Ramanujan’s Master Theorem and two formulas for zero-order Hankel transform

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Abstract

Using Ramanujan’s Master Theorem, two formulas are derived which define the Hankel transforms of order zero with even functions by inverse Mellin transforms, provided these functions and their derivatives obey special conditions. Their validity is illustrated by a number of examples. With a help of one of these formulas, one as yet unknown parametric improper integral of the Bessel function \( J_0(x) \) is calculated.

1 Introduction

The Hankel transform,

\[
\mathcal{H}_\nu = \int_0^\infty J_\nu(qx)f(x) \, x \, dx,
\]

where \( J_\nu(x) \) is the Bessel function of order \( \nu \), solves a number of problems in mathematical physics [1], [2], [3] and high energy nuclear and particle physics [4], [5], [6].

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Often it is necessary to know an asymptotics of this integral as \( q \to \infty \). A method of obtaining such an asymptotics were done in [7] (see also [1]), where the following results were obtained:

\[
\int_0^\infty J_0(qx)f(x)\,dx = \frac{f(0)}{q} - \frac{1}{2} \frac{f^{(2)}(0)}{q^3} + \frac{1}{2^2 \cdot 2!} \frac{f^{(4)}(0)}{q^5} - \frac{1 \cdot 3 \cdot 5}{2^3 \cdot 3!} \frac{f^{(6)}(0)}{q^7} + \cdots, \tag{2}
\]

\[
\int_0^\infty J_1(qx)f(x)\,dx = \frac{f(0)}{q} + \frac{f^{(1)}(0)}{q^2} - \frac{1}{2} \frac{f^{(3)}(0)}{q^4} + \frac{1}{2^2 \cdot 2!} \frac{f^{(5)}(0)}{q^6} + \cdots. \tag{3}
\]

Note that integrands in integrals (2), (3), in contrast to (1), do not contain a factor \( x \). Later on, asymptotic expansions for \( H_\nu \) as \( q \to \infty \) have been obtained by several authors [8]-[13].

It follows from (2) that for \( \nu = 0 \) Hankel transform (1) is presented by an asymptotic series which contains derivatives of odd orders only:

\[
H_0(q) = \frac{1}{q^3} \sum_{m=0}^\infty (-1)^{m+1} \frac{\Gamma(2m+2)}{\Gamma^2(m+1)} f^{(2m+1)}(0) (2q)^{-2m}. \tag{4}
\]

It is clear that a case when \( f^{(2m+1)}(0) = 0 \) for all \( m = 0, 1, 2, \ldots \) should be considered separately, as was mentioned in [14]. Then one expects that the Hankel transform \( H_0(q) \) decreases as \( q \to \infty \) more quickly than any inverse power of \( q \). For instance, it could take place if \( f(x) \) is an even function, \( f(x) = g(x^2) \).

The main goal of this paper is to derive two representations for the Hankel transform of order zero with even functions which are valid not only for \( q \to \infty \) but for all \( q > q_0 > 0 \). As a byproduct, one as yet unknown parametric improper integral of \( J_0(x) \) will be calculated.

## 2 Hankel transform of order zero

Let us study the Hankel transform of order zero with an even function:

\[
A(q) = H_0(q) = \int_0^\infty J_0(qx)f(x)\,dx, \tag{5}
\]
where \( f(x) = g(x^2) \).  

**Theorem 1.** The Hankel transform of order zero \(^3\) with the function \( f(x) = g(x^2) \) and \( q > 0 \) may be expressed in the form

\[
A(q) = \frac{1}{\pi i q^2} \int_{\alpha - i\infty}^{\alpha + i\infty} ds \tilde{g}^{(s)}(0) \Gamma(s + 1) \left(\frac{q^2}{4}\right)^{-s},
\]

for \(-1 < \alpha < 0\), provided that

1. \( g(z) \) is a regular function and its Taylor series at \( z = 0 \) has the form
   \[
g(z) = \sum_{m=0}^{\infty} \frac{\tilde{g}^{(m)}(0)}{m!} (-z)^m;
   \]
2. \( g(z) = O(z^{-d}) \), as \( z \to \infty \), for \( d > 1/4 \);
3. \( \tilde{g}^{(s)}(0) \) is a regular (single-valued) function defined on a half-plane
   \[
   H(\delta) = \{ s \in \mathbb{C} : \text{Re } s \geq -\delta \}
   \]
   for some \( 1/4 < \delta < 1 \) and satisfies the growth condition
   \[
   |\tilde{g}^{(s)}(0)| < C e^{Pv+|A|w}
   \]
   for some \( A < \pi/2 \) and all \( s = v + iw \in H(\delta) \).

