THE THIRD ORDER HELICITY OF MAGNETIC FIELDS VIA LINK MAPS II.

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Abstract. In this sequel we extend the derivation of the third order helicity to magnetic fields supported on unlinked domains in 3-space. The formula is expressed in terms of generators of the deRham cohomology of the configuration space of three points in \( \mathbb{R}^3 \), which is a more practical domain from the perspective of applications. It also admits an ergodic interpretation as the asymptotic Milnor \( \bar{\mu}_{123} \)-invariant of orbits of the field.

1. Introduction

In the recent work [10] author derived a formula for the third order helicity \( H_{123}(B; T) \) of a volume preserving vector field \( B \) on unlinked domains \( T \) in \( S^3 \) (illustrated on Figure 1). A purpose of this note is to extend these formulas to perhaps more natural, from the perspective of fluid dynamics, setting of 3-space \( \mathbb{R}^3 \). It may seem at first like a minor improvement since \( S^3 = \mathbb{R}^3 \cup \{\infty\} \), but from perspective of applications there is a qualitative difference, because the new formula involves familiar Green forms which are generators of the cohomology ring of the configuration space \( \text{Conf}_3(\mathbb{R}^3) \) of three points in flat \( \mathbb{R}^3 \). As a crucial step we obtain, a very practical integral for Milnor \( \bar{\mu}_{123} \)-invariant of 3-component Borromean links in \( \mathbb{R}^3 \). The helicity invariants measure topological complexity of the flow and are relevant in the context of e.g. plasma physics where \( B \) is a magnetic field frozen in the velocity field \( v \) of plasma because it satisfies

\[
\frac{dB}{dt} = \nabla \times (v \times B).
\]

We refer the reader to [2] for the background material on helicity invariants and more specifically to Problem 7.18 posed by Arnold and Khesin in [2, p. 176]. Our approach to formulation of \( H_{123}(B; T) \) is to interpret this invariant as an asymptotic Milnor \( \bar{\mu}_{123} \)-invariant of triples of orbits of the vector field \( B \), in the style of Arnold (c.f. [1]).

Throughout the article we use the convenient language of differential forms and work in smooth category, unless stated otherwise. We begin by introducing necessary facts about topology of the configuration space \( \text{Conf}_3(\mathbb{R}^3) \) in Section 2, then we turn to a derivation of suitable integrals for the Whitehead products in \( \text{Conf}_3(\mathbb{R}^3) \) which in Section 4 lead us to a new formula for \( \bar{\mu}_{123} \). In the last section we review the method of paper [10] in the new setting.

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2. Background on $\text{Conf}_3(\mathbb{R}^3)$.

We denote by $\text{Conf}_n(X)$ a configuration space of ordered distinct $n$ points in a space $X$

$$\text{Conf}_n(X) := \{(x_1, \ldots, x_n) \in X^n; x_i \neq x_j \text{ if } i \neq j\}.$$ 

In this note we almost entirely focus on the case $X = \mathbb{R}^3$. Following [6] we set $e$ to be the unit vector $(1, 0, 0)$ in $\mathbb{R}^3$ and define

$$q_1 = (0, 0, 0), \quad q_i = q_1 + 4(i-1)e, \quad Q_i = \{q_1, \ldots, q_i\}, \quad Q_0 = \emptyset.$$ 

The following spherical cycles on $\text{Conf}_3(\mathbb{R}^3)$ are of fundamental importance

$$A_{ij} : S^2 \to \text{Conf}_3(\mathbb{R}^3), \quad 1 \leq j < i \leq 3$$

$$A_{21} : \xi \mapsto (q_1, q_1 + \xi, q_3), \quad A_{32} : \xi \mapsto (q_1, q_2, q_2 + \xi), \quad A_{31} : \xi \mapsto (q_1, q_2, q_1 + \xi),$$

and we denote their respective homotopy classes in $\pi_2(\text{Conf}(\mathbb{R}^3))$ by $\alpha_{ji}$. Consider projections

$$\Pi_i : \text{Conf}_3(\mathbb{R}^3) \to \text{Conf}_2(\mathbb{R}^3),$$

$$\Pi_i(x_1, x_2, x_3) = (\ldots, \hat{x}_i, \ldots), \quad i = 1, 2, 3,$$

defined by skipping the $i$-th coordinate factor. Because $\text{Conf}_2(\mathbb{R}^3)$ is diffeomorphic to $\mathbb{R}^3 \times (\mathbb{R}^3 - \{0\})$, (via $(x_1, x_2) \mapsto (x_1, x_2 - x_1)$) it has a homotopy type of $S^2$. Directly from the definition $\Pi_k \circ A_{ij}$ are degree one maps when $i, j \neq k$, or null homotopic whenever $i = k$ or $j = k$. Results in [6, 4] tell us that every $\Pi_i$ is a fibration which admits a section. In particular choosing $i = 3$, we obtain the fibration diagram

$$\begin{array}{c} 
\text{Conf}_3(\mathbb{R}^3) \\
\Pi_3 \\
\text{Conf}_2(\mathbb{R}^3) \cong S_{21}
\end{array}$$

