RELATIONSHIP BETWEEN MULLINEUX INVOLUTION AND THE GENERALIZED REGULARIZATION, I

ALLEN WANG AND GUANGYI YUE

Abstract. In this paper, we study the Mullineux transpose map and the generalized column regularization on partitions and prove a condition under which the two maps are exactly the same. The results generalize the work of Bessenrodt, Olsson and Xu.

Contents

1. Introduction 1
2. Preliminaries 2
  2.1. Mullineux Transpose 3
  2.2. Regularization and Column Regularization 5
  2.3. Cores and Quotients 7
3. Rectangular Decomposition and Proof to the Main Theorem 10
4. Conjectures 16
References 17

1. INTRODUCTION

Mullineux involution appears in the study of modular representation of symmetric groups, where the irreducible \( p \)-modular representations of the symmetric group \( S_n \) are labeled by the \( p \)-regular partitions of \( n \). Given an irreducible representation \( \rho_\lambda \), the new representation obtained by taking the tensor product with the one-dimensional sign representation exactly corresponds to the Mullineux involution of \( \lambda \).

Definition 1.1. The Mullineux map \( M_p \) is the involution on the set of \( p \)-regular partitions satisfying

\[
\rho_{\lambda M_p} = \rho_\lambda \otimes \text{sgn}
\]

where \( \text{sgn} \) is the sign representation.

There are a few combinatorial ways to define \( M_p \) in \([\text{Kle96}, \text{FK97}]\), where \( p \) is not necessarily prime. Since we are studying the combinatorics, from now on we will use the parameter \( b \) instead of \( p \). Walker, Bessenrodt, Olsson, Xu, and Fayers studied the combinatorial properties of Mullineux involution by relating it to another operation named (column) regularization \( \text{Reg}_b \) (Colreg\( _b \)). Walker proved in \([\text{Wal94, Wal96}]\) that in case the partition \( \lambda \) is horizontal or row-stable, there is \( \lambda^{M_b T} = \lambda^{\text{Colreg}_b} \). Later, Bessenrodt, Olsson, and Xu showed in \([\text{BOX99}]\) that Walker’s conditions can be broadened to those short-legged (or shallow) partitions, namely for every hook in the partition divisible by \( b \), the length of the corresponding arm is at least \((b-1)\) times that of the leg, and those partitions are the only ones satisfying \( \lambda^{M_b T} = \lambda^{\text{Colreg}_b} \). Fayers studied a generalized version.

(A.Wang) Acton Boxborough Regional High School, 36 Charter Rd, Acton, MA, 01720, USA
(G. Yue) Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA, 02139, USA
E-mail addresses: (A. Wang) AllenWang271@gmail.com, (G. Yue) gyyue@mit.edu.
of the problem in [Fay08] that $\lambda^{\text{Reg}_b M_b} = \lambda^{T \text{Reg}_b}$ and the corresponding conditions for hooks are generalized to be either shallow or steep. In all the above work, the notion of $\text{Reg}_b$ and $\text{Colreg}_b$ only involves a single parameter $b$.

A second parameter was added by Dimakis and the second author in [DY18] where the parameters of (column) regularization $\text{Reg}_{a,b}$ ($\text{Colreg}_{a,b}$) are extended to any rational number $\frac{a}{b}$ in the unit interval. In [DY18], the composition of a certain series of column regularization and Mullineux transpose are shown to be same when applying to the one-row partition, giving a series of monotone (decreasing) partitions. This result on the one hand gives a special situation where the simpler operation column regularization can be used to understand the Mullineux map, whose combinatorial definition is more convoluted; on the other hand it proves a special case of Bezrukavnikov’s conjecture stated in the appendix of [DY18]. In this paper, we continue the idea of choosing a suitable parameter $a$ and find the condition under which Mullineux transpose is identical to the generalized column regularization, generalizing the result of Bessenrodt, Olsson, and Xu in [BOX99], meanwhile shedding light on other cases of Bezrukavnikov’s conjecture. The main theorem is as follows:

**Theorem 1.2.** Given positive integers $a < b$ and a partition $\lambda$. Suppose $\lambda^{\text{Colreg}_{a,b}} \in \mathcal{P}$ and all hooks $H_{i,j}$ in $\lambda$ with $b \mid H_{i,j}$ satisfy:

$$\left(\frac{b}{a} - 1\right) l_{i,j} < a_{i,j} + 1,$$

then $\lambda$ is $b$-regular and $\lambda^{M_b T} = \lambda^{\text{Colreg}_{a,b}}$.

The theorem is proved combinatorially in Section 3 by analyzing the Young diagrams. We completely characterize the shape of the partitions satisfying the inequalities in Equation (1).

This paper begins a series of papers aiming to interpret the Mullineux map and the generalized regularization in different aspects, both combinatorial and representation-theoretic, with the final goal of solving Bezrukavnikov’s conjecture. The rest of the paper is organized as follows. Section 2 is an overview of preliminaries. In Section 3, we present a detailed description of partitions satisfying the conditions in Theorem 1.2 and then prove Theorem 1.2. In the last section, we conjecture the reverse direction of the main theorem when the two parameters $a$ and $b$ are co-prime and provide another conjecture which is an analogue of Fayers’ theorem in [Fay08].

**Acknowledgements.** The authors would like to thank Roman Bezrukavnikov and Richard Stanley for useful conversations and comments and would like to thank Panagiotis Dimakis for discussions and carefully proofreading the manuscript. The first author would also like to thank the MIT-PRIMES program for facilitating a part of this research.

## 2. Preliminaries

In this section, we review the vocabulary and introduce some notations needed for the rest of the paper.

A partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of $n \in \mathbb{N}$ is a tuple of non-increasing positive integers, i.e. $\lambda_1 \geq \cdots \geq \lambda_k > 0$ and $|\lambda| = \sum_{i=1}^{k} \lambda_i = n$ is called the size of $\lambda$. Sometimes we also append infinite zeros at the end of $\lambda$, i.e., $\lambda_{k+1} = \lambda_{k+2} = \cdots = 0$. The exponential version of a partition is $(1^{i_1} 2^{i_2} \cdots)$ where the superscript $i_k$ indicates the number of repetitions of the part $k$. We denote $\mathcal{P}$ to be the set of all partitions including the empty partition $\emptyset$ of size 0 and $\mathcal{P}_n$ to be the set of all partitions of size $n$. Denote $l(\lambda) = k$, which is the number of nonzero parts of $\lambda$. Given two indices $i \leq j$, denote $\lambda_{[i,j]}$ to be the subpartition $(\lambda_i, \ldots, \lambda_j)$. For two partitions, $\lambda$ and $\mu$, we define their concatenation as the tuple $\lambda \oplus \mu = (\lambda_1, \ldots, \lambda_{l(\lambda)}, \mu_1, \ldots, \mu_{l(\mu)})$, which is a partition iff $\mu_1 \leq \lambda_{l(\lambda)}$. Given a positive integer $b$, we say $\lambda$ is $b$-regular if there is no index $i$ such that $\lambda_i = \cdots = \lambda_{i+b-1} > 0$.

We associate a Young diagram, a set of unit boxes in the plane, to each partition. In this paper, except Section 2.3, we adopt the English convention for the Young diagram (except in the discussion of cores and quotients) : rotate the plane to orient the positive $x$-axis pointing south and
the positive y-axis pointing east. To avoid ambiguity, we will use the cardinal directions most of the time throughout this paper, i.e. “south” as opposed to “positive x-direction”. The southeast vertices of the boxes of the Young diagram associated to $\lambda$ are given by
\[
\{(i, j) \in \mathbb{N}_2 | 1 \leq i, j \leq \lambda_i \}.
\]
For the sake of notation, we identify a box and its southeast vertex by the same name $(i, j)$. So a box $(i, j) \in \lambda$ iff $j \leq \lambda_i$.

The transpose $\lambda^T$ of a Young diagram $\lambda$ is given by:
\[
\{(i, j) \in \mathbb{N}_2 | 1 \leq j, 1 \leq i \leq \lambda_j \}.
\]
For example, $(5, 5, 3, 2, 1)^T = (5, 4, 3, 2, 2)$.

A box $A = (i, j) \in \lambda$ is called a removable box of $\lambda$ if $\lambda \setminus A \in \mathcal{P}_{n-1}$. A box $B \not\in \lambda$ is called an addable box of $\lambda$, if $\lambda \cup B \in \mathcal{P}_{n+1}$. Fix a positive integer $b$, the residue of $A$ with respect to $b$, denoted by $\text{res}_b A$, is defined to be the residue class $(j - i) \mod b$.

Given a box $(i, j) \in \lambda$, the corresponding arm $a_{i,j} = a_{i,j}(\lambda)$ is the set of boxes $(i, j') \in \lambda$ with $j < j'$. We use $a_{i,j}$ to denote either this set of boxes or the number of elements of the above set interchangeably. Similarly, the leg $l_{i,j} = l_{i,j}(\lambda)$ is the set of boxes $(i', j) \in \lambda$ with $i < i'$. We use $l_{i,j}$ to denote either the above set or the number of elements of the above set interchangeably as well. Finally the hook $H_{i,j} = H_{i,j}(\lambda)$ is the union of sets \{$(i, j)$ \cup $a_{i,j}$ \cup $l_{i,j}$\}. Again, the number of elements of the hook is also denoted by $H_{i,j}$ and is equal to $1 + a_{i,j} + l_{i,j}$. The northeast (rest. southwest) most box in $H_{i,j}$ is called the hand (resp. foot) box associated to $(i, j)$, denoted by $h_{i,j} = h_{i,j}(\lambda) = (i + b_{i,j}, j)$ (resp. $f_{i,j} = f_{i,j}(\lambda) = (i, j + b_{i,j})$).

