ON DE GIORGI CONJECTURE IN DIMENSION $N \geq 9$

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Abstract. A celebrated conjecture due to De Giorgi states that any bounded solution of the equation $\Delta u + (1 - u^2)u = 0$ in $\mathbb{R}^N$ with $\partial_y N u > 0$ must be such that its level sets $\{u = \lambda\}$ are all hyperplanes, at least for dimension $N \leq 8$. A counterexample for $N \geq 9$ has long been believed to exist. Based on a minimal graph $\Gamma$ which is not a hyperplane, found by Bombieri, De Giorgi and Giusti in $\mathbb{R}^N$, $N \geq 9$, we prove that for any small $\alpha > 0$ there is a bounded solution $u_\alpha(y)$ with $\partial_y N u_\alpha > 0$, which resembles $\tanh \left( \frac{t}{\sqrt{2}} \right)$, where $t = t(y)$ denotes a choice of signed distance to the blown-up minimal graph $\Gamma_\alpha := \alpha^{-1}\Gamma$. This solution constitutes a counterexample to De Giorgi conjecture for $N \geq 9$.

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1. Introduction

This paper deals with entire solutions of the Allen-Cahn equation
\begin{equation}
\Delta u + (1-u^2)u = 0 \quad \text{in } \mathbb{R}^N.
\end{equation}
Equation (1.1) arises in the gradient theory of phase transitions by Cahn-Hilliard and Allen-Cahn, in connection with the energy functional in bounded domains \( \Omega \)
\begin{equation}
J_\varepsilon(u) = \varepsilon^2 \int_\Omega |\nabla u|^2 + \frac{1}{4\varepsilon} \int_\Omega (1-u^2)^2, \quad \varepsilon > 0
\end{equation}
whose Euler-Lagrange equation corresponds precisely to a \( \varepsilon \)-scaling of equation (1.1) in the expanding domain \( \varepsilon^{-1}\Omega \). The theory of \( \Gamma \)-convergence developed in the 70s and 80s, showed a deep connection between this problem and the theory of minimal surfaces, see Modica, Mortola, Kohn, Sternberg, [18, 23, 24, 25, 31]. In fact, it is known that for a family \( u_\varepsilon \) of local minimizers of \( u_\varepsilon \) with uniformly bounded energy must converge, up to subsequences, in \( L^1 \)-sense to a function of the form \( \chi_E - \chi_{E^c} \) where \( \chi \) denotes characteristic function, and \( \partial E \) has minimal perimeter. Thus the interface between the stable phases \( u = 1 \) and \( u = -1 \), represented by the sets \( \{u_\varepsilon = \lambda\} \) with \( |\lambda| < 1 \) approach a minimal hypersurface, see Caffarelli and C´ordoba [7, 8] (also R¨oger and Tonegawa [27]) for stronger convergence and uniform regularity results on these level surfaces.

The above described connection led E. De Giorgi [9] to formulate in 1978 the following celebrated conjecture concerning entire solutions of equation (1.1).

De Giorgi’s Conjecture: Let \( u \) be a bounded solution of equation (1.1) such that \( \partial_{\varepsilon_N} u > 0 \). Then the level sets \( \{u = \lambda\} \) are all hyperplanes, at least for dimension \( N \leq 8 \).

Equivalently, \( u \) depends on just one Euclidean variable so that it must have the form
\begin{equation}
u(x) = \tanh \left( \frac{x \cdot a - b}{\sqrt{2}} \right),
\end{equation}
for some \( b \in \mathbb{R} \) and some \( a \) with \( |a| = 1 \) and \( a_N > 0 \). We observe that the function \( w(t) = \tanh \left( t/\sqrt{2} \right) \) is the unique solution of the one-dimensional problem,
\[w'' + (1-w^2)w = 0, \quad w(0) = 0 \quad w(\pm\infty) = \pm 1.\]

The monotonicity assumption in \( u \) makes the scalings \( u(x/\varepsilon) \) local minimizers in suitable sense of \( J_\varepsilon \), moreover the level sets of \( u \) are all graphs. In this setting, De Giorgi’s conjecture is a natural, parallel statement to Bernstein theorem for minimal graphs, which in its most general form, due to Simons [30], states that any minimal hypersurface in \( \mathbb{R}^N \), which is also a graph of a function of \( N - 1 \) variables, must be a hyperplane if \( N \leq 8 \). Strikingly, Bombieri, De Giorgi and Giusti [5]
proved that this fact is false in dimension $N \geq 9$. This was most certainly the reason for the particle at least in De Giorgi’s statement.

Great advance in De Giorgi conjecture has been achieved in recent years, having been fully established in dimensions $N = 2$ by Ghoussoub and Gui [15] and for $N = 3$ by Ambrosio and Cabrè [2]. More recently Savin [28] established its validity for $4 \leq N \leq 8$ under the following additional assumption (see [1] for a discussion of this condition)

\begin{equation}
\lim_{x_N \to \pm \infty} u(x', x_N) = \pm 1.
\end{equation}

Condition (1.4) is related to the so-called Gibbons’ Conjecture:

**Gibbons’ Conjecture:** Let $u$ be a bounded solution of equation (1.1) satisfying

\begin{equation}
\lim_{x_N \to \pm \infty} u(x', x_N) = \pm 1, \text{ uniformly in } x'.
\end{equation}

Then the level sets $\{u = \lambda\}$ are all hyperplanes.

Gibbons’ Conjecture has been proved in all dimensions with different methods by Caffarelli and Córdoba [8], Farina [13], Barlow, Bass and Gui [3], and Berestycki, Hamel, and Monneau [4]. In references [8, 3] it is proven that the conjecture is true for any solution that has one level set which is a globally Lipschitz graph. If the uniformity in (1.5) is dropped, a counterexample can be built using the method by Pacard and the authors in [12], so that Savin’s result is nearly optimal.

A counterexample to De Giorgi’s Conjecture in dimension $N \geq 9$ has long been believed to exist, but the issue has remained elusive. Partial progress in this direction has been achieved by Jerison and Monneau [17] and by Cabrè and Terra [6]. See the survey article by Farina and Valdinoci [14].

In this paper we disprove De Giorgi’s conjecture in dimension $N \geq 9$ by constructing a bounded solution of equation (1.1) which is monotone in one direction whose level sets are not hyperplanes. The basis of our construction is a minimal graph different from a hyperplane built by Bombieri, De Giorgi and Giusti [5]. In this work a solution of the zero mean curvature equation

\begin{equation}
\nabla \cdot \left( \frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = 0 \quad \text{in } \mathbb{R}^{N-1}
\end{equation}

different from a linear affine function was built, provided that $N \geq 9$, in other words a non-trivial minimal graph in $\mathbb{R}^N$. Let us observe that if $F(x')$ solves equation (1.6) then so does

$$F_\alpha(x') := \alpha^{-1} F(\alpha x'), \quad \alpha > 0,$$

and hence

\begin{equation}
\Gamma_\alpha = \{(x', x_N) / x' \in \mathbb{R}^{N-1} , x_N = F_\alpha(x')\}
\end{equation}

is a minimal graph in $\mathbb{R}^N$.

Our main result states as follows:

**Theorem 1.1.** Let $N \geq 9$. There is a solution $F$ to equation (1.6) which is not a linear affine function, such that for all $\alpha > 0$ sufficiently small there exists a bounded solution $u_\alpha(y)$ of equation (1.1) such that $u_\alpha(0) = 0$,

$$\partial_{yN} u_\alpha(y) > 0 \quad \text{for all } y \in \mathbb{R}^N,$$
and

\[(1.8) \quad |u_\alpha(y)| \to 1 \quad \text{as} \quad \text{dist}(y, \Gamma_\alpha) \to +\infty.\]

uniformly in small \(\alpha > 0\), where \(\Gamma_\alpha\) is given by (1.7).

Property (1.8) implies that the 0 level set of \(u_\alpha\) lies inside the region \(\text{dist}(y, \Gamma_\alpha) < R\) for some fixed \(R > 0\) and all small \(\alpha\), and hence it cannot be a hyperplane. Much more accurate information on the solution will be drawn from the proof. The idea is simple. If \(t(y)\) denotes a choice of signed distance to the graph \(\Gamma_\alpha\) then, for a small fixed number \(\delta > 0\), our solution looks like

\[u_\alpha(y) \sim \tanh \left( \frac{t}{\sqrt{2}} \right) \quad \text{if} \quad |t| < \frac{\delta}{\alpha}.\]

As we have mentioned, a key ingredient of our proof is the construction of a non-trivial solution of equation (1.6) carried out in [5]. We shall derive accurate information on its asymptotic behavior, which in particular will help us to find global estimates for its derivatives. This is a crucial step since the mean curvature operator yields in general poor gradient estimates. In addition we shall derive a similar theory of existence and uniform estimates for the Jacobi operator around the minimal graph thus found. This work is carried out in sections §2 and §3. In §4 a suitable first approximation for a solution is built, around which we linearize and carry out an infinite-dimensional Lyapunov Schmidt reduction, which eventually reduces the full problem to one of solving a nonlinear, nonlocal equation which involves as a main term the Jacobi operator of the minimal graph. Schemes of this type have been successful in capturing solutions to singular perturbation elliptic problems in various settings, while in finding concentrating solutions on higher dimensional objects many difficulties arise. For the Allen-Cahn equation in compact situations this has been done in the works del Pino, Kowalczyk and Wei [11], Kowalczyk [19], Pacard and Ritore [26]. In particular in [26] solutions concentrating on a minimal submanifold of a compact Riemannian manifold are found through an argument that shares some similarities with the one used here. In the non-compact settings for nonlinear Schrödinger equation solutions have been constructed in del Pino, Kowalczyk and Wei [10], del Pino, Kowalczyk, Pacard and Wei [12], and Malchiodi [22]. See also Malchiodi and Montenegro [20, 21]. We should emphasize here the importance of our earlier work [12] in the context of the present paper, and especially the idea of constructing solutions concentrating on a family of unbounded sets, all coming from a suitably rescaled basic set. While in [12] the concentration set was determined by solving a Toda system and the rescaling was the one appropriate to this system, here the concentration set is the minimal graph and the rescaling is the one that leaves invariant the mean curvature operator.

Let us observe that a counterexample to De Giorgi conjecture in \(N = 9\) induces one in \(R^N = R^9 \times R^{N-9}\) for any \(N > 9\), by just extending the solution in \(R^9\) to the remaining variables in a constant manner. For this reason, in what follows of this paper we shall assume \(N = 9\).

2. Preliminaries and an outline of the argument

In this section after introducing the necessary notations we will outline our argument.
Let us recall that for any \( \alpha > 0 \) the minimal surface \( \Gamma_\alpha \) found in [5] is given as the graph of the function \( F_\alpha \), which is in a natural way obtained as scaling of the function \( F \), a solution of (1.6), namely \( F_\alpha(x') = \alpha^{-1} F(\alpha x') \). In addition the function has \( F \) the following properties

\[
(2.9) \quad x_9 = F(u, v), \quad u = (x_1^2 + \cdots + x_4^2)^{1/2}, \quad v = (x_5^2 + \cdots + x_8^2)^{1/2},
\]

and additionally satisfies

\[
(2.10) \quad F(u, v) = -F(v, u).
\]

We observe that \( \Gamma_\alpha \) is an embedded manifold and so it has a natural differential structure inherited from \( \mathbb{R}^9 \). The first and second covariant derivatives on \( \Gamma_\alpha \) will be denoted, respectively by:

\[
\nabla_{\Gamma_\alpha} \text{ and } \nabla^2_{\Gamma_\alpha}.
\]

In order to introduce the ansatz we fix an orientation of \( \Gamma_\alpha \) and introduce the Fermi coordinates in a neighborhood of \( \Gamma_\alpha \):

\[
(2.11) \quad x \mapsto (y, z), \text{ where } x = y + zn(y), \quad y = (y_1, \ldots, y_8, F_\alpha(u', v')) \in \Gamma_\alpha, \quad u' = ((y_1)^2 + \cdots + (y_4)^2)^{1/2}, \quad v' = ((y_5)^2 + \cdots + (y_8)^2)^{1/2},
\]

and \( n(y) \) is the unit normal to \( \Gamma_\alpha \) at \( y \). Let us denote:

\[
(2.12) \quad \Pi_{R^9}(y) = (y_1, \ldots, y_8), \quad r_\alpha(y) = \sqrt{1 + \alpha^2 |\Pi_{R^9}(y)|^2}, \quad y \in \Gamma_\alpha.
\]

We notice that the Fermi coordinates are well defined in the following neighborhood of \( \Gamma_\alpha \):

\[
(2.13) \quad U_{\theta_0} = \{ x \in \mathbb{R}^9 \mid x = y + zn(y), y \in \Gamma_\alpha, |z| < \frac{\theta_0 r_\alpha}{\alpha} \},
\]

where \( \theta_0 > 0 \) is a fixed small number independent of \( \alpha \). This fact will be proven later (see section 4, formula (4.24) and the argument that follows). One advantage of working with the Fermi coordinates is the fact that the Laplacian in \( \mathbb{R}^9 \) has a particularly simple expression in these coordinates. To explain this let us denote \( H_{\Gamma_{\alpha, z}} \) the mean curvature of the surface \( \Gamma_{\alpha, z} \) obtained from \( \Gamma_\alpha \) after translating it by \( z \) in the direction of the normal. Then we have

\[
(2.14) \quad \Delta = \Delta_{\Gamma_{\alpha, z}} + \partial_z^2 - H_{\Gamma_{\alpha, z}} \partial_z,
\]

where \( \Delta_{\Gamma_{\alpha, z}} \) is the Laplace-Beltrami operator on \( \Gamma_{\alpha, z} \).

Intuitively, near \( \Gamma_\alpha \) the solution of (1.1) we are after should resemble a function of the form

\[
(2.15) \quad u(x) \sim w(z - h_\alpha), \text{ where } h_\alpha = h_\alpha(y), \quad y \in \Gamma_\alpha,
\]

and \( w(t) = \tanh \left( \frac{t}{\sqrt{2}} \right) \) is the heteroclinic solution of one dimensional version of (1.1). The introduction of the new, unknown function \( h_\alpha \) reflects the fact that while we expect the profile of the solution in the direction transversal to \( \Gamma_\alpha \) to be similar to the one dimensional heteroclinic, however its zero level set is not expected to coincide with \( \Gamma_\alpha \) but to be its small perturbation. Later on we will see that this perturbation, represented by \( h_\alpha \) is actually a small quantity, of order \( o(1) \), as \( \alpha \to 0 \). Finally we observe that we can identify \( h_\alpha \) as a function on \( \Gamma_\alpha \) with a function defined on the original surface \( \Gamma \) via the formula:

\[
(2.16) \quad h(\alpha y) = h_\alpha(y), \quad y \in \Gamma_\alpha.
\]
where \( h : \Gamma \to \mathbb{R} \) is a given function. This identification will be very useful in the sequel and used without further reference whenever it does not cause confusion.

Now, let \( \chi \) be a smooth cutoff function such that \( \chi(s) = 1, -1 \leq s \leq 1, \) and \( \chi(s) = 0, |s| > 2. \) Given the hypersurface \( \Gamma_\alpha \) and the local Fermi coordinates \( x \mapsto (y, z) \in U_{\theta_0}, \) as above we set:

\[
(2.16) \quad w(x) = \begin{cases} 
\chi\left(\frac{\alpha z}{\alpha r}c\right)\left(w(z - h_\alpha) + 1\right) - 1, & z < 0, \\
\chi\left(\frac{\alpha z}{\alpha r}c\right)\left(w(z - h_\alpha) - 1\right) + 1, & 0 \leq z,
\end{cases}
\]

Notice that function \( w \) is well defined in an expanding (as a function of \( r_\alpha \)) neighborhood \( U_{\theta_0} \) of \( \Gamma_\alpha. \) To complete this definition we need to extend \( w(x) \) as a smooth function to the whole \( \mathbb{R}^3, \) Noting that \( \mathbb{R}^3 \setminus U_{\theta_0} \) consists of two disjoint components (since \( \Gamma_\alpha \) is a graph) we define \( w \) to be the smooth function which satisfies \( (2.16) \) and takes only values \( \pm 1 \) in \( \mathbb{R}^3 \setminus U_{\theta_0}. \) We will use the same symbol \( w \) for the extended function.

We will look for the solution of \((1.1)\) in the form:

\[
(2.17) \quad u_\alpha = w(x) + \phi(x).
\]

Substituting in \((1.1)\) we get for the function \( \phi \)

\[
(2.18) \quad \Delta \phi + f'(w)\phi = S[w] + N(\phi),
\]

where

\[
(2.19) \quad S[w] = -\Delta w - f(w), \quad N(\phi) = -[f(w + \phi) - f(w) - f'(w)\phi], \quad f(w) = w(1 - w^2).
\]

For future references let us denote as well:

\[
(2.20) \quad \mathcal{L}(\phi) = \Delta \phi + f'(w)\phi.
\]

The remaining part of this paper is devoted to solving \((2.18)\) and in particular to showing that \( \mathcal{L} \) has a uniformly (in small \( \alpha \)) bounded inverse in a suitable function space. To explain the theory we will need let us observe that locally, that is near \( \Gamma_\alpha, \) for \( \alpha \) small \( \mathcal{L} \) resembles the following operator

\[
L(\phi) = \Delta_{\Gamma_\alpha} \phi + \partial_{zz}\phi + f'(w)\phi,
\]

where \( \Delta_{\Gamma_\alpha} \) is the Laplace-Beltrami operator on \( \Gamma_\alpha. \) This follows from the fact that \( w(x) \sim w(z - h_\alpha), \) and also \((2.14)\) since \( \Delta_{\Gamma_{\alpha,z}} \sim \Delta_{\Gamma_\alpha} \) and \( H_{\Gamma_{\alpha,z}} \sim H_{\Gamma_\alpha}. \) Immediately we notice that function \( w_z(z - h_\alpha) \) satisfies

\[
L(w(z - h_\alpha)) = -\Delta_{\Gamma_\alpha} h_\alpha w_z(z - h_\alpha) + |\nabla_{\Gamma_\alpha} h_\alpha|^2 w_z(z - h_\alpha)
= o(1),
\]

and consequently we do not expect to find a uniformly bounded inverse of \( \mathcal{L} \) without introducing some restriction on its range. In this paper we deal with this difficulty using a version of infinite dimensional Lyapunov-Schmidt reduction (c.f [10, 11, 12]). The essence of this method is to introduce a function \( c(y), y \in \Gamma_\alpha \) and consider the following problem:

\[
(2.21) \quad L(\phi) = S[w] + N(\phi) + c(y)w_z(z - h_\alpha), \quad (y, z) \in \Gamma_\alpha \times \mathbb{R},
\]

\[
\int_\mathbb{R} \phi(y, z)w_z(z - h_\alpha) \, dz = 0, \quad \forall y \in \Gamma_\alpha.
\]
Recall that the ansatz $w$ depends on, still undetermined, function $h_\alpha$. Solving (2.21) for a given $h_\alpha$ and then adjusting $h_\alpha$ in such a way that
\[ c(y, h_\alpha) = 0, \quad \forall y \in \Gamma_\alpha, \]
we get the solution of (2.18). Actually, the following extra steps are needed to solve (2.18): (1) gluing the local (inner) solution of (2.21) and a suitable outer solution; (2) a fixed point argument; but the main point of the method is to solve (2.21). Equation (2.22), called here the reduced problem, is a nonlocal PDE for $h_\alpha$ and its solvability is in itself a nontrivial step extensively treated in this paper (sections 8 and 9).

2.1. Improvement of the initial approximation. Let us consider more carefully $S[w]$. Near the surface $\Gamma_\alpha$ (say in the set $U_{\theta_0/4}$), where the Fermi coordinates are well defined and where $w = w$ we have, using (2.14):

\[
S[w] = -\Delta_{\Gamma_{\alpha,z}} w - \partial_z^2 w + H_{\Gamma_{\alpha,z}} \partial_z w - f(w) = -\Delta_{\Gamma_{\alpha,z}} w + H_{\Gamma_{\alpha,z}} \partial_z w.
\]

As we will see later the first term above is of relatively small size. The second term is of the leading order and it has to be treated separately. To see this we use Taylor expansion of $H_{\Gamma_{\alpha,z}}$ around $z = 0$:

\[
H_{\Gamma_{\alpha,z}}(y) = z|A_{\Gamma_\alpha}(y)|^2 + z^2 R_\alpha(y, z), \quad y \in \Gamma_\alpha,
\]

where $|A_{\Gamma_\alpha}(y)|$ is the norm of the second fundamental form on $\Gamma_\alpha$ and $R_\alpha$ is the remainder in the Taylor’s expansion. Here, for future references, we observe that:

\[
\int_{\mathbb{R}} \bar{z}|A_{\Gamma_\alpha}|^2 w_1^2 d\bar{z} = 0,
\]

hence there exists a unique solution $w_1$ of the problem

\[
\partial_z^2 w_1 + f'(w)w_1 = \bar{z}|A_{\Gamma_\alpha}|^2 \partial_z w_1, \quad \bar{z} \in \mathbb{R},
\]

which is explicitly given by:

\[
w_1(\bar{z}) = -|A_{\Gamma_\alpha}|^2 w'(\bar{z}) \int_0^\bar{z} \frac{d\sigma}{(w'(\sigma))^{3/2}} \int_{-\infty}^\sigma \eta(w'(\eta))^2 d\eta.
\]

We define now

\[
\mathfrak{w}_1(x) = \chi(\frac{4\alpha z}{\theta_0 R_\alpha}) w_1(\bar{z}).
\]
Function $w_1$ gives an improvement of the initial approximation, and in some sense it represents the first term in the asymptotic expansion of $\phi$ in (2.18). Let us explain this. Going back to (2.23)–(2.24) we see, again formally, that the error of the new approximation is:

$$S[w + w_1] = S[w] + S[w_1] - [f(w + w_1) - f(w) - f(w_1)]$$

(2.32)

$$\sim S[w] - \partial_{\bar{z}}^2 w_1 - f'(w)w_1$$

$$\sim (\Delta_{\Gamma^\alpha} h_\alpha + |A_{\Gamma^\alpha}|^2 h_\alpha)w_\bar{z} + \bar{z}^2 R_\alpha(y, 0)w_\bar{z}.$$

In (2.32) we have neglected those terms that are decaying fast in $r_\alpha^3$, which means here faster than $r_\alpha^{-3}$. This agrees with our expectation that we should have $R_\alpha(y, 0) \sim r_\alpha^{-3}$. In turn, this will imply that, if we write $u_\alpha = w + w_1 + \phi$, then this "new" $\phi \sim r_\alpha^{-3}$, while $w_1 \sim r_\alpha^{-2}$, which apparently is a technical, but rather crucial point in our analysis.

Now, let us recall that $w_\bar{z}$ is an element of the "approximate" kernel of the operator $L$ appearing in (2.21) and thus the problem of bounded solvability of (2.21) depends, roughly speaking, on the orthogonality of the right hand side of (2.21) and the function $w_\bar{z}$. This in turn be achieved if

$$\int_{\mathbb{R}} \left[(\Delta_{\Gamma^\alpha} + |A_{\Gamma^\alpha}|^2) h_\alpha + \bar{z}^2 R_\alpha(y, 0)\right]w_\bar{z}^2 d\bar{z} \approx 0,$$

(2.33)

which is equivalent to (2.22). Let us now summarize what is needed in order to reduce the full nonlinear problem (2.21) to the reduced problem (2.33) and solve (1.1) at the end.

(i) The linearized operator $L$ has a bounded inverse in the space of functions satisfying the orthogonality condition in (2.21).

(ii) Problem of the form:

$$(\Delta_{\Gamma^\alpha} + |A_{\Gamma^\alpha}|^2) h_\alpha = f_\alpha \sim O(r_\alpha^{-3}),$$

(2.34)

given on the manifold $\Gamma^\alpha$ can be solved in a suitable function space whose norm takes into account the decay of its elements in $r_\alpha$.

The rest of this paper is devoted to addressing (i) and (ii) above. In fact, since these two issues are not quite independent it is convenient to first treat (ii) and then later deal with (i). One of the main steps required to carry out the plan outlined above is a refinement of the existence result of Bombieri-De Giorgi and Giusti [5] (section 3). In addition we need to find precise decay estimates for the minimal graph $F_\alpha$ and its derivatives (up to order 3) which amounts to a refinement of a result of Simon [29] (section 4).

3. THE MINIMAL SURFACE EQUATION

In this section we will consider only one fixed minimal graph, denoted here by $F$, since as we have pointed out $\Gamma^\alpha$ is obtained as a graph of $F_\alpha(x') = \alpha^{-1} F(\alpha x')$, $x' \in \mathbb{R}^8$. Thus, we consider the mean curvature equation in $\mathbb{R}^8$

$$\sum_{i=1}^8 \partial_{x_i} \left( \frac{F_{x_i}}{\sqrt{1 + |\nabla F|^2}} \right) = 0 \quad \text{in} \; \mathbb{R}^8.$$

(3.1)
Bombieri, De Giorgi and Giusti [5] found that this equation possesses a non-constant entire solution if \( N \geq 8 \), therefore a minimal graph different from a hyperplane exists in dimensions 9 or higher. The solution found in [5] enjoys some simple symmetries and also is a function of the variables \((u, v)\) defined above. It is straightforward to check that the mean curvature operator written in terms of these variables becomes

\[
H[F] := \frac{1}{(uv)^3} \nabla \cdot \left( \frac{(uv)^3 \nabla F}{\sqrt{1 + |\nabla F|^2}} \right), \quad \nabla F = (F_u, F_v),
\]

while the equation (3.1) reads

\[
\frac{1}{u^3} \partial_u \left( \frac{u^3 F_u}{\sqrt{1 + |\nabla F|^2}} \right) + \frac{1}{v^3} \partial_v \left( \frac{v^3 F_v}{\sqrt{1 + |\nabla F|^2}} \right) = 0.
\]

Since the solution in [5] satisfies

\[ F(u, v) = -F(v, u) \quad \text{if} \ u < v, \]

and in particular \( F = 0 \) on the cone \( u = u \), therefore it is sufficient to consider (3.3) in the region (see figure 1).

Figure 1. Schematic view of the function \( F(u, v) \) representing \( \Gamma \) in the sector \( T = \{u > 0, v > 0, u < v\} \).

Let us introduce polar coordinates in \( T \), setting

\[ u = r \cos \theta, \quad v = r \sin \theta, \quad \theta \in (\frac{\pi}{4}, \frac{\pi}{2}), \]
so that \( r = |u|, \ u = (u,v) \). First, we will show that the solution in [5] can be described at main order as
\[
F(r, \theta) \sim r^3 g(\theta) \quad \text{as } r \to \infty,
\]
where \( g \) satisfies \( g(\frac{\pi}{2}) = 0, \ g_\theta(\frac{\pi}{2}) > 0, \ g_\theta(\frac{\pi}{4}) = 0 \). Intuitively \( g(\theta) \) resembles \( -\cos 2\theta, \pi/4 \leq \theta \leq \pi/2 \). In the sequel we will denote \( F_0 = r^3 g(\theta) \).

Second, we introduce coordinates \((s,t)\) in \( T \) which are adapted to \( F_0 \) and play a fundamental role in this paper.

3.1. Equation for \( g \). Since we expect
\[
F(u,v) \sim F_0(u,v) = r^3 g(\theta), \quad r \gg 1,
\]
therefore it is reasonable to require that \( F_0 \) should be a solution of
\[
\nabla \cdot \left( \frac{\nabla F_0}{|\nabla F_0|} \right) = 0, \quad \nabla = (\partial_u, \partial_v).
\]
Assuming that \( F_0 = r^3 g(\theta) \geq 0 \) in the sector \( T \) we get the following equation for the positive function \( g(\theta) \)
\[(3.5) \quad \frac{21 \sin^4(2\theta) g}{\sqrt{9g^2 + g_\theta^2}} + \left( \frac{\sin^3(2\theta) g_\theta}{\sqrt{9g^2 + g_\theta^2}} \right)_\theta = 0, \quad \theta \in (\pi/4, \pi/2),\]
with the boundary conditions
\[(3.6) \quad g(\pi/4) = 0, \quad g_\theta(\pi/2) = 0.
\]
The boundary conditions (3.6) follow from the symmetries of \( F \).

Let us observe that if \( g(\theta) \) is a solution of (3.5) then so is \( Cg(\theta) \), for any constant \( C \). The following lemma proves the existence of solutions to (3.5).

Lemma 3.1. Problem (3.5) has a solution such that:
\[(3.7) \quad g(\theta) \geq 0, \quad g_\theta(\theta) \leq 0, \quad g_\theta(\theta) \geq 0,
and the last inequality is strict for \( \theta \in [\pi/4, \pi/2] \).

Proof. If \( g \) is a solution to (3.5) then function
\[
\psi(\theta) = \frac{g_\theta(\theta)}{g(\theta)}, \quad g(\theta) \neq 0,
\]
satisfies the following equation:
\[(3.8) \quad 9\psi' + (9 + \psi^2)[21 + 6 \cot(2\theta)\psi] = 0.
\]
Our strategy is to solve (3.8) first and then find the function \( g \). To this end we will look for a solution of (3.8) in the interval \( I = (\pi/4, \pi/2) \) with
\[(3.9) \quad \psi(\pi/2) = 0.
\]
In order to define the function \( g \) we also need \( \psi \) to be defined and positive in the whole interval \( (\pi/4, \pi/2) \) and \( \lim_{\theta \to \pi/2} - \psi' = +\infty \). Let \( (\theta^*, \pi/2), \pi/4 \leq \theta^* \) be the maximal interval for which the solution of (3.8) exists.

We set \( \psi_+(\theta) = -11 \tan(2\theta) \). Then we have
\[
9\psi_+' + (9 + \psi_+^2)[21 + 6 \cot(2\theta)\psi_] < 0, \quad \theta \in (\pi/4, \pi/2),
\]
\[
\psi_+(\pi/2) = 0 = \psi(\pi/2), \quad \psi_+'(\pi/2) = -22 < -21 = \psi'(\pi/2).
\]
Substituting $\psi_- (\theta) = -2 \tan(2\theta)$ for $\psi$ in (3.8) we get:

$$9\psi'_- + (9 + \psi^2_-)[21 + 6 \cot(2\theta)\psi_-] > 0.$$  
(3.10)

We have $\psi(\pi/2) = \psi_-(\pi/2) = 0$ and, from (3.8),

$$\psi'_-(\pi/2) = -21 < -4 = \psi'_-(\pi/2).$$

From this we get that the maximal solution of (3.8) satisfies:

$$\psi_+(\theta) = -11 \tan(2\theta) > \psi(\theta) \geq \psi_-(\theta) = -2 \tan(2\theta) \geq 0, \quad \theta \in (\theta^*, \pi/2),$$

and that $\theta^* = \frac{\pi}{4}$. Let us now define

$$g(\theta) = \exp\left\{-\int_0^{\pi/2} \psi(t) \, dt\right\},$$

where $\psi$ is the unique solution of (3.8). Clearly we have $g_0(\pi/2) = 0$ and from (3.11) it follows $g(\pi/4) = 0$. Thus $g$ defined in (3.12) is a solution of (3.5)–(3.6).

We have $g_0 > 0$ in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$, since $g_0 = \hat{g}\psi$. To show that $g_0(\frac{\pi}{4}) > 0$ we will improve the upper bound on $\psi$. Let us define:

$$\psi_1 = -2 \tan(2\theta) + \hat{\psi}, \quad \text{where } \hat{\psi} = A\left(-\tan(2\theta)\right)^\eta,$$

and $\frac{2}{3} < \eta < 1$, $A > 1$ are to be chosen. Direct calculations give:

$$9\psi'_1 + (9 + \psi^2_1)[21 + 6\psi_1 \cot(2\theta)] = 9\psi'\cos^2(2\theta) + 45 \cos^2(2\theta)$$

$$+ 6\psi \cot(2\theta)[4 + 5 \cos^2(2\theta)] + 36\psi \sin(2\theta)(-\cos(2\theta))$$

$$+ 9\psi^2 \cos^2(2\theta) + 6\psi^2 \cot(2\theta)[4 \sin(2\theta)(-\cos(2\theta)) + \psi \cos^2(2\theta)].$$

Using the definition of $\hat{\psi}$, after some calculation we find that the last expression is negative for $\theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ when

$$0 > -18A\eta + 45(-\tan(2\theta))^{1-\eta} \cos^2(2\theta) - 6A[4 + 5 \cos^2(2\theta)] + 36A \sin^2(2\theta)$$

$$- 15A(-\tan(2\theta))^{1-\eta} \cos^2(2\theta) - 6A^3(-\tan(2\theta))^{1-2\eta} \sin(2\theta)(-\cos(2\theta)),$$

which can be achieved if $\frac{2}{3} < \eta < 1$ and $A$ is chosen sufficiently large. Since $\eta < 1$ it follows that

$$\psi(\theta) \leq \psi_1(\theta), \quad \theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right),$$

hence, for certain constant $C > 0$,

$$-C \cos(2\theta) \leq g(\theta) \leq -\cos(2\theta), \quad \theta \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right].$$

In fact the inequalities in (3.7) are strict for $\theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. It follows in addition that:

$$g_0(\theta) \geq C \sin(2\theta), \quad \theta \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right].$$

This shows in particular $g_0 > 0$ in $\left[\frac{\pi}{4}, \frac{\pi}{2}\right)$. The remaining estimate for $g_{\theta \theta}$ follows from the second order equation for $g$.

