General form of the renormalized, perturbed energy density via interacting quantum fields in cosmological spacetimes

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ABSTRACT: A covariant description of quantum matter fields in the early universe underpins models for the origin of species, e.g. baryogenesis and dark matter production. In nearly all cases the relevant cosmological observables are computed in a general approximation, via the standard irreducible representations found in the operator formalism of particle physics, where intricacies related to a renormalized stress-energy tensor in a non-stationary spacetime are ignored. Models of the early universe also include a dense environment of quantum fields where far-from-equilibrium interactions manifest expressions for observables with substantive corrections to the leading terms. A more detailed treatment of these cosmological observables may be carried out within the alternative framework of algebraic quantum field theory in curved spacetime, where the field theoretic model of quantum matter is compatible with the classical effects of general relativity. Here, we take the first step towards computing such an observable. We employ the algebraic formalism while considering far-from-equilibrium interactions in a dense environment under the influence of a classical, yet non-stationary, spacetime to derive an expression for the time-dependent energy density as a component of the renormalized stress-energy tensor associated with common proposals for quantum matter production in the early universe. We derive the analogous expression in the standard operator formalism and show that the results, presented here for the first time, reduce to those of the standard approach to non-equilibrium quantum field theory in the stationary spacetime limit.

KEYWORDS: Theoretical, Quantum Field Theory

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1 Introduction

A covariant description of quantum matter fields in the dynamical spacetime of the early universe is essential to proposals for models of baryogenesis and dark matter production. Calculation of cosmological observables involving the interactions of these fields are usually carried out at tree-level in the standard particle physics approach to quantum field theory, i.e. classical Boltzmann equations augmented with thermally averaged S-matrix derived interaction rates quantifying particle production in a covariant generalization of a non-stationary spacetime background (see refs. [1, 2] for a pedagogical treatment of the standard operator formulation of kinetic theory in a cosmological setting). However, in a non-stationary Friedmann–Robertson–Walker (FRW) spacetime the lack of time-translation symmetry, among other concerns, makes the applicability of the particle approach during periods of rapid expansion suspect, e.g. there is no notion of a global or preferred vacuum state serving as a basis of the Fock space formulation (see refs. [3, 4] for detailed treatment of the strengths and weaknesses of the operator formulation of quantum fields in curved spacetime). In addition, the current paradigm of modeling the origins of observed inhomogeneity of the universe as well as the observed matter content today presupposes that during some earlier period all quantum fields participated in both near and far from
equilibrium interactions with respect to a thermal plasma where the standard quantum formalism, as found in refs. [5, 6] for instance, can give rise to appreciable loop-level corrections to the aforementioned interaction rates [7–12].

A more detailed treatment may be carried out within the algebraic formulation of locally covariant quantum field theory as presented, for example, in ref. [13] (see refs. [14, 15] for a general introduction to the algebraic approach in the context of curved spacetime). This mathematically rigorous formalism is in general useful for clarifying conceptual issues related to and/or providing a foundation for the calculation of observables with traditionally heuristic justifications. In this work, however, we propose a non-traditional application of the formalism inspired by numerical calculations such as those found in refs. [16, 17] where algebraic quantum field theory is employed in computing and characterizing the energy density of a free scalar field propagating in a non-stationary FRW spacetime. In other words, we seek to employ the established algebraic formalism in a concrete numerical calculation of a cosmological observable and not in the traditional pursuit of a rigorous proof of theorem. Though this numerical calculation may be computationally expensive, as compared to the standard formalism, meeting the requirement that cosmological observables be compatible with the semiclassical Einstein equation; i.e. the stress-energy tensor is the expectation value of a quantum state back-reacting on the metric of general relativity, would seem to justify the cost [18–21].

In the algebraic framework finite time intervals are essential to formulating the physical states of interest given that gravitationally induced excitations of quantum matter fields generally accompany non-stationary spacetimes [22–25]. Furthermore, the usual notions of thermal equilibrium and non-equilibrium dynamics become somewhat ambiguous. For example, the work in refs. [26–28] suggests observables computed in a manner consistent with the standard formulation of thermal field theory in Minkowski spacetime may serve only as a reference for the properties of the observed state. Hence, we take the first step towards probing for corrections to the standard particle physics approach by deriving an expression, via algebraic quantum field theory in curved spacetime, that is at least in principle amenable to numerical calculation, for the renormalized energy density of a free scalar field subjected during a finite time interval to the influence of a dense background of other perturbatively coupled scalar fields propagating in a classical, yet non-stationary, FRW spacetime.

In order to derive this expression for the energy density we must begin with the general evolution of the algebraic state over the finite time interval. As there is no Poincaré invariance present in our cosmological model, we make use of a two-parameter family of propagators, as defined in ref. [29], resulting in a method analogous to the Schwinger–Keldysh closed-time-path [30, 31], however extended to non-stationary spacetimes. The state will then encode both the semiclassical gravitationally induced excitations, in accordance with renormalization constraints developed in the literature cited above and the perturbative influence of the dense background of coupled fields. Components of the cosmological stress-energy tensor must then be computed from the resulting expression for the perturbed state where renormalization is carried out via the geometric constructs of the algebraic formalism.
To this end we allocate Sec. 2 to the development of a general algebraic model of neutral scalars in which we assume a ΛCDM cosmology with an initial period of inflation per the results found in refs. [32, 33]. Here, we derive eq. (2.95) as the main result of this work; i.e. the general form of the expectation value of the renormalized, perturbed quantum energy density given the influence to leading order in the coupling of both a dense background of other scalar fields and the non-stationary spacetime. In Sec. 3 we review the derivation of the evolution equation in an effective field theory approach to non-equilibrium dynamics in the framework of the standard operator formalism. Sec. 4 serves to provide a check on our result such that, here, we reduce the algebraic form of eq. (2.95), in the stationary limit of Minkowski spacetime, to a standard expression found in the effective field theory approach of the previous section. We then discuss the result and future works in Sec. 5.

2 The General Model in the Algebraic Formalism

We consider a theory of a neutral scalar \( \phi \) on a globally hyperbolic spacetime \((\mathcal{M}_\Sigma, g)\), i.e. a Lorentzian manifold \( \mathcal{M}_\Sigma \) with Cauchy surface \( \Sigma \) and metric \( g \), via the classical free Lagrangian

\[
\mathcal{L}_0 = -\frac{1}{2} \left( g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + m^2 \phi^2 + \xi R \phi^2 \right)
\]

(2.1)
given \( R \) as the Ricci scalar on \( \mathcal{M}_\Sigma \), \( m \) as the field’s mass, and \( \xi \) its coupling to gravity. Canonical quantization is realized by constructing the Borchers–Uhlmann algebra, a topological \( \ast \)-algebra (with unit) defined as

\[
\mathcal{A}(\mathcal{M}_\Sigma, g) := \mathcal{A}_0(\mathcal{M}_\Sigma, g)/\mathcal{I}(\mathcal{M}_\Sigma, g)
\]

(2.2)

where \( \mathcal{A}_0(\mathcal{M}_\Sigma, g) = \bigoplus_{n=0}^{\infty} D(\mathcal{M}_\Sigma^n) \) given \( D(\mathcal{M}_\Sigma^n) = \mathbb{C} \), is the free tensor algebra over \( D(\mathcal{M}_\Sigma) \) as the space of compactly supported real-valued functions \( f \) on \( \mathcal{M}_\Sigma \) such that \( \text{supp} f \subset \Sigma^{(\epsilon)}_t = \{(t, \vec{x})| -\epsilon < t < \epsilon; \epsilon > 0\} \) and \( \mathcal{I}(\mathcal{M}_\Sigma, g) \) the \( \ast \)-ideal. The field \( \phi \), now denoted by the formal expression \( A(f) \), generates the free algebra such that \( f \rightarrow A(f) \) is \( \mathbb{R} \)-linear and

\[
A(f)^\ast = A(f)
\]

(2.3)
\[
A(\hat{p} f) = 0
\]

(2.4)
\[
[A(f), A(g)] = iE(f, g)
\]

(2.5)

\( \forall A \in \mathcal{A}(\mathcal{M}_\Sigma) \) and \( f, g \in D(\mathcal{M}_\Sigma) \); while \( \mathcal{I}(\mathcal{M}_\Sigma, g) \) is generated by elements including \( \hat{p} f \) and the causal propagator \( E := E^> - E^< \) defined via the unique advanced(\( > \)) and retarded(\( < \)) fundamental solutions of the Klein–Gordon operator

\[
\hat{p} = (\Box_g + m^2 + \xi R).
\]

(2.6)

Here, \( \mathcal{M}_\Sigma := \mathbb{R} \times \Sigma_0 \) is a spatially flat FRW spacetime with metric written in the familiar form

\[
ds^2 = dt^2 - a_t^2d\Sigma_0^2,
\]

(2.7)
such that
\[ \square_g = \partial_t^2 + 3H_t\partial_t + \frac{\nabla^2}{a_t^2} \] (2.8)
and
\[ R = 6\left(\frac{\dot{a}_t}{a_t} + \frac{\dot{a}_t^2}{a_t^2}\right) \] (2.9)
given \( a : \mathbb{R} \to \mathbb{R} \) as the scale factor and \( H_t := \dot{a}_t/a_t \) as the Hubble parameter. We pass to conformal time \( \eta \) via the relation
\[ dt = a_t d\eta \] (2.10)
where the metric becomes
\[ ds^2 = a_t^2 [d\eta^2 - d\Sigma^2_0]. \] (2.11)
and the Klein–Gordon operator of eq. (2.6) is rewritten as
\[ \hat{P}_\eta = \frac{1}{a_t^2} \left[ \partial^2_{\eta} - \nabla^2 + a_t^2 m^2 + a_t^2 \left( \xi - \frac{1}{6} \right) R \right] \] (2.12)
This transformation allows for the expansion of the domain of \( a_t \) into \(( -\infty, \eta_0 )\) with an asymptotically de Sitter (dS) spacetime \(( \tilde{M}_\Sigma, \tilde{g} )\) where \( \tilde{g} = (\Omega/a_t)^2 g \), given \( \Omega : M_\Sigma \to \mathbb{R}^+ \), such that \( a_t = \exp (H_\Lambda t) \) for \( \eta \in [ -\infty, \eta_0 ] \) corresponds to an early period of inflation with \( H_\Lambda \) a constant. \( \tilde{M}_\Sigma \) then contains a cosmological past horizon \( J^- \) as a boundary, i.e. a smooth hypersurface diffeomorphic to \( \mathbb{R} \times S^2 \) at \( \eta \to -\infty \), with coordinates \( (v = t + r, \theta, \phi) \) and metric of Bondi form
\[ \tilde{ds}^2|_{J^-} = 2d\Omega dv + dS^2. \] (2.13)

### 2.1 Homogeneous and Isotropic States

The algebraic states \( \omega : A \to \mathbb{C} \), where \( \omega(a^*A) \geq 0 \) and \( \omega(1) = 1 \) \( \forall A \in A \) define the \( n \)-point functions \( \omega(A_1A_2...A_n) \). In the case of quasifree states, i.e. the Gaussian states
\[ \omega(A_1A_2...A_n) = \begin{cases} \sum X \prod_{\{i,j\} \in X} \omega(A_i^*A_j) & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \] (2.14)
\[ X \equiv \text{the set of all possible pairings \{i, j\} where i < j}, \]
we require the two-point function \( \omega(A_iA_j) \) be of the physically admissible Hadamard form
\[ \omega(A_xA_y) = \lim_{\epsilon \downarrow 0} \frac{1}{8\pi^2} \left[ \frac{U(x_\mu, y_\mu)}{\sigma_\epsilon(x_\mu, y_\mu)} + V(x_\mu, y_\mu) \log \left( \frac{\sigma_\epsilon(x_\mu, y_\mu)}{L^2} \right) + W(x_\mu, y_\mu) \right] \] (2.15)
where the functions \( U, V, \) and \( W \) are smooth real–valued bi-distributions and
\[ \sigma_\epsilon(x_\mu, y_\mu) \equiv \sigma(x_\mu, y_\mu) + 2i\epsilon [\tau(x_\mu) - \tau(y_\mu)] + \epsilon^2, \] (2.16)
with \( \sigma(x_\mu, y_\mu) \) the signed squared geodesic distance; while \( \tau : M_\Sigma \to \mathbb{R} \) is an arbitrary time function and \( L \) the length scale. This allows us to extend the factored \(*\)-algebra to \( W(M_\Sigma, g) \) such that \( A \subset W \) where renormalization up to mass and curvature ambiguities
is carried out by local and covariant Hadamard point-splitting regularization, i.e. Wick products are defined in the coincidence limit
\[
\omega(\mathcal{A}_x) := \lim_{y \to x} [\omega(\mathcal{A}_x \mathcal{A}_y) - \mathcal{H}(x_\mu, y_\mu)]
\] (2.17)
given the purely geometric Hadamard parametrix \(\mathcal{H}(x_\mu, y_\mu)\) as the first two terms in eq. (2.15). Hence, the time ordered products of free and composite fields necessary to define perturbative interactions as well as prove the spin-statistics and CPT theorems allow for reliable cosmological observables [21, 35–38].