There are two special classes of hooks of particular interest as:

**Definition 2.1.** Given two positive integers $a < b$ and a partition $\lambda$. A hook $H_{i,j}$ in $\lambda$ is $(a, b)$-shallow if it satisfies:
\[
\left(\frac{b}{a} - 1\right) l_{i,j} < a_{i,j} + 1. \tag{2}
\]
Dually, a hook $H_{i,j}$ is $(a, b)$-steep if it satisfies:
\[
\left(\frac{b}{a} - 1\right) a_{i,j} < l_{i,j} + 1. \tag{3}
\]

**Remark 2.2.** When $b \mid H_{i,j}, H_{i,j}$ is $(a, b)$-shallow iff $l_{i,j} \leq ta - 1$ and $a_{i,j} \geq t(b-a)$ for some $t \in \mathbb{N}_0$; $H_{i,j}$ is $(a, b)$-steep iff $a_{i,j} \leq ta - 1$ and $l_{i,j} \geq t(b-a)$ for some $t \in \mathbb{N}_0$.

**Lemma 2.3.** A hook can be both $(a, b)$-shallow and $(a, b)$-steep only if $b < 2a$.

**Proof.** By the above remark, we know $t(b-a) \leq a_{i,j}, l_{i,j} \leq ta - 1$ for some $t \in \mathbb{N}_0$. Hence $t(b-a) < ta - 1$ is satisfied, and we obtain $b < 2a$. \hfill $\square$

Finally for two partitions $\lambda$ and $\mu$, the dominance order is defined as $\lambda \leq \mu$ if $\sum_{i=1}^{k} \lambda_i \leq \sum_{i=1}^{k} \mu_i$ is satisfied for all $k$. Note that $\lambda \leq \mu$ iff $\mu^T \leq \lambda^T$.

2.1. **Mullineux Transpose.** We abbreviate the composition of Mullineux involution and transpose as Mullineux transpose. There are multiple definitions for Mullineux transpose, and here we will follow Bessenrodt, Olsson and Xu as in [BOX99].

**Definition 2.4.** The rim of a partition $\lambda$ is the set of boxes $\{(i, j) \in \lambda | (i + 1, j + 1) \notin \lambda\}$. If $\lambda$ is $b$-regular, we define its $b$-rim to be the subset of its rim obtained through the following procedure:

The $b$-rim consists of the several pieces where each piece, except possibly the last one, contains $b$ boxes. We choose the first $b$ boxes from the rim, beginning with the rightmost box of the first row and moving contiguously southwestwards in the rim of $\lambda$. If the last box of this piece is chosen
from the $i_0$-th row of $\lambda$, then we choose the second piece of $b$ boxes beginning with the rightmost box of the next row $i_0 + 1$. Continue this procedure until we reach the last piece ending in the last row.

We call a maximal set of contiguous boxes of the $b$-rim a \textit{segment}. When the number of boxes in this segment is a multiple $b$, it is a \textit{$b$-segment}. Otherwise, it is called a \textit{$b'$-segment}. By construction, every $\lambda$ has at most one $b'$-segment.

Finally, denote $\lambda^{b}$ to be the partition obtained by removing the $b$-rim from $\lambda$.

\textbf{Definition 2.5.} Given $\lambda = (\lambda_1, \ldots, \lambda_k)$, if $\lambda^{b} = (\mu_1, \ldots, \mu_k)$, where some of the $\mu_i$ in the end are allowed to be zero, and $\phi(\lambda) = |\lambda| - |\lambda^{b}|$, define

$$\lambda^{b} := (\mu_1 + 1, \ldots, \mu_{k-1} + 1, \mu_k + \delta)$$

where

$$\delta = \begin{cases} 0 & \text{if } b \nmid \phi(\lambda) \\ 1 & \text{if } b \mid \phi(\lambda) \end{cases}$$

We call $\lambda \setminus \lambda^{b}$ the \textit{truncated $b$-rim} of $\lambda$.

\textbf{Definition 2.6.} For a $b$-regular partition $\lambda$, the operator $X_b (= M_b T)$ is defined as $\lambda^{X_b} := (j_1, \ldots, j_l)$, where

$$j_i = |\lambda^{M_b^{-i}}| - |\lambda^{M_b^{i}}|.$$  

The \textit{Mullineux map} $M_b$ for a $b$-regular partition is defined to be $\lambda^{M_b} = \lambda^{X_b}T$.

Recursively, we can write

$$\lambda^{M_bT} = (|\lambda| - |\lambda^{b}|) \oplus \lambda^{j_1, \ldots, j_{k-1}, k}.$$  

(4)

\textbf{Proposition 2.7} (\cite{FK97, BOX99}). The definition of Mullineux map $M_b$ in Definition 2.6 and Definition 1.1 are equivalent when $b$ is a prime number.

\textbf{Remark 2.8.} There are two other equivalent combinatorial ways to define the Mullineux map (or Mullineux transpose), see\cite{FK97, DY18} for details. It is easy to see Mullineux map is an involution from Definition 1.1, but this is not so obvious from Definition 2.6.

\textbf{Example 2.9.} The truncated 3-rims for $(7, 5, 1, 1)$ and $(7, 2, 1)$ are shaded while the 3-rims are labeled by integers in $[1, 3]$. Thus, $(7, 5, 1, 1)^{J_3} = (5, 3, 1)$ and $(7, 2, 1)^{J_3} = (5, 1, 1)$, which is illustrated in Figure 1.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{The truncated 3-rims for $(7, 5, 1, 1)$ and $(7, 2, 1)$.}
\end{figure}

In the proofs of the main theorems, we need to fully characterize the shape of the $b$-rim of certain partitions. The concept of rectangular decomposition allows for a simple description.

\textbf{Definition 2.10.} Given a partition $\lambda$, we label the boxes in its $b$-rim with positive integers from $1$ to $N$ in order from northeast to southwest. Suppose $N = bk - r$ where $k \in \mathbb{N}$ and $0 \leq r < b$. The \textit{$b$-rectangular decomposition} associated to the $b$-rim is the sequence of rectangles (of boxes) $r_1, \ldots, r_k$ such that each $r_i$ is the smallest rectangle containing the consecutive boxes labelled with $b(i - 1) + 1, \ldots, \min\{bi, bk - r\}$.
Example 2.11. The 5-rectangular decomposition of $(12, 9, 9, 7, 5, 2, 2, 1)$ is the sequence of rectangles $r_1, r_2, r_3, r_4$ as in Figure 2. Note that boxes from 1 to 10 form a $b$-segment, and boxes from 11 to 18 form a $b'$-segment. Also note that the rectangles may have at most one overlap in the columns, but they occupy all rows without overlap.

![Figure 2. The 5-rectangular decomposition of (12, 9, 9, 7, 5, 2, 2, 1).](image)

Remark 2.12. We will use $r_i^x$ (resp. $r_i^y$) to denote the dimension of the rectangle $r_i$ in the north-south (resp. west-east) direction. By definition, for every $i$ except possibly the last one (when $\lambda$ contains a $b'$-segment), there is $r_i^x + r_i^y = b + 1$.

2.2. Regularization and Column Regularization.

Definition 2.13. For two nonnegative integers $a < b$ and a partition $\lambda$, we define the column regularization $\lambda^{\text{Colreg}_{a,b}}$ and regularization $\lambda^{\text{Reg}_{a,b}}$ of a partition $\lambda$ through the following procedure (here we identify $\lambda$ as a set of integer points in the plane):

First of all, for any box $A = (x, y) \in \mathbb{Z}^2$ (not necessarily in $\lambda$ or the first quadrant), ladders passing through $A$ are defined to be:

$$\ell_A = \ell_{x,y} := \{(i, j) \in \mathbb{N}^2 \mid \exists t \in \mathbb{Z}, i = x - ta, j = y + t(b - a)\}.$$ 

They underneath line segment is defined by the equation:

$$\ell_c : y + \frac{b - a}{a}x = \frac{c}{a}, \quad x \in \mathbb{R}_{>0}, y \in \mathbb{R}_{>0}$$

for some $c \in \mathbb{Z}$.

For each ladder $\ell = \ell_A, A \in \mathbb{Z}^2$, if $\lambda \cap \ell \neq \emptyset$, slide those boxes in the intersection southwards (down) on the ladder $\ell$ to the bottom. The resulting set of boxes is $\lambda^{\text{Colreg}_{a,b}}$, which is not necessarily a partition.

