Solutions of the Two-Wave Interactions in Quadratic Nonlinear Media

Lazhar Bougoffa and Smail Bougouffa

1 Department of Mathematics, College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), P.O. Box 90950, Riyadh 11623, Saudi Arabia
2 Department of Physics, College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), P.O. Box 90950, Riyadh 11623, Saudi Arabia; sbougouffa@imamu.edu.sa

Received: 11 September 2020; Accepted: 20 October 2020; Published: 26 October 2020

Abstract: In this paper, we propose a reliable treatment for studying the two-wave (symbiotic) solitons of interactions in nonlinear quadratic media. We investigate the Schauder’s fixed point theorem for proving the existence theorem. Additionally, the uniqueness solution for this system is proved. Also, a highly accurate approximate solution is presented via an iteration algorithm.

Keywords: two-wave solitons; existence and uniqueness solutions; exact solution; approximate solution

1. Introduction

In the simplest case of type-I second harmonic generation (SHG) without walk-off between harmonic waves soliton evolution is described by the normalized system [1–6]:

\[
\begin{align*}
\frac{i}{\kappa} \frac{\partial \psi}{\partial z} + r \frac{\partial^2 \psi}{\partial x^2} - \psi + \psi^* \phi &= 0, \\
i\sigma \frac{\partial \phi}{\partial z} + s \frac{\partial^2 \phi}{\partial x^2} - \alpha \psi + \frac{1}{2} \phi^2 &= 0,
\end{align*}
\]

where \(\alpha\) is the rescaled soliton parameter and satisfies \(\alpha = \sigma (2\beta + \lambda) / \beta\), the dimensionless parameter \(\beta\) is the normalized nonlinearity-induced shift to the propagation constant of the fundamental harmonic wave, \(\sigma\) and \(\lambda\) are the coefficient and phase mismatch parameter, respectively. This system represents the generic model of \(\chi^2\) solitons. There are other types of normalization also used in the literature see e.g., [7–9]. The solutions of this system have been discussed for one dimensional in [3–5,10–13] and multi-dimensional cases in [14]. Under suitable assumptions, the problem of the two-wave (symbiotic) solitons can be reduced to the solution of the following coupled system [1–4,6,13,15]

\[
\begin{align*}
\frac{r}{\kappa} \frac{\partial^2 \phi}{\partial x^2} - \phi + \phi \psi &= 0, \\
\frac{s}{\sigma^2} \frac{\partial^2 \psi}{\partial x^2} - \alpha \psi + \frac{1}{2} \phi^2 &= 0,
\end{align*}
\]

The properties of solitons described by system (2) are well known see [3–5], where the authors Buryak and Kivshar [3–5] looked for stationary (i.e., z-independent) localized solutions of the normalized system in the form of an asymptotic series in the parameter \(\alpha^{-1}\) and found the real functions \(\phi(x)\) and \(\psi(x)\) in the form of asymptotic series:

\[
\phi(x) = 2a \frac{1}{x} \operatorname{sech}(x) + 4sa^{-\frac{1}{2}} \tanh^2(x) \text{sech}(x) + ..., \tag{3}
\]

\[
\psi(x) = 2 \text{sech}^2(x) + sa^{-1} \left( 16 \text{sech}^2(x) - 20 \text{sech}^4(x) \right) + ..., \tag{4}
\]
for bright solitons at \( r = +1 \), and

\[
\phi(\tau) = \sqrt{2}a^{\frac{1}{2}} \tanh(\tau) \sqrt{2}a^{-\frac{1}{2}} \left( \tau \sech^2(\tau) - \tanh(\tau) \sech^2(\tau) \right) + ..., \tag{5}
\]

\[
\psi(x) = \tanh^2(\tau) + sa^{-\frac{1}{2}} \left( 2r \tanh(\tau) \sech^2(\tau) - 4 \sech^2(\tau) + 5 \sech^4(\tau) \right) + ..., \tag{6}
\]

where \( \tau = \frac{x}{\sqrt{2}} \), for dark solitons at \( r = -1 \).

