Universal detector efficiency of a mesoscopic capacitor

Simon E. Nigg and Markus B"uttiker

D"epartement de Physique Th"eorique, Universit"e de Gen"eve, CH-1211 Gen"eve 4, Switzerland

(Dated: February 4, 2009)

We investigate theoretically a novel type of high frequency quantum detector based on the mesoscopic capacitor recently realized by Gabelli et al., [Science 313, 499 (2006)], which consists of a quantum dot connected via a single channel quantum point contact to a single lead. We show that the state of a double quantum dot charge qubit capacitively coupled to this detector can be read out in the GHz frequency regime with near quantum limited efficiency. To leading order, the quantum efficiency is found to be universal owing to the universality of the charge relaxation resistance of the mesoscopic capacitor.

PACS numbers:

The measurement problem is probably one of the oldest topics in quantum physics, which is still of prime interest to researchers nowadays. With the advent of mesoscopic physics, fundamental issues related to Von Neumann’s notion of the instantaneous wave function collapse can now be addressed experimentally. Indeed it has recently become possible to engineer systems in which parts of the measurement device are themselves unambiguously quantum. In the weak coupling regime the dynamics of the wave function collapse itself can be probed and sometimes even reversed [2, 3]. Questions such as “how long does it take to acquire the desired information?” and “how fast does the measurement decohere the state of the measured system?” become of relevance. This is in particular true in the emergent field of quantum information processing, where one wishes to both manipulate and read-out quantum bits (qubits) with the highest possible efficiencies.

An important figure of merit of any quantum detector is its Heisenberg efficiency. Loosely speaking it is the ratio of how fast to how invasive a given detector is. By “fast” we mean how quickly to two different states of the measured system can be distinguished from one another and by “invasive” we mean how strong is the back-action of the detector onto the state of the measured system. The Heisenberg uncertainty relation implies that one cannot acquire information about the system faster than one dephases it during the measurement process [4, 5, 6, 7]. Hence the Heisenberg efficiency is bounded from above. An important task is thus to find and characterize detectors which reach the maximum allowed Heisenberg efficiency.

Several such systems have been described in the literature. In the DC regime Refs. [8, 9, 10, 11] investigate the quantum point contact (QPC) detector. Refs. [4, 5] discuss two terminal scattering detectors capacitively coupled to a double dot charge qubit. In both cases, the average current through the detector functions as a meter, since the electron transmission probability is sensitive to the position of the charge in the qubit. Due to 1/f noise DC detectors are generically plagued by a large dephasing rate. To circumvent this, Schoelkopf et al. [12] introduced the radio-frequency single-electron transistor (rf-SET). The idea there, is to measure the damping of an oscillator circuit in which the SET is embedded. In this letter we present a novel quantum detector based on the mesoscopic capacitor [13], which consists of a quantum dot connected via a single channel QPC to a single lead. At temperatures low compared with the charging energy, such a system exhibits a universal [13, 14, 15] charge relaxation resistance $R_q = \hbar/(2e^2)$. We show that this system embedded in an LC tank circuit with impedance $L$ and capacitance $C$, can be operated as a high frequency detector near the quantum limit despite the presence of intrinsic dissipation. At the resonant frequency $\omega_0 = 1/\sqrt{LC}$ we find to leading order, a universal Heisenberg efficiency

$$\eta = \frac{L/C}{L/C + R_q Z_0},$$

where $Z_0$ is the characteristic impedance of the transmission line connected to the tank circuit.

![Diagram](image-url)

FIG. 1: (Color online) Detector and qubit system (a). Equivalent circuit in the adiabatic approximation (b). Incoming photons are reflected ($R$) and detected or dissipated ($T$).
The system we consider is depicted in Fig. 1(a). Let us first consider the system without the LC resonator and transmission line. The part being measured; the double dot charge qubit, plus the capacitive coupling term are described by the Hamiltonian

$$H_{qb} = \frac{1}{2} \left( \epsilon \sigma_z + \Delta \sigma_x + \kappa \sigma_z \hat{N} \right) \text{ with } \kappa = \frac{e^2}{C_v}. \quad (2)$$