Given the function $g$ let us define:

$$\cos \phi = \frac{3g}{\sqrt{9g^2 + g_0^2}}, \quad \sin \phi = \frac{g_0}{\sqrt{9g^2 + g_0^2}}.$$  
(3.14)

We see from Lemma 3.11 that $\phi$ satisfies:

$$\phi' + 7 + 6 \cot(2\theta) \tan \phi = 0, \quad \phi\left(\frac{\pi}{4}\right) = \frac{\pi}{2}, \quad \phi\left(\frac{\pi}{2}\right) = 0.$$  
(3.15)
We need the following lemma:

**Lemma 3.2.** It holds

\[
\phi'(\frac{\pi}{4}) = -3, \quad \phi'\left(\frac{\pi}{2}\right) = -\frac{7}{4}, \quad \phi'(\theta) > -3 \text{ for } \theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right).
\]

**Proof.** To prove the first identity we observe that

\[
\tan \phi = \frac{1}{3} \psi
\]

which after differentiation yields:

\[
\phi' = \frac{1}{3} \psi' \cos^2 \phi = -\frac{1}{3} \left[21 + 6 \cot(2\theta) \psi\right] \geq -3,
\]

since \(\psi(\theta) \geq -2 \tan(2\theta)\). Now considering (3.15) we see that when \(\theta \to \pi/4^+\) we can have \(\phi'(\pi/4^+) = -3\) or \(\phi'(\pi/4^+) = -4\). From (3.17) we get the required formula.

The second identity follows from simple analysis near \(\theta = \frac{\pi}{2}\).

To prove the last estimate, we suppose that there exists a point \(\theta_1 \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)\) such that \(\phi'(\theta_1) = -3\). We claim that \(\phi''(\theta_1) < 0\). This gives a contradiction. (We may take \(\theta_1\) to be the point closest to \(\frac{\pi}{2}\). Then necessarily \(\phi'(\theta_1) \geq 0\).) In fact, from (3.15), we deduce that

\[
2 \sin(2\theta_1) \cos \phi + 3 \cos(2\theta_1) \sin \phi = 0,
\]

which is equivalent to

\[
5 \sin(2\theta_1 + \phi) = \sin(2\theta_1 - \phi).
\]

Note that \(2\theta - \phi \in (0, \pi)\) and hence \(0 < 2\theta - \phi < 2\theta + \phi < \pi\). Now we compute

\[
\phi''(\theta_1) = \frac{6}{\sin^2 \theta_1 \cos^2 \phi} \left(\sin 2\phi - \frac{1}{2} \sin 4\theta_1 \phi'\right) = \frac{6}{\sin^2 \theta_1 \cos^2 \phi} \sin(2\theta_1 - \phi) \cos(2\theta_1) \cos \phi < 0,
\]

which completes the proof. \(\square\)

### 3.2. A new system of coordinates.

In this section we will introduce a system of coordinates in the sector \(T\) (see (3.4)) which depends on the function \(F_0\) defined above. The idea is that the coordinate lines on the (2 dimensional) surface given by the graph of \(F_0\) are orthogonal. As we will see this property is extremely useful in further developments.

**Lemma 3.3.** There exists a diffeomorphism \(\Phi : Q \to T\), where \(Q = \{(t, s) \mid t > 0, s > 0\}\) such that \(\Phi(t, s) = (u(t, s), v(t, s))\) and \(u\) satisfies the coupled system of differential equations

\[
\frac{\partial u}{\partial t} = \frac{\nabla F_0}{|\nabla F_0|^2}, \quad \frac{\partial u}{\partial s} = \frac{1}{(uv)^3} \frac{\nabla F_0^\perp}{|\nabla F_0|},
\]

where we denote

\[
\nabla F = (F_u, F_v), \quad \nabla F_0^\perp = (F_v, -F_u).
\]

Moreover \(\Phi\) maps \((t = 0, s)\) onto the line \(u = v\) and \((t, s = 0)\) onto \((u = 0, v)\).
Proof. Introducing polar coordinates

\[ u = r \cos \theta, \quad v = r \sin \theta, \]

and using (3.19) we find:

\[
\begin{align*}
\frac{\partial r}{\partial t} &= \frac{F_{0,r}}{\sqrt{|\nabla F_0|^2}}, \quad \frac{\partial \theta}{\partial t} = \frac{g_\theta}{r(9g^2 + g_\theta^2)}, \\
\frac{\partial r}{\partial s} &= \frac{8F_{0,\theta}}{r^7 \sin^2(2\theta)|\nabla F_0|}, \quad \frac{\partial \theta}{\partial s} = \frac{g_\theta r^3}{r^7 \sin^2(2\theta)|\nabla F_0|},
\end{align*}
\]

(3.20)

Using the formal relations

\[
\begin{bmatrix}
\begin{bmatrix} r & \theta \\ \frac{r}{s} & \frac{\theta}{s} \end{bmatrix}

\end{bmatrix}
\begin{bmatrix}
\begin{bmatrix} r & r_s \\ \theta & \theta_s \end{bmatrix}

\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

we arrive in particular at the equations for

\[
\begin{align*}
3gs_r + \frac{g_\theta}{r}s_\theta &= 0, \\
\frac{8gs_r s_\theta}{r^3 \sin^3(2\theta) \sqrt{9g^2 + g_\theta^2}} &= \frac{24gs_\theta}{r^7 \sin^3(2\theta) \sqrt{9g^2 + g_\theta^2}} = 1,
\end{align*}
\]

or

\[
\begin{align*}
\frac{\partial s}{\partial \theta} &= -\frac{3r^7 \sin^3(2\theta) g_\theta}{8 \sqrt{9g^2 + g_\theta^2}}, \\
\frac{\partial s}{\partial r} &= \frac{r^6 \sin^3(2\theta) g_\theta}{8 \sqrt{9g^2 + g_\theta^2}},
\end{align*}
\]

(3.21)

which are satisfied by the function

\[ s = \frac{r^7 \sin^3(2\theta) g_\theta}{56 \sqrt{9g^2 + g_\theta^2}} \]

because of the equation satisfied by \( g \). Similarly we obtain the solution for \( t = r^3 g(\theta) \).

(3.22)

Using the properties of the function \( g \) we can directly check that function given by the formulas (3.21)–(3.22) is a diffeomorphism with the required properties. \end{proof}

For future references let us keep in mind that setting \( \sin \phi, \cos \phi \) as in formula (3.14), we find simply

\[
\begin{align*}
\frac{\partial r}{\partial s} &= \frac{r}{7s} \sin^2 \phi, \\
\frac{\partial \theta}{\partial s} &= -\frac{1}{14s} \sin(2\phi),
\end{align*}
\]

(3.23)

and

\[
\begin{align*}
\frac{\partial r}{\partial t} &= \frac{r}{3t} \cos^2 \phi, \\
\frac{\partial \theta}{\partial t} &= \frac{1}{6t} \sin(2\phi).
\end{align*}
\]

(3.24)

Our next goal is to express the mean curvature operator (3.2) in terms of the variables \((t,s)\). Denoting by \( u' \) the matrix \((u_t, u_s)\) problem (3.3) is transformed to

\[
(uv)^{-3} \frac{1}{\sqrt{\det u'u'^T}} \nabla_{t,s} \left( \frac{(uv)^3 \sqrt{\det u'u'^T}}{\sqrt{1 + |\nabla F|^2}} (u'u'^T)^{-1} \nabla_{t,s} F \right) = 0. \]

(3.25)
From Lemma (3.3) we find

\[ (3.26) \quad \langle u_t, u_t \rangle = \frac{1}{|\nabla F_0|^2}, \quad \langle u_t, u_s \rangle = 0, \quad \langle u_s, u_s \rangle = \frac{1}{(uv)^6} := \rho^2, \]

hence we compute

\[ (3.27) \quad \det u' = -\frac{\rho}{|\nabla F_0|}, \quad (u'u'^T)^{-1} = \begin{pmatrix} |\nabla F_0|^2 & 0 \\ 0 & \rho^{-2} \end{pmatrix}. \]

Then equation (3.25) becomes

\[ (3.28) \quad |\nabla F_0| \partial_t \left( \frac{|\nabla F|^2}{\sqrt{1 + |\nabla F|^2}} \right) + |\nabla F_0| \partial_s \left( -\frac{\rho^{-2} \partial_s F}{|\nabla F_0| \sqrt{1 + |\nabla F|^2}} \right) = 0. \]

Let us observe that:

\[ \nabla F = \left\langle \frac{\nabla F_0}{|\nabla F_0|} \right\rangle \frac{|\nabla F_0|}{|\nabla F_0|} + \left\langle \frac{\nabla F_0}{|\nabla F_0|} \right\rangle \frac{\nabla F_0^\perp}{|\nabla F_0|} = F_t \nabla F_0 + \rho^{-1} F_s \frac{\nabla F_0^\perp}{|\nabla F_0|}. \]

From this we have

\[ 1 + |\nabla F|^2 = 1 + |\nabla F_0|^2 \left( F_t^2 + \frac{\rho^{-2} F_s^2}{|\nabla F_0|^2} \right) = |\nabla F_0|^2 \left( \frac{1}{|\nabla F_0|^2} + F_t^2 + \frac{\rho^{-2} F_s^2}{|\nabla F_0|^2} \right). \]

Denoting by \( Q(\nabla_{t,s} F) \) the function

\[ Q(\nabla_{t,s} F) = \frac{1}{|\nabla F_0|^2} + F_t^2 + \frac{\rho^{-2} F_s^2}{|\nabla F_0|^2}, \]

we see the mean curvature equation is equivalent to

\[ H[F] = \frac{|\nabla F_0|}{Q^{3/2}(\nabla_{t,s} F)} G[F] = 0 \]

where

\[ G[F] = Q(\nabla_{t,s} F) F_{tt} - \frac{1}{2} \partial_t Q(\nabla_{t,s} F) F_t + Q(\nabla_{t,s} F) \partial_s \left( \frac{\rho^{-2} F_s}{|\nabla F_0|^2} \right) - \frac{1}{2} \partial_s Q(\nabla_{t,s} F) \rho^{-2} F_s. \]

Now we derive the mean curvature operator for functions of the form

\[ F = F_0 + A \varphi(t, s) = t + A \varphi(t, s), \]

where \( A \) is constant parameter. Our goal is to write the resulting equation in the form of a polynomial in \( A \). In general we assume that for \( r \gg 1 \),

\[ (3.30) \quad |\varphi_t| + \frac{|\varphi_s \rho^{-1}|}{|F_0|} = o(1). \]
We compute
\[
\nabla F = \nabla F_0 + \left\langle \nabla \varphi, \frac{\nabla F_0}{|\nabla F_0|^2} \right\rangle \nabla F_0 + \left\langle \nabla \varphi, \frac{\nabla F_0^T}{|\nabla F_0|^2} \right\rangle \nabla F_0^T
\]
\[
= \nabla F_0 + \varphi_t \nabla F_0 + \rho^{-1} \varphi_s \frac{\nabla F_0}{|\nabla F_0|^2}.
\]
Then we have
\[
1 + |\nabla F|^2 = 1 + |\nabla F_0|^2 \left[ (1 + A\varphi_t)^2 + A^2 \frac{\rho^{-2} \varphi_s^2}{|\nabla F_0|^2} \right]
\]
\[
= |\nabla F_0|^2 \left[ 1 + \frac{1}{|\nabla F_0|^2} + 2A\varphi_t + A^2 R_1 \right],
\]
where we denote
\[
R_1 = \varphi_t^2 + \frac{\rho^{-2} \varphi_s^2}{|\nabla F_0|^2}.
\]
It is convenient to introduce
\[
R = \left( 1 + \frac{1}{|\nabla F_0|^2} + 2A\varphi_t + A^2 R_1 \right).
\]
With these notations we have:
\[
|\nabla F_0|^{-1} R^{3/2} H[F_0 + A\varphi]
\]
\[
= \left[ A R \partial_t^2 \varphi - \frac{1}{2} (1 + A \partial_t \varphi) \partial_t R + A R \partial_s \left( \frac{\rho^{-2} \partial_s \varphi}{|\nabla F_0|^2} \right) - \frac{1}{2} A \rho^{-2} \partial_s \varphi \partial_t R \right]
\]
\[
= \frac{1}{2} \partial_t |\nabla F_0|^{-2} + A \left[ |\nabla F_0|^{-2} \partial_t^2 \varphi - \frac{1}{2} \partial_t |\nabla F_0|^{-2} \partial_t \varphi \right]
\]
\[
+ A^2 \left[ \partial_t \varphi \partial_t^2 \varphi - \frac{1}{2} \partial_t R_1 + 2 \partial_t \varphi \partial_s \left( \frac{\rho^{-2} \partial_s \varphi}{|\nabla F_0|^2} \right) - \left( \frac{\rho^{-2} \partial_s \varphi}{|\nabla F_0|^2} \right) \partial_t^2 \varphi \right]
\]
\[
+ A^3 \left[ R_1 \partial_t^3 \varphi - \frac{1}{2} \partial_t \varphi \partial_t R_1 + R_1 \partial_s \left( \frac{\rho^{-2} \partial_s \varphi}{|\nabla F_0|^2} \right) - \frac{1}{2} \left( \rho^{-2} \partial_s \varphi \right) \partial_t R_1 \right].
\]
In the sequel we will refer to the consecutive term in (3.31) as the $A^0$, $A^1$, $A^2$ and $A^3$ terms respectively. For future references we observe that the $A^0$ term can be written as
\[
(3.32) \quad - \frac{1}{2} \partial_t |\nabla F_0|^{-2} = |\nabla F_0|^{-1} (1 + |\nabla F_0|^{-2})^{3/2} H[F_0],
\]
and the $A^1$ term can be written as
\[
(3.33) \quad \tilde{L}_0[\varphi] = |\nabla F_0|^{-1} \tilde{L}_0[\varphi]
\]
\[
- \frac{3}{2} \partial_t |\nabla F_0|^{-2} \partial_t \varphi + |\nabla F_0|^{-2} \partial_s \left( \frac{\rho^{-2} \partial_s \varphi}{|\nabla F_0|^2} \right) - \frac{1}{2} \left( \rho^{-2} \partial_s \varphi \right) \partial_t |\nabla F_0|^{-2},
\]
where
\[
(3.34) \quad \tilde{L}_0[\varphi] = |\nabla F_0| \left[ \partial_t \left( \frac{\partial_t \varphi}{|\nabla F_0|^2} \right) + \partial_s \left( \frac{\rho^{-2} \partial_s \varphi}{|\nabla F_0|^2} \right) \right].
\]
3.3. A refinement of the result of Bombieri, De Giorgi and Giusti. In this section, taking the existence result in [5] as the point of departure, we find the asymptotic behavior of the minimal graph. Our goal is prove the following theorem:

**Theorem 3.1.** There exists a solution $F = F(u,v)$ to the mean curvature equation with the following properties

\[ F_0 \leq F \leq F_0 + \frac{C}{r^\sigma} \min\{F_0, 1\}, \quad r > R_0, \]

where $0 < \sigma < 1$, $C \geq 1$, and $R_0$, are positive constants.

The rest of this section is devoted to the proof of the theorem. Our approach, which is based on a comparison principle, relies on a refinement of the supersolution/subsolution in [5]. We need the following comparison principle:

**Lemma 3.4.** Let $\Omega$ be a smooth and open bounded domain. If $F_1$ and $F_2$ satisfies

\[ H[F_1] \leq H[F_2] \text{ in } \Omega, \quad F_1 \geq F_2 \text{ on } \partial \Omega \]

Then

\[ F_1 \leq F_2 \text{ in } \Omega. \]

**Proof.** The proof is simple since

\[ H[F_1] - H[F_2] = \sum_{i,j} a_{ij} \frac{\partial^2}{\partial x_i x_j} (F_1 - F_2) \]

where the matrix $(a_{ij})$ is uniformly elliptic in $\Omega$. By the usual Maximum Principle, we obtain the desired result. \qed

Let us observe that from (3.13) we have

\[ \min(-\cos(2\theta) g(\theta)) \geq 1, \theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right). \]

Thus for $F_0 = r^3 g(\theta)$ it holds

\[ F_0 = r^3 g(\theta) \leq (v^2 - u^2)(v^2 + u^2)^{\frac{3}{2}}. \]

We will now construct a subsolution to the mean curvature equation.

**Lemma 3.5.** Let $H[F]$ denote the mean curvature operator. We have

\[ H[F_0] \geq 0. \]

It holds as well:

\[ H[F_0] = O(r^{-5}). \]

**Proof.** Since $H[F]$ and $G[F]$ (defined in (3.29)) differ only by a nonnegative factor it suffices to show that

\[ G[F_0] \geq 0. \]

In fact, let $F = F_0 = t$, we then have

\[ G[F_0] = -\frac{1}{2} \partial_t Q(\nabla_t s F_0) = -\frac{1}{2} \partial_t \left( \frac{1}{|\nabla F_0|^2} \right), \]

where $Q$ is the quadratic form.
where
\[ \frac{1}{|\nabla F_0|^2} \frac{1}{r^4(9g^2 + g_0^2)} = \frac{r^2 \cos^2 \phi}{9r^2}. \]

By the formulas (3.24), we have
\[ -\partial_t \left( \frac{r^2 \cos^2 \phi}{9t^2} \right) = \frac{r^2}{9t^3} \left( 2 \cos^2 \phi - \frac{2tr_1 \cos^2 \phi}{r} + t\phi_t \sin(2\phi) \right) \]
\[ = \frac{2r^2 \cos^2 \phi}{9t^3} \left( \frac{2}{3} \cos^2 \phi + \frac{1}{3} \sin^2 \phi(\phi' + 3) \right) \]
\[ \geq 0, \]
where we have used the fact that \( \phi'(\theta) \geq -3. \) Estimate (3.41) follows easily from this. This ends the proof.

By the standard theory of the mean curvature equation for each fixed \( R > 0, \) there exists a unique solution to the following problem
\[ (3.44) \quad \frac{1}{(uv)^3} \nabla \cdot \left( \frac{(uv)^3 \nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = 0 \text{ in } \Gamma_R, \quad F = F_0 \text{ on } \partial \Gamma_R \]
where \( \Gamma_R = B_R \cap T, T = \{u, v > 0, u < v\}. \) Let us denote the solution to (3.44) by \( F_R. \)

Using (3.39), the comparison principle and the supersolution found in [5], we have
\[ (3.45) \quad F_0 \leq F_R \leq H\left((v^2 - u^2) + (v^2 - u^2)(u^2 + v^2)^{1/2}(1 + A(|\cos(2\theta)|)^{\lambda-1})\right) \]
where
\[ H(t) := \int_t^\infty \exp \left( B \int w^2 - \lambda (1 + t^2)dw - \lambda - 2\alpha \right) \]
\( \lambda > 1 \) is a positive fixed number, \( \alpha = \frac{3}{2}, \) and \( A, B \) are sufficiently large positive constants. This inequality, combined with standard elliptic estimates, imply that as \( R \to +\infty, F_R \to F \) which is a solution to the mean curvature equation \( H[F] = 0 \)

\[ (3.46) \quad F_0 \leq F \leq H\left((v^2 - u^2) + (v^2 - u^2)(u^2 + v^2)^{1/2}(1 + A(|\cos(2\theta)|)^{\lambda-1})\right). \]

Next we need the following key lemma:

**Lemma 3.6.** There exists \( \sigma_0 \in (0, 1) \) such that for each \( \sigma \in (0, \sigma_0) \) there exists \( a_0 > 1 \) such that for each sufficiently large \( A \geq 1, \) we have
\[ (3.47) \quad H[F_0 + \frac{\tilde{A}F_0}{\alpha^r}] \leq 0, \quad \text{for } r > a_0. \]

Moreover, under the same assumptions for each sufficiently large \( A \geq 1 \) we have
\[ (3.48) \quad H[F_0 + \frac{A}{\sqrt{r}}] \leq 0, \quad \text{for } r > a_0 A^{1+\sigma}. \]

**Proof.** We will consider (3.47) first. We will use formula (3.31) to write \( H[F_0 + \frac{\tilde{A}F_0}{\alpha^r}] \)
multiplied by a nonnegative factor as a polynomial in \( A. \) Explicit computations (3.31) yield
\[ |\nabla F_0|^{-1} R^{3/2} H[F_0 + \frac{\tilde{A}F_0}{\alpha^r}] = H_0 + \tilde{A}H_1 + \tilde{A}^2H_2 + \tilde{A}^3H_3, \]
where

\begin{equation}
H_0 = |\nabla F_0|^{-1}(1 + |\nabla F_0|^{-2}) H_0 = \frac{r^2 \cos^2 \phi}{g^3} \left[ \frac{2}{3} \cos^2 \phi + \frac{1}{3} \sin^2 \phi (\phi' + 3) \right],
\end{equation}

\begin{equation}
H_1 = \frac{-7 \sigma \cos^2 \phi}{9tr^\sigma} (7 + (2\phi' - \sigma) \sin^2 \phi) + \frac{\cos^2 \phi}{tr^\sigma} O(r^{-4}).
\end{equation}

In the Appendix A we show in addition that

\begin{equation}
H_2 = \frac{\cos^2 \phi}{tr^\sigma} O(r^{-\sigma}) \leq 0,
\end{equation}

(3.50)

\begin{equation}
H_3 = \frac{\cos^2 \phi}{tr^\sigma} O(r^{-2\sigma}) \leq 0.
\end{equation}

Let us observe that the first term in (3.49) is bounded by

\begin{equation}
H_0 \leq c_1 r^2 \cos^4 \phi \leq c_1 \cos^2 \phi \frac{t}{r^3}.
\end{equation}

(3.51)

Estimate (3.51) follows from (3.49) and the fact that \( \phi(\pi/4) = \pi/2, \phi'(\pi/4) = -3, \phi''(\pi/4) = 0. \) Summarizing, we have

\begin{equation}
H[F_0 + \tilde{A}F_0] \leq H_0 + \tilde{A}H_1
\end{equation}

(3.52)

\begin{equation}
\leq \frac{-7 \tilde{A} \sigma \cos^2 \phi}{9tr^\sigma} (7 + (2\phi' - \sigma) \sin^2 \phi) + \frac{\cos^2 \phi}{tr^\sigma} O(r^{-4+\sigma}) \leq 0.
\end{equation}

To prove (3.48) we use a similar argument. Writing \( H[F_0 + \tilde{A}] \) as a polynomial in \( \tilde{A} \) we get that the \( \tilde{A}^0 \) term is equal to \( H_0 \) in (3.49) and:

\begin{equation}
H_1 = \frac{-7 \sigma \cos^2 \phi}{9g^2(\theta)r^{\theta+\sigma}} (7 + (2\phi' - \sigma) \sin^2 \phi) + \frac{1}{r^{\theta+\sigma}} O(r^{-1}).
\end{equation}

(3.53)

The other terms satisfy:

\begin{equation}
H_2 = \frac{1}{r^{\theta+\sigma}} O(r^{-3-\sigma}), \quad (A^2 \text{ term}),
\end{equation}

\begin{equation}
H_3 = \frac{1}{r^{\theta+\sigma}} O(r^{-6-2\sigma}), \quad (A^3 \text{ term}).
\end{equation}

Since \( H_0 = O(r^{-7}) \) the lemma follows by combing the above estimates. \( \square \)

Now we can prove Theorem 3.1: In fact, from (3.45), we have

\begin{equation}
F_0 \leq F_R \leq F_0 + \tilde{A}F_0, \quad \text{for } r = a_0,
\end{equation}

(3.54)

if we choose \( \tilde{A} \geq 1 \) such that

\begin{equation}
\max_{\theta} \frac{\mathcal{H} \left( a_0(-\cos(\theta)) + a_0^{3/2}(-\cos(\theta))(1 + \Lambda(|\cos(2\theta)|)^{\lambda-1}) \right)}{(a_0^3 + Aa_0^{3-\sigma})g(\theta)} \leq 1,
\end{equation}

(3.55)

which is possible since \( \frac{|\cos(2\theta)|}{g(\theta)} < \infty \) (this follows from (3.13) and the fact that \( g_\theta(\frac{\pi}{4}) > 0 \)). Note that (3.55) holds for any \( \tilde{A} \) large.
By comparison principle in the domain $\Gamma_R \setminus B_{a_0}$ (noting that the function $F_0 + \frac{\tilde{A}F_0}{r^\sigma}$ is a super-solution for $r > a_0$ by Lemma 3.6 and the function $F_0$ is a sub-solution by Lemma 3.5), we deduce that

\[(3.56) \quad F_0 \leq F_R \leq F_0 + \frac{\tilde{A}F_0}{r^\sigma}, \quad \text{in } \Gamma_R \setminus B_{a_0},\]

and hence

\[(3.57) \quad F_0 \leq F_R \leq F_0 + \tilde{A}r^{3-\sigma}, \quad \text{in } \Gamma_R \setminus B_{a_0},\]

for $\tilde{A}$ large.

Let $A \geq 1$ be a constant to be chosen later and let us consider the region $\Gamma_R \cap \{r > R_0\}$, where $R_0 = a_0 A^{\frac{1}{3+\sigma}}$. From (3.57), we then have

\[(3.58) \quad F_0 \leq F_R \leq F_0 + \tilde{A} R_0^{1-\sigma} \leq F_0 + \frac{A}{R_0^\sigma}, \quad \text{for } r = R_0\]

if we choose

\[(3.59) \quad \tilde{A} \leq \frac{A}{R_0^\sigma} = \frac{A}{a_0^3 A^{3+\sigma}} = a_0^{-3} A^{3+\sigma} .\]

By comparison principle applied now in $\Gamma_R \cap \{r > R_0\}$, using Lemma 3.6 we then obtain

\[(3.60) \quad F_0 \leq F_R \leq F_0 + \frac{A}{r^\sigma}, \quad \text{for } r \geq R_0 = a_0 A^{\frac{1}{3+\sigma}} .\]

The assertion of the Theorem follows now by combining (3.56) and (3.60) and letting $R \to \infty$.

The second Theorem of this section improves the super-solution and further refines the estimate on $F$.

**Theorem 3.2.** There exists $\sigma_0 \in (0, 1)$ such that for each $\sigma \in (0, \sigma_0)$ there exists $a_0 > 1$ such that for each sufficiently large $A \geq 1$, we have

\[(3.61) \quad H[F_0 + \frac{A\tanh(F_0 r^{-1})}{r^\sigma}] \leq 0, \quad \text{for } r > a_0 A^{\frac{1}{3+\sigma}} .\]

As a consequence there are constants $C, R_0$ such that the solution to the mean curvature equation described in Theorem 3.1 satisfies:

\[(3.62) \quad F_0 \leq F \leq F_0 + \frac{C\tanh(F_0 r^{-1})}{r^\sigma}, \quad \text{for } r > R_0 .\]

**Proof.** Let us prove (3.61) first. We will denote

\[F = F_0 + \frac{A}{r^\sigma} \varphi(t, s), \quad \varphi(t, s) = \tanh(t/r).\]
Note that the $A^0$ and $A^1$ terms in (3.31) are:

\[-\frac{1}{2} \partial_t \left( \frac{1}{|\nabla F_0|^2} \right) + \frac{A}{|\nabla F_0|^2} \partial_t^2 \varphi - \frac{A}{2} \partial_t \left( \frac{1}{|\nabla F_0|^2} \right) \partial_t \varphi + A \left( 1 + \frac{1}{|\nabla F_0|^2} \right) \partial_s \left( \frac{\rho^{-2} \varphi_s}{|\nabla F_0|^2} \right) - \frac{A}{2} \partial_s \left( \frac{1}{|\nabla F_0|^2} \right) \frac{\rho^{-2} \varphi_s}{|\nabla F_0|^2} \right]

(3.63)

\[= |\nabla F_0|^{-1} H[\nabla F_0] + A|\nabla F_0|^{-1} \left[ |\nabla F_0| \partial_t \left( \frac{\varphi_t}{|\nabla F_0|^2} \right) + |\nabla F_0| \partial_s \left( \frac{\rho^{-2} \varphi_s}{|\nabla F_0|^2} \right) \right] - A \left[ \frac{3}{2} \partial_t \left( \frac{1}{|\nabla F_0|^2} \right) \partial_t \varphi - \frac{1}{|\nabla F_0|^2} \partial_s \left( \frac{\rho^{-2} \varphi_s}{|\nabla F_0|^2} \right) + \frac{1}{2} \partial_s \left( \frac{1}{|\nabla F_0|^2} \right) \frac{\rho^{-2} \varphi_s}{|\nabla F_0|^2} \right].

We have by (3.51)

(3.64)

\[H_0 = |\nabla F_0|^{-1} H[\nabla F_0] \leq c_1 \frac{\cos^2 \phi}{tr^4} \leq c_1 \frac{\cos \phi}{r^7}.

Now we will deal with the first $A^1$ term in (3.63). This term is given explicitly in (3.33). We recall here that in (3.34) we have defined the following operator:

(3.65)

\[\tilde{L}_0[\varphi] := |\nabla F_0| \partial_t \left( \frac{\varphi_t}{|\nabla F_0|^2} \right) + |\nabla F_0| \partial_s \left( \frac{\rho^{-2} \varphi_s}{|\nabla F_0|^2} \right).

We will prove the following Lemma:

**Lemma 3.7.** There exists $\sigma_0 > 0$ such that for each $\sigma \in (0, \sigma_0)$ there exist $a_0 > 0$ and $c_0 > 0$ such that

(3.66)

\[\tilde{L}_0[r^{-\sigma} \tanh(t/r)] \leq -\frac{c_0}{r^{4+\sigma}} \min\{1, t/r\}, \quad r > a_0.

**Proof.** Let us denote:

\[\beta(\eta) = \tanh(\eta), \quad \eta = \frac{t}{r}, \quad \beta_1(\eta) = \beta(\eta) - \frac{1}{\sigma} \beta(\eta), \quad \beta_0(\eta) = \beta(\eta) - \frac{1}{\sigma} \beta(\eta),

and

(3.67)

\[\varphi = \beta(\eta)r^{-\sigma}, \quad \sigma > 0.

