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Nilpotent Lie Groups: Fourier Inversion and Prime Ideals

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Abstract We establish a Fourier inversion theorem for general connected, simply connected nilpotent Lie groups $G = \exp(g)$ by showing that operator fields defined on suitable sub-manifolds of $g^*$ are images of Schwartz functions under the Fourier transform. As an application of this result, we provide a complete characterisation of a large class of invariant prime closed two-sided ideals of $L^1(G)$ as kernels of sets of irreducible representations of $G$.

Keywords nilpotent Lie group · Irreducible representation · Co-adjoint orbit · Fourier inversion · Retract · Compact group action

Mathematics Subject Classification 22E30 · 22E27 · 43A20

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1 Introduction

For a connected, simply connected, nilpotent Lie group $G$, the description of its spectrum and the Fourier inversion theorem are due to Kirillov [3], who showed that the dual space $\widehat{G}$ of $G$ is in one-to-one correspondence with the space $g^*/G$ of co-adjoint orbits of $G$. R. Howe proved in [2] that for every irreducible unitary representation $(\pi, \mathcal{H}_\pi)$ of $G$ and every smooth linear operator $a$ on $\mathcal{H}_\pi$ there exists a Schwartz function $f_a$ on $G$ such that $\pi(f_a) = a$. He also showed that the mapping $a \mapsto f_a$ is linear and continuous with respect to the Fréchet topology of the space $B^\infty(\mathcal{H}_\pi)$ of smooth linear operators on $\mathcal{H}_\pi$ and the Fréchet topology on the space $S(G)$ of Schwartz functions on $G$. In [12], N. Pedersen gave a precise construction on the map $a \mapsto f_a$, for a fixed representation, using the trace function.

In this paper, we study a general version of the Fourier inversion theorem for nilpotent Lie groups. More precisely, we generalise the result of Howe’s mentioned above by constructing a continuous retract from the space of adapted smooth kernel functions defined on a smooth $G$-invariant sub-manifold $M$ of $g^*$ and supported in a subset $G \cdot \mathcal{M}$ of $M$, where $\mathcal{M}$ is a relatively compact open subset of $M$, into the space $S(G)$. Note that since we work with various representations at the same time, we cannot apply the construction of Pedersen’s, unless the manifold is an open subset of $g^*$. The main difficulty with this retract construction is that the spectrum of nilpotent Lie groups is not Hausdorff, thus given a smooth operator field defined on a small (Hausdorff) region in the spectrum, one cannot take an extension as in the usual locally compact Hausdorff space case. We will apply the variable group techniques developed in [11] to prove this result, which we call the Retract Theorem, by induction on the length $|\mathcal{I}|$ of the largest index set $\mathcal{I}$ for which $(B \times g^*)_I \cap M \neq \emptyset$ (defined in Sect. 2.4).

Once we have the Retract Theorem, we can apply it to study the $G$-prime ideals of the Banach algebra $L^1(G)$. Here $G$ denotes a Lie subgroup of the automorphism group of $G$ with the property that the $G$-orbits in $g^*$ are all locally closed. The Retract Theorem implies that the Schwartz functions contained in the kernel of a $G$-orbit $\Omega$ in $\widehat{G}$ are dense in the $L^1(G)$-kernel of $\Omega$. Using the methods in [7], it follows that every $G$-prime ideal in $L^1(G)$ is the kernel of such a $G$-orbit $\Omega$. This result can be used, for instance, in the study of bounded irreducible representations $(\pi, X)$ of a Lie group $G$ on a Banach space $X$. Restricting the representation $\pi$ to the nilradical $G$, one obtains the kernel $\ker(\pi_{|G})$ of $\pi_{|G}$ in the algebra $L^1(G)$. The ideal $\ker(\pi_{|G})$ is then $G$-prime. If $\ker(\pi_{|G})$ is given as the kernel in $L^1(G)$ of a $G$-orbit $G \cdot \pi_0 \subset \widehat{G}$ for some $\pi_0 \in \widehat{G}$, then one can use $\pi_0$ to make an analysis of $\pi$ as Mackey did in the case of unitary representations.

Prime ideals in $L^1(G)$ have been studied by various authors in different settings. For connected, simply connected, nilpotent Lie groups, J. Ludwig showed in [6] that the closed prime ideals of $L^1(G)$ coincide with the kernels of the irreducible unitary representations. In 1984, D. Poguntke studied the action of an abelian compact group $K$ on a nilpotent Lie group [13] and characterised the $K$-prime ideals as kernels of $K$-orbits. In [4], R. Lahiani and C. Molitor-Braun identified the $K$-prime ideals with hull contained in the generic part of the dual space of $G$ for a general compact Lie subgroup $K$ of the automorphism group of $G$. In [7] and [8], it was shown that for an
exponential Lie group $G$, the $G$-prime ideals are also kernels of $G$-orbits. In this way the bounded irreducible Banach space representations of an exponential Lie group could be determined.

The paper is organised in the following way: in Sect. 2 we recall the definition of induced representations and of kernel functions, we explain the notion of variable nilpotent Lie groups and their Lie algebras, of index sets for co-adjoint orbits and of adapted kernel functions on a $G$-invariant sub-manifold of $\mathfrak{g}^*$. In Sect. 3, we state our main theorem of the paper, the Retract Theorem, and in Sect. 4 we present the proof of the theorem, dividing it into several steps. As an application of the Retract Theorem, in the last section (Sect. 5) we show that every $G$-prime ideal in $L^1(G)$ is the kernel of a $G$-orbit.

2 Notations and Generalities

2.1 Representations and Kernel Functions

Let $G = \exp(\mathfrak{g})$ be a connected, simply connected, nilpotent Lie group and $\mathfrak{g}$ be its Lie algebra. All the irreducible unitary representations of $G$ (and hence of $L^1(G)$) are obtained (up to equivalence) in the following way: Let $l \in \mathfrak{g}^*$ and $\mathfrak{p} = \mathfrak{p}(l)$ be an arbitrary polarisation of $l$ in $\mathfrak{g}$ (a maximal isotropic subalgebra of $\mathfrak{g}$ for the bilinear form $(X, Y) \mapsto \langle l, [X, Y] \rangle$). Let $P(l) = \exp(\mathfrak{p}(l))$. The induced representation denoted by $\pi_l := \text{ind}_{\mathfrak{p}(l)}^G \chi_l$ on the Hilbert space $H_l$,

$$H_l = L^2(G/P(l), \chi_l) := \{ \xi : G \to \mathbb{C}; \xi \text{ measurable,} \xi(gp) = \chi_l(p^{-1})\xi(g), g \in G, p \in P(l), \|\xi\|_2^2 = \int_{G/P(l)} |\xi(g)|^2 d\hat{g} < \infty \},$$

where $d\hat{g}$ is the invariant measure on $G/P(l)$, is unitary and irreducible. Here $\chi_l$ is the character defined on $P(l)$ by $\chi_l(g) = e^{-i(l, \log g)}$ for all $g \in P(l)$. Two different polarisations for the same $l$ give equivalent representations. The same is true for the case of two linear forms $l$ and $l'$ belonging to the same co-adjoint orbit.

One particular way to obtain a polarisation is the following: Let $\{Z_1, \ldots, Z_n\}$ denote a Jordan-Hölder basis of $\mathfrak{g}$, for $1 \leq k \leq n$, let $\mathfrak{g}_k := \text{span}\{Z_k, \ldots, Z_n\}$ be the linear span of $Z_k, \ldots, Z_n$ and $l_k = l|_{\mathfrak{g}_k}$ for all $l \in \mathfrak{g}^*$. The polarisation $\mathfrak{p}(l) = \mathfrak{p}(l) := \sum_{k=1}^n \mathfrak{g}_k(l_k)$ of $l$ in $\mathfrak{g}$, with $\mathfrak{g}_k(l_k) := \{ U \in \mathfrak{g}_k; \langle l, [U, \mathfrak{g}_k] \rangle = 0 \}$, is called the Vergne polarisation at $l$ with respect to the basis $Z_1, \ldots, Z_n$. We refer to [1] for more details on the theory of irreducible representations of nilpotent Lie groups.

Let $\pi_l = \text{ind}_{\mathfrak{p}(l)}^G \chi_l$. The corresponding representation of $L^1(G)$, also denoted by $\pi_l$, is obtained via the formula $\pi_l(f)\xi := \int_G f(x)(\pi_l(x)\xi)dx$, for all $\xi \in H_l$. If $f \in L^1(G)$, then $\pi_l(f)$ is a kernel operator, i.e. it is of the form

$$\pi_l(f)\xi(g) = \int_{G/P(l)} F(l, g, u)\xi(u)du,$$
where $F$ is the operator kernel given by

$$F(l, g, u) = \int_{P(l)} f(ghu^{-1})\chi_l(h)dh$$

for $g, u \in G$.

If $f$ is a Schwartz function, then the kernel function $F$ belongs to $C^{\infty}$ and satisfies the covariance relation

$$F(l, gh, g'h') = \chi_l(h)\chi_l(h')F(l, g, g')$$

for $h, h' \in P(l)$ and $g, g' \in G$, and is a Schwartz function on $G/P(l) \times G/P(l)$.

### 2.2 Group Actions

Let $G = \exp(g)$ be a connected, simply connected, nilpotent Lie group and $A$ be a Lie subgroup of the automorphism group $\text{Aut}(G)$ of $G$ acting smoothly on $G$. This action will be denoted by

$$A \times G \mapsto G$$

$$(a, x) \mapsto a \cdot x.$$

The action of $A$ on $G$ induces naturally actions of $A$ on $g, g^*, \hat{G}, L^1(G)$, and on $\mathcal{S}(G)$. These group actions will lead to examples for our retract theory and provide an important application of retracts.

### 2.3 Variable Lie Algebras and Groups

We will prove our main theorem by induction; in our proofs, new parameters and new variations will appear. This may be handled most easily by the concept of variable Lie structures. Such structures were already considered in [5], [11], [10] and [9], among others.

**Definition 2.3.1** Let $g$ be a real vector space of finite dimension $n$ and $B$ be an arbitrary nonempty set. We say that $(B, g)$ is a variable (nilpotent) Lie algebra if

(a) For every $\beta \in B$, there exists a Lie bracket $[\cdot, \cdot]_\beta$ defined on $g$ such that $g_\beta := (g, [\cdot, \cdot]_\beta)$ is a nilpotent Lie algebra.

(b) There exists a fixed basis $Z = Z^0 = \{Z_1 = Z_1^0, \ldots, Z_n = Z_n^0\}$ of $g$ such that the structure constants $a_{ij}^k(\beta)$ defined by

$$[Z_i, Z_j]_\beta := \sum_{k=1}^n a_{ij}^k(\beta)Z_k$$

satisfy the following property: For all $\beta \in B$ and $k \leq \max(i, j)$, $a_{ij}^k(\beta) = 0$. This means that $\{Z_1, \ldots, Z_n\}$ is a Jordan-Hölder basis for $g_\beta = (g, [\cdot, \cdot]_\beta)$. 
2 Assume that $B$ is a smooth manifold. If the structure constants $a_{ij}^k(\beta)$ vary smoothly on $B$, we say that $(B, g)$ is a smooth variable (nilpotent) Lie algebra. We will denote $(B, g) = (g, [\cdot, \cdot]_\beta)_{\beta \in B}$ for the variable Lie algebra.

For the rest of the paper we will assume that all variable Lie algebras are smooth. If $B$ is reduced to a singleton, we have in fact no dependency on $\beta$ in $B$ but a fixed Lie algebra. To each variable Lie algebra, we associate a variable Lie group $G_\beta$. The variable Lie group $G := (G_\beta)_\beta$ may be identified with the collection of Lie algebras $(g, [\cdot, \cdot]_\beta)$ equipped with the corresponding Campbell-Baker-Hausdorff multiplications. If $G = (G_\beta)_\beta$ is a (smooth) variable Lie group endowed with a fixed Jordan-Hölder basis, then the corresponding Vergne polarisations, induced representations and operator kernels all depend on $\beta \in B$ and $l \in g^*$.

### 2.4 Ludwig-Zahir Indices

Let $(B, g)$ be a smooth variable Lie algebra. We assume that $g$ is equipped with a fixed basis $\mathcal{Z} = \mathcal{Z}^0 = \{Z_1 = Z_1^0, \ldots, Z_n = Z_n^0\}$, which is a Jordan-Hölder basis for every $(g, [\cdot, \cdot]_\beta)$.

Let $(\beta, I) \in B \times g^*$. The Ludwig-Zahir indices $I(\beta, I)$ defined in [11] can be obtained in the following way: Let $g_\beta(l) := \{U \in g; [l, [U, g]]_\beta \equiv 0\}$ be the stabiliser of $l$ in $g_\beta = (g, [\cdot, \cdot]_\beta)$ and let $a_\beta(l)$ be the maximal ideal contained in $g_\beta(l)$. If $a_\beta(l) = g_\beta(l) = g$, then $\chi(\beta, l)(x) := e^{-i(l \cdot \log x)}$ is a character on $G_\beta$ and nothing has to be done. In this case, there are no Ludwig-Zahir indices, i.e. $I(\beta, I) = \emptyset$. Otherwise, let

$$j_1(\beta, I) = \max\{j \in \{1, \ldots, n\}; Z_j^0 \not\in a_\beta(l)\}, \quad \text{and}$$

$$k_1(\beta, I) = \max\{k \in \{1, \ldots, n\}; [l, [Z_k^0(\beta, I), Z_k^0(\beta, I)]_{\beta}] \not= 0\}.$$

We let

$$X_1(\beta, I) := Z_{k_1(\beta, I)}^0,$$

$$Y_1(\beta, I) := Z_{j_1(\beta, I)}^0,$$

$$Z_1(\beta, I) := [Z_{k_1(\beta, I)}^0, Z_{j_1(\beta, I)}^0]_\beta,$$

$$c(\beta, I) := [l, Z_1(\beta, I)].$$

We then consider

$$g_1(\beta, I) := \{U \in g; [l, [Y_1(\beta, I)]_\beta] = 0\} \quad \text{(2.1)}$$

which is an ideal of co-dimension one in $g_\beta$.

