Temperley-Lieb $R$-matrices from generalized Hadamard matrices

Jean Avan$^a$, Tiago Fonseca$^b$, Luc Frappat$^b$
Petr Kulish$^c$, Eric Ragoucy$^b$ and Geneviève Rollet$^a$

$^a$ Laboratoire de Physique Théorique et Modélisation (CNRS UMR 8089), Université de Cergy-Pontoise, F-95302 Cergy-Pontoise, France

$^b$ LAPTh, CNRS and Université de Savoie, 9 Chemin de Bellevue, BP 110, F-74941 Annecy le Vieux Cedex

$^c$ St. Petersburg Department of Steklov Mathematical Institute Fontanka 27, 191023, St. Petersburg, Russia

Abstract

New sets of rank $n$-representations of Temperley-Lieb algebra $T_L(q)$ are constructed. They are characterized by two matrices obeying a generalization of the complex Hadamard property. Partial classifications for the two matrices are given, in particular when they reduce to Fourier or Butson matrices.

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1Emails: avan@u-cergy.fr, tiago.dinis@lapth.cnrs.fr, frappat@lapth.cnrs.fr, kulish@pdmi.ras.ru, ragoucy@lapth.cnrs.fr, rollet@u-cergy.fr
1 Introduction

The Temperley-Lieb algebra (hereafter denoted TL) $TL_N(q)$ \cite{1,2}, has been used extensively as a powerful algebraic tool in the construction and derivation of quantum integrable models of great interest in statistical mechanics and solid state physics (see e.g. \cite{2,3}). Special representations of the TL algebra where the generators are copies of a single endomorphism acting on a tensor product $V \otimes V$, $V$ being an $n$-dimensional vector space, give rise to constant solutions $R$ of the Yang-Baxter equation. Yang-Baxterization procedures are then systematically available (see e.g. \cite{4}). From such Yang-Baxterized $R$-matrices one then may in particular construct integrable quantum spin chains \cite{5} on the space of states $\mathcal{H} = (\mathbb{C}^n)^{\otimes N}$ for any integer $n$. These spin chains are very similar to the spin $1/2$ XXZ-model.

Specific representations of TL algebra were introduced in e.g. \cite{5,6}: they are parametrized by a single bivector $b$ yielding a rank-1 projector on $V \otimes V$. The Temperley-Lieb parameter $q$ to be defined hereafter was identified by $q + \frac{1}{q} \equiv -\text{tr}(bb^t)$. A classification of solutions to the reflection equation associated to the derived $R$-matrix was proposed in \cite{4}, aiming at building quantum integrable open spin chains.

An extension of these representations involving $n$ such bivectors was proposed in \cite{7,8} as relevant in the context of entanglement and quantum computing. The TL parameter or “loop index” is then identified by $q + \frac{1}{q} \equiv \sqrt{n}$. The matrices, originally parametrized by $n$ bivectors, were naturally written as $n^2 \times n^2$ matrices as in e.g. the $n = 3$ case of \cite{8}:

\[
U^{(II)} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \omega & 0 & 1 & 0 \\
0 & 1 & 0 & \omega & 0 & 0 & 0 & 0 & \omega \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & \omega^2 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
\omega^2 & 0 & 0 & 0 & 0 & 1 & 0 & \omega^2 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & \omega & 0 & 1 & 0 \\
0 & \omega^2 & 0 & 1 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

where $\omega^2 + \omega + 1 = 0$. It turns out (see below) that these new TL representations can more appropriately be reformulated in terms of a sum of $n^2$ ordinary tensor products of two $n \times n$ matrices, namely:

\[
T_i = \sum_{a,b=1}^{n} \mathbb{1}^{\otimes (i-1)} \otimes e_{ab} \otimes M^{n_a-n_b} \otimes \mathbb{1}^{\otimes (N-i-1)}, \quad i = 1, \ldots, N
\]  

(1.1)

where $e_{ab}$ denotes the canonical form of the generators of $n \times n$ matrices, $M$ is a single invertible $n \times n$ matrix and $n_a$ are integers.

Precisely the representation $U^{(II)}$ in \cite{8} takes the form (1.1) with:

\[
M = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & \omega \\
1 & 0 & 0
\end{pmatrix}
\]

In an explicit way $U^{(II)}$ reads:

\[
U^{(II)} = \begin{pmatrix}
M^0 & M^2 & M \\
M^{-2} & M^0 & M^{-1} \\
M^{-1} & M & M^0
\end{pmatrix}
\]  

(1.2)
As usual in such representations the \( i \)-th generator \( T_i \) of TL acts non trivially only on the two copies of \( V \) labeled by resp. \( i \) and \( i + 1 \) in the full tensorized representation space \( \bigotimes_{k=1}^{N} V_{(k)} \). The \( R \) matrix deduced from such an object is simply \( R_{i,i+1} = \Pi_{i,i+1}(q \mathbb{I} \otimes \mathbb{I} + T_i) \) where \( \Pi_{i,j} \) generically denotes the permutation operator on tensorized spaces \( V_i \otimes V_j \), and \( \mathbb{I} \) the identity. In our study, the \( M \) matrix will be restricted to be diagonalizable. Jordan-reducible matrices shall be considered elsewhere.

This provides us with an interesting example of rank-\( n \) realizations of the TL algebra and motivates our current investigation of such generic realizations. The study of associated scalar reflection matrices can be achieved on lines following [4] but will be left for another paper. One may expect that the new solutions which we propose here may be of interest, again in the description of quantum entanglement effects, or more canonically as building blocks for closed or open spin-chain like models after Yang-Baxterization. We shall come back to this in our conclusion.

The presentation runs as follows. In Section 2 we prepare the necessary notations, introduce precisely the Temperley-Lieb algebra and the rank-\( n \) Ansatz which we use. We then derive the relevant equations to be solved for a complete resolution based on this Ansatz [14].