Then we compute

\[\partial_s \varphi = -\frac{\sigma r^{-\sigma} \sin^2 \phi}{t^2} \beta_1,

hence

\[\partial_s \left( \frac{\rho^{-2} \partial_s \varphi}{|\nabla F_0|^2} \right) = -c_1 \sigma \partial_s \left( \frac{r^{-\sigma} \cos^2 \beta_1}{t^2} \right)

where $c_1 > 0$. From now on, by $c_i > 0$ we will denote generic positive constants. We obtain

(3.68)

\[\partial_s \left( \frac{\rho^{-2} \partial_s \varphi}{|\nabla F_0|^2} \right)

= -\frac{c_1 \sigma r^{-6-\sigma}}{9g^2 + 9g} \left\{ \beta_1 \left[ 1 + \frac{2 \sin^2 \phi}{7} \left( -\frac{\sigma}{2} + \beta^1 \right) \right] - \frac{\eta \sin^2 \phi}{7} \beta_1 \right\}

On the other hand, we have

\[\partial_t \varphi_0 = -\frac{\sigma r^{-\sigma}}{3t} \beta \cos^2 \phi + \beta (1 - \frac{\cos^2 \phi}{3}) r^{-\sigma-1},\]
\[
\frac{\partial_t \phi_0}{|\nabla F_0|^2} = \frac{r^{1-\sigma} \cos^2 \phi}{9t^2} \left[ -\sigma \left( \frac{\beta}{\eta} \right)^2 \cos^2 \phi + \left( 1 - \cos^2 \phi \right) \beta' \right],
\]

hence
\[
(3.69)
\]
\[
\partial_t \left( \frac{1}{|\nabla F_0|^2} \partial_t \phi \right) = \left[ -\frac{2r^{1-\sigma} \cos^2 \phi}{9t^2} + \frac{r^{1-\sigma} \sin^2(2\phi) \phi'}{6t^3} \right] \left( 1 - \frac{\cos^2 \phi}{3} \right) \beta' \\
+ \frac{r^{-\sigma} \cos^2 \phi}{9t^2} \left[ -\sigma \left( \frac{\beta}{\eta} \right)' \cos^2 \phi + \left( 1 - \frac{\cos^2 \phi}{3} \right) \beta'' \right] \left( 1 - \frac{\cos^2 \phi}{3} \right) \\
+ O \left( \frac{\cos \phi}{r^{6+\sigma}} \right).
\]

The first term in (3.69) is negative. The second term can be estimated as follows
\[
\frac{r^{-\sigma} \cos^2 \phi}{9t^2} \left[ -\sigma \left( \frac{\beta}{\eta} \right)' \cos^2 \phi + \left( 1 - \frac{\cos^2 \phi}{3} \right) \beta'' \right] \left( 1 - \frac{\cos^2 \phi}{3} \right) \\
\leq \frac{c_2}{r^{6+\sigma}} \left[ -\sigma \left( \frac{\beta}{\eta} \right)' \cos^2 \phi + \frac{2}{3} \beta'' \right].
\]

Combining (3.68) and (3.70), we have
\[
\tilde{L}_0[\phi] \leq \frac{c_3}{r^{4+\sigma}} \left\{ -\sigma \beta_1 \left[ 1 + \frac{2 \sin^2 \phi}{t} \left( \frac{-\sigma}{2} + \phi' \right) \right] + \frac{\eta \sigma \sin^2 \phi}{t} \beta_1' \\
+ \left[ -\sigma \left( \frac{\beta}{\eta} \right)' \cos^2 \phi + \frac{2}{3} \beta'' \right] \right\} + O \left( \frac{\cos \phi}{r^{6+\sigma}} \right).
\]

Denoting the term in brackets above by \( \tilde{a} \) we can estimate as follows:
\[
\tilde{a} \leq \beta'' \left( c_4 \eta^2 \sin^2 \phi + \frac{2}{3} \right) - c_5 \sigma \left[ \beta - c_6 |\beta'| \eta - c_7 \left( \frac{\beta}{\eta} \right) \right] \leq \varepsilon_0, \quad \eta \geq \eta_0,
\]

where \( \beta \in (0, \sigma_0) \) with \( \sigma_0 > 0 \) small. Finally let us consider the last term in (3.71).

When \( \eta \leq 1 \) then
\[
\cos \phi \leq \frac{c_{12} \eta}{r^{6+\sigma}},
\]
while when \( 1 \leq \eta \) then
\[
\cos \phi \leq \frac{1}{r^{6+\sigma}}.
\]
Summarizing the above and (3.71)-(3.73) we have that for each \( \sigma \in (0, \sigma_0) \), where \( \sigma_0 \) is small, there exists \( r_0 > 0 \), \( c_0 \) such that :

\[
\tilde{L}_0[\varphi] \leq -\left( \frac{c_{13}}{r^{4+\sigma}} - \frac{c_{14}}{r^{6+\sigma}} \right) \min\{1, \eta\} \\
\leq -\frac{c_0}{r^{4+\sigma}} \min\{1, \eta\}, \quad r > r_0.
\]

(3.74)

\[\Box\]

**Remark 3.1.** We observe that Lemma \( \text{3.7} \) remains true with \( \beta(\eta) = \tanh(\eta) \) replaced by \( \beta(\eta) = 1 - e^{-\eta}, \eta > 0 \) with no change in the proof.

Continuing the proof of Theorem \( \text{3.2} \) we notice that

\[
-\frac{3}{2} \partial_t(\frac{1}{|\nabla F_0|^2}) \partial_t \varphi \leq -\frac{c_{15}}{r^{8+\sigma}} \cos \phi \leq \frac{c_{15}}{r^{8+\sigma}} \min\{\eta, 1\}
\]

(3.75)

since \( \cos \phi \leq \frac{2}{r^2} \), and

\[
\frac{1}{|\nabla F_0|^2} \partial_s(\frac{\rho^{-2} \varphi_s}{|\nabla F_0|^2}) \leq \frac{c_{16}}{r^{8+\sigma}} \min\{\eta, 1\}
\]

(3.76)

\[-\frac{1}{2} \partial_s(\frac{\rho^{-2} \varphi_s}{|\nabla F_0|^2}) \leq \frac{c_{17}}{r^{10+\sigma}} \leq \frac{c_{17}}{r^{10+\sigma}} \min\{\eta, 1\}.
\]

(3.77)

We analyze the \( A^2 \)-term and \( A^3 \) terms in the expansion of \( H[F_0 + Ar^{-\sigma} \tanh(t/r)] \).

A typical term in (3.31) is

\[
-\frac{1}{2} \partial_t(\frac{\rho^{-2} F_s^2}{|\nabla F_0|^2}) = \frac{\sigma^2 \sin^2 \phi \cos^2 \phi}{27 r^3 \beta^3} \left( 3 \cos^2 \phi - 3 + \cos(2\phi) \phi' \right) \beta_1^2 \\
-\frac{\sigma^2 \sin^2 \phi \cos^2 \phi}{9 r^3 \beta^3} 2\beta_1 \beta'_1 \eta (1 - \cos^2 \phi) \leq \sin^2 \phi \min\{\eta, 1\} O(r^{-7-2\sigma}).
\]

(3.78)

Other \( A^2 \) terms are estimated in a similar way. Direct calculations show that \( A^3 \)-term satisfy

\[
H_3 = \sin^2 \phi \min\{\eta, 1\} O(r^{-8-3\sigma}).
\]

(3.79)

In conclusion, we have

\[
H[F_0 + Ar^{-\sigma} \tanh(t/r)] \leq \left( \frac{c_1}{r^7} - \frac{c_0 A}{r^{10+\sigma}} + \frac{c_{18} A^2}{r^{7+2\sigma}} + \frac{c_{19} A^3}{r^{7+3\sigma}} \right) \min\{1, \eta\} \\
\leq 0,
\]

if we choose \( a_0 \) large and \( r \geq a_0 A^{-1}. \) This proves (3.61).

Now we will show (3.62). From (3.57), we have

\[
F_0 \leq F_R \leq F_0 + \tilde{A} F_0 r^{-\sigma}, \quad \text{for} \ r \geq a_0
\]

(3.81)

for some \( \tilde{A} \geq 1. \)

Let us consider the region

\[
\Sigma := B_R \cap \{ v > u \} \cap \{ r > R_0 \} \cap \{ 0 \leq \frac{F_0}{r} < 1 \},
\]

(3.82)
DE GIORGI CONJECTURE IN DIMENSION $N \geq 9$

where $R_0 = a_0 A^{\frac{1}{1 + \sigma}}$, and $A$ is to be chosen. From (3.57), we have in $\Sigma$:

$$F_0 \leq F_R \leq F_0 + \tilde{A} F_0 R_0^{-\sigma} \leq F_0 + \frac{A \tanh(F_0 R_0^{-1})}{R_0^\sigma}, \quad \text{for } r = R_0,$$

if we choose

$$\tilde{A} \leq \frac{A \tanh(F_0 R_0^{-1})}{R_0 F_0 R_0^{-1}} = \frac{A^{\sigma/(1 + \sigma)}}{a_0^{-1} \sup_{|\eta| < 1} \tanh \eta}. \tag{3.82}$$

Consider now the boundary $\{ \frac{F_0}{r} = 1 \}$. We have by (3.60):

$$F_0 \leq F_R \leq F_0 + \frac{A \tanh(F_0/r)}{r^\sigma}, \quad \text{for } r \geq R_0 \geq a_0 (\tanh(1) A)^{\frac{1}{1 + \sigma}}, \quad \text{and } F_0/r = 1,$$

if we chose (c.f. (3.59)):

$$\tilde{A} \leq a_0^{-3} (\tanh(1) A)^{\frac{1}{1 + \sigma}}. \tag{3.83}$$

Choosing $A$ larger if necessary we can assume that in addition to (3.83) also (3.85) is satisfied. By comparison principle applied to $\Sigma$, we then obtain

$$F_0 \leq F_R \leq F_0 + \frac{A \tanh(F_0/r)}{r^\sigma}, \quad \text{for } r \geq R_0.$$

Passing to the limit $R \to \infty$ we then get:

$$F_0 \leq F \leq F_0 + \frac{A \tanh(F_0/r)}{r^\sigma}, \quad \text{for } r \geq R_0,$$

in $\Sigma$. Combining this with the statement of Theorem 3.1 to estimate $F$ for $r > R_0$ in the complement of $\Sigma$ we complete the proof. 

□

4. Local coordinates for the minimal graph

The minimal graph of Bombieri, De Giorgi and Giusti, $\Gamma = \{ x_9 = F(x') \}$ can also be represented locally as a graph over its tangent hyperplane $T_{p_0} \Gamma$ at $p_0 = (x_0, F(x_0))$, with $|x_0| = R$. In other words, for each fixed $p_0 \in \Gamma$ there is a function $G(t)$ such that, for some $\rho, a > 0$,

$$\Gamma \cap B_\rho(p_0) = p_0 + \{(t, G(t)) \mid |t| < a \}$$

where $t = (t_1, \ldots, t_8)$ are the Euclidean coordinates on $T_{p_0} \Gamma$. More precisely, $F(x)$ and $G(t)$ are linked through the following relation:

$$\begin{bmatrix} x \\ F(x) \end{bmatrix} = \begin{bmatrix} x_0 \\ F(x_0) \end{bmatrix} + \Pi t + G(t)n(p_0). \tag{4.1}$$

Here

$$\Pi t = \sum_{j=1}^{8} t_j \Pi_j, \quad t \in \mathbb{R}^8,$$

where $\{\Pi_1, \Pi_2, \ldots, \Pi_8\}$ is a choice of an orthonormal basis for the tangent space to the minimal graph at the point $p_0 = (x_0, F(x_0))$, and

$$n(p_0) = \frac{1}{\sqrt{1 + |\nabla F(x_0)|^2}} \begin{bmatrix} \nabla F(x_0) \\ -1 \end{bmatrix}.$$
so that

\[ G(t) = \frac{1}{\sqrt{1 + |\nabla F(x_0)|^2}} (F(x) - F(x_0) - \nabla F(x_0) \cdot (x - x_0)). \]

The implicit function theorem implies that \( G \) and \( x \), given in equation (4.1), are smooth functions of \( t \), at least while \( |t| < a \) for a sufficiently small number \( a > 0 \). Clearly when \( p_0 \) is restricted to some fixed compact set than there exists a \( \theta > 0 \) such that

\[ a = \theta(1 + R), \quad R = |x_0|. \]

To show a similar bound for all \( p_0 \in \Gamma \) we will assume \( |x_0| = R > 1 \). The bound we are seeking amounts to estimating (from below) the largest \( a \) so that

\[ \sup_{|t| < a} |D_t G(t)| < +\infty. \]

Here and below by \( D_t, D^2_t \) etc. we will denote the derivatives with respect to the local variable \( t \). Let \( n(z) \) denote unit normal at the point \( z = (t, G(t)) \) (with some abuse of notation \( n(p_0) \equiv n(0) \)). Let us set

\[ \hat{t} = \frac{t}{|t|} \]

and consider the following curve on the minimal surface:

\[ r \mapsto \gamma(r) := (r \hat{t}, G(r \hat{t})), \quad 0 < r \leq |t|. \]

Then,

\[ \partial_r n(\gamma(r)) = A_\Gamma(\gamma(r))[\hat{t}, D_t G(r \hat{t}) \cdot \hat{t}] \]

where \( A_\Gamma \) is the second fundamental form on \( \Gamma \) and \( D_t G(r \hat{t}) = D_t G(t) |_{t=r \hat{t}} \). Thus

\[ |n(\gamma(r)) - n(0)| \leq \sup_{0 < s < r} |A_\Gamma(\gamma(s))| \int_0^r (1 + |G'(st)|) \, ds. \]

We will now make use of Simon's estimate (Theorem 4, p. 673 and Remark 2, p. 674 in [29]) which yields:

\[ \sup_{0 < s < r} |A_\Gamma(\gamma(s))| < \frac{c}{R}, \]

since we can assume that \( |t| < \theta R \), with some small \( \theta > 0 \). In addition we have that

\[ |n(\gamma(r)) - n(0)| \geq \frac{|D_t G(r \hat{t})|}{1 + |D_t G(r \hat{t})|}. \]

hence

\[ \frac{|D_t G(r \hat{t})|}{1 + |D_t G(r \hat{t})|} \leq \frac{c}{R} \int_0^r (1 + |D_t G(st)|) \, ds. \]

Let us write \( \varepsilon = \frac{c}{R} \) and

\[ \psi(r) := \int_0^r (1 + |D_t G(st)|) \, ds. \]

The above inequality reads

\[ 1 - \frac{1}{\psi(r)} \leq \varepsilon \psi(r), \]

or

\[ (1 - \varepsilon \psi(r)) \psi'(r) \leq 1, \]
so that for all sufficiently small (relative to the size of $\varepsilon$) $r > 0$ we have that
$$1 - (1 - \varepsilon\psi(r))^2 \leq 2\varepsilon r.$$  

Since $\psi(0) = 0$ it follows that
$$(1 - 2\varepsilon r)^{\frac{1}{2}} \leq (1 - \varepsilon\psi(r)),$$

hence
$$1 - \frac{1}{1 + |D_t G(rt)|} \leq \varepsilon\psi(r) \leq 1 - (1 - 2\varepsilon r)^{\frac{1}{2}},$$

which implies
$$|D_t G(t)| \leq (1 - 2\varepsilon|t|)^{-\frac{1}{2}} - 1 \leq 8\varepsilon|t|,$$

provided that $\varepsilon|t| < \frac{1}{4}$. Hence we have established that there are positive numbers $\theta$, $c$, independent of $R$ such that

$$|D_t x(t)| \leq \frac{c}{R} |t| \quad \text{for all} \quad |t| < \theta R.$$  

In particular, we obtain a uniform bound on $D_t G(t)$ for $|t| \leq \theta R$, while at the same time

$$|n(t, G(t)) - n(0)| \leq \frac{c}{R} |t| \quad \text{for all} \quad |t| < \theta R.$$  

This guarantees the fact that our minimal surfaces indeed defines a graph over the tangent plane at $p_0$, at least for $|t| \leq \theta R$. The quantities $x(t)$ and $G(t)$ linked by equation (4.1) are thus well-defined, provided that $|t| < \theta R$. The implicit function theorem yields in addition their differentiability. We have

$$\left[ \frac{D_t x(t)}{\nabla F(x) \cdot D_t x(t)} \right] = \Pi + D_t G(t)n(p_0),$$

and in particular $|D_t x(t)|$ is uniformly bounded in $|t| < \theta R$. The above relation also tells us that

$$|D_t^m x(t)| \leq |D_t^m G(t)|, \quad m \geq 2, \quad |t| < \theta R.$$  

Let us estimate now the derivatives of $G$. Since $G(t)$ represents a minimal graph, we have that

$$H[G] = \nabla_t \cdot \left( \frac{\nabla_t G}{\sqrt{1 + |\nabla_t G|^2}} \right) = 0 \quad \text{in} \quad B(0, \theta R) \subset \mathbb{R}^8.$$  

Let us consider now the change of variable

$$\tilde{G}(t) = \frac{1}{R} G(Rt),$$

and observe that $\tilde{G}$ is bounded and satisfies

$$H[\tilde{G}] = \nabla_t \cdot \left( \frac{\nabla_t \tilde{G}}{\sqrt{1 + |\nabla_t \tilde{G}|^2}} \right) = 0 \quad \text{in} \quad B(0, \theta).$$  

In fact from (4.2) we have

$$|\tilde{G}(t)| \leq C \quad \text{for all} \quad |t| \leq \theta,$$

hence, potentially reducing $\theta$, from standard estimates for the minimal surface equation (see for instance [18]) we find

$$|D_t \tilde{G}(t)| \leq C \quad \text{for all} \quad |t| \leq \theta,$$
with a similar estimate for \( D^2_G \), and in general the same bound for \( D^m_G \), \( m \geq 2 \) in this region. As a conclusion, using also (4.5) we obtain
\[
|D^m_t x(t)| + |D^m_t G(t)| \leq \frac{C}{R^{m-1}} \quad \text{for all} \quad |t| \leq \theta R
\]
for \( m = 2, 3, \ldots \). Summarizing, we have established:

**Lemma 4.1.** There exists a constant \( \theta > 0 \) such that for each \( p_0 = (x_0, F(x_0)) \in \Gamma \) the surface \( \Gamma \cap B(p_0, \theta(1 + R)) \), \( R = |x_0| \), can be represented as a graph over \( T_{p_0} \Gamma \) of a smooth function \( G(t) \). Moreover, denoting \( \Gamma \cap B(p_0, \theta(1 + R)) = \{(t, G(t)) \mid t \in T_{p_0} \Gamma \} \), we have whenever \( |t| \leq \theta(1 + R) \):
\[
|D_t G(t)| \leq \frac{c|t|}{1 + R},
\]
and
\[
|D^m_t G(t)| \leq \frac{c}{1 + R^{m-1}},
\]
with some universal constant \( c \).

We want to estimate with higher accuracy derivatives of \( G \), in their relation with the approximated minimal graph \( \Gamma_0 \), \( x_0 = F_0(x) \). We shall establish next that in the situation considered above we also have that \( \Gamma_0 \) can be represented as the graph of a function \( G_0(t) \) over the tangent plane to \( \Gamma \) at the point \( p_0 \), at least in a ball on that plane of radius \( \theta R \) for a sufficiently small, fixed \( \theta > 0 \) and for all large \( R \). Below we let \( \nu \) and \( n \) denote respective normal vectors to \( \Gamma_0 \) and \( \Gamma \), with the convention \( \nu \cdot n \geq 0 \). For convenience the situation is presented schematically in Figure 2.

To prove the above claim we will show that for fixed, sufficiently small \( \theta \) we have the estimate
\[
|\nu(q) - n(p_0)| < C\theta \quad \text{for all} \quad q \in \Gamma_0 \cap B(p_0, \theta R).
\]

Since by Theorem 3.1
\[
F(x) - F_0(x) = O(|x|^{-\sigma}), \quad \text{some} \quad \sigma \in (0, 1),
\]
we have that the points \( p_0 = (x_0, F(x_0)) \) and \( q_0 = (x_0, F_0(x_0)) \) satisfy
\[
|p_0 - q_0| \leq \frac{C}{R^\sigma}.
\]

Let \( T_{p_0} \Gamma, T_{q_0} \Gamma_0 \) be the corresponding tangent hyperplanes, namely
\[
T_{p_0} \Gamma = \{z \in \mathbb{R}^9 \mid (z - p_0) \cdot n(p_0) = 0\},
\]
\[
T_{q_0} \Gamma_0 = \{z \in \mathbb{R}^9 \mid (z - q_0) \cdot \nu(q_0) = 0\}.
\]
We assume that \( n(p_0) \cdot \nu(q_0) \geq 0 \). We claim that there is a number \( M > 0 \) such that for all large \( R \),
\[
|n(p_0) - \nu(q_0)| \leq \frac{5M}{R}.
\]

Let us assume the opposite and let us consider a point \( z \in T_{q_0} \Gamma_0 \) with
\[
\theta R > |z - q_0| > \frac{\theta}{2} R,
\]
and
\[
|n(p_0) - \nu(q_0)| \leq \frac{5M}{R}.
\]
with $\theta > 0$ as in (4.2). Let us write $\cos \alpha = n(p_0) \cdot \nu(q_0)$ with $0 \leq \alpha \leq \frac{\pi}{2}$. Then, using (4.13) we get
\[
(4.15) \quad \text{dist} (z, T_{p_0} \Gamma) \geq |z - p_0| \sin \alpha \geq \left(\frac{\theta}{2} R - R^{-\sigma}\right)|n(p_0) - \nu(q_0)| \geq M\theta.
\]
Let now $\tilde{q} \in \Gamma_0$ be the point whose projection onto $T_{q_0} \Gamma_0$ is $z$. Point $\tilde{q}$ is unique by the analog of (4.3) for the surface $\Gamma_0$. Let us denote $\tilde{q} = (\tilde{x}, F_0(\tilde{x}))$. Notice that $|\tilde{x}| \sim R$. We will also set $\tilde{p} = (\tilde{x}, F(\tilde{x})) \in \Gamma$. Since the second fundamental form of the surface $\Gamma_0$ satisfies an estimate similar to the one for $\Gamma$ we may assume, reducing $\theta$ if necessary,
\[
\text{dist} (\tilde{q}, T_{q_0} \Gamma_0) \leq c\theta.
\]
Now, estimate (4.2) implies that
\[
\text{dist} (\tilde{p}, T_{p_0} \Gamma) \leq c\theta.
\]
If $M$ is fixed so that $M\theta$ is sufficiently large, the above two relations and (4.13) are not compatible with (4.15), indeed we get:
\[
M\theta \leq \text{dist} (z, T_{p_0} \Gamma) \leq \text{dist} (\tilde{p}, \tilde{q}) + \text{dist} (\tilde{p}, T_{p_0} \Gamma_0) + \text{dist} (\tilde{q}, T_{q_0} \Gamma_0) \\
\leq c\theta + \text{dist} (\tilde{p}, \tilde{q}) \\
\leq \frac{C}{R^\sigma} + c\theta,
\]
hence (4.14) holds. Moreover, using estimate (4.3) and the analogous estimate for the variation of $\nu$ we have the validity of the estimate
\[
|\nu(q) - \nu(q_0)| + |n(p) - n(p_0)| < C\theta \quad \forall p \in \Gamma \cap B(p_0, \theta R), \forall q \in \Gamma_0 \cap B(q_0, \theta R).
\]
Furthermore, we observe that analog of the estimate (4.3) implies that in the set \( \Gamma_0 \cap B(q_0, \theta R) \) the distance between \( \Gamma_0 \) and its tangent plane at \( q_0 \) varies by no more that \( c\theta \). From this and (4.13) and (4.14) the desired conclusion (4.12) immediately follows (taking \( \theta \) smaller if necessary). Hence the function \( G_0(t) \) is well-defined for \( |t| < \theta R \).

Let us observe that \( F_0 \) and \( G_0 \) are linked through the following relation:

\[
(4.16) \quad \begin{bmatrix} \tilde{x} \\ F_0(\tilde{x}) \end{bmatrix} = \begin{bmatrix} x_0 \\ F(x_0) \end{bmatrix} + \Pi t + G_0(t)n(p_0)
\]

By the implicit function theorem, \( \tilde{x} \) and \( G_0(t) \) define differentiable functions of \( t \) for \( |t| \leq \theta R \). We shall establish derivative estimates for \( G_0 \) similar to those found for \( G \). We claim that

\[
(4.17) \quad |D^m \tilde{x}(t)| + |D^m G_0(t)| \leq \frac{C}{R^{m-1}} \quad \text{for all} \quad |t| \leq \theta R,
\]

for \( m = 1, 2, \ldots \). Differentiation of relation (4.16) yields

\[
(4.18) \quad \begin{bmatrix} \partial_j \tilde{x} \\ \nabla F_0(\tilde{x}) \partial_j \tilde{x} \end{bmatrix} = \Pi_j + \partial_j G_0 n(p_0).
\]

Let \( q = (\tilde{x}, F_0(\tilde{x})) \) and

\[
\nu(q) = \frac{1}{\sqrt{1 + |\nabla F_0(\tilde{x})|^2}} \begin{bmatrix} \nabla F_0(\tilde{x}) \\ -1 \end{bmatrix}.
\]

From (4.18) and the fact that \( \nu(q) \cdot n(p_0) \geq c > 0 \) we then get

\[
|\partial_j G_0(t)| \leq C|\Pi_j \cdot \nu(q)| \leq C.
\]

Using again relation (4.18) we also get

\[
|\partial_j \tilde{x}(t)| \leq C.
\]

Let us differentiate again. Now we get

\[
(4.19) \quad \begin{bmatrix} \partial_j k \tilde{x} \\ \nabla F_0(\tilde{x}) \partial_j k \tilde{x} \end{bmatrix} + \begin{bmatrix} 0 \\ D^2 F_0(\tilde{x})[\partial_j \tilde{x}, \partial_j \tilde{x}] \end{bmatrix} = \partial_{jk} G_0 n(p_0).
\]

Again, taking the dot product against \( \nu(p) \) we get

\[
|\partial_{jk} G_0(t)| \leq C \frac{|D^2 F_0(\tilde{x})|}{\sqrt{1 + |\nabla F_0(\tilde{x})|^2}} \leq \frac{C}{R}
\]

and thus

\[
|\partial_{jk} \tilde{x}(t)| \leq \frac{C}{R}
\]

Iterating this argument, using that

\[
|D^m F_0(\tilde{x})| \leq CR^{3-m}, \quad m = 1, 2, \ldots
\]

the desired result (4.17) follows.

Let us write

\[
G(t) = G_0(t) + h(t).
\]

We will estimate first the size of \( h(t) \) in the ball \( |t| \leq \theta R \). We claim that we have

\[
(4.20) \quad |h(t)| \leq CR^{1-\sigma} \quad \text{for all} \quad |t| \leq \theta R.
\]
First observation we make is that when \( t = 0 \) we have:

\[(4.20)\]

\[|h(0)| = |G_0(0)| \leq \frac{C}{R^{1+\sigma}}.\]

To show this let \( \tilde{x} \) be such that

\[
\begin{bmatrix}
\tilde{x} \\
F_0(\tilde{x})
\end{bmatrix} = \begin{bmatrix} x_0 \\
F(x_0) \end{bmatrix} + G_0(0)n(p_0),
\]

and let \( \tilde{t} \) be such that

\[
\begin{bmatrix}
\tilde{x} \\
F(\tilde{x})
\end{bmatrix} = \begin{bmatrix} x_0 \\
F(x_0) \end{bmatrix} + \Pi \tilde{t} + G(\tilde{t})n(p_0).
\]

Comparing these two expressions and using \(|F(\tilde{x}) - F_0(\tilde{x})| \sim R^{-\sigma}\) we see that \(|\tilde{t}| \sim R^{-\sigma}\) hence, by (4.10), we get that \(|G(\tilde{t})| \sim R^{-1-2\sigma}\). Now multiplying the above relations by \( n(p_0) \) and subtracting them we infer (4.21) since by Theorem 4 p.673 and Theorem 5 p.680 [29], we have that

\[
|n_0(p_0)| = \frac{1}{\sqrt{1 + |\nabla F(p_0)|^2}} \leq \frac{C}{R}.
\]

To prove (4.20) now we let \( p_1 = (x_1, F(x_1)) \in \Gamma \cap B(p_0, \theta R) \) so that:

\[ p_1 = p_0 + \Pi t + G(t)n(p_0), \quad |t| \leq \theta R.\]

Then \(|G(t) - G_0(t)|\) corresponds to the length of the segment with direction \( n(p_0) \) starting at \( p_1 \), with end on the surface \( \Gamma_0 \). Let \( p_2 = (x_1, F_0(x_1)) \). Then

\[ |p_1 - p_2| \leq CR^{-\sigma}.\]

Let us consider the tangent plane \( T_{p_2} \Gamma_0 \) to \( \Gamma_0 \) at \( p_2 \), with normal \( \nu(p_2) \). Then, \( \Gamma_0 \cap B(p_2, CR^{-\sigma}) \) lies within a distance \( O(R^{-1-\sigma}) \) from \( T_{p_2} \Gamma_0 \), more precisely,

\[ \Gamma_0 \cap B(p_2, CR^{-\sigma}) \subset \mathcal{C}_R, \]

where \( \mathcal{C}_R \) is the cylinder

\[ \mathcal{C}_R = \{ \tilde{x} + sv(p_2) | \tilde{x} \in T_{p_2} \Gamma_0, |\tilde{x} - p_2| \leq CR^{-\sigma}, |s| \leq CR^{-1-\sigma} \}. \]

Using (4.21) we may assume that \( p_1 \in \mathcal{C}_R \). In particular, the line starting from \( p_1 \) with direction \( n(p_1) \) intersects \( \Gamma_0 \) inside this cylinder. Since \( n(p_1) \cdot \nu(p_2) \geq c > 0 \), the length of this segment is of the same order as the height of the cylinder, and we then get

\[ |G(t) - G_0(t)| \leq CR^{1-\sigma}, \]

hence (4.20) holds.

Next we shall improve the previous estimate. We claim that we have

\[(4.22)\]

\[|D_t^m h(t)| \leq \frac{c}{R^{m+1+\sigma}} \quad \text{in} \quad |t| < \theta R,\]

for \( m = 0, 1, 2, \ldots \). Let us set

\[
\dot{G}(t) = \frac{1}{R} G(Rt), \quad \dot{G}_0(t) = \frac{1}{R} G_0(Rt), \quad \dot{h}(t) = \frac{1}{R} h(Rt).
\]

We compute (for brevity dropping the subscript in the derivatives):

\[
\sqrt{1 + |\nabla \dot{G}|^2} \cdot H[\dot{G}] = \Delta \dot{G} - \frac{D^2 \dot{G}[\nabla \dot{G}, \nabla \dot{G}]}{1 + |\nabla \dot{G}|^2} = 0.
\]
Now,
\[
\frac{D^2\tilde{G}[\nabla\tilde{G},\nabla\tilde{G}]}{1+|\nabla\tilde{G}|^2} = \frac{D^2h[\nabla\tilde{G},\nabla\tilde{G}]}{1+|\nabla\tilde{G}|^2} + \frac{D^2\tilde{G}_0[\nabla\tilde{G},\nabla\tilde{G}]}{1+|\nabla\tilde{G}|^2},
\]
and
\[
\frac{D^2h[\nabla\tilde{G},\nabla\tilde{G}]}{1+|\nabla\tilde{G}|^2} = \frac{D^2\tilde{G}_0[\nabla\tilde{G}_0,\nabla\tilde{G}_0]}{1+|\nabla\tilde{G}_0|^2} + \frac{D^2\tilde{G}_0[2\nabla\tilde{G}_0 + \nabla h, \nabla h]}{1+|\nabla\tilde{G}_0|^2}.
\]
Furthermore,
\[
\frac{D^2\tilde{G}_0[\nabla\tilde{G}_0,\nabla\tilde{G}_0]}{1+|\nabla\tilde{G}_0|^2} = \frac{D^2\tilde{G}_0[\nabla\tilde{G}_0,\nabla\tilde{G}_0]}{1+|\nabla\tilde{G}_0|^2}
- \frac{D^2\tilde{G}_0[\nabla\tilde{G}_0,\nabla\tilde{G}_0](2\nabla\tilde{G}_0 + \nabla h) \cdot \nabla h}{(1+|\nabla\tilde{G}_0|^2)(1+|\nabla\tilde{G}|^2)}.
\]
Collecting terms we see that \( h \) satisfies the equation
\[
\Delta h - \frac{D^2h[\nabla\tilde{G},\nabla\tilde{G}]}{1+|\nabla\tilde{G}|^2} + b \cdot \nabla h + E = 0, \quad \text{in } B(0, \theta),
\]
where
\[
E = \Delta \tilde{G}_0 - \frac{D^2\tilde{G}_0[\nabla\tilde{G}_0,\nabla\tilde{G}_0]}{1+|\nabla\tilde{G}_0|^2} = \sqrt{1+|\nabla\tilde{G}_0|^2}H(\tilde{G}_0),
\]
and
\[
b = -\frac{D^2\tilde{G}_0[\nabla\tilde{G}_0,\nabla\tilde{G}_0](2\nabla\tilde{G}_0 + \nabla h)}{(1+|\nabla\tilde{G}_0|^2)(1+|\nabla\tilde{G}|^2)} + \frac{D^2\tilde{G}_0[2\nabla\tilde{G}_0 + \nabla h]}{1+|\nabla\tilde{G}|^2}.
\]
Notice that:
\[
|\nabla\tilde{G}(t)| \leq C, \quad |\tilde{h}(t)| \leq C R^{2-\sigma} \quad \text{in } |t| < \theta.
\]
Also by (3.41) the mean curvature of \( \Gamma_0 \) decays like \( R^{-5} \). From
\[
|E(t)| = R\left| \Delta \tilde{G}_0 - \frac{D^2\tilde{G}_0[\nabla\tilde{G}_0,\nabla\tilde{G}_0]}{1+|\nabla\tilde{G}_0|^2} \right|(Rt)
= R\sqrt{1+|\nabla\tilde{G}_0|^2}H(\tilde{G}_0)(Rt)
= R\sqrt{1+|\nabla\tilde{G}_0(Rt)|^2}H[F_0](\tilde{x}(Rt)),
\]
(in the notation of (4.16)) we then find
\[
|E(t)| = O(R^{-4}),
\]
and, as a conclusion, reducing \( \theta \) if needed,
\[
|D_t \tilde{h}(t)| \leq \frac{c}{R^{2+\sigma}} \quad \text{in } |t| < \theta,
\]
so that for \( h \) we get accordingly
\[
|D_t h(t)| \leq \frac{c}{R^{2+\sigma}} \quad \text{in } |t| < \theta R.
\]
On the other hand, using (4.17) we have for instance that
\[
D_t H[G_0](t) = D_x H[F_0](\tilde{x}(t))D_t \tilde{x}(t) = O(R^{-6}),
\]
hence
\[
|D_t E(t)| = O(R^{-4}).
\]
More generally, since
\[
D_t^m H[F_0](x) = O(|x|^{-5-m}),
\]
we get
\[ D^m_t E(t) = O(R^{-4}). \]
This, estimates (4.17), (4.9) and standard higher regularity elliptic estimates yield
\[ |D^m_t \tilde{h}(t)| \leq \frac{c}{R^{2+\sigma}} \quad \text{in } |t| < \theta R. \]
Hence
\[ |D^m_t h(t)| \leq \frac{c}{R^{m+1+\sigma}} \quad \text{in } |t| < \theta R. \]
for \( m \geq 1 \), as desired.

