Comparative study of high order methods for subsonic turbulence simulation with stochastic forcing

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Abstract. A class of spatially seventh-order nonlinear filter methods with adaptive dissipation control developed by Yee & Sjögreen [1, 2] is tested on three-dimensional subsonic turbulence simulations with stochastic forcing. The Euler equations are solved using the Strang operator splitting of the homogeneous part of the equations and the stochastic forcing term, with an ODE solver used to integrate the latter. Both Ducros et al. and Kennedy-Gruber skew-symmetric split formulations of the inviscid flux derivatives are considered to minimize the use of numerical dissipation. The nonlinear filter methods are shown to be numerically stable for this application at least up to an rms Mach number of 0.6. The performance and accuracy of this numerical approach are compared with those of second order TVD and fifth and seventh order WENO methods. The nonlinear filter methods are shown to be substantially more computationally efficient, delivering a superior spectral bandwidth compared to the standalone TVD and WENO methods.

1. Introduction
Developments in [3, 4, 1, 2] and studies in [1, 5, 6] indicated that a class of high order accurate nonlinear filter methods is efficient at suppressing spurious oscillations at discontinuities and high frequency oscillations that occur in long time integration of highly coupled nonlinear equations. The idea of nonlinear filtering was first articulated by Yee et al. [3] and tested using an artificial compression method of Harten [7] as the flow sensor. Smart flow sensors were developed at a later stage by Sjögreen & Yee, Yee & Sjögreen and Kotov et al. [4, 8, 2, 5, 6]. The smart flow sensor provides the locations and the estimated strength of numerical dissipation needed at these locations and leaves the rest of the flow field free of shock-capturing dissipation. This nonlinear filter approach requires one Riemann solver per time step per grid point for each spatial direction and it is completely independent of the time discretization employed. However, hybrid schemes (switching between high order non-dissipative methods and high order shock-capturing methods) would require four Riemann solvers per time step per grid point for each
spatial direction, if for instance a four-stage Runge-Kutta time discretization is used. Unlike the hybrid methods, the nonlinear filter approach is highly parallelizable and does not rely on switching between schemes to avoid the associated numerical stability issues and violation of conservation properties at switching locations.

Starting from the early 1980s, the use of skew-symmetric split forms of the inviscid flux derivatives in conjunction with central schemes was shown to improve their numerical stability for long time integration. It was found that certain split approximations can provide a stable $L_2$ energy-like norm estimate for smooth solutions of the Euler equations. Other skew-symmetric formulations can maintain a discrete momentum conservation or have a discrete kinetic energy preservation property. See articles by Arakawa, Blaisdell et al., Yee et al., Yee & Sjögreen, Sjögreen & Yee, Kotov et al., and Pirozzoli [9, 10, 11, 12, 13, 14, 8, 2, 15, 5, 6, 16, 17] for discussion of the performance of various skew-symmetric splitting approaches in applications to DNS andLES. See also Sjögreen et al. [18] for an overview of the high order formulations for the Euler and magnetohydrodynamic equations.

All of the methods mentioned above were developed for turbulent flows without any source terms, while insufficient attention has been paid to the development of numerical methods for problems with time-varying stochastic forcing, which is often included in simulations of statistically stationary turbulence. Here, we apply methods developed in [19, 20, 21, 22, 23] to simulate subsonic stochastically driven turbulence in a periodic domain (see Bauer et al. [24] for earlier low-resolution simulations with discontinuous Galerkin methods, using a similar setup). For the 3D Euler equations with a source term, describing the external stochastic force, we use the Strang operator splitting approach to solve the homogeneous part of the Euler equations and the forcing separately [19, 20]. The spatially seventh-order nonlinear filter methods are used to solve the homogeneous system and an ODE solver is used to treat the stochastic forcing. The nonlinear filter method consists of a full time step of a spatially eighth-order central base scheme step, using a third-order total variation diminishing (TVD) Runge-Kutta time integration. The solution computed with the base scheme is then nonlinearly filtered, using an adaptive flow sensor and the dissipative portion of a seventh-order weighted essentially non-oscillatory (WENO) scheme with the Roe Riemann solver. In order to improve the nonlinear stability of the nondissipative base scheme without adding excessive numerical dissipation (modeling turbulent flows requires stable long time integration), the eighth-order central base scheme discretizes a skew-symmetric split form of the inviscid flux derivatives. Both Ducros et al. [14] and Kennedy-Gruber [25, 17] skew-symmetric split formulations are considered in the numerical studies discussed below.

