Uncertainty and Analyticity

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Abstract. We describe a connection between minimal uncertainty states and holomorphy-type conditions on the images of the respective wavelet transforms. The most familiar example is the Fock–Segal–Bargmann transform generated by the Gaussian, however, this also occurs under more general assumptions.

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1. Introduction

There are two and a half main examples of reproducing kernel spaces of analytic function. One is the Fock–Segal–Bargmann (FSB) space and others (one and a half)—the Bergman and Hardy spaces. The first space is generated by the Heisenberg group [2 § 1.6; 5 § 7.3], two others—by the group SU(1, 1) [5 § 4.2] (this explains our way of counting).

Those spaces have the following properties, which make their study particularly pleasant and fruitful:

i. There is a group, which acts transitively on functions’ domain.
ii. There is a reproducing kernel.
iii. The space consists of holomorphic functions.

Furthermore, for FSB space there is the following property:

iv. The reproducing kernel is generated by a function, which minimises the uncertainty for coordinate and momentum observables.

It is known, that a transformation group is responsible for the appearance of the reproducing kernel [1 Thm. 8.1.3]. This paper shows that the last two properties are equivalent and connected to the group as well.

On leave from Odessa University.
2. The Uncertainty Relation

In quantum mechanics \[2, \text{§ 1.1}\], an observable (self-adjoint operator on a Hilbert space \(H\)) \(A\) produces the expectation value \(\bar{A}\) on a state (a unit vector) \(\phi \in H\) by \(\bar{A} = \langle A\phi, \phi \rangle\). Then, the dispersion is evaluated as follows:

\[
\Delta_{\phi}^{2}(A) = \langle (A - \bar{A})^{2}\phi, \phi \rangle = \langle (A - \bar{A})\phi, (A - \bar{A})\phi \rangle = \| (A - \bar{A})\phi \|^{2}.
\]

(1)

The next theorem links obstructions of exact simultaneous measurements with non-commutativity of observables.

**Theorem 1 (The Uncertainty relation).** If \(A\) and \(B\) are self-adjoint operators on a Hilbert space \(H\), then

\[
\| (A - a)u \| \| (B - b)u \| \geq \frac{1}{\pi} |\langle (AB - BA)u, u \rangle|,
\]

(2)

for any \(u \in H\) from the domains of \(AB\) and \(BA\) and \(a, b \in \mathbb{R}\). Equality holds precisely when \(u\) is a solution of \((A - a) + ir(B - b)u = 0\) for some real \(r\).

**Proof.** The proof is well-known \[2, \text{§ 1.3}\], but it is short, instructive and relevant for the following discussion, thus we include it in full. We start from simple algebraic transformations:

\[
\langle (AB - BA)u, u \rangle = \langle (A - a)(B - b) - (B - b)(A - a)u, u \rangle
\]

\[
= \langle (B - b)u, (A - a)u \rangle - \langle (A - a)u, (B - b)u \rangle
\]

\[
= 2i\Im \langle (B - b)u, (A - a)u \rangle
\]

(3)

Then by the Cauchy–Schwartz inequality:

\[
\frac{1}{2} \langle (AB - BA)u, u \rangle \leq |\langle (B - b)u, (A - a)u \rangle| \leq \| (B - b)u \| \| (A - a)u \|.
\]

The equality holds if and only if \((B - b)u\) and \((A - a)u\) are proportional by a purely imaginary scalar. \(\Box\)

The famous application of the above theorem is the following fundamental relation in quantum mechanics. Recall \[2, \text{§ 1.2}\], that the one-dimensional Heisenberg group \(\mathbb{H}^{1}\) consists of points \((s, x, y) \in \mathbb{R}^{3}\), with the group law:

\[
(s, x, y) * (s', x', y') = (s + s' + \frac{1}{2}(xy' - x'y), x + x', y + y').
\]

(4)

This is a nilpotent step two Lie group. By the Stone–von Neumann theorem \[2, \text{§ 1.5}\], any infinite-dimensional unitary irreducible representation of \(\mathbb{H}^{1}\) is unitary equivalent to the Schrödinger representation \(\rho_{\hbar}\) in \(L_{2}(\mathbb{R})\) parametrised by the Planck constant \(\hbar \in \mathbb{R} \setminus \{0\}\). A physically consistent form of \(\rho_{\hbar}\) is \[\text{(3.5)}\]:

\[
[\rho_{\hbar}(s, x, y) f](q) = e^{-2\pi i\hbar(s+xy/2)} - 2\pi i\hbar f(q + \hbar y).
\]