In Section 3 we separate these equations into a polynomial equation (denoted Master Polynomial equation) controlling the eigenvalues of \( M \) and a matrix equation controlling the eigenvectors of \( M \). Remarkably both sets are characterized by \( n \times n \) matrices obeying an extension which we define (General Hadamard Condition or GHC) of the Complex Hadamard property [9, 10]. We then discuss the explicit classification of eigenvalues and eigenvectors based on these relations. The eigenvalues are encapsulated into a Master Matrix obeying the general Hadamard condition. It however involves delicate issues not yet fully clarified, since the Hadamard condition is here necessary but not sufficient. The eigenvectors by contrast are entirely determined by the choice of an arbitrary generalized Hadamard matrix once the Master Matrix is known.

A partial set of solutions to the Complex Hadamard condition and its generalization is given in Section 4. The representation \( U^{(1)} \) in [7] is a simple example of a slightly more general set of objects which is discussed in Section 5. Finally we give some conclusions and perspectives.

2 General properties and equations

Let us first recall the general context of our discussion and obtain the equations to be solved to get at least a partial classification of the solutions.

2.1 Hecke and Temperley-Lieb Algebras

The braid group \( B_N \) is generated by \((N-1)\) generators \( \tilde{R}_j \), \( j = 1, 2, \ldots, N - 1 \), their inverses \( \tilde{R}_j^{-1} \) and the relations (see [11]):

\[
\tilde{R}_j \tilde{R}_k \tilde{R}_j = \tilde{R}_k \tilde{R}_j \tilde{R}_k, \quad \text{for } |j - k| = 1 \quad \text{and} \quad \tilde{R}_j \tilde{R}_k = \tilde{R}_k \tilde{R}_j, \quad \text{for } |j - k| > 1. \tag{2.1}
\]

Both Hecke algebra \( H_N(q) \) and Temperley-Lieb algebra \( TL_N(q) \) are quotients of the group algebra of \( B_N \):
The Hecke algebra $H_N(q)$ is obtained by adding to these relations the following constraints obeyed by each generator $\hat{R}_j$ ($q$-deformation of the symmetric group):

$$\left(\hat{R}_j - q\mathbb{I}\right)\left(\hat{R}_j + 1/q\mathbb{I}\right) = 0.\tag{2.2}$$

where $\mathbb{I}$ denotes the identity in the Hecke algebra. Equation (2.2) is equivalent to write $\hat{R}_j$ in term of some idempotent $X_j$, namely:

$$\hat{R}_j = q\mathbb{I} + X_j\tag{2.3}$$

with

$$X_j^2 = -\left(q + \frac{1}{q}\right)X_j.\tag{2.4}$$

The braid group relations (2.1) read in terms of the idempotents $X_j$ and $X_k$ such that $|j - k| = 1$:

$$X_jX_kX_j - X_j = X_kX_jX_k - X_k.\tag{2.5}$$

The TL algebra $TL_N(q)$ is obtained as the quotient algebra of the Hecke algebra $H_N(q)$ by the set of equations requiring that each side of (2.5) be zero. To sum up, $TL_N(q)$ is defined by the generators $X_j$, $j = 1, 2, \ldots, N - 1$ and their relations:

$$X_j^2 = -\nu(q)X_j,\tag{2.6}$$

$$X_jX_kX_j = X_j, \quad |j - k| = 1,\tag{2.7}$$

$$X_jX_k = X_kX_j, \quad |j - k| > 1\tag{2.8}$$

with $\nu(q) = q + 1/q$.

In connection with integrable spin systems we will be interested in representations of $TL_N(q)$ on the tensor product space $\mathcal{H} = (\mathbb{C}^n)^{\otimes N}$. The $\hat{R}_j$ generators are now represented in terms of endomorphisms on $\mathcal{H}$ acting non trivially on a pair $(j, j + 1)$ of adjacent spaces $V$. These endomorphisms are self-explanatorily denoted as $\hat{R}_{j,j+1}$. Conditions (2.1) are then represented as the braided Yang-Baxter equation:

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{21}\hat{R}_{12}\hat{R}_{23}.$$\tag{2.9}

The $R$-matrix is immediately defined from this representation of the braid group generators by $R_{j,j+1} = \Pi_{j,j+1}\hat{R}_{j,j+1}$, with $\Pi(v \otimes v') = v' \otimes v$ for any pair of vectors of $\mathbb{C}^n$. The indexation $(j, j + 1)$ of $\Pi$ is again self-explanatory. The notation $R_{j,j+1}$ is then straightforwardly extended to define general endomorphisms $R_{ij}$ of $\mathcal{H}$ labeled by any non-adjacent pair of “site indices” $(i, j)$, using the time-honored notation [12] for such elements of $End(\mathcal{H})$ with indices labelling the spaces.

Equation (2.9) then immediately becomes the Yang-Baxter equation for $R$:

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}.$$\tag{2.10}

Of course any matrix realization of the YB algebra (2.10) can be gauged to another matrix realization by the conjugation $R_{ij} = g \otimes g R_{ij} g^{-1} \otimes g^{-1}$ where $g$ is any invertible $n \times n$ matrix. This gauging freedom, naturally also valid for the considered TL representations, will be used in our reformulation of the Ansatz for TL representations.

