ALGEBRAIC $K$-THEORY AND SUMS-OF-SQUARES FORMULAS

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Abstract. We prove a result about the existence of certain ‘sums-of-squares’ formulas over a field $F$. A classical theorem uses topological $K$-theory to prove that if such a formula exists over $\mathbb{R}$, then certain powers of 2 must divide certain binomial coefficients. While it has been known that this result works over all characteristic 0 fields, the characteristic $p$ case has remained open. In this paper we prove the result for all fields, using algebraic $K$-theory in place of topological $K$-theory.

1. Introduction

Let $F$ be a field. A classical problem asks for what values of $r$, $s$, and $n$ do there exist identities of the form

$$(x_1^2 + \cdots + x_r^2) (y_1^2 + \cdots + y_s^2) = z_1^2 + \cdots + z_n^2$$

in the polynomial ring $F[x_1, \ldots, x_r, y_1, \ldots, y_s]$, where the $z_i$’s are bilinear expressions in the $x$’s and $y$’s. Such an identity is called a sums-of-squares formula of type $[r, s, n]$. For the history of this problem, see the expository papers [L, Sh].

The main theorem of this paper is the following:

Theorem 1.1. Suppose that the characteristic of $F$ is not 2. If a sums-of-squares formula of type $[r, s, n]$ exists over $F$, then $2^{\lceil \frac{s-1}{2} \rceil - 1} + 1$ divides $(n)$ for $n - r < i \leq \lceil \frac{s-1}{2} \rceil$.

As one specific application, the theorem shows that a formula of type $[13, 13, 16]$ cannot exist over any field of characteristic not equal to 2. Previously this had only been known in characteristic zero. (Note that the case $\text{char}(F) = 2$, which is not covered by the theorem, is rather trivial: formulas of type $[r, s, 1]$ always exist).

In the case $F = \mathbb{R}$, the above theorem was essentially proven by Atiyah [A] as an early application of complex $K$-theory; the relevance of Atiyah’s paper to the sums-of-squares problem was only later pointed out by Yuzvinsky [Y]. The result for characteristic zero fields can be deduced from the case $F = \mathbb{R}$ by an algebraic argument due to K. Y. Lam and T. Y. Lam (see [Sh]). Thus, our contribution is the extension to fields of non-zero characteristic. In this sense the present paper is a natural sequel to [D], which extended another classical condition about sums-of-squares.

Our proof of Theorem 1.1 given in Section 2 is a modification of Atiyah’s original argument. The existence of a sums-of-squares formula allows one to make conclusions about the geometric dimension of certain algebraic vector bundles. A computation of algebraic $K$-theory (in fact just algebraic $K^0$), given in Section 3, determines restrictions on what that geometric dimension can be—and this yields the theorem.
Atiyah’s result for $F = \mathbb{R}$ is actually slightly better than our Theorem 1.1. The use of topological $KO$-theory rather than complex $K$-theory yields an extra power of 2 dividing some of the binomial coefficients. It seems likely that this stronger result holds in non-zero characteristic as well and that it can be proved with Hermitian algebraic $K$-theory.

1.2. Restatement of the main theorem. The condition on binomial coefficients from Theorem 1.1 can be reformulated in a slightly different way. This second formulation surfaces often, and it’s what arises naturally in our proof. We record it here for the reader’s convenience. Each of the following observations is a consequence of the previous one:

- By repeated use of Pascal’s identity \( \binom{r}{d} = \binom{r-1}{d-1} + \binom{r-1}{d} \), the number \( \binom{n+i-1}{k+i} \) equals a $\mathbb{Z}$-linear combination of the numbers \( \binom{n}{k+1}, \binom{n}{k+2}, \ldots, \binom{n}{k+i} \). Similarly, \( \binom{n}{k+i} \) is a $\mathbb{Z}$-linear combination of \( \binom{n+1}{k+1}, \binom{n+1}{k+2}, \ldots, \binom{n+i-1}{k+i} \).
- An integer $b$ divides the numbers \( \binom{n}{k+1}, \binom{n}{k+2}, \ldots, \binom{n}{k+i} \) if and only if it divides the numbers \( \binom{n+1}{k+1}, \binom{n+1}{k+2}, \ldots, \binom{n+i-1}{k+i} \).
- The series of statements

\[
2^N \mid \binom{n}{k+1}, 2^{N-1} \mid \binom{n}{k+2}, \ldots, 2^{N-i} \mid \binom{n}{k+i}
\]

is equivalent to the series of statements

\[
2^N \mid \binom{n+1}{k+1}, 2^{N-1} \mid \binom{n+1}{k+2}, \ldots, 2^{N-i} \mid \binom{n+i-1}{k+i}.
\]