2. General numerical studies of forced turbulence

The nonlinear filter methods described above were tested on DNS of two- and three-dimensional periodic-box turbulence at rms Mach numbers up to 1.0, using either time-correlated Ornstein-Uhlenbeck forcing or random white-in-time forcing [5, 26, 24, 18]. Our tests used various combinations of algorithms, including (i) three flow sensors (artificial compression method (ACM) by Yee et al [3], Ducros et al. [27], and multiresolution wavelet [4]), (ii) three skew-symmetric split formulations of the flux derivatives [13, 14, 25, 17], (iii) three standard shock-capturing methods (2nd-order Harten-Yee method with four different flux limiters (minmod, van Albada, superbee and Collela & Woodward), 5th- and 7th-order WENO methods (hereafter, WENO5 and WENO7), and (iv) the corresponding 2nd-order TVD, 5th- and 7th-order WENO nonlinear filter methods, using the dissipative portion of TVD, WENO5, and WENO7 in combination with the three flow sensors listed in item (i) above. Besides TVD, we included two Balsara & Shu [28] variants of the WENO scheme of the fifth- and seventh-order of accuracy. More specifically, the WENO formulation given by equation (2.10a) in [28] was used in conjunction with the Roe Riemann solver [29].

In our earlier studies of two-dimensional compressible turbulence with an intermediate-scale random white-in-time forcing [5, 26], we tested the Ducros sensor [27], which proved very robust and stable up to an rms Mach number $M_{\text{rms}} \equiv \sqrt{\langle u \cdot u \rangle / c_s^2} \approx 0.4$, where $u$ is the fluid velocity, $c_s$ is the sound speed, and angular brackets denote spatial average. However, with large-scale time-correlated stochastic forcing
in three dimensions (3D), the simulations turned numerically unstable already after a few sound-crossing times at $M_{\text{rms}} \approx 0.2$. A reduction of the Courant-Friedrich-Lewy (CFL) number to 0.2 did not help to restore schemes’ stability with this sensor.

In what follows, we focus on 3D numerical experiments with time-correlated stochastic forcing similar to those considered by Bauer et al. [24], see also [18]. We tested numerical stability of nonlinear filter schemes with the entropy [13], Ducros [14], and Kennedy-Gruber (KG) [25, 17] splittings and found that all these approaches render the method stable and yield similar power spectra and spectral energy fluxes in a range of target Mach numbers $M \equiv U/c_v$, where $U$ is the target rms velocity. The KG split form is slightly more computationally expensive compared to the Ducros or entropy splitting. At the same time, at $M > 0.5$, the KG splitting rendered the base scheme more stable than Ducros splitting. Schemes with the entropy splitting remain stable at least up to $M = 0.8$ (schemes using the KG splitting turn unstable at $M = 0.7$), but do not conserve mass or momentum exactly. Turbulence spectra and spectral fluxes are very similar in tests with all three splitting strategies. Thus, keeping in mind the stability and conservation properties, our fiducial scheme for stochastically forced turbulence at $M \in [0.2, 0.5]$ uses the KG splitting, the multi-resolution wavelet sensor, and WENO-based filters applied after the full third-order TVD Runge-Kutta time step. This combination of method elements demonstrated stable execution at target Mach numbers up to $M = 0.6$ with CFL = 0.8.

Hereafter, we will use the notation WENO7fi for the fiducial scheme. For comparison, we used numerical solutions obtained with the second-order Harten-Yee TVD scheme [30] utilizing the Colella & Woodward limiter [31] and second-order explicit Runge-Kutta time integration, hereafter the TVD scheme, as well as the WENO5 and WENO7 solutions, using the fifth- and seventh-order WENO schemes detailed above.

3. Simulations with stochastic forcing

Test case setup The compressible Euler equations are solved in a triply-periodic domain of linear size $L = 1$ covered by a uniform Cartesian grid with $64^3$, $128^3$, $256^3$, or $512^3$ points. To achieve near-isothermal conditions, as in [24], the ideal gas equation of state $p = \gamma \rho e$ (here $p$ is the pressure, $\rho$ – density, and $e$ – specific internal energy) is parametrized by a specific heats ratio $\gamma \equiv c_p/c_v = 1.01$. The initial conditions are uniform and static with the density $\rho(x, 0) = 1$, pressure $p(x, 0) = 1$, and velocity $u(x, 0) = 0$. A random external body force is used to generate and support a statistically stationary homogeneous turbulence in the box. We picked two values of the target Mach number: $M = 0.2$ and 0.5, which resulted in stationary turbulence at $M_{\text{rms}} \approx 0.24$ and 0.59, respectively.

