Chiral Thermodynamic Model of QCD and its Critical Behavior
in the Closed-Time-Path Green Function Approach

Da Huang∗ and Yue-Liang Wu†

State Key Laboratory of Theoretical Physics (SKLTP)
Kavli Institute for Theoretical Physics China (KITPC)
Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing,100190, P.R.China
(Dated: March 1, 2012)

Abstract

By applying the closed-time-path Green function formalism to the chiral dynamical model based
on an effective Lagrangian of chiral quarks with the nonlinear-realized meson fields as bosonized
auxiliary fields, we then arrive at a chiral thermodynamic model for the meson fields after integrat-
ing out the quark fields. Particular attention is paid to the spontaneous chiral symmetry breaking
and restoration from the dynamically generated effective composite Higgs potential of meson fields
at finite temperature. It is shown that the minimal condition of the effective composite Higgs
potential of meson fields leads to the thermodynamic gap equation at finite temperature, which
enables us to investigate the critical behavior of the effective chiral thermodynamical model and
to explore the QCD phase transition. After fixing the free parameters in the effective chiral La-
grangian at low energies with zero temperature, we determine the critical temperature of the chiral
symmetry restoration and present a consistent prediction for the thermodynamical behavior of
several physically interesting quantities, which include the vacuum expectation value $v_o(T)$, quark
condensate $<\bar{q}q>(T)$, pion decay constant $f_\pi(T)$ and pion meson mass $m_\pi(T)$. In particular, it
is shown that the thermodynamic scaling behavior of these quantities becomes the same near the
critical point of phase transition.

PACS numbers: 12.39.Fe,12.38.Aw,11.30.Rd,12.38.Lg,12.40.Ee

∗Electronic address: dahuang@itp.ac.cn
†Electronic address: ylwu@itp.ac.cn
I. INTRODUCTION

Thermodynamics of quantum chromodynamics (QCD) has been attracted a lot of attention during the last three decades. Many interesting physical phenomena are related to it, such as the equation of state of quark gluon plasma (QGP), chiral symmetry breaking and restoration, deconfinement phase transition, and so on. Thus, the study of the QCD thermodynamics becomes a basic problem in our understanding of the strong interaction. Particularly, the deconfined QGP is expected to be formed in ultrarelativistic heavy-ion collisions (HIC) \cite{1-8} and many experiments like those at the Relativistic Heavy Ion Collision (RHIC) and at the Large Hadron Collider (LHC) have been built to explore the nature of the QGP and to search for the critical point of the phase transition, which enables many theoretical ideas testable and makes the research area more exciting.

In this paper, we are going to study the thermodynamic properties of the effective chiral dynamical model (CDM) \cite{9} of low energy QCD with spontaneous symmetry breaking via the dynamically generated effective composite Higgs potential of meson fields. The main assumption in such an effective CDM was based on an effective Lagrangian of chiral quarks with effective nonlinear-realized meson fields as the bosonized auxiliary fields, which may be resulted from the Nambu-Jona-Lasinio (NJL) four quark interaction \cite{10} due to the strong interactions of nonperturbative QCD at the low energy scale, then after considering the quantum loop contributions of quark fields by integrating over the quark fields with the loop regularization (LORE) method \cite{11,12}, the resulting effective chiral Lagrangian for the meson fields in the CDM \cite{9} has been turned out to provide a dynamically generated spontaneous symmetry breaking mechanism for the $SU(3)_L \times SU(3)_R$ chiral symmetry. The key point for deriving the dynamically generated spontaneous symmetry breaking mechanism is the use of the LORE method which keeps the physically meaningful finite quadratic term and meanwhile preserves the symmetries of original theory. More specifically, the advantage of the LORE method is the introduction of two intrinsic energy scales without spoiling the basic symmetries of original theory, such intrinsic energy scales play the role of the characteristic energy scale $M_c$ and the sliding energy scale $\mu_s$. Here $M_c$ is the characterizing energy scale of nonperturbative QCD below which the effective quantum field theory becomes meaningful to describe the low energy dynamics of QCD, and $\mu_s$ reflects the low energy scale of QCD on which the interesting physics processes are concerned. As a consequence, it was shown that the resulting effective CDM of low energy QCD can lead to the consistent predictions for the light quark masses, quark condensate, pseudoscalar meson masses and the lowest nonet scalar meson masses as well as their mixing at the leading order \cite{9}. Based on the success of the effective CDM at zero temperature, we will show in this paper how the CDM can be extended to an effective chiral thermodynamic model (CTDM) at finite temperature by applying the closed-time-path Green function (CTPGF) approach \cite{14-19}, and how the CTDM enables us to describe the critical behavior of the low energy dynamics of QCD, and in particular to determine the critical temperature of the chiral symmetry breaking and restoration. This comes to our main motivation in the present paper. For our present purpose, we will investigate the CTDM with considering only two flavors and ignoring the possible instanton effect which is known to account for the the anomalous $U(1)_A$ symmetry breaking.

Following the almost same procedure in deriving the CDM for the composite meson fields, we will arrive at an effective CTDM at finite temperature by adopting the CTPGF formalism. The key step in the derivation is the replacement of the zero-temperature propagator for the quark field with its finite temperature counterpart in the CTPGF formalism which has been shown to be more suitable for characterizing the nonequilibrium statistical processes.
We then obtain the dynamically generated effective composite Higgs potential at finite temperature, its minimal condition leads to the gap equation at finite temperature and enables us to explore the critical behavior and temperature for the chiral symmetry breaking and restoration. After fixing the free parameters in the effective Lagrangian, we present our numerical predictions for the temperature dependence of physically interesting quantities, such as the vacuum expectation value $v_o(T)$, the pion decay constant $f_\pi(T)$, and the masses of pseudoscalar and scalar mesons. The resulting critical temperature is found to be $T_c \simeq 200$ MeV, which is consistent with the NJL model prediction\[20–23\].

II. OUTLINE ON CHIRAL DYNAMICAL MODEL OF LOW ENERGY QCD

Before exploring the thermodynamic behavior of the chiral dynamical model, it is useful to have a brief review on its derivation of the effective chiral Lagrangian at zero-temperature. For our present purpose with paying attention to the investigation of the chiral symmetry restoration at finite temperature, we shall consider the simple case with only two flavors of light quarks $u$ and $d$, and ignore the instanton effects. For simplicity, we also assume the exact isospin symmetry for two light quarks $q = (u, d)^T$ by taking the equal mass $m_u = m_d = m$. Our discussion here is mainly following the previous paper by Dai and Wu[9].

Let us begin with the QCD Lagrangian for two light quarks

$$L_{QCD} = \bar{q}i\gamma_\mu(i\partial_\mu + g_s G^{a}_\mu T^a)q - \bar{q}Mq - \frac{1}{2}trG_{\mu\nu}G^{\mu\nu}$$

where $q = (u, d)^T$ denotes the SU(2) doublet of two light quarks and the summation over color degrees of freedom is understood. $G^{a}_\mu$ are the gluon fields with SU(3) gauge symmetry and $g_s$ is the running coupling constant. $M$ is the light quark mass matrix $M = \text{diag}(m_1, m_2) \equiv \text{diag}(m_u, m_d)$. In the limit $m_i \to 0 \ (i = 1, 2)$, the Lagrangian has the global $U(2)_L \times U(2)_R$ chiral symmetry. Due to the instanton effect, the chiral symmetry $U(1)_L \times U(1)_R$ is broken down to the diagonal $U(1)_V$ symmetry. This instanton effect can be expressed by the effective interaction\[24, 25\]

$$L^{\text{inst}} = \kappa^{\text{inst}}e^{i\theta^{\text{inst}}} \det(-\bar{q}_Rq_L) + h.c.$$ (2)

where $\kappa^{\text{inst}}$ is the constant containing the factor $e^{-8\pi^2/g^2}$. Obviously, such an instanton term breaks the $U(1)_V$ chiral symmetry. As our present consideration is paid to the phenomenon of the $SU(2)_L \times SU(2)_R$ chiral symmetry breaking and its restoration at finite temperature, we will switch off the instanton effect by simply setting $\kappa^{\text{inst}} = 0$ in the following discussions.

The basic assumption of the CDM is that at the chiral symmetry breaking scale ($\sim 1 GeV$) the effective Lagrangian contains not only the quark fields but also the effective meson fields describing bound states of strong interactions of gluons and quarks. After integrating over the gluon fields at high energy scales, the effective Lagrangian at low energy scale is expected to have the following general form when keeping only the lowest order nontrivial terms:

$$L^{\text{eff}}(q, \bar{q}, \Phi) = \bar{q}i\gamma_\mu(i\partial_\mu + \bar{q}_L\gamma_\mu\mathcal{A}^{\mu}_Lq_L + \bar{q}_R\gamma_\mu\mathcal{A}^{\mu}_Rq_R - [\bar{q}_L(\Phi - M)q_R + h.c.] + \mu^2_\nu tr(\Phi M^\dagger + M\Phi^\dagger) - \mu^2_\nu tr(\Phi\Phi^\dagger)$$ (3)

where $\Phi_{ij}$ are the effective meson fields which basically correspond to the composite operators $\bar{q}_Rj_1q_Li_2$. $\mathcal{A}_L$ and $\mathcal{A}_R$ are introduced as the external source fields. It is noticed in Eq.(3) that
the effective meson fields $\Phi_{ij}$ are the auxiliary fields in the sense that there is no kinetic term for them, which may explicitly be seen by integrating out $\Phi_{ij}$, we then obtain the following effective Lagrangian of quarks:

$$\mathcal{L}_{\text{eff}}^{\text{NJL}}(q, \bar{q}) = \bar{q} \gamma^\mu i \partial_\mu q + \bar{q}_L \gamma_\mu A^\mu_L q_L + \bar{q}_R \gamma_\mu A^\mu_R q_R$$

$$-(\frac{\mu_m^2}{\mu_f^2} - 1)(\bar{q}_L M q_R + \bar{q}_R M^\dagger q_L) + \frac{1}{\mu_f^2} \bar{q}_L q_R \bar{q}_R q_L$$

which arrives exactly at the Nambu-Jona-Lasinio(NJL) model of effective four-quark interaction with the quark mass matrix $(\frac{\mu_m^2}{\mu_f^2} - 1)M$. When the mass term is understood as the well-defined current quark mass term in the QCD Lagrangian Eq.(1), it is then clear that $\mu_m^2/\mu_f^2 = 2$. Note that the high order terms with the dimension above two for the effective meson fields are not included, which are assumed to be small and generated in loop diagrams, so only the lowest order nontrivial fermionic interaction terms are taken into account after integrating over the gluon field.

For our present purpose, we only focus on the scalar and pseudoscalar mesons while the vector and axial vector sectors will not be considered here. As the pseudoscalar mesons are known to be the would-be Goldstone bosons, the effective chiral field theory is naturally to be realized as a nonlinear model. Thus we may express the effective meson fields $\Phi(x)$ into the following $2 \times 2$ complex matrix form:

$$\Phi(x) \equiv \xi_L(x) \phi(x) \xi_R^\dagger(x), \quad U(x) \equiv \xi_L(x) \xi_R^\dagger(x) = \xi^2_L(x) = e^{2i f(x)}$$

$$\phi^\dagger(x) = \phi(x) = \sum_{a=0}^{3} \phi^a(x) T^a, \quad \Pi^\dagger(x) = \Pi(x) = \sum_{a=0}^{3} \Pi^a(x) T^a$$

where $T^a (a = 0, 1, 2, 3)$ with $[T^a, T^b] = if^{abc} T^c$ and $2tr T^a T^b = \delta_{ab}$ are the four generators of $U(2)$ group. The fields $\Pi^a(x)$ represent the pseudoscalar mesons and $\phi^a(x)$ the corresponding scalar chiral partners. Where $f$ is known as the decay constant with mass dimension.

