Dynamics of open quantum systems

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Fig. 1: A small quantum system coupled to several infinitely large reservoirs via energy and/or particle exchange. The reservoirs are characterized by temperatures $T_\alpha$ and chemical potentials $\mu_\alpha$.

1 Introduction

An open quantum system consists of a local quantummechanical system of fixed size coupled to infinitely large reservoirs in statistical equilibrium via a well-defined interaction, see Fig. 1 for a sketch of the system. The analysis of the time evolution of the reduced density matrix of the local system is of fundamental importance for nonequilibrium statistical mechanics. Of particular interest is the interplay of quantum coherence in the local quantum system and dissipation generated by the reservoirs, which is the reason why this field is called dissipative quantum mechanics. Many topics are here of current interest: (1) The development of a microscopic theory for irreversible time evolution of the local density matrix; (2) The characterization of the reduced dynamics, in particular by generic features independent of the microscopic details of high-energy processes; (3) The calculation of typical relaxation and decoherence rates; (4) The analysis of quantum fluctuations induced by the system-reservoir interaction beyond perturbation theory; (5) The analysis of the influence of strong correlations in the local quantum system, induced by charging energies; (6) The analysis of the influence of inhomogeneous boundary conditions, induced by different temperatures and/or chemical potentials of several reservoirs, leading to energy, particle, and spin currents; (7) The analysis of deviations of the stationary local density matrix from a grandcanonical one, induced by quantum fluctuations from the system-reservoir coupling or by the presence of several reservoirs; (8) The development of nonequilibrium renormalization group (RG) methods capable of resumming logarithmic divergencies occuring in higher-order perturbation theory in the system-reservoir coupling, either at high energies w.r.t. the band width of the reservoirs or at low energies w.r.t. the inverse time in the long-time limit; (9) The study of non-Markovian dynamics leading to additional terms in the time evolution with unexpected oscillation frequencies and decay rates together with non-exponential time evolution; (10) The crossover between coherent and incoherent dynamics induced by the sign and size of the system-reservoir coupling and other tunable parameters. It is the purpose of this tutorial introduction to present a microscopic theory for the time evolution of open quantum systems, to discuss some of the above aspects from a generic point of view, and to characterize explicitly the reduced dynamics for elementary 2-level quantum systems coupled via energy, particle or spin exchange to external reservoirs.

Although the field of dissipative quantum mechanics has a long history, the field has regained an enormous interest in the last decades due to its relevance in quantum transport phenomena in nanoelectronic systems and quantum information processing, and due to the controlled realization of low-dimensional quantum systems in cold atom gases. To describe the time evolution of the reduced density matrix of the local quantum system microscopically, one starts from the von
Neumann equation for the total density matrix $\rho_{tot}(t)$ of the full system (we set $\hbar = e = k = 1$)

$$i \dot{\rho}_{tot}(t) = [H_{tot}(t), \rho_{tot}(t)] = L_{tot}(t) \rho_{tot}(t),$$

(1)

where $L_{tot}(t)$ is the so-called Liouville operator, a superoperator which acts on an arbitrary operator via $L_{tot}(t)A = [H_{tot}(t), A]$. The central idea is always to integrate out the reservoir degrees of freedom and to set up a formally exact kinetic equation for the local density matrix $\rho(t) = Tr_{res}\rho_{tot}(t)$, defined by the trace $Tr_{res}$ over the reservoir degrees of freedom of the total density matrix. This kinetic equation has the form

$$i \dot{\rho}(t) = \int_{t_0}^{t} dt' L(t, t') \rho(t'),$$

(2)

where $t_0$ is the initial time and $L(t, t')$ is an effective Liouville operator acting only on operators of the local quantum system. This superoperator contains all the information of the reservoir degrees of freedom and the system-reservoir interaction. For a time-translational invariant Hamiltonian, $L(t, t') = L(t - t')$ depends only on the relative time difference. The effective Liouvillian $L(t, t')$ is only defined for $t > t'$, i.e. it acts as a response function relating the density matrix at time $t'$ to the one at the later time $t$. This accounts for memory effects and leads to non-Markovian dynamics. The only assumption needed to derive the kinetic equation (2) is the factorization of the total density matrix at the initial time $t_0$ in an arbitrary local part $\rho(t_0)$ and an equilibrium part for the reservoirs

$$\rho_{tot}(t_0) = \rho(t_0) \rho_{res}^{eq}, \quad \rho_{res}^{eq} = \prod_{\alpha} \rho_{\alpha}^{eq}, \quad \rho_{\alpha}^{eq} = \frac{1}{Z_{\alpha}} e^{-(H_{\alpha} - \mu_{\alpha} N_{\alpha})/T_{\alpha}},$$

(3)

where $T_{\alpha}, \mu_{\alpha}, H_{\alpha}, N_{\alpha},$ and $Z_{\alpha}$ are the temperature, the chemical potential, the Hamiltonian, the particle number, and the partition function of reservoir $\alpha$, respectively. However, by changing the Hamiltonian at a certain quench time $t_q > t_0$ abruptly, other initial conditions can be realized where system and reservoirs are correlated. Various techniques have been developed to calculate the effective Liouvillian $L(t, t')$. The traditional ones are projection operator techniques [1] and functional integrals [2]. Recently, a quantum field theoretical approach has been developed, which allows for a systematic classification of all processes in all orders of perturbation theory in the system-reservoir coupling [3]. With this method, it is possible to go beyond bare perturbation theory which is necessary at low temperatures due to various logarithmic divergencies at high and low energies. The method is capable of identifying these logarithmic divergencies very effectively and an RG method in nonequilibrium has been set up to resum them. This allows a systematic weak-coupling expansion in the renormalized coupling constants to be formulated with which the time evolution on all time scales even when the reservoirs have different chemical potentials or temperatures can be discussed. This technique has been applied successfully to the Kondo model [4, 5, 6], the interacting resonant level model (IRLM) [7, 8, 9], and the ohmic spin boson model [10]. In particular, it has turned out that the RG formulation is most effective for the calculation of the time evolution if the Fourier variable $E$ conjugate to the time $t$ is used as flow parameter, i.e. as the parameter w.r.t. which derivatives of the various quantities of interest are taken to obtain differential equations (the so-called RG equations). This technique is called the E-RTGR method [6, 10]. The models treated so far fall into the special class where the density of states
in the reservoirs and the frequency dependence of the system-reservoir couplings is weak and varies on the scale of the high-energy cutoff $D$. Physically, the high-energy cutoff can either be the band width of the reservoirs or it is some internal high-energy scale of the local quantum system, like e.g. charging energies, arising when effective models are used by integrating out high-energy processes (e.g. quantum dots in the Coulomb blockade regime where charge degrees of freedom can be eliminated, see the lecture B3 by T. Costi). For such models it is often possible to find universal physics where the special form of the high-energy cutoff function is not important and influences only the value of certain low-energy scales (e.g. the Kondo temperature for the Kondo model). In such a case the high-energy cutoff $D$ does no longer occur explicitly. Furthermore, for a wide class of time-translational invariant models it turns out that the effective Liouvillian has the form

$$L(E) = L_{\Delta}(E) + EL'(E),$$

where $L(E) = \int_0^{\infty} dt e^{iEt} L(t)$ is the Fourier-transform of the response function $L(t - t') = L(t, t') \theta(t - t')$. In this decomposition $L_{\Delta}(E)$ and $L'(E)$ are slowly varying logarithmic functions, where $L_{\Delta}(E)$ is proportional to some energy scale $\Delta$ of the model which can be anything except for the Fourier variable $E$. This form will be shown by the RG analysis in Section 5 for the concrete models under consideration but it remains an interesting question for the future how generic this form is. A large part of this tutorial deals with the technical details of calculating the appearing functions $L_{\Delta}$ and $L'$. Before we do that, we will first investigate the physical consequences for the time evolution in Section 3. We will see that when $L(E)$ has the form (4) the time evolution can generically be decomposed as

$$\rho(t) = \sum_n F_n(t) e^{-iz_n t} \rho_{t=0},$$

where $z_n = \pm \Omega_n - i \Gamma_n$, with $\Omega_n, \Gamma_n \geq 0$, determine the oscillation frequencies and decay rates of exponential decay, and $F_n(t)$ are pre-exponential functions, which typically consist of power-laws $\sim 1/t^k$ ($k = 1, 2, \ldots$) and logarithmic corrections in the long-time limit $t \gg 1/|z_n|$. At least one of the exponential scales is zero $z_{st} = 0$, which determines the stationary state.

It is the purpose of the present article to first discuss the generic physics of the time evolution on the basis of the form (4) of the effective Liouvillian, and with this motivation discuss the E-RTRG method for the calculation of $L(E)$ and its decomposition into (4). Then we will summarize the results for the time evolution of the Kondo model, the ohmic spin boson model, and the IRLM. We note that other RG methods have been developed recently to discuss the time evolution of open quantum systems. The most important ones are the flow-equation method [11] and the functional RG [12]. The latter will be introduced in the lecture B7 by V. Meden and is a method where one expands systematically in the short-ranged renormalized interaction parameter present in the local system but not in the system-reservoir coupling, i.e. it is complementary to the RTRG technique where arbitrary local interactions can be treated but an expansion in the renormalized system-reservoir coupling is needed. Besides the analytical RG methods, there is also an extensive research going on to develop numerical methods to describe the time evolution, like e.g. time-dependent numerical renormalization group [13], time-dependent density matrix renormalization group [14], iterative stochastic path integrals [15], and quantum Monte Carlo [16]. Furthermore, for special models, field-theoretical methods have been used to find exact results [17].
2 Basic models

We start with the description of the basic models under consideration, where the quantum system consists of 2 states coupled via spin (Kondo model), charge and potential (IRLM), or energy (spin boson) fluctuations to a noninteracting environment. The total Hamiltonian is assumed to be time-translational invariant and consists of three parts

\[ H_{\text{tot}} = H + H_{\text{res}} + V \], \[ H_{\text{res}} = \sum_{\alpha} H_{\alpha} \], \[ H_{\alpha} = \sum_{k, \sigma} \varepsilon_{\alpha k} a^{\dagger}_{\alpha k} a_{\alpha k} \] (6)

where \( H \) is the Hamiltonian of the local quantum system, \( V \) is the system-reservoir interaction, and \( H_{\text{res}} \) describes the noninteracting (fermionic or bosonic) reservoirs with field operators \( a_{\alpha k} \). \( \alpha \) is the reservoir index, \( \sigma \) the channel index (e.g. spin), and the quantum number \( k \) characterizes the energy. For convenience, for given \( \alpha \) and \( \sigma \), we will denote by \( \omega = \epsilon_{\alpha k} - \mu_\alpha \) the energy of the reservoir states measured relative to the chemical potential, and we assume that the relation between \( \omega \) and \( k \) is unique. As a consequence, the field operators of the reservoirs can be characterized by the multi-index \( 1 \equiv \eta\alpha\sigma\omega \), where \( \eta = \pm \) distinguishes between creation \( (\eta = +) \) and annihilation operators \( (\eta = -) \). Depending on the model under consideration, we will define below convenient forms of the field operators \( a_1 \equiv a_{\eta\alpha\sigma} (\omega) \) in continuum notation, such that the commutation relations read (the upper/lower case refers always to bosons/fermions)

\[ [a_{\alpha\sigma}(\omega), a_{\alpha\sigma}^\dagger(\omega')]_\pm = \delta_{\alpha\alpha'} \delta_{\sigma\sigma'} \delta(\omega - \omega') \rho_{\alpha\sigma}(\omega) \] , (7)

where \([\cdot, \cdot]_\pm\) denotes the commutator/anticommutator for bosons/fermions. As defined below the spectral function \( \rho_{\alpha\sigma}(\omega) \) contains the d.o.s. of the reservoirs and possibly frequency-dependencies of the system-reservoir couplings. Together with the commutation relations

\[ [a_1, H_\alpha] = -\eta(\omega + \mu_\alpha) a_1 \] , \[ [a_1, N_\alpha] = -\eta a_1 \] , (8)

it follows that the contraction of two reservoir field operators w.r.t. the equilibrium distribution is given by

\[ a_1 a'_{1'} = \text{Tr}_{\text{res}} a_1 a'_{1'} \rho_{\text{eq}}^{\text{res}} = \delta_{11'} \rho_{\alpha\sigma}(\omega) f_\alpha(\omega) f_{\alpha}(\eta\omega) = \delta_{11'} \rho_{\alpha\sigma}(\omega) f_\alpha(\eta\omega) \] , (9)

where \( \tilde{\alpha} \equiv -\eta\alpha\sigma\omega \) is defined by reversing the sign of \( \eta \), \( \delta_{12} = \delta_{\eta \eta_2} \delta_{\alpha_1 \alpha_2} \delta(\omega_1 - \omega_2) \), \( f_\alpha^+(\omega) = f_\alpha(\omega) \), \( f_\alpha^-(\omega) = 1 \pm f_\alpha(\omega) \), and \( f_\alpha(\omega) = (e^{\omega/T_\sigma} \mp 1)^{-1} \) is the Bose/Fermi distribution.

In terms of the continuum field operators, the system-reservoir interaction \( V \) is generically written as a sum of terms of the form

\[ V = \frac{1}{n!} \left\{ \frac{1}{\eta_1 \eta_2 \ldots \eta_n} : a_n a_{n-1} \ldots a_1 : g_{12 \ldots n} \rightarrow \frac{1}{n!} g_{12 \ldots n} : a_1 a_2 \ldots a_n : \right\} \] , (10)

where \( n = 1, 2, \ldots \) is any integer, implicit summation/integration is assumed over the multi-indices \( i \equiv \eta_i \alpha_i \sigma_i \omega_i \), the operator \( g_{12 \ldots n} \) acts only on the local system, and \( : \cdots : \) denotes normal-ordering w.r.t. to the equilibrium distribution \( \{3\} \) of the reservoirs (i.e. in any Wick-decomposition contractions are not allowed within the normal-ordered expression). We call the operators \( g_{1, \ldots, n} \) \( n \)-point vertex operators since, together with the corresponding superoperators \( (55) \), they will appear in the diagrammatic technique as vertices with \( n \) reservoir lines, see
Fig. 2: A sketch of the nonequilibrium Kondo model. A local spin is coupled via isotropic exchange couplings \( J_{\alpha\alpha'}^{(0)} \) to the reservoir spins. The two reservoirs are characterized by the same temperature \( T \) but the chemical potentials \( \mu_L \) and \( \mu_R \) can be different defining the voltage \( V = \mu_L - \mu_R \) across the system. The nondiagonal exchange couplings \( J_{LR}^{(0)} = J_{RL}^{(0)} \) describe spin exchange processes where a particle is transferred between the reservoirs, giving rise to a current.

Section 4. For bosons the two forms for \( V \) shown in (10) are the same. For fermions, the first form is needed for the definition of the vertex operators and, for \( n \) odd, \( g_{1...n} \) is of fermionic nature and anticommutes with the reservoir field operators. However, it can be shown \([3]\) that, for the calculation of any local observables, the second form for \( V \) can be used and local and reservoir operators can be taken as if they commute. The vertex operators have the properties

\[
g_{1...j...n} = \pm g_{1...j...n}, \quad g_{1...n}^\dagger = g_{n...1}.
\]

The first relation can always be achieved by (anti-)symmetrization of the reservoir field operators within the normal-ordering in (10), whereas the second one is necessary for the property \( V = V^\dagger \). In the following we will specify the definition of the continuum reservoir field operators \( a_1 \), the spectral density \( \rho_{\alpha\sigma}(\omega) \) and the vertex operators \( g_{1...n} \) for the various models.

The Kondo model. In its most basic form the Kondo model describes a local spin-\( \frac{1}{2} \) system coupled via short-ranged and isotropic exchange couplings to fermionic reservoir spins, see Fig. 2 for a sketch of the system. It is a model system to describe local spin fluctuations. For the case of a single channel the Hamiltonian reads

\[
H = h^{(0)} S_z, \quad V = \sum_{\alpha\alpha'} \frac{J_{\alpha\alpha'}^{(0)}}{\sqrt{\rho_{\alpha}^{(0)} \rho_{\alpha'}^{(0)}}} \sum_{\sigma\sigma'} \frac{1}{2} a_{\alpha\sigma k}^\dagger a_{\alpha'\sigma' k'}^\dagger : S : \ , \quad (12)
\]

where the isotropic exchange couplings \( J_{\alpha\alpha'}^{(0)} = J_{\alpha'\alpha}^{(0)} \) are real and dimensionless, \( \sigma = (\sigma_x, \sigma_y, \sigma_z) \) are the Pauli matrices, \( S \) is the local spin, and \( h^{(0)} \) is the local bare magnetic field. \( \rho_{\alpha}^{(0)} \) denotes the d.o.s. of the reservoirs at the Fermi level. The Kondo model can be derived via a Schrieffer-Wolff transformation from the single-impurity Anderson model (see the lecture B3 by T. Costi), in which case the exchange couplings fulfil the relation

\[
J_{\alpha\alpha'}^{(0)} = 2 \sqrt{x_\alpha x_{\alpha'}} J^{(0)}, \quad \sum_{\alpha} x_\alpha = 1, \quad 0 < x_\alpha < 1, \quad (13)
\]

where \( x_\alpha \) are asymmetry factors weighting the energy broadening of the local level from reservoir \( \alpha \). Defining the continuum field operator by \( a_1 = \frac{1}{\sqrt{\rho_{\alpha}^{(0)}}} \sum_k \delta(\omega - \epsilon_{\alpha k} + \mu_\alpha) a_{\alpha k} \), with
Fig. 3: A sketch of the IRLM. A local level without spin is coupled via tunneling and Coulomb interaction to several reservoirs.

\[ 1 \equiv \eta_{\alpha \sigma \omega}, \text{ and assuming a flat d.o.s. in the reservoirs, we obtain for the spectral function and the vertex operator} \]

\[ \rho_{\alpha \sigma}(\omega) = \rho(\omega) = \frac{1}{\rho_{\alpha}^{(0)}} \sum_{k} \delta(\omega - \epsilon_{\alpha \sigma k} + \mu_{\alpha}) = \frac{D^2}{D^2 + \omega^2}, \tag{14} \]

\[ g_{1 \nu} = \frac{1}{2} \sum_{\alpha \sigma \sigma'} \sum_{\omega} \delta(\omega - \omega_{\alpha \sigma 

where \( D \) is the band width of the reservoirs and, for convenience, we have chosen a Lorentzian for the high-energy cutoff function. The case \( \eta = -\eta' = - \) is obtained from \( g_{1 \nu} = -g_{1 \nu} \).