Now we will derive some consequences of the above estimates. First we will
consider the Fermi coordinates near \( \Gamma \). Let \( x = (x', x_9) \in \mathbb{R}^9 \) be a point in a
neighborhood of \( \Gamma \) and let \( p(x) \) be its projection on \( \Gamma \) in the direction of \( n(p) \). If
\( \text{dist}(x, \Gamma) \) is sufficiently small then \( p(x) \) is unique and we can write:
\[ x = p + zn(p), \]
where \( z = z(x) \). These \((p, z)\) are the Fermi coordinates of \( x \). They are defined as
long as the function \( x \mapsto (p, z) \) is invertible. We claim that this the case, and that
the Fermi coordinates are well defined as long as
\[ |z| \leq \theta |A_\Gamma(p)|^{-1}, \]
where \( R = |\Pi_{\mathbb{R}^8}(p)| \) is the the distance of the projection of \( p \) onto \( \mathbb{R}^8 \) from the
origin, and \( \theta \) is chosen to be a small number. Because of the symmetry of the
surface \( \Gamma \), it is enough to consider the situation in which, for certain \( x = (x', x_9) \)
such that \( x' \in T \) we have the existence of two different points \( p_1, p_2 \in \Gamma \cap T \times [0, \infty) \)
such that
\[ x = p_i + zn(p_i), \quad i = 1, 2, \]
with \( z \) satisfying (4.24). We may assume that \( |\Pi_{\mathbb{R}^8}(p_1)| \) is large. Then it
follows:
\[ |p_1 - p_2| \leq |z| |n(p_1) - n(p_2)| \leq \theta |A_\Gamma(p_1)|^{-1}. \]
In the portion of \( \Gamma \) where (4.26) holds we have in fact:
\[ |p_1 - p_2| \leq |z| |n(p_1) - n(p_2)| \]
\[ \leq \theta |A_\Gamma(p_1)|^{-1} \sup_{|p_1 - p| \leq \theta |A_\Gamma(p_1)|} |A_\Gamma(p)||p_1 - p_2|. \]
(4.27)
\[ \leq C \theta R_1 + |p_1 - p_2|. \]
Now the claim follows if we take \( \theta > 0 \) to be sufficiently small. From (4.24) we get
that the Fermi coordinates are well defined in an expanding neighborhood \( \mathcal{U}_{\theta_0} \) of
\( \Gamma_\alpha \) defined in (2.13).
Second we will compute the derivative of \( z(x) \) with respect to \( x_9 \). Since \( p \in \Gamma \)
therefore we can write
\[ p = (y', F(y')), \quad y' \in \mathbb{R}^8. \]
Then, taking the derivative with respect to \( x_9 \) and using (4.23) we get:
\[ e_9 = (\partial_{x_9} y', \nabla F(y') \cdot \partial_{x_9} y') + \partial_{x_9} zn(p), \quad \text{at } z = 0. \]
Notice that
\[ \tau(p) = (\partial_{x_0} y', \nabla F(y') \cdot \partial_{x_0} y') \in T_p \Gamma. \]

Multiplying (4.28) by \( \nu(q) \), where \( q \in \Gamma_0 \) is a point on the segment in the direction of \( n(p) \) and \( \nu(q) \) is the unit normal at this point we get
\[ (4.29) \quad \frac{-1}{\sqrt{1 + |\nabla F_0(q)|^2}} = \tau(p) \cdot \nu(q) + \partial_{x_0} z n(p) \cdot \nu(q). \]

From (4.28) we get as well:
\[ (4.30) \quad |\tau(p)| \leq 1 + |\partial_{x_0} z|. \]

Now let \((t, G(t))\) and \((t, G_0(t))\) be the local coordinates centered respectively at \( p \in \Gamma \) and \( q \in \Gamma_0 \). From the above discussion we conclude that any two unit tangent vectors in the same direction, say \( t_i \) differ by a factor proportional to \( D_t h(t) \). This means
\[ |\tau(p) \cdot \nu(q)| \leq C|D_t h(t)| \leq \frac{C}{R^{2+\sigma}}, \]

hence using (4.29) we get
\[ |\partial_{x_0} z| |n(p) \cdot \nu(q)| = \frac{1}{\sqrt{1 + |\nabla F_0(q)|^2}} + O\left( \frac{1}{R^{2+\sigma}} \right). \]

Since \( |n(p) \cdot \nu(q)| > c > 0 \) uniformly, it follows:
\[ (4.31) \quad \frac{C^{-1}}{1 + R^2} \leq |\partial_{x_0} z| \leq \frac{C}{1 + R^2}. \]

Notice that as a byproduct we get
\[ n_0(p) = \partial_{x_0} z, \]

hence
\[ (4.32) \quad \frac{C^{-1}}{1 + R^2} \leq \frac{1}{\sqrt{1 + |\nabla F(p)|^2}} \leq \frac{C}{1 + R^2}. \]

which is a special case of the estimate in Theorem 5 p. 679 in [29].

Next we will discuss the expressions for the Laplace-Beltrami operators on \( \Gamma \) and \( \Gamma_0 \) in terms of the local coordinates associated with \( G(t) \) and \( G_0(t) \). Let us observe that the metric tensor \( g \) in \( \Gamma \cap B(p_0, \theta R) \) satisfies the following relation:
\[ (4.33) \quad g_{ij} = \delta_{ij} + \partial_i G(t) \partial_j G(t) = \delta_{ij} + O(|t| R^{-2}), \quad |t| \leq \theta R, \]

where \( \partial_i = \partial_{t_i} \). Similar estimates hold for the metric tensor \( g_0 \) on the surface \( \Gamma_0 \) expressed in the same local coordinates. In fact we have:
\[ (4.34) \quad g_{0,ij} = \delta_{ij} + \partial_i G_0(t) \partial_j G_0(t) = g_{ij} - \partial_i G(t) \partial_j h(t) - \partial_j G(t) \partial_i h(t) + \partial_i h(t) \partial_j h(t) \]

where \( \partial_i = \partial_{t_i} \). Similar estimates hold for the metric tensor \( g_0 \) on the surface \( \Gamma_0 \) expressed in the same local coordinates. In fact we have:
\[ (4.35) \quad \tilde{f}(\tilde{x}) = f(t), \quad \tilde{x} = \tilde{z}(t). \]
Then using (4.34) we have

\[(4.36) \quad |\Delta f - \Delta \tilde{f}| \leq C|\nabla^2 f| R^2 + C|\nabla f| R^3 \sigma,\]

as long as $|t| \leq \theta R$. In the sequel we will denote functions on $\Gamma$ and on $\Gamma_0$ by the same symbol taking into account (4.35).

Finally, let us consider the second fundamental form on $\Gamma$, $A_\Gamma$, and the second fundamental form on $\Gamma_0$ denoted by $A_{\Gamma_0}$. We observe that in the local coordinates associated with the graph $G(t)$ over the tangent space we have:

\[|A_\Gamma(t)| = |D^2 t G(t)|, \quad at \quad t = 0.\]

Thus

\[(4.37) \quad |A_{\Gamma_0}(t)| = |A_\Gamma(t)| + O(|D^2 t h(t)|) = |A_\Gamma| + O(R^{-3-\sigma}), \quad at \quad t = 0.\]

It follows:

\[(4.38) \quad |A_\Gamma - A_{\Gamma_0}| \leq \frac{C}{1 + R^{3+\sigma}}.\]

5. Linear theory

In this section we will consider the basic linearized operator and we will derive a solvability theory for the operator

\[(5.1) \quad L(\phi) = \Delta \phi + f'(w)\phi,\]

already defined in (2.20).

5.1. Nondegeneracy of the approximate solution. To begin with we will review some basic facts about the one dimensional version of $L$. By $w$ we will denote the heteroclinic solution to $w'' + w - w^3 = 0$ such that $w(0) = 0, w(\pm \infty) = \pm 1$ namely

\[w(z) = \tanh \left( \frac{z}{\sqrt{2}} \right).\]

Consider the one-dimensional linear operator

\[L_0(\phi) = \phi_{zz} + f'(w)\phi, \quad f'(w) = 1 - 3w^2.\]

We recall some well known facts about $L_0$. First notice that $L_0(w_z) = 0$. Second, writing $\phi = w_z \psi$ we get that

\[L_0(\phi) = L_0(\psi w_z) = w_z \psi_{zz} + 2w_z w_{zz} + w_{zzz} \psi + f'(w)w_z \psi = w_z^{-1} (w_z^2 \psi_z)_z,\]

hence assuming that $\phi(z)$ and its derivative decay fast enough as $|z| \to +\infty$, we get the identity

\[\int_{\mathbb{R}} L_0(\phi) dz = \int_{\mathbb{R}} [\phi_z^2 - f'(w) \phi_z^2] dz = \int_{\mathbb{R}} w_z^2 |\psi_z|^2 dz.\]

Since $f'(w) \to -2$ as $|z| \to +\infty$ guarantees exponential decay of any bounded solution of $L_0(\phi) = 0$ therefore any such solution must be of the form $\phi = Cw_z$, $C \in \mathbb{R}$. Next we set, for $\phi(y, z)$ defined in $\mathbb{R}^m \times \mathbb{R}$, (where $m = 8$ in our case):

\[L(\phi) = \Delta_y \phi + \phi_{zz} + f'(w)\phi = \Delta_y \phi + L_0(\phi).\]
The equation $L(\phi) = 0$, has the obvious bounded solution $\phi(y, z) = w_z(z)$. Less obvious, but also true, is the converse:

**Lemma 5.1.** Let $\phi$ be a bounded, smooth solution of the problem

\begin{equation}
L(\phi) = 0 \quad \text{in } \mathbb{R}^m \times \mathbb{R}.
\end{equation}

Then $\phi(y, z) = Cw_z(z)$ for some $C \in \mathbb{R}$.

**Proof.** Let $\phi$ be a bounded solution of equation (5.2). We claim that $\phi$ has exponential decay in $z$, uniform in $y$. Let $0 < \sigma < 1$ and let us fix $z_0 > 0$ such that for all $|z| > z_0$ we have that

\[ f'(w) < -2 + \sigma^2. \]

In addition, consider the following function

\[ g_\delta(z, y) = e^{-\sigma(|z| - z_0)} + \delta \sum_{i=1}^m \cosh(\sigma y_i). \]

for $\delta > 0$. Then for $|z| > z_0$ we get that

\[ L(g_\delta) = \sigma^2 g_\delta + f'(w) g_\delta, \]

so that if $\sigma > 0$ is fixed small enough, we get

\[ L(g_\delta) < 0 \quad \text{if } |z| > z_0. \]

As a conclusion, using maximum principle, we get

\[ |\phi| \leq \|\phi\|_\infty g_\delta \quad \text{if } |z| > z_0, \]

and letting $\delta \to 0$ we then get

\[ |\phi(y, z)| \leq C\|\phi\|_\infty e^{-\sigma|z|} \quad \text{if } |z| > z_0. \]

Let us observe the following fact: the function

\[ \tilde{\phi}(y, z) = \phi(y, z) - \left( \int_{\mathbb{R}} w_z(\zeta) \phi(y, \zeta) d\zeta \right) \frac{w_z(z)}{\int_{\mathbb{R}} w^2_z} \]

also satisfies $L(\tilde{\phi}) = 0$ and, in addition,

\begin{equation}
\int_{\mathbb{R}} w_z(z) \tilde{\phi}(y, z) dz = 0 \quad \text{for all } y.
\end{equation}

In view of the above discussion, it turns out that the function

\[ \varphi(y) := \int_{\mathbb{R}} \tilde{\phi}^2(y, z) dz \]

is well defined. In fact so are its first and second derivatives by elliptic regularity theory applied to $\phi$, and differentiation under the integral sign is thus justified. Now, observe that

\[ \Delta_y \varphi(y) = 2 \int_{\mathbb{R}} \Delta_y \tilde{\phi} \cdot \tilde{\phi} dz + 2 \int_{\mathbb{R}} |\nabla_y \tilde{\phi}|^2 \]

and hence

\begin{equation}
0 = \int_{\mathbb{R}} (L_0 \tilde{\phi} \cdot \tilde{\phi}) \quad (5.4)
\end{equation}

\[ = \frac{1}{2} \Delta_y \varphi - \int_{\mathbb{R}} |\nabla_y \tilde{\phi}|^2 dz - \int_{\mathbb{R}} (|\tilde{\phi}|^2 - f'(w) \tilde{\phi}^2) dz. \]
Let us observe that because of relation (5.3), we have that
\[ \int_{\mathbb{R}} (|\tilde{\phi}_z|^2 - f'(w)\tilde{\phi}^2) \, dz \geq \gamma \varphi. \]
It follows then that
\[ \Delta_y \varphi - \gamma \varphi \geq 0 \]
Since \( \varphi \) is bounded, from maximum principle we find that \( \varphi \) must be identically equal to zero. But this means
\[ \phi(y, z) = \left( \int_{\mathbb{R}} w_\zeta(\zeta) \phi(y, \zeta) \, d\zeta \right) \frac{w_z(z)}{w_\zeta}. \]
Then the bounded function
\[ g(y) = \int_{\mathbb{R}} w_\zeta(\zeta) \phi(y, \zeta) \, d\zeta \]
satisfies the equation
\[ \Delta_y g = 0, \text{ in } \mathbb{R}^m. \]
Liouville’s theorem then implies that \( g \equiv \text{constant} \). Relation (5.5) then yields
\[ \phi = C w_z(z) \text{ for some } C. \] This concludes the proof. \( \square \)

5.2. Linearized problem near \( \Gamma_\alpha \). Let \( \theta_0 > 0 \) be a number such that the Fermi coordinates are well defined for \( x \in \mathbb{R}^9 \) satisfying \( d(x, \Gamma) < \theta_0 \). Here \( \theta_0 > 0 \) may be taken to be the same as in the definition of the expanding neighborhood \( U_{\theta_0} \), see (2.13). We will define the \( \delta \)-neighborhood of \( \Gamma_\alpha \) to be
\[ \mathcal{N}_\delta = \left\{ x \in \mathbb{R}^9 \mid d(x, \Gamma_\alpha) < \frac{\delta}{\alpha} \right\}. \]
Now, we let \( \delta > 0 \) be such that \( 4\delta < \theta_0 \) and consider neighborhood of the form \( \mathcal{N}_{4\delta} \). Observe that with this \( \delta \) fixed the approximate solution \( w \) defined in (2.16) satisfies \( w(x) = w(z - h_\alpha(y)) \), where \( (y, z) \) are the Fermi coordinates of \( x \in \mathcal{N}_{4\delta} \).
We will also denote
\[ \Gamma_{\alpha, z} = \left\{ x \in \mathbb{R}^9 \mid d(x, \Gamma_\alpha) = z \right\}, \quad |z| < \frac{4\delta}{\alpha}. \]
In \( \mathcal{N}_{4\delta} \) we can write the Laplacian in the local Fermi coordinates:
\[ \Delta = \Delta_{\Gamma_{\alpha, z}} + \partial_z^2 - H_{\Gamma_{\alpha, z}} \partial_z, \]
where \( H_{\Gamma_{\alpha, z}} \) denotes the mean curvature of the surface \( \Gamma_{\alpha, z} \). Expression (5.7) is valid only in \( \mathcal{N}_{4\delta} \) however it is convenient to extend it in such a way that it makes sense for all \( z \in \mathbb{R} \). To this end let \( \eta(\tau) \) be a smooth cut-off function with \( \eta(\tau) = 1 \), for \( |\tau| < 1 \) and \( \eta(\tau) = 0 \), \( |\tau| > 2 \), and let us denote:
\[ \eta_0^{\alpha} = \eta\left(\frac{\alpha z}{\delta}\right). \]
Then we define \( \mathcal{A}_0^{\alpha} \) to be the following operator
\[ \mathcal{A}_0^{\alpha} = \eta_0^{\alpha} (\Delta_{\Gamma_{\alpha, z}} + \partial_z^2 - H_{\Gamma_{\alpha, z}} \partial_z) + (1 - \eta_0^{\alpha})(\Delta_{\Gamma_{\alpha}} + \partial_z^2). \]
This operator is defined in the set \( \Gamma_\alpha \times \mathbb{R} \) and not just in the set \( \mathcal{N}_{15} \). We notice that \( \Gamma_\alpha \times \mathbb{R} \) can be parametrized by \( \mathbb{R}^9 \) and it is equipped with the natural product metric. In the sequel we will write

\[
A_\alpha^\delta(\phi) = \Delta_{\Gamma_\alpha} \phi + \partial_\alpha^2 \phi + B_\alpha^\delta(\phi),
\]

where

\[
B_\alpha^\delta(\phi) = \eta_\delta^\alpha (\Delta_{\Gamma_\alpha,z} - \Delta_{\Gamma_\alpha} - \Gamma_{\alpha,z} \partial_z)(\phi).
\]

We will introduce now some norms for functions defined in \( \Gamma_\alpha \times \mathbb{R} \). By \( dV_{\Gamma_\alpha} \), we will denote the volume element of \( \Gamma_\alpha \). In the sequel we will write

\[
\mathcal{L}^p_{\text{loc}}(\Gamma_\alpha), \quad 1 < p < \infty,
\]

where

\[
\mathcal{L}^\infty_{\text{loc}}(\Gamma_\alpha) \subset \mathcal{K}_{\text{comp}}.
\]

Similarly we define \( \mathcal{L}^\infty_{\text{loc}}(\Gamma_\alpha) \) to be the set of locally bounded functions on \( \Gamma_\alpha \). In \( \mathcal{L}^p_{\text{loc}}(\Gamma_\alpha \times \mathbb{R}) \), \( 1 < p < \infty \) we will introduce the following weighted norms:

\[
\|g\|_{p,\sigma} := \sup_{(y,z) \in \Gamma_\alpha \times \mathbb{R}} e^{\sigma \cdot |z|} \left\| g \right\|_{L^p(\Gamma_\alpha \cap B(y,1) \times (z-1,z+1))},
\]

where \( \sigma \) is such that \( 0 < \sigma < \sqrt{2} \). In the sequel \( \sigma \) will be taken to be smaller as necessary but always \( \alpha \)-independent. Similarly, we define

\[
\|g\|_{\infty,\sigma} := \sup_{(y,z) \in \Gamma_\alpha \times \mathbb{R}} e^{\sigma \cdot |z|} |g(y,z)|, \quad g \in \mathcal{L}^\infty_{\text{loc}}(\Gamma_\alpha \times \mathbb{R}).
\]

Observe that these definitions are consistent in the sense that if we take formally \( p = \infty \) in the definition of the norm \( \| \cdot \|_{p,\sigma} \) then the resulting norm will be equivalent to the \( \| \cdot \|_{\infty,\sigma} \) we have actually defined.

Our next goal is to establish a solvability theory for the following problem:

\[
\Delta_{\Gamma_\alpha} \phi + B_\alpha(\phi) + \partial_\alpha^2 \phi + f'(w)\phi = g + c_\alpha w, \quad \text{in} \ \Gamma_\alpha \times \mathbb{R},
\]

\begin{equation}
\int_{\mathbb{R}} \phi(y,z)w(z) \, dz = 0, \quad \forall y \in \Gamma_\alpha.
\end{equation}

(5.9)

where \( B_\alpha(\phi) \) is a second order differential operator of the form:

\[
B_\alpha(\phi) = \nabla_{\Gamma_\alpha} \phi_z \cdot a_{\alpha 1} + \phi_z b_{\alpha 2}.
\]

We assume additionally that \( a_{\alpha 1} : \Gamma_\alpha \to T\Gamma_\alpha \) and \( b_{\alpha 2} : \Gamma_\alpha \to \mathbb{R} \) are continuous functions such that

\begin{equation}
\|a_{\alpha 1}\|_{L^\infty(\Gamma_\alpha)} + \|b_{\alpha 2}\|_{L^\infty(\Gamma_\alpha)} \leq C\alpha.
\end{equation}

(5.10)

These conditions arise in a natural way as can be seen from (5.23) and the argument that follows. We observe that function \( c_\alpha : \Gamma_\alpha \to \mathbb{R} \) in (5.9) is a parameter to be determined using the orthogonality condition. We will show the following a priori estimate:

**Lemma 5.2.** There exists \( C > 0 \) such that for all sufficiently small \( \alpha \) and all \( g \in \mathcal{L}^p_{\text{loc}}(\Gamma_\alpha \times \mathbb{R}) \), \( 9 < p \leq \infty \) any solution \( \phi \) of problem (5.9) with \( \|\phi\|_{\infty,\sigma} < +\infty \) satisfies

\[
\|D^2 \phi\|_{p,\sigma} + \|D\phi\|_{\infty,\sigma} + \|\phi\|_{\infty,\sigma} \leq C\|g\|_{p,\sigma},
\]

(5.12)

**Proof.** A remark we make is that multiplying the equation by \( w(z) \), integrating by parts, and using the orthogonality assumption we readily get

\[
c_\alpha(y) \int_{\mathbb{R}} w_z^2 = \int_{\mathbb{R}} g(y,z)w_z(z) \, dz - \int_{\mathbb{R}} B_\alpha(\phi)w_z \, dz, \quad \forall y \in \Gamma_\alpha.
\]
In particular, we easily check that the function \(c_n(y)w_z(z)\) satisfies
\[
\|c_n w_z\|_{p,\sigma} \leq C\|g\|_{p,\sigma} + O(\alpha)\|D^2 \phi\|_{p,\sigma},
\]
hence for the purpose of the proof we do not lose generality in assuming simply that \(c_n \equiv 0\). Next, we will prove the existence of \(C\) for which
\[
\|\phi\|_{\infty,\sigma} \leq C\|g\|_{p,\sigma}.
\]
To establish this assertion we argue by contradiction. Let us assume that we have sequences \(\{\alpha_n\}, \{g_n\}, \{\phi_n\}\) for which problem (5.9) is satisfied (now with \(c_n \equiv 0\), and
\[
\|\phi_n\|_{\infty,\sigma} = 1, \quad \|g_n\|_{p,\sigma} \to 0, \quad \alpha_n \to 0.
\]
This means that there exists a sequence \((y_n, z_n) \in \Gamma_{\alpha_n} \times \mathbb{R}\) such that
\[
e^{\sigma|z_n|} |\phi_n(y_n, z_n)| \to 1.
\]
We consider two cases:

1. Sequence \(|z_n|\) is bounded.
2. \(\lim_{n \to \infty} |z_n| = \infty\).

**Case 1.** From Lemma 4.1 we know that there exists a \(\rho > 0\) such that for each \(n\) the surface \(\Gamma_{\alpha_n} \cap B(y_n, \rho\alpha_n^{-1})\) can be represented as a graph of a smooth function \(G_n : T_{y_n} \Gamma_{\alpha_n} \to \mathbb{R}\) and that moreover
\[
|G_n(t)| \leq \alpha_n |t|, \quad |t| \leq \rho\alpha_n^{-1}, t \in T_{y_n} \Gamma_{\alpha_n}.
\]
In local coordinates of \(\Gamma_{\alpha_n}\) given by the graph of \(G_n\) we can write
\[
\phi_n(y, z) = \hat{\phi}_n(t, z), \quad y \in \Gamma_{\alpha_n} \cap B(y_n, \rho\alpha_n^{-1}), \quad y = (t, G_n(t)).
\]
Using estimate (4.10) we get that the metric tensor on \(\Gamma_{\alpha_n} \cap B(y_n, \rho\alpha_n^{-1})\) expressed in terms of the coordinates \(t\) satisfies:
\[
g_n(t) = I + O(\alpha_n^2),
\]
where \(I\) is the identity matrix. Therefore, over compacts of \(\mathbb{R}^8 \equiv T_{y_n} \Gamma_{\alpha_n}\), we have that
\[
\sqrt{\det g_n(t)} \to 1, \quad g_n^{ij}(t) \to \delta_{ij},
\]
uniformly, together with its derivatives. Writing now the equation satisfied by \(\hat{\phi}_n\) in the local coordinates we get that
\[
\frac{1}{\sqrt{\det g_n}} \partial_t (g_n^{ij} \sqrt{\det g_n} \partial_{ij} \hat{\phi}_n) + g_n^{ij} b_{n1,j} \partial_{t,z} \hat{\phi}_n + b_{n2} \partial_z^2 \hat{\phi}_n + \partial_z^2 \hat{\phi}_n + f(w) \hat{\phi}_n = \tilde{g}_n,
\]
in \(B(0, \rho\alpha_n^{-1}) \times \mathbb{R}\),
\[
\int_{\mathbb{R}} \hat{\phi}_n(t, z) w_z \, dz = 0, \quad \text{in } B(0, \rho\alpha_n^{-1}).
\]
We have that
\[
\tilde{g}_n \to 0, \quad \text{in } L^p_{loc}(\mathbb{R}^8 \times \mathbb{R}),
\]
\[
|b_{n1}| + |b_{n2}| \to 0, \quad \text{uniformly over compacts},
\]
\[
0 < c \leq |\hat{\phi}_n(t, z)| \leq C, \quad \text{in } B(0, \rho\alpha_n^{-1}) \times \mathbb{R},
\]
and by standard elliptic regularity we get that
\[ \tilde{\phi}_n \to \tilde{\phi} \neq 0, \]
over compacts of \( \mathbb{R}^8 \times \mathbb{R} \). Moreover, \( \tilde{\phi} \) is a bounded, non-zero solution of
\[ \Delta_t \tilde{\phi} + \bar{\partial}_z \tilde{\phi} + f'(w)\tilde{\phi} = 0, \quad \text{in} \ \mathbb{R}^8 \times \mathbb{R}, \]
\[ \int_{\mathbb{R}} \tilde{\phi}(t,z)w_z \, dz = 0, \quad \text{in} \ \mathbb{R}, \]
which, by Lemma 5.1, implies that \( \tilde{\phi} \equiv 0 \), a contradiction.

Case 2. In this case the proof is similar, except that we define
\[ \tilde{\phi}_n(t,z) = \cosh(\sigma(z_n + z))\phi_n(y,z_n + z). \]
Then a similar limiting argument can be used to show that \( \tilde{\phi} \) satisfies
\[ \Delta_t \tilde{\phi} + \partial_z^2 \tilde{\phi} + \sigma a_1(z)\partial_z \tilde{\phi} - (2 - \sigma^2 a_2(z))\tilde{\phi} = 0, \quad \text{in} \ \mathbb{R}^8 \times \mathbb{R}, \]
where \( a_j(z) \) are bounded functions. Then, if \( \sigma \) is sufficiently small, by maximum principle, we get that \( \tilde{\phi} \equiv 0 \). We have reached a contradiction again and the proof of estimate (5.13) is concluded.

The next lemma establishes existence of a unique solution of problem (5.9) as in Lemma 5.2.

**Lemma 5.3.** For all sufficiently small \( \alpha \) and all \( g \) with \( \|g\|_{p,\sigma} < +\infty, \ 9 < p \leq \infty \) there exists a unique solution \( \phi \) of problem (5.9) with \( \|\phi\|_{\infty,\sigma} < +\infty \). This solution satisfies
\[ \|D^2\phi\|_{p,\sigma} + \|D\phi\|_{\infty,\sigma} + \|\phi\|_{\infty,\sigma} \leq C \|g\|_{p,\sigma}. \]

**Proof.** We assume initially that \( \|g\|_{\infty,\sigma} < \infty \). The general case of \( g \) with \( \|g\|_{p,\sigma} < \infty, \ 9 < p < \infty \) will follow by taking a suitable sequence of functions approximating \( g \). First we will show that there the assertion of the Lemma holds for the following problem:
\[ \Delta_{\Gamma,\alpha} \phi + \partial_z^2 \phi + f'(w)\phi = g + c_\alpha w_z, \quad \text{in} \ \Gamma_{\alpha} \times \mathbb{R}, \]
\[ \int_{\mathbb{R}} \phi(y,z)w_z(z) \, dz = 0, \quad \text{in} \ \Gamma_{\alpha}. \]
(5.17)

We shall argue by approximations. Let us replace \( g \) by
\[ g_R(y,z) = \begin{cases} g(y,z) & \text{in} \ B_R(0) \cap (\Gamma_{\alpha} \times \mathbb{R}) \\ 0 & \text{in} \ \Gamma_{\alpha} \times \mathbb{R} \setminus B_R(0) \end{cases}. \]

Then problem (5.17) corresponds to finding a minimizer of the functional
\[ J(\phi) = \frac{1}{2} \int_{\Gamma_{\alpha} \times \mathbb{R}} \left( |\nabla_{\Gamma,\alpha} \phi|^2 + |\phi_z|^2 + f'(w)\phi^2 \right) \, dV_{\alpha} \, dz - \int_{\Gamma_{\alpha} \times \mathbb{R}} g_R \phi \, dV_{\alpha} \, dz. \]
in the space \( H \) of all functions \( \phi \in H^1_{\text{loc}}(\Gamma_{\alpha} \times \mathbb{R}) \) for which
\[ \|\phi\|^2_H := \int_{\Gamma_{\alpha} \times \mathbb{R}} (|\nabla_{\Gamma,\alpha} \phi|^2 + |\phi_z|^2 + \phi^2) \, dV_{\alpha} \, dz < +\infty \]
and
\[ \int_{\mathbb{R}} \phi(y, z) w(z) \, dz = 0, \quad \forall y \in \Gamma_\alpha. \]

Since the orthogonality assumption implies that
\[ \int_{\Gamma_\alpha \times \mathbb{R}} (|\phi^2 - f'(w)|^2) \, dV_{\alpha} = \int_{\Gamma_\alpha \times \mathbb{R}} \phi^2 \, dV_{\alpha}, \]

therefore $J$ is coercive in $H$ by a standard argument. Consequently, the functional has a minimizer $\phi_R$ in $H$. Since the truncation $g_R$ has compact support, $\phi_R$ can be approximated by minimizing $J$ on the set of functions in $H$ which vanish outside a ball $B_n(0)$ with $n \gg R$. Calling $\phi_{R,n}$ this minimizer, we see that $\phi_{R,n}$ approaches $\phi_R$ in the sense of the $H$-norm. Applying elliptic estimates to the equation satisfied by $\phi_{R,n}$, we obtain that $\phi_{R,n}$ is in fact locally bounded, uniformly in $n$. Outside the support of $g_R$ the equation
\[ \Delta_{\Gamma_\alpha} \phi_{R,n} + |\alpha|^2 \phi_{R,n} + f'(w) \phi_{R,n} = 0 \]
is satisfied. Now, we recall the fact that
\[ \bar{h}_\alpha(x) = \frac{1}{\sqrt{1 + |\nabla F_\alpha|^2}}, \quad F_\alpha(x) = \frac{1}{\alpha} F(\alpha x), \quad x \in \mathbb{R}^8, \]
is a positive supersolution for the Laplace-Beltrami operator, indeed
\[ -\Delta_{\Gamma_\alpha} \bar{h}_\alpha = |\alpha|^2 \bar{h}_\alpha > 0. \]

Observe that $\psi_\alpha = e^{-\sigma(|z| - |z_0|)} \bar{h}_\alpha(x)$ satisfies
\[ \Delta_{\Gamma_\alpha} \psi_\alpha + |\alpha|^2 \psi_\alpha + f'(w) \psi_\alpha \leq |\sigma^2 + f'(w)| \psi_\alpha - |\alpha|^2 \psi_\alpha < 0 \]
for $|z| > |z_0|$ and $|z_0|$ large. Using comparison principle we then obtain that
\[ |\phi_{R,n}| \leq C_R \psi_\alpha, \quad \text{in } B_n(0) \subset \Gamma_\alpha \times \mathbb{R}, \]
where constant $C_R$ depends on $\|g\|_{\infty, \sigma}$, the uniform bound on $\|\phi_{R,n}\|_{\infty, 0}$, and $R$ only. Letting $n \to \infty$ we obtain that the same estimate is valid for $\phi_R$. We conclude that $\|\phi_R\|_{\infty, \sigma} < +\infty$. It follows from Lemma 5.2 that we have the uniform control
\[ \|D^2 \phi_R\|_{p, \sigma} + \|D \phi_R\|_{\infty, \sigma} + \|\phi_R\|_{\infty, \sigma} \leq C \|g_R\|_{p, \sigma} \leq C \|g\|_{p, \sigma} \]
and thus we can pass to the limit $R \to \infty$, obtaining a function $\phi$, which solves problem (5.17) with $\|\phi\|_{\infty, \sigma} < +\infty$. Then it follows from Lemma 5.2 that
\[ \|D^2 \phi\|_{p, \sigma} + \|D \phi\|_{\infty, \sigma} + \|\phi\|_{\infty, \sigma} \leq C \|g\|_{p, \sigma} \]
The general result for (5.9) follows now from a straightforward perturbation argument. □

We need to introduce now a norm that involves decay with respect $y \in \Gamma_\alpha$. We recall the definition of $r_\alpha$:
\[ r_\alpha(y) = \sqrt{1 + \alpha^2 |\Pi_{3B_1}(y)|^2}, \quad y \in \Gamma_\alpha, \]
where $\Pi_{3B_1} : \Gamma_\alpha \to \mathbb{R}^8$ is the projection onto $\mathbb{R}^8$ of the (embedded) graph $\Gamma_\alpha \subset \mathbb{R}^9$. Let us consider the new norms for $g$,
\[ \|g\|_{p, \sigma, \nu} := \sup_{(y, z) \in \Gamma_\alpha \times \mathbb{R}} e^{\sigma |z|} \|r_\alpha^\nu g\|_{L^p(\Gamma_\alpha \cap B(y, 1) \times (z, z + 1))}, \quad 1 < p < \infty, \nu \geq 1, \]
and
\[ \|g\|_{\infty, \sigma, \nu} := \sup_{(y, z) \in \Gamma_\alpha \times \mathbb{R}} r_\alpha^\nu(y) e^{\sigma |z|} |g(y, z)|, \quad \nu \geq 1. \]
Then we have the following a priori estimate.