- If $N$ is a fixed integer, then $2^{N-i}$ divides \( \binom{n}{r} \) for $n - r < i \leq N$ if and only if $2^{N-i-1}$ divides \( \binom{r+i-1}{i} \) for $n - r < i \leq N$.

The last observation shows that Theorem 1.1 is equivalent to the theorem below. This is the form in which we’ll actually prove the result.

**Theorem 1.3.** Suppose \( \text{char}(F) \neq 2 \). If a sums-of-squares formula of type \( [r, s, n] \) exists over $F$, then $2^{\lfloor \frac{r-1}{i} \rfloor}$ divides $\binom{r+i-1}{i}$ for $n - r < i \leq \lfloor \frac{r-1}{i} \rfloor$.

2. The main proof

Let $q$ be the quadratic form on $\mathbb{A}^k$ defined by $q(x) = \sum_{i=1}^{k} x_i^2$. A sums-of-squares formula of type \( [r, s, n] \) gives a bilinear map $\phi : \mathbb{A}^r \times \mathbb{A}^s \to \mathbb{A}^n$ such that $q(x)q(y) = q(\phi(x, y))$. We claim that $\phi$ induces a map

\[
f : \mathbb{P}^{s-1} - V_q \to \text{Gr}_r(\mathbb{A}^n)
\]

where $V_q \hookrightarrow \mathbb{P}^{s-1}$ is the subvariety defined by $q(x) = 0$ and $\text{Gr}_r(\mathbb{A}^n)$ is the Grassmannian variety of $r$-planes in affine space $\mathbb{A}^n$. Given $y \in \mathbb{P}^{s-1} - V_q$, $f(y)$ is the $r$-plane spanned by the vectors $\phi(e_1, y), \phi(e_2, y), \ldots, \phi(e_r, y)$ of $\mathbb{A}^n$, where $e_1, \ldots, e_r$ is the standard basis of $\mathbb{A}^r$. To see that these vectors are linearly independent, note that the sums-of-squares identity implies that

\[
\langle \phi(x, y), \phi(x', y) \rangle = q(y) \langle x, x' \rangle
\]

for any $x$ and $x'$ in $\mathbb{A}^r$, where $\langle -, - \rangle$ denotes the ‘dot product’ on $\mathbb{A}^k$ (for any $k$). If one had $\phi(x, y) = 0$ then the above formula shows $q(y) \langle x, x' \rangle = 0$ for every $x'$; but since $q(y) \neq 0$, this can only happen when $x = 0$.

To verify that our description really gives a map of schemes, one can restrict to a standard open subvariety of $\mathbb{P}^{s-1}$ intersected with $\mathbb{P}^{s-1} - V_q$, i.e., the subvariety $U_t$ of those $y = [y_1 : y_2 : \cdots : y_r]$ in $\mathbb{P}^{s-1}$ such that $q(y) \neq 0$ and $y_t \neq 0$. It is
Proposition 2.1. by \( \zeta \) there is an algebraic vector bundle \( P \) on \( \mathbb{A}^n \) as elements of the Grothendieck group \( K \).

Proof. \( P \) restricts to \( \mathbb{A}^r \) by the tautological \( r \)-plane bundle over the Grassmannian, and let \( \eta \) be the restriction to \( \mathbb{P}^s - V_q \) of the tautological line bundle \( O(1) \) of \( \mathbb{P}^s \). We claim the map \( \phi \) induces a map of bundles \( \tilde{f} : r\xi \to \eta \) covering the map \( f \). The map \( \tilde{f} \) takes an element of \( r\xi \), i.e., an element \( y \) of \( \mathbb{P}^s - V_q \) together with \( r \) non-zero scalar multiples \( \alpha_1, \ldots, \alpha_r \) of \( y \), to the vector \( \phi_*(e_1 + \cdots + \alpha_r e_r, y) \). This vector lies in the fiber of \( \eta \) over the \( r \)-plane \( f(y) \) of \( Gr_r(\mathbb{A}^n) \).