Exact solutions of system (2) have been found at \( \alpha = 1 \) for \( r = s = 1 \), \( r = s = -1 \) and \( r = -s = -1 \) in [10,13]. Another solution of (2) in the case \( r = 1 \), \( s = -1 \) and \( \alpha = 2 \) is provided in an explicit analytical form [11]. Also, different analytical approximation methods have been proposed to deal with the system (2). For example, an accurate approximate solution with the help of the variational method in quadratic media, where the problem is formulated in the context of two nonlinear coupled differential equations in one dimension. Then, in Section 3, we solve the coupled system by an appropriate technique with suitable boundary conditions. A systematic numerical procedure is proposed in Section 4. Finally, we conclude with some remarks in Section 5.

2. An Existence and Uniqueness Theorem

Rewrite (2) in the following system

\[
\begin{cases}
\frac{d^2 \varphi}{dx^2} = f_1(\varphi, \psi), \\
\frac{d^2 \psi}{dx^2} = f_2(\varphi, \psi),
\end{cases} \tag{8}
\]

where \( f_i : [l_1, l_2] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, i = 1, 2 \) are defined by \( f_1(\varphi, \psi) = \frac{1}{r}(\varphi - \varphi \psi) \) and \( f_2(\varphi, \psi) = \frac{1}{s} (a\varphi - \frac{1}{2} \varphi^2) \).

**Existence** In this section, we shall deal with the existence of solutions of the BVP (2) with (7). First off all, we shall prove the following lemmas, which are useful tools in the proof of the existence and uniqueness theorem.

**Lemma 1.** If we assume that \( \varphi, \psi \in C[l_1, l_2], l_2 > l_1 \). Then, the functions \( f_i(\varphi, \psi), i = 1, 2 \) are Lipschitz continuous functions of \( \varphi \) and \( \psi \).

**Proof.** From the definition of \( f_i(\varphi, \psi) \), we have

\[
| f_1(\varphi_1, \psi_1) - f_1(\varphi_2, \psi_2) | = \frac{1}{|r|} | (\varphi_1 - \varphi_2) + \psi_2(\varphi_2 - \varphi_1) + \varphi_1(\psi_2 - \psi_1) |. \tag{9}
\]
Since \( \varphi_i, \psi_i \in \mathbb{C}[I_i, I_2] \), then there exist \( M_i, M_i^* > 0 \), \( i = 1, 2 \) such that \(| \varphi_i(x) | \leq M_i \) and \(| \psi_i(x) | \leq M^*_i \), \( M_i, M^*_i > 0 \), \( i = 1, 2 \) for all \( x \in [I_1, I_2] \).

\[
| f_1(\varphi_1, \psi_1) - f_1(\varphi_2, \psi_2) | \leq \frac{1}{|r|} \left[ | \varphi_2 - \varphi_1 | + M^*_2 | \varphi_2 - \varphi_1 | + M_1 | \psi_2 - \psi_1 | \right].
\]

(10)

Hence,

\[
| f_1(\varphi_1, \psi_1) - f_1(\varphi_2, \psi_2) | \leq L_{1,1} | \varphi_2 - \varphi_1 | + L_{1,2} | \psi_2 - \psi_1 |.
\]

(11)

Similarly, we obtain

\[
| f_2(\varphi_1, \psi_1) - f_2(\varphi_2, \psi_2) | \leq L_{2,1} | \varphi_2 - \varphi_1 | + L_{2,2} | \psi_2 - \psi_1 |,
\]

(12)

where \( L_{1,1} = \frac{1}{|r|} (1 + M^*_2) \), \( L_{1,2} = \frac{M_1}{|r|} \), \( L_{2,1} = \frac{M_2 + M_*}{2|r|} \) and \( L_{2,2} = \frac{a}{|r|} \).

