WEAKLY \((m, n)\)-CLOSED IDEALS AND \((m, n)\)-VON NEUMANN REGULAR RINGS

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WEAKLY \((m,n)\)-CLOSED IDEALS AND \((m,n)\)-VON NEUMANN REGULAR RINGS

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Abstract. Let \(R\) be a commutative ring with \(1 \neq 0\), \(I\) a proper ideal of \(R\), and \(m\) and \(n\) positive integers. In this paper, we define \(I\) to be a weakly \((m,n)\)-closed ideal if \(0 \neq x^m \in I\) for \(x \in R\) implies \(x^n \in I\), and \(R\) to be an \((m,n)\)-von Neumann regular ring if for every \(x \in R\), there is an \(r \in R\) such that \(x^m r = x^n\). A number of results concerning weakly \((m,n)\)-closed ideals and \((m,n)\)-von Neumann regular rings are given.

1. Introduction

Let \(R\) be a commutative ring with \(1 \neq 0\), \(I\) a proper ideal of \(R\), and \(n\) a positive integer. As in [2], \(I\) is an \(n\)-absorbing (resp., strongly \(n\)-absorbing) ideal of \(R\) if whenever \(x_1 \cdots x_{n+1} \in I\) for \(x_1, \ldots, x_{n+1} \in R\) (resp., \(I_1 \cdots I_{n+1} \subseteq I\) for ideals \(I_1, \ldots, I_{n+1}\) of \(R\)), then there are \(n\) of the \(x_i\)'s (resp., \(n\) of the \(I_i\)'s) whose product is in \(I\). As in [4], \(I\) is a semi-\(n\)-absorbing ideal of \(R\) if \(x^{n+1} \in I\) for \(x \in R\) implies \(x^n \in I\); and for positive integers \(m\) and \(n\), \(I\) is an \((m,n)\)-closed ideal of \(R\) if \(x^m \in I\) for \(x \in R\) implies \(x^n \in I\). And, as in [15], \(I\) is a weakly \(n\)-absorbing (resp., strongly weakly \(n\)-absorbing) ideal of \(R\) if whenever \(0 \neq x_1 \cdots x_{n+1} \in I\) for \(x_1, \ldots, x_{n+1} \in R\) (resp., \(0 \neq I_1 \cdots I_{n+1} \subseteq I\) for ideals \(I_1, \ldots, I_{n+1}\) of \(R\)), then there are \(n\) of the \(x_i\)'s (resp., \(n\) of the \(I_i\)'s) whose product is in \(I\).

In this paper, we define \(I\) to be a weakly semi-\(n\)-absorbing ideal of \(R\) if \(0 \neq x^{n+1} \in I\) for \(x \in R\) implies \(x^n \in I\). More generally, for positive integers \(m\) and \(n\), we define \(I\) to be a weakly \((m,n)\)-closed ideal of \(R\) if \(0 \neq x^m \in I\) for \(x \in R\) implies \(x^n \in I\). Thus \(I\) is a weakly semi-\(n\)-absorbing ideal if and only if \(I\) is a weakly \((n+1,n)\)-closed ideal. Moreover, an \((m,n)\)-closed ideal is a weakly \((m,n)\)-closed ideal, and the two concepts agree when \(R\) is reduced. Every proper ideal is weakly \((m,n)\)-closed for \(m \leq n\); so we usually assume that \(m > n\).

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The above definitions all concern generalizations of prime ideals. A 1-absorbing ideal is just a prime ideal, and a weakly 1-absorbing ideal is just a weakly prime ideal (a proper ideal $I$ of $R$ is a weakly prime ideal if $0 \neq xy \in I$ for $x, y \in R$ implies $x \in I$ or $y \in I$). A proper ideal is a radical ideal if and only if it is $(2, 1)$-closed. However, a weakly $(2, 1)$-closed ideal need not be a weakly radical ideal (a proper ideal $I$ of $R$ is a weakly radical ideal if $0 \neq x^n \in I$ for $x \in R$ and $n$ a positive integer implies $x \in I$) (see Example 2.3(b)).

Weakly prime ideals and weakly radical ideals were studied in [1], and weakly radical (semiprime) ideals have been studied in more detail in [6]. The concept of 2-absorbing ideals was introduced in [5] and then extended to $n$-absorbing ideals in [2]. Related concepts include 2-absorbing primary ideals (see [9]), weakly 2-absorbing ideals (see [11]), weakly 2-absorbing primary ideals (see [10]), and $(m, n)$-closed ideals (see [4]). Other generalizations and related concepts are investigated in [1], [6], [8], [11], [12], [13], and [15]. For a survey on $n$-absorbing ideals, see [7].

Let $R$ be a commutative ring and $m$ and $n$ positive integers. We define $R$ to be an $(m, n)$-von Neumann regular ring if for every $x \in R$, there is an $r \in R$ such that $x^m r = x^n$. Thus a $(2, 1)$-von Neumann regular ring is just a von Neumann regular ring. In this paper, we study weakly $(m, n)$-closed ideals, $(m, n)$-von Neumann regular rings, and the connections between the two concepts.

