Non-polynomial realizations of $\mathcal{W}$-algebras

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Abstract

Relaxing first-class constraint conditions in the usual Drinfeld-Sokolov Hamiltonian reduction leads, after symmetry fixing, to realizations of $\mathcal{W}$ algebras expressed in terms of all the $J$-current components.

General results are given for $\mathcal{G}$ a non exceptional simple (finite and affine) algebra. Such calculations directly provide the commutant, in the (closure of) $\mathcal{G}$ enveloping algebra, of the nilpotent subalgebra $\mathcal{G}_-$, where the subscript refers to the chosen gradation in $\mathcal{G}$. In the affine case, explicit expressions are presented for the Virasoro, $\mathcal{W}_3$, and Bershadsky algebras at the quantum level.

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1 Introduction

In a recent paper \[1\], the construction of a particular class of finite \(W\) algebras has been carried out in a way slightly different from the usual Hamiltonian reduction (H.R.). The method consists in following the same steps as the ones considered in the canonical H.R., except that no more constraints are imposed on the current components. Let us be more explicit by reviewing rapidly these technics.

As widely used these recent years \[2\], the determination of a \(W\) algebra through H.R. first demands to decompose the simple Lie algebra \(G\) under interest into its graded parts relative to an \(\mathfrak{s}\ell(2)\) embedding in \(G\):

\[
G = G_- \oplus G_0 \oplus G_+ \quad (1.1)
\]

The general element of \(G\) will be denoted

\[
J = J^a t_a = J^\alpha t_\alpha + J^i t_i + J^{\bar{\alpha}} t_{\bar{\alpha}} \quad (1.2)
\]

with the \(t_\alpha\)'s and \(t_{\bar{\alpha}}\)'s \((\alpha, \bar{\alpha} = 1, \ldots, \dim G_\pm = \dim G_-)\) generating \(G_+\) and \(G_-\) respectively, while the \(t_i\)'s \((i = 1, \ldots, \dim G_0)\) form a basis of \(G_0\). The \(J^a\) components generate the dual basis in \(G^*\), on which can be naturally set a Poisson Kirillov Lie algebra structure \(\{., .\}_{PB}\). Then one imposes the \(J^\alpha\) components to take constant fixed (non zero and zero) real values in such a way that these constraints constitute a set of (Dirac) first class constraints. This means that the Poisson bracket of any couple of them weakly commute, or in other words, the result of their Poisson commutator vanishes when imposing the constraint conditions. These first class constraints on \(G^*_-\) induce a gauge invariance on the \(J^a\)'s which can be reformulated on the constrained matrix

\[
J = t_- + J^i t_i + J^\alpha t_\alpha \quad (1.3)
\]

as follows:

\[
J \rightarrow J^g = g J g^{-1} + k \partial g . g^{-1} \quad (1.4)
\]

with \(g\) element of the \(G\) subgroup \(G_+\), the Lie algebra of which is \(G_+\). In the above equation, we have considered the \(J^a\)'s as functions of a \(z\)-variable, \(z \in \mathbb{C}\) and \(\partial \equiv \frac{\partial}{\partial z}\). In the case where the \(J^\alpha\)'s are constant, the gauge transformation reduces naturally to a conjugation.

By a suitable gauge fixation, one can transform \(J\) into:

\[
J^g = t_- + W^k t_k \quad (1.5)
\]

with the \(W^k\)'s polynomials in the \(J^i, J^\alpha\) and their derivatives, and the \(t_k\)'s constituting a \(G\)-subset of \(\dim G_0\) generators. The \(W^k\)'s are gauge invariant; they weakly P.B. commute with the constraints and form a basis of the \(W\) algebra one wishes to construct. Such a \(W\) algebra will be a finite \(W\) algebra \[3\] when the \(J^\alpha\)'s are \(z\) independent: in this last case the \(W\) generators can be related to the zero modes of the \(z\)-dependent \(W^k\) ones.

\[^1\]More precisely, with respect to the Cartan generator \(H\) of an \(\mathfrak{s}\ell(2)\)-subalgebra, \(G\) decomposes as:

\[
G = \oplus_{i=h}^{\infty} G_i \quad \text{with} \quad [H, X_i] = i X_i \quad \forall X_i \in G_i
\]

We will denote by \(G_{\pm}\) the subalgebras \(G_+ = \oplus_{i>0} G_i\) and \(G_- = \oplus_{i<0} G_i\).
As emphasized in [1], if one ignores the constraints on $G^*$ and acts with the $G_+$ subgroup as in (1.4) on the unconstrained $J$ given in (1.2), one will then get by a symmetry fixing $\tilde{W}$ quantities which strongly Poisson commute with the $J^\alpha$’s. In [1], we have focussed our attention on finite $W$ algebras -as above mentioned, the coadjoint action (1.4) is then identical to the adjoint one- and also to the case where $G_-$ (resp. $G_+$) is Abelian. The so obtained $\tilde{W}$ show up as functions $P(J^a)/Q(J^\alpha)$ with $P$ a polynomial in all the $J^a$’s and $Q$ a smooth function in the $J^\alpha$’s of $G^*_-$ only. Since $G_-$ is Abelian and the $\tilde{W}$ strongly Poisson commute with $G^*_-$, the $P(J^a)$ are in the commutant of $G^*_-$ only. However, one must remark that the P.B. of two $\tilde{W}$ a priori closes on a function of the form: $P(\tilde{W}, J^\alpha)/Q(J^\alpha)$. That is the price one has to pay to get a realisation of the $\tilde{W}$ generators in terms of all the $J^a$ components. Let us add that by a direct quantization, one thus obtains the commutant $\text{Com}(G_-)$ of the subalgebra $G_-$ in the closure $U(G)$ of the $G$-enveloping algebra $U(G)$. By closure we mean that we will allow formal series in the generators of $G_-; G_+$ instead of only polynomials. Then, fractions and square roots (for instance) will be in this closure.

In addition to the new obtained realization of a $W$ algebra, and the determination of the commutant in $U(G)$ of a nilpotent subalgebra $G_-$, there is another interesting consequence of the relaxing of the constraints. Indeed, let us suppose we have a given realization $(R)$ of the algebra $G$ in which the $G_-$ generators are represented by -commuting- variables and the other ones in $G \setminus G_-$ by differential operators in these variables, or coordinates. Then the knowledge of the $\text{Com}(G_-)$ elements, with their explicit expressions in terms of the $G$ generators, allows to deduce from the canonical representation $(R)$ new realizations of $G$. Such an approach has been exploited in [1] on the $Sp(2d, \mathbb{R})$ algebras in order to reformulate the Heisenberg quantization for a system of two identical particles in $d = 1$ and 2 dimensions, and to recognize also in $d = 2$ a suitable anyonic operator.

In the present paper, we pursue our investigations and calculations on any classical simple Lie algebra $G$, with a graded decomposition relative to any of its $s\ell(2)$ embeddings. We widely develop the finite case, before considering, in the last section, the affine one with explicit calculations on $G = s\ell(2)$ and $s\ell(3)$.

The first technical problem we are faced with concerns a suitable symmetry fixing. Indeed the usual Drinfeld-Sokolov highest weight gauge does not appear adapted in the general case if we do not impose constraints. We first present in section 2 our symmetry fixing for $G = s\ell(n, \mathbb{R})$ and $G_-$ given by the gradation of the principal $s\ell(2, \mathbb{R})$, leaving for an appendix the treatment of an orthogonal or symplectic Lie algebra $G$ submitted to the principal gradation. Then we generalize the symmetry fixing for any $s\ell(2)$ gradation of a classical simple Lie algebra (section 3). One can note that our presentation is facilitated by a graphical matrix description. The proof is rather detailed in the case of $s\ell(n)$ algebras, and we use the folding operation allowing to go from the unitary to the orthogonal and symplectic algebras to study these last ones. Once again, the obtained $\tilde{W}$ elements are quotients $P(J^a)/Q(J^\alpha)$ with the $Q(J^\alpha)$ (PB-)commuting with the $G^*_-$ elements and also with the $P$’s. By recalling in section 3 the direct quantization procedure for finite $W$ algebras, we are naturally led to consider the commutants $\text{Com}(G_-)$ of the nilpotent subalgebras $G_-$.

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Actually, the $Q$’s belong to the center of the obtained commutant. Rather unexpected, the set of all the different $\text{Com}(\mathcal{G}^H)$, each relative to an $\text{sl}(2,\mathbb{R})$ itself principal in a regular subalgebra $\mathcal{H}$ of $\mathcal{G}$, does not provide all the different $\mathcal{W}(\mathcal{G}, \mathcal{H})$ algebras. As developed in the section 4, each $\text{Com}(\mathcal{G}^H)$ is isomorphic, up to a center part, to the $\mathcal{W}(\mathcal{G}, \mu A_1)$ algebra, where $\mu$ is the maximal number of $A_1$ subalgebras contained in the just defined $\mathcal{H}$ subalgebra. Following the type of considered $\mathcal{G}$ algebra, one has to separate the cases where the $A_1$’s are all $\text{sl}(2,\mathbb{R})$ - $\text{sl}(n)$ type- from the ones where some of them is an $\text{so}(3)$ - $\text{so}(n)$ type- or $\text{sp}(2,\mathbb{R})$ - $\text{sp}(2n)$ type- algebra. For the reader surprised by this result - i.e. that all the $\mathcal{W}(\mathcal{G}, \mathcal{H})$ are not obtained in this way-, one may refer to figure 1 of section 2 where we have tried to graphically represent our symmetry fixing in the $\mathcal{G} = \text{sl}(n)$ case: there, the nested squares in the $\mathcal{G}$ matrix characterize the commuting $\text{sl}(2,\mathbb{R})$ subalgebras, the corresponding matrix elements of which are lying at the corners of each square.

Now, one may wonder if there is a possibility to get from a given $\mathcal{G}$ the other $\mathcal{W}$ algebras which have not been obtained. This point is discussed in section 6 and illustrated in section 7 by a detailed study of the $\text{sl}(3)$ and $\text{sl}(4)$ algebras. We show on these examples that the not yet obtained $\mathcal{W}$ algebras can be reached either by enlarging the conjugation (1.4) to a $\mathcal{G}$ subalgebra bigger than $\mathcal{G}_+$, and no more nilpotent, or by carrying out a secondary reduction process [4, 5].

This last operation consists in applying the same technics of symmetry transformation, now with the help of $\mathcal{W}$ generators, on an already obtained $\mathcal{W}$ algebra.

Our symmetry fixing procedure can directly be extended to the affine case. This is remarked in section 8 where, after presenting the different steps of all calculation, we give our explicit realization of the quantum affine $\mathcal{W}$ algebras relative to $\text{sl}(2)$ (Virasoro case) and to $\text{sl}(3)$ ($\mathcal{W}_3$ and Bershadsky ones).

2 Symmetry fixing: the case of the maximal nilpotent subalgebra in $\text{sl}(n)$

As emphasized in the introduction, we start with the most general element\footnote{Strictly speaking, we take an $n \times n$ matrix whose entries on the lower part of the anti-diagonal are non-zero, i.e. $E_{i,n+1-i} \neq 0$ for $i = 1, ..., [n/2]$} of $\mathcal{G}$, as given in eq. (1.2). Limiting our study for the moment to $J^g$’s which do not depend on any $z$-variable, the coadjoint transformation given in (1.4) reduces to:

\[ J^g = gJg^{-1} \quad \text{with} \quad g \in G_+ \quad \text{and} \quad \text{Lie}G_+ = \mathcal{G}_+ \quad (2.1) \]

To give a simple idea of the symmetry fixing that we have chosen, we first consider the case $\mathcal{G} = \text{sl}(n,\mathbb{R})$ and $H$ its principal gradation. Then, $\mathcal{G}_+$ is the subalgebra generated by all the positive roots of $\text{sl}(n)$ (maximal nilpotent subalgebra of $\text{sl}(n)$). Considering the fundamental representation of $\text{sl}(n)$, the elements are $n \times n$ matrices, and $\mathcal{G}_+$ is the set of upper triangular matrices (with zeros on the diagonal).
2.1 Direct calculation

We propose to prove the following property

Property 1 Starting with a general element $J$ of $\mathfrak{sl}(n)$, one can in an unique way determine the parameters of the transformations associated to the maximal nilpotent subalgebra $G_+$ by requiring $J^g$ (transformed of $J$) to have zeros in its whole lower triangular part ($G_{\leq 0}$ part), except on the anti-diagonal.