$$\begin{array}{c} 
\longrightarrow \\
\longrightarrow \\
\text{Conf}_1(\mathbb{R}^3 - Q_2) = \mathbb{R}^3 - Q_2 \cong S_{31} \lor S_{32}
\end{array}$$

$$\text{Conf}_3(\mathbb{R}^3) \leftarrow \text{Conf}_1(\mathbb{R}^3 - Q_2) = \mathbb{R}^3 - Q_2 \cong S_{31} \lor S_{32}$$

$$\Pi_3$$

$$\text{Conf}_2(\mathbb{R}^3) \cong S_{21}$$

where by $S_{ij}$ we denote the images: $A_{ij}(S^2) \subset \text{Conf}_3(\mathbb{R}^3)$, obviously we may choose to fiber over each $S_{ij}$ separately. As an immediate consequence, we obtain [15, p. 189]

$$\pi_k(\text{Conf}_3(\mathbb{R}^3)) \cong \pi_k(S_{21}) \oplus \pi_k(S_{31} \lor S_{32}),$$

thus in particular for $k = 2$, we conclude that $\alpha_{ij}$ generate $\pi_2(\text{Conf}_3(\mathbb{R}^3)) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. Next, we describe a structure of the deRham cohomology ring of the configuration space, [4]. For $x = (x, y, z) \in \mathbb{R}^3 \setminus \{0\}$ we define a closed differential 2-form

$$\omega(x) := \frac{1}{4\pi} \frac{x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy}{|x|^3},$$
which is a volume form on the unit sphere in $S^2 \subset \mathbb{R}^3$, normalized so that $\int_{S^2} \omega(x) = 1$. Define the Green form $\omega_{ij}$ by

$$\omega_{ij} := \omega(x_i - x_j), \quad i > j,$$

which is a closed 2-form on $\text{Conf}_3(\mathbb{R}^3)$. In the vector notation

$$\omega_{ij}(x_1, x_2, x_3)(X, Y) = \frac{\langle x_i - x_j, X, Y \rangle}{|x_i - x_j|^3}, \quad X, Y \in T\mathbb{R}^3.$$

![Sample unlinked handlebodies](image)

**Figure 1.** Sample unlinked handlebodies $\{T_1, T_2, T_3\}$.

Every $\omega_{ij}$ defines an integral cohomology class $\psi_{ij} := [\omega_{ij}]$ and is dual to cycles $[A_{ij}]$. It is known, [4], that the cohomology ring $H^*_d(\text{Conf}_n(\mathbb{R}^3))$ is generated integrally by $\psi_{ij}$, $1 \leq j < i \leq 3$ with relations

$$\psi_{ij}^2 = 0, \quad \psi_{ij} = -\psi_{ji},$$

$$\psi_{31}\psi_{32} = \psi_{21}(\psi_{32} - \psi_{31}),$$

(see also [6, p. 101]). The last relation on representatives $\omega_{ij}$ reads

$$\omega_{21} \land \omega_{32} - \omega_{32} \land \omega_{31} - \omega_{31} \land \omega_{21} =$$

$$\omega_{12} \land \omega_{23} + \omega_{23} \land \omega_{31} + \omega_{31} \land \omega_{12} = d\phi_{123},$$

for some smooth 3-form $\phi_{123}$.

### 3. Integrals for the Whitehead products in $\text{Conf}_3(\mathbb{R}^3)$

Our goal in the later section is to understand $\pi_3(\text{Conf}_3(\mathbb{R}^3))$ in the context of so called link maps (see Section 4). Among relevant generators of this group are the Whitehead products of $\alpha_{ij}$’s (c.f. [13]), thanks to the decomposition in (2.4). In this section we aim to obtain suitable integrals for these Whitehead products and we begin by introducing appropriate definitions.
Let $D^p$ denote a $p$ dimensional disk in $\mathbb{R}^{p+1}$, given two continuous maps

$$f_k : (D^{p_k}, \partial D^{p_k}) \rightarrow (X, x_0), \quad k = 1, 2$$

into a pointed topological space $(X, x_0)$, the Whitehead product of $f_1$ and $f_2$ is the map (c.f. [15])

$$[f_1, f_2] : \partial(D^{p_1} \times D^{p_2}) \cong S^{p_1+p_2-1} \rightarrow (X, x_0),$$

defined as follows

\begin{equation}
[f_1, f_2](x_1, x_2) := \begin{cases} 
  f_1(x_1), & x_2 \in \partial D^{p_2}, \\
  f_2(x_2), & x_1 \in \partial D^{p_1}, 
\end{cases}
\end{equation}

(recall $\partial(D^{p_1} \times D^{p_2}) = \partial D^{p_1} \times D^{p_2} \cup D^{p_1} \times \partial D^{p_2}$). The operation $[, ] : \pi_{p_1}(X) \times \pi_{p_2}(X) \rightarrow \pi_{p_1+p_2-1}(X)$ is well defined and turns the vector space $\pi_*(X) \otimes \mathbb{R}$ into a graded Lie algebra over $\mathbb{R}$ (c.f. [1]). In the following proposition we extend calculations in [7] to define an integral detecting certain Whitehead products in the configuration space $\text{Conf}_3(\mathbb{R}^3)$, (this calculation can be considered as a precursor of more advanced theory of analytical homotopy periods developed by Novikov in [13]).

