Research Article

The Study of a Predator-Prey Model with Fear Effect Based on State-Dependent Harvesting Strategy

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In presence of predator population, the prey population may significantly change their behavior. Fear for predator population enhances the survival probability of prey population, and it can greatly reduce the reproduction of prey population. In this study, we propose a predator-prey fishery model introducing the cost of fear into prey reproduction with Holling type-II functional response and prey-dependent harvesting and investigate the global dynamics of the proposed model. For the system without harvest, it is shown that the level of fear may alter the stability of the positive equilibrium, and an expression of fear critical level is characterized. For the harvest system, the existence of the semitrivial order-1 periodic solution and positive order-q \((q \geq 1)\) periodic solution is discussed by the construction of a Poincaré map on the phase set, and the threshold conditions are given, which can not only transform state-dependent harvesting into a cycle one but also provide a possibility to determine the harvest frequency. In addition, to ensure a certain robustness of the adopted harvest policy, the threshold condition for the stability of the order-\(q\) periodic solution is given. Meanwhile, to achieve a good economic profit, an optimization problem is formulated and the optimum harvest level is obtained. Mathematical findings have been validated in numerical simulation by MATLAB. Different effects of different harvest levels and different fear levels have been demonstrated by depicting figures in numerical simulation using MATLAB.

1. Introduction

Prey-predator interaction is a crucial topic in theoretical ecology and evolutionary biology. The history of the study about the prey-predator interactions dates back long. The pioneering work to describe the prey-predator interactions in mathematics belongs to the Lotka-Volterra model [1, 2]. Subsequently the model was improved by adding logistic growth term for the prey and variety of population-dependent response functions [3–15]. A prototype model that captures the prey-predator interaction takes the form

\[
\begin{align*}
\frac{dx}{dt} &= bx - dx - cx^2 - yp (x, y), \\
\frac{dy}{dt} &= e\phi(p(x, y))y - my,
\end{align*}
\]

where \(x(t)\) and \(y(t)\) represent the densities of prey and predator population, respectively, \(b, d, (d < b)\) and \(c\) represent the birth rate, natural death rate, and density-dependent decay rate due to the intraspecies competition, respectively, \(p(x, y)\) represents the functional response, \(e\) is the efficiency of conversion, \(m\) is natural mortality of predator, and \(\phi\) is a monotonically increasing function.

Due to prey-predator interactions, predators always have an impact, direct, indirect, or both, on prey population. In model (1), the term \(yp (x, y)\) models the direct impact of predator on prey by catching and killing behavior. Meanwhile fear of predation risk can be regarded as the indirect impact of predator on prey, and some theoretical ecologists and biologists have realised that a prey-predator model should involve not only direct killing but also the fear [16, 17]. The fieldwork of Zanette et al. [18] on song sparrows observed the impact of fear and found a reduction in reproduction by 40% in the number of offspring due to the fear
of predation. Based on this phenomenon, Wang et al. [19] incorporated a predator-dependent fear factor into the birth rate of prey in model (1) (i.e., replace \( b(y) = b/(1 + ky) \)) with linear and Holling type-II functional response to explore the effect of fear on population dynamics. The results show that high level of fear could stabilize the system. Das and Samanta [20] investigated the impact of fear in exponential form on a stochastic prey-predator system when the predator is provided additional food. Sahoo and Samanta [21] investigated a two prey-one predator model by including the cost of fear into prey reproduction and switching mechanism in predation. Das et al. [22] developed and explored a predator-prey model incorporating the cost of perceived fear into the birth and death rates of prey species with Holling type-II functional response. Sarkar and Khajanchi [23] and Kumar and Kumari [24] incorporated a form of fear factor into the birth rate of prey by assuming a nonzero minimum cost of fear. The impact of fear has also been investigated on prey-predator systems with prey refuge [25–27], Allee effect [26], hunting cooperation [28], and additional food resource for predator [20, 29].

The study of resource management including fisheries, forestry, and wildlife management has great importance. It is necessary to harvest the population but harvesting should be regulated in such a way that ecological sustainability as well as conservation of the species can be implemented in a long run. Besides, it is always hoped that the sustained ability can be achieved at a high level of productivity and good economic profit. In the past decade, scholars considered different kinds of harvest on the dynamics of the predator-prey system such as continuous harvesting [30–33] and intermittent harvesting [34–37]. Compared to fixed time harvest, the state-dependent harvesting strategy takes the existing resources of species into full consideration and can maintain the sustainability of species in certain level. State-dependent harvested system can be described by the impulsive semidynamical system [38–45]. Recently, Lai et al. [46] proposed and studied a Lotka-Volterra predator-prey system incorporating both continuous harvesting and fear effect; that is,

\[
\begin{align*}
\frac{dx}{dt} &= \frac{bx}{1 + ky} - dx - cx^2 - pxy - \frac{qEx}{a_1E + a_2x}, \\
\frac{dy}{dt} &= epxy - my,
\end{align*}
\]

(2)

where \( k \) is the level of fear, \( E \) is the fishing effort used to harvest, \( q \) is the catchability coefficient, and \( a_1 \) and \( a_2 \) are constants. The harvest in (2) is continuous and in Michaelis-Menten type. However, in reality, the harvest of species should consider the aspect of ecological sustainability as well as conservation. Thus, in most cases, species are caught intermittently, not continuously.

To the best of our knowledge, to this day, still no scholars investigated the dynamic behavior of the predator-prey system incorporating both fear effect and intermittent harvesting, which motivated us to study a predator-prey model incorporating fear effect based on state-dependent harvesting strategy. The aim of this study is to check the influence of fear level on the stability of the positive steady state of the system without harvest. Meanwhile, for the harvest system, it mainly discusses the existence of the order-\( q \) (\( q > 0 \)) periodic solution, since it provides a possibility to transform the state-dependent harvesting into a cycle one. Meanwhile, in order to make a maximum economic profit in the harvest process, the optimal control problem is discussed. The organization of this study is as follows. In the next section, we introduce the mathematical model for predator-prey system with fear effect based on state-dependent harvesting strategy. In the same section, we present some preliminaries used in the discussion of the system dynamics. Section 3 is dedicated to the existence and stability of semitrivial order-1 and positive order-1 periodic solution. We also study the existence of order-2 and order-3 periodic solution. In Section 4, we demonstrate different effects of different harvest levels and different fear levels by depicting figures in numerical simulation using MATLAB. The paper concludes in Section 5, in which we briefly summarize the biological indications of our analytical findings.

