Abstract: One introduces here the natural join $P \oplus Q$ of graded posets $\langle P, \leq_P \rangle$ and $\langle Q, \leq_Q \rangle$ with correspondingly maximal (in $P$) and minimal (in $Q$) sets being identical as expressed by ordinal sum $P \oplus Q$ (see [1]) apart from other definition in [2,3,4,5] and due to that one arrives at a simple proof of the Möbius function formula for cobweb posets.

We also quote the other authors explicit formulas for the zeta matrix and the inverse of zeta matrix for any graded posets with the finite set of minimal elements from [2] and [3,4,5]. These formulas are based on the formulas for cobweb posets and their Hasse diagrams - graphs named KoDAGs, which are interpreted as chains of binary complete (or universal) relations - joined by the natural join operation.

Natural join of two independent sets is therefore the ordinal sum [1] of this trivially ordered posets represented also by directed bi-clique named di-biclique and correspondingly by their Hasse diagrams - graphs named KoDAGs.

Such cobweb posets and equivalently their Hasse diagrams - graphs named KoDAGs - are also encoded by discrete hyper-boxes and the natural join operation of such discrete hyper-boxes is just Cartesian product of them accompanied with projection out of sine qua non - common faces. All graded posets with no mute vertices in their Hasse diagrams [2] (i.e. no vertex has in-degree or out-degree equal zero) are natural join of chain of relations and may be at the same time interpreted an $n$-ary, $n \in N \cup \{\infty\}$ relation. The Whitney numbers and characteristic polynomials explicit formulas for cobweb posets are derived.

Key Words: graded digraphs, cobweb posets, natural join, ordinal sum

AMS Classification Numbers: 06A06 ,05B20, 05C75

This is The Internet Gian-Carlo Rota Polish Seminar article, No 8, Subject 4, 2009-07-13, http://ii.uwb.edu.pl/akk/sem/sem_rota.htm

1 Reference information on ordinal sum and natural join of graded ordered sets and Möbius function formula.

1.1. Ponderables. [2,3,4,5]

We shall here take for granted the notation and the results of [2,3]. In particular $(\Pi, \leq)$ denotes cobweb partial order set (cobweb poset) while $I(\Pi, R)$ denotes its incidence algebra over the ring $R$. Correspondingly $(P, \leq)$ denotes arbitrary
graded poset while $I(P,R)$ denotes its incidence algebra over the ring $R$. $[n] \equiv \{1,2,\ldots,n\}$, for example $[k_F] \equiv \{1,2,\ldots,k_F\}$.

$R$ might be taken to be Boolean algebra $2^{(1)}$, the field $Z_2 = \{0,1\}$, the ring of integers $Z$ or real or complex or $p$-adic fields. The present article is the next one in a series of papers listed in order of appearance and these are: [5],[4],[3],[2].

The abbreviation DAG $\equiv$ Directed Acyclic Graph.

Inspired by Gaussian integers sequence notation $\{n_q\}_{n \geq 0}$ - the authors upside down notation is used throughout this paper i.e. $F_n \equiv n_F$. The Upside Down Notation was used since last century effectively (see [2-22] and [29-48]). Through all the paper $F$ denotes a natural numbers valued sequence $\{n_F\}_{n \geq 0}$ sometimes specified to be Fibonacci or others - if needed. Among many consequences of this is that graded posets (iff their cover relation digraphs $\iff$ Hasse diagrams) are connected and sets of their minimal elements are finite.

**Definition 1**

Let $F = (k_F)_{k=0}^n$ be an arbitrary natural numbers valued sequence, where $n \in N \cup \{0\} \cup \{\infty\}$. We say that the graded poset $P = (\Phi,\leq)$ is **denominated** (encoded=labeled) by $F$ iff $|\Phi_k| = k_F$ for $k = 0,1,\ldots,n$. We shall also use the expression - $F$-graded poset.

**Definition 2**

Let $n \in N \cup \{0\} \cup \{\infty\}$. Let $r,s \in N \cup \{0\}$. Let $\Pi_n$ be the graded partial ordered set (poset) i.e. $\Pi_n = (\Phi_n,\leq) = (\cup_{k=0}^n \Phi_k,\leq)$ and $\langle \Phi_k \rangle_{k=0}^n$ constitutes ordered partition of $\Pi_n$. A graded poset $\Pi_n$ with finite set of minimal elements is called **cobweb poset** iff

$$\forall x,y \in \Phi \text{ i.e. } x \in \Phi_r \text{ and } y \in \Phi_s \, r \neq s \Rightarrow x < y \text{ or } y < x,$$

$\Pi_\infty \equiv \Pi$.

**Note.** By definition of $\Pi_n$ being graded its levels $\Phi_r \in \{\Phi_k\}_{k=0}^n$ are independent sets, $n \in N \cup \{0\} \cup \{\infty\}$.

The Definition 2 is the reason for calling Hasse digraph $D = (\Phi,\leq \cdot)$ of the poset $\Pi = (\Phi,\leq)$ a **KoDAG** as in Professor Kazimierz Kuratowski native language one word **Komplet** means complete ensemble - see more in [2,3] and for the history of this name see: The Internet Gian-Carlo Polish Seminar Subject 1. oDAGs and KoDAGs in Company (Dec. 2008). Examples - see Fig.1 and Fig.3. and consult also [2,3,4,5] and references therein.

**Definition 3**

$$\langle \Phi_k \rightarrow \Phi_n \rangle \equiv \oplus_{n \geq s \geq k} \Phi_s$$

is called the layer of the cobweb poset [2,3,4,5,6.7,8.9,10.11] where $\oplus$ denotes ordinal sum of posets while $\Phi_k$ stay for independent sets (see below) hence

$$\Pi = \equiv \oplus_{n \geq 0} \Phi_s$$

.

**Comment 1.** Colligate and make identifications of graded DAGs with $n$-ary relations digraph representation as in [2,3,4,5]:

2
Figure 1: Display of the natural join of bipartite layers \( \langle \Phi_k \rightarrow \Phi_{k+1} \rangle \) \( F = N \).

\[ \leq = \Phi_0 \times \Phi_1 \times \ldots \times \Phi_n \iff \text{cobweb poset} \iff \text{KoDAG}, \]

for the natural join of di-bicliques and similarly for \( \leq \) being natural join of any sequence binary relations [see Fig. 1, 3]

\[ \leq \leq \Phi_0 \times \Phi_1 \times \ldots \times \Phi_n \iff \text{cobweb poset} \iff \text{KoDAG}. \]

**Warning.** Note that **not for all** \( F \)-graded posets their partial orders may be consequently identified with \( n \)-ary relations, where \( F = (k_F)_{k=1}^n \) while \( n \in N \cup \{\infty\} \). This is possible iff no biadjacency matrices entering the natural join for \( \leq \) has a zero column or a zero row. If a vertex \( m \in \Phi_k \) has not either incoming or outgoing arcs then we shall call it the **mute** node [2]. This naming being adopted we may say now:

**F-graded poset may be identified with n-ary relation as above iff it is F-graded poset with no mute nodes.**

Equivalently - zero columns or rows in bi-adjacency matrices are forbidden. See and compare with figures below.

Figure 2: Display of the natural join \( \oplus \rightarrow \) of bipartite digraphs with one mute node.

### 1.2. Ordinal sum and natural join.

Let us recall that the ordinal sum [linear sum] of two disjoint ordered sets \( P \) and \( Q \), denoted by \( P \oplus Q \), is the union of \( P \) and \( Q \), with \( P \)'s elements ordered as in \( P \) while \( Q \)'s elements are correspondingly ordered as in \( Q \), and for each \( x \in P \) and \( y \in Q \) we put \( x \leq y \). The Hasse diagram of \( P \oplus Q \) we construct placing \( Q \)'s diagram just above \( P \)'s diagram and with an edge between each minimal element of \( Q \) and each maximal element of \( P \).
Figure 3: Display of the layer $(\Phi_1 \rightarrow \Phi_4) = \text{the subposet } \Pi_4 \text{ of the } F = \text{Gaussian integers sequence } (q = 2) \text{ } F\text{-cobweb poset and } \sigma \Pi_4 \text{ subposet of the } \sigma \text{ permuted Gaussian } (q = 2) \text{ } F\text{-cobweb poset.}

Here we propose to add to the standard operations 1, 2, 3 on ordered sets below from [1] (definitions and properties see: [1] Chapter III) i.e. operations:

1. dual $P^*$ of $P$;
2. the disjoint union $P + Q = \text{cardinal sum}$;
3. the ordinal sum $P \oplus Q$.

Namely - we propose to add the following binary operation on \textit{graded} posets:

4. this is \textbf{natural join} [5,4,3,2] : $P \oplus \rightarrow Q$, which here now is expressed by the ordinal sum $P \oplus Q$ below.

5. Let $P = \Pi_1 \oplus \Pi_2$ and $Q = \Pi_2 \oplus \Pi_3$ then we define $\oplus \rightarrow$ via identity

$$P \oplus \rightarrow Q \equiv \Pi_1 \oplus \Pi_2 \oplus \Pi_3$$

6. Therefore the cobweb poset identified till now with the natural join of chain of bipartite complete digraphs ["di-biqliques"] $B_k = \Phi_k \oplus \Phi_{k+1}$ - see Fig.1. - (contact [2,3,4,5] and references therein) is now defined equivalently as the ordinal sum [linear sum] of chain of trivially ordered sets.

Cobweb poset $\Pi$ is a linear sum of trivially ordered sets $\{\Phi_k\}_{k \geq 0}$. Nethertheless, the definition of cobweb poset as natural join of relations binary, ternary, etc.) as in [5,4,3,2] - by no means - provides additionally advantages - visual, based on sight interpretation included. \textit{All graded posets with no mute vertices} in their Hasse diagrams (i.e. no vertex has in-degree or out-degree equal zero - consult Fig.2 and compare with Fig.1) are natural join of chain of relations and may be at the same time interpreted an $n$ -- \textit{ary}, $n \in N \cup \{\infty\}$ relation. As for cobweb posets - these have also discrete hyper-boxes representation [11,2,49,16].

Illustration - see Fig.4, Fig.8, Fig.7, Fig.6.

$\Pi = \oplus_{k \geq 0} \Phi_k$

$\{\Phi_k\}_{k \geq 0}$ are then independent sets of $\Pi$. See Fig.4. and Fig.5.

\textbf{1.3. Colligation of the above with hyper-boxes from [11]}


\[ \langle \Phi_2 \rightarrow \Phi_4 \rangle = \langle \Phi_2 \rightarrow \Phi_3 \rangle \oplus \langle \Phi_3 \rightarrow \Phi_4 \rangle \]

Figure 4: Display of the natural join of bipartite layers \( \langle \Phi_k \rightarrow \Phi_{k+1} \rangle \) \( F = N \), resulting in \( 2 \cdot 3 \cdot 4 \) maximal chains and equivalent hyper-box \( V_{2,4} \) with \( 2 \cdot 3 \cdot 4 \) white circle-dots.

Recall [2]: \( C_{\text{max}}(\Pi_n) \) is the set of all maximal chains of \( \Pi_n \). Recall [2]:

\[ C_{\text{max}}^{k,n} = \{ \text{maximal chains in } \langle \Phi_k \rightarrow \Phi_n \rangle \} \]

Consult Section 3. in [11] in order to view \( C_{\text{max}}(\Pi_n) \) or \( C_{\text{max}}^{k,n} \) as the hyper-box of points.

Namely [11,2] denoting with \( V_{k,n} \) the discrete finite rectangular \( F \)-hyper-box or \( (k, n) - F \)-hyper-box or in everyday parlance just \( (k, n) \)-box

\[ V_{k,n} = [k_F] \times [(k + 1)_F] \times \ldots \times [n_F] \]

we identify (see Figure 7.) the following two just by agreement according to the \( F \)-natural identification:

\[ C_{\text{max}}^{k,n} \equiv V_{k,n} \]

i.e.

\[ C_{\text{max}}^{k,n} = \{ \text{maximal chains in } \langle \Phi_k \rightarrow \Phi_n \rangle \} \equiv V_{k,n}. \]

Illustration - see Fig.8, Fig.4, Fig.7, Fig.6 and for more see [11], [47], [12] and [2] with specific indication on [16].

Important. Accordingly the natural join operation of discrete hyper-boxes - (cobweb posets are encoded by discrete hyper-boxes! [11]) is just Cartesian product of them accompanied with projection out of sine qua non common faces (see Fig.7) which is schematically represented by an sample case below [note - cobweb poset might be defined also as the ordinal sum of its independence sets \( \{ \Phi_k \}_{k \geq 1} \)],

\[ (\Phi_k \times \Phi_{k+1}) \oplus \rightarrow (\Phi_{k+1} \times \Phi_{k+2}) \equiv \Phi_k \times \Phi_{k+1} \times \Phi_{k+2} \]
1.4. Cobweb posets’ Möbius function

In order to find out the Möbius function $\mu$ of the cobweb poset $\Pi$ note the following obvious statements ($|\Phi_k| = k_F$)

Obvious statement: $\mu(x, x) = 1$, $\mu(x, y) = -1$ for $x \prec y$, and $\mu(x, z) = k_F - 1$ for $[x, z] = x \oplus \Phi_k \oplus z$.

Obvious statement: $|\Phi_k| = k_F$ for $[x, z] = x \oplus \Phi_k \oplus \Phi_k + 1 \oplus z$, $x \in \Phi_{k-1}$, $z \in \Phi_{k+1}$

Obvious statement:

$$\mu(x, z) = [k_F - 1].$$

Obvious statement ($|\Phi_k| = k_F$): for $[x, z] = x \oplus \Phi_k \oplus \Phi_k + 1 \oplus z \equiv x \oplus B_k \oplus z$, $x \in \Phi_{k-1}$, $z \in \Phi_{k+2}$
Figure 6: Bipartite layer \( \langle \Phi_3 \rightarrow \Phi_4 \rangle \) with six maximal chains and equivalent hyper-box \( V_{2,3} \) with six white circle-dots

\[
\mu(x, z) = -[k_F - 1]((k + 1)_F - 1].
\]

Illustration (see Fig.9 and Fig.10).

Hence via induction for \( x \in \Phi_r, z \in \Phi_s \) and for \( s > r \), \( x \oplus z = x \oplus \Phi_{r+1} \oplus ... \oplus \Phi_{s-1} \oplus z \)

we have the final obvious statement:

\[
\mu(x, y) = (-1)^{s-r} \prod_{k=r+1}^{s-1} [k_F - 1].
\]

Compare the above formula as derived and discussed in [2] and declared in [46, 24] for Fibonacci sequence. Compare also with [25-28].

Naturally the values of \( \mu(x, y) \) depend only on the rank of its arguments - here \( r(x) = r \) and \( r(y) = s \) - which is the reason of coding matrix existence for Möbius function in a natural labelling representation (see: examples in Section 4 and Definition 7). The rank function is here defined as follows: \( r(x) = r \) if \( x \in \Phi_r \).

2 Combinatorial interpretation.

For combinatorial interpretation of cobweb posets via their cover relation digraphs (Hasse diagrams) called KoDAGs see [7,6,2,3,5]. The recent equivalent formulation of this combinatorial interpretation is to be found in [6] (Feb 2009) or [8] from which we quote it here down.

Definition 4 F-nomial coefficients are defined as follows

\[
\binom{n}{k}_F = \frac{n_F!}{k_F!(n-k)!_F} = \frac{n_F \cdot (n-1)_F \cdot ... \cdot (n-k+1)_F}{1_F \cdot 2_F \cdot ... \cdot k_F} = \frac{n_F^k}{k_F!}
\]

while \( n, k \in \mathbb{N} \) and \( 0_F! = 0 \) with \( n_F^k \equiv \frac{n_F^k}{k_F!} \) staying for falling factorial. \( F \) is called F-graded poset admissible sequence iff \( \binom{n}{k}_F \in \mathbb{N} \cup \{0\} \) (In particular we shall use the expression - F-cobweb admissible sequence).
Definition 5

\[ C_{\text{max}}(\Pi_n) \equiv \{ c = \langle x_0, x_1, \ldots, x_n \rangle, x_s \in \Phi_s, s = 0, \ldots, n \} \]

i.e. \( C_{\text{max}}(\Pi_n) \) is the set of all maximal chains of \( \Pi_n \)

and consequently (see Section 2 in [11] on Cobweb posets’ coding via \( N^{\infty} \) lattice boxes)

Definition 6 \( (C_{\text{max}}^{k,n}) \) Let

\[ C_{\text{max}}(\Phi_k \rightarrow \Phi_n) \equiv \{ c = \langle x_k, x_{k+1}, \ldots, x_n \rangle, x_s \in \Phi_s, s = k, \ldots, n \} \equiv \]

\[ \equiv \{ \text{maximal chains in } (\Phi_k \rightarrow \Phi_n) \} \equiv C_{\text{max}}((\Phi_k \rightarrow \Phi_n)) \equiv C_{\text{max}}^{k,n}. \]

Note. The \( C_{\text{max}}((\Phi_k \rightarrow \Phi_n)) \equiv C_{\text{max}}^{k,n} \) is the hyper-box points’ set [11] of Hasse sub-diagram corresponding maximal chains and it defines biunivocously the layer \( (\Phi_k \rightarrow \Phi_n) = \bigcup_{s=k}^n \Phi_s \) as the set of maximal chains’ nodes (and vice versa) - for these arbitrary \( F \)-denominated graded DAGs (KoDAGs included).
The equivalent to that of [7,6,8] formulation of the fractals reminiscent combinatorial interpretation of cobweb posets via their cover relation digraphs (Hasse diagrams) is the following.

