MODULI OF TANGO STRUCTURES AND DORMANT MIURA OPERS

YASUHIRO WAKABAYASHI

Abstract. The purpose of the present paper is to develop the theory of (pre-)Tango structures and (dormant generic) Miura $\mathfrak{g}$-opers (for a semisimple Lie algebra $\mathfrak{g}$) defined on pointed stable curves in positive characteristic. A (pre-)Tango structure is a certain line bundle of an algebraic curve in positive characteristic, which gives some pathological (relative to zero characteristic) phenomena. In the present paper, we construct the moduli spaces of (pre-)Tango structures and (dormant generic) Miura $\mathfrak{g}$-opers respectively and prove certain properties of them. One of the main results of the present paper states that there exists a bijective correspondence between the (pre-)Tango structures (of prescribed monodromies) and the dormant generic Miura $\mathfrak{sl}_2$-opers (of prescribed exponents). By using this correspondence, we achieve a detailed understanding of the moduli stack of (pre-)Tango structures. As an application, we construct a family of algebraic surfaces in positive characteristic parametrized by a higher dimensional base space whose fibers are pairwise non-isomorphic and violate the Kodaira vanishing theorem.

Contents

Introduction 2
1. Preliminaries 6
2. Cartan connections on a pointed stable curve 14
3. Miura $\mathfrak{g}$-opers on pointed stable curves 25
4. Miura $\text{GL}_n$ opers on pointed stable curves 38
5. Pre-Tango structures on a log-curve 46
6. Deformations of (dormant) Miura opers 55
7. Pathology in positive characteristic 65
References 72
INTRODUCTION

The purpose of the present paper is to develop the moduli theory of (pre-)Tango structures and Miura $\mathfrak{g}$-opers (for a semisimple Lie algebra $\mathfrak{g}$) defined on (families of) pointed stable curves in positive characteristic. One of the main results of the present paper states (cf. Theorem 5.4.1) that there exists a bijective correspondence between the (pre-)Tango structures (of prescribed monodromies) and the dormant generic Miura $\mathfrak{sl}_2$-opers (of prescribed exponents). By means of this correspondence, we achieve a detailed understanding of the moduli stack of (pre-)Tango structures (cf. Theorem 6.3.2). As an application, we construct a family of algebraic surfaces in positive characteristic parametrized by a higher dimensional base space whose fibers are pairwise non-isomorphic and violate the Kodaira vanishing theorem (cf. Corollary 7.2.3).

In the rest of this Introduction, we shall provide more detailed discussions, including the content of the present paper.

0.1. First, recall the notion of a Tango structure on an algebraic curve, which is one of the central objects in the present paper. Let $p$ be an odd prime, $k$ an algebraically closed field of characteristic $p$, and $X$ a proper smooth curve over $k$ of genus $g > 1$. Denote by $F_X : X \to X$ the absolute (i.e., $p$-th power) Frobenius endomorphism of $X$. In [32], H. Tango studied the injectivity of the map

$$F_X^* : H^1(X, \mathcal{V}) \to H^1(X, F_X^*(\mathcal{V}))$$

induced by $F_X$ between the first cohomology groups of a vector bundle $\mathcal{V}$ and its pull-back $F_X^*(\mathcal{V})$. In the case where $\mathcal{V} = \mathcal{O}_X(-D)$ with some effective divisor $D$, he described the kernel of the map $F_X^*$ in terms of exact differentials, and characterized the injectivity by means of a certain numerical invariant which is now called the Tango-invariant. The Tango-invariant is defined as

$$n(X) := \max \left\{ \deg \left[ \frac{(df)}{p} \right] \in \mathbb{Z} \bigg| f \in K(X) \setminus K(X)^p \right\}.$$

Here, $K(X)$ denotes the function field of $X$, $K(X)^p$ denotes the subfield of $K(X)$ consisting of $p$-th powers (i.e., $K(X)^p := \{f^p \mid f \in K(X)\}$), $(df)$ denotes the divisor $\sum_{x \in X} v_x(df)x$ (where $v_x$ denotes the valuation associated with $x$), and $\lfloor \cdot \rfloor$ denotes round down of coefficients. One verifies that the inequalities $0 \leq n(X) \leq \frac{2g-2}{p}$ hold (cf. [32], Lemma 12). Moreover, the inequality $n(X) > 0$ (resp., the equality $n(X) = \frac{2g-2}{p}$) implies that there exists an ample divisor $D$ on $X$ with $(df) \geq pD$ (resp., $(df) = pD$) for some $f \in K(X) \setminus K(X)^p$. We refer to the line bundle $\mathcal{L} := \mathcal{O}_X(D)$ for such a divisor $D$ as a pre-Tango structure (resp., a Tango structure) on $X$. (This definition of a Tango structure coincide with the definition described in Definition 5.1.1.) Then, according to [32], Theorem 15, for each pre-Tango structure $\mathcal{L}$ on $X$, the
map $F^*: H^1(X, \mathcal{L}^\vee) \to H^1(X, F^*_X/k(\mathcal{L}^\vee))$ (where $\mathcal{L}^\vee$ denotes the dual of $\mathcal{L}$) is not injective. Furthermore, the notion of a Tango structure has an impotence in the study of pathology of algebraic geometry in positive characteristic, as discussed in §7 later. The existence of a Tango structure implies a strong restriction to the genus $g$ of the underlying curve, i.e., $p$ must divide $2g - 2$. At any rate, it will be natural to ask how many curves admitting a (pre-)Tango structure exist, or whether such curves are really exceptional or not. These questions lead us to study the moduli stack

$$\mathcal{T}_{an}$$

classifying proper smooth curves over $k$ of genus $g$ together with a Tango structure. For instance, we want to understand the image of the forgetting morphism $\mathcal{T}_{an} \to \mathcal{M}_g$, where $\mathcal{M}_g$ denotes the moduli stack classifying proper smooth curves over $k$.

0.2. In the present paper, we deal with (pre-)Tango structures defined in a more general setting, i.e., (pre-)Tango structures on families of pointed stable curves. Let $(g, r)$ be a pair of nonnegative integers with $2g - 2 + r > 0$. Write $\mathcal{M}_{g,r}$ for the moduli stack classifying $r$-pointed stable curves over $k$ of genus $g$ and $\mathcal{M}_{g,r}$ for its dense open substack classifying smooth curves. Suppose that we are given an $r$-pointed genus $g$ stable curve $X := (X, \{\sigma_i\}_{i=1}^r)$ classified by a $k$-rational point of $\mathcal{M}_{g,r}$, where $X$ denotes a proper nodal curve and $\{\sigma_i\}_{i=1}^r$ denotes an ordered set consisting of $r$ marked points in $X$. Then, $X$ admits a log structure in a natural manner (cf. §1.1) and we denote by $X^{\log}$ the resulting log scheme. A pre-Tango structure on $X$ is defined (cf. Definition 5.3.1) as a logarithmic connection on the sheaf of logarithmic 1-forms $\Omega_{X^{\log}}^1/k$ with vanishing $p$-curvature whose horizontal sections are contained in the kernel of the Cartier operator. If $r = 0$ (i.e., $\{\sigma_i\}_{i=1}^r = 0$) and the underlying curve is smooth, then this definition of a pre-Tango structure is equivalent to the definition of a Tango structure mentioned in the previous subsection (cf. Proposition 5.3.2). Notice that our definition does not require a type of condition corresponding to the inequality $n(X) > 0$ as required in the classical case. But, we can proceed to our discussion regardless of whether such a condition should be imposed or not.

Denote by $\mathbb{F}_p^\times$ the product of $r$ copies of $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$ and let $\vec{\mu} \in \mathbb{F}_p^\times$, where $\vec{\mu} := 0$ if $r = 0$. We shall write

$$\mathcal{T}_{an_{g,r,\vec{\mu}}}$$

(cf. (191)) for the moduli stack classifying $r$-pointed stable curves over $k$ of genus $g$ together with a pre-Tango structure on it of monodromies $\vec{\mu}$. In particular, we have $\mathcal{T}_{an_g} \cong \mathcal{T}_{an_{g,0,\vec{0}}} \times \mathcal{M}_{g,0}$. 

0.3. In the present paper, we deal with (pre-)Tango structures defined in a more general setting, i.e., (pre-)Tango structures on families of pointed stable curves. Let $(g, r)$ be a pair of nonnegative integers with $2g - 2 + r > 0$. Write $\mathcal{M}_{g,r}$ for the moduli stack classifying $r$-pointed stable curves over $k$ of genus $g$ and $\mathcal{M}_{g,r}$ for its dense open substack classifying smooth curves. Suppose that we are given an $r$-pointed genus $g$ stable curve $X := (X, \{\sigma_i\}_{i=1}^r)$ classified by a $k$-rational point of $\mathcal{M}_{g,r}$, where $X$ denotes a proper nodal curve and $\{\sigma_i\}_{i=1}^r$ denotes an ordered set consisting of $r$ marked points in $X$. Then, $X$ admits a log structure in a natural manner (cf. §1.1) and we denote by $X^{\log}$ the resulting log scheme. A pre-Tango structure on $X$ is defined (cf. Definition 5.3.1) as a logarithmic connection on the sheaf of logarithmic 1-forms $\Omega_{X^{\log}}^1/k$ with vanishing $p$-curvature whose horizontal sections are contained in the kernel of the Cartier operator. If $r = 0$ (i.e., $\{\sigma_i\}_{i=1}^r = 0$) and the underlying curve is smooth, then this definition of a pre-Tango structure is equivalent to the definition of a Tango structure mentioned in the previous subsection (cf. Proposition 5.3.2). Notice that our definition does not require a type of condition corresponding to the inequality $n(X) > 0$ as required in the classical case. But, we can proceed to our discussion regardless of whether such a condition should be imposed or not.

Denote by $\mathbb{F}_p^\times$ the product of $r$ copies of $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$ and let $\vec{\mu} \in \mathbb{F}_p^\times$, where $\vec{\mu} := 0$ if $r = 0$. We shall write

$$\mathcal{T}_{an_{g,r,\vec{\mu}}}$$

(cf. (191)) for the moduli stack classifying $r$-pointed stable curves over $k$ of genus $g$ together with a pre-Tango structure on it of monodromies $\vec{\mu}$. In particular, we have $\mathcal{T}_{an_g} \cong \mathcal{T}_{an_{g,0,\vec{0}}} \times \mathcal{M}_{g,0}$. 

0.3. In the present paper, we deal with (pre-)Tango structures defined in a more general setting, i.e., (pre-)Tango structures on families of pointed stable curves. Let $(g, r)$ be a pair of nonnegative integers with $2g - 2 + r > 0$. Write $\mathcal{M}_{g,r}$ for the moduli stack classifying $r$-pointed stable curves over $k$ of genus $g$ and $\mathcal{M}_{g,r}$ for its dense open substack classifying smooth curves. Suppose that we are given an $r$-pointed genus $g$ stable curve $X := (X, \{\sigma_i\}_{i=1}^r)$ classified by a $k$-rational point of $\mathcal{M}_{g,r}$, where $X$ denotes a proper nodal curve and $\{\sigma_i\}_{i=1}^r$ denotes an ordered set consisting of $r$ marked points in $X$. Then, $X$ admits a log structure in a natural manner (cf. §1.1) and we denote by $X^{\log}$ the resulting log scheme. A pre-Tango structure on $X$ is defined (cf. Definition 5.3.1) as a logarithmic connection on the sheaf of logarithmic 1-forms $\Omega_{X^{\log}}^1/k$ with vanishing $p$-curvature whose horizontal sections are contained in the kernel of the Cartier operator. If $r = 0$ (i.e., $\{\sigma_i\}_{i=1}^r = 0$) and the underlying curve is smooth, then this definition of a pre-Tango structure is equivalent to the definition of a Tango structure mentioned in the previous subsection (cf. Proposition 5.3.2). Notice that our definition does not require a type of condition corresponding to the inequality $n(X) > 0$ as required in the classical case. But, we can proceed to our discussion regardless of whether such a condition should be imposed or not.

Denote by $\mathbb{F}_p^\times$ the product of $r$ copies of $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$ and let $\vec{\mu} \in \mathbb{F}_p^\times$, where $\vec{\mu} := 0$ if $r = 0$. We shall write

$$\mathcal{T}_{an_{g,r,\vec{\mu}}}$$

(cf. (191)) for the moduli stack classifying $r$-pointed stable curves over $k$ of genus $g$ together with a pre-Tango structure on it of monodromies $\vec{\mu}$. In particular, we have $\mathcal{T}_{an_g} \cong \mathcal{T}_{an_{g,0,\vec{0}}} \times \mathcal{M}_{g,0}$. 

0.3. In the present paper, we deal with (pre-)Tango structures defined in a more general setting, i.e., (pre-)Tango structures on families of pointed stable curves. Let $(g, r)$ be a pair of nonnegative integers with $2g - 2 + r > 0$. Write $\mathcal{M}_{g,r}$ for the moduli stack classifying $r$-pointed stable curves over $k$ of genus $g$ and $\mathcal{M}_{g,r}$ for its dense open substack classifying smooth curves. Suppose that we are given an $r$-pointed genus $g$ stable curve $X := (X, \{\sigma_i\}_{i=1}^r)$ classified by a $k$-rational point of $\mathcal{M}_{g,r}$, where $X$ denotes a proper nodal curve and $\{\sigma_i\}_{i=1}^r$ denotes an ordered set consisting of $r$ marked points in $X$. Then, $X$ admits a log structure in a natural manner (cf. §1.1) and we denote by $X^{\log}$ the resulting log scheme. A pre-Tango structure on $X$ is defined (cf. Definition 5.3.1) as a logarithmic connection on the sheaf of logarithmic 1-forms $\Omega_{X^{\log}}^1/k$ with vanishing $p$-curvature whose horizontal sections are contained in the kernel of the Cartier operator. If $r = 0$ (i.e., $\{\sigma_i\}_{i=1}^r = 0$) and the underlying curve is smooth, then this definition of a pre-Tango structure is equivalent to the definition of a Tango structure mentioned in the previous subsection (cf. Proposition 5.3.2). Notice that our definition does not require a type of condition corresponding to the inequality $n(X) > 0$ as required in the classical case. But, we can proceed to our discussion regardless of whether such a condition should be imposed or not.

Denote by $\mathbb{F}_p^\times$ the product of $r$ copies of $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$ and let $\vec{\mu} \in \mathbb{F}_p^\times$, where $\vec{\mu} := 0$ if $r = 0$. We shall write

$$\mathcal{T}_{an_{g,r,\vec{\mu}}}$$

(cf. (191)) for the moduli stack classifying $r$-pointed stable curves over $k$ of genus $g$ together with a pre-Tango structure on it of monodromies $\vec{\mu}$. In particular, we have $\mathcal{T}_{an_g} \cong \mathcal{T}_{an_{g,0,\vec{0}}} \times \mathcal{M}_{g,0}$.
0.3. On the other hand, we recall the notion of a Miura $\mathfrak{g}$-oper for a semismple Lie algebra $\mathfrak{g}$. A Miura $\mathfrak{g}$-oper is a $\mathfrak{g}$-oper equipped with an additional data, that is to say, a $G$-torsor (where $G$ denotes the identity component of the group of Lie algebra automorphisms of $\mathfrak{g}$) equipped with two Borel reductions and a flat connection satisfying some condition, including a certain transversality condition. For example, $\mathfrak{sl}_2$-opers and Miura $\mathfrak{sl}_2$-opers may be identified with projective and affine connections respectively. The Miura opers (over the field of complex numbers $C$) plays an essential role in integrable systems and representation theory of loop algebra, including the Drinfeld-Sokolov reduction and geometric Langlands correspondence (cf. [4], [6], and [9]). The solutions of the Bethe Ansatz equations may be described by means of Miura opers. (cf. [7], [8]).

0.4. Let $k$, $\mathfrak{g}$, and $G$ be as above, and suppose that either one of the two conditions (Char)$_p$, (Char)$_{\mathfrak{sl}_p}$ described in §2.1 is satisfied (in particular, $k$ has characteristic $p > 0$). Let $t$ be the Lie algebra of a split maximal torus of $G$, and let $\vec{z} \in t(k)^{x r}$, where $\vec{z} := \emptyset$ if $r = 0$. Denote by $\mathfrak{M}_{\mathfrak{g}, g, r, \vec{z}}$ (cf. (111)) the moduli stack classifying $r$-pointed stable curves over $k$ of genus $g$ paired with a generic Miura $\mathfrak{g}$-oper on it of exponents $\vec{z}$. As shown in Proposition 3.7.1, it may be represented by a Deligne-Mumford stack over $k$. Although there is no generic Miura $\mathfrak{g}$-oper on a proper smooth curve over $C$ of genus $g > 1$, this moduli stack may not be empty. One may find the locus $\mathfrak{M}_{\mathfrak{g}, g, r, \vec{z}}$ (cf. (118)) of $\mathfrak{M}_{\mathfrak{g}, g, r, \vec{z}}$ classifying generic Miura $\mathfrak{g}$-opers satisfying a nice condition (regarding $p$-curvature) called dormant generic Miura $\mathfrak{g}$-opers. It follows from Theorem 3.8.3 (i) and (ii) that $\mathfrak{M}_{\mathfrak{g}, g, r, \vec{z}}$ is empty unless $\vec{z} \in t(F_p)^{x r}$ or $\vec{z} = \emptyset$, and is finite over $\mathfrak{M}_g$. If, moreover, $\mathfrak{g} = \mathfrak{sl}_2$, then one may obtain the following assertion, by which a dormant generic Miura $\mathfrak{g}$-oper may be thought of as a generalization of a (pre-)Tango structure.

Theorem A (cf. Theorem 5.4.1, Theorem 6.3.2). Let $\vec{z} := \vec{z} \in t(k)^{x r}$, where $\vec{z} := \emptyset$ if $r = 0$.

(i) Both $\mathfrak{M}_{\mathfrak{g}, g, r, \vec{z}}$ and $\mathfrak{M}_{\mathfrak{sl}_2, g, r, [\vec{z}]}$ (cf. Theorem 5.4.1 and 166) for the definitions of $-\vec{z}$ and $[\vec{z}]$ respectively may be represented by a (possibly empty) smooth proper Deligne-Mumford stack over $k$. Also, there exists a canonical isomorphism

$$\mathfrak{T}_{\mathfrak{g}, g, r, \vec{z}} \sim \mathfrak{M}_{\mathfrak{g}, g, r, \vec{z}}$$

over $\mathfrak{M}_{g, r}$. 

(ii) Write

$$\mathcal{MOp}_{z_{sl2, g, r}}^{z_{zz...}}(\varepsilon) := \mathcal{MOp}_{z_{sl2, g, r}}^{z_{zz...}} \times \mathbb{P}_g, \mathcal{M}_{g, r},$$

and let $\tau$ denote the natural bijection $\{0, 1, \cdots, p-1\} \rightarrow \mathbb{P}_g$. Then, $\mathcal{MOp}_{z_{sl2, g, r}}^{z_{zz...}}(\varepsilon)$ is empty if $2g - 2 + 2g - 2 + 2\sum_{i=1}^{r}r^{-1}(\varepsilon_i) < 0$. On the other hand, if $2g - 2 + 2g - 2 + 2\sum_{i=1}^{r}r^{-1}(\varepsilon_i) \in \mathbb{Z}_{\geq 0}$, then any irreducible component $\mathcal{N}$ of $\mathcal{MOp}_{z_{sl2, g, r}}^{z_{zz...}}(\varepsilon)$ with $\mathcal{N} \times \mathbb{P}_g, \mathcal{M}_{g, r} \neq \emptyset$ is equidimensional of dimension $2g - 2 + 2g - 2 + 2\sum_{i=1}^{r}r^{-1}(\varepsilon_i)$.

Notice that if $r = 0$ and $g = \frac{(lp-1)(lp-2)}{2}$ for some integer $l$ with $lp \geq 4$, then $\mathcal{MOp}_{z_{sl2, 0, 0}}^{z_{zz...}}(\varepsilon)$ is nonempty (cf. Remark 6.3.3 (i)). As a corollary of the above theorem, one may conclude the following assertion concerning the structure of the moduli stack $\mathcal{M}_g$.

**Theorem B.**

If $\mathcal{Tang}_g \neq \emptyset$ (e.g., $g = \frac{(lp-1)(lp-2)}{2}$ for some integer $l$ with $lp \geq 4$), then $p|(g-1)$ and $\mathcal{Tang}_g$ may be represented by an equidimensional smooth Deligne-Mumford stack over $k$ of dimension $2g - 2 + \frac{2g-2}{p}$ which is finite over $\mathcal{M}_g$.

In particular, the locus of $\mathcal{M}_g$ classifying proper smooth curves admitting a Tango structure (i.e., the scheme-theoretic image of the projection $\mathcal{Tang}_g \rightarrow \mathcal{M}_g$) forms, if it is nonempty, an equidimensional closed substack of dimension $2g - 2 + \frac{2g-2}{p}$.

**0.5.** In the last section of the present paper, we study the pathology of algebraic geometry in positive characteristic, which is of certain interest, since pathology reveals some completely different geometric phenomena from those in complex geometry. It is well-known that the Kodaira vanishing theorem does not always hold if the characteristic of the base field is positive. M. Raynaud has given (in [30]) its counterexamples on smooth algebraic surfaces in positive characteristic. He constructed a smooth polarized surface $(X, \mathcal{Z})$ with $H^1(X, \mathcal{Z}^\vee) \neq 0$. S. Mukai (cf. [26]) generalized Raynaud’s construction to obtain polarized smooth projective varieties $(X, \mathcal{Z})$ of any dimension with $H^1(X, \mathcal{Z}^\vee) \neq 0$. The construction similar to Mukai’s construction has been also studied by Y. Takeda (cf. [34], [35]) and P. Russell (cf. [31]). The key ingredient of these constructions is the use of a Tango structure on an algebraic curve. By applying one of these construction and Theorem B above, we obtain, as described in Theorem C below, a family of algebraic varieties violating the Kodaira vanishing theorem parametrized by a higher dimensional variety. This
result may be thought of as a refinement of the result given in [36], Theorem 4.1 (or [37].)

**Theorem C** (cf. Corollary 7.2.3).
Suppose that \( p > 3, p(p-1)|2g-2 \), and \( 4|p-3 \). Then, there exists a flat family \( \mathcal{Y} \to \mathcal{X} \) of proper smooth algebraic surfaces of general type parametrized by a Deligne-Mumford stack \( \mathcal{X} \) over \( k \) of dimension \( \geq g-2 + \frac{2g-2}{p-1} \), all of whose fibers are pairwise non-isomorphic and have the automorphism group schemes being non-reduced.

**Acknowledgement**
The author cannot express enough his sincere and deep gratitude to Professors Shinichi Mochizuki and Kirti Joshi. Without their philosophies and amazing insights, his study of mathematics would have remained “dormant”. The author would like to thank all those who gave the opportunity or impart the great joy of studying mathematics to him; the author wrote the present paper like a gratitude letter to them. Also, the author would like to thank the referee for reading carefully his manuscript. Finally, special thanks go to the moduli stack of dormant opers, who has guided him to the beautiful world of mathematics. The author was partially supported by the FMSP program at the Graduate School of Mathematical Sciences of the University of Tokyo, and the Grant-in-Aid for Scientific Research (KAKENHI No. 18K13385).

1. **Preliminaries**

In this section, we recall some definitions and notation concerning the theory of logarithmic connections on a torsor defined over a log-curve. Basic references for the notion of a log scheme (or, more generally, a log stack) are [16], [14], and [15].

Throughout the present paper, we shall fix a perfect field \( k \) and a pair of nonnegative integers \((g, r)\) with \( 2g - 2 + r > 0 \). For each stack \( \mathcal{M} \) over \( k \), we shall denote by \( \mathbf{Sch}_{/\mathcal{M}} \) the category of \( k \)-schemes equipped with a \( k \)-morphism \( S \to \mathcal{M} \). For a log stack (resp., a morphism of log stacks) indicated, say, by \( Y^{\log} \) (resp., \( f^{\log} : Y^{\log} \to Z^{\log} \)), we shall write \( Y \) (resp., \( f : Y \to Z \)) for the underlying stack of \( Y^{\log} \) (resp., the underlying morphism of stacks of \( f^{\log} \)). If \( Y^{\log} \) is as above and \( Z^{\log} \) is a log stack over \( Y^{\log} \), then we shall write \( \Omega_{Z^{\log}/Y^{\log}} \) for the sheaf of logarithmic 1-forms on \( Z^{\log} \) over \( Y^{\log} \) and write \( T_{Z^{\log}/Y^{\log}} := \Omega_{Z^{\log}/Y^{\log}}^\vee \), i.e., its dual. Also, we shall write \( d \) for the universal (logarithmic) derivation \( \mathcal{O}_Z \to \Omega_{Z^{\log}/Y^{\log}} \).
1.1. Log-curves.

Let $T_{\log}$ be an fs log scheme over $k$. A log-curve over $T_{\log}$ (cf. [1], Definition 4.5) is, by definition, a log smooth integrable morphism $f_{\log} : U_{\log} \to T_{\log}$ of fs log schemes such that the geometric fibers of the underlying morphism of schemes $f : U \to T$ are reduced and connected 1-dimensional schemes. In particular, both $\Omega_{U_{\log}/T_{\log}}$ and $T_{U_{\log}/T_{\log}}$ are line bundles.

Denote by $\overline{M}_{g,r}$ the moduli stack of $r$-pointed stable curves (cf. [20], Definition 1.1) over $k$ of genus $g$, and by $\overline{f}_{\tau} : \overline{C}_{g,r} \to \overline{M}_{g,r}$ the tautological curve over $\overline{M}_{g,r}$, equipped with its $r$ marked points $\sigma_{\tau,1}, \cdots, \sigma_{\tau,r} : \overline{M}_{g,r} \to \overline{C}_{g,r}$. Recall (cf. [20], Corollary 2.6 and Theorem 2.7; [3], § 5) that $\overline{M}_{g,r}$ may be represented by a geometrically connected, proper, and smooth Deligne-Mumford stack over $k$ of dimension $3g - 3 + r$. Denote by $D_{g,r}$ the divisor at infinity. In particular, its complement $M_{g,r} := \overline{M}_{g,r} \setminus D_{g,r}$ in $\overline{M}_{g,r}$ classifies smooth curves; we denote the tautological smooth curve over $M_{g,r}$ by $f_{\tau} : C_{g,r} \to M_{g,r}$ (i.e., $C_{g,r} := \overline{C}_{g,r} \times_{\overline{M}_{g,r}} M_{g,r}$). $\overline{M}_{g,r}$ has a natural log structure given by $D_{g,r}$ (cf. [15], Theorem 4.5), where we shall denote the resulting log stack by $\overline{M}_{g,r}^{\log}$. Also, by taking the divisor which is the union of the $\sigma_{\tau,i}$’s and the pull-back of $D_{g,r}$, we obtain a log structure on $\overline{C}_{g,r}$; we denote the resulting log stack by $\overline{C}_{g,r}^{\log}$. The structure morphism $\overline{f}_{\tau} : \overline{C}_{g,r} \to \overline{M}_{g,r}$ extends naturally to a morphism $\overline{f}_{\tau}^{\log} : \overline{C}_{g,r}^{\log} \to \overline{M}_{g,r}^{\log}$ of log stacks.

Next, let $S$ be a scheme over $k$, or more generally, a stack over $k$. Also, let $X := (f : X \to S, \{\sigma_i : S \to X\}_{i=1}^r)$ be an $r$-pointed stable curve over $S$ of genus $g$, where $f : X \to S$ denotes a proper nodal curve over $S$ of genus $g$ and $\{\sigma_i\}_{i=1}^r$ denotes an ordered set of $r$ marked points in $X$. Then, $X$ determines its classifying morphism $c_X : S \to \overline{M}_{g,r}$ and an isomorphism $X \cong S \times c_X^{-1} \overline{M}_{g,r}$. By pulling-back the log structures of $\overline{M}_{g,r}^{\log}$ and $\overline{C}_{g,r}^{\log}$, we obtain log structures on $S$ and $X$ respectively; we denote the resulting log stacks by $S^{X-\log}$ and $X^{X-\log}$ respectively. If there is no fear of causing confusion, we write $S^{\log}$ and $X^{\log}$ instead of $S^{X-\log}$ and $X^{X-\log}$ respectively. The structure morphism $f : X \to S$ extends to a
morphism \( f^{\log} : X^{\log} \to S^{\log} \) of log stacks, by which \( X^{\log} \) determines a log-curve over \( S^{\log} \) (cf. [16], §3; [15], Theorem 2.6). For each \( i = 1, \ldots, r \), there exists a canonical isomorphism (i.e., the so-called residue map)

\[
\text{triu}_{X,i} : \sigma_i^*(\Omega_{X^{\log}/S^{\log}}) \cong \mathcal{O}_S,
\]

(13)

(cf. [39], §1.6, (80)) which maps any local section of the form \( \sigma_i^*(\text{dlog}(x)) \in \sigma_i^*(\Omega_{X^{\log}/S^{\log}}) \) (for a local function \( x \) defining the closed subscheme \( \sigma_i : S \to X \) of \( X \)) to \( 1 \in \mathcal{O}_S \).

If, moreover, \( k \) has characteristic \( p > 0 \), then we shall denote by \( \mathcal{M}^{\text{ord}}_{g,r} \) the locus in \( \mathcal{M}_{g,r} \) classifying pointed proper smooth curves \( X := (X/S, \{\sigma_i\}_{i=1}^r) \) such that the underlying curve \( X/S \) is ordinary (i.e., the \( p \)-rank of its Jacobian is maximal). It is well-known that \( \mathcal{M}^{\text{ord}}_{g,r} \) forms a dense open substack of \( \mathcal{M}_{g,r} \).

1.2. Logarithmic connections on a vector bundle.

Let \( T^{\log} \) be an fs log scheme over \( k \) and \( f^{\log} : U^{\log} \to T^{\log} \) a log-curve over \( T^{\log} \). Also, let \( V \) be a vector bundle on \( U \) (i.e., a locally free \( \mathcal{O}_U \)-module of finite rank). By a \( T^{\log} \)-connection on \( V \), we mean an \( f^{-1}(\mathcal{O}_T) \)-linear morphism \( \nabla_V : V \to \Omega_{U^{\log}/T^{\log}} \otimes V \) satisfying the Leibniz rule: \( \nabla_V(a \cdot v) = da \otimes v + a \cdot \nabla_V(v) \), where \( a \) and \( v \) denote any local sections of \( \mathcal{O}_U \) and \( V \) respectively. A log flat bundle on \( U^{\log}/T^{\log} \) is a pair

\[
V^\flat := (V, \nabla_V)
\]

consisting of a vector bundle \( V \) on \( U \) and a \( T^{\log} \)-connection \( \nabla_V \) on \( V \). A log flat line bundle on \( U^{\log}/T^{\log} \) is a log flat bundle \( V^\flat := (V, \nabla_V) \) such that \( V \) is of rank one. We shall write

\[
\mathcal{O}_U^\flat := (\mathcal{O}_U, d : \mathcal{O}_U \to \Omega_{U^{\log}/T^{\log}}).
\]

Next, let \( V^\flat := (V, \nabla_V) \) and \( V'^\flat := (V', \nabla_{V'}) \) be log flat bundles on \( U^{\log}/T^{\log} \). An isomorphism of log flat bundles from \( V^\flat \) to \( V'^\flat \) is an isomorphism of vector bundles \( V \to V' \) that is compatible with the respective \( T^{\log} \)-connections \( \nabla_V \) and \( \nabla_{V'} \). If \( \nabla_V \otimes \nabla_{V'} \) denotes the \( T^{\log} \)-connection on the tensor product \( V \otimes V' \) induced from \( \nabla_V \) and \( \nabla_{V'} \) (i.e., given by \( v \otimes v' \mapsto \nabla_V(v) \otimes v' + v \otimes \nabla_{V'}(v') \)), then we shall write \( V^\flat \otimes V'^\flat \) for the tensor product of \( V^\flat \) and \( V'^\flat \), i.e., the log flat bundle

\[
V^\flat \otimes V'^\flat := (V \otimes V', \nabla_V \otimes \nabla_{V'})
\]

on \( U^{\log}/T^{\log} \).

Definition 1.2.1.

Let \( V^\flat := (V, \nabla_V) \) and \( V'^\flat := (V', \nabla_{V'}) \) be log flat bundles on \( U^{\log}/T^{\log} \). We shall say that \( V^\flat \) is \( \mathbb{G}_m \)-equivalent to \( V'^\flat \) if there exists a log flat line bundle \( L^\flat := (L, \nabla_L) \) on \( U^{\log}/T^{\log} \) such that \( V^\flat \otimes L^\flat \) is isomorphic to \( V'^\flat \).
1.3. Logarithmic connections on a torsor.

Let $U^\log/T^\log$ be as above, $G$ a connected smooth algebraic group over $k$, and $\pi : E \to U$ be a right $G$-torsor over $U$ in the étale topology. If $\mathfrak{h}$ is a $k$-vector space equipped with a left $G$-action, then we shall write $\mathfrak{h}_E$ for the vector bundle on $U$ associated with the relative affine space $E \times_k \mathfrak{h} := (E \times_k \mathfrak{h})/G$ over $U$.

Let us equip $E$ with a log structure pulled-back from $U^\log$ via $\pi : E \to U$; we denote the resulting log scheme by $E^{\log}$. The projection $\pi$ extends to a morphism $E^{\log} \to U^{\log}$, whose differential gives rise to the following short exact sequence:

$$0 \to g_E \to \tilde{T}_{E^{\log}}/T^{\log} \xrightarrow{\log} T_{U^{\log}}/T^{\log} \to 0,$$

where $\tilde{T}_{E^{\log}}/T^{\log}$ denotes the subsheaf $\pi_* (T_{E^{\log}}/T^{\log})^G$ of $G$-invariant sections of $\pi_* (T_{E^{\log}}/T^{\log})$ (cf. [39], §1.2, (31)). By a $T^{\log}$-connection on $E$, we mean an $\mathcal{O}_E$-linear morphism $\nabla_E : T_{U^{\log}}/T^{\log} \to \tilde{T}_{E^{\log}}/T^{\log}$ such that $\log \circ \nabla_E = \text{id}_{T_{U^{\log}}/T^{\log}}$.

Also, by a log flat $G$-torsor over $E^{\log}/T^{\log}$, we mean a pair $(E, \nabla_E)$ consisting of a right $G$-torsor $E$ over $U$ and a $T^{\log}$-connection $\nabla_E$ on $E$.

1.4. Monodromy of a logarithmic connection.

Let $X := (X/S, \{ \sigma_i \}_{i=1}^r)$ be an $r$-pointed stable curve of genus $g$ over a $k$-scheme $S$. Unless otherwise stated, we suppose, in this subsection, that $r > 0$.

Recall from [39], Definition 1.6.1, that, to each log flat $G$-torsor $(E, \nabla_E)$ over $X^{\log}/S^{\log}$ and each $i \in \{ 1, \cdots, r \}$, one may associate an element

$$\mu_i^{(E, \nabla_E)} \in \Gamma(S, \sigma_i^*(\mathfrak{g}_E))$$

called the monodromy of $(E, \nabla_E)$ at $\sigma_i$.

Definition 1.4.1.

Let $\bar{\mu} := (\mu_i)_{i=1}^r$ be an element of $\prod_{i=1}^r \Gamma(S, \sigma_i^*(\mathfrak{g}_E))$ and $(E, \nabla_E)$ a log flat $G$-torsor over $X^{\log}/S^{\log}$. Then, we shall say that $(E, \nabla_E)$ is of monodromies $\bar{\mu}$ if $\mu_i^{(E, \nabla_E)} = \mu_i$ for any $i \in \{ 1, \cdots, r \}$. If $r = 0$, then we shall refer to any log flat $G$-torsor as being of monodromy $\emptyset$.

Remark 1.4.2.

(i) Let us consider the case where $G = \text{GL}_n$ (for some positive integer $n$). Let $U^{\log}/T^{\log}$ be as in §1.3. Recall that giving a $\text{GL}_n$-torsor over $U$ is equivalent to giving a rank $n$ vector bundle on $U$. This equivalence may be given by assigning $E \mapsto (k^{\pm n})_E$. Now, let $E$ be a $\text{GL}_n$-torsor over $U$.
and \( \mathcal{V} \) the rank \( n \) vector bundle on \( U \) corresponding to \( \mathcal{E} \). Then, there exists a canonical isomorphism

\[
\mathcal{E}nd_{\mathcal{O}_S}(\mathcal{V}) \xrightarrow{\sim} (\mathfrak{gl}_n)_{\mathcal{E}}.
\]

Moreover, the notion of a \( T^{log} \)-connection on \( \mathcal{E} \) (defined above) coincides, via this equivalence, with the notion of a \( T^{log} \)-connection on \( \mathcal{V} \) (defined at the beginning of \S 1.2). We refer to [39], \S 4.2, for a detailed discussion.