**Lemma 2.** (See pp. 70–71 [17]). Let \( g : [a, b] \to \mathbb{R} \) be a continuous function. The unique solution \( u \) of the following boundary value problem

\[
u'' = g(x)
\]

(13)

subject to the Dirichlet boundary conditions \( u(a) = u(b) = 0 \)

is given by

\[
u(x) = \int_a^b G(x, y) g(y) dy,
\]

(14)

where \( G(x, y) \) is the Green function given by

\[
G(x, y) = \begin{cases}
\frac{1}{r} (x - a)(b - y), & a \leq x \leq y \leq b, \\
\frac{1}{r} (y - a)(b - x), & a \leq y \leq x \leq b
\end{cases}
\]

(15)

and \( \int_a^b | G(x, y) | dy \leq \frac{(b - a)^2}{8} \).

Replacing \( g(x) \), \( a \) and \( b \) by \( f_1(\varphi, \psi), l_1 \) and \( l_2 \) in Lemma 2, respectively, we obtain an equivalent integral system

\[
\begin{align*}
\varphi(x) &= \int_{l_1}^{l_2} G(x, s) f_1(\varphi(s), \psi(s)) ds, \ x \in [l_1, l_2], \\
\psi(x) &= \int_{l_1}^{l_2} G(x, s) f_2(\varphi(s), \psi(s)) ds, \ x \in [l_1, l_2].
\end{align*}
\]

(16)

Define the Banach space \( X = \mathbb{C}[l_1, l_2] \) with norm \( \|(\varphi, \psi)\| = \|\varphi\| + \|\psi\| \), where \( \|u\| = \max_{1 \leq x \leq 2} | u(x) | \) and the operator \( T : X \to X \) by \( T(\varphi, \psi) = (T_1(\varphi, \psi), T_2(\varphi, \psi)) \), where

\[
T_1(\varphi, \psi) = \int_{l_1}^{l_2} G(x, s) f_1(\varphi(s), \psi(s)) ds
\]

(17)

and

\[
T_2(\varphi, \psi) = \int_{l_1}^{l_2} G(x, s) f_2(\varphi(s), \psi(s)) ds.
\]

(18)

Since \( \varphi(x) \leq M \) and \( \psi(x) \leq M^* \), \( x \in [l_1, l_2] \), consider the closed and convex set

\[
S = \{ (\varphi, \psi) \in X : \|(\varphi, \psi)\| \leq M + M^* \}.
\]

(19)

Furthermore, assume that \( l_2 - l_1 = \sqrt{\frac{8|rs|/(M+M^*)}{|r|(M+M^*) + |r| \alpha M^* + M^*}} \).
In the theory of differential equations, there are a lot of methods to establish the existence of solutions. Theorems concerning the existence and properties of fixed points are known as fixed-point theorems. Such theorems are the most important tools for proving the existence and uniqueness of the solution. The fundamental theorem used in this theory is Schauder’s theorem. In order to make use of this theorem, it is sufficient to prove the following lemma.

**Lemma 3.** For any \((\varphi, \psi) \in \mathcal{S}\), \(T(\varphi, \psi)\) is contained in \(\mathcal{S}\).

**Proof.** It follows by the definition of \(T(\varphi, \psi)\) that

\[
| T_1(\varphi(x), \psi(x)) | \leq \int_{l_1}^{l_2} | G(x, s) | | f_1(\varphi(s), \psi(s)) | \, ds \leq \frac{(l_2 - l_1)^3}{8} (M + MM^*),
\]

\[
| T_2(\varphi(x), \psi(x)) | \leq \int_{l_1}^{l_2} | G(x, s) | | f_2(\varphi(s), \psi(s)) | \, ds \leq \frac{(l_2 - l_1)^3}{8} (aM^* + \frac{M^2}{2}).
\]

Thus

\[
| T(\varphi(x), \psi(x)) | \leq \frac{(l_2 - l_1)^3}{8} \left[ \frac{M + MM^*}{r} + \frac{aM^* + \frac{M^2}{2}}{s} \right].
\]

Since \(l_2 - l_1\) is defined by the above condition, thus \(| T(\varphi(x), \psi(x)) | \leq M + M^*\). On account of the continuity of \(f_1(\varphi(x), \psi(x)), \varphi\) and \(\psi\), it follows that \(T(\varphi(x), \psi(x))\) is continuous. This shows that \(T(\varphi(x), \psi(x))\) is also contained in \(\mathcal{S}\).

