A Local Information Criterion for Dynamical Systems

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Abstract

Encoding a sequence of observations is an essential task with many applications. The encoding can become highly efficient when the observations are generated by a dynamical system. A dynamical system imposes regularities on the observations that can be leveraged to achieve a more efficient code. We propose a method to encode a given or learned dynamical system. Apart from its application for encoding a sequence of observations, we propose to use the compression achieved by this encoding as a criterion for model selection. Given a dataset, different learning algorithms result in different models. But not all learned models are equally good. We show that the proposed encoding approach can be used to choose the learned model which is closer to the true underlying dynamics. We provide experiments for both encoding and model selection, and theoretical results that shed light on why the approach works.

1 Introduction

Objects are of various complexities in nature. A round stone looks simpler than a convoluted rough piece of rock; a constant beep-like sound is simpler than an orchestra. We humans have internal ideas about how complex are objects. The complexity can also be defined for abstract objects such as mathematical creatures. The focus of this paper is on the complexity of dynamical systems that model the laws of nature \[1\]. To our eyes, a dynamical system is nothing more than a temporal sequence of observations. We might use the data sequence to infer a model. But what is the better representation of the dynamical system – the data or the model, and which model should we use? In this paper, we take a closer look at efficient encoding of dynamical systems and, based on that, propose a model selection criterion with practical use in empirical inference.

For illustration, assume the following scenario: Alice and Bob are friends and they are talking over the phone. Alice is watching a dynamical system and wants to share her experience with Bob. Alice knows what Bob knows about the nature, math, etc. She is watching a temporal sequence of observations \([x(1), x(2), \ldots]\) caused by an underlying mathematical expression \(\dot{x}(t) = f(x(t))\). Unfortunately, the transmission cord from Alice to Bob charges her for every voltage pulse. Therefore, Alice would like to transmit her experience to Bob with the least phone cost. Due to the physical constraints, Alice can observe samples from the model with sampling frequency \(f_s = 1/T_s\) where \(T_s\) is the time interval between two consecutive observations. Assume the phone call starts at time \(t_0\) and Alice can describe each of her observations with \(m\) bits. One trivial solution is that Alice talks constantly with Bob and tells him every observation at each time instant \(\{t_0, t_0 + T_s, t_0 + 2T_s, \ldots, t_0 + nT_s\}\) for an indeterminate amount of time. Despite its simplicity,
Model selection

Change the number of partitions

\[ \text{Compute total L-ICDS} \]

\[ \hat{f}(x) \]

\[ \text{Learning Machine} \]

\[ f(x) \]

\[ \text{Simple} \]

\[ \text{Complex} \]

\[ \text{Simple} \]

\[ \text{Complex} \]

\[ \text{Smaller} \]

\[ \text{Selected Model} \]

\[ \text{Smaller} \]

\[ \text{Selected Model} \]

Figure 1: Two cartoons for two ideas presented in this paper. (a) Encoding a time series by first learning a model from the observations (bottom) and then locally encoding the model by L-ICDS. (b) Using the L-ICDS score as an information criterion to perform model selection. Two different neural network architectures are trained from a time series and L-ICDS score is computed for each.

this approach will cost Alice a horrible amount \( mn \) that increases without bound as \( n \to \infty \). More cleverly, Alice can use her prior assumptions about nature and her belief that her observations are not totally random. Hence, she is able to infer the underlying dynamics by a nonparametric model \( \hat{f} \) from her observations \( \mathcal{S} \). Assume this model is chosen from a hypothesis set \( \mathcal{H} \) and both Alice and Bob agree on the members of \( \mathcal{H} \). Thus, Alice only needs to inform Bob about the initial state \( \hat{x}(0) = x(0) \) of the system and the model \( \hat{f} \) she has learned about the dynamics. Bob can reconstruct the sequence \( \hat{S} \approx S \) on his side by running \( \hat{x}(t) = f(\hat{x}(t)) \) starting from the initial state \( \hat{x}(0) \). Notice that the state dynamics may cover only a small subset of the state space, which removes the need to model \( f \) on its whole domain. We use this property of dynamical systems for compressing their information and obtaining an optimal local trade-off between model complexity and prediction accuracy.

The underlying questions of this example are highly relevant also for artificial intelligent systems. Imagine autonomous vehicles flying or driving in a formation [2], or multiple robots coordinating their actions [3]. These systems need to know of each other; that is, agents need to transmit dynamics information between each other. An intelligent agents, however, will use its resources wisely and thus communicate only when and what is necessary. In this scenario, better encoding of dynamical systems means reduced communication, lower bandwidth requirements, and thus reduced cost. Likewise, intelligent agents may store various internal models for the purpose of simulation, prediction, or control [4]. Better representations here may mean improved performance, reduced memory requirements, and faster computation.

Contribution — In this paper, we propose to encode dynamical systems through local representations, which are computed to yield (locally) an optimal trade-off between model complexity and predictive performance. The criterion automatically chooses the ‘right’ complexity – locally simple dynamics are represented by low-order models, while higher-order representations are automatically taken in areas with more complex dynamics. Because the representation thus adapts to the local information content of the dynamics, the proposed encoding scheme represents a novel information criterion for dynamical systems, which we call Local Information Criterion for Dynamical Systems (L-ICDS).

L-ICDS is motivated by compressing information through local representations. Since there are theories and empirical evidence in machine learning confirming the relation between generalization and compression [5][4], we hypothesized L-ICDS is also useful for model selection. Indeed, we show that the information criterion can be used (in addition to efficient encoding) to choose among different models learned from a given dataset. In particular, we show that it is possible to choose between different architectures of neural networks (NNs) and to compare different types of learned models (e.g., NNs versus GPs) solely with the aid of the compression score and not with test data. We extend our empirical findings with theoretical results, which confirm the correctness of our method for certain function classes and provide insight into why L-ICDS is a useful criterion for model selection. Fig. [1] illustrate the two proposed applications of L-ICDS: encoding and model selection.

