Which tunnel faster across a quantum Hall strip: fractional charges or electrons?

Assa Auerbach

Department of Physics, Technion, Haifa 32000, Israel.

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The tunneling rate $t_{1/3}$ of fractional charge across a $\nu = 1/3$ Laughlin state on the cylinder is computed numerically using Laughlin states on the cylinder. The decay with strip width $Y$ is fitted to $t_{1/3} \propto \exp(-\alpha Y^2/(12\lambda^2))$ where $\lambda$ is the Landau length, and $\alpha \simeq 1.0$. This rate is exponentially larger than the electron tunneling rate $t_1 \propto \exp(-Y^2/(4\lambda^2))$, and can be interpreted by analogy to a superfluid vortex tunneling problem. Experimental implications include the “law of corresponding states”, periodicity of Aharonov-Bohm resistance oscillations and charge measurements by quantum shot noise.

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Transport of fractional charges in quantum Hall systems has important experimental manifestations. Quasiparticles and edges of an incompressible Hall liquid strip of bulk filling fraction $\nu = 1/m$, ($m$ is an odd integer $\square$), have quantized fractional charges whose values determine the following effects: (i) The leading powers (of current and temperature) of the longitudinal voltage drop depend on the tunneling charge $Q$, as shown by Wen using Luttinger liquid edge theory $\square$. (ii) The Aharonov-Bohm (AB) flux periodicity $\Delta \phi$ of the current oscillations in a Corbino disk is related to the tunneling charge by $Q = e\phi_0/\Delta \phi$ $\square$ (iii) The charges which dominate the backscattering current $I_B$ can be measured by the magnitude of the quantum shot noise $S$ by $S = 2QI_B$ $\square$. Recent experiments in the $\nu = 1/3$ phase, report excellent fits to fractional charge $Q = e/3$ $\square$.

Impurities allow tunneling of both fractional charges and electrons across the bulk of a Hall fluid. The question of the title is therefore relevant to the observations of experimental probes (i)-(iii), since the latter depend on the relative rates of fractional versus integer charge tunneling. Kane and Fisher (KF) $\square$ calculated the renormalization group flows of the tunneling coupling constants due to low lying Luttinger liquid edge excitations. They found that for $\nu < 1$, electron tunneling becomes irrelevant at low enough temperatures while fractional charge tunneling flows to strong coupling as the infra-red cut-off is reduced. In KF theory, however, the inter-edge scattering parameter is an undetermined parameter, which leaves the possibility that it might be undetectable at experimental temperatures.

This Letter presents a microscopic calculation of fractional charge tunneling rates across a Hall fluid. The numerical results show that that the fractional charge tunneling rate is much larger than the electron charge rate at large strip widths. Subsequently, we connect the microscopic tunneling matrix element to the inter-edge scattering parameter of KF theory, and discuss its experimental implications.

Our domain is the open cylinder $x \in [0, 2\pi R)$, $-\infty < y < \infty$, with $N$ electrons, and a radially penetrating field $B = \frac{e\phi_0}{2\pi$, where $\lambda$, the Landau length, is henceforth our unit of distance. This geometry can describe a quantum Hall liquid strip with two symmetric edges.

The free electron states of the lowest Landau level (LLL) are labelled by momenta $k = \gamma n$, $n$ integer, and $\gamma \equiv 1/R$. The wavefunctions are

$$\psi_k(x, y) = \sqrt{\frac{\gamma}{2\pi}} \exp \left( ikx - \frac{(y-k)^2}{2} \right).$$

The Laughlin state of filling fraction $\nu = \frac{1}{m}$ on the cylinder was given by Thouless $\square$.

$$\psi^{1/m} = \prod_{i<j} (e^{ik(x_i+y_j)} - e^{ik(x_i+y_j)})^m \prod_i e^{-\gamma^2/2}$$

It is the ground state of a suitably defined pseudopotential Hamiltonian $\square$. The expansion of $\psi^{1/m}$ in the LLL Fock basis is

$$\Psi^{1/m}_L = \sum_{|k|} A[|k|/\gamma] \exp \left( \sum_i k_i^2 \right) |k|$$

where $|k| = |k_1, \ldots, k_N|$ and $k_i \in [0, Y]$. $Y = m\gamma(N-1)$ is defined as the width of the Hall liquid strip (the width of the area partially occupied by electrons depicted between the horizontal solid lines in Fig. $\square$).

