Principle of Relativity for Quantum Theory

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In non relativistic physics it is assumed that both chronological ordering and causal ordering of events (telling whether there exists a causal relationship between two events or not) are absolute, observer independent properties. In relativistic physics on the other hand chronological ordering depends on the observer who assigns space-time coordinates to physical events and only causal ordering is regarded as an observer independent property. In this paper it is shown that quantum theory can be considered as a physical theory in which causal (as well as chronological) ordering of probabilistic events happening in experiments may be regarded as an observer dependent property. We then argue that this result has connections with the problem of dark energy in cosmology.

I. INTRODUCTION

The most notable attempts in formulating a theory that unifies quantum theory and general relativity are String Theory and Loop Quantum Gravity [1, 2]. The lack of experiments that could verify or falsify any of the predictions of the two theories leaves physicists with the consciousness that something is missing in our current understanding of nature at the fundamental level. Despite the formulation of both theories mentioned above depart from very reasonable starting points, they remain naïve about giving a foundational principle to explain the mathematical formalism of quantum theory. This means that they retain superposition principle, non-locality and all the counterintuitive features manifested by quantum theory as natural facts and do not try to give a motivation for them. This attitude is perhaps justified by the fact that quantum theory is extremely successful in making predictions. Until now, no experimental situation has been found in which the predictions of quantum theory are not satisfied. Such an extraordinary predicting power has led many physicists to think that it is not necessary to have a physical intuition of what is going on at atomic and subatomic scales, it is sufficient to have a model that can predict whatever can occur in an experiment. This pragmatic attitude would be the right one if theoretical physics accommodated all phenomena experienced in nature in a unique and coherent model. Despite the many successes of the Standard Model and the potentiality of string theory and loop quantum gravity, there is large consensus among physicists that we are far away to have such a unified picture. This has recently led physicists to turn the attention back to the problem of foundations quantum theory with a new slant given by the emergence of quantum information [3, 4]. In this paper it is analyzed the mathematical structure of quantum theory (as used in the field of quantum information) from a novel point of view enlighting the interplay between quantum features and causal structure of space-time events. In quantum theory events correspond to probabilistic outcomes and the only predictable and verifiable statements regard correlations between outcomes happening on different devices located in distinct regions of space. Since these outcomes are also thought to be events happening in space-time, it is always assumed an absolute causal ordering for them. A set of physical events \( E \), like those that can happen in a quantum experiment, possesses a causal ordering if, for these events, it is defined a causal structure. This means that for any pair of events, \( \chi_a, \chi_b \in E \), one of the following must hold:

- \( \chi_a \) causes \( \chi_b \)
- \( \chi_b \) causes \( \chi_a \)
- \( \chi_a \) does not cause \( \chi_b \) and \( \chi_b \) does not cause \( \chi_a \) (they are space-like events)

For example, \( \chi_a \) could be a preparation contained in a preparations ensemble for a quantum system of a certain type while \( \chi_b \) could be an outcome of a measurement caused by that preparation. In this case \( \chi_a \) causes \( \chi_b \). \( \chi_a \) and \( \chi_b \) could also be two outcomes obtained respectively in two measurements performed in parallel on a bipartite state of a composite system. In this case \( \chi_a \) and \( \chi_b \) are indeed two space-like separated events. The main result of this paper is that, in quantum theory, any experimental situation of the former type mentioned above can be considered as equivalent to a situation of the latter type. This equivalence is such that the two experiments can be interpreted as the same experiment viewed by two different observers that make two different assumptions regarding the causal ordering of events happening in the experiment. To prove this it is shown in section [5] that, in a generic quantum experiment involving two sets of random outcomes happening on distinct devices, the mathematical expression of the joint probability of any two outcomes calculated by one observer, can be mapped, by means of a simple transformation rule, into the expression for the joint probability of the same two outcomes calculated by another observer that assumes a different causal ordering of events with respect to the first. After having generalized this concept to experiments involving more sets of random outcomes we are led to introduce a new physical principle, the Principle of Relativity of

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Causal Structure\textsuperscript{\textcopyright}, and to put it as a foundational principle for quantum theory. From this principle we understand that a possible way to move towards a theory of quantum gravity is to retain causal structure of physical events as an observer dependent property. Here we take a first step in this direction comparing the idea that causal structure is an observer dependent property with the role causal structure plays in general relativity (see section \textsection{III}). It is argued that the situation in general relativity theory is somewhat opposite to the one outlined in quantum theory. If, in quantum theory, causal ordering of probabilistic events can be regarded as an observer dependent property, this clearly cannot hold in general relativity. In general relativity, the causal ordering of two events is represented by the value of the metric function evaluated at the two space-time points representing those events. Einstein’s equations relate the metric function to the stress-energy tensor representing energy density in the portion of universe including the two events. This implies that, in general relativity, whether it exists a causal influence between two events or not, ultimately depends on energy density that is an objective, physically measurable quantity and hence cannot be regarded as an observer dependent property. Elevating the principle of relativity of causal structure to universal principle finally leads us to consider dark energy not as a conceptual problem but as an essential ingredient of our current understanding of the universe (see section \textsection{V}).

This research is important for two reasons. The first is that it gives a new foundational principle to motivate the mathematical structure of quantum theory. The second is that, in doing this, it is possible to argue that one of the most puzzling features of modern theoretical physics, dark energy, could be explained elevating the above foundational principle for quantum theory to a universal principle. Clearly this would imply that Einstein’s theory of general relativity should be definitely abandoned and should be elaborated a deeper theory of the cosmos to explain observational data.

\section*{II. SPACE-TIME AND CAUSAL STRUCTURE}

A space-time is, roughly speaking, a mathematical representation of physical events. For any set of physical events \(E\), given two events \(p, q \in E\) one of the three mutually exclusive alternatives must hold:

- \(p\) is the cause of \(q\)
- \(q\) is the cause of \(p\)
- \(p\) is not the cause of \(q\) and vice versa.

Specifying one of the three alternatives for every pair of events leads to define the causal structure of the set \(E\). The first of the above alternatives means that \(q\) is in the future of \(p\) while the second means that \(p\) is in the future of \(q\). This, in turn, is equivalent to say that it exists a physical system that is present in correspondence with both events \(p\) and \(q\). The third indeed means that it is impossible for a physical system to be present in correspondence with both events \(p\) and \(q\) (i.e. \(p\) and \(q\) are causally independent).

In non relativistic (or newtonian) space-time, given an event \(p\) for all other events \(q\) it must hold one of the following alternatives: (i) \(q\) is in the future of \(p\); (ii) \(q\) in the past of \(p\); (iii) \(q\) happens at the same time of (is simultaneous with) \(p\). Regarding this latter case, the events simultaneous with \(p\) constitute points of a three dimensional euclidean space. This distinction comes from the fact that, in non relativistic space-time, the chronological ordering of events is the same as their causal ordering. If \(p\) and \(q\) are one the cause of the other then necessarily one must happen before the other while if \(p\) and \(q\) are causally independent then they must necessarily happen at the same time.

In relativistic space-time the latter fact above mentioned does not hold anymore. In particular, two causally independent events can be simultaneous for some observers and have a different chronological ordering for another observer. From this fact the set of events \(q \in E\) that constitutes the past and future of \(p\) are represented respectively as points of a four dimensional cone while the set of events that are not in past nor in the future of \(p\) are represented by points outside those two cones embedded in euclidean four dimensional space.

Both in non relativistic and relativistic physics, two different observers can in principle assign different coordinates to a physical event \(p\) because they move relatively to one another. In newtonian space-time if observer \(O\) labels \(p\) with coordinates \((t, x, y, z)\) and \(O’\) moves with velocity \(v\) in the \(x\) direction passing \(O\) at \(t = x = y = 0\) then the coordinate labels assigned to \(p\) by \(O’\) are \(t’ = t, x’ = x - vt, y’ = y, z’ = z\). In special relativity, i.e. if \(v\) is sufficiently close to the speed of light \(c\), those relations become \(t’ = (t - vx/c^2)/(1 - v^2/c^2)^{1/2}, x’ = (x - vt)/(1 - v^2/c^2)^{1/2}, y’ = y, z’ = z\). Since two different observers looking at the same physical process must describe the same physics independently of their state of motion relative to one another, it is clear that the above transformations of coordinates leave unaffected any significant physical property. This implies that coordinate labels do not have any intrinsic physical significance since they only depend on which observer labels physical events.