In order to prove that \(T(\varphi(x), \psi(x))\) is equicontinuous, it is easy to see from its definition that

\[
| T(\varphi(x), \psi(x)) - T(\varphi'(x), \psi'(x)) | \leq K | x - x' |, \quad \text{for any } x, x' \in [l_1, l_2].
\]

where \(K = \frac{(l_2 - l_1)^2}{8} \left[ \frac{M + MM^*}{r} + \frac{aM^* + \frac{M^2}{2}}{s} \right] \).

Therefore \(T\) is compact by the classical Ascoli lemma, and Schauder’s fixed point theorem yield the fixed point of \(T\). Thus, we have proved:

**Theorem 1.** There exists a continuous solution \((\varphi, \psi)\) which satisfies system Equations (2) and (7) with the condition on \(l_1\) and \(l_2\).

**Uniqueness-** A uniqueness theorem can also be obtained from the Lipschitz continuous in \(\varphi\) and \(\psi\).

**Theorem 2.** If \(\max \{\max (L_{1,1}, L_{2,1}), \max (L_{1,2}, L_{2,2})\} < \frac{8}{(l_2 - l_1)^2}\), then the system Equation (2) with (7) has a unique solution \((\varphi(x), \psi(x))\).

**Proof.** Let \((\varphi_1, \psi_1)\) and \((\varphi_2, \psi_2)\) be two solutions of (2)–(7). Then, for \(x \in [l_1, l_2]\),

\[
| \varphi_2 - \varphi_1 | \leq \frac{(l_2 - l_1)^2}{8} \left[ L_{1,1} \max_{l_1 \leq x \leq l_2} | \varphi_2(x) - \varphi_1(x) | + L_{1,2} \max_{l_1 \leq x \leq l_2} | \psi_2(x) - \psi_1(x) | \right],
\]

\[
| \psi_2 - \psi_1 | \leq \frac{(l_2 - l_1)^2}{8} \left[ L_{2,1} \max_{l_1 \leq x \leq l_2} | \varphi_2(x) - \varphi_1(x) | + L_{2,2} \max_{l_1 \leq x \leq l_2} | \psi_2(x) - \psi_1(x) | \right].
\]

Consequently,

\[
\| \varphi_2 - \varphi_1 \| + \| \psi_2 - \psi_1 \| \leq A \| \varphi_2 - \varphi_1 \| + \| \psi_2 - \psi_1 \|,
\]

where \(A = \frac{(l_2 - l_1)^2}{8} \max \{\max (L_{1,1}, L_{2,1}), \max (L_{1,2}, L_{2,2})\}\). We now apply the condition \(A < 1\) to this inequality, we get \(\| \varphi_2 - \varphi_1 \| = 0\) and \(\| \psi_2 - \psi_1 \| = 0\). Therefore \((\varphi_1, \psi_1) = (\varphi_2, \psi_2)\). \(\square\)
The proof is complete.

3. On the Decoupling of the System (2)

In this section, first of all, we are concerned with the norms estimate for the functions $\varphi$ and $\psi$ when $r = s = 1$.

Lemma 4. Let $\varphi$ and $\psi$ be two functions in $L^2(I)$, where $I = [l_1, l_2]$. Then

$$
\begin{cases}
\|\varphi\|_1 < \sqrt{2}\|\psi\|_1, & \text{if } \alpha < 1, \\
\|\varphi\|_1 = \sqrt{2}\|\psi\|_1, & \text{if } \alpha = 1, \\
\|\varphi\|_1 > \sqrt{2}\|\psi\|_1, & \text{if } \alpha > 1,
\end{cases}
$$

(27)

where $\|\cdot\|_1$ is the norm defined in the Sobolev space $H^1_0(I)$ by

$$
\|u\|_1^2 = \int_{l_1}^{l_2} \left( u^2(x) + u'^2(x) \right) dx, \quad \text{with } u(l_1) = u(l_2) = 0.
$$

