Problems for observing the inflaton potential

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Abstract. Robustness of the solutions to the inflaton potential inverse problem based on the slow-roll approximation is addressed. Here robustness is defined as the convergence of solutions to a unique functional form for the potential while increasing the order of the underlying expansion. For the analysis we introduce a measure of the difference of the outputs obtained using first and second order in the horizon-flow expansion. The evolution of this measure is determined by a second-order linear non-autonomous non-homogeneous differential equation. Boundedness of the general solutions to this equation is analysed. It is shown that they diverge for most of the physically meaningful cases. Examples for typical inflationary models are presented which confirm this result. The consequence is that the reconstructed inflationary potential depends in all its physical characteristics on the order of the approximation. It is argued that this lack of robustness is due to the limitations of the slow-roll expansion for probing the scale dependence of the inflationary spectra.

Keywords: CMBR experiments, cosmological perturbation theory, inflation, power spectrum

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1. Introduction

The relatively recent release of the analysis of the first-year WMAP data [1] and of a significant amount of very precise data on the power spectrum of the matter distribution at large scales [2] caused turmoil in the cosmological community. The feasibility of obtaining reliable information about our universe when it was younger than $10^{-49}$ s accounts for a large part of that excitement. Data confirmed that the primordial power spectrum of curvature fluctuations is consistent with a scale-invariant, Gaussian and adiabatic spectrum. The simplest and most elegant mechanism known to causally produce primordial spectra with those properties is cosmological inflation.

In the simplest scenarios with an exit from inflation the entity responsible for the accelerated expansion of the early universe is the *inflaton* $\varphi$, a single scalar field with equations of motion

$$\ddot{\varphi} + 3H \dot{\varphi} + V'(\varphi) = 0,$$

$$H^2 = \frac{1}{3} \left[ \frac{\dot{\varphi}^2}{2} + V(\varphi) \right].$$

Here $V(\varphi)$ is the potential energy density of the inflaton field, a dot stands for a derivative with respect to cosmic time $t$ and a prime denotes a derivative with respect to $\varphi$. We use Planck units, $8\pi G = c = \hbar = 1$.

The Hubble horizon $d_H = 1/H$, is roughly the size of the region where causal processes can take place during one Hubble time $1/H$. During inflation the comoving $d_H$ decreased allowing comoving scales to cross out of the causal horizon. Physical information of the inflaton and of the space-time quantum fluctuations contained on those scales was effectively ‘frozen’. In this way, when the scales crossed $d_H$ back later during the era of standard (non-accelerated) expansion, the whole observable universe was seeded with statistically uniform curvature perturbations for large-scale structure formation. These seeds served also as initial conditions for the evolution of CMBR anisotropies.

The spectra of quantum fluctuations of density and metrics during cosmological inflation can be predicted [3]. It is necessary to know the behaviour of the Hubble horizon
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as a function of the logarithm of the scale factor, \( N \equiv \ln a(t) \). In the inflaton scenario this means solving equations (1) and (2). For most potentials that is a difficult task that can be simplified using the horizon-flow functions which are defined recursively as [4]

\[
\epsilon_{m+1} = \frac{d \ln |\epsilon_m|}{dN}, \quad \forall m \geq 0, \quad \epsilon_0 = \frac{H(N)}{H(N)},
\]

(3)

where \( N_i \) denotes an arbitrary ‘initial’ moment. The necessary condition for inflation to take place is \( \epsilon_1 < 1 \), and if the weak energy condition and null energy condition hold, \( \epsilon_1 \geq 0 \). In general, for the primordial spectra to be nearly scale invariant, it is sufficient to assume \( |\epsilon_m| < 1 \) allowing us, in this way, to expand all the interesting inflationary quantities in terms of the horizon-flow functions\(^1\). The slow-roll approximation is a particular case where all the \( \epsilon_m \) are assumed to have the same order of magnitude.