The causal structure of any set of events \(E\) is incorporated in any space-time that can be used to represent those events. Moreover, it constitutes an absolute, observer independent property, contrary to the space-time coordinates assigned to them. For this reason, in both newtonian and relativistic space-time there exist specific quantities represented by functions of the coordinates of any two points \(p\) and \(q\), that remain unchanged in changing point of view from one observer to another. In newtonian physics this function is the time interval \(\Delta t = t_p - t_q\). In special relativity this function is \(M = -((\Delta t)^2 + 1/c^2(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2))\). In gen-
eral relativity this function is represented by the metric
tensor associated to a manifold representing a solution of
Einstein's equations. The value of these functions eval-
uated at every pair of points \((p, q)\) encodes the causal
structure of events.

We can thus say that both newtonian and relativistic
space-time are different mathematical ways to model a
set of events with an absolutely (i.e. independently of
observers) defined causal structure.

Outcomes happening on devices in quantum experi-
ments are supposed to be events in space-time. From this
fact they possess a definite, observer independent causal
structure. In the next section we are going to show that,
although an absolute causal structure of events is a back-
ground assumption in the usual formulation of quantum
theory, the quantum formalism permits to compute cor-
relations for events happening in experiments in such a
way that their causal structure can be regarded as an
observer dependent property.

III. CAUSAL STRUCTURE IN QUANTUM
THEORY

In what follows we are going to show that causal struc-
ture in quantum theory may be regarded as an observer
dependent property rather than fixed in an absolute way.

A. Experiments involving two sets of random
outcomes

Consider the quantum experiment involving a polar-
ized photon shown in figure 1.

![Figure 1: Scheme for an experiment involving a single polarized photon](image)

We have two polarizers \(P_A\) and \(P_B\), the former aligned
at an angle \(\alpha\) and the latter aligned at an angle \(\beta\). A pho-
ton passes first through \(P_A\) is reflected by a mirror and
then passes through \(P_B\). For the experiment to take place
the photon must either be transmitted or be reflected
by polarizer \(P_A\). Hence associated to \(P_A\) we have two
possible mutually exclusive outcomes that we indicate
\(\{a_r, a_t\}\). After the mirror reflection the photon enters
\(P_B\) and then is absorbed by some photon counter. In order
to be counted the photon must either be transmitted or
be reflected by \(P_B\). Hence also associated to \(P_B\) we have
two mutually exclusive outcomes that we call \(\{b_r, b_t\}\).
The information contained in the experiment is repre-
sented by the joint probability distribution \(p(a_i, b_j)\)
with \((a_i, b_j) \in \{a_r, a_t\} \times \{b_r, b_t\}\). The arrows linking the vari-
dious devices represent the path followed by the photon. In
particular the arrow pointing out of \(P_A\) means that the
photon is an output system for polarizer \(P_A\). The arrow
pointing inside \(P_B\) means that the photon is an input
system for \(P_B\). The lightcone and the arrow of time are
drawn to remark that two events associated to any pair
of outcomes \((a_i, b_j)\) are one the cause of the other. In-
deed there is a physical system, i.e. the photon, that
cares the information regarding the probability distribu-
tion \(\{p(a_i)\}_{a_i \in \{a_r, a_t\}}\) from \(P_A\) to \(P_B\). This means that
if the probability distribution \(\{p(a_i)\}_{a_i \in \{a_r, a_t\}}\) changes
and becomes \(\{q(a_i)\}_{a_i \in \{a_r, a_t\}}\) then also the probability
distribution \(\{p'(b_j)\}_{b_j \in \{b_r, b_t\}}\) changes. The above dis-
cussion implies that any pair of outcomes \((a_i, b_j)\) is such
that \(a_i\) causes \(b_j\) and the correlations between the sets
of random outcomes \(\{a_i\}\) and \(\{b_j\}\) are due to a causal
influence.

Consider now the experiment shown in figure 2.

![Figure 2: Scheme for an experiment involving maximally
entangled photons](image)

We have the same polarizers \(P_A\) and \(P_B\) involved in
the previous experiment and for simplicity we assumed
they are aligned in the same direction as before. Two
photons in an entangled state of zero total angular momentum start from a source of entangled photons, $M'$, and reach independently $P_A$ and $P_B$ respectively. After they have passed the polarizers they are absorbed by two photon counters placed after $P_A$ and $P_B$ respectively. For the experiment to take place, both the photons must be either transmitted or reflected by the respective polarizers before being detected. Hence also in this case, associated to both $P_A$ and $P_B$, there are two sets of mutually exclusive outcomes $\{a_r, a_t\}$ and $\{b_r, b_t\}$ and these represent the same outcomes as in the previous experiment. The joint probability distribution $p(a_i, b_j)$ with $(a_i, b_j) \in \{a_r, a_t\} \times \{b_r, b_t\}$ contains the information about the experiment. In figure 2 there are two arrows pointing inside polarizers $P_A$ and $P_B$ respectively. Also in this case are drawn the lightcone and the arrow of time to help visualizing that any pair of outcomes $(a_i, b_j) \in \{a_r, a_t\} \times \{b_r, b_t\}$ represents two space-like events.

The two experiments described above seem very different. The latter involves, for each repetition of the experiment, a pair of entangled photons while the former involves a single photon. This difference in their physical description is due to the fact that in each run of the experiment, it is assumed in one case that the pair of outcomes $(a_i, b_j)$ are one the cause of the other (the casual relationship being represented by a photon travelling from $P_A$ to $P_B$) and in the other case that they are two space-like events (since they are due to two causally independent systems). We can thus say that the main difference in the two above experiments relies on how, each run of the experiment, the outcomes $(a_i, b_j) \in \{a_{r,t}\} \times \{b_{r,t}\}$ are embedded in spacetime. The setup in figure 1 involves three devices, the two polarizers $P_A$ and $P_B$ and a mirror $M$. The experiment in figure 2 also involves three devices, two of them are the same polarizers as before while the third device, $M'$ is a source of entangled photons. For the experiment in figure 1 the photon is an output system for $P_A$, it is an input and an output for $M$ while it is an input system for $P_B$. For the experiment in figure 2 the photons involved may be regarded as two outputs for $M'$ and as two input systems for $P_A$ and $P_B$ respectively. Hence the difference between the two experiments is that a photon is seen as an output system for $P_A$ (and in consequence as an input for $M$) in the experiment of figure 1 while it is seen as an input system for $P_A$ (and in consequence as an output for $M'$) in the experiment of figure 2. From the above discussion we can say that the existence of a causal relationship between the region where lies $P_A$ (where happen outcomes $\{a_r, a_t\}$) and the region where lies $P_B$ (where happen $\{b_r, b_t\}$) is equivalent to assign a specific input/output structure for the devices involved in the experiment. We can thus say that the input/output structure of the devices involved in the experiment is equivalent to the causal structure assigned to the outcomes associated to those devices.

In both the situations described above it is assumed a definite causal structure between the region of space where lies $P_A$ and that where lies $P_B$. This means that it is assumed in an absolute way either that between region $P_A$ and region $P_B$ there exists a causal relationship or that regions $P_A$ and $P_B$ are space-like separated. On the other hand, every experiment in quantum theory is intrinsically probabilistic and whatever an observer might experience reduces to correlations between outcomes happening on two devices in distinct regions. This observation suggests that a definite causal structure between region $P_A$ and region $P_B$ could not be significant in predicting joint probabilities for events happening in these two regions. Since correlations between events is the only observable and physically predictable property in quantum theory, it could be the case that the two experiments described in figure 1 and 2 are simply a different way to describe the same experiment. Indeed they both define a joint probability distribution between the values of the same pair of observables (polarizations along $\alpha$ and $\beta$), they refer to the same type of system (the photon) and differ only because in the former it is assumed a causal relationship between regions $P_A$ and $P_B$ while in the latter it is assumed that regions $P_A$ and $P_B$ are space-like separated. In what follows we will show that the mathematical formalism of quantum theory is consistent with the above suggestion.

