APÉRY LIMITS AND MAHLER MEASURES

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To the blessed memory of Asmus L. Schmidt

Abstract. It is the first paper which relates Apéry limits to Mahler measures.

1. Introduction

There seems to be no paper yet in which the two notions in the title, Apéry limits and Mahler measures, interplay. Though we marry them here somewhat artificially, there is a story behind the marriage.

Main theorem. Consider the second order polynomial recursion

\[
32(n + 1)(2n + 1)(2n + 3)(4n + 3)(4n + 5)(8n + 3)^2(8n + 5)^2(8n + 7)^2(8n + 9)^2 \\
\times (654311424n^8 - 763363328n^7 + 336592896n^6 - 62390272n^5 + 1949696n^4 \\
+ 706560n^3 - 43520n^2 - 2880n + 225)r_{n+1} \\
- 3(2n + 1)(101168862029250822140n^{20} + 6913205571998806179840n^{19} \\
+ 20264522984105850175488n^{18} + 32694809236419354034176n^{17} \\
+ 30311863475685170348032n^{16} + 1394098194461731823616n^{15} \\
- 511604730729009774592n^{14} - 397361701783984930816n^{13} \\
- 1526200168532215332864n^{12} + 19473726075079110144n^{11} \\
+ 26987025941865168896n^{10} + 37629485093249613824n^9 \\
- 17598923437928087552n^8 - 5481696915139592192n^7 + 199010711963172864n^6 \\
+ 237194722753118208n^5 + 18366301200549888n^4 - 2095780639795200n^3 \\
- 198344332843200n^2 + 7359342480000n + 776998726875)r_n \\
- 1536n^3(6n - 1)(6n + 1)(8n + 1)^2(8n - 1)^2(8n - 3)^2(8n - 5)^2 \\
\times (654311424n^8 + 4471128064n^7 + 13313769472n^6 + 22567976960n^5 + 23822974976n^4 \\
+ 16040183808n^3 + 6728855040n^2 + 1608382656n + 167760801)r_{n-1} = 0
\]

and its two solutions \(\{q_n\}_{n=0}^\infty, \{p_n\}_{n=0}^\infty\) defined through the (rational!) initial values

\[
q_0 = p_0 = 1, \quad q_1 = \frac{1289}{160}, \quad p_1 = \frac{136185509}{15876000}.
\]

Then

\[
\lim_{n \to \infty} \frac{p_n}{q_n} = -\frac{1}{16\sqrt{2}} \int_0^1 \int_0^1 \log |(x^4 + 1)y^2 - 2(x^4 - 4x^2 + 1)y + x^4 + 1| \frac{dx}{x} \frac{dy}{y}.
\]
To clarify the matters, limits of quotients of solutions to linear recurrence equations, like the one featured in the left-hand side of (1), are usually called Apéry limits [11,6] (see also Section 3 below). The double integral on the right-hand side in (1) represents (up to a multiple) the (logarithmic) Mahler measure [2,5]

\[ m(P(x, y)) = \frac{1}{(2\pi i)^2} \iiint_{|x|=|y|=1} \log |P(x, y)| \frac{dx}{x} \frac{dy}{y} \]

of a particular polynomial \( P(x, y) \in \mathbb{Z}[x, y] \).

2. Manifestation

In our proof of the main theorem we simply show that the two sides in (1) represent the same quantity. This number is somewhat exceptional (as it is one of a few that can be realised simultaneously as the Apéry limit of a second order recursion and as a Mahler measure), so it serves as the main hero of the story.