Let $m$ and $n$ be positive integers with $m > n$. Among the many results in this paper, we show in Theorem 2.6 that if $I$ is a weakly $(m, n)$-closed, but not $(m, n)$-closed, ideal of $R$, then $I \subseteq \text{Nil}(R)$. In Theorem 2.11, we determine when a proper ideal of $R_1 \times R_2$ is weakly $(m, n)$-closed, but not $(m, n)$-closed; and in Theorem 2.12, we investigate when a proper ideal of $R(+)M$ is weakly $(m, n)$-closed, but not $(m, n)$-closed. In Section 3, we introduce and investigate $(m, n)$-von Neumann regular elements and $(m, n)$-von Neumann regular rings. It is shown in Theorem 3.5 that every proper ideal of $R$ is weakly $(m, n)$-closed if and only if every non-nilpotent element of $R$ is $(m, n)$-von Neumann regular and $w^m = 0$ for every $w \in \text{Nil}(R)$. In Theorem 3.7, we show that every proper ideal of $R$ is $(m, n)$-closed if and only if $R$ is $(m, n)$-von Neumann regular. Finally, we define the concepts of $n$-regular and $\omega$-regular commutative rings as a way to measure how far a zero-dimensional commutative ring is from being von Neumann regular.

We assume throughout this paper that all rings are commutative with $1 \neq 0$, all $R$-modules are unitary, and $f(1) = 1$ for all ring homomorphisms $f : R \to T$. For such a ring $R$, let $\text{Nil}(R)$ be its ideal of nilpotent elements, $Z(R)$ its set of zero-divisors, $U(R)$ its group of units, $\text{char}(R)$ its characteristic, and $\dim(R)$ its (Krull) dimension. Then $R$ is reduced if $\text{Nil}(R) = \{0\}$ and $R$ is quasilocal if it has exactly one maximal ideal. As usual, $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Z}_n$ will denote the positive integers, integers, and integers modulo $n$, respectively. Several of our results use the $R(+)M$ construction as in [14]. Let $R$ be a commutative ring and $M$ an $R$-module. Then $R(+)M = R \times M$ is a commutative ring with identity $(1, 0)$.
under addition defined by \((r,m) + (s,n) = (r + s, m + n)\) and multiplication defined by \((r,m)(s,n) = (rs, rn + sm)\). Note that \(\{0\}(+)M^2 = \{0\}\); so \(\{0\}(+)M \subseteq \text{Nil}(R(+)M)\).

2. Properties of weakly \((m, n)\)-closed ideals

In this section, we give some basic properties of weakly \((m, n)\)-closed ideals and investigate weakly \((m, n)\)-closed ideals in several classes of commutative rings. We start by recalling the definitions of weakly semi-\(n\)-absorbing and weakly \((m, n)\)-closed ideals.

Definition 2.1. Let \(R\) be a commutative ring, \(I\) a proper ideal of \(R\), and \(m\) and \(n\) positive integers.

(1) \(I\) is a weakly semi-\(n\)-absorbing ideal of \(R\) if \(0 \neq x^{n+1} \in I\) for \(x \in R\) implies \(x^n \in I\).

(2) \(I\) is a weakly \((m, n)\)-closed ideal of \(R\) if \(0 \neq x^m \in I\) for \(x \in R\) implies \(x^n \in I\).

The proof of the next result follows easily from the definitions, and thus will be omitted.

Theorem 2.2. Let \(R\) be a commutative ring and \(m\) and \(n\) positive integers.

(1) If \(I\) is a weakly \(n\)-absorbing ideal of \(R\), then \(I\) is weakly semi-\(n\)-absorbing (i.e., weakly \((n+1, n)\)-closed).

(2) If \(I\) is a weakly \((m, n)\)-closed ideal of \(R\), then \(I\) is weakly \((m, n')\)-closed for every positive integer \(n' \geq n\).

(3) If \(I\) is a weakly \(n\)-absorbing ideal of \(R\), then \(I\) is weakly \((m, n)\)-closed for every positive integer \(m\).

(4) An intersection of weakly \((m, n)\)-closed ideals of \(R\) is weakly \((m, n)\)-closed.

While an \((m, n)\)-closed ideal is always weakly \((m, n)\)-closed, the converse need not hold. If an ideal is \((m, n)\)-closed, then it is also \((m', n')\)-closed for all positive integers \(m' \leq m\) and \(n' \geq n\) [4, Theorem 2.1(3)]. However, a weakly \((m, n)\)-closed ideal need not be weakly \((m', n)\)-closed for \(m' < m\). We next give two examples to illustrate these differences.

Example 2.3. (a) Let \(R = \mathbb{Z}_8\) and \(I = \{0, 4\}\). Then \(I\) is weakly \((3, 1)\)-closed since \(x^3 = 0\) for every nonunit \(x\) in \(R\). However, \(I\) is not \((3, 1)\)-closed since \(2^3 = 0 \in I\) and \(2 \notin I\), and \(I\) is not weakly \((2, 1)\)-closed since \(0 \neq 2^2 = 4 \in I\) and \(2 \notin I\).

(b) Let \(R = \mathbb{Z}_{16}\) and \(I = \{0, 8\}\). Then \(I\) is weakly \((2, 1)\)-closed since \(8\) is not a square in \(\mathbb{Z}_{16}\). However, \(I\) is not \((2, 1)\)-closed since \(4^2 = 0 \in I\) and \(4 \notin I\), and \(I\) is not a weakly radical ideal (and thus not weakly prime) since \(0 \neq 2^2 = 8 \in I\) and \(2 \notin I\).

The following definition will be useful for studying weakly \((m, n)\)-closed ideals that are not \((m, n)\)-closed (cf. [6, Definition 2.2]).
**Definition 2.4.** Let $R$ be a commutative ring, $m$ and $n$ positive integers, and $I$ a weakly $(m,n)$-closed ideal of $R$. Then $a \in R$ is an $(m,n)$-unbreakable-zero element of $I$ if $a^m = 0$ and $a^n \notin I$. (Thus $I$ has an $(m,n)$-unbreakable-zero element if and only if $I$ is not $(m,n)$-closed.)