As before, to construct \( \tilde{f} \) as a map of schemes one first constructs it over the open subvarieties \( U_i \) described above. The bundle \( r\xi \) is trivial over each \( U_i \), so this is straightforward.

One consequence of the existence of \( \tilde{f} \) is that if we pull back the bundle \( \eta \) along \( f \) to obtain a bundle \( f^*\eta_r \) on \( \mathbb{P}^s - V_q \), this bundle is isomorphic to \( r\xi \).

Proposition 2.1. If a sums-of-squares identity of type \( [r,s,n] \) exists over \( F \), then there is an algebraic vector bundle \( \zeta \) on \( \mathbb{P}^s - V_q \) of rank \( n - r \) such that

\[
[r] + [\zeta] = n
\]

as elements of the Grothendieck group \( K^0(\mathbb{P}^s - V_q) \) of locally free coherent sheaves on \( \mathbb{P}^s - V_q \).

Proof. The bundle \( \eta \) is a subbundle of the rank \( n \) trivial bundle, which we denote by \( n \). Consider the quotient \( n/\eta \), and set \( \zeta = f^*(n/\eta) \). Since \( n = [\eta] + [n/\eta] \) in \( K^0(Gr_r(\mathbb{A}^n)) \), application of \( f^* \) gives \( n = [f^*\eta] + [\zeta] \) in \( K^0(\mathbb{P}^s - V_q) \). Now recall that \( f^*\eta \cong r\xi \).

The next task is to compute the Grothendieck group \( K^0(\mathbb{P}^s - V_q) \). This becomes significantly easier if we assume that \( F \) contains a square root of \(-1\). The reason for this is made clear in the next section.

Proposition 2.2. Suppose that \( F \) contains a square root of \(-1\), and \( \text{char}(F) \neq 2 \). Let \( c = \sqrt{-1} \). Then \( \mathbb{C} \langle [\xi] \rangle \) is isomorphic to \( \mathbb{Z}[\nu]/(2\nu, \nu^2 = -2\nu) \), where \( \nu = [\xi] - 1 \) generates the reduced Grothendieck group \( \tilde{K}^0(\mathbb{P}^s - V_q) \cong \mathbb{Z}/2\nu \).

The proof of the above result will be deferred until the next section. Note that \( K^0(\mathbb{P}^s - V_q) \) has the same form as the complex \( K \)-theory of real projective space \( \mathbb{R}P^s \) [Ad].

By accepting the above proposition for the moment, we can finish the

Proof of Theorem Recall that one has operations \( \gamma^t \) on \( \tilde{K}^0(X) \) for any scheme \( X \) [SGA6, Exp. V] (see also [AT] for a very clear explanation). If \( \gamma_1 = 1 + \gamma_t + \gamma_2t^2 + \cdots \) denotes the generating function, then the basic properties are:

(i) \( \gamma_1(ab) = \gamma_1(a)\gamma_1(b) \).

(ii) For a line bundle \( L \) on \( X \) one has \( \gamma_1([L] - 1) = 1 + t([L] - 1) \).
(iii) If $E$ is an algebraic vector bundle on $X$ of rank $k$ then $\gamma^i([E] - k) = 0$ for $i > k$.

The third property follows from the preceding two via the splitting principle.

If a sums-of-squares identity of type $[r, s, n]$ exists over a field $F$, then it also exists over any field containing $F$. So we may assume $F$ contains a square root of $-1$. If we write $X = \mathbb{P}^{s-1} - V_q$, then by Proposition 2.2 there is a rank $n - r$ bundle $\zeta$ on $X$ such that $r[\xi] + [\zeta] = n$ in $K^0(X)$. This may also be written as $r([\xi] - 1) + ([\xi] - (n - r)) = 0$ in $\tilde{K}_0(X)$. Setting $\nu = [\xi] - 1$ and applying the operation $\gamma^r$ we have

