Global existence of strong solutions to the planar compressible magnetohydrodynamic equations with large initial data in unbounded domains

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Abstract

In one-dimensional unbounded domains, we consider the equations of a planar compressible magnetohydrodynamic (MHD) flow with constant viscosity and heat conductivity. More precisely, we prove the global existence of strong solutions to the MHD equations with large initial data satisfying the same conditions as those of Kazhikhov’s theory in bounded domains (Kazhikhov 1987 Boundary Value Problems for Equations of Mathematical Physics (Krasnoyarsk)). In particular, our result generalizes the Kazhikhov’s theory for the initial boundary value problem in bounded domains to the unbounded case.

Keywords. Magnetohydrodynamics; Global strong solutions; Large initial data; Unbounded domains

Math Subject Classification: 35Q35; 76N10.

1 Introduction

Magnetohydrodynamics (MHD) is concerned with the study of the interaction between the magnetic fields and electrically conducting fluids. It is widely applied to astrophysics, geophysics and plasma physics in practice, see [2,6,13,15,20,21,26] and the references therein. We are concerned with the governing equations of a planar magnetohydrodynamic compressible flow written in the Lagrange variables

\[ v_t = u_x, \]  
\[ u_t + (P + \frac{1}{2}|b|^2)_x = \left( \mu \frac{u_x}{v} \right)_x, \]  
\[ w_t - b_x = \left( \lambda \frac{w_x}{v} \right)_x, \]  
\[ (v b)_t - w_x = \left( \nu \frac{b_x}{v} \right)_x. \]
\[
\left( e + \frac{u^2 + |w|^2 + v|b|^2}{2} \right)_t + \left( u \left( P + \frac{1}{2} |b|^2 \right) - w \cdot b \right)_x = \left( \frac{\theta_x}{v} + \frac{uu_x}{v} + \frac{w \cdot w_x}{v} + \frac{b \cdot b_x}{v} \right)_x,
\]

where \( t > 0 \) is time, \( x \in \Omega \subset \mathbb{R} = (-\infty, +\infty) \) denotes the Lagrange mass coordinate, and the unknown functions \( v > 0, u, w \in \mathbb{R}^2, b \in \mathbb{R}^2, e > 0, \theta > 0, \) and \( P \) are, respectively, the specific volume of the gas, longitudinal velocity, transverse velocity, transverse magnetic field, internal energy, absolute temperature and pressure. \( \mu \) and \( \lambda \) are the viscosity of the flow, \( \nu \) is the magnetic diffusivity of the magnetic field, and \( \kappa \) is the heat conductivity. In this paper, we consider a perfect gas for magnetohydrodynamic flow, that is, \( P \) and \( e \) satisfy

\[
P = R\theta/v, \quad e = c_v\theta + \text{const},
\]

where both specific gas constant \( R \) and heat capacity at constant volume \( c_v \) are positive constants. We also assume that \( \lambda \) and \( \nu \) are positive constants, and that \( \mu, \kappa \) satisfy

\[
\mu = \tilde{\mu}_1 + \tilde{\mu}_2 v^{-\alpha}, \quad \kappa = \tilde{\kappa}\theta^\beta,
\]

with constants \( \tilde{\mu}_1 > 0, \tilde{\mu}_2 \geq 0, \tilde{\kappa} > 0, \) and \( \alpha, \beta \geq 0 \).

The system (1.1)-(1.6) is supplemented with the initial conditions

\[
(v, u, \theta, b, w)(x, 0) = (v_0, u_0, \theta_0, b_0, w_0)(x), \quad x \in \Omega,
\]

and one of three types of far-field and boundary ones:

1) Cauchy problem

\[
\Omega = \mathbb{R}, \quad \lim_{|x| \to \infty} (v, u, \theta, b, w) = (1, 0, 1, 0, 0), \quad t > 0;
\]

2) boundary and far-field conditions for \( \Omega = (0, \infty) \)

\[
u(0, t) = 0, \quad \theta(0, t) = 1, \quad b(0, t) = w(0, t) = 0, \quad \lim_{x \to \infty} (v, u, \theta, b, w) = (1, 0, 1, 0, 0), \quad t > 0;
\]

3) boundary and far-field conditions for \( \Omega = (0, \infty) \)

\[
u(0, t) = \theta_x(0, t) = 0, \quad b(0, t) = w(0, t) = 0, \quad \lim_{x \to \infty} (v, u, \theta, b, w) = (1, 0, 1, 0, 0), \quad t > 0.
\]

There is huge literature on the studies of the global existence and large time behavior of solutions to the compressible MHD system. In particular, when \( w = b = 0, \) the MHD system (1.1)-(1.5) reduces to the Navier-Stokes equations, which have been studied extensively in [4, 5, 7, 9, 11, 14, 17, 19, 22, 23, 25, 26] and the references therein. Here, we recall briefly some results which are more relative with our problem. For constant coefficients \( \alpha = \beta = 0 \) and large initial data, Kazhikhov-Shelukhin [19] first obtained the global existence of solutions to the initial boundary value problem in bounded domains. When \( \alpha = 0 \) and \( \beta > 0 \) in (1.7), Huang-Shi [11] obtained the global strong solutions to the initial-boundary-value problem with the initial data \((v_0, u_0, \theta_0) \in H^1 \times H^2 \times H^2 \). For the Cauchy
problem in the unbounded domain, Kazhikhov [17] obtained the global existence of strong solutions with constant coefficients $\alpha = \beta = 0$, which recently be refined by Li-Shu-Xu [23] to the general case with $\alpha = 0$ and $\beta > 0$.

Now, let's go back to the MHD system (1.1)-(1.5). For the initial-boundary-value problem in bounded domains, the global existence of strong solutions with large initial data was obtained by Kazhikhov [18] (see also Amosov-Zlotnik [1]) for the constant coefficients $\alpha = \beta = 0$ and by Huang-Shi-Sun [12] for $\alpha \geq 0$ and $\beta > 0$ (see also Hu-Ju [10] with $\alpha = 0$ and $\beta > 0$). Concerning the unbounded domains, Cao-Peng-Sun [3] established the global existence of strong solutions with large initial data. It should be pointed that the method in [3] depends heavily on the assumption of $\beta > 0$ and thus cannot be adapted to the case of constant coefficients $\alpha = \beta = 0$. Therefore, the main aim of this paper is to prove the global existence of strong solutions with constant coefficients $\alpha = \beta = 0$ in unbounded domains, which generalized Kazhikhov’s result [18] to the case of unbounded domains. That is, our main result is as follows.