And $\lambda^{\text{Reg}_{a,b}}$ is defined to be $\lambda^{\text{TColreg}_{a,b}}$. Equivalently, we define the dual ladders passing through $A = (x, y) \in \mathbb{Z}^2$:

$$\hat{\ell}_A = \hat{\ell}_{x,y} := \{(i, j) \in \mathbb{N}^2 \mid \exists t \in \mathbb{Z}, i = x - t(b - a), j = y + ta\}.$$ 

The corresponding underneath line segment is defined by the equation:

$$\hat{\ell}_{c'} : y + \frac{a}{b - a}x = \frac{c'}{b - a}, \quad x \in \mathbb{R}_{>0}, y \in \mathbb{R}_{>0}$$

for some $c' \in \mathbb{Z}$. And $\lambda^{\text{Reg}_{a,b}}$ is exactly sliding boxes in each $\lambda \cap \hat{\ell}_A$ northwards (up) the dual ladders $\hat{\ell}_A$ to the top if the intersection with the partition is nonempty for all $A \in \mathbb{Z}^2$.

Remark 2.14. Note that only when $a$ and $b$ is co-prime, the (dual) ladder coincide with the set of all the positive integer points on its underneath line segments.

All the boxes on a (dual) ladder has the same $b$-residue, we say this residue to be the residue of the (dual) ladder, denoted as $\text{res}_b \ell$ ($\text{res}_b \hat{\ell}$).
Example 2.15. In the diagrams of Figure 3, the partition $(3, 2, 2, 1)$ becomes $(2, 2, 2, 1, 1) \in \mathcal{P}$ under Colreg$_{2,3}$, but Colreg$_{2,3}$ maps $(3, 2, 2)$ to $(2, 1, 2, 1, 1) \notin \mathcal{P}$.

![Figure 3. An example of a Colreg$_{2,3}$-valid partition $(3, 2, 2, 1)$ and an Colreg$_{2,3}$-invalid one $(3, 2, 2)$.

We say a partition $\lambda$ is Colreg$_{a,b}$-valid (resp. Reg$_{a,b}$-valid) if $\lambda^\text{Colreg}_{a,b} \in \mathcal{P}$ (resp. $\lambda^\text{Reg}_{a,b} \in \mathcal{P}$). By definition, $\lambda$ is Reg$_{a,b}$-valid exactly when $\lambda^T$ is Colreg$_{a,b}$-valid, so we only specify the criterion for $\lambda$ being Colreg$_{a,b}$-valid.

Lemma 2.16. Given $\lambda \in \mathcal{P}$ and positive integers $a < b$, $\lambda$ is Colreg$_{a,b}$-valid if and only if for all $y \in [1, \lambda_1]$, either $\ell_{1,y} \subset \lambda$, or there exists a box $(i, j) \in \lambda \cap \ell_{0,y}$ such that $(i + 1, y) \notin \ell_{1,y}$. In the first case, we say the ladder $\ell_{1,y}$ is full, and in the latter case, we say $\ell_{0,y}$ gains a box at $(i, j)$.

Proof. $\lambda^\text{Colreg}_{a,b} \in \mathcal{P}$ is equivalent to saying for every $(i, j) \in \lambda^\text{Colreg}_{a,b}$, there is $(i - 1, j) \in \lambda^\text{Colreg}_{a,b}$ if $i \geq 2$ and $(i, j - 1) \in \lambda^\text{Colreg}_{a,b}$ if $j \geq 2$.

For every $(x, y) \in \lambda$, consider two ladders $\ell_{x,y}$ and $\ell_{x,y-1}$. We have $\#\ell_{x,y-1} = \#\ell_{x,y} - 1$ or $\#\ell_{x,y-1} = \#\ell_{x,y} - 1$ (there may be an integer point in the first column on $\ell_{x,y}$). And $\#(\ell_{x,y-1} \cap \lambda) \geq \#(\ell_{x,y} \cap \lambda)$ if $\#\ell_{x,y-1} = \#\ell_{x,y}$ and $\#(\ell_{x,y-1} \cap \lambda) \geq \#(\ell_{x,y} \cap \lambda) - 1$ if $\#\ell_{x,y-1} = \#\ell_{x,y} - 1$. This indicates for any $(i, j) \in \lambda^\text{Colreg}_{a,b}$, $(i, j - 1) \in \lambda^\text{Colreg}_{a,b}$ if $j \geq 2$.

On the other hand, for every $(x, y) \in \lambda$, consider $\ell_{x,y}$ and $\ell_{x-1,y}$. Similarly, we have $\#\ell_{x-1,y} = \#\ell_{x,y} - 1$ or $\#\ell_{x-1,y} = \#\ell_{x,y} - 1$ depending on whether there is an integer box on $\ell_{x,y}$ at the first row. And $\#(\ell_{x-1,y} \cap \lambda) \geq \#(\ell_{x,y} \cap \lambda)$ if $\#\ell_{x-1,y} = \#\ell_{x,y}$ and $\#(\ell_{x-1,y} \cap \lambda) \geq \#(\ell_{x,y} \cap \lambda) - 1$ if $\#\ell_{x-1,y} = \#\ell_{x,y} - 1$. Since performing Colseg$_{a,b}$ is sliding boxes southwestwards, $\lambda^\text{Colseg}_{a,b} \in \mathcal{P}$ happens only in the latter case and $\#(\ell_{x-1,y} \cap \lambda) = \#(\ell_{x,y} \cap \lambda) - 1$. Without loss of generality, suppose $x = 1$ and $y \in [1, \lambda_1]$. And this condition is equivalent to saying $\ell_{1,y} \not\subset \lambda$, and pairs of boxes $(i, j) \in \ell_{1,y}$, $i \geq 2$ and $(i - 1, y) \in \ell_{0,y}$ are either in $\lambda$ or not in $\lambda$ simultaneously. Then the box $(1, y) \in \ell_{1,y}$ slides to a position where the box immediate north to it is not in $\lambda^\text{Colseg}_{a,b}$.

Remark 2.17. Referring to Example 2.15, there is $(3, 2) \in \ell_{0,3} \cap (3, 2, 2)^\text{Colreg}_{2,3}$ while $(2, 2) \notin (3, 2, 2)^\text{Colreg}_{2,3}$. In fact, in $(3, 2, 2)$, there is no box $(i, j) \in \ell_{0,3} \cap \lambda$ where $(i + 1, j) \notin \lambda$, so it is not Colreg$_{a,b}$-valid.

Definition 2.18. Given positive integers $a < b$, a partition is called $(a, b)$-regular if $\lambda^{\text{Reg}_{a,b}} = \lambda$.

Remark 2.19. $(1, b)$-regular is equivalent to the notion of $b$-regular. Also $(a, b)$-regular implies $b$-regular for any $a$ with $0 < a < b$.

We now rewrite Colreg$_{a,b}$ in a recursive way parallel to that of the Mullineux transpose as in Equation (4).

For $0 < a < b$, we define a middle-step operator called the column semi-regulation Colseg$_{a,b}$ in order to compute Colreg$_{a,b}$ in a row-by-row fashion: $\lambda_i^{\text{Colreg}_{a,b}} = |\lambda^{\text{Colseg}_{a,b}^{i-1}}| - |\lambda^{\text{Colseg}_{a,b}^i}|$, or recursively,

$$\lambda^{\text{Colreg}_{a,b}} = (|\lambda| - |\lambda^{\text{Colseg}_{a,b}^i}|) + \lambda^{\text{Colseg}_{a,b}} \cdot \text{Colreg}_{a,b}.$$
Definition 2.20. Given a Colreg_{a,b}-valid partition \( \lambda \), for all \((1, y) \in \lambda \), if \( \ell_{1,y} \not\subset \lambda \), slide \((1, y)\) to the north-most position \((i, j)\) in \( \ell_{1,y} \setminus \lambda \) such that \((i - 1, j) \in \lambda \) (i.e. \( \ell_{0,y} \) gains a box at \((i - 1, j)\)). Then remove the first row and the resulting partition is defined to be \( \lambda^{\text{Colseg}_{a,b}} \).

It is clear that \( \text{Colseg}_{a,b} \) is well-defined by Lemma 2.16. And \( \lambda^{\text{Colseg}_{a,b}} \) is \( \text{Colreg}_{a,b} \)-valid iff \( \lambda \) is. Note that we only perform \( \text{Colseg}_{a,b} \) to those \( \text{Colreg}_{a,b} \)-valid partitions.

Example 2.21. In the diagram shown in Figure 4, the boxes \((1, 12)\) and \((1, 13)\) shift under \( \text{Colseg}_{2,5} \) to \((3, 10)\) and \((5, 6)\) respectively. We have

\[
(13, 10, 9, 7, 5, 2, 2, 1)^{\text{Colseg}_{2,5}} = (10, 10, 7, 6, 2, 2, 1).
\]

![Figure 4. An example of column semi-regulation Colseg_{2,5}:
(13, 10, 9, 7, 5, 2, 2, 1)^{\text{Colseg}_{2,5}} = (10, 10, 7, 6, 2, 2, 1).](image)

2.3. Cores and Quotients. Cores and Quotients are crucial concepts of partitions, and many attempts have been made to understand the Mullineux map in terms of cores and quotients. In this section we first explain two definitions of cores and quotients and then prove that (column) regularization fixes the core of a partition. The first definition is following [Hai02] (note that there is a slight difference as in [Hai02] the residue of a box \((i, j)\) is defined to be \((i - j) \mod b\) but we are using \((j - i) \mod b\).