The IRLM. The IRLM is a basic model to describe charge and potential fluctuations. It consists of a single fermionic level, which is coupled to fermionic reservoirs via tunneling and a local Coulomb interaction, see Fig. 3 for a sketch of the system. Disregarding the spin, the Hamiltonian is defined by

\[ H = \epsilon c^\dagger c, \quad V = \sum_{\alpha} \frac{t_{\alpha}}{\sqrt{\rho_{\alpha}^{(0)}}} \sum_{k} \left( a_{\alpha k}^\dagger c + \text{h.c.} \right) + \sum_{\alpha} \frac{U_{\alpha}}{\rho_{\alpha}^{(0)}} \left( c^\dagger c - \frac{1}{2} \right) \sum_{kk'} : a_{\alpha k}^\dagger a_{\alpha k'} : , \tag{16} \]

where \( c \) is the field operator annihilating a particle on the local system, \( t_{\alpha} \) are the tunneling matrix elements (in units of \( 1/\sqrt{\rho_{\alpha}^{(0)}} \)), \( U_{\alpha} \) denote the dimensionless Coulomb couplings, and \( \epsilon \) is the bare energy of the local level. At \( \epsilon = 0 \) the model fulfills particle-hole symmetry. Defining the continuum field operators as for the Kondo model, with \( 1 \equiv \eta_{\alpha \omega} \) (i.e. omitting the spin index), we find the same result (14) for the spectral function, and the vertex operators are given by

\[ g_{1} = t_{\alpha} \left\{ \begin{array}{l} c \quad \text{for} \quad \eta = + \\ c^\dagger \quad \text{for} \quad \eta = - \end{array} \right., \quad g_{1 \nu} = \delta_{\eta_{\nu} \eta} \delta_{\alpha \alpha'} \eta U_{\alpha} (c^\dagger c - \frac{1}{2}) . \tag{17} \]

The spin boson model. The spin boson model describes energy fluctuations, where a 2-level system is coupled linearly to a phonon bath, see Fig. 4 for a sketch of the system. The Hamiltonian is given by

\[ H_{\text{res}} = \sum_{k} \omega_{k} a_{k}^\dagger a_{k}, \quad H = \frac{1}{2} \epsilon \sigma_{z} - \frac{1}{2} \Delta \sigma_{x}, \quad V = \frac{1}{2} \sigma_{z} \sum_{k} \alpha_{k} (a_{k} + a_{k}^\dagger) , \tag{18} \]

where \( \epsilon \) and \( \Delta \) denote the bias and the tunneling of the local 2-level system, respectively. The phonon frequencies \( \omega_{k} > 0 \) are positive, and the equilibrium phonon distribution is character-
Fig. 4: A sketch of the spin boson model. A 2-level system, characterized by tunneling $\Delta$ and bias $\varepsilon$ is coupled linearly via the dimensionless coupling constant $\alpha$ to a phonon bath of harmonic oscillators.

ized by temperature $T$. The $k$-dependence of the real coupling constants $\alpha_k$ and the phonon frequencies $\omega_k$ is considered by defining the continuum field operators by $a_1 = \sum_k \alpha_k \delta(\omega - \omega_k) a_\eta k$ with $1 \equiv \eta \omega$. This leads to the following spectral function and vertex operator

$$\rho(\omega) = \sum_k \alpha_k^2 \delta(\omega - \omega_k) = 2\alpha \omega \left(\frac{\omega}{D}\right)^{s-1} \theta(\omega) \frac{D^2}{D^2 + \omega^2}, \quad g_1 = \frac{1}{2} \sigma_z,$$  

where $\alpha$ is a dimensionless coupling constant, and we have again chosen a Lorentzian high-energy cutoff function with band width $D$. The special form chosen for $\rho(\omega)$ describes the ohmic case for $s = 1$ considered in this article, whereas $s < 1$ ($s > 1$) define the sub-ohmic (super-ohmic) cases. For the special case $\Delta = 0$ the spin boson model can be solved exactly \[2\] with the result

$$\langle \sigma_{x,y}(t) \rangle = e^{-h(t)} \langle \sigma_{x,y} \rangle_{t=0}, \quad \langle \sigma_z(t) \rangle = \langle \sigma_z \rangle_{t=0},$$  

with $h(t) = -\int d\omega (\rho(\omega)/\omega)(1 - \cos(\omega t))(1 + 2f(\omega))$, where $f(\omega) = (\exp(\omega/T) - 1)^{-1}$ is the Bose function.

For the special case of $\alpha$ close to $\frac{1}{2}$, the ohmic spin boson model can be mapped on the IRLM with a single reservoir (with $\mu = 0$) \[2\]. The parameters $U$ and $t$ of the IRLM are related to $\alpha$ and $\Delta$ of the ohmic spin boson model in the following way

$$U = 1 - \sqrt{2\alpha}, \quad \Gamma^{(0)} = 2\pi t^2 = \frac{\Delta^2}{D}.$$  

The local occupation $\langle n \rangle(t) = \langle c^\dagger c \rangle(t)$ of the IRLM is related to the expectation value $\langle \sigma_z \rangle(t)$ of the ohmic spin boson model via

$$2 \langle n \rangle(t) - 1 = \langle \sigma_z \rangle(t),$$  

whereas the expectation value $\langle \sigma_{x,y} \rangle(t)$ of the spin boson model is related to expectation values of highly nonlinear operators involving reservoir degrees of freedom in the IRLM. The value $\alpha = \frac{1}{2}$ is of special importance since, at this point, the time evolution of $\langle \sigma_z \rangle(t)$ changes from an oscillating one (for $\alpha < \frac{1}{2}$) to a purely decaying one (for $\alpha > \frac{1}{2}$) \[2, 8, 9\]. Correspondingly, for the IRLM, this crossover occurs when the sign of the Coulomb interaction $U$ is changed.
3 Kinetic equation and time evolution

In this section we aim at discussing the time evolution from a generic point of view based on the general form (2) of the kinetic equation and the form (4) of the effective Liouvillian for the case of a time-translational invariant Hamiltonian. Using $L(t, t') = L(t - t')$ the kinetic equation reads

$$i \dot{\rho}(t) = \int_{0}^{t} dt' L(t - t') \rho(t') \quad ,$$

where, for convenience, we have set the initial time $t_0 = 0$. The reduced density matrix $\rho(t)$ acts only in local space, i.e. has matrix elements $\rho(t)_{s,s'} = \langle s|\rho(t)|s'\rangle$, where $s$ and $s'$ are states of the local quantum system. In contrast, the superoperator $L(t)$ acts on local operators $A$, i.e. the matrix elements can be written as $L_{s_1 s_2,s'_1 s'_2} = \langle s_1 s_2|L(t)|s'_1 s'_2\rangle$, where $|ss'\rangle = |s\rangle\langle s'|$ are the basis elements (= operators) in Liouville space and $\langle ss'|A = \langle s|A|s'\rangle$ are the corresponding dual vectors. The density matrix fulfils the property of conservation of probability $\text{Tr} \rho(t) = 1$ and is self-adjoint $\rho(t) = \rho(t)\dagger$. It is straightforward to show that the kinetic equation respects these properties if and only if the effective Liouville operator fulfils the properties

$$\text{Tr} L(t) = \sum_s L(t)_{ss,\cdot} = 0 \quad , \quad L(t)^c = -L(t) \quad ,$$

where the c-transform is defined by $L(t)^c_{s_1 s_2,s'_1 s'_2} = L(t)_{s_2 s_1,s'_2 s'_1}$ and fulfills the useful property $(L(t)A)^\dagger = L(t)^c A\dagger$. In Fourier space $L(E) = \int_{0}^{\infty} dt e^{iEt} L(t)$ this means

$$\text{Tr} L(E) = 0 \quad , \quad L(E)^c = -L(-E^*) \quad ,$$

or for the quantities $L_\Delta(E)$ and $L'(E)$ appearing in the decomposition (4)

$$\text{Tr} L_\Delta(E) = \text{Tr} L'(E) = 0 \quad , \quad L_\Delta(E)^c = -L_\Delta(-E^*) \quad , \quad L'(E)^c = L'(-E^*) \quad .$$

With $\rho(E) = \int_{0}^{\infty} dt e^{iEt} \rho(t)$, the kinetic equation reads in Fourier space $E\rho(E) - i\rho_{t=0} = L(E)\rho(E)$ leading to the formal solution

$$\rho(E) = i R(E) \rho_{t=0} \quad , \quad R(E) = \frac{1}{E - L(E)} \quad .$$

We now investigate the consequences of the generic form (4) of the effective Liouvillian $L(E)$. Using inverse Fourier transform, the time evolution can be calculated for $t > 0$ from

$$\rho(t) = \frac{i}{2\pi} \int_{-\infty+i0^+}^{\infty+i0^+} dE e^{-iEt} R(E) \rho_{t=0} = \frac{i}{2\pi} \int_{-\infty+i0^+}^{\infty+i0^+} dE e^{-iEt} \tilde{R}(E) Z'(E) \rho_{t=0} \quad ,$$

where we have defined

$$\tilde{R}(E) = \frac{1}{E - L_\Delta(E)} \quad , \quad \tilde{L}_\Delta(E) = Z'(E) L_\Delta(E) \quad , \quad Z'(E) = \frac{1}{1 - L'(E)} \quad .$$

By convention, $Z'(E)$ is called the $Z'$-factor operator. The last form of (28) is very helpful for the evaluation of the energy integral because it explicitly exhibits the slowly varying logarithmic
functions $\tilde{L}_\Delta(E)$ and $Z'(E)$. The energy integral $\int dE$ is calculated by closing the integration contour in the lower half of the complex plane and deforming the contour such that the poles and branch cuts of the integrand are enclosed, see Fig. 5. To identify the singularities of the integrand we use the spectral decomposition of the Liouvillian $\tilde{L}_\Delta(E)$ in terms of its eigenvalues $\lambda_k(E)$ and corresponding projectors $P_k(E)$

$$\tilde{L}_\Delta(E) = \sum_k \lambda_k(E) P_k(E).$$

(30)

Since we deal with a non-hermitian superoperator, we have to distinguish the left and right eigenvectors, which we denote in Dirac notation by $|x_k(E)\rangle$ and $\langle \bar{x}_k(E)|$,

$$\tilde{L}_\Delta(E) |x_k(E)\rangle = \lambda_k(E) |x_k(E)\rangle, \quad \langle \bar{x}_k(E)| \tilde{L}_\Delta(E) = \langle \bar{x}_k(E)| \lambda_k(E).$$

(31)

The eigenvectors fulfil the orthonormalization condition $\langle \bar{x}_k(E)|x_{k'}(E)\rangle = \delta_{kk'}$ and the projectors are given by $P_k(E) = |x_k(E)\rangle\langle \bar{x}_k(E)|$ with $\sum_k P_k(E) = 1$.

Due to the condition Tr$\tilde{L}_\Delta(E) = 0$, we obtain either $\lambda_k(E) = 0$ or Tr$|x_k(E)\rangle = 0$. Therefore, the Liouvillian has always an eigenvalue zero, which we characterize by the index $k = \text{st}$ since it corresponds to the stationary state (see below). The other eigenvalues are numerated by $k = 0, \pm 1, \pm 2, \ldots$. Normalizing the eigenvector with $k = \text{st}$ according to Tr$|x_{\text{st}}(E)\rangle = 1$ and using $\langle \bar{x}_{\text{st}}(E)| = \text{Tr}$, we get

$$\text{Tr} |x_{\text{st}}(E)\rangle = \sum_s \langle ss |x_{\text{st}}(E)\rangle = 1, \quad \langle \bar{x}_{\text{st}}(E)|ss\rangle = 1$$

(32)

$$\text{Tr} |x_k(E)\rangle = \sum_s \langle ss |x_k(E)\rangle = 0, \quad \text{for} \quad k = 0, \pm 1, \pm 2, \ldots$$

(33)

As a consequence we get

$$P_{\text{st}}(E) = |x_{\text{st}}(E)\rangle \text{Tr},$$

(34)
and the property $\text{Tr}L_\Delta(E) = \text{Tr}L'(E) = 0$ can also be written as

$$P_\alpha(E) Z'(E) = P_\alpha(E) , \quad P_\alpha(E) L_\Delta(E) = 0 . \quad (35)$$

Due to the condition $\tilde{L}_\Delta(E)^c = -\tilde{L}_\Delta(-E^*)$, the eigenvalues and projectors occur always in pairs (except for $k = 0$, st where we define $k \equiv -k$) with

$$\lambda_{-k}(E) = -\lambda_k(-E^*), \quad P_{-k}(E) = P_k(-E^*)^c . \quad (36)$$

Using the spectral representation, the time evolution can be written as

$$\rho(t) = \frac{i}{2\pi} \sum_k \int_\gamma dE e^{-iEt} \frac{1}{E - \lambda_k(E)} P_k(E) Z'(E) \rho_{t=0} , \quad (37)$$

where $\gamma$ is an integration contour which encloses the lower half of the complex plane including the real axis. Poles are located at $E = z_k^p = \lambda_k(z_k^p) = \pm \Omega_k - i\Gamma_k$, with $\Omega_k, \Gamma_k \geq 0$, where $z_k^p = 0$ is a pole at the origin, see Fig.5. At zero temperature, which we consider from now on, additional nonanalytic features occur from branch cuts since $\lambda_k(E), P_k(E)$ and $Z'(E)$ depend logarithmically via terms $\sim \ln(\frac{D}{E-z_n})$ generated by the ultraviolet divergencies from the high-energy cutoff $D$ (at finite temperature the branch cuts turn into an infinite number of discrete poles separated by $2\pi T$). From the structure of the perturbation theory (see below) it can be seen that the singularities $z_n$ are associated with poles of the resolvents $\tilde{R}(E_{1...n})$, where

$$E_{1...n} = E + \bar{\mu}_{1...n}, \quad \bar{\mu}_{1...n} = \bar{\mu}_1 + \cdots + \bar{\mu}_n, \quad \bar{\mu}_1 = \eta_1 \mu_{\alpha_1} , \quad (38)$$

i.e. are located at $z_n = E$ with $E_{1...n} = E + \bar{\mu}_{1...n} = z_k^p$. Therefore, the singularities $z_n = z_k^p - \bar{\mu}_{1...n}$ are generically given by the poles shifted by some linear combination of the chemical potentials of the reservoirs.

In Section5 we will see how $\tilde{L}_\Delta(E)$ and $Z'(E)$ can be determined from differential equations, see Eq. (111), where we differentiate w.r.t the Fourier variable $E$. These differential equations are defined in the whole complex plane and will be the RG equations of the E-RTRG method. $E$ is called the flow parameter and a solution of the RG equations along a certain path is called the RG flow. The particular advantage is that these RG equations can be solved along the paths $E = z_n + i\Lambda \pm O^+$, with $\Lambda$ real, starting at some high value $\Lambda \sim D$ down to $\Lambda = -\infty$. Since no singularities are present on these paths, it can even be numerically enforced that the branch cuts start at $z_n$ and point into the direction of the negative imaginary axis. Furthermore, the jump of the Liouvillian at the branch cuts can be determined from the difference of the two solutions and the integrals around the branch cuts can be calculated. The choice that the branch cuts point into the direction of the negative imaginary axis is very convenient since $e^{-iEt} = e^{-iz_n t} e^{-xt}$ is exponentially decaying in $xt$, which allows an analytical discussion of the long-time limit (see below). Using $E = z_n - ix \pm O^+$, the integration around a particular branch cut (including the case when the branching point is a pole) gives the contribution $\rho_n(t) = F_n(t) e^{-iz_n t} \rho_{t=0}$ to the time evolution with

$$F_n(t) = \frac{1}{2\pi} \int_0^\infty dx e^{-xt} \left\{ R(z_n - ix + 0^+) - R(z_n - ix - 0^+) \right\} \quad (39)$$
such that the total time evolution can be written in the form \((5)\)

\[
\rho(t) = \sum_n \rho_n(t) = \sum_n F_n(t) e^{-iz_n t} \rho_{t=0} \quad .
\]

(40)

For the further evaluation of \(F_n(t)\) it is important to distinguish between the cases when the branching point is a pole or not. We label the contributions from branching poles \(z_k^p\) by \(F_k^p(t)\) and \(\rho_k^p(t)\) and the others by \(F_k^b(t)\) and \(\rho_k^b(t)\), such that \((40)\) reads

\[
\rho(t) = \sum_k \rho_k^p(t) + \sum_n \rho_n^b(t)
\]

\[
= \sum_k F_k^p(t) e^{-iz_k^p t} \rho_{t=0} + \sum_n F_n^b(t) e^{-iz_n^b t} \rho_{t=0} \quad .
\]

(41)

Thereby we note that the same singularity \(z_k^p = z_k^b\) can appear as a branching pole and as a branching point, since a certain term involving \(\lambda_k(E)\) in \((37)\) can have a branch cut at \(z_n = z_k^p\)

with \(k' \neq k\). Generically, for weakly coupled system-reservoir systems, the contributions \(\rho_k^b(t)\) are smaller since they are proportional to the system-reservoir coupling (see below Eq. \((38)\)). However, if the decay rates occuring in \(z_n\) are smaller than those ones of \(z_k^p\), the relative order of the various terms can change as function of time, as discussed e.g. in detail in Refs. \([8, 9]\) for the IRLM with positive Coulomb interaction or the ohmic spin boson model for \(\alpha\) close but slightly below the value \(\alpha = \frac{1}{2}\). In the Markovian approximation, only the contributions \(\rho_k^p(t)\) remain and the pre-exponential functions are approximated by constants of \(O(1)\).

**Time-evolution regimes.** Using the general expressions \((28)\) and \((37)\), one can discuss the qualitative form of the time evolution in different time regimes. For short times \(t \ll 1/|z_n|\), only high frequencies \(E \sim 1/t \gg |z_n|\) matter in Eq. \((28)\), i.e. the cutoff scales \(z_n\) in the logarithmic terms are unimportant and can be neglected. Furthermore, to leading order, we can replace \(E \rightarrow 1/t\) in the logarithmic parts, and we obtain from \((28)\)

\[
\rho(t) = \frac{i}{2\pi} \int dE e^{-iEt} \frac{1}{E - L_\Delta(1/t) Z'(1/t)} Z'(1/t) \rho_{t=0} = e^{-iL_\Delta(1/t)t} Z'(1/t) \rho_{t=0} \quad .
\]

(42)

Expanding the exponential one finds in leading order that the logarithmic dependence of \(Z'(1/t)\) and \(L_\Delta(1/t)\) at high energies determine the short time behavior. This means that the RG equations are cut off at the large energy scale \(E = 1/t\), which is the poor man scaling regime, where all the cutoff scales \(z_n\) are unimportant. In this regime the time evolution is determined by the scaling of \(Z'(1/t)\) and \(L_\Delta(1/t)\). If, in addition, \(t \gg 1/D\), where \(D\) is the high-energy cutoff, one obtains universal time evolution in the short-time regime. It means that all leading logarithmic divergencies \(\sim (\alpha \ln(Dt))^n\) have been resummed in the functions \(Z'(1/t)\) and \(L_\Delta(1/t)\), where \(\alpha \ll 1\) is some small dimensionless coupling parameter. Based on this unified picture the universal short-time behaviour has been derived in Refs. \([5, 9, 10]\) for the Kondo model, the IRLM, and the ohmic spin boson model, in accordance with similar results of previous literature.

For intermediate and long times \(t \gtrsim 1/|z_n|\), we have to study the contributions from the poles and branch cuts in detail, based on the decomposition \((41)\). We start with the contributions from the branch cuts starting at a pole \(z_k^p\), which we evaluate by using the form \((37)\). For the branch cut integral we set \(E = z_k^p - ix \pm 0^+\) and replace in leading order \(\lambda_k(E) \rightarrow z_k^p\) and the
logarithmic function $P_k(E)Z'(E)$ by its average $\bar{P}_k(z^p_k - ix)Z'(z^p_k - ix)$ over the branch cut, where $\bar{A}(E) = \frac{1}{2}(A(E + 0^+) + A(E - 0^-))$. Furthermore, in leading order, we can use $x \to 1/t$ in the logarithmic functions. This gives the result

$$F^p_k(t) \approx \frac{1}{2\pi} \int_0^\infty dx \, e^{-xt} \left( \frac{1}{-ix + 0^+} - \frac{1}{-ix - 0^+} \right) \bar{P}_k(z^p_k - i/t) Z'(z^p_k - i/t) .$$  \hspace{1cm} (43)

Using $\frac{1}{1 + x} - \frac{1}{1 - x} = 2\pi \delta(x)$, we obtain the following contribution to the total time evolution

$$\rho^p_k(t) \approx \bar{P}_k(z^p_k - i/t) Z'(z^p_k - i/t) e^{-iz^p_k t} \rho_{st=0} ,$$  \hspace{1cm} (44)

i.e., for $z^p_k = \pm \Omega_k - i\Gamma_k$, an exponential one with oscillation $\Omega_k$ and decay rate $\Gamma_k$, modulated by a logarithmic function. For the special term $k = st$, where $z^p_{st} = 0$, $P_{st}(E) = |x_{st}(E)| \text{Tr}$ and $P_{st}(E)Z'(E) = P_{st}(E)$, we get the following contribution to the time evolution

$$\rho^p_{st}(t) \approx |x_{st}(-i/t)| \lim_{t \to \infty} \rho_{st} = |x_{st}(0)| ,$$  \hspace{1cm} (45)

i.e. we see that for $t \to \infty$ one always gets the stationary distribution $\rho_{st}$ but, if $z^p_{st}$ is a branching pole, logarithmic corrections can occur for the time evolution which do not decay exponentially.

We note that for the models discussed here, there is no logarithmic term in the diagrammatic series involving the pole $z^p_{st}$. In addition, there is no accidental pole $z^p_{k\neq st} = 0$, and therefore the pole at $E = 0$ is isolated and has no attached branch cuts.