Proof: We prove this property by a direct calculation, using the principal gradation $H$ of $\mathfrak{sl}(n)$. We first consider the root of lowest grade: it is the lowest root ($-\psi_0$) of $\mathfrak{sl}(n)$, and its grade is $-h = -(n-1)$. Then, we define $G_+^{(0)}$ as the subalgebra generated by the elements of $\mathfrak{g}_+$ that act on $E - \psi_0$:

$$G_+^{(0)} = \{X \in \mathfrak{g}_+ \text{ s.t. } [X, E - \psi_0] \neq 0\} \quad (2.2)$$

$G_+^{(0)}$ is a subalgebra of the symmetry group $G_+$ (see section 2.2). As a first step, we show that one can determine the parameters of the coadjoint transformations associated to $G_+^{(0)}$ by their action on $E - \psi_0$.

There is a natural decomposition of $G_+^{(0)}$ w.r.t. the principal gradation:

$$G_+^{(0)} = \oplus_{i>0} G_i^{(0)} \quad \text{with} \quad G_i^{(0)} = G_+^{(0)} \cap G_i$$

Now, if we denote by $G_1^{(0)} = [E - \psi_0, G_1^{(0)}]$, the definition of $G_+^{(0)}$ insures that $\dim G_1^{(0)} = \dim G_1^{(0)}$. Moreover, the fact that $-\psi_0$ is the root of lowest grade guarantees that the elements of $G_1^{(0)}$ are the only elements of $G_+^{(0)}$ that lead to $G_1^{(0)}$ by coadjoint action. Thus, the condition

$$J^g|_{G_1^{(0)}_{-h}} = 0$$

fixes in an unique way the parameters of $G_1^{(0)}$ through linear equations. This condition still hold after the action of the other generators of $G_+$.

Once $G_1^{(0)}$ is fixed, we consider the action of $G_2^{(0)}$. Defining $G_2^{(0)} = [E - \psi_0, G_2^{(0)}]$, it is clear that only the elements of $G_2^{(0)}$ will lead by coadjoint action to $G_2^{(0)}$ (the elements of $G_1^{(0)}$ have not to be considered now). Then, as above, we will fix linearly (and in an unique way) the parameters of $G_2^{(0)}$ through the condition

$$J^g|_{G_2^{(0)}_{-h}} = 0$$

Recursively, we will fix the parameters of $G_i^{(0)}$ through the condition

$$J^g|_{G_i^{(0)}_{-h}} = 0 \quad \text{where} \quad G_i^{(0)}_{i-h} = [E - \psi_0, G_i^{(0)}] \quad i = 1, \ldots, n - 1 \quad (2.3)$$

Once the parameters associated to $G_j^{(0)}$ ($j < i$) are fixed, the condition (2.3) fixes in an unique way (and linearly) the parameters of $G_i^{(0)}$. 
Thus, we have fixed all the parameters of $G_+^{(0)}$ through the condition
\[ J^g|_{G_{\leq 0}^{(0)}} = 0 \quad \text{where} \quad G_{\leq 0}^{(0)} = [E_{-\psi_0}, G_+^{(0)}] \] (2.4)

As we get linear equations, the parameters will be fractions. However, due to the definition of $G_+^{(0)}$, the denominators will be built on $J^{-\psi_0}$ only, and we have supposed that it is not zero.

Considering the $n \times n$ matrix of $s\ell(n)$, it is not difficult to see that $E_{-\psi_0}$ is in the lower-left corner of the matrix, while $G_+^{(0)}$ constitute the "edge" of $G_+$, i.e. elements of the form $E_{1,j}$ and $E_{i,n}$. $G_{\leq 0}^{(0)}$ is almost the edge of $G_-$: if $H_{\psi_0}$ is the Cartan generator based on $\psi_0$, $G_{\leq 0}^{(0)}$ is the edge of $G_-$ with $E_{-\psi_0}$ removed, but $H_{\psi_0}$ included.

Altogether, if we gather $G_+^{(0)}$, $G_{\leq 0}^{(0)}$ and $E_{-\psi_0}$ we obtain the whole border of the $n \times n$ matrix. Then, we are left with a $(n-2) \times (n-2)$ matrix (that stands in the middle of the original matrix) and it should be clear that the second step will deal with this submatrix. Once again, we determine the root of lowest weight ($-\psi_1$) in the submatrix: by construction of $G_+^{(0)}$, $E_{-\psi_0}$ is in the center of $(G_+ \setminus G_+^{(0)})$, so that at the second step $E_{-\psi_1}$ really plays the role of $E_{-\psi_0}$ at the first step. Then we look at the subalgebra of elements that act on $E_{-\psi_1}$:

\[ G_+^{(1)} = \{X \in (G_+ \setminus G_+^{(0)}) \text{ s.t. } [X, E_{-\psi_1}] \neq 0\} \] (2.5)

and we will fix the associated parameters through the condition:
\[ J^g|_{G_{\leq 0}^{(1)}} = 0 \quad \text{where} \quad G_{\leq 0}^{(1)} = [E_{-\psi_1}, G_+^{(1)}] \] (2.6)

With the same decomposition w.r.t. the principal gradation $(G_+^{(1)} = \bigoplus_{i>0} G_i^{(1)}$ and $G_i^{(1)} = G_+^{(1)} \cap G_i)$, we get, grade by grade, linear equations that fix in an unique way the parameters of $G_+^{(1)}$. As in the first step, the denominators of the fractions appearing in the fields are built only on $J^{-\psi_0}$ and $(J^{-\psi_1})^g$, where $(J^{-\psi_1})^g$ is the transformed of $J^{-\psi_1}$ under $G_+^{(0)}$. Thus, the fields are well-defined as soon as $J^{-\psi_0}$ and $(J^{-\psi_1})^g$ are not zero.

Once again, we have studied the border of the $(n-2) \times (n-2)$ submatrix, and we are left with an $(n-4) \times (n-4)$ submatrix that stands in the middle of the $(n-2) \times (n-2)$ matrix. Of course, we continue until we exhaust all $G_+$. This ends the proof of the property.

\[ \square \]

To visualise the procedure, let us remark that we have divided the $n \times n$ matrix $J$ into "nested boxes" as drawn in picture [I]. In the lower part (below the diagonal) of the resulting matrix $J^g$ we have only zeros, except where stand the different roots ($-\psi_j$), that is in the lower-left corner of each "box". In these "corners" and in the upper part, we have fractions whose denominators are well-defined as soon as $J^{-\psi_i} \neq 0$.

The generalization is quite easy when one realizes that the structure of "nested boxes" really reflects a series of mutually embedded subalgebras. To be pedagogical, we first rephrase the above calculation.
2.2 Formal calculation

Thus, we come back to the root of lowest grade \((-\psi_0)\). The first thing to remark is that we have
\[ G_0(-\psi_0) \subset \text{Im}(adE_{-\psi_0}) \quad \text{and} \quad G^0(+\psi_0) \subset \text{Im}(adE_{+\psi_0}) \]
(2.7)
\[ \text{Im}(adE_{-\psi_0}) \cap \text{Im}(adE_{+\psi_0}) = \text{Vect}(H_{\psi_0}) \]
where \(\text{Vect}(H_{\psi_0})\) is the vector space spanned by \(H_{\psi_0}\). This implies that the "box" formed by \(G_0(-\psi_0), G^0(+\psi_0),\) and \(E_{-\psi_0}\) is exactly
\[ B_0 = \text{Im}(adE_{-\psi_0}) + \text{Im}(adE_{+\psi_0}) \]
(2.8)
In fact, we have
\[ B_0 = G_0^0(-\psi_0) \oplus G^{00}(\pm 0) \oplus \text{Vect}(E_{-\psi_0}) \]
Then, it is a simple exercise to prove that the inside of this box \(B_0\) \((\text{i.e. the } (n - 2) \times (n - 2) \text{ submatrix})\) is the subalgebra \(S_0\) defined by:
\[ S_0 = \text{Ker}(adE_{\psi_0}) \cap \text{Ker}(adE_{-\psi_0}) \]
As vector space, it satisfies:
\[ \mathcal{G} = B_0 \oplus S_0 \]
Moreover, we have the basic property:
\[ G^0(+) = \text{Im}(adE_{\psi_0}) \cap \text{Ker}(adE_{+\psi_0}) \]
which can be taken as a definition for \(G^0(+\psi_0)\). Note that this definition immediately proves that \(G^0(+\psi_0)\) is a subalgebra (since \(\text{Im}(adX) \cap \text{Ker}(adX)\) is always a subalgebra for any \(X \in \mathcal{G}\)).
Finally, we define recursively the submatrices at the level \(i\) by
\[ S_{-1} = \mathcal{G} \quad ; \quad S_i = \bigcap_{j=0}^{i} \left[ \text{Ker}(adE_{\psi_j}) \cap \text{Ker}(adE_{-\psi_j}) \right] \quad i \geq 0 \]
(2.10)
Indeed, if we compute the root of lowest grade \( -\psi_i \), we will define the symmetry group at the level \( i \) by

\[
G_+^{(i)} = [\text{Im}(adE_{\psi_i}) \cap \text{Ker}(adE_{\psi_i})] \cap S_{i-1}
\]

and its fixation by

\[
J^g|_{G_{\leq 0}^{(i)}} = 0 \quad \text{with} \quad G_{\leq 0}^{(i)} = [E_{-\psi_i}, G_+^{(i)}] \cap S_{i-1}
\]

Therefore, after this fixing of the parameters of \( G_+^{(i)} \), we will have to work on

\[
S_{i-1} \cap [\text{Ker}(adE_{\psi_i}) \cap \text{Ker}(adE_{-\psi_i})] \equiv S_i
\]

while the "box"

\[
B_i = G_+^{(i)} \oplus G_{\leq 0}^{(i)} \oplus \text{Vect}(E_{-\psi_i})
\]

has been completely treated. We have also the property

\[
S_{i-1} = B_i \oplus S_i
\]

Although the definitions (2.10), (2.11) and (2.12) are more formal than the previously studied, they insure that at each step the "box" is well-defined and is a subalgebra. They also prove that at each step \( i \) we have subalgebras \( G_+^{(i)} \). Moreover, these definitions allow a straightforward generalization as we see in the section \( \text{S} \).

3 Symmetry fixing: the \( G_- \) general case in a simple Lie algebra

Let us consider now the general case of a classical simple Lie algebra \( G \). We grade \( G \) w.r.t. a Cartan generator \( H \), itself defined through a \( \text{sl}(2) \) subalgebra. We recall that, up to few exceptions (which we discard) and that occur for \( G = \text{so}(2m) \), this \( \text{sl}(2) \) subalgebra can be considered as principal in a regular \( G \)-subalgebra \( \mathcal{H} \). The nilpotent subalgebra \( \mathcal{N} \) whose commutant we are looking for, is defined with the help of this gradation through \( \mathcal{N} = G_- \).

The technics to compute the commutant of \( \mathcal{N} \) in \( \mathcal{U} \) will be the same as in the previous case: we prove that one can find recursively a form for \( J^g \) that fixes in an unique way and linearly the parameters of the transformation associated to \( G_+ \). As the procedure is very similar to the one given in section 2.2, we just mention the different steps that allow the fixing through some lemmas and properties.

