Research article

Idealization of functionally graded porous tubes for buckling modelling of bone structures

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1. Introduction

Bone structures are hierarchical composites consisting of a dense stiff external cortical bone (the compact bone), to a porous or cancellous bone (the sponge bone). The bone structures can have ability to withstand bending, buckling and any types of fracture, depending on several effects such as bone-structural organization, ageing pathologies and oncology [1, 2, 3]. Moreover, the mechanical properties in terms of bone stiffness and strength also depend on the geometry of microstructure, density associated with porous size and the stiffness of the bone tissue [4, 5, 6, 7]. The useful methods for obtaining the bone properties are the non-destructive imaging techniques such as X-ray and micro-CT. The relations of X-ray and CT structural parameters can be used further with the experimental testing or numerical simulation. There exist other methodologies, ultra-sound and indentation techniques, to gain bone properties available in the literature [8, 9, 10]. In the part of sponge area of the bone, the properties and structural analysis are idealized by using the models of open and close cells with high porosities [11, 12].

The concept of composite materials can be utilized and considered as a useful background for developing modellings of animal and human bones, tree rings and other natural structures, especially functionally graded materials (FGMs). The FGMs are novel class of composites in which whose material constituents are assumed to vary continuously across the desired direction [12, 13, 14, 15, 16]. The concept of FGMs has been applied to construct the material formulations for porous materials in different forms of porous distributions. In structural analysis due to various possible scenarios of the porous structures being under the action of loadings, Chen et al. [17] gave the explorations on bending and buckling of functionally graded (FG) porous beams using the approximate approach and this group of researchers also extended their research work for free and forced vibration of such beams [18]. Other works related to FG porous beams were found in Refs. [19, 20] for different mechanical problems, using for evaluating deformation of the beams. Some studies focused on sandwich-like structures which have high stiffness or high density materials at the covering layers (top and bottom faces) to protect the softer materials at the core. Recently, the FGM concept also plays an important role at this state to produce sand-
wich structures with FG core. Chen et al. [21] investigated non-linear vibration of sandwich beams with FG porous core using Timoshenko beam theory for taking into account shear deformation effect. Then, Srikaran et al. [22] applied the third order shear deformation theory (TSST) to deal with non-linear bending of sandwich beams with various patterns of FG porous cores. Wang et al. [23] proposed the investigation on transient response of sandwich beams with FG porous core under the action of non-uniformly distributed moving mass. This investigation found that porous core can reduce the flexural resistance of the beams under dynamic loading. Li et al. [24] proposed the discussion and investigation on the use of artificial porous materials for human bone repair. Ramtani applied beam theories, Timoshenko and Euler-Bernoulli hypothesis, to govern behaviour of bone-column buckling [25].

According to above literature review, the development of FG porous materials has been broadly applied to various scientific and engineering fields. The idealization of bone structures can be characterized by assuming that the outer ring is full of high density and strong material (without porosities) and the inner tube surrounded by the outer ring and inside area of bone's marrow is made of FG porous material. The refined beam theory is utilized in this study for describing buckling behaviour of bone-like structures. Several parametric studies such as bone properties and bone-geometry ratio, which have significant impact of buckling results, are brought to consider in this investigation. The organization of this study can be outlined as follows: Section 1 is for introduction and literature review, Section 2 includes mechanical properties of bone structures, Section 3 illustrates the details of mathematical modelling of buckling of bone structures, Section 4 is given for results and discussions and the last one is the concluding section.

2. Properties of bone structures

In this study, there are some assumptions used for idealizing the formulations for bone-like structures as follows:

- The bone structures are composed of three main parts including compact bone, sponge bone and bone’s marrow.
- The stress concentration and interfacial stresses between circular layers are discarded.
- The complex shape of the bone-like structures is simplified and idealized to general geometry of circle in each part of the bone components.
- The compressive force (F) is acting through the neutral axis of the idealized bone structures without carrying any bending moments.

