Superspace Dynamics

Perturbations around “Emptiness”

Chopin Soo$^1$ & Lay Nam Chang$^2$

Institute for High Energy Physics
Virginia Polytechnic Institute and State University
Blacksburg, Virginia 24061-0435

Abstract

Superspace parametrized by gauge potentials instead of metric three-geometries is discussed in the context of the Ashtekar variables. Among other things, an “internal clock” for the full theory can be identified. Gauge-fixing conditions which lead to the natural geometrical separation of physical from gauge modes are derived with the use of the supermetric in connection-superspace. A perturbation scheme about an unconventional background which is inaccessible to conventional variables is presented. The resultant expansion retains much of the simplicity of Ashtekar’s formulation of General Relativity.

Revised 9/93

---

1 Present address: Center for Gravitational Physics and Geometry, Penn. State University, University Park, PA 16802-6300; e-mail: soo@phys.psu.edu

2 Present address: Theoretical Physics, NSF, Washington, DC; e-mail: lchang@nsf.gov
In the ADM formalism[1], the superhamiltonian constraint can be written as

\[ \mathcal{H} \equiv (16\pi G) G_{ijkl} \pi^{ij} \pi^{kl} + \frac{\sqrt{g}}{16\pi G} 3R = 0 \]  

(1)

where \( \sigma \) takes the value of +1 for spacetimes of Euclidean signature and −1 for spacetimes of Lorenztian signature. As noted by the authors of Ref. 2, the theory has an interesting strong coupling limit or zero signature limit at which the potential term vanishes and only the kinetic term which is quadratic in the momenta remains. \( G_{ijkl} \) can be assumed to be the metric of superspace (the space of 3-geometries described by the equivalence classes of spatial metrics under 3D-diffeomorphisms) and it has the form[3]

\[ G_{ijkl} = \frac{1}{2\sqrt{|g|}} (g_{ik} g_{jl} + g_{il} g_{jk} - g_{ij} g_{kl}) \]  

(2)

with inverse

\[ G^{ijkl} = \frac{1}{2} \sqrt{|g|} (g^{ik} g^{jl} + g^{il} g^{jk} - 2g^{ij} g^{kl}) \]  

(3)

The supermetrics are ultralocal in the spatial metric variables. Moreover, an intrinsic time parameter which is proportional to \( \ln|g| \) can be identified since the supermetric

\[ \delta S^2(\vec{x}) = \frac{1}{2} \sqrt{|g|} \left( g^{ik} g^{jl} + g^{il} g^{jk} - 2g^{ij} g^{kl} \right) \delta g_{ij} \delta g_{kl} \]  

(4)

has hyperbolic signature \((-+,+,+,+,+,+)[3]\). This suggests that in quantum gravity, especially in the context of spatially compact manifolds, a preferred degree of freedom of the theory can be singled out as the “internal clock” relative to which other degrees of freedom of the theory evolve according to the dynamics governed by the Wheeler-DeWitt Equation. The adoption of such an approach could lead to a resolution of the issue of time in quantum gravity. (For a discussion on the “issue of time” in quantum gravity in the context of the connection variables we will be focussing on, see Chap. 12 of Ref. 7). With expression (4) as the metric of superspace, in the strong coupling (zero signature) limit, the superhamiltonian constraint can be interpreted to be the free Klein-Gordon equation[2].

Ashtekar has achieved remarkable simplifications of the constraints of General Relativity by introducing \( \text{SO}(3) \) gauge potentials as fundamental variables[4–7]. In terms of the new variables, the constraints for pure gravity read

\[ G^a \equiv D_i \tilde{\sigma}^a = 0 \]  

(5a)

\[ H_i \equiv \varepsilon_{ijk} \tilde{B}^{jb} \tilde{\sigma}^a = 0 \]  

(5b)

\[ H \equiv \varepsilon_{ijk} \varepsilon_{abc} \tilde{B}^{kc} \tilde{\sigma}^a \tilde{\sigma}^b = 0 \]  

(5c)
where the magnetic field

\[ \tilde{B}^{ia} \equiv \frac{1}{2} \tilde{\epsilon}^{ijk} \left\{ \partial_j A^a_k - \partial_k A^a_j + \epsilon^a_{bc} A^b_j A^c_k \right\} \]  

(6)

The tildes above and below the variables denote the fact that they are tensor densities of weight +1 and −1 respectively. In the above, lower case Latin indices from \(a\) to \(c\) denote internal SO(3) indices while indices from \(i\) onwards are spatial indices. All these indices run from 1 to 3. We can replace \(\epsilon_{abc}\) by the tensor density \(\tilde{\epsilon}_{abc}\) in (5c) so that rather than being of weight two, the superhamiltonian is of weight one as is the case for \(H\) with the ADM variables. This makes the supermetric in connection superspace, expression (16) below, gauge and 3D-diffeomorphism-invariant without the introduction of metric or triad variables.

