Exact multiple inverses of the quadratic planar map

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Abstract. We developed a method for exactly determining the multiple inverses of quadratic planar maps. The quadratic planar map was transformed through an affine transformation into its conjugate domain, where all inverses can be exactly determined, and transform the inverses back. We showed that for all quadratic planar maps with non-vanishing critical curves, except for two exceptional cases of the map with a degenerate critical curve, there exists at least one available transformation. For both exceptional cases and for those maps with vanishing critical curves, the inverses are determinable without any transformation. This result can be applied to the calculation of stable manifolds of the quadratic planar map. It may help us further understand the dynamics, the basins of attraction, and the bifurcations associated with the quadratic planar map under iteration.

1. Introduction
The quadratic planar map is an important class of the two-dimensional non-invertible map called planar endomorphism. The most remarkable feature of planar endomorphism is the non-invertibility. Determination for preimages (inverses) of a point under the map plays an important role in studying the dynamical behaviour of nonlinear systems described by the iteration of the map [1-2]. For example, the basin boundary of attraction is often organized by the stable manifold associated with a saddle fixed point under the inverse of the map [3-4]. It is impossible to write down the inverses of such maps explicitly and the determination of the preimage often resorts to the Newton method numerically. However, whether the Newton method converges to the root (preimage) depends on the initial guess. The dynamics of the Newton method often settle down to periodic orbits or chaos when the root is near the critical point of the map. Furthermore, the Newton method gives no guarantee that how many preimages exist even though the root can be found very quickly. This work arises from the study on the calculation of the stable manifold of quadratic planar maps [5]. Kostelich et al. [6] have given a plot of the stable manifold of a non-invertible map, but still leave the problem of how to trace the stable manifold to traverse the critical curve of the map unresolved.

In this paper, we successfully developed an analytical method for determining the exact multiple inverses of quadratic planar maps. Generally speaking, the inverses of all quadratic planar maps with non-vanishing critical curves can be determined exactly by an affine transformation except for two exceptional cases of maps with critical curves degenerating to straight lines, for which no available transformation could be found. However, for both exceptional cases and for those maps with vanishing critical curve, inverses are determinable without any transformation. This result can be applied to the calculation of stable manifolds of the quadratic planar map. It may help us further understand the dynamics, the basins of attraction, and the bifurcations associated with the quadratic planar map under iteration.
2. Critical curve of the quadratic planar map

Consider the quadratic planar map \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \)

\[
T(x) = \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix} = \begin{pmatrix} a_{11}x^2 + a_{12}xy + a_{13}y^2 + b_{11}x + b_{12}y + c_1 \\ a_{21}x^2 + a_{22}xy + a_{23}y^2 + b_{21}x + b_{22}y + c_2 \end{pmatrix},
\]

(1) where \( x = (x, y)^T \) is a point in \( \mathbb{R}^2 \) and all the coefficients are real. The critical curve (LC) of \( T \) is the locus of points such that \( \det J_T(x) = 0 \). Here, by \( \det J_T(x) \), we mean the determinant of the Jacobian matrix evaluated at the point \( x \). After a brief calculation, \( \det J_T(x) \) can be written as

\[
\det J_T(x) = x^T A x + \beta x + \gamma,
\]

(2) where \( A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \beta = \begin{pmatrix} 2a_{11}a_{22} - a_{12}a_{21} \\ 2a_{12}a_{21} - a_{11}a_{22} \end{pmatrix} \), and \( \gamma = \begin{pmatrix} b_{11} \\ b_{22} \end{pmatrix} \). By the change in variable \( x = Q \tilde{x} + r \), \( \det Q \neq 0 \), the above equation can be rewritten as

\[
\det J_T(x) = x^{T} \Lambda \tilde{x} + \beta^{T} \tilde{x} + \gamma,
\]