We remind the reader that the appropriate cosmological observables in this case are the ratios of expectation values of the quantum energy densities of a field
\[
\epsilon_{A,x} := \omega(\langle A_x^2 \rangle)
\] (2.18)
to the so called critical density
\[
\epsilon_{c,x} := \omega\left(\sum_i \langle A_x^2 \rangle^{(i)}\right)
\] (2.19)
consistent with the local and covariant semiclassical Einstein equation
\[
\left( R_{\mu\nu}(x_\mu) - \frac{1}{2} R \, g_{\mu\nu}(x_\mu) \right) = -8\pi G \omega\left(\sum_i \langle A_x^2 \rangle^{(i)}\right)
\] (2.20)
where \(R_{\mu\nu}\) is the Ricci tensor, \(G\) is Newton’s constant, and \(\omega(\sum_i \langle A_x^2 \rangle^{(i)})\) is interpreted as the expectation value of the stress–energy tensor \(T_{\mu\nu}\) corresponding to all the quantum matter fields \(A_x^{(i)}\) in the theory. Following the formulation of ref. [23] with explicit constructions found in ref. [39] the quasifree homogeneous and isotropic states in FRW spacetimes are expressed
\[
\omega(A_x A_y) = \int \frac{d^3k}{(2\pi)^3} \mathcal{X}_k \left\{ X_k(\eta_x) X_k(\eta_y) + X_k(\eta_x) X_k(\eta_y) \right\} \exp(ik \cdot [\vec{x} - \vec{y}]),
\] (2.21)
where the mode functions \(X_k(\eta)\) satisfy
\[
X_k(\eta) X_k(\eta)' - X_k(\eta)' X_k(\eta) = i
\] (2.22)
given \(X_k(\eta)\)' as the derivative with respect to \(\eta\). Members of the set of unitarily equivalent mode functions satisfying eq. (2.22) are expressed as a Bogoliubov transformation such that
\[
X_k(\eta) = p_k T_k(\eta) + q_k T_k(\eta)
\] (2.23)
with \(|p_k|^2 - |q_k|^2 = 1\) and \(T_k(\eta)\) an arbitrary reference mode that satisfies the time portion of \(\hat{P}_\eta T_k(\eta) = 0\). Here, \(X_k \geq 1/2\) is polynomially bounded in \(k\) such that equality obtains the pure state while inequality corresponds to the generic mixed state, i.e. the convex combination
\[
\omega(A_x A_y) = \sum_n w_n \omega_n(A_x A_y); \ w_n \geq 0, \ \sum_n w_n = 1
\] (2.24)
of at least two other mixed states
\[ \omega_n(A_x A_y) := \int \frac{d^3k}{(2\pi)^3} \mathcal{M}^{(n)}_k \left\{ X_k(\eta_x) X_k(\eta_y) + X_k(\eta_x) X_k(\eta_y) \right\} \exp(i\vec{k} \cdot [\vec{x} - \vec{y}]) \] (2.25)

where \( \mathcal{M}^{(m)}_k \neq \mathcal{M}^{(n)}_k \). We may now express the function \( X^\omega_k \) via the series
\[ X^\omega_k = \sum_{n\in\mathbb{N}^+} w_n \mathcal{M}^{(n)}_k; \quad \forall w \in \left\{ w_1, w_2, w_3, ... \mid w_n \in [0, 1], \sum_{n\in\mathbb{N}^+} w_n = 1 \right\} \] (2.26)
such that \( X^\omega_k = \mathcal{M}^{(1)}_k = 1/2 \), with \( w_1 = 1 \) and \( w_{n>1} = 0 \), recovers the pure state while \( X^\omega_k \neq \mathcal{M}^{(n)}_k \), with \( w_n \neq 1 \) \( \forall n \in \mathbb{N}^+ \), corresponds to the generic mixed state of eq. (2.24).

Given the perturbative interacting fields in our model are real scalars, there is a Gel’fand–Naimark–Segal (GNS)–representation \( \pi_\omega : \mathcal{A} \to \mathcal{T}(\mathcal{D}) \), where \( \mathcal{T}(\mathcal{D}) \) is the Banach space of linear operators on a dense domain \( \mathcal{D} \) of the Hilbert space \( \mathcal{H}_\omega \), with cyclic vector \( \Omega_\omega \in \mathcal{D} \subset \mathcal{H}_\omega \) such that
\[ \omega(A) = \langle \Omega | \pi_\omega(A) | \Omega \rangle , \] (2.27)
where the irreducible representations \( \pi_\omega(A) \) are in one-to-one correspondence with the pure algebraic states and contain the usual annihilation and creation operators over \( \mathcal{D} \) as the bosonic Fock space over the one–particle space \( \mathcal{H}^{(1)}_\omega \). However, for more robust models that include interacting fields of perturbative Yang–Mills theory in a general non-stationary spacetime; an equivalent correspondence with \( \pm \)-helicity one-particle states of the electromagnetic field is not possible \([15, 36]\). Hence, we continue in the algebraic framework without regard to a Hilbert space representation.

### 2.2 Ground States as States of Low Energy

We now propose generalized ground states from states of low energy (SLE) as put forward in ref. \([24]\) with explicit constructions in FRW spacetimes found in refs. \([16, 17]\). Here, we focus on a massive minimally coupled, i.e. \( \xi = 0 \), free scalar field. These quasifree pure homogeneous and isotropic states are specified by mode functions that minimize the energy density per mode
\[ \varepsilon^\omega(\vec{k}) = \frac{1}{2a^4(2\pi)^3} \left( |X^\prime(\eta)|^2 - a_t H_t (|X(\eta)|^2)' + (k^2 + a^2 m^2 + a^2 H^2_t) |X_k(\eta)|^2 \right) \] (2.28)

via the Bogoliubov coefficients of eq. (2.23) such that
\[ p^\omega_\vec{k} = \exp\left( i[\pi - \arg c_2(\vec{k})] \right) \sqrt{\frac{c_1(\vec{k})}{2\sqrt{c_1^2(\vec{k}) - |c_2(\vec{k})|^2}}} + \frac{1}{2} \] (2.29)
\[ q^\omega_\vec{k} = \sqrt{\frac{c_1(\vec{k})}{2\sqrt{c_1^2(\vec{k}) - |c_2(\vec{k})|^2}} - \frac{1}{2}} \] (2.30)
where, for a comoving observer,

\[
c_1(\vec{k}) := \frac{1}{2} \int_{t_i}^{t_f} dt \, g^2(t) \left\{ |X_k'(\eta)|^2 - a_t H_t |X_k(\eta)|^2 \right\} \\
+ \left( k^2 + a_t^2 m^2 + a_t^2 H_t^2 \right) |X_k(\eta)|^2 \right\}
\]

(2.31)

and

\[
c_2(\vec{k}) := \frac{1}{2} \int_{t_i}^{t_f} dt \, g^2(t) \left\{ |X_k''(\eta)| - a_t H_t |X_k'(\eta)| \right\} \\
+ \left( k^2 + a_t^2 m^2 + a_t^2 H_t^2 \right) X_k(\eta)^2 \right\}.
\]

(2.32)

Convolution with the sampling function \(g(t)\) is then taken over a finite interval of cosmological time, i.e. \(t_i, t_f \in I_t \subset \mathbb{R}\). In what follows we take as our reference

\[
T_{\vec{k}}(\eta) := \frac{1}{\sqrt{2\Omega_{\vec{k}}(\eta)}} \exp \left( i \int_{\eta_0}^{\eta} d\bar{\eta} \Omega_{\vec{k}}(\bar{\eta}) \right).
\]

(2.33)

where

\[
\Omega_{\vec{k}}(\bar{\eta}) = \sqrt{k^2 + a_t^2 m^2}
\]

(2.34)

such that as we approach the asymptotically dS spacetime \(\tilde{M}_\Sigma\)

\[
\lim_{\eta \to -\infty} T_{\vec{k}}(\eta) = \frac{1}{\sqrt{2k}} \exp(-ik\eta)
\]

(2.35)

gives the Bunch–Davies vacuum. This is consistent with a bulk–to–boundary correspondence via the injective \(*\)-homomorphism \(\alpha_f : \mathcal{A}(\tilde{M}_\Sigma) \to \mathcal{A}(\tilde{3}^-)\) in order to construct an induced Hadamard ground state, i.e. Bunch–Davies, on the bulk FRW spacetime \([40, 41]\).

### 2.3 Excited States as Generalized Hadamard States

The formulation of a general excited state in the algebraic framework,

\[
\omega^B(A_x A_y) := \frac{\omega(B_x A_x A_y B_y)}{\omega(B_x B_y)}
\]

(2.36)

follows from a generalized Hadamard condition such that any finite excitation of a Hadamard state is itself a Hadamard state \([44]\). For example, the thermal Kubo–Martin–Schwinger (KMS) state is indeed Hadamard and invariant under the \(*\)-automorphisms \(\alpha_t\) where

\[
A_t = \alpha_t(A)
\]

(2.37)

such that

\[
\omega(\alpha_t(A) B_t) = \omega(B_t \alpha_{t-i\beta}(A))
\]

(2.38)

given the global temperature parameter \(\beta^{-1}\). We direct the reader to refs. \([45, 46]\) for a rigorous and extensive treatment of both the vacuum and KMS state, constructed at
a finite time in a Hamiltonian approach to perturbative algebraic quantum field theory in Minkowski spacetime via a distinguished time-direction using a one-parameter group of automorphisms $\alpha_t$, where the interacting dynamics are related to free dynamics by a co-cycle in the algebra of the free field. However, in FRW spacetimes there is no time translation invariance and hence no abelian one-parameter group of automorphisms $\alpha_t$ implemented as unitary operators on a corresponding Fock space, i.e. there is no well defined Hamiltonian as the generator of time translations and no strict notion of local thermal equilibrium in non-stationary spacetimes. This has led to several innovative and interesting frameworks, e.g. the Almost Equilibrium States of ref. [42], Local $S_x$ Thermal Equilibrium States found in refs. [26, 39], and the Bulk-to-Boundary Almost KMS States in ref. [34].

