MODULI SPACE OF M-MINIMAL-DOMINANT RATIONAL CURVES ON LOW DEGREE COMPLETE INTERSECTIONS

XUANYU PAN

Abstract. This is the second in a sequence of papers on the geometry of spaces of minimal-dominant rational curves on a smooth complete intersection $X \subseteq \mathbb{P}^n$. For a smooth complete intersection $X$, we consider a general fiber $F$ of the following evaluation map $ev$ of Kontsevich moduli space $\overline{\mathcal{M}}_{0,m}(X, m)$:

$$ev : \overline{\mathcal{M}}_{0,m}(X, m) \to X^m$$

and the forgetful functor $F : F \to \overline{\mathcal{M}}_{0,m}$. We prove that a general fiber of the map $F$ is a smooth complete intersection variety if $X$ is of low degree. As a result, we answer questions relating to:

1. Rational connectedness of moduli space
2. Enumerative geometry
3. Search for a new 2-Fano variety
4. Picard group of moduli space

Contents

1. Introduction 1
2. Setup 4
3. Example 5
4. Preliminary 6
5. Classification of Degeneration Type of Stable Maps 10
6. Geometry of Rational Normal Curve 13
7. Embedding Map 15
8. Cycle Relation 20
9. Main Theorem 24
10. Applications 27
References 34

1. Introduction

There is a well-developed analogue of path connectedness: A nonempty, projective, complex variety is rationally connected if each pair of closed points is contained in a rational curve, cf. [18]. In the paper [6], A. de Jong and J. Starr introduce a notation of rational simple connectedness and strongly rational simple connectedness. They prove that low degree complete intersections are (strongly) rationally simply connected.

Date: October 15, 2013.
Theorem 1.1. Let $X$ be a smooth complete intersection in $\mathbb{P}^n$ of type $(d_1, \ldots, d_c)$. For simplicity, assume all $d_i \geq 2$. The variety $X$ is strongly rationally simply connected if
\[ n + 1 \geq \sum_{i=1}^{c} (2d_i^2 - d_i) \]
and $X$ is not a quadric hypersurface.

One motivation of proving a projective variety to be strongly rationally simply connected is due to a beautiful theorem of B. Hassett relating the strongly rational simple connectedness to weak approximation, namely,

Theorem 1.2. Let $K$ be the function field of a curve over an algebraically closed field of characteristic 0. A smooth, projective $K$-scheme $X$ satisfies weak approximation if the geometric generic fiber $X \otimes_K \overline{K}$ is strongly rationally simply connected.

Let $X$ be a smooth complete intersection variety. Suppose $X$ is neither a linear variety nor a quadric hypersurface. In the paper [6, Lemma 6.5], it introduces a notation of rational curves of minimal degree, namely, $m$-minimal dominant curve class, c.f [6, Definition 5.2]. Suppose $h^0(X, \mathcal{O}_X(1))$ is at least $m$, by [6, Lemma 5.5], it proves that $\mathcal{O}_X(1)$-degree $m$ curve class is $m$-minimal dominant if the evaluation map
\[ ev : \overline{M}_{0,m}(X, m) \to X^m. \]
is dominant. Suppose the fiber $F$ is a general fiber of the evaluation map $ev$. The fiber $F$ parameterizing the $m$-minimal dominant curves is a smooth projective variety, c.f [6]. In the paper [6], A. de Jong and J. Starr only use the fact that the fiber $F$, when $m=2$ and $X$ is low degree complete intersection, is Fano to prove their main theorem [11].

In M. Deland’s thesis, see [7], he proves that a smooth cubic hypersurface in $\mathbb{P}^e$ is strongly rationally simply connected if $e$ is at least 9. He claims that $e \geq 9$ is the best bound. But he did not give a proof about this claim. So it is quite interesting to consider the following question.

Question 1.3. Let $F$ be a general fiber of the evaluation map
\[ ev : \overline{M}_{0,3}(X_3, 3) \to X \times X \times X. \]
Is the fiber $F$ corresponding to a smooth cubic hypersurface $X_3$ in $\mathbb{P}^8$ rationally connected?

In general, we can ask a more general question as following:

Question 1.4. Is the smooth projective variety $F$ rationally connected if $X$ is a low degree complete intersection?

It is the lack of studying rational connectedness of the moduli space $F$. This paper completes the studying of this direction.

On the other hand, there is an interesting question arising from enumerative geometry.
Question 1.5. How many twist cubics passing through three general points on a complete intersection $X$ if the number of these twist cubics is finite?

In the paper [2], A. Beauville answer this question by using quantum cohomology. The answer gives a little bit information about the cycles and their intersection numbers of the general fiber $\mathcal{F}$ when $m = 3$. But what is a geometrical reason behind this answer provided by A. Beauville?

Another motivation of studying $\mathcal{F}$ is to search examples of 2-Fano variety. In the paper [5], it introduces a new notation of higher Fano variety which generalizes Fano varieties, namely, 2-Fano, c.f [5] for the definition. Unfortunately, there are a few 2-Fano examples which are known. The main examples are

(1) low degree complete intersections,
(2) some Grassmannians,
(3) some hypersurfaces in some Grassmannians.

See [5] and [1] for the details. Theorem 1.4 in [1] states that a polarized minimal family of rational curves passing through a general point $x$ in a smooth projective variety $Y$ is 2-Fano variety if the variety $Y$ is 3-Fano. It suggests that some moduli space of rational curves on a $n$-Fano complete intersection variety of $X$ are $(n-1)$-Fano. Therefore, when $X$ is a low degree complete intersection, the smooth variety $\mathcal{F}$ is reasonable to be a candidate for a 2-Fano variety. It is quite natural to asking the following question.

Question 1.6. Is the smooth variety $\mathcal{F}$ a new type (different from the known 3 cases above) 2-Fano variety if $X$ is a low degree complete intersection(like 3-Fano)?

For $m = 1$, the answer is negative. In this case, even though the smooth variety $\mathcal{F}$ is 2-Fano variety, it is a complete intersection, see [4] Page 83 ,(1)]. For $m = 2$, the main theorem of the paper [23] still gives a negative answer to this question. The main theorem of this paper also gives a negative answer for $m \geq 3$.

For a moduli space, we can ask the following question:

Question 1.7. What is the Picard group of $\mathcal{F}$?

At the end of this paper, we prove that the Picard group of $F$ is finite generated and it rank is at least 2 if $m$ is at least 4.

In summary, the motivation of this paper is to study the moduli space $\mathcal{F}$ which parametrizes $m$-minimal dominant curves passing through $m$ general points on a complete intersection for

(1) Rational connectedness of moduli space.
(2) Enumerative geometry.
(3) Search for a new 2-Fano variety.
(4) Picard group of moduli space.

We state our main theorem of this paper.
Theorem 1.8. Suppose $X$ is a smooth complete intersection variety of type $(d_1, \ldots, d_c)$, $(d_i \geq 2)$ in $\mathbb{P}^n$ but not a quadratic hypersurface, $m \geq 3$, and

$$n + m(c - \sum_{i=1}^{c} d_i) - c \geq 1.$$ 

Let the natural forgetful functor be $F : \mathcal{F} \to \mathcal{M}_{0,m}$. For a general point $t \in \mathcal{M}_{0,m}$, we have a line bundle $\lambda|_{\mathcal{F}_t}$ on the fiber $\mathcal{F}_t$ of $F$ over $t$ whose corresponding complete linear system $|\lambda|_{\mathcal{F}_t}$ defines an embedding

$$|\lambda|_{\mathcal{F}_t} : \mathcal{F}_t \hookrightarrow \mathbb{P}^N = \mathbb{P}^{n-m(c-1)}.$$ 

Via this embedding, the smooth variety $\mathcal{F}_t$ is a complete intersection in $\mathbb{P}^N$ of type

$$T_1(d_1, m) = \begin{pmatrix} 2 & \ldots & d_1 - 1 \\ \vdots & \vdots & \vdots \\ 2 & \ldots & d_1 - 1 \end{pmatrix}, \ldots, T_1(d_c, m) = \begin{pmatrix} 2 & \ldots & d_c - 1 \\ \vdots & \vdots & \vdots \\ 2 & \ldots & d_c - 1 \end{pmatrix}.$$ 

Structure of Paper. In Section 3, we give an example to illustrate the main theorem. In Section 4, we provide necessary preparations. In Section 5, we classify all the possibility of the degeneration types of stable maps with arithmetic genus 0 passing through $m$ general points, roughly speaking, the components of stable maps corresponding points in $\mathcal{F}$ are rational normal curves. The argument is elementary but very delicate. In Section 6, we discuss some geometry properties of rational normal curves, the main tools are a theorem due to Z. Ran to analyze the normal bundles of rational normal curves and theory of Hirzebruch surface. In Section 7, we use the results in section 5 and 6 to give a deformation argument for showing that we can naturally embed $\mathcal{F}$ into a projective space by using a natural linear system. In Section 8, we use an idea following from the paper [17] to interpret the moduli space $\mathcal{F}_t$ as constructed by blowing up. In this way, we prove some cycle relations on the moduli space $\mathcal{F}_t$. In Section 9 and 10, we prove the main theorem of this paper and give some applications to questions arising from rational simple connectedness, enumerative geometry and the Picard group of moduli space.

Acknowledgments. I am very grateful for my advisor Prof. A. de Jong for suggesting this project to me and teaching me a lot of moduli techniques. I thank Prof. Chiu-Chu (Melissa) Liu and Prof. Fedorchuk for pointing to me some references. I also thank Prof. J. Starr and his students Z. Tian and Y. Zhu for useful conversation.

2. Setup

We work over the complex numbers $\mathbb{C}$. Suppose $X$ is a projective variety in $\mathbb{P}^n$, the Kontsevich moduli space $\mathcal{M}_{0,m}(X, e)$ parameterizes data $(C, f, x_1, \ldots, x_m)$ of

(i) a proper, connected, at-worst-nodal, arithmetic genus 0 curve $C$,
(ii) an ordered collection $x_1, \ldots, x_m$ of distinct smooth points of $C$,
(iii) and a morphism $f : C \to X$ whose image has degree $e$ in $\mathbb{P}^n$

such that $(C, f, x_1, \ldots, x_m)$ has only finitely many automorphisms. There is an evaluation morphism

$$\text{ev} : \mathcal{M}_{0,m}(X, e) \to X^m, \quad (C, f, x_1, \ldots, x_m) \mapsto (f(x_1), \ldots, f(x_m)).$$
The Kontsevich moduli space $M_{0,m}(X,e)$ is a Deligne-Mumford stack, we refer [3] for the construction. Let $n,m,c,d_1,\ldots,d_c$ be numbers such that $n,m,c,d_i \in \mathbb{N}$, $n \geq m \geq 3$, $c \leq n$, and $d_i \geq 2$. Let $X$ be a smooth complete intersection of type $(d_1,d_2,\ldots,d_c)$ in $\mathbb{P}^n$. Let the points $p_1,\ldots,p_m$ be points on $X$ in general position (see [13, Page 7] for details) and $F$ be the fiber $ev^{-1}(p_1,\ldots,p_m)$ such that $F$ is a general fiber of the evaluation map $ev$.

$$F = ev^{-1}(p_1,\ldots,p_m) \subseteq M_{0,m}(X,m)$$

In this paper, we fix the notations $n,m,d_i,c,ev,p_i,X$ and $F$. We also assume $X$ is not a quadric hypersurface in $\mathbb{P}^n$.

3. Example

Example 3.1. Suppose a non-singular projective variety $X_{22}$ is a complete intersection of two quadric hypersurfaces $F_1$ and $F_2$ in $\mathbb{P}^n$. We consider a general fiber $F_3$ of the evaluation map,

$$ev_3 : \overline{M}_{0,3}(X_{22},3) \to X_{22} \times X_{22} \times X_{22}$$

where $F_3$ is over $(p,q,r)$. Since the intersection of $F_1$ and $F_2$ on the $\mathbb{P}^2 = \text{Span}(p,q,r)$ is four points, we can assume the fourth point differ from $p,q,r$ is $w$. Since $p,q,r$ are general points of $X_{22}$, we can assume $w$ is a general point.

Suppose $(C,f,x_1,x_2,x_3)$ is a point in $F$, hence, the projective space $\text{Span}(C)$ is $\mathbb{P}^3$. The intersection of $X_{22}$ and $\text{Span}(C)$ is a curve of degree 4 containing $f(C)$, hence, the residual curve is a line passing through $w$, i.e,

$$X_{22} \cap \text{Span}(C) = f(C) \cup l$$

where $l$ is a line and $w \in l$. Conversely, give a line $l$ on $X_{22}$ passing through $w$, we can span $l$ and $p,q,r$ to get a $P = \mathbb{P}^3$. Since the intersection of $X_{22}$ and $P$ is a curve of degree 4 containing $l$, the residual curve $f(C)$ is a curve passing through $p,q,r$.
of degree 3. Therefore, we can associate to the line \( l \) a point in \( F_3 \) corresponding to the curve \( f(C) \). It is easy to see it is an one to one correspondence, hence,

\[ F_3 \cong F_1 = \text{the space of line passing through } w \]

where \( F_1 \) is the fiber of \( ev_1 \) over \( w \),

\[ ev_1 : \mathcal{M}_{0,1}(X_{22}, 1) \to X_{22} \]

By [3 Page 83(1)], the fiber \( F_1 \) is a complete intersection of type \((2, 2)\) in \( \mathbb{P}^{n-3} \). Therefore, the fiber \( F_3 \) is a complete intersection variety of type \((2, 2)\) in \( \mathbb{P}^{n-3} \).

4. Preliminary

**Lemma 4.1.** [6 Lemma 5.1] With the notations as in Section 3, if every point in a general fiber of \( ev \) parametrizes a curve whose irreducible components are all free, then a (non-empty) general fiber \( F \) of \( ev \) is smooth of the expected dimension,

\[(c + 2 - \sum_{i=1}^{c} d_i)m + n - c - 3\]

and the intersection with the boundary is a simple normal crossings divisor.