Suppose that an experimenter sets up one of the two experiments illustrated above, say the one in figure 1 for definiteness. Two observers look at this experiment without knowing the nature of device $M$ and the actual input/output structure between the devices. The observers experience the correlations between the set of outcomes $\{a_r, a_t\}$ associated to $P_A$ and the set $\{b_r, b_t\}$ associated to $P_B$. To one observer it is said that $M$ is a mirror and that the setup is actually the one in figure 1. To the other observer it is indeed said that $M$ constitutes a source of maximally entangled photons and that the setup corresponds to the one in figure 2. We will call the former observer $O_1$ and the latter observer $O_2$. Comparing figure 1 and 2 we can readily understand that $O_1$ assumes that photons constitute outputs for $P_A$ and inputs for $P_B$ while $O_2$ assumes that photons constitute inputs for both $P_A$ and $P_B$. These two assumptions cannot be verified (or falsified) by the two observers experiencing correlations between $\{a_r, a_t\}$ and $\{b_r, b_t\}$. Hence they can calculate the joint probability distribution $\{p(a_i, b_j)\}$ with $(a_i, b_j) \in \{a_{r,t}\} \times \{b_{r,t}\}$ on the basis of the information they respectively have regarding causal structure. We will now show that for all $(a_i, b_j) \in \{a_{r,t}\} \times \{b_{r,t}\}$, the probability calculations of observers $O_1$ and $O_2$, although apparently different, reduce to the same calculation and give rise to the same probability value. According to this we may conclude that the two experiments in figures 1,2 are the same experiment seen by two different observers who assume a different causal structure between the regions where are situated polarizers $P_A$ and $P_B$.
$O_1$ assumes that the polarizer $P_A$ prepares an ensemble represented by $p(a_r)\langle a_r | + (1 - p)\langle a_t | a_t \rangle$. By now, let us assume $p = 1/2$ for simplicity. The probability of seeing outcome $b_t$ in correspondence of $P_B$ given that it is prepared a photon in state $a_r$ is $p(b_t|a_r) = |\langle b_t | a_r \rangle|^2$ thus the joint probability is:

$$p_{O_1}(a_r, b_t) = 1/2 |\langle b_t | a_r \rangle|^2$$

(1)

$O_2$ indeed assumes that $M$ is a source of entangled photons in state $|\psi\rangle = 1/\sqrt{2}(|a_r a_r \rangle + |a_t a_t \rangle)$. The joint probability of seeing outcomes $a_r$ and $b_t$ calculated by $O_2$ is:

$$p(a_r, b_t) = |\langle a_r | b_t \rangle 1/\sqrt{2}(|a_r a_r \rangle + |a_t a_t \rangle)|^2$$

(2)

But the above equation actually reduces to (1). Expliciting (2) we have:

$$p_{O_2}(a_r, b_t) = 1/2(\langle a_r | a_r \rangle^2 |\langle b_t | a_r \rangle|^2 + |\langle a_r | a_t \rangle|^2 |\langle b_t | a_t \rangle|^2 +$$

$$+ 2 |\langle a_r | a_r \rangle| |\langle b_t | a_t \rangle| |\langle a_r | a_t \rangle| |\langle b_t | a_r \rangle|)$$

(3)

and all terms in (3) are zero except the first thus we can write:

$$p_{O_2}(a_r, b_t) = 1/2 |\langle b_t | a_r \rangle|^2$$

(4)

Clearly the above reasoning is true for every pair $(a_r, b_t) \in \{a_r, a_t\} \times \{b_r, b_t\}$. Moreover it is simple to convince ourselves that nothing would change if we assumed that the set up prepared by the experimenter at which $O_1$ and $O_2$ both look was that in figure 2 in place of the one in figure 1. This simple example shows that the assumptions of $O_1$ and $O_2$ regarding causal structure of regions $P_A$ and $P_B$ are equivalent for the purpose of calculating joint probabilities. Whatever an observer of anyone of the above experiments can experience are correlations between outcomes in region $P_A$ and outcomes in region $P_B$, and whatever he can predict are joint probabilities for the outcomes in those regions. Hence, the fact that between those two regions there exists a causal relationship or not is a property that depends on the assumption of an observer and cannot be fixed absolutely for all observers in any way.

Note that the equivalence stated above derives from the fact that (3) is an alternative way of writing (1). If it were not so then causal structure could not be an observer dependent property. Indeed the correlations between region $P_A$ and region $P_B$ depend on the probability distribution $\{p(a_i, b_j)\}$ and if the probability distribution calculated by observer $O_2$ was different from that calculated by observer $O_1$ then one of the observers, $O_2$, would predict wrong probabilities and would become aware, after comparing his calculations with those of $O_1$, that correlations are effectively due to a causal relationship between $P_A$ and $P_B$. This implies that the equivalence of the two above situations is a consequence of how in quantum theory are performed probability calculations for the experiments illustrated in figure 1 and 2.

The two situations considered above are far from being the most general experiments correlating random outcomes in two regions of space. The equivalence of (1) and (2) could in fact be a numerical coincidence. In the remaining part of this section we will prove that the above property is a general feature of quantum theory. We will consider a generic quantum experiment in which two devices $D_A$ and $D_B$ display two sets of random outcomes $\{a_i\}_{i \in X}$ and $\{b_j\}_{j \in Y}$ respectively with $X$ and $Y$ two sets of outcomes. The information on such correlations is contained in the joint probability distribution $\{p(a_i, b_j)\}_{(i,j) \in X \times Y}$. As in the previous example, we suppose that two observers $O_1$ and $O_2$ are looking at the experiment; $O_1$ assumes that correlations between $D_A$ and $D_B$ are due to a system causally correlating the outcomes in $\{a_i\}_{i \in X}$ to those in $\{b_j\}_{j \in Y}$ while $O_2$ assumes that $D_A$ and $D_B$ lie in space-like separated regions.

Observer $O_1$

$O_1$ assumes that correlations are due to a causal relationship. In this case a system $\mathcal{S}$ carries the information of the probability distribution of one of the sets of outcomes, say $\{a_i\}_{i \in X}$ with probability distribution $\{p_i\}_{i \in X}$, from device $D_A$ to device $D_B$. The experiment seen by $O_1$ is represented in figure 3.

![Figure 3](image_url)

**Figure 3:** Scheme for a generic quantum experiment involving two sets of random outcomes. The outcome on device $D_A$ causes the outcome on device $D_B$.