(ii) Suppose further that \( U^{log}/T^{log} = X^{log}/S^{log} \) for a pointed stable curve \( X := (X/S, \{\sigma_i\}_{i=1}^r) \). Let \( \nabla_\mathcal{E} \) be an \( S^{log} \)-connection on \( \mathcal{E} \), and denote by \( \nabla_\mathcal{V} \) the \( S^{log} \)-connection on \( \mathcal{V} \) corresponding to \( \nabla_\mathcal{E} \). For each \( i \in \{1, \cdots, r\} \), we shall consider the composite

\[
\mathcal{V} \xrightarrow{\nabla_\mathcal{V}} \Omega_{X^{log}/S^{log}} \otimes \mathcal{V} \twoheadrightarrow \sigma_i^*(\Omega_{X^{log}/S^{log}}) \otimes \sigma_i^*(\mathcal{V}) \xrightarrow{\sim} \sigma_i^*(\mathcal{V}),
\]

where the second arrow arises from the adjunction relation \( \sigma_i^*(-) \dashv \sigma_i^*(-) \) (i.e., “the functor \( \sigma_i^*(-) \) is left adjoint to the functor \( \sigma_i^*(-) \)) and the third arrow arises from (13). This composite corresponds (via the relation \( \sigma_i^*(-) \dashv \sigma_i^*(-) \) again) to an \( \mathcal{O}_S \)-linear endomorphism \( \sigma_i^*(\mathcal{V}) \to \sigma_i^*(\mathcal{V}) \). Thus, we obtain an element

\[
\mu_i^{(\mathcal{V}, \nabla_\mathcal{V})} \in \Gamma(S, \mathcal{E}nd_{\mathcal{O}_S}(\sigma_i^*(\mathcal{V}))) = \Gamma(S, \sigma_i^*(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{V}))),
\]

which we shall refer to as the monodromy of \( (\mathcal{V}, \nabla_\mathcal{V}) \) (or just, of \( \nabla_\mathcal{V} \)) at \( \sigma_i \). This element \( \mu_i^{(\mathcal{V}, \nabla_\mathcal{V})} \) coincides, via (19), with the monodromy \( \mu_i^{(\mathcal{E}, \nabla_\mathcal{E})} \) of \( (\mathcal{E}, \nabla_\mathcal{E}) \). In the present paper, we shall not distinguish between these notions of monodromy.

**Remark 1.4.3.**

Let \( \mathcal{L}^\flat := (\mathcal{L}, \nabla_\mathcal{L}) \) be a log flat line bundle on \( X \). Then, \( \mathcal{E}nd_{\mathcal{O}_S}(\sigma_i^*(\mathcal{L})) \cong \mathcal{O}_S \), and hence, \( \mu_i^{\mathcal{L}^\flat} \) \((i = 1, \cdots, r)\) may be thought of as an element of \( \Gamma(S, \mathcal{O}_S) \). In particular, it makes sense to ask whether \( \mu_i^{\mathcal{L}^\flat} \) lies in \( k \) \((\subseteq \Gamma(S, \mathcal{O}_S)) \) or not.

### 1.5. Moduli of logarithmic connections.

We shall write

\[
\mathcal{C}_{g,r} \equiv \mathcal{C}_{g,r}
\]

for the set-valued contravariant functor on \( \mathfrak{Sch}_{/\mathcal{M}_{g,r}} \) which, to any object \( c_X : S \to \mathcal{M}_{g,r} \) of \( \mathfrak{Sch}_{/\mathcal{M}_{g,r}} \) classifying a pointed stable curve \( X \), assigns the set of \( S^{log} \)-connections on the line bundle \( \Omega_{X^{log}/S^{log}} \). Also, we write

\[
\mathcal{C}_{g,r} := \mathcal{C}_{g,r} \times_{\mathcal{M}_{g,r}} \mathcal{M}_{g,r}.
\]
It follows from a routine argument (or, an argument similar to the argument in the proof of Proposition 2.2.2 described later) that $\mathcal{C}o_{g,r}$ and $\mathcal{C}o_{g,r}$ may be represented by relative schemes of finite type over $\overline{\mathcal{M}}_{g,r}$ and $\mathcal{M}_{g,r}$ respectively.

Next, let $\vec{\mu} := (\mu_i)_{i=1}^r$ be an element of $k^{\times r}$ (= the product of $r$ copies of $k$), where $\vec{\mu} := \emptyset$ if $r = 0$. We shall write $\mathcal{C}o_{g,r,\vec{\mu}}$ for the closed substack of $\mathcal{C}o_{g,r}$ classifying connections of monodromies $\vec{\mu}$.

1.6. $p$-curvature.

In the rest of this section, we discuss logarithmic connections in positive characteristic. Suppose that $\text{char}(k) = p > 0$. In the following, we shall recall (cf. [39], §3.2) the definition of the $p$-curvature of a connection. Let $\mathbb{G}$ and $\mathfrak{X}$ be as before and let $(\mathcal{E}, \nabla_\mathcal{E})$ be a log flat $\mathbb{G}$-torsor over $X^{\log}/S^{\log}$. Then, we obtain an $\mathcal{O}_X$-linear morphism $\mathcal{T}^{\otimes p}_{X^{\log}/S^{\log}} \to \mathfrak{g}_\mathcal{E} (\subseteq \mathfrak{T}_{X^{\log}/S^{\log}})$ given by $\partial^{\otimes p} \mapsto (\nabla_\mathcal{E}(\partial)^{[p]} - \nabla_\mathcal{E}(\partial^{[p]}))$, where $\partial$ is any local section of $\mathcal{T}^{\otimes p}_{X^{\log}/S^{\log}}$ and $\partial^{[p]}$ denotes the $p$-th symbolic power of $\partial$ (i.e., “$\partial \mapsto \partial^{(p)}$” asserted in [28], Proposition 1.2.1). This morphism corresponds to an element

$$\psi(\mathcal{E}, \nabla_\mathcal{E}) \in \Gamma(X, \Omega^{\otimes p}_{X^{\log}/S^{\log}} \otimes \mathfrak{g}_\mathcal{E}),$$

which we shall refer to as the $p$-curvature of $(\mathcal{E}, \nabla_\mathcal{E})$ (cf. [39], Definition 3.2.1).

If $\mathbb{G} = \text{GL}_n$ and $(\mathcal{V}, \nabla_\mathcal{V})$ denotes the log flat vector bundle corresponds to $(\mathcal{E}, \nabla_\mathcal{E})$, then $\psi(\mathcal{E}, \nabla_\mathcal{E})$ coincides, via (19), with the classical definition of the $p$-curvature of $(\mathcal{V}, \nabla_\mathcal{V})$ (cf., e.g., [40], §1.5).

1.7. Canonical connections arising from the Frobenius morphism.

We shall recall the canonical connection arising from pull-back via Frobenius morphisms. Let $Y$ be an $S$-scheme with structure morphism $f : Y \to S$. Denote by $F_S : S \to S$ (resp., $F_Y : Y \to Y$) the absolute (i.e., $p$-th power) Frobenius endomorphism of $S$ (resp., $Y$). The Frobenius twist of $Y$ over $S$ is, by definition, the base-change $Y^{(1)}_S := Y \times_{S,F_S} S$ of $Y$ via $F_S : S \to S$. Denote by $f^{(1)} : Y^{(1)}_S \to S$ the structure morphism of $Y^{(1)}_S$. The relative Frobenius morphism of $Y$ over $S$ is the unique morphism $F_Y/S : Y \to Y^{(1)}_S$.
over $S$ that makes the following diagram commute:

\[
\begin{array}{ccc}
Y & \xrightarrow{F_Y} & Y \\
\downarrow{Y/S} & & \downarrow{(1)} \\
S & \xrightarrow{S} & S.
\end{array}
\]

Now, let $X := (f : X \rightarrow S, \{\sigma_i\}_{i=1}^r)$ be as before and $\mathcal{U}$ a vector bundle on $X^{(1)}_S$. Then, one may construct (cf. [39], §3.3) an $S^{\log}$-connection

\[\nabla_{\mathcal{U}} \text{can} : F^*_X/S(\mathcal{U}) \rightarrow \Omega_X^{\log}/S \otimes F^*_X/S(\mathcal{U})\]

on the pull-back $F^*_X/S(\mathcal{U})$ of $\mathcal{U}$ which is uniquely determined by the condition that the sections of the subsheaf $F^{-1}_X/S(\mathcal{U}) (\subset F^*_X/S(\mathcal{U}))$ are contained in $\ker(\nabla_{\mathcal{U}} \text{can})$. We shall refer to $\nabla_{\mathcal{U}} \text{can}$ as the canonical $S^{\log}$-connection on $F^*_X/S(\mathcal{U})$. One verifies immediately that

\[\text{Im}(\nabla_{\mathcal{U}} \text{can}) \subset \Omega_X/S \otimes F^*_X/S(\mathcal{U}) \subset \Omega_X^{\log}/S \otimes F^*_X/S(\mathcal{U})\]

(i.e., $\nabla_{\mathcal{U}} \text{can}$ comes from a non-logarithmic connection on $F^*_X/S(\mathcal{U})$). Moreover, we have

\[\psi(F^*_X/S(\mathcal{U}), \nabla_{\mathcal{U}} \text{can}) = 0\]

and (under the assumption that $r > 0$)

\[\mu_i(F^*_X/S(\mathcal{U}), \nabla_{\mathcal{U}} \text{can}) = 0\]

for any $i \in \{1, \ldots, r\}$.

1.8. Moduli of connections on a line bundle.

We shall write

\[\overline{p}_r := \{0, 1, \ldots, p-1\} \subset \mathbb{Z},\]

and write $\tau$ for the natural composite bijection

\[\tau : \overline{p}_r \hookrightarrow \mathbb{Z} \twoheadrightarrow \mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}.\]

Let $\bar{\mu} := (\mu_i)_{i=1}^r$ be an element of $k^{\times r}$ (where $\bar{\mu} := \emptyset$ if $r = 0$), $d$ an integer, and $\mathcal{L}$ a line bundle on the tautological curve $\overline{\mathcal{C}}_{g,r}$ (resp., $\mathcal{C}_{g,r}$) whose restriction to any fiber of $\overline{f}_\text{tau} : \overline{\mathcal{C}}_{g,r} \rightarrow \overline{\mathcal{M}}_{g,r}$ (resp., $f_\text{tau} : \mathcal{C}_{g,r} \rightarrow \mathcal{M}_{g,r}$) has degree $d$. We shall denote by

\[\overline{\mathcal{C}}_{g,\bar{\mu},\bar{\nu}}^{\psi=0} \text{ (resp., } \mathcal{C}_{g,\bar{\mu},\bar{\nu}}^{\psi=0})\]
the set-valued contravariant functor on $\mathcal{S}ch_{/\mathcal{M}_{g,r}}$ (resp., $\mathcal{S}ch_{/\mathfrak{M}_{g,r}}$) which, to any object $S \to \mathcal{M}_{g,r}$ (resp., $S \to \mathfrak{M}_{g,r}$) (where we denote by $X := (X/S, \{\sigma_i\}_{i=1}^r)$ the pointed curve classified by this object), assigns the set of $S^{x, \log}$-connections on $\mathcal{L}|_X$ with vanishing $p$-curvature. If $\mathcal{L} = \Omega_{\mathcal{L}^{\log}/\mathcal{M}_{g,r}}$, then we shall write

$$\mathcal{C}_0^{\psi=0} := \mathcal{C}_0^{\psi=0}_{\mathcal{L}, g, r, \mu} \ (\text{resp., } \mathcal{C}_0^{\psi=0} := \mathcal{C}_0^{\psi=0}_{\mathcal{L}, g, r, \mu})$$

for simplicity.

**Proposition 1.8.1.**

Let $\mathcal{L}$ be as above and $\mu := (\mu_i)_{i=1}^r \in \mathbb{F}_p^r$ (where $\mu := \emptyset$ if $r = 0$).

(i) $\mathcal{C}_0^{\psi=0}_{\mathcal{L}, g, r, \mu}$ is nonempty if and only if $p|(d + \sum_{i=1}^r \tau^{-1}(\mu_i))$.

(ii) Suppose that $\mathcal{C}_0^{\psi=0}_{\mathcal{L}, g, r, \mu}$ is nonempty. Then, $\mathcal{C}_0^{\psi=0}_{\mathcal{L}, g, r, \mu}$ may be represented by a Deligne-Mumford stack over $k$ which is finite and faithfully flat over $\mathfrak{M}_{g,r}$ of degree $p^d$. Moreover, the open substack $\mathcal{C}_0^{\psi=0}_{\mathcal{L}, g, r, \mu} \times_{\mathfrak{M}_{g,r}} \mathcal{M}_{g,r}^{\text{ord}}$ is étale over $\mathcal{M}_{g,r}^{\text{ord}}$.

**Proof.** Let $S$ be a $k$-scheme and $X := (X/S, \{\sigma_i\}_{i=1}^r)$ an $r$-pointed proper smooth curve over $S$ of genus $g$. Denote by $c_X : S \to \mathcal{M}_{g,r}$ the classifying morphism of $X$. For each $d' \in \mathbb{Q}$, denote by $\text{Pic}_{X/S}^{d'}$ (resp., $\text{Pic}_{X_S^{(1)}/S}^{d'}$) the relative Picard scheme of $X/S$ (resp., $X_S^{(1)}/S$) classifying the set of (equivalence classes, relative to the equivalence relation determined by tensoring with a line bundle pulled-back from the base $S$) of degree $d'$ line bundles on $X$ (resp., on $X_S^{(1)}$). (Here, we take $\text{Pic}_{X_S^{(1)}/S}^{d'} := \emptyset$ if $d' \notin \mathbb{Q} \setminus \mathbb{Z}$.) Denote by $c_{\mathcal{L}, \mu} : S \to \text{Pic}_{X_S^{(1)}/S}^{d+\sum_{i=1}^r \tau^{-1}(\mu_i)}$ the classifying morphism of the line bundle $\mathcal{L}(\sum_{i=1}^r \tau^{-1}(\mu_i) \cdot \sigma_{\text{am}}(c_{\mathcal{L}, \mu}))$ restricted to $X$ via $c_X$, i.e., $\mathcal{L}|_X(\sum_{i=1}^r \tau^{-1}(\mu_i) \cdot \sigma_i)$. Let us consider the morphism

$$\mathcal{N} : \text{Pic}_{X_S^{(1)}/S}^{d+\sum_{i=1}^r \tau^{-1}(\mu_i)} \to \text{Pic}_{X_S^{(1)}/S}^{d+\sum_{i=1}^r \tau^{-1}(\mu_i)}$$

$$\nu \mapsto \nu$$

$$[\mathcal{N}] \mapsto [F_{X/S}(\mathcal{N})]$$

determined by pull-back via $F_{X/S}$. In what follows, we shall prove the claim that $\mathcal{C}_0^{\psi=0}_{\mathcal{L}, g, r, \mu} \times_{\mathfrak{M}_{g,r}, c_X} S$ is isomorphic to the inverse image $\mathcal{N}^{-1}(c_{\mathcal{L}, \mu})$ of $c_{\mathcal{L}, \mu}$ via the morphism $\mathcal{N}$.

First, let $\mathcal{N}$ be a line bundle on $X_S^{(1)}$ classified by $\mathcal{N}^{-1}(c_{\mathcal{L}, \mu})$, which admits, by definition, an isomorphism

$$F_{X/S}(\mathcal{N}) \simeq \mathcal{L}|_X(\sum_{i=1}^r \tau^{-1}(\mu_i) \cdot \sigma_i).$$
Consider the $S$-connection on $L|_{X}(\sum_{i=1}^{r} \tau^{-1}(\mu_{i}) \cdot \sigma_{i})$ corresponding, via (36), to the canonical (logarithmic) $S$-connection $\nabla_{L|X}^{\text{log}}$; it restricts to an $S$-connection $\nabla_{L|X}$ on $L|_{X}$, which has vanishing $p$-curvature and monodromies $\bar{\mu}$. The assignment $N \mapsto \nabla_{L|X}$ is functorial with respect to $S$, and hence, determines a morphism
\[ \text{Ver}^{-1}(c_{L,\bar{\mu}}) \to \mathcal{O}_{L,g,r,\bar{\mu}}^{\psi=0} \times \mathfrak{g}_{r,c_{X}} S \]
over $S$.

On the other hand, let us take an $S$-connection $\nabla_{L|X}$ on $L|_{X}$ with vanishing $p$-curvature and of monodromies $\bar{\mu}$. One may find a unique $S$-connection $\nabla_{L|X,\pm \bar{\mu}}$ on $L|_{X}(\sum_{i=1}^{r} \tau^{-1}(\mu_{i}) \cdot \sigma_{i})$ whose restriction to $L|_{X}$ coincides with $\nabla_{L|X}$. Let us regard $F_{X/S*}(\text{Ker}(\nabla_{L|X,\pm \bar{\mu}}))$ as an $\mathcal{O}_{X^{\dag}}$-module. Then, the natural inclusion $F_{X/S*}(\text{Ker}(\nabla_{L|X,\pm \bar{\mu}})) \hookrightarrow F_{X/S*}(L|_{X}(\sum_{i=1}^{r} \tau^{-1}(\mu_{i}) \cdot \sigma_{i}))$ is $\mathcal{O}_{X^{\dag}}$-linear and corresponds, via the adjunction relation “$F_{X/S}(\pm) \dashv F_{*}(\pm)$”, to a morphism
\[ F_{X/S*}(F_{X/S*}(\text{Ker}(\nabla_{L|X,\pm \bar{\mu}}))) \to L|_{X}(\sum_{i=1}^{r} \tau^{-1}(\mu_{i}) \cdot \sigma_{i}). \]
Since $\nabla_{L|X,\pm \bar{\mu}}$ has monodromies $(0,0,\cdots,0) \in \mathbb{F}_{p}^{r}$, it turns out to be an isomorphism. By this isomorphism, $F_{X/S*}(\text{Ker}(\nabla_{L|X,\pm \bar{\mu}}))$ specifies a morphism $S \to \text{Ver}(c_{L,\bar{\mu}})$ as its classifying morphism. The resulting assignment $\nabla_{L|X} \mapsto F_{X/S*}(\text{Ker}(\nabla_{L|X,\pm \bar{\mu}}))$ determines a morphism
\[ \mathcal{O}_{L,g,r,\bar{\mu}}^{\psi=0} \times \mathfrak{g}_{r,c_{X}} S \to \text{Ver}^{-1}(c_{L,\bar{\mu}}) \]
over $S$. One verifies that (37) and (39) are inverses to each other, and hence, $\mathcal{O}_{L,g,r,\bar{\mu}}^{\psi=0} \times \mathfrak{g}_{r,c_{X}} S$ is isomorphic to $\text{Ver}^{-1}(c_{L,\bar{\mu}})$. This completes the proof of the claim. Then, assertions (i) and (ii) follow from the claim just proved and the well-known fact that the morphism $\text{Ver}$ is finite and faithfully flat of degree $p^{r}$ and moreover étale if $c_{X}$ lies in $\mathfrak{M}_{g,r}^{\text{ord}}$. \[ \square \]

2. Cartan connections on a pointed stable curve

In this section, we shall discuss logarithmic connections on a certain torsor (i.e., $L_{T_{1},U_{1}}^{\text{log}}/\mathcal{O}_{T_{1}}^{\text{log}}$ defined in (14)) called Cartan connections (cf. Definition 2.3.1). As shown in (54) such connections correspond bijectively to generic Miura opers; this result may be thought of as a generalization of [1], Proposition 8.2.2.

Denote by $\mathbb{G}_{m}$ the multiplicative group over $k$. In the following, for a scheme (or more generally, a stack) $Y$ and an $\mathcal{O}_{Y}$-module $\mathcal{V}$, we shall write $\mathcal{V}(\mathcal{V})$ for the relative affine space over $Y$ associated with $\mathcal{V}$.
2.1. Algebraic groups and Lie algebras.

Let \( \mathcal{G} \) be a split connected semisimple algebraic group of adjoint type over \( k \). Assume that either one of the following three conditions \((\text{Char})_0, (\text{Char})_p, \) and \((\text{Char})_p^d\) is satisfied:

\[(\text{Char})_0 : \text{char}(k) = 0;\]
\[(\text{Char})_p : \text{char}(k) = p > 2 \cdot h, \text{where} \ h \text{denotes the Coxeter number of} \ \mathcal{G};\]
\[(\text{Char})_p^d : \text{char}(k) = p > 0 \text{ and} \ \mathcal{G} = \text{PGL}_n \text{ for a positive integer} \ n \text{ with} \ n < p.\]

In particular, \((\text{Char})_p\) and \((\text{Char})_p^d\) imply respectively that \( p \) does not divide the order of the Weyl group of \( \mathcal{G} \) (and is very good for \( \mathcal{G} \)).

Let us fix a split maximal torus \( T_G \) of \( \mathcal{G} \), a Borel subgroup \( B_G \) of \( \mathcal{G} \) containing \( T_G \). If there is no fear of causing confusion, then we shall write \( T := T_G, B := B_G \) for simplicity. Denote by \( \Phi^+ \) the set of positive roots in \( B \) with respect to \( T \) and by \( \Phi^- \) the set of negative roots. Also, denote by \( \Gamma := (\subseteq \Phi^+) \) the set of simple positive roots. Denote by \( \mathfrak{g}, \mathfrak{b}, \) and \( \mathfrak{t} \) the Lie algebras of \( \mathcal{G}, B, \) and \( T \) respectively (hence \( \mathfrak{t} \subseteq \mathfrak{b} \subseteq \mathfrak{g} \)). For each \( \alpha \in \Phi^+ \cup \Phi^- \), we write

\[(40) \quad \mathfrak{g}^\alpha := \{ x \in \mathfrak{g} \mid \text{ad}(t)(x) = \alpha(t) \cdot x \text{ for all} \ t \in T \}.\]

\( \mathfrak{g} \) admits a canonical decomposition

\[(41) \quad \mathfrak{g} = \mathfrak{t} \oplus \left( \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}^\alpha \right) \oplus \left( \bigoplus_{\beta \in \Phi^-} \mathfrak{g}^\beta \right) \]

(which restricts to a decomposition \( \mathfrak{b} = \mathfrak{t} \oplus \left( \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}^\alpha \right) \) of \( \mathfrak{b} \)). By means of this decomposition, we obtain a unique decreasing filtration \( \{\mathfrak{g}^j\}_{j \in \mathbb{Z}} \) on \( \mathfrak{g} \) such that

- \( \mathfrak{g}^0 = \mathfrak{b}, \mathfrak{g}^0/\mathfrak{g}^1 = \mathfrak{t}, \mathfrak{g}^{-1}/\mathfrak{g}^0 = \bigoplus_{\alpha \in \Gamma} \mathfrak{g}^{-\alpha};\)
- \( [\mathfrak{g}^{j_1}, \mathfrak{g}^{j_2}] \leq \mathfrak{g}^{j_1+j_2} \) for \( j_1, j_2 \in \mathbb{Z}.\)

We shall write \( \hat{\rho} \) for the one-parameter subgroup \( \mathbb{G}_m \rightarrow T \) of \( T \) defined to be the sum \( \sum_{\alpha \in \Gamma} \hat{\omega}_\alpha \), where each \( \hat{\omega}_\alpha \) (\( \alpha \in \Gamma \)) denotes the fundamental coweight of \( \alpha \). By passing to differentiation, we consider \( \hat{\rho} \) as an element of \( \mathfrak{t} \).

Denote by \( \mathbb{W} \) the Weyl group of \((\mathcal{G}, T)\), i.e., \( \mathbb{W} := N_G(T)/T \), where \( N_G(T) \) denotes the normalizer of \( T \) in \( \mathcal{G} \). We shall identify \( \mathfrak{g} \) and \( \mathfrak{t} \) with \( \text{Spec}(\mathbb{S}_k(\mathfrak{g}')) \) and \( \text{Spec}(\mathbb{S}_k(\mathfrak{t}')) \) respectively, where for a \( k \)-vector space \( \mathfrak{a} \) we denote by \( \mathbb{S}_k(\mathfrak{a}) \) the symmetric algebra on \( \mathfrak{a} \) over \( k \). Consider the GIT quotient \( \mathfrak{g}/\mathcal{G} \) (resp., \( \mathfrak{t}/\mathbb{W} \)) of \( \mathfrak{g} \) (resp., \( \mathfrak{t} \)) by the adjoint action of \( \mathcal{G} \) (resp., \( \mathbb{W} \)), i.e., the spectrum of the ring of polynomial invariants \( \mathbb{S}_k(\mathfrak{g}')^\mathcal{G} \) (resp., \( \mathbb{S}_k(\mathfrak{t}')^\mathbb{W} \)) on \( \mathfrak{g} \) (resp., \( \mathfrak{t} \)). Let us write

\[(42) \quad c_G := \mathfrak{t}/\mathbb{W}.\]

If there is no fear of causing confusion, then we shall write \( c := c_G \) for simplicity. A Chevalley’s theorem asserts (cf. [27], Theorem 1.1.1; [19], Chap. VI, Theorem...
8.2) that the natural morphism \( S_k(g^\vee)^G \rightarrow S_k(t^\vee)^W \) is an isomorphism. Thus, one may define a morphism

\[
\chi : g \rightarrow c_G
\]

of \( k \)-schemes to be the composite of the natural quotient \( g \rightarrow g//G \) and the inverse of the resulting isomorphism \( c_G \xrightarrow{\sim} g//G \). \( \chi \) factors through the quotient \( g \rightarrow [g//G] \). (Here, \([g//G]\) is the quotient stack representing the contravariant functor which, to any \( k \)-scheme \( T \), assigns the groupoid of pairs \((\mathcal{F}, R)\) consisting of a right \( G \)-torsor \( \mathcal{F} \) over \( T \) and \( R \in \Gamma(T, g_\mathcal{F}) \).) We shall equip \( c_G \) with the \( G_m \)-action that comes from the homotheties on \( g \) (i.e., the natural grading on \( S_k(g^\vee) \)).

### 2.2. Cartan connections.

Let \( T^{\log} \) be an fs log scheme over \( k \) and \( f^{\log} : U^{\log} \rightarrow T^{\log} \) a log-curve over \( T^{\log} \). We shall denote by \( E^T_{U^{\log}/T^{\log}} \) the \( T \)-torsor over \( U \) defined to be

\[
E^T_{U^{\log}/T^{\log}} := (\Omega_{U^{\log}/T^{\log}})^{\times} \times_{G_m, \hat{\rho}} T,
\]

where \((\Omega_{U^{\log}/T^{\log}})^{\times}\) denotes the \( G_m \)-torsor over \( U \) corresponding to the line bundle \( \Omega_{U^{\log}/T^{\log}} \). If there is no fear of causing confusion, then we shall write \( E^T := E^T_{U^{\log}/T^{\log}} \) for simplicity.

**Definition 2.2.1.**

A \( g \)-Cartan connection on \( U^{\log}/T^{\log} \) is a \( T^{\log} \)-connection on \( E^T_{U^{\log}/T^{\log}} \). If \( U^{\log}/T^{\log} = X^{X^{\log}/S^{X^{\log}}} \) for some pointed stable curve \( \mathfrak{X} := (X/S, \{\sigma_i\}_{i=1}^r) \), then we shall refer to any \( g \)-Cartan connection on \( X^{X^{\log}/S^{X^{\log}}} \) as a \( g \)-Cartan connection on \( \mathfrak{X} \).

Next, let \( \mathfrak{X} := (f : X \rightarrow S, \{\sigma_i\}_{i=1}^r) \) be an \( r \)-pointed stable curve of genus \( g \) over a \( k \)-scheme \( S \), and suppose that \( r > 0 \). Let us fix \( i \in \{1, \cdots, r\} \). By the definition of \( E^T_i := E^T_{X^{X^{\log}/S^{X^{\log}}}} \), we have a sequence of isomorphisms

\[
\sigma_i^*(E^T_i) \xrightarrow{\sim} (\sigma_i^*(\Omega_{X^{X^{\log}/S^{X^{\log}}}}))^{\times} \times_{G_m, \hat{\rho}} T \xrightarrow{\sim} (S \times_k G_m) \times_{G_m, \hat{\rho}} T \xrightarrow{\sim} S \times_k T,
\]

where the second arrow arises from (13). It follows that the \( T \)-torsor \( \sigma_i^*(E^T_i) \) is trivial and we have a sequence of isomorphisms:

\[
\Gamma(S, \sigma_i^*(t_{E^T_i})) \xrightarrow{\sim} \Gamma(S, t_{\sigma_i^*(E^T_i)}) \xrightarrow{\sim} \Gamma(S, ts_{E^T_i}) \xrightarrow{\sim} t(S).
\]

The monodromy of any \( S^{\log} \)-connection on \( E^T_i \) (at each marked point \( \sigma_i \)) may be thought, via (14), of as an element of \( t(S)^{\times r} \), it makes sense to speak of a \( g \)-Cartan connection on \( \mathfrak{X} \) of monodromies \( \bar{\mu} \).
Let \( \bar{\mu} \in \mathfrak{t}(k)^{\times r} \) (where if \( r = 0 \), then we take \( \bar{\mu} := \emptyset \)). We shall write
\[
(47) \quad \mathcal{C}\mathcal{T}_{0,g,r} \quad \text{(resp., } \mathcal{C}\mathcal{T}_{\bar{g},g,r,\bar{\mu}}) \]
for the set-valued contravariant functor on \( \mathcal{S}ch_{/\mathcal{M}_{g,r}} \) which, to each object \( S \to \mathcal{M}_{g,r} \) of \( \mathcal{S}ch_{/\mathcal{M}_{g,r}} \) classifying a pointed stable curve \( \mathfrak{X} \), assigns the set of \( g \)-Cartan connections on \( \mathfrak{X} \) (resp., the set of \( g \)-Cartan connections on \( \mathfrak{X} \) of monodromies \( \bar{\mu} \)). Also, we shall write
\[
(48) \quad \mathcal{C}\mathcal{C}_{0,g,r} := \mathcal{C}\mathcal{T}_{0,g,r} \times \mathcal{M}_{g,r}, \quad \mathcal{C}\mathcal{C}_{\bar{g},g,r,\bar{\mu}} := \mathcal{C}\mathcal{T}_{\bar{g},g,r,\bar{\mu}} \times \mathcal{M}_{g,r}. \]
Then, the following proposition holds. (Notice that we will apply, in advance, the result of Proposition 2.6.1 described later in order to prove assertion (iii).)

**Proposition 2.2.2.**

(i) Both \( \mathcal{C}\mathcal{T}_{0,g,r} \) and \( \mathcal{C}\mathcal{T}_{\bar{g},g,r,\bar{\mu}} \) may be represented by (possibly empty) relative affine schemes over \( \mathcal{M}_{g,r} \) of finite type. In particular, these moduli functors may be represented by Deligne-Mumford stacks over \( k \).

(ii) Assume that \( r > 0 \). Then, \( \mathcal{C}\mathcal{T}_{0,g,r} \) may be represented by a relative affine space over \( \mathcal{M}_{g,r} \) modeled on \( \mathcal{V}(\mathfrak{f}_{\tau \mspace{1mu} \text{aux}}(\Omega_{\mathfrak{X}_{\text{log} / \mathcal{S}^{\text{log}}}) \otimes_k t) \). In particular, the projection \( \mathcal{C}\mathcal{T}_{0,g,r} \to \mathcal{M}_{g,r} \) is smooth of relative dimension \( (g - 1 + r) \cdot \text{rk}(\mathfrak{g}) \), where \( \text{rk}(\mathfrak{g}) \) denotes the rank of \( \mathfrak{g} \).

(iii) Assume that the \( \text{char}(k) = p > 0, r = 0, \) and \( p | (2g - 2) \). Then, \( \mathcal{C}\mathcal{T}_{0,g,0} \) may be represented by a relative affine space over \( \mathcal{M}_{g,0} \) modeled on \( \mathcal{V}(\mathfrak{f}_{\tau \mspace{1mu} \text{aux}}(\Omega_{\mathfrak{X}_{\text{log} / \mathcal{S}^{\text{log}}}) \otimes_k t) \). In particular, the projection \( \mathcal{C}\mathcal{T}_{0,g,0} \to \mathcal{M}_{g,0} \) is smooth of relative dimension \( g \cdot \text{rk}(\mathfrak{g}) \).

**Proof.** To begin with, let us fix an object \( c_X : S \to \mathcal{M}_{g,r} \) of \( \mathcal{S}ch_{/\mathcal{M}_{g,r}} \), which classifies a pointed stable curve \( \mathfrak{X} := (f : X \to S, \{\sigma_i\}_{i=1}) \). We shall write
\[
(49) \quad \mathcal{C}\mathcal{T}_{0,g,x} := \mathcal{C}\mathcal{T}_{0,g,r} \times \mathcal{M}_{g,r} \times_X S. \]

First, we shall prove assertion (i). Since \( \mathcal{C}\mathcal{T}_{0,g,r,\bar{\mu}} \) is a closed substack of \( \mathcal{C}\mathcal{T}_{0,g,r} \), it suffices to prove that \( \mathcal{C}\mathcal{T}_{0,g,x} \) may be represented by a (possibly empty) relative affine scheme over \( S \) of finite type. Consider the \( \mathcal{O}_X \)-linear morphism
\[
(50) \quad \text{Hom}_{\mathcal{O}_X}(\mathcal{T}_{X^{\text{log} / \mathcal{S}^{\text{log}}}, \mathfrak{F}^{\text{log} / \mathcal{S}^{\text{log}}}_\mathfrak{X}}) \to \text{End}_{\mathcal{O}_X}(\mathcal{T}_{X^{\text{log} / \mathcal{S}^{\text{log}}}, \mathfrak{F}^{\text{log} / \mathcal{S}^{\text{log}}}_\mathfrak{X}}) \cong \mathcal{O}_X \]
obtained by composition with \( \mathfrak{a}^{\text{log}}_{\mathfrak{X}} : \mathfrak{F}^{\text{log} / \mathcal{S}^{\text{log}}}_\mathfrak{X} \to \mathcal{T}_{X^{\text{log} / \mathcal{S}^{\text{log}}}} \). This morphism gives rise to a morphism
\[
(51) \quad \mathcal{V}(f_*(\text{Hom}_{\mathcal{O}_X}(\mathcal{T}_{X^{\text{log} / \mathcal{S}^{\text{log}}}, \mathfrak{F}^{\text{log} / \mathcal{S}^{\text{log}}}_\mathfrak{X})))) \to \mathcal{V}(f_*(\mathcal{O}_X)) = \mathcal{V}(\mathcal{O}_S) \]
of \( S \)-schemes. By the definition of an \( \mathcal{S}^{\text{log}} \)-connection, \( \mathcal{C}\mathcal{T}_{0,g,x} \) is isomorphic to the inverse image, via (51), of the section \( S \to \mathcal{V}(\mathcal{O}_S) \) corresponding to
$1 \in \Gamma(S, O_S)$. In particular, $\mathcal{E}\mathcal{O}_{g,X}$ may be represented by a closed subscheme of $\bigvee (f_* (\text{Hom}_{O_X}(\mathcal{T}_{X/S}, \mathcal{T}_{\tilde{E}/S}))$ and admits a free transitive action of

\begin{equation}
\bigvee (f_* (\text{Hom}_{O_X}(\mathcal{T}_{X/S}, \mathcal{T}_{\tilde{E}/S}))) \cong \bigvee (f_* (\text{Hom}_{O_X}(\mathcal{T}_{X/S}, \mathcal{T}_{\tilde{E}/S})))
\end{equation}

This completes the proof of assertion (i).

Next, we shall prove assertion (ii). By the above discussion, it suffices to prove that $\mathcal{E}\mathcal{O}_{g,X}$ admits, locally on $S$, a section $S \to \mathcal{E}\mathcal{O}_{g,X}$. One may assume, without loss of generality, that $S = \text{Spec}(R)$ for some $k$-algebra $R$.

The inverse image of the section $1 \in \Gamma(X, O_X)$ via $\mathcal{T}_{X/S}$ has, locally on $X$, a section. That is to say, there exists a collection $\{(U_i, \nabla_i)\}_{i \in I}$ indexed by a set $I$, where $\{U_i\}_{i \in I}$ is an open covering of $X$ and each $\nabla_i$ denotes an element of $\Gamma(U_i, \text{Hom}_{O_X}(\mathcal{T}_{X/S}, \mathcal{T}_{\tilde{E}/S}))$ whose image via $\mathcal{T}_{X/S}$ coincides with $1 \in \Gamma(U_i, O_X)$. Hence, $\nabla_{I\times I} := \{\nabla_i | U_i \cap U_j - \nabla_j | U_i \cap U_j\}_{(i,j) \in I \times I}$ specifies a Čech 1-cocycle of $\{U_i\}_{i \in I}$ with coefficients in $\text{Hom}_{O_X}(\mathcal{T}_{X/S}, \mathcal{T}_{\tilde{E}/S})$ ($\cong \Omega_{X/S} \otimes_k t$). Denote by $\nabla_{I \times I}$ the element of $H^1(X, \Omega_{X/S} \otimes_k t)$ represented by $\nabla_{I \times I}$. By Serre duality, we obtain the following sequence of isomorphisms of $R$-modules

\begin{equation}
H^1(X, \Omega_{X/S} \otimes_k t) \xrightarrow{\sim} H^1(X, \Omega_{X/S} \otimes_k t)^{\oplus \text{rk}(g)}
\end{equation}

\begin{equation}
\cong (H^0(X, \Omega_{X/S} \otimes_k t)^{\vee})^{\oplus \text{rk}(g)}
\end{equation}

\begin{equation}
\cong (H^0(X, O_X(-D_X))^\vee)^{\oplus \text{rk}(g)}
\end{equation}

\begin{equation}
= 0,
\end{equation}

where $D_X$ denotes the relative effective divisor on $X$ defined to be the union of the marked points $\sigma_i$ ($i = 1, \ldots, r$) and $\omega_{X/S}$ denotes the dualizing sheaf of $X$ over $S$, which is isomorphic to $\Omega_{X/S} \otimes_k t$. Hence, we have $\nabla_{I \times I} = 0$.

