POLYNOMIAL MOMENTS WITH A WEIGHTED ZETA SQUARE MEASURE ON THE CRITICAL LINE

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Abstract. We prove closed-form identities for the sequence of moments

\[ \int t^{2n} \left| \frac{\Gamma(s)}{\zeta(s)} \right|^2 dt \]

on the whole critical line \( s = 1/2 + it \). They are finite sums involving binomial coefficients, Bernoulli numbers, Stirling numbers and \( \pi \), especially featuring the numbers \( \zeta(n)B_n/n \) unveiled by Bettin and Conrey in [BC13a, BC13b]. Their main power series identity [BC13b], together with [DH21a], allows for a short proof of an auxiliary result: the computation of the \( k \)-th derivatives at \( 1 \) of the "exponential auto-correlation" function studied in [DH21a]. We also provide an elementary and self-contained proof of this secondary result. The starting point of our work is a remarkable identity proven by Ramanujan in 1915. The sequence of moments studied here, not to be confused with the moments of the Riemann zeta function, entirely characterizes \( \left| \zeta \right| \) on the critical line. They arise in some generalizations of the Nyman-Beurling criterion, but might be of independent interest for the numerous connections concerning the above mentioned numbers.

1. Introduction

1.1. Main result. The Gamma function and the Riemann zeta function are defined as usual as:

\[
\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx, \quad \Re(s) > 0,
\]

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re(s) > 1,
\]

and by analytic continuation. We often write \( \Gamma \zeta(s) = \Gamma(s) \zeta(s) \), for some complex number \( s \), with real part \( \Re(s) \). Throughout the paper, we use the following notations and convention:

- \( N, n, k, j \) are natural integers, possibly 0 or \(-1\); The convention \( \sum_{n=2}^{0,1} = \sum_{\emptyset} = 0 \) holds;
- \( i^2 = -1 \), and \( \gamma = 0.577215664 \cdots \) is the Euler constant;
- \( B_j \) denotes the \( j \)-th Bernoulli number (See Section 2.1);
- \( S(n,k) \) are the Stirling numbers of the second kind (\( S(n,k) = 0 \) if \( k > n \), see Section 2.2);
- The notation for binomial coefficients is standard.

The main result of this paper is the following

Theorem 1.1. For all \( N \geq 0 \),

\[
\frac{(-4)^N}{2\pi} \int_{-\infty}^{\infty} t^{2N} \left| \frac{\Gamma(s)}{\zeta(s)} \right|^2 dt = \log(2\pi) - \gamma - 4N + \left( \frac{4N}{2} - 1 \right) B_{2N} + \sum_{j=2}^{2N} T_{2N,j} \frac{\zeta(j)B_j}{j},
\]

where for all \( j \geq 2 \),

\[
T_{N,j} = (j-1)! \sum_{n=2}^{N} \binom{N}{n} 2^n \left[ (-1)^n S(n+1,j) + (-1)^j S(n,j-1) \right].
\]

Let us make four comments.

(i) Notice that the \( T_{2N,j} \)'s are integers and the numbers \( \zeta(j)B_j/j \) only involve \( \pi \), \( j \) and \( B_{2j} \), since \( 2(2j)! \zeta(2j) = (-1)^{j+1} B_{2j}(2\pi)^{2j} \) and \( B_{2j+1} = 0, j \geq 1 \). We keep this elegant notation in

Keywords: Riemann zeta function; Weighted moments; Gamma function; Polynomials; Bernoulli numbers; Stirling numbers; Nyman-Beurling criterion.
reference to the connection with the fundamental works of Bettin and Conrey \cite{BC13a, BC13b} (see below for more details).

(ii) The quantities under study

\[ M_k^{\Gamma} = \int_{-\infty}^{\infty} t^k \left| \Gamma \left( \frac{1}{2} + it \right) \right|^2 dt, \quad k \geq 0, \]

are moments related to a measure with a density where \( |\zeta|^2 \) appears. These moments are not to be confused with the moments of the Riemann zeta function

\[ \int_0^T \zeta \left( \frac{1}{2} + it \right)^{2k} dt, \quad T > 0, \quad k \geq 1, \]

which constitute a very important topic in analytic number theory, and has been the subject of huge research works; See recently e.g. \cite{CK19, HRS19, Naj20} and the fundamental references and connections therein.

(iii) The sequence \( M_k^{\Gamma} \) entirely characterizes \( |\zeta| \) on the critical line. Indeed, on one hand, \( |\Gamma(s)|^2 \) is entirely known due to Euler’s reflection formula (where we set \( s = 1/2 + it \) for real \( t \)):

\[ |\Gamma(s)|^2 = \Gamma(1-s) = \frac{\pi}{\sin(\pi s)} = \frac{2\pi}{e^{\pi t} + e^{-\pi t}}. \]

We could have written \( 1/ \cosh(\pi t) \) within the measure instead of \( \Gamma \), but we keep this notation in reference to its basic relationship with Mellin-Plancherel isometry (see Lemma 2.2). It is also known that the measure \( dt/ \cosh(\pi t) \) satisfies remarkable self-reciprocity properties regarding Fourier transform, see e.g. \cite{God15, p.125} and \cite{Coh15, p.125}.

On the other hand, using \( |\Gamma(s)|^2 = O(e^{-\pi t}) \) and the crude bound \( \zeta(s) = O(t) \) as \( t \to +\infty \) (see e.g. \cite{Ten22}), one has

\[ \int_{-\infty}^{\infty} |t|^k |\Gamma \zeta(s)|^2 dt = O \left( \int_0^{\infty} t^k e^{-t} dt \right) = O(k!) \]

so the Hamburger moment problem \( (-\infty, \infty) \) is determinate (\cite[Prop.1.5 p.88]{Sim98}), i.e. \( |\Gamma \zeta(s)|^2 dt \) is the only measure verifying Theorem 1.1. Notice that \( M_k^{\Gamma} = 0 \) since \( t \mapsto |\Gamma \zeta(s)| \) is even.

(iv) The right hand side of the main formula is an alternate quantity, which is the \( 2N \)-th derivative at \( 0 \) of the following function \( G \) unveiled by Ramanujan in 1915:

\begin{align}
\text{Theorem 1.2} \quad & \quad \text{\cite[Eq. (22) p .97]{Ram15}. For all real} \ v, \\
(1.1) \quad & \quad \int_0^\infty \left| \Gamma \left( -\frac{1}{4} + it \right) \right|^2 \Xi \left( \frac{t}{2} \right)^2 \frac{\cos(zt)}{1+tz} \ dt = \pi \sqrt{\pi} \ G(v), \\
\text{where} \ & \quad \Xi(t) = \zeta(\frac{t}{2} + it) = \frac{1}{2} s(s - 1) \pi^{-s/2} \Gamma(s/2) \zeta(s), \quad \text{and} \\
(1.2) \quad & \quad G(v) = \int_0^{\infty} \left( \frac{1}{ex^v - 1} - \frac{1}{xe^v} \right) \left( \frac{1}{x} - \frac{1}{e^{-x}} \right) dx.
\end{align}

The left hand side of our main formula is basically the \( 2N \)-th derivative at \( 0 \) of the left hand side of (1.1) (See Section 2.3).

The function \( G \) is related by a simple change of variable to the "exponential auto-correlation" function \( A \) introduced in \cite{DH21a}:

\[ A(w) = \int_0^\infty \left( \frac{1}{xw} - \frac{1}{e^{xw} - 1} \right) \left( \frac{1}{x} - \frac{1}{e^{x} - 1} \right) dx, \quad w > 0. \]
1.2. Secondary result. We can now state our secondary result on which Theorem 1.1 heavily relies. As usual, $\delta_{k,j}$ denotes the Kronecker symbol, and let $H_k$ be the harmonic number:

$$H_k = \sum_{j=1}^{k} \frac{1}{j}, \quad k \geq 1,$$

with the convention $H_{-1} = H_0 = 0$. Set

$$(1.3) \quad C = \frac{\log(2\pi) - \gamma}{2} = 0.6303307 \ldots$$

**Theorem 1.3.** For all $k \geq 0$,

$$A^{(k)}(1) = (-1)^k k! \left( (1 + \delta_{k,0})C - \frac{1}{2(k+1)} - \frac{H_{k-1}}{2} + \sum_{j=2}^{k} \left( \frac{k}{j-1} \right)^2 \frac{\zeta(j) B_j}{j} \right).$$

The last sum term of this equality is exactly the same term as in Theorem 1 – Lemma 1 in [BC13b], since $A(x)$ is related to their "period" function $\psi$ up to some $1/x$ and $\log(x)$ factors, as shown in [DH21a].