In this work, we invoke the notion of a propagator-family [29, 43] as a non-commutative two-parameter family of automorphisms $\alpha_{t,s}$ such that

$$A_t = \alpha_{t,s}(A_s)$$

(2.39)

with

$$\alpha_{t,r} = \alpha_{t,s} \circ \alpha_{s,r}$$

(2.40)

and

$$\omega_t(A) := \omega_s(A) \circ \beta_{s,t}$$

(2.41)

where

$$\beta_{r,t} = \beta_{r,s} \circ \beta_{s,t}$$

(2.42)

such that

$$\omega_s(A) \circ \beta_{s,t} = \omega(\alpha_{t,s}(A_s))$$

(2.43)

Additionally, the following group automorphism properties are imposed to ensure the dynamics are consistent with a causal propagator:

$$\alpha_{t,t} = 1$$

(2.44)

$$\alpha_{t,s}^{-1} = \alpha_{s,t}$$

(2.45)

$$\alpha_{t,s}(A_s B_s) = \alpha_{t,s}(A_s) \alpha_{t,s}(B_s).$$

(2.46)

The infinitesimal generators of time shifts are then defined via the relations

$$\dot{\alpha}_{t,s} = d_t \circ \alpha_{t,s}$$

(2.47)

$$\dot{\beta}_{s,t} = \beta_{s,t} \circ \delta_t$$

(2.48)

where

$$d_t := \lim_{\Delta t \to 0} \frac{\alpha_{t+\Delta t,t} - \alpha_{t,t}}{\Delta t}$$

(2.49)

$$\delta_t := \lim_{\Delta t \to 0} \frac{\beta_{t,t+\Delta t} - \beta_{t,t}}{\Delta t}$$

(2.50)

such that

$$\dot{\alpha}_t(A B) = \dot{\alpha}_t(A) B + A \dot{\alpha}_{t,s}(B).$$

(2.51)
We may not equate eq. (2.49) with the Heisenberg equation of motion in non-stationary spacetimes; however, we may define a generator of a perturbed time shift via the relation

\[ \delta^P_t (A) := [iP_t, A] \]  

(2.52)
given

\[ \dot{\beta}_{s,t}^P = \beta_{s,t}^P \circ (\delta_t + \delta_t^P). \]  

(2.53)

with \( \beta_{s,t}^P = \mathbb{1} \) and time dependent perturbation \( P_t \) defined via an interaction Lagrangian \( \mathcal{L}_I \), i.e. of the form

\[ P_t = \int d^3x \, f(t, \vec{x}). \mathcal{L}_I. \]  

(2.54)

Though we do not rigorously prove the existence of the generator \( P_t \), we invoke the analysis performed in ref. [21] such that for a general perturbative interaction of the form

\[ \mathcal{L}_I := \kappa J x \phi x \]  

(2.55)

when \( \kappa \) is the perturbative coupling parameter and \( J_t \) a c-number “external current” that sources the field \( \phi \), we may rely on the existence of time ordered products and a conserved stress-energy tensor. Hence, we let

\[ \beta_{s,t}^P(A_s) := \Omega(t_f, t_i)^{-1} \beta_{s,t}(A_s) \Omega(t_f, t_i) \]  

(2.56)

where

\[ \Omega(t_f, t_i) := T^+ \left[ \exp \left( -i \int_{t_i}^{t_f} \, dt \, \beta_{s,t}(P_s) \right) \right] \]  

(2.57)
given \( \Omega(t_f, t_i)^{-1} := \Omega(t_i, t_f) \) with \( T^+ (T^-) [...] \) the time(anti–time) ordering operation such that

\[ \omega^P(A) := \omega(\beta_{s,t}^P(A_s)) = \omega(\alpha_{t,f}(A_s)) \circ \gamma_{s,t} \]  

(2.58)

with \( \gamma_{s,t} := \text{Ad} \, \Omega(t_f, t_i)^{-1} \). The generalized perturbed state may now be written

\[ \omega^P(A_{t_f}A_{t_f'}) = \frac{\omega(\alpha_{t_f,t_i}(A_{t_i}) \alpha_{t_f',t_i'}(A_{t_i})) \circ \gamma_{t_i,t_f} \circ \gamma_{t_i,t_f'}}{\omega(\mathbb{1}) \circ \gamma_{t_i,t_f} \circ \gamma_{t_i,t_f'}} \]  

(2.59)

via the group automorphism properties of \( \alpha_{t_f,t_i} \) and \( \gamma_{t_i,t_f} \), given \( t_i < t' < t < t' < t_f \in \mathcal{I}_t \), such that

\[ \omega^P(A_{t_f}A_{t_f'}) = \frac{\omega \left( \Omega^{-1}(t_i, t_f) \, \Omega^{-1}(t_i, t_f') \, A_{t_f}A_{t_f'} \, \Omega(t_i, t_f') \, \Omega(t_i, t_f) \right)}{\omega \left( \Omega^{-1}(t_i, t_f) \, \Omega^{-1}(t_i, t_f') \, \Omega(t_i, t_f') \, \Omega(t_i, t_f) \right)} \]  

(2.60)
2.4 The General Form of a Perturbed State

We now derive and examine the general expression for the perturbed state $\omega^P(A_x A_y)$. Let the algebraic perturbation take the form

$$ P_t = \kappa J_t W_t $$

(2.61)

given the element $W(f) \in W(M_\Sigma, g)$. We choose this form so that we may make the direct comparison with the standard operator formalism in the next section. The propagator may then be expressed

$$ \mathcal{U}(t_f, t_i) = T^+ \left[ \exp \left( -i \kappa \int_{t_i}^{t_f} dt J_t W_t \right) \right] . $$

(2.62)

This allows for the definition of a generating functional $Z_\omega[J^+, J^-]$ via the functional derivative $D_{W}^\pm$ such that

$$ i D_{W}^\pm := (-i)^2 \frac{1}{Z_\omega[0, 0]} \left. \frac{\delta^2 Z_\omega[J^+, J^-]}{\delta J^\pm \delta J^\pm} \right|_{J^\pm = J^\pm = 0} $$

(2.63)

for

$$ Z_\omega[J^+, J^-] := \omega(T^+ [\exp(\ k J W^\pm)] \ T^+ [\exp(\ k J W)]) ) $$

$$ = \omega(1) - \frac{\kappa^2}{2} \left( J^+ J^+ \omega(W^+ W^+) - J^- J^+ \omega(W^- W^+) 
- J^+ J^- \omega(W^+ W^-) + J^- J^- \omega(W^- W^-) \right) + \mathcal{O}(\kappa^4) + ... $$

(2.64)

or in the more familiar form

$$ Z_\omega[J^+, J^-] = \exp \left( -\frac{\kappa^2}{2} \int_{t_i}^{t_f} dt_x \int_{\Sigma_x} d^3 x \int_{t_i}^{t_x} dt_y \int_{\Sigma_y} d^3 y \left[ J^x_+ J^-_x \right] i D_W(x, y) \left[ J^x_+ J^-_x \right] \right) $$

(2.65)

where

$$ D_W(x, y) = \begin{bmatrix} \omega(T^+ [W_x W_y]) - \omega(W_x) \omega(W_y) & \omega(W_y W_x) - \omega(W_x) \omega(W_y) \\
\omega(W_x W_y) - \omega(W_x) \omega(W_y) & \omega(T^- [W_x W_y]) - \omega(W_y) \omega(W_x) \end{bmatrix} $$

(2.66)

for the quasifree Hadamard states we consider. Equivalence of the Schwinger–Keldysh “in–in” formalism with the CPT theorem in FRW spacetimes, i.e. an in-state in an expanding universe is related to an in-state in the corresponding contracting universe [38], is exploited below. Now, we let the source the source term $J_t$ become the element $A_t$ and write out the composite element $W_t$ as the product $B_tC_t$ such that the perturbed state $\omega^P(A_{t_f} A_{t_f}')$
of eq. \( (2.60) \) may be written

\[
\omega^P(A_t, A'_t) = \left\{ \omega \left( T^+ \left[ \exp \left( i \kappa \int_{t_i}^{t_f} dt A_t B_t C_t \right) \right] A_t \right) T^+ \left[ \exp \left( -i \kappa \int_{t_i}^{t_f} dt A_t B_t C_t \right) \right] \right. \\
\left. - \left[ A_t^+ A_t^- B_t^+ B_t^- C_t^+ C_t^- + A_t^- A_t^+ B_t^- B_t^+ C_t^- C_t^+ \\
+ A_t^+ A_t^- B_t^+ B_t^- C_t^+ C_t^- - A_t^- A_t^+ B_t^- B_t^+ C_t^- C_t^+ \right] + O(\kappa^4) + ... \right\}
\]

\[ \times \left\{ \omega(1) + \omega \left( \frac{\kappa^2}{2} \int_{t_i}^{t_f} dt \int_{t_i}^{t'} dt' \left[ \\
- A_t^+ A_t^- B_t^+ B_t^- C_t^+ C_t^- + A_t^- A_t^+ B_t^- B_t^+ C_t^- C_t^+ \\
+ A_t^+ A_t^- B_t^+ B_t^- C_t^+ C_t^- - A_t^- A_t^+ B_t^- B_t^+ C_t^- C_t^+ \right] + O(\kappa^4) + ... \right\}^{-1} \quad (2.67) \]

for the quasifree states that vanish with odd products of the fields. Here, the time-ordering index \( \pm \) corresponds to the forward(+) and backward(-) branch of the closed-time-path depicted in Fig. 1.

**Figure 1.** Closed–time–path evolution of the field \( A_t \) via the perturbation \( P_t = \kappa A_t B_t C_t \) for a finite macroscopic cosmological time interval \( t_i, t_f \in \mathcal{I}_t \), given \( t_i < t' < t < t_f \) on the forward(+) branch \( t_f < t < t' < t_i \) on the backward(-) branch.

Notice the numerator of eq. \( (2.68) \) is separated into a sum including an unperturbed term, evolving only via the mode functions with Bogoliubov coefficients given by the SLE
formulation, i.e. $X_k(t_i)$ evolves to $X_k(t_f)$, and the perturbed term evolving under the influence of the fields $B_t$ and $C_t$ via the closed-time-path; while the denominator, as the perturbed normalization factor, is immediately recognizable as the generating functional of eq. (2.64) in analogy with field strength renormalization in the standard formalisms. We now write the perturbed state as

$$\omega^P(A_{t,f},A_{t,f}') = \frac{1}{Z_{\omega|A^+,A^-}} \left\{ \omega(A_{t,f}A_{t,f}') + \omega(A_{t,j}A_{t,j}') \int_{t_i}^{t_f} dt \int_{t_i}^{t_f} dt' \left[ \right. \right.$$

$$- A^+_t A^+_t B^+_t B^{+_t}_t C^+_t C^{+t}_t + A^-_t A^+_t B^-_t B^{+_t}_t C^-_t C^{+t}_t$$

$$\left. + A^-_t A^+_t B^-_t B^{+_t}_t C^-_t C^{+t}_t - A^-_t A^-_t B^-_t B^-_t C^-_t C^-_t + \ldots \right] \right\}, \quad (2.69)$$

or equivalently

$$\omega^P(A_{t,f},A_{t,f}') = \int d^3x \int d^3x' \int \frac{d^3k}{(2\pi)^3} \mathcal{X}_k^P \left\{ X_k(t_f)X_k(t_f') + X_k(t_f)\overline{X_k(t_f')} \right\} \exp[i\vec{k} \cdot (\vec{x} - \vec{x}')] \quad (2.70)$$

where $\mathcal{X}_k^P$ is now the perturbed function corresponding to $X_k$ of eq. (2.26) and $X_k(t_f)$ the mode functions defined in eq. (2.23). This constitutes imposing the “algebraic adiabatic limit” at the boundaries $t_i, t_f \in I$ and though we do not rigorously prove the existence of the adiabatic limit in FRW spacetimes we take $f(t,\vec{x}) = \varphi(t)h(\vec{x})$ such that $h(\vec{x}) \to 1$ and $\varphi(t) = 0 \forall t \notin I$, corresponds to a model dependent cosmological scenario in which interactions are observationally irrelevant outside the specified time interval. In this limit we may combine eq. (2.69) and eq. (2.70) such that

$$\int d^3x \int d^3x' \int \frac{d^3k}{(2\pi)^3} \mathcal{X}_k^P \left\{ X_k(t_f)X_k(t_f') + X_k(t_f)\overline{X_k(t_f')} \right\} \exp[i\vec{k} \cdot (\vec{x} - \vec{x}')]$$

$$= \frac{1}{Z_{\omega|A^+,A^-}} \left\{ \omega(A_{t,f}A_{t,f}') + \omega(A_{t,j}A_{t,j}') \frac{\kappa^2}{2} \int_{t_i}^{t_f} dt \int_{t_i}^{t_f} dt' \left[ \right. \right.$$

$$\left. - \omega(A^+_t A^+_t) \omega(B^+_t B^{+_t}_t) \omega(C^+_t C^{+t}_t) + \omega(A^-_t A^+_t) \omega(B^-_t B^{+_t}_t) \omega(C^-_t C^{+t}_t) \right.$$

$$\left. + \omega(A^-_t A^-_t) \omega(B^-_t B^{+_t}_t) \omega(C^-_t C^-_t) - \omega(A^-_t A^-_t) \omega(B^-_t B^-_t) \omega(C^-_t C^-_t) + \ldots \right] \right\}; \quad (2.71)$$

where

$$\omega(A_{t,f}A_{t,f}') := \int d^3x \int d^3x' \int \frac{d^3k}{(2\pi)^3} \mathcal{X}_k \left\{ X_k(t_f)X_k(t_f') + X_k(t_f)\overline{X_k(t_f')} \right\} \exp[i\vec{k} \cdot (\vec{x} - \vec{x}')] \quad (2.72)$$
\[ \Omega_{A} := \frac{\mathcal{E}_A}{\mathcal{E}_c} \]  