Equations (1) and (2) relate the horizon-flow functions to the inflaton potential \( V(\varphi) \) and its derivatives with respect to the inflaton field. For instance, the first two flow functions are given approximately by [6]

\[
\epsilon_1 \approx \frac{1}{2} \left( \frac{V'}{V} \right)^2, \quad \epsilon_2 \approx 2 \left( \left( \frac{V'}{V} \right)^2 - \frac{V''}{V} \right).
\]

(4)

In turn, the behaviour of \( d_H(N) \) near a suitable pivot point \( N_* \) is described by

\[
d_H(N) = d_H(N_*) \left[ 1 + \epsilon_1(N - N_*) + \frac{1}{2} (\epsilon_1^2 + \epsilon_1 \epsilon_2)(N - N_*)^2 + \cdots \right].
\]

(5)

Thus, approximated expressions for the inflationary spectra can be derived. To first order for the horizon-flow functions\(^2\) they were found by Stewart and Lyth [7] using the slow-roll approximation to truncate the argument of Bessel functions evaluated at the time of horizon crossing. Their accuracy was tested against numerical solutions in [8,9,5]. In the same approximation but using Green’s functions, second-order expressions were derived in [10,6] and tested by Schwarz and Terrero [9]. The approach based on Bessel functions was then modified [4,9] to derive expressions at any order for two wide classes of inflationary models that do not necessarily satisfy the slow-roll condition. Up to third order, the corresponding expressions were tested in [9]. Several other methods were introduced to allow for relaxing the standard slow-roll condition [11]–[13]. In general, their predictions seem to match the accuracy of those methods tested against numerical results. All these tests yielded that the predictions of inflationary spectra using terms up to second order in the horizon-flow expansion are accurate enough to match the precision of current and near future observations.

In this way, given an inflaton potential as input in equations (1) and (2), the horizon-flow functions (3) can be evaluated at any moment, the behaviour of the Hubble radius can be described by means of (5), the spectra of the inflaton and metrics fluctuations accurately predicted, used as initial conditions for the calculation of the evolution of scalar and tensor modes of anisotropies of the CMBR, and the resulting spectra compared with the observational data to test the reliability of the given inflationary model.

\(^1\) Note that according to recent results, that condition may be unnecessary [5].

\(^2\) Since the order is not the same for all the observables, we follow here the convention in [9] of denoting the order by that of the first term in the Taylor parametrization of the primordial spectra.
‘Observing’ the inflaton potential means solving the related inverse problem, i.e., that of looking for the functional form of the inflaton potential corresponding to given CMBR spectra. Herein, we will refer to this task as the inverse problem for the inflaton potential (IP$^2$). Starting with the seminal paper by Hodges and Blumenthal [14], during the last 15 years there have been introduced several methods for solving the IP$^2$. Depending on the way the output is given, they can be classified as parametric [14]–[18], perturbative [15, 19, 20], fully numerical [21] and stochastic [22] methods.

Assuming that the initial conditions for the calculation of CMBR spectra have been already fitted to the observational data, what all the methods for solving the IP$^2$ do (with the remarkable exception of the fully numerical method) is, essentially, to start by looking for the combinations of $\epsilon_m$ that give the best fit to those primordial spectra. These combinations are constrained by the form and order of the approximated expressions for the spectra of density and metrics fluctuations. Note that, following definitions (3), the number $m$ of horizon-flow functions is directly related to the order of the corresponding expressions. In its turn, the order of the predicted spectra is constrained by the finite precision of data. Now, according with expansion (5), specifying the set \{$\epsilon_m(N_*)$\} is equivalent to specifying $d_H(N)$. So, the unavoidable truncation of \{$\epsilon_m(N_*)$\} corresponds to an incomplete knowledge of the evolution of the Hubble horizon. The next step in the IP$^2$ solution is to use equations (1) and (2), or the approximations (4), to find the functional form of the inflaton potential corresponding to that approximated behaviour of $d_H(N)$.