Let us finally formulate the Yang-Baxterization procedure of these $R$-matrices. In fact the Yang-Baxterization procedure is already valid at the stage of abstract Hecke algebra generators. Indeed if one defines the spectral parameter-dependent $R$-matrix as [13]

$$\hat{R}_j(u) = u\hat{R}_j - \frac{1}{u}\hat{R}_j^{-1} = (u - \frac{1}{u})\hat{R}_j + \frac{\omega(q)}{u}\mathbb{I}; \quad \omega(q) = q - \frac{1}{q}$$\tag{2.11}
one sees that it obeys the cubic equation in braid group form with multiplicative spectral parameter \( u \) (additive spectral parameter is of course obtained as \( u \equiv e^{\lambda} \)):

\[
\hat{R}_j(u)\hat{R}_k(uw)\hat{R}_j(w) = \hat{R}_k(w)\hat{R}_j(uw)\hat{R}_k(u), \quad \text{for } |j - k| = 1.
\]

(2.12)

Now once the generators \( \hat{R} \) of the Hecke algebra \( H_N(q) \) itself have been represented as \( R \)-matrices acting on some tensor product of two finite-dimensional vector spaces, this procedure will immediately (see [4]) give rise to solutions of the non-constant braided Yang-Baxter equation with multiplicative spectral parameters:

\[
\hat{R}_{12}(u)\hat{R}_{23}(uw)\hat{R}_{12}(w) = \hat{R}_{23}(w)\hat{R}_{12}(uw)\hat{R}_{23}(u).
\]

(2.13)

### 2.2 The rank-\( n \) Ansatz and the master equation

The initial construction of a rank-1 TL representation was proposed in [5]. The \( U \) generators are represented by copies of a projector onto a single bivector in \( \mathbb{R}^n \) with multiplication by \( \sqrt{n} \). To eliminate these awkward \( \sqrt{n} \) factors we redefine the generators \( U_i \) by an overall multiplication by \( \sqrt{n} \). The one-loop equation (2.6) then gets a factor \( n \) and the equation (2.7) acquires a factor \( n \) on the r.h.s. It is these renormalized equations that we shall study from now on.

**Lemma 2.1.** Let \( M \) be an invertible diagonalizable \( n \times n \) matrix: \( M = \Lambda P \Lambda^{-1} \), where \( \Lambda \) is diagonal, \( \Lambda = \text{diag}(\lambda_1, ..., \lambda_n) \). Then, the matrices

\[
T_{i,i+1} = \sum_{a,b=1}^{n} \mathbb{I}^\otimes(i-1) \otimes e_{ab} \otimes M^{a-n_b} \otimes \mathbb{I}^\otimes(N-i-1), \quad i = 1, ..., N
\]

(2.14)

obey the TL algebra if and only if

\[
\forall \{i,j,u\} \subset \{1, \cdots, n\}, \quad \left( \sum_r \left( \frac{\lambda_j}{\lambda_i} \right)^{n_r} \right) \left( \sum_{k,l} P^{-1}_{i,k} P_{j,l} \lambda_u^{n_k-n_l} \right) = n \delta_{i,j}.
\]

(2.15)

**Proof:** Note that the generic gauge covariance of such TL representations \( T_{ij} \rightarrow g_i g_j T_{ij} g_i^{-1} g_j^{-1} \) allows us to reorder the indices \( 1, ..., n \) in such a way that \( n_a \geq n_b \) when \( a \geq b \). Moreover, since only the differences \( n_a - n_b \) play a role, up to a global shift, we can always assume that \( n_a \geq 0, \forall n_a \).

The form (2.13) automatically solves the one-loop condition (2.6), so that we only need to consider the second condition (2.7). It reads:

\[
\sum_{i,j,k,l,r} e_{i,j} \otimes M^{n_i-n_r} e_{k,l} M^{n_j-n_l} \otimes M^{n_k-n_i} = n \sum_{i,j} e_{i,j} \otimes M^{n_i-n_j} \otimes \mathbb{I}
\]

(2.16)

which is equivalent to:

\[
\sum_{r,k,l} M^{-n_r} e_{k,l} M^{n_l} \otimes M^{n_k-n_l} = n \mathbb{I} \otimes \mathbb{I}.
\]

(2.17)
We shall now restrict ourselves to matrices $M$ being invertible and diagonalizable. Hence we set $M = P\Lambda P^{-1}$, where $\Lambda$ is an invertible diagonal matrix.

Then the equation becomes

$$\sum_{r,k,l} \Lambda^{-n_r} P^{-1} e_k, l P^\Lambda \Lambda^{n_k-n_l} = n I \otimes I$$

or equivalently (2.15) by projecting on $e_{ij} \otimes e_{uu}$.

3 Resolution of the TL condition

We now extract from Eqs. (2.15) the master equations for eigenvalues and eigenvectors of the $M$ matrix. We first need to give some general key definitions for objects which we will come across in the course of this discussion.

3.1 Hadamard matrices and master equation

Definition 3.1. • A Complex Hadamard Matrix (CHM) is an $n \times n$ invertible matrix $U$ obeying

$$|U_{ij}| = 1, \quad \forall i, j = 1, ..., n$$

$$U = n (U^{-1})^\dagger,$$ (3.1) (3.2)

• A Generalized complex Hadamard Matrix (GHM) is an $n \times n$ invertible matrix $U$ with all its entries non-zero and obeying the single condition

$$U^{-H} = n (U^{-1})^t,$$ (3.3)

where $U^{-H}$ is the Hadamard inverse: $(U^{-H})_{i,j} = \frac{1}{U_{ij}}$.

• A complex (or generalized) Hadamard matrix $H$ is called dephased when all the entries of its first column and first row are equal to one, $H_{1,j} = H_{j,1} = 1, \forall j$.

Remark that the relation (3.3) is equivalent to

$$n U_{ij} (U^{-1})_{ji} = 1 \quad \forall i, j = 1, ..., n$$

Real Hadamard matrices (definition 3.1 with real entries $\pm 1$) date back to works of Sylvester [14]. Complex Hadamard matrices with entries restricted to be roots of unity are also known as Butson matrices, introduced in [15]. The situation with generic unimodular entries is described in e.g. [16]. The notion of Hadamard-type criterion for matrices with non-unimodular complex entries, which we introduce in Definition 2, is to the best of our knowledge a new one.

Note that this denomination of “Generalized Hadamard matrices” that we have introduced here to denote matrices satisfying (3.3) must not be confused with the (unfortunately) similarly-named notion in [15] which involved particular complex Hadamard matrices with an extra free parameter $k \neq n$: $U^* = k (U^{-1})^t$, and was later dropped to become “Butson matrices”.