As stated in the assumptions, the bone structures as shown in Fig. 1 have three main parts [26] that are the part of compact bone located at the outside ring ($R_1 < r < R_2$), the sponge bone located at the middle ring ($R_2 < r < R_3$) and the bone’s marrow is of the inside circular area with the radius of $R_3$.

The elastic modulus of the compact bone, which is high density zone, is represented by the constant value of $E_C = E_1$. The softest part is of the bone’s marrow that has much lower modulus than compact bone, defined by the constant value of $E_M = E_1$. However, for the middle ring which is the area for the sponge bone, the elastic modulus is not a constant value but it is varied according to idea of FGMS with the porous coefficient ($e_p$) in the radial direction. The range of the porous coefficient is 0 ≤ $e_p$ ≤ 1 if $e_p = 0$ means that there is no porosity. Notice that $e_p$ is related to the mass density coefficient ($e_m$) of $e_m = 1 - (1 - e_p)^2$ in which $e_m$ is defined as $e_m = 1 - \rho_o/\rho_I$, where $\rho_o$ and $\rho_I$ are the minimum and maximum values of the mass density, respectively [18]. The equation used for predicting the modulus of sponge bone can be obtained as follows:

$$E_s(r) = E_s[1 - e_p \cos(\zeta + \pi/4)]$$

$$\zeta = \frac{\pi}{2(R_2 - R_3)} \left( r - \frac{R_2 + R_3}{2} \right),$$  

where $E_s$ is the elastic modulus of sponge bone. Based on Eq. (1), at the sponge-bone zone, there is no pore at $R_3$ and the number of pores increases functionally to be maximum at $R_2$. Based on literature survey [27, 28], $E_s$ is recognized as the Poisson’s ratio ($\nu$) of the compact bone ($\nu_C = \nu_1$), sponge bone ($\nu_S = \nu_2$) and bone’s marrow ($\nu_M = \nu_3$) is constant in which the value of the parameter for each type of bones will be given in Section 4.

3. Mathematical modelling

As shown in Fig. 1, the bone geometry is assumed to be in form of circular-cylindrical beams or columns composing of homogenous bone at outside ring and porous bone at the inner tube. The x-axis of Cartesian coordinate system is lined along the neutral axis of the cylindrical beams and y- and z-axes of cross section are perpendicular to the x-axis.

The displacements in relation to the coordinates (x, y, z) are defined as $(u, v, w)$, respectively. The relationship between the Cartesian and polar coordinate systems can be expressed as:

$$y = r \cos \theta, z = r \sin \theta$$

in which $r = \sqrt{x^2 + z^2}$. Thus, the displacements based on the polar coordinate are ($u_r$, $u_\theta$, $u_z$) according to ($r$, $\theta$), respectively. $u_r$ is discarded due to no consideration of warping deformation and $u_\theta$ is of the following form:

$$u_\theta = v \cos \theta + w \sin \theta.$$  

(3)

For beam structures of circular cross section [29], the deformation in y-axis (v) is relatively small and it is independent on the variation of distance in x direction. Therefore, we consider only for the longitudinal (u) and transverse (w) displacements which can be written as [29]:

$$u(x, y, z, t) = z \psi - f(y, z) \left( \frac{\partial w}{\partial x} + \psi \right),$$

(4a)

$$w(x, z, t) = u_\theta(x, t),$$

(4b)

where $f(y, z) = \frac{z^2 \psi}{2R_2^2}$, $\psi$ is the rotation of the cross section at the neutral axis, $t$ is time if the dynamic behaviour of the structures is considered. By using the main displacement field of Eqs. (3 and 4a-4b), the normal strain ($\varepsilon_{xx}$) and shear strain components ($\gamma_{xy}$, $\gamma_{xz}$, $\gamma_{yz}$) can be written as follows [29]:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = 2z \frac{\partial f}{\partial x} - f^{(3)}$$

(5)

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial w}{\partial x} = \frac{\partial f}{\partial y}$$

(6)

$$\gamma_{xz} = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = \frac{\partial f}{\partial z}$$

