Longest Cycles in Sparse Random Digraphs

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Received 14 February 2011; accepted 30 March 2012
Published online 28 May 2012 in Wiley Online Library (wileyonlinelibrary.com).
DOI 10.1002/rsa.20435

ABSTRACT: Long paths and cycles in sparse random graphs and digraphs were studied intensively in the 1980’s. It was finally shown by Frieze in 1986 that the random graph $G(n, p)$ with $p = c/n$ has a cycle on at all but at most $(1 + \varepsilon)ce^{-\varepsilon}n$ vertices with high probability, where $\varepsilon = \varepsilon(c) \to 0$ as $c \to \infty$. This estimate on the number of uncovered vertices is essentially tight due to vertices of degree 1. However, for the random digraph $D(n, p)$ no tight result was known and the best estimate was a factor of $c/2$ away from the corresponding lower bound. In this work we close this gap and show that the random digraph $D(n, p)$ with $p = c/n$ has a cycle containing all but $(2 + \varepsilon)e^{-\varepsilon}n$ vertices w.h.p., where $\varepsilon = \varepsilon(c) \to 0$ as $c \to \infty$. This is essentially tight since w.h.p. such a random digraph contains $(2e^{-c} - o(1))n$ vertices with zero in-degree or out-degree. © 2012 Wiley Periodicals, Inc. Random Struct. Alg., 43, 1–15, 2013

Keywords: directed graphs; random graphs; cycles

1. INTRODUCTION

In this paper we consider long cycles in random directed graphs, aiming to obtain estimates analogous to those derived for the undirected case. Formally, a random graph $G(n, p)$ is a probability space of all graphs with vertex set $[n]$, where each pair of vertices $1 \leq i < j \leq n$
We say that a sequence of events \((A_n)\) in a random (di)graph model occurs with high probability, or w.h.p. for brevity, if the probability of \(A_n\) tends to 1 as the number of vertices \(n\) tends to infinity.

**The proof of the theorem is given in the next section.** In certain similarity to Frieze’s argument in [9] we proceed by first filtering out vertices of zero in-degree or out-degree as well as some vertices close to them. The so obtained digraph typically retains all but a negligible fraction of the vertices of positive in-degrees and out-degrees; it is then upgraded to another random digraph, containing an almost spanning cycle, by sprinkling a few more random directed edges.

Before we embark into the technicalities of the proof, we provide its outline, aiming to help the reader to parse the proof’s details.

The proof has two components/stages: filtering and factoring. The filtering stage aims to filter out vertices of in- or out-degree zero and possibly some other vertices and to
produce an induced subgraph $D_0$ of $D \sim \mathcal{D}(n, p)$, containing most of the vertices of positive degree; moreover, $D_0$ is constructed in a way making it rather straightforward to show that it typically contains a factor of directed cycles. In order to produce $D_0$, we define the following iterative process. Let $Y = \{v : d_D(v) = 0\}$, and let $Z = \{v : d_D(v) \leq 3\}$, where $d_D(v) = \min[d_D^+(v), d_D^-(v)]$. We start with $X_0 = \emptyset$, and then for $k \geq 1$ we obtain $X_k$ by including vertices not in $\bigcup_{i<k} X_i$, lying on a short path (of length at most 4) connecting two vertices $x, y \in \bigcup_{i<k} X_i \cup Z$. We repeat this process till it stabilizes and set $X = \bigcup_{k} X_k$. Finally, the subgraph $D_0$ is defined by $D_0 = D[V - (X \cup Y)]$. Observe that for a given vertex $v$ the probability that $d_D(v)$ is at most some absolute constant (independent of $c$) is at most $\text{poly}(c)e^{-c}$. Thus, for a given $v$ the probability of having two vertices of small degree at a constant distance from $v$ is $\text{poly}(c)e^{-2c}$. Hence, we can expect to have eventually $|X| \leq \text{poly}(c)e^{-2c}n$, and this is indeed what we prove. To facilitate the proof, we first get rid of short cycles (of length $O((1/c) \log n)$) in the underlying undirected graph $G$ of $D$—they typically touch very few vertices. Analyzing the filtering process in a large girth graph is easier—for every $v \in X_k$ there should be an evidence for its association with $X_k$ in the form of a tree $T$, rooted at $v$, of prescribed order, depth and with $\ell$ leaves, where $k \leq \ell \leq 2^k$. This is proven in Lemma 2.6. Using this lemma we can bound the size of the set $X_k$ (or rather of a set $X'_k$ closely related to $X_k$ and defined through an analogous filtering process) by the number of labeled rooted trees meeting these requirements. This is done by first truncating unusually deep trees (Lemma 2.7) and then by bounding from above the expected number of trees of bounded depth. The final argument invokes martingales to show the concentration of the corresponding random variable around its mean and to bound its upper tail. All this is done in Theorem 2.2.

The factoring stage takes the induced subgraph $D_0$, the output of the filtering stage, as an input. By Theorem 2.2 we know that with high probability $D_0$ contains all but a suitably small part of the vertices of $D$ of positive degree. Moreover, one can prove (Lemma 2.3) that all vertices in $D_0$ have positive degree, and in addition every two vertices $u, v$ with $d_{D_0}(u), d_{D_0}(v) \leq 2$ are at undirected distance at least 5. We then form an auxiliary bipartite graph $H_0$ with parts $L, R$, corresponding to two copies of the vertices of $D_0$, where a directed edge $(x, y) \in D_0$ becomes an edge $x_L, y_R \in E(H_0)$. It is quite easy to see that the existence of a perfect matching in $H_0$ implies the existence of a spanning subgraph of $D_0$ composed of directed cycles. The probable existence of a perfect matching in $H_0$ is shown in Lemma 2.4 using Hall’s condition and standard density/expansion arguments for random (di)graphs. The next step is to trade the factor of directed cycles in $D_0$ for one nearly spanning cycle using extra random edges, about $O(n/\sqrt{\log n})$ of them—a negligible quantity easily absorbed into the original random digraph. This is done using rather standard random graph arguments and extremal statements guaranteeing the existence of a long cycle in a highly connected digraph (see Lemma 2.5). The factoring stage is treated in Theorem 2.1.

The next section contains the full details of the proof of the main result, and is followed by concluding remarks in Section 3.