2. Model Formulation and Preliminaries

2.1. Model Formulation. In presence of predator population, the prey population may significantly change their behavior. Fear for predator population enhances the survival probability of prey population, and it can greatly reduce the reproduction of prey population [23]. In this study, we consider a predator-prey model introducing the cost of fear into prey reproduction with Holling type-II functional response and a saturation function \( \phi \) in equation (1); that is,

\[
\begin{align*}
\frac{dx}{dt} &= \frac{bx}{1 + ky} - dx - cx^2 - pxy - \frac{qEx}{a_1E + a_2x}, \\
\frac{dy}{dt} &= epxy - my, \\
&\quad \text{where the variables, model parameters, and their units/dimensions are given in Table 1. To achieve the commercial purpose of the fishery, it is necessary to harvest the population in such a way that ecological sustainability as well as conservation of the species can be implemented in a long run. The harvest can be continuous or intermittent. In this work, a state-dependent harvest strategy is considered. Let } l \text{ be the harvest level of prey population; that is, when the density of prey population reaches level } l, \text{ the harvest is implemented, resulting in a portion of prey and predator being caught. Let } E \text{ denote the harvest effort, which is dependent on the harvest level } l, \text{ and let } q_1 \text{ and } q_2 \text{ be the catchability coefficients of prey and predator populations. In addition, to avoid the extinction of predator, it is necessary to release a quantity of predator pups, denoted by } r, \text{ which is also dependent on level } l. \text{ Based on this consideration, the model with state-dependent harvesting takes the following form:}
\end{align*}
\]
Table 1: The description of the model parameters and variables and their units/dimensions.

| Symbol | Description                                      | Units/dimensions |
|--------|--------------------------------------------------|------------------|
| $x$    | Density of prey population                       | Mass             |
| $y$    | Density of predator population                   | Mass             |
| $b$    | Birth rate of prey population not affected by predators | 1/time           |
| $d$    | Natural death rate of prey population            | 1/time           |
| $c$    | Decay rate due to intraspecies competition       | 1/mass.1/time    |
| $p$    | Rate of predation                                | 1/mass.1/time    |
| $h_1$  | Handling time                                    | 1/mass           |
| $e$    | Conversion rate of prey biomass to predator biomass | Dimensionless    |
| $k$    | Conversion time                                  | 1/mass           |
| $m$    | Death rate of predator                           | 1/time           |
| $l$    | Harvesting level of prey population              | Mass             |
| $E$    | Harvesting effort                                | Dimensionless    |
| $q_1$  | Catchability coefficient of prey                 | Dimensionless    |
| $q_2$  | Catchability coefficient of predator             | Dimensionless    |
| $\tau$ | Quantity of predator pups released               | Mass             |

\[
\begin{align*}
\frac{dx}{dt} &= \frac{bx}{1+ky} - cx^2 - \frac{pxy}{1+h_1x}, \quad x \neq l, \\
\frac{dy}{dt} &= \frac{epy}{1+(h_1+ph_1)x} - my, \\
\Delta x &= -q_1Ex, \\
\Delta y &= -q_2Ey + \tau,
\end{align*}
\]

Denote $K \triangleq (b-d)/c$. Then $K$ is the carrying capacity of prey population in absence of predator. System (4) is considered in the domain $S = \{(x,y)|0 \leq x \leq K, y \geq 0\}$ for ecological practices. The purpose of this paper is to analyze the dynamics of system (4). Besides, it is always hoped that the harvest can be achieved at a good economic profit, and this requires determining an optimal harvest level $l$. Next, some preliminaries are listed for the analysis of the harvest model (4).

Let $z(t) = (x(t), y(t))$ be the solution of system (5) with initial value $z(0) = z_0$. Denote $\gamma(z, z_0) = z(t)_{|t \geq t_0}$ with $z(t_0) = z_0$, also denoted as $\gamma(z)$ in short. Denote $z_k = z(t^*_k) \in \gamma(z)$, where $t^*_k \in \prod \{t_k|k = 1, 2, \ldots\}$ with $z(t_k) \in M_{\text{IMP}}$. Definition 1 (priodic solution [47–49]). The solution $z = \bar{z}(t)$ of system (5) is said to be periodic if there exists a positive integer $m \geq 1$ such that $\bar{z}_m = \bar{z}_0$. Denote $k \triangleq \min\{m \in \mathbb{N}, \bar{z}_m = \bar{z}_0\}$; then orbit $\gamma(\bar{z})$ is said to be an order-$k$ periodic orbit of system (5).

2.2. Preliminaries. Let us consider a general planar system:

\[
\begin{align*}
\frac{dx}{dt} &= P(x, y), \\
\frac{dy}{dt} &= Q(x, y), \\
\Delta x &= a(x, y), \\
\Delta y &= b(x, y),
\end{align*}
\]

where $(x, y) \in \Omega \subset \mathbb{R}^2$, and $\gamma(x, y) = 0$ describes the states at which the harvest is implemented; $a$ and $b$ describe the effects of the harvest strategy. $P(x, y)$ and $Q(x, y)$ are arbitrarily defined with respect to $(x, y) \in \Omega; \gamma, \alpha, \beta$ are linearly dependent on $x$ and $y$; that is, $\chi, x, y, \alpha, \beta$ are constant.

The dynamic system constituted by the solution mapping defined by system (5) is called an impulsive semi-continuous dynamic system, denoted as $(\Omega, \pi; I, M_{\text{IMP}})$, where $\pi = (\pi_1, \pi_2)$; $\Omega \times \mathbb{R} \rightarrow \Omega$, $M_{\text{IMP}} \triangleq \{(x, y)| \gamma(x, y) = 0\}$, and

\[
I: M_{\text{IMP}} \rightarrow N_{\text{PHA}} = I(M_{\text{IMP}}) = \{(x', y')|x' = x + \alpha(x, y), y' = y + \beta(x, y), (x, y) \in M_{\text{IMP}}\}.
\]

Definition 2 (orbitally stable [47–49]). An orbit $\bar{z}(t)$ is said to be orbitally stable if, for any $\varepsilon > 0$, there is a neighborhood $V$ of $\bar{z}$ so that, for all $z \in V$, there is a reparametrization of time (a smooth, monotonic function) $\bar{t}(t)$ such that $|z(t) - \bar{z}(\bar{t}(t))| < \varepsilon$ for all $t \geq t_0$. Definition 3 (asymptotic orbital stability [47–49]). $\gamma(\bar{z})$ is said to be asymptotically orbitally stable if it is orbitally stable and additionally $V$ may be chosen so that, for all $z \in V$, there is a constant $\tau(z)$ such that $|z(t) - \bar{z}(t - \tau(z))| \longrightarrow 0$ as $t \rightarrow \infty$. 

Theorem 1. There are three equilibria for system (3) where \( e > e_M \): two boundary saddles \((0,0)\) and \(E(K,0)\) and one positive equilibrium \(E^*(x^*,y^*)\). Moreover, one of the two following cases holds:

(i) \(E^*(x^*,y^*)\) is a stable focus or node in case of \( k > k^* \)

(ii) \(E^*(x^*,y^*)\) is unstable in case of \( k < k^* \), and a unique stable limit cycle exists, denoted by \( \Gamma_{1C} \)
It is obvious that $O(0, 0)$ and $E(K, 0)$ are saddles. At the equilibrium $E^*(x^*, y^*)$, the characteristic equation $\lambda^2 - T^* \lambda + D^* = 0$. If $y^* < y_{M} < a(1 + h_1 x^*)^2/\rho_1$, then $T^* < 0$. By equation (6), $y^* < y_{M}$ if and only if $K > k^*$. Thus, the positive equilibrium $E^*(x^*, y^*)$ is locally asymptotically stable in case of $K > k^*$ and unstable in case of $K < k^*$. In this case, there exists a unique stable limit cycle for system (3).