In particular, among the simplifications achieved by Ashtekar, there is remarkably no potential term in the superhamiltonian constraint of the full theory, if we treat \(\tilde{\sigma}^{ia}\) as the momentum variable, and adopt the natural choice of

\[ G_{iajb} \equiv \epsilon_{ijk} \epsilon_{abc} \tilde{B}^{kc} \]  

(7)

as the contravariant metric for the space of “gauge-invariant 3-geometries” described by the equivalence classes of Ashtekar connections under gauge transformations and 3D-diffeomorphisms. This definition of the supermetric does not involve the variables \(\tilde{\sigma}^{ia}\) or \(g_{ij}\). In terms of vielbeins

\[ G_{iajb} \equiv \frac{1}{3} (e^c)_i a (e^d)_j b + (e^d)_i a (e^c)_j b \]  

\[ \equiv (e^0)_i a (e^0)_j b - (e^c)_i a (e^d)_j b \]  

(8a) (8b)

where the vielbeins can be taken to be

\[ \sqrt{\frac{1}{3}} (e^c)_i a = (e^0)_i a = (\frac{2}{3} \tilde{B})^{1/2} b_{ia} \]  

\[ \pm i (\bar{e}^c)_i a = (e^c)_i a = \tilde{B}^{\frac{1}{2}} \left\{ \pm \frac{1}{3} b_{ia} \delta^c d \mp b^c_i \delta a d \right\} \]  

(9a) (9b)

Here \(b_{ia}\) denotes the inverse of the magnetic field and

\[ \tilde{B} \equiv \frac{1}{3!} \epsilon_{ijk} \epsilon_{abc} \tilde{B}^{ia} \tilde{B}^{jb} \tilde{B}^{kc} \]  

(10)
is the determinant of $\tilde{B}^{ia}$. Notice however, that unlike expression (2), the supermetric (8) is not ultralocal in $A$. See Ref. 8 for some comments on the trade off between an ultralocal supermetric with a local potential term in the ADM formalism and a local supermetric without any potential in the Ashtekar formalism. The covariant supermetric, the inverse of $\tilde{G}_{iajb}$ in the Ashtekar formalism, is readily computed to be

$$\tilde{G}^{iajb} \equiv \tilde{B}^{-1} \left\{ \frac{1}{2} \tilde{B}^{ia} \tilde{B}^{jb} - \tilde{B}^{ib} \tilde{B}^{ja} \right\}$$

(11)

with

$$\tilde{G}^{iakc}(\bar{x})G_{kcjb}(\bar{x}) = G_{jbkc}(\bar{x})\tilde{G}^{kcja}(\bar{x}) = \delta^i_j \delta^a_b$$

(12)

$\bar{x}$ denotes the coordinate on an initial-value hypersurface. Since

$$\det(\tilde{G}^{iajb}) = \left\{ \det(\tilde{G}^{iajb}) \right\}^{-1} = -2\tilde{B}^3$$

(13)

the inverse (covariant) supermetric exists if and only if the magnetic field is non-degenerate. In this report, unless stated otherwise, we shall deal only with Ashtekar variables that are real and side-step the reality conditions[4–7] that have to be imposed on the Ashtekar variables. For space-times with Euclidean signature, it is consistent to assume that all the variables are real.

In superspace parametrized by gauge potentials, the supermetric takes the form

$$\delta S^2(\bar{x}) = \tilde{G}^{iajb}(\bar{x})\delta A_{ia}(\bar{x})\delta A_{jb}(\bar{x})$$

(14a)

$$= \tilde{B}^{-1} \left\{ \frac{1}{2} (\tilde{B}^{ia} \delta A_{ia})(\tilde{B}^{jb} \delta A_{jb}) - (\tilde{B}^{ia} \delta A_{ia})(\tilde{B}^{jb} \delta A_{ja}) \right\}$$

(14b)

$$= \tilde{B}^{-1} \left\{ \frac{1}{6} (\delta X^a_a)^2 - \delta X^a_b \delta X^b_a \right\}$$

(14c)

where the local coordinates $X^a_b$ are defined by

$$\delta X^a_b \equiv \tilde{B}^{ia} \delta A_{ib}; \quad \delta X^a_b \equiv \delta X^a_b - \frac{1}{3} \delta^a_{b}(\delta X^c_c)$$

(15)

Curvature of superspace obstructs the integrability of the local coordinates. The supermetric

$$\delta S^2 = \int d^3 \bar{x} \tilde{G}^{iajb}(\bar{x})\delta A_{ia}(\bar{x})\delta A_{jb}(\bar{x})$$

(16)
is clearly diffeomorphism and SO(3) gauge-invariant and does not require the metric variables $\tilde{\sigma}^{ia}$ or $g_{ij}$ for it to be defined. The supermetric has signature

$$\text{sign}(\tilde{B}^{-1})(+, +, +, +, -, -, -, -, -)$$

since a straightforward decomposition of $\delta \bar{X}$ into symmetric traceless and anti-symmetric parts

$$\delta X^a_b \equiv \delta S^a_b + \frac{1}{2} \epsilon^a_{bc} \delta T^c$$

yields

$$\delta S^2(\bar{x}) = \tilde{B}^{-1} \left\{ \frac{1}{6} (\delta X^a_a)^2 - \delta X^a_b \delta X^b_a \right\}$$

$$\quad = \tilde{B}^{-1} \left\{ \frac{1}{6} (\delta X^a_a)^2 + \frac{1}{2} (\delta T^c)(\delta T_c) \right\}$$

$$\quad - \left\{ 2(\delta S^1_1)^2 + 2(\delta S^2_3)^2 + 2(\delta S^3_1)^2 + \frac{3}{2}(\delta S^1_1 + \delta S^2_2)^2 + \frac{1}{2}(\delta S^1_1 - \delta S^2_2)^2 \right\}$$

It will be shown that when restricted to the physical subspace modulo the constraint (5b), the hyperbolic supermetric has signature $\text{sign}(\tilde{B}^{-1})(+, -, -, -, -, -)$ and the anti-symmetric matrices $\epsilon^a_{bc} \delta T^c$ drop out.