2. PROOF OF MAIN RESULT

2.1. Filtering and Factoring

If $D$ is a directed graph we use the notation $d_D(v)$ to denote $\min[d_D^+(v), d_D^-(v)]$, where as usual $d_D^+(v), d_D^-(v)$ stand for the out-degree and the in-degree of a vertex $v \in V(D)$ in

Random Structures and Algorithms DOI 10.1002/tsa
a digraph $D$. Similarly, we let $N_D(v) = N_D^+(v) \cup N_D^-(v)$ and in both cases may omit the subscript $D$ when there is no danger of confusion.

For an undirected graph $G$ and a special subset of its vertices $Z$ we define a filtering process which produces a sequence $\{X_k\}$ of disjoint subsets of the vertices as follows:

$$X_0 = \emptyset,$$

$$X_k = \{ v \notin \bigcup_{j<k} X_j : \exists x, y \in \bigcup_{j<k} X_j \cup Z \text{ such that } x \neq y \text{ and } v \text{ is on a path of length } l \leq 4 \text{ between } x, y, \text{ i.e. } v \in \{ x = u_0, u_1, \ldots, u_l = y \} \text{ with } u_i u_{i+1} \in E(G) \} \text{ for } k \geq 1,$$

$$X = \bigcup_k X_k. \tag{2.1}$$

The first ingredient in the proof is showing that w.h.p., once we filter the set $X$ from the graph along with the vertices with zero in/out degree, the remaining vertices may be factored into large cycles and thereafter combined into a single long cycle while losing only a negligible number of vertices in the process. This is shown in the next theorem whose proof appears in Section 2.2.

**Theorem 2.1.** Let $D \sim D(n, p)$ where $p = \frac{c}{n}$ for $c > 1$ fixed and let $G$ be the undirected underlying graph of $D$. Let $Y = \{ v : d_0(v) = 0 \}$, $Z = \{ v : d_0(v) \leq 3 \}$, and set $X(G, Z)$ as in (2.1). Let $D_0$ be the induced subgraph of $D$ on $V(D) \setminus (X \cup Y)$ and let $D'_0$ be its union with a random digraph $D(|D_0|, (n \sqrt{\log n})^{-1})$. If $|D_0| > n/5$ then w.h.p. $D'_0$ contains a directed cycle on $|D_0| - o(n)$ vertices.

The following theorem, which we prove in Section 2.3, estimates the size of the filtered subset $X$ w.r.t. vertices of low in/out degree in $D$.

**Theorem 2.2.** Let $D \sim D(n, p)$ be a random digraph with edge probability $p = \frac{c}{n}$ for fixed $c > 1$. Let $Z = \{ v : d_0(v) \leq 3 \}$ and define $X = X(G, Z)$ as in (2.1) where $G$ is the undirected underlying graph of $D$. If $c$ is sufficiently large then with high probability $|X| \leq (2c)^{10} e^{-2c} n$.

From the above two theorems we can immediately derive our main result.

**Proof of Theorem 1.** Let $Y = \{ v : d_0(v) = 0 \}$. Note that w.h.p. $|Y| = (2e^{-c} + o(1))n$ since $d^+(v), d^-(v) \sim \text{Bin}(n-1, c/n)$. For a sufficiently large $c$ we obtain from Theorem 2.2 that w.h.p. $|X \cup Y| \leq (2e^{-c} + (2c)^{10} e^{-2c} + o(1))n$. In particular, for large $c$ and $n$ we have that w.h.p. $D_0$, the induced subgraph on $V(D) \setminus (X \cup Y)$, has at least $n/5$ vertices (with room to spare) and we deduce from Theorem 2.1 that w.h.p. $D(n, p')$ has a cycle missing at most $|X \cup Y| + o(n)$ vertices, where $p' = (c/n) + (n \sqrt{\log n})^{-1} = (c + o(1))/n$. This establishes the required result for a choice of, say, $\varepsilon(c) = 2(2c)^{10} e^{-2c}$, which makes up for $|X|$ with an extra factor of 2 that readily absorbs the additive $o(n)$-term in $|X \cup Y|$ as well as the $o(1)$-term in $p'$.

## 2.2. Long Cycles in the Filtered Graph

To prove Theorem 2.1 we first need to establish several properties of the graph $D_0$ stemming from the definition of $X$ and the geometry of the random digraph $D$.

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**Lemma 2.3.** Let $D_0$ be the induced subgraph on $V(D) \setminus (X \cup Y)$ and let $G_0$ be its undirected underlying graph. Then

(i) Every $u \in D_0$ has $d_{D_0}(u) \geq 1$.

(ii) Every $u, v \in D_0$ with $d_{D_0}(u), d_{D_0}(v) \leq 2$ have $\text{dist}_{G_0}(u, v) \geq 5$.

**Proof.** To prove Part (i) assume that some $u \in V(D_0)$ has $d_{D_0}(u) = 0$ and assume without loss of generality that $d_{D_0}(u) = 0$.

First consider the case where $d_{D_0}^+(u) \geq 2$. Observe that in this case there exist distinct $x, y \notin D_0$ such that $(u, x), (u, y) \in E(D)$. Thus $u$ is on a path of length 2 between $x, y \in X \cup Y \subset X \cup Z$, implying that $u \in X$ by definition and contradicting the fact that $u \in V(D_0)$. The case where $d_{D_0}^+(u) = 1$ is treated similarly: Here there is some vertex $v \notin V(D_0)$ such that $(u, v), (v, y) \in E(D)$, $v \in X \cup Y \subset X \cup Z$, and furthermore $u \in Z$ by definition. Hence, $u$ is on a path of length 1 between two distinct vertices $u \neq v$ in $X \cup Z$ and must thus also belong to $X$, in contradiction to the fact that $u \in V(D_0)$.

To prove Part (ii) let $u, v$ be vertices satisfying $d_{D_0}(u) \leq 2$ and $d_{D_0}(v) \leq 2$. If $d_D(u) \geq 4$ then it necessarily lost at least 2 neighboring (in/out) vertices in $X \cup Y$ and hence must also belong to $X$. We thus conclude that $d_D(u) \leq 3$ and similarly that $d_D(v) \leq 3$.