This implies that after possibly replacing each $\nabla_i$ by $\nabla_i + \delta_i$ (for some $\delta_i \in \Gamma(U_i, \text{Hom}_{O_X}(\mathcal{T}_{X/S}, \mathcal{T}_{\tilde{E}/S})))$, the sections $\nabla_i$ may be glued together to obtain a global section $\nabla$ of $\text{Hom}_{O_X}(\mathcal{T}_{X/S}, \mathcal{T}_{\tilde{E}/S})$ mapped to $1 \in \Gamma(X, O_X)$ (i.e., $\nabla$ forms a g-Cartan connection on $\tilde{X}$). $\nabla$ specifies a section $S \to \mathcal{E}\mathcal{O}_{g,X}$, and this completes the proof of assertion (ii).

Finally, assertion (iii) follows from Proposition [2.6.1](i), which implies that after base-changing via the finite and faithfully flat morphism $\mathcal{C}\mathcal{E}_{g,0} \to \mathcal{M}_{g,0}$ (cf. [70] for the definition of $\mathcal{C}\mathcal{E}_{g,0}$), the projection $\mathcal{E}\mathcal{O}_{g,0} \to \mathcal{M}_{g,0}$ admits a global section (i.e., the section corresponding to the closed immersion $\mathcal{E}\mathcal{O}_{g,0} \to \mathcal{E}\mathcal{O}_{g,0}$).
2.3. GL$_n$-Cartan connections.

In this subsection, we restrict ourselves to the case where $G = \text{PGL}_n$. (In particular, either one of the two conditions $(\text{Char})_0$, $(\text{Char})_p^\text{st}$ is satisfied.) Let $S$ be a $k$-scheme and $X := (X/S, \{\sigma_i\}_{i=1}^r)$ an $r$-pointed stable curve of genus $g$ over $S$. For each line bundle $N$ on $X$ and each integer $l$, we shall write

$$F^\dagger_{N,l} := T^\otimes_l X^{\log}/S^{\log} \otimes N$$

(hence, $F^\dagger_{N,0} = N$). Also, write

$$F^{[n]}_{N} := \bigoplus_{l=0}^{n-1} F^\dagger_{N,l}.$$ 

In particular, the $\mathcal{O}_X$-module $F^{[n]}_{N}$ admits, by definition, a natural grading.

**Definition 2.3.1.**

(i) A GL$_n$-Cartan connection on $X$ is a log flat bundle on $X^{\log}/S^{\log}$ of the form

$$\hat{F}^\bullet := (F^{[n]}_{N}, \bigoplus_{l=0}^{n-1} \nabla_l) = \bigoplus_{l=0}^{n-1} (F^\dagger_{N,l}, \nabla_l),$$

where $N$ is a line bundle on $X$ and each $\nabla_l$ ($l = 0, \ldots, n - 1$) is an $S^{\log}$-connection on the $l$-th component $F^\dagger_{N,l}$ of $F^{[n]}_{N}$.

(ii) Let $N^\flat := (N, \nabla_N)$ be a log flat line bundle on $X^{\log}/S^{\log}$. A (GL$_n$, $N^\flat$)-Cartan connection on $X$ is a GL$_n$-Cartan connection $\hat{F}^\bullet := (F^{[n]}_{N}, \bigoplus_{l=0}^{n-1} \nabla_l)$ on $X$ satisfying the equality $(F^{\dagger}_{N,0}, \nabla_0) = N^\flat$. (Hence, any GL$_n$-Cartan connection is a (GL$_n$, $N^\flat$)-Cartan connection for some log flat line bundle $N^\flat$.)

**Remark 2.3.2.**

Let $\hat{F}^\bullet := (F^{[n]}_{N}, \bigoplus_{l=0}^{n-1} \nabla_l)$ be a (GL$_n$, $N^\flat$)-Cartan connection on $X$ for some log flat line bundle $N^\flat = (N, \nabla_N)$. If $L^\flat := (L, \nabla_L)$ is a log flat line bundle on $X^{\log}/S^{\log}$, then $F^{[n]}_{N \otimes L}$ may be canonically identified with $F^{[n]}_{N} \otimes L$ and, via this identification, the tensor product

$$\hat{F}^\bullet \otimes L^\flat = (F^{[n]}_{N} \otimes L, \bigoplus_{l=0}^{n-1} \nabla_l \otimes \nabla_L)$$

forms a (GL$_n$, $N^\flat \otimes L^\flat$)-Cartan connection.

Denote by $\mathbb{T}_{\text{GL}_n}$ the maximal torus of GL$_n$ consisting of diagonal matrices. We take the maximal torus $\mathbb{T}_{\text{PGL}_n}$ of PGL$_n$ to be the image of diagonal matrices via the quotient $\text{GL}_n \to \text{PGL}_n$. In particular, we obtain a natural projection
\( T_{GL_n} \rightarrow T_{PGL_n} \) and an isomorphism \( \mathbb{G}_m^{\times n} \sim T_{GL_n} \) (where \( \mathbb{G}_m^{\times n} \) denotes the product over \( k \) of \( r \) copies of \( \mathbb{G}_m \)) which, to any element \((a_1, \ldots, a_n) \in \mathbb{G}_m^{\times n}\), assigns the diagonal matrix in \( T_{PGL_n} \) with entries \( a_1, \ldots, a_n \).

Now, let us take a \( GL_n \)-Cartan connection \( \widehat{F}^\bullet \) on \( X \), which determines (since it is a direct sum of \( n \) log flat line bundles) a log flat \( \mathbb{G}_m^{\times n} \)-torsor. One verifies that its underlying \( \mathbb{G}_m^{\times n} \)-torsor determines the \( TPGL_n \)-torsor \( E^\bullet \) via change of structure group via the composite \( \mathbb{G}_m^{\times n} \sim T_{GL_n} \rightarrow T_{PGL_n} \). That is to say, \( \widehat{F}^\bullet \) induces an \( sl_n (= pgl_n) \)-Cartan connection \( \nabla^\widehat{F}^\bullet \) on \( X \).

**Proposition 2.3.3.**

(i) The following maps of sets are bijective:

\[
\begin{align*}
(\text{the set of } S^{\log} \text{-connections}) \times (n-1) \\
\xrightarrow{\sim} \left( \text{the set of } (GL_n, \mathcal{O}_X^\bullet) \text{-Cartan connections on } X \right) \\
\xrightarrow{\sim} \left( \text{the set of } sl_n \text{-Cartan connections on } X \right),
\end{align*}
\]

where the first and second maps are given by assigning \((\nabla_i)_{i=1}^{n-1} \mapsto (\mathcal{F}^{[n]}_X, \bigoplus_{j=0}^{n-1}(\bigotimes_{j=1}^{r} \nabla_j^\vee)) \) (where \( \bigotimes_{j=1}^{r} \nabla_j^\vee := d \)) and \( \widehat{F}^\bullet \mapsto \nabla^\widehat{F}^\bullet \) respectively. Moreover, these bijections are functorial with respect to \( S \).

(ii) The composite of the bijections in (58) induces an isomorphism

\[
\overline{\mathcal{O}}_{g,r}^{\times (n-1)} \sim \mathcal{C}C \mathcal{O}_{sl_n,g,r}
\]

over \( \overline{\mathcal{M}}_{g,r} \), where the left-hand side denotes the product over \( \overline{\mathcal{M}}_{g,r} \) of \( n-1 \) copies of \( \mathcal{O}_{g,r} \).

**Proof.** The assertions follow immediately from the various definitions involved. \( \Box \)

Next, denote by \( t_{GL_n} \) and \( t_{PGL_n} \) the Lie algebras of \( T_{GL_n} \) and \( T_{PGL_n} \) respectively. The set \( t_{GL_n}(S) \) of \( S \)-rational points of \( t_{GL_n} \) may be identified, via the isomorphism \( \mathbb{G}_m^{\times n} \rightarrow T_{GL_n} \) mentioned above, with \( \Gamma(S, \mathcal{O}_S)^{\oplus n} \). Also, the set \( t_{PGL_n}(S) \) may be identified with \( \text{Coker}(\Delta) \), where \( \Delta \) denotes the diagonal embedding \( \Gamma(S, \mathcal{O}_S) \rightarrow \Gamma(S, \mathcal{O}_S)^{\oplus n} \). If

\[
(60) \quad \pi : t_{GL_n}(S) \rightarrow t_{PGL_n}(S)
\]

denotes the quotient arising from the natural quotient \( GL_n \rightarrow PGL_n \), then it may be identified, under the identifications just discussed, with the natural quotient \( \Gamma(S, \mathcal{O}_S)^{\oplus n} \rightarrow \text{Coker}(\Delta) \). For each \( \bar{\mu} := ((\mu_{i1}, \mu_{i2}, \ldots, \mu_{in}))_{i=1}^r \in \mathcal{C}C \mathcal{O}_{sl_n,g,r} \),
(Γ(S, OS)⊗n)×r ( = tGLn(S)×r), we shall write
(61) \[ \pi(\vec{\mu}) := (\pi(\mu_1, \mu_2, \cdots, \mu_n))_{i=1}^r \in tGL_n(S)^{\times r}. \]

If \( \vec{\mu} := \emptyset \), then we write \( \pi(\vec{\mu}) := \emptyset \). By the construction of (59), the following proposition holds.

**Proposition 2.3.4.**
Assume that \( r > 0 \).

(i) Let \( \vec{\mu} := ((\mu_1, \mu_2, \cdots, \mu_n))_{i=1}^r \in (\Gamma(S, OS)⊗n)^{\times r} \). Then, the bijections

\[
\prod_{l=1}^{n-1} \begin{pmatrix} \text{the set of } S^\log \text{-connections on } \Omega_X^\log / S^\log \text{ of monodromies } (\mu_l - \mu_{i+1})_{i=1}^r \\ \text{on } \Omega_X^\log / S^\log \end{pmatrix}
\]

\[ \sim \begin{pmatrix} \text{the set of } (GL_n, OS_X)\text{-Cartan connections on } \mathcal{X} \text{ of monodromies } ((0, \mu_2 - \mu_1, \cdots, \mu_n - \mu_1))_{i=1}^r \\ \text{connections on } \mathcal{X} \end{pmatrix} \]

\[ \sim \begin{pmatrix} \text{the set of } \mathfrak{sl}_n\text{-Cartan connections on } \mathcal{X} \text{ of monodromies } \pi(\vec{\mu}) \\ \text{of monodromies } \pi(\vec{\mu}) \end{pmatrix} \].

(ii) Let \( \vec{\mu} := ((\mu_1, \mu_2, \cdots, \mu_n))_{i=1}^r \in (k^{\times n})^{\times r} \). Then, the composite of the bijections in (62) induces an isomorphism

\[
\prod_{l=1}^{n-1} \begin{pmatrix} \mathfrak{B}_{g,r,((\mu_1, \mu_2, \cdots, \mu_n))_{i=1}^r} \\ \mathfrak{B}_{g,r,((\mu_1, \mu_2, \cdots, \mu_n))_{i=1}^r} \end{pmatrix}
\]

(63) \[ \sim \begin{pmatrix} \mathfrak{C}_{\mathfrak{b}_{g,r,((\mu_1, \mu_2, \cdots, \mu_n))_{i=1}^r}} \end{pmatrix} \]

over \( \mathfrak{M}_{g,r} \), where the left-hand side is the product over \( \mathfrak{M}_{g,r} \).

2.4. The relative affine spaces \( \mathfrak{X}_{g,r} \) over \( \mathfrak{M}_{g,r} \).

Now, let us go back to the general case but, in the rest of this section, we assume that either one of two conditions \((\text{Char})_p, (\text{Char})^d_p\) is satisfied. Moreover, assume that \( \mathcal{G}, \mathcal{B}, \text{ and } \mathcal{T} \) are all defined over \( \mathbb{F}_p \). In particular, it makes sense to speak of the subset \( t(\mathbb{F}_p) \) of \( t(k) \) (cf. Proposition 2.5.2). We shall write

\[
\mathfrak{X}_{g,r} := \mathcal{V}(\mathcal{T}_{\tau^\ast}(\Omega_{\mathcal{E}_{g,r}}^{\log} \otimes \mathfrak{t}_{\mathfrak{e}^\dagger})),
\]

\[
\mathfrak{X}_{g,r} := \mathcal{V}(\mathcal{T}_{\tau^\ast}(\Omega_{\mathcal{E}_{g,r}}^{\log} \otimes k \mathfrak{t})),
\]

where \( \mathcal{E}^\dagger := (\Omega_{\mathcal{E}_{g,r}}^{\log} \otimes k \mathfrak{t}) \times \mathbb{G}_{m,\mathbb{R}} \) (cf. (44)). Also, we shall write

\[
\mathfrak{X}_{g,r}
\]
for the relative scheme over \(\mathcal{M}_{g,r}\) (cf. [5], Theorem 5.23) representing the set-valued contravariant functor on \(\mathcal{Sch}/\mathcal{M}_{g,r}\) which, to each object \(S \to \mathcal{M}_{g,r}\) of \(\mathcal{Sch}/\mathcal{M}_{g,r}\) classifying pointed stable curve \(X := (X/S, \{\sigma_i\}_{i=1}^r)\), assigns the set of morphisms \(X \to (\text{\text{\varOmega}}^{\otimes}_{X^\log/S^\log}) \times \mathbb{G}_m\) over \(X\). The composite \(b \hookrightarrow g \twoheadrightarrow c\) and the projection \(t \twoheadrightarrow c\) induce morphisms
\[
\bigoplus \otimes b \to c\text{, }g,r
\]
\[
\bigoplus \otimes c\text{, }g,r
\]
(66)
respectively. Also, we shall write
\[
[0]_{c,g,r} : \mathcal{M}_{g,r} \to \bigoplus \otimes c\text{, }g,r
\]
(67)
for the zero sections.

2.5. Cartan connections with vanishing \(p\)-curvature.

We shall introduce the definitions of a dormant Cartan connections and a \(p\)-nilpotent Cartan connection (cf. Definition 2.5.1).

The assignment from each \(g\)-Cartan connection to its \(p\)-curvature determines a morphism
\[
\Psi_{b,g,r} : \mathcal{T}_{b,g,r} \to \bigotimes_{t,g,r}
\]
over \(\mathcal{M}_{g,r}\), and we obtain the composite
\[
\Psi^\chi_{b,g,r} : \mathcal{T}_{b,g,r} \to \bigotimes_{t,g,r} \bigoplus \otimes t \to c\text{, }g,r
\]
(69)
Let us write
\[
\mathcal{T}_{b,g,r}(\text{resp., } \mathcal{T}_{b,g,r}^{p-nilp})
\]
for the inverse image via \(\Psi_{b,g,r}\) (resp., \(\Psi^\chi_{b,g,r}\)) of the zero section \([0]_{t,g,r}\) (resp., \([0]_{c,g,r}\)).

Definition 2.5.1.
We shall say that a \(g\)-Cartan connection is dormant (resp., \(p\)-nilpotent) if it is classified by the closed substack \(\mathcal{T}_{b,g,r}(\text{resp., } \mathcal{T}_{b,g,r}^{p-nilp})\). (In particular, a dormant \(g\)-Cartan connection is a \(g\)-Cartan connection with vanishing \(p\)-curvature.)

Proposition 2.5.2.
Assume that \(r > 0\). Let \(S\) be a connected (resp., reduced and connected) \(k\)-scheme, \(X := (X/S, \{\sigma_i\}_{i=1}^r)\) an \(r\)-pointed stable curve over \(S\) of genus \(g\), and \(\nabla\) a dormant (resp., \(p\)-nilpotent) \(g\)-Cartan connection on \(X\). Then, the monodromy \(\mu_i^{(\nabla)} \in \mathfrak{t}(S)\) of \(\nabla\) at any marked point \(\sigma_i\) \((i = 1, \cdots, r)\) lies in \(\mathfrak{t}(\mathbb{F}_p)\).
Proof. Let us fix \( i \in \{1, \cdots, r\} \). The restriction \( \sigma_i^*(\psi(\xi_i, \nabla)) \) of the \( p \)-curvature \( \psi(\xi_i, \nabla) \in \Gamma(X, \Omega^{\oplus p}_{\text{log}/\text{Spec}(k)} \otimes t_{\xi_i}) \) may be thought of as an element of \( t(S) \) via the following composite isomorphism

\[
\Gamma(S, \sigma_i^*(\Omega^{\oplus p}_{\text{log}/\text{Spec}(k)} \otimes k t)) \xrightarrow{\sim} \Gamma(S, \sigma_i^*(\Omega^{\oplus p}_{\text{log}/\text{Spec}(k)} \otimes_k t)) \xrightarrow{\sim} t(S),
\]

where the second isomorphism arises from \( \text{triv}_X \) (cf. (13)). By the definition of \( p \)-curvature, the following equality holds:

\[
\sigma_i^*(\psi(\xi_i, \nabla)) = (\mu_i^{(\xi_i, \nabla)})^{[p]} - \mu_i^{(\xi_i, \nabla)}, \tag{72}
\]

where \( (\mu_i^{(\xi_i, \nabla)})^{[p]} \) denotes the image of \( \mu_i^{(\xi_i, \nabla)} \in t(S) \) via the \( p \)-th power operation \((-)^{[p]}\) on the Lie algebra \( t \) (cf. [39], Remark 3.2.2). Since \( t \) (considered as a scheme) is defined over \( \mathbb{F}_p \), it may be identified with \( t_k^{(1)} \) via the isomorphism \( \text{id}_k \times F_{\text{Spec}(k)} : t_k^{(1)} \xrightarrow{\sim} t \). Under this identification, the \( p \)-th power operation on \( t \) coincides with the relative Frobenius morphism \( F_{t/k} : t \to t = t_k^{(1)} \). Hence, (72) induces the equality

\[
\sigma_i^*(\psi(\xi_i, \nabla)) = F_{t/k} \circ \mu_i^{(\xi_i, \nabla)} - \mu_i^{(\xi_i, \nabla)}. \tag{73}
\]

Since \( F_{t/k} \circ \mu_i^{(\xi_i, \nabla)} = \mu_i^{(\xi_i, \nabla)} \) if and only if \( \mu_i^{(\xi_i, \nabla)} \in t(\mathbb{F}_p) \), the non-resp’d assertion follows from (73).

Next, let us consider the resp’d assertion. Since \( \chi \circ \sigma_i^*(\psi(\xi_i, \nabla)) = \chi(0) \), the element \( \sigma_i^*(\psi(\xi_i, \nabla)) \) of \( t(S) \) lies in \( (t \times_{\chi(0)} \text{Spec}(k))(S) \). But, the closed subscheme \( (t \times_{\chi(0)} \text{Spec}(k))_{\text{red}} \) of \( t \) is isomorphic to the point \( 0 \in t(k) \). Hence, under the assumption that \( S \) is reduced and connected, the equality \( \sigma_i^*(\psi(\xi_i, \nabla)) = 0 \) holds. By applying the same discussion as above, we have \( \mu_i^{(\xi_i, \nabla)} \in t(\mathbb{F}_p) \). This completes the proof of the resp’d assertion. \( \square \)

**Corollary 2.5.3.**

Let \( \bar{\mu} \in t(k)^{x_r} \). Then, both \( \mathcal{C} \mathcal{T}_{0, g, r, \bar{\mu}}^{\text{Zas...}} \) and \( \mathcal{C} \mathcal{T}_{0, g, r, \bar{\mu}}^{\text{p-nilp}} \) are empty unless \( \bar{\mu} \) lies in \( t(\mathbb{F}_p)^{x_r} \). Moreover, \( \mathcal{C} \mathcal{T}_{0, g, r}^{\text{Zas...}} \) decomposes as

\[
\mathcal{C} \mathcal{T}_{0, g, r}^{\text{Zas...}} = \coprod_{\bar{\mu} \in t(\mathbb{F}_p)^{x_r}} \mathcal{C} \mathcal{T}_{0, g, r, \bar{\mu}}^{\text{Zas...}}. \tag{74}
\]
2.6. Structure of the moduli stacks $\mathcal{C}Co^{\text{Zas}}_{g,g,r,\vec{\mu}}, \mathcal{C}Co^{\text{p-nilp}}_{g,g,r,\vec{\mu}}$

For each $\vec{\mu} \in t(k)^{\times r}$ (where $\vec{\mu} := \emptyset$ if $r = 0$), we shall write

$$\mathcal{C}Co^{\text{Zas}}_{g,g,r,\vec{\mu}} := \mathcal{C}Co^{\text{Zas}}_{g,g,r,\vec{\mu}} \times_{\mathbb{M}_{g,r}} \mathbb{M}_{g,r}, \quad \mathcal{C}Co^{\text{p-nilp}}_{g,g,r,\vec{\mu}} := \mathcal{C}Co^{\text{p-nilp}}_{g,g,r,\vec{\mu}} \times_{\mathbb{M}_{g,r}} \mathbb{M}_{g,r}. \quad (75)$$

**Proposition 2.6.1.** Let $\vec{\mu} \in t(\mathbb{F}_p)^{\times r}$ (where $\vec{\mu} := \emptyset$ if $r = 0$).

(i) $\mathcal{C}Co^{\text{Zas}}_{g,g,r,\vec{\mu}}$ may be represented by either the empty stack or a Deligne-Mumford stack over $k$ which is finite and faithfully flat over $\mathbb{M}_{g,r}$ of degree $p^{\text{rk}(\vec{\mu})}$. If $\mathcal{C}Co^{\text{Zas}}_{g,g,r,\vec{\mu}} \neq \emptyset$, then the open substack $\mathcal{C}Co^{\text{ord}}_{g,r,\vec{n}}$ is étale over $\mathbb{M}_{g,r}$.

(ii) If $r = 0$ and $p | 2g - 2$, then $\mathcal{C}Co^{\text{Zas}}_{g,g,0,\emptyset}$ is nonempty.

**Proof.** Let us consider assertion (i). For each integer $m$, we shall write $L^m := \Omega^{\otimes m}_{\mathcal{C}o_{g,r}/\mathbb{M}_{g,r}}$. Observe that the $\mathbb{T}$-torsor $\mathcal{E}^\dagger_\mathbb{T}$ for the tautological family of curves $\mathcal{C}_g / \mathbb{M}_{g,r}$ is isomorphic to the product of $\mathcal{G}_m$-torsors corresponding to $L^m$'s (where $j = 1, \cdots, \text{rk}(\vec{\mu})$ and each $m_j$ is an integer). Hence, there exists an isomorphism $\mathcal{C}Co_{g,g,r} \iso \prod_{j=1}^{\text{rk}(\vec{\mu})} \mathcal{C}o_{L^{m_j}_{g,r}}$. This isomorphism restricts to an isomorphism

$$\mathcal{C}Co^{\text{Zas}}_{g,g,r,\vec{\mu}} \iso \prod_{j=1}^{\text{rk}(\vec{\mu})} \mathcal{C}o_{L^{m_j}_{g,r},\vec{n}_j}, \quad (76)$$

where each $\vec{n}_j$ is an element of $\mathbb{F}_p^{\times r}$ (cf. Remark 1.4.3). Thus, the assertion follows from Proposition 1.8.1 (ii). Assertion (ii) follows from (76) and Proposition 1.8.1 (i) (together with the fact that $(2g - 2) | \text{deg}(L^{m_j})$ for any $j = 1, \cdots, \text{rk}(\vec{\mu})$).

**Corollary 2.6.2.** Let $\vec{\mu} \in t(\mathbb{F}_p)^{\times r}$ (where $\vec{\mu} := \emptyset$ if $r = 0$). Then, $\mathcal{C}Co^{\text{p-nilp}}_{g,g,r,\vec{\mu}}$ may be represented by a (possibly empty) Deligne-Mumford stack over $k$ which is finite over $\mathbb{M}_{g,r}$.

**Proof.** Let $\mathcal{N}$ be the closed substack of $\mathcal{X}_{t,g,r}$ defined to be $\mathcal{X}^{-1}_{t-g,r}([0]_{g,r})$. The morphism $\Psi_{g,g,r}$ (cf. (68)) restricts to a morphism $\mathcal{C}Co^{\text{p-nilp}}_{g,g,r,\vec{\mu}} \to \mathcal{N}$. Here, observe that the reduced stack $\mathcal{N}_{\text{red}}$ associated with $\mathcal{N}$ is isomorphic to the closed substack $[0]_{t,g,r}$. It follows that

$$\mathcal{C}Co^{\text{p-nilp}}_{g,g,r,\vec{\mu}} \times_{\mathcal{N}} \mathcal{N}_{\text{red}} \iso \mathcal{C}Co^{\text{Zas}}_{g,g,r,\vec{\mu}}. \quad (77)$$

Since $\mathcal{C}Co^{\text{p-nilp}}_{g,g,r,\vec{\mu}}$ is of finite type over $\mathbb{M}_{g,r}$, (77) and Proposition 2.6.1 (i) imply that $\mathcal{C}Co^{\text{p-nilp}}_{g,g,r,\vec{\mu}}$ is finite over $\mathbb{M}_{g,r}$.

□
Also, in the case where $G = \text{PGL}_n$ (hence $\mathfrak{g} = \mathfrak{pgl}_n = \mathfrak{sl}_n$), the following proposition holds.

**Proposition 2.6.3.**

(i) We have the following commutative square diagram:

$$
\begin{array}{ccc}
(Co_{\psi=0})_{g,r} \times (n-1) & \sim & CCo_{\text{ssl},g,r}^{\text{Zax}} \\
\text{open imm.} & & \text{open imm.} \\
(Co_{\psi=0})_{g,r} \times (n-1) & \sim & CCo_{\text{ssl},g,r}^{\text{Zax}} \\
\end{array}
$$

(78)

where the upper and lower horizontal arrows are isomorphisms obtained by restricting (77).

(ii) Assume further that $r > 0$, and let $\bar{\mu}, \pi(\bar{\mu})$ be as in Proposition 2.3.4. Then, by restricting square diagram (78) above, we have the following commutative square diagram:

$$
\begin{array}{ccc}
\prod_{l=1}^{n-1} Co_{\psi=0}^{\mu_l, \mu_{l+1}}_{g,r} & \sim & CCo_{\text{ssl},g,r,\pi(\bar{\mu})}^{\text{Zax}} \\
\text{open imm.} & & \text{open imm.} \\
\prod_{l=1}^{n-1} Co_{\psi=0}^{\mu_l, \mu_{l+1}}_{g,r} & \sim & CCo_{\text{ssl},g,r,\pi(\bar{\mu})}^{\text{Zax}} \\
\end{array}
$$

(79)

where the upper and lower horizontal arrows are isomorphisms and the two stacks in the left-hand side are the products over $\overline{M}_{g,r}$. In particular, if $\mu_l \in \mathbb{F}_p$ for any pair $(i, l)$ and $p \mid (2g-2 + \sum_{i=1}^{r} \tau^{-1}(\mu_l - \mu_{l+1}))$ for any $l = 1, \cdots, n-1$, then both $CCo_{\text{ssl},g,r,\pi(\bar{\mu})}^{\text{Zax}}$ and $CCo_{\text{ssl},g,r,\pi(\bar{\mu})}^{\text{Zax}}$ are nonempty.

**Proof.** The assertions follow from Proposition 1.8.1 (i) and the construction of (63). (Indeed, if two connections $\nabla_1$ and $\nabla_2$ have vanishing $p$-curvature, then the tensor product $\nabla_1 \otimes \nabla_2^\vee$ of $\nabla_1$ and the dual $\nabla_2^\vee$ of $\nabla_2$ has vanishing $p$-curvature.)

3. Miura $\mathfrak{g}$-opers on pointed stable curves

In this section, we discuss the definition and some basic properties of Miura $\mathfrak{g}$-opers on a family of pointed stable curves. It will be shown that generic Miura $\mathfrak{g}$-opers correspond bijectively to $\mathfrak{g}$-Cartan connections. (This result may be thought of as a global version of [9], Proposition 8.2.2.) In particular, the moduli functor, denoted by $\overline{\text{MP}}_{\mathfrak{g},g,r,\vec{e}}$ (cf. (110) and (111)), classifying pointed stable curves paired with a generic Miura $\mathfrak{g}$-oper (of prescribed exponents $\vec{e}$) may be represented by a Deligne-Mumford stack (cf. Proposition 3.7.1). In
the case of positive characteristic, we introduce two kinds of Miura $\mathfrak{g}$-opers, called *dormant generic Miura $\mathfrak{g}$-opers* and *$p$-nilpotent generic Miura $\mathfrak{g}$-opers*. The bijective correspondence mentioned above restricts to a correspondence between $p$-nilpotent Miura $\mathfrak{g}$-opers and $p$-nilpotent $\mathfrak{g}$-Cartan connections (cf. Theorem 3.8.3).

### 3.1. $\mathfrak{g}$-opers.

Let us keep the notation and assumptions in §2.1. First, we shall recall the definition of a $\mathfrak{g}$-oper (on a log-curve). Let $T^{\log}$ be an fs log scheme over $k$, $U^{\log}$ a log-curve over $T^{\log}$, and $\pi_B : \mathcal{E}_B \to U$ a right $B$-torsor over $U$. Denote by $\pi_G : (\mathcal{E}_B \times_B G =:) \mathcal{E}_G \to U$ the right $G$-torsor over $U$ obtained by change of structure group via the inclusion $B \hookrightarrow G$. For each $j \in \mathbb{Z}$, $\mathfrak{g}^j$ is closed under the adjoint action of $B$ on $\mathfrak{g}$, and hence, induces an $O_U$-submodule $\mathfrak{g}^j_{E_g}$ of $\mathfrak{g}_{E_g}$ ($\subseteq \tilde{T}_{E_G^{\log}/T^{\log}}$). If we write

$$
\tilde{T}^j_{E_G^{\log}/T^{\log}} := \iota(\tilde{T}_{E_G^{\log}/T^{\log}}) + \mathfrak{g}^j_{E_g}
$$

($j \in \mathbb{Z}$), where $\iota$ denotes the injection $\tilde{T}_{E_G^{\log}/T^{\log}} \hookrightarrow \tilde{T}_{E_G^{\log}/T^{\log}}$ induced by the natural inclusion $E_B \hookrightarrow E_G$, then the collection $\{\tilde{T}^j_{E_G^{\log}/T^{\log}}\}_{j \in \mathbb{Z}}$ forms a decreasing filtration on $\tilde{T}_{E_G^{\log}/T^{\log}}$. Since each $\mathfrak{g}^{-\alpha}$ ($\alpha \in \Gamma$) is closed under the action of $B$ (defined to be the composite $B \to T \xrightarrow{\text{adj.rep.}} \text{Aut}(\mathfrak{g}^{-\alpha})$), we have a decomposition

$$
\tilde{T}^{-1}_{E_G^{\log}/T^{\log}}/\tilde{T}^{0}_{E_G^{\log}/T^{\log}} \sim \bigoplus_{\alpha \in \Gamma} \mathfrak{g}^{-\alpha}_{E_g}
$$

arising from the decomposition $\mathfrak{g}^{-1}/\mathfrak{g}^0 = \bigoplus_{\alpha \in \Gamma} \mathfrak{g}^{-\alpha}$. We recall from [39], Definition 2.2.1, the definition of a $\mathfrak{g}$-oper, as follows.

**Definition 3.1.1.**

(i) Let

$$
\mathcal{E}^\bullet := (\pi_B : \mathcal{E}_B \to U, \nabla_\mathcal{E} : \mathcal{T}^{\log}_{U^{\log}/T^{\log}} \to \tilde{T}^{\log}_{E_G^{\log}/T^{\log}})
$$

be a pair consisting of a right $B$-torsor $\mathcal{E}_B$ over $U$ and a $T^{\log}$-connection $\nabla_\mathcal{E}$ on the right $G$-torsor $\pi_G : \mathcal{E}_G \to U$ induced by $\mathcal{E}_B$. We shall say that the pair $\mathcal{E}^\bullet = (\mathcal{E}_B, \nabla_\mathcal{E})$ is a *$\mathfrak{g}$-oper* on $U^{\log}/T^{\log}$ if it satisfies the following two conditions:

- $\nabla_\mathcal{E}(\mathcal{T}^{\log}_{U^{\log}/T^{\log}}) \subseteq \tilde{T}^{-1}_{E_G^{\log}/T^{\log}}$;
- For any $\alpha \in \Gamma$, the composite

$$
\mathcal{K}^\alpha: \mathcal{T}^{\log}_{U^{\log}/T^{\log}} \xrightarrow{\nabla_\mathcal{E}} \tilde{T}^{-1}_{E_G^{\log}/T^{\log}} \xrightarrow{\tilde{T}^{-1}_{E_G^{\log}/T^{\log}}} \tilde{T}^0_{E_G^{\log}/T^{\log}} \to \mathfrak{g}_{E_g}^\alpha.
$$


is an isomorphism, where the third arrow denotes the natural projection relative to the decomposition (81).

(ii) Let \( \mathcal{E}^\bullet := (\mathcal{E}_B, \nabla_\mathcal{E}) \), \( \mathcal{F}^\bullet := (\mathcal{F}_B, \nabla_\mathcal{F}) \) be \( \mathfrak{g} \)-opers on \( U^{\log}/T^{\log} \). An isomorphism of \( \mathfrak{g} \)-opers from \( \mathcal{E}^\bullet \) to \( \mathcal{F}^\bullet \) is an isomorphism \( \mathcal{E}_B \sim \mathcal{F}_B \) of right \( \mathbb{B} \)-torsors such that the induced isomorphism \( \mathcal{E}_G \sim \mathcal{F}_G \) of right \( \mathbb{G} \)-torsors is compatible with the respective \( T^{\log} \)-connections \( \nabla_\mathcal{E} \) and \( \nabla_\mathcal{F} \).

3.2. Miura \( \mathfrak{g} \)-opers.

Next, we shall introduce the definition of a Miura \( \mathfrak{g} \)-oper on a log-curve.

**Definition 3.2.1.**

(i) A Miura \( \mathfrak{g} \)-oper on \( U^{\log}/T^{\log} \) is a collection of data

\[
\hat{\mathcal{E}}^\bullet := (\mathcal{E}_B, \nabla_\mathcal{E}, \mathcal{E}'_B, \eta_\mathcal{E})
\]

consisting of

- a \( \mathfrak{g} \)-oper \( (\mathcal{E}_B, \nabla_\mathcal{E}) \) on \( U^{\log}/T^{\log} \);
- a \( \mathbb{B} \)-torsor \( \mathcal{E}'_B \) over \( U \), where we shall write \( \mathcal{E}'_B := \mathcal{E}'_B \times^\mathbb{B} \mathbb{G} \);
- an isomorphism \( \eta_\mathcal{E} : \mathcal{E}'_B \sim \mathcal{E}_B \) of \( \mathbb{G} \)-torsors such that if \( d\eta_\mathcal{E} \) denotes the \( \mathcal{O}_U \)-linear isomorphism \( \tilde{T}_{\mathcal{E}'_B}^{\log} \subseteq \tilde{T}_{\mathcal{E}'_B}^{\log} \rightarrow \tilde{T}_{\mathcal{E}'_B}^{\log} \) obtained by differentiating \( \eta_\mathcal{E} \), then we have

\[
\nabla_\mathcal{E}(\tilde{T}_{U^{\log}/T^{\log}}) \subseteq d\eta_\mathcal{E}(\tilde{T}_{\mathcal{E}'_B}^{\log}).
\]

The \( T^{\log} \)-connection \( \nabla_\mathcal{E} \) specifies a \( T^{\log} \)-connection \( \nabla_{\mathcal{E}'_B} : \mathcal{T}_{U^{\log}/T^{\log}} \rightarrow \tilde{T}_{\mathcal{E}'_B}^{\log}/T^{\log} \) on \( \mathcal{E}'_B \) in such a way that the composite

\[
\mathcal{T}_{U^{\log}/T^{\log}} \nabla_{\mathcal{E}'_B} \tilde{T}_{\mathcal{E}'_B}^{\log}/T^{\log} \rightarrow \tilde{T}_{\mathcal{E}'_B}^{\log}/T^{\log} \nabla_{\tilde{T}_{\mathcal{E}'_B}^{\log}/T^{\log}} \nabla_\mathcal{E} \tilde{T}_{\mathcal{E}'_B}^{\log}/T^{\log}
\]

coincides with \( \nabla_\mathcal{E} \). We shall refer to \( (\mathcal{E}'_B, \nabla_{\mathcal{E}'_B}) \) as the log flat \( \mathbb{B} \)-torsor associated with \( \hat{\mathcal{E}}^\bullet \). Also, we shall refer to \( \mathcal{E}^\bullet := (\mathcal{E}_B, \nabla_\mathcal{E}) \) as the underlying \( \mathfrak{g} \)-oper of \( \hat{\mathcal{E}}^\bullet \). If \( U^{log}/T^{log} = X^{x-log}/S^{x-log} \) for some pointed stable curve \( X := (X/S, \{\sigma_i\}_{i=1}^r) \), then we shall refer to any Miura \( \mathfrak{g} \)-oper on \( X^{x-log}/S^{x-log} \) as a Miura \( \mathfrak{g} \)-oper on \( X \).

