Quark-Antiquark and Diquark Condensates in Vacuum in a 2D Two-Flavor Gross–Neveu Model

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Abstract The analysis based on the renormalized effective potential indicates that, similar to in the 4D two-flavor Nambu–Jona-Lasinio (NJL) model, in a 2D two-flavor Gross–Neveu model, the interplay between the quark-antiquark and the diquark condensates in vacuum also depends on $G_S/H_S$, the ratio of the coupling constants in scalar quark-antiquark and scalar diquark channel. Only the pure quark-antiquark condensates exist if $G_S/H_S > 2/3$, which is just the ratio of the color numbers of the quarks participating in the diquark and quark-antiquark condensates. The two condensates will coexist if $0 < G_S/H_S < 2/3$. However, different from the 4D NJL model, the pure diquark condensates arise only at $G_S/H_S = 0$ and are not in a possibly finite region of $G_S/H_S$ below 2/3.

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1 Introduction

It is well known that the four-fermion interaction models including the Nambu–Jona-Lasinio (NJL) model\cite{1} in 4D space-time and the Gross–Neveu (GN) models\cite{2} in 2D and 3D space-time are useful means to explore dynamical symmetry breaking. The NJL model may be used in research not only on chiral symmetry breaking in Quantum Chromodynamics (QCD), which is connected to the quark-antiquark condensates\cite{1,3-6} but also on color superconducting at low temperature and moderate baryonic density, which is related to the diquark condensates.\cite{7} In fact, owing to that the four-fermion interactions of $(qq)^2$-form and $(qq)^2$-form may be related to each other via the Fierz transformation, in any four-fermion model, the above two forms of interactions are always allowed to exist simultaneously. As a result, one will face to interplay between the quark-antiquark condensates and the diquark condensates even at temperature $T = 0$ and the quark chemical potential $\mu = 0$, i.e. in vacuum. To clarify such interplay theoretically is certainly interesting for deeper understanding of feature of the four-fermion interaction models.

In a preceding paper\cite{8}, we have researched a 4D two-flavor NJL model and proven that the mutual competition between the quark-antiquark and the diquark condensates does occur in vacuum and, depending on the ratio of the coupling constants $G_S$ and $H_S$ in scalar quark-antiquark and scalar diquark channel, the diquark condensates could either exist alone, or coexist with the quark-antiquark condensates, or not be formed completely. Then a natural question is whether similar conclusions can be reached from the similar four-fermion interaction models in other space-time dimensions, i.e. whether these conclusions are of generality for this kind of models. To answer this question, we will generalize the discussions in Ref.\cite{8} to a 2D GN model in this paper. For convenience of making a comparison with the results in the 4D NJL model given in Ref.\cite{8}, we will take the fermions (still called quarks) with zero bare-masses, two flavors and three colors and assume that the diquark condensates will come from the attractive interactions in the color anti-triplet channel.

The paper is arranged as follows. In Sec. 2 we will give the 2D two-flavor GN model with diquark interactions, and analyze its discrete symmetries and possible spontaneous breaking induced by the condensates. In Sec. 3 we will derive the renormalized effective potential in the space-time dimension regularization approach. In Sec. 4, by means of the effective potential, the ground states of the model will be determined and the mutual competition between the quark-antiquark and the diquark condensates will be expounded. Finally in Sec. 5 our conclusions follow.

2 2D Two-Flavor Gross–Neveu Model with Diquark Interactions

The Lagrangian of the model is expressed by

$$
\mathcal{L} = \bar{q}_i \gamma^\mu \partial_\mu q_i + G_S (\bar{q}_i q_i)^2 + (\bar{q}_i \gamma_5 \tau_i q_i)^2 \\
+ H_S (\bar{q}_i \gamma_5 \tau_i \lambda_i q_i) (\bar{q}_i \gamma_5 \tau_i \lambda_i q_i),
$$

(1)

where in 2D space-time, $q \equiv q_{\alpha k}(t, x)$ is two-component spinor field with the two flavors $\alpha = u, d$ and three colors $k = r$ (red), $g$ (green), $b$ (blue) and the matrices

