Beyond the ABCDs: A projective geometry treatment of paraxial ray tracing using homogeneous coordinates

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Abstract

Homogeneous coordinates are a projective geometry tool particularly well suited to paraxial geometric optics. They are useful because they allow the expression of rotations, translations, affine transformations, and projective transformations as linear operators (matrices). While these techniques are common in the computer graphics community, they are not well-known to physicists. Here we apply them to paraxial ray tracing.

Geometric optics is often implemented by tracing the paths of non-diffracting rays through an optical system. In the paraxial limit rays can be represented through their height and slope, and traced through the optical system using ray transfer (colloquially, “ABCD”) matrices. The homogeneous representation allows us to expand our set of operations with translations and rotations, allowing us to consider decentered and rotated optical elements. The homogeneous representation also distinguishes the orientation of rays (forwards or backwards). Using projective duality, we also show how to image points through an optical system, once the homogeneous ray transfer matrix of the system is known. We demonstrate the usefulness of these methods with several examples, and discuss future directions to expand this technique.

1 Introduction

Geometric optics describes light as non-diffracting rays traveling through media and surfaces using the laws of reflection and refraction. In the paraxial limit, the effect of optical elements may be approximated by linear functions of a ray’s height and direction. This approximation is often the first step in designing optical systems and is also one of the first optics topics presented to students (the Gaussian and Newtonian image equations, lensmaker’s equation, etc.). Assuming the optical system of lenses, mirrors, etc. has a rotational axis of symmetry, we can define the paraxial approximation as being the limit where the ray height $h$ relative to the axis is much smaller in magnitude than other lengths in the system, such as radii of curvature or focal lengths, and the slope $m$ the ray makes with the axis has magnitude much smaller than one. Under these conditions, the laws of reflection and refraction at planar and spherical surfaces may be replaced by their Taylor series approximations to first order in $h$ and $m$, and therefore may be expressed as matrix equations[1–4]. These matrices are known as ray transfer matrices (RTM), or “ABCD” matrices after their typical parameterization. In this paper we will introduce a fuller geometric description of rays and ray tracing by using homogeneous coordinate representations of lines and points.
Homogeneous coordinates are a well known tool in the computer graphics community because they allow the expression of rotations, translations, affine transformations, and perspective transforms as linear operators (matrices) in homogeneous space[5]. Homogeneous coordinates are also well suited to paraxial optics, because they are the natural setting for projective geometry (particularly oriented projective geometry[6]) which itself underlies optics in the paraxial limit[7]. Most optics work in this regard has been on the tracing of rays through the optical system[8]. Here we treat rays as oriented lines in projective geometry and identify the projective group elements that correspond to common optical elements. This takes us beyond the augmented ABCD matrices used by some authors[2, 8–14] to a fully consistent geometric treatment. Additionally, projective duality allows us to show how points are imaged through an optical system once we know the ray transfer matrix. The ideas here are implicit in, for example, the geometric optics derivations of Born and Wolf[15] but are not presented there in a readily applicable form. A better understanding of the geometry underlying the transfer matrices may also be helpful in solving the inverse problem of finding a set of optical elements needed to produce a given optical transformation[16, 17].

We begin in Section 2 with a review of the paraxial ray transfer matrices as typically used in two-dimensional or axially symmetric optical systems. Next we show how RTMs are a disguised form of general linear operators on the vector space of lines represented in homogeneous coordinates (Section 3). Because we are using homogeneous coordinates, we can also consider optical elements decentered or rotated with respect to the optical axis, which we discuss in Section 4 along with an example. Careful treatment of algebraic signs allows us to preserve the propagation direction of our rays, which we explore in Section 5 to derive orientation-preserving RTMs for reflective elements. By requiring a coincident point and line to remain so after imaging, we then find a key new result, the point transfer matrices (PTMs) (Section 6), and provide some examples demonstrating how PTMs simplify imaging calculations and the analysis of optical systems (Section 7). Finally, we close in Section 8 with a discussion of what would be required to extend these results to three-dimensional systems and the complications that arise therein. An appendix is included giving an alternative derivation of the PTMs using Grassmann exterior algebra.

2 Ray transfer matrices

Before introducing our new concepts, we will briefly review the ray transfer matrices as conventionally used. We refer readers to Refs. [1–4] for detailed derivations. Several conventions exist in the literature regarding the ordering of elements and the implementation of the index of refraction. We will follow the textbook by the Pedrottis[3].

We define an incoming ray vector \( r = (h, m)^T \) representing a ray with slope \( m \) relative to the optical axis and crossing the input plane of the optical system at height \( h \). (We use \( T \) to indicate matrix transposition.) The coordinate origins of the object/image spaces are centered on the first/last surfaces of the optical system, as shown in Fig 1. The incoming ray transforms into the outgoing ray \( r' = (h', m')^T \) through the relation

\[
\begin{pmatrix} h' \\
 m' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} h \\
 m \end{pmatrix},
\]

(1)

where \( A, B, C, D \) are real-valued constants obeying \( AD - BC \neq 0 \) such that this ray transfer matrix is invertible. In particular, the matrix determinant is equal to the ratio of the incoming and outgoing indices of refraction, \( AD - BC = n/n' \), which is often unity for common optical systems. This makes the determinant a useful check on computational results. The relationship described by Eq. (1) is shown schematically in Fig. 1. Note that some authors, notably Hecht[4], use different conventions for the ray vectors and matrices. A common choice is to multiply the slope in the ray vector by the index of refraction, yielding the “reduced slope.” We prefer to use the geometric slope to avoid complications in our transformation operators (Sec. 5).