The evaluation of a branch cut starting at a branching point $z^b_n$ which is not a pole is more subtle since both $\lambda_k(E)$ and $P_k(E)Z'(E)$ can be discontinuous and cancellations can occur between the two contributions. Therefore, it is more convenient to start from the first expression of (28) involving the resolvent $R(E)$. Denoting by $\delta A = A_+ - A_-$ the jump across the branch and by $\bar{A} = \frac{1}{2}(A_+ + A_-)$ the average value, with $A_\pm = A(E \pm 0^+) = \bar{A} \pm \frac{1}{2}\delta A$, one finds for the jump of the resolvent expanding in the small quantity $\delta L$ (leading to higher orders in the renormalized coupling constants)

$$\delta R(E) = R_+ \delta L R_- = \frac{1}{E - L} \delta L \frac{1}{E - L} + O(\delta L^2) .$$  \hspace{1cm} (46)

Using $\bar{A} \bar{B} - \bar{A} \bar{B} = \frac{1}{4} \delta A \delta B$, we get

$$\frac{1}{E - L} = \frac{1}{E - L} + O(\delta L^2) = \sum_k \frac{1}{E - \lambda_k} P_k Z' + O(\delta L^2) = \sum_k \frac{1}{E - \lambda_k} \tilde{P}_k \tilde{Z}' + O(\delta L^2)$$

Inserting this in (46), neglecting $O(\delta L^2)$, and approximating $E = z^b_n - ix \to z^b_n - i/t$ in the logarithmic functions $\lambda_k$, $\tilde{P}_k$ and $\tilde{Z}'$, we get the following result for the branch cut integral

$$F^n_k(t) \approx \frac{1}{2\pi} \sum_{z^b_n, z^p_k \neq z^b_n} \int_0^\infty dx \, e^{-xt} \frac{1}{z^b_n - ix - \lambda^n_k} \bar{P}_k^n \tilde{Z}'^m \delta L(z^b_n - ix) \frac{1}{z^b_n - ix - \lambda^n_{k'}} \bar{P}_k^n \tilde{Z}'^m ,$$

where $\lambda^n_k = \bar{\lambda}(z^b_n - i/t)$, $\bar{P}_k^n = \bar{P}_k(z^b_n - i/t)$ and $\tilde{Z}'^m = \tilde{Z}'(z^b_n - i/t)$. We have omitted the cases $z^p_k = z^b_n$ or $z^p_{k'} = z^b_n$ since we consider a branching point and not a branching pole. Since $\lambda^n_k \approx z^p_k$, we can neglect $x$ in the denominators of the resolvents for times $t \sim 1/x \gg$
$$1/|z_n^b - z_{k,k'}^p|.$$ In this case, the long-time scaling is determined by the scaling of $\delta L(z_n^b - ix)\sim 1/\ln(1/\Lambda)$ for small $x$. Besides additional logarithmic corrections (which again can be treated by replacing $x \to 1/t$), we will show in Section [5] that

$$\delta L(z_n^b - ix) \sim \theta(x)$$

for models with charge fluctuations (like the IRLM) and

$$\delta L(z_n^b - ix) \sim x\theta(x)$$

for models with spin/orbital or energy fluctuations (like the Kondo and the ohmic spin boson model), see Eq. (120). Therefore, if $x$ can be neglected in the resolvents of the integrand of (48), we obtain (up to logarithmic corrections) $\rho_n^b(t) \sim 1/t$ for charge fluctuations and $\delta L_{\Lambda}(t) \sim 1/t^2$ for spin/orbital and energy fluctuations. For special resonant cases, where $z_n^b$ comes close to $z_{k,k'}^p$, one can also define time regimes $1/|z_n^b| \lesssim t \ll 1/|z_n^b - z_{k,k'}^p|$, where $x$ dominates in the denominators for certain values of $k$ or $k'$, leading to different scaling. If $x$ is not neglected in (88), the integral can also be calculated exactly, leading typically to exponential integrals from which the whole crossover behaviour from intermediate $t \sim 1/|z_n|$ to long times $t \gg 1/|z_n|$ can be calculated.

In the regime of intermediate to long times the cutoff scales $z_n$ are very important. Each term of the series (41) has to be treated separately, leading to different scaling of the individual terms (in contrast to the short-time regime, where all exponentials can be approximated by one and only the sum of all pre-exponential functions matters). As we have seen above, various functions $K(z_n - i/t)$ with logarithmic scaling occur in the projectors, the $Z$-factors, and the jump of the Liouvillian. In bare perturbation theory, the logarithmic functions $K(E)$ will contain powers of terms $\sim \alpha \ln \frac{D}{E - z_n}$. To get rid of the high-energy cutoff $D$, a standard technique is to resum first all leading logarithmic divergencies $\sim (\alpha \ln \frac{D}{E - z_n})^n$, where $\Lambda \gg |z_n|$ is some maximal physical low energy scale. Technically, this can be achieved by cutting off the RG flow at $\Lambda$, defining renormalized coupling constants $\alpha_c$ at this point, and expanding the full solution for $|E| \lesssim |z_n|$ in $\alpha_c$. This is possible if $\alpha_c$ is small, i.e. if $\Lambda$ is much larger than the strong coupling scale $\Lambda^*$, where the coupling constants become of $O(1)$. As a result, $K(E)$ will contain powers of logarithmic terms $\sim \alpha_c \ln \frac{\Lambda}{E - z_n}$. For $E = z_n - i/t$ the most dangerous case is $n = m$, leading to powers in the time-dependent parameter $\alpha_t \sim \alpha_c \ln \alpha_c t$. Since $\alpha_c \ll 1$, this parameter is small $\alpha_t \ll 1$, unless time is exponentially large. Therefore, it can be treated perturbatively, leading to logarithmic corrections $\sim \alpha_t$ in the pre-exponential functions. This strategy has been used in Refs. [5, 7, 8, 9, 10] to determine the time evolution at intermediate and long times (but not exponentially large times) for the Kondo model, the IRLM, and the spin boson model.

Finally, the most complicated time regime is the one at exponentially large times, where $\alpha_t \sim O(1)$. In this regime, a perturbative treatment is no longer possible and all powers of $\alpha_t$ are important. These logarithmic divergencies at low energies are independent of those at large energies and can even arise if there is no logarithmic divergence at high energies. Their occurrence is related to the fact that, concerning the time evolution, the final cutoff scale at low energies is set by inverse time $1/t$ and not by decay rates. The latter holds only for the calculation of stationary properties, see Refs. [3, 4, 18]. The E-RTRG method is unique in the sense that it is also capable of resumming the logarithmic divergencies at low energies, provided the renormalized coupling constants remain small when $E$ approaches one of the singularities.
z_n. Recently, this has been achieved in a controlled way for the ohmic spin boson model [10], where deviations from previously predicted scaling behaviour have been found. Results for the Kondo model and the IRLM are still under investigation in this regime. In particular for the Kondo model, the problem is that the renormalized coupling constants become of \(O(1)\) when approaching one of the singularities although they might be small for the calculation of stationary quantities at \(E = 0\). Thus, weak-coupling problems for stationary quantities can turn into strong-coupling ones for the calculation of the long-time behaviour at exponentially large times.

4 Diagrammatic expansion

**Effective Liouvillian.** In this section we will derive a quantum field theoretical diagrammatic representation of the effective Liouvillian by expanding in the system-reservoir interaction \(V\), following Refs. [3, 19]. Although this can be done for the general case of an explicitly time-dependent Hamiltonian [9], here we will restrict ourselves to the more simpler case of a time-translational invariant Hamiltonian. To find a diagrammatic expansion of the effective Liouvillian \(L(E)\) in Fourier space, we try to bring the local density matrix \(\rho(E)\) into the form (27). We start from the formal solution of the von Neumann equation (1) for the total density matrix, use the initial condition (3), and obtain by expanding in the system-reservoir interaction

\[
\rho(E) = \int_0^\infty dt e^{iEt} Tr_{res} \rho_{tot}(t) = \int_0^\infty dt e^{iEt} Tr_{res} e^{-iL_{tot}t} \rho_{tot}(t = 0)
\]

\[
= Tr_{res} \frac{i}{E - L_{tot}} \rho_{tot}(t = 0) = Tr_{res} \frac{i}{E - L(0)} - L_{res} - L_V \rho_{tot}(t = 0) \rho_{eq}^{res}
\]

\[
= i Tr_{res} C(E - L_{res}) \sum_{k=0}^\infty (L_V C(E - L_{res}))^k \rho_{tot}(t = 0) \rho_{eq}^{res},
\]

where we have defined

\[
C(E) = \frac{1}{E - L(0)}, \quad L(0) = [H, \cdot], \quad L_{res} = [H_{res}, \cdot], \quad L_V = [V, \cdot].
\]

Using the form (10) of the system-reservoir interaction, a similar form can be derived for the Liouville superoperator \(L_V\)

\[
L_V = \frac{1}{n!} \sum_{p=\pm} \sigma_p^{res} A_1^p \ldots A_n^p.
\]

Here, \(p = \pm\) is the so-called Keldysh index, which indicates whether the interaction \(V\) arises from the first or the second part of the commutator \(L_V b = V b - b V\) (\(b\) is an arbitrary operator). \(A_1^p\) are reservoir field superoperators in Liouville space defined by

\[
A_1^p b = \sigma_p^{res} \begin{cases} a_1 b & \text{for } p = + \\ b a_1 & \text{for } p = - \end{cases},
\]

and \(G_{1\ldots n}^{(0)p\ldots p}\) is a superoperator acting in Liouville space of the local quantum system defined by

\[
G_{1\ldots n}^{(0)p\ldots p} b = \begin{cases} 1 & \text{for } n \text{ even} \\ \sigma^p & \text{for } n \text{ odd} \end{cases} \begin{cases} g_{1\ldots n} b & \text{for } p = + \\ -b g_{1\ldots n} & \text{for } p = - \end{cases}.
\]
\(\sigma^p\) and \(\sigma^p_{res}\) are convenient sign superoperators which account for fermionic signs and measure the parity of the fermionic particle number difference \(N_s - N_{s'}\) of intermediate states \(|ss'\rangle\) in Liouville space via the definition (\(N_s\) denotes the particle number of state \(s\) and \(\pm\) refers to bosons/fermions)

\[
\sigma^+ = 1 , \quad \sigma^-_{s_1s_2s'_1s'_2} = \delta_{s_1s'_1} \delta_{s_2s'_2} (\pm)^{N_s - N_{s'}} ,
\]

and a corresponding definition for \(\sigma^p_{res}\) by replacing local states \(s\) by reservoir states. Since the total parity (local system plus reservoirs) of all intermediate states must be even in Liouville space for fermions (note that it is impossible to prepare a nondiagonal matrix element of the total density matrix where the total fermionic particle number difference is odd, see Refs. [19, 20] for a detailed discussion and the consequences of this point), we obtain the important property

\[
\sigma^p \sigma^p_{res} = 1 .
\]

From the definition of the reservoir field superoperators one can straightforwardly derive how the product \(A^p_1 \ldots A^p_n\) occuring in Eq. (55) acts in Liouville space

\[
: A^p_1 \ldots A^p_n : b = \left\{ \begin{array}{l} 1 \quad \text{for } n \text{ even} \\
 \sigma^p_{res} \quad \text{for } n \text{ odd} \end{array} \right\} \left\{ \begin{array}{l} : a_1 \ldots a_n : b \quad \text{for } p = + \\
 b : a_1 \ldots a_n : \quad \text{for } p = - \end{array} \right\} ,
\]

i.e. similar to \(G^{(0)p\ldots p}_{1\ldots n}\) but a minus sign is missing for \(p = -\). Taking this equation together with (55) and using the property (57), one can easily prove the representation (53) for \(L_V\).

Most importantly, the reservoir field superoperators are defined such that the usual Wick theorem can be applied (see Ref. [19] for an elegant proof), i.e. the average \(\text{Tr}_{res} A^p_1 \ldots A^p_n \rho_{res}\) decomposes into a product of pair contractions and the sum has to be taken over all combinations, with the usual definition of a fermionic sign to disentangle the various contractions. Using (9) a single contraction is given by the expression

\[
\eta^{p_1' \ldots p_n'}_{11'} = \overline{A^p_1 A^p_n} = \text{Tr}_{res} A^p_1 A^p_n \rho_{eq}^{\delta_1} = \delta_{11'} \left\{ \begin{array}{l} 1 \\
 p' \end{array} \right\} \rho_{\alpha\sigma}(\omega) f^{\eta\sigma}_\alpha(\omega) = \delta_{11'} \left\{ \begin{array}{l} 1 \\
 \eta \end{array} \right\} \rho_{\alpha\sigma}(\omega) f_\alpha(p'\eta\omega) .
\]

Using the form (53) in (51) one can shift all reservoir field superoperators \(A^p_n\) to the right by using the analog of the commutation relation (7) in Liouville space

\[
A^p_1 L_{res} = (L_{res} - \eta(\omega + \mu_\alpha)) A^p_1 .
\]

This means that by shifting a certain field superoperator \(A^p_1\) through all resolvents to the right, we shift all reservoir Liouville operators \(L_{res}\) standing right to \(A^p_1\) by \(-\eta(\omega + \mu_\alpha)\), where \(1 \equiv \eta\alpha\omega\). We note that, with the second form (10) of the interaction, there is no fermionic sign when commuting local and reservoir operators. Shifting all reservoir field superoperators to the right and using the notation

\[
X_{1\ldots n} = \tilde{\omega}_{1\ldots n} + \tilde{\mu}_{1\ldots n} , \quad \tilde{\omega}_{1\ldots n} = \omega_1 + \ldots \omega_n , \quad \tilde{\omega}_1 = \eta_1 \omega_1 , \\
\tilde{\mu}_{1\ldots n} = \mu_1 + \ldots \mu_n , \quad \tilde{\mu}_1 = \eta_1 \mu_1 ,
\]

\[
A^p_1 L_{res} = (L_{res} - \eta(\omega + \mu_\alpha)) A^p_1 .
\]

This means that by shifting a certain field superoperator \(A^p_1\) through all resolvents to the right, we shift all reservoir Liouville operators \(L_{res}\) standing right to \(A^p_1\) by \(-\eta(\omega + \mu_\alpha)\), where \(1 \equiv \eta\alpha\omega\). We note that, with the second form (10) of the interaction, there is no fermionic sign when commuting local and reservoir operators. Shifting all reservoir field superoperators to the right and using the notation

\[
X_{1\ldots n} = \tilde{\omega}_{1\ldots n} + \tilde{\mu}_{1\ldots n} , \quad \tilde{\omega}_{1\ldots n} = \omega_1 + \ldots \omega_n , \quad \tilde{\omega}_1 = \eta_1 \omega_1 , \\
\tilde{\mu}_{1\ldots n} = \mu_1 + \ldots \mu_n , \quad \tilde{\mu}_1 = \eta_1 \mu_1 ,
\]
we obtain for (51) the form

\[
\rho(E) = i \sum_{k=0}^{\infty} \text{Tr}_{\text{res}} \left( R^{(0)}(E - L_{\text{res}}) \left( \frac{1}{n!} G^{(0)} \right) R^{(0)}(E + X_{M_1} - L_{\text{res}}) \right) \\
\times \left( \frac{1}{n!} G^{(0)} \right) R^{(0)}(E + X_{M_2} - L_{\text{res}}) \ldots \left( \frac{1}{n!} G^{(0)} \right) R^{(0)}(E + X_{M_k} - L_{\text{res}}) \rho_{t=0} \\
\times \left( : A \ldots A : \right) \left( A \ldots A \right) \rho_{\text{res}}^{(eq)}
\]

\[
= i \sum_{k=0}^{\infty} R^{(0)}(E) \left( \frac{1}{n!} G^{(0)} \right) R^{(0)}(E + X_{M_1}) \ldots \left( \frac{1}{n!} G^{(0)} \right) R^{(0)}(E + X_{M_k}) \rho_{t=0} \\
\times \text{Tr}_{\text{res}} \left\{ \left( : A \ldots A : \right) \ldots \left( A \ldots A \right) \right\} \rho_{\text{res}}^{(eq)},
\] (62)

where we have used \( \text{Tr}_{\text{res}} L_{\text{res}} = 0 \) in the last step. Thereby, the set \( M_i \) includes those indices of reservoir field superoperators which were standing left to the corresponding resolvent in the original expression. As a result the local and reservoir degrees of freedom have been decoupled and the trace over the reservoir degrees of freedom can be performed by the application of Wick’s theorem in Liouville space. Since all diagrams give the same contribution when the indices of a particular vertex \( G^{(0)}_{i_1 \ldots i_n} \) are permuted, the factor \( \frac{1}{m!} \) is cancelled, except for the case when two vertices are connected by \( m \) contractions, leaving a symmetry factor \( \frac{1}{m!} \). This leads to a sum of diagrams which symbolically are translated by the rule

\[
\rho(E) \to i \left( \prod \gamma \right) \left( \sum_{N_p} \frac{(\pm)^{N_p}}{S} \right) R^{(0)}(E) G^{(0)} R^{(0)}(E + X_{M_1}) \ldots G^{(0)} R^{(0)}(E + X_{M_k}) \rho_{t=0},
\] (63)

where \( \prod \gamma \) denotes the product over all contractions \( \langle 59 \rangle \), \( N_p \) is the number of permutations of reservoir field superoperators to disentangle the fermionic contractions, and \( S = \prod m_i! \) is a symmetry factor arising for the case when pairs of vertices are connected by \( m_i \) contractions.

The determination of the shift variables \( X_{M_i} \) is simplified by noting that, according to \( \langle 59 \rangle \), a single contraction \( \gamma^{p_1 p_2}_{12} \) between \( A^{p_1}_{1} \) and \( A^{p_2}_{2} \) is only possible for \( \eta_1 = -\eta_2, \alpha_1 = \alpha_2 \) and \( \omega_1 = \omega_2 \). This gives \( \bar{\omega}_{12} = \bar{\mu}_{12} = 0 \), i.e. if the two indices fall both into the same set \( M_i \), there is no contribution to the shift \( X_{M_i} \). As a consequence, the left index 1 of a contraction \( \gamma^{p_1 p_2}_{12} \) will contribute only to those resolvents, which stand between the two field operators \( A^{p_1}_{1} \) and \( A^{p_2}_{2} \) in the original series. For this reason, the last resolvent in \( \langle 63 \rangle \) has no shift \( X_{M_k} = 0 \) and is given by \( R^{(0)}(E) \).

With the diagrammatic rules it is straightforward to translate a particular diagram, which we visualize as follows:

\[
\rho(E) \to i \gamma_{12}^{p_1 p_2} \gamma_{36}^{p_3 p_6} \gamma_{45}^{p_4 p_5} R^{(0)}(E) G_1^{(0)p_1} R^{(0)}(E_1 + \bar{\omega}_1) G_2^{(0)p_2} R^{(0)}(E) \\
G_3^{(0)p_4} R^{(0)}(E_3 + \bar{\omega}_3) G_4^{(0)p_5} R^{(0)}(E_3) \rho_{t=0},
\] (64)

where we used the notation \( E_{1\ldots n} = E + \bar{\mu}_{1\ldots n} \), see (38). In the diagrams, the green lines are the contractions, the circles denote the vertices, and the black lines connecting the vertices represent the resolvents \( R^{(0)} \) describing the dot propagation in Fourier space. The indices of the shift
variables of a particular resolvent can be determined by drawing a vertical line at the position of that resolvent and taking the left indices of all contractions which cut through this line. We note that we do not distinguish between diagrams which differ only by a permutation of the contractions connected to a certain vertex, i.e. the permutation of the two green lines connected to the indices 3 and 4 in the above example does not lead to a new diagram.