Related Work — The subject of obtaining a representation of a dynamical system from its input-output data is known as system identification [7][8] or model learning [9]. Two major approaches in system identification are gray-box and black-box approaches [7][8]. Gray-box methods learn the parameters of a known model [10], where parameters typically have a physical interpretation.
However, black-box methods need to identify both the structure and parameters of the model. In black-box system identification, or machine learning in general, choosing the appropriate structure is usually done by investigating model performance on a left-out validation set. Information criterion is a different approach to model selection by taking into account model complexity and data explanation at the same time. Many information criteria have been proposed and used for supervised and unsupervised learning. Despite some recent work on the information criterion approach towards dynamical systems, the field is not explored well yet. This work is proposing a compression method for dynamical systems that can be used as an information criterion and for model selection as well.

Models of dynamical systems take very different representations. On the one end of the spectrum, there are classical parametric models such as linear transfer functions and state-space models, as well as nonlinear gray-box models with known structure (e.g., from first principles) and some free parameters. In these, the model structure is relatively rigid and information is encoded in a small number of parameters, often with some physical interpretation. Neural networks (NNs) can also learn model structure and encode information in a large number of weight parameters without direct physical interpretation. Fuzzy models such as Takagi-Sugeno encode dynamical systems as a set of fuzzy rules or sets and associated models. Nonparametric methods such as Gaussian process (GP) models and classical time- or frequency-domain methods represent dynamical system information essentially in a dataset (in time or frequency domain). Herein, we propose to encode dynamical systems in local models whose complexity is adapted to the data stream. Our encoding thus provides a middle ground between encoding in a dataset and a single (global) parametric model.

The benefits of local modeling approaches for dynamical systems have long been recognized. These include, in particular, the abilities to learn fast and incrementally from a continuous stream of (possibly large) data, while allowing for non-stationary distributions. This is critical in real-time learning such as robot control. While in these works the complexity of the local model must be chosen a priori (most often, locally linear models are used), our method allows for determining the optimal model complexity, which is adapted to the data.

The proposed encoding scheme for dynamical systems was first considered, but in a different context from the one herein. While in, the true dynamics are assumed known and L-ICDS is used to efficiently communicate state information between agents in a networked system, we consider encoding of dynamics models learned from data. Moreover, the proposal of L-ICDS for model selection (Sec. 4), and all theoretical (Theorems 1-3) and empirical results (Sec. 3 and 5) are novel contributions.

## 2 Proposed Local Encoding

In this section, we explain our proposed encoding scheme for dynamical systems

\[
\dot{x}(t) = f(x(t)), x(0) = x_0, t \in [0, T].
\]  

We present our idea based on the concepts of algorithmic complexity, Universal Turing Machines (UTM), and minimum message length. UTM is a programmable machine that receives a message as input and produces the desired output. Minimum length of the input message can be seen as the complexity of the output and is called algorithmic complexity (AC). The MML principle chooses a model for the observed data where the joint AC of the tuple (model, data) is minimized. The detailed prerequisite definitions are delegated to the supplementary material.

**General notion**—Our aim is to construct a brief and efficient explanation for the observed data \(x(t)\) from the model. The explanation is a message consisting of two parts. The first part encodes some general assertion (theory) \(\hat{\theta}\) about the source of data and the second part is the explanation for the data were the assertion is correct. Throughout this paper, we assume the data takes finite discrete values with certain precision. Hence, each data example can be encoded to a finite sequence of ‘0s’ and ‘1s’. This is a reasonable assumption because, in practice, values are usually stored in a quantized way on digital computers, and we shall consider a finite horizon hereafter.

**Alice’s encoding of a dynamical system**—Assume Alice is given a long sequence of observations \(S = [x(1), x(2), \ldots, x(N)]\) to be transmitted to Bob over the phone. Alice thinks of a message \(I = I_1, I_2\) consisting of two parts. The first part \(I_1\) encodes her belief about the dynamical system
Algorithm 1 L-ICDS: Computing the efficient code

Input: Dynamical function $f$, initial state $x_0 = \tilde{x}(0)$, global time horizon $T_{\text{global}}$, maximum number of partitions $m_{\text{max}}$, maximum number of expansion terms for each local model $k_{\text{max}}$

Output: Approximated states $\tilde{x}$, optimal total cost $L^*_\text{total}$, optimal total complexity $k^*_\text{total}$, optimal number of partitions $m^*$

1: $x_{\text{exact}}(t) \leftarrow \tilde{x} = f(\bar{x}(t))$: Observations
2: $L^\text{total}_i \leftarrow 0$: Optimal total cost
3: $k^*_i \leftarrow 0$: Optimal total complexity
4: $m^* \leftarrow 1$: Optimal number of partitions
5: $T_\text{local} \leftarrow T_{\text{global}}/m$: Local time horizon
6: for $m \in \{1, \ldots, m_{\text{max}}\}$ do
7: Reset $L^\text{total}_\text{local}$ to 0
8: Reset $k^\text{total}_\text{local}$ to 0
9: for $i \in \{1, \ldots, m\}$ do
10: $x_0 \leftarrow x_{\text{exact}}((i-1) \times T_\text{local} + 1)$
11: $\tilde{x}, L^*_i, k^*_i \leftarrow \text{LMS}(f, i, T_\text{local}, \lambda, x_0, k_{\text{max}})$
12: $\bar{x} \leftarrow \tilde{x}, \bar{x}$
13: $L^\text{total}_\text{local} \leftarrow L^\text{total}_\text{local} + L^*_{\text{total}}$
14: $k^\text{total}_\text{local} \leftarrow k^\text{total}_\text{local} + k^*_i$
15: end for
16: if $m = 1$ then
17: $L^\text{total}_\text{local} \leftarrow L^\text{total}_\text{local}$
18: $k^\text{total}_\text{local} \leftarrow k^\text{total}_\text{local}$
19: else if $m > 1$ and $L^\text{total}_\text{total} < L^*_\text{total}$ then
20: $L^\text{total}_\text{total} \leftarrow L^\text{total}_\text{total}$
21: $k^\text{total}_\text{total} \leftarrow k^\text{total}_\text{total}$
22: $m^* \leftarrow m$
23: end if
24: end if
25: return $[\tilde{x}, L^*_\text{total}, k^*_\text{total}, m^*]$