Here \( \sigma_z = |\uparrow\rangle \langle \uparrow| - |\downarrow\rangle \langle \downarrow| \) and \( \sigma_x = |\uparrow\rangle \langle \downarrow| + |\downarrow\rangle \langle \uparrow| \). In the state \(|\uparrow\rangle\langle \uparrow|\) the excess charge is located on the upper (lower) dot. The energy difference \( \epsilon \) and the coupling \( \Delta \) between these two states can be tuned by the gate voltages \( V_{g1} \) and \( V_{g2} \) (see Fig. 1(a)). \( \hat{Q} = e\hat{N} = e\sum_i d_i^d d_i \) is the excess charge on the quantum dot (QD) of the mesoscopic capacitor. The latter is described by the Hamiltonian

$$H_D = \sum_i \varepsilon_i d_i^d d_i + \frac{\hat{Q}^2}{2C_S}. \quad (3)$$

Here the first term describes the unperturbed level spectrum while the second term gives the Coulomb interaction. \( \sigma = \{1/C_v + 1/C_2 + 1/C_0\}^{-1} \) is the total series capacitance. Finally the QD of the capacitor is coupled to the lead via the tunneling Hamiltonian \( H_T = \sum_{i k} t_{i k} c_i^\dagger c_k + h.c. \), where \( t_{i k} \) is the tunneling matrix element between state \( i \) of the dot and state \( k \) of the lead and can be tuned with the gate voltage \( V_{qpc} \) (see Fig. 1(a)). The lead, where we neglect the electron-electron interaction, is described by \( H_L = \sum_k E_k c_k^\dagger c_k \). The entire system is described by the Hamiltonian

$$H = H_{qb} + H_D + H_L + H_T. \quad (4)$$

If not for the tunneling term \( H_T \), which changes the charge on the dot of the capacitor, a qubit prepared in one of the eigenstates of \( H_{qb} \) for a given charge \( \hat{Q} \) would remain in this state under the time evolution. Because of \( H_T \) however, the charge on the dot fluctuates leading to a modulation in time of the level splitting of the qubit. If this modulation is slow enough though, the qubit will remain in an instantaneous eigenstate of \( H_{qb} \) at all times. To derive the necessary conditions for this to be true, we follow [10], and apply a unitary transformation onto \( \hat{H} \), which diagonalizes \( H_{qb} \) in each subspace of fixed \( \hat{N} \).

$$H' = U(\hat{N})^\dagger H U(\hat{N}). \quad (5)$$

With \( \eta_{\Omega} = \arccot \left( \frac{\kappa}{\Omega} \right) \), the unitary operator to second order in the coupling \( \kappa \) is explicitly given by \( U(\hat{N}) = \hat{a}_0 U_0 + \hat{a}_1 U_1 \), with

$$\hat{a}_0 = 1 - \frac{\Delta^2}{8\Omega_0^2} \hat{N}^2 \quad \text{and} \quad \hat{a}_1 = \frac{\kappa \hat{N}}{2\Omega_0^2} \left( 1 - \frac{\kappa}{\Omega^2} \right), \quad (6)$$

where \( \Omega_0 = \sqrt{\omega^2 + \Delta^2} \) is the bare Rabi frequency and

$$U_0 = \left( \begin{array}{cc} \cos \frac{\Omega_0}{2} & -\sin \frac{\Omega_0}{2} \\ \sin \frac{\Omega_0}{2} & \cos \frac{\Omega_0}{2} \end{array} \right), \quad U_1 = \left( \begin{array}{cc} \sin \frac{\Omega_0}{2} & \cos \frac{\Omega_0}{2} \\ -\cos \frac{\Omega_0}{2} & \sin \frac{\Omega_0}{2} \end{array} \right) \quad (7)$$

Note that \( U_0^\dagger U_0 = U_1^\dagger U_1 = 1 \) while \( U_0^\dagger U_1 = -U_1^\dagger U_0 = i\sigma_y \). Using the fact that \([[H_T, \hat{N}], \hat{N}] = H_T\), we finally obtain

$$U^\dagger H_T U = H_T + i\sigma_y \frac{\kappa \Delta}{2\Omega_0^2} [H_T, \hat{N}] + O(\kappa^3). \quad (8)$$

where we have neglected a small \( O(\kappa^2) \) renormalization of the tunneling amplitudes \( t_{i k} \), which is insensitive to the state of the qubit. In the linear response regime, the time scale on which \( \langle \hat{N}(t) \rangle \) fluctuates is set by the inverse of the drive frequency \( \omega \). Therefore the energy available for making a real transition between the qubit eigenstates, which is given by the second term on the right-hand side of Eq. (8), is proportional to \( \hbar \omega \kappa \Delta/(2\Omega_0^2) \). Demanding that this energy be small compared to the level splitting \( \Omega_0 \) of the qubit leads us to the following adiabatic condition on the drive frequency