To bring the density matrix \( \rho(E) \) into the form (27) and to identify the effective Liouvillian \( L(E) \), we note that each diagram consists of a sequence of connected blocks, defined by the property that each vertical line will at least hit one contraction, connected by resolvents \( R^{(0)}(E) \). E.g., the diagram (64) consists of a sequence of two blocks. Denoting the sum of all connected diagrams by \( \Sigma(E) \), the diagrammatic series can be written as

\[
\rho(E) = i \left\{ R^{(0)}(E) + R^{(0)}(E) \Sigma(E) R^{(0)}(E) + \right. \\
\left. + R^{(0)}(E) \Sigma(E) R^{(0)}(E) \Sigma(E) R^{(0)}(E) + \ldots \right\} \rho_{t=0} = \frac{i}{E - L^{(0)} - \Sigma(E)} \rho_{t=0} .
\] (65)

Comparing to (27), we see that the effective Liouvillian is given by

\[
L(E) = L^{(0)} + \Sigma(E) ,
\] (66)

and \( \Sigma(E) \) consists of the sum of all connected diagrams with translation rule

\[
\Sigma(E) \rightarrow \frac{(\pm)^{N_p}}{S} \left( \prod \gamma \right)_{\text{con}} G^{(0)} R^{(0)}(E_{M_1} + \bar{\omega}_{M_1}) \ldots G^{(0)} R^{(0)}(E_{M_k} + \bar{\omega}_{M_k}) G^{(0)} ,
\] (67)

where \( \left( \prod \gamma \right)_{\text{con}} \) means that only connected diagrams are considered. E.g. some of the lowest order diagrams of \( \Sigma(E) \) are given by

\[
\Sigma(E) = \begin{array}{c}
\begin{array}{c}
\text{diagram 1} \\
\text{diagram 2} \\
\text{diagram 3}
\end{array}
\end{array} + \ldots
\] (68)

\( \Sigma(E) \) is the dissipative part of the effective Liouvillian, which contains the whole information of the coupling to the reservoirs and leads to irreversible time evolution. In time space we obtain \( L(t) = L^{(0)} \delta(t - 0^+) + \Sigma(t) \), such that the kinetic equation (23) reads

\[
i \dot{\rho}(t) = L^{(0)} \rho(t) + \int_0^t dt' \Sigma(t - t') \rho(t') .
\] (69)

The first term describes the von Neumann equation in the absence of the reservoirs, whereas the second one is the dissipative part. The two terms are the analog of the “flow” and the “collision” term of quantum Boltzmann equations.

**Local observables.** From the density matrix \( \rho(E) \) the time evolution of all averages of local observables can be calculated. The diagrammatic expansion can also be formulated for the calculation of arbitrary observables containing reservoir degrees of freedom or correlation functions. E.g., if an observable \( I \) of the generic form (10) is taken

\[
I = \frac{1}{n!} \left\{ \frac{1}{\eta_1 \eta_2 \ldots \eta_n} \right\} : a_n a_{n-1} \ldots a_1 : i_{12 \ldots n} \rightarrow \frac{1}{n!} i_{12 \ldots n} : a_1 a_2 \ldots a_n : ,
\] (70)
we define a corresponding superoperator $L_I$ by the anticommutator
\[ L_I = \frac{i}{2} [I, \cdot]_+ = \frac{1}{n!} \sum_{p=\pm} f^{(0)p...p}_{1...n} : A^p_1 \ldots A^p_n : , \] (71)
with
\[ f^{(0)p...p}_{1...n} b = \frac{i}{2} \left\{ \begin{array}{ll} 1 & \text{for } n \text{ even} \\ \sigma^p & \text{for } n \text{ odd} \end{array} \right\} \begin{cases} i b_{1...n} & \text{for } p = + \\ b_{1...n} & \text{for } p = - \end{cases} , \] (72)
such that the average can be written as
\[ \langle I \rangle(t) = \text{Tr}_{\text{tot}} I \rho_{\text{tot}}(t) = -i \text{Tr} \Sigma_{\text{res}}^I e^{-iL_{\text{tot}} t} \rho_{t=0}^{\text{eq}} \] . (73)
This expression has a formal similarity to
\[ \dot{\rho}(t) = -i \text{Tr}_{\text{res}} L_{\text{tot}} e^{-iL_{\text{tot}} t} \rho_{\text{tot}}(t = 0) = -i \text{Tr}_{\text{res}} (L^{(0)} + L_{\text{res}} + L_V) e^{-iL_{\text{tot}} t} \rho_{t=0}^{\text{eq}} \]
(74)
where $\text{Tr}_{\text{res}} L_{\text{res}} = 0$ has been used in the last line. Comparing to the kinetic equation (69), we find
\[ - i \text{Tr}_{\text{res}} L_V e^{-iL_{\text{tot}} t} \rho_{t=0}^{\text{eq}} = -i \int_0^t dt' \Sigma(t - t') \rho(t') \] . (75)
Therefore, when applying the same perturbative expansion to (73), we obtain the result
\[ \langle I \rangle(t) = -i \int_0^t dt' \text{Tr} \Sigma_I(t - t') \rho(t') \] , \hspace{1em} \langle I \rangle(E) = -i \text{Tr} \Sigma_I(E) \rho(E) \] , (76)
with the only difference that the first vertex of the kernel $\Sigma_I(E)$ has to be the vertex $I^{(0)}$ instead of $G^{(0)}$, i.e. the diagrammatic rule (67) changes to
\[ \Sigma_I(E) \rightarrow \frac{\pm N_p}{S_p} \left( \prod_{\text{con}} \gamma \right) \begin{cases} I^{(0)} R^{(0)}(E_{M_1} + \omega_{M_1}) G^{(0)} R^{(0)}(E_{M_2} + \omega_{M_2}) \ldots G^{(0)} R^{(0)}(E_{M_k} + \omega_{M_k}) G^{(0)} \end{cases} \] . (77)
A prominent example for an observable is the particle current operator flowing from reservoir $\alpha$ into the local system defined by
\[ I_{\alpha} = -\frac{d}{dt} N_{\alpha} = -i [H_{\text{tot}}, N_{\alpha}] = -i [V, N_{\alpha}] \] . (78)
Inserting the form (10) of $V$, one finds after some straightforward manipulations
\[ (i_{\alpha})_{1...n} = i \sum_{k=1}^n \eta_k \delta_{\alpha_k \alpha} g_{1...n} \] , \hspace{1em} \begin{cases} (I_{\alpha})_{1...n}^{(0)p...p} = -\frac{1}{2} \sum_{k=1}^n \eta_k \delta_{\alpha_k \alpha} p G_{1...n}^{(0)p...p} \end{cases} \] . (79)
Using the di"{a}grammatic expansion (67) and (77), the kernels $\Sigma(E)$ and $\Sigma_I(E)$ can be calculated in perturbation theory w.r.t. the bare vertices $G^{(0)}$, and the local density matrix $\rho(E)$ and the average $\langle I \rangle(E)$ of any observable $I$ follow from (65) and (76) in Fourier space. Using inverse
Fourier transform the time evolution can finally be discussed following the strategy described in Section 3. Stationary quantities are obtained from

\[
\rho_{st} = \lim_{t \to \infty} \rho(t) = \lim_{E \to 0^+} (-iE) \rho(E) \quad \iff \quad L(E = 0^+) \rho_{st} = 0 \tag{80}
\]

\[
\langle I \rangle_{st} = \lim_{t \to \infty} \langle I \rangle(t) = \lim_{E \to 0^+} (-iE) \langle I \rangle(E) = -i \text{Tr} \Sigma_{f}(E = 0^+) \rho_{st} \quad . \tag{81}
\]

Applications of these perturbative schemes for the calculation of transport properties will be discussed in the lecture C7 by M. Wegewijs. Similar schemes have also been developed to calculate correlation functions [21] and to consider explicitly time-dependent Hamiltonians [9, 22]. Concerning the latter first applications have considered adiabatic response [22] and quantum quenches [9] for the IRLM.

**Analytic properties.** From the perturbative expansion one finds that the effective Liouvillian \( L(E) \) has a branch cut on the real axis and is analytic in the upper and lower half of the complex plane. This can be seen from the resolvents \( R^{(0)}(E_M + \bar{\omega}_M) \) since \( L^{(0)} = [H^{(0)}, \cdot] \) is a self-adjoint superoperator with real eigenvalues and all frequency variables \( \bar{\omega} \) are integrated over the real axis. The same analytic property holds for the resolvent \( R(E) = \frac{1}{E - L(E)} \), since, due to [27], we get for any initial density matrix \( \rho_{st} \)

\[
R(E) \rho_{t=0} = -i \rho(E) = -i \int_0^\infty dt e^{iE t} \rho(t) = -i \int_0^\infty dt e^{iE t} \text{Tr}_{\text{res}} \rho_{\text{tot}}(t)
\]

\[
= -i \int_0^\infty dt e^{iE t} \text{Tr}_{\text{res}} \rho_{\text{tot}}(t = 0) = \text{Tr}_{\text{res}} \frac{1}{E - L_{\text{tot}}} \rho_{\text{tot}}(t = 0) \quad . \tag{82}
\]

This function can only have a branch cut on the real axis since \( L_{\text{tot}} = [H_{\text{tot}}, \cdot] \) is a self-adjoint superoperator with real eigenvalues. To calculate the time evolution we have seen from (28) that the integration \( \int dE \) is slightly above the real axis and has to be closed in the lower half of the complex plane (due to \( t > 0 \)). It is very inconvenient to calculate this integral by enclosing the branch cut of the integrand on the real axis due to the rapidly oscillating function \( e^{-iE t} \) in the integrand on the scale \( 1/t \). Therefore, in analogy to the standard procedure for response functions, one tries to find an appropriate analytic continuation of the functions \( L(E) \) and \( R(E) \) into the lower half of the complex plane such that all branch cuts point into the direction of the negative imaginary axis starting at certain singularities \( z_n \) with \( \text{Im} z_n \leq 0 \). We achieve this in two steps. First, we will transform the perturbative series for \( L(E) \) into a self-consistent equation by resumming all blocks of connected diagrams on the propagators connecting the vertices. The diagrammatic representation allows this to be done in a unique way and, as a result, the bare resolvents \( R^{(0)}(E_M + \bar{\omega}_M) \) are replaced by the full ones \( R(E_M + \bar{\omega}_M) \) and no diagrams are allowed with connected sub-blocks without any free lines on the propagators which we indicate by \( (\prod \gamma)_{\text{irr}} \)

\[
\Sigma(E) \rightarrow \frac{(\pm)^{N_p}}{S} (\prod \gamma)_{\text{irr}} G^{(0)}(E_{M_1} + \bar{\omega}_{M_1}) \ldots G^{(0)}(E_{M_k} + \bar{\omega}_{M_k}) G^{(0)} \quad . \tag{83}
\]

For \( \text{Im} E > 0 \), all resolvents are analytic functions w.r.t. the integration variables \( \bar{\omega}_i \) in the upper half of the complex plane. Therefore, in the second step, we can close all integration contours in the upper half and have to enclose only the nonanalytic features arising from the spectral function \( \rho_{\text{tot}}(\omega) \) and the Bose/Fermi distribution \( f_{n_p}(\rho' \eta \omega) \) in the contraction \( \gamma'_{\gamma p} \), defined in (59). Thereby we assume that the frequency dependence of the vertices \( G^{(0)}_{1_{\ldots n_p}} \) can be neglected.
Decomposing the Bose/Fermi distribution in symmetric and antisymmetric parts and using the representation in terms of the Masubara frequencies \( \omega_n^\alpha = 2n\pi T_\alpha \) (\( \omega_n^\alpha = (2n + 1)\pi T_\alpha \)) for bosons (fermions), we can write the contraction in the form

\[
\gamma_{pp'}_{11'} = \delta_{11'} p' \left\{ \frac{\eta}{1} \right\} \tilde{\rho}_{\alpha \sigma}(\tilde{\omega}) \left\{ \mp \frac{1}{2} + p' (f_\alpha(\tilde{\omega}) \pm \frac{1}{2}) \right\} = \delta_{11'} (p' \gamma^s_1 + \gamma^a_1) ,
\]

(84)

\[
\gamma^s_1 = \mp \frac{1}{2} \left\{ \frac{\eta}{1} \right\} \tilde{\rho}_{\alpha \sigma}(\tilde{\omega}) ,
\]

(85)

\[
\gamma^a_1 = \left\{ \frac{\eta}{1} \right\} \tilde{\rho}_{\alpha \sigma}(\tilde{\omega}) \left\{ f_\alpha(\tilde{\omega}) \pm \frac{1}{2} \right\}
\]

\[
= \left\{ \frac{\eta}{1} \right\} \tilde{\rho}_{\alpha \sigma}(\tilde{\omega}) T_\alpha \frac{1}{2} \sum_n \left( \frac{1}{\omega - i\omega_n^\alpha} + \frac{1}{\omega + i\omega_n^\alpha} \right) ,
\]

(86)

where \( \tilde{\rho}_{\alpha \sigma}(\tilde{\omega}) = \rho_{\alpha \sigma}(\omega) \). Thus, after performing all integrations \( \int d\tilde{\omega} \), and assuming for the moment that the spectral function \( \tilde{\rho}_{\alpha \sigma}(\tilde{\omega}) \) is an analytic function in the upper half, the quantities \( \tilde{\omega}_M \) occurring in the resolvents \( R(\tilde{E}_M + \tilde{\omega}_M) \) will consist of a sum of positive Matsubara frequencies \( i \sum_{j \in M} |\alpha_{n_j}^\alpha| \). As a consequence, the analytic continuation w.r.t. \( E \) of this result for \( L(E) \) into the lower half of the complex plane will lead to nonanalytic features at \( \tilde{E}_M + i \sum_{j \in M} |\alpha_{n_j}^\alpha| = z_k^p \), where \( z_k^p \) are the poles of the resolvent \( R(E) \) after the analytic continuation into the lower half of the complex plane. Since \( R(E) \) is analytic in the upper half, the poles \( z_k^p \) have to lie in the lower half, and we find that \( L(E) \) has an infinite series of poles in the lower half located at

\[
E = z_k^p - \tilde{\mu}_M - i \sum_{j \in M} |\alpha_{n_j}^\alpha| ,
\]

(87)

which, at zero temperature, turn into a series of branch cuts in the direction of the negative imaginary axis with branching points \( z_n \) located at the poles \( z_k^p \) shifted by any combinations of the chemical potentials of the reservoirs

\[
z_n = z_k^p - \tilde{\mu}_M .
\]

(88)

This result has formed the basis for the generic discussion of the time evolution in Section 3.

**Influence of spectral function.** We note that the spectral function \( \tilde{\rho}_{\alpha \sigma}(\tilde{\omega}) \) of the models introduced in Section 2 does not change this picture. For quantum dots coupled to Fermi liquid leads, like the Kondo model or the IRLM, the spectral function \( \tilde{\rho}_{\alpha \sigma}(\tilde{\omega}) = \frac{D^2}{D^2 + \tilde{\omega}^2} \) defines just a high-energy cutoff function with pole at \( \tilde{\omega} = iD \) in the upper half and residuum \(-iD/2\). The contribution of this pole to the frequency integration leads for \( D \to \infty \) either to a vanishing or to a regular contribution in \( E \). For the ohmic spin boson model we get from (19) that \( \tilde{\rho}(\tilde{\omega}) = 2\alpha |\tilde{\omega}| \theta(\eta \tilde{\omega}) \frac{D^2}{D^2 + \tilde{\omega}^2} \), which has a branch cut on the whole imaginary axis. However, since the vertex \( g_1 = \frac{1}{2} \sigma_z \) is independent of \( \eta \), we can sum the contraction \( \gamma_{pp'}_{11'} \) over \( \eta \) and \( \eta' \) at fixed \( \tilde{\omega} \) and \( \tilde{\omega}' \) (which are the integration variables) and get from (59) for the case of bosons the effective contraction

\[
\gamma_{pp'}_{11'} = \delta(\tilde{\omega} + \tilde{\omega}') p' 2\alpha \tilde{\omega} \frac{D^2}{D^2 + \tilde{\omega}^2} f(p' \tilde{\omega}) = \delta(\tilde{\omega} + \tilde{\omega}') (p' \gamma^s_1 + \gamma^a_1) ,
\]

(89)

with

\[
\gamma^s_1 = -\alpha \tilde{\omega} \frac{D^2}{D^2 + \tilde{\omega}^2} , \quad \gamma^a_1 = \alpha \tilde{\omega} \frac{D^2}{D^2 + \tilde{\omega}^2} (2f(\tilde{\omega}) + 1) ,
\]

(90)
where the index $1 \equiv \bar{\omega}$ contains only the frequency variable. Since $\bar{\omega}$ is an analytic function, there is no change of the analytic structure of $L(E)$ for the ohmic spin boson model. For a generic frequency dependence of the spectral function, the analytic structure might change. If $\bar{\rho}(\bar{\omega})$ has a branch cut in the upper half in the direction of the positive imaginary axis starting at $\Delta_\rho + i\gamma_\rho$, with $\gamma_\rho \geq 0$, the position (88) of the branching points of $L(E)$ can be shifted by multiples of $-\Delta_\rho - i\gamma_\rho$. This can e.g. happen for superconducting leads, where $\Delta_\rho$ corresponds to the superconducting gap and $\gamma_\rho = 0$. For sub- or super-ohmic spin boson models there is no change of the analytic properties since the branch cuts of the spectral function start at the origin.

**Symmetric part of the contraction.** We note that the part $\gamma_1^s$ of the contraction (84) involving the symmetric part of the Bose/Fermi distribution plays a special role. It is the only part of the contraction $\gamma_{11'}^{pp'}$ which depends on the Keldysh indices via $p'$ and it depends on the frequency only via the spectral function. In particular for a spectral function of the form $\bar{\rho}(\bar{\omega}) = D^2 + \bar{\omega}$, i.e. if it just acts as a high-energy cutoff function but has no other special form, the frequency integration $\int d\bar{\omega}$ will involve only the pole of the spectral function at $\bar{\omega} = iD$ when closed in the upper half. In the limit $D \to \infty$ this means that this integration gives either zero (if more than one resolvent involves $\bar{\omega}$) or a constant if this contraction connects two consecutive vertices

$$\int d\bar{\omega} \frac{D^2}{D^2 + \bar{\omega}} R(E_{1...n} + \bar{\omega}_{1...n}) = \pi D R(E_{1...n} + \bar{\omega}_{2...n} + iD) \xrightarrow{D \to \infty} -i \pi . \quad (91)$$

As a result, the symmetric part of the contraction can be integrated out analytically and can be incorporated in an effective vertex by taking the two consecutive vertices together to a single one. The same can be shown for the ohmic spin boson model [10] due to its special algebra, whereas for more general spectral functions with nonanalytic features in the upper half this is not the case. If it holds, one important consequence of this property is that the special pole at $E_M + \bar{\omega}_M = z_k^b = 0$ of the resolvents $R(E_M + \bar{\omega}_M)$ leads to regular contributions in the limit $D \to \infty$ and does not contribute to the branch cuts of $L(E)$. The reason is the special form $P_{st}(E) = |x_{st}(E)\rangle \text{Tr}$ for the projector of the mode $k = st$ (see Eq. (34)) together with the property

$$\sum_p \text{Tr} G^{(0)pp...p}_{1...n} = 0, \quad (92)$$

which follows straightforwardly from the definition [55]. Since the contractions $\gamma_{11'}^{pp'}$ are independent of the first Keldysh index $p$ (see Eq. (59)), this means that if the projector $P_{st}(E_M + \bar{\omega}_M)$ is inserted between two consecutive vertices, at least one of the contractions $\gamma_{11'}^{pp'}$ associated with the right vertex must point into the left direction and only its $p'$-dependent symmetric part $p'\gamma_1^s$ will contribute

$$\gamma_1^s$$

As shown above this means that this contraction has to connect the two consecutive vertices and the frequency integration gives a constant. Thus, the case $k = st$ does not contribute to the positions (88) of branch cuts for $L(E)$ or $R(E)$. There might be other accidental poles $z_k^p = 0$ for $k \neq st$, like e.g. for multi-channel Kondo models with non-Fermi liquid behaviour [6], but
for the models discussed in Section 2, this is not the case. Therefore, for these models, the pole at \( z_{m} = 0 \) is isolated, as already stated in Section 3 after Eq. (45).