(5)

Elements of the Lie algebra \(\mathfrak{h}_{1}\), corresponding to the infinitesimal generators \(X\) and \(Y\) of one-parameters subgroups \((0, t/(2\pi), 0)\) and \((0, 0, t)\) in \(\mathbb{H}^{1}\), are represented in \[\text{[(5)]}\] by the (unbounded) operators \(M\) and \(D\) on \(L_{2}(\mathbb{R})\):

\[
M = -iq, \quad D = \hbar \frac{d}{dq}, \quad \text{with the commutator} \quad [M, D] = i\hbar I.
\]

(6)
In the Schrödinger model of quantum mechanics, $f(q) \in \mathcal{L}_2(\mathbb{R})$ is interpreted as a wave function (a state) of a particle, with $M$ and $D$ are the observables of its coordinate and momentum.

**Corollary 2 (Heisenberg–Kennard uncertainty relation).** For the coordinate $M$ and momentum $D$ observables we have the Heisenberg–Kennard uncertainty relation:

$$\Delta_\phi(M) \cdot \Delta_\phi(D) \geq \frac{\hbar}{2}.$$  
(7)

The equality holds if and only if $\phi(q) = e^{-cq^2}$, $c \in \mathbb{R}_+$ is the vacuum state in the Schrödinger model.

**Proof.** The relation follows from the commutator $[M, D] = i\hbar I$, which, in turn, is the representation of the Lie algebra $h_1$ of the Heisenberg group. The minimal uncertainty state in the Schrödinger representation is a solution of the differential equation: $(M - i r D) \phi = 0$ for some $r \in \mathbb{R}$, or, explicitly:

$$(M - i r D) \phi = -i \left(q + r\hbar \frac{d}{dq}\right) \phi(q) = 0.$$  
(8)

The solution is the Gaussian $\phi(q) = e^{-cq^2}$, $c = \frac{1}{2\pi r\hbar}$. For $c > 0$, this function is in the state space $\mathcal{L}_2(\mathbb{R})$. □

It is common to say that the Gaussian $\phi(q) = e^{-cq^2}$ represents the ground state, which minimises the uncertainty of coordinate and momentum.

3. Wavelet transform and analyticity

3.1. Induced wavelet transform

The following object is common in quantum mechanics [4], signal processing, harmonic analysis [8], operator theory [7,9] and many other areas [5]. Therefore, it has various names [1]: coherent states, wavelets, matrix coefficients, etc. In the most fundamental situation [1, Ch. 8], we start from an irreducible unitary representation $\rho$ of a Lie group $G$ in a Hilbert space $\mathcal{H}$. For a vector $f \in \mathcal{H}$ (called mother wavelet, vacuum state, etc.), we define the map $W_f$ from $\mathcal{H}$ to a space of functions on $G$ by:

$$[W_f v](g) = \tilde{v}(g) := \langle v, \rho(g)f \rangle.$$  
(9)

Under the above assumptions, $\tilde{v}(g)$ is a bounded continuous function on $G$. The map $W_f$ intertwines $\rho(g)$ with the left shifts on $G$:

$$W_f \circ \rho(g) = \Lambda(g) \circ W_f,$$

where $\Lambda(g) : \tilde{v}(g') \mapsto \tilde{v}(g^{-1}g')$.  
(10)

Thus, the image $W_f \mathcal{H}$ is invariant under the left shifts on $G$. If $\rho$ is square integrable and $f$ is admissible [1, § 8.1], then $\tilde{v}(g)$ is square-integrable with respect to the Haar measure on $G$. At this point, none of admissible vectors has an advantage over others.

It is common [5, § 5.1], that there exists a closed subgroup $H \subset G$ and a respective $f \in \mathcal{H}$ such that $\rho(h)f = \chi(h)f$ for some character $\chi$ of $H$. In
this case, it is enough to know values of $\tilde{v}(s(x))$, for any continuous section $s$ from the homogeneous space $X = G/H$ to $G$. The map $v \mapsto \tilde{v}(x) = \tilde{v}(s(x))$ intertwines $\rho$ with the representation $\rho_\chi$ in a certain function space on $X$ induced by the character $\chi$ of $H$ [3, § 13.2]. We call the map $\mathcal{W}_f : v \mapsto \tilde{v}(x)$ the \textit{induced wavelet transform} [5, § 5.1].