Indeed our object generalizes the notion of a complex Hadamard matrix by replacing the complex conjugation (an idempotent operation on each matrix element) by the...
number-inverse, a similarly idempotent operation naturally extending it to non-unimodular complex numbers. The transposition operation on the matrix is kept. The complex Hadamard condition is then that the inverse of $U$ be given by the transposed of the complex conjugate matrix $U^*$. Any complex Hadamard matrix is therefore a generalized Hadamard matrix. The reciprocal problem will be addressed (but not solved) later: can any GHM be obtained by some well-defined procedure from a CHM?

**Lemma 3.2.**
- If $H$ is a CHM (resp. GHM) then $H' = \sigma_1 D_1 H D_2 \sigma_2$ is also a CHM (resp. GHM), where $D_j$, $j = 1, 2$ are unitary (resp. invertible) diagonal matrices and $\sigma_j$, $j = 1, 2$ are permutation matrices. Two complex (generalized) Hadamard matrices $H$ and $H'$ related in such a way are called equivalent.
- Any CHM (resp. GHM) is equivalent to a dephased CHM (resp. GHM).

These properties of CHM are to be found in e.g. [10]. Their extension to GHM is trivial.

We are now in a position to delve into our issue. Let us first introduce the matrix $\Omega$ with entries

$$\Omega_{i,j} = \lambda_i^{n_j}, \quad i, j = 1, ..., n$$

hereafter denoted *Master Matrix*.

**Proposition 3.3.** The Master matrix solving (2.15) must be a GHM:

$$\Omega^{-H} = n (\Omega^{-1})^t.$$  \hspace{1cm} (3.6)

Moreover, all the $n_a$'s have to be different, and the spectrum of $M$ must be simple.

**Proof:** Equation (2.15) can be rewritten in terms of the matrices $\Omega$ and $P$ as

$$\forall i, j, u \quad (\Omega^{-H} \Omega^t)_{i,j} (P^{-1} \Omega^t)_{i,u} (\Omega^{-H} P)_{u,j} = n \delta_{i,j}$$  \hspace{1cm} (3.7)

Summing equation (3.7) over $i$ or $j$ yields:

$$\forall i, u, \quad (P^{-1} \Omega^t)_{i,u} (\Omega^{-H} P (\Omega^{-H})^t)_{i,u} = n \quad (\text{summed over } j)$$  \hspace{1cm} (3.8)

$$\forall j, u, \quad (\Omega^{-H} P)_{u,j} (\Omega (\Omega^{-H})^t P^{-1} \Omega^t)_{j,u} = n \quad (\text{summed over } i)$$  \hspace{1cm} (3.9)

Therefore the two matrices $P^{-1} \Omega^t$ and $\Omega^{-H} P$ are full, i.e. all their entries are non-zero.

It is always consistent to write $\Omega^{-H} \Omega^t = n I_n + K$, where $K$ is some matrix with zero diagonal. Indeed one trivially sees from the definition of $\Omega$ that $(\Omega^{-H} \Omega^t)_{i,i} = n$ and $K$ therefore measures how far $\Omega^{-H}$ is from being the matrix inverse (if any) of $\Omega^t$.

From equation (3.7) one then gets:

$$\forall i, j, u \quad K_{i,j} (P^{-1} \Omega^t)_{i,u} (\Omega^{-H} P)_{u,j} = 0$$  \hspace{1cm} (3.10)

Since we have already established that both matrices $P^{-1} \Omega^t$ and $\Omega^{-H} P$ are full, one has necessarily $K_{i,j} = 0$. Hence $\Omega^{-H} \Omega^t = n I_n$, that is $\Omega$ is invertible and obeys (3.6).

Note immediately that any two integers $n_a$'s have to be distinct otherwise the matrix $\Omega$ would have at least two identical columns and would not be invertible.

A dual necessary condition is that no two distinctly labeled eigenvalues are equal (which would imply two identical lines in $\Omega$). In other words, no degeneracy of eigenvalues is allowed in a realization of the TL condition by diagonalizable $M$ matrices.
The TL condition (2.15), or equivalently (3.7), therefore factorizes completely into two sets of equations:

- The one obtained for \( i \neq j \) (and trivial at \( i = j \)) is the polynomial condition expressing that the Master Matrix is a GHM:

\[
\Omega^{-\mu} = n(\Omega^{-1})^t \quad \text{that is} \quad \sum_{a=1}^{n} \left( \frac{\lambda_i}{\lambda_j} \right)^{n_a} = n \delta_{ij} \quad (3.11)
\]

Solving this condition on the Master Matrix will yield simultaneously consistent sets of powers \( n_a \) for \( T \) and sets of eigenvalues \( \lambda_i \) for \( M \).

- The one obtained for \( i = j \) that yields a single condition for the \( P \) eigenvector matrices:

\[
\forall i, u, \quad \left( P^{-1} \Omega^t \right)_{i,u} \left( \Omega^{-\mu} P \right)_{u,i} = 1 \quad (3.12)
\]

But since \( \Omega^{-\mu} \) is \( n \) times the inverse of \( \Omega^t \) then \( \Omega^{-\mu} P = n \left( P^{-1} \Omega^t \right)^{-1} \) and therefore (3.12) actually means that the matrix \( \Omega^{-\mu} P \) is a generalized Hadamard matrix in the same sense as before (including the \( n \) factor). We shall denote it \( H \). Hence once the eigenvalues are determined by solving the condition (3.11), the associated consistent \( P \) matrices are obtained directly from the Master Matrix \( \Omega \) once a classification of generalized Hadamard matrices is available.

The problem therefore boils down to two issues, both related to the notion of generalized Hadamard matrices:

1. Find a classification of the generalized Hadamard matrices \( H \) (with complex entries) themselves (to get \( P \) from \( \Omega \) using \( H \)).