**Breakdown of perturbation theory for time-evolution problems.** Finally we note that it is very important to use the perturbative expansion of \( L(E) \) in the self-consistent form (83) in order to find the right position of the branching points \( z_{n} \) of the branch cuts of \( R(E) \). For the original series (67) involving the bare resolvent \( R^{(0)}(E_{M} + \tilde{\omega}_{M}) \), the same considerations as above lead to branch cuts of \( L(E) \) starting on the real axis at the value \( z_{n} = \lambda^{(0)} - \mu_{M} \), where \( \lambda^{(0)} \) is a real eigenvalue of the bare Liouvillian \( L^{(0)} \). This would have a dramatic effect on the long-time evolution because it leads to non-exponential decay. However, this result is not correct since perturbation theory is very dangerous in the regime \( |E - z_{n}| \sim \Gamma \), where \( \Gamma \) is a typical decay rate. At low frequencies, the resolvents can then become very large of the order of the inverse coupling constant, raising serious questions about convergence. In particular for the original series (67) involving \( R^{(0)} \) this effect is most dramatic since a series of connected sub-blocks contains an arbitrary number of resolvents with exactly the same argument, i.e. the singularity at low frequency appears to an arbitrary power. Such a series is certainly not convergent and it is necessary to resum it first to the self-consistent version (83) before determining the position of the branching points. E.g. consider a contribution to the effective Liouvillian of the form \( (\alpha \text{ is some small dimensionless coupling constant and } \Delta \text{ denotes a typical low-energy scale}) \)

\[
\alpha (iE + \alpha \Delta) \ln \frac{D}{-iE + \alpha \Delta} = \alpha (iE + \alpha \Delta) \ln \frac{D}{-iE} - \alpha^{2} \Delta - \frac{1}{2} \alpha \Delta^{2} + O(\alpha^{4}) .
\]

The logarithm on the l.h.s. has a branching point at \( E = -i\alpha \Delta \) but the expanded form gives in \( O(\alpha) \) and \( O(\alpha^{2}) \) a branching point at \( E = 0 \). The mistake can only be seen by considering higher orders in \( \alpha \), where an infinite series of terms with a pole at \( E = 0 \) is obtained. Due to the factor \( -iE + \alpha \Delta \) in front of the logarithm, this artifact is even not visible in \( O(\alpha^{2}) \) but starts in \( O(\alpha^{3}) \) or higher. The form (94) arises e.g. for the Kondo model and for the ohmic spin boson model, where we will show in Section 5 that \( \frac{d}{dt} L(E) \) must be a slowly varying logarithmic function leading to typical terms of the form (94), see also Eq. (105). For the ohmic spin boson model previous calculations [2] have predicted terms with non-exponential decay which have been corrected recently [10, 23, 24].

### 5 Renormalization group

**General remarks.** At low temperatures the perturbative calculation of the effective Liouvillian can break down even at small reservoir-system coupling for two reasons. First, at high energies (the so-called ultraviolett regime), the frequency integrals are typically logarithmic leading to logarithmic contributions \( \sim \alpha^{k} (\ln \frac{D}{E - z_{m}})^{l} \) in higher-order perturbation theory, with \( k \geq l \), where \( z_{n} \) are the branching points (88) of the resolvent \( R(E) \) and \( \alpha \) is an appropriate dimensionless coupling constant. Secondly, even if perturbation theory does not contain ultraviolett logarithmic divergencies in the limit \( D \to \infty \), it may contain logarithmic terms \( \sim \alpha \ln \frac{E - z_{m}}{E - z_{m}} \), which, for \( E \to z_{n} \), turn into the form \( \sim \alpha \ln \frac{E - z_{m}}{z_{n} - z_{m}} \), which can lead to a breakdown of perturbation theory at low energies (the so-called infrared regime). Therefore, a method is needed capable of reorganizing perturbation theory such that all ultraviolett and infrared logarithmic divergencies are resummed. Concerning high energies, resumming all logarithmic contributions...
$\sim \alpha^k (\ln \frac{D}{E-z_n})^l$ with $l = k, k-1, k-2, \ldots$ is called leading order, sub-leading order, sub-sub leading order, etc. approximation (sometimes also referred to as 1-loop, 2-loop, 3-loop, etc.). In traditional (so-called poor man scaling) RG methods, one tries to perform this resummation by integrating out high-energy scales, i.e. the band width $D$ is successively reduced in infinitesimal steps and the physical quantity of interest is kept invariant by renormalizing the coupling constants and other energy scales. Provided that the renormalized coupling constants remain small (the so-called weak-coupling regime), a well-controlled truncation scheme can be set up by neglecting higher-order terms in the renormalized couplings. This strategy has also been used for calculating stationary quantities of nonequilibrium problems, but it turns out that for the calculation of the effective Liouvillian $L(E)$ it is very hard to set up a systematic truncation scheme. This has been improved by using a high-energy cutoff on the imaginary axis by cutting off the Matsubara frequencies of the Bose/Fermi distribution function and applied to various models. Here we will follow another route by describing the E-RTRG method, which is unique in the sense that it is capable of dealing with all logarithmic divergencies at high and low energies. Technically, this is achieved by considering the perturbation theory not for $L(E)$ but for its first or second derivative w.r.t. the Fourier variable $E$ together with a proper resummation in terms of effective vertices. Whether a first or a second derivative is needed depends on the model under consideration. This leads to a series where all frequency integrals converge at high energies and the limit $D \to \infty$ can be performed in all orders. As a consequence one obtains a universal differential equation (called RG equation) for $L(E)$ independent of the specific choice of the high-energy cutoff function. Furthermore, the RG equation for $L(E)$ turns out to be such that the divergence at low energies for $E \to z_n$ is at most $\frac{1}{E-z_n}$ multiplied with a perturbative series in terms of effective vertices which exists in the limit $E \to z_n$. This allows for a systematic solution at low energies as well. Besides the effective Liouvillian also effective vertices will appear in the RG equation due to the resummation procedure, for which similar universal RG equations can be derived. Provided that the effective vertices stay small (so-called weak coupling problems) the RG equations can be systematically truncated and well-controlled universal properties can be determined at high as well as at low energies. According to the discussion in Section 3 this allows a well-controlled discussion of the time evolution at short and long times together with the crossover behaviour. The high-energy cutoff $D$ will only appear in the initial condition for the various quantities which are calculated by a well-controlled perturbation theory in the bare couplings at $E = iD$. This procedure has the advantage that by construction only the universal properties of the model are obtained, although it is also possible to keep $D$ fixed and solve the RG equations for a given high-energy cutoff function. Furthermore, the use of a physical scale $E$ as flow parameter of the RG equations has the advantage that at each stage of the flow the solution $L(E)$ provides a result for a physical quantity. Moreover, since $E$ is a complex flow parameter, the flow can be solved on any path in the complex plane which is very helpful to find appropriate analytic continuations of retarded functions into the lower half of the complex plane, even by using numerical methods, see also the discussion in Section 3 after Eq. (38).

**Derivation of the E-RTRG equations.** To illustrate the general strategy for the derivation of the RG equations within the E-RTRG method we consider here, for simplicity, a spectral function of the form $\tilde{\rho}(\bar{\omega}) = \frac{D^2}{D^2+\bar{\omega}^2}$, which arises typically for fermionic metallic reservoirs where the d.o.s. is approximately a constant in the physically relevant energy regime. Therefore, we consider only the fermionic case in the following. Furthermore, we assume that only 1- and 2-point vertices occur in the original model and that the frequency dependence of the bare
vertices $g_1$ and $g_{12}$ can be neglected. This applies to the Kondo model and the IRLM introduced in Section 2. For the ohmic spin boson model, a similar procedure can be used to derive the RG equations, see Ref. [10]. For such models, one obtains a problem with convergence at high energies if the number of frequency integrations is larger or equal to the number of resolvents where the frequencies occur. For models with 1- and 2-point vertices this means that diagrammatic sub-elements of the form

\begin{equation}
\begin{array}{c}
\text{(95)}
\end{array}
\end{equation}

lead to problems at high energies and have to be avoided. This can be achieved by taking a single or a double derivative w.r.t. $E$ of the resolvents occurring in these diagrams. Therefore, the idea is to consider a perturbative expansion for the derivatives $\frac{\partial}{\partial E} L(E)$ or $\frac{\partial^2}{\partial E^2} L(E)$ and to resum the series such that no sub-elements of the form (95) remain. The procedure is quite straightforward and we illustrate it for the case of a model where only 2-point vertices occur, like e.g. the Kondo model. Here, to guarantee convergence we consider two derivatives w.r.t. $E$ of the diagrammatic series (83) of $L(E)$ in the self-consistent form. Since the $E$-dependence occurs only in the resolvents $R(E_M + \tilde{\omega}_M)$, we can either take two derivatives of a single resolvent or two single derivatives of different resolvents. Fixing the positions of the resolvents we can then resum all remaining diagrams in a unique way such that the bare 2-point vertices $G^{(0)pp}_{12}$ are replaced by full effective 2-point vertices $G^{pp_{12}}_{12}(E)$, which are defined as the sum of all connected diagrams with 2 external reservoir lines. With the convention that these two external lines are directed to the right, it turns out that the energy argument of an effective vertex is identical to the one of the preceding resolvent, i.e. only the combination $R(E_M + \tilde{\omega}_M)G^{pp_{12}}_{12}(E_M + \tilde{\omega}_M)$ can occur in the diagrammatic expansion. Furthermore, for all diagrams contributing to the effective vertex $G^{pp_{12}}_{12}(E)$, where the two external lines have the sequence 21, a fermionic sign has to be added. After this resummation, the diagrammatic series for $\frac{\partial^2}{\partial E^2} L(E)$ up to third order in the effective vertices reads

\begin{equation}
\frac{1}{2} \frac{\partial^2}{\partial E^2} L(E) = \frac{1}{2} \frac{1}{2} \begin{array}{c}
\text{(96)}
\end{array}
\end{equation}

where the red slash indicates a derivative $\frac{\partial}{\partial E}$ of the corresponding resolvent (two slashes indicate the second derivative $\frac{\partial^2}{\partial E^2}$). This is one of the central equations in the E-RTRG approach. Prefactors arising from the symmetry factor $\frac{1}{S}$ have explicitly been indicated and all vertices are full effective 2-point vertices from now on. All frequency integrations are convergent even if one neglects the frequency-dependence of the effective vertices (those can only enhance convergence). Therefore on the r.h.s. of this differential equation we can take the limit $D \to \infty$. This property holds in all orders since, by construction, all diagrammatic sub-elements (95) leading to a divergence in the infinite-$D$ limit have been eliminated by the resummation procedure. To close the equation one can also derive in the same way a differential equation for the effective
2-point vertex

\[
\frac{\partial}{\partial E} G^{p_1 p_2}_{12}(E) = \left[ \begin{array}{c}
\begin{array}{c}
\circ \\
1
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\circ \\
2
\end{array}
\end{array} - (1 \leftrightarrow 2) \\
\begin{array}{c}
\circ \\
1
\end{array}
\begin{array}{c}
\circ \\
2
\end{array} \right]
+ \frac{1}{2} \begin{array}{c}
\begin{array}{c}
\circ \\
1
\end{array}
\begin{array}{c}
\circ \\
2
\end{array} + \frac{1}{2}
\begin{array}{c}
\begin{array}{c}
\circ \\
1
\end{array}
\begin{array}{c}
\circ \\
2
\end{array} \end{array} - (1 \leftrightarrow 2) + O(G^4). \quad (97)
\]

After the limit \(D \to \infty\) has been taken, the symmetric part \(85\) of the contraction becomes an analytic function and does not contribute to the frequency integration when closing the integration in the upper half of the complex plane. This means that the Keldysh indices no longer appear explicitly in the RG equations, i.e. only the effective 2-point vertices averaged over the Keldysh indices are needed, which we denote by \(G_{12}(E) = \sum_{p_1 p_2} G^{p_1 p_2}_{12}(E)\). This simplifies the analysis considerably. As a result all contractions can be replaced by the antisymmetric part given by \(86\)

\[
\gamma_{11}^{pp'} \to \gamma_1^p = f_\alpha(\bar{\omega}) - \frac{1}{2}, \quad (98)
\]

where we have already taken the limit \(D \to \infty\) and integrated out the trivial part \(\delta_{11}\) of all contractions in the RG diagrams. By convention, \(\bar{\omega}\) is always the frequency variable of the left vertex.

**Frequency dependence.** To calculate the integrals over the internal frequencies in the RG diagrams, it is necessary to know the frequency dependence of the effective vertices and the Liouvillian. This can be treated systematically by the formalism. Provided that the bare vertices are frequency-independent, one finds for the vertices that the diagrammatic series for the difference \(G_{12}(E) - G_{12}(E)_{\bar{\omega}_1 = \bar{\omega}_2 = 0}\) can be resummed by a similar procedure in terms of effective 2-point vertices such that the limit \(D \to \infty\) is well-defined. The reason is that at least one resolvent in the original perturbative series must involve the difference \(R(E_M + \bar{\omega}_M + \bar{\omega}_M) - R(E_M + \bar{\omega}_M)\), where \(M_{ex}\) contains some of the external indices \(\{1, 2\}\). Fixing this resolvent and resumming the rest of the diagram in terms of effective 2-point vertices yields in lowest order the equation

\[
\begin{array}{c}
\begin{array}{c}
\circ \\
1
\end{array}
\begin{array}{c}
\circ \\
2
\end{array} = \begin{array}{c}
\begin{array}{c}
\circ \\
1
\end{array}
\begin{array}{c}
\circ \\
2
\end{array} + \begin{array}{c}
\begin{array}{c}
\circ \\
1
\end{array}
\begin{array}{c}
\circ \\
2
\end{array} \end{array} - (1 \leftrightarrow 2) + O(G^3), \quad (99)
\end{array}
\]

where the filled double dots represent the effective vertices at zero frequency. This is the second key equation in the E-RTRG approach. A contraction with an open circle and external frequency \(\bar{\omega}_1\) indicates that the resolvent corresponding to the vertical cut at the position of that circle has to be replaced by the difference \(R(E_M + \bar{\omega}_M + \bar{\omega}_1) - R(E_M + \bar{\omega}_M)\). This difference falls off \(\sim (\bar{\omega}_M)^2\) w.r.t. the internal frequency integration variables \(\bar{\omega}_M\) and, therefore, all frequency integrations are convergent in the limit \(D \to \infty\). For the frequency dependence of the Liouvillian \(L(E_M + \bar{\omega})\) it turns out that the similar diagrammatic series for the difference \(L(E + \bar{\omega}) - L(E)\) does not exist in the limit \(D \to \infty\), similar to the fact that two derivatives are needed for convergence (see above). Therefore, one defines a discrete version of the second derivative \(\Delta^2_{\bar{\omega}} L(E)\) via

\[
L(E + \bar{\omega}) = L(E) + \frac{\partial}{\partial E} L(E) \bar{\omega} + \Delta^2_{\bar{\omega}} L(E) = L(E) + \frac{\partial}{\partial E} L(E) \bar{\omega} + O(G^2) \quad , (100)
\]
and finds that \( \Delta^2 L(E) \) exists in the limit \( D \to \infty \) and is at least of \( O(G^2) \) since it involves second and higher-order derivatives of the Liouvillian. Neglecting \( O(G^2) \) (note that this contributes \( O(G^4) \) to the RG equations (96) and (97)), the resolvents occurring in the RG diagrams of (96) and (97) can be written as

\[
R(E_M + \omega_M) = \frac{1}{\omega_M + \chi(E_M)} Z(E_M) + O(G^2) \quad ,
\]

with

\[
\chi(E) = Z(E) (E - L(E)) \quad , \quad Z(E) = \frac{1}{1 - \frac{\partial}{\partial E} L(E)} \quad ,
\]

where the RG equation for \( Z(E) \) follows from the one for \( \frac{\partial^2}{\partial E^2} L(E) \) by

\[
\frac{\partial}{\partial E} Z(E) = Z(E) \left\{ \frac{\partial^2}{\partial E^2} L(E) \right\} Z(E) \sim O(G^2) \quad .
\]

Inserting (99) and (101) into the RG equations (96) and (97), calculating all frequency integrations and neglecting all terms of \( O(G^4) \), one obtains a closed set of RG equations for \( G_{12}(E) \big|_{\omega_1 = \omega_2 = 0} \) and \( L(E) \), which can be easily solved numerically. These constitute the basic equations of the E-RTRG approach. The crucial step in the formalism is the parametrization of the frequency dependence, otherwise a numerical solution would be very time consuming. Truncating the RG equations at \( O(G^2) \) provides the solution up to leading order, whereas a truncation at \( O(G^3) \) includes all sub-leading terms. An important check for the reliability of the solution is whether these two truncation schemes lead approximately to the same universal solution. For the nonequilibrium Kondo model at zero magnetic field, the equations have been solved in Ref. [6] to calculate the stationary conductance with reliable results even in the strong coupling regime. Similar RG equations can be set up for the IRLM and the spin boson model which have been studied in Refs. [7, 10].

**RG equations for the slowly varying parts of the Liouvillian.** We are now ready to show how the decomposition (4) can be derived together with RG equations for the slowly varying functions \( L_\Delta(E) \) and \( L'(E) \). First of all, one can see from the RG equations (96) and (97) that \( \frac{\partial}{\partial E} L(E) \) and \( G_{12}(E) \) are slowly varying logarithmic functions. At large \( E \) we find from dimensional arguments that

\[
\frac{\partial^2}{\partial E^2} L(E) , \quad \frac{\partial}{\partial E} G_{12}(E) \sim \frac{1}{E} \left( 1 + O(\frac{\Delta}{E}) \right) \quad ,
\]

where \( \Delta \) is some physical scale except \( E \). For large \( E \), we can neglect the higher orders \( \sim O(\frac{\Delta}{E}) \) and we see that, due to the factor \( \frac{1}{E} \), logarithmic functions are generated by integrating over \( E \). For \( E \) close to some branching point \( z_n \), we find, that even in the worst case when all resolvents contain the same branching point, that \( \frac{\partial^2}{\partial E^2} L(E) \) and \( \frac{\partial}{\partial E} G_{12}(E) \) can at most diverge \( \sim \frac{1}{E - z_n} \) for \( E \to z_n \). As a result, also for \( E \to z_n \), \( \frac{\partial}{\partial E} L(E) \) and \( G_{12}(E) \) are slowly varying logarithmic functions of \( E - z_n \). This can only be the case if \( L(E) \) consists of terms

\[
L(E) \sim (E - z_n) K_n(E - z_n) = -z_n K_n(E - z_n) + E K_n(E - z_n) \quad ,
\]

where \( K(E) \) is a slowly varying function, or, more precisely, \( z_n = \frac{E}{\lambda_k(E_M) - \bar{\mu}_M} \) will be replaced by \( \lambda_k(E_M) - \bar{\mu}_M \) if \( E \) is not close to one of the singularities. Therefore, we see that \( L(E) \) can be decomposed in the form (4),

\[
L(E) = L_\Delta(E) + E L'(E) \quad ,
\]
with slowly varying functions \( L_\Delta(E) \sim -z_n K_n(E - z_n) \) and \( L'(E) \sim K_n(E - z_n) \). We note that we used precisely this form at the end of Section 4 in Eq. (94). It shows that \( L_\Delta(E) \) and \( L'(E) \) have a quite similar structure.

We note that the property that \( \frac{\partial}{\partial E} L(E) \) and \( G_{12}(E) \) are slowly varying logarithmic functions can also be seen directly from the original perturbative expansion (83) since in all orders of perturbation theory the number of frequency integrations is identical to the number of resolvents. This leads to logarithmic integrals at large and low energies even if all resolvents contain the same cutoff scale at low energies. For the proof it is essential that the perturbation theory is taken in the self-consistent form (83) since this leads to the property that all resolvents involve a different combination of the frequencies. The same can be shown for the ohmic spin boson model where the decomposition (4) holds also in all orders of perturbation theory. For models with 1-point vertices and a flat spectral function (like e.g. quantum dot models in the charge fluctuation regime), the number of resolvents can be arbitrarily larger than the number of frequency integrations. Here, to show the logarithmic scaling at low energies in all orders of perturbation theory, it is very important that the resolvents do not only have different frequency combinations but many of them have also different cutoff scales at low energies. In contrast to models with spin/orbital fluctuations, it turns out that already the first derivative \( \frac{\partial}{\partial E} L(E) \) exists in the limit \( D \to \infty \), see e.g. the first diagram of (95). This means that \( \frac{\partial}{\partial E} L(E) \sim \frac{\Gamma}{E - z_n} \) multiplied with a well-controlled series with no divergence at high or low energies. This part influences only the function \( L_\Delta(E) \) but not \( EL'(E) \). The systematic treatment of all orders in the tunneling for models with charge fluctuations is still an issue of ongoing research.