Algorithm 2 LMS: Local Model Selection

Input: Dynamical function $f$, index of local window $i$, local time horizon $T_\text{local}$, relative weight $\lambda$, initial state of local time horizon $x_0$, and the maximum allowed complexity $k_{\text{max}}$

Output: Local approximate of state trajectory $\tilde{x}$, optimal local cost $L^\text{local}_i$, optimal local complexity $k^\text{local}_i$

1: $x(t) \leftarrow \tilde{x} = f(\bar{x}(t))$; $x(0) = x_0$: Observations
2: $L^\text{local}_i \leftarrow 0$: Optimal local cost
3: $k^\text{local}_i \leftarrow 0$: Optimal local complexity
4: $t_{\text{start}} \leftarrow (i-1) \times T_\text{local}$
5: $t_{\text{stop}} \leftarrow i \times T_\text{local}$
6: for $k = 1, \ldots, k_{\text{max}}$ do
7: $\tilde{x}(t) \leftarrow \bar{x} = f(\bar{x}(t))$; $x(0) = x_0$
8: $L_k = \lambda k + \int_{t_{\text{start}}}^{t_{\text{stop}}} \|\tilde{x}(t) - x(t)\|dt$
9: if $k = 1$ then
10: $L^\text{local}_i \leftarrow L_k$
11: else if $k > 1$ and $L_k < L^*_\text{local}$ then
12: $L^\text{local}_i \leftarrow L_k$
13: $k^\text{local}_i \leftarrow k$
14: end if
15: end for
16: return $[\tilde{x}, L^\text{local}_i, k^\text{local}_i]$

that has generated the sequence, and the second part $I_2$ encodes the unexplained portion of the data by the assumed dynamical system. In this setting, Bob has a UTM that decodes $I$ and reconstructs the original sequence. The first part of the message $I_1$ teaches Bob Alice’s belief $f$ about the source dynamical system, and the second part $I_2$ teaches Bob how to recover the observations given this dynamical system. Assume that Alice and Bob have agreed on a finite set of functions $\mathcal{H}$ from which the dynamics $f$ is chosen. Therefore, the first part of the message takes $\log |\mathcal{H}|$ bits to choose one member of this set. The second part of the message $I_2$ encodes the initial point $x(0)$, from which the dynamical system starts evolving.

Again, we assume that state values are discrete, finite and chosen from alphabet set $\mathcal{X}$. This assumption is valid by assuming bounded value and finite precision for states. This requires $\log |\mathcal{X}|$ bits to encode the initial point. In total, the number of bits required to encode the sequence of observations can be seen as an Information Criterion for Dynamical Systems (ICDS). For a deterministic dynamical system, having $(f, x(0))$ suffices to recover $x(t)$ for all $t > 0$ (within the assumed precision). Therefore, ICDS number of bits is sufficient information to recover the sequence $S$.

Can Alice do better?—The states of a dynamical system move along a certain trajectory in the state space depending on $f$ and $x(0)$. Therefore, we do not need to encode $f$ for its entire input space. If $f$ takes a simple shape around the working point, we can save many bits by encoding $f$ locally rather than globally. This idea results in Local Information Criterion for Dynamical Systems (L-ICDS). Assume the state space is adaptively partitioned into $m$ pieces along the state trajectory and the complexity of the system within each partition is also adaptively chosen. The input tape of the UTM is formed as a concatenation of several messages (instead of two as before), i.e., as $I = I_1, I_2, \ldots, I_m$, where each tuple $(I_{2j-1}, I_{2j})$, $j \in 1, 2, \ldots$ corresponds to the local partition $j$ in the state space of the dynamical system of Eq. (1). In each tuple, $I_{2j-1}$ reprograms the UTM into the simulator of the $j$th local approximation to $f$ and decodes $I_{2j}$ to its corresponding initial point from the Observations; It means that the local trajectory corresponding to each local model starts from a point belonging to
the correct trajectory to prevent propagating error from one local model to the next one. Formally speaking, we look for a local representation of a function based on a finite set of basis functions

\[ \dot{x}(t) = f(x, t) \quad \Rightarrow \quad \hat{x}_j(t) = \tilde{f}_j(\tilde{x}_j(t)) = \sum_{i=0}^{k_{\text{max}}} \alpha_i \phi_i(x, t), \]

where \( \tilde{f}_j \) is the local approximation to \( f \) around \( j \)th working point \( x^{*j} \). In other words, \( \tilde{f}_j \) approximates the function \( f \) in its \( j \)th local partition of the state space to which \( x^{*j} \) belongs. The set \( \{ \phi_i \} \) is chosen from the hypothesis space \( \Phi \) with cardinality \( |\Phi| = k_{\text{max}} \). The set \( \Phi \) is chosen rich enough such that it contains basis functions that are able to approximate \( f \) arbitrarily well as \( k_{\text{max}} \to \infty \). Different classes of basis functions can be used, e.g., Taylor expansion, Fourier series, Legendre polynomials, etc. In this paper, we use Taylor expansion to showcase our points, but the concepts are generally applicable to other expansions as well. Let us next assume the coefficients are chosen from a finite discrete set \( \mathcal{A} \). The coefficients are bounded because we approximate the dynamics function by a smooth function (e.g., NN with tanh nonlinearity or GP) and the derivatives are bounded. In addition, the coefficient are continuous quantities, but we again assume they are represented by finite precision (as represented on a computer). Therefore, each local message \( (I_{2j-1}, I_{2j}) \) requires \( N_j \) bits code as follows: \( N_j = |I_j| + |I_{j+1}| = k_{\text{max}} \log |\mathcal{A}| + \log |\mathcal{X}| \). On the other hand, if \( f \) is encoded globally, we have \( N_g = |I_1| + |I_2| = k_g \log |\mathcal{A}| + \log |\mathcal{X}| \) that encodes \( f \) on its whole input domain. The idea of this section is that in many practical dynamical systems, \( k_g \) needs to be much larger than \( k_{\text{max}} \) to give a good approximation to \( f \) on its whole domain, which may result in \( N_g > \sum_j N_j \) (see Fig. 2).