There is an infinite family of other degenerate groundstates labelled by the total momentum $P = \sum_{i} k_i$, which are given by uniformly shifting the momenta $k_i$ moving the electron density up or down the cylinder. A weak ($v \to 0$) confining potential $V(y) = \frac{\gamma}{2}(y - Y/2)^2$, selects $\square$ as the ground state.

The expansion coefficients $A$ are given by

$$A[n] = \frac{1}{N!} \sum_{r^1, \ldots, r^m} (-1)^{\sum_{i=1}^m r^i} p[r^i] \prod_{i=1}^N \delta \left( n_i - \sum_{l=1}^m r^i_l \right)$$

where $r^i$ is a permutation of the set 0, 1, \ldots, $N-1$, and $p$ is the parity of a permutation.
The coefficients $A$ have a complicated structure \[ A(k_0,\mu) \neq 0 \] but it is useful to note that the components with $A(k_0,\mu) \neq 0$ can be derived from a single parent Tao-Thouless (TT) state \[ |k^{TT}\rangle = |0, m\gamma, 2m\gamma, \ldots, Y\rangle \] (5)

For this state $A(k^{TT}/\gamma) = 1$. All other $|k\rangle$ components are given by successively squeezing pairs of momenta toward each other. Rezayi and Haldane have shown \[ \text{that in the regime } 1 << Y << N \text{ the occupation number is constant for } k \text{ far from the edges, i.e. } n_k = \langle c_k^\dagger c_k \rangle = 1/m \text{ for } 1 < k < N - 1. \]

An impurity potential in the LLL Fock representation is

$$\mathcal{V} = \sum_{k,k'} V_{k,k'} c_k^\dagger c_{k'}$$

where $c_k^\dagger$ creates an electron in state $\phi_k$. The ground state to ground state tunneling rate of charge $qe$ between the edges, to leading order in $\mathcal{V}$, is

$$t_q = \langle \Psi | \mathcal{V}^{(mq)} | \Psi \rangle,$$

(7)

where $U$ is the unitary phase operator which translates all the single particle momenta by one interval

$$U^\dagger c_k U = c_{k+\gamma_k}.$$ (8)

$U^\dagger$ moves a fractional charge $1/m$ from the $p = -1$ to the $p = +1$ edge (see Fig. \[ ]), and thus increases the total momentum of the Hall state by $P \rightarrow P + Q$, where $Q = N\gamma$.

For a weak impurity potential, the tunneling rate of a fractional charge is thus given by

$$t_{1/m} = \langle \Psi | \mathcal{V} U | \Psi \rangle = \sum_k V_{k,k+Q} M_{k,k+Q}$$

$$M_{k,k+Q} = \frac{1}{Z} \sum_{k,k'} A(k/\gamma) A(k'/\gamma) e^{-\frac{1}{2} (k^2 + k'^2)}$$

$$\times \prod_{n=1}^N \delta^N(k + Q(n), k' - \gamma 1)$$

(9)

where $Z = \langle \Psi | \Psi \rangle$, $Q(n) = Q\delta_{n1}$, and $1_i = 1$. $M_{k,k+Q}$ reflects the many body overlap of the relatively displaced Laughlin states. Its weighted sum $M(Q) = V^{-1} \sum_k V_{k,k+Q} M_{k,k+Q}$ was computed numerically for local impurity potentials $V \delta(x)$ and $V \delta(x) \delta(y - Y/2)$. The calculation was carried out for $\nu = 1/3$ states with 5 up to 8 electrons. As shown in Figs. \[ 6 \] at large widths we find the asymptotic decay

$$|M(Q)| \propto \exp\left(-\frac{\alpha}{2} Q^2\right)$$

(10)

where $\alpha \approx 1.0$, and independent of the number of electrons. Combining (10) with $V_{0,Q} \propto \exp(-\frac{1}{2} Q^2)$ yields the tunneling rate’s asymptotic dependence on width

$$t_{1/3} \approx \exp(-\alpha Y^2/12)$$

(11)

