Effective description of the LOFF phase of QCD

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Abstract

We present an effective field theory for the crystalline color superconductivity phase of QCD. It is known that at high density and at low temperature QCD exhibits a transition to a color superconducting phase characterized by energy gaps in the fermion spectra. Under specific circumstances the gap parameter has a crystalline pattern, breaking translational and rotational invariance. The corresponding phase is the crystalline color superconductive phase (or LOFF phase). We compute the parameters of the low energy effective lagrangian describing the motion of the free phonon in the high density medium and derive the phonon dispersion law.

1 Introduction

In this talk We will discuss the recently proposed crystalline color superconducting phase of QCD. This state is the QCD analogous to a state studied in QED by Larkin and Ovchinnikov and Fulde and Ferrell. Therefore it has been named after them LOFF state. It is known that at very high density and at low temperature quark matter exhibits color superconductivity. Color superconductivity is a phenomenon
analogous to the BCS superconductivity of QED. According to the BCS theory, whenever there is an arbitrary small attractive interaction between electrons, the Fermi sea becomes unstable with respect to the formation of bound pairs of electrons. Indeed the creation of a Cooper pair costs no free energy, because each electron is on its own Fermi surface. On the other hand there is an energy gain due to the attraction of electrons. We call \( \Delta \) the binding energy of each pair with respect to the Fermi level. It is possible to show that the most energetically favored arrangement of electrons is made of two electrons with opposite momenta and spins. That is the condensate has spin zero and momentum zero. This is a normal superconductor. Now let us see what is a LOFF superconductor.

2 \textbf{LOFF phase in QED}

Let us consider a system made of a ferromagnetic alloy containing paramagnetic impurities. In first approximation the action of the impurities on the electrons may be viewed as a constant self-consistent exchange field. Due to this field the Fermi surfaces of up and down electrons (with respect to the direction of polarization of the medium) split, see figure (1). The half-separation of the Fermi surfaces will be

\[ I = N S a, \]

where \( N \) is the concentration of impurities, \( S \) is their spin and \( a \) is the integral of the exchange potential. To determine the energetically favored state we have to compare the BCS free energy with the free energy of the unpaired state. One has

\[ F_{BCS} - F_{\text{free}} \propto I^2 - \Delta^2/2. \]  

(2)

Therefore for \( I > \Delta/\sqrt{2} \) the BCS state is no more energetically favorite. For \( I = \Delta/\sqrt{2} \) one expects that a first order phase transition takes place from the BCS to the normal state. Or at least this is what one should naively expect. Instead it has been shown that for

\[ \frac{\Delta}{\sqrt{2}} < I < .75 \Delta, \]

(3)

it is energetically favorite a superconducting phase characterized by a condensate of spin zero but with non zero total momentum. We call \( \Delta_{LOFF} \equiv \)
Figure 1: Fermi surfaces of down (d) and up (u) electrons (quarks). Black dots are electrons (quarks). The BCS pair, in a), gains energy $\Delta$ but has $p_F^d = p_F^u$ and therefore pays free energy price. Indeed one of the two electrons (quarks) of the pair has to stay far from its own Fermi surface. The LOFF pair, in b), gains energy $\Delta_{LOFF} \ll \Delta$ but does not pay any free energy price, because each electron (quark) is on its own Fermi surface. For more details on figure b) see the text below.

$\Delta_{LOFF}(I)$ the electron binding energy when the distance between the Fermi surfaces is $2I$. Increasing the half-distance between the Fermi spheres, from $\frac{\Delta}{\sqrt{2}}$ to $.75\Delta$, the LOFF gap decreases to zero. Therefore, for $I \simeq .75\Delta$, a second order phase transition from the LOFF phase to the normal phase takes place. In the LOFF phase pairs are made of fermions which remain close to their own Fermi surfaces as shown in figure [1]. Therefore the creation of a pair costs no free energy. The pair has total momentum $2\vec{q}$, given by the formula

$$\vec{K}_u + \vec{K}_d = 2\vec{q},$$  (4)

where $\vec{K}_u$ and $\vec{K}_d$ are the momenta of up and down electrons. The magnitude of $q$ is determined by minimizing the free energy of the LOFF state. The direction of $\vec{q}$ is chosen spontaneously by the condensate. We observe that not all electrons can condense in LOFF pairs. Only the electrons of a restricted region of the phase space can pair because of equation (3). Therefore $\Delta_{LOFF} \ll \Delta_{BCS}$ because of the reduced phase space available for coupling.
Moreover $\Delta_{LOFF}$ turns out to be a periodic function of the coordinates and we have a crystalline structure described by $\Delta_{LOFF}(\vec{r})$.

