Abstract—A learning method is proposed for Koopman operator-based models with the goal of improving closed-loop control behavior. A neural network-based approach is used to discover a space of observables in which nonlinear dynamics is linearly embedded. While accurate state predictions can be expected with the use of such complex state-to-observable maps, undesirable side-effects may be introduced when the model is deployed in a closed-loop environment. This is because of modeling or residual error in the linear embedding process, which can manifest itself in a different manner compared to the state prediction. To this end, a technique is proposed to refine the originally trained model with the goal of improving the closed-loop behavior of the model while retaining the state-prediction accuracy obtained in the initial learning. Finally, a simple data sampling strategy is proposed to use inputs deterministically sampled from continuous functions, leading to additional improvements in the controller performance for nonlinear dynamical systems. Several numerical examples are provided to show the efficacy of the proposed method.

I. INTRODUCTION

The Koopman operator describes the evolution of a nonlinear dynamical system in terms of a possibly infinite-dimensional but linear operator in a lifted space of observables. Starting from applications in dimension reduction of high-dimensional nonlinear systems such as turbulent flows [1], [2], the Koopman operator has gained high popularity in recent years as a data-driven modeling approach for dynamical systems. Among a number of applications is that of data-driven control. On the basis of finite-dimensional approximation of the Koopman operator derived from Extended Dynamic Mode Decomposition (EDMD) [3], which yields a Linear Time-Invariant (LTI) data-driven model, several linear controllers have been applied such as Linear Quadratic Regulator (LQR) [4]–[6] and Model Predictive Control (MPC) [7]–[11]. While there are many frameworks and methods in literature to incorporate the Koopman operator into data-driven control, not much attention has been paid to how the modeling error behaves while implementing state-prediction or feedback control, which pertains to practically unavoidable discrepancy between theories and actual implementations.

It has been recognized that the convergence property of the EDMD algorithm [12] does not hold for non-autonomous systems [7]. This motivates the use of neural networks to learn the observable functions themselves along with the finite-dimensional approximation of the Koopman operator. Especially, while the model structure we employ in this paper is restrictive so that we can realize LTI systems in the embedded space, which allows the use of linear controller designs even if the dynamics is nonlinear, it is shown that obtaining high state-predictive accuracy is achievable if the dynamics is linear with respect to input and one has access to enough data and computational resources that afford high-dimensional and complex feature maps.

On the other hand, we show that the modeling error of the Koopman models interacts with the closed-loop system in a different way from the state-prediction and we exemplify that the controller performance can greatly suffer from the modeling error, on which the complexity and dimension of observables have a large influence. To improve the possibly undesirable closed-loop behavior induced by Koopman-based control models, a control-consistent method is proposed, in which the model is refined after the initial training with additional use of data points sampled from closed-loop dynamics. This modification of the model aims to directly reduce the impact of the modeling error on the controller performance. Moreover, with the same intent as the control-consistent learning method, we also present a simple yet effective data sampling strategy that only uses inputs deterministically sampled from continuous functions.

This paper is organized as follows. In Section II, the Koopman operator framework for non-autonomous systems is presented. In Section III, we discuss the manifestation of the modeling error on both prediction and control and propose a control-consistent learning method along with a data sampling strategy about inputs to improve the actual closed-loop behavior. Finally, several dynamical systems are tested to show the effectiveness of the proposed method in Section IV.

II. KOOPMAN OPERATOR THEORY FOR NON-AUTONOMOUS SYSTEMS

A. Koopman Operator for General Non-Autonomous Systems

In this section, the Koopman operator is formally introduced for general non-autonomous systems, which forms the basis of observations made in Section III and the proposed method. Consider a dynamical system:

\[ \dot{x}(t) = f(x(t), u(t)), \]

where \( x(t) \in \mathcal{X} \subseteq \mathbb{R}^n \), \( u(t) \in \mathcal{U} \subseteq \mathbb{R}^p \), and \( f : \mathcal{X} \times \mathcal{U} \rightarrow \mathbb{R}^n \) are the state, the input, and the possibly nonlinear mapping describing dynamics of the system, respectively. Throughout the paper, we assume the solution \( x(t) \) to (1) to be continuous with respect to \( t \). With a first-order time discretization, (1) yields the following difference equation:

\[ x_{k+1} = F(x_k, u_k), \]

where \( x_k := x(k\Delta t) \), \( u_k := u(k\Delta t) \), and \( \Delta t \) denotes the sampling period. On the assumption \( \Delta t \ll 1 \), we consider (2) as the discrete-time system whose dynamics is equivalent to that of (1). It is assumed that \( f \) (i.e., \( F \)) is unknown and its dynamics is modeled in a data-driven manner.

In the Koopman operator formalism, the dynamics is characterized through functions called observables, which are mappings from the state-space into \( \mathbb{R} \). While the Koopman operator was first introduced in the context of autonomous systems, there have been also several efforts extending it to...
non-autonomous systems with control inputs [7], [13], [14]. In a formal extension [7], the state-space is extended to the augmented space $X \times l(U)$, where

$$l(U) := \{ U := (u_1, u_2, \cdots) \mid u_i \in U, \forall i \in \mathbb{N} \},$$

is the space of sequences of inputs, and the observables $g$ are of the form:

$$g : X \times l(U) \to \mathbb{R} : (x, U) \mapsto g(x, U).$$

In practice, the observables $g$ may be considered as the feature maps that are either specified by users or learned from data in the modeling procedure.