(ii) Let \( \hat{\mathcal{E}}^\bullet := (\mathcal{E}_B, \nabla_\mathcal{E}, \mathcal{E}'_B, \eta_\mathcal{E}) \), \( \hat{\mathcal{F}}^\bullet := (\mathcal{F}_B, \nabla_\mathcal{F}, \mathcal{F}'_B, \eta_\mathcal{F}) \) be Miura \( \mathfrak{g} \)-opers on \( U^{\log}/T^{\log} \). An isomorphism of Miura \( \mathfrak{g} \)-opers from \( \hat{\mathcal{E}}^\bullet \) to \( \hat{\mathcal{F}}^\bullet \) is a pair

\[
(\alpha_\mathcal{B}, \alpha'_\mathcal{B})
\]

consisting of
• an isomorphism $\alpha_B : (\mathcal{E}_B, \nabla_\mathcal{E}) \xrightarrow{\sim} (\mathcal{F}_B, \nabla_F)$ of $\mathfrak{g}$-opers (i.e., an isomorphism $\alpha_B : \mathcal{E}_B \xrightarrow{\sim} \mathcal{F}_B$ of $B$-torsor respecting the structures of $T^{\log}$-connection);
• an isomorphism $\alpha'_B : \mathcal{E}'_B \xrightarrow{\sim} \mathcal{F}'_B$ of right $B$-torsors such that the induced isomorphism $\alpha'_G : \mathcal{E}'_G \xrightarrow{\sim} \mathcal{F}'_G$ of right $G$-torsors satisfies the equality $\alpha_G \circ \eta_\mathcal{E} = \eta_F \circ \alpha'_G$.

Proposition 3.2.2.
Any Miura $\mathfrak{g}$-oper on $U^{\log}/T^{\log}$ does not have nontrivial automorphisms.

Proof. Each automorphism of a Miura $\mathfrak{g}$-oper determines and is determined by an automorphism of its underlying $\mathfrak{g}$-oper. Hence, the assertion follows directly from [39], Proposition 2.2.5. □

3.3. Generic Miura $\mathfrak{g}$-opers.

We shall define the notion of a generic Miura $\mathfrak{g}$-oper. Let $\hat{\mathcal{E}}^\bullet := (\mathcal{E}_B, \nabla_\mathcal{E}, \mathcal{E}'_B, \eta_\mathcal{E})$ be a Miura $\mathfrak{g}$-oper on $U^{\log}/T^{\log}$. Recall that the flag variety associated with $G$ is the quotient $G/B$, which classifies all Borel subgroups of $G$. (Indeed, the point of $G/B$ represented by $h \in G$ classifies the Borel subgroup $Ad_h(B)$, where $Ad_h$ denotes the automorphism of $G$ given by conjugation by $h$.) The $B$-orbits (with respect to the left $B$-action on $G$) in the flag variety $G/B$ are parametrized by the Weyl group $W$. That is to say, we have a decomposition $G = \coprod_{w \in W} BwB$, which is known as the Bruhat decomposition. (More precisely, for each $w \in W = N_G(T)/T$ we choose a representative $\check{w} \in N_G(T)$ of $w$ and consider the double coset $B\check{w}B$. Since $B\check{w}B$ is independent of the choice of the representative $\check{w}$ of $w$, we simply denote it by $BwB$.) Let $w_0$ be the longest element of $W$. The orbit $Bw_0B \subseteq G/B$ is a dense open subscheme of $G/B$ (called the big cell). The morphism $N \to G/B$ given by assigning $h \mapsto h\check{w}_0B$ defines an isomorphism $N \xrightarrow{\sim} Bw_0B$ of $k$-schemes. By passing to this isomorphism, we shall regard $N$ as a dense open subscheme of $G/B$ that is closed under the left $B$-action on $G/B$. By twisting $G/B$ by the right $B$-torsor $\mathcal{E}_B$, we obtain a proper $U$-scheme

\begin{equation}
\mathcal{E}_B \times^B (G/B),
\end{equation}

which contains a dense open subscheme $\mathcal{E}_B \times^B N$. It follows from the definition of $\mathcal{E}_B \times^B (G/B)$ that the image of $\mathcal{E}'_B$ via the isomorphism $\eta_\mathcal{E} : \mathcal{E}'_G \xrightarrow{\sim} \mathcal{E}_G$ determines its classifying morphism

\begin{equation}
\sigma_{\hat{\mathcal{E}}^\bullet} : U \to \mathcal{E}_B \times^B (G/B).
\end{equation}
**Definition 3.3.1.**

We shall say that a Miura $\mathfrak{g}$-oper $\mathcal{E}^\bullet$ is **generic** if the image of the morphism $\sigma$ (for simplicity).

**(3.4)** **Special Miura $\mathfrak{g}$-opers.**

Next, we define the notion of a *special* Miura $\mathfrak{g}$-oper. Write

\[ (90) \quad \mathcal{E}^{\dagger}_{B,U^{\log}/T^{\log}} := \mathcal{E}^{\dagger}_{T,U^{\log}/T^{\log}} \times^T B, \quad \mathcal{E}^{\dagger}_{G,U^{\log}/T^{\log}} := \mathcal{E}^{\dagger}_{T,U^{\log}/T^{\log}} \times^T G. \]

(cf. (91) for the definition of $\mathcal{E}^{\dagger}_{T,U^{\log}/T^{\log}}$). The subscheme $w_0 B$ of $G$ is closed under the left action by $T$. Hence, we have a $B$-torsor

\[ (91) \quad \mathcal{E}^{\dagger}_{B,U^{\log}/T^{\log}} := \mathcal{E}^{\dagger}_{T,U^{\log}/T^{\log}} \times^T w_0 B, \]

which admits a natural isomorphism

\[ (92) \quad \eta^\dagger_B : \mathcal{E}^{\dagger}_{B,U^{\log}/T^{\log}} \times^B G \sim \mathcal{E}^{\dagger}_{G,U^{\log}/T^{\log}} \]

of $G$-torsors. If there is no fear of causing confusion, then we shall write

\[ (93) \quad \mathcal{E}^{\dagger}_B := \mathcal{E}^{\dagger}_{B,U^{\log}/T^{\log}}, \quad \mathcal{E}^{\dagger}_G := \mathcal{E}^{\dagger}_{G,U^{\log}/T^{\log}}, \quad \mathcal{E}^{\dagger}_B := \mathcal{E}^{\dagger}_{B,U^{\log}/T^{\log}} \]

for simplicity.

For each $\alpha \in \Gamma$, let us fix a generator $x_\alpha$ of $\mathfrak{g}^\alpha$. Write $p_1 := \sum_{\alpha \in \Gamma} x_\alpha$. Then, one may find a unique collection $(y_\alpha)_{\alpha \in \Gamma}$, where each $y_\alpha$ is a generator of $\mathfrak{g}^{-\alpha}$, such that if we write

\[ (94) \quad p_{-1} := \sum_{\alpha \in \Gamma} y_\alpha \in \mathfrak{g}_{-1}, \]

then the set $\{p_{-1}, 2\hat{p}, p_1\}$ forms an $\mathfrak{sl}_2$-triple (cf. §2.1 for the definition of $\hat{p}$).

For each $\alpha \in \Gamma$, we shall write

\[ (95) \quad \eta^\alpha : \mathfrak{g}^{-\alpha}_{\mathcal{E}^{\dagger}_{B,U^{\log}/T^{\log}}} \sim \mathcal{E}^{\dagger}_{T,U^{\log}/T^{\log}} \]

for the isomorphism determined uniquely by the following condition: for each local trivialization $\tau : \mathcal{O}_V \sim \mathcal{T}^{\log}_U \langle T \rangle^\log \mid V$ of $\mathcal{T}^{\log}_U \langle T \rangle^\log$ (where $V$ denotes an open subscheme of $U$), the composite isomorphism

\[ (96) \quad \mathfrak{g}^{-\alpha}_{\mathcal{E}^{\dagger}_{B,U^{\log}/T^{\log}}} \mid V \sim \mathfrak{g}^{-\alpha}_{\mathcal{E}^{\dagger}_{T,U^{\log}/T^{\log}}} \mid V \sim (k \cdot y_\alpha) \mathcal{O}_{U^{\log}/T^{\log}} \mid V \times^G \mathfrak{g}^{-\alpha}_{T^\times} \]

\[ \sim (k \cdot y_\alpha) \mathcal{O}_{U^{\log}/T^{\log}} \mid V \sim \mathcal{O}_V \sim \mathcal{T}^{\log}_U \langle T \rangle^\log \mid V \]

coincides with $\eta^\alpha \mid V$, where the third isomorphism arises from the dual isomorphism $\Omega_{U^{\log}/T^{\log}} \mid V \sim \mathcal{O}_V$ of $\tau$ and the fourth isomorphism is given by $a \cdot y_\alpha \mapsto a$ (for any $a \in k$).
Remark 3.4.1.
In the case where $\mathcal{G} = \text{PGL}_n$ (hence $\mathfrak{g} = \mathfrak{pgl}_n = \mathfrak{sl}_n$), we fix an $\mathfrak{sl}_n$-triple $\{p_1, 2p, p\}$ given by

\[(97)\]

$$
p_{-1} := \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \quad 2p := \begin{pmatrix} n - 1 & 0 & 0 & \cdots & 0 \\ 0 & n - 3 & 0 & \cdots & 0 \\ 0 & 0 & n - 5 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -(n - 1) \end{pmatrix},
$$

$$
p_1 := \begin{pmatrix} 0 & n - 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2(n - 2) & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3(n - 2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n - 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.
$$

Definition 3.4.2.
A $p_{-1}$-special Miura $\mathfrak{g}$-oper on $U^{\log}/T^{\log}$ is a Miura $\mathfrak{g}$-oper on $U^{\log}/T^{\log}$ of the form

\[(98)\]

$$
\hat{E}^{\bullet} := (E_{\mathfrak{B},U^{\log}/T^{\log}}, \nabla_{\mathfrak{B}}, \mathcal{E}_{\mathfrak{B},U^{\log}/T^{\log}}, \eta_{\mathfrak{B}})$$

(for some $T^{\log}$-connection $\nabla_{\mathfrak{B}}$ on $E_{\mathfrak{G},U^{\log}/T^{\log}}$) satisfying the equality $\eta^{\alpha} \circ \mathcal{KS}_{\hat{E}^{\bullet}}^{\mathfrak{B}} = \text{id}_{T^{\log}/T^{\log}}$ for any $\alpha \in \Gamma$, where $\mathcal{E}^{\bullet} := (E_{\mathfrak{B},U^{\log}/T^{\log}}, \nabla_{\mathfrak{B}})$. (In particular, any $p_{-1}$-special Miura $\mathfrak{g}$-oper is a generic Miura $\mathfrak{g}$-oper.)

Proposition 3.4.3.
For any generic Miura $\mathfrak{g}$-oper $\hat{E}^{\bullet} := (E_{\mathfrak{B}}, \nabla_{\mathfrak{B}}, \mathcal{E}_{\mathfrak{B}}, \eta_{\mathfrak{B}})$ on $U^{\log}/T^{\log}$, there exists a unique pair $(\hat{E}^{\bullet}, \iota_{\hat{E}^{\bullet}})$ consisting of a $p_{-1}$-special Miura $\mathfrak{g}$-oper on $U^{\log}/T^{\log}$ and an isomorphism $\iota_{\hat{E}^{\bullet}} : \hat{E}^{\bullet} \sim \hat{E}^{\bullet}$ of Miura $\mathfrak{g}$-opers.

Proof. Since Miura $\mathfrak{g}$-opers may be constructed by means of descent with respect to étale morphisms, we are (by taking account of Proposition 2.2.2) free to replace $U$ with its étale covering. Thus, we may assume that there exists a global section $\partial \in \Gamma(U, \Omega_{U^{\log}/T^{\log}})$ with $\mathcal{O}_U \cdot \partial = \mathcal{T}_{U^{\log}/T^{\log}}$. Also, we may assume that $E_{\mathfrak{B}} = U \times_k \mathbb{B}$ and $E_{\mathfrak{B}} = U \times_k \mathbb{B}$.

First, let us consider the existence assertion. It follows from [39], Lemma 2.2.4, that there exists a unique $U$-rational point $h : U \to \mathbb{T}$ of $\mathbb{T}$ such that the underlying $\mathfrak{g}$-oper $\mathcal{E}^{\bullet}$ of $\hat{E}^{\bullet}$ is of precanonical type relative to the triple $(U, x, I_h)$.
Strictly speaking, there exists an automorphism $\alpha$ determined by $h : U \to T$ ($\subseteq \mathbb{G}$). Let us write
\begin{equation}
\mathcal{E}^\mathbf{a} := (U \times_k \mathbb{B}, U^0_h(\nabla_E), U \times_k w_0\mathbb{B}, \text{id}_{U \times_k \mathbb{G}}),
\end{equation}
where $U^0_h(\nabla_E)$ denotes the $T^{\log}$-connection on $U \times_k \mathbb{G}$ obtained from $\nabla_E$ via pull-back by $h$. Then, the pair $(\mathcal{E}^\mathbf{a}, \iota_{\mathcal{E}^\mathbf{a}})$ specifies a desired pair, where $\iota_{\mathcal{E}^\mathbf{a}} := (h|_{U \times_k \mathbb{B}}, h|_{U \times_k w_0\mathbb{B}})$. This completes the existence assertion.

Next, let us prove the uniqueness assertion. To this end, it suffices to prove the claim that if $\mathcal{E}_1^\mathbf{a}$ and $\mathcal{E}_2^\mathbf{a}$ are $p_{-1}$-special Miura $\mathfrak{g}$-opers on $U^{\log}/T^{\log}$ admitting an isomorphism $(\alpha_{\mathfrak{g}}, \alpha_{\mathfrak{g}}') : \mathcal{E}_1^\mathbf{a} \to \mathcal{E}_2^\mathbf{a}$, then $\mathcal{E}_1^\mathbf{a}$ is identical to $\mathcal{E}_2^\mathbf{a}$. Since the automorphisms $\alpha_{\mathfrak{g}} : U \times_k \mathbb{B} \to U \times_k \mathbb{B}$ and $\alpha_{\mathfrak{g}}' : U \times_k w_0\mathbb{B} \to U \times_k w_0\mathbb{B}$ are compatible in the evident sense, they come from an automorphism of $U \times_k T$. Strictly speaking, there exists an automorphism $\alpha_T$ of $U \times_k T$ (i.e., a left-translation on $U \times_k T$ by some $U \to T$) which induces both $\alpha_{\mathfrak{g}}$ and $\alpha_{\mathfrak{g}}'$. But, because of the equalities $\eta^a \circ \mathcal{K} \mathcal{S}^a_{\mathfrak{g}} = \text{id}_{T^{\log}/T^{\log}}$ and $\eta^a \circ \mathcal{K} \mathcal{S}^a_{E_2^{\mathfrak{g}}} = \text{id}_{T^{\log}/T^{\log}}$ (for every $a \in \Gamma$), $\alpha_T$ must be the identity morphism (cf. the proof of [39], Lemma 2.2.4). Hence, we have $\mathcal{E}_1^\mathbf{a} = \mathcal{E}_2^\mathbf{a}$. This completes the proof of the uniqueness assertion, and hence, Proposition 3.4.4.

**Definition 3.4.4.**
For each generic Miura $\mathfrak{g}$-oper $\mathcal{E}^\mathbf{a}$ on $U^{\log}/T^{\log}$, we shall refer to the pair $(\mathcal{E}^\mathbf{a}, \iota_{\mathcal{E}^\mathbf{a}})$ obtained by applying Proposition 3.4.3 to $\mathcal{E}^\mathbf{a}$ as the $p_{-1}$-specialization of $\mathcal{E}^\mathbf{a}$.

### 3.5. From $\mathfrak{g}$-Cartan connections to special Miura $\mathfrak{g}$-opers.
In what follows, let us construct a bijective correspondence between the $\mathfrak{g}$-Cartan connections and the generic Miura $\mathfrak{g}$-opers (cf. Proposition 3.5.1). To begin with, we shall write
\begin{equation}
\iota : \tilde{T}_{\mathcal{E}_1^1 log}/T^{\log} \oplus \bigoplus_{\mathfrak{a} \in \Gamma} \mathfrak{g}_{E^1_1}^{-\mathfrak{a}} \to \tilde{T}_{\mathcal{E}_2^1 log}/T^{\log}
\end{equation}
for the $\mathfrak{O}_U$-linear morphism determined by the natural inclusions $\tilde{T}_{\mathcal{E}_1^1 log}/T^{\log} \hookrightarrow \tilde{T}_{\mathcal{E}_1^1 log}/T^{\log}$ (obtained by differentiating the inclusion $\mathcal{E}_1^1 \to \mathcal{E}_2^1$) and $\mathfrak{g}_{E^1_2}^{-\mathfrak{a}} \hookrightarrow \tilde{T}_{\mathcal{E}_1^1 log}/T^{\log}$ (for $\mathfrak{a} \in \Gamma$).

Now, let $\nabla : \mathcal{T}^{log}/T^{log} \to \tilde{T}_{\mathcal{E}_2^1 log}/T^{\log}$ be a $\mathfrak{g}$-Cartan connection on $U^{log}/T^{log}$. The composite
\begin{equation}
\nabla_E := \iota \circ (\nabla \oplus \bigoplus_{\mathfrak{a} \in \Gamma} \eta^\mathfrak{a}) : \mathcal{T}^{log}/T^{\log} \to \tilde{T}_{\mathcal{E}_2^1 log}/T^{\log}
\end{equation}
specifies a $T^{\log}$-connection, and the quadruple
\[ (\mathcal{E}^{\blacklozenge}_{\nabla}, \nabla \mathcal{E}, \mathcal{E}^!_{B}, \eta^!_{\mathcal{E}}) \]
forms a $p_{-1}$-special Miura $\mathfrak{g}$-oper on $U^{\log}/T^{\log}$.

Conversely, let us take a generic Miura $\mathfrak{g}$-oper $\mathcal{E}^{\blacklozenge}$ on $U^{\log}/T^{\log}$. Denote by $(\mathcal{E}^{\blacklozenge}_{\nabla}, \iota_{\mathcal{E}^{\blacklozenge}})$ the $p_{-1}$-specialization of $\mathcal{E}^{\blacklozenge}$. Also, denote by $(\mathcal{E}^!_{B}, \nabla \mathcal{E}^!_{B})$ the log flat $B$-torsor associated with $\mathcal{E}^{\blacklozenge}_{\nabla}$. Then, the composite
\[ \nabla \mathcal{E}^{\blacklozenge}_{\nabla} : T_{U^{\log}/T^{\log}} \xrightarrow{\nabla \mathcal{E}^!_{B}} T_{\mathcal{E}^{\log}_{\mathcal{E}}/T^{\log}} \rightarrow T_{\mathcal{E}^{\log}_{\mathcal{E}}/T^{\log}} \]
forms a $\mathfrak{g}$-Cartan connection on $U^{\log}/T^{\log}$, where the second arrow denotes the surjection arising from the quotient $B \rightarrow T$. The $\mathfrak{g}$-Cartan connection $\nabla \mathcal{E}^{\blacklozenge}_{\nabla}$ depends only on the isomorphism class of $\mathcal{E}^{\blacklozenge}$. Moreover, the following proposition holds.

**Proposition 3.5.1.**
The assignments $\nabla \mapsto \mathcal{E}^{\blacklozenge}_{\nabla}$ and $\mathcal{E}^{\blacklozenge} \mapsto \nabla \mathcal{E}^{\blacklozenge}$ discussed above define the following bijection:
\[ \left( \text{the set of } \mathfrak{g} \text{-Cartan connections on } U^{\log}/T^{\log} \right) \cong \left( \text{the set of isomorphism classes of generic Miura } \mathfrak{g} \text{-opers on } U^{\log}/T^{\log} \right). \]
Moreover, this bijection is functorial with respect to $S$.

**Proof.** The assertion follows immediately from the definitions of the assignments involved. \qed

### 3.6. Miura $\mathfrak{g}$-opers of prescribed exponents.

Let $S$ be a $k$-scheme and $\mathfrak{X} := (\mathfrak{X}/S, \{\sigma_i\}_{i=1}^r)$ an $r$-pointed stable curve over $S$ of genus $g$. Denote by $B^{-} (\subseteq G)$ the opposite Borel subgroup of $B$ relative to $T$, which is, by definition, a unique Borel subgroup satisfying that $B \cap B^{-} = T$. Now, suppose that $r > 0$, and let us fix $i \in \{1, \ldots, r\}$. By change of structure group, the composite isomorphism induces an isomorphism $\sigma^*(\mathcal{E}^!_{B}) \rightarrow S \times_k w_0B$, and hence, induces a sequence of isomorphisms
\[ \Gamma(S, \sigma^*_i(b_{\mathcal{E}^!_{B}})) \rightarrow \Gamma(S, b_{\sigma^*_i(\mathcal{E}^!_{B})}) \rightarrow \Gamma(S, b_{S \times_k w_0B}) \rightarrow b^{-}(S), \]
where $b^{-}$ denotes the Lie algebra of $B^{-}$. In particular, we obtain a surjection
\[ \Gamma(S, \sigma^*_i(b_{\mathcal{E}^!_{B}})) \rightarrow b^{-}(S) \rightarrow \mathfrak{t}(S). \]

Now, let $\mathcal{E}^{\blacklozenge} : (\mathcal{E}_B, \nabla \mathcal{E}, \mathcal{E}^!_B, \eta \mathcal{E})$ be a generic Miura $\mathfrak{g}$-oper on $\mathfrak{X}$. Denote by $(\mathcal{E}^{\blacklozenge}_{\nabla}, \iota_{\mathcal{E}^{\blacklozenge}_{\nabla}})$ the $p_{-1}$-specialization of $\mathcal{E}^{\blacklozenge}$. The monodromy of the flat $B$-torsor
associated with $\hat{E}^{\bullet}$ at each marked point $\sigma_i$ ($i = 1, \cdots, r$) is sent, via the composite (106), to an element

$$\varepsilon_i^{\hat{E}^{\bullet}} \in t(S).$$

**Definition 3.6.1.**

(i) For each $i \in \{1, \cdots, r\}$, we shall refer to $\varepsilon_i^{\hat{E}^{\bullet}}$ as the exponent of $\hat{E}^{\bullet}$ at $\sigma_i$.

(ii) Let $\vec{\varepsilon} := (\varepsilon_i)_{i=1}^r \in t(S)^{\times r}$, and let $\hat{E}^{\bullet}$ be a generic Miura $g$-oper on $X$. We shall say that $\hat{E}^{\bullet}$ is of exponents $\vec{\varepsilon}$ if $\varepsilon_i^{\hat{E}^{\bullet}} = \varepsilon_i$ for any $i \in \{1, \cdots, r\}$. If $r = 0$, then we shall refer to any generic Miura $g$-oper as being of exponent $\emptyset$.

**Proposition 3.6.2.**

Let $\vec{\varepsilon}^\prime$ be an element of $t(S)^{\times r}$ (where $\vec{\varepsilon}^\prime := \emptyset$ if $r = 0$). Then, the bijection (104) (of the case where $U^{\log}/T^{\log}$ is taken to be $X^{x-log}/S^{x-log}$) restricts to a functorial (with respect to $S$) bijection:

$$(\text{the set of } g\text{-Cartan connections on } X \text{ of monodromies } \vec{\varepsilon}^\prime) \Rightarrow \left( \text{the set of isomorphism classes of generic Miura } g\text{-opers on } X \text{ of exponents } \vec{\varepsilon}^\prime \right).$$

**Remark 3.6.3.**

Let $\hat{E}^{\bullet}$ be a generic Miura $g$-oper on $X$ of exponents $\vec{\varepsilon} := (\varepsilon_i)_{i=1}^r \in t(S)^{\times r}$, where each $\varepsilon_i$ ($i \in \{1, \cdots, r\}$) is supposed to be regular. Then, one verifies that the radii (cf. [39], Definition 2.9.2 (i)) of the underlying $g$-oper $E^{\bullet}$ of $\hat{E}^{\bullet}$ is given by

$$\chi(\vec{\varepsilon}) := (\chi(\varepsilon_i))_{i=1}^r \in \varepsilon(S)^{\times r}.$$ 

3.7. Moduli of generic Miura $g$-opers.

We shall write

$$\mathcal{M} \mathcal{Op}_{g,g,r}$$

for the set-valued contravariant functor on $\mathcal{S}ch_{/\mathcal{M}g,r}$ which, to any object $S \to \mathcal{M}g,r$ of $\mathcal{S}ch_{/\mathcal{M}g,r}$ classifying a pointed stable curve $\mathcal{X}$, assigns the set of isomorphism classes of generic Miura $g$-opers on $\mathcal{X}$. Also, for each $\vec{\varepsilon} \in t(k)^{\times r}$ (where $\vec{\varepsilon} := \emptyset$ if $r = 0$), we shall write

$$\mathcal{M} \mathcal{Op}_{g,g,r,\vec{\varepsilon}}$$
for the subfunctor of $\mathcal{M}_{\mathfrak{g},g,r}$ classifying generic Miura $\mathfrak{g}$-opers of exponents $\vec{\varepsilon}$. Then, the following proposition holds.

**Proposition 3.7.1.**
The functorial bijection (104) determines an isomorphism
\[ \Xi_{g,g,r,p} : \mathcal{C}O_{g,g,r} \sim \mathcal{M}_{\mathfrak{g},g,r} \]
over $\mathcal{M}_{g,r}$. Moreover, if $\vec{\varepsilon} \in \mathfrak{t}(k)^{\times r}$, then the isomorphisms (112) restricts to an isomorphism
\[ \Xi_{g,g,r,\vec{\varepsilon},p} : \mathcal{C}O_{g,g,r,\vec{\varepsilon}} \sim \mathcal{M}_{\mathfrak{g},g,r,\vec{\varepsilon}}. \]

In particular, both $\mathcal{M}_{\mathfrak{g},g,r}$ and $\mathcal{M}_{\mathfrak{g},g,r,\vec{\varepsilon}}$ (for any $\vec{\varepsilon}$) may be represented by Deligne-Mumford stacks over $k$.

**Proof.** The last assertion follows from Proposition 2.2.2 (i). □

**Remark 3.7.2.**
If $\text{char}(k) = 0$, then we have
\[ \mathcal{M}_{\mathfrak{sl}_2,g,0} \times_{\mathcal{M}_{g,0}} \mathcal{M}_{\mathfrak{g},0} \cong \mathcal{C}O_{\mathfrak{sl}_2,g,0} \cong \mathcal{C}O_{g,0} = \emptyset. \]
Indeed, it is well-known (cf. [12], Remark 1.2.10) that if $k$ is an algebraically closed field of characteristic zero and $X$ is a proper smooth curve over $k$ of genus $g > 1$, then the degree of any line bundle on $X$ admitting a $k$-connection must be 0. On the other hand, $\Omega_{X/k}$ has positive degree (more precisely, $\deg(\Omega_{X/k}) = 2g - 2 > 0$). It follows that there is no generic Miura $\mathfrak{sl}_2$-oper on $X$. That is to say, the stack $\mathcal{M}_{\mathfrak{sl}_2,g,0}$ turns out to be empty.

### 3.8. Dormant and $p$-nilpotent Miura opers.

In this subsection, we consider Miura $\mathfrak{g}$-opers in positive characteristic. Assume that either $(\text{Char})_p$ or $(\text{Char})_{\mathfrak{sl}_2}$ is satisfied.

Let $\mathcal{E}^\bullet_{\mathfrak{g}}$ be a generic Miura $\mathfrak{g}$-oper on a pointed stable curve $X$, $(\mathcal{E}^\bullet_{\mathfrak{g}}$, $\iota_{\mathcal{E}^\bullet_{\mathfrak{g}}})$ the specialization of $\mathcal{E}^\bullet_{\mathfrak{g}}$, and $(\mathcal{E}'$, $\nabla_{\mathcal{E}'})$ the log flat $\mathcal{B}$-torsor associated with $\mathcal{E}^\bullet_{\mathfrak{g}}$. Then, the assignment $\mathcal{E}^\bullet_{\mathfrak{g}} \mapsto \psi((\mathcal{E}'^\bullet, \nabla_{\mathcal{E}'})$ determines a morphism
\[ \kappa_{b,g,r} : \mathcal{M}_{\mathfrak{g},g,r} \rightarrow \mathfrak{O}_{b,g,r} \]
over $\mathcal{M}_{g,r}$. We shall write
\[ \kappa_{b,g,r}^{\mathfrak{H}} : \mathcal{M}_{\mathfrak{g},g,r} \rightarrow \mathfrak{O}_{\mathfrak{t},g,r} \]
for the composite $\otimes_{b \rightarrow \mathfrak{t},g,r} \circ \kappa_{b,g,r}$ (cf. (56)).
Denote by
\[ M_{op}^{Z_{\text{as...}} g, g, r} \] (resp., \( M_{op}^{p-nilp g, g, r} \))
the closed substack of \( \mathcal{M}_{g, r} \) defined to be the inverse image of the zero section \([0]_{g, r}\) (resp., \([0]_{g, r}\)) (cf. (57)) via the morphism \( \kappa_{b, g, r} \) (resp., \( \kappa_{H-M g, r} \)).

Also, for each \( \vec{\epsilon} \in t(k)^r \) (where \( \vec{\epsilon} := \emptyset \) if \( r = 0 \)), we shall write
\[ M_{op}^{Z_{\text{as...}} g, g, r, \vec{\epsilon}} := M_{op}^{Z_{\text{as...}} g, g, r} \times_{\mathcal{M}_{g, r}} M_{op}^{g, g, r, \vec{\epsilon}} \] (resp., \( M_{op}^{p-nilp g, g, r, \vec{\epsilon}} := M_{op}^{p-nilp g, g, r} \times_{\mathcal{M}_{g, r}} M_{op}^{g, g, r, \vec{\epsilon}} \)).

**Definition 3.8.1.**
Let \( S \) be a \( k \)-scheme and \( X := (X/S, \{\sigma_i\}_{i=1}^r) \) an \( r \)-pointed stable curve over \( S \) of genus \( g \). We shall say that a generic Miura \( g \)-oper on \( X \) is **dormant** (resp., \( p \)-nilpotent) if it is classified by \( M_{op}^{Z_{\text{as...}} g, g, r} \) (resp., \( M_{op}^{p-nilp g, g, r} \)). In other words, a dormant (resp., \( p \)-nilpotent) generic Miura \( g \)-oper is a generic Miura \( g \)-oper whose underlying \( g \)-oper is dormant (resp., \( p \)-nilpotent) (cf. [39], Definition 3.6.1 and Definition 3.8.3).

**Remark 3.8.2.**
In [39], §3.12, the moduli stack
\[ \mathcal{D}_{g, r, \vec{\rho}} \] (resp., \( \mathcal{D}_{g, r, \vec{\rho}}^{Z_{\text{as...}} g, g, r} \); resp., \( \mathcal{D}_{g, r, \vec{\rho}}^{p-nilp g, g, r} \)),
where \( \vec{\rho} \in c(k)^r \), classifying \( r \)-pointed genus \( g \) stable curves over \( k \) together with a \( g \)-oper (resp., a dormant \( g \)-oper; resp., a \( p \)-nilpotent \( g \)-oper) of radii \( \vec{\rho} \) was introduced. (Notice that in loc. cit., we used the notations \( D_{g, r, \vec{\rho}} \), \( D_{g, r, \vec{\rho}}^{Z_{\text{as...}} g, g, r} \), and \( D_{g, r, \vec{\rho}}^{p-nilp g, g, r} \) to denote these moduli stacks respectively. We refer to [39] for the study of these moduli stacks.

Let \( \vec{\epsilon} := (\epsilon_i)_{i=1}^r \in t(k)^r \) (where \( \vec{\epsilon} := \emptyset \) if \( r = 0 \)), and suppose that each \( \epsilon_i \) (\( i \in \{1, \cdots, r\} \)) is regular. According to the discussion in Remark 3.6.3, the assignment from each generic Miura \( g \)-oper to its underlying \( g \)-oper determines a morphism
\[ g, g, r, \vec{\epsilon} : (\mathcal{M}_{g, r, \vec{\epsilon}}^{Z_{\text{as...}} g, g, r, \vec{\epsilon}} \xrightarrow{\cong} \mathcal{M}_{g, r, \vec{\epsilon}}^{p-nilp g, g, r, \vec{\epsilon}}) \to \mathcal{M}_{g, r, \vec{\epsilon}} \to \mathcal{D}_{g, r, \vec{\epsilon}} \]
over \( \mathcal{M}_{g, r} \). This morphism restricts to morphisms
\[ g, g, r, \vec{\epsilon} : M_{op}^{Z_{\text{as...}} g, g, r, \vec{\epsilon}} \to M_{op}^{p-nilp g, g, r, \vec{\epsilon}} \]
We shall refer to \( g, g, r, \vec{\epsilon} \) (resp., \( g, g, r, \vec{\epsilon}^{\text{as...}}, \vec{\epsilon}^{\text{as...}}, \vec{\epsilon}^{\text{p-nilp}}, \vec{\epsilon}^{\text{p-nilp}} \)) as the **universal Miura transformation** (resp., the **universal dormant Miura transformation**;
resp., the universal $p$-nilpotent Miura transformation. Moreover, these morphisms fit into the following commutative diagram:

$$
\begin{array}{ccc}
\mathcal{MP}^{\text{zar}}_{g,g,r,z} & \longrightarrow & \mathcal{MP}^{\text{p-nilp}}_{g,g,r,z} \\
\downarrow & & \downarrow \\
\mathcal{MP}^{\text{zar}}_{g,g,r,x(z)} & \longrightarrow & \mathcal{MP}^{\text{p-nilp}}_{g,g,r,x(z)}
\end{array}
$$

(122)

where all the horizontal arrows are closed immersions and both sides of square diagrams are cartesian.

**Theorem 3.8.3.**

(i) The isomorphism $\Xi^{p-\text{nilp}}_{g,g,r,z,p-1}$ restricts to an isomorphism

$$
\Xi^{p-\text{nilp}}_{g,g,r,z,p-1} : \mathcal{CTo}^{p-\text{nilp}}_{g,g,r,z} \sim \mathcal{MP}^{p-\text{nilp}}_{g,g,r,z}.
$$

In particular, both $\mathcal{MP}^{p-\text{nilp}}_{g,g,r,z}$ and $\mathcal{MP}^{\text{zar}}_{g,g,r,z}$ are finite over $\mathcal{M}_{g,r}.$

(ii) We have the decomposition

$$
\mathcal{MP}^{\text{zar}}_{g,g,r,z} = \bigoplus_{\bar{z} \in t(F_p)^{\times r}} \mathcal{MP}^{\text{zar}}_{g,g,r,z}(\bar{z}).
$$

In particular, $\mathcal{MP}^{\text{zar}}_{g,g,r,z}$ is empty unless $\bar{z} \in t(F_p)^{\times r}$ (or $\bar{z} = \emptyset$).

**Proof.** Let us consider assertion (i). By the various definitions involved, the square diagram

$$
\begin{array}{ccc}
\mathcal{CTo}^{p-\text{nilp}}_{g,g,r,z} & \longrightarrow & \mathcal{MP}^{p-\text{nilp}}_{g,g,r,z} \\
\downarrow & & \downarrow \\
\mathcal{CTo}^{p-\text{nilp}}_{g,g,r,z} \ast & \ast & \mathcal{MP}^{p-\text{nilp}}_{g,g,r,z}
\end{array}
$$

(125)

is commutative, where $\Psi^{p-\text{nilp}}_{g,g,r,z} := \Psi^{p-\text{nilp}}_{g,g,r,z} \ast$ (cf. (68)) and $\Psi^{p-\text{nilp}}_{g,g,r,z} := \Psi^{p-\text{nilp}}_{g,g,r,z} \ast$ (cf. (115)). On the other hand, $\mathcal{CTo}^{p-\text{nilp}}_{g,g,r,z}$ and $\mathcal{MP}^{p-\text{nilp}}_{g,g,r,z}$ are the inverse images of $\otimes_{t,g,r}^{-1}(\{0\}_{t,g,r})$ (which is a closed substack of $\otimes_{t,g,r}^{-1}$) via $\Psi^{p-\text{nilp}}_{g,g,r,z}$ and $\otimes_{b \rightarrow t,g,r} \ast$ respectively. Hence, by the commutativity of (125), the isomorphism $\Xi^{p-\text{nilp}}_{g,g,r,z,p-1}$ restricts to an isomorphism $\Xi^{p-\text{nilp}}_{g,g,r,z,p-1}$ as of (123). Moreover, since $\mathcal{MP}^{p-\text{nilp}}_{g,g,r,z}$ is a closed substack of $\mathcal{MP}^{\text{zar}}_{g,g,r,z},$ it follows from the isomorphism $\Xi^{p-\text{nilp}}_{g,g,r,z,p-1}$ and the result in Corollary 2.6.2 that both $\mathcal{MP}^{p-\text{nilp}}_{g,g,r,z}$ and $\mathcal{MP}^{\text{zar}}_{g,g,r,z}$ are finite over $\mathcal{M}_{g,r}.$ This completes the proof of assertion (i).

