Extremal quantum correlations for \( N \) parties with two dichotomic observables per site

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Consider a scenario where \( N \) separated quantum systems are measured, each with one among two possible dichotomic observables. Assume that the \( N \) events corresponding to the choice and performance of the measurement in each site are space-like separated. In the present paper, the correlations among the measurement outcomes that arise in this scenario are analyzed. It is shown that all extreme points of this convex set are attainable by measuring \( N \)-qubit pure-states with projective observables. This result allows the possibility of using known algorithms in order decide whether some correlations are achievable within quantum mechanics or not. It is also proven that if an \( N \)-partite state \( \rho \) violates a given Bell inequality, then, \( \rho \) can be transformed by stochastic local operations into an \( N \)-qubit state that violates the same Bell inequality by an equal or larger amount.

I. INTRODUCTION AND RESULTS

A privileged method for contrasting quantum and classical physics is by comparing the correlations among space-like separated events that each theory predicts. This is so because one can find constraints on the correlations predicted by each theory which are independent of any model for the experiment. For instance, Bell inequalities \cite{1} are constraints on the correlations that emerge from any possible experiment described by classical physics. Analogously, the quantum Bell-type inequalities \cite{2} are constraints on the probability distributions generated by measuring quantum systems, whatever the kind of systems and measurements involved.

Let us characterize the set of distributions that can be generated within quantum theory. Suppose the \( n^{th} \) party has a system with Hilbert space \( \mathcal{H}_n \), which is measured with the \( M \) generalized measurements \( \{ A_n(a|x) : a = 1, \ldots K \} \) for \( x = 1, \ldots M \). These POVMs satisfy \( A_n(a|x) \geq 0 \) for \( a = 1, \ldots K \) and \( \sum_{a=1}^{K} A_n(a|x) = \mathbb{I}_n \), for \( x = 1, \ldots M \) and \( n = 1, \ldots N \), where \( \mathbb{I}_n \) is the identity matrix acting on \( \mathcal{H}_n \). For an introduction to the formalism of generalized measurements see \cite{4}. The distributions predictable by quantum theory are the ones that can be written as

\[
P(a_1 \ldots a_N|x_1 \ldots x_N) = \text{tr} \left[ \rho \bigotimes_{n=1}^{N} A_n(a_n|x_n) \right], \tag{2}
\]

where \( \rho \) is a positive semidefinite matrix acting on \( \mathcal{H} = \bigotimes_{n=1}^{N} \mathcal{H}_n \) with \( \text{tr} \rho = 1 \). Fixed \( (N, M, K) \) to some finite values, the set of distributions \( P \) that can be written as (2) is convex, but not a polytope \cite{2}. These sets could also be characterized by Bell-type nonlinear inequalities, but little is known about them \cite{5, 6}. However, if the dimension of the local Hilbert spaces \( \mathcal{H}_n \) are fixed to a finite number, deciding whether a given distribution \( P \) can be written as (2) or not (up to a chosen precision) is an algorithmic task \cite{7}. Unfortunately, it is not known how to bound the dimension of the local Hilbert spaces.
given \((N,M,K)\). In this paper it is shown that for the case \(M = K = 2\) the extreme points of the set \((2)\) are attainable with \(\mathcal{H}_n = \mathbb{C}^2\). This allows for using the algorithms of [7] in order to decide whether a given distribution \(P\) is quantum or not.

From a fundamental point of view, it is also interesting to have a finite characterization for quantum correlations. Actually, this problem is proposed in the web page “Some Open Problems in Quantum Information Theory” (problem 26.A) [8]. In particular, they rise the question whether the minimal dimension sufficient to generate all quantum correlations for a given \((N,M,K)\) is \(K\). Here, this question is answered for the case \(M = K = 2\) and arbitrary \(N\).