A Jordan-Hölder basis of $(g_1(\beta, I), [\cdot, \cdot]_\beta)$ is given by $Z_1(\beta, I) = \{Z_i^1(\beta, I); i \not= k_1(\beta, I)\}$ with

$$Z_i^1(\beta, I) := Z_i^0 - \frac{[l, [Z_i^0, Y_1(\beta, I)]_\beta]}{c(\beta, I)} X_1(\beta, I), \quad i \not= k_1(\beta, I). \quad \text{(2.2)}$$
One sees that $Z_i^1(\beta, l) = Z_i^0$, if $i > k_1(\beta, l)$. As previously we may now compute the indices $j_2(\beta, l), k_2(\beta, l)$ of $l_1 := l_{1|1}(\beta, l)$ with respect to this new basis and construct the corresponding subalgebra $g_2(\beta, l)$ with its associated basis $(Z_2^i(\beta, l); i \neq k_1(\beta, l), k_2(\beta, l))$. This procedure stops after a finite number $r$ of steps. Let

$$I_{\mathcal{Z}}(\beta, l) = I(\beta, l) = ((j_1(\beta, l), k_1(\beta, l)), \ldots, (j_r(\beta, l), k_r(\beta, l)))$$

which is called the Ludwig-Zahir index of $l$ in $g_\beta$ with respect to the basis $\{Z_1, \ldots, Z_n\}$. The construction in [11] shows that the final subalgebra $g_r(\beta, l)$ obtained by this construction coincides with the Vergne polarisation of $l$ in $g_\beta$ with respect to the basis $\mathcal{Z}^0$ (see also [10], [9]). Note that the length $|I| = 2r$ of the index set $I = I(\beta, l)$ gives us the dimension of the co-adjoint orbit $Ad^*(G_\beta)\ell$. The vectors $Y_1(\beta, l), \ldots, Y_r(\beta, l)$ together with the stabiliser $g_\beta(l)$ of $l$ in $g_\beta$ span the polarisation $p_\beta(l) = g_r(\beta, l)$ and

$$g = \bigoplus_{i=1}^r \mathbb{R}X_i(\beta, l) \oplus \bigoplus_{i=1}^r \mathbb{R}Y_i(\beta, l) \oplus g_\beta(l).$$

Let us introduce the following notations: For any index set $I \in (\mathbb{N}^2)^r \equiv \mathbb{N}^{2r}$ with $r = 0, \ldots, \dim(g)/2$, we let

$$(\mathcal{B} \times g^*)_I := \{((\beta, l) \in \mathcal{B} \times g^*; I(\beta, l) = I\} \quad \text{and} \quad (\mathcal{B} \times g^*)_I \cap \Sigma_I := \{((\beta, l) \in (\mathcal{B} \times g^*)_I; l(Z_{j_i}) = l(Z_{k_i}) = 0 \text{ for } 1 \leq i \leq r\}.$$  

This last line corresponds to the Pukanszky section associated to the index $I$. In fact, in [9] it was proved that the indices $j_s(\beta, l), k_s(\beta, l)$ coincide with the Pukanszky indices of the given layer (if one does not make any distinction between the $j$’s and the $k$’s). For many $I$’s, the subset $(\mathcal{B} \times g^*)_I$ is empty. Hence it is reasonable to define

$$\mathcal{I} := \left\{I \in \bigcup_{j=0}^{\dim(g)/2} (\mathbb{N}^2)^j; (\mathcal{B} \times g^*)_I \neq \emptyset \right\} \quad \text{and} \quad B \times g^* = \bigcup_{I \in \mathcal{I}} (\mathcal{B} \times g^*)_I.$$  

This gives a partition of $B \times g^*$ into the different layers $(\mathcal{B} \times g^*)_I$. The set $\mathcal{I}$ may be ordered lexicographically: if $I = ((j_1, k_1), \ldots, (j_r, k_r)), I' = ((j_1', k_1'), \ldots, (j_r', k_r')) \in \mathcal{I}$, we say that $I < I'$ if either $2r = |I| < |I'| = 2r'$ or there exists $a \in \{1, \ldots, r\}$ such that

$$(j_s, k_s) = (j_s', k_s') \text{ if } s < a \text{ and } (j_a, k_a) < (j_a', k_a').$$

which means that

either $j_a < j_a'$ or $(j_a = j_a' \text{ and } k_a < k_a').$

This allows us to define

$$(\mathcal{B} \times g^*)_{\leq I} := \{((\beta, l) \in (\mathcal{B} \times g^*)_J; J \leq I\} = \bigcup_{J \leq I} (\mathcal{B} \times g^*)_J.$$

\[\text{Birkhäuser}\]
By induction on the length of the index sets, it is easy to see that for every \( I \in \mathcal{I} \) there exists a smooth function \( P_I \) on \( B \times g^* \), which is polynomial in \( l \) for fixed \( \beta \in B \) such that

\[
(B \times g^*)_I = \{ (\beta, l); P_I'(\beta, l) = 0 \text{ for } I' > I \text{ and } P_I(\beta, l) \neq 0 \}. \tag{2.3}
\]

### 2.5 Co-adjoint Orbits

For any index set \( I \), we consider the subspace \( s_I \) of \( g^* \) which is given by

\[
s_I = \text{span}\{ Z_j^*; j \in I \}.
\]

For each \( \beta \in B \), let

\[
\Sigma_{\beta,I} := \{ (\beta, l) \in (\{\beta\} \times g^*)_I; l \in s_I \}.
\]

Then \( \Sigma_{\beta,I} \) is locally closed in \( s_I \), since we have the smooth functions \( P_{I'}, I' \in \mathcal{I} \), defined on \( B \times g^* \) as in (2.3).

Let \( d := |I| \). For \( l \in g^* \), let

\[
\Omega_{\beta,I} = \{ Ad^*_\beta(g)l; g \in G \}
\]

be the \( G_\beta \)-orbit of \( l \). Then

\[
\dim(\Omega_{\beta,I}) = d \quad \text{for} \quad l \in (B \times g^*)_I.
\]

There exist functions \( p_j : (B \times g^*)_I \times \mathbb{R}^d \to \mathbb{R}, j = 1, \ldots, n \), which are rational in \( l \in g^* \) and polynomial in \( z \in \mathbb{R}^d \) for fixed \( \beta \in B \) such that for every \( (\beta, l) \in (B \times g^*)_I \),

\[
\Omega_{\beta,I} = \left\{ \sum_{i=1}^n p_i(\beta, l, z)Z_i^*; z \in \mathbb{R}^d \right\}.
\]

Furthermore if we write \( I = \{i_1 < \cdots < i_d\} \), then

\[
p_{ij}(\beta, l, z) = z_j \quad \text{for} \quad j = 1, \ldots, d,
\]

and for \( i \notin I \), we have

\[
p_i(\beta, l, z) = \langle l, Z_i \rangle + p'_i(\beta, l, z_1, \ldots, z_j), \quad i_j < i < i_{j+1}.
\]

**Definition 2.5.1** A subset \( M \) of \( B \times g^* \) is called \( G \)-invariant if for every \( (\beta, l) \in M \) the element \( g \cdot (\beta, l) := (\beta, Ad^*_\beta(g)l) \) is also contained in \( M \).
2.6 Schwartz Functions

Let $r \in \mathbb{N}$, we define the space of (generalised) Schwartz functions $S(\mathbb{R}^r, \mathcal{B}, G) \equiv S(\mathbb{R}^r, \mathcal{B}, g) \equiv S(\mathbb{R}^r, \mathcal{B}, \mathbb{R}^n)$ to be the set of all functions $f$ from $\mathbb{R}^r \times \mathcal{B} \times G$ to $\mathbb{C}$ such that the function $\hat{f}$ defined by

$$\hat{f}(\alpha, \beta, (x_1, \ldots, x_n)) := f(\alpha, \beta, \exp(\beta(x_1Z_1 + \cdots + x_nZ_n))) \quad \text{for} \quad \alpha \in \mathbb{R}^r, \beta \in \mathcal{B}$$

is smooth on $\mathbb{R}^r \times \mathcal{B} \times \mathbb{R}^n$ and that

$$\|\hat{f}\|_{K,T_1,\ldots,T_s;A_1,A_2,B_1,B_2} = \sup_{\beta \in K; \alpha \in \mathbb{R}^r; x \in \mathbb{R}^n} \left[ \sup_{|r_i| \leq A_i; |s_j| \leq B_j; i,j \in \{1,2\}} |\alpha^{r_1} x^{s_1} T_1T_2 \cdots T_s \frac{\partial^{r_2}}{\partial \alpha^{r_2}} \frac{\partial^{s_2}}{\partial x^{s_2}} \hat{f}(\alpha, \beta, (x_1, \ldots, x_n))| \right]$$

$$< +\infty,$$

for any compact subset $K$ of $\mathcal{B}$, any finite collection $T_1, \ldots, T_s$ of smooth vector fields defined on the manifold $\mathcal{B}$, and any $A_1, A_2, B_1, B_2 \in \mathbb{N}$. The function space $S(\mathbb{R}^r, \mathcal{B}, G)$ is equipped with the topology defined by the collection of all these semi-norms. One may of course also use coordinates of the second kind to define the space $S(\mathbb{R}^r, \mathcal{B}, \mathbb{R}^n)$ does not depend on the choice of the Jordan-Hölder basis.

2.7 Kernel Functions

Let $S$ be a subset of $\mathcal{B} \times g^*$ and $L$ be a smooth manifold. We say that a mapping $F : S \to L$ is smooth, if the restriction of $F$ to any smooth manifold $N$ contained in $S$ is smooth.

Let $\mathcal{B} \times g^*$ be a smooth variable nilpotent Lie group with Jordan-Hölder basis $Z$. For any $(\beta, l) \in \mathcal{B} \times g^*$, denote the Vergne polarisation $p_{\beta, l}$ at $(\beta, l)$ associated to $Z$. We put $\pi(\beta, l) := \text{ind}^G_{P(\beta, l) \chi}$, with $P(\beta, l) := \exp(p_{\beta, l})$, for the corresponding family of induced unitary representations. Then the mapping $(\beta, l) \mapsto p(\beta, l)$ is smooth on each subset $(\mathcal{B} \times g^*)_l$. For each index set $I$ with length $d_I$ and $(\beta, l) \in \mathcal{B} \times g^*$, choose a Malcev basis $R(\beta, l) = \{R_1(\beta, l), \ldots, R_{d_I}(\beta, l)\}$ of $g$ relative to $p(\beta, l)$, such that the mappings $(\beta, l) \mapsto R(\beta, l)$ are smooth on the different layers $(\mathcal{B} \times g^*)_l$.

Definition 2.7.1 Let $M$ be any smooth $G$-invariant manifold of $\mathcal{B} \times g^*$ and let $r \in \mathbb{N}$. We denote by $\mathcal{D}^r_{M,G}$ the space of all functions $F : \mathbb{R}^r \times M \times G \times G \to \mathbb{C}$ satisfying the following conditions.

1. $F$ satisfies the covariance condition for every $(\beta, l) \in M$ with respect to $p(\beta, l)$, i.e.

   $$F(\alpha, (\beta, l), x \cdot_{\beta} p, y \cdot_{\beta} q) = \overline{\chi_l(p)} \chi_l(q) F(\alpha, (\beta, l), x, y)$$

   for all $\alpha \in \mathbb{R}^r, p, q \in P(\beta, l)$ and $x, y \in G$.  

\[\text{Birkhäuser}\]
2. The function $F$ satisfies the following compatibility condition

$$F(\alpha, (\beta, \text{Ad}_B^\alpha(g))l, x, y) = F(\alpha, (\beta, l), x \cdot \beta g, y \cdot \beta g),$$

for $\alpha \in \mathbb{R}^r$, $(\beta, l) \in M$ and $x, y, g \in G$. This compatibility condition reflects the unitary equivalence of the representations $\pi(\beta, l)$ and $\pi(\beta, \text{Ad}_B^\alpha(g)l)$. 

3. The support of $F$ in $(\beta, l)$ is compact modulo $G$, i.e. there exists a compact subset $C$ of $M$ such that $F(\cdot, (\beta, l), \cdot, \cdot)$ is 0 outside the subset of $G \cdot C$.

4. The function $F$ has the Schwartz space property, i.e. for any $I \in \mathcal{I}$ the function $F|_{\mathbb{R}^r \times M \setminus (B \times B)} \times G \times G$ is smooth and that

$$\|F\|_{DA_1A_2B_1B_2C_1C_2} := \sup_{(\beta, l) \in M, \alpha \in \mathbb{R}^r, x, x' \in \mathbb{R}^r} \sup_{|r| \leq A_1, |s| \leq B_1, |x| \leq C_1} \sup_{i, j \in \{1, 2\}}|\alpha' r^1 x^1 (x')^{l_1} D(\beta, l) \frac{\partial^r}{\partial \alpha^r} \frac{\partial^s}{\partial x^s} \frac{\partial^l}{\partial (x')^l} \tilde{F}(\alpha, (\beta, l), x, x')| < \infty,$$ (2.4)

where

$$\tilde{F}(\alpha, (\beta, l), x, x') := F(\alpha, (\beta, l), \exp_\beta(x_1 R_1) \cdots \exp_\beta(x_r R_r), \exp_\beta(x'_1 R_1) \cdots \exp_\beta(x'_r R_r)),$$

for any smooth differential operator $D = D(\beta, l)$ on the manifold $M$, and any $A_1, A_2, B_1, B_2, C_1, C_2 \in \mathbb{N}$.

The space $D_M^{r,I}$ will be equipped with the topology defined by the collection of all these semi-norms. This does of course not depend on the choice of the smooth Malcev basis of $\mathfrak{g}$ with respect to the smooth family of Vergne polarisations.

**Definition 2.7.2** Let $M \subset B \times \mathfrak{g}^*$. A field $F = (F(\beta, l))_{(\beta, l) \in M}$ of kernel functions is called adapted if it satisfies the conditions in Definition 2.7.1.

For an adapted field of kernel functions $F$ on $M$, denote by $\text{op}_F$ the field of smooth operators defined through their kernel functions. For $(\beta, l) \in M$, the operator $\text{op}_F(\beta, l)$ acts on the space $L^2(G/P(\beta, l), \chi(\beta, l))$ in the following way:

$$\text{op}_F(\beta, l) \xi(g) = \int_{G/P(\beta, l)} F(\beta, l)(g, x) \xi(x) dx.$$  

**Remarks 2.7.2.1**

a) If we impose the condition that the support of $(\beta, l)$ be contained in the set $G \cdot C_0$ for a fixed subset $C_0$ of $M$, then we will denote the space of kernel functions by $D_M^{C_0}$.

b) One has a similar definition of the kernel functions if one takes another smooth family of polarisations together with a smooth family of Malcev bases.
3 The Retract Theorem

In this section, we state the main theorem of the paper which will be proved in the next section.