Let $G$ be the underlying undirected graph of $D$. Recalling that $u, v \in Z$ by the definition of $Z$, there cannot be a path of length at most 4 between $u, v$ in $G$, as such a path would imply that $u, v$ must both belong to $X$. In particular, the induced subgraph $G_0 \subset G$ also satisfies $\text{dist}_{G_0}(u, v) \geq 5$, completing the proof.

We now define an auxiliary bipartite graph $H_0$. Its parts $L$ and $R$ are two copies of $V(D_0)$: $L = \{x_L : x \in V(D_0)\}$, $R = \{y_R : y \in V(D_0)\}$, and for each directed edge $(x, y) \in E(D_0)$ we put an undirected edge $x_L y_R$ in $H_0$.

**Lemma 2.4.** Let $H_0$ be as defined above. Then w.h.p. $H_0$ has a perfect matching.

**Proof.** Recall that by Part (i) of Lemma 2.3 there are no isolated vertices in $H_0$ (neither in $L$ nor in $R$). Furthermore, by Part (ii) of that lemma we know that if $x, y \in L$ have degree 1 in $H_0$ then $N(x) \cap N(y) = \emptyset$ (otherwise $D_0$ would have two vertices with out-degree 1 and an undirected distance of at most 2 between them) and similarly for $x, y \in R$ with degree 1 in $H_0$. It follows that the set $M_0$ of edges of $H_0$ touching vertices of degree 1 in $H_0$ forms a matching in $H_0$. Let $H_1$ denote the bipartite graph obtained by deleting the vertices of $M_0$ from $H_0$, i.e. $H_1 = H_0 \setminus V(M_0)$. We now claim that $H_1$ has minimum degree at least 2. To see this, suppose that $d_{H_1}(u) \leq 1$ and argue as follows.

First, we must have $d_{H_0}(u) > 1$ otherwise $u \in V(M_0)$ and hence it does not belong to $H_1$. If $d_{H_0}(u) = 2$ then there must be some $w \in V(M_0)$ such that $uw \in E(H_0)$. In particular, either $w$ has degree 1 in $H_0$ or it is a neighbor of such a vertex, and either way we have that there exists some degree-1 vertex $v \in H_0$ whose distance from $u$ is at most 2. The vertices corresponding to $u$ and $v$ in $D_0$ thus satisfy $d_{D_0}(u) \leq 2$ and $d_{D_0}(v) \leq 1$ while the undirected distance between them is at most 2, contradicting Part (ii) of Lemma 2.3.

It thus remains to treat the case $d_{H_0}(u) \geq 3$. In this case $u$ has two neighbors $w_1, w_2 \in V(M_0)$, giving rise to $v_1, v_2 \in V(M_0)$ whose distance from $u$ is at most 2 and with $d_{H_0}(v_1) = d_{H_0}(v_2) = 1$. These correspond to two vertices $v_1, v_2$ in $D_0$ satisfying $d(v_1), d(v_2) \leq 1$ while the undirected distance between them is at most 4, again contradicting Part (ii) of Lemma 2.3.
We have thus obtained that $H_1$ has minimum degree of 2, and will now derive from this fact the existence of a perfect matching on $H_1$. It suffices to show that w.h.p. every set $S \subset V(H_1) \cap L$ of size at most $n/2$ has $|N(S)| \geq |S|$, as the same conclusion will carry by symmetry to all sets $S \subset V(H_1) \cap R$ of size at most $n/2$, which would in turn imply Hall’s condition for sets $S \subset V(H_1) \cap L$ of size larger than $n/2$.

Let $S$ be a subset of $V(H_1) \cap L$ of size $s \leq n/5$ let $T = N(S)$ in $H_1$ and assume that $T$ has size $t < s$. Identifying these vertices with those of the original digraph $D$ we have that $e(S, T) \geq 2s$ by the definition of $H_1$ and the fact that it has minimum degree 2.

Moreover, observe that every $u \in S$ has at most 2 neighbors in $V(D) \setminus T$. Indeed, since $T$ includes all the neighbors of $S$ corresponding to vertices of $H_1$, any other neighbor $v \in N^+(u) \setminus T$ must belong either to $X \cup Y$ or to the vertices corresponding to $V(M_0)$, and these satisfy:

1. The vertex $u$ cannot have two distinct neighbors in $X \cup Y$ otherwise it would belong to $X$ by definition and hence would be excluded from $D_0$.
2. The vertex $u$ cannot have two distinct neighbors in $V(M_0)$ otherwise there would exist some $x, y$ with $d_{D_0}(x) = d_{D_0}(y) = 1$ and $\text{dist}_{G_0}(x, y) \leq 4$, contradicting Part (ii) of Lemma 2.3.

Combining these arguments we conclude that $|N^+(u) \setminus T| \leq 2$, and note that for a given vertex $u$ and subset $T$ the probability of this event is at most

$$\mathbb{P}(\text{Bin}(n - t, p) \leq 2) \leq 3\left(\frac{n - t}{2}\right)p^2(1 - p)^{n - t - 2} \leq 2c^2e^{-\frac{t}{5}np},$$

where the last inequality used the fact $t < s \leq n/5$ and holds for any sufficiently large $n$ as the $(1 + O(p))$-factor was absorbed into the leading constant. Further note that the event that $|N^+(u) \setminus T| \leq 2$ depends only on the edges from $u$ to $T$ and therefore for distinct vertices these events are independent.

At this point, the following straightforward first moment argument shows that w.h.p. $D$ cannot contain sets $S, T$ of the above sizes where $S$ has at least $2|S|$ edges going to $T$ and every $u \in S$ has at most 2 edges going elsewhere. Indeed, the probability that such sets exist in $D$ for any given such $s, t$ is at most

$$\left(\frac{n}{s}\right)\left(\frac{n}{t}\right)\left(\frac{st}{2s}\right)p^{2s} \left(2c^2e^{-\frac{ts}{5}np}\right)^s \leq \left[\frac{en}{s}\left(\frac{en}{t}\right)^{t/s}\left(\frac{et}{2}\right)^2 \frac{c^2}{n^2} 2c^2e^{-4pn/5}\right]^s \leq ((e^4/2)c^4e^{-4c/5})(t/n)^{1-\frac{t}{s}} \leq ((e^4/2)c^4e^{-4c/5})^{s-1} \frac{1}{n} =: \Delta(s, t),$$

where we used the inequality $\binom{n}{s} \leq (ea/b)^b$ and the fact that $t < s \leq n/5$. For large enough $c$ we have $c^4e^{-4c/5} < 2e^{-5}$ and so

$$\Delta(s, t) < e^{-t}(s-1)/n,$$

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an edge from $S$ follows immediately from the fact that w.h.p. every two sets
respond to a spanning subgraph of $\mathcal{P}$. The edges of the matching provided w.h.p. by Lemma 2.4 corre-
Proof of Theorem 2.1. The digraph $H$ is a random digraph on $n/\Theta(\log n)$ vertices with edge probability
implies that $\rho = (1 - o(1)) n/\log^0.9 n$. We thus

\begin{equation}
\left(\frac{n}{n/5}\right)^2 (1 - \rho)^{n/5} \leq \left[(5e)^{2} e^{-c/5}\right]^{n/5} = o(1),
\end{equation}

where the last inequality holds for a sufficiently large $c$.