**Lemma 5.4.** There exists a number $C > 0$ such that for all sufficiently small $\alpha$ the following holds: Given $g$ with $\|g\|_{p,\sigma} < +\infty$, $9 < p \leq \infty$, $0 \leq \nu$ there exists a unique solution $\phi$ of equation $(5.9)$ with $\|\phi\|_{\infty,\nu} < +\infty$. This solution satisfies

$$\|D^2\phi\|_{p,\sigma,\nu} + \|D\phi\|_{\infty,\sigma,\nu} + \|\phi\|_{\infty,\sigma,\nu} \leq C \|g\|_{p,\sigma,\nu}.$$  

**Proof.** This result is in fact a direct corollary (with obvious modifications for the respective norms) of the previous lemma. We will set

$$\phi = r_\alpha^{-\nu}\psi,$$

and use $(5.9)$ to find the equation satisfied by $\psi$:

$$\Delta_{\Gamma_{\alpha}} \psi + r_\alpha^{\nu} \hat{B}_\alpha(r_\alpha^{-\nu} \psi) + \partial_z^2 \psi + f'(w)\psi = r_\alpha^{\nu} g + r_\alpha^{\nu} c_\alpha w_z, \quad \text{in } \Gamma_{\alpha} \times \mathbb{R},$$

where $\hat{B}_\alpha$ is a second order differential operator. Let us observe that

$$\|\Delta_{\Gamma_{\alpha}} r_\alpha\|_{L_{\infty}(\Gamma_{\alpha})} + \alpha \|\nabla_{\Gamma_{\alpha}} r_\alpha\|_{L_{\infty}(\Gamma_{\alpha})} \leq C \alpha^2.$$  

This means that essentially the same argument as in the proof of Lemma 5.2 applies to yield the $L^\infty$ estimate for $\psi$ in terms of $\|g\|_{p,\sigma,\nu}$ and then local elliptic estimates give the estimate for the derivatives. We omit the details. \hfill \Box

The theory developed in this section allows us to define an operator $T_\alpha(g) := \phi$ where $\phi$ is a solution of equation $(5.9)$. In particular, with this definition, we have

$$\|D^2T_\alpha(g)\|_{p,\sigma,\nu} + \|DT_\alpha(g)\|_{\infty,\sigma,\nu} + \|T_\alpha(g)\|_{\infty,\sigma,\nu} \leq C \|g\|_{p,\sigma,\nu}.$$  

### 5.3. Full linearized operator

We will now use the solvability theory for the problem $(5.9)$ to treat the full linearized operator $L(\phi) = \Delta\phi + f'(w)\phi$, defined in $(5.1)$. Thus we will consider the following problem:

$$(5.21) \quad L(\phi) = g, \quad \text{in } N_{\delta}. $$

We recall that

$$u(x) = \begin{cases} 
\chi\left(\frac{4a_2}{\theta_0} r_\alpha\right) \left(w(z - h_\alpha) + 1\right), & z < 0, \\
\chi\left(\frac{4a_2}{\theta_0} r_\alpha\right) \left(w(z - h_\alpha) - 1\right) + 1, & 0 \leq z,
\end{cases} $$

where $h_\alpha$ is a function defined on $\Gamma_{\alpha}$. In addition we will assume that, with some $\mu > 8/p$ and $\nu \geq 2$:

$$\alpha^2 \|h_\alpha\|_{\infty,\nu-2} + \alpha \|\nabla_{\Gamma_{\alpha}} h_\alpha\|_{\infty,\nu-1} + \alpha^{8/p} \|\nabla^2_{\Gamma_{\alpha}} h_\alpha\|_{p,\nu} := \|h_\alpha\|_{s,p,\nu} \leq C \alpha^{2+\mu},$$

where $p \in (9, \infty)$, and

$$\|h_\alpha\|_{p,\nu} := \sup_{y \in \Gamma_{\alpha}} \|r_\alpha^{\nu} h_\alpha\|_{L^p(\Gamma_{\alpha} \cap B(y,\theta_0 a^{-1}))}, \quad 9 < p < \infty, \quad \nu \geq 2,$$

$$\|h_\alpha\|_{\infty,\nu} := \sup_{y \in \Gamma_{\alpha}} \|r_\alpha(y)^\nu |h_\alpha(y)|\}, \quad \nu \geq 2.$$
This means that we can assume that in $\mathcal{N}_{4\delta}$ we have $u(x) = w(z - h_\alpha)$. We recall that we can identify functions defined on $\Gamma_\alpha$ and those defined on $\Gamma$ through the relation:

$$h_\alpha(y) = h(ay), \quad y \in \Gamma,$$

where $h : \Gamma \to \mathbb{R}$. This justifies the definition of the norms and the assumption (5.22).

Using the operator defined in (5.8) we can write

$$\mathcal{L}(\phi) = \Delta_{\Gamma_\alpha}\phi + \partial^2_z\phi + B^\alpha_\delta(\phi) + f'(u)\phi.$$

Now we need to make a change of variables in $\mathcal{N}_{4\delta}$:

$$\bar{z} = z - h_\alpha.$$

We will denote $\phi(y, z) = \tilde{\phi}(y, z - h_\alpha(y))$. Then we have

$$\Delta_{\Gamma_\alpha}\phi(y, z) = \Delta_{\Gamma_\alpha}\tilde{\phi}(y, \bar{z}) + B_1(\tilde{\phi}),$$

where $B_1(\tilde{\phi})$ is a linear second order differential operator (in variables $(y, \bar{z})$):

$$B_1(\tilde{\phi}) = -2\nabla_{\Gamma_\alpha}\tilde{\phi}_\bar{z} \cdot \nabla_{\Gamma_\alpha}h_\alpha + \tilde{\phi}_\bar{z}\bar{z}\nabla^2_{\Gamma_\alpha}h_\alpha - \tilde{\phi}_\bar{z}\Delta_{\Gamma_\alpha}h_\alpha.$$

We will separate the term whose coefficients depend on the first derivatives of $h_\alpha$ from the rest. Thus we will denote:

$$B_{1t}(\tilde{\phi}) = -2\nabla_{\Gamma_\alpha}\tilde{\phi}_\bar{z} \cdot \nabla_{\Gamma_\alpha}h_\alpha + \tilde{\phi}_\bar{z}\bar{z}\nabla^2_{\Gamma_\alpha}h_\alpha.$$

Notice that $B_{1t}$ is an operator of the same form as $B_\alpha$ in (5.10), and whose coefficients satisfy the analog of (5.11) due to the assumption (5.22).

With the same change of variables we now analyze:

$$B^\alpha_\delta(\phi) = \eta^\alpha_\delta(\Delta_{\Gamma_\alpha} - \Delta_{\Gamma_\alpha} - H_{\Gamma_{\alpha,z}}\bar{z})(\phi).$$

Let us fix a $y \in \Gamma_\alpha$ and let $g_{\alpha,z}$ be the metric tensor in local coordinates around $y$. Let $g_{\alpha,z}$ be the corresponding metric tensor on $\Gamma_{\alpha,z}$ (i.e. around the point $y_z = y + zn_{\alpha}(y) \in \Gamma_{\alpha,z}$, with $n_{\alpha}(y)$ denoting the normal vector). Then we can write, keeping in mind that $z = \bar{z} + h_\alpha$, $g_{\alpha,z} = g_{\alpha,z}$ and:

\[
\begin{aligned}
\Delta_{\Gamma_{\alpha,z}}\phi &= \frac{1}{\sqrt{\det(g_{\alpha,z})}} \partial_i \left( g^{ij}_{\alpha,z} \sqrt{\det(g_{\alpha,z})} \partial_j \phi \right) \\
&\quad - \frac{1}{\sqrt{\det(g_{\alpha,z})}} \partial_i \left( g^{ij}_{\alpha,z} \sqrt{\det(g_{\alpha,z})} \partial_j h_\alpha \right) \partial_i \phi \\
&\quad + \frac{1}{\sqrt{\det(g_{\alpha,z})}} \partial_i \left( g^{ij}_{\alpha,z} \sqrt{\det(g_{\alpha,z})} \partial_j h_\alpha \partial_i \phi \right) \\
&\quad + g^{ij}_{\alpha,z} \partial_i h_\alpha \partial_i h_\alpha \partial_j \partial_i \phi - g^{ij}_{\alpha,z} \partial_i h_\alpha \partial_j \phi - g^{ij}_{\alpha,z} \partial_i h_\alpha \partial_j \phi \\
&\quad - \frac{1}{\sqrt{\det(g_{\alpha,z})}} \partial_i \left( g^{ij}_{\alpha,z} \sqrt{\det(g_{\alpha,z})} \partial_j h_\alpha \partial_i \phi \right).
\end{aligned}
\] (5.25)

We notice if in $g_{\alpha,z}$ we set $z = 0$ then the above operator is equal to $\Delta_{\Gamma_\alpha}\phi(y, z)$. Thus we need to "expand" (5.25) in powers of $z$. To this end let us use the local system of coordinates around $y$ given by $(t, G_\alpha(t))$, where $G_\alpha(t) = \alpha^{-1}G(\alpha t)$, and $G$ given in Lemma 4.1 and $|t| \leq \rho \alpha^{-1}$. Since we are interested in the size of the local norms defined above we only need to consider $|t| \leq C$, where $C > 0$ is a
constant independent on $\alpha$. By direct calculation, using Lemma 4.1 and (4.33) we get that
\[
g_{\alpha, z} = I + \frac{1}{r_\alpha(y)} O(\alpha^2 |t|^2) + \frac{z}{r_\alpha(y)} O(\alpha) = g_{\alpha, 0} + O(\alpha^2 |t|^2) + \frac{z}{r_\alpha(y)} O(\alpha)
\]
(5.26)
with similar estimates for the derivatives. It follows:
\[
\bar{g}_{ij}^{\alpha, z} = g_{ij}^{\alpha, 0} + \frac{\bar{z} + \bar{h}_\alpha}{r_\alpha(y)} O(\alpha)
\]
(5.27)
Setting, with some abuse of notation,
\[
\Delta_{\Gamma_{\alpha, z}} \bar{\phi} = \frac{1}{r_\alpha(y)} \partial_i \left( \bar{g}_{ij}^{\alpha, z} \sqrt{\det(\bar{g}^{\alpha, z})} \partial_j \bar{h}_0 \right)
\]
(5.28)
we see that
\[
(\Delta_{\Gamma_{\alpha, z}} - \Delta_{\Gamma_0}) \bar{\phi} = B_{2\alpha}(\bar{\phi}) - [(\Delta_{\Gamma_{\alpha, z}} - \Delta_{\Gamma_0}) h_\alpha] \partial_\bar{z} \bar{\phi},
\]
where $B_{2\alpha}(\bar{\phi})$ is a second order linear differential operator in $\bar{\phi}$ whose coefficients depend on $h_\alpha, \nabla h_\alpha, z$ and that can be estimated as follows:
\[
|B_{2\alpha}(\bar{\phi})| \leq \frac{O(\alpha)(|z| + |h_\alpha|)}{1 + r_\alpha(y)} + |\nabla h_\alpha| (|\nabla \bar{\phi}| + |D^2 \bar{\phi}|).
\]
(5.30)
We will consider now the term:
\[
H_{\Gamma_{\alpha, z}} \partial_\bar{z} \phi = |A_{\Gamma_{\alpha}}|^2 \partial_\bar{z} \phi + z^2 R_\alpha \partial_\bar{z} \bar{\phi}
\]
(5.31)
where $|A_{\Gamma_{\alpha}}|^2$ is the norm of the second fundamental form on $\Gamma_\alpha$, which satisfies:
\[
|A_{\Gamma_{\alpha}}|^2 = \alpha^2 |A_\Gamma|^2 \leq \frac{C_\alpha^2}{r_\alpha^2(y)}.
\]
(5.32)
(Above $|A_\Gamma|$ is the norm of the second fundamental form on $\Gamma$). Term $R_\alpha$ comes from the Taylor expansion of $H_{\Gamma_{\alpha, z}}$ (see (2.24)–(2.25)) and it has the form:
\[
R_\alpha = \frac{1}{z} [H_{\Gamma_{\alpha, z}} - z |A_{\Gamma_{\alpha}}|^2]
\]
(5.33)
\[
= \frac{1}{z^2} \sum_{i=1}^{8} \left( \frac{\kappa_i}{1 - z \kappa_i} - z \kappa_i^2 \right)
\]
\[
= \sum_{i=1}^{8} \left[ \kappa_i^3 + O(|z| \kappa_i^4) \right],
\]
and can be bounded as follows:
\[
|R_\alpha| \leq \frac{C_\alpha^3}{r_\alpha^3(y)}.
\]
(5.34)
From this we see that \((5.21)\) can be equivalently written in the form:

\[
\Delta_{\Gamma_\alpha} \tilde{\phi} + \partial_{z}^2 \tilde{\phi} + B_{1\alpha}(\tilde{\phi}) + f'(w) \tilde{\phi} = \bar{g} + \partial_{z} \tilde{\phi} \Delta_{\Gamma_\alpha} h_{\alpha} - \eta_{0}^{\alpha} B_{2\alpha}(\tilde{\phi}) \\
+ \eta_{0}^{\alpha} [(\Delta_{\Gamma_\alpha} - \Delta_{\Gamma_{\alpha^*}}) h_{\alpha}] \\
+ \eta_{0}^{\alpha} (\bar{z} + h_{\alpha}) |A_{\Gamma_\alpha}|^2 \partial_{z} \tilde{\phi} \\
+ \eta_{0}^{\alpha} (\bar{z} + h_{\alpha})^2 R_{\alpha} \partial_{z} \tilde{\phi},
\]

\((5.35)\)
in \(N_{4\delta}\). It follows that it is natural to extend \(L\) outside of \(N_{4\delta}\) by letting

\[
L(\phi) = \Delta_{\Gamma_\alpha} \tilde{\phi} + \eta_{0}^{\alpha} B_{1\alpha}(\tilde{\phi}) + \partial_{z}^2 \tilde{\phi} + f'(w) \tilde{\phi}, \quad \text{in} \ \Gamma_\alpha \times \mathbb{R} \setminus N_{4\delta},
\]
since \(\eta_{0}^{\alpha} \equiv 1\) in \(N_{4\delta}\). We will denote the extended operator \(L\) by \(\tilde{L}\). With this extension we can consider \((5.35)\) as an equivalent problem to \(\tilde{L}(\phi) = \bar{g}\) in \(\Gamma_\alpha \times \mathbb{R}\). Now we can use the results of the previous subsection to show the following analog of Lemma \(5.4\).

**Proposition 5.1.** There exists a number \(C > 0\) such that for all sufficiently small \(\alpha\) and \(\delta\) the following holds: Given \(\bar{g}\) with \(\|\bar{g}\|_{p,\sigma,\nu} < +\infty\), \(9 < p < \infty\), \(2 \leq \nu\) there exists a unique solution \(\tilde{\phi}\) of

\[
\tilde{L}(\tilde{\phi}) = \bar{g} + \bar{c}_{\alpha} \bar{w}_{\bar{z}} , \quad \text{in} \ N_{\infty},
\]

\((5.36)\)
with \(\|\tilde{\phi}\|_{\infty,\sigma,\nu} < +\infty\). This solution satisfies

\[
\|D^2 \tilde{\phi}\|_{p,\sigma,\nu} + \|D \tilde{\phi}\|_{\infty,\sigma,\nu} + \|\tilde{\phi}\|_{\infty,\sigma,\nu} \leq C \|\bar{g}\|_{p,\sigma,\nu}.
\]

**Proof.** Using estimates \((5.28)-(5.35)\) we see that the right hand side of \((5.35)\) can be estimated by:

\[
\|\bar{g}\|_{p,\sigma,\nu} + C(\delta + \alpha) \|D^2 \tilde{\phi}\|_{p,\sigma,\nu} + \|D \tilde{\phi}\|_{\infty,\sigma,\nu} + \|\tilde{\phi}\|_{\infty,\sigma,\nu}.
\]

Using this and a straightforward fixed point argument we obtain our result. The details are left to the reader. \(\square\)

6. The Nonlinear Problem

6.1. The Gluing. Let us recall (see section 2) that we are looking for the solution of \((1.1)\) in the form:

\[(6.1)\]
\[u_{\alpha} = u(x) + \phi(x).\]

Substituting in \((1.1)\) we get for the function \(\phi\)

\[(6.2)\]
\[L(\phi) = S[u] + N(\phi),\]

where

\[(6.3)\]
\[S[u] = -\Delta u - f(u), \quad N(\phi) = -[f(u + \phi) - f(u) - f'(u)\phi], \quad f(u) = u(1 - u^2).\]

Let us recall (see section 2.1) that we introduced an improvement of the initial approximation, namely the function \(w_1\) defined in \((2.30)-(2.31)\). We look for a solution \(\tilde{\phi}\) of equation \((6.2)\) in the form

\[(6.4)\]
\[\tilde{\phi} = w_1 + \eta_{0}^{\alpha} \phi + \psi,\]
so that the following must be satisfied:

\begin{equation}
\eta_{23}^{0}(\Delta \phi + f'(u)\phi) + 2\nabla \eta_{23}^{0} \nabla \phi + \phi \Delta \eta_{23}^{0} + \Delta \psi + f'(u)\psi = S[u] + L(u_1)
\end{equation}

where

\[ L(u_1) = -\Delta u_1 - f'(u)u_1. \]

For brevity in what follows we will write:

\[ N(\phi) = N(u_1 + \phi) = -[f(u + u_1 + \phi) - f(u) - f'(u)(u_1 + \phi)], \]

\[ S(u, u_1) = S[u] + L(u_1). \]

This equation is satisfied provided that \((\phi, \psi)\) satisfies a coupled system of nonlinear elliptic equations:

\begin{align}
\Delta \phi + f'(u)\phi &= \eta_{23}^{0} [S(u, u_1) + N(\phi + \psi) - (2 + f'(u))\psi], \quad \text{in } \mathcal{N}_{45}, \\
\Delta \psi - 2\psi &= (1 - \eta_{23}^{0}) [S(u, u_1) + N(\eta_{23}^{0}\phi + \psi) - (2 + f'(u))\psi] \\
&\quad - 2\nabla \eta_{23}^{0} \nabla \phi - \phi \Delta \eta_{23}^{0}, \quad \text{in } \mathbb{R}^9.
\end{align}

Let us consider the extension \( \tilde{L} \) of the linear operator \( L \) introduced in the previous section. Then equation \((6.6)\) is equivalent to:

\begin{equation}
\tilde{L}(\phi) = \eta_{23}^{0} [S(w, w_1) + N(\phi + \psi) - (2 + f'(w))\psi], \quad \text{in } \Gamma_\alpha \times \mathbb{R},
\end{equation}

where

\[ \phi(y, z) = \phi(y, z + h_\alpha), \quad \tilde{\psi} = \psi(y, z + h_\alpha), \]

according with the change of variables \( z = z - h_\alpha \) introduced in the previous section. The error term \( S(w, w_1) \) expressed in these variables is denoted above by \( S(w, w_1) \) (observe that in the support of \( \eta_{23}^{0} \) we have \( w = w, w_1 = w_1 \)).

Now we will recast the system \((6.6)\)--\((6.7)\) as a fixed point problem for \( \tilde{\phi} \). Let us consider a given function \( \phi \) such that \( \| \nabla \phi \|_{\infty, \sigma, \nu} + \| \phi \|_{\infty, \sigma, \nu} \) is small. We set \( \phi = \phi(y, z - h_\alpha) \) in \( \mathcal{N}_{45} \) and \( \phi \equiv 0 \) in \( \mathbb{R}^9 \setminus \mathcal{N}_{45} \) and write \((6.7)\) as

\begin{equation}
\Delta \psi - 2\psi = M(\psi, \phi) + P(\phi), \quad \text{in } \mathbb{R}^9,
\end{equation}

where

\[ P(\phi) = (1 - \eta_{23}^{0})(S(w, w_1) + N(\eta_{23}^{0}\phi)) + 2\nabla \eta_{23}^{0} \nabla \phi + \phi \Delta \eta_{23}^{0}, \]

\[ M(\psi, \phi) = (1 - \eta_{23}^{0})[(2 + f'(w))\psi + N(\eta_{23}^{0}\phi + \psi) - N(\eta_{23}^{0}\phi)]. \]

For functions in \( \mathbb{R}^9 \) we define the following norms:

\[ \| Q \|_{p, \nu} := \sup_{x \in \mathbb{R}^9} \| r_\alpha Q \|_{L^p(B(x, 1))} < +\infty, \quad r_\alpha(x) = \sqrt{1 + \alpha^2 \| \Pi_{\mathbb{R}^3}(x) \|^2}. \]

We notice that the function \( P \) above satisfies the following estimate:

\begin{equation}
\| P(\phi) \|_{p, \nu} \leq C e^{-\delta \alpha/\alpha} \| S(u, u_1) \|_{p, \sigma, \nu} + e^{-\delta \alpha/\alpha} (\| \nabla \phi \|_{\infty, \sigma, \nu} + \| \phi \|_{\infty, \sigma, \nu}) \]
\[ \leq C e^{-\delta \alpha/\alpha} \| S(u, u_1) \|_{p, \sigma, \nu} + e^{-\delta \alpha/\alpha} (\| \nabla \phi \|_{\infty, \sigma, \nu} + \| \phi \|_{\infty, \sigma, \nu}). \]

Motivated by \((6.10)\) we will establish our next result:
Lemma 6.1. Let us consider the linear problem

\[ \Delta \psi - 2\psi = Q(x) \quad \text{in } \mathbb{R}^9. \]

There exists a number \( C > 0 \) such that if \( Q(x) \) satisfies

\[ \|Q\|_{p,0} < +\infty, \]

with \( 9 < p < \infty \), then equation \( 6.11 \) has a unique bounded solution \( \psi \), which defines a linear operator in \( Q \) and satisfies

\[ \|\nabla \psi\|_{\infty,0} + \|\psi\|_{\infty,0} \leq C\|Q\|_{p,0}. \]

Proof. Let us write

\[ \mu(x) = r_\alpha^{-\nu}(x), \quad \psi(x) = \mu(x)\tilde{\psi}(x). \]

Then the equation in terms of \( \tilde{\psi} \) reads

\[ \Delta \tilde{\psi} - 2\tilde{\psi} + \mu^{-1}(\tilde{\psi}\Delta \mu + \nabla \mu \cdot \nabla \tilde{\psi}) = \mu^{-1}Q. \]

We notice that:

\[ |\mu^{-1}\Delta \mu| \leq C\alpha^2 r_\alpha^{-2}, \quad \|\tilde{\psi}\|_{\infty,0} \leq C\|\psi\|_{\infty,0}. \]

Thus the equation becomes,

\[ \Delta \tilde{\psi} - 2\tilde{\psi} + a_1 \cdot \nabla \tilde{\psi} + a_2 \tilde{\psi} = \tilde{Q}(y), \quad \tilde{Q} = \mu^{-1}Q, \]

where \( a_1 = o(1) \), as \( \alpha \to 0 \), uniformly. Let us approximate \( \tilde{Q} \) by bounded functions,

\[ \tilde{Q}_n = \min\{|\tilde{Q}|, n\} \operatorname{sign} \tilde{Q}, \]

and consider the unique bounded solution \( \tilde{\psi}_n \) of

\[ \Delta \tilde{\psi}_n - 2\tilde{\psi}_n + a_1 \cdot \nabla \tilde{\psi}_n + a_2 \tilde{\psi}_n = \tilde{Q}_n(y). \]

We claim that there exists a \( C > 0 \), independent of \( \tilde{Q}_n \) and \( \alpha \) such that

\[ (6.12) \quad \|\tilde{\psi}_n\|_{\infty} + \|\nabla \tilde{\psi}_n\|_{\infty} \leq C\|\tilde{Q}_n\|_{p,0} \leq C\|\tilde{Q}\|_{p,0}. \]

Assuming the opposite, we have sequences \( \tilde{\psi}_n, \tilde{Q}_n \) such that

\[ \|\tilde{\psi}_n\|_{\infty} + \|\nabla \tilde{\psi}_n\|_{\infty} = 1, \quad \|\tilde{Q}_n\|_{p,0} \to 0, \quad n \to \infty. \]

Let \( x_n \) be such that

\[ (|\tilde{\psi}_n| + |\nabla \tilde{\psi}_n|(x_n)) \geq \frac{1}{2} \]

Then, by local elliptic estimates, we get that the function \( x \mapsto \psi_n(x_n + x) \) converges locally over compacts to a nontrivial, bounded solution \( \tilde{\psi} \) of the equation

\[ \Delta \tilde{\psi} - 2\tilde{\psi} = 0. \]

We have reached a contradiction that proves the estimate \( 6.12 \). Passing to the limit, the lemma readily follows. \( \square \)

Using the above lemma, we can apply contraction mapping principle to conclude that there is a unique solution \( \psi = \psi(\phi) \) of equation \( 6.9 \), which in addition satisfies

\[ (6.13) \quad \|\nabla \psi\|_{\infty,0} + \|\psi\|_{\infty,0} \leq Ce^{-\sigma_0} \left[ |||S(\psi, \psi_1)|||_{p,\sigma,0} + e^{-\sigma_0} \left( \|\phi\|_{\infty,\sigma,0} + \|\nabla \phi\|_{\infty,\sigma,0} \right) \right]. \]
In addition, we will check that $\psi$ is a Lipschitz function in the considered norms, both in $\phi$ and in $h_\alpha$. Substituting in equation (6.8), we get that our full problem has been reduced to solving the nonlinear, nonlocal problem
\begin{equation}
\hat{L}(\tilde{\phi}) = \eta_0^a \left[ \hat{S}(w, w_1) + N(\tilde{\phi} + \tilde{\psi}(\tilde{\phi})) - (2 + f'(w)) \tilde{\psi}(\tilde{\phi}) \right], \quad \text{in } \Gamma_\alpha \times \mathbb{R}.
\end{equation}

6.2. The projected nonlinear problem. We consider the projected version of problem (6.14)
\begin{equation}
\hat{L}(\tilde{\phi}) = c_\alpha w \tilde{z} + \eta_0^a \hat{S}(w, w_1) + \tilde{N}(\tilde{\phi}), \quad \text{in } \Gamma_\alpha \times \mathbb{R},
\end{equation}
where for convenience we have denoted:
\begin{equation}
\tilde{N}(\tilde{\phi}) = \eta_0^a \left[ N(\tilde{\phi} + \tilde{\psi}(\tilde{\phi})) - (2 + f'(w)) \tilde{\psi}(\tilde{\phi}) \right].
\end{equation}
In the sequel we will use notation:
\begin{equation}
\|\tilde{\psi}\|_{*,p,\sigma,\nu} = \|D^2 \tilde{\psi}\|_{*,p,\sigma,\nu} + \|\nabla \tilde{\psi}\|_{\infty,\sigma,\nu} + \|\tilde{\psi}\|_{\infty,\sigma,\nu}, \quad 9 < p < \infty.
\end{equation}

Lemma 6.2. Let $\tilde{\phi}$ be a given function such that $\bar{\psi} \leq \tilde{\psi} \leq \bar{\psi}$, $\|\bar{\psi}\|_{\infty,\sigma,\nu}$ is small, see Proposition 6.1 to follow. This shows (6.18).

\begin{equation}
\|\tilde{\psi}\|_{*,p,\sigma,\nu} < \infty.
\end{equation}

Then mapping $\tilde{N}(\tilde{\phi})$ satisfies:
\begin{equation}
\|\tilde{N}(\tilde{\phi})\|_{p,\sigma,\nu} \leq \|\tilde{E}\|_{p,\sigma,\nu} + \|\nabla \tilde{\psi}\|_{\infty,\sigma,\nu} + \|\tilde{\psi}\|_{\infty,\sigma,\nu}.
\end{equation}

In addition for any functions $\tilde{\phi}_k, k = 1, 2$, satisfying (6.17) we have:
\begin{equation}
\|\tilde{N}(\tilde{\phi}_1) - \tilde{N}(\tilde{\phi}_2)\|_{p,\sigma,\nu} \leq C\|\tilde{E}\|_{p,\sigma,\nu} + \|\nabla \tilde{\psi}\|_{\infty,\sigma,\nu} + \|\tilde{\psi}\|_{\infty,\sigma,\nu}.
\end{equation}

Proof. Let us consider the solution of (6.9) denoted, after the change of variables, by $\tilde{\psi}(\tilde{\phi})$. Using (6.13) we get
\begin{equation}
\|\tilde{N}(\tilde{\phi})\|_{p,\sigma,\nu} \leq C\|\tilde{E}\|_{p,\sigma,\nu} + \|\nabla \tilde{\psi}\|_{\infty,\sigma,\nu} + \|\tilde{\psi}\|_{\infty,\sigma,\nu}.
\end{equation}

Estimate (6.19) follows readily from this.

We will now show the main result of this section.