We start as in the previous case by considering the roots of highest grade \( \psi_p \) (so that \( -\psi_p \) is lowest grade). The difference here is that there can exist more than one such root at each

\[\text{Be careful that the grading operator has not changed from the very beginning: it is still the Cartan generator of the principal } \text{sl}(2).\]
step. However, generalizing the definition (2.9), we consider the subgroup associated to

\[ G_+^{(0)} = \left( \bigcap_{p=1}^{m} \ker(adE_{\psi_p}^{(0)}) \right) \cap \left( \bigcup_{p=1}^{m} \text{Im}(adE_{\psi_p}^{(0)}) \right) \cap G_+ \] (3.1)

It is quite easy to see that \( G_+^{(0)} \) is indeed a subalgebra of \( G_+ \) (using the fact that each \( \psi_p \) is highest grade). One can also prove:

**Property 2** Let \( G \) be a semi-simple Lie algebra, graded w.r.t. a \( \mathfrak{sl}(2) \)-Cartan generator, and \( \mathcal{H} \) the regular subalgebra defining this \( \mathfrak{sl}(2) \) embedding. We introduce \( \Lambda = \{ \psi_p, \ p = 1, \ldots, m \} \) the set of roots of highest grade, and

\[ \Lambda = \{ \psi \in \Lambda \text{ such that } E_\psi \in \mathcal{H} \} = \{ \psi_a, \ a = 1, \ldots, m \} \]

Then, the element

\[ E_{-\varphi_0} = \sum_{a=1}^{m} E_{-\psi_a} \]

is such that

\[ \ker(adE_{-\varphi_0}) = \bigcap_{p=1}^{m} \ker(adE_{-\psi_p}) \quad \text{and} \quad \text{Im}(adE_{-\varphi_0}) = \bigcup_{p=1}^{m} \text{Im}(adE_{-\psi_p}) \] (3.2)

**Proof:** We will prove this property for \( \mathcal{G}=\mathfrak{sl}(n) \), and extend it to \( \mathcal{G} = \mathfrak{so}(m) \) and \( \mathfrak{sp}(2n) \) by folding. We start by remarking that we have the following inclusions

\[ V_- \subset \left[ \bigcap_{p=1}^{m} \ker(adE_{-\psi_p}) \right] \subset \ker(adE_{-\varphi_0}) \]

\[ \text{Im}(adE_{-\varphi_0}) \subset \left[ \bigcup_{p=1}^{m} \text{Im}(adE_{-\psi_p}) \right] \] (3.3)

where we have introduced \( V_\pm = \text{Vect}(E_{\pm\psi_p}, \ p = 1, \ldots, m) \) the vector space (commutative \( G \)-subalgebra) generated by the root generators \( E_{\pm\psi_p} \).

Moreover, it is easy to see that it is enough to prove one of the equalities in (3.2), since we have:

\[ \ker(adE_{-\varphi_0}) = \bigcap_{p=1}^{m} \ker(adE_{-\psi_p}) \Leftrightarrow \text{Im}(adE_{+\varphi_0}) = \bigcup_{p=1}^{m} \text{Im}(adE_{+\psi_p}) \]

\[ \Leftrightarrow \ker(adE_{+\varphi_0}) = \bigcap_{p=1}^{m} \ker(adE_{+\psi_p}) \Leftrightarrow \text{Im}(adE_{-\varphi_0}) = \bigcup_{p=1}^{m} \text{Im}(adE_{-\psi_p}) \] (3.4)

We prove the property by constructing an element \( E_{-\varphi_0} \) such that \( \text{Im}(adE_{-\varphi_0}) = \bigcup_{p=1}^{m} \text{Im}(adE_{-\psi_p}) \)
We take \( \mathcal{G} = \mathfrak{sl}(n) \) and use a graphical method. First, we note that if \( e_{ij} \) is the element \((i, j)\) in a \( n \times n \) matrix, then \( \text{Im}(\text{ad}e_{ij}) \) is formed by the \( i^{th} \) row and the \( j^{th} \) column. Moreover, if the subalgebra \( \mathcal{H} \) which defines the grading is decomposed as

\[
\mathcal{H} = \bigoplus_{j=1}^{d} \alpha_j \mathfrak{sl}(p_j) \quad \text{with} \quad p_1 > p_2 > \ldots > p_d > 1
\]

there will be more than one root of highest grade if and only if \( \alpha_1 \geq 2 \). Thus, for the demonstration of the property \( \mathbb{2} \) we suppose that \( \alpha_1 \geq 2 \).

Then, we decompose the set of roots of lowest grade \( \Lambda \) into two orthogonal subsets:

\[
\Lambda_{\parallel} = \{ \psi \in \Lambda \text{ such that } E_{-\psi} \in \mathcal{H} \} \quad \text{and} \quad \Lambda_{\perp} = \{ \psi \in \Lambda \text{ such that } E_{-\psi} \notin \mathcal{H} \}
\]

with of course \( m_{\parallel} + m_{\perp} = m \). As a notation, we use \( p, q, r, \ldots \) for the indices of all the roots of highest grade, and \( a, b, c, \ldots \) (\( \bar{a}, \bar{b}, \bar{c}, \ldots \) resp.) to label the roots that belong to \( \Lambda_{\parallel} \) (resp. \( \Lambda_{\perp} \)). We want to show that the generator \( E_{-\psi_{\bar{a}}} = \sum_{a=1}^{m_{\parallel}} E_{-\psi_a} \) obeys to the property \( \mathbb{2} \).

Indeed, it is not difficult to see that any element \( E_{-\psi_{\bar{a}}} \) stands at the crossing of a row belonging to a set \( \text{Im}(\text{ad}E_{-\psi_a}) \), with a column belonging to a set \( \text{Im}(\text{ad}E_{-\psi_{\bar{a}}}) \), as it is drawn in the figure \( \mathbb{3} \).

\[
\forall \psi_{\bar{a}} \in \Lambda_{\perp}, \exists \{ \psi_a, \psi_b \} \in \Lambda_{\parallel} \text{ such that } E_{-\psi_{\bar{a}}} \in \text{Im}(\text{ad}E_{-\psi_a}) \cap \text{Im}(\text{ad}E_{-\psi_b})
\]

Mathematically, this means

\[
\forall \psi_{\bar{a}} \in \Lambda_{\perp}, \exists \{ \psi_a, \psi_b \} \in \Lambda_{\parallel} \text{ such that } E_{-\psi_{\bar{a}}} \in \text{Im}(\text{ad}E_{-\psi_a}) \cap \text{Im}(\text{ad}E_{-\psi_b})
\]

Since \( \text{Im}(\text{ad}E_{-\psi_a}) \) is formed by two lines (as described above), this immediately implies that this last set is also part of \( \text{Im}(\text{ad}E_{-\psi_a}) \cup \text{Im}(\text{ad}E_{-\psi_{\bar{a}}}) \), so that

\[
\bigcup_{p=1}^{m} \text{Im}(\text{ad}E_{-\psi_p}) = \bigcup_{a=1}^{m_{\parallel}} \text{Im}(\text{ad}E_{-\psi_a})
\]

Figure 2: Location of \( \psi_a \) and \( \psi_{\bar{a}} \)
Now, as the sum in (3.5) is direct, it is clear that

$$\text{Im}(adE_{-\varphi}) = \bigoplus_{a=1}^{m_{||}} \text{Im}(adE_{-\psi_a})$$

(3.9)

which ends the proof for \(s\ell(n)\).

For \(so(m)\) and \(sp(2m)\), the proof is essentially the same: the only difference relies on the fact that root generators have more than one non-zero entries in the matrices of the fundamental representation. Indeed, we begin with the matrix realization that we obtain from the folding of \(s\ell(n)\) matrices (see appendix). The matrices of \(so(m)\) and \(sp(2m)\) are (graded) symmetric w.r.t. the anti-diagonal. Moreover, up to few exceptional embeddings that we discard in the case \(so(2m)\), any \(s\ell(2)\) embedding in \(so(m)\) or \(sp(2m)\) can be viewed as the principal embedding of the regular subalgebra \(H\). This subalgebra itself can always be taken of the form \(\epsilon so(p) \oplus_i s\ell(p_i)\) for \(so(m)\) and \(\epsilon sp(p) \oplus_i s\ell(p_i)\) for \(sp(2m)\) with \(\epsilon = 0\) or 1. Then, \(H\) can be considered as resulting from the folding of \(\epsilon s\ell(p) \oplus_i 2 s\ell(p_i)\) in \(s\ell(n)\), so that we can use the graphical method of \(s\ell(n)\) together with the folding.

Thus, for a root generator \(E_{-\psi}\), \(\text{Im}(adE_{-\psi})\) will be included in two lines and two columns. With a careful identification of the elements of these lines (and columns) that are not in \(\text{Im}(adE_{-\psi})\), one can do the same calculation as above (using the proof made for \(s\ell(n)\), thanks to the folding property of classical \(W\)-algebras), so that the proof for \(so(m)\) and \(sp(2n)\) can be carried on.

\[ \square \]

Strictly speaking, the above calculation is valid for the first step only (with the same notations as above), \(i.e.\) when considering the full \(G\) algebra. For the other steps \(i\), where we deal with \(S_{i-1} = \text{Ker}(adE_{-\varphi_{i-1}}) \cap \text{Ker}(adE_{+\varphi_{i-1}})\) instead of \(G\), \(H\) is not a subalgebra of \(S_{i-1}\). However, one has just to consider the restriction \(H^{(i)}\) of \(H\) to the \(G\)-subalgebra \(S_{i-1}\), \(H^{(i)} = H \cap S_{i-1}\), as well as the space \(\Lambda^{(i)}\) of roots of lowest grade in \(S_{i-1}\). Then, decomposing \(H^{(i)}\) into \(s\ell(2)\)-representations, \(H^{(i)} = \bigoplus_{j=1}^{d} \alpha_j' s\ell(p_j')\), we will get more than one root of highest grade if and only if \(\alpha_1' \geq 2\). In that case, we form the subset of \(\Lambda^{(i)}\) associated to \(H^{(i)}\):

\[ \Lambda^{(i)} = \{ \psi^{(i)} \in \Lambda^{(i)} \text{ such that } E_{-\psi^{(i)}} \in H^{(i)} \} = \{ \psi^{(i)}_a, a = 1, \ldots, m^{(i)}_{||} \} \]

and the following generator

\[ E_{-\varphi_1} = \sum_{a=1}^{m^{(i)}_{||}} E_{-\psi^{(i)}_a} \]

will satisfy the property \(\square\) at the step \((i)\).

With the help of property \(\square\) we can now determine a fixing for the parameters of \(G^{(i)}_+\) (where \(i\) denotes as above the step we are considering). At each step, we work on the subalgebra \(S_{i-1}\) of \(G\). In this subalgebra, we fix the parameters of the transformations associated to \(G^{(i)}_+\), defined as the \(G_+\)-subalgebra of generators acting on \(E_{-\varphi_1}\):

\[ G^{(i)}_+ = [ \text{Im}(adE_{\varphi_1}) \cap \text{Ker}(adE_{\varphi_1}) ] \cap S_{i-1} \]

(3.10)
As above, we introduce $G^{(i)}_{\leq 0} = [E_{-\varphi_i}, G^{(i)}_+]$, and we fix the parameters through $J^g|G^{(i)}_{\leq 0} = 0$. It is easy to see that this condition fixes in an unique way all the parameters in $G^{(i)}_+$. Moreover, due to the property 2, we can show the following lemma:

**Lemma 1** Let $S_i = Ker(adE_{\varphi_i}) \cap Ker(adE_{-\varphi_i})$ with $i \geq 0$, and $S_{-1} = G$. Then, for any $i \geq 0$, we can decompose $S_{i-1}$ as:

$$S_{i-1} = S_i \oplus B_i \quad \text{with} \quad B_i = G^{(i)}_+ \oplus G^{(i)}_{\leq 0} \oplus V^{(i)}$$

(3.11)

where we have introduced the subalgebras

$$G^{(i)}_+ = [Im(adE_{\varphi_i}) \cap Ker(adE_{-\varphi_i})] \cap S_{i-1}, \quad G^{(i)}_{\leq 0} = [E_{-\varphi_i}, G^{(i)}_+]$$

$$V^{(i)}_\pm = \text{Vect}\{E_{\pm \psi_p^{(i)}}, \quad p = 1, \ldots, m\}$$

This lemma ensures that we can perform recursively the fixing of the parameters. In particular, as we have $V^{(i)}_\pm \subset B_i$, the roots of highest grade in $S_i$ will be different from the roots $\psi_p^{(i)}$ of $S_{i-1}$.