3. The left-hand side

During this author’s visit in the University of Copenhagen in 2004, A. Schmidt suggested to look for second order polynomial recursions giving reasonable approximations to the following Dirichlet \( L \)-series at the point \( s = 2 \):

\[ L(\chi_{-8}, s) = \sum_{n=1}^{\infty} \frac{\chi_{-8}(n)}{n^s}, \]

where the (odd) quadratic character \( \chi_{-8} = (\frac{-8}{\cdot}) \) modulo 8 is determined by the data \( \chi_{-8}(1) = \chi_{-8}(3) = 1, \chi_{-8}(5) = \chi_{-8}(7) = -1 \). The value \( L(\chi_{-8}, 2) \) appears quite natural in a problem of approximations of complex numbers by elements from \( \mathbb{Z}[\sqrt{-2}] \) (see [10, 14]; the \( L \)-value is not featured there but its appearance is clarified in [8, Theorem 3.1]). In the corresponding problems of approximations of real numbers by rationals and of complex numbers by elements from \( \mathbb{Z}[\sqrt{-1}] \), the (‘ergodic’) analogues of \( L(\chi_{-8}, 2) \) are \( \pi^2/12 \) and \( G = L(\chi_{-4}, 2) \) (Catalan’s constant).\(^2\) The two latter constants admit second order Apéry-like recursions; there were several constructions at that time known, which appeared in print later [16,19]. But \( L(\chi_{-8}, 2) \) looked quite challenging, and it is surprising to see that it did not show up as such an Apéry limit since then.

The Apéry-limit realisation of \( L(\chi_{-8}, 2) \) comes from mixing the hypergeometric constructions [17,19]. Executing Zeilberger’s creative telescoping on the series

\[ r_n = \sum_{\nu=1}^{\infty} \frac{d}{dt} \left( \frac{(-1)^{n+1}(2t + n - \frac{1}{2}) \prod_{j=1}^{n}(t - j)^2(t + n + j - \frac{1}{2})^2}{2^9 \prod_{l=0}^{n}(t + l + \frac{1}{8})(t + l - \frac{1}{8})(t + l - \frac{3}{8})(t + l - \frac{5}{8})} \right) \bigg|_{t=0} \]

we find out that \( r_n = q_n L(\chi_{-8}, 2) - p_n \), together with the coefficient sequences \( q_n \) and \( p_n \), satisfy the the linear homogeneous difference equation displayed in the main theorem. The characteristic polynomial of the equation is

\[ \lambda^2 - 270\lambda - 27 = (\lambda - (135 + 78\sqrt{3}))(\lambda - (135 - 78\sqrt{3})), \]

\(^1\)The notion is ‘canonical’ for the second order recurrence equations (with coefficients from \( \mathbb{Q}[n] \)), because there are two linear independent rational-valued solutions in this case and the quotient is well defined up to a linear-fractional transformation. Here we deal with such second order recursions.

\(^2\)Schmidt famously developed generalisations of the theory of continued fractions to the complex numbers [12,13] and gave numerous explicit results for diophantine approximations in imaginary quadratic fields.
and a standard analysis leads to the asymptotics
\[
\lim_{n \to \infty} q_n^{1/n} = \lim_{n \to \infty} p_n^{1/n} = 78\sqrt{3} + 135, \quad \lim_{n \to \infty} |r_n|^{1/n} = 78\sqrt{3} - 135 = 0.09996299\ldots.
\] (2)

In particular, this demonstrates that \(p_n/q_n\to L(\chi_{-8}, 2)\) as \(n\to\infty\) and decodes the left-hand side of (1).