In the dichotomic case \((K = 2)\), one can reduce the amount of experimental data by considering full-correlation functions

\[
C(x_1 \ldots x_N) = \sum_{a_1=1}^{2} \ldots \sum_{a_N=1}^{2} (-1)^{\sum_{n=1}^{N} a_n} P(a_1 \ldots a_N | x_1 \ldots x_N) ,
\]

instead of all the information \(P(a_1 \ldots a_N | x_1 \ldots x_N)\). That is, for each experimental setting \((x_1 \ldots x_N)\) all the information is summarized in the single number \(C(x_1 \ldots x_N)\). Notice that in the general case \(2^N - 1\) numbers are necessary. In [9], the set of extremal quantum full-correlation functions (3) is obtained for the case \(M = K = 2\). Here, the extremal points are obtained for the general case, where all experimental data is considered.

As a corollary of the results proven in this paper, the following is shown. If an \(N\)-partite state \(\rho\) violates a given Bell inequality (in the setting \(M = K = 2\)), then \(\rho\) can be transformed by stochastic local operations into an \(N\)-qubit state \(\tilde{\rho}\) which violates the same Bell inequality by an equal or larger amount. Here, by stochastic it is meant that the operation can fail with some probability.

**II. EXTREMAL QUANTUM CORRELATIONS FOR \(M = K = 2\)**

In this section we prove the main results of the paper. Here and in the rest of this document only the case \(M = K = 2\) is considered.

**A. Projective measurements are enough**

**Lemma 1.** In the case \(K = M = 2\), all extreme points are attainable by measuring pure states with orthogonal observables.

**Proof.** By the linearity of (2) with respect to \(\rho\) it is clear that all extreme points can be expressed with \(\rho\) being pure. The next holds for each party. A nice fact about dichotomic POVMs \(A(1|x) + A(2|x) = \mathbb{I}\) is that both operators can be diagonalized in the same basis. Suppose that \(|v\rangle\) is a simultaneous eigenvector of \(A(1|1)\) and \(A(2|1)\), that is \(A(a|1)|v\rangle = \lambda_a |v\rangle\) for \(a = 1,2\), and \(\lambda_1 + \lambda_2 = 1\). Then we can write \(A(a|1) = \lambda_a |v\rangle \langle v| + \tilde{A}(a|1)\), where \(\tilde{A}(a|1)\) is a positive operator orthogonal to \(|v\rangle \langle v|\), for \(a = 1,2\). It is clear that the POVM \(\{A(1|1), A(2|1)\}\) can be written as a convex combination of the two POVMs \(\{|v\rangle \langle v| + \tilde{A}(1|1), \tilde{A}(2|1)\}\) and \(\{A(1|1), |v\rangle \langle v| + \tilde{A}(2|1)\}\) with the respective weights \(\lambda_1\) and \(\lambda_2\). Continuing this procedure with the rest of simultaneous eigenvectors one can express the POVM \(\{A(1|1), A(2|1)\}\) as a convex combination of projective POVMs. \(\Box\)

**B. Non-factorizable extreme points**

Some extreme distributions have the property that a party can be factorized, for instance

\[
P_1(a_1|x_1)P(a_2 \ldots a_N|x_2 \ldots x_N) .
\]

We are not interested in such extreme points, because they reduce to the case of \(N - 1\) parties, which is already considered in the \(N\)-partite case. We say that an extreme point is non-factorizable if it cannot be written like (4), for any of the parties. Notice that a non-factorizable extreme point can be factorized in groups containing more than one party.

**Lemma 2.** All non-factorizable extreme points are achieved with observables \(\{A_n(a|x)\}\) such that, every non-zero vector \(|v\rangle_n \in \mathcal{H}_n\) belongs to at most one of the four subspaces \(\text{range} A_n(a|x) : a, x = 1,2\), for \(n = 1, \ldots N\).

**Remark.** This implies that all vectors in the range of \(A(1|1)\) have nonzero overlap with both, \(A(1|2)\) and \(A(2|2)\). Loosely speaking, the two observables \(A(a|1)\) and \(A(a|2)\) “do not commute for each possible direction”.