$$\hbar \omega \ll \frac{2\Omega_0^2}{\kappa \Delta}. \quad (9)$$

Let us briefly discuss this condition. We see that for \( \Delta = 0 \), we can drive the system as fast as we wish if provided \( \epsilon \neq 0 \). This simply reflects the fact that for \( \Delta = 0 \) \( \ll \epsilon \) the two eigenstates of the qubit, which in fact are the charge states in this limit, are decoupled from one another. We also see that the weaker the coupling, the faster we may drive the system without inducing transitions, which is intuitively reasonable. For realistic values of the parameters; \( \Delta = \Omega_0 = 5 \mu eV, \epsilon = 0 \) and \( \kappa = 50 \mu eV \), we find \( 2\Omega_0^2/(\kappa \Delta) \gtrsim 1.5 \cdot 10^{12} \text{ Hz} \), so that even for drive frequencies in the GHz regime we are still safely in the adiabatic regime.

In the adiabatic approximation and for weak coupling, i.e. \( \kappa \ll \Omega_0 \), the dynamics of the system is thus appropriately described to second order in \( \kappa \) by the purely longitudinal effective Hamiltonian \( H_{\text{eff}} = H_+ |+\rangle \langle +| + H_- |-\rangle \langle -| \), where

$$H_{\pm} = \pm \frac{\Omega_0}{2} + \sum_i \varepsilon_i^\pm d_i^d d_i + \frac{e^2}{2C_{\text{eff}}} \hat{N}^2 + H_L + H_T. \quad (10)$$

Here \(|\pm\rangle\) are the adiabatic eigenstates of \( H_{qb} \). The presence of the qubit appears thus as a renormalization of the spectrum of the QD of the detector: \( \varepsilon_i^\pm = \varepsilon_i \pm \kappa \Omega_0/(2\Omega_0) \), and a renormalization of the geometric capacitance \( C_{\text{eff}}^\pm \) of the dot vis-à-vis the gate \( V_{g1} = 1/C_{\text{eff}}^\pm = 1/C_2 \pm \kappa^2 \Delta^2/(2\Omega_0^2) \). Formally, the effective Hamiltonian we have just derived is exactly the same as the one of a mesoscopic capacitor with a single level spectrum \( \varepsilon_i^\pm \) and a geometric capacitance \( C_{\text{eff}}^\pm \). Within the self-consistent Hartree approximation \( \epsilon \ll \Omega_0 \), the linear response of a mesoscopic capacitor to an applied AC voltage is known [13, 10]. For short RC times \( \tau_{RC}^\pm = R_q C_{\text{eff}}^\pm \) such that \( \omega_{\tau_{RC}}^\pm \ll 1 \), the mesoscopic capacitor is equivalent to an RC circuit with the impedance \( Z_{\text{eff}}^\pm (\omega) = R_q + i/(\omega C_{\text{eff}}^\pm) \). Here \( R_q \) is the
charge relaxation resistance, which at zero temperature and for a single channel capacitor is universal and given by half a resistance quantum, i.e. \( R_q = h/(2e^2) \). The electrochemical capacitance \( C^\pm_{\mu} \) however depends on \( C^\pm_{\text{eff}} \) and on the density of states (DOS) of the capacitor and is thus sensitive to the state of the qubit. Explicitly one finds \[ \frac{1}{C^\pm_{\mu}} = \frac{1}{C^\pm_{\text{eff}}} + \frac{1}{\epsilon^2 \nu_\pm(E_F)}. \] (11)

Here \( \nu_\pm(E_F) \) is the DOS at the Fermi-energy of the QD with the shifted spectrum \( \{\varepsilon_\pm^\varepsilon\} \). The electrochemical capacitance thus acts like the pointer of a measurement device. At the degeneracy point \( \epsilon = 0 \), the shift of the levels vanishes, while the correction to the capacitance is maximal. If to the contrary \( \epsilon \gg \Delta \) then the correction to the capacitance vanishes while the dot spectrum is maximally shifted by the amount \( \pm \kappa/2 \).

Let us now discuss a way of probing the electrochemical capacitance in the high frequency regime. Using a dispersive read-out scheme similar to \[12, 16\], we embed our effective capacitor into an LC tank-circuit and via a standard homodyne detection scheme \[17\], probe the phase shift of waves reflected from the tank-circuit (see Fig. 1 (b)). We here do not want to measure the resistance of our effective capacitor. Indeed, owing to the universality of the charge relaxation resistance in the single channel limit, this quantity is actually insensitive to the state of the qubit. Instead, we propose to detect the phase shift of a reflected signal, which is determined by the non-dissipative part of the response of the mesoscopic capacitor.

