Quantum complex scalar field in the two-dimensional spacetime with closed timelike curves and a time-machine problem

Sushkov S V*
Department of Geometry, Kazan State Pedagogical Institute, Mezhlauk, 1, Kazan 420021, Tatarstan, RUSSIA

May 11, 2021

Abstract

It is considered the quantum complex scalar field which obeys the authomorphic condition in the two-dimensional spacetime with closed timelike curves and the chronology horizon. The renormalized stress-energy tensor is obtained. It is shown that the value of the stress-energy tensor is regular at the chronology horizon for specific authomorphic parameters. Thus the particular example of field configuration is given for which the Hawking’s chronology protection conjecture is violated.

1 Introduction

The important feature of general relativity is the possible existence of spacetimes with a nontrivial topological and causal global structure. In particular the spacetimes can exists which have closed timelike curves (CTC’s) in a bounded region of the spacetime, such as the wormhole model [1], the Misner universe [2], and the Gott spacetime of two moving infinite cosmic strings

*e-mail: sushkov@univex.kazan.su
In such a spacetime the chronology horizon exists. It separates chronal regions of spacetime, which are free of CTC’s, and nonchronal ones, which have CTC’s. The chronology horizon is a special type of Cauchy horizon (see for more details Ref.[4]). The spacetime with CTC’s is multiply connected.

In the spacetime with chronology horizon the creation of time machine is possible [1] because the causality will be violated for travellers using CTC’s. To prevent the violation of causality, Hawking has proposed the chronology protection conjecture which states that the laws of physics prevent the formation of CTC’s [4]. At present it seems that the best mechanism providing the chronology protection is the possible quantum instability of a chronology horizon. Note that the complete solution of the problem of chronology horizon stability is only possible in the framework of quantum gravity. However, while the theory of quantum gravity has not built, one can use the semiclassical approximation to examine the behaviour of quantum fields near the chronology horizon. The different cases of quantum scalar fields in various spacetimes with CTC’s and chronology horizons has been successfully studied in series papers [6-13]. The general result, which was obtained there, is that the renormalized vacuum stress-energy tensor of scalar field is diverged at the chronology horizon. This result is interpreted as pointing out to the quantum instability of the chronology horizon, that can prevent the formation of CTC’s.

We may ask a question: How general is the conclusion about divergent character of the vacuum stress-energy tensor near the chronology horizon? In order to answer to this question we must consider all kinds of matter fields. Previously the real nontwisted scalar field had been investigated. Now I want to consider a complex scalar field.

In this paper we will consider the quantum field theory of complex massless scalar field in two-dimensional model of spacetime with the chronology horizon.

In Sec.2 I give some basic ideas about a covering-space approach for studying the quantum field theory in multiply connected spacetimes [14-18], and show that the complex scalar field has to obey the generalized periodic (or authomorphic) conditions in these spacetimes. The Frolov’s model of two-dimensional locally static multiply connected spacetime [4] is given in Sec.3. The vacuum stress-energy tensor for a complex scalar field is computed in Sec.4, and it is analized in Sec.5. In Sec.6 main conclusions are discussed. The units $c = \hbar = G = 1$ are used through the paper.
2 Field theory in multiply connected spacetime

The covering-space method developed by Schulman, Dowker and Banach [14-18] is convenient to build a quantum field theory in a multiply connected spacetime. The main idea of this method is to regard a field theory on a multiply connected spacetime $M$ as a field theory on the universal covering space $\tilde{M}$ obeying certain conditions. Let us briefly consider some basic details of the covering-space approach.

Consider a multiply connected manifold $M$. Let $\tilde{M}$ is an universal (i.e., simply connected) covering space of $M$. $\Gamma$ is a discrete group of isometry on $\tilde{M}$ which is isomorphic to the fundamental group $\pi_1(M)$ on $M$. $\gamma$ is elements of $\Gamma$. The quotient $\tilde{M}/\Gamma$ coincides with the original spacetime $M$.

Now let us consider the free physical field $\phi$ described by the Lagrangian $L[\phi(X)]$. In order to quantize the field $\phi$ in $M$ we will use the covering-space method, i.e., we will assume that the field $\phi$ is defined on $\tilde{M}$ and obeyed a certain condition. This condition is obtained from demand that the Lagrangian is invariant under the symmetry transformations $\gamma$ of spacetime $\tilde{M}$:

$$\gamma L[\phi(X)] = L[\phi(\gamma X)] = L[\phi(X)].$$

(1)

The Lagrangian of free fields is quadratic in $\phi$ so the next field transformation follows from (1)

$$\phi(\gamma X) = a(\gamma) \phi(X)$$

(2)

where $a^2(\gamma) = 1$ and $a(\gamma_1 \gamma_2) = a(\gamma_1)a(\gamma_2)$ (the group property). (For the sake of simplicity we assume that $a(\gamma)$ does not depend on a point $X$, what means the global symmetry.) The fields $\phi$ on $\tilde{M}$ obeying Eq.(2) are known as authomorphic fields and the condition (2) is called the authomorphic one.

The Lagrangian of conformally coupled complex massless scalar field has the form

$$\mathcal{L} = -g^{\mu\nu} \phi_{,\mu} \phi^{*,\nu} - \xi R \phi^*,$$

(3)

where $R$ is a scalar curvature and the parameter $\xi$ is equal to $\frac{1}{6}$ or 0 for four- or two-dimensional cases correspondingly. The authomorphic condition [4]
for complex fields reads

$$\phi(\gamma X) = e^{2\pi i \alpha} \phi(X),$$

(4)

where $\alpha$ is an arbitrary real parameter, $0 \leq \alpha \leq 1/2$ (we will call it as an authomorphic parameter). There are two particular special cases $\alpha = 0$ and $\alpha = 1/2$ for which the field transformation rule (4) takes forms $\phi(\gamma X) = \phi(X)$ (nontwisted field) and $\phi(\gamma X) = -\phi(X)$ (twisted field).

So, below we will study the theory of authomorphic field $\phi$ on the universal covering space $\tilde{M}$, which obeys the authomorphic condition (4), instead of the complex field theory on $M$.