The reliability of solving the IP$^2$ for a potential as constrained by measurement errors was directly tested in [20, 23]. There are also several studies on the impact of the observational uncertainty on the form of the output potential. They give us complementary information about possible problems for observing the potential (see [24, 6, 9], [25]–[27] and references therein). Though the results are in general encouraging, it must be noted that each of them uses in their analysis a fixed order for the slow-roll expansion of the primordial spectra. This is a formal expansion, and hence there is no rigorous proof that it converges, so there is no certainty about the solution of the inverse problem based on this expansion being robust. By robust we mean that, while increasing the order of the underlying expressions, the IP$^2$ solution must converge to a unique functional form. This robustness is required to make definite statements about the high energy physics linked to the observed inflaton potential. Nevertheless, the preliminary analysis presented in [28] gave us a warning about the possible existence of problems for the robust solution of the IP$^2$ when the slow-roll approximation is used.

The aim of this paper is to step forward in the analysis of the robustness of the reconstruction of the inflaton potential as regards the convergence of the horizon-flow expansion for the spectra of inflationary perturbations under the slow-roll approximation. With that in mind, in the next section we will derive the basic equation for our analysis, a second-order linear non-autonomous non-homogeneous differential equation obtained from the comparison between equations for the inflationary perturbations to first and second order in the horizon-flow expansion. In section 3 we will present mathematical evidence pointing to the non-existence of bounded solutions for this equation if the time variation of the coefficients is required to be consistent with the inflationary cosmology. We reinforce that conclusion in section 4, on the basis of the analysis of inflationary models belonging to the prototypical classes introduced in [9]. Finally, in section 5 we discuss our results.
2. The basic equation

For finding the functional form of the inflaton potential all the methods mentioned in
the introduction use the same kind of expressions for the spectra of scalar perturbations.
Nevertheless, early in the research on the IP$^2$ it was realized that the role of tensor
perturbations also deserves attention if a unique solution to that problem is desired [23].
The information on these modes in the available data is so far rather poor [1, 2]. That is
why all the studies but the so-called Stewart–Lyth inverse problem [18] avoided dealing
with tensor equations. However, the quality of the data on the tensor modes could be
radically improved if a number of interesting experiments manage to see the primordial
light (see for instances [29]). Furthermore, studies based on the parametric method have
shown that even poor information on the spectrum of the primordial gravitational waves
allows for breaking the degeneracy in the solution of the IP$^2$ and for obtaining interesting
results [30]–[32]. In particular, in [32] it was shown how using the ratio of the amplitudes
of the tensor and scalar perturbations,

\[ r \equiv \alpha \frac{A_T^2}{A_S^2}, \quad (\alpha = \text{constant}), \]

as input for the IP$^2$ can be a very fruitful way of taking into account the available
observational information on both types of spectrum to observe the convexity of the
potential during inflation. To do that it is necessary to solve the differential equation
(recall the definitions (3) for the $\epsilon_m$)

\[ \frac{\ln r}{r_0} = \ln \epsilon_1 + C \epsilon_1 \epsilon_2 + C_1 \epsilon_1 \epsilon_2^2 + C_2 \epsilon_2^2 + C_3 \epsilon_2 \epsilon_3, \]  

(6)

where $r_0$ is a constant and

\[ C \approx -0.7296, \]  

(7)

\[ C_1 \equiv -\left( \frac{\pi^2}{2} + 5 + C \right) \approx -0.6644, \]  

(8)

\[ C_2 \equiv -\left( \frac{\pi^2}{8} + 1 \right) \approx -0.2337, \]  

(9)

\[ C_3 \equiv -\left( \frac{\pi^2}{24} + \frac{C^2}{2} \right) \approx -0.1451. \]  

(10)

This equation can be derived from the second-order expressions for the amplitudes of the
scalar and tensor perturbations obtained using the slow-roll approximation [10, 6]. (It can
also be derived following the lines described in [32].)