Proposition 6.1. Under the assumption $2 \leq \nu \leq 3$ we have that
\begin{equation}
\|\tilde{S}(w, w_1)\|_{p,\sigma,\nu} \leq C\alpha^2, \quad 9 < p < \infty.
\end{equation}

As a consequence, for all $\alpha$ sufficiently small, problem (6.15) has a unique solution $\tilde{\phi}$ with
\begin{equation}
\|\tilde{\phi}\|_{*,p,\sigma,\nu} \leq C\alpha^2.
\end{equation}
In addition, \( \bar{\phi} \) depends in a Lipschitz way on \( h_\alpha \) in natural norms, namely we have:

\[
\| \bar{\phi}^{(1)} - \bar{\phi}^{(2)} \|_{*,p,\sigma,\nu} \leq C \alpha^{-8/p} \| h_\alpha^{(1)} - h_\alpha^{(2)} \|_{*,p,\nu},
\]

where \( \bar{\phi}^{(k)} \), \( k = 1, 2 \) are solutions of (6.13) with \( h_\alpha = \bar{h}_\alpha^{(k)} \).

Proof. We begin by proving (6.22). Let us write:

\[
\mathcal{S}(w, w_1) = \eta_\alpha^8 \mathcal{S}(w, w_1) + (1 - \eta_\alpha^8) \mathcal{S}(w, w_1)
\]

\[
:= -E_1 - E_2.
\]

Notice that in terms of the original Fermi coordinates of \( \Gamma_\alpha \) we have in \( N_3 \):

\[
E_1 = \Delta w + f(w) + \Delta w_1 + f'(w)w_1
\]

\[
= \Delta_{\Gamma_{\alpha,z}} (w + w_1) - H_{\Gamma_{\alpha,z}} \partial_z (w + w_1) + \partial^2_z (w + w_1) + f(w) + f'(w)w_1,
\]

where \( w = w(z - h_\alpha) \). We will decompose:

\[
E_1 = \Delta_{\Gamma_{\alpha,z}} (w + w_1) + [-H_{\Gamma_{\alpha,z}} \partial_z w + \| \partial_z | A_{\Gamma_{\alpha}} |^2 w_z \|_1 - H_{\Gamma_{\alpha,z}} \partial_z w_1
\]

\[
:= E_{11} + E_{12} + E_{13}.
\]

To estimate \( E_{11} \) we use the expression of the Laplace-Beltrami operator in local coordinates as in (5.25)–(5.26). Thus we get, changing to \( \tilde{z} = z - h_\alpha \), denoting \( \bar{w} = w + w_1 \) and remembering that \( w + w_1 = w(\tilde{z}) + w_1(\tilde{z}) \):

\[
\Delta_{\Gamma_{\alpha,z}} \bar{w} = -\frac{1}{\sqrt{\det(g_{\alpha,z})}} \partial_i \left( \frac{g_{\alpha,z}^{ij}}{\sqrt{\det(g_{\alpha,z})}} \partial_j h_\alpha \right) \partial_z \bar{w}
\]

\[
+ \frac{1}{\sqrt{\det(g_{\alpha,z})}} \partial_z \left( \frac{g_{\alpha,z}^{ij}}{\sqrt{\det(g_{\alpha,z})}} \partial_i h_\alpha \partial_j h_\alpha \partial_z \bar{w}
\]

\[
+ \bar{g}_{\alpha,z}^{ij} \partial_j h_\alpha \partial_i h_\alpha \partial^2_z \bar{w},
\]

We observe that, fixing a \( y \in \Gamma_\alpha \) and considering the norms in the local variables \( (t, G_\alpha(t)) \) around \( y \), we get:

\[
e^{|z|} r_\alpha^\mu \partial_j h_\alpha \tilde{w}_\alpha \|_{L^p(B(0,1))} \leq C \alpha^{2+\mu-8/p} \leq C \alpha^2,
\]

because the assumption (5.22) we have made about \( h_\alpha \) and the fact that \( \mu > p/8 \).

Likewise we get:

\[
e^{|z|} r_\alpha^\mu \partial_i h_\alpha \partial_j h_\alpha \partial_z w \|_{L^p(B(0,1))} \leq C \alpha^{2+2\mu}.
\]

This and (5.26) gives:

\[
\| E_{11} \|_{p,\sigma,\nu} \leq C \alpha^2.
\]

Now we turn our attention to \( E_{12} \). Using (5.31)–(5.34) we get locally in \( N_3 \):

\[
H_{\Gamma_{\alpha,z}} \partial_z w - \| w_z \|_1 A_{\Gamma_{\alpha}}^2 w_z = h_\alpha | A_{\Gamma_{\alpha}} |^2 \partial_z w + (\tilde{z} + h_\alpha)^2 R_{\alpha} \partial_z w
\]

\[
\leq \frac{C \alpha^2 h_\alpha + \alpha r_\alpha^{-1} e^{-\sigma |z|}}{r_\alpha^2 (y)},
\]

hence using the assumption \( 2 \leq \nu \leq 3 \) we get:

\[
\| E_{12} \|_{p,\sigma,\nu} \leq C \alpha^2.
\]

Since

\[
| w_1 | \leq C \alpha^2 e^{-\sigma |z|},
\]
therefore we get immediately

\[(6.31) \quad \| E_{13} \|_{p,\sigma,v} \leq C\alpha^2.\]

In order to estimate \(E_2\) we will assume that \(\bar{\varepsilon} + h_\alpha > 0\) and write:

\[
w = \chi\left(\frac{4\alpha (\bar{\varepsilon} + h_\alpha)}{\theta_0 r_\alpha}\right)w(\bar{\varepsilon}) + 1 - \chi\left(\frac{4\alpha (\bar{\varepsilon} + h_\alpha)}{\theta_0 r_\alpha}\right).\]

Using the Fermi coordinates we can write:

\[
(1 - \eta_\delta^2)(\Delta w + f(w)) = (1 - \eta_\delta^2)[\partial_\varepsilon^2 w + f(w)] + (1 - \eta_\delta^2)\Delta \gamma_{\alpha,z} w
\]

\[
- (1 - \eta_\delta^2)H_{\Gamma_{\alpha,z}} \partial_\varepsilon w
\]

\[
= (1 - \eta_\delta^2)(E_{21} + E_{22} + E_{23}).
\]

Let us consider the term denoted by \(E_{21}\). We have

\[
E_{21} = f\left(\frac{4\alpha (\bar{\varepsilon} + h_\alpha)}{\theta_0 r_\alpha}\right)w(\bar{\varepsilon}) + 1 - \chi\left(\frac{4\alpha (\bar{\varepsilon} + h_\alpha)}{\theta_0 r_\alpha}\right) - \chi\left(\frac{4\alpha (\bar{\varepsilon} + h_\alpha)}{\theta_0 r_\alpha}\right) f(w(\bar{\varepsilon}))
\]

\[
+ 2\partial_\varepsilon \chi\left(\frac{4\alpha (\bar{\varepsilon} + h_\alpha)}{\theta_0 r_\alpha}\right)\partial_\varepsilon w(\bar{\varepsilon})
\]

\[
+ \partial_\varepsilon^2 \chi\left(\frac{4\alpha (\bar{\varepsilon} + h_\alpha)}{\theta_0 r_\alpha}\right)[w(\bar{\varepsilon}) - 1]
\]

\[
= A_1 + A_2 + A_3.
\]

To estimate \(A_1\) we write

\[
\chi\left(\frac{4\alpha (\bar{\varepsilon} + h_\alpha)}{\theta_0 r_\alpha}\right)w(\bar{\varepsilon}) + 1 - \chi\left(\frac{4\alpha (\bar{\varepsilon} + h_\alpha)}{\theta_0 r_\alpha}\right)
\]

\[
= w(\bar{\varepsilon}) + [1 - \chi\left(\frac{4\alpha (\bar{\varepsilon} + h_\alpha)}{\theta_0 r_\alpha}\right)](1 - w(\bar{\varepsilon}))
\]

\[
= w(\bar{\varepsilon}) + [1 - \chi\left(\frac{4\alpha (\bar{\varepsilon} + h_\alpha)}{\theta_0 r_\alpha}\right)](1 - w(\bar{\varepsilon})) O(e^{-\sqrt{\bar{\varepsilon}}})
\]

\[
= w(\bar{\varepsilon}) + O(e^{-\sigma |\bar{\varepsilon}|})e^{-(\sqrt{\bar{\varepsilon}} - \sigma)|\theta_{0,r_\alpha}|/\alpha}.
\]

Notice that, with some \(\tilde{\sigma} > 0\) we have

\[
e^{-\sqrt{\bar{\varepsilon}} - \sigma)|\theta_{0,r_\alpha}|/\alpha} \leq C e^{-\tilde{\sigma}|\theta_{0,r_\alpha}|/\alpha} e^{-\tilde{\sigma}h_\alpha/\alpha},
\]

hence, using the fact that \(f(w)\) is exponentially small as well in the support of the function \(1 - \chi\left(\frac{4\alpha (\bar{\varepsilon} + h_\alpha)}{\theta_0 r_\alpha}\right)\), we get:

\[
|A_1| \leq C e^{-\sigma |\bar{\varepsilon}|}e^{-\tilde{\sigma}|\theta_{0,r_\alpha}|/\alpha} e^{-\tilde{\sigma}h_\alpha/\alpha},
\]

form which it follows

\[(6.33) \quad \| A_1 \|_{p,\sigma,v} \leq e^{-\tilde{\sigma}h_\alpha/4\alpha} \leq C\alpha^2.
\]

Terms \(A_2\) and \(A_3\) above are estimated in a similar way. To estimate the remaining term in \(E_2\), namely \(E_{22}\) and \(E_{23}\), we use the same general approach. The key point here is the fact that, with \(k \geq 1:\)

\[
\| \nabla Gamma_{\alpha} r_\alpha^{-k} \|_{\infty,2} \leq C\alpha,
\]

\[
\| \nabla^2 Gamma_{\alpha} r_\alpha^{-k} \|_{p,2} \leq C\alpha^{2-8/p},
\]

\[(6.34) \quad \| E_{22} \|_{p,\sigma,v} \leq C\alpha^2.
\]
and the exponential smallness of \( w(\bar{z}) \pm 1 \) in the support of \( \chi'(\frac{4\alpha(\bar{z} + \Delta \alpha)}{\theta_0 r_\alpha}) \). Finally, the remaining terms in \( E_2 \) are handled similarly since \( w_1(\bar{z}) \sim e^{-\sigma |\bar{z}|} \). The details are omitted.

Now, using (6.22), Lemma 6.2 and Proposition 5.1 we show the existence of a unique solution \( \phi \) of (6.15) by a fixed point argument. The estimate (6.23) is deduced from this as well.

Next we will prove that \( \tilde{\phi} \) is Lipschitz as a function of \( h_\alpha \). To apply the general theory developed and in particular Proposition 5.1 let us fix functions \( h_\alpha^{(k)}, k = 1, 2 \) satisfying (5.22) and denote by \( \tilde{\phi}^{(k)} \) solutions of the respective nonlinear projected problems (6.15). We notice that the functions \( \tilde{\phi}^{(k)} \) are defined in the same domain \( \Gamma_\alpha \times \mathbb{R} \) however the linear parts of the equations they solve are different, since the coefficients of the differential operators involved expressed in local coordinates depend on \( h_\alpha^{(k)} \) as well. Thus we will denote the respective linear operators by \( \tilde{\mathcal{L}}^{(k)} \).

We will also write \( \phi = \tilde{\phi}^{(1)} - \tilde{\phi}^{(2)} \). With these notations we have that \( \phi \) is a solution of:

\[
\begin{align*}
\tilde{\mathcal{L}}^{(1)} \phi &= \{ \eta_\alpha^\alpha (\bar{z} + h_\alpha^{(1)}) \mathcal{S}(w^{(1)} + w_1^{(1)}) - \eta_\alpha^\alpha (\bar{z} + h_\alpha^{(2)}) \mathcal{S}(w^{(2)} + w_1^{(2)}) \} \\
&\quad + (\tilde{\mathcal{L}}^{(2)} - \tilde{\mathcal{L}}^{(1)}) \tilde{\phi}^{(2)} + (\tilde{c}_\alpha^{(1)} - \tilde{c}_\alpha^{(2)}) w_{\bar{z}} \\
&\quad + \mathbf{N}(\tilde{\phi}^{(1)}) - \mathbf{N}(\tilde{\phi}^{(2)}), \quad \text{in } \Gamma_\alpha \times \mathbb{R}, \\
\int_{\mathbb{R}} \tilde{\phi}(y, \bar{z}) w_{\bar{z}}(\bar{z}) \, d\bar{z} = 0, \quad \text{in } \Gamma_\alpha.
\end{align*}
\]

We will begin with estimating the following term:

\[
\tilde{E} = \mathcal{S}(w^{(1)} + w_1^{(1)}) - \mathcal{S}(w^{(2)} + w_1^{(2)}).
\]

This term is particularly important because the Lipschitz character of \( \tilde{\psi} \) follows from the Lipschitz property of \( \tilde{E} \). Let us further decompose:

\[
\begin{align*}
\tilde{E} &= \chi \left( \frac{8\alpha \bar{z}}{\theta_0 r_\alpha} \right) \{ \mathcal{S}(w^{(1)} + w_1^{(1)}) - \mathcal{S}(w^{(2)} + w_1^{(2)}) \} \\
&\quad + [1 - \chi \left( \frac{8\alpha \bar{z}}{\theta_0 r_\alpha} \right)] \{ \mathcal{S}(w^{(1)} + w_1^{(1)}) - \mathcal{S}(w^{(2)} + w_1^{(2)}) \} \\
&= \tilde{E}_1 + \tilde{E}_2.
\end{align*}
\]

Notice that in the support of \( \chi \left( \frac{8\alpha \bar{z}}{\theta_0 r_\alpha} \right) \) we can assume (since \( h_\alpha^{(k)} \) is small) that \( w = w(\bar{z}), w_1 = w_1 \). Then we get:

\[
\begin{align*}
\tilde{E}_1 &= \chi \left( \frac{\alpha \bar{z}}{8\theta_0 r_\alpha} \right) \{ \Delta_{\alpha, \bar{z}}^{(1)} - \Delta_{\alpha, \bar{z}}^{(2)} \} (w + w_1) \\
&\quad - \chi \left( \frac{\alpha \bar{z}}{8\theta_0 r_\alpha} \right) \{ H_{\alpha, \bar{z}}^{(1)} - H_{\alpha, \bar{z}}^{(2)} \} \partial_{\bar{z}} w \\
&= \tilde{E}_{11} + \tilde{E}_{12}.
\end{align*}
\]
Using formula (6.25) we get:

\[ (6.36) \]
\[ \| \hat{E}_{11} \|_{p,\sigma,\nu} \leq C\| \nabla \tilde{h} \|_{1} \| \tilde{h}^{(2)} \|_{1,\sigma,\nu} \]
\[
+ C(\| \nabla \tilde{h} \|_{\infty,\nu-1} + \| \nabla \tilde{h} \|_{\infty,\nu-1}) \| \tilde{h} \|_{\infty,\nu-1}
\]
\[
+ C(\| \nabla^{2} \tilde{h} \|_{p,\nu} + \| \nabla^{2} \tilde{h} \|_{p,\nu}) \| \hat{h} \|_{1,\sigma,\nu-2}
\]
\[
\leq C(\| \nabla^{2} \tilde{h} \|_{1} \| \hat{h} \|_{1,\sigma,\nu-2})
\]
\[
+ C\alpha^{1+\mu} \| \nabla \tilde{h} \|_{\infty,\nu-1} + C\alpha^{2-8/p+\mu} \| \hat{h} \|_{1,\sigma,\nu-2}
\]
\[
\leq C\alpha^{-8/p} \| \hat{h} \|_{1,\sigma,\nu-2}
\]

(see (5.22) for the definition of \( \| \cdot \|_{s,\sigma,\nu} \)). Using similar argument as in (6.29) we get as well:

\[ (6.37) \]
\[ \| \hat{E}_{12} \|_{p,\sigma,\nu} \leq C\alpha^{2} \| \hat{h} \|_{1,\sigma,\nu-2}. \]

To estimate \( \hat{E}_{2} \) we follow the same approach, again using (6.34) and the exponential smallness of \( w(z) \) \( \pm 1 \) in the support of \( 1 - \chi(\frac{8a^2}{\hbar r}) \). As a consequence we get that

\[ (6.38) \]
\[ \| \tilde{S}(w^{(1)} + \tilde{w}^{(2)}) - \tilde{S}(w^{(1)} + \tilde{w}^{(1)}) \|_{p,\sigma,\nu} \leq C\alpha^{-8/p} \| \hat{h} \|_{1,\sigma,\nu-2}. \]

From this and (6.13), denoting \( \tilde{\psi} = \psi^{(1)} - \psi^{(2)} \), we get

\[ (6.39) \]
\[ \| \tilde{\psi} \|_{\infty,\nu} + \| \tilde{\psi} \|_{\infty,\nu} \leq C e^{-\sigma/\alpha} \| S^{(1)}[w] - S^{(2)}[w] \|_{p,\sigma,\nu}
\]
\[
+ C e^{-2\sigma/\alpha} \| \| \tilde{\phi} \|_{\infty,\sigma,\nu} + \| \nabla \tilde{\phi} \|_{\infty,\sigma,\nu} \]
\[
\leq C e^{-\sigma/\alpha} \alpha^{-8/p} \| \hat{h} \|_{1,\sigma,\nu-2}
\]
\[
+ C e^{-2\sigma/\alpha} \| \| \tilde{\phi} \|_{\infty,\sigma,\nu} + \| \nabla \tilde{\phi} \|_{\infty,\sigma,\nu} \].

Another important term to estimate in (6.35) is

\[ (6.40) \]
\[ \hat{E}_{3} = (\tilde{\phi}^{(2)} - \tilde{\phi}^{(1)}) \tilde{\phi}^{(2)}. \]

It is a matter of rather tedious but standard calculations to show that:

\[ (6.41) \]
\[ \| \hat{E}_{3} \|_{p,\sigma,\nu} \leq C\alpha^{-8/p} \| \hat{h} \|_{1,\sigma,\nu-2} \| \tilde{\phi} \|_{1,\sigma,\nu-2}
\]
\[
\leq C\alpha^{-8/p} \| \hat{h} \|_{1,\sigma,\nu-2} \| \tilde{\phi} \|_{1,\sigma,\nu-2}
\]

Here we use the fact that the coefficients of the derivatives in the expressions in local coordinates for \( \tilde{\phi}^{(k)} \) are smooth functions of \( \hat{h}^{(k)} \) and that all terms involved have a total of at most 3 derivatives summing up both derivatives of \( \hat{h}^{(k)} \) and \( \tilde{\phi}^{(2)} \).

Using the a priori estimate for \( \tilde{\phi}^{(1)} \) to estimate \( \tilde{\phi} \) in (6.35) we obtain the required estimate from (6.38)–(6.41) and the Lipschitz character of the nonlinear term \( \mathcal{N}(\tilde{\phi}^{(1)}) - \mathcal{N}(\tilde{\phi}^{(2)}) \). This ends the proof.

\[ \Box \]

This results of Proposition 6.1 allow us to reduce the full nonlinear problem to one dependent on \( h_\alpha \). Indeed, using the definition of \( \phi, \psi \) and the fact that \( u_\alpha = w + \eta^{(2)}_\alpha \phi + \psi \) (see (6.1)–(6.4)) we see that instead of the nonlinear problem (1.1) we have found, for given \( h^{(1)} \) functions \( u_\alpha, c_\alpha \) such that

\[ (6.42) \]
\[ \Delta u_\alpha + f(u_\alpha) = \eta^{(2)} \tilde{c}_\alpha w_\xi (\tilde{z}), \quad \tilde{z} = z - h_\alpha, \quad \text{in } \mathbb{R}^3. \]
If we can adjust $h_\alpha$ in such a way that

$$\bar{c}_\alpha = 0,$$

then $u_\alpha$ in (6.42) is a solution we are looking for. The theory we have already derived allows to derive a relatively simple form of the reduced problem (6.43). In the next section we will see that it amounts to a nonlocal PDE for $h_\alpha$ which involves the Jacobi operator on $\Gamma_\alpha$ applied to $h_\alpha$ as its leading term.

7. Derivation of the reduced problem

To derive the reduced problem we will go back to (6.15). Multiplying the equation by $w_\bar{z}(\bar{z})$ and integrating over $\mathbb{R}$ with respect to $\bar{z}$ we get the following identity:

$$\int_{\mathbb{R}} \bar{\mathcal{L}}(\bar{\phi})w_\bar{z} d\bar{z} = \bar{c}_\alpha \int_{\mathbb{R}} w_\bar{z}^2 d\bar{z} + \int_{\mathbb{R}} \bar{\eta}_\alpha \bar{S}(w + w_1)w_\bar{z} d\bar{z} + \int_{\mathbb{R}} \bar{\mathbf{N}}(\bar{\phi})w_\bar{z} d\bar{z},$$

hence (6.43) is equivalent to:

$$\int_{\mathbb{R}} \eta_\alpha \delta_{\bar{S}}(w + w_1)w_\bar{z} d\bar{z} = - \int_{\mathbb{R}} \bar{\mathcal{L}}(\bar{\phi})w_\bar{z} d\bar{z} + \int_{\mathbb{R}} \bar{\mathbf{N}}(\bar{\phi})w_\bar{z} d\bar{z}.$$  

We will now calculate more explicitly all terms involved in (7.1).

We will begin with:

$$\int_{\mathbb{R}} \eta_\alpha \delta_{\bar{S}}(w + w_1)w_\bar{z} d\bar{z} = \int_{\mathbb{R}} \eta_\alpha \Delta_{\Gamma_{\alpha,z}} w w_\bar{z} d\bar{z} = \int_{\mathbb{R}} \eta_\alpha ^2 \delta_{\bar{H}_{\Gamma_{\alpha,z}}} - \bar{z} |A_{\Gamma_{\alpha,z}}|^2 w_\bar{z}^2 + \int_{\mathbb{R}} \eta_\alpha ^2 \Delta_{\Gamma_{\alpha,z}} w_1 w_\bar{z} d\bar{z} = M_1 + M_2 + M_3.$$

Using the local representation for $\Delta_{\Gamma_{\alpha,z}} w$ given in (6.25) we get

$$M_1 = - \int_{\mathbb{R}} \frac{1}{\sqrt{\det(g_{\alpha,\bar{z}})}} \partial_i \left( g^{ij}_{\alpha,\bar{z}} \sqrt{\det(g_{\alpha,\bar{z}})} \partial_j h_\alpha \right) w_\bar{z}^2 d\bar{z}$$

$$+ \int_{\mathbb{R}} \frac{1}{\sqrt{\det(g_{\alpha,\bar{z}})}} \partial_\bar{z} \left( g^{ij}_{\alpha,\bar{z}} \sqrt{\det(g_{\alpha,\bar{z}})} \right) \partial_i h_\alpha \partial_j h_\alpha w_\bar{z}^2 d\bar{z}$$

$$+ \int_{\mathbb{R}} g^{ij}_{\alpha,\bar{z}} \partial_j h_\alpha \partial_i h_\alpha w_\bar{z}^2 d\bar{z} = M_{11} + M_{12} + M_{13}.$$

We will start with:

$$M_{11} = - c_0 \Delta_{\Gamma_{\alpha}} h_\alpha + B_{\alpha_1}(h_\alpha), \quad c_0 = \int_{\mathbb{R}} w_\bar{z}^2 d\bar{z}.$$
Let us fix \( y_0 \in \Gamma_\alpha \). The local norm of the second order differential operator \( B_{\alpha 1} \) can be estimated as follows:

\[
\| r^{\nu+1}_{\alpha} B_{\alpha 1}(h_\alpha) \|_{L^p(\Gamma_\alpha \cap B(y_0, r_\alpha^{-1}))}^p \leq C \alpha \int_{B(0, 2r_\alpha^{-1})} r^{p(\nu+1)}(y(t)) |\nabla_{\Gamma_\alpha} h_\alpha|^2 |\frac{dt}{1 + r_\alpha(y(t))}|^p \tag{7.3}
\]

\[
+ C \alpha \int_{B(0, 2r_\alpha^{-1})} r^{p(\nu+1)}(y(t)) |\nabla_{\Gamma_\alpha} h_\alpha|^p |\frac{dt}{1 + r_\alpha(y(t))}|^p \nonumber \tag{7.4}
\]

\[
+ C \alpha \int_{B(0, 2r_\alpha^{-1})} r^{p(\nu+1)}(y(t)) |\nabla_{\Gamma_\alpha} h_\alpha|^p |\frac{dt}{1 + r_\alpha(y(t))}|^{2p}. \nonumber \tag{7.5}
\]

Notice that in the ball \( B(0, 2r_\alpha^{-1}) \) we have:

\[
1 + r_\alpha(y(t)) \leq C, \tag{7.6}
\]

by Lemma 4.1. Hence, from the definition of the \( \| \cdot \|_{*,p,\nu} \)-norm and the assumption we have made on \( h_\alpha \), see (5.22), we get that:

\[
|B_{\alpha 1}(h_\alpha)|_{p,\nu+1} \leq C \alpha^{1-8/p} \| h_\alpha \|_{*,p,\nu}. \tag{7.7}
\]

Similarly we have, setting \( B_{\alpha 2}(h_\alpha) = M_{12} + M_{13} \):

\[
|B_{\alpha 2}(h_\alpha)|_{p,\nu} \leq C \alpha^{1-8/p} \| h_\alpha \|_{*,p,\nu}. \tag{7.8}
\]

To estimate \( M_2 \) we first use the expansion (5.31) to find:

\[
M_2 = -h_\alpha |A_{\Gamma_\alpha}|^2 \int_\mathbb{R} w_\alpha^2 d\bar{z} - \int_\mathbb{R} (\bar{z} + h_\alpha)^2 \mathcal{R}_\alpha w_\alpha^2 d\bar{z} \nonumber \tag{7.9}
\]

\[
= -c_0 h_\alpha |A_{\Gamma_\alpha}|^2 - \int_\mathbb{R} \bar{z}^2 \mathcal{R}_\alpha w_\alpha^2 d\bar{z} + \int_\mathbb{R} (2\bar{z}h_\alpha + h_\alpha^2) \mathcal{R}_\alpha w_\alpha^2 d\bar{z} \nonumber \tag{7.10}
\]

Observe that here \( \mathcal{R}_\alpha = \mathcal{R}_\alpha(y, \bar{z} + h_\alpha) \). We can further Taylor expand this function in terms of \( \bar{z} + h_\alpha \) to get

\[
\mathcal{R}_\alpha(y, \bar{z} + h_\alpha) = \mathcal{R}_{1,\alpha}(y) + (\bar{z} + h_\alpha)\mathcal{R}_{2,\alpha}(y, \bar{z} + h_\alpha), \tag{7.11}
\]

where

\[
\mathcal{R}_{1,\alpha} \sim \frac{\alpha^3}{(1 + r_\alpha^4)}, \quad \mathcal{R}_{2,\alpha} \sim \frac{\alpha^4}{1 + r_\alpha^4}, \tag{7.12}
\]

by formula (2.25) and Lemma 4.1. Then we can write, denoting \( c_1 = \int_\mathbb{R} \bar{z}^2 w_\alpha^2 \),

\[
M_2 = -c_0 h_\alpha |A_{\Gamma_\alpha}|^2 - c_1 \mathcal{R}_{1,\alpha} \nonumber \tag{7.13}
\]

\[
- \int_\mathbb{R} [\bar{z}^2(\bar{z} - h_\alpha)\mathcal{R}_{2,\alpha} + (2\bar{z}h_\alpha + h_\alpha^2)\mathcal{R}_\alpha] w_\alpha^2 d\bar{z} \nonumber \tag{7.14}
\]

\[
= -c_0 h_\alpha |A_{\Gamma_\alpha}|^2 - c_1 \mathcal{R}_{1,\alpha} + B_{\alpha 3}(h_\alpha) \nonumber \tag{7.15}
\]

We observe that since \( \nu \in [2, 3] \) therefore, from (7.10), we have:

\[
\| \mathcal{R}_{1,\alpha} \|_{p,\nu} \leq C \alpha^{3-8/p}. \tag{7.16}
\]

We notice that this is the only term that is of order in \( r_\alpha^\nu \), since the rest of the terms computed so far (and those evaluated below) have weights \( r_\alpha^{\nu+1} \) in their norms.

From (7.12) we get:

\[
\| B_{\alpha 3}(h_\alpha) \|_{p,\nu+1} \leq C \alpha^{4-8/p} + C \alpha^{1-8/p} \| h_\alpha \|_{*,p,\nu}, \tag{7.17}
\]
Now we will estimate the terms involved in the projection of $\bar{\mathcal{L}}(\bar{\phi})$ onto $w_\mathcal{Z}$. Using the same notation as in in (5.35) we get, after integration by parts and also using the orthogonality condition
\[
\int_R \mathcal{L}(\bar{\phi}) w_\mathcal{Z} d\bar{Z} = -\int_R B_{1\alpha}(\bar{\phi}) w_\mathcal{Z} d\bar{Z} + \frac{1}{2} \int_0^1 \partial_\bar{Z} \bar{\phi} \Delta \Gamma_{\alpha} h_\alpha w_\mathcal{Z} d\bar{Z} + \int_\mathcal{Z} \eta_\alpha^2 B_{2\alpha}(\bar{\phi}) w_\mathcal{Z} d\bar{Z}
\]
(7.12)
\[
- \int_R \eta_\alpha^2 (\Delta \Gamma_{\alpha,\bar{Z}} h_\alpha + (\bar{Z} + h_\alpha) |A\Gamma_{\alpha}|^2) \partial_\bar{Z} \bar{\phi} w_\mathcal{Z} d\bar{Z}
\]
\[
- \int_R \eta_\alpha^2 (\bar{Z} + h_\alpha)^2 R_{\alpha} \partial_\bar{Z} \bar{\phi} w_\mathcal{Z} d\bar{Z}
\]
= $I_1 + I_2 + I_3 + I_4$.

