SPECTRAL INEQUALITIES FOR ANISOTROPIC SHUBIN OPERATORS

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Abstract. In this paper, new spectral inequalities for finite combinations of eigenfunctions of anisotropic Shubin operators are presented. Given a subset $\omega$ and an energy level, we provide an explicit control of the ratio of the $L^2(\mathbb{R}^d)$-norm over the $L^2(\omega)$-norm with respect to the energy level. The proofs are based on recent uncertainty principles holding in Gelfand-Shilov spaces and Bernstein type estimates deduced from quantitative smoothing effects proved by Paul Alphonse. These spectral inequalities allow to derive the null-controllability in any positive time from any control subset with positive Lebesgue measure of evolution equations associated to anisotropic Shubin operators, except for the harmonic oscillator.

1. Introduction

This work aims at providing quantitative spectral inequalities for anisotropic Shubin operators and null-controllability results for the evolution equations associated to these operators. Anisotropic Shubin operators are defined as the following self-adjoint elliptic operators

$$H_{k,m} = (-\Delta_x)^m + |x|^{2k}, \quad x \in \mathbb{R}^d$$

equipped with the domain

$$D(H_{k,m}) = \{ g \in L^2(\mathbb{R}^d) : H_{k,m}g \in L^2(\mathbb{R}^d) \},$$

where $k, m \geq 1$ are positive integers. The operators $H_{k,m}$ are known to admit a Hilbert basis $(\psi_{k,m}^n)_{n \in \mathbb{N}}$ of $L^2(\mathbb{R}^d)$, where $\psi_{k,m}^n$ is associated to some eigenvalue $\lambda_{k,m}^n > 0$. When $k = m = 1$, the operator $H_{1,1}$ is the harmonic oscillator

$$\mathcal{H} = -\Delta_x + |x|^2$$

and the Hilbert basis $(\psi_{1,1}^n)$ can be taken equal to the Hermite basis $(\Phi_\alpha)_{\alpha \in \mathbb{N}^d}$ given by

$$\forall \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d, \forall x = (x_1, \ldots, x_d) \in \mathbb{R}^d, \quad \Phi_\alpha(x) = \prod_{i=1}^d H_i(x_i)e^{-\frac{x_i^2}{2}},$$

where for all $n \in \mathbb{N}$, $H_n$ is the $n$th-Hermite polynomial. In this particular case, spectral inequalities have been studied by Pravda-Starov and the author in [13]:

Theorem 1.1 ([13] Theorem 2.1). Let $\rho : \mathbb{R}^d \rightarrow (0, +\infty)$ be a $\frac{1}{2}$-Lipschitz positive function with $\mathbb{R}^d$ being equipped with the Euclidean norm, such that there exist some positive constants $0 \leq \delta < 1$, $m > 0$, $R > 0$ such that

$$\forall x \in \mathbb{R}^d, \quad 0 < m \leq \rho(x) \leq R(x)$$

Let $\omega$ be a measurable subset of $\mathbb{R}^d$ which is $\gamma$-thick with respect to the density $\rho$, that is,

$$\exists \gamma \leq 1, \forall x \in \mathbb{R}^d, \quad |\omega \cap B(x, \rho(x))| \geq \gamma |B(x, \rho(x))|,$$
where $B(y, r)$ denotes the Euclidean ball centered at $y \in \mathbb{R}^d$ with radius $r > 0$, and $| \cdot |$ denotes the Lebesgue measure. Then, there exist some positive constant $\kappa_d(m, R, \gamma, \delta) > 0$, $C_d(\delta, R) > 0$ and a positive universal constant $K_d > 0$ such that

$$\forall N \geq 1, \forall f \in \mathcal{E}_N, \quad \|f\|_{L^2(\mathbb{R}^d)} \leq \kappa_d(m, R, \gamma, \delta) \left( \frac{K_d}{\gamma} \right) C_d(\delta, R) N^{\frac{1+\delta}{2}} \|f\|_{L^2(\omega)},$$

with $\mathcal{E}_N$ being the finite dimensional vector space spanned by the Hermite functions $(\Phi_\alpha)_{|\alpha| \leq N}$.

This Theorem provides quantitative upper bounds on the ratio

$$C_N(\omega) = \sup_{f \in \mathcal{E}_N \setminus \{0\}} \frac{\|f\|_{L^2(\mathbb{R}^d)}}{\|f\|_{L^2(\omega)}}$$

and explicitly describes how this upper bound depends on the energy level $N$ and the growth of the density appearing in (1.1) ruled by the parameter $\delta$. In a previous work [2], Pravda-Starov with Beauchard and Jaming, established such spectral inequalities in the case when $\delta = 0$ with the same growth of the constant $C_N(\omega)$ with respect to the energy level $N$, but also under other geometric conditions for $\omega$:

**Theorem 1.2** ([2, Theorem 2.1]). The following spectral inequalities hold:

(i) If $\omega$ is a non empty open subset of $\mathbb{R}^d$, then there exists a positive constant $C = C(\omega) > 1$ such that

$$C_N(\omega) \leq C e^{\frac{1}{2}N \ln(N+1) + CN}.$$

(ii) If the measurable subset $\omega \subset \mathbb{R}^d$ satisfies

$$\liminf_{R \to +\infty} \frac{|\omega \cap B(0, R)|}{|B(0, R)|} > 0,$$

then there exists a positive constant $C = C(\omega) > 1$ such that

$$C_N(\omega) \leq C e^{CN}.$$

(iii) If the measurable subset $\omega \subset \mathbb{R}^d$ satisfies

$$\exists 0 < \gamma \leq 1, \exists R > 0, \forall x \in \mathbb{R}^d, \quad |\omega \cap B(x, R)| \geq \gamma |B(x, R)|,$$

then there exist a positive constant $C = C(R, \gamma, d) > 0$ depending on the dimension $d \geq 1$ and the parameters $\gamma, R > 0$, and a universal positive constant $\kappa = \kappa(d) > 0$ only depending on the dimension such that

$$C_N(\omega) \leq C \left( \frac{K}{R} \right)^{\kappa R \sqrt{N}}.$$

Let us mention that Dicke, Seelmann and Veselić have recently generalised the spectral inequalities given by Theorem 1.1. In [7, Theorem 2.7], these authors consider a more general class of measurable subset of $\mathbb{R}^d$, called sensor sets, which satisfies

$$\exists 0 < \gamma \leq 1, \forall x \in \mathbb{R}^d, \quad |\omega \cap B(x, \rho(x))| \geq \gamma^{1 + |x| \varepsilon'} |B(x, \rho(x))|,$$

where $\rho : \mathbb{R}^d \to (0, +\infty)$ is a continuous density satisfying

$$\exists R > 0, \forall x \in \mathbb{R}^d, \quad 0 < \rho(x) \leq R(x)^{1-\varepsilon},$$

with $0 \leq \varepsilon' < \varepsilon \leq 1$. Under this new geometric condition, they established ([7, Theorem 2.7]) that the following spectral inequalities hold

$$\exists K > 1, \forall N \in \mathbb{N}, \forall f \in \mathcal{E}_N, \quad \|f\|_{L^2(\mathbb{R}^d)}^2 \leq \frac{1}{3} \left( \frac{K}{\gamma} \right)^{K^{1+\delta}d^{(11+3\delta)/2}(1+R^2)N^{1-\varepsilon}} \|f\|_{L^2(\omega)}^2.$$
In this work, we will not consider the general condition (1.3) since ε′ will be always taken equal to 0.

In the case when \( m = 1 \) and \( k \geq 1 \), the Shubin operator
\[
H_{1,k} = -\Delta_x + |x|^{2k}
\]
is an anharmonic oscillator and Miller established in [14] spectral inequalities for finite sum of eigenfunctions associated to these operators when \( \omega \) is a cone:

**Theorem 1.3** ([14, Corollary 1.6]). Let \( k \in \mathbb{N}^* \). For any non empty open cone
\[
\omega = \{ x \in \mathbb{R}^d; |x| > r_0, x/|x| \in \Omega_0 \},
\]
where \( r_0 \geq 0 \) and \( \Omega_0 \) is an open subset of the unit sphere, there is a constant \( C > 0 \) such that
\[
\forall \lambda \geq 0, \forall g \in \mathcal{E}^{1,k}_\lambda, \quad \int_{\mathbb{R}^d} |g(x)|^2 dx \leq C e^{C \lambda \frac{1}{2} (1 + \frac{1}{k})} \int_\omega |g(x)|^2 dx.
\]

Notice that the case \( k = 1 \) is a particular case of Theorem 1.2 since the cone \( \omega \) appearing in Theorem 1.3 satisfies the condition required in assertion (ii) and the spectrum of the oscillator harmonic is given by
\[
\sigma(\mathcal{H}) = \{ 2n + d; n \in \mathbb{N} \}.
\]

Since the work [10], where the authors developed the so-called Lebeau-Robbiano method in order to establish a null-controllability result for the heat equation posed on a bounded domain \( \Omega \) of \( \mathbb{R}^d \), spectral inequalities have known a great interest in the control theory. This classical method in control theory requires to have a control on the \( L^2 \)-norm of finite combinations of eigenfunctions associated to the operator by their \( L^2(\omega) \)-norms, where \( \omega \subset \Omega \) is a subset of the domain. Following this strategy, thanks to the Theorem 1.1 the authors established in [13] Theorem 2.5] that, given \( \frac{1}{2} < s \leq 1 \), the fractional harmonic heat equation
\[
\begin{aligned}
\left\{ \begin{array}{ll}
\partial_t f(t,x) + \mathcal{H}^s f(t,x) = u(t,x)1_{\omega}(x), & x \in \mathbb{R}^d, t > 0, \\
f|_{t=0} = f_0 & \in L^2(\mathbb{R}^d),
\end{array} \right.
\end{aligned}
\tag{1.5}
\]
is null-controllable in any time \( T > 0 \) from any measurable subset \( \omega \subset \mathbb{R}^d \) satisfying (1.1) when \( \rho : \mathbb{R}^d \to (0, +\infty) \) is a \( \frac{1}{2} \)-Lipschitz function and verifies
\[
\forall x \in \mathbb{R}^d, \quad 0 < m \leq \rho(x) \leq R(x)^{\delta},
\]
with \( 0 \leq \delta < 2s - 1 \). By using the new spectral inequalities (1.4), the authors of [7 Theorem] have shown that sensor sets defined by (1.3) are efficient control subsets for fractional harmonic heat equations. In particular, these authors have constructed control subsets of finite Lebesgue measure from which the null-controllability of the harmonic heat equation holds. Regarding the case when \( s > 1 \), by using the spectral inequalities given in Theorem 1.2 assertion (iii), Alphonse proved in [11 Theorem 2.9] that the evolution equation (1.5) is null-controllable in any time \( T > 0 \) as soon as \( \omega \) is a nonzero Lebesgue measure subset of \( \mathbb{R}^d \).