**Lemma 4.2.** Suppose \( n \geq m(\sum_{i=1}^{c} d_i - c - 2) + c + 3 \). The general fiber \( F \) is a smooth projective variety of expected dimension.

Proof. Due to the Lemma 4.1 and [6 Corollary 5.11]. □

**Lemma 4.3.** Suppose \( X \) is a complete intersection of type \((d_1, d_2, \ldots, d_c)\) for \( d_i \geq 2 \) in \( \mathbb{P}^n \) (it could be a quadric hypersurface) and \( \dim X = n - c \geq 1 \), we have that

\[ H^0(X, \mathcal{O}_X(1)) = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) = n + 1. \]

Proof. Since \( X \) is a complete intersection of type \((d_1, d_2, \ldots, d_c)\), we have the following exact Koszul complex,

\[ 0 \to \mathcal{O}_{\mathbb{P}^n}(-d_1-\ldots-d_c) \to \ldots \to \bigoplus_{i<j} \mathcal{O}_{\mathbb{P}^n}(-d_i-d_j) \to \bigoplus_{i=1}^{c} \mathcal{O}(-d_i) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to 0 \]

We split this long exact sequence to short exact sequences,

\[ 0 \to I_1 \to \bigoplus_{c} \mathcal{O}_{\mathbb{P}^n}(-d_i) \to \mathcal{O}_X \to 0 \]

\[ 0 \to I_2 \to \bigoplus_{i=1}^{c} \mathcal{O}_{\mathbb{P}^n}(-d_i) \to I_1 \to 0 \]

\[ \vdots \]

\[ 0 \to \mathcal{O}_{\mathbb{P}^n}(-d_1-\ldots-d_c) \to \bigoplus_{k=1}^{c} \mathcal{O}_{\mathbb{P}^n}(-d_1-\ldots-d_k-\ldots-d_c) \to I_{c-1} \to 0 \]

take \( H^0(\ast \otimes \mathcal{O}_{\mathbb{P}^n}(1)) \) on these short exact sequences, by syzygy and \( d_i \geq 2 \), we have

\[ H^1(I_1(1)) = H^2(I_2(1)) = \ldots = H^c(\mathcal{O}_{\mathbb{P}^n}(-d_1-\ldots-d_c)) = 0 \]

\[ H^1(I_2(1)) = H^2(I_3(1)) = \ldots = H^{c-1}(\mathcal{O}_{\mathbb{P}^n}(-d_1-\ldots-d_c)) = 0 \]

Hence, we have that \( H^0(\mathbb{P}^n, I_1(1)) = H^1(\mathbb{P}^n, I_1(1)) = 0 \), which implies

\[ H^0(X, \mathcal{O}_X(1)) = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) = n + 1 \]

\[ \square \]
Corollary 4.4. With the conditions as Lemma 4.3, the smooth projective variety $X$ is linearly non-degenerated, i.e., the variety $X$ is not in any hyperplane.

Proof. Due to Lemma 4.3, we have that $H^0(X, I_X(1)) = 0$, where $I_X$ is the ideal sheaf of $X$. It means that no linear form is vanishing on $X$, i.e., the variety $X$ is not in any hyperplane. □

Corollary 4.5. If $n \geq m$, then we can choose general $m$ points in $X \subseteq \mathbb{P}^n$ such that they are in general position.

Proof. We pick up a general point in $X$ first. Secondly, we pick up the second general point and get a line by spanning those two points. By the above lemma, we can pick up the third general point which is not on this line. Since $n \geq m$, we can process in this way to produce $m$ general points of $X$ which are in general position by Corollary 4.4. □

Definition 4.6. From [6, the proof of Lemma 6.4], we have a rational map $\Phi: \mathbb{P}^{n-m} = \mathbb{P}^n / \text{Span}(p_1, \ldots, p_m)$, which associates to a point $(f, C, x_1, \ldots, x_m) \in \mathcal{F}$ a point $\text{Span}(f(C)) \in \mathbb{P}^n / \text{Span}(p_1, \ldots, p_m)$

Excluding the following cases, the map $\Phi$ is a morphism.

(1) $c = 2, d_1 = d_2 = 2, m \geq 6$
(2) $c = 1, d_1 = 3, m \geq 5$
(3) $c = 1, d_1 = 2$

We denote $\Phi^*(\mathcal{O}_{\mathbb{P}^{n-m}(1)})$ by $\mathcal{O}_{\mathcal{F}}(\lambda)$ where $\lambda$ is a divisor on $\mathcal{F}$.

Remark 4.7. If $n$ is equal to $m - 1$, let $Y$ be the parameter space of stable maps of degree $m$ mapping into $X \cap \text{Span}(p_1, \ldots, p_m) = X \cap \mathbb{P}^n$, then the dimension of $Y$ equals $(c + 3 - d_1 - \ldots - d_c)m - c - 4$ by Lemma 4.4. By elementary analysis and the fact $d_i \geq 2$, the dimension of $Y$ is non-negative if $c, d_i$ satisfy the condition in Definition 4.6. Moreover, the dimension of $Y$ is

(1) $m - 6$ if $c = 2, d_1 = d_2 = 2$
(2) $m - 5$ if $c = 1, d_1 = 3$
(3) $2m - 5$ if $c = 1, d_1 = 2$

In particular, the map $\Phi$ is not always defined on entire $\mathcal{F}$. In fact, it is a rational map and the dimension of indeterminate locus of $\Phi$ does not exceed the dimension of $Y$.

In the paper [6], the hypothesis 6.3 excludes the cases

(1) $c = 1, d_1 = 3, m \geq 5$
(2) $c = 2, d_1 = d_2 = 2, m \geq 6$

Therefore, when we exclude these cases, the dimension of $Y$ is negative, i.e., $Y = \emptyset$, hence, the map $\Phi$ is a morphism.

Definition 4.8. We have a forgetful morphism

$F : \mathcal{F} \to \overline{\mathcal{M}}_{0,m}$

associate to a point $(C, f, x_1, \ldots, x_m) \in \mathcal{F}$ a point $(C', x_1, \ldots, x_m) \in \overline{\mathcal{M}}_{0,m}$. We refer the detail to [8].
Lemma 4.9. Let \((C, f, x_1, \ldots, x_m)\) be a stable map corresponding to a point in \(\mathcal{F}\). The domain \(C\) consists of a comb with \(m\) rational teeth and the map \(f\) collapse the handle and maps the teeth to lines meeting at a point in \(X\). (For the notations, see [18, page 156, Definition 7.7]). We call this map is of maximal degeneration type. Then we have that the forgetful morphism \(F\) has a section \(\sigma\).

Proof. We describe the section \(\sigma\) pointwise. From the hypothesis, there are \(m\) lines \(l_1, \ldots, l_m\) in \(X\), they pass through \(m\) general points \(p_1, \ldots, p_m\) and intersect at a distinct point \(q \in X\). We denote pairs \((l_i, p_i)\) where \(p_i \in l_i\). Let \((C, x_1, \ldots, x_m)\) be a point in \(\overline{M}_{0,m}\). By the [3, Theorem 3.6], there exist a stable map as following:
\[
f : C \cup l_1 \cup \cdots \cup l_m \to X
\]
where the domain \(C \cup l_1 \cup \cdots \cup l_m\) is union of \(C\) and \(l_i\) with identifying \(x_i\) with \(p_i\) and the map \(f\) is an identity on each \(l_i\) and collapse \(C\) to the point \(q\). The section \(\sigma\) associates to a point \((C, x_1, \ldots, x_m)\) a point \((C \cup l_1 \cup \cdots \cup l_m, f, x_1, \ldots, x_m)\) in \(\mathcal{F}\). It is easy to check it is a section of \(\mathcal{F}\).

\(\square\)

Definition 4.10. In the following sections, we denote a general fiber of the map \(F\) over a general point \(t\) in \(\overline{M}_{0,m}\).

Corollary 4.11. With the same hypothesis as Lemma 4.9. The fibers \(F_t\) are connected. Moreover, the fibers \(F_t\) of \(F\) are smooth projective varieties.

Proof. By the stein factorization [15] and Lemma 4.9 we have the following factorization,

\[
\begin{array}{ccc}
\overline{M}_{0,m} & \xrightarrow{\sigma} & Y \\
\downarrow & & \downarrow h \\
\mathcal{F} & \xrightarrow{s} & \overline{M}_{0,m}
\end{array}
\]

where \(Y\) is a normal variety, the morphism \(h\) is finite over \(\overline{M}_{0,m}\), the section \(\sigma\) is a section of \(F\) in Lemma 4.9 and the fibers of \(s\) is connected. Therefore, the composition of \(\sigma\) and \(s\) is a section of \(h\). Since \(h\) is a finite morphism between two varieties, the composition \(s \circ \sigma\) is an isomorphism, i.e, the variety \(Y\) is isomorphic to \(\overline{M}_{0,m}\). We get the first assertion. By the generic smoothness theorem, we have the second assertion.

Lemma 4.12. We have the following diagram

\[
\begin{array}{ccc}
\mathcal{F}_t & \xrightarrow{\Phi|_{\mathcal{F}_t}} & \mathbb{P}^{n-m} \\
\downarrow & & \downarrow \\
t & \in & \overline{M}_{0,m}
\end{array}
\]

Suppose \(X\) is not a quadric hypersurface, the restriction map \(\Phi|_{\mathcal{F}_t}\) is a morphism when \(t\) is a general point of \(\overline{M}_{0,m}\).
Proof. By Remark 4.7, the dimension of the parameter space $Y$ parameterizing stable maps of degree $m$ which map into the intersection $X \cap \text{Span}(p_1, \ldots, p_m)$ is at most

1. $m - 6$ if $c = 2, d_1 = d_2 = 2$,
2. $m - 5$ if $c = 1, d_1 = 3$.

Since the dimension of $\mathcal{M}_{0,m}$ is $m - 3$, for a general point $t \in \mathcal{M}_{0,m}$, the general fiber $\mathcal{F}_t$ does not intersect $Y$. Therefore, the indeterminate locus of the rational map $\Phi$ does not intersect $\mathcal{F}_t$, i.e, the restriction map $\Phi|_{\mathcal{F}_t}$ is a morphism. □

A geometric interpretation of Lemma 4.12 is following corollary.

Corollary 4.13. Suppose $X$ is not a quadric hypersurface, for the fiber $\mathcal{F}_i$ corresponding to general $k$ points ($k \leq n$) of $X$, we have that a stable map of degree $k$ (corresponding to a point in $\mathcal{F}_i$) spans a projective space $\mathbb{P}^k$.

Proof. It is just a geometric interpretation of the well-defined locus of the map $\Phi$ in the Lemma 4.12 □

Since we have following diagram,

\[
\begin{bmatrix}
\sigma_i
\end{bmatrix}
\xymatrix{
\mathcal{U} \ar[r]^{f} \ar[d]_{\pi} & X \\
\mathcal{F}
}
\]

it induces a map $f^*\Omega_X \to \Omega_{\mathcal{U}}$, where $\mathcal{U}$ is the universal bundle and $\sigma_i$ is the universal section induced by the $i$-th pointed point of the universal bundle. Since there is a canonical map $\Omega_{\mathcal{U}} \to \Omega_{\mathcal{U}/\mathcal{F}}$, the composition of these two maps gives $f^*\Omega_X \to \Omega_{\mathcal{U}/\mathcal{F}}$. It induces a morphism.

\[
\sigma_i^*f^*\Omega_X \to \sigma_i^*\Omega_{\mathcal{U}/\mathcal{F}}
\]

Since the image of $\sigma_i$ is in the locus of the smooth point of $\pi$ and the composition $f \circ \sigma_i$ is a constant map with value $p_i$, it gives a map

\[
(\mathcal{T}_{p_i}X) \cong \mathcal{O}_X \to \sigma_i^*\omega_{\mathcal{U}/\mathcal{F}}
\]

where $\omega_{\mathcal{U}/\mathcal{F}}$ is the dualizing sheaf of $\pi$. It is easy to see it is surjective, i.e, the map $f$ is unramified. Hence, this map gives the following morphism $\pi_{p_i}$.

Definition 4.14. We define a morphism

\[
\pi_{p_i} : \mathcal{F} \to \mathbb{P}(\mathcal{T}_{p_i}X) = \mathbb{P}^{n-c-1}
\]

associates to a point $(C, f, x_1, \ldots, x_m) \in \mathcal{F}$ a tangent direction $T_{f(C), p_i} = df_a(T_{C, x_i}) \in \mathbb{P}(\mathcal{T}_{p_i}X)$, where the point $p_i$ is $f(x_i)$.

Remark 4.15. If $\text{Span}(f(C)) = \mathbb{P}^m$, then the tangent direction $T_{f(C), p_i}$ points out of $\text{Span}(p_1, \ldots, p_m) = \mathbb{P}^{m-1}$. In fact, if it is not the case, the degree of $\text{Span}(p_1, \ldots, p_m) \cap f(C)$ is at least $m + 1$, therefore, the image curve $f(C)$ is in $\mathbb{P}^{m-1}$. It is a contradiction.

Lemma 4.16. If the map $\Phi$ is well-defined on $\mathcal{F}$, then we have that

\[
\pi_{p_i}^*(\mathcal{O}_{\mathbb{P}^{n-c-1}}(1)) \simeq \Phi^*(\mathcal{O}_{\mathbb{P}^{n-m}}(1)).
\]

Similarly, if the map $\Phi|_{\mathcal{F}_i}$ is well-defined on $\mathcal{F}_i$, then we have that

\[
(\pi_{p_i}|_{\mathcal{F}_i})^*(\mathcal{O}_{\mathbb{P}^{n-c-1}}(1)) \simeq (\Phi|_{\mathcal{F}_i})^*(\mathcal{O}_{\mathbb{P}^{n-m}}(1)).
\]
Proposition 5.2. We exclude the following cases:

$p \in P$ and the intersection of \[1\], the curve $C$

For every point \((x, f, x_1, \ldots, x_m) \in F_i\) and \(t\) is a general point in \(\mathcal{M}_{0,m}\), then the same statement holds for \(f(C)\).