**Theorem 1.1.** Suppose that $\alpha = \beta = 0$ and the initial data $(v_0, u_0, \theta_0, b_0, w_0)$ satisfies

\[(v_0 - 1, u_0, \theta_0 - 1, b_0, w_0) \in H^1(\Omega), \quad (1.12)\]

and

\[\inf_{x \in \Omega} v_0(x) > 0, \quad \inf_{x \in \Omega} \theta_0(x) > 0, \quad (1.13)\]

and are compatible with (1.10), (1.11). Then there exists a unique global strong solution $(v, u, \theta, b, w)$ to the initial-boundary-value problem (1.1)-(1.5), or (1.1)-(1.8) (1.10), or (1.1)-(1.8) (1.11) satisfying for any $T > 0$,

\[
\begin{cases}
  v - 1, u, \theta - 1, b, w \in L^\infty(0, T; H^1(\Omega)), \\
  v_t \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \\
  u_t, \theta_t, b_t, w_t, u_{xx}, \theta_{xx}, b_{xx}, w_{xx} \in L^2((\Omega) \times (0, T)),
\end{cases}
\]

and for each $(x, t) \in \Omega \times [0, T]$

\[C^{-1} \leq v(x, t) \leq C, \quad C^{-1} \leq \theta(x, t) \leq C, \quad (1.15)\]

where $C > 0$ is a constant depending on the data and $T$.

**Remark 1.1.** Our result can be regarded as a natural generalization of Kazhikhov’s result [18] in the bounded domains to the case of unbounded domains.

**Remark 1.2.** It should be mentioned here that Theorem 1.1 still holds for the case of $w = b = 0$. This in particular yields that our result also establishes the global strong solutions for compressible Navier-Stokes equations which has been considered in Kazhikhov [17].

We now comment on the analysis of this paper. To extend the local strong solutions whose existence is guaranteed by lemma 2.1 to be global, the main issue is to establish some necessary global a priori estimates of solutions. Motivated by Kazhikhov [18] (see also [11, 12]), we first obtain a key representation of $v$ (see (2.8)), which together with the standard energetic estimates (2.1) derives the lower bound of $v$ (2.3). Then, following the similar arguments as those in [23], multiplying the temperature equation (2.2) by $\theta^{-2}(\theta^{-1} - 2)_+^p$ and using the boundedness of domains ($\theta < 1/2)(t)$ (see (2.14)),
we can also obtain the lower bound of $\theta$ (see (2.13)). Next, we will prove the key upper bound of $v$. It should be mentioned here that the method in [3] for the case $\beta > 0$ is not valid in the case $\beta = 0$ due to lack of the estimates on $L^1(0, T; L^\infty(\Omega))$-norm of $\theta$.

In this paper, modifying slightly the idea due to Amosov-Zlotnik [1], we can prove the key upper bound of $v$ (see (2.15)) by controlling the $L^\infty(0, T; L^2(\Omega))$-norm of $(\ln v)_x$ (see (2.39)), see Lemma 2.5. Finally, using the similar arguments as those in [3, 12, 23], one can derive the necessary a priori estimates of the solutions, see Lemmas 2.6–2.8. The whole procedure will be carried out in the next section.

2 Proof of Theorem 1.1

We first state the following existence and uniqueness of local solutions which can be obtained by using the Banach theorem and the contractivity of the operator defined by the linearization of the problem on a small time interval (c.f. [16, 24, 27]).

Lemma 2.1. Under the assumptions in Theorem 1.1, there exists some $T_0 > 0$ such that the initial-boundary-value problem (1.1)–(1.9), or (1.1)–(1.8)–(1.10), or (1.1)–(1.8)–(1.11) has a unique strong solution $(v, u, \theta, b, w)$ with positive $v(x, t)$ and $\theta(x, t)$ satisfying (1.14).

Then, to finish the proof of Theorem 1.1 it only remains to obtain some a priori estimates (see (2.3), (2.13), (2.15), (2.16), (2.36), (2.46), and (2.55)), where the constants depend only on $T$ and the data of the problem. Thus, one can use the a priori estimates to continue the local solutions to the whole interval $[0, T]$.

Next, without loss of generality, we assume that $\lambda = \nu = \mu = \tilde{\kappa} = R = c_v = 1$. We first state the following basic energy estimates.

Lemma 2.2. It holds that for any $(x, t) \in \Omega \times [0, T],

$$\sup_{0 \leq t \leq T} \int_{\Omega} \left( \frac{u^2 + |w|^2 + v|b|^2}{2} + (v - \ln v - 1) + (\theta - \ln \theta - 1) \right) dx + \int_0^T W(t) dt \leq e_0,$$

where

$$W(t) \equiv \int_{\Omega} \left( \frac{\theta^2}{v \theta^2} + \frac{u_x^2 + |w_x|^2 + |b_x|^2}{v \theta} \right) dx,$$

and

$$e_0 \equiv 2 \int_{\Omega} \left( \frac{u_0^2 + |w_0|^2 + v_0|b_0|^2}{2} + (v_0 - \ln v_0 - 1) + (\theta_0 - \ln \theta_0 - 1) \right) dx.$$

Proof. Using (1.1)–(1.3), the energy equation (1.5) can be rewritten as

$$\theta_t + \frac{\theta}{v} u_x = \left( \frac{\theta_x}{v} \right)_x + \frac{u_x^2 + |w_x|^2 + |b_x|^2}{v}.$$  (2.2)

Multiplying (1.1) by $1 - v^{-1}$, (1.2) by $u$, (1.3) by $w$, (1.4) by $b$, and (2.2) by $1 - \theta^{-1}$,
respectively, one obtains after adding the resultant equalities altogether that

\[
\left( \frac{u^2 + |w|^2 + v|b|^2}{2} + (\theta - \ln \theta - 1) + (v - \ln v - 1) \right)_t \\
+ \frac{\theta^2}{v\theta^2} + \frac{u_x^2 + |w_x|^2 + |b_x|^2}{v\theta} \\
= \left( \frac{\theta x}{v} + \frac{uu_x}{v} + \frac{w \cdot w_x}{v} + \frac{b \cdot b_x}{v} \right)_x + u_x \\
- \left( u \left( \frac{\theta}{v} + \frac{1}{2} |b|^2 \right) - w \cdot b \right)_x - \left( \frac{\theta x}{v\theta} \right)_x,
\]

which along with (1.9) or (1.10) or (1.11) yields (2.1) and completes the proof Lemma 2.2.