Such null-controllability questions for evolution equations associated to anharmonic oscillators
\[
\begin{aligned}
\left\{ \begin{array}{ll}
\partial_t f(t,x) + (\Delta_x + |x|^{2k}) f(t,x) = u(t,x)1_{\omega}(x), & x \in \mathbb{R}^d, t > 0, \\
f|_{t=0} = f_0 & \in L^2(\mathbb{R}^d),
\end{array} \right.
\end{aligned}
\tag{1.6}
\]
have been investigated by Miller in [14]. He obtained in [14 Theorem 1.10] that for \( k > 1 \), the evolution equation (1.6) is null-controllable in any time \( T > 0 \) from any open cone \( \omega \) defined in Theorem 1.3. However, when \( k = 1 \), Miller established that this fact can not hold since the harmonic heat equation (1.5) is not null-controllable for any time \( T > 0 \) when \( \omega \) is contained in a half space.
This paper aims at generalizing spectral inequalities stated in Theorem 1.1, Theorem 1.2, and Theorem 1.3. More precisely, we will consider measurable subsets which are said to be weakly thick:

**Definition 1.4.** Let $0 \leq \delta \leq 1$ and $\omega \subset \mathbb{R}^d$ be a measurable subset. $\omega$ is said to be $\delta$-weakly thick if and only if there exists some positive constants $0 < \gamma \leq 1$, $R > 0$ such that

$$\forall x \in \mathbb{R}^d, \quad |\omega \cap B(x, R(x)^\delta)| \geq \gamma |B(x, R(x)^\delta)|.$$ 

Some examples and facts about weak thick sets are given in Section 3. As in Theorem 1.1, Theorem 1.2, and Theorem 1.3, the spectral inequalities established in this work (Theorem 2.1) are quantitative and provide explicit constants which depend on energy levels, the parameter $\delta$ appearing in the weak thickness definition and the indexes $k, m \geq 1$ appearing in the definition of the Shubin operator $H_{k,m}$. As an application, we derive new null-controllability results for evolution equations associated to fractional anisotropic Shubin operators

$$\left\{ \begin{array}{ll}
\partial_t f(t, x) + \left( (-\Delta_x)^m + |x|^{2k} \right) f(t, x) = u(t, x) \mathbb{1}_\omega(x), & x \in \mathbb{R}^d, \ t > 0, \\
f|_{t=0} = f_0 \in L^2(\mathbb{R}^d), & \end{array} \right.$$ 

from any subset with positive Lebesgue measure (Theorem 2.5). The null-controllability of these equations will be obtained by a Lebeau-Robbiano strategy and these results generalize and complete previous null-controllability results for these equations.

### 2. Statement of the main results

This section is devoted to present the main results of this work. The first part states new spectral inequalities for finite combinations of eigenfunctions associated to anisotropic Shubin operators. The second one deals with the null-controllability of evolution equations associated to these operators.

#### 2.1. Spectral inequalities for anisotropic Shubin operators.

This section is devoted to state spectral inequalities for finite combinations of eigenfunctions associated to anisotropic Shubin operators. Let us define, for all $\lambda > 0$, the finite dimensional subspace

$$\mathcal{E}^{k,m}_{\lambda} = \mathbb{1}_{H_{k,m} \leq \lambda} \left( L^2(\mathbb{R}^d) \right)$$

where

$$\forall g \in L^2(\mathbb{R}^d), \quad \mathbb{1}_{H_{k,m} \leq \lambda} g = \sum_{\substack{n \in \mathbb{N}, \ \lambda^n_{k,m} \leq \lambda}} (g, \psi_{n}^{k,m})_{L^2(\mathbb{R}^d)} \psi_{n}^{k,m}.$$ 

The main result of this section is the following:

**Theorem 2.1.** Let $k, m \geq 1$ be positive integers and $0 \leq \delta \leq 1$. Let $\omega \subset \mathbb{R}^d$ be a measurable subset.

1) If $\omega$ is $\delta$-weakly thick, then there exists a positive constant $K = K(\gamma, R, \delta, k, m) > 0$ such that

$$\forall \lambda > 0, \forall f \in \mathcal{E}^{k,m}_{\lambda}, \quad \|f\|_{L^2(\mathbb{R}^d)} \leq K e^{K \gamma \lambda^{\frac{1}{\delta} + \frac{1}{k}} \log \lambda} \|f\|_{L^2(\omega)},$$

where $0 < \gamma \leq 1$ and $R > 0$ are the constants appearing in Definition 1.4.

2) If $\omega$ satisfies $|\omega| > 0$, then there exists a positive constant $K = K(\omega) > 0$ such that

$$\forall \lambda > 0, \forall f \in \mathcal{E}^{k,m}_{\lambda}, \quad \|f\|_{L^2(\mathbb{R}^d)} \leq K e^{K \gamma \lambda^{\frac{1}{\delta} + \frac{1}{k}} |\log \lambda|} \|f\|_{L^2(\omega)}.$$
Theorem 2.1 establishes unique continuation properties by providing an explicit upper bound on the quantity
\[ C_{\lambda}^{k,m}(\omega) = \sup_{f \in \mathcal{E}_{\lambda}^{k,m} \setminus \{0\}} \frac{\|f\|_{L^2(\mathbb{R}^d)}}{\|f\|_{L^2(\omega)}}. \]

This upper bound naturally depends on \( k, m \) and the measurable subset \( \omega \). In the case when \( \omega \) is \( \delta \)-weakly thick for some \( 0 \leq \delta \leq 1 \), the growth of this upper bound with respect to the energy level \( \lambda \) is explicitly described and highlights a competition between \( \delta \) and the parameter \( k \). In the limit case \( \delta = 1 \), this growth is ruled by the quantity \( \frac{1}{2} \left( \frac{1}{k} + \frac{1}{m} \right) \) which is strictly smaller than 1 if and only if \( (k, m) \neq (1, 1) \). On the other hand, when \( \omega \) is only a nonzero Lebesgue measure subset, this growth is comparable to the first case with \( \delta = 1 \) but needs to be slightly modified by a factor \(|\log \lambda|\).

Notice that the results of Theorem 1.2 and Theorem 1.3 are contained in the Theorem 2.1. Indeed, as it is explained in the Section 3, the condition \( \left( \frac{1}{k} + \frac{1}{m} \right) \) is equivalent to the 1-weak thick condition and the cone appearing in the Theorem 1.3 is 1-weakly thick. Moreover, this result covers more general geometric conditions, since the subset \( \omega \) is not necessarily open contrary to Theorem 1.3 and Theorem 1.2 assertion (i), but also deals with general anisotropic Shubin operators. Let us mention that in the work [14], the author wonders whether such inequalities could hold when \( \omega \) is a bounded open subset and with \( \lambda \frac{1}{2}(1+k) \log \lambda \) instead of \( \lambda \frac{1}{2}(1+k) \). This paper positively addresses this question.

The proof of the Theorem 2.1 is given in Section 4.2 and is based on Bernstein type estimates and on uncertainty principles holding in Gelfand-Shilov spaces established by the author in [12]. As an application of these spectral inequalities, we derive new null-controllability results for evolution equations associated to these operators in the next section.

2.2. Application to the null-controllability. In this section, we present null-controllability results for the evolution equations associated to fractional anisotropic Shubin operators
\[
\begin{align*}
\partial_t f(t, x) + ((-\Delta_x)^m + |x|^2k)^s f(t, x) &= u(t, x) \mathbb{1}_\omega(x), \quad x \in \mathbb{R}^d, \ t > 0, \\
f|_{t=0} &= f_0 \in L^2(\mathbb{R}^d),
\end{align*}
\]

Fractional Shubin operators are defined as follow
\[
\forall g \in D(H_{k,m}^s), \quad H_{k,m}^s g = \sum_{n=0}^{+\infty} \langle \lambda_n^{k,m} \rangle^s \langle g, \psi_n^{k,m} \rangle_{L^2(\mathbb{R}^d)} \psi_n^{k,m}
\]
equipped with the domain
\[
D(H_{k,m}^s) = \{ g \in L^2(\mathbb{R}^d) : \sum_{n=0}^{+\infty} \langle \lambda_n^{k,m} \rangle^{2s} \langle g, \psi_n^{k,m} \rangle^2 < +\infty \}
\]

where \( k, m \geq 1 \) are positive integers and \( s > 0 \) is a positive real parameter. These operators then generate a strongly continuous contraction semi-group on \( L^2(\mathbb{R}^d) \) given by
\[
\forall g \in L^2(\mathbb{R}^d), \forall t \geq 0, \quad e^{-tH_{k,m}^s} g = \sum_{n=0}^{+\infty} e^{-t\langle \lambda_n^{k,m} \rangle^s} \langle g, \psi_n^{k,m} \rangle \psi_n^{k,m}.
\]

The notion of null-controllability is defined as follows:

**Definition 2.2** (Null-controllability). Let \( P \) be a closed operator on \( L^2(\mathbb{R}^d) \), which is the infinitesimal generator of a strongly continuous semigroup \( (e^{-tP})_{t \geq 0} \) on \( L^2(\mathbb{R}^d) \), \( T > 0 \) and \( \omega \) be a measurable subset of \( \mathbb{R}^d \). The evolution equation
\[
\begin{align*}
(\partial_t + P)f(t, x) &= u(t, x) \mathbb{1}_\omega(x), \quad x \in \mathbb{R}^d, \ t > 0, \\
f|_{t=0} &= f_0 \in L^2(\mathbb{R}^d),
\end{align*}
\]
Theorem 2.5. \textit{Controllability result:} \( \text{the evolution equation (2.2) is equivalent to the observability of the adjoint system} \)

\[
(\forall a < b \text{ positive constants with } P \text{ measurable subset of } \Omega) \exists \text{ a positive constant } C \text{ (Observability)}
\]

\[
\mathcal{D}(2.3) \text{ evolution equation} \quad \text{null-controllable from the set } \omega \text{ is said to be}
\]

\[
\mathcal{H}(2.4) \text{ Theorem 2.1:} \text{semigroup case with fixed control supports and weaker dissipation estimates than in [4]}
\]

\[
\text{Robbiano method which is a simplified formulation of Theorem 3.2 in [3] limited to the estimates on the adjoint systems. In this work, we will apply the following adapted Lebeau-Robbiano method which is a simplified formulation of Theorem 3.2 in [3] limited to the semigroup case with fixed control supports and weaker dissipation estimates than in [4] (Theorem 2.1):}
\]