These last facts then allow for a short proof of Theorem 1.3, see Section 3.1. We also provide a self-contained and elementary proof, whose various techniques might be of independent interest, showing e.g. how Bettin and Conrey’s numbers arise from a combinatorial and real analysis setting.

1.3. Previous works and motivation. The so-called second moment of zeta $\int_0^T |\zeta(s)|^2 dt$ appears in a variety of contexts and is well understood since Hardy and Littlewood in 1916, see [Tit86, Chap. VII]. Integrating on $(0, \infty)$ requires a weight, and we encounter the denomination "weighted moment" in the literature. For instance, asymptotic expansion of $\int_0^\infty |\zeta(s)|^2 e^{-\delta t} dt$, $\delta > 0$, can be obtained, see e.g. [Tit86, Theorem 7.15 p.164], and [BC13a] for new remarkable formulas with convergent asymptotic series (See also [BC13c]). Second moments are also used and studied for other Dirichlet series, see e.g. [BISS18, ABBRS19] for various applications combining interesting tools. Higher weighted moments of zeta, especially the fourth one, are also studied, see e.g. [Tit86, Chap. VII], [IM06], [BC13a], and references therein.

The motivation for studying $MT^G$ stems from some generalization of the Nyman-Beurling criterion (NB) for the Riemann hypothesis (RH). NB is an approximation problem of the indicator function of $(0, 1)$ in $L^2(0, \infty)$ by linear combination of functions $t \mapsto \{\theta_k t\} \{\theta_j t\}$ where $\theta_k \in (0, 1)$, and $\{ \cdot \}$ is the fractional part function. Báez-Duarte [BD03] showed one can take $\theta_k = 1/k$ in NB. We refer to [DR07, DFMR13] where the authors consider generalizations of the zeta function.

RH then reads as a geometric problem studying a square distance in the Hilbert space $L^2(0, \infty)$; One then has to consider the scalar products

$$(1.4) \quad G_{k,j} = \int_0^\infty \left\{ \frac{1}{kt} \right\} \left\{ \frac{1}{jt} \right\} dt, \quad 1 \leq k, j \leq n,$$

where $n$ is the size of the corresponding Gram matrix. See [BDLS05] for a fine study of the corresponding auto-correlation function. Vasyumin [Vas95] proved that $G_{k,j}$ is a finite cotangent sum, which is connected to a variety of important objects and topics: Estermann function, reciprocity formulas, modular forms, Lewis–Zagier theory, see [BCH85, LZ01, BC13a, MR16, ABB17].

Replacing the $\theta_k$ in NB by random variables (r.v.) $X_{k,n}$, $1 \leq k \leq n$, produced new characterizations and structures, for instance using functions $t \mapsto E\{X_{k,n}/t\}$ (EZ is the expectation of a r.v. $Z$). See [DH21b] and an important generalization in [ADH22]. Two main frameworks arise:

- **Dilation** — Take e.g. exponential r.v. $X_k = X_1/k \sim E(k)$; The corresponding square distance written with Mellin isometry involves Dirichlet polynomials, as in the original criterion, but a smoothing effect appears: see the auto-correlation function in [DH21a].

- **Concentration** — Take e.g. Gamma distributed r.v. $X_{k,n}$ concentrated in $n$ around $1/k$; The square distance involves polynomials and the structure of zeta is then contained within the measure $|\Gamma(\zeta(s))|^2 dt$. The moments directly appear in the Gram matrix (see [ADH22]), a block Hankel matrix with a symbol having power-like singularities (see e.g. [Kra07]).
Surprisingly, the auto-correlation function $A$ plays a role in both scalar products: through its value at rational numbers $A(k/j)$ in (d), or its derivatives $A^{(k)}(1)$ in (c).

1.4. Outline. In Section 2, we first gather useful information and properties on Bernoulli and Stirling numbers that we will use throughout the paper. Second, we relate the polynomial moments with the derivatives of the remarkable function unveiled by Ramanujan in 1915. Third, we set the tools to differentiate this function.

In Section 3, we prove Theorem 1.1, especially including the short proof of Theorem 1.3 based on [BC13b] and [DH21a]. Section 4 is devoted to the elementary proof of Theorem 1.3, based on several tools developed in Section 2. Finally Section 5 exposes numerical results.

2. Reminders and Preliminary results

2.1. Bernoulli numbers. The sequence of Bernoulli numbers $(B_n)_{n \geq 0}$ can be defined by its exponential generating function:

\[
\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad |t| < 2\pi.
\]

(2.5)

The generating function for the Bernoulli polynomials $B_n(\cdot)$ reads

\[
\frac{t e^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad |t| < 2\pi.
\]

(2.6)

The $N$–th Bernoulli polynomial reads

\[
B_N(x) = \sum_{n=0}^{N} \binom{N}{n} B_n x^{N-n}.
\]

The first Bernoulli numbers are

\[
B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}, \quad B_5 = 0, \quad B_6 = \frac{1}{42}, \ldots
\]

For $0 \leq j \leq k$, we set:

\[
K_{k,j} = \frac{1}{k} \binom{k}{j} B_{k-j} + \delta_{j,k-1}.
\]

(2.7)

2.2. Stirling numbers.

2.2.1. Change of basis. We use the notation $(x)_n$ for the falling factorial polynomial:

\[
(x)_n = x(x-1) \cdots (x-n+1), \quad n \geq 1,
\]

with $(x)_0 = 1$. Both families $(1, x, x^2, \ldots, x^n)$ and $((x)_0, (x)_1, \ldots, (x)_n)$ are bases of the linear space of polynomial of degree at most $n$. The change of basis is ruled by Stirling numbers.

The Stirling numbers of the first kind (signed), resp. second kind, is the family of coefficients $(s(n,k))_{n \geq 1, 1 \leq k \leq n}$, resp. $(S(n,k))_{n \geq 1, 0 \leq k \leq n}$, defined by

\[
(x)_n = \sum_{k=0}^{n} s(n,k) x^k,
\]

\[
x^n = \sum_{k=0}^{n} S(n,k)(x)_k.
\]

If $k \leq 0$ or $k > n$ we set $s(n,k) = S(n,k) = 0$. Stirling numbers satisfy the recurrence relations:

\[
s(n+1,k) = s(n,k-1) - n s(n,k)
\]
\[
S(n+1,k) = S(n,k-1) + k S(n,k).
\]
The interplay between Stirling numbers of the first and second kind transcripts with matrices. Let $S_1 = (s(n,k))_{0 \leq n,k \leq N}$ and $S_2 = (S(n,k))_{0 \leq n,k \leq N}$. As an example, for $N = 4$,

$$S_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 1 & 3 & 1 & 0 \\
0 & 1 & 7 & 6 & 1
\end{pmatrix}, \quad S_1 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 \\
0 & 2 & -3 & 1 & 0 \\
0 & -6 & 11 & -6 & 1
\end{pmatrix}.$$

One then have the fundamental relation $S_2 = S_1^{-1}$. By computing each coefficient of $S_1 S_2 = \text{Id}_N$, one obtains that for any couple of indices $j \leq k$:

$$\sum_{p=j}^{k} S_2(k,p) s(p,j) = \delta_{j,k}.$$

The following identity, see [GKP94, (6.100)], will be used in the elementary proof of Theorem 1.3:

$$(2.8) \quad \sum_{p=j}^{k} \frac{S(k,p)s(p,j)}{p} = K_{k,j}.$$  

This formula is obtained writing a finite sum $\sum_{k} n^k$ using two times the change of basis by means of Stirling numbers, and then by identification using the Faulhaber formula.

2.2.2. Operator and generating function. Let $(u_k)_{k \geq 0}$ be a real sequence such that the power series

$$F_u(x) = \sum_{p \geq 0} u_p x^p$$

has radius of convergence $r > 0$. Denote by $\mathcal{E}(u)$ the sequence defined by:

$$\mathcal{E}(u)_n = \sum_{k=0}^{n} S(n,k)(-1)^k k! u_k,$$

and define the sequence $U$ by

$$U_n = \frac{(-1)^n}{n!} \mathcal{E}(u)_n.$$

The following lemma uses standard techniques of exponential generating functions, and will be very useful in the sequel.

