$$
f(u^k) \to f(u) \quad \text{weakly in } L^q(t-2, t; L^q(\Omega));
$$

(4.21)

$$
a(l(u^k))u^k \to a(l(u))u \quad \text{weakly in } L^2(t-2, t; H^1_0(\Omega));
$$

(4.22)

$$
\partial_t u^k \to \partial_t u \quad \text{weakly in } L^2(t-2, t; \mathcal{H}_t(\Omega));
$$

(4.23)

$$u^k \to u \quad \text{a.e. } (x, t) \in \Omega \times [\tau, +\infty),
$$

(4.24)

for any $\tau_k \leq \tau_1(t, \hat{D})$. Then by the similar proof to Theorem 3.5, we obtain that $u \in C([t-2, t]; \mathcal{H}_t(\Omega))$, and from (4.19) – (4.24), we can derive that $u$ satisfies Definition 3.1 in $(t-2, t)$. 

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In order to prove the lemma, we only need to check that \( u^k \to u \) strongly in \( C([t-2,t]; \mathcal{H}_t(\Omega)) \). We establish it by contradiction. Suppose there is a constant \( \mu > 0 \) and a sequence \( t_k \in [t-2,t] \) such that when \( t_k \to t_\ast \),

\[
|u^k(t_k) - u(t_\ast)| \geq \mu, \quad \forall k \in \mathbb{N}^+.
\] (4.25)

By similar calculations to (3.11), we derive

\[
\|e(s)\|_{\mathcal{H}_t}^2 < \|e(r)\|_{\mathcal{H}_t}^2 + 2C_0(s-r) + \frac{2\xi^2}{m\lambda_1} \int_r^s \|h(x,z)\|^2 dz, \quad \forall t-2 \leq r \leq s \leq t,
\] (4.26)

where the function \( e \) can be replaced by \( u \) or \( u^k \).

Now we define the following functions

\[
Q_k(s) = \|u^k(s)\|_{\mathcal{H}_t}^2 - 2C_0s - \frac{2\xi^2}{m\lambda_1} \int_{t-2}^s \|h(x,r)\|^2 dr
\] (4.27)

and

\[
Q(s) = \|u(s)\|_{\mathcal{H}_t}^2 - 2C_0s - \frac{2\xi^2}{m\lambda_1} \int_{t-2}^s \|h(x,r)\|^2 dr.
\] (4.28)

From the smoothness of \( u \) and \( u^k \), we obtain that the functions \( Q_k(s) \) and \( Q(s) \) are continuous on \([t-2,t]\). Then using the above inequality, we obtain that \( Q_k(s) \) and \( Q(s) \) are non-increasing on \([t-2,t]\) and from (1.19) – (1.24), we can derive

\[
Q_k(s) \to Q(s) \quad \text{a.e. } s \in (t-2,t). \] (4.29)

Assume that there is a sequence \( \{\tilde{t}_n\} \) \((t-2,t_\ast)\) such that \( \tilde{t}_n \to t_\ast \) when \( k \to \infty \) and the above convergence holds. Since the function \( Q(s) \) is continuous on \([t-2,t]\), there exists a positive integer \( n(\tilde{\epsilon}) \geq 1 \) such that

\[
|Q(\tilde{t}_n) - Q(t_\ast)| < \frac{\tilde{\epsilon}}{2}, \quad \forall n \geq n(\tilde{\epsilon}),
\] (4.30)

where \( \tilde{\epsilon} \) is a positive constant.

From (4.29), we can derive that there exists a constant \( k(\tilde{\epsilon}) \in \mathbb{N}^+ \) such that \( t_k \geq \tilde{t}_n(\tilde{\epsilon}) \) and

\[
|Q_k(\tilde{t}_n(\tilde{\epsilon})) - Q(\tilde{t}_n(\tilde{\epsilon}))| < \frac{\tilde{\epsilon}}{2} \quad \text{for all } k \geq k(\tilde{\epsilon}).
\]

Furthermore, notice that \( Q_k(s) \) is a non-increasing function, then by the Cauchy inequalities, we can conclude that for all \( k \geq k(\tilde{\epsilon}) \) the following inequality holds

\[
Q_k(t_k) - Q(t_\ast) \leq |Q_k(\tilde{t}_n(\tilde{\epsilon})) - Q(t_\ast)|
\]
\[
\leq |Q_k(\tilde{t}_n(\tilde{\epsilon})) - Q(\tilde{t}_n(\tilde{\epsilon}))| + |Q(\tilde{t}_n(\tilde{\epsilon})) - Q(t_\ast)|
\]
\[
\leq \frac{\tilde{\epsilon}}{2} + \frac{\tilde{\epsilon}}{2} = \tilde{\epsilon}.
\] (4.31)
Since the arbitrariness of $\bar{\varepsilon} > 0$, it follows that $\lim_{k \to \infty} \sup Q_k(t_k) \leq Q(t_*)$ as $\bar{\varepsilon} \to 0$. Then we can conclude that $\lim_{k \to \infty} \| u^k(t_k) \|_{\mathcal{H}_t} \leq \| u(t_*) \|_{\mathcal{H}_t}$, which contradicts with (4.25). Therefore, the proof is complete. □

From the above proofs, it follows the following theorem about the existence of the minimal time-dependent pullback $\mathcal{D}_{\sigma}^{H_t}$-attractors.