Assertion (ii) follows directly from assertion (i) and Corollary 2.5.3. □
3.9. From Miura $\mathfrak{sl}_2$-opers to Miura $\mathfrak{g}$-opers.

We shall construct a morphism from the moduli stack of generic Miura $\mathfrak{sl}_2$-opers to the moduli stack of generic $\mathfrak{g}$-opers.

Let us write
\[
\begin{align*}
p^0_0 &:= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & \quad p^0 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & \quad p^0_{-1} &:= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\end{align*}
\]

The $\mathfrak{sl}_2$-triple $\{p_{-1}, 2\hat{\rho}, p_1\}$ given in §3.4 determines a Lie algebra homomorphism
\[
(\mathfrak{g} \rightarrow \mathfrak{g})
\]
in such a way that $\iota_{\mathfrak{g}}(p^0_0) = 2\hat{\rho}$, $\iota_{\mathfrak{g}}(p^1_0) = p_1$, and $\iota_{\mathfrak{g}}(p^0_{-1}) = p_{-1}$. It follows from [39], Proposition 2.5.1, that there exists a unique homomorphism $\iota_{\mathfrak{g}} : \mathbb{B}_{\text{PGL}_2} \rightarrow \mathbb{B}$ of algebraic groups that is compatible, in the natural sense, with $\iota_{\mathfrak{g}}$. Then, $\iota_{\mathfrak{g}}$ restricts to a homomorphism $\iota_{\mathfrak{T}} : \mathbb{T}_{\text{PGL}_2} \rightarrow \mathbb{T}$.

Let $\mathcal{T}^\log$ be an fs log scheme over $k$ and $\mathcal{U}^{\log}$ a log-curve over $\mathcal{T}^{\log}$. By the definition of $\mathcal{E}_{\mathfrak{T}}^{\mathfrak{T}} := \mathcal{U}_{\mathfrak{T}^{\log}}^{\log}/\mathcal{T}^{\log}$, there exists a canonical isomorphism $\mathcal{E}_{\mathfrak{T}_{\text{PGL}_2}}^{\mathfrak{T}} \times_{\mathbb{T}} \mathcal{T} \sim \mathcal{E}_{\mathfrak{T}}^{\mathfrak{T}}$. Moreover, the collection of data $(\mathcal{E}_{\mathfrak{B}_{\text{PGL}_2}}^{\mathfrak{T}}, \mathcal{E}_{\mathfrak{g}_{\text{PGL}_2}}^{\mathfrak{T}}, \eta_{\mathfrak{g}_{\text{PGL}_2}}^{\mathfrak{T}})$ becomes $(\mathcal{E}_{\mathfrak{B}}^{\mathfrak{B}}, \mathcal{E}_{\mathfrak{g}}^{\mathfrak{g}}, \eta_{\mathfrak{g}}^{\mathfrak{g}})$ after change of structure group by $\iota_{\mathfrak{T}}$.

Now, let us fix $\varepsilon := (\varepsilon_t)_{t=1}^r \in \mathfrak{t}_{\text{PGL}_2}(k)^{\times r}$ if $r > 0$ (resp., $\varepsilon := \emptyset$ if $r = 0$). Write $\iota_{\mathfrak{g}}(\varepsilon) := (\iota_{\mathfrak{g}}(\varepsilon_t))_{t=1}^r \in \mathfrak{t}(k)^{\times r}$ (resp., $\iota_{\mathfrak{g}}(\varepsilon) := \emptyset$). Suppose further that we are given a $p_{-1}$-special Miura $\mathfrak{g}$-oper $\hat{\mathcal{E}}^{\mathfrak{g}} := (\mathcal{E}_{\mathfrak{B}_{\text{PGL}_2}}^{\mathfrak{B}}, \nabla_{\mathfrak{g}}, \mathcal{E}_{\mathfrak{g}_{\text{PGL}_2}}^{\mathfrak{g}}, \eta_{\mathfrak{g}_{\text{PGL}_2}}^{\mathfrak{g}})$ on $\mathcal{U}_{\mathfrak{T}}^{\log}/\mathcal{T}^{\log}$ of exponents $\varepsilon$. According to [39], the discussion following Definition 2.7.1, the underlying $\mathfrak{sl}_2$-oper $(\mathcal{E}_{\mathfrak{B}_{\text{PGL}_2}}^{\mathfrak{B}}, \nabla_{\mathfrak{g}})$ induces a $\mathfrak{g}$-oper $(\mathcal{E}_{\mathfrak{B}}^{\mathfrak{B}}, \iota_{\mathfrak{g}}(\nabla_{\mathfrak{g}}))$ on $\mathcal{U}_{\mathfrak{T}}^{\log}/\mathcal{T}^{\log}$ by change of structure group. Moreover, the collection of data
\[
\iota_{\mathfrak{g}}(\hat{\mathcal{E}}^{\mathfrak{g}}) := (\mathcal{E}_{\mathfrak{B}}^{\mathfrak{B}}, \iota_{\mathfrak{g}}(\nabla_{\mathfrak{g}}), \mathcal{E}_{\mathfrak{g}}^{\mathfrak{g}}, \eta_{\mathfrak{g}}^{\mathfrak{g}})
\]
forms a $p_{-1}$-spacial Miura $\mathfrak{g}$-oper on $\mathfrak{X}$ of exponents $\iota_{\mathfrak{g}}(\varepsilon)$.

**Proposition 3.9.1.**

Let $\varepsilon$ be as above. Then, the assignment $\hat{\mathcal{E}}^{\mathfrak{g}} \mapsto \iota_{\mathfrak{g}}(\hat{\mathcal{E}}^{\mathfrak{g}})$ is compatible with base-change over $S$, and hence, determines a morphism
\[
\iota_{\mathfrak{g}}^{\mathfrak{g}} : \mathfrak{M}_{\mathfrak{B}_{\text{PGL}_2}, g, r, \varepsilon} \rightarrow \mathfrak{M}_{\mathfrak{B}, g, r, \varepsilon}(\varepsilon)
\]
over $\mathfrak{M}_{g, r}$, which restricts to a morphism
\[
\iota_{\mathfrak{g}}^{\mathfrak{g}} : \mathfrak{M}_{\mathfrak{B}_{\text{PGL}_2}, g, r, \varepsilon} \rightarrow \mathfrak{M}_{\mathfrak{B}, g, r, \varepsilon}(\varepsilon).
\]

In particular, $\mathfrak{M}_{\mathfrak{B}_{\text{PGL}_2}, g, r, \varepsilon}(\varepsilon)$ is nonempty if $\mathfrak{M}_{\mathfrak{B}, g, r, \varepsilon}(\varepsilon)$ is nonempty (cf. Remark 3.3.3, in which we will discuss the case where $\mathfrak{M}_{\mathfrak{B}_{\text{PGL}_2}, g, r, \varepsilon}(\varepsilon)$ is nonempty).

**Proof.** The assertion follows from the above discussion and [39], Proposition 3.2.3.

\[\square\]
In this section, we describe Miura $\mathfrak{sl}_n$-opers in terms of vector bundles. Let us fix a positive integer $n$, and suppose that the characteristic $\text{char}(k)$ of $k$ is either 0 or a prime $p$ with $n < p$. Given a vector bundle $\mathcal{F}$ on a scheme, an integer $l$ with $0 \leq l \leq n - 1$, and an $n$-step decreasing filtration $f := \{\mathcal{F}^j\}_{j=0}^n$ on $\mathcal{F}$, we shall write $\text{gr}^l_f := \mathcal{F}^l / \mathcal{F}^{l+1}$. Also, write $\text{gr}^f := \bigoplus_{l=0}^{n-1} \text{gr}^l_f$.

4.1. $\text{GL}_n$-opers on a log-curve.

We first recall the definition of a $\text{GL}_n$-oper on a log-curve (cf. [39], Definition 4.3.1). Let $T^\text{log}$ be an fs log scheme over $k$, and $U^\text{log}$ a log-curve over $T^\text{log}$.

**Definition 4.1.1.**

(i) A $\text{GL}_n$-oper on $U^\text{log}/T^\text{log}$ is a collection of data $\mathcal{F}^\varnothing := (\mathcal{F}, \nabla_\mathcal{F}, f)$, where

- $\mathcal{F}$ denotes a vector bundle on $U$ of rank $n$;
- $\nabla_\mathcal{F}$ denotes a $T^\text{log}$-connection $\mathcal{F} \to \Omega_{U^\text{log}/T^\text{log}} \otimes \mathcal{F}$ on $\mathcal{F}$;
- $f$ denotes an $n$-step decreasing filtration $\{\mathcal{F}^j\}_{j=0}^n$ on $\mathcal{F}$ consisting of subbundles $0 = \mathcal{F}^n \subseteq \mathcal{F}^{n-1} \subseteq \cdots \subseteq \mathcal{F}^0 = \mathcal{F}$, satisfying the following three conditions:
  1. The subquotients $\text{gr}^l_f (= \mathcal{F}^j / \mathcal{F}^{j+1})$ ($0 \leq j \leq n-1$) are line bundles;
  2. $\nabla_\mathcal{F}(\mathcal{F}^j) \subseteq \Omega_{U^\text{log}/T^\text{log}} \otimes \mathcal{F}^{j-1}$ ($1 \leq j \leq n-1$);
  3. The $\mathcal{O}_X$-linear morphism

\begin{equation}
\mathcal{K}S^j_{\mathcal{F}^\varnothing} : \text{gr}^l_f \to \Omega_{U^\text{log}/T^\text{log}} \otimes \text{gr}^{l-1}_f
\end{equation}

defined by assigning $\sigma \mapsto \nabla_{\mathcal{F}^j}(\sigma)$ for any local section $\sigma \in \mathcal{F}^j$ (where $(-)'$'s denote the image in the respective quotients), which is well-defined by virtue of the second condition, is an isomorphism.

If $U^\text{log}/T^\text{log} = X^{\text{X-log}} / S^{X-\text{log}}$ for some pointed stable curve $\mathcal{X} := (X/S, \{\sigma_i\}_{i=1}^r)$, then we shall refer to any $\text{GL}_n$-oper on $X^{\text{X-log}} / S^{X-\text{log}}$ as a $\text{GL}_n$-oper on $\mathcal{X}$.

(ii) Let $\mathcal{F}^\varnothing := (\mathcal{F}, \nabla_\mathcal{F}, f)$ and $\mathcal{G}^\varnothing := (\mathcal{G}, \nabla_\mathcal{G}, g)$ be $\text{GL}_n$-opers on $U^\text{log}/T^\text{log}$. An **isomorphism of $\text{GL}_n$-opers** from $\mathcal{F}^\varnothing$ to $\mathcal{G}^\varnothing$ is an isomorphism $(\mathcal{F}, \nabla_\mathcal{F}) \to (\mathcal{G}, \nabla_\mathcal{G})$ of flat bundles compatible with the respective filtrations $f$ and $g$. 
4.2. Miura GL\(_n\)-opers.

Next, we define the notion of a Miura GL\(_n\)-oper. Let us consider the triple
\[(\mathcal{F}, f, f^-)\]
consisting of a rank \(n\) vector bundle \(\mathcal{F}\) on \(U\) and two \(n\)-step decreasing filtrations \(f := \{\mathcal{F}^j\}_{j=0}^n, f^- := \{\mathcal{F}^{-j}\}_{j=0}^n\) on \(\mathcal{F}\) such that for each \(j \in \{0, \cdots, n\}\), the composite \(\alpha_j : \mathcal{F}^{-j} \hookrightarrow \mathcal{F} \to \mathcal{F}/\mathcal{F}^{-n-j}\) is an isomorphism, equivalently, the composite \(\beta_j : \mathcal{F}^j \hookrightarrow \mathcal{F} \to \mathcal{F}/\mathcal{F}^{-n-j}\) is an isomorphism. The composite
\[(\mathcal{F}, f, f^-) \sim \mathcal{F}/\mathcal{F}^{-n-j} \to \mathcal{F}/\mathcal{F}^{-n-j} \oplus \mathcal{F}^{-j} \sim \mathcal{F}^{-j}
\]
(where \(j \in \{0, \cdots, n\}\)) determines a split surjection of the short exact sequence
\[0 \to \mathcal{F}^{-j} \to \mathcal{F}/\mathcal{F}^{-n-j} \to \text{gr} f^{-j-1} \to 0,\]
which gives rise to a decomposition
\[\gamma_j : \mathcal{F}^{-j-1} \sim \mathcal{F}^{-j} \oplus \text{gr} f^{-j-1}.
\]
Also, this fact implies that the kernel of the surjection \(\mathcal{F}/\mathcal{F}^{-n-j} \to \mathcal{F}/\mathcal{F}^{-n-j}\)
(appeared as the second arrow in \((135)\)) is isomorphic to \(\text{gr} f^{-j-1}\). That is to say, we have a canonical isomorphism
\[\delta_{j,f,f^-} : \text{gr} f^{-j} \sim \text{gr} f^{-j-1}.
\]
Notice that \(\mathcal{F}\) decomposes into the direct sum of \(n\) line bundles \(\{\text{gr} f^{-j}\}_{j=0}^{n-1}\) by means of the composite isomorphism
\[\gamma_{(\mathcal{F}, f, f^-)} : \mathcal{F} \sim \mathcal{F}^{-1} \oplus \text{gr} f^{-1} \sim \mathcal{F}^{-2} \oplus \text{gr} f^{-2} \oplus \text{gr} f^{-1} \sim \cdots \sim \bigoplus_{l=0}^{n-1} \text{gr} f^{-l} \quad (= \text{gr} f^{-\cdot}),
\]
where the \(j\)-th isomorphism (for each \(j \in \{0, \cdots, n\}\)) arises from \(\gamma_j\). In the following, we shall consider \(\mathcal{F}\) as being equipped with a grading (indexed by \(\{0, \cdots, n-1\}\)) by means of \(\gamma_{(\mathcal{F}, f, f^-)}\), i.e., a grading whose \(j\)-th component is \(\text{gr} f^{-j}\).

Conversely, let us consider a rank \(n\) vector bundle \(\mathcal{F}\) on \(U\) of the form \(\mathcal{F} = \bigoplus_{l=0}^{n-1} \mathcal{F}_l,\) where each \(\mathcal{F}_l\) \((l \in \{0, \cdots, n-1\})\) is a line bundle. For each subset \(I\) of \(\{0, \cdots, n-1\}\), we shall consider \(\bigoplus_{l \in I} \mathcal{F}_l\) as an \(\mathcal{O}_U\)-submodule of \(\mathcal{F}\) in the evident manner. For each \(j \in \{0, \cdots, n-1\}\), we shall write
\[\mathcal{F}^j := \bigoplus_{l=0}^{n-1-j} \mathcal{F}_l, \quad \mathcal{F}^{-j} := \bigoplus_{l=j}^{n-1} \mathcal{F}_l.
\]
and write \(\mathcal{F}^n := 0, \mathcal{F}^{-n} := 0\). The vector bundle \(\mathcal{F}\) admits two decreasing filtrations \(f, f^-\) on \(\mathcal{F}\) defined as follows:
\[f := \{\mathcal{F}^j\}_{j=0}^n, \quad f^- := \{\mathcal{F}^{-j}\}_{j=0}^n.
\]
In particular, the composite $\mathcal{F}^{-j} \hookrightarrow \mathcal{F} \twoheadrightarrow \mathcal{F}/\mathcal{F}^{n-j}$ (for each $j \in \{0, \cdots, n\}$) is an isomorphism. In this manner, one may identify each triple $(\mathcal{F}, f, f^-)$ as in (134) with a rank $n$ vector bundle decomposing into a direct sum of $n$ line bundles.

**Definition 4.2.1.**

(i) A (generic) Miura GL$_n$-oper on $U^\text{log}/T^\text{log}$ is a quadruple

$$\hat{\mathcal{F}}^\triangleright := (\mathcal{F}, \nabla_\mathcal{F}, f, f^-),$$

where the triple $(\mathcal{F}, f, f^-)$ is as in (134) and $\nabla_\mathcal{F}$ denotes a $T^\text{log}$-connection on $\mathcal{F}$, such that

$$\nabla_\mathcal{F}(\mathcal{F}^{-j}) \subseteq \Omega_{U^\text{log}/T^\text{log}} \otimes (\mathcal{F}^{-j})$$

for any $j \in \{0, \cdots, n\}$ and the triple

$$\mathcal{F}^\triangleright := (\mathcal{F}, \nabla_\mathcal{F}, f)$$

forms a GL$_n$-oper on $U^\text{log}/T^\text{log}$. We shall refer to $\mathcal{F}^\triangleright$ as the underlying GL$_n$-oper of $\hat{\mathcal{F}}^\triangleright$. If $U^\text{log}/T^\text{log} = X^\text{log}/S^\text{log}$ for some pointed stable curve $\hat{X} := (X/S, \{\sigma_i\}_{i=1}^n)$, then we shall refer to any (generic) Miura GL$_n$-oper on $X^\text{log}/S^\text{log}$ as a (generic) Miura GL$_n$-oper on $\hat{X}$.

(ii) Let $\hat{\mathcal{F}}^\triangleright := (\mathcal{F}, \nabla_\mathcal{F}, f, f^-)$ and $\hat{\mathcal{F}}^\triangleright := (\mathcal{F}', \nabla_\mathcal{F}', f', f'^-)$ be Miura GL$_n$-opers on $U^\text{log}/T^\text{log}$. An isomorphism of (generic) Miura GL$_n$-opers from $\hat{\mathcal{F}}^\triangleright$ to $\hat{\mathcal{F}}^\triangleright$ is an isomorphism $\alpha : (\mathcal{F}, \nabla_\mathcal{F}) \simto (\mathcal{F}', \nabla_\mathcal{F}')$ of log flat bundles which restricts to isomorphisms $\alpha|_{\mathcal{F}^j} : \mathcal{F}^j \simto \mathcal{F}'^j$ and $\alpha|_{\mathcal{F}^{-j}} : \mathcal{F}^{-j} \simto F'^{-j}$ for any $j \in \{0, \cdots, n\}$.

**Definition 4.2.2.**

Assume that $k$ has characteristic $p$ with $n < p$. We shall say that a GL$_n$-oper $\mathcal{F}^\triangleright := (\mathcal{F}, \nabla_\mathcal{F}, f)$ (resp., a Miura GL$_n$-oper $\hat{\mathcal{F}}^\triangleright := (\mathcal{F}, \nabla_\mathcal{F}, f, f^-)$) on $\hat{X}$ is dormant if $\psi(\mathcal{F}, \nabla_\mathcal{F}) = 0$.

**Remark 4.2.3.**

If $\hat{\mathcal{F}}^\triangleright := (\mathcal{F}, \nabla_\mathcal{F}, f, f^-)$ is a Miura GL$_n$-oper on $U^\text{log}/T^\text{log}$. By the condition (142), the $T^\text{log}$-connection $\nabla_\mathcal{F}$ induces, for each $l \in \{0, \cdots, n-1\}$, a $T^\text{log}$-connection

$$\text{gr}_{f^-}(\nabla_\mathcal{F}) : \text{gr}_{f^-} \rightarrow \Omega_{U^\text{log}/T^\text{log}} \otimes \text{gr}_{f^-}$$

on the line bundle $\text{gr}_{f^-}$. In particular, we have a log flat bundle

$$\text{gr}_{f^-}(\hat{\mathcal{F}}^\triangleright) := (\text{gr}_{f^-}, \bigoplus_{l=0}^{n-1} \text{gr}_{f^-}(\nabla_\mathcal{F})) = \bigoplus_{l=0}^{n-1} (\text{gr}_{f^-}, \text{gr}_{f^-}(\nabla_\mathcal{F}))$$
and satisfies the following two conditions:

\[ \text{Miura GL} \]

Let \( \alpha \) be another Miura GL-opper on \( U^{\log}/T^{\log} \) and \( \alpha : \mathcal{F}^{\diamond} \rightarrow \tilde{\mathcal{F}}^{\diamond} \) an isomorphism of Miura GL-opers. By taking the gradings \( \text{gr}_F \) and \( \text{gr}_{F^\dagger} \) of \( F^- \) and \( F^\dagger \) respectively, we obtain an isomorphism

\[ \text{gr}(\alpha) : \text{gr}_F(\tilde{\mathcal{F}}^{\diamond}) \rightarrow \text{gr}_{F^\dagger}(\tilde{\mathcal{F}}^{\diamond}) \]

of log flat bundles which makes the following square diagram of vector bundles commute:

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\alpha} & \mathcal{F}' \\
\downarrow{\gamma_{(F,F^-)}} & & \downarrow{\gamma_{(F',F'^-)}} \\
\text{gr}_F & \xrightarrow{\sim} & \text{gr}_{F^\dagger}
\end{array}
\]

(147)

4.3. Special Miura GL\(_n\)-opers.

Let \( U^{\log}/T^{\log} \) be as before and \( \mathcal{N} \) a line bundle on \( U \). Recall the rank \( n \) vector bundle \( \mathcal{F}_{\mathcal{N}}^{[n] \dagger} := \bigoplus_{j=0}^{n-1} \mathcal{F}_{\mathcal{N},j}^{\dagger} \) (cf. (55)). We shall write

\[ (\mathcal{F}_{\mathcal{N}}^{[n] \dagger}, f_{\mathcal{N}}, f_{\mathcal{N}}^\dagger) \]

for the triple corresponding (via the identification mentioned in the italicized comment preceding Definition 4.2.1) to the direct sum \( \mathcal{F}_{\mathcal{N}}^{[n] \dagger} \) of \( n \) line bundles (cf. (55)). In particular,

\[ \text{gr}_{\mathcal{F}_{\mathcal{N}}^{[n] \dagger}} := \mathcal{F}_{\mathcal{N},n-j-1}^{\dagger} = \mathcal{T}_{U^{\log}/T^{\log}}^{\dagger \times (n-j-1)} \otimes \mathcal{N}, \quad \text{gr}_{\mathcal{F}_{\mathcal{N}}^{[n] \dagger}} := \mathcal{F}_{\mathcal{N},j}^\dagger = \mathcal{T}_{U^{\log}/T^{\log}}^\dagger \otimes \mathcal{N}.
\]

Definition 4.3.1.

Let \( \mathcal{N}^\diamond := (\mathcal{N}, \nabla_\mathcal{N}) \) be a log flat line bundle on \( U^{\log}/T^{\log} \). We shall say that a Miura GL\(_n\)-opper on \( U^{\log}/T^{\log} \) is \( \mathcal{N}^\diamond \)-special if it is of the form

\[ \tilde{\mathcal{F}}_{\mathcal{N}}^{\diamond} := (\mathcal{F}_{\mathcal{N}}^{[n] \dagger}, \nabla_\mathcal{F}, f_{\mathcal{N}}^\dagger, f_{\mathcal{N}}^\dagger) \]

and satisfies the following two conditions:

- The equality \( \mathcal{N}^\diamond = (\text{gr}_{f_{\mathcal{F}}} \otimes \text{gr}_{f_{\mathcal{F}}}(\nabla_\mathcal{F})) \) holds;
- For each \( j \in \{0, \cdots, n-1\} \), the automorphism of \( \mathcal{T}_{U^{\log}/T^{\log}}^{\otimes (n-j-1)} \otimes \mathcal{N} \) obtained from the isomorphism

\[ \mathcal{KS}_{\tilde{\mathcal{F}}_{\mathcal{N}}^{\diamond}}^j : \text{gr}_{\mathcal{F}_{\mathcal{N}}^{[n] \dagger}}^j \rightarrow \mathcal{O}_{U^{\log}/T^{\log}} \otimes \text{gr}_{\mathcal{F}_{\mathcal{N}}^{[n] \dagger}}^{j-1} \quad (\text{cf. (133)}) \]

(cf. (133)) coincides with the identity morphism, where \( \mathcal{F}_{\mathcal{N}}^{\diamond} \) denotes the underlying GL\(_n\)-opper of \( \tilde{\mathcal{F}}_{\mathcal{N}}^{\diamond} \).
Remark 4.3.2.
If $\overset{\circ}{F}^\circ := (F_n^{[1]}, \nabla_F, f^t, f^{-1})$ is an $N^\circ$-special Miura $GL_n$-oper, then the log flat bundle $gr_{f^{-1}}(\overset{\circ}{F}^\circ)$ (cf. Remark [4.2.3]) forms a $(GL_n, N^\circ)$-Cartan connection.

Proposition 4.3.3.
Let $\overset{\circ}{F}^\circ := (F, \nabla_F, f, f^{-})$ be a Miura $GL_n$-oper on $U^\log/T^\log$. Write $N := gr^0_f$ and $N^\circ := (N, gr^0_f(\nabla_F))$. Then, there exists a unique pair $(\overset{\circ}{F}^\circ, \iota_{\overset{\circ}{F}^\circ})$ consisting of

- an $N^\circ$-special Miura $GL_n$-oper $\overset{\circ}{F}^\circ := (F_N^{[1]}, \nabla_F, f^t, f^{-1})$ on $U^\log/T^\log$, and
- an isomorphism $\iota_{\overset{\circ}{F}^\circ} : \overset{\circ}{F}^\circ \sim \overset{\circ}{F}^\circ$ of Miura $GL_n$-opers such that the automorphism of $N^\circ$ defined as the 0-th component of $gr(\iota_{\overset{\circ}{F}^\circ})$ coincides with the identity morphism.

Proof. First, we shall consider the existence assertion. Let us fix $j \in \{1, \cdots, n-1\}$. For each $l \in \{0, \cdots, j-1\}$, we shall write $t^l_j$ for the composite

$$t^l_j : T_{U^\log/T^\log} \otimes gr^0_f \to T_{U^\log/T^\log} \otimes gr^0_f$$

where the first and third arrow denote $id_{T_{U^\log/T^\log} \otimes \delta^{-1}_{j-l+1, f^{-}}} \otimes \delta_{j-l, f^{-}}$ respectively and the second arrow denotes $id_{T_{U^\log/T^\log} \otimes (KS_{\overset{\circ}{F}^\circ})^{-1}}$. Then, we obtain the following composite isomorphism

$$t_j := t^{j-1}_j \circ t^{j-2}_j \circ \cdots \circ t^0_j : gr^0_f \to T_{U^\log/T^\log} \otimes gr^0_f \approx F^{[n]}_{N,j}.$$ 

Write $t^0 : id_{gr^0_f}$ and write

$$t_{\overset{\circ}{F}^\circ} := \bigoplus_{j=0}^{n-1} t_j \circ \gamma_{(F, f^{-})} : F \sim \overset{\circ}{F}^\circ$$

Also, write $\nabla_{F^{[1]}}$ for the $T^\log$-connection on $F^{[n]}_{N}$ corresponding, via $t_{\overset{\circ}{F}^\circ}$, to $\nabla_{F}$. It follows from the definitions of $t_{\overset{\circ}{F}^\circ}$ and $\nabla_{F^{[1]}}$ that the quadruple

$$\overset{\circ}{F}^\circ := (F^{[n]}_{N}, \nabla_{F^{[1]}, f^t, f^{-1}})$$

specifies a Miura $GL_n$-oper on $U^\log/T^\log$ and $t_{\overset{\circ}{F}^\circ}$ specifies an isomorphism $\overset{\circ}{F}^\circ \sim \overset{\circ}{F}^\circ$ of Miura $GL_n$-opers. Moreover, if $F^\circ$ denotes the underlying
GL_n-oper of $\hat{\mathcal{F}}^{\diamond}$, then for each $j \in \{1, \cdots, n-1\}$ the following sequence of equalities hold:

\[(156)\]

\[\mathcal{K}S^j_{\mathcal{F}^{\diamond}} = (\text{id}_{\Omega(U) \log / T} \otimes t_{n-j}^{-1}) \circ (\text{id}_{\Omega(U) \log / T} \otimes \delta_{n-j+1,f,f^-}) \circ \mathcal{K}S^j_{\mathcal{F}^{\diamond}} \circ \delta_{n-j,f,f^-}^{-1} \circ t_{n-j-1}^{-1} = (\text{id}_{\Omega(U) \log / T} \otimes t_{n-j}^{-1}) \circ (\text{id}_{\Omega(U) \log / T} \otimes t_{n-j}^{-1}) \circ (\text{id}_{\Omega(U) \log / T} \otimes t_{n-j}^{-1}) \circ \cdots \circ (\text{id}_{\Omega(U) \log / T} \otimes t_{n-j}^{-1}) = \cdots = \text{id}_{\Omega(U) \log / T}^{-1},\]

where the second equality follows from the equality

\[(157)\]

\[\text{id}_{\Omega(U) \log / T} \otimes t_{n-j}^{-1} = (\text{id}_{\Omega(U) \log / T} \otimes \delta_{n-j+1,f,f^-}) \circ \mathcal{K}S^j_{\mathcal{F}^{\diamond}} \circ \delta_{n-j,f,f^-}^{-1}\]

and all the equalities after the third equality follow from the equalities $\text{id}_{\Omega(U) \log / T} \otimes t_{n-j}^{-1}$ $(l = 0, \cdots, n - j - 2)$. This implies that the Miura GL_n-oper $\hat{\mathcal{F}}^{\diamond}$ is $N^0$-special, and hence, completes the existence assertion.

The uniqueness assertion may be immediately verified. Indeed, let $(\hat{\mathcal{F}}^{\diamond}, t_{\hat{\mathcal{F}}^{\diamond}})$ (where $\hat{\mathcal{F}}^{\diamond} := (\mathcal{F}_N^{[n]}, \nabla_{\mathcal{F}}, f^+, f^-)$) be a desired pair. Since $\mathcal{K}S^j_{\mathcal{F}^{\diamond}}$ and $\mathcal{K}S^j_{\mathcal{F}^{\diamond}}$ (for any $j$) are compatible, $t_{\hat{\mathcal{F}}^{\diamond}}$ turns out to coincide with the isomorphism “$t_{\hat{\mathcal{F}}^{\diamond}}$” as defined by (154). Also, $\nabla_{\mathcal{F}}$ must be the $T^{\log}$-connection corresponding to $\nabla_{\mathcal{F}}$ via $t_{\hat{\mathcal{F}}^{\diamond}}$. Thus, $(\hat{\mathcal{F}}^{\diamond}, t_{\hat{\mathcal{F}}^{\diamond}})$ is uniquely determined. This completes the proof of Proposition 4.3.3 \(\square\)

**Definition 4.3.4.**

For each Miura GL_n-oper $\hat{\mathcal{F}}^{\diamond}$ on $U^{\log}/T^{\log}$, we shall refer to the pair $(\hat{\mathcal{F}}^{\diamond}, t_{\hat{\mathcal{F}}^{\diamond}})$ obtained by applying Proposition 4.3.3 to $\hat{\mathcal{F}}^{\diamond}$ as the specialization of $\hat{\mathcal{F}}^{\diamond}$.

Next, let us assume that $n > 1$ and $r > 0$. Let $S$ be a $k$-scheme, $\mathfrak{X} := (X/S, \{\sigma_i\}_{i=1}^r)$ an $r$-pointed stable curve of genus $g$ over $S$, and $\hat{\mathcal{F}}^{\diamond} := (\mathcal{F}, \nabla_{\mathcal{F}}, f, f^-)$ a Miura GL_n-oper on $\mathfrak{X}$. Denote by $(\hat{\mathcal{F}}^{\diamond}, t_{\hat{\mathcal{F}}^{\diamond}})$ the specialization of $\hat{\mathcal{F}}^{\diamond}$, where $\hat{\mathcal{F}}^{\diamond} := (\mathcal{F}_N^{[n]}, \nabla_{\mathcal{F}}, f^+, f^-)$. Then, for each $i \in \{1, \cdots, r\}$, we obtain an element

\[(158)\]

\[\epsilon_i^{\hat{\mathcal{F}}^{\diamond}} := \mu_i^{(g_{\nabla_{\mathcal{F}}^{[n]}}^+(\nabla_{\mathcal{F}}))} \in \Gamma(S, \mathcal{O}_S)^{\otimes n} (= \text{t}_{\text{GL}_n}(S)).\]
Definition 4.3.5. Let \( \tilde{\epsilon} := (\epsilon_i)_{i=1}^r \in \mathfrak{t}_{\text{GL}_n}(S)^{\times r} \). We shall say that a Miura GL\(_n\)-oper \( \hat{\mathcal{F}}^{\diamond} \) is of exponents \( \tilde{\epsilon} \) if the equality \( \epsilon_i^{\hat{\mathcal{F}}^{\diamond}} = \epsilon_i \) holds for any \( i \in \{1, \ldots, r\} \). If \( r = 0 \), then we shall refer to any Miura GL\(_n\)-oper as being of exponent \( \emptyset \).

4.4. From Miura GL\(_n\)-opers to Miura sl\(_n\)-opers.

In what follows (cf. Theorem 4.4.1), we construct bijections between the set of special Miura GL\(_n\)-opers and other objects defined so far. Let \( S \) be a \( k \)-scheme and \( X := (X/S, \{\sigma_i\}_{i=1}^r) \) an \( r \)-pointed stable curve over \( S \) of genus \( g \). In this subsection, let \( G := \text{PGL}_n \) and denote by \( \mathbb{B} \) the Borel subgroup of \( G \) defined to be the image of the upper triangular matrices via the quotient \( \text{GL}_n \rightarrow \text{PGL}_n \).

First, let \( \hat{\mathcal{F}}^{\diamond} := (\mathcal{F}, \nabla_{\mathcal{F}}, f, f^-) \) be an \( \mathcal{O}_X \)-special Miura GL\(_n\)-oper on \( X \). The underlying GL\(_n\)-oper \( (\mathcal{F}, \nabla_{\mathcal{F}}, f) \) induces naturally an sl\(_n\) (= \text{pgl}_n) -oper \( (E, \nabla_E) \) on \( X \) (cf. [39], the discussion following Lemma 4.4.1). Also, the filtration \( f^- \) determines a \( \mathbb{B} \)-reduction of \( E_G := E_B \times_{\mathbb{B}} \mathbb{G}_m \), that is to say, a \( \mathbb{B} \)-torsor \( E' \) together with an isomorphism \( \eta_E : E'_B \times_{\mathbb{B}} \mathbb{G}_m \rightarrow E_G \). It follows from the definition of a Miura GL\(_n\)-oper that the quadruple \( \hat{\mathcal{F}}^{\diamond \bullet} := (E_B, \nabla_E, E'_B, \eta_E) \) forms a generic Miura sl\(_n\)-oper on \( X \). Hence, we obtain an assignment \( \hat{\mathcal{F}}^{\diamond} \rightarrow \hat{\mathcal{F}}^{\diamond \bullet} \) from each \( \mathcal{O}_X \)-special Miura GL\(_n\)-opers on \( X \) to a generic Miura sl\(_n\)-opers on \( X \).