**Theorem 3.1** Let $\mathcal{B} \times G$ be a smooth variable nilpotent Lie group, $I = \{(j_1, k_1) < \cdots < (j_r, k_r)\}$ be an index set and let $M$ be a smooth $G$-invariant sub-manifold of $\mathcal{B} \times \mathfrak{g}^*$ contained in $(\mathcal{B} \times \mathfrak{g}^*)_{\leq 1}$ such that $M_1 := M \cap (\mathcal{B} \times \mathfrak{g}^*)_1 \neq \emptyset$. Let $\pi(\beta, l)$ be defined as previously from the smooth family of Vergne polarisations for $(\beta, l) \in M$. Then there exists an open nonempty relatively compact subset $\mathcal{M} \subset M_1$ with closure $\overline{\mathcal{M}}$ contained in $M_1$ such that the following holds: For any adapted kernel function $F \in \mathcal{D}^M_M$, there is a function $f$ in the Schwartz space $\mathcal{S}(\mathbb{R}^r, \mathcal{B}, G)$ such that $\pi(\beta, l)(f(\alpha, \beta, \cdot))$ has $F(\alpha, (\beta, l), \cdot, \cdot)$ as an operator kernel for all $(\alpha, (\beta, l)) \in \mathbb{R}^r \times M$. Moreover the mapping $F \mapsto f$ is continuous with respect to the corresponding function space topologies.

If the variation is trivial, then we get the following theorem.

**Theorem 3.2** Let $\mathfrak{g}$ be a nilpotent Lie algebra with Jordan-Hölder basis $Z$. Let $M$ be a smooth $G$-invariant sub-manifold of $\mathfrak{g}^*$ and $I : = \max\{J \in \mathcal{I}_Z : M \cap \mathfrak{g}^*_J \neq \emptyset\}$. Let $\pi_1 = \pi(1)$ be defined as previously from the smooth family of Vergne polarisations for $l \in M$. Then there exists an open, relatively compact nonempty subset $\mathcal{M} \subset \mathfrak{g}^*_1$ of $M$ such that $\mathcal{M} \subset \overline{\mathcal{M}} \subset M_1$, $\overline{\mathcal{M}}$ is compact and that the following holds: For any kernel function $F \in \mathcal{D}^M_M$, there is a function $f$ in the Schwartz space $\mathcal{S}(G)$ such that $\pi_1(f)$ has $F(l, \cdot, \cdot, \cdot)$ as an operator kernel for all $l \in M$. Moreover, the Schwartz function $f$ may be constructed such that the mapping $F \mapsto f$ is continuous with respect to the corresponding function space topologies.

**Remark 3.2.1** If $M$ is contained in $\mathfrak{g}^*_{\max}$, where $I_{\max}$ is the maximal index set in $\mathcal{I}$, then we have the following (well-known) result.

**Theorem 3.3** Let $\mathcal{B} \times G$ be a simply connected, connected smooth variable nilpotent Lie group and $M = (\mathcal{B} \times \mathfrak{g}^*)_{\gen} : = (\mathcal{B} \times \mathfrak{g}^*)_{I_{\max}}$ be the space of generic co-adjoint orbits. Let $\mathcal{M}$ be an open relatively compact subset of $M$ such that $\overline{\mathcal{M}} \subset M$. For every adapted field of kernel functions $F \in \mathcal{D}^M_M$, there exists a unique Schwartz function $f = R(F) : G \to \mathbb{C}$ such that

$$\pi(\beta, l)(f) = op_{F(\beta, l)} \text{ for any } (\beta, l) \in \mathcal{B} \times \mathfrak{g}^*,$$

and the mapping $F \mapsto R(F)$ is continuous.

**Proof** It suffices to apply the Fourier inversion formula. For each $F \in \mathcal{D}^M_M$, let

$$f(\beta, g) = R(F)(\beta, g) := \int_{\Sigma_{\beta, I_{\max}}} tr(\pi(\beta, l)(g) \circ op_{F(\beta, l)})(\alpha) |P_{\alpha}(\beta, l)| dl, \ g \in G,$$

where $P_{\alpha}(\beta, l)$ is the Pfaffian of the polynomial $Q(l) = \det (\langle l, [Z_i, Z_j]\rangle_{i,j \in I_{\max}})$. It follows from [11] that the function $f$ is Schwartz and the Fourier inversion theorem tells us that $\pi(\beta, l)(f) = op_{F(\beta, l)}$ for any $(\beta, l) \in \mathcal{B} \times \mathfrak{g}^*$.

\[\text{Birkhäuser}\]
4 Proof of the Retract Theorem

The proof of Theorem 3.1 proceeds by induction on the length $|I|$ of the largest index set $I$ for which $(B \times g^*)_I \cap M \neq \emptyset$ and it will be done in several steps.

4.1 The case $I = \emptyset$

Suppose that all the elements $(\beta, l) \in M$ are characters of $g_{\beta}$, which means that their index sets are empty.

Let us replace the variable group $(B, G)$ by the group $(C, G)$, where $C = B$ as a manifold, and the multiplications coming from $C$ are abelian, i.e. $[U, V]_\gamma = 0$ for every $U, V \in g$ and $\gamma \in C$. We identify now the group $G$ with its Lie algebra and then $U \cdot_\gamma V = U + V$ for every $U, V \in g$ and $\gamma \in C$. This also means that $\chi_l$ is a character on $G_\gamma = \exp_\gamma g$, for all $(\gamma, l) \in C \times g^*$. Now take $M = M$. Let $F \in S(\mathbb{R}^r \times M)$ be a kernel function with compact support in the variables $(\gamma, l)$. As $\mathbb{R}^r \times M$ is a sub-manifold of $\mathbb{R}^r \times C \times g^*$, the function $F$ may be extended to a Schwartz function $\tilde{F}$ (in the sense of Sect. 2.6 and 2.7) on $\mathbb{R}^r \times C \times g^*$ with compact support in the variables $(\gamma, l)$. Let $f := (2\pi)^n F^{-1} \tilde{F}$, where $F^{-1}$ denotes the partial inverse Fourier transform in the variable $l$ which is the third variable in $\mathbb{R}^r \times C \times g^*$. Then $f \in S(\mathbb{R}^r \times C \times g^*)$. For all $(\alpha, (\gamma, l)) \in \mathbb{R}^r \times M$, we have

$$
\pi_{(\gamma, l)}(f(\alpha, \gamma, \cdot)) = \hat{f}^3(\alpha, \gamma, l) = (2\pi)^n F_3 F^{-1}_3 F(\alpha, (\gamma, l)) = F(\alpha, (\gamma, l)).
$$

In particular, $\pi_{(\gamma, l)}(f(\alpha, \gamma, \cdot)) = 0$ if $(\alpha, (\gamma, l)) \in \mathbb{R}^r \times (M \setminus C)$. The continuity of the map $F \mapsto f$ is obvious. This proves the first step in the induction procedure.

4.2 Reducing $B$

There are two cases where we can reduce the manifold $B$.

1. Suppose that there exists a smooth function $\varphi : B \to \mathbb{R}_+$ which is not constant on the subset $B_M := p_B(M)$, where $p_B : B \times g^* \to B$ is the canonical projection. Let $\beta_0 \in B$ such that $\varphi(\beta_0) \in [a, b]$ for some $b > a > 0$ and let $B_0 := \{\beta \in B; \frac{a}{2} < \varphi(\beta) < 2b\}$ and $M_0 := \{(\beta, l) \in M; \beta \in B_0\}$. Suppose that the theorem holds for the pair $(B_0, M_0)$. Let us show that the result remains true for the pair $(B, M)$. Let $M_0$ be an open relatively compact subset as in the theorem for $(B_0, M_0)$. We let $M := \{(\beta, l) \in M; a < \varphi(\beta) < b\} \cap M_0$. We will show that $M$ works for $(B, M)$. Note that since $M_0$ is open in $M_0$, we have that $M$ is open in $M$.

Let $F$ be a kernel function defined on $\mathbb{R}^r \times M \times G \times G$ such that its support in $M$ is contained in $G \cdot M \subseteq M_0$. By assumption, there exists $f \in S(\mathbb{R}^r \times B_0 \times G)$ such that $\pi_{(\beta, l)}(f(\cdot, \beta, \cdot))$ admits $F(\cdot, (\beta, l), \cdot, \cdot)$ as an operator kernel if $(\beta, l) \in M_0$. In particular, $\pi_{(\beta, l)}(f(\cdot, \beta, \cdot)) = 0$ if $(\beta, l) \in M_0 \setminus G \cdot M_0$. As $B_0$ is a sub-manifold
of $B$, we may extend $f$ to a function in $S(\mathbb{R}^r \times B \times G)$ which we denote also by $f$. Choose $\vartheta \in \mathcal{C}_c^\infty(\mathbb{R})$ with $0 \leq \vartheta \leq 1$, $\vartheta \equiv 1$ on $[a, b]$ and $\vartheta \equiv 0$ on $[0, \frac{b}{2}] \cup [2b, +\infty[$. We define $\phi \in \mathcal{C}_c^\infty(M)$ by $\phi(\beta, l) := \vartheta(\varphi(\beta))$. Then $\phi \equiv 1$ on $G \cdot M_0$ and $\phi \equiv 0$ on $M \setminus G \cdot M_0$. By taking $g := \phi \cdot f$, we have that

$$
\pi_{(\beta, l)}(g(\cdot, \beta, \cdot)) = \phi(\beta, l) \cdot \pi_{(\beta, l)}(f(\cdot, \beta, \cdot)).
$$

If $(\beta, l) \in M \subset M_0$, then $\pi_{(\beta, l)}(g(\cdot, \beta, \cdot)) = \pi_{(\beta, l)}(f(\cdot, \beta, \cdot))$ and it admits $F(\cdot, (\beta, l), \cdot, \cdot)$ as an operator kernel. If $(\beta, l) \in M_0 \setminus G \cdot M_0$, then $\pi_{(\beta, l)}(f(\cdot, \beta, \cdot)) = 0$ and $\pi_{(\beta, l)}(g(\cdot, \beta, \cdot)) = 0$. If $(\beta, l) \in M \setminus M_0$, then $\varphi(\beta) \in [0, \frac{b}{2}] \cup [2b, +\infty[$, hence $\phi(\beta, l) = 0$ and so $\pi_{(\beta, l)}(g(\cdot, \beta, \cdot)) = 0$. Therefore the result is true for the function $g$.

2. If there exists a smooth sub-manifold $B_0$ of $B$ such that $p_B(M) \subset B_0$, then we can apply our theorem to the pair $(B_0, M)$. Since every smooth function $f_0$ on $B_0 \times G$ can be extended to a smooth function $f$ on $B \times G$, the Retract Theorem also holds for $(B, M)$.

**Remark 4.2.1** Let $B$ and $M$ be given as in the statement of the theorem. Let

$$
p_B : M \to B; \quad p_B(\beta, l) = \beta,
$$

be the canonical projection. If we denote by $M^{max}$ the subset of $M$ consisting of all $(\beta, l) \in M$ for which the rank of $dp_B(\beta, l)$ is maximal, then $M^{max}$ is open in $M$ and the subset $p_B(M^{max})$ of $B$ is a smooth sub-manifold of $B$. If $p_B(M^{max})$ contains at least two elements, by the reasoning in Sect. 4.2, using a non-constant smooth function $\varphi_0$ on $p_B(M^{max})$, which can be extended to a smooth function $\varphi$ of $B$, we reduce $B$ to $B^{max}$ and we can always assume in this way that $p_B(M)$ is a smooth sub-manifold of $B$. If $p_B(M^{max})$ is a singleton $\{\beta_0\}$, then $M = M^{max}$ and $p_B(M)$ is obviously a smooth sub-manifold of $B$.

### 4.3 Reducing to smoothly varying subspaces depending on $B$

Let $M \subset B \times g^*$ be a smooth $G$-invariant sub-manifold of $B \times g^*$. Let us fix the largest index

$$
I_M = I = ((j_1, k_1), \cdots, (j_r, k_r)) = (j_1, k_1) \times I_1,
$$

where $I_1 = ((j_2, k_2), \cdots, (j_r, k_r))$ is the index set of $(\beta, l|_{g^{(\beta, l)}})$, such that the open subset $M_I := (B \times g^*)_I \cap M$ of $M$ is nonempty. Let $p_B : M \to B; (\beta, l) \mapsto \beta$, be the projection onto the first variable and set

$$
B_M := p_B(M),
$$

which is a smooth sub-manifold of $B$ by Remark 4.2.1.