2.1 Practical Algorithm

In this section, we present a practical algorithm to implement the above-mentioned idea of encoding (the exposition of this subsection follows [28]). Taylor expansion is used as the method for local approximation and error in the prediction of states. It can also be interpreted from an information theoretic perspective: Assume the values of the coefficients of the Taylor expansion come from a Gaussian distribution, i.e., \( a_j \sim N(\mu, \sigma^2) \). In the optimal coding scheme, the number of bits required to encode the coefficients equals the Shannon entropy of the normal distribution, \( \log(\sigma\sqrt{2\pi e}) \). Thus, \( \lambda \) is log-proportional to the variance of coefficients that is caused by the
fluctuations of the dynamics functions. In the current version, we manually choose λ such that two terms of Eq. 1 are of the same order.

The proposed method is summarized in Alg. 1 and the schematic in Fig. 1(a). The general message is that a sequence of approximations ŵ(0) = ŵ(x(0)) over the sequence of partitions ∪i=1 mΩ (m > 1) leads to a better encoding, i.e., Ltotal(0) = mL(0) < Lm=1(f) (see Sec. 4.1 of supplementary document for an illustrative example).

2.2 Theoretical Results

In this section, we will prove that it is possible to control the error introduced by local approximations. We distinguish here between two objects, the states x(.) and the dynamics ŵ(x(.)), both as a function of time t. We rely heavily on the identity in Eq. 1, which adds a lot of regularity to this problem. Therefore, we can derive statements of the type: if f and ŵ are close in some sense, then the state trajectories x and ŵ are close as well. And even better, the opposite is also true – close states imply close dynamical functions. This guarantees sufficiently accurate state prediction, while being able to reduce model complexity. Furthermore, we will elaborate later on the other direction in order to deploy ICDS as a model selection criterion.

First, we show a result that accurate local approximations imply precise state estimations. The proof of this and all following theorems are given in the supplementary material.

**Theorem 1.** Consider Eq. 1 with f Lipschitz-continuous on [0, T]. Furthermore, assume a Lipschitz-continuous approximation ŵ is used to obtain state approximations ŵ. Then, for ŵ(0) = x(0) = x0,

\[ \| x(.) - ŵ(.) \|_{L^2(0,T)} \leq T^2 \| f(x(.)) - ŵ(x(.)) \|_{L^2(0,T)}. \]  

In particular, this implies: if \( \| f(x(.)) - ŵ(x(.)) \|_{L^2(0,T)} \leq \epsilon \), then \( \| x(.) - ŵ(.) \|_{L^2(0,T)} \leq T^2 \epsilon. \)

Next, we show the opposite direction: close state trajectories imply close dynamical systems.

**Theorem 2.** Let the assumptions of Theorem 1 hold, and assume \( \sup_{t \in (0,T)} x(t) = M < \infty \) and \( \sup_{t \in (0,T)} x'(t) \leq \bar{M} \). Then, we have

\[ \| f(x(.)) - ŵ(x(.)) \|_{L^1(0,T)} \leq C \| x(.) - ŵ(.) \|_{L^1(0,T)}. \]  

This implies: if \( \| x(.) - ŵ(.) \|_{L^1(0,T)} \leq \delta \) then \( \| f(x(.)) - ŵ(x(.)) \|_{L^1(0,T)} \leq C\delta. \)

3 Experiments: Encoding Dynamical Systems

In this section, we illustrate how the encoding scheme proposed in Sec. 2 looks in practice. We elaborate in detail on the algorithm with the aid of two examples. More descriptive examples are in the supplementary material. We consider: 1) the one-dimensional system \( \ddot{x}(t) = -\tanh(x(t)) + 0.01\epsilon(t) \); and 2) a pendulum with two states \( \dot{x}_1 = x_2 + 0.01\epsilon_1(t) \) and \( \dot{x}_2 = -x_2 - 0.81\sin(x_1) + \)
We propose a new way to compare learned functions, which is based on L-ICDS and facilitates model selection. However, determining the depth of the NN and finding a suitable kernel function for the GP can be challenging. Again, we consider the three objects \( f, \hat{f}, \tilde{f} \), which are, respectively, the true dynamical function, the output of an arbitrary learning algorithm, and the local encoding. Assume we are given a collection \( \mathcal{S}_i \) of observed sequences \( \{S_i\} \) all generated by the underlying dynamical system \( \mathcal{F} \). Based on this dataset, we can deploy several different learning algorithms in order to obtain approximations to the true function \( f \). These approximations will most likely differ in their quality, which gives rise to the question, which of the learned functions should be selected.

Frequently used methods to learn dynamical functions are, for example, NNs and GPs (cf. ‘Related work’). However, determining the depth of the NN and finding a suitable kernel function for the GP can be non-trivial tasks. Ill-considered choices can lead to overfitting and bad performance and hence, should be discarded as soon as possible. For example, an over-parameterized NN may overfit to the training data and result in zero training error while being far from the correct dynamics function \( f \).

We propose a new way to compare learned functions, which is based on L-ICDS and facilitates choosing among them. In particular, we claim that, for a certain class of functions, the function with the smallest \( L \) is closest to the true dynamics. We quantify this statement in the following theorem.

**Theorem 3 (Model Selection).** Let the assumptions of Theorems 4 and 2 hold, \( L(f_1) \leq L(\tilde{f}_2) \) and
\[
\| \hat{f}_1(x(.)) - \hat{f}_1(x(.)) \|_{L^1([0,T])} \leq C \lambda k_1^r,
\]
then
\[
\| f(x(.)) - \hat{f}_1(x(.)) \|_{L^1([0,T])} \leq E + \xi \| f(x(.)) - \tilde{f}_2(x(.)) \|_{L^2([0,T])},
\]
where \( E, \xi, C \geq 0 \) are constants, which depend on certain properties of the dynamical systems.
Remark 1. The proposed algorithm works well for certain types of systems, which we confirm with empirical results. However, the theorem does not guarantee that it works for all systems; clearly, for large $E$ and $\xi$, the above theorem is not very meaningful. The condition in Eq. 26 is an interesting starting point to investigate the suitable class of functions, for which the theorem yields a meaningful bound. Finally, since Theorem 1 can also be stated in $L^1$ with slightly different assumptions, it is also possible to derive a result similar to Theorem 3 purely in the $L^1$ norm.

