On semi-symmetric non-metric connexion

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Abstract

H. A. Hayden [1] introduced the idea of semi-symmetric non-metric connection on a Riemannian manifold in (1932). Agashe and Chafle [1] defined and studied semi-symmetric non-metric connection on a Riemannian manifold. In the present paper, we define a new type of semi-symmetric non-metric connexion in an almost contact metric manifold and studied its properties. In the end, we have studied some properties of the covariant almost analytic vector field equipped with semi-symmetric non-metric connection.

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1 Preliminaries

If there are a tensor field $F$ of type $(1, 1)$, a vector field $T$, a 1-form $A$ and metric $g$, satisfying next equations for arbitrary vector fields $X, Y, Z \in T_p, p \in M_n$

\[ \overline{X} + X = A(X)T, \]

(1)

\[^{1}\text{E-mail: sk22.math@yahoo.co.in}\]
\[ A(\overline{X}) = 0, \] \hspace{1cm} (2)

\[ 'F(X,Y) \overset{\text{def}}{=} g(\overline{X}, Y), \] \hspace{1cm} (3)

\[ g(\overline{X}, \overline{Y}) = g(X,Y) - A(X)A(Y), \] \hspace{1cm} (4)

\[ \overline{X} \overset{\text{def}}{=} F X, \]

then the n-dimensional differentiable manifold \( M_n \) is called an almost contact metric manifold \[6\].

An almost contact metric manifold satisfying

\[ (D_X'F)(Y,Z) = A(Y)(D_XA)(\overline{Z}) - A(Z)(D_XA)(\overline{Y}) \] \hspace{1cm} (5)

\[ D_X'F)(Y, Z) + (D_Y'F)(Z, X) + (D_Z'F)(X, Y) \]

\[ = A(X)[(D_YA)(\overline{Z}) - (D_ZA)(\overline{Y})] + A(Y)[(D_ZA)(\overline{X}) - (D_XA)(\overline{Y})] \]

\[ - (D_XA)(\overline{Z}) + A(Z)[(D_XA)(\overline{Y}) - (D_YA)(\overline{X})] \] \hspace{1cm} (6)

are called generalized co-symplectic manifold and generalized quasi-Sasakian manifold respectively \[5\].

The Nijenhuis tensor in generalized cosymplectic manifold is given by

\[ 'N(X, Y, Z) = (D_X'F)(Y, Z) - (D_Y'F)(X, Z) \]

\[ + (D_X'F)(Y, \overline{Z}) - (D_Y'F)(X, \overline{Z}) \] \hspace{1cm} (7)

If on any manifold \( T \) satisfies

\[ (D_XA)(\overline{Y}) = - (D_XA)(\overline{Y}) = (D_YA)(\overline{X}) \] \hspace{1cm} (8)

\[ \iff (D_XA)(Y) = (D_XA)(\overline{Y}) = -(D_YA)(X), \]

\[ D_T F = 0, \]

then \( T \) is said to be of first class and the manifold is said to be of first class \[5\]. If on an almost contact metric manifold \( T \) satisfies

\[ (D_XA)(\overline{Y}) = (D_XA)(Y) = -(D_YA)(\overline{X}) \] \hspace{1cm} (9)
\[ (D_X A)(Y) = -(D_{X\overline{Y}})(\overline{Y}) = -(D_{Y\overline{X}})(X), \]
\[ D_T F = 0, \]
then \( T \) is said to be of second class and the manifold is said to be of second class \[5\].

## 2 Semi-symmetric non-metric connexion

### Definition 2.1. Let \( D \) be a Riemannian connexion, then we define an affine connexion \( \tilde{B} \) as

\[ \tilde{B}_X Y = D_X Y + f(X, Y) T \tag{10} \]

satisfying
\[ \tilde{S}(X, Y) = 2f(X, Y) T \tag{11} \]

and
\[ (\tilde{B}_X g)(Y, Z) = -A(Y) f(X, Z) - A(Z) f(X, Y) \tag{12} \]

is called a semi-symmetric non-metric connexion. Also,
\[ \tilde{S}(X, Y, Z) \overset{\text{def}}{=} g(\tilde{S}(X, Y), Z) = 2A(Z) f(X, Y) \tag{13} \]

\[ (\tilde{B}_X F)(Y) = (D_X F)(Y) + g(\overline{X}, \overline{Y}) T \tag{14} \]

\[ (\tilde{B}_X A)(Y) = (D_X A)(Y) - g(\overline{X}, Y) \tag{15} \]

### Theorem 2.1. If \( D \) be a Riemannian connexion and \( \tilde{B} \) be a semi-symmetric non-metric connexion, then on an almost contact metric manifold \( \tilde{S} \) is hybrid.