$$\gamma^r(\nu)^r \cdot \gamma^r([\xi] - (n - r)) = 1$$

or

$$\gamma^r([\xi] - (n - r)) = \gamma^r(\nu)^{-r} = (1 + t\nu)^{-r}.$$ The coefficient of $t^i$ on the right-hand-side is $(-1)^i (r+i-1)\nu^i$, which is the same as $-2^{i-1} (r+i-1)\nu$ using the relation $\nu^2 = -2\nu$. Finally, since $\zeta$ has rank $n - r$ we know that $\gamma^i([\xi] - (n - r)) = 0$ for $i > n - r$. In light of Proposition 2.2 this means that $2^c$ divides $2^{i-1} (r+i-1)$ for $i > n - r$, where $c = \lfloor \frac{\nu}{2} \rfloor$. When $i - 1 < c$, we can rearrange the powers of 2 to conclude that $2^{c-i+1}$ divides $(r+i-1)$ for $n - r < i \leq c$.

3. $K$-THEORY OF DELETED QUADRICS

The rest of the paper deals with the $K$-theoretic computation stated in Proposition 2.2.

Let $Q_{n-1} \hookrightarrow \mathbb{P}^n$ be the split quadric defined by one of the equations

$$a_1b_1 + \cdots + a_kb_k = 0 \quad (n = 2k - 1) \quad \text{or} \quad a_1b_1 + \cdots + a_kb_k + c^2 = 0 \quad (n = 2k).$$

Beware that in general $Q_{n-1}$ is not the same as the variety $V_q$ of the previous section. However, if $F$ contains a square root $i$ of $-1$, then one can write $x^2 + y^2 = (x + iy)(x - iy)$. After a change of variables the quadric $V_q$ becomes isomorphic to $Q_{n-1}$. These ‘split’ quadrics $Q_{n-1}$ are simpler to compute with, and we can analyze the $K$-theory of these varieties even if $F$ does not contain a square root of $-1$.

Write $DQ_n = \mathbb{P}^n - Q_{n-1}$, and let $\xi$ be the restriction to $DQ_n$ of the tautological line bundle $\mathcal{O}(-1)$ of $\mathbb{P}^n$. In this section we calculate $K^0(DQ_n)$ over any ground field $F$ of characteristic not 2. Proposition 2.2 is an immediate corollary of this more general result:

**Theorem 3.1.** Let $F$ be a field of characteristic not 2. The ring $K^0(DQ_n)$ is isomorphic to $\mathbb{Z}[\nu]/(2^c\nu, \nu^2 = -2\nu)$, where $\nu = [\xi] - 1$ generates the reduced group $\tilde{K}^0(DQ_n) \cong \mathbb{Z}/2^c$ and $c = \lfloor \frac{n}{2} \rfloor$.

Note to the reader: Below we will write $K^i(X)$ for what is usually (but unfortunately) denoted $K_{-i}(X)$ in the algebraic $K$-theory literature.

3.2. Basic facts about $K$-theory. Let $X$ be a scheme. As usual $K^0(X)$ denotes the Grothendieck group of locally free coherent sheaves, and $G_0(X)$ (also called $K'_0(X)$) is the Grothendieck group of coherent sheaves [Q, Section 7]. Topologically, $K^0(X)$ corresponds to the usual complex $K$-theory functor $KU^0(-)$, whereas $G_0$ is something like a Borel-Moore version of $KU$-homology.
Note that there is an obvious map \( \alpha : K^0(X) \to G_0(X) \) coming from the inclusion of locally free coherent sheaves into all coherent sheaves. When \( X \) is nonsingular, \( \alpha \) is an isomorphism whose inverse \( \beta : G_0(X) \to K^0(X) \) is constructed in the following way [H] Exercise III.6.9. If \( F \) is a coherent sheaf on \( X \), there exists a resolution
\[
0 \to E_n \to \cdots \to E_0 \to F \to 0
\]
in which the \( E_i \)'s are locally free and coherent. One defines \( \beta(F) = \sum_i (-1)^i [E_i] \).
This does not depend on the choice of resolution, and now \( \alpha \beta \) and \( \beta \alpha \) are obviously the identities. This is 'Poincaré duality' for \( K \)-theory.