The ray transfer matrices are particularly useful because whole optical systems may be summarized by the product of the matrices of their components, with the sequence written from right to left (first element
3 RAYS AND RAY TRANSFER MATRICES IN HOMOGENEOUS COORDINATES

Figure 1: Ray transfer matrix. The incoming ray \( r \) enters the optical system (gray box) and is transformed into the outgoing ray \( r' \) by the according the matrix in Eq. (1). Note that the incoming coordinate system \((x, y)\) has its origin at the first surface of the system and that the outgoing coordinate system \((x', y')\) has its origin at the final surface of the system. The horizontal line represents the optical axis of the system.

Figure 2: (Color online only.) Representation of the ray \((c, a, b)^T\) as the oriented line \( ax + by + c = 0 \). The coefficients of the equation can be used to calculate the \(x\)- and \(y\)-intercepts, slope, and distance from the origin, as shown. Note that for this particular oriented line, \(a, b > 0\) and \(c < 0\).

right-most). If we exchange the input and output ends of the optical system (or equivalently, reverse time), the resulting ray transfer matrix is the matrix inverse of the original system.

3 Rays and ray transfer matrices in homogeneous coordinates

Given a ray vector \( r = (h, m)^T \), we could equivalently say that the ray falls along the line given by the equation \( y = mx + h \). More broadly, we define the homogeneous coordinates of a line obeying the equation \( ax + by + c = 0 \) as \( r = (c, a, b)^T \) (note the order of the coefficients, which we’ve chosen to facilitate comparisons of our matrices with the conventional version). Our standard ray above is described by the equation \(-mx + y - h = 0\), corresponding to the ray vector \( r = (-h, -m, 1)^T \). Multiplying \( r \) by a positive scalar does not change the line it represents, so this is indeed a homogeneous representation of the line (Fig. 2). Although it is not necessary for our calculations, if we wish to normalize \( r \), a convenient choice is to require \(a^2 + b^2 = 1\). With this normalization \(a\) and \(b\) equal the direction cosines of the line to the \(x\) and \(y\) axes, respectively, and \(|c|\) is the perpendicular distance from the line to the coordinate origin. If care is taken with signs during calculations (see Sec. 5 for how reflections must be handled), the rays expressed this way are oriented (have a well-defined forward direction). For a ray \((c, a, b)^T\), the direction of propagation, measured counter-clockwise from the forward positive \(x\)-axis, is the angle \(\phi = \arctan2(a, -b)\), where \(\arctan2(y, x)\) is the two-argument form of the inverse tangent function with range \(\phi \in (-\pi, \pi]\), used to disambiguate between opposite quadrants on the unit circle. The propagation direction of paraxial rays in this convention is given by the sign of \(b\): left-to-right rays have \(b > 0\) and right-to-left rays have \(b < 0\) (see Sec. 5 for more detail).

If the conventional \(2 \times 2\) ray transfer matrix of an optical system is known, the corresponding ray transfer matrix in homogeneous coordinates for nonreflecting systems (see Sec. 5 for the reflecting case) is

\[
M = \begin{pmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]
Table 1: Summary of ray transfer and point transfer matrices

| Element                                                                 | RTM                                                                 | PTM                                                                 |
|------------------------------------------------------------------------|----------------------------------------------------------------------|----------------------------------------------------------------------|
| General form for centered elements                                      | \[
\begin{pmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{pmatrix}
\] | \[
\begin{pmatrix} D & -C & 0 \\ -B & A & 0 \\ 0 & 0 & AD - BC \end{pmatrix}
\] |
| Thin lens of focal length \( f \)                                       | \[
\begin{pmatrix} 1 & 0 & 0 \\ -1/f & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\] | \[
\begin{pmatrix} 1 & 1/f & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\] |
| Free-space propagation over distance \( d \)                            | \[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\] | \[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\] |
| Refraction at a flat normal surface from index of refraction \( n \) to \( n' \) | \[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & n/n' & 0 \\ 0 & 0 & 1 \end{pmatrix}
\] | \[
\begin{pmatrix} n/n' & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\] |
| Refraction at a curved surface centered on the axis from index of refraction \( n \) to \( n' \) with radius of curvature \( R \) (\( R > 0 \) is convex) | \[
\begin{pmatrix} 1 & 0 & 0 \\ \frac{n-n'}{R} & \frac{n}{n'} & 0 \\ 0 & 0 & 1 \end{pmatrix}
\] | \[
\begin{pmatrix} \frac{n}{n'} & \frac{n-n'}{R} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\] |
| Reflection at a plane surface                                           | \[
\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}
\] | \[
\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}
\] |
| Reflection at a spherical surface with radius of curvature \( R \) (\( R > 0 \) is convex) | \[
\begin{pmatrix} -1 & 0 & 0 \\ 2/R & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}
\] | \[
\begin{pmatrix} -1 & 2/R & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}
\] |
| Translation by displacement \( (u, v) \)                               | \[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\] | \[
\begin{pmatrix} 1 & 0 & 0 \\ u & 1 & 0 \\ v & 0 & 1 \end{pmatrix}
\] |
| Rotation by angle \( \theta \)                                         | \[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix}
\] | \[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix}
\] |

Using the height-slope definition of the ray vectors, the ray transfer equation reads

\[
\begin{pmatrix} -h' \\ -m' \\ 1 \end{pmatrix} = \begin{pmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -h \\ -m \\ 1 \end{pmatrix}.
\]

Compare this with Eq. 1. The RTMs for common optical elements are summarized in Table 1.