**Lemma 2.1.** For all $t \in [0, \log(1+r)]$,

$$F_U(t) = F_u(1-e^{-t}).$$

**Proof.** Let $t \in [0, \log(1+r)]$, i.e. $0 \leq e^t - 1 < r$ and $0 \leq 1 - e^{-t} < r$. We have, since $S(n,k) \geq 0$,

$$\sum_{n \geq 0} \sum_{k=0}^{n} \frac{1}{n!} S(n,k) k! |u_k| t^n = \sum_{k \geq 0} |u_k| \left( k! \sum_{n \geq k} S(n,k) \frac{t^n}{n!} \right) = \sum_{k \geq 0} |u_k| (e^t - 1)^k < \infty.$$

Therefore, by Fubini Theorem,

$$\sum_{n \geq 0} U_n t^n = \sum_{k \geq 0} (-1)^k u_k \left( k! \sum_{n \geq k} S(n,k)(-1)^n \frac{t^n}{n!} \right) = \sum_{k \geq 0} (-1)^k u_k (e^{-t} - 1)^k = \sum_{k \geq 0} u_k (1 - e^{-t})^k,$$

as desired. \[\square\]
2.3. Ramanujan’s identity and the polynomial moments. In [Ram15], Ramanujan obtained remarkable identities concerning the \( \Xi \) and \( \xi \) functions, see [D11, Kim16] for many investigations. Fourier type transform involving the \( \Xi \)-function (and not its square) is a vast subject, see e.g. [Tit86, 2.6 p.35], and we refer to [DRRZ15] and references therein for recent applications.

The identity (1.1), which is Eq. (22) p.97 in [Ram15], is today interpreted as a consequence of Mellin-Plancherel isometry (See [Kim16] for generalizations and the full treatment of Ramanujan’s identities contained in [Ram15]). For the sake of completeness, we recall the Mellin-Plancherel isometry argument to obtain the following lemma, which is only a reformulation of Ramanujan’s identity (1.1) in terms of \( \zeta \) and \( \Gamma \):

**Lemma 2.2** (Reformulation of Eq.(22) in [Ram15]). For all real \( v \),

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \cos(2vt)|\Gamma(s)\zeta(s)|^2 \, dt = G(v) = \int_{0}^{\infty} \left( \frac{1}{e^{xe^s} - 1} - \frac{1}{xe^s} \right) \left( \frac{1}{e^{xe^{-s}} - 1} - \frac{1}{xe^{-s}} \right) \, dx.
\]

Proof. Let us recall the fundamental formula

\[
\int_{0}^{\infty} \left( \frac{1}{e^x - 1} - \frac{1}{x} \right) x^{w-1} \, dx = \Gamma(w)\zeta(w), \quad 0 < \Re(w) < 1.
\]

Set for real \( v \),

\[
f_v(x) = \frac{1}{e^{xe^s} - 1} - \frac{1}{xe^s}, \quad x > 0.
\]

Therefore, we have in particular for \( s = \frac{1}{2} + it \):

\[
\hat{f}_v(s) = \int_{0}^{\infty} \left( \frac{1}{e^{xe^s} - 1} - \frac{1}{xe^s} \right) x^{s-1} \, dx = e^{-vs}f_0(s) = e^{-vs}\Gamma(s)\zeta(s).
\]

Then, by Mellin-Plancherel isometry:

\[
G(v) = \int_{0}^{\infty} f_v(x)f_{-v}(x) \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}_v(s)\hat{f}_{-v}(\overline{s}) \, dt
= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-vs+it\overline{s}}|\Gamma(s)\zeta(s)|^2 \, dt
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos(2vt)|\Gamma(s)\zeta(s)|^2 \, dt,
\]

since \( -vs + it\overline{s} = -v\left( \frac{1}{2} + it \right) + v\left( \frac{1}{2} - it \right) = -2vt \), and \( t \mapsto \sin(2vt)|\Gamma(s)\zeta(s)|^2 \) is odd. \( \square \)

We can now differentiate under the integral sign to obtain

\[
G^{(2N)}(v) = \frac{(-1)^N}{2\pi} e^{2\pi i N} \int_{-\infty}^{\infty} \cos(2vt)|\Gamma(s)\zeta(s)|^2 \, dt,
\]

and then taking \( v = 0 \), we get the immediate

**Corollary 2.3.** For all \( N \geq 0 \),

\[
\int_{-\infty}^{\infty} t^{2N} \left| \Gamma \zeta \left( \frac{1}{2} + it \right) \right|^2 \, dt = 2\pi (-1)^N 2^{-2N} G^{(2N)}(0).
\]

The change of variable \( y = xe^{-v} \) in \( G \) directly yields:

**Lemma 2.4.** For all real \( v \),

\[
G(v) = e^v A(e^{2v}).
\]

The main point of the paper is to differentiate \( G \). To do so for the composed function \( v \mapsto A(e^{2v}) \), we will need the two lemmas of the following subsection.
2.4. Differentiating functions composed with the exponential. Let us consider the differential operator $D : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$:

$$(D\varphi)(x) = x\varphi'(x),$$

which is a basic example of a Cauchy-Euler operator. We set $D^{n+1} = D \circ D^n$ for all $n \geq 0$ with $D^0 = \text{Id}$. The following lemma is the first question of Exercise 13 p. 300 in [GKP94]:

**Lemma 2.5.** For all $\varphi \in C^\infty$ and $n \geq 0$, we have

$$D^n\varphi(x) = \sum_{k=0}^{n} S(n,k) x^k \varphi^{(k)}(x).$$

**Proof.** We check that $D^0 \varphi(x) = S(0,0) \varphi^{(0)}(x) = \varphi(x)$ and $D^1 \varphi(x) = x \varphi'(x) = S(1,1)x^1 \varphi^{(1)}(x)$. Assume the equality holds for some $n \geq 1$. Since $D$ is linear, we have

$$D^{n+1}\varphi(x) = \sum_{k=0}^{n} S(n,k) x^k \varphi^{(k)}(x) + x S(n,n) x^n \varphi^{(n)}(x),$$

where we used $S(n,n) = 1 = S(n+1,n+1)$, $S(n,0) = 0 = S(n+1,0)$ (since $n \geq 1$) and $S(n,1) = 1 = S(n+1,1).$ \hfill $\square$

**Lemma 2.6.** Let $\phi(x) = \varphi(e^x)$ for any real number $x$. Then, for all $n \geq 1$,

$$\phi^{(n)}(x) = (D^n \varphi)(e^x) = \sum_{k=0}^{n} S(n,k) e^{kx} \varphi^{(k)}(e^x).$$

**Proof.** We have $\phi'(x) = e^x \varphi'(e^x) = (D \varphi)(e^x)$. By induction, assume that for some $n \geq 1$, $\phi^{(n)}(x) = (D^n \varphi)(e^x)$. Then

$$\phi^{(n+1)}(x) = e^x (D^n \varphi)'(e^x) = (D(D^n \varphi))(e^x),$$

and the conclusion follows. \hfill $\square$

3. Proof of the main result Theorem 1.1

In view of Corollary 2.3 and Lemma 2.4, we need to compute the derivatives of $A$ (Theorem 1.3), and then those of $G$.

3.1. Computation of the derivatives of $A$. We give a short proof of Theorem 1.3 based on Bettin and Conrey’s power series identity for the function $\psi$ below.

Following [BC13a, BC13b], consider, for $\Im(z) > 0$,

$$E_1(z) = 1 - 4 \sum_{n \geq 1} d(n) e^{2\pi inz}$$

$$\psi(z) = E_1(z) - 1/z E_1(-1/z),$$

and their analytic continuation to $\mathbb{C} \setminus (-\infty, 0]$ ($d(n)$ is the number of divisors of $n$).
For $|z| < 1$, Bettin and Conrey [BC13b, Lemma 1, Part two] prove

\[ \psi(1 + z) = \frac{2i}{\pi} \sum_{k \geq 0} \psi_k z^k, \]

with

\[ \psi_k = \frac{(-1)^k}{k + 1} + 2 \sum_{j=1}^{k-1} (-1)^{k-j} \binom{k}{j} \frac{\zeta(j + 1)B_{j+1}}{j + 1}, \quad k \geq 2, \]

and $\psi_0 = 1$, $\psi_1 = -1/2$.

Moreover, comparing the theorems [BC13b, Theorem 1] and [DH21a, Theorem 1], we happily found a connection between $\psi$ and $A$ (See [DH21a] and Lemma 3.1 below). This turns out to be an almost direct consequence of [BC13b, Lemma 1, Part one, p.57 17]. It is not really obvious that the inverse Mellin transform of their function $Q(s) = \zeta(s)\zeta(1-s)/\sin(\pi s)$ is $A$, up to a multiplicative constant, while it is straightforward to see that the Mellin transform of $A$ in the critical strip is basically $Q$.