Using the explicit formula for $B_{1\alpha}(\bar{\phi})$ given in (5.24) we get after integrating by parts once with respect to $\bar{Z}$:
\[
\left| \int_R B_{1\alpha}(\bar{\phi}) w_\mathcal{Z} d\bar{Z} \right| \leq C \left| \nabla \Gamma_{\alpha} h_\alpha \right| \int_R \left| D\bar{\phi} \right|\left| w_\mathcal{Z} \right| d\bar{Z},
\]
It follows that if $\nu \in [2,3]$ then:
\[
\| r^{\nu+1}_\alpha \int_R B_{1\alpha}(\bar{\phi}) w_\mathcal{Z} d\bar{Z}\|_{L^p(\Gamma_{\alpha} \cap B(y_0, \theta_0 \alpha^{-1} ))} \leq C \| r^{\nu-1}_\alpha \nabla \Gamma_{\alpha} h_\alpha \| \left\{ \int_{B(0, \theta_0 \alpha^{-1})} r^{\nu}_\alpha (y(t)) \sup_{\bar{Z}} \left[ e^{\nu [\bar{Z}]} \left| D\bar{\phi} \right| \right] dt \right\}^{1/p}.
\]
\[
\leq C \alpha^{-1-8/p} \| h_\alpha \|_{*, p, \nu} \| \bar{\phi} \|_{*, p, \sigma, \nu},
\]
Similarly, we get:
\[
\| r^{\nu+1}_\alpha \int_R \partial_\bar{Z} \bar{\phi} \Delta \Gamma_{\alpha} h_\alpha w_\mathcal{Z} d\bar{Z}\|_{L^p(\Gamma_{\alpha} \cap B(y_0, \theta_0 \alpha^{-1} ))} \leq C \| h_\alpha \|_{*, p, \nu} \| \bar{\phi} \|_{*, p, \sigma, \nu},
\]
hence,
\[
\| r^{\nu+1}_\alpha I_1 \|_{L^p(\Gamma_{\alpha} \cap B(y_0, \theta_0 \alpha^{-1} ))} \leq C \| h_\alpha \|_{*, p, \nu} \| \bar{\phi} \|_{*, p, \sigma, \nu} \leq C \alpha^{-1-8/p} \| h_\alpha \|_{*, p, \nu} \| \bar{\phi} \|_{*, p, \sigma, \nu}.
\]
Using (5.30) we get as well
\[
\| r^{\nu+1}_\alpha I_2 \|_{L^p(\Gamma_{\alpha} \cap B(y_0, \theta_0 \alpha^{-1} ))} \leq C \alpha^{-1-8/p} (1 + \alpha^{-2} \| h_\alpha \|_{*, p, \nu}) \| \bar{\phi} \|_{*, p, \sigma, \nu}.
\]
We can also estimate jointly:
\[
\| r^{\nu+1}_\alpha I_3 \|_{L^p(\Gamma_{\alpha} \cap B(y_0, \theta_0 \alpha^{-1} ))} + \| r^{\nu+1}_\alpha I_4 \|_{L^p(\Gamma_{\alpha} \cap B(y_0, \theta_0 \alpha^{-1} ))} \leq C \alpha^{-1-8/p} \| \bar{\phi} \|_{*, p, \sigma, \nu} \leq C \alpha^{-8/p} \| \bar{\phi} \|_{*, p, \sigma, \nu} \| h_\alpha \|_{*, p, \nu}.
\]
Finally, denoting
\[
\int R \mathcal{N}(\bar{\phi}) w_\mathcal{Z} d\bar{Z} = I_5,
\]
we get that
\[
\| r^{\nu+1}_\alpha I_5 \|_{L^p(\Gamma_{\alpha} \cap B(y_0, \theta_0 \alpha^{-1} ))} \leq C \| \bar{\phi} \|^2_{*, p, \sigma, \nu}.
\]
Summarizing (7.1)–(7.18) we get that $h_\alpha$ must be a solution of the following problem:

\[ \Delta_{\Gamma_\alpha} h_\alpha + |A_{\Gamma_\alpha}|^2 h_\alpha = c_1 R_{1,\alpha} + F_\alpha(h_\alpha, \nabla_{\Gamma_\alpha} h_\alpha, \nabla^2_{\Gamma_\alpha} h_\alpha), \]  

where the first term on the right hand side of (7.19) satisfies (7.10) and formula (2.25) and Lemma 4.1 it is explicitly given by:

\[ R_{1,\alpha}(y) = \sum_{i}^8 \kappa_i^3(y), \]  

and $F_\alpha$ is a nonlinear and nonlocal function of $h_\alpha$ and its first and second derivatives that satisfies:

\[ \|F_\alpha\|_{p,\nu + 1} \leq C\alpha^{1-8/p}\|h_\alpha\|_{\ast,p,\nu} + C\alpha^{3-8/p}. \]  

The rest of this paper is devoted to solving the reduced problem (7.19). A natural way to do this is to argue by approximations on expanding balls $B_{R_i}$, as we have done before in this paper. However an extra difficulty in the case of the reduced problem is to derive a priori estimates (independent on $R$) for the Jacobi operator in (7.19).

To deal with this problem we will consider an approximate Jacobi operator, which is the mean curvature linearized around $\Gamma_{0,\alpha} = \{ x_9 = \frac{1}{\alpha} F_0(\alpha x') \}$, rather than $\Gamma_\alpha$.

At this point we will also use the symmetry of the minimal graph. Let us recall that in reality $\Gamma_\alpha$ is a graph of a function $F_\alpha$ that satisfies

\[ F_\alpha(u,v) = -F_\alpha(v,u), \quad u^2 = x_1^2 + \cdots + x_4^2, \quad v^2 = x_5^2 + \cdots + x_8^2. \]

It is therefore natural to make the following assumption on $h_\alpha$:

\[ h_\alpha(u,v) = -h_\alpha(v,u). \]  

Observe that in particular the principal curvatures of $\Gamma$ satisfy

\[ \kappa_i(u,v) = -\kappa_i(v,u), \]

hence:

\[ R_{1,\alpha}(u,v) = -R_{1,\alpha}(v,u). \]

Notice that the Fermi coordinate $z$ depends on $x$ only through $(u, v, x_9)$ and:

\[ z(u, v, x_9) = -z(v, u, -x_9). \]

From this it follows:

\[ w(u, v, x_9) = -w(v, u, -x_9), \quad w_1(u, v, x_9) = -w_1(v, u, -x_9), \]

and

\[ c_\alpha(u, v) = -c_\alpha(v, u), \quad \phi(u, v, x_9) = -\phi(v, u, -x_9). \]

In all, the right hand side of (7.19) has the same type of symmetry as $h_\alpha$:

\[ F_\alpha(u, v) = -F_\alpha(v, u). \]

To put it differently: the procedure that leads to determining $u_\alpha$ for a given $h_\alpha$ can be done in the sector $T\{ u > 0, v > 0, v > u \}$ first and then the resulting solution can be extended to the whole space by using the natural symmetries of the minimal graph.
8. Solvability theory for the Jacobi operator

8.1. The approximate Jacobi operator. In this and the following section we will consider the Jacobi operator associated to a fixed minimal surface setting the scaling parameter $\alpha = 1$. We will denote:

$$\Gamma = \{ x_9 = F(x') \}.$$  

By $A_\Gamma$ we will denote the second fundamental form on $\Gamma$. The Jacobi operator $J$ is given by:

$$\mathcal{J}(h) = \Delta h + |A_\Gamma|^2 h.$$  

A convenient form of the Jacobi operator is obtained using the natural parametrization of $\Gamma$ given by $\Gamma = \{(x', F(x')) \mid x' \in \mathbb{R}^8 \}$. In these coordinates we get

$$\mathcal{J}(h) = H[F](h\sqrt{1 + |\nabla F|^2}),$$  

where $H[F]$ is the mean curvature operator and $H'[F]$ is its linearization around $F$, namely:

$$H'[F](\varphi) = \nabla \cdot \left( \frac{\nabla \varphi}{1 + |\nabla F|^2} - \frac{\nabla F(\nabla F \cdot \nabla \varphi)}{(1 + |\nabla F|^2)^{3/2}} \right).$$  

We will define the norms:

$$\|f\|_{p, \nu} := \sup_{y \in \Gamma} \|r^\nu f\|_{L^p(\Gamma \cap B(y, \delta_0))}, \quad 9 < p < \infty, \quad \nu \geq 2,$$

$$\|f\|_{\infty, \nu} := \sup_{y \in \Gamma} r^\nu |f(y)|, \quad \nu \geq 2,$$

which are analogous to the weighted norms defined above, and will be useful to treat question of invertibility of the Jacobi operator.

We will go back now to the expression of $H$ in terms of the variables $(t, s)$ introduced in section 3.2. We recall that:

$$H[F] = |\nabla F_0| \partial_t \left( \frac{|\nabla F_0| \partial_t F}{1 + |\nabla F|^2} \right) + |\nabla F_0| \partial_s \left( \frac{\rho^{-2} \partial_s F}{|\nabla F_0| \sqrt{1 + |\nabla F|^2}} \right),$$

where

$$\nabla F = F_t \nabla F_0 + \rho^{-1} F_s \frac{\nabla F_0}{|\nabla F_0|}, \quad \rho = \frac{1}{(uv)^2},$$

(see Lemma 3.3 for the definition of the coordinates $u = u(t, s), v = v(t, s)$). The linearized mean curvature operator expressed in these variables takes form:

$$H'[F](\varphi) = |\nabla F_0| \partial_t \left( \frac{\partial_t \varphi (1 + |\nabla F|^2) - \partial_t F(\nabla F \cdot \nabla \varphi)}{(1 + |\nabla F|^2)^{3/2}} \right)$$

$$+ |\nabla F_0| \partial_s \left( \frac{\rho^{-2} \partial_s \varphi (1 + |\nabla F|^2) - \partial_s F(\nabla F \cdot \nabla \varphi)}{(1 + |\nabla F|^2)^{3/2}} \right).$$

Let us now consider the linearized of mean curvature operator obtained by linearizing around the surface $\Gamma_0 = \{(x', F_0(x')) \mid x' \in \mathbb{R}^8 \}$, namely:

$$H'[F_0](\varphi) = |\nabla F_0| \partial_t \left( \frac{|\nabla F_0| \partial_t \varphi}{1 + |\nabla F_0|^2} \right) + |\nabla F_0| \partial_s \left( \frac{\rho^{-2} \partial_s \varphi}{|\nabla F_0| \sqrt{1 + |\nabla F_0|^2}} \right),$$

where we have used the fact that

$$\partial_t F_0 = 1, \quad \partial_s F_0 = 0, \quad \nabla \varphi \cdot \nabla F_0 = \partial_t \varphi |\nabla F_0|^2.$$
We notice that for $\Gamma_0$ the relation analogous to (8.3) holds, namely:
\begin{equation}
\Delta_{\Gamma_0} h + |A_{\Gamma_0}|^2 h = H'[F_0](h\sqrt{1 + |\nabla F_0|^2}),
\end{equation}
where $A_{\Gamma_0}$ is the second fundamental form on $\Gamma_0$. We will refer to the operator defined above as the approximate Jacobi operator and denote it by $\mathcal{J}_0$. The reader should keep in mind that $\mathcal{J}_0$ as the Jacobi operator associated to $\Gamma_0$ "approximates" $\mathcal{J}$.

8.2. Supersolutions for the operator $\mathcal{J}_0$. In this section, we obtain supersolutions for the operator $\mathcal{J}_0(h)$ which is equivalent to finding supersolutions for $H'[F_0](\varphi)$, where $\varphi = h\sqrt{1 + |\nabla F_0|^2}$. Let us recall the definition of the set $T$:
\begin{equation}
T = \{(u,v) \mid u > 0, v > 0, u < v\},
\end{equation}
(see (3.4)) and the fact that in Lemma 3.3 we have associated $T$ in the set $Q = \{t > 0, s > 0\}.$

**Lemma 8.1.** For $\sigma \in (-1, 0)$ and $\sigma_1 \in [0, 1]$, there exist $r_0$ and $C > 0$ such that in the set $T \cap \{R > r_0\}$ we have:
\begin{equation}
H'[F_0](r^\sigma t^{\sigma_1}) + \frac{C(g(\theta))^{\sigma_1}}{r^{4-\sigma-\sigma_1}} \leq 0.
\end{equation}
Likewise for $\sigma \in (-1, 0)$ and $\sigma_1 \in (0, 1)$, there holds in $T \cap \{r > R_0\}$
\begin{equation}
H'[F_0](r^\sigma t^{\sigma_1}) + \frac{C}{r^{4-\sigma-\sigma_1}} \leq 0.
\end{equation}

**Remark 8.1.** Note that $4-\sigma-3\sigma_1 \in (1,5)$ with the choice of $\sigma, \sigma_1$, while $4-\sigma-\sigma_1 \in (3,5)$.

**Proof.** Let us write
\begin{equation}
H'[F_0](\varphi) = \tilde{L}_0(\varphi) + \tilde{L}_1(\varphi),
\end{equation}
where
\begin{equation}
\tilde{L}_0(\varphi) = |\nabla F_0|\partial_t \left( \frac{\partial \varphi}{|\nabla F_0|^2} \right) + |\nabla F_0|\partial_r \left( \frac{\rho^{-2} \partial \varphi}{|\nabla F_0|^2} \right),
\end{equation}
\begin{equation}
\tilde{L}_1(\varphi) = \tilde{L}_0(\varphi) - H'[F_0](\varphi).
\end{equation}
We have
\[
\partial_t (r^\sigma t^{\sigma_1}) = \sigma r^{\sigma-1} t^{\sigma_1} \frac{\partial r}{\partial t} + \sigma_1 r^\sigma t^{\sigma_1-1}
\]
\[
= \sigma r^{\sigma-1} t^{\sigma_1} \frac{\cos^2 \phi}{3t} + \sigma_1 r^\sigma t^{\sigma_1-1}
\]
\[
= \sigma r^\sigma t^{\sigma_1} \frac{1}{3t} \cos^2 \phi + \sigma_1 r^\sigma t^{\sigma_1-1}
\]
and
\[
\partial_t^2 (r^\sigma t^{\sigma_1}) = r^\sigma (\sigma_1 - 1) t^{\sigma_1-2} + O(r^{\sigma-6} t^{\sigma_1})
\]
Hence
\begin{equation}
|\nabla F_0|\partial_t \left( \frac{1}{|\nabla F_0|^2} \partial_t^2 (r^\sigma t^{\sigma_1}) \right) = \frac{1}{|\nabla F_0|^2} \partial_t (r^\sigma t^{\sigma_1}) + |\nabla F_0|\partial_t \left( \frac{1}{|\nabla F_0|^2} \partial_t (r^\sigma t^{\sigma_1}) \right)
\]
\[
= r^{\sigma-2} \sigma_1 (\sigma_1 - 1) t^{\sigma_1-2} + O(r^{\sigma-8} t^{\sigma_1}),
\]
where we have used the fact that $|\nabla F_0| \partial_t (\frac{1}{|\nabla F_0|^2}) \sim -\frac{\cos \phi}{r^3}$. Then we compute

$$|\nabla F_0| \partial_s \left( \frac{\rho^{-2}}{|\nabla F_0|^2} \partial_s r^\sigma \right) = |\nabla F_0| \partial_s \left( \frac{\rho^{-2}}{|\nabla F_0|^2} r^{\sigma-1} \partial_s r \right).$$

Using formula (3.21) the above quantity thus equals

$$C_0 \sigma |\nabla F_0| \partial_s \left( \frac{r^{12} \sin^6 2\theta}{r^{4}(9g^2 + g_0^2)} r^{\sigma-1} \frac{r}{s} \sin^2 \phi \right) = C_1 \sigma |\nabla F_0| \partial_s \left( \frac{sr^\sigma}{t^2} \cos^2 \phi \right),$$

where $C_1 > 0$ are generic positive constants, from now on. Now, using (3.16), we obtain

$$\partial_s \left( \frac{sr^\sigma}{t^2} \cos^2 \phi \right) = \frac{1}{t^2} \left( r^\sigma \cos^2 \phi + s \sigma r^{\sigma-1} \frac{r}{t} \sin^2 \phi \cos \phi + s \sigma r^{\sigma} (-\sin 2\phi)(-\sin 2\phi) \frac{\phi'}{14s} \right)$$

$$= \frac{r^\sigma}{t^2} \left( \cos^2 \phi + \frac{\sigma}{7} \sin^2 \phi \cos \phi + \frac{\sin^2 2\phi}{14} \phi' \right)$$

$$= \frac{r^\sigma}{t^2} \cos^2 \phi \left( 1 + \frac{\sigma}{7} \sin^2 \phi + \frac{2\sin^2 \phi}{7} \phi' \right),$$

hence

$$t^\sigma \left| \nabla F_0 \right| \partial_s \left( \frac{\rho^{-2}}{|\nabla F_0|^2} \partial_s r^\sigma \right)$$

$$= \sigma C_3 \frac{r^{2+\sigma}}{t^{2-\sigma}} \cos^2 \phi \sqrt{g^2 + g_0^2} \left( 1 + \frac{\sigma}{7} \sin^2 \phi + \frac{2\sin^2 \phi}{7} \phi' \right),$$

and finally

$$|\nabla F_0| \partial_s \left( \frac{\rho^{-2}}{|\nabla F_0|^2} \partial_s r^\sigma \right) = \sigma C_3 \frac{t^{\sigma_1}}{r^{4-\sigma}} \frac{1}{\sqrt{g^2 + g_0^2}} \left( 1 + \frac{\sigma}{7} \sin^2 \phi + \frac{2\sin^2 \phi}{7} \phi' \right)$$

$$= \sigma C_3 \frac{t^{\sigma_1}}{r^{4-\sigma}} a_1(\theta),$$

where $a_1(\theta) > 0$. Then we obtain:

$$\tilde{L}_0[\rho^\sigma r^{\sigma_1}] = r^{-4+\sigma+3\sigma_1} g^{\sigma_1} \left( \frac{\sigma_1(1 - \sigma_1)}{r^4 g^2} + \sigma C_3 a_1(\theta) \right) + O(r^{-8+\sigma+3\sigma_1}) g^{\sigma_1}$$

$$\leq -\frac{C g^{\sigma_1}}{r^{4-3\sigma_1-\sigma}},$$

which proves (8.9) with $H'[F_0]$, replaced by $\tilde{L}_0$.

For $-1 < \sigma < 0$ we have $\phi' \geq -3$, hence

$$|\nabla F_0| \partial_s \left( \frac{\rho^{-2}}{|\nabla F_0|^2} \partial_s r^\sigma \right) \leq -\frac{C_4}{r^{4-\sigma}}.$$
Combining this and (8.12), we obtain
\[ \tilde{L}_0[\tau^\sigma t^\sigma] \leq -C \left( r^{-2+\sigma} \tau^{\sigma_1-2} + \frac{\tau^{\sigma_1}}{r^{4-\sigma}} \right) \]
(8.16)
\[ \leq -\frac{C}{r^{4-3\sigma_1-\sigma}} \left( (\cos \phi)^{\sigma_1-2} + (\cos \phi)^{\sigma_1} \right) \]
\[ \leq -\frac{C_1}{r^{4-\sigma-\sigma_1}}. \]
This proves (8.10) with \( H'[F_0] \) replaced by \( \tilde{L}_0 \).

To finish the proof one needs to estimate \( \tilde{L}_1[\tau^{\sigma} t^{\sigma_1}] \) and show that this term is of smaller order than \( \tilde{L}_0[\tau^{\sigma} t^{\sigma_1}] \). This is straightforward since
\[ \frac{1}{|\nabla F_0|} - \frac{1}{\sqrt{1 + |\nabla F_0|^2}} \sim \frac{1}{|\nabla F_0|^3}, \quad r \gg 1. \]
We leave the details to the reader. \( \square \)

By \( \tilde{T} \) we denote the following sector:
\[ \tilde{T} = \{(u, v) : v > 0, |u| < v\} \subset \mathbb{R}^8. \]
Since all functions involved in the proof of Lemma 8.1 are even with respect to \( u \) in the set \( \tilde{T} \) we immediately obtain:

**Corollary 8.1.** For \( \sigma \in (-1, 0) \) and \( \sigma_1 \in [0, 1] \), there exist \( r_0 \) and \( C > 0 \) such that in the set \( \tilde{T} \cap \{r > r_0\} \) we have:
\[ J_0((1 + |\nabla F_0|^2)^{-1/2} r^{\sigma} t^{\sigma_1}) + \frac{C_1 (g(\theta))^{\sigma_1}}{r^{4-\sigma-3\sigma_1}} \leq 0. \]
Likewise for \( \sigma \in (-1, 0) \) and \( \sigma_1 \in (0, 1) \), there holds in \( \tilde{T} \cap \{r > R_0\} \)
\[ J_0((1 + |\nabla F_0|^2)^{-1/2} r^{\sigma} t^{\sigma_1}) + \frac{C_1}{r^{4-\sigma-\sigma_1}} \leq 0. \]

8.3. The outer problem for \( J_0 \). We will use the supersolutions derived above to treat the following problem:
\[ J_0(h) = f, \quad \text{in} \ \tilde{T} \cap \{r > R_0 + 1\}, \]
\[ h = 0, \quad \text{on} \ \partial(\tilde{T} \cap \{r > R_0 + 1\}), \]
where \( R_0 > r_0 \) is fixed. We will solve this problem by an approximation scheme in extending domains:
\[ J_0(h_R) = f, \quad \text{in} \ \tilde{T} \cap \{R > r > R_0 + 1\}, \]
\[ h_R = 0, \quad \text{on} \ \partial(\tilde{T} \cap \{R > r > R_0 + 1\}). \]
In this section we will consider the weighted norms defined in (8.5) with \( \Gamma \) replaced by \( \Gamma_0 \). As for the right hand side of (8.20) we assume that one of the following holds:
(1) Either \( \nu = 3 \)
\[ \|f\|_{\infty, \nu} < \infty, \quad \text{and} \quad |f| \leq C \frac{g(\theta)^{\sigma_1}}{r^3}, \quad r > R_0, \]
with some \( \sigma_1 \in (1/3, 2/3) \);
(2) or \( \nu \geq 3 + \mu, \mu \in (\frac{2}{3}, 1) \) and
\begin{equation}
\|f\|_{\infty, \nu} < \infty.
\end{equation}

**Lemma 8.2.** Let \( f \) be such that at least one of the two conditions (8.22) or (8.23) is satisfied. Then there exists a solution \( h \) of (8.20) such that:
\begin{equation}
\|h\|_{\infty, \nu'} - 2 + \|\nabla \Gamma_0 h\|_{\infty, \nu' - 1} + \|\nabla^2 \Gamma_0 h\|_{p, \nu'} \leq C\|f\|_{\infty, \nu},
\end{equation}
where \( \nu' \leq \nu \) satisfies:
\begin{equation}
\nu' = \begin{cases} 
3, & \text{if (8.22) holds,} \\
3 + \mu', & 0 < \mu' < 3\mu - 2, & \text{if (8.23) holds.}
\end{cases}
\end{equation}

**Proof.** We will solve (8.21) and then take the limit \( R \to \infty \). To fix attention we will consider \( f \) such that (8.22) holds, the other case being similar.

We observe that an easy consequence of Lemma 8.1 is that (8.21) has a unique solution for all \( R > R_0 + 1 \). Indeed taking \( \sigma, \sigma_1 \) such that \( \sigma + 3\sigma_1 = 2 \) (say \( \sigma = -\frac{1}{2}, \sigma_1 = \frac{1}{6} \)) we see that there is a bounded, positive supersolution of (8.21) of the form:
\begin{equation}
h_{\sigma, \sigma_1} = \frac{r^{\sigma} t^{\sigma_1}}{\sqrt{1 + |\nabla F_0|^2}}.
\end{equation}
This means that the homogeneous version of (8.21) has only a trivial solution. By a similar argument we can prove that the operator \( J_0 \) is non-degenerate for the outer problem (8.20). This means that the only vanishing at \( \infty \) solution of the homogeneous version of (8.20) is necessarily equal to 0.

Now let \( h_R \) be a solution of (8.21). We claim that there exists \( C > 0 \), independent on \( R \) such that
\begin{equation}
\|h_R\|_{\infty, 1} \leq C\|f\|_{\infty, 3}.
\end{equation}
We will argue by contradiction. If (8.27) does not hold then there exist sequences \( R_n, h_{R_n} \) and \( f_n \) such that:
\begin{equation}
\|f_n\|_{\infty, 3} \to 0, \quad \text{while} \quad \|h_{R_n}\|_{\infty, 1} = 1.
\end{equation}
Taking function
\[ h^+ = C(1 + \|f\|_{\infty, 3})h_{\sigma, \sigma_1}, \quad \sigma + 3\sigma_1 = 1, \]
with a suitable constant \( C \) (dependent on \( R_0 \) only) as a supersolution we see that for all \( R_n \) sufficiently large the supremum of \( rh_{R_n} \) must be attained in a fixed compact set. Passing now to the limit we obtain a nontrivial solution of (8.20) which contradicts the non-degeneracy of \( J_0 \). This proves estimate (8.27). The assertion of the Lemma follows now by elliptic estimates applied to the function \( h_R = rh_R \). The proof is complete. \( \square \)

**8.4. An approximation scheme for the Jacobi operator.** We will consider the following problem
\begin{equation}
J(h) = f, \quad \text{in} \ \hat{T},
\end{equation}
\begin{equation}
h = 0, \quad \text{on} \ \partial \hat{T}.
\end{equation}
In this section we will in general assume that:
\begin{equation}
\|f\|_{\infty, \nu} < \infty, \quad \text{with some} \ \nu > 4.
\end{equation}
We will solve (8.29) by approximations. For each sufficiently large \( R \) we will consider:

\[
\begin{align*}
\mathcal{J}(h_R) &= f, \quad \text{in } \tilde{T} \cap B_R(0), \\
h_R &= 0, \quad \text{on } \partial(\tilde{T} \cap B_R(0)).
\end{align*}
\]

(8.31)

Our goal is to show the following:

**Proposition 8.1.** Consider a family of solutions of (8.31), \( \{h_R\} \) with \( f \) satisfying (8.30). As \( R \to \infty \), \( h_R \) converges along a subsequence to a solution of (8.29). Moreover, denoting this solution by \( h \), there holds:

\[
\|h\|_{\infty, \nu - 2} + \|
abla \Gamma h\|_{\infty, \nu - 1} + \|
abla^2 \Gamma h\|_{p, \nu'} \leq C\|f\|_{\infty, \nu},
\]

where \( 4 < \nu' < \nu \) and \( p \in (9, \infty) \).

The proof of this Proposition follows from a series of Lemmas. First we will use Corollary 8.1 and in particular (8.19) to show:

**Lemma 8.3.** There exists \( r_1 > 0 \) such that for \( \sigma \in (-1, 0) \) and \( \sigma_1 \in (0, 1) \), we have in \( \tilde{T} \cap \{ r > r_1 \} \)

\[
\mathcal{J}\left( (1 + |\nabla F|^2)^{-1/2} r^{\sigma_1} \right) + \frac{C_1}{r^{4 - \sigma - \sigma_1}} \leq 0.
\]

(8.33)

**Proof.** The proof is based on comparing the expressions for \( \mathcal{J} \) and \( \mathcal{J}_0 \) in local coordinates and using formulas (4.36) and (4.38). We omit the details.

Next we will show that the operator \( \mathcal{J} \) is non-degenerate:

**Lemma 8.4.** Let \( h \), such that

\[
\|h\|_{\infty, 2 + \mu} < \infty, \quad \text{for some } \mu > 0,
\]

be a solution of the following problem:

\[
\begin{align*}
\mathcal{J}(h) &= 0, \quad \text{in } \tilde{T}, \\
h &= 0, \quad \text{on } \partial\tilde{T}.
\end{align*}
\]

(8.35)

Then we have \( h \equiv 0 \).

**Proof.** Let \( \varepsilon > 0 \) be a small number and let

\[
h_\varepsilon(x) = \frac{\varepsilon}{\sqrt{1 + |\nabla F(x)|^2}}.
\]

(8.36)

Observe that \( h_\varepsilon \) satisfies:

\[
\mathcal{J}(h_\varepsilon) = 0,
\]

and also, by (4.32),

\[
h_\varepsilon \geq \frac{C\varepsilon}{1 + r^2}.
\]

It follows that for all sufficiently large \( R \) we have

\[
h_\varepsilon \geq h, \quad \text{on } \partial(\tilde{T} \cap B_R(0)),
\]

hence by comparison principle we have

\[
h_\varepsilon \geq h, \quad \text{in } \tilde{T}.
\]

Taking \( \varepsilon \to 0 \) the assertion of the Lemma follows.
Proof. (of the Proposition 8.1) Since for $R > 0$ the function $h_\varepsilon$ defined (8.36) is a positive supersolution of (8.31) therefore for each $R > 0$ there exists a unique solution $h_R$ of (8.31).

We claim that there exists $C > 0$ independent of $R$ such that for all $f$ we have:

$$\|h_R\|_{\infty,2+\mu'} \leq C \|f\|_{\infty,4+\mu},$$

where $\nu = 4+\mu$ and $0 < \mu' < \mu$. To prove the claim we will argue by contradiction. Assume then that there exist $f_n$, $R_n$ and $h_R^n$ such that $h_R^n$ is a solution of (8.31) in $\tilde{T} \cap B_{R_n}(0)$ with $f = f_n$ and that

$$\|f_n\|_{\infty,4+\mu} \to 0, \quad \text{while} \quad \|h_R^n\|_{\infty,2+\mu'} = 1.
\tag{8.38}$$

Using Lemma 8.3 we will construct a supersolution of (8.31) in the set $\tilde{T} \cap \{R_0 < r < R\}$. Let $0 < \mu' < \mu$ be given and let:

$$0 < \varepsilon = \frac{\mu - \mu'}{2},$$

be fixed. Further let:

$$\sigma_1 = \varepsilon \frac{\varepsilon}{2}, \quad \sigma = -\mu + \frac{\varepsilon}{2}.$$

Then we have $0 > \sigma > -1$, $1 > \sigma_1 > 0$, and

$$\sigma + 3\sigma_1 = -\mu + 2\varepsilon = \mu'. \tag{8.39}$$

With $\sigma$, $\sigma_1$ as above function

$$\tilde{h}_R^n = C(1 + \|f_n\|_{\infty,4+\mu}) \frac{r^\sigma t^{\sigma_1}}{\sqrt{1 + |\nabla F_0|^2}}, \tag{8.40}$$

with some constant $C > 0$ is a positive supersolution of (8.31) in $\tilde{T} \cap \{R_0 < r < R_n\}$. This means that there exists $R' > R_0$ such that for all sufficiently large $R_n$ we have

$$|h_R^n| \leq \frac{1}{2r^2+\varepsilon^2}, \quad \text{for} \quad R' < r < R_n,$$

which means that the supremum of $|h_R^n r^{2+\mu'}|$ is taken on a compact set contained in $B_{R'+1}(0)$. This allows us to pass to the limit $n \to \infty$ and conclude that the limiting function $\tilde{h}$ satisfies the assumptions of Lemma 8.4 and hence $\tilde{h} \equiv 0$. This is a contradiction with the fact that $\|\tilde{h}\|_{\infty,2+\mu'} = 1$. The proof of the claim complete.

The assertion of the Proposition follows now by a standard argument. We omit the details.