3 Two-dimensional model of spacetime with chronology horizon

In order to study the behaviour of complex field near the chronology horizon we will take the two-dimensional model of spacetime which was proposed by Frolov for investigation of the ordinary (i.e., real nontwisted) massless scalar field. Let us consider this model. (For more details, see Ref.[6].) It is the two-dimensional locally static spacetime with the metric

$$ds^2 = -e^{-2Wl}dt^2 + dl^2,$$

(5)

where $W = \text{const}$ is a parameter, $t \in (-\infty, \infty)$, the proper distance coordinate $l$ changes from $l = 0$ to $l = L$, and the boundary points 0 and $L$ being considering identical. The regularity of the spacetime $M$ requires that the metric (5) is identical at both boundaries: $\gamma^-, l = 0$ and $\gamma^+, l = L$. It is guaranteed if the time parameters of point $(t, 0)$ of $\gamma^-$ and $(t', L)$ of $\gamma^+$, which are to be identified, are related as follows:

$$t' = At, \quad A \equiv e^{WL}.\quad (6)$$

Therefore, the next identification rule: $(t, 0) \leftrightarrow (At, L)$ is assumed.

In $M$ the local (defined in any simply connected region $U \subset M$) uniquely defined (up to normalization) nonvanishing timelike Killing vector field $\xi^\mu(1, 0)$ exists. But it is not defined globally. Really, the norm of Killing vector $|\xi^2|^{1/2} = e^{-Wl}$ cannot be fixed in a whole of spacetime $M$. It changes on
the quantity \( A = e^{WL} \) in one pass along a closed spatial path, where the coordinate \( l \) changes from 0 to \( L \). The definition of gravitational potential \( \varphi \) in the spacetime \( \mathcal{M} \) reads
\[
e^\varphi = \left| \xi^2 \right|^{1/2} = e^{-Wl}, \tag{7}\]
and it is clear that the potential \( \varphi \) cannot be also defined globally in \( \mathcal{M} \). Such gravitational field is called nonpotential. The degree of nonpotentiality is characterized by the value \( A \).

For looking into the causal structure of \( \mathcal{M} \) let us take null coordinates
\[
u = W^{-1}e^{Wl} - t, \quad v = W^{-1}e^{Wl} + t. \tag{8}\]
The equations \( u = const \) and \( v = const \) determine null geodesics in spacetime \( \mathcal{M} \). The lines \( u = 0 \) and \( v = 0 \) are closed null geodesics. In the chronal region \( R_+ : \nu v > 0 \) there are no CTC’s, while CTC’s are possible in the nonchronal regions lying beyond \( R_+ \). Hence lines \( u = 0 \) and \( v = 0 \) form chronology horizons \( H_+ \) and \( H_- \) correspondingly.

4 Vacuum stress-energy tensor

Let us begin now the calculation of renormalized stress-energy tensor for the complex scalar field. The universal covering space \( \tilde{\mathcal{M}} \) for \( \mathcal{M} \) is a spacetime with the metric
\[
d\tilde{s}^2 = -e^{-2Wl}dt^2 + dl^2 \tag{9}\]
where \( t \in (-\infty, \infty) \), \( l \in (-\infty, \infty) \). By using the dimensionless coordinates
\[
\eta = Wt; \quad \xi = \exp(Wl), \tag{10}\]
this metric can be rewritten in the form
\[
d\tilde{s}^2 = (W\xi)^{-2} (-d\eta^2 + d\xi^2) . \tag{11}\]
The action of operator \( \gamma_n \equiv (\gamma)^n \) (where \( \gamma \) is generators of the fundamental group \( \Gamma \)) on \( \tilde{\mathcal{M}} \) is described by the relations
\[
\gamma_n \eta = A^n \eta, \quad \gamma_n \xi = A^n \xi. \tag{12}\]
The strip $\xi \in (1,A)$, $\eta \in (-\infty, \infty)$ is a fundamental domain.

Consider in $\tilde{M}$ a complex massless scalar field $\phi$ with the Lagrangian (3) obeying the wave equation

$$\Box\phi = 0.$$  \hspace{1cm} (13)

(For the complex conjugated field $\phi^*$ the wave equation is the same, so, as usually, we will consider the case of $\phi$ only.) In null coordinates

$$\zeta_- \equiv W u = \xi - \eta; \quad \zeta_+ \equiv W v = \xi + \eta$$  \hspace{1cm} (14)

the field equation (13) reads

$$\partial_+ \phi - \partial_- \phi = 0,$$  \hspace{1cm} (15)

where $\partial_+ = \partial_+ \partial_-$ and $\partial_\pm = \partial / \partial \zeta_\pm$. A general positive-frequency solution of this equation can be written as follows

$$u(\eta, \xi) = \int_0^\infty \frac{d\omega}{\omega} a(\omega) \left[ e^{i\omega \zeta_-} - e^{-i\omega \zeta_+} \right].$$  \hspace{1cm} (16)

This solution provides a fulfillment of the boundary condition

$$u_{\xi=0} = 0.$$  \hspace{1cm} (17)

(It is easy to show that regular solutions of massive scalar field equation vanish at $\xi = 0$ for arbitrary small value of mass.)

The positive-frequency solutions

$$U_\omega = (4\pi \omega)^{-1/2} \left[ e^{i\omega \zeta_-} - e^{-i\omega \zeta_+} \right]$$  \hspace{1cm} (18)

obey the normalization conditions

$$\langle U_\omega U_{\omega'} \rangle = -i \int_{\eta = \text{const}} \left( U_\omega \overline{U}_{\omega'} - \overline{U}_\omega U_{\omega'} \right) d\xi = \delta(\omega - \omega'),$$  \hspace{1cm} (19)

where the dot means $d/d\eta$, and form a basis in $H_{\tilde{M}}$.