The Lagrangian Eq.(3) with the definition of composite meson fields $\Phi_{ij}(x)$ Eq.(5) is our starting point for the derivation of the effective chiral Lagrangian for mesons. The Lagrangian can be obtained by integrating over the quark fields and the procedure for the derivation can be formally expressed in terms of the generating functionals via the following relations

$$\frac{1}{Z} \int [dG_\mu][dq][d\bar{q}] e^{i \int d^4x \mathcal{L}_{\text{QCD}}} = \frac{1}{Z} \int [d\Phi][dq][d\bar{q}] e^{i \int d^4x \mathcal{L}_{\text{eff}}(\Phi)(q, \bar{q})}$$

$$= \frac{1}{Z_{\text{eff}}} \int [d\Phi] e^{i \int d^4x \mathcal{L}_{\text{eff}}(\Phi)}$$

Let us first demonstrate the derivation of $\mathcal{L}_{\text{eff}}(\Phi)$. In order to obtain the effective chiral Lagrangian for the meson fields, we need to integrate over the quark fields (which is equivalent to calculate the Feynman diagrams of quark loops) from the following chiral Lagrangian

$$\mathcal{L}_{\text{eff}}^q(q, \bar{q}) = \bar{q} \gamma^\mu i \partial_\mu q + \bar{q}_L \gamma_\mu A^\mu_L q_L + \bar{q}_R \gamma_\mu A^\mu_R q_R - [\bar{q}_L (\Phi - M) q_R + h.c.] + \chi \bar{q} q$$

where we have introduced the real source field $\chi(x)$ for the composite operator $\bar{q} q$ and the source term $\chi \bar{q} q$ which will be useful for the derivation of the chiral thermodynamic model, while eventually the source field is taken to be zero $\chi = 0$. 


With the method of path integral, the effective Lagrangian of the meson fields is evaluated by integrating over the quark fields

$$\int [d\Phi] \exp\{i \int d^4x L^M \} = Z_0^{-1} \int [d\Phi][d\bar{q}]\exp\{i \int d^4x L^q_{eff} \}$$  \hspace{1cm} (8)

The functional integral of the right hand side is known as the determination of the Dirac operator

$$\int [dq][d\bar{q}]\exp\{i \int d^4x L^q_{eff} \} = \det(iD^\chi)$$  \hspace{1cm} (9)

To obtain the effective action, it is useful to go to Euclidean space via the Wick rotation

$$\gamma_0 \rightarrow i\gamma_4, \quad G_0 \rightarrow iG_4, \quad x_0 \rightarrow -ix_4$$  \hspace{1cm} (10)

and to define the Hermitian operator

$$S^M_E = \int d^4x_E L^M_E = \ln \det(iD^\chi_E)$$

$$= \frac{1}{2} [\ln \det(iD^\chi_E) + \ln \det((iD^\chi_E)^\dagger)] + \frac{1}{2} [\ln \det(iD^\chi_E) - \ln \det((iD^\chi_E)^\dagger)]$$

$$\equiv S^M_{Re} + S^M_{Im}$$  \hspace{1cm} (11)

with

$$S^M_{Re} = \int d^4x_E L^M_{Re} = \frac{1}{2} \ln \det(iD^\chi_E((iD^\chi_E)^\dagger)) \equiv \frac{1}{2} \ln \det \Delta^\chi - \ln Z_0$$  \hspace{1cm} (12)

$$S^M_{Im} = \int d^4x_E L^M_{Im} = \frac{1}{2} \ln \det((iD^\chi_E/(iD^\chi_E)^\dagger)) \equiv \frac{1}{2} \ln \det \Theta_E$$  \hspace{1cm} (13)

where the imaginary part $L^M_{Im}$ appears as a phase which is related to the anomalous terms and will not be discussed in the present paper. The operators in the Euclidean space are given by

$$iD^\chi_E = -i\gamma \cdot \partial - \gamma \cdot A_L P_L - \gamma \cdot A_R P_R + \hat{\Phi} P_R + \hat{\Phi}^\dagger P_L + \chi$$

$$= iD_E + \chi$$

$$(iD^\chi_E)^\dagger = i\gamma \cdot \partial + \gamma \cdot A_R P_L + \gamma \cdot A_L P_R + \hat{\Phi} P_R + \hat{\Phi}^\dagger P_L + \chi$$

$$= (iD_E)^\dagger + \chi$$  \hspace{1cm} (14)

with $\hat{\Phi} = \Phi - M$ and $P_\pm = (1 \pm \gamma_5)/2$. $iD^\chi_E$, $(iD^\chi_E)^\dagger$ and $\Delta^\chi_E$ are regarded as matrices in coordinate space, internal symmetry space and spin space. Noticing the following identity

$$\ln \det O = Tr \ln O$$  \hspace{1cm} (15)

with $Tr$ being understood as the trace defined via

$$TrO = tr \int d^4x < x|O|y > |_{x=y}$$  \hspace{1cm} (16)

Here $tr$ is the trace for the internal symmetry space and $< x|O|y >$ is the coordinate matrix element defined as

$$< x|O_{ij}|y > = O_{ij}^k(x)\delta^4(x-y), \quad \delta^4(x-y) = \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} e^{ik(x-y)}$$  \hspace{1cm} (17)
For the derivative operator, one has in the coordinate space

\[ <x|\partial^\mu|y> = \delta^4(x-y)(-ik^\mu + \partial_y^\mu) \]  

(18)

With the above definitions, the operators \(iD_E^k\), \((iD_E^k)\dagger\) and \(\Delta_E^k\) in the Euclidean space are given by

\[
\begin{align*}
iD_E^k &= -\gamma \cdot k - i\gamma \cdot \partial - \gamma \cdot A_L P_L - \gamma \cdot A_R P_R + \hat{\Phi} P_R + \hat{\Phi}^\dagger P_L + \chi \\
(iD_E^k)\dagger &= \gamma \cdot k - i\gamma \cdot \partial + \gamma \cdot A_R P_L + \gamma \cdot A_L P_R + \hat{\Phi}^\dagger P_R + \hat{\Phi} P_L + \chi \\
\Delta_E^k &= k^2 + \hat{\Phi}^\dagger P_R + \hat{\Phi}^\dagger P_L - i\gamma \cdot D_E \Phi P_L - i\gamma \cdot D_E \Phi^\dagger P_R \\
&\quad - \sigma_{\mu\nu} F_R^{\mu\nu} P_L - \sigma_{\mu\nu} F_L^{\mu\nu} P_R + (iD_{E\mu})(iD_{E\nu}) + 2k \cdot (iD_E) \\
&\quad - i\gamma \cdot \partial \chi + \chi [iD_E + (iD_E)\dagger] + \chi^2 \\
&= k^2 + \Delta^\chi = k^2 + \Delta_E + \Delta^\chi
\end{align*}
\]

(19-21)

where

\[
iD_E \Phi = i\partial \Phi + A_L \Phi - \Phi A_R \\
iD_E = i\partial + A_R P_L + A_L P_R
\]

(22-23)

and

\[
\begin{align*}
\Delta_E &\equiv \hat{\Phi}^\dagger P_R + \hat{\Phi}^\dagger P_L - i\gamma \cdot D_E \Phi P_L - i\gamma \cdot D_E \Phi^\dagger P_R \\
&\quad - \sigma_{\mu\nu} F_R^{\mu\nu} P_L - \sigma_{\mu\nu} F_L^{\mu\nu} P_R + (iD_{E\mu})(iD_{E\nu}) + 2k \cdot (iD_E) \\
\Delta^\chi &\equiv -i\gamma \cdot \partial \chi + \chi [iD_E + (iD_E)\dagger] + \chi^2 \\
\Delta_E &\equiv \Delta_E + \Delta^\chi
\end{align*}
\]

(24-26)

Thus the effective action is obtained as

\[
S^M_{ERe} = \frac{N_c}{2} \int d^4x_E \int \frac{d^4k}{(2\pi)^4} e^{i(kx-y)} tr_{SF} \ln(k^2 + \Delta_E^\chi) |_{x=y} - \ln Z_0
\]

(27)

where the subscripts \(SF\) refer to the trace over the spin and flavor indices and \(N_c\) is the color number.

Before proceeding, we would like to mention some formulae which will be useful for the derivation of chiral thermodynamic model late on. By taking the functional derivative of Eq. (27) with respect to the source field \(\chi\) at \(\chi = 0\), we have

\[
\frac{\delta S^M_{ERe}}{\delta \chi(x)|_{\chi=0}} = \frac{N_c}{2} \int d^4x_E tr_{SF} \int \frac{d^4k}{(2\pi)^4} e^{i(kx-y)} \left(-\gamma \cdot k + [iD_E + (iD_E)\dagger]\right) \frac{1}{k^2 + \Delta_E}|_{x=y}
\]

(28)

from the right hand side one may pick up the quark propagator as \(\chi\) is the source field for the quark operator \(\bar{q}(x)q(x)\). This can explicitly be shown by differentiating the effective action with respect to \(\chi\), which gives the coinciding limit of the quark propagator \(\lim_{x\to y} tr_{SF} \langle T[q(x)\bar{q}(y)] \rangle\).
Alternatively, if taking the functional derivative of Eq.(27) with respect to the operator $\Delta_0$, we obtain another form of the formulae

$$\frac{\delta S^M_{ERe}}{\delta (\Delta_0)^{ij}} = \frac{N_c}{2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 + \Delta_E + \Delta_0)^{ji}}. \quad (29)$$

In other word, when functionally integrating over $\Delta_0$ and summing over the flavor and spin degrees of freedom, we are led to the effective action Eq.(27). The physical action is yielded by taking the source field to be zero $\chi(x) = 0$

$$S^M_{ERe} = \frac{N_c}{2} \int d^4x E \int \frac{d^4k}{(2\pi)^4} tr_{SF} \ln(k^2 + \Delta_E) - \ln Z_0 \quad (30)$$

To derive the effective chiral Lagrangian for the meson fields, we shall make the following redefinition for $\Delta_E^k \equiv k^2 + \Delta_E$

$$\Delta_0 = k^2 + \bar{M}^2 \quad (31)$$

$$\bar{\Delta}_E \equiv (\hat{\Phi}\hat{\Phi}^\dagger - \bar{M}^2)P_R + (\hat{\Phi}\hat{\Phi} - \bar{M}^2)P_L - i\gamma \cdot D_E\Phi P_L - i\gamma \cdot D_E\Phi^\dagger P_R$$

$$-\sigma_{\mu\nu} F_{R\mu\nu} P_L - \sigma_{\mu\nu} F_{L\mu\nu} P_R + (iD_{E\mu})(iD_{E\mu}) + 2k \cdot (iD_E) \quad (33)$$

where $\bar{M}$ is the supposed vacuum expectation values (VEVs) of $\hat{\Phi}$, i.e., $<\hat{\Phi}> = \bar{M} = \text{diag.}(\bar{m}_u, \bar{m}_d)$ with $\bar{m}_u = \bar{m}_d = \bar{m}$ under the exact isospin symmetry. Here $\bar{m}_i = v_i - m_i$ is regarded as the dynamical quark masses, and $v_i$ is supposed to be the VEVs of the scalar fields, i.e., $<\phi> = V = \text{diag.}(v_1, v_2)$ which will be determined from the minimal conditions of the effective potential in the effective chiral Lagrangian $L_{eff}(\Phi)$. With this convention, it is seen that the minimal conditions of the effective potential are completely determined by the lowest order terms up to the dimension four $(\hat{\Phi}\hat{\Phi}^\dagger - \bar{M}^2)^2$ in the effective chiral field theory of mesons.