**Proposition 3.1.** For any $f : S^3 \rightarrow \text{Conf}_3(\mathbb{R}^3)$ define

\begin{equation}
I(f) := \int_{S^3} (f^* \omega_{12} \wedge \eta_{23} + f^* \omega_{23} \wedge \eta_{31} + f^* \omega_{31} \wedge \eta_{12}) - \int_{S^3} f^* \phi_{123},
\end{equation}

where forms $f^* \omega_{ij}$ are exact and $d\eta_{ij} := f^* \omega_{ij}$. Then,

(i) $I$ is independent of the choice of “potentials” $\eta_{ij}$.

(ii) $I \in \text{Hom}(\pi_3(\text{Conf}_3(\mathbb{R}^3)), \mathbb{R})$ and satisfies

\begin{equation}
I([\alpha_{12}, \alpha_{23}]) = I([\alpha_{23}, \alpha_{31}]) = I([\alpha_{31}, \alpha_{12}]) = 1,
\end{equation}

\begin{equation}
I([\alpha_{ij}, \alpha_{ij}]) = 0, \quad 1 \leq j < i \leq 3.
\end{equation}

**Proof.** First we show that $I$ is independent of choices of potentials $\eta_{ij}$’s, let $\eta_{ij}'$ be different potentials, then $d(\eta_{ij} - \eta_{ij}') = f^* \omega_{ij} - f^* \omega_{ij} = 0$ and we may calculate

$$I(f) - I'(f) = \sum_{i,j,k} \int_{S^3} f^* \omega_{ij} \wedge (\eta_{jk} - \eta_{jk}') = \sum_{i,j,k} \int_{S^3} d\eta_{ij} \wedge (\eta_{jk} - \eta_{jk}')$$

$$= \sum_{i,j,k} \int_{S^3} \eta_{ij} \wedge d(\eta_{jk} - \eta_{jk}') = 0,$$

where in the third identity we applied Stokes Theorem. To show that $I$ is a well defined homomorphism we show invariance under homotopies. Let $F : I \times S^3 \rightarrow \text{Conf}_3(\mathbb{R}^3)$ be a homotopy between $F_0$ and $F_1$, define $\tilde{\eta}_{ij} = d^{-1} F^* \omega_{ij}$ on $S^3 \times I$. By Stokes Theorem and (i),
The sum under integral is abbreviated by $\Sigma$:

\[ \mathcal{I}(F_1) - \mathcal{I}(F_0) = \int_{S^3 \times \{1\}} \left( \sum F^*_{ij} \omega_{ij} \land \eta_{jk} - F^*_1 \phi_{123} \right) - \int_{S^3 \times \{0\}} \left( \sum F^*_{0} \omega_{ij} \land \eta'_{jk} - F^*_0 \phi_{123} \right) \]

\[ = \int_{S^3 \times I} dF^* \omega_{12} \land \tilde{\eta}_{23} + F^* \omega_{23} \land \tilde{\eta}_{13} + F^* \omega_{13} \land \tilde{\eta}_{12} - F^* \phi_{123} \]

\[ = \int_{F(S^3 \times I)} \left( \omega_{12} \land d\tilde{\eta}_{23} + \omega_{23} \land d\tilde{\eta}_{13} + \omega_{13} \land d\tilde{\eta}_{12} - d\phi_{123} \right) = 0, \]

because of $d\eta_{ij} = \omega_{ij}$ and Equation (2.7). Since additivity is a direct consequence of additivity for integrals and the definition of $+$ in $\pi_n(\cdot)$, $\mathcal{I}$ is a well defined element of $\text{Hom}(\pi_3(\text{Conf}_3(\mathbb{R}^3)), \mathbb{R})$.

For Equations (3.3), consider $f = [f_1, f_2]$ as defined in (3.1) for $p_1 = p_2 = 2$, and let $\pi_i : D_1 \times D_2 \mapsto D_1$, $i = 1, 2$ be projections onto each factor $D_i \cong D^2$, and $j : \partial(D_1 \times D_2) \mapsto D_1 \times D_2$ be a natural inclusion. According to (3.1) we have

\[ f^* \omega_{ij} = f^* (\pi_1^* f_1^* \omega_{ij} + \pi_2^* f_2^* \omega_{ij}), \]

since every 2-form $\pi_1^* f_1^* \omega_{ij} + \pi_2^* f_2^* \omega_{ij}$ is closed and thus exact on $D_1 \times D_2$. Let $\eta_{ij}$ define a smooth potential

\[ d\eta_{ij} = \pi_1^* f_1^* \omega_{ij} + \pi_2^* f_2^* \omega_{ij}, \]

clearly $dj^* \eta_{ij} = f^* \omega_{ij}$. We calculate by applying Stokes Theorem and (3.5)

\[ \mathcal{I}(f) = \int_{\partial(D_1 \times D_2)} \left( f^* \omega_{12} \land \eta_{23} + f^* \omega_{23} \land \eta_{13} + f^* \omega_{13} \land \eta_{12} - f^* \phi_{123} \right) \]