(7)

$$\gamma_{yz} = \frac{\partial u}{\partial r} + \frac{\partial w}{\partial x} = \frac{\partial f}{\partial r}$$

(8)
where

\[
\left\{ \begin{array}{l}
\varepsilon_{xx}^{(1)}
\varepsilon_{yy}^{(1)}
\gamma_{xy}^{(1)}
\end{array} \right\} = \left\{ \begin{array}{l}
\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial v}{\partial z} \\
\frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} + \frac{\partial v}{\partial x} \\
\varepsilon_{xy}^{(1)} + \frac{1}{2} \varepsilon_{yy}^{(1)} - \frac{1}{2} \varepsilon_{xx}^{(1)}
\end{array} \right\}
\]

\[
\{ \varepsilon_{xx}^{(2)}, \varepsilon_{yy}^{(2)}, \gamma_{xy}^{(2)}, \gamma_{xz}^{(2)} \} = \left\{ \begin{array}{l}
\frac{\partial u_0}{\partial x} + \psi \\
\frac{\partial u_0}{\partial y} + \psi \\
\frac{\partial u_0}{\partial z} + \gamma
\end{array} \right\}
\]

(9a)

(9b)

Then, the constitutive relationship between the normal stress \(\sigma_{xx}\) and shear stress components \(\tau_{xy}, \tau_{xz}, \tau_{yy}\) with all above strain components are expressed as [29]:

\[
\begin{bmatrix}
\sigma_{xx} \\
\tau_{xy} \\
\tau_{xz} \\
\tau_{yy} \\
\end{bmatrix} = \begin{bmatrix}
Q_{x1}^{(r)}(r) & 0 & 0 & 0 \\
0 & Q_{y1}^{(r)}(r) & 0 & 0 \\
0 & 0 & Q_{yy}^{(r)}(r) & 0 \\
0 & 0 & 0 & Q_{zz}^{(r)}(r) \\
\end{bmatrix} \begin{bmatrix}
\varepsilon_{xx} \\
\gamma_{xy} \\
\gamma_{xz} \\
\gamma_{yz} \\
\end{bmatrix}
\]

(10)

in which \(Q_{x1}^{(r)}(r) = E_x(r)\) and \(Q_{y1}^{(r)}(r) = Q_{yy}^{(r)}(r) = G_y(r) = G_{xy}(r) = \frac{E_x(r)}{2(1+v_y)}\).

To construct the governing equation for buckling analysis, the strain energy is required which can be expressed as [29]:

\[
U_s = \frac{1}{2} \int_A \left( \sigma_{xx} \varepsilon_{xx} + \tau_{xy} \gamma_{xy} + \tau_{xz} \gamma_{xz} + \tau_{yy} \gamma_{yz} \right) dA dx
\]

(11)

Substituting all stress and strain components into Eq. (11), one can obtain another form of the strain energy as:

\[
U_s = \frac{1}{2} \int_0^L \left[ E_{00} \left( \frac{\partial \psi}{\partial x} \right)^2 - 2E_{11} \frac{\partial \psi}{\partial x} \frac{\partial \delta \psi}{\partial x} + F_{11} \left( \frac{\partial^2 \psi}{\partial x^2} \right)^2 \right] + G_{00} \left( \frac{\partial \psi}{\partial x} \right)^2 dA dx
\]

(12)

The details of the coefficients in relation to the strength of the structures in Eq. (12) are:

\[
E_{00} = B_{11} - 2D_{11} + F_{11}, \\
E_{11} = D_{11} - F_{11}, \\
G_{00} = A_{55} - 2B_{55} + D_{55} + D_{66} + D_{77}.
\]

(13a)\-(13c)

The coefficients in Eqs. (13a-13c) can be obtained from the stiffness components:

\[
\left\{ \begin{array}{l}
B_{11}, D_{11}, F_{11} \\
A_{55}, B_{55}, D_{55} \\
D_{66} \\
D_{77}
\end{array} \right\} = \sum_{n=1}^{R_a} \int_{R_{a-1}}^{R_a} Q_{x1}^{(r)}(r) \left\{ c_1 r^3, c_2 r^5, c_3 r^7 \right\} dr
\]