By regarding $\bar{\Delta}_E$ as the perturbative interaction term and taking $Z_0 = (\det \Delta_0)^{\frac{1}{2}}$, the effective action in the Euclidean space can be written as

$$S^M_{ERe} = \frac{N_c}{2} \int d^4x E \int \frac{d^4k}{(2\pi)^4} tr_{SF} \ln[(\Delta_0 + \bar{\Delta}_E) - \ln \Delta_0]$$

$$= \frac{N_c}{2} \int d^4x E \int \frac{d^4k}{(2\pi)^4} tr_{SF} \ln(1 + \frac{1}{\Delta_0}\bar{\Delta}_E)$$

$$= \frac{N_c}{2} \int d^4x E \int \frac{d^4k}{(2\pi)^4} tr_{SF} \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n} \frac{1}{\Delta_0^n} \bar{\Delta}_E^n$$

$$\simeq \frac{N_c}{2} \int d^4x E \int \frac{d^4k}{(2\pi)^4} tr_{SF} \left( \frac{1}{\Delta_0} \bar{\Delta}_E - \frac{1}{2\Delta_0^2} \bar{\Delta}_E^2 \right) \quad (34)$$

Note that we only keep the first two terms in the expansion over $\bar{\Delta}_E$ since these are the only divergent terms in the integration over the internal momentum $k$. Particularly, the divergence degree of the first integral is quadratical while the second logarithmic. In order to maintain the gauge invariance and meanwhile keep the divergence behavior of the integral, we adopt the loop regularization (LORE) method proposed in [11, 12] for the momentum.
in Eq. (3), we finally arrive at the following effective chiral Lagrangian at zero temperature

\[ I_2 = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + M^2} \rightarrow I_2^R = \frac{M_c^2}{16\pi^2} L_2(\frac{\mu_i^2}{M_c^2}) \quad (35) \]

\[ I_0 = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 + M^2)^2} \rightarrow I_0^R = \frac{1}{16\pi^2} L_0(\frac{\mu_i^2}{M_c^2}) \quad (36) \]

with the consistent conditions for the tensor type divergent integrals

\[ I_{2\mu\nu}^R = \frac{1}{2} g_{\mu\nu} I_2^R, \quad I_{0\mu\nu}^R = \frac{1}{4} g_{\mu\nu} I_0^R \quad (37) \]

where

\[ I_{2\mu\nu} = \int \frac{d^4k}{(2\pi)^4} \frac{k_\mu k_\nu}{(k^2 + M^2)^2}, \quad I_{0\mu\nu} = \int \frac{d^4k}{(2\pi)^4} \frac{k_\mu k_\nu}{(k^2 + M^2)^3} \quad (38) \]

The two diagonal matrices \( L_0 = diag.(L_0^{(1)}, L_0^{(2)}) \) and \( L_2 = diag.(L_2^{(1)}, L_2^{(2)}) \) are given by the following form:

\[ L_0^{(i)} = \ln \frac{M^2}{\mu_i^2} - \gamma_\omega + y_0(\frac{\mu_i^2}{M_c^2}) \quad (39) \]

\[ L_2^{(i)} = 1 - \frac{\mu_i^2}{M_c^2}[\ln \frac{M^2}{\mu_i^2} - \gamma_\omega + 1 + y_2(\frac{\mu_i^2}{M_c^2})] \quad (40) \]

with

\[ y_0(x) = \int_0^x d\sigma \frac{1 - e^{-\sigma}}{\sigma}, \quad y_1(x) = \frac{1}{x} (e^{-x} - 1 + x), \quad y_2(x) = y_0(x) - y_1(x) \quad (41) \]

Note that \( M_c^2 \) is the characteristic energy scale from which the nonperturbative QCD effects start to play an important role and the effective chiral field theory is considered to be valid below the scale \( M_c \). We have also introduced the definitions

\[ \mu_i^2 = \mu_s^2 + \bar{m}_i^2, \quad \bar{m}_i = v_i - m_i \quad (42) \]

with \( \mu_i^2 \) the sliding energy scale. It is usually taken to be at the energy scale at which the physical processes take place, which is expected to be around the QCD scale \( \Lambda_{QCD} \) for our present consideration.

With these analysis, the effective chiral Lagrangian can be systematically obtained to be

\[ S_{ERe}^M = \frac{N_c}{16\pi^2} \int d^4x E_{tr} \left\{ M_c^2 L_2[(\hat{\Phi}^\dagger \hat{\Phi} - \bar{M}^2) + (\hat{\Phi}^\dagger \hat{\Phi} - \bar{M}^2)] \right. \]

\[ \left. - \frac{1}{2} L_0[D_{E}{\hat{\Phi}} \cdot D_{E}{\hat{\Phi}} + D_{E}{\hat{\Phi}}^\dagger \cdot D_{E}{\hat{\Phi}}^\dagger + (\hat{\Phi}^\dagger - \bar{M}^2)^2 + (\hat{\Phi} - \bar{M}^2)^2] \right\} \quad (43) \]

where the trace over the spin indices give the factor 2 since our quark fields defined in Eq. (3) are Weyl fermion fields, each of which has 2 degrees of freedom.

By transforming back to the Minkowski spacetime signature and adding the extra terms in Eq. (3), we finally arrive at the following effective chiral Lagrangian at zero temperature

\[ \mathcal{L}_{eff}(\Phi) = \frac{1}{2} \frac{N_c}{16\pi^2} E_{tr} L_0[D_{\mu}{\hat{\Phi}}^\dagger D_{\mu}{\hat{\Phi}} + D_{\mu}{\hat{\Phi}} D_{\mu}{\hat{\Phi}}^\dagger - (\hat{\Phi}^\dagger \hat{\Phi} - \bar{M}^2)^2 - (\hat{\Phi}^\dagger \hat{\Phi} - \bar{M}^2)^2] \]

\[ + \frac{N_c}{16\pi^2} M_c^2 E_{tr} L_2[(\hat{\Phi}^\dagger \hat{\Phi} - \bar{M}^2) + (\hat{\Phi}^\dagger - \bar{M}^2)] \]

\[ + \mu_m^2 E_{tr}(\Phi M^\dagger + M \Phi^\dagger) - \mu_i^2 E_{tr} \Phi^\dagger \Phi^\dagger. \quad (44) \]
The derivation of the above effective chiral Lagrangian in an equivalent rotated basis\cite{13} is given in Appendix B, which may be more transparent for the spontaneous symmetry breaking with the composite Higgs-like scalar mesons.

III. DERIVATION OF CHIRAL THERMODYNAMIC MODEL OF QCD

After a brief outline for the derivation of the effective chiral Lagrangian of the CDM for mesons at zero temperature, it is straightforward to incorporate the finite temperature effects into the effective Lagrangian. The method for the derivation of finite temperature effective Lagrangian is similar by applying for the closed-time-path Green function (CTPGF) formalism. The CTPGF formalism, developed by Schwinger \cite{14} and Keldysh \cite{15}, has been used to solve lots of interesting problems in statistical physics and condensed matter theory \cite{16,17}. It is generally believed that this technique is quite efficient in investigating the nonequilibrium and finite temperature dynamical systems as this formalism simultaneously incorporates both the statistical and dynamical properties\cite{16,17}. It has also been used to treat a system of self-interacting bosons described by $\lambda \phi^4$ scalar fields \cite{26}. A brief introduction to this formalism is given in Appendix A. Readers who are not familiar with the CTPGF formalism are referred to the excellent review articles \cite{17,27} and monographs \cite{18,19}.

As shown in the Appendix A, the main step for the derivation of the effective action with finite temperature in the CTPGF formalism is to replace the field propagator with its finite temperature counterpart\cite{16,17,27,28}. Applying the CTPGF formalism to the propagator of the quark fields in Eq.(28), we arrive at the following result

$$\frac{\delta S_M}{\delta \chi(x)}|_{x=0} = \frac{N_c}{2} \int \frac{d^4k}{(2\pi)^4} tr_{SF} \left[ -\gamma \cdot k + \left(iD_E + (iD_E)^\dagger\right) \right]$$

$$\left[ \frac{1}{k^2 + \Delta_E} - 2\pi i \theta(k_4) \frac{\delta(k^2 + \Delta_E)}{k^2 + \Delta_E} \right] - 2\pi i n_F(\omega) \delta(k^2 + \Delta_E) \left( \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array} \right)$$

where $\omega$ is defined as the effective energy $\omega \equiv \sqrt{k^2 + \Delta_E}$ and $n_F(\omega)$ represents the Fermi-Dirac distribution function

$$n_F(\omega) \equiv \frac{1}{e^{\beta \omega} + 1} \quad (45)$$

For our present purpose, we only need to calculate the first component of the effective action since it is the only one which is related to the causal propagation\cite{27,28}. For convenience, we may adopt the following alternative formula which is similar to Eq.(29) at zero temperature

$$\frac{\delta S_{M_{Re}}}{\delta \Delta \chi} = \frac{N_c}{2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + \Delta_E + \Delta \chi} - 2\pi i n_F(\omega) \delta(k^2 + \Delta_E + \Delta \chi)$$

$$= \frac{N_c}{2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + \Delta_E + \Delta \chi} - \int \frac{d^3k}{(2\pi)^3} n_F(\omega) \frac{1}{\sqrt{k^2 + \Delta_E + \Delta \chi}} \quad (47)$$

By functionally integrating over $\Delta \chi$ and summing over the spin and flavor indices, we then obtain the effective action for the CTDM

$$S_{E_{Re}}^M = \frac{N_c}{2} \int d^4 x E tr_{SF} \left[ \int \frac{d^4k}{(2\pi)^4} \ln(k^2 + \Delta_E) + \frac{1}{\beta} \int \frac{d^3k}{(2\pi)^3} \ln(1 + e^{-\beta \sqrt{k^2 + \Delta_E}}) \right]$$

$$- \ln Z_0 \quad (48)$$
where we have put the source field $\chi(x) = 0$ in the end of the calculation.

Separating $\Delta^k_E$ as in Eq.(31) and identifying $\tilde{\Delta}_E$ as the perturbation part, we then make the expansion in terms of $\tilde{\Delta}_E$. Here we only keep the lowest order terms as they are the only ones relevant to our present discussion. Also, we take

\[
\ln Z_0 = \frac{N_c}{2} \int d^4x_E tr SF[\int \frac{d^4k}{(2\pi)^4} \ln(k^2 + \tilde{M}^2) + \frac{1}{\beta} \int \frac{d^3k}{(2\pi)^3} \ln(1 + e^{-\beta \sqrt{k^2 + \tilde{M}^2}})]
\]

(49)

which provides the cancelation for the infinite zero-point energy.

Thus, the effective action with leading terms in the Euclidean space can be written as

\[
S_{E}^{M} \approx \frac{N_c}{2} \int d^4x_E tr SF[\int \frac{d^4k}{(2\pi)^4} \ln(k^2 + \tilde{M}^2) + \frac{1}{\beta} \int \frac{d^3k}{(2\pi)^3} \ln(1 + e^{-\beta \sqrt{k^2 + \tilde{M}^2}})]
\]

\[
= \frac{N_c}{16\pi^2} \int d^4x_E tr F\{\frac{M_c^2}{16}(\Delta_{\tilde{E}} + \tilde{L}_2(T)) + (\tilde{\Phi}^+ \tilde{\Phi} - \tilde{M}^2)^2 + (\tilde{\Phi}^+ \tilde{\Phi} - \tilde{M}^2)^2\}
\]

(50)

where we have defined

\[
L_0(T) \equiv L_0\left(\frac{\mu^2}{M_c^2}\right) - \frac{1}{\pi} \int d^3k \frac{\beta \sqrt{k^2 + \tilde{M}^2} e^{\beta \sqrt{k^2 + \tilde{M}^2}} + e^{\beta \sqrt{k^2 + \tilde{M}^2}} + 1}{(k^2 + \tilde{M}^2)^{3/2}(e^{\beta \sqrt{k^2 + \tilde{M}^2}} + 1)^2}
\]

\[
L_2(T) \equiv L_2\left(\frac{\mu^2}{M_c^2}\right) - \frac{4}{\pi M_c^2} \int d^3k \frac{1}{\sqrt{k^2 + \tilde{M}^2}(e^{\beta \sqrt{k^2 + \tilde{M}^2}} + 1)}
\]

(51)

and used the results

\[
tr S\tilde{\Delta}_E = 2[(\hat{\Phi}^+ \hat{\Phi}^+ - \tilde{M}^2) + (\hat{\Phi}^+ \hat{\Phi} - \tilde{M}^2)]
\]

(52)

\[
tr S\tilde{\Delta}_E^2 = 2[D_E \hat{\Phi} \cdot D_E \hat{\Phi}^+ + D_E \hat{\Phi}^+ \cdot D_E \hat{\Phi} + (\hat{\Phi}^+ \hat{\Phi} - \tilde{M}^2)^2 + (\hat{\Phi}^+ \hat{\Phi} - \tilde{M}^2)^2]
\]

(53)

Finally, transforming the action to the Minkowski spacetime and adding the extra terms in Eq.(33), we arrive at the following effective chiral Lagrangian at finite temperature for the composite meson fields

\[
\mathcal{L}_{eff}(\Phi) = \frac{1}{16\pi^2} \int d^4x_E tr F L_0(T)\{D_\mu \hat{\Phi}^+ D^\mu \hat{\Phi} + D_\mu \hat{\Phi}^+ D^\mu \hat{\Phi}^+ - (\hat{\Phi}^+ \hat{\Phi} - \tilde{M}^2)^2 - (\hat{\Phi}^+ \hat{\Phi} - \tilde{M}^2)^2\}
\]

\[
+ \frac{N_c}{16\pi^2} M_c^2 tr F L_2(T)\{\hat{\Phi}^+ \hat{\Phi} - \tilde{M}^2\} + (\hat{\Phi}^+ \hat{\Phi}^+ - \tilde{M}^2)^2
\]

\[
+ \mu_m^2(T) tr F (\hat{M}^T + M \hat{\Phi}^T) - \mu_m^2(T) tr F \hat{\Phi}^T
\]