\textbf{Theorem 2.4 (Beauchar, Egidi & Pravda-Starov).} \textit{Let } \Omega \text{ be an open subset of } \mathbb{R}^d, \omega \text{ be a measurable subset of } \Omega, (\pi_k)_{k \geq 1} \text{ be a family of orthogonal projections on } L^2(\Omega), (e^{-tA})_{t \geq 0} \text{ be a strongly continuous contraction semigroup on } L^2(\Omega); c_1, c_2, c'_1, c'_2, a, b, t_0, m_1 > 0 \text{ be positive constants with } a < b; m_2 \geq 0. \text{ If the following spectral inequality}
\]

\[
\forall g \in L^2(\Omega), \forall k \geq 1, \|\pi_k g\|_{L^2(\Omega)} \leq c'_1 e^{c_1 k^a} \|\pi_k g\|_{L^2(\omega)},
\]

\text{and the following dissipation estimate with controlled blow-up}

\[
\forall g \in L^2(\Omega), \forall k \geq 1, \forall 0 < t < t_0, \| (1 - \pi_k) (e^{-tA} g) \|_{L^2(\Omega)} \leq \frac{e^{-c_2 t m_1 k^b}}{c'_2 t m_2} \|g\|_{L^2(\Omega)}
\]

\text{hold, then there exists a positive constant } C > 1 \text{ such that the following observability estimate holds}

\[
\forall T > 0, \forall g \in L^2(\Omega), \|e^{-T A} g\|_{L^2(\Omega)}^2 \leq C \exp \left( \frac{C}{T^{1 - s}} \right) \int_0^T \|e^{-t A} g\|_{L^2(\omega)}^2 dt.
\]

\text{From the spectral inequalities stated by the Theorem 2.1 we deduce the following null-controllability result:}

\textbf{Theorem 2.5. Let } k, m \geq 1 \text{ be positive integers and } s > \frac{1}{2} \left( \frac{1}{k} + \frac{1}{m} \right). \text{ Let } \omega \subset \mathbb{R}^d \text{ be a measurable subset of positive Lebesgue measure } |\omega| > 0. \text{ The evolution system}
\]

\[
\left\{ \begin{array}{l}
\partial_t f(t, x) + ((-\Delta_x)^m + |x|^{2k})^s f(t, x) = u(t, x) \mathbb{1}_\omega(x), \quad x \in \mathbb{R}^d, \ t > 0,

f|_{t=0} = f_0 \in L^2(\mathbb{R}^d),
\end{array} \right.
\]

\text{is null-controllable from } \omega \text{ at any time } T > 0.
Let \( \delta \) be a positive number such that \( \delta < \frac{1}{2} \left( \frac{1}{k} + \frac{1}{m} \right) \). In particular, when the couple \( (k, m) \neq (1, 1) \), then \( 0 \leq s^* < 1 \) and the evolution equation associated to \( H_{k,m} \) is null-controllable at any time \( T > 0 \) from \( \omega \subset \mathbb{R}^d \) as soon as \( |\omega| > 0 \). This critical index \( s^* \) turns out to be equal to 1 if and only if \( k = m = 1 \). In this case, the operator associated is the harmonic oscillator and Theorem 2.5 provides that evolution equations associated to fractional harmonic oscillators \( H^s \), with \( s > 1 \) is null-controllable at any time \( T > 0 \) from any positive Lebesgue measure subsets. This particular case was already proved by Alphonse in [1] Theorem 2.9.

The proof of Theorem 2.6 is given in Section 5.1. It follows from the adapted Lebeau-Robbiano method (Theorem 2.4) together with the spectral inequalities given in the Theorem 2.1.

Regarding the case when \( s \) is smaller than the critical index, spectral inequalities can be used to deduce the following null-controllability results:

**Theorem 2.6.** Let \( k, m \geq 1 \) be positive integers, \( \frac{1}{2m} < s \leq \frac{1}{2} \left( \frac{1}{k} + \frac{1}{m} \right) \) and \( 0 \leq \delta < \frac{k}{m}(2sm - 1) \). If the measurable subset \( \omega \subset \mathbb{R}^d \) is \( \delta \)-weakly thick, then the evolution system

\[
\begin{aligned}
\partial_t f(t, x) + ((-\Delta_x)^m + |x|^{2k})^s f(t, x) &= u(t, x) \mathbb{1}_\omega(x), \quad x \in \mathbb{R}^d, \ t > 0, \\
|f|_{t=0} &= f_0 \in L^2(\mathbb{R}^d),
\end{aligned}
\]

is null-controllable from \( \omega \) at any time \( T > 0 \).

Theorem 2.6 provides non-trivial measurable subsets from which evolution equations associated to fractional Shubin operators are null-controllable. Actually, this Theorem is already proved by the author in [12] Corollary 2.12 and therefore the proof of Theorem 2.6 is omitted in this work. The strategy developed by the author in [12] consists in establishing quantitative uncertainty principles in Gelfand-Shilov spaces (see [12] Theorem 2.2 and 2.3) in order to derive observability estimates for abstract evolution equations enjoying Gelfand-Shilov smoothing effects.

### 3. The thickness property

This section is devoted to present some facts about the weak thickness property.

#### 3.1. Some properties of the weak thickness condition

The first result establishes that the \( \delta \)-weak thickness property is non-decreasing with respect to the parameter \( \delta \).

**Proposition 3.1.** Let \( 0 \leq \delta \leq \delta' \leq 1 \) and \( \omega \subset \mathbb{R}^d \) be a measurable subset. If \( \omega \) is \( \delta \)-weakly thick, then it is \( \delta' \)-weakly thick.

**Proof.** Let us assume that \( \omega \) is \( \delta \)-weakly thick. By definition, there exists \( R > 0 \) and \( 0 < \gamma \leq 1 \) such that

\[
\forall x \in \mathbb{R}^d, \ |\omega \cap B(x, R\langle x \rangle^\delta)| \geq \gamma |B(x, R\langle x \rangle^{\delta'})|.
\]

Let \( x \in \mathbb{R}^d \). Let us first notice that

\[
B(x, 3R\langle x \rangle^{\delta'}) \subset \bigcup_{y \in B(x, 3R\langle x \rangle^{\delta'})} B(y, R\langle y \rangle^{\delta})
\]

Indeed, if \( y \in B(x, 3R\langle x \rangle^{\delta'}) \) and \( \langle y \rangle^{\delta'} > 3\langle x \rangle^{\delta'} \), then the continuous function defined for all \( t \in [0, 1] \) by \( f(t) = \langle ty + (1 - t)x \rangle^{\delta'} \) satisfies \( f(0) = \langle x \rangle^{\delta'} \leq \langle x \rangle^{\delta'} \) and \( f(1) = \langle y \rangle^{\delta'} > 3\langle x \rangle^{\delta'} \).
It follows that there exists $0 < t_0 < 1$ such that $\langle z \rangle^d = 3\langle x \rangle^d$ with $z = t_0 y + (1 - t_0)x \in B(x, 3R\langle x \rangle^d)$ and $y \in B(z, R\langle z \rangle^d)$, as

$$\|z - x\| = t_0 \|x - y\| < 3R\langle x \rangle^d, \quad \|y - z\| = (1 - t_0)\|x - y\| < 3R\langle x \rangle^d = R\langle z \rangle^d.$$  
It follows that there exists a finite sequence $(x_{ik})_{0 \leq k \leq N}$ of $B(x, 3R\langle x \rangle^d)$ such that

$$\tag{3.2} B(x, 3R\langle x \rangle^d) \subset \bigcup_{k=0}^{N} B(x_{ik}, R\langle x_{ik} \rangle^d) \quad \text{and} \quad \forall 0 \leq k \leq N, \quad \langle x_{ik} \rangle^d \leq 3\langle x \rangle^d.$$

We can now use the following covering lemma [17] (Lemma 7.3):

**Lemma 3.2** (Vitali covering lemma). Let $(y_i)_{0 \leq i \leq N}$ be a finite sequence of $\mathbb{R}^n$ and $(r_i)_{0 \leq i \leq N} \subset (0, +\infty)^{N+1}$. There exists a subset $S \subset \{0, \ldots, N\}$ such that

(i) The balls $(B(y_i, r_i))_{i \in S}$ are two by two disjoint

(ii) $\bigcup_{i \in S} B(y_i, r_i) \subset \bigcup_{i \in S} B(y_i, 3r_i)$

It follows from Lemma 3.2 and (3.2) that there exists a subset $S \subset \{0, \ldots, N\}$ such that the balls $(B(x_{ik}, R\langle x_{ik} \rangle^d))_{k \in S}$ are two by two disjoint and satisfy

$$\tag{3.3} B(x, 3R\langle x \rangle^d) \subset \bigcup_{k \in S} B(x_{ik}, 3R\langle x_{ik} \rangle^d).$$

We also notice that

$$\tag{3.4} \bigcup_{k \in S} B(x_{ik}, R\langle x_{ik} \rangle^d) \subset B(x, 6R\langle x \rangle^d),$$

since, if $y \in B(x_{ik}, R\langle x_{ik} \rangle^d)$ then

$$\|y - x\| \leq \|y - x_{ik}\| + \|x_{ik} - x\| < R\langle x_{ik} \rangle^d + 3R\langle x \rangle^d \leq 6R\langle x \rangle^d.$$  

It follows from (3.1), (3.3) and (3.4) that

$$\left|\omega \cap B(x, 6R\langle x \rangle^d)\right| \geq \sum_{k \in S} \left|\omega \cap B(x_{ik}, R\langle x_{ik} \rangle^d)\right| \geq \gamma \sum_{k \in S} \left|B(x_{ik}, R\langle x_{ik} \rangle^d)\right| = \frac{\gamma}{3^d} \sum_{k \in S} \left|B(x_{ik}, 3R\langle x_{ik} \rangle^d)\right| \geq \frac{\gamma}{3^d} \left|\bigcup_{k \in S} B(x_{ik}, 3R\langle x_{ik} \rangle^d)\right| \geq \frac{\gamma}{3^d} \left|B(x, 3R\langle x \rangle^d)\right| = \frac{\gamma}{6^d} \left|B(x, 6R\langle x \rangle^d)\right|.$$  

This ends the proof of Proposition 3.1.\[\square\]

The following proposition provides a characterization of the 1-weak thickness property and will be used in the proof of Theorem 2.1.