3.1. Semitrivial Order-1 Periodic Solution for $\tau = 0$. When $\tau = 0$, there is $y(t) \equiv 0$ for $t > 0$ with $y(0) = 0$. In this case, system (3) is reduced to the following system:

$$\begin{align*}
\frac{dx}{dt} &= (b - d)x\left(1 - \frac{x}{K}\right), \quad x \neq l, \\
\Delta x &= -q_1 E x, \quad x = l.
\end{align*}$$

(12)

which is orbitally asymptotically stable when $R_0 < 1$, where

$$R_0 \equiv \left(1 - q_1 E\right)\left(\frac{1 + (h_1 + h_2 p)}{1 + (1 - q_1 E)(h_1 + h_2 p)}\right)\left(\frac{K - (1 - q_1 E)l}{K - (1 - q_1 E)l}\right)^{\frac{1}{m(b - d)}}.$$

(16)

Proof. To discuss the stability of $(\xi(t), \eta(t))$, let us consider a small disturbance $\delta_0$. The trajectory starting from $B_1((1 - q_1 E)l, \delta_0)$ is denoted by $(\tilde{\xi}(t), \tilde{\eta}(t))$. This disturbed trajectory first intersects the harvest set $M_{\text{imp}}$ at point $B_2(l, \eta_1)$ when $t = T + \delta t$, and then it jumps to point $B_1^*((1 - q_1 E)l, \delta_1)$. Thus, there is

$$\tilde{\xi}(0) = (1 - q_1 E)l, \quad \tilde{\eta}(0) = \delta_0, \quad \tilde{\xi}(T + \delta t) = l, \quad \tilde{\eta}(T + \delta t) = \eta_1.$$  

(17)

Let $\delta x = \xi(t) - \overline{\xi}(t)$ and $\delta y = \eta(t) - \overline{\eta}(t)$. Then $\delta x_0 = \xi(0) - \overline{\xi}(0) = 0$, and $\delta y_0 = \eta(0) - \overline{\eta}(0) = \delta_0$. Setting $\delta y_1 = \eta_1$, for $0 < t < T$, the variables $\delta x$ and $\delta y$ can be expressed by the relation

$$\Phi_2(T) = \left(\frac{1 + (h_1 + h_2 p)}{1 + (1 - q_1 E)(h_1 + h_2 p)}\right)\left(\frac{K - (1 - q_1 E)l}{K - (1 - q_1 E)l}\right)^{\frac{1}{m(b - d)}}.$$  

(20)

Setting $x_0 = (1 - q_1 E)l$, the solution of equation $dx/dt = (b - d)x(1 - x/K)$ with $x(0) = x_0$ is

$$x(t) = \frac{K(1 - q_1 E)l \exp((b - d)t)}{(K - (1 - q_1 E)l) + (1 - q_1 E)l \exp((b - d)t)}.$$  

(13)

Let

$$T \equiv \frac{1}{\tau} \ln\left(\frac{K - (1 - q_1 E)l}{(1 - q_1 E)(K - l)}\right).$$  

(14)

Then there is $x(T) = l$ and $x(T^*) = (1 - q_1 E)l = x_0$ by impulse effect. Thus, the following result holds.

Theorem 2. System (4) with $\tau = 0$ has a semitrivial order-1 periodic solution for $(n - 1)T < t \leq nT$.
By impulse effect, there is $\delta_t = (1 - q_2 E) \delta y_1 = (1 - q_2 E) \phi_{22} (T) \delta_0$. Thus, if inequality (10) holds, there is $\delta < \delta_0$. By the arbitrary of $\delta_t$, it concludes that the order-1 semitrivial periodic solution is orbitally asymptotically stable.

**Corollary 1.** The semitrivial order-1 periodic solution $(x(t), y(t))$ is orbitally asymptotically stable if one of the two following cases is satisfied: (i) $l < x^*$ and (ii) $l > x^*$ and $E > E^* \equiv \max \{0, (\phi_{22} (T) - 1)/\phi_{22} (T) q_2\}$. 

\[ y^-_{L}(x) \equiv \frac{\sqrt{[p + k(d + cx)(1 + h_1 x)]^2 + 4pk(b - d - cx)(1 + h_1 x) - [p + k(d + cx)(1 + h_1 x)]}}{2pk} \]  

Let $N_0$ and $M_0$ denote the intersection point between $y = y^-_{L}(x)$ and the phase set $N_{\text{PHA}}$ and the harvest set $M_{\text{IMP}}$, respectively; $G_0$ denotes the intersection point between $y = \tau$ and the phase set $N_{\text{PHA}}$; in general $\tau \leq \tau_{\text{max}} \equiv y^-_{N_0} = y^-_{L}(1 - q_1 E)l$. For a point $S$ on $N_{\text{PHA}}$ with $0 \leq y_s \leq y_{N_0}$, if the trajectory of system (4) starting from $S$ intersects the harvest set $M_{\text{IMP}}$, then it defines a function relationship between $y$ and $x$ for $(1 - q_1 E)l \leq x \leq l$, denoted by $y = y(x, y_s)$, which satisfies  

\[ \frac{dy}{dx} = \frac{epx y/1 + (h_1 + ph_2) x - my}{bx/1 + ky - dx - cx^2 - pxy/1 + h_1 x} \equiv \kappa(x, y), \]  

\[ y((1 - q_1 E)l, y_s) = y_s. \]  

By equation (22), the function $y = y(x, y_s)$ can be expressed as follows:

\[ y(x, y_s) = y_s + \int_{(1 - q_1 E)l}^{x} \kappa(u, y(u, y_s))du. \]  

**Property 1.** For system (4), when $l \leq x^*$, the Poincaré map $\phi_N$ defined by equation (24) has the following properties:

(i) $\phi_N$ is continuous on $[0, +\infty)$. Moreover, $\phi_N$ is increasing on $[0, y^-_{L}(1 - q_1 E)l]$ and decreasing on $(y^-_{L}(1 - q_1 E)l, +\infty)$

(ii) $\phi_N$ is continuously differentiable on $[0, +\infty)$, and $\phi_N$ is concave on $[0, y^-_{L}(1 - q_1 E)l]]$

(iii) There exists a horizontal asymptote $\phi_N = \tau$; that is, $\phi_N(y_s) \to \tau$ when $y_s \to +\infty$

Define $f_N(y) \equiv \phi_N(y) - y$. The following result holds.

**3.2. Positive Order-K Periodic Solution for $\tau > 0$.** Since the harvest may cause the extinction of predator when $\tau = 0$, in order to keep the predator species from going extinct, it is necessary to reduce the harvest strength and release a certain quantity of predator pups.

For $0 \leq x \leq K$, define

\[ \phi_N(y_s) = \begin{cases} 
(1 - q_2 E)y(l, y_s) + \tau, & 0 \leq y_s \leq y_{N_0}, \\
(1 - q_2 E)y(l, \max_2 ((1 - q_1 E)l, y_s)) + \tau, & y_s > y_{N_0}. \end{cases} \]

**Theorem 3.** There exists a unique positive order-1 periodic solution for system (4) when $0 \leq l \leq x^*$ and $0 < \tau \leq \tau_{\text{max}}$

**Proof.** By Remark 1, the existence of order-1 periodic solution is equivalent to the existence of a point $L \in N_{\text{PHA}}$ such that $y_L$ is a fixed point of $\phi_N$. By Property 1 (i), $f_N$ is continuous on $[0, +\infty)$. Since $f_N(0) = \phi_N(0) = \tau > 0$ and $f_N(y) = \phi_N(y) - y \to -\infty$ as $y \to +\infty$, by the intermediary property of continuous function, there exists at least one $y_L > 0$ such that $f_N(y_L) = 0$, that is, $\phi_N(y_L) = y_L$. Thus, the trajectory of system (4)
starting from \( L((1 - q_1)E,y_L) \) forms an order-1 periodic orbit.