(54)

where $L_0(T)$ and $L_2(T)$ are given in Eq.(33). Note that the initial mass scale $\mu_f$ ($\mu_m$) characterizes the nonperturbative gluon effect at zero temperature. At the finite temperature, it is expected that the mass scale $\mu_f(\mu_m)$ is in general temperature dependent, which will be seen more clear below.
IV. DYNAMICAL SYMMETRY BREAKING AND THERMODYNAMIC PROPERTIES OF CTDM

Let us now focus on the dynamically generated effective composite Higgs potential of meson fields, which can be reexpressed as the following general form

\[ V_{eff}(\Phi) = -tr_F \tilde{\mu}_m^2(T)(\Phi M^\dagger + M \Phi^\dagger) + \frac{1}{2} tr_F \tilde{\mu}_j^2(T)(\Phi \Phi^\dagger + \Phi^\dagger \Phi) \]

\[ + \frac{1}{2} tr_F \lambda(T)[(\Phi \tilde{\Phi})^2 + (\Phi^\dagger \tilde{\Phi})^2] \]  

with \( \tilde{\mu}_j^2(T), \tilde{\mu}_m^2(T) \) and \( \lambda(T) \) the three diagonal matrices

\[ \tilde{\mu}_j^2(T) \equiv \mu_j^2(T) - \frac{N_c}{8\pi^2} (M_c^2 L_2(T) + \bar{M}^2 L_0(T)) \]  

\[ \tilde{\mu}_m^2(T) \equiv \mu_m^2(T) - \frac{N_c}{8\pi^2} (M_c^2 L_2(T) + \bar{M}^2 L_0(T)) \]

\[ \lambda(T) \equiv \frac{N_c}{16\pi^2} L_0(T) \]  

Taking the nonlinear realization \( \Phi(x) = \xi_L(x) \phi(x) \xi_R^\dagger(x) \) with supposing that the minimal of the above effective potential occurs at the point \( \langle \phi \rangle = V(T) = \text{diag.}(v_1(T), v_2(T)) \), we can write the scalar fields as follows

\[ \phi(x) = V(T) + \varphi(x) \]  

where the VEVs may be written in terms of the following general form:

\[ v_i(T) = v_o(T) + \beta_o m_i \quad i = 1, 2 \quad \text{or} \quad i = u, d \]  

For the equal mass \( m_u = m_d = m \) considered in our present case, it leads to the general VEVs \( v_1(T) = v_2(T) = v(T) \) and the single form \( v(T) = v_o(T) + \beta_o m \).

By differentiating the effective composite Higgs potential of the scalar meson field at the VEV \( v(T) \), we then obtain the minimal conditions:

\[ -\tilde{\mu}_j^2(T) v(T)_i + \tilde{\mu}_m^2(T) m_i - 2\lambda(T) \tilde{m}^3(T)_i = 0 \]  

with equal mass of two flavor quarks, it reduces to one minimal condition. For convenience of discussions, it is helpful to decompose \( \mu^2(T), \tilde{\mu}_j^2(T), \tilde{\mu}_m^2(T) \) and \( \lambda(T) \) into two parts with one part independent of the current quark mass \( m \). Practically, it can be done by making an expansion with respect to the current quark masses

\[ \mu^2(T) = \mu_o^2(T) + 2(\beta_o - 1)v_o(T)\tilde{m}, \quad \mu_o^2(T) = \mu_s^2 + v_o^2(T), \]

\[ \tilde{m}(T) = m[1 + (\beta_o - 1)m/(2v_o(T))] \]  

\[ \tilde{\mu}_j^2(T) = \tilde{\mu}_j^2(T) + 2\mu_f(T)\tilde{m}(T)[1 + \sum_{k=1} \alpha_k(T) (\frac{\tilde{m}(T)}{\mu_o(T)})^k (\beta_o - 1)^k] \]

\[ \tilde{\mu}_m^2(T) = \tilde{\mu}_m^2(T) + 2\mu_f(T)\tilde{m}(T)[1 + \sum_{k=1} \alpha_k(T) (\frac{\tilde{m}(T)}{\mu_o(T)})^k (\beta_o - 1)^k] \]

\[ \lambda(T) = \lambda(T) - \lambda_o \sum_{k=1} \beta_k(T) (\frac{\tilde{m}(T)}{\mu_o(T)})^k (\beta_o - 1)^k, \quad \lambda_o = \frac{N_c}{16\pi^2} \]
By keeping only the nonzero leading terms in the expansion of current quark masses, we then obtain the following constraints from the minimal condition Eq. (61)

$$\bar{\mu}_f^2(T) + 2\bar{\lambda}(T)v_o^2(T) = 0$$  \hspace{1cm} (66)

Here the temperature-dependent parameters $\bar{\mu}_m^2(T)$, $\bar{\mu}_f^2(T)$ and $\bar{\lambda}(T)$ are related to the initial parameters in the effective potential and the characteristic energy scale via the following relations

$$\bar{\mu}_f^2(T) = \mu_f^2(T) - \frac{N_c}{8\pi^2}(M_c^2\bar{L}_2(T) + v_o^2(T)\bar{L}_0(T))$$  \hspace{1cm} (67)

$$\bar{\mu}_m^2(T) = \mu_m^2(T) - \frac{N_c}{8\pi^2}(M_c^2\bar{L}_2(T) + v_o^2(T)\bar{L}_0(T))$$  \hspace{1cm} (68)

$$\bar{\lambda}(T) = \frac{N_c}{16\pi^2}L_0(T)$$  \hspace{1cm} (69)

where $L_0(T)$ and $L_2(T)$ represent the leading order expansion of $L_0(T)$ and $L_2(T)$ with respect to $m$. Explicitly, they are given by

$$L_0(T) \equiv L_0\left(\frac{\mu_o^2(T)}{M_c^2}\right) - \frac{1}{\pi} \int d^3k \frac{\beta\sqrt{k^2 + v_o^2(T)}e^{\beta\sqrt{k^2 + v_o^2(T)}} + e^{\beta\sqrt{k^2 + v_o^2(T)}} + 1}{(k^2 + v_o^2(T))^{3/2}(e^{\beta\sqrt{k^2 + v_o^2(T)}} + 1)^2}$$  \hspace{1cm} (70)

$$L_2(T) \equiv L_2\left(\frac{\mu_o^2(T)}{M_c^2}\right) - \frac{4}{\pi M_c^2} \int d^3k \frac{1}{\sqrt{k^2 + v_o^2(T)}(e^{\beta\sqrt{k^2 + v_o^2(T)}} + 1)}$$  \hspace{1cm} (71)

With these definitions of parameters, the minimal condition Eq. (66) is transformed into the following form

$$\bar{\mu}_f^2(T) = \frac{N_c}{8\pi^2}M_c^2\bar{L}_2(T)$$

$$= \frac{N_c}{8\pi^2}[M_c^2 - \mu_o^2(T)(\ln\frac{M_c^2}{\mu_o^2(T)} - \gamma_o + 1 + y_2(\frac{\mu_o^2(T)}{M_c^2}))]
- \frac{2N_c}{\pi^2} \int_0^{\infty} dk \frac{k^2}{\sqrt{k^2 + v_o^2(T)}(e^{\beta\sqrt{k^2 + v_o^2(T)}} + 1)}$$  \hspace{1cm} (72)

which is the gap equation at finite temperature. In order to obtain the critical temperature, let us make a simple assumption for the temperature dependence of the mass scale $\mu_f^2(T)$

$$\mu_f^2(T) = \gamma_o v_o^2(T)$$  \hspace{1cm} (73)

with $\gamma_o$ a temperature independent constant. The reason for this assumption will become manifest below from the thermodynamic property of the pion meson mass. In fact, recall that the appearance of $\mu_f^2$ in the effective Lagrangian Eq. (5) can be traced back to the integration over the gluon fields. In principle, it can be calculated from QCD and given in terms of the QCD parameters $g_s(\mu)$ and $\Lambda_{QCD}$. So the thermodynamic property of $\mu_f^2$ is expected to be obtained from the detailed analysis of gluon dynamics at finite temperature. The above simple assumption means that the gluon thermodynamics has the same temperature dependence as the chiral thermodynamics of quark condensate.

With such an assumption and based on the chiral thermodynamic gap equation, we are able to calculate the critical temperature for the chiral symmetry restoration. Suppose that
at the critical temperature the VEV \( v_\omega(T) \) approaches to vanish, so does the \( \mu_j^2(T) \), then
the gap equation becomes
\[
0 = \frac{N_c}{8\pi^2} [M_c^2 - \mu_s^2(\ln \frac{M_c^2}{\mu_s^2} - \gamma_\omega + 1 + y_2(\frac{\mu_s^2}{M_c^2})) - \frac{2N_c}{\pi^2} \int_0^\infty dk \frac{k}{e^{\beta k} + 1}]
\]
\[
= \frac{N_c}{8\pi^2} [M_c^2 - \mu_s^2(\ln \frac{M_c^2}{\mu_s^2} - \gamma_\omega + 1 + y_2(\frac{\mu_s^2}{M_c^2})) - \frac{2N_c}{\pi^2} T^2 \int_0^\infty dk' \frac{k'}{e^{\beta k'} + 1}]
\]
\[
= \frac{N_c}{8\pi^2} [M_c^2 - \mu_s^2(\ln \frac{M_c^2}{\mu_s^2} - \gamma_\omega + 1 + y_2(\frac{\mu_s^2}{M_c^2})) - \frac{2N_c}{12} T_c^2
\]
where \( k' \) is defined as \( k' = \beta k \) and we have used the result
\[
\int_0^\infty dk' \frac{k'}{e^{k'} + 1} = \frac{\pi^2}{12}
\]
Thus, the critical temperature for the chiral symmetry restoration is given by
\[
T_c = \sqrt{\frac{3}{4\pi^2} [M_c^2 - \mu_s^2(\ln \frac{M_c^2}{\mu_s^2} - \gamma_\omega + 1 + y_2(\frac{\mu_s^2}{M_c^2}))]}
\]
which shows that the critical temperature is characterized by the quadratic term evaluated
in the LORE method, which differs from the dimensional regularization where the quadratic
term is in general suppressed.

So far, we have explicitly shown the mechanism of dynamical spontaneous chiral symme-
try breaking and its restoration at finite temperature.

We are now going to present the explicit expressions for the masses of the scalar mesons,
pseudoscalar mesons and/or light quarks. To be manifest, let us first write down the scalar
and pseudoscalar meson matrices
\[
\sqrt{2}\varphi = \left( \begin{array}{cc} a_0^0 + \frac{\sigma}{\sqrt{2}} & a_0^+ \\ a_0^- & -a_0^0 + \frac{\sigma}{\sqrt{2}} \end{array} \right)
\]
and
\[
\sqrt{2}\Pi = \left( \begin{array}{cc} \pi^0 + \frac{\eta}{\sqrt{2}} & \pi^+ \\ \pi^- & -\pi^0 + \frac{\eta}{\sqrt{2}} \end{array} \right)
\]
Keeping to the leading order of the current quark masses, we have
\[
m_{0,\pm}^2(T) = m_{\eta,\pm}^2(T) \simeq \frac{2\mu_\eta^2(T)}{f_\pi^2(T)}(m_u + m_d) = \frac{4\mu_\eta^2(T)}{f_\pi^2(T)} m
\]
for the pseudoscalar mesons, and
\[
m_{0,\pm}^2(T) = m_{\sigma,\pm}^2(T) \simeq 3(\bar{m}_u(T) + \bar{m}_d(T)) = 6\bar{m}_s^2(T)
\]
for the scalar mesons. Where \( \mu_\eta^3 \) is given by
\[
\mu_\eta^3(T) = (\bar{m}_m(T) + 2\lambda(T)v_\eta^2(T))v_\eta(T) = \mu_\eta^3(T)v_\eta(T) = \gamma_\eta v_\eta^3(T)
\]
where we have used the minimal condition Eq. (76) and the relation Eq. (73).