For example, if $G = \mathbb{H}^1$, $H = \{(s, 0, 0) \in \mathbb{H}^1 : s \in \mathbb{R}\}$ and its character $\chi_h(s, 0, 0) = e^{2\pi i hs}$, then any vector $f \in L^2(\mathbb{R})$ satisfies $\rho_\chi(s, 0, 0)f = \chi_h(s)f$ for the representation [5]. Thus, we still do not have a reason to prefer any admisible vector to others.

3.2. Right shifts and analyticity

To discover some preferable mother wavelets, we use the following a general result from [5, § 5]. Let $G$ be a locally compact group and $\rho$ be its representation in a Hilbert space $\mathcal{H}$. Let $[\mathcal{W}_f v](g) = \langle v, \rho(g)f \rangle$ be the wavelet transform defined by a vacuum state $f \in \mathcal{H}$. Then, the right shift $R(g) : [\mathcal{W}_f v](g') \mapsto [\mathcal{W}_f v](g'g)$ for $g \in G$ coincides with the wavelet transform $[\mathcal{W}_{f_g} v](g') = \langle v, \rho(g')f_g \rangle$ defined by the vacuum state $f_g = \rho(g)f$. In other words, the covariant transform intertwines right shifts on the group $G$ with the associated action $\rho$ on vacuum states, cf. (10):

$$R(g) \circ \mathcal{W}_f = \mathcal{W}_{\rho(g)}f.$$  \hspace{1cm} (11)

Although, the above observation is almost trivial, applications of the following corollary are not.

\textbf{Corollary 3 (Analyticity of the wavelet transform, [5, § 5]).} Let $G$ be a group and $dg$ be a measure on $G$. Let $\rho$ be a unitary representation of $G$, which can be extended by integration to a vector space $V$ of functions or distributions on $G$. Let a mother wavelet $f \in \mathcal{H}$ satisfy the equation

$$\int_G a(g) \rho(g)f \, dg = 0,$$

for a fixed distribution $a(g) \in V$. Then any wavelet transform $\tilde{v}(g) = \langle v, \rho(g)f \rangle$ obeys the condition:

$$D\tilde{v} = 0,$$

where $D = \int_G \tilde{a}(g) R(g) \, dg$, \hspace{1cm} (12)

with $R$ being the right regular representation of $G$.

Some applications (including discrete one) produced by the $ax+b$ group can be found in [3, § 6]. We turn to the Heisenberg group now.

\textbf{Example 4 (Gaussian and FSB transform).} The Gaussian $\phi(x) = e^{-c q^2/2}$ is a null-solution of the operator $\hbar c M - i D$. For the centre $Z = \{(s, 0, 0) : s \in \mathbb{R}\} \subset \mathbb{H}^1$, we define the section $s : \mathbb{H}^1/Z \to \mathbb{H}^1$ by $s(x, y) = (0, x, y)$. Then, the corresponding induced wavelet transform is:

$$\tilde{v}(x, y) = \langle v, \rho(s(x, y))f \rangle = \int_{\mathbb{R}} v(q) e^{\pi i y x y - 2\pi i x q} e^{-c(q + hy)^2/2} \, dq.$$  \hspace{1cm} (13)
The infinitesimal generators $X$ and $Y$ of one-parameters subgroups $(0, t/(2\pi), 0)$ and $(0, 0, t)$ are represented through the right shift in (1) by

$$R_s(X) = -\frac{1}{4\pi} y \partial_s + \frac{1}{2\pi} \partial_x, \quad R_s(Y) = \frac{1}{2} x \partial_s + \partial_y.$$  

For the representation induced by the character $\chi_h(s, 0, 0) = e^{2\pi ih s}$ we have $\partial_s = 2\pi i h I$. Cor. 3 ensures that the operator

$$\hbar c \cdot R_s(X) + i \cdot R_s(Y) = -\frac{\hbar}{2} (2\pi x + i\hbar cy) + \frac{\hbar c}{2\pi} \partial_x + i \partial_y$$