The Koopman operator corresponding to the non-autonomous system (2) is defined as an infinite-dimensional linear operator $K : F \to F$ (F: space of functions $g$) s.t.

$$Kg = g \circ \tilde{F} \iff (Kg)(x_k, U) = g(\tilde{F}(x_k, U)),$$

where the mapping $\tilde{F} : X \times l(U) \to X \times l(U)$ is defined by

$$\tilde{F}(x_k, U) := (F(x_k, u_k), SU) = (x_{k+1}, SU),$$

with the notation $U = (u_k, u_{k+1}, \cdots)$, and $S$ denotes the shift operator s.t.

$$SU = S(u_k, u_{k+1}, \cdots) := (u_{k+1}, u_{k+2}, \cdots).$$

It is easily inferred from (5) that $K$ is a linear operator. Note that since (2) only specifies the evolution of $x_k$, it is required to introduce the sequence $U = (u_k, u_{k+1}, \cdots)$ of inputs, which can be also interpreted as an input signal (i.e., a function) $U : \mathbb{Z}_{\geq 0} \to U$, to formally define the Koopman operator $K$. The equation (5) along with the definition (6) can be viewed as the evolution of the dynamics (2) through the observable $g$.

B. Finite Dimensional Approximation of the Koopman Operator and Data-Driven Koopman Models

To apply the Koopman operator formalism to dynamical systems modeling, a finite-dimensional approximation $K$ of the Koopman operator $K$ is introduced as follows.

**Proposition 1:** Given observables $g_i \in F$ \((i = 1, \cdots, D)\), let $g$ be an arbitrary element of $\text{span}(g_1, \cdots, g_D)$. Then, $Kg \in \text{span}(g_1, \cdots, g_D)$, i.e., $\text{span}(g_1, \cdots, g_D)$ is an invariant subspace under the action of the Koopman operator $K : F \to F$, if and only if there exists $K \in \mathbb{R}^{D \times D}$ s.t.

$$[K_{g_1} \cdots K_{g_D}]^T = K[g_1 \cdots g_D]^T.$$  \hspace{1cm} (8)

**Proof:** See Appendix A.

From an engineering perspective, it is of great interest to introduce the observables such that they allow practical models for control applications. One major choice of $g_i$ for the Koopman control problem takes the following structure of observables [10], [11], [15]:

$$[g_1(x_k, U) \cdots g_D(x_k, U)]^T = [x_k^T \hat{g}(x_k)^T u_k]^T,$$  \hspace{1cm} (9)

where $D = n + N + p$ and $\hat{g}(x_k) \in \mathbb{R}^N$ represents a vector-valued function from $X$ into $\mathbb{R}^N$ for some $N \in \mathbb{N}$. Note that only the first element $u_k$ in the sequence $U = (u_k, u_{k+1}, \cdots)$ appears in the definition (9), which leads to a practical form of data-driven models consistent with many linear controller designs such as LQR and MPC. On the assumption that we have access to $x_k$ and $u_k$ as data, we consider the following finite-dimensional approximation $K_c \in \mathbb{R}^{(n+N+p) \times (n+N+p)}$ of the Koopman operator $K$:

$$\begin{bmatrix} x_{k+1} \\ \hat{g}(x_{k+1}) \\ u_{k+1} \end{bmatrix} \approx \begin{bmatrix} A & B \\ \ast & \hat{g}(x_k) \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix} : = K_c,$$  \hspace{1cm} (10)

where matrices $A \in \mathbb{R}^{(n+N) \times (n+N)}$ and $B \in \mathbb{R}^{(n+N) \times p}$ are to be learned along with the feature maps $\hat{g}$. Note that (10) is approximate since $\text{span}(g_1, \cdots, g_D)$ defined by (9) may not be invariant under the action of $K_c$.

Noticing that the first $n + N$ rows of (10) are enough to specify the evolution of the state $x_k$ s.t.

$$\begin{bmatrix} x_{k+1} \\ \hat{g}(x_{k+1}) \end{bmatrix} \approx A \begin{bmatrix} x_k \\ \hat{g}(x_k) \end{bmatrix} + Bu_k,$$  \hspace{1cm} (11)

we are only interested in learning (11) and the last $p$ rows of $K_c$ in (10) are ignored in the proceeding formulations. They (approximately) represent $u_{k+1}$ given $x_k$ and $u_k$, which corresponds to the fact that the Koopman operator shifts the sequence of inputs according to (7). From (11), the modeling error is defined as:

$$r(x, u) := \begin{bmatrix} F(x, u) \\ \hat{g}(F(x, u)) \end{bmatrix} - \begin{bmatrix} A & B \\ \ast & \hat{g}(x) \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + Bu,$$  \hspace{1cm} (12)

and its norm:

$$\|r\|_{L_2} = \sqrt{\int_{X \times U} \|r(x, u)\|^2 dx du},$$  \hspace{1cm} (13)

may be used as a characteristic to evaluate the model, e.g., (11) is exact almost everywhere if (13) is well-defined and $\|r\|_{L_2} = 0$.