System $\mathcal{S}_A$ is the output system for $D_A$ while $\mathcal{S}_B$ is the input system for $D_B$. Of course they may be the same system and we distinguish them only for the purpose of distinguishing the arrow associated to $D_A$ from that associated to $D_B$ in the above diagram. An outcome $a_{i_0} \in \{a_i\}_{i \in X}$ is a preparation belonging to the preparations ensemble $\{a_i\}_{i \in X}$ with associated probability distribution $\{p_i\}_{i \in X}$. The ensemble is represented by
a density matrix $\rho$ and a POVM $\{a_i\}_{i \in X}$ as follows:

$$\rho = \sum_{i \in A} \text{Tr}[a_i \rho] \frac{\sqrt{\rho} a_i \sqrt{\rho}}{\text{Tr}[a_i \rho]} \quad (5)$$

To achieve as much generality as we can, we will not make any restriction on $\rho$ apart from assuming that it does not represent a pure state since otherwise the outcomes on device $D_A$ would not be random anymore contrary to our initial assumptions. The ensemble represented by $\rho$ causes probabilistically an outcome $b_{j_0} \in \{b_j\}_{j \in Y}$ on device $D_B$. In the most general case, this is represented by an element of a POVM $\{b_j\}_{j \in Y}$ for hilbert space $\mathcal{H}_{\mathcal{I}B}$. The ensemble represented by $\rho$ before causing outcome $b_{j_0}$ will eventually undergo an evolution that is generically represented by a Completely Positive Trace Preserving (CPTP) map $\mathcal{T}$. Its Kraus decomposition is $\sum_m K^m \otimes K^{m_1}$ with $K^m = \sum_{e,f,c,d} K^m_{e,f,c,d} |e\rangle_B |f\rangle_A \langle c| \otimes \langle d|$ Kraus operator $K$ where $\{ |e\rangle \}_{e=1}^m, \{ |f\rangle \}_{f=1}^m$ are orthonormal basis for hilbert space $\mathcal{H}_{\mathcal{I}A}$ and $\mathcal{H}_{\mathcal{I}B}$ respectively. We now explicit the evolution of ensemble $\rho$ by means of transformation $\mathcal{T}$. The density matrix obtained after the evolution is:

$$\mathcal{T}(\rho) = \sum_{m,e,f,c,d} K^m_{e,f,c,d} K^{m_1}_{c,d} |e\rangle_B |f\rangle_A \langle c| \otimes \langle d| \quad (6)$$

Using the fact that $\sum_m K^m \otimes K^{m_1}$ can be written as:

$$\sum_{m,e,f,c,d} K^m_{e,f,c,d} K^{m_1}_{c,d} |e\rangle_B |f\rangle_A \langle c| \otimes \langle d| = \sum_{m,e,f,c,d} K^m_{e,f,c,d} \sqrt{\rho} |e\rangle_B |f\rangle_A \langle c| \otimes \langle d| \quad (7)$$

and the polar decomposition of $\rho$ we have:

$$\mathcal{F}(\rho) = \text{Tr}_A[ \sum_{m,e,f,c,d} K^m_{e,f,c,d} \sqrt{\rho} |e\rangle_B |f\rangle_A \langle c| \otimes \langle d|] \quad (8)$$

Note that, for the polar decomposition of $\rho$ to be uniquely defined, one must assume $\rho$ to be full rank in $\mathcal{H}_{\mathcal{I}A}$. The density matrix obtained after the evolution can thus be written as $\mathcal{T}(\rho) = \text{Tr}_A[ \mathcal{F}_\rho]$ where we define:

$$\mathcal{F}_\rho := \sqrt{\rho} \otimes I_B \left[ \sum_m (K^m \otimes K^{m_1}) \right] \sqrt{\rho} \otimes I_B \quad (9)$$

where $I_B$ is the identity matrix on $\mathcal{H}_{\mathcal{I}B}$. From (4) we see that the evolution of ensemble $\rho$ can be represented as an operator acting on $\mathcal{H}_{\mathcal{I}A} \otimes \mathcal{H}_{\mathcal{I}B}$. The probability calculated by observer 1 is then:

$$p_1(a_{i_0}, b_{j_0}) = \text{Tr}_B[ b_{j_0} \text{Tr}_A[ \mathcal{F}_\rho a_{i_0} ]] \quad (10)$$

Observer $O_2$

$O_2$ indeed assumes that correlations are not due to a causal relationship. This means that the two sets of outcomes constitute two measurements performed in parallel on two copies of system $\mathcal{S}$. In figure 4 it is represented the same experiment of figure 3 as seen by observer $O_2$ assuming that the regions in which are situated $D_A$ and $D_B$ are space-like separated.

Assumptions of observers $O_1$ and $O_2$ are equivalent

We are now going to prove the following statement: Given the mathematical objects used to calculate joint probabilities of the outcomes by $O_1$, there exists a unique choice of mathematical objects that permits $O_2$ to calculate the same joint probability distribution of outcomes. Before proving the above statement we recall the discussion regarding the equivalence between input/output structure and causal structure in quantum experiments. The only difference between the experiment seen by $O_1$ and the experiment seen by $O_2$ is that $\mathcal{F}_A$ is assumed as an output for $D_A$ by $O_1$ while is assumed as input for $D_A$ by $O_2$. This becomes apparent comparing figure 3 with figure 4. Based on this observation, we now give the rule that permits to prove the statement done at the beginning of this paragraph.

Transformation Rule: If a system $\mathcal{S}$, with hilbert space $\mathcal{H}_{\mathcal{S}}$ is an input (output) for $O_1$ and an output (input) for $O_2$, then the operators involving $\mathcal{H}_{\mathcal{S}}$ used by $O_1$ are the transposed on $\mathcal{H}_{\mathcal{S}}$ of those used by $O_2$. 

**Figure 4**: Scheme for a generic quantum experiment involving two sets of random outcomes. The outcome on device $D_A$ and the outcome on $D_B$ are space-like events.
From the above rule, if $a_{i_0}$ represents an element of the preparation ensemble $\rho$ of $O_1$, $a_i^T$ represents the corresponding measurement outcome for $O_2$. For the same reason, the bipartite state $\tau_{AB}$ has the following expression:

$$\tau_{AB} = \mathcal{T}^{T_A} = \sqrt{\rho^T} \otimes I_B \frac{1}{N} \sum_{m} (K^m \otimes K'^m)^{T_A} \sqrt{\rho^T} \otimes I_2$$

(12)

Where $T_A$ denotes partial transposition on hilbert space $\mathcal{H}_{/A}$. First we have to prove that (12) is a normalized bipartite state. This can be seen defining the normalized bipartite state on two copies of $\mathcal{H}_{/A}$, $|\Phi\rangle_{AA'}$:

$$|\Phi\rangle_{AA'} = \sqrt{\rho^T} \otimes I_{A'} \sum_{j} |j\rangle_A \otimes |j\rangle_{A'}$$

(13)

where $\{|j\rangle\}^d$ is an orthonormal basis for space $\mathcal{H}_{/A}$. Exploiting (13) we can write:

$$\mathcal{I} \otimes T(|\Phi\rangle \langle \Phi|) = \tau_{AB}$$

(14)

where $\mathcal{I}$ is the identity map on $\mathcal{H}_{/A}$ and $T$ represents the evolution defined above. From (13) we can see that $\tau_{AB}$ is a normalized bipartite state since $\mathcal{T}$ is a TPCP map acting on system $\mathcal{T}_A$ and $|\Phi\rangle \langle \Phi|$ is a normalized bipartite state. The probability $p_1(a_{i_0}, b_{j_0})$ expressed in (14) calculated by $O_1$ is then equal to the probability $p_2(a_{i_0}, b_{j_0})$ calculated by $O_2$, namely:

$$p_2(a_{i_0}, b_{j_0}) = \text{Tr}_{AB}[a_{i_0}^T \otimes b_{i_0}^T \mathcal{T}_{\rho}^{T_A}] = p_1(a_{i_0}, b_{j_0})$$

(15)

This expression represents the probability for a given pair of outcomes $(a_{i_0}, b_{j_0}) \in \{a_i, b_j\}_{(i,j) \in X \times Y}$ to jointly happen. This proves the statement done at the beginning of this paragraph.

In conclusion, every experiment in quantum theory is intrinsically probabilistic and whenever it correlates two sets of random outcomes displayed by two devices in two distinct regions of space, an observer can only experience correlations between these two sets of outcomes and can only predict their joint probabilities. The causal structure of these two regions, namely whether the correlations have a causal origin or not, is always assumed a priori and cannot be subject to a physical verification. From this fact it follows that if two observers look at one such experiment and for some reason an observer assumes that correlations are due to a causal relationship and the other observer assumes that they are not, they cannot become aware of differences between their respective probabilistic predictions and the joint probabilities originated by the experiment.