**Proof.** Suppose that a distribution \(P\) is obtained by measuring \(\rho\) with the observables \(\{A_n(a|x)\}\). Let us consider the first party \(n = 1\). By Lemma 1 we assume that the four operators \(A_1(a|x)\) are projectors, hence their ranges are the subspaces spanned by their corresponding eigenvectors with unit eigenvalue. By orthogonality, no single non-zero vector belongs to both, \(\text{range} A_1(1|x)\) and \(\text{range} A_1(2|x)\). Suppose that there is a non-zero vector \(|v\rangle \in \mathcal{H}_1\) that belongs to \(\text{range} A_1(1|1)\) and \(\text{range} A_1(1|2)\). Then we can write the four projectors as

\[
A_1(a|x) = \delta_{a,1} |v\rangle \langle v| + \tilde{A}_1(a|x) ,
\]
where each \( \tilde{A}_i(a|x) \) is a projector orthogonal to \(|v_i^r\rangle\). Let us define the probability \( \pi = \text{tr}[|v_i^r\rangle\langle v_i^r|\rho] \) and the normalized states

\[
\rho_v = \frac{1}{\pi} |v_i^r\rangle\langle v_i^r| ,
\]

acting respectively on \( \bigotimes_{n=2}^N \mathcal{H}_n \) and \( \bigotimes_{n=1}^N \mathcal{H}_n \). Clearly, the original correlations —for instance (2)— can be expressed as the mixture

\[
\begin{align*}
\text{tr} & \left[ \rho \bigotimes_{n=1}^N A_n(a_n|x_n) \right] \\
= & \pi \delta_{a,1} \text{tr} \left[ \rho_v \bigotimes_{n=2}^N A_n(a_n|x_n) \right] \\
+ & (1 - \pi) \text{tr} \left[ \tilde{\rho} \tilde{A}_1(a|x) \bigotimes_{n=2}^N A_n(a_n|x_n) \right].
\end{align*}
\]

The first term in the right-hand side is factorizable, hence, we ignore it. In the second term, neither the matrix \( \tilde{\rho} \) nor the operators \( \tilde{A}_1(a|x) \) have any overlap with \(|v_i^r\rangle\). Now, relabel \( \tilde{\rho} \rightarrow \rho \) and \( A_1(a|x) \rightarrow A_1(a|x) \), and consider the second term in the right-hand side of (9). We can repeat the process until no single vector in range \( A_1(a|1) \) is contained in range \( A_1(a|2) \). The same can be done to the other three pairs of operators: \( \{ A_1(1|1), A_1(2|2) \}, \{ A_1(2|1), A_1(1|2) \}, \{ A_2(1|1), A_2(2|2) \} \), and also to the rest of parties \( n = 2, \ldots, N \). If the initial correlations \( P \) are non-factorizable extreme points, after all this procedure, we obtain an extreme point with the property stated in Lemma 2.

**Lemma 3.** All non-factorizable extreme points are attainable with a state acting on \( \bigotimes_{n=1}^N \mathcal{H}_n \), where every local Hilbert space \( \mathcal{H}_n \) has even dimension, and \( \text{rank} A_n(a|x) = \dim \mathcal{H}_n/2 \) for \( a, x = 1, 2 \) and \( n = 1, \ldots, N \).