(14a)\-(14d)

\[
D_{66} = \sum_{n=1}^{R_a} \int \frac{R_c}{2} c_2 r^5 Q_{x1}^{(r)}(r) dr
\]

\[
D_{77} = \sum_{n=1}^{R_a} \int \frac{R_c}{2} c_3 r^7 Q_{x1}^{(r)}(r) dr
\]

where \(R_0 = 0, R_1 = R, R_2 = R\) and \(R_3 = R\) and \(c_1 = \pi, c_2 = \frac{\pi}{2 R_c}\) and \(c_3 = \frac{\pi}{2 R_c}\).

To consider buckling analysis of the bone structures, the potential energy due to the compressive force \(F\) is of the form [17]:

\[
U_F = -\frac{1}{2} F \int_0^L \left( \frac{\partial \psi}{\partial x} \right)^2 dx.
\]

(15)

The total energy of the structures under the compressive force can be established as:

\[
\Pi = U_s + U_F
\]

(16)

By using the total energy of Eq. (16), we can generate the equations for governing the buckling of bone structures using the following procedure of solution.

The governing equations can be obtained from the principle of stationary of the system associated with the variation of \(\delta \psi\) and \(\delta \psi\). The expression of the principle is [19]:

\[
\delta \Pi = \delta U_s + \delta U_F.
\]

(17)

Then we have,

\[
\begin{align*}
0 &= \int_0^L \left[ E_{00} \frac{\partial \psi}{\partial x} \frac{\partial \delta \psi}{\partial x} - E_{11} \frac{\partial \psi}{\partial x} \frac{\partial^2 \delta \psi}{\partial x^2} - E_{11} \frac{\partial^2 \delta \psi}{\partial x^2} \frac{\partial \psi}{\partial x} \\
&\quad + F_{11} \frac{\partial^2 \delta \psi}{\partial x^2} \frac{\partial^2 \psi}{\partial x^2} \right] dx
\end{align*}
\]

(18)

Integrating by parts to relieve the virtual displacements, \(\delta \psi\) and \(\delta \psi\), in Eq. (18), and collecting only the terms in the integrals according to the mentioned-virtual displacements result in the following form:

\[
\begin{align*}
0 &= \int_0^L \left[ -E_{00} \frac{\partial^2 \delta \psi}{\partial x^2} + E_{11} \frac{\partial^2 \psi}{\partial x^2} + G_{00} \frac{\partial \psi}{\partial x} + G_{00} \frac{\partial \delta \psi}{\partial x} \\
&\quad - G_{00} \frac{\partial \psi}{\partial x} \frac{\partial \delta \psi}{\partial x} + \frac{F_{11} \partial^2 \psi}{\partial x^2} \right] dx
\end{align*}
\]

(19)

From Eq. (19), we will have the two governing equations in relation the variations of \(\delta \psi\) and \(\delta \psi\) as:

\[
\begin{align*}
&- E_{00} \frac{\partial^2 \psi}{\partial x^2} + E_{11} \frac{\partial^2 \psi}{\partial x^2} + G_{00} \frac{\partial \psi}{\partial x} + G_{00} \frac{\partial \psi}{\partial x} = 0, \\
&- E_{11} \frac{\partial^2 \psi}{\partial x^2} - G_{00} \frac{\partial \psi}{\partial x} = 0
\end{align*}
\]

(20a)\-(20b)

To solve the governing equations in Eqs. (20a-20b), we can assume that the bone structures are jointed at both ends in which the conditions of simply supported ends are applied for this problem. Thus the admissible displacements for simply supported boundary condition are used in this study, which can be expressed as [31]:

\[
\psi(x) = \sum_{n=1}^{\infty} \Psi_m \cos(\alpha_m x)
\]

(21a)

\[
\psi'_0(x) = \sum_{n=1}^{\infty} W_m \sin(\alpha_m x)
\]