Next, the location and uniqueness of the order-1 periodic orbit will be analyzed. By Property 1 (i), \( \phi_N \) achieves its maximum at \( y = y_{N_0} = y_{T_1}^* (1 - q_1)E \). It is obvious that \( f_N (\tau) > 0 \).

If \( \tau = \tau_f \), then \( \phi_N (y_{N_0}) = y_{N_0} \).

If \( \tau > \tau_f \), then \( f_N (y_{N_0}) < 0 \), which means that \( y_L \in (\tau, y_{N_0}) \), as shown in Figure 1(a). Since \( \phi_N (y) \) is concave on \( [0, y_{N_0}] \), \( y_1 \) is unique.

If \( \tau > \tau_f \), there is \( f_N (y_{N_0}) > 0 \); that is, \( \phi_N (y_{N_0}) > y_{N_0} \). Since \( \phi_N \) is decreasing on \( [y_{N_0}, +\infty) \), \( \phi_N (y_{N_0}) < \phi_N (y_{N_0}) < 0 \). Besides, define \( y_1^+ = \max \{ y|\phi_N (y) = y_{N_0} \} \) and \( y_2^+ = \min \{ y|\phi_N (y) = y_{N_0} \} \). Denote \( y_{\min} \equiv \min \{ y^1, \phi_N (y) \} \). Then there is \( y_L \in (y_2^+ (1 - q_1)E)y_{N_0} \) and \( y_L \) is unique, as shown in Figure 1(b). \( \square \)

Case I: \( l > x^* \): in case of \( x^* < l \leq l \), the trajectory of system (4) starting from \( N_0 \) will intersect the harvest set \( M_{\text{IMP}} \). When \( l > l \), the trajectory starting from point \( N_0 \) does not intersect the harvest set \( M_{\text{IMP}} \). Denote \( y_{M_1} = \max \{ y|\phi_N (y) = y_{N_0} \} \) and \( y_{M_1} = \min \{ y|\phi_N (y) = y_{N_0} \} \). Then the domain of \( \phi_N \) is \( [0, y_{M_1}] \cup (y_{M_1}, +\infty) \). Define \( \tau_{M_1} \equiv y_{M_1} - (1 - q_2)E) \) and \( \tau_{M_2} \equiv y_{M_1} - (1 - q_2)E) \).

\[ \Theta (\eta_1) \equiv \ln \left( \frac{(1 - q_2)E)\eta_1 + \tau}{(1 - q_2)E} \right) \left[ \frac{(b/1 + k((1 - q_2)E)\eta_1 + \tau)}{(1 + h_1))} \right] - c(\xi (t)) \right) dt \leq \Theta (\eta_1). \] (26)

**Theorem 6.** The order-2T-periodic solution \( (\xi (t), \eta (t)) \) is orbitally asymptotically stable if

\[ \frac{\partial P}{\partial x} = \frac{b}{1 + ky} - d - cx^2 - \frac{pxy}{1 + h_1x}, \]

\[ \frac{\partial P}{\partial y} = \frac{epx}{1 + (h_1 + ph_2)x} - m, \]

\[ \frac{\partial Q}{\partial x} = 1, \]

\[ \frac{\partial Q}{\partial y} = 0, \]

\[ \frac{\partial \alpha}{\partial x} = -q_1E, \]

\[ \frac{\partial \alpha}{\partial y} = 0, \]

\[ \frac{\partial \beta}{\partial x} = 0, \]

\[ \frac{\partial \beta}{\partial y} = -q_2E. \]

With a direct calculation, there is

**Theorem 4.** There exists a positive order-1 periodic solution for system (4) when (i) \( l \leq \tau \in (0, \tau_{\text{max}}) \) or (ii) \( l > \tau \in (0, \tau_{\text{max}}) \) or \( \tau_{\text{max}} \).

**Proof.** When \( x^* < l \leq l \), similar to the proof of Theorem 3, system (4) admits an order-1 periodic solution. For \( l > l \), if \( y_{M_1} > (1 - q_2)E)\eta_1 \), then, for \( \tau \in (0, \tau_{\text{max}}) \), there is \( \phi_N (y_{M_1}) < \phi_N (y_{M_1}) \). Combining with \( \phi_N (y_{G_0}) > y_{G_0} \), it can be concluded that system (4) admits an order-1 periodic solution. For \( \tau \in (\tau_{\text{max}}) \), there is \( \phi_N (y_{M_1}) \geq y_{M_1} \) and \( \phi_N (y_{M_1}) < \phi_N (y_{M_2}) \); thus, there exists \( y_L \in [y_{M_2}, \phi_N (y_{M_1})] \) such that \( \phi_N (y_L) = y_L \); that is, system (4) admits an order-1 periodic solution. \( \square \)

Case II: \( k < k^* \): in this case, the trajectory of system (4) starting from \( N_0 \) will intersect the harvest set \( M_{\text{IMP}} \).

**Theorem 5.** There exists a positive order-1 periodic solution for system (4) when (i) \( l \leq \tau \in (0, \tau_{\text{max}}) \) or (ii) \( l > \tau \in (0, \tau_{\text{max}}) \). Moreover, the order-1 periodic solution is unique when \( \tau \leq l \).

3.2.2. Stability of the Order-1 Periodic Solution. Let \( (\xi (t), \eta (t)) (0 \leq t \leq T) \) be an order-1 periodic solution of system (4). Denote \( \xi_0 = \xi (0), \xi_1 = \xi (T), \eta_0 = \eta (0), \) and \( \eta_1 = \eta (T) \). Define

\[ \Theta (\eta_1) \equiv \ln \left( \frac{(1 - q_2)E)\eta_1 + \tau}{(1 - q_2)E} \right) \left[ \frac{(b/1 + k((1 - q_2)E)\eta_1 + \tau)}{(1 + h_1))} \right] - c(\xi (t)) \right) dt \leq \Theta (\eta_1). \] (25)
Thus, in case of (26), there is $\mu_1 < 1$; then, by Lemma 1, the order-1 $T$-periodic solution $(\xi(t), \eta(t))$ is orbitally asymptotically stable.

**Theorem 7.** For $l \leq x^*$, if $\tau \leq \tau_f$, the order-1 periodic solution for system (4) is globally orbitally asymptotically stable.