Hence, they will satisfy $gr(-\psi_p^{(i)}) < gr(-\psi_q^{(i+1)}) < 0$: this guarantees that the procedure will end up in a finite number of steps.

Let us summarize the results of sections 2 and 3 in the following property

**Property 3** Let $G = \mathfrak{sl}(n)$, $\mathfrak{so}(n)$, or $\mathfrak{sp}(2n)$, graded w.r.t. the Cartan generator of a $\mathfrak{sl}(2)$ subalgebra in $G$. Let $K = \bigoplus_i G^{(i)}_{\leq 0}$, where the subalgebras $G^{(i)}_{\leq 0}$ are defined through the property 2 and the lemma 1. Then, imposing $J^g$ to obey the condition $J^g|K = 0$ determines in an unique way the parameters of the symmetry transformations

$$J \rightarrow J^g = gJg^{-1} \quad \text{where} \quad g \in G_+ \quad \text{with} \quad \text{Lie}G_+ = G_+$$

By this procedure, the non-zero entries appearing in the final form of $J^g$ have vanishing PB with the $J^a$'s. They generate the commutant of these elements in the closure of the enveloping algebra of the Poisson-Kirillov one generated by the $J$'s. With some abuse of notations, we will denote this set $\text{Com}(G^*_\pm)$.

Now, we can formulate two important remarks.

The first one concerns the quantum analogue of the obtained algebra. The direct quantization technics that we recall hereafter directly leads to the determination of the commutant $\text{Com}(G_\pm)$ of the subalgebra $G_\pm$.

The second remarks concerns the affine case. The gauge transformation $g$ belonging to $G_+$, so is the quantity $\partial gg^{-1}$. Since the above symmetry fixing does not impose any condition on the $G_+$ component of $J^g$, it directly extends to the affine case, by simple adjonction of the $\partial gg^{-1}$ term (and restauration of the $z$-dependence).
4 Quantization

Up to now, we have presented a classical version (i.e. with Poisson brackets) of the commutants and $\mathcal{W}$-algebras. The quantization of the preceeding approach in the case of finite $\mathcal{W}$-algebras has been already developed in [1]. We just recall here the main points and refer to [1] for more details.

To each generator $J^a$ we associate an operator $\hat{J}^a$, with as usual the PB replaced by the commutator. If we stay at the level of the Lie algebra $\mathfrak{g}$, this quantization is a Lie algebra isomorphism. The problem is to extend this isomorphism to the whole envelopping algebra $\mathcal{U}$. In that case, we have on the one hand a commutative product with an algebraic structure defined by Poisson brackets (classical version); and on the other hand a (non-commutative) operator algebra. Clearly, as soon as the (classical) product law enters into the game, we have problems to represent it at the quantum level. These problems are (partially) cured by replacing each classical product by a symmetrized product at the quantum level. Indeed, with this rule, one can map the commutant of a generator of $\mathfrak{g}$ (or the commutant of a Lie subalgebra) into its quantum analogue. This property is sufficient for our purpose. Note however that this rule does not avoid the fact that the (vector space) isomorphism at the level of envelopping algebras is not an algebra isomorphism anymore. Therefore, it may happen that structure constants of the quantum commutant differ from those of the classical one, even if the realizations (in terms of Lie algebra generators) are similar. We gather the technics in the following scheme:

| Classical | $J^a$ | $\mathfrak{g}^*$ | $\{J^a, J^b\}$ | $\mathcal{U}$ | $J^{a_1} J^{a_2} \ldots J^{a_k}$ | $\text{Com}(\mathfrak{g}_\ast)$ |
|-----------|-------|-----------------|----------------|--------------|----------------|-----------------|
| Quantum   | $\hat{J}^a$ | $\hat{\mathfrak{g}} \equiv \mathcal{G}$ | $[\hat{J}^a, \hat{J}^b]$ | $\hat{\mathcal{U}} \sim \mathcal{U}$ | $s_k(\hat{J}^{a_1}, \hat{J}^{a_2}, \ldots, \hat{J}^{a_k})$ | $\text{Com}(\mathcal{G}_\ast)$ |
| Isomorphism | $\hat{\mathfrak{g}} \sim \mathfrak{g}^*$ | $\hat{\mathcal{U}} \sim \mathcal{U}$ | $\text{Com}(\mathcal{G}_\ast) \sim \text{Com}(\mathfrak{g}^*)$ |

where we have indicated in subscripts when the isomorphisms are algebra (alg.) or simply vector space (v.s.) isomorphisms. $s_k$ denotes the symmetrized product of $k$ generators. It contains $k!$ terms and is normalized in such a way that $s_k(\hat{J}^a, \hat{J}^a, \ldots, \hat{J}^a) = (\hat{J}^a)^k$.

Another problem relies on the quotients we form. In the classical approach, it is possible to define the inverse of $J^a$, since the product law is commutative. At the quantum level, it is in general difficult to deal with the inverse of an operator $\hat{J}^a$. Fortunately, the inverse we are considering all belong to $\mathfrak{g}_\ast$ (see lemma [2]). Then, the operators will be constant on each representation of $\text{Com}(\mathfrak{g}_\ast)$, and thus it is licite to introduce their inverse.

In the following section, we present the generalization of the previous approach in a classical framework, but the above remark still applies, so that we have also the quantization of our finite $\mathcal{W}$-algebras (see also examples in section [7]).
5 Commutants and \( W \)-algebras

5.1 Generalities

Now that we know how to compute \( \text{Com}(G^* -) \) (and \( \text{Com}(G -) \)), the problem is to identify the algebraic structure of this algebra. In particular, one has to remark that this algebra is built on quotients of fields. At the classical level, we can deal with such objects, but at the quantum level we have to define fractions of operators. Fortunately, all the fractions that appear in our algebra have denominators which are in the center of the commutant \( \text{Com}(G^*_-) \).

**Lemma 2** All the fields \((J^-)g\) are in the center of \( \text{Com}(G^*_-) \). These fields are the only ones appearing as denominators in the fractions we have considered.

**Proof:** From the conditions given in (2.12), it is clear that the parameters will depend on \( G_- \) elements only. Moreover, by construction, \((J^-)g\) is in the commutant of \( G_- \) (since it is a non-vanishing matrix element of \( J^g \)). On the other hand, as all the parameters belong to \( G_- \), and since \((J^-)g\) is itself an element of \( G_- \), we deduce that \((J^-)g\) is a function of \( G_- \) elements only. Thus, \((J^-)g\) is in the commutant of \( G_- \) and formed from \( G_- \)-elements only: this implies that the set of all \((J^-)g\) is in the center of \( \text{Com}(G^*_-) \). Since we have already seen in the proof of property 1 that the denominators are functions of the fields \((J^-)g\) only, this also proves that these denominators are in the center of \( \text{Com}(G^*_-) \), and thus are scalar in each representation of the algebra.

Let us remark that since all the denominators are themselves in \( \text{Com}(G^*_-) \), one can choose a finite dimensional basis of \( \text{Com}(G^*_-) \) whose elements (say \( P_\alpha \)) are polynomials in the \( J^\alpha \)'s and \( J^i \)'s. Using this basis, one can consider the subalgebra of \( \text{Com}(G^*_-) \) generated by polynomials in the \( P_\alpha \)'s. In this subalgebra, we have only polynomials in the \( J^\alpha \)'s and \( J^i \)'s, and the PB are also manifestly polynomials in the \( P_\alpha \)'s. This algebra is then a polynomial subalgebra not only of \( U \) but also of \( U \), the usual enveloping algebra of \( G \). However, one has to be careful that there are polynomials in the \( J^\alpha \)'s and \( J^i \)'s that belongs to \( \text{Com}(G^*_-) \) but that are not polynomials in the \( P_\alpha \)'s. In other words, the \( \text{Com}(G^*_-) \)-basis formed by the \( P_\alpha \)'s do not generate polynomially the commutant of \( G_- \) in \( U \) (see example in [6] and also below).

On the other hand, let \( Q_\alpha \) be the elements of the (rational) basis one obtains directly through the fixing of the symmetry. Then, any polynomial in the \( J^\alpha \)'s and \( J^i \)'s that belongs to \( \text{Com}(G^*_-) \) will be expressed as a polynomial in the \( Q_\alpha \)'s. Therefore, the subalgebra formed by the polynomials in the \( Q_\alpha \)'s will contain the commutant of \( G_- \) in \( U \), the usual enveloping algebra of \( G \). Thus, we believe that for the study of the commutant of \( G_- \) the \( Q_\alpha \)'s are more "natural" than the \( P_\alpha \)'s.

We have seen that \( \text{Com}(G^*_-) \) possesses a center. Apart from this center, we should identify the non-trivial part of \( \text{Com}(G^*_-) \). For such a purpose, and as for the calculation of \( \text{Com}(G^*_-) \), we start with the simplest case, namely the maximal nilpotent subalgebra in \( \mathfrak{sl}(n) \).
5.2 Maximal nilpotent subalgebra in $\mathfrak{sl}(n)$

**Property 4** If we decompose $G = \mathfrak{sl}(n)$ w.r.t. its principal gradation, we have the following algebra isomorphism:

$$\text{Com}(G^*) \sim \mathcal{W}(\mathfrak{sl}(n), \mu \mathfrak{sl}(2)) \oplus \mathfrak{g}^\mu \quad \text{with} \quad \mu = \left\lfloor \frac{n}{2} \right\rfloor \quad (5.1)$$

Before proving this proposition, let us mention that the center of $\text{Com}(G^*)$ contains at least $\mu$ different fields (hence is at least $\mathfrak{g}^\mu$), since it contains all the $(J^{-\psi})^g$ fields. Moreover, it is clear that $\text{Com}(G^*)$ contains $\dim G - \dim G_+ = \left\lfloor \frac{n(n+1)}{2} \right\rfloor - \mu - 1$ generators: a direct calculation shows that there are exactly $\left\lfloor \frac{n(n+1)}{2} \right\rfloor - \mu - 1$ highest weights for the diagonal $\mathfrak{sl}(2)$ in $\mu \mathfrak{sl}(2) \subset_{\text{reg}} \mathfrak{sl}(n)$. Therefore, the dimensions of the two algebras described in (5.1) are equal.

**Proof:** From the final form of $J^g$, it is natural to consider the $\mathfrak{sl}(2)$ subalgebras built on the root generators $E_{\pm \psi}$. These subalgebras mutually commute (due to the construction of the $E_{\psi_i}$), so that we get $\mu$ regular $\mathfrak{sl}(2)$ subalgebras in $\mathfrak{sl}(n)$. To these $\mu \mathfrak{sl}(2)$, we can associate a natural gradation, corresponding to the Cartan generator in the diagonal $\mathfrak{sl}(2)$, namely $G = G^{-1} \oplus G^0 \oplus G^1$, so that we have a bigrading of $G$. For instance, under the new gradation, the symmetry group $G_+$ is divided in two parts $G_+ = G^0_+ \oplus G^1_+$, where we have indicated in superscript the new grading. In the same way, we get $G_- = G^0_- \oplus G^1_-$, and $G_0 = G^0_0$.

Using this decomposition, we make the symmetry group $G_+$ acting in two steps.

In the first step, we consider $G^0_+$. Using the bigrading, it is easy to see that it is a subalgebra. We use the associated subgroup to set $J$ into the form:

$$J' = \begin{pmatrix} 0 & * & \cdot & \cdot \\ * & 0 & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \quad (5.3)$$

Let us first remark that the bigrading implies $[G^0_+, G^-] \subset G^-$. Moreover, $G^0_+$ has dimension $\mu^2 - \mu$ while $\dim G^- = \mu^2$. As the anti-diagonal in $G^-$ is not fixed, we need $\mu^2 - \mu$ group
generators to get the form \((5.3)\), and it is easy to convince one-self by a direct calculation that \(G'_0\) indeed acts correctly on \(G_-\) for that purpose.