The addition of the extra row and column may seem excessive as they have no apparent function yet, but they provide additional degrees of freedom that we will exploit for coordinate transformations to represent decentered and rotated optical elements (Sec. 4), consistent handling of orientation (Sec. 5), and imaging of points (Sec. 6).
4 Coordinate transformations

The $2 \times 2$ ray transfer matrices assume that the optical system has axial symmetry. In our expanded $3 \times 3$ formulation we can incorporate off-axis and rotated elements[2, 11, 12]. We add to our repertoire of ray transfer matrices a rotation operator $R_\theta$ of angle $\theta$ about the coordinate origin and a translation operator $T_{u,v}$ of displacement $u$ in the $x$-direction and $v$ in the $y$-direction.

\[
R_\theta = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta
\end{pmatrix}, \\
T_{u,v} = \begin{pmatrix}
1 & -u & -v \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\] (2)

In particular, a translation matrix with only horizontal displacement is equivalent to the propagation matrix from Table 1. These coordinate transformation matrices are exact in the sense that they are not linear approximations with respect to their arguments.

The ray transfer matrices given earlier in Table 1 assume the elements are located at the origin and normal to the optical axis. We can use the $T$ and $R$ matrices above to walk between optical elements without the restriction that they lie on the optical axis or that they are oriented normal to the original optical axis. We can represent an element $M$ that is rotated and then translated into a new position by

\[
M' = MR_\theta^{-1}T_{u,v}^{-1},
\] (3)

where the outgoing coordinate axes are centered and aligned with the axis of the optical element, rather than the original optical axis. Note that the inverses are simply $R_\theta^{-1} = R_{-\theta}$ and $T_{u,v}^{-1} = T_{-u,-v}$. Eq. (3) respects the traditional application of the ray transfer matrices, where the input rays and output rays are specified in different coordinate systems. The input rays are measured relative to a coordinate origin located at the first surface of the optical system, and the output rays are measured relative to a coordinate origin located at the final surface. As a simple example of Eq. (3), take $M$ to be the identity matrix and then translate along the axis by a distance $d$. The resulting matrix is

\[
M' = IT_{d,0}^{-1} = \begin{pmatrix}
1 & d & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

which is exactly the expanded form of the propagation matrix in Table 1.

In some applications (e.g. when mechanical dimensions are needed for prototyping an optomechanical layout) it is preferable to restore the original coordinate system. This choice of output coordinates is accomplished by

\[
M' = T_{u,v}R_\theta MR_\theta^{-1}T_{u,v}^{-1}.
\] (4)

This construction places the image space coordinate system to be coincident with the object space coordinate system, which contrasts with the typical convention for ray transfer matrices that treats the object space and image space independently.

Several expressions other than Eq. (2) are found in the literature for handling decentered or tilted optical elements[2, 8–14]. For example, the rotation operator is often given as [2, 11]

\[
R'_\theta = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & -\theta \\
0 & 0 & 1
\end{pmatrix}.
\]

This operator is in fact not a rotation but a shear.
Figure 3: (Color online only.) Tilted window of thickness $d$ and index of refraction $n$ at an angle $\theta$ relative to normal incidence. The angle of incidence inside the window is denoted $\theta'$. The path of the ray is shown as a thick red line, with the original optical axis as a dashed line.

The benefit we gain over other implementations is that our Eq. (2) does not require small parameters, for example, $|\theta| \ll 1$ in the rotation matrix, because we are transforming the whole coordinate system rather than just the rays themselves. In particular, plane mirrors of any orientation may be exactly modelled (see Sec. 5), allowing the model to include, for example, path-folding mirrors along a beam. So long as our rays do not stray too far from our transformed optical axis, the paraxial approximation is still valid even if the mechanical layout of the system is not along a single axis. A similar method is described by Lin[8].

4.1 Example: tilted window

As an example, consider a window of thickness $d$ and index of refraction $n$ tilted by a small angle $\theta$ (Fig. 3). The ray transfer matrix for this system referenced back to the incoming coordinate system is

\[
M = \begin{pmatrix}
1 & -d \left(1 - \frac{1}{n}\right) & d\theta \left(1 - \frac{1}{n}\right) \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

where $\theta^2$ and higher-orders terms have been discarded in keeping with the paraxial approximation (rays must stay near the transformed optical axis). Applying this RTM to an input ray $(0, 0, 1)^T$ coincident with the input optical axis gives an output ray

\[
\begin{pmatrix}
1 & -d \left(1 - \frac{1}{n}\right) & d\theta \left(1 - \frac{1}{n}\right) \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -d\theta \left(1 - \frac{1}{n}\right) \\ 0 \\ 1 \end{pmatrix},
\]

showing that the outgoing ray is parallel to the original optical axis and has been displaced upward by a distance $d\theta \left(1 - \frac{1}{n}\right)$. This agrees to within the paraxial limit with the direct calculation of the displacement of a ray parallel to the optical axis through a tilted plate using Snell’s Law (e.g. Problem 2-8 of Pedrotti[3]), which is

\[
\Delta h = \frac{d \sin(\theta - \theta')}{\cos(\theta')} \approx d\theta \left(1 - \frac{1}{n}\right),
\]

where the angle of incidence of the ray inside the window, $\theta'$, is given by $\sin(\theta') = \sin(\theta)/n$, and we use the small angle approximation in keeping with the paraxial limit needed for consistency with the refracting surfaces.
5 Orientation and Reflections

Reflections require special attention if we wish to include orientation in our homogeneous representation\[6, 18\]. The convention of direction introduced in Section 3 can be summarized: for the homogeneous ray \( r = (c, a, b)^T \) representing the line \( ax + by + c = 0 \), the ray is oriented left-to-right if \( b > 0 \) and right-to-left if \( b < 0 \). (This explains our choice of signs for the ray vectors in homogeneous coordinates.) The case \( b = 0 \) (vertical rays) does not occur in the paraxial limit, but we may still identify rays with \( a > 0 \) as going down and rays with \( a < 0 \) as going up. The final case is \( a = b = 0 \), representing the line infinitely far away encircling the plane. For these ideal lines \( c > 0 \) circulates counter-clockwise and \( c < 0 \) goes clockwise. The null ray vector \((0, 0, 0)^T\) is undefined geometrically.