**Theorem 1** [10,8,7,2] (Kwaśniewski) For F-cobweb admissible sequences F-nominal coefficient \( \binom{n}{k}_F \) is the cardinality of the family of equipotent to \( C_{\text{max}}(P_m) \) mutually disjoint maximal chains sets, all together partitioning the set of maximal chains \( C_{\text{max}}(\Phi_{k+1} \rightarrow \Phi_n) \) of the layer \( \langle \Phi_{k+1} \rightarrow \Phi_n \rangle \), where \( m = n - k \).

For environment needed and then simple combinatorial proof see [7,8,3,4,5] easily accessible via Arxiv.

**Comment 2.** For the above Kwaśniewski combinatorial interpretation of F-nominals' array it does not matter of course whether the diagram is being directed or not, as this combinatorial interpretation is equally valid for partitions of the family of \( \text{SimplePath}_{\text{max}}(\Phi_k - \Phi_n) \) in comparability graph of the Hasse digraph with self-explanatory notation used on the way. The other insight into this irrelevance for combinatoric interpretation is [9]: colligate the coding of \( C_{\text{max}}^{k,n} \) by hyper-boxes. (More on that soon). And to this end recall what really also matters here: a poset is graded if and only if every connected component of its comparability graph is graded. We are concerned here with connected graded graphs and digraphs.

For the relevant recent developments see [9] while [10] is their all source paper as well as those reporting on the broader affiliated research (see [11-22,24-28,46-48] and references therein). The inspiration for "philosophy" of notation in mathematics as that in Knuth's from [23] - in the case of "upside-downs" has been driven by Gauss' "q-Natural numbers" \( N_q = \{ n_q = q^0 + q^1 + ... + q^{n-1} \} \) where \( n \geq 0 \).
from finite geometries of linear subspaces lattices over Galois fields. As for the earlier use and origins of the use of this author’s upside down notation see [29-45].

In discrete hyper-boxes language the combinatorial interpretation reads:

**Theorem 2** (Kwaśniewski) For $F$-cobweb admissible sequences $F$-nomial coefficient $\binom{n}{k}_F$ is the cardinality of the family of equipotent to $V_{0,m}$ mutually disjoint discrete hyper-boxes, all together partitioning the discrete hyper-box $V_{k+1,n} \equiv$ to the layer $\langle \Phi_{k+1} \rightarrow \Phi_n \rangle$, where $m = n - k$.

**Comment 3.** Recall: all graded posets with no mute vertices in their Hasse diagrams [2] (i.e. no vertex has in-degree or out-degree equal zero) are natural join of chain of relations and may be at the same time interpreted an $n$-ary, $n \in N \cup \{\infty\}$ relation.

Colligate any binary relation $R$ with Hasse digraph cover relation $\prec \cdot$ and identify as in [3,2] $\zeta(R) \equiv R^*$ with incidence algebra zeta function and with zeta matrix of the poset associated to its Hasse digraph, where the reflexive reachability relation $\zeta(R) \equiv R^*$ is defined as

$$R^* = R^0 \cup R^1 \cup R^2 \cup \ldots \cup R^n \cup \ldots \bigcup_{k>0} R^k = R^\infty \cup I_A =$$

$$\equiv \text{transitive and reflexive closure of } R$$
Figure 10: M"obius 2 level function for \( [x, z] = x \oplus \Phi_k \oplus \Phi_{k+1} \oplus z \), \( x \in \Phi_{k-1} \), \( z \in \Phi_{k+2} \).

\[
\mu(x,z) = -[k_F - 1][k_F - 1]
\]

where \( k_F = |\Phi_k| \)

\[
\Leftrightarrow A(R^\infty) = A(R)^{\otimes 0} \vee A(R)^{\otimes 1} \vee A(R)^{\otimes 2} \vee \ldots \vee A(R)^{\otimes n} \vee \ldots,
\]

where \( A(R) \) is the Boolean adjacency matrix of the relation \( R \) simple digraph and \( \otimes \) stays for Boolean product.

Then colligate and/or recall from [3] the resulting schemes. **Schemes:**

\[
\leq = \prec \cdot \infty = \text{connectivity of } \prec \cdot \\
\leq = \prec \cdot * = \text{reflexive reachability of } \prec \cdot \\
\prec \cdot * = \zeta(\prec \cdot).
\]

**Remark 1.** Obvious. Needed also for the next Section. Compare with the Observation 3. below.

The \( \zeta \) matrix (\( \equiv \) the algebra structure coding element of the incidence algebra \( I(P, \leq) \)) is the characteristic function \( \chi \) of a partial order relation \( \leq \) for any given \( F \)-graded poset including \( F \)-cobweb posets \( \Pi \) :
\[ \zeta = \chi (\leq). \]

The consequent (customary-like notation included) notation of other algebra \( I(P, \leq) \) important elements then - for the any fixed order \( \leq \) - is the following \([3,2,1]\):

\[ \zeta = \chi (\leq) = \zeta_< + \delta, \]

\[ \zeta_< = \chi (<) = \zeta - \delta \equiv \rho, \text{ (reachability = connectivity),} \]

\[ \zeta_<. = \chi (< \cdot) \equiv \kappa, \text{ (cover),} \]

\[ \zeta_{\leq} = \chi (\leq \cdot) = \kappa + \delta \equiv \eta, \text{ (reflexive”’cover”’).} \]

\[ \eta = \kappa + \delta = \begin{bmatrix}
  I_1 & B_1 & \text{zeros} \\
  I_2 & B_2 & \text{zeros} \\
  I_3 & B_3 & \text{zeros} \\
  \vdots & \vdots & \vdots \\
  I_n & B_n & \text{zeros}
\end{bmatrix} \]

\[ \eta^{-1} = (\delta + \kappa)^{-1} = \sum_{k \geq 0} (-\kappa)^k = \begin{bmatrix}
  I_1 & -B_1 & B_1B_2 & \cdots \\
  I_2 & -B_2 & \cdots \\
  I_3 & -B_3 & \cdots \\
  \vdots & \vdots & \vdots \\
  I_n & -B_n & \cdots
\end{bmatrix} \]

Recall from [3] : \( B(A) \) is the biadjacency i.e cover relation \( \prec \cdot \) matrix of the adjacency matrix \( A \).

**Note:** biadjacency and cover relation \( \prec \cdot \) matrix for bipartite digraphs coincide.

By extension - we call **cover relation** \( \prec \cdot \) matrix \( \kappa \) the bi-adjacency matrix too in order to keep reminiscent convocations going on.

As a consequence - quoting [3] - we have:

\[ B(\oplus_{i=1}^n G_i) \equiv B[\oplus_{i=1}^n A(G_i)] = \oplus_{i=1}^n B[A(G_i)] \equiv \text{diag}(B_1, B_2, \ldots, B_n) = \begin{bmatrix}
  B_1 \\
  B_2 \\
  \vdots \\
  B_3 \\
  \vdots \\
  B_n
\end{bmatrix}, \]

or equivalently

\[ \kappa \equiv \chi (\oplus_{i=1}^n \prec \cdot_i) \equiv \begin{bmatrix}
  0 & B_1 & 0 & \cdots \\
  0 & 0 & B_2 & \cdots \\
  \vdots & \vdots & \vdots & \ddots \\
  0 & \cdots & 0 & B_3 \\
  0 & \cdots & \cdots & 0 & B_n
\end{bmatrix}, \]
\( n \in N \cup \{ \infty \} \)

In view of the above the following is obvious:

\((A \oplus B)^{-1} \neq A^{-1} \oplus B^{-1}\)

except for the trivial case.

Anticipating considerations of Section III and customarily allowing for the identifications: \(\chi \langle \prec \cdot \rangle \equiv \prec \cdot \equiv \kappa\) - consider [Max] \(\in I(P,R)\):

\[ [\text{Max}] = (I^{-1} \cdot \prec \cdot)^{-1} = \prec \cdot^{-1} + \prec \cdot^{-2} + ... + \prec \cdot^k + ... = \sum_{k \geq 0} \kappa^k \]

in order to note that \((x \in \Phi_t \equiv x = x_t \in \Phi_t)\)

\[ [\text{Max}]_{s,t} = \text{the number of all maximal chains in the poset interval } [x_s, x_t] \equiv [s, t] \]

where \(x_s \in \Phi_s\) and \(x_t \in \Phi_t\) for, say, \(s \leq t\) with the reflexivity (loop) convention adopted i.e. \([\text{Max}]_{t,t} = 1\).

**Sub-Remark 1.1.** It is now a good - prepared for - place to note further relevant properties of constructs as to be used in the sequel. These are the following.

\[ C_{\max} (\langle \Phi_r \to \Phi_k \rangle \oplus \to \langle \Phi_k \to \Phi_s \rangle) = C_{\max} (\Phi_r \to \Phi_s) , \]

for \(r \leq k \leq s\) while \(|\Phi_n| \equiv n_F\).

Let \(C_{\max}^{k,n} \equiv C^{k,n}\). Then for \(F\)-cobweb posets (what about just \(F\)-graded?) we note that

\[ C^{r,k}C^{k,s} = k_FC^{r,s} , \]

hence

\[ C^{r,k}C^{k,s} = C^{r,s} \text{ iff } k_F = 1 \]

for \(r \leq k \leq s\) while

\[ C^{r,k}C^{k+1,s} = C^{r,s} \]

for \(r \leq k < s\) while \(|\Phi_n| \equiv n_F\). Let us now see in more detail how this kind (Q.M.? of mimics of Markov property) is intrinsic for natural joins of digraphs. For that to do consider levels i.e. independent (stable) sets \(\Phi_k = \{x_{k,i}\}_{i=1}^{k_F}\) and extend the notation accordingly so as to encompass

\(\langle \Phi_r \to x_{k,i} \rangle = \{c = x_r, x_{r+1}, ..., x_{k-1}, x_{k,i}, x_s \in \Phi_s, s = r, ..., k - 1\}\).

Let

\[ |C_{\max} (\Phi_r \to x_{k,i})| \equiv C^{r,k,i} . \]

Then

\[ \sum_{i=1}^{k_F} C^{r,k,i}C^{s,k,i} = C^{r,s} \]
for \( r \leq k < s \), ... (for \( r \leq k \leq s \) ?). In the case of cobweb posets (what about just \( F \)-graded?) the numbers \( C^{s,k,i} \) are the same for each \( i = 1, \ldots, k_F \) therefore we have for cobwebs

\[
k_F C^{r,k,i} C^{s,k,i} = C^{r,s}
\]

which in view of \( k_F C^{r,k,i} = C^{r,k} \) is of course consistent with \( C^{r,k} C^{k,s} = k_F C^{r,s} \).

We consequently notice that - with self-evident extension of notation:

\[
\langle \Phi_k \to \Phi_n \rangle = \bigcup_{i,j=1}^{k_F,n_F} \langle x_{k,i} \to x_{n,j} \rangle.
\]

The frequently used block matrices are: 1) \( I(s \times k) \) which denotes \((s \times k)\) matrix of ones i.e. \([I(s \times k)]_{ij} = 1; 1 \leq i \leq s, 1 \leq j \leq k \). and \( n \in N \cup \{\infty\} \), 2) and \( B(s \times k) \) which stays for \((s \times k)\) matrix of ones and zeros accordingly to the \( F \)-graded poset has been fixed - see Observation 2.

In the block matrices language the above Markov property for cobweb posets (what about just \( F \)-graded?) reads as follows (to be used in Section 3) for example:

\[
I (r_F \times (r+1)_F) I ((r+1)_F \times (r+2)_F) = (r+1)_F I (r_F \times (r+2)_F).
\]

Well, what about then just \( F \)-graded? - See Comment 3 and its Warning.

3 Zeta and Möbius functions formulas and Matrices in natural labeling with examples

Examples of \( \zeta(\leq) \in I(\Pi, Z) \)

Let \( F \) denotes arbitrary natural numbers valued sequence. Let \( A_N \) be the Hasse matrix i.e. adjacency matrix of cover relation \( \prec \) digraph denominated by sequence \( N \) [1]. Then the zeta matrix \( \zeta = (1 - A_N)^{-1} \) for the denominated by \( F = N \) cobweb poset is of the form [3] (see also [17-22,6]):
Example 1. \( \zeta_N \). The incidence matrix for the N-cobweb poset.

Note that the matrix \( \zeta \) representing uniquely its corresponding cobweb poset does exhibits a staircase structure of zeros above the diagonal (see above, see below) which is characteristic to Hasse diagrams of all cobweb posets and for graded posets it is characteristic too.

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Example 2. \( \zeta_F \). The matrix \( \zeta \) for the Fibonacci cobweb poset associated to \( F \)-KoDAG Hasse digraph.

The above remarks are visualized as below [17-22,6]. Namely - apart from \( F \)-label, the another label and simultaneously visual code of cobweb graded poset

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
is its "La scala" descending down there to infinity with picture which looks like that below.

Example.3 La Scala di Fibonacci. The staircase structure of incidence matrix $\zeta_F$ for $F=$Fibonacci sequence

Note. The picture above is drawn for the sequence $F = \langle F_1, F_2, F_3, ..., F_n, ... \rangle$, where $F_k$ are Fibonacci numbers.

Description of the Figure "La Scala di Fibonacci" following [17-22,6]. If one defines (see: [17-22] and for earlier references therein as well as in all [3-10]) the Fibonacci poset $\Pi = \langle P, \leq \rangle$ with help of its incidence matrix $\zeta$ representing $P$ uniquely then one arrives at $\zeta$ with easily recognizable staircase-like structure
- of zeros in the upper part of this upper triangle matrix $\zeta$. This structure is depicted by the Figure "La Scala di Fibonacci" where: empty places mean zero values (under diagonal) and filled with $-$ places mean values one (above the diagonal).

**Advice. Simultaneous perpetual Exercises.** How the all above and coming figures, formulas and expressions change (simplify) in the case of $2^{(1)}$ replacing the ring $Z$ of integers in $I(\Pi, Z)$.

**Comment 4.** The given $F$-denominated staircase zeros structure above the diagonal of zeta matrix $\zeta$ is the unique characteristics of its corresponding $F$-KoDAG Hasse digraphs, where $F$ denotes any natural numbers valued sequence as shown below.

For that to deliver we use the Gaussian coefficients inherited upside down notation i.e. $F_n \equiv n_F$ (see [2-18], [29-32], and the Appendix in [9] extracted from [34]) and recall the Upside Down Notation Principle.

Let us also easier the portraying task putting $n_F = 1$. Then - apart from $F$-label, the another label and simultaneously visual code of cobweb graded poset is its "La scala" descending down there to infinity with picture which looks like that below, where recall the $F = \langle k_F \rangle_{k=0}$ is an arbitrary natural numbers valued sequence finite or infinite as $n \in N \cup \{0\} \cup \{\infty\}$.

Example.4 La scala F-Generale. The assumptive, perspicacious staircase structure of the incidence matrix $\zeta_F$ for any $F$ natural numbers valued sequence

Another special case Example is delivered below.
Example 5 $\zeta_F$. The matrix $\zeta$ for $(0_F = 1_F = 1$ and $n_F = 3$ for $n \geq 2)$ the special sequence $F$ constituting the label sequence denominating cobweb poset associated to $F$-KoDAG Hasse digraph.

**Advice.** Simultaneous perpetual Exercises. How the all above and coming picture Examples, figures, formulas and expressions change (simplify) in the case of $2^{(1)}$ replacing the ring $Z$ of integers in $I(\Pi, Z)$.

**Graded Posets’ $\zeta$ matrix formula.**

Recall now following [3] that any graded poset with the finite set of minimal elements is an $F$-sequence denominated sub-poset of its corresponding cobweb poset. The Observation 2 in SNACK supplies the simple recipe for the biadjacency (reduced adjacency) matrix of Hasse digraph coding any given graded poset with the finite set of minimal elements. The recipe for zeta matrix is then standard. We illustrate this by the [3] source example; the source example as the adjacency matrices i.e. zeta matrices of any given graded poset with the finite set of minimal elements are sub-matrices of their corresponding cobweb posets and as such have the same block matrix structure and differ ”only” by eventual additional zeros in upper triangle matrix part while staying to be of the same cobweb poset block type.