Proof. The statement is clear from the following commutative diagram:

\[
\begin{array}{ccc}
F_i & \xrightarrow{\pi_p |_{F_i}} & \mathbb{P}(T_p, X) \\
& \Phi|_{F_i} \downarrow & \\
& \mathbb{P}^n / \text{Span}(p_1, \ldots, p_m) & \xrightarrow{\downarrow L} \\
\end{array}
\]

where \(L\) is a projection map as following,

\[
L : \mathbb{P}(T_p, X) \longrightarrow \mathbb{P}^n / \text{Span}(p_1, \ldots, p_m)
\]

which associates to a point \([v] \in \mathbb{P}(T_p, X)\) a point (in \(\mathbb{P}^n - m\)) corresponding to \(\text{Span}(v, p_1, \ldots, p_m) / \text{Span}(p_1, \ldots, p_m)\). The indeterminate locus of \(L\) is the points in \(\mathbb{P}(T_p, X)\) which can be represented by a nonzero vector \(v \in T_p X\) lying in \(\text{Span}(p_1, \ldots, p_m)\), hence, the image of \(\pi_p\) is outside of the indeterminate locus of \(L\) by Remark 4.15. The composition \(L \circ \pi_p |_{F_i}\) is a morphism and equals to \(\Phi|_{F_i}\).

The same argument works for the \(F\).

5. Classification of Degeneration Type of Stable Maps

In this section, we suppose that \(X \subseteq \mathbb{P}^n\) and \(n \geq m\) as in the Section 2, by Corollary 4.15 we can choose \(m\) general points in \(X\) which are in general position.

Lemma 5.1. A reduced curve \(C\) in \(\mathbb{P}^n\) which passes through \(m\) points in general position has degree is at least \(m - 1\). The equality holds iff the curve \(C \subseteq \text{Span}(p_1, \ldots, p_m) = \mathbb{P}^{m-1}\) and \(C\) is a rational normal curve in \(\text{Span}(p_1, \ldots, p_m)\).

Proof. Since \(\text{Span}(p_1, \ldots, p_m) \cap C\) contains \(m\) points, by the intersection theory, the curve \(C \subseteq \text{Span}(p_1, \ldots, p_m) = \mathbb{P}^{m-1}\) if the degree of \(C\) is \(m - 1\). Hence, by [11], the curve \(C\) is a rational normal curve in \(\text{Span}(p_1, \ldots, p_m)\).

Suppose the degree of \(C\) is at most \(m - 2\), the points \(p_1, \ldots, p_{m-1}\) span a \(\mathbb{P}^{m-2}\) and the intersection of \(\mathbb{P}^{m-2}\) and \(C\) contains \(m - 1\) points, therefore \(C \subseteq \mathbb{P}^{m-2}\). But \(p_m \in C\) is not in the \(\text{Span}(p_1, \ldots, p_{m-1})\). It contradicts.

Proposition 5.2. We exclude the following cases:

(1) \(c = 1, d_1 = 3, m \geq 5\)
(2) \(c = 2, d_1 = d_2 = 2, m \geq 6\)
(3) \(c = 1, d_1 = 2\), i.e, the variety \(X\) is a quadric hypersurface.

For every point \((C, f, x_1, \ldots, x_m) \in F_i\), we have that

(1) the image \(f(C) = C_1 \cup \ldots \cup C_k\) where \(C_i\) is a rational normal curves of degree \(n_i\),

(2) there are \(n_i\) distinct points among points \(\{p_1, \ldots, p_m\}\) on \(C_i\) and union of these points is the points \(\{p_1, \ldots, p_m\}\), in particular, \(\sum_{i=1}^{k} n_i = m\).

Moreover, if the complete intersection \(X\) is not a quadric hypersurface, then , for a point \((C, f, x_1, \ldots, x_m) \in F_i\) and \(t\) is a general point in \(\mathcal{M}_{0,m}\), then the same statement holds for \(f(C)\).
Figure 2 $m = 6$

**Proof.** Let $k$ be an integer such that $1 \leq k \leq m$. Suppose $S$ is a general fiber of the evaluation map as following:

$$S = ev^{-1}(p_1, \ldots, p_k) \subseteq \mathcal{M}_{0,k}(X, k-1)$$

$$p = (p_1, \ldots, p_k) \in X^k$$

We can assume the map $ev$ is dominated, otherwise the fiber $S = \emptyset$. Therefore, stable maps in the general fiber $S$ is unobstructed by [18, Theorem II.7.6], i.e, the Kontsevich moduli space $\mathcal{M}_{0,k}(X, k-1)$ is smooth and of expected dimension at the point of $S$. Therefore, by the dimension calculation, we have that

$$\dim S = (c + 2 - d_1 - \ldots - d_c)(k-1) + k - 4 - c$$

If $\dim S \geq 0$, then we have that the numbers $c$ and $d_i$ are one of the following cases:

1. $c = 1, d_1 = 3, \dim S = k - 5$
2. $c = 1, d_1 = 2, \dim S = 2k - 6$
3. $c = 2, d_1 = d_2 = 2, \dim S = k - 6$

Since we have $1 \leq k \leq m$ and the hypothesis of the proposition excludes these cases, the dimension of $S$ is negative. Hence, we have that $S = \emptyset$. In other words, for $k$ general points in $X$, there is no stable map of degree $k - 1$ whose pointed points on the domain are mapped to these $k$ general points respectively.

By Lemma 5.1, every reduced curves on $X$ passing through $k$ general points is of degree is at least $k$. Suppose that

$$f(C) = C_1 \cup \ldots \cup C_k$$

where $C_i$ is a reduced curve with $k_i$ distinct points among the points $\{p_1, \ldots, p_m\}$.

Hence, by $\deg f(C) = m$ and $\sum_{i=1}^{k} k_i = m$, we have that the deg $C_i = k_i$. Moreover,
the spanning space \( \text{Span}(C_i) \) is a projective space \( \mathbb{P}^{s_i} \) with \( s_i \leq k_i \). In fact, we
have an equality \( s_i = k_i \), otherwise, the spanning space \( \text{Span}(f(C)) = \text{Span}(C_i \cup C_2 \cup \ldots \cup C_k) \) is a projective space \( \mathbb{P}^l \) with \( l \leq m - 1 \). Therefore, \( C_i \) is a rational
normal curves of degree \( k_i \) with \( k_i \) points among the points \( \{p_1, \ldots, p_m\} \). It proves
the first assertion.

If \( f(C) = C_1 \cup C_2 \cup \ldots \cup C_m \) and \( p_i \in C \), then it is easy to see \( C_i \) is a line.
Therefore, to prove the second assertion, we can suppose \( f(C) \) has at least two
irreducible components, namely, the image \( f(C) \) of the stable map is equal to
\( A \cup B \) where there are \( l(\geq 2) \) distinct points among the points \( \{p_1, \ldots, p_m\} \) on
\( B \) and the curve \( B \) is irreducible. I claim that for general \( l \) points on \( X \), there
is no stable map \( (C, f, x_1, \ldots, x_m) \in \overline{\mathcal{M}}_{0,l}(X, l - 1) \) such that \( f(x_1) \) is \( p_i \) and
\( F((C, f, x_1, \ldots, x_m)) \in \overline{\mathcal{M}}_{0,l} \) is a general point in an open set \( U \subseteq \overline{\mathcal{M}}_{0,l} \). In fact,
by the dimension calculation of \( S \) as above and the dimension of \( \overline{\mathcal{M}}_{0,l} = l - 3 \), the
claim is clear since the dimension of \( S \) is less than the dimension of \( \overline{\mathcal{M}}_{0,l} \).

Therefore, if the degree of \( B \) is less than \( l \), the degree of \( B \) is \( l - 1 \) by the first
assertion of Lemma 5.1 But according to the claim above, it is impossible if \( t \) is a
general point in an open set \( V \) of \( \overline{\mathcal{M}}_{0,m} \), where \( V \) satisfies the following diagram,

\[
\begin{array}{ccc}
\overline{\mathcal{M}}_{0,m} & \xrightarrow{\pi} & \overline{\mathcal{M}}_{0,l} \\
U & \cup & U \\
V & \longrightarrow & U
\end{array}
\]

The morphism \( \pi \) is the forgetful functor to forget points, therefore , the degree of
\( B \) equals to \( l \). By induction of the degree of \( A \), we get the proposition.

For the general fiber \( \mathcal{F}_i \), the proof is similar. \( \square \)

**Corollary 5.3.** Let \( (C, f, x_1, \ldots, x_m) \) be a point in \( \mathcal{F}_i \). If \( X \) is not a quadric
hypersurface, then we have

1. the curve \( C = C_1 \cup l_1 \ldots \cup l_k \) where \( C \) is a comb with handle \( C_1 \) and teeth \( l_i \).
   Moreover, the pointed point \( x_j \) is on \( l_i \) and the rest pointed points among
   the points \( \{x_1, \ldots, x_m\} \) are on \( C_1 \),
2. the image \( f(l_i) \) of \( l_i \) is a line on \( X \) passing through \( p_j \), and \( f(C_1) \) is a
   rational normal curve of degree \( m - k \), where \( 0 \leq k \leq m \). In particular, if
   \( k \neq m \), then \( (C, f, x_1, \ldots, x_m) \in \mathcal{F}_i \) is an embedded curve. See the following
   figure.

**Remark 5.4.** For the case \( k = m \), it may be that the stable map \( f \) collapses the
handle \( C_1 \) to a point \( p \) and maps the tooth \( l_i \) isomorphic to a line passing through
\( p \) and \( p_i \).

**Proof.** Since a point \( t \in \overline{\mathcal{M}}_{0,m} \) is general, the image \( F(C, f, x_1, \ldots, x_m) \) corresponds
to a smooth rational point with \( m \) pointed points. Hence, the stabilization process
of the forgetful map \( F \) ensures the first assertion, namely, \( C = C_1 \cup l_1 \ldots \cup l_k \). The
second assertion is due to the previous proposition. \( \square \)
6. Geometry of Rational Normal Curve

Definition 6.1. Suppose $C$ is a smooth rational curves and $C \subseteq \mathbb{P}^n$, by the Grothendieck theorem, the normal bundle of the rational curve $C$ is

$$N_{C/\mathbb{P}^n} = \bigoplus_{i=1}^{n-1} \mathcal{O}_C(a_i)$$

with $a_1 \leq a_2 \leq \ldots \leq a_{n-1}$. We call the rational curve $C$ is almost balanced if $a_{n-1} - a_1 \leq 1$.

In the paper [21], Z.Ran gives a careful analysis about the balanced property of rational curves in the projective space. We cite one result from the paper [21].

Theorem 6.2. [21] Theorem 6.1 (Sacchiero) A generic rational curve of degree $d \geq n$ in $\mathbb{P}^n$ is almost balanced.

Corollary 6.3. If $C \subseteq \mathbb{P}^n$ is a rational normal curve, then the normal bundle is

$$N_{C/\mathbb{P}^n} = \bigoplus_{i=1}^{n-1} \mathcal{O}_C(n+2).$$

Proof. Since we have two short exact sequences

$$0 \to T_C \to T_{\mathbb{P}^n}|_C \to N_{C/\mathbb{P}^n} = \bigoplus_{i=1}^{n-1} \mathcal{O}_C(a_i) \to 0,$$

$$0 \to \mathcal{O}_C \to \bigoplus_{i=1}^{n+1} \mathcal{O}_{\mathbb{P}^n}(1)|_C = \bigoplus_{i=1}^{n+1} \mathcal{O}_C(n) \to T_{\mathbb{P}^n}|_C \to 0,$$

we have degree equalities,

$$(1) \ deg N_{C/\mathbb{P}^n} = \sum_{i=1}^{n-1} a_i$$
Suppose \( a_1 \leq a_2 \ldots \leq a_{n-1} \), since every rational normal curve is projectively linearly isomorphic to each other, by the theorem above, the rational normal curves are almost balanced \( a_{n-1} - a_1 \leq 1 \), hence, we have the equality \( a_i = n + 2 \). \( \square \)

**Proposition 6.4.** Suppose two curves \( C, C' \subseteq \mathbb{P}^n \) are rational normal curves and \( p_1, \ldots, p_n \) are \( n \) points in general position, if \( C \) and \( C' \) satisfy the following conditions,

1. both curves \( C \) and \( C' \) pass through points \( p_1, \ldots, p_n \)
2. \( T_{p_i}C = T_{p_i}C' \) for all \( i \)

where \( T_{p_i} \) is the tangent direction of \( C \) at point \( p_i \), the similar for \( C' \), then we have that \( C = C' \).

**Proof.** We prove it by induction. Take \( n=3 \) first, we project \( C \) and \( C' \) from \( p_1 \) to a plane \( \mathbb{P}^2 \) and denote this projection by \( \pi \). Hence, the image \( \pi(C) \) and \( \pi(C') \) are conics passing through the point \( Q = \pi(p_1) = T_{p_1}C \cap \mathbb{P}^2 \). They are tangent to two distinct lines \( l_1 = \pi(T_{p_1}C) \) and \( l_2 = \pi(T_{p_1}C') \). It is obvious that \( Q \) is neither on these two lines. By an elementary calculation, there is a unique conic \( S \) passing through \( Q \) and with tangent lines \( l_1 \) and \( l_2 \). In fact, any conic which is tangent to \( x_0 = 0 \) and \( x_1 = 0 \) can be described by an equation

\[
ax_2^2 + bx_0x_1 = 0
\]

where \( a \) and \( b \) are determined by the coordinates of \( Q \). By the discussion above, we know that \( C, C' \subseteq ConeS \) where \( ConeS \) is the projective cone of \( S \) which is a singular quadric with the singular point is \( p_1 \). For the point \( p_2 \), we have the same conclusion which implies that \( C \) and \( C' \) are in another singular quadric \( ConeS' \). Therefore,

\[
C \cup C' \subseteq ConeS' \cap ConeS
\]

Since the intersection of \( ConeS' \) and \( ConeS \) is a curve of degree 4, we have that \( C = C' \).