\[ = \int_{D_1 \times D_2} \left( \sum (\pi_1^* f_1^* \omega_{ij} + \pi_2^* f_2^* \omega_{ij}) \land (\pi_1^* f_1^* \omega_{jk} + \pi_2^* f_2^* \omega_{jk}) - (\pi_1^* f_1^* d\phi_{123} + \pi_2^* f_2^* d\phi_{123}) \right) \]

\[ = \int_{D_1 \times D_2} \left( \sum (\pi_1^* f_1^* (\omega_{ij} \land \omega_{jk}) + \pi_2^* f_2^* (\omega_{ij} \land \omega_{jk}) - (\pi_1^* f_1^* d\phi_{123} + \pi_2^* f_2^* d\phi_{123}) \right) \]

\[ + \int_{D_1 \times D_2} \left( \sum (\pi_1^* f_1^* (\omega_{ij} \land \omega_{jk}) + \pi_2^* f_2^* (\omega_{ij} \land \omega_{jk}) \right). \]

The first integral in the above identity vanishes because of Equation (2.7). The second integral is equal to

\[ \mathcal{I}(f) = \sum \left( \int_{D_1} f_1^* \omega_{ij} \int_{D_2} f_2^* \omega_{jk} + (-1)^4 \int_{D_1} f_1^* \omega_{jk} \int_{D_2} f_2^* \omega_{ij} \right) \]

\[ = (\omega_{12}(f_1)\omega_{23}(f_2) + \omega_{23}(f_1)\omega_{12}(f_2)) + (\omega_{23}(f_1)\omega_{13}(f_2) + \omega_{13}(f_1)\omega_{23}(f_2)) \]

\[ + (\omega_{12}(f_1)\omega_{13}(f_2) + \omega_{13}(f_1)\omega_{12}(f_2)), \]

where $\omega_{ij}(f) = \int_{S^2} f^* \omega_{ij}$. Identities in (3.3), now follow from the definition of $A_{ij}$ in (2.1) and (2.6). \( \square \)
Remark 3.2. The argument in [7] “runs” as follows let $\omega_1$ and $\omega_2$ be closed differential forms of degree $p_1$ and $p_2$, such that $\omega_1 \wedge \omega_2 = 0$. Consider spherical cycles $f_k : S^{pk} \to X$, $k = 1, 2$, it is shown in [7] that for any $f : S^{p_1+p_2-1} \to X$ the integral

$$J_{(\omega_1, \omega_2)}(f) := \int_{S^{p_1+p_2-1}} f^* \omega_1 \wedge \eta_2 = \int_{S^{p_1+p_2-1}} f^* \omega_2 \wedge \eta_1,$$

where $f^* \omega_k = d\eta_k$, $k = 1, 2$, defines an element of $\text{Hom}(\pi_{p_1+p_2-1}(X); \mathbb{R})$ satisfying

(3.6) $$J_{(\omega_1, \omega_2)}([f_1, f_2]) = \omega_1(f_1)\omega_2(f_2) + (-1)^{p_1p_2}\omega_1(f_2)\omega_2(f_1),$$

In particular for $X = S^p$ let $\omega$ be a volume form, such that $\int_{S^p} \omega = 1$, and let $f : S^p \to S^p$ be a degree one map. Then Equation (3.6) implies

(3.7) $$J_{(\omega, \omega)}([f, f]) = \begin{cases} 0, & p = 2k+1, \\ 2, & p = 2k, \end{cases}$$

Thus for even $p$, $[f, f] : S^{2p-1} \to S^p$ is twice the Hopf map.

4. Milnor $\bar{\mu}_{123}$-invariant for Borromean links in $\mathbb{R}^3$.

Denote a parametrized $n$-component link in $\mathbb{R}^3$ by $L = \{L_1, L_2, \ldots, L_n\}$, (where $L_i : S^1 \to \mathbb{R}^3$, such that $L_i(S^1) \cap L_k(S^1) = O$, $i \neq k$), and let $\text{Links}(n)$ be the set of such $n$-component links. We define a link map, (c.f [9, 11]) as follows

$$F : \text{Links}(n) \to \text{Maps}((S^1)^n, \text{Conf}_n(\mathbb{R}^3))$$

(4.1) $$F(L) = F_L : (S^1)^n \to \text{Conf}_n(\mathbb{R}^3), \quad F_L := L_1 \times \ldots \times L_n.$$

Clearly, deformations of $L$ in $\mathbb{R}^3$ yield homotopies of the associated $F_L$, as a result homotopy invariants of $F_L$ provide invariants of $L$, since $\text{Conf}_n(\mathbb{R}^3)$ is 1-connected the set of based homotopy classes is in bijective correspondence with the set of base point free homotopy classes, [15]. One may in fact obtain the classical Milnor $\bar{\mu}$-invariant this way, to demonstrate we review the Gauss formula for the linking number of a 2-component link $L = \{L_1, L_2\}$ in $\mathbb{R}^3$. Denote parameterizations of components by $L_1 = \{x(s)\}, L_2 = \{y(t)\}$ and consider

$$F_L : S^1 \times S^1 \xrightarrow{L} \text{Conf}_2(\mathbb{R}^3) \xrightarrow{r} S^2, \quad L(s, t) = (x(s), y(t)).$$

where $r(x, y) = \frac{x - y}{\|x - y\|}$ is a retraction of $\text{Conf}_2(\mathbb{R}^3)$ onto $S^2$. This yields the classical Gauss linking number formula