The explicit expression for zeta matrix $\zeta_F$ of cobweb posets via known blocks of zeros and ones for arbitrary natural numbers valued $F$-sequence was given in [3] due to more than mnemonic efficiency of the up-side-down notation being applied (see [1] and references therein). With this notation inspired by Gauss and replacing $k$-natural numbers with ”$k_F$” numbers (Note. The Upside Down Notation Principle has been used in [1]) one gets:

$$
A_F = \begin{bmatrix}
0_{1_F \times 1_F} & I(1_F \times 2_F) & 0_{1_F \times \infty} \\
0_{2_F \times 1_F} & 0_{2_F \times 2_F} & I(2_F \times 3_F) & 0_{2_F \times \infty} \\
0_{3_F \times 1_F} & 0_{3_F \times 2_F} & 0_{3_F \times 3_F} & I(3_F \times 4_F) & 0_{3_F \times \infty} \\
0_{4_F \times 1_F} & 0_{4_F \times 2_F} & 0_{4_F \times 3_F} & 0_{4_F \times 4_F} & I(4_F \times 5_F) & 0_{4_F \times \infty} \\
... & etc & ... & ... & ... & ...
\end{bmatrix}
$$

and

$$
\zeta_F = \exp(\mathbb{C})[A_F] \equiv (1 - A_F)^{-\mathbb{C}} \equiv I_{\infty \times \infty} + A_F + A_F^{\mathbb{C}^2} + ... =
$$
\[
\begin{pmatrix}
I_{1_F \times 1_F} & I(1_F \times \infty) \\
O_{2_F \times 1_F} & I_{2_F \times 2_F} & I(2_F \times \infty) \\
O_{3_F \times 1_F} & O_{3_F \times 2_F} & I_{3_F \times 3_F} & I(3_F \times \infty) \\
O_{4_F \times 1_F} & O_{4_F \times 2_F} & O_{4_F \times 3_F} & I_{4_F \times 4_F} & I(4_F \times \infty)
\end{pmatrix}
\]

where \(I(s \times k)\) stays for \((s \times k)\) matrix of ones i.e. \([I(s \times k)]_{ij} = 1; 1 \leq i \leq s, 1 \leq j \leq k\) and \(n \in N \cup \{\infty\}\).

Particular examples of the above block structure of \(\zeta\) matrix (resulting from \(\zeta\) being a result of natural join operations on the way) are supplied by Examples 1,2,3,4,5 above and Examples 6, 7, 8 represented by Fig.4, Fig.5, Fig.6 below. As a matter of fact - all elements \(\sigma\) of the incidence algebra \(I(P,R)\) including \(\zeta\) i.e. characteristic function of the partial order \(\leq\) or Möbius function \(\mu = \zeta^{-1}\) (as exemplified with Examples 9,10,11,12 below) have the same block structure encoded by \(F\) sequence chosen. Recall that \(R\) from \(I(P,R)\) denotes commutative ring and for example \(R\) might be taken to be Boolean algebra \(2^{(1)}\), the field \(Z_2 = \{0,1\}\) the ring \(Z\) of integers or real or complex or \(p\)-adic numbers.

Namely, arbitrary \(\sigma \in I(P,R)\) is of the form

\[
\sigma = \begin{pmatrix}
D_{1_F \times 1_F} & M(1_F \times \infty) \\
O_{2_F \times 1_F} & D_{2_F \times 2_F} & M(2_F \times \infty) \\
O_{3_F \times 1_F} & O_{3_F \times 2_F} & D_{3_F \times 3_F} & M(3_F \times \infty) \\
O_{4_F \times 1_F} & O_{4_F \times 2_F} & O_{4_F \times 3_F} & D_{4_F \times 4_F} & M(4_F \times \infty)
\end{pmatrix}
\]

where \(D_{k_F \times k_F}\) denotes diagonal \(k_F \times k_F\) matrix while \(M(n_F \times \infty)\) stays for arbitrary \(n_F \times \infty\) matrix and both with matrix elements from the ring \(R=2^{(1)}\), \(Z_2 = \{0,1\}, Z\) etc.

In more detail: it is trivial to note that all elements \(\sigma \in I(P,R)\) - including \(\zeta^{-1}\) for which \(D_{k_F \times k_F} = I_{k_F \times k_F}\) - are of matrix block form resulting from \(\oplus\) of the subsequent bipartite digraphs \(\langle \Phi_k \cup \Phi_{k+1}, R \rangle\), \(R \subseteq \Phi_k \times \Phi_{k+1}\), \(|\Phi_k| = k_F\) i.e.

\[
\sigma = \begin{pmatrix}
D_{1_F \times 1_F} & M(1_F \times 2_F) & M(1_F \times 3_F) & M(1_F \times 4_F) & M(1_F \times 5_F) & M(1_F \times 6_F) \\
0_{1_F \times 1_F} & D_{2_F \times 2_F} & M(2_F \times 3_F) & M(2_F \times 4_F) & M(2_F \times 5_F) & M(2_F \times 6_F) \\
0_{1_F \times 1_F} & 0_{3_F \times 2_F} & D_{3_F \times 3_F} & M(3_F \times 4_F) & M(3_F \times 5_F) & M(3_F \times 6_F) \\
0_{4_F \times 1_F} & 0_{4_F \times 2_F} & 0_{4_F \times 3_F} & D_{4_F \times 4_F} & M(4_F \times 5_F) & M(4_F \times 6_F)
\end{pmatrix}
\]

where \(M(k_F \times (k+1)_F)\) denote corresponding \(k_F \times (k+1)_F\) matrices with matrix elements from the ring \(R=2^{(1)}\), \(Z_2 = \{0,1\}, Z\) etc. However... for some seemingly most useful of them ...

**The New Name:** \(\oplus\)\(\rightarrow\)natural \(\Leftrightarrow M(k_F \times (k+1)_F)_{r,s} = c_{r,s}B(k_F \times (k+1)_F)_{r,s}\).

In the case of \(\zeta\) or August Ferdinand Möbius matrices motivating examples of specifically natural elements \(\sigma \in I(P,R)\) (i.e. \(\oplus\)\(\rightarrow\)natural including those obtained via the ruling formula) - so in the case of such type elements \(\sigma \in I(P,R)\) we ascertain and may prove via just see it - that
\[ M(k_F \times (k + 1)_F)_{r,s} = c_{r,s} B(k_F \times (k + 1)_F)_{r,s}, \]

where the rectangular "zero-one" \( B(k_F \times (k + 1)_F) \) matrices from Observation 2. are obtained from the \( F \)-cobweb poset matrices \( I(k_F \times (k + 1)_F) \) by replacing some ones by zeros. Moreover (see Observation 3) - in the case of Möbius \( \mu = \zeta^{-1} \) matrix as it is obligatory \( c_{r,r+1} = -1 \).

The motivating example of \( \oplus \rightarrow \text{natural} \) element of the incidence algebra is \( \zeta_F \) due to the algorithm of the ruling formula considered over the \( R = 2^{(1)} \) ring in this particular case element:

\[ \zeta_F = I_{\infty \times \infty} + A_F + A_F^{(2)} + ... = (1 - A_F)^{-1} \]

where

\[
A_F = \begin{bmatrix}
0_{1_F \times 1_F} & B(1_F \times 2_F) & 0_{1_F \times \infty} \\
0_{2_F \times 1_F} & 0_{2_F \times 2_F} & B(2_F \times 3_F) & 0_{2_F \times \infty} \\
0_{3_F \times 1_F} & 0_{3_F \times 2_F} & 0_{3_F \times 3_F} & B(3_F \times 4_F) & 0_{3_F \times \infty} \\
0_{4_F \times 1_F} & 0_{4_F \times 2_F} & 0_{4_F \times 3_F} & 0_{4_F \times 4_F} & B(4_F \times 5_F) & 0_{4_F \times \infty} \\
... & etc & ... & and so on & ... \\
\end{bmatrix}
\]

and where \( B(k_F \times (k + 1)_F) \) are introduced by the Observation 2.

For the other example of \( \oplus \rightarrow \text{natural} \) element is \( [\text{Max}] \) given by the algorithm of the ruling formula over the \( R = Z \) ring see further on in below.

For the sake of the forthcoming Observation 1 we introduce the set of corresponding Hasse diagram maximal chains called the layer of the graded DAG called KoDAG to be just this \([8,6,5,4]:\)

\[ \langle \Phi_k \rightarrow \Phi_n \rangle = \{ c =< x_k, x_{k+1}, ..., x_n >, x_s \in \Phi_s, s = k, ..., n \} . \]

**Observation 1 (see [3] - and consult the Remark 1).** Let us denote by \( \langle \Phi_k \rightarrow \Phi_{k+1} \rangle \) the di-bicliques denominated by subsequent levels \( \Phi_k, \Phi_{k+1} \) of the graded \( F \)-poset \( P(D) = (\Phi, \leq) \) i.e. levels \( \Phi_k, \Phi_{k+1} \) of its cover relation graded digraph \( D = (\Phi, \prec) \) [Hasse diagram]. Then

\[ B(\oplus_{k=1}^{n} \langle \Phi_k \rightarrow \Phi_{k+1} \rangle) = \text{diag}(I_1, I_2, ..., I_n) = \]

\[
\begin{bmatrix}
I(1_F \times 2_F) \\
I(2_F \times 3_F) \\
I(3_F \times 4_F) \\
... \\
I(n_F \times (n+1)_F)
\end{bmatrix}
\]

where \( I_k \equiv I(k_F \times (k + 1)_F), k = 1, ..., n \) and where - recall - \( I(s \times k) \) stays for \( (s \times k) \) matrix of ones i.e. \( [I(s \times k)]_{ij} = 1; 1 \leq i \leq s, 1 \leq j \leq k. \) and \( n \in N \cup \{ \infty \}. \)
The binary natural join operation $\oplus \to$ being defined for such pairs of arguments (matrices, digraphs, graphs, relations of varying arity,..) which do satisfy the natural join condition (see [3] and [5,4,2]) is associative of course iff performable and obviously $\oplus \to$ is noncommutative.

The recipe for any connected - hence $F$-denominated - the recipe for any given graded poset with a finite minimal elements set is supplied via the following observation.

**Observation 2** (see [3] and consult the Remark 1). Consider bigraphs’ chain obtained from the above di-bicliques’ chain via deleting or no arcs making thus [if deleting arcs] some or all of the di-bicliques (\(\Phi_k \to \Phi_{k+1}\)) not di-bicliques; denote them as \(G_k\). Let \(B_k = B(G_k)\) denotes their biadjacency matrices correspondingly. Then for any such $F$-denominated chain [hence any chain] of bipartite digraphs \(G_k\) the general formula is:

\[
B (\oplus_{i=1}^n G_i) \equiv B(\oplus_{i=1}^n A(G_i)) = \oplus_{i=1}^n B[A(G_i)] \equiv \text{diag}(B_1, B_2, ..., B_n) = \begin{bmatrix} B_1 & B_2 & B_3 & \ldots & B_n \end{bmatrix}
\]

\(n \in N \cup \{\infty\}\).

Let us recall that \(\zeta\) is defined for any poset as follows \((p, q \in P)\):

\[
\zeta(p, q) = \begin{cases} 1 & \text{for } p \leq q, \\ 0 & \text{otherwise.} \end{cases}
\]

This is the reason why in the above **ruling formula**:

\[
\zeta_F = I_{\infty \times \infty} + A_F + A_F^2 + \ldots = (1 - A_F)^{-1}
\]

the Boolean powers are used. If this rule is applied with $Z$-ring or other ring $R, Z \subseteq R$ powers then we get

\[
[\text{Max}]_F = A_F^0 + A_F^1 + A_F^2 + \ldots = (1 - A_F)^{-1} = \begin{bmatrix} I_{1_F \times 1_F} & B(1_F \times 2_F) & B(1_F \times 3_F) & B(1_F \times 4_F) & B(1_F \times 5_F) & \ldots \\ 0_{2_F \times 1_F} & I_{2_F \times 2_F} & B(2_F \times 3_F) & B(2_F \times 4_F) & B(2_F \times 5_F) & \ldots \\ 0_{3_F \times 1_F} & 0_{3_F \times 2_F} & I_{3_F \times 3_F} & B(3_F \times 4_F) & B(3_F \times 5_F) & \ldots \\ 0_{4_F \times 1_F} & 0_{4_F \times 2_F} & 0_{4_F \times 3_F} & I_{4_F \times 4_F} & B(4_F \times 5_F) & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix}
\]

where $B(k_F \times (k + 1)_F)$ are introduced by the Observation 2.

It is a matter of simple observation and induction to see that

\[
B(r_F \times (r + 2)_F) = B(r_F \times (r + 1)_F)B((r + 1)_F \times (r + 2)_F)
\]

and consequently for \(s > r + 2\)

\[
B(r_F \times s_F) = B(r_F \times (r + 1)_F)B((r + 1)_F \times (r + 2)_F) \ldots B((s-2)_F \times (s-1)_F)B((s-1)_F \times s_F).
\]
In the case of $F$-cobweb posets - replace $B(r_F \times s_F)$ by $I(r_F \times s_F)$ and then one may use the "Markov" property.

What about then just $F$-graded posets case? - See Comment 3 and its Warning.

**Remark 2.** $F$-graded poset construction - summary. The knowledge of $\zeta$ matrix explicit form enables one to construct (calculate) via standard algorithms the Möbius matrix $\mu = \zeta^{-1}$ and other typical elements of incidence algebra perfectly suitable for calculating number of chains, of maximal chains etc. in finite sub-posets of $P$. Right from the definition of $P$ via its Hasse diagram. The way the $\zeta$ is written above underlines the fact that this is the staircase structure encoding formula for any natural numbers valued sequence $F$. Recall: this sequence $F$ serves as the label encoding all resulting digraphs and combinatorial objects.

The subsequent di-biclique of bipartite digraph adjoining via natural join $\oplus \to$ is in one to one correspondence with adjoining another subsequent one step down of La Scala. In another words: one more step down La Scala - one more di-biclique $\oplus \to$-adjoint.

To this end define the $L$-Logic function as follows:

$$L([\text{Max}]_F) = \zeta_F, \quad \zeta_{r,s} = \begin{cases} 1 & [\text{Max}]_{r,s} > 0 \\ 0 & [\text{Max}]_{r,s} = 0 \end{cases}.\$$

This completes the natural join $\oplus \to$ structural description of $\zeta_F$ matrix construction for any $F$-graded poset and will be of use as a guide while looking for the similar form of Möbius matrix $\mu = \zeta^{-1}$ bearing in mind that for $s > r$

$$B(r_F \times s_F) = \prod_{i=r}^{s-1} B(i_F \times (i+1)_F).$$

**Remark 3.** The choice of $F$-poset II labeling and then Knuth notation. If one defines any graded $F$ poset $P$ with help of its incidence matrix $\zeta$ representing $P$ uniquely then in case of cobweb posets one arrives at $\zeta$ with Type characterization La Scala code of zeros in the upper part of this upper triangle matrix $\zeta$ due to implicit natural for right-handed oriented choice of nodes labeling. See all figures above. In the case of arbitrary $F$-graded poset $P$ apart from La Scala additional zeros appear. These are the fixed zeros of $B(i_F \times (i+1)_F)$ yielding all the other zeros from $B(r_F \times s_F)$ in the upper block triangle of $\zeta$ matrix via the product formula above. Let us make now this choice of labeling explicit. For that to do it is enough to focus on any cobweb poset II as a sample case.

**Remark 3.1.**

A bit of history. The matrix elements of $\zeta(x,y)$ matrix for Fibonacci cobweb poset were given in 2003 ([18,22] Kwaśniewski) using $x, y \in \mathbb{N} \cup \{0\}$ labels of vertices in their "natural" order i.e. applying the natural labeling as in [52] - see Fig.11. Namely:

1. set $k = 0$,
2. then label subsequent vertices - from the left to the right - along the level $k$,
3. repeat 2. for $k \to k + 1$ until $k = n + 1$; $n \in \mathbb{N} \cup \{\infty\}$
As the result we obtain the $\zeta$ matrix for Fibonacci sequence as presented by the Fig. *La Scala di Fibonacci* dating back to 2003 [18,22].