The proposed method quantifies the model accuracy along a trajectory, which depends on the initial point $x(0)$. Since we are interested in obtaining results, which are representative in the whole domain of the training data $S_c$, we propose to randomize $x(0)$ within $S_c$ and average over the obtained results to make them meaningful for the whole domain.

5 Experiments: Model Selection for Dynamical Systems

After discussing the capabilities of L-ICDS as a model selection criterion, we also present empirical results in order to provide more evidence to our claims. First, consider a dynamical system as in Eq. 1 with $f(x) = -\tanh(x) + 0.1 \sin(5x)$ and additive white noise $0.01\epsilon(t)$. This system is used to generate 10 noisy trajectories starting from randomly chosen initial points and 100 data points each are sampled with the aid of the Euler–Maruyama method. This results in a training set $S_c$ with total size of 1000 samples. In Fig. 3, the learned functions $f_{NN}$ are depicted together with the true function $f$. Figure 3 clearly shows the connection between the L score and the respective fit of the different NN architectures as the least score gives the best fit.

Next, we consider dynamical systems of Eq. 1 for some benchmark problems. Similar benchmark problems are considered, for example, in [21] and [35], which we used to shape the nonlinearities for the problems considered herein, which are summarized in Table 1. After generating noisy data based on the dynamical system, we deploy several NNs with different depth and width and a GP to capture the behaviour of the system (details of the learning procedures in the supplementary material). The results in terms of the $L^2$ score and actual distance to the underlying function (in the $L^2$ norm sense) are shown in Table 1. We emphasize that the best learned function achieves the lowest $L$ score.

The proposed method does not aim at improving any of the model learning methods. Instead, we provide a structured way to post-process learned models and select the best among several candidates. However, the presented ideas might be incorporated into improving also the training process.

6 Discussion

In this paper, we proposed L-ICDS as a method to efficiently encode information of a dynamical system, which is either known or learned from a sequence of observations. We built the encoding scheme on top of the minimum message length principle and came up with a practical method to approximate the algorithmic complexity of dynamical systems by means of local approximations. In addition to efficient encoding, we showed through experiments and theorems that the proposed encoding criterion can be used for model selection likewise. By comparing L-ICDS scores for different learned models (e.g., NN and GP), the model that is closer to the underlying dynamics can be selected. For future work, we aim to apply L-ICDS for efficient communication in networked multi-agent systems. Also, we seek to characterize more precisely the class of dynamical systems, for which L-ICDS is effective (cf. Remark 1), and investigate extensions to stochastic systems.

Table 1: Results for the model selection experiment. The proposed criterion L-ICDS is applied to the dynamical functions $f$ shown in the first column. Columns 2–4 show results for learning NNs (NN[\#number of hidden units of each layer]), and column 5 the GP. Per model, the two numbers are the L-ICDS score (left) and the ground truth as the $L^2$ norm between the learned and the true function. Best scores are highlighted in bold.

| $-\tanh(x)$ | NN[1] | NN[10] | NN[40] | GP |
|-------------|-------|--------|--------|----|
| $-\tanh(x) + 0.5 x$ | 4.40e−5 | 5.84e−5 | 7.43e−5 | 6.15e−4 |
| $-x/(1 + x^2)$ | 4.69e−3 | 7.30e−3 | 8.69e−2 | 2.1e−3 |
| $-\tanh(x) + 0.5 \sin(x)$ | 1.05e−1 | 1.9e−1 | 9.16e−1 | 2.48e0 |

\footnotesize

In the supplementary material, we provide a structured way to post-process learned models and select the best among several candidates. However, the presented ideas might be incorporated into improving also the training process.

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| $-\tanh(x)$ | NN[1] | NN[10] | NN[40] | GP |
|-------------|-------|--------|--------|----|
| $-\tanh(x) + 0.5 x$ | 4.40e−5 | 5.84e−5 | 7.43e−5 | 6.15e−4 |
| $-x/(1 + x^2)$ | 4.69e−3 | 7.30e−3 | 8.69e−2 | 2.1e−3 |
| $-\tanh(x) + 0.5 \sin(x)$ | 1.05e−1 | 1.9e−1 | 9.16e−1 | 2.48e0 |
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Appendices

A Algorithmic Complexity and Universal Turing Machine

The concepts of algorithmic complexity and universal Turing machine were briefly introduced in paper and will be presented in more details here. Using the Alice-Bob scenario, we briefly review some necessary terms. The message is a sequence of symbols chosen from a set of alphabets. Each symbol is encoded by a binary sub-sequence called word that forms the entire message when the code for all symbols are pieced together. Shannon’s information theory considers the message as a sequence of outcomes of a random process \( \{a_i : i = 1, \ldots, N\} \) and the probability of word \( a_i \) be \( p_i \) for all \( i \). The goal is to find a code that maps each word \( a_i \) to a binary string with length \( l_i \) such that the expected length of the string which is defined as

\[
E(l) = \sum_i p_i l_i
\]

is minimized. It can be proved that the optimal code in this sense will be obtained if \( l_i = - \log p_i \) where this value is also known as Shannon’s entropy. We consider the base of the logarithm 2 throughout this paper. The length \( l_i \) of the code of a word \( a_i \) can be taken as a measure of information content of word \( a_i \) represented by \( \text{Info}(a_i) \). Nonetheless, the major limitation of Shannon’s approach to information is its explicit dependence on probabilistic source of the message. Algorithmic Complexity (AC) is a different approach that removes this assumption and gives a more generic idea of information. To present the core idea of AC, some preliminary definitions are required which will be briefed in the following

Universal Turing Machine—A Turing machine (TM) is a machine with

1. A clock that synchronizes all activities of the machine.
2. A finite set of internal states indexed by \( \{1, 2, \ldots, S\} \). The machine may change its state at the clock tick.
3. A binary work tape which can be moved to the right or left and be updated by the machine.
4. A one-way binary input tape which forms the input to the machine. The input tape cannot be moved backward.
5. A one-way binary output tape that carries the machine’s output.
6. An instruction list that determines the action of the machine at each clock tick depending on the current value of the input tape, work tape and the internal state of the machine. The action may include moving the input tape, updating and moving the work tape, updating and moving the output tape, or moving to a new internal state.