Let $c_1 := g_{j_1+1} = \text{span}(Z_{j_1+1}, \ldots, Z_n) \subset g$ and let

$$
n_p^1 := [g, c_1]_\beta + [Z_{j_1}, g_{k_1+1}]_\beta \subset c_1 \text{ for } \beta \in B.
$$

(4.5)
Then, by the definition of the indices \((j_1, k_1)\), we have

\[
n_1^\beta \subset \ker(l) \cap c_1 \subset a_\beta(l) \quad \text{if} \quad (\beta, l) \in (B \times g^*)_{\leq I}.
\] (4.6)

It is easy to see that \(n_1^\beta\) is an ideal in \(g\). Let

\[
Z_\beta := [Z_{k_1}, Z_{j_1}]_\beta \quad \text{for} \quad \beta \in B.
\]

We fix a scalar product \(\langle \cdot, \cdot \rangle\) on \(g\) such that \(\{Z_1, \ldots, Z_n\}\) is an orthonormal basis and we identify \(c_1^\ast\) with \(c_1\) by identifying \(\sum_{r=1}^n a_r Z_r^\ast \in c_1^\ast\) with the element \(\sum_{r=j_1+1}^n a_r Z_r\) of \(c_1\). Denote by \(\| \cdot \|_2\) the Euclidean norm on \(c_1\) (and hence on \(c_1^\ast\)) with respect to the given scalar product. We also identify

\[
(n_1^\beta)^\perp := \{q \in c_1^\ast; \langle q, n_1^\beta \rangle = \{0\}\}
\]

with a subspace of \(c_1\). For all \(\beta \in B\), we write \(c_1 = n_1^\beta \oplus (n_1^\beta)^\perp\) and define \(p_\beta\) to be the orthogonal projection of \(c_1\) onto \((n_1^\beta)^\perp\). For each \(\beta \in B\), a generating subset of \(n_1^\beta\) is given by

\[
V(\beta) = \{v_1(\beta), \ldots, v_s(\beta)\}
\]
\[
:= \{[Z_a, Z_{a'}]_\beta; a = 1, \ldots, n, a' = j_1 + 1, \ldots, n\}
\]
\[
\cup \{[Z_b, Z_{j_1}]_\beta; b = k_1 + 1, \ldots, n\}.
\]

Let

\[
a_{j,j'}(\beta) := \langle v_j(\beta), v_{j'}(\beta) \rangle \quad \text{for} \quad 1 \leq j, j' \leq s.
\]

Fix \(0 \leq k \leq s\), let \(I_k = \{J \subset \{1, \ldots, s\}; |J| = k\}\) and for \(\beta \in B\), let

\[
h_k(\beta) := \sum_{J \in I_k} \det ((a_{j,j'}(\beta))_{j,j' \in J})^2.
\]

It is easy to check that

\[
h_k(\beta) \neq 0 \iff v_1(\beta), \ldots, v_s(\beta) \text{ have at least rank } k,
\]
\[
h_k(\beta) = 0 \iff v_1(\beta), \ldots, v_s(\beta) \text{ have rank } r < k.
\]

Let \(n_1 \in \mathbb{N}\) and put \(f_0 := h_{n_1+1}\) and \(f_1 := h_{n_1}\). Let

\[
B_{\leq n_1} = \{\beta \in B; f_0(\beta) = 0\},
\]
\[
B_{\geq n_1} = \{\beta \in B; f_1(\beta) \neq 0\},
\]
\[
B_{n_1} = \{\beta \in B; f_0(\beta) = 0 \text{ and } f_1(\beta) \neq 0\}.
\]

\(\text{Birkhäuser}\)
One sees that $B_{\geq n_1}$ is open in $B$, and hence is a sub-manifold of $B$. Again, according to the reduction argument in Sect. 4.2 we can assume that $f_1(\beta) \neq 0$ for all $\beta \in B$. On the other hand, let $n_1 := \max_{\beta \in B_M} \dim(n_1^1)$, then we have

\[ B_{\leq n_1} := \{ \beta \in B; \dim(n_1^1) \leq n_1 \}, \]
\[ B^{n_1} := \{ \beta \in B; \dim(n_1^1) = n_1 \}, \]
\[ B_{\geq n_1} := \{ \beta \in B; \dim(n_1^1) \geq n_1 \}. \]

Note that if we want $n_1^1$ to be of fixed dimension and to have $n_1^1, (n_1^1)_{\perp}$ and $p_\beta$ to vary smoothly with respect to $\beta$, we must restrict to $B^{n_1}$. But in general $B^{n_1}$ is not a sub-manifold of $B$. Therefore we must find a smooth sub-manifold inside $B^{n_1}$ containing an open subset of the smooth manifold $B_M = p_B(M)$. We have to distinguish the following two cases:

**Case 1:** If the differential $df_0$ is not identically zero on $B_M$, we may define

\[ B_M^{\text{max}} := \{ \beta \in B_M; df_0(\beta) \neq 0 \} \quad \text{and} \quad B^{\text{max}} := \{ \beta \in B; df_0(\beta) \neq 0 \}. \]

By assumption, $B^{\text{max}}$ is a nonempty open subset of $B$.

**Case 2:** Assume that $df_0$ is identically zero on $B_M$.

If $\dim(B_M) < \dim(B)$, we may build a function $\gamma \in C^\infty(B)$ such that $\gamma \equiv 0$ on $B_M$ and $d\gamma$ is not identically zero on $B_M$. We put $\tilde{f}_0 = f_0 + \gamma$. Then $\tilde{f}_0 \equiv 0$ on $B_M$ and $d\tilde{f}_0$ is not identically zero on $B_M$. We then define

\[ B_M^{\text{max}} := \{ \beta \in B_M; d\tilde{f}_0(\beta) \neq 0 \}, \quad B^{\text{max}} := \{ \beta \in B; d\tilde{f}_0(\beta) \neq 0 \}. \]

By the construction of $\tilde{f}_0$, we have again that $B^{\text{max}}$ is an open subset of $B$ and $B_M^{\text{max}} \subset B^{\text{max}}$.

If $\dim(B_M) = \dim(B)$, then $B_M$ is open in $B$ and we take a smooth function $\tilde{f} \neq 0$ in $B$ supported on $B_M$. Let

\[ B^{\text{max}} := \{ \beta \in B; \tilde{f}(\beta) \neq 0 \} \quad \text{and} \quad B_M^{\text{max}} := \{ \beta \in B_M; \tilde{f}(\beta) \neq 0 \}. \]

In both cases, the ideals $n_1^1$ vary smoothly on the smooth sub-manifold $B^{\text{max}}$ of $B$, since $\dim(n_1^1) = n_1$ on $B_M^{\text{max}}$. The projection $p_\beta$ also varies smoothly on $B^{\text{max}}$.

**Remark 4.3.1** According to Remark 4.2.1, we can now assume that

\[ B = B^{\text{max}}. \]

Furthermore, since the function $\beta \mapsto \|p_\beta(Z_\beta)\|_2^2$ is smooth on $B$, we can take $\beta^0 \in B$ and $0 < \delta < R < \infty$ such that $\delta < \|p_\beta^0(Z_\beta^0)\|_2 < R$ and by using the reduction argument, we can then assume that the number $\|p_\beta(Z_\beta)\|_2$ is contained in the interval $[\delta, R]$ for any $\beta \in B$. 
4.3.1 On the Manifold $M$

Let us focus on the manifold $M$ again. Let $(\beta_0, l_0) \in M$ be fixed, but arbitrary. There exist $0 < \delta < R < \infty$ such that

$$0 < \delta < \min\{\|l_0, Z_{\beta_0}\|, \|p_{\beta_0}(Z_{\beta_0})\|_2\} < \max\{\|l_0, Z_{\beta_0}\|, \|p_{\beta_0}(Z_{\beta_0})\|_2\} < R.$$

This is due to the fact that $M \subset (\mathcal{B} \times \mathfrak{g}^*)_I$. According to Remark 4.3.1 we can now assume that

$$0 < \delta < \min\{\|p_{\beta}(Z_{\beta})\|_2\} < \max\{\|p_{\beta}(Z_{\beta})\|_2\} < R$$

for all $\beta \in \mathcal{B}$. We define

$$M^{\delta, R} = M^{red} \subseteq M^{\delta, R} := \{(\beta, l) \in M; 0 < \delta < \min\{\|l, Z_{\beta}\|\} < \max\{\|l, Z_{\beta}\|\} < R\}.$$

Obviously, $M^{red}$ is open in $M$ and thus is a smooth sub-manifold of $M$. On the other hand, we define

$$(\mathcal{B} \times \mathfrak{g}^*)_{\leq \delta, \leq R, \delta} = \left\{(\beta, l) \in (\mathcal{B} \times \mathfrak{g}^*)_{\leq \delta}; \frac{1}{2} \delta < \min\{\|l, Z_{\beta}\|\} < \max\{\|l, Z_{\beta}\|\} < \frac{3}{2} R\right\}.$$

4.3.2 Reducing $M$

Now we claim that if the Retract Theorem holds for $(\mathcal{B}, M^{red})$, then it remains true for $(\mathcal{B}, M)$.

Assume that the result is true for $(\mathcal{B}, M^{red})$. Let $\mathcal{M}$ be the open subset in $M^{red}$ given by the assumption. We will show that one may take the same manifold $\mathcal{M}$ for $(\mathcal{B}, M)$ such that the theorem remains true for $(\mathcal{B}, M)$. As $M^{red}$ is open in $M$, the set $\mathcal{M}$ also has a nonempty interior in $M$. Moreover, $p_{\mathcal{B}}(\mathcal{M}) \subset p_{\mathcal{B}}(M^{red}) \subset B^{n_1} \subset B^{\geq n_1}$. Let $\emptyset \neq C \subset \mathcal{M}$ be compact and let $F$ be a kernel function defined on $\mathbb{R}^r \times M \times G \times G$ whose support in $(\beta, l)$ is contained in $G \cdot C$. The restriction of $F$ to $\mathbb{R}^r \times M^{red} \times G \times G$ is a kernel function for $(\mathcal{B}, M^{red})$.

By assumption, there exists $f \in \mathcal{S}(\mathbb{R}^r \times \mathcal{B} \times G)$ such that $\pi_{(\beta, l)}(f(\cdot, \beta, \cdot))$ admits $F(\cdot, (\beta, l), \cdot, \cdot)$ as an operator kernel if $(\beta, l) \in M^{red}$. In particular, $\pi_{(\beta, l)}(f(\cdot, \beta, \cdot)) = 0$ if $(\beta, l) \in M^{red} \setminus C$. As $\emptyset \neq C \subset \mathcal{M}$ is compact, there exist $\delta_1, R_1 \in \mathbb{R}_+$ such that

$$0 < \delta < \delta_1 \leq \min\{\|l, Z_{\beta}\|\} \leq \max\{\|l, Z_{\beta}\|\} \leq R_1 < R$$

for all $(\beta, l) \in C$, as $C \subset \mathcal{M} \subset M \subset (\mathcal{B} \times \mathfrak{g}^*)_I$. Let $u \in \mathcal{C}_c^\infty(\mathbb{R})$ be odd such that $u \equiv 1$ on $[\delta_1, R_1]$ and $u \equiv 0$ on $[0, \delta] \cup [R, +\infty]$. There exists $\chi \in \mathcal{S}(\mathbb{R})$ such that $\hat{\chi} = u$. 
Let us define a function $\psi$ on $\bigcup \beta \{ \beta \} \times \exp(\mathbb{R}Z_{\beta})$ by $\psi(\beta, \exp(sZ_{\beta})) := \chi(s)$. For $(\beta, l) \in (B \times g^*)_J$ with $J \leq I$, we have $Z_{\beta} \in a_{\beta}(l) \subset g_{\beta}(l)$ and $\pi(\beta, l)(\psi(\beta, \cdot)) = \hat{F}(l, Z_{\beta})I_{\beta}(\beta, l) = u(l, Z_{\beta})I_{\beta}(\beta, l)$.

We define a function $g$ on $\mathbb{R}^r \times B \times G$ by

$$g(\cdot, \beta, \cdot) := f(\cdot, \beta, \cdot) \ast \psi(\beta, \cdot).$$

This implies that

$$\pi(\beta, l)(g(\cdot, \beta, \cdot)) = u(l, Z_{\beta})\pi(\beta, l)(f(\cdot, \beta, \cdot)).$$

If $(\beta, l) \in C$, then $u(l, Z_{\beta}) = 1$ and $\pi(\beta, l)(g(\cdot, \beta, \cdot)) = \pi(\beta, l)(f(\cdot, \beta, \cdot))$ admits $F(\cdot, (\beta, l), \cdot, \cdot)$ as an operator kernel. If $(\beta, l) \in M_{\text{red}} \setminus C$, then $\pi(\beta, l)(f(\cdot, \beta, \cdot)) = 0$, hence $\pi(\beta, l)(g(\cdot, \beta, \cdot)) = 0$ and $F(\cdot, (\beta, l), \cdot, \cdot) = 0$. If $(\beta, l) \in M \setminus M_{\text{red}}$, then $|l, Z_{\beta})| \not\in [0, R]$, i.e. $u(l, Z_{\beta}) = 0$, which implies that $\pi(\beta, l)(g(\cdot, \beta, \cdot)) = 0$. Hence, the mapping $F \mapsto g$ satisfies the property of the retract for $(B, M)$.

### 4.4 Construction of a New Variable Group

We start this section with an example which will demonstrate the use of a variable group and its variable algebra in the retract construction.

#### 4.4.1 An Example

In this subsection, we will consider the free two-step nilpotent Lie group on four generators.

Let $g = f_{4,2}$ be the free two-step nilpotent Lie algebra with four generators. This algebra has a Jordan-Hölder basis $\{Y_1, \cdots, Y_{10}\}$, where

$$[Y_1, Y_2] = Y_5, \quad [Y_2, Y_3] = Y_8, \quad [Y_1, Y_3] = Y_6, \quad [Y_2, Y_4] = Y_9, \quad [Y_1, Y_4] = Y_7, \quad [Y_3, Y_4] = Y_{10}.$$ 

The centre $Z$ of $g$ is the subspace $Z = \text{span}\{Y_j \mid j = 5, \cdots, 10\}$. Let $G = F_{4,2} = \exp(f_{4,2})$ be the simply connected nilpotent Lie group with Lie algebra $f_{4,2}$.