### Theorem 2.2. If the manifold is of first class with respect to the Riemannian connexion \( D \), then it is also first class with respect to the semi-symmetric non-metric connexion \( \tilde{B} \)
**Proof.** From (15)

\[(D_X A)(Y) = (\check{B}_X A)(Y) + g(X, Y)\] (16)

and
\[(D_X A)(Y) = (\check{B}_X A)(Y) - g(X, Y)\] (17)

Adding (16) and (17) and then using (4), we have
\[(D_X A)(Y) + (D_X A)(\check{Y}) = (\check{B}_X A)(Y) + (\check{B}_X A)(\check{Y})\] (18)

In view of (8), (18) becomes
\[(\check{B}_X A)(Y) = -(\check{B}_X A)(\check{X})\] (19)

Similarly, we have from (15)
\[(\check{B}_X A)(\check{Y}) = (\check{B}_Y A)(X)\] (20)

Equations (19) and (20) give
\[(\check{B}_X A)(Y) = -(\check{B}_X A)(Y) = (\check{B}_Y A)(X)\] (21)

Now taking covariant derivative of \(FY = \check{Y}\) with respect to \(\check{B}\) and using (8) and (10), we obtain
\[\check{B}_T F = 0\]

Hence the theorem. \(\square\)

**Theorem 2.3.** Let \(D\) be a Riemannian connexion and \(\check{B}\) be a semi-symmetric non-metric connexion. Then an almost contact metric manifold is a generalized quasi-Sasakian manifold of the first kind if
\[(\check{B}_X F)(Y, Z) + (\check{B}_Y F)(Z, X) + (\check{B}_Z F)(X, Y) = 0\]
Proof. We have,

\[ X(\mathbf{F}(Y,Z)) = (\tilde{B}X\mathbf{F})(Y,Z) + F(Y,\tilde{B}XZ) \]

\[ = (D_X\mathbf{F})(Y,Z) + F(D_XY,Z) + F(Y,DXZ) \]

Then,

\[ (D_X\mathbf{F})(Y,Z) = (\tilde{B}X\mathbf{F})(Y,Z) + F(\tilde{B}XY - DXY, Z) + F(Y,\tilde{B}XZ - DXZ) \]

Using (10) and (3), we get

\[ (D_X\mathbf{F})(Y,Z) = (\tilde{B}X\mathbf{F})(Y,Z) \] (22)

Taking covariant derivative of \( A(Z) = 0 \) with respect to the Riemannian connexion \( D \) and using (10), we obtain

\[ (D_XA)(Z) = (\tilde{B}XA)(Z) + g(X,Z) \] (23)

Using (8), (22) and (23), (6) gives the required result.

Theorem 2.4. A quasi-Sasakian manifold be normal if and only if

\[ (\tilde{B}X\mathbf{F})(Y,Z) = A(Y)[(\tilde{B}ZA)(X) + g(X,Z)] \]

\[ + A(Z)[(\tilde{B}XA)(Y) + g(X,Y)], \]

where \( \tilde{B} \) being semi-symmetric non-metric connexion.