Next, let us consider the natural decomposition
\[
\text{End}_{\mathcal{O}_X}(\mathcal{F}^{[n]}_\mathcal{O}_X, \Omega^1_{X, \log/S, \log} \otimes \mathcal{F}^{[n]}_\mathcal{O}_X) = \bigoplus_{l,l' \in \mathbb{N}} \text{End}_{\mathcal{O}_X}(\mathcal{F}^{l}_{\mathcal{O}_X,l'} \Omega^1_{X, \log/S, \log} \otimes \mathcal{F}^{l}_{\mathcal{O}_X,l'}). \tag{160}
\]
The \((l'', l'' + 1)\)-th component (for each \( l'' \in \{0, \ldots, n - 2\} \)) in the right-hand side of (160) admits a natural identification
\[
\text{End}_{\mathcal{O}_X}(\mathcal{F}^{l}_{\mathcal{O}_X,l''} \Omega^1_{X, \log/S, \log} \otimes \mathcal{F}^{l}_{\mathcal{O}_X,l''+1}) = \text{End}_{\mathcal{O}_X}(\mathcal{T}^{l}_{X, \log/S, \log} \otimes \mathcal{T}^{l}_{X, \log/S, \log}) \cong \Gamma(X, \mathcal{O}_X) = k. \tag{161}
\]
Let
\[
(\epsilon_{l,l'})_{l,l' \in \mathbb{N}} \in \bigoplus_{l,l' \in \mathbb{N}} \text{End}_{\mathcal{O}_X}(\mathcal{F}^{l}_{\mathcal{O}_X,l} \Omega^1_{X, \log/S, \log} \otimes \mathcal{F}^{l}_{\mathcal{O}_X,l'}). \tag{162}
\]
be the element defined in such a way that $e_{l,l'} = 1$ (via the identification (161)) if $l + 1 = l'$ and $e_{l,l'} = 0$ if otherwise. The element $(e_{l,l'})_{l,l'}$ corresponds, via (160), to an $O_X$-linear morphism

\[ \nabla_0 : F^{[n]}_{O_X} \to \Omega_{X}^{log/S^{log}} \otimes F^{[n]}_{O_X}. \]

(163)

Now, let $\tilde{\mathcal{F}}^\bullet := (F^{[n]}_{O_X}, \bigoplus_{l=0}^{n-1} \nabla_l)$ be a $(GL_n, O_X^*)$-Cartan connection on $X$. The sum $\nabla_0 + \bigoplus_{l=0}^{n-1} \nabla_l$ specifies an $S^{log}$-connection on $F^{[n]}_{O_X}$, and moreover, the quadruple

\[ \tilde{\mathcal{F}}^\otimes := (F^{[n]}_{O_X}, \nabla_0 + \bigoplus_{l=0}^{n-1} \nabla_l, f^l_{O_X}, f^{-l}_{O_X}) \]

forms an $O_X^*$-special Miura $GL_n$-oper on $X$. One verifies immediately that the assignment $\tilde{\mathcal{F}}^\bullet \mapsto \tilde{\mathcal{F}}^\otimes$ determines a functorial (with respect to $S$) bijection from the set of $(GL_n, O_X^*)$-Cartan connections on $X$ to the set of isomorphism classes of $O_X^*$-special Miura $GL_n$-opers on $X$. Indeed, the inverse map is given by assigning, to each $O_X^*$-special Miura $GL_n$-oper $\mathcal{F}^\otimes$, the $(GL_n, O_X^*)$-Cartan connection $\text{gr}_{\mathcal{F}^\otimes}$ (cf. Remark 13.2), where let $(\mathcal{F}^\otimes, \iota_{\mathcal{F}^\otimes})$ be the specialization of $\mathcal{F}^\otimes$.

Here, for each $\varepsilon^r := ((\varepsilon_{i_1}, \varepsilon_{i_2}, \cdots, \varepsilon_{i_{(n-1)}}))_{i=1}^{(n)} \in (\Gamma(S, O_S))^{\otimes(n-1)}$, we write

\[ \varepsilon^{+0} := ((0, \varepsilon_{i_1}, \varepsilon_{i_2}, \cdots, \varepsilon_{i_{(n-1)}}))_{i=1}^{(n)} \in (\Gamma(S, O_S))^{\otimes(n)} \]

and

\[ \varepsilon[\varepsilon^r] := \pi(\varepsilon^{+0}) \]

(166) (cf. (61) for the definition of $\pi$). If $r = 0$ and $\varepsilon := \emptyset$, then we write $\varepsilon^{+0} := \emptyset$ and $[\varepsilon] := \emptyset$.

**Theorem 4.4.1.**

Let $\varepsilon \in (\Gamma(S, O_S))^{\otimes(n-1)}$ (where $\varepsilon := \emptyset$ if $r = 0$). Then, the following square diagram is commutative (cf. Remark 3.4.1 for the construction of the lower horizontal morphism):

\[
\begin{array}{ccc}
\text{the set of (GL}_n, O_X^*)\text{-Cartan connections on X of monodromies } \varepsilon^{+0} \quad & \xrightarrow{\sim} & \quad \text{the set of isomorphism classes of } O_X^*\text{-special Miura GL}_n\text{-opers on X of exponents } \varepsilon^{+0} \\
\text{of monodromies } [\varepsilon] \quad & \xrightarrow{\sim} & \quad \text{of isomorphism classes of Miura } \mathfrak{sl}_n\text{-opers on X of exponents } [\varepsilon] \\
\end{array}
\]
where the left-hand vertical arrow denotes the second arrow in (62). In particular, the right-hand vertical arrow $\hat{F}^\circ \hookrightarrow \hat{F}^\circ\star$ turns out to be bijective.

**Proof.** The assertion follows from the various definitions of maps involved.  

5. **Pre-Tango structures on a log-curve**

In this section, we recall the notion of a *Tango structures* on a smooth curve (cf., e.g., [32], [33] for various discussions concerning Tango structures) and study its generalization, i.e., a *pre-Tango structure on a pointed stable curve*. Our purpose is to understand the relationship between pre-Tango structures and generic Miura opers. The main result of this section (cf. Theorem 5.4.1) asserts that there exists a canonical bijective correspondence between the set of pre-Tango structures (of monodromies $\vec{\varepsilon}$) and the set of isomorphism classes of dormant generic Miura $\mathfrak{sl}_2$-opers (of exponents $\vec{\varepsilon}$).

5.1. **Tango structures on a smooth curve.**

In this section, we assume that $k$ has characteristic $p > 2$. We first recall the definition of a Tango structure on a smooth curve. Let $T$ be a $k$-scheme and $U$ a smooth curve over $T$. Denote by $\mathcal{B}_U/T (\subseteq \Omega_{U/T})$ the sheaf of locally exact 1-forms on $U$ relative to $T$. The direct image $F_{U/T*}(\mathcal{B}_U/T)$ (cf. §1.7 for the definition of $F_{U/T*}$) forms a vector bundle on $U_T^{(1)}$ of rank $p - 1$.

Now, let $\mathcal{L}$ be a line subbundle of $F_{U/T*}(\mathcal{B}_U/T)$. Consider the $\mathcal{O}_{U_T^{(1)}}$-linear composite

\begin{equation}
\mathcal{L} \hookrightarrow F_{U/S*}(\mathcal{B}_U/S) \hookrightarrow F_{U/S*}(\Omega_{U/S}),
\end{equation}

where the first arrow denotes the natural inclusion and the second arrow denotes the morphism obtained by applying the functor $F_{U/S*}(\mathcal{B}_U/S)$ to the natural inclusion $\mathcal{B}_U/S \hookrightarrow \Omega_{U/S}$. This composite corresponds to a morphism

\begin{equation}
\xi_\mathcal{L} : F_{U/T*}(\mathcal{L}) \to \Omega_{U/T}
\end{equation}

via the adjunction relation “$F_{U/T*}(\mathcal{L}) \Rightarrow F_{U/T*}(\mathcal{L})$”.

**Definition 5.1.1.**

We shall say that $\mathcal{L}$ is a *Tango structure* on $U/T$ if the morphism $\xi_\mathcal{L}$ is an isomorphism.
Remark 5.1.2. Suppose that $X$ is a geometrically connected, proper, and smooth curve over $k$ of genus $g \ (> 1)$. Also, suppose that $X/k$ admits a Tango structure $\mathcal{L} \subseteq F_{X/k}^*(\mathcal{B}_{X/k})$. Then, since $F_{X/k}^*(\mathcal{L}) \cong \Omega_{X/k}$, the following equalities hold:

\[(170) \quad \deg(\mathcal{L}) = \frac{1}{p} \cdot \deg(F_{X/k}^*(\mathcal{L})) = \frac{1}{p} \cdot \deg(\Omega_{X/k}) = \frac{2(g - 1)}{p}.\]

In particular, if $p \nmid g - 1$, then there is no Tango structure on $X/k$.

Remark 5.1.3. Let $X$ be as in Remark 5.1.2 above. Then, a Tango structure $\mathcal{L} \subseteq F_{X/k}^*(\mathcal{B}_{X/k})$ on $X/k$ is completely determined the isomorphism class $[\mathcal{L}]$ of the underlying line bundle $\mathcal{L}$. Indeed, if $\mathcal{L} \subseteq F_{X/k}^*(\mathcal{B}_{X/k})$ and $\mathcal{L}' \subseteq F_{X/k}^*(\mathcal{B}_{X/k})$ are Tango structures on $X/k$ admitting an isomorphism $\mathcal{L} \cong \mathcal{L}'$. The map of sets $\text{Isom}_{\mathcal{O}_U}(\mathcal{L}, \mathcal{L}') \to \text{Isom}_{\mathcal{O}_X}(F_{X/k}^*(\mathcal{L}), F_{X/k}^*(\mathcal{L}'))$ induced by pull-back via $F_{X/k}$ is bijective because it may be identified with the $p$-power map on $k^\times$ (under an identification $\mathcal{L} \cong \mathcal{L}'$). Hence, the isomorphism $\xi_{\mathcal{L}}^{-1} \circ \xi_{\mathcal{L}'} : F_{X/k}^*(\mathcal{L}) \cong F_{X/k}^*(\mathcal{L}')$ comes from a unique isomorphism $\xi : \mathcal{L} \cong \mathcal{L}'$, which is verified (by the adjunction relation ”$F_{X/k}^*(\mathcal{L}) \ad F_{X/k}^*(\mathcal{L}')$") to be compatible with the respective inclusions $\mathcal{L} \subseteq F_{X/k}^*(\mathcal{B}_{X/k})$ and $\mathcal{L}' \subseteq F_{X/k}^*(\mathcal{B}_{X/k})$. This implies that $\mathcal{L}$ and $\mathcal{L}'$ specify the same Tango structure.

5.2. From Tango structures to dormant Miura GL$_2$-opers.

In what follows, we shall discuss a construction of dormant Miura GL$_2$-oper on $U/T$ by means of a Tango structure. Let $\mathcal{L} \subseteq F_{U/T*}(\mathcal{B}_{U/T})$ be a Tango structure on $U/T$. Denote by $\mathcal{G}_\mathcal{L}$ the inverse image of $\mathcal{L}$ via the surjection $F_{U/T*}(\mathcal{O}_U) \to F_{U/T*}(\mathcal{B}_{U/T})$ induced from the universal derivation $d : \mathcal{O}_U \to \Omega_{U/T}$. We have an inclusion of short exact sequences:

\[(171) \quad 0 \to \mathcal{O}_{U/T} \to \mathcal{G}_\mathcal{L} \to \mathcal{L} \to 0.\]
Consider the following inclusion of short exact sequences defined to be the pull-back of (171) via $F_{U/T}$:

\[
\begin{array}{c}
0 \\ \| \\
\| \\
0 \\
\end{array} 
\begin{array}{cccc}
\mathcal{O}_U & \rightarrow & F^*_U(T(\mathcal{G}_\mathcal{L})) & \rightarrow & F^*_U(T(\mathcal{L})) & \rightarrow & 0 \\
\& & \cap & \cap & \& \\
0 & \rightarrow & \mathcal{O}_U & \rightarrow & F^*_U(T(F_{U/T}(\mathcal{O}_U))) & \rightarrow & F^*_U(T(F_{U/T}(\mathcal{B}_{U/T}))) & \rightarrow & 0
\end{array}
\]

(172)

Then, the morphism \( F^*_U(T(\mathcal{O}_U)) \rightarrow \mathcal{O}_U \) corresponding, via the adjunction relation "\( F^*_U(T(-) \dashv F_{U/T}(-) \)”, to the identity morphism \( F_{U/T}(\mathcal{O}_U) \rightarrow F_{U/T}(\mathcal{O}_U) \) is verified to be a split surjection of the lower sequence in (172). This split surjection determines a decomposition

\[
F^*_U(T(\mathcal{F}_{U/T}(\mathcal{G}_\mathcal{L})) \rightarrow F^*_U(T(\mathcal{L})) \oplus \mathcal{O}_U.
\]

(173)

Hence, we obtain a composite isomorphism

\[
F^*_U(T(\mathcal{G}_\mathcal{L} \otimes \mathcal{L}^\vee)) \rightarrow F^*_U(T(\mathcal{G}_\mathcal{L})) \otimes F^*_U(T(\mathcal{L}^\vee))
\]

(175)

where the second and fourth arrows arise from (176) and \( \xi_\mathcal{L} \) respectively and the fifth arrow \( \varsigma \) arises from both \( \text{id}_{\mathcal{O}_U} \) and the automorphism of \( \mathcal{T}_{U/T} \) determined by multiplication by \((-1)\). Denote by \( \nabla_\mathcal{L} \) the \( T \)-connection on \( \mathcal{F}^{|2|}_{\mathcal{O}_U} \) corresponding to the \( T \)-connection \( \nabla^\text{can}_{\mathcal{G}_\mathcal{L} \otimes \mathcal{L}^\vee} \) (resp., (27)) via the composite isomorphism (176). Thus, we obtain a quadruple

\[
\hat{\text{Tan}}^\otimes_{\mathcal{L}} := (\mathcal{F}^{|2|}_{\mathcal{O}_U}, \nabla_\mathcal{L}, f_{\mathcal{O}_U}, f_{\mathcal{O}_U})
\]

(177)

Proposition 5.2.1.

The quadruple \( \hat{\text{Tan}}^\otimes_{\mathcal{L}} \) forms a dormant \( \mathcal{O}_X \)-special Miura GL_{2,\bullet}-oper on \( U/T \).

Moreover, let us write

\[
\hat{\mathcal{F}}^\otimes_{\mathcal{L}} := (\mathcal{F}^{|2|}_{\mathcal{O}_U}, d \oplus \xi^*_\mathcal{L}((\nabla^\text{can}_{\mathcal{L}^\vee})))
\]

(178)
which specifies a \((\text{GL}_2, \mathcal{O}_U)\)-Cartan connection on \(U/T\). Then, there exists a natural isomorphism

\[
\text{gr}_{\mathcal{O}_U}(\hat{T}\text{an}_{L}^\diamondsuit) \cong \hat{\mathcal{F}}_L^\bullet.
\]

In particular, \(\hat{T}\text{an}_{L}^\diamondsuit\) is isomorphic to \(\hat{\mathcal{F}}_L^\diamondsuit\) (cf. (164)).

**Proof.** Let us prove the first assertion. Denote by \(\hat{T}\text{an}_{L}^\diamondsuit\) the \(\text{GL}_n\)-oper defined as \(\hat{T}\text{an}_{L}^\diamondsuit\) (i.e., the underlying \(\text{GL}_n\)-oper of \(\hat{T}\text{an}_{L}^\diamondsuit\)) tensored with the log flat line bundle \((\Omega_{U/T}, \xi_L^*)\). In the following, let us use the various notations (e.g., \(\alpha, \beta, \gamma, \cdots\)) defined in Lemma 5.2.2 below. By the definition of \(\hat{T}\text{an}_{L}^\diamondsuit\), the following sequence of equalities holds:

\[
\begin{align*}
\mathcal{K}S^1_{\hat{T}\text{an}_{L}^\diamondsuit} &= \epsilon^{-1} \circ \beta|_{\Omega_{U/T} \otimes F_{U/T}(\mathcal{O}_U)} \circ \nabla_{\xi_L}^{\text{can}} \circ \alpha|_{F_{U/T}(\mathcal{O}_U)} \circ \xi_L^{-1} \\
&= \epsilon^{-1} \circ \beta \circ \nabla_{\xi_L}^{\text{can}} \circ \alpha \circ \text{incl} \circ \xi_L^{-1} \\
&= \delta \circ \gamma \circ \text{incl} \circ \xi_L^{-1} \\
&= \text{id}_{\Omega_{U/T}},
\end{align*}
\]

where \(\text{incl}\) denotes the inclusion \(F_{U/T}^*(\mathcal{L}) \hookrightarrow F_{U/T}^*(F_{U/T}(\mathcal{B}_{U/T}))\), the third equality follows from Lemma 5.2.2 below, and the last equality follows from the definition of \(\xi_L\). This implies the equality \(\mathcal{K}S^1_{\hat{T}\text{an}_{L}^\diamondsuit} = \text{id}_{\mathcal{O}_U}\), and hence, completes the first assertion.

The second assertion follows directly from the definition of \(\hat{T}\text{an}_{L}^\diamondsuit\). Also, the third assertion follows from the bijectivity of the upper horizontal arrow in (177).

The following lemma was used to prove the above proposition.

**Lemma 5.2.2.**

The following diagram is commutative:

\[
\begin{array}{ccc}
F_{U/T}^*(F_{U/T}(\mathcal{B}_{U/T})) & \xrightarrow{\alpha} & F_{U/T}^*(F_{U/T}(\mathcal{O}_U)) \\
\downarrow \gamma & & \downarrow \jmath \\
F_{U/T}^*(\Omega_{U/T}) & \xrightarrow{\delta} & \Omega_{U/T} \otimes \mathcal{O}_U \quad (= \Omega_{U/T}),
\end{array}
\]

where

- \(\alpha\) denotes the split injection of the lower sequence in (172) corresponding to the split surjection (173);
β denotes the tensor product of (173) and the identity morphism of $\Omega_{U/T}$;

γ denotes the injection arising from the natural inclusion $B_{U/T} \hookrightarrow \Omega_{U/T}$;

δ denotes the morphism corresponding to the identity morphism $F_{U/S*}(\Omega_{U/T}) \to F_{U/S*}(\Omega_{U/T})$ via the adjunction relation “$F_{U/T}(\cdot) \dashv F_{U/T*}(\cdot)$”;

ε denotes the automorphism given by multiplication by $(-1)$.

Proof. Since $U$ is flat over $T$, it suffices to prove Lemma 5.2.2 after restricting $U$ to its fibers over various geometric points of $T$. Hence, one may assume that $k$ is algebraically closed and $T = \text{Spec}(k)$. Moreover, as the statement is of local nature, one may replace $U$ by $\text{Spec}(k[[x]])$ and $\Omega_{U/T}$ by the $O_U$-module given by the $k[[x]]$-module $k[[x]]dx$. Observe that for each $O_U$-module $M$ obtained from some $k[[x]]$-module $M$, we have

\[ \Gamma(U, F_{U/T*}(\Omega_{U/T})) = k[[x]] \otimes_{k[[x]]} M. \]

Then, $F_{U/T*}(F_{U/T*}(B_{U/T}))$ corresponds to the $k[[x]]$-modules $\bigoplus_{l=0}^{p-1} k[[x]] : 1 \otimes x^l dx$, and the kernel of (173) corresponds to $\bigoplus_{l=1}^{p-1} k[[x]](1 \otimes x^l - x^l \otimes 1)$. The morphisms $\alpha$ and $\nabla_{F_{U/T*}(O_U)}$ are given by assigning $1 \otimes x^l dx \mapsto \frac{1}{l+1} \cdot (1 \otimes x^{l+1} - x^{l+1} \otimes 1)$ and $a \otimes b \mapsto da \otimes b$ (for any $a, b \in k[[x]]$) respectively. Hence, the following equalities hold:

\[ \beta \left( \nabla_{F_{U/T*}(O_U)} \left( \alpha \left( 1 \otimes x^l dx \right) \right) \right) = \beta \left( \frac{1}{l+1} \cdot (1 \otimes x^{l+1} - x^{l+1} \otimes 1) \right) \]
\[ = \beta(-x^l dx \otimes 1) \]
\[ = -x^l dx. \]

On the other hand, we have

\[ \epsilon(\delta(\gamma(1 \otimes x^l))) = \epsilon(\delta(1 \otimes x^l dx)) = \epsilon(x^l dx) = -x^l dx. \]

Thus, the equality $\beta \circ \nabla_{F_{U/T*}(O_U)} \circ \alpha = \epsilon \circ \delta \circ \gamma$ holds. This completes the proof of Lemma 5.2.2. \qed

5.3. Pre-Tango structures on a log-curve.

In this section, we shall discuss the definition of a pre-Tango structure on a pointed stable curve. Let $T^{\log}$ be an fs log scheme over $k$ and $U^{\log}$ a log-curve over $T^{\log}$.

Here, recall the Cartier operator

\[ C_{U^{\log}/T^{\log}} : F_{U/T*}(\Omega_{U^{\log}/T^{\log}}) \to \Omega_{U^{(1)\log}/T^{\log}} \]
of $U^{\log}/T^{\log}$. That is to say, $C_{U^{\log}/T^{\log}}$ is a unique $\mathcal{O}_{U^{(1)}}$-linear morphism whose composite with the inclusion $\Omega_{U^{(1)}}^{\log}/T^{\log} \to \Omega_{U_{T}^{(1)}}^{\log}/T^{\log} \otimes F_{U/T^{*}}(\mathcal{O}_{U})$ induced by the natural injection $\mathcal{O}_{U^{(1)}} \to F_{U/T^{*}}(\mathcal{O}_{U})$ coincides with the Cartier operator associated with $(\mathcal{O}_{U}, d)$ in the sense of [28], Proposition 1.2.4.

**Definition 5.3.1.**
A pre-Tango structure on $U^{\log}/T^{\log}$ is a $T^{\log}$-connection $\nabla_{\Omega}$ on $\Omega_{U^{\log}/T^{\log}}$ with vanishing $p$-curvature satisfying that $F_{U/T^{*}}(\text{Ker}(\nabla_{\Omega})) \subseteq \text{Ker}(C_{U^{\log}/T^{\log}})$.

If $U^{\log}/T^{\log} = X^{\log}/S^{\log}$ for a pointed stable curve $X := (X/S, \{\sigma_{i}\}_{i=1}^{r})$, then we shall refer to a pre-Tango structure on $X^{\log}/S^{\log}$ as a pre-Tango structure on $X$.

If the curve $U^{\log}/T^{\log}$ under consideration is a non-logarithmic smooth curve, then the notion of a pre-Tango structure is equivalent to the notion of a Tango structure defined in Definition 5.1.1, as verified in the following proposition.

**Proposition 5.3.2.**
Let $U/T$ be as in §§5.1-5.2. Then, the assignment $\nabla_{\Omega} \mapsto F_{U/T^{*}}(\text{Ker}(\nabla_{\Omega}))$ determines a bijection of sets

\[
(186) \quad \left( \text{the set of pre-Tango structures on } U/T \right) \sim \left( \text{the set of Tango structures on } U/T \right).
\]

**Proof.** First, we shall prove the claim that the assignment $\nabla_{\Omega} \mapsto F_{U/T^{*}}(\text{Ker}(\nabla_{\Omega}))$ defines a map from the set of pre-Tango structures on $U/T$ to the set of Tango structures on $U/T$. Recall from [17], §5, p. 190, Theorem 5.1, that the assignments $V \mapsto (F_{U/T^{*}}(V), \nabla_{V}^{\text{can}})$ and $(\mathcal{F}, \nabla_{\mathcal{F}}) \mapsto F_{U/T^{*}}(\text{Ker}(\nabla_{\mathcal{F}}))$ determines an equivalence of categories

\[
(187) \quad \left( \text{the category of vector bundles on } U_{T}^{(1)} \right) \sim \left( \text{the category of flat bundles on } U/T \right. \left. \text{with vanishing } p\text{-curvature} \right).
\]

Now, let $\nabla_{\Omega}$ be a pre-Tango structure on $U/T$. The equivalence of categories recalled above implies that $F_{U/T^{*}}(\text{Ker}(\nabla_{\Omega}))$ is a line bundle on $U_{T}^{(1)}$ and the morphism

\[
(188) \quad F_{U/T^{*}}^{*}(F_{U/T^{*}}(\text{Ker}(\nabla_{\Omega}))) \to \Omega_{U/T}
\]

corresponding to the inclusion $F_{U/T^{*}}(\text{Ker}(\nabla_{\Omega})) \hookrightarrow F_{U/T^{*}}(\Omega_{U/T})$ (via the adjunction relation “$F_{U/T}(-) \dashv F_{U/T^{*}}(-)$”) is an isomorphism. Also, since the sequence

\[
(189) \quad 0 \to F_{U/T^{*}}(\mathcal{B}_{U/T}) \to F_{U/T^{*}}(\Omega_{U/T}) \xrightarrow{C_{U/T}} \Omega_{U_{T}^{(1)}/T} \to 0
\]
is exact, the inclusion \( F_{U/T*}(\text{Ker}(\nabla_\Omega)) \hookrightarrow F_{U/T*}(\Omega_{U/T}) \) factors through the inclusion \( F_{U/T*}(B_{U/T}) \to F_{U/T*}(\Omega_{U/T}) \). By taking account of the resulting injection \( F_{U/T*}(\text{Ker}(\nabla_\Omega)) \hookrightarrow F_{U/T*}(B_{U/T}) \), one may regard \( F_{U/T*}(\text{Ker}(\nabla_\Omega)) \) as an \( \mathcal{O}_{U/T} \)-submodule of \( F_{U/T*}(B_{U/T}) \). Now, consider the natural exact sequence

(190)

\[
0 \to F_{U/T*}(\Omega_{U/T})/F_{U/T*}(\text{Ker}(\nabla_\Omega)) \to F_{U/T*}(\Omega_{U/T}^{\otimes 2}) \to F_{U/T*}(\text{Coker}(\nabla_\Omega)) \to 0,
\]

where the second arrow arises from \( \nabla_\Omega : \Omega_{U/T} \to \Omega_{U/T}^{\otimes 2} \). By Proposition 6.8.3, \( F_{U/T*}(\text{Coker}(\nabla_\Omega)) \) turns out to be a vector bundle on \( U_T^{(1)} \). Since \( F_{U/T*}(\Omega_{U/T}^{\otimes 2}) \) is a vector bundle, the exactness of (190) implies that the quotient \( F_{U/T*}(\Omega_{U/T})/F_{U/T*}(\text{Ker}(\nabla_\Omega)) \) is a vector bundle. Moreover, let us consider the short exact sequence

(191)

\[
0 \to F_{U/T*}(B_{U/T})/F_{U/T*}(\text{Ker}(\nabla_\Omega)) \to F_{U/T*}(\Omega_{U/T})/F_{U/T*}(\text{Ker}(\nabla_\Omega)) \to \Omega_{U_T^{(1)}/T} \to 0
\]

induced from (190) via taking quotients by \( F_{U/T*}(\text{Ker}(\nabla_\Omega)) \). Since both \( \Omega_{U_T^{(1)}/T} \) and \( F_{U/T*}(\Omega_{U/T})/F_{U/T*}(\text{Ker}(\nabla_\Omega)) \) are vector bundles, \( F_{U/T*}(B_{U/T})/F_{U/T*}(\text{Ker}(\nabla_\Omega)) \) is verified to be a vector bundle, namely, \( F_{U/T*}(\text{Ker}(\nabla_\Omega)) \) is a line subbundle of \( F_{U/T*}(B_{U/T}) \). Thus, \( F_{U/T*}(\text{Ker}(\nabla_\Omega)) \) specifies a Tango structure on \( U/T \). This completes the proof of the claim.

Next, let \( \mathcal{L} (\subseteq F_{X/S*}(B_{T/T})) \) be a Tango structure on \( U/T \). Denote by \( \xi_{\mathcal{L}*}(\nabla_\mathcal{L}^{\text{can}}) \) the \( T \)-connection on \( \Omega_{U/T} \) corresponding to \( \nabla_\mathcal{L}^{\text{can}} \) via \( \xi_{\mathcal{L}} \), which has vanishing \( p \)-curvature. By the equivalence of categories displayed in (187), \( F_{U/T*}(\text{Ker}(\xi_{\mathcal{L}*}(\nabla_\mathcal{L}^{\text{can}}))) \subseteq F_{U/T*}(\Omega_{U/T}) \) turns out to coincide with the image of the composite \( \mathcal{L} \hookrightarrow F_{U/T*}(B_{U/T}) \hookrightarrow F_{U/T*}(\Omega_{U/T}) \). But, since (189) is exact, the composite

(192)

\[
F_{U/T*}(\text{Ker}(\xi_{\mathcal{L}*}(\nabla_\mathcal{L}^{\text{can}}))) \hookrightarrow F_{U/T*}(\Omega_{U/T}) \xrightarrow{c_{U/T}} \Omega_{U_T^{(1)}/T}
\]

is the zero map. Hence, \( \xi_{\mathcal{L}*}(\nabla_\mathcal{L}^{\text{can}}) \) specifies a pre-Tango structure on \( U/T \). One verifies that the assignment \( \mathcal{L} \mapsto \xi_{\mathcal{L}*}(\nabla_\mathcal{L}^{\text{can}}) \) determines the inverse to the map \( \nabla_\Omega \mapsto F_{U/T*}(\text{Ker}(\nabla_\Omega)) \) discussed above. Consequently, we obtain the desired bijection. \( \square \)

**Example 5.3.3.**

We shall consider pre-Tango structures on an (ordinary) elliptic curve. Let \( X \) be a geometrically connected, proper, and smooth curve over \( k \) of genus 1. Suppose that \( X \) is ordinary, i.e., the \( p \)-rank of its Jacobian is maximal. One may find an invariant differential \( \delta \in \Gamma(X, \Omega_{X/k}) \) with \( C_{X/k}(\delta) = \delta \), which induces an identification \( \mathcal{O}_X \xrightarrow{\sim} \Omega_{X/k} \) (given by assigning \( s \mapsto s \cdot \delta \) for any local section \( s \in \mathcal{O}_X \)). Now, let \( \nabla \) be a \( k \)-connection on \( \Omega_{X/k} \) with vanishing \( p \)-curvature.
By means of the above identification, $\nabla$ may be considered as a $k$-connection on $O_X$, and hence, expressed as $\nabla = d + u \cdot \delta$ (for some $u \in k = \Gamma(X, O_X)$). It follows from [18], Proposition 7.1.2, that $\psi(O_X, \nabla) = 0$ implies the equality $(\text{id}_X \times F_{\text{Spec}(k)})^*(u \cdot \delta) = C_{X/k}(u \cdot \delta) (= u^{1/p} \cdot C_{X/k}(\delta))$. Hence, we have $u = u^{1/p}$, or equivalently, $u \in \mathbb{F}_p$.

Here, we shall suppose that $u \neq 0$. If $h$ is a local section of $\text{Ker}(\nabla) (\subseteq O_X)$, i.e., a local function satisfying the equality $dh = -uh\delta$, then $C_{X/k}(h \cdot \delta) = -u^{-\frac{1}{p}} \cdot C_{X/k}(dh) = -u^{-\frac{1}{p}} \cdot 0 = 0$. This implies that $F_{X/k*}(\text{Ker}(\nabla)) \subseteq \text{Ker}(C_{X/k})$, that is to say, $\nabla$ forms a pre-Tango structure on $X/k$. Next, suppose that $u = 0$, i.e., $\nabla = d$. Then, $F_{X/k*}(\text{Ker}(\nabla))$ coincides with $O_{X/k} \cdot \delta$. For any nonzero local section $v$ of $O_X$, $C_{X/k}(v^p \cdot \delta) = v \cdot C_{X/k}(\delta) = v\delta \neq 0$. Hence, $F_{X/k*}(\text{Ker}(\nabla)) \not\subseteq \text{Ker}(C_{X/k})$, and $\nabla$ is not a pre-Tango structure.

Consequently, we have obtained the fact that the set of pre-Tango structures on an ordinary elliptic curve $X/k$ is in bijection with the set of $k$-connections $\nabla$ on $O_X$ with vanishing $p$-curvature which is not equal to the universal derivation $d$. In particular, the number of pre-Tango structures on $X/k$ is exactly $p - 1$ ($= \sharp(\mathbb{F}_p \setminus \{0\})$).

5.4. Pre-Tango structures vs. dormant Miura $\mathfrak{sl}_2$-opers.

Denote by

$$\overline{\text{Tan}}_{g,r}$$

the set-valued contravariant functor on $\mathfrak{Sch}_{/\overline{M}_{g,r}}$ which, to any object $S \rightarrow \overline{M}_{g,r}$ classifying a pointed stable curve $X$, assigns the set of pre-Tango structures on $X$. Also, for each $\varepsilon \in k^{nr}$ (where we take $\varepsilon := \emptyset$ if $r = 0$), we shall write

$$\overline{\text{Tan}}_{g,r,\varepsilon}$$

for the subfunctor of $\overline{\text{Tan}}_{g,r}$ classifying pre-Tango structures of monodromies $\varepsilon$. $\overline{\text{Tan}}_{g,r}$ and $\overline{\text{Tan}}_{g,r,\varepsilon}$ may be represented by closed substacks of $\overline{\text{Co}}_{g,r}$ and $\overline{\text{Co}}_{g,r,\varepsilon}$ respectively. One verifies (from an argument similar to the argument in the proof of the non-resp’d assertion in Proposition 2.5.2) that $\overline{\text{Tan}}_{g,r,\varepsilon}$ is empty unless $\varepsilon$ lies $\mathbb{F}_p^{nr}$ (or $\varepsilon = \emptyset$), and $\overline{\text{Tan}}_{g,r}$ decomposes into the disjoint union

$$\overline{\text{Tan}}_{g,r} = \coprod_{\varepsilon \notin \mathbb{F}_p^{nr}} \overline{\text{Tan}}_{g,r,\varepsilon}. \tag{195}$$

Moreover, the following Theorem 5.4.1 holds; by this proposition, the notion of a dormant generic Miura $\mathfrak{g}$-oper may be thought of as a generalization of the notion of a Tango structure.
Theorem 5.4.1 (Theorem 5.4.1 (i)).
Let $\varepsilon := (\varepsilon_i)_{i=1}^r \in k^{\times r}$, and write $-\varepsilon' := (-\varepsilon_i)_{i=1}^r$ (where $\varepsilon := 0$ and $-\varepsilon' := 0$ if $r = 0$). Then, the composite isomorphism

$$\tag{196} \mathcal{T}_{\overline{g},g,\varepsilon} \xrightarrow{\sim} \mathcal{T}_{\overline{g},g,\varepsilon} \xrightarrow{\sim} \mathcal{M}_{\overline{p}_{\mathcal{B}_{2,q,g},r}[\varepsilon]}$$

restricts to an isomorphism

$$\tag{197} \mathcal{F}_{\mathcal{O}_{\mathcal{X}},g,\varepsilon} \xrightarrow{\sim} \mathcal{M}_{\overline{p}_{\mathcal{B}_{2,q,g},r}[\varepsilon]}$$

over $\mathcal{M}_{\mathcal{g},g}$. 

Proof. Let $S$ be a $k$-scheme and $\mathcal{X} := (X/S, \{\sigma_i\}_{i=1}^r)$ an $r$-pointed stable curve over $S$ of genus $g$. Also, let $\nabla$ be an $S_{\log}$-connection on $\Omega^n_{X/S}$ classified by $\mathcal{T}_{\overline{g},g,\varepsilon}$. Denote by $\mathcal{F}_{\mathcal{T}_{\mathcal{O}_{\mathcal{X}}},g,\varepsilon}$ the $\mathcal{O}_{\mathcal{X}}$-special Miura GL2-oper on $\mathcal{X}$ determined by $\nabla$ via the composite of (196) and the inverse of the right-hand vertical arrow in (167). In order to complete the proof, it suffices to prove the claim that $\nabla$ specifies a pre-Tango structure on $X$ if and only if $\mathcal{F}_{\mathcal{T}_{\mathcal{O}_{\mathcal{X}}},g,\varepsilon}$ is dormant.

To begin with, denote by $U$ the smooth locus of $X \setminus \text{Supp}(D_X)$ relative to $S$, where $D_X$ denotes the effective relative divisor on $X$ defined to be the union of the image of the marked points $\sigma_i$ ($i = 1, \cdots, r$). We equip $U$ with the log structure pulled-back from $X_{\log}$ via the open immersion $U \hookrightarrow X$. Denote by $U_{\log}$ the resulting log scheme. Since the natural projection $U_{\log} \to S_{\log}$ is strict (cf. [14], §1.2), each pre-Tango structure (resp., Miura GL2-oper) on $U_{\log}/S_{\log}$ may be identified with a pre-Tango structure (resp., Miura GL2-oper) on $U/S$.

Now, suppose that $\nabla$ specifies a pre-Tango structure. By Proposition 5.3.2, the restriction $\nabla|_U$ corresponds (via (186)) to a Tango structure $\mathcal{L} \subseteq F_{U/S}(\mathcal{B}_{U/S})$ on $U/S$. Moreover, by the last assertion of Proposition 5.2.1, the restriction $\mathcal{F}_{\mathcal{O}_{\mathcal{X}}|_U}$ to $U$ is isomorphic to the dormant Miura GL2-oper $\mathcal{F}_{\mathcal{O}_{\mathcal{X}}|_U}$, i.e., isomorphic to $\mathcal{F}_{\mathcal{T}_{\mathcal{O}_{\mathcal{X}}}}$. Hence, since $U$ is scheme-theoretically dense in $X$, $\mathcal{F}_{\mathcal{T}_{\mathcal{O}_{\mathcal{X}}}}$ itself turns out to be dormant, as desired.