(21b)

where \(\alpha_m = m \pi / L\). Substituting the admissible displacements of Eqs. (21a-21b) into the governing equations of Eqs. (20a-20b) results in the following equations:

\[
\begin{align*}
&\sum_{n=1}^{\infty} \left[ (E_{00} \alpha_m^2 + G_{00}) \Psi_m - (E_{11} \alpha_m^2 - G_{00} \alpha_m) W_m \right] \cos \alpha_m x = 0
\end{align*}
\]

(22a)

\[
\begin{align*}
&\sum_{n=1}^{\infty} \left[ - (E_{11} \alpha_m^2 - G_{00} \alpha_m) \Psi_m + (F_{11} \alpha_m^2 + G_{00} \alpha_m^2) W_m - F \alpha_m^2 W_m \right] \sin \alpha_m x = 0
\end{align*}
\]

(22b)

The expressions in Eqs. (22a-22b) can be rearranged into the matrix form as follows:
Table 1. Buckling forces of compact-bone column for various modes.

| $L/R_o$ | Source | Modes | 1 | 2 | 3 | 4 | 5 |
|---------|--------|-------|---|---|---|---|---|
| 35      | Present| 11.7156 | 46.2242 | 101.6975 | 175.3532 | 263.7879 |
| 31      | [31]   | 11.7697 | 47.0789 | 105.9276 | 188.3157 | 294.2432 |
| 32      | Present| 16.7511 | 65.9169 | 144.4087 | 247.6069 | 370.0023 |
| 30      | [31]   | 16.8437 | 67.3750 | 151.5937 | 269.5000 | 421.0937 |
| 28      | Present| 21.6685 | 85.0784 | 185.7304 | 316.9923 | 471.1192 |
| 25      | [31]   | 21.8049 | 87.2195 | 196.2439 | 348.8780 | 545.1219 |
| 27      | Present| 28.5286 | 111.7115 | 242.8338 | 412.1668 | 608.6429 |
| 20      | [31]   | 28.7347 | 114.9388 | 258.6122 | 459.7551 | 718.3073 |
| 25      | Present| 44.8086 | 174.5340 | 376.2683 | 631.9400 | 921.9687 |
| 20      | [31]   | 45.2146 | 180.8584 | 406.9313 | 723.4335 | 1130.3648 |
| 30      | Present| 108.8462 | 417.8933 | 881.3173 | 1404.5762 | 2040.0965 |
| 20      | [31]   | 110.3872 | 441.5488 | 993.4847 | 1766.1950 | 2759.6797 |

Table 2. Dimensionless buckling force of bones composing of compact, sponge, and marrow bone components ($L/R_o = 28$).

| $R_o/R_c$ $(R_c - R_o)/R_c$ | $e_o$ | 0.50 | 0.60 | 0.70 | 0.80 | 0.90 |
|------------------------------|-------|------|------|------|------|------|
| 0.70                         | 0.00  | 0.74638 | 0.74638 | 0.74638 | 0.74638 | 0.74638 |
| 0.05                         | 0.69952 | 0.69687 | 0.69422 | 0.69158 | 0.68893 |
| 0.10                         | 0.64349 | 0.63778 | 0.63206 | 0.62634 | 0.62062 |
| 0.15                         | 0.57715 | 0.56791 | 0.55867 | 0.54942 | 0.54016 |
| 0.75                         | 0.00  | 0.69999 | 0.69999 | 0.69999 | 0.69999 |
| 0.05                         | 0.61308 | 0.60986 | 0.60664 | 0.60342 | 0.60019 |
| 0.10                         | 0.54941 | 0.53890 | 0.53199 | 0.52507 | 0.51815 |
| 0.15                         | 0.46697 | 0.45587 | 0.44476 | 0.43364 | 0.42251 |
| 0.80                         | 0.00  | 0.57727 | 0.57727 | 0.57727 | 0.57727 |
| 0.05                         | 0.50902 | 0.50516 | 0.50129 | 0.49743 | 0.49356 |
| 0.10                         | 0.42917 | 0.42092 | 0.41267 | 0.40441 | 0.39615 |
| 0.15                         | 0.33639 | 0.32520 | 0.31000 | 0.29679 | 0.28357 |
| 0.85                         | 0.00  | 0.46616 | 0.46616 | 0.46616 | 0.46616 |
| 0.05                         | 0.38524 | 0.38065 | 0.37607 | 0.37148 | 0.36694 |
| 0.10                         | 0.29137 | 0.28162 | 0.27188 | 0.26213 | 0.25227 |
| 0.15                         | 0.18305 | 0.16752 | 0.15200 | 0.13647 | 0.12093 |