The Hadamard function $\widetilde{G}^1$ in $\tilde{M}$ is

$$\widetilde{G}^1(X, X') = \sum_\omega \left( U_\omega(X) \overline{U}_{\omega}(X') + U_\omega(X') \overline{U}_\omega(X) \right) =$$  \hspace{1cm} (20)
\[ \gamma u(\eta, \xi) = u(\gamma \eta, \gamma \xi) = u(A\eta, A\xi) = e^{2\pi i \alpha} u(\eta, \xi) \] (22)

must be fulfilled. It is fulfilled provided

\[ a(\omega) = e^{2\pi i \alpha} a(A\omega). \] (23)

The general solution of this functional equation is

\[ a(\omega) = \sum_n c_n \omega^{-2\pi i \beta(n+\alpha)}, \] (24)

where \( \beta \equiv (\ln A)^{-1} = (WL)^{-1} \), \( c_n \) are constants, and the summation is taken over all integer numbers \( n \). By using the formula [19]

\[ \int_0^\infty \frac{d\omega}{\omega} \omega^{2\pi i \beta(n+\alpha)} e^{\pm i\omega \zeta} = \pm i e^{\pm i\pi/2} e^{\pm \pi \beta(n+\alpha)} \Gamma(2\pi i \beta(n+\alpha)) (\zeta \pm i0)^{\pm 2\pi i \beta(n+\alpha)} \] (25)

one can show from (16) that the positive-frequency authomorphic solution allows the representation

\[ u = \sum_n \bar{c}_n u_n, \] (26)

where

\[ u_n = b_n \left[ e^{\pi i \beta(n+\alpha)} (\zeta_+ - i0)^{-2\pi i \beta(n+\alpha)} - e^{-\pi i \beta(n+\alpha)} (\zeta_- + i0)^{-2\pi i \beta(n+\alpha)} \right]. \] (27)

The scalar product for authomorphic solutions (27), defined in \( M \), does not depend on the particular choice of the Cauchy surface in the fundamental domain and we may choose the section \( \eta = 0 \) as such a surface and write

\[ \langle u^1, u^2 \rangle = -i \int_1^A d\xi \left( u^1 \dot{u}^2 - \dot{u}^1 u^2 \right)_{\eta=0}. \] (28)

The solutions (27) form an orthonormal basis in \( H_M \):

\[ \langle u_n, u_{n'} \rangle = \delta_{nn'}. \] (29)
Using the formula (5.3.2) from [20], the Hadamard function $G^1$ defined in $M$ is

$$G^1(X, X') = \sum_n (u_n(X)\bar{u}_n(X') + u_n(x')\bar{u}_n(X)) =$$

$$= \sum_n b_n^2 \left\{ e^{\chi(n+\alpha)} \left( \frac{\zeta_+}{\zeta_-} \right)^{i\mu(n+\alpha)} + e^{-\chi(n+\alpha)} \left( \frac{\zeta_-}{\zeta_+} \right)^{i\mu(n+\alpha)} - \left( \frac{\zeta_+}{\zeta_-} \right)^{i\mu(n+\alpha)} - \left( \frac{\zeta_-}{\zeta_+} \right)^{i\mu(n+\alpha)} \right\} + (c.c.),$$

where $\mu = 2\pi\beta$, $\zeta_\pm \equiv \zeta_\pm \mp i0$, $\bar{\zeta}_\pm = \bar{\zeta}_\pm \pm i0$. In the region $R_+$ where $\zeta_+ > 0$ and $\zeta_- > 0$, the values $\zeta_\pm$, $\zeta'_\pm$ and their complex conjugate coincide. The Hadamard function for this case can be rewritten as

$$G^1(X, X') = \frac{1}{4\pi} \sum_{n=-\infty}^{\infty} \frac{e^{\chi(n+\alpha)} \cos(n+\alpha) y_\epsilon}{(n+\alpha) \sinh(\chi(n+\alpha))} -$$

$$- \frac{1}{4\pi} \sum_{n=-\infty}^{\infty} \cos \left[ \frac{\mu(n+\alpha) \ln(\zeta_+ / \zeta_-)}{(n+\alpha) \sinh(\chi(n+\alpha))} \right] - \frac{1}{4\pi} \sum_{n=-\infty}^{\infty} \cos \left[ \frac{\mu(n+\alpha) \ln(\zeta_- / \zeta_+)}{(n+\alpha) \sinh(\chi(n+\alpha))} \right],$$

where $y_\pm = \mu \ln(\zeta_+ / \zeta_\pm)$. Consider separately the first term in the expression (32). It can be given as follows

$$\frac{1}{4\pi} \sum_{n=-\infty}^{\infty} \frac{e^{\chi(n+\alpha)} \cos(n+\alpha) y_\epsilon}{(n+\alpha) \sinh(\chi(n+\alpha))} = \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{\cos(n+\alpha) y_\epsilon}{(n+\alpha)} +$$

$$+ \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{\cos(n+\alpha) y_\epsilon}{(n+\alpha) [e^{2\chi(n+\alpha)} - 1]} + \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{\cos(n-\alpha) y_\epsilon}{(n-\alpha) [e^{2\chi(n-\alpha)} - 1]}.$$  

Using the formula (5.3.2) from [20]

$$\sum_{n=0}^{\infty} \frac{1}{n+\alpha} \left\{ \sin yn \cos yn \right\} = \beta(\alpha) \left\{ \sin(\pi - y) \alpha \cos(\pi - y) \alpha \right\} +$$

$$+ \frac{1}{2} \int_0^\pi \left\{ \sin[-y\alpha + (\alpha - 1/2)t] \cos[-y\alpha + (\alpha - 1/2)t] \right\} \csc \frac{t}{2} dt.$$  

8
where $\beta(\alpha) = \frac{1}{2} \left[ \psi \left( \frac{\alpha+1}{2} \right) - \psi \left( \frac{\alpha}{2} \right) \right]$; $\psi(z) = \Gamma'(z)/\Gamma(z)$ is a psi-function, one can rewrite the first series in (33) as follows

$$
\frac{1}{2\pi} \sum_{n=0}^{\infty} \cos(n+\alpha) y_{\epsilon} \left( \frac{n+1}{n+\alpha} \right) = \frac{\beta(\alpha)}{\pi} \cos \pi \alpha + \frac{1}{4\pi} \sum_{\epsilon=+,-} \int_{y_{\epsilon}}^{\pi} \frac{\cos(\alpha - 1/2) t}{\sin(t/2)} dt. \tag{35}
$$