Suppose for \( n = s \geq 3 \) the proposition holds. When \( n = s + 1 \), as \( n = 3 \), we project the curves \( C, C' \) from the point \( p_1 \) to a hyperplane \( \mathbb{P}^s \), the projection is denoted by \( \pi \). The projections \( \pi(C), \pi(C') \subseteq \mathbb{P}^s \) are rational normal curves of degree \( s \) with the same tangent direction \( \pi(T_{p_i}C) = \pi(T_{p_i}C') \) at \( \pi(p_i) \) for \( i = 2, 3, \ldots, s + 1 \). By induction, we have \( \pi(C) = \pi(C') = D \), hence, both \( C \) and \( C' \) are in \( ConeD = \overline{p_1D} \).

We blow up the unique singular point \( p_1 \) of \( ConeD \), by [14, Chapter IV],

\[
Bl_{p_1}(ConeD) = \mathbb{P}(O \oplus O(-s)) \xrightarrow{|B+sf|/B} ConeD \subseteq \mathbb{P}^{s+1}
\]

where \( B \) is the curve class of section \( \sigma(\mathbb{P}^1) \). It is unique among the curve classes satisfying the self-intersection equal to \(-s \). Let \( f \) be a fiber of \( pr \). See the following figure.
I claim that
\[
\widetilde{C} = \widetilde{C}' = B + (s + 1)f \in \text{Pic}((\mathcal{O} \oplus \mathcal{O}(-s)))
\]
where \(\widetilde{C}\) and \(\widetilde{C}'\) are proper transform of \(C\) and \(C'\) respectively. In fact, we can assume that \(\widetilde{C} = aB + bf\), since \(\text{Bl}(\widetilde{C}) = C\), we have that
\[
\widetilde{C} \cdot (B + sf) = B = \text{Bl}(\widetilde{C}) \cdot \mathcal{O}_{p^{s+1}}(1) = s + 1
\]
The second equation is due to the projection formula, see [9, Chapter 8]. Hence
\[
b = s + 1
\]
It is obvious that \(B\) and \(\widetilde{C}\) is transversal at a point. Therefore
\[
B \cdot \widetilde{C} = -sa + s + 1 = 1
\]
hence, the similar argument works for \(\widetilde{C}'\), the claim is clear.

By the claim, we have that
\[
\widetilde{C} \cdot \widetilde{C}' = (B + (s + 1)f)^2 = -s + 2(s + 1) = s + 2
\]
Since \(\widetilde{C}\) and \(\widetilde{C}'\) are tangent at \(p_2, \ldots, p_{s+1}\) and intersect with \(B\) at the same point, if \(\widetilde{C} \neq \widetilde{C}'\), then we have that
\[
\widetilde{C} \cdot \widetilde{C}' \geq 2s + 1 > s + 2
\]
It contradicts. Hence, we have that
\[
C = \text{Bl}(\widetilde{C}) = \text{Bl}(\widetilde{C}') = C'
\]
Lemma 6.5. If $C$ is a rational normal curve of degree $m$ in $\mathbb{P}^m$, then any $m + 1$ distinct points on $C$ are in general position.

Proof. It is a direct calculation on standard rational normal curve, see [13, Chapter I].

Lemma 6.6. If $C$ is a rational normal curve of degree $m$ in $\mathbb{P}^m$ passing through the points $p_1, \ldots, p_m$, then we have that

$$\text{Span}(p_1, \ldots, p_m, T_p C) = \mathbb{P}^m$$

and for $1 < k < m$ and $m \geq 2$

$$\text{Span}(Q_1, \ldots, Q_k, T_Q C, \ldots, T_{Q_k} C) = \mathbb{P}^l$$

for any distinct $k$ points $Q_1, \ldots, Q_k$ on $C$, then it implies that $l$ is great than $k$.

Proof. Since $\text{Span}(C)$ is a projective space $\mathbb{P}^m$ and $\text{Span}(p_1, \ldots, p_m, T_p C)$ is tangent to $C$, we have

$\deg(C \cap \text{Span}(p_1, \ldots, p_m, T_p C)) > m$

If $\text{Span}(p_1, \ldots, p_m, T_p C) = \mathbb{P}^{m-1}$, it has a contradiction with $\deg C = m$. Since $\text{Span}(Q_1, \ldots, Q_k, T_Q C, \ldots, T_{Q_k} C)$ contains $k$ points and at least one tangent direction of $C$, similar as above, we have

$$\text{Span}(Q_1, \ldots, Q_k, T_Q C, \ldots, T_{Q_k} C) = \mathbb{P}^l$$

where $l \geq k$.

Suppose $\text{Span}(Q_1, \ldots, Q_k, T_Q C, \ldots, T_{Q_k} C) = \mathbb{P}^k$, we can pick up $m - k - 1$ points on $C$ but not in $\text{Span}(Q_1, \ldots, Q_k, T_Q C, \ldots, T_{Q_k} C) = \mathbb{P}^k$, by Lemma 6.5 $\text{Span}(Q_1, \ldots, Q_k, T_Q C, \ldots, T_{Q_k} C)$ and those $m - k - 1$ points on $C$ span a projective space $\mathbb{P}^{m-1}$. Hence, we have

$$\deg(\mathbb{P}^{m-1} \cap C) \geq m - k - 1 + 2k = m + k - 1 > m$$

It contradicts. Hence, we have $l$ is great than $k$. □

Definition 6.7. Suppose $X$ is not a quadric hypersurface, we call two points in $\mathcal{F}_t$ $(C, f, x_1, \ldots, x_m)$ and $(C', f', x'_1, \ldots, x'_m)$ are of the same degeneration type if it satisfies the following property:

$x_i$ is on the handle of $C$ iff $x'_i$ is on the handle of $C'$.

Lemma 6.8. Excluding the case $c = 1, d_1 = 2$ and suppose that stable maps $(C, f, x_1, \ldots, x_m)$ and $(C', f', x'_1, \ldots, x'_m)$ correspond two points in $\mathcal{F}_t$ $(m \geq 3)$. If we have

$$\pi_{p_i}(C, f, x_1, \ldots, x_m) = \pi_{p_i}(C', f', x'_1, \ldots, x'_m)$$

for all $i$, then $C$ and $C'$ have the same degeneration type.

Proof. The condition

$$\pi_{p_i}(C, f, x_1, \ldots, x_m) = \pi_{p_i}(C', f', x'_1, \ldots, x'_m)$$

just means $T_{p_i}f(C) = T_{p_i}f'(C')$, where $T_{p_i}f(C)$ is the tangent direction of $f(C)$ at the point $p_i$, similarly, for $T_{p_i}f'(C')$.

By Corollary 5.3 we can suppose that

$$f(C) = C_1 \cup L_1 \cup \ldots \cup L_k$$

and $f'(C') = C'_1 \cup L'_1 \cup \ldots \cup L'_s$
where \(0 \leq k, s \leq m\), the components \(C_1, C'_1\) are rational normal curves and the components \(L_i, L'_i\) are lines. In the maximal degeneration case (i.e. \(k = m\) in Corollary 5.3), the curve \(C_1\) or \(C_2\) could be a point, but the argument in this case is trivial. Suppose the curve \(C\) is smooth, by the second assertion of Lemma 6.6, the curve \(C'\) is smooth, i.e., \(s = 0\). Hence, we can suppose the numbers \(k, s\) are at least 1.

If both lines \(L_1\) and \(L'_1\) are passing through the same pointed point \(p_1\) and the tangent directions \(T_{p_1} L_1 = T_{p_1} L'_1\), then we have that
\[
L_1 = L'_1.
\]
We can take out the \(L_1 = L'_1\) from these two stable maps and use induction to prove the assertion.

Therefore, we can assume that the curve \(C'_1\) contains points \(p_1(\in L_1), \ldots, p_k(\in L_k)\), we have that the intersection \(C \cap C'\) is \((m - s) - k(\geq 0)\) points which are in general position.

We can assume the intersection \(C \cap C'\) is a set of points \(\{p'_1, p'_2, \ldots, p'_m - s - k\} \subseteq \{p_1, \ldots, p_m\}\). By Corollary 5.3, we have that
\[
\text{Span}\, C_1 = \mathbb{P}^{m-k} \quad \text{and} \quad \text{Span}\, C'_1 = \mathbb{P}^{m-s}
\]
Since the curve \(C'_1\) contains the points \(\{p_1, \ldots, p_k\}\), we have that \(\text{Span}(C_1 \cup C'_1) = \mathbb{P}^m\). Therefore, it concludes that
\[
\text{Span}\, C_1 \cap \text{Span}\, C'_1 = \mathbb{P}^{(m-s)+(m-k)-m} = \mathbb{P}^{m-s-k}
\]
By Lemma 6.6, if \(m - s - k\) is at least 2, then we know that
\[
\text{Span}(p'_1, \ldots, p'_m - s - k, T_{p'_1} C, \ldots, T_{p'_m - s - k} C) = \mathbb{P}^l
\]
where \(l\) is great than \(m - s - k\). But \(\text{Span}(p'_1, \ldots, p'_m - s - k, T_{p'_1} C, \ldots, T_{p'_m - s - k} C) \subseteq \text{Span}\, C'_1 \cap \text{Span}\, C_1 = \mathbb{P}^{m-s-k}\)
It is a contradiction. Hence, we have \(m - s - k\) is at most 1.

If \(m - s - k = 0\), then the intersection \(\text{Span}\, C_1 \cap \text{Span}\, C'_1\) is a point. Since the curve \(C_1\) contains the points \(\{p_1, \ldots, p_k\}\), the spanning space \(\text{Span} C\) contains \(L_1, \ldots, L_k\). Since the lines \(L_1, \ldots, L_k\) intersect \(C_1\) at \(k\) points, it implies that \(k\) is equal to 1. By symmetry, we also have that \(s = 1\). Therefore, it concludes that \(m\) is equal to 2. It is a contradiction.

If \(m - s - k = 1\), then the intersection \(\text{Span}\, C_1 \cap \text{Span}\, C'_1\) is \(\mathbb{P}^1\). It also implies that the intersection \(C_1 \cap C'_1\) is a point. I claim that the space \(\text{Span}\, C'_1\) intersects \(C_1\) at exactly two points and one of them is \(C_1 \cap C'_1\).

In fact, if the projective space \(\text{Span}\, C'_1\) intersects the curve \(C_1\) at least 3 points, by Lemma 6.5, the intersection \(\text{Span}\, C_1 \cap \text{Span}\, C'_1\) contains a projective plane \(\mathbb{P}^2\). It is impossible, so the claim is clear.

Since the lines \(L_1, \ldots, L_k(\subseteq \text{Span}\, C'_1)\) intersect \(C_1\) at \(k\) points. Since there is no point of the intersection \(C_1 \cap C'_1\) which belongs to these \(k\) points, it implies that these \(k\) points are other points of \(\text{Span}\, C'_1 \cap C\) by the claim. Therefore, we have that \(k\) is equal to 1. By symmetry, it implies that \(s = 1\). Therefore, we have \(m = 1 + k + s = 3\). In this case, by some elementary analysis of the degeneration type, it is obvious that they have the same degeneration type.

\[\square\]
7. Embedding Map

Excluding the case $c = 1$, $d_1 = 2$, by Lemma 4.12, the restriction $\Phi|_{\mathcal{F}_t} : \mathcal{F}_t \to \mathbb{P}^{n-m}$ is a morphism. In this section, we prove the complete linear system induced by $\Phi|_{\mathcal{F}_t}$ is a closed embedding.

**Proposition 7.1.** Let $\lambda|_{\mathcal{F}_t}$ be $(\Phi|_{\mathcal{F}_t})^*\mathcal{O}_{\mathbb{P}^{n-m}}(1)$, the complete linear system $|\lambda|_{\mathcal{F}_t}$ on $\mathcal{F}_t$ separates points.

**Proof.** By Lemma 4.16, we know the maps $\pi_{p_1}|_{\mathcal{F}_t}, \pi_{p_2}|_{\mathcal{F}_t}, \ldots, \pi_{p_m}|_{\mathcal{F}_t}$ induce sublinear systems of $|\lambda|$. I claim we can separate points by these sub-linear systems. In fact, the claim is equivalent to say

$$(C, f, x_1, \ldots, x_m) = (C', f', x'_1, \ldots, x'_m)$$

if they satisfy the following properties:

1. $(C, f, x_1, \ldots, x_m), (C', f', x'_1, \ldots, x'_m) \in \mathcal{F}_t$
2. $\pi_{p_i}(C, f, x_1, \ldots, x_m) = \pi_{p_i}(C', f', x'_1, \ldots, x'_m)$

By Lemma 6.8, the stable maps $(C, f, x_1, \ldots, x_m)$ and $(C', f', x'_1, \ldots, x'_m)$ have the same degeneration type. So, by Proposition 6.4, these two stable maps

$$(C, f, x_1, \ldots, x_m) = (C', f', x'_1, \ldots, x'_m)$$

if the curve $C$ or $C'$ is smooth (non-degenerate).

In general, by Corollary 5.3, the image curves $f(C)$ and $f'(C')$ are union of a rational normal curve and several lines. The configuration of these lines are uniquely determined by the degeneration type and the tangent directions at those corresponding pointed points. By induction on the number of components of $C$ and $C'$, we get the proposition.

**Proposition 7.2.** With the notations as Proposition 7.1. The complete linear system of $\lambda|_{\mathcal{F}_t}$ separates tangent vectors of $\mathcal{F}_t$, i.e., the differential of the map induced by the complete linear system $|\lambda|_{\mathcal{F}_t}$ is injective. In particular, together with the above proposition, it implies that $|\lambda|_{\mathcal{F}_t}$ is a closed embedding.