(14)

annihilate any $\tilde{v}(x, y)$ from (13). The integral (13) is known as Fock–Segal–Bargmann (FSB) transform and in the most common case the values $\hbar = 1$ and $c = 2\pi$ are used. For these, operator (14) becomes $-\pi(x + iy) + (\partial_x + i\partial_y) = -\pi z + 2\partial_z$ with $z = x + iy$. Then the function $V(z) = e^{\pi z^2/2} \tilde{v}(z) = e^{\pi(x^2+y^2)/2} \tilde{v}(x, y)$ satisfies the Cauchy–Riemann equation $\partial_z V(z) = 0$.

This example shows, that the Gaussian is a preferred vacuum state (as producing analytic functions through FSB transform) exactly for the same reason as being the minimal uncertainty state: the both are derived from the identity $(\hbar c M + iD)e^{-cq^2/2} = 0$.

### 3.3. Uncertainty and analyticity

The main result of this paper is a generalisation of the previous observation, which bridges together Cor. 3 and Thm. 1. Let $G$, $H$, $\rho$ and $\mathcal{H}$ be as before. Assume, that the homogeneous space $X = G/H$ has a (quasi-)invariant measure $d\mu(x)$ \[3 \S 13.2\]. Then, for a function (or a suitable distribution) $k$ on $X$ we can define the integrated representation:

$$\rho(k) = \int_X k(x) \rho(s(x)) d\mu(x),$$

(15)

which is (possibly, unbounded) operators on (possibly, dense subspace of) $\mathcal{H}$. In particular, $R(k)$ denotes the integrated right shifts, for $H = \{e\}$.

**Theorem 5.** Let $k_1$ and $k_2$ be two distributions on $X$ with the respective integrated representations $\rho(k_1)$ and $\rho(k_2)$. The following are equivalent:

i. A vector $f \in \mathcal{H}$ satisfies the identity

$$\Delta_f(\rho(k_1)) \cdot \Delta_f(\rho(k_2)) = |\langle \rho(k_1), \rho(k_1) \rangle f, f \rangle|.$$

ii. The image of the wavelet transform $W_f : v \mapsto \tilde{v}(g) = \langle v, \rho(g)f \rangle$ consists of functions satisfying the equation $R(k_1 + ik_2)\tilde{v} = 0$ for some $r \in \mathbb{R}$, where $R$ is the integrated form (13) of the right regular representation on $G$.

**Proof.** This is an immediate consequence of a combination of Thm. 1 and Cor. 3. \[3 \]

Example 1 is a particular case of this theorem with $k_1(x, y) = \delta'_x(x, y)$ and $k_2(x, y) = \delta'_y(x, y)$ (partial derivatives of the delta function), which represent vectors $X$ and $Y$ from the Lie algebra $\mathfrak{h}_1$. The next example will be of this type as well.
3.4. Hardy space

Let SU(1, 1) be the group of $2 \times 2$ complex matrices of the form

\[
\begin{pmatrix}
\alpha & \beta \\
\bar{\beta} & \bar{\alpha}
\end{pmatrix}
\]

with the unit determinant $|\alpha|^2 - |\beta|^2 = 1$. A standard basis in the Lie algebra $\mathfrak{su}_{1,1}$ is

\[
A = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.
\]

The respective one-dimensional subgroups consist of matrices:

\[
e^{tA} = \begin{pmatrix} \cosh \frac{t}{2} & -i \sinh \frac{t}{2} \\ i \sinh \frac{t}{2} & \cosh \frac{t}{2} \end{pmatrix}, \quad e^{tB} = \begin{pmatrix} \cosh \frac{t}{2} & \sinh \frac{t}{2} \\ \sinh \frac{t}{2} & \cosh \frac{t}{2} \end{pmatrix}, \quad e^{tZ} = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}.
\]

The last subgroup—the maximal compact subgroup of SU(1, 1)—is usually denoted by $K$. The commutators of the $\mathfrak{su}_{1,1}$ basis elements are

\[
\begin{align*}
[Z, A] &= 2B, \\
[Z, B] &= -2A, \\
[A, B] &= -\frac{1}{2}Z. \quad (16)
\end{align*}
\]