Note that in obtaining the above results for the scalar and pseudoscalar meson masses,
the SU(2) triplet and singlet mesons have the common masses: \( m_{0,\pm}^2 = m_\sigma^2 \) and \( m_{\eta,\pm}^2 = m_\eta^2 \),
which is the reflection of the present assumption of the exact \( U(1)_A \) symmetry. However,
as we discussed in Sec. 2, in the real world such a symmetry gets quantum anomalous and
will be broken down by the instanton effects, which is ignored in our present consideration.
V. PREDICTIONS WITH INPUT PARAMETERS AT LOW ENERGY AND CRITICAL TEMPERATURE IN CTDM

In order to make numerical predictions for the temperature dependence of the masses of the light scalar and pseudoscalar mesons, it needs to fix the values of input parameters in the effective chiral Lagrangian with finite temperature. There are in general five parameters: $\mu^2_f (\mu^2_m), M^2_c, \mu^2_s, v_o,$ and a universal current quark mass $m$. To fix the parameters, we shall use the constraints at low energy with zero temperature.

In general, the minimal condition Eq.$(61)$ with different quark masses will lead to two constraints by expanding the equation with respect to the current quark mass up to the order of $m^2$. For the equal mass case, we get the following minimal condition

$$\bar{\mu}^2_f + 2\bar{\lambda}v^2_o = 0 \quad (82)$$

with

$$\frac{\lambda_o}{\lambda}[(\frac{2v^2_o}{\mu^2_o} - 1)(1 - \frac{v^2_o}{3\mu^2_o}) - \frac{2v^2_o}{3\mu_o}\alpha_1(1 - r) + r] = 1 \quad (83)$$

$$r \equiv \frac{\mu^2_s}{\mu^2_o} - \frac{\mu^2_o}{M^2_c}[1 - \frac{\mu^2_s}{\mu^2_o} + O(\frac{\mu^2_o}{M^2_c})] \quad (84)$$

$$\alpha_1(1 - r) \equiv \frac{2v^2_o}{\mu_o}[\frac{\mu^2_s}{2\mu^2_o} + O(\frac{\mu^2_o}{M^2_c})] \quad (85)$$

Note that in obtaining the above result one needs to keep to the order of $m^2$ in the current quark mass expansion.

As we have shown in Sec.II that in order to have well-defined QCD current quark masses, it requires that

$$\left(\frac{\mu^2_m}{\mu^2_f} - 1\right)M = M, \quad i.e. \quad \mu^2_m = 2\mu^2_f \quad (86)$$

which fixes the parameter

$$\beta_o = \frac{\mu^2_m}{\mu^2_f} = 2 \quad (87)$$

Also, from the original Lagrangian of chiral dynamical model Eq.$(3)$, the auxiliary fields $\Phi_{ij}$ are found to be given by the quark fields as follows

$$\Phi_{ij} = -\frac{1}{\mu^2_f}q^R_j q^L_i + \frac{\mu^2_m}{\mu^2_f}M_{ij} \quad (88)$$

By assuming that the quark condensation is almost flavor independent, i.e., $<\bar{u}u> \approx <\bar{d}d>$, and combining the condition Eq.$(86)$, then the dynamical quark masses take the simple form

$$\bar{m} = v - m = v_o + (\beta_o - 1)m = v_o + m \quad (89)$$

which may be identified with the expected constituent quark masses after dynamically spontaneous symmetry breaking, and $v_o$ is caused by the quark condensation

$$v_o = -\frac{1}{2\mu^2_f}\langle\bar{q}q\rangle, \quad q = u, d \quad (90)$$
To determine the remaining parameters, we consider the following constraints. There are two constraints arising from the pseudoscalar sector. One is from the normalization of the kinetic terms.

$$\bar{\lambda} v_0^2 = \frac{f_\pi^2}{4}$$  \hspace{1cm} (91)

After some manipulation, the equation can be transformed into the following form

$$L_0(\mu^2/M_c^2)v_0^2 = \frac{(4\pi f_\pi)^2}{4N_c} \equiv \bar{\Lambda}_f^2 \simeq (340\text{MeV})^2$$  \hspace{1cm} (92)

where we have used the pion decay constant $f_\pi \simeq 94\text{MeV}$.

The other comes from the current quark mass and pion mass via the relation Eq. (79).

Taking the pion mass $m_\pi^0 \simeq 139\text{MeV}$ and the VEV $v_0 = 350^{+20}_{-20}\text{MeV}$ or alternatively the current quark mass $m = 4.76^{+0.04}_{-0.08}\text{MeV}$ as the inputs, we obtain the following relation

$$2\mu_f^2 v_0 = -\langle \bar{q}q \rangle = \frac{m_\pi^2+f_\pi^2}{2m} = (262^{+1}_{-2}\text{MeV})^3$$  \hspace{1cm} (93)

With the above relations and constraints Eqs. (82), (83), (86), (90), (92), (93), all the parameters can be completely determined

$$v_0 \simeq 350^{+20}_{-20}\text{MeV}$$
$$M_c \simeq 881^{+32}_{-37}\text{MeV}, \quad \mu_s \simeq 312^{+3}_{-11}\text{MeV}$$
$$\mu_m^2 = 2\mu_f^2 = (226^{+5}_{-7}\text{MeV})^2$$
$$\beta_o = 2, \quad \gamma_o = \frac{\mu_f^2}{v_0^2} = 0.209^{+0.031}_{-0.041}$$
$$\langle \bar{q}q \rangle = -(262^{+1}_{-2}\text{MeV})^3$$  \hspace{1cm} (94)

With these parameters, we can immediately obtain the critical temperature for the chiral symmetry restoration

$$T_c = \sqrt{\frac{6}{8\pi^4}[M_c^2 - \mu_s^2(\ln \frac{M_c^2}{\mu_s^2} - \gamma_0 + 1 + y_2(\frac{\mu_f^2}{M_c^2})]} \simeq 200^{+9}_{-15}\text{MeV}$$  \hspace{1cm} (95)

which is consistent with NJL model prediction [20–23].

VI. CHIRAL SYMMETRY RESTORATION AND CRITICAL PHASE TRANSITION OF LOW ENERGY QCD

In this section, we will present numerical predictions based on the CTDM. Especially we will show the thermodynamic behavior of the VEV $v_0(T)$, pion decay constant $f_\pi(T)$, the quark condensate $\langle \bar{q}q \rangle(T)$ and the masses of pseudoscalar mesons $m_{\pi^0,\pm}(T)$. Their temperature dependence and the properties of critical phase transition are plotted in all the diagrams with adopting the central values of the quantities listed in Eq. (94).

From the gap equation Eq. (72), we can numerically solve the vacuum expectation value $v_0(T)$ at any finite temperature until the critical temperature where $v_0(T)$ approaches to vanish. The result is shown in Fig. (1).
By the normalization of kinetic terms of pseudoscalar sector Eq. (91), we can obtain the expression determining the pion decay constant at finite temperature

\[
f_{\pi}(T) = \sqrt{4\lambda(T)v_0^2(T)} = 2v_o(T)\sqrt{\frac{N_c}{16\pi^2}}\bar{L}_0(T) \tag{96}
\]

which is presented in Fig. (2)

Furthermore, the quark condensate \(\langle \bar{q}q \rangle(T)\) is given by:

\[
\langle \bar{q}q \rangle(T) = -2\mu_f^2(T)v_o(T) = -2\gamma_o v_0^3(T) \tag{97}
\]

its variation with respect to temperature is displayed in Fig. (3)

The leading order approximation of the pseudoscalar meson mass \(m_{\pi0,\pm}\) with respect to current quark mass \(m\) are expressed in Eq. (79), which is shown in Fig. (4).
Let us now turn to thermodynamic property of the pseudoscalar meson mass Eq. (79):

\[
m^2_{\pi^0,\pm}(T) \simeq \frac{4\mu_p^3(T)}{f_\pi^2(T)} m = \frac{4v_\omega(T)\mu_f^2(T)}{f_\pi^2(T)} m = \frac{\mu_f^2(T)}{\lambda(T)v_\omega(T)} m
\]  

(98)

which explicitly shows that when keeping the mass scale \(\mu_f^2\) to be a temperature-independent constant, the thermodynamic mass of the pseudoscalar meson becomes divergent near the critical temperature \(T_c\) as \(v_\omega(T_c) = 0\), which is obviously contrary to our intuition. This is a manifest reason why we should make an assumption for the temperature dependence of \(\mu_f^2(T)\) given in Eq. (73), which can lead to the expected thermodynamic behavior for the pseudoscalar meson mass near the critical point

\[
m^2_{\pi^0,\pm}(T) = \frac{\gamma_0 v_\omega(T)}{\lambda(T)} m,
\]  

(99)
which is shown in Fig. (4).

According to the above derivation, we see that the phase transition of chiral symmetry restoration is second order in our simple model. Thus, it is natural to determine the critical behavior of all the quantities discussed previously. Now we would like to find the the scaling behavior of the vacuum expectation value $v_o(T)$ near the critical point. By expanding our gap equation Eq.(72) around the critical temperature $T_c$ with respect to the small value of VEV $v_o(T)^2$ up to the order of $v_o(T)^2$, we can obtain:

$$\frac{1}{6}N_c(T_c^2 - T^2) - Cv_o(T)^2 = 0,$$

where $C = \frac{N_c}{8\pi^2}\gamma(0, \frac{\mu^2}{M^2}) + \frac{1}{4} - \gamma_o$ and $\gamma(s, x) \equiv \int_0^x t^{s-1}e^{-t}dt$ is the lower incomplete gamma function. Given above equation, we can easily obtain:

$$v_o = \sqrt{\frac{N_c}{6C}(T_c^2 - T^2)} \propto (T_c - T)^{\frac{1}{2}}.$$  

(101)

Thus, the critical dimension is $\beta = 0.5$. Other quantities such as $f_\pi(T)$, $m_{\pi^0, \pm}(T)$ and $(-\langle \bar{q}q(T) \rangle)^{1/3}$ all have the same scaling behavior. Such a critical behavior is not accidental, which can actually be understood from the fact that near the critical temperature all these quantities are proportional to the VEV $v_o(T)$ with $\lambda$ keeping fixed to $\lambda(T_c) \neq 0$.

VII. CONCLUSIONS AND REMARKS

In this paper, we have extended the chiral dynamical model to the chiral thermodynamic model by adopting the CTPGF approach. The resulting effective chiral Lagrangian for the composite meson fields at finite temperature is similar to the one of the CDM, but all the couplings and mass scales become temperature dependent. We have discussed in detail the finite temperature behavior of CTDM. Much attention has been paid to the thermodynamic chiral symmetry breaking and its restoration at finite temperature. After fixing the free parameters in the effective chiral Lagrangian, we have determined the critical temperature for the chiral symmetry restoration, its value has been found to be around $T_c \simeq 200$ MeV which is consistent with other predictions based on the NJL model[20–23]. We have also explicitly presented the thermodynamic behavior of several interesting quantities which include the vacuum expectation value VEV $v_o(T)$, the pion decay constant $f_\pi(T)$, the quark condensate $\langle \bar{q}q(T) \rangle$ and the pseudoscalar meson mass $m_{\pi^0, \pm}(T)$, they all display the property of the chiral symmetry restoration at the critical temperature $T_c$. From the numerical calculations, we have shown that they all have the same scaling behavior near the critical point. It is interesting to note that the mass scale $\mu_f$ for the four quark interaction in the NJL model should be temperature dependent at finite temperature as expected from the gluon thermodynamics, its thermodynamic behavior near the critical point is required to be same as the one of the chiral symmetry breaking. As a consequence, we are led to the assumption that $\mu_f^2(T) = \gamma_o v_o^2(T)$ in order to yield the expected thermodynamic behavior of the pion meson mass and to avoid the divergent behavior near the critical point of phase transition. Finally, we would like to remark that as limited from our main purpose in the present paper we have only considered two flavor quarks and ignored the important instanton effects and $U(1)_A$ anomalous effect, which prevents us to discuss some other interesting properties, such as the large strange quark mass effects and the anomalous $U(1)_A$ symmetry restoration at finite temperature, we shall investigate those interesting effects elsewhere.
Acknowledgement

The authors would like to thank L.X. Cui and Y.B. Yang for useful discussions. This work was supported in part by the National Science Foundation of China (NSFC) under Grant #No. 10821504, 10975170 and the Project of Knowledge Innovation Program (PKIP) of the Chinese Academy of Science.
Appendix A: Brief Outline on Closed-Time-Path Green Function (CTPGF) Formalism

The formalism used in zero-temperature quantum field theory is suitable to describe observables (e.g. cross-sections) measured in empty space-time, as particle interactions in an accelerator. However, at high temperature, the environment has a non-negligible density of matter which makes the assumption of zero-temperature field theories inapplicable. Namely, under those circumstances, the methods of zero-temperature field theories are not sufficient any more and should be replaced by others, which is closer to thermodynamics where the background state is a thermal bath. Therefore we shall develop quantum field theory with finite temperature which is extremely useful to study all phenomena due to the collective effects, such as: phase transitions, early universes, etc. There are several approaches for the finite temperature field theories, in this appendix we shall focus on the closed-time-path Green function (CTPGF) formalism which is simply applicable in our case. The CTPGF formalism, developed by Schwinger [14] and Keldysh [15], has been used to solve lots of interesting problems in statistical physics and condensed matter theory [17]. It is generally believed that this technique is quite efficient in investigating the nonequilibrium and finite temperature dynamical systems, this is because such a formalism naturally incorporates both the statistical and dynamical information[16, 17]. Excellent review articles [17, 27] and monographs [18, 19] have described different aspects of these issues. In this appendix, we will briefly outline the main method with the Schwinger-Keldysh propagators and the universal Feynman rules for the general theory.

