HAUPT’S THEOREM
FOR STRATA OF ABELIAN DIFFERENTIALS

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ABSTRACT

Let $S$ be a closed topological surface. Haupt’s theorem provides necessary and sufficient conditions for a complex-valued character of the first integer homology group of $S$ to be realized by integration against a complex-valued 1-form that is holomorphic with respect to some complex structure on $S$. We prove a refinement of this theorem that takes into account the divisor data of the 1-form.

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1. Introduction

Let \( S \) be an oriented connected topological surface without boundary having genus \( g \geq 2 \). We say that a character \( \chi : H_1(S;\mathbb{Z}) \rightarrow \mathbb{C} \) is realized by a complex-valued 1-form \( \omega \) if and only if for each integral cycle \( \gamma \) we have

\[
\int_{\gamma} \omega = \chi(\gamma).
\]

In this case, the image \( \Lambda_\chi \) of \( \chi \) is the set of periods of \( \omega \).

In 1920, O. Haupt [Hpt20] determined those characters that are realized by some 1-form that is holomorphic with respect to some complex structure on \( S \). More recently, M. Kapovich [Kpv17] rediscovered Haupt’s characterization in the following form: A character \( \chi \) is realized by a holomorphic 1-form \( \omega \) if and only if

1. its area
   \[
   A(\chi) := \text{Im} \sum \overline{\chi(a_i)} \chi(b_i)
   \]
   is positive where \( \{a_i, b_i\} \) is a symplectic basis of \( H_1(S;\mathbb{Z}) \), and

2. if \( \Lambda_\chi \) is discrete, then \( \Lambda_\chi \) is a lattice and the induced homotopy class of maps from \( S \) to the torus \( \mathbb{C}/\Lambda_\chi \) has degree \( d_\chi \) strictly greater than 1.

In addition, if \( \Lambda_\chi \) is discrete, then the induced map is realized by a branched covering \( p : S \rightarrow \mathbb{C}/\Lambda_\chi \) and the pullback \( p^*(dz) \) realizes \( \chi \).

In this note we provide a refinement of Haupt’s theorem that involves the divisor data of the 1-form. To be precise, let

\[
Z(\omega) = \{z_1, z_2, \ldots, z_k\}
\]

be the set of zeros of a nontrivial holomorphic 1-form \( \omega \), and for each \( i \) let \( \alpha_i \) denote the multiplicity of the zero \( z_i \). The divisor data, \( \alpha(\omega) \), is the unordered \( n \)-tuple \( (\alpha_1, \ldots, \alpha_k) \), whose sum is

\[
\alpha_1 + \alpha_2 + \cdots + \alpha_k = 2g - 2.
\]

**Theorem 1.1:** A character \( \chi : H_1(S,\mathbb{Z}) \rightarrow \mathbb{C} \) is realized by a 1-form \( \omega \) with divisor data \( \alpha(\omega) = (\alpha_1, \ldots, \alpha_k) \) if and only if

1. \( A(\chi) \) is positive, and

2. if \( \Lambda_\chi \) is discrete, then the induced map \( S \rightarrow \mathbb{C}/\Lambda_\chi \) has degree
   \[
   d_\chi > \max \{\alpha_i\}.
   \]
The proof of the sufficiency is immediate. Indeed, one applies Haupt’s theorem and notes that the Riemann–Hurwitz formula shows that the degree of an induced branched covering is at least $1 + \max \{\alpha_i\}$.

To prove the necessity, we will recast the problem in terms of the moduli space theory of 1-forms (see §2). The Hodge bundle $\Omega \mathcal{M}_g$ is the moduli space of complex-valued 1-forms that are holomorphic with respect to some complex structure on $S$. It is a disjoint union of the strata $\Omega \mathcal{M}_g(\alpha)$, consisting of forms with divisor data $\alpha$. A connected component of the set of 1-forms that have a prescribed set of periods constitutes a leaf of the ‘isoperiodic foliation’. Cal-samiglia, Deroin, and Francaviglia [CDF15] classified the closures of the leaves of the isoperiodic foliation. We use this classification to prove the following.

**Theorem 1.2:** If $L$ is an isoperiodic leaf whose associated set of periods is not a lattice, then $L$ intersects each connected component of each stratum of the Hodge bundle.

To prove Theorem 1.1, one combines Theorem 1.2 with the following proposition.

**Proposition 1.3:** Let $\Gamma$ be a lattice in $\mathbb{C}$. For each connected component $K$ of each stratum $\Omega \mathcal{M}_g(\beta)$ of $\Omega \mathcal{M}_g$ and for each integer $d > \max \{\beta_k\}$, there exists a primitive degree $d$ branched covering $p: S \to \mathbb{C}/\Gamma$ such that $(S, p^*(dz))$ belongs to $K$.

Recall that a branched cover of a torus is primitive if the induced map on homology is surjective.

In §2, we construct the Hodge bundle over Teichmüller space, define the isoperiodic foliation, recall the main result of [CDF15], and prove Theorem 1.2. In §3, we prove Proposition 1.3.

Soon after we posted this paper on the arXiv, Thomas Le Fils shared a preprint [LFs20] containing his independent proof of Theorem 1.1. His proof differs from ours in that it does not pass through Theorem 1.2 and instead uses a study of the mapping class group action on the space of characters in the spirit of [Kpv17]. We note that his paper does not consider connected components of strata.

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2. The Hodge bundle and the isoperiodic foliation

In this section we describe the Hodge bundle and the absolute and relative period mappings. We define the isoperiodic foliation and show that each leaf that passes near a stratum must intersect the stratum. We use this to prove Theorem 1.2. Finally we prove Theorem 1.1 modulo the proof of Proposition 1.3.

We begin by describing the Hodge bundle as a bundle over Teichmüller space. A marked Riemann surface is a closed Riemann surface $X$ together with an orientation-preserving homeomorphism $f: S \rightarrow X$. Two marked surfaces $(f_1, X_1)$ and $(f_2, X_2)$ are considered to be equivalent if $f_2 \circ f_1^{-1}$ is isotopic to a conformal map. The set of equivalence classes of marked genus $g$ surfaces may be given the structure of a complex manifold homeomorphic to $\mathbb{C}^{3g-3}$ called the Teichmüller space $T_g$.

The Hodge bundle $\Omega T_g \rightarrow T_g$ is the (trivial) vector bundle over $T_g$ whose fiber above $(f, X)$ consists of (equivalence classes of) holomorphic 1-forms on $X$. In other words, $\Omega T_g$ is the space of triples $(f, X, \omega)$ up to natural equivalence.

The total space of $\Omega T_g$ is naturally a complex manifold of dimension $4g - 3$. The absolute period map $P: \Omega T_g \rightarrow H^1(S; \mathbb{C})$ is the holomorphic map that assigns to each triple $(f, X, \omega)$ the cohomology class $f^*(\omega)$.

Let $\Omega^* T_g \subset \Omega T_g$ denote the set of one-forms that do not vanish identically. The map that assigns divisor data to each 1-form defines a stratification of $\Omega^* T_g$. In particular, for each partition $\alpha = (\alpha_1, \ldots, \alpha_k)$ of $2g - 2$, we define the stratum $\Omega T_g(\alpha)$ to consist of those triples $(f, X, \omega)$ such that the divisor data of $\omega$ equals $\alpha$.

One may also define a relative period map in a neighborhood of each non-trivial marked one-form $(f_0, X_0, \omega_0)$ in the stratum $\Omega T_g(\alpha)$. Let $Z \subset S$ be a set of $k$ marked points. Over a contractible neighborhood $U \subset \Omega T_g(\alpha)$ of $(f_0, X_0, \omega_0)$, one may choose representative marking maps to identify $Z$ with the zero sets $Z(\omega)$. Pulling back by these marking maps the class $[\omega] \in H^1(X, Z(\omega); \mathbb{C})$

then defines the relative period map

$$P_{rel}: U \rightarrow H^1(S, Z; \mathbb{C}).$$

The relative period map is well-known to be a local biholomorphism [Vch90]. Moreover, the relative and absolute period maps are related by $P|_U = r \circ P_{rel}$ where $r$ is the natural map from $H^1(S, Z; \mathbb{C})$ to $H^1(S; \mathbb{C})$. By considering the
long exact sequence in cohomology, one finds that $r$ is surjective, and hence $P|_U$ is a submersion. Since every non-trivial one-form lies in some stratum, we have the following.

**Lemma 2.1:** The restriction of the absolute period map $P$ to $\Omega^* T_g$ is a submersion, as is its restriction to any stratum in $\Omega T_g$.