Proof. From (13),

\[ (D_XA)(Y) = (\tilde{B}XA)(Y) - g(X,Y) \] (24)

The necessary and sufficient condition that a quasi-Sasakian manifold be normal [5] is

\[ (D_X\mathbf{F})(Y,Z) = A(Y)(D_ZA)(X) + A(Z)(D_XA)(Y) \] (25)

Using (22), (23) and (24) in (25), we obtain the required result.
Theorem 2.5. An almost contact metric manifold to be generalized cosymplectic manifold if

\[(\tilde{B}_X F)(Y, Z) = A(Y)[(\tilde{B}_X A)(Z) + g(X, Z)] - A(Z)[(\tilde{B}_X A)(Y) + g(X, Y)]\]

(26)

Proof. From (15)

\[(D_X A)(Z) = (\tilde{B}_X A)(Z) + g(X, Z)\]

(27)

Using (22) and (27) in (5), we obtain the required result.

\[\square\]

Theorem 2.6. On generalized co-symplectic manifold, \(F\) is killing with respect to \(\tilde{B}\) if and only if

\[(\tilde{B}_X A)(Z) + g(X, Z) = 0\]

Proof. Since \(F\) is killing with respect to semi-symmetric non-metric connexion \(\tilde{B}\), we have

\[(\tilde{B}_X F)(Y, Z) + (\tilde{B}_Y F)(X, Z) = 0\]

(28)

Using (26), (28) becomes

\[A(X)[(\tilde{B}_Y F)(T, Z)] + A(Y)[(\tilde{B}_X F)(T, Z)] + 2A(Z)g(X, Y) = 0\]

or,

\[2A(Z)g(X, Y) + A(X)[(\tilde{B}_Y A)(Z) + g(Y, Z)] + A(Y)[(\tilde{B}_X A)(Z) + g(X, Z)] = 0\]

(29)

Putting \(T\) for \(X\) in this equation, we obtain

\[A(X)(\tilde{B}_T A)(Z) + (\tilde{B}_X A)(Z) + g(X, Z) + 2A(Z)A(X) = 0\]

(30)

Putting \(T\) for \(X\) in this equation, we get

\[(\tilde{B}_T A)(Z) + A(Z) = 0\]

(31)

From (30) and (31), we get the result. Converse part is obvious.

\[\square\]
Theorem 2.7. If $U$ is killing, then on generalized co-symplectic manifold

$$'N(X,Y,Z) - d'F(X,Y,Z) = 2A(Z)(\tilde{B}_Y A)(X)$$

Proof. From (7) and (22), we get

$$'N(X,Y,Z) - d'F(X,Y,Z) = (\tilde{B}_Y' F)(Y,Z) - (\tilde{B}_X' F)(Y,Z) + (\tilde{B}_X' F)(X,Z) - (\tilde{B}_Y' F)(X,Z)$$

Using (20) in the above equation, we have

$$'N(X,Y,Z) - d'F(X,Y,Z) = -A(X)[(\tilde{B}_Y A)(Z) + (\tilde{B}_Z A)(Y)] + A(Y)[(\tilde{B}_X A)(Z) + (\tilde{B}_Z A)(X)] + A(Z)[(\tilde{B}_X A)(Y) + (\tilde{B}_Y A)(X)]$$

Since $U$ is killing, then

$$'N(X,Y,Z) - d'F(X,Y,Z) = 2A(Z)(\tilde{B}_Y A)(X)$$

Corollary 2.1. If $'F$ is closed, then

$$'N(X,Y,Z) = 0$$

Theorem 2.8. A generalized co-symplectic manifold is quasi-Sasakian if

$$(\tilde{B}_X' F)(Y,T) = (\tilde{B}_Y' F)(X,T),$$

where $\tilde{B}$ being a semi-symmetric non-metric connexion.
Proof. From (22) and (26), we have

\[
(D_X F)(Y, Z) + (D_Y F)(Z, X) + (D_Z F)(X, Y) = \]

\[
A(X)[(\tilde{B}_Z A)(Y) - (\tilde{B}_Y A)(Z)] + A(Y)[(\tilde{B}_X A)(Z)] - (\tilde{B}_Z A)(\overline{X}) + A(Z)[(\tilde{B}_Y A)(\overline{X})] =
\]

\[
(\tilde{B}_Z A)(\overline{Y}) + A(Z)[(\tilde{B}_Y A)(\overline{X}) - (\tilde{B}_X A)(\overline{Y})] =
\]

\[
A(X)[(\tilde{B}_Z F)(T, Y) - g(\overline{Y}, Z) - (\tilde{B}_Y F)(T, Z) + g(\overline{Y}, Z)] + A(Y)[(\tilde{B}_X F)(T, Z) - g(\overline{X}, Z)] + A(Z)[(\tilde{B}_X F)(Y, T) + g(\overline{X}, \overline{Y})] =
\]

\[
A(X)[(\tilde{B}_Z F)(T, Y) - (\tilde{B}_Y F)(T, Z)] + A(Y)[(\tilde{B}_X F)(T, Z)] - (\tilde{B}_Z F)(T, X)] + A(Z)[(\tilde{B}_X F)(Y, T) - (\tilde{B}_Y F)(X, T)] =
\]

\[
0,
\]

which proved the statement.