Now consider the action of a plane mirror coincident with the \( y \) axis on an incoming ray \((c, a, b)^T = (-h, -m, 1)^T\) (Fig. 4). We require both the slope of the ray and its orientation to change signs. The RTM equation that satisfies this is

\[
\begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
-h \\
-m \\
1
\end{pmatrix} =
\begin{pmatrix}
h \\
-m \\
-1
\end{pmatrix}.
\]

The presence of the negative coefficient in the bottom right element of the RTM is the indicator of reflection. Generalizing, we see that the recipe for converting the standard \( 2 \times 2 \) RTM of reflective elements into an RTM for oriented homogeneous coordinates is

\[
M_{\text{reflective}} = (-1) \begin{pmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

Specifically, the RTMs for a plane mirror normal to the axis and a spherical mirror with center of curvature on the axis are

\[
M_{\text{plane mirror}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad M_{\text{sph. mirror}} = \begin{pmatrix} -1 & 0 & 0 \\ 2/R & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},
\]

where \( R > 0 \) for a convex mirror.

5.1 Example: Retroreflector

A retroreflector can be made from a pair of plane mirrors intersecting at right angles (Fig. 5). Using our rotation operator from Eq. (2) we can build a system ray transfer matrix that also respects the orientation of the rays. Note that our rotation angles are \( \pm 45^\circ \), which are not small. We’ll assume that the incoming ray \( r_0 = (-h, -m, 1)^T \) first strikes the upper mirror and then strikes the lower mirror. The RTM for the upper
mirror ($M_1$) is generated by a $45^\circ$ rotation of a plane mirror situated at the origin:

$$M_1 = R_{45^\circ}M_{\text{plane mirror}}R_{45^\circ}^{-1},$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix},$$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (5)$$

Similarly, the second mirror ($M_2$) has the RTM

$$M_2 = R_{-45^\circ}M_{\text{plane mirror}}R_{-45^\circ}^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}. \quad (6)$$

We choose an incoming ray $r_0 = (-h, -m, 1)^T$ with $h > 0$ such that it will strike mirror $M_1$ first, yielding the reflected ray

$$r_1 = M_1r_0 = (h, 1, -m)^T.$$

This ray follows the line $h + x - my = 0$, propagating from right to left ($b < 0$). After the second reflection in $M_2$ the final ray is

$$r_2 = M_2r_1 = (-h, m, -1)^T,$$

which is antiparallel to the incoming ray, as expected, with a $y$ intercept of $-h$ and propagating right to left ($b < 0$).

Note also that the method works for any angular separation between the mirrors by including the appropriate angles in Eqs. (5) and (6).

### 6 Point transfer matrices

In this section we show that if the ray transfer matrix $M$ for a centered, paraxial optical system in standard notation is given by

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

then the corresponding point transfer matrix (PTM) $\overline{M}$ is

$$\overline{M} = \begin{bmatrix} D & -C & 0 \\ -B & A & 0 \\ 0 & 0 & (AD - BC) \end{bmatrix},$$
which acts on a point measured relative to the input surface of the optical system with \( x \) along the axis and \( y \) perpendicular to it (see Fig. 1). We express points using the homogeneous coordinates: \( p = [w, x, y]^T \). (We will use square brackets to denote point vectors and round brackets to denote ray vectors to avoid ambiguity.) For \( w \neq 0 \), the corresponding physical point is located at position \( \left[ x/w, y/w \right] \). We’ll describe the points as "normalized" if \( w = 1 \). For \( w = 0 \), we interpret the point to be in the limit \( w \to 0^+ \): a point infinitely far away in the direction of slope \( y/x \). The inclusion of infinite (or "ideal") points is a key feature of homogeneous coordinates and projective geometry. In our choice of coordinates, object points with \( x/w < 0 \) correspond to the region before the input surface of the optical system and image points with \( x'/w' > 0 \) are those past the last surface. Explicitly, we have the matrix equation \( p' = \overline{M} p \), or

\[
\begin{bmatrix}
w' \\
x' \\
y'
\end{bmatrix} =
\begin{bmatrix}
D & -C & 0 \\
-B & A & 0 \\
0 & 0 & (AD - BC)
\end{bmatrix}
\begin{bmatrix}
w \\
x \\
y
\end{bmatrix}.
\]

(7)

To demonstrate the plausibility of our definition for point transfer matrices, Eq. (7), consider a ray \( r = (c, a, b)^T \) that is coincident with a point \( [x, y] \), represented by a homogeneous column vector \( p = [1, x, y]^T \). The coincidence relationship can be stated

\[
c + ax + by = 0, \\
p^T r = 0.
\]

Now, let’s image this ray and point through an optical system with ray transfer matrix \( M \) and yet-to-be-determined point transfer matrix \( \overline{M} \):

\[
r' = Mr, \\
p' = \overline{M} p.
\]

The resulting image-space ray \( r' \) and point \( p' \) should also be coincident. This requires

\[
p'^T r' = 0, \\
(\overline{M} p)^T (Mr) = 0, \\
p'^T (\overline{M}^T M) r = 0.
\]