Let us begin by the general discussion of statistical physics. A dynamical system can be characterized by its Hamiltonian $H$ and a statistical ensemble of this system in equilibrium at a finite temperature $T = \frac{1}{\beta}$ (in units of Boltzmann constant) is described in terms of a partition function

$$Z(\beta) = \text{Tr} \rho(\beta) = \text{Tr} e^{-\beta H}$$  \hspace{1cm} (A1)

Here $\rho(\beta)$ is known as the density matrix operator and $H$ can be thought of as the generalized Hamiltonian of the system. For example, for a canonical ensemble in which the system can only exchange energy with the heat bath, $\mathcal{H}$ is defined as:

$$\mathcal{H} = H$$  \hspace{1cm} (A2)

while for a grand canonical ensemble in which the system can not only exchange energy with the heat bath but also exchange particles with the reservoir, $\mathcal{H}$ is taken as

$$\mathcal{H} = H - \mu N$$  \hspace{1cm} (A3)

where $\mu$ is the chemical potential and $N$ represents the particle number operator.

A observable in a statistical ensemble is the ensemble average for any operator

$$\langle \mathcal{O} \rangle_\beta = \frac{1}{Z(\beta)} \text{Tr} \rho(\beta) \mathcal{O}$$  \hspace{1cm} (A4)

Since the partition function and ensemble averages involve a trace operation, this feature
leads to the famous KMS (Kubo-Martin-Schwinger) relation.

\[
\langle O_1(t)O_2(t') \rangle_\beta = \frac{1}{Z(\beta)} \text{Tr} e^{-\beta H} O_1(t)O_2(t')
\]

\[
= \frac{1}{Z(\beta)} \text{Tr} e^{-\beta H} O_2(t')e^{-\beta H}O_1(t)e^{\beta H}
\]

\[
= \frac{1}{Z(\beta)} \text{Tr} e^{-\beta H} O_2(t')O_1(t-i\beta)
\]

\[
= \langle O_2(t')O_1(t-i\beta) \rangle_\beta (A5)
\]

Note that KMS relation only rely on the trace operation and does not depend on any periodicity property of operators along the temperature (imaginary time) interval.

Now it is easily seen that the operator \(e^{-\beta H}\) in the definition of the partition function is very similar to the time evolution operator in the imaginary time axis \(e^{-i(-i\beta)H}\) \[29\]. We promote this similarity and analytically extend the time variable to the complex plane. So the operator \(e^{-\beta H}\) would live on the line interval which is parallel to the negative imaginary time-axis, with its length \(\beta\). The analogy implies that we can define our theory on some certain contour on the complex t-plane.

The contour should satisfy the following several requirement: (i) The two endpoints of the contour must be fixed in an interval in a line parallel to the imaginary axis with its length \(\beta\). The stating point A (corresponding to time \(t_i\)) and the ending point B (corresponding to \(t_f = t_i - i\beta\)) are identified, and one requires that \(O|_B = O|_A\) if \(O\) is bosonic and \(O|_B = -O|_A\) if \(O\) is fermionic; (ii) For a system whose spectrum of the Hamiltonian is semi-positive (at least bounded below due to the stability of the system), the contour needs to have a monotonically decreasing or constant imaginary part for the reason of the convergence of the complete partition function. The same result can also be obtained by analyzing the convergence of the two-point Green function \[28\]; (iii) The contour needs to pass the whole real axis of t-plane on which the field operators \(O_i(t)\) are defined (Real-time Formalism. Otherwise, like the imaginary time formalism, the field operators need to be analytically extended to the imaginary axis first).

The particular family of such real time contours is depicted in Fig. 5 where the contour

![Contour used in the real time formalism](img)

\(\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4\). The contour \(\mathcal{C}_1\) goes from the initial time \(t_i\) to the final time \(t_f\)
$t_f$, $C_3$ from $t_f$ to $t_f - i\sigma$, with $0 \leq \sigma \leq \beta$, $C_2$ from $t_f - i\sigma$ to $t_i - i\sigma$, and $C_4$ from $t_i - i\sigma$ to $t_i - i\beta$. Different choices of $\sigma$ lead to an equivalence class of quantum field theories at finite temperature. Our preferred choice is the Schwinger-Keldysh one with $\sigma = 0$.

With this specific contour, the action of a field configuration is the sum of contributions from the three parts,

$$ S = \int_\mathcal{C} dt \, L(t) = \int_{t_i}^{t_f} dt \, L(t) - \int_{t_i}^{t_f} dt \, L(t) - i \int_0^{\beta} d\tau \, L(t_i - i\tau) $$

(A6)

where

$$ L(t) = \int d\vec{x} \mathcal{L}[\phi(t, \vec{x})], $$

(A7)

and $\mathcal{L}$ is the Lagrangian density. In the following we will take the theory of a scalar field $\phi(x)$ as an example. However, in the limit $t_i \to -\infty$ and $t_f \to \infty$, it can be shown that the third branch gets decoupled from the other two (the factors in the propagators connecting such branches are asymptotically damped). Consequently, in this limit, we are effectively dealing with two branches leading to the name ”closed time path formalism”. In this contour, then, the time integration has to be thought of as

$$ \int_\mathcal{C} dt = \int_{-\infty}^{\infty} dt_+ - \int_{-\infty}^{\infty} dt_- $$

(A8)

where the relative negative sign arises because time is decreasing in the second branch of the time contour.

The advantage of introducing the contour $\mathcal{C}$ is that one can introduce the sources coupled to the field $\phi$ which is not vanishing on the two Minkowski parts of the contour. This procedure would give us the generating functional

$$ Z[J_1, J_2] = \int \mathcal{D}\phi \exp \left( i S + i \int_{-\infty}^{\infty} dt_+ \int d\vec{x} J_1(x) \phi_1(x) - i \int_{-\infty}^{\infty} dt_- \int d\vec{x} J_2(x) \phi_2(x) \right) . $$

(A9)

Here $J_{1,2}$ and $\phi_{1,2}$ are the sources and fields on the two Minkowski parts of the contour, i.e.,

$$ J_1(t, \vec{x}) = J_+(t, \vec{x}) , \quad \phi_1(t, \vec{x}) = \phi_+(t, \vec{x}) , $$

$$ J_2(t, \vec{x}) = J_-(t, \vec{x}) , \quad \phi_2(t, \vec{x}) = \phi_-(t, \vec{x}) . $$

(A10a)

(A10b)

By taking second variations of $Z$ with respect to the source $\phi$ one finds the Schwinger-Keldysh propagator

$$ iG_{ab}(x - y) = \frac{1}{i^2} \frac{\delta^2 \ln Z[J_1, J_2]}{\delta J_a(x) \delta J_b(y)} = i \begin{pmatrix} G_{11} & -G_{12} \\ -G_{21} & G_{22} \end{pmatrix} . $$

(A11)

In the operator formalism, the Schwinger-Keldysh propagator corresponds to the contour-ordered correlation function. In the single time representation\cite{17}, this means:

$$ iG_{11}(t, \vec{x}) = \langle T \phi_1(t, \vec{x}) \phi_1(0) \rangle_\beta , \quad iG_{12}(t, \vec{x}) = \langle \phi_2(0) \phi_1(t, \vec{x}) \rangle_\beta , $$

$$ iG_{21}(t, \vec{x}) = \langle \phi_2(t, \vec{x}) \phi_1(0) \rangle_\beta , \quad iG_{22}(t, \vec{x}) = \langle T \phi_2(t, \vec{x}) \phi_2(0) \rangle_\beta . $$

(A12)
where $T$ ($\bar{T}$) denotes normal (reversed) time ordering, and

\[
\phi_1(t, \bar{x}) = e^{iHt-i\bar{P}\cdot \bar{x}}\phi(0)e^{-iHt+i\bar{P}\cdot \bar{x}}, \\
\phi_2(t, \bar{x}) = e^{iH(t-i\sigma)-i\bar{P}\cdot \bar{x}}\phi(0)e^{-iH(t-i\sigma)+i\bar{P}\cdot \bar{x}}.
\] (A13a)

(A13b)

Let us now consider the free real scalar theory. If one goes to the momentum space and by inserting the complete set of states into the definitions (A12), one finds the explicit form of the previously defined Schwinger-Keldysh propagator.

\[
iG_{11}(k) = \frac{i}{k^2 - m^2 + i\epsilon} + 2\pi n_B(\omega)\delta(k^2 - m^2), \quad \omega = |k_0| \quad (A14)
\]

\[
iG_{12}(k) = 2\pi[n_B(\omega) + \theta(-k_0)]\delta(k^2 - m^2), \quad (A15)
\]

\[
iG_{21}(k) = 2\pi[n_B(\omega) + \theta(k_0)]\delta(k^2 - m^2), \quad (A16)
\]

\[
iG_{22}(k) = -\frac{i}{k^2 - m^2 + i\epsilon} + 2\pi n_B(\omega)\delta(k^2 - m^2). \quad (A17)
\]

Or in the matrix form:

\[
iG_\beta(k) = \begin{pmatrix}
i + \frac{i}{k^2 - m^2 + i\epsilon} & 2\pi \theta(-k_0)\delta(k^2 - m^2) \\
2\pi \theta(k_0)\delta(k^2 - m^2) & -\frac{i}{k^2 - m^2 - i\epsilon}
\end{pmatrix}
\]

\[
+ 2\pi n_B(\omega)\delta(k^2 - m^2) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad (A18)
\]

where $n_B(\omega) \equiv \frac{1}{e^{\omega/2} - 1}$ stands for the Bose-Einstein distribution function. Note that the propagator is a $2 \times 2$ matrix, a consequence of the doubling of the degrees of freedom. However, the propagators (12), (21) and (22) are unphysical since at least one of their time arguments is on the negative branch. They are required for the consistency of the theory. The only physical propagator is the (11) component shown in Eq. (A14).

For perturbative calculations we need to know the complete Feynman rules besides of the propagators. From the generating functional Eq. (A9) and the action Eq. (A6) defined on contour $\mathcal{C}$ Eq. (A8), we see that the complete theory contains two types of vertices- type-1 for the original fields $\phi_1(x)$ while type-2 for the doubled fields $\phi_2(x)$. The vertices for the partner fields will have a relative negative sign corresponding to the original vertices, because time is decreasing in the negative branch. The four possible propagators, (11), (12), (21) and (22) defined above connect them. All of them have to be considered for the consistency of the theory. The golden rule is that: physical legs must always be attached to type 1 vertices.[28]

For other Feynman rules, including the integration measure, the symmetry factors involved in Feynman diagrams, the topology of the Feynman diagrams, etc. are all the same as the zero-temperature field theory.