Next, we will give out the following lower bound of \( v \).

**Lemma 2.3.** There exists a positive constant \( C \) such that for any \( (x, t) \in \Omega \times [0, T] \),

\[
v(x, t) \geq C,
\]

where (and in what follows) \( C \) denotes a generic positive constant depending only on \( T, \| (v_0 - 1, u_0, \theta_0 - 1, b_0, w_0) \|_{H^1(\Omega)}, \inf_{x \in \Omega} v_0(x), \) and \( \inf_{x \in \Omega} \theta_0(x) \).

**Proof.** Letting

\[
\sigma \triangleq \frac{u_x}{v} - \left( \frac{\theta}{v} + \frac{1}{2} |b|^2 \right) = (\ln v)_t - \left( \frac{\theta}{v} + \frac{1}{2} |b|^2 \right),
\]

owing to (1.1), we write (1.2) as

\[
u_t = \sigma_x.
\]

For any \( x \in \Omega \), denoting \( N = [x] \), one obtains after integrating (2.4) over \([N, x] \times [0, t] \) that

\[
\int_N^x udy - \int_N^x u_0dy = \ln v - \ln v_0 \\
- \int_0^t \left( \frac{\theta}{v} + \frac{1}{2} |b|^2 \right) d\tau - \int_0^t \sigma(N, \tau) d\tau,
\]

which implies

\[
v(x, t) = B_N(x, t)Y_N(t) \exp \left\{ \int_0^t \left( \frac{\theta}{v} + \frac{1}{2} |b|^2 \right) d\tau \right\},
\]

where

\[
B_N(x, t) \triangleq v_0 \exp \left\{ \int_N^x udy - \int_N^x u_0dy \right\},
\]

and

\[
Y_N(t) \triangleq \exp \left\{ \int_0^t \sigma(N, \tau) d\tau \right\}.
\]

Denoting by

\[
g(x, t) = \int_0^t \left( \frac{\theta}{v} + \frac{1}{2} |v|^2 \right) d\tau,
\]
it deduces from (2.5) that
\[ g_t = \frac{\theta + \frac{1}{2}v|b|^2}{v} = \frac{\theta + \frac{1}{2}v|b|^2}{B_N(x,t)Y_N(t) \exp \{ g \}}, \]
which gives
\[ \exp \{ g \} = 1 + \int_0^t \frac{\theta + \frac{1}{2}v|b|^2}{B_N(x,\tau)Y_N(\tau)} d\tau. \]
Combining this with (2.5) leads to
\[ v(x,t) = B_N(x,t)Y_N(t) \left( 1 + \int_0^t \frac{\theta + \frac{1}{2}v|b|^2}{B_N(x,\tau)Y_N(\tau)} d\tau \right). \quad (2.8) \]
On the one hand, it follows from (2.1) that
\[ \left| \int_N^x (u(y,t) - u_0(y)) dy \right| \leq \left( \int_N^{N+1} u^2 dy \right)^{\frac{1}{2}} + \left( \int_N^{N+1} u_0^2 dy \right)^{\frac{1}{2}} \leq C, \]
which together with (2.6) implies
\[ C^{-1} \leq B_N(x,t) \leq C, \quad (2.9) \]
where, and in what follows, \( C \) is a constant independent of \( N \).
On the other hand, it follows from (2.1) that
\[ \int_N^{N+1} (v - \ln v - 1 + \theta - \ln \theta - 1) dx \leq e_0, \]
which together with Jensen’s inequality yields that for any \( t \in [0,T] \)
\[ \alpha_1 \leq \int_N^{N+1} v(x,t) dx \leq \alpha_2, \quad \alpha_1 \leq \int_N^{N+1} \theta(x,t) dx \leq \alpha_2, \quad (2.10) \]
where \( 0 < \alpha_1 < \alpha_2 \) are two roots of \( z - \ln z - 1 = e_0 \).
Furthermore, multiplying (2.8) by \( \frac{1}{Y_N(t)} \) and integrating the resultant equality over \([N,N+1]\), we obtain after using (2.10), (2.11), and (2.9) that
\[ \frac{1}{Y_N(t)} \int_N^{N+1} v(x,t) dx \leq C \int_N^{N+1} B_N(x,t) \left( 1 + \int_0^t \frac{\theta + \frac{1}{2}v|b|^2}{B_N(x,\tau)Y_N(\tau)} d\tau \right) dx \]
\[ \leq C + C \int_0^t \frac{1}{Y_N(\tau)} \int_N^{N+1} \left( \theta + \frac{1}{2}v|b|^2 \right) dx d\tau \]
\[ \leq C + C \int_0^t \frac{1}{Y_N(\tau)} d\tau, \quad (2.11) \]
which combined with Grönwall’s inequality and (2.10) shows that for any \( t \in [0,T] \),
\[ \frac{1}{Y_N(t)} \leq C. \quad (2.12) \]
This together with (2.8), (2.9), and the fact that \( C \) is independent of \( N \) yields (2.3) and thus finishes the proof of Lemma 2.3.

Now, with the similar arguments in [3, 23], we can obtain the following lower bound of the temperature \( \theta \).
Lemma 2.4. There exists a positive constant $C$ such that for any $(x, t) \in \Omega \times [0, T],
\theta \geq C. \tag{2.13}$

Proof. The proof is similar as those in \cite{3,23}. We will sketch them here for completeness. Denoting by
\[(\theta > 2) (t) = \{ x \in \Omega | \theta(x, t) > 2 \}, \quad (\theta < 1/2) (t) = \{ x \in \Omega | \theta(x, t) < 1/2 \}, \]
it follows from (2.1) that
\[
\max_{\Omega} C \text{ constant which along with (2.14) leads to}
\]
which shows that for any $t \in [0, T],
\[|(\theta < 1/2) (t)| + |(\theta > 2) (t)| \leq \frac{2e_0}{2\ln 2 - 1}. \tag{2.14}\]