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\[ \gamma^0(\mu = 0, 1), \gamma_5 \text{ and } C \text{ will be } 2 \times 2 \text{ matrices, which are defined by} \]
\[ \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -C, \]
\[ \gamma_5 = \gamma^0\gamma^1, \]
\[ (2) \]
and obey the relations
\[ \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad \{\gamma_5, \gamma^\mu\} = 0, \quad C\gamma^\mu T C = \gamma^\mu. \]
\[ (3) \]
The matrix \( C \) is related to the charge conjugates of the quark fields \( q \) by
\[ q^c = Cq^T, \quad \bar{q}^c = \bar{q}^T C. \]
\[ (4) \]
The Pauli matrices \( \tau \) act in the two-flavor space and \( \tau_2 = (\tau_2, \tau_3) \) are the flavor-triplet symmetric matrices. The \( \lambda_4 \) are color-triplet antisymmetric Gell-Mann matrices acting in the three-color space. It is indicated that the product matrix \( C\gamma_5\tau_2\lambda_4 \) is antisymmetric and the term with \( H_S \) in Eq. (1) corresponds to the diquark interactions, which could lead to scalar color-antitriplet diquark condensates.

Since a continuous symmetry can never be spontaneously broken in 2D space-time by the Mermin–Wagner–Coleman theorem,\[8\] we are interested only in the discrete symmetries of the model and their possible spontaneous breaking. In this respect, it is easy to check that the Lagrangian (1) has the following discrete symmetries: \( R: q(t, x) \xrightarrow{R} -q(t, x), \) parity \( P: q(t, x) \xrightarrow{P} \gamma^0q(-t, -x), \) time reversal \( T: q(t, x) \xrightarrow{T} \gamma^0q(-t, x), \) charge conjugate \( C: q(t, x) \xrightarrow{C} \bar{q}^c(t, x), \) discrete chiral symmetry \( \chi_0: q(t, x) \xrightarrow{\chi_0} \gamma_5\gamma(t, x), \) special parity \( P_1: q(t, x) \xrightarrow{P_1} \gamma_1q(t, -x), \) and the center \( Z_2^f \) of \( SU_f(2): \]
\[ q(t, x) \xrightarrow{Z_2^f} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} q(t, x). \]
\[ \text{and the center } Z_2^f \text{ of } SU_f(2): \]
\[ q(t, x) \xrightarrow{Z_2^f} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} q(t, x). \]
The subscripts \( c \) and \( f \) indicate that the corresponding matrices act in flavor and color spaces respectively.

Assume that the four-fermion interactions could lead to the scalar quark-antiquark condensates
\[ \langle \bar{q}q \rangle = \phi, \]
\[ (5) \]
which will break the discrete symmetries \( \chi_D \) and \( \mathcal{P}_1, \) and the scalar color-antitriplet diquark condensates
\[ \langle \bar{q}^c \gamma_5 \gamma_1 \gamma_2 q \rangle = \delta, \]
\[ (6) \]
which will break, besides \( \chi_D \) and \( \mathcal{P}_1, \) the symmetry \( Z_2^f \) down to \( Z_2^c \) corresponding to \( r \) and \( q \) color degree of freedom. For the condensates \( \delta, \) we have made appropriate transformations respectively in flavor and color space and rotated it into \( \tau_5 = \tau_0 = 1 \) and \( A = 2 \) direction.