**Proof.** The following analysis can be applied to every party, and we omit the subindex \( n \). Suppose that \( \{ |u_i\rangle, \ldots, |u_r\rangle \} \) is an orthonormal basis for the subspace range \( A_1(1|1) \), where \( r = \text{rank} A(1|1) \). Because the direct sum of range \( A(1|2) \) plus range \( A(2|2) \) is the local Hilbert space \( \mathcal{H}_n \), every vector in this basis can be expressed as a direct sum \( |u_i\rangle = |u_i^1\rangle + |u_i^2\rangle \), where \( |u_i^1\rangle = A(a|2)|u_i\rangle \in \text{range } A(a|2) \) for \( i = 1, \ldots, r \). According to Lemma 2, both \( |u_i^1\rangle \) and \( |u_i^2\rangle \) are not null, otherwise \( |u_i\rangle \) would belong to range \( A(1|2) \) or range \( A(2|2) \). If \( \dim \langle \text{span}\{|u_i^1\rangle, \ldots, |u_r^1\rangle\} \rangle < r \) there exists a set of coefficients \( \{c_1, \ldots, c_r\} \), not all being zero, such that \( \sum_{i=1}^r c_i |u_i^1\rangle = 0 \). This implies that \( \sum_{i=1}^r c_i |u_i\rangle = 0 \), and consequently that \( \sum_{i=1}^r c_i |u_i\rangle \) belongs to range \( A(1|1) \) and range \( A(2|2) \), against Lemma 2. Therefore, it must be the case that \( \dim \langle \text{span}\{|u_1^1\rangle, \ldots, |u_r^1\rangle\} \rangle = r \). This implies that \( \text{rank} A(1|1) \leq \text{rank} A(1|2) \), but applying the same argument from the point of view of \( A(1|2) \) we obtain \( \text{rank} A(1|2) \leq \text{rank} A(1|1) \), so both ranks are equal. One can repeat this argument for the pairs of projectors \( \{ A(1|1), A(2|2) \} \) and \( \{ A(1|2), A(2|1) \} \), concluding that \( \text{rank} A(a|x) = r \) for \( a, x = 1, 2 \). We finish the proof by noticing that, by construction \( \dim \mathcal{H}_n = 2r \), which is an even number. \( \square \)

**C. Qubits are enough**

The main result of the paper is the following

**Theorem 4:** In the case \( K = M = 2 \), all quantum extreme points (2) are achievable by measuring \( N \)-qubit pure states with projective observables.

**Proof:** Suppose that the distribution \( P(a_1 \ldots a_N|x_1 \ldots x_N) \) is obtained by measuring \( \rho \) with the set of observables \( \{ A_n(a|x) \} \). Here we assume that the observables \( A_n(a|x) \) are of the form specified in Lemma 1, Lemma 2 and Lemma 3. The following analysis can be applied to every party, and thus, we omit the subindex \( n \). Define \( r = \text{rank} A(1|1) \) and the two matrices \( G_a = A(1|1)A(a|2)A(1|1) \) for \( a = 1, 2 \). Because \( A(1|1) \) is the identity in the subspace range \( A(1|1) \) and \( G_a + G_1 = A(1|1) \), there exists a simultaneous eigenbasis for \( G_0 \) and \( G_1 \) in the subspace range \( A(1|1) \), denoted by \( \{ |v_1\rangle, \ldots, |v_r\rangle \} \). Define the pair of vectors \( |v_k^a\rangle = A(a|2)|v_k\rangle \) and the two-dimensional subspace \( E_k^a = \text{span}\{|v_k^1\rangle, |v_k^2\rangle\} \) for \( k = 1, \ldots, r \). Because \( |v_k\rangle \perp G_a|v_k\rangle = A(1|1)A(a|2)A(1|1)|v_k\rangle \) then \( |v_k\rangle \perp A(1|1)|v_k^a\rangle \), which implies that for any \( |v\rangle \in E_k^a \) we have \( A(1|1)|v\rangle \perp A(1|1)|v_k\rangle \). Denote by \( |w_k^a\rangle \) any non-zero vector in \( E_k^a \) orthogonal to \( |v_k\rangle \). Due to the above discussion we have that \( |w_k^1\rangle = [A(1|1) + A(2|1)]|v_k^1\rangle = A(2|1)|v_k^1\rangle \), and then \( |v_k^1\rangle \in \text{range } A(2|1) \). Summarizing, the subspace \( E_k^a \) contains one, and only one, vector (up to a constant factor) from each of the four spaces range \( A(1|1) \), range \( A(2|1) \), range \( A(1|2) \), range \( A(2|2) \). These vectors are respectively \( |v_k\rangle, |v_k^1\rangle, |v_k^2\rangle, |w_k^a\rangle \).