Next, for any $p > 2$, integrating (2.2) multiplied by $\theta^{-2} (\theta^{-1} - 2)^p_+ \frac{\Delta}{\theta^{-1} - 2} + (\theta^{-1} - 2)^p_+$ with $\max \{ \theta^{-1} - 2, 0 \}$ over $\Omega$, we obtain after using (2.3) that
\[
\frac{1}{p + 1} \frac{d}{dt} \int_\Omega (\theta^{-1} - 2)^{p+1} + \int_\Omega \frac{u^2_x}{v \theta^2} + |b_x|^2 (\theta^{-1} - 2)_+^p \frac{\Delta}{\theta^{-1} - 2} + (\theta^{-1} - 2)_+^p \frac{\Delta}{\theta^{-1} - 2} dx
\]
\[\leq \int_\Omega \frac{u_x}{v \theta} (\theta^{-1} - 2)_+^p dx
\]
\[\leq \frac{1}{2} \int_\Omega \frac{u_x^2}{v \theta^2} (\theta^{-1} - 2)_+^p dx + \frac{1}{2} \int_\Omega \frac{1}{v} (\theta^{-1} - 2)_+^p dx
\]
\[\leq \frac{1}{2} \int_\Omega \frac{u_x^2}{v \theta^2} (\theta^{-1} - 2)_+^p dx + C \int_{\Omega} (\theta^{-1} - 2)_+^p dx,
\]
which along with (2.14) leads to
\[
\left\| (\theta^{-1} - 2)_+ \right\|_{L^{p+1}(\Omega)}^p \frac{d}{dt} \left\| (\theta^{-1} - 2)_+ \right\|_{L^{p+1}(\Omega)} \leq C \left\| (\theta^{-1} - 2)_+ \right\|_{L^{p+1}(\Omega)}^p
\]
with $C$ independent of $p$. This, in particular, implies that there exists some positive constant $C$ independent of $p$ such that
\[
\sup_{0 \leq t \leq T} \left\| (\theta^{-1} - 2)_+ \right\|_{L^{p+1}(\Omega)} \leq C.
\]
Letting $p \rightarrow +\infty$ and using (2.14), it holds that
\[
\sup_{0 \leq t \leq T} \left\| (\theta^{-1} - 2)_+ \right\|_{L^\infty(\Omega)} \leq C,
\]
which derives (2.13) and thus finishes the proof of Lemma 2.4 \hfill \Box

Now, we will prove the the following upper bound of $v$, which is crucial for deducing the desired a priori estimates.
Lemma 2.5. There exists a positive constant $C$ such that for any $(x, t) \in \Omega \times [0, T],$
\[ v(x, t) \leq C. \quad (2.15) \]

Moreover, it holds that
\[ \sup_{0 \leq t \leq T} \int_\Omega (v_x^2 + |b|^2) dx + \int_0^T \int_\Omega ((1 + \theta)v_x^2 + u_x^2 + |b_x|^2 + |w_x|^2) dx dt \leq C. \quad (2.16) \]

\textbf{Proof.} First, multiplying (1.3) by $w$ and integrating the resultant equality over $\Omega$ by parts, it holds that
\[ \frac{1}{2} \frac{d}{dt} \int_\Omega |w|^2 dx + \int_\Omega \frac{|w_x|^2}{v} dx - C \int_\Omega |w_x| |b| dx \leq C \left( \left( \int_\Omega \frac{|w_x|^2}{v} dx \right)^{1/2} \left( \int_\Omega |b|^2 dx \right)^{1/2} \right) \]
\[ \leq \frac{1}{2} \int_\Omega \frac{|w_x|^2}{v} dx + C, \quad (2.17) \]
where in the last inequality one has used (2.1). Denoting by
\[ \tilde{V}(t) \equiv \int_\Omega \frac{|w_x|^2}{v} dx + V(t) + 1 \quad (2.18) \]
where
\[ V(t) \equiv \int_\Omega \left( \frac{v^2 + |w|^2 + v|b|^2}{2} \right) dx + \int_\Omega \left( \frac{\theta^2}{v \theta^2} + \frac{u_x^2}{v \theta^2} + \frac{|w_x|^2 + |b_x|^2}{v \theta} \right) dx, \quad (2.19) \]
it deduces from (2.17) and (2.1) that
\[ \int_0^T \tilde{V}(t) dt \leq C. \quad (2.20) \]

Next, using (1.1), we rewrite (1.2) as follows
\[ \ln v_x = u_t + \left( \frac{\theta}{v} \right)_x + b \cdot b_x. \quad (2.21) \]

Adding (2.21) multiplied by $(\ln v)_x$ and (1.4) multiplied by $v b$ together, one obtains after integrating the resultant equality over $\Omega$ by parts that
\[ \frac{1}{2} \frac{d}{dt} \int_\Omega \left( (\ln v)_x^2 + v^2 |b|^2 \right) dx + \int_\Omega \left( |b_x|^2 + \frac{\theta^2}{v \theta^2} (\ln v)_x^2 \right) dx \]
\[ = \frac{d}{dt} \int_\Omega u(\ln v)_x dx + \int_\Omega \frac{u_x^2}{v} dx + \int_\Omega \frac{\theta_x (\ln v)_x}{v} dx + \int_\Omega v w_x \cdot b dx. \quad (2.22) \]

Setting
\[ M_v(t) \equiv 1 + \max_{x \in \Omega} v(x, t), \quad (2.23) \]
the last term on the righthand side of (2.22) can be estimated as follows
\[ \left| \int_\Omega v w_x \cdot b dx \right| \leq CM_v(t) \int_\Omega |w_x| |b| dx \leq CM_v(t) \tilde{V}(t) \quad (2.24) \]
due to (2.17).