**The origin - of effectiveness.** Inspired [29-44] by Gauss finite geometries numbers and in the spirit of Knuth "notationlogy" [23] we shall refer here also to the upside down notation effectiveness as in [2-6] or earlier in [29-44]. As for that upside down attitude $F_n \equiv n_F$ being much more than "just a convention" to be used substantially in what follows as well as for the reader’s convenience - let us recall it just here quoting it as The Principle according to Kwaśniewski [2] where this rule has been formulated as an "'of course'" Principle i.e. simultaneously trivial and powerful statement.

**The Upside Down Notation Principle.**
1. Let the statement $s(F)$ depends only on the fact that $F$ is a natural numbers valued statement.
2. Then if one proves that $s(N) \equiv s(n_{\bar{n}}_{n \in N})$ is true - the statement $s(F) \equiv s(n_F_{n \in N})$ is also true. Formally - use equivalence relation classes induced by co-images of $s : \{F\} \mapsto 2^{(1)}$ and proceed in a standard way.

In order to proceed further let us now recall-rewrite purposely here Kwaśniewski 2003 - formula for $\zeta$ function of arbitrary cobweb poset in order to see that its’ algorithm rules automatically make it valid for all $F$-cobweb posets where $F$ is any natural numbers valued sequence i.e. with $F_0 > 0$. $I(\Pi, R)$ stays for the incidence algebra of the poset $\Pi$ over the commutative ring $R$ where $x, y, k, s \in N \cup \{0\}$.

$$\zeta(x, y) = \zeta_1(x, y) - \zeta_0(x, y)$$

$$\zeta_1(x, y) = \sum_{k=0}^{\infty} \delta(x + k, y)$$

$$\zeta_0(x, y) = \sum_{k \geq 1} \sum_{s \geq 0} \delta(x, F_s + k) \sum_{r=1}^{F_s-k-1} \delta(k + F_s + r, y)$$
and naturally
\[ \delta(x, y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}. \]

The above formula for \( \zeta \in I(\Pi, R) \) rewritten in \((F_s \equiv s_F)\) upside down notation equivalent form as below is of course valid for all cobweb posets \((x, y, k, s \in N \cup \{0\})\).

\[ \zeta(x, y) = \zeta_1(x, y) - \zeta_0(x, y), \]

\[ \zeta_1(x, y) = \sum_{k=0}^{\infty} \delta(x + k, y), \]

\[ \zeta_0(x, y) = \sum_{s \geq 1} \sum_{k \geq 1} \delta(x, k + s_F) \sum_{r=1}^{(s-1)_F - k - 1} \delta(x + r, y). \]

**Note.** \(+\zeta_1\) "produces the Pacific ocean of 1's" in the whole upper triangle part of a would be incidence algebra \(\sigma \in I(\Pi, R)\) matrix elements with then \((-\zeta_0)\) resulting zeros and ones multiplying arbitrary \(\sigma\) choice fixed elements of \(R\).

**Note.** \(-\zeta_0\) cuts out 0's i.e. thus producing "'zeros' F-La Scala staircase" in the 1's delivered by \(+\zeta_1\).

This results exactly in forming 0's rectangular triangles: \(s_F - 1\) of them at the start of subsequent stair and then down to one 0 till - after \(s_F - 1\) rows passed by one reaches a half-line of 1's which is running to the right- right to infinity and thus marks the next in order stair of the F- La Scala.

The \(\zeta\) matrix explicit formula was given for arbitrary graded posets with the finite set of minimal in terms of natural join of bipartite digraphs in [3].

**Recall 1. Recapitulation - the La Scala Mantra.**

What was said is equivalent to the fact that the cobweb poset coding La Scala is of the natural join operation origin thus producing \(\zeta\) matrix [5,4,3] with one down step of La Scala being equivalent to \(\oplus\to\) - adjoining the subsequent bipartite digraph and what results in: (quote from [3], see: Subsection 2.6.) The explicit expression for zeta matrix \(\zeta_F\) of cobweb posets via known blocks of zeros and ones for arbitrary natural numbers valued \(F\)- sequence was given in [3] due to more than mnemonic efficiency of the up-side-down notation being applied (see [6] [v6] Feb 2009 and references therein). With this notation inspired by Gauss and replacing \(k\) - natural numbers with "\(k_F\)" numbers - elements of the \(F\)-sequence one gets

\[
A_F = \begin{bmatrix}
0_{1_F \times 1_F} & I(1_F \times 2_F) & 0_{1_F \times \infty} \\
0_{2_F \times 1_F} & 0_{2_F \times 2_F} & I(2_F \times 3_F) & 0_{2_F \times \infty} \\
0_{3_F \times 1_F} & 0_{3_F \times 2_F} & 0_{3_F \times 3_F} & I(3_F \times 4_F) & 0_{3_F \times \infty} \\
0_{4_F \times 1_F} & 0_{4_F \times 2_F} & 0_{4_F \times 3_F} & 0_{4_F \times 4_F} & I(4_F \times 5_F) & 0_{4_F \times \infty} \\
... & etc & ... & ... & ... & ... & and so on & ...
\end{bmatrix}
\]
\[ \zeta_F = \exp[\mathbf{A}_F] \equiv (1 - \mathbf{A}_F)^{-1} \equiv I_{\infty \times \infty} + \mathbf{A}_F + \mathbf{A}_F^2 + \ldots = \begin{bmatrix} I_{1_F \times 1_F} & I(1_F \times \infty) \\ O_{2_F \times 1_F} & I(2_F \times \infty) \\ O_{3_F \times 1_F} & O_{3_F \times 2_F} & I(3_F \times \infty) \\ O_{4_F \times 1_F} & O_{4_F \times 2_F} & O_{4_F \times 3_F} & I(4_F \times \infty) \\ \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{bmatrix} \]

where \( I(s \times k) \) stays for \((s \times k)\) matrix of ones i.e. \([I(s \times k)]_{ij} = 1; 1 \leq i \leq s, 1 \leq j \leq k. \) and \( n \in N \cup \{\infty\} \)

In the \( \zeta_F \) formula from \([5,4,3]\) \( \odot \) denotes the Boolean product, hence exactly this product is meant while Boolean powers enter formulas. We readily recognize from its block structure that \( F \)-La Scala is formed by upper zeros of block-diagonal matrices \( I_{k_F \times k_F} \) which sacrifice these their zeros to constitute the \( k \)-th subsequent stair in the \( F \)-La Scala descending and descending far away down to infinity. Thus the cobweb poset coding La Scala is due to the natural join origin of \( \zeta \) matrix. In general case of any \( F \)-graded poset (with as in Remark 3.1 labeling fixed) one naturally encounters - apart from obligatory La Scala zeros generated via the ruling formula from (Remark.1.) those of \( B(A) \) which is biadjacency i.e cover relation \( \prec \cdot \) matrix of the adjacency matrix \( A \) of the \( F \)-graded poset.

**Note:** biadjacency and cover relation \( \prec \cdot \) matrix for bipartite digraphs coincide.

By extension - we call cover relation \( \prec \cdot \) matrix \( \kappa \) the biadjacency matrix too in order to keep reminiscent convocations going on.

Note now that because of \( \delta \)'s under summations in the former \( \zeta \) formula the following is obvious:

\[ 1 \leq r = y - x \leq (s - 1)_F - k - 1 = 1 \leq r = y - k - s_F \leq s - 1)_F - k - 1 \equiv 1 \leq r = y \leq s_F - (s - 1)_F - 1. \]

Because of that the above last expression of the \( \zeta \) expressed in terms of \( \delta \in I(\Pi, R) \) may be still simplified [for the sake of verification and portraying via computer simple program implementation]. Namely the following is true:

\[ \zeta(x, y) = \zeta_1(x, y) - \zeta_0(x, y), \]

where

\[ \zeta_1(x, y) = \sum_{k=0}^{\infty} \delta(x + k, y), \]

[- note: +\( \zeta_1 \) ’’produces the Pacific ocean of 1’s’’ in the whole upper triangle part of a would be incidence algebra \( \sigma \in I(\Pi, R) \) matrix elements with then \((-\zeta_0\) resulting zeros and ones multiplying arbitrary \( \sigma \) choice fixed elements of \( R\),]
and where (where \( x, y, k, s \in N \cup \{0\} \))

\[
\zeta_0(x,y) = \sum_{s \geq 1} \sum_{k \geq 1} \delta(x, k + sF) \sum_{r \geq 1} \delta(r, y),
\]

[- note then again that \(-\zeta_0\) cuts out "one’s F-La Scala staircase" in the 1’s provided by \(+\zeta_1\)].

Note, that for \( F = \text{Fibonacci} \) this still more simplifies as then

\[
s_F + (s - 1)_F - 1 = (s + 1)_F.
\]

**Remark 3.2. ad Knuth notation** [23].

In the wise "notationology" Knuth’s note [23] one finds among others the notation just for the purpose here (see [6] [v6] Fri, 20 Feb 2009)

\[
[s] = \begin{cases} 
 1 & \text{if } s \text{ is true}, \\
 0 & \text{otherwise}.
\end{cases}
\]

Consequently for any set or class

\[
[x = y] \equiv \delta(x, y).
\]

Consequently for any set with addition (group, free group, semi-group, ring,...):

\[
[x < y] \equiv \sum_{k \geq 1} \delta(x + k, y),
\]

\[
[x \leq y] \equiv \sum_{k \geq 0} \delta(x + k, y).
\]

Using this makes my last above expression of the \( \zeta \) in terms of \( \delta \) still more transparent and handy if rewritten in Donald Ervin Knuth’s notation [23]. Namely:

\[
\zeta(x, y) = \zeta_1(x, y) - \zeta_0(x, y)
\]

\[
\zeta_1(x, y) = [x \leq y]
\]

\[
\zeta_0(x, y) = \sum_{s \geq 1} \sum_{k \geq 1} [x = k + sF][1 \leq y \leq sF + (s - 1)_F - 1].
\]

\[
\zeta_0(x, y) = \sum_{s \geq 1} [x > sF][1 \leq y \leq sF + (s - 1)_F - 1],
\]

where, let us recall: \( x, y, k, s \in N \cup \{0\} \).

Note, that for \( F = \text{Fibonacci} \) this still more simplifies as then

\[
s_F + (s - 1)_F - 1 = (s + 1)_F.
\]
Remark 3.3. Knuth notation [23] - and Dziemiańczuk’s guess

It was remarked by my Gdańsk University Student Coworker Maciej Dziemiańczuk - that my \( \zeta \in I(\Pi, R) \) (equivalent) expressions are valid according to him only for \( F = \text{Fibonacci sequence} \). In view of the Upside Down Notation Principle if any of these is proved valid for any particular natural numbers valued sequence \( F \) using no other particular properties of \( F \) then it should be true for all of the kind.

His this being doubtful - has led him to invention of his own - in the course of our The Internet Gian Carlo Rota Polish Seminar discussions with me (see [6] 20 Feb 2009).

Here comes the formula postulated by him in the course the The Internet Seminar e-mail discussions (see then resulting now Comment 5 referring to \textbf{Krot}).

\[
\zeta(x, y) = [x \leq y] - [x < y] \sum_{n \geq 0} [(x > S(n)) [y \leq S(n + 1)],
\]

where

\[
S(n) = \sum_{k \geq 1} k_F
\]

**Exercise.** My todays reply to his guess (compare [19] 20 Feb 2009) is the following Exercise.

Let \( x, y \in N \cup \{0\} \) be the labels of vertices in their "natural" linear order as explained earlier.

Prove the true claim:

\textbf{Dziemiańczuk guess is equivalent to Kwaśniewski formulas.}

- What is for? My "for" is the Socratic Method question. Why not use the arguments in favor of

\[
\zeta_0(x, y) = \sum_{s \geq 1} \sum_{k \geq 1} \delta(x, k + s_F) \sum_{r \geq 1} \delta(r, y),
\]

Hint. Use the same argumentation. Hint. Then - contact Comment 5.

Remark 4. Ewa Krot Choice. While the above is established it is a matter of simple observation by inspection to find out how does the the Möbius matrix \( \mu = \zeta^{-1} \) looks like . Using in [24,25] this author example and expression for \( \zeta \) matrix this has been accomplished first (see also [27]) for Fibonacci sequence and then the same formula was declared to be valid for \( F \) sequences as above in [26,27]. Namely the author of [28] states that the Möbius function for the Fibonacci sequence designated cobweb poset can be easily extended to the whole family of all cobweb posets with indication to the reference [28] where one neither finds the proof except for declaration that the validity far all cobweb posets is OK. From the todays perspective the present author should say that it is not so automatic if definition of cobweb poset via ordinal sum is not uncovered as in \textbf{1.2} above i.e. \( \Pi = \oplus_{k \geq 0} \Phi_k \). For that to see follow what follows.
By now here is her formula for the cobweb posets’ Möbius function (see: (6) in [24] then it is recommended to consult Comment 5).

Let \( x = \langle s, t \rangle \) and \( y = \langle u, v \rangle \) where \( 1 \leq s \leq F_t, \ 1 \leq u \leq F_v \) while \( t, v \in \mathbb{N} \).

These are descriptive and extra external with respect to the Krot formula below conditions imposed in order to stay in accordance with the zeros’ ”La Scala di Fibonacci”” structure of the present author ”‘discovered”’ in 2003 [18,19]. This Ewa Krot brave independence declaration step formula was since now on presented by the author of [25-28] in opposition (?) to the Kwa´sniewski’s choice which makes these conditions being automatically inherited from \( \zeta \) matrix formula by the present author (see 2003 [19] and consequently all relevant papers of Kwa´sniewski later on till today).

If these external with respect to formula conditions are assumed then \( \mu \) Möbius function for Fibonacci cobweb Krot formula reads ([24])

\[
\mu(x, y) = \mu(\langle s, t \rangle, \langle u, v \rangle) = \delta(s, u)\delta(t, v) - \delta(t+1, v) + \sum_{k=2}^{\infty} \delta(t+k, v)(-1)^k \prod_{i=t+1}^{v-1} (F_i - 1)
\]

In particular for Fibonacci sequence either \( F = \langle 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ... \rangle \) or \( F = \langle 1, 1, 1, 2, 3, 5, 8, 13, 21, 34, ... \rangle \) we get the right number

\[
\mu(\langle 1, 1 \rangle, \langle 2, 1 \rangle) = -1, \quad \text{as} \quad \prod_{i=1}^{v-1} (F_i - 1) = \prod_{i=0}^{t-1} (F_i - 1) = 0.
\]

The same is right for \( F = \mathbb{N} \). We shall see also by inspection via Examples below that this is a obviously decisive sensitive starting point in applying the recurrent definition of Möbius function matrix \( \mu \) and its descendant - the block structure of Möbius function coding matrix \( C(\mu) \) - with this latter recurrence for \( C(\mu) \) allowing simple solution simultaneously with combinatorial interpretation of Krot on matrix \( K = (K_s(r_F)) \), where \( K_s(r_F) = |C(\mu)_{r,s}| \).

Now bearing in mind the Upside Down Notation Principle let start to prepare the formula for all connected graded posets \( F \)-cobweb posets included) with \( F_0 > 0 \) (as it should be for natural numbers valued sequences) and of course for other natural numbers valued sequences \( F \).

Note the condition resulting from \( F_0 > 0 \) unavoidable convention: Fibonacci means since now on \( F = \langle 1, 1, 1, 2, 3, 5, 8, 13, 21, 34, ... \rangle \)

At first the First Step. Let us formulate equivalent versions of the above Krot formula in coordinate grid \( \mathbb{Z} \times \mathbb{Z} \) adequately to to the task of verifying it in the case of Fibonacci sequence \( F = \langle 1, 1, 2, 3, 5, 8, 13, 21, 34, ... \rangle \). This has been done we arrive at what follows.

Let \( x = \langle s, t \rangle \) and \( y = \langle u, v \rangle \) where \( 1 \leq s \leq t_F, \ 1 \leq u \leq v_F \) while \( t, v \in \mathbb{N} \).

Then (versions equivalent to Krot formula)

\[
\mu(x, y) = \mu(\langle s, t \rangle, \langle u, v \rangle) = [(s = u)[t = v] - [t+1 = v] + \sum_{k=2}^{\infty} [t+k = v](-1)^k \prod_{i=t+1}^{v-1} (i_F - 1)]
\]
\[ \mu(x, y) = \mu(\langle s, t \rangle, \langle u, v \rangle) = [(s = u)[t = v] - [t + 1 = v] + [t + 1 < v]( -1)^{v-t} \prod_{i=t+1}^{v-1} (i_F - 1) \]

or with sine qua non conditions being implemented in there:

\[ \mu(x, y) = \mu(\langle s, t \rangle, \langle u, v \rangle) = [(s = u)[t = v] - [t + 1 = v] + [t + 1 < v][1 \leq s \leq t_F][1 \leq u \leq v_F](-1)^k \prod_{i=t+1}^{v-1} (i_F - 1). \]

The above Möbius function re-formulas if proved valid for \( F = N \) thanks to no more than the assumption \( n_N \in N \) then it should be literally valid for all natural numbers valued sequences \( F \).