Conversely, suppose that $\mathcal{F}_{\mathcal{T}_{\mathcal{O}_{\mathcal{X}}}}$ is dormant. The dormant Miura GL2-oper on $U/S$ defined as the restriction of $\mathcal{F}_{\mathcal{T}_{\mathcal{O}_{\mathcal{X}}}}$ to $U$ comes from a Tango-structure $\mathcal{L}$ (cf. Proposition 5.2.1), i.e., isomorphic to $\mathcal{F}_{\mathcal{T}_{\mathcal{O}_{\mathcal{X}}}}$. By the various definitions involved, the pre-Tango structure on $U/S$ corresponding to $\mathcal{L}$ via (186) coincides with $\nabla|_U$. By Proposition 5.3.2, in particular, both the $p$-curvature of $\nabla$ and the composite

$$\tag{198} F_{X/S}(\text{Ker}(\nabla)) \to F_{X/S}(\Omega^n_{X/S}) \xrightarrow{\text{C}_{X_{\log}/S_{\log}}} \Omega^n_{X_{\log}/S_{\log}}$$

vanishes on $U$. But, since $U$ is scheme-theoretically dense in $X$, both the $p$-curvature of $\nabla$ and the composite (198) vanishes identically on $X$. That is to say, $\nabla$ specifies a pre-Tango structure on $\mathcal{X}$. This completes the proof of Theorem 5.4.1. \qed
6. Deformations of (dormant) Miura opers

In this section, we describe the deformation space of a given (dormant) generic Miura \( g \)-oper in terms of de Rham cohomology of complexes (cf. Propositions 6.1.2 and 6.2.2). By applying these descriptions, one may prove (cf. Theorem 6.3.2) that the moduli stack of dormant generic Miura \( \mathfrak{sl}_2 \)-opers is smooth.

6.1. The tangent bundle of \( \mathcal{M} \bar{O}_{g,r} \).

First, assume that either one of the three conditions \((\text{Char})_0, (\text{Char})_p, \) and \((\text{Char})^d_\mathbb{P} \) (cf. §2.1) is satisfied. The moduli stack \( \mathcal{M} \bar{O}_{g,r} \) admits a log structure pulled-back from the log structure of \( \mathcal{M} \bar{O}^\log_{g,r} \); we denote the resulting log stack by

\[ \mathcal{M} \bar{O}_{g,r}^\log. \]

The forgetting morphism \( \mathcal{M} \bar{O}_{g,r} \to \mathcal{M} \bar{O}^\log_{g,r} \) extends to a morphism \( \mathcal{M} \bar{O}_{g,r}^\log \to \mathcal{M} \bar{O}^\log_{g,r} \).

Let \( S \) be a scheme over \( k \) and \( \mathfrak{X} := (f : X \to S, \{\sigma_i\}_{i=1}^r) \) an \( r \)-pointed stable curve of genus \( g \) over \( S \). If \( \nabla : K^0 \to K^1 \) is a morphism of sheaves of abelian groups on \( X \), then it may be thought of as a complex concentrated in degrees 0 and 1; we denote this complex by

\[ K^\bullet[\nabla] \]

(200)

where \( K^i[\nabla] := K^i \) for \( i = 0, 1 \). Also, for \( i = 0, 1, \ldots \), we obtain the sheaf

\[ \mathbb{R}^i f_* (K^\bullet[\nabla]) \]

(201)

on \( S \), where \( \mathbb{R}^i f_* (-) \) is the \( i \)-th hyper-derived functor of \( \mathbb{R}^0 f_* (-) \) (cf. [17], (2.0)). In particular, \( \mathbb{R}^0 f_* (K^\bullet[\nabla]) = f_* (\text{Ker(\nabla)}) \).

Now, let \( \mathcal{E}^{\otimes} := (E^\sigma_B, \nabla_E, E^{\sigma_1}_B, \eta^{\sigma_1}_B) \) be a \( p_{-1} \)-special Miura \( g \)-oper on \( \mathfrak{X} \). We shall write \( (E^\sigma_B, \nabla_E) \) for the log flat \( \mathbb{B} \)-torsor associated with \( \mathcal{E}^{\otimes} \) and

\[ \nabla^{ad}_{\mathcal{E}^\sigma_B} : \mathcal{E}^\sigma_B \to \Omega_{X^{\log}/S^{\log}} \otimes \mathfrak{X}^{\sigma_1}_B \]

for the \( S^{\log} \)-connection on the vector bundle \( \mathcal{E}^\sigma_B \) induced from \( \nabla_{\mathcal{E}^\sigma_B} \) via the adjoint representation \( \mathbb{B} \to \text{GL}(\mathfrak{b}) \). If we write \( \mathfrak{g}_j := \mathfrak{g}^j/\mathfrak{g}^{j+1} \) (for each \( j \in \mathbb{Z} \)), then it has a \( T \)-action induced from the \( T \)-action on \( \mathfrak{g}^j \), and hence, we have an \( \mathcal{O}_X \)-module \( (\mathfrak{g}_j)_{\mathcal{E}^\sigma_T} \). The decomposition displayed in (41) gives rise to a decomposition

\[ \mathfrak{g}^j_{\mathcal{E}^\sigma_T} \cong \bigoplus_{j \in \mathbb{Z}} (\mathfrak{g}_j)_{\mathcal{E}^\sigma_T} \]
(which restricts to a decomposition \( b_{c_1} \cong \bigoplus_{j \leq 0} (g_j)_{c_1} \) on \( g_{c_1} \). We shall write
\[
\mathfrak{g}_{c_1}^{-1/1} := \{ (g_{-1})_{c_1} \oplus (g_0)_{c_1} \} \quad \left( = \mathfrak{g}_{c_1}^{-1}/\mathfrak{g}_{c_1}^1 \right)
\]
and regard it as an \( \mathcal{O}_X \)-submodule of \( b_{c_1} \). Consider the \( f^{-1}(\mathcal{O}_S) \)-linear morphism
\[
\nabla_{\text{ad}} : \tilde{T}_{c_1}^{\text{tr} \log}/\mathcal{S}_{\text{log}} \to \Omega_{X_1}^{\text{log}}/\mathcal{S}_{\text{log}} \otimes \tilde{T}_{c_1}^{\text{tr} \log}/\mathcal{S}_{\text{log}}
\]
determined uniquely by the condition that
\[
\langle \partial, \nabla_{\text{ad}}(s) \rangle = [\nabla_{\text{ad}}(\partial), s] - \nabla_{\text{ad}}([\partial, a_{\log}(s)])
\]
where
\[
\begin{align*}
\bullet & \text{ and } \partial \text{ denote arbitrary local sections of } \tilde{T}_{c_1}^{\text{tr} \log}/\mathcal{S}_{\text{log}} \text{ and } \mathcal{T}_{X_1}^{\text{log}}/\mathcal{S}_{\text{log}} \text{ respectively;} \\
\bullet & \langle - , - \rangle \text{ denotes the } \mathcal{O}_X \text{-bilinear pairing }
\end{align*}
\]
\[
\tilde{T}_{X_1}^{\text{log}}/\mathcal{S}_{\text{log}} \times (\Omega_{X_1}^{\text{log}}/\mathcal{S}_{\text{log}} \otimes \tilde{T}_{c_1}^{\text{tr} \log}) \to \tilde{T}_{c_1}^{\text{tr} \log}/\mathcal{S}_{\text{log}}
\]
induced by the natural pairing \( \tilde{T}_{X_1}^{\text{log}}/\mathcal{S}_{\text{log}} \times \Omega_{X_1}^{\text{log}}/\mathcal{S}_{\text{log}} \to \mathcal{O}_X \);
\[
\bullet [ , - ]'s \text{ denote the Lie bracket operators in the respective tangent bundles.}
\]
One verifies that the restriction of \( \nabla_{\text{ad}} \) to \( b_{c_1} \) \((\subseteq \tilde{T}_{c_1}^{\text{tr} \log}/\mathcal{S}_{\text{log}})\) coincides with \( \nabla_{\text{ad}} \). Moreover, the following assertion holds.

**Lemma 6.1.1.**
The image of \( \nabla_{\text{ad}} \) is contained in \( \Omega_{X_1}^{\text{log}}/\mathcal{S}_{\text{log}} \otimes \mathfrak{g}_{c_1}^{-1/1} \).

**Proof.** For any local sections \( s \) and \( \partial \) of \( \tilde{T}_{c_1}^{\text{tr} \log}/\mathcal{S}_{\text{log}} \) and \( \mathcal{T}_{X_1}^{\text{log}}/\mathcal{S}_{\text{log}} \) respectively, the following sequence of equalities holds:
\[
\langle \partial, (\text{id}_{\Omega_{X_1}^{\text{log}}/\mathcal{S}_{\text{log}}} \otimes a_{\log}) \nabla_{\text{ad}}(s) \rangle \\
= a_{\log}^{\mathfrak{g}_{c_1}}(\langle \partial, \nabla_{\text{ad}}(s) \rangle) \\
= a_{\log}^{\mathfrak{g}_{c_1}}([\nabla_{\text{ad}}(\partial), s]) - (a_{\log}^{\mathfrak{g}_{c_1}} \circ \nabla_{\text{ad}})([\partial, a_{\log}(s)]) \\
= ([a_{\log}^{\mathfrak{g}_{c_1}} \circ \nabla_{\text{ad}}](\partial), a_{\log}(s)) - \text{id}_{\Omega_{X_1}^{\text{log}}/\mathcal{S}_{\text{log}}}(\langle \partial, a_{\log}(s) \rangle) \\
= [\partial, a_{\log}(s)] - [\partial, a_{\log}(s)] \\
= 0.
\]
This implies that the image of \( \nabla^{ad}_{\mathcal{E}^T} \) is contained in \( \Omega^{X^{log}/S^{log}} \otimes \mathfrak{g}_{\mathcal{E}^T} \) (\( \simeq \text{Ker} (\text{id}_{\mathcal{N}^{X^{log}/S^{log}}} \otimes \mathfrak{a}^{log}_{\mathcal{E}^T}) \)). Moreover, by the definition of a \( g \)-oper, the image of \( \nabla^{ad}_{\mathcal{E}^T} \) is contained in \( \Omega^{X^{log}/S^{log}} \otimes \mathfrak{T}_{\mathcal{E}^T}^{-1} \otimes \mathfrak{g}_{\mathcal{E}^T}^{-1} \), and hence, in \( \Omega^{X^{log}/S^{log}} \otimes \mathfrak{g}_{\mathcal{E}^T}^{-1} \) (\( \simeq \Omega^{X^{log}/S^{log}} \otimes (\mathfrak{T}_{\mathcal{E}^T}^{-1} \otimes \mathfrak{g}_{\mathcal{E}^T}^{-1} \otimes \mathfrak{T}_{\mathcal{E}^T}^{-1} \otimes \mathfrak{g}_{\mathcal{E}^T}^{-1}) \)). This completes the proof of Lemma 6.1.1. \( \square \)

Because of the above lemma, \( \nabla^{ad}_{\mathcal{E}^T} \) restricts to an \( f^{-1}(\mathcal{O}_S) \)-linear morphism

\[
\nabla^{ad,-1/1}_{\mathcal{E}^T} : \mathfrak{T}_{\mathcal{E}^T}^{log} \rightarrow \Omega^{X^{log}/S^{log}} \otimes \mathfrak{g}^{-1/1}_{\mathcal{E}^T}.
\]

The pair of the morphism \( \mathfrak{a}^{log}_{\mathcal{E}^T} : \mathfrak{T}_{\mathcal{E}^T}^{log} \rightarrow \mathfrak{T}_{\mathcal{E}^T}^{log} \) and the zero map \( \Omega^{X^{log}/S^{log}} \otimes \mathfrak{g}_{\mathcal{E}^T}^{-1/1} \rightarrow 0 \) specifies a morphism

\[
\mathcal{K}^*[\nabla^{ad,-1/1}_{\mathcal{E}^T}] \rightarrow \mathfrak{T}_{\mathcal{E}^T}^{log}[0]
\]

of complexes, where for any abelian sheaf \( \mathcal{F} \) we shall write \( \mathcal{F}[0] \) for \( \mathcal{F} \) considered as a complex concentrated at degree 0.

**Proposition 6.1.2.**

Let \( c_{\mathfrak{X},\mathfrak{E}^{\circ}} : S \rightarrow \mathfrak{M}_{\mathfrak{p}_{g,r}} \) (resp., \( c_{\mathfrak{X}} : S \rightarrow \mathfrak{M}_{g,r} \)) be the \( S \)-rational point of \( \mathfrak{M}_{\mathfrak{p}_{g,r}} \) (resp., \( \mathfrak{M}_{g,r} \)) classifying the pair \( (\mathfrak{X}, \mathfrak{E}^{\circ}) \) (resp., \( \mathfrak{X} \)). Then, there exists a natural isomorphism

\[
c^s_{\mathfrak{X},\mathfrak{E}^{\circ}}(\mathfrak{T}_{\mathfrak{M}_{\mathfrak{p}_{g,r}}}^{log}/k) \xrightarrow{\text{[211]}} \mathbb{R}^1 f_*(\mathcal{K}^*[\nabla^{ad,-1/1}_{\mathcal{E}^T}])
\]

of \( \mathcal{O}_S \)-modules, which makes the square diagram

\[
\begin{array}{ccc}
c^s_{\mathfrak{X},\mathfrak{E}^{\circ}}(\mathfrak{T}_{\mathfrak{M}_{\mathfrak{p}_{g,r}}}^{log}/k) & \xrightarrow{\text{[211]}} & \mathbb{R}^1 f_*(\mathcal{K}^*[\nabla^{ad,-1/1}_{\mathcal{E}^T}]) \\
\downarrow & & \downarrow \\
c^s_{\mathfrak{X}}(\mathfrak{T}_{\mathfrak{M}_{g,r}}^{log}/k) & \xrightarrow{\sim} & \mathbb{R}^1 f_*(\mathfrak{T}_{\mathcal{E}^T}^{log}/S^{log})
\end{array}
\]

commute, where the right-hand vertical arrow denotes the morphism obtained by applying the functor \( \mathbb{R}^1 f_* \), the left-hand vertical arrow denotes the morphism arising from the forgetting morphism \( \mathfrak{M}_{\mathfrak{p}_{g,r}} \rightarrow \mathfrak{M}_{g,r} \), and the lower horizontal arrow denotes the isomorphism defined as the Kodaira-Spencer map of \( \mathfrak{X} \).

**Proof.** The assertion follows from an argument (in the case where \( \mathfrak{X} \) is a pointed stable curve over an arbitrary scheme \( S \)) similar to the argument (in the case where \( \mathfrak{X} \) is an unpointed smooth curve over \( \mathbb{C} \)) given in [2]. Indeed, by the explicit description of the hyperchomology sheaf \( \mathbb{R}^1 f_*(\mathcal{K}^*[\nabla^{ad,-1/1}_{\mathcal{E}^T}]) \).
in terms of the Čech double complex associated with $\mathcal{K}^\bullet[\tilde{\nabla}_{E_t}^{-1}]$, one verifies that $\mathbb{R}^1 f_*(\mathcal{K}^\bullet[\tilde{\nabla}_{E_t}^{-1}])$ may be naturally identified with the deformation space (relative to $S$) of $(X, \tilde{\mathcal{E}}\circ\hat{\mathcal{E}})$, i.e., the pair consisting of the pointed stable curve $X$ and (the isomorphism class of) the generic Miura $\mathfrak{g}$-oper $\tilde{\mathcal{E}}\circ\hat{\mathcal{E}}$ on it. That is to say, $\mathbb{R}^1 f_*(\mathcal{K}^\bullet[\tilde{\nabla}_{E_t}^{-1}])$ is canonically isomorphic to the $\mathcal{O}_S$-module $c^*_{X,\tilde{\mathcal{E}}\circ\hat{\mathcal{E}}}(\mathcal{T}_{\mathcal{M}_{\text{log}}^{\text{op}}_g,r}/k)$. In particular, for completing the proof of Proposition 6.1.2, we refer to [2], Propositions 4.1.3 and 4.3.1.

6.2. The tangent bundle of $\mathcal{M}_{\text{g,g,r}}$.

Next, suppose further that $\text{char}(k) = p > 0$, i.e., either one of the two conditions (Char)$_p$, and (Char)$_p^{\text{st}}$ is satisfied. Let us consider the natural composite

$$
\omega : \Omega_{X_{\text{log}}/\mathcal{S}_{\text{log}}} \otimes \frac{\mathfrak{g}_{E_t}}{\frac{1}{2}} \hookrightarrow \Omega_{X_{\text{log}}/\mathcal{S}_{\text{log}}} \otimes \mathfrak{b}_{E_t} \twoheadrightarrow \text{Coker}(\nabla_{E_t}^{\text{ad}}).
$$

By Lemma 6.2.1 below, $\tilde{\nabla}_{E_t}^{-1}$ induces (by restricting its codomain) an $f^{-1}(\mathcal{O}_S)$-linear morphism

$$
\tilde{\nabla}_{E_t}^{\text{ad},\omega} : \tilde{T}_{E_t^{\text{log}}/\mathcal{S}_{\text{log}}} \to \text{Ker}(\omega).
$$

Lemma 6.2.1.

The image of $\tilde{\nabla}_{E_t}^{\text{ad},-1}$ is contained in $\text{Ker}(\omega)$.

Proof. Let us consider the $\mathcal{O}_X$-linear endomorphism $\zeta$ of $\tilde{T}_{E_t^{\text{log}}/\mathcal{S}_{\text{log}}}$ determined by assigning $s \mapsto s - \nabla_{E_t} \circ a_{E_t}^{\text{log}}(s)$ for any local section $s \in \tilde{T}_{E_t^{\text{log}}/\mathcal{S}_{\text{log}}}$. Then, for any local section $s$ of $\tilde{T}_{E_t^{\text{log}}/\mathcal{S}_{\text{log}}}$, the following equalities hold:

$$
a_{E_t}^{\text{log}}(\zeta(s)) = a_{E_t}^{\text{log}}(s - \nabla_{E_t} \circ a_{E_t}^{\text{log}}(s)) \\
= a_{E_t}^{\text{log}}(s) - (a_{E_t}^{\text{log}} \circ \nabla_{E_t})(a_{E_t}^{\text{log}}(s)) \\
= a_{E_t}^{\text{log}}(s) - a_{E_t}^{\text{log}}(s) \\
= 0.
$$
Hence, we have \( \text{Im}(\zeta) \subseteq \text{Ker}(a_{\log E}^{\log})(= b_{E_T}) \). Moreover, let us observe the following equalities:

\[
\langle \partial, \nabla^{ad}(\zeta(s)) \rangle = \langle \partial, \nabla^{ad,-1/1}(\zeta(s)) \rangle = [\nabla E(\partial), \zeta(s)] = \langle \partial, \tilde{\nabla}^{ad}(\zeta(s)) \rangle = \langle \partial, \nabla^{ad}(s) \rangle.
\]

This implies that the image of \( \tilde{\nabla}^{ad} \) is contained in the image of \( \nabla^{ad} \), which coincides tautologically with the kernel of the quotient \( \Omega^{\log SS}/S \log \otimes b_{E_T} \rightarrow \text{Coker}(\nabla^{ad}) \). Thus, this observation and the definition of \( \tilde{\nabla}^{ad,-1/1} \) implies the validity of Lemma 6.2.1.

Here, we shall denote by

\[
\mathcal{MOp}_{g,\log}^{Zas...}
\]

the log stack defined to be the stack \( \mathcal{MOp}_{g,\log}^{Zas...} \) equipped with the log structure pulled-back from the log structure of \( \mathcal{M}_{g,\log}^{Zas...} \) via the forgetful morphism \( \mathcal{MOp}_{g,\log}^{Zas...} \rightarrow \mathcal{M}_{g,\log}^{Zas...} \).

**Proposition 6.2.2.**

Let \( c_{X,\tilde{E},\diamond} \) and \( c_X \) be as in Proposition 6.1.2. Suppose further that \( \tilde{E}_{\log} \) is dormant, i.e., \( c_{X,\tilde{E},\diamond} \) factors through the closed immersion \( \mathcal{MOp}_{g,\log}^{Zas...} \rightarrow \mathcal{MOp}_{g,\log}^{Zas...} \). Denote by \( \overline{c}_{X,\tilde{E},\diamond} : S \rightarrow \mathcal{MOp}_{g,\log}^{Zas...} \) the resulting \( S \)-rational point of \( \mathcal{MOp}_{g,\log}^{Zas...} \). Then, there exists a natural isomorphism

\[
\overline{c}_{X,\tilde{E},\diamond}(\mathcal{T}_{\mathcal{MOp}_{g,\log}^{Zas...}/\mathcal{O}_{X,\log}}) \cong f_* f^*(K^* [\tilde{\nabla}^{ad}])
\]

which makes the square diagram

\[
\begin{array}{ccc}
\overline{c}_{X,\tilde{E},\diamond}(\mathcal{T}_{\mathcal{MOp}_{g,\log}^{Zas...}/\mathcal{O}_{X,\log}^F}) & \cong & f_*(K^* [\tilde{\nabla}^{ad}]) \\
\cong & & \cong \\
\overline{c}_{X,\tilde{E},\diamond}(\mathcal{T}_{\mathcal{MOp}_{g,\log}^{Zas...}/\mathcal{O}_{X,\log}^F}) & \cong & f_*(K^* [\tilde{\nabla}^{ad}])
\end{array}
\]
The assertion follows from \[39\], Proposition 6.8.1, and the various definitions involved.

6.3. The smoothness of \( \mathcal{M}_{\text{Op}} \)

In what follows, let us consider the case where \( g = \mathfrak{sl}_2 \) and prove the smoothness of the moduli stack \( \mathcal{M}_{\text{Op}} \) of dormant generic Miura \( \mathfrak{sl}_2 \)-opers.

Lemma 6.3.1.
Let us keep the above notation, and suppose further that \( g = \mathfrak{sl}_2 \) and \( S = \text{Spec}(k) \). Denote by \( H^j(X, \mathcal{K}^\bullet[\tilde{\nabla}_{\nabla}] \) the \( j \)-th hypercohomology group of the complex \( \mathcal{K}^\bullet[\tilde{\nabla}_{\nabla}] \).

(i) The following equalities hold:
\[
H^0(X, \mathcal{K}^\bullet[\tilde{\nabla}_{\nabla}] = H^2(X, \mathcal{K}^\bullet[\tilde{\nabla}_{\nabla}]) = 0.
\]

In particular, any 1-st order deformation of \((X, \hat{E})\) (i.e., the pair consisting of the pointed stable curve \( X \) and the dormant generic Miura \( \mathfrak{sl}_2 \)-oper \( \hat{E} \) on it) is unobstructed. That is to say, \( \mathcal{M}_{\text{Op}} \) is smooth at the point \((X, \hat{E})\).

(ii) Let \( \tilde{e} := (\varepsilon_i)_{i=1}^r \in \mathbb{F}^r \) (where \( \tilde{e} := 0 \) if \( r = 0 \)). Assume further that \( X \) is smooth over \( k \) and \( \hat{E} \) is of exponents \( [\tilde{e}] \). Then, \( H^1(X, \mathcal{K}^\bullet[\tilde{\nabla}_{\nabla}]) \) is a \( k \)-vector space of dimension \( 2g - 2 + \frac{2g - 2 + \sum_{i=1}^r r_i - 1}{2} \) (cf. (32) for the definition of \( \tau \)).

Proof. First, we shall prove assertion (i). Since the latter assertion follows from the former assertion (and Proposition 6.2.2), it suffices to prove the former assertion (i.e., the equalities in (220)).

Let \( \zeta : \mathcal{T}_{\hat{E}}^{\log} \to \mathcal{B}_{\mathfrak{g}^\tau} \) be the morphism obtained from \( \zeta \) (defined in the proof of Lemma 6.2.1) by restricting its domain and codomain. One verifies from the definition of a Miura \( \mathfrak{g} \)-oper (in particular, of \( \nabla \)) that \( \zeta \) is surjective. Both \( \mathcal{T}_{\hat{E}}^{\log} \) and \( \mathcal{B}_{\mathfrak{g}^\tau} \) are rank 2 vector bundles, and hence, \( \zeta \) turns out to be an isomorphism. Since \( \text{Im}(\zeta) \subseteq b_{\mathfrak{g}^\tau} \) and the composite equality (216) holds, the pair of \( \zeta \) and the identity morphism of \( \Omega_{\hat{E}}^{\log} \otimes b_{\mathfrak{g}^\tau} \) specifies an isomorphism \( \mathcal{K}^\bullet[\tilde{\nabla}_{\nabla}^{-1/2}] \simeq \mathcal{K}^\bullet[\tilde{\nabla}_{\nabla}] \). This isomorphism restricts to an
isomorphism

\[(221) \quad \mathcal{K}^\bullet[\nabla_{\mathcal{E}_T}] \cong \mathcal{K}^\bullet[\nabla_{\mathcal{E}_T}] ,\]

where $\nabla_{\mathcal{E}_T}$ denotes the morphism $\mathcal{b}_{\mathcal{E}_T} \to \text{Im}(\nabla_{\mathcal{E}_T})$ obtained by restricting the codomain of $\mathcal{K}^\bullet[\nabla_{\mathcal{E}_T}]$. $\mathcal{K}^\bullet[\nabla_{\mathcal{E}_T}]$ is quasi-isomorphic to $\text{Ker}(\nabla_{\mathcal{E}_T})[0]$ via the natural inclusion $\text{Ker}(\nabla_{\mathcal{E}_T})[0] \to \mathcal{K}^\bullet[\nabla_{\mathcal{E}_T}]$. Hence, we have the composite isomorphism

\[(222) \quad \mathbb{H}^j(X, \mathcal{K}^\bullet[\nabla_{\mathcal{E}_T}]) \cong \mathbb{H}^j(X, \mathcal{K}^\bullet[\nabla_{\mathcal{E}_T}]) \cong H^j(X, \text{Ker}(\nabla_{\mathcal{E}_T})).\]

In particular, (since $\dim(X) = 1$) the equality $\mathbb{H}^2(X, \mathcal{K}^\bullet[\nabla_{\mathcal{E}_T}]) = 0$ holds.

Next, let us prove that $(\mathbb{H}^0(X, \mathcal{K}^\bullet[\nabla_{\mathcal{E}_T}]) \cong H^0(X, \text{Ker}(\nabla_{\mathcal{E}_T}))) = 0$. Let

\[(223) \quad \kappa : F^\bullet_{X/k}(\text{Ker}(\nabla_{\mathcal{E}_T}))) \to \mathcal{b}_{\mathcal{E}_T}\]

be the $\mathcal{O}_X$-linear morphism corresponding, via the adjunction relation “$F^\bullet_{X/k}(-) \dashv F^\bullet_{X/k}(\text{Ker}(\nabla_{\mathcal{E}_T})))$", to the natural inclusion $F^\bullet_{X/k}(\text{Ker}(\nabla_{\mathcal{E}_T})) \hookrightarrow F^\bullet_{X/k}(\mathcal{b}_{\mathcal{E}_T})$. Since $\nabla_{\mathcal{E}_T}$ has vanishing $p$-curvature, $\kappa$ becomes an isomorphism when restricted to $X \setminus \bigcup_{i=1}^{n} \{\text{Im}(\sigma_i)\}$ (cf. [17], §5, p. 190, Theorem 5.1). Also, $\kappa$ is compatible with the respective $k$-connections $\nabla_{\mathcal{E}_T}^{can}$ and $\nabla_{\mathcal{E}_T}^{ad}$. Now, suppose that $H^0(X, \text{Ker}(\nabla_{\mathcal{E}_T})) (= H^0(X^{(1)}_k, F^\bullet_{X/k}(\text{Ker}(\nabla_{\mathcal{E}_T}))) \neq 0$, i.e., that there exists a nonzero element of $H^0(X^{(1)}_k, F^\bullet_{X/k}(\text{Ker}(\nabla_{\mathcal{E}_T})))$. This element determines an injection $\mathcal{O}_{X^{(1)}_k} \hookrightarrow F^\bullet_{X/k}(\text{Ker}(\nabla_{\mathcal{E}_T})))$. By pulling-back via $F_{X/k}$, we obtain an injection $\mathcal{O}_X \hookrightarrow F^\bullet_{X/k}(F^\bullet_{X/k}(\text{Ker}(\nabla_{\mathcal{E}_T})))$ compatible with the respective $k$-connections $d$ and $\nabla_{\mathcal{E}_T}^{can}$. Denote by $\kappa' : \mathcal{O}_X \hookrightarrow \mathcal{b}_{\mathcal{E}_T}$ the composite of that injection and $\kappa$. Here, recall that $\mathcal{b}_{\mathcal{E}_T} = \mathcal{g}_{\mathcal{E}_T}^0 \oplus \mathcal{g}_{\mathcal{E}_T}^{-1} \cong \mathcal{O}_X \oplus T_{X^{(1)}_k}$ (cf. (3)). Since $\deg(T_{X^{(1)}_k}) < 0$, the composite of $\kappa'$ and the second projection $\mathcal{b}_{\mathcal{E}_T} (= \mathcal{g}_{\mathcal{E}_T}^0 \oplus \mathcal{g}_{\mathcal{E}_T}^{-1}) \to \mathcal{g}_{\mathcal{E}_T}^{-1}$ is zero. Hence, the image of $\kappa'$ lies in the first component $\mathcal{g}_{\mathcal{E}_T}^0 \subseteq \mathcal{b}_{\mathcal{E}_T}$. But, $\mathcal{g}_{\mathcal{E}_T}^0$ is (by the definition of $\nabla_{\mathcal{E}_T}$) not closed under $\nabla_{\mathcal{E}_T}^{-1}$. This is a contradiction, and consequently, the equality $H^0(X^{(1)}_k, F^\bullet_{X/k}(\text{Ker}(\nabla_{\mathcal{E}_T}))) = 0$ holds. This completes the proof of assertion (i).

Next, let us consider assertion (ii). Let $X$ and $\mathcal{E}^{\bullet \circ \bullet} \mathcal{O}$ be as assumed in the statement of (ii). By the definition of a Miura $\mathfrak{g}$-oper, $\nabla_{\mathcal{E}_T}^{-1}$ may restrict to a $k$-connection $\nabla_{-1}$ on $\mathcal{g}_{\mathcal{E}_T}^{-1}$. Also, we obtain a $k$-connection $\nabla_{0}$ on $\mathcal{g}_{\mathcal{E}_T}^{0}$ induced from $\nabla_{\mathcal{E}_T}^{-1}$ via the projection $\mathcal{b}_{\mathcal{E}_T} \to (\mathcal{b}_{\mathcal{E}_T} \mathcal{g}_{\mathcal{E}_T}^{-1} = \mathcal{g}_{\mathcal{E}_T}^{0})$. For each $i \in \{1, \cdots, r\}$, the
monodromy of \((\mathcal{E}_B', \nabla_{\mathcal{E}_B'})\) at \(\sigma_i\) coincides with \(\left(\begin{array}{cc} -\frac{\varepsilon_i}{2} & 0 \\ 1 & \frac{\varepsilon_i}{2} \end{array}\right) \in \Gamma(S, \sigma_i^*(b_{E'\mathcal{B}^\dag})) = b^-(k) (\subseteq \mathfrak{sl}_2(k))\). It follows that the monodromies at \(\sigma_i\) of \((g_{E'\mathcal{B}^\dag}^{-1}, \nabla_{-1})\) and \((g_{E'\mathcal{B}^\dag}^0, \nabla_0)\) are \(\varepsilon_i\) and 0 respectively. (Indeed, the monodromy of \(\nabla_{ad} E'\) at \(\sigma_i\) may be expressed, by means of the basis \(\left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\rangle\) of \(b^-(k)\), as the matrix \(\begin{pmatrix} 0 & 0 \\ 2 & \varepsilon_i \end{pmatrix}\).) Hence, if \(\kappa_{-1} : F^*_X/k(F_X/k^*(\text{Ker}(\nabla_{-1}))) \hookrightarrow g_{E'\mathcal{B}^\dag}^{-1}\) and \(\kappa_0 : F^*_X/k(F_X/k^*(\text{Ker}(\nabla_0))) \hookrightarrow g_{E'\mathcal{B}^\dag}^0\) are the injections obtained in the same manner as \(\kappa\), then \(\kappa_0\) is an isomorphism and \(\text{Coker}(\kappa_{-1}) \cong \bigoplus_{i=1}^r \Lambda_i\), where each \(\Lambda_i (i = 1, \cdots, r)\) is an \(\mathcal{O}_X\)-module supported on \(\text{Im}(\sigma_i)\) of length \(\tau^{-1}(-\varepsilon_i) \in \tilde{F}_p\).

On the other hand, by [29], Corollary 3.2.2 (and the fact that \(F^*_X/k\) is flat), the following natural sequence turns out to be exact:

\[
0 \to F^*_X/k(F_X/k^*(\text{Ker}(\nabla_{-1}))) \to F^*_X/k(F_X/k^*(\text{Ker}(\nabla_{E'\mathcal{B}^\dag}))) \to F^*_X/k(F_X/k^*(\text{Ker}(\nabla_0))) \to 0
\]

Thus, the following sequence of equalities holds:

\[
\text{deg}(F^*_X/k^*(\text{Ker}(\nabla_{E'\mathcal{B}^\dag}))) = \frac{1}{p} \cdot \text{deg}(F^*_X/k^*(\text{Ker}(\nabla_{E'\mathcal{B}^\dag})))
\]

\[
= \frac{1}{p} \cdot \left(\text{deg}(F^*_X/k(F_X/k^*(\text{Ker}(\nabla_{\mathcal{E}_B'})))) + \text{deg}(F^*_X/k(F_X/k^*(\text{Ker}(\nabla_{-1}))))\right)
\]

\[
= \frac{1}{p} \cdot \left(\text{deg}(g_{E'\mathcal{B}^\dag}^0) + \left(\text{deg}(g_{E'\mathcal{B}^\dag}^{-1}) - \sum_{i=1}^r \text{length}(\Lambda_i)\right)\right)
\]

\[
= \frac{1}{p} \cdot \left(\text{deg}(\mathcal{O}_X) + \text{deg}(\mathcal{T}_{X^{\text{log}}/k}) - \sum_{i=1}^r \tau^{-1}(-\varepsilon_i)\right)
\]

\[
= \frac{-2g - 2 + r + \sum_{i=1}^r \tau^{-1}(-\varepsilon_i)}{p}.
\]
Since $X$ was assumed to be smooth (hence $F_{X/k*}(\text{Ker}(\nabla^{\text{ad}}_{E^\prime}B))$ is a rank 2 vector bundle), the following sequence of equalities holds:

\begin{equation}
\dim(H^1(X, \text{Ker}(\nabla^{\text{ad}}_{E^\prime}B))) = \dim(H^1(X, F_{X/k*}(\text{Ker}(\nabla^{\text{ad}}_{E^\prime}B)))) = \dim(H^0(X, F_{X/k*}(\text{Ker}(\nabla^{\text{ad}}_{E^\prime}B)))) - \text{rk}(F_{X/k*}(\text{Ker}(\nabla^{\text{ad}}_{E^\prime}B))) \cdot (1 - g) - \deg(F_{X/k*}(\text{Ker}(\nabla^{\text{ad}}_{E^\prime}B))) - 2 \cdot (1 - g) + 2g - 2 + \sum_{i=1}^{r} \tau^{-1}(-\epsilon_i) \sum_{i=1}^{r} \tau^{-1}(-\epsilon_i),
\end{equation}

where the second equality follows from the Riemann-Roch theorem and the third equality follows from assertion (i) and (225). This completes the proof of assertion (ii). □

We shall write

\begin{equation}
\mathcal{M}_\text{Op}^{Z_{\text{zzz}...}sl_2,g,r,\vec{\epsilon}} := \mathcal{M}\mathcal{D}_\text{sl}_{2,g,r,\vec{\epsilon}} \times \mathcal{M}_{g,r}
\end{equation}

(where $\vec{\epsilon} \in \mathfrak{t}_{\text{PGL}_2}(\mathbb{F}_p)^r$ or $\vec{\epsilon} = \emptyset$). By Lemma 6.3.1 (and Theorem 3.8.3 (ii)), the following assertion holds.

**Theorem 6.3.2.**

Let $\vec{\epsilon} \in \mathbb{F}_p^r$ (where $\vec{\epsilon} := \emptyset$ if $r = 0$). Then, the stack $\mathcal{M}\mathcal{D}_{\text{sl}_{2,g,r,\vec{\epsilon}}}$ is a (possibly empty) smooth proper Deligne-Mumford stack over $k$.

If $2g - 2 + \frac{2g - 2 + r + \sum_{i=1}^{p} \tau^{-1}(-\epsilon_i)}{p} < 0$, then $\mathcal{M}\mathcal{D}_{\text{sl}_{2,g,r,\vec{\epsilon}}}$ is empty. Moreover, if $2g - 2 + \frac{2g - 2 + r + \sum_{i=1}^{p} \tau^{-1}(-\epsilon_i)}{p} \in \mathbb{Z}_{\geq 0}$, then any irreducible component $\mathcal{N}$ of $\mathcal{M}\mathcal{D}_{\text{sl}_{2,g,r,\vec{\epsilon}}}$ with $\mathcal{N} \times \mathcal{M}_{g,r} \neq \emptyset$ is equidimensional of dimension $2g - 2 + \frac{2g - 2 + r + \sum_{i=1}^{p} \tau^{-1}(-\epsilon_i)}{p}$.

In particular, the above corollary (of the case where $r = 0$) and Remark 6.3.3 (i) below imply Theorem B.

**Remark 6.3.3.**

In this remark, let us mention two facts concerning the non-emptiness of $\mathcal{M}\mathcal{D}_{\text{sl}_{2,g,r}}$ deduced from the previous works.