Let $\mathbb{T}$ denote the unit circle in $\mathbb{C}$ with the rotation-invariant measure. The mock discrete representation of SU(1, 1) \cite[§ VI.6]{H} acts on $L_2(\mathbb{T})$ by unitary transformations

\[
[\rho_1(g)f](z) = \frac{1}{(\beta z + \bar{\alpha})} f \left( \frac{\alpha z + \beta}{\beta z + \bar{\alpha}} \right), \quad g^{-1} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}. \quad (17)
\]

The respective derived representation $\rho_{1*}$ of the $\mathfrak{su}_{1,1}$ basis is:

\[
\rho_{1*}^A = \frac{1}{2}(z + (z^2 + 1)\partial_z), \quad \rho_{1*}^B = \frac{1}{2}(z + (z^2 - 1)\partial_z), \quad \rho_{1*}^Z = -iI - 2iz\partial_z.
\]

Thus, $\rho_{1*}^{B+A} = -\partial_z$ and the function $f_+(z) \equiv 1$ satisfies $\rho_{1*}^{B+A}f_+ = 0$. Recalling the commutator $[A, B] = -\frac{1}{2}Z$ we note that $\rho_1(e^{itZ})f_+ = e^{it}f_+$. Therefore, there is the following identity for dispersions on this state:

\[
\Delta_{f_+}(\rho_{1*}^A) \cdot \Delta_{f_+}(\rho_{1*}^B) = \frac{1}{2},
\]

with the minimal value of uncertainty among all eigenvectors of the operator $\rho_1(e^{itZ})$.

Furthermore, the vacuum state $f_+$ generates the induced wavelet transform for the subgroup $K = \{e^{itZ} \mid t \in \mathbb{R}\}$. We identify SU(1, 1)/$K$ with the open unit disk $D = \{w \in \mathbb{C} \mid |w| < 1\}$ \cite[§ 5.5; 9]{H}. The map $s :$ SU(1, 1)/$K \to$ SU(1, 1) is defined as $s(w) = \frac{1}{\sqrt{1-|w|^2}} \begin{pmatrix} 1 & w \\ \bar{w} & 1 \end{pmatrix}$. Then, the induced wavelet transform is:

\[
\tilde{\nu}(w) = \langle v, \rho_1(s(w))f_+ \rangle = \frac{1}{2\pi \sqrt{1-|w|^2}} \int_{\mathbb{T}} \frac{v(e^{i\theta}) \, d\theta}{1 - we^{-i\theta}} = \frac{1}{2\pi i \sqrt{1-|w|^2}} \int_{\mathbb{T}} \frac{v(e^{i\theta}) \, de^{i\theta}}{e^{i\theta} - w}.
\]

Clearly, this is the Cauchy integral up to the factor $\frac{1}{\sqrt{1-|w|^2}}$, which presents the conformal metric on the unit disk. Similarly, we can consider the operator
\[ \rho B^{1} - i A^{1} = z + z^2 \partial_z \] and the function \( f_-(z) = \frac{1}{z} \) simultaneously solving the equations \( \rho B^{1} - i A^{1} f_- = 0 \) and \( \rho_1(e^{tZ}) f_- = e^{-it} f_- \). It produces the integral with the conjugated Cauchy kernel.

Finally, we can calculate the operator \( L^{B-IA} \) annihilating the image of the wavelet transform. In the coordinates \((w, t) \in (SU(1,1)/K) \times K\), the restriction to the induced subrepresentation is, cf. [10, \S IX.5]:

\[ L^{B-IA} = e^{2it}(\frac{1}{2}w + (1 - |w|^2) \partial \bar{w}). \]

Furthermore, if \( L^{B-IA} \tilde{v}(w) = 0 \), then \( \partial \bar{w}(\sqrt{1 - w \bar{w}} \cdot \tilde{v}(w)) = 0 \). That is, \( V(w) = \sqrt{1 - w \bar{w}} \cdot \tilde{v}(w) \) is a holomorphic function on the unit disk.

Similarly, we can treat representations of \( SU(1,1) \) in the space of square integrable functions on the unit disk. The irreducible components of this representation are isometrically isomorphic to weighted Bergman spaces of (purely poly-)analytic functions on the unit, cf. [11].

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