Since we will only be dealing with smooth schemes, we are now going to blur the distinction between \( G \) and \( K \). If \( F \) is a coherent sheaf on \( X \), we will write \([F] \) for the class that it represents in \( K^0(X) \), although we more literally mean \( \beta([F]) \). As an easy exercise, check that if \( i : U \to X \) is an open immersion then the image of \([F] \) under \( i^*: K^0(X) \to K^0(U) \) is the same as \([F] \mid_U \). We will use this fact often.

If \( j: Z \to X \) is a smooth embedding and \( i: X - Z \to X \) is the complement, there is a Gysin sequence [Q] Prop. 7.3.2
\[
0 \leftarrow K^0(X - Z) \xrightarrow{i_*} K^0(X) \xrightarrow{j^*} K^0(Z) \leftarrow K^{-1}(X - Z) \leftarrow \cdots
\]
(because of our degree conventions, \( K^n \) vanishes for smooth schemes when \( n > 0 \)). The map \( j_! \) is known as the Gysin map. If \( F \) is a coherent sheaf, then \( j_!( [F] ) \) equals the class of its pushforward \( j_*(F) \) (also known as extension by zero). Note that the pushforward of coherent sheaves is exact for closed immersions.

3.3. Basic facts about \( \mathbb{P}^n \). If \( Z \) is a degree \( d \) hypersurface in \( \mathbb{P}^n \), then the structure sheaf \( \mathcal{O}_Z \) can be pushed forward to \( \mathbb{P}^n \) along the inclusion \( Z \to \mathbb{P}^n \); we will still write this pushforward as \( \mathcal{O}_Z \). It has a very simple resolution of the form
\[
0 \to \mathcal{O}(−d) \to \mathcal{O} \to \mathcal{O}_Z \to 0,
\]
where \( \mathcal{O} \) is the trivial rank 1 bundle on \( \mathbb{P}^n \) and \( \mathcal{O}(−d) \) is the \( d \)-fold tensor power of the tautological line bundle \( \mathcal{O}(−1) \) on \( \mathbb{P}^n \). So \( [\mathcal{O}_Z] \) equals \([\mathcal{O}] - [\mathcal{O}(−d)] \) in \( K^0(\mathbb{P}^n) \). From now on we'll write \([\mathcal{O}] = 1 \).

Now suppose that \( Z \to \mathbb{P}^n \) is a complete intersection, defined by the regular sequence of homogeneous equations \( f_1, \ldots, f_r \in k[x_0, \ldots, x_n] \). Let \( f_i \) have degree \( d_i \). The module \( k[x_0, \ldots, x_n]/(f_1, \ldots, f_r) \) is resolved by the Koszul complex, which gives a locally free resolution of \( \mathcal{O}_Z \). It follows that
\[
[\mathcal{O}_Z] = (1 - [\mathcal{O}(−d_1)])(1 - [\mathcal{O}(−d_2)] \cdots (1 - [\mathcal{O}(−d_r)])
\]
in \( K^0(\mathbb{P}^n) \). In particular, note that
\[
[\mathcal{O}_P] = (1 - [\mathcal{O}(−1)])^{n-i}
\]
where \( \mathbb{P}^i \to \mathbb{P}^n \) is a linear subspace because \( \mathbb{P}^i \) is defined by \( n - i \) linear equations.

One can compute that \( K^0(\mathbb{P}^n) \cong \mathbb{Z}^{n+1} \), with generators \([\mathcal{O}_P], [\mathcal{O}_{P^1}], \ldots, [\mathcal{O}_{P^n}] \) (see [Q] Th. 8.2.1, as one source). If \( t = 1 - [\mathcal{O}(−1)] \), then the previous paragraph tells us that \( K^0(\mathbb{P}^n) \cong \mathbb{Z}[t]/(t^n) \) as rings. Here \( t^k \) corresponds to \([\mathcal{O}_{P^n-k}] \).

