A Result for Orthogonal Plus Rank-1 Matrices

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Abstract
In this paper the sum of an orthogonal matrix and an outer product
is studied, and a relation between the norms of the vectors forming the
outer product and the singular values of the resulting matrix is presented.
The main result may be found in Theorem 1.

Preliminaries
We start by proving Lemma 1 below, which will later be used in the proof
of Lemma 2. The proof of Lemma 1 relies only on well-known properties of
inner products and norms (in particular the Cauchy-Schwarz inequality
and the triangle inequality). The necessary background material may be found in
Trefethen and Bau (1997, Chapter 3) or Renardy and Rogers (2004, Chapter 6).
Throughout this paper, \( \| x \| \) denotes the Euclidean norm of \( x \).

Lemma 1. Suppose \( x, y \in \mathbb{R}^n \) with \( \| x \| = 1 \). Then
\[
\| x + y \|^2 + \| y \| \| 2x + y \| \geq 1 \geq \| x + y \|^2 - \| y \| \| 2x + y \|. \tag{1}
\]
Proof. Using the triangle inequality followed by the reverse triangle inequality,
\[
2 \leq \| y \| + | 2 - \| y \| | \leq \| y \| + \| 2x + y \|. \tag{2}
\]
The first part of (1) now follows, since
\[
1 \leq 1 + \| y \| (\| y \| + \| 2x + y \| - 2) \tag{3}
= 1 - 2\| y \| + \| y \|^2 + \| y \| \| 2x + y \| \tag{4}
\leq 1 - 2\| x^T y \| + \| y \|^2 + \| y \| \| 2x + y \| \tag{5}
\leq 1 + 2x^T y + \| y \|^2 + \| y \| \| 2x + y \| \tag{6}
= \| x + y \|^2 + \| y \| \| 2x + y \|. \tag{7}
\]
The second part of (1) follows as
\[
1 = 1 + 2x^T y + \| y \|^2 - 2x^T y - \| y \|^2 \tag{8}
= \| x + y \|^2 - y^T (2x + y) \tag{9}
\geq \| x + y \|^2 - \| y \| \| 2x + y \|. \tag{10}
\]
This concludes the proof. □
Main Result

The main result in this paper concerns the singular values of a special kind of matrices. Recommended background material on the Singular Value Decomposition (SVD) and its properties may be found in Golub and Van Loan (1996, Section 2.5.3) and Trefethen and Bau (1997, Chapters 4–5). The present paper proves the following theorem.

**Theorem 1.** Let \( A = Q + ab^T \) for an orthogonal \( n \times n \) matrix \( Q \) and \( a, b \in \mathbb{R}^n \). Let \( \sigma_1, \ldots, \sigma_r \) be the singular values of \( A \), and if \( r < n \) additionally let \( \sigma_n = 0 \). Then

\[
\sigma_1 - \text{sign}(1 + a^T Q b) \sigma_n = \|a\| \|b\|. 
\]  

(11)

The theorem is obtained from Lemma 2 below by multiplication from any side by \( Q \) and renaming of the variables. This is possible since multiplication by an orthogonal matrix does not change the singular values.

**Lemma 2.** Let \( A = I + ab^T \) with \( a, b \in \mathbb{R}^n \). Let \( \sigma_1, \ldots, \sigma_r \) be the singular values of \( A \), and if \( r < n \) additionally let \( \sigma_n = 0 \). Then

\[
\sigma_1 - \text{sign}(1 + a^T b) \sigma_n = \|a\| \|b\|. 
\]  

(12)

**Proof.** If either \( a \) or \( b \) is zero, the proposition follows by inspection, as both sides in (12) evaluate to zero. Suppose therefore that neither \( a \) nor \( b \) is zero, and introduce

\[
x = \frac{1}{\|a\|} a \quad \text{and} \quad y = \|a\| b. 
\]  

(13)

Now

\[
A^T A = (I + xy^T)^T (I + xy^T) = I + xy^T + yx^T + yy^T, 
\]  

(14)

and we may write

\[
A^T A - I = (x + y)(x + y)^T - xx^T. 
\]  

(15)

We see here that precisely two singular values of \( A \) are different from 1, except when \( x \) and \( y \) are parallel, in which case only one of them is different from 1.