(28)

Proof. Multiplying both sides of the first equation of system (2) by $\varphi$ and integrating from $l_1$ to $l_2$, we obtain

$$
\int_{l_1}^{l_2} \varphi''(x)\varphi(x)dx - \int_{l_1}^{l_2} \varphi^2(x)dx + \int_{l_1}^{l_2} \psi(x)\varphi^2(x) = 0.
$$

(29)

From the second equation of system (2), we have

$$
\varphi^2 = -2\frac{d^2\psi}{dx^2} + 2\alpha\psi.
$$

(30)

By substitution into the last term of (29), we obtain

$$
\int_{l_1}^{l_2} \varphi''(x)\varphi(x)dx - \int_{l_1}^{l_2} \varphi^2(x)dx - 2\int_{l_1}^{l_2} \varphi'(x)\psi(x)dx + 2\alpha\int_{l_1}^{l_2} \psi^2(x)dx = 0.
$$

(31)

Integrating by parts and taking into account the given boundary conditions, we obtain

$$
\int_{l_1}^{l_2} \varphi^2(x)dx + \int_{l_1}^{l_2} \varphi'^2(x)dx = 2\alpha\int_{l_1}^{l_2} \psi^2(x)dx + 2\int_{l_1}^{l_2} \varphi'^2(x)dx.
$$

(32)

This gives (27).

Let us now consider the case $\alpha = 1$ [6,10,13] and in view of Lemma 6 if $\varphi(x) = \sqrt{2}\psi(x)$, then it may be shown that the two equations of system (2) can be separated into the following nonlinear equation

$$
\psi''(x) - \psi(x) + \psi^2(x) = 0.
$$

(33)

The exact solution to Equation (33) follows by simply multiplying both sides of Equation (33) by $\psi'$.

$$
\psi'\psi'' - \psi\psi' + \psi^2 = 0,
$$

(34)

which can be written as follows

$$
\frac{1}{2} \frac{d}{dx} (\psi')^2 - \frac{1}{2} \frac{d}{dx} (\psi)^2 + \frac{1}{3} \frac{d}{dx} (\psi)^3 = 0.
$$

(35)

and integrating with respect to $x$, we obtain

$$
(\psi')^2 - \psi^2 + \frac{2}{3}\psi^3 = c_1,
$$

(36)
where $c_1$ is an arbitrary constant of integration. Thus

$$
\psi' = \pm \sqrt{\psi^2 - \frac{2}{3} \psi^3 + c_1}.
$$

(37)

In view of $\psi'(x) = \frac{d\psi}{dx}$, we have

$$
\pm \frac{d\psi}{\sqrt{\psi^2 - \frac{2}{3} \psi^3 + c_1}} = dx.
$$

(38)

Consequently,

$$
\pm \int \frac{d\psi}{\sqrt{\psi^2 - \frac{2}{3} \psi^3 + c_1}} = x + c_2,
$$

(39)

where $c_2$ is also a constant of integration.

The LHS of Equation (39) can be evaluated directly from the integrals of irrational functions. Indeed, if we choose $c_1 = 0$, then

$$
\int \frac{d\psi}{\sqrt{\psi^2 - \frac{2}{3} \psi^3}} = \pm 2 \tanh^{-1} \left( \sqrt{1 - \frac{2}{3} \psi} \right) = x + c_2.
$$

(40)

Since $\tanh^2(\mp x) = \tanh^2(x)$. Hence, a simple computation leads to the implicit solution

$$
(\varphi, \psi) = \left( \frac{3}{2} \left( 1 - \tanh^2 \left( \frac{1}{2} (x + c_2) \right) \right), \frac{3}{\sqrt{2}} \left( 1 - \tanh^2 \left( \frac{1}{2} (x + c_2) \right) \right) \right).
$$

(41)

Thus, we have

**Lemma 5.** The system (2) can be decoupled without increasing the order of the system into the nonlinear Equation (33) when $\alpha = 1$. Furthermore, the solution $(\varphi, \psi)$ is given by (41).