**Theorem 2.5** (cf. [6, Theorem 2.3]). Let $R$ be a commutative ring, $m$ and $n$ positive integers, and $I$ a weakly $(m,n)$-closed ideal of $R$. If $a$ is an $(m,n)$-unbreakable-zero element of $I$, then $(a + i)^m = 0$ for every $i \in I$.

Proof. Let $i \in I$. Then

$$(a + i)^m = a^m + \sum_{k=1}^{m} \binom{m}{k} a^{m-k} i^k = 0 + \sum_{k=1}^{m} \binom{m}{k} a^{m-k} i^k \in I,$$

and similarly, $(a + i)^n \notin I$ since $a^n \notin I$. Thus $(a + i)^m = 0$ since $I$ is weakly $(m,n)$-closed. \[\square\]

**Theorem 2.6** (cf. [1, p. 839] and [6, Theorems 2.4 and 2.5]). Let $R$ be a commutative ring, $m$ and $n$ positive integers, and $I$ a weakly $(m,n)$-closed ideal of $R$. If $I$ is not $(m,n)$-closed, then $I \subseteq \text{Nil}(R)$. Moreover, if $I$ is not $(m,n)$-closed and $\text{char}(R) = m$ is prime, then $i^m = 0$ for every $i \in I$.

Proof. Since $I$ is a weakly $(m,n)$-closed ideal of $R$ that is not $(m,n)$-closed, $I$ has an $(m,n)$-unbreakable-zero element $a$. Let $i \in I$. Then $a^m = 0$, and $(a + i)^m = 0$ by Theorem 2.5; so $a, a + i \in \text{Nil}(R)$. Thus $i = (a + i) - a \in \text{Nil}(R)$; so $I \subseteq \text{Nil}(R)$.

The “moreover” statement is clear since $0 = (a + i)^m = a^m + i^m = i^m$ when $\text{char}(R) = m$ is prime. \[\square\]

The next two theorems are the analogs of the results for $(m,n)$-closed ideals in [4, Theorem 2.8] and [4, Theorem 2.10], respectively. Their proofs are similar, and thus will be omitted.

**Theorem 2.7.** Let $R$ be a commutative ring, $I$ a proper ideal of $R$, $S \subseteq R \setminus \{0\}$ a multiplicative set, and $m$ and $n$ positive integers. If $I$ is a weakly $(m,n)$-closed ideal of $R$, then $I_S$ is a weakly $(m,n)$-closed ideal of $R_S$.

**Theorem 2.8.** Let $f : R \rightarrow T$ be a homomorphism of commutative rings and $m$ and $n$ positive integers.

1. If $f$ is injective and $J$ is a weakly $(m,n)$-closed ideal of $T$, then $f^{-1}(J)$ is a weakly $(m,n)$-closed ideal of $R$. In particular, if $R$ is a subring of $T$ and $J$ is a weakly $(m,n)$-closed ideal of $T$, then $J \cap R$ is a weakly $(m,n)$-closed ideal of $R$.

2. If $f$ is surjective and $I$ is a weakly $(m,n)$-closed ideal of $R$ containing $\text{ker} f$, then $f(I)$ is a weakly $(m,n)$-closed ideal of $T$. In particular, if $I$ is a weakly $(m,n)$-closed ideal of $R$ and $J \subseteq I$ is an ideal of $R$, then $I/J$ is a weakly $(m,n)$-closed ideal of $R/J$. 
In the following theorems, we determine when an ideal of $R_1 \times R_2$ is weakly $(m,n)$-closed, but not $(m,n)$-closed. (Recall that an ideal of $R_1 \times R_2$ has the form $I_1 \times I_2$ for ideals $I_1$ of $R_1$ and $I_2$ of $R_2$.) It is easy to determine when an ideal of $R_1 \times R_2$ is $(m,n)$-closed.

**Theorem 2.9** (cf. [4, Theorem 2.12]). Let $R = R_1 \times R_2$, where $R_1$ and $R_2$ are commutative rings, $J$ a proper ideal of $R$, and $m$ and $n$ positive integers. Then the following statements are equivalent.

1. $J$ is an $(m,n)$-closed ideal of $R$.
2. $J = I_1 \times R_2$, $R_1 \times I_2$, or $I_1 \times I_2$ for $(m,n)$-closed ideals $I_1$ of $R_1$ and $I_2$ of $R_2$.

**Proof.** This follows directly from the definitions. \hfill $\square$

The analog of (1) $\Rightarrow$ (2) of Theorem 2.9 clearly holds for weakly $(m,n)$-closed ideals by Theorem 2.8(2), but our next theorem shows that the analog of (2) does not hold for weakly $(m,n)$-closed ideals.

**Theorem 2.10.** Let $R = R_1 \times R_2$, where $R_1$ and $R_2$ are commutative rings, $I_1$ a proper ideal of $R_1$, and $m$ and $n$ positive integers. Then the following statements are equivalent.

1. $I_1 \times R_2$ is a weakly $(m,n)$-closed ideal of $R$.
2. $I_1$ is an $(m,n)$-closed ideal of $R_1$.
3. $I_1 \times R_2$ is an $(m,n)$-closed ideal of $R$.

A similar result holds for $R_1 \times I_2$ when $I_2$ is a proper ideal of $R_2$.

**Proof.** (1) $\Rightarrow$ (2) $I_1$ is a weakly $(m,n)$-closed ideal of $R_1$ by Theorem 2.8(2). If $I_1$ is not an $(m,n)$-closed ideal of $R_1$, then $I_1$ has an $(m,n)$-unbreakable-zero element $a$. Thus $(0,0) \neq (a,1)^m \in I_1 \times R_2$, but $(a,1)^n \notin I_1 \times R_2$, a contradiction. Hence $I_1$ is an $(m,n)$-closed ideal of $R_1$.