**Proof.** The main tool of the proof will be the power Ramanujan Master Theorem (see, for instance, its proof provided by Hardy in \(^\text{15}\)). It provides an analytic expression for the Mellin transform of an analytic function. We will use the following version of Ramanujan’s master theorem \(^\text{16}\):

\[
\int_{0}^{\infty} x^{s-1} \left[ \varphi(0) - x \varphi(1) + x^2 \varphi(2) - \ldots \right] dx = \frac{\pi}{\sin(s\pi)} \varphi(-s),
\]

where \( 0 < \text{Re } s < \delta \). Formula \(^\text{10}\) is valid if the function \( \varphi(s) \) in \(^\text{10}\) satisfies condition 3 of the Theorem 1 with \( A < \pi \).

\(^\text{1}\)A special case \( f(x) = e^{-x^2} \varphi(x^2) \) was considered in \(^\text{14}\).
Let us put

$$\varphi(s) = \frac{\bar{g}^{(s)}(0)}{\Gamma(s+1)}$$

in (10). One can see that this function obeys all conditions of Ramanujan’s Master Theorem, and we have

$$\int_{0}^{\infty} x^{s-1} \left[ \sum_{m=0}^{\infty} \frac{\bar{g}^{(m)}(0)}{\Gamma(m+1)} (-x)^m \right] = \frac{\pi}{\sin(s\pi)} \frac{\bar{g}^{(-s)}(0)}{\Gamma(-s+1)}. \tag{12}$$

Then the inverse Mellin transform gives immediately \((1/4 < c < \delta)\)

$$g(z) = \frac{1}{2i} \int_{c-i\infty}^{c+i\infty} ds \frac{\bar{g}^{(-s)}(0)}{\sin(s\pi) \Gamma(-s+1)} z^{-s}$$

$$= -\frac{1}{2i} \int_{-c-i\infty}^{-c+i\infty} ds \frac{\bar{g}^{(s)}(0)}{\sin(s\pi) \Gamma(s+1)} z^{s}. \tag{13}$$

Now we put \(z = x^2\) and replace \(g(x^2)\) in (5) by its integral representation (13). As a result, we obtain the formula

$$A(q) = -\frac{1}{2i} \int_{-c-i\infty}^{-c+i\infty} ds \frac{\bar{g}^{(s)}(0)}{\sin(s\pi) \Gamma(s+1)} \int_{0}^{\infty} dx \ x^{2s+1} J_0(qx). \tag{14}$$

Using equation (17)

$$\int_{0}^{\infty} dx \ x^{2s+1} J_0(qx) = 2^{2s+1} q^{-2s-2} \frac{\Gamma(1+s)}{\Gamma(-s)}, \tag{15}$$

valid for \(-1 < -\delta < \Re s < -1/4\), as well as equation (18)

$$\frac{1}{\sin(s\pi) \Gamma(-s)} = -\frac{1}{\pi} \frac{\Gamma(s+1)}{\Gamma(s)}, \tag{16}$$

we come to formula (6). Q.E.D.
Let us see how does our formula (6) work? It is worth it to consider a number of examples (everywhere below it is assumed that $q > 0$).

A.1. \( f(z) = e^{-a^2x^2} \), \( a > 0 \), then \( \bar{g}^{(m)}(0) = a^{2m} \), and we get from (6) \( \alpha > -1 \)

\[
A_1(q) = \int_0^\infty xe^{-a^2x^2}J_0(qx) \, dx = \frac{1}{\pi i q^2} \int_{\alpha-i\infty}^{\alpha+i\infty} ds \Gamma(1 + s) \left( \frac{q^2}{4a^2} \right)^{-s} = \frac{1}{2a^2} e^{-q^2/4a^2}, \tag{17}
\]

in accordance with eq. 2.12.9.3. in [19]. To get this result, we used formula 7.3(1) from ref. [20] \((\text{Re} \, c > 0)\):