**Lemma 3.1** (Reformulation of Lemma 1, Part one, in [BC13b]). For all $x > 0$

\[ A(x) = \frac{i\pi}{4} \psi(x) + r(x), \]

where

\[ r(x) = C \left( 1 + \frac{1}{x} \right) - \frac{1}{2} \left( 1 - \frac{1}{x} \right) \log(x). \]

We can now differentiate $A$. On one hand,

\[ r^{(k)}(x) = \frac{(-1)^k k!}{x^{k+1}} C - \frac{1}{2} \sum_{j=0}^{k} \binom{k}{j} (1 - 1/x)^{(j)} \log(x)^{(k-j)} \]

\[ = \frac{(-1)^k k!}{x^{k+1}} C + \frac{1}{2} \sum_{j=1}^{k-1} \binom{k}{j} \frac{(-1)^j j!}{x^{j+1}} \frac{(-1)^{k-j-1} (k-j-1)!}{x^{k-j}} \]

\[ - \frac{1}{2} \frac{1}{1 - 1/x} \frac{(-1)^{k-1} (k-1)!}{x^{k}} + \frac{1}{2} \frac{(-1)^k k!}{x^{k+1}} \log(x). \]

Therefore

\[ r^{(k)}(1) = (-1)^k k! C - \frac{(-1)^k k!}{2} \sum_{j=1}^{k-1} \frac{1}{k-j} \]

\[ = (-1)^k k! \left( C - \frac{H_{k-1}}{2} \right), \]

with the convention $H_0 = 0$. On the other hand, with Taylor formula,

\[ \frac{i\pi}{4} \psi^{(k)}(1) = \frac{i\pi}{4} \cdot \frac{2i}{\pi} \psi_k k! \]

\[ = - \frac{1}{2} \frac{(-1)^k k!}{k + 1} - (-1)^k k! \sum_{j=2}^{k} \frac{(-1)^{j-1} \binom{k}{j-1} \zeta(j)B_{j}}{j}, \]

which yields to the desired expression in Theorem 1.3, noting that $(-1)^j B_j = B_j$ for $j \geq 2$. 
3.2. Computation of the derivatives of $G$. Let $u$ be a real sequence, recall the definition of $\mathcal{E}(u)$ and define the sequence $\mathcal{L}(u)$:

$$
\mathcal{E}(u)_n = \sum_{k=0}^{n} S(n, k)(-1)^k k! u_k
$$

$$
\mathcal{L}(u)_N = \sum_{n=0}^{N} \binom{N}{n} 2^n u_n.
$$

The $\mathcal{E}$ is used to suggest that it is related to differentiating a function composed by an exponential. We use the notation $\mathcal{L}$ to refer to the Leibniz rule to differentiate a product.

**Lemma 3.2.** For all $N \geq 1$,

$$
G^{(N)}(0) = \left( \mathcal{L} \circ \mathcal{E} \left( c - \frac{1}{2}(\iota + \eta) + \beta \right) \right)_N,
$$

where we define for all $k \geq 0$,

$$
c_k = C(1 + \delta_{k,0})
$$

$$
\iota_k = \frac{1}{k + 1}
$$

$$
\eta_k = H_{k-1}
$$

$$
\beta_k = \sum_{j=2}^{k} \left( \frac{k}{j-1} \right) B_j \zeta(j).
$$

**Proof.** Recall that

$$
G(v) = e^v A(e^{2v}).
$$

Using the Leibniz rule, Lemma 2.6 together with the composition $x \mapsto 2x$, we obtain

$$
G^{(N)}(x) = \sum_{n=0}^{N} \binom{N}{n} 2^n \sum_{k=0}^{n} S(n, k)e^{2kx} A^{(k)}(e^{2x}),
$$

and then

$$
G^{(N)}(0) = \sum_{n=0}^{N} \binom{N}{n} 2^n \sum_{k=0}^{n} S(n, k) A^{(k)}(1).
$$

Writing

$$
A^{(k)}(1) = (-1)^k k! \left( c_k - \frac{1}{2}(\iota_k + \eta_k) + \beta_k \right),
$$

and using the notations, yields the desired expression. \hfill \square

In view of Corollary 2.3 and the linearity of the operator $\mathcal{L} \circ \mathcal{E}$, what remains to establish Theorem 1.1 is to compute each of the four quantities $\mathcal{L} \circ \mathcal{E}(c)$, $\mathcal{L} \circ \mathcal{E}(\iota)$, $\mathcal{L} \circ \mathcal{E}(\eta)$ and $\mathcal{L} \circ \mathcal{E}(\beta)$. This task is accomplished in the next four lemmas.

**Lemma 3.3.** We have

$$
\mathcal{L} \circ \mathcal{E}(c)_N = C \sum_{n=0}^{N} \binom{N}{n} 2^n \sum_{k=0}^{n} S(n, k)(-1)^k k!(1 + \delta_{k,0}) = (1 + (-1)^N)C.
$$

**Proof.** First, we have due to the definition of $S(n, k)$:

$$
\sum_{k=0}^{n} S(n, k)(-1)^k k! = \sum_{k=0}^{n} S(n, k)(-1)_k = (-1)^n,
$$

and
and
\[
\sum_{n=0}^{N} \binom{N}{n} 2^n (-1)^n = (-1)^N.
\]

Second, notice that
\[
\sum_{n=0}^{N} \binom{N}{n} 2^n \sum_{k=0}^{n} S(n,k)(-1)^k k! \delta_{k,0} = \sum_{n=0}^{N} \binom{N}{n} 2^n S(n,0) = \binom{N}{0} S(0,0) = 1.
\]

\[\square\]

**Lemma 3.4.** We have
\[
\mathcal{E}(\iota)_n = B_n \quad \mathcal{L} \circ \mathcal{E}(\iota)_N = (2 - 2^N)B_N.
\]

**Proof.** First:
\[
F_\iota(x) = \sum_{k=0}^{\infty} \frac{x^k}{k+1} = -\frac{\ln(1-x)}{x}, \quad 0 < x < 1.
\]

Therefore, noting that \(B_1 + 1 = (-1)^1 B_1\) and \(B_n = (-1)^n B_n\) for \(n \geq 2\),
\[
F_\iota(1 - e^{-t}) = \frac{t}{1 - e^{-t}} - \left(1 - \frac{1}{e^t - 1}\right) = \sum_{n=0}^{\infty} (-1)^n B_n \frac{t^n}{n!}.
\]

But \(F_\iota(1 - e^{-t}) = F_\iota(t)\) with \(I_n = (-1)^n e^{(\iota)_n}\). Therefore, by identification,
\[
\mathcal{E}(\iota)_n = B_n.
\]

Second: Recall that the \(N^{th}\) Bernoulli polynomial is
\[
B_N(x) = \sum_{n=0}^{N} \binom{N}{n} B_n x^{N-n},
\]
and \(B_N(0) = B_N\). Thus
\[
B_N\left(\frac{1}{2}\right) = \frac{1}{2^N} \sum_{n=0}^{N} \binom{N}{n} B_n 2^n
\]
\[
= \frac{1}{2^N} \mathcal{L} \circ \mathcal{E}(\iota)_N.
\]

But it is known that (see e.g. Corollary 9.1.5. p.5-6 in [Coh07])
\[
B_N\left(\frac{1}{2}\right) = \left(2 - 2^{N-1}\right) B_N.
\]

Hence
\[
\mathcal{L} \circ \mathcal{E}(\iota)_N = (2 - 2^N)B_N.
\]

\[\square\]

**Lemma 3.5.** We have
\[
\mathcal{E}(\eta)_n = (-1)^n n + \delta_{n,1} \quad \mathcal{L} \circ \mathcal{E}(\eta)_N = 2N(1 + (-1)^N).
\]

**Proof.** First, we compute \(\mathcal{E}(\iota)_n\). Since \(H_{-1} = H_0 = 0\),
\[
F_\eta(x) = \sum_{k \geq 0} H_{k-1} x^k = \sum_{k \geq 2} H_{k-1} x^k = \sum_{k \geq 1} H_k x^{k+1}, \quad 0 < x < 1.
\]
Thus, writing $H_k$ and using the Fubini–Tonelli theorem,

$$F_{\eta}(x) = x \sum_{k=1}^{\infty} \sum_{1 \leq p \leq k} \frac{x^k}{p} = x \sum_{p \geq 1} \frac{1}{p} \sum_{k \geq p} x^k = \frac{x}{1-x} \sum_{p \geq 1} \frac{x^p}{p} = -\frac{x \ln(1-x)}{1-x}.$$ 

Then

$$F_{\eta}(1-e^{-t}) = -\frac{(1-e^{-t})(-t)}{e^{-t}} = t(e^t - 1) = \sum_{n \geq 1} \frac{t^{n+1}}{n!} = \sum_{n \geq 2} \frac{t^n}{(n-1)!}.$$ 