(2) $\Rightarrow$ (3) This is clear (cf. [4, Theorem 2.12]).

(3) $\Rightarrow$ (1) This is clear by definition. \hfill $\square$

**Theorem 2.11.** Let $R = R_1 \times R_2$, where $R_1$ and $R_2$ are commutative rings, $J$ a proper ideal of $R$, and $m$ and $n$ positive integers. Then the following statements are equivalent.

1. $J$ is a weakly $(m,n)$-closed ideal of $R$ that is not $(m,n)$-closed.
2. $J = I_1 \times I_2$ for proper ideals $I_1$ of $R_1$ and $I_2$ of $R_2$ such that either
   (a) $I_1$ is a weakly $(m,n)$-closed ideal of $R_1$ that is not $(m,n)$-closed, $y^m = 0$ whenever $y^m \in I_2$ for $y \in R_2$ (in particular, $x^m = 0$ for every $i \in I_2$), and if $0 \neq z^n \in I_1$ for some $x \in R_1$, then $I_2$ is an $(m,n)$-closed ideal of $R_2$, or
   (b) $I_2$ is a weakly $(m,n)$-closed ideal of $R_2$ that is not $(m,n)$-closed, $y^m = 0$ whenever $y^m \in I_1$ for $y \in R_1$ (in particular, $x^m = 0$ for every $i \in I_1$), and if $0 \neq z^n \in I_2$ for some $x \in R_2$, then $I_1$ is an $(m,n)$-closed ideal of $R_1$. 

\hfill $\square$
Proof. (1) $\Rightarrow$ (2) Since $J$ is not an $(m, n)$-closed ideal of $R$, by Theorem 2.10 we have $J = I_1 \times I_2$, where $I_1$ is a proper ideal of $R_1$ and $I_2$ is a proper ideal of $R_2$. Since $J$ is not an $(m, n)$-closed ideal of $R$, either $I_1$ is a weakly $(m, n)$-closed ideal of $R_1$ that is not $(m, n)$-closed or $I_2$ is a weakly $(m, n)$-closed ideal of $R_2$ that is not $(m, n)$-closed. Assume that $I_1$ is a weakly $(m, n)$-closed ideal of $R_1$ that is not $(m, n)$-closed. Thus $I_1$ has an $(m, n)$-unbreakable-zero element $a$. Assume that $y^m \in I_2$ for $y \in R_2$. Since $a$ is an $(m, n)$-unbreakable-zero element of $I_1$ and $(a, y)^m \in J$, we have $(a, y)^m = (0, 0)$. Hence $y^m = 0$ (in particular, $i^m = 0$ for every $i \in I_2$). Now assume that $0 \neq x^m \in I_1$ for some $x \in R_1$. Let $y \in R_2$ such that $y^m \in I_2$. Then $(0, 0) \neq (x, y)^m \in J$. Thus $y^n \in I_2$, and hence $I_2$ is an $(m, n)$-closed ideal of $R_2$. Similarly, if $I_2$ is a weakly $(m, n)$-closed ideal of $R_2$ that is not $(m, n)$-closed, then $y^m = 0$ whenever $y^m \in I_1$ for $y \in R_1$ (in particular, $i^m = 0$ for every $i \in I_1$), and if $0 \neq x^m \in I_2$ for some $x \in R_2$, then $I_1$ is an $(m, n)$-closed ideal of $R_1$.

(2) $\Rightarrow$ (1) Suppose that $I_1$ is a weakly $(m, n)$-closed proper ideal of $R_1$ that is not $(m, n)$-closed, $y^m = 0$ whenever $y^m \in I_2$ for $y \in R_2$ (in particular, $i^m = 0$ for every $i \in I_2$), and if $0 \neq x^m \in I_1$ for some $x \in R_1$, then $I_2$ is an $(m, n)$-closed ideal of $R_2$. Let $a$ be an $(m, n)$-unbreakable-zero element of $I_1$. Then $(a, 0)$ is an $(m, n)$-unbreakable-zero element of $J$. Thus $J$ is not an $(m, n)$-closed ideal of $R$. Now assume that $(0, 0) \neq (x, y)^m = (x^m, y^m) \in J$ for $x \in R_1$ and $y \in R_2$. Then $(0, 0) \neq (x, y)^m = (x^m, 0) \in J$ and $0 \neq x^m \in I_1$. Since $I_1$ is a weakly $(m, n)$-closed ideal of $R_1$ and $I_2$ is an $(m, n)$-closed ideal of $R_2$, we have $(x, y)^m \in J$. Similarly, assume that $I_2$ is a weakly $(m, n)$-closed ideal of $R_2$ that is not $(m, n)$-closed, $y^m = 0$ whenever $y^m \in I_1$ for $y \in R_1$ (in particular, $i^m = 0$ for every $i \in I_1$), and if $0 \neq x^m \in I_2$ for some $x \in R_2$, then $I_1$ is an $(m, n)$-closed ideal of $R_1$. Then again, $J$ is a weakly $(m, n)$-closed ideal of $R$ that is not $(m, n)$-closed. □

We next consider when certain ideals of $R(+)M$ are weakly $(m, n)$-closed.

Theorem 2.12. Let $R$ be a commutative ring, $I$ a proper ideal of $R$, $M$ an $R$-module, and $m$ and $n$ positive integers. Then the following statements are equivalent.

1. $I(+)M$ is a weakly $(m, n)$-closed ideal of $R(+)M$ that is not $(m, n)$-closed.