\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \, (ax)^{-s} \Gamma(s) = e^{-ax}. \tag{18}
\]

A.2. \( f(x) = e^{-a^2x^2} J_0(cx) \), \( a, c > 0 \). The derivatives \( \bar{g}^{(m)}(0) \) are calculated in Appendix A to be

\[
\bar{g}^{(m)}(0) = a^{2m} L_m \left( -\frac{c^2}{4a^2} \right), \tag{19}
\]

where \( L_n(z) \) is the Laguerre polynomial [17]. Note that \( L_n(-z) > 0 \) for \( n \geq 0 \) if \( z > 0 \). It means that \( \bar{g}^{(m)}(0) > 0 \) for all \( m \geq 0 \).

We have the relation [18]

\[
L_m(-z) = e^{z/2}(-z)^{-1/2} M_{m+1/2,0}(-z), \tag{20}
\]

where \( M_{\lambda,\chi}(z) \) is the Whittaker function of the first kind [18]. Thus, we obtain \((-1 < \alpha < 0)\)

\[
A_2(q) = \int_0^\infty xe^{-a^2x^2}J_0(cx)J_0(qx) \, dx = \left( -\frac{c^2}{4a^2} \right)^{-1/2} e^{-c^2/8a^2} \times \frac{1}{\pi i q^2} \int_{\alpha-i\infty}^{\alpha+i\infty} ds \Gamma(s + 1) M_{s+1/2,0} \left( -\frac{c^2}{4a^2} \right) \left( \frac{q^2}{4a^2} \right)^{-s}. \tag{21}
\]
Now we can apply formula 7.5(19) from [20]

\[ \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \Gamma(s + \nu + 1/2) M_{s,\nu}(y) x^{-s} \]

\[ = \Gamma(2\nu + 1)(xy)^{1/2} e^{-x+y/2} J_{2\nu}(2(xy)^{1/2}) , \]

(22)

valid for \( \text{Re}(c+\nu) > -1/2 \). After replacement \( s + 1/2 = s' \) in (21) and taking into account that \( J_0(iz) = I_0(z) \), we find (see eq. 2.12.39.3. in [19])

\[ A_2(q) = \frac{1}{2a^2} \exp\left(-\frac{q^2 + c^2}{4a^2}\right) I_0\left(\frac{qc}{4a^2}\right) , \]

(23)

where \( I_\nu(z) \) is the modified Bessel function of the first kind.

As it turns out, our formula (6) gives correct results even if the condition 3 of the Theorem 1 is violated (namely, if \( A \) is equal to \( \pi/2 \) in (9)).

Let us illustrate this statement by several examples.

A.3. \( f(x) = 1/(x^2 + a^2)^{n+1}, a > 0, \) integer \( n \geq 0 \). Then

\[ A_3(q) = \int_0^\infty \frac{x}{(x^2 + a^2)^{n+1}} J_0(qx) \, dx . \]

(24)

We have \( \bar{g}(s)(0) = a^{-2m-2n-2} \Gamma(m + n + 1)/\Gamma(n + 1) \)

\[ \text{then we obtain from (6) (for } -1 < \alpha < 0) \]

\[ A_3(q) = \frac{1}{a^{2n+2}q^2} \frac{1}{\Gamma(n+1)} \frac{1}{\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} ds \Gamma(s + 1) \Gamma(s + n + 1) \left(\frac{qa}{2}\right)^{-2s} \]

\[ = \frac{1}{\Gamma(n+1)} \left(\frac{q}{2a}\right)^n K_n(qa) , \]

(25)

where \( K_\nu(z) \) is the modified Bessel function of the second kind (Macdonald function), in accordance with eq. 2.12.4.28. in [19]. We used formula 6.8(26) in [20] (Re \( c > |\text{Re } \nu|) \)

\[ \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds 2^{s-2} \Gamma\left(\frac{s + \nu}{2}\right) \Gamma\left(\frac{s - \nu}{2}\right) (ax)^{-s} = K_\nu(ax) . \]

(26)

\(^2\)The tabulated expression for this contour integral presented in [20] has a misprint. Namely, the factor \( x^{1/2} \) is missing in it.