3.4. Computations. Let \( n = 2k \). Recall that \( Q_{2k-1} \) denotes the quadric in \( \mathbb{P}^{2k} \) defined by \( a_1 b_1 + \cdots + a_k b_k + c^2 = 0 \). The Chow ring \( CH^*(Q_{2k-1}) \) consists of a copy of \( \mathbb{Z} \) in every dimension (see [H] Appendix A, for example). The generators in dimensions \( k \) through \( 2k - 1 \) are represented by subvarieties of \( Q_{2k-1} \) which correspond to linear subvarieties \( \mathbb{P}^{k-1}, \mathbb{P}^{k-2}, \ldots, \mathbb{P}^0 \) under the embedding \( Q_{2k-1} \to \mathbb{P}^{2k} \). In terms of equations, the \( \mathbb{P}^{k-1} \) is defined by \( c = b_1 = \cdots = b_k = 0 \) together with \( 0 = a_k = a_{k-1} = \cdots = a_{k-i+2} \). The generators of the Chow ring in degrees 0
through \( k - 1 \) are represented by subvarieties \( Z_i \hookrightarrow \mathbb{P}^{2k} \) \((k \leq i \leq 2k - 1)\), where \( Z_i \) is defined by the equations
\[
0 = b_1 = b_2 = \cdots = b_{2k-1-i}, \quad a_1b_1 + \cdots + a_kb_k + c^2 = 0.
\]
Note that \( Z_{2k-1} = Q_{2k-1} \).

**Proposition 3.5.** The group \( K^0(Q_{2k-1}) \) is isomorphic to \( \mathbb{Z}^{2k} \), with generators \([\mathcal{O}_{p_0}], \ldots, [\mathcal{O}_{p_{k-1}}]\) and \([\mathcal{O}_{Z_0}], \ldots, [\mathcal{O}_{Z_{2k-1}}]\).

This result is classical; see [S Thm. 13.1] and [J Rem. 2.5.3]. We include a brief proof for completeness. The proof is similar to the computation of \( CH^*(Q_{2k-1}) \) given in [DI] (which goes back at least to Hodge and Pedoe [HP]), but the details are somewhat different. It is worth noting that to prove Theorem 3.1 we won’t actually need to know that \( K^0(Q_{2k-1}) \) is free—all we’ll need is the list of generators.

**Proof.** The argument is by induction. When \( k = 1 \) we have \( Q_1 \cong \mathbb{P}^1 \), and \( K^0(\mathbb{P}^1) \) has the desired form. Suppose \( k > 1 \), and write \( Q = Q_{2k-1} \) and \( Z = Z_{2k-2} \). If \(*\) denotes the point \([1,0,0,\ldots,0]\) (i.e., \( a_1 = 1 \) and all other coordinates equal to zero), consider the projection \( Z - * \hookrightarrow Q_{2k-3} \) which forgets \( a_1 \) and \( b_1 \). This is a locally trivial fiber bundle with fiber \( \mathbb{A}^1 \); hence \( K^0(Q_{2k-3}) \to K^0(Z - *) \) is an isomorphism. The closed inclusion \( Z - * \hookrightarrow Q - * \) induces a localization sequence in \( K \)-theory of the form
\[
0 \leftarrow Z \leftarrow K^0(Q - *) \xleftarrow{\delta} K^0(Z - *) \xleftarrow{\delta^{-1}} K^0(\mathbb{A}^{2k-1}) \leftarrow \cdots
\]
Note that the pullback map \( K^0(Q - *) \to K^0(Q - Z) \cong Z \) sends \([\mathcal{O}_{Q - *}]\) to \([\mathcal{O}_{Q - Z}]\), which is the generator.

We know by the computation in [DI] that \( CH^*(Q - *) \) is free of rank \( 2k - 1 \). The Chern character isomorphism \( K^0(Q - \ast) \otimes \mathbb{Q} \cong CH^*(Q - \ast) \otimes \mathbb{Q} \) then shows that the rank of \( K^0(Q - \ast) \) is \( 2k - 1 \). Since we know by induction that \( K^0(Z - *) \cong K^0(Q_{2k-3}) \) is free of rank \( 2k - 2 \), the boundary map \( \delta \) must be zero. Thus, \( K^0(Q - *\) \) is free as well.