In each of the subspaces \( E_k^a \) we define the pair of projective measurements

\[
\begin{align*}
A(1|1)^{k} &= |v_k^1\rangle \langle v_k^1| \\
A(2|1)^{k} &= |v_k^2\rangle \langle v_k^2|, \\
A(1|2)^{k} &= |v_k^1\rangle \langle v_k^1| \\
A(2|2)^{k} &= |v_k^2\rangle \langle v_k^2|.
\end{align*}
\]

Suppose that the \( N \) parties have made this procedure. That is, from \( n = 1, \ldots, N \), the \( n \)-th party has the \( r_n \) bidimensional subspaces \( E_n^{k} \) and pairs of observables \( A_n^{k}(a|x) \) for \( k = 1, \ldots, r_n \). We also denote
by \( E_n^k \) the projector onto the subspace \( E_n^k \). Define the probability distribution

\[
\pi[k_1 \ldots k_N] = \text{tr}[\rho E_{i_1}^{k_1} \otimes \cdots \otimes E_{i_N}^{k_N}],
\]

(12)

and the normalized \( N \)-qubit states

\[
\rho[k_1 \ldots k_N] = \frac{E_i^{k_1} \otimes \cdots \otimes E_i^{k_N} \rho E_i^{k_1} \otimes \cdots \otimes E_i^{k_N}}{\pi[k_1 \ldots k_N]},
\]

(13)

for \( k_n = 1, \ldots r_n \) and \( n = 1, \ldots N \). Due to the fact that for each party \( n \), the subspaces \( E_1^r \), \ldots, \( E_{r_n}^r \) are mutually orthogonal and add up to the whole local Hilbert space \( \mathcal{H}_n = \bigoplus_{k=1}^{r_n} E_k^r \), we can conclude the following. The original distribution \( P(a_1 \ldots a_N|x_1 \ldots x_N) \) can be written as the mixture

\[
P(a_1 \ldots a_N|x_1 \ldots x_N) = \sum_{k_1=1}^{r_1} \cdots \sum_{k_N=1}^{r_N} \pi[k_1 \ldots k_N] P[k_1 \ldots k_N](a_1 \ldots a_N|x_1 \ldots x_N),
\]

(14)

in terms of the more extreme distributions

\[
P[k_1 \ldots k_N](a_1 \ldots a_N|x_1 \ldots x_N) = \text{tr} \left[ \rho[k_1 \ldots k_N] \bigotimes_{n=1}^N A_n^{k_n}(a_n|x_n) \right].
\]

(15)

Each distribution \( P[k_1 \ldots k_N] \) is obtained by measuring an \( N \)-qubit state with projective observables. Concluding, if the original distribution \( P(a_1 \ldots a_N|x_1 \ldots x_N) \) is an extreme point, it can be obtained by measuring an \( N \)-qubit state with projective observables. \( \square \)

III. VIOLATION OF BELL INEQUALITIES AFTER LOCC

In this section we derive a corollary that follows from the previous results. Given an \( N \)-partite state \( \rho \), consider the \( N \)-qubit states \( \tilde{\rho} \) that can be obtained from \( \rho \) with some probability, when the parties perform protocols consisting of local operations and classical communication (LOCC). We do not care about the success probability as long as it is nonzero. This set of transformations is called stochastic-LOCC or SLOCC.

Consider the Bell inequality specified by the coefficients \( \{ \beta(a_1 \ldots a_N|x_1 \ldots x_N) \} \). That is, all distributions of the form (1) satisfy

\[
\sum_{a_1, x_1=1}^{2} \cdots \sum_{a_N, x_N=1}^{2} \beta(a_1 \ldots a_N|x_1 \ldots x_N) \times P(a_1 \ldots a_N|x_1 \ldots x_N) \geq 0.
\]

(16)

And suppose that the \( N \)-partite state \( \rho \) violates this inequality when measured with the observables \( \{ A_n(a_n|x_n) \} \)

\[
\sum_{a_1=x_1=1}^{2} \cdots \sum_{a_N=x_N=1}^{2} \beta(a_1 \ldots a_N|x_1 \ldots x_N) \times \text{tr} \left[ \rho \bigotimes_{n=1}^N A_n(a_n|x_n) \right] < 0.
\]

(17)

Let us apply the methods used in the proofs of Lemma 2 and Theorem 4 to prove the following result.