(4.2) $$\bar{\mu}_{12}(L_1, L_2) = \deg(F_L), \quad \deg(F_L) = \int_{S^1 \times S^1} F_L^*(\nu),$$

where $\nu \in \Omega^2(S^2)$ is the area form on $S^2$. Consequently, the linking number $\text{lk}(L_1, L_2)$, also known as the Milnor $\bar{\mu}_{12}$-invariant, can be obtained as the homotopy invariant of the map $F_L$ associated to $L$. In the following we focus exclusively on the 3-component case, in the context of recent results, [10, 5] obtained for 3-component links in $\mathbb{R}^3$.

Consider a 3-component link $L = \{L_1, L_2, L_3\}$ in $S^3$ parametrized by $\{x(s), y(t), z(u)\}$ and the link map

(4.3) $$F_L : S^1 \times S^1 \times S^1 \xrightarrow{L} \text{Conf}_3(S^3) \xrightarrow{H} S^2, \quad L(s, t, u) = (x(s), y(t), z(u)).$$
where $H$ is a projection on the second factor of $\text{Conf}_3(S^3) \cong S^3 \times S^2$, and can be defined with help of quaternionic structure of $S^3$ as follows \cite{5}

\begin{equation}
\text{Conf}_3(S^3) \ni (x, y, z) \xrightarrow{H} \frac{\text{pr}(x^{-1} \cdot y) - \text{pr}(x^{-1} \cdot z)}{\|\text{pr}(x^{-1} \cdot y) - \text{pr}(x^{-1} \cdot z)\|} \in S^2,
\end{equation}

where $\cdot$ stands for the quaternionic multiplication, $^{-1}$ is the quaternionic inverse, and $\text{pr} : S^3 \to \mathbb{R}^3$ the stereographic projection from 1.

**Theorem 4.1** (\cite{10, 5}). Let $L = \{L_1, L_2, L_3\}$ be a 3-component Borromean link in $S^3$, then the associated map to $F_L : S^1 \times S^1 \times S^1 \to S^2$ defined in (4.1) is homotopic to $\pm 2\bar{\mu}_{123} \times$ the Hopf map, (where the sign depends on the orientation of components).

The next theorem of this section extends the above result to links in $\mathbb{R}^3$.

**Theorem 4.2.** Let $L = \{L_1, L_2, L_3\}$ be a 3-component link in $S^3$, then the associated link map $F_L : S^1 \times S^1 \times S^1 \to \text{Conf}_3(\mathbb{R}^3)$ defined in (4.1) satisfies

\begin{equation}
\deg(F_L|_{S^1 \times S^1}) = \text{lk}(L_i, L_j).
\end{equation}

In addition, whenever $L$ is Borromean (i.e. $\text{lk}(L_i, L_j) = 0$, $1 \leq i \neq j \leq 3$) $F_L$ is homotopic to $\bar{\mu}_{123}$ to one of the Whitehead products $[\alpha_{32}, \alpha_{31}], [\alpha_{31}, \alpha_{21}], -[\alpha_{32}, \alpha_{21}]$. As a result the integral defined in (3.2) yields

$$I(F_L) = \pm \bar{\mu}_{123}(L).$$

**Proof.** The proof of (1.3) is immediate from the definition of the linking number in (4.2).

To prove the second assertion we consider the inclusion

$$j : \mathbb{R}^3 \to \mathbb{R}^3 \cup \{\infty\} \cong S^3,$

which induces the inclusion on configuration spaces

$$\widehat{j} : \text{Conf}_3(\mathbb{R}^3) \to \text{Conf}_3(S^3).$$

As a first step, using the methodology in \cite{4, 6}, we will calculate the induced homomorphism

$$\pi_2(\widehat{j}) : \pi_2(\text{Conf}_3(\mathbb{R}^3)) \to \pi_2(\text{Conf}_3(S^3)).$$

Recall that $\pi_2(\text{Conf}_3(\mathbb{R}^3)) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ is generated by classes $\alpha_{ij}$ represented by $A_{ij}$ and $\pi_2(\text{Conf}_3(S^3)) \cong \mathbb{Z}$ is generated just by one class $\alpha$ represented by $S^2$ factor in $\text{Conf}_3(S^3) \cong S^3 \times S^2$. Let us denote

\begin{equation}
\beta_{ij} = \pi_2(\widehat{j})(\alpha_{ij}),
\end{equation}

clearly, $\beta_{ij} = a_{ij} \alpha$, $a_{ij} \in \mathbb{Z}$ and we must determine the coefficients $a_{ij}$.