The factor $\tilde{B}^{-1}$ would have been a conformal factor in $\delta S^2(\bar{x})$ were it to be positive definite. We make no such restrictions; but note that the signature of the supermetric is determined by the sign of $\tilde{B}^{-1}$, so that there is a switching of time-like and space-like coordinates in connection-superspace when $\tilde{B}$ reverses sign. This situation is much akin to what happens in spacetime when one crosses the horizon of a black hole. This parallel is more than a mere analogy. In superspace, the crossover in the sign of $\tilde{B}$ occurs at vanishing $\tilde{B}$. A computation of the Ashtekar connection one-forms for the classical Schwarzschild solution yields[9]

$$A_1 = \pm \frac{m}{r^2} idt \pm \cos \theta d\phi$$

$$A_2 = -\sqrt{1 - \frac{2m}{r}} \sin \theta d\phi$$

$$A_3 = \sqrt{1 - \frac{2m}{r}} d\theta$$

if the vierbein one-forms $(g_{\mu\nu} dx^\mu dx^\nu = e_A e_A)$ are taken to be

$$e_A = \left\{ \pm \sqrt{1 - \frac{2m}{r}} idt, \frac{dr}{\sqrt{1 - \frac{2m}{r}}}, r d\theta, r \sin \theta d\phi \right\}$$
(As emphasized in Refs. 9, the classical Ashtekar connections are the anti-self-dual part of the spin connection and therefore depends on the orientation of the vierbeins). It can be seen that at the horizon \( r = 2m \), the Ashtekar connection becomes abelian and \( \tilde{B} \) vanishes. Moreover, it can be checked that \( \tilde{B} \) remains real and changes sign when we cross the horizon. It is appropriate here to interject a word of caution. The expressions (20) and (21) are for the Schwarzschild solution with Lorentzian signature. To obtain the Euclidean Schwarzschild solution, we can make the continuation \( t \rightarrow -i\tau \), but the Euclidean Schwarzschild solution exists only for \( r \geq 2m[10] \), which also follows from requiring the Ashtekar potentials to be real. \( \tilde{B} \) still vanishes at the horizon of course. In general, for real connections, the gauge-invariant condition \( \tilde{B}(\vec{x}) = 0 \) has codimension one and thus naturally defines a two-dimensional surface on which it holds. (For complex Ashtekar connections which corresponds to spacetimes with Lorentzian signature, reality conditions have to be imposed on the variables but as we have discussed, for the Schwarzschild solution, \( \tilde{B} \) of the Ashtekar connection remains real and switches sign at the horizon).

The covariant supermetric in connection superspace exists if and only if \( \tilde{B} \) is non-vanishing. Thus configurations with vanishing \( \tilde{B} \) correspond to points in superspace at which the supermetric (9) is singular. Some examples are manifolds with horizons and manifolds described by Ashtekar potentials which are abelian anti-instantons[9]. In non-perturbative quantum gravity, such configurations stand out and can be expected to play crucial roles.

The Gauss’ Law and supermomentum constraints are first order in the momenta and generate SO(3) gauge transformations and gauge-covariant 3D diffeomorphisms respectively. Their constraint algebra closes without structure functions which depend on dynamical variables. This suggests that these constraints are to be treated differently from the superhamiltonian constraint which is quadratic in the momenta. Physically, it can be interpreted that the theory depends only on gauge-invariant 3-geometries and as is suggested by the form of the supermetric, the superhamiltonian constraint can then be used to determine the dynamical evolution of the theory with respect to an intrinsic time parameter.

The existence of a supermetric allows a local decomposition of the cotangent space into gauge directions and their orthogonal complement with respect to the supermetric. There is thus a natural geometrical separation of the physical from the gauge degrees of freedom through the derived gauge-fixing conditions. Some recent works on the relevance
of such gauge-fixing conditions in gravity using conventional variables can be found in Ref. 11. With \( \tilde{G}^{iaj}_{b} \) as the supermetric, the gauge-fixing conditions are obtained from

\[
\int d^{3}\vec{x}\tilde{G}^{iaj}_{b}\delta A_{ia}^{\perp}(\delta A_{j}^{b}) = 0 \tag{22}
\]

where \( \delta A^{\perp} \) and \( \delta A^{g} \) are the physical modes and gauge directions respectively. Using the explicit form of the contraints, we obtain

\[
\int d^{3}\vec{x}\tilde{G}^{iaj}_{b}\delta A_{ia}^{\perp}(-D_{j}\eta)_{b} = 0 \tag{23}
\]

and

\[
\int d^{3}\vec{x}\tilde{G}^{iaj}_{b}\delta A_{ia}^{\perp}(\varepsilon_{jkl}\bar{B}^{kb}\xi^{l}) = \int d^{3}\vec{x}2\xi^{l}c^{ac}_{\ b}b_{lc}\bar{B}^{ib}\delta A_{ia}^{\perp} = 0 \tag{24}
\]

where \( b_{ia} \equiv \frac{1}{2!} \bar{B}^{-1}_{\ ijk}\varepsilon_{abc}\bar{B}^{jb}\bar{B}^{kc} \).

The six gauge-fixing conditions are derived from the requirement that Eqns. 23 and 24 hold for arbitrary \( \eta_{a}(\vec{x}) \) and \( \xi^{i}(\vec{x}) \). Thus for non-degenerate magnetic fields, the gauge-fixing conditions for 3D diffeomorphisms lead to

\[
\varepsilon^{ac}_{\ b}\bar{B}^{ib}\delta A_{ia}^{\perp} = 0 \tag{25}
\]

In terms of the local coordinates of Eqn. 15, this is the same as the requirement that \( \delta X^{a}_{\ b} \) be symmetric. Thus the unphysical modes \( \delta T^{a} \) can be set to zero and we are left with a supermetric of signature \( \text{sign}(\tilde{B}^{-1})(+, -, -, -, -, -) \) which picks out \( X^{a}_{\ a} \) as the preferred intrinsic time-like coordinate in superspace. Notice that the time-like coordinate has the interesting property of

\[
\int d^{3}\vec{x}\delta X^{a}_{\ a} = \delta C \tag{26}
\]

where \( C \) is the Chern-Simons functional. It should be noted that with the Ashtekar variables, it is the supermomentum rather than the Gauss’ Law constraint which eliminates the subspace of the supermetric which corresponds to the antisymmetric part of \( \tilde{B}^{ia}\delta A_{ib} \).