Now, in the second step, we will deal with the subgroup \(G'_+\). This subgroup is exactly the symmetry group associated to the reduction w.r.t. \(\mu \text{ sl}(2)\), and the matrix (5.3) has just the form of the corresponding constrained matrix (up to the grade \(-1\) terms \(J_\psi, i\) which have been left free). This kind of reduction (with an Abelian symmetry group) has been studied in [1], and we know that the resulting algebra is just the \(W(\text{sl}(n), \mu \text{ sl}(2))\) algebra plus a center that corresponds to the \(J_\psi, i\)’s themselves. This ends the proof. 

\[\Box\]

Let us remark that the decomposition \(G' = G'_0 \oplus G'_+\) divides \(G'_+\) in a part \((G'_0)\) that acts on the \(J_\psi, i\)'s, and an abelian part \((G'_+)\) that does not transform them.

### 5.3 The other nilpotent subalgebras in \(\text{sl}(n)\)

The above procedure can be applied to any \(G_-, \) subalgebra of \(G\) relative to a given \(\text{sl}(2)\)-embedding. We introduce a new gradation (written in superscript) for \(G\): \(G = G^{-1} \oplus G^0 \oplus G^{+1}\) that is directly related to the \(W\)-algebra we will recognize inside the commutant. Thus, we have a bi-gradation of \(G\) (cf figure 5.2) and the property:

**Property 5** Let \(\rho\) be the total number of roots \(\psi_p^{(i)} (\forall p, \forall i)\), and \(\mu \leq \rho\) the number of roots \(\psi_a^{(i)}\) that enter in the decomposition of \(\varphi_i\)’s:

\[
\rho = \sum_i m^{(i)} \quad \text{and} \quad \mu = \sum_i m^{(i)}_\parallel
\]

Then, we have the following isomorphism

\[\text{Com}(G^-) \sim W(\text{sl}(n), \mu \text{ sl}(2)) \oplus \mathfrak{C}^\nu \quad \text{with} \quad \mu \leq \nu = 2\dim G^+ + \dim G_+ \leq \rho\]

and \(\mu\) will be also the maximal number of regular \(\text{sl}(2)\) that one can simultaneously embed in \(\mathcal{H}\), the algebra that defines the gradation.

**Proof:** We have first to identify the \(\text{sl}(2)\) embeddings we consider. Since at each step of the process we may have several highest grade roots, we use the property 2 to distinguish one from the others, i.e. we select the \(E_{\pm \varphi_i}\)’s that satisfy the equation (3.2). Then, it is clear that the \(\text{sl}(2)\) algebras built on these roots will commute, since \(B_i\) is in direct sum with \(B_j\) in \(G\) as soon as \(i \neq j\). Now, we have to calculate the number of regular \(\text{sl}(2)\) subalgebras we get. It is not simply the number of \(E_{\varphi_i}\) generators because these elements may be not regular in \(G\). Thus, one has to consider the regular generators \(E_{\psi_p}\). Therefore, at each step, we have just the number of root generators that enter in the decomposition of \(E_{\varphi_i}\) (since these roots mutually commute). Thus, at each step we get \(m^{(i)}_\parallel\) \(\text{sl}(2)\) subalgebras (same notation as in property 2), and the total number is \(m^{(i)} = \sum_i m^{(i)}_\parallel\).

Now, using the gradation associated to the diagonal \(\text{sl}(2)\), one remarks that each \(G^{(i)}_+\) divides into \((G^{(i)}_+ \cap G^0) \oplus (G^{(i)}_+ \cap G^+\) and that the associated subgroup does not interact in the
determination of the parameters (since the lowest grade roots have a grade -1 w.r.t. this new gradation). Therefore, we can make \((G_+ \cap G^0)\) act first, and use it to set \(J\) into the form
\[
J \to J_{-1} + J_{\geq 0}
\]
where the index denotes the grade.

Then, the group associated to \(G_+ = (G_+ \cap G^+),\) which is Abelian, is simply the group corresponding to the reduction w.r.t. \(\mu \ s\ell(2)\) and we end as in property \(\mathbb{[4]}\).

The dimension of the center subset \(C^\nu\) is then determined by a counting argument. The symmetry group has dimension \(\dim G_+\), so that \(\text{Com}(G^\nu)\) has dimension \(\dim G - \dim G_+\). Therefore, this center part has dimension \((\dim G - \dim G_+) - (\dim G - 2\dim G^+_+) = 2\dim G^+_+ - \dim G_+\).

Let us remark that in the case of the maximal nilpotent subalgebra, we have \(\mu = \nu = \rho\), the number of roots \(\psi_i\) since there is only one lowest grade root at each step.

We end this paragraph by emphasizing that the \(\mathcal{W}\)-algebra we obtain from the commutant \(\text{Com}(G^\nu)\) by disregarding (a part of) its center is entirely determined by \(G_-,\) the maximal abelian subalgebra of \(G_-\).

### 5.4 Commutants in \(sp(2n)\) and \(so(m)\)

The study of commutants in \(sp(2n)\) can be immediately deduced from the \(s\ell(n)\) case. Indeed, using the folding of \(s\ell(2n)\), one realizes that all the technics developed for \(s\ell(2n)\) also applies to \(sp(2n)\).

The calculation for \(so(m)\) algebras is slightly different. Indeed, if we consider the folding of \(s\ell(2n+1)\) to get \(so(2n+1)\), we obtain a matrix representation with zeros on the anti-diagonal. Then, it is clear that the fixed form for \(s\ell(2n+1)\) will not be correct for the \(so(2n+1)\). What one has to do is to first fold the non-fixed matrix, and then compute the form one has to take (see appendix).

Altogether, we have:

**Property 6** Let \(G\) be \(so(n)\) or \(sp(2n)\), graded with respect to some \(s\ell(2)\)-Cartan generator. Let \(\rho\) be the total number of roots \(\psi^{(i)}_p (\forall p, \forall i)\), and \(\mu \leq \nu\) the number of roots \(\psi_a^{(i)}\) that enter in the decomposition of \(\varphi_i\)'s:

\[
\rho = \sum_i m_i^{(i)} \quad \text{and} \quad \mu = \sum_i m_i^{(i)}
\]

Then, we have the following isomorphisms

For \(G = sp(2n)\) : \(\text{Com}(G^\nu) \sim \mathcal{W}(sp(2n), \mu_1 s\ell(2) \oplus \mu_2 sp(2)) \oplus C^\nu\)

For \(G = so(n)\) : \(\text{Com}(G^\nu) \sim \mathcal{W}(so(n), \mu_1 s\ell(2) \oplus \mu_2 so(3)) \oplus C^\nu\)

with \(\mu_2 = 0\ or \ 1; \ \mu = \mu_1 + \mu_2 \leq \nu \leq \rho\) and \(\mu\) will be also the maximal number of regular \(s\ell(2)\) and \(sp(2)\) (resp. \(so(3)\)) that one can simultaneously embedd in \(\mathcal{H}\), the algebra that defines the gradation.
Note that in the property, we have used the fact that in $sp(2m)$ ($so(m)$ respectively), the regular subalgebras that classify the $sl(2)$ embeddings are sums of $sl(n)$’s and eventually one $sp(2)$ (respectively $so(3)$) subalgebra.

### 5.5 Casimirs of $W$-algebras

We propose to show that $det(J^g - \lambda I)$ with $J^g = gJg^{-1} = W^k t_k$, provides directly the Casimirs of the $W$-algebra. For such a purpose, let us calculate $det(J^g - \lambda I)$ in two different ways.

On the one hand, anticipating on the results of section 6.1, where the elimination of the center is considered, $J^g$ is a matrix containing, apart from 0 and 1-entries, the $W$ generators. Therefore, $det(J^g - \lambda I)$ is a polynomial in $\lambda$ whose coefficients are expressed as polynomials in the $W$ generators.

On the other hand, $det(J^g - \lambda I) = det(gJg^{-1} - \lambda I) = det((J - \lambda I)g^{-1}) = det(J - \lambda I)$. This is a polynomial in $\lambda$ whose coefficients commute with all the generators of the Lie algebra. In the $sl(n)$ case, $det(J - \lambda I)$ can be written in the form $(-1)^n \lambda^n + \sum_{i=0}^{n-2} C_{n-i} \lambda^i$ where $C_i, i = 2, \ldots, n$ are the Casimirs of $sl(n)$ (note that the coefficient of $\lambda^{n-1}$ is zero since $Tr(J) = 0$).

Thus, we have constructed some polynomials in the $W$ generators that can be expressed as the Casimirs of the Lie algebra. Since the Casimirs commute with all the generators of the Lie algebra and since the $W$ generators are all expressed in terms of the Lie algebra generators, the polynomials we have constructed commute finally with all the generators of the $W$-algebra.

We remark that the above property can be used to replace some of the $W$ generators by the Casimirs by inverting (when it is possible) the expression of the $C_i$’s in terms of the $W$’s. Then, the $C_i$’s decouple and participate to the center (of the $W$-algebra), which thus appears naturally.

### 6 Commutant of non-nilpotent subalgebras

Up to now, we have considered the case where $\mathcal{N}$ is nilpotent. We have obtained in this way many but not all of the known $W$-algebras. We want to show that the other $W$-algebras can be also obtained as commutant of some $G$-subalgebra if one relaxes the condition of nilpotency, or in other words take $\mathcal{N}$ as a subalgebra of the full $G$-algebra, instead of $G_+$ only. Before coming to the general case, we show that relaxing the condition of nilpotency already helps a lot for the calculation of the $W$-algebras we have obtained.

#### 6.1 Use of the Cartan subalgebra

Let $G_-^-$ be an abelian subalgebra, and $G_-$ the maximal nilpotent $G$-subalgebra whose maximal abelian subalgebra is $G^-_-$. Then, even in this case where the $G_-^- \setminus G^-_-$-part is the biggest one can get, $Com_y(G_-)$ still contains some central elements, in addition to those of
\( \mathcal{W}(\mathfrak{sl}(n), \mu s\ell(2)) \). To get rid of this center part, the idea is to extent the symmetry transformation \( \mathcal{G}_+ \) with some Cartan generators.

Actually, the elements of this center are polynomials, formed with \( \mathcal{G}_- \) generators, i.e. only with negative roots. So there exist Cartan generators such that their eigenvalues under these polynomials are not all zero. One can choose suitably a set \( \hat{\mathcal{H}} \) of \((2 \dim \mathcal{G}_- - \dim \mathcal{G}_-)\) such elements and use them as symmetry generators. As a result, one will get the commutant in \( \mathcal{U} \) of \((\mathcal{G}_- \oplus \hat{\mathcal{H}})\), and the undesired center will drop out. One must note that the current components which undergo such Cartan transformations can be fixed to 1 but not to 0.

Note that there may be different sets \( \hat{\mathcal{H}} \); to each choice of \( \hat{\mathcal{H}} \) will correspond a different normalization of the remaining \( W \) generators. Such a phenomenon will be considered in the \( s\ell(3) \) case for the algebra noted \( \mathcal{W}_3^{(2)} \) (see section 7.1.1).

### 6.2 The general case

In the previous construction of \( \mathcal{W}(\mathfrak{sl}(n), \mu s\ell(2)) \), apart from the subalgebra \( \mathcal{G}^+ \), we act with some other elements, belonging to \( \mathcal{G}^0 \). In fact, \( \mathcal{G}^0 \) can be used to transform the \( J_{-1} \)-part into a constrained form, identical to the one used in the Hamiltonian reduction formalism. Consequently, in order to obtain \( \mathcal{W}(\mathcal{G}, \mathcal{H}) \)-algebras, with \( \mathcal{H} \neq \mu s\ell(2) \), the idea is to act on \( J \) first with elements which do not belong to \( \mathcal{G}^+ \) in order to recover the constrained form for the \( J_{-1} \)-part, and then with \( \mathcal{G}^+ \), using for instance the highest weight gauge. The \( \mathcal{W} \)-algebra is then viewed as the commutant in \( \mathcal{U} \) of a subalgebra \( \tilde{\mathcal{G}} \) of \( \mathcal{G} \), formed by \( \mathcal{G}_- \) and a subset of \( \mathcal{G}^0 \), noted \( \tilde{\mathcal{G}} \geq 0 \).