**In the parallel case** \( y = \mu x \) it is clear from (14) that any vector orthogonal to \( x \) is an eigenvector to \( A^T A \) with eigenvalue 1, and any vector parallel to \( x \) is an eigenvector with eigenvalue

\[
\lambda = 1 + 2\mu + \mu^2 = (1 + \mu)^2 = (1 + x^T y)^2. 
\]  

(16)

Now

\[
(\lambda - 1 - x^T y)^2 = (\lambda - 1 - \mu)^2 = (\mu + \mu^2)^2 = \mu^2(1 + \mu)^2 = \mu^2 \lambda, 
\]  

(17)

and

\[
|\mu|^2 = \frac{(\lambda - 1 - x^T y)^2}{\lambda} = \left( \sqrt{\lambda} - \frac{1 + x^T y}{\sqrt{\lambda}} \right)^2 = \left( \sqrt{\lambda} - \text{sign}(1 + x^T y) \right)^2. 
\]  

(18)

Here \( \sqrt{\lambda} \) is equal to either \( \sigma_1 \) or \( \sigma_n \), depending on whether \( \sqrt{\lambda} \) is larger or smaller than 1. In both cases it is readily verified that

\[
\sigma_1 - \text{sign}(1 + a^T b) \sigma_n = \sigma_1 - \text{sign}(1 + x^T y) \sigma_n = |\mu| = \|a\| \|b\|. 
\]  

(19)
In the non-parallel case it is clear from (14) that any vector orthogonal to both \( x \) and \( y \) is an eigenvector to \( A^T A \) with eigenvalue 1, and we may choose \( n-2 \) linearly independent such vectors. It follows that the remaining two eigenvectors are linear combinations of \( x \) and \( y \). One finds that
\[
A^T A(x + sy) = (1 + x^T y + s\|y\|^2)x + (s + 1 + sx^T y + x^T y + s\|y\|^2)y, \tag{20}
\]
so in order for \( x + sy \) to be an eigenvector one must have
\[
\begin{align*}
1 + x^T y + s\|y\|^2 &= \lambda, \\
s + sx^T y + 1 + x^T y + s\|y\|^2 &= \lambda s.
\end{align*}
\tag{21}
\]
This means that
\[
s + sx^T y + 1 + x^T y + s\|y\|^2 = (1 + x^T y + s\|y\|^2)s \iff s^2 - s - \frac{1 + x^T y}{\|y\|^2} = 0 \iff s = \frac{1}{2} \pm \frac{\|2x + y\|}{2\|y\|},
\tag{22}
\]
and together with (21) this gives the two eigenvalues
\[
\begin{align*}
\lambda_1 &= \frac{1}{2} + \frac{\|x + y\|^2}{2} + \frac{\|y\|\|2x + y\|}{2}, \\
\lambda_2 &= \frac{1}{2} + \frac{\|x + y\|^2}{2} - \frac{\|y\|\|2x + y\|}{2}.
\end{align*}
\tag{23}
\]
Using Lemma 1 and the fact that \( \lambda_1 \neq 1 \) and \( \lambda_2 \neq 1 \) we conclude that \( \lambda_1 = \sigma_1^2 \) and \( \lambda_2 = \sigma_2^2 \). It can be verified that
\[
\lambda_1\lambda_2 = (1 + x^T y)^2, \tag{24}
\]
and some further computations show that
\[
(\lambda_1 - 1 - x^T y)^2 = \|y\|^2\lambda_1. \tag{25}
\]
Dividing both sides by \( \lambda_1 \), we see that
\[
\|y\|^2 = \frac{(\lambda_1 - 1 - x^T y)^2}{\lambda_1} = \left( \frac{\sqrt{\lambda_1}}{\sqrt{\lambda_1}} - \frac{1 + x^T y}{\sqrt{\lambda_1}} \right)^2
= \left( \sqrt{\lambda_1} - \text{sign}(1 + x^T y)\sqrt{\lambda_2} \right)^2.
\tag{26}
\]
As \( \lambda_1 > \lambda_2 \), it is clear that
\[
\|y\| = \sigma_1 - \text{sign}(1 + x^T y)\|a\| = \sigma_1 - \text{sign}(1 + a^T b)\sigma_n = \|a\|\|b\|, \tag{27}
\]
which is what we wanted to prove. \( \blacksquare \)

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