But $F_{\eta}(1-e^{-t}) = F_{\mathcal{H}}(t)$ with $\mathcal{H}_n = \frac{(-1)^n}{n!} \mathcal{E}(\eta)_n$. Therefore $\mathcal{H}_0 = \mathcal{H}_1 = 0$ and

$$\frac{(-1)^n}{n!} \mathcal{E}(\eta)_n = \frac{1}{(n-1)!}, \quad n \geq 2.$$ 

Hence

$$\mathcal{E}(\eta)_n = (-1)^n n, \quad n \geq 2,$$

and $\mathcal{E}(\eta)_0 = \mathcal{E}(\eta)_1 = 0$, which we can write for all $n \geq 0$,

$$\mathcal{E}(\eta)_n = (-1)^n n + \delta_{n,1}.$$ 

Second:

$$L \circ \mathcal{E}(\eta)_N = \sum_{n=0}^{N} \binom{N}{n} (-1)^n n + \delta_{n,1}$$

But

$$\sum_{n=0}^{N} \binom{N}{n} (-2)^n n = \sum_{n=1}^{N} n \binom{N}{n} (-2)^n = N \sum_{n=1}^{N} \binom{N-1}{n-1} (-2)^n = N \sum_{n=0}^{N-1} \binom{N-1}{n} (-2)^{n+1} = -2N(-2+1)^{N-1} = 2(-1)^N N.$$ 

Finally,

$$L \circ \mathcal{E}(\eta)_N = 2N(1 + (-1)^N).$$

\[\square\]

**Lemma 3.6.** We have

$$L \circ \mathcal{E}(\beta)_N = \sum_{j=2}^{N} T_{N,j} \frac{\zeta(j)B_j}{j},$$

where

$$T_{N,j} = (j-1)! \sum_{n=2}^{N} \binom{N}{n} 2^n \left[ (-1)^n S(n+1,j) + (-1)^j S(n,j-1) \right].$$
Proof. Let us expand and rewrite
\[
\mathcal{L} \circ \mathcal{E}(\beta)_N = \sum_{n=0}^{N} \binom{N}{n} 2^n \sum_{k=0}^{n} S(n,k)(-1)^k k! \sum_{j=2}^{k} \binom{k}{j-1} \frac{\zeta(j) B_j}{j}
\]
\[
= \sum_{j=2}^{N} \sum_{n=2}^{N} \binom{N}{n} 2^n W_{n,j} \frac{\zeta(j) B_j}{j} ,
\]
where \( W_{n,j} = 0 \) if \( n < j \), and for \( n \geq j \),
\[
W_{n,j} = \sum_{k=j}^{n} S(n,k)(-1)^k k! \binom{k}{j-1}
\]
\[
= \sum_{k=j-1}^{n} S(n,k)(-1)^k k! \binom{k}{j-1} - (-1)^{j-1}(j-1)!S(n,j-1).
\]
Fix \( j \geq 2 \). Then
\[
\sum_{k=j-1}^{n} S(n,k)(-1)^k k! \binom{k}{j-1} = \mathcal{E}(u)_n ,
\]
where \( u_k = \binom{k}{j-1} 1_{k \geq j-1} \). We have
\[
F_u(x) = \sum_{k=0}^{\infty} u_k x^k = \sum_{k=j-1}^{\infty} \binom{k}{j-1} x^k
\]
\[
= x^{j-1} \sum_{k=j-1}^{\infty} \binom{k}{j-1} x^{k-j+1}
\]
\[
= \frac{x^{j-1}}{(1 - x)^{j}}.
\]
Thus
\[
F_u(1 - e^{-t}) = e^{t \sum_{j=1}^{\infty} \binom{k}{j-1} x^{k-j+1}}
\]
\[
= e^{t(1 - 1)^j-1}
\]
\[
= (e^t - 1)^j - (e^t - 1)^{j-1}.
\]
Since
\[
(e^t - 1)^j = j! \sum_{n \geq j} \frac{S(n,j)}{n!} t^n
\]
\[
(e^t - 1)^{j-1} = (j-1)! \sum_{n \geq j-1} \frac{S(n,j-1)}{n!} t^n,
\]
we deduce that for \( n \geq j \):
\[
U_{n,j}(-1)^n = (j-1)! [jS(n,j) + S(n,j-1)] = (j-1)!S(n+1,j).
\]
Hence
\[
W_{n,j} = (j-1)! \left[ (-1)^n S(n+1,j) + (-1)^j S(n,j-1) \right],
\]
as desired. \( \square \)
4. Elementary Proof of Theorem 1.3

4.1. Decomposition of $A^{(k)}(1)$. Set

\[
  h(x) = \frac{1}{e^x - 1}, \quad x > 0.
\]

We can differentiate $A$ under the integral sign:

\[
A^{(k)}(1) = \int_0^\infty \left( \frac{(-1)^{k} k!}{x} - x^k h^{(k)}(x) \right) \left( \frac{1}{x} - \frac{1}{e^x - 1} \right) \, dx,
\]

by grouping the divergent terms as $x \to 0$ inside the integral of $A^{(k)}(v)$ and using e.g. Lemma 4.4 with a uniform bound for $v \in [1 - \eta, 1 + \eta]$ (small $\eta > 0$). We can then develop and obtain:

**Lemma 4.1.** For all $k \geq 1$,

\[
A^{(k)}(1) = D_{3,k}^\varepsilon + D_{2,k}^\varepsilon - D_{1,k}^\varepsilon,
\]

where

\[
D_{1,k}^\varepsilon = \int_\varepsilon^\infty x^{k-1} h^{(k)}(x) \, dx \\
D_{2,k}^\varepsilon = \int_\varepsilon^\infty \frac{x^k h^{(k)}(x)}{e^x - 1} \, dx \\
D_{3,k}^\varepsilon = (-1)^k k! \int_\varepsilon^\infty \frac{1}{x^k} \left( \frac{1}{x} - \frac{1}{e^x - 1} \right) \, dx.
\]

4.2. The derivatives of $h$. Set

\[
\alpha_{k,p} = (-1)^p S(k,p)! 
\]

The integrals $D_{1,k}^\varepsilon$ and $D_{2,k}^\varepsilon$ involve the derivatives of $h$. The paper [GQ14] gives some expressions of these ones and interesting applications. For our purpose, we need a different formula, especially having an $e^x$ within the numerator:

**Lemma 4.2.** For all $k \geq 1$,

\[
h^{(k)}(x) = \sum_{p=1}^{k} \alpha_{k,p} \frac{e^{px}}{(e^x - 1)^{p+1}}.
\]

**Proof.** Setting $\varphi(x) = \frac{1}{x-1}$, we have for all $p \geq 1$,

\[
\varphi^{(p)}(x) = \frac{(-1)^p p!}{(x-1)^{p+1}}.
\]

Noting that $h(x) = \varphi(e^x)$, we apply Lemma 2.6 to deduce

\[
h^{(k)}(x) = \sum_{p=1}^{k} S(k,p) e^{px} \frac{(-1)^p p!}{(e^x - 1)^{p+1}},
\]

as claimed. $\Box$

**Lemma 4.3.** For $k \geq 0$ and $p \geq 1$,

\[
\alpha_{k,p-1} - \alpha_{k,p} = -\frac{\alpha_{k+1,p}}{p}.
\]

**Proof.** This only requires the recursive definition of $S(k+1,p)$:

\[
\alpha_{k,p-1} - \alpha_{k,p} = (-1)^{p-1} S(k,p-1)(p-1)! - (-1)^p S(k,p)! \\
= (-1)^{p+1} (p-1)! S(k,p-1) + p S(k,p) \\
= (-1)^{p+1} (p-1)! S(k+1,p),
\]

and the claim follows. $\Box$
4.3. Asymptotic expansions involving $h$. We gather here several asymptotic expansions, useful for the sequel.

**Lemma 4.4.** For all $a \geq 0$, as $\varepsilon \to 0$,

$$e^a \frac{e^{a\varepsilon}}{(e^\varepsilon - 1)^{a+1}} = \frac{1}{\varepsilon} + \frac{a - 1}{2} + o(1).$$

*Proof.* We have

$$e^a \frac{e^{a\varepsilon}}{(e^\varepsilon - 1)^{a+1}} = \frac{1}{\varepsilon} \frac{1 + a\varepsilon + o(\varepsilon)}{(1 + \varepsilon/2 + o(\varepsilon))^{a+1}}.$$ 

But

$$(1 + \varepsilon/2 + o(\varepsilon))^{-(a+1)} = 1 - \frac{a + 1}{2} \varepsilon + o(\varepsilon),$$

and the conclusion follows. □

**Lemma 4.5.** As $\varepsilon \to 0$,

$$I(\varepsilon) = \int_{\varepsilon}^{\infty} \frac{1}{t} \left(\frac{1}{t} - \frac{1}{e^t - 1}\right) dt = \frac{1}{2} \log \varepsilon + C + o(1),$$

where

$$C = 1 - \int_{0}^{1} \left(\frac{1}{t} - \frac{1}{t^2} + \frac{1}{2t}\right) dt - \int_{1}^{\infty} \frac{dt}{t(e^t - 1)}.$$ 

*Proof.* In [Bal18], Balazard identified the constant $C = \frac{1}{2} (\log 2\pi - \gamma)$, see [DH21a] for his proof.