Let us be more precise. If \( J_{\text{const}} \) denotes a set of constraints that leads to \( \mathcal{W}(\mathcal{G}, \mathcal{H}) \) in the usual Hamiltonian reduction, we will determine the \( \tilde{\mathcal{G}} \geq 0 \) subset by requiring

\[
J^g_{<0} = J_{\text{const}} \quad \text{with} \quad g \in \tilde{\mathcal{G}} \leq 0 \tag{6.1}
\]

where \( J^g \) is the transformed current under symmetry transformations belonging to \((\tilde{\mathcal{G}} \geq 0)^* = \tilde{\mathcal{G}} \leq 0 \). Moreover, \( \tilde{\mathcal{G}} \geq 0 \) must also be such that \( \tilde{\mathcal{G}} \geq 0 \oplus \mathcal{G}_- \) is a \( \mathcal{G} \)-subalgebra.

As an example, we consider the \( \mathcal{W}(\mathfrak{sl}(n), s\ell(n)) \) case. The \( \mathcal{G}_- \) part consists of elements below the main diagonal of the \( n \times n \) matrix. The \( \tilde{\mathcal{G}} \geq 0 \) part can be made with all the other generators except the ones expressed by the matrices \( E_{i,n} \) with \( i = 1, \ldots, n - 1 \). The \( \tilde{\mathcal{G}} \leq 0 \) transformation parameters will be determined through the condition

\[
(J^g)_{ij} = \delta_{i,j+1} \quad \forall \ j < i \quad \text{with} \quad g \in \tilde{\mathcal{G}} \leq 0 \tag{6.2}
\]

### 6.3 Commutants of second order

Since \( \mathcal{W} \)-algebras can be viewed as deformations of semi-simple Lie algebras, we may think of performing on those objects the same kind of transformations we use on Lie algebras to obtain \( \mathcal{W} \)-algebras. In the framework of usual Hamiltonian reductions, this leads to the so-called secondary reduction \([4, 5]\). Here in the same way as in the previous sections, we replace the Hamiltonian reduction by the computation of a commutant. More precisely, considering
the closure $\overline{W}$ of a $W$-algebra, we compute the commutant of a given nilpotent (not necessarily Lie-) $W$-subalgebra $W_-$: in the following, we will denote this commutant $Com_{\overline{W}}(W_-)$.

Examples of such secondary paths will be given in section 7 for the $\mathfrak{sl}(3)$ and $\mathfrak{sl}(4)$ cases.

7 Commutants and $W$-algebras in $\mathfrak{sl}(3)$ and $\mathfrak{sl}(4)$

We deal with the $\mathfrak{sl}(3)$ and $\mathfrak{sl}(4)$ cases in a detailed way. The $\mathfrak{sl}(3)$ case is the most well-known and the easiest one to handle with. Although calculations are rather heavy, we decide to present the $\mathfrak{sl}(4)$ case because it seems to be a good example providing computation of commutants of nilpotent and non-nilpotent subalgebras and commutants of second order. At the classical (quantum) level, generators are denoted by $J$ (respectively $\hat{J}$). In what follows, we will present the quantum version of each $W$-algebra, since the classical one can be deduced from the former one by eliminating terms corresponding to quantum corrections; these last ones can be recognized easily: they have lower degree than the other terms. In the quantum framework, we will always consider the symmetrized product $s_k(, , )$.

7.1 $\mathfrak{sl}(3)$ case

Let $\alpha, \beta$ be the simple positive roots. The most general element of $\mathfrak{sl}(3)$ can be written as:

$$J = \begin{pmatrix} J_1 + \frac{d_2}{2} & J_\alpha & J_{\alpha+\beta} \\ J_\alpha & -J_2 & J_\beta \\ J_{\alpha-\beta} & J_\beta & -J_1 + \frac{d_2}{2} \end{pmatrix}.$$ 

In $\mathfrak{sl}(3)$, there are two types of regular subalgebras: $\mathcal{H} = \mathfrak{sl}(2), \mathfrak{sl}(3)$, and so two different $\mathfrak{sl}(2)$-embeddings, which induce two different gradations of $\mathfrak{sl}(3)$, $\mathcal{G} = \mathcal{G}_\mathcal{H}^\mathcal{H} \oplus \mathcal{G}_0^\mathcal{H} \oplus \mathcal{G}_1^\mathcal{H}$. In the usual Hamiltonian reductions, one gets the following $W$-algebras:

$$W_3^{(1)} = W(\mathfrak{sl}(3), \mathfrak{sl}(3)) \quad W_3^{(2)} = W(\mathfrak{sl}(3), \mathfrak{sl}(2)).$$

We present explicitly the computation of $Com(\mathcal{G}_\mathcal{H}^\mathcal{H})$, for $\mathcal{H} = \mathfrak{sl}(2), \mathfrak{sl}(3)$, and show that the resulting algebra is $W_3^{(2)}$ up to a center, as expected from section 3. We also display a normalization procedure in order to get rid of this center. Then we characterize the $W_3^{(1)}$ algebra as a commutant of an $\mathfrak{sl}(3)$ subalgebra as well as the commutant of a $W_3^{(2)}$ subalgebra.

7.1.1 $\mathcal{H} = \mathfrak{sl}(3)$

In this case $\mathcal{G}_\mathcal{H}^\mathcal{H} = \{J_\alpha, J_\beta, J_{\alpha-\beta}\}$. We consider a coadjoint transformation $J^g = gJg^{-1}$ under a $G^+$-element:

$$g = \begin{pmatrix} 1 & A & C \\ 0 & 1 & B \\ 0 & 0 & 1 \end{pmatrix}.$$ 

The generator $J_{\alpha-\beta}$ associated to the root of lowest grade with respect to this principal gradation is invariant under $G^+$ transformations. The conditions

$$J_2^{g_{2,1}} = 0, \quad J_3^{g_{3,2}} = 0, \quad J_1^{g_{1,1}} = J_3^{g_{3,3}}$$

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uniquely fix the parameters $A, B, C$ through linear equations and $J^9$ has the following form:

$$J^9 = \begin{pmatrix} W_0 & W_a & W_{ab} \\ 0 & -2W_0 & W_b \\ J_{-\alpha-\beta} & 0 & W_0 \end{pmatrix}.$$ 

$J_{-\alpha-\beta}$ commutes with all the generators, so the commutant writes

$$\text{Com}(J_{-\alpha}, J_{-\beta}, J_{-\alpha-\beta}) = \{W_0, W_a, W_b, W_{ab}\} \oplus \{J_{-\alpha-\beta}\}$$

where denominators of $W_0, W_a, W_b, W_{ab}$ only contain the element $J_{-\alpha-\beta}$.

$$W_0 = \frac{P_0}{2J_{-\alpha-\beta}}, \quad W_a = \frac{P_a}{2J^2_{-\alpha-\beta}}, \quad W_b = \frac{P_b}{2J^2_{-\alpha-\beta}}, \quad W_{ab} = \frac{P_{ab}}{4J^3_{-\alpha-\beta}} \quad (7.3)$$

Using the quantization procedure recalled in section [4] (i.e. symmetrisation) we get the finite quantum $\mathcal{W}$-algebra generated by:

$$\begin{align*}
\hat{P}_0 &= s_2(\hat{J}_2, \hat{J}_{-\alpha-\beta}) + s_2(\hat{J}_\alpha, \hat{J}_\beta) \\
\hat{P}_a &= -2s_3(\hat{J}_1, \hat{J}_{-\beta}, \hat{J}_{-\alpha-\beta}) - 3s_3(\hat{J}_2, \hat{J}_{-\beta}, \hat{J}_{-\alpha-\beta}) - 2s_3(\hat{J}_\alpha, \hat{J}_{-\beta}, \hat{J}_{-\alpha-\beta}) + 2s_3(\hat{J}_\alpha, \hat{J}_{-\alpha-\beta}, \hat{J}_{-\alpha-\beta}) \\
\hat{P}_b &= 2s_3(\hat{J}_1, \hat{J}_{-\alpha}, \hat{J}_{-\alpha-\beta}) - 3s_3(\hat{J}_2, \hat{J}_{-\alpha}, \hat{J}_{-\alpha-\beta}) - 2s_3(\hat{J}_\alpha, \hat{J}_{-\alpha}, \hat{J}_{-\alpha-\beta}) + 2s_3(\hat{J}_\beta, \hat{J}_{-\alpha-\beta}, \hat{J}_{-\alpha-\beta}) \\
\hat{P}_{ab} &= 4s_4(\hat{J}_1, \hat{J}_{-\alpha-\beta}, \hat{J}_{-\alpha}), 6s_4(\hat{J}_2, \hat{J}_{-\beta}, \hat{J}_{-\alpha}, \hat{J}_{-\alpha-\beta}) - 3s_4(\hat{J}_{-\beta}, \hat{J}_{-\alpha}, \hat{J}_{-\alpha}), \hat{J}_\alpha) + \\
&\quad + 4s_4(\hat{J}_{-\beta}, \hat{J}_{-\alpha}, \hat{J}_{-\alpha}, \hat{J}_{-\alpha-\beta}) + 4s_4(\hat{J}_{\alpha}, \hat{J}_{-\alpha}, \hat{J}_{-\alpha}, \hat{J}_{-\alpha-\beta}) + \\
&\quad + 4s_4(\hat{J}_{\beta}, \hat{J}_{-\alpha}, \hat{J}_{-\alpha}, \hat{J}_{-\alpha-\beta})
\end{align*}$$

By a simple change of basis, $\hat{W}_{ab}$ can be replaced by the central element $\hat{C}_2$,

$$\hat{C}_2 = 3\hat{W}_0^2 + \hat{J}_{-\alpha-\beta}\hat{W}_{ab} + \frac{53}{48},$$

using determinant equalities\footnote{Beware: the determinant property works at the classical level; we quantize afterwards.} (see section [3.3]). Actually, $\hat{C}_2$ is second degree $s\ell(3)$ Casimir (its explicit expression will be given below). We note that the constant term $\frac{53}{48}$ comes from quantum corrections. In the basis of polynomials $\{\hat{P}_0, \hat{P}_a, \hat{P}_b, \hat{C}_2, \hat{J}_{-\alpha-\beta}\}$, commutation relations take the form:

$$[\hat{P}_0, \hat{P}_a] = \hat{J}_{-\alpha-\beta}\hat{P}_a, \quad [\hat{P}_0, \hat{P}_b] = -\hat{J}_{-\alpha-\beta}\hat{P}_b$$

$$[\hat{P}_a, \hat{P}_b] = -12\hat{J}_{-\alpha-\beta}\hat{P}_0^2 + \hat{J}_{-\alpha-\beta}^3(4\hat{C}_2 + 3).$$

Normalization (see section [6.1]) consists in eliminating $\hat{J}_{-\alpha-\beta}$ in those equations. Since it commutes with all the other quantities, we can redefine without ambiguities

$$\hat{Y} = \frac{\hat{P}_0}{\hat{J}_{-\alpha-\beta}}, \quad \hat{W}_+ = \frac{-\hat{P}_a}{2\hat{J}^2_{-\alpha-\beta}}, \quad \hat{W}_- = \frac{\hat{P}_b}{2\hat{J}^2_{-\alpha-\beta}} \quad \text{with} \quad m \in \mathbb{R}.$$
In this form we then recognize the commutation relations of the algebra \( \mathcal{W}_3^{(2)} \), seen as a deformed \( sl(2) \) algebra plus a central \( \hat{C}_2 \) element:

\[
[\hat{Y}, \hat{W}_\pm] = \pm \hat{W}_\pm \quad [\hat{W}_+, \hat{W}_-] = 3\hat{Y}^2 - (\hat{C}_2 + \frac{3}{4})
\]  

(7.4)

The analogue of the Casimir for this \( \mathcal{W} \)-algebra can be found thanks to the determinant property and is indeed equal in this realization to \( \hat{C}_3 \) the third order Casimir of \( sl(3) \):

\[
\hat{C}_3 = s_2(\hat{W}_+, \hat{W}_-) + \hat{Y}^3 - \hat{Y}(\hat{C}_2 + \frac{1}{2}).
\]

When \( m \) takes the value \( \frac{3}{2} \), the quantities \( \hat{Y}, \hat{W}_\pm, \hat{C}_2 \) commute with \( \hat{J}_1 \). Actually, they generate \( Com(\hat{J}_-\alpha, \hat{J}_-\beta, \hat{J}_-\alpha-\beta, \hat{J}_1) \) that can be obtained by action of a coadjoint transformation

\[
g = \begin{pmatrix} D & A & C \\ 0 & 1 & B \\ 0 & 0 & \frac{1}{D} \end{pmatrix}
\]  

and imposing

\[
J^g = \begin{pmatrix} Y & W_+ & C_2 + Y^2 \\ 0 & -2Y & W_- \\ 1 & 0 & Y \end{pmatrix}
\]  

In this case, the degree of the four generators \( Y, W_\pm, C_2 \) are respectively \( 1, \frac{3}{2}, \frac{3}{2}, 2 \), and corresponds exactly to the spin of the generators of the Bershadsky algebra in the affine case.