In order to handle the third term on the righthand side of (2.22), integration by parts together with (2.2) and (1.1) gives

$$
\int_\Omega \frac{\theta_x (\ln v)}{v} dx = - \int_\Omega \ln v \left( \theta_x \right)_x dx
= - \int_\Omega \ln v \left( (\theta - 1) \tau + \frac{\theta}{v} u_x - \frac{u_x^2 + |w_x|^2 + |b_x|^2}{v} \right) dx
= - \left( \int_\Omega (\theta - 1) \ln v dx \right) + \int_\Omega \frac{(\theta - 1) u_x}{v} dx
- \int_\Omega \frac{\theta u_x}{v} \ln v dx + \int_\Omega \frac{u_x^2 + |w_x|^2 + |b_x|^2}{v} \ln v dx.
$$

On the one hand, it holds

$$
\int_\Omega \frac{(\theta - 1)}{v} u_x dx \leq C \int_\Omega \frac{u_x^2}{v} dx + C \int_\Omega (\theta - 1)^2 dx
\leq C \int_\Omega \frac{u_x^2}{v} dx + C \max_{x \in \Omega} \left( \theta^{1/2} - 2^{1/2} \right)^2 \int_\Omega \theta dx
\leq C \int_\Omega \frac{u_x^2}{v} dx + C \int_\Omega \frac{\theta^2}{\theta^2 v} dx \int_{(\theta > 2)(t)} v \theta dx
\leq C \int_\Omega \frac{u_x^2}{v} dx + CV(t)M_v(t) + C,
$$

where one has used (2.23) and the following estimates

$$
\int_{(\theta \leq 2)(t)} (\theta - 1)^2 dx + \int_{(\theta > 2)(t)} \theta dx \leq C \int_\Omega (\theta - \ln \theta - 1) dx \leq C
$$

owing to (2.1) and (2.14). On the other hand, the straight calculations yield

$$
- \int_\Omega \frac{\theta}{v} u_x \ln v dx = - \int_\Omega \frac{\theta - 1}{v} u_x \ln v dx - \int_\Omega \frac{u_x \ln v}{v} dx
\leq C \left( \int_\Omega \frac{u_x^2}{v} dx + \int_\Omega (\theta - 1)^2 dx \right) \ln M_v(t)
+ C \int_\Omega \frac{u_x^2}{v} dx + C \int_\Omega \ln^2 v dx
\leq C \left( \int_\Omega \frac{u_x^2}{v} dx + V(t)M_v(t) + 1 \right) \ln M_v(t),
$$

where one has used (2.23), (2.26), and the following fact

$$
\int_{(v \leq 2)(t)} v dx \leq C \int_{(v \leq 2)(t)} (v - 1)^2 dx + C \int_{(v > 2)(t)} v dx \leq C \int_\Omega (v - \ln v - 1) dx \leq C
$$

(2.29)
due to (2.1) and (2.3).

Putting (2.26) and (2.28) into (2.25) gives
\[
\int \frac{\theta_x (\ln v)_x}{v} dx \leq \frac{d}{dt} \int \frac{u^2}{v} dx + C \left( \int \frac{u^2 + |b_x|^2}{v} dx + \hat{V}(t) M_v(t) \right) \ln M_v(t),
\]
which together with (2.22) and (2.24) leads to
\[
\frac{1}{2} \frac{d}{dt} \int (\ln v)^2 dx + \int |b|^2 dx + \int \left( \frac{\theta}{v} (\ln v_x)_x^2 + \hat{V}(t) M_v(t) \right) \ln M_v(t),
\]
where
\[
B(t) \triangleq \int u (\ln v_x)_x dx - \int (\theta - 1) \ln v dx
\]
satisfies
\[
B(t) \leq \frac{1}{8} \int (\ln v)_x^2 dx + C \int u^2 dx + \int |\theta - 1| |\ln v| dx
\]
\[
\leq \frac{1}{8} \int (\ln v)_x^2 dx + C \int_{(\theta \leq 2)(t)} (\theta - 1)^2 dx
\]
\[
+ C \int \ln^2 v dx + \int_{(\theta > 2)(t)} \theta dx \ln M_v(t)
\]
\[
\leq \frac{1}{8} \int (\ln v)_x^2 dx + C \ln \int (\ln v)_x^2 dx
\]
\[
\leq \frac{1}{4} \int (\ln v)_x^2 dx + C
\]
due to (2.1), (2.27), (2.29), and the following fact
\[
M_v(t) \leq C + C \int (\ln v)_x^2 dx.
\]
Indeed, the direct calculations combined with (2.29) imply that
\[
(\ln v)_x^2 \leq C \left( \int_{(\ln v)_x^2 > 2} v^2 dx \right)^{1/2} \left( \int \frac{v^2}{v_x^2} dx \right)^{1/2}
\]
\[
\leq C \left( \int_{(\ln v)_x^2 > 2} v dx M_v(t) \right)^{1/2} \left( \int (\ln v)_x^2 dx \right)^{1/2}
\]
\[
\leq CM_v^{1/2}(t) \left( \int (\ln v)_x^2 dx \right)^{1/2},
\]
which along with Young's inequality leads to (2.32).

Now, adding (1.2) multiplied by \(u\) and (1.4) by \(b\) together, integrating the resultant
equality over \((0,1) \times (0,t)\), it follows from (2.3), (2.26), and (2.29) that
\[
\sup_{0 \leq s \leq t} \int_\Omega (u^2 + v|b|^2) \, dx + \int_0^t \int_\Omega \frac{u_x^2 + |b_x|^2}{v} \, dx \, ds \\
\leq C + C \left( \int_0^t \int_\Omega \frac{\theta u_x}{v} \, dx \, ds \right) \\
\leq C + C \left( \int_0^t \int_\Omega \left( \frac{(\theta - 1)u_x}{v} - \frac{(v - 1)u_x}{v} + u_x \right) \, dx \, ds \right) \\
\leq C + \frac{1}{2} \int_0^t \int_\Omega \frac{u_x^2}{v} \, dx \, ds + C \int_0^t \int_\Omega (\theta - 1)^2 \, dx \, ds + C \int_0^t \int_\Omega (v - 1)^2 \, dx \, ds \\
\leq C + \frac{1}{2} \int_0^t \int_\Omega \frac{u_x^2}{v} \, dx \, ds + C \int_0^t \tilde{V}(s) M_v(s) \, ds \\
+ C \int_0^t \int_{\{v \leq 2\}(t)} (v - 1)^2 \, dx + M_v(s) \int_{\{v > 2\}(t)} v \, dx \, ds \\
\leq C + \frac{1}{2} \int_0^t \int_\Omega \frac{u_x^2}{v} \, dx \, ds + C \int_0^t \tilde{V}(s) M_v(s) \, ds.
\]
This gives directly
\[
\int_0^t \int_\Omega \frac{u_x^2 + |b_x|^2}{v} \, dx \, ds \leq C + C \int_0^t \tilde{V}(s) M_v(s) \, ds, 
\tag{2.34}
\]
which combined with (2.30) and (2.31) yields
\[
\sup_{0 \leq s \leq t} \int_\Omega (\ln v)^2 \, dx + \int_0^t \int_\Omega \left( |b_x|^2 + \frac{\theta}{v} (\ln v)^2 \right) \, dx \, ds \\
\leq C \ln \sup_{0 \leq s \leq t} M_v(s) + C \int_0^t \tilde{V}(s) M_v(s) \, ds \ln \sup_{0 \leq s \leq t} M_v(s) + C 
\tag{2.35} \\
\leq C \left( 2 + \int_0^t \tilde{V}(s) M_v(s) \, ds \right) \ln \left( 2 + \int_0^t \tilde{V}(s) M_v(s) \, ds \right),
\]
where in the second inequality one has used the following fact
\[
f \leq e^{f} - f - 1 + (1 + g) \ln(1 + g) - g, \quad \text{for any } f \geq 0, \ g \geq 0,
\]
with
\[
f = \frac{1}{2} \ln \sup_{0 \leq s \leq t} M_v(s), \quad g = 2C \int_0^t \tilde{V}(s) M_v(s) \, ds.
\]