We are now in a position to prove Theorem 2.1.

\textbf{Proof of Theorem 2.1}. The edges of the matching provided w.h.p. by Lemma 2.4 corre-
respond to a spanning subgraph of $D_0$ comprised of disjoint directed cycles. Our first step is
to delete from $D_0$ all cycles of length less than $1/2 \log n$. Note that the number of vertices
participating in such cycles in the original digraph $D$ is w.h.p. at most

\[
\sum_{t < \frac{1}{2} \log n} n^t p^t \leq (\log n) \sum_{t \leq \frac{1}{2} \log n} c^t = O(n^{1/2} \log n) = o(n).
\]

The remaining disjoint directed cycles, denoted by $C_1, \ldots, C_m$, thus contain $|D_0| - o(n)$ vertices. Note also that the total number of cycles $m$ satisfies: $m \leq n/(\frac{1}{2} \log n) = O(n/\log n)$.

Let $\mathcal{P} = \{P_i\}_{i=1}^m$ be a maximum collection of vertex disjoint directed paths, each of
length precisely $\lfloor \log^0.9 n \rfloor$, formed from the edges of $\{C_i\}_{i=1}^m$. Since the number of vertices
uncovered by $\mathcal{P}$ in each $C_i$ is at most $\lfloor \log^0.9 n \rfloor$ it follows that $\mathcal{P}$ covers all but at most $m \log^0.9 n = O(n/\log^0.9 n)$ vertices of $D_0$. Furthermore, recalling that $n/5 \leq |D_0| \leq n$, this implies that

\[
(\frac{3}{5} - o(1)) n/\log^0.9 n \leq t \leq n/\log^0.9 n.
\]
need to prove that such a random digraph contains w.h.p. an almost spanning cycle. This is an established fact, and here we derive it from the following lemma of [3], whose short proof is included for completeness.

**Lemma 2.5** [3]. Let \( D = (V, E) \) be a directed graph on \( t \) vertices in which for every ordered pair \( A, B \) of disjoint vertex subsets \( A, B \subset V \) of size \(|A| = |B| = k\) there is an edge from \( A \) to \( B \). Then \( D \) contains a path of length at least \( t - 2k \) and a cycle of length at least \( t - 4k \).

**Proof.** Fix an arbitrary order \( \sigma \) on the vertices of \( D \) and run the DFS (Depth First Search) on \( D \), guided by \( \sigma \). The DFS maintains three sets of vertices: Let \( S \) be the set of vertices which we have completed exploring, \( T \) be the set of unvisited vertices, and \( U = V(G) - (S \cup T) \), where the vertices of \( U \) are kept in a stack (a last in, first out data structure). The DFS starts with \( S = U = \emptyset \) and \( T = V(D) \), and at each stage moves a vertex from \( T \) to \( U \) (an unvisited vertex with an incoming edge from the top of the stack \( U \)) or from \( U \) to \( S \) until eventually all vertices are in \( S \). As such, at some point in the course of the algorithm we must have \(|S| = |T|\); consider that point, and observe crucially that all the vertices in \( U \) form a directed path, and that there are no edges from \( S \) to \( T \). We conclude that \(|S| = |T| \leq k - 1\), and therefore \(|U| \geq t - 2k + 2\), so there is a directed path with \( t - 2k + 1 \) edges in \( D \), as required. To get a directed cycle of the desired length, take a path as above and use a directed edge from its last \( k \) vertices to its first \( k \) vertices to close a cycle. \( \blacksquare \)

In order to apply the above lemma, take \( k = \lfloor n/ \log n \rfloor \) while recalling that \( H \sim \mathcal{D}(t, \rho) \) with \( t \) satisfying (2.2) and \( \rho = (1 + o(1))(\log^{1.1} n)/n \). As \( t \leq n/ \log^{0.9} n \), the probability that \( H \) has two disjoint vertex sets \( A, B \) of cardinality \( k \) each with no edges from \( A \) to \( B \) is at most

\[
\binom{n}{k}(1 - \rho)^{k^2} \leq [(et/k) e^{-n/k}]^k \leq [(e + o(1))(\log n)^{0.1} \cdot e^{-(1-o(1)) \log^{0.1} n}]^k = o(1),
\]

thus w.h.p. \( H \) satisfies the conditions of Lemma 2.5 and in turn it contains w.h.p. a cycle of length at least \( t - 4k = (1 - o(1))t \). As explained above, it follows that w.h.p. the digraph \( D_0 \cup D_1 \) contains a directed cycle covering all but \( O(k \log^{0.9} n) + O(n/ \log^{0.1} n) = o(n) \) vertices, as required. \( \blacksquare \)

### 2.3. Controlling the Effect of the Filtering Process

**Proof of Theorem 2.2.** An important element in the proof would be to analyze the set \( X \) with respect to a subgraph of \( D \sim \mathcal{D}(n, p) \) with a reasonably large undirected girth. To this end we need the following lemma.

**Lemma 2.6.** Let \( G \) be an undirected graph with girth \( g \) and let \( Z \) be a subset of its vertices. Define \( X(G, Z) \) as in (2.1). For every \( 1 \leq k \leq g/8 \) and \( v \in X_k \) there is a tree \( T_v \subset G \) rooted at \( v \) whose leaves are in \( Z \) and interior vertices are in \( \bigcup_{j \leq k} X_j \). Moreover, \( T_v \) has at most \( 5(|T_v \cap Z| - 1) \) vertices, at most \( 4k \) levels (including the root) and its number of leaves \( \ell \) satisfies \( k < \ell \leq 2^k \).