The model (11) is an LTI system in the new coordinates $[x_k^T \hat{g}(x_k)^T]^T$ and linear controller designs can be applied to control (2). In this paper, the following feedback controller with a static gain $K \in \mathbb{R}^{p \times (n+N)}$ is considered:

$$u_k = K(x_k^T \hat{g}(x_k)^T)^T.$$  \hspace{1cm} (14)

III. CONTROL-CONSISTENT LEARNING OF KOOPMAN EMBEDDING

A. Motivating Example

We consider the following one-dimensional system as a guiding example to motivate the proposed control-consistent learning.

$$x_{k+1} = x_k^2 e^{-x_k} + u_k,$$  \hspace{1cm} (15)

Suppose that we create the Model 1 defined as:

$$\begin{bmatrix} x_{k+1} \\ x_k^2 e^{-x_k} \end{bmatrix} \approx \begin{bmatrix} x_k \\ x_k^2 e^{-x_k} \end{bmatrix} + Bu_k, (\hat{g}(x_k) = x_k^2 e^{-x_k}).$$  \hspace{1cm} (16)

From Proposition 2 in Section III-D, perfect state prediction with no error is possible with $A$ and $B$ given as the following forms:

$$A = \begin{bmatrix} 1 & 0 \\ \alpha_1 & \alpha_2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ \alpha_3 \end{bmatrix}, \alpha_i \in \mathbb{R}.$$  \hspace{1cm} (17)

The modeling error (12) is then represented by

$$r(x, u_k) = \begin{bmatrix} 0 \\ (x_k^2 e^{-x_k} + u_k)^2 \exp(-x_k^2 e^{-x_k} - u_k) \\ -\alpha_1 x_k - \alpha_2 x_k^2 e^{-x_k} - \alpha_3 u_k \end{bmatrix}.$$  \hspace{1cm} (18)
Several methods have been proposed to learn the model (11), among of which the most straightforward one is to first specify the feature maps $\tilde{g}$ and infer $A$ and $B$ from data. This is reduced to a linear regression problem and a unique solution can be obtained in the same manner as the Dynamic Mode Decomposition (MDM) [1], [2]. This learning algorithm with nonlinear observables is called the Extended Dynamic Mode Decomposition (EDMD) [3], based on which many data-driven Koopman controller designs are developed [5]–[11], [15]–[17]. An important feature of the EDMD algorithm is its convergence property. Letting $\mathcal{F} = L_2(\mathcal{X} \times l(\mathcal{U}))$, the approximation obtained by EDMD is, under several assumptions, shown to converge to the true Koopman operator in the strong operator topology as the number $M$ of data points and the number $D$ of observables $g_i$ tend to infinity. Specifically, this convergence property is stated as follows: for $\forall g \in L_2(\mathcal{X} \times l(\mathcal{U}))$,

$$
\lim_{D \to \infty} \int_{\mathcal{X} \times l(\mathcal{U})} |(P_{\mu}^D K)P_{\mu}^D g - K g|^2 d\mu = 0,
$$

(22)

where $P_{\mu}^D$ and $\mu$ are the $L_2$ projection onto $\operatorname{span}(g_1, \ldots, g_D)$ and a measure with which $L_2(\mathcal{X} \times l(\mathcal{U}))$ is endowed, respectively. The operator $P_{\mu}^D K$ in (22) is related to the finite dimensional approximation $\tilde{K}$ computed by EDMD in the following manner:

$$
\lim_{M \to \infty} \| a^T [g_1 \cdots g_D] - P_{\mu}^D K g \| = 0,
$$

(23)

where $\| \cdot \|$ is an arbitrary norm on $\operatorname{span}(g_1, \ldots, g_D)$. The convergence property (22) implies that data-driven Koopman models can provide reasonable approximation with sufficiently large $M$ and $D$. However, it is not the case for non-autonomous systems as seen in the following remark.

Remark 1: Given feature maps $\tilde{g}(x_k)$, the approximation $K_e$ in (10) does not possess the the convergence property (22) since for (22) to hold, it is necessary that $(g_1)(1) = 0$ is an orthonormal basis of $L_2(\mathcal{X} \times l(\mathcal{U}))$ as $D \to \infty$ [12]. As described in [7], no elements of $U = \{ u_k, u_{k+1}, \ldots \}$ except for the first one $u_k$ depend on the definition (9) and it is obvious that they cannot form any basis of $L_2(\mathcal{X} \times l(\mathcal{U}))$.

No convergence property of the model (10) or (11) implies $\|r\|_{L_2}$ may not be negligibly small even if $D$ and $M$ are sufficiently large. To this end, we adopt another learning formulation, where the nonlinear feature maps $\tilde{g}$ are also learned from data along with matrices $A$ and $B$.

C. Initial Training: Simultaneous Learning of the Feature Maps and the System Matrices

Another class of methods to learn model (11) estimates the system matrices $A$, $B$, and the feature maps $\tilde{g}$ simultaneously. The resulting models are expected to achieve better predictive accuracy than those of the linear formulations such as EDMD since they can have greater model expressivity with the feature maps $\tilde{g}$ also learned from data along with the matrices $A$ and $B$. Especially, the use of neural networks has been shown to be promising to incorporate into the Koopman operator-based modeling, analyses, and control [19]–[23].