2. $I$ is a weakly $(m, n)$-closed ideal of $R$ that is not $(m, n)$-closed and $m(a^{m-1}M) = 0$ for every $(m, n)$-unbreakable-zero element $a$ of $I$.

Proof. (1) $\Rightarrow$ (2) Let $J = I(+)M$. Assume that $0 \neq r^m \in I$ for $r \in R$. Thus $(0, 0) \neq (r, 0)^m = (r^m, 0) \in J$. Hence $(r, 0)^n = (r^n, 0) \in J$; so $r^n \in I$. Thus $I$ is a weakly $(m, n)$-closed ideal of $R$. Since $J$ is not $(m, n)$-closed, $J$, and hence $I$, has an $(m, n)$-unbreakable-zero element; so $I$ is not $(m, n)$-closed. Let $a$ be an $(m, n)$-unbreakable-zero element of $I$ and $x \in M$. Then $(a, x)^m = (a^m, m(a^{m-1}x)) \in J$. Since $a^n \notin I$, we have $(a, x)^m = (a^m, m(a^{m-1}x)) = (0, 0)$. Thus $m(a^{m-1}M) = 0$. 

(2) $\Rightarrow$ (1) Let $J = I(+)M$. Assume that $0 \neq r^m \in I$ for $r \in R$. Thus $(0, 0) \neq (r, 0)^m = (r^m, 0) \in J$. Hence $(r, 0)^n = (r^n, 0) \in J$; so $r^n \in I$. Thus $I$ is a weakly $(m, n)$-closed ideal of $R$. Since $J$ is not $(m, n)$-closed, $J$, and hence $I$, has an $(m, n)$-unbreakable-zero element; so $I$ is not $(m, n)$-closed. Let $a$ be an $(m, n)$-unbreakable-zero element of $I$ and $x \in M$. Then $(a, x)^m = (a^m, m(a^{m-1}x)) \in J$. Since $a^n \notin I$, we have $(a, x)^m = (a^m, m(a^{m-1}x)) = (0, 0)$. Thus $m(a^{m-1}M) = 0$. 


(2) ⇒ (1) Since $I$ is a weakly $(m, n)$-closed ideal of $R$ that is not $(m, n)$-closed, $I$ has an $(m, n)$-unbreakable-zero element $a$. Hence $(a, 0)$ is an $(m, n)$-unbreakable-zero element of $J = I(+M)$. Thus $J$ is not an $(m, n)$-closed ideal of $A$. Suppose that $(0, 0) \neq (r, y)^m = (r^m, m(r^{n-1}y)) \in J$. Then $r$ is not an $(m, n)$-unbreakable-zero element of $I$ by hypothesis. Hence $(r^n, n(r^{m-1}y)) = (r, y)^n \in J$; so $J$ is a weakly $(m, n)$-closed ideal of $A$ that is not $(m, n)$-closed. □

We end this section with another way to construct weakly $(m, n)$-closed ideals that are not $(m, n)$-closed. See [4, Theorems 3.1 and 3.8] for similar results for $(m, n)$-closed ideals.

**Theorem 2.13.** Let $R$ be an integral domain and $I = p^kR$ a principal ideal of $R$, where $p$ is a prime element of $R$ and $k$ a positive integer. Let $m$ be a positive integer such that $m < k$, and write $k = mq + r$ for integers $q, r$, where $q \geq 1$ and $0 \leq r < m$. Then $J = I/p^kR$ is a weakly $(m, n)$-closed ideal of $R/p^kR$ that is not $(m, n)$-closed for positive integers $n < m$ and $c \geq k + 1$ if and only if $r \neq 0, k + 1 \leq c \leq m(q + 1)$, and $n(q + 1) < k$.

**Proof.** Suppose that $J$ is a weakly $(m, n)$-closed ideal of $R/p^kR$ that is not $(m, n)$-closed for positive integers $n < m$ and $c \geq k + 1$. It is clear that $r \neq 0$, for if $r = 0$, then $0 \neq (p^i)^m + p^eR \in J$, but $(p^i)^n + p^eR \notin J$. Since $q + 1$ is the smallest positive integer such that $(p^{(q+1)}m + p^eR \in J$ and $J$ is not $(m, n)$-closed, we have $0 = (p^{(q+1)})^m + p^eR \in J$ and $(p^{(q+1)})^n + p^eR \notin J$. Thus $n(q + 1) < k$ and $k + 1 \leq c \leq (q + 1)m$.

Conversely, assume that $r \neq 0, k + 1 \leq c \leq m(q + 1)$, and $n(q + 1) < k$. Let $x \in R/p^kR$ such that $x^m \in J$. Then $x = p^i y + p^e R$ for some $y \in R$ such that $p^{(i+1)}y \in R$. Since $x^m = (p^i)^m + p^eR \in J$, we have $i \geq q + 1$. Thus by hypothesis, $x^m = 0$ in $R/p^eR$. Since $0 = (p^{(q+1)})^m + p^eR \in J$ and $n(q + 1) < k$, we have $(p^{(q+1)})^n + p^eR \notin J$. Hence $J$ is not $(m, n)$-closed. □

**Example 2.14.** (a) Let $R = \mathbb{Z}$, $I = 2^{12}\mathbb{Z}$, and $J = I/2^{13}\mathbb{Z}$. Then by Theorem 2.13, $J$ is a weakly $(5, 3)$-closed ideal of $\mathbb{Z}/2^{13}\mathbb{Z}$ that is not $(5, 3)$-closed.

(b) Let $R$, $I$, and $J$ be as in part (a) above. Then $J(+)I$ is a weakly $(5, 3)$-closed ideal of $\mathbb{Z}/2^{13}\mathbb{Z}(+)I$ that is not $(5, 3)$-closed by Theorem 2.12.