**Proof.** We proceed by induction on \( k \). For the induction base recall that if \( v \in X_1 \) then there are 2 vertices \( x, y \in Z \) such that \( v \) is on a path of length at most 4 between \( x, y \) in \( G \). Treat this path as a tree \( T_v \) rooted at \( v \), and notice that it has 2 leaves, at most 4 levels

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including the root (as dist(v, x), dist(v, y) ≤ 3) and the induced subgraph on it in G is a tree by the girth assumption on G. Furthermore, \( T_v \) has at most 5 ≤ 5(\(|T_v \cap Z| - 1\)) vertices since \(|T_v \cap Z| \geq 2\), thus satisfying the statement of the lemma.

Next, let \( k > 1 \) and let \( v \in X_k \). Let \( x, y \in Z \cup \bigcup_{j<k} X_j \) be the endpoints of a shortest path \( P \) containing \( v \) (by definition (2.1) the path \( P \) has length at most 4). Suppose first that one of these vertices belongs to \( Z \), i.e. without loss of generality \( x \in Z \) whereas \( y \in X_{k-1} \) (otherwise \( v \) would have belonged to some \( X_j \) with \( j < k \)). Define the tree \( T_v \) as a path \( P_\ell \) of length dist\(_G\)(v, y) from the root \( v \) to the sub-tree \( T_v \), provided by the induction, together with another path \( P_y \) of length dist\(_G\)(v, x) from \( v \) to \( x \). On one hand, the paths \( P_v, P_y \) are disjoint by definition, and furthermore, excluding their endpoints, their vertices do not belong to \( Z \cup \bigcup_{j<k} X_j \) by the minimality of \( P \) and in particular do not belong to \( T_v \). On the other hand, if the path \( P_v \) does intersect \( T_y \), which was guaranteed to have at most \( 4(k-1) \) levels by induction, then together with \( P_y \) they complete a cycle of length at most \( 4(k-1) + 4 < 4k \leq g/2 \) in \( G \) contradicting its girth assumption. We conclude that \( T_v \) is indeed a tree, with at most \( 4(k-1) + 4 = 4k \) levels including the root. Finally, \(|T_v \cap Z| = |T_v \cap Z| + 1\), hence the induction hypothesis and the fact that \( T_v \) adds at most 4 vertices to \( T_v \) together imply that

\[
|T_v| \leq |T_v| + 4 \leq 5(|T_v \cap Z| - 1) + 4 < 5(|T_v \cap Z| - 1).
\]

It remains to treat the case where \( x \in X_j \) for some \( j < k \) while \( y \in X_{k-1} \). As before, if \( T_v \cap T_y \neq \emptyset \) then together with the path \( P \) we obtain a cycle of length at most \( 8(k-1) + 4 < 8k \leq g \) in \( G \), contradicting the girth assumption. Otherwise, \(|T_v \cap Z| = |T_v \cap Z| + |T_y \cap Z|\) and so our hypothesis on \( T_x, T_y \) gives that

\[
|T_v| \leq |T_x| + |T_y| + 3 \leq 5(|T_x \cap Z| - 1) + 5(|T_y \cap Z| - 1) + 3 < 5(|T_v \cap Z| - 1).
\]

Noting that the number of leaves \( \ell(T_x) \) was either \( \ell(T_v) + 1 \) or \( \ell(T_x) + \ell(T_y) \) immediately implies that \( k + 1 \leq \ell(T_x) \leq 2^k \) and completes the proof of the lemma.

Let \( G \) denote the undirected underlying graph of \( D \sim \mathcal{D}(n, p) \), and define \( C \subset V(G) \) to be comprised of all vertices that belong to cycles of length at most

\[
R = (20/c) \log n
\]

in \( G \). Since each edge appears in \( G \) with probability at most \( 2p \) independently of other edges, the expected number of cycles of length \( r \) in \( G \) is at most \( n' (2p)^r / r \) and thus

\[
\mathbb{E}|C| \leq \sum_{r < R} (2c)^r \leq \frac{(2c)^R}{c-1} < n^{1/5},
\]

where we used the fact that \( c / \log(2c) > 100 \) for sufficiently large \( c \). In particular, \( |C| < n^{1/4} \) w.h.p.

Define \( Z' = C \cup Z \) and let \( D' \) be the graph obtained by deleting all inner edges between vertices of \( C \) (i.e. all edges of the induced subgraph on \( C \)). Let \( G' \) denote the undirected underlying graph of \( D' \) and for all \( k \) let \( X'_k \) denote the set \( X_k(G', Z') \) defined via (2.1). A key observation is that

\[
X \subset X' \cup C. \tag{2.3}
\]
To see this, recall that \( X_0 = \emptyset \) and assume by induction that
\[
\bigcup_{j < k} X_j \subset \left( \bigcup_{j < k} X_j \right) \cup C \] for some \( k \geq 1 \).

Let \( v \in X_k \setminus C \). Let \( P \) be a shortest path containing \( v \) in \( G \) with endpoints \( x, y \in (\bigcup_{j < k} X_j) \cup Z \). By the definition of \( X_j \) and the minimality of \( P \) we know that \( P \) has \( 1 \leq l \leq 4 \) edges and none of its interior vertices belongs to \( \bigcup_{j < k} X_j \cup Z \). Consider the two sub-paths from \( v \) to \( x \), \( y \) (one of which is possibly empty) and let \( x', y' \) be the first vertices on these respective paths that belong to \( \bigcup_{j < k} X_j \cup C \cup Z \). Since \( x, y \) clearly belong to this set, this defines a sub-path \( P' \) of length \( 1 \leq l' \leq l \leq 4 \) that contains \( v \) in \( G \). Crucially, since \( v \notin C \) the path \( P' \) has no interior vertices in \( C \) and therefore all of its edges belong to \( G' \). Finally, the induction hypothesis ensures that \( x', y' \in (\bigcup_{j < k} X_j') \cup Z' \) and we conclude that \( v \in (\bigcup_{j < k} X_j') \cup Z' \), completing the induction.