Alternatively, if we order the supermomentum constraint in the connection-representation as

\[
\varepsilon_{ijk}\bar{B}^{ia}\frac{\delta}{\delta A_{ka}}\Phi[X(A)] = 0 \tag{27}
\]

and use

\[
\frac{\delta}{\delta A_{ia}(\vec{x})}\Phi[X(A)] = \int d^{3}\vec{y}\frac{\delta X^{b}_{\ c}(\vec{y})}{\delta A_{ia}(\vec{x})}\frac{\delta \Phi}{\delta X^{b}_{\ c}(\vec{y})} = \bar{B}^{ib}(\vec{x})\frac{\delta \Phi}{\delta X^{b}_{\ a}(\vec{x})} \tag{28}
\]
we have
\[ \varepsilon_{ijk} \tilde{B}^{ja} \tilde{B}^{kb} \varepsilon_{abc} \frac{\delta \Phi}{\delta T_c} = 0 \] (29)
which, for non-degenerate magnetic fields, implies \( \frac{\delta \Phi}{\delta T_c} = 0 \).

Three other gauge modes which correspond to the SO(3) gauge-invariance of the Ashtekar variables can be eliminated using (23) which leads to the gauge-fixing condition
\[ [D_i (\tilde{G}^{i, jb} \delta A_{jb}^\perp)]^a = \partial_i (\tilde{G}^{ia j b} \delta A_{j b}^\perp) + \varepsilon_{abc} A^b_i \tilde{G}^{ic jd} \delta A_{jd}^\perp = 0 \] (30)
With the non-trivial supermetric, this is a natural generalization. For ordinary SO(3) gauge theory in flat space-time, this gauge-fixing condition reduces to the usual covariant Coulomb gauge condition
\[ (D^i \delta A^\perp_i)^a = 0 \] (31)
since the supermetric for this particular instance is flat and is of the form \( \tilde{G}^{ia j b} = \delta^{ij} \delta^{ab} \).

The gauge-fixing condition (30) however involves the gauge-invariant intrinsic time parameter relative to which other degrees of freedom are to evolve. A more reasonable alternative gauge-fixing procedure is to use the supermetric for the subspace complement to the intrinsic time coordinate to eliminate the gauge degrees of freedom of the subspace. Since we can also write
\[ (\delta S)^2 = \left\{ [(E^0)^{ia} \delta A_{ia}]^2 - \tilde{G}^{ia j b} \delta A_{ia} \delta A_{jb} \right\} \] (32)
the supermetric for the desired subspace has the form
\[ \tilde{G}^{ia j b} \equiv (\tilde{E}^b_c)^{ia} (\tilde{E}^a_c)^{jb} \]
\[ = \tilde{B}^{-1} \left\{ \tilde{B}^{ib} \tilde{B}^{ja} - \frac{1}{3} \tilde{B}^{ia} \tilde{B}^{jb} \right\} \] (33b)
with vielbeins
\[ (E^0)^{ia} = (6 \tilde{B})^{-\frac{1}{2}} \tilde{B}^{ia} \] (34a)
\[ (\tilde{E}^b_c)^{ia} = \tilde{B}^{-\frac{1}{2}} \left\{ \pm \tilde{B}^{ia} \delta^b_c + \frac{1}{3} \tilde{B}^{ib} \delta^a_c \right\} \] (34b)
The resultant gauge-fixing conditions from
\[ \int d^3 x \tilde{G}^{ia j b} \delta A^\perp_{ia} (\delta A^g_{jb}) = 0 \] (35)
are as before
\[ \varepsilon^{ac} \delta^b \tilde{B}^{ib} \delta A^\perp_{ia} = 0 \quad i.e. \delta \tilde{X}_{a}^i \text{is symmetric} \] (36)
and

\[ [D_i (G^{i,jb} \delta A^{\perp}_{jb})]^a = 0 \]  (37)

The gauge-fixing conditions that we have discussed so far are good only locally in superspace and there can be subtleties associated with Gribov copies[12]. Moreover, the gauge-fixing conditions are derived for regions in superspace where the supermetric is assumed to be regular. As we have discussed, this means that we stay away from singular points with vanishing \( \tilde{B} \). In the context of perturbation theory, this restriction may not be unreasonable. Typically, as in the case of abelian anti-instantons, such configurations have more symmetry than neighbouring configurations and are thus singular in the gauge and diffeomorphism-invariant moduli space. The full quantum theory must of course take into account these intriguing configurations.