B. Experiments involving more sets of random outcomes

Generalizing the result obtained above to experiments involving more than two sets of outcomes presents some subtleties. Consider an experiment involving three sets of random outcomes appearing in three distinct regions of space, say regions $A,B,C$, such that the outcomes in $A$ cause the outcomes in $B$ and these in turns cause the outcomes in $C$. Let us suppose that the random outcomes happening in $A,B,C$ are $\{a_i\}, \{b_j\}, \{c_k\}$ respectively. A physical system $\mathcal{I}$ passing through the three regions constitutes the causal influence propagating from $A$ to $B$ and then from $B$ to $C$. From an operational point of view $\mathcal{I}$ is the output of region $A$, the input and the output of region $B$ and the input of region $C$. An outcome in region $B$ thus represents a possible evolution of $\mathcal{I}$. In quantum theory a system evolution is represented by a CPTP map and is a deterministic notion. The only way to take into account randomness in region $B$ is thus to consider convex combinations of CP maps that decrease the trace of states. An observer assuming an input/output structure of regions $A,B,C$ modified with respect to the one given above, does never arrive to assume $A,B,C$ as three space-like separated regions. Conversely, an experiment where $A,B,C$ are space-like separated regions and in which the outcomes in the three regions are correlated, is due to a tripartite entangled state. An observer assuming, for this experiment, a different input/output structure, can never arrive to assume that $A,B,C$ are such that outcomes in $A$ cause outcomes in $B$ and that these in turns cause outcomes in $C$. From these examples we see that when we take into account three regions of space $A,B,C$, displaying correlated random outcomes, if an observer is able to calculate joint probabilities of the outcomes assuming these three regions as space-like separated, there cannot exist an observer assuming that outcomes on $A$ cause outcomes on $B$ that in turns cause outcomes on $C$. In order to generalize the result in the previous section to experiments involving more than two sets of random outcomes we thus simply consider that different observers of the same experiment can in principle assume a different input/output structure for the devices involved. Suppose now to have an experiment in which there are three devices, $D_A$, $D_B$, $D_C$ in regions $A,B,C$ respectively displaying random correlated outcomes and that an observer $O_2$, in order to predict the joint probabilities of the outcomes, assumes that $A,B,C$ are three space-like separated regions. Let the set of outcomes on the three devices be $\{a_i\}_{i \in X} \times \{b_j\}_{j \in Y} \times \{c_k\}_{k \in Z}$ and the associated joint probability distribution be $\{p(a_i, b_j, c_k)\}_{i,j,k \in X \times Y \times Z}$. Let $\mathcal{I}_A, \mathcal{I}_B, \mathcal{I}_C$ be the systems to which the outcomes on $D_A, D_B, D_C$, refer respectively. $O_2$ assumes that $\mathcal{I}_A, \mathcal{I}_B, \mathcal{I}_C$ are respectively three inputs for devices $D_A, D_B$ and $D_C$. This is represented in figure 5.
Another observer, $O_1$, assumes that systems $A$ and $B$ are inputs for $D_A$ and $D_B$ respectively and system $C$ is an output for $D_C$. This is represented in figure 6.

It is easy to see that this situation is not different from the one analyzed in the above sections. $O_1$ assumes the outcomes on devices $D_C$ as representing preparations belonging to some preparation ensemble represented by a density matrix $\rho$:

$$\rho = \sum_{k \in Z} \frac{\text{Tr}[c_k \rho] \sqrt{\rho} c_k \sqrt{\rho}}{\text{Tr}[c_k \rho]}$$

Moreover he assumes that outcomes on devices $D_A$ and $D_B$ are POVMs $\{a_i\}_{i \in X}$, $\{b_j\}_{j \in Y}$. The ensemble $\rho$ undergoes an evolution represented by a CPTP map $\mathcal{F}$ with Kraus decomposition $\sum_{m} K^m \otimes K^m*$ resulting in a density matrix $\mathcal{F}(\rho)$ having the following expression:

$$\mathcal{F}(\rho) = \text{Tr}_C[\mathcal{F}_\rho]$$

where

$$\mathcal{F}_\rho = \sum_{m,e,f,cd} K_{ef}^m K_{cd}^m \sqrt{\rho} |e\rangle_C \langle f| \sqrt{\rho} \otimes |e\rangle_{AB} \langle f|$$

We see that the only difference between (18) and (8) explicitely refers to the hilbert space of a composite system $H_{AB}$. From the transformation rule stated in the previous section, $O_2$ assumes that outcomes on devices $D_A$, $D_B$ and $D_C$ are respectively represented by the POVMs $\{a_i\}_{i \in X}$, $\{b_j\}_{j \in Y}$, $\{c_k\}_{k \in Z}$ where $T$ denotes transposition. The three devices seen by $O_2$ are indeed correlated by a tripartite entangled state $\tau_{ABC}$ that, according to the transformation rule of the previous section, is written as:

$$\tau_{ABC} = \mathcal{F}_\rho^T_C$$

$O_1$ and $O_2$ experience the same joint probability distribution since:

$$\text{Tr}_{ABC}[\tau_{ABC} a_i \otimes b_j \otimes c_k] = \text{Tr}_{AB}[a_i \otimes b_j \text{Tr}_{C}[\mathcal{F}_k c_k]]$$

In the same way they can be treated all the cases in which different observers assume different input/output labels for $A$, $B$ and $C$. Based on these arguments it can be seen that analogous results hold for generic experiments in which an arbitrary number of devices display correlated random outcomes.

C. Related work

The work presented here has connections with three other works by Hardy [11], Oreshkov-Costa-Bruckner [12] and Leifer-Spekkens [13]. All these works present formulations of quantum theory in which calculations of joint probabilities for sets of outcomes in distinct regions of space can be performed with a mathematical formalism that is not sensitive of the causal structure imposed to the regions. The mathematical objects that permit this to be done are called Causaloid, Process Matrix and Quantum Conditional State for the three works cited above respectively. Note that quantum theory, as is currently regarded, is a formalism that is sensitive to what causal structure is imposed to different correlated regions. For two devices in two regions of space displaying correlated random outcomes such that the outcomes on one device cause those on the other, we have the following mathematical representation: one set of outcomes is represented by a density matrix for a single system that is subject to some evolution represented by a linear map; the other set of outcomes is represented by a set of positive operators that sum to the identity. For two devices displaying correlated random outcomes in two space-like separated regions we have indeed the following mathematical representation: the two sets of outcomes are represented by two sets of positive operators that sum to the
identity; a state for the composite system, represented by a density matrix for this system, originates the correlations between the outcomes. On the other hand the analysis done in this paper suggests that this may not be the proper way to approach the theory. Indeed, investigating more deeply quantum theory from this point of view we have shown that the two above mathematical representations are more similar than one could expect.

Here we enlight similarities and analogies of this paper with \[8,9\]. The operator defined as \(\mathcal{T}_\rho\) in \[8\], i.e. the evolution by means of map \(\mathcal{T}\) of ensemble \(\rho\), has a lot of analogies with a process matrix \[12\]. Indeed they both represent a way to calculate joint probabilities for outcomes happening in different regions of space that is insensitive to what causal structure is assumed for the regions. This is because the operator \(\tau_{AB}\) establishing correlations for outcomes in space-like separated regions is a mathematical object of the same nature of \(\mathcal{T}_\rho\) (being simply its partial transposition). The main difference between the situation depicted in the previous section and the process matrix formalism is that in the former case, outcomes are represented by POVM elements while in the latter case they are represented by quantum operations. Hence we could regard \(\mathcal{T}_\rho\) as a process matrix for POVM elements. There are even more strict similarities with the work in \[13\]. To see this note that \(\tau_{AB}\) in \[13\] is simply the joint state obtained with an acausal conditional state (see equation (9) in \[13\]). \(\mathcal{T}_\rho\) in \[8\], on the other hand is a causal joint state, i.e. the joint state obtained with a causal conditional state (see section IIIE in \[13\]). Relationships of the work in \[8\] with the work presented here (as with the other two works) are less explicit. The work in \[8\] has the remarkable feature of being formulated in a general probabilistic framework. To achieve such generality it becomes necessarily more abstract and the formulation of quantum theory in this framework suffers of such abstractness. The main idea of the causaloid is that embedding probabilistic physical processes in space-time (hence giving to probabilistic events a causal structure) is an instance of compression of information. The starting point to reach this conclusion is that causal structure and space-time in physics may not be regarded as something really existing in an objective way. Indeed this is very close to the starting point we adopted in the previous section and to the idea that causal structure of probabilistic outcomes is an observer dependent property.