Finally, the expression (32) for the Hadamard function takes the form

$$
G^1(X, X') = \frac{\beta(\alpha)}{\pi} \cos \pi \alpha + \frac{1}{4\pi} \sum_{\epsilon=+,-} \int_{y_{\epsilon}}^{\pi} \frac{\cos(\alpha - 1/2) t}{\sin(t/2)} dt +
$$

$$
\frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{\cos(n+\alpha) y_{\epsilon} \left( \frac{n+1}{n+\alpha} \right)}{(n+\alpha) \left[ e^{2\chi(n+\alpha)} - 1 \right]} + \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{\cos(n-\alpha) y_{\epsilon} \left( \frac{n+1}{n+\alpha} \right)}{(n-\alpha) \left[ e^{2\chi(n-\alpha)} - 1 \right]} -
$$

$$
\frac{1}{4\pi} \sum_{n=-\infty}^{\infty} \frac{\cos \left[ \mu(n+\alpha) \ln(\zeta_+ / \zeta_-) \right]}{(n+\alpha) \sinh \chi(n+\alpha)} - \frac{1}{4\pi} \sum_{n=-\infty}^{\infty} \frac{\cos \left[ \mu(n+\alpha) \ln(\zeta_- / \zeta_+) \right]}{(n+\alpha) \sinh \chi(n+\alpha)}. \tag{36}
$$

In order to calculate the renormalized stress-energy tensor one can use the standard point-splitting method [21,22]. But in our case due to the high symmetry of the universal covering space it is possible to simplify the calculations. The space $M$ is locally isometric to $\tilde{M}$. Thus the terms which are to be subtracted in order to renormalize the vacuum expectation value of the stress-energy tensor are the same in the both spaces and we can write

$$
\langle T_{\mu\nu} \rangle^\text{ren}_M = T^1_{\mu\nu} + \langle T_{\mu\nu} \rangle_\tilde{M}^\text{ren}, \tag{37}
$$

where $\langle T_{\mu\nu} \rangle^\text{ren}_M$ and $\langle T_{\mu\nu} \rangle_\tilde{M}^\text{ren}$ are the renormalized stress-energy tensors in $M$ and $\tilde{M}$ correspondingly. For the scalar field the point-splitting method gives (see Ref. [21])

$$
\langle T_{\mu\nu} \rangle^\text{ren} = \lim_{X \to X'} D_{\mu\nu} G^1, \tag{38}
$$

where

$$
D_{\mu\nu} = \frac{1}{4} \left( \nabla_\mu \nabla_\nu + \nabla_\nu \nabla_\mu - g_{\mu\nu} g^{\rho\sigma} \nabla_\rho \nabla_\sigma \right). 
$$

Thus we have the following expression for $T^1_{\mu\nu}$

$$
T^1_{\mu\nu} = \lim_{X \to X'} D_{\mu\nu} [G^1(X, X') - \tilde{G}^1(X, X')], \tag{39}
$$

9
The expression in the square brackets on the right-hand side of (39) is a regular function. It is easy to see that \( T_{\pm \mp}^1 = T_{\mp \pm}^1 = 0 \), where we use the notations \( T_{\zeta \pm \zeta \mp}^1 \). While the nonvanishing components \( T_{\pm \pm}^1 \) and \( T_{\mp \mp}^1 \) are

\[
T_{\pm \pm}^1 = \lim_{X \to X'} \partial_{\pm} \partial'_{\pm} [G^1(X,X') - \tilde{G}^1(X,X')] .
\]

(40)

Differentiating the Hadamard function (36) we obtain

\[
\partial_{\pm} \partial'_{\pm} G^1(X,X') = \frac{\mu^2}{\zeta_\pm \zeta'_\pm} \left\{ -\frac{1}{8\pi} \frac{\cos(y_\pm/2) \cos(\alpha - 1/2)y_\pm}{\sin^2(y_\pm/2)} - \frac{\alpha - 1/2}{4\pi} \frac{\sin(\alpha - 1/2)y_\pm}{\sin(y_\pm/2)} + \frac{1}{2\pi} \sum_{\epsilon = +,-}^{\infty} \frac{(n + \alpha) \cos(n + \alpha) y_\epsilon}{e^{2\chi(n+\alpha)} - 1} \right\} .
\]

(41)

By using this relation and the expression (21) for the Hadamard function \( \tilde{G}^1(X,X') \) in \( \tilde{M} \) one can obtain for \( T_{\mu \nu}^1 \) in the limit of coinciding points \( X \to X' \):

\[
T_{\pm \pm}^1 = \frac{1}{\zeta'_\pm} F_\alpha(\beta),
\]

(42)

where a function \( F_\alpha(\beta) \) is defined as follows

\[
F_\alpha(\beta) = -\frac{1}{24\pi} + \pi \beta^2 \left[ \frac{1}{12} - (\alpha - \frac{1}{2})^2 \right] +
\]

\[
+ 2\pi \beta^2 \sum_{n=0}^{\infty} \frac{n + \alpha}{e^{2\chi(n+\alpha)} - 1} + 2\pi \beta^2 \sum_{n=1}^{\infty} \frac{n - \alpha}{e^{2\chi(n-\alpha)} - 1} .
\]

(43)

The renormalized stress-energy tensor \( \langle T_{\mu \nu}^1 \rangle_{\tilde{M}}^{\text{ren}} = -g_{\mu \nu}/24\pi \), where \( g_{\mu \nu} \) is the metric in \( \tilde{M} \), was obtained in Ref.\[4\] for a real scalar field. In the case of complex field this result must be multiplied by two, so that

\[
\langle T_{\mu \nu}^1 \rangle_{\tilde{M}}^{\text{ren}} = -g_{\mu \nu}/12\pi .
\]

(44)
Combining expressions \((1\text{2})\) and \((1\text{4})\) we can write finally for the renormalized stress-energy tensor of authomorphic complex scalar field:

\[
\langle T_{\mu\nu}\rangle_M^{\text{ren}} = \left( \frac{1}{\zeta^+} \delta^{\mu+}_{\nu+} + \frac{1}{\zeta^-} \delta^{\mu-}_{\nu-} \right) F_\alpha(\beta) - \frac{1}{12\pi} g_{\mu\nu}. \tag{45}
\]