**Corollary 5.** If an \( N \)-partite state \( \rho \) violates the Bell inequality given by the coefficients \( \{ \beta(a_1 \ldots a_N|x_1 \ldots x_N) \} \), then \( \rho \) can be transformed by SLOCC into an \( N \)-qubit state \( \tilde{\rho} \) that violates the inequality \( \{ \beta(a_1 \ldots a_N|x_1 \ldots x_N) \} \) by an equal or larger amount.

**Proof.** To prove this statement, let us show how to construct a rank-two projector for each party \( (X_n \) for \( n = 1, \ldots N) \), such that the \( N \)-qubit state

\[
\tilde{\rho} = \frac{X_1 \otimes \cdots \otimes X_N \rho X_1 \otimes \cdots \otimes X_N}{\text{tr}[\rho X_1 \otimes \cdots \otimes X_N]},
\]

(18)

violates the Bell inequality \( \beta(a_1 \ldots a_N|x_1 \ldots x_N) \) by an equal or larger amount.

In the proof of Lemma 2, each party keeps removing a particular kind of vectors \( |\psi_n \rangle \) from its Hilbert space \( \mathcal{H}_n \). At the end of this procedure the final Hilbert space \( \mathcal{H}_n' \) is a subspace of \( \mathcal{H}_n \). The projection of the observables \( \{ A_n(a|x) \} \) onto \( \mathcal{H}_n' \) satisfy the properties stated in Lemma 2. The rank-two projector \( X_n \) has support on \( \mathcal{H}_n' \), and is specified in the next. In the proof of Theorem 4 we define the family of \( N \)-qubit states \( \rho[k_1 \ldots k_N] \). Each of them is obtainable from \( \rho \) by performing the SLOCC transformation (18) with projectors \( X_n = E_n^k \). By equation (14), it is clear that if \( P(a_1 \ldots a_N|x_1 \ldots x_N) \) violates a Bell inequality, there must exist one distribution \( P[k_1 \ldots k_N](a_1 \ldots a_N|x_1 \ldots x_N) \) which also does. The corresponding \( N \)-qubit state \( \tilde{\rho} = \rho[k_1 \ldots k_N] \) is what we are looking for. By convexity, the violation attained by \( \tilde{\rho} \) is never smaller than the one by \( \rho \). \( \square \)

For some authors, Bell inequalities need not to be facets of the classical polytope (1). For them, any linear inequality satisfied by all distributions of the form (1) is a Bell inequality. Remarkably, Corollary 5 also holds for these more general definition of Bell inequalities.
IV. CONCLUSIONS

We have considered a Bell-experiment scenario for $N$ parties, each with two dichotomic observables. The correlations that arise when measuring quantum systems in such scenario form a convex set. We have proven that all the extreme points of this set are achievable by measuring $N$-qubit pure states with projective observables. This answers the question risen in Problem 26.A of [8] for the case $M = K = 2$. It would be very interesting to prove that the minimal local dimension sufficient for generating all the extreme points of the quantum set $(N, M, K)$ is always $K$. Unfortunately, the techniques used in our proofs are not directly applicable for larger values of $M$ or $K$.

More practically, the obtained characterization allows for an algorithmic procedure to decide whether a particular distribution $P$ is predictable by quantum mechanics or not [7].

We have also shown that if a state $\rho$ violates a given Bell inequality, then $\rho$ can be transformed by stochastic local operations into an $N$-qubit state which violates the same Bell inequality by an equal or larger amount. This result has interesting consequences when considering the violation of Bell inequalities after LOCC, and in the regime where a large number of copies of the state ($\rho^{\otimes n}$) are jointly measured. This will be investigated in a forthcoming paper.

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