Hence, the proposed method characterizes the feature maps $\tilde{g}$ in (11) as a neural network aiming at high predictive

Suppose we also have the Model 2, which has richer features:

$$
\begin{bmatrix}
  x_{k+1} \\
  x_{k+1}^2 + 1
\end{bmatrix}
\approx
\begin{bmatrix}
  x_k \\
  x_k^2 + e^{-2x_k}
\end{bmatrix}
+ B u_k, \quad \tilde{g}(x_k) = \begin{bmatrix} x_k^2 e^{-x_k} \\ x_k^2 \end{bmatrix}.
$$

(19)

Perfect state prediction can be also achieved with

$$
A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \beta_5 & \beta_6 & \beta_7 & \beta_8 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ \beta_7 \\ \beta_8 \end{bmatrix}, \quad \beta_i \in \mathbb{R}.
$$

(20)

In this case, however, the modeling error $r(x_k, u_k)$ takes a different form:

$$
r(x_k, u_k) =
\begin{bmatrix}
  0 \\
  \frac{(x_k^2 e^{-x_k} + u_k)^2 \exp(-x_k^2 e^{-x_k} - u_k)}{-\beta_1 x_k - \beta_2 x_k^2 e^{-x_k} - \beta_3 x_k^2 - \beta_7 u_k} \\
  \frac{x_k^2 e^{-x_k} + u_k}{-\beta_5 x_k - \beta_6 x_k^2 e^{-x_k} - \beta_7 x_k^2 - \beta_8 u_k}
\end{bmatrix}.
$$

(21)

Figures 1a and 1b show the heat maps of $\|r(x_k, u_k)\|_2$, where the model parameters $\alpha_i$ and $\beta_i$ are obtained by EDMD [7]. Model 2 in (19) is highly erroneous for $x_k < 0$ compared to the Model 1 in (16) and the modeling error accumulates according to (38) in Section III-D leading to undesirable closed-loop behavior. Figure 1c shows the controller performance of both models, where $K$ is computed as an LQR gain with the cost function $\sum_{k=0}^{\infty} x_k^2 + u_k^2$. Despite the fact that both models achieve precisely zero state-prediction error, the controller designed for Model 2 causes an undesirable oscillation even as the open-loop dynamics quickly converges to the origin after $k = 1$. Herein, it is also emphasized that from the state-prediction point of view, complex and large models such as the Model 2 may be preferable in general since it is more likely to achieve better state-prediction by Corollary 1 in Section III-D. To deal with this degradation of the controller performance, which is not revealed from the state-prediction accuracy, a control-consistent learning approach is proposed along with a simple data sampling strategy.
implemented as follows:

\[
\{ g, A, B \} = \text{argmin} \ J(g, A, B),
\]

where \( J(g, A, B) := \lambda_1 \| AG_x + BU - G_y \|_F^2 + \lambda_2 \| (AG_x + BU) - Y \|_F^2, \)

\[
G_x := \{ g(x_1) \cdots g(x_M) \}, \quad U := [u_1 \cdots u_M], \\
G_y := \{ g(y_1) \cdots g(y_M) \}, \quad Y := [y_1 \cdots y_M], \\
y_k = F(x_k, u_k), \quad k = 1, \ldots, M, \\
W := [I \ 0], \\
g: \mathcal{X} \to \mathbb{R}^{n+P}: x_k \mapsto \begin{bmatrix} x_k \\ \hat{g}(x_k) \end{bmatrix},
\]

\[\hat{g}(x_k) = \text{NN}(x_k; w) \text{ (a neural network)}.\] (28)

accuracy and solves a nonlinear regression problem to learn \(A, B,\) and \(\hat{g},\) which is formulated as Step 1.

The loss function \(J(g, A, B)\) in (25) consists of two terms. The first one multiplied by a hyperparameter \(\lambda_1\) accounts for (approximately) minimizing \(\|r\|_{L_2}\) in (13). The other one with a hyperparameter \(\lambda_2\) intends to directly minimize the state-reconstruction error by applying the decoder \(W\) to the model prediction in order to compare it with the state \(y_k.\) While the decoder may be also characterized by a neural network in general, the specific structure (9) of observables, which explicitly includes the state \(x_k,\) allows an analytical expression of the decoder \(W\) in (26).

The nonlinear feature maps \(\hat{g}\) are defined as a fully-connected feed-forward neural network:

\[
\text{NN}(x_k; w) := \sigma(\Theta_0 \sigma(\cdots \Theta_2 \sigma(\Theta_1 x_k + b_1) + b_2) + b_3),
\]

\[w := \{ \Theta_i, b_i \}_{i=1}^n,\] (29)

where \(\Theta_i, b_i,\) and \(\sigma\) are a kernel, a bias, and an activation function, respectively. In this paper, we only consider continuous activation functions so that (29) is continuous.