**Theorem 4.7** Under the assumptions of Theorems 3.5 and 3.6 and assume that the function $h(x, t)$ satisfies (4.11), then there exists the minimal time-dependent pullback $\mathcal{D}_{\sigma}^{H_t}$-attractor $A_{\mathcal{D}_{\sigma}^{H_t}} = \{ A_{\mathcal{D}_{\sigma}^{H_t}}(t) : t \in \mathbb{R} \}$ and the minimal time-dependent pullback $\mathcal{D}_{\sigma}^{H_t}$-attractors $A_{\mathcal{D}_{\sigma}^{H_t}} = \{ A_{\mathcal{D}_{\sigma}^{H_t}}(t) : t \in \mathbb{R} \}$ for the process $\{ U(t, \tau) \}_{t \geq \tau}$ on $\mathcal{H}_t(\Omega)$ of problem (1.1). Moreover, the family $A_{\mathcal{D}_{\sigma}^{H_t}}$ belongs to $\mathcal{D}_{\sigma}^{H_t}$ and for any $t \in \mathbb{R}$ the following relationships hold:

$$A_{\mathcal{D}_{\sigma}^{H_t}}(t) \subset A_{\mathcal{D}_{\sigma}^{H_t}}(t) \subset \bar{B}_{\mathcal{H}_t} \left( 0, \rho_{\xi}^2(t) \right).$$

(4.32)

In addition, if $h(x, t)$ satisfies

$$\sup_{r \leq 0} \left( e^{-\sigma r} \int_{-\infty}^{r} e^{\sigma s} \| h(x, s) \|^2 ds \right) < +\infty,$$

then $A_{\mathcal{D}_{\sigma}^{H_t}}(t) = A_{\mathcal{D}_{\sigma}^{H_t}}(t)$ for any $t \in \mathbb{R}$.

**Proof.** From Definition 2.12, Theorems 3.5 - 3.6 and Lemmas 4.1 - 4.6, it follows the existence of above minimal time-dependent pullback $\mathcal{D}_{\sigma}^{H_t}$-attractors. Besides, from Lemma 2.14 and Corollaries 2.15 - 2.16, we can derive that (4.32) and $A_{\mathcal{D}_{\sigma}^{H_t}}(t) = A_{\mathcal{D}_{\sigma}^{H_t}}(t)$. □

5  Existence of time-dependent pullback attractors $\{ A_{\xi}(t) \}_{t \in \mathbb{R}}$ and upper semicontinuity of $\{ A_{\xi}(t) \}_{t \in \mathbb{R}}$ and the global attractor $A$

In this section, we will establish the existence of time-dependent pullback attractors $\{ A_{\xi}(t) \}_{t \in \mathbb{R}}$ and the upper semicontinuity of $\{ A_{\xi}(t) \}_{t \in \mathbb{R}}$ and the global attractor $A$ of equation (1.1) with $\xi = 0$.

First of all, we decompose the solution $U_{\xi}(t, \tau)u_\tau = u(t)$ of problem (1.1) with initial data $u_\tau \in \mathcal{H}_t(\Omega)$ as follows:

$$U_{\xi}(t, \tau)u_\tau = U_{1, \xi}(t, \tau)u_\tau + U_{2, \xi}(t, \tau)u_\tau,$$
where $U_{1,\xi}(t, \tau)u_{\tau} = v(t)$ and $U_{2,\xi}(t, \tau)u_{\tau} = g(t)$ solve, respectively,

$$
\begin{align*}
 v_t - \varepsilon(t) \Delta v_t - a(l(u)) \Delta v &= f_0(v) \quad \text{in } \Omega \times (\tau, \infty), \\
v &= 0 \quad \text{on } \partial \Omega \times (\tau, \infty), \\
v(x, \tau) &= u_{\tau}(x) \quad x \in \Omega,
\end{align*}
$$

and

$$
\begin{align*}
 g_t - \varepsilon(t) \Delta g_t - a(l(u)) \Delta g &= f(u) - f_0(v) + \xi h(x, t) \quad \text{in } \Omega \times (\tau, \infty), \\
g &= 0 \quad \text{on } \partial \Omega \times (\tau, \infty), \\
g(x, \tau) &= 0 \quad x \in \Omega.
\end{align*}
$$