To find RG equations for \( L_\Delta(E) \) and \( L'(E) \), we try to bring the RG equation (96) for \( \frac{\partial^2}{\partial E^2} L(E) \) into the form

\[
\frac{\partial^2}{\partial E^2} L(E) = \frac{\partial}{\partial E} L'(E) + \frac{\partial}{\partial E} \left\{ \frac{\partial}{\partial E} L_\Delta(E) + E \frac{\partial}{\partial E} L'(E) \right\} ,
\]

such that \( \frac{\partial}{\partial E} L_\Delta(E) \) and \( \frac{\partial}{\partial E} L'(E) \) can be identified and that \( L_\Delta(E) \) is proportional to some physical scale \( \Delta \) except \( E \). For simplicity we show the procedure only up to \( O(G^2) \), for \( O(G^3) \) see Ref. (29). Taking only the first term on the r.h.s. of the RG equation (96), replacing the vertices by the ones at zero frequency via (99), and shifting the two derivatives of the resolvent \( \frac{\partial}{\partial E} R(E_{12} + \omega_{12}) = \frac{\partial}{\partial \omega_{12}} \frac{\partial}{\partial \omega_{12}} R(E_{12} + \omega_{12}) \) via two partial integrations to the contractions, we obtain

\[
\frac{\partial^2}{\partial E^2} L(E) = \frac{1}{2} \begin{array}{c}
\text{cross} \\
\end{array} + \frac{\partial}{\partial E} \left\{ \frac{1}{2} \begin{array}{c}
\text{cross} \\
\end{array} \right\} + O(G^3) ,
\]

where a cross at a contraction denotes the derivative \( \frac{\partial}{\partial \omega} T_1^a = \frac{\partial}{\partial \omega} f_a(\bar{\omega}) \), see (28). The dashed line in the second term indicates that the resolvent is replaced by the \( Z' \)-factor \( R(E_M + \bar{\omega}_M) \to Z'(E_M + \bar{\omega}) \), defined in (29). Therefore, this term is of \( O(G^3) \) and can be added without violating the consistency of the truncation scheme up to \( O(G^2) \). The term has been added in such a way that when identifying (108) with (107), the derivative \( \frac{\partial}{\partial E} L_\Delta(E) \) will become proportional to a physical scale \( \Delta \). Together with the relation

\[
Z'(E_M + \bar{\omega}_M) - E R(E_M + \bar{\omega}_M) = \chi_\Delta(E, \bar{\mu}_M + \bar{\omega}_M) R(E_M + \bar{\omega}_M) ,
\]

\[
\chi_\Delta(E, \bar{\mu}_M + \bar{\omega}_M) = \bar{\mu}_M + \bar{\omega}_M - \bar{L}_\Delta(E_M + \bar{\omega}_M) ,
\]
which follows from (29) with the definition $\tilde{L}_\Delta(E) = Z'(E) L_\Delta(E)$, we obtain
\[
\frac{\partial}{\partial E} L_\Delta(E) = \frac{1}{2} \begin{array}{c}
\chi_{\Delta}\\ \times\\ \chi_{\Delta}
\end{array} + O(G^3), \quad \frac{\partial}{\partial E} L'(E) = \frac{1}{2} \begin{array}{c}
\chi_{\Delta}\\ \times\\ \chi_{\Delta}
\end{array} + O(G^3),
\]
where the symbol $\chi_{\Delta}$ at the resolvent means that the resolvent multiplied with $\chi_{\Delta}(E, \bar{\mu}_M + \bar{\omega}_M)$ has to be taken. Obviously, $\chi_{\Delta}$ is proportional to a physical scale, since $\bar{\mu}_M, \bar{\omega}_M$ and $L_\Delta(E_M + \bar{\omega})$ have this property. For $\omega_M$ this follows from the fact that the RG equations contain only the derivatives $\frac{\partial}{\partial E} = \frac{\partial}{\partial \sigma} f_\alpha(\bar{\omega})$ of the contractions, such that $|\omega_1| \lesssim T_n$. In contrast to the RG equation (96) for the full Liouvillean $L(E)$, the RG equations (111) for $L_\Delta(E)$ and $L'(E)$ are first order differential equations. Therefore, the differences $L_\Delta(E_M + \bar{\omega}_M) - L_\Delta(E_M)$ and $L'(E_M + \bar{\omega}_M) - L'(E_M)$ are of $O(G^2)$ such that the frequency dependence of the resolvent and $\chi_{\Delta}$ entering the RG equations (111) can be approximated by
\[
\begin{align*}
R(E_M + \bar{\omega}_M) &= \frac{1}{\bar{\omega}_M + E_M - L_\Delta(E_M)} Z'(E_M) + O(G^2) \quad (112) \\
\chi_{\Delta}(E, \bar{\mu}_M + \bar{\omega}_M) &= \bar{\mu}_M + \bar{\omega}_M - L_\Delta(E_M) + O(G^2) \quad . (113)
\end{align*}
\]
As a consequence, all frequency integrations can be straightforwardly performed such that the differential-integro equations (111) are converted into differential equations. E.g., at zero temperature, the two frequency integrations in (111) are trivial leading to the explicit expression
\[
\begin{align*}
\frac{\partial}{\partial E} L_\Delta(E) &= \frac{1}{2} G_{12}(E) \frac{\bar{\mu}_{12} - L_\Delta(E_{12})}{E_{12} - L_\Delta(E_{12})} Z'(E_{12}) G_{21}(E_{12}) \quad , (114) \\
\frac{\partial}{\partial E} L'(E) &= \frac{1}{2} G_{12}(E) \frac{1}{E_{12} - L_\Delta(E_{12})} Z'(E_{12}) G_{21}(E_{12}) \quad , (115)
\end{align*}
\]
\[\text{together with the RG equation for the vertex which follows from the lowest order term of (97) as}\]
\[
\frac{\partial}{\partial E} G_{12}(E) = G_{13}(E) \frac{1}{E_{13} - L_\Delta(E_{13})} Z'(E_{13}) G_{32}(E_{13}) - (1 \leftrightarrow 2) \quad . (116)
\]

**Solution of approximate E-RTRG equations.** The first-order RG equations (114) and (115) for $L_\Delta(E)$ and $L'(E)$ provide the most convenient starting point for an analytical solution of the RG equations at least in that regime of the complex plane where the effective vertices stay small, see Refs. [10, 26] for details. The strategy is to solve the RG equations approximately in three different energy regimes by expanding in the effective vertices but keeping large logarithmic terms (either at large or low energies) to all orders, and matching the different solutions to fix the integration constants. Denoting the small dimensionless coupling constant by $\alpha$, we distinguish the following regimes: (1) The regime of high energies $|E| \gg |z_n|$, where the RG resums all ultraviolet logarithmic terms $\sim (\alpha \ln \frac{Z}{E})^k$; (2) The regime of intermediate and small energies $|E - z_n| \lesssim O(|z_n|)$ but $E$ not too close to the branching points such that one can expand in the small parameter $\alpha \ln \frac{|z_n|}{|E - z_n|} \ll 1$; (3) The regime of small energies exponentially close to some of the branching points, i.e. $|E - z_n| \ll O(|z_n|)$ and $\alpha \ln \frac{|z_n|}{|E - z_n|} \sim 1$, where the RG resums all infrared logarithmic terms $\sim (\alpha \ln \frac{|z_n|}{|E - z_n|})^k$. In particular for the ohmic spin boson model and the IRLM, we will see in Section 6 that the coupling constant $\alpha$ stays small in the whole complex plane such that a well-controlled analytical solution is possible for
all $E$, showing that the resummation of logarithmic terms for high and low energies gives very different results. For the Kondo problem a weak-coupling solution is only possible for high, intermediate and small energies, but not for exponentially small energies where the coupling constant $\alpha \sim O(1)$.

**Initial conditions.** The initial conditions for the RG flow at large energies are set up at the value $E = iD$, where $D \gg |z_n|$ is the high-energy cutoff. The motivation for the choice $E = iD$ lies in the fact that, for $D \gg |E| \gg |z_n|$, the bare perturbation series for $L(E)$ and $L'(E)$ contain logarithmic terms $\sim (\ln \frac{D}{|E|})^k$ of all powers $k$ (we have chosen $-iE$ in the argument, such that the branch cut is directed towards the negative imaginary axis). All other terms $\sim (\frac{|E|}{D})^n$ are neglected since they vanish in the limit $D \to \infty$ and thus do not contribute to the universal solution which is independent of the cutoff details. Extrapolating this result up to $E = iD$ has the effect that all logarithmic terms vanish, which sets the initial point for the universal RG flow. The calculation of the initial values can be done by bare perturbation theory for $D \gg |E| \gg |z_n|$ and omitting all logarithmic contributions. For small bare coupling constants it is sufficient to take the lowest order term if it is universal, otherwise one takes zero for the initial condition. We note that this procedure works well to determine the universal initial condition for $L(E)$ and $L'(E)$ at $E = iD$ but fails for the initial condition of $L(E)$ since $L(E)$ contains terms linear in $E$ which are very large for $E = iD$. Therefore, for the RG equation (96) one either has to keep the high-energy cutoff function in the RG equations and start the RG flow at $|E| \gg D$ (where one can take the bare values as initial condition), or one has to find a reference point at low energies where $L(E)$ is known from exact results, see e.g. the solution of the Kondo model in strong coupling in Ref. [6].

**RG for the Liouvillian discontinuity jumps.** Finally we show how RG equations can be derived for the jump $\delta L$ of the Liouvillian at a particular branch cut, as this jump is needed to evaluate the branch cut contributions to the time evolution, see Eq. (48). As described in Section 4 the branch cuts of $L(E)$ occur only at zero temperature and can be identified in the perturbative expansion (83) by closing all frequency integrations in the upper half of the complex plane and considering the branch cuts of the Fermi distribution functions on the positive imaginary axis. This means that the frequencies are shifted to the positive imaginary axis $\bar{\omega}_M \to i|\bar{\omega}_M|$. In leading order, a given branch cut at $E = z_n - ix \pm 0^+$ is generated by some resolvent which is resonant, i.e. the jump of this resolvent across the branch cut becomes a $\delta$-function. If the resolvent contains the eigenvalue $\lambda_k(E_M + i|\bar{\omega}_M|)$ the resonance occurs if $z_n = z^p_k - \bar{\mu}_M$. With $E_M = z_n + \bar{\mu}_M - ix \pm 0^+ = z^p_k - ix \pm 0^+$, we replace approximately $\lambda_k \to z^p_k$, $P_k \to P_k(z^p_k - ix)$ and $Z' \to \tilde{Z}'(z^p_k - ix)$, which gives for the jump of the resolvent the following $\delta$-function

$$
\delta R = \left( \frac{1}{-ix + i\bar{\omega}_1\ldots + 0^+} - \frac{1}{-ix + i|\bar{\omega}_M| - 0^+} \right) \tilde{P}_k(z^p_k - ix) \tilde{Z}'(z^p_k - ix) = 2\pi\delta(|\bar{\omega}_M| - x) \tilde{P}_k(z^p_k - ix) \tilde{Z}'(z^p_k - ix) .
$$

(117)

As expected, the frequency integrals give only a contribution for $x > 0$, since this is the region where the branch cut starts. The RG equation for $\frac{\partial L}{\partial E}(z_n - ix)$ is obtained by fixing the resonant resolvent together with the resolvent where the $E$-derivative is taken and resumming the rest of the perturbative series in terms of effective 2-point vertices. If both resolvent are the same, the $E$-derivative is replaced by a frequency derivative $\frac{\partial}{\partial |\omega_i|}$ and is shifted via partial integration to
the derivative of some contraction crossing over the resolvent. Thus, we obtain in leading order

\[
\frac{\partial \delta L}{\partial E}(z_n - ix) = i \frac{\partial}{\partial x} \delta L(z_n - ix) = -\frac{1}{2} \text{[Diagram]} \tag{118}
\]

where the symbol \( \delta \) at the resolvent means that we replace the resolvent by its jump \( \delta R \) given by (117). The initial condition for the RG equation is \( \delta L(z_n) = 0 \). At zero temperature both frequency integrations are trivial. The contraction with the cross gives

\[
\int d\bar{\omega} f'_a(\bar{\omega}) \{ \ldots \} = -\{ \ldots \} \bar{\omega} = 0, \tag{119}
\]

where we have used that the jump of the sign-function at the branch cut is given by 2. Therefore, (118) gives explicitly

\[
\frac{\partial}{\partial x} \delta L(z_n - ix) = -\pi \theta(x) \bar{G}_{12}(z_k^p - \bar{\mu}_{12} - ix) \bar{P}_k(z_k^p - ix) \bar{Z}'(z_k^p - ix) \bar{G}_{21}(z_k^p - ix), \tag{120}
\]

where we have replaced all vertices that are discontinuous across the branch cut by their average value \( \bar{G} \). Up to the corrections from the weak \( x \)-dependence of the logarithmic functions on the r.h.s. of this equation, we obtain

\[
\delta L(z_n - ix) \sim x \theta(x), \tag{121}
\]

giving Eq. (50) used in Section 3. Therefore, if \( x \) can be neglected in the resolvents of the integrand of (88), we obtain (up to logarithmic corrections) \( \rho^{n,b}_{t} \sim 1/t^2 \) in the long-time limit for all models with spin or orbital fluctuations, like e.g. the Kondo model. Similar considerations show that the same holds for the ohmic spin boson model, whereas for models with charge fluctuations (like the IRLM), one obtains \( \delta L(z_n - ix) \sim \theta(x) \).

\section{Results}

In this section we will discuss the application of the formalism to the models introduced in Section 2. Since the Kondo model has the simplest algebra, we will take this model as a tutorial example to discuss the solution of the RG equations and the consequences for the time evolution in all detail. Since the general strategy is always the same, we will then show briefly the results for the ohmic spin boson model and the IRLM, and will concentrate on interesting features which are different from the ones for the Kondo model.

\subsection{Kondo model}

We consider the nonequilibrium Kondo model at zero magnetic field \( h^{(0)} = 0 \) and zero temperature \( T = 0 \) for the antiferromagnetic case \( J^{(0)} > 0 \). We assume that the local spin-\( \frac{1}{2} \) is coupled to several reservoirs with chemical potentials \( \mu_\alpha \). For the special case of two reservoirs \( \alpha \equiv L/R \equiv \pm \), we take \( \mu_\alpha = \alpha V \), where \( V \) denotes the bias voltage across the system. Following Refs. [5, 6] our aim is to calculate the time evolution of the local spin \( \langle \vec{S} \rangle(t) \).
The model is spin rotational invariant and therefore the effective Liouvillian $L(E)$ should be an invariant under spin rotations. Defining two basis spinoperators $L_{\pm}$ in Liouville space by ($A$ is an arbitrary local operator) $L_{+}A = aA$ and $L_{-}A = -Aa$, the only two invariants are given by the identity and $L_{+} \cdot L_{-}$. Since the Liouvillian must also fulfill $\text{Tr}L(E) = 0$, we find that the Liouvillian can be parametrized by

$$L(E) = -i \Gamma(E) L^{a} \ , \quad L^{a} = \frac{3}{4} + L^{+} \cdot L^{-} \ . \quad (122)$$

$\Gamma(E)$ is the energy dependent spin relaxation rate. The Liouvillian has one zero eigenvalue with projector $P_{st} = 1 - L^{a}$ and three degenerate eigenvalues at $-i \Gamma(E)$ with projector $1 - P_{st} = L^{a}$. Therefore, by using (28), we can write for the time evolution of the local density matrix

$$\rho(t) = \rho_{st} + \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{dE}{iE} e^{-iEt} \frac{1}{E + i\Gamma(E)} L^{a} \rho_{t=0} \ , \quad (123)$$

where $\rho_{st} = (1 - L^{a})\rho_{t=0}$ is the diagonal stationary density matrix with equal probabilities for both spin directions ($\rho_{stxx} = \frac{1}{2}$). Using $\text{Tr}S\rho(t) = \text{Tr}SL^{a}\rho(t)$ and $(L^{a})^{2} = L^{a}$, we find that the spin relaxation rate $\Gamma(E)$ determines the spin dynamics via

$$\langle S \rangle(t) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{dE}{iE} e^{-iEt} \frac{1}{E + i\Gamma(E)} \langle S \rangle_{t=0} \ . \quad (124)$$

As a consequence, the operator structure of the Liouvillian is no longer important, and we can use all formulas derived in Section 28 for the time evolution with the replacement $L(E) \rightarrow -i\Gamma(E)$ or $R(E) \rightarrow \frac{1}{E + i\Gamma(E)}$. This means that all projectors $P_{k}$ can be left out and $Z'(E)$ can be used from the decomposition

$$\Gamma(E) = \Gamma_{\Delta}(E) + E \Gamma'(E) \ , \quad R(E) = \frac{1}{E + i\Gamma(E)} = \frac{1}{E + i\Gamma_{\Delta}(E)} Z'(E) \ , \quad (125)$$

with

$$\tilde{\Gamma}_{\Delta}(E) = Z'(E) \Gamma_{\Delta}(E) \ , \quad Z'(E) = \frac{1}{1 + i\Gamma'(E)} \ . \quad (126)$$

Analysing the functions $\tilde{\Gamma}_{\Delta}(E)$ and $Z'(E)$ in leading order from the RG equations (114) and (115), we will show below that the branching poles and branching points of the resolvent $R(E)$ are given by

$$z_{0}^{p} = -i\Gamma^{*} \ , \quad z_{\alpha\alpha'}^{b} = -i\Gamma^{*} + \mu_{\alpha} - \mu_{\alpha'} \quad \text{for} \quad \alpha \neq \alpha' \quad (127)$$

where we used the notation of Eq. (41) and assumed that all reservoirs have different chemical potentials (otherwise they can be taken together). $z_{0}^{p} = i\tilde{\Gamma}_{\Delta}(z_{0}^{p})$ is the pole of the resolvent $R(E)$ and $\Gamma^{*}$ is called the Korringa rate which, as will be shown below, is given by

$$\Gamma^{*} = 2\pi J_{V}^{2} \sum_{\alpha \neq \alpha'} x_{\alpha} x_{\alpha'} |\mu_{\alpha} - \mu_{\alpha'}| \ , \quad J_{V} = \frac{1}{2 \ln \frac{1}{T_{K}}} \ , \quad T_{K} = D e^{-1/(2J^{(0)})} \ , \quad (128)$$

where we have used the notation of Eq. (13). $J_{V}$ is the renormalized exchange coupling at the scale $V = \max_{\alpha\alpha'} \{|\mu_{\alpha} - \mu_{\alpha'}|\}$ (which is the bias voltage for two reservoirs) and $T_{K}$ is called the Kondo temperature. The result for the Korringa rate holds in the weak-coupling case $V \gg T_{K}$.
which we will consider from now on. In this case, we get \( J_V \ll 1 \) and \( \Gamma^* \ll V \). Using (41) and (127), we get the following general form for the time evolution of the local spin

\[
\langle S \rangle(t) = F_0^\eta(t) e^{-\Gamma t} \langle S \rangle(t=0) + \sum_{\alpha \neq \alpha'} F_{\alpha\alpha'}^b(t) e^{-i(\mu_\alpha - \mu_{\alpha'})t} \langle S \rangle(t=0).
\]  

(129)

For intermediate and long times \( t \gtrsim \frac{1}{t} \), the pre-exponential functions can be calculated from (45) and (48) as

\[
F_0^\eta(t) = Z'(i\Gamma^* - i/t),
\]

(130)

\[
F_{\alpha\alpha'}^b(t) = -\frac{i}{2\pi} \int_0^\infty dx e^{ixt} \frac{Z'_{\alpha\alpha'}(x) - i/t}{(\mu_\alpha - \mu_{\alpha'} - ix)^2},
\]

(131)

where we have neglected \(-i\Gamma^* + i\tilde{\Delta}(\hat{z}_{\alpha\alpha'} - ix)\) in the dominator of the integrand of the last equation. This can be done for \( |\mu_\alpha - \mu_{\alpha'}| \gg \Gamma^* \) for all \( \alpha \neq \alpha' \), i.e. if the branch cuts are sufficiently apart from each other (other cases can be treated as well but need a special procedure [5]). For short times \( t \ll \frac{1}{t} \), we can use (42) and get

\[
\langle S \rangle(t) = Z'(1/t) e^{-\tilde{\Delta}(1/\xi)t} \langle S \rangle(t=0).
\]  

(132)

To evaluate (130), (131) and (132) explicitly, we need the functions \( \tilde{\Delta}(E), Z'(E) \) and \( \delta \Gamma(z_{\alpha\alpha'} - ix) \), which we will derive in the following by considering the RG equations (114), (115) and (120), together with the RG equation (116) for the vertex.