where

$$
\begin{bmatrix}
K_{11}^S & K_{12}^S & \Psi_m m \\
K_{21}^S & K_{22}^S & W_m
\end{bmatrix} - \frac{E_c}{E_c} \begin{bmatrix}
K_{11}^E & K_{12}^E & \Psi_m m \\
K_{21}^E & K_{22}^E & W_m
\end{bmatrix} = 0
$$

(23)

4. Results and discussion

This section is given to show the analytical results of the proposed modelling for buckling of bone in form of composite beams with circular cross-section or columns. The significant properties of bones in terms of the elastic modulus and Poisson’s ratio are available in the literature. For compact bone, we can obtain the properties from Ref. [27, 30] that can be used for metaphysis, epiphysis and diaphysis, because they are of the same structure. Thus the properties of the compact bone are, $E_c = 18.6$ GPa, $\nu_c = 0.3$. Similarly, for sponge bone of any parts in bone structures, the properties are $E_s = 10.4$ GPa, $\nu_s = 0.3$, Ref. [27, 30]. Then, in case of bone’s marrow, we obtain the properties of Ref. [28, 32] which are $E_m = 1$ kPa, $\nu_m = 0.5$. The bone’s length of $L = 35$ cm is used throughout this study. Now, we can proceed to find out buckling analysis of bone structures with above properties and parameters.

To verify our modelling, we begin with the comparison between the results obtained from the proposed modelling with those computed by using the simple modelling of Ref. [31]. The simple modelling based on Euler-Bernoulli hypothesis of Ref. [31] is limited to the case of isotropic column (full of compact bone) only with high slenderness; while, our modelling is effective for any composites (compact bone covering sponge bone and bone’s marrow) and arbitrary values of slenderness. Owing to this reason, Table 1 shows the comparison between the results of two modellings to carry out the buckling forces (in unit of kN) from 1st to 5th modes of buckling in which the 1st mode is the critical buckling force. In this table, it is assumed that the column is full of compact bone. According to the comparison, an agreement of the results is observed, especially the cases of high slenderness of the column ($L/R_o \geq 30$).

After validation of our modelling for accuracy in Table 1, we further investigate the following analysis with the bone structures composing of compact bone at the outer ring covering the inside tube made of sponge bone at the middle layer and bone’s marrow at the inside area of the cross-section. Therefore, the critical buckling forces of the bones composing of three main parts are tabulated in Table 2 for various values of $R_o/R_c$ and $(R_o - R_j)/R_c$ ratios and the porous coefficient ($e_o$) ranging from $e_o = 0.5 \rightarrow 0.9$. For convenient, all of the following investigations as well as the results in this table are presented in the dimensionless form of $F_{cr} = F_{cr}^L L^2/x^2 E_c I$ where $I = x R_o^4/4$. From numerical analy-
sis, it is clearly seen that the dimensionless buckling force is reduced as the increase of the porous coefficient and the area of sponge bone indicating by \((R_p - R_i)/R_o\) ratio. Increasing the \(R_i/R_o\) ratio also causes the significant reduction of the buckling force results for every value of \((R_p - R_i)/R_o\) ratio.

In what follows, Table 3 shows the results of the bone columns with the variation of \(L/R_o\) ratio. It is indicated that the reduction of the buckling force is found when the ratio is increased for every value of \(e_o\). The differences between the critical buckling forces of the bones carrying different values of \(e_o\) and \(L/R_o\) ratio are illustrated graphically in Figs. 2a-2c. It can be observed that the considerable changes of the results are very large in the range of \(L/R_o = 5\) to \(15\) before changing slightly after that range of \(L/R_o\) ratio.