**Proposition 3.3.** Let $\omega \subset \mathbb{R}^d$ be a measurable subset. The set $\omega$ is 1-weakly thick if and only if

$$\liminf_{R \to +\infty} \frac{|\omega \cap B(0, R)|}{|B(0, R)|} > 0.$$
Proof. Let us first assume that \( \omega \) is 1-weakly thick. There exists \( 0 < \gamma \leq 1, R_0 > 0 \) such that

\[
\forall x \in \mathbb{R}^d, \quad |\omega \cap B(x, R_0 \langle x \rangle)| \geq \gamma |B(x, R_0 \langle x \rangle)|.
\]

Let \( R \geq R_0 \) and \( x_R \in \mathbb{R}^d \) such that

\[
|x_R| + R_0 \langle x_R \rangle = R.
\]

We first notice that

\[
|x_R| \leq R \quad \text{and} \quad \langle x_R \rangle \geq \frac{R}{R_0 + 1}
\]

and therefore, for all \( R \geq R_0 + 1, \)

\[
\frac{R^2 - (R_0 + 1)^2}{(R_0 + 1)^2} \leq |x_R|^2 \leq R^2.
\]

Since \( |x_R| + R_0 \langle x_R \rangle = R \), the following inclusion holds

\[
B(x_R, R_0 \langle x_R \rangle) \subset B(0, R).
\]

We deduce that

\[
|\omega \cap B(0, R)| \geq |\omega \cap B(x_R, R_0 \langle x_R \rangle)|
\]

\[
\geq \gamma |B(x_R, R_0 \langle x_R \rangle)| = \gamma |B(0, 1)| R_0^d \langle x_R \rangle^d
\]

\[
\geq \frac{\gamma R_0^d}{(R_0 + 1)^d} R_0^d |B(0, 1)| = \frac{\gamma R_0^d}{(R_0 + 1)^d} |B(0, R)|,
\]

and it readily follows that

\[
\liminf_{R \to +\infty} \frac{|\omega \cap B(0, R)|}{|B(0, R)|} \geq \frac{\gamma R_0^d}{R_0^d + 1} > 0.
\]

Conversely, let us assume that

\[
\liminf_{R \to +\infty} \frac{|\omega \cap B(0, R)|}{|B(0, R)|} > 0.
\]

It provides some constants \( \delta > 0 \) and \( R_0 > 0 \) such that

\[
\forall R \geq R_0, \quad |\omega \cap B(0, R)| \geq \delta |B(0, R)|.
\]

Let \( x \in \mathbb{R}^d \). Since \( |x| \leq \langle x \rangle \), we have the following inclusion

\[
B(0, R_0 \langle x \rangle) \subset B(x, (R_0 + 1) \langle x \rangle).
\]

Since \( R_0 \langle x \rangle \geq R_0 \), we therefore deduce that

\[
|\omega \cap B(x, (R_0 + 1) \langle x \rangle)| \geq |\omega \cap B(0, R_0 \langle x \rangle)|
\]

\[
\geq \delta |B(0, R_0 \langle x \rangle)|
\]

\[
= \frac{\delta R_0^d}{(R_0 + 1)^d} |B(x, (R_0 + 1) \langle x \rangle)|.
\]

This establishes that \( \omega \) is 1-weakly thick and this ends the proof of Proposition 3.3. \( \square \)
3.2. Some examples of weak thick sets. We end this section by giving some examples of weak thick sets in the one and two-dimensional cases, which can be easily generalized to higher dimensions. Of course, any subset $\omega$ containing one of the following examples also satisfies a weak thickness condition.

In the one-dimensional case, let us consider the following measurable subset

$$\omega_\delta = \bigcup_{n \in \mathbb{N}} \left[ n^{\frac{1}{1-\delta}}, \frac{n^{\frac{1}{1-\delta}} + (n+1)^{\frac{1}{1-\delta}}}{2} \right] \cup \left[ -\frac{n^{\frac{1}{1-\delta}} + (n+1)^{\frac{1}{1-\delta}}}{2}, -n^{\frac{1}{1-\delta}} \right],$$

with $0 \leq \delta < 1$. The set $\omega_\delta$ is $\delta$-weakly thick. However, when $\delta > 0$, it fails to satisfy the $\delta'$-weak thickness condition, for any $0 \leq \delta' < \delta$, since the complementary of $\omega_\delta$ contains some intervals of the following form

$$\left[ n^{\frac{1}{1-\delta}} + (n+1)^{\frac{1}{1-\delta}}, (n+1)^{\frac{1}{1-\delta}} \right], \quad n \in \mathbb{N},$$

of length

$$\frac{(n+1)^{\frac{1}{1-\delta}} - n^{\frac{1}{1-\delta}}}{2} \sim_{n \to +\infty} \frac{n^{\frac{\delta}{1-\delta}}}{2(1-\delta)}.$$

![Figure 1. The subset $\omega_4$.](image1.png)

![Figure 2. The subset $\omega_0$.](image2.png)

![Figure 3. The subset $\omega_{\delta,R}$ with $0 < \delta \leq 1$ and $R > 0$.](image3.png)
In the two-dimensional case, we define the following measurable subset of $\mathbb{R}^2$
\[ \omega_{\delta,R} = \left\{ (x, y) \in \mathbb{R}^2; \ |y| > R|x|^\delta \right\}, \]
with $0 \leq \delta \leq 1$ and $R > 0$. The subset $\omega_{\delta,R}$ is $\delta$-weakly thick. However, when $\delta > 0$, its complement contains some ball of the form
\[ B(z, R'(z)^{\delta'}), \quad z \in \mathbb{R}^2, \]
for any $R' > 0$ and $0 \leq \delta' < \delta$. Then, this set can not be $\delta'$-weakly thick, for any $0 \leq \delta' < \delta$.

![Figure 4. The subset $\omega_{0,1}$.](image)

We end this section by noticing that contrary to the case $\delta < 1$, the 1-weak thickness property allows the subset $\omega$ to be contained in a half-space. Indeed, Proposition 3.3 ensures that any cone of the following form
\[ C_\theta = \left\{ (r \cos t, r \sin t) \in \mathbb{R}^2 : \ r > 0, \ |t| \leq \frac{\pi}{2} - \theta \right\}, \]
where $0 \leq \theta < \frac{\pi}{2}$, defines a 1-weak thick set.

![Figure 5. The cone $C_\theta$.](image)
4. Proof of the results

This section aims at establishing the spectral inequalities given by Theorem 2.1. In Subsection 4.1, we establish some Bernstein type estimates which are used in the proof of Theorem 2.1 given in Subsection 4.2. In the following, for all multi-indexes $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$, we use the notation $|\alpha| = \alpha_1 + \cdots + \alpha_d$.

4.1. Smoothing properties and Bernstein type estimates. This section is devoted to the proof of the following Bernstein type estimates which play a key role in the proof of Theorem 2.1.

**Proposition 4.1.** Let $k, m \geq 1$ be positive integers and $0 < s \leq \frac{1}{2} \left( \frac{1}{k} + \frac{1}{m} \right)$. There exist positive constants $C = C_{s,k,m,d} \geq 1$, $C' = C'_{s,k,m,d} \geq 1$, $C'' = C''_{s,k,m,d} \geq 1$, $\eta = \eta_{s,k,m,d} > 0$ and $\eta' = \eta'_{s,k,m,d} > 0$ such that

\begin{align*}
\forall \lambda > 0, \forall g \in \mathcal{E}^{k,m}_\lambda, \forall \beta \in \mathbb{N}^d, \\
\| (x)^{p} \partial_x^{\beta} g \|_{L^2(\mathbb{R}^d)} \leq C^{1+|\beta|}(p!) \frac{1}{2\pi} (|\beta|!) \frac{1}{2^{2m}} e^{\eta' \lambda^s} \|g\|_{L^2(\mathbb{R}^d)},
\end{align*}

(4.1)

\begin{align*}
\forall \lambda > 0, \forall g \in \mathcal{E}^{k,m}_\lambda, \forall \beta \in \mathbb{N}^d, \\
\| e^{\eta(x)^{2\lambda}} \partial_x^{\beta} g \|_{L^2(\mathbb{R}^d)} \leq C'^{1+|\beta|}(|\beta|!) \frac{1}{2^{2m}} e^{\eta' \lambda^s} \|g\|_{L^2(\mathbb{R}^d)},
\end{align*}

(4.2)

\begin{align*}
\forall \lambda > 0, \forall g \in \mathcal{E}^{k,m}_\lambda, \forall \beta \in \mathbb{N}^d, \\
\| \partial_x^{\beta} g \|_{L^\infty(\mathbb{R}^d)} \leq C''^{1+|\beta|}(|\beta|!) \frac{1}{2^{2m}} e^{\eta' \lambda^s} \|g\|_{L^2(\mathbb{R}^d)},
\end{align*}

(4.3)

where $\mathcal{E}^{k,m}_\lambda$ is defined in (2.1).

These estimates are derived from quantitative regularizing effects of the semigroup associated to anisotropic Shubin operators. Alphonse investigated the smoothing effects of these semigroups and established the following quantitative estimates in (1) (estimates (2.3)).

**Theorem 4.2.** Let $k, m \geq 1$ be positive integers and $s > 0$. Setting $s^* = \frac{1}{2} \left( \frac{1}{k} + \frac{1}{m} \right) \leq 1$, when $s \leq s^*$, there exist two positive constants $0 < t_s \leq 1$ and $C_s \geq 1$ such that

\[ \forall 0 < t \leq t_s, \forall g \in L^2(\mathbb{R}^d), \forall \alpha, \beta \in \mathbb{N}^d, \]

\[ \| x^\alpha \partial_x^\beta (e^{-t\mathcal{L}^*_{k,m}} g) \|_{L^2(\mathbb{R}^d)} \leq C_s^{1+|\alpha|+|\beta|} \left( \frac{2^{|\beta|}}{t^{\frac{|\beta|}{k+m}} + \frac{|\beta|}{2^m}} \right) (\alpha!) \frac{1}{2^{2m}} \|g\|_{L^2(\mathbb{R}^d)}, \]

whereas, when $s \geq s^*$, there exist two positive constants $0 < t_s \leq 1$ and $C_s \geq 1$ such that

\[ \forall 0 < t \leq t_s, \forall g \in L^2(\mathbb{R}^d), \forall \alpha, \beta \in \mathbb{N}^d, \]

\[ \| x^\alpha \partial_x^\beta (e^{-t\mathcal{L}^*_{k,m}} g) \|_{L^2(\mathbb{R}^d)} \leq C_s^{1+|\alpha|+|\beta|} \left( \frac{2^{|\beta|}}{t^{\frac{|\beta|}{k+m}} + \frac{|\beta|}{2^m}} \right) \frac{m^{|\beta|}}{k+m} (\beta!) \frac{k}{2^{2m}} \|g\|_{L^2(\mathbb{R}^d)}. \]

Let $k, m \geq 1$ be positive integers, and $0 < s \leq \frac{1}{2} \left( \frac{1}{k} + \frac{1}{m} \right)$. According to Theorem 1.2, there exist two positive constants $0 < t_s \leq 1$ and $C_s \geq 1$ such that

\[ \forall 0 < t \leq t_s, \forall g \in L^2(\mathbb{R}^d), \forall \alpha, \beta \in \mathbb{N}^d, \]

\[ \| x^\alpha \partial_x^\beta (e^{-t\mathcal{L}^*_{k,m}} g) \|_{L^2(\mathbb{R}^d)} \leq C_s^{1+|\alpha|+|\beta|} \left( \frac{2^{|\beta|}}{t^{\frac{|\beta|}{k+m}} + \frac{|\beta|}{2^m}} \right) (\alpha!) \frac{1}{2^{2m}} (\beta!) \frac{k}{2^{2m}} \|g\|_{L^2(\mathbb{R}^d)}. \]