\(^3\)Note that \( \Gamma(s + n + 1) \) grows as \( \exp(A\Im s) \) with \( A = \pi/2 \) when \( \Im s \to -\infty \).
A.4. \( f(x) = 1/(x^2 + a^2)^{n+3/2}, a > 0, \) integer \( n \geq 0. \) Then we get the integral

\[
A_4(q) = \int_0^\infty \frac{x}{(x^2 + a^2)^{n+3/2}} J_0(qx) \, dx . \tag{27}
\]

The derivatives of \( g(z) \) are:

\[
\bar{g}'^{(m)}(0) = a^{-2m-2n-3}\Gamma(m+n+3/2)/\Gamma(n+3/2). \tag{31}
\]

We obtain from (6) \((-1 < \alpha < 0)\)

\[
A_4(q) = 1/\Gamma(n+3/2) (q^2/a^2)^{n+1/2} K_{n+1/2}(qa) , \tag{29}
\]

By using formula (26), after change of variable \( s + 1 = (s'/2 - n - 1/2)/2, \)

we find that

\[
A_4(q) = 1/\Gamma(n+3/2) (q^2/a^2)^{n+1/2} K_{n+1/2}(qa) , \tag{29}
\]

see eq. 2.12.4.28. in [19].

A.5. \( f(x) = J_0(ax)/(x^2 + c^2), \) where \( q > a > 0, c > 0. \) Then we come to the integral

\[
A_5(q) = \int_0^\infty \frac{x}{x^2 + c^2} J_0(ax) J_0(qx) \, dx . \tag{30}
\]

The derivatives of \( g(z) \) are the following:

\[
\bar{g}'^{(m)}(0) = c^{-2m-2}\Gamma(m+1) \sum_{k=0}^m \frac{1}{(k!)^2} \left( \frac{ac}{2} \right)^{2k} - \frac{1}{(m+1)\Gamma(m+2)} {}_2F_1 \left( 1; m+2, m+2; \frac{a^2c^2}{4}; \frac{a}{2} \right)^{2m+2} . \tag{31}
\]

Here and in what follows \( _pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) \) is the generalized hypergeometric series [18]. Then we obtain from (6), (31)

\[
A_5(q) = A_5^{(1)}(q) + A_5^{(2)}(q) , \tag{32}
\]

where (see eq. 7.3(17) in [20])

\[
A_5^{(1)}(q) = \frac{I_0(ac)}{q^2c^2} \frac{1}{\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} ds \Gamma^2(s+1) \left( \frac{q^2c^2}{4} \right)^{-s} = I_0(ac)K_0(qc) , \tag{33}
\]

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and

$$A_5^{(2)}(q) = -\frac{1}{\pi i q^2} \int_{\alpha - i\infty}^{\alpha + i\infty} ds \frac{1}{(s + 1)^2} F_2 \left( 1; s + 2, s + 2; \frac{a^2 c^2}{4} \right) \left( \frac{q^2}{a^2} \right)^{-s},$$

(34)

with $-1 < \alpha < 0$. The integrand in (34) is a regular function in the half-plane $\text{Res} > -1$. Moreover, since $q > a$, it decreases very rapidly as $|s| \to \infty$ in the half-plane $\text{Res} > -1$. According to the Cauchy integral theorem, $A_5^{(2)}(q) = 0$.

As a result,

$$A_5(q) = I_0(ac) K_0(qc),$$

(35)

see also eq. 2.12.32.11. in [19].

Our formula [6] enables one to calculate new improper parametric integrals of Bessel functions which are not yet presented in the most complete tables of integrals [19], [21].