Observe that the symmetric group $\Sigma_3$ acts on both $\text{Conf}_3(\mathbb{R}^3)$ and $\text{Conf}_3(S^3)$ by permuting the coordinate factors, and the inclusion map $j$ is equivariant with respect to this action. For our purposes we only need to find out the action of the transposition $(23)$ on $\alpha_{31}$ and $\alpha_{32}$. We claim the following identities

\begin{equation}
(23)\alpha_{32} = -\alpha_{32} \quad (23)\alpha_{31} = \alpha_{21}.
\end{equation}
To justify we calculate on representatives in (2.1)

\[(23)A_{32} : \xi \longrightarrow (23)(q_1, q_2, q_2 + \xi) = (q_1, q_2 + \xi, q_2)\]

\[(23)A_{31} : \xi \longrightarrow (23)(q_1, q_2, q_1 + \xi) = (q_1, q_1 + \xi, q_2) .\]

Consider the homotopy

\[G : t \longrightarrow (q_1, q_2 + (1 - t)\xi, q_2 - t\xi), \quad t \in [0, 1]\]

Notice that \(G\) is well defined in \(\text{Conf}_3(\mathbb{R}^3)\), because \(q_2 + (1 - t)\xi = q_2 - t\xi\) implies \(\xi = 0\) which contradicts \(\xi \in S^2\). This homotopy connects \(G_0 = (q_1, q_2 + \xi, q_2) = (23)A_{32}\) and \(G_1 = (q_1, q_2, q_2 - \xi) = -A_{32}\), which shows the first identity in (4.7). To see the second identity consider the homotopy: \(G' : t \longrightarrow (q_1, q_1 + \xi, (1 - t)q_2 + tq_3), t \in [0, 1]\).

Recall the fibration Diagram (2.3), obviously the image of the fiber \(\mathbb{R}^3 - \{q_1, q_2\}\) under inclusion \(j\) is in \((\mathbb{R}^3 \cup \{\infty\}) - \{q_1, q_2\} \subset \text{Conf}_3(S^3)\) implying the following relation

\[\beta_{31} + \beta_{32} = 0, \quad \text{in} \quad \pi_2(\text{Conf}_3(S^3)) .\]

Applying (4.7) to the above equation we obtain \(\beta_{21} - \beta_{32} = 0,\) hence

\[\beta_{21} = \beta_{32} = -\beta_{31} \quad \text{in} \quad \pi_2(\text{Conf}_3(S^3)) .\]

Thanks to the map defined in (4.4) we observe that \(\beta_{21} = \alpha\) and

\[\pi_2(\hat{j})(\alpha_{21}) = \alpha, \quad \pi_2(\hat{j})(\alpha_{32}) = \alpha, \quad \pi_2(\hat{j})(\alpha_{31}) = -\alpha,\]

thus coefficients in (4.9) are \(\alpha_{21} = \alpha_{32} = -\alpha_{31} = 1\). Let \(h : S^3 \hookrightarrow \text{Conf}_3(S^3) \cong S^3 \times S^2\) be a map such that \(p_1 \circ h\) is null and \(p_2 \circ h\) is homotopic to the the Hopf map (where \(p_i\) is the projection onto the \(i\)th factor in \(S^3 \times S^2\)). Because the Whitehead product is natural, [15, p. 473], we obtain

\[(4.8) \quad \pi_3(\hat{j})([\alpha_{32}, \alpha_{31}]) = [\alpha, -\alpha] = -2[h], \quad \pi_3(\hat{j})([\alpha_{32}, \alpha_{21}]) = [\alpha, \alpha] = 2[h],
\]

\[\pi_3(\hat{j})([\alpha_{31}, \alpha_{21}]) = [-\alpha, \alpha] = -2[h],\]

where we used the fact shown in Equation (3.7). In \(\pi_3(\text{Conf}_3(\mathbb{R}^3))\) the following identity, known as Yang-Baxter relation, [9], holds

\[(4.9) \quad [\alpha_{32}, \alpha_{31} + \alpha_{32}] = 0,\]

just consider a map \(\phi : S^2 \times S^2 \longrightarrow \text{Conf}_3(\mathbb{R}^3)\), \(\phi(\xi_1, \xi_2) = (q_1, q_1 + \xi_1, q_1 + 5\xi_2),\) [9]. It is easy to see from the formulas in (2.1) that

\[i_1 := \phi |_{S^2 \times \{+\}} \cong \alpha_{21}, \quad i_2 := \phi |_{\{+\} \times S^1} \cong \alpha_{31} + \alpha_{32},\]

therefore (4.9) immediately follows from the naturality of the Whitehead product and the fact that \(i_1, i_2 = 0\) in \(\pi_3(S^2 \times S^2)\). Thanks to (4.9) and (4.7) we obtain

\[(4.10) \quad [\alpha_{32}, \alpha_{31}] = [\alpha_{31}, \alpha_{21}] = -[\alpha_{32}, \alpha_{21}] .\]