In Figure 1, we display the variation of the exact solutions Equation (41) in terms of the independent variables $x$ for different values of the constant of integration $c_2$ (Equation (39)). It can be seen that this constant of integration shifted left or right the distribution away from the origin with negative or positive values of $c_2$, respectively. Also, it does not affect the behavior of the solutions, and the maximum value of the solution remains unchanged. Thus it can be chosen $c_2 = 0$.

![Figure 1](image-url)
4. Numerical Analysis

An integral system equivalent to Equation (2) with Equation (7) can be derived. Indeed, integrating (2) twice from \( l_1 \) to \( x \) and taking into account the boundary conditions \( \varphi(l_1) = \psi(l_1) = 0 \), we obtain

\[
\begin{align*}
\varphi &= \beta(x - l_1) + \int_{l_1}^{x} f_1(s) dsdy, \\
\psi &= \gamma(x - l_1) + \int_{l_1}^{x} f_2(s) dsdy,
\end{align*}
\]

(42)

where \( \beta = \varphi'(l_1) \) and \( \gamma = \psi'(l_1) \) are unknown constants to be determined from the second boundary conditions \( \varphi(l_2) = \psi(l_2) = 0 \).

We now construct a sequence of approximation of the solution that converges to the solution. The components \((\varphi_n, \psi_n)\) can be elegantly determined by setting the recursion scheme

\[
\begin{align*}
\varphi_0 &= \beta(x - l_1), \quad \psi_0 = \gamma(x - l_1), \\
\varphi_n &= \varphi_0 + \int_{l_1}^{x} f_1(s) \varphi_{n-1}(s) dsdy, \quad n \geq 1, \\
\psi_n &= \psi_0 + \int_{l_1}^{x} f_2(s) \psi_{n-1}(s) dsdy, \quad n \geq 1.
\end{align*}
\]

(43)

Theorem 3. The sequence \((\varphi_n, \psi_n)\) defined by (43) converges uniformly on \( I \) to the unique solution \((\varphi, \psi)\).

Proof. We shall construct an upper bound for \(|(\varphi_{n+1}, \psi_{n+1}) - (\varphi_n, \psi_n)|\) by induction.

\[
\begin{align*}
|\varphi_1 - \varphi_0| &= \int_{l_1}^{x} f_1(s) \psi_0(s) dsdy \leq K_1 \frac{(x-l_1)^2}{2}, \\
|\psi_1 - \psi_0| &= \int_{l_1}^{x} f_2(s) \varphi_0(s) dsdy \leq K_2 \frac{(x-l_1)^2}{2},
\end{align*}
\]

(44)

where \( K_1 = \frac{M_1 + M_2}{|\gamma|} \) and \( K_2 = \frac{M_1 + M_2^2}{|\gamma|} \). Proceeding in the same manner, we obtain by induction

\[
\begin{align*}
|\varphi_{n+1} - \varphi_n| &\leq K_1 \frac{(x-l_1)^{n+2}}{(n+2)!}, \\
|\psi_{n+1} - \psi_n| &\leq K_2 \frac{(x-l_1)^{n+2}}{(n+2)!}.
\end{align*}
\]

(45)

The two series \( \sum_{n=0}^{\infty} K_1 \frac{(x-l_1)^{n+2}}{(n+2)!} \) and \( \sum_{n=0}^{\infty} K_2 \frac{(x-l_1)^{n+2}}{(n+2)!} \) are absolutely convergent series. Moreover, these series dominate the series \( \sum_{n=0}^{\infty} |\varphi_{n+1} - \varphi_n| \) and \( \sum_{n=0}^{\infty} |\psi_{n+1} - \psi_n| \). Hence, by the Weierstrass test, the last two infinite series converge absolutely and uniformly on \( I \). If we consider the \( m - \text{th} \) partial sum of these series, we see that \( \sum_{n=0}^{m} |(\varphi_{n+1} - \varphi_n)| = |\varphi_{m+1} - \varphi_0| \) and \( \sum_{n=0}^{m} |(\psi_{n+1} - \psi_n)| = |\psi_{m+1} - \psi_0| \), that is, \((\varphi_n, \psi_n)\) converges absolutely and uniformly on \( I \). If we now define \((\varphi, \psi) = \lim_{n \to \infty} (\varphi_n, \psi_n)\), then taking the limit as \( n \to \infty \), we obtain