These formulas for Möbius function appear suitable [check] for Fibonacci sequence \( F = \langle 1, 1, 2, 3, 5, 8, 13, 21, 34, ... \rangle \) as well [check] as in the case of \( F = N \) (Example 9) and as well as in the case of Example 12 (see both below).

As a matter of fact this might be already expected from the following simple check using any of the equivalent formulas:

\[ \mu(\langle 1, 1 \rangle, \langle 2, 1 \rangle) = -1 \]

which is the right number for Fibonacci sequence (or see Example 11) as well as

\[ \mu(\langle 1, 1 \rangle, \langle 2, 1 \rangle) = (\langle 1, 1 \rangle, \langle 2, 2 \rangle) - 1 \]

is right the number for \( F = N \) natural number sequence (or see Example 12).

The reason for that fact is at hand just by inspection of Hasse digraphs of cobweb posets under consideration. And these checks are crucial at the start in view of recurrent form of August Ferdinand Möbius matrix \( \mu \) formula. However...

**However we are in need of The Proof!** of this Krot Formula for Möbius function for any one - hence for all of the relevant sequences \( F \). Here let us call back (hail) the mantra: The Upside Down Notation Principle.

Also let us recall here that due to obvious observation of this article that the natural join \( P \oplus Q \) of graded posets \( \langle P, \leq_P \rangle \) and \( \langle Q, \leq_Q \rangle \) with correspondingly maximal (in \( P \)) and minimal (in \( Q \)) sets being identical is expressed by ordinal sum \( P \oplus Q \) (see [1]) one arrives in 1.4 at a very simple proof of the Möbius function formula for cobweb posets.

So - as now we see it - one is in need of the **Second Step**. In the sequel this is to be done and we shall use formulas for Möbius function with the structure inferred from the fact that incidence algebra \( I(\Pi, R) \) elements arise in the sequential natural join of di-bicliques or bipartite digraphs in the general case of \( F \)-graded posets as to be exemplified and derived below. Then implementation of the recurrent definition of Möbius function matrix \( \mu \) gives birth to daughter descendant of \( \mu \) i.e. the block structure of Möbius function coding matrix \( C(\mu) \) implying for \( C(\mu) \) an recurrence allowing simple solution simultaneously
with combinatorial interpretation of Krot matrix \( K = (K_s(r_F)) \), where \( K_s(r_F) = |C(\mu)_{r,s}| \).
And this is to be this Second Step.

Before doing this in the next section - to this end - lets for now continue “the Krot and Krot-Sieniawska contribution subject”. The author of [24] introduces parallelly also another form of \( \zeta \) function formula and since now on - except for [24,46] - in subsequent papers [25,27,28] their author uses the formula for \( \zeta \) function in this another form. Namely - this other form formula for \( \zeta \) function in the present authors’ grid coordinate system description of the cobweb posets was given by Krot in her note on Möbius function and Möbius inversion formula for Fibonacci cobweb poset [25] with \( F \) designating the Fibonacci cobweb posets. In [26] the formula the Krot formula for the Möbius function for Fibonacci sequence \( F \) was declared as valid for all cobweb posets i.e. for all natural numbers valued sequences \( F \) denominated cobweb posets. (Consult also so the recent note “On Characteristic Polynomials of the Family of Cobweb Posets” [28] and see also Comment 5.).

Comment 5. ad \( \zeta \)ta and \( \mu \). Back to 03 Feb 2004 preprint [46] for to see the source of the past in the future which is present in 1.4 subsection due to the definition of natural join of graded cobweb posets via ordinal sum of independent sets of these cobweb posets.

In [46] the deliberate task was to consider just the case of Fibonacci sequence in order to to find the inverse matrix \( \zeta^{-1} \) of the \( \zeta \) from [17] (November 2003) using the present author \( \zeta \) matrix expression in terms of the infinite Kronecker delta matrix \( \delta \) from [18] (November 2003) and [19] (December 2003). Why Fibonacci? See the formula (5) page 9 in [46]. Applying (5) to the conditions on the top of the page 9 above the relevant formula (5) one arrives at (6) which afterwords - in this author coordinate grid description reads:

\[
x = (s,t), \ y = (u,v), \text{ where } 1 \leq s \leq F_t, \ 1 \leq u \leq F_v, \text{ while } t,v \in \mathbb{N}.
\]

Nevertheless already in Ewa Krot preprint [46] (see the top of the page 9 above the relevant formulas (5) and (6) ...) already there the **general case** conditions are stated which in notation of the present author labeling and upside down notation as well as due to Dziemiańczuk’s observed Knuth notation now simply read as follows:

\[
[(x > S(n))[y \leq S(n+1)]
\]

where

\[
S(n) = \sum_{k \geq 1} k_F
\]

and accordingly we now infer \( (x, y, k, s, n \in N \cup \{0\} \) [Remark 2.1.])

\[
\zeta(x, y) = [x \leq y] - [x < y] \sum_{n \geq 0} [(x > S(n))[y \leq S(n+1)].
\]

30
Note. The author of [24-28,46] consequently avoids the upside down notation. However she had used this notation then in her Rota and cobweb posets related dissertation that she had defended with distinction on 30 September 2008 [8]. 

The end of comment.

No doubt the \( \zeta \) function formulas - the former (Kwaśniewski) and the latter (Krot) are valid for all natural numbers valued sequences \( F \).

(Let us recall here that due to obvious observation of this article that the natural join \( P \oplus \rightarrow Q \) of graded posets \( \langle P, \leq_P \rangle \) and \( \langle Q, \leq_Q \rangle \) with correspondingly maximal (in \( P \)) and minimal (in \( Q \)) sets being identical is expressed by ordinal sum \( P \oplus Q \) (see [1]) one arrives in 1.4 at a very simple proof of the Möbius function formula for cobweb posets).

Well, here is this other latter form of Krot formula for \( \zeta \) function (see: (7) in [24] or (1) in [26]).

Let \( x = \langle s,t \rangle \) and \( y = \langle u,v \rangle \) where \( 1 \leq s \leq F_t, 1 \leq u \leq F_v \) while \( t,v \in \mathbb{N} \).

Then

\[
\zeta(x,y) = \zeta(\langle s,t \rangle, \langle u,v \rangle) = \delta(s,u)\delta(t,v) + \sum_{k=1}^{\infty} \delta(t+k,v)
\]

where here - recall \( (a,b) \in \mathbb{Z} \):

\[
\delta(a,b) = \begin{cases} 
1 & \text{for } a = b, \\
0 & \text{otherwise.}
\end{cases}
\]

In February 2009 - in the course of The Internet Gian Carlo Rota Polish Seminar e-mail discussions with the present author - still another \( \zeta \)-matrix formula was postulated by Dziemiańczuk - in Knuth notation. See - below. We claim : all are - up to the equivalence of description - the same. See then Comment 5.

Remark 4.1. again on \( \zeta \) formulas.

Let us compare the above Krot formula for \( \zeta \) with those by Kwaśniewski equivalent to the one from the Remark 3.1. \( (x,y) \in \mathbb{N} \) i.e. with

\[
\zeta(x,y) = [x \leq y] - \sum_{s \geq 1} \sum_{k \geq 1} [x = k + sF][1 \leq y \leq sF + (s-1)F - 1],
\]

\[
\zeta(x,y) = [x \leq y] - \sum_{s \geq 1} [x > sF][1 \leq y \leq sF + (s-1)F - 1],
\]

where, let us recall: \( k,s \in \mathbb{N} \cup \{0\} \).

Let us rewrite the above Krot formula in Knuth notation keeping in mind the conditions

\[ 1 \leq s \leq F_t, 1 \leq u \leq F_v, t,v \in \mathbb{N}, \]

which should have been imposed altogether with:

\[
\zeta(\langle s,t \rangle, \langle u,v \rangle) = [s = u][t = v] + [v > t].
\]
The above formula with sine qua non conditions being implemented in there reads:

\[ \zeta(<s,t>,<u,v>) = [s = u][t = v] + [v > t][1 \leq s \leq t_F][1 \leq u \leq v_F] \]

and so, if written with \( \delta \)'s it contains three subsequent summations as in the Kwaśniewski formula from 2003.

4 The formula of inverse zeta matrix for graded posets with the finite set of minimal elements via natural join of matrices and digraphs technique.

Training in relabeling - Exercise.

As we were and are to compare formulas from papers using different labeling - write and/or learn to see formulas from the above and below Observations, definitions etc. as for \( x, y, k, s \in \mathbb{N} \cup \{0\} \) on one hand and as for \( x, y, k, s \in \mathbb{N} \) on the other hand. Because of the comparisons reason we shall tolerate and use both being indicated explicitly.

Let us start with picture Examples 9,10,11 of inverse zeta matrices subsequently corresponding to picture Examples 1,2,5. For that to do it is enough for now to use the recurrent definition of the Möbius function

\[ \mu(x,y) = \begin{cases} 1 & x = y \\ -\sum_{x \leq z < y} \mu(x,z), & x < y \end{cases} \]

Before doing that note that we deal with \( F \)-graded posets and contact Remark 1 for notation and typical relations relevant below.

Recall What form of the August Ferdinand Möbius matrix we do expect by now.

Recall: (see Observation 3) - in the case of Möbius \( \mu = \zeta^{-1} \) matrix as it is obligatory \( c_{r,r+1} = -1 \).

Recall (Remark 1) Markov property and observe by inspection that - in the case of Möbius \( \mu = \zeta^{-1} \) matrix for cobweb posets it is obligatory to put

\[ M(r_F \times (r+2)_F) = -[I(r_F \times (r+1)_F)I((r+1)_F \times (r+2)_F) - I(r_F \times (r+2)_F)] \]

i.e.

\[ M(r_F \times (r+2)_F) = -[(r+1)_F - 1]I(r_F \times (r+2)_F), \]

thereby :

\[ c_{r,r+2} = -[(r+1)_F - 1]c_{r,r+1}, \quad c_{r,r+1} = -1. \]

- What about then with arbitrary \( F \)-graded posets \((P, \leq)\) ?

In what follows we consider (consult the Remark 1.) motivating examples and then representative Examples 9,10,11,12 of Möbius matrix. After that the looked for **Theorem 4.** is stated for arbitrary \( F \)-graded posets \((P, \leq)\).
Motivating examples.

Example 1. Let \( i = 1, \ldots, r \), \( k = 1, \ldots, (r + 1) \), \( j = 1, \ldots, (r + 2) \) as now we consider (Remark 1.) \( x_{r,i} < x_{r+1,k} \) where \( \{x_{r,i}\} = \Phi_r \) and \( \{x_{r+1,k}\} = \Phi_{r+1} \) are independent sets. Then

\[
\mu(x_{r,i}, x_{r+2,j}) = - \sum_{x_{r,i} \leq z < x_{r+2,j}} \mu(x_{r,i}, z) = - \left( 1 + \sum_{k=1}^{(r+1)_F} \mu(x_{r,i}, x_{r+1,k}) \right),
\]

i.e.

\[
\mu(x_{r,i}, x_{r+2,j}) = +[(r + 1)_F - 1] = c_{r,r+2} = -[(r + 1)_F - 1]c_{r,r+1}.
\]

Example 2. From Example 1 we infer that as \( \mu(x_{r,i}, x_{r+2,j}) = \mu(x_{r}, x_{r+2}) \) then it is now enough to consider what follows \( (x_{r}, x_{r+3} \) any fixed):

\[
\mu(x_{r}, x_{r+3}) = - \sum_{x_{r} \leq z < x_{r+3}} \mu(x_{r}, z) = - \left( 1 + \sum_{x_{r+1} \leq z < x_{r+3}} \mu(x_{r}, z) \right) =
\]

\[= - \left( 1 + (r + 1)_F \mu(x_{r}, x_{r+1}) + \sum_{x_{r+2} \leq z < x_{r+3}} \mu(x_{r}, z) \right) = - (1 - (r + 1)_F + (r + 2)_F \mu(x_{r}, x_{r+2})),
\]

i.e.

\[
\mu(x_{r}, x_{r+3}) = -[(r + 2)_F - 1]c_{r,r+2} = -[(r + 2)_F - 1][r + 1)_F - 1].
\]

Via straightforward induction we conclude that now for arbitrary \( r, s \in N \cup \{0\} \) and for any cobweb poset - in accordance with subsection 1.4. induction - the following is true.

**Theorem 3 for cobweb posets.** \( (N \cup \{0\}). \)

\[
c_{r,s} = [s = r] - [s = r+1] + [s > r+1][-1]^{s-r} [(s - r - 1)_F - 1] \ldots (3_F - 1) (+1) =
\]

\[= [s = r] - [s = r + 1] + [s > r + 1][-1]^{s-r} \prod_{i=r+1}^{s-1} (i_F - 1).
\]

Let us see now how it works and how this theorem may be extended to general case of arbitrary \( F \)-denominated poset. At first the representative Examples 9,10,11,12 of Möbius matrix follow which might be derived right from the recurrent definition of Möbius function without even referring to the above theorem.
Example 9. $\zeta_N^{-1}$. The Möbius function matrix $\mu = \zeta^{-1}$ for the natural numbers i.e. $N$ - cobweb poset.

$$
\begin{bmatrix}
1 & -1 & -1 & +1 & +1 & -2 & -2 & -2 & +6 & +6 & +6 & +6 & -24 & \cdots \\
0 & 1 & 0 & -1 & -1 & -1 & +2 & +2 & +2 & -6 & -6 & -6 & -6 & +24 & \cdots \\
0 & 0 & 1 & -1 & -1 & +2 & +2 & +2 & -6 & -6 & -6 & -6 & +24 & \cdots \\
0 & 0 & 0 & 1 & 0 & -1 & -1 & -1 & +3 & +3 & +3 & +3 & -12 & \cdots \\
0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & +3 & +3 & +3 & +3 & -12 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & +4 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & +4 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 -1 & -1 & -1 -1 & +4 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \cdots \\
\end{bmatrix}
$$

Note. $\mu$ has of course natural join inherited structure, of course.

Fig.9a $\mu_N = \zeta_N^{-1}$. The block presentation of the Möbius function matrix $\mu = \zeta^{-1}$ for the natural numbers i.e. $N$ - cobweb poset.

The code for this KoDAG is given by its KoDAG self-evident code-triangle of the coding matrix $C(\mu_F)$ (a starting part of it shown below):

$$
C(\mu_N) = \begin{bmatrix}
+1 & -1 & +1 & -2 & +6 & 24 \\
-0 & +1 & -1 & +2 & -6 & 24 \\
+0 & -0 & +1 & -1 & +3 & 12 \\
-0 & +0 & -0 & +1 & -1 & 4 \\
+0 & -0 & +0 & -0 & +1 & -1 \\
\end{bmatrix}
$$

34
Example 10 $\zeta_f^{-1}$. The M"{o}bius function matrix $\mu = \zeta^{-1}$ for $F=$Fibonacci sequence.

$$\mu_F = \begin{bmatrix}
I_{1 \times 1} & -I(1 \times 1) & 0I(1 \times 1) & 0I(1 \times 2) & 0I(1 \times 3) \\
0I_{1 \times 1} & I_{1 \times 1} & -I(1 \times 1) & 0I(1 \times 2) & 0I(1 \times 3) \\
O_{1 \times 1} & O_{1 \times 1} & I_{1 \times 1} & -I(1 \times 2) & +I(1 \times 3) \\
O_{2 \times 1} & O_{2 \times 1} & O_{2 \times 1} & I_{2 \times 2} & -I(2 \times 3) \\
O_{3 \times 1} & O_{3 \times 1} & O_{3 \times 1} & 0_{3 \times 2} & I_{3 \times 3} \\
\vdots & \text{etc} & \vdots & \text{and so on} & \vdots
\end{bmatrix}$$

Example 10a $\zeta_f^{-1}$. The block presentation of the M"{o}bius function matrix $\mu = \zeta^{-1}$ for $F=$Fibonacci sequence.