The co-adjoint orbits and the Pukanszky layers of the orbit spaces of the group $G$ have been determined in the paper [4]. The non-generic index sets are given by the conditions

$$I = \{(i, j) \mid 1 \leq i < j \leq 4\}$$

and

$$g_{(i, j)}^* = \{l \in g^* \mid \langle l, [Y_i, Y_j] \rangle = \langle l, Y_k \rangle \neq 0, \langle l, Y_{k'} \rangle = 0 \text{ for } k' > k\},$$

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where $Y_k = [Y_i, Y_j]$. Consider now the layer $\mathfrak{g}^*_3(3, 4)$, it is given precisely by

$$\mathfrak{g}^*_3(3, 4) = \{ l = \sum_{i=1}^{10} a_i Y_i^* \in \mathfrak{g}^* | a_5 a_{10} + a_8 a_7 - a_6 a_9 = 0, a_{10} \neq 0 \}.$$ 

Hence the section $\Sigma_3(3, 4)$ of $\mathfrak{g}^*_3(3, 4)$ is given by

$$\Sigma_3(3, 4) = \left\{ a_1 Y_1^* + a_2 Y_2^* + \left( \frac{a_6 a_9 - a_7 a_8}{a_{10}} \right) Y_5^* + \sum_{i=6}^{10} a_i Y_i^* | a_1, a_2, a_i \in \mathbb{R}, i = 6, \ldots, 10 \right\}.$$ 

The co-oadjoint orbit of $l \in \Sigma_3(3, 4)$ is the subset

$$\Omega_l = \left\{ \left( a_1 + \frac{a_6}{a_{10}} u_2 + \frac{a_7}{a_{10}} u_1 \right) Y_1^* + \left( a_2 + \frac{a_8}{a_{10}} u_2 + \frac{a_9}{a_{10}} u_1 \right) Y_2^* + u_1 Y_3^* + u_2 Y_4^* + \left( \frac{a_6 a_9 - a_7 a_8}{a_{10}} \right) Y_5^* + \sum_{i=6}^{10} a_i Y_i^* | u_1, u_2 \in \mathbb{R} \right\}.$$ 

For $l \in \mathfrak{g}^*_3(3, 4)$, the stabiliser $\mathfrak{g}(l)$ of $l$ is spanned by

$$S_1(q) := Y_1 - \frac{a_7}{a_{10}} Y_3 + \frac{a_6}{a_{10}} Y_4, \quad S_2(q) := Y_2 - \frac{a_9}{a_{10}} Y_3 + \frac{a_8}{a_{10}} Y_4$$ 

and the center $3$, for $q := l|_3$, and the Vergne polarisation $\mathfrak{p}(l)$ of $l$ is $\mathfrak{g}(l) + \mathbb{R} Y_4$. By the construction in Sect. 2.4, we obtain the new basis vectors of $\mathfrak{g}$ adapted to $\mathfrak{p}(l)$ as the following:

$$Z_1(l) = Y_3,$$
$$Z_2(l) = Y_1 - \frac{a_7}{a_{10}} Y_3 + \frac{a_6}{a_{10}} Y_4,$$
$$Z_3(l) = Y_2 - \frac{a_9}{a_{10}} Y_3 + \frac{a_8}{a_{10}} Y_4,$$
$$Z_k(l) = Y_k \text{ for } k = 4, \ldots, 10.$$ 

Let $\mathcal{B} = \{(a_5, \ldots, a_{10}) \in \mathbb{R}^6 | a_5 a_{10} + a_8 a_7 - a_6 a_9 = 0, a_{10} \neq 0 \}$. Now we define the new variable Lie algebra $(\mathfrak{g}, \mathcal{B})$ by fixing a basis $\{Z_1, \ldots, Z_{10}\} \subseteq \mathfrak{g} \simeq \mathbb{R}^{10}$ and we equip $\mathbb{R}^{10}$ with the brackets $[\cdot, \cdot]_q$, for $q \in \mathcal{B}$, satisfying the following:
\[ [Z_1, Z_2]_q = [Y_3, Y_1 - \frac{a_7}{a_{10}} Y_3 + \frac{a_6}{a_{10}} Y_4] = -Y_6 + \frac{a_6}{a_{10}} Y_{10}, \]

\[ [Z_1, Z_3]_q = [Y_3, Y_2 - \frac{a_9}{a_{10}} Y_3 + \frac{a_8}{a_{10}} Y_{10}] = -Y_8 + \frac{a_8}{a_{10}} Y_{10}, \]

\[ [Z_1, Z_4]_q = Y_{10}, \]

\[ [Z_2, Z_3]_q = Y_5 - \frac{a_9}{a_{10}} Y_6 + (\frac{a_7}{a_{10}} + \frac{a_8}{a_{10}}) Y_8 - \frac{a_6}{a_{10}} Y_9 + \frac{a_6 a_{10}}{a_2^2} Y_{10}, \]

\[ [Z_2, Z_4]_q = Y_7 - \frac{a_7}{a_{10}} Y_{10}, \]

\[ [Z_3, Z_4]_q = Y_9 - \frac{a_9}{a_{10}} Y_{10}. \]

For any \( q = (a_5, \ldots, a_{10}) \in \mathcal{B} \), the linear functional \( l_q \) on \( \mathfrak{g}^* \) is defined by

\[
\begin{align*}
l_q(Z_1) &= a_3, \\
l_q(Z_2) &= \langle l, Y_1 - \frac{a_7}{a_{10}} Y_3 + \frac{a_6}{a_{10}} Y_4\rangle = a_1 - \frac{a_3 a_7}{a_{10}} + \frac{a_4 a_6}{a_{10}}, \\
l_q(Z_3) &= \langle l, Y_2 - \frac{a_9}{a_{10}} Y_3 + \frac{a_8}{a_{10}} Y_4\rangle = a_2 - \frac{a_3 a_9}{a_{10}} + \frac{a_4 a_8}{a_{10}}, \\
l_q(Z_j) &= a_j \text{ for } j = 4, \ldots, 10.
\end{align*}
\]

Let \( (\mathfrak{g}_1(q), \mathcal{B}) \) be the subalgebra of \( (\mathfrak{g}, \mathcal{B}) \) given by \( \text{span}\{Z_2, \ldots, Z_{10}\} \). We have then \([\mathfrak{g}_1(q), \mathfrak{g}_1]\) \( \subset \mathfrak{g}_1 \). By the construction of variable Lie algebras given in Sect. 2.3, we have \([\mathfrak{g}_1(q), [\cdot, \cdot]]_q \simeq (\mathfrak{p}(l_q), [\cdot, \cdot])\) and hence \([l_q, [\mathfrak{g}_1, \mathfrak{g}_1]_q] = 0\). Note that we will prove the retract theorem by induction on the dimension of the variable group. The induction procedure transforms a given variable Lie algebra \( (\mathfrak{g}, \mathcal{B}) \) into a new variable one \( (\mathfrak{g}_1, \mathcal{B}_1) \) of lower dimension in a consistent manner.

Let \( M \) be a smooth \( G \)-invariant manifold of \( \mathfrak{g}^* \) contained in \( \bigcup_{(i, j) \leq (3, 4)} \mathfrak{g}^*_{(i, j)} \) such that \( M \cap \mathfrak{g}^*_{(3, 4)} \neq \emptyset \). Choose \( \delta > 0 \) so that

\[
M^\text{red} := \{ l \in M \mid \langle l, Y_{10}\rangle > \delta \} \cap \Sigma_{(3, 4)}
\]

is non-empty. For \( l \in M^\text{red} \), let

\[
\pi_l := \text{ind}_{\rho(l)}^G \chi_l.
\]

For \( l \in \Sigma_{(3, 4)} \), we define also the diffeomorphism \( \Phi_l \) from \( \mathbb{R} \times \mathbb{R}^9 \) onto \( G \) by

\[
\Phi_l(y_3, r) := \exp(y_3 Y_3) \cdot \exp\left(\sum_{j=2}^{10} z_j Z_j(l)\right) \text{ for } (y_3, r) \in \mathbb{R} \times \mathbb{R}^9.
\]

Then for any \( u, v \in \mathbb{R}^{10} \) and \( l \in \Sigma_{(3, 4)} \), we have that

\[
\Phi_l(u \cdot v) = \Phi_l(u) \cdot \Phi_l(v).
\]
For $f \in S(G)$, $y_3 \in \mathbb{R}$, $l \in \Sigma_{(3,4)}$ and $q = l|_{3}$, let 

$$\hat{f}(l, y_3, q) := \hat{f} \circ \Phi_l^2 (y_3, l_q |_{\mathbb{R}^9}),$$

where $\hat{h}^2$ is defined for all $h \in C^\infty(g_{(3,4)}^*, S(\mathbb{R}^{10}))$ by

$$\hat{h}^2(l, y_3, r) := \int_{\mathbb{R}^9} e^{-ir(u)} h(l, y_3, u) du, \ r \in (\mathbb{R}^9)^*, \ y_3 \in \mathbb{R}, \ l \in \Sigma_{(3,4)}.$$

Then for $\xi \in L^2(\mathbb{R})$ and $u \in \mathbb{R}$, we have that

$$\pi_l(f)\xi(u) = \int_{\mathbb{R}} \hat{f}^2(l, u - y_3, (a_1 - \frac{a_3a_7}{a_{10}} + \frac{a_4a_6}{a_{10}}, a_2 - \frac{a_3a_9}{a_{10}} + \frac{a_4a_8}{a_{10}}, a_4 + y_3a_{10}, l|_{3})) \xi(y_3) dy_3.$$

We must find a smooth sub-manifold $\mathcal{M}$ inside $M^{\text{red}}$ such that for a given smooth compactly supported operator field $F(l)$ on $\mathcal{M}$, we have a Schwartz function $f$ defined on $G$ such that

$$\hat{f}^2\left(l, u - y_3, (a_1 - \frac{a_3a_7}{a_{10}} + \frac{a_4a_6}{a_{10}}, a_2 - \frac{a_3a_9}{a_{10}} + \frac{a_4a_8}{a_{10}}, a_4 + y_3a_{10}, l|_{3})\right) = k_f(l, u, y_3)$$

for all $u, y_3 \in \mathbb{R}$ and $l$ in $\mathcal{M}$ (here $k_f$ denotes the kernel function of the operator $F(l)$), and $\pi_{l'}(f) = 0$ for all $l'$ in $g_{(i,j)}^*$ with $(i, j) < (3, 4)$.

For simplicity of the notations, let

$$g_{1}^0 := \text{span}\{Z_1, Z_2, \bar{z}\} \subset g_{1}.$$

Then we can consider $\mathcal{M}$ to be a smooth sub-manifold of $(g_{1}^0)^*$. The construction of the manifold $\mathcal{M}$ and of the retract function in the general case will be shown in Sect. 4.5.

In our example, we take a smooth kernel function $k(l, u, v), l \in \mathcal{M}$ and $u, v \in \mathbb{R}$, compactly supported in $\mathcal{M}$ and we extend it to a smooth function $K(l, u, v)$ on $(g_{1}^0)^*$ which is compactly supported in $l \in (g_{1}^0)^*$, vanishes for $|l(Y_{10})| \leq \delta$ and is Schwartz in the variables $(u, v)$. We can then find a function $h \in S(g_{1}^0 \times \mathbb{R}^2)$ such that

$$\hat{h}^0(l, u, v) = |l(Y_{10})| K(l, u, v) \text{ for } u, v \in \mathbb{R}, l \in (g_{1}^0)^*.$$

Now let

$$\tilde{h}(q, g_1, u, v) := \int_{\mathbb{R} \times \mathbb{R}^*} h(g_1 + yY_4 + V, u, v) e^{-i(q, V)} dydV \text{ for } q \in \mathbb{R}^*, g_1 \in g_1.$$
Define the function \( \tilde{f} : G \times s^* \to \mathbb{C} \) by
\[
\tilde{f}(\exp(y_3 Y_3) \cdot \exp(s_1 S_1(q) + s_2 S_2(q)) \cdot \exp(y_4 Y_4) \cdot \exp(Z), q) := e^{i q(Z)} \int_{\mathbb{R}} \tilde{h}(q, \text{Ad}_q(\exp(v Y_3))(s_1 S_1(q) + s_2 S_2(q)), y_3, v) e^{-ia_1 q v} dv.
\]
Finally let
\[
f(g) := \int_{s^*} \tilde{f}(g, q) dq \quad \text{for } g \in G.
\]
Then the function \( f \) has the required properties (by the computations in Sect. 4.5).

4.4.2 The Mapping \( \alpha(\beta, l) \)

For \((\beta, l) \in (\mathcal{B} \times \mathfrak{g}^*)_{\leq I}, \) we have seen in (4.6) that \( n_{\beta}^1 \subset \ker(l) \). Let \( q := l|_{c_1} \in (n_{\beta}^1)^\perp \) and \([l, Z_\beta] = (l, p_\beta(Z_\beta)) = (q, p_\beta(Z_\beta))\). For \((\beta, l) \in (\mathcal{B} \times \mathfrak{g}^*)_{\leq I, R, \delta}, \) we have that \(|l, [Z_{k_1}, Z_{j_1}]_\beta| = |l, Z_\beta| > \frac{\delta}{2} > 0 \) implies \([l, [Z_{k_1}, Z_{j_1}]_\beta] \neq 0 \) and \( j_1(\beta, l) = j_1, k_1(\beta, l) = k_1. \)

Take an odd function \( \varphi \in C^\infty(\mathbb{R}) \) with the properties that \( \varphi(s) = 0 \) for \( 0 \leq s < \delta/4 \) and \( s > 2R, 1 > \varphi(s) > 0 \) for \( s \in [\delta/4, \delta/2] \cup [3R/2, 2R] \) and \( \varphi(s) = 1 \) for \( 3R/2 \geq s \geq \delta/2. \) For every \((\beta, q) \in \mathcal{B} \times c_1^*, \) we construct the vector \( \alpha(\beta, q) \in c_1 \simeq (c_1^*)^* \) by
\[
\alpha(\beta, q) := \varphi(\|p_\beta(Z_\beta)\|_2) \varphi((q, p_\beta(Z_\beta))) p_\beta(q) + (1 - \varphi(\|p_\beta(Z_\beta)\|_2) p_\beta(Z_\beta)) \varphi((q, p_\beta(Z_\beta))) p_\beta(Z_\beta).
\]
By the construction, \( \alpha(\beta, q) \in (n_{\beta}^1)^\perp \subset c_1^* \equiv c_1 \) for every \((\beta, q) \in \mathcal{B} \times c_1^*. \) On the other hand, for \((\beta, l) \in (\mathcal{B} \times \mathfrak{g}^*)_{\leq I, R, \delta} \) and \( q = l|_{c_1}, \) we have that
\[
\alpha(\beta, q) = \varphi(\|p_\beta(Z_\beta)\|_2) \varphi((q, p_\beta(Z_\beta))) p_\beta(q) + (1 - \varphi(\|p_\beta(Z_\beta)\|_2) \varphi((q, p_\beta(Z_\beta))) p_\beta(Z_\beta)
\]
\[
= \pm p_\beta(q) + (1 - 1)p_\beta(Z_\beta)
\]
\[
= \pm q.
\]
This is due to the fact that \( p_\beta(q) = q \) as \( n_{\beta}^1 \subset \ker(q) \) for \( q = l|_{c_1}, \) if \((\beta, l) \in (\mathcal{B} \times \mathfrak{g}^*)_{\leq I}. \) We will show that
\[
\langle \alpha(\beta, q), Z_\beta \rangle = \varphi(\|p_\beta(Z_\beta)\|_2) \varphi((q, p_\beta(Z_\beta))) \langle p_\beta(q), Z_\beta \rangle + (1 - \varphi(\|p_\beta(Z_\beta)\|_2) \varphi((q, p_\beta(Z_\beta))) \|p_\beta(Z_\beta)\|_2^2 > 0
\]
on \( \mathcal{B} \times c_1^*. \) In fact, let us first notice that \( \langle p_\beta(q), Z_\beta \rangle = \langle q, p_\beta(Z_\beta) \rangle. \) As \( \varphi \) is an odd function and \( \varphi \geq 0 \) on \( \mathbb{R}_+, \) we have
\[
A := \varphi(\|p_\beta(Z_\beta)\|_2) \varphi((q, p_\beta(Z_\beta))) \langle p_\beta(q), Z_\beta \rangle \geq 0.
\]
Since \(0 \leq \varphi \leq 1\) on \(\mathbb{R}_+\),

\[
B := (1 - \varphi(\|p_\beta(Z_\beta)\|_2))\varphi(\|\langle q, p_\beta(Z_\beta) \rangle\|_2) \|p_\beta(Z_\beta)\|_2^2 \geq 0.
\]

If none of the \(\varphi(\cdot)\)'s is equal to zero and if \(\langle p_\beta(q), Z_\beta \rangle \neq 0\), then \(A > 0\). If \(\langle p_\beta(q), Z_\beta \rangle = 0\), then \(\varphi(|\langle p_\beta(q), Z_\beta \rangle|) = 0\) and thus \(B > 0\), as by Remark 4.3.1 \(\|p_\beta(Z_\beta)\|_2 > 0\). If one of the \(\varphi(\cdot)\)'s is equal to zero, then again \(B > 0\).