D. Second Training: Modification of the Initial Model

1) State-Prediction Accuracy:

Characterizing observables by a neural network allows greater expressivity, which can lead to higher accuracy of the data-driven model if the optimization problem is feasible. On the other hand, from the controller design perspective, including high-order nonlinearities in the observables is not preferable since it may introduce unexpected or undesirable effect on the closed-loop system due to the modeling error, which could even alter the actual closed-loop system unstable, while it is easier for the state prediction to eliminate the modeling error. Specifically, the state prediction is implemented as follows:

\[
x_{k+1}^{\text{est}} = [I \ 0] \begin{bmatrix} x_k^{\text{est}} \\ \hat{g}(x_k) \end{bmatrix} + B u_k, \quad k = 0, 1, \ldots,
\]

(31)

where \(x_k^{\text{est}}\) denotes the state prediction at time \(k\) and \(x_0^{\text{est}} = x_0\) is given. From (31), the modeling error in the state prediction is evaluated as:

\[
x_{k+1} = [I \ 0] \begin{bmatrix} x_k \\ \hat{g}(x_k) \end{bmatrix} + B u_k + [I \ 0] r(x_k, u_k).
\]

(32)

Proposition 2: Let \(x_k\) and \(u_k\) be arbitrary. There exist \(\hat{g}, A_1 \in \mathbb{R}^{n \times n}, A_2 \in \mathbb{R}^{n \times N}\) and \(B_1 \in \mathbb{R}^{n \times p}\) s.t.

\[
F(x_k, u_k) = A_1 x_k + A_2 \hat{g}(x_k) + B_1 u_k,
\]

if and only if

\[
[I \ 0] r(x_k, u_k) = 0,
\]

i.e., (31) has no state prediction error.

Proof: Let \([A_1 A_2] \in \mathbb{R}^{n \times (n+N)}\) and \(B_1 \in \mathbb{R}^{n \times p}\) be the first \(n\) rows of \(A\) and \(B\) in (32), respectively. From (2), the equation (32) reads

\[
F(x_k, u_k) = A_1 x_k + A_2 \hat{g}(x_k) + B_1 u_k + [I \ 0] r(x_k, u_k),
\]

which implies the statement of the proposition.

Note that there exist \(\hat{g}, A_1, A_2,\) and \(B_1\) that satisfy (33) with \(A_2 = 0\) or \(\hat{g}(x_k) \equiv 0\) if and only if the original dynamics (2) is linear.

Corollary 1: If (2) is of the linear form w.r.t. input:

\[
F(x_k, u_k) = F(x_k) + B u_k \quad (B \in \mathbb{R}^{n \times p}),
\]

including an enough number of different features to reconstruct \(F\) is necessary and sufficient for (31) to be able to achieve zero state-prediction error. In case of control-affine dynamics s.t. \(B = \hat{B}(x_k)\) is dependent on \(x_k,\) the same argument holds if the state-prediction is implemented with \(u_k \equiv 0.\)

Although the specific model structure we adopt in (11) limits the validity of state-prediction to certain classes of nonlinear dynamics as seen in Corollary 1, it is considered as a preferable property from the controller design perspective since it allows to utilize linear systems theories in the new coordinates \([x_k^T \ \hat{g}(x_k)^T]^T\) to control possibly nonlinear dynamics. It should be emphasized that the choice of model structure is a trade-off relation between the simple and practical controller designs and applicability of the method to actual nonlinear systems, i.e., such linear controller designs in the embedded space will result in poor control performance if the model has a large modeling error.

2) Accuracy in Terms of Closed-Loop Dynamics:

While the accuracy of the state prediction of the model (11) may be evaluated by Proposition 2, its accuracy in terms of controller design is characterized in a different way. Note that the system to be controlled by the feedback controller \(K\) in (14) is assumed to be a linear time-invariant system:

\[
\xi_{k+1} = A \xi_k + B u_k, \quad \xi_k \in \mathbb{R}^{n+N},
\]

(36)

and we can only ensure properties of the closed-loop system:

\[
\xi_{k+1} = (A + B K) \xi_k = (A + B K)^{k+1} \xi_0.
\]

(37)

Clearly, (36) is identical to the Koopman control model (11) if \([r]_2 = 0\) and \(\xi_0 = [x_0^T \ \hat{g}(x_0)^T]^T.\) However, in general cases where \(r(x_k, u_k) \neq 0,\) the modeling error may persist at any time and accumulate as follows:
if there exists a finite-dimensional model that can embed the original dynamics linearly in the subspace spanned by observables. In this regard, in addition to the modeling error defined in (12), one can also consider evaluating to what extent the subspace in which the model coordinates are defined is close to being invariant to measure the accuracy of modeling. Specifically, given some finite-dimensional subspace $\mathcal{F}_{\text{sub}} \subset \mathcal{F}$, it is obvious by Proposition 1 that if $\mathcal{F}_{\text{sub}}$ is shown to be invariant under the action of $K$, the realization of an exact model without any modeling error is possible with an arbitrary choice of observables from $\mathcal{F}_{\text{sub}}$. Also, finding such a model should be relatively easy since it is reduced to a linear regression problem after we choose observables from $\mathcal{F}_{\text{sub}}$.

Therefore, the invariance property of the chosen subspace may be considered as a more general characteristic to evaluate the modeling procedure itself than the modeling error $r(x,u)$ defined in (12). In fact, if we change the observables of the Model 2 in the example of Section III-A from $[x_k \ x_k^2 e^{-x_k} \ x_k^2]^T$ to $[\alpha x_k \ \alpha^2 x_k e^{-x_k} \ \alpha x_k]^T$ for some $\alpha \in \mathbb{R}$, norms of the modeling error $r(x,u)$ result in different values as shown in Fig. 2 despite the fact that both models should be considered as essentially the same ones since the choices of observables are the same up to multiplicity by $\alpha$. This result may be also inferred from the following error bounds analysis.