**Proof.** To prove the complete linear system $|\lambda|_{\mathcal{F}_t}$ separates the tangents of $\mathcal{F}_t$, it is sufficient to prove the sub-linear systems induced by

$$\pi_{p_1}, \pi_{p_2}, \ldots, \pi_{p_m}$$

can separate tangent vectors of $\mathcal{F}$. Namely, it is sufficient to prove the kernel of the differential of $\pi_{p_1} \times \pi_{p_2} \times \ldots \times \pi_{p_m}$ is equal to 0, where the map $\pi_{p_1} \times \pi_{p_2} \times \ldots \times \pi_{p_m}$ is

$$\mathcal{F} \to \mathbb{P}(T_{p_1}X) \times \cdots \times \mathbb{P}(T_{p_m}X)$$

By the contangent complex calculation [10] or [10, Page 61], if $C$ is an embedded rational curve of the degree $m$, then we have that

$$T_{X_{0,0}(X,m),[C]} = H^0(C, N_C/X)$$

where $N_C/X$ is the normal bundle of $C$ in $X$ and $T_{X_{0,0}(X,m),[C]}$ is the space of first order deformation of the stable map $(C, C \subseteq X, p_1, \ldots, p_m)$.
Since $\mathcal{F}$ is smooth, the first order deformation space of a stable map $(C, C \subseteq X, p_1, \ldots, p_m)$ which fixes the $p_i$ for all $i$ is the tangent space of $\mathcal{F}$ at the point $(C, C \subseteq X, p_1, \ldots, p_m)$. By the argument above, the space of first order deformation of the stable map $(C, C \subseteq X, p_1, \ldots, p_m)$ which preserves the $p_i$ is $H^0(C, N_{C/X}(\sum_{i=1}^{m} -p_i))$, therefore, we have that

\begin{equation}
T_{\mathcal{F}, [C]} = H^0(C, N_{C/X}(\sum_{i=1}^{m} -p_i)) \tag{7.1}
\end{equation}

where $T_{\mathcal{F}, [C]}$ is the tangent space of $\mathcal{F}$ at the point $(C, C \subseteq X, p_1, \ldots, p_m)$. Since we have a short exact sequence

\[\sum_{i=1}^{m} p_i \longrightarrow \sum_{i=1}^{m} p_i \longrightarrow \sum_{i=1}^{m} p_i \] \(\xrightarrow{\text{ker}}\) \(\sum_{i=1}^{m} p_i \)

the kernel of $d(\pi_{p_1} \times \pi_{p_2} \times \ldots, \pi_{p_m})$ at a point $(C, C \subseteq X, p_1, \ldots, p_m)$ is the first order deformation space of the stable map $(C, C \subseteq X, p_1, \ldots, p_m)$ which fixes the point $p_i$ and the tangent direction at $p_i$ for all $i$. Similarly as (7.1), the first order deformation space is $H^0(C, N_{C/X}(\sum_{i=1}^{m} -2p_i))$, therefore, we have that

\[\text{Ker}(d(\pi_{p_1} \times \pi_{p_2} \times \ldots, \pi_{p_m})) = H^0(C, N_{C/X}(\sum_{i=1}^{m} -2p_i))\]

On the other hand, we have a short exact sequence

\[0 \rightarrow N_{C/X} \rightarrow N_{C/P^n} \rightarrow N_{X/P^n}|_C \rightarrow 0.\]

Suppose the curve $C$ is smooth, we have that

\[H^0(C, N_{C/X}(\sum_{i=1}^{m} -2p_i)) \subseteq H^0(C, N_{C/P^n}(\sum_{i=1}^{m} -2p_i)) = H^0(C, N_{C/P^n}(-2m))\]

Since we have a short exact sequence

\[0 \rightarrow N_{C/P^n} \rightarrow N_{C/P^n} \rightarrow N_{X/P^n}|_C \rightarrow 0\]

where $C \subseteq \mathbb{P}^m \simeq \text{Span}(C)$. We know that the normal bundle $N_{p_m/P^n} = \bigoplus O_{p_m-n}(1)$ and $C$ is a rational normal curve. It implies that $N_{p_m/P^n}|_C(-2m) = \bigoplus O_{p_m-n}(-m)$. Therefore,

\[H^0(C, N_{C/P^n}(-2m)) = H^0(C, N_{C/P^n}(-2m)) = H^0(C, N_{C/P^n})(-2m)\]

By Corollary 6.3 and $m \geq 3$, we know that

\[H^0(C, N_{C/P^n}(-2m)) = H^0(C, \bigoplus_{i=1}^{m-1} O_C(2 - m)) = 0\]

Therefore, it concludes that

\[\text{Ker}(d(\pi_{p_1} \times \pi_{p_2} \times \ldots, \pi_{p_m})) = 0\]

If the curve $C$ is not smooth, then each component of $C$ is either a rational normal curve or a line. Since a line passing through a fixed point is uniquely determined by the tangent direction, by the above argument, the first order deformation of a rational normal curve $C_1$ of degree $k$ is trivial if we does not change $k$ general points.
on $C_1$ and tangent directions at these points. Therefore, the first order deformation of $(C, f, p_1, \ldots, p_m)$ which preserves points $f(x_1), \ldots, f(x_n)$ and tangent directions $df(T_{C, x_1}), \ldots, df(T_{C, x_n})$ is trivial. The assertion follows.

**Question 7.3.** What is the dimension of $|\lambda|_{\mathcal{F}}$? At this time, even to calculate dimension of $|\lambda|_{\mathcal{F}}$ is still painful. We need more preparation to calculate the dimension.

8. **Cycle Relation**

**Lemma 8.1.** [6, Lemma 5.1] The boundary divisor $\Delta$ of $\mathcal{F}$ is a simple normal crossings divisor in $\mathcal{F}$.

**Definition 8.2.** We denote $\mathcal{F}_i \cap \Delta$ by $\Delta_i$. For $i \in \{1, 2, \ldots, m\}$, the divisor $\Delta_{i,i} (\subseteq \Delta_i)$ is the divisor its general points are parameterizing a degeneration stable map of type whose image is union of a line containing $p_i$ and a rational normal curve.

**Corollary 8.3.** The divisor $\Delta_t$ is a simple normal crossing divisor in $\mathcal{F}_t$.

**Proof.** Since $t \in \mathcal{M}_{0,m}$ is a general point, the corollary is due to Lemma 8.1 and the generic smoothness theorem.

**Definition 8.4.** Let $s, d$ be two natural numbers. We say a family of homogeneous polynomials of $\mathbb{P}^e$ is of type $I$ if these polynomials consist of the union of one equation of degree $d$, $s$ equations of degree $2$, $s$ equations of degree $3$, $\ldots$, and $s$ equations of degree $d - 1$. We denote it by

$$T_1(d, s) = \begin{pmatrix} 2 & 3 & \ldots & d - 1 \\ \vdots & \vdots & \ddots & \vdots \\ 2 & 3 & \ldots & d - 1 \end{pmatrix}$$

We call a family of homogeneous polynomials of $\mathbb{P}^e$ is of type $II$ if the equations consist of the union of one equation of degree $d$, $s$ equations of degree $1$, $s$ equations of degree $2$, $\ldots$, and $s$ equations of degree $d - 1$. We can denote it by

$$T_2(d, s) = \begin{pmatrix} 1 & 2 & \ldots & d - 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & \ldots & d - 1 \end{pmatrix}$$

The union of families of equations means the disjoint union of the two equations belong to these families.

**Lemma 8.5.** With the notations as in (4.1) and Lemma 4.16, then we have that $\lambda = \sigma_i^* \omega_{U/F}$.

**Proof.** Since the image of $\sigma_i$ is the smooth points of $\pi$, we have that

$$\sigma_i^* \omega_{U/F} = \sigma_i^* \Omega_{U/F} = \sigma_i^* \mathcal{T}_{U/F}$$

From Definition 4.14 of $\pi_{p_i}$, we know that the line bundle $\sigma_i^* \mathcal{T}_{U/F} = \pi_{p_i}^* \mathcal{O}_{\mathbb{P}(\mathcal{F}(p_i, \lambda))}(-1)$, by Lemma 4.16, the equality $\sigma_i^* \mathcal{T}_{U/F} = \mathcal{O}(-\lambda)$ implies this lemma.

□
Lemma 8.6. Suppose $B$ is a smooth variety and we have the following diagram,

$$\xymatrix{ C \ar[r]^{\text{Bl}} & B \times \mathbb{P}^1 \\
 B \ar[u]^{\sigma_0} \ar[r]_{\text{pr}_2} & \mathbb{P}^1 \ar[u]_{\sigma_0} }
$$

where the map $\sigma_0$ associate to a point $b \in B$ a point $(b, 0) \in B \times \mathbb{P}^1$. Let $\Delta$ be a smooth divisor of $B$. We denote the image $\sigma_0(\Delta)$ by $Z$. The total space $C$ is a blow-up of $B \times \mathbb{P}^1$ along $Z$ and the blow-up map is $\text{Bl}$. Let $\tilde{\sigma}_0$ be the section of $q$ such that $\text{Bl} \cdot \tilde{\sigma}_0 = \sigma_0$, then we have that

1. The variety $C$ is a smooth variety
2. $\tilde{\sigma}_0^* \omega_{C/B} = \tilde{\sigma}_0^* \mathcal{O}_C(Y)$

where $Y$ is the strict transformation of $Z$, i.e., $\text{Bl}^{-1}(Z) = Y$ and $\omega_{C/B}$ is the relative dualizing sheaf of $q$.

Proof. The variety $C$ is smooth since we will get a smooth variety if we blow up a smooth variety along a smooth subvariety. So it is not difficult to see the morphism $q$ is a local complete intersection map, see [9]. By [14] for the definition, the relative dualizing sheaf $\omega_{C/B}$ equals to

$$K_C \otimes q^*(K_B)^{-1} = \text{Bl}^*(pr_1^* K_B \otimes pr_2^* \mathcal{O}_{\mathbb{P}^1}) \otimes \mathcal{O}_C(Y) \otimes q^*(K_B)^{-1}$$

where $K_B$ and $K_C$ are canonical bundles of $B$ and $C$ respectively. Since $pr_1 \cdot \text{Bl} = q$, the relative dualizing sheaf $\omega_{C/B}$ can be simplified to

$$\text{Bl}^* pr_2^* \mathcal{O}_{\mathbb{P}^1} \otimes \mathcal{O}_C(Y)$$

Since $pr_2 \cdot \text{Bl} \cdot \tilde{\sigma}_0$ is constant, we have that $\tilde{\sigma}_0^* \omega_{C/B} = \tilde{\sigma}_0^* \mathcal{O}_C(Y)$. □

Proposition 8.7. Suppose $X$ is not a quadric hypersurface, the divisor $\Delta_{i,t}$ is linearly equivalent to $\lambda|_{\mathcal{F}_t}$. In particular, $\Delta_{i,t}$ is an ample divisor on $\mathcal{F}_t$.

Proof. Suppose the general point $t \in \mathcal{M}_{0,m}$ represents a rational curve $C$ with $m$ pointed points as $(0, 1, \infty, t_1, \ldots, t_{m-3})$. It is obvious that we can can construct the universal family $\mathcal{U}$ by blowing up $\mathcal{F} \times \mathbb{P}^1$, more precisely, we have the following commutative diagram for $\mathcal{U}|_{\mathcal{F}_t}$,

$$\xymatrix{ \mathcal{U}|_{\mathcal{F}_t} \ar[r]^{\text{Bl}} & \mathcal{F}_t \times \mathbb{P}^1 \\
 \mathcal{F}_t \ar[u]^{\sigma_{0}} \ar[r]_{\text{pr}_1} & \mathcal{F}_t \ar[u]_{\pi} \ar[r]_{\text{pr}_2} & \mathbb{P}^1 \ar[u]_{\sigma_{0}} }
$$

where $\sigma_i$ associates to a point $y \in \mathcal{F}_t$ a pair $(y, i)$, the index $i \in \{0, 1, \infty, t_1, \ldots, t_{m-3}\}$. The map $\text{Bl}$ blows up the variety $\mathcal{F}_t \times \mathbb{P}^1$ along the subvariety $\sigma'_i(\Delta_{s(i),t})$ where the number $s(i)$ represents the order of $i$ in $(0, 1, \infty, t_1, \ldots, t_{m-3})$.

In particular, we have that $\text{Bl}(\sigma'_i) = \sigma_i$. Since we are going to compare two divisors on a smooth variety $\mathcal{F}_t$, we can exclude any codimension 2 subvariety in
it does not change the result. Therefore, we can assume $\Delta_{i,t}$ are smooth and disjoint.

Applying Lemma 8.5 and Lemma 8.6, we have that

$$\lambda = \sigma_i^* \omega_{U/F} = \sigma_i^* O_U(Y_i)$$

where $Y_i$ is the strict transformation of $\sigma_i'(\Delta_{s(i),t})$. I claim that $\sigma_i^*(Y_i) = \Delta_{s(i),t}$.

In fact, the transformation $Y_i$ is just supported on $\pi^{-1}(\Delta_{s(i),t})$. The intersection $Y_i \cap \pi^{-1}(C,f,x_1,\ldots,x_m)$ is the component of $C$ whose image of $f$ is a line containing $p_{s(i)}$. Hence, the intersection of the image $\sigma_i(F_t)$ and $Y_i$ (scheme theoretically) is $\Delta_{s(i),t}$. The claim is clear.

□

Remark 8.8. There is no such "blow-up" interpretation of $U$ for conics case, i.e, $m=2$, that is why, in the paper [23], we need to use the Grothendieck-R.R theorem to prove a similar result.

Lemma 8.9. Suppose $Z$ is a projective algebraic set in $\mathbb{P}^n$ and we have that $Z = A \cup B$ where $A$ and $B$ are closed subsets of $Z$. We also assume that the subset $A$ does not belong to the subset $B$, the subset $B$ does not belong to the subset $A$. For any hypeplane $H = \mathbb{P}^{n-1}$ which does not contain $A \cap B$, we have that the intersection $H \cap Z$ has at least two irreducible components.