If the reconstruction of the inflaton potential were robust, then we would expect that,
given $r/r_0$ as input into the IP$^2$, the output using up to first order in equation (6) (see [32]
for an example of such an output) will differ only in small features from the output of
the problem with the same input but taking into account all terms in (6). To assess this
small departure, following [28], we introduce a quantity $\delta(N)$ such that

\[ \delta \epsilon_1 = \epsilon_1 (1 + \delta), \]  

(11)
where the ‘s’ denotes the horizon-flow functions in the set to be found if second-order terms in equation (6) are used. The standard notation remains for the set found with up to first-order terms. Next, using definitions (3) it can be obtained that

\[ s_1 = \epsilon_1 + \frac{d\delta}{dN}, \]

\[ s_2 = \epsilon_2 + \frac{d^2\delta}{dN^2}. \]

Now, by definition \( \epsilon_1, \epsilon_2 \) and \( \epsilon_3 \) satisfy identically the first-order version of equation (6) and we will demand \( \delta(N) \) to remain very small as \( N \) increases. Substituting (11), (12) and (13) into (6), and keeping only linear terms in \( \delta(N) \) and its derivatives, the following second-order linear non-autonomous non-homogeneous differential equation [28],

\[ \frac{d^2\delta(N)}{dN^2} + \alpha(N)\frac{d\delta(N)}{dN} - \omega^2\delta(N) = f(N), \]

is obtained, where

\[ \alpha(N) = \frac{(C + C_1\epsilon_1 + 2C_2\epsilon_2)}{C_3}, \]

\[ \omega^2 = -\frac{1}{C_3} \approx 6.893, \]

\[ f(N) = -\frac{(C_1\epsilon_1\epsilon_2 + C_2\epsilon_2^2 + C_3\epsilon_2\epsilon_3)}{C_3}. \]

For the inverse problem of the inflaton potential to be robust, the mildest requirement we could ask for, for the solution of this equation, is for it to be bounded for all possible initial conditions \( \delta(N_i) \). As we will show in the next two sections, that seems to be unlikely for all \( \alpha(N) \) and \( f(N) \) that are physically meaningful.

3. Mathematical evidence

Equations like (14) are usually found in mechanical problems like in the study of vibrations. As in mechanics, \( \alpha(N) \) is going to be called here the ‘damping’ of the system, \(-\omega^2\) the ‘stiffness’, while \( f(N) \) will be referred to as the ‘forcing’.

First of all, we recall that the formal solution of equation (14) is [33]

\[ \delta(N) = g^N\delta(N_i) + \int_{N_i}^N (g^s)^{-1} f(s) \, ds, \]

where \( g^N : \mathbb{R}^2 \to \mathbb{R}^2 \) is the linear \( (N_i, N) \)-advance mapping for the homogeneous system

\[ \frac{d^2\delta(N)}{dN^2} + \alpha(N)\frac{d\delta(N)}{dN} - \omega^2\delta(N) = 0. \]

As we already mentioned, for most of the cosmologically interesting cases, \( |\epsilon_m| < 1 \) for \( m > 0 \). According to definitions (3) this means that they vary slowly with \( N \). So, without loss of generality, we can assume the forcing \( f(N) \) to be bounded. In that case, boundedness of (18) requires boundedness of solutions to equation (19). The necessary
conditions for the existence of bounded solutions of this equation with undetermined coefficients is still an open problem. Nevertheless, there are inferences that can be drawn for the case of a stiffness strictly negative for all \( N \). Such a stiffness is usually considered as a signal of instability of the trivial solution of the above equation. The reason is that for the planar system corresponding to equation (19),

\[
\frac{d\Delta}{dN} = \Delta^T \begin{pmatrix} 0 & 1 \\ \omega^2 & -\alpha(N) \end{pmatrix} \Delta \quad \text{with} \quad \Delta^T \equiv \left| \delta(N) \frac{d\delta(N)}{dN} \right|,
\]

the eigenvalues \( \lambda^\pm(N) \propto (-\alpha(N) \pm \sqrt{\alpha(N)^2 + 4\omega^2}) \) point to a saddle-like instability for any given value of \( N \). If these eigenvalues are slowly changing, then the system is assumed to have no bounded (finite) solutions. A rigorous foundation of this folk theorem is still missing together with a precise definition of ‘slowly changing’. However, Lyapunov’s direct method reveals that, for the existence of bounded solutions to equation (19) for all \( \delta(N_i) \), the following conditions are close to necessary [34]: the damping \( \alpha(N) \) should be positive (at least asymptotically) and should increase at an exponential rate. As was already mentioned, this last condition is incompatible with all physically meaningful inflationary scenarios.