(3) where \( \Lambda = Q^{T} A Q \), \( \beta = Q^{T} (2 A r + \beta) \) and \( \gamma = \det J_T(r) \). Since the matrix \( A \) is real-symmetric (we exclude the trivial case \( A = 0 \) in which the inverse can be determined directly), it can always be diagonalized such that \( A = \Lambda^{T} A \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \) with \( \lambda_1 \) and \( \lambda_2 \) the two real eigenvalues of \( A \). When \( \det A = \lambda_1 \lambda_2 \neq 0 \), we choose \( r = -2^{-1} \Lambda^{-1} \beta \) such that (3) becomes

\[
\det J_T(x) \bigg|_{x = -2^{-1} \Lambda^{-1} \beta} = \lambda_1 \tilde{x}^2 + \lambda_2 \tilde{y}^2 + \xi,
\]

(4) where \( \xi = \det J_T(r) \bigg|_{r = -2^{-1} \Lambda^{-1} \beta} \). In this way, the LC of \( T \) can be classified into the following categories.

- **Elliptic type** (\( \det A > 0 \)): The LC is generally an ellipse, but degenerates into a point as \( \xi = 0 \) or vanishes as \( \lambda_1 \xi = 0 \).

- **Hyperbolic type** (\( \det A < 0 \)): The LC is generally a hyperbola, but degenerates into two crossing lines as \( \xi = 0 \).

However, when \( \det A = \lambda_1 \lambda_2 = 0 \), one of the eigenvalues \( \lambda_1 \) equals zero (since \( A \neq 0 \), \( \lambda_2 \neq 0 \)). Similarly, (3) is reduced to (leave \( r \) to be arbitrary rather than \( -2^{-1} \Lambda^{-1} \beta \))

\[
\det J_T(x) \bigg|_{x = -2^{-1} \Lambda^{-1} \beta} = \lambda_1 \tilde{x}^2 + \lambda_2 \tilde{y}^2 + \bar{\beta}_1 \tilde{x} + \bar{\beta}_2 \tilde{y} + \bar{\gamma}.
\]

(5) Thus the LC of \( T \) can be classified into the third category.

- **Parabolic type** (\( \det A = 0 \)): The LC is generally a parabola except that \( \bar{\beta}_1 = 0 \). In such a case, the LC degenerates into two parallel lines as \( \bar{\beta}_1^2 - 4 \lambda_2 \tilde{y} > 0 \), a single line as \( \bar{\beta}_1^2 - 4 \lambda_2 \tilde{y} = 0 \), or vanishes as \( \bar{\beta}_1^2 - 4 \lambda_2 \tilde{y} < 0 \).

3. Determination of the inverses of the quadratic planar map

It is convenient to rewrite (1) as

\[
T(x) = a x^2 + b x + c,
\]

(6)
where \( \mathbf{a} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \), \( \mathbf{x}^2 = (x^2 \ y^2) \), \( \mathbf{b} = \begin{bmatrix} b_{11} \\ b_{21} \\ b_{12} \\ b_{22} \end{bmatrix} \), and \( \mathbf{c} = (c_1 \ c_2)^T \) all have real entries. Our goal is to find the exact multiple values of \( T^{-1}(x') : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) for any given \( x' = (x' \ y')^T \in \mathbb{R}^2 \). That is, we wish to find all the points \( x \) such that \( x = T(x) \) which takes the form explicitly

\[
\begin{align*}
    x' &= a_{11}x^2 + a_{12}xy + a_{13}y^2 + b_{11}x + b_{12}y + c_1 \\
    y' &= a_{21}x^2 + a_{22}xy + a_{23}y^2 + b_{21}x + b_{22}y + c_2.
\end{align*}
\]

Generally, it is impossible to write down the inverses explicitly. Our aim is to solve this general problem exactly by an analytical method.