3 Renormalization and Effective Potential

The coupling constants \( G_S \) and \( H_S \) in Eq. (1) are dimensionless, thus the theory is perturbatively renormalizable. We will conduct the renormalization operation in the dimension regularization approach.\[10\] To this end, on the basis of the standard procedure, we change the dimension of space-time from 2 to
\[ D = 2 - 2\varepsilon, \]
\[ (7) \]
and write the renormalized Lagrangian by
\[ \mathcal{L}_D = \bar{q}i\gamma^\mu\partial_\mu q + G_3M^{2-D}Z_G\left(\langle \bar{q}q \rangle^2 + \langle \bar{q}\gamma_5\gamma_0q \rangle^2\right) + H_3M^{2-D}Z_H\left(\langle \bar{q}\gamma_5\gamma_0\gamma_2q \rangle^2 + \langle \bar{q}\gamma_5\gamma_0\gamma_2q \rangle^2\right). \]
\[ (8) \]
where \( M \) is a scale parameter with mass dimension and the \( \gamma^\mu \) are now \( 2^{D/2} \times 2^{D/2} \) matrices. \( Z_G \) and \( Z_H \) are renormalization constants, which will be appropriately selected to cancel the ultraviolet (UV) divergences in the loop integrations. Our main interest lies in interplay between the two condensates \( \phi \) and \( \delta, \) so we will derive the renormalized effective potential of the model containing the order parameters from the two condensates. Define the order parameters
\[ \sigma = -2G_3M^{2-D}Z_G\phi, \quad \Delta = -2H_3M^{2-D}Z_H\delta, \]
\[ (9) \]
then in the mean-field approximation,\[11\] which is equivalent to the linearization of the interaction terms in the existence of the corresponding condensates,\[12\] i.e.
\[ \langle \bar{q}q \rangle^2 \lesssim 2\phi q^2 - \phi^2, \]
\[ \langle \bar{q}\gamma_5\gamma_0\gamma_2q \rangle^2 \lesssim \delta(\bar{q}\gamma_5\gamma_0\gamma_2q)\langle \bar{q}\gamma_5\gamma_0\gamma_2q \rangle^2 + \langle \bar{q}\gamma_5\gamma_0\gamma_2q \rangle^2\delta^2 + |\delta|^2, \]
and in the Nambu–Gorkov basis\[13\] with the bilinear fields
\[ \Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{q} \phi \\ \bar{q}^c \end{pmatrix}, \quad \bar{\Psi} = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{q} \phi^c \end{pmatrix}, \]
\[ (10) \]
we can write
\[ \mathcal{L}_D = \bar{\Psi}(x)S^{-1}(x)\Psi(x) - \frac{\alpha^2}{4G_3M^{2-D}Z_G} - \frac{|\Delta|^2}{4H_3M^{2-D}Z_H}. \]
In the momentum space, the inverse propagator \( S^{-1}(x) \) for the quark fields will be changed into
\[ S^{-1}(p) = \begin{pmatrix} \not{p} - \sigma & -i\gamma_5\gamma_2\Delta \not{p} \not{\sigma} \\ i\gamma_5\gamma_2\Delta & \not{p} - \sigma \end{pmatrix}, \quad \not{p} = \gamma^\mu p_\mu. \]
\[ (11) \]
The effective potential in vacuum corresponding to \( \mathcal{L}_D \) can be expressed by
\[ V(\sigma, |\Delta|) = \frac{\alpha^2}{4G_3M^{2-D}Z_G} + \frac{|\Delta|^2}{4H_3M^{2-D}Z_H} + i \int \frac{d^d q}{(2\pi)^{d/2}} \text{Tr} \ln S^{-1}(p)S_0(p), \]
\[ (12) \]
where
\[ S_0(p) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \]
represents the propagator for massless quark fields (up to a factor \( i \)) and the \( \text{Tr} \) is taken over flavor, color, Nambu-Gorkov and Dirac spin degrees of freedom.