Since $P$ is a submersion, it defines a holomorphic foliation of $\Omega^* T_g$ called the **isoperiodic** (or **Rel**) foliation. Each isoperiodic leaf is a connected component of a level set of $P$.

The mapping class group $\text{Mod}(S)$ naturally acts biholomorphically and properly discontinuously on the Hodge bundle. The quotient of this action is the classical Hodge bundle $\Omega M_g \to M_g$ where the base $M_g$ is the moduli space of Riemann surfaces. In particular, each point in $\Omega M_g$ may be regarded as (the equivalence class of) a pair $(X, \omega)$ where $X$ is a Riemann surface and $\omega$ is a holomorphic 1-form on $X$.

If $\varphi \in \text{Mod}(S)$ then we have

$$P(\varphi^*(\omega)) = \varphi^*(P(\omega)).$$

It follows that the isoperiodic foliation descends to a foliation of $\Omega M_g$ that we will also refer to as the isoperiodic foliation. Moreover, we have a well-defined map from the set of leaves to the orbit space $H^1(S; \mathbb{C})/\text{Mod}(S)$, and the set of periods

$$\Lambda_L := \left\{ \int_\gamma \omega : \gamma \in H_1(S; \mathbb{Z}) \right\}$$

depends only on the isoperiodic leaf $L$ to which $\omega$ belongs.

Each stratum $\Omega T_g(\alpha)$ is invariant under the action of $\text{Mod}(S)$. Each quotient,

$$\Omega M_g(\alpha) := \Omega T_g(\alpha)/\text{Mod}(S),$$

is the **stratum** that consists of pairs $(X, \omega)$ with divisor data $\alpha$.

**Proposition 2.2:** Let $K$ be a connected component of a stratum. There exists a neighborhood $Z \subset \Omega M_g$ of $K$ such that if an isoperiodic leaf $L$ intersects $Z$, then $L$ also intersects $K$.

**Proof.** Let $\tilde{K}$ be a connected component of the preimage of $K$ in $\Omega^* T_g$. By Lemma 2.1, the map $P$ is a holomorphic submersion from the $4g - 3$ dimensional complex manifold $\Omega^* T_g$ onto the complex vector space $H^1(S; \mathbb{C})$ which
has dimension $2g$. Thus, given $(f, X, \omega)$, the inverse function theorem provides an open ball $B^{2g-3} \subset \mathbb{C}^{2g-3}$, an open ball $B^g \subset H^1(S; \mathbb{C})$, and a biholomorphism $\varphi$ from $B^{2g-3} \times B^g$ onto a neighborhood $U$ of $(f, X, \omega)$ so that

$$P \circ \varphi(z, w) = w.$$ 

Suppose that $(f, X, \omega)$ lies in $\tilde{K}$. Since the restriction of $P$ to $\tilde{K}$ is a submersion, the image $V := P(U \cap \tilde{K})$ is open. Note that

$$(P \circ \varphi)^{-1}(V) = B^{2g-3} \times V.$$

If $L$ is a connected component of $P^{-1}(\chi)$ that intersects

$$W := \varphi(B^{2g-3} \times V),$$

then $\chi \in V$ and $L \cap U = \varphi(B^{2g-3} \times \{\chi\})$. In particular, $L$ intersects $\tilde{K}$.

The neighborhood $Z$ is constructed by taking the image in $\Omega \mathcal{M}_g$ of the union of all such neighborhoods $W$ as $(f, X, \omega)$ varies over $\tilde{K}$. $lacksquare$

Next, we describe the result of Casamiglia, Deroin, and Francaviglia [CDF15] that classifies the closures of leaves $L$ in terms of the associated set of periods $\Lambda_L$. The closure, $\overline{\Lambda}_L$, is a closed real Lie subgroup of $\mathbb{C} \cong \mathbb{R}^2$. Thus, $\overline{\Lambda}_L$ is either equal to $\mathbb{C}$, is isomorphic to $\mathbb{Z} \oplus \mathbb{R}$, or is discrete.

Let $\Omega_1 \mathcal{M}_g \subset \Omega \mathcal{M}_g$ denote the locus of unit-area forms. Since the area functional

$$A(\omega) = \frac{i}{2} \int_S \omega \wedge \overline{\omega}$$

depends only on absolute periods, $\Omega_1 \mathcal{M}_g$ is saturated by leaves of the isoperiodic foliation.

Given any closed subgroup $\Gamma \subset \mathbb{C}$, let $\Omega_1^\Gamma \mathcal{M}_g \subset \Omega_1 \mathcal{M}_g$ denote the union of the leaves $L$ such that there exists a connected subgroup $\Gamma' \subset \Gamma$ with $\Gamma = \overline{\Lambda}_L + \Gamma'$. If $\Gamma = \mathbb{C}$, then

$$\Omega_1^\Gamma \mathcal{M}_g = \Omega_1 \mathcal{M}_g.$$ 

If $\Gamma$ is isomorphic to $\mathbb{R} + \sqrt{-1} \cdot \mathbb{Z}$, then $L \subset \Omega_1^\Gamma \mathcal{M}_g$ if either $\overline{\Lambda}_L = \Gamma$ or $\overline{\Lambda}_L$ is a discrete subgroup of $\Gamma$ with ‘primitive imaginary part’. If $\Gamma$ is discrete, then $\Omega_1^\Gamma \mathcal{M}_g$ is nonempty only if $\Gamma$ has covolume $1/d$ for some integer $d > 1$, in which case $\Omega_1^\Gamma \mathcal{M}_g$ is a closed isoperiodic leaf which parameterizes primitive degree $d$ branched covers of $\mathbb{C}/\Gamma$. 

Proposition 2.3: If $\Gamma$ is a lattice, then the space $\Omega^\Gamma_{1, M_g}$ is connected.

Proof. By Theorem 9.2 of [GabKaz87], given two primitive, simply branched coverings $p : S \to \mathbb{C}/\Gamma$ and $q : S \to \mathbb{C}/\Gamma$ of the same degree, there exists a homeomorphism $h : S \to S$ and a homeomorphism $k : \mathbb{C}/\Gamma \to \mathbb{C}/\Gamma$ isotopic to the identity so that $k \circ p = q \circ h$. Let $k_t$ be the isotopy with $k_0 = k$ and $k_1 = \text{id}$. For each $t$, the $1$-form $(k_t \circ p)^*(dz)$ is holomorphic with respect to the pulled-back complex structure. We have

$$(k_0 \circ p)^*(dz) = h^*(q^*(dz)) \quad \text{and} \quad (k_1 \circ p)^*(dz) = p^*(dz).$$

Hence the path in $\Omega^\Gamma_{1, M_g}$ associated to $(k_t \circ p)^*(dz)$ joins the point represented by $q^*(dz)$ to the point represented by $p^*(dz)$. Since simply branched coverings are generic, the space $\Omega^\Gamma_{1, M_g}$ is connected.

Because $\Omega^\Gamma_{1, M_g}$ is connected, we may simplify the statement of the main theorem of [CDF15].

Theorem 2.4 ([CDF15]): Let $L \subset \Omega_{1, M_g}$ be a leaf of the isoperiodic foliation and let $\Gamma = \overline{\Lambda}_L$. If $g > 2$, then the closure of $L$ is $\Omega^\Gamma_{1, M_g}$. If $g = 2$, then either the closure of $L$ is $\Omega^\Gamma_{1, M_2}$ or $L$ lies in the eigenform locus $E \subset \Omega_{1, M_2}$.

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. We first suppose that $g > 2$ or $g = 2$ and $L \not\subset E$. By assumption, $L$ is an isoperiodic leaf such that $\Lambda_L$ is not a lattice, and so $\overline{\Lambda}_L$ either equals $\mathbb{C}$ or equals $\mathbb{R} \cdot z_1 \oplus \mathbb{Z} \cdot z_2$ where $z_1 \in \mathbb{C}$. By Lemma 2.1 the restriction of the absolute period map to a given component $K$ of a given stratum is an open map. It follows that there exists $(X, \omega) \in K$ of area $1$ so that the periods of $\omega$ lie in $\mathbb{Q} \cdot z_1 \oplus \mathbb{Q} \cdot z_2$. In particular, the set of periods constitute a lattice and there exists $A \in SL_2(\mathbb{R})$ so that the periods of $A \cdot (X, \omega)$ lie in $\overline{\Lambda}_L$. Hence $A \cdot (X, \omega)$ lies in the closure $\overline{L}$ by Theorem 2.4. Thus $K$ intersects $\overline{L}$, and hence $K$ intersects $L$ by Proposition 2.2.