To second order in \( C^\pm_{\mu}/C \), the shifted resonance frequency of the tank-circuit is given by

\[ \omega^\pm_{\text{osc}} \approx \omega_0 \left( 1 - \frac{1}{2} \frac{C^\pm_{\mu}}{C} \right) - i \omega_0 \frac{C^\pm_{\mu}}{2C} \nu^\pm_{\text{RC}}, \] (12)

where \( \omega_0 = 1/\sqrt{LC} \) is the bare oscillator resonance frequency. Notice that because of the finite resistance \( R_q \), the oscillation of the LC circuit is damped. This is reflected in the non-vanishing imaginary part of \( \omega^\pm_{\text{osc}} \) in Eq. (12). In other words, photons coming down the transmission line toward the LC-tank circuit, will be dissipated with some finite probability. The reflected photons however will experience a phase shift, which depends on the state of the qubit. It is this phase shift which we propose to measure.

The impedance of the tank-circuit which terminates the transmission line is \( Z_{\pm}(\omega) = iL(\omega^2_{\text{osc}} - \omega^2)/\omega \). From this, we can calculate the complex reflection coefficient \( R_{\pm} \) of the transmission line with characteristic impedance \( Z_0 \), relating incoming and outgoing modes via

\[ a^\pm_{\text{out}}(\omega) = R_{\pm}(\omega) a^\pm_{\text{in}}(\omega). \]

We find

\[ R_{\pm}(\omega) = \frac{Z_0 - Z_{\pm}(\omega)}{Z_0 + Z_{\pm}(\omega)} = \frac{\omega^2 - (\omega^\pm_{\text{osc}})^2 - i\eta_0 \omega}{\omega^2 - (\omega^\pm_{\text{osc}})^2 + i\eta_0 \omega}, \] (13)

with \( \eta_0 = Z_0/L \). Because \( (\omega^\pm_{\text{osc}})^2 \) has a non-vanishing imaginary part, \( R_{\pm} \) is not unitary. At the bare resonance frequency, we obtain, \( R_{\pm}(\omega) = \gamma_{\pm} e^{i\phi_{\pm}} \), with

\[ \gamma_{\pm} = 1 - 2 \frac{R_q}{Z_0} \left( \frac{C^\pm_{\mu}}{C} \right)^2 + O \left( \left( \frac{C^\pm_{\mu}}{C} \right)^3 \right), \] (14)

and

\[ \phi_{\pm} = Q_0 \frac{C^\pm_{\mu}}{C} \left( 2 - \frac{1}{2} \left( \frac{C^\pm_{\mu}}{C} \right) \right) + O \left( \left( \frac{C^\pm_{\mu}}{C} \right)^3 \right). \] (15)

Here we have introduced the quality factor \( Q_0 = \sqrt{L/C}/Z_0 \) of the resonator plus transmission line circuit. To leading order, the probability of a photon to be dissipated is thus given by \( 1 - \gamma_0^2 = 4(R_q/Z_0)(C^\pm_{\mu}/C)^2 \). Also, we remark that the leading order correction to the reflection phase due to a finite \( R_q \) is of order \( (C^\pm_{\mu}/C)^3 \). Finally note that the leading order correction to \( \gamma_{\pm} \) is independent of \( L \). This is ultimately the reason why we can achieve a large Heisenberg efficiency; increasing \( L \) increases the signal without increasing the dissipation.

We next derive expressions for the measurement and dephasing rates \[16, 20, 21\]. The measured quantity is the number of photons reflected from the load in time \( T \). By mixing this signal with a strong signal from a local oscillator driven at the same frequency \( \omega_0 \) as the drive and afterwards taking the average, the measured number of photons \( n_{\pm}(T) \) becomes sensitive to the reflection phase shift, which in turn depends on the state of the qubit. The two eigenstates are said to be resolved, when \( \Delta n(T) = n_{+}(T) - n_{-}(T) \) becomes larger than the noise. The time when this happens defines the measurement time \( T_m \). Let us consider a monochromatic coherent state input with amplitude \( \beta_0 \).