(4.4)
Let $\lambda > 0$ and $g_0 \in e^{k,m}_\lambda$. Let us define an auxiliary function
\[ g = e^{k,m}_\lambda \mathcal{H}_x^k g_0 := \sum_{n \in \mathbb{N}} e^{\mathcal{L}_x^*(\lambda_{n,k}^{k,m})} (g_0, \psi_{n,k}^{k,m})_{L^2(\mathbb{R}^d)} \psi_{n,k}^{k,m}. \]

Notice that
\[ e^{-\mathcal{L}_x} \mathcal{H}_x^k g_0 = g_0 \quad \text{and} \quad \|g\|_{L^2(\mathbb{R}^d)} \leq e^{\mathcal{L}_x^*(\lambda_{n,k}^{k,m})} \|g_0\|_{L^2(\mathbb{R}^d)}. \]

By applying (4.3) and the above remark, it follows that for all $\alpha, \beta \in \mathbb{N}^d$,
\begin{align*}
\|x^\alpha \partial_x^\beta g_0\|_{L^2(\mathbb{R}^d)} & \leq \frac{C_s^{1+|\alpha|+|\beta|}}{t_s^{1+|\alpha|/2} \pi^{d/2} \beta!} \frac{1}{2\pi} e^{\mathcal{L}_x^*(\lambda_{n,k}^{k,m})} \|g_0\|_{L^2(\mathbb{R}^d)} \\
& \leq C_s^{1+|\alpha|+|\beta|} (\alpha!) \frac{1}{\beta!} e^{\mathcal{L}_x^*(\lambda_{n,k}^{k,m})} \|g_0\|_{L^2(\mathbb{R}^d)}
\end{align*}

with
\[ C_s^{1+|\alpha|+|\beta|} (\alpha!) \frac{1}{\beta!} \geq 1. \]

We can now apply the technical Lemmas 6.1 and 6.2, stated and proved in Appendix 6.1. First, it follows from Lemma 6.1 that for all $p \in \mathbb{N}$, $\beta \in \mathbb{N}^d$,
\[ \| \langle x \rangle^p \partial_x^\beta g_0 \|_{L^2(\mathbb{R}^d)} \leq (d + 1) \frac{1}{2} C_s^{1+|\beta|} (\beta!) \frac{1}{2\pi} e^{\mathcal{L}_x^*(\lambda_{n,k}^{k,m})} \|g_0\|_{L^2(\mathbb{R}^d)}. \]

It proves the estimates (4.1). Regarding the estimates (4.2), we deduce from the fact that $0 < s \leq 1$, (4.4) and Lemma 6.2 that for all $p \in \mathbb{N}$, $\beta \in \mathbb{N}^d$,
\[ \| \langle x \rangle^{skp} \partial_x^\beta g_0 \|_{L^2(\mathbb{R}^d)} \leq C_s^{skp \beta} (\beta!) \frac{1}{2\pi} e^{\mathcal{L}_x^*(\lambda_{n,k}^{k,m})} \|g_0\|_{L^2(\mathbb{R}^d)}. \]

By direct computations, we have for all $p \in \mathbb{N}$,
\[ (kp)! \leq (kp)^{kp} = (k^p)^p \leq (ke)^p, \]

since
\[ \forall p \in \mathbb{N}, \quad p^p \leq p!e^p. \]

The above estimate, together with (4.5), imply that there exists a new constant $C' = C_s^{skp \beta} \geq 1$ such that for all $p \in \mathbb{N}$, $\beta \in \mathbb{N}^d$,
\[ \| \langle x \rangle^{skp} \partial_x^\beta g_0 \|_{L^2(\mathbb{R}^d)} \leq C'^{skp \beta} (\beta!) \frac{1}{2\pi} e^{\mathcal{L}_x^*(\lambda_{n,k}^{k,m})} \|g_0\|_{L^2(\mathbb{R}^d)}. \]

Let us define $\eta = \frac{1}{4C_s^{2}}$. We deduce that for all $\beta \in \mathbb{N}^d$,
\[ \| e^{\eta \langle x \rangle^{skp} \partial_x^\beta} g_0 \|_{L^2(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} e^{2\eta \langle x \rangle^{skp} \partial_x^\beta} |\partial_x^\beta g_0(x)|^2 \, dx \]
\begin{align*}
& = \sum_{p=0}^{+\infty} (2\eta)^p \| \langle x \rangle^{skp} \partial_x^\beta g_0 \|_{L^2(\mathbb{R}^d)}^2 \\
& \leq C'^{2(1+|\beta|) (\beta!)} \frac{1}{2\pi} e^{2\mathcal{L}_x^*(\lambda_{n,k}^{k,m})} \|g_0\|_{L^2(\mathbb{R}^d)}^2 \sum_{p=0}^{+\infty} \frac{1}{2p} \\
& \leq 2C'^{2(1+|\beta|) (\beta!)} \frac{1}{2\pi} e^{2\mathcal{L}_x^*(\lambda_{n,k}^{k,m})} \|g_0\|_{L^2(\mathbb{R}^d)}^2,
\end{align*}

which proves the estimates (4.2). It remains to establish (4.3). First, by the multinomial formula, we have
\begin{align*}
\forall j \in \mathbb{N}, \quad (-\Delta_x)^j &= ( - \partial_{x_1}^2 - \cdots - \partial_{x_d}^2 )^j = \sum_{\gamma \in \mathbb{N}^d} \frac{j!}{\gamma!} (-1)^{\gamma} \partial_x^{2\gamma},
\end{align*}
We deduce from the above equality and (4.5) that for all $j \in \mathbb{N}$,

\begin{equation}
\|(\Delta_x)^j g_0\|_{L^2(\mathbb{R}^d)} \leq \sum_{\gamma \in \mathbb{N}^d} \frac{j!}{\gamma!} \|\partial_x^{2\gamma} g_0\|_{L^2(\mathbb{R}^d)} \leq \sum_{\gamma \in \mathbb{N}^d} \frac{j!}{\gamma!} C_{s,k,m,d}^{1+2\gamma} (2\gamma!) \frac{1}{2\pi m} e^{\frac{1}{2}\lambda^*} \|g_0\|_{L^2(\mathbb{R}^d)}.
\end{equation}

Since

$$(2j)! = \binom{2j}{j} (j!)^2 \leq 2^{2j}(j!)^2.$$ 

Thanks to the Fourier inversion formula and Plancherel’s formula, it follows that for all $\beta \in \mathbb{N}^d$

\begin{equation}
\|\partial_x^\beta g_0\|_{L^\infty(\mathbb{R}^d)} \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\xi^\beta \widehat{g}_0(\xi)| \, d\xi \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{|\xi|^{|\beta|} + |\xi||\beta| + d}{1 + |\xi|^d} |\widehat{g}_0(\xi)| \, d\xi = \frac{1}{(2\pi)^d} \left( \int_{\mathbb{R}^d} \frac{1}{(1 + |\xi|^d)^2} \, d\xi \right)^\frac{1}{2} \left( \| |\xi|^\beta \widehat{g}_0\|_{L^2(\mathbb{R}^d)} + \| |\xi||\beta| + d \widehat{g}_0\|_{L^2(\mathbb{R}^d)} \right).
\end{equation}

We therefore deduce from (4.8), together with (4.7), that for all $\beta \in \mathbb{N}^d$,

$$\|\partial_x^\beta g_0\|_{L^\infty(\mathbb{R}^d)} \leq \frac{1}{(2\pi)^d} \left( \int_{\mathbb{R}^d} \frac{1}{(1 + |\xi|^d)^2} \, d\xi \right)^\frac{1}{2} \left( 2^{1+|\beta|} C_{s,k,m,d}^{1+|\beta|+d} (|\beta| + d)! \frac{1}{2\pi m} e^{\frac{1}{2}\lambda^*} \|g_0\|_{L^2(\mathbb{R}^d)} \right).$$

Since for all $\beta \in \mathbb{N}^d$, $(|\beta| + d)! \leq 2^{2|\beta|+d} |\beta|!$, it follows that there exists a new constant $C_{s,k,m,d}^{\prime} \geq 1$ such that

$$\forall \beta \in \mathbb{N}^d, \quad \|\partial_x^\beta g_0\|_{L^\infty(\mathbb{R}^d)} \leq C_{s,k,m,d}^{\prime \beta + d} (|\beta|! \frac{1}{2\pi m} e^{\frac{1}{2}\lambda^*} \|g_0\|_{L^2(\mathbb{R}^d)}.$$ 

This ends the proof of Proposition 4.1.

4.2. Proof of Theorem 2.1 This section is devoted to the proof of the spectral inequalities given by Theorem 2.1. Before presenting its proof, we begin by presenting an elementary lemma:

**Lemma 4.3.** Let $k, m \geq 1$ be positive integers and $\omega \subset \mathbb{R}^d$ be a measurable subset of positive Lebesgue measure $|\omega| > 0$. For all $\lambda > 0$, there exists a positive constant $C_{k,m,\lambda}(\omega) > 0$ such that

$$\forall f \in \mathcal{E}_\lambda^{k,m}, \quad \|f\|_{L^2(\mathbb{R}^d)} \leq C_{k,m,\lambda}(\omega) \|f\|_{L^2(\omega)}.$$
Proof. Let \( \lambda > 0 \). Since \( \mathcal{E}_\lambda^{k,m} \) is a \( \mathbb{C} \)-vector space of finite dimension, it is sufficient to show that \( \| \cdot \|_{L^2(\omega)} \) is a norm on this space. Let \( f \in \mathcal{E}_\lambda^{k,m} \) such that \( \| f \|_{L^2(\omega)} = 0 \). By applying (4.3) in Proposition 4.1 with \( s = \frac{1}{2}(\frac{k}{k} + \frac{m}{m}) \), we deduce that \( f \) is analytic on \( \mathbb{R}^d \), since \( \frac{k}{k} + \frac{m}{m} = \frac{1}{k} + \frac{1}{m} < 1 \). It follows that \( f = 0 \) on \( \omega \) and therefore, \( f = 0 \), since \( |\omega| > 0 \). \( \square \)

4.3. First case: \( \omega \) is \( \delta \)-weakly thick with \( 0 \leq \delta < 1 \). In this case, we use the following uncertainty principle with error term holding in Gelfand-Shilov spaces stated in [12, Theorem 2.3]:

**Theorem 4.4.** Let \( A \geq 1 \), \( 0 < \gamma \leq 1 \), \( 0 < \mu \leq 1 \), \( \nu > 0 \) with \( \mu + \nu \geq 1 \). Let \( \rho: \mathbb{R}^d \to (0, +\infty) \) be a positive \( \frac{1}{2} \)-Lipschitz function such that there exist \( m > 0 \), \( R > 0 \) satisfying

\[
\forall x \in \mathbb{R}^d, \quad 0 < m \leq \rho(x) \leq R|x|^{-m}.
\]

Let \( \omega \subset \mathbb{R}^d \) be a measurable subset. If \( \rho \) is \( \gamma \)-thick with respect to \( \rho \), that is,

\[
\forall x \in \mathbb{R}^d, \quad |\omega \cap B(x, \rho(x))| \geq \gamma |B(x, \rho(x))|,
\]

then for all \( 0 < \varepsilon < 1 \), there exist a positive constant \( C_{e,A} > 0 \) such that for all \( f \in \mathcal{H}(\mathbb{R}^d) \),

\[
\| f \|_{L^2(\mathbb{R}^d)} \leq C_{e,A} \| f \|_{L^2(\omega)} + \varepsilon \sup_{\rho \in \mathbb{R}, \beta \in \mathbb{N}^d} \left( \frac{\| (x)p\beta^2 f \|_{L^2(\mathbb{R}^d)}}{A^{p+1}\beta! \nu((\beta!\nu)^{1/\nu})} \right)^2,
\]

where there exists a positive constant \( K = K(d, \gamma, \rho, \mu, \nu) \geq 1 \) depending on the dimension \( d \), \( \rho \) and \( \nu \) such that

\[
0 < C_{e,A} \leq e^{K(1-\log \varepsilon + \log A)e^{Kd^2}}.
\]

Let \( \omega \subset \mathbb{R}^d \) be a measurable subset \( \delta \)-weakly thick. By definition, there exist \( R > 0 \) and \( 0 < \gamma \leq 1 \) such that

\[
\forall x \in \mathbb{R}^d, \quad |\omega \cap B(x, R|x|^\delta)| \geq \gamma |B(x, R|x|^\delta)|.
\]

Let us check that \( \omega \) satisfies the assumptions of Theorem 4.4. Notice that

\[
\nabla(R|x|^\delta) = R\delta x|\omega|^{2-\delta} |x| \to 0,
\]

and therefore, there exists \( L = L(R, \delta) > 0 \) such that

\[
\forall |x| \geq L, \quad |\nabla(R|x|^\delta)| \leq \frac{1}{2}.
\]

We define an auxiliary density by

\[
\rho(x) = \begin{cases} \frac{R|x|^\delta}{L} & \text{if } |x| \geq L, \\ \frac{R(1+L^2)^{\frac{\delta}{2}}}{|x|^2} & \text{if } |x| \leq L. \end{cases}
\]

Thanks to (4.10), we can readily check that \( \rho \) is a \( \frac{1}{2} \)-Lipschitz function and moreover, we have

\[
\forall x \in \mathbb{R}^d, \quad 0 \leq \rho \leq \frac{1}{\gamma} R |x|^\delta \leq R(1+L^2)^{\frac{\delta}{2}} |x|^\delta.
\]

To apply Theorem 4.4 it remains to check that there exists \( 0 < \tilde{\gamma} \leq 1 \) such that

\[
\forall x \in \mathbb{R}^d, \quad |\omega \cap B(x, \rho(x))| \geq \tilde{\gamma} |B(x, \rho(x))|.
\]

Let \( x \in \mathbb{R}^d \). We deduce from (4.9) and (4.11) that

\[
|\omega \cap B(x, \rho(x))| \geq |\omega \cap B(x, R|x|^\delta)| \geq \gamma |B(x, R|x|^\delta)| \geq \frac{\gamma}{(1+L^2)^{\frac{\delta}{2}}} |B(x, \rho(x))|.
\]
We can therefore apply Theorem 4.4 with the constants $A = C$, $s = \frac{1}{2} \left( \frac{k}{m} + \frac{1}{m} \right)$, $0 < \mu = \frac{2k}{2\mu m} = \frac{k}{2m + k} \leq 1$ and $\nu = \frac{1}{2\mu k} = \frac{m}{2m + k} > 0$. Since $\delta = \frac{1-\mu}{\mu}$, we obtain that there exist some positive constants $K = K_{R,k,m,\delta,d} \geq 1$ and $D = D_{R,k,m,\delta,d} \geq 1$ such that for all $0 < \varepsilon \leq 1$, $\lambda > 0$, $f \in \mathcal{E}_{\lambda}^{k,m}$

$$
\|f\|_{L^2(\mathbb{R}^d)}^2 \leq e^{K(1-\log \varepsilon + \log A)e^{K\lambda A}} \|f\|_{L^2(\mathbb{R}^d)}^2 + \varepsilon \sup_{p \in \mathbb{N}, \beta \in \mathbb{N}^d} \frac{\|\langle x \rangle^p \partial_x^\beta f\|_{L^2(\mathbb{R}^d)}^2}{C^{2(\nu + |\beta|)}(2\nu (|\beta|)!)^{2\mu}}
$$

$$
\leq e^{K(1-\log \varepsilon + \log A)e^{K\lambda A}} \|f\|_{L^2(\mathbb{R}^d)}^2 + C^2 e^{2\eta' \lambda^*} \varepsilon \|f\|_{L^2(\mathbb{R}^d)}^2,
$$

where the last inequality is obtained by using (4.11). By setting, $\varepsilon = \frac{1}{2\nu C} e^{-2\eta' \lambda^*}$, we deduce that there exists a new constant $K' = K_{R,k,m,\delta,d} \geq 1$ so that

$$
\forall \lambda > 0, \forall f \in \mathcal{E}_{\lambda}^{k,m}, \quad \|f\|_{L^2(\mathbb{R}^d)}^2 \leq K' \lambda \left(1 + \lambda^*\right) \|f\|_{L^2(\mathbb{R}^d)}^2.
$$

By recalling that $s = \frac{1}{2} \left( \frac{k}{m} + \frac{1}{m} \right)$, the result follows.

4.4. Second case: $\omega$ is 1-weakly thick. In this case, since $\omega$ is 1-weakly thick, we deduce from Proposition 3.3 that there exist $\gamma > 0$ and $R_0 > 0$ such that

$$
\forall R \geq R_0, \quad |\omega \cap B(0,R)| \geq \gamma |B(0,R)|.
$$

As a first step, we establish that there exists a positive constant $c_{d,k,m} \geq 1$ such that

$$
\forall \lambda \geq 1, \forall f \in \mathcal{E}_{\lambda}^{k,m}, \quad \|f\|_{L^2(\mathbb{R}^d)}^2 \leq 2\|f\|_{L^2(B(0,c_{d,k,m},\lambda R_0))}^2.
$$

Let $a > 0$. Thanks to Proposition 4.1 with $s = \frac{1}{2} (\frac{k}{m} + \frac{1}{m})$, there exist some positive constants $\eta = \eta(d,k,m) > 01$, $\eta' = \eta'(d,k,m) > 0$ and $C = C(d,k,m) \geq 1$ such that:

$$
\forall \lambda > 0, \forall f \in \mathcal{E}_{\lambda}^{k,m}, \forall \beta \in \mathbb{N}^d,
$$

$$
\|e^{\eta(\langle x \rangle^*) \frac{1}{(a+1)^{\frac{1}{2}+\frac{1}{d}}}} \partial_x^\beta f\|_{L^2(\mathbb{R}^d)} \leq C^{1+|\beta| (|\beta|)!} e^{\eta' \lambda^* \frac{1}{(a+1)^{\frac{1}{2}+\frac{1}{d}}}} \|f\|_{L^2(\mathbb{R}^d)}
$$

and

$$
\|\partial_x^\beta f\|_{L^\infty(\mathbb{R}^d)} \leq C^{1+|\beta| (|\beta|)!} e^{\eta' \lambda^* \frac{1}{(a+1)^{\frac{1}{2}+\frac{1}{d}}}} \|f\|_{L^2(\mathbb{R}^d)}.
$$

It follows that for all $\lambda > 0$ and $f \in \mathcal{E}_{\lambda}^{k,m}$

$$
\|f\|_{L^2(\mathbb{R}^d)}^2 = \|f\|_{L^2(B(0,a))}^2 + \|f\|_{L^2(\mathbb{R}^d \setminus B(0,a))}^2
$$

$$
\leq \|f\|_{L^2(B(0,a))}^2 + e^{-2\eta(\langle x \rangle^*) \frac{1}{(a+1)^{\frac{1}{2}+\frac{1}{d}}}} \|e^{\eta(\langle x \rangle^*) \frac{1}{(a+1)^{\frac{1}{2}+\frac{1}{d}}}} f\|_{L^2(\mathbb{R}^d)}^2
$$

$$
\leq \|f\|_{L^2(B(0,a))}^2 + e^{-2\eta(\langle x \rangle^*) \frac{1}{(a+1)^{\frac{1}{2}+\frac{1}{d}}}} C^2 e^{2\eta' \lambda^* \frac{1}{(a+1)^{\frac{1}{2}+\frac{1}{d}}}} \|f\|_{L^2(\mathbb{R}^d)}^2.
$$

Let $c_{d,k,m} > 0$ be a positive constant so that

$$
\forall \lambda \geq 1, \quad e^{-2\eta(\langle x \rangle^*) \frac{1}{(a+1)^{\frac{1}{2}+\frac{1}{d}}}} C^2 e^{2\eta' \lambda^* \frac{1}{(a+1)^{\frac{1}{2}+\frac{1}{d}}}} \leq \frac{1}{2},
$$