D. Relativity of causal structure and no-signalling

In this subsection we will discuss the no-signalling principle in light of the result obtained so far. Although it could seem at first sight that our result contradicts no-signalling, we will show that the principle of relativity of causal structure is indeed consistent with it. In particular we are going to show that there is no contradiction in relativity of causal ordering of probabilistic outcomes in a quantum experiment even when it is established in an absolute way that the correlations among those outcomes are either "signalling" or "no-signalling".

No-signalling is the name of a condition usually formulated in the context of foundations of quantum theory for outcomes correlations between two space-like separated devices due to an entangled state. This condition is elevated to a principle because it ensures that non-local correlations in quantum mechanics do not allow instantaneous signalling between two parties. The following provides the definition of no-signalling condition for outcomes correlations of two space-like separated devices:

Definition 1 No-signalling condition

Suppose to have two generic sets of outcomes happening respectively on two devices \(D_A\) and \(D_B\) due to an entangled state \(\tau\) of a composite system. We say that the outcomes correlations obey the no-signalling condition if the following are both satisfied:

- for all \(b_{j_0} \in \{b_j\}_{j \in Y}\), where \(\{b_j\}_{j \in Y}\) is any set of outcomes on \(D_B\), it holds:
  
  \[
p(b_{j_0}|\tau) = \sum_{a_i} p(a_i, b_{j_0}|\tau) = \sum_{a_i'} p(a_i', b_{j_0}|\tau) \tag{21}
  \]

  for all possible different pairs of sets of outcomes \(\{a_i\}_{i \in X}, \{a_{i'}\}_{i' \in X'}\) happening on \(D_A\).

- for all \(a_{i_0} \in \{a_i\}_{i \in X}, \{b_j\}_{j \in Y}\) is any set of outcomes on \(D_A\), it holds:
  
  \[
p(a_{i_0}|\tau) = \sum_{b_j} p(a_{i_0}, b_j|\tau) = \sum_{b_{j'}} p(a_{i_0}, b_{j'}|\tau) \tag{22}
  \]

  for all possible different pairs of sets of outcomes \(\{b_j\}_{j \in Y}, \{b_{j'}\}_{j' \in Y}\) happening on \(D_B\).

In \[14\] it is showed that it is possible to formulate models in which this condition holds and where the outcomes correlations are stronger than those originated by maximally entangled states. This implies that no-signalling condition alone cannot be put as a foundational constraint for quantum correlations. In \[15\] they are explored the consequences of assuming this condition alone, for a generic non-local probabilistic theory.

The meaning of the above condition is the following. If the probability distribution and the outcomes are changed on device \(D_A\) (\(D_B\)) then the probability distribution of the outcomes on \(D_B\) (\(D_A\)) is not affected. This is the case since the probability of any outcome \(a_{i_0}(b_{j_0})\) that happens on \(D_A\) (\(D_B\)) is the joint probability of \(a_{i_0}(b_{j_0})\) with the outcome corresponding to the coarse graining of all the outcomes that can possibly happen on \(D_B\) (\(D_A\)). No-signalling is thus guaranteed by the fact the correlations are such that changing something in one of the two devices cannot result in any statistical change in the outcomes on the other device. Such changes, if possible, would be due to an instantaneous influence since outcomes correlations are instantaneous, and this would permit instantaneous signalling from one device to the other.
The apparent tension between no-signalling and relativity of causal ordering is due to the fact that no-signalling conditions are formulated using joint probabilities of outcomes and thus provide absolute statements about the (im)possibility of influencing probability distributions on one device manipulating a different and space-like separated device. Provided that causal ordering of probabilistic outcomes is not an absolute property one could in fact imagine the following misleading scenario. An observer assumes that two sets of correlated outcomes are such that the outcomes in one set cause those in the other set; in this case correlations can be signalling and one of the two conditions in definition 1 can be violated. A second observer assuming the same outcomes to be space-like separated would then experience signalling correlations. But this would mean that from his point of view there could be an instantaneous influence from one device to another. In what follows we are going to show that this situation is never attained. To see it is so suppose that an observer $O_1$ is looking to an experiment in which they are displayed correlated random outcomes on two devices, $D_A$ and $D_B$ and that he assumes that outcomes on device $D_A$ cause those of device $D_B$. Accordingly he assumes the specific input/output structure depicted in figure 3. From his point of view the correlations between the outcomes on $D_A$ and those on $D_B$ can be signalling. This means that $O_1$ can devise situations in which:

$$\sum_{i \in X} p(a_i, b_{j_0}) \neq \sum_{i' \in X'} p(a'_i, b_{j_0}).$$  \tag{23}

This means that the preparations ensemble $\{a_i\}_{i \in X}$ with probability distribution $\{p_i\}_{i \in X}$ represented by a density matrix $\rho$ is changed into the preparation ensemble $\{a'_i\}_{i' \in X'}$ with probability distribution $\{p'_i\}_{i' \in X'}$ represented by a density matrix $\rho'$ different from $\rho$. An observer $O_2$ on the other hand assumes that the two devices are indeed space-like separated thus assuming the input/output structure of figure 4. From his point of view (23) is by no means paradoxical. Indeed if we write the explicit expression for (23) as calculated by $O_1$ we have, using (10):

$$\sum_{i \in X} \text{Tr}_{AB}[b_j \otimes a_i \mathcal{T}_\rho] \neq \sum_{i \in X} \text{Tr}_{AB}[b_j \otimes a'_i \mathcal{T}_{\rho'}].$$  \tag{24}

where $\mathcal{T}_\rho$ represents the evolution by means of TPCP map $\mathcal{T}$ of ensemble $\rho$ and similarly for $\mathcal{T}_{\rho'}$. According to observer $O_2$ following the transformation rule stated in section 3.2 (24) is rewritten as follows:

$$\sum_{i \in X} \text{Tr}_{AB}[b_j \otimes a_i^T \mathcal{T}_\rho^{T_A}] \neq \sum_{i \in X} \text{Tr}_{AB}[b_j \otimes a'_i^T \mathcal{T}_{\rho'}^{T_A}].$$  \tag{25}

This expression means that the bipartite state $\tau = \mathcal{T}_\rho^{T_A}$ has changed into the bipartite state $\tau' = \mathcal{T}_{\rho'}^{T_A}$ and the verification of the above inequality may not be ascribed to an instantaneous influence between two space-like separated devices but simply to a change in the bipartite state correlating the two devices.

On the other hand, for observer $O_2$ assuming that correlations of the outcomes on $D_A$ and $D_B$ are due to a bipartite state $\tau$ we must have that definition 3 holds. However, there is nothing that prevents observer $O_1$ to assume that outcomes on $D_A$ cause the outcomes on $D_B$. This is the case since no signalling conditions are only a set of necessary conditions that the outcomes correlations of two devices must satisfy if the devices are space-like separated. Indeed, from the Choi-Jamiolkowsky isomorphism \cite{16,17}, observer $O_1$ can always interpret every bipartite state $\tau$ as the evolution by means of a CPTP map $\mathcal{T}$ of an ensemble $\rho$ thus assuming no-signalling correlations between two sets of outcomes as due to a system carrying a causal influence from one device to another.

E. Two principles for quantum theory

The work presented here, compared to those reviewed above, has, in our opinion, a deeper foundational value since it poses a new physical principle, the observer dependence of causal structure, as a foundational principle for quantum theory. This is achieved recognizing the equivalence of input/output structure and causal structure and showing that the mathematical formalism of quantum theory is consistent with the assumption that input/output structure is an observer dependent property.

We can thus summarize the work done in this section saying that quantum theory is consistent with the two following principles:

Principle of causality The input/output structure of the devices involved in a quantum experiment defines the causal structure of the outcomes happening on those devices.