In leading order it can be shown that the effective vertex \( G_{11'}(E) \) can be parametrized in the same form as the initial vertex \( G_{11'}^{(0)} \), defined by (15) and (55). This gives the form

\[
G_{11'}(E) = -J_{\alpha\alpha'}(E) \frac{1}{2} \cdot \varphi_{\alpha\alpha'}, \text{ for } \eta = -\eta' = +, \quad L^2 = -\frac{1}{2} (L^+ + L^-),
\]

(133)

together with \( G_{11'}(E) = -G_{11'}(E) \) for \( \eta = -\eta' = - \). Using this ansatz together with the form (122) for the Liouvillian in the RG equations and omitting the terms \( \sim L^a \) in the RG equation for the vertex (which generate higher orders), one finds after some straightforward algebra [6] the following RG equations

\[
\frac{\partial}{\partial E} \Gamma_{\Delta}(E) = i \chi_{\Delta}(E, \hat{\mu}_{\alpha\alpha'}) R(\hat{E}_{\alpha\alpha'}) J_{\alpha\alpha'}(E) J_{\alpha'\alpha}(\hat{E}_{\alpha\alpha'}) ,
\]

(134)

\[
\frac{\partial}{\partial E} Z'(E) = Z'(E)^2 R(\hat{E}_{\alpha\alpha'}) J_{\alpha\alpha'}(E) J_{\alpha'\alpha}(\hat{E}_{\alpha\alpha'}) ,
\]

(135)

\[
\frac{\partial}{\partial E} J_{\alpha\alpha'}(E) = -\frac{1}{2} R(\hat{E}_{\alpha\alpha'}) J_{\alpha\alpha'}(E) J_{\alpha'\alpha}(\hat{E}_{\alpha\alpha'}) - \frac{1}{2} R(\hat{E}_{\alpha'\alpha'}) J_{\alpha\alpha'}(E) J_{\alpha\alpha'}(\hat{E}_{\alpha'\alpha}) ,
\]

(136)

where \( \hat{E}_{\alpha\alpha'} = E + \hat{\mu}_{\alpha\alpha'}, \hat{\mu}_{\alpha\alpha'} = \mu_\alpha - \mu_{\alpha'}, \chi_{\Delta}(E, \hat{\mu}_{\alpha\alpha'}) = \hat{\mu}_{\alpha\alpha'} + i\tilde{\Delta}(\hat{E}_{\alpha\alpha'}), \) and \( Z'(E) \) and \( R(E) \) have been defined in (125) and (126). The initial conditions at \( E = iD \) are \( \Gamma_{\Delta} = 0, \quad Z' = 1 \) and \( J_{\alpha\alpha'} = 2\sqrt{x_\alpha x_{\alpha'}} J^{(0)} \).
We first start with the analytic solution in the regime of \textbf{high energies} $|E| \gg V$. Neglecting $\mu_{\alpha\alpha'}$ everywhere gives the solution $\Gamma_\Delta(E) \approx 0$ and $J_{\alpha\alpha'}(E) \approx 2\sqrt{x_\alpha x_{\alpha'}} J(E)$, together with

\begin{align}
\frac{\partial}{\partial E} Z'(E) &= \frac{4}{E} Z'(E)^2 J(E)^2 = \frac{4}{E} J(E)^2 + O(J^3) , \\
\frac{\partial}{\partial E} J(E) &= -\frac{2}{E} Z'(E) J(E)^2 = -\frac{2}{E} J(E)^2 + O(J^3) ,
\end{align}

where we have used $Z' = 1 + O(J)$ on the r.h.s. (see Eq. (139)). We find that $-1/2J(E) + \ln(-iE) \equiv \ln T_K$ is an invariant and $\partial J / \partial E = -2$, which gives the solution

\begin{equation}
Z'(E) = 1 - 2(J(E) - J^{(0)}) \to 1 - 2J(E) , \quad J(E) = \frac{1}{2\ln\left(\frac{iE}{T_K}\right)} ,
\end{equation}

where the Kondo temperature $T_K$ is defined in (128) and we used the scaling limit in the first equation, defined by $D \to \infty$, $J^{(0)} \to 0$, such that $T_K$ remains a constant. We have chosen $-iE$ in the argument of the logarithm to define a real value for the Kondo temperature at $E = iD$ and since we want the branch cut of the logarithm to point into the direction of the negative imaginary axis. In the solution (139) all logarithmic terms $\sim (J^{(0)}\ln \frac{D}{iE})^k$ have been resummed. $J(E)$ is the poor man scaling solution of the Kondo model, already introduced in the lecture B3 by T. Costi, but with the difference that $-iE$ plays now the role of the effective energy scale. Most importantly, the solution would diverge at $E = iT_K$ when extrapolated to small energies, indicating an increase of antiferromagnetic spin fluctuations at the scale of the Kondo temperature. However, in this regime the solution can not be used since the RG flow becomes very different for $|E| \lesssim V$. The solution at high energies can be used to evaluate the universal solution (132) for short times $t \ll \frac{1}{V}$

\begin{equation}
\langle S \rangle(t) = \left(1 - \frac{1}{\ln(T_K t)}\right) \langle S \rangle(t=0) .
\end{equation}

In this result all logarithmic terms $\sim (J^{(0)}\ln(Dt))^k$ have been resummed, which can be seen from $1/\ln(T_K) = -2J^{(0)}/(1 - 2J^{(0)}\ln(Dt))$. Sub-leading terms are not included but can be taken into account by truncating the RG equations at $O(G^3)$ [5]. For ferromagnetic Kondo models the universal short time behaviour has also been discussed in Ref. [27] using flow equation methods.

To find the solution at \textbf{intermediate and small energies} $|E| \lesssim V$ but not exponentially close to the singularites such that $J(E) \ll 1$ is still fulfilled (we state below what this precisely means), we first set the initial value by expanding the solution (139) at high energies for the case when $E$ starts to approach $V$ such that $|E| \gg V$ is still fulfilled but $J_V = \ln \left(\frac{iE}{V}\right) \ll 1$, where $J_V = 1/(2\ln(V/T_K))$ is the exchange coupling at high energies evaluated at the scale $-iE = V$, as introduced in (128). This gives

\begin{equation}
J_{\alpha\alpha'}(E) \approx 2\sqrt{x_\alpha x_{\alpha'}} J_V (1 - 2J_V \ln \frac{-iE}{V}) , \quad Z'(E) \approx 1 - 2J_V + 4J_V^2 \ln \frac{-iE}{V} .
\end{equation}

Using the first term of this expansion in the r.h.s. of the full RG equations (134), (135) and (136),
we can easily integrate the equations up to $O(J_0^2)$ by using $R(E) = \frac{\partial}{\partial E} \ln \frac{-i(E + i\tilde{\Delta}(E))}{V} + O(J_0^2)$

$$
\hat{\Gamma}_\Delta(E) = i 4 x_\alpha x_\alpha' J_\nu^2 (\hat{\mu}_{\alpha\alpha'} + i \hat{\Gamma}_\Delta(\hat{E}_{\alpha\alpha'})) \ln \frac{-i(\hat{E}_{\alpha\alpha'} + i \hat{\Gamma}_\Delta(\hat{E}_{\alpha\alpha'}))}{V},
$$

$$
Z'(E) = 1 - 2 J_\nu + 4 x_\alpha x_\alpha' J_\nu^2 \ln \frac{-i(\hat{E}_{\alpha\alpha'} + i \hat{\Gamma}_\Delta(\hat{E}_{\alpha\alpha'}))}{V},
$$

$$
J_{\alpha\alpha'}(E) = 2 \sqrt{x_\alpha x_\alpha'} J_\nu \left\{ 1 - x_{\alpha\alpha'} J_\nu \ln \frac{-i(\hat{E}_{\alpha\alpha'} + i \hat{\Gamma}_\Delta(\hat{E}_{\alpha\alpha'}))}{V}
- x_{\alpha\alpha'} J_\nu \ln \frac{-i(\hat{E}_{\alpha\alpha'} + i \hat{\Gamma}_\Delta(\hat{E}_{\alpha\alpha'}))}{V} \right\}.
$$

The integration constants have been chosen such that for $|E| \gg V$ and $J_\nu \ln \frac{1}{|E - z_n|} \ll 1$, the result \((141)\) at high energies is reproduced. From the solution we can see that branch cuts appear starting at the singularities $z_n = -i \Gamma^* + \hat{\mu}_{\alpha\alpha'}$, as stated in \((127)\). Furthermore, we can see that the expansion is well-defined provided that $J_\nu \ln \frac{1}{|E - z_n|} \ll 1$, which is the precise condition that $\hat{E}$ should not be exponentially close to the branching points. This is the reason why the scale $\hat{V} \sim |z_n|$ has been chosen as reference scale in the logarithm to integrate the RG equations perturbatively for intermediate and small energies. In the solution all logarithmic terms $\sim (J(0) \ln \frac{1}{V})^k$ have been resummed in $J_\nu$, whereas a perturbative treatment has been used for the logarithmic terms $J_\nu \ln \frac{1}{|E - z_n|} \ll 1$.

Since $\hat{\Gamma}_\Delta(E)$ is a weakly varying function for $|E| \ll |z_n|$, we can replace $\hat{\Gamma}_\Delta(\hat{E}_{\alpha\alpha'}) \to \Gamma^*$ in the above equations and neglect the term $\sim J_0^2 \Gamma^* \sim J_0^2$ in \((142)\). Inserting $E = -i \Gamma^* + \delta$ in \((142)\) (where $|\delta| \ll \Gamma^*$ is a small scale to exclude an exponentially small region around $z_0^p = -i \Gamma^*$), we find straightforwardly the result \((128)\) for $\Gamma^*$. Inserting the solution for $Z'(E)$ in \((130)\), we can calculate the pre-exponential function for the contribution from the branching pole at $E = z_0^p = -i \Gamma^*$. For long times $t \gg \frac{1}{\Gamma^*}$ we obtain

$$
F_0^p(t) = 1 - 2 J_\nu - 4 \sum_\alpha x_\alpha^2 J_\nu^2 \ln(Vt),
$$

whereas, for intermediate times $t \sim \frac{1}{\Gamma^*}$, the contribution in $O(J_0^2)$ is not logarithmic and unimportant (the precise coefficient is also influenced by other sub-leading terms). Several interesting features appear in this result. The first term is the result from a Markov approximation, where only the pole without the residuum is considered. We note that the pole position is also influenced by non-Markovian contributions arising when \((142)\) is solved self-consistently for $\Gamma^*$. Here, this is a very weak effect occuring in $O(J_0^2)$. For quantum dot models such non-Markovian contributions have been dicussed perturbatively in Ref. \[28\]. All other terms of \((145)\) are of pure non-Markovian nature arising from the term linear in $E$ of the effective Liouvillian (leading to the $Z'$-factor). The second term linear in $J_\nu$ can not be obtained from perturbation theory since this term of the $Z'$-factor involves the difference $J_\nu - J(0)$ (see Eq. \((139)\)), which reduces to $J_\nu$ only in the scaling limit. It arises from a resummation of a series of logarithmic terms $\sim (J(0) \ln \frac{1}{V})^k$ which starts at $k = 2$, i.e. the $k = 1$ term is absent. The last term $\sim J_0^2 \ln(Vt)$ is logarithmic in time and becomes of $O(1)$ for exponentially large times $t \sim \frac{1}{\Gamma^*} e^{1/J_0^2}$. In this regime the solution can no longer be used since it corresponds to the regime of energies $E$ exponentially close to $z_0^p$. In this regime, the full RG equation \((136)\) for the vertex
shows that $J(E)$ does not stay small, i.e. a strong coupling problem arises and the truncation scheme is no longer controlled. As a consequence we see that, concerning the long-time evolution at exponentially large times, even in the regime $V \gg T_K$, a strong coupling method is needed to calculate pre-exponential functions. On the other hand, the exponential decay $e^{-\Gamma t}$ leads to a very small contribution for exponentially large times, so that it is of no practical use to know the pre-exponential function in this regime. However, for other problems with quantum critical points, like e.g. multi-channel Kondo models or the sub-ohmic spin boson model, it happens that the pole $z_{0b}^b = 0$ lies at the origin such that no exponential decay appears. For such models, it is an interesting subject for the future to calculate the precise form of $F_0^p(t)$ for exponentially large times. E.g., for a multi-channel Kondo model with many channels $N \gg 1$, which turns out to be a weak-coupling problem in the whole complex plane, it has been shown in Ref. [9] that $F_0^p(t) \sim (1/T_K t)^{4/N}$.

Finally, to calculate the branch cut contribution (131), we need also the jump $\delta \Gamma(z_{\alpha\alpha'}^b - i\epsilon)$ for $x \sim 1/t \lesssim V$. This is obtained from the RG equation (120), which reads with $J_{\alpha\alpha'}(E) \rightarrow 2/x_{\alpha\alpha'}J_V$

$$\frac{\partial}{\partial x} \delta \Gamma(z_{\alpha\alpha'}^b - i\epsilon) = -8\pi i x_{\alpha\alpha'} J^2 V \theta(x) .$$

(146)

This leads to $\delta \Gamma(z_{\alpha\alpha'}^b - i\epsilon) = -8\pi i x_{\alpha\alpha'} J^2 V \theta(x)$. Inserting this result in (131) and using $Z' = 1 + O(J_V^2)$, we obtain for long times $t \gg 1/|\mu_{\alpha} - \mu_{\alpha'}|$ (note that $\alpha \neq \alpha'$ and we assumed that $|\mu_{\alpha} - \mu_{\alpha'}| \gg \Gamma^*$)

$$F_{\alpha\alpha'}^b(t) = -4x_{\alpha\alpha'} J^2 V \left(\frac{1}{(\mu_{\alpha} - \mu_{\alpha'})^t}\right)^2 .$$

(147)

Other time regimes $t \sim 1/|\mu_{\alpha} - \mu_{\alpha'}|$ can also be studied leading to exponential integrals [5]. Inserting (147) in (129) we get an oscillating term $\sim J^2 V \left(\frac{1}{(\mu_{\alpha} - \mu_{\alpha'})^t}\right)^2 e^{-\Gamma^* t} e^{-i(\mu_{\alpha} - \mu_{\alpha'}) t}$ for the time evolution of the local spin. It appears in second order in $J_V$ and is again of non-Markovian nature. In contrast to the Markov contribution it oscillates with a frequency set by the differences of chemical potentials and the pre-exponential function decays as a power law $\sim 1/t^2$ for long times. This behaviour is quite generic for models with spin or orbital fluctuations. In higher orders the oscillation frequencies are set by the renormalized excitation energies of the system associated with certain processes. E.g. a process where a particle is transferred from reservoir $\alpha$ to reservoir $\alpha'$ involves an energy cost $\mu_{\alpha'} - \mu_{\alpha}$, which gives the oscillation frequency. In the presence of a local magnetic field $h^{(0)}$, the same process costs the energy $\mu_{\alpha'} - \mu_{\alpha} \pm \hbar$ if the local spin is flipped, where $\hbar$ is the renormalized magnetic field. As a consequence, these scales define further oscillation frequencies. In addition, each process has its own decay rate, setting the scale of the exponential decay. These issues have been discussed in detail in Ref. [5] for the case of the anisotropic Kondo model at finite magnetic field.

### 6.2 Ohmic spin boson model

Here, we consider the ohmic spin boson model at zero bias $\epsilon = 0$ and zero temperature $T = 0$. We will follow Ref. [10] where the model has recently been solved for weak damping $\alpha \ll 1$ by a systematic RG analysis using the E-RTRG method. In contrast to the Kondo model it turns out that the effective vertex $G(E) = G_1(E)_{\alpha=0}$ at zero frequency stays small in the whole complex plane allowing for a full solution of the problem on all time scales. We show here only the solution since the derivation is very similar to the one for the Kondo model, except that the
algebra is more involved and the solution of the RG equations can also be derived for $E$ close to the branching points $z_n$. It turns out that the resolvent $R(E)$ has four poles at

$$z_{st} = 0 \quad , \quad z_0 = -i \Gamma \quad , \quad z_{\pm} = \pm \tilde{\Delta} - i \Gamma / 2 \quad ,$$

where

$$\tilde{\Delta} = \Delta \left( \frac{\Delta}{D} \right)^{-\alpha} = \Delta \left( \frac{\Delta}{D} \right)^{\frac{\alpha}{1-\alpha}} \quad , \quad \Gamma = \pi \alpha \tilde{\Delta} \quad .$$

$\tilde{\Delta}$ is called the renormalized tunneling which is kept fixed in the scaling limit $D \to \infty$ and $\alpha \to 0$. In leading order truncation at $O(\alpha)$ it turns out that no branching poles appear, i.e. all poles are isolated. In addition, the eigenvalue $\lambda_0(E)$ has two branch cuts starting at $z_\pm$ and the eigenvalues $\lambda_{\pm}(E)$ have a branch cut starting at $z_0$. Therefore, according to the general expression (41) we get

$$\rho(t) = \rho_{st} + \sum_{k=0,\pm} \left( F^p_k(t) + F^b_k(t) \right) e^{-iz_k t} \rho_{t=0} \quad ,$$

i.e. all singularities can either act as a pole or as a branch cut. Ordering the four possible states in Liouville space by $++, --, +-, --$, where $\pm$ are the two local states, one can show that the stationary density matrix is given by $\rho_{st} = \frac{1}{2} \left( 1, \frac{1}{\Delta}, \frac{1}{\Delta}, 0 \right)$.