\[
\begin{align*}
\varphi &= \beta(x - l_1) + \int_{l_1}^{x} f_1(s) \psi(s) dsdy, \\
\psi &= \gamma(x - l_1) + \int_{l_1}^{x} f_2(s) \varphi(s) dsdy.
\end{align*}
\]

(46)

It follows that upon differentiation of this system that \((\varphi, \psi)\) is the solution of Eq. (2). Furthermore, it is clear that \( \varphi(l_1) = \psi(l_1) = 0, \ i = 1, 2. \)

In view of (43), the numerical solutions are then given by

\[
\begin{align*}
\varphi &= \beta(x - l_1) + \frac{1}{2} \left[ \frac{\beta(x-l_1)^3}{3!} - \frac{\beta \gamma (x-l_1)^4}{4!} \right] + ..., \\
\psi &= \gamma(x - l_1) + \frac{1}{2} \left[ \frac{\gamma(x-l_1)^3}{3!} - \frac{\beta \gamma (x-l_1)^4}{4!} \right] + ....
\end{align*}
\]

(47)
If we match \((\varphi_1, \psi_1)\) at \(x = l_2\), then we need to solve
\[
\begin{align*}
\beta (l_2 - l_1) + \frac{1}{3} \left[ \beta (l_2 - l_1)^3 \right] - \beta \gamma (l_2 - l_1)^4 &= 0, \\
\gamma (l_2 - l_1) + \frac{1}{3} \left[ \alpha \gamma (l_2 - l_1)^3 \right] - \frac{\beta^2}{2} (l_2 - l_1)^4 &= 0,
\end{align*}
\]
(48)
we obtain \(\beta = \sqrt{2} \frac{r s}{(l_2 - l_1)^3} \) and \(\gamma = 4! r \frac{(l_2 - l_1)^4}{(l_2 - l_1)^3} \).

In Figure 2, we present the variation of the numerical solutions Equation (47) against of the independent variables \(x\) for different values of the rescaled soliton parameter \(\alpha\).

![Figure 2](image)

**Figure 2.** (Color online) The numerical solutions of Equation (47) for different values of the rescaled soliton parameter \(\alpha\). (a) for \(\alpha = 1\), and (b) for \(\alpha = 0.1\). The other parameters are \(r = 1\) and \(s = 1\). The solid red line represents the solution \(\varphi\) while the black dashed is for \(\psi\).

Now if we insert the solutions of Equation (47) in the second member of Equation (43) and performing the integrals, then using the boundary condition for \((\varphi_2, \psi_2)\) at \(x = l_2\), we can obtain the second solutions \(\varphi_2\) and \(\psi_2\). As the mathematical expressions are more cumbersome, we plot the numerical solutions in Figure 3. It is clear from these plots that when we increase the order of the recurrence, the numerical solutions converge rapidly to the exact solutions.

![Figure 3](image)

**Figure 3.** (Color online) The numerical solutions of Equation (47) for different values of the integration constant \(\alpha\). (a) for \(\alpha = 1\), and (b) for \(\alpha = 0.1\). The other parameters are \(r = 1\) and \(s = 1\). The solid red line represents the solution \(\varphi(x)\) while the black dashed is for \(\psi(x)\).
5. Conclusions

This paper is concerned with the treatment of the interaction of two-wave solitons in nonlinear quadratic media. These kinds of problems appear in different applications of nonlinear physics and play a crucial role in the stability problems of solitary waves. The problem is presented within the framework of two coupled nonlinear differential equations, which can be solved numerically with specific boundary conditions. But the generalization to any type of boundary condition constitutes a great challenge.

