GAP-LABELLING FOR QUASI-CRYSTALS
(PROVING A CONJECTURE BY J. BELLISSARD)

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Abstract. In the present paper, we summarize a proof of the Bellissard gap-labelling conjecture for quasi-crystals. Our main tools are the measured index theorem for laminations together with the naturality of the longitudinal Chern character.

1. Introduction

Let us consider a Schrödinger operator on $\mathbb{R}^p$

$$H = -\frac{\hbar^2}{2m} \Delta + V,$$

where $\Delta$ is the Laplacian on $\mathbb{R}^p$ and $V$ is a potential. The set of observables affiliated to this Schrödinger operator is a $C^*$-algebra. It must contain at least the $C^*$-algebra generated by operators $(H - z \text{Id})^{-1}$ where $z$ belongs to the resolvent of $H$. If $H$ describes a particle in a homogeneous media, the physical properties of this media do not depend upon the choice of an origin in $\mathbb{R}^p$. In particular, the algebra of observables must also contain the $C^*$-algebra $C^*(H)$ generated by $T_a(H - z \text{Id})^{-1}T_{-a}$ where $T_a$ is the operator of translation by $a \in \mathbb{R}^p$.

In [2], J. Bellissard has attached to this Schrödinger operator $H$ a compact space $\Omega_H$ equipped with a minimal action of $\mathbb{R}^p$ such that the crossed product algebra $C(\Omega_H) \rtimes \mathbb{R}^p$ contains $C^*(H)$. The space $\Omega_H$ is called the hull of $H$ and is by definition the strong closure in $B(L^2(\mathbb{R}^p))$ of the family

$$\{T_a(H - z \text{Id})^{-1}T_{-a}; a \in \mathbb{R}^p\}$$

where we have fixed an element $z$ in the resolvent of $H$. The action of $\mathbb{R}^p$ on $\Omega_H$ is induced by translations and up to a $\mathbb{R}^p$-equivariant homeomorphism, the space $\Omega_H$ is independent on the choice of $z$ in the resolvent of $H$.

A typical potential for motion of conduction electrons is given by:

$$V(x) = \sum_{y \in L} v(x - y),$$

where $L$ is the point set of equilibrium positions of atoms and $v$ is the effective potential for a valence electron near an atom (see [3]). This set $L$ is uniformly discrete, i.e. there exists a positive number $r$ such that any ball of radius $r$ contains at most one point. In [3], J. Bellissard, D. J. L. Hermann and M. Zarrouati had attached to this point set $L$ a compact space $\Omega_L$, called the hull of $L$ equipped with a minimal action of $\mathbb{R}^p$. The space $\Omega_L$ is called the hull of $L$ and is constructed as follows: Let $\nu_L$ be the measure on $\mathbb{R}^p$ defined for every compactly supported
continuous function $f$ by $\nu_L(f) = \sum_{y \in L} f(y)$. Then $\Omega_L$ is by definition the weak-$*$ closure (with respect to the compactly supported continuous functions on $\mathbb{R}^p$) of the family of translations of $\nu_L$ by the elements of $\mathbb{R}^p$. Note that $V = \nu_L * v$ and more generally, for $\nu \in \Omega_L$ we set:

$$H_\nu = -\frac{\hbar^2}{2m} \Delta + \nu * v.$$

For all $z$ in the resolvent of the operator $H$, the map

$$\nu \in \Omega_L \mapsto (H_\nu - z \text{Id})^{-1} \in \Omega_H$$

is continuous, equivariant and surjective (see [3]). Thus, the crossed product algebra $C(\Omega_H) \rtimes \mathbb{R}^p$ lies in the crossed product algebra $C(\Omega_L) \rtimes \mathbb{R}^p$ and in particular $C(\Omega_L) \rtimes \mathbb{R}^p$ contains the algebra $C^*(H)$. The main advantage of dealing with $\Omega_L$ rather than $\Omega_H$ is that $\Omega_L$ only depends on the geometry of $L$: For instance, if $L$ is given by a rank $p$ lattice $\mathcal{R}$ in $\mathbb{R}^p$, then the hull of $L$ is $\mathbb{R}^p / \mathcal{R}$. Actually in this case $C^*(H)$ can be computed by using Bloch theory [4]: we can check that $C^*(H) = C(B) \otimes \mathcal{K}$, where $\mathcal{K}$ is the algebra of compact operator on a separable Hilbert space and $B$ is the Brillouin zone, defined by $B = \mathbb{R}^p / \mathcal{R}^*$, where $\mathcal{R}^*$ is the reciprocal lattice of $\mathcal{R}$.

To the Schrodinger operator $H$ is associated the integrated density of states (see [2]) $E \mapsto \mathcal{N}(E)$, where $\mathcal{N}(E)$ is defined as the number of states by unit of volume with eigenvalues less or equal to $E$. The remarkable result of J. Bellissard, D. J. L. Herrmann and M. Zardouati in [3] is that the integrated density of states on gaps must take value in a countable subgroup of $\mathbb{R}$ that only depends on the point set $L$. This computation goes as follows. If $\mathbb{P}$ is a $\mathbb{R}^p-$invariant probability measure on $\Omega_L$, then $\mathbb{P}$ induces a trace $\tau^p$ on $C(\Omega_L) \rtimes \mathbb{R}^p$ and this trace extends to the von-Neumann algebra of $C(\Omega_L) \rtimes \mathbb{R}^p$. Let us denote by $\chi_{[-\infty, E]}$ the characteristic function of the set $[-\infty, E]$. Then $\chi_{[-\infty, E]}(H)$ belongs to the von-Neumann algebra of $C(\Omega_L) \rtimes \mathbb{R}^p$. For an ergoic probability $\mathbb{P}$, J. Bellissard stated in [2] the so-called Shubin’s formula for $H$:

$$\mathcal{N}(E) = \tau^p(\chi_{[-\infty, E]}(H)).$$

In particular, if $E$ is in a spectral gap of $H$, then $\chi_{[-\infty, E]}$ is a continuous function on the spectrum of $H$ and thus, $\chi_{[-\infty, E]}(H)$ is an idempotent in $C(\Omega_L) \rtimes \mathbb{R}^p$ (recall that $H$ is bounded below). In consequence, according to the Shubin formula, the value of $\mathcal{N}$ on spectral gaps of $H$ belongs to the image of the additive map

$$\tau^p : K_0(C(\Omega_L) \rtimes \mathbb{R}^p) \rightarrow \mathbb{R},$$

where $\tau^p$ is the morphism induced by the trace $\tau^p$ in $K$-theory.

We now focus on the case where the point set $L$ is a quasicrystal which is obtain by the cut-and-project method (see [1] and also [5]). To such a point set, J. Bellissard, E. Contensous and A. Legrand have associated in [5] (see also [2]) a canonical minimal dynamical system $(\mathcal{T}_L, \mathbb{Z}^p)$ which is Morita equivalent to $(\Omega_L, \mathbb{R}^p)$ and such that $\mathcal{T}_L$ is a Cantor set. Moreover, there is a canonical ergoic invariant probability measure $\mu$ on $\mathcal{T}_L$ such that

$$\tau^p(K_0(C(\Omega_L) \rtimes \mathbb{R}^p)) = \tau^\mu(K_0(C(\mathbb{T}_L) \rtimes \mathbb{Z}^p)),$$

where $\tau^\mu$ is the trace on $C(\mathbb{T}_L) \rtimes \mathbb{Z}^p$ induced by $\mu$. Eventually, the image

$$\tau^p(K_0(C(\Omega_L) \rtimes \mathbb{R}^p))$$

is predicted by the following conjecture of J. Bellissard [5]:

$$\tau^p(K_0(C(\Omega_L) \rtimes \mathbb{R}^p))$$
Let \( \Omega \) be a Cantor set equipped with an action of \( \mathbb{Z}^p \) and with a \( \mathbb{Z}^p \)-invariant measure \( \mu \). The measure \( \mu \) induces a trace \( \tau^\mu \) on the crossed product \( C^*(\Omega) \rtimes \mathbb{Z}^p \). Let us denote by \( \mathbb{Z}[\mu] \) the additive subgroup of \( \mathbb{R} \) generated by \( \mu \)-measures of compact-open subsets of \( \Omega \). We make the assumption that \( \Omega \) has no non-trivial compact-open invariant subsets (this is clearly the case if the action of \( \mathbb{Z}^p \) is minimal).

**Conjecture 1. (The Bellissard gap-labelling conjecture)**

\[
\tau^\mu(K_0(C(\Omega) \rtimes \mathbb{Z}^p)) = \mathbb{Z}[\mu].
\]

The goal of the present paper is to give a proof of the Bellissard conjecture. The method adopted uses the measured index theorem for laminations as proved in [13].

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### 2. The mapping torus

Let \( \Omega \) be a Cantor set. Assume that the group \( \mathbb{Z}^p \) acts on \( \Omega \) by homeomorphisms and that there exists a \( \mathbb{Z}^p \)-invariant measure \( \mu \) on \( \Omega \). We assume that \( \Omega \) has no compact-open \( \mathbb{Z}^p \)-invariant subset except \( \emptyset \) and \( \Omega \). This is the case in particular if the action of \( \mathbb{Z}^p \) on \( \Omega \) is minimal.

The action of \( \mathbb{Z}^p \) on \( \Omega \) induces an action of \( \mathbb{Z}^p \) on the \( C^* \)-algebra \( C(\Omega) \) and thus, we can form the crossed product \( C^* \)-algebra \( C(\Omega) \rtimes \mathbb{Z}^p \). The measure \( \mu \) induces a trace \( \tau^\mu \) on the \( C^* \)-algebra \( C(\Omega) \rtimes \mathbb{Z}^p \) and we obtain in this way a group morphism \( \tau^\mu : K_0(C(\Omega) \rtimes \mathbb{Z}^p) \to \mathbb{R} \). In what follows, we shall denote by \( \mathbb{Z}[\mu] \) the additive subgroup of \( \mathbb{R} \) generated by the \( \mu \)-measures of compact-open subsets of \( \Omega \) and by \( C(\Omega, \mathbb{Z}) \) the coinvariants of the action of \( \mathbb{Z}^p \) on \( C(\Omega, \mathbb{Z}) \), i.e. the quotient of \( C(\Omega, \mathbb{Z}) \) by the subgroup generated by elements of the form \( n(f) - f \), where \( f \in C(\Omega, \mathbb{Z}) \) and \( n \in \mathbb{Z}^p \).

We are interested in computing the image of \( K_0(C(\Omega) \rtimes \mathbb{Z}^p) \) under \( \tau^\mu \).