Our next goal is to provide an upper bound on \( X' \) which is linear in \( n \) (with a suitably small coefficient), absorbing the negligible contribution to it from vertices in \( C \). Observe that by definition the girth of \( G' \) is larger than \( R = (20/c) \log n \) and set
\[
K = \lfloor (2/c) \log n \rfloor . \tag{2.4}
\]

Invoking Lemma 2.6 w.r.t. \( G' \), for each \( k \leq K \) we can bound \( |X'_k| \) from above by \( |T'_k| \) where
\[
T'_k = \left\{ T \subset G' : \begin{array}{l}
\text{labeled rooted tree with } \ell \text{ leaves, } k < \ell \leq 2^k, \text{ all belonging to } Z', \\
\text{a total of } t \text{ vertices for } t \leq 5(\ell - 1) \text{ and at most } 4k \text{ levels.}
\end{array} \right\} . \tag{2.5}
\]

In particular, we will be able to assert that \( X'_k = \emptyset \) by showing that \( T'_k \) is empty, as the next lemma establishes.

**Lemma 2.7.** Set \( K \) as in (2.4). With high probability \( X'_k = \emptyset \).

**Proof.** In what follows let \( L(T) \) denote the set of leaves of a tree \( T \) and recall that if \( T \in T'_k \) then \( L(T) \subset Z' = C \cup Z \) by definition.

Let \( T'_k \) be the set of all trees in \( T'_k \) where at least \( \ell - 1 \) of the leaves belong to \( Z \). We have \( \binom{\ell}{t} \) choices for the vertices of \( T \in T'_k \) on \( t \) vertices, and the well-known Cayley formula asserts that the number of labeled rooted trees on \( t \) vertices is \( t^{t-1} \). The probability that a given labeled tree on \( t \) vertices is in \( G \) (an upper bound on the probability that it belongs to \( G' \subset G \)) is exactly \((2p)^{t-1}\). Finally, if \( u \in L(T) \cap Z \) then by definition \( d_G(u) \leq 3 \) and in particular \( d_{G,T}(u) \leq 3 \). Crucially, the events \( \{d_{G,T}(u) \leq 3\} \) for \( u \in L(T) \) are mutually independent as well as independent of all the interior edges of \( T \) (accounted for in the probability that \( T \subset G \)). Altogether, for all \( k \leq K \),
\[
\mathbb{E}[T'_k] \leq \sum_{\ell=k+1}^{2k} \sum_{t \leq 5(\ell - 1)} \binom{n}{t} t^{t-1} (2p)^{t-1} \ell (2\mathbb{P}(\text{Bin}(n-t,p) \leq 3))^{\ell-1} .
\]
\[
\leq \sum_{\ell=k+1}^{2k} \sum_{t \leq 5(\ell - 1)} e(2ec)^{t-1} (2\mathbb{P}(\text{Bin}(n-t,p) \leq 3))^{\ell-1} n . \tag{2.6}
\]
where in the last inequality we used the facts that \( \binom{n}{2} \leq (en/t) \) and \( t > \ell \). If \( c \) is sufficiently large then \( n - t = (1 - o(1))n \) as \( t < 5 \cdot 2^k = o(n) \) and in particular \( P(Bin(n - t, p) \leq 3) \leq \frac{1}{2}e^{-c}. \) Plugging this in (2.6) gives that for sufficiently large \( n \),

\[
\mathbb{E}|T_k^*| \leq \sum_{\ell=0}^{\log n} \sum_{t \leq 5(\ell - 1)} (2e)^{\ell} (e^{-c})^{\ell - 1} n
\]

\[
\leq \sum_{\ell=0}^{\log n} ((2e)^{\ell} e^{-c})^{\ell - 1} n \leq ne^{-\frac{1}{2}c^\ell},
\]

(2.7)

where the last inequality holds for large enough \( c \) and \( n \). Substituting \( k = K = [2/c \log n] \) now gives

\[
\mathbb{E}|T_k^*| \leq ne^{-\frac{1}{2}c^k} = O(n^{-1/2}) = o(1),
\]

hence w.h.p. \( T_k^* = \emptyset. \)

Now consider \( T \in T_k \setminus T_k^* \). Here there exist distinct \( u_i, u_j \in L(T) \cap C \). As \( T \) has at most 4\( K \) levels and connects \( u_i, u_j \) in \( G' \) (where the inner edges between the vertices of \( C \) are absent) this implies the existence of a subgraph \( F \subset G \) with \( m \) vertices and at least \( m + 1 \) edges such that

\[
m \leq 2R + 8K + 2 \leq 160/c \log n
\]

(accounting for \( u_i \) and \( u_j \), a path of length at most 8\( K \) between them and up to 2 cycles in \( C \), with the last inequality holding for large enough \( c \)). When \( c \) is sufficiently large, the probability that such a graph \( F \) belongs to \( G \) is at most

\[
\binom{n}{m} \binom{m}{m+1} (2p)^{m+1} \leq \binom{en}{m} \left(\frac{em}{2}\right)^m \frac{m+1}{2} (2e/n)^{m+1} \leq \frac{m}{n} (e^2c)^{m+1} = O(1/\sqrt{n}) = o(1)
\]

(2.8)

implying that \( T_k \setminus T_k^* \), and hence also \( T_k^* \), is w.h.p. empty. By (2.5) and the remark following that definition it now follows that \( X_k = \emptyset \) w.h.p., as required.

It remains to estimate \( |\bigcup_{k<K} X_k| \). To this end, let \( B(C, R/2) \) be the set of all vertices whose undirected distance from \( C \) in \( G \) is less than \( R/2 = (10/c) \log n \). Consider some \( v \in X_k \) for some \( k \leq K \), let \( T_v \) be the corresponding tree provided by Lemma 2.6 and suppose first that some leaf \( u \) in \( T_v \) belongs to \( C \) (recall that every leaf of \( T_v \) is in \( C \cup Z \) by (2.5)). Since by definition \( T_v \) has at most 4\( k \leq 4K \leq (8/c) \log n \) levels it follows that \( T_v \subset B(C, R/2) \) and in particular \( v \in B(C, R/2) \). Due to this argument, if we let \( T_v \) denote the set of rooted trees in \( T_k^* \) where all leaves belong to \( Z \) and let \( Y_k \) denote the number of vertices serving as roots of such trees, i.e.