3. $(m, n)$-von Neumann regular rings

In this section, we introduce the concepts of $(m, n)$-von Neumann regular elements and $(m, n)$-von Neumann regular rings and use them to determine when every proper ideal of $R$ is $(m, n)$-closed or weakly $(m, n)$-closed. We also define the related concepts of $n$-regular and $\omega$-regular commutative rings. First, we handle the case for ideals contained in $\text{Nil}(R)$.

**Theorem 3.1.** Let $R$ be a commutative ring and $m$ and $n$ positive integers with $m > n$. Then every ideal of $R$ contained in $\text{Nil}(R)$ is weakly $(m, n)$-closed if and only if $w^m = 0$ for every $w \in \text{Nil}(R)$. 
Proof. Suppose that every ideal of $R$ contained in $\text{Nil}(R)$ is weakly $(m,n)$-closed, but $w^m \neq 0$ for some $w \in \text{Nil}(R)$. Let $J = w^mR \subseteq \text{Nil}(R)$. Then $J$ is weakly $(m,n)$-closed and $0 \neq w^m \in J$; so $w^n \in J$ and $w^n \neq 0$ since $n < m$. Thus $w^n = w^m a$ for some $a \in R$, and hence $w^n(1 - w^{m-n}a) = 0$. Then $1 - w^{m-n}a \in U(R)$ since $w^{m-n}a \in \text{Nil}(R)$; so $w^n = 0$, a contradiction. Thus $w^m = 0$ for every $w \in \text{Nil}(R)$.

Conversely, suppose that $w^m = 0$ for every $w \in \text{Nil}(R)$. Then every ideal of $R$ contained in $\text{Nil}(R)$ is weakly $(m,n)$-closed by definition. 

Recall that $x \in R$ is a von Neumann regular element of $R$ if $x^2r = x$ for some $r \in R$. Similarly, $x \in R$ is a $\pi$-regular element of $R$ if $x^{2n}=x^n$ for some $r \in R$ and positive integer $n$. Thus $R$ is a von Neumann regular ring (resp., $\pi$-regular ring) if and only if every element of $R$ is von Neumann regular (resp., $\pi$-regular). It is well known that $R$ is $\pi$-regular (resp., von Neumann regular) if and only if dim$(R) = 0$ (resp., $R$ is reduced and dim$(R) = 0$) [14, Theorem 3.1, p. 10]. A ring $R$ is a strongly $\pi$-regular ring if there is a positive integer $n$ such that for every $x \in R$, we have $x^{2n}=x^n$ for some $r \in R$. For a recent article on von Neumann regular and related elements of a commutative ring, see [3]. These concepts are generalized in the next definition.

**Definition 3.2.** Let $R$ be a commutative ring and $m$ and $n$ positive integers. Then $x \in R$ is an $(m,n)$-von Neumann regular element of $R$ (or $(m,n)$-vnr for short) if $x^{mn} = x^n$ for some $r \in R$. If every element of $R$ is $(m,n)$-vnr, then $R$ is an $(m,n)$-von Neumann regular ring.

Thus a commutative ring $R$ is von Neumann regular if and only if it is $(2,1)$-von Neumann regular, and $R$ is strongly $\pi$-regular if and only if it is $(2n,n)$-von Neumann regular for some positive integer $n$. The next theorem gives some basic facts about $(m,n)$-vnr elements.

**Theorem 3.3.** Let $R$ be a commutative ring, $x \in R$, and $m$ and $n$ positive integers.

1. $x$ is $(m,n)$-vnr for $m \leq n$ (so we usually assume that $m > n$).
2. If $x$ is $(m,n)$-vnr, then $x$ is $(m',n')$-vnr for all positive integers $m' \leq m$ and $n' \geq n$.
3. If $x \in U(R)$ or $x = 0$, then $x$ is $(m,n)$-vnr for all positive integers $m$ and $n$.
4. If $x \in R \setminus (Z(R) \cup U(R))$, then $x$ is $(m,n)$-vnr if and only if $m \leq n$.
5. If $x^n = 0$, then $x$ is $(m,n)$-vnr for every positive integer $m$.
6. If $x^k = 0$ and $x^{k-1} \neq 0$ for an integer $k \geq 2$, then $x$ is $(m,n)$-vnr if and only if $m \leq n$ or $n \geq k$.
7. If $x$ is $(m,n)$-vnr with $m > n$, then $x$ is $(m+1,n)$-vnr. Moreover, in this case, $x$ is $(m',n')$-vnr for all positive integers $m'$ and $n' \geq n$. Thus $R$ is von Neumann regular if and only if $R$ is $(m,n)$-von Neumann regular for all positive integers $m$ and $n$. 

Corollary 3.4. Let \( m > n \). Then the following statements are equivalent.

1. \( \pi \)-closed and 0 \( \neq x \) for some \( r \in R \).
2. \( \pi \)-closed.
3. \( \pi \)-closed.
4. \( \pi \)-closed.
5. \( \pi \)-closed.
6. \( \pi \)-closed.
7. \( \pi \)-closed.

\( m \neq n \) implies \( x^m r = x^n r = x^n (x^m - n^m r) = x^n (x^{m+1} r^2) \) with \( x^{m+1} r^2 \in R \). Thus \( x \) is \( (m+1, n) \)-vnr. The “moreover” statement follows by induction and (2).

\( \square \)

Corollary 3.6. Let \( R \) be a commutative ring and \( m \) and \( n \) positive integers with \( m > n \). Then \( R \) is \( (m, n) \)-vnr Neumann regular if and only if \( R \) is \( (m', n') \)-vnr Neumann regular for all positive integers \( m' \) and \( n' \geq n \). In particular, if \( R \) is \( (m, n) \)-vnr Neumann regular, then \( R \) is strongly \( \pi \)-regular, and thus \( \dim(R) = 0 \).