Precisely because the contravariant superspace metric is chosen to be \( G_{iajb} \equiv \epsilon_{ijkl} \epsilon_{abc} \tilde{B}^{kc} \), the superhamiltonian constraint can be interpreted as the free Klein-Gordon equation in curved superspace with covariant metric

\[ \tilde{G}^{iajb} \equiv \tilde{B}^{-1} \left\{ \frac{1}{2} \tilde{B}^{ia} \tilde{B}^{jb} - \tilde{B}^{ib} \tilde{B}^{ja} \right\} \]  (38)

This provides a natural ordering for the “Ashtekar-Wheeler-DeWitt Equation”

\[ \tilde{B}^{3/2} \tilde{\sigma}^{ia} \left\{ \tilde{B}^{-3/2} \epsilon_{ijk} \epsilon_{abc} \tilde{B}^{kc} \right\} \tilde{\sigma}^{jb} \Phi[A] = 0 \]  (39)

which in the connection-representation is equivalent to

\[ (\det \tilde{G})^{-\frac{1}{2}} \frac{\delta}{\delta A_{jb}} (\det \tilde{G})^{\frac{1}{2}} G_{iajb} \frac{\delta}{\delta A_{ia}} \Phi[A] = 0 \]  (40)

(The equations are for wavefunctionals which are harmonic zero-forms in superspace. They can be generalized for instance, to the case of wavefunctionals of weight \( \frac{1}{2} \) in superspace, by appropriate insertions of powers of \( \det \tilde{G} \).) Notice that in the classical context, (39) reduces to

\[ H \equiv \epsilon_{ijk} \epsilon_{abc} \tilde{B}^{kc} \tilde{\sigma}^{ia} \tilde{\sigma}^{jb} = 0 \]  (41)

and in local coordinates, the Ashtekar-Wheeler-DeWitt Equation can be written as

\[ \left( \frac{\delta}{\delta x^a_b} + \Omega^b_a \right) \frac{\delta}{\delta x^b_a} \Phi[A] = 0 \]  (42)
where
\[
\frac{\delta}{\delta x^b_c} \Phi \equiv (e^c_b)_{ia} \frac{\delta}{\delta A_{ia}} \Phi
\] (43)
and
\[
\Omega^b_c \equiv B^{\frac{3}{2}} A_{ia} \left\{ B^{-\frac{3}{2}} (e^b_c)_{ia} \right\}
\] (44)

The reduced configuration space can be interpreted to be the light-cone in curved superspace subject to the gauge-fixing conditions for SO(3) gauge transformations and 3D diffeomorphisms. A natural way to order the remaining constraints, which are first order in the momenta, is to place the momentum operator on the extreme right of the constraints, which then read
\[
(\partial_i \delta \delta A_{ia} + \epsilon_{abc} A^b_i \delta A_{ic}) \Phi[A] = 0
\] (45a)
\[
\epsilon_{ijk} \tilde{B}^{ja} \delta \delta A_{ak} \Phi[A] = 0
\] (45b)
in the \( A \)-representation. The chosen ordering implies that the wavefunctionals which satisfy the constraints are invariant under infinitesimal SO(3) gauge transformations and 3D diffeomorphisms since
\[
\Phi[A + \delta A^g] = \Phi[A] + \int d^3 \vec{x} \delta A_{ia}^g (\vec{x}) \frac{\delta}{\delta A_{ia}^g (\vec{x})} \Phi[A] = \Phi[A]
\] (46)
is ensured by the ordering[13]. Issues related to possible anomalies in the associated quantum constraint algebra will be taken up in a future work. It has been argued by others that without proper regularizations, the closure of the quantum constraint algebra cannot be addressed meaningfully[14].

With the gauge-fixing conditions and supermetric in hand, we can now study various limits of the theory and consider perturbations about backgrounds \( (A^*_a, \tilde{\sigma}_*^a) \) which satisfy all the constraints. The fields can be decomposed as
\[
A_{ia} = A^*_a + a_{ia}, \tilde{\sigma}_*^a = \tilde{\sigma}_a^ia + \tilde{e}_a^ia
\] (47)
where \( a_{ia} \) and \( \tilde{e}_a^ia \) are the fluctuations relative to the background \( (A^*_a, \tilde{\sigma}_*^a) \). To lowest order in the fluctuations, the linearized constraints are
\[
D^*_{ia} \tilde{e}_a^ia + \epsilon_{abc} a_b^ia \tilde{\sigma}_*^ic = 0
\] (48a)
\[
\epsilon_{ijk} (\tilde{B})^k_{ja} \tilde{e}_a^ja + \tilde{\sigma}_*^ja [D^*_{ia} a_ia - D^*_{ja} a_{ia}] = 0
\] (48b)
\(D^\ast\) denotes the covariant derivative with respect to the background gauge connection. Expanding about a background which is compatible with the constraints, the superhamiltonian constraint can be written as

\[
2\xi_{ijk}\epsilon_{abc}\tilde{B}_*^k\varepsilon^ia\tilde{e}^jb + \xi_{ijk}\epsilon_{klm}\epsilon_{abc}(D^\ast l a_m)c\tilde{\sigma}_i^a\tilde{\sigma}_j^b + \\
\xi_{ijk}\epsilon_{abc}\tilde{B}_*^k\varepsilon^ia\tilde{e}^jb + 2\xi_{ijk}\epsilon_{klm}\epsilon_{abc}(D^\ast l a_m)c\tilde{\sigma}_j^b + \\
\epsilon_{abc}\epsilon^{cde}d_iad_j\tilde{\sigma}_i^a\tilde{\sigma}_j^b + \text{higher order terms}
\]

\[
= 0 \quad (49)
\]