In general, given (rational) \(0 < \alpha, \beta, \gamma < 1\) with \(\alpha \neq \beta\) and \(\alpha + \beta \neq \gamma\), the choice
\[
R_n(t) = (-1)^{n+1}(2t + n - \gamma) \prod_{j=1}^{n}(t - j)^2(t + n + j - \gamma)^2 / \prod_{l=0}^{n}(t + l - \alpha)(t + l - \beta)(t + l + \alpha - \gamma)(t + l + \beta - \gamma)
\]
and launch of Zeilberger’s creative telescoping result in a second order recursion (whose coefficients are polynomials of degree 23 in \(n\)) for the quantities
\[
r_n = \sum_{\nu=1}^{\infty} \frac{dR_n(t)}{dt} \bigg|_{t=\nu} = q_n C(\alpha, \beta, \gamma) - p_n
\]
approximating the Apéry limit
\[
C(\alpha, \beta, \gamma) = \psi_1(1 - \alpha) - \psi_1(1 - \beta) + \psi_1(1 + \alpha - \gamma) - \psi_1(1 + \beta - \gamma),
\]
where
\[
\psi_1(x) = \frac{d^2}{dx^2} \log \Gamma(x) = \sum_{n=0}^{\infty} \frac{1}{(n + x)^2}
\]
is the so-called trigamma function. Notice that the characteristic polynomial of the recursion and asymptotics (2) remain independent of the data \(\alpha, \beta, \gamma\). Since
\[
\psi_1(x) + \psi_1(1 - x) = \frac{\pi^2}{\sin^2 \pi x},
\]
the quantity
\[
C(\alpha, \beta, 1) = \pi^2 \left( \frac{1}{\sin^2 \pi \alpha} - \frac{1}{\sin^2 \pi \beta} \right)
\]
is an algebraic multiple of \(\pi^2\); here are examples of such Apéry limits when \(\gamma = 1\):
\[
\frac{1}{4} C\left(\frac{1}{8}, \frac{3}{8}, 1\right) = \pi^2 \sqrt{2}, \quad \frac{1}{8} C\left(\frac{1}{12}, \frac{5}{12}, 1\right) = \pi^2 \sqrt{3},
\]
\[
\frac{5}{4} C\left(\frac{1}{5}, \frac{2}{5}, 1\right) = \pi^2 \sqrt{5}, \quad \frac{1}{4} C\left(\frac{1}{10}, \frac{3}{10}, 1\right) = \pi^2 \sqrt{5},
\]
\[
\frac{1}{8} C\left(\frac{1}{9}, \frac{4}{9}, 1\right) = \pi^2 \cos \frac{\pi}{9}, \quad \frac{1}{8} C\left(\frac{1}{9}, \frac{2}{9}, 1\right) = \pi^2 \cos \frac{2\pi}{9}, \quad \frac{1}{8} C\left(\frac{2}{9}, \frac{4}{9}, 1\right) = \pi^2 \cos \frac{4\pi}{9},
\]
\[
\frac{1}{4} C\left(\frac{1}{15}, \frac{4}{15}, 1\right) = \pi^2 \sqrt{3} \sqrt{5} + 2 \sqrt{5}, \quad \frac{1}{4} C\left(\frac{2}{15}, \frac{7}{15}, 1\right) = \pi^2 \sqrt{3} \sqrt{5} - 2 \sqrt{5},
\]
\[
\frac{1}{12} C\left(\frac{1}{10}, \frac{2}{5}, 1\right) = \pi^2 \left( \frac{1}{3} + \frac{1}{\sqrt{5}} \right), \quad \frac{1}{8} C\left(\frac{1}{10}, \frac{1}{5}, 1\right) = \pi^2 \left( \frac{1}{2} + \frac{1}{\sqrt{5}} \right),
\]
\[
\frac{1}{12} C\left(\frac{3}{10}, \frac{1}{5}, 1\right) = \pi^2 \left( \frac{1}{3} - \frac{1}{\sqrt{5}} \right), \quad \frac{1}{8} C\left(\frac{3}{10}, \frac{2}{5}, 1\right) = \pi^2 \left( \frac{1}{2} - \frac{1}{\sqrt{5}} \right).
\]
When \(\gamma \neq 1\), we find only two instances that can be identified with Dirichlet \(L\)-values (for odd characters):
\[
\frac{1}{160} C\left(\frac{3}{12}, \frac{1}{12}, \frac{1}{2}\right) - \frac{1}{160} = L(\chi_{-4}, 2), \quad \frac{1}{64} C\left(\frac{3}{8}, \frac{1}{8}, \frac{1}{2}\right) - \frac{1}{64} = L(\chi_{-8}, 2).
\]
Numerous realisations of Catalan’s constant \( L(\chi_{-4}, 2) \) as the Apéry limits of second order recursions are already known \([16, 17]\), while the coverage of \( L(\chi_{-8}, 2) \) explicitly given in the main theorem is new. In a certain sense, this explains the uniqueness of the latter quantity.