For \((\beta, y) \in B \times c_1^*\), let

\[
\delta(\beta, q) := \text{ad}_{p_\beta}^*(Z_{j_1})\alpha(\beta, q) \in \mathfrak{g}^*.
\]

We have that

\[
\langle \delta(\beta, q), Z_{k_1} \rangle = \langle \alpha(\beta, q), Z_\beta \rangle > 0
\]

and

\[
\langle \delta(\beta, q), [\mathfrak{g}, \mathfrak{g}]_\beta \rangle = \langle \alpha(\beta, q), [[\mathfrak{g}, \mathfrak{g}]_\beta, Z_{j_1}]_\beta \rangle \subset \langle \alpha(\beta, q), [\mathfrak{g}, [\mathfrak{g}, Z_{j_1}]_\beta]_\beta \rangle
\]

\[
\subset \langle \alpha(\beta, q), [\mathfrak{g}, c_1]_\beta \rangle \subset \langle \alpha(\beta, q), n_1^\perp \rangle = \{0\},
\]

by the definition of \(\alpha(\beta, q)\) in (4.7). This means that \(\delta(\beta, q)\) is an algebra homomorphism of \(\mathfrak{g}_\beta = (\mathfrak{g}, [\cdot, \cdot]_\beta)\) which does not vanish at the vector \(Z_{k_1}\). Hence the subspace

\[
\mathfrak{g}^1(\beta, q) := \ker(\delta(\beta, q))
\]

is an ideal of \(\mathfrak{g}_\beta\) of co-dimension one and

\[
\mathfrak{g} = \mathbb{R}Z_{k_1} \oplus \mathfrak{g}^1(\beta, q).
\]  

Furthermore \(\mathfrak{g}^1(\beta, q)\) contains \(c_1\) for any \((\beta, y) \in B \times c_1^*\). In fact,

\[
\langle \delta(\beta, q), c_1 \rangle = \langle \alpha(\beta, q), [c_1, Z_{j_1}]_\beta \rangle = 0
\]

as \(\alpha(\beta, q) \in (n_1^\perp)\) and \([c_1, Z_{j_1}]_\beta \subset n_1^\perp\).

4.4.3 The New Variable Group \((B_1, G_1)\)

In order to construct a new variation in the induction procedure, we put

\[
B_1 := B \times \mathbb{R} \times c_1^*.
\]

For \((\beta, y, q) \in B_1\), we define a Jordan-Hölder basis

\[
\tilde{Z}^1(\beta, y, q) = \{\tilde{Z}_1^1(\beta, q), \cdots, \tilde{Z}_{n-1}^1(\beta, q)\}
\]
of $g_1(\beta, y, q) = \ker(\delta(\beta, q))$ by

$$\alpha^\beta_{k, y, q} = \alpha^\beta_k := \frac{\langle \alpha(\beta, q), [Z_k, Z_j]_\beta \rangle}{\langle \alpha(\beta, q), Z_\beta \rangle}$$

and

$$\tilde{Z}^1(\beta, y, q) = \tilde{Z}^1(\beta, q) := \{ Z_1 - \alpha^\beta_1 Z_k, \ldots, Z_{k_1 - 1} - \alpha^\beta_{k_1 - 1} Z_k, Z_{k_1 + 1}, \ldots, Z_n \}$$

$$= \{ \tilde{Z}_1^1(\beta, q), \ldots, \tilde{Z}_{n-1}^1(\beta, q) \}.$$ 

In particular, for $(\beta, l) \in (B \times g^\ast_{\leq I, R, \delta})$ we have by (2.1) that

$$g_1(\beta, y, l|_{c_1}) = g_1(\beta, l) = g_1(\beta, l|_{c_1}).$$

In fact, in this case $j_1(\beta, l) = j_1$, $k_1(\beta, l) = k_1$ and

$$g_1(\beta, y, l|_{c_1}) = \{ U \in g; (\delta(\beta, l|_{c_1}), U) = 0 \} = \{ U \in g; (l, [U, Z_{j_1}]_\beta) = 0 \} = g_1(\beta, l),$$

as $\alpha(\beta, l|_{c_1}) = \varepsilon \cdot |_{c_1}$ with $\varepsilon = \pm 1$ provided $(\beta, l) \in (B \times g^\ast_{\leq I, R, \delta})$.

For each $k$, we also have that

$$\alpha^\beta_k = \frac{\langle l, [Z_k, Z_{j_1}]_\beta \rangle}{\langle l, Z_\beta \rangle}.$$

(4.10)

This new basis $\tilde{Z}^1(\beta, y, q)$ coincides then, up to normalisation, with the basis obtained in Sect. 2.4, both procedures and bases generate the same indices. Furthermore by (4.10), for $(\beta, l) \in (B \times g^\ast_{\leq I, R, \delta})$, we have

$$\tilde{Z}^1(\beta, y, l_1) = Z^1(\beta, l),$$

(4.11)

where $Z^1(\beta, l)$ is defined in Sect. 2.4 and $l_1 = l|_{c_1}$.

For any $(\beta, y, q) \in B_1$, let us write

$$[\tilde{Z}^1_u(\beta, q), \tilde{Z}^1_v(\beta, q)] = \sum_{w=1}^{n-1} \gamma(\beta, q)^u,v \tilde{Z}^1_w(\beta, q) \text{ for } u < v \text{ in } \{1, \ldots, n-1\}.$$ 

We obtain in this way a new variable Lie algebra $(B_1, g_1)$, where

$$g_1 = R^{n-1}, B_1 = B \times R \times c_1^\ast$$

and the canonical basis $Z^1 = \{ Z^1_1, \ldots, Z^1_{n-1} \}$ of $g_1$ satisfies, by definition,

$$[Z^1_u, Z^1_v]_{(\beta, q)} = \sum_{w=1}^{n-1} \gamma(\beta, q)^u,v Z^1_w \text{ for } u < v \text{ in } \{1, \ldots, n-1\}.$$
This means that the new variable Lie algebra \((\mathcal{B}_1, g_1)\) with \(g_1 = g^1(\beta, q)\) is defined such that \((g_1, \{\cdot, \cdot\}_{\beta, q}) = (g^1(\beta, q), \{\cdot, \cdot\}_{\beta, q})\).

Given \((\beta, l) \in \mathcal{B} \times g^*,\) let us define \(l_1 \in g^*_1\) by \(l_1(Z_i^1) = l(Z_i^1(\beta, q))\) for all \(i \in \{1, \ldots, n - 1\}\). One has \(l_1(Z_i^1) = l(Z_{i+1})\) if \(i \geq k_1\). We also define a map

\[
\iota_1 : \mathcal{B} \times g^* \rightarrow \mathcal{B}_1 \times g^*_1
\]

\[
(\beta, l) \mapsto ((\beta, \langle l, Z_{k_1}\rangle, \alpha(\beta, l_{|_{\xi_1}})), l_1),
\]

(4.12)

where \(l_1 = L_{\beta, q}^\dagger \). We see that \(\iota_1\) is obviously smooth, injective and is a diffeomorphism onto its image.

Using (4.9) we can identify every \(l \in g^*\) with the pair \((v, l_1)\) where \(v := \langle l, Z_{k_1}\rangle\) and \(l_1 := L_{\beta, q}^\dagger \). We can then transfer the natural action of \(G\) on \(\mathcal{B} \times g^*\) to \(\mathcal{B}_1 \times g^*_1\) using the mapping \(\iota_1\). This gives us

\[
g \cdot ((\beta, v, q), l_1) = ((\beta, v + \langle Ad^*_{\beta}(g)l_1, Z_{k_1}\rangle, Ad^*_{\beta}(g)q), Ad^*_{\beta}(g)l_1).
\]

We have automatically the relation

\[
\iota_1(g \cdot (\beta, l)) = g \cdot (\iota_1(\beta, l))
\]

for any \(g \in G\) and \((\beta, l) \in \mathcal{B} \times g^*\).

Consider now the smooth manifold

\[
(\mathcal{B} \times g^*)^0_{\leq I, R, \delta} := \{ (\beta, l) \in (\mathcal{B} \times g^*)_{\leq I, R, \delta}; \langle l, Z_{k_1}\rangle = \langle l, Z_{j_1}\rangle = 0 \}.
\]

Obviously the smooth manifold \((\mathcal{B} \times g^*)_{\leq I, R, \delta}\) is diffeomorphic with the manifold \(\mathbb{R}^2 \times (\mathcal{B} \times g^*)^0_{\leq I, R, \delta}\). The mapping

\[
\Phi : \mathbb{R}^2 \times (\mathcal{B} \times g^*)^0_{\leq I, R, \delta} \rightarrow (\mathcal{B} \times g^*)_{\leq I, R, \delta}
\]

given by

\[
\Phi(s, t, (\beta, l)) := \left(\beta, \text{Ad}^*(\exp\left(\frac{s}{\langle l, Z_{k_1}\rangle} Z_{j_1}\right) \exp\left(\frac{t}{\langle l, Z_{k_1}\rangle} Z_{k_1}\right))l\right)
\]

is such a diffeomorphism. Hence every smooth \(G\)-invariant sub-manifold \(M\) of \((\mathcal{B} \times g^*)_{\leq I, R, \delta}\) can be decomposed into a direct product of \(\mathbb{R}^2\) with the smooth manifold \(M^0\), where

\[
M^0 := \{ (\beta, l) \in M; \langle l, Z_{k_1}\rangle = \langle l, Z_{j_1}\rangle = 0 \}.
\]

For \((\beta, l) \in \mathcal{B} \times g^*\), one has \(l_1(Z_i^1) = l(Z_i)\) if \(i < k_1\) and \(l_1(Z_i^1) = l(Z_{i+1})\) if \(i \geq k_1\). We remark that for \((\beta, l)\) and \((\beta, l')\) in \(M\) with \(\iota_1(\beta, l) = \iota_1(\beta, l')\) we have...
that $l$ and $l'$ have the same restriction to $g_1(\beta, l) = g_1(\beta, l')$, so they are on the same co-adjoint orbit and $l' = \text{Ad}^*(y)l$ for some $y \in P(\beta, l)$ and hence

$$F(\beta, l) = F(\beta, l')$$

by the conditions on the operator fields defined over $M$ given in Definition 2.7.1.

We denote the new variable Lie group by $G_1 = (B_1, G_1)$, where $G_1 = (\exp_{[\beta]} g_1)_{\beta \in B_1}$ and $\exp_{[\beta]} g_1$ is the connected, simply connected, nilpotent Lie group associated to the Lie algebra $(g_1, [\cdot, \cdot]_{\beta_1})$.

## 4.5 Induction Step

To simplify notations, from now on we will omit the subscript $\beta$ in the notations of the multiplication and the exponential map, unless the subscript is crucial for the understanding. There are two preliminary steps to check.

### 4.5.1 Induction Hypothesis

In this subsection, we will prove the result for $(B, M)$ using induction. Let $M_1 = \iota_1(M)$, where $\iota_1$ is constructed in (4.12). Let us recall that $l_1 = l_1(\beta, l_{c_1}) \equiv l_{1|g_1(\beta, l_{c_1})}$ for $(\beta, l) \in (B \times g^*) \leq I$. The Vergne polarisation $\pi(\beta, l)$ for $l \in (g_1, [\cdot, \cdot]_{\beta})$, obtained by the procedure of Ludwig-Zahir (see [11], [9]), is also the Vergne polarisation for $l_1$ in $(g_1, [\cdot, \cdot]_{\beta})$. Let us denote by $P(\beta, l) = \exp_{[\beta]} p(\beta, l))$ the corresponding subgroup. The associated induced representations will be denoted by $\pi_{(\beta, l)} := \text{ind}_{P(\beta, l)} G \chi_l$, respectively, $\tilde{\pi}_{(\beta, l_{c_1})} := \text{ind}_{P(\beta, l)} G \chi_l$. Then $\pi_{(\beta, l)} \cong \text{ind}_{G_1} \tilde{\pi}_{(\beta, l_{c_1})}$, as usual.

Since $M$ is $G$-invariant, the manifold $M_1 = \iota_1(M)$ is also $G$-invariant in $B_1 \times g_1^*$. Hence we can write $M_1$ as a direct product manifold $R^2 \times M_1^0$, where

$$M_1^0 := \{(\beta, 0, q), l_1); \langle l_1, Z_{j_1} \rangle = 0\}$$

is $G_1$-invariant. Note that $M_1^0$ is contained in $(B_1 \times g_1^*) l_1$ and for every $((\beta, v, l_{c_1}), l_1) \in M_1$ we have that $\infty > R > |\langle l_1, Z_{\beta} \rangle| > \delta > 0$. The induction hypothesis in $B_1 \times g_1^*$ and $M_1^0 \subset (B_1 \times g_1^*) l_1$ gives us an open relatively compact non-empty subset $M_1^0$ of $M_1^0$ with the required properties of the theorem.