Definition 2.22. The \( b \)-core \( \text{Core}_b(\lambda) \) of any partition \( \lambda \) is the partition that remains after removing as many \( b \)-ribbons in succession as possible. It is well-known that the result is independent of the choice of removals. If \( \lambda = \text{Core}_b(\lambda) \), then \( \lambda \) itself is called a \( b \)-core.

Definition 2.23. Given a partition \( \lambda \), and for each residue class \( k \), the collection of boxes \( A \in \lambda \) satisfying \( b \mid H_A \), \( \text{res}_b h_A = k \), \( \text{res}_b f_A = k + 1 \) forms an “exploded” copy of a partition which we denote as \( \lambda_k \). The \( b \)-quotient of a partition \( \lambda \) is defined to be the \( b \)-tuple of partitions, \( \text{Quot}_b(\lambda) = (\lambda_0, \lambda_1, ..., \lambda_{b-1}) \). The \( b \)-weight of \( \lambda \) is defined to be \( \left| \lambda \right|_b = \sum_{k=0}^{b-1} |\lambda_k| \).

The second definition of cores and quotients given in [JK] needs the notion of Maya diagram and abacus. We use the Russian version of Young diagrams in this section which is obtained from the English version by rotating 135 degrees counterclockwise. Now the positive \( x \)-axis is pointing northeastwards and positive \( y \)-axis is pointing northwestwards. We scale the measure on the horizontal line passing through the origin by \( \sqrt{2} \), and place black beads positions at positions \( \lambda^T_k - k + \frac{1}{2} \) for all \( k \geq 1 \), and white beads at other positions in \( \mathbb{Z} + 1/2 \). Indeed, this is projecting each down-sloping (NW-SE) segment on the edge of the diagram to a black bead and up-sloping (SW-NE) segment to a white bead. The resulting infinite bead sequence with a labelling of the origin is called the Maya diagram of \( \lambda \). In fact, each bead sequence (without the information of the origin), such that there are positions \( k_{\text{black}}, k \in \mathbb{Z} + \frac{1}{2} \), such that every half integer point smaller
than $k_{\text{black}}$ are all placed with black beads and larger than $k_{\text{white}}$ are all white beads, uniquely determines a partition. This is essentially the correspondence between partitions and semi-infinite wedge products. We refer to [JK] and [Kac94, Chapter 14] for details.

We then put the beads into groups of $b$ starting at the origin, and rotate each group 90 degrees counterclockwise, and compressing them to get $b$ rows. This figure is called the $b$-abacus of $\lambda$, where each row is called a runner. And we label the runners from top to bottom by $R_0, R_1, \ldots, R_{b-1}$.

**Example 2.24.** Figures in 5 are the Maya diagram and 4-abacus for $(4, 2, 2, 1)$. We have

$\text{Core}_4((4, 2, 2, 1)) = (4, 1), \quad \text{Quot}_4((4, 2, 2, 1)) = ((1), \emptyset, \emptyset, \emptyset)$.

![Figure 5. Maya diagram and 4-abacus of (4, 2, 2, 1) ](image)

Indeed each runner $R_i$ of the $b$-abacus, up to some (unique) proper shifting (of the origin), is the Maya diagram of a partition, we name it $\lambda_i$. Also we could exchange the position of beads on each runner finitely many times to make black beads appears to the left of the white ones (on the same runner), but not changing between runners, and the result diagram is the $b$-abacus of a new partition $\mu$. We then have the following result, which gives a new definition of cores and quotients.

**Proposition 2.25.** With the above notation, we have $\mu = \text{Core}_b(\lambda)$ and $(\lambda_0, \ldots, \lambda_{b-1}) = \text{Quot}_b(\lambda)$.

Adding an addable box $A$ to $\lambda$ corresponds to exchanging a black bead with a white one immediately to its right. If we consider the $b$-abacus, the changed black bead originally appears in runner $R_{r+1}$ and the white one originally appears in runner $R_r$, where subscripts are taken mod $b$ and $r = \text{res} A$. Conversely, removing a removable box $B$ from $\lambda$ is exchanging a white bead with a black bead immediately to its right in the Maya diagram. In the $b$-abacus, the moved black bead originally appears in some $R_{r'}$ and the white one is in $R_{r'+1}$ where $r' = \text{res} B$. This is illustrated in the following example:

**Example 2.26.** We remove the box $(3, 2)$ and add $(1, 5)$ to the partition $(4, 2, 2, 1)$ to obtain $(5, 2, 1, 1)$. The changes to the Maya diagram and 4-abacus is shown in Figure 6.

The following is well-known:
Figure 6. Changes to the Maya diagram and 4-abacus when adding (1, 5) and removing (3, 2) from (4, 2, 2, 1). The numbers in each boxes are its 4-residues. And numbers labelled on the boundary of the partition denote the indices of the runners where the corresponding beads are located.

Proposition 2.27. For a partition $\lambda$, there is

$$|\lambda| = |\text{Core}_b(\lambda)| + b|\lambda|_b.$$  

In particular, $|\lambda|_b$ is also the number of hooks in $\lambda$ that is divisible by $b$.

Moreover, if we fix a $b$-core $\nu$, there is a bijection from $\{\lambda \mid \text{Core}_b(\lambda) = \nu\}$ to $b$-tuples of partitions.

Lemma 2.28. For a $\text{Reg}_{a,b}$-valid (resp. $\text{Colreg}_{a,b}$-valid) partition $\lambda$, we have

$$\text{Core}_b(\lambda) = \text{Core}_b(\lambda) \quad (\text{resp.} \quad \text{Core}_b(\lambda) = \text{Core}_b(\lambda))$$

And in particular, there is $|\lambda|_{\text{Reg}_{a,b}} = |\lambda|_b$ (resp. $|\lambda|_{\text{Colreg}_{a,b}} = |\lambda|_b$).

Proof. For a $\text{Reg}_{a,b}$-valid (resp. $\text{Colreg}_{a,b}$-valid) partition $\lambda$, denote $\tilde{\lambda} = \lambda \cap \lambda_{\text{Reg}_{a,b}} \in P$.

Suppose we have $\lambda \setminus \lambda_{\text{Reg}_{a,b}} = \{A_1, \ldots, A_m\}$, such that $A_j$ is a removable box in $\lambda^{(j-1)}$, where $\lambda = \lambda^{(0)}, \lambda^{(1)} = \lambda^{(0)} \setminus A_1, \ldots, \lambda^{(j)} = \lambda^{(j-1)} \setminus A_j, \ldots, \lambda^{(m)} = \tilde{\lambda}$. Then at each step of removal, we are exchanging a white bead in runner $R_{r_j+1}$ with a black bead in $R_{r_j}$ in the $b$-abacus, where $r_j = \text{res}_b A_j$.

Now we build $\lambda_{\text{Reg}_{a,b}}$ by adding boxes in $\lambda_{\text{Reg}_{a,b}} \setminus \lambda = \{B_1, \ldots, B_m\}$ in order such that $\tilde{\lambda} = \lambda^{(0)}, \lambda^{(1)} = \lambda^{(0)} \cup B_1, \ldots, \lambda^{(j)} = \lambda^{(j-1)} \setminus B_j, \ldots, \lambda^{(m)} = \lambda_{\text{Reg}_{a,b}}$ and $B_j$ is an addable box of $\lambda^{(j-1)}$. Similarly, at each step, we are exchanging a white bead in runner $R_{r_j'}$ with a black bead in $R_{r_j'}$ in the $b$-abacus, where $r_j' = \text{res}_b B_j$.

Since we know every box on each dual ladder (resp. ladder) has the same residue, there is $\{r_1, \ldots, r_m\} = \{r_1', \ldots, r_m'\}$. By the second definition of $b$-core, we know $\text{Core}_b(\lambda_{\text{Reg}_{a,b}}) = \text{Core}_b(\lambda)$ (resp. $\text{Core}_b(\lambda_{\text{Colreg}_{a,b}}) = \text{Core}_b(\lambda)$).

Mullineux transpose also fixes the core of a $b$-regular partition, as shown in [FK97], which is compatible with Theorem 1.2.
3. Rectangular Decomposition and Proof to the Main Theorem

In this section we will completely characterize the shape of $\lambda$ satisfying the conditions restraining hook shape from Equations (1). The following comparison lemma is essential.

**Lemma 3.1.** Given a partition $\lambda$, suppose $r_\alpha$ and $r_\beta$ are two smallest rectangles, each containing $b$ consecutive boxes of the rim of $\lambda$ ($r_\alpha = r_\beta = b + 1$, and $r_\alpha > a \geq r_\beta^y$). Also the box $B_0 = (i_0, j_0)$, furthest northeast in $r_\alpha$, satisfies that $(i_0, j_0 + 1) \notin \lambda$. If $r_\beta$ contains a box strictly southwest of $r_\alpha$, then $\lambda$ contains a hook with leg length $a$ and arm length $b - a - 1$.

**Proof.** Let $B_c = (i_c, j_c)$ be the box furthest southwest in $r_\beta$. $j_c < j_0$ and $i_c > i_0$ since $r_\alpha^x > a \geq r_\beta^y$.