It is seen from the structure of the matrix \( \lambda_2 \) that the

\[
V(\sigma, |\Delta|) = \frac{\sigma^2}{4G_S M^{2-D} Z_G} + \frac{|\Delta|^2}{4H_S M^{2-D} Z_H} + i D^{D/2-1} \int \frac{d^D \bar{p}}{(2\pi)^D} \left( 4\ln \frac{p^2 - \sigma^2 - |\Delta|^2 + i\epsilon}{p^2 + i\epsilon} + 2\ln \frac{p^2 - \sigma^2 + i\epsilon}{p^2 + i\epsilon} \right),
\]

(13)

where in the third term of the right-handed side, the factor 4 comes from (the flavor number) \( \times \) (the color number) of the red and green quarks participating in the diquark condensates, the factor 2 from the flavor number of the blue quarks and the factor \( 2^{D/2} \) is due to the representation’s dimension of the \( \gamma^\mu \) matrix. For the momentum integrations, we may make the Wick rotation, use the Euclidean variable \( \bar{p}^0 = ip^0, \bar{p}^i = p^i \) \( (i = 1, \ldots, D - 1) \) and define the integration

\[
I(a^{2}) = \int \frac{d^D \bar{p}}{(2\pi)^D} \ln \frac{\bar{p}^2 + a^2}{\bar{p}^2}
\]

with \( I(a^{2}) = 0 \),

(14)

then it is easy to calculate

\[
I(a^{2}) = \int_0^{a^2} du^2 \frac{dI(u^{2})}{du^2} = \int_0^{a^2} du^2 \int \frac{d^D \bar{p}}{(2\pi)^D} \frac{1}{\bar{p}^2 + u^2} = \frac{\Gamma(1 - D/2)}{(4\pi)^{D/2}} \frac{(a^{2})^{D/2}}{D/2}.
\]

(15)

By means of Eqs. (14), (15), and (7) we can transform Eq. (13) to

\[
V(\sigma, |\Delta|) = \frac{\sigma^2}{4G_S M^{2-D}} \left\{ Z_G^{-1} - 1 - \frac{6G_S}{\pi} \right\} + \frac{|\Delta|^2}{4H_S M^{2-D}} \left\{ Z_H^{-1} - 1 - \frac{4H_S}{\pi} \right\}
\]

(16)

where \( \gamma = 0.5772 \) is the Euler constant. To eliminate the UV divergent term with the pole \( 1/\epsilon \) (up to one-loop order), we use the minimal subtraction scheme\(^{[10]} \) and define the renormalization constants \( Z_G \) and \( Z_H \) by

\[
Z_G = 1 - \frac{6G_S}{\pi} \frac{1}{\epsilon}, \tag{17}
\]

\[
Z_H = 1 - \frac{4H_S}{\pi} \frac{1}{\epsilon}. \tag{18}
\]

In this way, the renormalized effective potential up to one-loop order (after the limit \( D \to 2 \) is taken) becomes

\[
V(\sigma, |\Delta|) = \frac{\sigma^2}{4G_S} - \frac{\sigma^2}{2\pi} \left( 2\ln \frac{M^2}{\sigma^2 + |\Delta|^2} + \ln \frac{M^2}{\sigma^2} + 3 \right) + \frac{|\Delta|^2}{4H_S} \frac{|\Delta|^2}{\pi} \left( \ln \frac{M^2}{\sigma^2 + |\Delta|^2} + 1 \right), \tag{19}
\]

where the denotation

\[
M^2 = 2\pi e^{-\gamma} M^2
\]

has been used.

### 4 Ground States

The effective potential \( V(\sigma, |\Delta|) \) depends on the two real order parameters \( \sigma \) and \( |\Delta| \). Its minimum points can be found out analytically, then the ground states of the blue quarks do not participate in the diquark condensates and irrelevant to the order parameter \( \Delta \), thus we may conduct similar derivation to the one made in Ref. [8] and obtain

\[
K = \begin{vmatrix} A & B \\ B & C \end{vmatrix}
\]

(23)