\[ |\psi\rangle_{\text{in}} = \exp \left[ T(\beta_0 a^\dagger_{L}(\omega_0) - \beta_0^* a_{L}(\omega_0)) \right] |0\rangle, \] (16)

where \( a^\dagger_{L}(\omega) \) creates an incoming photon at frequency \( \omega \). Using the same definition for the signal to noise ratio as in \[21\], we find a measurement rate given by

\[ \Gamma_m \equiv T_m^{-1} = |\beta_0|^2 \frac{(\gamma_+ + \gamma_-)^2}{\gamma_+^2 + \gamma_-^2} \sin^2(\Delta \phi/2), \] (17)

where \( \Delta \phi = \phi_+ - \phi_- \). We note that \( \Gamma_m \) is bounded from above by \( 2|\beta_0|^2 \), or twice the photon injection rate. Incidentally, this is the maximally achievable measurement rate in the absence of dissipation, where \( \gamma_{\pm} = 1 \).

To derive the dephasing rate, we essentially follow the quantum information theoretic argument of \[21\] and
adapt it to a dissipative system. The resistor is replaced by a semi-infinite transmission line with characteristic impedance \( R_q \) (see Fig. 1(b)), which is then quantized \[22\]. Hence we determine the transmission coefficient \( T^\pm \) for photons to be dissipated. The measurement can be represented as the entangling process

\[
(\alpha \ket{+} + \beta \ket{-}) \ket{\beta_0} \rightarrow \alpha' \ket{+} \ket{\beta_+} + \beta' \ket{-} \ket{\beta_-}, \tag{18}
\]

where detector states after the scattering are given by a product of phase shifted and damped coherent states as

\[
\ket{\beta_\pm} = \ket{\beta_0 \gamma_\pm e^{i\phi_\pm}} \otimes \ket{\beta_0 \sqrt{1 - \gamma_\pm^2 e^{i\phi_\pm}}} \tag{19}
\]

Here \( \theta_\pm = \arg(T^\pm) \) is the phase shift of the dissipated photons. The off-diagonal elements of the reduced density matrix of the qubit are proportional to the overlap of the detector states, i.e. \( \ket{\rho_{12}} \sim \ket{\beta_+ \ket{\beta_-}} \). For long times, these elements decay exponentially defining the dephasing rate by \( \ket{\rho_{12}} \sim \exp[-\Gamma_\phi T] \). We find explicitly

\[
\Gamma_\phi = |\beta_0|^2 \left[ 1 - D_1 \cos(\Delta \phi) - D_2 \cos(\Delta \theta) \right], \tag{20}
\]

with \( D_1 = \gamma_+ \gamma_- \) and \( D_2 = \sqrt{(1 - \gamma_+^2)(1 - \gamma_-^2)} \). From Eqs. \[17\] and \[20\] we finally obtain the Heisenberg efficiency of our detector

\[
\eta \equiv \frac{\Gamma_m}{\Gamma_\phi} = \frac{L/C}{L/C + R_q \omega_0} + O \left( (C^\mu / C)^2 \right). \tag{21}
\]

Thus we find that to leading order the Heisenberg efficiency does not depend on \( C^\mu \). This result holds as long as \( \omega_0 \gamma_\pm^2 R_C \ll 1 \ll C^\mu \). To reach acceptable efficiency, we need to have \( L/C \gg R_q \omega_0 \). Fig. 2 shows the Heisenberg efficiency as a function of \( L \) for realistic parameters. Decreasing \( C \), which increases \( \Delta \phi \), increases the efficiency and at the same time increases the measurement frequency \( \omega_0 = 1 / \sqrt{LC} \). For example for \( L = 10 \mu \text{H} \), and \( C = 100 \text{fF} \), we have \( \omega_0 = 1 \text{GHz} \) and \( \eta = 99.4\% \) (see full thick (red) curves on Fig. 2).

In conclusion, we have shown that the mesoscopic capacitor can in principle be operated as an efficient detector in the GHz regime. We find that to leading order its efficiency is universal, i.e. independent of the microscopic details of the detector and qubit. This universality can be directly traced back to the experimentally demonstrated \[14\] universality \[13\] of the charge relaxation resistance of a mesoscopic capacitor.

This work is supported by the Swiss NSF, MaNEP and the STREP project SUBTLE.

\[\begin{align*}
\text{FIG. 2: (Color online) Efficiency } \eta \text{ as a function of inductance } L. \text{ The inset shows } \eta(\omega) \text{ for } L = 10 \mu \text{H and } C = 10^{-13} \text{ F.}
\end{align*}\]