By introducing an affine transformation \( \mathbf{x} = h(\overline{\mathbf{x}}) = \mathbf{A}_\psi \mathbf{x} + \mathbf{b} \) where \( \mathbf{A}_\psi = \begin{bmatrix} \psi_{11} & \psi_{12} & \psi_{13} \\ \psi_{21} & \psi_{22} & \psi_{23} \end{bmatrix} \in \mathbb{R}^{2 \times 3} \), \( \overline{\mathbf{x}} = (\overline{x} \ \overline{y})^T \in \mathbb{R}^2 \), \( \mathbf{x} = (\varphi_1 \ \varphi_2)^T \in \mathbb{R}^2 \), and det \( \psi \neq 0 \), the original equation \( x' = T(x) \) is transformed to \( \overline{x}' = T(\overline{x}) \) or

\[
\overline{x}' = \overline{T}(\overline{x}) \equiv h^{-1}T(h(\overline{x})),
\]

where \( \overline{x}' = (\overline{x}' \ \overline{y}')^T = h^{-1}(x') = \psi^{-1}(x' - \varphi) \). We say that the map \( \overline{T}(\overline{x}) \) is conjugate to the map \( T(x) \) and the two equations \( \overline{x}' = \overline{T}(\overline{x}) \) and \( x' = T(x) \) are equivalent. If we can find the value of \( \overline{T}^{-1}(\overline{x}') \), we can also find the value of \( T^{-1}(x') = h\overline{T}^{-1}h^{-1}(x') \) as shown in Figure 1. Note that the Taylor series expansion of any quadratic planar map is also a quadratic planar map and all the coefficients associated with the quadratic terms remain unchanged. We have \( T(h(\overline{x})) = T(\psi \overline{x} + \mathbf{b}) = T(\mathbf{0} + J_y(\varphi)\psi \overline{x} + \mathbf{b}) \) and, through a brief calculation, \( (\psi \overline{x})^2 = \mathbf{A}_\psi \overline{x}^2 \) where

\[
\mathbf{A}_\psi = \begin{bmatrix} \psi_{11} & 2\psi_{12} & \psi_{13} \\ \psi_{21} & 2\psi_{22} & \psi_{23} \end{bmatrix}.
\]

Therefore, the conjugate map of \( T \) has the form

\[
\overline{T}(\overline{x}) = \overline{\mathbf{a}} \overline{x}^2 + \overline{\mathbf{b}} \overline{x} + \overline{\mathbf{c}},
\]

where \( \overline{\mathbf{a}} = \psi^{-1}a \mathbf{A}_\psi \), \( \overline{\mathbf{b}} = \psi^{-1}J_y(\varphi)\psi \), and \( \overline{\mathbf{c}} = \psi^{-1}(T(\mathbf{0}) - \varphi) = (\overline{c}_1 \ \overline{c}_2)^T \).

3.1. A special case of the quadratic planar equation

There is a special case of the quadratic planar equation \( \overline{x}' = \overline{T}(\overline{x}) \) that we can find its inverses exactly. Consider \( \overline{\mathbf{b}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) and \( \overline{x}' = \overline{c}_1 \), that is, the equation takes the form

\[
\begin{align*}
    0 &= a_{11}\overline{x}^2 + a_{12}\overline{x}\overline{y} + a_{13}\overline{y}^2 \\
    \overline{y}' &= a_{21}\overline{x}^2 + a_{22}\overline{x}\overline{y} + a_{23}\overline{y}^2 + \lambda \overline{x} + \overline{c}_2.
\end{align*}
\]

Figure 1. We can find the inverses \( T^{-1}(x') = h\overline{T}^{-1}h^{-1}(x') \) by the transformation \( h \).
It is straightforward to obtain the relation $\bar{x} = (2a_{11})^{-1}(a_{12} + \sqrt{a_{12}^2 - 4a_{11}a_{13}})\bar{y}$ from the first scalar equation (if $a_{11} = 0$, then either $\bar{y} = 0$ or $a_{13}\bar{x} + a_{11}\bar{y} = 0$). Thus the inverses can be readily solved by substituting this relation into the second scalar equation.

### 3.2. Quadratic planar map with a non-degenerate critical curve

In this subsection, we present the main result of this paper about how to find the affine transformation $\mathbf{x} = \psi \bar{x} + \phi$ such that the equivalent equation in the conjugate domain satisfies (11) and prove that there exists at least one available transformation for all quadratic planar maps with non-degenerate LCs.