Given a binary string \( A \) representing some data or information, the amount of information, a.k.a Algorithmic Complexity (AC), in \( A \) given a particular Turing Machine (TM), is the shortest input tape \( I \) which will cause TM to output \( A \), i.e., \( AC(A) = I \). It is obvious from this definition that the information content of a message \( A \) depends on the chosen TM. The concept of Universal Turing Machine (UTM) comes as an assistance here. Apart from its detailed definition that can be looked up in \[37\], a UTM has the interesting property of being programmable. Meaning that the input tape \( I \) may consist of two concatenated parts \( I = I_1.I_2 \) such that \( I_1 \) pushes the the initial Turing machine TM\(_0\) into a state from that state on, the UTM behaves as another Turing machine TM\(_1\). The second part of the input tape \( I_2 \) is then decoded by TM\(_1\) rather than TM\(_0\). This capacity of UTM enables us to achieve a universal measure for complexity or information content. In the next section we discuss how information content of dynamical systems can be described in the framework of a UTM.

B Proofs of Theoretical Results

We provide here the proofs to our theoretical results:

**Theorem 4.** Consider Eq. (1) with \( f \) Lipschitz-continuous on \([0, T]\). Furthermore, assume a Lipschitz-continuous approximation \( \hat{f} \) is used to obtain state approximations \( \hat{x} \). Then, for \( \hat{x}(0) = x(0) = x_0 \),

\[
\|x(.) - \hat{x}(.))\|^2_{L^2([0,T])} \leq T^2\|f(x(.)) - \hat{f}(x(.))\|^2_{L^2([0,T])},
\]

(10)
In particular, this implies: if $\|f(x(.)) - \hat{f}(x(.))\|_{L^2([0,T])}^2 \leq \epsilon$, then $\|x(.) - \hat{x}(.)\|_{L^2([0,T])}^2 \leq T^2\epsilon$.

**Proof of Theorem 1.** We start the proof by showing that there exists a well defined solution $x(t)$ to the considered ODE, which is due to the Picard–Lindelöf theorem.

Next, we show how to bound a function against its derivative, which is frequently done in Poincaré inequalities. Depending on the given assumptions, these results all look slightly different. Here, we use $z \in C^1((0, T))$, $z(0) = 0$ and proof $\|z(.)\|_{L^2([0,T])}^2 \leq T^2\|z'(.)\|_{L^2([0,T])}^2$.

We start with the fundamental theorem of calculus and obtain

$$z(t) = \int_0^t z'(s) \, ds, \forall t \in [0, T].$$

(11)

Hence, we obtain for the absolute value

$$|z(t)| \leq \int_0^t |z'(s)| \, ds.$$  

(12)

Now we assume a multiplicative one and apply the Cauchy-Schwarz inequality

$$\int_0^t |z'(s)| \, ds \leq \sqrt{\int_0^t |z'(s)|^2 \, ds} \sqrt{\int_0^t 1 \, ds}.$$  

(13)

Since $t \leq T$ and everything is nonnegative, we obtain

$$|z(t)| \leq \sqrt{T} \sqrt{\int_0^t |z'(s)|^2 \, ds}.$$  

(14)

Taking the square and integrating does not change the inequality, since the right hand side is not dependent on $t$ anymore. This yields the final result

$$\int_0^T |z(s)|^2 \, ds \leq T^2 \int_0^T |z'(s)|^2 \, ds.$$  

(15)

Now we substitute $z(t) = x(t) - \hat{x}(t)$ and obtain

$$\|x(.) - \hat{x}(.)\|_{L^2([0,T])}^2 \leq T^2 \|f(x(.)) - \hat{f}(x(.))\|_{L^2([0,T])}^2 \leq T^2\epsilon.$$  

(16)

**Lemma 1.** Assume $x(t) \in C^1((0, T))$ and $\sup_{t \in (0, T)} |x'(t)| \leq \tilde{M}$ on the domain $[0, T]$. We can show

$$\|x(.)\|_{L^\infty([0,T])} \leq K\|x(.)\|_{L^1([0,T])},$$  

(17)

**Proof.** Since we consider a bounded domain and the derivative is bounded we conclude that $\sup_{t \in (0, T)} |x(t)| = \tilde{M}$ is bounded as well. We proof the statement by considering the worst case scenario, which is a triangle for this case. What essentially can happen is that the support of the function shrinks, while the maximum remains constant. However, by bounding the derivative we have control over the growth of the area beneath the function. Therefore, the extreme case is a triangle $\phi_\Delta$ with the maximal slope $\tilde{M}$ and peak point $M$. This yields

$$\|\phi_\Delta\|_{L^\infty} = M$$  

(18)

and

$$\|\phi_\Delta\|_{L^1} = \frac{1}{2} \left( \frac{M^2}{\tilde{M}^2} + M \right).$$  

(19)

Hence for $\frac{1}{K} \leq \frac{1}{2} \left( \frac{M^2}{\tilde{M}^2} + 1 \right)$ we obtain our claim.  

**Lemma 2.** Assume the function $x(t)$ is monotonically increasing in $[0, T]$. Then the variation of $x(t)$ is given by

$$V(x, [0, T]) = x(T) - x(0).$$  

(20)
Proof. The proof is straightforward and follows immediately with a monotonicity and telescope sum argument. \(\square\)

**Theorem 5.** Let the assumptions of Theorem 2 hold. Additionally, assume \(\|x(\cdot) - \tilde{x}(\cdot)\|_{L^1([0,T])} \leq \delta\). Then, we have

\[
\|f(x(\cdot)) - \tilde{f}(x(\cdot))\|_{L^1([0,T])} \leq C\|x(\cdot) - \tilde{x}(\cdot)\|_{L^1([0,T])}. \tag{21}
\]

This implies: if \(\|x(\cdot) - \tilde{x}(\cdot)\|_{L^1([0,T])} \leq \delta\) then \(\|f(x(\cdot)) - \tilde{f}(x(\cdot))\|_{L^1([0,T])} \leq C\delta\).