**A.6.** As an example, let us consider the following integral ($a, c > 0$)

$$A_6(q) = \int_0^\infty \frac{x}{x^2 + c^2} e^{-a^2 x^2} J_0(qx) \, dx,$$

(36)

i.e. zero-order Hankel transform with the function $f(x) = e^{-a^2 x^2}/(x^2 + c^2)$. We find that

$$\bar{g}^{(m)}(0) = e^{-2m-2} \Gamma(m + 1) \sum_{k=0}^{m} \frac{(ac)^{2k}}{k!}.$$

(37)

Using relation

$$\sum_{k=0}^{m} \frac{(ac)^{2k}}{k!} = e^{(ac)^2} - (ac)^2 \sum_{p=0}^{\infty} \frac{(ac)^{2p}}{\Gamma(p + m + 2)},$$

(38)

we obtain

$$A_6(q) = A_6^{(1)}(q) + A_6^{(2)}(q),$$

(39)

where (see eq. 33)

$$A_6^{(1)}(q) = \frac{e^{(ac)^2}}{\pi i (qc)^2} \int_{\alpha - i\infty}^{\alpha + i\infty} ds \Gamma^2(s + 1) \left( \frac{q^2 c^2}{4} \right)^{-s} = e^{(ac)^2} K_0(qc),$$

(40)
and
\[
A_6^{(2)}(q) = -\frac{a^2}{\pi i q^2} \sum_{p=0}^{\infty} (ac)^{2p} \int_{\alpha-i\infty}^{\alpha+i\infty} ds \frac{\Gamma^2(s+1)}{\Gamma(s+p+2)} \left( \frac{q^2}{4a^2} \right)^{-s},
\]
with \(-1 < \alpha < 0\). According to eqs. 7.3(43) from [20] and 5.6(6) from [18], we have
\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \frac{\Gamma^2(s+1)}{\Gamma(s+p+2)} \left( \frac{q^2}{4a^2} \right)^{-s} = \frac{q}{2a} e^{-q^2/8a^2} W_{p-1/2,0} \left( \frac{q^2}{4a^2} \right),
\]
where \(W_{\chi,\mu}(x)\) is the Whittaker function of the second kind [18]. We can apply the relation 6.9(5) from [18]:
\[
W_{\chi,\mu}(x) = e^{-x/2} x^{\chi/2} F_0 \left( \frac{1}{2} - \chi + \mu, \frac{1}{2} - \chi - \mu; -\frac{1}{x} \right).
\]
After that we find the analytic expression for (36)
\[
A_6(q) = e^{(ac)^2} K_0(qc) - \frac{2a^2 q}{4a^2} e^{-q^2/4a^2} \times \sum_{p=0}^{\infty} \left( \frac{2a^2 c}{q} \right)^{2p} F_0 \left( p + 1, p + 1; -\frac{4a^2}{q^2} \right).
\]
For \(a = 0\) it coincides with the known integral (see eq. 2.12.4.23 in [19])
\[
A_6(q) \bigg|_{a=0} = \int_0^{\infty} x J_0(qx) dx = K_0(qc).
\]
In the limit \(c \rightarrow 0\) the leading term of (36),
\[
A_6(q) \bigg|_{c \rightarrow 0} \simeq \int_0^{\infty} \frac{y}{y^2 + 1} J_0(qcy) dy = K_0(qc),
\]
is also reproduced by eq. (44).
Let us demonstrate that our formula (44) gives a correct result for \(q = 0\). To do this, we use the following equation [18]
\[
W_{\chi,\mu}(x) = e^{-x/2} x^{\mu+1/2} \Psi \left( \frac{1}{2} - \chi + \mu, 2\mu + 1; x \right),
\]
where $Ψ(a, c, x)$ is the confluent hypergeometric series of the second kind \[18\].

Then we get another representation for (36)

\[
A_6(q) = e^{(ac)^2} K_0(qc) - \frac{1}{2} e^{-q^2/4a^2} \sum_{p=0}^{\infty} (ac)^{2p} Ψ(p + 1, 1; \frac{q^2}{4a^2}) .
\] (48)

The function $Ψ(a, 1, x)$ has the following asymptotic behavior as $x \to 0$ \[22\]:

\[
Ψ(a, 1, x) = -\frac{1}{\Gamma(a)} [\ln x + ψ(1) + 2γ_E] + O(x \ln x) ,
\] (49)

where $ψ(x)$ is the psi function, and $γ_E$ is the Euler–Mascheroni constant. As a result, we get in the limit $q \to 0$

\[
-\frac{1}{2} \sum_{p=0}^{\infty} (ac)^{2p} Ψ(p + 1, 1; \frac{q^2}{4a^2})
\]

\[
= e^{(ac)^2} [\ln(q/2a) + γ_E] + \frac{1}{2} \sum_{p=0}^{\infty} (ac)^{2p} \frac{ψ(p + 1)}{p!} .
\] (50)