Then, the combination of (2.32) with (2.35) gives
\[
M_v(t) \leq C \left( 2 + \int_0^t \tilde{V}(s) M_v(s) \, ds \right) \ln \left( 2 + \int_0^t \tilde{V}(s) M_v(s) \, ds \right),
\]
which along with (2.20) and Grönwall’s inequality derives (2.15). Furthermore, (2.16) is deduced directly from (2.14), (2.35), and (2.21). The proof Lemma 2.5 is completed.

**Lemma 2.6.** There exists a positive constant \(C\) such that
\[
\sup_{0 \leq t \leq T} \int_\Omega \left( |b_x|^2 + |w_x|^2 \right) \, dx \\
+ \int_0^T \int_\Omega \left( |b_t|^2 + |b_{xx}|^2 + |w_t|^2 + |w_{xx}|^2 \right) \, dx \, dt \leq C. 
\tag{2.36}
\]
Proof. First, rewriting (1.3) as
\[ w_t = \frac{w_{xx}}{v} - \frac{w_x v_x}{v^2} + b_x, \] (2.37)
multiplying (2.37) by \( w_{xx} \), and integrating the resultant equality over \( \Omega \times (0, T) \) by parts, we obtain after using (2.16) and Cauchy inequality that
\[
\frac{1}{2} \int_{\Omega} |w_x|^2 \, dx + \int_0^T \int_{\Omega} \frac{|w_{xx}|^2}{v} \, dx \, dt
\leq C + \int_0^T \int_{\Omega} \left( |b_x|^2 + |w_x|^2 |v_x|^2 \right) \, dx \, dt
\]
\[
\leq C + \frac{1}{4} \int_0^T \int_{\Omega} \frac{|w_{xx}|^2}{v} \, dx \, dt + \int_0^T \max_{x \in \Omega} |w_x|^2 \, dt \quad (2.38)
\]
\[
\leq C + \frac{1}{4} \int_0^T \int_{\Omega} \frac{|w_{xx}|^2}{v} \, dx \, dt + C \int_0^T \left( \int_{\Omega} |w_x|^2 \, dx \right)^{1/2} \left( \int_{\Omega} |w_{xx}|^2 \, dx \right)^{1/2} \, dt
\]
\[
\leq C + \frac{1}{2} \int_0^T \int_{\Omega} \frac{|w_{xx}|^2}{v} \, dx \, dt,
\]
where in the third inequality one has used the following fact
\[
|f|^2 = \int_x^\infty (|f|^2)_y \, dy \leq C \int_{\Omega} |f| |f_x| \, dx \leq C \left( \int_{\Omega} |f|^2 \, dx \right)^{1/2} \left( \int_{\Omega} |f_x|^2 \, dx \right)^{1/2} \quad (2.39)
\]
for any \( f \in H^1(\Omega) \).

The combination of (2.38) with (2.15) leads to
\[
\sup_{0 \leq t \leq T} \int_{\Omega} |w_t|^2 \, dx + \int_0^T \int_{\Omega} |w_{xx}|^2 \, dx \, dt \leq C. \quad (2.40)
\]

Thus, it follows from (2.40), (2.37), and (2.16) that
\[
\int_0^T \int_{\Omega} |w_t|^2 \, dx \, dt \leq C \int_0^T \int_{\Omega} \left( |b_x|^2 + |w_{xx}|^2 + v_x^2 |w_x|^2 \right) \, dx \, dt
\]
\[
\leq C + C \int_0^T \max_{x \in \Omega} |w_x|^2 \, dt
\]
\[
\leq C. \quad (2.41)
\]

Next, rewriting (1.4) as
\[
b_t = \frac{w_x}{v} + \frac{b_{xx}}{v^2} - \frac{b_x v_x}{v^3} - \frac{b u_x}{v}, \quad (2.42)
\]
multiplying (2.42) by \( b_{xx} \) and integrating the resultant equality over \( \Omega \times (0, T) \) by
parts, we deduce from (2.16), (2.3), (2.1), (2.39), and Cauchy inequality that
\[
\frac{1}{2} \int_\Omega |b_t|^2 dx + \int_0^T \int_\Omega \frac{|b_{xx}|^2}{v^2} dx dt \\
\leq C + \frac{1}{2} \int_0^T \int_\Omega \frac{|b_{xx}|^2}{v^2} dx dt + C \int_0^T \int_\Omega (|b_x|^2 v_x^2 + u_x^2 |b|^2 + |w_x|^2) dx dt \\
\leq C + \frac{1}{2} \int_0^T \int_\Omega \frac{|b_{xx}|^2}{v^2} dx dt + C \int_0^T \max_{x \in \Omega} |b_x|^2 dt + \max_{(x,t) \in \Omega \times [0,T]} |b|^2 \\
\leq C + \frac{3}{4} \int_0^T \int_\Omega \frac{|b_{xx}|^2}{v^2} dx dt + C \int_0^T \int_\Omega |b_x|^2 dx dt \\
+ C \sup_{0 \leq t \leq T} \int_\Omega |b|^2 dx + \frac{1}{4} \sup_{0 \leq t \leq T} \int_\Omega |b_x|^2 dx \\
\leq C + \frac{3}{4} \int_0^T \int_\Omega \frac{|b_{xx}|^2}{v^2} dx dt + \frac{1}{8} \sup_{0 \leq t \leq T} \int_\Omega |b_x|^2 dx,
\]
which along with (2.43) implies
\[
\sup_{0 \leq t \leq T} \int_\Omega |b_x|^2 dx + \int_0^T \int_\Omega |b_{xx}|^2 dx dt \leq C. \tag{2.44}
\]
This combined with (2.39) and (2.16) yields
\[
\max_{(x,t) \in \Omega \times [0,T]} |b|^2 \leq C \sup_{0 \leq t \leq T} \left( \int_\Omega |b|^2 dx \right)^{1/2} \left( \int_\Omega |b_x|^2 dx \right)^{1/2} \leq C. \tag{2.45}
\]
Finally, it follows from (2.42), (2.3), (2.44), (2.16), (2.45), and (2.39) that
\[
\int_0^T \int_\Omega |b|^2 dx dt \leq C \int_0^T \int_\Omega \left( |b_{xx}|^2 + |b_x|^2 v_x^2 + |w_x|^2 + |b|^2 u_x^2 \right) dx dt \\
\leq C + C \int_0^T \max_{x \in \Omega} |b_x|^2 \int_\Omega v_x^2 dx dt \\
\leq C + C \int_0^T \int_\Omega \left( |b_x|^2 + |b_{xx}|^2 \right) dx dt \\
\leq C.
\]
This together with (2.40), (2.41), and (2.44) gives (2.36) and thus finishes the proof of Lemma 2.6.