\footnote{It is a direct consequence of the definition of the Whitehead product as an attaching map of the 4-cell to the 2-skeleton \(S^2 \cap S^2\) in \(S^2 \times S^2\).}
Next we need to prove why the link map $F_L$ associated to a Borromean link $L$ is represented by the above Whitehead products. Because of $(4.5)$, and $\text{lk}(L_i, L_j) = 0$, $F_L$ can be homotopied to a map $\tilde{F}_L$ constant on the 2-skeleton of $(S^1)^3$ and thus represents a class $\tilde{f}_L$ in $\pi_3(\text{Conf}_3(\mathbb{R}^3))$. Since any $\Pi_i \circ F_L$ is null homotopic we obtain

\begin{equation}
\tilde{f}_L \in \ker(\pi_3(\Pi)) \quad \pi_3(\Pi) : \pi_3(\text{Conf}_3(\mathbb{R}^3)) \to \pi_3((\text{Conf}_2(\mathbb{R}^3))^3),
\end{equation}

where $\Pi = \Pi_1 \times \Pi_2 \times \Pi_3$, and $\Pi_i$ were defined in $(2.2)$. Recall that fibration $\Pi_i$ admits a section thus $\pi_3(\text{Conf}_3(\mathbb{R}^3)) \cong \pi_3(S(21)) \oplus \pi_3(S(31) \vee S(32)), \text{[15, p. 189].}$ Group $\pi_3(S(21)) \cong \mathbb{Z}$ is generated by a Hopf map $h_{21}$. The group $\pi_3(S(31) \vee S(32)) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ is generated by Hopf maps $h_{31}, h_{32}$ and the Whitehead product $[\alpha_{32}, \alpha_{31}]$ (which immediately follows from the split short exact sequence $0 \to \pi_4(S^2 \times S^2, S^2 \vee S^2) \to \pi_3(S^2 \vee S^2) \to \pi_3(S^2 \times S^2)$, $\text{[15, p. 492]}$). We expand

\begin{equation}
\tilde{f}_L = c_1 h_{21} + c_2 h_{31} + c_3 h_{32} + c_4 [\alpha_{32}, \alpha_{31}], \quad c_k \in \mathbb{Z}.
\end{equation}

Since $\pi_3(\Pi_i)(h_{ij}) = h_{ij}$ for all $1 \leq j < i \leq 3$, $(4.11)$ tells us that $c_1 = c_2 = c_3 = 0$ and

\begin{equation}
\tilde{f}_L = c_4 [\alpha_{32}, \alpha_{31}].
\end{equation}

Theorem $4.1$ and Identities $(4.8)$ yield

\begin{align*}
\pm 2 \mu_{123}[h] &= \pi_3(\tilde{f}_L) = 2 \ c_4 \ [h], \\
&\Rightarrow \ c_4 = \pm \mu_{123}.
\end{align*}

Claims of Theorem $4.2$ now follow from $(4.10)$ and Lemma $3.1$. By analogous argument as in $(3.4)$ we may evaluate the integral $(3.2)$ directly on the torus $(S^1)^3$ to obtain

\begin{equation}
\mu_{123} = \pm \mathcal{I}(F_L) = \int_{(S^1)^3} \left(F_L^* \omega_{12} \wedge \alpha_{23} + F_L^* \omega_{23} \wedge \eta_{31} + F_L^* \omega_{31} \wedge \eta_{12} - F_L^* \phi_{123}\right),
\end{equation}

where $\eta_{ij}$ satisfy $d\eta_{ij} = F_L^* \omega_{ij}$. \hfill \square

Remark 4.3. By applying standard identities for Chen iterated integrals we may easily see that $(4.12)$ represents Chen’s iterated integral proposed by Toshitake Kohno in $\text{[9, p. 155].}$

5. The Third Order Helicity on Unlinked Invariant Domains in $\mathbb{R}^3$

Methodology developed in $\text{[10]}$ can be now directly applied to the integral in $(4.12)$ to obtain the following formula for the third order helicity

\begin{equation}
H_{123}(B; \mathcal{T}) = \int_{\mathcal{T}} \omega_{123} \wedge t_{B_1} \mu_2 \wedge t_{B_2} \mu_1 \wedge t_{B_3} \mu_3, \\
\omega_{123} = \omega_{12} \wedge \eta_{23} + \omega_{23} \wedge \eta_{31} + \omega_{31} \wedge \eta_{12} - \phi_{123}, \\
d\alpha_{ij} = \omega_{ij}, \quad \text{on} \quad \mathcal{T}_i \times \mathcal{T}_j
\end{equation}

The domain $\mathcal{T}$ of integration in $(5.1)$ is assumed to be an invariant unlinked domain of $B$, $\text{[10]}$. In the simplest case $\mathcal{T}$ is a product of disjoint compact solid handlebodies $\mathcal{T}_i \subset \mathbb{R}^3$ (as on Figure 1) such that every pair of 1 dimensional cycles in $H^1(\mathcal{T})$ has linking number zero. Generally, an $\Phi$-invariant set is defined in $\text{[10]}$ to be a union of products of orbits of $B$ in...
An invariant unlinked domain is an arbitrary $\Phi$-invariant set, with topological closure $\mathcal{T}$ which belongs to a larger product of disjoint open sets $\mathcal{T} = \mathcal{T}_1 \times \mathcal{T}_2 \times \mathcal{T}_3$ which satisfies

(A) $\mathcal{T}$ admits a short path system $\mathcal{S}$,

(B) every pair of 1 dimensional cycles in $H_1(\mathcal{T})$ has linking number zero.