For the application to the present paper, we also need to know the Schwinger-Keldysh propagator for fermions as the quark fields here are represented as the chiral fermion fields

\[
iS_\beta(k) = (\slashed{k} + m)[\begin{pmatrix}
i + \frac{i}{k^2 - m^2 + i\epsilon} & 2\pi \theta(-k_0)\delta(k^2 - m^2) \\
2\pi \theta(k_0)\delta(k^2 - m^2) & -\frac{i}{k^2 - m^2 - i\epsilon}
\end{pmatrix}
\]

\[
-2\pi n_F(\omega)\delta(k^2 - m^2) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad (A19)
\]

where $n_F(\omega) \equiv \frac{1}{e^{\omega/2} + 1}$ stands for the Fermi-Dirac distribution function.
When transforming into the Euclidean spacetime the Schwinger-Keldysh propagator defined above becomes

\[ iS_{E\beta}(k) = (-i)(k_E^2 + m)[\begin{pmatrix} \frac{1}{k_E^2 + m^2} & 2\pi i\theta(-k_E^2)\delta(k_E^2 + m^2) \\ 2\pi i\theta(-k_E^2)\delta(k_E^2 + m^2) & -\frac{1}{k_E^2 + m^2} \end{pmatrix} \]

\[ -2\pi i\epsilon_F(\omega)\delta(k_E^2 + m^2) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \]  

(A20)

In Sec. 3 the factor \(-i\) is canceled by the factor \(i\) in the integration measure transformation \(d^4k \to id^4k_E\).

**Appendix B: Derivation of Chiral Dynamical Model in the Chiral Rotated Basis**

In this appendix, we will derive the effective chiral Lagrangian for mesons in the so-called chiral “Rotated Basis” \[13\]. Although the obtained effective chiral Lagrangian will not change, it is more transparent to see the chiral symmetries and their spontaneous breaking in this derivation. Let us begin with the effective Lagrangian

\[ \mathcal{L}_{\text{eff}}^\theta(q, \bar{q}) = \bar{q}\gamma^\mu i\partial_\mu q + \bar{q}L\gamma_\mu A_\mu^T q_L + \bar{q}R\gamma_\mu A_\mu^R q_R - [\bar{q}L(\Phi - M)q_R + h.c.] \]  

(B1)

where the auxiliary meson fields \(\Phi(x)\) is defined as in Eq.(5)

\[ \Phi(x) \equiv \xi_L(x)\phi(x)\xi_R^\dagger(x), \quad U(x) \equiv \xi_L(x)\xi_R^\dagger(x) = e^{i2\pi(x)} \]

\[ \phi^\dagger(x) = \phi(x) = \sum_{a=0}^{3} \phi^a(x)T^a, \quad \Pi^\dagger(x) = \Pi(x) = \sum_{a=0}^{3} \Pi^a(x)T^a, \]  

(B2)

where \(\Pi(x)\) and \(\phi(x)\) represent the pseudoscalar and scalar mesons respectively. Note that except for the mass term or the source term, the Lagrangian is invariant under the transformation of the local \(U(2)_L \times U(2)_R\) chiral symmetry:

\[ g_L(x) \to g_L(x)q_L(x), \quad g_R(x) \to g_R(x)q_R(x); \quad \Phi(x) \to g_L(x)\Phi(x)g_R^\dagger(x), \]

\[ A_{\mu L} \to g_L^\dagger A_{\mu L}g_L(x) - ig_L^\dagger\partial_\mu g_L(x), \quad A_{\mu R} \to g_R^\dagger A_{R\mu}g_R(x) - ig_R^\dagger\partial_\mu g_R(x), \]  

(B3)

The transformation for \(\Phi(x)\) can also be written in terms of the fields \(\phi(x)\) and \(\xi_L(x)\) as:

\[ \phi(x) \to h(x)\phi(x)h^\dagger(x), \quad \xi_L(x) \to g_L(x)\xi_L(x)h^\dagger(x) = h(x)\xi_L(x)g_R^\dagger(x). \]  

(B4)

Let us now introduce new quark fields, which is referred to the chiral “Rotated Basis” in \[13\].

\[ q_L = \xi_L Q_L, \quad \bar{q}_L = \bar{Q}_L\xi_L^\dagger, \]

\[ q_R = \xi_R^\dagger Q_R, \quad \bar{q}_R = \bar{Q}_R\xi_L. \]  

(B5)

With this new quark basis, we can rewrite the Lagrangian Eq.(B1) in the following form

\[ \mathcal{L}_{\text{eff}}^Q(Q, \bar{Q}) = \bar{Q}\gamma^\mu i\partial_\mu Q + \bar{Q}_L\gamma^\mu L_\mu Q_L + \bar{Q}_R\gamma^\mu R_\mu Q_R - [\bar{Q}_L(\phi - \mathcal{M})Q_R + h.c.], \]  

(B6)

where the fields \(L_\mu, R_\mu\) and \(\mathcal{M}\) are defined as

\[ L_\mu \equiv \xi_L^\dagger A_{\mu L}\xi_L + i\xi_L^\dagger\partial_\mu \xi_L, \quad R_\mu \equiv \xi_L A_{R\mu}^\dagger \xi_L + i\xi_L\partial_\mu \xi_L^\dagger, \]

\[ \mathcal{M} \equiv \xi_L^\dagger M\xi_L, \quad \mathcal{M}^\dagger \equiv \xi_L M^\dagger \xi_L. \]  

(B7)
In the above “rotated basis”, the quark fields $Q_{L(R)}(x)$ transform only under the diagonal $U_V(2)$ symmetry:

$$Q_L(x) \rightarrow h(x)Q_L(x), \quad Q_R(x) \rightarrow h(x)Q_R(x).$$  \hfill (B8)

Thus, the quark fields $Q_{L(R)}(x)$ are much like the “constituent quark” defined in the non-relativistic quark model \cite{30}. When the chiral symmetry is spontaneous breaking and the meson field $\phi(x)$ acquires the vacuum expectation value (VEV) $<\phi(x)> = V$, $Q_{L(R)}$ will obtain a mass term $\bar{Q}_L(V-M)Q_R + h.c.$ In the case where each quark flavor possesses the universal current mass $m$, the VEV matrix is diagonal $V = v \cdot I$, here $I$ is the identity matrix in the flavor space. The mass term is $(v-m)Q_LQ_R + h.c.$, namely the mass of quarks $Q(x)_{L(R)}$ is the dynamical quark mass $\bar{m} = v - m$ defined in Sec.\ref{sec:4}

Note that the Lagrangian Eq.(B6) has the same structure as the original one Eq.(B1) except for the definition of the mass and gauge fields. Thus, we may expect that the effective chiral Lagrangian for the meson fields has the same structure as Eq.(13). Indeed, by integrating over the quark fields following the procedure from Eq.(7) to Eq.(30), we obtain

$$S^M_{E_{Re}} = \frac{N_c}{2} \int d^4x E \int \frac{d^4k}{(2\pi)^4} tr_{SF} \ln(k^2 + \Delta'_E) - \ln Z_0,$$  \hfill (B9)

where $\Delta'_E$ above is defined as

$$\Delta'_E = \hat{\phi} \hat{\bar{\phi}} P_R + \hat{\phi} \hat{\bar{\phi}} P_L - i\gamma \cdot D'_{E} \phi P_L - i\gamma \cdot D'_{E} \phi P_R$$

$$- \sigma_{\mu\nu} \mathcal{F}_{R\mu\nu} P_L - \sigma_{\mu\nu} \mathcal{F}_{L\mu\nu} P_R + (iD'_{E\mu})(iD'_{E\mu}) + 2k \cdot (iD'_{E}),$$  \hfill (B10)

and

$$iD'_{E\mu} \phi = i\partial_{\mu} \phi + L_{\mu} \phi - \phi R_{\mu},$$  \hfill (B11)

$$iD'_{E\mu} = i\partial_{\mu} + R_{\mu} P_L + L_{\mu} P_R,$$  \hfill (B12)

$$\hat{\phi} \equiv \phi - \mathcal{M}.$$  \hfill (B13)

In order to derive the effective action for meson field, we redefine $\Delta_{E}^{k'} \equiv k^2 + \Delta'_E$ to the following two terms:

$$\Delta_{E}^{k'} \equiv k^2 + \Delta'_E = \Delta_0 + \tilde{\Delta}'_E,$$  \hfill (B14)

with

$$\Delta_0 = k^2 + \bar{M}^2,$$

$$\tilde{\Delta}'_E = [(\hat{\phi} \hat{\bar{\phi}} - \bar{\mathcal{M}} \mathcal{M}^\dagger) P_R + (\hat{\phi} \hat{\bar{\phi}} - \mathcal{M}^\dagger \bar{\mathcal{M}}) P_L - i\gamma \cdot D'_{E} \phi P_L - i\gamma \cdot D'_{E} \phi P_R$$

$$- \sigma_{\mu\nu} \mathcal{F}_{R\mu\nu} P_L - \sigma_{\mu\nu} \mathcal{F}_{L\mu\nu} P_R + (iD'_{E\mu})(iD'_{E\mu}) + 2k \cdot (iD'_{E})]$$

$$+ (\mathcal{M} \bar{\mathcal{M}}^\dagger - \bar{M}^2) P_R + (\mathcal{M}^\dagger \bar{\mathcal{M}} - \mathcal{M} \bar{M}) P_L,$$  \hfill (B15)

where $\bar{M}$ is supposed vacuum expectation values (VEVs) of $\hat{\phi}$, i.e., $<\hat{\phi}> = \bar{M}$ which is real and $\bar{\mathcal{M}} \equiv \xi_L^\dagger \bar{M} \xi_L^\dagger$. As we will see that the terms in the third line of the definition of $\Delta'_E$ is crucial to prove the equivalence of the obtained effective action for mesons in this rotated basis to the one given in Eq.(13).
Now if we regard $\tilde{\Delta}'$ as the perturbation and take $Z_0 = (\det \Delta_0)^{\frac{1}{2}}$ as before, we can expand the effective action Eq. (B19) according to $\tilde{\Delta}'$:

$$S_{E_R}^M = \frac{N_c}{2} \int d^4 x_E \int \frac{d^4 k}{(2\pi)^4} tr_{SF} \left[ \frac{1}{\Delta_0} \tilde{\Delta}' - \frac{1}{2 \Delta_0^2} (\tilde{\Delta}')^2 \right]$$

$$= \frac{N_c}{16\pi^2} \int d^4 x_E tr_F \left\{ M_2^2 L_2 \left[ (\hat{\phi} \hat{\phi}^\dagger - \bar{\mathcal{M}} \mathcal{M}^\dagger) + (\hat{\phi} \hat{\phi}^\dagger - \bar{\mathcal{M}} \mathcal{M}^\dagger)^2 \right] + M_2^2 L_2 \left[ (\bar{\mathcal{M}} \mathcal{M}^\dagger - \bar{\mathcal{M}} \mathcal{M}) (\mathcal{M}^\dagger \mathcal{M} - \bar{\mathcal{M}} \mathcal{M} - \bar{\mathcal{M}} \mathcal{M}) + (\mathcal{M}^\dagger \mathcal{M} - \bar{\mathcal{M}} \mathcal{M}) (\mathcal{M}^\dagger \mathcal{M} - \bar{\mathcal{M}} \mathcal{M}) \right] \right\},$$

(B17)

where the matrix $L_0$ and $L_2$ are defined in Eq. (39). It is easy to see that the last four lines of terms vanish if the different flavors of quarks have the same current mass which leads to the same vacuum expectation value for each flavor:

$$\mathcal{M} \mathcal{M}^\dagger - \bar{\mathcal{M}} \mathcal{M}^\dagger - \bar{\mathcal{M}} \mathcal{M}^\dagger = (\xi_L^\dagger \bar{M} \xi_L)(\xi_L \bar{M} \xi_L) = \bar{M}^2 = \xi_L^\dagger \bar{M}^2 \xi_L - \bar{M}^2 = 0.$$  

(B18)

The last line is valid since the mass matrix $\bar{\mathcal{M}}$ is diagonal with the same eigenvalues and $\bar{\mathcal{M}}$ can commute with the SU(2) matrix $\xi_L$.