We choose now a relatively compact open subset $M_1$ of $M_1$ such that $\overline{M_1} \subset M_1$ and $M_1$ is contained in $G \cdot M_1^0$. Let

$$\mathcal{M} := \iota_1^{-1}(M_1) \quad \text{and} \quad \mathcal{M}^0 := \iota_1^{-1}(M_1^0).$$

Then $\mathcal{M}$ is non-empty open with its closure $\overline{\mathcal{M}}$ contained in $M$ and $\mathcal{M}$ is contained in $G \cdot \mathcal{M}^0$. We take a kernel function $F \in D'_{\mathcal{M}}$ such that its support is contained in $R^2 \times G \cdot \mathcal{M} \times G \times G$.

Given the kernel function $F$, we will now define a kernel function for the variable group $(B_1, G_1)$. For simplicity, we will omit the subscripts $\beta$ or $(\beta, v, l_{c_1})$ in the
notations of the multiplication and the exponential map, and we will identify \( g_1, g_1' \in G_1 = ((G_1)_{(\beta_1)})_{\beta_1 \in B_1} = g_1 \) with the corresponding elements in \( G_1 \). In the following computations, the parameters \( \beta \) and \( (\beta, v, l|_{\epsilon_1}) \) will indicate how to multiply group elements or how to decompose smoothly the group elements, even if it is not marked explicitly. For \( \iota_1(\beta, l) = ((\beta, (l, Z_{k_1}), l|_{\epsilon_1}), l_1) \in M_1 \), we put

\[
F_1(\alpha, u, t, ((\beta, (l, Z_{k_1}), l|_{\epsilon_1}), l_1), g_1, g_1') := (2\pi)^{-j_1+1} \cdot |c(\beta, l)| \cdot F(\alpha, (\beta, l), \exp((u + t)X), g_1 \cdot \exp(tX) \cdot g_1'),
\]

for \( \alpha \in \mathbb{R}^r, u, t \in \mathbb{R} \) and \( g_1, g_1' \in G_1 \), where \( c(\beta, l) := \langle l, [Z_{k_1}, Z_{j_1}]_\beta \rangle \neq 0 \) and \( X = Z_{k_1} \). This function \( F_1 \) has its support \( S_1 := \iota_1(S) \) contained in \( G \cdot M_1 \), and belongs to \( D_{M_1}^c \). The operator field \( F_1 \) is smooth on \( M_1 \), since the mappings \( F \) and \( c \) are both smooth. By the induction hypothesis, there exists \( h \in S(\mathbb{R}^{r+2}, B_1, G_1) \) such that \( \tilde{\pi}_{((\beta_1, l|_{\epsilon_1}), l_1)}(h(\alpha, u, t, \beta_1, \cdot)) \) admits \( F_1(\alpha, u, t, (\beta_1, l_1), \cdot, \cdot) \) as an operator kernel for all \( (\beta_1, l_1) \in M_1^0 \). The construction of the retraction function \( f \) will now be done in several steps.

### 4.5.2 Definition of the Retract Function on the Original Group

For \( (\beta, v, q) \in B_1 \), let us first define \( \tilde{h} \) by

\[
\tilde{h}(\alpha, u, t, (\beta, v, q), g_1) := \int_{\mathbb{R}} \int_{\mathbb{C}_1} h(\alpha, u, t, (\beta, v, q), g_1 \cdot \exp(yY) \cdot \exp(Z)) e^{-iq(Z)} dZ dy,
\]

where \( Y = Z_{j_1} \) and \( Z = Z_\beta = [X, Y]_\beta \) with \( X = Z_{k_1} \). The integral converges, as \( h \) is Schwartz in \( g_1 \) (for fixed \( \beta_1 \)), and it is rapidly decreasing in \( q \in (\epsilon_1)^* \), because it is a Fourier transform in \( Z \). For all \( (\beta, v, q) \in B \times \mathbb{R} \times \epsilon_1^* \), we then define

\[
\tilde{f}(\alpha, (\beta, v, q), \exp(uX) \cdot g_1 \cdot \exp(yY) \cdot \exp(Z)) = e^{iq(Z)} \int_{\mathbb{R}} \tilde{h}(\alpha, u, t, (\beta, v, q), g_1^{-t}) e^{-ityq([X,Y]_\beta)} dt
\]

with \( g = \exp(uX) \cdot g_1 \) and \( g_1^{-t} := \exp(-tX) \cdot g_1 \cdot \exp(tX) \). The function \( \tilde{f} \) is smooth on \( \mathbb{R}^r \times (B \times \epsilon_1^*) \times G \). As \( f \) is of rapid decrease in \( q \in \epsilon_1^* \) by construction, we may define \( f \) by

\[
f(\alpha, \beta, g) := \int_{\epsilon_1^*} \tilde{f}(\alpha, (\beta, 0, q), g) dq, \quad \alpha \in \mathbb{R}^r, \beta \in B, g \in G.
\]

One can see that \( f \in S(\mathbb{R}^r, B, G) \) (in the sense of Sect. 2.6).
4.5.3 The Retract Property

Let us now compute $\pi_{(\beta, l)}(f(\alpha, \beta, \cdot))$ for $(\beta, l) \in M$. Since the manifold $M$ is contained in $(\mathcal{B} \times \mathfrak{g}^*)_{\leq 1}$, we have that $c_1 \subset \alpha_\beta(l) \subset \mathfrak{g}_\beta(l)$. If we identify $\exp(c_1)$ and $c_1$, as well as $\exp(Z)$ and $Z$, for any function $\tilde{\xi}(\beta) \in \mathcal{F}_{(\beta, l)}$ (the representation space of $\pi_{(\beta, l)}$) and any $\tilde{g} \in G$, we have that

$$
\left( \pi_{(\beta, l)}(f(\alpha, \beta, \cdot))\tilde{\xi}(\beta) \right)(\tilde{g}) = \int_{G_{\beta}/c_1} \int_{c_1} f(\alpha, \beta, g \cdot Z) \left( \pi_{(\beta, l)}(g) \pi_{(\beta, l)}(Z) \tilde{\xi}(\beta) \right)(\tilde{g}) dZ d\tilde{g} 
$$

$$
= \int_{G_{\beta}/c_1} \int_{c_1} f(\alpha, \beta, g \cdot Z) e^{-ilZ} \left( \pi_{(\beta, l)}(g) \tilde{\xi}(\beta) \right)(\tilde{g}) dZ d\tilde{g} 
$$

$$
= \int_{G_{\beta}/c_1} \int_{c_1} \int_{\mathbb{C}^*} \tilde{f}(\alpha, \beta, 0, q, g \cdot Z) e^{-ilZ} \left( \pi_{(\beta, l)}(g) \tilde{\xi}(\beta) \right)(\tilde{g}) dq dZ d\tilde{g} 
$$

$$
= \int_{G_{\beta}/c_1} \int_{c_1} \int_{\mathbb{C}^*} \tilde{f}(\alpha, \beta, 0, q) e^{iqZ} e^{-ilZ} \left( \pi_{(\beta, l)}(g) \tilde{\xi}(\beta) \right)(\tilde{g}) dq dZ d\tilde{g} 
$$

$$
= \left( \frac{1}{2\pi} \right)^{n-j} \int_{G_{\beta}/c_1} \tilde{f}(\alpha, \beta, 0, l|c_1), g) \left( \pi_{(\beta, l)}(g) \tilde{\xi}(\beta) \right)(\tilde{g}) d\tilde{g}. 
$$

We use the following smooth decomposition: $X = Z_{k_1}, g_1 = g_1(\beta, l)$ which gives us

$$
\tilde{g} = \exp(sX) \cdot \tilde{g}_1 \quad \text{with} \quad s = s(\tilde{g}, \beta, l|c_1), \tilde{g}_1 = \tilde{g}_1(\tilde{g}, \beta, l|c_1).
$$

We then obtain (using the fact that $c_1 \subset \alpha_\beta(l)$ for all our $(\beta, l)$’s that:

$$
(2\pi)^{n-j} \cdot \left( \pi_{(\beta, l)}(f(\alpha, \beta, \cdot))\tilde{\xi}(\beta) \right)(\exp(sX) \cdot \tilde{g}_1) 
$$

$$
= \int_{\mathbb{R}} \int_{G_{1}/c_1} \tilde{f}(\alpha, \beta, 0, l|c_1), \exp(uX) \cdot g_1) \cdot \tilde{\xi}(\beta)(g_1^{-1} \cdot \exp((s-u)X) \cdot \tilde{g}_1) d\tilde{g}_1 du 
$$

$$
= \int_{\mathbb{R}} \int_{G_{1}/c_1} \tilde{f}(\alpha, \beta, 0, l|c_1), \exp((s-r)X) \cdot g_1) \cdot \tilde{\xi}(\beta)(l|c_1)(v) := \tilde{\xi}(\beta)(\exp(vX) \cdot g_1) 
$$

$$
= \int_{\mathbb{R}} \int_{G_{1}/c_1} \tilde{f}(\alpha, \beta, 0, l|c_1), \exp((s-r)X) \cdot g_1) \cdot \tilde{\xi}(\beta)(l|c_1)(r) \left( \exp(-rX) \cdot g_1 \cdot \exp(rX) \right)^{-1} \cdot \tilde{g}_1 d\tilde{g}_1 dr 
$$

(with $s-u = r$)
\[
J \text{ Fourier Anal Appl } = \int_\mathbb{R} \int_{G_1/c_1} \tilde{f}(\alpha, (\beta, 0, l|_{c_1}) \cdot \exp((s - r)X) \cdot g_1^r \cdot \tilde{g}(\beta, 0, l|_{c_1}))(r)(g_1^{-1} \cdot \tilde{g}_1) d\tilde{g}_1 dr
\]

with \( g_1^r = \exp(rX) \cdot g_1 \cdot \exp(-rX) \)

\[
= \int_\mathbb{R} \int_{G_1/c_1} \tilde{f}(\alpha, (\beta, 0, l|_{c_1}) \cdot \exp((s - r)X) \cdot g_1^r \cdot \tilde{g}(\beta, 0, l|_{c_1}))(r)(\tilde{g}_1) d\tilde{g}_1 dr
\]

with \( l_1 \equiv l|_{g_1} \)

\[
= \int_\mathbb{R} \int_{G_1/\exp(\mathbb{R}Y) - c_1} \int_\mathbb{R} \int_{G_1/\exp(\mathbb{R}Y) - c_1} \tilde{h}(\alpha, s - r - t, (\beta, 0, l|_{c_1}), w_1^{r-t}) (\tilde{\pi}(\beta, 0, l|_{c_1}, l_1)(w_1) \tilde{\xi}(\beta, 0, l|_{c_1}))(r)(\tilde{g}_1) dtd\tilde{g}_1 dr
\]

\[
= \int_\mathbb{R} \int_{G_1/\exp(\mathbb{R}Y) - c_1} \int_\mathbb{R} \int_{G_1/\exp(\mathbb{R}Y) - c_1} \tilde{h}(\alpha, s - r - t, (\beta, 0, l|_{c_1}), w_1^{r-t}) (\tilde{\pi}(\beta, 0, l|_{c_1}, l_1)(w_1) \tilde{\xi}(\beta, 0, l|_{c_1}))(r)(\tilde{g}_1) dtd\tilde{g}_1 dr
\]

\[
\text{(as } l(Y) = 0, \text{ for } l_1 \equiv l|_{\mathbb{R}(\beta, 0, l|_{c_1}))})
\]

\[
= \frac{1}{|c(\beta, l)|} \int_\mathbb{R} \int_{G_1/\exp(\mathbb{R}Y) - c_1} \int_\mathbb{R} \int_{G_1/\exp(\mathbb{R}Y) - c_1} \tilde{h}(\alpha, s - r - t, (\beta, 0, l|_{c_1}), w_1^{r-t}) \cdot \text{exp}(c(\beta, l)^{-1} \tilde{Y}Y) \cdot e^{i(r-t)c(\beta, l)y}(\tilde{\pi}(\beta, 0, l|_{c_1}, l_1)(w_1) \tilde{\xi}(\beta, 0, l|_{c_1}))(r)(\tilde{g}_1) dtd\tilde{g}_1 dr
\]

\[
\text{(with } \tilde{Y} = c(\beta, l)y) \]

\[
= \frac{1}{|c(\beta, l)|} \int_\mathbb{R} \int_{G_1/\exp(\mathbb{R}Y) - c_1} \int_\mathbb{R} \int_{G_1/\exp(\mathbb{R}Y) - c_1} \int_\mathbb{R} \int_{G_1/\exp(\mathbb{R}Y) - c_1} \int_{\mathbb{R}} \int_{\mathbb{R}} \tilde{h}(\alpha, s - r - t, (\beta, 0, l|_{c_1}), w_1^{r-t}) \cdot \text{exp}(c(\beta, l)^{-1} \tilde{Y}Y) \cdot \text{exp}(y'Y) \cdot \text{Z} \cdot e^{-ilZ} e^{i\tilde{Y}Y} e^{-i\tilde{Y}Y}
\]

\[
(\tilde{\pi}(\beta, 0, l|_{c_1}, l_1)(w_1) \tilde{\xi}(\beta, 0, l|_{c_1}))(r)(\tilde{g}_1) dZ dy'dt d\tilde{Y} d\tilde{Y} dr
\]

\[
= \frac{1}{|c(\beta, l)|} \int_\mathbb{R} \int_{G_1/\exp(\mathbb{R}Y) - c_1} \int_\mathbb{R} \int_{G_1/\exp(\mathbb{R}Y) - c_1} \int_\mathbb{R} \int_{G_1/\exp(\mathbb{R}Y) - c_1} \int_{\mathbb{R}} \int_{\mathbb{R}} \tilde{h}(\alpha, s - r - t, (\beta, 0, l|_{c_1}), w_1^{r-t}) \cdot \text{exp}(y''Y) \cdot \text{Z}
\]

\[
\cdot e^{-ilZ} e^{i\tilde{Y}Y} e^{-i\tilde{Y}Y}(\tilde{\pi}(\beta_1, l_1)(w_1) \tilde{\xi}(\beta, 0, l|_{c_1}))(r)(\tilde{g}_1) dZ dy'' dt d\tilde{Y} d\tilde{Y} dr
\]

\[
\text{(for } y'' = y' + c(\beta, l)^{-1} \tilde{Y}) \]

\[
= \frac{1}{2\pi} \frac{1}{|c(\beta, l)|} \int_\mathbb{R} \int_{G_1/\exp(\mathbb{R}Y) - c_1} \int_\mathbb{R} \int_{G_1/\exp(\mathbb{R}Y) - c_1} \int_{\mathbb{R}} \int_{\mathbb{R}} \tilde{h}(\alpha, s - r - t, (\beta, 0, l|_{c_1}), w_1^{r-t})
\]

\[
\cdot \text{exp}(y''Y) \cdot \text{Z}
\]

\[
\cdot e^{-ilZ} e^{i\tilde{Y}Y} e^{-i\tilde{Y}Y}(\tilde{\pi}(\beta_1, l_1)(w_1) \tilde{\xi}(\beta, 0, l|_{c_1}))(r)(\tilde{g}_1) dZ dy'' dt d\tilde{Y} d\tilde{Y} dr
\]
the topology of our function spaces. So the retract map 
\[ (\beta, H) \rightarrow (\xi, \beta, 0, l|_{c_1}) \]
Hence for every 
\[ l \in L \]
where 
\[ G \]
Let 
\[ l \]
be a Lie group of automorphisms of a connected, simply connected, nilpotent Lie group \( G = \exp(g) \) containing the inner automorphisms of \( G \). For instance take any simply connected Lie group \( G \) and let \( G \) be the nilradical of \( G \).