We next determine when every proper ideal of \( R \) is weakly \((m, n)\)-closed.

Theorem 3.5. Let \( R \) be a commutative ring and \( m \) and \( n \) positive integers with \( m > n \). Then the following statements are equivalent.

1. Every proper ideal of \( R \) is weakly \((m, n)\)-closed.
2. Every non-nilpotent element of \( R \) is \((m, n)\)-vnr and \( w^m = 0 \) for every \( w \in \Nil(R) \).

Proof. (1) \( \Rightarrow \) (2) Since every ideal of \( R \) contained in \( \Nil(R) \) is weakly \((m, n)\)-closed, \( w^m = 0 \) for every \( w \in \Nil(R) \) by Theorem 3.1. Let \( x \in U(R) \), then \( x \) is \((m, n)\)-vnr by Theorem 3.3(3). If \( x \notin U(R) \), then \( I = x^m R \) is weakly \((m, n)\)-closed and \( 0 \neq x^m \in I \); so \( x^m \in I \). Thus \( x^n = x^n r \) for some \( r \in R \), and hence \( x \) is \((m, n)\)-vnr.

(2) \( \Rightarrow \) (1) Let \( I \) be a proper ideal of \( R \) and \( 0 \neq x^m \in I \) for \( x \in R \). Then \( x \notin \Nil(R) \); so \( x \) is \((m, n)\)-vnr. Thus \( x^m r = x^n r \) for some \( r \in R \), so \( x^n = x^m r \in I \). Hence \( I \) is weakly \((m, n)\)-closed.

\( \square \)

In view of Theorem 3.5, we have the following result.

Corollary 3.6. Let \( R \) be a reduced commutative ring and \( m \) and \( n \) positive integers. Then the following statements are equivalent.

1. Every proper ideal of \( R \) is weakly \((m, n)\)-closed.
2. Every proper ideal of \( R \) is \((m, n)\)-closed.
3. \( R \) is \((m, n)\)-vnr Neumann regular.

The following result is the analog of Theorem 3.5 for \((m, n)\)-closed ideals.

Theorem 3.7. Let \( R \) be a commutative ring and \( m \) and \( n \) positive integers. Then the following statements are equivalent.

1. Every proper ideal of \( R \) is \((m, n)\)-closed.
(2) $R$ is $(m,n)$-von Neumann regular.

Proof. (1) $\Rightarrow$ (2) Let $x \in R$. If $x \in U(R)$, then $x$ is $(m,n)$-vnr by Theorem 3.3(3). If $x \notin U(R)$, then $I = x^mR$ is $(m,n)$-closed and $x^m \in I$. Thus $x^n \in I$; so $x^n = x^nr$ for some $r \in R$. Hence $x$ is $(m,n)$-vnr, and thus $R$ is $(m,n)$-von Neumann regular.

(2) $\Rightarrow$ (1) Let $I$ be a proper ideal of $R$ and $x^m \in I$ for $x \in R$. Since $x$ is $(m,n)$-vnr, $x^nr = x^n$ for some $r \in R$. Thus $x^n = x^nr \in I$; so $I$ is $(m,n)$-closed. $\square$

Of course, we are mainly interested in the case when $m > n$. The next theorem incorporates Theorem 3.7 with another characterization ([4, Theorem 2.14]) of when every proper ideal is $(m,n)$-closed. Note that in Theorem 3.8(3) below, there are no conditions on $m$ other than $m > n$.

**Theorem 3.8.** Let $R$ be a commutative ring and $m$ and $n$ positive integers with $m > n$. Then the following statements are equivalent.

1. Every proper ideal of $R$ is $(m,n)$-closed.
2. $R$ is $(m,n)$-von Neumann regular.
3. $\dim(R) = 0$ and $w^n = 0$ for every $w \in \text{Nil}(R)$.

Proof. (1) $\Leftrightarrow$ (2) is Theorem 3.7 and (1) $\Leftrightarrow$ (3) is [4, Theorem 2.14]. $\square$

Theorem 3.8 gives a nice ring-theoretic characterization of $(m,n)$-von Neumann regular rings (for $m > n$). This can now be used to give a characterization of strongly $\pi$-regular commutative rings which strengthens Corollary 3.4.

**Theorem 3.9.** Let $R$ be a commutative ring. Then the following statements are equivalent.

1. $R$ is strongly $\pi$-regular.
2. There are positive integers $m$ and $n$ with $m > n$ such that $R$ is $(m,n)$-von Neumann regular.
3. There is a positive integer $n$ such that $R$ is $(m,n)$-von Neumann regular for every positive integer $m$.
4. $\dim(R) = 0$ and there is a positive integer $n$ such that $w^n = 0$ for every $w \in \text{Nil}(R)$.

Proof. (1) $\Rightarrow$ (2) A strongly $\pi$-regular ring is $(2n,n)$-von Neumann regular for some positive integer $n$.

(2) $\Rightarrow$ (3) This follows from Corollary 3.4.

(3) $\Rightarrow$ (1) In particular, $R$ is $(2n,n)$-von Neumann regular, and thus strongly $\pi$-regular.

(2) $\Leftrightarrow$ (4) This is just (2) $\Leftrightarrow$ (3) of Theorem 3.8. $\square$

We next investigate in more detail the pairs $(m,n)$ for which a commutative ring $R$ or an $x \in R$ is $(m,n)$-von Neumann regular.
Definition 3.10. Let $R$ be a commutative ring, $x \in R$, and $k$ a positive integer.