For discussion of the feature of the extreme value points of \( V(\sigma, |\Delta|) \) we define the determinant

\[
A = \frac{\partial^2 V(\sigma, |\Delta|)}{\partial \sigma^2} = \frac{1}{2G_S} \left( 1 - \frac{4G_S}{\pi} \ln \frac{M^2}{\sigma^2 + |\Delta|^2} - \frac{2G_S}{\pi} \ln \frac{M^2}{\sigma^2} \right) + \frac{2}{\pi} \left( 1 + \frac{2\pi^2}{\sigma^2 + |\Delta|^2} \right)
\]

\[
B = \frac{\partial^2 V(\sigma, |\Delta|)}{\partial \sigma \partial |\Delta|} = \frac{\partial^2 V(\sigma, |\Delta|)}{\partial |\Delta| \partial \sigma} = \frac{4}{\pi} \frac{\sigma |\Delta|}{\sigma^2 + |\Delta|^2},
\]

\[
\text{model will be determined. From the extreme value conditions } \partial V(\sigma, |\Delta|)/\partial \sigma = 0 \text{ and } \partial V(\sigma, |\Delta|)/\partial |\Delta| = 0 \text{ we get the equations that }
\]

\[
\sigma \left[ 1 - \frac{2G_S}{\pi} \left( 2\ln \frac{M^2}{\sigma^2 + |\Delta|^2} + \ln \frac{M^2}{\sigma^2} + 3 \right) \right] = 0 \tag{21}
\]

and

\[
|\Delta| \left[ 1 - \frac{4H_S}{\pi} \ln \frac{M^2}{\sigma^2 + |\Delta|^2} \right] = 0. \tag{22}
\]

It is seen from Eqs. (21) and (22) that the non-zero solutions of \( \sigma \) and \( |\Delta| \) will depend on the coupling constants \( G_S \) and \( H_S \). Since 2D NG model is an asymptotically free theory and \( G_S \) and \( H_S \) can become running, hence it seems that the non-zero \( \sigma \) and \( |\Delta| \) will depend on the scale parameter \( M \). However, we will prove in Appendix that the non-zero \( \sigma \) and \( |\Delta| \) are practically scale-independent. The same result also arose in the 2D GN model without the diquark interactions\(^{[14]} \) and it simply reflects the characteristic of the mean-field approximation.
\begin{equation}
C = \frac{\partial^2 V(\sigma, |\Delta|)}{\partial |\Delta|^2} = \frac{1}{2H_S} \left( 1 - \frac{4H_S}{\pi} \ln \frac{M^2}{\sigma^2 + |\Delta|^2} \right) + \frac{4}{\pi} \frac{|\Delta|^2}{\sigma^2 + |\Delta|^2}.
\end{equation}

Equations (21) and (22) have the following four different solutions, which will be discussed successively.

i) \((\sigma, |\Delta|) = (0, 0)\). In this case we can calculate and get
\(A = -\infty, \quad B = 0, \quad C = -\infty, \quad \text{and} \quad K = \infty^2 > 0\), hence \((0, 0)\) is a maximum point of \(V(\sigma, |\Delta|)\).

ii) \((\sigma, |\Delta|) = (\sigma_1, 0)\). By means of the equation obeyed by non-zero \(\sigma_1\),
\[
1 - \frac{6G_S}{\pi} \ln \frac{M^2}{\sigma_1^2} = 0,
\]
it is easy to obtain that
\[
A = \frac{6}{\pi}, \quad K = \left( \frac{1}{2H_S} - \frac{1}{3G_S} \right) A.
\]
Obviously, \((\sigma_1, 0)\) is a minimum point of \(V(\sigma, |\Delta|)\) if \(G_S/H_S > 2/3\) and it will be neither a maximum nor a minimum point of \(V(\sigma, |\Delta|)\) if \(G_S/H_S \leq 2/3\).

iii) \((\sigma, |\Delta|) = (0, \Delta_1)\). Now by using the equation satisfied by non-zero \(\Delta_1\),
\[
1 - \frac{4H_S}{\pi} \ln \frac{M^2}{\Delta_1^2} = 0,
\]
we get
\[
A = \frac{1}{2G_S} - \frac{3}{4H_S} + \frac{2}{\pi} \frac{1}{\ln |\Delta_1^2|} \bigg|_{|\sigma| \to 0}, \quad K = \frac{4}{\pi} A.
\]
If \(G_S \neq 0\) we will have \(A = -\infty, K = -\infty\) thus \((0, \Delta_1)\) is not an extreme value point of \(V(\sigma, |\Delta|)\). Only if \(G_S = 0\), it is just possible that \(A > 0\) and \(K > 0\) thus \((0, \Delta_1)\) becomes a minimum point of \(V(\sigma, |\Delta|)\). It is indicated that when \(G_S = 0\) we will have \(\sigma \equiv 0\) by Eq. (9) and \(V(\sigma, |\Delta|)\) will reduce to \(V(0, |\Delta|)\) with the single order parameter \(|\Delta|\) coming from the pure diquark interactions.