### 7.1.2 \( H = sl(2) \)

In this case, after halving, \( G_- = \{ J_-\alpha, J_-\alpha-\beta \} \) is abelian and has gradation \(-1\). We will act on \( J \) with coadjoint transformation of the form

\[
g = \begin{pmatrix} 1 & A' & B' \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]  

Now we have more than one root of lowest weight. Following notations of section [3] we have \( \Lambda_\parallel = \{ \alpha + \beta \} \). The symmetry fixing is determined by \( J^g_{3,2} = 0, J^g_{1,1} = J^g_{3,3} \):

\[
J^g = \begin{pmatrix} W'_0 & W'_a & W'_b \\ J^-\alpha & -2W'_0 & W'_b \\ J^-\alpha-\beta & 0 & W'_b \end{pmatrix},
\]

the generators \( \{ J_-\alpha, J_-\alpha-\beta \} \) commuting with the other \( W' \) generators. Moreover, the \( W' \) generators can be related to the \( W \) ones previously defined in eq. (7.3):

\[
W'_0 = W_0 \quad W'_a = W_a \quad W'_b = W_b + 3\frac{J^-\alpha}{J^-\alpha-\beta}W_0 \quad W'_{ab} = W_{ab} - \frac{J^-\alpha}{J^-\alpha-\beta}W_a
\]

Thus we recognize the \( \mathcal{W}_3^{(2)} \) algebra and

\[
Com(\hat{J}_-\alpha, \hat{J}_-\alpha-\beta) = \mathcal{W}_3^{(2)} \oplus \{ J_-\alpha, J_-\alpha-\beta \}.
\]
7.1.3 $\mathcal{W}_3^{(1)}$ algebra

We conclude this paragraph by recalling that the $\mathcal{W}_3^{(1)}$ algebra can itself be seen as the commutant of the whole $\mathfrak{sl}(3)$ algebra, and even simply as $\text{Com}(\hat{J}_1, \hat{J}_2, \hat{J}_a, \hat{J}_-\alpha, \hat{J}_-\beta, \hat{J}_-\alpha\beta)$. So the $\mathcal{G}^{\mathfrak{sl}(3)}_0$ part has been enlarged with the help of three other generators (see section 6.2).

We recall the two $\mathcal{W}_3^{(1)}$ generators (i.e. $\mathfrak{sl}(3)$ Casimirs):

\[
\hat{C}_2 = s_2(\hat{J}_1, \hat{J}_1) + \frac{3}{4} s_2(\hat{J}_2, \hat{J}_2) + s_2(\hat{J}_a, \hat{J}_a) + s_2(\hat{J}_-\alpha, \hat{J}_-\alpha) + s_2(\hat{J}_-\beta, \hat{J}_-\beta)
\]

\[
\hat{C}_3 = \frac{1}{4} s_3(\hat{J}_2, \hat{J}_2, \hat{J}_2) - s_3(\hat{J}_1, \hat{J}_1, \hat{J}_2) - s_3(\hat{J}_a\beta, \hat{J}_-\alpha, \hat{J}_-\beta) - s_3(\hat{J}_-\alpha\beta, \hat{J}_a, \hat{J}_\beta) + s_3(\hat{J}_a, \hat{J}_-\alpha, \hat{J}_-\beta) + s_3(\hat{J}_2, \hat{J}_\beta, \hat{J}_-\beta) + s_3(\hat{J}_2, \hat{J}_\alpha, \hat{J}_-\beta) - s_3(\hat{J}_2, \hat{J}_a\beta, \hat{J}_-\alpha).
\]

It is worthwhile to note that $\mathcal{W}_3^{(1)}$ can also be obtained by performing a symmetry fixing on the $\mathcal{W}_3^{(2)}$ algebra in a way analogous to the one used in $\mathcal{G}$ algebra (see section 5.3). Such a technics has been used in \[4\] in the case of classical affine $\mathcal{W}$ algebras in presence of constraints. This method of secondary reduction had been also denoted a gauging of a $\mathcal{W}$ algebra. A natural gradation shows up on the C.R. (7.4) of the $\mathcal{W}_3^{(2)}$ algebra, with the zero graded part generated by $\hat{Y}$ and $\hat{C}_2$. Performing first computations at the classical level, one will finally obtain $\mathcal{W}_3^{(1)}$ as the commutant of the $\mathcal{W}_3^{(2)}$ subalgebra generated by $\hat{Y}$ and $\hat{W}_-$, i.e.

\[
\mathcal{W}_3^{(1)} = \text{Com}_{\mathcal{W}_3^{(2)}}(\hat{Y}, \hat{W}_-).
\]

7.2 $\mathfrak{sl}(4)$ case

Let $\alpha, \beta, \gamma$ be the simple positive roots.

\[
J = \left( \begin{array}{cccc}
\frac{J_1 + J_2}{2} & J_\alpha & J_{\alpha + \beta} & J_{\alpha + \beta + \gamma} \\
\frac{J_1 + J_2}{2} & J_\beta & J_{\beta + \gamma} \\
J_\alpha & J_{\alpha - \beta} & J_{\beta - \gamma} \\
J_{\alpha - \beta - \gamma} & J_{\beta - \gamma} & \frac{-J_2 + J_3}{2} 
\end{array} \right)
\]

In $\mathfrak{sl}(4)$, there are four different types of regular subalgebras (up to conjugation):

\[
\mathcal{H} = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2), \mathfrak{sl}(3), \mathfrak{sl}(4).
\]

They correspond to four different $\mathfrak{sl}(2)$-embeddings and so to four different gradations, $\mathcal{G} = \mathcal{G}_0^\mathcal{H} \oplus \mathcal{G}_0^\mathcal{H} \oplus \mathcal{G}_+^\mathcal{H}$. We denote the corresponding $\mathcal{W}(\mathcal{G}, \mathcal{H})$ algebra as:

\[
\mathcal{W}_4^{(1)}(\mathfrak{sl}(4), \mathfrak{sl}(4)) \quad \mathcal{W}_4^{(2)}(\mathfrak{sl}(4), \mathfrak{sl}(3)) \quad \mathcal{W}_4^{(3)}(\mathfrak{sl}(4), \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)) \quad \mathcal{W}_4^{(4)}(\mathfrak{sl}(4), \mathfrak{sl}(2)).
\]

In the following, we will present the quantum version, so dealing only with $\hat{J}$ quantities. The Casimir of $\mathfrak{sl}(4)$ are denoted by $\hat{C}_2, \hat{C}_3, \hat{C}_4$. 

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7.2.1 \( \text{Com}(\mathcal{G}_H^2) \)

For each \( s\ell(2) \)-embedding, we give:

- the corresponding regular subalgebra \( \mathcal{H} \),
- the corresponding negative graded part of \( \mathcal{G} \): \( \mathcal{G}_-^H \),
- the number of steps needed for the symmetry fixing (see section 3), and also the elements \( E_{-\varphi_i} \) of lowest grade,
- an Abelian subalgebra of \( \mathcal{G}_-^H \) with a maximal number of generators: \( \mathcal{G}_- \)
- and finally the commutant of \( \mathcal{G}_-^H \) identified as a \( \mathcal{W} \)-algebra plus the central part (after having performed a change of basis).

| \( \mathcal{H} \) | \( \mathcal{G}_-^H \) | steps | \( E_{-\varphi_i} \) | \( \mathcal{G}_- \) | \( \text{Com}(\mathcal{G}_-^H) \) |
|---|---|---|---|---|---|
| \( s\ell(2) \) | \( \hat{J}_{-\alpha}, \hat{J}_{-\alpha-\beta}, \hat{J}_{-\alpha-\beta-\gamma} \) | 1 | \( E_{-\alpha-\beta-\gamma} \) | \( \hat{J}_{-\alpha}, \hat{J}_{-\alpha-\beta} \) | \( \mathcal{W}_4^{(4)} \oplus \mathcal{G}_- \) |
| \( 2 \ s\ell(2) \) | \( \hat{J}_{-\beta}, \hat{J}_{-\alpha-\beta}, \hat{J}_{-\beta-\gamma} \) | 1 | \( E_{-\alpha-\beta} + E_{-\beta-\gamma} \) | \( \hat{J}_{-\beta}, \hat{J}_{-\alpha-\beta} \) | \( \mathcal{W}_4^{(3)} \oplus \mathcal{G}_- \) |
| \( s\ell(3) \) | \( \hat{J}_{-\alpha}, \hat{J}_{-\beta}, \hat{J}_{-\beta-\gamma} \) | 1 | \( E_{-\alpha-\beta-\gamma} \) | \( \hat{J}_{-\alpha}, \hat{J}_{-\alpha-\beta} \) | \( \mathcal{W}_4^{(4)} \oplus \{ \hat{J}_{-\alpha-\beta-\gamma} \} \) |
| \( s\ell(4) \) | \( \hat{J}_{-\alpha}, \hat{J}_{-\beta}, \hat{J}_{-\beta-\gamma} \) | 2 | \( E_{-\beta}, E_{-\alpha-\beta-\gamma} \) | \( \hat{J}_{-\beta}, \hat{J}_{-\alpha-\beta} \) | \( \mathcal{W}_4^{(3)} \oplus \{ \hat{J}_{-\alpha-\beta-\gamma}, C_w \} \) |

where \( C_w = \hat{J}_{-\beta} \hat{J}_{-\alpha-\beta-\gamma} - \hat{J}_{-\alpha-\beta} \hat{J}_{-\beta-\gamma} \). For \( \mathcal{H} = s\ell(4) \), \( \mathcal{G}_-^H \) is the maximal nilpotent algebra in \( s\ell(4) \). As expected from section 3, by considering the commutant of the \( \mathcal{G}_-^H \)-part, we have obtained up to center only the algebra \( \mathcal{W}_4^{(4)} \) for \( \mathcal{H} = s\ell(2), s\ell(3), \) and \( \mathcal{W}_4^{(3)} \) for \( \mathcal{H} = 2s\ell(2), s\ell(4) \). The difference between the case \( \mathcal{H} = s\ell(2) \) and \( \mathcal{H} = s\ell(3) \) (respectively \( \mathcal{H} = 2s\ell(2) \) and \( \mathcal{H} = s\ell(4) \)) stands in the center: the bigger \( \mathcal{H} \), the smaller the center.

7.2.2 \( \text{Com}(\mathcal{N}) \)

Here we consider some other nilpotent algebras \( \mathcal{N} \), not associated to an \( s\ell(2) \) embedding.

- \( \mathcal{N}^{(I)} = \{ \hat{J}_{-\alpha}, \hat{J}_{-\alpha-\beta-\gamma}, \hat{J}_{-\beta-\gamma} \} \)

  The commutant of \( \mathcal{N}^{(I)} \) is \( \mathcal{W}_4^{(4)} \oplus \mathcal{N}^{(I)} \). This has to be compared with the commutant of \( \mathcal{G}^{s\ell(2)}_2 \): \( \mathcal{N}^{(I)} \) and \( \mathcal{G}^{s\ell(2)}_2 \) are isomorphic nilpotent (abelian) algebras, but they are not conjugated.