Stochastic forcing Kinetic energy injection by the forcing incites turbulence during an initial transient and then supports a statistically stationary turbulent flow with homogeneous statistics in which the production of turbulence is balanced by the dissipation. To compute a specific force (acceleration) $\alpha(x, t)$ smoothly varying in time, we employ an algorithm suggested by Eswaran & Pope in [32] and further detailed in Refs. [33, 34, 35, 36]. The calculation is based on a spectral representation of the acceleration field governed by an asymptotically stationary Ornstein-Uhlenbeck process controlled by the forcing autocorrelation time $\tau$ and variance $\sigma^2(k)$

$$\hat{a}(k, t + \delta t) = \hat{a}(k, t)e^{-\delta t/\tau} + A_0\sigma(k)\sqrt{1 - e^{-2\delta t/\tau}} P(\zeta, k) \cdot \hat{N}(k).$$ (1)

Here $\hat{\cdot}$ indicates a Fourier transform, $k$ is the wave vector and the symmetric projection tensor $P_{ij}(\zeta, k) = \zeta \delta_{ij} + (1 - 2\zeta) k_i k_j / k^2$ depends on the spectral weight $\zeta$ of the solenoidal component of the force. In all test cases considered here, the force is purely solenoidal ($\zeta = 1$), but the algorithm allows for an arbitrary $\zeta \in [0, 1]$, with purely dilatational forcing corresponding to $\zeta = 0$. Finally, $\hat{N}(k)$ is a random complex vector generated for each cell $k$ in the Fourier cube with both real and imaginary parts distributed normally with zero mean and a variance of unity. These random phases are periodically updated after a certain fraction of the autocorrelation time has elapsed, namely after
\( \delta t = 0.1 \tau \). The dimensionless variance \( \sigma^2(k) \) determines the symmetries of the random force and is chosen to be a function of wavenumber \( k = |k| \), corresponding to an isotropic force field in physical space. We further define the spectral profile of the acceleration in a polynomial form, with \( \sigma(k) = \sigma_0[1 - 4(k - k_0)^2/(k_{\text{max}} - k_{\text{min}})^2] \), if \( k \in [k_{\text{min}}, k_{\text{max}}] \), and \( \sigma(k) = 0 \) otherwise. This choice corresponds to a paraboloid centered at the forcing wave number \( k_f = (k_{\text{max}} + k_{\text{min}})/2 \), and we used \( k_{\text{min}} = 6.27, k_{\text{max}} = 12.57 \) in all our test cases. The normalization constant \( \sigma_0 \) is determined by a requirement \( \Sigma_{ijk}\sigma^2(k_{ijk}) = 1/2 \). The amplitude of the force \( A_0 = a_{\text{rms}}(1 - 2\zeta + 3\zeta^2)^{-1/2} = a_{\text{rms}}/\sqrt{2} \) is properly rescaled to achieve the target Mach number \( M \), using \( a_{\text{rms}} = U/\tau \), where \( U \) is the target rms velocity, \( \tau = \lambda_f/U = \lambda_f/(Me_0) \) is the autocorrelation time of the force, and \( \lambda_f = 2\pi/k_f \) is the effective forcing scale.

Tests cases We carried out many simulations, probing the stability of various nonlinear filter methods, but here we present results for only 20 selected test cases. To compare methods, for each of the two target Mach numbers, we run a set of four 128\(^3\) and four 256\(^3\) cases with WENO7fi, WENO7, WENO5, and TVD schemes. In addition, to demonstrate convergence with grid resolution, we run 64\(^3\) and 512\(^3\) tests with WENO7fi.