iv) \((\sigma, |\Delta|) = (\sigma_2, \Delta_2)\), where the non-zero \(\sigma_2\) and \(\Delta_2\) are solutions of the equations
\[
1 - \frac{2G_S}{\pi} \left( 2 \ln \frac{M^2}{\sigma_2^2 + \Delta_2^2} + \ln \frac{M^2}{\sigma_2^2} \right) = 0,
\]
\[
1 - \frac{4H_S}{\pi} \ln \frac{M^2}{\sigma_2^2 + \Delta_2^2} = 0.
\]
By means of the two equations we may obtain
\[
A = \frac{2}{\pi} \left( 1 + \frac{2\sigma_2^2}{\sigma_2^2 + \Delta_2^2} \right) > 0,
\]
\[
K = \frac{8}{\pi^2} \frac{\Delta_2^2}{\sigma_2^2 + \Delta_2^2} > 0.
\]
Hence \((\sigma_2, \Delta_2)\) is certainly a minimum point of \(V(\sigma, |\Delta|)\) as long as the solution exists. For seeking the condition in which the solution \((\sigma_2, \Delta_2)\) emerges, we find out from Eqs. (25) and (26) that
\[
\sigma_2^2 = M^2 \exp \left[ \frac{\pi}{2} \left( \frac{1}{H_S} - \frac{1}{G_S} \right) \right],
\]
\[
\Delta_2^2 = M^2 \exp \left[ -\frac{\pi}{4H_S} \left( 1 - \exp \left[ \frac{3\pi}{4} \left( \frac{1}{H_S} - \frac{2}{3G_S} \right) \right] \right) \right].
\]
Obviously, the solution \((\sigma_2, \Delta_2)\) exists only if \(G_S/H_S < 2/3\).

The total results above can be summarized as follows.

In the 2D GN model given by the Lagrangian (1), the locations of the minimum points of the effective potential \(V(\sigma, |\Delta|)\) depend on the ratios of \(G_S\) and \(H_S\) and will be

\[
(\sigma, |\Delta|) = \begin{cases} (0, \Delta_1), & \text{if } G_S/H_S = 0, \\ (\sigma_2, \Delta_2), & \text{if } 0 < G_S/H_S < 2/3, \\ (\sigma_1, 0), & \text{if } G_S/H_S > 2/3. \\
\end{cases}
\]

These results are very similar to the ones obtained in the 4D two-flavor NJL model with both quark-antiquark and diquark interactions,\[8\] except for the only difference that in present model the pure diquark condensate solution \((0, \Delta_1)\) arises only if \(G_S = 0\) however in the 4D model it emerges in a limited region of \(G_S/H_S\): \(0 < G_S/H_S < f(H_S) < 2/3\). The critical value 2/3 of \(G_S/H_S\) below which the diquark condensates emerge is also due to the same fact that only the two-colors (red and green) of quarks participate in the diquark condensates but all the three colors of quarks get into the quark-antiquark condensates.