It is known that

\[
\sum_{p=0}^{\infty} \frac{ψ(p + 1)}{Γ(p + 1)} x^p = e^x [Γ(0, x) + \ln x] ,
\] (51)

where $Γ(a, x)$ is the incomplete gamma function \[17\]. Since

\[
K_0(x) = -γ_E - \ln(x/2) + O(x^2) ,
\] (52)

we obtain from eqs. (48), (50), (51):

\[
A_6(0) = \frac{1}{2} e^{(ac)^2} Γ(0, (ac)^2) ,
\] (53)

in full accordance with eq. 2.3.4.3 from \[23\]

\[
A_6(q)\bigg|_{q=0} = \int_{0}^{\infty} \frac{x}{x^2 + c^2} e^{-a^2x^2} dx = \frac{1}{2} e^{(ac)^2} Γ(0, (ac)^2) .
\] (54)

\[\text{In the similar formula 6.8(5) in } \[18\] \text{ the term } 2γ_E \text{ has the wrong sign.}\]
For large $a$ or $c$ we have to take $\bar{g}^{(m)}(0) = e^{-2a^2}$, then

$$A_6(q) \bigg|_{a \gg 1} \simeq \frac{1}{\pi i (qc)^2} \int_{\alpha-i\infty}^{\alpha+i\infty} ds \Gamma(s+1) \left( \frac{q^2}{4a^2} \right)^{-s} = \frac{1}{2(ac)^2} e^{-q^2/4a^2}, \quad (55)$$

where $-1 < \alpha < 0$. On the other hand,

$$A_6(q) \bigg|_{ac \gg 1} \simeq \frac{1}{(ac)^2} \int_0^\infty z e^{-x^2} J_0(qz/a) \, dz = \frac{1}{2(ac)^2} e^{-q^2/4a^2}. \quad (56)$$

Finally, we predict the following asymptotic behavior

$$A_6(q) \bigg|_{q \gg 1} \overset{\alpha s}{\sim} e^{(ac)^2} K_0(qc). \quad (57)$$

Let us underline that integrals $A_i(q) \ (i = 1, 2, \ldots 6)$ are exponentially decreasing functions at large $q$. It is in agreement with the suggestion made in [14] about the integral

$$I_f(q) = \int_0^\infty x e^{-x^2} \varphi(x^2) J_0(qx) \, dx. \quad (58)$$

It says that if functions $\varphi(z)$ are entire or meromorphic, $I_f(q) = O(e^{-\gamma q^2})$ and $I_f(q) = O(e^{-\delta q})$, respectively, as $q \to \infty$. See asymptotics of our integrals in the examples A.1-A.2 and A.3-A.6, correspondingly.

**Theorem 2.** The Hankel transform [5] with the function $f(x) = h(x^4)$ and $q > 0$ may be expressed in the form

$$A(q) = \frac{1}{q^2 \sqrt{\pi i}} \int_{\alpha-i\infty}^{\alpha+i\infty} ds \bar{h}(s) \frac{\Gamma(2s+1)}{\Gamma(1/2-s)} 2^{6s+1} q^{-4s}, \quad (59)$$

for $-1/2 < \alpha < 0$, provided that

1. $h(z)$ is a regular function and its Taylor series at $z = 0$ has the form

$$h(z) = \sum_{m=0}^\infty \frac{\bar{h}^{(m)}(0)}{m!} (-z)^m; \quad (60)$$
\( h(z) = O(z^{-d}), \) as \( z \to \infty, \) for \( d > 1/8; \)

(3) \( \bar{h}^{(s)}(0) \) is a regular (single-valued) function defined on a half-plane

\[
H(\delta) = \{ s \in \mathbb{C}: \text{Re} \, s \geq -\delta \}
\]

for some \( 1/8 < \delta < 1/2 \) and satisfies the growth condition

\[
|\bar{g}^{(s)}(0)| < Ce^{Pv+A|w|}
\]

for some \( A < \pi/2 \) and all \( s = v + iw \in H(\delta). \)

**Proof.** The proof proceeds as the proof of the Theorem 1.

Let us consider one example.