It remains to consider the case where $g = 2$ and $L \subset E$. In this case, Theorem 1.2 follows from work of McMullen [McM03, McM05]. Indeed, $\Omega_{1, M_2}$ consists of two strata, the principal stratum $\Omega_{1, M_2}(1, 1)$ and the stratum $\Omega_{1, M_2}(2)$, and both of these strata are connected. McMullen shows that the eigenform locus $E \subset \Omega_{1, M_2}$ is a countable union of orbifolds $\Omega_{1, E_D}$ where $D$ belongs to a subset of the positive integers. Moreover, each $\Omega_{1, E_D}$ is saturated by leaves of
the isoperiodic foliation. The intersection $\Omega_1E_D \cap \Omega_1M_2(2)$ is his “Weierstrass curve” $\Omega_1W_D$. The eigenform locus $\Omega_1E_D$ is a circle bundle over a Hilbert modular surface, which is covered by $\mathbb{H} \times \mathbb{H}$. In this covering, the isoperiodic foliation is simply the “vertical” foliation with leaves $\{c\} \times \mathbb{H}$. Each component of the Weierstrass curve is covered by a graph of a holomorphic function $\mathbb{H} \to \mathbb{H}$ which a fortiori must intersect each vertical leaf, and hence every isoperiodic leaf in $\Omega_1E_D$ must intersect $\Omega_1W_D$. Finally, each $\Omega_1W_D$ is nonempty unless $D = 4$, in which case $\Omega_1E_4$ parameterizes degree 2 torus-covers, a case that is excluded by hypothesis.

We remark that if $\Lambda_L$ is a lattice, then the associated space $\Omega_1^{\Lambda_L}M_2$ need not intersect every stratum $\Omega_1M_2(\alpha)$. Indeed, for such an intersection to be nonempty, it is necessary for the covolume of $\Lambda_L$ to be strictly less than $1/\max \alpha_i$. Proposition 1.3 implies that this condition is also sufficient.

Finally, we prove our variant of Haupt’s theorem modulo Proposition 1.3.

**Proof of Theorem 1.1.** Suppose that $\chi \in \text{Hom}(H^1(S;\mathbb{Z}),\mathbb{C}) \cong H^1(S;\mathbb{C})$ is a character which satisfies the hypotheses of Theorem 1.1. By applying a real rescaling, we may assume moreover that $A(\chi) = 1$. Haupt’s theorem then provides a unit-area holomorphic 1-form $(X,\omega) \in \Omega_1T_g$ representing $\chi$. Note that each 1-form that lies in the isoperiodic leaf, $L$, that contains $(X,\omega)$ also represents $\chi$. Hence it suffices to show that $\pi(L)$ intersects $\Omega_1M(\alpha)$.

If $\Gamma := \overline{\Lambda_L}$ is a lattice, then $\omega = p^*(dz)$ for some degree $d$ primitive branched covering $p : X \to \mathbb{C}/\Gamma$. By Proposition 1.3, there exists a degree $d$ primitive branched covering $q : X' \to \mathbb{C}/\Gamma$ so that

$$q^*(dz) \in \Omega_1M(\alpha).$$

In particular, both $\pi(X',q^*(dz))$ and $\pi(X,p^*(dz))$ lie in $\Omega_1^\Gamma M$. Proposition 2.3 implies that

$$\pi(L) = \Omega_1^\Gamma M,$$

and so $\pi(X',q^*(dz))$ lies in $\pi(L) \cap \Omega_1M(\alpha)$.

If $\Gamma$ is not a lattice and $g > 2$, then Theorem 1.2 implies that the projection $\pi(L)$ is dense in $\Omega_1^\Gamma M_g$, and hence $\pi(L)$ intersects $\Omega_1M(\alpha)$ by Proposition 2.2.

If $\Gamma$ is not a lattice and $g = 2$, then one can directly construct a 1-form in $\Omega_1M(1,1)$ (resp. $\Omega_1M(2)$) that represents $\chi$ by gluing together two well-chosen slit tori (resp. gluing a cylinder to a slit torus).
3. Primitive torus covers

In this section we complete the proof of Theorem 1.1 by proving Proposition 1.3. That is, for each connected component $K$ of a stratum $\Omega \mathcal{M}(\alpha)$, we construct a primitive branched torus covering $p: S \to \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ so that $p^*(dz)$ lies in $K$. Since the component $K$ is invariant under the $GL_2^+(\mathbb{R})$ action, Proposition 1.3 follows.

To prove Theorem 1.1, we will explicitly construct torus coverings that lie in connected components of strata having one or two zeros, and then we apply a sequence of ‘surgeries’ to obtain torus coverings with additional zeros. In §3.1 we construct torus coverings for each connected component of each minimal stratum $\Omega \mathcal{M}(2g - 2)$. In §3.2 we construct covers for each component of $\Omega \mathcal{M}_g(g - 1, g - 1)$. In §3.3 we introduce surgeries that add zeros to a torus cover while preserving the degree, and we check the effect of surgery on the spin parity. In §3.4 we construct torus covers such that the 1-form has exactly two zeros and each zero has odd order. We use surgeries to construct torus covers when $\max \alpha_i$ is odd. In §3.5 we describe the algorithm that can be used to construct a torus cover any desired connected component. We also provide some examples.

In what follows we will let $T$ denote the ‘unit square’ torus $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$.

According to [KoZo03], the connected components of strata are distinguished by hyperellipticity and spin parity. To be precise, we will need to determine whether a torus covering $p: S \to T$ admits a hyperelliptic involution, a holomorphic involution $\tau: S \to S$ such that the quotient $S/\langle \tau \rangle$ is a sphere. Because $\tau^*(\omega) = -\omega$,

a hyperelliptic involution maps each vertical (resp. horizontal) cylinder to a vertical (resp. horizontal) cylinder. Moreover, if $\tau$ preserves a vertical or horizontal cylinder $C$, then $\tau$ preserves the central curve of the cylinder and fixes exactly two points on the central curve. The Riemann–Hurwitz formula implies that $\tau$ has exactly $2g + 2$ fixed points.

We will also need to check the spin parity of a holomorphic 1-form. Given a Riemann surface $X$ with a holomorphic one-form $\omega$ and a loop $\gamma: S^1 \to X$ disjoint from the zeros of $\omega$, the Gauss map $G_\gamma: S^1 \to S^1$ is defined by

$$G_\gamma(t) = \frac{\omega(\gamma'(t))}{|\omega(\gamma'(t))|}.$$
The index of $\gamma$ is the degree of $G_\gamma$. Note that if $\gamma$ is a geodesic with respect to the natural flat structure on the surface, then $G_\gamma$ is a constant map and hence $\text{ind}(\gamma) = 0$.

Following Thurston and Johnson [Jns80], Kontsevich and Zorich [KoZo03] gave the following formula for the spin parity of a holomorphic 1-form $\omega$ all of whose zeros have even order. Given a symplectic basis $a_1, b_1, \ldots, a_g, b_g$ for $H_1(X; \mathbb{Z})$ consisting of curves that do not pass through a zero, the spin parity of $\omega$ equals

$$
\left( \sum_{i=1}^{g} (\text{ind}(a_i) + 1)(\text{ind}(b_i) + 1) \right) \pmod{2}.
$$

In particular, this invariant of a holomorphic 1-form with zeros of even order lies in $\mathbb{Z}/2\mathbb{Z}$. We refer to a 1-form as even if its spin parity equals 0 mod 2, and as odd otherwise.

3.1. Minimal Strata. In this subsection, for each $d > 2g - 2$, we construct a degree $d$ primitive branched torus covering for each connected component of the ‘minimal stratum’ $\Omega \mathcal{M}_g(2g - 2)$. For $g \geq 4$, the minimal stratum has exactly three connected components [KoZo03]:

- hyperelliptic: The 1-forms in $\Omega \mathcal{M}_g(2g - 2)$ that are canonical double covers of meromorphic quadratic differentials on the Riemann sphere with one zero of order $2g - 3$ and $2g + 1$ simple poles.
- even: The non-hyperelliptic 1-forms with even spin parity.
- odd: The non-hyperelliptic 1-forms with odd spin parity.

Denote these components by

$$
\Omega \mathcal{M}_g(2g - 2)^{\text{hyp}}, \quad \Omega \mathcal{M}_g(2g - 2)^{\text{odd}} \quad \text{and} \quad \Omega \mathcal{M}_g(2g - 2)^{\text{even}}.
$$

In the case $g = 3$, there is no even component, and in the case $g = 2$, there is only the hyperelliptic component [KoZo03].