2. Find a characterization and/or a classification of all generalized Hadamard matrices which can be realized as Master Matrices, i.e. under the form (3.5), in order to get all consistent sets of \( \lambda_i \) and \( n_a \) obeying (3.11) and the associated master matrix \( \Omega \).

The integers \( n_a \) define a polynomial

\[
p(z) = \sum_{a=1}^{n} z^{n_a} \quad (3.13)
\]

hereafter called the **master polynomial**, and the condition (3.11) expresses that ratios of any two distinct eigenvalues of \( M \) are zeroes of \( p(z) \).

From these data one then reconstructs all \( M \) matrices as:

\[
M = \Omega^t H \Lambda H^{-1} \Omega^{-\mu}. \quad (3.14)
\]

We are now going to partially tackle these two issues.

### 3.2 Solving the generalized Hadamard condition

It must immediately be emphasized at this point that even in the much more studied case of complex Hadamard matrices no general classification exists. We are thus going to restrict ourselves to a description of the state of the art in this case, and a formulation of some exact results allowing to extend it to the generalized Hadamard condition.

Let us now focus on complex Hadamard matrices (\( |H_{ij}| = 1 \)). A quite complete picture of the current situation can be found in [16], see also [10]. To give a few salient facts:
- The classification is done for \( n = 2, 3, 4, 5 \);
- At \( n = 2, 3 \) and 5, only Fourier matrices \( \Omega_{ab} \equiv \omega^{(a-1)(b-1)} \) (where \( \omega = e^{\frac{2\pi i}{n}} \)) realize CHM (up to equivalence);
- At \( n = 4 \) an extra one-parameter family exists;
- At \( n = 6 \) several families (including a possibly quasi-all-encompassing 4 parameters family) exist [17];
- Conjectures [16] are available for partial classifications for \( n = p^k \), \( p \) prime; \( n = ab^k \), \( a, b \) prime; and many individual values of \( n \) [10].

The first issue now is to try to extend some of these conclusions to dephased generalized Hadamard matrices. Direct solution of the Generalized Hadamard property, by analytic or computer computations, are available for \( n = 1...4 \) and we shall presently give the results of these studies. They unfortunately become very cumbersome as soon as \( n \geq 5 \).

We have however identified a powerful, easily handled tool which generates GHM from CHM (sufficient condition):

**The thickening procedure**[4] consists in starting from any parametrized set \( M(a) \) of CHM such that the complex Hadamard criterion is satisfied solely due to the algebraic relations \( a_i \bar{a}_i = 1 \) for all parameters \( a_i \). If one substitutes in \( M \) the quantity \( \bar{a}_i \) by \( 1/a_i \) and relax the conditions \( |a_i| = 1 \), the resulting set of matrices obeys the generalized Hadamard criterion. This procedure is in particular valid for all families of parametric complex Hadamard matrices in dimension 4 and 6.

As an example let us consider the family \( F_4 \) of one parameter \( n = 4 \) complex Hadamard matrices. They are parametrized as:

\[
\Omega = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & \bar{a} & \bar{1} & \bar{a} \\
1 & a & \bar{1} & -a \\
1 & -a & \bar{1} & a
\end{pmatrix}
\text{ where } |a| = 1.
\] (3.15)

If now \( a \) is any non-zero complex number, these matrices then become generalized Hadamard matrices.

This procedure may be combined with several classical constructions described hereafter, used for the CHM, to get many more examples of GHM.

Let us conclude with the cases of dimension 2, 3, 4 where we have been able to get a full classification of GHM by explicit resolution of the equations.

- at \( d = 2, 3 \), GHM are identical to CHM;
- at \( d = 4 \), they are all obtained by thickening of CHM.

We have yet no such result at \( d = 5 \), in particular to get GHM matrices not identical to the Fourier-type CHM (the only such case existing at \( d = 5 \)).

### 4 GHM, master matrices and master polynomials

In this section we explain how to generate larger GHM, with special attention to the construction of master matrices. Our procedure is based on Dităş’s construction of complex Hadamard matrices, which is a generalization of the tensoring procedure.

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[1] We borrow this formulation from the notion of “thickened contours” used by Yu. I. Manin in e.g. Riemann-Hilbert procedures.
4.1 General constructions

4.1.1 Fourier matrices

There exists a general construction that provides one (up to equivalence) CHM which is also a master matrix. The construction can be done in any dimension, and the corresponding matrices are called Fourier matrices.

Let $\omega$ be a primitive $n$-th root of unity, i.e. $\omega = e^{i\ell\pi/n}$ with $\ell$ prime with $n$. The Fourier matrix is defined by

$$\Omega_{ab} = \omega^{(a-1)(b-1)}, \quad a, b = 1, \ldots, n. \quad (4.1)$$

A master matrix being of the form $\lambda n^b$, it is natural to identify $\lambda = \omega^{a-b}$ and $n^b = b - 1$. Notice that this is not the only solution, for example $n^b = k b_n + b - 1$ for some $k_b \in \mathbb{N}$ is also an acceptable identification.

We can then build the master polynomial:

$$F_n(z) = \sum_{b=1}^{n} z^{n^b} = 1 + z + \ldots + z^{n-1} = \frac{z^n - 1}{z - 1}. \quad (4.2)$$

The roots of this polynomial are $\frac{\lambda_a}{\lambda_b} = \omega^{a-b}$ for $a \neq b$, as expected. The solutions proposed in [7, 8] belong to this class.