Considering (8) and (10), if we choose $\phi$ to be a critical point of the map (i.e., $\det J_\phi = 0$), then $\bar{x}'$ always has two eigenvalues $\lambda_1 = 0$ and $\lambda_2 = \lambda \in R$ with eigenvectors $e_1$ and $e_2$. Thus $\bar{b}$ can always be diagonalized such that $\bar{b} = \psi^{-1} J_\phi(\phi) \psi = \begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix}$ with $\psi = [e_1, e_2]$. Note that there are infinitely many critical points for the quadratic planar map with a non-degenerate LC. Thus, without loss of generality, from now on, we only need to deal with the equation in a simpler form

$$
\begin{cases}
    x' = a_{11}x^2 + a_{12}xy + a_{13}y^2 + c_1 \\
    y' = a_{21}x^2 + a_{22}xy + a_{23}y^2 + \lambda y + c_2
\end{cases}
$$

(12)

where $x' \neq c_1$. In order to find the required affine transformation, we must find the critical point $\phi$ by imposing an additional constraint that, after the transformation, the constant term of the first scalar equation of $\mathbf{x}' = \bar{f}(\mathbf{x})$ is zero (i.e., $x' = c_1$). Our main result is as follows.

**Theorem 1.** The required affine transformation is determined by choosing a critical point $\phi = (\phi_1, \phi_2)^T$ which can be parameterized as

$$
\begin{cases}
\phi_1 &= \frac{\lambda(a_{12} - a_{22}\rho)}{R_1\rho^2 + R_2\rho + R_3} \\
\phi_2 &= -\frac{2\lambda(a_{22} - a_{12}\rho)}{R_1\rho^2 + R_2\rho + R_3}
\end{cases}
$$

(13)

where $R_1 = 4a_{11}a_{13} - a_{12}^2$, $R_2 = 2a_{12}a_{22} - 4a_{11}a_{23} - 4a_{13}a_{21}$, $R_3 = 4a_{21}a_{23} - a_{22}^2$, and the parameter $\rho$ satisfies the following equation

$$
A\rho^3 + B\rho^2 + C\rho + D = 0
$$

(14)

where $A = (c_1 - x')R_1$, $B = (c_1 - x')R_2 - (c_2 - y')R_1$, $C = (c_1 - x')R_3 - (c_2 - y')R_2 - \lambda^2 a_{11}$, and $D = \lambda^2 a_{21} - (c_2 - y')R_1$.

**Proof.** Since $\psi = [e_1, e_2]$, $\bar{x}' = (\bar{x}' - \bar{x})^T = \psi^{-1}(x' - \phi)$, $\bar{x} = (e_1, e_2)^T = \psi^{-1}(T(\phi) - \phi)$, hence $x' - \phi = \psi \bar{x}' = \bar{x}'e_1 + \bar{x}'e_2$ and $T(\phi) - \phi = \psi \bar{x} = \bar{x}'e_1 + \bar{x}'e_2$. Accordingly, we have $T(\phi) - x' = (\bar{x}' - \bar{x})e_2$ due to $\bar{x}' = \bar{x}'$, which means $T(\phi) - x' = 0$. Since $e_2$ is the eigenvector of $J_\phi(\phi) = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$ for the eigenvalue $\lambda_2 = \phi_{11} + \phi_{22}$, thus $-\phi_{11} - \phi_{12} \phi_{21} - \phi_{22} = 0$. We thus obtain $e_2 \begin{bmatrix} \phi_{11} \\ \phi_{21} \end{bmatrix} = 0$, which also means $T(\phi) - x' \begin{bmatrix} \phi_{11} \\ \phi_{21} \end{bmatrix} = 0$.