The final line reproduces the original coincidence relation provided that \( \overline{M}^T M \) is a scalar times the identity matrix, or equivalently, that \( \overline{M}^T \) is a scalar times \( M^{-1} \). A more rigorous derivation (see the Appendix) shows that the proper scalar factor is \( \det(M) \), yielding a point transfer matrix

\[
\overline{M} = \det(M) (M^{-1})^T.
\]

(8)

\( \overline{M} \) can be calculated using this expression because the ray transfer matrix \( M \) is necessarily invertible. If the 2 × 2 RTM is known, the corresponding PTM is

\[
\overline{M} =
\begin{bmatrix}
D & -C & 0 \\
-B & A & 0 \\
0 & 0 & (AD - BC)
\end{bmatrix}.
\]

In the more general case (e.g. using the coordinate transformations of Sec. 5), Eq. (8) holds.a

One can also show that \( \overline{M} \) is equal to be the matrix adjugate[19] of the ray transfer matrix \( M \), which can be calculated even in the (unphysical) case \( \det(M) = 0 \) using cofactor expansion. When combining elements

\[
(M_2 M_1) = \det(M_2 M_1) ((M_2 M_1)^{-1})^T = \det(M_2) \det(M_1) (M_1^{-1} M_2^{-1})^T,
\]

\[
= \det(M_2) (M_2^{-1})^T \det(M_1) (M_1^{-1})^T = \overline{M}_2 \overline{M}_1,
\]

so the PTMs get multiplied in order from right to left, just like the RTMs.
7 Point transfer examples

Here we present four examples showing how the PTMs simplify calculations.

7.1 Deriving Gauss’s lens equation

As a simple example employing the point-transfer matrix consider a thin lens of focal length $f$. Reading the ray transfer matrix elements $ABCD$ from Table 1 and inserting them into the point transfer matrix, we get the equation

$$
\begin{bmatrix}
  w' \\
  x' \\
  y'
\end{bmatrix} =
\begin{bmatrix}
  1 & 1/f & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  1 + \frac{x}{f} \\
  x \\
  y
\end{bmatrix} =
\begin{bmatrix}
  1 \\
  1 + \frac{x}{f} \\
  y
\end{bmatrix},
$$

(9)

where the input point is the normalized point at position $[x, y]$. (Recall our sign convention for the coordinates $x$ and $x'$ increase along the direction of propagation, so that a real object has $x < 0$ and a real image has $x' > 0$.) The resulting output is the normalized point

$$
\hat{x}' = \frac{x'}{w'} = \frac{x}{1 + (x/f)} = \frac{1}{(1/x) + (1/f)},
$$

$$
\hat{y}' = \frac{y'}{w'} = \frac{y}{1 + (x/f)} = \frac{x'}{x} y.
$$

The first line agrees with the Gaussian lens equation giving the location of the image, and the second line gives the image magnification $\hat{x}'/x$. The sign of $w'$ before normalization gives the orientation of the image: $w' > 0$ indicates upright and $w' < 0$ indicates inverted. Our formulation also easily handles the special case when the object (image) point is infinitely far away (the homogeneous weight $w (w')$ goes to zero) as well as virtual objects ($x > 0$) and virtual images ($x' < 0$).

7.2 Imaging an infinite object point

To see how the PTMs are used in numerical calculations, consider a simple single-lens “telescope” pointed at the horizon to view a star infinitely far away. The parallel rays from the star will converge at the back focal plane of the lens, at a height equal to minus the angular height of the star, times the focal length of the lens. For concreteness, let’s show this using our point transfer matrix for a star 10 milliradians above the horizon (Fig. 6) and an $f = 50$ mm lens. The resulting image of the star may be found using the point transfer matrix for a thin lens:

$$
\begin{bmatrix}
  1 & 1/50 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  0 \\
  -1 \\
  .01
\end{bmatrix} =
\begin{bmatrix}
  -0.02 \\
  -1 \\
  0.01
\end{bmatrix} \equiv
\begin{bmatrix}
  1 \\
  50 \\
  -0.5
\end{bmatrix},
$$

with implied length units of millimeters. The last step is normalization of the point vector. In this example, the object point is infinitely far away in the direction of slope $= \arctan(0.01/ -1) \approx -0.01$ (so the star is above the optical axis ($y > 0$) and before the lens ($x < 0$)). The final normalized image point has height $y' = -0.5$ mm and is located at $x' = 50$ mm after the lens, corresponding with the back focal plane of the lens as expected. Note that we do not need any explicit reference to the light rays to perform the calculation. This example demonstrates one advantage of this method: using homogeneous coordinates allows us to treat both finite and infinite objects/images in the same way without resorting to special cases. A naïve application of Gauss’s lens equation with an infinite object distance would not tell us the vertical position of the image because the linear magnification is undefined in this instance.
Figure 6: (Color online only.) Calculating the image of a star by treating the star as an object point infinitely far away. The location of the star in homogeneous coordinates is $[0, -1, 0.01]^T$, the lens has a focal length of $f = 50\, \text{mm}$, and the star’s image is at $[1, x, y]^T = [1, 50, -0.5]^T$ (after normalization). The length units of the vectors are millimeters.

7.3 Analysis of a compound lens

For our next example (adapted from Examples 6.6 and 6.7 of Hecht[4] – note that we use a different convention for the ray transfer matrices than Hecht), consider a compound lens system with the ray transfer matrix (length units of cm)

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0.867 & 1.338 \\ -0.198 & 0.848 \end{pmatrix},$$

and an object 20 cm in front of the lens with a height of 0.1 cm. We would like to locate the resulting image position and height.