In comparison, a unit charge tunneling rate, which is proportional to the potential matrix element, is

$$t_1 = \langle \Psi | U^{3} | \Psi \rangle = n_0^2 V(0,Y)$$

(12)

For a localized potential of the form $V \delta(x) \delta(y - Y/2)$,

$$t_1 \sim \gamma^2 \exp(-Y^2/4)$$

(13)

where $n_0 \approx \gamma^3$ is appropriate for a density profile which vanishes as a power law $n_k \approx k^3$ at the edge. (The numerical results for $n_k$ in the Laughlin state \[ ] up to 8 particles is $\beta = 1.0$.) Thus, the tunneling exponent is three times larger for quasiparticles than for electrons.

This result could be understood using the superfluid description of the fractional Hall phase, which can be derived by the Chern-Simons Ginzburg-Landau functional (CSGL). \[ ] At the mean field level, the ground state is a Bose superfluid of density $\rho_s = \frac{1}{m} B/\phi_0$. The dissipation of current involves tunneling of vortices between opposite edges, where a vortex of unit circulation carries a fractional electric charge of $e/m$. Ignoring auxiliary gauge field fluctuations, and interactions at the core size, the vortex dynamics are governed by a Magnus force $e\phi_0 \rho_s \mathbf{v} \times \mathbf{\hat{z}}$. Thus they are quantized as particles with charge $e$ in the lowest Landau level of an effective field $\mathbf{B} = \phi_0 \rho_s$, and Landau length $\lambda = \lambda_0 \pi R$, with wave functions given by \[ ] . For $m = 3$, the matrix element of $\mathcal{V}$ between two vortex wavefunctions at the edges readily recovers (11), with $\alpha = 1$.

**How do tunneling matrix elements couple to edge excitations?** A half-strip density operator is defined as follows:

$$\rho_p(q) = \int_{Y/2}^{Y} dy \int_{0}^{2\pi R} dxe^{iqx} \rho(x,y)$$

$$\approx \sum_k \theta_{p,k} \theta_{p,k+q} c_{k+q}^\dagger c_k + \mathcal{O}(q^2)$$

(14)

where $p = \pm 1$, $Y_p = (1 + p)Y/2$ are the two edge $y$-coordinates, and

$$\theta_{pk} = \begin{cases} 1 & p(k - Y/2) > 0 \\ 0 & p(k - Y/2) < 0 \end{cases}$$

(15)

The last approximation in (14) applies to the long wavelength regime $q << 1$. The commutation relations of $\rho_p(q)$ are

$$[\rho_p(q), \rho_{p'}(q')] = \delta_{pp'} \delta_{qq'} \sum_k \theta_{pk} \theta_{p,k+q} (n_{p,k} - n_{p,k+q})$$

$$+ \delta_{pp'} \{q \neq -q'\}$$

(16)

Since excitations in the bulk have an energy gap $\Delta_B$, the low energy sector includes only particle-hole excitations.
near the edges, i.e. \( c_k^\dagger c_{k+q} \Psi \) with \( k \approx Y_p \), and energies \( \omega_q = \nu q \), where \( \nu \) is the gradient of the confining potential. \( \{ q \neq q' \} \) terms in \([16]\) create excitations deep in the bulk which introduce corrections suppressed by factors of \( \omega_q/\Delta_B \) and \( q/Y \). Also, in this sector \( \nu_{p,k} \) is approximately diagonal

\[
n_{p,k} \simeq \begin{cases} \nu & p(Y_p - k) >> 1 \\ 0 & (Y_p - k) >> 1 \end{cases}
\]

(17)

Thus, Wen’s Kac-Moody algebra of edge bosons \([3]\) is recovered:

\[
[\rho_p(q), \rho_{p'}(q')] \simeq \delta_{pp'} \delta_{q,-q'} \gamma^{-1} \nu q
\]