4.1.2 Diţă’s construction

As for complex Hadamard matrices, if $A$ and $B$ are two generalized Hadamard matrices then $A \otimes B$ is also a generalized Hadamard matrix. Diţă generalized this construction:

**Lemma 4.1.** Let $A$ be an $n \times n$ complex Hadamard matrix and $\{B^{(1)}, \ldots, B^{(n)}\}$ be a family of $m \times m$ complex Hadamard matrices. Then the $nm \times nm$ matrix:

$$C = \begin{pmatrix}
A_{11}B^{(1)} & A_{12}B^{(1)} & \ldots & A_{1n}B^{(1)} \\
A_{21}B^{(2)} & A_{22}B^{(2)} & \ldots & A_{2n}B^{(2)} \\
\vdots & \vdots & \ddots & \vdots \\
A_{n1}B^{(n)} & A_{n2}B^{(n)} & \ldots & A_{nn}B^{(n)}
\end{pmatrix} \quad (4.3)$$

is also a complex Hadamard matrix.

This statement is also true for generalized Hadamard matrix.

The proof can be found in [18] for CHM and extends trivially to GHM.

4.2 Two examples

Because of lemma 3.2 we will work with dephased matrices.

4.2.1 $F_4$ family of complex Hadamard matrices

The single one-parameter family of complex Hadamard matrices of rank 4 can be represented by master matrices whenever the parameter $a$ is any root of unity. Let $\Omega$ be the matrix given in (3.15). It can be associated to the master polynomial $p(z) = (1+z)(1+z^{2k}) = 1 + z + z^{2k} + z^{2k+1}$. Let

$$\begin{align*}
\lambda_1 &= 1 \\
\lambda_2 &= -1 \\
\lambda_3 &= e^{i\pi/2k} = a \\
\lambda_4 &= -e^{i\pi/2k} = -a
\end{align*} \quad (4.4)$$
where \( m \) is odd. The master matrix reads \( \Omega_{ij} = \lambda_i^{n_j} \), where \( n_j \) are the exponents that appear in \( p(z) \), i.e.

\[
\{n_1, n_2, n_3, n_4\} = \{0, 1, 2k, 2k + 1\} \tag{4.5}
\]

Notice that varying \( m \) and \( k \) we get a dense set of \( a \in S^1 \).

### 4.2.2 \( F_6 \) family of complex Hadamard matrices

The two-parameter family \( F_6 \) complex Hadamard matrices of rank 6

\[
\Omega = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & \omega^2 & \omega^4 & 1 & \omega^2 & \omega^4 \\
1 & \omega^4 & \omega^2 & 1 & \omega^2 & \omega^4 \\
1 & a & b & -1 & -a & -b \\
1 & a\omega^2 & b\omega^4 & -1 & -a\omega^2 & -b\omega^4 \\
1 & a\omega^4 & b\omega^2 & -1 & -a\omega^4 & -b\omega^2 \\
\end{pmatrix} \tag{4.6}
\]

can be represented by master matrices whenever the parameters \( a, b \) are both any root of unity. We remind that in (4.6), \( \omega \) is a 6th root of unity.

We fix three integers \( k, r \) and \( s \) such that \( 0 < r, s < k \), and consider the polynomial

\[
p(z) = (1 + z^{3r+1} + z^{3s+2})(1 + z^{3k}) \tag{4.7}
\]

then the exponents \( n_i \) are

\[
\{n_1, n_2, n_3, n_4, n_5, n_6\} = \{0, 3r + 1, 3s + 2, 3k, 3k + 3r + 1, 3k + 3s + 2\} \tag{4.8}
\]

We chose the values of \( \lambda_i \) to be

\[
\lambda_1 = 1 \quad \lambda_3 = \omega^2 \quad \lambda_5 = \omega^4 \\
\lambda_2 = e^{i\frac{\pi}{6}} \quad \lambda_4 = \omega^2 e^{i\frac{\pi}{6}} \quad \lambda_6 = \omega^4 e^{i\frac{\pi}{6}}
\]

It is easy to check that all ratios \( \lambda_i/\lambda_j \) \( (i \neq j) \) are roots of \( p(z) \). The master matrix associated to these \( \lambda_i \) is exactly (4.6) with \( a = \lambda_2^{3r+1} \) and \( b = \lambda_2^{3s+2} \). Varying now \( k, r \) and \( s \) we get a dense set in \( S^1 \times S^1 \).

In the context of GHM, we allow \( a \) and \( b \) to be any non-zero complex number. However we cannot identify the resulting matrix with a master matrix, since for instance one should have \( \lambda_4^{n_2 n_4} = a^{n_4} = (-1)^{n_2} \) and therefore \( a \) must be a root of unity.

### 4.3 Nesting Fourier matrices

Both of these examples can be written using Dita’s construction (lemma 4.1). For instance, the second one corresponds to the Fourier matrices of size \( 2 \times 2 \) and \( 3 \times 3 \) and a diagonal matrix \( D \):

\[
A = \begin{pmatrix}
1 & 1 \\
1 & -1 \\
\end{pmatrix} \quad B = \begin{pmatrix}
1 & 1 & 1 \\
1 & \omega^2 & \omega^4 \\
1 & \omega^4 & \omega^2 \\
\end{pmatrix} \quad D = \begin{pmatrix}
1 & 0 & 0 \\
0 & a & 0 \\
0 & 0 & b \\
\end{pmatrix} \tag{4.9}
\]

where we set \( B^{(1)} = B \) and \( B^{(2)} = BD \).

This process of nesting is already manifest in the way we build the master polynomial. In what follows we show how to build new solutions nesting smaller solutions, the small block always being Fourier matrices. This will construct a very large class of solutions.
Let
\[ F_{p_1}(z) = \sum_{i=1}^{p_1} z^{g_{1i}p_1+i-1}, \]
(4.10)
where \( g_{1i} \in \mathbb{N} \). Pick the polynomial’s root \( \omega_1 = e^{\frac{2\pi i}{p_1}} \) and chose \( \lambda_i = \omega_1^{g_{1i}p_1+i-1} \). Then the associated master matrix is the Fourier matrix \( \Omega^{(p_1)}_{ij} = \omega_1^{(i-1)(j-1)} \).