8.5. A gluing procedure for the reduced problem. Given $f$ such that either (8.22) or (8.23) is satisfied we consider

$$\mathcal{J}(h) = f, \quad \text{in} \quad \tilde{T},
\tag{8.41}$$

$$h = 0, \quad \text{on} \quad \partial \tilde{T}.$$
Proposition 8.2. There exists a solution of problem (8.41) such that
\[ \|h\|_{\infty,\nu'} - 2 + \|\nabla^{2}\Gamma h\|_{p,\nu'} \leq C\|f\|_{\infty,\nu}, \]
where
\[ \nu' = \begin{cases} 3, & \text{if (8.22) holds,} \\ 3 + \mu', & 0 < \mu' < 3\mu - 2, \end{cases} \]
(8.43)

Proof. Let \( \tilde{h} \) be the solution of the outer problem (8.20). We will look for a solution of (8.41) in the form:
\[ h = \eta R \tilde{h} + h, \]
where \( \eta R \) is a cut of function such that \( \eta R(r) = 0 \) if \( r < R \) and \( \eta R(r) = 1 \) for \( r > R + 1 \), for some fixed \( R > R_0 + 2 \). Notice that in principle function \( \tilde{h} \) is defined on \( \Gamma_0 \) rather than on \( \Gamma \) but using the \( (u,v) \) coordinates we can assume that \( \tilde{h} \) is a function on \( \Gamma \). Then we have:
\[ \mathcal{J}(h) = f - \mathcal{J}(\eta R \tilde{h}), \text{ in } \tilde{T}, \]
\[ h = 0, \quad \partial\tilde{T}. \]
(8.45)
We have:
\[ f - \mathcal{J}(\eta R \tilde{h}) = \eta R(\mathcal{J}_0(\tilde{h}) - \mathcal{J}(\tilde{h})) - \tilde{h} \Delta_{\Gamma} \eta R - 2\nabla_{\Gamma} \tilde{h} \cdot \nabla_{\Gamma} \eta R + (1 - \eta R)f \]
(8.46)
Observe that the last three terms in (8.46) are compactly supported. On the other hand, using (4.36) and (4.38), we get that if \( \|\tilde{h}\|_{\infty,\nu'} < \infty \) then
\[ \|\eta R(\mathcal{J}_0(\tilde{h}) - \mathcal{J}(\tilde{h}))\|_{\infty,\nu' + 1 + \zeta} < \infty, \]
with some \( \zeta > 0 \). This means that
\[ \|f - \mathcal{J}(\eta R \tilde{h})\|_{\infty,\nu} < \infty, \]
with some \( \nu > 4 \). This allows to use Proposition 8.1 to solve (8.45). Combining this with the results of Proposition 8.2 we end the proof. \( \square \)

8.6. The inverse of the Jacobi operator in \( L^{p,\nu} \). Notice that so far we have assumed that the right hand sides of the problems involving the operators \( \mathcal{J}, \mathcal{J}_0 \) are bounded in \( L^{\infty,\nu} \). However in the case \( \nu \geq 4 \) we have to deal with the right hand sides in \( L^{p,\nu} \), where \( p > 9 \). Now we will show how to overcome this technical difficulty. We will prove first:

Lemma 8.5. Let us consider problem (8.20) but now assuming that with some \( \nu \geq 3 + \mu, \mu \in \left(\frac{3}{2}, 1\right) \) we have
\[ \|f\|_{p,\nu} < \infty, \quad p > 9. \]
(8.49)
There exists a number \( C > 0 \) such that for each \( f \) with \( \|f\|_{p,\nu} < \infty \) there is a solution \( h \) to problem (8.20) with \( \|h\|_{p,\nu'} - 2 < +\infty \), where \( \nu' \leq \nu \) satisfies:
\[ \nu' = 3 + \mu', \quad 0 < \mu' < 3\mu - 2, \]
(8.50)
This solution satisfies the estimate
\[ \|\nabla^2_{\Gamma_0} h\|_{p,\nu'} + \|\nabla_{\Gamma_0} h\|_{\infty,\nu' - 1} + \|h\|_{\infty,\nu' - 2} \leq C\|f\|_{p,\nu}. \]
(8.51)
Proof. Let us set
\[ h = r^{-\nu}\psi, \tilde{f} = r^\nu f, \]
so that Problem (8.20) reads
\begin{equation}
\mathcal{J}_0(\psi) + r^\nu \psi\Delta_{\Gamma_0} r^{-\nu} + 2r^\nu \nabla_{\Gamma_0} \psi \cdot \nabla_{\Gamma_0} r^{-\nu} = \tilde{f} \quad \text{in} \quad \tilde{T} \cap \{ r > R_0 + 1 \},
\end{equation}
\begin{equation}
\psi = 0 \quad \text{on} \quad \partial(\tilde{T} \cap \{ r > R_0 + 1 \}).
\end{equation}

We will denote:
\[ \tilde{J}(\psi) = \mathcal{J}_0(\psi) + r^\nu \psi\Delta_{\Gamma_0} r^{-\nu} + 2r^\nu \nabla_{\Gamma_0} \psi \cdot \nabla_{\Gamma_0} r^{-\nu}. \]

Let us consider now the following problem for \( \tilde{f} \in L^\infty \).
\begin{equation}
\lambda^2 \tilde{J}(\psi) - M\psi = \tilde{f} \quad \text{in} \quad \tilde{T} \cap \{ r > R_0 + 1 \},
\end{equation}
\begin{equation}
\psi = 0 \quad \text{on} \quad \partial(\tilde{T} \cap \{ r > R_0 + 1 \}),
\end{equation}
where \( \lambda > 0 \), and \( M > 0 \) is such that
\[ \sup_{\Gamma_0} |A_{\Gamma_0}|^2 < \frac{M}{2}. \]

We easily check that there is a \( \lambda_0 > 0 \) such that whenever \( \lambda < \lambda_0 \)
\[ \lambda^2 \tilde{J}(1) - M < -\frac{M}{2} \quad \text{in} \quad \tilde{T} \cap \{ r > R_0 + 1 \}, \]
and hence problem (8.53) has a unique, bounded solution. Let us scale out \( \lambda \) by setting \( \psi_\lambda(y) = \psi(\lambda y) \), where \( y \in \Gamma_0,\lambda \). Then equation (8.53) takes the form,
\begin{equation}
\tilde{J}_\lambda(\psi_\lambda) - M\psi_\lambda = \tilde{f}_\lambda \quad \text{in} \quad \tilde{T}_\lambda \cap \{ r > \lambda^{-1}(R_0 + 1) \},
\end{equation}
\begin{equation}
\psi = 0 \quad \text{on} \quad \partial(\tilde{T}_\lambda \cap \{ r > \lambda^{-1}(R_0 + 1) \}),
\end{equation}
where
\[ \tilde{J}_\lambda(\psi_\lambda) = \Delta_{\Gamma_0,\lambda} \psi_\lambda + |A_{\Gamma_0,\lambda}|^2 \psi_\lambda + r^\nu \psi\Delta_{\Gamma_0,\lambda} r^{-\nu} + 2r^\nu \nabla_{\Gamma_0,\lambda} \psi \cdot \nabla_{\Gamma_0,\lambda} r^{-\nu}. \]

We claim that there exists a number \( C > 0 \) such that for all sufficiently small \( \lambda \) the following holds: any bounded solution \( \psi_\lambda \) of problem (8.54) satisfies the a priori estimate
\begin{equation}
\lambda^{-8/p}\|\nabla_{\Gamma_0,\lambda} \psi_\lambda\|_{p,0} + \lambda\|\nabla_{\Gamma_0,\lambda} \psi_\lambda\|_{\infty,0} + \lambda^2\|\psi_\lambda\|_{\infty,0} \leq C\lambda^{2-8/p}\|\tilde{f}_\lambda\|_{p,0}.
\end{equation}

We will prove the existence of \( C \) for which
\begin{equation}
\|\psi_\lambda\|_{\infty,0} \leq C\lambda^{-8/p}\|\tilde{f}_\lambda\|_{p,0}.
\end{equation}

Assuming the opposite, we have sequences \( \lambda = \lambda_n, \tilde{f}_n, \psi_n \) for which problem (8.54) is satisfied and
\[ \|\psi_n\|_{\infty,0} = 1, \quad \lambda_n^{-8/p}\|\tilde{f}_n\|_{p,0} \to 0, \quad \lambda_n \to 0. \]

Let us assume that \( y_n \in \Gamma_0,\lambda_n \cap \{ r > \lambda_n^{-1}(R_0 + 1) \} \) is such that
\[ |\psi_n(y_n)| \to 1. \]

Let us consider the local system of coordinates around \( y_n \) given by the graph of the function \( G_{0,n}(t) \), i.e
\[ \tilde{\Gamma}_n \cap B(y_n, |y_n|) = \{ (t, G_{0,n}(t) \} \quad |t| < \theta_0|y_n| \}
\]
and define
\[ \tilde{\psi}_n(t) = \psi_n(y_n + (t, G_{0,n}(t)). \]
Let us observe that the components of the metric tensor associated with these local coordinates satisfy:

\[ g_{\lambda n} = I + \frac{1}{r^2(\lambda_n y_n)} O(\lambda_n^2 |t|^2), \]

hence, locally over compacts:

\[ g_{ij}^{\lambda n} \to \delta_{ij}. \]

uniformly. Let us observe further that:

\[ |A_{\Gamma_0,\lambda_n}|^2 \leq M^2, \]

because of the definition of \( M \). Thus we can assume that, as \( \lambda_n \to 0 \) we have uniformly over compact sets:

\[ \lim_{n \to \infty} |A_{\Gamma_0,\lambda_n}|^2 \to \tilde{a}^*(t), \]

where

\[ |\tilde{a}^*(t)| \leq \frac{M}{2}. \]

Furthermore, expressing the other coefficients in the definition of \( \tilde{J}_\lambda \) in local coordinates we get that:

\[ |r^{\nu} \Delta_{\Gamma_0,\lambda_n} r^{-\nu}| + |r^{\nu} \nabla_{\Gamma_0,\lambda_n} r^{-\nu}| \leq \frac{C\lambda_n}{\tau(\lambda_n y_n)}, \]

uniformly over compact sets.

Standard elliptic estimates give local uniform \( C^1 \) bound for \( \tilde{\psi}_n \), which implies that we may assume

\[ \tilde{\psi}_n \to \tilde{\psi} \neq 0, \]

locally in \( C^1 \)-sense over compacts. We get in the limit the equation

\[ \Delta \tilde{\psi} + \tilde{a}^*(t) \tilde{\psi} - M \tilde{\psi} = 0, \]  
\[ \text{in } \mathbb{R}^8. \]  
\[ (8.57) \]

Since by our assumption \( \tilde{\psi} \) is bounded, maximum principle yields \( \tilde{\psi} = 0 \). We have reached a contradiction, hence estimate \( (8.56) \) holds true. The estimates for first and second derivatives follow from local elliptic \( L^p \)-theory, and the proof of the a priori estimate \( (8.55) \) is concluded.

Now, given \( \tilde{f} \) with \( \|\tilde{f}\|_{p,\nu} < +\infty \), existence of a solution to problem \( (8.53) \) which satisfies estimate \( (8.55) \) follows by approximating \( \tilde{f} \) by a sequence of bounded functions whose \( \| \cdot \|_{p,0} \)-norm is controlled by that of \( \tilde{f} \). The a priori estimate itself yields uniqueness of such a solution. Let us translate the result obtained in terms of \( \psi = r^{\nu} h \) in original variable. We have found that, fixed \( \lambda > 0 \) sufficiently small, there is a \( C = C_\lambda > 0 \) such that given \( f \) with \( \|f\|_{p,\nu} < +\infty \), there is a unique solution \( h := \tau(f) \) with \( \|h\|_{\infty,\nu} < +\infty \) to the problem

\[ \lambda^2 J_0(h) - Mh = f, \quad \text{in } T \cap \{ r > R_0 + 1 \}, \]
\[ h = 0, \quad \text{on } \partial(T \cap \{ r > R_0 + 1 \}), \]  
\[ (8.58) \]

that satisfies the estimate

\[ \|\nabla_{\Gamma_0}^2 h\|_{p,\nu} + \|\nabla_{\Gamma_0} h\|_{\infty,\nu} + \|h\|_{\infty,\nu} \leq C_\lambda \|f\|_{p,\nu}. \]  
\[ (8.59) \]

Let us consider now our original problem \( (8.20) \), and let us decompose

\[ h = \lambda^2 \tau(f) + \tilde{h}. \]
Then:

\[ \mathcal{J}_0(h) = \lambda^2 \mathcal{J}_0(\tau(f)) + \mathcal{J}_0(\tilde{h}) = \mathcal{J}_0(\tilde{h}) + M\tau(f) + f. \]

The equation in terms of \( \tilde{h} \) reads

\[ \mathcal{J}(\tilde{h}) = -M\tau(f) \quad \text{in } \bar{T} \cap \{ r > R_0 + 1 \}, \]
\[ h = 0 \quad \text{on } \partial(\bar{T} \cap \{ r > R_0 + 1 \}). \] (8.60)

Since \( \|\tau(g)\|_{\infty, \nu} < +\infty \), Lemma 8.2 yield the existence of a unique solution \( \tilde{h} \) to problem (8.60) such that

\[ \|\nabla^2 \Gamma h\|_{p, \nu'} + \|\nabla \Gamma h\|_{\infty, \nu' - 1} + \|\tilde{h}\|_{\infty, \nu' - 2} \leq CM\|\tau(f)\|_{\infty, \nu}. \]

This and the a priori estimate (8.59) concludes the proof of the proposition. □

Proposition 8.3. Let us consider problem (8.41) but now assuming that with some \( \nu \geq 3 + \mu, \mu \in \left( \frac{2}{3}, 1 \right) \) we have

\[ \|f\|_{p, \nu} < \infty, \quad p > 9. \] (8.61)

There exists a solution of problem (8.41) such that

\[ \|h\|_{\infty, \nu' - 2} + \|\nabla h\|_{\infty, \nu' - 1} + \|\nabla^2 h\|_{p, \nu'} \leq C\|f\|_{p, \nu}, \]

where

\[ \nu' = 3 + \mu', \quad 0 < \mu' < 3\mu - 2. \] (8.62)

Proof. In order to establish (8.62) we first solve the outer problem (8.20) using the regularization procedure described in the Lemma 8.5. Second, we have to deal with the problem:

\[ \mathcal{J}(h) = f, \quad \text{in } \bar{T}, \]
\[ h = 0, \quad \text{on } \partial\bar{T}. \] (8.63)

where now \( f \) satisfies:

\[ \|f\|_{p, \nu + 1 + \sigma} < \infty, \]

with some \( \sigma > 0 \). Notice that this is basically the same problem as (8.41) except that the right hand side is only an \( L_{p, \nu' + 1 + \sigma} \) function. At this point we need to use the regularization procedure similar to the one described in Lemma 8.5. Namely we solve:

\[ \lambda^2 \mathcal{J}(h) - Mh = f, \quad \text{in } \bar{T}, \]
\[ h = 0, \quad \text{on } \partial\bar{T}. \] (8.64)

The existence of a solution of this problem, such that:

\[ \|\nabla^2 h\|_{p, \nu + 1 + \sigma} + \|\nabla h\|_{\infty, \nu + 1 + \sigma} + \|h\|_{\infty, \nu + 1 + \sigma} \leq C_{\lambda}\|f\|_{p, \nu + 1 + \sigma} \]

can be shown using essentially the same argument as the one in the proof of Lemma 8.5. We omit the details. After this step we conclude the proof of the proposition. □
We will now summarize our results stating them in the form more suitable for the reduced problem on $\Gamma_\alpha$. Let us recall (see (7.19)) that the basic problem we need to solve is:

\begin{align}
J_\alpha &= f_\alpha, \quad \text{in } \tilde{T}, \\
h_\alpha &= 0, \quad \text{on } \partial \tilde{T},
\end{align}

where

\begin{equation}
J_\alpha(h_\alpha) = \Delta_{\Gamma_\alpha} h_\alpha + |A_{\Gamma_\alpha}|^2 h_\alpha.
\end{equation}

As for the right hand side of (8.66) we assume that one of the following holds:

1. Either $\nu = 3$ and $\|f_\alpha\|_{\infty} < C_\alpha^{3}$,

   with some $\sigma_1 \in (1/3, 2/3)$;

2. or $\nu = 3$, and

   $\|f_\alpha\|_{p,\nu+1} < C_\alpha^{3-8/p}$,

which is consistent with (7.21).

First we state a counterpart of Proposition:

**Proposition 8.4.** Let us assume that (8.68) holds. There exists a solution of problem (8.66) such that

\begin{equation}
\alpha^2 \|h_\alpha\|_{1,\infty} + \alpha \|\nabla_{\Gamma_\alpha} h_\alpha\|_{2,\infty} + \alpha^{-8/p} \|\nabla^2_{\Gamma_\alpha} h_\alpha\|_{3,p} \leq C \|f_\alpha\|_{\infty,3}.
\end{equation}

Next we state a suitable modification of Proposition 8.3.

**Proposition 8.5.** Let us consider problem (8.66) but now assuming that (8.69) holds. There exists a solution of problem (8.66) such that

\begin{equation}
\|h_\alpha\|_{3,\infty,3} \leq C \alpha^{-8/p} \|f_\alpha\|_{p,4},
\end{equation}

where

\begin{equation}
\|h_\alpha\|_{3,\infty,3} = \alpha^2 \|h_\alpha\|_{1,\infty} + \alpha \|\nabla_{\Gamma_\alpha} h_\alpha\|_{2,\infty} + \alpha^{-8/p} \|\nabla^2_{\Gamma_\alpha} h_\alpha\|_{3,p},
\end{equation}

(c.f. definition of $\|\cdot\|_{3,\infty,\nu}$ in (5.22)).

9. Resolution of the reduced problem

9.1. Improvement of the initial approximation of $h_\alpha$. Let us recall the reduced problem derived in Section 7 (see (7.19), (7.22)):

\begin{align}
\mathcal{J}(h_\alpha) &= c_1 R_{1,\alpha} + \mathcal{F}_\alpha(h_\alpha, \nabla_{\Gamma_\alpha} h_\alpha, \nabla^2_{\Gamma_\alpha} h_\alpha), \quad \text{in } \tilde{T}, \\
h_\alpha &= 0, \quad \text{on } \partial \tilde{T}.
\end{align}

Notice that the boundary conditions imposed above allow to solve (7.22) with the symmetry condition (7.22) simply by extending the solution of (9.1) to the whole space.

Our plan is to solve (9.1) in two steps:

1. We find the leading order term in the expansion of $h_\alpha$.
2. We use a fixed point argument to determine $h_\alpha$. 

In this section we will perform Step 1. To begin with let $\Gamma_{0,\alpha}$ be the surface:

$$\Gamma_{0,\alpha} = \varphi = \frac{1}{\alpha} F_0(\alpha x'),$$

and

$$\Gamma_{0,\alpha,z} = \{ x \in \mathbb{R}^9 \mid \text{dist}(x, \Gamma_{0,\alpha}) = z \}.$$ 

The mean curvature of $\Gamma_{0,\alpha,z}$ can be expanded as follows:

$$H_{\Gamma_{0,\alpha,z}} = H_{\Gamma_{0,\alpha}} + z \left| \tilde{A}_{\Gamma_{0,\alpha}} \right|^2 + \frac{1}{2} z^2 R_{01,\alpha} + O \left( \frac{\alpha^4 |z|^3}{1 + r_\alpha^4} \right).$$

Let us consider $R_{01,\alpha}$. This term is given by

$$\sum_{i=0}^{8} \kappa^3_{0i,\alpha},$$

where principal $\kappa^3_{0i,\alpha}$ are the principal curvatures of $\Gamma_{0,\alpha}$. It follows that it has the same symmetry as the function $F_0$, in other words:

$$R_{01,\alpha}(u,v) = -R_{01,\alpha}(v,u),$$

hence $R_{01,\alpha}(u,u) = 0$ and there exists a $\sigma_1 \in \left( \frac{1}{3}, \frac{2}{3} \right)$ such that

$$|R_{01,\alpha}| \leq \frac{C \alpha^3 g(\theta)^\sigma}{1 + r_\alpha^4}.$$ 

Also we notice that a similar term in the expansion of $H_{\Gamma_{0,\alpha,z}}$, denoted above by $R_{1,\alpha}$ is also equal to the sum of the cubes the principal curvatures of $\Gamma_{\alpha}$. It follows that (see (7.20), (7.23)):

$$|R_{1,\alpha} - R_{01,\alpha}| \leq \frac{C \alpha^{4+\sigma}}{1 + r_\alpha^{4+\sigma}},$$

with some $\sigma > 0$, using (4.22).

Now going back to (9.1) we let:

$$h_{\alpha} = \tilde{h}_{\alpha} + h_{\alpha},$$

where

$$J(\tilde{h}_{\alpha}) = c_1 R_{1,\alpha}, \quad \text{in } \tilde{T},$$

$$\tilde{h}_{\alpha} = 0, \quad \text{on } \partial \tilde{T}.$$ 

**Lemma 9.1.** For each sufficiently small $\alpha$ there exists a solution of (9.6) such that

$$\|\tilde{h}_{\alpha}\|_{*,p,3} \leq C \alpha^3,$$

provided that $p > 9$ is taken sufficiently large.

**Proof.** We will write $\tilde{h}_{\alpha} = \tilde{h}_{1,\alpha} + \tilde{h}_{2,\alpha}$, where

$$J(\tilde{h}_{1,\alpha}) = c_1 R_{01,\alpha}, \quad \text{in } \tilde{T},$$

$$h_{1,\alpha} = 0, \quad \text{on } \tilde{T},$$

and

$$J(\tilde{h}_{2,\alpha}) = c_1 (R_{1,\alpha} - R_{01,\alpha}), \quad \text{in } \tilde{T},$$

$$h_{2,\alpha} = 0, \quad \text{on } \tilde{T}. $$
We notice that problem (9.8) has a solution satisfying (9.7) because of Proposition 8.4 and estimate (9.3). In addition problem (9.9) has a solution satisfying:
\[
\|\tilde{h}_{2,\alpha}\|_{*,p,3} \leq C\alpha^{4+\sigma-8/p} \leq C\alpha^{4}
\]
if \(p\) is large, by Proposition 8.5. This completes the proof of the Lemma. \(\Box\)

9.2. The fixed point argument. With \(\tilde{h}_\alpha\) given by Lemma 9.1 we will use the theory of solvability for the Jacobi operator to define a map \(T_\alpha\) on a subset of a space of function whose \(\|\cdot\|_{*,p,3}\) norm is bounded into itself. Let \(\mu > 0\) be a fixed small number and let \(p > 9\) be large so that
\[
\mu < 1 - \frac{32}{p}.
\]
Let us set:
\[
B_{\alpha^{2+\mu}} = \{f_\alpha | \|f_\alpha\|_{*,p,3} \leq \alpha^{2+\mu}\}.
\]
Given an \(f \in B_{\alpha^{2+\mu}}\) we let \(h_\alpha\) to be a solution of:
\[
J(h_\alpha) = \tilde{F}_\alpha(f_\alpha, \nabla_{\Gamma_\alpha} f_\alpha, \nabla^2_{\Gamma_\alpha} f_\alpha), \quad \text{in } \tilde{T},
\]
\[
h_\alpha = 0, \quad \text{on } \partial \tilde{T},
\]
where:
\[
\tilde{F}_\alpha(f_\alpha, \nabla_{\Gamma_\alpha} f_\alpha, \nabla^2_{\Gamma_\alpha} f_\alpha) = F_\alpha(f_\alpha + \tilde{h}_\alpha, \nabla_{\Gamma_\alpha}(f_\alpha + \tilde{h}_\alpha), \nabla^2_{\Gamma_\alpha}(f_\alpha + \tilde{h}_\alpha)).
\]
Now we define:
\[
T_\alpha(f_\alpha) = h_\alpha.
\]
We observe that by (7.21) we have:
\[
\|\tilde{F}_\alpha\|_{p,4} \leq C\alpha^{1-8/p}\|f_\alpha\|_{*,p,3} + C\alpha^{1-8/p}\|\tilde{h}_\alpha\|_{*,p,3} + C\alpha^{3-8/p}
\]
\[
\leq C\alpha^{3+\mu-8/p} + C\alpha^{4-8/p} + C\alpha^{3-8/p}
\]
\[
\leq \alpha^{2+\mu+16/p},
\]
provided that \(\alpha\) is taken sufficiently small, since \(3 - \frac{32}{p} > 2 + \mu\). Then Proposition 8.5 implies that \(T_\alpha\) is well defined, indeed since by (8.71) we have:
\[
\|h_\alpha\|_{*,p,3} \leq C\alpha^{-8/p}\|\tilde{F}_\alpha\|_{p,4}
\]
\[
\leq C\alpha^{2+\mu+8/p}
\]
\[
< \frac{1}{2}\alpha^{2+\mu},
\]
again by the choice of \(p\), taking \(\alpha\) small enough. We will now prove:

**Lemma 9.2.** Mapping \(T_\alpha\) has a unique fixed point in \(B_{\alpha^{2+\mu}}\).

**Proof.** In view of (9.12) to use Banach fixed point theorem we need to show that \(T_\alpha\) is a contraction map. Let \(f_\alpha^{(j)} \in B_{\alpha^{2+\mu}}\) be fixed and \(h_\alpha^{(j)} = T_\alpha(f_\alpha^{(j)}), j = 1, 2\). We claim that
\[
|||\tilde{F}_\alpha(f_\alpha^{(1)}) - \tilde{F}_\alpha(f_\alpha^{(2)})|||_{p,4} \leq C\alpha^{1-8/p}\|h_\alpha^{(1)} - h_\alpha^{(2)}\|_{*,p,3}.
\]
This amounts to calculations similar as in the Section 7 but taking into account two solutions \(\phi^{(j)}, j = 1, 2\) of the projected nonlinear problem and subtracting the resulting projections. The key estimates are (6.21) and also (6.40)–(6.41). The somewhat tedious details are omitted here. From (9.13) we conclude that \(T_\alpha\) is
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Lipschitz with a constant proportional to $\alpha^{1-8/p}$. Taking $\alpha$ smaller if necessary we show that $T_\alpha$ is a contraction map.

\[ \square \]

10. CONCLUSION OF THE PROOF OF THE MAIN THEOREM

Let us summarize the results of our considerations so far. Given the solution to the nonlinear projected problem $\phi$ and the corresponding solution $h_\alpha$ to the reduced problem found above we have found a function $u_\alpha$ such that

\[ u_\alpha = w + w_1 + \eta_{\alpha}^2 \phi + \psi(\phi), \]

and

\[ \Delta u_\alpha + u_\alpha (1 - u_\alpha) = 0, \quad \text{in } \mathbb{R}^9. \]

Clearly $u_\alpha$ is a bounded function. Also $u_\alpha$ obeys the symmetry of the minimal graph $\Gamma_\alpha$:

\[ u_\alpha(u, v, x_9) = -u(v, u, -x_9), \quad (10.14) \]

from which it follows in particular

\[ u_\alpha(0) = 0. \]

To finish the result of the theorem, we need to prove that the solution $u_\alpha$ of the Allen-Cahn equation obtained this way is in fact monotone in the $x_9$-direction.

Observe that the function $\psi_\alpha := \partial_{x_9} u_\alpha$ is a solution of the linear problem

\[ \Delta \psi_\alpha + f'(u_\alpha) \psi_\alpha = 0. \]

We claim that the construction yields that inside any bounded neighborhood of $\Gamma_\alpha$ of the form $N_M = \{ \text{dist } (x, \Gamma_\alpha) < M \}$ we have $\psi_\alpha > 0$. Indeed,

\[ \partial_{x_9} u_\alpha = \partial_{x_9} w(z - h_\alpha) + O\left( \frac{\alpha^2}{1 + r_\alpha^2} \right) \]

\[ = w'(z - h_\alpha) \partial_{x_9} z + O\left( \frac{\alpha^2}{1 + r_\alpha^2} \right), \]

where $z$ is the Fermi coordinate of $\Gamma_\alpha$. We see that if $|z|$ is bounded then

\[ \partial_{x_9} z \sim \frac{1}{\sqrt{1 + |\nabla \Gamma_\alpha|^2}} = O\left( \frac{1}{1 + r_\alpha^2} \right), \]

by (4.31). This shows our claim. Taking $M$ sufficiently large (but independent on $\alpha$) we can achieve $f'(u_\alpha) > -3/2$ outside of $N_M$. We claim that we cannot have that $\psi_\alpha < 0$ in $N_M$. Indeed, a non-positive local minimum of $\psi_\alpha$ is discarded by maximum principle. If there were a sequence of points $x_n \in \mathbb{R}^9$, such that

\[ \psi_\alpha(x_n) < 0, \]

$|x_n| \to \infty$, and at the same time $\text{dist } (x_n, \Gamma_\alpha) > M$, for some large $M$, the usual compactness argument would give us a nontrivial bounded solution of

\[ \Delta \psi - c \psi = 0 \quad \text{in } \mathbb{R}^9, \quad c(x) > 1, \]

hence a contradiction. We conclude that $\psi_\alpha > 0$ in entire $\mathbb{R}^9$ and the proof of the theorem is concluded.

\[ \square \]
11. Appendix A

In this appendix we will provide the details of the computations needed in the proof of Lemma 3.6. We will collect first some terms appearing in the expansion formula \(3.31\). We have \(\varphi(t, s) = tr^{-\sigma}\) and

\[
\begin{align*}
\partial_t \varphi &= \frac{1}{r^{\sigma}}(1 - \frac{\sigma}{3} \cos^2 \phi), \quad \partial_s \varphi = -\frac{\sigma t \sin^2 \phi}{7 r^{\sigma} s} \\
\partial_t^2 \varphi &= \frac{C\sigma}{9r^{\sigma}t} \cos^2 \phi [\sigma \cos^2 \phi - 3 + 2\phi' \sin^2 \phi] \\
\partial_{ts} \varphi &= -\frac{\sigma \sin^2 \phi}{7r^{\sigma}s} \left(1 + \frac{2\phi' \cos^2 \phi}{3} - \frac{\sigma \cos^2 \phi}{3}\right)
\end{align*}
\]

We also have

\[
\begin{align*}
\rho^{-2} \varphi_s &= -\frac{7s \sigma \cos^2 \phi}{|\nabla F_0|^2} \\
\partial_s \left(\frac{\rho^{-2} \varphi_s}{|\nabla F_0|^2}\right) &= -\frac{7\sigma \cos^2 \phi}{9tr^{\sigma}} \left[1 + \frac{\sin^2 \phi}{7}(2\phi' - \sigma)\right] \\
\rho^{-2} \varphi_s^2 &= \frac{\sigma^2 \sin^2 \phi \cos^2 \phi}{9t^{2\sigma}} \\
\partial_s \left(\frac{\rho^{-2} \varphi_s^2}{|\nabla F_0|^2}\right) &= -\frac{2\sigma^2 \sin^2 \phi \cos^2 \phi}{63t^{2\sigma} s} (\sigma \sin^2 \phi + \phi' \cos (2\phi)) \\
\partial_t \left(\frac{\rho^{-2} \varphi_s^2}{|\nabla F_0|^2}\right) &= \frac{2\sigma^2 \sin^2 \phi \cos^2 \phi}{27t^{2\sigma}} [-\sigma \cos^2 \phi + \phi' \cos (2\phi)]
\end{align*}
\]

Using formula \(3.31\) we get

\[
\begin{align*}
H_2 &= -\frac{1}{2} \partial_t \left(\frac{\rho^{-2} \varphi_s^2}{|\nabla F_0|^2}\right) + 2\partial_t \varphi \partial_s \left(\frac{\rho^{-2} \varphi_s}{|\nabla F_0|^2}\right) - \left(\frac{\rho^{-2} \varphi_s}{|\nabla F_0|^2}\right) \partial_{ts} \varphi \\
H_3 &= \left(\frac{\rho^{-2} \varphi_s^2}{|\nabla F_0|^2}\right) \partial_t^2 \varphi - \frac{1}{2} \partial_t \varphi \partial_s \left(\frac{\rho^{-2} \varphi_s^2}{|\nabla F_0|^2}\right) + \left(\partial_t \varphi\right)^2 + \left(\frac{\rho^{-2} \varphi_s^2}{|\nabla F_0|^2}\right) \partial_s \left(\frac{\rho^{-2} \varphi_s}{|\nabla F_0|^2}\right) \\
&- \partial_t \varphi \partial_{ts} \varphi \left(\frac{\rho^{-2} \varphi_s^2}{|\nabla F_0|^2}\right) - \frac{1}{2} \left(\frac{\rho^{-2} \varphi_s}{|\nabla F_0|^2}\right) \partial_s \left(\frac{\rho^{-2} \varphi_s^2}{|\nabla F_0|^2}\right)
\end{align*}
\]

From \((11.1)\)–\((11.3)\) we get by direct calculation

\[
\begin{align*}
H_2 &= \frac{\sigma^2 \sin^2 \phi \cos^2 \phi}{27t^{2\sigma}} [\sigma \cos^2 \phi - \cos (2\phi) \phi'] \\
&- \frac{2\sigma^2 \cos^2 \phi}{27t^{2\sigma}} (3 - \sigma \cos^2 \phi)[7 + (2\phi' - \sigma) \sin^2 \phi] \\
&+ \frac{\sigma \cos^2 \phi \sin^2 \phi}{27t^{2\sigma}} (3 - \sigma \cos^2 \phi + 2 \cos^2 \phi' \phi') \\
&= \frac{\sigma \cos^2 \phi}{27t^{2\sigma}} [-6(7 + 2\phi' \sin^2 \phi) + (3 + 2 \cos^2 \phi' \sin^2 \phi + O(\sigma)] \\
&= \frac{\sigma \cos^2 \phi}{27t^{2\sigma}} [-42 + \sin^2 \phi(-12\phi' + 3 + 2\phi' \cos^2 \phi) + O(\sigma)] < 0.
\end{align*}
\]
and

\[ H_3 = \frac{\sigma^2 \sin^2 \phi \cos^2 \phi}{81 \sigma^{3\sigma}} \left[ \sigma \cos^2 \phi - 3 \cos 2\phi \phi' \right. \\
- \frac{\sigma \cos^2 \phi}{81 \sigma^{3\sigma}} (9 - 6\sigma \cos^2 \phi + \sigma^2 \cos^2 \phi)(7 + (2\phi' - \sigma) \sin^2 \phi) \\
+ \frac{\sigma^3 \sin^2 \phi \cos^4 \phi}{81 \sigma^{3\sigma}} \left( \sigma \sin^2 \phi + \cos(2\phi)\phi' \right) \\
+ \frac{\sigma \sin^2 \phi \cos^2 \phi}{81 \sigma^{3\sigma}} (3 - \sigma \cos^2 \phi)(3 - \sigma \cos^2 \phi + 2\cos^2 \phi \phi') \\
\left. \right] \\
= \frac{\sigma \cos^2 \phi}{27 \sigma^{3\sigma}} \left[ \sin^2 \phi(3 + 2\cos^2 \phi \phi' - 3(7 + (2\phi' - \sigma) \sin^2 \phi) \\
- \sigma \sin^2 \cos(2\phi)\phi' + O(\sigma^2 \cos^2 \phi) \right] \\
= \frac{\sigma \cos^2 \phi}{27 \sigma^{3\sigma}} \left[ -21 \cos^2 \phi - (6 - \sigma) \sin^2 \phi(\phi' + 3) \\
+ (2 - 2\sigma) \cos^2 \phi \sin^2 \phi \phi' + O(\sigma^2 \cos^2 \phi) \right] \leq 0, \]

when \( \sigma > 0 \) is sufficiently small. From this we get \((3.47)\). The proof of \((3.48)\) is similar.

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