Rewriting $C$, we obtain

$$C = 1 - \int_{0}^{1} \left(\frac{1}{t} - \frac{1}{t^2} + \frac{1}{2t}\right) dt - \int_{1}^{\infty} \frac{dt}{t(e^t - 1)}$$

$$= \int_{0}^{1} \left(\frac{1}{t} - \frac{1}{t^2} + \frac{1}{2t}\right) dt + \int_{1}^{\infty} \frac{1}{t} \left(\frac{1}{t} - \frac{1}{e^t - 1}\right) dt$$

$$= I(\varepsilon) - \int_{\varepsilon}^{1} \frac{dt}{2t} + o(1)$$

$$= I(\varepsilon) + \frac{\log \varepsilon}{2} + o(1).$$

The asymptotic development of $I(\varepsilon)$ and the second quantity then follow. Moreover

$$\int_{\varepsilon}^{\infty} \frac{-1}{(e^t - 1)^2} dt = \int_{\varepsilon}^{\infty} \frac{e^t - 1}{(e^t - 1)^2} dt - \int_{\varepsilon}^{\infty} \frac{e^t}{(e^t - 1)^2} dt$$

$$= \lim_{\varepsilon \to 0} \left(\log(1 - e^{-t})\right) + \left[\frac{1}{e^t - 1}\right]_{\varepsilon}^{\infty}$$

$$= - \log(1 - e^{-\varepsilon}) - \frac{1}{e^\varepsilon - 1}$$

$$= - \log(\varepsilon) - \frac{1}{\varepsilon} + \frac{1}{2} + o(1),$$

as desired. □

The following elementary quantities will be useful in the next section.

**Lemma 4.6.** As $\varepsilon \to 0$,

$$\sum_{q \geq 1} \frac{e^{-\varepsilon q}}{q} = - \log \varepsilon + o(1).$$

(4.21)
and for all \( a \geq 1 \),
\[
\varepsilon^a \sum_{q \geq 1} q^{a-1} e^{-\varepsilon q} = (a-1)! + o(1) \tag{4.22}
\]
\[
\varepsilon^a \sum_{q \geq 1} q^a e^{-\varepsilon q} = \frac{a!}{\varepsilon} + o(1). \tag{4.23}
\]

Proof. We have
\[
\sum_{q \geq 1} \frac{e^{-\varepsilon q}}{q} = \int_{\varepsilon}^{\infty} \frac{dt}{e^t - 1} = [\log(1 - e^{-t})]_{\varepsilon}^{\infty} = -\log(1 - e^{-\varepsilon}),
\]
and we obtain the first expansion.

For \( a \geq 2 \), we have
\[
\varepsilon^a \sum_{q \geq 1} q^{a-1} e^{-\varepsilon q} = (-1)^{a-1} \varepsilon^a h^{(a-1)}(\varepsilon).
\]

But
\[
\varepsilon^a h^{(a-1)}(\varepsilon) = \varepsilon^a \sum_{p=1}^{a-1} S(a-1, p) e^{px} \frac{(-1)^p p!}{(e^x - 1)^{p+1}}
\]
\[
= (-1)^{a-1} (a-1)! \frac{\varepsilon^a e^{(a-1)x}}{(e^x - 1)^a} + o(1)
\]
\[
= (-1)^{a-1} (a-1)! + o(1),
\]
and the conclusion follows noticing that the identity also holds for \( a = 1 \).

Finally, for \( a \geq 1 \),
\[
\varepsilon^a \sum_{q \geq 1} q^a e^{-\varepsilon q} = (-1)^a \varepsilon^a h^{(a)}(\varepsilon).
\]

Notice that for \( a \geq 2 \), as \( \varepsilon \to 0 \),
\[
\varepsilon^a h^{(a)}(\varepsilon) = \varepsilon^a \sum_{p=1}^{a} S(a, p) e^{px} \frac{(-1)^p p!}{(e^x - 1)^{p+1}}
\]
\[
= (-1)^{a-1} (a-1)! S(a, a-1) \frac{\varepsilon^a e^{(a-1)x}}{(e^x - 1)^a} + (-1)^a a! \frac{\varepsilon^a e^{ax}}{(e^x - 1)^{a+1}} + o(1)
\]
\[
= (-1)^{a-1} (a-1)! S(a, a-1) + (-1)^a a! \left( \frac{1}{\varepsilon} + \frac{a-1}{2} \right) + o(1).
\]

Since
\[
(a-1)! S(a, a-1) = (a-1)! \left( \frac{a}{2} \right) = \frac{a-1}{2} a!
\]
we deduce
\[
\varepsilon^a \sum_{q \geq 1} q^a e^{-\varepsilon q} = \frac{a!}{\varepsilon} + o(1).
\]
and the conclusion follows noticing that the identity also holds for \( a = 1 \). \qed

4.4. **Incomplete integrals related to** \( D_{1,k}^\varepsilon \) **and** \( D_{2,k}^\varepsilon **.** The following quantities will be involved in the computation of \( D_{1,k}^\varepsilon \) and \( D_{2,k}^\varepsilon **.** Set
\[
J_1^\varepsilon (k, p) = \int_{\varepsilon}^{\infty} x^k \frac{e^{px}}{(e^x - 1)^{p+1}} dx
\]
\[
J_2^\varepsilon (k, p) = \int_{\varepsilon}^{\infty} x^k \frac{e^{px}}{(e^x - 1)^{p+2}} dx.
\]
and

\[ Z_\varepsilon(k, j) = \sum_{q=1}^{\infty} q^j \int_\varepsilon^\infty e^{-qy} y^k dy. \]

**Lemma 4.7.** For all \( k, p \geq 0 \),

\[ J_\varepsilon^2(k, p) = J_\varepsilon^1(k, p + 1) - J_\varepsilon^1(k, p), \]

**Proof.** A simple manipulation gives

\[ J_\varepsilon^2(k, p) = \int_\varepsilon^\infty x^k \frac{e^{px}}{(e^x - 1)^{p+2}} dx = -\int_\varepsilon^\infty x^k \frac{e^{px}(e^x - 1 - e^x)}{(e^x - 1)^{p+2}} dx \]

\[ = -\int_\varepsilon^\infty x^k \frac{e^{px}}{(e^x - 1)^{p+1}} dx + \int_\varepsilon^\infty x^k \frac{e^{(p+1)x}}{(e^x - 1)^{p+2}} dx, \]

as claimed. \( \square \)

**Lemma 4.8.** For all \( k, p \geq 0 \),

\[ J_\varepsilon^1(k, p) = \int_\varepsilon^\infty x^k \frac{e^{px}}{(e^x - 1)^{p+1}} dx = \frac{1}{p^!} \sum_{j=0}^{p} (-1)^{p+j} s(p, j) Z_\varepsilon(k, j), \]

where

\[ Z_\varepsilon(k, j) = k! \sum_{a=0}^{k} \frac{\varepsilon^a}{a!} \sum_{q=1}^{\infty} \frac{e^{-\varepsilon q}}{q^{k+j+1-a}}. \]

If \( j < k \), i.e. \( k - j + 1 \geq 2 \), then

\[ Z_\varepsilon(k, j) = k! \zeta(k-j+1) + o(1). \]

**Proof.** Using changes of variables, we obtain

\[ J_\varepsilon^1(k, p) = \int_\varepsilon^\infty x^k \frac{e^{px}}{(e^x - 1)^{p+1}} dx \quad (u = e^x, \ x = \log u) \]

\[ = \int_\varepsilon^\infty u^{p-1} \log u \frac{k}{u-1}^{p+1} du \quad (u = 1/x) \]

\[ = (-1)^k \int_0^{e^{-\varepsilon}} \log^k x \frac{k}{(1-x)^{p+1}} dx. \]

We have the expansion

\[ \frac{1}{(1-x)^{p+1}} = \frac{1}{p^!} \sum_{q=1}^{\infty} \frac{(q+p-1)!}{(q-1)!} x^{q-1}. \]