4. Results

4.1. Method comparison

To compare performance of WENO7fi, WENO7, WENO5, and TVD schemes, we selected a set of relevant turbulence statistics. First, time evolution of the rms Mach number is evaluated and then we compare power spectra, small-scale quantities, spectral energy fluxes, and gas density probability density functions (PDFs). With these diagnostics, we show that all methods are stable and robust for long integration times. However, their effective spectral bandwidths differ within a factor of \( \approx 2 \). These differences stem from two factors: (i) the method’s order of accuracy and (ii) the order of effective numerical dissipation of the method. More accurate methods generally provide higher spectral bandwidth. However, close-to-quadratic effective dissipation can compromise the high-order advantage by stretching the extent of the dissipation range at the expense of the inertial interval.

Turbulence Mach number evolution In all test cases, the rms Mach number, \( M_{\text{rms}} \), peaks on a dynamical time scale \( \tau \) and then saturates in an oscillatory regime at a level slightly above the target Mach number \( M \). Note that \( \rho_0\sigma^2_{\text{rms}}M_{\text{rms}}^2/2 \) is a good measure of the kinetic energy of the fluid \( K = \langle \rho u \cdot u \rangle /2 \) in our subsonic near-isothermal simulations. The rms Mach number evolution in all test cases indicated that all schemes achieved energy saturation after the initial transient, with WENO5 and WENO7fi bracketing the time-average \( \langle M_{\text{rms}} \rangle \in [0.22, 0.23] \) in the low-Mach number test cases. At \( M = 0.5 \), instead WENO5 and TVD restricted the range, \( \langle M_{\text{rms}} \rangle \in [0.56, 0.60] \).

Velocity and density power spectra Time-average power spectra are shown in Figs. 1 and 2 for 128\(^3\) (left) and 256\(^3\) (right) tests at \( M = 0.2 \) (top) and 0.5 (bottom). Overall, at \( M = 0.2 \), the methods seem to converge to a Kolmogorov-like spectrum \( P(\upsilon, k) \propto k^{-5/3} \) at large scales. Indeed, the velocity scaling in very weakly compressible turbulence dominated by eddies should be similar to the incompressible case. Note that all four schemes display a characteristic bottleneck in the spectra [37]. However, with TVD and WENO7fi, the bottleneck is particularly strong, indicating the hyper-viscous nature of their numerical dissipation. At the same time, the two WENO schemes show only mild bottleneck bumps. It is worth noting that at these modest grid resolutions, the inertial range is expected to be very narrow, if any, and hence the obtained spectral slopes should not be overinterpreted.

At \( M = 0.5 \), the bottleneck range in the spectra obtained with both TVD and WENO7fi is somewhat more extended, but at 256\(^3\) all schemes show similar spectra. The scaling of the density spectrum largely reflects that of the velocity at these grid resolutions. Note that the convergence of density spectra with
grid resolution is particularly slow due to a more extended forcing range (density is not forced directly in these tests) and a generally weak coupling of the solenoidal and dilatational modes at low Mach numbers. As a result, with WENO7fi, grid resolutions higher than $1024^3$ are required to capture the $k^{-2}$ scaling of the density spectrum expected for the acoustic energy cascade [38]. For the study of turbulence stationarity, see the expanded journal version of this article [39].

Spectral bandwidths One can use an effective Kolmogorov scale $\eta_{\text{eff}} \sim \sqrt[4]{\nu_{\text{eff}}^3 / \epsilon_f} \propto \langle \omega^2 \rangle^{-3/4}$ to compare relative standing of the methods in terms of their spectral bandwidths. (Here $\nu_{\text{eff}}$ is the effective kinematic viscosity, $\omega = \nabla \times u$ is the vorticity vector, and $\epsilon_f$ is the energy injection rate.) At $M = 0.2$, shock dissipation can be neglected, and a trivial calculation shows that the WENO7fi scheme has an effective Kolmogorov scale $\approx 2\times$ times smaller than that of the TVD scheme. Similar factors for the WENO5 and WENO7 schemes are 1.3 and 1.6, respectively. At $M = 0.5$, shock dissipation cannot be ignored, hence we can compare $\eta_{\text{eff}}^{(1)}$ and $\eta_{\text{eff}}^{(2)}$ separately representing incompressible (vortical) and compressible (shock) dissipation. The relative standing of the WENO7fi, WENO7, and WENO5 schemes compared to the TVD scheme for eddy-dissipation is 1.8, 1.4, and 1.1; whereas in terms of shock dissipation we get 1.5, 1.14, and 0.8, respectively. These data demonstrate clear advantage of the nonlinear filter schemes over regular WENO or TVD alternatives of comparable or lower order accuracy.