Before investigation of the obtained result we consider the case when the nonpotential gravitational field is weak, i.e., \(A = e^{W L} \to 1\). Then we have \(\delta \equiv A - 1 \ll 1\), \(W \simeq \delta/L\) and \(\beta = (\ln A)^{-1} \simeq \delta^{-1} \gg 1\). In the \(\delta \to 0\) limit, the terms of \((1\text{3})\) that contain series are of the order \(\delta^{-2} \exp(-4\pi^2/\delta)\), and it can be neglected. Thus we have \(F_\alpha(\beta) \simeq \pi\delta^{-2}[1/12 - (\alpha - 1/2)^2]\). For fixed values of \(l\) and \(t\), one has \(u \simeq v \simeq L/\delta\), and hence up to the terms which vanish in the \(\delta \to 0\) limit one has

\[
\langle T_{uu}\rangle_M^{\text{ren}} = \langle T_{vv}\rangle_M^{\text{ren}} = \frac{\pi}{L^2} \left[ \frac{1}{12} - \left( \alpha - \frac{1}{2} \right)^2 \right];
\]

\[
\langle T_{uv}\rangle_M^{\text{ren}} = \langle T_{vu}\rangle_M^{\text{ren}} = 0. \tag{46}
\]

These expressions correctly reproduce the value of the renormalized stress-energy tensor for the authomorphic complex conformal massless scalar field in a two-dimensional cylindrical spacetime (see, e.g., Ref. [23]).

In particular cases when \(\alpha = 0\) (the ordinary field) and \(\alpha = 1/2\) (the twisted field) we have \(\langle T_{uu}\rangle_M^{\text{ren}} = \langle T_{vv}\rangle_M^{\text{ren}} = -\pi/6L^2\) and \(+\pi/12L^2\). As one can see the value of stress-energy tensor changes himself sign depending on values of the authomorphic parameter. Note also that the special value

\[
\alpha_0 = \frac{1}{2} - \frac{1}{\sqrt{12}} \tag{47}
\]

exists for which the stress-energy tensor \((1\text{6})\) is equal to zero for any \(L\).

5 The behaviour of stress-energy tensor near the chronology horizon

Consider now the general case of nonpotential gravitational field with an arbitrary value of \(\beta\). As it was mentioned before the equations \(u = 0\) and \(v = 0\) describe closed null geodesics which form the chronology horizons
The leading term of $\langle T_{\mu\nu} \rangle_{M}^{\text{ren}}$ near the chronology horizon $H_+$ is

$$\langle T_{uu} \rangle_{M}^{\text{ren}} = F_{\alpha}(\beta) \frac{k_{\mu} k_{\nu}}{u^2}, \; k_{\mu} = \nabla_{\mu} u$$

if only the function $F_{\alpha}(\beta)$ is not became equal to zero (this case we will consider especially). Then, the expression (48) shows that near $H_+$ there exists an infinitely growing flux of energy density. The sign of energy density is determined by the sign of the function $F_{\alpha}(\beta)$. So let us investigate it in details.

At first we take the case $\alpha = 0$ corresponding to the nontwisted scalar field. The next expression for $F_{\alpha}(\beta)$ follows from (43):

$$F_{\alpha=0}(\beta) = -\frac{1}{24\pi} + \frac{\beta}{2\pi} - \frac{\pi \beta^2}{6} + 4\pi \beta^2 \sum_{n=0}^{\infty} \frac{n}{e^{2\pi n} - 1}.$$  

(49)

It coincides with the formula for $F_{\alpha=0}(\beta)$ obtained in [6]. It was also shown in [6] that the function $F_{\alpha=0}(\beta)$ allows another representation:

$$F_{\alpha=0}(\beta) = -\frac{1}{4\pi} \sum_{n=0}^{\infty} \frac{1}{\sinh^2(n/2\beta)}.$$  

(50)

from which one can clearly see that $F_{\alpha=0}(\beta)$ is a negative defined function for all values of $\beta$. It means that the renormalized stress-energy tensor of the scalar field with $\alpha = 0$ is divergent near the chronology horizon for any value of nonpotentiality of the gravitational field. And then, the infinitely growing flux of negative energy density appears which is propagating in the direction of the gravitational potential decrease.

Consider now the case $\alpha = 1/2$ corresponding to the twisted scalar field. We have from (43):

$$F_{\alpha=1/2}(\beta) = -\frac{1}{24\pi} + \frac{\pi \beta^2}{12} + 4\pi \beta^2 \sum_{n=0}^{\infty} \frac{n + 1/2}{e^{2\pi(n+1/2)} - 1}.$$  

(51)

As it is shown in the appendix A the function $F_{\alpha=1/2}(\beta)$ can be rewritten in another equivalent form

$$F_{\alpha=1/2}(\beta) = -\frac{1}{24\pi} + \frac{1}{3\pi} K^2(m) \beta^2(m + 1),$$  

(52)
where $m$ is a parameter, $0 \leq m \leq 1$; $K(m)$ is an elliptic integral \cite{24}: \[ K(m) = \int_0^{\pi/2} d\theta (1 - m \sin^2 \theta)^{-1/2} \]. The parameter $m$ is connected with the nonpotentiality parameter $\beta$ as follows (see appendix A): \[ 2\pi \beta = K(1 - m)/K(m) \]. It is also shown in appendix A that the function $F_{\alpha = 1/2}(\beta)$ defined by the relation \cite{12} is positive for all values of $\beta$. It means that the renormalized stress-energy tensor of twisted scalar field is divergent near the chronology horizon for any values of nonpotentiality of a gravitational field. And then, the infinitely growing flux of positive energy density appears which is propagating in the direction of the gravitational potential increase.

Thus we have obtained that the function $F_{\alpha}(\beta)$ is negative defined for the ordinary scalar field and positive defined for the twisted one. As the consequence of this fact the infinitely growing flux of negative or positive energy density exists near the chronology horizon. This result can be interpreted as an evidence of quantum nonstability of the chronology horizon.