Hecht’s solution has several steps: (1) Extend the system matrix $M$ with a propagation matrix of length 20 cm in front of the lens (for the object position) and a propagation of unknown length $d$ after the lens (for the image position) so that the system matrix now includes both the object and image points. (2) Algebraically set the $B$ element of the resulting product matrix to zero. This enforces the condition that the final ray height is independent of the incoming ray slope (i.e. an image forms). With this constraint, one can solve for the unknown length $d$ to locate the image plane. (3) Identify the magnification as the $A$ element of the resulting matrix. This is true because we set $B = 0$ above, meaning that the image ray height only depends on the object ray height, which gives us the magnification.

Our solution is much more direct. First, we construct the point transfer matrix from the original ray transfer matrix (without additional propagations) and multiply this into the homogeneous position vector for the object point:

$$\begin{pmatrix} D & -C & 0 \\ -B & A & 0 \\ 0 & 0 & (AD - BC) \end{pmatrix} \begin{pmatrix} 1 \\ x \\ y \end{pmatrix} = \begin{pmatrix} 0.848 & 0.198 & 0 \\ -1.338 & 0.867 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -20.0 \\ 0.1 \end{pmatrix},$$

$$= \begin{pmatrix} -3.112 \\ -18.678 \\ 0.1 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 6.002 \\ -0.032 \end{pmatrix}.$$

After performing the matrix multiplication, we only need to normalize the resulting image point vector (final step above) to read off the coordinates of the image: 6.002 cm past the lens at a height of $-0.032\, \text{cm}$. We’ve been able to replace algebraic multiplication of matrices containing unknown variables and solving for an unknown distance with a direct numerical calculation.

Taking this example one step farther, we can also calculate the front and back focal points of the lens system easily. The back focal point (BFP) is the image of an ideal (infinite) object point on the axis: $[0, -1, 0]^T$. The image of this point is

$$\begin{pmatrix} 0.848 & 0.198 & 0 \\ -1.338 & 0.867 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -0.198 \\ -0.867 \\ 0 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 4.38 \\ 0 \end{pmatrix},$$
where the last step is normalization. Reading off the coordinates, the BFP is located 4.38 cm past the final optical surface. To find the front focal point (FFP), we invert the point-transfer matrix of the system (i.e. time reversal) and then follow the same procedure using a ray coming from the reverse direction, \([0, 1, 0]^T\).

### 7.4 Misaligned thin lens

A common laboratory task is to take a collimated laser beam and focus it through a pinhole located at the back focal point of a converging lens (for example, to build a spatial filter or to inject the beam into a fiber). If the pinhole is in a fixed position, but the beam doesn’t quite hit the hole, a common remedy is to move the lens slightly in the transverse direction. Here we analyze this configuration for small displacements and small tilts of the lens.

Let’s model our laser beam as a point source infinitely far away, parallel to the optical axis and ignore diffraction effects. The corresponding (infinite) point vector is \(p = [0, 1, 0]^T\). In essence, we’re approximating the laser beam as a pencil of parallel rays. We’ll assume a converging lens of focal length \(f\). First, let’s consider what happens to the image if we translate the lens by a distance \(d\) perpendicular to the optical axis. The system PTM is:

\[
\overline{M} = \overline{T}_{0,d} \overline{M}_{\text{thin lens}}(f) \overline{T}_{0,d}^{-1},
\]

\[
= \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
d & 0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 1/f & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-1 & d & 1
\end{bmatrix} = \begin{bmatrix}
1 & 1/f & 0 \\
0 & 1 & 0 \\
0 & d/f & 1
\end{bmatrix},
\]

and the image point \(p'\) is

\[
p' = \overline{M}p = \begin{bmatrix}
1 & 1/f & 0 \\
0 & 1 & 0 \\
0 & d/f & 1
\end{bmatrix} \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix} = \begin{bmatrix}
1/f \\
1 \\
d/f
\end{bmatrix} \equiv \begin{bmatrix}
f \\
1 \\
0
\end{bmatrix},
\]

where the last step is normalization of the point vector. We see that the beam is still focused in the back focal plane \((x' = f)\), but the focus point is displaced transversely by the same distance as the lens, \(d\).

If we tilt the lens, the system PTM is

\[
\overline{M} = R_\theta \overline{M}_{\text{thin lens}}(f) R_\theta^{-1},
\]

\[
= \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta
\end{bmatrix} \begin{bmatrix}
1 & 1/f & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta & \sin \theta \\
0 & -\sin \theta & \cos \theta
\end{bmatrix} = \begin{bmatrix}
1 & \cos(\theta)/f & \sin(\theta)/f \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix},
\]

Applying this to the object point, we get

\[
p' = \overline{M}p = \begin{bmatrix}
1 & \cos(\theta)/f & \sin(\theta)/f \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
\cos(\theta)/f \\
1 \\
0
\end{bmatrix} \equiv \begin{bmatrix}
1 \\
f/\cos(\theta) \\
0
\end{bmatrix}.
\]

So, up to first order in the angle \(\theta\), tilting the lens has no effect on the location of the beam focus.
8 Outlook and Conclusion

The results we’ve presented (summarized in Table 1) are restricted to two-dimensional systems or three-dimensional systems with axial symmetry. To consider general three-dimensional systems, we will need a different approach. The homogeneous ray transfer matrices are linearizations of the propagation equations of Hamiltonian optics[15, 20], restricted to the paraxial regime. A similar approach can be used to expand the work here to three-dimensional optics. Rays in three dimensions may be represented homogeneously in 3D using a set of 6 Plücker coordinates[6, 14, 21]. These may be further reduced to 5 non-trivial coordinates for paraxial rays in 3D (e.g. near the z axis)[13]. Geometric Algebra[18, 22–26], following Dorst’s description of the algebra of lines[27] and earlier work on 2D optical systems by Sugon and McNamara[28–31], is an intriguing mathematical system for approaching this problem that we are looking into.