A more rigorous way of measuring the spectral bandwidth of a method requires a calculation of the spectral energy flux. For homogeneous isothermal compressible turbulence, this can be done using an exact relation for energy transfer derived in Ref. [40] and verified numerically in [38]. We refer the
Figure 2. Same as Fig. 1, but for the density power spectra.

interested reader to Sections IV.B and C in [40] for definitions and technical details of the derivation while keeping the same notation here. The spectral flux is expected to be constant and equal to the energy injection rate in the inertial interval of scales and thus provides a direct measure of the method’s bandwidth. An alternative approach based on the energy spectra adopted in [41, 42] cannot be effective for all methods employed here due to strong bottleneck.

Figure 3 shows spectral fluxes for test cases at grid resolutions 128$^3$ (dash-dotted lines) and 256$^3$ (solid lines) computed with the TVD (violet), WENO5 (green), WENO7 (brown), and WENO7fi (red) methods at $M = 0.2$ (upper panel) and $M = 0.5$ (lower panel). A 20% tolerance level in the normalized flux $\Pi(k)/\Pi_{\text{max}}$ is indicated with a horizontal dotted line. It is convenient to define the bandwidth as the width of the flux plateau, corresponding to the chosen tolerance level.

At $M = 0.2$, WENO7fi flux from the 128$^3$ test (red dash-dotted line) essentially overlaps with TVD flux at 256$^3$ (violet solid line) while leaving slightly behind the 256$^3$ WENO5 flux (green solid line). Hence with WENO7fi, it is sufficient to have a two times smaller linear grid resolution to get scale separation comparable to that in the TVD or WENO5 tests. The WENO7 result falls right in between of the WENO5 and WENO7fi bandwidths. Note that the bandwidth improvement with the grid resolution is not the same for all methods. For instance, the bandwidth of WENO7fi grows by a factor of 2 as the grid resolution doubles, while with the TVD and WENO5 methods the improvement is 1.7 and 1.8, respectively. This reflects the fact that the more extended dissipation range of the TVD and WENO5 schemes is not fully resolved in 128$^3$ tests.

Our results for $M = 0.5$ are qualitatively similar to those at $M = 0.2$ and generally show the same ranking of methods in terms of the spectral bandwidth. In this case, the WENO7fi flux at 128$^3$ is very close to that of WENO5 at 256$^3$, while the computational cost of one time step with WENO7fi at the same grid resolution is just $\approx 7\%$ higher. This implies that WENO7fi is $\sim 15$ times more efficient in 3D than WENO5. Since the TVD scheme has a larger bandwidth compared to WENO5 and is $\approx 2.9$ times
Figure 3. Normalized spectral fluxes of total energy $\Pi(k)/\Pi_{\text{max}}$ for $128^3$ (dash-dotted lines) and $256^3$ (solid lines) test cases with different schemes at $M = 0.2$ (top) and 0.5 (bottom). Time averaging intervals: $t \in [20, 60]$ and $t \in [20, 40]$, respectively.

The analysis of spectral fluxes, thus, shows a clear advantage of nonlinear filter schemes for subsonic turbulence simulation with stochastic forcing.

Density PDFs Figure 4 shows time-average PDFs of the fluid density for all $128^3$ and $256^3$ test cases. At $M = 0.2$ (top panels of Fig. 4), the four methods produce very similar results for the core of the distribution, with some limited variation in the tails. In the low-density tail, which is sensitive to small-scale numerical dissipation as it represents cores of vortex filaments, the performance is strongly method-dependent. In this respect, the TVD scheme shows large deviations from the WENO-based schemes on both $128^3$ and $256^3$ grids. This implies slow convergence of the TVD results at low Mach numbers.
and grid resolutions up to $256^3$ points. One possible explanation for the slow convergence could be lower effective Reynolds numbers in our low Mach number cases. If that is the case, slow convergence would indicate that we are way off the asymptotic regime of fully developed turbulence with the TVD scheme. Another possibility is that in this weakly compressible regime the TVD scheme without viscous stabilization struggles to handle the pseudo-sound pressure component, which is known to dominate the small scale structure in DNS at $M < 0.4$ [43]. At $M \geq 0.4$, instead the acoustic component dominates across the whole spectrum, changing the shape of the low- and high-density tails of the PDF.