Denote by $\mathcal{B}$ the set of consecutive boxes in the rim of $\lambda$ starting at $B_0$ and ending at $B_c$. Let

$$\mathcal{S} = \{(i, j) \in \mathcal{B} | (i + 1, j) \notin \lambda\}$$
$$\mathcal{E} = \{(i + a, j - (b - a - 1)) | (i, j) \in \mathcal{B}, (i, j + 1) \notin \lambda\}.$$

We claim $\mathcal{S} \cap \mathcal{E} \neq \emptyset$. By construction, $\mathcal{S}$ contains exactly one box for every $y$-coordinate in $[j_c + 1, j_0]$, and $\mathcal{E}$ contains one box for every $x$-coordinate in $[i_0 + a, i_c + a]$.

We look at the boxes in $\mathcal{S}$ and $\mathcal{E}$ from northeast to southwest in order. The first box in $\mathcal{S}$ is $E_{i_0+a} = (i_0 + a, j_0 - (b - a - 1))$ by our assumption on $B_0$. Denote the southwest-most box of $r_\alpha$ by $A = (i_0 + r_\alpha^x - 1, j_0 - r_\alpha^y + 1)$, we know that $E_{i_0+a}$ is northwest to $A$ (not necessarily strict), and they two lie one a slope 1 line. Since $j_c < j_0 - (b - a - 1) \leq y_A = j_0 - r_\alpha^y + 1$, the unique box $(i', j_0 - (b - a - 1)) \in \mathcal{S}$ in the same column of $E_{i_0+a}$ is south to $E_{i_0+a}$. Hence, $i' \geq i_0 + a$.

Next, we consider the unique box $E_{i_c} = (i_c, j_c')$ in $\mathcal{E}$, which has $x$-coordinate $i_c$. Then $j' > j_c$ since $r_\beta^x \geq b - a + 1$. Next, we find the unique box $(i'', j'') \in \mathcal{S}$ in the same column of $E_{i_c}$. Then $i'' \leq i_c$ since $r_\beta$ is the smallest rectangle containing a piece of the rim.

We now consider boxes in $\mathcal{S}$ and $\mathcal{E}$ inside the big rectangle whose northeast most box is $E_{i_0+a}$ and southwest most box is $E_{i_c}$. Boxes in $\mathcal{E}$ divides the rows into:

$$x_0 = i_0 + a < x_1 < \ldots < x_k = i_c + 1$$

such that the boxes in $\mathcal{E}$ on rows $[x_{j-1}, x_j - 1]$ lies in the same column $\varphi(j)$ and $\varphi(j) < \varphi(j - 1)$ for all $j = 1, \ldots, k$. And we consider the unique box $(\psi(j), \varphi(j)) \in \mathcal{S}$ in column $\varphi(j)$.

Suppose on the contrary that $\mathcal{S} \cap \mathcal{E} = \emptyset$. Then the above gives boundary conditions $\psi(1) \geq x_1$ and $\psi(k) \leq x_{k-1}$. Also there is $\psi(j) \geq \psi(j - 1)$ because $\varphi(j) < \varphi(j - 1)$. We show by induction that $\psi(j) \geq x_j$. The base case is already known. Then suppose this is true for indices $< j$. So

$$\psi(j) \geq \psi(j - 1) \geq x_{j-1}$$

But we know $(x, \varphi(j)) \in \mathcal{E}$ for $x \in [x_{j-1}, x_j - 1]$. So $\psi(j) \geq x_j$. In particular, $\psi(k) \geq x_k = i_c + 1$, which is a contradiction, so $\mathcal{S} \cap \mathcal{E} \neq \emptyset$. Figure 7 is an illustration.

Pick $(i, j) \in \mathcal{S} \cap \mathcal{E}$. Then, $(i, j) \in \lambda$, and $(i + 1, j) \notin \lambda$. Also, $(i - a, j + (b - a - 1)) \in \lambda$, and $(i - a, j + b - a) \notin \lambda$. Hence, $H_{i-a,j}$ has corresponding leg length $a$ and arm length $b - a - 1$, as desired.

**Example 3.2.** We demonstrate the argument of the proof of Lemma 3.1 on $(8, 8, 7, 6, 6, 1)$ and take $a = 2, b = 5$, see Figure 10. The rectangle $r_\alpha$ has $r_\alpha^x = 3$ and $r_\alpha^y = 3$ with the northeast most box $(1,8)$. The rectangle $r_\beta$ is determined by the last 5 boxes of the rim and has $r_\beta^x = 2$ and $r_\beta^y = 4$. The two rectangles are both colored in yellow. $\mathcal{S}$ is shaded with black lines and $\mathcal{E}$ is shaded with red lines. They intersect at $(5, 5)$, which determines a hook $H_{3,5}$ with $a_{3,5} = b - a - 1 = 2$ and $l_{3,5} = a = 2$ (labelled in thick blue lines).
Here we point out there is a special type of hooks in the shape:

\[ H_{i,j} = tb, \quad l_{i,j} = ta \quad \text{for some} \quad t \in \mathbb{N}_{>0}. \]  

Note that this type of hooks are not \((a,b)\)-shallow, in case of \(b > 2a\), they are not \((a,b)\)-steep as well.

**Lemma 3.3.** Consider a Colreg\(_{a,b}\)-valid partition \(\lambda\) containing no hook of shape (6). Then for any ladder \(\ell_{1,y}, y \in [1,\lambda_1]\), that’s not full (i.e. \(\ell_{1,y} \not\subset \lambda\)), the furthest north box \((i,j)\) in \(\ell_{1,y} \setminus \lambda\) satisfies \((i-1,j) \in \lambda\).

**Proof.** We denote \(\ell_{1,y}, \ell_{1,y+1}, \ldots, \ell_{1,\lambda_1}\) to be all the ladders passing through a box in the first row of \(\lambda\) that’s not fully contained in \(\lambda\) (if there are any). And we proceed by induction.

For the ladder \(\ell_{1,y_1}\), let \((i,j)\) \((i>1)\) be the box furthest north on \(\ell_{1,y_1}\) but not in \(\lambda\). Suppose \((i-1,j) \not\in \lambda\). Since \(\lambda\) is Colreg\(_{a,b}\)-valid, Lemma 2.16 implies \(\exists t \in \mathbb{N}\) such that \((i+ta,j-t(b-a)) \not\in \lambda\) and \((i+ta-1,j-t(b-a)) \in \lambda\). Since \(\ell_{1,y_1-1} \subset \lambda\), \((i-1,j-1) \in \lambda\). Then, \(H_{i-1,j-t(b-a)}\) has \(l_{i-1,j-t(b-a)} = ta\) and \(a_{i-1,j-t(b-a)} = t(b-a) - 1\), which is disallowed by hypothesis on \(\lambda\). Hence \((i-1,j) \in \lambda\). Figure 9 is an illustration for this part of reasoning.

Now suppose the result is true for all \(\ell_{1,y}, y \in [y_1,y']\) and denote \((i,j)\) \((i>1)\) to be the furthest north on \(\ell_{1,y'+1}\) that’s not in \(\lambda\). Suppose \((i-1,j) \not\in \lambda\). Note that all boxes on \(\ell_{1,y'+1}\) strictly
above \((i, j)\) (if there are any) are in \(\lambda\), so all boxes on \(\ell_{1,y}\) strictly above \((i, j - 1)\) (if there are any) are in \(\lambda\). If \((i, j - 1) \in \lambda\), then \((i - 1, j - 1) \in \lambda\); if not, then \((i, j - 1)\) is the north-most box in \(\ell_{1,y} \setminus \lambda\), and the inductive hypothesis also guarantees \((i - 1, j - 1) \in \lambda\). Then we can find a hook \(H_{i-1,j-t(b-a)} = tb\) with leg length \(ta\) following the similar argument as in the base case, which is not allowed, so we have a contradiction and obtain \((i - 1, j) \in \lambda\).

Consider the \(b\)-rectangular decomposition of \(\lambda\). Define \(\omega(\lambda)\) to be largest index such that

\[ r^x_{\omega(\lambda)} = a, \quad r^y_{\omega(\lambda)} = b - a + 1, \]

and define \(\psi(\lambda) = \sum_{i=1}^{\omega(\lambda)} r^x_i\). When rectangle of such shape does not exist, we let \(\omega(\lambda) = 0\).

**Example 3.4.** When \(a = 2\) and \(b = 5\), \(\omega((2,1,1)) = \psi((2,1,1)) = 0\); and \(\omega((8,4,1)) = 2, \psi((8,4,1)) = 3\), which is shown in Figure 10.

![Figure 10](image-url)

Using the above definitions we give a description of the general shape of \(\lambda\) satisfying the conditions of Theorems 1.2.

**Proposition 3.5.** Given a Colreg\(_{ab}\)-valid partition \(\lambda\) containing no hook of shape (6). Denote by \(r_1, \ldots, r_k\) the rectangles in order (northeast to southwest) in its \(b\)-rectangular decomposition with \(r^x_\alpha + r^y_\alpha = b + 1, \ \alpha = 1, \ldots, k\) (i.e., removing the last rectangle if there is a \(b'\)-segment). Then \(r^x_\alpha > a\) for \(\alpha > \omega(\lambda)\), and if \(\omega(\lambda) > 0\), there is \(r^x_\alpha = a\) for every \(\alpha \in [1, \omega(\lambda)]\).