3 LOFF phase in QCD

Let us consider quark matter at asymptotic density and at low temperature. In the previous section we have learnt that the LOFF phase may arise when fermions have different chemical potentials, and the potential difference lies inside a certain window\cite{1, 4, 5, 6, 7, 8, 9}. Crystalline color superconductivity is also expected to occur in case of different quark masses\cite{11}. In what follows we shall consider matter made of massless quarks and electrons, in weak equilibrium and electrically neutral. If we impose weak equilibrium we have $\mu_u = \bar{\mu} - \frac{2}{3}\mu_e$ and $\mu_{d,s} = \bar{\mu} + \frac{1}{3}\mu_e$ where $\mu_{u,d,s,e}$ are quarks and electrons chemical potential. Requiring electrical neutrality we get:

$$\frac{2}{3}N_u - \frac{1}{3}N_d - \frac{1}{3}N_s - N_e = 0,$$

(5)

where $N_{u,d,s,e}$ are the concentrations of quarks and electrons. When $m_s = 0$ we get $N_u = N_d = N_s, N_e = 0$ and we expect matter to be in the color-flavor locked phase (CFL)\cite{5}. For $m_s \sim \bar{\mu}$ we have $\mu_u = \bar{\mu} - \delta \mu, \mu_d = \bar{\mu} + \delta \mu$, and there are just two active flavors.

4 Velocity dependent effective lagrangian

In this Section we give a brief review of the effective lagrangian approach for the LOFF phase\cite{7}, based on velocity-dependent fields. We perform a Fourier transformation of the fermion field

$$\psi(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \psi(p),$$

(6)

and we decompose the fermion momentum as

$$p_i^\mu = \mu_i v_i^\mu + \ell_i^\mu,$$

(7)

where $i = 1, 2$ is a flavor index, $v_i^\mu = (0, \vec{v}_i)$ and $\vec{v}_i$ the Fermi velocity (for massless fermions $|\vec{v}| = 1$); finally $\ell_i^\mu$ is a residual momentum. By the decomposition (7) only the positive energy component $\psi_+$ of the fermion field
survives in the lagrangian in the $\mu \to \infty$ limit while the negative energy component $\psi_-$ can be integrated out. These effective fields are velocity dependent and are related to the original fields by

$$
\psi(x) = \sum_{\vec{v}} e^{-i\mu v \cdot x} [\psi_+(x) + \psi_-(x)],
$$

where

$$
\psi_{\pm}(x) = \frac{1 \pm \vec{\alpha} \cdot \vec{v}}{2} \int \frac{d\ell}{(2\pi)^3} \int_{-\infty}^{+\infty} \frac{d\ell_0}{2\pi} e^{-i\ell x} \psi(\ell).
$$

Here $\sum_{\vec{v}}$ means an average over the Fermi velocities and

$$
\psi_{\pm}(x) \equiv \psi_{\pm,\vec{v}}(x)
$$

are velocity-dependent fields.

Now we write the condensates. We have a scalar condensate (sc)

$$
<\epsilon_{ij}\epsilon_{\alpha\beta}\psi^i_{\alpha}(\vec{x})C\psi^j_{\beta}(\vec{x})> \propto \Delta_A e^{2i\vec{q}\cdot\vec{x}}
$$

and a spin 1 vector condensate (vc)

$$
<\sigma^1_{ij}\epsilon_{\alpha\beta}\psi^i_{\alpha}(\vec{x})C\sigma^0_{j\beta}(\vec{x})> \propto \Delta_B e^{2i\vec{q}\cdot\vec{x}}.
$$