**A.7.** \( f(x) = 1/\sqrt{x^4 + a^4}, \ a > 0. \) Then we have the following integral

\[
A_7(q) = \int_0^\infty x J_0(qx) \sqrt{x^4 + a^4} \, dx.
\]

Since \( \bar{h}^{(m)}(0) = (\pi)^{-1/2}a^{-4m-2} \Gamma(m+1/2), \) we find from (59) that

\[
A_7(q) = \frac{1}{(qa)^2} \frac{1}{\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} ds' \frac{\Gamma(2s + 1)\Gamma(s + 1/2)}{\Gamma(1/2 - s)} (qa)^{-4s'}.
\]

(64)

Let us put \( s = s'/4 - 1/2, \) then \( 0 < \alpha' < 2 \)

\[
A_7(q) = \frac{1}{2\pi i} \int_{\alpha' - i\infty}^{\alpha' + i\infty} ds' \frac{\Gamma(s'/2)\Gamma(s'/4)}{\Gamma(1-s'/4)} \left( \frac{qa}{\sqrt{2}} \right)^{-s'}.
\]

(65)

Now we can apply formula 6.8(41) from ref. [20] to find that

\[
A_7(q) = J_0\left( \frac{qa}{\sqrt{2}} \right) K_0\left( \frac{qa}{\sqrt{2}} \right),
\]

in accordance with eq. 2.12.5.2. in [19]. Again, \( A_7(q) \) is an exponentially decreasing function at large \( q, \) up to damped oscillations of \( J_0(qa/\sqrt{2}). \)
3 Conclusions

With the use of Ramanujan’s Master Theorem, we have derived two formulas which define the Hankel transforms of order zero $H_0(q)$ (5) with the even functions $g(x^2)$, $h(x^4)$ in terms of the inverse Mellin transforms (6), (59), provided that functions $g(z)$, $h(z)$ and their derivatives satisfy special conditions. The representation obtained is valid not only for $q \to \infty$ but for all $q > q_0 > 0$, if these conditions are satisfied. The validity of our formulas is illustrated by a number of examples. It is shown that formula (6) can be useful in calculating as yet unknown parametric improper integrals of the Bessel function $J_0(x)$ (see integral (36) and eqs. (44), (48)).

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Appendix A

Let us calculate the derivatives of the function $(n = 1, 2, \ldots)$,

$$g(z) = e^{-a^2z}J_{2n}(c\sqrt{z}) ,$$

(A.1)

taken at $z = 0$. We start from the expression

$$g^{(m)}(z) = \sum_{p=0}^{m} \binom{m}{p} [J_{2n}(c\sqrt{z})]^{(p)} \left[ e^{-a^2z} \right]^{(m-p)} ,$$

(A.2)

where

$$[J_{2n}(c\sqrt{z})]^{(p)} \bigg|_{z=0} = \frac{(-1)^p p!}{(p-n)! (p+n)!} \left( \frac{c^2}{4} \right)^p .$$

(A.3)

As a result, we obtain that $g^{(m)}(0) = 0$ for $0 \leq m \leq n - 1$, while for $m \geq n$

$$g^{(m)}(0) = (-1)^m a^{2m} \sum_{p=n}^{m} \binom{m}{p} \frac{p!}{(p-n)! (p+n)!} \left( \frac{c^2}{4a^2} \right)^p$$

$$= (-1)^m \frac{m!}{(m+n)!} a^{2m} \left( -\frac{c^2}{4a^2} \right)^n L_{m-n} \left( -\frac{c^2}{4a^2} \right) ,$$

(A.4)

$^5$A particular value of $q_0$ depends on the form of $g(z)$ or $h(z)$, see examples A.1-A.7.
where

\[ L_\alpha^n(z) = \sum_{k=0}^{m} \binom{n + \alpha}{n - k} \frac{(-z)^k}{k!} \]  

(A.5)

is the generalized Laguerre polynomial, \( \alpha > -1 \) \[17\]. In particular, we find for \( n = 0 \)

\[ g^{(m)}(0) = (-1)^m a^{2m} L_m \left( -\frac{c^2}{4a^2} \right), \]  

(A.6)

where \( L_n(x) = L_0^n(x) \).

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