**Lemma 5.1** Under the assumptions of Theorems 3.5 and 3.6 and Lemma 4.1, if for any $t \in \mathbb{R}$ the function $h(x, t)$ satisfies (4.11), then for any bounded set $B \subset \mathcal{H}_t(\Omega)$, there exists $T(t, B) > 0$ such that

$$
\|U_{\xi}(t, t - \tau)u_{t-\tau}\|_{\mathcal{H}_t}^2 \leq \rho_{\xi}(t) \quad \text{for all } \tau \geq T(t, B) \text{ and all } u_{t-\tau} \in B,
$$

where $\rho_{\xi}(t)$ is the same as (4.12).

**Proof.** From Lemmas 4.1 and 4.4 the desired result follows easily. \hfill \Box

Let $D_{\xi}(t) = \{u \in \mathcal{H}_t(\Omega) \mid \|u\|_{\mathcal{H}_t}^2 \leq \rho_{\xi}(t)\}$. It is easy to check that the family $D_{\xi} = \{D_{\xi}(t)\}_{t \in \mathbb{R}}$ is pullback absorbing in $\mathcal{H}_t(\Omega)$. Moreover, we derive

$$
\lim_{t \to -\infty} e^{\sigma t} \rho_{\xi}(t) = 0 \quad \text{for any } \xi > 0.
$$

**Lemma 5.2** Let $\rho_{\xi}(t), D_{\xi}(t)$ given as the above. For any $t \in \mathbb{R}$, the solution $v(t)$ of problem (5.1) satisfies

$$
\|v(t)\|_{\mathcal{H}_t}^2 = \|U_{1,\xi}(t, t - \tau)u_{t-\tau}\|_{\mathcal{H}_t}^2 \leq (1 + 2\lambda_1) e^{-\sigma t} \rho_{\xi}(t - \tau)
$$

for all $\tau \geq 0$, $u_{t-\tau} \in D_{\xi}(t - \tau)$ and $0 < \sigma < \min \left\{ \eta, \frac{-\varepsilon'(t)}{\varepsilon(t)}, \frac{1}{(1 + \lambda_1)\lambda_1} \right\} \text{ with } 0 < \eta < m\lambda_1$.

**Proof.** Multiplying (5.1) by $v$, then we arrive at

$$
\frac{d}{dt}(\|v\|^2 + \varepsilon(t)\|\nabla v\|^2) + (2a(l(u)) - \varepsilon'(t))\|\nabla v\|^2 = 2(f_0(v), v).
$$

Then from (1.10), we can derive

$$
\frac{d}{dt}(\|v\|^2 + \varepsilon(t)\|\nabla v\|^2) + \frac{\lambda_1}{1 + \lambda_1} (\|v\|^2 + \varepsilon(t)\|\nabla v\|^2) \leq 0
$$

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which, by taking $0 < \sigma < \min \left\{ \eta, \frac{\varepsilon(t)}{\varepsilon(t)}, \frac{1}{1+\lambda_1} \right\}$ with $0 < \eta < m\lambda_1$, gives

$$\frac{d}{dt} \left( e^{\sigma t} \left( \|v\|^2 + \varepsilon(t) \|\nabla v\|^2 \right) \right) + \left( \frac{1}{1+\lambda_1} \right) e^{\sigma t} \left( \|v\|^2 + \varepsilon(t) \|\nabla v\|^2 \right) \leq 0. \quad (5.7)$$

Integrating (5.7) over $[t_0 - \tau, t_0]$, then by Lemmas 4.1 and 5.1, we conclude

$$\|v(t_0)\|^2 + \varepsilon(t) \|\nabla v(t_0)\|^2 \leq e^{-\sigma \tau} \left( \|v_{t_0-\tau}\|^2 + \varepsilon(t_0 - \tau) \|\nabla v_{t_0-\tau}\|^2 \right) \leq e^{-\sigma \tau} (\|v(t_0 - \tau)\|^2 + \varepsilon(t_0 - \tau) \|\nabla v(t_0 - \tau)\|^2) \leq (1 + 2\lambda_1) e^{-\sigma \tau} \rho_{\xi}(t - \tau),$$

which completes the proof. \qed

**Lemma 5.3** For any $t \in \mathbb{R}$, there exists $T(t, D_{\xi}) > 0$ and $R_{\xi}(t) > 0$, such that

$$\|U_{2,\xi}(t, t - \tau) u_{t-\tau}\|^2_{H_1^{1+a}} = \|A^{\frac{\mu}{2}} g\|^2 + \varepsilon(t) \|A^{\frac{1+\alpha}{2}} g\|^2 \leq R_{\xi}(t)$$