**Proof of Theorem 2.** We use bounded variation type arguments here. In particular, we start again with

\[
\int_0^T |f(x(t))|dt = \int_0^T |x'(t)|dt. \tag{22}
\]

It is well known fact in analysis that the quantity \(\int_0^T |x'(t)|dt\) can be used to compute the total variation of a smooth function. We will use an equivalent approach to quantify the total variation and use this to bound the derivative with the states. We use

\[
\int_0^T |x'(t)|dt = \sup_{P \in \mathcal{P}} \sum_{i=0}^{n_P-1} |x(t_{i+1}) - x(t_i)|, \tag{23}
\]

where we take the supremum over all possible grids, which are not necessarily equidistant. Hence, \(n_P\) is the number of grid points, which can in general go to infinity. This is even possible for functions with bounded variation, as long as the function value decays fast enough. The assumption \(\sup_{t \in (0,T)} |x'(t)| < \infty\) ensures a finite number of oscillations on a bounded domain, which combined with Lemma 2 yields that \(n_P < \infty\). Hence, there exists an optimal grid with a finite number of points \(n_P\) and we can use the bound

\[
\sup_{P \in \mathcal{P}} \sum_{i=0}^{n_P-1} |x(t_{i+1}) - x(t_i)| \leq n_P \sup_{t \in (0,T)} |x(t_{i+1}) - x(t_i)|. \tag{24}
\]

We can split this apart with the triangle inequality and make the quantity even bigger by dropping the grid. Hence, we obtain

\[
\|f(x(\cdot))\|_{L^1([0,T])} \leq 2n_P \|x(\cdot)\|_{L^\infty([0,T])}. \tag{25}
\]

With the aid of Lemma 1, \(C = 2n_P K\) and the same argument as in the end of the proof of Theorem 1 we conclude this proof. \(\square\)

**Theorem 6 (Model Selection).** If the previous assumptions hold, \(L(\hat{f}_1) \leq L(\hat{f}_2)\) and

\[
\|\hat{f}_1(x(\cdot)) - \hat{f}_1(x(\cdot))\|_{L^1([0,T])} \leq C\lambda k^1, \tag{26}
\]

then

\[
\|f(x(\cdot)) - \hat{f}_1(x(\cdot))\|_{L^1([0,T])} \leq E + \|f(x(\cdot)) - \hat{f}_2(x(\cdot))\|_{L^2([0,T])}, \tag{27}
\]

where \(E, \xi, C \geq 0\) are constants, which depend on certain properties of the dynamical systems.

**Proof of Theorem 3.** We consider three objects in this proof - the true dynamical function \(f\), an approximation \(\hat{f}\), which is most likely obtained from a learning algorithm and the local approximation \(\tilde{f}\), obtained through a local expansion, e.g. Taylor.

We start by inserting \(+\hat{f} - \tilde{f}\) and obtain with the triangle inequality

\[
\|f(x(\cdot)) - \tilde{f}_1(x(\cdot))\|_{L^1([0,T])} \leq \nu_1 + \|\tilde{f}_1(x(\cdot))\|_{L^1([0,T])}. \tag{28}
\]

Now, we use Theorem 2 and obtain

\[
\nu_1 + \|f(x(\cdot)) - \tilde{f}_1(x(\cdot))\|_{L^1([0,T])} \leq \nu_1 + C\|x(\cdot) - \tilde{x}_1(\cdot)\|_{L^1}. \tag{29}
\]
The assumption $L(\hat{f}_1) \leq L(\hat{f}_2)$ expands to
\[ \lambda k_1^2 + \|x(.) - \tilde{x}_1(.)\|_{L^1([0,T])} \leq \lambda k_2^2 + \|x(.) - \tilde{x}_2(.)\|_{L^1([0,T])}. \]
Hence, for $\nu_1 \leq C\lambda k_1^2$ we obtain
\[ \nu_1 + C\|x(.) - \tilde{x}_1(.)\|_{L^1} \leq C\lambda k_2^2 + C\|x(.) - \tilde{x}_2(.)\|_{L^1([0,T])}. \]
Now we apply the Cauchy-Schwarz inequality to transform the $L^1$ norm into the $L^2$ norm and obtain
\[ C\lambda k_2^2 + C\|x(.) - \tilde{x}_2(.)\|_{L^1([0,T])} \leq C\lambda k_2^2 + TC\|x(.) - \tilde{x}_2(.)\|_{L^2([0,T])}. \]
With the aid of Theorem 1 it follows that
\[ C\lambda k_2^2 + TC\|x(.) - \tilde{x}_2(.)\|_{L^2([0,T])} \leq C\lambda k_2^2 + T^2C\|f(x(.)) - \hat{f}_2(x(.)\|_{L^2([0,T])}. \]
With the aid of the triangle inequality we can again show
\[ C\lambda k_2^2 + T^2C\|f(x(.)) - \hat{f}_2(x(.)\|_{L^2([0,T])} \leq C\lambda k_2^2 + T^2C\nu_2 + T^2C\|f(x(.)) - \hat{f}_2(x(.)\|_{L^2([0,T])}. \]

C Learning dynamical systems

The more detailed description of the method we used to learn dynamics function $\hat{f}$ from observational data is presented here. We use a simple black-box approach to learn the dynamical system from a set of trajectories.

Learning $\hat{f}$ by Neural network— Assume we are given a collection of sequences of observations $S_c = \{S_1, S_2, \ldots\}$. Each sequence $S_i$ covers a trajectory in the state space starting from some starting point $x(0)$. We use each sequence $S_i$ as a mini-batch of observations and train $f(x; \theta)$ by the following simple relationship between its input/output pairs:
\[ \dot{x}(t) = f(x(t)) \implies \dot{x}(t) = f(x(t); \theta) \]
\[ \implies \frac{\Delta x}{\Delta t} = f(x(t); \theta) \]
\[ \implies x((k + 1)T_s) - x(kT_s) = \hat{f}(x(kT_s); \theta) \]

The reason for using a collection $S_c$ instead of a single sequence $S$ is clear. A single sequence starting from an initial point $x(0)$ is unlikely to be representative enough so that $\hat{f}$ is learned as a good approximation to $f$. Once $\hat{f}$ is learned, we can use automatic differentiation to compute its derivative w.r.t. the input $\theta$.