Proof. Let $A_1$ be $A \cap H$ and $B_1$ be $B \cap H$, since the hypeplane $H = \mathbb{P}^{n-1}$ does not contain $A \cap B$, the intersection $A_1$ has an irreducible component $A_2$ which is not contained in $B$, similarly, the subset $B$ has an irreducible component $B_2$ which is not contained in $A$. I claim that $A_2$ and $B_2$ belong to two different irreducible components of $Z \cap H$. Otherwise, there is an irreducible subset $K$ of $H \cap Z$ containing $A_1$ and $B_1$. Therefore, we have that $K = (K \cap A) \cup (K \cap B)$ which implies $K \subseteq A$ or $K \subseteq B$. It is a contradiction with the fact that the subset $K$ contains $A_2$ and $B_2$.

□

Corollary 8.10. Suppose $Y$ is the locus in $F_t$ corresponding to stable maps of maximal degeneration type in $F_t$, i.e, the image of the stable map is union of $m$ lines which intersect at a point. See the following figure.

If $\dim Y \geq 1$, then the divisor $\Delta_{i,t}$ is irreducible and the locus $Y$ is a smooth variety. More generally, the intersection $\bigcap_{i=1}^{k} \Delta_{i,t}$ is irreducible for any $1 \leq k \leq m$.

In particular, the intersection $\bigcap_{i=1}^{k} \Delta_{i,t}$ is a smooth projective variety. The dimension of $Y$ is $n + m(c - \sum_{i=1}^{c} d_i) - c$.

Via the map $\Phi|_Y : Y \hookrightarrow \mathbb{P}^{n-m}$, the variety $Y$ is a complete intersection in $\mathbb{P}^{n-m}$ defined by equations of the following type:

$$T_1(d_1,m) = \left( \begin{array}{cccc}
2 & 3 & \cdots & d_1 - 1 \\
\vdots & \vdots & \ddots & \vdots \\
2 & 3 & \cdots & d_1 - 1
\end{array} \right)$$
union

\[
T_2(d_2, m) = \begin{pmatrix}
1 & 2 & \cdots & d_2 - 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 2 & \cdots & d_2 - 1
\end{pmatrix}
\]

\[
T_2(d_c, m) = \begin{pmatrix}
1 & 2 & \cdots & d_c - 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 2 & \cdots & d_c - 1
\end{pmatrix}
\]

Proof. As in the [4, Page 83 (2)], let \(D\) be the space of \(m\) lines on \(X\) with each line containing exactly one point among the points \(p_1, \ldots, p_m\) and intersecting at a point is a non-singular complete intersection in \(\mathbb{P}^n\) of type

(8.2)

\[
T_2(d_1, m) = \begin{pmatrix}
1 & \cdots & d_1 - 1 \\
\vdots & \ddots & \vdots \\
1 & \cdots & d_1 - 1
\end{pmatrix}, \ldots, T_2(d_c, m) = \begin{pmatrix}
1 & \cdots & d_c - 1 \\
\vdots & \ddots & \vdots \\
1 & \cdots & d_c - 1
\end{pmatrix}
\]

We denote it by \(D\). Hence, the dimension of \(D\) is \(n - m(\sum d_i - c) - c\).

Suppose a point \(t \in \mathcal{M}_{0,m}\) represents a smooth rational curve with \(m\) pointed points, we denote it by \((R, y_1, \ldots, y_m)\), where \(R\) is a rational curve and \(y_i\) are points on \(R\). For each point \(u \in D\), it represents a union of lines \(l_1 \cup l_2 \cup \cdots \cup l_m\) that the point \(p_i \in l_i\) and the lines \(l_i\) intersect at \(Q\), we can canonically associate to it a stable map \((C, f, x_1, \ldots, x_m)\) of maximal degeneration type in \(\mathcal{F}_t\) as following,

1. the domain \(C\) is the disjoint union of \(R \coprod l_1 \coprod l_2 \coprod \cdots \coprod l_m\) mod the relations \(Q(\in l_i) \sim y_i\)
2. set the points \(x_i = p_i\)
3. the map \(f\) maps \(l_i\) identically to \(l_i \subseteq X\) and collapse \(R\) to point \(Q\)

Therefore, we have a morphism

\[
\psi : D \to Y = \bigcap_{i=1}^{m} \Delta_{i,t} \subseteq \mathcal{F}_t
\]

It is easy to check this morphism is bijective. Since \(Y = \Delta_{1,t} \cap \Delta_{2,t} \cap \cdots \cap \Delta_{m,t}\) and \(\psi\) is surjective, the intersection \(\bigcap_{i=1}^{m} \Delta_{i,t} \subseteq \mathcal{F}_t\) is irreducible.

Via the embedding map \(\Phi\) and Proposition 8.7, the divisor \(\Delta_{i,t}\) is a hyperplane section. Let \(\bigcap_{i=1}^{k} \Delta_{i,t}\) be \(Z\) and \(\Delta_{j,t}\) be \(H\), we apply Lemma 8.9. Since we already prove the intersections \(\bigcap_{i=1}^{m} \Delta_{i,t} \subseteq \mathcal{F}_t\) are all irreducible, the intersection \(\bigcap_{i=1}^{k} \Delta_{i,t} \subseteq \mathcal{F}_t\) is irreducible for \(1 \leq k \leq m\) by Lemma 8.9. In particular, the divisor \(\Delta_{i,t}\) is irreducible for all \(i = 1, 2, \ldots, m\). Since \(\Delta_t\) is a simply normal crossing divisor, the variety \(Y\) and the intersection \(\bigcap_{i=1}^{k} \Delta_{i,t}\) are smooth by the argument above. In summary, we have that
(1) the morphism \( \psi \) is an isomorphism,
(2) the maximal degeneration locus \( Y = D \) is a smooth complete intersection variety.

From (8.2), we know that the locus \( D \), inside the underlying projective space \( \mathbb{P}^n \),
is in the \( m \) transversal hyperplanes and denote it by \( P \). To prove the last assertion,
we observe the following commutative diagram,
\[
\begin{array}{ccc}
Y = D & \subseteq & \mathbb{P}^n \\
\Phi|_Y & \downarrow & \downarrow pr \\
& \mathbb{P}^{n-m} & \simeq P
\end{array}
\]
where \( pr \) is a projection from \( \text{Span}(p_1, \ldots, p_m) = \mathbb{P}^{m-1} \) to \( \mathbb{P}^{n-m} \simeq P \). The map \( \Phi|_Y \) associates to a point \( Q \in Z = Y \) a point
\[ \text{Span}(p_1, \ldots, p_m, Q) \cap P = a \text{ point } \in P(= \mathbb{P}^{n-m}) \]
Therefore, the morphism \( \Phi|_Y \) is an inclusion which makes \( Y \) be a complete intersection \( \mathbb{P}^{n-m} \) of type (8.1), i.e., just take out \( m \) hyperplanes of the first matrix of (8.2).

\[ \square \]

9. Main Theorem

**Proposition 9.1.** \[23] Proposition 7.1\] Suppose that we have smooth projective varieties \( \Delta \subseteq F \) in \( \mathbb{P}^N \),
(1) the variety \( \Delta \) is a smooth divisor of a smooth projective variety \( F \),
(2) \( \dim \Delta \geq 1 \),
(3) the divisor \( \Delta \) is a complete intersection in \( \mathbb{P}^N \) of type \( (d_1, \ldots, d_c) \) where \( d_i \geq 1 \),
(4) the divisor \( \Delta \) is defined by a homogeneous polynomial of degree \( d_1 \) restricted to \( F \),

\[ \begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{Figure 5}
\end{figure} \]
then the smooth variety $F$ is a complete intersection of type $(d_2, \ldots, d_c)$ in $\mathbb{P}^N$.

**Proposition 9.2.** Suppose $N = \dim |\lambda|_{F_t}$ and $n + m(c - \sum_{i=1}^{c} d_i) - c \geq 1$. Then

$$|\lambda|_{F_t} : F_t \to \mathbb{P}^N$$

is an embedding and $\dim H^0(Y, \mathcal{O}_Y(1)) = n - mc$, where $Y$ is the same as in Corollary 8.10.

Moreover, the varieties $F_t$ and $\bigcap_{i=1}^{k} \Delta_{i,t}$ $(1 \leq k \leq m)$ are smooth complete intersections via this embedding.

**Proof.** By Corollary 8.10 we have an embedding which induces

$$\Phi|_Y : Y \hookrightarrow \mathbb{P}^{n-m}$$

whose the image is in a $\mathbb{P}^{n-m}$. Moreover, it is a complete intersection of type of the union of some hyperplanes and types (9.1). Since $Y = \bigcap_{i=1}^{m} \Delta_{i,t}$, the inequality

$$n + m(c - \sum_{i=1}^{c} d_i) - c \geq 1$$

implies $\dim Y \geq 1$. The intersection $\bigcap_{i=1}^{k} \Delta_{i,t}$ is a smooth projective variety for $1 \leq k \leq m$, hence, by Proposition 8.7, we can apply Proposition 9.1 to $\bigcap_{i=1}^{k} \Delta_{i,t}$ inductively, from $k = m$ to $k = 0$, where we set $\bigcap_{i=1}^{0} \Delta_{i,t}$ be $F_t$ if $k = 0$.

Therefore, the intersection $\bigcap_{i=1}^{k} \Delta_{i,t}$ and $F_t$ are complete intersections in $\mathbb{P}^N$. Their complete intersection types are union of some hyperplanes and types of (9.1). □

**Proposition 9.3.** With the same hypothesis as in Proposition 9.2, $N = \dim |\lambda|_{F_t} = n - mc - (c - 1)$.

**Proof.** We define $Y_k$ to be $\bigcap_{i=1}^{k} \Delta_{i,t}$ and $Y_0 = F_t$. Since $Y_k$ is a smooth projective variety by Corollary 8.10 and the divisor $\Delta_{i,t}$ is linear equivalent to $\lambda|_{F_t} = \mathcal{O}_{Y_k}(1)$
by Proposition 8.7, we have the following short exact sequence
\[ 0 \to \mathcal{O}_{Y_k}(-\lambda|_{Y_k}) \to \mathcal{O}_{Y_k} \to j_* \mathcal{O}_{Y_{k+1}} \to 0, \]

where \( j \) is the inclusion \( Y_{k+1} \subseteq Y_k \). Since \( Y_k \) is a complete intersection of dimension \( \geq 1 \), we have \( H^1(Y_k, \mathcal{O}_{Y_k}) = 0 \). We tensor the short exact sequence above with \( \mathcal{O}_{Y_k}(1) \) and take the \( H^i(\_) \) to get a long sequence as following
\[ 0 \to H^0(Y_k, \mathcal{O}_{Y_k}) \to H^0(Y_k, \mathcal{O}_{Y_k}(1)) \to H^0(Y_{k+1}, \mathcal{O}_{Y_{k+1}}(1)) \to H^1(Y_k, \mathcal{O}_{Y_k}) = 0. \]

Therefore, it concludes that
\[ \dim |\lambda|_{\mathcal{F}_t} = \dim H^0(\mathcal{F}_t, \mathcal{O}_{\mathcal{F}_t}(1)) = \dim H^0(Y_m, \mathcal{O}_{Y_m}(1)) + m = n - mc + m \]

the last equality follows from Proposition 9.2.

\[ \square \]

**Theorem 9.4.** Let \( X \) be a smooth complete intersection in \( \mathbb{P}^n \) of type \((d_1, \ldots, d_c)\), where \( d_i \) is at least 2, \( m \geq 3 \) and \( n \geq m \). If \( X \) is not a quadric hypersurface and
\[ n + m(c - \sum_{i=1}^c d_i) - c \geq 1 \]

as in Definition 4.6, we have a line bundle \( \lambda|_{\mathcal{F}_t} \) on \( \mathcal{F}_t \), then the corresponding complete linear system \( |\lambda|_{\mathcal{F}_t} \) defines a map
\[ |\lambda|_{\mathcal{F}_t} : \mathcal{F}_t \to \mathbb{P}^N = \mathbb{P}^{n-m(c-1)}, \]

via this map, the smooth variety \( \mathcal{F}_t \) is a complete intersection in \( \mathbb{P}^N \) of type
\[ T_1(d_1, m) = \begin{pmatrix} 2 & \ldots & d_1 - 1 \\ \vdots & \ddots & \vdots \\ 2 & \ldots & d_1 - 1 \end{pmatrix}, \ldots, T_1(d_c, m) = \begin{pmatrix} 2 & \ldots & d_c - 1 \\ \vdots & \ddots & \vdots \\ 2 & \ldots & d_c - 1 \end{pmatrix} \]

**Proof.** As at the end of the proof of Proposition 9.2, the general fiber \( \mathcal{F}_t \subseteq \mathbb{P}^N \) is a complete intersection. Its type is union of type 9.1 and \( s \) hyperplanes. Since we have
\[ \dim \mathcal{F}_t = \dim \mathcal{F} - \dim \mathcal{M}_{0,m} = (c + 1 - \sum_{i=1}^c d_i)m + n - c \]

and \( N = n - m(c-1) \) and type 8.2 consist of \( m(\sum_{i=1}^c d_i - 2c) \) equations, by dimension counting, we have that
\[ s = N - \dim \mathcal{F}_t - m(\sum_{i=1}^c d_i - 2c) - c = 0 \]

Therefore, the theorem is clear. 

\[ \square \]
10. Applications

We have three interesting applications of our main Theorem. The first one is related to rational connectedness of moduli space of rational curves on varieties. It is arising from some arithmetic problems, such as weak approximation (see the introduction of the paper [6]) and the existence of rational points on the variety over a function field (see the papers [4] and [10]).

The second application towards to enumerative geometry. We give a proof of a classical formula to count the number of twist cubic curves on a complete intersection (see the paper [2] and [8]). Moreover, we provide a new formula for counting the number of two crossing conics on a complete intersection.