for any $\tau > T(t, D_{\xi})$ and $u_{t-\tau} \in D_{\xi}(t - \tau)$, where $R_{\xi}(t) = C \rho_{\xi}^{2p+2}(t)$ and $0 < \alpha < \min\{1, \frac{4-(N-2)p}{2}\}$. \n
**Proof.** Multiplying (5.2) by $e^{\delta t} A^\alpha g$, then we obtain

$$(g_k, e^{\delta t} A^\alpha g) - (\varepsilon(t) \Delta g_k, e^{\delta t} A^\alpha g) - (a(l(u)) \Delta g, e^{\delta t} A^\alpha g) = (f(u) - f_0(v)) + \xi h(x, t), e^{\delta t} A^\alpha g), \quad (5.10)$$

where $0 < \delta < \min \left\{ (p + 1) \sigma, \frac{16m - 2 - \varepsilon(t)}{8(\lambda_1 + \varepsilon(t))} \right\}$.

By simple calculations, we derive the following equalities

$$(g_k, e^{\delta t} A^\alpha g) = \frac{1}{2} \frac{d}{dt} \left( e^{\delta t} \|A^{\frac{\mu}{2}} g\|^2 \right) - \frac{1}{2} \delta e^{\delta t} \|A^{\frac{\mu}{2}} g\|^2, \quad (5.11)$$

and

$$-(\varepsilon(t) \Delta g_k, e^{\delta t} A^\alpha g) = \frac{1}{2} \frac{d}{dt} \left( e^{\delta t} \|A^{\frac{1+\alpha}{2}} g\|^2 \right) - \frac{1}{2} \varepsilon(t) e^{\delta t} \|A^{\frac{1+\alpha}{2}} g\|^2 - \frac{1}{2} \delta \varepsilon(t) e^{\delta t} \|A^{\frac{1+\alpha}{2}} g\|^2 \quad (5.12)$$

and

$$-(a(l(u)) \Delta g, e^{\delta t} A^\alpha g) = a(l(u)) e^{\delta t} \|A^{\frac{1+\alpha}{2}} g\|^2. \quad (5.13)$$

From (1.8), we conclude

$$(f(u) - f_0(v), e^{\delta t} A^\alpha g) = e^{\delta t} (f(u) - f(v), A^\alpha g) + e^{\delta t} (f_1(v), A^\alpha g). \quad (5.14)$$

Inserting (5.11) – (5.14) into (5.10), we obtain

$$\frac{d}{dt} \left( e^{\delta t} \|A^{\frac{\mu}{2}} g\|^2 + \varepsilon(t) \|A^{\frac{1+\alpha}{2}} g\|^2 \right) - \varepsilon'(t) e^{\delta t} \|A^{\frac{1+\alpha}{2}} g\|^2 + 2a(l(u)) e^{\delta t} \|A^{\frac{1+\alpha}{2}} g\|^2$$

$$= \delta e^{\delta t} \|A^{\frac{\mu}{2}} g\|^2 + \varepsilon(t) \|A^{\frac{1+\alpha}{2}} g\|^2 + 2e^{\delta t} (f(u) - f(v), A^\alpha g) + 2e^{\delta t} (f_1(v), A^\alpha g)$$

$$+ 2 \xi e^{\delta t} (h, A^\alpha g). \quad (5.15)$$
By similar calculations to Lemma 3.3 in [31], noticing that \( p \leq \frac{4}{N-2} \), from assumptions (1.8), (1.11) and Lemma 2.1, then assuming \( 0 < \alpha < \min\{1, \frac{4-(N-2)p}{2}\} \), we can derive the following inequalities

\[
|e^{\delta t}(f(u) - f(v), A^\alpha g)| \leq C e^{\delta t}(\|A^{\frac{1}{2}}u\|^2 + \|A^{\frac{1}{2}}v\|^2 + 1)\|A^{\frac{1}{2}}g\|^2 + \frac{1}{16} e^{\delta t}\|A^{\frac{1}{2}+\alpha}g\|^2, \tag{5.16}
\]

\[
|e^{\delta t}(f_1(v), A^\alpha g)| \leq C e^{\delta t}(1 + \|A^{\frac{1}{2}}v\|^2 + 1)\|A^{\frac{1}{2}}g\|^2 \tag{5.17}
\]

and

\[
|\langle \xi h(x, t), A^\alpha g \rangle| \leq 4\xi \|h(x, t)\|^2 + \frac{1}{16} \xi \|A^{\frac{1}{2}+\alpha}g\|^2. \tag{5.18}
\]