To clearly understand the variation of buckling force of the bones due to the changes of above considered parameters, it is worth to make an illustration in 3-D plot of the dimensionless critical buckling force against the changes in values of \(L/R_o\) and \(R_i/R_o\) ratios with several values of \((R_p - R_i)/R_o\), as shown in Fig. 3.

Additionally, all above investigations are of the fixed values of modulus of elasticity for \(E_C\) and \(E_S\). Now, we need to explore more investigations on the mechanical properties of the bone structures if the ratio of the modulus is varied. Fig. 4 shows the variations of the critical buckling force versus the changes in value of \(e_o\). The buckling force is very high if the bones have equivalent modulus of elasticity for compact and sponge bones \((E_C/E_S = 1.0)\) and it is decreased as the increase of the ratio. Moreover, the influence of the modulus ratio on the variation of the critical buckling force is also studied in Fig. 5 with the changes in value of \(L/R_o\).

5. Conclusion

This study presents the mathematical modelling for buckling of bone structures idealized from FG porous tubes at the area of sponge bone which is covered by the compact bone at the outer ring and the in-

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Table 3. Dimensionless buckling force of bones composing of compact, sponge, and marrow bone components \((R_i/R_o = 0.75, (R_p - R_i)/R_o = 0.05)\).

| \(L/R_o\) | 0.50  | 0.60  | 0.70  | 0.80  | 0.90  |
|-----------|-------|-------|-------|-------|-------|
| 35        | 0.61779| 0.61457| 0.61135| 0.60812| 0.60490|
| 32        | 0.61614| 0.61292| 0.60969| 0.60647| 0.60325|
| 30        | 0.61476| 0.61154| 0.60831| 0.60509| 0.60187|
| 28        | 0.61308| 0.60986| 0.60664| 0.60342| 0.60019|
| 25        | 0.60979| 0.60657| 0.60335| 0.60013| 0.59691|
| 20        | 0.60085| 0.59764| 0.59443| 0.59122| 0.58800|

---

Fig. 2. Dimensionless buckling force of bones composing of compact, sponge, and marrow bone components using \(R_i/R_o = 0.75, (R_p - R_i)/R_o = 0.05\) in Fig. 2(a), \((R_p - R_i)/R_o = 0.10\) in Fig. 2(b) and \((R_p - R_i)/R_o = 0.15\) in Fig. 2(c).

Fig. 3. Three dimensionless critical buckling force of bones composing of compact, sponge, and marrow bone components \((e_o = 0.70)\).
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Fig. 4. Dimensionless buckling force of bones composing of compact, sponge, and marrow bone components \((R_c/R_s = 0.75, (R_p - R_c)/R_s = 0.05, L/R_o = 28)\).

Fig. 5. Dimensionless buckling force of bones composing of compact, sponge, and marrow bone components \((R_c/R_s = 0.75, (R_p - R_c)/R_s = 0.05, c_p = 0.70)\).

side area is of bone’s marrow. The modelling is based on the refined theory including all necessary stress and strain components that have been neglected in the simple modellings of previous studies. The close form solutions are provided for illustrating several influences of bone strength, bone-geometry ratios and the porous coefficient on the buckling behaviour of bone structures. Based on the numerical exercises, it can be revealed that the critical buckling force is reduced as the increase of the porous coefficient and the area of sponge bone indicating by \((R_p - R_c)/R_s\) ratio. Additionally, \(R_c/R_s\) and \(L/R_o\) of the bones are significant parameters that have considerable impact on the buckling force of the bone. Reductions of the critical buckling are caused by the increase of the ratios for every value of the porous coefficient and the area of sponge bone.

Declarations

Author contribution statement

Nuttikarn Nokkaew: Performed the experiments; Contributed reagents, materials, analysis tools or data; Wrote the paper.

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The authors declare no conflict of interest.

Additional information

No additional information is available for this paper.

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