**Proof.** Without loss of generality, suppose \(k \geq 0\).

Claim: \(r^x_1 \geq a\).

We suppose the converse where \(r^x_1 \leq a - 1\) and \(r^y_1 \geq b - a + 2\). Consider the ladder \(\ell_{1,\lambda_1}\) and \(\ell_{0,\lambda_1}\). The box \((a + 1, \lambda_1 - (b - a)) \in \ell_{1,\lambda_1} \setminus \lambda\) and \((a, \lambda_1 - (b - a)) \in \ell_{0,\lambda_1} \setminus \lambda\) because of the dimension constraints of \(r_1\). But this contradicts with Lemma 3.3. So \(r^x_1 \geq a\).

Case 1. \(r^x_1 > a\). From Lemma 3.1, we know all \(\alpha > 1\) have \(r^x_\alpha > a\), and \(\omega(\lambda) = 0\).

Case 2. \(r^x_1 = a\). Then \(\omega(\lambda) \geq 1\).

1. We first apply induction to show \(r^x_\alpha = a\) for \(\alpha \in [1, \omega(\lambda)]\).
   a. We know that \(r^x_\alpha \leq a\) for \(\alpha \in [1, \omega(\lambda)]\) by applying Lemma 3.1 to \(r^x_{\omega(\lambda)}\) and \(r_\alpha\).
(b) Now suppose that \( r_\alpha^x = a \) for \( \alpha \in [1, \alpha'] \) with \( 1 \leq \alpha' < \omega(\lambda) \) and consider \( r_{\alpha'+1} \). Denote the northeast most box in \( r_{\alpha'+1} \) by \( A = (x_A, y_A) \). The inductive hypothesis indicates all boxes in \( \ell_A \) and north to \( A \) is in \( \lambda \). Figure 11 is an illustration. Similar as above, if \( r_{\alpha'+1}^x < a \), the two boxes \( (x_A + a, y_A - (b - a)) \) and \( (x_A + a - 1, y_A - (b - a)) \) are both not in \( \lambda \), but all the boxes on the same ladder and north to them are in \( \lambda \), giving a contradiction to Lemma 3.3. So we know \( r_\alpha^x \geq a \) for \( \alpha \in [1, \omega(\lambda)] \).

![Figure 11](image.jpg)

(2) The above (1)(b) actually shows that \( r_\omega^{\omega(\lambda)+1} \geq a \) since all rectangles above have \( x \)-dimension \( a \). By maximality of \( \omega(\lambda) \), we know \( r_\omega^{\omega(\lambda)+1} > a \). Similar as Case 1, we know \( r_\alpha^x > a \) for all \( \alpha > \omega(\lambda) \).

**Lemma 3.6.** Given a Colseg\(_{a,b}\)-valid partition \( \lambda \) containing no hook of shape (6). Then any hook \( H_{i,j} \) divisible by \( b \) with \( i > \psi(\lambda) \) is not \((a,b)\)-shallow.

**Proof.** Pick any \( H_{i,j} = tb \) for some \( t \) and \( i > \psi(\lambda) \). The existence of this and of hook indicates that \( r_{\omega(\lambda)+1}^x \) has \( r_{\omega(\lambda)+1}^x + r^y_{\omega(\lambda)+1} = b + 1 \) and Proposition 3.5 says \( r_{\omega(\lambda)+1}^x > a \).

Now we look at the \( tb \) boxes in the rim corresponding to \( H_{i,j} \), and denote \( s_\beta, \beta = 1, \ldots, t \) to be the smallest rectangles, containing the \((\beta - 1)b + 1\)-th box to the \((\beta b)\)-th box (in the order from northeast to southwest) in this part of the rim. Suppose \( H_{i,j} \) is \((a,b)\)-shallow, which is equivalent to \( l_{i,j} \leq at - 1 \), we have \( 1 + \sum_{\beta=1}^t (s_\beta^x - 1) \leq at \). Hence there exists a \( \beta_0 \), such that \( s_{\beta_0}^x \leq a \). Applying Lemma 3.1 to \( r_{\omega(\lambda)+1}^x \) and \( r_{\beta_0}^x \) gives a contradiction.

**Lemma 3.7.** Given a Colseg\(_{a,b}\)-valid and \( b \)-regular partition \( \lambda \) containing no hook of shape (6), there is

\[
\lambda_{\text{Colseg}_{a,b}} = \lambda_{[1,\psi(\lambda)]} \oplus \lambda_{[\psi(\lambda)+2, l(\lambda)]}.
\]

**Proof.** We first claim that \( \ell_{\psi(\lambda)+1, \Lambda_{\psi(\lambda)+1}} \subset \lambda \) (we take \( \lambda_{\psi(\lambda)+1} = 0 \) if \( \psi(\lambda) + 1 > l(\lambda) \)).

By Proposition 3.5, we know that all boxes on \( \ell_{\psi(\lambda)+1, \Lambda_{\psi(\lambda)+1}} \) and strictly north to \( (\psi(\lambda) + 1, \Lambda_{\psi(\lambda)+1}) \) lies in \( \lambda \). If \( \lambda_{\psi(\lambda)+1} = 0 \), then the claim is immediate. Otherwise suppose the contrary and we can pick the north-most box \( (i, j) \in \ell_{\psi(\lambda)+1, \Lambda_{\psi(\lambda)+1}} \setminus \lambda \). By Lemma 3.3, we know that \( (i - 1, j) \in \lambda \). So the hook \( H_{\psi(\lambda)+1,j} = tb \) for some \( t \in \mathbb{N}_{>0} \) has \( \ell_{\psi(\lambda)+1,j} = ta - 1 \), hence it is \((a,b)\)-shallow, which contradicts Lemma 3.6. Hence \( \ell_{\psi(\lambda)+1, \Lambda_{\psi(\lambda)+1}} \subset \lambda \).

In case \( \omega(\lambda) = \psi(\lambda) = 0 \), there is no sliding movement when performing Colseg\(_{a,b}\), hence

\[
\lambda_{\text{Colseg}_{a,b}} = \lambda_{[2,l(\lambda)]}.
\]

Now we only need to consider the case \( \omega(\lambda) \geq 1 \) and \( \psi(\lambda) \geq 1 \), we know from above that no changes happen underneath row \( \psi(\lambda) + 1 \). And because of \( \ell_{\psi(\lambda)+1, \Lambda_{\psi(\lambda)+1}} \subset \lambda \), there is

\[
\lambda_{\text{Colseg}_{a,b}} = \lambda_{[\psi(\lambda)+1, l(\lambda_{\text{Colseg}_{a,b}})]} = \lambda_{[\psi(\lambda)+2, l(\lambda)]}.
\]
Then consider precisely about the \( b \)-rectangular decomposition stated in Proposition 3.5. The northeast-most box in \( r_\alpha \) (also in \( \lambda \)), \( \alpha = 1, \ldots, \omega(\lambda) \), has coordinates \((1 + (\alpha - 1)a, \lambda_{1+(\alpha-1)a})\) \( \alpha = 1, \ldots, \omega(\lambda) \).

We also know from the shape of \( r_\alpha \)’s that
\[
\lambda_{1+\alpha a} \leq \lambda_{1+(\alpha-1)a} - (b - a), \quad \alpha = 1, \ldots, \omega(\lambda).
\]

Denote
\[
\delta_\alpha = \lambda_{1+(\alpha-1)a} - (b - a) - \lambda_{1+aa}, \quad \alpha = 1, \ldots, \omega(\lambda).
\]

We define the \( b \)-gaps of \( \lambda \) as a collection of boxes in the first row of \( \psi(\lambda) + 1 \) rows of \( \lambda \) and immediately southeast to a box in \( \{ \text{rim of } \lambda \} \setminus \{ b\text{-rim of } \lambda \} \):

\[
\Gamma(\lambda) = \{(i, j) \mid i \leq \psi(\lambda) + 1, (i - 1, j - 1) \in \text{rim of } \lambda \}\setminus \text{b-rim of } \lambda.
\]

Note also that ladders passing through boxes in the first row also passes through some box in the first row of \( r_\alpha \), \( \alpha = 1, \ldots, \omega(\lambda) \), and row \( \psi(\lambda) + 1 \). Hence \( \Gamma(\lambda) \) are exactly the boxes \((1 + a\alpha, \lambda_{1+aa} + k_\alpha) \) for \( k_\alpha = 1, \ldots, \delta_\alpha \) if \( \delta_\alpha \geq 1 \), for \( \alpha = 1, \ldots, \omega(\lambda) \). Then performing \( \text{Colseg}_{a,b} \) to \( \lambda \) is exactly moving boxes \((1, \lambda_1), \ldots, (1, \lambda_1 - \sum_{\beta=1}^{\omega(\lambda)} \delta_\beta) \) in order to \( G_1, \ldots, G_{\sum_{\beta=1}^{\omega(\lambda)} \delta_\beta} \) respectively, which are all boxes in \( \Gamma(\lambda) \) ordered from northeast to southwest, and then removing the first row.