**Proof.** By Theorem 3, system (4) admits a unique order-1 periodic solution when $l \leq x^*$. If $\tau \leq \tau_f$, there exists a unique $y_L \in (r, y_{N_0})$ such that $\phi_N(y_L) = y_L$. Thus, for any $y_0$, a sequence $\{y_k\}_{k=1,2,\ldots}$ is obtained under $\phi_N$; that is, $y_k = \phi_N(y_{k-1})$. If $y_0 < y_L$, then $\{y_k\}$ is a monotonically increasing sequence with $y_k < y_L$, so the limit is $y_L$. Similarly, if $y_0 \in (y_L, y_{N_0})$, then $\{y_k\}$ is a monotonically decreasing sequence, and the limit is $y_L$. If $y_0 > y_{N_0}$, then $\phi_N(y_0) \in (0, y_{N_0})$; thus $\{y_k\}_{k=1,2,\ldots}$ is a monotonically bounded sequence with limit $y_L$. To sum up, by the arbitrariness of $y_0$, the order-1 periodic solution is globally attractive and so is globally orbitally asymptotically stable.

3.2.3. Existence of Order-$q$ $(q \geq 2)$ Periodic Solution. For $l \leq l^*$, by Theorem 3, if $\tau \leq \tau_f$, the order-1 periodic solution is orbitally asymptotically stable and globally attractive, which means that system (4) does not admit order-$q$ $(q \geq 2)$ periodic solution. For $\tau > \tau_f$, there exists unique $y_L \in (y_{N_0}, \phi_N(y_{N_0}))$ such that $\phi_N(y_L) = y_L$. Let $y_{L_1} \in (0, y_{N_0})$ such that $\phi_N(y_{L_1}) = y_{L_1}$. Then $\phi_N(y_{L_1}) = y_{L_1}$. Meanwhile, let $y_{N_1} \in (0, y_{L_1})$ and $y_{N_2} \in (y_{L_1}, +\infty)$ such that $\phi_N(y_{N_1}) = \phi_N(y_{N_2}) = y_{N_1}$.

**Theorem 8.** For $l \leq l^*$ and $\tau > \tau_f$, if (i) $\phi_N^2(y_{N_1}) < y_{N_1}$, or (ii) $\phi_N^3(y_{N_1}) \geq y_{N_1}$ and $\mu_2 > 1$ holds, system (4) admits an order-2 periodic solution.

**Proof.** Obviously, $\phi_N^2(y_{N_1}) = \phi_N(y_{N_1}) = \phi_N(y_{N_1})$. It can be easily checked that $\phi_N^2(y)$ is increasing on $[0, y_{N_0}]$ and $[y_{N_0}, y_{N_1}]$, and $\phi_N^3(y)$ is decreasing on $[y_{N_1}, y_{N_1}]$ and $[y_{N_1}, +\infty)$. 

(i) $\phi_N^2(y_{N_1}) < y_{N_1}$. In this case, there is $\phi_N(y_{N_1}) > y_{N_1}$; that is, $\phi_N^2(y_{N_1}) > y_{N_1}$; then $\phi_N^3(\phi_N(y_{N_1})) < \phi_N(y_{N_1})$. Besides, there is $\phi_N^3(y_{L_1}) = y_{L_1} > y_{L_1}$. Thus,
there exist $y_P \in (y_L, y_H)$ and $y_P \in [y_N, \phi_N(y_{N0})]$ such that $\phi_N(y_P) = y_P$, and $\phi_N(y_P) = y_P$. Moreover, there are $\phi_N(y_P) = y_P$, and $\phi_N(y_P) = y_P$.

(ii) $\phi_N(y_{N0}) \geq y_{N0}$. In this case, there is $\phi_N(y_{N0}) \geq y_{N0}$; that is, $\phi_N(y_{N0}) \leq y_{N0}$. For any $y \in [y_{N0}, \phi_N(y_{N0})]$, there is $y_{N0} \leq \phi_N(y_{N0}) \leq \phi_N(y_{N0}) \leq \phi_N(y_{N0})$. Next, it mainly discusses the property of $\phi_N$ on $[y_{N0}, \phi_N(y_{N0})]$. Let $y_0 = y_{N0}$. Then $y_1 = \phi_N(y_0) = \phi_N(y_{N0}) > y_{N0}$. $y_2 = \phi_N(y_1) = \phi_N(y_{N0}) > y_{N0}$, and $y_3 = \phi_N(y_2) = \phi_N(y_{N0}) = y_1$. Then, under $\phi_N$, a sequence $\{y_k\}$ is obtained, where

$$y_0 < y_2 < y_4 < \cdots < y_{2k} < \cdots < y_{2} < y_{3} < y_{1}.$$  \hspace{1cm} (31)

Denote $y_{P1} = \lim_{k \to \infty} y_{2k}$ and $y_{P2} = \lim_{k \to \infty} y_{2k+1}$. It is obvious that $y_{P1} \leq y_{P2} \leq y_{P2}$. Since $\mu_2 > 1$, then $y_{P1} < y_{P2} \leq y_{P2}$. Moreover, there is $\phi_N(y_{P1}) = y_{P1}$, and $\phi_N(y_{P2}) = y_{P2}$. It can be concluded that $\mu_2 < 1$, that is, the order-2 periodic solution is orbitally asymptotically stable and globally attractive.

**Theorem 9.** For $l \leq \text{land} > \tau_f$, system (4) admits an order-3 periodic solution if and only if $T_1(y_{N0}) < y_{N0}$. Moreover, there is at least one order-3 periodic solution when $\phi_N(y_{N0}) = y_{N0}$, and there are at least two order-3 periodic solutions when $\phi_N(y_{N0}) < y_{N0}$.

**Proof.** “Necessity.” Proof by contradiction. Assume that $\phi_N(y_{N0}) > y_{N0}$. Since $\phi_N(y_{N0}) < y_{L1}$, if $\phi_N(y_{N0}) \geq y_{N0}$, system (4) admits a stable order-1 periodic solution or a stable order-2 periodic solution. Moreover, there is $\phi_N(y_{N0}) \leq y_{N0}$. If $\tau \leq y_{N0}$, that is, $\phi_N(y_{N0}) \leq y_{N0}$. If $\tau \geq y_{N0}$, there exist $y_{R1} \in [0, y_{N0}]$ and $y_{R2} \in (y_{N0}, +\infty)$ such that $\phi_N(y_{R1}) = \phi_N(y_{R2}) = y_{N0}$. It can be easily checked that $\phi_N$ is increasing on $[0, y_{R1}]$, $[y_{R1}, y_{N0}]$, and $[y_{N0}, y_{R2}]$, and $\phi_N$ is decreasing on $[y_{R2}, y_{N0}]$, $[y_{N0}, y_{R1}]$, and $[y_{R1}, +\infty)$. If $\tau > y_{N0}$, then $R_1$ does not exist, $\phi_N$ is increasing on $[y_{N0}, y_{R2}]$ and $[y_{N0}, y_{R2}]$, and $\phi_N$ is decreasing on $[0, y_{N0}]$, $[y_{N0}, y_{R2}]$, and $[y_{R2}, +\infty)$. In any case, since $\phi_N(y_{N0}) = \phi_N(y_{N0}) \geq y_{N0} \geq y_{N0}$, $\phi_N(y_{N0}) = \phi_N(y_{N0}) = y_{N0}$, and $\phi_N(y_{R2}) = \phi_N(y_{R2}) < y_{R2}$, it
Figure 4: Illustration of the Poincaré map $\phi_N$ of system (4) for $\tau_2 = 2$ and different harvest level $l$ with parameters given in numerical section: (a) $l = 30\%K$; (b) $l = 50\%K$; (c) $l = 60\%K$. The dotted line represents $r = y$ and the intersection point is the fixed point of the Poincaré map $\phi_N$. For $l = 30\%K$ and $l = 50\%K$, the Poincaré map $\phi_N$ has a unique fixed point; that is, system (4) admits a unique order-1 periodic solution. For $l = 60\%K$, the Poincaré map $\phi_N$ does not have fixed point; that is, system (4) does not admit order-1 periodic solution.