In the case \( p = 1 \), E. Contensous proved in [2], using the Pimsner-Voiculescu six term exact sequence, that the inclusion \( C(\Omega) \hookrightarrow C(\Omega) \rtimes \mathbb{Z} \) induces an isomorphism \( C(\Omega, \mathbb{Z}) \iso K_0(C(\Omega) \rtimes \mathbb{Z}) \) and thus,

\[
\tau^\mu(K_0(C(\Omega) \rtimes \mathbb{Z})) = \mathbb{Z}[\mu].
\]

In the case \( p = 2 \), the computation of \( K_0(C(\Omega) \rtimes \mathbb{Z}^2) \) was carried out by J. Bellissard, E. Contensous and A. Legrand in [3], using the Kasparov spectral sequence. The result is:

\[
K_0(C(\Omega) \rtimes \mathbb{Z}^2) \iso C(\Omega, \mathbb{Z}) \oplus \mathbb{Z}.
\]

More precisely:

- The inclusion \( \mathbb{Z} \hookrightarrow K_0(C(\Omega) \rtimes \mathbb{Z}^2) \) maps the canonical generator of \( \mathbb{Z} \) to the image, under the morphism induced by the inclusion \( C^*_r(\mathbb{Z}^2) \to C(\Omega) \rtimes \mathbb{Z}^2 \), of the Bott generator in the \( K \)-theory of the \( C^* \)-algebra \( C^*_r(\mathbb{Z}^2) \iso C(\mathbb{T}^2) \) of the discrete group \( \mathbb{Z}^2 \).
- The inclusion \( C(\Omega, \mathbb{Z}) \iso K_0(C(\Omega) \rtimes \mathbb{Z}^2) \) is induced by the inclusion \( C(\Omega) \hookrightarrow C(\Omega) \rtimes \mathbb{Z}^2 \).
Recall that the Bott generator of $K_0(C(T^2))$ is the unique element with Chern character equal to the volume form of $T^2$. In particular, it is traceless and therefore, the above computation gives:

$$\tau^p_0(K_0(C(\Omega \times \mathbb{Z}^2))) = \mathbb{Z}[\mu] .$$

In the case $p = 3$, the computation of the image of $K_0(C(\Omega \times \mathbb{Z}^3))$ under the trace has been recently performed by J. Bellissard, J. Kellendonk and A. Légrand \[4\] and gave the same result. These computations lead J. Bellissard to state conjecture \[3\]. We can without loss of generality assume that $p$ is even (see \[3\]). From now on, $p$ will denote an even integer.

The group $K_0(C(\Omega \times \mathbb{Z}^p))$ can be computed in term of Kasparov cycles out of the mapping torus isomorphism. We recall in the end of this section the construction of this isomorphism.

The mapping torus is by definition the space

$$V_\Omega = \frac{\Omega \times \mathbb{R}^p}{\mathbb{Z}^p} .$$

This space is a foliated space with leaves given by the projections of $\{\omega\} \times \mathbb{R}^p$ in $\Omega \times \mathbb{R}^p \to V_\Omega$, where $\omega$ is an element of $\Omega$.

Let $\partial : C^\infty(\mathbb{R}^p) \otimes S^+ \to C^\infty(\mathbb{R}^p) \otimes S^-$ be the Dirac operator on $\mathbb{R}^p$ where $S^+$ and $S^-$ are the two irreducible spin representations of $\text{Spin}(p)$. The Dirac operator $\partial$ is $\mathbb{Z}^p$-equivariant. Let $e$ be a projection of $M_n(C(V_\Omega))$ and let $\tilde{e}$ be the $\mathbb{Z}^p$-invariant projection of $C(\Omega \times \mathbb{R}^p) \otimes M_n(C)$ corresponding to $e$ under the projection $\Omega \times \mathbb{R}^p \to V_\Omega$. If we assume $\tilde{e}$ smooth in the $\mathbb{R}^p$-direction, then the operator

$$\tilde{e} \circ (I_{C(\Omega)} \otimes \partial \otimes I_n) : \tilde{e}(C(\Omega) \otimes C^\infty(\mathbb{R}^p) \otimes S^+ \otimes \mathbb{C}^n) \to \tilde{e}(C(\Omega) \otimes C^\infty(\mathbb{R}^p) \otimes S^- \otimes \mathbb{C}^n)$$

is a $\mathbb{Z}^p$-equivariant elliptic differential operator in the $\mathbb{R}^p$-direction. Hence it induces a longitudinal elliptic differential operator $\partial^p_{\Omega,\mathbb{R}^p}$ on $V_\Omega$. According to \[13\], the operator $\partial^p_{\Omega,\mathbb{R}^p}$ admits a $K$-theory index $\text{Ind}_{V_\Omega} \partial^p_{\Omega,\mathbb{R}^p}$ which belongs to the $K$-theory group $K_0(C(\Omega) \times \mathbb{Z}^p)$.

We can easily check that the map $e \mapsto \text{Ind}_{V_\Omega} \partial^p_{\Omega,\mathbb{R}^p}$ induces a well defined morphism from $K_0(C_0(V_\Omega))$ to $K_0(C(\Omega) \times \mathbb{Z}^p)$. The following theorem is due to A. Connes (see \[3\]).

**Theorem 1.** The map $e \mapsto \text{Ind}_{V_\Omega} \partial^p_{\Omega,\mathbb{R}^p}$ induces an isomorphism:

$$\mu^p_{\Omega} : K_0(C_0(V_\Omega)) \xrightarrow{\cong} K_0(C(\Omega) \times \mathbb{Z}^p) .$$

In consequence of this theorem, it is enough for proving the Bellissard conjecture to check that $\tau^p_0(\text{Ind}_{V_\Omega} \partial^p_{\Omega,\mathbb{R}^p})$ belongs to $\mathbb{Z}[\mu]$. 