(2.21)

The cosmological observable of interest is the ratio

\[ \Omega_{A} := \frac{\mathcal{E}_A}{\mathcal{E}_c} \]  

at the finite final cosmological time \( t_f \) corresponding to the SLE construction above. Here, \( \mathcal{E}_A \) is subject to the so-called back-reaction problem via the semi-classical Friedmann equation

\[ H_t^2 = \frac{8\pi G}{3} \omega_P \left( \sum_i T_{00}(: A_i^2 :) \right) \]  

(2.22)

such that the critical density is given by the standard expression

\[ \mathcal{E}_c = \frac{3H_t^2}{8\pi G} \]  

(2.23)

with \( H_t \) found via the observations in ref. [32]. However, we may neglect solving the full back-reaction problem by imposing the solution for a vacuum (Λ), radiation (r), or matter (m) dominated epoch, i.e., we will take \( a(t) \) to maintain the fixed form \( a(t) \propto (t-t_i)^{1/2} \) or \( a(t) \propto (t-t_i)^{3/2} \) during the respective epoch.

Following the formulations and results in refs. [16, 17, 39], we now review the general form of the energy density of a minimally coupled, i.e., \( \xi = 0 \), free scalar field perturbed to an arbitrary mixed state in a non-stationary FRW spacetime background. We begin with the expectation value of the renormalized stress-energy tensor, taken in the adiabatic limit, as

\[ \omega^P( T_{\mu\nu}(: A_x^2 :) ) = \left\{ \omega^P \left( \mathcal{D}_{x,y}(A_x A_y) - \mathcal{D}_{x,y} \mathbb{H}^1(x_\mu, y_\mu) + \frac{1}{3} \mathcal{P}_x \mathbb{H}^1(x_\mu, y_\mu) + C_{\mu\nu}(x_\mu) \right) \right\} |_{x_\mu = y_\mu} \]  

(2.24)
Here, the bi–differential operator $\hat{D}$ is defined
\[
\hat{D}_{a,b} := \frac{1}{2} \left( \partial_a \partial_b + \frac{1}{a_t^2} \nabla^a \nabla^b + m^2 \right)
\]  
(2.80)
and the purely geometric Hadamard parametrix is expressed
\[
\mathbb{H}_n(x_\mu, y_\mu) = \lim_{\epsilon \downarrow 0} \frac{1}{4\pi^2} \left[ \frac{1}{\sigma(x_\mu, y_\mu)} + \frac{1}{L^2} \sum_{m=1}^{n} V_m \left( \frac{\sigma(x_\mu, y_\mu)}{L^2} \right)^m \log \left( \frac{\sigma(x_\mu, y_\mu)}{L^2} \right) \right],
\]
(2.81)
where $V_m$ satisfies the so called Hadamard recursion relations (see e.g. ref. [39]).

\[
\mathbb{H}_1(x_\mu, y_\mu) := \frac{1}{2} \left( \mathbb{H}_1(x_\mu, y_\mu) + \mathbb{H}_1(y_\mu, x_\mu) + i[A_x, A_y] \right)
\]  
(2.82)
is then the symmetric Hadamard bi–distribution truncated to order $n = 1$ where
\[
V_1 = -\frac{1}{3} \hat{P}_x \mathbb{H}_1^s(x_\mu, y_\mu)
\]  
(2.83)
and $C_{\mu\nu}(x_\mu)$ carries the renormalization freedom of Wick products contained in a conserved stress-energy tensor.

This is a nuanced expression that we briefly examine term by term. The divergent, perturbed energy density $\tilde{\varepsilon}_P^\rho(x_\mu, y_\mu)$ is given by the state $\omega_P^\rho(\hat{D}_{x,y}[A_x^\rho A_y])$ restricted to the diagonal $\eta(t) := \eta_x = \eta_y$ such that
\[
\varepsilon_P^\rho(x_\mu, y_\mu) := \omega_P^\rho \left( \hat{D}_{x,y}[A_x^\rho A_y] \right) \bigg|_{\eta(t)}
\]
\[
= \frac{1}{a_t^4} \int_0^\infty \frac{dk}{2\pi^2} k^2 X_k^P \left[ |X_k^P(\eta)|^2 - a_t H_t(|X_k^P(\eta)|^2)' + (k^2 + a_t^2 m^2 + a_t^2 H_t^2)|X_k^P(\eta)|^2 \right];
\]
(2.84)
now expressed as the divergent mode integral with mode functions $X_k^P(\eta)$ found via SLE minimized energy density of the ground state and the polynomially bounded function $X_k^P$ determined by the perturbation $P_\eta(t)$ of the previous subsection.

\[
\hat{D}_{x,y} \mathbb{H}_1^s(x_\mu, y_\mu) \bigg|_{\eta(t)} = \frac{1}{4\pi^2} \left[ -\frac{1}{a_t^4} \frac{2}{r_+^2} + \frac{m^2 + H_t^2}{2a_t^2} \frac{1}{r_+^2} \right.
\]
\[
+ \left( \frac{m^4}{16} - \frac{2m^2 H_t^2}{16} + \frac{2H_t H_{16}^2}{16} + \frac{6H_t^2 H_{16}^2}{16} - \frac{H_t^2}{16} \right) \left( \log \left( a_t^2 \right) \right)
\]
\[
+ \frac{\Box_g R}{120} + m^2 \left( \frac{7H_t^2}{24} + \frac{H_t}{4} \right) - \frac{m^4 H_t^2}{80} + \frac{H_t^4}{80} - \frac{11H_t H_{16}}{120}
\]
\[
- \frac{61H_t^2 H_{16}}{120} - \frac{19H_t^2}{240} \right],
\]

(2.85)
where the singular counter–terms given by the symmetric distributions \( r_+^4, r_+^2, \) and \( l_0 \) are defined via the convolutions

\[
\frac{2}{r_+^4} \equiv \lim_{\epsilon \to +0} \int_{\mathbb{R}^3} d^3x \frac{\nabla f(x)}{x^2 + \epsilon^2}, \tag{2.86}
\]

\[
\frac{1}{r_+^2} \equiv \lim_{\epsilon \to +0} \int_{\mathbb{R}^3} d^3x \frac{f(x)}{x^2 + \epsilon^2}, \tag{2.87}
\]

\[
l_0 \equiv \int_{\mathbb{R}^3} d^3x f(x) \log(x^2), \tag{2.88}
\]

for a fixed \( f \in C_0^{\infty} (\mathbb{R}^3) \) are the geometric contribution of the parametrix. Here, the sum of the singular terms may then be rewritten as a mode integral, i.e.

\[
\lim_{\epsilon \to 0} \int \frac{dk}{2\pi^2} k^2 I(k) \exp(ik \cdot \vec{x}) \exp(-k\epsilon) =
\]

\[
\frac{1}{2\pi^2} \left\{ -\mathcal{C}_{-1} \frac{2}{r_+^4} + \mathcal{C}_0 \frac{1}{r_+^2} + \mathcal{C}_1 l_0 \right. 
\]

\[
+4\pi \int_{\mathbb{R}^3} d^3x f(x) \lim_{M \to \infty} \left[ \int_0^M dk \int_0^M K(k) I(k) \right.
\]

\[
\left. -\mathcal{C}_{-1} k^2 - \mathcal{C}_0 \right) - \mathcal{C}_1 \left( \log(ML) - 1 + \gamma_{EM} \right) \right\}, \tag{2.89}
\]

where \( \gamma_{EM} \) is the Euler–Mascheroni constant and the integrand \( I(k) \) has asymptotic behavior

\[
I(k \to \infty) = \sum_{m=-1}^{1} \frac{\mathcal{C}_m}{k^{2m+1}} + \mathcal{O}(k^{-5}), \tag{2.90}
\]

such that the subtraction of singular terms may occur inside the mode integral of eq. (2.84); and

\[
\frac{1}{3} \tilde{P}_x \mathbf{H}_1^t(x_\mu, y_\mu) \left| \eta(t) \right. = \frac{1}{4\pi^2} \left( \frac{3\dot{H}_t^2}{40} + \frac{\ddot{H}_t}{20} + \frac{7H_t^2 \dot{H}_t}{60} + \frac{7H_t \dddot{H}_t}{20} \right.
\]

\[
-\frac{29H_t^4}{60} - \frac{m^4}{8} + \frac{m^2 H_t^2}{2} + \frac{m^2 \dddot{H}_t}{4}. \tag{2.91}
\]

\[
C_{00}(\eta(t)) := c_1 m^4 g_{00} + c_2 m^2 G_{00} + (3c_3 + c_4)(6\dot{H}_t^2 - 12\dot{H}_t H_t - 36\dddot{H}_t H_t^2) \tag{2.92}
\]

allows for a renormalization freedom via the coefficients \( c_{\{1,2,3,4\}} \), which are not fixed \textit{a priori} in the theory. However, they may be constrained either by experiment or physical arguments. This is to say that \( c_1 \) and \( c_2 \) correspond to a renormalization of the cosmological constant and Newton’s constant respectively, as quantities appearing in Einstein’s equation, while the sum \( (3c_3 + c_4) \) is constrained by higher order derivative corrections to the semiclassical approximation. In this work we take the position that \( c_{\{2,3,4\}} \) are not free
parameters at the length scale, $L$ of eq. (2.15), probed by current experiments that support the ΛCDM model and we omit the afforded freedom. However, we do embrace renormalization of the vacuum energy density where the requirement that this scheme reduces to normal ordering [16, 21, 37], i.e. subtraction of $\tilde{E}_{A,t}^0$ as the reference state in Minkowski spacetime

$$\tilde{E}_{A,t}^0 = \frac{1}{2} \int_0^\infty \frac{d^3k}{(2\pi)^3} \left\{ |T_k(t)|^2 + \Omega_k |T_k(t)|^2 \right\}$$ (2.93)

fixes $c_1$ as a function of $L$ such that

$$c_1(L) = -\frac{m^4}{32\pi^2} \left( \log(mL) - \log(2) - \frac{3}{4} + \gamma_{EM} \right) g_{00}.$$ (2.94)