Figure 5: The time series evolution of prey population $x(t)$, predator population $y(t)$, and the phase portrait diagram of model (4) for $l = 60\%K$ and $\tau_2 = 2$ with parameters given in numerical section. The phase portrait diagram shows that the trajectory of system (4) will eventually tend to the positive equilibrium $E^*$. In this case, the positive equilibrium $E^*$ is globally asymptotically stable.

Figure 6: The phase portrait diagram of model (4) for $\tau_2 = 9$ and different harvest level $l$ with parameters given in numerical section: (a) $l = 60\%K$; (b) $l = 80\%K$. For $l = 60\%K$, there is $\tau_2 > \tau_{M1}$, and system (4) admits an order-1 periodic solution; for $l = 80\%K$, there is $\tau_2 \in (\tau_{M1}, \tau_{M2})$, and the trajectory of system (4) will tend to the positive equilibrium $E^*$ after finite impulses.
Figure 7: Illustration of the successor function $f_N(y) = \phi_N(y) - y$ for $q_1 = 0$ with parameters given in numerical section. The result indicates that the function is always greater than zero, which means that the Poincaré map $\phi_N$ of system (4) does not have fixed point and the trajectory will tend to the positive equilibrium $E^*$.

Figure 8: Illustration of the Poincaré maps $\phi_N$ and $\phi_N^2$ of system (4) for $l = 25\% K$ and $\tau_2 = 8$ and different values of $q_1$ with parameters given in numerical section. For $q_1 = 0.8$, system (4) admits a unique orbitally asymptotically stable order-1 periodic solution; for $q_1 = 0.5$, there is $\phi_N^2(4.92) > 4.92$ and $\mu_1 > 1$; in this case, system (4) admits a stable order-2 periodic solution; (c) for $q_1 = 0.2$, there is $\phi_N^2(4.92) < 4.92$, and system (4) admits a stable order-2 periodic solution.

Figure 9: Illustration of the Poincaré map $\phi_N$ of system (4) for $l = 25\% K$ and different values of $\tau_2$ with parameters given in numerical section. The results indicate that $\phi_N$ has a unique fixed point for any $\tau_2$; that is, system (4) admits a unique order-1 periodic solution.
can be concluded that \( \phi_3^N(y) = y \) if and only if \( y = y_{L2} \); that is, the order-3 periodic solution does not exist.

If \( y_{N1} < \phi_2^N(y_{N0}) < y_{N0} \), system (4) simultaneously admits an order-1 periodic solution and order-2 periodic solution. Moreover, there is \( \phi_N(y_{N1}) > y_{N2} \). If \( r \leq y_{N1} \), that is, \( \phi_N(r) \leq y_{N1} \), there exist \( y_{R1} \in [y_{N1}, y_{N2}] \), \( y_{R2} \in (y_{L1}, y_{N2}) \), \( y_{R3} \in (y_{N2}, y_{L2}) \), and \( y_{R4} \in (y_{N2}, +\infty) \) such that \( \phi_N(y_{R1}) = \phi_N^2(y_{R2}) = \phi_N^3(y_{R3}) = \phi_N^4(y_{R4}) = y_{N2} \). It can be
easily checked that $\phi_N^2$ is increasing on $[0, y_R]$, $[y_N, y_R]$, $[y_N, y_R]$, and $[y_N, y_R]$, and $\phi_N^3$ is decreasing on $[y_R, y_N]$, $[y_R, y_N]$, $[y_R, y_N]$, and $[y_R, +\infty)$. If $\tau > y_N$, then $R_1$ does not exist, $\phi_N$ is increasing on $[y_N, y_R]$, $[y_N, y_R]$, and $[y_N, y_R]$, and $\phi_N$ is decreasing on $[0, y_N]$, $[y_R, y_N]$, $[y_R, y_N]$, and $[y_R, +\infty)$. Since $\phi_N^3(y_N) = \phi_N^2(y_N) > y_N$, and $\phi_N^2(y_R) = \phi_N(y_N) < y_R$, it can be concluded that $\phi_N^2(y) = y$ if and only if $y = y_L$; that is, the order-3 periodic solution does not exist.

“Sufficiency.” If $\phi_N^2(y_N) \leq y_N$, then there is $\phi_N(y_R) = \phi_N(y_R) > 0$ and $\phi_N^2(y_N) = \phi_N^2(y_N) \leq y_N$. If $\phi_N(y_N) = y_N$, then there exists an order-3 periodic solution since $\phi_N(y_N) \neq y_N$. If $\phi_N(y_N) < y_N$, then there exists at least one $y_f \in (y_R, y_N)$ such that $\phi_N^3(y_f) = y_f$ and $\phi_N^3(y_f) > y_f$; that is, system (4) admits an order-3 periodic solution.

Next, the number of order-3 periodic solutions will be discussed:

(i) When $\phi_N^3(y_N) = y_N$, there are $\phi_N^3(y_N) = y_N$, $\phi_N^3(y_N) = y_N$, and $\phi_N^3(\phi_N(y_N)) = \phi_N(y_N)$. Moreover, there are $y_N = \phi_N(y_N)$ and $y_N = \phi_N(y_N)$; that is, system (4) admits at least one order-3 periodic solution.

(ii) When $\phi_N^3(y_N) < y_N$, there are $\phi_N^3(y_N) = \phi_N^3(y_N) = y_N$, $\phi_N^3(y_N) = y_N$, and $\phi_N^3(\phi_N(y_N)) = \phi_N(y_N)$. Moreover, there are $y_N = \phi_N(y_N)$ and $y_N = \phi_N(y_N)$; that is, system (4) admits at least two order-3 periodic solutions.

3.3. Optimal Harvest Level Determination. To achieve the commercial purpose of the fishery, it is necessary to harvest the population, and it is always hoped that the sustained ability can be achieved at a good economic profit. For the harvest problem, it is necessary to determine the controlled values $E$ and $\tau$ and harvest level $l$, and this in general involves the optimization theory [48, 49].

Let $l$ be the harvest level, which is a decision variable. Theorems 3 and 4 show that system (4) admits an order-1 periodic solution when $l \leq \overline{l}$ or $l > \overline{l}$ with $\tau \leq y_M - (1 - q_1)Ey_L(l)$. Since the harvest effort and yield of released
predator are dependent on the harvest level, then it is assumed that \( E(l) \) and \( \tau(l) \) take the following forms:

\[
E(l) = E_1 + (E_2 - E_1) \frac{l - l_1}{l_2 - l_1},
\]

\[
\tau(l) = \tau_1 + (\tau_2 - \tau_1) \frac{l - l_1}{l_2 - l_1},
\]

where \( l_1 \leq l \leq l_2 \), \( l_1 \) and \( l_2 \) are minimum and maximum of the harvest level, and \( E_1 \) (\( E_2 \)) and \( \tau_1 \) (\( \tau_2 \)) are the harvest effort and yield of released predator at the harvest level \( l_1 \) (\( l_2 \)).