\[
T_k = \left\{ T \subset G' : \text{labeled rooted tree with } \ell \text{ leaves, } k < \ell \leq 2^k, \text{ all belonging to } Z, \text{ a total of } t \text{ vertices for } t \leq 5(\ell - 1) \text{ and at most } 4k \text{ levels.} \right\}
\]

\[
Y_k = \#\{v \in V(G) : v \text{ is the root of } T \text{ for some } T \in T_k \}
\]

(notice that clearly \( Y_k \leq |T_k| \) for any \( k \)), then

\[
|\bigcup_{k<K} X_k| \leq |B(C, R/2)| + \sum_{k<K} Y_k.
\]

Random Structures and Algorithms DOI 10.1002/tsa
To estimate the size of $B(C, R/2)$ observe that each vertex $v$ in this set corresponds to a graph on $m < 3R/2$ vertices and at least $m$ edges. We can therefore repeat the calculation in (2.8) to get that

$$
\mathbb{E}|B(C, R/2)| \leq \sum_{m < 3R/2} \binom{n}{m} \binom{m}{2} (2p)^m \leq \sum_{m < 3R/2} (e^c m)^m < \sqrt{n},
$$

where the last inequality is valid for large $c$. In particular, $|B(C, R/2)| < n^{3/4}$ w.h.p. and it remains to estimate $\sum_{k < K} Y_k$.

Consider $|T_i|$, counting rooted labeled trees in $G$ with $2$ leaves (i.e. paths with a distinguished vertex) and at most $5$ vertices and where both leaves are in $Z$. Conditioned on the existence of a given labeled path $P$ in $G$, the probability that its endpoints are in $Z$ is less than the probability that each endpoint has an at most $3$ in-neighbors or at most $3$ out-neighbors in $D \setminus P$. Altogether,

$$
\mathbb{E}Y_1 \leq \mathbb{E}|T_i| \leq \sum_{2 \leq t \leq 5} m^t (2p)^{t-1}(2\mathbb{P}(\text{Bin}(n-t, p) \leq 3))^2
\leq 4 \cdot 5(2c)^4(2\mathbb{P}(\text{Bin}(n-5, p) \leq 3))^2n \leq 20(2c)^4(c^3 e^{-c})^2n < 600 c^{10}e^{-2c}n,
$$

where the last inequality holds for sufficiently large $n$.

Next examine $|T_k|$ for $2 \leq k < K$, which counts trees with at most $4k$ levels, $\ell \in \{k+1, \ldots, 2^k\}$ leaves and a total of $t \leq 5(\ell - 1)$ vertices, where all leaves are in $Z$. The calculation in (2.6), (2.7), with the single change that now all leaves (rather than $\ell - 1$) belong to $Z$, yields

$$
\mathbb{E}Y_k \leq \mathbb{E}|T_k| \leq \sum_{\ell = k+1}^{2^k} (2e^5 c^8 e^{-c})^\ell n \leq e^{-\frac{3}{2}c(k+1)}n,
$$

and combining the above inequalities we deduce that for large enough $c$ and $n$ we have

$$
\sum_{k < K} \mathbb{E}Y_k \leq 1000 c^{10}e^{-2c}n.
$$

(2.10)

To assess the deviation of the $Y_k$’s from their mean, set $K_0 = \lfloor \log \log n \rfloor$ and observe that

$$
\sum_{K_0 \leq k < K} \mathbb{E}Y_k \leq 2e^{-\frac{3}{2}cK_0}n < n/ \log^2 n,
$$

with the last inequality easily holding for $c$ large. Applying Markov’s inequality we deduce that

$$
\mathbb{P}\left(\sum_{K_0 \leq k < K} Y_k \geq n/ \log n\right) \leq 1/ \log n = o(1).
$$

(2.11)

It remains to estimate the $Y_k$’s for $k < K_0$. To this end, define $Y'_k$ to be the number of roots of trees $T \in \mathcal{T}_k$ such that every vertex in $T$ has degree less than $\log^2 n$ in $G$:

$$
Y'_k = \#\{v \in V(G) : v \text{ is the root of } T \text{ for some } T \in \mathcal{T}_k \text{ and } d(u) < \log^2 n \text{ for all } u \in T\}.
$$
Recall that the underlying graph $G$ is obtained from $D$ by erasing its edge directions. Therefore, $G$ itself is a random undirected graph $\mathcal{G}(n, p')$ with edge probability $p' = 1 - (1 - p)^2 = (1 + o(1))2p$. As we will formally state later, $G \sim \mathcal{G}(n, p')$ has maximum degree less than $\log^2 n$ except with extremely low (super-polynomial) probability, and so $Y_k = Y_k^1$ w.h.p. We will show that $Y_k^1$ is concentrated about its mean and then use it to derive concentration for $Y_k$.

Let $(M_i)$ be the edge-exposure Doob’s martingale for $D$; that is, let $e_1, \ldots, e_{|E|}$ be an arbitrary ordering of the edges of the complete graph on $n$ vertices and set $M_t = \mathbb{E}[Y_k^1 | \mathcal{F}_t]$ where $\mathcal{F}_t$ is the $\sigma$-algebra corresponding to revealing the indicators $\{1_{\{e \in E(D)\}} : i \leq t\}$. We are interested in bounds on the increments of the martingale $(M_i)$ in $L^\infty$ and $L^2$.

Consider the effect of modifying one of the indicators $1_{\{e \in E\}}$; clearly this can create or destroy a tree $T \in \mathcal{T}_k$ only if that tree includes an endpoint of $e$ as one of its vertices. Since $Y_k^1$ counts roots of such trees where every vertex has degree less than $\log 2 n$ and by the definition of $\mathcal{T}_k$ each such tree has at most $4k$ levels (including the root), it follows that modifying $e$ can alter $Y_k^1$ by at most $(\log^2 n)^k$. In other words, $Y_k^1$ is $B$-Lipschitz as a function of the edges of $D$, where

$$B = (\log^2 n)^{4k} \leq (\log^2 n)^{4K_0} \leq \exp(8(\log \log n)^2) = n^{o(1)}.$$  

It is a well-known (and easy to show) corollary that in this case $|M_{t+1} - M_i| \leq B$ for all $t$ (see, e.g. [2] for the standard coupling argument deriving this for Doob’s martingale of Lipschitz functions).