For each of the above connected components we will first construct a degree $2g - 1$ primitive branched cover $p$ so that $p^*(dz)$ lies in the component. A slight modification of the construction will provide primitive branched coverings of each degree $d > 2g - 2$.

For a torus covering to lie in the minimal stratum, it is necessary that it be branched over a single point. To describe such coverings, consider the unbranched covers of the punctured torus $\mathbb{C}/((\mathbb{Z} + i\mathbb{Z}) \setminus \{0\})$. Each such degree $d$
covering corresponds to a homomorphism $\rho$ from the fundamental group of the once punctured torus to the symmetric group on $d$ letters (the ‘monodromy representation’). The fundamental group of the once punctured torus is freely generated by the central curve $h$ of the horizontal cylinder and the central curve $v$ of the vertical cylinder. It follows that each degree $d$ covering that is branched over 0 is determined by $\rho(h)$ and $\rho(v)$. In sum, each branched covering is determined by a pair of permutations that we will denote $h$ and $v$ respectively. This description is unique up to simultaneous conjugation of $h$ and $v$.

There is a one-to-one correspondence between the zeros of $p^*(dz)$ and the nontrivial cycles of the commutator $[h, v]$. Each cycle of length 1 in $[h, v]$ corresponds to a point in the fiber above $[0]$ that is not ramified. In particular, since in this section, we wish to construct torus coverings with a single ramification point of degree $2g - 1$, we will need to check that $[h, v]$ has one cycle of length $2g - 1$ and $d - (2g - 1)$ cycles of length 1.

Torus coverings branched over one point are often called square-tiled surfaces. Indeed, given a pair of permutations $h, v$ of $\{1, \ldots, d\}$, we can construct the covering by gluing together $d$ disjoint unit squares labeled $1, \ldots, d$ as follows: Glue the right side of square $i$ to the left side of square $h(i)$ and the top of square $i$ to the bottom of square $v(i)$. Note that the group generated by $h$ and $v$ must act transitively on $\{1, 2, \ldots, d\}$ for the surface to be connected.

3.1.1. The hyperelliptic component. Let $p: H_g \to T$ be the degree $d = 2g - 1$ torus covering branched over one point that is defined by the following permutations on $2g - 1$ letters (in cycle notation)

$$
\begin{align*}
h &= (1, 2)(3, 4) \cdots (2g - 3, 2g - 2)(2g - 1), \\
v &= (1)(2, 3)(4, 5) \cdots (2g - 2, 2g - 1).
\end{align*}
$$

See Figure 1. The commutator $[h, v]$ has order $2g - 1$ and so $p$ has only one ramification point, and thus $p^*(dz)$ has exactly one zero $z$ of order $2g - 2$. Hence each vertical edge (resp. horizontal edge) of each unit square is a 1-cycle in $H_1(H_g; \mathbb{Z})$, and the covering map sends this 1-cycle to the standard vertical (resp. horizontal) generator of $H_1(\mathbb{C}/\mathbb{Z}^2; \mathbb{Z})$. Hence $p$ is primitive.

The 1-form $p^*(dz)$ admits a unique hyperelliptic involution $\tau$. Indeed, the map $\tau$ may be constructed by rotating each square in Figure 1 about its center by $\pi$ radians. The involution $\tau$ has $2g + 2$ fixed points consisting of the zero of $p^*(dz)$, the centers of each of the $2g - 1$ squares, the midpoint of the top
Figure 1. A hyperelliptic surface, \( H_g \), in the minimal stratum that is a degree \( 2g - 1 \) primitive branched covering of the torus.

(and bottom) edge of square 1, and the midpoint of the left (and right) edge of square \( 2g - 1 \). The quotient \( H_g/\langle \tau \rangle \) is a sphere and it follows that \( p^*(dz) \) is hyperelliptic.

To construct primitive branched covers of degree \( d > 2g - 1 \), we lengthen one of the vertical cylinders by placing \( d - (2g - 1) \) additional squares on top of the square \( 2g - 1 \) in Figure 1. To be precise, let \( p: H_g^d \to \mathbb{C}/\mathbb{Z} \) be the covering determined by the permutations

\[
\begin{align*}
h &= (1, 2)(3, 4) \cdots (2g - 3, 2g - 2)(2g - 1)(2g - 2) \cdots (d - 1)(d), \\
v &= (1)(2, 3)(4, 5) \cdots (2g - 2, 2g - 1, \ldots, d - 1, d).
\end{align*}
\]

The commutator \([h, v]\) has one cycle of length \( 2g - 1 \) and \( d - (2g - 1) \) cycles of length 1. In other words, \( p^*(z) \) has a single zero of order \( 2g - 2 \). The covering \( p \) is primitive for the same reason that the covering \( H_g \to T \) is primitive.

The surface \( H_g^d \) admits a hyperelliptic involution \( \tau \) which rotates by \( \pi \) each of the squares labeled 1 through \( 2g - 2 \) about their respective centers. The involution \( \tau \) preserves the horizontal Euclidean cylinder \( C \) consisting of the
squares $2g - 1, \ldots, d$, and its restriction to $C$ has two fixed points. The only remaining fixed point of $\tau$ is the unique zero of $p^*(dz)$.

3.1.2. The odd component. Let $p: O_g \to T$ be the degree $d = 2g - 1$ torus covering branched over one point that is defined by the following permutations on $2g - 1$ letters (in cycle notation):

$$h = (1, 3, 5, \ldots, 2g - 1) \cdot (2) \cdot (4) \cdots (2g - 2),$$

$$v = (1, 2)(3, 4) \cdots (2g - 3, 2g - 2).$$

See Figure 2.

![Figure 2. The odd spin parity torus cover $O_g$ in the minimal stratum. The odd numbered squares together form a horizontal cylinder of length $g$ and each even numbered square corresponds to a horizontal cylinder of length 1.](image)

To see that the surface is not hyperelliptic, we suppose that $O_g$ admits a hyperelliptic involution $\tau$ and then derive a contradiction. The horizontal cylinder $C$ that consists of the odd numbered squares is the only horizontal cylinder of length greater than 1, and hence $\tau(C) = C$.\footnote{In fact, for each 1-form in the minimal stratum, each cylinder is preserved by the hyperelliptic involution. See, for example, the proof of Lemma 8 in [KoZo03].} In particular, the map $\tau$ preserves the union $V$ of the vertical saddle connections that are contained in $C$. The complement of $V$ consists of the vertical cylinder corresponding to the square labeled $2g - 1$ and the $g - 1$ slit tori $S_i$ corresponding to the cycles $(i, i + 1)$ for $i$ odd. If $\tau(S_i) = S_j$ for some $i \neq j$, then the sphere $O_g/\langle \tau \rangle$ would contain a once holed torus. This is not possible, and so $\tau(S_i) = S_i$ for each $i$. In
particular, $\tau$ preserves each odd numbered square, and the center of each odd numbered square is a fixed point. That is, $\tau$ has at least $g - 1 > 2$ fixed points. But each (non-null homologous) cylinder $C$ has exactly 2 fixed points, and this is the desired contradiction.

To see that the spin parity of $O_g$ is odd, we exhibit $O_g$ as the slit tori decomposition mentioned in the previous paragraph. See Figure 3. Here we have chosen a symplectic basis $\{a_i, b_i\}$ for the first homology of $O_g$. The curve $a_g$ ‘turns’ once as it traverses each slit torus and hence has index equal to $g - 1$. All other curves in this symplectic basis are geodesics and hence have index equal to zero. Thus, it follows from formula (1) that the spin parity equals $2g - 1$ mod 2.

The map $p$ is primitive because, for example, the classes $p_*(a_1)$ and $p_*(b_g)$ generate the first homology of $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$.

Figure 3. The simple closed curves $\{a_i, b_i\}$ form a symplectic basis for the first homology of $O_g$. Note that the curve $a_g$ intersects each slit torus, and each intersection contributes 1 to the index of $a_g$.