In the case of complex authomorphic scalar field the parameter $\alpha$ can take values within the interval $[0, 1/2]$. The function $F_{\alpha}(\beta)$ is negative or positive defined for the extreme values $\alpha = 0$ and $\alpha = 1/2$ correspondingly. It is clear that the value $\alpha$ must exist when $F_{\alpha}(\beta)$ is equal to zero (at least for some $\beta$). In order to analyze the behaviour of function $F_{\alpha}(\beta)$ we will build it graph for different values of $\alpha$. The form of $F_{\alpha}(\beta)$ is defined by the expression \cite{13}. The series in this expression is fast convergent for large values of $\beta$ and slowly convergent for small ones. In the appendix B the other representation of $F_{\alpha}(\beta)$ is obtained which is convenient for the numerical analysis in the case of small $\beta$:

\[ F_{\alpha}(\beta) = -\frac{1}{4\pi} \sum_{n=1}^{\infty} \frac{\cos 2\pi n \alpha}{\sinh^2(n/2\beta)}. \] (53)

The graph of function $F_{\alpha}(\beta)$ defined by the formulae (13) and (53), is given in Fig.1.

From Fig.1 one can see the next characteristic features of the behaviour of the function $F_{\alpha}(\beta)$. If $0 \leq \alpha < \alpha_0$, $\alpha_0 = \frac{1}{2} - \frac{1}{\sqrt{2}}$ then $F_{\alpha}(\beta)$ is less than zero for all values of $\beta$. In the special case $\alpha = \alpha_0$ the graph of $F_{\alpha}(\beta)$ is asymptotically approached to the straight line $-\frac{1}{24\pi}$. If $\alpha = \alpha_*$, where $\alpha_0 < \alpha_* < 1/2$, then $F_{\alpha}(\beta)$ is a function with alternating signs; it is less than zero if $0 < \beta < \beta_*$ and greater than zero if $\beta > \beta_*$. At the point $\beta = \beta_*$ the function $F_{\alpha}(\beta)$ is equal to zero; the value $\beta_*$ is defined from the equation $F_{\alpha_*}(\beta_*) = 0$. And, at last, $F_{\alpha}(\beta)$ is greater than zero everywhere for $\alpha = 1/2$. 

13
Figure 1: The graph of $F_\alpha(\beta)$. The curve a corresponds to the value $0 \leq \alpha < \alpha_0$; the curve b is obtained if $\alpha = \alpha_0$; the curves c, d are obtained if $\alpha_0 < \alpha < 1/2$; and the curve e corresponds to the value $\alpha = 1/2$.

(Note that at the point $\beta = 0$ the function $F_\alpha(\beta)$ is equal to zero for all values of parameter $\alpha$. The value $\beta = 0$ corresponds to the infinite nonpotentiality of gravitational field, $W \to \infty$.)

Now we can do the complete characterization of stress-energy tensor (45) near the chronology horizon. There are two qualitatively different cases. (i) The first case, when the function $F_\alpha(\beta)$ is not equal to zero, then the leading term in expression (45) near the chronology horizon takes the form (48), and at the chronology horizon the stress-energy tensor is divergent. For various values $\alpha$ and $\beta$ the function $F_\alpha(\beta)$ can be negative or positive. In consequence of this fact the infinitely growing flux of negative or positive energy density exists near the horizon. (ii) And second case, when $F_\alpha(\beta)$ is equal to zero. It is possible if the authomorphic parameter $\alpha = \alpha_*$ takes its values within the interval $(\alpha_0, 1/2)$, then such parameter $\beta = \beta_*$ exists that $F_{\alpha_*}(\beta_*) = 0$. In this case the expression (45) for $(T_{\mu\nu})^\text{ren}_M$ takes the form

$$\langle T_{\mu\nu} \rangle^\text{ren}_M = -g_{\mu\nu}/12\pi,$$

and one can see that the stress-energy tensor is regular at the chronology horizons $H_+$ and $H_-$.  

14
6 Summary and concluding remarks

In this paper we have considered the complex massless scalar field in two-dimensional model of spacetime $M$ with closed timelike curves. We have obtained the expression for renormalized vacuum stress-energy tensor of scalar field (see Eq.(45)) which depends on the authomorphic parameter $\alpha$, defining the transformation law of scalar field under the symmetry transformations of spacetime $M$, and the parameter $\beta$, characterizing the degree of nonpotentiality of the gravitational field. The analysis of behaviour of $\langle T_{\mu\nu} \rangle_{M}^{ren}$ near the chronology horizon shows that there are two qualitatively different cases. In first case (in particular, when $\alpha = 0$ (nontwisted field) and $\alpha = 1/2$ (twisted field)) the value of stress-energy tensor is diverged near the chronology horizon and the infinitely growing flux of negative or positive energy density exists there. This fact can be interpreted as pointing out to the quantum instability of the chronology horizon. Note that the obtained result extends conclusions of works [6-13] for the case of complex field.

The essentially new important result, which is obtained in this paper, is the fact that the stress-energy tensor of authomorphic scalar field is finite at the chronology horizon for some values of parameters $\alpha$ and $\beta$. It means that the formation of chronology horizon is possible for this field configuration. Hence, the Hawking’s chronology protection conjecture does not work for the case of authomorphic fields.

A question remains: How can we determine and fix the values of parameters $\alpha$ and $\beta$? I think that the answer is only possible in the framework of selfconsistent investigation of the problem.

Acknowledgment

This work was presented at 7th Marsel Grossmann Meeting [27] due to the support of International Science Foundation.

Appendix A

Here we will consider the expression (51) for the function $F_{\alpha=1/2}(\beta)$:
\[ F_{n=1/2}(\beta) = -\frac{1}{24\pi} + \frac{\pi \beta^2}{12} + 4\pi \beta^2 \sum_{n=0}^\infty \frac{n + 1/2}{e^{4\pi \beta(n+1/2)} - 1} , \] (A.1)

and will obtain the other representation which is more convenient for an analysis.