Recall then and note here up and below the block structure.

$$\sigma = \begin{bmatrix}
I_{1_F \times 1_F} & B(1_F \times 2_F) & B(1_F \times 3_F) & B(1_F \times 4_F) & B(1_F \times 5_F) & B(1_F \times 6_F) \\
0_{2_F \times 1_F} & I_{2_F \times 2_F} & B(2_F \times 3_F) & B(2_F \times 4_F) & B(2_F \times 5_F) & B(2_F \times 6_F) \\
0_{1_F \times 1_F} & 0_{1_F \times 2_F} & I_{3_F \times 3_F} & B(3_F \times 4_F) & B(3_F \times 5_F) & B(3_F \times 6_F) \\
0_{4_F \times 1_F} & 0_{4_F \times 2_F} & 0_{4_F \times 3_F} & I_{4_F \times 4_F} & B(4_F \times 5_F) & B(4_F \times 6_F) \\
\vdots & \text{etc} & \vdots & \text{and so on} & \vdots
\end{bmatrix}$$

where $B(k_F \times (k+1)_F)$ denote corresponding constant $k_F \times (k+1)_F$ matrices in the case of $\zeta$ or $\zeta^{-1}$ matrices for example, with matrix elements from the ring $R=2^{(1)}, Z_2 = \{0,1\}, Z$ etc.
Example 11.\( \zeta^{-1}_F \). The Möbius function matrix \( \mu = \zeta^{-1} \) for \((1_F = 2_F = 1 \text{ and } n_F = 3 \text{ for } n \geq 2)\) the \( F = \text{Fibonacci} \) relative special sequence \( F \) constituting the label sequence denominating cobweb poset associated to \( F\text{-KoDAG} \) Hasse digraph

\[
\mu_F = \begin{bmatrix}
I_{3 \times 1} & -I(1 \times 1) & +0I(1 \times 3) & -0I(1 \times 3) & +0I(1 \times 3) \\
O_{1 \times 1} & +I_{1 \times 1} & -I(1 \times 3) & +2I(1 \times 3) & -4I(1 \times 3) \\
O_{3 \times 1} & -O_{3 \times 1} & +I_{3 \times 3} & -I(3 \times 3) & +2I(3 \times 3) \\
O_{3 \times 1} & +O_{3 \times 1} & -O_{3 \times 3} & +I_{3 \times 3} & -I(3 \times 3) \\
... & etc & ... & and so on & ...
\end{bmatrix}
\]

Example 11a.\( \zeta^{-1}_F \). The block presentation of the Möbius function matrix \( \mu = \zeta^{-1} \) for \((1_F = 2_F = 1 \text{ and } n_F = 3 \text{ for } n \geq 2)\) the \( F = \text{Fibonacci} \) relative special sequence \( F \) constituting the label sequence denominating cobweb poset associated to \( F\text{-KoDAG} \) Hasse digraph

The code for this KoDAG is given by its KoDAG self-evident code-triangle of the coding matrix \( C(\mu_F) \) (a starting part of it shown below):

\[
C(\mu_F) = \begin{bmatrix}
1 & -1 & +0 & -0 & +0 & -0 \\
0 & +1 & -1 & +2 & -4 & +8 \\
0 & -0 & +1 & -1 & +2 & -4 \\
0 & +0 & -0 & +1 & -1 & +2 \\
0 & -0 & +0 & -0 & +1 & -1 \\
... & ... & ... & ...
\end{bmatrix}
\]
From Observation 2 we infer what follows as obvious.

**Example.12** $\zeta_F^{-1}$. The Möbius function matrix $\mu = \zeta^{-1}$ for $(1_F = 1$ and $n_F = 3$ for $n \geq 2)$ the $N$ relative special sequence $F$ constituting the label sequence denominating cobweb poset associated to $F$-KoDAG Hasse digraph

\[
\begin{bmatrix}
I_{1 \times 1} & -I(1 \times 1) & +2I(1 \times 3) & -4I(1 \times 3) & +8I(1 \times 3) \\
O_{1 \times 1} & +I_{1 \times 1} & -I(1 \times 3) & +2I(1 \times 3) & -4I(1 \times 3) \\
O_{3 \times 1} & -O_{3 \times 1} & +I_{3 \times 3} & -I(3 \times 3) & +2I(3 \times 3) \\
O_{3 \times 1} & +O_{3 \times 1} & -O_{3 \times 3} & +I_{3 \times 3} & -I(3 \times 3) \\
O_{3 \times 1} & -O_{3 \times 1} & +O_{3 \times 3} & -I_{3 \times 3} & +I_{3 \times 3} \\
... & etc & ... & and so on & ...
\end{bmatrix}
\]

**Example.12a** $\zeta_F^{-1}$. The block presentation of the Möbius function matrix $\mu = \zeta^{-1}$ for $(1_F = 1$ and $n_F = 3$ for $n \geq 2)$ the $N$ relative special sequence $F$ constituting the label sequence denominating cobweb poset associated to $F$-KoDAG Hasse digraph

The code for this KoDAG is given by its KoDAG self-evident code-triangle of the coding matrix $C(\mu_F)$ (a starting part of it shown below):

\[
C(\mu_F) =
\begin{bmatrix}
1 & -1 & +2 & -4 & +8 & -16 \\
0 & +1 & -1 & +2 & -4 & +8 \\
0 & -0 & +1 & -1 & +2 & -4 \\
0 & +0 & -0 & +1 & -1 & +2 \\
0 & -0 & +0 & -0 & +1 & -1 \\
. & . & . & . & . & .
\end{bmatrix}
\]
Observation 3  Compare with the Remark 1. The block structure of $\zeta$ and consequently the block structure of $\mu$ for any graded poset with finite set of minimal elements (including cobwebs) is of the type:

$$\zeta = \begin{bmatrix}
I_1, B_1, & I_2, B_2, & \cdots & I_3, B_3, & \cdots & I_n, B_n,
\end{bmatrix}$$

$$\mu = \begin{bmatrix}
I_1, -B_1, & I_2, -B_2, & \cdots & I_3, -B_3, & \cdots & I_n, -B_n,
\end{bmatrix}$$

$n \in N \cup \{\infty\}, \zeta, \mu \in \Pi(\Pi; R)$ where $I_r = I_{r_F \times r_F}$ and $B_r = B(r_f \times (r+1)_F)$ as introduced by Observation 2.

Recall then and note here up and below the block structure $\zeta$ and consequently the block structure of $\mu$ for any graded poset $P$ with finite set of minimal elements (including cobwebs) which is proprietary characteristic for any $\sigma \in I(P; R)$ where the ring $R = 2^{(1)}$, $Z_2 = \{0, 1\}, Z$ etc.

$$\sigma = \begin{bmatrix}
I_{1_F \times 1_F} & M(1_F \times 2_F) & M(1_F \times 3_F) & M(1_F \times 4_F) & M(1_F \times 5_F) & M(1_F \times 6_F)
\end{bmatrix}$$

where in the case of $\oplus \rightarrow$-natural $\zeta$ or $\zeta^{-1}$ matrices, with matrix elements from the ring $R = 2^{(1)}$, $Z_2 = \{0, 1\}, Z$ etc the rectangle non-zero block matrices $M(k_F \times (k+1)_F)$ denote corresponding connected graded poset characteristic $k_F \times (k+1)_F$ matrices.

Note then that $M(k_F \times (k+1)_F)_{i,j} = c_{i,j,k}B(k_F \times (k+1)_F)_{i,j}, i = 1, \ldots, k_F$ and $i = 1, \ldots, (k+1)_F$ where the rectangular ”zero-one” $B(k_F \times (k+1)_F)$ matrices were introduced by the Observation 2. Consult Remark 1. - apart from the motivating examples - for $i = 1, \ldots, k_F$ and $i = 1, \ldots, (k+1)_F$ as the layer $(\Phi_k \rightarrow \Phi_{k+1})$ variables.

Note now the important fact. The relation

$$M(k_F \times (k+1)_F)_{i,j} = c_{i,j,k}B(k_F \times (k+1)_F)_{i,j},$$

where

$$i = 1, \ldots, k_F, \quad i = 1, \ldots, (k+1)_F$$

does not fix uniquely the layer $(\Phi_k \rightarrow \Phi_{k+1})$ coding matrix $C_{k,k+1} = (c_{i,j,k}), i = 1, \ldots, k_F, i = 1, \ldots, (k+1)_F$ for $F$-denominated arbitrary graded poset - except for cobweb posets for which $B(k_F \times (k+1)_F) = I(k_F \times (k+1)_F)$.

In order to delimit this layer coding matrix uniquely we define en bloc the coding matrix $C(\mu_F)$ for all layers.
Definition 7 F-graded poset $\langle \Phi, \mu_F \rangle$ coding matrix $C(\mu_F)$.
Let $k, r, s \in N \cup \{0\}$. Then we define $C(\mu_F)$ via $\oplus \rightarrow$ originated blocks as follows:

$$C(\mu_F) = (c_{r,s})$$

where $c_{r,s}$ are coding matrix elements for $F$-denominated cobweb poset, hence

$$\mu_F = ([r = s]I_{rF \times rF} + [s > r]c_{r,s}B(rF \times sF)), \quad \text{and where} \quad c_{i,j,k} \equiv M(kF \times (k + 1)F)_{i,j} = c_{i,j}B(kF \times (k + 1)F)_{i,j}.$$ 

thus the following identifications are self-evident:

$$\langle \Phi, \mu_F \rangle \equiv \langle \Phi, \zeta_F \rangle \equiv \langle \Phi, \leq \rangle \equiv \langle \Phi, C(\mu_F) \rangle.$$ 

Result: $C(\mu_F)$ as well as block sub-matrices $M(kF \times (k + 1)F) = (c_{i,j,k})$ where $k \in N \cup \{0\}$ are defined i.e are given unambiguously.

Specifically, in cobweb posets case: for $\zeta$ function (matrix) we have $M(kF \times (k + 1)F) = I(kF \times (k + 1)F)$, while for $\zeta^{-1} = \mu$ Möbius function (matrix) - from already considered examples’ prompt we have already deduced these unambiguous $c_{r,s}$ (see Theorem 2 for cobweb posets - above). Namely:

$$M(rF \times (r + 1)F) = c_{r,r+1}I(rF \times (r + 1)F).$$

What about any $F$-denominated graded posets then? The answer now is of course secured now to be the same as for $F$-cobweb posets. The answer is automatically secured by the Definitions 7,8. Just replace in the above Theorem 3 for cobweb posets $I(rF \times (r + 1)F)$ by $B(rF \times (r + 1)F)$ and-or see the Theorem 4 below for the corresponding recurrence - the recurrence equivalent to the recurrence relation definition of $c_{r,s}$.

In order to be complete also with the next section content another important example - the example of cover relation $\kappa_{\Pi} \in I(\Pi, R)$ matrix follows. Recall for that purpose now Observation 1 and the Remark 1 as to conclude what follows.

Observation 4 $(n \in N \cup \{\infty\})$ The block structure of cover relation $\kappa_{\Pi} \in I(\Pi, R)$ $(\chi (\prec \Pi) \equiv \kappa_{\Pi})$ is the following

$$\kappa_{\Pi} = \oplus \rightarrow_{k=1}^{n} \kappa_{k} =$$

$$= \begin{bmatrix}
0_{1F \times 1F} & I(1F \times 2F) & 0_{1F \times \infty} \\
0_{2F \times 1F} & 0_{2F \times 2F} & I(2F \times 3F) & 0_{2F \times \infty} \\
0_{3F \times 1F} & 0_{3F \times 2F} & 0_{3F \times 3F} & I(3F \times 4F) & 0_{3F \times \infty} \\
0_{4F \times 1F} & \ldots & 0_{nF \times nF} & I(nF \times (n + 1)F) & 0_{nF \times \infty}
\end{bmatrix}$$

where $\kappa_{k}$ is a cover relation of di-biclique $\langle \Phi_k \rightarrow \Phi_{k+1}, I_k \equiv I(kF \times (k + 1)F), k = 1, \ldots, n \text{ and where - recall - } I(s \times k) \text{ stays for } (s \times k) \text{ matrix of ones i.e. } [I(s \times k)]_{ij} = 1; 1 \leq i \leq s, 1 \leq j \leq k. \text{ while } n \in N \cup \{\infty\}.$
and consequently the block structure of reflexive cover relation $\eta_\Pi \in I(\Pi, R)$ $(\chi (\leq \Pi) = \chi \Pi + \delta \equiv \Pi)$ is given by

$$
\begin{bmatrix}
I_{1_F \times 1_F} & I(1_F \times 2_F) & 0_{1_F \times \infty} \\
0_{2_F \times 1_F} & I(2_F \times 3_F) & 0_{2_F \times \infty} \\
0_{3_F \times 1_F} & 0_{3_F \times 2_F} & I(3_F \times 4_F) \\
0_{n_F \times 1_F} & \ldots & I(n_F \times (n+1)_F) \end{bmatrix}
$$

Specifically, if restricting to cobweb posets: for $\zeta$ function (matrix) we have $B(k_F \times (k+1)_F) = I(k_F \times (k+1)_F)$, while for $\zeta^{-1} = \mu$ Möbius function (matrix) we would expect

$$
B(r_F \times (r+1)_F) = c_{r, r+1} I(r_F \times (r+1)_F)
$$

where $c_{k,k+1} = [C(\mu_F)]_{k,(k+1)}$.

What is then the explicit formula for $c_{k,k+1}$? It is of course equivalent to the question: what is then the explicit formula for $c_{r,s}$? Let us recapitulate our experience till now in order to infer the closing answer Theorem 4. and its equivalent proof method.

**Training in relabeling - Exercise.**

As we were and are to compare formulas from papers using different labeling - write and learn to see formulas from the above and below Observations as for $x,y,k,s \in N \cup \{0\}$ on one hand and as for $x,y,k,s \in N$ on the other hand. Because of the comparisons reason we shall tolerate and use both being indicated explicitly if needed.

**Recapitulation 4.1. ; notation and The Formula.** The code $C(\mu_F)$ matrix no more secret.

**Notation.** Upside down notation development continuation.

Recall:

$$n_F = n(n+1)(n+2)\ldots(n+k-1),$$

Denote:

$$n_F \equiv n_F(n+1)_F(n+2)_F\ldots(n+k-1)_F$$

Denote (valid whenever defined for corresponding functions $f$ of the natural number argument or of an argument from any chosen ring):

$$f(r_F)^{\overline{n}} = f(r_F)f([r+1]_F)\ldots f([r+k-1]_F), \quad n^{\overline{0}} \equiv 1, \quad n \in N \cup \{0\}, \quad Z, R, etc.,$$

$$f(r_F)^{\overline{n}} = f(r_F)f([r-1]_F)\ldots f([r-k+1]_F), \quad n^{\overline{0}} \equiv 1, \quad n \in N \cup \{0\}, \quad Z, R, etc..$$

Define Krot-on-shift-functions $K_s, s,r,i \in N\cup\{0\}$ or **Kroton** functions in brief -(Kroton = Croton = Codiaeum).
Definition 8 \((\mathbb{N} \cup \{0\}) \text{ labels}\)

\[ K_s(r_F) = [s > r][r + 1)_F - 1]^{s-r} \]

These of course constitute an upper triangle matrix with zeros on the diagonal for \(s, r \in \mathbb{N} \cup \{0\}\), \((r = \text{labels rows})\).

Note two cases:

Let \(s - r - 1 \neq 0\). Then

\[ K_s(r_F) = [s > r] \prod_{i=r+1}^{s-1} (i_F - 1) \]

Let \(s - r - 1 = 0\). Then

\[ K_s(r_F) = [s > r]. \]

Now - with this \(\mathbb{N} \cup \{0\}\) labeling as established in this note (Remark 2.1.) - perform simple calculations. Fibonacci sequence \(F = \langle 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots \rangle\)

case Example.

\[ K_2(1_F) = 1, K_s(1_F) = 0 \text{ for } s > 2; \]
\[ K_3(2_F) = 1, K_s(2_F) = 0 \text{ for } s > 3; \]
\[ K_4(3_F) = 1, K_5(3_F) = 1, K_6(3_F) = 2, K_7(3_F) = 2 \cdot 4 = 8, K_8(3_F) = 8 \cdot 12 = 96, \]
\[ K_9(3_F) = 96 \cdot 20 = 1920, \text{ and so on,} \]
\[ K_5(4_F) = 1, K_6(4_F) = 1 \cdot 4, K_7(4_F) = 4 \cdot 7 = 14, K_8(4_F) = 14 \cdot 12 = 168, \]
\[ K_9(4_F) = 168 \cdot [F_8 - 1] = ?, K_{10}(4_F) = 3360 \cdot [9_F - 1] = ?, \text{ and so on.} \]

Note that in the course of the above the following was used \((\mathbb{N} \cup \{0\})\) - labeling).

Lemma 4.1 \((r, s \in \mathbb{N} \cup \{0\})\). Obvious

\[ K_{s+1}(r_F) = K_s(r_F) \cdot [s_F - 1], \quad K_{r+1}(r_F) = 1, \]

\(\mathbb{N}\) sequence case Example. This exercise has obvious outcomes in view of the Lemma 2.1. For the just check results see absolute values of coding matrix matrix elements from the Example 9.

The next fact we mark as Lemma because of its importance.

Lemma 4.2 (Obvious - recapitulation.)