Assume first that $\Omega = \{pt\}$ is just a point. The mapping torus is then $V_\Omega = T_p$, the usual $p$-torus. If $e$ is a smooth projector in $M_n(T_p)$, then $\partial^p_{\{pt\},\mathbb{R}^p}$ is just the Dirac operator of the $p$-torus $T_p$ with coefficients in the vector bundle over $T_p$ associated with $e$. The Atiyah index formula for coverings (see \[13\]) asserts then that

$$\tau_e(\text{Ind}_{\{pt\},\mathbb{R}^p}) = \langle \text{ch}(e), [T_p] \rangle$$

where $\tau$ is the canonical trace on $C^*(\mathbb{Z}^p)$, $[T_p] \in H^p(T_p, \mathbb{R})$ is the fundamental class of $T_p$ and $\text{ch}(e) \in H^*(T_p, \mathbb{R})$ is the Chern character of $[e] \in K_0(C_0(V_\Omega))$.

The Connes measured longitudinal index theorem (see \[8\]), or rather its extension to foliated spaces (see \[13\]) provides such a formula for $\partial^p_{\Omega,\mathbb{R}^p}$. To state this formula for $V_\Omega$, we need to define suitable characteristic classes and a cycle to integrate these.
classes. This will be done in the two following sections. The proof of the Bellissard
conjecture will then be given in Section 6.

3. LONGITUDINAL CHARACTERISTIC CLASSES

The support for the characteristic classes involved in the measure index theorem
for foliation is the longitudinal cohomology of $V_\Omega$. We give in this section, first
the definition of this cohomology and then, the construction of the longitudinal Chern
character on $K_0(C(V_\Omega))$ valued in this longitudinal cohomology.

3.1. The longitudinal de Rham complex. Let $\Omega^k(\mathbb{R}^p)$ be the space of $k$-
differential forms on the vector space $\mathbb{R}^p$, endowed with its usual Frechet topology.

**Definition 1.** A (real) longitudinal differential $k$-form on $V_\Omega$ is a $\mathbb{Z}^p$-equivariant
continuous map $\phi : \Omega \to \Omega^k(\mathbb{R}^p)$.

We denote by $\Omega^k_c(U,\mathbb{R})$ the space of longitudinal differential $k-$forms on $V_\Omega$.
If $\phi$ is a longitudinal $k-$form, its longitudinal differential $d_\ell(\phi)$ is by definition
the longitudinal $(k+1)$-differential form which is given by the $\mathbb{Z}^p$-equivariant map
$\omega \mapsto d(\phi(\omega))$ where $d$ is the de Rham differential on $\mathbb{R}^p$. Hence:

$$d_\ell : \Omega^k_c(U,\mathbb{R}) \to \Omega^{k+1}_c(U,\mathbb{R})$$

provides a differential structure on the graded vector space $\Omega^*_c(U,\mathbb{R}) = \bigoplus \Omega^k_c(U,\mathbb{R})$
and satisfies $d_\ell \circ d_\ell = 0$. In what follows the superscripts $c$ and $o$ mean respectively
even and odd forms or classes of forms. The cohomology of the complex
$(\Omega^*_c(U,\mathbb{R}), d_\ell)$ will be denoted by

$$H^*_c(U,\mathbb{R}) = \bigoplus_{k \geq 0} H^k_c(U,\mathbb{R}) = H^*_c(U,\mathbb{R}) \bigoplus H^*_o(U,\mathbb{R}).$$

**Remark 1.** We can check (see [13]) that $H^*_c(U,\mathbb{R})$ is the cohomology of the sheaf of
continuous functions which are locally constant in the leaf direction or equivalently
the sheaf of continuous and equivariant functions on equivariant open subsets of
$\Omega \times \mathbb{R}^p$, constant in the $\mathbb{R}^p-$direction. If $H^*(V_\Omega,\mathbb{R})$ denotes the Cech cohomology
groups of $V_\Omega$ with real coefficients, then we have a well defined morphism:

$$H^*(V_\Omega,\mathbb{R}) \to H^*_c(U,\mathbb{R})$$

induced by the natural morphism of sheaves.

We define the support of a longitudinal form $\phi$ as the image of the support of
$(\omega, t) \mapsto \phi(\omega, t)$ under the projection $\Omega \times \mathbb{R}^p \to V_\Omega$. The support of $\phi$ is thus a
compact subset of $V_\Omega$. Let now $U$ be an open subset of $V_\Omega$ and let $\Omega^*_c(U,\mathbb{R}) = \bigoplus \Omega^k_c(U,\mathbb{R})$ be the space of longitudinal differential $k-$forms with support in $U$.

The restriction of the longitudinal differential $d_\ell$ to $\Omega^*_c(U,\mathbb{R})$ preserves it and
$(\Omega^*_c(U,\mathbb{R}), d_\ell)$ is a subcomplex of the longitudinal complex $(\Omega^*_c(U,\mathbb{R}), d_\ell)$. We
denote by

$$H^*_c(U,\mathbb{R}) = \bigoplus H^k_c(U,\mathbb{R}) = H^*_{c-o}(U,\mathbb{R}) \bigoplus H^*_o(U,\mathbb{R})$$

the cohomology of this subcomplex. Similarly, we can also define the relative
longitudinal cohomology for relative open pairs. A relative open pair $(U_0,U_1)$ is by
definition given by two open subsets $U_0$ and $U_1$ of $V_\Omega$ such that $U_0 \subset U_1$. From
such a pair $(U_0,U_1)$, we can define the differential complex $(\Omega^*_c(U_0,U_1,\mathbb{R}), d_\ell)$ as
the quotient differential complex in the exact sequence:
0 → \Omega^{*,k}_{\ell,c}(U_0, \mathbb{R}) \to \Omega^{*}_{\ell,c}(U_1, \mathbb{R}) \to \Omega^{*}_{\ell,c}(U_0, U_1, \mathbb{R}) → 0.