with

$$
a = c_{d,k,m} \lambda R_0.
$$

It follows from (4.13) that

$$
\forall \lambda \geq 1, \forall f \in \mathcal{E}_{\lambda}^{k,m}, \quad \|f\|_{L^2(\mathbb{R}^d)}^2 \leq 2\|f\|_{L^2(B(0,c_{d,k,m},\lambda R_0))}^2.
$$
Let $\lambda \geq 1$ and $f \in \mathcal{E}_{\lambda}^{k,m} \setminus \{0\}$. We set
\[
\forall x \in B\left(0, c_{d,k,m}\lambda^{\frac{1}{d+k}}\right), \quad g(x) = \frac{f(x)}{\sqrt{2}C_{d,k,m}\lambda^{\frac{1}{d+k}}\|f\|_{L^2\left(B\left(0,c_{d,k,m}\lambda^{\frac{1}{d+k}}\right)\right)}}.
\]
By using
\[
\|f\|_{L^2\left(B\left(0,c_{d,k,m}\lambda^{\frac{1}{d+k}}\right)\right)} \leq \sqrt{\left|B\left(0, c_{d,k,m}\lambda^{\frac{1}{d+k}}\right)\right|}\|f\|_{L^\infty\left(B\left(0,c_{d,k,m}\lambda^{\frac{1}{d+k}}\right)\right)},
\]
it follows that
\[
\|g\|_{L^\infty\left(B\left(0,c_{d,k,m}\lambda^{\frac{1}{d+k}}\right)\right)} \geq \frac{e^{-\eta'\lambda^{\frac{1}{d+k}}\left(1 + \frac{1}{d+k}\right)}}{\sqrt{2\left|B\left(0, 1\right)\right|c_{d,k,m}\lambda^{\frac{1}{d+k}}}} \geq e^{-\eta''\lambda^{\frac{1}{d+k}}\left(1 + \frac{1}{d+k}\right)},
\]
for a suitable constant $\eta'' = \eta''(d,k,m) \geq 1$ independent on $\lambda \geq 1$. Furthermore, we deduce from the estimates (4.12) and (4.14) that
\[
\forall \beta \in \mathbb{N}^d, \quad \|\partial_\beta g\|_{L^\infty\left(B\left(0,c_{d,k,m}\lambda^{\frac{1}{d+k}}\right)\right)} \leq \frac{\|\partial_\beta f\|_{L^\infty(B^d)} \sqrt{2}C_{d,k,m}\lambda^{\frac{1}{d+k}}\|f\|_{L^2\left(B\left(0,c_{d,k,m}\lambda^{\frac{1}{d+k}}\right)\right)}}{\sqrt{2\left|B\left(0, 1\right)\right|c_{d,k,m}\lambda^{\frac{1}{d+k}}}},
\]
and
\[
\|\|\|_{L^\infty(B(0,R))} \geq t, \quad \text{the following estimate holds}
\]
\[
\|f\|_{L^2(B(0,R))} \leq C_{t,A,s,R,\gamma,d}\|f\|_{L^2(E)},
\]
where
\[
0 < C_{t,A,s,R,\gamma,d} \leq \left(\frac{K}{\gamma}\right)^{K \left(1 - \log l + (AR)^{1-s}\right)}.
\]

We can now use the following proposition established in [12, Example 11]:

**Proposition 4.5.** Let $0 < s < 1$, $A \geq 1$, $R > 0$, $d \geq 1$, $0 < t \leq 1$ and $0 < \gamma \leq 1$. Let $E \subset B(0,R)$ be a measurable subset of the Euclidean ball centered at 0 with radius $R$ such that $|E| \geq \gamma |B(0,R)|$. There exists a constant $K = K(s,d) \geq 1$ such that for all $f \in C^\infty(B(0,R))$ satisfying
\[
\forall \beta \in \mathbb{N}^d, \quad \|\partial_\beta g\|_{L^\infty(B(0,R))} \leq A^{|\beta|}(1 + |\beta|)^s
\]
and $\|f\|_{L^\infty(B(0,R))} \geq t$, the following estimate holds
\[
\|f\|_{L^2(B(0,R))} \leq C_{t,A,s,R,\gamma,d}\|f\|_{L^2(E)},
\]
where
\[
0 < C_{t,A,s,R,\gamma,d} \leq \left(\frac{K}{\gamma}\right)^{K \left(1 - \log l + (AR)^{1-s}\right)}.
\]

We deduce that there exists a new positive constant $\tilde{K} = \tilde{K}(d,k,m) \geq 1$ such that for all $\lambda \geq \lambda_0 = \max \left(\frac{R_0}{c_{d,k,m}}, 1\right)^{2k}$
\[
\|g\|_{L^2\left(B\left(0,c_{d,k,m}\lambda^{\frac{1}{d+k}}\right)\right)} \leq \left(\frac{\tilde{K}}{\gamma}\right)^{\tilde{K}^{\frac{1}{d+k}}}\|g\|_{L^2\left(\omega \cap B\left(0,c_{d,k,m}\lambda^{\frac{1}{d+k}}\right)\right)},
\]
and
It follows from (4.14) that
\[ \|f\|_{L^2(\mathbb{R}^d)} \leq \sqrt{2}\|f\|_{L^2(B(0, c_{d,k,m} \lambda^{\frac{1}{e}r}))} \]
\[ \leq \sqrt{2} \left( \frac{K}{\gamma} \right) K^{\frac{1}{2}} \lambda^{\frac{1}{2}} \|f\|_{L^2(\omega \cap B(0, c_{d,k,m} \lambda^{\frac{1}{e}r}))} \]
\[ \leq \sqrt{2} \left( \frac{K}{\gamma} \right) K^{\frac{1}{2}} \lambda^{\frac{1}{2}} \|f\|_{L^2(\omega)}. \]

When \( 0 < \lambda \leq \lambda_0 = \max \left( \frac{R_0}{c_{d,k,m}}, 1 \right)^{2k} \), Lemma 4.3 implies that there exists \( C_{k,m,\lambda_0}(\omega) > 0 \) such that
\[ \forall f \in \mathcal{E}_{\lambda}^{k,m}, \quad \|f\|_{L^2(\mathbb{R}^d)} \leq C_{k,m,\lambda_0}(\omega) \|f\|_{L^2(\omega)}, \]
since
\[ \forall 0 < \lambda \leq \lambda_0, \quad \mathcal{E}_{\lambda}^{k,m} \subset \mathcal{E}_{\lambda_0}^{k,m}. \]

By choosing \( \tilde{K} = \max(C_{k,m,\lambda_0}(\omega), \sqrt{2}) \), it follows that
\[ \forall f \in \mathcal{E}_{\lambda}^{k,m}, \quad \|f\|_{L^2(\mathbb{R}^d)} \leq \tilde{K} \left( \frac{K}{\gamma} \right) \lambda^{\frac{1}{2}} \|f\|_{L^2(\omega)}. \]

4.5. Third case: \( |\omega| > 0 \). Since \( 0 < |\omega| \leq +\infty \), there exist \( R_0 > 0, \sigma > 0 \) such that
\[ \forall R \geq R_0, \quad |\omega \cap B(0, R)| \geq \sigma > 0. \]

With the same notation as in Section 4.4, Proposition 4.5 implies once again that there exists a positive constant \( \tilde{K} = \tilde{K}(d, k, m) \geq 1 \) such that for all \( \lambda \geq \lambda_0 = \max \left( \frac{R_0}{c_{d,k,m}}, 1 \right)^{2k} \), \( f \in \mathcal{E}_{\lambda}^{k,m}, \)
\[ \|f\|_{L^2(\mathbb{R}^d)} \leq \sqrt{2} \left( \frac{\tilde{K}|B(0, 1)||c_{d,k,m}^{d/2}}{\sigma} \right) \tilde{K}^{\frac{1}{2}} \lambda^{\frac{1}{2}} \|f\|_{L^2(\omega \cap B(0, c_{d,k,m} \lambda^{\frac{1}{e}r}))} \]
\[ \leq \sqrt{2} \left( \frac{\tilde{K}|B(0, 1)||c_{d,k,m}^{d/2}}{\sigma} \right) \tilde{K}^{\frac{1}{2}} \lambda^{\frac{1}{2}} \|f\|_{L^2(\omega)}. \]

It follows that there exists a positive constant \( \tilde{K}' = \tilde{K}'(\omega, d, k, m) \geq 1 \) such that for all \( \lambda \geq \lambda_0 \) and \( f \in \mathcal{E}_{\lambda}^{k,m}, \)
\[ \|f\|_{L^2(\mathbb{R}^d)} \leq \sqrt{2} \left( \frac{\tilde{K}'|B(0, 1)||c_{d,k,m}^{d/2}}{\sigma} \right) \tilde{K}'^{\frac{1}{2}} \lambda^{\frac{1}{2}} \|f\|_{L^2(\omega)} \]
\[ \leq \left( \tilde{K}' \right)^{\frac{1}{2}} \lambda^{\frac{1}{2}} \frac{c_{d,k,m}^{d/2}}{\sigma} \left( \frac{k + \frac{1}{e}}{\lambda^{\frac{1}{2}}/\log \lambda} \right) \|f\|_{L^2(\omega)}. \]

It provides a new positive constant, still denoting by \( \tilde{K}' \geq 1 \) such that for all \( \lambda \geq \max(\lambda_0, e) \) and \( f \in \mathcal{E}_{\lambda}^{k,m}, \)
\[ \|f\|_{L^2(\mathbb{R}^d)} \leq \left( \tilde{K}' \right)^{\frac{1}{2}} \lambda^{\frac{1}{2}} \frac{c_{d,k,m}^{d/2}}{\sigma} \left( \frac{k + \frac{1}{e}}{\lambda^{\frac{1}{2}}/\log \lambda} \right) \|f\|_{L^2(\omega)}. \]

By using the very same lines as in the previous case, we deduce that for all \( \lambda > 0 \) and \( f \in \mathcal{E}_{\lambda}^{k,m}, \)
\[ \|f\|_{L^2(\mathbb{R}^d)} \leq \tilde{K} \left( \frac{K}{\gamma} \right) \lambda^{\frac{1}{2}} \|f\|_{L^2(\omega)}, \]
with \( \tilde{K} = \max(C_{k,m,\lambda_0}(\omega), 1) \). This ends the proof of Theorem 2.1.
5. Null-controllability for evolution equations associated to anisotropic Shubin operators

This section is devoted to the proof of the null-controllability result given by Theorem 2.5.

5.1. Proof of the Theorem 2.5

Let $k, m \geq 1$, $s > 0$ and $\omega \subset \mathbb{R}^d$ be a measurable subset of positive Lebesgue measure $|\omega| > 0$. Thanks to the Hilbert Uniqueness Method, the null-controllability of the evolution equation

\[
\begin{aligned}
\partial_t f(t, x) + ((-\Delta)^m + |x|^{2k})^s f(t, x) &= u(t, x) \mathbb{1}_\omega(x), \quad x \in \mathbb{R}^d, \quad t > 0, \\
\|f\|_{t=0} &= f_0 \in L^2(\mathbb{R}^d),
\end{aligned}
\]

is equivalent to the final state observability of the adjoint system

\[
\begin{aligned}
\partial_t g(t, x) + ((-\Delta)^m + |x|^{2k})^s g(t, x) &= 0, \quad x \in \mathbb{R}^d, \quad t > 0, \\
g|_{t=0} &= g_0 \in L^2(\mathbb{R}^d),
\end{aligned}
\]

Let us assume that $s > \frac{1}{2}(\frac{1}{k} + \frac{1}{m})$. Thanks to Theorem 2.1, there exists a positive constant $K = K(\omega, d, k, m) \geq 1$ such that

\[\forall \lambda \geq 1, \forall f \in \mathcal{E}_\lambda^{k,m}, \quad \|f\|_{L^2(\mathbb{R}^d)} \leq K e^{K\lambda \frac{1}{2} \left(\frac{1}{k} + \frac{1}{m}\right)} \|\lambda\| f\|_{L^2(\omega)} \] .