We define \( F_{p_1p_2}(z) = F_{p_1}(z)F_{p_2}(z^{\eta_2}) \), where \( \eta_2 = k_1p_1 \) for some positive integer \( k_1 \), with the second polynomial being defined in the same way:
\[ F_{p_2}(z) = \sum_{i=1}^{p_2} z^{g_{2i}p_2+i-1}. \]
(4.11)

Let \( \omega_2 = e^{\frac{2\pi i}{\eta_2}} \) and chose
\[ \lambda_{i,j} = \omega_1^{f_{1i}p_1+i-1} \omega_2^{f_{2j}p_2+j-1}. \]
(4.12)

It is not difficult to show that
\[ F_{p_1p_2}\left(\frac{\lambda_{i,j}}{\lambda_{k,\ell}}\right) = n\delta_{ik}\delta_{j\ell}, \]
(4.13)
where \( n = p_1p_2 \).

The master matrix associated to the polynomial \( F_{p_1p_2} \) can be constructed using Dita’s construction:
\[ \Omega^{(p_1p_2)}_{ijk,\ell} = \lambda_{ij}^{(g_{1k}p_1+k-1)+\eta_2(g_{2\ell}p_2+\ell-1)} = \omega_2^{\eta_2(j-1)(\ell-1)} \Omega^{(p_1)}_{ik}D\left(\omega_2^{f_{2\ell}p_2+j-1}\right) \]
(4.14)
where \( D(z) \) is the diagonal matrix:
\[ (D(z))_{k\ell} = \delta_{k\ell}z^{g_{1k}p_1+k-1}. \]
(4.15)

This process can now be iterated\(^2\), the size of the final matrix being \( n = \prod_i p_i \). In that way, we obtain a large number of examples, including all examples that we were able to construct from known complex Hadamard examples. An interesting question to tackle would be to understand if this method is complete or to find a counter-example.

Notice that all the entries of the matrix are roots of unity, but the free parameters \( f_{ij} \) and \( g_{ij} \) allow us to create a dense set on \( S^1 \), when varying \( k_i \). Therefore, proving that all examples are obtainable using this method would imply that any master matrix is a CHM, the entries of which are restricted to be roots of unity, i.e. a Butson matrix.

An alternative approach is through the master polynomial. One can wonder whether it is possible to find a polynomial \( F(z) \) with coefficients in \( \{0, 1\} \), such that the two following conditions are satisfied: \( F(1) = n \) and there is a subset of its roots, \( \{\alpha_1, \ldots, \alpha_m\} \), that obeys relations of the type \( \alpha_i\alpha_j = \alpha_k \). Such problems have been studied in [19], though not exactly in our formulation.

### 4.4 Limitations

There are several limitations of this method.

- Although it provides a wide spectrum of master matrices and polynomials, we have no proof that it is exhaustive.

\(^2\)Define \( \eta_j = \prod_{i<j} k_ip_i \), where \( k_i \in \mathbb{N} \).

12
• In the construction of the master polynomials, not all of them correspond to distinct master matrices. For example:

$$F(z) = 1 + z^2 + z^3 + z^4 + z^6$$

also corresponds to the Fourier matrix based on the root $e^{j\frac{2\pi}{4}}$.

• Using this construction, we only construct master matrices composed solely by roots of unity. We must add that none of the thickened matrices in $d = 4, 6$ with matrix elements of module different from 1 are identified as master matrices for any polynomial. For example, if one considers a matrix of the form (4.6), only when $a = b$ and $b$ are module-1 complex numbers does $\Omega$ take the form of a Master Matrix. The same goes if we try to thicken CHM constructed by the above method.

4.5 Non-master Complex Hadamard matrices

It is important to note that not all complex Hadamard matrices are master matrices. Two examples:

$$H_0 = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & j & j & j^2 & j^2 \\
1 & j & 1 & j^2 & j^2 & j \\
1 & j & j^2 & 1 & j & j^2 \\
1 & j^2 & j^2 & j & 1 & j \\
1 & j^2 & j & j^2 & j & 1
\end{pmatrix} \quad H_1 = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & i & -i & -i & i \\
1 & i & -1 & a & -a & -i \\
1 & -i & -\bar{a} & -1 & i & \bar{a} \\
1 & -i & \bar{a} & i & -1 & -\bar{a} \\
1 & i & -i & -a & a & -1
\end{pmatrix}$$

where $j$ is a primitive cubic root of unity, and $a$ is a non-zero complex number.

We prove that $H_0$ is not a Master Matrix. Suppose that $(H_0)_{ij} = \lambda_i^{n_j}$, where $n_1, \ldots, n_6$ have no common divisor. All entries of $H_0$ are a third root of unity, and therefore $\lambda_i$ is a third root of unity. But there are only three different third roots of unity, which is in contradiction to the fact that $H_0$ has six different rows.

In a similar way we can prove that $H_1$ is not a master matrix either.

5 Generalized rank-$n$ Ansatz

We propose finally (and briefly) a generalization of the initial Ansatz. Indeed the rank-$n$ Ansatz which we started from (1.1) can be rewritten in a very illuminating form as:

$$T = \left( \sum_{i=1}^{n} e_{ii} \otimes M^{n_i} \right) \left( \Gamma \otimes I \right) \left( \sum_{j=1}^{n} e_{jj} \otimes M^{n_j} \right)^{-1}$$

where $\Gamma$ is the particular rank-1 projector $\Gamma \equiv v.v^t$, and $v$ is the $n$-vector with all components equal to 1.