The cohomology of the resulting complex \((\Omega^{*}_{\ell,c}(U_0, U_1, \mathbb{R}), d_\ell)\) is called the relative longitudinal cohomology of the relative open pair \((U_0, U_1)\) and is denoted by:

\[ H^*_{\ell,c}(U_0, U_1, \mathbb{R}) = \bigoplus H^k_{\ell,c}(U_0, U_1, \mathbb{R}) = H^k_{\ell,c}(U_0, U_1, \mathbb{R}) \bigoplus H^k_{\ell,c}(U_0, U_1, \mathbb{R}). \]

The following long exact sequence in longitudinal cohomology holds:

\[ \cdots \to H^k_{\ell,c}(U_0, \mathbb{R}) \to H^k_{\ell,c}(U_1, \mathbb{R}) \to H^k_{\ell,c}(U_0, U_1, \mathbb{R}) \to H^{k+1}_{\ell,c}(U_0, \mathbb{R}) \to \cdots \]

### 3.2. The longitudinal Chern character.

We give in this subsection the definition and main properties of the longitudinal Chern character:

\[ ch_\ell : K^*(V_\Omega) \to H^1_{\ell}(V_\Omega, \mathbb{R}), \quad *=0,1, \]

where \(H^*_\ell = \oplus_{j \in \mathbb{Z}} H^{*+2j}_{\ell}(V_\Omega, \mathbb{R})\).

**Definition 2.** A continuous function \(f\) on \(V_\Omega\) is longitudinally smooth if, viewed as a \(\mathbb{Z}p\)-invariant function on \(\mathbb{R}^p \times \Omega\), it is smooth in the \(\mathbb{R}^p\)-direction (i.e. an element of \(\Omega^*_p(V_\Omega, \mathbb{R}) \otimes \mathbb{C}\)). We denote by \(C^{\infty, 0}(V_\Omega)\) the algebra of continuous longitudinally smooth functions on \(V_\Omega\).

The algebra \(C^{\infty, 0}(V_\Omega)\) is a dense subalgebra of the algebra \(C(V_\Omega)\) of continuous functions on \(V_\Omega\), which is stable under holomorphic functional calculus. Hence the inclusion \(i : C^{\infty, 0}(V_\Omega) \hookrightarrow C(V_\Omega)\) induces an isomorphism:

\[ i_* : K_0(C^{\infty, 0}(V_\Omega)) \cong K_0(C(V_\Omega)). \]

Let \(e\) be a projector in \(M_n(C^{\infty, 0}(V_\Omega))\). Let us denote by \(\tilde{e}\) the smooth in the \(\mathbb{R}^p\)-direction \(\mathbb{Z}p\)-invariant \(M_n(C)\)-valued map on \(\mathbb{R}^p \times \Omega\), defined by the projection \(e\). Then \(\text{Tr} \tilde{e} \exp \left( \frac{d\tilde{e}d\tilde{e}^\ell}{2i\pi} \right)\) is an element of \(\Omega^*_p(V_\Omega, \mathbb{R})\). We have the following proposition:

**Proposition 2.** If \(e\) is a projector in \(M_n(C^{\infty, 0}(V_\Omega))\), then

1. \(\text{Tr} \tilde{e} \exp \left( \frac{d\tilde{e}d\tilde{e}^\ell}{2i\pi} \right)\) is a \(d_\ell\)-closed differential form;
2. If \([e]\) and \([e']\) define the same class in \(K_0(C^{\infty, 0}(V_\Omega))\), then

\[
\text{Tr} \tilde{e} \exp \left( \frac{d\tilde{e}d\tilde{e}^\ell}{2i\pi} \right) - \text{Tr} \tilde{e'} \exp \left( \frac{d\tilde{e'}d\tilde{e'}^\ell}{2i\pi} \right)
\]

is a coboundary of \(\Omega^*_p(V_\Omega, \mathbb{R})\).

As a consequence of this proposition, the class of \(\text{Tr} \tilde{e} \exp \left( \frac{d\tilde{e}d\tilde{e}^\ell}{2i\pi} \right)\) in \(H^*_\ell(V_\Omega, \mathbb{R})\) only depends on the class of \(e\) in \(K_0(C^{\infty, 0}(V_\Omega))\).

**Corollary 1.** There is a morphism \(ch_\ell : K_0(C(V_\Omega)) \to H^*_\ell(V_\Omega, \mathbb{R})\) such that for every smooth projector \(e\) of \(M_n(C^{\infty, 0}(V_\Omega))\), \(ch_\ell([e]) = \text{Tr} \tilde{e} \exp \left( \frac{d\tilde{e}d\tilde{e}^\ell}{2i\pi} \right)\).
The morphism $\text{ch}_l$ is called the longitudinal Chern character. The longitudinal Chern character can also be defined for the odd $K-$theory of $C(V_\Omega)$. Hence we obtain in this way a morphism

$$\text{ch}_l : K_\ast(C(V_\Omega)) = K_0(C(V_\Omega)) \oplus K_1(C(V_\Omega)) \to H^\ast_l(V_\Omega, \mathbb{R}) = H^\ast_l(V_\Omega, \mathbb{R}) \oplus H^\ast_l(V_\Omega, \mathbb{R}).$$

If $U$ is an open subset of $V_\Omega$, we can in the same way define a longitudinal Chern character $\text{ch}_l : K_\ast(C_0(U)) \to H^\ast_l(U, \mathbb{R})$. Moreover, the longitudinal Chern character admits a relative version: for every relative open pair $(U_0, U_1)$ of $V_\Omega$, there is a map $\text{ch}_l : K_\ast(C_0(U_0 \setminus U_1)) \to H^\ast_l(U_0, U_1, \mathbb{R})$. The crucial point for the proof of the Bellissard conjecture is that this Chern character is compatible with the long exact sequences associated with a relative open pair $(U_0, U_1)$ of $V_\Omega$.