Let us rewrite the series in (A.1) in the other form:

\[ \sum_{n=0}^\infty \frac{n + 1/2}{e^{4\pi \beta(n+1/2)} - 1} = \frac{1}{2} \lim_{y \to 0} \sum_{n=0}^\infty \frac{(2n + 1) \cos(2n + 1)y}{e^{2\pi \beta(2n+1)} - 1} . \] (A.2)

(We can do it because the functional series in the right-hand side of the expression (A.2) is uniformly converged for any values of \( y \).) Now the next transformation can be executed:

\[ \sum_{n=0}^\infty \frac{(2n + 1) \cos(2n + 1)y}{e^{2\pi \beta(2n+1)} - 1} = -d\left( \frac{\sum_{n=0}^\infty q^{2n+1} \sin(2n + 1)y}{1 - q^{2n+1}} \right), \] (A.3)

where \( q = e^{-2\pi^2 \beta} \). Then we will use the formula (16.23.10) from Ref.[20]:

\[ \sum_{n=0}^\infty q^{2n+1} \sin(2n + 1)y \left( \frac{1}{1 - q^{2n+1}} \right) = \frac{1}{4} \csc y - \frac{K}{2\pi} \text{ns} \left( \frac{2K}{\pi} y|m \right) . \] (A.4)

Here and below \( ds(z|m), \ cn(z|m), \ ns(z|m), \ dn(z|m), \ cn(z|m), \ sn(z|m) \) are the Jacobian elliptic functions; \( m \) is a parameter, \( 0 \leq m \leq 1; \ K \equiv K(m) = \int_0^{\pi/2} d\theta (1 - m \sin^2 \theta)^{-1/2} \) is the complete elliptic integral [24]. The parameters \( m \) and \( q \) are connected by the relation

\[ q = \exp(-\pi K(1 - m)/K(m)) . \] (A.5)

Taking into account that \( q = e^{-2\pi^2 \beta} \) we can obtain the connection between parameters \( m \) and \( \beta \):

\[ 2\pi \beta = K(1 - m)/K(m) . \] (A.6)

Let us write out the differentiation formulae of the Jacobian functions [24]:

\[ \begin{align*}
    ds'(z|m) &= -\text{cs}(z|m) \text{ns}(z|m); \quad \text{cs}'(z|m) = -\text{ns}(z|m) ds(z|m); \\
    \text{ns}'(z|m) &= -ds(z|m) \text{cs}(z|m);
\end{align*} \] (A.7)
take into account the relations [24]:

\[
\begin{align*}
\text{ds}(z|m) &= \frac{\text{dn}(z|m)}{\text{sn}(z|m)}; \\
\text{cs}(z|m) &= \frac{\text{cn}(z|m)}{\text{sn}(z|m)};
\end{align*}
\]  

(A.8)

and allow for the expansion of elliptic functions in the degrees of \( z \) [24]:

\[
\begin{align*}
\text{dn}(z|m) &= 1 - m z^2 + O(z^4); \\
\text{cn}(z|m) &= 1 - \frac{z^2}{2!} + O(z^4); \\
\text{sn}(z|m) &= z - (1 + m) \frac{z^3}{3!} + O(z^5).
\end{align*}
\]  

(A.9)

Now, substituting (A.2) into (A.1) and using consecutively the relations (A.3-9) one can easy obtain the next representation for the function \( F_{\alpha=1/2}(\beta) \):

\[
F_{\alpha=1/2}(\beta) = -\frac{1}{24\pi} + \frac{1}{3\pi} K^2 \beta^2 (1 + m).
\]  

(A.10)

In order to determine the region of change of the function \( F_{\alpha=1/2}(\beta) \) we take into account that the function \( K(m) \) is monotonically increasing from \( \frac{\pi}{2} \) to \( +\infty \) if \( m \) changes from 0 to 1. Then, it is easy to see from (A.6) that the quantity \( \beta K(m) = K(1 - m)/2\pi \) takes its values within an interval \((+\infty, \frac{1}{4})\). And finally, we can see from (A.10) that the function \( F_{\alpha=1/2}(\beta) \) is positive defined and its region of change is an interval from \( +\infty \) to 0.

**Appendix B**

Let us consider the expression (41) for the function \( F_{\alpha}(\beta) \):

\[
F_{\alpha}(\beta) = -\frac{1}{24\pi} + \pi \beta^2 \left[ \frac{1}{12} - \left( \alpha - \frac{1}{2} \right)^2 \right] +
\]

\[
+ 2\pi \beta^2 \sum_{n=0}^{\infty} \frac{n + \alpha}{e^{4\pi^2 \beta(n+\alpha)} - 1} + 2\pi \beta^2 \sum_{n=0}^{\infty} \frac{n + \overline{\alpha}}{e^{4\pi^2 \beta(n+\overline{\alpha})} - 1},
\]  

(B.1)

where \( \overline{\alpha} = 1 - \alpha \). The series in (B.1) are converged fastly for big values of \( \beta \) and slowly for small ones. With the aim to make the numerical analysis of the function \( F_{\alpha}(\beta) \) we will obtain the other representation for the expression (B.1).
By using the simplest form of the Euler-Maclaurin formula for a summation \[25\] we can rewrite the series in (B.1) as follows

$$
\sum_{n=0}^{\infty} \frac{n + \alpha}{e^{4\pi^2\beta(n+\alpha)} - 1} = \int_0^\infty dn \frac{n + \alpha}{e^{4\pi^2\beta(n+\alpha)} - 1} + \frac{1}{2} \frac{\alpha}{e^{4\pi^2\beta\alpha} - 1} + \\
+ \int_0^\infty dn \{n - [n] - 1/2\} \frac{d}{dn} \frac{n + \alpha}{e^{4\pi^2\beta(n+\alpha)} - 1}.
$$

(B.2)

It is not difficult to transform the expression (B.2) into the next form:

$$
\frac{n + \alpha}{e^{4\pi^2\beta(n+\alpha)} - 1} = \frac{1}{96\pi^2\beta^2} + \int_0^\infty dx \{x - \alpha - [x - \alpha] - 1/2\} \frac{d}{dx} \frac{x}{e^{4\pi^2\beta x} - 1}.
$$

(B.3)