To construct a primitive branched cover $p : O_g^d \to \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ of degree $d > 2g - 1$, replace the cycle $(2g - 1)$ in the horizontal permutation $h$ with $(2g - 1, 2g, \ldots, d - 1, d)$. This is equivalent to replacing the vertical cylinder that corresponds to the square labeled $2g - 1$ with a vertical cylinder of width $d - (2g - 2)$. 
3.1.3. The even component. For $g \geq 4$, let $p: E_g \to T$ be the degree $d = 2g - 1$ branched covering of $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ defined by the permutations

$$h = (1, 3, 5, \ldots, 2g - 1, 4),$$
$$v = (1, 2)(3, 4) \cdots (2g - 3, 2g - 2)(2g - 1).$$

See Figure 4. The surface $E_g$ differs from $O_g$ in the way that the squares labeled 3 and 4 are attached. Arguments similar to the ones given in §3.1.2 show that $p$ is not hyperelliptic, is of even spin parity, and is primitive. For example, the horizontal cylinder $C$ consisting of the square labeled 4 and the odd numbered squares would be preserved by a hyperelliptic involution, and one can use this to argue that $E_g$ is not hyperelliptic. All of the elements in the symplectic basis in Figure 4 have index zero except for $a_2$ and $a_g$ which have indices 1 and $g - 1$ respectively. In particular, the spin parity is $2g \mod 2$.

![Figure 4](image)

Figure 4. The even parity torus cover $E_g$ in the minimal stratum. The simple closed curves $\{a_i, b_i\}$ form a symplectic basis for the first homology of $O_g$. The intersection of $a_g$ with each vertical cylinder of height two contributes 1 to the index of $a_g$. All other basis elements have index 0 except for $a_2$ which has index 1.

To obtain a degree $d$ cover $p: E_g^d \to \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$, replace the vertical cylinder of width 1 corresponding to the square labeled $2g - 1$ with a vertical cylinder of width $d - (2g - 2)$. In other words, replace the cycle $(2g - 1)$ that appears in $v$ with the cycle $(2g - 1, 2g, \ldots, d - 1, d)$. 
3.2. The strata with two zeros of equal order. According to [KoZo03], if \( g \geq 5 \) is odd, then the stratum \( \Omega M(g - 1, g - 1) \) has three connected components: hyperelliptic; even spin parity and non-hyperelliptic; and odd parity and non-hyperelliptic. When \( g = 2, 3 \) or \( g \geq 4 \) and even, the stratum has exactly two components: hyperelliptic and non-hyperelliptic. In §3.2.1 we exhibit a surface in each hyperelliptic component, regardless of the parity of \( g \), and then in §3.2.2 we construct examples in the remaining non-hyperelliptic component(s). Our constructions will be based on gluing together surfaces with slits.

3.2.1. \( \Omega M(g - 1, g - 1)^{hyp} \). If \( g = 2m \) is even, we construct a degree \( g \) hyperelliptic torus cover as follows. First, create a genus two surface by gluing together two copies of \( \mathbb{C}/\mathbb{Z}^2 \) that each have a horizontal slit. Take \( m \) distinct copies, \( S_1, \ldots, S_m \), of this genus two surface. Each genus two surface \( S_i \) has exactly four horizontal saddle connections, two that correspond to the slits and two that do not. See Figure 5.

From both \( S_1 \) and \( S_m \) remove one of the ones that do not correspond to a slit and from each of the remaining genus two surfaces, \( S_2, \ldots, S_{m-1} \), remove both of the horizontal saddle connections that do not correspond to a slit. Glue the top (resp. bottom) of the new slit on \( S_1 \) to the bottom (resp. top) of one of the (new) slits on \( S_2 \). Then, inductively, glue the top (resp. bottom) of the remaining slit on \( S_i \) to the bottom (resp. top) of one of the slits on \( S_{i+1} \). Let \( X_g \) denote the resulting degree \( g \) cover of \( \mathbb{C}/(\mathbb{Z} + i\mathbb{Z}) \) when \( g \) is even.

If \( g = 2m + 1 \) is odd, then remove the horizontal saddle connection of \( X_{2m} \) that lies in \( S_m \) and then glue in an additional horizontally slit torus to obtain the torus cover \( X_{2m+1} \). The surfaces \( X_6 \) and \( X_7 \) are described in Figure 6.

![Figure 5. A genus two surface constructed from two slit tori has four horizontal saddle connections.](image-url)
A torus cover $X_g^d$ of degree $d = k + g - 1$ can be constructed in the same way if one replaces a slit copy of $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ in the construction of the genus two surface $S_1$ with a slit copy of $\mathbb{C}/(k\mathbb{Z} + i\mathbb{Z})$. The hyperelliptic involution on $X_g^d$ corresponds to the elliptic involution of each slit torus that fixes the center of each slit. A vertical curve in $S_1$ (resp. horizontal curve in $S_2$) is mapped to the standard vertical (resp. horizontal) generator of $H_1(\mathbb{C}/(\mathbb{Z} + i\mathbb{Z}), \mathbb{Z})$. Hence the covering is primitive.

**Remark 3.1:** A degree $g$, primitive, hyperelliptic torus covering can also be defined in terms of the classical Chebyshev polynomial $P_g$, the unique polynomial satisfying

$$P_g(\cos \theta) = \cos(g \cdot \theta)$$

for each $\theta \in \mathbb{R}$. Given $a \in (0, 1)$ such that $P_g(a) \neq \pm 1$, let $q$ be the unique quadratic differential on the Riemann sphere $\hat{\mathbb{C}}$ with simple poles at $\{\pm 1, \pm a\}$. The set $P_g^{-1}\{+1, -1\}$ consists of all $g - 1$ critical points of degree two together with the two additional points at which $P_g$ is not branched. The map $P_g$ is not branched at any of the $2g$ points in $P_g^{-1}\{+a, -a\}$. It follows that $P_g^*(q)$ has $2 + 2g$ simple poles and one zero of degree $2g - 2$ at $\infty \in \hat{\mathbb{C}}$. Let $(X, \omega)$ and $(\mathbb{C}/\Lambda, dz)$ be the respective canonical double covers of $(\hat{\mathbb{C}}, P_g^*(q))$ and $(\hat{\mathbb{C}}, q)$. The map $P_g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ lifts to a primitive degree $g$ branched cover $\tilde{P}_g : X \to \mathbb{C}/\Lambda$. 

Figure 6. Primitive degree $g$ torus covers in $\Omega M_g(g-1, g-1)^{hyp}$ in the cases $g = 6$ and $g = 7$. Each square corresponds to a slit torus.
so that $\tilde{P}_g^*(dz) = \omega$. It follows that $(X, \omega)$ lies in the hyperelliptic component of $\Omega M(g - 1, g - 1)$.

3.2.2. Non-hyperelliptic components of $\Omega M_g(g - 1, g - 1)$. Recall that if $g = 3$ or $g \geq 4$ and $g$ is even, then there is exactly one non-hyperelliptic component. If $g \geq 5$ and $g$ is odd, then there are exactly two non-hyperelliptic components, one consisting of odd spin parity 1-forms and one consisting of even spin parity 1-forms. We first construct a torus covering that is non-hyperelliptic in each genus and then observe that if $g$ is odd, then its spin parity is odd. Then we separately construct an even spin torus covering for $g$ odd.

For each $g \geq 2$, define a degree $g$ torus cover $X_g$ by cyclically gluing together distinct horizontally slit tori $S_1, \ldots, S_g$. To be more precise, glue the top of the slit on $S_i$ to the bottom of the slit on $S_{i+1}$. The case of $g = 5$ is illustrated in Figure 7.

![Figure 7. A cyclically glued $g$-slit torus cover $X_g$ when $g = 5$.](image)

To prove that the surface $X_g$ is not hyperelliptic, let us assume to the contrary that a hyperelliptic involution $\tau$ exists and derive a contradiction. Let $C$ be the vertical cylinder that contains each of the slits $s_i \subset S_i$. The cylinder $C$ is the only vertical cylinder that has length greater than one, and hence it would be preserved by a hyperelliptic involution $\tau$. Thus, $\tau$ would preserve the union of horizontal saddle connections that belong to $C$, and hence would preserve the complement $A$, that is the disjoint union of the slit tori $S_i$. If $\tau$ were to map one slit torus $S_i$ onto a distinct slit torus $S_j$, then the quotient $X_g/\langle \tau \rangle$ would contain the embedded one-holed torus $S_i \cup S_j/\langle \tau \rangle$, and hence the quotient would not be a sphere. Thus the hyperelliptic involution $\tau$ would have to preserve each $S_i$, and hence would act as an elliptic involution on each $S_i$. It follows that the involution $\tau|_{S_i}$ has a fixed point $x_i \in C$. Hence $C$ contains $g$ fixed points, and since $g \geq 3$, this is the desired contradiction.
When $g$ is odd, then the spin parity of $X_g$ is well-defined, and a straightforward argument shows that the spin parity of $X_g$ is odd. Indeed, choose a homology basis for each slit torus $S_i$ consisting of a vertical and a horizontal curve. The index of each of these curves is zero. Thus, the spin parity of $X_g$ is $\sum_{i=1}^{\frac{g}{2}} 1 \equiv g \mod 2$.