**Theorem 2.** The longitudinal Chern character intertwines the two following exact sequences:

$$
\begin{array}{ccc}
K_0(C^\infty_0(U_0)) & \to & K_0(C^\infty_0(U_1)) \\
\downarrow & & \downarrow \\
K_1(C^\infty_0(U_0, U_1)) & \leftarrow & K_1(C^\infty_0(U_1))
\end{array}
$$

and

$$
\begin{array}{ccc}
H^\ast_l(U_0, \mathbb{R}) & \to & H^\ast_l(U_1, \mathbb{R}) \\
\downarrow & & \downarrow \\
H^\ast_l(U_0, U_1, \mathbb{R}) & \leftarrow & H^\ast_l(U_1, \mathbb{R})
\end{array}
$$

4. **The Fundamental Cycle**

We give in this section the definition of the Ruelle-Sullivan cycle associated with the $\mathbb{Z}^p$-invariant measure $\mu$ on the foliated bundle $V_\Omega$. This cycle allows to integrate the longitudinal top dimensional classes. Let $\chi$ be the characteristic function of the open set $U = [0, 1]^p$ in $\mathbb{R}^p$, we define $C_{\mu, \mathbb{Z}^p} : \Omega^p_l(V_\Omega, \mathbb{R}) \to \mathbb{R}$, by:

$$C_{\mu, \mathbb{Z}^p}(\phi) := \left\langle \mu \otimes \int_{\mathbb{R}^p} \chi \phi \right\rangle.$$

The next proposition shows that actually, the map $C_{\mu, \mathbb{Z}^p}$ induces a well defined map on $H^\ast_l(V_\Omega, \mathbb{R})$.

**Proposition 3.** If $\phi$ is a $d_l-$coboundary, then $< C_{\mu, \mathbb{Z}^p}, \phi > = 0$.

We are now in position to state the measured index theorem for the longitudinal Dirac operator with coefficients in a continuous longitudinally smooth vector bundle.

**Theorem 3.** The measured index of the longitudinal Dirac operator $\partial^\ast_l(V_\Omega, \mathbb{R})$ with coefficients in the vector bundle $E$ associated with a projector $e$ of $M_n(C(V_\Omega))$ is given by:

$$\text{Ind}_{\mathbb{Z}^p, \mu}(\partial^\ast_l(V_\Omega, \mathbb{R})) = < \text{ch}_l([e]), [C_{\mu, \mathbb{Z}^p}] >.$$
We end this section by identifying the top dimension group of longitudinal cohomology with the coinvariants of the algebra \( C(\Omega, \mathbb{R}) \) of real valued continuous function on \( \Omega \). Recall that the coinvariants \( C(\Omega, \mathbb{R})_{\mathbb{Z}^p} \) of the action of \( \mathbb{Z}^p \) on \( C(\Omega, \mathbb{R}) \) is the quotient of \( C(\Omega, \mathbb{R}) \) by the subgroup generated by elements of the form \( \alpha(f) - f \), where \( f \in C(\Omega, \mathbb{R}) \) and \( \alpha \in \mathbb{Z}^p \).

**Proposition 4.** \( \boxdot \) If \( \phi \) is an element of \( \Omega^p_p(V_\Omega, \mathbb{R}) \), we define the continuous function \( \psi_{\mathbb{Z}^p}(\phi) \) of \( C(\Omega, \mathbb{R}) \) by setting:

\[
\psi_{\mathbb{Z}^p}(\phi)(\omega) := \int_{[0,1]^p} \phi(\omega, x) \, dx, \quad \forall \omega \in \Omega.
\]

Then, \( \phi \to \psi_{\mathbb{Z}^p}(\phi) \) induces a map from \( H^p_p(V_\Omega, \mathbb{R}) \) to the coinvariants \( C(\Omega, \mathbb{R})_{\mathbb{Z}^p} \).

Moreover, this map identifies \( H^p_p(V_\Omega, \mathbb{R}) \) with \( C(\Omega, \mathbb{R})_{\mathbb{Z}^p} \):

**Theorem 4.** \( \boxdot \) The transform \( \psi_{\mathbb{Z}^p} \) is an isomorphism, i.e.

\[
H^p_p(V_\Omega, \mathbb{R}) \cong C(\Omega, \mathbb{R})_{\mathbb{Z}^p}.
\]

Since the measure \( \mu \) is \( \mathbb{Z}^p \)-invariant, it induces a linear form on \( C(\Omega, \mathbb{R})_{\mathbb{Z}^p} \) that we shall denote by \( \Phi_{\mathbb{Z}^p, \mu} \). Under the above identification between \( C(\Omega, \mathbb{R})_{\mathbb{Z}^p} \) and \( H^p_p(V_\Omega, \mathbb{R}) \), the map \( \Phi_{\mathbb{Z}^p, \mu} \) corresponds to the Ruelle-Sullivan cycle \( C_{\mu, \mathbb{Z}^p} \):

**Proposition 5.** \( \boxdot \) The following diagram is commutative:

\[
\begin{array}{ccc}
H^p_p(V_\Omega, \mathbb{R}) & \xrightarrow{\psi_{\mathbb{Z}^p}} & C(\Omega, \mathbb{R})_{\mathbb{Z}^p} \\
C_{\mu, \mathbb{Z}^p} & \xrightarrow{\Phi_{\mathbb{Z}^p, \mu}} & \mathbb{R}
\end{array}
\]

5. Proof of the Bellissard conjecture

We give in this section a proof of the Bellissard conjecture. We first recall the construction of the Kasparov spectral sequence associated with the mapping torus \( V_\Omega \) (see [12]).