\[\blacksquare\]

**Lemma 2.7.** There exists a positive constant \( C \) such that
\[
\sup_{0 \leq t \leq T} \int_\Omega u_x^2 dx + \int_0^T \int_\Omega \left( u_t^2 + u_{xx}^2 \right) dx dt \leq C. \tag{2.46}
\]

**Proof.** First, we rewrite (1.2) as follows
\[
u_t = \frac{u_{xx}}{v} - \frac{u_x v_x}{v^2} - \frac{\theta x}{v^2} + \frac{\theta v_x}{v^2} - b \cdot b_x. \tag{2.47}
\]
Furthermore, it follows from (2.1) and (2.15) that

\[
\frac{1}{2} \int_{\Omega} u_x^2 \, dx + \int_0^T \int_{\Omega} \frac{u_{xx}^2}{v} \, dx \, dt \\
\leq C + \frac{1}{2} \int_0^T \int_{\Omega} \frac{u_{xx}^2}{v} \, dx \, dt + C \int_0^T \int_{\Omega} \frac{u_{x}^2}{v} \, dx \, dt + \frac{C}{2} \int_0^T \int_{\Omega} \frac{u_{x}^2}{v} \, dx \, dt \\
\leq C + \frac{1}{2} \int_0^T \int_{\Omega} \frac{u_{xx}^2}{v} \, dx \, dt + C \int_0^T \int_{\Omega} \frac{u_{x}^2}{v} \, dx \, dt + C \int_0^T \int_{\Omega} \theta^2 \, dx \, dt \\
+ C \sup_{(x,t) \in \Omega \times [0,T]} |b|^2 \int_0^T \int_{\Omega} |b_x|^2 \, dx \, dt + C \int_0^T \sup_{x \in \Omega} u_x^2 \int_{\Omega} v^2 \, dx \, dt \\
\leq C + \frac{1}{2} \int_0^T \int_{\Omega} \frac{u_{xx}^2}{v} \, dx \, dt + \frac{3}{4} \int_0^T \int_{\Omega} \frac{u_{x}^2}{v} \, dx \, dt + C \int_0^T \int_{\Omega} \theta^2 \, dx \, dt,
\]

where one has used (2.16), (2.36), (2.45), (2.39), and the following estimates:

\[
\sup_{x \in \Omega} (\theta - 2)_+^2 = \sup_{x \in \Omega} \left( \int_{x}^{\infty} \partial_y (\theta - 2)_+ \, dy \right)^2 \leq \left( \int_{(\theta > 2)_+} |\theta_y| \, dy \right)^2 \leq C \int_{\Omega} \theta^2 \, dx
\]

owing to (2.14).

Then, motivated by [22], integrating (2.2) multiplied by \((\theta - 2)_+ \frac{\Delta}{\max \{\theta - 2, 0\}}\) over \(\Omega \times (0, T)\), one obtains after using (2.36) and (2.3) that

\[
\frac{1}{2} \int_{\Omega} \frac{(\theta - 2)_+^2}{v} \, dx + \int_0^T \int_{(\theta > 2)_+} \frac{\theta_x^2}{v} \, dx \, dt \\
= \frac{1}{2} \int_{\Omega} \frac{(\theta_0 - 2)_+^2}{v} \, dx + \int_0^T \int_{\Omega} \frac{u_x^2}{v} + \frac{\|w_x\|^2}{v} + \frac{|b|^2}{v} (\theta - 2)_+ \, dx \, dt \\
- \int_0^T \int_{\Omega} \frac{\theta (\theta - 2)_+}{v} u_x \, dx \, dt \\
\leq C + C \int_0^T \sup_{x \in \Omega} \theta \int_{\Omega} \left( |b_x|^2 + |w_x|^2 + (\theta - 2)_+^2 + u_x^2 \right) \, dx \, dt \\
\leq C + C \int_0^T \sup_{x \in \Omega} \theta \left( \int_{\Omega} (\theta - 2)_+^2 \, dx + \int_{\Omega} u_x^2 \, dx + 1 \right) \, dt.
\]

Furthermore, it follows from (2.11) and (2.15) that

\[
\int_0^T \int_{\Omega} \theta_x^2 \, dx \, dt = \int_0^T \int_{(\theta > 2)_+} \theta_x^2 \, dx \, dt + \int_0^T \int_{(\theta \leq 2)_+} \theta_x^2 \, dx \, dt \\
\leq C \int_0^T \int_{(\theta > 2)_+} \frac{\theta_x^2}{v} \, dx \, dt + C \int_0^T \int_{(\theta \leq 2)_+} \frac{\theta_x^2}{v} \, dx \, dt \\
\leq C \int_0^T \int_{(\theta > 2)_+} \frac{\theta_x^2}{v} \, dx \, dt + C,
\]

where one has used (2.16), (2.36), (2.45), and the following estimates:
which together with (2.50) yields
\[
\frac{1}{2} \int_\Omega (\theta - 2)_+^2 \, dx + C_2 \int_0^T \int_\Omega \theta_x^2 \, dx \, dt \leq C + C \int_0^T \sup_{x \in \Omega} \left( \int_\Omega (\theta - 2)_+^2 \, dx + \int_\Omega u_x^2 \, dx + 1 \right) \, dt. \tag{2.51}
\]