By a system of short paths on $\mathcal{T}$, [14, 2], we understand a collection of curves $\mathcal{S} = \{\sigma_i(x, y)\}$ on each open set $\mathcal{T}_i$ such that for every pair of points $x, y \in \mathcal{T}_i$ there is a connecting curve $\sigma_i(x, y) : I \mapsto \mathcal{T}_i$, $\sigma_i(0) = x$ and $\sigma_i(1) = y$, and the lengths of curves in $\mathcal{S}$ are uniformly bounded above by a common constant.

Given $T > 0$ we introduce the following notation

$$O^B_{iT}(x) = \{\Phi_{i}(x)| 0 \leq t \leq T\}, \quad \tilde{O}^B_{iT}(x) := O^B_{iT}(x) \cup \sigma(x, \Phi^i(x, T)),$$

where $\sigma(x, y) \in \mathcal{S}$. Because the methodology in [10] is independent of a particular integral for $\tilde{\mu}_{123}$, as a result we may reiterate the following theorem from in the new context.

**Theorem 5.1.** On every unlinked invariant domain $\mathcal{T}$ of $B$ in $\mathbb{R}^3$, $H_{123}(B; \mathcal{T})$ is

(i) independent of a choice of the potentials $\alpha_{ij}$.

(ii) invariant under the action of $\text{SDiff}_0(S^3)$, i.e. for every $g \in \text{SDiff}_0(\mathbb{R}^3)$:

$$H_{123}(B; \mathcal{T}) = H_{123}(g_B; g(\mathcal{T})).$$

(iii) Moreover, $H_{123}(B; \mathcal{T})$ admits the following ergodic interpretation as an asymptotic $\tilde{\mu}_{123}$-invariant of orbits of $B$. For almost every $(x, y, z) \in \mathcal{T}$ the following limit exist:

$$\bar{m}_B(x, y, z) = \lim_{T \to \infty} \frac{1}{T^3}\tilde{\mu}_{123}(\tilde{\mathcal{O}}^{B_1}_T(x), \tilde{\mathcal{O}}^{B_2}_T(y), \tilde{\mathcal{O}}^{B_3}_T(y)),

\text{and}

$$H_{123}(B; \mathcal{T}) = \int_{\mathcal{T}} \bar{m}_B(x, y, z) \mu(x) \land \mu(y) \land \mu(z).$$

For the detailed proof of above theorem we refer the reader to [10]. We comment that the Formula (5.1) leads to the $L^2$-energy bound for $B$ in $\mathbb{R}^3$ which can be derived analogously as in [10].

**References**

[1] V. Arnold. The asymptotic Hopf invariant and its applications. *Selecta Math. Soviet.*, 5(4):327–345, 1986. Selected translations.

[2] V. Arnold and B. Khesin. *Topological methods in hydrodynamics*, volume 125 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1998.

[3] A. Y. K. Chui and H. K. Moffatt. The energy and helicity of knotted magnetic flux tubes. *Proc. Roy. Soc. London Ser. A*, 451(1943):609–629, 1995.

[4] F. Cohen, T. J. Lada, and J. P. May. *The homology of iterated loop spaces*. Springer-Verlag, Berlin, 1976. Lecture Notes in Mathematics, Vol. 533.

[5] D. DeTurck, H. Gluck, R. Komendarczyk, P. Melvin, C. Shonkwiler, and D. Vela-Vick. Triple linking numbers, Hopf invariants and Integral formulas for three-component links. (preprint arXiv:0901.1612), 2009.

[6] E. R. Fadell and S. Y. Husseini. *Geometry and topology of configuration spaces*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2001.
[7] A. Haefliger. Whitehead products and differential forms. In Differential topology, foliations and Gelfand-Fuks cohomology (Proc. Sympos., Pontifícia Univ. Católica, Rio de Janeiro, 1976), volume 652 of Lecture Notes in Math., pages 13–24. Springer, Berlin, 1978.

[8] B. Khesin. Topological fluid dynamics. Notices Amer. Math. Soc., 52(1):9–19, 2005.

[9] T. Kohno. Loop spaces of configuration spaces and finite type invariants. In Invariants of knots and 3-manifolds (Kyoto, 2001), volume 4 of Geom. Topol. Monogr., pages 143–160 (electronic). Geom. Topol. Publ., Coventry, 2002.

[10] R. Komendarczyk. The third order helicity of magnetic fields via link maps. to appear in Comm. Math. Phys. (avail. at arXiv:0808.1533), 2009.

[11] U. Koschorke. A generalization of Milnor’s $\mu$-invariants to higher-dimensional link maps. Topology, 36(2):301–324, 1997.

[12] U. Koschorke. Link homotopy in $S^n \times \mathbb{R}^{m-n}$ and higher order $\mu$-invariants. J. Knot Theory Ramifications, 13(7):917–938, 2004.

[13] S. P. Novikov. Analytical theory of homotopy groups. In Topology and geometry—Rohlin Seminar, volume 1346 of Lecture Notes in Math., pages 99–112. Springer, Berlin, 1988.

[14] T. Vogel. On the asymptotic linking number. Proc. Amer. Math. Soc., 131(7):2289–2297 (electronic), 2003.

[15] G. W. Whitehead. Elements of homotopy theory, volume 61 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1978.