Let again \( T_p \) be the \( p \)-torus and let us denote for \( j \geq 1 \) by \( D_j \) the unit open disk in \( \mathbb{R}^j \). Let \( Y_0 \subset Y_1 \subset \cdots \subset Y_p = T_p \) be a filtration of \( T_p \) by the skeletons of some triangulation of \( T_p \), where \( Y_j \) is the \( j \)-skeleton of \( T_p \). We associate to \( \{ Y_i \}_{0 \leq i \leq p} \) an open filtration \( \{ Z_i \}_{0 \leq i \leq p} \) of \( T_p \) in the following way:

- \( Z_0 \) is an open neighbourhood of \( Y_0 \) which is homeomorphic to \( Y_0 \times D_p \).
- Assume that for \( 1 \leq i \leq p \), the open set \( Z_{i-1} \) is already constructed. Then \( Z_i \) is an open neighbourhood of \( Y_i \) containing \( Z_{i-1} \) and such that \( Z_i \setminus Z_{i-1} \) is a disjoint union:

\[
Z_i \setminus Z_{i-1} \cong \bigoplus_{\sigma \text{ simplex}} \sigma' \times D_{p-i}.
\]

where \( \sigma' \) is obtained from \( \sigma \) by a contractive homothety centered at the center of \( \sigma \).

Let \( W_i \) be the inverse image of \( Z_i \) by the projection \( V_\Omega \to T_p \). Then \( \{ W_i \}_{i} \) is an open filtration of the space \( V_\Omega \) which satisfies:

\[
\forall i \in \{ 1, \cdots, p \}, W_i \setminus W_{i-1} \cong \bigoplus_{\sigma \text{ simplex}} \Omega \times \sigma' \times D_{p-i}.
\]
To the filtration \((W_i)_i\) are associated two spectral sequences converging respectively to \(K_*(C(V_\Omega))\) and \(H^*_{\text{c}}(V,\mathbb{R})\) with \(E^2\)—tens respectively equal to \(\oplus_j H_j\mathbb{Z}^p, C(\Omega, \mathbb{Z}))\) and \(\oplus_j H_j\mathbb{Z}^p, C(\Omega, \mathbb{R}))\). Note that the coefficient in the first homology group comes from the identification \(K_0(C(\Omega)) \cong C(\Omega, \mathbb{Z})\) for totally disconnected compact spaces and that the existence of the second spectral sequence is a consequence of proposition 4.

**Proposition 6.** \(\mathbb{1}\) The homology group \(H_*(\mathbb{Z}^p, C(\Omega, \mathbb{Z})) = \oplus_j H_j(\mathbb{Z}^p, C(\Omega, \mathbb{Z}))\) is torsion free.

Recall that there exist a Chern character in Cech cohomology \(K^*(X) \rightarrow H^*(X, \mathbb{R})\) for every topological space \(X\). According to remark \(\mathbb{1}\), this induces a natural morphism \(\text{ch} : K^*(U) \rightarrow H^*_{\text{c}}(U, \mathbb{R})\) for every open subset \(U\) of \(V_\Omega\) that we shall call again the Chern character.

Moreover, according to theorem \(\mathbb{2}\), the longitudinal Chern character induces a morphism between these two spectral sequences. For the \(E^2\)—term, the longitudinal Chern character \(\text{ch}_{\ell}\) and the Chern character \(\text{ch}\) both induce the inclusion \(\oplus_j H_j(\mathbb{Z}^p, C(\Omega, \mathbb{Z})) \hookrightarrow \oplus_j H_j(\mathbb{Z}^p, C(\Omega, \mathbb{R}))\). In particular, we get:

**Proposition 7.** The two morphisms \(\text{ch} : K^*(V_\Omega) \rightarrow H^*_\ell(V_\Omega, \mathbb{R})\) and \(\text{ch}_{\ell} : K^*(V_\Omega) \rightarrow H^*_\ell(V_\Omega, \mathbb{R})\) coincide.

We can check by using spectral sequences associated to the above filtration that the map \(H^*(V_\Omega, \mathbb{R}) \rightarrow H^*_\ell(V_\Omega, \mathbb{R})\) of remark \(\mathbb{2}\) is an isomorphism. Actually, we can check that the two above spectral sequences collapse at the \(E^2\) terms. In particular, since the graduate associated to the filtration

\[\langle \text{Im}(K_*(C_0(W_k)) \rightarrow K_*(C(V_\Omega))) \rangle_{0 \leq k \leq p}\]

of \(K_*(C(V_\Omega))\) is \(\oplus_k H_k(\mathbb{Z}^p, C(\Omega, \mathbb{Z}))\) and this group being torsion-free by proposition \(\mathbb{1}\) the abelian group \(K_*(C(V_\Omega))\) is torsion free. To be more accurate:

**Theorem 5.** \(\mathbb{2}\)

- The morphism \(\text{ch} : K_*(V_\Omega) \rightarrow H^*_\ell(V_\Omega, \mathbb{R})\) is injective with homogeneous image (i.e. its image is graded by the grading of \(H^*_\ell(V_\Omega, \mathbb{R})\)).
- Moreover, the image of \(\text{ch} : K_*(V_\Omega) \rightarrow H^*_\ell(V_\Omega, \mathbb{R})\) is isomorphic to \(\oplus_j H_j(\mathbb{Z}^p, C(\Omega, \mathbb{Z}))\).

To prove the Bellissard conjecture, we shall need the following result of integrality for the top dimension Chern character.

**Proposition 8.** \(\mathbb{3}\) Let us denote by \(\text{ch}^k_{\ell}\) the \(k\)—component of the longitudinal Chern character lying in \(H^k_{\ell}(V_\Omega, \mathbb{R})\). Then \(\psi_{2p}(\text{ch}^k_{\ell}(K_0(C(V_\Omega))) \rightarrow C(\Omega, \mathbb{Z})_{2p}\).