Let us now extend this construction to a more general case of rank-1 projector $\Gamma \equiv v.w^t$ where $v$ and $w$ are any two $n$-vectors such that $\sum_{i=1}^{n} v_i w_i \equiv \alpha \neq 0$ (i.e. $\Gamma^2 = \alpha \Gamma$). Remark that in this construction, one sees immediately that $T$ is of rank $n$:

$$\text{rank}(T) = \text{rank}(\Gamma \otimes I) = \text{rank}(\Gamma) \text{rank}(I) = n.$$  (5.2)

The TL generators now read, generalizing (1.1):

$$T_i = \sum_{a, b=1}^{n} v_a w_b \Gamma^{i-1} \otimes e_{ab} \otimes M^{n_a-n_b} \otimes I^{\otimes (N-i-1)}, \quad i = 1, \ldots, N$$  (5.3)
In this generalized situation the whole derivation works out identically to realize representations of the TL algebra $T\!L_N(\sqrt{\alpha})$ by the Ansatz (5.1) at least in the case of diagonalizable $M$ matrices. Keeping the exact definition of the master matrix $\Omega$ as in (3.5) it appears that we must now solve a weighted generalized Hadamard condition for $\Omega$

$$\Omega^{-H}VW = \alpha(\Omega^{-1})^t$$  \hspace{1cm} (5.4)

Here $V, W$ are Cartan-algebra representations of the vectors $v, w$: $V \equiv \Sigma v_i e_{ii}$ and $W \equiv \Sigma w_i e_{ii}$.

A quasi-exact (up to replacing $n$ by $\alpha$) Hadamard condition will determine the $P$ matrix but this time for a “twisted” combination involving $V$ and $W$:

$$(P^{-1}V \Omega^t)_{iu}(\Omega^{-H}WP)_{ui} = 1$$  \hspace{1cm} (5.5)

General resolution of the weighted Hadamard condition (5.4) will be left for further studies.

The representation proposed in [7] takes exactly the form (5.1) or equivalently (5.3) albeit with more general vectors $v, w$ once the spurious parameters $q_1, q_2$ are gauged away using the standard gauge covariance for the TL conditions $T_{12} \rightarrow g_1 g_2 T_{12}(g_1 g_2)^{-1}$.

In [7], after getting rid of the gauge generated by:

$$g = \begin{pmatrix} q_2 & 0 & 0 \\ 0 & q_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

one obtains:

$$U^{(I)} = \begin{pmatrix} 1 & 0 & 0 & 0 & \omega & 0 & 0 & 0 & \omega^2 \\ 0 & 1 & 0 & 0 & \omega^2 & \omega & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ \omega^2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \omega \\ 0 & \omega & 0 & 0 & 0 & 1 & \omega^2 & 0 & 0 \\ 0 & \omega^2 & 0 & 0 & 0 & \omega & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ \omega & 0 & 0 & 0 & \omega^2 & 0 & 0 & 0 & 1 \end{pmatrix}$$

that takes the form (5.3) with :

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega \\ \omega^2 & 0 & 0 \end{pmatrix}$$

In a compact form, $U^{(I)}$ reads:

$$U^{(I)} = \begin{pmatrix} M^0 & \omega M & \omega M^2 \\ \omega^2 M^{-1} & M^0 & M \\ \omega^2 M^{-2} & M^{-1} & M^0 \end{pmatrix}$$  \hspace{1cm} (5.6)

The extra vectors $v, w$ have however the simplifying feature that their associated diagonal matrices obey $VW = 1$ hence the Master Matrix condition (5.4) is not modified. More precisely:

$$V = \begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$  \hspace{1cm} (5.7)
The condition (5.5) associated to $P$ can in this case be rewritten as a non-twisted condition (3.12) for the matrix $\tilde{P} = V^{-1}PV$. The solutions in [7] are thus very closely related to, but not identical with, matrices $M$ deduced from canonical Fourier-type solutions of the Hadamard conditions.

However due to the degeneracy condition $VW = 1$ this form actually becomes gauge-equivalent in the canonical TL sense (i.e. $T_{12} \rightarrow g_1g_2 T_{12} (g_1g_2)^{-1}$) to the original, pure-power form (2.14) with a conjugated $M$ matrix $\tilde{M} = g Mg^{-1}$. This situation is actually generic: whenever the diagonal matrices $V$ and $W$, built from the vectors $v$ and $w$, are inverse of each other, the “general” rank-$n$ Ansatz with $v$ and $w$ is TL-gauge equivalent to the standard one.

6 Conclusion

We have established an explicit construction of all diagonalizable building blocks $M$ for the Temperley-Lieb representation Ansatz (1.1). Complex Hadamard matrices and their generalization feature prominently in this construction, both in characterizing the set of eigenvalues (Master Matrix $\Omega$) and the set of eigenvectors (matrix $P$). It is interesting to remark that the original proposition for such generators of TL algebra [7, 8] stemmed from considerations on quantum entanglement: indeed Complex Hadamard matrices arise in particular in issues related to quantum computation and discrete matrix Fourier transform (in this last case most specifically Fourier matrices): they define so-called Walsh-Hadamard gates or more general quantum gates (see e.g. [20]). It is thus not a big surprise to see such a connection between TL representations and Hadamard matrices.

While eigenvectors are parametrized by GHM, it appears at this stage that all master matrices $\Omega$, encapsulating the eigenvalues of the matrix $M$, constructed explicitly in the previous sections, are complex Hadamard matrices of Butson type (i.e. entries are roots of unity) [15]. It is an open question whether more general master matrices of GHM type may occur; and to determine some sufficient criterion for a GHM to be rewritten as a Master Matrix.

The Butson matrices are the ones that are directly relevant to consideration on quantum entanglement and quantum computations issues [10]. The GHM however are at this stage not known to have any particular relationship to such problematics. The issue of their relevance and the relevance of the derived TL representations (with at least eigenvectors described by GHM instead of CHM) to some “generalized quantum computing” should be addressed.

A number of technical issues have been left for further analysis. The most pregnant one is probably the question of non-diagonalizable (Jordan-like) $M$ matrices. Very preliminary results [21] indicate that the notion of master polynomial survives for the non-degenerate eigenvalues (simple zeroes of the minimal polynomial). The formulation of TL conditions however is much more complicated due to the occurrence of off-diagonal contributions entangling with the pure eigenvalue-dependent equations.

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