Note that $T(\phi) = \begin{bmatrix} f_1(\phi_1, \phi_2) \\ f_2(\phi_1, \phi_2) \end{bmatrix} = \begin{bmatrix} 2^{-1} (\phi_{11}\phi_1 + \phi_{12}\phi_2) + c_1 \\ 2^{-1} (\phi_{21}\phi_1 + \phi_{22}\phi_2) + 2^{-1}\lambda \phi_2 + c_2 \end{bmatrix}$.
Hence

\[
T(\varphi) - x' = \begin{vmatrix}
\varphi_{11} & \varphi_{11} \\
\varphi_{21} & \varphi_{21}
\end{vmatrix} = \begin{pmatrix}
2^{-1}(\varphi_{11}\varphi_{11} + \varphi_{12}\varphi_{2}) + c_{1} - x' & \varphi_{11} \\
2^{-1}(\varphi_{21}\varphi_{11} + \varphi_{22}\varphi_{2}) + 2^{-1}\lambda\varphi_{2} + c_{2} - y' & \varphi_{21}
\end{pmatrix}
\]

\[
= \frac{1}{2} \begin{vmatrix}
\varphi_{11} & \varphi_{11} \\
\varphi_{22} & \varphi_{22}
\end{vmatrix} + \frac{1}{2} \begin{vmatrix}
\varphi_{12} & \varphi_{11} \\
\varphi_{22} & \varphi_{22}
\end{vmatrix} + \begin{vmatrix}
c_{1} - x' & \varphi_{11} \\
2^{-1}\lambda\varphi_{2} + c_{2} - y' & \varphi_{21}
\end{vmatrix}
\]

\[
= \begin{vmatrix}
c_{1} - x' & \varphi_{11} \\
2^{-1}\lambda\varphi_{2} + c_{2} - y' & \varphi_{21}
\end{vmatrix}
\]

The second determinant in the second line of the above equation vanishes because of \(\text{det } \mathbf{J}_\varphi(x) = 0\). Thus we obtain

\[
\begin{vmatrix}
c_{1} - x' & \varphi_{11} \\
2^{-1}\lambda\varphi_{2} + c_{2} - y' & \varphi_{21}
\end{vmatrix} = 0
\]  \( \text{(15)} \)

Solving (15) by using \(\text{det } \mathbf{J}_\varphi(x) = 0\), we can obtain the point \(\varphi\). Generally, we can let \(\varphi_{21} = \rho\varphi_{11}\), which implies \(\varphi_{22} = \rho\varphi_{12}\) from \(\text{det } \mathbf{J}_\varphi(x) = 0\) (we assert that \(\varphi_{11} \neq 0\), because \(\varphi_{11} = 0\) implies \(\varphi_{12} = 0\) or \(\varphi_{21} = 0\) both lead to \(x' = c_1\), which we have assumed unequal). From the above two relations (i.e., \(\varphi_{21} = \rho\varphi_{11}\) and \(\varphi_{22} = \rho\varphi_{12}\)), \(\varphi\) can be parameterized as (13). By substituting (13) into (14), we have (15). The proof is completed.

When \(R_1 \neq 0\), (14) is a third-order equation for \(\rho\). According to the standard theory of the cubic equation [7], the solutions to (14) are

\[
\begin{align*}
\rho_1 &= A' + B' - 3^{-1}p \\
\rho_2 &= -2^{-1}(A' + B' - (-3)^{1/2}(A'' - B'')) - 3^{-1}p \\
\rho_3 &= -2^{-1}(A' + B' + (-3)^{1/2}(A'' - B'')) - 3^{-1}p
\end{align*}
\]  \( \text{(16)} \)

where \(A' = (-2^{-1}b + \sqrt{s})^{1/3}\), \(B' = (-2^{-1}b - \sqrt{s})^{1/3}\), \(s = 4^{-1}b^2 + 27^{-1}a^3\), \(a = 3^{-1}(3q - p^2)\), \(b = 27^{-1}(2p^3 - 9qp + 27r)\), \(p = A'B\), \(q = A'C\), and \(r = A''D\). There are generally three roots for \(\rho\), among which the first root \(\rho_1\) is always real. In what follows, we give the sufficient condition of the existence of such an affine transformation as described in Theorem 1.