**Proposition 3:** Let $g$ be observables defined in (27) and suppose $\mu_x$ and $\mu_u$ are positive measures with compact supports s.t. $(\mathcal{X}, \mathcal{A}_x, \mu_x)$ and $(\mathcal{U}, \mathcal{A}_u, \mu_u)$ are measure spaces with any appropriate $\sigma$-algebras $\mathcal{A}_x$ and $\mathcal{A}_u$, respectively. Also, let $\mu := \mu_x \mu_u$ be their product measure. If $g$ is measurable and continuous, the following holds:

$$\tag{43} \|r\|_{L^2}^2 \leq \|g \circ F\|_{L^2}^2 + \|\mu A B\| \left\{ \|\mu A\| \|\mu h\|_{L^2} + 2\|\mu A B\| \right\} \{\|\mu h\|_{L^2} \},$$

where $h : \mathcal{X} \times \mathcal{U} \rightarrow \mathbb{R}^{n+p}$ is a vector-valued function s.t.

$$\tag{44} h(x,u) := \begin{bmatrix} g(x) \\ u \end{bmatrix},$$

and the function $\|(g \circ F)(\cdot)\|_{L^2} : \mathbb{R}^{n+p} \rightarrow \mathbb{R}_{\geq 0} : (x,u) \mapsto \|(g \circ F)(x,u)\|_{L^2}$ denotes an $l_2$ norm on $\mathbb{R}^{n+p}$, so does $\|h(\cdot)\|_{L^2}$.

**Proof:** See Appendix B.

Note that the measures $\mu_x$ and $\mu_u$ in Proposition 3 correspond to probability distributions from which we sample data points $x_k$ and $u_k$, respectively.

**Remark 2:** Given a model of the form (11), if we construct another model with the same observables but multiplied by some $\alpha \neq 0$ s.t. $[\alpha x_k \ \alpha^2 x_k e^{-x_k} \ \alpha x_k]^T$, error bounds given by Proposition 3 can be different for these essentially same models. For instance, by replacing $A$ in (11) with $(1/\alpha)A$, both models have the exact same dynamics, while the corresponding error bounds may be different, e.g., $\|g \circ F\|_{L^2}$ and $\|A B\|$ in (43) become $|\alpha|\|g \circ F\|_{L^2}$ and $\|((1/\alpha)A) B\|$, etc.

Thus, it should be emphasized that the modeling error $r(x,u)$ we adopt in this paper is specific to individual learning results and error analyses may not be robust to unessential variations of models. However, it is useful to incorporate into the


In this paper, we propose to only use deterministic \( u_k \) sampled from continuous functions. First, the solution \( x(t) \) to \( (1) \) is assumed to be continuous and \( \tilde{g}(x) \) defined by \( (29) \) is also continuous, the control inputs \( (14) \) are always discretized points sampled from continuous functions. Hence, for the same reason as that of the modification of the initial model in Section III-D, we only use \( u_k \) sampled from continuous functions so that Steps 1 and 2 can minimize losses over possible regime of dynamics realized by the controller \( (14) \) only.

IV. Numerical Examples

In this section, state-prediction is only implemented with \( u_k \equiv 0 \) for simplicity. Also, the control objective is defined as stabilizing the system at the origin while minimizing the cost and LQR is used for the controller design.

A. Simple Pendulum

The simple pendulum is considered as the first example:

\[
\ddot{\theta} = -\sin \theta + u, \quad (x_1 := \theta, x_2 := \dot{\theta}).
\]

We collect 300 data sets generated by \( (46) \), each of which consists of a single trajectory of a length of 50 steps with the sampling period \( \Delta t = 0.1 \) starting from an initial condition \( x_0 \sim \text{Uniform}[-3,3]^2 \). The Runge-Kutta method is used to solve \( (46) \) with a step size of 0.01. We include one nonlinear feature \( \tilde{g}(x_k) \in \mathbb{R} \) \((N = 1)\) in the model, which is a neural network with a single hidden layer consisting of 10 neurons and the swish function is used as the activation. Step 1 is implemented in TensorFlow.

The following two types of \( u_k \) are considered to evaluate the efficacy of the sampling strategy in Section III-F:

\[
\begin{align*}
\begin{aligned}
&u_k \sim \text{Uniform}[-1,1], \quad \omega_i := 20i, \quad i = 0, 1, \cdots, 5, \quad (47)
\end{aligned}
\end{align*}
\]

In the proposed method that adopts \( (48) \), each single trajectory data set is split evenly into six groups \( \mathcal{D}_i \) and \( u_k \) with \( \omega_i \) is included in \( \mathcal{D}_i \), \( i = 0, 1, \cdots, 5 \).