To obtain non-hyperelliptic covers $p: X_d^g \rightarrow T$ of degree $d = g - 1 + k$, one may modify the construction by replacing, for example, $S_1$ with the slit torus obtained by removing a horizontal slit $s$ from the torus $\mathbb{C}/(k\mathbb{Z} + i\mathbb{Z})$. Similar arguments show that $X_d^g$ is not hyperelliptic and has spin parity equal to $g \mod 2$.

It remains to construct, for each odd $g \geq 5$ and each $d \geq g$, a non-hyperelliptic, even spin parity, torus cover in $\Omega M(g-1,g-1)$ of degree $d$. To construct it for degree $d = g$, we will perform a surgery to the surfaces $X_2$ and $X_{g-2}$. More precisely, we begin with the disjoint union of slit tori $S_1, \ldots, S_g$ as above. We construct the surface $X_2$ by cyclically gluing together $S_1$ and $S_2$ as above. We construct the surface $X_{g-2}$ by cyclically gluing the slit tori $S_3, \ldots, S_{g-1}$.

Let $\delta_2 \subset S_2 \subset X_2$ denote the unique horizontal saddle connection that is parallel but disjoint from the horizontal saddle connections associated to the slit $\sigma_2$. Let $\delta_3 \subset S_3 \subset X_{g-3}$ denote the unique horizontal saddle connection that is parallel but disjoint from the horizontal saddle connections associated to the slit $\sigma_3$. Remove $\delta_2$ from $X_2$ and remove $\delta_3$ from $X_{g-2}$. Glue the top (resp. bottom) of $\delta_2$ to the bottom (resp. top) of $\delta_3$. See Figure 8 for the case of $g = 5$. The resulting surface $Y_g$ covers $T$, and using a homology basis like the one illustrated in Figure 8, one finds that the spin parity is $g - 2 + 2 + 3 \equiv g + 3 \mod 2$.

![Figure 8](image.png)

Figure 8. A genus 5 torus covering $Y_g$ in the even spin parity component of $\Omega M(4,4)$. The top of the slit $\delta_2$ and the bottom of the slit $\delta_3$ are labeled with $\alpha$, and the bottom of $\delta_2$ and the top of $\delta_3$ are labeled by $\beta$. 
To obtain torus covers of higher degree one need only, as above, replace one of the slit tori with the slit torus coming from \( \mathbb{C}/(k\mathbb{Z} + i\mathbb{Z}) \). To see that the surface \( Y_g \) is not hyperelliptic, apply the argument used for \( X_g \) to the unique vertical cylinder \( C \) in \( X_{g-2} \) that has circumference greater than 2.

3.3. Surgeries that add zeros and preserve degree. Thus far, we have produced torus coverings in each connected component of both the minimal stratum \( \Omega \mathcal{M}_g(2g - 2) \) and the stratum \( \Omega \mathcal{M}_g(g - 1, g - 1) \). To obtain torus covers in all connected components of all other strata, we will perform certain ‘surgeries’ on the torus covers \( E_d^g \) and \( O_d^g \) in the minimal strata as well as a variant, \( Z_d^g \), of these that will be described in §3.4. Each surgery described here modifies the torus covering by adding zeros, increasing genus, and preserving degree. Each surgery can be performed on a torus cover branched over one point that has at least one vertical cylinder of circumference one and that has sufficiently many vertical cylinders of circumference at least two.

To be precise, let \( p : X \to T \) be a torus covering of degree \( d \) such that there exists a vertical (open) cylinder \( C \subset T \) that does not contain a branch point of \( p \).\(^2\) We will say that the torus covering \( p \) is surgery admissible with respect to \( C \) and \( k \) if the components of \( p^{-1}(C) \) consist of

- at least \( k \) cylinders each having circumferences at least two, and
- at least one nonseparating cylinder whose circumference equals one.

In particular, to be surgery admissible \( p \) must have degree \( d \geq 2k + 1 \).

In §3.3.1, we show how to add a zero of order \( 2k \) to a surgery admissible covering, and in §3.3.2 we show how to add a pair of odd order zeros. Each of these surgeries produces a surgery admissible torus covering. Therefore, we may apply any finite sequence of these surgeries. In §3.3.3, we show how to calculate the change in the spin parity so as to be sure that we can obtain a torus covering in each connected component of a stratum.

According to Theorem 1 in [KoZo03], components consisting of hyperelliptic surfaces only occur in the strata \( \Omega \mathcal{M}(2g - 2) \) and \( \Omega \mathcal{M}(g - 1, g - 1) \). Thus, we will not need to consider the effect of surgeries on hyperellipticity.

\(^2\) The boundary of \( C \) may contain a branch point.
3.3.1. Adding a zero of order $2k$. Let $p : X \to T$ be a surgery admissible torus cover. In this subsection, we describe a ‘surgery’ on this torus covering that yields a surgery admissible torus covering $\overline{p} : \overline{X} \to T$ with the same degree $d$ and an additional zero of order $2k$.

Let $C \subset T$ be the vertical (open) cylinder that does not contain a branch point of $p$. Let $C_0$ be a component of $p^{-1}(C)$ that has circumference 1, and let $C_1, \ldots, C_k$ be components of circumference at least two. Choose a vertical closed geodesic $\sigma \subset C$ and choose $P \in \sigma$. The inverse image $p^{-1}(\sigma)$ consists of disjoint closed geodesics. The set $p^{-1}(\sigma \setminus \{P\})$ consists of $d$ disjoint vertical segments. Choose exactly one segment $\sigma_i$ from each cylinder $C_i$. See Figure 9. Cut along each $\sigma_i$ and glue the left side of $\sigma_i$ to the right side of $\sigma_{i+1}$. Let $\overline{X}$ be the resulting surface.

![Figure 9](image-url)  

Figure 9. Adding a zero of order $2k$. Cut along each $\sigma_i$ and identify the left side of $\sigma_i$ to the right side of $\sigma_j$. The red endpoints are thus all identified with one another and they represent a ramification point of local index $2k + 1$ over $P$.

The covering $p$ determines a surgery admissible torus covering $\overline{p} : \overline{X} \to T$ of degree $d$ that is branched over $0$ and $P$. Moreover, the 1-form $\overline{p}'(dz)$ has an additional zero of order $2k$, and the genus of $\overline{X}$ is $k$ greater than the genus of $X$.

3.3.2. Adding a pair of zeros of odd order. In this subsection we describe a surgery on $p : X \to T$ that adds a zero of order $2k - 1$ and a zero of order $2k' - 1$ where $k' \leq k$.\footnote{The orders of zeros of a holomorphic 1-form $\omega$ on a genus $g$ Riemann surface must sum to $2g - 2$, and so $\omega$ has even number of zeros of odd order. Thus, any surgery that increases}$^3$ We will first assume that $k' = k$ and then show how to modify this surgery when $k' < k$. 

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To add a pair of zeros that have the same order $2k - 1$, choose a horizontal segment $\tau$ that lies in $C$ and has length strictly less than the width of $C$. Let $\tau_0$ be the unique component of $p^{-1}(\tau)$ that lies in $C_0$. For each $i \in \{1, \ldots, k - 1\}$, choose two connected components, $\tau_{2i-1}$ and $\tau_{2i}$, of $p^{-1}(\tau)$ that lie in $C_i$, and choose one component, $\tau_{2k-1}$, of $p^{-1}(\tau)$ that lies in $C_k$. See Figure 10. Cut the surface $X$ along each $\tau_i$. Then identify the top of $\tau_i$ with the bottom of $\tau_{i+1}$. The resulting surface is a degree $d$ torus cover with two new zeros of order $2k-1$.

![Figure 10](image)

Figure 10. The white points are ramification points over one endpoint of $\tau$, and the red points are ramifications over the other endpoint of $\tau$.