Principle of relativity of causal structure Two observers looking at a given quantum experiment and assuming a different causal structure for the outcomes involved in the experiment cannot become aware of differences in their respective probabilistic predictions.

In the next section, the principle of relativity of causal structure will be compared with the role causal structure plays in general relativity. In particular it is argued that the situation in general relativity is somewhat opposite to the one outlined above. This is the case since, in general relativity, whether two events in two distinct regions of universe are space-like or not is determined by the metric that, in turn, is related to the stress energy tensor via Einstein’s equations. This implies that in general relativity causal structure depends on a (in principle) measurable physical quantity, energy density, and cannot be regarded as an observer dependent property.
IV. CAUSAL STRUCTURE IN GENERAL RELATIVITY

In this section we briefly examine the role causal structure of events has in general relativity. The main equations of general relativity are Einstein’s equations relating the metric of a portion of space-time manifold describing a given portion of universe with the mass/energy content of that portion of universe. They are often expressed in the following compact form [18]:

\[ G_{\mu\nu} = kT_{\mu\nu} \]  

(26)

On the r.h.s. \( k \) is a constant and \( T_{\mu\nu} \) is the stress-energy tensor; on the l.h.s \( G_{\mu\nu} \) is the Einstein’s tensor and has the following expression:

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} \]  

(27)

where \( g_{\mu\nu} \) is the metric, \( R_{\mu\nu} \) is the Ricci tensor, \( R \) is the Ricci scalar and \( \Lambda \) is the cosmological constant. On a manifold, \((M, g)\), a geodesic is a path \( x^\mu(\lambda) \) characterized by the following equation [18, 19]:

\[ \frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0 \]  

(28)

In the above equation \( \Gamma^\mu_{\rho\sigma} \) are the coefficients of the Levi-Civita connection associated to the metric of the manifold (in general one can use any connection but in general relativity it is used only the Levi-Civita one). This is written as follows:

\[ \Gamma^\mu_{\rho\sigma} = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \]  

(29)

where \( \partial_\mu \) denotes partial derivative, \( g_{\nu\sigma} \) is the metric and \( g^{\rho\sigma} \) is its inverse. Equation (28) can be interpreted as the vanishing of the covariant derivative of \( x^\mu(\lambda) \). This means that any vector on \( x^\mu(\lambda) \) is transported parallel to itself along this path. The tangent vector to a point of the geodesic describes an interval between two points in the tangent space. If the manifold is a solution of Einstein’s Equations, such interval can be time-like, null or space-like depending on the norm of the vector. Since a geodesic describes a path along which a tangent vector of the manifold is parallel transported, we have that if the tangent vector on a given point of the geodesic is time-like, null or space-like, the tangent vector on any other point of the geodesic will preserve this property. From this, one interprets geodesics where the tangent vector is time-like or null as paths followed respectively by freely falling material particles or photons. On the other hand if the tangent vector is space-like, then there is no physical system that can follow the path corresponding to the geodesics. From this fact we can state that, in general relativity, given two points in space-time \( x_a, x_b \), pertaining to two different regions of universe \( R_A, R_B \) respectively, it can exist a causal relationship between them (namely it is possible for a material or light particle to start at \( x_a \) and cause an effect at \( x_b \)) if the two points lie on a time-like or null geodesic. On the other hand it cannot exist a causal relationship between the two points if they lie on a space-like geodesic. From (28) and (29) we see that, in last analysis, the metric tensor is the object characterizing geodesics. This together with Einstein’s equations imply that the stress-energy tensor representing the energy density in a given portion of universe establishes whether between two space-time points it can exist a causal relationship.

According to general relativity we thus have that the existence (or non existence) of a causal relationship between two events depends on the energy density of the portion of universe in which the events happen and thus on an objective physical quantity. This means that causal structure in general relativity should (in principle) be inferred in an objective way by whatever observer by means of energy density measurements. We used the conditional because it is well known that, on large cosmological scales, to explain at best observational data it must be introduced dark energy and this poses various problems from the theoretical point of view. In the following section we will briefly review these problematic issues. After that we will discuss the possible relationship that could exist between these problems and the fact that causal structure in quantum theory may be regarded as an observer dependent property.

V. DARK ENERGY

In this section we first give a brief review on dark energy. This material is mostly taken from a review on the subject done by Carroll [20]. We then discuss the conclusions reached in this review in relationship with the observer dependence of causal structure for outcomes happening in quantum experiments.

The standard assumption in cosmology is that universe is homogenous and isotropic. Since in general relativity, universe is described by a manifold \( M \), these two assumptions translate into formal statements regarding the geometry of \( M \). Homogeneity means that given two points \( p, q \) in \( M \) there exists an isometry that takes \( p \) into \( q \). Isotropy means that given a point \( p \) in \( M \), for any two vectors \( v \) and \( w \) in \( T_p M \), there exists an isometry such that the pushforward of \( w \) under the isometry is parallel to \( v \). Since the universe is not static, we infer that it is homogeneous and isotropic in space but not in time. This and the above assumptions imply that the universe can be foliated in space-like slices such that each slice is homogeneous and isotropic. Based only on these considerations it can be shown [18, 19] that the metric of the universe must have the following form:

\[ ds^2 = -dt^2 + a^2(t) d\sigma^2(k) \]  

(30)

where \( a(t) \) is the scale factor and \( d\sigma^2(k) \) is a metric for
three space which depends on the curvature parameter $k$. The metric in (34) is called the Friedmann Robertson Lemaître Walker (FRLW) metric. Note that Einstein’s equations are not taken into account to derive (34) since its derivation is based on purely geometrical arguments. Einstein’s equations are used to find the functional form for $a(t)$. In order to do so it must be made the assumption that matter and energy on large cosmological scale can be modelled as a perfect fluid and it is chosen an equation of state relating pressure $p$ to matter and energy density $\rho$ of the type $p = w\rho$ with $w$ constant. Putting the metric in (34) into Einstein’s equations and using the above assumption leads to write Friedman equations [18, 19], i.e. a set of differential equations establishing the evolution of scale factor in relationship with curvature, pressure and energy density:

$$\frac{\dot{a}}{a} = \frac{4\pi G}{3} (\rho + 3p)$$  \hspace{1cm} (31)

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho - \frac{k}{a^2}$$  \hspace{1cm} (32)

The quantity on the l.h.s of (32) is the square of the Hubble parameter $H = \frac{\dot{a}}{a}$ and can be used to define the value of the critical density:

$$\rho_c = \frac{3H^2}{8\pi G}$$  \hspace{1cm} (33)

The critical density is the value of energy density solving Friedman’s equations for zero spatial curvature, i.e. for a flat universe. Exploiting $\rho_c$ one can define the density parameter $\Omega = \frac{\rho}{\rho_c}$ by means of which (32) can be written as:

$$\Omega - 1 = \frac{k}{H^2 a^2}$$  \hspace{1cm} (34)

This shows that whether $k = +1, 0, -1$ depends on the magnitude of the actual (i.e. observed) energy density $\rho$ with respect to critical density $\rho_c$. If $\Omega < 1$ then $k < 0$ and the universe is described by a three dimensional manifold with constant negative curvature. On the contrary, if $\Omega > 1$ then $k > 0$ and the universe is described by a three dimensional manifold with constant positive curvature (the analog in three dimension of a sphere). Finally $\Omega = 1$ implies $k = 0$ and describes a flat universe the associated manifold being simply a three dimensional euclidean space.

There are three forms of energy density usually considered. The first is called dust $\rho_d$ and is composed of non relativistic matter whose pressure is negligible with respect to its energy density. The second is called radiation $\rho_r$ and is composed of photons and other relativistic particles moving approximately at the speed of light. The third is dark energy $\rho_{\Lambda}$ coming from the introduction of the cosmological constant in Einstein’s equations. There are strong evidences [24] that the amount of total energy density $\rho$ due to dust is negligible with respect to the amount due to matter ($\rho_{\text{m}}/\rho_d = 10^6$). We thus say that we live in a matter dominated universe and the relevant contributions to total energy density come from $\rho_d$ and $\rho_{\Lambda}$.