Although the path to describing rays is straight-forward, the key difficulty in extending the work here to 3D is that points in 3D do not necessarily image to other points. A converging cylindrical lens is a simple example: it images a point source of light onto a line. The general behavior in 3D requires defining a point as the intersection of three non-coplanar lines (not two lines, as one might naively assume)[27], and then studying how those lines transform, noting that the three output lines may not converge to the same image point (e.g. because of astigmatism). This leads to the geometric theory of line complexes[32]. As a tantalizing bonus, Arnaud’s representation of Gaussian laser beams as complex-valued rays[9, 33] may also be interpreted as skew line complexes[34]. We are still investigating how to use this correspondence to build a geometric representation of the paraxial optics of Gaussian beams that is more flexible than the complex beam parameter formalism[11] and the related expression of diffraction integrals using RTMs described by Collins[35, 36].

In conclusion, our goal here was to show that enhancing the ABCD ray transfer matrices (RTM) by applying them in a homogeneous coordinate system greatly expands their usefulness. With a fairly modest change in interpretation of the RTM to better capture their geometric content we added the capability to describe translated and rotated optical elements and keep track of the ray propagation directions. A simple mathematical operation converts the RTMs into point transfer matrices (PTM), simplifying imaging calculations. We hope that our examples demonstrate the advantages of using this system and can be a jumping-off point for more advanced applications.

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Appendix: Derivation using Grassmann exterior algebra

This appendix will introduce the machinery needed to derive the point transfer matrices more directly using Grassmann exterior algebra[37, 38], by applying it to map the ray transfer transformation \( M \) in the vector space of rays to a corresponding transformation \( \hat{M} \) in the vector space of (homogeneous) points. The inclusion of exterior algebra invites an expanded set of interpretations. As an example of this, we’ll close with a nearly trivial derivation of the Scheimpflug principle used, for example, to describe tilt-shift photography.

We define a graded vector algebra \( \bigwedge^3 V \) with basis vectors \( e_1, e_2, e_0 \). The vector \( e_2 \) represents the nominal optical axis, \( e_1 \) represents the input plane of the optical system, and \( e_0 \) stands in for the ideal projective line at infinity (Fig. 7). Higher grades of elements are built up using Grassmann exterior (wedge) products of vectors, which obey simply

\[
a \wedge b = -b \wedge a.
\]
This implies \( a \wedge a = 0 \). More generally, the exterior product of any set of linearly dependent vectors will be zero. Other than this simplification for linearly dependent vectors, in general the exterior products do not reduce further.

The unit bivectors \( e_{01}, e_{20}, e_{12} \), using the abbreviation \( e_{ij} = e_i \wedge e_j \), represent the +y direction (or ideal point along the +y axis), +x direction, and origin, respectively. The trivector \( e_{012} = e_0 \wedge e_1 \wedge e_2 \) represents the whole \( xy \) plane. (The order of indices above is chosen to ensure consistent signs in what follows.)

More concretely, a line obeying the equation \( ax + by + c = 0 \) is described in this notation by the vector

\[
 r = ce_0 + ae_1 + be_2,
\]

and a finite point with homogeneous coordinates \([w, x, y]^T\) is described by the bivector

\[
 P = we_{12} + xe_{20} + ye_{01}.
\]

As with homogeneous coordinates, bivectors with \( w = 0 \) represent points infinitely far away in the corresponding direction.

The definitions of points and lines are congruent, including orientation, up to an overall scalar factor: e.g. \( P \) and \( cP \), with \( c \) a positive scalar constant, represent the same geometric point and similarly with lines. For a general finite point, Eq. (11), we define the norm \( \|P\| \) to be the coefficient of \( e_{12} \). For a general line, Eq. (10), we will choose the norm to be \( \|r\| = \sqrt{a^2 + b^2} \). We define normalized elements \( \hat{a} = a/\|a\| \). Notably, two lines \( r_1 \) and \( r_2 \) intersect at the point \( r_1 \wedge r_2 \) (the projective “meet” operation), which may be an infinite point in the case of parallel lines. If three lines \( r_{1,2,3} \) meet at a common point then \( r_1 \wedge r_2 \wedge r_3 = 0 \). If we extend our Grassmann algebra to the Cayley-Grassmann algebra by inclusion of the regressive product \( \vee \), then we get the full projective duality between lines and points in two dimensions. The regressive product of two distinct points yields the line connecting them (the projective “join”) \([38]\). If those two points are normalized, the norm of the resulting line gives the distance between the original points.

Our next task is to rewrite the ray transfer matrix equation, Eq. (1), in this new notation. A ray with height \( h \) at the \( y \) axis and slope \( m \) obeys the algebraic equation \( y = mx + h \). Applying Eq. 10 we can associate this ray with a (un-normalized) line

\[
 r = -he_0 - me_1 + e_2,
\]

Figure 7: (Color online only.) (Top) The unit vectors for our 2d projective space. \( e_1 \) and \( e_2 \) are lines perpendicular to the \( x \) and \( y \) axes, resp. \( e_0 \) is the ideal line infinitely far away from the origin. (Bottom) The unit bi-vectors for our space. \( e_{20} \) and \( e_{01} \) are the directions (ideal points) along the \( x \) and \( y \) axes, resp. \( e_{12} \) represents the point at the origin.
which may be written as a column vector \( r = (-h, -m, 1)^T \) on the basis \( \{e_0, e_1, e_2\} \) (note the order of the basis elements). Similarly, the outgoing ray is the column vector \( r' = (-h', -m', 1)^T \). A simple comparison shows that the ray transfer matrix equation Eq. (1) in this new representation becomes

\[
    r' = Mr,
    \begin{pmatrix}
    -h' \\
    -m' \\
    1
    \end{pmatrix} =
    \begin{pmatrix}
    A & B & 0 \\
    C & D & 0 \\
    0 & 0 & 1
    \end{pmatrix}
    \begin{pmatrix}
    -h \\
    -m \\
    1
    \end{pmatrix}.
\]

The purpose of this rewriting is to facilitate our next step: finding the matrix representation of the ray transfer matrix in the bivector basis. First, we’ll take a small interlude to demonstrate an example.