### 4. The right-hand side

The Mahler measure evaluation

\[
m((x^4 + 1)y^2 - 2(x^4 - 4x^2 + 1)y + x^4 + 1) = L'(\chi_{-8}, -1)
\]

for the right-hand side in \([1]\) is borrowed from the paper by G. Ray \([11, \text{Proposition 2}]\). The connection with \( L(\chi_{-8}, 2) \) comes from the general formula

\[
L(\chi_{-N}, 2) = \frac{4\pi}{N\sqrt{N}} L'(\chi_{-N}, -1),
\]

which is a consequence of the functional equation for \( L(\chi_{-N}, s) \). Ray’s formula can be replaced with the evaluations

\[
m((x + 1)^4 y - (x - 1)^2(x^2 + 1)) = L'(\chi_{-8}, -1),
\]

\[
m((x + 1)^2(x^2 + x + 1)y - (x^2 + 1)^2) = \frac{2}{3} L'(\chi_{-8}, -1),
\]

\[
m((x + 1)^{12} y - (x - 1)^8(x^4 - x^3 + x^2 - x + 1)) = \frac{16}{5} L'(\chi_{-8}, -1)
\]
given by D. Boyd and F. Rodriguez-Villegas in \([3]\), or

\[
m((x^2 + 1)y + (x + 1)^2(x^2 - x + 1)) = \frac{1}{3} L'(\chi_{-8}, -1)
\]
of F. Brunault \([4]\). An interest in casting the \( L \)-values \( L'(\chi_{-N}, -1) \) corresponding to odd Dirichlet characters \( \chi_{-N} \) as two-variable Mahler measures has its roots in the first such example

\[
m(x + y + 1) = L'(\chi_{-3}, -1)
\]
established by C. Smyth \([15]\) and a circulated conjecture of T. Chinburg \([7]\). The latter suggests that, given an odd Dirichlet character \( \chi_{-N} \), there should be a polynomial with integer coefficients \( P_N(x, y) \) for which \( L'(\chi_{-N}, -1)/m(P_N) \) is a rational number.\(^3\) It seems more plausible to expect the existence of polynomial \( P_N(x, y) \in \mathbb{Z}[x, y] \) such that

\[
m(P_N) = r L'(\chi_{-N}, -1) + \log |s|
\]

for some rational \( r \neq 0 \) and algebraic \( s \neq 0 \). A statement of this type is shown to be true for the \( L \)-value \( L'(E, 0) \) of an elliptic curve \( E \) over \( \mathbb{Q} \) with complex multiplication by R. Pengo \([9, \text{Theorem 4.7}]\).

\(^3\)Chinburg also states in \([7]\) a general conjecture for \( d \)-variable Mahler measures but he shows its truth for \( d = 1 \) only and discusses some evidence for \( d = 2 \). Unfortunately for the latter, there is an irreparable mistake in his proof of Theorem 2 (namely, the statement of \([7, \text{Lemma 1}]\) is incorrect).
5. Final touch

A number $C$ is called an Apéry limit (of order $d$) if there is an irreducible homogeneous linear difference equation

$$a_d(n)r(n + d) + a_{d-1}(n)r(n + d - 1) + \cdots + a_1(n)r(n + 1) + a_0(n)r(n) = 0$$

with coefficients $a_d(n), \ldots, a_1(n), a_0(n) \in \mathbb{Z}[n], a_d(n)a_0(n) \neq 0$, and its two rational-valued solutions $\{q(n)\}_{n \geq n_0}$ and $\{p(n)\}_{n \geq n_0}$ such that $p(n)/q(n) \to C$ as $n \to \infty$. Though we only discuss above the case $d = 2$, some constructions are known [6, 18] which demonstrate that many ‘fundamental’ constants, like the values of Riemann’s zeta function at positive integers, are Apéry limits. In analogy with Chinburg’s general conjecture from [7], we would expect that every $L$-value $L(\chi, k)$, for $k \geq 2$, associated with a quadratic character $\chi$ is an Apéry limit.

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