This ends the proof of Theorem 2.5.

6. Appendix

This appendix is devoted to recall basic properties about Gelfand-Shilov spaces and to present technical lemmas used in the proof of Proposition 4.1.

6.1. Gelfand-Shilov regularity. We refer the reader to the works [8, 9, 16, 18] and the references herein for extensive expositions of the Gelfand-Shilov regularity theory. The Gelfand-Shilov spaces $S^\mu_\nu(\mathbb{R}^d)$, with $\mu, \nu > 0$, $\mu + \nu \geq 1$, are defined as the spaces of smooth functions $f \in C^\infty(\mathbb{R}^d)$ satisfying the estimates

\[\exists A, C > 0, \quad |D_x^\alpha f(x)| \leq C A^{[\alpha]}(\alpha!)^{\mu} e^{\frac{1}{4} |x|^{2\nu}}, \quad x \in \mathbb{R}^d, \quad \alpha \in \mathbb{N}^d,
\]

or, equivalently

\[\exists A, C > 0, \quad \sup_{x \in \mathbb{R}^d} |x^\beta D_x^\alpha f(x)| \leq C A^{[\alpha]+[\beta]}(\alpha!)^{\mu} (\beta!)^\nu, \quad \alpha, \beta \in \mathbb{N}^d.
\]
with \( \alpha! = (\alpha_1! \ldots (\alpha_d!) \text{ if } \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d \). These Gelfand-Shilov spaces \( S^\mu_\nu(\mathbb{R}^d) \) may also be characterized as the spaces of Schwartz functions \( f \in \mathcal{S}(\mathbb{R}^d) \) satisfying the estimates

\[
\exists C > 0, \varepsilon > 0, \quad |f(x)| \leq C e^{-\varepsilon|x|^{\frac{1}{\nu}}}, \quad x \in \mathbb{R}^d; \quad |\hat{f}(\xi)| \leq C e^{-\varepsilon|\xi|^{\frac{1}{\nu}}}, \quad \xi \in \mathbb{R}^d.
\]

In particular, we notice that Hermite functions belong to the symmetric Gelfand-Shilov space \( S^1_1(\mathbb{R}^d) \). More generally, the symmetric Gelfand-Shilov spaces \( S^\mu_\nu(\mathbb{R}^d) \), with \( \mu \geq 1/2 \), can be nicely characterized through the decomposition into the orthonormal basis of eigenfunctions of

\[
\lambda
\]

where \( \lambda \) is a rational number, the Gelfand-Shilov space \( S^\mu_\nu(\mathbb{R}^d) \) can also be nicely characterized through the decomposition into the orthonormal basis of eigenfunctions of a class of anisotropic Shubin operators, whose basic model is the operator

\[
\mathcal{H}_{k,m} = (-\Delta_x + |x|^2)^m, \quad x \in \mathbb{R}^d,
\]

with \( k, m \geq 1 \) two positive integers. Let \( (\psi_n^{k,m})_{n \geq 1} \) be an orthonormal basis of \( L^2(\mathbb{R}^d) \) composed of eigenfunctions of the above operator \( \mathcal{H}_{k,m} \). Given a real number \( t \geq 1 \), the Gelfand-Shilov space \( S^\mu_\nu(\mathbb{R}^d) \), with

\[
\mu = \frac{kt}{k + m} \quad \text{and} \quad \nu = \frac{mt}{k + m},
\]

are characterized in the following way, thanks to the result \([5, \text{Theorem 1.4}]\) by Cappiello, Gramchev, Pilipović and Rodino,

\[
f \in S^\mu_\nu(\mathbb{R}^d) \iff \exists \varepsilon > 0, \quad \sum_{n=1}^{+\infty} |\langle f, \psi_n^{k,m} \rangle_{L^2(\mathbb{R}^d)}|^2 e^{-\varepsilon|n|^{\frac{k+m}{k+m}}} < +\infty,
\]

where \( \lambda_n > 0 \) is the eigenvalue associated to the eigenfunction \( \psi_n^{k,m} \in L^2(\mathbb{R}^d) \) for all \( n \geq 1 \). Such a characterization in the case when \( \mu/\nu \notin \mathbb{Q} \) has not been found yet. We end this section by proving two technical lemmas:

**Lemma 6.1.** Let \( \mu, \nu > 0 \) such that \( \mu + \nu \geq 1 \), \( C > 0 \) and \( A \geq 1 \). If \( f \in S^\mu_\nu(\mathbb{R}^d) \) satisfies

\[
(6.1) \quad \forall \alpha \in \mathbb{N}^d, \forall \beta \in \mathbb{N}^d, \quad \|x^\alpha \partial_x^\beta f\|_{L^2(\mathbb{R}^d)} \leq CA^{|\alpha|+|\beta|}(\alpha!)^\nu(\beta!)^\mu,
\]

then, it satisfies

\[
(6.2) \quad \forall p \in \mathbb{N}, \forall \beta \in \mathbb{N}^d, \quad \|x^p \partial_\beta^\beta f\|_{L^2(\mathbb{R}^d)} \leq C(d+1)^2 A^{p+|\beta|}(p!)^\nu(|\beta|!)^\mu.
\]

**Proof.** Let \( f \in S^\mu_\nu(\mathbb{R}^d) \) satisfying the estimates \((6.1)\). By using Newton formula, we obtain that for all \( p \in \mathbb{N}, \beta \in \mathbb{N}^d \),

\[
(6.2) \quad \|x^p \partial_\beta^\beta f\|_{L^2(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} \left(1 + \sum_{i=1}^{d} x_i^2\right)^p |\partial_\beta^\beta f(x)|^2 dx = \int_{\mathbb{R}^d} \sum_{\gamma \in \mathbb{N}^{d+1}, |\gamma|=p} \frac{p!}{\gamma!} x_1^{2|\gamma|} |\partial_\beta^\gamma f(x)|^2 dx = \sum_{\gamma \in \mathbb{N}^{d+1}, |\gamma|=p} \frac{p!}{\gamma!} \|x^{\gamma} \partial_\beta^\gamma f\|_{L^2(\mathbb{R}^d)}^2.
\]
where we denote $\tilde{r} = (\gamma_1, \ldots, \gamma_d) \in \mathbb{N}^d$ if $\gamma = (\gamma_1, \ldots, \gamma_{d+1}) \in \mathbb{N}^{d+1}$. Since for all $\alpha \in \mathbb{N}^d$, $\alpha ! \leq (|\alpha|)!$, it follows from (6.1) and (6.2) that

$$\| \langle x \rangle^{p \partial_x^\beta} f \|^2_{L^2(\mathbb{R}^d)} \leq C^2 \sum_{\gamma \in \mathbb{N}^{d+1}} \frac{p!}{|\gamma|!} A^{2(|\tilde{r}|+|\beta|)}(|\tilde{r}|!)^{2\nu}(|\beta|!)^{2\mu},$$

since

$$\sum_{\gamma \in \mathbb{N}^{d+1}} \frac{p!}{|\gamma|!} = (d+1)^p.$$

Lemma 6.2. Let $\mu, \nu > 0$ such that $\mu + \nu \geq 1$, $0 \leq \delta \leq 1$, $C > 0$ and $A \geq 1$. If $f \in S^0_p(\mathbb{R}^d)$ satisfies

$$\forall p \in \mathbb{N}, \forall \beta \in \mathbb{N}^d, \quad \| \langle x \rangle^p \partial_x^\beta f \|^2_{L^2(\mathbb{R}^d)} \leq CA^{p+|\beta|}(p!)^\mu(|\beta|!)^\mu,$$

then, it satisfies

$$\forall p \in \mathbb{N}, \forall \beta \in \mathbb{N}^d, \quad \| \langle x \rangle^p \partial_x^\beta f \|^2_{L^2(\mathbb{R}^d)} \leq C(8^\nu \nu A)^{p+|\beta|}(p!)^\mu(|\beta|!)^\mu.$$

Proof. Let $f \in S^0_p(\mathbb{R}^d)$ satisfying the estimates (6.3). It follows from Hölder inequality that for all $r \in (0, +\infty) \setminus \mathbb{N}$ and $\beta \in \mathbb{N}^d$,

(6.4)  $$\| \langle x \rangle^r \partial_x^\beta f \|^2_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \left( \langle x \rangle^{r[1]} \| \partial_x^\beta f(x) \|^2 \right)^{1-r} \left( \langle x \rangle^{2(r+1)} \| \partial_x^\beta f(x) \|^2 \right)^{r} dx \leq \frac{\| \langle x \rangle^{r[1]} \partial_x^\beta f \|^2_{L^2(\mathbb{R}^d)}}{\| \langle x \rangle^{r[1]} \|^2_{L^2(\mathbb{R}^d)}} \| \langle x \rangle^{(r+1)} \partial_x^\beta f \|^2_{L^2(\mathbb{R}^d)}.$$

where $\cdot !$ denotes the floor function. Since the above inequality clearly holds for $r \in \mathbb{N}$, we deduce from (6.3) and (6.4) that for all $r \geq 0$ and $\beta \in \mathbb{N}^d$,

(6.5)  $$\| \langle x \rangle^r \partial_x^\beta f \|^2_{L^2(\mathbb{R}^d)} \leq C A^{r+|\beta|} \left( \langle r \rangle! \right)^{(r+1)} (\langle r \rangle + 1)^{\nu} \langle \beta \rangle!^\mu \leq C A^{r+|\beta|} \left( \langle r \rangle + 1 \right)^{\nu} \langle \beta \rangle!^\mu \leq C A^{r+|\beta|} \left( \langle r \rangle + 1 \right)^{1+\mu} \langle \beta \rangle!^\mu \leq C A^{r+|\beta|} \left( \langle r \rangle + 1 \right)^{\nu} \langle \beta \rangle!^\mu.$$

It follows from (6.5) that for all $p \in \mathbb{N}^*$, $\beta \in \mathbb{N}^d$,

(6.6)  $$\| \langle x \rangle^p \partial_x^\beta f \|^2_{L^2(\mathbb{R}^d)} \leq C A^{p+|\beta|} \langle p+1 \rangle^\nu \langle \beta \rangle!^\mu \leq C A^{p+|\beta|} (2p)^{(p+1)\nu} \langle \beta \rangle!^\mu \leq C (2^\nu \nu A)^{p+|\beta|}(p!)^\mu \langle \beta \rangle!^\mu \leq C (8^\nu \nu A)^{p+|\beta|}(p!)^\mu \langle \beta \rangle!^\mu,$$

since for all positive integer $p \geq 1$,

$$p+1 \leq 2p \leq 2^p \quad \text{and} \quad p^p \leq e^p p!.$$  

Notice that from (6.3), since $8^\nu \nu \geq 1$, estimates (6.6) also hold for $p = 0$. This ends the proof of Lemma 6.2. \hfill \Box

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