Let \( c_1 \) and \( c_2 \) be the unit selling prices of prey and predator, let \( c_3 \) be the unit cost of harvest, and let \( c_4 \) be the unit cost in breeding predator. Then the benefits from harvest can be described as \( F_{\text{benefit}}(l) = c_1 q_1 E(l) + c_2 q_2 E(l) \eta(T(l)) - c_3 E^2(l) - c_4 \tau(l) \). The objective is to maximize the unit benefits; that is,

\[
\max \frac{F_{\text{benefit}}(l)}{T(l)} \quad \text{such that} \quad l_1 \leq l \leq l_2,
\]

\[
l \leq yM_1 - (1 - q_1 E) y^*_M(l).
\]

4. Numerical Simulations and Optimization

In this section, we compute some numerical simulations regarding the existence and stability of the periodic solution for the predator-prey model (4). It is quite difficult to verify the mathematical model simulations with realistic parameter values. We take a hypothetical set of parameter values to illustrate our analytical findings. The model parameters are as follows: \( b = 0.7 \), \( d = 0.2 \), \( c = 0.005 \), \( p = 0.1 \), \( h_1 = 0.036 \), \( h_2 = 1.44 \), \( e = 0.44 \), and \( m = 0.2 \). The control parameters are as follows: \( E_1 = 0.2 \), \( E_2 = 1 \), and \( \tau_1 = 0 \).

4.1. Numerical Simulations. Since \( e > e_M \), the boundary equilibrium \( E(100, 0) \) is unstable, and system (3) has a positive equilibrium \( E^* \). From equation (6), it can be observed that the fear effect factor \( k \) only affects the value \( y^* \). Figure 2 illustrates the dependence of isoline \( dx/dt = 0 \) on \( x \) for different fear level \( k \). As illustrated, the positive equilibrium becomes stable from unstable with increasing of \( k \). By equation (6), there is \( k^* = 0.0376 \). To verify the theoretical results obtained in the above section, the simulations are implemented by considering different combinations of \( k \), \( q_1 \), \( q_2 \), \( \tau_2 \), and \( l \).
It can be observed that system (4) admits an order-1 periodic solution for any \( \tau \) value of \( \phi_N(y) \) in the positive equilibrium. Meanwhile, for \( \tau = 0 \), the semitrivial order-1 periodic solution is stable, as presented in Figure 3.

Figure 14: Illustration of function \( \phi_N(y) \) of system (4) for \( I = 50\%K \) and \( \tau = 3.6 \) with parameters given in numerical section. The time series evolution of prey population \( x(t) \), predator population \( y(t) \), and the phase portrait diagram demonstrate the order-1 periodic solution.

Case I: \( k = 0.04 \), \( q_1 = 0.8 \), and \( q_2 = 0.6 \).

I-(1): \( \tau_2 = 0 \). By Theorem 2, system (4) admits an order-1 semitrivial periodic solution for any \( I \leq K = 100 \), which is expressed by equation (10). Moreover, for \( I \leq l = 34.45 \), by Corollary 1, the semitrivial order-1 periodic solution is orbitally asymptotically stable. Meanwhile, for \( I = 50\%K \), by Theorem 2, there is \( R_0 = 0.5673 < 1 \); that is, the order-1 semitrivial periodic solution is orbitally asymptotically stable, as presented in Figure 3.

I-(2): \( \tau_2 > 0 \). Firstly, for \( \tau_2 = 2 \), there is \( l = 34.45\%K \). For \( I = 25\%K \leq x^* \), Theorem 4 and Theorem 7 indicate that system (4) admits a globally asymptotically stable positive order-1 periodic solution, as illustrated in Figure 1. Meanwhile, for \( I > x^* \), function \( \phi_N \) of system (4) for \( I = 30\%K \), \( I = 50\%K \), and \( I = 60\%K \) is presented in Figure 4. It can be observed that system (4) admits an order-1 periodic solution for \( I = 30\%K \) and \( I = 50\%K \) since the inequality \( \tau < y_M - (1 - q_2E)y_L(l) \) in Theorem 4 holds. When \( I = 60\%K \), the direction of the inequality has changed, and the trajectory of system (4) will tend to the positive equilibrium \( E^* (25, 4.93) \) after finite impulses, as shown in Figure 5. But this is not always true; it is dependent on the value of \( \tau_2 \). As \( \tau_2 \) goes up to 9, the inequality \( \tau > y_M - (1 - q_2E)y_L(l) \) in Theorem 4 holds; then system (4) admits a positive order-1 periodic solution, as shown in Figure 6(a). However, for \( I = 80\%K \), the direction of the inequality is changed again, and the trajectory of system (4) will tend to the positive equilibrium \( E^* (25, 4.93) \) after finite impulses, as shown in Figure 6(b).

Case II. \( k = 0.04 \) and \( q_2 = 0 \).

II-(1): \( \tau_2 = 0 \) and \( q_1 = 0.8 \). Notice from Figure 1 that a higher catching rate for predators (e.g., \( q_2 = 0.6 \)) will cause the predator species extinction for \( I = 50\%K \). When the catch for the predator is very small or ignored, that is, \( q_2 = 0 \), the order-1 semitrivial periodic solution is unstable by Corollary 1 (i.e., \( R_0 = 1.02 > 1 \)). The function \( f_N(y) = \phi_N(y) - y \) is presented in Figure 7 and it can be observed that \( f_N(y) > 0 \), and, for any initial condition, the trajectory of system (4) will eventually tend to the positive equilibrium \( E^* (25, 4.93) \).

II-(2): \( \tau_2 > 0 \). For \( \tau_2 = 8 \) and \( q_1 = 0.8 \), by Theorem 3, system (4) admits a unique positive order-1 periodic solution for \( I = 25\%K \). To show the existence of order-2 periodic solution, the catching rate for prey \( q_1 \) is selected as a key parameter to verify how does the dynamic behavior of the system change. Function \( \phi_N \) for different catching rate for prey is presented in Figure 8. It can be observed that system (4) admits a unique globally asymptotically stable order-1 periodic solution for a higher catching rate for prey, for example, \( q_1 = 0.8 \), as shown in Figure 8(a). As the catching rate for prey goes down, for example \( q_1 = 0.5 \), condition (ii) \( \phi^*_N(y_{N_0}) \geq y_{N_0} \) and \( \mu_2 = \)
Figure 15: Illustration of functions $\phi^l_N(y)$ and $\phi_N(y)$ of system (4) for $l = 50\%K$ and $\tau_e = 4.2$ with parameters given in numerical section. The time series evolution of prey population $x(t)$, predator population $y(t)$, and the phase portrait diagram demonstrate the order-4 periodic solution.

Figure 16: The dependence of period $T$ and benefit $F_{\text{benefit}}$ on the harvest level $l$ for $k = 0.04$ with parameters given in numerical section.
1.2612 > 1 in Theorem 8 holds; then system (4) admits a stable order-2 periodic solution, as shown in Figure 8(b). Meanwhile, for \( q_1 = 0.2 \), condition (i) \( \phi_3^2(y_N) < y_N \) in Theorem 8 holds, and then system (4) admits an order-2 periodic solution, as shown in Figure 8(c).