Inserting (5.16) – (5.18) into (5.15), we derive

\[
\frac{d}{dt}(e^{\delta t}(\|A^{\frac{1}{2}}u\|^2 + \varepsilon(t)\|A^{\frac{1}{2}+\alpha}g\|^2)) - \varepsilon'(t)e^{\delta t}\|A^{\frac{1}{2}+\alpha}g\|^2.
\]

\[
\leq C e^{\delta t}(\|A^{\frac{1}{2}}u\|^2 + \|A^{\frac{1}{2}}v\|^2 + 1)\|A^{\frac{1}{2}}g\|^2 + C e^{\delta t}(1 + \|A^{\frac{1}{2}}v\|^2 + 1)\|A^{\frac{1}{2}}g\|^2 + c\xi e^{\delta t}\|h(x, t)\|^2.
\tag{5.19}
\]

From \( u = v + g \), we conclude

\[
(\|A^{\frac{1}{2}}u\|^2 + \|A^{\frac{1}{2}}v\|^2 + 1)\|A^{\frac{1}{2}}g\|^2
\]

\[
\leq C (\|A^{\frac{1}{2}}u\|^2 + \|A^{\frac{1}{2}}v\|^2 + 1)\|A^{\frac{1}{2}}v\|^2 + 1)
\tag{5.20}
\]

Inserting (5.20) into (5.19) and by the Poincaré inequality, we conclude

\[
\frac{d}{dt}(e^{\delta t}(\|A^{\frac{1}{2}}u\|^2 + \varepsilon(t)\|A^{\frac{1}{2}+\alpha}g\|^2)) - \varepsilon'(t)e^{\delta t}\|A^{\frac{1}{2}+\alpha}g\|^2
\]

\[
\leq C e^{\delta t}(\|A^{\frac{1}{2}}u\|^{2p+2} + \|A^{\frac{1}{2}}v\|^{2p+2} + 1) + C\xi e^{\delta t}\|h(x, t)\|^2.
\tag{5.21}
\]

From (4.1), it follows that

\[
\|u\|^2_\mathcal{H}_t \leq Ce^{-\sigma t}\|u_{t-}\|^2_\mathcal{H}_t + C \left(\xi e^{-\sigma t}\int_{-\infty}^{t} e^{\sigma s}\|h(x, s)\|^2 ds + 1\right). \tag{5.22}
\]

Multiplying (5.22) by \( e^{\sigma t} \), then we obtain

\[
e^{\sigma t}\|u\|^2_\mathcal{H}_t \leq Ce^{\sigma(t_0-\tau)}\|u_{t_0-\tau}\|^2_\mathcal{H}_t + C\xi \int_{-\infty}^{t_0} e^{\sigma t}\|h(x, t)\|^2 dt + Ce^{\sigma t}. \tag{5.23}
\]

Calculating the \( p + 1 \) power on both sides of (5.23), we conclude

\[
e^{(p+1)\sigma t}\|u\|^{2p+2}_\mathcal{H}_t \leq Ce^{(p+1)\sigma t}\|u_{t_0-\tau}\|^{2p+2}_\mathcal{H}_t + C\xi \left(\int_{-\infty}^{t_0} e^{\sigma t}\|h(x, t)\|^2 dt\right)^{2p+2} + Ce^{(p+1)\sigma t} + C. \tag{5.24}
\]

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Multiplying (5.24) by $e^{(\delta-(p+1)\sigma)t}$, then we derive
\[
e^{\delta t} \|u\|_{H_t}^{2p+2} \leq Ce^{(p+1)\sigma(t_0-\tau)}e^{(\delta-(p+1)\sigma)t}\|u_{t_0-\tau}\|_{H_t}^{2p+2} + C\xi e^{(\delta-(p+1)\sigma)t} \left(\int_{-\infty}^{t_0} e^{\sigma t}\|h(x,t)\|^2 dt\right)^{2p+2} + Ce^{\delta t} + Ce^{(\delta-(p+1)\sigma)t}.
\] (5.25)

Integrating (5.25) over $[t_0-\tau,t_0]$, we conclude
\[
\int_{-\infty}^{t_0} e^{\delta t} \|u(t)\|_{H_t}^{2p+2} dt \leq Ce^{(p+1)\sigma(t_0-\tau)}e^{(\delta-(p+1)\sigma)t}\|u_{t_0-\tau}\|_{H_t}^{2p+2} + C\xi e^{(\delta-(p+1)\sigma)t_0} \left(\int_{-\infty}^{t_0} e^{\sigma t}\|h(x,t)\|^2 dt\right)^{p+1} + Ce^{\delta t_0} + Ce^{(\delta-(p+1)\sigma)t_0}.
\] (5.26)