Hence, we find as our main result the general expression for $\tilde{E}_{A,\eta}^0(t)$ as the renormalized, perturbed energy density of a massive, minimally coupled scalar field in the adiabatic limit to be

$$\tilde{E}_{\Phi}^P(t) = \frac{1}{a_t^4} \int_0^\infty \frac{d^3k}{(2\pi)^3} \left\{ X_k^P \left( |\partial_t X^{\Phi}_k(t)|^2 + \Omega_k |X^{\Phi}_k(t)|^2 \right) \right\}$$

$$- \frac{1}{4\pi^2} \left[ -\frac{1}{a_t^4} \frac{m^2}{r_+^4} + \frac{H_t^2 + m^2}{2a_t^2} \frac{1}{r_+^2} \right.$$ 

$$+ \left( \frac{m^4}{16} - \frac{2m^2H_t^2}{16} + \frac{2\dot{H}_t H_t}{16} + \frac{6\dot{H}_t H_t^2}{16} - \frac{\dot{H}_t^2}{16} \right) \left( \log + \log(a_t^2) \right)$$

$$+ \frac{\Box g R}{120} + m^2 \left( \frac{7H_t^2}{24} + \frac{\dot{H}_t}{4} \right) - \frac{m^4}{8} + \frac{H_t^4}{80} - \frac{11H_t \dot{H}_t}{120}$$

$$- \frac{61H_t^2 \dot{H}_t}{120} - \frac{19H_t^2}{240} \right]$$

$$+ \frac{1}{4\pi^2} \left( \frac{3H_t^2}{40} + \frac{\dot{H}_t}{20} + \frac{7H_t^2 \dot{H}_t}{60} + \frac{7H_t \dot{H}_t}{20} \right.$$ 

$$- \frac{29H_t^4}{60} - \frac{m^4}{8} + \frac{m^2H_t^2}{2} + \frac{m^2 \dot{H}_t}{4} \right)$$

$$- \frac{m^4}{32\pi^2} \left( \log(mL) - \log(2) - \frac{3}{4} + \gamma_{EM} \right) g_{00}.$$ (2.95)

Finding the exact form of $X_k^P$ is, however, a nontrivial task. We must first determine the form of $X^{\Phi}_k$, $\mathcal{Y}_\eta$, and $\mathcal{Z}_q$ which as discussed above may not simply be assumed to be that of the KMS state. We defer this task to future work. The remainder of this work is dedicated to deriving an analogous expression in the standard operator formalism to ensure that the result presented in eq. (2.95) reduces to that of the standard approach to non-equilibrium quantum field theory in the stationary spacetime limit $a_t \to 0$ where the FRW metric becomes that of Minkowski.

### 3 The General Model in the Operator Formalism

In order to validate the main result of the previous section we now treat our general model as an open quantum system in an effective field theory approach to the standard operator
formalism (see, for example, refs. [47, 48] for an overview of open quantum systems near and far from equilibrium). In this approximation the evolution of the perturbed state may be described via the time evolution of the density operator \( \hat{\rho}(t) \) of a given species under the influence of an environment, i.e. coupled constituents of the primordial plasma.

We take \( \phi_x \) as the field of interest, represented by the field operator

\[
\hat{\phi}_x(x, t) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\Omega_k}} (\hat{a}_k \exp[-i\Omega_k(t-t_i)] + \hat{a}^\dagger_{-k} \exp[i\Omega_k(t-t_i)]) \exp[i\vec{k} \cdot \vec{x}]
\]

(3.1)

with the standard dispersion relation

\[
\Omega_k := \sqrt{k^2 + m_i^2}.
\]

(3.2)

The coupled fields \( B_x \) and \( C_x \) are similarly defined by \( \hat{\phi}_j(x, t) \) and \( \hat{\phi}_k(x, t) \) respectively and will proxy the primordial plasma where they are collectively represented by the composite operator

\[
\hat{\psi}(x, t) := \hat{\phi}_j(x, t) \hat{\phi}_k(x, t).
\]

(3.3)

We take a standard Lagrangian of the form

\[
\mathcal{L}[\phi, \psi] = \sum_n \mathcal{L}_0[\hat{\phi}_n(x, t)] - \kappa \hat{\phi}_i(x, t) \hat{\psi}(x, t)
\]

(3.4)

where,

\[
\mathcal{L}_0[\hat{\phi}_n(x, t)] := \frac{1}{2} \partial_\mu \hat{\phi}_n(x, t) \partial^\mu \hat{\phi}_n(x, t) - \frac{1}{2} m_n^2 \hat{\phi}_n^2(x, t).
\]

(3.5)

such that

\[
(\Box + m_n^2) \hat{\phi}_n = 0.
\]

(3.6)

The factorizing Schrödinger picture density operator including all three species is given at the initial time \( t_i \) as

\[
\hat{\rho}_{\phi,\psi}(t_i) = \hat{\rho}_{\phi}(t_i) \otimes \hat{\rho}_{\psi}(t_i),
\]

(3.7)

where \( \hat{\rho}_{\phi}(t) \) is the density operator for the subsystem of \( \phi \)-field states \( |\phi^{(i)}_x\rangle \) in causal contact with the subsystem of \( \psi \)-field states \( |\psi_x\rangle := |\phi^{(j)}_x\rangle \otimes |\phi^{(k)}_x\rangle \); from here forward referred to as the \( \phi \)-system and its \( \psi \)-environment respectively.

While the tensor product in eq. (3.7) represents the pure state of the composite density operator, we take the initial state of the \( \psi \)-environment to be of the KMS form. Hence, the density operator in the Born approximation will maintain the a general form

\[
\hat{\rho}_\psi(t_i) := \frac{1}{Z} \exp[-\beta \hat{H}_\psi]
\]

(3.8)

where the partition function is defined

\[
Z := \text{Tr} \exp[-\beta \hat{H}_\psi].
\]

(3.9)

Here, \( \hat{H}_\psi \) is the free Hamiltonian operator with \( \beta \) the KMS parameter interpreted as the inverse temperature of the plasma. The reduced density operator of the \( \phi \)-system at the initial time \( t_i \) is then given by the partial trace

\[
\hat{\rho}_\phi(t_i) = \text{Tr}_\psi \hat{\rho}_{\phi,\psi}(t_i)
\]

(3.10)
where $\text{Tr}_{\psi}: \mathcal{T}(\mathcal{H}_{\phi} \otimes \mathcal{H}_{\psi}) \to \mathcal{T}(\mathcal{H}_{\phi})$.

Passing to the interaction picture, the unitary time evolution of the composite system from the common reference time $t_i = 0$ to a final time $t_f$ is then

$$\hat{\rho}_{\phi,\psi}(t_f) = \hat{U}^\dagger(t_f,0)\hat{\rho}_{\phi,\psi}(0)\hat{U}(t_f,0) \quad (3.11)$$

where

$$\hat{U}(t_f,0) = \hat{T}\{ \exp[-i(\hat{H}_{\phi,0} + \hat{H}_{\psi,0})t_f - i\kappa \int_0^{t_f} dt_x \int d^3x \hat{\phi}(\vec{x},t_x)\hat{\psi}(\vec{x},t_x)] \}. \quad (3.12)$$

In accordance with the partition function the defining generating functional for some macroscopic time $t_f$ is given by the partial trace

$$Z[\hat{\phi}^+,\hat{\phi}^-;t_f] := \text{Tr}_{\psi}\left[ \hat{U}(\hat{\phi}^+;t_f,0)\hat{\rho}_{\phi,\psi}(0)\hat{U}(\hat{\phi}^-;0,t_f) \right] \quad (3.13)$$

such that $\hat{\rho}_{\phi}(t_f) = Z[\hat{\phi}^+,\hat{\phi}^-;t_f]$ by eq. (3.10). Here,

$$\hat{\phi} \to \begin{bmatrix} \hat{\phi}^+ \\ \hat{\phi}^- \end{bmatrix} \quad \text{and} \quad \hat{\psi} \to \begin{bmatrix} \hat{\psi}^+ \\ \hat{\psi}^- \end{bmatrix} \quad (3.14)$$

while $\hat{U}(\hat{\phi}^+;t_f,0)$ and $\hat{U}(\hat{\phi}^-;0,t_f)$ are interpreted as forward and backward time evolution operators on the Schwinger–Keldysh closed–time–path contour $C$ of Fig 2.

### 3.1 Path Integral Representation

Invoking the path integral representation, we may put the generating functional into the standard form

$$Z_C[\kappa \hat{\phi}] = \int \mathcal{D}[\psi(\vec{x},t_x)] \exp(i \int_0^{t_f} dt_x \int d^3x \left\{ \mathcal{L}_0[\hat{\psi}(\vec{x},t_x)] + \kappa \hat{\phi}(\vec{x},t_x)\hat{\psi}(\vec{x},t_x) \right\})$$

$$= \exp\left(-i\frac{\kappa^2}{2} \int_0^{t_f} dt_x \int_0^{t_f} dt_y \int d^3x \int d^3y \hat{\phi}_C(\vec{x},t_x)D_C(\vec{x},t_x;\vec{y},t_y)\hat{\phi}_C(\vec{y},t_y) \right) \quad (3.15)$$
where $\mathcal{L}[\hat{\psi}(\vec{x}, t_x)]$ is the free Lagrangian and $\kappa \phi(\vec{x}, t_x)$ is treated as the external source term. Given the closed-time-path contour $\mathcal{C}$

$$D_{\mathcal{C}}(\vec{x}, t_x; \vec{y}, t_y) := \begin{bmatrix} D^+_{++}(\vec{x}, t_x; \vec{y}, t_y) & D^+_{+-}(\vec{x}, t_x; \vec{y}, t_y) \\ D^-_{-+}(\vec{x}, t_x; \vec{y}, t_y) & D^-_{--}(\vec{x}, t_x; \vec{y}, t_y) \end{bmatrix},$$

and

$$\hat{\phi}_{\mathcal{C}}(\vec{x}, t_x) := \begin{bmatrix} \hat{\phi}^+(\vec{x}, t_x), \hat{\phi}^-(\vec{x}, t_x) \end{bmatrix},$$

We may find $D_{\pm\pm}(\vec{x}, t_x; \vec{y}, t_y)$ via the functional derivatives

$$iD_{\pm\pm}(\vec{x}, t_x; \vec{y}, t_y) = (-i)^2 \frac{1}{Z(0)} \frac{\delta^2 Z[\kappa \hat{\phi}^+, \kappa \hat{\phi}^-]}{\delta \kappa \hat{\phi}^+(\vec{x}, t_x) \delta \kappa \hat{\phi}^-(\vec{y}, t_y)} \bigg|_{\kappa \phi=0}$$

such that

$$D^+_{++}(\vec{x} - \vec{y}; t_x - t_y) + D^-_{--}(\vec{x} - \vec{y}; t_x - t_y) = D^+_{-+}(\vec{x} - \vec{y}; t_x - t_y) + D^-_{+-}(\vec{x} - \vec{y}; t_x - t_y)$$

for

$$iD^+_{++}(\vec{x} - \vec{y}; t_x - t_y) = \langle \hat{T} \hat{\psi}(\vec{x}, t_x) \hat{\psi}(\vec{y}, t_y) \rangle - \langle \hat{\psi}(\vec{x}, t_x) \rangle \langle \hat{\psi}(\vec{y}, t_y) \rangle; t_x, t_y \in [+i \epsilon, t_f]$$

$$iD^-_{--}(\vec{x} - \vec{y}; t_x - t_y) = \langle \hat{T} \hat{\psi}(\vec{x}, t_x) \hat{\psi}(\vec{y}, t_y) \rangle - \langle \hat{\psi}(\vec{y}, t_y) \rangle \langle \hat{\psi}(\vec{x}, t_x) \rangle; t_x, t_y \in [t_f, -i \epsilon]$$

$$iD^+_{-+}(\vec{x} - \vec{y}; t_x - t_y) = \langle \hat{\psi}(\vec{y}, t_y) \hat{\psi}(\vec{x}, t_x) \rangle - \langle \hat{\psi}(\vec{y}, t_y) \rangle \langle \hat{\psi}(\vec{x}, t_x) \rangle; t_x \in [+i \epsilon, t_f], t_y \in [t_f, -i \epsilon]$$

$$iD^-_{+-}(\vec{x} - \vec{y}; t_x - t_y) = \langle \hat{\psi}(\vec{x}, t_x) \hat{\psi}(\vec{y}, t_y) \rangle - \langle \hat{\psi}(\vec{x}, t_x) \rangle \langle \hat{\psi}(\vec{y}, t_y) \rangle; t_x \in [t_f, -i \epsilon], t_y \in [+i \epsilon, t_f]$$

where

$$\langle \hat{\psi}(\vec{x}, t_x) \hat{\psi}(\vec{y}, t_y) \rangle := \text{Tr}_{\psi}[\hat{\psi}(\vec{x}, t_x) \hat{\psi}(\vec{y}, t_y) \hat{\rho}_{\psi}(t_y)].$$