Learning $\hat{f}$ Gaussian Process— The discretization of the nonlinear dynamics function is done just like above. We used a vanilla GP without any sparse approximations and a squared exponential kernel function.

D More experiments

Some parts of the experiment sections are delegated to here from the main text. It includes more sophisticated experiments with higher dimensional and physical dynamical systems.

D.1 Enlarged version of the illustrative example

As an illustrative example, let’s assume the dynamical system $\dot{x}(t) = f(x(t))$ with $f(x) = -\text{sat}(x)$ depicted in Fig. 4a. This dynamical system is stable and the evolution of its state is depicted in Fig. 4. If the initial point $x(0)$ resides in the positive region of the state space, it never leaves the non-negative side of the state space. Hence, encoding $f$ for the negative domain is not necessary and we can safely only encode $f$ in its positive domain. This can be formalized in terms of algorithmic complexity as
\[ AC(x(0), f) < AC(x(0)) + AC(f) \]
meaning that knowing $x(0)$ allows to design a better code for $f$. 
Figure 4: (a) Shows that the negative region of the state space is not visited when the initial state is positive. (b) Shows the learned dynamics function \( \hat{f} \) when only some trajectories of the state dynamics governed by \( f \) is available.

Figure 5: The effect of learning \( f \) by \( \hat{f} \) then approximating \( \hat{f} \) by \( \tilde{f} \) on state evolution of the dynamical system. The optimal trade-off is found by L-ICDS score in (f) for \( m = 2 \) and the corresponding state evolution is shown in (b). The number of employed basis functions \( k \) in each local region is written over that region.

D.2 Illustrative example:

Here is more figures related to sec.3(Fig.2) of the main text. The assumed dynamical system is \( \dot{x}(t) = -\tanh(x(t)) \). The state evolution for different number of partitions is depicted in Fig. 5. Fig. 5(f) shows that L-ICDS score finds a non-trivial local encoding of the space for \( m > 1 \).

D.3 Local approximations to the learned function:

In this section, the system \( \dot{x}(t) = \tanh(x(t)) + 0.1 \sin(5x(t)) \) is used to generate samples based on the method explained for the experiment in the main text. The dynamics function is then learned and depicted in Fig. 6(a). Once the function is learned, multiple local Taylor approximations is computed and shown in Fig. 6(b-e) corresponding to different number of partitions. Figure 6(f) shows the \( L \) score is optimal for a non-trivial \( m > 1 \) case.

D.4 Pendulum dynamics:

Part of this experiment is in the main text. This is the complete version. We consider a realistic physical system of a pendulum with two dimensional dynamics \( \dot{x}_1 = x_2 \) and \( \dot{x}_2 = -x_2 - 9.81 \sin(x_1) \). The result for how \( \hat{f} \) is learned from \( f \) and how \( \tilde{f} \) approximates \( \hat{f} \) is depicted in Fig. 7. The corresponding result in the state space is shown in Fig. 8 where \( x_1 \) is the angular position and \( x_2 \) the angular velocity of the pendulum. Again, L-ICDS finds a good trade-off between model complexity and prediction accuracy.
We test the model selection capability of L-ICDS for Lorenz system which is a three dimensional chaotic system described below.

\[
\begin{align*}
\dot{x}(t) &= \sigma(y - x) \\
\dot{y}(t) &= x(\rho - z) - y \\
\dot{z}(t) &= xy - \beta z
\end{align*}
\]
Figure 7: (a) shows the dynamics function for two dimensional dynamics of a pendulum. The dynamics functions of two dimensions are drawn on a mutual domain where both $x_1$ and $x_2$ ranges over $[-10, 10]$ (b) Taylor approximation of the learned dynamics function around working point $[x_1; x_2] = [2, 2]$.

Figure 8: State evolution of a two dimensional physical pendulum dynamical system with various number of local partitions. The minimum value of $L_{\text{total}}$ where $L(m = \{1, 2, 3, 4\}) = [0.067, 0.048, 0.0035, 0.044]$ occurs when $m = 3$ as a good trade-off between accuracy of the states and total complexity of the local models. The number of bases used to approximate each local region is also depicted over the figures.

with parameters $[\sigma, \beta, \rho] = [10, 8/3, 28]$ for which the system exhibits chaotic behaviour. We used two neural networks with the same number of neurons but different architectures to learn the dynamics of the system. Fig. 10 refers to the neural network that achieves a closer approximation to the dynamics. The value of L-ICDS score for both architectures are computed. As can be seen in the subcaptions, the optimal value of the L-ICDS score for the correct architecture (top row) that occurs for $m = 2$ is less than the optimum value of L-ICDS for the incorrect architecture (bottom row) that occurs for $m = 2$. This confirmed our hypothesis that the best encoding by MML principle is a sign of the correct joint description of the model and data. We tested the accuracy of the learned $f$ for a left-out subset of $S_c$ as the validation set and observed that the least validation error is concurrent with the least L-ICDS score. This confirms the link between generalization and compression.
Figure 9: Quadrotor dynamics. Top four figures show the effect of suboptimal \( m = 1 \), left) and optimal number of partitions \( m = 2 \), right) on two different states of the system. As can be seen, having single partition in the state space imposes a large value of the prediction error when the system evolves in time. Partitioning the state space into two local regions gives a fair trade-off between model complexity and total inaccuracy in the prediction of states. The diagram of (e) shows this trade-off due to the first and second terms of \( L_{\text{total}} \) with parameters \( \lambda = 2 \) and \( T_{\text{global}} = 2 \) and sampling time step \( \Delta T = 0.01 \). The overlaying numbers in each area show the number of Taylor terms chosen to model the function restricted to that area.

Figure 10: Each figure shows the states of the true dynamics \( f \) (solid), states of the learned dynamics \( \hat{f} \) and states of the locally approximated dynamics \( \tilde{f} \) (blue dashed). The neural network architecture \( \text{NN} = [50] \) for the top row and \( \text{NN} = [10, 10] \) for the bottom row.