At $M = 0.5$ (Fig. 4, bottom), the PDFs are again very similar in the core of the distribution, although TVD tends to overproduce the high tail at low resolution. Systematically wider PDFs have been seen before in simulations with diffusive shock-capturing methods, see, e.g., Ref. [44]. However, all three WENO-based schemes, including WENO7fi, demonstrate reasonable convergence already at $128^3$. It is remarkable that at a grid resolution of $256^3$, the PDFs obtained with all four methods converge so well. WENO7fi additionally shows reasonable self-convergence with the grid resolution [39]. A lognormal high-density tail associated with shocks is building up with increased resolution, as expected for isothermal fluids [45].

4.2. Resolution study for WENO7fi

We track the variation in select turbulence statistics obtained with the WENO7fi scheme, using a set of test cases with grid resolution varying from $64^3$ to $512^3$ points, for both low and high target Mach numbers. Note that we use the term convergence in a weak sense in this work, as we discuss numerical solutions to the Euler equations without explicit viscous terms. Hence we do not expect statistical diagnostics related to small-scale dissipative structures (e.g., the enstrophy $\langle \omega^2 \rangle$ or the mean-square
Figure 5. Velocity power spectra for low (left) and high (right) Mach number test cases computed with WENO7fi on grids with $64^3, 128^3, 256^3,$ and $512^3$ nodes. The spectra were averaged over time intervals corresponding to statistically stationary states at $t \in [20, 60]$ and $t \in [20, 40]$, respectively. As a reference, dashed lines indicate Kolmogorov’s slope of $-5/3$.

dilatation $(\theta^2)$ to converge for different methods or different grid sizes. However, we expect convergence of such diagnostics as power spectra or spectral energy fluxes at scales in the emerging inertial range. The rms Mach number evolution (figure not shown) shows similar steady oscillatory regimes for all grid sizes without any discernible long-term trend.

Spectra The velocity spectra shown in Fig. 5 exhibit convergence to Kolmogorov’s scaling in a range of scales sufficiently separated from both the injection scale ($k_f \approx 9$, independent of resolution) and the dissipation range (determined solely by the numerical scheme in the absence of physical viscosity). The inertial range spectral slopes predictably converge to the incompressible velocity scaling [46] and similar density scaling (not shown here, but see [39]) in the range of grid resolutions explored here, supporting a conjecture that the statistical properties of turbulence in the inertial range may be universal and independent of the dissipation mechanism. This has been suggested earlier based on a combination of high-resolution ($1856^3$) simulation of weakly compressible turbulence at $M_{rms} = 0.3$ [47] (using the piecewise-parabolic method [31]) and a DNS of incompressible turbulence at a similar Taylor microscale Reynolds number, $R_{\lambda} \sim 460$ [48]. As far as the density scaling is concerned, higher resolution simulations ($1024^3$ or higher with WENO7fi) are needed to first see the $k^{-2}$ scaling expected for acoustic turbulence. We will report on these simulations with better scale separation in Ref. [38]; more detail of this study can be found in [39].

5. Summary

We tested performance of the nonlinear filter schemes using weakly compressible turbulence simulations at grid resolutions up to $512^3$. At rms Mach numbers up to 0.6, among the considered method combinations, it is found that the best performer is an eighth-order central scheme with Kennedy-Gruber skew-symmetric split approximation of the inviscid flux derivative, a wavelet-based local flow sensor, a nonlinear filter using the dissipative part of the seventh-order WENO scheme with the Roe Riemann solver, and third-order TVD Runge-Kutta time integration.

Global small-scale quantities, such as the enstrophy and mean-square dilatation, as well as turbulence power spectra indicate robust nonlinear stability of all tested methods for long-time integration. Overall, the methods demonstrate reasonably good convergence of spectral scaling in the inertial range in agreement with theoretical predictions for subsonic turbulence.

Spectral energy fluxes computed using an exact relation for energy transfer in compressible turbulence yield a direct measure of the spectral bandwidth for each method. Comparing the bandwidths and taking
into account the computational cost associated with each method, we show that to achieve a given scale separation in simulations of forced 3D turbulence at an rms Mach number ~ 0.6 with the nonlinear filter method it would take approximately $15 \times$ and $8 \times$ less time compared to fifth and seventh order WENO schemes, respectively, and $3.6 \times$ less time compared to the second order TVD scheme.

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