3 Covariant almost analytic vector field

If 1-form \( w \) satisfies

\[
w( (D_X F)(Y) - (D_Y F)(X) ) = (D_{\overline{X}} w)(Y) - (D_{\overline{X}} w)(\overline{Y})
\]

then 1-form \( w \) is said to be covariant almost analytic vector field [6]. Here \( D \) is the Riemannian connection.

Theorem 3.1. On an almost contact metric manifold if 1-form \( w \) is covariant almost analytic vector field with respect to the Riemannian connexion \( D \), then the semisymmetric non-metric connexion \( \tilde{B} \) coincide with \( D \) if \( w(\rho) = 0 \).

Proof. We have,

\[
(\tilde{B}_X F)(Y) = (D_X F)(Y) + g(\overline{X}, \overline{Y})\rho - g(\overline{X}, Y)p
\]
Interchanging $X$ and $Y$, we have

$$(\bar{B}_Y F)(X) = (D_Y F)(X) + g(\overline{Y}, \overline{X})\rho - g(\overline{Y}, X)\overline{\rho}$$  \hspace{1cm} (34)$$

Subtracting (34) from (33) and using (4) and (2), we obtain

$$w((D_X F)(Y) - (D_Y F)(X)) = w((\bar{B}_X F)(Y) - (\bar{B}_Y F)(X)) + 2g(\overline{X}, Y)w(\overline{\rho})$$  \hspace{1cm} (35)$$

Again,

$$(\bar{B}_X w)(Y) = (D_X w)(Y) - w(\rho)g(\overline{X}, Y)$$  \hspace{1cm} (36)$$

and

$$(\bar{B}_X w)(\overline{Y}) = (D_X w)(\overline{Y}) - w(\rho)g(\overline{X}, \overline{Y})$$  \hspace{1cm} (37)$$

From last two expressions, we get

$$(D_X w)(Y) - (D_X w)(\overline{Y}) = (\bar{B}_X w)(Y) - (\bar{B}_X w)(\overline{Y})$$

$$+ w(\rho)[g(\overline{X}, Y) - g(\overline{X}, \overline{Y})]$$  \hspace{1cm} (38)$$

Now subtracting (38) from (35) and using (32), we obtain that $w$ is covariant almost analytic vector field with respect to semi-symmetric non-metric connexion $\bar{B}$ if and only if

$$w(\rho)g(\overline{X}, \overline{Y}) = 0$$

But in general, $g(\overline{X}, \overline{Y}) = 0$ is not possible, therefore $w(\rho) = 0$. \hfill \Box$

**Theorem 3.2.** On an almost contact metric manifold if 1-form $A$ is covariant almost analytic vector field with respect to the connexion $D$, then

$$\tilde{d}A(X, Y) = dA(X, Y) + 2g(X, \overline{Y})$$  \hspace{1cm} (39)$$

where

$$dA(X, Y) \overset{\text{def}}{=} (D_X A)(Y) - (D_Y A)(X)$$  \hspace{1cm} (40)$$

$$\tilde{d}A(X, Y) \overset{\text{def}}{=} (\bar{B}_X A)(Y) - (\bar{B}_Y A)(X)$$  \hspace{1cm} (41)$$
Proof. Interchanging $X$ and $Y$ in (15), we get

$$(\tilde{B}_Y A)(X) = (D_Y A)(X) - g(X, Y)$$

(42)

From (15) and (42), we obtain

$$(\tilde{B}_X A)(Y) - (\tilde{B}_Y A)(X) = (D_X A)(Y) - (D_Y A)(X) + 2g(X, Y)$$

(43)

In view of (40), (41), (43) gives (39).

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