By (5.8) and the Poincaré inequality, we conclude
\[
e^{\sigma t} \|v(t)\|_{H_t}^2 \leq Ce^{\sigma(t_0-\tau)}\|u_{t_0-\tau}\|_{H_t}^2.
\] (5.27)

Then calculating the $p+1$ power on both sides of (5.27), we obtain
\[
e^{(p+1)\sigma t} \|v(t)\|_{H_t}^2 \leq Ce^{(p+1)\sigma(t_0-\tau)}\|u_{t_0-\tau}\|_{H_t}^{2p+2}.
\] (5.28)

Calculating similarly to deriving (5.25) from (5.24), we can derive
\[
e^{\delta t} \|v(t)\|_{H_t}^{2p+2} \leq Ce^{(\delta-(p+1)\sigma)t}e^{(p+1)\sigma(t_0-\tau)}\|u_{t_0-\tau}\|_{H_t}^{2p+2}.
\] (5.29)

Then integrating (5.29) over $[t_0-\tau,t_0]$, we obtain
\[
\int_{-\infty}^{t_0} e^{\sigma t} \|v(t)\|_{H_t}^{2p+2} dt \leq Ce^{\sigma t_0}e^{-(p+1)\sigma\tau} \|u_{t_0-\tau}\|_{H_t}^{2p+2}.
\] (5.30)

Integrating (5.21) over $[t_0-\tau,t_0]$ and noting that $\varepsilon(t)$ is a decreasing function, we obtain
\[
e^{\delta t_0}(\|A^{\frac{\alpha}{2}}g(t_0)\|^2 + \varepsilon(t_0)\|A^{\frac{1+\alpha}{2}}g(t_0)\|^2) \leq e^{\delta(t_0-\tau)}(\|A^{\frac{\alpha}{2}}g_{t_0-\tau}\|^2 + \varepsilon(t_0-\tau)\|A^{\frac{1+\alpha}{2}}g_{t_0-\tau}\|^2)
+ C \int_{t_0-\tau}^{t_0} e^{\delta t}(\|A^{\frac{1}{2}}u\|^{2p+2} + \|A^{\frac{1}{2}}v\|^{2p+2} + 1)dt
+ C\xi \int_{t_0-\tau}^{t_0} e^{\delta t}\|h(x,t)\|^2 dt + C.
\] (5.31)

Then we can derive
\[
\|A^{\frac{\alpha}{2}}g(t_0)\|^2 + \varepsilon(t_0)\|A^{\frac{1+\alpha}{2}}g(t_0)\|^2 \leq Ce^{-\delta t_0}\int_{t_0-\tau}^{t_0} e^{\delta t}(\|A^{\frac{1}{2}}u\|^{2p+2} + \|A^{\frac{1}{2}}v\|^{2p+2} + 1)dt
+ C\xi e^{-\delta t_0} \int_{t_0-\tau}^{t_0} e^{\delta t}\|h(x,t)\|^2 dt + C.
\] (5.32)
Multiplying (5.26) by $e^{-\delta t_0}$ and by $\delta > (p + 1)\sigma$, we obtain
\begin{align*}
e^{-\delta t_0} \int_{-\infty}^{t_0} e^{\delta t} \|u(t)\|_{\mathcal{H}_t}^{2p+2} dt &\leq C e^{-\delta t_0} e^{(p+1)\sigma(t_0-\tau)} e^{(\delta-(p+1)\sigma)t} \|u_{t_0-\tau}\|_{\mathcal{H}_t}^{2p+2} \\
&+ C\xi e^{(\delta-(p+1)\sigma)t_0} e^{-\delta t_0} \left( \int_{-\infty}^{t_0} e^{\sigma t} \|h(x,t)\|^2 dt \right)^{p+1} + C e^{-(p+1)\sigma t_0} + C \\
&\leq C e^{-(p+1)\sigma\tau} \|u_{t_0-\tau}\|_{\mathcal{H}_t}^{2p+2} + C\xi e^{-(p+1)\sigma t_0} \left( \int_{-\infty}^{t_0} e^{\sigma t} \|h(x,t)\|^2 dt \right)^{p+1} + C \\
&\leq C e^{-\delta\tau} \|u_{t_0-\tau}\|_{\mathcal{H}_t}^{2p+2} + C\xi e^{-\delta t_0} \left( \int_{-\infty}^{t_0} e^{\sigma t} \|h(x,t)\|^2 dt \right)^{p+1} + C.
\end{align*}
(5.33)

Besides, multiplying (5.30) by $e^{-\delta t_0}$, we conclude
\begin{align*}
e^{-\delta t_0} \int_{-\infty}^{t_0} e^{\sigma t} \|v(t)\|_{\mathcal{H}_t}^{2p+2} dt &\leq e^{-(p+1)\sigma\tau} \|v_{t_0-\tau}\|_{\mathcal{H}_t}^{2p+2} \leq e^{-\delta\tau} \|v_{t_0-\tau}\|_{\mathcal{H}_t}^{2p+2}.
\end{align*}
(5.34)