The pre-exponential functions for long times $\tilde{\Delta} t \gg 1$ (note that this includes the important regime $\Gamma t \sim O(1)$ where the exponentials are of $O(1)$) are given by [10]

$$F^p_0(t) = \frac{\tilde{\Delta}}{\Delta} \begin{pmatrix} 0 & 0 \\ -1 & \tilde{\Delta} / \Delta \end{pmatrix} \otimes \tau_+ \quad , \quad F^p_{\pm}(t) = \frac{1}{2} \begin{pmatrix} 1 & \pm \tilde{\Delta} / \Delta \\ \pm \tilde{\Delta} / \Delta & (\Delta / \Delta^2) \end{pmatrix} \otimes \tau_- \quad ,$$

$$F^b_0(t) = -2\alpha \frac{1}{(\Delta t)^2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \tau_- \quad , \quad F^b_{\pm}(t) = -\alpha \frac{s(t)}{(\Delta t)^2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \tau_+ \quad ,$$

where $\tau_\pm = \frac{1}{2} (1 \pm \sigma_x)$, and, for two $2 \times 2$-matrices $A$ and $B$, we have defined the $4 \times 4$-matrix

$$A \otimes B \equiv \begin{pmatrix} A_{11}B & A_{12}B \\ A_{21}B & A_{22}B \end{pmatrix} \quad .$$

Furthermore, the logarithmic function $s(t)$ is defined by

$$s(t) = \frac{1}{\left( 1 + \alpha \ln(\tilde{\Delta} t) \right) \left[ 1 - \ln(1 + \alpha \ln(\tilde{\Delta} t)) \right]} \quad .$$

In terms of the expectation values of the Pauli matrices $\langle \sigma_i \rangle = \text{Tr} \rho(t)$, these equations can also be written as

$$\begin{pmatrix} 1 \\ \langle \sigma_z \rangle(t) \end{pmatrix} = \begin{pmatrix} 1 \\ \langle \sigma_z \rangle(t) \end{pmatrix} + \frac{\tilde{\Delta}}{\Delta} \begin{pmatrix} 0 & 0 \\ -1 & \tilde{\Delta} / \Delta \end{pmatrix} \begin{pmatrix} 1 \\ \langle \sigma_z \rangle(t=0) \end{pmatrix} e^{-iz_0 t}$$

$$- \alpha \frac{s(t)}{(\Delta t)^2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \langle \sigma_z \rangle(t=0) \end{pmatrix} \sum_{\sigma=\pm} e^{-iz_{\sigma} t} \quad ,$$

$$\begin{pmatrix} \langle \sigma_z \rangle(t) \\ -i \langle \sigma_y \rangle(t) \end{pmatrix} = \frac{1}{2} \sum_{\sigma=\pm} \begin{pmatrix} 1 \\ \sigma \tilde{\Delta} / \Delta \end{pmatrix} \begin{pmatrix} \sigma \tilde{\Delta} / \Delta \\ (\tilde{\Delta} / \Delta^2) \end{pmatrix} \begin{pmatrix} \langle \sigma_z \rangle(t=0) \\ \langle i \sigma_y \rangle(t=0) \end{pmatrix} e^{-iz_{\sigma} t} \quad ,$$

$$- 2\alpha \frac{1}{(\Delta t)^2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \langle \sigma_z \rangle(t=0) \\ -i \langle \sigma_y \rangle(t=0) \end{pmatrix} e^{-iz_0 t} \quad .$$
In this result all logarithmic terms at high energies \( \sim (\alpha \ln \frac{D}{\Delta})^k \) have been resummed in the renormalized tunneling \( \tilde{\Delta} \), and all logarithmic terms at low energies (or large times) \( \sim (\alpha \ln(\tilde{\Delta} t))^k \) are contained in \( s(t) \). For the pre-exponential function \( F_0^b(t) \) it turns out that, in leading order, no logarithmic terms are present at large times. This has to be contrasted to the solution within the noninteracting blip approximation (NIBA) \cite{2}, where, for \( \langle \sigma_z \rangle_{t=0} = 1 \) and \( \langle \sigma_y \rangle_{t=0} = 0 \), one obtains

\[
\langle \sigma_z \rangle_{t=NIBA} = e^{-\frac{\tilde{\Delta} t}{\Gamma}} \cos(\Delta t) - 2\alpha \frac{1}{(\Delta t)^{2-2\alpha}},
\]

whereas the correct result from (156) reads

\[
\langle \sigma_z \rangle_{t} = e^{-\frac{\tilde{\Delta} t}{\Gamma}} \cos(\Delta t) - 2\alpha \frac{1}{(\Delta t)^2} e^{-\Gamma t}.
\]

Besides the missing exponential part in the second term, which has already been discussed at the end of Section \[4\], the NIBA predicts a different exponent for the pre-exponential power law. This shows that power-law exponents of pre-exponential functions can only be calculated by resumming consistently all logarithmic terms for long times. The E-RTRG method predicts that no such logarithmic terms are present for \( \langle \sigma_{y,z} \rangle_{t} \) but they appear for \( \langle \sigma_z \rangle_{t} \) within the logarithmic function \( s(t) \). The leading power-law behaviour \( \sim (\frac{t}{\Gamma})^2 \) of the pre-exponential function is the same as for the Kondo model and can also be obtained from perturbative calculations \cite{29}.

There are always two terms with different decay rates \( \Gamma \) and \( \Gamma/2 \) for the time evolution. If one transforms to the exact eigenbasis \( \epsilon_{1/2} = \frac{1}{\sqrt{2}}(\ket{+} \pm \ket{-}) \) of the local system, the expectation values \( \langle \gamma_i \rangle \) of the Pauli matrices in the new basis are related to the ones of the original basis by \( \langle \gamma_x \rangle = \langle \sigma_z \rangle, \langle \gamma_y \rangle = -\langle \sigma_y \rangle \) and \( \langle \gamma_z \rangle = \langle \sigma_x \rangle \). Thus the Markovian term \( \sim e^{-\Gamma t} \) from the pole contribution describes the decay of the diagonal matrix elements of the density matrix in the new basis, whereas the one \( \sim e^{-(\Gamma/2)t} e^{\pm \Delta t} \) corresponds to the decay of the nondiagonal matrix elements. Therefore, \( \Gamma \) is called the relaxation rate, whereas \( \Gamma/2 \) is the decoherence rate, in accordance with the general rule that, in the absence of pure dephasing, the relaxation rate is always twice as large as the decoherence rate.

For large energies \( |E| \gg \tilde{\Delta} \), one needs the function \( Z'(E) \) to determine the regime of short times \( t \ll \frac{1}{\Delta} \) from (42) (the contribution from the exponential is a small correction and can be neglected). One obtains the result

\[
Z'(E) = \sum_{\sigma=\pm} \left( \begin{array}{cc} 1 & 0 \\ 0 & Z_\sigma(E) \end{array} \right) \otimes \tau_\sigma, \quad Z_\pm(E) \approx \left( -i E \frac{D}{\alpha} \right)^{2\alpha}.
\]

This gives rise to the universal short time behavior

\[
\rho(t) = \left( \begin{array}{cc} 1 & 0 \\ 0 & (\frac{1}{D t})^{2\alpha} \end{array} \right) \otimes 1 \rho_{t=0}
\]

or

\[
\langle \sigma_{x,y} \rangle_{t} = (\frac{1}{D t})^{2\alpha} \langle \sigma_{x,y} \rangle_{t=0}, \quad \langle \sigma_z \rangle_{t} = \langle \sigma_z \rangle_{t=0}.
\]

This agrees with previous predictions and can also be obtained from the exact solution (20) at \( \Delta = T = 0 \) in the universal regime \( t \gg \frac{1}{\Delta} \). Again we can see that all logarithmic terms \( \sim (\alpha \ln(D t))^k \) have been resummed in this result.
6.3 Interacting resonant level model

Finally we discuss the IRLM for the special case of a single reservoir with chemical potential \( \mu = 0 \) and zero level position \( \epsilon = 0 \) (i.e. in resonance with the reservoir). As discussed in Section 2 this model can be mapped to the ohmic spin boson model close to the exactly solvable point \( \alpha = \frac{1}{2} \). In particular, we want to understand where the crossover from coherent to incoherent time evolution by changing the sign of \( U = 1 - \sqrt{2\alpha} \) comes from. We follow Refs. \cite{8, 9}, where the IRLM has been studied by using E-RTRG and functional RG.

We concentrate on the time evolution of the occupation \( \langle n \rangle(t) \) of the local level which is related via Eq. (22) to the expectation \( \langle \sigma_z \rangle(t) \) within the spin boson model by \( 2 \langle n \rangle(t) - 1 = \langle \sigma_z \rangle(t) \). For \( \langle \sigma_z \rangle(t) \), one can show that the result can be written in the form

\[
P(t) = \frac{i}{2\pi} \int_{-\infty+i0^+}^{\infty+i0^+} dE e^{-iEt} \frac{1}{E + i\Gamma_1(E)}.
\]

(162)

For this special case there is no \( Z' \)-factor and \( \Gamma_1(E) \) is a slowly varying logarithmic function describing the energy dependent charge relaxation rate. It is determined from the RG equations

\[
\frac{\partial}{\partial E} \Gamma_1(E) = -g R_2(E) \Gamma_1(E), \quad \frac{\partial}{\partial E} \Gamma_2(E) = -g R_1(E) \Gamma_1(E),
\]

(163)

where \( g = 2U - U^2 = 1 - 2\alpha \) and the resolvents \( R_{1/2}(E) \) are defined by

\[
R_1(E) = \frac{1}{E + i\Gamma_1(E)}, \quad R_2(E) = \frac{1}{E + i\Gamma_2(E)/2}.
\]

(164)

The initial conditions are given by \( \Gamma_{1/2}(E = iD) = \Gamma^{(0)} \). \( \Gamma_2(E)/2 \) is also a slowly varying logarithmic function and describes the energy dependent broadening of the local level corresponding to the decoherence mode for nondiagonal matrix elements of the local density matrix w.r.t. the charge states (note, however, that such elements can not be prepared). As we will see below the subtle coupling of the two RG equations for \( \Gamma_{1/2}(E) \) leads to the interesting effect that, for \( g > 0 \), the resolvent \( R_1(E) \) can have poles with a finite real part although the local system has no finite excitation energy.

We start by solving the RG equations at high energies \( E \gg \Gamma_{1/2}(E) \). Neglecting \( \Gamma_{1/2}(E) \) on the r.h.s. of the RG equations, we find the solution

\[
\Gamma_{1/2}(E) = \Gamma^{(0)} \left( \frac{D}{-iE} \right)^g = \tilde{\Delta} \left( \frac{\tilde{\Delta}}{-iE} \right)^g,
\]

(165)

where

\[
\tilde{\Delta} = \Gamma^{(0)} \left( \frac{D}{\Delta} \right)^g = \Gamma^{(0)} \left( \frac{D}{\Gamma^{(0)}} \right)^{g/(1+g)}
\]

(166)

is the renormalized tunneling which is kept fixed in the scaling limit \( D \to \infty \) and \( \alpha, \Gamma^{(0)} \to 0 \). Using the relation \( g = 1 - 2\alpha \) and \( \Gamma^{(0)} = \frac{\Delta^2}{D} \) to the spin boson model, one can see that it is identical to the definition (149) of the renormalized tunneling for the spin boson model. As discussed in detail in Refs. \cite{7, 9}, the solution at high energies contains all leading logarithmic
charge relaxation mode, whereas for the spin boson model at small poles the decoherence mode for the IRLM and its imaginary part is half of the one of the relaxation decoherence mode w.r.t. the exact eigenstates of the local system. Therefore, for intermediate and small energies we can calculate with (42) the time evolution for short times $t \ll 1/\Delta$ as

$$P(t) \approx e^{-\Gamma_1(1/0)t} \approx e^{-(\Delta t)^{1+g}},$$

i.e. the relaxation rate in the exponent is cut off at the energy scale $1/t$. In contrast to the spin boson model at small $\alpha$ and the Kondo model, there is no $Z'$-factor and therefore the exponential provides the leading order. Expanding the exponential we find $P(t) = 1 - (\Delta t)^{1+g}$ in agreement with previous results [2]. Since $(\Delta t)^{1+g} = (Dt)^g \Gamma(0,t)$, we see again that all logarithmic terms $\sim (g \ln(Dt))^k$ have been resummed for small times.

Next we study the analytic structure of the resolvent $R_1(E)$ to find the time evolution for intermediate and long times. As we will show below, for positive $g > 0$, $R_1(E)$ has two poles at $z_\pm$ (followed by a branch cut with jump of $O(g^2)$ which can be neglected) and one branch cut starting at $z_0$ (with jump of $O(g)$), where the singularities $z_n, n=0, \pm$, are given by

$$z_0 = -i \frac{\tilde{\Delta}}{2}, \quad z_\pm = \pm \Omega - i \tilde{\Delta}, \quad \Omega = \pi g \tilde{\Delta}.$$ (168)

For $g < 0$, there is only a branch cut starting at $z_0$. Thereby, $z_0$ is the position of the pole of the resolvent $R_2(E)$, i.e. $z_0$ and $z_\pm$ can be determined from the equations

$$z_\pm + i \Gamma_1(z_\pm) = 0, \quad z_0 + i \Gamma_2(z_0)/2 = 0.$$ (169)

Note that, in contrast to the singularities (138) for the spin boson model at small $\alpha$, for the IRLM (or the spin boson model at $\alpha = 1/2$) the renormalized tunneling determines the rate and not the oscillation frequency. Furthermore, we note that the pole of $R_1(E)$ describes the charge relaxation mode, whereas for the spin boson model at small $\alpha$ it corresponds to the decoherence mode w.r.t. the exact eigenstates of the local system. Therefore, $z_0$ corresponds to the decoherence mode for the IRLM and its imaginary part is half of the one of the relaxation poles $z_\pm$. To derive the result for the positions of the singularities we solve the RG equations for intermediate and small energies $|E| \ll \tilde{\Delta}$ but $g \ln \frac{\tilde{\Delta}}{|E-z_n|} \ll 1$, i.e. $E$ should not be exponentially close to the singularities. Expanding in the small parameter $g \ln \frac{\tilde{\Delta}}{|E-z_n|} \ll 1$ and fixing the integration constants by comparing with the solution (165) at high energies in the usual way, we find

$$\Gamma_1(E)/\tilde{\Delta} \approx 1 - g \ln \frac{-iE + \Gamma_2(E)/2}{\Delta}, \quad \Gamma_2(E)/\tilde{\Delta} \approx 1 - g \ln \frac{-iE + \Gamma_1(E)}{\Delta}.$$ (170)

In contrast to the corresponding equation (142) for the Kondo model, there is a subtle coupling of the singularities of $\Gamma_1(E)$ and $\Gamma_2(E)$, which leads to the new feature that $z_\pm$ obtains a finite real part for $g > 0$. We note that although the equations cannot be used for $E$ exponentially close to the singularities, they can be used for $|E - z_n| \sim g^2$ since $g \ln(g) \ll 1$ for $g \ll 1$. Therefore, the equations can be used to determine the positions of the branching points of $\Gamma_1(E)$ up to $O(g)$. From the equations we can see that $\Gamma_1(E)$ ($\Gamma_2(E)$) have a branch cut with jump of $O(g)$ starting at the branching point of the logarithmic function where (169) is fulfilled, i.e. at $z_0$ ($z_\pm$). Thereby, the branch cut of $\Gamma_2(E)$ starting at $z_\pm$ leads also to a branch cut for $\Gamma_1(E)$ at the same position but this branch cut has a jump of $O(g^2)$ and can be neglected.
Inserting the leading order results $\Gamma_{1/2}(E) \approx \tilde{\Delta}$, $z_0 \approx -i\tilde{\Delta}/2$ and $z_\pm \approx -i\tilde{\Delta}$ on the r.h.s. of (170), we find for the position of the singularities the result (168)

$$2iz_0/\tilde{\Delta} = \Gamma_2(z_0)/\tilde{\Delta} \approx 1 - g \ln(-iz_0/\tilde{\Delta} + 1) \approx 1 - g \ln\left(-\frac{1}{2} + 1\right) \approx 1 \quad ,$$

$$iz_\pm/\tilde{\Delta} = \Gamma_1(z_\pm)/\tilde{\Delta} \approx 1 - g \ln\left(-iz_\pm/\tilde{\Delta} + \frac{1}{2}\right) \approx 1 - g \ln\left(-1 \mp i\Omega/\tilde{\Delta} + \frac{1}{2}\right) \approx 1 \pm i\pi g \quad .$$ (172)

Due to the analytic structure of the resolvent $R_1(E)$ the time evolution can be written as

$$P(t) = \theta(g) \sum_{\sigma=\pm} e^{-iz_\sigma t} + F_0^b(t) e^{-iz_0 t} = \theta(g) 2 \cos(\Omega t) e^{-\tilde{\Delta} t} + F_0^b(t) e^{-\tilde{\Delta}/2 t} \quad ,$$ (173)

where the first term involves the contribution from the isolated poles (we have neglected corrections of $O(g)$ to the residuum) and the second term involves the analog of the branch cut integral (48), which can be written as

$$F_0^b(t) = \frac{1}{\pi} \text{Im} \int_0^\infty dx e^{-xt} e^{-i(t_0 + 1/t + 0^+) - x} \quad .$$ (174)

For intermediate and long times $t \gtrsim \frac{1}{\Delta}$ but $g \ln(\tilde{\Delta} t) \ll 1$, $F_0^b(t)$ can be evaluated by using the result (170), where we obtain $-i\Gamma_1(z_0 - i/t + 0^+) = z_\pm(1 + O(g \ln(\tilde{\Delta} t)))$. In particular one has to consider the fact that $x \sim 1/t$ can not be neglected compared to the difference $|z_0 - z_\pm| \sim \tilde{\Delta}$ for intermediate times $t \sim \frac{1}{\Delta}$. This time regime is of particular interest here since the exponentials of the time evolution (173) decay on the time scale $\frac{1}{\Delta}$. Therefore, the integral (174) has to be calculated more carefully in terms of the exponential integral $E_1(z)$

$$F_0^b(t) = -\frac{1}{\pi} \text{Im} \left\{ e^{-i(z_0 - z_\pm) t} E_1(-i(z_0 - z_\pm) t) \right\} \quad .$$ (175)

This result has been used in Refs. [8, 9] to discuss the competition between the oscillating (i.e. coherent) and the purely decaying (i.e. incoherent) term of the time evolution in Eq. (173). Since the incoherent term decays on a longer time scale it turns out that it wins very rapidly such that the coherent term leads only to a few number of oscillations, in contrast to the physics of a classical damped harmonic oscillator. For long times $t \gg \frac{1}{\Delta}$ but still $g \ln(\tilde{\Delta} t) \ll 1$, the incoherent term dominates and, using the asymptotic expansion $E_1(z) \approx e^z/z$ of the exponential integral, one obtains

$$F_0^b(t) \approx -4g \frac{1}{\tilde{\Delta} t} \quad ,$$ (176)

i.e. a power law $\sim 1/t$ typical for models with charge fluctuations.

Finally, for exponentially large times $g \ln(\tilde{\Delta} t) \sim O(1)$, we need the solution for $\Gamma_1(E)$ for energies $E$ exponentially close to the branching point $z_0$. In this regime, we can replace $\Gamma_2(E)/2 \rightarrow iz_0$ on the r.h.s. of the RG equation (163) for $\Gamma_1(E)$, which gives the solution

$$\Gamma_1(E) = \tilde{\Delta} \left( \frac{-\tilde{\Delta}}{-i(E - z_0)} \right)^g \quad ,$$ (177)
where the integration constant has been fixed by comparison with the solution (170) at intermediate and small energies. Using this solution for the evaluation of the branch cut integral (174) for exponentially large times, we can neglect $x$ in the denominator and find with $\Gamma_1(z_0 - i/t + 0^+) = \tilde{\Delta}(\tilde{\Delta}t)^g + 2\pi ig\tilde{\Delta}(\tilde{\Delta}t)^g$ the result (neglecting terms of $O(g^2)$ in the denominator)

$$F^b_0(t) \approx -g \frac{1}{(1/2 - (\tilde{\Delta}t)^g)^2 (\tilde{\Delta}t)^{1-g}}.$$  

This result holds for all times $t \gg \frac{1}{2}$. For long times with $g \ln(\tilde{\Delta}t) \ll 1$ it reduces to the result (176). However, for exponentially large times where $(\tilde{\Delta}t)^g$ is some number of $O(1)$, the result changes. In the extreme regime $(\tilde{\Delta}t)^g \gg 1$, it reduces to

$$F^b_0(t) \approx -g [1 + 3\theta(-g)] \frac{1}{(\tilde{\Delta}t)^{1+|g|}}.$$  

This result agrees with the prediction of the NIBA [2] and its improved version [24] (where the exponential term $e^{-(\tilde{\Delta}/2)t}$ has also been obtained, see Eq. (173)). However, as we have seen, it holds only for extremely long times and, for $g > 0$, the prefactor is different from the result (176) for more realistically long times $\tilde{\Delta}t \gg 1$ with $g \ln(\tilde{\Delta}t) \ll 1$. Therefore, we see that the regime of long times is very subtle and the result can change significantly by entering the regime of exponentially large times.

Finally, as already mentioned in Section 5, it has not yet been studied to a full extent how the RG equation (163) looks like in higher orders in the tunneling. There is some evidence that all higher order terms in $\Gamma$ are of the form

$$U^n \left( \frac{\Gamma_i}{\tilde{\Delta}} \right)^k (\ln \frac{E - z_n}{\Delta})^l \text{ for } n = 1, 2, \quad U^n \frac{\Gamma_i}{E - z_n} \left( \frac{\Gamma_i}{\tilde{\Delta}} \right)^k \text{ for } n > 2,$$  

i.e., after integration, either vanish in the limit $E \rightarrow z_n$ or contribute to higher orders in $U$, but this is still under investigation. Furthermore, the results have been compared to functional RG in Refs. [8, 9], where all orders in the tunneling have been resummed keeping only the lowest order term in the Coulomb interaction. The numerical results of functional RG agree quite nicely with the analytical result (175) for intermediate and long times and, in particular for extremely long times, the result (179) has been confirmed analytically by functional RG. Therefore, there is good evidence that also within E-RTRG higher orders in the tunneling will not change the results at least in leading order in $U$.

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