Figure 3 shows the results of the models obtained by Step 1, where the state predictions (Figs. 3a and 3c) are implemented according to \( (31) \) and the control simulations (Figs. 3b and 3d) use LQR gains computed with the cost \( \sum_{k=0}^{\infty} 100x_{k1}^2 + x_{k2}^2 + u_k^2 \). Note that if the discretized dynamics \( F \) that is considered as equivalent to the original dynamics \( (46) \) can be obtained by the forward Euler discretization, \( F \) is linear w.r.t. input and high predictive accuracy is expected by Corollary 1 with the use of sufficiently rich feature maps. Whereas the state-prediction accuracy barely changes
where \( A(x_2, x_3, x_4) = mLx_4^2 \sin x_3 - \delta x_2, m = 1, M = 5, L = 2, g = -10, \) and \( \delta = 1. \) Step 1 is applied with the same conditions as the first example except for the number of hidden layers, which is changed to 25. Also, only the deterministic sampling (48) for \( u_k \) is considered in this example. For the controller design, the cost is defined as \( \sum_{k=0}^{\infty} 100x_1^2 + x_2^2 + 100x_3^2 + x_4^2 + u_k^2. \) The governing equation (49) can yield its equivalent discretized dynamics as the control-affine form with the forward Euler and Corollary 1 is applicable to the state-prediction with \( u_k = 0. \)

While the initially learned model has reasonable predictive accuracy as in Fig. 4a, the controller performance suffers from the undesirable modeling error effect, which is shown in Fig. 4e. Thus, Step 2 is implemented to modify the model, for which we additionally collect data points in the same way as Step 1. The TensorFlow Constrained Optimization module [25] is used to solve the optimization problem in Step 2 with \( \epsilon_A = \epsilon_B = 0.1. \) The LQR gain is also recomputed with the updated model parameters \( (A + \Delta A, B + \Delta B). \)

It is shown in Fig. 4f that the modified model achieves the control objective. As a more quantitative analysis, we estimate the basin of attraction of the closed-loop systems by testing various initial conditions, whose results are shown in Figs. 4g and 4h. Some initial conditions do not converge to the origin for the closed-loop system formed by the initial model, while it is shown that the given set of initial conditions is a basin of attraction for the one formed by the modified model. Moreover, the modified model retains good state-prediction accuracy thanks to the constraints on \( \Delta A \) and \( \Delta B \) and it is comparable to that of the initial model (Figs. 4a and 4b). Finally, the profiles of the state prediction errors are evaluated. Figures 4c and 4d show the heat maps of \( ||(1, 0)r(x, 0)||_2 \) with \( x_2 = x_4 = 0, \) which are overlaid with the data points used in Step 1. It is confirmed that both models have quite similar error profiles and the good state-prediction accuracy of the initial model is successfully preserved after the modification.

V. CONCLUSION

A control-consistent learning method is proposed along with a simple but effective data sampling strategy for Koopman operator-based control. The initial learning of the observables and the operator matrices is augmented by a second step to improve the controller performance. The use of deterministically-sampled input data from continuous functions improves the controller performance and the proposed two-stage learning contributes to improved closed-loop behavior by updating the operator matrices while retaining high state-prediction accuracy obtained by the initially learned model.

APPENDIX

A. Proof of Proposition 1

Suppose \( \text{span}(g_1, \cdots, g_D) \) is an invariant subspace under the action of the Koopman operator \( \mathcal{K}, \) which implies

\[
\mathcal{K}g = \sum_{j=1}^{D} a_j (\mathcal{K}g_j),
\]

is in \( \text{span}(g_1, \cdots, g_D) \) for \( \forall g \in \text{span}(g_1, \cdots, g_D), \) where \( g = \sum_{j=1}^{D} a_j g_j \) for some \( a_j \in \mathbb{R}. \) By taking \( g = g, \)
i = 1, \ldots, D, we have

\[ \exists k_{i1}, \ldots, k_{iD} \in \mathbb{R} \text{ s.t. } Kg_i = \sum_{j=1}^{D} k_{ij}g_j, \text{ for } i = 1, \ldots, D, \]

\[ \iff \exists K \in \mathbb{R}^{D \times D} \text{ s.t. } \begin{bmatrix} K g_1 \\ \vdots \\ K g_D \end{bmatrix} = K \begin{bmatrix} g_1 \\ \vdots \\ g_D \end{bmatrix}. \] (51)

Conversely, if there exists $K \in \mathbb{R}^{D \times D}$ satisfying (8), for all $g = \sum_{i=1}^{D} a_i g_i \in \text{span}(g_1, \ldots, g_D)$,

\[ Kg = \sum_{i=1}^{D} a_i(Kg_i) = \sum_{i=1}^{D} a_i \sum_{j=1}^{D} k_{ij}g_j \quad \text{(for some } k_{ij} \in \mathbb{R}) \]

\[ = \sum_{j=1}^{D} \left( \sum_{i=1}^{D} a_i k_{ij} \right) g_j \in \text{span}(g_1, \ldots, g_D). \] (52)

\[ \blacksquare \]