Chasing through the isomorphisms in the above argument, \( K^0(Q - *) \) is generated by the classes \([\mathcal{O}_{(W - *)}]\) where \( W \) ranges through the subvarieties
\[
\mathbb{P}^1, \ldots, \mathbb{P}^{k-1}, Z_k, \ldots, Z_{2k-1}.
\]
Finally, one analyzes the localization sequence for the closed inclusion \(*\hookrightarrow Q:\)
\[
0 \leftarrow K^0(Q - *) \leftarrow K^0(Q) \xleftarrow{\delta} K^0(*\) \xleftarrow{\delta^{-1}} K^0(Q - *) \leftarrow \cdots
\]
We know that \( K^0(Q - *) \) is a free abelian group of rank \( 2k - 1 \), and we know that the rank of \( K^0(Q) \) must be \( 2k \) by comparing rational \( K \)-theory to the rational Chow groups using the Chern character isomorphism for \( Q \). It follows again that \( \delta \) is zero, and \( K^0(Q) \) is free of rank \( 2k \). The map \( j_1 \) takes the generator of \( K^0(\ast) \) to \([\mathcal{O}_{p^{2k}}]\). This class together with the classes \([\mathcal{O}_W]\) where \( W \) ranges over the subvarieties in (3.5) are a free basis for \( K^0(Q) \). \( \square \)

**Proof of Theorem 3.1 when \( n \) is even.** Set \( n = 2k \). To calculate \( K^0(DQ_{2k}) \) we must analyze the localization sequence
\[
0 \leftarrow K^0(DQ_{2k}) \leftarrow K^0(\mathbb{P}^{2k}) \xleftarrow{\delta} K^0(Q_{2k-1}).
\]
The image of $j_i : K^0(Q_{2k-1}) \to K^0({\mathbb P}^k)$ is precisely the subgroup generated by $[O_{P_0}], \ldots, [O_{P_{k-1}}]$ and $[O_{Q_{2k}}], \ldots, [O_{Q_{2k-1}}]$. Since $\mathbb{P}^k$ is a complete intersection defined by $2k - i$ linear equations, formula (3.3) tells us that $[O_{P_i}] = t^{2k-i}$ for $0 \leq i \leq k-1$.

Now, $Z_{2k-1}$ is a degree 2 hypersurface in $\mathbb{P}^k$, and so $[O_{Z_{2k-1}}] = 1 - [O(-2)]$. Note that

$$1 - [O(-2)] = 2(1 - [O(1)]) - (1 - [O(-2)])^2 = 2t - t^2.$$

In a similar way one notes that $Z_i$ is a complete intersection defined by $2k - 1 - i$ linear equations and one degree 2 equation, so formula (3.3) tells us that

$$[O_Z] = (1 - [O(1)])^{2k-1-i} \cdot (1 - [O(-2)]) = t^{2k-1-i}(2t - t^2).$$

The calculations in the previous two paragraphs imply that the kernel of the map $K^0(\mathbb{P}^k) \to K^0(DQ_{2k})$ is the ideal generated by $2t - t^2$ and $t^{k+1}$. This ideal is equal to the ideal generated by $2t - t^2$ and $2kt$, so $K^0(DQ_{2k})$ is isomorphic to $\mathbb{Z}[t]/(2kt, 2t - t^2)$. If we substitute $\nu = [t] - 1 = 2t$, we find $\nu^2 = -2\nu$.

To find $K^0(DQ_{2k})$, we just have to take the (additive) quotient of $K^0(DQ_{2k})$ by the subgroup generated by 1. This quotient is isomorphic to $\mathbb{Z}/2k$ and is generated by $\nu$.

This completes the proof of Theorem 8.1 in the case where $n$ is even. The computation when $n$ is odd is very similar; we will only briefly outline the differences.

**Sketch proof of Theorem 8.1 when $n$ is odd.** In this case $Q_{n-1}$ is defined by the equation $a_1b_1 + \cdots + a_kb_k = 0$ with $k = \frac{n-1}{2}$. The Chow ring $\text{CH}^*(Q_{n-1})$ consists of $\mathbb{Z}$ in every dimension except for $k - 1$, which is $\mathbb{Z} \oplus \mathbb{Z}$. The ‘extra’ generator in this dimension is the projective space $P$ defined by $0 = b_1 = b_2 = \cdots = b_{k-1} = a_k$. One finds that $K^0(Q_{n-1})$ is free of rank $n + 1$, on the same generators as before plus this ‘extra’ generator $[O_P]$. The map $j_i : K^0(Q_{n-1}) \to K^0({\mathbb P}^n)$ sends both $[O_P]$ and $[O_{P_{k-1}}]$ to $t^{k-1}$, so this doesn’t affect the computation of $K^0(DQ_n)$.

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