Using the definition of \( J_b \), we know that \( \lambda_i^{J_b} = \lambda_{i+1} \) for \( i \in \{2, \ldots, \psi(\lambda) + 1\} \) and \( i \neq \beta a \) for all \( \beta = 1, \ldots, \omega(\lambda) \) and \( \lambda_i^{J_b} = \lambda_{\beta a + 1} + \delta_\beta, \beta = 1, \ldots, \omega(\lambda) \). Hence \( \lambda_{\psi(\lambda)}^{\text{Colseg}_{a,b}} = \lambda_{\psi(\lambda)}^{J_b} \).

Since they match in every part, \( \lambda_{\psi(\lambda)}^{\text{Colseg}_{a,b}} = \lambda_{\psi(\lambda)}^{J_b} \oplus \lambda_{\psi(\lambda) + 2t(\lambda)} \), as desired. \( \square \)

**Example 3.8.** We demonstrate the argument in the above proof on \( \lambda = (13, 10, 9, 7, 5, 2, 2, 1) \) when \( a = 2 \) and \( b = 5 \). In the left diagram of Figure 12, we are performing \( \text{Colseg}_{2,5} \), and on the right is \( \lambda \) with the 5-rim labeled. (On the right, for the sake of illustration, we remove the box furthest south in each column of \( \lambda_{\psi(\lambda)+1,\psi(\lambda)+1} \), which is equivalent to removing \( \lambda_{\psi(\lambda)+1,\psi(\lambda)+1} \).)

And both operations gives \((10, 10, 7, 6, 2, 2, 1) \).

![Figure 12](image)

*Figure 12.* Demonstration of the argument in Lemma 3.7 of \( \lambda = (13, 10, 9, 7, 5, 2, 2, 1) \), where \( a = 2 \) and \( b = 5 \). The shaded boxes are removed in the corresponding procedure.

Now we are ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** We proceed by induction on \(|\lambda|\). For the base case, we have \((1)^{M_{a,b}} = (1)^{\text{Colreg}_{a,b}} = (1)\) for all \( a, b \).

Now for \( \lambda \) being \( \text{Colreg}_{a,b} \)-valid and all hooks divisible by \( b \) being \((a, b)\)-shallow, we assume that \( \mu^{M_{b,T}} = \mu^{\text{Colreg}_{a,b}} \) holds for all \( \mu \) with the same constraints and \(|\mu| < |\lambda|\).
By Lemma 3.6 and the condition that all hooks divisible by \( b \) is \((a, b)\)-shallow, we know strictly below row \( \psi(\lambda) \), the rectangles in the \( b \)-rectangular decomposition corresponds to a single \( b' \)-segment. Hence \( \lambda^b_{[\psi(\lambda)+1,t(\lambda^b)]} = \lambda^b_{[\psi(\lambda)+2,t(\lambda)]} \). Lemma 3.7 then indicates \( \lambda^b = \lambda^{Colseg_{a,b}} \) and 
\[
(\lambda^{Colreg})_1 = |\lambda| - |\lambda^{Colseg_{a,b}}| = |\lambda| - |\lambda^b| = (\lambda^{M_1 T})_1 = \{|y| \ 1 \leq y \leq \lambda_1, \ell_1, y < \lambda\}.
\]

In order to apply the inductive hypothesis, we need to show that \( \lambda^b = \lambda^{Colseg_{a,b}} \) is \( Colseg_{a,b} \)-valid and all hooks divisible by \( b \) are \((a, b)\)-shallow. \( Colseg_{a,b} \)-validity is straightforward as \( Colseg_{a,b} \) is an intermediate for \( Colseg_{a,b} \) and \( \lambda \) is \( Colseg_{a,b} \)-valid. Now consider all hooks \( H_{i,j}(\lambda^{Colseg_{a,b}}) = tb \) for some \( t \in \mathbb{N}_{>0} \) and compare it with \( H_{i+1,j}(\lambda) \) (note that when performing \( Colseg_{a,b} \) the first row is removed, so there is an index shifting). They are different only in the following three situations:

1. The foot box \( f_{i,j}(\lambda^{Colseg_{a,b}}) = (i', j) \) comes from a box in the first row of \( \lambda \), but the hand box does not.
   
   Suppose \( H_{i,j}(\lambda^{Colseg_{a,b}}) \) is not \((a, b)\)-shallow, the corresponding leg \( l_{i,j}(\lambda^{Colseg_{a,b}}) \geq at \). Hence \( (i'-ta,j+t(b-a)) \notin \lambda^{Colseg_{a,b}} \), contradicting to Lemma 3.3 and the definition of \( Colseg_{a,b} \). So \( H_{i,j}(\lambda^{Colseg_{a,b}}) \) is \((a, b)\)-shallow.

2. The hand box \( h_{i,j}(\lambda^{Colseg_{a,b}}) = (i, j') \) comes from a box in the first row of \( \lambda \), but the foot box does not.
   
   In this case \( i = ka \) for \( k \geq 1 \). Suppose \( H_{i,j}(\lambda^{Colseg_{a,b}}) \) is not \((a, b)\)-shallow, then we claim that \( H_{i-a+1,j}(\lambda) \) is also divisible by \( b \) and not \((a, b)\)-shallow. \( H_{i-a+1,j}(\lambda) \) is also divisible by \( b \) is clear because the rectangle in the \( b \)-rectangular decomposition occupying row \( i-a+1 \) to row \( i \) in \( \lambda \) has \( x \)-dimension \( a \) and \( y \)-dimension \( b-a+1 \). Then \( H_{i-a+1,j}(\lambda) = (t+1)b \), which is illustrated in the following Figure 13.

![Figure 13](image-url)

**Figure 13.** The hand box \( h_{i,j}(\lambda^{Colseg_{a,b}}) = (i, j') \) comes from a box in the first row of \( \lambda \), but the foot box does not. The gap boxes are shaded in the figure, which comes from the first row of \( \lambda \). \( H_{i,j}(\lambda^{Colseg_{a,b}}) \) is labelled by thick black lines and \( H_{i-a+1,j}(\lambda) \) is colored in red.

3. Both the foot box \( f_{i,j}(\lambda^{Colseg_{a,b}}) = (i', j) \) and the hand box \( h_{i,j}(\lambda^{Colseg_{a,b}}) = (i, j') \) comes from a box in the first row of \( \lambda \).

   Proposition 3.5 indicates that in this case both \( i \) and \( i' \) are multiples of \( a \), say \( l_{i,j}(\lambda^{Colseg_{a,b}}) = i'-i = sa \) and \( a_{i,j}(\lambda^{Colseg_{a,b}}) = j'-j \geq s(b-a)+1 \). Hence we have \( s < t \), and \( H_{i,j}(\lambda^{Colseg_{a,b}}) \) is \((a, b)\)-shallow.

Now we can apply the inductive hypothesis: \( \lambda^b M_b T = \lambda^{Colseg_{a,b}} \). By Equation (5) and Equation (4), \( \lambda^{Colseg_{a,b}} = (|\lambda| - |\lambda^{Colseg_{a,b}}|) \oplus \lambda^{Colseg_{a,b}} \) and \( \lambda M_b T = (|\lambda| - |\lambda^b|) \oplus \lambda^b M_b T \), we find that \( \lambda M_b T = \lambda^{Colseg_{a,b}} \).

**Remark 3.9.** Note that the proofs in this section do not require \( a \) and \( b \) to be co-prime.
4. Conjectures

In this section we state the following two conjectures, where the case of \( a = 1 \) is proved by Bessenrodt, Olsson, and Xu in [BOX99] and Fayers in [Fay08] respectively:

**Conjecture 4.1.** Given positive co-prime integers \( a < b \) and a partition \( \lambda \) such that \( \lambda^{M_b}T = \lambda^{\text{Colreg}_{a,b}} \), then all hooks \( H_{i,j} \) in \( \lambda \) with \( b \mid H_{i,j} \) must satisfy equation (1):

\[
\left( \frac{b}{a} - 1 \right) l_{i,j} < a_{i,j} + 1.
\]

**Conjecture 4.2.** Given positive integers \( a, b \) with \( 2a < b \) and a partition \( \lambda \) that is both \( \text{Reg}_{a,b} \)-valid and \( \text{Colreg}_{a,b} \)-valid, we have the following:

(i) If all hooks \( H_{i,j} \) in \( \lambda \) with \( b \mid H_{i,j} \) satisfy:

\[
either \left( \frac{b}{a} - 1 \right) l_{i,j} < a_{i,j} + 1 \quad \text{or} \quad \left( \frac{b}{a} - 1 \right) a_{i,j} < l_{i,j} + 1,
\]

then we have \( \lambda^{\text{Reg}_{a,b}M_b} = \lambda^{T\text{Reg}_{a,b}} \) (i.e. \( \lambda^{\text{Reg}_{a,b}M_b} = \lambda^{\text{Colreg}_{a,b}} \)).

(ii) If \( a \) and \( b \) are co-prime, and \( \lambda \) satisfies \( \lambda^{\text{Reg}_{a,b}M_b} = \lambda^{T\text{Reg}_{a,b}} \), then for all hooks divisible by \( b \), Equation (7) hold.