(18)

The edge charge operator is \( N_p = \sum_k \theta_{pk} n_{p,k} \), which is conjugate to the edge phase operators \( U_p \):

\[
[N_p, U_p^\dagger] = pm_p, \gamma/2 U_p \simeq \nu p U_p^\dagger
\]

(19)

The total phase operator \([3]\) is \( U^\dagger = U_1^\dagger U_{-1} \). The edge quasiparticle creation operator is constructed following Haldane \([3]\):

\[
\phi_p = p\gamma \left( xN_p/2 + i \sum_{q \neq 0} \theta(-pq)e^{-iqx}/q \rho_p(q) \right)
\]

\[
\psi_p^\dagger(x) = Ae^{i\varphi_p x}e^{iqx/\nu} U_p^\dagger e^{i\varphi_p(x)}
\]

(20)

where \( A \) is an undetermined normalization constant. \( \psi_p^\dagger(x) \) creates a localized edge excitation of extra charge \( \nu \) as evidenced by the commutator with \( \rho_p(x) = \sum_q e^{iqx} \rho_p(q) \):

\[
[\rho_p(x), \psi_{p'}^\dagger(x')] = \nu \delta_{pp'} \delta(x - x') \psi_{p'}^\dagger(x)
\]

(21)

The impurity potential operator in the low energy sector, simply transfers a localized fractional charge between the edges. It must therefore be proportional to the normal ordered operator

\[
\mathcal{V}(x) = : \psi_p^\dagger(x) \psi_{-p}(x) : + \text{H.c.}
\]

\[
= A^2 e^{i\varphi_p x} e^{i \sum_p \phi_p^\dagger(x)} U_p^\dagger e^{i \sum_p \phi_p(x)} + \text{H.c.}
\]

(22)

The normalization \( A^2 \) is precisely the bare fractional charge tunneling parameter amplitude in KF theory \([3]\). It can now be determined by sandwiching both sides of Eq. \([22]\) between the relatively displaced ground states leading to

\[
A^2 = \langle \Psi | \mathcal{V} U | \Psi \rangle = t_{1/m}
\]

(23)

Two comments. (i) Tao and Haldane \([3]\) have shown that in the absence of an impurity potential, the quantum Hall ground state of \( \nu = 1/m \) on the torus has an \( m \) fold degeneracy. A time dependent AB flux threading the torus, moves the ground state between the \( m \) different states of this manifold, and the Hall conductance is precisely \( \sigma_{xy} = \frac{1}{4\pi v^2} \). An impurity potential couples between degenerate ground states, as it does on the cylinder, opening a minigap \( \Delta \) between the ground state and the first excited state \([\beta]\). The quantum Hall effect can be observed provided the flux does not vary extremely slowly \([\beta]\), i.e. \( \Delta/\nu << \phi_0/V_x \), where \( V_x \) is the induced electromotive force.

(ii) For the infinite plane geometry, the tunneling exponent between two \emph{localized} quasiparticle states centered on delta function impurities at \( r_1, r_2 \) is \([\alpha]\)

\[
S(r_1 - r_2) \propto \exp \left( -\frac{B}{4m\phi_0} |r_1 - r_2|^2 \right)
\]

(24)

The scaling of the tunneling action with \( B/m \) was argued to be a general property of low elementary excitations with fractional charge \( 1/m \). Generalizing this idea further, Jain, Kivelson and Trivedi \([17]\) have formulated a “law of corresponding states” which relate the dissipative response of quantum Hall liquids at different filling fractions using a conjecture that they scale with \( \frac{B}{m} \). The results reported here are consistent with this law, provided we restrict ourselves to \textit{wide} Hall strips in the presence of \textit{weak} impurity potentials.

\textbf{Discussion}: KF have shown that edge excitations enhance the fractional charge and suppress the unit charge contributions to the backscattering current \([3]\). Thus the renormalization group flow enhances the bare tunneling ratio, which strengthens the experimental relevance of KF theory.