**Lemma 1.** If the critical curve of the quadratic planar map is non-degenerate, (14) must be a cubic equation.

**Proof.** Without loss of generality, the quadratic planar map with a non-vanishing LC can be represented as

\[
T(x) = \begin{cases}
T_1(x, y) = \begin{pmatrix} f_{11}(x, y) \\ f_{12}(x, y) \end{pmatrix},
\end{cases}
\]  \( \text{(17)} \)

The Jacobian matrix is \(\mathbf{J}_T(x) = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}\), where \(f_{11} = 2a_{11}x + a_{12}y\), \(f_{12} = a_{13}x + 2a_{13}y\), \(f_{21} = 2a_{21}x + a_{22}y,\) and \(f_{22} = a_{23}x + 2a_{23}y + \lambda\). Considering (14), if \(R_1 = 4a_{11}a_{13} - a_{12}^2 = 0\), it means \(f_{11}\) and \(f_{12}\) are linear dependent and there exists a constant \(s \neq 0\) such that \(f_{22} = sf_{11}\). Therefore, \(\text{det } \mathbf{J}_T(x) = f_{11}(f_{22} - sf_{11})\) and the LC will be either \(f_{11} = 0\) or \(f_{22} - sf_{11} = 0\), both of which represent a straight line. The proof is completed.
However, it must be pointed out that there might exist certain exceptional situation that (14) has only one real root $\rho_1$ which happens to leave the denominator in (13) zero. Here we give the condition that excludes this possibility.

**Lemma 2.** If the critical curve of the quadratic planar map is non-degenerate, there exist no common roots between (14) and the denominator in (13).

**Proof.** Let us consider (13) and (14). Note that

$$A\rho^3 + B\rho^2 + C\rho + D = (c_2 - y'(c_1 - x')\rho)(R_1\rho^2 + R_2\rho + R_3) - \lambda_2(a_{11} - a_{11}\rho)$$  \hspace{1cm} (18)

Suppose $\rho^*$ is a common root between (14) and $R_1\rho^2 + R_2\rho + R_3 = 0$. By substituting $\rho^*$ into (18), we have $\lambda_2(a_{11} - a_{11}\rho^*) = 0$. It can be readily obtained that either $a_{11} = a_{21} = 0$ or $\rho^* = a_{11}^{-1}a_{21}$. For the former case, obviously $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0$. For the latter case, after a brief calculation, it can be obtained that $R_1\rho^2 + R_2\rho + R_3 = 0$ is equivalent to $(a_{11}a_{22} - a_{12}a_{21})^2 = 0$ or $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0$. Then following the proof in Lemma 1, Lemma 2 can be proved.

From Lemma 1 and Lemma 2, we readily conclude the following result.

**Theorem 2.** For all quadratic planar maps with non-degenerate critical curves, there exists at least one available affine transformation such that the multiple inverses of the map can be exactly determined via (13) and (14).

### 3.3. Quadratic planar map with a degenerate critical curve

Thus far we have ensured the existence of the transformation such that the multiple inverses of the quadratic planar map with a non-degenerate LC can be determined analytically. There left the maps with straight lines of LCs (hyperbolic and parabolic types), with an isolated single point of LC (elliptic type), and with vanishing LCs which we cannot guarantee the existence of the transformation. Essentially, for the maps with LCs degenerating to straight lines, we still can find an available transformation via (13) and (14), but, under certain $x'$, two exceptional cases leading to no available transformation may occur. First, it is possible that $R_1 \neq 0$ and (14) has only one real root $\rho_1$ which happens to leave the denominator in (13) zero. Secondly, it is possible that $R_1 = 0$ and the reduced second order equation of (14) has no real roots. However, these two exceptions are not problems. To the contray, we can prove that for these two exceptional cases and the case of vanishing LCs, inverses are determinable without any transformation. As for the map with an isolated single critical point $\Phi$, we can also prove that the exact inverse can be obtained analytically by a translation $x = h(x) = x + \Phi$. Since the details of the proof are lengthy, we will not include them in this conference paper and wish to present the thorough details for all quadratic planar maps in a possible publication of a journal paper.