So \( F_\alpha(\beta) \) is

$$
F_\alpha(\beta) = \pi \beta^2 \left[ \frac{1}{12} - \left( \alpha - \frac{1}{2} \right)^2 \right] + \\
+2\pi \beta^2 \int_0^\infty dx \{(x - \alpha - [x - \alpha] - 1/2) + (x - \overline{\alpha} - [x - \overline{\alpha}] - 1/2)\} \frac{d}{dx} \frac{x}{e^{4\pi^2\beta x} - 1}.
$$

(B.4)

The periodic function \( S(x) = x - [x] - 1/2 \) possesses the Fourier series representation \[26\]:

$$
S(x) = x - [x] - 1/2 = -\sum_{n=1}^{\infty} \sin \frac{2\pi n x}{\pi n}.
$$

(B.5)

By using this representation we can transform the integrals in (B.4) as follows

$$
2\pi \beta^2 \int_0^\infty dx \{(x - \alpha - [x - \alpha] - 1/2) + (x - \overline{\alpha} - [x - \overline{\alpha}] - 1/2)\} \frac{d}{dx} \frac{x}{e^{4\pi^2\beta x} - 1} = \\
= \frac{\beta}{2\pi} \int_0^\infty dy \{(z - \alpha - [z - \alpha] - 1/2) + (z - \overline{\alpha} - [z - \overline{\alpha}] - 1/2)\} \frac{d}{dy} \frac{y}{e^{\pi^2\beta y} - 1} = \\
= -\frac{\beta}{2\pi} \sum_{n=1}^{\infty} \int_0^\infty dy \frac{\sin 2\pi n(z - \alpha) + \sin 2\pi n(z - \overline{\alpha})}{\pi n} \frac{d}{dy} \frac{y}{e^{\pi^2\beta y} - 1},
$$

(B.6)
where \( z = y/4\pi^2 \beta \). Substituting the expression (B.6) into (B.4) and computing the integral we obtain the next expression for \( F_\alpha(\beta) \):

\[
F_\alpha(\beta) = \pi \beta^2 \left[ \frac{1}{12} - \left( \alpha - \frac{1}{2} \right)^2 \right] + \frac{\beta^2}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2\pi n \alpha}{n^2} - \frac{1}{4\pi} \sum_{n=1}^{\infty} \frac{\cos 2\pi n \alpha}{\sinh^2(n/2\beta)}.
\]

(B.7)

Using the formula (2.5.34) from Ref. [20]:

\[
\sum_{n=1}^{\infty} \frac{\cos nx}{n^2} = \frac{1}{4}x^2 - \frac{\pi x}{2} + \frac{\pi^2}{6},
\]

(B.8)

we have finally

\[
F_\alpha(\beta) = -\frac{1}{4\pi} \sum_{n=1}^{\infty} \frac{\cos 2\pi n \alpha}{\sinh^2(n/2\beta)}.
\]

(B.9)

The representation (B.9) for the function \( F_\alpha(\beta) \) contains the series which is fastly convered for small values of \( \beta \).

Note also that in the case \( \alpha = 0 \) (nontwisted scalar field)

\[
F_{\alpha=0}(\beta) = -\frac{1}{4\pi} \sum_{n=1}^{\infty} \frac{1}{\sinh^2(n/2\beta)},
\]

(B.10)

and this expression coincides with one obtained in Ref. [3].
References

[1] Morris M S, Thorne K S and Yurtsever U 1988 Phys.Rev.Lett. 61 1446
[2] Misner C W 1967, in Relativity Theory an Astrophysics I: Relativity and Cosmology, Lectures in Applied Mathematics, edited by J.Ehlers (American Mathematical Society, Providence), pp.160-169
[3] Gott III J R 1991 Phys.Rev.Lett. 66 1126
[4] Hawking S W 1992 Phys.Rev.D 46 603
[5] Thorne K S 1993, in General Relativity and Gravitation 1992, Proceedings of the Thirteenth International Conference on General Relativity and Gravitation, edited by Gleiser R J, Kozameh C N, and Moreschi O M (Institute of Physics, Bristol), p.295
[6] Frolov V P 1991 Phys.Rev.D 43 3878
[7] Kim S W and Thorne K S 1991 Phys.Rev.D 43 3929
[8] Yurtsever U 1991 Class.Quantum Grav. 8 1127
[9] Klinkhammer G 1992 Phys.Rev.D 46 3388
[10] Boulware D G 1992 Phys.Rev.D 46 4421
[11] Grant J D E 1993 Phys.Rev.D 47 2388
[12] Hiscock W A and Konkowski D A 1982 Phys.Rev.D 26 1125
[13] Tanaka T and Hiscock W A 1994 Phys.Rev.D 49 5240
[14] Schulman L S 1968 Phys. Rev. 176 1558
[15] Dowker J S 1972 J.Phys.A 5 936
[16] Dowker J S and Banach R 1978 J.Phys.A 11 2255
[17] Banach R and Dowker J S 1979 J.Phys. A 12 2527
[18] Banach R and Dowker J S 1979 J.Phys. A 12 2545
[19] Gel’fand I M and Shilov G E 1958 Generalized Functions and Operations with Them (Fizmatgiz, Moscow) [in Russian]
[20] Prudnikov A P, Brychkov Yu A and Marichev O I 1981 Integrals and Series (Nauka, Moscow), Vol.1
[21] Christensen S M 1978 Phys.Rev.D 17 946
[22] Birrell N D and Davies P S V 1984 Quantum Fields in Curved Space (Cambridge University Press, Cambridge)
[23] Mostepanenko V M and Trunov N N 1988 Usp.Fiz.Nauk 156 385 [1988 Sov.Phys.Usp 31 965]
[24] Abramowitz M and Stegun I A 1964 Handbook of Mathematical Functions (U.S.Natl.Bur.Stds., Washington, D.C.)
[25] Olver F W J 1974 Asymptotics and Special Functions (Academic Press, New York)
[26] Korn G A and Korn T M 1961 Mathematical Handbook (McGraw-Hill Book Company, Inc. NewYork Toronto London)
[27] Sushkov S V 1994, in Proceeding of 7th Marsel Grossmann Meeting (WSP, Singapure) (to be published)
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/gr-qc/9410008v1