Now assume that we have exposed $\mathbb{1}_{\{e \in E\}}, \ldots, \mathbb{1}_{\{|E|\}}$ and are about to reveal whether or not $e_{t+1} \in E$. We wish to bound $\operatorname{Var}(M_{t+1} | \mathcal{F}_t)$. If we let $\theta = \mathbb{E}[Y_k^1 | \mathcal{F}_t, e_{t+1} \notin E]$ then the shifted variable $Q = M_{t+1} - \theta$ satisfies $\mathbb{P}(Q \neq 0 | \mathcal{F}_t) = \mathbb{P}(e_{t+1} \in E | \mathcal{F}_t) = p' \leq 2p$ whereas $|Q| \leq B$ by the assumption that $\mathbb{P}(|M_{t+1} - M_i| \leq B) = 1$ (in fact, even more precisely, one has $|Q| \leq B$ due to the $B$-Lipschitz property of $Y_k^1$). Thus,

$$\operatorname{Var}(M_{t+1} | \mathcal{F}_t) = \operatorname{Var}(Q | \mathcal{F}_t) \leq 2p(B)^2 = n^{-1+o(1)} \,,$$

and we conclude that for some $L = n^{1+o(1)}$ we have $\sum_i \operatorname{Var}(M_i | \mathcal{F}_{i-1}) \leq L$ with probability 1.

We are now in a position to apply the following large-deviation inequality which is a special case of a result of Freedman [8, Theorem 1.6] (see also [12, Theorem 3.15]):

**Theorem 2.8.** Let $(S_0, S_1, \ldots, S_N)$ be a martingale with respect to the filter $(\mathcal{F}_i)$. Assume that $S_{i+1} - S_i \leq B$ for all $i$ and that $\sum_{i=1}^N \operatorname{Var}(S_i | \mathcal{F}_{i-1}) \leq L$ with probability 1 for some $L > 0$. Then for any $s > 0$ we have $\mathbb{P}(\bigcup_{i=1}^N \{S_i \geq S_0 + s\}) \leq \exp[-\frac{1}{2}s^2/(L + Bs)]$.

Plugging in our estimate for $|M_{t+1} - M_i|$ and $\sum_i \operatorname{Var}(M_i | \mathcal{F}_{i-1})$ while recalling that by the definition of the Doob martingale $M_0 = \mathbb{E}[Y_k^1]$ whereas $M(\mathcal{F}_i) = Y_k^i$ it now follows that

$$\mathbb{P}\left(|Y_k^i - \mathbb{E}Y_k^i| > s\right) \leq 2 \exp\left[-\frac{1}{2}s^2/(n^{1+o(1)} + n^{o(1)}s)\right],$$

and in particular

$$\mathbb{P}\left(|Y_k^i - \mathbb{E}Y_k^i| > n^{3/4}\right) \leq \exp(n^{-1/2+o(1)}).$$  \hspace{1cm} (2.12)

To complete the proof, recall that the probability that any vertex in $G \sim \mathcal{G}(n, p')$ would have degree at least $\log^2 n$ is at most

$$n \mathbb{P}(\operatorname{Bin}(n, 2c/n) \geq \log^2 n) \leq n \exp(-c' \log^2 n) < n^{-10},$$

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where the last inequality holds for large enough \( n \). In particular, \( Y'_k = Y_k \) except with probability \( \frac{1}{10} + o(1) \). Combining these inequalities with (2.12) now gives

\[
P(\left| Y_k - E[Y_k] \right| > 2n^{3/4}) \leq 2n^{-10},
\]

where the extra factors of 2 absorbed the \( O(n^{-9}) \) and \( \exp(n^{-1/2+o(1)}) \) error terms. In particular, taking a union bound over the \( K_0 \leq \log \log n \) values of \( k \) we deduce that w.h.p.

\[
\sum_{k < K_0} Y_k - \sum_{k < K_0} E[Y_k] \leq 2n^{3/4} \log \log n.
\]

Finally, combining this inequality with (2.10) and (2.11) we conclude that w.h.p.

\[
\sum_{k < K} Y_k \leq 1000c^{10}e^{-2c}n + 2n^{3/4} \log \log n + n/\log n = (1000 + o(1))c^{10}e^{-2c}n,
\]

where the last inequality is valid for any sufficiently large \( n \), as required.

3. CONCLUDING REMARKS

We have proved that a random directed graph \( D(n, c/n) \) contains with high probability a directed cycle including all but at most \( (2+\varepsilon)e^{-\varepsilon}n \) vertices, where \( \varepsilon = \varepsilon(c) \to 0 \) as \( c \to \infty \). In fact, our proof shows that the relative error term \( \varepsilon(c) \) is exponentially small in \( c \), namely \( \varepsilon(c) \leq \text{poly}(c)e^{-c} \). The main term in the result is asymptotically optimal as such a random digraph typically contains \( (2e^{-\varepsilon} - o(1))n \) vertices with zero in-degree or out-degree.

It would be very interesting to derive accurate estimates for the length of a longest cycle in \( D(n, c/n) \) for small(er) values of the constant \( c \), starting perhaps as low as the threshold for the appearance of a linear length cycle in such a random digraph. See the related work [10] where Łuczak studied the length of the longest cycle in the undirected random graph near its critical window, showing lower and upper bounds that are tight up to a factor of \( 1 + \log(3/2) \approx 1.41 \).

Compared to the situation in undirected graphs, the toolkit available for the case of directed graphs is rather poor at present, thus making the progress in a variety of questions about directed random and pseudo-random graphs much harder to achieve. In particular, the absence of any form of a direct analogue of the famed Pósa’s rotation-extension technique, widely applied for undirected graphs, is felt throughout. It would be very useful to derive some directed version of it.

In general, the field of random and pseudo-random directed graphs is largely an uncharted territory, compared to the situation for the undirected case. Although this is certainly partly due to its relative difficulty, we believe enough knowledge and technology have been accumulated now to start exploring it in a systematic way. One recent such result is the paper [4].

*Random Structures and Algorithms* DOI 10.1002/rsa
where global resilience type results with respect to long cycles have been derived for sparse random and pseudo-random directed graphs. It would be interesting to explore further resilience type questions in directed graphs.

ACKNOWLEDGMENTS

A major part of this work was carried out when the first and the third authors were visiting Microsoft Research at Redmond, WA. They would like to thank the Theory Group at Microsoft Research for hospitality and for creating a stimulating research environment.

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