Thus, one obtains after adding (2.51) multiplied by \(2C_2^{-1}C_1\) to (2.48) that
\[
\int_\Omega (u_x^2 + (\theta - 2)_+^2) \, dx + \int_0^T \int_\Omega (u_{xx}^2 + \theta_x^2) \, dx \, dt \leq C + C \int_0^T \sup_{x \in \Omega} \left( \int_\Omega (\theta - 2)_+^2 \, dx + \int_\Omega u_x^2 \, dx + 1 \right) \, dt. \tag{2.52}
\]

The straight calculations together with (2.15) and (2.27) imply that
\[
\theta^{1/2} \leq \left( \theta^{1/2} - 2^{1/2} \right) + C \leq \left( \int_{(\theta \geq 2)(t)} \frac{|\theta_x^2|}{\theta_x^2} \, dx + C \right)^{1/2} \left( \int_{(\theta \geq 2)(t)} \frac{\theta_x^2}{v \theta^2} \, dx \right)^{1/2} + C
\]
\[
\leq C \left( \int_{(\theta \geq 2)(t)} \frac{\theta_x^2}{v \theta^2} \, dx \right)^{1/2} + C,
\]
which along with (2.1) yields
\[
\int_0^T \sup_{x \in \Omega} \theta \, dt \leq C + C \int_0^T \int_\Omega \frac{\theta_x^2}{v \theta^2} \, dx \, dt \leq C. \tag{2.53}
\]

The Grönwall’s inequality together with (2.52) and (2.53) leads to
\[
\sup_{0 \leq t \leq T} \int_{\Omega} \left( u_x^2 + (\theta - 2)_+^2 \right) \, dx + \int_0^T \int_{\Omega} \left( u_{xx}^2 + \theta_x^2 \right) \, dx \, dt \leq C. \tag{2.54}
\]

Finally, it is easy to derive from (2.47), (2.31), (2.54), (2.16), (2.45), (2.39), (2.49), and (2.36) that
\[
\int_0^T \int_{\Omega} u_t^2 \, dx \, dt \leq C \int_0^T \int_{\Omega} \left( u_{xx}^2 + u_x^2 \, v_x^2 + \theta_x^2 + \theta^2 \, v_x^2 + |b|^2 |b_x|^2 \right) \, dx \, dt \leq C + C \int_0^T \left( \int_{\Omega} u_x^2 \, dx + \int_{\Omega} u_{xx}^2 \, dx + \int_\Omega \theta_x^2 \, dx \right) \, dt \leq C,
\]
which combined with (2.54) gives (2.46) and thus completes the proof of Lemma 2.7. \(\square\)

**Lemma 2.8.** There exists a positive constant \(C\) such that
\[
\sup_{(x,t) \in \Omega \times [0,T]} \theta(x,t) + \sup_{0 \leq t \leq T} \int_{\Omega} \theta_x^2 \, dx + \int_0^T \int_{\Omega} \left( \theta_t^2 + \theta_{xx}^2 \right) \, dx \, dt \leq C. \tag{2.55}
\]
Proof. First, multiplying (2.2) by \( \theta_t \) and integrating the resultant equality over \( \Omega \), it holds
\[
\int_\Omega \theta_t^2 \, dx + \frac{1}{2} \left( \int_\Omega \frac{\theta_t^2}{v} \, dx \right)_t = -\frac{1}{2} \int_\Omega \frac{\theta_x^2 u_x}{v^2} \, dx + \int_\Omega \frac{\theta_t \left( -\theta u_x + u_x^2 + |w_x|^2 + |b_x|^2 \right)}{v} \, dx
\]
\[
\leq C \sup_{x \in \Omega} (|u_x|) \int_\Omega \theta_x^2 \, dx + \frac{1}{2} \int_\Omega \theta_x^2 \, dx + C \int_\Omega \theta_x^2 \, dx + C \int_\Omega \left( u_x^4 + |w_x|^4 + |b_x|^4 \right) \, dx
\]
\[
\leq C \left( \int_\Omega \theta_x^2 \, dx \right)^2 + \frac{1}{2} \int_\Omega \theta_x^2 \, dx + C \int_\Omega \left( u_x^4 + |w_x|^4 + |b_x|^4 \right) \, dx + C,
\]
where one has used (2.3), (2.49), (2.39), (2.36), and (2.54). Thus, Grönwall’s inequality together with (2.56), (2.54), and (2.36) implies that
\[
\sup_{0 \leq t \leq T} \int_\Omega \theta_x^2 \, dx + \int_0^T \int_\Omega \theta_t^2 \, dx \, dt \leq C,
\]
which combined with (2.49) gives
\[
\sup_{(x,t) \in \Omega \times [0,T]} \theta(x,t) \leq C.
\]
Finally, it follows from (2.2) that
\[
\frac{\theta_{xx}}{v} = \frac{\theta_x v_x}{v^2} = \frac{u_x^2 + |b_x|^2}{v} + \frac{\theta u_x}{v} + \theta_t,
\]
which together with (2.15), (2.3), (2.58), (2.16), (2.36), (2.46), (2.39), and (2.57) yields
\[
\int_0^T \int_\Omega \theta_{xx}^2 \, dx \, dt \leq C \int_0^T \int_\Omega \left( \theta_x^2 v_x^2 + u_x^4 + |b_x|^4 + |w_x|^4 + u_x^2 + \theta_t^2 \right) \, dx \, dt
\]
\[
\leq C + C \int_0^T \sup_{x \in \Omega} \theta_x^2 \, dt + \int_0^T \sup_{x \in \Omega} \left( |u_x|^2 + |b_x|^2 + |w_x|^2 \right) \, dt
\]
\[
\leq C + \frac{1}{2} \int_0^T \int_\Omega \theta_{xx}^2 \, dx \, dt.
\]
Combining this with (2.57)–(2.58) proves (2.55) and finishes the proof of Lemma 2.8.

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