Let \(R = \mathbb{N}, \mathbb{Z}, \ldots\) any commutative ring. For any graded \(F\)-denominated poset (hence connected) i.e for any chain of subsequent natural joins of bipartite digraphs (di-bicliques for KoDAGs) and with the linear labeling of nodes fixed \((s, r \in \mathbb{N} \cup \{0\}\) as in Remark 2.1. or \(s, r \in \mathbb{N}\)):

\[ \mu = (\delta_{r,s} I_{r_F \times r_F} + [s > r]C(\mu_F)_{r,s}B(r_F \times s_F)) \]

where \(C(\mu_F)_{r,s} \in R\) are given by Definition 6. while \(B(r_F \times s_F)\) are nonzero matrices introduced in the Observation 2.

Bearing in mind Definitions 7 and 8 and the the above Lemma 4.2. we see that the Theorem 3 for cobweb posets extends to be true for all \(F\)-denominated posets.
Theorem 4 (Kwaśniewski)
Let $F$ be any natural numbers valued sequence. Then for arbitrary $F$-denominated graded poset (cobweb posets included)

$$C(\mu_F)_{r,s} = c_{r,s} = [r = s] + K_s(r_F)(-1)^{s-r} = [r = s] + [s > r](-1)^{s-r}[(r+1)_F-1]^{s-r}$$

i.e.

$$C(\mu_F)_{r,s} = c_{r,s} = [s \geq r](-1)^{s-r}[(r + 1)_F - 1]^{s-r},$$

i.e.

$$C(\mu_F)_{r,s} = c_{r,s} = [s \geq r](-1)^{s-r} K_s(r_F),$$

with matrix elements from $N$ or the ring $R = 2 \{1\}, Z_2 = \{0, 1\}, Z$ etc.

i.e. for cobweb posets

$$\mu = (\delta_{r,s}I_{F \times F} + (-1)^{s-r} K_s(r_F) I(r_F \times s_F))$$

i.e.

$$\mu = \begin{bmatrix}
I_{1_F \times 1_F} & c_{1,2}I(1_F \times 2_F) & c_{1,3}I(1_F \times 3_F) & c_{1,4}I(1_F \times 4_F) & c_{1,5}I(1_F \times 5_F) & c_{1,6}I(1_F \times 6_F) \\
0_{2_F \times 1_F} & I_{2_F \times 2_F} & c_{2,3}I(2_F \times 3_F) & c_{2,4}I(2_F \times 4_F) & c_{2,5}I(2_F \times 5_F) & c_{2,6}I(2_F \times 6_F) \\
0_{3_F \times 1_F} & 0_{3_F \times 2_F} & I_{3_F \times 3_F} & c_{3,4}I(3_F \times 4_F) & c_{3,5}I(3_F \times 5_F) & c_{3,6}I(3_F \times 6_F) \\
0_{4_F \times 1_F} & 0_{4_F \times 2_F} & 0_{4_F \times 3_F} & I_{4_F \times 4_F} & c_{4,5}I(4_F \times 5_F) & c_{4,6}I(4_F \times 6_F) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix}
$$

where $I_{k_F \times (k+1)_F}$ denotes (recall) $k_F \times (k+1)_F$ matrix of all entries equal to one. For any $F$-denominated poset replace $I_{k_F \times (k+1)_F}$ by $B(k_F \times (k+1)_F)$ obtained from $I_{k_F \times (k+1)_F}$ via replacing adequately (in accordance with Hasse digraph) corresponding ones by zeros.

Another Proof : One may prove the above also as follows.

From motivating examples we know that $\mu(x_{r,i}, x_{s,j}) = \mu(x_r, x_s)$. Observe then how the recurrent definition of M"obius function matrix $\mu$ gives birth to daughter descendant of $\mu$ i.e. the block structure of M"obius function coding matrix $C(\mu)$ implying for $C(\mu)$ a recurrence allowing simple solution simultaneously with combinatorial interpretation of Krot on matrix $K = (K_s(r_F)) \equiv (K_{r,s})$, where $K_s(r_F) = |C(\mu)_{r,s}|$.

For that to call back the recurrent definition of the M"obius function where $x, y \in \Phi$ for $\Pi = (\Phi, \leq)$ and where - note: $\mu(x, y) = -1$ for $x < y$ :

$$\mu(x, y) = \begin{cases}
1, & x = y \\
- \sum_{x \leq z < y} \mu(x, z), & x < y
\end{cases}$$

The above recurrent definition M"obius function becomes - after linear order labeling has been applied - either $r, s, i \in N \cup \{0\}$ - as fixed-stated in this note, Remark 2. and/or fact that $r, s \in N$ - whereby $r, s$ are block-row and block-column indexes correspondingly - say it again - the above recurrent definition M"obius function in the case of $F$-denominated graded posets becomes ( $c_{r, r+1} = -1$ )

42
For that to see note that \( \forall x, y, z \in \Phi, \exists r, s, i \in N \) such that \( x_r \in \Phi_r \), \( y_s \in \Phi_s \), \( z_i \in \Phi_i \), hence for \( x_r < y_s \equiv r < s \) where \((\text{Important})\) \( r, s, i \) stay now for \textit{labels of independent sets} (levels) \( \{ \Phi_k \} \) i.e. label steps of La Scala i.e. label blocks. Thereby

\[
c_{r,s} = \mu(x_r, y_s) = - \sum_{x_r \leq z < y_s} \mu(x_r, z) = - \sum_{x_r \leq z_i < y_s} \mu(x_r, z_i) = \sum_{r \leq i < s} c_{r,i}.
\]

(Bear in mind Lemma 2.2. in order to get back to \( \mu \) make an inspection via example \((r \text{ stays for block - row label and } k > 1)\) and \(2\) use the Russian babushka in Babushka inspection i.e. apply the recurrent relation above subsequently till the end - till the smallest of size 1 babushka is encountered which is here \( c_{r,r+1} = -1 \). Use then trivial induction to state the validity of what follows below for all relevant values of variables \( r, s \in N \).

\[
c_{1,k} = - \sum_{1 \leq i < k} c_{1,i} = \left( - \sum_{1 \leq i < k_F} \right) \left( - \sum_{1 \leq i < (k-1)_F} \right) \ldots \left( - \sum_{1 \leq i < 3_F} \right) c_{1,2},
\]

i.e.

\[
c_{1,k} = (-1)^{k-1} \left( \sum_{1 \leq i < k_F} \right) \ldots \left( \sum_{1 \leq i < 4_F} \right) \left( \sum_{1 \leq i < 3_F} \right) (+1),
\]

i.e.

\[
c_{1,k} = -[1 + 1 = k] + [k > 2](-1)^{k-1} (k_F - 1)) \ldots (3_F - 1)) \ (1) =
\]

\[
= -[1 + 1 = k] + [k > 2](-1)^{k-1} \prod_{i=2+1}^{k} (i_F - 1).
\]

Similarly we conclude that now for arbitrary \( r, s \in N \)

\[
c_{r,s} = [s = r] - [s = r + 1] + [s > r + 1](-1)^{s-r} (s_F - 1)) \ldots (3_F - 1)) \ (1) =
\]

\[
= [s = r] - [s = r + 1] + [s > r + 1](-1)^{s-r} \prod_{i=r+2}^{s} (i_F - 1),
\]

Equivalently we conclude that now for arbitrary \( r, s \in N \cup \{0\} \)

\[
c_{r,s} = [s = r] - [s = r+1]+[s > r+1](-1)^{s-r} ((s - r - 1)_F - 1)) \ldots (3_F - 1)) \ (1) =
\]

\[
= [s = r] - [s = r + 1] + [s > r + 1](-1)^{s-r} \prod_{i=r+1}^{s-1} (i_F - 1),
\]

43
To colligate and to imagine hint. Starting from the left upper corner of La Scala of $\zeta$, $\mu, \ldots, \sigma \in \mathcal{I}(\Pi, R)$ down $\downarrow$ is biunivoquely starting from the "'bottom'" or "'root'" minimal elements level $\Phi_0$ up $\uparrow$ the Hasse digraph $(\Pi, \prec \cdot)$ uniquely representing the "'much, much more cobwebbed tree'" - the digraph $(\Pi, \leq)$.

**Descriptive - combinatorial interpretation:** Once the formula has been observed-derived as above the following turns out perceptible. Namely note that

1. for $F = N, [s \neq r]$, the Kroton matrix element $|C(\mu_N)_{r,s}|$, where

$$C(\mu_N)_{r,s} = c_{r,s} = [s > r](-1)^{s-r}[(r+1)N-1]^{s-r}$$

is equal to the number of heads’ dispositions of maximal chains tailed at one vertex of the $r$-th level and headed up at one vertex of the $s$-th level. This biunivoquely corresponds to the number of summands $|C(\mu_N)_{r,s}|$ entering the recurrence calculation of the $C(\mu_N)$ matrix ("the Russian babushka in Babushka introspection" with interchangeable signs) being in one to one correspondence with climbing up Hasse digraph i.e. descending down the matrix $\mu$ La Scala along the way uniquely encoded by the subjected to their heads disposition maximal chains

$$c = <x_r, x_{r+1}, \ldots, x_{s-1}, x_s>, x_i \in \Phi_i, i = r, r+1, \ldots, s-1, s$$

with the tail $r$ and the head $s$ fixed as start and the end points of the descending down the La Scala blocks trip ($\equiv$ climbing up the levels of the graded Hasse digraph $\langle \Phi, \prec \cdot \rangle$).

2. For the same interpretation in the general $F$-case apply the Upside Down Notation Principle.

According to and from the above one extracts the obvious now property of Kroton functions i.e. matrix elements of Kroton matrix $K = (K_s(r_F)) \equiv (K_{r,s})$

**Lemma 4.3** ($r, s \in N \cup \{0\}$).

$$K_{s+1}(r_F) = K_s(r_F) \bullet [s_F - 1], \ K_{r+1}(r_F) = 1$$

is equivalent to

$$K_{r,s} = -\sum_{r \leq i < s} (-1)^{s-i}K_{r,i} \quad K_{r+1}(r_F) = 1, \ s > r.$$ 

**Exercise.** Deliver the descriptive combinatorial interpretation of Kroton matrix in the language of hyper-boxes from [11].

**Compare.** All the above may be now compared with 1.4. Cobweb posets’ Möbius function where it has been proved that for $x \in \Phi_r, z \in \Phi_s$ and for $s > r, [x, z] = x \oplus \Phi_{r+1} \oplus \ldots \oplus \Phi_{s-1} \oplus z$

$$\mu(x, y) = (-1)^{s-r} \prod_{k=r+1}^{s-1} [k_F - 1].$$
**Whitney numbers.** Let us remind the notation: \( r_F = |\Phi_r| \). Let us then recall (see [49]) definitions of Whitney numbers of the first kind \( w_r(P) \) and Whitney numbers of second kind \( W_r(P) \) where \( P \) is any given graded poset with bounded independence sets - i.e. \( |\Phi_r| \in N \) for \( r \in N \cup \{0\} \) and we assume that \( |\Phi_0| = 1 = 0_F \) hence \( \Phi_0 = \{0\} \); \( 0 \in \Phi \) stays for minimal element of the poset \( P \). Now here are these definitions.

\[
w_r(P) = \sum_{x \in P, r(x) = r} \mu(0, x),
\]

\[
W_r(P) = \sum_{x \in P, r(x) = r} 1 = |\{x \in P : r(x) = r\}|.
\]

It is obvious just by notation that

\[ W_r(P) = r_F. \]

Of course in the general case of finite posets \( P_n = \bigcup_{k=0}^{n} \Phi_k(n) \) the number \( |\Phi_k| \) might depend on \( n \) and then we end up with an array \((W_k(P_n))\) of Whitney numbers as it is the case with binomials or Gaussian binomials for example - (see further classical examples in [49]).

According to Theorem 4 we have for **cobweb posets** i.e. for \( \Pi \)'s

\[
C(\mu_F)_{0,s} = c_{0,s} = [s \geq 0](-1)^s(1_F - 1)^s,
\]

or equivalently - as the values of \( \mu(x, y) \) depend only on the rank of its arguments

\[
\mu_F(0, x) = [r(x) \geq 0](-1)^{r(x)}(1_F - 1)^{r(x)},
\]

or equivalently (compare all this with (3) in [28])

\[
\mu_F(0, x) = [x = 0] - [r(x) = 1](1_F - 1) + [r(x) > 1](-1)^{r(x)} \prod_{k=1}^{r(x)-1} (k_F - 1),
\]

or equivalently just

\[
\mu_F(0, x) = [r(x) \geq 0](-1)^{r(x)}K_r(0_F).
\]

Consequently - as the values of \( \mu(x, y) \) depend only on the rank of its arguments - the Whitney numbers of the first kind for the denominated by \( F \) **cobweb poset** \( \Pi \) may be calculated along the formula

\[
w_r(\Pi) = \sum_{\{x \in \Pi : r(x) = r\}} \mu_F(0, x) = r_F \cdot \mu_F(0, x)
\]

i.e.

\[
w_r(\Pi) = r_F \cdot (-1)^rK_r(0_F).
\]

Naturally \( w_0(\Pi) = 1 \). Compare the above with (4) in [28].

Of course in the **general case** of finite posets \( P_n = \bigcup_{k=0}^{n} \Phi_k(n) \) the number \( |\Phi_k(n)| \) might depend on \( n \) and then we end up with an array \((w_k(P_n))\) of
Whitney numbers of the first kind as it is the case with binomials or Gaussian binomials for example - (see further classical examples in [49]).

To this end - consequently - let us consider characteristic polynomials $\chi_{P_n}(t)$, $n \geq 0$ defined as ([50,51],[28])

$$\chi_{P_n}(t) = \sum_{x \in P_n} \mu(0,x) t^{n-\tau(x)} = \sum_{k=0}^{n} w_k(P_n) t^{n-k}.$$ 

The formula for characteristic polynomials - here for specific $F$-denominated finite cobweb sub-posets $P_n = \bigoplus_{k=0}^{\infty} \Phi_k$ (i.e. $|\Phi_k(n)|$ does not depend on $n$)-namely - this formula for characteristic polynomials obviously is of the form

$$\chi_{P_n}(t) = \sum_{k=0}^{n} (-1)^k \cdot k_F \cdot x^{n-k} \cdot K_k(0_F)$$

or equivalently (compare with Theorem 3.1 in [28])

$$\chi_{P_n}(t) = x^n - x^{n-1} 1_F(1_F - 1) + \sum_{k=2}^{n} (-1)^k \cdot k_F \cdot x^{n-k} \cdot \prod_{r=1}^{k-1} (r_F - 1).$$

Recapitulation 4.2. natural join.

Recall that both $\leq$ partial order and $\prec \cdot$ cover relations are natural join of their bipartite correspondent chains, and this is exactly the reason and the very source of the Theorem 2 validity and shape. This is also the obvious clue statement for what follows. Note also that all on structure of any $P$ poset’s information is coded by the $\zeta$ matrix - a characteristic function of $\leq \in P = (\Phi, \leq)$. In short: $\zeta$ and equivalently $\mu = \zeta^{-1}$ are the Incidence algebra of $P$ coding elements. In brief - recall - the following identifications are self-evident:

$$\langle \Phi, \mu_F \rangle \equiv \langle \Phi, \zeta_F \rangle \equiv \langle \Phi, \leq \rangle \equiv \langle \Phi, C(\mu_F) \rangle.$$

5 $F$-nomial coefficients and $[Max]$ matrix of the $N$ weighted reflexive reachability relation

Call back now the Remark 1. Then consider the incidence algebra of the cobweb poset $\Pi$ as the algebra over (simultaneously) the ring $R$ and the Boolean algebra $2 \{1\}$. Denote this incidence algebra by $I(\Pi, R, 2 \{1\})$.

In the case $R = 2 \{1\}$ denote it by $I(\Pi, 2 \{1\}) \equiv I(\Pi, 2 \{1\}, 2 \{1\})$. Then for $\zeta \in I(\Pi, 2 \{1\})$ we have of course $\zeta^{-1} = \zeta$ ("reflexive reachability"), $\zeta_{\leq}^{-1} = \zeta_{\leq}$ (reflexive "cover") and so on. This is of course true for any poset relevant algebra i.e. for $I(P, 2 \{1\})$ - graded posets with finite set of minimal elements - included.