4. Physical evidence

To consolidate the analysis in the previous section, let us see how the solutions of equation (14) behave for physical situations. To cover as much as possible of the space of inflationary scenarios we will use one model from each of the classes introduced in [9]. We choose that classification because it involves only exact expressions. Then, the classifying criteria are not dependent on the order of the horizon-flow expansion, contrary to what happens in the more frequently used small-field/large-field/hybrid models classification [24]. This was realized first by Schwarz and Terrero [9] and recently confirmed by Kinney and Riotto [27]. Classification in [9] is unambiguous because it is strictly based on physical criteria, namely the behaviour of the kinetic and total energy densities of the inflaton field. It consists of three classes: (I) hidden-exit inflation \( (\epsilon_2 \leq 0) \), (II) toward-exit inflation with general initial conditions \( (0 < \epsilon_2 \leq 2\epsilon_1) \) and (III) toward-exit inflation with special initial conditions \( (0 < 2\epsilon_1 < \epsilon_2) \).

Monomial potentials like

\[
V(\varphi) = \lambda \varphi^n
\]

with chaotic initial conditions give rise to models of the class II kind [35]. Using relations (4) it is found that for such potentials

\[
\epsilon_1(\varphi) \approx \frac{n^2}{2} \frac{1}{\varphi^2}.
\]

To convert from dependence on \( \varphi \) into dependence on the e-fold number, we use

\[
\frac{d\varphi}{dN} = \pm \sqrt{2\epsilon_1},
\]

\[
\frac{d\Delta}{dN} = \Delta^T \begin{pmatrix} 0 & 1 \\ \omega^2 & -\alpha(N) \end{pmatrix} \Delta \quad \text{with} \quad \Delta^T \equiv \left| \delta(N) \frac{d\delta(N)}{dN} \right|,
\]
which can be derived using definition (3) for $\epsilon_1$ and the Friedmann equation (2) for the inflaton cosmology. After doing the conversion we obtain

$$\epsilon_1 \approx \frac{n}{4} (N_f - N), \quad \epsilon_2 = \epsilon_3 = \frac{4}{n} \epsilon_1.$$  

So, the class criterion is met for $n \geq 2$. $N_f$ is the number of e-folds by the end of inflation.

Substituting the above results into (16) and (17) the following expressions are obtained:

$$\alpha(N) = \frac{n C_1 + 8 C_2}{4 C_3 (N_f - N)} + \frac{C}{C_3},$$  

$$f(N) = -\frac{n C_1 + 4 C_2 + 4 C_3}{4 C_3 (N_f - N)^2}.$$  

Next, we derive the analogous expressions for class III (toward-exit inflation with special initial conditions). Typical for this class are models where the inflationary phase starts near a false vacuum. An example arising in superstring theory is [36]

$$V = V_0 - \frac{m^2 \varphi^2}{2}.$$  

Here, according to (4), in the slow-roll phase we have

$$\epsilon_1(\varphi) \approx \frac{1}{2} \frac{m^4 \varphi^2}{(V_0 - (m^2 \varphi^2/2))^2}, \quad \epsilon_2 \approx 4 \frac{V_0}{m^2 \varphi^2} \epsilon_1.$$  

Such models exit inflation gracefully when reaching $\epsilon_1(N_f) = 1$ so, for simplicity, we set the condition $\varphi(N_f) = \sqrt{2} V_0/m^2$ (implying $V_0 \leq 1$) and use (21) to obtain

$$\epsilon_1 \approx \exp \left[ -\frac{2m^2}{V_0} (N_f - N) \right], \quad \epsilon_2 = \frac{2m^2}{V_0}, \quad \epsilon_3 = 0.$$  

Then, the corresponding damping (16) and forcing (17) are given by

$$\alpha(N) = \frac{C_1}{C_3} \exp \left[ -\frac{2m^2}{V_0} (N_f - N) \right] + \frac{4C_2}{C_3} \frac{2m^2}{V_0} + \frac{C}{C_3},$$  

$$f(N) = -\frac{2C_1}{C_3} \frac{m^2}{V_0} \exp \left[ -\frac{2m^2}{V_0} (N_f - N) \right] - \frac{4C_2}{C_3} \frac{m^4}{V_0^2}.$$  