Then inserting (5.33) and (5.34) into (5.32) and by Lemma 2.1 we derive
\begin{align*}
\|A_{\mathbb{H}}^\frac{4}{N-2} g(t_0)\|^2 + \varepsilon(t_0) \|A_{\mathbb{H}}^{\frac{4}{N-2}} g(t_0)\|^2 &\leq C + C e^{-\delta t_0} \int_{-\infty}^{t_0} e^{\delta t} \|A_{\mathbb{H}}^{\frac{4}{N-2}} g(t_0)\|^2 dt \\
+ C e^{-\delta t_0} \int_{-\infty}^{t_0} e^{\delta t} \|A_{\mathbb{H}}^{\frac{4}{N-2}} v(t)\|^2 dt + C e^{-\delta t_0} \int_{-\infty}^{t_0} e^{\delta t} \|h(x,t)\|^2 dt \\
&\leq C e^{-\delta\tau} \|v_{t_0-\tau}\|_{\mathcal{H}_t}^{2p+2} + C\xi e^{-\delta t_0} \int_{-\infty}^{t_0} e^{\delta t} \|h(x,t)\|^2 dt + C.
\end{align*}
(5.35)

From $p \leq \frac{4}{N-2}$, it follows that $p + 1 \leq 5$. Then by Lemma 5.1 we derive
\begin{align*}
\|A_{\mathbb{H}}^\frac{4}{N-2} g(t_0)\|^2 + \varepsilon(t_0) \|A_{\mathbb{H}}^{\frac{4}{N-2}} g(t_0)\|^2 &\leq R_\xi(t),
\end{align*}
(5.36)
where $R_\xi(t) = C\rho_{\xi}^{2p+2}(t)$.

As a conclusion of (5.30), (5.31) follows directly. \hfill \Box

**Remark 5.4** In Lemma 5.2 we assume $0 < \sigma < \min\left\{\eta, \frac{\varepsilon'(t)}{\varepsilon(t)}, \frac{1}{(1+\lambda_1)\lambda_1^{-1}}\right\}$ with $0 < \eta < m\lambda_1$ and in this lemma we assume that $0 < \delta < \min\left\{(p+1)\sigma, \frac{16m-2-\xi}{8(\lambda_1^{-1}+\varepsilon(t))}\right\}$, then it follows the more accurate range of $\sigma$:
\begin{align*}
0 < \sigma < \min\left\{m\lambda_1, \frac{\varepsilon'(t)}{\varepsilon(t)}, \frac{1}{(1+\lambda_1)\lambda_1^{-1}} \frac{\delta}{p+1}\right\}.
\end{align*}
(5.37)

**Lemma 5.5** For any $t_0 \in \mathbb{R}$ and $\tau > 0$, if $u_0$ is a subset of bounded set $B \subset \mathcal{H}_t(\Omega)$, the solution $u^\xi(t_0,t_0-\tau)u_0$ of problem (1.1) converges to the solution $u(\tau)u_0$ of the following unperturbed equation of (1.1):
\begin{align*}
\begin{cases}
 u_t - \varepsilon(t)\Delta u_t - a(l(u))\Delta u = f(u) &\text{in } \Omega \times (\tau, \infty), \\
 u = 0 &\text{on } \partial\Omega \times (\tau, \infty), \\
 u(x, \tau) = u_\tau &x \in \Omega,
\end{cases}
\end{align*}
(5.38)
that is,

\[
\lim \sup_{\xi \to 0^+, \omega \in B} \|u^\xi(t_0, t_0 - \tau)u_0 - u(\tau)u_0\|_{H^1} = 0. 
\]  

(5.39)

**Proof.** Let \( \omega^\xi = u^\xi(t, t_0 - \tau)u_0 - u(t - t_0 + \tau)u_0 \), then \( \omega^\xi \) satisfies

\[
\begin{align*}
\begin{cases}
   w_t^\xi - \varepsilon(t)\Delta w_t^\xi - a(l(u^\xi))\Delta u^\xi + a(l(u))\Delta u = f(u^\xi) - f(u) + \xi h(x, t), & \text{in } \Omega \times (\tau, \infty), \\
   \omega^\xi = 0 & \text{on } \partial \Omega \times (\tau, \infty), \\
   \omega^\xi(t_0 - \tau) = 0 & x \in \Omega.
\end{cases}
\end{align*}
\]

