A Note on the Intermediate Region in Turbulent Boundary Layers

G. I. Barenblatt, 1 A. J. Chorin 1 and V. M. Prostokishin 2

1Department of Mathematics and
Lawrence Berkeley National Laboratory
University of California
Berkeley, California 94720

2P. P. Shirshov Institute of Oceanology
Russian Academy of Sciences
36, Nakhimov Prospect
Moscow 117218 Russia

Abstract.

We demonstrate that the processing of the experimental data for the average velocity profiles obtained by J. M. Österlund (www.mesh.kth.se/~jens/zpg/) presented in [1] was incorrect. Properly processed these data lead to the opposite conclusion: they confirm the Reynolds-number-dependent scaling law and disprove the conclusion that the flow in the intermediate (‘overlap’) region is Reynolds-number-independent.
In a recent issue of the Physics of Fluids Österlund et al. [1] presented results of the processing of experimental data for the average velocity field in the zero-pressure-gradient turbulent boundary layer, published by the first author of [1] on the Internet (http://www.mesh.kth.x/∼jens/zpg/). The conclusion of [1] is very definite (p. 1): ‘Contrary to the conclusions of some earlier publications, careful analysis of the data reveals no significant Reynolds number dependence for the parameters describing the overlap region using the classical logarithmic relation’.

In the present note we show that the processing of the experimental data in [1] is incorrect. A correct processing is performed, and the result is the opposite: there is significant Reynolds number dependence for the parameters describing the intermediate region and the scaling (power) law is valid. We also show where the authors of [1] went wrong.

The correct processing of the Österlund data. If a scaling (power) law relates the dimensionless velocity $U^+$ and dimensionless distance from the wall $y^+$ (in the notations of [1]):

(I) \[ U^+ = A(y^+)^\alpha \]  

then in the coordinates $\lg y^+$, $\lg U^+$ the experimental points should lie within experimental accuracy along a straight line: $\lg U^+ = \lg A + \alpha \lg y^+$. Therefore our first step was to plot the data from all 70 runs available on the Internet in these coordinates. All 70 runs corresponding to different $Re_\theta$ yield the same pattern: in the intermediate region between the viscous sublayer and free stream the average velocity distribution consists of two straight lines. Three examples are presented in Figure 1 and the Table, all the remaining ones are similar and can be found in our detailed report [2]. Thus the intermediate structure (between the viscous sublayer and free stream) consists of two self-similar layers: the scaling law (I) is valid for the layer adjacent to the viscous sublayer, and the scaling law

(II) \[ \bar{U}^+ = B(y^+)^\beta \]  

holds in the layer adjacent to the free stream. This means, in particular, that some scaling law with Reynolds-number-dependent coefficients is valid in the region (I). The coefficients $A$ and $\alpha$, constant for every run, depend on the Reynolds number which is different for different runs, and their variation is substantial (see the Table).

Once this is established, we investigate whether this scaling law can be represented in the form

\[ \bar{U}^+ = \left( \frac{1}{\sqrt{3}} \ln Re + \frac{5}{2} \right) (y^+)^{3 \ln Re} \]  

obtained by us earlier for flow in pipes. In the case of pipe flows $Re$ was $\bar{u}d/\nu$, where $\bar{u}$ is
the mean velocity (bulk flux divided by cross-section area), and \( d \) is the diameter. But what is \( Re \) for the boundary layer?

The effective Reynolds number \( Re \) should have the form \( Re = U \Lambda / \nu \), where \( U \) is the free stream velocity, \( \nu \) the kinematic viscosity, and \( \Lambda \) a length scale which we cannot \( \text{à priori} \) identify with the momentum thickness \( \theta \), as there is no rationale for such identification. So, the basic question is whether one can find for each run a length scale \( \Lambda \) so that the scaling law (3) will be valid for the mean velocity distribution in the first intermediate region. If this scale exists then the law (3) is not specific to flows in pipes but may be a general law for wall-bounded shear flows at large Reynolds numbers.

To answer this question we took the values \( A \) and \( \alpha \) for each run, obtained by standard statistical processing of the experimental data in the first intermediate scaling region, and then calculated two values \( \ln Re_1 \) and \( \ln Re_2 \) by solving two equations suggested by the law (3),

\[
\frac{1}{\sqrt{3}} \ln Re_1 + \frac{5}{2} = A, \quad \frac{3}{2 \ln Re_2} = \alpha.
\]

If the values \( \ln Re_1 \) and \( \ln Re_2 \) obtained by solving these two different equations (4) coincide within experimental accuracy, then the unique length scale \( \Lambda \) can be determined so that the experimental scaling law in the region (1) coincides with the law (3). Indeed these values are close — for all \( Re_\theta > 10,000 \), the difference \( \Delta = (\ln Re_2 - \ln Re_1) / \ln Re \) does not exceed 3%, see the table for the examples in Figure 1 and in \([2]\) for all runs. This allows one to introduce, for large Reynolds numbers, an effective Reynolds number \( Re \), for example by the relation

\[
\ln Re_1 = \frac{1}{2} (\ln Re_1 + \ln Re_2), \ \text{or} \ Re = \sqrt{Re_1 Re_2},
\]

the geometric mean of \( Re_1 \) and \( Re_2 \). This Reynolds number defines the effective length scale \( \Lambda \), which plays for boundary layer flow the same role as the pipe diameter for flow in pipes. Remember that the momentum thickness is calculated by integration of the velocity profile obtained experimentally: the calculation of the length scale on the basis of the measured velocity profile is not more complicated. Furthermore, the scaling law (3) can be reduced to a universal form