**B. Proof of Proposition 3**

Since $\mu_x$ and $\mu_u$ have compact supports and $g$ is both measurable and continuous, $g \in L_p(X, A_x, \mu_x)$ for all $1 \leq p < \infty$. Specifically, we have $g \in L_1(X, A_x, \mu_x)$ and $g \in L_2(X, A_x, \mu_x)$. The function $h$ is well-defined as $h \in L_2$. Indeed,

\[ \int_{X \times U} \left\| g(x) \frac{u}{\mu(x \, du)} \right\|_2^2 \mu(dx du) = \int_X \left\| g(x) \mu_x(dx) \right\|_2 \int_U \left\| \frac{u}{\mu_u(du)} \right\|_2^2 \mu_u(du) \]

\[ < \infty, \] (53)

where the last inequality holds since $a = \|g\|_2 < \infty$ and $b$, $c$, and $d$ are also finite since the supports of $\mu_x$ and $\mu_u$ are bounded. Using the triangle inequality, we have

\[ \|r(x, u)\|_2 = \left\| g(F(x, u)) - [A B] \begin{bmatrix} g(x) \\ u \end{bmatrix} \right\|_2 \]

\[ \leq \left\| g(F(x, u))\right\|_2 + \left\| [A B] \right\| \left\| \begin{bmatrix} g(x) \\ u \end{bmatrix} \right\|_2 \]

\[ \iff \|r(x, u)\|_2^2 \leq \left\| g(F(x, u))\right\|_2^2 + \left\| [A B]\right\| \left\| [A B]\right\| \left\| \begin{bmatrix} g(x) \\ u \end{bmatrix} \right\|_2^2 + 2\left\| g(F(x, u))\right\|_2 \left\| \begin{bmatrix} g(x) \\ u \end{bmatrix} \right\|_2. \] (54)

Define

\[ \hat{h} : X \times U \to \mathbb{R}^{n+p} : (x, u) \mapsto [A B] \begin{bmatrix} g(x) \\ u \end{bmatrix}. \] (55)

It is confirmed that $\hat{h} \in L_2$ as

\[ \int_{X \times U} \left\| [A B] \begin{bmatrix} g(x) \\ u \end{bmatrix} \right\|_2^2 \mu(dx du) = \int_{X \times U} \left\| \sum_{i=1}^{N} \sum_{l=1}^{N} a_{l,i}g_l(x) + \sum_{l=1}^{p} b_{l,1}u_l \right\|_2^2 \mu(dx du) \]

\[ = \sum_{i=1}^{N} \left\{ \sum_{l=1}^{N} a_{l,i}^2 \left\| g_l \right\|_{L_2}^2 \int_U \mu_u(du) \right\} + 2\sum_{i=1}^{N} \sum_{j=1}^{N} a_{i,j} \int_X g_j(x) \mu_x(dx) \int_U u_j \mu_u(du) \]

\[ + \sum_{i=1}^{p} b_{i,1}^2 \int_{X \times U} u_i \mu(dx du) \]

\[ + 2\sum_{i=1}^{p} \sum_{j=1}^{p} b_{i,1} b_{i,j} \int_{X \times U} u_i u_j \mu(dx du) \] (56)

\[ < \infty, \] (57)

where any integration in (56) is finite since the supports of $\mu_x$ and $\mu_u$ are bounded and $g \in L_1$, i.e., $g$ is integrable. Noticing that $r$ is a linear combination of $g \circ F \in L_2$ and $\hat{h} \in L_2$, we have $r \in L_2$ by the same argument as above. Also, $L_2$ norms:

\[ N_1(\cdot) := \| (g \circ F)(\cdot) \|_{L_2} : \mathbb{R}^{n+p} \to \mathbb{R}_{\geq 0} \]

\[ : (x, u) \mapsto \|(g \circ F)(x, u)\|_2, \] (58)

\[ N_2(\cdot) := \| h(\cdot) \|_{L_2} : \mathbb{R}^{n+p} \to \mathbb{R}_{\geq 0} : (x, u) \mapsto \| h(x, u)\|_2, \] (59)

belong to $L_2$ as

\[ \int_{X \times U} |N_1(x, u)|^2 \mu(dx du) = \int_{X \times U} \| (g \circ F)(x, u)\|_{L_2}^2 \mu(dx du) \]

\[ = \| g \circ F \|_{L_2}^2 \]

\[ < \infty, \] (60)

\[ \int_{X \times U} |N_2(x, u)|^2 \mu(dx du) = \int_{X \times U} \| h(x, u)\|_{L_2}^2 \mu(dx du) \]

\[ = \| h \|_{L_2}^2 \]

\[ < \infty, \] (61)

so that the following quantity is well-defined as the inner product of $L_2$:

\[ \int_{X \times U} \left\| g(F(x, u))\right\|_2 \left\| \begin{bmatrix} g(x) \\ u \end{bmatrix} \right\|_2 dx du = \left( \| (g \circ F)(\cdot)\|_{L_2}, \| h(\cdot)\|_{L_2} \right)_{L_2}. \] (62)
Therefore, (54) implies the following:

\[
||r||_{L_2}^2 = \int_{X \times U} ||r(x,u)||^2 dx du \\
\leq \int_{X \times U} ||g(F(x,u))||_{L_2}^2 dx du \\
+ \|[[A B]]\| \left\{ \|[[A B]]\| \int_{X \times U} \left\| \frac{g(x)}{u} \right\|_2^2 dx du \\
+ 2 \int_{X \times U} \left\| \frac{g(F(x,u))}{u} \right\|_2 \left\| \frac{g(x)}{u} \right\|_2 dx du \right\} \\
= ||g \circ F||_{L_2}^2 + ||[A B]]|| \left\{ ||[A B]]|| ||A||_{L_2}^2 \\
+ 2 \left( ||g \circ F||_{L_2}^2, ||\mathcal{H}||_{L_2} \right) \right\}.
\]

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