Case III: \( k = 0.01 \) and \( q_1 = q_2 = 0.6 \). The positive equilibrium \( E^* \) becomes unstable when \( k = 0.01 \), and system (3) admits a limit cycle \( l_{IC} \). Since the existence and stability of semitrivial order-1 periodic solution for \( \tau = 0 \) do not depend on the fear factor \( k \), the results are the same as those in Case I-(1) and are omitted hereby. So it mainly discusses the dynamic behavior of system (4) for \( \tau_2 > 0 \). It is easily checked that \( l = 32.72\% \).

Firstly, for \( l = 25\% K \leq x^* \), by Theorem 3, system (4) admits a positive order-1 periodic solution for any \( \tau \), as shown in Figure 9.

Next, let us consider \( l = 50\% K > l \). It is easily checked that \( \tau_M = 1.2495 \) (i.e., \( \tau_2 = 2.1867 \)); then, by Theorem 4, there exists an order-1 periodic solution for system (4) when \( \tau_2 \leq 2.1867 \). Here it should be pointed out that the condition given in Theorem 4 is only a sufficient one; in fact, as long as \( \tau_2 \leq 2.55 \), system (4) admits an order-1 periodic solution, as illustrated in Figure 10.

With an increase of \( \tau_2 \), the existence of order-1 periodic solution cannot be guaranteed. For example, for any \( \tau_2 \in [2.56, 3.58] \), there does not exist order-1 periodic solution for system (4). System (4) admits an order-3 periodic solution for \( \tau_2 = 2.9 \) (Figure 11), an order-5 periodic solution for \( \tau_2 = 3 \) (Figure 12), an order-2 periodic solution for \( \tau_2 = 3.1 \) (Figure 13), an order-1 periodic solution for \( \tau_2 = 3.6 \) (Figure 14), and an order-4 periodic solution for \( \tau_2 = 4.2 \) (Figure 15).

4.2. Optimization. To achieve a good economic profit, it is necessary to find a level \( I^* \) at which the benefits from harvest are maximal. Assume that \( c_1 = 100, c_2 = 5c_1 = 500, \) and \( c_3 = 20\% c_2 = 100 \). Denote \( \sigma = c_3/c_1 \). In order to sustain the harvest, the releasing yield of predator should not be too large, so, in this part, it is assumed that \( \tau_2 = 1 \).

For \( k = 0.04 \), system (4) admits an order-1 periodic solution when \( I \leq 70\% K \); the dependencies of period \( T \) and benefit \( F_{benefit} \) on the harvest level \( I \) are presented in Figure 16. It can be seen that period \( T \) increases as \( I \) increases. Meanwhile the benefit function \( F_{benefit} \) climbs up and then declines as \( I \) increases. For different \( \sigma \), \( F_{benefit} \) achieves its maximum at different \( I_{\sigma}^* \). When the cost of harvest is ignored, that is, \( \sigma = 0 \), there is \( I_{\sigma}^* = 40\% K \). As \( \sigma \) goes up, \( F_{benefit} \) goes down. When \( \sigma = 8 \), \( F_{benefit} \) achieves its maximum at \( I_{\sigma}^* = 50\% K \).

For the case of \( k = 0.01 \), system (4) admits an order-1 periodic solution when \( I \leq I_2 \), and the dependencies of period \( T \) and benefit \( F_{benefit} \) on the harvest level \( I \) are presented in Figure 17. The benefit function \( F_{benefit} \) climbs up and then declines as \( I \) increases. For different \( \sigma \), \( F_{benefit} \) almost achieves its maximum at \( I_{\sigma}^* = 50\% K \).

5. Conclusion and Discussions

In this paper, we have discussed the dynamics of a harvested prey-predator model, where the prey is provided with fear effect. For the system without harvest (3), there exists a unique positive equilibrium. To verify the stability of the equilibrium, a critical level of fear factor \( k^* \) is characterized (i.e., equation (6)). When the impact of fear on prey is small, that is, \( 0 \leq k < k^* \), the positive equilibrium is unstable and a limit cycle exists. As the impact of fear grows and exceeds \( k^* \), the positive equilibrium becomes stable and the limit cycle disappears (Figure 2). In any case, predators coexist with prey and the system is persistent.

For the system with harvest (4), if we do not consider the release of predator (i.e., \( \tau = 0 \), system (4) admits a semitrivial order-1 periodic solution for any harvest level (Figure 3). Moreover, the semitrivial order-1 periodic solution is orbitally stable when the harvest level is not higher than the first equilibrium component (i.e., \( I \leq x^* \)). Meanwhile, for the case of \( I > x^* \), the semitrivial order-1 periodic solution is orbitally stable when a strong harvest intensity is implemented. This means that the system can be disrupted, and
predators will go extinct if the harvest is not properly planned. To maintain the ecological health and avoid the extinction of predator populations, it is necessary to reduce the catch rate of predators (Figure 7) or release a certain quantity of predator pups. In the second case, that is, \( r > 0 \), system (4) admits a positive order-1 periodic solution when \( I \leq I \) (Figures 1, 4, and 9). Moreover, the order-1 periodic solution is orbitally asymptotically stable and globally attractive when \( I \leq x^* \) and \( \tau \leq \tau_f \). Meanwhile, for \( r > r_f \), system (4) admits an order-2 periodic solution in case of \( \phi_{N}^{\max} (y_{N_1}) \geq y_{N_1} \), and \( p_1^2 > 1 \) (Figure 8). In case of \( I > I_0 \) system (4) also admits a positive order-1 periodic solution for \( \tau \leq \tau_{M_1} \) (Figures 4 and 10) or \( \tau \geq \tau_{M_2} \) (Figure 6). But, for \( \tau \in (\tau_{M_1}, \tau_{M_2}) \), when \( k > k^* \), the trajectory of system (4) will tend to the positive equilibrium \( E^* (25, 4.93) \) after finite impulses (Figure 5). Meanwhile, in case of \( k < k^* \), the dynamic behavior of system (4) depends heavily on parameter \( \tau \). For different value of \( \tau \), system (4) may admit an order-\( k = 1, 2, 3, 4, 5 \) periodic solution (Figures 11–15).

To achieve a good economic profit, optimization with \( \tau^* = 1 \) is carried out and the results show that the benefits from harvest depend on the unit selling prices of prey and predator, as well as the unit cost of harvest. For given \( c_1 = 100, c_2 = 500, \) and \( c_4 = 100, \) the benefit function first climbs up and then declines as \( I \) increases. For \( k = 0.04, \) the economic profit \( F_{\text{benefit}} \) achieves its maximum at \( I^* = 40\%K \) when \( \sigma = 2 \). As \( \sigma \) goes up, \( F_{\text{benefit}} \) goes down, and \( F_{\text{benefit}} \) achieves its maximum at \( I^* = 50\%K \) when \( \sigma = 8 \). Meanwhile, in case of \( k = 0.01, \) \( F_{\text{benefit}} \) almost achieves its maximum at \( I^* = 50\%K \).

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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