The usual perturbation analysis is to consider fluctuations about the flat Euclidean \(\vec{x}\)-independent background \((A_i^*, \tilde{\sigma}_i^a) = (0, \tilde{\sigma}_i^a)\). Conventional perturbation analysis is carried out in the metric or vierbein representation. In the \(\tilde{\sigma}\) -representation, the covariant supermetric is the coefficient of the quadratic term in \(A_i^a\) (the variable conjugate to \(\tilde{\sigma}_i^a\)), in the superhamiltonian constraint. This covariant supermetric has the form \((\det \tilde{\sigma})^{-\frac{1}{2}}(\tilde{\sigma}_i^a\tilde{\sigma}_j^b - \tilde{\sigma}_i^b\tilde{\sigma}_j^a)\) which implies that the metric of superspace parametrized by \(\tilde{\sigma}_i^a\) takes the form

\[
\delta S^2(\vec{x}) = (\det \tilde{\sigma})^{\frac{1}{2}}(\frac{1}{2}\tilde{E}_i^a\tilde{E}_j^b - \tilde{E}_i^b\tilde{E}_j^a)\tilde{\sigma}_i^a\tilde{\sigma}_j^b
\]

\[
= (\det \tilde{\sigma})^{\frac{1}{2}}\left\{\frac{1}{6}(\tilde{E}_i^a\delta\tilde{\sigma}_i^a)^2 - (\tilde{E}_i^b\delta\tilde{\sigma}_i^a)(\tilde{E}_j^a\delta\tilde{\sigma}_j^b)\right\} \quad (50a)
\]

where \(\tilde{E}_i^a\) is the inverse of \(\tilde{\sigma}_i^a\) and \((\tilde{E}_i^b\delta\tilde{\sigma}_i^a)\) is the traceless part of \((\tilde{E}_i^b\delta\tilde{\sigma}_i^a)\). Demanding, as before, that the physical modes of the subspace with supermetric \((\det \tilde{\sigma})\)\((\frac{1}{3}\tilde{E}_i^a\tilde{E}_j^b - \tilde{E}_i^b\tilde{E}_j^a)\) be orthogonal to the gauge directions, the gauge-fixing conditions are

\[
\epsilon_{abc}\tilde{\varepsilon}^{bc} = 0 \quad (51)
\]

and

\[
\partial_i\tilde{\varepsilon}^i = 0 \quad (52)
\]

where \(\tilde{\varepsilon}^i\) is the traceless part of \((\delta\tilde{\sigma}_i^a)\). The linearized constraints from \((49a)\) and \((49b)\) are

\[
\partial_i\tilde{\varepsilon}^i + \epsilon^i_a a^b_i a^b = 0 \quad (53a)
\]

\[
\partial_i a^b_j - \partial_j a^b_i = 0 \quad (53b)
\]
and to lowest order, the superhamiltonian constraint is

$$\epsilon^{ab}_c \partial_a a^c_b = 0$$  \hspace{1cm} (54)$$

The constraints and gauge-fixing conditions are solved by $a_{ia}$ and $\tilde{e}^{ia}$ being transverse, symmetric and traceless. These two local degrees of freedom of the theory linearized about the flat background can be identified with the usual gravitons[16]. In the asymptotically flat context, the boundary Hamiltonian generates asymptotic time translation and dynamical evolution for the theory. However, in the case of spatially compact manifolds, the supermetric (51) suggests that the intrinsic time parameter is proportional to $\ln(\det \tilde{\sigma})$ since $E_{ia} \delta \tilde{\sigma}^{ia} = \delta \ln(\det \tilde{\sigma})$. This is in agreement with the previous analyses based on the supermetric (4). In the linearized limit, the intrinsic time coordinate is proportional to $\text{tr}(\tilde{\sigma})$ since

$$\left(\delta S\right)^2_{\tilde{\sigma}} = \frac{1}{6} (\text{tr}\tilde{\sigma})^2 - \tilde{\sigma}^{ab} \tilde{\sigma}_{ba}$$  \hspace{1cm} (55)$$

Keeping the fluctuations to second order in the superhamiltonian constraint, we have

$$a^b_i a^c_a - a^b_c a^c_b + \epsilon^{ab}_c \partial_a a^c_b + 4 \epsilon^{cb}_a (\partial_c a_{db}) \tilde{e}^{da} = 0$$  \hspace{1cm} (56)$$

As can be expected from the form of superhamiltonian constraint of the full theory, in the $\tilde{\sigma}$-representation, we do not end up with a free Klein-Gordon equation (see also Ref. 15 for connection and loop-representations perturbation analyses with the flat background). Note that expression (11), the supermetric of connection-superspace, is singular for this flat Euclidean background configuration! In the connection-representation, if we wish to consider the superhamiltonian constraint as the Klein-Gordon Equation in superspace, perturbing about the flat background is highly unnatural if not impossible.

We now consider perturbations about an unconventional background field which will exhibit many of the salient features associated with the full theory in the connection-representation. Consider the $\tilde{x}$-independent background

$$(A^*_{ia}, \tilde{\sigma}^{ia}_*) = (\delta_{ia}, 0)$$  \hspace{1cm} (57)$$

This choice of background leads to $\tilde{B}^{ia}_* = \delta^{ia}$ and the supermetric takes on a simple form for this configuration. It is as natural to consider such a background in the $A$-representation as it is to use the flat background with vanishing Ashtekar connection in the metric representation with supermetric (3) or in the $\tilde{\sigma}$-representation with supermetric.
The background with vanishing $\tilde{\sigma}^{ia}$ is also an extremum of superhamiltonian $H$. What is remarkable about this background is that it is considered to be unphysical in the context of ADM variables because for vanishing densitized triads, the ADM variable $g_{ij}$ (considered as derived from $\tilde{\sigma}^{ia}$ through $g^{ij} = (\text{det}\tilde{\sigma})^{-1}(\tilde{\sigma}^{ia}\tilde{\sigma}^{ja})$) is not even well-defined. The supermetric (2) is singular at such a configuration and the constraints for the ADM variables are not defined for degenerate metrics. The situation is however very different with the Ashtekar variables since nowhere in the Ashtekar constraints is there a requirement that the conjugate variable $\tilde{\sigma}^{ia}$ be non-degenerate. Thus it makes perfect sense to consider the perturbation about zero momenta. Notice also that this background satisfies all the constraints. Indeed in the $A$-representation, the condition $\tilde{\sigma}^{ia} = 0$ translates into

$$\frac{\delta}{\delta A^{ia}} \Phi[A] = 0$$

which has the interpretation that $\Phi[A]$ is a topological invariant of $A^{ia}$. With the ordering of the constraints discussed previously, the condition (58) is sufficient for all the quantum constraints to hold and a state which satisfies it is a possible quantum state of the theory[13]. In particular, a quantum state with this property is the non-abelian Ray-Singer torsion $\Phi = \int DA \exp(ikC)$ discussed in Ref. 16.