We next modify the construction to show how to add one zero of order $2k - 1$ and one zero of order $2k' - 1$ where $k' < k$. Roughly speaking, the surgery is a combination of the surgery that adds two zeros of order $2k' - 1$ and the surgery that adds a zero of order $2(k - k')$. To be precise, let $\sigma$ be a vertical closed geodesic that lies in $C$ and let $\tau$ be a horizontal segment in $C$ that has one endpoint $P$ on $\sigma$. Let $\tau_0$ be the component of $p^{-1}(\tau)$ that lies in $C_0$ and let $\sigma_0$ be the lift of $\sigma$ to $C_0$. For $i \in \{1, \ldots, k' - 1\}$ choose two components, $\tau_{2i-1}$ and $\tau_{2i}$, of $p^{-1}(\tau)$ that lie in $C_i$, choose one component, $\tau_{2k'-1}$, of $p^{-1}(\tau)$ that lies in $C_{k'}$. For $i \in \{k' + 1, \ldots, k\}$ choose one component, $\sigma_i$, of $p^{-1}(\sigma \setminus \{P\})$ that lies in $C_i$. Cut along each $\sigma_i$ and each $\tau_i$, cyclically reglue the $\sigma_i$, and cyclically reglue the $\tau_i$. The new zero that corresponds to the point $P$ has order $2k - 1$ and the new zero that corresponds to the other endpoint, $Q$, of $\tau$ has order $2k' - 1$. See Figure 11 for an example of this construction.

the number of odd order zeros will necessarily increase the number of odd order zeros by an even integer.
3.3.3. Change of parity computations. In this subsection we consider how the spin parity changes when the surgery described in §3.3.1 is applied. In particular, we find that adding a zero of order $2k$ preserves the spin parity if $k$ is even and it changes the spin parity if $k$ is odd. Recall that the spin parity is not defined for 1-forms with zeros of odd order, and hence we will not consider the surgery of §3.3.2.

Let $p : X \to T$ be a surgery admissible torus covering, and let $\overline{p} : \overline{X} \to T$ be the result of applying the surgery of §3.3.1.

**Lemma 3.2:** If $k$ is even, then the spin parity of $p^*(dz)$ equals the spin parity of $\overline{p}^*(dz)$. If $k$ is odd, then the spin parity of $p^*(dz)$ does not equal the spin parity of $\overline{p}^*(dz)$.

**Proof.** Let $C, C_0, C_1, \ldots, C_k, \sigma, \sigma_0, \sigma_1, \ldots, \sigma_k,$ and $P$ be as in §3.3.1.

We first prove the statement in the special case of $k = 1$. Let $b$ be a vertical closed geodesic in $C$ that is disjoint from $\sigma$ and let $b_0$ be the component of $p^{-1}(b)$ that lies in $C_0$. Since $X$ is surgery admissible, the simple closed curve $b_0$ is not null-homologous. Let $a_0$ be a simple closed curve on $X$ so that the geometric intersection number $i(a_0, b_0) = 1$, so that $a_0$ does not intersect a ramification point of $p$, and so that $a_0 \cap C_0$ is a horizontal segment. We further suppose that $a_0$ intersects $\sigma_1$ orthogonally at a point in $p^{-1}(a_0 \cap \sigma)$. Thus, after cutting...
along $\sigma_0$ and $\sigma_1$ and regluing as described in §3.3.1, the closed curve $a_0$ becomes two simple closed curves, $a_0^+$ and $a_0^-$. Let $a_0^+$ be the resulting simple closed curve that intersects $b_0^+ := b_0$ and let $a_0^-$ be the other curve. Let $b_0^-$ be a vertical geodesic in $C_0$ that intersects $a_0^-$. See Figure 12.

**Figure 12.** The first four elements of symplectic basis for the surface that results from adding one zero of order two.

Complete $\{a_0, b_0\}$ to a symplectic basis $\{a_0, b_0, \ldots, a_{g-1}, b_{g-1}\}$ for $H_1(X;\mathbb{Z})$ so that no $a_i$ nor $b_i$ intersects a ramification point or $\sigma_1$ if $i > 0$. Then the collection

$$\{a_0^+, b_0^+, a_0^-, b_0^-, a_1, b_1, \ldots, a_{g-1}, b_{g-1}\}$$

is a symplectic basis for the surface that results from the surgery. The curves $a_i$ and $b_i$ do not change if $i > 0$ and hence their indices do not change. The curves $b_0 = b_0^+$ and $b_0^-$ are geodesics and hence their indices equal zero. The index of $a_0$ equals the sum of the indices of $a_0^+$ and $a_0^-$. It follows that the spin parity ‘increases’ by 1. Hence the claim is proven in the case $k = 1$. 
To prove the claim for \( k > 1 \), we will consider, for \( i \leq k \), the result \( \overline{p}_i : \overline{X}_i \to T \) of adding a zero of order \( 2i \) using \( C_0, C_1, \ldots, C_i \) and curves \( \sigma_0, \ldots, \sigma_i \), and we will consider the result \( \overline{q}_i : \overline{Y}_i \to T \) of adding a zero of order \( 2i \) at \( \overline{X}_{i-1} \). It suffices to show that for each \( i \leq k \), the spin parity of \( \overline{p}_i^* (dz) \) equals the spin parity of \( \overline{q}_i^* (dz) \). Indeed, an inductive argument using the case \( k = 1 \) would then imply the claim.

To prove that the spin parity \( \overline{p}_i^* (dz) \) equals the spin parity of \( \overline{q}_i^* (dz) \), we realize \( \overline{Y}_i \) as an arbitrarily small perturbation of \( \overline{X}_i \). In particular, we choose \( \delta > 0 \) and add a zero of order two to \( \overline{X}_{i-1} \) as follows: Let \( \sigma' \subset C \) be the vertical geodesic to the ‘right’ of \( \sigma \) such that the distance between \( \sigma \) and \( \sigma' \) equals \( \delta \). Let \( \alpha \subset T \) be the horizontal geodesic that intersects \( \sigma \) at \( P \). Let \( P' \) denote the intersection point of \( \alpha \) and \( \sigma' \). Let \( \sigma'_0 \) be the component of \( \{ \sigma' \} \cap C_0 \) that has distance \( \delta \) from \( \sigma_0 \). Let \( R_i \) be the connected component of \( \{ p^{-1}(C \setminus \alpha) \} \cap C_i \) that contains \( \sigma_i \), and let \( \sigma'_i \) be the component of \( \{ p^{-1}(\sigma' \setminus P) \} \) contained in \( R_i \) that has distance \( \delta \) from \( \sigma_i \). Cut along \( \sigma'_0 \) and \( \sigma'_1 \) and glue the left (resp. right) side of \( \sigma'_0 \) to the right (resp. left) side of \( \sigma'_1 \). The resulting torus covering is

\[
\overline{q}_i : \overline{Y}_i \to T.
\]

We next construct a piecewise differentiable homeomorphism \( f : \overline{Y}_i \to \overline{X}_i \) as follows. Define \( f \) to be the identity on the complement of \( C_0 \cup C_i \). The right side of the segment \( \sigma_i \) corresponds to a simple closed curve \( \gamma \subset \overline{X}_i \). Let \( A \) be the annular neighborhood of \( \gamma \) consisting of points of distance at most \( \delta/2 \) from \( \gamma \). Let \( A^+ \) be the connected component of \( A \setminus \gamma \) that lies in \( C_0 \cup C_i \). Define \( f \) so that it maps the annulus \( A' \subset C_0 \) bounded by \( \sigma_0 \) and the right by \( \sigma'_0 \) onto the annulus \( A^+ \). Define \( f \) so that it maps the cylinder in \( C_0 \) that lies to the right of \( \sigma'_0 \) to the part of the cylinder in \( C_0 \) that lies to the right of \( \sigma_0 \) that is exterior to \( A^+ \). Define \( f \) to map the thrice holed sphere \( C_i \setminus \sigma'_i \) to the thrice holed sphere \( C_i \setminus A^+ \). See Figure 13.

The construction of \( f \) can be made to depend continuously on the parameter \( \delta \). By pulling back the 1-form \( \overline{p}^* (dz) \) using the inverse \( f^{-1} \) we obtain a continuous family of 1-forms \( \omega_\delta \) on \( \overline{X}_i \). Each zero of \( \omega_\delta \) defines a simple arc on \( \overline{X}_i \) that is parametrized by \( \delta \). These arcs are disjoint, and hence we may choose a symplectic basis for \( H_1 (\overline{X}_i; \mathbb{Z}) \) that avoids these arcs. It follows that the spin parity of \( \omega_\delta \) is constant in \( \delta \). Thus, for each \( \delta \), the spin parity of \( \overline{p}^* (dz) \) equals the spin parity of \( \overline{p}^* (\delta) \).
3.4. Surgery admissible torus covers with highest order odd. In the next subsection, we will describe an algorithm which produces a degree $d$ torus cover in a prescribed connected component of a stratum that satisfies the hypotheses of Proposition 1.3. The algorithm will be based on the surgeries described above. If the highest order of a zero in the prescribed stratum is even, then we will choose the initial surface to be either the torus covering $E_d^g$ or the torus covering $O_d^g$ (see §3.1) depending on the desired spin parity. In this section we construct a surgery admissible degree $d$ torus covering $q: Z_{m,n}^d \to T$ which will be the starting point of the algorithm when the zero of highest order in the prescribed stratum is odd. The surface $Z_{m,n}^d$ will have genus $m + n$ and the associated 1-form $q^∗(dz)$ will have exactly two zeros, one with order $2m − 1$ and the other with order $2n − 1$. We will assume that $m \geq n$.