5 Conclusions
We have proven by effective potential approach that, similar to the case of the 4D two-flavor NJL model, in the 2D two-flavor GN model with diquark interactions, the interplay between the quark-antiquark and the diquark condensates depends on the ratio \(G_S/H_S\) of the coupling constants in scalar quark-antiquark and scalar diquark channels. In the ground state of the model, only pure quark-antiquark condensates could emerge if \(G_S/H_S > 2/3\) and the two condensates could coexist if \(0 < G_S/H_S < 2/3\). However, the pure diquark condensates could be formed only if \(G_S/H_S = 0\), which is different from the 4D model, where the pure diquark condensates could arise in a finite region of \(G_S/H_S\). In any way, the critical value 2/3 of \(G_S/H_S\), below which the diquark condensates could emerge, is the same for the 2D GN model and the 4D NJL model and it represents the ratio of the color numbers of the quarks participating in the diquark and the quark-antiquark condensates. This is probably a common characteristic of this kind of two-flavor four-fermion models. It is also indicate that if the model is considered as some simulation of 2D QCD and the four-fermion interactions are assumed to come from the heavy color gluon exchange interactions \(\sim g(\bar{q}_1^\mu \lambda^a q_2^\nu)^2 (\mu = 0, 1; a = 1, \ldots, 8)\) via the Fierz
transformation,\cite{[11]} then one will obtain $G_S/H_S = 4/3$ for two-flavor and three-color case. Hence based on the above results, similar to the 4D NJL model case, we will have only the pure quark-antiquark condensates surviving and no diquark condensates could appear in the ground state of the 2D two-flavor GN model in vacuum.

**Appendix**

We will prove that in the mean-field approximation, in the considered 2D GN model the non-zero order parameters $\sigma_2$ and $\Delta_2$ satisfying extreme value conditions of the effective potential $V(\sigma, |\Delta|)$ are scale-independent. First rewrite Eqs. (27) and (28) coming from the extreme value Eqs. (25) and (26) by

$$
\sigma_2^2 = M^2 \exp \left[ \frac{\pi}{4} \left( \frac{1}{H_S} - \frac{1}{G_S} \right) \right],
$$

$$
\Delta_2^2 = M^2 \exp \left[ - \frac{\pi}{4H_S} \right] - \sigma_2^2, \quad (A1)
$$

then we will replace the coupling constants $G_S$ and $H_S$ by respective running form. For this end, define the bare couplings

$$
G_S^0 = G_SM^{2-D}Z_G, \quad H_S^0 = H_SM^{2-D}Z_H. \quad (A2)
$$

When viewed as independent variables, $G_S^0$ and $H_S^0$ do not depend on the scale parameter $M$, i.e. they obey the equations

$$
\frac{dG_S^0}{dM} = \frac{dH_S^0}{dM} = 0. \quad (A3)
$$

By means of Eqs. (17) and (18), which define $Z_G$ and $Z_H$, and the standard renormalization group method,\cite{[10]} we may derive the equations

$$
M \frac{dG_S}{dM} = M \frac{dG_S}{dM} = - \frac{12}{\pi} G_S^2, \\
M \frac{dH_S}{dM} = M \frac{dH_S}{dM} = - \frac{8}{\pi} H_S^2, \quad (A4)
$$

noting that $M$ differs from $M$ only by a constant factor. Equation (A4) will give the running coupling constants,

$$
\frac{1}{G_S(M)} = \frac{1}{G_S(M_0)} = \frac{6}{\pi} \ln \frac{M^2}{M_0^2} + \frac{1}{G_S(M_0)}, \\
\frac{1}{H_S(M)} = \frac{1}{H_S(M_0)} = \frac{4}{\pi} \ln \frac{M^2}{M_0^2} + \frac{1}{H_S(M_0)}, \quad (A5)
$$

where $M_0$ is an arbitrarily fixed momentum scale. Now in Eq. (A1) we make the replacements

$$
G_S \rightarrow G_S(M), \quad H_S \rightarrow H_S(M),
$$

which will lead to the corresponding substitutions

$$
\sigma_2 \rightarrow \sigma_2(M), \quad \Delta_2 \rightarrow \Delta_2(M),
$$

Then by using Eq. (A5), it is easy to verify that

$$
\sigma_2(M^2) = \sigma_2(M_0^2), \quad \Delta_2(M^2) = \Delta_2(M_0^2), \quad (A6)
$$

i.e. $\sigma_2$ and $\Delta_2$ are scale-independent indeed.

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