With this unconventional background, the linearized Gauss’ Law and supermomentum constraints take the form of

$$\partial_i \tilde{e}^{ia} = 0$$

$$\varepsilon_{ija} \tilde{e}^{ja} = 0$$

Using

$$\{a_{ia}(\vec{x}), \tilde{e}^{jb}(\vec{y})\}_{P.B.} = \frac{1}{2} \delta^{a}{}_{b} \delta^{j}{}_{i} \delta^{3} \delta(\vec{x} - \vec{y})$$

the linearized constraints generate

$$\delta \tilde{a}_{ia} = \left\{ \tilde{a}_{ia}, \int d^{3}x (\eta_{b} \partial_{j} \tilde{e}^{jb} + \xi_{j} \varepsilon_{jkb} \tilde{e}^{kb}) \right\}_{P.B.}$$

$$= \frac{1}{2} (-\partial_{i} \eta_{a} + \frac{1}{3} \delta_{ia} \partial_{b} \eta^{b} + \xi_{j} \varepsilon_{jia})$$

where $\tilde{a}_{ia}$ is the traceless part of $a_{ia}$. Thus the constraints preserve the tracelessness of $\tilde{a}_{ia}$ and suggest that the longitudinal and anti-symmetric parts of $\tilde{a}_{ia}$ can be gauged away. Indeed, with $a_{ia} \equiv \delta A^{ia\perp}$, the gauge-fixing conditions, (25) and (37), yield

$$\tilde{e}^{ija} \tilde{a}_{ja} = 0$$

$$\partial_{b} \tilde{a}^{b}{}_{a} = 0$$

12
So the constraints and gauge-fixing conditions indicate that the physical degrees of freedom are in the transverse and symmetric parts of $\tilde{e}^{ia}$ and $\tilde{e}^{\bar{i}a}$. With this background, to lowest order (which is second order since there are no first order terms because the background with vanishing $\tilde{\sigma}^{ia}$ is an extremum of $H$) the superhamiltonian constraint reads

$$\tilde{e}^{aa}\tilde{e}^{bb} - \tilde{e}^{ab}\tilde{e}^{ba} = 0 \quad (63)$$

The perturbations can result in configurations with non-degenerate metrics when $\det(\tilde{e}^{ia})$ is non-vanishing. The two local degrees of freedom associated with $a_{ia}$ and $\tilde{e}^{ia}$ described by Equations (60), (63)-(64), are the “nonconventional gravitons”.

All the constraints including the superhamiltonian constraint commute among themselves. The supermetric at this background configuration is

$$\tilde{G}^{iaj}b|_{A^*_{ia}} = \left\{ \frac{1}{2} \delta^{ia} \delta^{jb} - \delta^{ib} \delta^{ja} \right\} \quad (64)$$

i.e.

$$(\delta S)^2|_{A^*_{ia}} = \left\{ \frac{1}{6} (a^c \epsilon^2 - \bar{a}^b_c a^c_b) \right\} \quad (65)$$

where $a_{ia} \equiv \delta A^\perp_{ia}$ and $\bar{a}_{ia}$ is symmetric, traceless and transverse. In the $a$-representation, as in the full theory, the superhamiltonian constraint translates into the free Klein-Gordon equation

$$\left\{ 6 \frac{\delta^2}{(\delta a^c \epsilon)^2} - \frac{\delta}{\delta a_\alpha } \frac{\delta}{\delta a^\alpha } \right\} \Phi[a] = 0 \quad (66)$$

where $\alpha = 1, 2, 3, +, -, \text{and the five conjugate variables to the traceless symmetric components of } \tilde{e}^{ab} \text{ can be written explicitly as}$

$$a_1 = \sqrt{2} a_{(23)}, \quad a_2 = \sqrt{2} a_{(31)}, \quad a_3 = \sqrt{2} a_{(12)}, \quad a_+ = \sqrt{\frac{3}{2}} (\bar{a}_{11} + \bar{a}_{22}), \quad a_- = \sqrt{\frac{1}{2}} (\bar{a}_{11} - \bar{a}_{22}) \quad (67a)$$

This suggests that the coordinate $\tau \equiv \sqrt{\frac{1}{6}} \text{tr}(a)$ should be identified as the intrinsic time in the quantum theory. The conserved density can be chosen as the usual one for Klein-Gordon wavefunctionals. With appropriate restriction to positive frequency modes, it will be positive definite. “Plane wave” solutions are given by $\exp(i \int \tilde{e}^\alpha a_\alpha - tr(\tilde{e})\tau d^3x)$. However, one can also proceed further and obtain the massless Dirac equation

$$\left\{ \gamma^0 \frac{\delta}{\delta \tau} + \gamma^\alpha \frac{\delta}{\delta a^\alpha } \right\} \Psi[\tau, a_\alpha] = 0 \quad (68)$$
with the conserved (in $\tau$ - evolution) positive-definite probability density

$$\rho = \Psi^\dagger[\tau, \tilde{a}]\Psi[\tau, \tilde{a}]$$

(69)

It is even possible to contemplate *quantum states which are chiral* in connection superspace.