Finally, typical examples from class I (hidden-exit inflation) are hybrid models where cosmic acceleration is driven by the inflaton and, in order to end inflation, there is yet another field which finally drives the vacuum energy $V_0$ to zero. Our example is the hybrid model with effective potential [37]

$$V = V_0 + \frac{m^2 \varphi^2}{2}.$$  

In this case from (4) we obtain

$$\epsilon_1(\varphi) \approx \frac{1}{2} \frac{m^4 \varphi^2}{(V_0 + (m^2 \varphi^2/2))^2}, \quad \epsilon_2 \approx -4 \frac{V_0}{m^2 \varphi^2} \epsilon_1.$$
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Figure 1. Plot of the solutions of equation (14) for the chaotic model $V(\varphi) = \lambda \varphi^4/4$. Each curve corresponds to a different solution starting at $N_i = 0$ (blue) and running towards $N_f = 100$ (red).

Similarly to in the previous example, but now in the opposite regime, to simplify the calculations we assume the initial conditions for the inflationary era\(^3\) $\epsilon_1(N_i) = 1$, $\varphi(N_i) = \sqrt{2}V_0/m^2$, which after using (21) leads to

$$\epsilon_1 \approx \exp \left[ -\frac{2m^2}{V_0} (N - N_i) \right], \quad \epsilon_2 = -\frac{2m^2}{V_0}, \quad \epsilon_3 = 0.$$  

The corresponding damping and forcing are then given by

$$\alpha(N) = \frac{C_1}{C_3} \exp \left[ -\frac{2m^2}{V_0} (N - N_i) \right] - \frac{4C_2}{C_3} \frac{2m^2}{V_0} + \frac{C}{C_3}, \quad (26)$$

$$f(N) = \frac{2C_1}{C_3} \frac{m^2}{V_0} \exp \left[ -\frac{2m^2}{V_0} (N - N_i) \right] - \frac{4C_2}{C_3} \frac{m^4}{V_0^2}. \quad (27)$$

After substituting in expressions (22) and (23), and putting in the numbers (10), one can look for numerical solutions to equation (14). In figure 1 we show the phase space for a potential given by equation (20) with $n = 4$. Note that the value of $n$ does not modify the qualitative behaviour of these solutions, while the value of $\lambda$ does not play any role at all. Hence, we can conclude that for such models the solution obtained to second order in the horizon-flow functions will typically depart from the lowest order solutions.

Analogously, it can be shown that the phase diagrams with the damping and forcing given correspondingly by (24) and (25), or by (26) and (27), are topologically equivalent\(^3\) There are uncountable other choices that will lead to similar results and we should expect our analysis to be independent of the initial conditions for inflation.
to that in figure 1. Therefore, we have confirmed the result of the previous section about solutions to equation (14) diverging around a saddle point near the origin, as the generic dynamical pattern for those cases which are interesting from the point of view of inflationary cosmology.

5. Discussion

The rigorous conclusion to be drawn from the results presented in this paper is that solutions of the inverse problem for the inflaton potential obtained using expressions up to second order in the slow-roll approximation will be generically very different from those solutions obtained using lower order expressions.

It may happen that including higher orders helps to get rid of this problem. However, the order of the expressions used to predict the inflationary spectra is determined by the accuracy of the data from the observations of the CMBR anisotropies and large-scale matter distribution. As has been pointed out, second-order expressions will indeed match the precision of current and near future measurements [9,5]. In other words, in the expressions for the amplitudes of the inflationary spectra, horizon-flow functions with orders higher than 2 would not contain any useful information. Therefore, including such orders would be irrelevant for the output of the inverse problem.