The usual description of the ray transfer matrices is that they are valid in the limit of paraxial rays, defined as those with small slope and small ray heights, such that the ray tracing equations may be approximated as linear equations. The ray transfer matrices also hold in a complementary approximation: the approximation that the optical elements themselves are linear. For example, we could consider an ideal thin lens that exhibits no aberrations even for rays far from the axis.

With this proviso we can derive the Scheimpflug principle (Fig. 8). We’ll consider the simplest case of an ideal thin lens. The Scheimpflug principle states when an extended object lies in a plane that is not normal to the optical axis, then the object plane, image plane, and the plane of the lens all meet at a point. Let the plane of the lens be the \( y \)-axis of our system, \( e_1 \). We place the object plane \( p \) such that it crosses the optical axis at a distance \( d \) from the lens and is tilted by an angle \( \theta \) relative to normal. Because we are working in two dimensions, this is equivalent to the line

\[
p = -d \cos \theta e_0 - \cos \theta e_1 + \sin \theta e_2.
\]

We find the image plane, \( p' \) by applying the thin lens ray transfer matrix to the object plane.

\[
p' = \begin{pmatrix}
    1 & 0 & 0 \\
    -1/f & 1 & 0 \\
    0 & 0 & 1
    \end{pmatrix}
    \begin{pmatrix}
    -d \cos \theta \\
    -\cos \theta \\
    \sin \theta
    \end{pmatrix} =
    \begin{pmatrix}
    -d \cos \theta \\
    (d/f - 1) \cos \theta \\
    \sin \theta
    \end{pmatrix}.
\]
With this, we can calculate the intersection point \( P \) of \( p \) and \( p' \) as their wedge product:

\[
P = p \wedge p' = -\frac{d^2 \cos^2 \theta}{f} e_{01} - \frac{d \sin \theta \cos \theta}{f} e_{12} \equiv \frac{d}{\tan \theta} e_{01} + e_{12},
\]

where in the last step a common factor is removed to normalize the point, showing that the intersection point is indeed on the \( y \) axis at a height of \( \frac{d}{\tan \theta} \). We could also have checked that the three planes intersect by showing \( p \wedge p' \wedge e_{1} = 0 \). We can also verify that the image plane crosses the optical axis \( e_{2} \) at the appropriate image point given by, e.g., Gauss’s lens equation.

\[
p' \wedge e_{2} = d \cos \theta e_{20} + \left( \frac{d}{f} - 1 \right) e_{12} \equiv \frac{df}{d-f} e_{20} + e_{12},
\]

where the last step is normalization to show the expected location of the image point.

Because points are represented by bivectors in our algebra, we wish to find a representation for how bivectors transform in 2D optical systems. This will be given by the outermorphism of the linear operator. Taking a linear operator \( L \) acting on vectors (lines) in our space, we will extend it to higher grades by defining the outermorphism

\[
L(a \wedge b) = L(a) \wedge L(b).
\]

A matrix representation of \( L \) on the vector space of column vectors may be constructed by acting \( L \) on each of the basis elements \( e_{01,2} \) and storing the results as the corresponding columns of a matrix. Similarly, a matrix representation on the bivector space can be constructed by acting \( L \) on each of the bivector basis elements through the outermorphism described above.

To begin, let’s see how the ray transfer matrix \( M \) transforms the basis vectors. The \( e_{1} \) element transforms into

\[
Me_{1} = \begin{pmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} B \\ D \end{pmatrix},
\]

\[
= Be_{0} + De_{1}.
\]

Similarly, \( Me_{0} = Ae_{0} + Ce_{1} \) and \( Me_{2} = e_{2} \). Not surprisingly, these are just the columns of \( M \).

By the outermorphism property Eq. (12), we can find the matrix representation in the bivector space by operating on each basis bivector:

\[
Me_{12} = Me_{1} \wedge Me_{2},
\]

\[
= (Be_{0} + De_{1}) \wedge e_{2} = De_{12} - Be_{20},
\]

\[
Me_{20}
\]

\[
= -Ce_{12} + Ae_{20},
\]

\[
Me_{01} = (AD - BC)e_{01}.
\]

Our (un-normalized) object-space point \( P = [w, x, y]^{T} \) on the bivector basis \( \{e_{12}, e_{20}, e_{01}\} \) (note order). The equations above for the transformation of the unit bivectors can be arranged as a matrix equation. The optical system generates a corresponding image-space point

\[
P' = \begin{bmatrix} w' \\ x' \\ y' \end{bmatrix} = \begin{bmatrix} D & -C & 0 \\ -B & A & 0 \\ 0 & 0 & AD - BC \end{bmatrix} \begin{bmatrix} w \\ x \\ y \end{bmatrix}.
\]

(13)

This confirms the result we presented earlier in Eq. (7), including the overall scalar factor.
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