Note also that in the case of 2 flavors with the universal current quark mass, after the chiral symmetry $SU_L(2) \times SU_R(2)$ is broken to $SU_V(2)$ and the field $\hat{\phi}(x)$ acquires vacuum expectation value (VEV) $V = v \cdot I$, we have:

$$iD_{E\mu}^{'\dagger} \hat{\phi} \approx (v - m) [L_\mu - R_\mu]$$

$$= (v - m) i[\xi_L^\dagger (\partial_\mu - iA_{L\mu}) \xi_L - \xi_L (\partial_\mu - iA_{R\mu}) \xi_L^\dagger]$$

$$= (v - m) i\xi_L^\dagger (D_{E\mu} U) \xi_L^\dagger = -(v - m) i\xi_L (D_{E\mu} U^\dagger) \xi_L.$$  

(B19)

In order to prove the last two equalities, we have to use the definition of $U \equiv \xi_L^\dagger$ and the identity $\partial_\mu \xi_L = -\xi_L (\partial_\mu \xi_L^\dagger) \xi_L$. Therefore, from the kinetic terms for $\hat{\phi}(x)$ in Eq. (B17), we can obtain the kinetic term for the pseudoscalar meson field $U \equiv e^{2i\Phi(x)/f_\pi}$:

$$- \frac{N_c (v - m)}{16\pi^2} \int d^4 x_E tr_E [L_0 (D_{E\mu} U)(D_{E\mu} U^\dagger)].$$  

(B20)
However, when the chiral symmetry is restored, as discussed in the context of CDTM, this term will disappear as the VEV of $\phi(x)$ vanishes. Thus, the discussion in this “rotated basis” gives a more transparent picture of Goldstone boson character of the pseudoscalar mesons.

Next we would like to prove the equivalence between the effective chiral Lagrangians Eq. (43) and Eq. (B17). According to the definition of the transformations Eq. (B7), we can easily obtain the following relations:

$$D_{E \mu}^\prime \hat{\Phi} = \xi_L^\dagger(D_{E \mu} \hat{\Phi}) \xi_L$$

where $D_{E \mu} \hat{\Phi}$ and $D_{E \mu}^\prime \hat{\Phi}^\dagger$ are defined as in Eq. (22). Thus, the first two lines of terms in Eq. (B17) can be written in a form with respect to $\Phi$:

$$S_{E \Re}^M = \frac{N_c}{2} \int d^4x E \tr \{ \xi_L^\dagger \xi_L \}
- \frac{1}{2} \left[ L_0 \xi_L^\dagger (D_E \hat{\Phi} \cdot D_E \hat{\Phi}^\dagger) \xi_L + \xi_L (D_E \hat{\Phi}^\dagger \cdot D_E \hat{\Phi}) \xi_L^\dagger
+ \xi_L^\dagger (\hat{\Phi} \hat{\Phi}^\dagger - \bar{M}^2) \xi_L + \xi_L (\hat{\Phi}^\dagger \hat{\Phi} - \bar{M}^2) \xi_L^\dagger \right]
= \frac{N_c}{2} \int d^4x E \tr \{ \xi_L^\dagger \xi_L \}
- \frac{1}{2} \left[ (\xi_L^\dagger L_0 \xi_L) \xi_L^\dagger (D_E \hat{\Phi} \cdot D_E \hat{\Phi}^\dagger) + (\hat{\Phi} \hat{\Phi}^\dagger - \bar{M}^2)^2 \right]
+ \left[ \xi_L^\dagger L_0 \xi_L \right] (D_E \hat{\Phi}^\dagger \cdot D_E \hat{\Phi} + (\hat{\Phi}^\dagger \hat{\Phi} - \bar{M}^2)^2) \}}.

In general the matrices $\xi_L$ and $L_0 \ (L_2)$ do not commute with each other when different flavors do not have the same current masses. So the part of effective chiral Lagrangian shown in Eq. (B22) is in general not equivalent to Eq. (43) by just comparing to the same truncated terms. In fact, by taking into account the higher-order terms in Eq. (B16) and the last four lines of terms in Eq. (B17), it is expected that the extra terms would cancel the unwanted terms in Eq. (B22) due to non-commutativity of $\xi_L$ and $L_0 \ (L_2)$. In our present case, as we only consider two flavors with the same current quark mass, $\xi_L$ can commute with $L_0$ and $L_2$, there is no such extra terms. Furthermore, as mentioned before, the last four lines of terms in Eq. (B17) will vanish due to commutativity of $\xi_L$ and $\bar{M}$. Thus, we arrive at the following effective chiral Lagrangian:

$$S_{E \Re}^M = \frac{N_c}{16\pi^2} \int d^4x E \tr \{ \xi_L^\dagger \xi_L \}
= \frac{N_c}{2} \left[ \xi_L^\dagger \xi_L \right] (D_E \hat{\Phi} \cdot D_E \hat{\Phi}^\dagger + (\hat{\Phi} \hat{\Phi}^\dagger - \bar{M}^2)^2) + (\hat{\Phi}^\dagger \hat{\Phi} - \bar{M}^2)^2) \}
= \frac{N_c}{2} \left[ \xi_L^\dagger \xi_L \right] (D_E \hat{\Phi} \cdot D_E \hat{\Phi}^\dagger + (\hat{\Phi} \hat{\Phi}^\dagger - \bar{M}^2)^2) + (\hat{\Phi}^\dagger \hat{\Phi} - \bar{M}^2)^2) \}
= \frac{N_c}{2} \left[ \xi_L^\dagger \xi_L \right] (D_E \hat{\Phi} \cdot D_E \hat{\Phi}^\dagger + (\hat{\Phi} \hat{\Phi}^\dagger - \bar{M}^2)^2) + (\hat{\Phi}^\dagger \hat{\Phi} - \bar{M}^2)^2) \}
= \frac{N_c}{2} \left[ \xi_L^\dagger \xi_L \right] (D_E \hat{\Phi} \cdot D_E \hat{\Phi}^\dagger + (\hat{\Phi} \hat{\Phi}^\dagger - \bar{M}^2)^2) + (\hat{\Phi}^\dagger \hat{\Phi} - \bar{M}^2)^2) \}
= \frac{N_c}{2} \left[ \xi_L^\dagger \xi_L \right] (D_E \hat{\Phi} \cdot D_E \hat{\Phi}^\dagger + (\hat{\Phi} \hat{\Phi}^\dagger - \bar{M}^2)^2) + (\hat{\Phi}^\dagger \hat{\Phi} - \bar{M}^2)^2) \}$$

which is exactly agree with Eq. (43) obtained in the original unrotated basis.

Following the procedure in Sec. III we can derive the effective chiral Lagrangian at finite
temperature in the chiral thermodynamic model (CTDM), which is:
\[
S_{Ere}^M \approx \frac{N_c}{16\pi^2} \int d^4x E tr_F \{M_c^2 L_2(T) \left[ (\hat{\phi}^\dagger \hat{\phi} - \bar{\mathcal{M}} \hat{\mathcal{M}}^\dagger) + (\hat{\phi}^\dagger \hat{\phi} - \bar{\mathcal{M}} \bar{\mathcal{M}}) \right] \\
- \frac{1}{2} L_0(T) \left[ D_E^* \hat{\phi} \cdot D_E \hat{\phi}^* + D_{E\phi}^* \hat{\phi} \cdot D_{E\phi} \hat{\phi} + (\hat{\phi}^\dagger \hat{\phi} - \bar{\mathcal{M}} \hat{\mathcal{M}}^\dagger)^2 + (\hat{\phi}^\dagger \hat{\phi} - \bar{\mathcal{M}} \bar{\mathcal{M}})^2 \right] \\
+ M_c^2 L_2(T) \left[ (\bar{\mathcal{M}} \hat{\mathcal{M}}^\dagger - \bar{\mathcal{M}}^2) + (\bar{\mathcal{M}}^\dagger \hat{\mathcal{M}} - \bar{\mathcal{M}}^2) \right] \\
- \frac{1}{2} L_0(T) \left[ (\hat{\phi}^\dagger \hat{\phi} - \bar{\mathcal{M}} \hat{\mathcal{M}})^2 (\bar{\mathcal{M}} \hat{\mathcal{M}}^\dagger - \bar{\mathcal{M}}^2) + (\hat{\phi}^\dagger \hat{\phi} - \bar{\mathcal{M}} \bar{\mathcal{M}}) (\bar{\mathcal{M}}^\dagger \hat{\mathcal{M}} - \bar{\mathcal{M}}^2) \right] \\
+ (\bar{\mathcal{M}} \hat{\mathcal{M}}^\dagger - \bar{\mathcal{M}}^2) (\hat{\phi}^\dagger \phi - \bar{\mathcal{M}} \hat{\mathcal{M}}^\dagger) + (\bar{\mathcal{M}}^\dagger \hat{\mathcal{M}} - \bar{\mathcal{M}}^2) (\hat{\phi}^\dagger \phi - \bar{\mathcal{M}} \bar{\mathcal{M}}) \\
(\bar{\mathcal{M}} \hat{\mathcal{M}}^\dagger - \bar{\mathcal{M}}^2)^2 + (\bar{\mathcal{M}}^\dagger \hat{\mathcal{M}} - \bar{\mathcal{M}}^2)^2 \right}. \tag{B24}
\]

By using the same argument as the one for the zero-temperature chiral dynamical model (CDM) analyzed in the text, we can prove the equivalence between Eq. (B24) and Eq. (50). Thus, based on the equivalent Lagrangians, the resulting physics, especially the spectrum of mesons, will not be changed.

[1] E. V. Shuryak, Prog. Part. Nucl. Phys. 53 (2004) 273.
[2] M. Gyulassy, L. McLerran, Nucl. Phys. A 750 (2005) 30.
[3] E. V. Shuryak, Nucl. Phys. A 750 (2005) 64.
[4] I. Arsene et al, Nucl. Phys. A 757 (2005) 1.
[5] B. B. Back et al, Nucl. Phys. A 757 (2005) 28.
[6] J. Adams et al, Nucl. Phys. A 757 (2005) 102.
[7] K. Adcox et al, Nucl. Phys. A 757 (2005) 184.
[8] J.-P. Blaizot, J. Phys. G 34 (2007) S243.
[9] Y. -B. Dai, Y. -L. Wu, Eur. Phys. J. C39, S1 (2004). [hep-ph/0304075].
[10] Y. Nambu, G. Jona-Lasinio, Phys. Rev. 122, 345-358 (1961).
[11] Y. L. Wu, Int. J. Mod. Phys. A 18, 5363 (2003) [arXiv:hep-th/0209021].
[12] Y. L. Wu, Mod. Phys. Lett. A 19, 2191 (2004) [arXiv:hep-th/0311082].
[13] D. Espriu, E. de Rafael and J. Taron, Nucl. Phys. B 345, 22 (1990) [Erratum-ibid. B 355, 278 (1991)].
[14] J. Schwinger, J. Math. Phys. 2 (1961) 407.
[15] L. V. Keldysh, Sov. Phys. JETP 20 (1965) 1018.
[16] G. Z. Zhou, Z. B. Su, B. L. Hao, L. Yu, Phys. Rev. B 22 (1980) 3385.
[17] K. C. Chou, Z. B. Su, B. L. Hao, L. Yu, Phys. Rep. 118 (1985) 1.
[18] E. Calzetta and B. L. Hu, NonEquilibrium Quantum Field Theory (Cambridge University Press, 2008)
[19] J. Rammer, Quantum Field Theory of Non-equilibrium States (Cambridge University Press, 2007)
[20] S. P. Klevansky, Rev. Mod. Phys. 64 (1992) 649.
[21] T. Hatsuda, T. Kunihiro, Phys. Lett. B 145 (1984) 7; Phys. Rep. 247 (1994) 221.
[22] R. Alkofer, H. Reinhardt, H. Weigel, Phys. Rep. 265 (1996) 139.
[23] M. Buballa, Phys. Rep. 407 (2005) 205.
[24] G. ‘t Hooft, Phys. Rev. Lett. 37 8(1976), Phys. Rev. D14 3432(1976); Err. Phys. Rev. D18 (1978) 2199.
[25] G. ’t Hooft, [hep-ph/9903189], 1999
[26] E. Calzetta, B. L. Hu, Phys. Rev. D 37 (1988) 2878.
[27] A. Das, In *Mitra, A.N. (ed.): Quantum field theory* 383-411. [hep-ph/0004125].
[28] M. Quiros, talk given at ICTP Summer School In High-Energy Physics And Cosmology, 29
    Jun - 17 Jul 1998, Miramare, Trieste, Italy [hep-ph/9901312].
[29] C. Bloch, Nucl. Phys. 7 (1958) 451.
[30] A. Manohar and H. Georgi, Nucl. Phys. B 234, 189 (1984).