We will describe the construction of $Z_{m,n}^d$ in the case where $d = 2m$. See Figure 14 for an example with $m = 3$. The surfaces for $d > 2m$ are obtained by adding additional squares as before. Let $p: O_m \to T$ be the degree $2m − 1$ torus covering described in §3.1.2. Let $S$ denote the disjoint union of $O_m$ and $T$. Define the degree $2m$ torus covering $\tilde{p}: S \to T$ by letting $\tilde{p}(z) = p(z)$ for each $z \in O_m$, and by letting $\tilde{p}(z) = z$ for each $z \in T$. 

Figure 13. The homeomorphism $f$ that maps $\overline{Y}_i$ to $\overline{X}_i$. The map $f$ is the identity on the complement of the cylinders $C_0$ and $C_i$. Each colored region in $C_0 \cup C_i$ is mapped to the region of the same color in $C_0 \cup C_i$. 

Figure 14.
Choose a horizontal segment $\tau$ in $T$ of length $\epsilon < 1$ that has one endpoint at the origin in $T$. The inverse image $\tilde{p}^{-1}(\tau)$ has $2m$ connected components. Let $\tau_{2n}$ denote the unique connected component of $\tilde{p}^{-1}(\tau)$ that lies in $T$, and choose components $\tau_1, \ldots, \tau_{2n-1}$, from among the remaining $2m-1$ components that lie in $O_m$.

We will cut along each $\tau_i$ and reglue, but the choice of gluing must be made with some care to ensure that the resulting 1-form has no more than two zeros. To describe precisely the gluing, we will suppose that for $i < 2n$, the $\tau_i$ are labeled in the order that they appear as one winds clockwise around the zero of degree $2m - 2$ on $O_m$. More precisely, suppose that $\gamma: \mathbb{R}/\mathbb{Z} \to T$ gives the standard clockwise parameterization of the circle of radius $\epsilon/2$ centered at the origin $0 \in T$. Then $\gamma$ lifts via $p$ to a map

$$\tilde{\gamma}: \mathbb{R}/(2m-1)\mathbb{Z} \to O_m,$$

and for each $i \in \{1, \ldots, 2n-1\}$, there exists a unique $t_i \in \mathbb{R}/(2m-1)\mathbb{Z}$ such that

$$\tilde{\gamma}(t_i) \in \tau_i.$$

By relabeling if necessary, we may assume that $i < j$ if and only if $t_i < t_j$.

Cut along each $\tau_i$ and reglue the bottom of $\tau_i$ to the top of $\tau_{i+1}$. Let

$$q: \mathbb{Z}^d_{m,n} \to \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$$

denote the resulting (connected) torus covering of degree $d$. Because $2n$ is even, the point $q^{-1}(0)$ is a zero of order $2m - 1$, and if $Q$ denotes the other endpoint of $\tau$, then $q^{-1}(Q)$ is a single zero of order $2n - 1$. The genus of $\mathbb{Z}^d_{m,n}$ is $m + n$.

### 3.5. AN ALGORITHM AND EXAMPLES

In this section we describe an algorithm for constructing a primitive torus cover of degree $d$ in any desired connected component $K$ of a stratum $\Omega M_g(\alpha)$. We then illustrate the algorithm with some examples.

Otherwise, we suppose that the desired divisor data $\alpha = (\alpha_1, \ldots, \alpha_n)$ satisfies

$$\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n,$$

and if each $\alpha_i$ is even, we define $\theta \in \mathbb{Z}/2\mathbb{Z}$ by

$$\theta := \text{spin} + \frac{\alpha_1}{2} + \frac{\alpha_2}{2} + \cdots + \frac{\alpha_{n-1}}{2} \mod 2$$

where ‘spin’ denotes the desired spin parity.
Figure 14. The degree 6 torus cover $Z_{3,2} \in \Omega M(5, 3)$ obtained from the surface $O_3 \in \Omega M(4)^{\text{odd}}$. The purple points correspond to the zero of order 5, and the white points correspond to the zero of order 3. The labeling of the $\tau_i$ for $i < 2n$ is induced by the simple closed curve $\tilde{\gamma}$ (in orange) on $O_3$ that winds clockwise around the pre-image of 0.

- If each $\alpha_i$ is an even integer, and
  - $n = 1$, then apply one of the constructions in §3.1,
  - $n = 2$ and $\alpha_1 = \alpha_2$, then apply one of the constructions in §3.2,
  - otherwise
    * if $\theta = 0 \mod 2$, then apply the surgery of §3.3.1 to the torus cover $E_g^d$ to add zeros of order $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$,
    * if $\theta = 1 \mod 2$, then apply the surgery of §3.3.1 to the torus cover $O_g^d$ to add zeros of order $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$;
- otherwise (when some $\alpha_i$ is odd)
  - if $\alpha_n$ is even, then apply the surgeries of §3.3.1 and §3.3.2 to the torus cover $O_g^d$ to add zeros of order $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$,
  - if $\alpha_n$ is odd, then some other zero, say $\alpha_j$, is odd. Begin with the torus cover $Z_{m,n}^d$ where $m = (\alpha_n + 1)/2$ and $n = (\alpha_j + 1)/2$, and apply the surgeries of §3.3.1 and §3.3.2 to add zeros of order $\alpha_i$ for $i \neq j$ or $n$. 
3.5.1. Torus covers in $\Omega M_4(1,2,3)$. Suppose that we wish to construct a degree 4 torus cover $p$ so that $p^*(dz)$ has a zero of order 3, a zero of order 2, and a zero of order 1. That is, we have $\alpha_3 = 3$ which is odd, and $\alpha_1 = 1$ is odd as well. Hence we begin with the torus cover $Z_{2,1}^4$ whose construction is described in §3.4. Then we perform the surgery described in §3.3.1 to add a zero of order two. See Figure 15. In more detail, the surface $Z_{2,1}^4$ is obtained by slitting the $L$-shaped surface $O_2$ that lies in $\Omega M_2(2)$ along the segment $\tau_2$, slitting a square torus along a segment $\tau_2$, and then gluing the top (resp. bottom) of $\tau_1$ to the bottom (resp. top) of $\tau_2$. The resulting surface is cut along the segments $\sigma_0$ and $\sigma_1$ and the the left (resp. right) of $\sigma_0$ to the right (resp. left) of $\sigma_1$.

Note that by adjoining $d - 4$ squares to the right—that is, by replacing the unit square torus with the rectangular torus $\mathbb{C}/((d - 3)\mathbb{Z} + i\mathbb{Z})$—one obtains a degree $d > 4$ torus covering in $\Omega M_4(1,2,3)$.

![Figure 15](image)

Figure 15. A degree 4 torus cover in the stratum $\Omega M_4(1,2,3)$. The blue points correspond to a zero of order 3, the white points correspond to a zero of order 2, and the red points correspond to a zero of order 1.

3.5.2. Torus covers in $\Omega M_7^{\text{odd}}(2,4,6)$. We describe the construction of a torus cover of degree 7 that has odd spin parity and has one zero of order 6, one zero of order 4, and one zero of order 2. Since $\theta = 1 + 2 + 1 = 0 \mod 2$, we begin with the degree 7 torus cover $p : E_4^7 \to T$ such that $p^*(dz) \in \Omega M_4(6)$, and we then we use the surgery of §3.3.1 to add zeros of order 2 and 4 while
preserving the degree. See Figure 16. In detail, ‘opposing sides’ of the polygon in Figure 16 are identified with the exception of the sides labeled $\gamma_i$ in which case we identify $\gamma_1$ with $\gamma_3$ and identify $\gamma_2$ with $\gamma_4$. To add the additional zero of order 4 (resp. 2) we cut along $\sigma_0$ and $\sigma_1$ (resp. $\sigma'_0$ and $\sigma'_1$) and reglue.

Figure 16. A degree 7 torus cover in the odd component of $\Omega M_7(2, 4, 6)$. The blue points correspond to the zero of order 6, the red points correspond to the zero of order 4, the white points correspond to the zero of order 2.

By adjoining $d - 7$ additional squares to the right, one obtains a degree $d > 7$ torus covering in $\Omega M_7^{\text{odd}}(2, 4, 6)$.

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