In momentum space, the two degrees of freedom can be isolated even more explicitly and the Fourier transform of $\bar{a}_{bc}(\vec{x})$ is

$$\bar{a}_{bc}(\vec{k}) = A^+(\vec{k})m_b m_c + A^-(\vec{k})\bar{m}_b \bar{m}_c$$

(70)

where (see, for instance, Ref. 16) the basis vectors satisfy

$$m_a m^a = \bar{m}_a \bar{m}^a = k_a m^a = k_a \bar{m}_a = 0, \quad m_a \bar{m}^a = 1$$

(71)

and $\bar{m}_a$ is the complex conjugate of $m_a$. The physical modes $A^\pm$ are the positive and negative helicity modes. The same decomposition can be done for the transverse, symmetric and traceless conjugate variable $\tilde{e}^{bc}$. As stated in the beginning, we have concentrated on real Ashtekar variables and this is consistent for spacetimes with Euclidean signature. The reality of $a_{ab}(\vec{x})$ (which need to be imposed only on the physical modes) is equivalent to the condition

$$[a_{bc}(\vec{k})] = a_{bc}(-\vec{k})$$

(72)

Similar reality conditions can be imposed on the physical modes of the conjugate variables.

It has been postulated by many authors that in quantum gravity, the signature of spacetime is not sacred and fluctuations of it can occur. Certainly for fluctuations about the background at which the metric is not even defined, it is rather unnatural to impose a set of reality conditions on the Ashtekar variables to restrict the configurations to correspond to spacetimes with Lorentzian signature. It is more natural to start with complex variables and demand the wavefunctionals to be holomorphic in the Ashtekar potentials[6,7]. Although it may no longer be true that for complex potentials, the signature of the supermetric is as in expression (17), the decomposition (19) can still be carried out and the gauge-fixing conditions will not be altered. If we start with the Ashtekar variable written as $A_{ia} = iK_{ia} - \frac{1}{2}\epsilon_{abc}\omega^i_{bc}$, it can be checked that provided the other constraints are satisfied, despite $\tilde{B}_{ia}$ being complex, the superhamiltonian constraint (5c) remains real if $\tilde{\sigma}_{ia}$ is real. This suggests that in the quantum theory, despite the complex Ashtekar potentials, we should require the Laplacian operator in superspace to be hermitian with respect to a suitable measure and inner product after gauge-fixing.
We emphasize that the “unconventional” background is inaccessible to conventional perturbation analyses with the ADM variables and cannot be perturbatively related to the flat background. This highly interesting limit of the theory is precisely the zero-momentum limit of quantum gravity with Ashtekar variables. Various interesting questions such as the perturbative renormalizability (or non-renormalizability) of the theory about this unconventional background, the physical implications of spin and chiral quantum states of gravity, the influence of matter fields on the stability of the theory, and the intriguing role of configurations with vanishing $\tilde{B}$ in the dynamics of the full theory immediately come to mind, and are being studied. We hope to address these issues in a future report.

The research for this work is supported in part by the DOE under Grant No.(72)DE-FG05-92ER40709. One of us (CS) acknowledges the support of a Cunningham Fellowship.
References

[1] R. Arnowitt, S. Deser and C. Misner, in *Gravitation: An introduction to Current Research*, ed. L. Witten (Wiley, New York, 1962); Phys. Rev. **117**, 1595 (1960).

[2] M. Henneaux, M. Pilati and C. Teitelboim, Phys. Lett. **110B**, 123 (1982); M. Pilati, Phys. Rev. **D26**, 2645 (1982); ibid. **D28**, 729 (1983).

[3] B. S. DeWitt, Phys. Rev. **160**, 1113 (1967).

[4] A. Ashtekar, Phys. Rev. Lett. **57**, 2244 (1986).

[5] A. Ashtekar, Phys. Rev. **D36**, 1587 (1987).

[6] A. Ashtekar, in *New Perspectives in Canonical Gravity* (Bibliopolis, Naples, 1988).

[7] A. Ashtekar, *Lectures on Non-perturbative Canonical Gravity*, Advanced Series in Astrophysics and Cosmology-Vol. 6. (World Scientific, Singapore, 1991).

[8] K. Kuchar, *Canonical Quantum Gravity*, gr-qc/9304012.

[9] L. N. Chang and C. P. Soo, *Einstein Manifolds in Ashtekar Variables: Explicit Examples*, hep-th/9207056, VPI-IHEP-92-5; C. P. Soo, *Classical and Quantum Gravity with Ashtekar Variables*. Ph. D. Thesis, VPI&SU (1992), VPI-IHEP-92-11.

[10] G. B. Gibbons and S. W. Hawking, Phys. Rev. **D15**, 2752 (1977).

[11] P. Mazur and E. Mottola, Nucl. Phys. **B341**, 187 (1990); Z. Bern, S. K. Blau and E. Mottola, Phys. Rev. **D43**, 1212 (1991); P. Mazur, Phys. Lett. **262B**, 405 (1991).

[12] V. N. Gribov, Nucl. Phys. **B139**, 1 (1978); I. M. Singer, Comm. Math. Phys. **60**, 7 (1978).

[13] T. Jacobson and L. Smolin, Nucl. Phys. **B299**, 295 (1988).

[14] J. L. Friedman and I. Jack, Phys. Rev. **D37**, 3495 (1987); N. C. Tsamis and R. P. Woodard, Phys. Rev. **D36**, 3641 (1987).

[15] A. Ashtekar, C. Rovelli and L. Smolin, Phys. Rev. **D44**, 1740 (1991).

[16] E. Witten, Comm. Math. Phys. **121**, 351 (1989).