The lagrangian for the (sc) is

$$
L_{\Delta}^{(s)} = -\frac{\Delta_A}{2} \sum_{\vec{v}_1, \vec{v}_2} \exp\{i\vec{x} \cdot \vec{\alpha}(\vec{v}_1, \vec{v}_2, \vec{q})\}\epsilon_{ij}\epsilon_{\alpha\beta}\psi_{-\vec{v}_1}(x)C\psi_{-\vec{v}_2}(x)
- (L \to R) + h.c.,
$$

where $\vec{\alpha}(\vec{v}_1, \vec{v}_2, \vec{q}) = 2\vec{q} - \mu_1 \vec{v}_1 - \mu_2 \vec{v}_2$. We take $\vec{q} = (0, 0, q)$, therefore the components of $\vec{\alpha}$ satisfy the equations

$$
\alpha_x = -\mu_1 \sin \alpha_1 \cos \phi_1 - \mu_2 \sin \alpha_2 \cos \phi_2,
\alpha_y = -\mu_1 \sin \alpha_1 \sin \phi_1 - \mu_2 \sin \alpha_2 \sin \phi_2,
\alpha_z = 2q - \mu_1 \cos \alpha_1 - \mu_2 \cos \alpha_2.
$$

In the limit $\mu_1, \mu_2 \to \infty$ we get

$$
\phi_2 = \phi_1 + \pi
$$

$$
\theta_q = \arccos \left( \frac{\delta \mu}{q} \right) + O\left( \frac{\delta \mu}{\mu} \right),
$$

$$
\alpha_1 = \alpha_2 + \pi + O\left( \frac{\delta \mu}{\mu} \right).
$$
Therefore the velocities are almost opposite $\vec{v}_1 \simeq -\vec{v}_2$, and we have to deal with a no more symmetric sum over velocities, because the angle $\alpha_2$ is fixed.

5 Phonon quark interaction

The condensates (11) and (12) explicitly break rotations and translations. We have an induced lattice structure given by parallel planes perpendicular to $\vec{n} = \vec{q}/|\vec{q}|$:

$$\vec{n} \cdot \vec{x} = \frac{\pi k}{q} \quad (k = 0, \pm 1, \pm 2, \ldots).$$

(18)

Lattice planes are allowed to fluctuate in two ways. We describe these fluctuations by means of two fields $\phi$ and $R$:

$$\vec{n} \cdot \vec{x} \to \vec{n} \cdot \vec{x} + \frac{\phi}{2qf} \equiv \frac{\Phi}{2q},$$

(19)

$$\vec{n} \to \vec{R}.$$  

(20)

But $\phi$ and $R$ are not independent, indeed it is possible to show that

$$\vec{R} = \frac{\vec{\nabla}\Phi}{|\vec{\nabla}\Phi|}.$$  

(21)

Therefore there is just one independent degree of freedom, i.e. the phonon.

6 Effective lagrangian

By a bosonization procedure it is possible to get the effective action for the phonon and to calculate the polarization tensor. At the lowest order in the momentum $\vec{p}$ of the phonon we have

$$\Pi(0) = \Pi(0)_{s.e.} + \Pi(0)_{tad} = 0,$$

(22)

$$\Pi(p) = -\frac{\mu^2}{4\pi^2 f^2} \left[ p_0^2 - v_1^2 (p_x^2 + p_y^2) - v_2^2 p_z^2 \right],$$

(23)
where

\[
v_{\perp}^2 = \frac{1}{2} \sin^2 \theta_q + O \left( \frac{\Delta^{(\nu)}}{q} \right)^2, \quad (24)
\]

\[
v_{\parallel}^2 = \cos^2 \theta_q, \quad (25)
\]

are the velocities perpendicular and parallel to \( \vec{q} \). From eq. (22) we see that the phonon is massless. From eq. (23) we have the dispersion law for the phonon:

\[
E(\vec{p}) = \sqrt{v_{\perp}^2 (p_x^2 + p_y^2) + v_{\parallel}^2 p_z^2}. \quad (26)
\]

In conclusion we can say that the dispersion law for the phonon is anisotropic in two ways. Indeed the velocity of propagation of the phonon in the plane perpendicular to \( \vec{q} \), i.e. \( v_{\perp} \), is not equal to \( v_{\parallel} \), that is the velocity of propagation of the phonon along \( \vec{q} \). Moreover \( p_z \) is a quasi-momentum.

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