With concordance to real physical problems, the boundary conditions can be chosen properly. Thus, within this framework, we have proved a theorem of existence and uniqueness for the two-wave solitons in nonlinear quadratic media. Furthermore, we have suggested a useful technique of separation of the coupled system, and we have revealed that the formalism leads to analytic solutions.

Moreover, we have explored an interesting numerical technique, and we have used it to obtain the numerical solutions of the coupled system with suitable boundary conditions. The obtained results are in good agreement with those analytically achieved.

These crucial results open a novel class of investigations, which involve solitary waves with more coupled differential equations and more coupling terms. Some other examples of two or more-coupled solitary waves can be treated with the proposed techniques, and the results will be reported elsewhere.

Author Contributions: Investigation, L.B. and S.B.; Methodology, L.B.; Writing—review and editing, L.B. and S.B.; writing—original draft preparation, L.B. and S.B.; Supervision, S.B. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

References
1. Buryak, A.V.; Akhmediev, N.N. Internal friction between solitons in near-integrable systems. Phys. Rev. E 1994, 50, 31263133.
2. Buryak, A.V.; Akhmediev, N.N. Influence of radiation on soliton dynamics in nonlinear fibre couplers. Opt. Commun. 1994, 110, 287–292.
3. Buryak, A.V.; Kivshar, Y.S. Spatial optical solitons governed by quadratic nonlinearity. Opt. Lett. 1994, 19, 1612–1614. Erratum in 1995, 20, 1080–1080.
4. Buryak, A.V.; Kivshar, Y.S. Solitons due to second harmonic generation. Phys. Lett. A 1995, 197, 407–412.
5. Buryak, A.V.; Kivshar, Y.S. Twin-hole dark solitons. Phys. Rev. A 1995, 51, R41-R44.
6. Buryak, A.V.; Trapani, P.D.; Skryabin, D.V.; Trillo, S. Optical solitons due to quadratic nonlinearities: From basic physics to futuristic applications. Phys. Rep. 2002, 370, 63–235.
7. He, H.; Werner, M.J.; Drummond, P.D. Simultaneous solitary-wave solutions in a nonlinear parametric waveguide. Phys. Rev. E 1996, 54, 896–911.
8. Skryabin, D.V.; Firth, W.J. Generation and stability of optical bullets in quadratic nonlinear media. Opt. Commun. 1998, 148, 79–84.
9. Peschel, T.; Peschel, U.; Lederer, F.; Malomed, B.A. Solitary waves in Bragg gratings with a quadratic nonlinearity. Phys. Rev. E 1997, 55, 4730–4739.
10. Hayata, K.; Koshiba, M. Multidimensional solitons in quadratic nonlinear media. Phys. Rev. Lett. 1993, 71, 3275.
11. Werner, M.J.; Drummond, P.D. Strongly coupled nonlinear parametric solitary waves. Opt. Lett. 1994, 19, 613–615.
12. Torner, L.; Wright, E.M. Soliton excitation and mutual locking of light beams in bulk quadratic nonlinear crystals. J. Opt. Soc. Am. B 1996, 13, 864–875.
13. Sukhorukov, A.A. Approximate solutions and scaling transformations for quadratic solitons. Phys. Rev. E 2000, 61, 4530–4539.
14. Firth, W.J.; Skryabin, D.V. Optical solitons carrying orbital angular momentum. Phys. Rev. Lett. 1997, 79, 2450.
15. Nikolov, N.I.; Neshev, D.; Bang, O.; Królikowski, W.Z. Quadratic solitons as nonlocal solitons. *Phys. Rev. E* **2003**, *68*, 036614.

16. Chen, J.; Ge, J.; Lu, D.; Hu, W. A simple approach to study the boundary-induced trajectory evolution of spatial nonlocal quadratic solitons: Based on the Green's function method. *Appl. Math. Lett.* **2020**, *102*, 106–108.

17. Hunter, J.K.; Nachtergaele, B. *Applied Analysis*; World Scientific Publishing: Singapore, 2001.

**Publisher's Note:** MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.

© 2020 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).