\[
\psi = \frac{1}{\alpha} \ln \left( \frac{2\alpha U^+}{\sqrt{3} + 5\alpha} \right) = \ln y^+
\]

where \( \alpha = \frac{3}{2 \ln Re} \). This formula gives another way to check the applicability of the Reynolds-number-dependent scaling law (3) in the intermediate region (1). Indeed, according to (6), in the coordinates \( \ln y^+, \psi \), all experimental points should collapse onto the bisectrix of the first quadrant. Figure 2 shows that all data for large Reynolds numbers \( (Re_\theta > 15,000, 24 \ \text{runs}) \) presented on the Internet collapse onto the bisectrix with accuracy sufficient to
give an additional confirmation to the Reynolds-number-dependent scaling law (3). For lesser values of $Re_\theta$ a small but systematic parallel shift is observed (see [2]). One possible reason is that in these cases the choice of $Re$ according to (5) may be insufficient because in particular at small Reynolds numbers the higher terms of the expansion of the coefficients of the scaling law over $1/\ln Re$ could have some influence (see the paper by Radhakrishnan Srinivasan [3]); another possibility is that the relation that was to be proved was used as fact in the calculation of the skin friction in equation (6) of [1].

**Why is the data processing in [1] incorrect?** The main argument against the power law used by the authors of [1] is the following. They introduce the “diagnostic function” (p.3, right)

$$\Gamma = \frac{y^+}{U^+} \frac{dU^+}{dy^+}. \quad (7)$$

Their statement, “The function $\Gamma$ should be a constant in a region governed by a power law” is correct for a fixed Reynolds number. However, this is not true for the ‘diagnostic function averaged for KTH data’, which is shown in their Figure 6.

We invite the reader to look at Figure 1 (the situation with all other runs is the same, see our report [2]). It is clear that for each run $\Gamma$, which is equal to $d(\ln U^+)/d\ln(y^+)$, is a constant—look at the straight lines in the first intermediate region! However, (see the Table) this constant is different for different runs because the slope of the straight lines is $Re$-dependent! Indeed, the slope in the first region decays with growing Reynolds number. It is clear why $\Gamma$ obtained by the authors based on averages for KTH data, is decreasing: the runs with larger Reynolds number and smaller slopes contribute more at larger $y^+$.

Due to this incorrect procedure the authors were unable to recognize the Reynolds-number-dependent power law in their rather rich database.

Their determination of the constants of the logarithmic law gives an additional illustration of the incorrectness of their procedure. They do it in two ways; it is enough to mention what they call the traditional procedure (p. 3, right). They take the data representation in the traditional $\ln y^+, U^+$ plane, and calculate the constant $\kappa$ in the logarithmic law

$$\bar{U}^+ = \frac{1}{\kappa} \ln(y^+) + B. \quad (8)$$

‘by fitting a log-law relation for each profile using the following traditional limits of the fit: $M_1 = 50$, and $M_0 = 0.15$’. The result of their fit is shown in Figure 3 (their Figure 5) which is far from convincing.

Apparently embarrassed by this result, the authors extended the upper boundary of the viscous sublayer to $M_1 = 200$ (the traditional value is 30-70), and obtained, again, with huge
scatter, the value $\kappa = 0.38$. For the constant $B$ they obtained the value 4.1. Both values are the lowest contenders among the values available in the literature (Nikuradze, $\kappa = 0.417$, $B=5.89$; Monin and Yaglom, $\kappa = 0.40$, $B=5.1$; Schlichting, $\kappa = 0.40$, $B=5.5$). However, if the law (8) is universal, the values of the constants should be identical for all high quality experiments!

Finally, according to the authors of [1], the logarithmic layer extends only over 1/6 of the boundary layer thickness. In fact, a better representation is provided by the power law in the first region. The upper boundary of this first region is always higher than the upper boundary of the logarithmic region presented in [1].

In conclusion, careful analysis of Österlund’s data, contrary to the claim in [1], reveals a significant Reynolds number dependence for the parameters describing the ‘overlap’ region and confirms the Reynolds-number-dependent scaling law.

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Table

| $Re_\theta$ | $\alpha$ | $A$ | $\beta$ | B | ln($Re_1$) | ln($Re_2$) | ln($Re$) | $\Delta,\%$ |
|-------------|----------|-----|---------|---|-------------|-------------|---------|-------------|
| 2,532       | 0.157    | 7.84| 0.226   | 5.32| 9.24        | 9.57        | 9.4     | 3.4         |
| 14,207      | 0.132    | 9.01| 0.191   | 5.87| 11.28       | 11.39       | 11.33   | 1.0         |
| 26,612      | 0.120    | 9.74| 0.177   | 6.24| 12.54       | 12.48       | 12.51   | 0.5         |
Figure 1.

\[ \text{Re}_\theta = 2,532 \]

\[ \text{Re}_\theta = 14,207 \]

\[ \text{Re}_\theta = 26,612 \]
Figure 2
Figure 3. The von Kármán constant determined by a least-squares fit, with the outer limit fixed at $\eta = 0.15$ and the inner limit at $M_i$; ○: KTH, $M_i = 50$. ●: KTH, $M_i = 200$. Dashed line: KTH, linear fit, $M_i = 50$. Solid line: KTH, linear fit, $M_i = 200$. □: IIT, $M_i = 50$. ■: IIT, $M_i = 200$. Dotted line: IIT, linear fit, $M_i = 50$. Dash dotted line: IIT, linear fit, $M_i = 200$. 

Figure 3.