Note now that equation (14) has the trivial solution $\delta(N) = 0$ only for the cases of de Sitter (where $\epsilon_m = 0$ for $m > 0$) and power law inflation [38]. In this last model $a \propto t^p$ (with a constant $p \gg 1$), implying $d_H \propto \exp(N/p)$, $\epsilon_1 = 1/p$, $\epsilon_m = 0$ for $m > 1$ and $V(\varphi) = \Lambda \exp(B\varphi)$. Here $B$ is a real constant which is zero for a universe dominated by a cosmological constant. Therefore, $\delta(N)$ actually measures the departure from an exponential of the functional form of the output potential. Power law inflation is the unique model which predicts exact power laws for the amplitudes of both the tensor and scalar spectra [39], i.e., $A(k) = A_* k^n$, where $k$ is the comoving wavenumber and $n$ is a constant called the spectral index. Moreover, during power law inflation the scalar index $n_S$ is equal to the tensor $n_T$, which implies a constant ratio of tensor to scalar amplitudes, $r$. Conversely, in [30] it was proved that using power law spectra as input in the inverse problem renders the exponential potential as the unique output. From this point of view, $\delta(N)$ measures the departure of the behaviour of the inflationary spectra from a power law.

With regard to the relation between the order of the slow-roll expansion and the scale dependence of the spectra, we recall that the primordial spectra are usually parametrized as Taylor series where, for instance, the coefficient of the third term corresponds to the running of the spectral index. Comparing approximated and numerical predictions for the three given models, in [9] it was noted that increasing the order of the horizon-flow functions beyond the quadratic in every term included in the Taylor parametrization does not significantly improve the accuracy of the approximations. In fact, adding terms in that parametrization actually seems to be more important for increasing the precision of the predictions. However, the number of terms in the parametrization depends on the capability of the observations to measure a significant departure from scale invariance in the primordial spectra. It is also worth recalling that using first-order expressions for $r$ and $n_S$, Dodelson et al were able to discriminate among inflationary models with decreasing and increasing kinetic energy density by distributing them in the $(r, n_S)$ plane [24]. As
was noted in [9], including higher order corrections makes it impossible to distinguish between these models on the basis of a $r-n_S$ plot, unless the running of the spectral index is known or constrained to be very small (see also the results in [27]).

We can thus conclude that the essence of the problems for observing the inflaton potential is the weak signal in the CMBR of the scale dependence of the primordial spectra and the difficulties of extracting that information from data using expressions for the inflationary perturbations based on the slow-roll approximation. At this point we nevertheless must mention that according to recent results obtained by Starobinsky [40], exact scale invariance (and hence weak scale dependence) may not even be a sufficient condition for the convergence of the solutions to the inflaton potential inverse problem.

Limitations from theoretical and observational sides to account for the scale dependence of the primordial spectra have been misunderstood as generic predictions of inflation. That this scale dependence is at the edge of observational capabilities does not mean that all the successful inflaton potentials must resemble an exponential during the inflationary era, just that we are unable so far to observe the difference. The findings by Hoffman and Turner [41] concerning power law inflation as an $\alpha$-attractor (attractor in the future) of the flow in the space of inflationary models were constrained by the order of the expressions in their analysis. As was shown in [42], departure from power law behaviour is nullified whenever the leading order is used. To first order, for instance, it remains true that the inflationary dynamics yielding a constant tensor index $n_T$ has power law inflation as an $\alpha$-attractor [30]. Nevertheless, it is also true that the inflationary dynamics characterized by a constant ratio $r$ has power law inflation as an $\omega$-attractor (attractor in the past or repellor) [31]. Furthermore, in [32] it was shown how a minimal scale dependence for $r$ (and information on such a scale dependence can be drawn from the difference between the two spectral indices) allows for models with power law inflation as a transient regime. On the basis of these facts, we conjecture here that, if when one uses higher orders in the slow-roll expansion for the analysis of the inflationary flow, power law inflation still arises as a fixed point, then it will do it like a saddle point, consistently with the results presented in this paper.

Summarizing, either the slow-roll expansion does not converge or it does it so slowly that a large number of terms in this expansion are required for probing the scale dependence of the primordial spectra. Nevertheless, information on this dependence seems to be a necessary (but perhaps not sufficient) ingredient for a robust solution to the inflaton potential inverse problem. This conclusion should encourage further development of the fully numerical method for the resolution of this inverse problem [21], as well as other analytical methods, for instance, those based on the WKB [11] approximation, on the uniform approximation [12] and on the inverse scattering theory [25].

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