Hence,

$$Z[\kappa \hat{\phi}^+, \kappa \hat{\phi}^-] = \exp \left( -\frac{\kappa^2}{2} \int_{+i \epsilon}^{t_f} dt_x \int_{+i \epsilon}^{t_x} dt_y \int d^3x \int d^3y \left\{ \hat{\phi}^+(\vec{x}, t_x) \hat{\phi}^+(\vec{y}, t_y) D^\to(\vec{x} - \vec{y}; t_x - t_y) + \hat{\phi}^-(\vec{x}, t_x) \hat{\phi}^-(\vec{y}, t_y) D^\le(\vec{x} - \vec{y}; t_x - t_y) \right. \right.$$  

$$\left. - \hat{\phi}^+(\vec{x}, t_x) \hat{\phi}^-(\vec{y}, t_y) D^\le(\vec{x} - \vec{y}; t_x - t_y) - \hat{\phi}^-(\vec{x}, t_x) \hat{\phi}^+(\vec{y}, t_y) D^\to(\vec{x} - \vec{y}; t_x - t_y) \right\} \right)$$

upon the relabeling of the connected correlation functions

$$D^\to(\vec{x} - \vec{y}; t_x - t_y) := iD_{\pm\pm}(\vec{x} - \vec{y}; t_x - t_y)$$

$$D^\le(\vec{x} - \vec{y}; t_x - t_y) := iD_{\pm\pm}(\vec{y} - \vec{x}; t_y - t_x).$$
To all orders, the dynamics of the $\psi$-environment are contained in the spectral function
\[
\sigma(\vec{k}, k_0; \beta) = \frac{\pi}{2} \int \frac{d^3p}{(2\pi)^3 \Omega \Omega} \left\{ 1 + n_\Phi + n_\Theta \right\} \left( \delta(k_0 - \Omega \Phi - \Omega \Theta) - \delta(k_0 + \Omega \Phi + \Omega \Theta) \right)
+ \left\{ [n_\Theta - n_\Phi] \left[ \delta(k_0 - \Omega \Phi + \Omega \Theta) - \delta(k_0 + \Omega \Phi - \Omega \Theta) \right] \right\} \right) \right)
\]
(3.29)
where
\[
n_{\Phi, \Theta} := \frac{1}{\exp[\beta \Omega_{\Phi, \Theta}] - 1}
\]
(3.30)
given $\vec{p}_\Theta = |\vec{p}_\Phi - \vec{k}|$. Here, the four delta functions correspond to all processes available in the plasma. Given
\[
\langle [\hat{\psi}(\vec{x}, t_x), \hat{\psi}(\vec{y}, t_y)] \rangle = \int \frac{d^3k}{(2\pi)^3} \int \frac{dk_0}{2\pi} \sigma(\vec{k}, k_0; \beta) \exp[-ik_0(t_x - t_y) + i\vec{k} \cdot (\vec{x} - \vec{y})]
\]
(3.31)
where
\[
\langle [\hat{\psi}(\vec{x}, t_x), \hat{\psi}(\vec{y}, t_y)] \rangle = D^>(\vec{x} - \vec{y}; t_x - t_y) - D^<(\vec{x} - \vec{y}; t_x - t_y)
\]
(3.32)
such that
\[
\sigma(\vec{k}, k_0; \beta) = D^>(k_0, \vec{k}; \beta) - D^<(k_0, \vec{k}; \beta)
\]
(3.33)
and
\[
D^>(\vec{k}, k_0; \beta) = D^<(\vec{k}, k_0; \beta) \exp[\beta k_0]
\]
(3.34)
satisfies the KMS relation, then
\[
D^>(\vec{k}, k_0; \beta) = \sigma(\vec{k}, k_0; \beta) [1 + n(k_0)]
\]
(3.35)
\[
D^<(\vec{k}, k_0; \beta) = \sigma(\vec{k}, k_0; \beta) n(k_0)
\]
(3.36)
for
\[
n(k_0) := \frac{1}{\exp[\beta k_0] - 1}.
\]
(3.37)
Following the formulation in ref. [49] we may now assign an effective action, to second order in $\kappa$, in the form of
\[
S_{Eff}[^{\hat{\phi}^+}, ^{\hat{\phi}^-}; \beta] := \int_{t_t + \kappa}^{t_f} dt_x \int d^3x \left\{ \mathcal{L}_0[^{\hat{\phi}^+}(\vec{x}, t_x)] - \mathcal{L}_0[^{\hat{\phi}^-}(\vec{x}, t_x)] \right\} + F[^{\hat{\phi}^+}, ^{\hat{\phi}^-}; \beta]
\]
(3.38)
with $F[^{\hat{\phi}^+}, ^{\hat{\phi}^-}; \beta]$ the so called influence phase of ref. [50]; such that
\[
Z[^{\hat{\phi}^+}, ^{\hat{\phi}^-}; \beta] = \exp \left( iF[^{\hat{\phi}^+}, ^{\hat{\phi}^-}; \beta] \right).
\]
(3.39)
Using the Fourier transforms
\[
\int d^3x \hat{\phi}^\pm(\vec{x}, t_x) \exp(i\vec{k} \cdot \vec{x}) = \hat{\phi}^\pm(\vec{k}, t_x)
\]
(3.40)
and
\[
\int d^3y \, \hat{\phi}^\pm (\vec{y}, t_y) \exp(-i \vec{k} \cdot \vec{y}) = \int \frac{d\Omega_{\vec{k}}}{2\pi} \hat{\phi}^\pm(-\vec{k}, \Omega_{\vec{k}}) \exp[-i \Omega_{\vec{k}} t_y]
\]  
(3.41)
produces the expression
\[
F[\hat{\phi}^+, \hat{\phi}^-; \beta] = \lim_{\epsilon \downarrow 0} \left\{ \int_{+i\epsilon}^{t_f} dt_x \int_{+i\epsilon}^{t_x} dt_y \int \frac{d^3k}{(2\pi)^3} \int \frac{d\Omega_{\vec{k}}}{2\pi} \exp[-i \Omega_{\vec{k}} t_x] \exp[i(\Omega_{\vec{k}} - k_0)(t_x - t_y)] \right\}
\]
\[
\hat{\phi}^+(-\vec{k}, t_x)\hat{\phi}^+(\vec{k}, \Omega_{\vec{k}})D^>(\vec{k}, k_0; \beta) + \hat{\phi}^-(\vec{k}, t_x)\hat{\phi}^-(\vec{k}, \Omega_{\vec{k}})D^<(-\vec{k}, k_0; \beta)
\]
\[- \hat{\phi}^+(-\vec{k}, t_x)\hat{\phi}^+(-\hat{k}, \Omega_{\vec{k}})D^>(\vec{k}, k_0; \beta) - \hat{\phi}^-(\vec{k}, t_x)\hat{\phi}^-(-\hat{k}, \Omega_{\vec{k}})D^<(-\vec{k}, k_0; \beta) \}
\]  
(3.42)
Setting \(t_i = 0\) in the integrals above establishes the common reference time as the moment of the plasma’s instantiation, i.e. the approximation of instantaneous reheating after inflation. Here,
\[
\lim_{\epsilon \downarrow 0} \int_{+i\epsilon}^{t_x} dt_y \exp[i(\Omega_{\vec{k}} - k_0)(t_x - t_y)] = \frac{i}{(\Omega_{\vec{k}} - k_0)} \left( 1 - \exp[i(\Omega_{\vec{k}} - k_0)t_x] \right)
\]  
(3.43)
such that for later convenience we may decompose the influence phase
\[
F[\phi^+, \phi^-; \beta] = F_1[\phi^+, \phi^-; \beta] + F_2[\phi^+, \phi^-; \beta]
\]  
(3.44)
into a unitary fluctuation term
\[
F_1[\hat{\phi}^+, \hat{\phi}^-; \beta] = \lim_{\epsilon \downarrow 0} \left\{ \int_{+i\epsilon}^{t_f} dt_x \int \frac{d^3k}{(2\pi)^3} \int \frac{d\Omega_{\vec{k}}}{2\pi} \exp[-i \Omega_{\vec{k}} t_x] \right\}
\]
\[
\hat{\phi}^+(-\vec{k}, t_x)\hat{\phi}^+(\vec{k}, \Omega_{\vec{k}})\Delta^>(\vec{k}, \Omega_{\vec{k}}; \beta) + \hat{\phi}^-(\vec{k}, t_x)\hat{\phi}^-(-\vec{k}, \Omega_{\vec{k}})\Delta^>(\vec{k}, \Omega_{\vec{k}}; \beta)
\]
\[- \hat{\phi}^+(-\vec{k}, t_x)\hat{\phi}^+(-\vec{k}, \Omega_{\vec{k}})\Delta^>(\vec{k}, \Omega_{\vec{k}}; \beta) - \hat{\phi}^-(-\vec{k}, t_x)\hat{\phi}^+(-\vec{k}, \Omega_{\vec{k}})\Delta^>(\vec{k}, \Omega_{\vec{k}}; \beta) \}
\]  
(3.45)
and a nonunitary dissipation term
\[
F_2[\hat{\phi}^+, \hat{\phi}^-; \beta] = \lim_{\epsilon \downarrow 0} \left\{ \int_{+i\epsilon}^{t_f} dt_x \int \frac{d^3k}{(2\pi)^3} \int \frac{d\Omega_{\vec{k}}}{2\pi} \exp[-i \Omega_{\vec{k}} t_x] \right\}
\]
\[
\hat{\phi}^+(-\vec{k}, t_x)\hat{\phi}^+(\vec{k}, \Omega_{\vec{k}})\Delta^>(\vec{k}, \Omega_{\vec{k}}; \beta) + \hat{\phi}^-(\vec{k}, t_x)\hat{\phi}^-(-\vec{k}, \Omega_{\vec{k}})\Delta^>(\vec{k}, \Omega_{\vec{k}}; \beta)
\]
\[- \hat{\phi}^+(-\vec{k}, t_x)\hat{\phi}^+(-\vec{k}, \Omega_{\vec{k}})\Delta^>(\vec{k}, \Omega_{\vec{k}}; \beta) - \hat{\phi}^-(-\vec{k}, t_x)\hat{\phi}^+(-\vec{k}, \Omega_{\vec{k}})\Delta^>(\vec{k}, \Omega_{\vec{k}}; \beta) \}
\]  
(3.46)
\[ 
\Delta_1^{>}(\vec{k}, \Omega; \beta) := i \int \frac{dk_0}{2\pi} \left( \frac{1 - \cos[(\Omega - k_0)t_x]}{\Omega - k_0} \right) \]

and

\[ 
\Delta_2^{>}(\vec{k}, \Omega; \beta) := \int \frac{dk_0}{2\pi} \left( \frac{\sin[(\Omega - k_0)t_x]}{\Omega - k_0} \right) \]