The third one is to prove the Picard group of the moduli space is finite generated. We also prove the first hodge numbers of $F$ are zero.

Rational Connectedness of Moduli Space

Proposition 10.1. With the hypothesis as in Theorem 9.4, if
\[
m \left( \sum_{i=1}^{c} d_i(d_i-1) \right) + \sum_{i=1}^{c} d_i \leq n
\]
then $F$ is rationally connected.

Proof. With the notations as in Theorem 9.4 the canonical bundle of the fiber $F_t$ is given by
\[
K_{F_t} = \mathcal{O}_{\mathbb{P}^n} \left( -N - 1 + m \left( \sum_{i=1}^{c} \frac{d_i(d_i-1)}{2} \right) + \sum_{i=1}^{c} d_i \right)
\]
where $t \in \overline{\mathcal{M}}_{0,m}$ is a general point as in Theorem 9.4. The inequality in the hypothesis of the proposition is equivalent to say $K_{F_t}$ is anti-ample. In particular, in this case, the fiber $F_t$ is a smooth projective Fano variety, hence, it is rationally connected, see [18, Chapter V]. I claim that the fiber $F$ is rationally connected.

In fact, let points $p$ and $q$ be two general points in $F$ such that $F(p)$ and $F(q)$ are general points in $\overline{\mathcal{M}}_{0,m}$. Since $\overline{\mathcal{M}}_{0,m}$ is a smooth projective rational variety, there is a rational curve $D$ in $\overline{\mathcal{M}}_{0,m}$ connecting $F(p)$ and $F(q)$ such that the fiber of $F$ over a general point $D$ is rationally connected. Therefore, by Corollary 1.3 in [10], the general fiber $F$ is rationally connected. □

Remark 10.2. By [6, Lemma 6.5], it is not hard to prove that the canonical bundle $K_F$ is trivial on some rational curve sitting inside the maximal degeneration locus. Therefore, it is not a Fano variety in general.

In M.Deland’s thesis [7], he proves that cubic hypersurfaces in $\mathbb{P}^n$ are strongly rationally simply connected if $n \geq 9$. The following corollary explains why $n \leq 8$ it fails to be strongly rationally simply connected.

Proposition 10.3. If $m = 3$, the general fiber $F$ is the complete intersection variety in $\mathbb{P}^{n-3(c-1)}$ of type as in Theorem 9.4.

Proof. Note that $\overline{\mathcal{M}}_{0,m}$ is a point when $m = 3$, so the proposition is obvious from Theorem 9.4. □
Corollary 10.4. Suppose $X_3$ is a smooth cubic hypersurface in $\mathbb{P}^8$, then the moduli fiber $F$ for $m = 3$ is a complete intersection Calabi-Yau 4-fold. Hence, the general fiber $F$ is not rationally connected.

Proof. By Proposition 10.3, we know the fiber $F$ is a complete intersection variety in $\mathbb{P}^8$ of type $(2, 2, 2, 3)$. By the adjunction formula, we have the canonical bundle of $F$ is trivial since
\[ K_F = O_{\mathbb{P}^8}(-9 + 2 + 2 + 2 + 3)|_F = O_F \]
It completes the proof. \square

Enumerative Geometry

Suppose a complete intersection variety $X'$ is cut out from a complete intersection variety $X$ by $n - s$ general hyperplanes and $s \geq m$. We denote the general fibers of the evaluation map corresponding to $X'$ by $F'$. It is clear that $F'$ is cut out from the general fiber $F$ by $n - s$ divisors which are linear equivalent to $\lambda$.

\[ (10.1) \]

If $F'$ is just discrete points, i.e, $\dim F' = 0$, then the number of these points is just the number of rational curves of degree $m$ passing through $m$ general points on $X'$. See the following propositions for more precise statements.

Proposition 10.5. \[2, \text{Collary, page 9}\] Let $X$ be a smooth complete intersection of degree $(d_1, \ldots, d_r)$ in $\mathbb{P}^{n+r}$, with $n = 3 \sum_{i=1}^{r} (d_i - 1) - 3$. Then the number of twist cubics in $X$ passing through 3 general points $(p, q, r)$ is
\[ \frac{1}{d^2} \prod_{i=1}^{r} (d_i)!^3 \]
where $d$ is the degree of $X$.

Proof. We can assume the variety $X$ is cut out by hyperplanes from a smooth complete intersection $Y$ of type $(d_1, \ldots, d_r)$ and $Y \subseteq \mathbb{P}^e$ has sufficiently large dimension. The degree of the general fiber $F \subseteq \mathbb{P}^e$ corresponding to $Y$ has an enumerative geometrical interpretation.

In fact, a point in the $\mathbb{P}^{e-3}$ corresponds to a projective space $(\mathbb{P}^3)$ in $\mathbb{P}^e$ containing the plane $pqr$. More generally, a sub-projective space $\mathbb{P}^k \subseteq \mathbb{P}^{e-3}$ corresponds to a $\mathbb{P}^{k+3}$ in $\mathbb{P}^e$. So if we take a $\mathbb{P}^l \subseteq \mathbb{P}^e$ with complementary dimension respect to the fiber $F$, i.e, $l + \dim(F) = e$, then the number of the intersection points counts the number of twist cubics in the $X = \mathbb{P}^{l+3} \cap Y$ passing through $p, q$ and $r$. In other words, the degree($F$) is
\[ \# \{\text{twist cubics in } X \text{ which is passing through three general points } p, q \text{ and } r\} \]
By Proposition 10.3, the degree of $F$ is
\[ \frac{1}{d^2} \prod_{i=1}^{r} (d_i)!^3 \]
It completes the proof. □

**Definition 10.6.** We call a curve \( C \) in a projective space \( \mathbb{P}^n \) is a linking conic if the following conditions are satisfied,

1. the curve \( C \) is a nodal curve with two smooth rational component, i.e,
   \[ C = C_1 \cup C_2 \]
   where \( C_1 \) and \( C_2 \) are rational curves \( \mathbb{P}^1 \).
2. each component \( C_i \) of \( C \) is a conic in \( \mathbb{P}^n \) for \( i = 1, 2 \).

See the following figure.

![Figure 6](image)

**Proposition 10.7.** Let \( X \) be a smooth complete intersection of degree \((d_1, \ldots, d_r)\) in \( \mathbb{P}^{n+r} \), with \( n = 4 \sum_{i=1}^{r} (d_i - 1) - 4 \). Let \( S \) be the set consisting of linking conics \( C \) (see Definition 10.6) in \( X \) which pass through 4 general points \( p_1, p_2, p_3, p_4 \) such that the points \( p_1, p_2 \) are on the same component and the points \( p_3, p_4 \) are on the other component. The cardinality of \( S \) is

\[
\frac{1}{d^3} \prod_{i=1}^{r} (d_i!)^4
\]

where \( d \) is the degree of \( X \).

**Proof.** As before, we can assume that \( X \) is cut out from a smooth complete intersection \( Y \subseteq \mathbb{P}^c \) of type \((d_1, \ldots, d_r)\) by some hyperplanes and the variety \( Y \) has sufficiently large dimension. Since \( m = 4 \), we have the following diagram

\[
\begin{array}{c}
\mathcal{M}_{0,4} = \mathbb{P}^1 \\
\downarrow \Phi \downarrow F \\
\mathcal{F} \\
\end{array}
\]
Suppose the point $0 \in \overline{M}_{0,4} = \mathbb{P}^1$ represents a data $(B, x_1, x_2, x_3, x_4)$ where $B$ is equal to $B_1 \cup B_2 = \mathbb{P}^1 \cup \mathbb{P}^1$ and the points $x_1, x_2$ are on $B_1$, the points $x_3, x_4$ are on $B_2$.

I claim that the image of any stable map corresponding to a point in $\mathcal{F}_0$ is a linking conic.

In fact, suppose the stable map $(C = C_1 \cup C_2, f, x_1, \ldots, x_4)$ corresponds a point $P$ in $\mathcal{F}_0$. Since the points $p_i$ are general, neither the image $f(C_1)$ nor $f(C_2)$ can not be a line, hence, the images are two conics which meet at least one point. If the intersection of these two conics contains more than one point, then the intersection of $\text{Span}(f(C_1)) = \mathbb{P}^2$ and $\text{Span}(f(C_2)) = \mathbb{P}^2$ contains a line $\mathbb{P}^1$, therefore, the space $\text{Span}(f(C))$ is a $3 - \text{dim}$ projective space $\mathbb{P}^3$ not $\mathbb{P}^4$, it is a contradiction with that the map $\Phi$ is well-defined on the point $P(\in \mathcal{F})$. Therefore, the image $f(C) = f(C_1) \cup f(C_2)$ in $\mathbb{P}^e$ is a linking conic.

As in the paper [8, Page 33] using deformation methods, one can prove that the multiplicity of the fiber $\mathcal{F}_0$ of $F$ over 0 is one. See the figure below.

As before, it is easy to see that the degree of $\lambda|_{\mathcal{F}_0}$ (i.e., it is $(\lambda|_{\mathcal{F}_0})^{\text{dim}\mathcal{F}_0}$) is equal to

\begin{figure}
\begin{center}
\begin{tikzpicture}
\fill[black] (1,1) circle (2pt);
\fill[black] (-1,1) circle (2pt);
\fill[black] (1,-1) circle (2pt);
\fill[black] (-1,-1) circle (2pt);
\draw[->] (0,0) to (2,2);
\draw[->] (0,0) to (-2,-2);
\draw[->] (0,0) to (0,2);
\draw[->] (0,0) to (0,-2);
\draw[->] (0,0) to (1,1);
\draw[->] (0,0) to (-1,1);
\draw[->] (0,0) to (1,-1);
\draw[->] (0,0) to (-1,-1);
\end{tikzpicture}
\end{center}
\caption{Figure 7}
\end{figure}

\# {linking conic in $X$ and the points $p_1$ and $p_2$ are on the same component of the linking conic, the other points $p_3$ and $p_4$ are on the other component.}

Since $\overline{M}_{0,4}$ is a smooth curve and $\mathcal{F}$ is a variety, the map $F$ is flat by [15, Chapter III, Proposition 9.7]. By the fact that a flat map preserves the intersection numbers of divisors, see [15, Chapter VI.2 Appendix], we have that

$$(\lambda|_{\mathcal{F}_0})^{\text{dim}\mathcal{F}_0} = (\lambda|_{\mathcal{F}_i})^{\text{dim}\mathcal{F}_i}$$
where, as before, the point \( t \) is a general point of \( \mathcal{M}_{0,4} \). By Theorem 9.4, we know

\[
(\lambda|_{F_t})^{dim F_t} = \frac{1}{d^3} \prod_{i=1}^{r}(d_i)!^4
\]

where \( d \) is the degree of \( X \). It completes the proof. \( \square \)

**Picard Group of Moduli Space**

**Lemma 10.8.** Consider a morphism \( h \) between two smooth varieties \( A \) and \( B \) over the complex number \( \mathbb{C} \)

\[ h : A \to B \]

Suppose the morphism \( h \) is proper and dominant. Let \( K \) be the function field of \( B \). If the following two conditions are satisfied

1. The generic fiber \( A_K \) of \( h \) is geometrical connected,
2. The Picard groups \( Pic(A_K) \) and \( Pic(B) \) finite generated,

then the Picard group \( Pic(A) \) is finite generated.

**Proof.** By EGA IV [12, Corollary 9.7.9] and the generic smoothness theorem, we can choose an open subset \( U \subseteq B \) such that

1. The morphism \( h|_{h^{-1}(U)} : h^{-1}(U) \to U \) is smooth.
2. The geometrical fiber \( h^{-1}(\mathbb{C}(p)) \) is irreducible for every \( p \in U \).

Suppose \( A - h^{-1}(U) \) is equal to \( X_1 \cup X_2 \ldots \cup X_k \) where \( X_i \) are the irreducible components of \( A - h^{-1}(U) \). We consider the following pull back\( r^* : Pic(A) \to Pic(A_K) \)

where \( K \) is the function field of \( B \) and the morphism \( r \) is induced by the following diagram,

\[
\begin{array}{ccc}
A_K & \xrightarrow{r} & A \\
\downarrow & & \downarrow \\
\text{Spec}(K) & \longrightarrow & B
\end{array}
\]

By the hypothesis (2) of the lemma, we conclude that the image \( Im(r^*) \) is finite generated. Suppose \( L = \mathcal{O}_X(D) \) is a line bundle in the kernel of \( r^* \), where \( D \) is a weil divisor. We assume that the divisor \( D \) equals to \( \sum n_i D_i \) where \( D_i \) are irreducible codimension 1 closed subvariety of \( F \). Since \( L \) is in the kernel of \( r^* \), it implies that the line bundle \( L|_{A_K} \) is isomorphic to \( \mathcal{O}_{A_K} \). Hence, there exists a rational function \( f \) in the function field \( K(A_K) \) of \( A_K \), which is equal to the function field \( K(X) \) of \( X \), such that

\[
\sum n_i D_i + \text{div}(f) = \sum n'_j D'_j
\]

where

1. The divisors \( D'_j \) are irreducible closed subvariety of \( F \) of codimension 1,
2. The restriction \( h|_{D'_j} : D'_j \to B \) does not dominate \( B \).

Suppose \( D'_j = \{ q_j \} \) where \( q_j \) is the generic point of \( D'_j \). We have the following two cases:
(1) Case I: If the point \( q_j \) is in \( A - h^{-1}(U) = \bigcup X_i \), then it implies that the point \( q_j \) is in some \( X_i \). Therefore, the divisor \( D'_j \) is equal to \( X_i \) for some \( i \).