Current oscillations in the Corbino disk in a slowly time dependent AB flux can measure the dominant transport mechanism between edge states \([\beta]\). Let us first consider \emph{open leads} in the Hall voltage terminals. The impurity potential opens minigaps between the adiabatic energy curves as a function of AB flux \( \phi_{AB} \). The lowest level crossing is at \( \phi_{AB} = \phi_0 \) between curves with minima separated by \( \phi_0 \), with a minigap of size \( t_1/3 \). A minigap \( t_1 \) opens at higher energies between curves with minima separated by \( 3\phi_0 \). Since \( t_1/3 >> t_1 \) we expect to see only a periodicity of \( \phi_0 \) for open leads. If, on the other hand, we short the two Hall terminals with a low resistance wire, electron tunneling between the edges is enabled. Thus, by varying the AB flux faster than the minigap \( t_1/3 \) we could observe a periodicity of \( 3\phi_0 \), reflecting transfer of unit charges through the short.

Finally, the quasiparticle charge which dominates the backscattering current between the edges can be measured by quantum shot noise at zero temperature and bias \([\beta]\). Recent experimental reports of measuring fractional charge in quantum shot noise of fractional quantum Hall systems \([\beta]\) are consistent with the expectation that fractional charges tunnel faster than electrons.

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[1] “The Quantum Hall Effect”, Eds. R.E. Prange and S.M. Girvin (Springer-Verlag, NY 1987).
[2] X.-G. Wen, Int. Jour. Mod. Physics A6, 1711 (1992).
[3] Y. Gefen and D.J. Thouless, Phys. Rev. B47 10423 (1993).
[4] P. Fendley, A.W.W. Ludwig and H. Saleur, Phys. Rev. Lett 75 2196 (1995).
[5] L. Saminadayar, D.C. Glattli, Y. Jin and B. Etienne, cond-mat/9706307; R. de-Picciotto, M. Reznikov, M. Heiblum, V. Umansky, G. Bunin, D. Mahalu cond-mat/9707289.
[6] C.L. Kane and M.P.A. Fisher, Phys. Rev. B46 15233 (1992).
[7] D.J. Thouless, Surf. Sci. 142, 147 (1984).
[8] E.H. Rezayi and F.D.M. Haldane, Phys. Rev. B50 17119 (1994).
[9] K. Takano and A. Isihara, Phys. Rev. B34 1399 (1986).
[10] P. Di Francesco, M. Gaudin, C. Itzykson and F. Lesage, Int. Jour. Mod. Physics A9, 4257 (1994).
[11] R. Tao and D.J. Thouless, Phys. Rev. B28 1142 (1983).
[12] S.-C. Zhang, T.H. Hansson and S.A. Kivelson, Phys. Rev. Lett 62 82 (1989).
[13] F.D.M. Haldane, J. Phys. C14 2585 (1981).
[14] R. Tao and F.D.M. Haldane, Phys. Rev. B33 3844 (1986). The value of the impurity minigap on the torus in their Eq. (4.9) differs by the factor of $M$ from our result (9) for the cylindrical geometry.
[15] In Eq. (4.9) of ref. [14], the estimate of the impurity matrix element does not include an exponential suppression due to the many-body factor (the analogue of $M(Q)$ of Eq. (10) for the torus).
[16] R. B. Laughlin, in Ref. [1], S.A. Kivelson and V.L. Pokrovsky, Phys. Rev. B40 1373 (1989);
[17] J.K. Jain, S.A. Kivelson and N. Trivedi, Phys. Rev. Lett 64 1297 (1990).

**FIG. 1.** Fractional charge tunneling depicted by two displaced Laughlin states of bulk density $\nu$ on the cylinder. $\Delta N_p$ are the edge charge differences, and $V$ is an impurity potential which enables a transition between the states.

**FIG. 2.** Numerical evaluation of the asymptotic decay of the many-body factor $M(Q)$, see Eq. (10) for the $\nu = 1/3$ Laughlin states. (a) and (b) show similar dependence for two different localized impurity potentials.
$V = \delta(x) \delta(y - Y/2)$

![Graph showing $d \log(M)/d Q^2$ versus $Q^2$ for different values of $N$.]