On the other hand,

\[ \frac{(q+p-1)!}{(q-1)!} = q(q+1) \cdots (q+p-1) \]

\[ = (-1)^p (-q)(-q-1) \cdots (-q-(p-1)) \]

\[ = (-1)^p \sum_{j=0}^{p} s(p, j)(-1)^j q^j, \]

where we recall that \( s(p, j) \) is a (signed) Stirling number of the first kind. Using the Fubini–Tonelli theorem and the change of variables \( y = -\log(x) \), we obtain

\[ J_\varepsilon^1(k, p) = \frac{(-1)^k}{p^!} \sum_{j=0}^{p} (-1)^{p+j} s(p, j) \sum_{q=1}^{\infty} q^j \int_0^{e^{-\varepsilon}} x^{q-1} \log^k x \ dx \]

\[ = \frac{1}{p^!} \sum_{j=0}^{p} (-1)^{p+j} s(p, j) \sum_{q=1}^{\infty} q^j \int_\varepsilon^\infty e^{-qy} y^k dy. \]
Let us now set and compute, using the particular form of the incomplete Gamma function $\int_{\varepsilon}^{\infty} e^{-x} x^k dx$ when $k$ is an integer:

$$Z_{\varepsilon}(k,j) = \sum_{q=1}^{\infty} q^j \int_{\varepsilon}^{\infty} e^{-\varepsilon q y} y^k dy$$

$$= \sum_{q=1}^{\infty} q^j \frac{1}{q^{k+1}} \int_{\varepsilon}^{\infty} e^{-x} x^k dx$$

$$= k! \sum_{q=1}^{\infty} e^{-\varepsilon q} \sum_{a=0}^{k} \frac{(\varepsilon q)^a}{a!}.$$

Finally notice that if $j < k$, the dominated convergence theorem yields

$$Z_{\varepsilon}(k,j) = k! \zeta(k-j+1) + o(1),$$

and the conclusion follows. \qed

**Lemma 4.9.** The following expansions hold:

(4.24) $Z_{\varepsilon}(k,k) = -k! \log(\varepsilon) + k! H_k + o(1), \quad k \geq 1$;

(4.25) $Z_{\varepsilon}(k,k+1) = \frac{(k+1)!}{\varepsilon} - \frac{k!}{2} + o(1) \quad k \geq 0$.

**Proof.** First, we compute

$$Z_{\varepsilon}(k,k) = k! \sum_{a=0}^{k} \frac{\varepsilon^a}{a!} \sum_{q \geq 1} q^{a-1} e^{-\varepsilon q}$$

$$= k! \left( \sum_{q \geq 1} q^{-1} e^{-\varepsilon q} + \sum_{a=1}^{k} \frac{1}{a!} \varepsilon^a \sum_{q \geq 1} q^{a-1} e^{-\varepsilon q} \right)$$

$$= k! \left( - \log(\varepsilon) + \sum_{a=1}^{k} \frac{1}{a} \right) + o(1).$$

Second,

$$Z_{\varepsilon}(k,k+1) = k! \sum_{a=0}^{k} \frac{\varepsilon^a}{a!} \sum_{q \geq 1} q^{a} e^{-\varepsilon q}$$

$$= k! \left( \sum_{q \geq 1} e^{-\varepsilon q} + \sum_{a=1}^{k} \frac{1}{a!} \varepsilon^a \sum_{q \geq 1} q^{a} e^{-\varepsilon q} \right)$$

$$= k! \left( \frac{1}{e^\varepsilon - 1} + \sum_{a=1}^{k} \frac{1}{a! \varepsilon} \right) + o(1)$$

$$= k! \left( \frac{1}{\varepsilon} - \frac{1}{2} + \frac{k}{\varepsilon} \right) + o(1),$$

as claimed. \qed

4.5. Asymptotic expansion of $D_{1,k}^\varepsilon$, $D_{2,k}^\varepsilon$, and conclusion.

**Lemma 4.10.** For all $k \geq 1$,

$$D_{1,k}^\varepsilon = (-1)^k \left( \frac{k!}{\varepsilon} - \frac{(k-1)!}{2} \right) + o(1).$$
Proof. We have for all $k \geq 1$,

$$D_{1,k}^\varepsilon \quad = \quad \int_\varepsilon^\infty x^{k-1}h(k)(x) \, dx$$

$$= \int_\varepsilon^\infty x^{k-1} \sum_{p=1}^{k} \alpha_{k,p} \frac{e^{px}}{(e^x - 1)^{p+1}} \, dx$$

$$= \sum_{p=1}^{k} \alpha_{k,p} J_1^\varepsilon(k-1,p)$$

$$= \sum_{p=1}^{k} (-1)^p S(k,p) \sum_{j=1}^{p} S_1(p,j) Z_\varepsilon(k-1,j)$$

$$= \sum_{j=1}^{k} (-1)^j \delta_{k,j} Z_\varepsilon(k-1,j)$$

$$= (-1)^k Z_\varepsilon(k-1,k),$$

and we use (4.25) replacing $k$ by $k-1$ to conclude. $\square$

Lemma 4.11. We have

$$D_{2,k}^\varepsilon \quad = \quad \sum_{j=1}^{k+1} (-1)^{j+1} K_{k+1,j} Z_\varepsilon(k,j).$$

Proof. We have

$$D_{2,k}^\varepsilon \quad = \quad \int_\varepsilon^\infty \frac{x^k h(k)(x)}{e^x - 1} \, dx$$

$$= \int_\varepsilon^\infty \frac{x^k}{e^x - 1} \sum_{p=1}^{k} \alpha_{k,p} \frac{e^{px}}{(e^x - 1)^{p+1}} \, dx$$

$$= \sum_{p=1}^{k} \alpha_{k,p} J_2^\varepsilon(k,p)$$

$$= \sum_{p=1}^{k} \alpha_{k,p} (J_1^\varepsilon(k,p+1) - J_1^\varepsilon(k,p))$$

$$= \sum_{p=2}^{k+1} \alpha_{k,p-1} J_1^\varepsilon(k,p) - \sum_{p=1}^{k} \alpha_{k,p} J_1^\varepsilon(k,p)$$

$$= \alpha_{k,k} J_1^\varepsilon(k,k+1) - \alpha_{k,1} J_1^\varepsilon(k,1) + \sum_{p=2}^{k} (\alpha_{k,p-1} - \alpha_{k,p}) J_1^\varepsilon(k,p)$$

$$= \alpha_{k,k} J_1^\varepsilon(k,k+1) - \alpha_{k,1} J_1^\varepsilon(k,1) - \sum_{p=2}^{k} \frac{\alpha_{k-1+p}}{p} J_1^\varepsilon(k,p)$$

$$= -\sum_{p=1}^{k+1} \frac{\alpha_{k+1,p}}{p} J_1^\varepsilon(k,p),$$
We have Lemma 4.12. Moreover

\[
\sum_{p=1}^{k+1} \frac{\alpha_{k+1,p}}{p} J_p(k, p) = \sum_{p=1}^{k+1} \frac{(-1)^p}{p} S(k+1, p) \sum_{j=1}^{p} (-1)^{p-j} S_1(p, j) Z_\varepsilon(k, j)
\]

\[
= \sum_{j=1}^{k+1} \sum_{p=j}^{k+1} \frac{(-1)^j}{p} S(k+1, p) S_1(p, j) Z_\varepsilon(k, j)
\]

\[
= \sum_{j=1}^{k+1} (-1)^j K_{k+1,j} Z_\varepsilon(k, j),
\]

which concludes the proof. \(\square\)

Therefore, by Lemma 4.9,

\[
\sum_{j=1}^{k} \frac{(-1)^j}{j} K_{k+1,j} Z_\varepsilon(k, j) \geq \sum_{j=1}^{k+1} (-1)^{k+1} K_{k+1,k+1} Z_\varepsilon(k, k+1)
\]

But

\[
K_{k+1,k} = \frac{1}{k+1} \binom{k+1}{k} B_1 + 1 = 1/2
\]

\[
K_{k+1,k+1} = \frac{1}{k+1} \binom{k+1}{k+1} B_0 = \frac{1}{k+1}
\]

Therefore, by Lemma 4.9,

\[
R_{2,k}^\varepsilon = (-1)^{k+1} K_{k+1,k} Z_\varepsilon(k, k) + (-1)^k K_{k+1,k+1} Z_\varepsilon(k, k+1)
\]

For \(k \geq 2\), \(D_{2,k}^\varepsilon\) contains convergent terms involving \(\zeta\) and a divergent term \(R_{2,k}^\varepsilon\), which reads

\[
R_{2,k}^\varepsilon = (-1)^{k+1} K_{k+1,k} Z_\varepsilon(k, k) + (-1)^k K_{k+1,k+1} Z_\varepsilon(k, k+1)
\]