Observations of the dynamics of galaxies and clusters have shown that a reasonable value for the density parameter referring to $\rho_d$, is $\Omega_d = 0.3 \pm 0.1$ [24]. On the other hand observations of the anisotropies of the cosmic microwave background are consistent with a nearly spatially flat universe [21]. Thus we infer $\Omega \approx 1$. This implies that the amount of $\rho_{\Lambda}$ to the total energy density is such that $\Omega_{\Lambda} \approx 0.7$. Measurements of the distance vs. redshift relation for Type Ia supernovae [22, 23] have provided evidences that the universe is accelerating i.e. that $\ddot{a} > 0$. Since conventional matter could not make the universe expansion accelerate it is inferred that the component of the energy density that is responsible for such acceleration is $\rho_{\Lambda}$. The most natural candidate component of energy density for $\rho_{\Lambda}$ is the vacuum energy $\rho_v$.

This is corroborated by the following argument. Let us write (32) as:

$$\ddot{a} = \frac{8\pi G}{3} a^2 \rho - k.$$  \hspace{1cm} (35)

If the universe is expanding, then $\rho_d$ must necessarily decrease as the particle number density is diluted by expansion, so $\rho_d \propto a^{-3}$. Hence the right-hand side of (35) will be decreasing in an expanding universe (since $a^2 \rho$ is decreasing, while $k$ is a constant), hence the derivative of $\dot{a}$ should be negative if one only takes into account the contribution of $\rho_d$. The supernova data therefore imply that, to make the universe accelerate, there must be a source of energy density that varies more slowly than $a^2 \rho$ i.e. more slowly than $a^{-2}$. Since the distinguishing feature of vacuum energy is that it is a minimum amount of energy density in any region, strictly constant throughout spacetime, the slow variation of $\rho_{\Lambda}$ corroborates the statement that vacuum energy be the source of energy density making the expansion of universe accelerate. To match the data, it is required a vacuum energy:

$$\rho_v \approx (10^{-3}\text{eV})^4 = 10^{-8}\text{ergs/cm}^3$$  \hspace{1cm} (36)

It is not possible to reliably calculate the expected vacuum energy in the universe, or even in some specific field theory such as the Standard Model of particle physics; at best they can be evaluated order-of-magnitude estimates for the contributions from different sectors. These estimates lead to the following value:

$$\rho_{\text{v}(\text{theory})} \sim (10^{27} \text{eV})^4 = 10^{112}\text{ergs/cm}^3.$$  \hspace{1cm} (37)

This value is 120 orders of magnitude (30 if we change units of measurement) greater than the value in (36). Such a huge discrepancy with observational data implies that the source of energy density responsible for the expansion of universe, $\rho_{\Lambda}$, should be something different
from the vacuum energy. This is known as the cosmological constant problem.

As already told the actual model for the universe has $\Omega_\Lambda = 0.7$ and $\Omega_d = 0.3$ but the relative balance of dark energy and matter changes rapidly as the universe expands:

$$\frac{\Omega_\Lambda}{\Omega_d} = \frac{\rho_\Lambda}{\rho_d} \propto a^{-3} \quad (38)$$

This is due to the facts pointed out above, namely, that $\rho_\Lambda$ should be almost constant while $\rho_d \propto a^{-3}$. As a consequence, at early times of the universe’s expansion, dark energy was negligible in comparison to matter and radiation, while at late times matter and radiation are negligible. There is only a brief epoch of the universe’s history during which it would be possible to witness the transition from domination by one type of component to another. On the other hand, from the fact that $\Omega_\Lambda = 0.7$ and $\Omega_d = 0.3$ we conclude that we actually live in such a transitional period. It seems remarkable that we live during the short transitional period between those two eras. The approximate coincidence between matter and dark energies in the current universe is called the coincidence problem.

Inferring the existence of a source of energy different from ordinary matter or radiation to explain observational data in cosmology is not, on its own, a conceptual problem. Problems arise because it is not possible to explain the origin of this source of energy in a scenario that is logically consistent with the current physical knowledge. Thus, the problematic issues of inferring the existence of dark energy lie in the fact that this inference leads to logical inconsistencies such as the cosmological constant problem and the coincidence problem.

From the above analysis we understood that different methods to measure the curvature of space-time give rise to different curvature estimations. Since in general relativity curvature is related to an objective physical quantity that should have a definite value, energy density, we have that curvature itself must be uniquely defined. For the latter fact to be consistent with the former we postulate the existence of dark energy (this assumption is corroborated but not proved by the observations of \[23\]).

The existence of a uniquely defined curvature is the consequence of a unique metric tensor. In general relativity metric tensor is unique because causal structure of space-time events is assumed to be fixed in an absolute way. This, as suggested in the previous section, is opposed to what we found in quantum theory where causal structure of events is relative to an observer.

Inferring dark energy is thus directly related to the assumption that causal structure of space-time events is absolutely defined. However, if we elevated relativity of causal structure to be a universal principle, we could not model our universe with a uniquely defined metric tensor anymore. From this in turn we could conjecture that two different methods of measuring curvature give rise to different estimations simply because an absolutely defined causal structure of space-time events is not physically defineable. This in turn could eliminate the problem of dark energy at all but would pose the deeper problem of formulating a theory of the universe completely different from the one we have at the moment.

VI. CONCLUSIONS

Quantum theory is an extraordinarily successful theory and still lacks a clear physical explanation. Moreover, the absence of experiments linking quantum theory with the geometry of space-time leaves physicists with the consciousness that something is missing in our current understanding of nature at a fundamental level. This has renewed efforts in finding foundational principles for quantum theory in order to find a more general theory.

In this paper it is analyzed the interplay between causal structure of space-time events and the probabilistic nature of quantum theory. This analysis leads us to state two principles that can be put as foundations of quantum theory:

**Principle of causality** The input/output structure of the devices involved in a quantum experiment defines the causal structure of the outcomes happening on those devices.

**Principle of relativity of causal structure** Two observers looking at a given quantum experiment and assuming a different causal structure for the outcomes involved in the experiment cannot become aware of differences in their respective probabilistic predictions.

Since the only thing that can be predicted and physically verified in quantum theory are probabilities, the last principle suggests that causal structure of outcomes happening in quantum experiments is an observer dependent property. This principle could be a guiding principle to construct a theory of quantum gravity for the following reason. Quantum theory and general relativity are both successful and problematic in different and somewhat opposite aspects. On one hand quantum theory is extremely successful in making predictions. Until now, no experimental situation has been found in which the predictions of quantum theory are not satisfied. However there are still difficulties, after almost 90 years from its birth, to understand its physical meaning. On the other hand general relativity is not completely satisfactory in making predictions at large cosmological scales. This is related to the need to introduce dark energy to explain observational data. General relativity is, by the way, founded on two extremely clear and intuitive physical principles, namely, the Einstein’s principles of relativity and equivalence. It is then likely that a theory
more fundamental than the ones we have at the moment will come from a physical principle that can be put as foundation of quantum theory on one hand and that can motivate the need to introduce dark energy to explain observational data at large cosmological scales on the other hand. The principle of relativity of causal structure is indeed such a principle as we discussed in the previous section. If dark energy was not necessary to explain cosmological observations, and we could estimate the sources of energy responsible for the inferred dynamics of the universe, then, in principle, two observers could not assume different perspectives regarding the existence of a causal connection between two regions of universe since this would be absolutely defined by energy density measurements. Elaborating a theory of quantum gravity starting from the conclusions of this work is an extremely hard task and its success is far from being certain. The main motivation to try to formulate a new theory according to the above principle is that, as far as we know, the most plausible proposal for a source of dark energy is the assumption of a "cosmic aether" permeating all space whose origin is unknown [24]. Clearly this cannot be satisfactory since we are forcing new physical degrees of freedom, motivated only by the fact that the current model of universe and the theory underlying it do not properly explain observations.

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