Let \( l_0 \in g^* \) be fixed, we consider the orbit \( \Omega = \Omega_{l_0} := G \cdot l_0 \) in \( g^* \), let \( O = O_{l_0} \) be the \( G \)-orbit of \( l_0 \). We assume that \( \Omega \) is locally closed in \( g^* \). In particular we can write 
\[ \Omega = \overline{\Omega} \cap U, \]
where \( \overline{\Omega} \) denotes the closure of \( \Omega \) in \( g^* \) and \( U \) is a \( G \)-invariant open subset of \( g^* \). It is then a \( G \)-invariant smooth sub-manifold of \( g^* \) diffeomorphic to the manifold \( G/G_{l_0} \),

Let us finish the computation for \((\beta, l) \in M\). It suffices to take \((\beta, l) \in M \). Then 
\[ ((\beta, 0, l|_{c_1}), l_1) \in M_{1}^{0} \] and by the induction hypothesis,
\[ \left( \pi((\beta, l))(f(\alpha, \beta, \cdot))\xi(\beta) \right)(g) \]
= \[ \left( \pi((\beta, l))(f(\alpha, \beta, \cdot))\xi(\beta) \right)(\exp(sX) \cdot g_1) \]
\[ = \left( \frac{1}{2\pi} \right)^{n-j_1+1} \cdot \frac{1}{|c(\beta, l)|} \int_{R} \int_{G_{1}/P(\beta, l)} F(\alpha, s-t, t, (\beta, 0, l|_{c_1}), l_1, g_1, g_1) \]
\[ \tilde{\xi}(\beta, 0, l|_{c_1})(t)(g_1)d\tilde{g}_1dt \]
= \[ \int_{R} \int_{G_{1}/P(\beta, l)} F(\alpha, (\beta, l), \exp(sX) \cdot \tilde{g}_1, \exp(tX) \cdot g_1)\xi(\beta)(\exp(tX) \cdot g_1)dg_1dt \]
= \[ \int_{G_{1}/P(\beta, l)} F(\alpha, (\beta, l), \tilde{g}, g)\xi(\beta)(g)dg. \]

Hence for every \((\beta, l) \in M\), we have the required result.

The algorithm used to build the retract function \( f \) respects the semi-norms defining the topology of our function spaces. So the retract map \( F \mapsto f \) is continuous.

5 G-prime ideals in \( L^1(G) \)

In this section, we will study the structure of the \( A \)-prime ideals in \( L^1(G) \) by using the Retract Theorem.

5.1 A Retract Defined on Closed Orbits

Let \( G \) be a Lie group of automorphisms of a connected, simply connected, nilpotent Lie group \( G = \exp(g) \) containing the inner automorphisms of \( G \). For instance take any simply connected Lie group \( G \) and let \( G \) be the nilradical of \( G \).

Let \( l_0 \in g^* \) be fixed, we consider the orbit \( \Omega = \Omega_{l_0} := G \cdot l_0 \) in \( g^* \), let \( O = O_{l_0} \) be the \( G \)-orbit of \( l_0 \). We assume that \( \Omega \) is locally closed in \( g^* \). In particular we can write 
\[ \Omega = \overline{\Omega} \cap U, \]
where \( \overline{\Omega} \) denotes the closure of \( \Omega \) in \( g^* \) and \( U \) is a \( G \)-invariant open subset of \( g^* \). It is then a \( G \)-invariant smooth sub-manifold of \( g^* \) diffeomorphic to the manifold \( G/G_{l_0} \),
where $G_{l_0}$ denotes the stabiliser $G_{l_0} := \{ \alpha \in G; \alpha \cdot l_0 = l_0 \}$. The $G$-orbit $G \cdot (G \cdot l_0)$ in the orbit space $g^* / G$ is then locally closed and homeomorphic to the quotient $G / G_{O}$, where $G_{O}$ is the stabiliser of the set $O$ in $G$. In fact, we have that $G_{O} = G \cdot G_{l_0}$.

For a Jordan-Hölder basis $\mathcal{Z} = \{ Z_1, \ldots, Z_n \}$ of $g$ and $g \in G$, let

$$g \cdot \mathcal{Z} := \{ \text{Ad}(g)Z_1, \ldots, \text{Ad}(g)Z_n \},$$

which is again a Jordan-Hölder basis of $g$. For every index set $I$, we have the following relation (see [9]):

$$\text{Ad}^*(g)g^*_I,\mathcal{Z} = g^*_I,\mathcal{Z}, \quad g \in G.$$  \hspace{1cm} (5.13)

For an index set $I$ and a Jordan-Hölder basis $\mathcal{Z}$ of $g$, recall that

$$s_I := \sum_{i \in I} \mathbb{R}Z^*_i, \quad \Sigma_I,\mathcal{Z} := s_I \cap g^*_I,\mathcal{Z},$$

and the mapping $E_I : \mathbb{R}^d \times \Sigma_I,\mathcal{Z} \to g^*_I,\mathcal{Z}$ is given by

$$E_I(s_I, t_I, \ldots, s_r, t_r; l) := \text{Ad}^*(\exp(s_I Z_{j_1})\exp(t_I Z_{k_1}) \cdots \exp(s_r Z_{j_r})\exp(t_r Z_{k_r})) l.$$

We have that $E_I$ is a bijection and $E_I(\mathbb{R}^d \times \{ l \})$ is the $G$-orbit of $l$. Let

$$\Upsilon : g^*_I,\mathcal{Z} \to \Sigma_I,\mathcal{Z}; \quad \Upsilon(l) := \text{Ad}^*(G)l \cap \Sigma_I,\mathcal{Z} = p_{\Sigma_I,\mathcal{Z}}(E_I^{-1}(l)),$$

where $p_{\Sigma_I,\mathcal{Z}}$ is the projection of $\mathbb{R}^d \times \Sigma_I,\mathcal{Z}$ onto $\Sigma_I,\mathcal{Z}$.

For the orbit $\Omega$, we need to construct a finite partition of unity $(\psi_i)_{i \in \Gamma}$ consisting of smooth $G$-invariant functions $\psi_i : \Omega \to \mathbb{R}^+$ such that for every $i \in \Gamma$ the support of each function $\psi_i$ is contained in an open subset of $g^*_I,\mathcal{Z}$ for some $g_i \in G$. In order to do that let $\varphi : \mathbb{R} \to \mathbb{R}^+$ be a smooth function with compact support and vanish in a neighbourhood of 0. We define a function $\psi : g^*_I \to \mathbb{R}^+$ by

$$\psi(l) := \varphi(P_I(\Upsilon(l))) \quad \text{if} \quad l \in g^*_I \quad \text{and} \quad \psi(l) := 0 \quad \text{if} \quad l \in g^*_I.$$

where $P_I$ is a smooth function on $B \times \mathbb{R}^*$ defined in Sect. 2.4. We see that $\psi$ is smooth (since $\varphi$ vanishes in a neighbourhood of 0) and is $G$-invariant by the construction. Let

$$U_I,\mathcal{Z} := \{ l \in \Omega; \psi(l) \neq 0 \}.$$

Now assume that $g^*_I = g^*_I,\mathcal{Z}$ be the maximal layer with respect to $\mathcal{Z}$ such that $\Omega \cap g^*_I,\mathcal{Z} \neq \emptyset$. We have that $\Omega \cap g^*_I,\mathcal{Z} \neq \emptyset$ but $\Omega \cap g^*_I,\mathcal{Z} \neq \emptyset$ for $g \in G$ and $I' > I$. Moreover, $U_I,\mathcal{Z}$ is a non-empty open subset of $\Omega$ contained in $g^*_I$ and

$$\Omega \subset \bigcup_{g \in G} \text{Ad}^*(g)U_I,\mathcal{Z}.$$
Let $C$ be a compact subset of $g^*$ contained in $\Omega$, then there exists a finite subset $\Gamma \subset G$ such that

$$C \subset \bigcup_{g \in \Gamma} \text{Ad}^*(g)U_{1,Z}.$$ 

Hence there is a finite partition of unity $(\psi_i)$ consisting of smooth $G$-invariant functions $\psi_i : \Omega \to \mathbb{R}_+$ such that the support of each function $\psi_i$ is contained in $\text{Ad}^*(g_i)U_{1,Z} \subset g_i^* g_i^{-1} Z$ for every $g_i \in \Gamma$.

Suppose we have a smooth adapted operator field $F$ on $\Omega$ supported on $G \cdot C$, we can write

$$F = \sum_{i \in \Gamma} \psi_i F.$$ 

According to the Retract Theorem, for each $i \in \Gamma$, there is a (retract) Schwartz function $f_i$ on $G$ such that

$$\pi_l(f_i) = op_{\psi_i F}(l)$$

for every $l \in \Omega$. Let $f := \sum_{i \in \Gamma} f_i$, we have that

$$\pi_l(f) = \sum_{i \in \Gamma} \pi_l(f_i) = \sum_{i \in \Gamma} \psi_i op_F(l) = op_F(l).$$

This is, for every smooth adapted kernel function supported on $G \cdot C$, we build a retract function.

### 5.2 G-Prime Ideals

As an application, we show that every $G$-prime ideal in $L^1(G)$ is the kernel of a $G$-orbit. Let us first recall the definition of $G$-prime ideals.

**Definition 5.2.1** A two-sided closed ideal $I$ in $L^1(G)$ is called $G$-prime, if $I$ is $G$-invariant and if, for all $G$-invariant two-sided ideals $I_1$ and $I_2$ of $L^1(G)$, the following implication holds

$$I_1 \ast I_2 \subset I \Rightarrow I_1 \subset I \text{ or } I_2 \subset I.$$ 

Denote by $Prim^*(G)$ the collection of all the kernels of irreducible unitary representations of $L^1(G)$. For a closed subset $C$ of $Prim^*(G)$, let

$$\ker(C) := \bigcap_{P \subset C} P.$$
For a subset $\mathcal{I}$ of $L^1(G)$, denote by $h(\mathcal{I})$ the subset

$$h(\mathcal{I}) := \{ P \in \text{Prim}^*(G); \mathcal{I} \subset P \}.$$ 

The set $h(\mathcal{I})$ is then closed in $\text{Prim}^*(G)$ with respect to the Fell topology.

We have the following result for $G$-prime ideals of $L^1(G)$ which can be viewed as an application of the Retract Theorem.

**Theorem 5.1** Let $G$ be a simply connected, connected nilpotent Lie group and let $G$ be a Lie group of automorphisms of $G$ containing the inner automorphisms, which acts smoothly on the group $G$, such that every $G$-orbit in $\mathfrak{g}^*$ is locally closed. If $\mathcal{I}$ is a proper $G$-prime ideal of $L^1(G)$, then there exists an $G$-orbit $\Omega_{l_0}$ in $\mathfrak{g}^*$ such that

$$\mathcal{I} = \ker(\Omega_{l_0}).$$

Moreover, the kernel of each $G$-orbit is a $G$-prime ideal.

**Proof** For any $G$-orbit $\Omega$ in $\mathfrak{g}^*$, the Retract Theorem tells us that the Schwartz functions contained in $\ker(\Omega)$ are dense in $\ker(\Omega)$ (see [7, proof of Proposition 4.1] and [4]). From the proof of [8, Theorem 1.2.12], it follows that the hull of a prime ideal $\mathcal{I}$ is the closure of a $G$-orbit in $\text{Prim}^*(G) \simeq \hat{G}$. On the other hand, the density of $S(G) \cap \ker(\Omega)$ implies that $\ker(\Omega)^N$ is contained in the minimal ideal $J(\Omega)$ with hull $\Omega$ for some $N \in \mathbb{N}$. This tells us that $\ker(\Omega)^N \subset J(\Omega) \subset \mathcal{I}$, since the minimal ideal with hull $\Omega$ is contained in every ideal with hull $\Omega$. Since $\mathcal{I}$ is $G$-prime, we have that $\mathcal{I} = \ker(\Omega)$.

Obviously the ideal $\ker(\Omega)$ is $G$-prime for any $G$-orbit $\Omega$ in $\mathfrak{g}^*$. To see this, let $\mathcal{I}_1$ and $\mathcal{I}_2$ be two $G$-invariant ideals of $L^1(G)$ such that $\mathcal{I}_1 \ast \mathcal{I}_2 \subset \ker(\Omega)$. This means that $\mathcal{I}_1 \ast \mathcal{I}_2 \subset \ker(\Omega) \subset \ker(\pi_l)$ for some $l \in \Omega$. We have then either $\mathcal{I}_1$ or $\mathcal{I}_2$ is contained in $\ker(\pi_l)$, since $\pi_l$ is irreducible. But if $\mathcal{I}_1$ is contained in $\ker(\pi_l)$, it is also contained in $\ker(\pi_{k,l})$ since $\mathcal{I}_1$ is $G$-invariant. Hence $\mathcal{I}_1 \subset \ker(\Omega)$ and the proof is thus complete.

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