Consider now the algebra $I(\Pi, \mathbb{Z}, 2 \{1\})$. We shall define now another characteristic matrix $[Max]$ as the matrix of the "$N$ weighted" reflexive reachability relation. For that to do recall that in case of $I(\Pi, 2 \{1\})$
\[ \leq = \prec \cdot \ast = \text{reflexive reachability of } \prec \cdot \]

\[ \prec \cdot \ast \equiv (I - \prec \cdot)^{-1} = \prec.0^\ast + \prec.1^\ast + \prec.2^\ast + \ldots + \prec.k^\ast + \ldots = \text{ } \bigcup_{k \geq 0} \prec.k^\ast, \]

where binary relations \( \leq \subset \Phi \times \Phi \) and \( \prec \cdot \subset \Phi \times \Phi \) etc., subsets are identified with their matrices (see [3,2]), for example \( \prec \cdot \equiv \kappa \). In the above the Boolean powers of \( \kappa \) were in action while here below this are to be powers over the \( R = N, Z, 2^{(1)} \), etc.

**Definition 9** The \([\text{Max}]\) matrix of the \( N \) weighted reflexive reachability relation is defined by the over the ring \( Z \) power series formula

\[ [\text{Max}] = (I - \prec \cdot)^{-1} = \prec.0^\ast + \prec.1^\ast + \prec.2^\ast + \ldots + \prec.k^\ast + \ldots = \sum_{k \geq 0} \kappa^k = (I - \kappa)^{-1} \]

Naturally

\[ [\text{Max}]^{-1} = \delta - \kappa = \begin{bmatrix} I_1 & -B_1 & \text{zeros} \\ I_2 & -B_2 & \text{zeros} \\ I_3 & -B_3 & \text{zeros} \\ \vdots & \vdots & \vdots \\ I_n & -B_n & \text{zeros} \end{bmatrix} \]

where (recall from Section I. 1.5)

\[ [\text{Max}]_F = A_0^F + A_1^F + A_2^F + \ldots = (1 - A_F)^{-1} = \begin{bmatrix} I_1 F \times 1_F & B(1_F \times 2_F) & B(1_F \times 3_F) & B(1_F \times 4_F) & B(1_F \times 5_F) & \ldots \\ 0_2 F \times 1_F & I_2 F \times 2_F & B(2_F \times 3_F) & B(2_F \times 4_F) & B(2_F \times 5_F) & \ldots \\ 0_3 F \times 1_F & 0_3 F \times 2_F & I_3 F \times 3_F & B(3_F \times 4_F) & B(3_F \times 5_F) & \ldots \\ 0_4 F \times 1_F & 0_4 F \times 2_F & 0_4 F \times 3_F & I_4 F \times 4_F & B(4_F \times 5_F) & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix} \]

**Comment 6.** Combinatorial interpretation of \([\text{Max}]\).

\[ [\text{Max}]_{s,t} = \text{the number of all maximal chains in the poset interval} \]

\[ [x_{s,i}, x_{t,j}] = [x_s, x_t] = [s, t]. \]

where \( x_{s,i}, x_s \in \Phi_s \) and \( x_{t,j}, x_t \in \Phi_t \) for, say, \( s \leq t \) with the reflexivity (loop) convention adopted i.e. \( [\text{Max}]_{t,t} = 1. \)

The above obvious statement being taken into the account, in view and in conformity with the environment of the Theorem 1 we arrive at the trivial and powerful Theorem 5.

**Theorem 5.**

Consider any \( F \)-cobweb poset with \( F \) being a natural numbers valued sequence. Let \( x_k \equiv k \in \Phi_k \) and \( x_t \equiv t \in \Phi_n. \) Then

\[ \sum_{i \in \Phi_n} [\text{Max}]_{k,i} = \sum_{i = 1}^{n_F} [\text{Max}]_{k,i} = [C_{max} (\Phi_{k+1} \rightarrow \Phi_n)] = n_{F}^{\Phi_{k}}, \]

47
where \( m = n - k \).

**Note** that \( k, m, n \) are level labels (vertical) while \( i = 1, \ldots, n_F \) stays for horizontal - along the fixed level - label. With this in mind fixed we observe what follows.

**Corollary 5.1.**
Consider any \( F \)-cobweb poset with \( F \) being a **cobweb admissible** sequence. Let \( x_k \equiv k \in \Phi_k \) and \( x_n \equiv n \in \Phi_n \). Let \( n \geq k \equiv (n - m) \geq 2 \). Then

\[
[Max]_{k,n} |\Phi_n| = \frac{n}{F}
\]

i.e.

\[
[Max]_{k,n} = \binom{n - 1}{k - 2} F (n - k + 1)!
\]

**Corollary 5.2.** **colligate with heads dispositions allied to the Theorem 3.**
Consider any \( F \)-cobweb poset with \( F \) being a **cobweb admissible** sequence. Let \( x_k \equiv k \in \Phi_k \) and \( x_m \equiv n \in \Phi_n \). Let \( l + 1 = n \geq k \equiv (n - m) \geq 2 \). Then

\[
[Max]_{k,n} |\Phi_n| = \frac{n}{F}
\]

i.e.

\[
\binom{n - 1}{n - 1 - k} F (n - k + 1)! = \binom{n - 1}{k} F (n - 1 - k)! = [Max]_{k-2,n}
\]

\[
\binom{n - 1}{n - 1 - k} F = \frac{[Max]_{k-2,n}}{(n - 1 - k)!} = \frac{[Max]_{k-2,l+1}}{(l - k)!}
\]

i.e. \( (n - 1 = l) \)

\[
\binom{l}{k} F = \binom{l}{l - k} F = \frac{[Max]_{k-2,l+1}}{(l - k)!}
\]

**Note** that \( k, m, n, l \) are level labels (vertical) and this is convention to be kept till the end of this note.

The above obvious statement being taken into the account, in view and in conformity with the environment of Theorems 1 and 3 we are prompt to extract the trivial and powerful statement as the Theorem 6.

**Theorem 6.**
Consider any \( F \)-cobweb poset with \( F \) being a **cobweb admissible** sequence. Let \( x_k \equiv k \in \Phi_k \) and \( x_m \equiv n \in \Phi_n \). Let \( (l + 1) \geq k \geq 2 \). Then

\[
\binom{l}{k} F = \binom{l}{l - k} F = \frac{[Max]_{k-2,l+1}}{(l - k)!}
\]

i.e.

\[
\binom{l}{k} F = (l - k)! \text{th fraction of the number of all maximal chains in the poset interval } [x_{k-2}, x_{l+1}],
\]
where \( x_l \in \Phi_l \) and \( x_k \in \Phi_k \) with the reflexivity (loop) convention adopted i.e. \( \text{Max}_{n,n} = 1 \).

Farewell Exercises.

**Problem-Exercise 5.1.** Rewrite Markov property in \( F \)-nomials language.

**Problem-Exercise 5.2.** Find the inverse of \( \binom{1}{F} \), using the Theorem 4 and the knowledge of \( \text{Max}^{-1} \). Compare with [11].

Acknowledgments Thanks are expressed here to the Student of Gdańsk University Maciej Dziemiańczuk for applying his skillful TeX-nology with respect most of my articles since three years as well as for his general assistance and cooperation on KoDAGs investigation.

The author expresses his gratitude also Dr Ewa Krot-Sieniawska for her several years’ cooperation and vivid application of the alike material deserving Students’ admiration for her being such a comprehensible and reliable Teacher before she as Independent Person was fired by local Białystok University local authorities exactly on the day she had defended Rota and cobweb posets related dissertation with distinction.

References

[1] Garrett Birkhoff ; 1967 (1940). Lattice Theory, 3rd ed. American Mathematical Society

[2] A. Krzysztof Kwasniewski *Graded posets inverse zeta matrix formula* arXiv:0903.2575 [v2] Tue, 7 Jul 2009 18:31:04 GMT

[3] A. Krzysztof Kwasniewski *Graded posets zeta matrix formula* arXiv:0901.0155 [v1] Thu, 1 Jan 2009 01:43:35 GMT (15 pages Sylvester Night paper)

[4] A.K. Kwaśniewski , *Some Cobweb Posets Digraphs’ Elementary Properties and Questions* arXiv:0812.4319 [v1] Tue, 23 Dec 2008 00:40:41 GMT

[5] A.K. Kwaśniewski , *Cobweb Posets and KoDAG Digraphs are Representing Natural Join of Relations, their di-Bigraphs and the Corresponding Adjacency Matrices*, arXiv:math/0812.4066v1,[v1] Sun, 21 Dec 2008 23:04:48 GMT

[6] A. Krzysztof Kwaśniewski, *Fibonomial cumulative connection constants* arXiv:math/0406000v2 [v6] Fri, 20 Feb 2009 02:26:21 GMT , upgrade of Bulletin of the ICA vol. 44 (2005) 81-92 paper.

[7] A. Krzysztof Kwaśniewski, *On cobweb posets and their combinatorially admissible sequences*, Adv. Studies Contemp. Math. Vol. 18 No 1, 2009 17-32 ArXiv:0512578v4 [v5] Mon, 19 Jan 2009 21:47:32 GMT

[8] A. Krzysztof Kwaśniewski, *How the work of Gian Carlo Rota had influenced my group research and life*, arXiv:0901.2571[v4] Tue, 10 Feb 2009 03:42:43 GMT

49
[23] Donald E. Knuth, *Two Notes on Notation*, Amer. Math. Monthly 99 (1992), no. 5, 403-422. [arXiv:math/9205211v1, [v1] Fri, 1 May 1992 00:00:00 GMT]

[24] Ewa Krot, *A note on Möbius function and Möbius inversion formula of fibonacci cobweb poset*, Bulletin de la Societe des Sciences et des Lettres de Lodz (54), Série: Recherches sur les Deformations Vol. 44 (2004), 39-44. [arXiv:math/0404158v2, [v1] Wed, 7 Apr 2004 10:23:38 GMT [v2] Wed, 28 Apr 2004 07:37:14 GMT]

[25] Ewa Krot, *The First Ascent into the Incidence Algebra of the Fibonacci Cobweb Poset*, Advanced Studies in Contemporary Mathematics 11 (2005), No. 2, 179-184. [arXiv:math/0411007v1, [v1] Sun, 31 Oct 2004 12:46:51 GMT]

[26] Ewa Krot-Sieniawska, *On incidence algebras description of cobweb posets*, [arXiv:0802.3703v1 [v1] Tue, 26 Feb 2008 13:12:43 GMT]

[27] Ewa Krot-Sieniawska, *Reduced Incidence algebras description of cobweb posets and KoDAGs*, [arXiv:0802.4329v1, [v1] Fri, 29 Feb 2008 05:43:27 GMT]

[28] Ewa Krot-Sieniawska, *On Characteristic Polynomials of the Family of Cobweb Posets*, Proc. Jangjeon Math. Soc. Vo 111 (2008) no. 2, pp.105-111. [arXiv:0802.2696v1 [v1] Tue, 19 Feb 2008 18:53:38 GMT]

[29] A.K.Kwaśniewski, *Towards ψ-extension of Finite Operator Calculus of Rota* Rep. Math. Phys. 48 (3), 305-342 (2001). [arXiv:math/0402078v1, [v1] Thu, 5 Feb 2008 13:02:30 GMT]

[30] A.K. Kwaśniewski, *On extended finite operator calculus of Rota and quantum groups* Integral Transforms and Special Functions 2(4), 333 (2001)

[31] A. Krzysztof Kwaśniewski, *On Simple Characterizations of Sheffer ψ-polynomials and Related Propositions of the Calculus of Sequences*, Bull. Soc. Sci. Lett. Lodz Ser. Rech. Deform. 52, Ser. Rech. Deform. 36 (2002): 45-65. [arXiv:math/0312397v1 [v1] Sat, 20 Dec 2003 23:21:51 GMT]

[32] A.K.Kwaśniewski, *Main theorems of extended finite operator calculus* Integral Transforms and Special Functions 14, 333 (2003).

[33] A.K.Kwaśniewski, *On basic Bernoulli-Ward polynomials* Bulletin de la Societe des Sciences et des Lettres de Lodz (54) Serie: Recherches sur les Deformations Vol. 45 (2004) 5-10, ArXiv: [math/0405577]v1 [v1] Sun, 30 May 2004 00:32:47 GMT

[34] A.K.Kwaśniewski, *ψ-Appell polynomials‘ solutions of an umbral difference nonhomogeneous equation*, Bulletin de la Societe des Sciences et des Lettres de Lodz (54) Serie: Recherches sur les Deformations Vol. 45 (2004) 11-15 [v2 Sat, 13 Nov 2004 05:02:24 GMT]

[35] A.K.Kwaśniewski, *The logarithmic Fib-binomial formula*, Adv. Stud. Contemp. Math. v.9 No.1 (2004) 19-26 [arXiv:math/0406258v1 [v1] Sun, 13 Jun 2004 17:24:54 GMT]
[36] A. K. Kwaśniewski, *Fibonacci-triad sequences* Advan. Stud. Contemp. Math. 9 (2) (2004), 109-118.

[37] A. K. Kwaśniewski, *Fibonacci q-Gauss sequences* Adv. Stud. Contemp. Math., 8 No 2 (2004), 121-124. ArXive: math.CO/0405591 31 May 2004.

[38] A.K.Kwaśniewski, *q-Poisson, q-Dobinski, Rota and coherent states-a fortieth anniversary memoir*, Proc. Jangjeon Math. Soc. Vol 7 (2), 2004 pp. 95-98. arXiv:math/0402254v2 Tue, 17 Feb 2004 21:49:39 GMT

[39] A.K.Kwaśniewski, *Cauchy \(\hat{q}\)-identity and \(\hat{q}\)-Fermat matrix via \(\hat{q}\)-muting variables for Extended Finite Operator Calculus* Proc. Jangjeon Math. Soc. Vol 8 (2005) no. 2, pp.191-196. Inst.Comp.Sci.UwB/Preprint No. 60 , December , (2003). arXiv:math/0403107v1 Fri, 5 Mar 2004 09:57:32 GMT

[40] A.K.Kwaśniewski, *Cauchy identity and -Fermat matrix via muting variables for Extended Finite Operator Calculus* Proc. Jangjeon Math. Soc. Vol 8 (2005) no. 2. pp.191-196 arXiv:math/0403107v1 Fri, 5 Mar 2004 09:57:32 GMT

[41] A. K. Kwaśniewski *ψ-Pascal and \(\hat{q}\)-Pascal matrices - an accessible factory of one source identities and resulting applications*, Advanced Stud. Contemp. Math. 10 No2 (2005), 111-120. ArXiv:math.CO/0403123 v1 7 March 2004

[42] A. K. Kwaśniewski: *Information on some recent applications of umbral extensions to discrete mathematics* Review Bulletin of Calcutta Mathematical Society Vol 13 (2005), 1-10 ArXiv:math.CO/0411145v1 [v2] Wed, 21 Sep 2005 14:12:33 GMT

[43] A.K.Kwaniewski *Extended finite operator calculus - an example of algebraization of analysis* Bulletin of the Allahabad Mathematical Society Vol 20, (2005): 1-24 arXiv:0812.5027 v1, [v1] Tue, 30 Dec 2008 08:09:29 GMT

[44] A. K. Kwaśniewski, *On umbral extensions of Stirling numbers and Dobinski-like formulas* Adv. Stud. Contemp. Math. 12(2006) no. 1, pp.73-100. arXiv:math/0411002v5 [v5] Thu, 20 Oct 2005 02:12:47 GMT

[45] A. Krzysztof Kwaśniewski, *First Observations on Prefab Posets Whitney Numbers*, Advances in Applied Clifford Algebras Volume 18, Number 1 / February, 2008, 57-73. arXiv:0802.1696v1, [v1] Tue, 12 Feb 2008 19:47:18 GMT

[46] Ewa Krot *Further developments in finite fibonomial calculus*, Inst. Comp. Sci. UwB/Preprint No64/Febuary/2004, arXiv:math/0410550v2 [v1] Tue, 26 Oct 2004 10:37:43 GMT, [v2] Wed, 27 Oct 2004 08:01:59 GMT

[47] M. Dzieniańczuk, *On Cobweb Posets and Discrete F-Boxes Tilings*. arXiv:0802.3473v2, Thu, 2 Apr 2009 11:05:55 GMT

[48] M. Dzieniańczuk, *Counting Bipartite, k-Colored and Directed Acyclic Multi Graphs Through F-nomial coefficients*. arXiv:0901.1337v1.. Sun, 11 Jan 2009 21:00:27 GMT
[49] Joni S.A., Rota. G.-C., Sagan B.: From Sets to Functions: Three Elementary Examples, Discrete Mathematics, 37 (1981), p.193-202.

[50] Richard P. Stanley Enumerative combinatorics, Volume 2, by, Cambridge University Press, Cambridge, 1999

[51] Bruce E. Sagan, Why the characteristic polynomial factors Bull. Amer. Math. Soc. 36 (1999), 113-133.

[52] Richard Stanley Ordered structures and partitions (revision of 1971 Harvard University thesis) Memoirs of the Amer. Math. Soc. no. 119 (1972)