(i) Let $l$ be an integer with $lp \geq 4$. According to [30, Example], it is verified that

\begin{equation}
\mathcal{M}\mathcal{D}_{\text{sl}_{2},(lp-1)(lp-2),0} \neq \emptyset.
\end{equation}
In order to prove this, (under the assumption that \( k \) is an algebraically closed field) let us consider the smooth projective curve \( X \) in \( \mathbb{P}^2 (= \text{Proj}(k[x, y, z])) \) defined by the equation \( x^{lp} - x^{lp-1} - y^{lp-1} = 0 \). The genus of \( X \) is given by \( \frac{(lp-1)(lp-2)}{2} \) (\( \geq 2 \)). One verifies that \( \Omega_{X/k} = \mathcal{O}_X(lp(lp - 3) \cdot P_\infty) \), where \( P_\infty = [0 : 0 : 1] \), and the line bundle \( \text{id}_X \times F_k^* \mathcal{O}_X(l((lp - 3) \cdot P_\infty)) \) on \( X_k^{(1)} \) specifies a Tango structure. The existence of this Tango structure implies \( (228) \).

(ii) Suppose that \( g \geq p \). Then, there exists an element \( \vec{e} \in \mathbb{F}^{xr} \) (or \( \vec{e} = \emptyset \)) such that \( \mathcal{MOp}_{sl_2, g, r, [\vec{e}]} \neq \emptyset \) and the composite projection

\[
(229) \quad \mathcal{MOp}_{sl_2, g, r, [\vec{e}]} \to \mathcal{M}_{g, r} \to \mathcal{M}_{g, r-1} \to \cdots \to \mathcal{M}_{g, 0}
\]

has dense image. Indeed, let us take a geometric generic point \( c : \text{Spec}(K) \to \mathcal{M}_{g, 0} \) of \( \mathcal{M}_{g, 0} \), where \( K \) denotes some algebraically closed field over \( k \). Denote by \( X := (X/K, \{\sigma_i\}_{i=1}^{r}) \) the (unpointed) proper smooth curve over \( K \) classified by \( c \). It follows from \( [33] \), Corollary 1.5, that there exist a (possibly empty) collection of \( K \)-rational points \( \sigma_1, \ldots, \sigma_r \) of \( X \), an element \( \vec{e} \) in \( \mathbb{F}^{xr} \), and a pre-Tango structure \( \nabla \) on \( X \) of monodromies \( -\vec{e} \). \( \nabla \) specifies a \( K \)-rational point of \( \mathcal{MOp}_{sl_2, g, r, [\vec{e}]} \) over \( c \). This implies that the image of the composite projection \( \mathcal{MOp}_{sl_2, g, r, [\vec{e}]} \to \mathcal{M}_{g, 0} \) is dense.

6.4. The case of \( g = 1 \).

In this last subsection, we study the moduli stack \( \mathcal{MOp}_{sl_2, g, r, [\vec{e}]} \) of the case where \( g = 1 \). To begin with, we shall observe the following assertion.

Proposition 6.4.1.

Let \( \vec{e} := (\varepsilon_i)_{i=1}^{r} \in \mathbb{F}^{xr} \) (where \( \vec{e} = \emptyset \) if \( r = 0 \)). For any positive integer \( s \), we shall write \( \vec{e}_{r+s} := (\varepsilon_1, \cdot \ldots, \varepsilon_r, 1, 1, \ldots, 1) \in \mathbb{F}^{(r+s)}_p \), where the last \( s \) factors are all 1. Then, there exists a canonical isomorphism

\[
(230) \quad \mathcal{MOp}_{sl_2, g, r, [\vec{e}]} \times \mathcal{M}_{g, r} \cong \mathcal{MOp}_{sl_2, g, r+s, [\vec{e}_{r+s}]} \]

over \( \mathcal{M}_{g, r+s} \). In particular,

\[
(231) \quad "\mathcal{MOp}_{sl_2, g, r, [\vec{e}]} \neq \emptyset" \iff "\mathcal{MOp}_{sl_2, g, r+s, [\vec{e}_{r+s}]} \neq \emptyset".
\]

Proof. Let \( S \) be a \( k \)-scheme and \( \bar{X} := (X/S, \{\sigma_i\}_{i=1}^{r}) \) an \((r+s)\)-pointed smooth curve over \( S \) of genus \( g \). Denote by \( \bar{X} := (X/S, \{\sigma_i\}_{i=1}^{r}) \) the pointed smooth curve obtained from \( \bar{X} \) by forgetting the last \( s \) marked points \( \{\sigma_i\}_{i=r+1} \) (hence, \( S^{\text{log}} = S/\log \rightarrow S \)). Suppose that we are given a pre-Tango structure \( \nabla \) on \( \bar{X} \) of monodromies \( -\vec{e} \). The pair \( (\nabla, \bar{X}) \) specifies an \( S \)-rational point of \( \mathcal{MOp}_{sl_2, g, r, [\vec{e}]} \times \mathcal{M}_{g, r} \). One may find a unique Tango structure \( \nabla_+ \) on \( \bar{X} \) whose
restriction to $\Omega^{X_{\log}/S} = \Omega^{X_{\log}/S}_{-\sum_{i=r+1}^s \sigma_i}$ coincides with $\nabla$. Moreover, the monodromy of $\nabla_+$ at $\sigma_i$ (for each $i = r + 1, \cdots, r + s$) coincides with $-1$. That is to say, $(\nabla_+, X)$ specifies an $S$-rational point of $\mathcal{T}an^\prime_{g,r+s,-(\varepsilon_{1+s})} \times \mathcal{M}_{g,r+s}$. One verifies immediately that the resulting morphism

$$\mathcal{T}an^\prime_{g,r+s,-(\varepsilon_{1+s})} \times \mathcal{M}_{g,r+s} \rightarrow \mathcal{T}an_{g,r+s,-(\varepsilon_{1+s})} \times \mathcal{M}_{g,r+s} \tag{232}$$

(i.e., the morphism given by $(\nabla, X) \mapsto (\nabla_+, X)$) determines an isomorphism over $\mathcal{M}_{g,r+s}$. Hence, the desired isomorphism may be obtained from this isomorphism and (197). □

**Corollary 6.4.2.**
Suppose that $r > 0$, and write $\vec{1}_r := (1, 1, \cdots, 1) \in \mathbb{F}_p^{\times r}$. Then, $\mathcal{M}Op^{\text{Zas...}}_{sl_2,1,r,\vec{1}_r}$ is a nonempty, geometrically connected, and smooth Deligne-Mumford stack over $k$ of dimension $r$. Moreover, the forgetting morphism $\mathcal{M}Op^{\text{Zas...}}_{sl_2,1,r,\vec{1}_r} \rightarrow \mathcal{M}_{1,r}$ is finite, surjective, and generically étale of degree $p - 1$.

**Proof.** According to Example 5.3.3, there exists a pre-Tango structure $\nabla$ on any ordinary elliptic curve $X/k$ (without marked points). It follows from Proposition 6.4.1 (more precisely, from the discussion in the proof of that proposition) that $\nabla$ gives rise to a pre-Tango structure of monodromies $-\vec{1}_r$ on $X/k$ with $r$ marked points. This implies that $\mathcal{M}Op^{\text{Zas...}}_{sl_2,1,r,\vec{1}_r}$ is nonempty. By Theorem 6.3.2, we can verify the smoothness and the calculation of the dimension, and moreover, the fact that the forgetting morphism $\mathcal{M}Op^{\text{Zas...}}_{sl_2,1,r,\vec{1}_r} \rightarrow \mathcal{M}_{1,r}$ is finite and surjective. Also, according to the discussion in Example 5.3.3 $\mathcal{M}Op^{\text{Zas...}}_{sl_2,1,r,\vec{1}_r}$ is isomorphic to $\mathcal{C}o_{\mathcal{O}_{1,r},1,r,(0,0,\cdots,0)} \setminus \mathcal{I}$, where $\mathcal{I}$ denotes the component classifying the trivial connections (i.e., the universal derivations) on curves. In particular, the forgetting morphism $\mathcal{M}Op^{\text{Zas...}}_{sl_2,1,r,\vec{1}_r} \rightarrow \mathcal{M}_{1,r}$ is generically étale and of degree $p - 1$. (More precisely, the forgetting morphism is étale at the points classifying pointed smooth curves whose underlying curves are ordinary.) Its fiber over the point classifying any supersingular curve consists precisely of one point. Hence, $\mathcal{M}Op^{\text{Zas...}}_{sl_2,1,r,\vec{1}_r} \cong \mathcal{C}o_{\mathcal{O}_{1,r},1,r,(0,0,\cdots,0)} \setminus \mathcal{I}$ is geometrically connected. This completes the proof of Corollary 6.4.2. □

7. **Pathology in positive characteristic**

In this last section, we study the pathology of algebraic geometry in positive characteristic, which is of certain interest, since pathology reveals some completely different geometric phenomena from those in complex geometry.
7.1. Generalized Tango curves.

In what follows, suppose that \( k \) is an algebraically closed field of characteristic \( p > 2 \).

**Definition 7.1.1.** [cf. 34, § 3]

Let \( S \) be a \( k \)-scheme and \( l \) a positive integer.

(i) A **generalized Tango curve of index** \( l \) over \( S \) is a quadruple
\[
\hat{X} := (X, \mathcal{L}, \mathcal{N}, \nu),
\]
where \( X \) denotes a geometrically connected, proper, and smooth curve over \( S \), \( \mathcal{L} \) denotes a Tango structure on \( X/S \), \( \mathcal{N} \) a line bundle on \( X^{(1)}_S \), and \( \nu \) denotes an isomorphism \( \nu : \mathcal{N} \otimes (lp - 1) \xrightarrow{\sim} \mathcal{L} \). (Notice that this definition is equivalent to the definition of a generalized Tango curve of index \( lp - 1 \) in the sense of [34], § 3.)

(ii) Let \( \hat{X} := (X, \mathcal{L}, \mathcal{N}, \nu) \) and \( \hat{X}' := (X', \mathcal{L}', \mathcal{N}', \nu') \) be generalized Tango curves of index \( l \) over \( S \). An **isomorphism of generalized Tango curves** from \( \hat{X} \) to \( \hat{X}' \) is a pair \((h_X, h_N)\) consisting of an isomorphism \( h_X : X \xrightarrow{\sim} X' \) over \( S \) (where we shall write \( h_X^{(1)} \) for the isomorphism \( X^{(1)}_S \xrightarrow{\sim} X'^{(1)}_S \) obtained from \( h_X \) via base-change by \( F_S \)) and an isomorphism \( h_N : h_X^{(1)}(\mathcal{N}') \xrightarrow{\sim} \mathcal{N} \) satisfying the following conditions:

- The isomorphism \( h_X^{(1)}(\mathcal{N}') \otimes (lp - 1) \xrightarrow{\sim} \mathcal{N} \otimes (lp - 1) \)

\[ h_X^{(1)}(\mathcal{N}') \xrightarrow{\sim} \mathcal{N} \]

is commutative.

Let \( g \) be an integer with \( g > 1 \). Since the pull-back of a generalized Tango curve by a morphism \( T \to S \) of \( k \)-schemes can be defined in a natural manner, we obtain a stack in groupoids
\[
\mathcal{GTan}_g
\]
over \( \mathcal{Sch}_{/\text{Spec}(k)} \) whose category of sections over an object \( S \) of \( \mathcal{Sch}_{/\text{Spec}(k)} \) is the groupoid of generalized Tango curves of index \( l \) over \( S \). According to [21] and [26], one may construct a family of algebraic varieties parametrized by
Remark 7.1.2. \(\mathfrak{GTan}_g^l\) may be represented by either the empty stack or an equidimensional smooth Deligne-Mumford over \(k\) of dimension \(2g - 2 + \frac{2g-2}{p}\), and the forgetting morphism \(\mathfrak{GTan}_g^l \rightarrow \mathfrak{Tan}_g\) (i.e., the morphism given by \((X, L, N, \nu) \mapsto (X, L)\)) is finite and étale. Indeed, for each integer \(d\), let us denote by \(\text{Pic}^d_g\) the Picard stack for the universal family of curves \(f_{\text{tut}}: C_g \rightarrow M_g\) classifying line bundles of relative degree \(d\). For each pair of integers \((d, e)\) with \(e \geq 1\), denote by \(\mu_{d,e}\) the morphism \(\text{Pic}^d_g \rightarrow \text{Pic}^{de}_g\) over \(M_g\) given by assigning \(L \mapsto L \otimes e\) (for any line bundle \(L\) of relative degree \(d\)). Then, we obtain a canonical isomorphism \(\mathfrak{GTan}_g^l \cong \mathfrak{Tan}_g \times \text{Pic}^N(2g - 2 + \frac{2g-2}{p})\), as desired.

7.2. Generalized Raynaud surfaces.

In what follows, we construct, by means of a certain closed substack of \(\mathfrak{GTan}_g^l\), a family of algebraic surfaces (of general type) which is parametrized by a high dimensional variety and each of whose fiber has the automorphism group scheme which is not reduced.

First, recall the generalized Raynaud surface associated with a Tango structure. Let \(S\) be a \(k\)-scheme, \(X\) a proper smooth curve over \(S\) of genus \(g\), and \(L(\subseteq F_{X/S}^\ast(B_{X/S}))\) a Tango structure on \(X/S\). Suppose that \(lp(p-1) = 2g - 2\) for some positive integer \(l\), and that there exists a line bundle \(N\) on \(X(1)\) of relative degree \(l\) (relative to \(S\)) admitting an isomorphism \(\nu : \text{Pic}^N \rightarrow \text{Pic}^{N(2g - 2)}\). In particular, the quadruple \(\hat{\mathcal{X}} := (X, L, N, \nu)\) specifies a generalized Tango curve of index 1 over \(S\).

Let us take an affine open covering \(\{U_{\alpha}\}_{\alpha \in I}\) of \(X\) such that there exists an isomorphism \(\xi_{\alpha} : \mathcal{O}(U_{\alpha}) \rightarrow \mathcal{N}|_{U_{\alpha}}\). Write \(I_2 := \{(\alpha, \beta) \in I \times I \mid U_{\alpha \beta} \neq \emptyset\}\) (where \(U_{\alpha \beta} := U_\alpha \cap U_\beta\)). For each pair \((\alpha, \beta) \in I_2\), the automorphism \(\xi_{\beta}|_{U_{\alpha \beta}} \circ \xi_{\alpha}^{-1}|_{U_{\alpha \beta}}\) of \(\mathcal{N}|_{U_{\alpha \beta}}\) is given by multiplication by some element
Denote by $P$ the resulting algebraic surface and refer to it as the \textbf{generalized Raynaud surface} associated with the generalized Tango curve $\hat{\mathcal{X}}$. Denote by $\Psi : P \to X^{(1)}_S$ the natural projection. For each point $s$ of $S$, we shall denote by $P_s$ and $X^{(1)}_s$ the fibers over $s$ of the composite projection $P \xrightarrow{\Psi} X^{(1)}_S \xrightarrow{\varphi} S$ and the projection $X^{(1)}_S \to S$ respectively. According to [30] (or, [36], Theorem 3.1), the fiber $P_s$ (for any $k$-rational point $s$ of $S$) is a proper smooth algebraic surface over $k$, and it is of general type (resp., a quasi-elliptic surface) if $p > 3$ (resp., $p = 3$). Moreover, by [35], Theorem 2.1, there exists an isomorphism

$\Gamma(P_s, \mathcal{T}_{P_s/k}) \cong \Gamma(X^{(1)}_s, \mathcal{N}|_{X^{(1)}_s})$. 

Here, we shall assume that $p > 3$ and $\Gamma(X^{(1)}_s, \mathcal{N}|_{X^{(1)}_s}) \neq 0$. Since $X^{(1)}_s$ is a surface of general type, the automorphism group scheme $\text{Aut}_k(X^{(1)}_s)$ of $X^{(1)}_s$ is finite. Hence, because of (240), $\text{Aut}_k(X_s)$ is not reduced; this fact may be thought of as a pathological phenomenon (relative to zero characteristic) of algebraic geometry in positive characteristic.

Denote by

$\mathcal{S}^1_{\hat{\mathcal{T}}an, p^r \neq 0}$
the closed substack of $\mathcal{T}an^1\Gamma$ (resp. $\mathcal{T}an_g \times_{\mathcal{Pic}_g} \mathcal{Pic}^{p(p-2)}_{\mathcal{Pic}_g}$) classifying generalized Tango curves $(X/k, \mathcal{L}, N, \nu)$ with $\Gamma(X^{(1)}_k, N) \neq 0$. The tautological family of generalized Tango curves over $\mathcal{T}an^1_{g, \Gamma \neq 0}$ induces, by means of the above discussion, a family

$$\mathcal{P} \to \mathcal{T}an^1_{g, \Gamma \neq 0}$$

of proper smooth surfaces of general type parametrized by $\mathcal{T}an^1_{g, \Gamma \neq 0}$ all of whose fibers have the automorphism group schemes being non-reduced.

**Theorem 7.2.1.**

Suppose that $p > 3$, $p(p-1)|2g-2$, and $4|p-3$. Then, $\mathcal{T}an^1_{g, \Gamma \neq 0}$ is a nonempty closed substack of $\mathcal{T}an^1_g$ of dimension $\geq g - 2 + \frac{2g-2}{p-1}$.

**Proof.** It follows from [36], Theorem 4.1, that $\mathcal{T}an^1_{g, \Gamma \neq 0}$ is nonempty. In the following, we shall calculate the dimension of $\mathcal{T}an^1_{g, \Gamma \neq 0}$. Let us take a $k$-scheme $T$ together with an étale surjective morphism $T \to \mathcal{T}an^1_g$; this morphism classifies a generalized Tango curve $(Y, \mathcal{L}_Y, N_Y, \nu_Y)$ of index 1 over $T$. Since $\mathcal{T}an^1_g$, as well as $T$, has dimension $2g - 2 + \frac{2g-2}{p}$ (cf. Remark 7.1.2), the relative Picard scheme $\mathcal{Pic}^{N}_{Y/T}$ of $Y/T$ (cf. the proof of Proposition 1.8.1), where $N := \frac{2g-2}{p(p-1)}$, has dimension $(2g - 2 + \frac{2g-2}{p} + g =) 3g - 2 + \frac{2g-2}{p}$. Denote by

$$\mathcal{Pic}^{N}_{Y/T, \Gamma \neq 0}$$

the closed subscheme of $\mathcal{Pic}^{N}_{Y/T}$ (resp., the closed substack of $\mathcal{Pic}^{N}_{Y/T} := \mathcal{Pic}^{N}_{g} \times_{\mathcal{Pic}_g} T$) classifying line bundles admitting a nontrivial global section. The isomorphism displayed in (237) (of the case where $l = 1$) restricts to an isomorphism

$$\mathcal{T}an^1_{g, \Gamma \neq 0} \to \mathcal{T}an_g \times_{\mathcal{Pic}_g} \mathcal{Pic}^{N}_{Y/T, \Gamma \neq 0}$$

Thus, it suffices to calculate the dimension of $\mathcal{T}an_g \times_{\mathcal{Pic}_g} \mathcal{Pic}^{N}_{Y/T, \Gamma \neq 0}$. By a well-known fact of the Brill-Noether theory (cf. [10], [22]), the closed subscheme $\mathcal{Pic}^{N}_{Y/T, \Gamma \neq 0}$ of $\mathcal{Pic}^{N}_{Y/T}$ is of codimension $\leq g - N$. On the other hand, the closed subscheme $T \times_{[\mathcal{L}_Y]} \mathcal{Pic}^{N}_{Y/T, \Gamma \neq 0}$ of $\mathcal{Pic}^{N}_{Y/T}$ is of codimension $g$, where $[\mathcal{L}_Y] : T \to \mathcal{Pic}^{N}_{Y/T}$ denotes the classifying morphism of $\mathcal{L}_Y$ and $\mathcal{Pic}^{N}_{Y/T, \Gamma \neq 0} : \mathcal{Pic}^{N}_{Y/T} \to \mathcal{Pic}^{N}_{Y/T, \Gamma \neq 0}$ denotes the finite étale morphism given by assigning $\mathcal{L}' \mapsto \mathcal{L}' \otimes (p-1)$ (for any line bundle $\mathcal{L}'$). Hence, their intersection

$$T \times_{[\mathcal{L}_Y]} \mathcal{Pic}^{N}_{Y/T, \Gamma \neq 0}$$

$$\simeq \left( T \times_{[\mathcal{L}_Y]} \mathcal{Pic}^{N}_{Y/T, \Gamma \neq 0} \right) \times_{\mathcal{Pic}^{N}_{Y/T}} \mathcal{Pic}^{N}_{Y/T, \Gamma \neq 0}$$

is of dimension $\geq g - N$. Since $\mathcal{T}an_g \times_{\mathcal{Pic}_g} \mathcal{Pic}^{N}_{Y/T, \Gamma \neq 0}$ is nonempty, the dimension of $\mathcal{T}an_g \times_{\mathcal{Pic}_g} \mathcal{Pic}^{N}_{Y/T, \Gamma \neq 0}$ is at least $g - N$. Thus, the dimension of $\mathcal{T}an^1_{g, \Gamma \neq 0}$ is at least $g - N + g - N = 2g - 2$. Since $\mathcal{T}an^1_{g, \Gamma \neq 0}$ is nonempty, the dimension of $\mathcal{T}an^1_{g, \Gamma \neq 0}$ is at least $2g - 2$. Therefore, the dimension of $\mathcal{T}an^1_{g, \Gamma \neq 0}$ is at least $g - 2 + \frac{2g-2}{p-1}$. Since $p > 3$, $p(p-1)|2g-2$, and $4|p-3$, the dimension of $\mathcal{T}an^1_{g, \Gamma \neq 0}$ is at least $g - 2 + \frac{2g-2}{p-1}$, as desired.
is of codimension \( \leq g + (g - N) = 2g - N \). That is to say, it has dimension 
\[ \geq 3g - 2 + \frac{2g - 2}{p - 1} - (2g - N) = g - 2 + \frac{2g - 2}{p - 1}. \]
Since the natural projections in the diagram
\[
\begin{array}{c}
T \times L_Y, \psi \nu \Rightarrow \Psi^N_{Y/T, \Gamma \neq 0} \quad \longrightarrow \quad T \times [L_Y, \psi \nu^N_{Y/T, \Gamma \neq 0}] \\
\xrightarrow{T \times \psi \nu^N_{Y/T, \Gamma \neq 0}} \Rightarrow \Psi^N_{Y/T, \Gamma \neq 0}
\end{array}
\]
are étale and surjective, \( \Rightarrow \) turns out to be of dimension \( \geq g - 2 + \frac{2g - 2}{p - 1} \). This completes the proof of Theorem 7.2.1

**Remark 7.2.2.**

Let us keep the notation and assumption in Theorem 7.2.1. Denote by \( \Xi_g \) the locus of \( \mathcal{M}_g \) classifying curves which admits a Tango structure of the form \( \mathcal{L} = N \otimes (p-1) \) for some line bundle \( N \) having a nontrivial global section. According to [36], Theorem 4.1, \( \Xi_g \) contains a variety of dimension \( \geq g - 1 \). Moreover, Tsuda’s method (cf. [37] or [36], Remark 4.2) gives a slightly better estimation:
\[
\dim(\Xi_g) \geq 2g - \frac{(g - 1)(p - 1)}{p}.
\]
According to our result, we can prove a lower bound estimation sharper than the bounds obtained previously. Indeed, since the stack-theoretic image of the projection \( \mathcal{G}^\Gamma_{g, \Gamma \neq 0} \rightarrow \mathcal{M}_g \) coincides with \( \Xi_g \), Theorem 7.2.1 implies that
\[
(247) \quad \dim(\Xi_g) \geq g - 2 + \frac{2g - 2}{p - 1}.
\]

Finally, we shall conclude the paper with the following corollary (i.e., Theorem C), which follows immediately from Theorem 7.2.1 and the construction of generalized Raynaud surfaces discussed at the beginning of §7.2.

**Corollary 7.2.3.**

Let us keep the assumption in Theorem 7.2.1. Then, there exists a flat family \( \mathcal{Y} \rightarrow \Xi \) (i.e., \( \mathcal{Y} \rightarrow \mathcal{G}^\Gamma_{g, \Gamma \neq 0} \) displayed in (241)) of proper smooth algebraic surfaces of general type parametrized by a Deligne-Mumford stack \( \Xi \) of dimension \( \geq g - 2 + \frac{2g - 2}{p - 1} \) all of whose fibers are pairwise non-isomorphic and have the automorphism group schemes being non-reduced.

**Proof.** The assertion follows from Theorem 7.2.1, the discussion preceding Theorem 7.2.1 and the following lemma. \( \square \)

**Lemma 7.2.4.**

Let \( \hat{\Xi} := (X, L, N, \nu) \) and \( \hat{\Xi}' := (X', L', N', \nu') \) be generalized Tango curves of index 1 over \( k \). Denote by \( P \) and \( P' \) the generalized Raynaud surfaces associated
with \( \hat{X} \) and \( \hat{X}' \) respectively. Suppose that \( P \) is isomorphic (as a scheme over \( k \)) to \( P' \). Then, \( \hat{X} \) is isomorphic to \( \hat{X}' \).

**Proof.** Denote by \( \Psi : P \to X_k^{(1)} \) and \( \Psi' : P' \to X_k^{(1)} \) the natural projections of \( P \) and \( P' \) respectively. Let us fix an isomorphism \( h_P : P \sim P' \). Recall that both \( X_k^{(1)} \) and \( X_k^{(1)} \) are of genus \( g > 1 \), and that the fibers of \( \Psi : P \to X_k^{(1)} \) and \( \Psi' : P' \to X_k^{(1)} \) are rational (cf. [36], Theorem 3.1). This implies that \( h_P \) maps each fiber of \( \Psi \) to a fiber of \( \Psi' \). Hence, \( h_P \) determines a homeomorphism \( |X_k^{(1)}| : \left| X_k^{(1)} \right| \sim \left| X_k^{(1)} \right| \) between the underlying topological spaces of \( X_k^{(1)} \) and \( X_k^{(1)} \) which is compatible, in a natural sense, with (the underlying homeomorphism of) \( h_P \) via \( \Psi \) and \( \Psi' \). Since \( \Psi \) and \( \Psi' \) induce isomorphisms \( O_{X_k^{(1)}} \sim \Psi_*(O_P) \) and \( O_{X_k^{(1)}} \sim \Psi'_*(O_P') \) respectively, \( |h_X^{(1)}| \) extends to an isomorphism \( h_X^{(1)} : X_k^{(1)} \sim X_k^{(1)} \) of \( k \)-schemes which makes the following square diagram commute:

\[
\begin{array}{ccc}
P & \xrightarrow{h_P} & P' \\
\Psi \downarrow & & \Psi' \downarrow \\
X_k^{(1)} & \xrightarrow{h_X^{(1)}} & X_k^{(1)}. \\
\end{array}
\]

(248)

As \( k \) is algebraically closed, one may find an isomorphism \( h_X : X \sim X' \) which induces \( h_X^{(1)} \), via base-change by \( F_{\text{Spec}(k)} \). Moreover, the isomorphism \( h_X \) induces an isomorphism

\[
h_X^{(1)}(F'_{k/\text{Spec}(k)}(\Omega_{X'/k})) \left( \cong F_X/k^*(h_X^*(\Omega_{X/k})) \right) \sim F_{X/k^*}(\Omega_{X/k}).
\]

(249)

Next, denote by \( P^{\text{sm}} \) and \( P'^{\text{sm}} \) the smooth loci in \( P \) and \( P' \) respectively relative to \( X_k^{(1)} \) and \( X_k^{(1)} \) respectively. \( h_P \) restricts to an isomorphism \( h_X^{\text{sm}} : P^{\text{sm}} \sim P'^{\text{sm}} \). It follows from [21], Lemma 1, (and the fact that both \( P^{\text{sm}} \) and \( P'^{\text{sm}} \) are relative affine spaces) that the projection \( \Psi|_{P^{\text{sm}}} : P^{\text{sm}} \to X_k^{(1)} \) (resp., \( \Psi'|_{P'^{\text{sm}}} : P'^{\text{sm}} \to X_k^{(1)} \)) admits a global section, and that the normal bundle of any global section of this projection is isomorphic to \( N' \) (resp., \( N' \)). Hence, by passing to \( h_P \) and \( h_X^{(1)} \), we obtain an isomorphism \( h_N' : h_X^{(1)}(N') \sim N' \). By the fact discussed in Remark 5.1.3, the images of the two composite injections

\[
h_X^{(1)}(N')^\otimes(p-1) \xrightarrow{h_X^{(1)}(p-1)} N^\otimes(p-1) \xrightarrow{\nu'_\text{incl}} F_{X/k^*}(\Omega_{X/k})
\]

(250)

and

\[
h_X^{(1)}(N')^\otimes(p-1) \xrightarrow{h_X^{(1)}(p-1)} h_X^{(1)}(\mathcal{L}') \xrightarrow{\text{incl}} h_X^{(1)}(F'_{X/k^*}(\Omega_{X'/k})) \xrightarrow{249} F_{X/k^*}(\Omega_{X/k})
\]

(251)
specify the same Tango structure on $X/k$. Thus, after possibly replacing $h_N$ with its composite with the automorphism of $N$ (given by multiplication by some element of $k^*$), $h_N$ makes the following square diagram commute:

$$
\begin{array}{ccc}
\hat{h}^*_{X^1}(N') \otimes (p-1) & \sim & h^*_{X^1}(N') \otimes (p-1) \\
h^*_{X^1}(\nu) \downarrow & & \downarrow \nu' \\
h^*_{X^1}(\mathcal{L}') & \sim & \mathcal{L},
\end{array}
$$

(252)

where the lower horizontal arrow is obtained by restricting (249). Consequently, the pair $(h_X, h_N)$ specifies an isomorphism $\hat{X} \sim \hat{X}'$ of generalized Tango curves. This completes the proof of Lemma 7.2.4. □

References

[1] D. Abramovich, Q. Chen, D. Gillam, Y. Huang, M. Olsson, M. Satriano, S. Sun. Logarithmic geometry and moduli. *Handbook of Moduli, Vol. I*, Adv. Lect. Math. 24, Int. Press, Somerville, MA, (2013), pp. 1-61.

[2] T. Chen, The associated map of the nonabelian Gauss-Manin connection. *Central European J. of Math.* 10 (2012), pp. 1-15.

[3] P. Deligne, D. Mumford, The irreducibility of the space of curves of given genus. *Publ. Math. I.H.E.S.* 36 (1969), pp. 75-110.

[4] V. Drinfeld, V. Sokolov, Lie algebras and KdV type equations. *J. Sov. Math.* 30 (1985), pp. 1975-2036.

[5] B. Fantechi, L. G"u"utsche, L. Illusie, S. Kleiman, N. Nitsure, A. Vistoli, Angelo, *Fundamental algebraic geometry. Grothendieck’s FGA explained*.. Mathematical Surveys and Monographs, 123 AMS (2005).

[6] B. Feigin, E. Frenkel, N. Reshetikhin, Gaudin model, Bethe Ansatz and Critical Level. *Commun. Math. Phys.* 166 (1994), pp. 29-62.

[7] E. Frenkel, Opers on the projective line, flag manifolds and Bethe Ansatz. *Mosc. Math. J.* 4 (2004), pp. 655-705.

[8] E. Frenkel, Gaudin model and opers. *Progress in Mathematics* 237 (2005), pp. 1-58.

[9] E. Frenkel, Langlands Correspondence for Loop Groups. Cambridge Studies in Advanced Mathematics 103 Cambridge Univ. Press (2007).

[10] D. Gieseker, Stable curves and special divisors: Petri’s conjecture. *Invent. Math.* 66 (1982), pp. 251-275.

[11] R. Hartshorne, *Algebraic Geometry*. Graduate Texts in Mathematics 52 Springer-Verlag, New York-Heidelberg (1977).

[12] R. Hotta, K. Takeuchi, T. Tanisaki, *D-modules, Perverse Sheaves, and Representation Theory*. Progress in Mathematics 236 Birkhäuser Boston Inc., Boston (2008), Translated from the 1995 Japanese edition by Takeuchi, xii + 407 pp.

[13] L. Illusie, Frobenius and Hodge degeneration. *Introduction to Hodge theory. Translated from the 1996 French original by James Lewis and Peters*, SMF/AMS Texts and Monographs, 8, Amer. Math. Soc., Providence, RI; Société Mathématique de France, Paris, (2002), x+232 pp.

[14] L. Illusie, An Overview of the Work of K. Fujiwara, K. Kato and C. Nakamura on Logarithmic Etale Cohomology. *Astérisque* 279 (2002), pp. 271-322.
[15] F. Kato, Log smooth deformation and moduli of log smooth curves. *Internat. J. Math.* **11**, (2000), pp. 215-232.
[16] K. Kato, Logarithmic structures of Fontaine-Illusie. *Algebraic analysis, geometry, and number theory*, John Hopkins Univ. Press, Baltimore, (1989), pp. 191-224.
[17] N. M. Katz, Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin. *Inst. Hautes Études Sci. Publ. Math.* **39**, (1970), 175-232.
[18] N. M. Katz, Algebraic solutions of differential equations (p-curvature and the Hodge filtration). *Invent. Math.* **18**, (1972), 1-118.
[19] R. Kiehl, R. Weissauer, Weil conjectures, Perverse Sheaves and l’adic Fourier Transform. *Ergeb. Math. Grenzgeb. (3)* **42** Springer (2001).
[20] F. F. Knudsen, The projectivity of the moduli space of stable curves. II. The stacks $M_{g,r}$. *Math. Scand.* **52** (1983), pp. 161-199.
[21] W. Lang, Examples of surfaces of general type with vector fields. *Arithmetic and Geometry, vol. II, Progress in Math.* **36** (1983), pp. 167-173.
[22] R Lazarsfeld, Brill-Noether-Petri without degenerations. *J. Diff. Geometry* **23** (1986), pp. 299-307.
[23] S. Mochizuki, A theory of ordinary p-adic curves. *Publ. RIMS* **32** (1996), pp. 957-1151.
[24] S. Mochizuki, *Foundations of p-adic Teichmüller theory*. American Mathematical Society, (1999).
[25] S. Mochizuki, Semi-graphs of Anabelioids. *Publ. RIMS* **42** (2006), pp. 221-322.
[26] S. Mukai, Counterexamples to Kodaira’s vanishing and Yau’s inequality in positive characteristics. *Kyoto J. Math.* **53** (2013), pp. 515-532.
[27] B. C. Ngô, Le lemme Fondamental pour les Algèbres de Lie. *Publ. Math. IHES* **111** (2010), pp. 1-271.
[28] A. Ogus, F-Crystals, Griffiths Transversality, and the Hodge Decomposition. *Astérisque* **221**, Soc. Math. de France, (1994).
[29] A. Ogus, Higgs cohomology, p-curvature, and the Cartier isomorphism. *Compositio. Math.* **140** (2004), pp. 145-164.
[30] M. Raynaud, Contre-exemple au “vanishing theorem” en caracteristique $p > 0$. *C.P. Ramanujan-a tribute, Tata Inst. Fund. Res. Studies in Math.* **8** Springer, Berlin-New York (1978), pp. 273-278.
[31] P. Russell, Factoring the Frobenius morphism of an algebraic surface. *Lecture Notes in Math., Springer-Verlag, Berlin, Heidelberg, New York, Tokyo* **1056** (1984), pp. 366-380.
[32] H. Tango, On the behavior of extensions of vector bundles under the Frobenius map. *Nagoya Math. J.* **48** (1972), pp. 73-89.
[33] Y. Takeda, K. Yokogawa, Pre-Tango Structures on curves. *Tohoku Math. J.* **54** (2002), pp. 227-237. Errata and addenda, **55** (2003), pp. 611-614.
[34] Y. Takeda, Fibrations with moving cuspidal singularities. *Nagoya Math. J.* **122** (1991), pp. 161-179.
[35] Y. Takeda, Vector fields and differential forms on generalized Raynaud surfaces. *Tohoku Math. J.* **44** (1992), pp. 359-364.
[36] Y. Takeda, Groups of Russell type and Tango structures. *Affine algebraic geometry, CRM Proc. Lecture Notes, Amer. Math. Soc., Providence, RI*, **54** (2011), pp. 327-334.
[37] N. Tsuda, Pre-Tango structures on hyperelliptic curves (Japanese). Master’s Thesis. *Osaka University, Osaka*, (2003).
[38] Y. Wakabayashi, An explicit formula for the generic number of dormant indigenous bundles. *Publ. Res. Inst. Math. Sci.* **50** (2014), pp. 383-409.
[39] Y. Wakabayashi, A theory of dormant opers on pointed stable curves—a proof of Joshi’s conjecture—. *arXiv: math. AG/1411.1208v3*, (2014).
[40] Y. Wakabayashi, Duality for dormant opers. *J. Math. Sci. Univ. Tokyo* **24** (2017), pp. 271-320.