### 3.2 Quantum Master Equation Approach

Having framed our effective field theory as an open quantum system we should then expect a Markovian master equation (MME) as a description of the entropically irreversible reduced dynamics of the \( \phi \)-system. Here, we introduce a quasifree quantum dynamical semigroup (QDS) as the family of maps \( \mu_t : T(\mathcal{H}_\phi) \to T(\mathcal{H}_\phi) \) such that

\[ 
\partial_t \hat{\rho}_\phi(t) = \hat{\mathcal{L}} \hat{\rho}_\phi(t); \quad (3.49) 
\]

where the unbounded operator \( \hat{\mathcal{L}} \), with dense domain \( \mathcal{D}(\hat{\mathcal{L}}) \subset \mathcal{H}_\phi \), is the generator of a one-parameter, completely positive Markov semigroup

\[ 
\mu_t = \exp(\hat{\mathcal{L}} t) \quad (3.50) 
\]

such that \( \mu_t \circ \mu_s \to \mu_{s+t} \). In this context \( \hat{\mathcal{L}} \) is of course the Linblad superoperator where the MME is of the well known Linblad form \([51]\)

\[ 
\hat{\mathcal{L}} \hat{\rho}_\phi(t) = -i[H_\phi, \hat{\rho}_\phi(t)] + \sum_n \hat{L}_n \hat{\rho}_\phi(t) \hat{L}_n^\dagger - \frac{1}{2} \sum_n \hat{L}_n^\dagger \hat{L}_n \hat{\rho}_\phi(t). \quad (3.51) 
\]

The complete positivity of \( \mu_t \) ensures the form

\[ 
\mu_t(\hat{\rho}_\phi(0)) = \sum_n \hat{M}_n \hat{\rho}_\phi(0) \hat{M}_n^\dagger \quad (3.52) 
\]

where the Kraus operators \( \hat{M}_n \) satisfy \( \sum_n \hat{M}_n \hat{M}_n^\dagger \leq 1 \), such that

\[ 
\mu_t(\hat{\rho}_\phi(0)) = \text{Tr}_\psi \left[ \hat{U}^{-1}(t,0) \hat{\rho}_\phi(0) \otimes \hat{\rho}_\psi \hat{U}(t,0) \right], \quad (3.53) 
\]

with the continued assumption of factorization throughout the unitarity evolution of the composite system \( \hat{\rho}_{\phi,\psi}(t) \). Given

\[ 
\exp(iF_1[\hat{\phi}^+, \hat{\phi}^-; \beta] + iF_2[\hat{\phi}^+, \hat{\phi}^-; \beta]) = \text{Tr}_\psi \left[ \hat{U}^{-1}(\hat{\phi}^-; t_f, t_i = 0) \hat{\rho}_{\phi,\psi}(t_i = 0) \hat{U}(\hat{\phi}^+; t_f, t_i = 0) \right] \quad (3.54) 
\]

established by Eqs. (3.13) and (3.39), we may review the main result of ref. [49] via the relation

\[ 
\hat{\rho}_\phi(t_f) = \exp(iF[\hat{\phi}^+, \hat{\phi}^-; \beta]). \quad (3.55) 
\]
such that to one-loop order eq. (3.55) may be brought to the time-local form of a Bloch–Redfield master equation

\[
\partial_t \hat{\rho}_\phi(t) = -\kappa^2 \int_0^t dt_y \int d^3y \int d^3x \left\{ \hat{\phi}^+(\bar{x}, t) \hat{\phi}^+(\bar{y}, t) D^>(\bar{x} - \bar{y}; t - t_y) \hat{\rho}_\phi(t) + \hat{\rho}_\phi(t) \hat{\phi}^-(\bar{x}, t) \hat{\phi}^-\bar{y}, t) D^<(\bar{x} - \bar{y}; t - t_y) - \hat{\rho}_\phi(t) \hat{\phi}^-(\bar{x}, t) \hat{\phi}^+(\bar{y}, t) D^<(\bar{x} - \bar{y}; t - t_y) - \hat{\phi}^+(\bar{x}, t) \hat{\phi}^-\bar{y}, t) D^>(\bar{x} - \bar{y}; t - t_y) \hat{\rho}_\phi(t) \right\}.
\]

(3.56)

To write eq. (3.56) as an MME of Linblad form we first transform to the spectral representation via the relations of the previous subsection and time order the source terms \( \hat{\phi}^\pm \) with respect to \( \hat{\rho}_\phi \), e.g. \( \hat{\phi}^+ \hat{\phi}^- \hat{\rho}_\phi \rightarrow \hat{\phi}^+ \hat{\rho}_\phi \hat{\phi}^- \). The Markov approximation is then made by first replacing the time coordinate \( t_y \) with the interval \( \delta t := t - t_y \), i.e. the interval over which memory effects may be ignored, and then carrying out the integral over \( dt_y \) with a memoryless upper bound \( t \rightarrow \infty \) such that

\[
\lim_{t \rightarrow \infty} \int_0^t dt_y \exp[-i(k_0 - \Omega_\vec{k})(t - \delta t)] = i \text{PV}[-] + \pi \delta(\Omega_\vec{k} - k_0).
\]

(3.57)

We may now write the Linblad master equation

\[
\partial_t \hat{\rho}_\phi(t) = \int \frac{d^3k}{(2\pi)^3} \left\{ -i\delta \Omega_\vec{k} \{ \hat{a}_k^\dagger \hat{a}_k, \hat{\rho}_\phi(t) \} 
- \frac{\Gamma_\vec{k}^>} {2} \left[ \hat{a}_k^\dagger \hat{a}_k \hat{\rho}_\phi(t) + \hat{\rho}_\phi(t) \hat{a}_k^\dagger \hat{a}_k - 2 \hat{a}_k \hat{\rho}_\phi(t) \hat{a}_k^\dagger \right] 
- \frac{\Gamma_\vec{k}^<} {2} \left[ \hat{a}_k \hat{a}_k^\dagger \hat{\rho}_\phi(t) + \hat{\rho}_\phi(t) \hat{a}_k \hat{a}_k^\dagger - 2 \hat{a}_k^\dagger \hat{\rho}_\phi(t) \hat{a}_k \right] \right\}.
\]

(3.58)

Here,

\[
\delta \Omega_\vec{k} = \text{PV} \left[ \frac{\kappa^2}{2\Omega_\vec{k}} \int \frac{dk_0}{2\pi} \frac{\sigma(k_0, \vec{k}; \beta)}{(\Omega_\vec{k} - k_0)} \right] = \frac{\text{Re} \Pi(\vec{k}; \vec{k}; \beta)}{2\Omega_\vec{k}}
\]

(3.59)

and

\[
\Gamma_\vec{k}^> = \frac{\kappa^2}{2\Omega_\vec{k}} \int \frac{dk_0}{2\pi} \delta(\Omega_\vec{k} - k_0) \sigma(k_0, \vec{k}; \beta) \left[ n(k_0) + \Xi^>(\vec{k}) \right] \pi \delta(\Omega_\vec{k} - k_0)

= - \frac{\text{Im} \Pi(\Omega_\vec{k}, \vec{k}; \beta)}{2\Omega_\vec{k}} \left[ n(\Omega_\vec{k}) + \Xi^>(\vec{k}) \right]
\]

(3.60)

where \( \Pi(\Omega_\vec{k}, \vec{k}; \beta) \) is the self energy and \( \Xi^>(\vec{k}) := 1(0) \) in accordance with the optical theorem

\[
\Gamma_\vec{k}^> = \Gamma_\vec{k}^< - \Gamma_\vec{k}^\leq = - \frac{\text{Im} \Pi(\Omega_\vec{k}, \vec{k}; \beta)}{\Omega_\vec{k}}.
\]

(3.61)
The secular approximation has also been invoked such that the contributions of rapidly dephasing terms proportional to \( \exp[\pm i\Omega_k(t + t_y)] \) were ignored. Here, the fluctuating Hamiltonian like term \( \delta \Omega_k \) and dissipative non-Hamiltonian terms \( \Gamma_k^{> (<)} \) correspond to the decomposed influence action terms \( F_1[\hat{\phi}^+, \hat{\phi}^-; \beta] \) and \( F_2[\hat{\phi}^+, \hat{\phi}^-; \beta] \) respectively. One sees immediately that though we have imposed the Born approximation along with the nontrivial approximation of secular, Markovian evolution there remain in this formulation dissipative, nonlocal terms valid for the descriptions of the nonunitary entropically irreversible evolution of the subsystem \( \hat{\rho}_\phi(0) \). Taking the derivative with respect to time of the expectation value of the number operator via the trace

\[
\partial_t \langle \hat{N}_k(t) \rangle = \text{Tr} \left[ \hat{a}^+_k \hat{a}_k \partial_t \hat{\rho}_\phi(t) \right]
\]

we find

\[
\partial_t \mathcal{M}_k(t) = \left[ \mathcal{M}_k(t) + 1 \right] \Gamma_k^{> (<)}(\Omega_k) - \mathcal{M}_k(t) \Gamma_k^{> (<)}(\Omega_k)
\]

where \( \mathcal{M}_k(t) := \langle \hat{N}_k(t) \rangle \) is the statistical number density of the mode \( \vec{k} \) in agreement with the standard kinetic theory [52, 53].

### 3.3 The Renormalized, Perturbed Energy Density in the Operator Formalism

We may now relate the observable evolved to some macroscopic time \( t \gtrsim \Gamma_k^{-1} \) to the observables of the asymptotic state through the operator relations

\[
\langle \hat{\phi}(\vec{x}, t) \hat{\phi}(\vec{x}, t) \rangle = \exp[-\Gamma_k t/2] \exp[i\delta \Omega_k t] \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\Omega_k} \left\{ 2 \langle \hat{N}_k \rangle + 1 \right\}
\]

where \( \hat{\phi}^0(\vec{x}, t) = \sqrt{Z} \hat{\phi}^R(\vec{x}, t) \) is the formally infinite field strength renormalization of the perturbative theory, i.e.

\[
\hat{\phi}^0(\vec{x}, t) = \sqrt{Z} \hat{\phi}^R(\vec{x}, t)
\]

Similarly, the expectation value of the time-dependent number operator is related to its renormalized expectation value via

\[
\langle \hat{N}_k(t) \rangle^R := \text{Tr} \left[ \hat{a}^+_k \hat{a}_k \hat{\rho}_\phi(t) \right]^R = Z \text{Tr} \left[ \hat{a}^+_k \hat{a}_k \hat{\rho}_\phi(0) \right]^R \exp[-\Gamma_k t].
\]

In the operator formalism the KMS state is written

\[
\langle \hat{\phi}(\vec{x}, t) \hat{\phi}(\vec{x}, t) \rangle_β = \langle \hat{\phi}(\vec{x}, t) \hat{\phi}(\vec{x}, t - i\beta) \rangle_β = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\Omega_k} \left\{ 2 \langle \hat{N}_k \rangle_β + 1 \right\},
\]

where we now construct the expression

\[
\langle \hat{N}_k(t) \rangle^R - \langle \hat{N}_k \rangle_β^R = Z \{ \langle \hat{N}_k(0) \rangle^R - \langle \hat{N}_k \rangle_β^R \} \exp[-\Gamma_k t].
\]
Here, $\Gamma_\vec{k}$ is the relaxation rate, the rate at which the $\phi$-system in an initial configuration $\langle N_\vec{k}(0) \rangle$ approaches equilibrium $\langle N_\vec{k} \rangle$. When ignoring the finite contribution from $Z \simeq 1 + \mathcal{O}(\kappa^2)$, we may write

$$\mathcal{M}_\vec{k}(t) = \mathcal{M}_{\vec{k},\beta}(t) + \{ \mathcal{M}_\vec{k}(t) - \mathcal{M}_{\vec{k},\beta}(t) \} \exp[ - \Gamma_\vec{k} t ]. \quad (3.70)$$

In the standard formalism the energy density of the free field $\hat{\phi}$ is the expectation value of the $T_{00}$ component of the Minkowski stress-energy tensor and is defined as the Hamiltonian density $\mathcal{H}_0$ found via the Legendre transform

$$\mathcal{E}_\phi := \mathcal{H}_0[\hat{\phi}] = \partial / \partial \dot{\hat{\phi}}(\mathcal{L}_0[\hat{\phi}]) - \mathcal{L}_0[\hat{\phi}]. \quad (3.71)$$

Hence,

$$\mathcal{E}_\phi(t) = \int \frac{d^3k}{(2\pi)^3} \Omega_\vec{k} \left( \mathcal{M}_\vec{k}(t) + \frac{1}{2} \right) \quad (3.72)$$

is finite when $\mathcal{M}_\vec{k}(t)$ vanishes faster than any polynomial in $k$ as $k \to \infty$. We now examine eq. (3.70) where we restrict our attention to the regime in which $N_\vec{k}(0) \simeq 0$, i.e. the regime of $\hat{\phi}$ particle production from the vacuum state, and find that indeed $\mathcal{M}_\vec{k}(t) \propto \mathcal{M}_{\vec{k},\beta}$ for all times $t \gtrsim \Gamma_\vec{k}^{-1}$.