(2) Case II: If the point \( q_j \) is in \( h^{-1}(U) \), then I claim that the codimension of the point \( h(q_j) \) is equal to 1. In fact, the codimension of \( h(q_j) \in B \) is at least 1 since the point \( h(q_j) \) is not the generic point of \( B \). In other words, the dimension of the fiber \( h^{-1}(h(q_j)) \) is at most \( \dim A - 1 \). Since \( h^{-1}(h(q_j)) \) is irreducible (by the second property of the points in \( U \)), we have that

\[
D'_j = \overline{\{ q_j \}} = h^{-1}(h(q_j)).
\]

Since the morphism \( h \) over \( U \) is smooth, the codimension of \( h(p_i) \) is equal to 1 by \( \dim D'_j = \dim A - 1 \). In this case, the line bundle \( \mathcal{O}_A(D'_j) \) is equal to \( h^*(\mathcal{O}_B(h(q_j))) \in h^*(\text{Pic}(B)) \).

In summary, the kernel of \( h \) is generated by \( h^*(\text{Pic}(B)) \) and \( \mathcal{O}_A(X_j) \) for those \( X_j \) with \( \text{codim}_X X_j = 1 \). Therefore, the Picard group \( \text{Pic}(A) \) is finite generated. \( \square \)

**Lemma 10.9.** Suppose \( U \) is a variety and we have the following diagram

\[
\begin{array}{ccc}
Y & \subseteq & \mathbb{P}^n_U \\
\downarrow g & & \downarrow \pi \\
U & \to & \mathbb{P}^n_K
\end{array}
\]

where \( g \) is a flat and projective morphism. We assume the fibers \( Y_s \) are complete intersections of the same type \( (d_1, \ldots, d_c) \) in \( \mathbb{P}^n_{\mathbb{C}(s)} \) for each \( s \in U(\mathbb{C}) \). If \( K \) is the function field of \( U \), then the generic fiber \( Y_K \) of \( h \) is a complete intersection of type \( (d_1, \ldots, d_c) \) in \( \mathbb{P}^n_K \).

**Proof.** We first notice that if a projective variety \( T \) in \( \mathbb{P}^e \) is a complete intersection of type \( (d_1, \ldots, d_g) \) defined by homogenous polynomials \( (F_1, F_2, \ldots, F_g) \), then, by Hilbert theory (see [15], Chapter I section 7), the polynomials \( (F_1, F_2, \ldots, F_g) \) are minimal generators of the ideal sheaf \( I_T \) and the dimension of \( H^0(T, I_T(m)) \) is only dependent on the type \( (d_1, \ldots, d_g) \).

Therefore, the dimension function

\[
s \mapsto \dim H^0(Y_s, I_{Y_s}(m))
\]

is a constant function where \( s \in U(\mathbb{C}) \). In particular,

\[
\dim H^0(Y_K, I_{Y_K}(m)) = \dim H^0(Y_s, I_{Y_s}(m))
\]

by [20] Page 48, Corollary 2]. This equality also implies that we can choose homogenous polynomials \( (F_1, F_2, \ldots, F_c) \) defined over \( K \) such that, on the domain \( V \) of the coefficients of \( F_i \), the fiber \( Y_s \) of \( g \) over a point \( s \in V(\mathbb{C}) \) is a complete intersection in \( \mathbb{P}^n_s \) defined by equations as following :

\[
(F_1, F_2, \ldots, F_c)|_{V^s}.
\]

It implies that the projective variety \( Y_K \) is a complete intersection in \( \mathbb{P}^n_K \) defined by \( (F_1, F_2, \ldots, F_c) \). It completes the proof.

Another possible way of proving the lemma is to prove the locus of complete intersection variety in the corresponding Hilbert scheme is open, it involves the deformation theory of a complete intersection variety, see [22], Chapter 2].

\( \square \)
Lemma 10.10. Suppose $K$ is a field and consider a projective morphism $h$ from a scheme $A$ to $\text{Spec}(K)$. If the following two conditions are satisfied,

1. the morphism $h$ has a section $\sigma$.

\[
\begin{array}{c}
A \\
\sigma \\
\downarrow h \\
\text{Spec}(K)
\end{array}
\]

2. the Picard group $\text{Pic}(A_K) = \mathbb{Z}[L|A_K] = \mathbb{Z}$ where $L$ is a line bundle on $A$ and $K$ is the algebraic closure of $K$.

Then, the Picard group $\text{Pic}(A) = \mathbb{Z} = \mathbb{Z}[L]$.

Proof. The hypothesis (1) of the lemma ensures that the Picard functor is representable by the Picard scheme $\text{Pic}_{A/K}$, see [19, section 5] for the details. Therefore, the Picard group $\text{Pic}(A)$ (resp. $\text{Pic}(A_K)$) is just the $K$ (resp. $K$)-points of $\text{Pic}_{A/K}$.

In other words, we have that

\[
\text{Pic}(A) = \text{Pic}_{A/K}(K)
\]

\[
\text{Pic}(A_K) = \text{Pic}_{A/K}(K)
\]

Therefore, it concludes that

\[
\text{Pic}(A) = \text{Pic}(A_K)^{\text{Gal}(K/K)} = \mathbb{Z}[L|A_K]^{\text{Gal}(K/K)} = \mathbb{Z}[L]
\]

where the last equality is due to the line bundle $L|A_K$ is $\text{Gal}(K/K)$ invariant. It completes the proof. \hfill \Box

Proposition 10.11. With the hypothesis as in Theorem 9.4, the Picard group of $F$ is finite generated.

Proof. Suppose $K$ is the function field of $\mathcal{M}_{0,m}$. By Theorem 9.4 and Lemma 10.9, the generic fiber $F_K$ of the forgetful map $F$ is a complete intersection and the hypothesis of Theorem 9.4 implies that $\dim F_K$ is at least 3.

By the equality

\[
\text{Pic}(F_K) = \mathbb{Z} = \mathcal{O}_Y(1) = \mathbb{Z}[\lambda_K],
\]

the Picard group $\text{Pic}(F_K)$ is $\mathbb{Z}[\lambda_K] = \mathbb{Z}$ by Lemma 10.10 where we have that

\[
F_K |_{\lambda_K} \rightarrow \mathbb{P}_K^N.
\]

If we take $F = A$ and $\mathcal{M}_{0,m} = B$ in Lemma 10.8, then we have

1. the Picard group $\text{Pic}(F)$ is finite generated.

2. the restriction morphism

\[
r^*: \text{Pic}(F) \rightarrow \text{Pic}(F_K)
\]

is surjective.

It completes the proof. \hfill \Box

Corollary 10.12. With the hypothesis as in Theorem 9.4, the first hodge numbers $h^{1,0}(F) = h^{0,1}(F)$ of $F$ are all vanished.
Proof. We suppose that $\text{Pic}^0(\mathcal{F})$ is the connected component of the Picard scheme $\text{Pic}(\mathcal{F})$ containing the trivial bundle. The scheme $\text{Pic}^0(\mathcal{F})$ is a group scheme of finite type over $\mathbb{C}$. By the paper [20, Theorem Page 95], the scheme $\text{Pic}^0(\mathcal{F})$ is smooth over $\mathbb{C}$. By Proposition 10.11, we have that $\text{Pic}^0(\mathcal{F})$ is just a reduced point, i.e., it is isomorphic to $\text{Spec}(\mathbb{C})$. By the paper [16], the first order deformation space of the trivial bundle is given by $H^1(\mathcal{F}, O_{\mathcal{F}})$, and this space is just the tangent space of $\text{Pic}^0(\mathcal{F})$. Therefore, we have

$$0 = T_{[O_{\mathcal{F}}]}\text{Pic}^0(\mathcal{F}) = H^1(\mathcal{F}, O_{\mathcal{F}})$$

where $T_{[O_{\mathcal{F}}]}\text{Pic}^0(\mathcal{F})$ is the tangent space of $\text{Pic}^0(\mathcal{F})$ at the point $[O_{\mathcal{F}}] \in \text{Pic}^0(\mathcal{F})$.

By Hodge theory and Dolbeault Isomorphism theorem, see [11, Chapter 0], we have

$$h^1,0(\mathcal{F}) = h^0,1(\mathcal{F}) = 0.$$ □

Corollary 10.13. With the hypothesis as in Theorem 9.4, the Picard group of $\mathcal{F}$ is equal to

$$\mathbb{Z}[\lambda] \bigoplus \text{Pic}(\overline{\mathcal{M}_{0,m}}) \bigoplus N$$

where $\text{Pic}(\overline{\mathcal{M}_{0,m}})$ is the Picard group of $\overline{\mathcal{M}_{0,m}}$ and $N$ is a finite generated abelian group.

Proof. By the result (2) at the end of the proof of Proposition 10.11 we have a surjection as following

$$h : \text{Pic}(\mathcal{F}) \to \text{Pic}(\mathcal{F}_K) \cong \mathbb{Z} \cong \mathbb{Z}[\lambda],$$

where $\mathcal{F}_K$ is the generic fiber of the forgetful map $F$. It is easy to check the image of the pull back $F^*$

$$F^* : \text{Pic}(\overline{\mathcal{M}_{0,m}}) \to \text{Pic}(\mathcal{F})$$

is in the kernel of $h$. By Lemma 4.9 we have the following splitting of $F^*$

$$\sigma^* : \text{Pic}(\mathcal{F}) \to \text{Pic}(\overline{\mathcal{M}_{0,m}}),$$

where $\sigma$ is a section of $F$ as in Lemma 4.9. Therefore, the map $F^*$ is injective and $\text{Pic}(\overline{\mathcal{M}_{0,m}})$ is a summand of $\text{Pic}(\mathcal{F})$. It completes the proof. □

Corollary 10.14. With the hypothesis as in Theorem 9.4, the smooth variety $\mathcal{F}$ is not a complete intersection variety if $m \geq 4$.

Proof. Since the hypothesis in the Theorem 9.4 implies that the dimension of $\mathcal{F}$ is at least 4, the Picard number of $\mathcal{F}$ is one if $\mathcal{F}$ is a complete intersection, see [11, Page 178] for the details. By Corollary 10.13 the Picard number of $\mathcal{F}$ is at least 2 if $m \geq 4$. It completes the proof. □

References

1. Carolina Araujo and Ana-Maria Castravet, Polarized minimal families of rational curves and higher Fano manifolds, Amer. J. Math. 134 (2012), no. 1, 87–107. MR 2876140
2. Arnaud Beauville, Quantum cohomology of complete intersections, Mat. Fiz. Anal. Geom. 2 (1995), no. 3-4, 384–398. MR 1484335 (98i:14053)
3. K. Behrend and Yu. Manin, Stacks of stable maps and Gromov-Witten invariants, Duke Math. J. 85 (1996), no. 1, 1–60. MR 1412436 (98i:14041)
4. A. J. de Jong, Xuhua He, and Jason Michael Starr, Families of rationally simply connected varieties over surfaces and torsors for semisimple groups, Publ. Math. Inst. Hautes Études Sci. (2011), no. 114, 1–85. MR 2854858
5. A. J. de Jong and Jason Starr, Higher Fano manifolds and rational surfaces, Duke Math. J. 139 (2007), no. 1, 173–183. MR 2322679 (2008j:14078)
6. A. J. de Jong and Jason Michael Starr, *Low degree complete intersections are rationally simply connected*, Preprint (2006).

7. Matthew F. Deland, *Geometry of rational curves on algebraic varieties*, ProQuest LLC, Ann Arbor, MI, 2009, Thesis (Ph.D.)--Columbia University. MR 2713614

8. W. Fulton and R. Pandharipande, *Notes on stable maps and quantum cohomology*, Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., vol. 62, Amer. Math. Soc., Providence, RI, 1997, pp. 45–96. MR 1492534 (98m:14025)

9. William Fulton, *Intersection theory*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 2, Springer-Verlag, Berlin, 1998. MR 1644323 (99d:14003)

10. Tom Graber, Joe Harris, and Jason Starr, *Families of rationally connected varieties*, J. Amer. Math. Soc. 16 (2003), no. 1, 57–67 (electronic). MR 1937199 (2003m:14081)

11. Phillip Griffiths and Joseph Harris, *Principles of algebraic geometry*, Wiley Classics Library, John Wiley & Sons Inc., New York, 1994, Reprint of the 1978 original. MR 1288523 (95d:14001)

12. A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV*, Inst. Hautes Études Sci. Publ. Math. (1967), no. 32, 361. MR 0238860 (39 #220)

13. Joe Harris, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 133, Springer-Verlag, New York, 1995, A first course, Corrected reprint of the 1992 original. MR 1416564 (97e:14001)

14. Robin Hartshorne, *Residues and duality*, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, Vol. 20, Springer-Verlag, Berlin, 1966. MR 0222093 (36 #5145)

15. Joe Harris, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52. MR 0463157 (57 #3116)

16. Luc Illusie, *Complexe cotangent et déformations. I*, Lecture Notes in Mathematics, Vol. 239, Springer-Verlag, Berlin, 1971. MR 0491680 (58 #10886a)

17. Sean Keel, *Intersection theory of moduli space of stable n-pointed curves of genus zero*, Trans. Amer. Math. Soc. 330 (1992), no. 2, 545–574. MR 1034665 (92f:14003)

18. János Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 32, Springer-Verlag, Berlin, 1996. MR 1440180 (98c:14001)

19. D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, third ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], vol. 34, Springer-Verlag, Berlin, 1994. MR 1304906 (95m:14012)

20. David Mumford, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics, vol. 5, Published for the Tata Institute of Fundamental Research, Bombay, 2006, With appendices by C. P. Ramanujam and Yuri Manin, Corrected reprint of the second (1974) edition. MR 2514037 (2010e:14040)

21. Ziv Ran, *Normal bundles of rational curves in projective spaces*, Asian J. Math. 11 (2007), no. 4, 567–608. MR 2402939 (2009e:14091)

22. Angelo Vistoli, *The deformation theory of local complete intersections*, Preprint.

23. Pan Xuanyu, *Moduli space of 2-minimal-dominant rational curves on low degree complete intersections*, Preprint (2012).

Department of Mathematics, Columbia University, New York, NY 10025

E-mail address: pan@math.columbia.edu