Choosing \( \omega^\xi \) as the test function of the above function in \( L^2(\Omega) \), we obtain

\[
\begin{align*}
   & \frac{d}{dt} \left( \|w^\xi\|^2 + \varepsilon(t)\|\nabla w^\xi\|^2 \right) - \varepsilon'(t)\|w^\xi\|^2 + 2a(l(u^\xi))\|\nabla w^\xi\|^2 \\
   = & 2(a(l(u)) - a(l(u^\xi)))(\nabla u, \nabla w^\xi) + 2(f(u^\xi) - f(u), w^\xi) + \xi (h(x, t), w^\xi).
\end{align*}
\]

(5.40)

Then the following inequalities follows by similar calculations to (5.25) and (5.26),

\[
2(a(l(u)) - a(l(u^\xi)))(\nabla u, \nabla w^\xi) \leq 2m\|\nabla u - \nabla u^\xi\|^2 + \frac{(La(R))^2\|l\|^2\|\nabla u\|^2\|u^\xi - u\|^2}{2m},
\]

(5.41)

and

\[
2(f(u^\xi) - f(u), \nabla w^\xi) \leq C \int_\Omega (1 + |u^\xi|^p + |u|^p)|\omega^\xi|^2 dx \\
\leq C \left( \int_\Omega (1 + |u^\xi|^p + |u|^p)^{\frac{2N}{N-2(p+1)}} dx \right)^{\frac{N-2(p+1)}{2N}} \left( \int_\Omega |\omega^\xi|^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{2N}} \\
\times \left( \int_\Omega |w^\xi|^2 dx \right)^{\frac{2N}{N-2(p+1)}} \\
\leq C(1 + \|A^{\frac{1}{2}}u^\xi\|^p + \|A^{\frac{1}{2}}u\|^p)(\|A^{\frac{1}{2}}u^\xi\|\|A^{\frac{1}{2}-\frac{1}{2}}w^\xi\|) \\
\leq C(1 + \|A^{\frac{1}{2}}u^\xi\|^p + \|A^{\frac{1}{2}}u\|^p)(\|w^\xi\|^2 + \varepsilon(t)\|\nabla w^\xi\|^2) \\
\leq C(1 + \|A^{\frac{1}{2}}u^\xi\|^{\frac{1}{2}N} + \|A^{\frac{1}{2}}u\|^{\frac{1}{2}N})\|w^\xi\|_{H^1}^2.
\]

(5.42)

By the Young inequality, we can derive

\[
2(\xi h(x, t), \omega^\xi) \leq \frac{\xi^2}{\lambda_1} \|h(x, t)\|^2 + \lambda_1 \|\omega^\xi\|^2.
\]

(5.43)

Inserting (5.41) – (5.43) into (5.40), from (1.4) and \( \varepsilon(t) \) is a decreasing function, then using the Sobolev embedding theorem, we conclude

\[
\begin{align*}
\frac{d}{dt}(\|w^\xi\|_{H^1}^2) &= \frac{d}{dt}(\|w^\xi\|^2 + \varepsilon(t)\|\nabla w^\xi\|^2) \\
\leq & C(1 + \|A^{\frac{1}{2}}u^\xi\|^{\frac{1}{2}N} + \|A^{\frac{1}{2}}u\|^{\frac{1}{2}N})\|w^\xi\|^2_{H^1} + C\varepsilon^2\|h(x, t)\|^2.
\end{align*}
\]

(5.44)
By the Gronwall inequality, we obtain
\[
\|w^\xi(t_0)\|_{ \mathcal{H}_t}^2 \leq C\xi^2 e^{\int_{t_0-\tau}^{t_0} (1+\|A_{1/2}u^\xi\|_{N/2}^4 + \|A_{1/2}u\|_{N/2}^4) dt} \cdot \int_{t_0-\tau}^{t_0} \|h(x,t)\|_{2}^2 dt.
\] (5.45)

Then the desired result follows directly. □

The main result of this section is as follows.

**Theorem 5.6** Under the assumptions of Theorems 3.3–3.6 and Lemmas 5.1–5.5, assume further that the function \( h(x,t) \) satisfies (4.11), then the pullback attractor \( A_\xi = \{ A_\xi(t) \}_{t \in \mathbb{R}} \) of problem (1.1) with \( \xi > 0 \) and the global attractor \( A \) for problem (1.1) with \( \xi = 0 \) satisfy
\[
\lim_{\xi \to 0^+} \text{dist}_{\mathcal{H}_t} (A_\xi(t), A) = 0,
\] (5.46)
for all \( t \in \mathbb{R} \).

**Proof.** From Lemma 5.1 to Lemma 5.5, we deduced that Lemma 2.18 holds. Then by Lemma 2.17, (5.46) follows directly. □

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**Conflict of interest statement**

The authors have no conflict of interest.
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