**Proof.** Let \(x\) be an element of \(K_0(C(V_\Omega))\). According to theorem \(\mathbb{3}\) there is an element \(x' \in K_0(C(V_\Omega))\) such that \(\text{ch}(x) = \text{ch}(x')\) is an homogeneous element of degree \(p\). Recall that \(Y_{p-1}\), is the \(p - 1\) skeleton of \(T_p\). Let us denote by \(V_\Omega^\prime\) the inverse image of \(Y_{p-1}\) under the fibration projection \(V_\Omega \rightarrow T_p\) and by \(y\) the image of \(x - x'\) under the map \(K_0(C(V_\Omega)) \rightarrow K_0(C(V_\Omega^\prime))\). We can check easily that the Chern character on \(V_\Omega\) has no component of degree \(p\) or higher. In particular, since the Chern character is natural, \(\text{ch}(y) \in H^*_\ell(V_\Omega^\prime, \mathbb{R})\) vanishes. By using the same trick as we did for \(V_\Omega\), we can check that \(K_0(C(V_\Omega^\prime))\) is torsion-free and that the Chern character of \(V_\Omega^\prime\) is injective. In particular, \(y = 0\) and \(x - x'\) lies in the image of \(K_0(C_0(V_\Omega \setminus V_\Omega^\prime)) \rightarrow K_0(C(V_\Omega))\). We have an identification \(V_\Omega \setminus V_\Omega^\prime \cong \Omega \times [0, 1]^p\).
Since \(ch_{\ell}^{p}(x') = 0\), under the above identification, \(\psi_{\mathbb{Z}^{p}}(ch_{\ell}^{p}(x))\) lies in the image of the composition
\[
K_{0}(C(\Omega)) \cong K_{0}(C(\Omega \times 0, 1[\mathbb{P}])) \to K_{0}(C(V_{\Omega})) \xrightarrow{ch_{\ell}^{p}} H_{\ell}^{p}(V_{\Omega}, \mathbb{R}) \xrightarrow{\psi_{\mathbb{Z}^{p}}} C(\Omega, \mathbb{R})_{\mathbb{Z}^{p}}.
\]
But we have an isomorphism \(K_{0}(C(\Omega)) \cong C(\Omega, \mathbb{Z})\) and under this isomorphism, it was shown in [6] that the above composition is equal to the following:
\[
C(\Omega, \mathbb{Z}) \to C(\Omega, \mathbb{Z})_{\mathbb{Z}^{p}} \to C(\Omega, \mathbb{R})_{\mathbb{Z}^{p}}.
\]
This conclude the proof. \(\square\)

We are now in position to prove the Bellissard conjecture.

**Theorem 6.** Let \((\Omega, \mathbb{Z}^{p})\) be a dynamical system with \(\Omega\) a Cantor set and let \(\mu\) be a \(\mathbb{Z}^{p}\)-invariant measure on \(\Omega\). Assume that \(\Omega\) has no non-trivial invariant compact-open subset. Let \(\tau_{\mu}^{c}\) be the additive map induced by the trace \(\tau^{c}\), associated with \(\mu\), on the \(K\)-theory of the \(C^{*}\)-algebra \(C(\Omega) \rtimes \mathbb{Z}^{p}\). Then we have:
\[
\tau_{\mu}^{c}(K_{0}(C(\Omega) \rtimes \mathbb{Z}^{p})) = \mathbb{Z}[\mu].
\]

**Proof.** As we said before, we can assume that \(p\) is even. From Theorem 3 we deduce that:
\[
\tau_{\mu}^{c}(K_{0}(C(\Omega) \rtimes \mathbb{Z}^{p})) = \{\text{Ind}_{\mathbb{Z}^{p}, \mu}(\partial_{\Omega}^{e}), \text{Ind}_{\mathbb{Z}^{p}, \mu}(\partial_{\Omega}^{e}), [e] - [e'] \in K_{0}(C(\mathbb{Z}^{p})).\}
\]
Using Theorem 5 we obtain:
\[
\tau_{\mu}^{c}(K_{0}(C(\Omega) \rtimes \mathbb{Z}^{p})) = < ch_{\ell}^{p}(K_{0}(V_{\Omega})), [C_{\mu, \mathbb{Z}^{p}}] >.
\]
Now, from the very definition of \([C_{\mu, \mathbb{Z}^{p}}]\) and \(\psi_{\mathbb{Z}^{p}}\), we have:
\[
<x, [C_{\mu, \mathbb{Z}^{p}}] >= < \psi_{\mathbb{Z}^{p}}(x), \mu > , \quad \forall x \in H_{\ell}^{p}(V_{\Omega}, \mathbb{R}).
\]
Therefore,
\[
< ch_{\ell}^{p}(e), [C_{\mu, \mathbb{Z}^{p}}] >= < \psi_{\mathbb{Z}^{p}}(ch_{\ell}^{p}(e)), \mu > .
\]
But according to Proposition 8, we have:
\[
\psi_{\mathbb{Z}^{p}}(ch_{\ell}^{p}(e)) \in C(\Omega, \mathbb{Z})_{\mathbb{Z}^{p}}.
\]
Hence:
\[
\tau_{\mu}^{c}(K_{0}(C(\Omega) \rtimes \mathbb{Z}^{p})) \subset \mu(C(\Omega, \mathbb{Z})_{\mathbb{Z}^{p}}) = \mathbb{Z}[\mu].
\]
Since the opposite inclusion is straightforward to check, the proof is complete. \(\square\)

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