### 4. Example

An example was provided to illustrate practically how to determine the inverses of the quadratic planar map. Consider a map taking the following form

$$T(x) = \begin{cases} f_1(x, y) = 2x^2 + 2xy + y^2 + 4x + 1 \\ f_2(x, y) = 3x^2 + 3xy + y^2 + 3x + y + 2 \end{cases}$$  \hspace{1cm} (19)

Following the procedures in section 2, we know that the LC is a non-degenerate hyperbola. For any given $x'$ (say $x' = (4 \ 8)'$), our goal is to determine the multiple inverses $T^{-1}(x')$. First, choose $\Phi,$
satisfying $\partial f_x/\partial x = \partial f_y/\partial y = 0$ and construct the affine transformation $h_1(\mathbf{x}) = \begin{bmatrix} \sqrt{10} & 0 \\ -3\sqrt{10} & 1 \end{bmatrix} \mathbf{x} + \begin{pmatrix} -2 \\ 2 \end{pmatrix}$ to transform the original equation to

$$
\begin{cases}
6\sqrt{10} = \sqrt{10}/2x^2 - 4xy + \sqrt{10}y^2 - \sqrt{10} \\
24 = 9/5x^2 + 15/\sqrt{10}xy + 4y^2 - y - 3
\end{cases} \quad (20)
$$

Since (20) does not satisfy (11), we must perform another transform once again. Since $R_1=4$, (14) is a cubic equation. It is enough for the purpose to take the first root $\rho = 1.1184$ according to (16). After a patient computation, the transformation is determined that $h_2(\mathbf{x}) = \begin{bmatrix} 0.7492 & 0.6666 \\ 0.6623 & 0.7454 \end{bmatrix} \mathbf{x} + \begin{pmatrix} -19.1633 \\ -4.5031 \end{pmatrix}$ and (20) is transformed to

$$
\begin{cases}
80.5917 = -0.4964x^2 - 1.2495xy - 0.7831y^2 + 80.5917 \\
-33.3698 = 0.9928x^2 + 2.4575xy + 1.5887y^2 + 11.2877y + 382.9361
\end{cases} \quad (21)
$$

According to (11), there are two real solutions $\mathbf{x} = (68.9758 - 58.5091)^T$ and $(117.624 - 99.7753)^T$. (the other two solutions are complex numbers with non-zero imaginary parts) to (21). Thus the solutions to the original equation are obtained by transform $\mathbf{x}$ back to $\mathbf{x} = h_1h_2(\mathbf{x}) = \psi_1(\psi_2(\mathbf{x}) + \varphi_1) + \varphi_2 = (-4.051 5.719)^T$ and $(-1.2236 -1.3043)^T$. It is straightforward to verify that the two points are indeed the preimages of $\mathbf{x}' = (4.8)^T$ under the map (19) and there exist only two preimages of $\mathbf{x}' = (4.8)^T$.

5. Conclusions
We developed an analytical method for determining the exact multiple inverses of quadratic planar maps. The quadratic planar map was transformed through an affine transformation into its conjugate domain, where all inverses can be exactly determined, and then transform the inverses back. Generally speaking, the inverses of all quadratic planar maps with non-vanishing LCS can be determined exactly by an affine transformation except for two exceptional cases of maps with LCS degenerating to straight lines, for which no available transformation could be found. However, for both exceptional cases and for those maps with vanishing LCS, inverses are determinable without any transformation. The result of this work can be applied to the calculation of the stable manifold of quadratic planar maps. It may help us further understand the dynamics, the basins of attraction, and their bifurcations of the quadratic planar map under iteration.

Acknowledgments
The author would like to acknowledge the support by MOST in Taiwan through Grants No. 105-2112-M-232-001-MY3.

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