1. $V(R,x) = \{(m,n) \in \mathbb{N} \times \mathbb{N} \mid x \text{ is } (m,n)\text{-vnr}\}$.
2. $V(R) = \{(m,n) \in \mathbb{N} \times \mathbb{N} \mid R \text{ is } (m,n)\text{-von Neumann regular}\}$.
3. $B_k = \{(m,n) \in \mathbb{N} \times \mathbb{N} \mid m \leq n \text{ or } n \geq k\}$.
4. $B_\omega = \{(m,n) \in \mathbb{N} \times \mathbb{N} \mid m \leq n\}$.

Then $V(R) = \bigcap_{x \in R} V(R,x)$ and $\mathbb{N} \times \mathbb{N} = B_1 \supseteq B_2 \supseteq \cdots \supseteq B_\omega$.

Theorem 3.11. Let $R$ be a commutative ring and $x \in R$.

1. $V(R,x) = B_k$, where $k$ is the smallest positive integer such that $(i,k) \in V(R,x)$ for some $i > k$. (Thus $k$ is the smallest positive integer such that $x$ is $(m,k)$-vnr for every positive integer $m$.) If no such $k$ exists, then $V(R,x) = B_\omega$.
2. $V(R) = B_k$, where $k$ is the smallest positive integer such that $(i,k) \in V(R,x)$ for some $i > k$ and every $x \in R$. (Thus $k$ is the smallest positive integer such that $x$ is $(m,k)$-vnr for every $x \in R$ and positive integer $m$.) If no such $k$ exists, then $V(R) = B_\omega$.

Proof. (1) follows directly from Theorem 3.3(7). Thus (2) holds by definition. □

These ideas can also be used to classify zero-dimensional commutative rings.

Definition 3.12. Let $R$ be a commutative ring and $n$ a positive integer.

1. $R$ is $n$-regular if $V(R) = B_n$, i.e., $n$ is the smallest positive integer such that for every $x \in R$ and positive integer $m$, $x^n = x^mr_m$ for some $r_m \in R$.
2. $R$ is $\omega$-regular if for every $x \in R$, $V(R,x) = B_{n_x}$ for some positive integer $n_x$, but $V(R) = B_\omega$.

A commutative ring $R$ is von Neumann regular if and only if it is 1-regular, and $R$ is strongly $\pi$-regular if and only if it is $n$-regular for some positive integer $n$. Note that $R$ is $\pi$-regular if and only if every $x \in R$ is $(m,n)$-vnr for some positive integers $m$ and $n$ with $m > n$, but a $\pi$-regular ring may be $\omega$-regular (see Example 3.13(d)). Thus $R$ is $\alpha$-regular for $\alpha$ a positive integer or $\omega$ if and only if $R$ is $\pi$-regular, if and only if $\dim(R) = 0$. So, in some sense, this concept measures how far a zero-dimensional commutative ring is from being von Neumann regular.

We next give several examples. In particular, we show that if $\alpha$ is any positive integer or $\omega$, there is a quasilocal commutative ring $R_\alpha$ that is $\alpha$-regular.

Example 3.13. Let $R$ be a commutative ring.
(a) Suppose that there is an $x \in R \setminus (Z(R) \cup U(R))$ (so $\dim(R) > 0$). Then $V(R) = V(R,x) = B_\omega$ by Theorem 3.3(4). Thus $R$ is not $\omega$-regular or $n$-regular for any positive integer $n$.

(b) Suppose that $R$ is quasi-local with maximal ideal $M = (x)$ with $x^k = 0$ and $x^{k-1} \neq 0$ for an integer $k \geq 2$. Then $V(R) = B_k$ by Theorem 3.3(3),(6); so $R$ is $k$-regular. This also holds for $k = 1$ since a field is von Neumann regular. In particular, for a prime $p$ and any positive integer $k$, $V(\mathbb{Z}_p^k) = B_k$, and thus $\mathbb{Z}_p^k$ is $k$-regular.

(c) Let $R_1$ and $R_2$ be commutative rings. Then $x = (x_1, x_2) \in R_1 \times R_2$ is $(m,n)$-vnr if and only if $x_1$ and $x_2$ are $(m,n)$-vnr in $R_1$ and $R_2$, respectively. Thus $V(R_1 \times R_2) = B_{k_1} \times B_{k_2}$, where $V(R_1) = B_{k_1}$, $V(R_2) = B_{k_2}$, and $k = \max\{k_1, k_2\}$; so $R_1 \times R_2$ is $\max\{k_1, k_2\}$-regular when $R_1$ and $R_2$ are $k_1$-regular and $k_2$-regular, respectively. In particular, for distinct primes $p_1, \ldots, p_r$, positive integers $k_1, \ldots, k_r$, and $k = \max\{k_1, \ldots, k_r\}$, $V(\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_r}^{k_r}) = B_k$, and hence $\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_r}^{k_r}$ is $k$-regular.

(d) Let $R = \mathbb{Z}_2[\{X_n\}_{n \in \mathbb{N}}]/\langle\{X_n+1\}_{n \in \mathbb{N}}\rangle = \mathbb{Z}_2[\{x_n\}_{n \in \mathbb{N}}]$. Then $R$ is a zero-dimensional quasi-local commutative ring with maximal ideal $Nil(R) = \langle\{x_n\}_{n \in \mathbb{N}}\rangle$; so $R$ is $\pi$-regular. Thus every $x \in R$ has $V(R,x) = B_k$ for some positive integer $k$ and $V(R,x_n) = B_{n+1}$ by Theorem 3.3(3),(6); so $V(R) = B_\omega$. Hence $R$ is $\omega$-regular.

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