4 The Algebraic Energy Density in the Stationary Spacetime Limit

In order to ensure that the result presented in eq. (2.95) reduces to the form of the standard formulation of the energy density when gravitational effects are ignored, we invoke the definitions

$$X_\vec{k}^P := \mathcal{M}_\vec{k} + \frac{1}{1 + \mathcal{M}_\vec{k}} \quad (4.1)$$

and

$$\mathcal{M}_\vec{k} := \frac{X_\vec{k}(\eta_x)X_\vec{k}(\eta_y)}{X_\vec{k}(\eta_x)X_\vec{k}(\eta_y)} \quad (4.2)$$

such that the perturbed state in eq. (2.69) may be written

$$\omega^P(A_{\eta_x},A_{\eta_y}) = \int \frac{d^3k}{(2\pi)^3} \left\{ \mathcal{M}_\vec{k} X_\vec{k}(\eta_x) X_\vec{k}(\eta_y) + \left( \mathcal{M}_\vec{k} + 1 \right) X_\vec{k}(\eta_x) X_\vec{k}(\eta_y) \right\} \exp[ - i \vec{k} \cdot (\vec{x} - \vec{y}) ] \quad (4.3)$$

Restricting to the diagonal $\eta_x = \eta_y = \eta$ we find the divergent, perturbed energy density to be of the form

$$\mathcal{E}_A^P = \int \frac{dk}{2\pi^2} \frac{k^2}{a_t^4} \left\{ \mathcal{M}_\vec{k} \left( |X_\vec{k}(\eta)|^2 - a_t H_t (|X_\vec{k}(\eta)|^2)' + (k^2 + a_t^2 m^2 + a_t^2 H_t^2 |X_\vec{k}(\eta)|^2) \right) \right\} \quad (4.4)$$

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where we may separate the integral into two terms:

\[ \tilde{\mathcal{E}}_{A,\eta}^P = \mathcal{E}_{A,\eta}^N + \tilde{\mathcal{E}}_{A,\eta}^0 \]  

such that \( \mathcal{E}_{A,\eta}^N \) contains the finite excitation \( \mathcal{N}_k \), assuming it vanishes faster than any polynomial in \( k \) for \( k \to \infty \), and \( \tilde{\mathcal{E}}_{A,\eta}^0 \) the divergent vacuum term.

### 4.1 Renormalizing the Perturbed Energy Density

We now focus on the vacuum term where we follow the established results for renormalization of the unperturbed state as summarized in ref. [17]. Here, the mode integral representation of the Hadamard parametrix, eq. (2.89), is implemented such that the renormalized vacuum energy density is

\[
\mathcal{E}_{A,\eta}^0 = \frac{1}{2} \int \frac{dk}{2\pi^2} \, k^2 \left\{ \frac{1}{a_t^4} \left( |X_k'(\eta)|^2 - |X_k(\eta)|^2 \right) + (k^2 + a_t^2 m^2 + a_t^2 H_t^2) |X_k(\eta)|^2 \right\} 
- \frac{k}{2a_t^4} - \frac{m^2 + H_t^2}{4a_t^2 k} + \Theta[k - a_t m] \frac{m^4}{16k^3} 
- \Theta[k - a_t m] \frac{1}{16k^3} \left( 2m^2 H_t^2 - 6H_t^2 \dot{H}_t + \ddot{H}_t^2 - 2H_t \dot{H}_t \right) 
+ \frac{1}{96\pi^2} \left[ 12H_t^4 + 18H_t^2 \dot{H}_t + 6\dot{H}_t^2 - m^2 H_t^2 \right] - m^4 \left( \frac{1 - 4\log 2}{128\pi^2} \right). 
\]  

(4.6)

In the stationary spacetime limit \( \dot{a}_t \to 0 \), i.e. \( a_t \simeq a_0 \), this expression reduces to

\[
\mathcal{E}_{A,\eta}^0(t) = \frac{1}{2a_0^4} \int \frac{dk}{2\pi^2} \, k^2 \left\{ |\partial_t T_k^*(t)|^2 + (k^2 + a_0^2 m^2) |T_k(t)|^2 \right\} 
- \frac{k}{2a_0^4} - \frac{a_0^2 m^2}{4k} + \Theta[k - a_0 m] \frac{a_0^4 m^4}{16k^3} - m^4 \left( \frac{1 - 4\log 2}{128\pi^2} \right) 
= \frac{1}{2a_0^4} \int \frac{d^3k}{(2\pi)^3} \left\{ |\partial_t T_k(t)|^2 + (k^2 + a_0^2 m^2) |T_k(t)|^2 \right\} 
- \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left\{ |\partial_t T_k(t)|^2 + \Omega_k |T_k(t)|^2 \right\} 
\]  

(4.7)

which vanishes according to the reduction to normal ordering condition of eq. (2.93) and eq. (2.94).
4.2 Validating the Finite Excitation

It now remains only to establish the form of $\mathcal{F}_\kappa$ in this limit. To this end we invoke definitions similar to eq. (4.1) and eq. (4.2), i.e.

$$\mathcal{F}_\kappa := n_\kappa + \frac{1}{1 + r_\kappa} \quad r_\kappa := \frac{X_\kappa(\eta_x)X_\kappa(\eta_y)}{X_\kappa(\eta_x)X_\kappa(\eta_y)}$$

(4.8)

$$\mathcal{V}_\rho := n_\rho + \frac{1}{1 + r_\rho} \quad r_\rho := \frac{Y_\rho(\eta_x)Y_\rho(\eta_y)}{Y_\rho(\eta_x)Y_\rho(\eta_y)}$$

(4.9)

$$\mathcal{Z}_q := n_q + \frac{1}{1 + r_q} \quad r_q := \frac{Z_q(\eta_x)Z_q(\eta_y)}{Z_q(\eta_x)Z_q(\eta_y)}$$

(4.10)

(4.11)

such that upon substitution the nonvacuum terms on the RHS of eq. (2.71) become

$$-\frac{k^2}{2} \int_0^{\eta_x} d\eta_x \int_0^{\eta_y} d\eta_y \int d^3x \int d^3y \int \frac{d^3k}{(2\pi)^3} \left\{ \mathcal{F}_\kappa \mathcal{V}_\rho \mathcal{Z}_q \exp[i\vec{k} \cdot (\vec{x} - \vec{y})] \right\}$$

$$\times \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \left\{ \int d^3x \int d^3y \left[ n_\kappa X_\kappa(\eta_x)X_\kappa(\eta_y) + (n_\kappa + 1)X_\kappa(\eta_x)X_\kappa(\eta_y) \right] \exp[i\vec{k} \cdot (\vec{x} - \vec{y})] \right\}$$

$$\times \int d^3x \int d^3y \left[ n_\rho Y_\rho(\eta_x)Y_\rho(\eta_y) + (n_\rho + 1)Y_\rho(\eta_x)Y_\rho(\eta_y) \right] \exp[i\vec{p} \cdot (\vec{x} - \vec{y})]$$

$$\times \int d^3x \int d^3y \left[ (n_\kappa + 1)X_\kappa(\eta_x)X_\kappa(\eta_y) + n_\kappa X_\kappa(\eta_x)X_\kappa(\eta_y) \right] \exp[i\vec{k} \cdot (\vec{x} - \vec{y})]$$

$$\times \int d^3x \int d^3y \left[ n_\rho Y_\rho(\eta_x)Y_\rho(\eta_y) + n_\rho Y_\rho(\eta_x)Y_\rho(\eta_y) \right] \exp[i\vec{p} \cdot (\vec{x} - \vec{y})]$$

$$\times \int d^3x \int d^3y \left[ (n_q + 1)Z_q(\eta_x)Z_q(\eta_y) + n_q Z_q(\eta_x)Z_q(\eta_y) \right] \exp[i\vec{q} \cdot (\vec{x} - \vec{y})]$$

(4.12)

Invoking once again the stationary limit of Minkowski spacetime we let $n_\rho$ and $n_q$ be the equilibrium distribution of the KMS state such that $n_\kappa$ takes the same form by the KMS condition. We then differentiate with respect to $\eta_x$ and carrying out the $\eta_y$ integral. Making use of the Fourier transforms and imposing the Born, Markov, and secular approximations of the previous section we bring this expression to the form

$$\int d^3x \int d^3y \int \frac{d^3k}{(2\pi)^3} \exp[i\vec{k} \cdot (\vec{x} - \vec{y})] \left\{ \mathcal{F}_\kappa \mathcal{V}_\rho \mathcal{Z}_q \left[ \frac{\mathcal{M}_\kappa}{\mathcal{M}_\kappa} - 2\Gamma_\kappa \mathcal{M}_\kappa \right] \right\} \left| \eta_x = \eta_y = \eta \right.$$

(4.13)
with \(\Gamma^>(<)\) defined as in eq. (3.60). Examining now the nonvacuum term on the LHS of eq. (2.71) differentiated with respect to \(\eta_x\) and restricted to the diagonal \(\bar{\eta}_x = \bar{\eta}_y = \eta\) we have

\[
\partial_{\eta_x} \int d^3x \int d^3y \int \frac{d^3k}{(2\pi)^3} \exp[i\vec{k} \cdot (\vec{x} - \vec{y})] \left\{ 2N_{\vec{k}} X_{\vec{k}}(\eta_x) X_{\vec{k}}(\eta_y) \right\} \bigg|_{\bar{\eta}_x = \bar{\eta}_y = \eta}^{\eta_x = \eta} \tag{4.14}
\]

such that we find the full expression for eq. (2.71) in the stationary spacetime limit to be

\[
\partial_{\eta_x} \mathcal{N}_{\vec{k}} = (\mathcal{N}_{\vec{k}} + 1) \bf{\Gamma}_{\vec{k}}^> - \mathcal{N}_{\vec{k}} \bf{\Gamma}_{\vec{k}}^<. \tag{4.15}
\]

Hence, eq. (2.95) as the renormalized, perturbed energy density derived in the algebraic approach to quantum field theory in a non-stationary FRW spacetime, the main result of this work, is found to reduce to the standard form in an effective field theory approach to non-equilibrium quantum field theory in the operator formalism.

5 Discussion

In this work we began with first principles of algebraic quantum field theory in curved spacetime and derived for the first time eq. (2.95) as an expression for the renormalized, perturbed energy density of a massive, minimally coupled free scalar field in the adiabatic limit of interacting fields propagating in a non-stationary FRW spacetime. We employed both the SLE construction of renormalizable ground states via finite time intervals and a two-parameter family of automorphisms in the approach to the non-equilibrium dynamics of a dense environment of interacting fields. We checked the validity of this result by taking the stationary spacetime limit, i.e. a reduction to Minkowski spacetime, and found that our result reduced to the appropriate form in an effective field theory approach to the standard operator formalism.

The exact form of the perturbed state in the non-stationary regime remains, however, an open question and thus additional work is required in order to faithfully capture an expression that allows us to compute observables. As we may not simply invoke the standard KMS state in regard to the dense environment of interacting fields a careful assessment of the behavior of the perturbed state in general models of matter genesis is necessary. We leave this for future work.

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