But

\[
K_{k+1,k} = \frac{1}{k+1} \binom{k+1}{k} B_1 + 1 = 1/2
\]

\[
K_{k+1,k+1} = \frac{1}{k+1} \binom{k+1}{k+1} B_0 = \frac{1}{k+1}
\]

Therefore, by Lemma 4.9,

\[
R_{2,k}^\varepsilon = (-1)^{k+1} \frac{k! \log(\varepsilon)}{2} + \frac{(-1)^k 1}{k+1} \binom{k+1}{k} \left( \frac{(k+1)!}{\varepsilon} - \frac{k!}{2} \right) + o(1).
\]

**Lemma 4.12.** We have

\[
D_{3,k}^\varepsilon = (-1)^{k+1} \frac{k! \log(\varepsilon)}{2} + (-1)^k k! C + o(1).
\]

We can now complete the proof of Theorem 1.3 using

\[
A^{(k)}(1) = D_{3,k}^\varepsilon + D_{2,k}^\varepsilon - D_{1,k}^\varepsilon.
\]

Therefore for all \(k \geq 1\), with the convention \(\sum_{j=1}^{0} = 0\),

\[
A^{(k)}(1) = (-1)^k k! \left( \frac{H_{k+1}}{2} + \frac{1}{2k} - \sum_{j=1}^{k-1} (-1)^{j+k} K_{k+1,j} \zeta(k-j+1) \right).
\]

Using for all \(j \leq k-1\)

\[
(4.26)
\]

\[
K_{k+1,j} = \frac{1}{k+1} \binom{k+1}{j} B_{k+1-j},
\]

symmetry and "pion" formula for binomial coefficients, and \((-1)^j B_j = B_j, j \geq 2\), we can write:

\[
\sum_{j=1}^{k-1} (-1)^{j+k} K_{k+1,j} \zeta(k-j+1) = \sum_{j=1}^{k-1} (-1)^{j+k} \frac{1}{k+1} \binom{k+1}{j} B_{k+1-j} \zeta(k-j+1)
\]

\[
= - \sum_{j=2}^{k} (-1)^j \frac{1}{k+1} \binom{k+1}{j} B_j \zeta(j)
\]

\[
= - \sum_{j=2}^{k} \frac{1}{j} \binom{k}{j-1} B_j \zeta(j)
\]
Hence, for all \( k \geq 1 \), with the convention \( H_0 = 0 \),

\[
A^{(k)}(1) = (-1)^k k! \left( C - \frac{1}{2(k+1)} - \frac{H_{k-1}}{2} + \sum_{j=2}^{k} \left( j - 1 \right) \frac{\zeta(j)B_j}{j} \right).
\]

To obtain the formula for \( k = 0 \), let us finally compute \( A(1) \).

**Lemma 4.13.** We have

\[
A(1) = 2C - \frac{1}{2}.
\]

**Proof.** We expand

\[
A(1) = \int_{\varepsilon}^{\infty} \left( \frac{1}{x} - \frac{1}{e^x - 1} \right) \left( \frac{1}{x} - \frac{1}{e^x - 1} \right) dx + o(1)
\]

\[
= I(\varepsilon) - \int_{\varepsilon}^{\infty} x^{-1} dx + \int_{\varepsilon}^{\infty} \frac{dx}{(e^x - 1)^2} + o(1)
\]

\[
= -\frac{\log \varepsilon}{2} + C - \frac{1}{\varepsilon} \frac{\log \varepsilon}{2} + C + \log(\varepsilon) + \frac{1}{\varepsilon} - \frac{1}{2} + o(1)
\]

\[
= 2C - \frac{1}{2} + o(1),
\]

and take the limit. \( \square \)

5. **Numerical results**

5.1. **Expression of the first** \( A^{(k)}(1) \). Recall that

\[
A(v) = \int_0^{\infty} \left( \frac{1}{x^v} - \frac{1}{e^{xv} - 1} \right) \left( \frac{1}{x} - \frac{1}{e^x - 1} \right) dx, \quad v > 0,
\]

and

\[
C = \frac{\log(2\pi) - \gamma}{2} = 0.6303307 \ldots
\]

We have \( A(1) = 2C - \frac{1}{2} \),

\[
A'(1) = -\left( C - \frac{H_2}{2} + \frac{1}{2} \right) = -C + \frac{1}{4},
\]

and

\[
A''(1) = 2 \left( C - \frac{H_3}{2} + \frac{1}{4} + \frac{2}{1} \frac{B_2\zeta(2)}{2} \right)
\]

\[
= 2C - \frac{4}{3} + \frac{1}{3} \zeta(2).
\]

To check these values, we directly differentiate \( A \) and evaluate at 1, for instance:

\[
A'(1) = \int_0^{\infty} \left( \frac{1}{x} + \frac{xe^x}{(e^x - 1)^2} \right) \left( \frac{1}{x} - \frac{1}{e^x - 1} \right) dx.
\]
5.2. Expression of the two first moments. The formula for $N = 1$ reads

$$\frac{-2}{\pi} \int_{-\infty}^{\infty} t^2 \left| \Gamma \left( \frac{1}{2} + it \right) \right|^2 \, dt = \log(2\pi) - \gamma - 4 + \left( \frac{4}{2} - 1 \right) B_4 + \frac{T_{2,2} \zeta(2) B_2}{2}$$

$$= \log(2\pi) - \gamma - 4 + \frac{23}{6} + \frac{4}{3} \zeta(2),$$

since

$$T_{2,2} = \left( \frac{2}{2} \right)^2 2^2 [S(3, 2) + S(2, 1)]$$

$$= 4(3 + 1) = 16.$$

The formula for $N = 2$ reads

$$\frac{-4}{2\pi} \int_{-\infty}^{\infty} t^4 \left| \Gamma \left( \frac{1}{2} + it \right) \right|^2 \, dt = \log(2\pi) - \gamma - 8 + \left( \frac{4^2}{2} - 1 \right) B_4 + \sum_{j=2}^{4} T_{4,j} \frac{\zeta(j) B_j}{j},$$

where

$$T_{4,j} = (j - 1)! \sum_{n=2}^{4} \left( \frac{4}{n} \right)^2 \left[ (-1)^n S(n + 1, j) + (-1)^j S(n, j - 1) \right].$$

Thus, we have

$$T_{1,2} = 24(3 + 1) + 32(-7 + 1) + 16(15 + 1)$$

$$= 160,$$

$$T_{1,4} = 6(0 + 0 + 16(10 + 6))$$

$$= 1536.$$

Hence

$$\frac{8}{\pi} \int_{-\infty}^{\infty} t^4 \left| \Gamma \left( \frac{1}{2} + it \right) \right|^2 \, dt = \log(2\pi) - \gamma - 8 - 7 \cdot \frac{1}{30} + 160 \frac{\zeta(2) / 6}{2} - 1536 \frac{\zeta(4) / 30}{4}$$

$$= \log(2\pi) - \gamma - \frac{247}{30} + \frac{40}{3} \zeta(2) - \frac{64}{5} \zeta(4).$$

5.3. Numerical values. The following table gives the first values of the $T(\ell, j)$:

| $\ell \backslash j$ | 2     | 3     | 4 | 5     | 6     | 7     | 8     |
|-------------------|-------|-------|---|-------|-------|-------|-------|
| 2                 | 16    |       |   |       |       |       |       |
| 3                 | 0     | -144  |   |       |       |       |       |
| 4                 | 160   | 0     | 1536 |       |       |       |       |
| 5                 | 0     | -5280 | 0 | -19200 |       |       |       |
| 6                 | 1456  | 0     | 145920 | 0 | 276480 |       |       |
| 7                 | 0     | -147504 | 0 | -3897600 | 0 | -4515840 |       |
| 8                 | 13120 | 0     | 9225216 | 0 | 105799680 | 0 | 82575360 |

The last one gives the first moments $M_k^{\Gamma \zeta}$:
Acknowledgement

The authors are very grateful to the anonymous referee for their report, comments and suggestions that improved the paper.

The authors thank Michel Balazard for noting the relation of Ramanujan’s formulas with Mellin-Plancherel theorems and communicating the reference [Kim16]. They thank Charles Bordenave for communicating [Sim98]. They are also grateful to René Adad, Séréna Pedon and Gérald Tenenbaum for corrections, and to René Adad and Joseph Najnudel for numerical experiments.

The first author warmly thanks Francois Alouges, Michel Balazard, Paul Bourgade, Christophe Delaunay, Persi Diaconis, Bryce Kerr, Oleksiy Klurman, Joseph Najnudel, Olivier Ramaré and Kristian Seip for interesting references and insightful discussions.

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