Conjecture 4.1 and the necessity direction of Conjecture 4.2 are closely related to the \( q \)-decomposition numbers in the basic \( U_q(\mathfrak{sl}_b) \)-representation because the unique the lower global crystal basis \( \{ G(\mu) \mid \mu \in \mathcal{P} \text{ is } b\text{-regular} \} \) are constructed using the ladder method, as shown in [LLT96].

We refer to [LLT96] for details quantized affine Lie algebra \( U_q(\mathfrak{sl}_b) \) and its action on the Fock space \( \mathcal{F} = \bigoplus_{\mu \in \mathcal{P}} \mathbb{Q}(q)^{\lambda} \), which is originally developed by Misra-Miwa [MM90] using work of Hayashi [Hay90]. And we state Kashiwara’s existence and uniqueness of crystal bases of the integrable highest weight modules of affine quantum algebra in [K+91] as follows:

**Theorem 4.3** ([K+91]). Denote \( M(\Lambda_0)_{\mathbb{Q}} = U_q^{-\emptyset} \) where \( U_q^{-\emptyset} \in U_q(\mathfrak{sl}_b) \) is the \( \mathbb{Q}[q,q^{-1}] \)-algebra generated by the Chevalley generators \( f_i^{(\alpha)} \)’s. Then there exists a unique \( \mathbb{Q}[q,q^{-1}] \)-basis \( \mathcal{G} = \{ G(\mu) \mid \mu \text{ is b-regular} \} \), which is called the lower crystal basis, such that the following holds:

(i) \( G(\mu) \equiv \mu \mod qL \);

(ii) \( G(\mu) = G(\mu) \).

The coordinates of \( G(\mu) \) for \( \mu \) being \( b \)-regular in the standard basis \( \{ \lambda \in \mathcal{P} \} \) is called the \( q \)-decomposition numbers. The following are its properties:

**Theorem 4.4** ([LLT96, VV+99, Sch00]).

\[
G(\mu) = \sum_{\lambda} d_{\lambda,\mu}(q)\lambda.
\]

The coefficients satisfies:

(i) \( d_{\lambda,\mu} \in \mathbb{N}[q] \);

(ii) \( d_{\lambda,\mu}(q) = 1 \) and \( d_{\lambda,\mu}(q) = 0 \) unless \( \lambda \leq \mu, |\lambda| = |\mu|, \) and \( \text{Core}_b(\lambda) = \text{Core}_b(\mu) \);

(iii) \( d_{\lambda,\mu}(q) = q^{|\mu|_b}d_{\lambda,\mu}(q^{-1}) \), where \( |\mu|_b \) is the \( b \)-weight of \( \mu \), which is the number of hooks in \( \mu \) that is divisible by \( b \).

The third property in the above theorem is important in relating the Mullineux involution with the \( q \)-decomposition numbers.

**Conjecture 4.5.** Suppose that \( \lambda \) and \( \mu \) are partitions of the same size and \( \lambda \) is \( \text{Reg}_{a,b} \)-valid and \( \mu \) is \((a,b)\)-regular. Then \( d_{\lambda,\mu}(q) = 0 \) if \( \lambda^{\text{Reg}_{a,b}} \not\subseteq \mu \) and \( d_{\lambda,\mu}(q) = q^s(\lambda) \), where \( s(\lambda) \) is the number of \((a,b)\)-steep hooks in \( \lambda \) that are divisible by \( b \).
Conjecture 4.5 is indeed Fayers’ [Fay07, Theorem 2.2] in terms of the generalized (column) regularization. In fact, this is the only ingredient needed for Conjecture 4.1 and the necessity direction of Conjecture 4.2.

**Proof of Conjecture 4.1 given Conjecture 4.5.** Suppose a Colreg$_{a,b}$-valid partition $\lambda$ satisfies $\lambda^{M_b}_T = \lambda^{\text{Colreg}_{a,b}}$, so $d_{\lambda^{T},\lambda^{M_b}}(q) = d_{\lambda^{T},\lambda^{\text{Reg}_{a,b}}}(q)$. By Conjecture 4.5, $d_{\lambda^{T},\lambda^{\text{Reg}_{a,b}}}(q) = q^{s(\lambda_T)}(q)$. But $s(\lambda_T)$ is also the number of shallow hooks in $\lambda$. From Theorem 4.4, we know that $d_{\lambda^{T},\lambda^{M_b}} = q^{\lambda|b}d_{\lambda^{T}}(q^{-1}) = q^{\lambda|b}$. Therefore, $|\lambda|_b = s(\lambda_T)$, so all hooks in $\lambda$ divisible by $b$ must be shallow. □

**Proof of necessity of Conjecture 4.2 given Conjecture 4.5.** Suppose we have a both Colreg$_{a,b}$-valid and Reg$_{a,b}$-valid partition $\lambda$ satisfying $\lambda^{\text{Reg}_{a,b}}M_b = \lambda^{T}\text{Reg}_{a,b}$. So $d_{\lambda^{T},\lambda^{\text{Reg}_{a,b}}M_b}(q) = d_{\lambda^{T},\lambda^{\text{Reg}_{a,b}}}(q)$.

On the one hand, by Conjecture 4.5, there is

$$d_{\lambda^{T},\lambda^{\text{Reg}_{a,b}}}(q) = q^{s(\lambda_T)}.$$ 

On the other hand,

$$d_{\lambda^{T},\lambda^{\text{Reg}_{a,b}}M_b}(q) = q^{\lambda^{\text{Reg}_{a,b}}|b}d_{\lambda^{T},\lambda^{\text{Reg}_{a,b}}}(q^{-1}) = q^{\lambda|b}d_{\lambda^{T},\lambda^{\text{Reg}_{a,b}}}(q^{-1}) = q^{\lambda|b}q^{-s(\lambda)}$$

where the first equality comes from Theorem 4.4, the second comes from Lemma 2.28 and the last one is due to Conjecture 4.5.

So $s(\lambda) + s(\lambda_T) = |\lambda|_b$. Lemma 2.3 indicates no hooks can be both shallow and steep in case of $2a < b$, which means all hooks divisible by $b$ must be either shallow or steep. □

Finally, we would like to point out that the $(a, b)$-regular partitions form an important collection, whose algebraic interpretations will be studied in the later papers of this series.

**References**

[BOX99] Christine Bessenrodt, Jørn B Olsson, and Maozhi Xu. On properties of the Mullineux map with an application to Schur modules. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 126, pages 443–459. Cambridge University Press, 1999.

[DY18] Panagiotis Dimakis and Guangyi Yue. Combinatorial wall-crossing and the Mullineux involution. *Journal of Algebraic Combinatorics*, Sep 2018.

[Fay07] Matthew Fayers. $q$-Analogues of regularisation theorems for linear and projective representations of the symmetric group. *Journal of Algebra*, 316(1):346–367, 2007.

[Fay08] Matthew Fayers. Regularisation and the Mullineux map. *The Electronic Journal of Combinatorics*, 15(1):142, 2008.

[FK97] Ben Ford and Alexander S Kleshchev. A proof of the Mullineux conjecture. *Mathematische Zeitschrift*, 226(2):267–308, 1997.

[Hai02] Mark Haiman. Combinatorics, symmetric functions, and Hilbert schemes. *Current developments in mathematics*, 2002:39–111, 2002.

[Hay90] Takahiro Hayashi. $Q$-Analogues of Clifford and Weyl algebras-spinor and oscillator representations of quantum enveloping algebras. *Communications in Mathematical Physics*, 127(1):129–144, 1990.

[JK] Gordon James and Adalbert Kerber. The representation theory of the symmetric group. With a foreword by PM Cohn. With an introduction by Gilbert de B. Robinson. *Encyclopedia of Mathematics and Its Applications*, 16.

[K+91] Masaki Kashiwara et al. On crystal bases of the $Q$-analogue of universal enveloping algebras. *Duke Mathematical Journal*, 63(2):465–516, 1991.

[Kac94] Victor Kac. *Infinite-dimensional Lie algebras*, volume 44. Cambridge university press, 1994.

[Kle96] Alexander S Kleshchev. Branching rules for modular representations of symmetric groups III: some corollaries and a problem of Mullineux. *Journal of the London Mathematical Society*, 54(1):25–38, 1996.

[LLT96] Alain Lascoux, Bernard Leclerc, and Jean-Yves Thibon. Hecke algebras at roots of unity and crystal bases of quantum affine algebras. *Communications in Mathematical Physics*, 181(1):205–263, Nov 1996.

[MM90] Kailash C Misra and Tetsuji Miwa. Crystal base for the basic representation of $U_q(\frak{sl}(n))$. *Communications in Mathematical Physics*, 134(1):79–88, 1990.

[Sch00] Olivier Schiffmann. The hall algebra of a cyclic quiver and canonical bases of fock spaces. *International Mathematics Research Notices*, 2000(8):413–440, 2000.
[VV+99] Michela Varagnolo, Eric Vasserot, et al. On the decomposition matrices of the quantized schur algebra. *Duke mathematical journal*, 100(2):267–297, 1999.

[Wal94] Grant Walker. Modular Schur functions. *Transactions of the American Mathematical Society*, 346(2):569–604, 1994.

[Wal96] Grant Walker. Horizontal partitions and Kleshchev’s algorithm. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 120, pages 55–60. Cambridge University Press, 1996.