- \( \mathcal{N}^{(II)} = \{ \hat{J}_{-\alpha}, \hat{J}_{-\alpha-\beta}, \hat{J}_{-\beta-\gamma}, \hat{J}_{-\alpha-\beta-\gamma} \} \)

  \( \mathcal{N}^{(II)} \) has the same number of generators as \( \mathcal{G}_2^{s\ell(2)} \). However, its commutant is not \( \mathcal{W}_4^{(3)} \) up to center, but \( \mathcal{W}_4^{(4)} \oplus \{ \hat{J}_{-\alpha-\beta-\gamma}, \hat{J}_{-\alpha-\beta} \} \). Note that in this case, an Abelian \( \mathcal{N} \)-subalgebra has at most three generators, as in the cases \( \mathcal{G}^{s\ell(2)}_2 \) and \( \mathcal{G}^{s\ell(3)}_2 \), leading to the same algebra \( \mathcal{W}_4^{(4)} \).
\( \mathcal{N}^{(III)} = \{ \hat{J}_{-\alpha}, \hat{J}_{-\beta}, \hat{J}_{-\alpha-\beta}, \hat{J}_{-\beta-\gamma}, \hat{J}_{-\alpha-\beta-\gamma} \} \)

\( \mathcal{N}^{(III)} \) has the same number of generators as \( g_{-}^{s\ell(3)} \), but the commutant of \( \mathcal{N}^{(III)} \) is \( W_4^{(3)} \oplus \{ \hat{J}_{-\alpha-\beta}, \hat{J}_{-\alpha-\beta-\gamma}, C_w \} \). As in the previous case, we note that the maximal abelian subalgebra in \( \mathcal{N}^{(III)} \) is the same as in the cases \( g_{-}^{s\ell(2)} \) and \( g_{-}^{s\ell(4)} \), where their commutant lead also to the algebra \( W_4^{(4)} \), up to center.

We remark that in each case, the commutant of a nilpotent algebra \( \mathcal{N} \) has a center formed by elements of \( \overline{U}(g_{-}) \), where \( g_{-} \) is an Abelian algebra contained in \( \mathcal{N} \) with a maximal number of generators. Note also that the dimension of this center is \( 2\dim(\mathcal{N}) - \dim(\mathcal{N}_{-}) \). We have seen that algebras associated to different \( \mathcal{N} \), but with same \( g_{-} \)-parts differ from central elements. Thus, we can say that the role of \( \mathcal{N} \backslash g_{-} \) is to select in \( \overline{U}(g_{-}) \) the elements that will form the center of the commutant.

### 7.2.3 \( W_4^{(i)} \) as \( \text{Com}(\mathcal{G}) \)

In the following, we describe all the \( \mathcal{W}(s\ell(4), \mathcal{H}) \) algebras in terms of the commutant of an \( s\ell(4) \) subalgebra: note that this subalgebra \( \mathcal{G} \) is not necessarily solvable. Following section 6.2, to compute this commutant, we act first with the dual of \( \mathcal{G}_{\geq 0} \). All \( J_{<0} \) are therefore fixed: the non-zero fixing conditions are presented in the column ”fixing”: they are identical to the set of constraints used in the Hamiltonian reduction framework. Then, we act with the \( g_{+} \)-part in a way similar to the so-called highest weight gauge. Finally, we enumerate the different generators, with their degree.

| \( \mathcal{W} \) | \( \mathcal{G} \) | \( \mathcal{G}_{>0} \) | fixing | generators | degree |
|---|---|---|---|---|---|
| \( W_4^{(4)} \) | \( J_{-\alpha}, \hat{J}_{-\gamma} \) \( \hat{J}_{-\alpha-\beta}, \hat{J}_{-\beta-\gamma}, \hat{J}_{-\alpha-\beta-\gamma} \) \( J_1 + J_2 + J_3 \) | \( J_{-\alpha}, \hat{J}_{-\gamma} \) \( J_1 + J_2 + J_3 \) | \( J_{3,1}^{0,1} = 1, J_{3,2}^{0,1} = 1 \) | \( H_1, H_2, E, F \) | 1 |
| \( \hat{C}_1, \hat{C}_1, \hat{C}_2, \hat{C}_2 \) | \( \hat{C}_2 \) | \( \frac{3}{2} \) |
| \( W_4^{(3)} \) | \( J_{-\alpha}, \hat{J}_{-\gamma} \) \( \hat{J}_{-\alpha-\beta}, \hat{J}_{-\beta-\gamma}, \hat{J}_{-\alpha-\beta-\gamma} \) \( J_1, J_2, J_3 \) | \( J_{-\alpha}, \hat{J}_{-\gamma} \) \( J_1, J_2, J_3 \) | \( J_{3,1}^{0,1} = 1, J_{3,2}^{0,1} = 1 \) | \( H, J^+, J^- \) | 1 |
| \( \hat{C}_2, \hat{S}_0, \hat{S}_+, \hat{S}_- \) | \( \hat{S}_- \) | 2 |
| \( W_4^{(2)} \) | \( J_{-\alpha}, \hat{J}_{-\gamma} \) \( \hat{J}_{-\alpha-\beta}, \hat{J}_{-\beta-\gamma}, \hat{J}_{-\alpha-\beta-\gamma} \) \( J_1, J_2, J_3 \) \( \hat{J}_1, \hat{J}_2, \hat{J}_3, \hat{J}_a \) | \( J_{-\alpha}, \hat{J}_{-\gamma} \) \( \hat{J}_1, \hat{J}_2, \hat{J}_3, \hat{J}_a \) | \( J_{3,1}^{0,1} = 1, J_{3,2}^{0,1} = 1 \) | \( \hat{U} \) | 1 |
| \( \hat{C}_2, \hat{C}_3, \hat{G}^+ \) | \( \hat{C}_3 \) | 2 |
| \( W_4^{(1)} \) | \( J_{-\alpha}, \hat{J}_{-\gamma} \) \( \hat{J}_{-\alpha-\beta}, \hat{J}_{-\beta-\gamma}, \hat{J}_{-\alpha-\beta-\gamma} \) \( \hat{J}_1, \hat{J}_2, \hat{J}_3 \) \( \hat{J}_1, \hat{J}_2, \hat{J}_3 \) \( \hat{J}_a, \hat{J}_b, \hat{J}_a+\hat{J}_b \) | \( J_{-\alpha}, \hat{J}_{-\gamma} \) \( \hat{J}_1, \hat{J}_2, \hat{J}_3 \) \( \hat{J}_1, \hat{J}_2, \hat{J}_3 \) \( \hat{J}_a, \hat{J}_b, \hat{J}_a+\hat{J}_b \) | \( J_{3,1}^{0,1} = 1, J_{3,2}^{0,1} = 1 \) | \( \hat{C}_2, \hat{C}_3, \hat{C}_4 \) | 2, 3, 4 |

We reproduce hereafter the C.R. of these \( s\ell(4) \) \( \mathcal{W} \) algebras \( \mathcal{U} \) and give the explicit expression of their Casimir-like generators, using our determinant trick.
\( \mathcal{W}_4^{(3)} \simeq \mathfrak{s}\ell(3)_{\text{def}} \oplus \hat{C}_2. \)

\( \{H_1, \tilde{E}, \tilde{F}\} \oplus \{\hat{H}_2\} \) forms a \( \mathfrak{s}\ell(2) \oplus \mathfrak{u}(1) \) subalgebra. \( \{\hat{G}_1^+, \hat{G}_2^+\} \) and \( \{\hat{G}_1^-, \hat{G}_2^-\} \) transform as two \( D_{\frac{3}{2}} \) representations under this \( \mathfrak{s}\ell(2) \), indices 1 and 2 refering to the \( \mathfrak{u}(1) \) charge. \( \hat{C}_2 \) is a central element.

\[
[\hat{G}_1^+, \hat{G}_2^+] = -9\hat{H}_2\hat{E} \quad [\hat{G}_1^+, \hat{G}_2^-] = \frac{3}{16}\hat{C}_2 - \frac{9}{2}\hat{C} - \frac{27}{4}\hat{H}_2^2 - 9\hat{H}_1\hat{H}_2
\]

\[
[\hat{G}_1^-, \hat{G}_2^-] = 9\hat{H}_2\hat{F} \quad [\hat{G}_1^-, \hat{G}_2^+] = \frac{3}{16}\hat{C}_2 - \frac{9}{2}\hat{C} - \frac{27}{4}\hat{H}_2^2 + 9\hat{H}_1\hat{H}_2
\]

where \( \hat{C} = 2\hat{H}_1^2 + \hat{E}\hat{F} + \hat{F}\hat{E} \) is the Casimir of the \( \mathfrak{s}\ell(2) \) subalgebra. This \( \mathcal{W} \)-algebra has two Casimir-like elements, which are proportional to \( \hat{C}_3 \) and \( \hat{C}_4 \), the third and fourth order Casimir of \( \mathfrak{s}\ell(4) \):

\[
\hat{C}_3 \propto 2s_2(\hat{G}_1^+, \hat{G}_2^-) + 2s_2(\hat{G}_1^-, \hat{G}_2^+) - 9\hat{H}_2\hat{C} + \frac{3}{8}\hat{H}_2\hat{C}_2 - \frac{9}{2}\hat{H}_2^3 - 9\hat{H}_2
\]

\[
\hat{C}_4 \propto 6s_3(\hat{H}_1, \hat{G}_1^+, \hat{G}_2^-) + 3s_3(\hat{H}_2, \hat{G}_1^+, \hat{G}_2^-) + 6s_3(\hat{H}_1, \hat{G}_1^-, \hat{G}_2^+) + 3s_3(\hat{H}_2, \hat{G}_1^-, \hat{G}_2^+) + 6s_3(\hat{E}, \hat{G}_1^+, \hat{G}_2^-) + 6s_3(\hat{F}, \hat{G}_1^+, \hat{G}_2^+) + \frac{9}{16}\hat{C}_2\hat{C} - \frac{9}{32}\hat{C}_2\hat{H}_2^2 + \frac{54}{8}\hat{C}_2^2 + \frac{27}{4}\hat{C}\hat{H}_2^2 + \frac{81}{16}\hat{H}_2^4 + 9\hat{C} - \frac{27}{4}\hat{H}_2^2
\]

\( \mathcal{W}_4^{(3)} \simeq (\mathfrak{s}\ell(2) \oplus \mathfrak{s}\ell(2))_{\text{def}} \oplus \hat{C}_2. \)

\( \{\hat{H}, \hat{J}^+, \hat{J}^-, \hat{u}\} \) forms a \( \mathfrak{s}\ell(2) \) subalgebra. \( \{\hat{S}_0, \hat{S}_+, \hat{S}_-\} \) transforms as a \( D_1 \) representation under the \( \mathfrak{s}\ell(2) \) subalgebra. \( \hat{C}_2 \) is a central element.

\[
[\hat{S}_0, \hat{S}_+] = (\hat{C}_2 - \hat{\mathcal{C}}')\hat{J}^+ \quad [\hat{S}_0, \hat{S}_-] = -(\hat{C}_2 - \hat{\mathcal{C}}')\hat{J}^- \quad [\hat{S}_+, \hat{S}_-] = 2(\hat{C}_2 - \hat{\mathcal{C}}')\hat{H}
\]

where \( \hat{\mathcal{C}}' = 2\hat{H}^2 + \hat{J}^+\hat{J}^- + \hat{J}^-\hat{J}^+ \) is the Casimir of the \( \mathfrak{s}\ell(2) \) subalgebra.

\[
\hat{C}_3 \propto \hat{H}\hat{S}_0 + \frac{1}{2}s_2(\hat{J}^-, \hat{S}_+) + \frac{1}{2}s_2(\hat{J}^+, \hat{S}_-)
\]

\[
\hat{C}_4 \propto 2s_2(\hat{S}_+, \hat{S}_-) + 2\hat{S}_0^2 + \hat{C}_2\hat{C}' - \frac{1}{2}\hat{C}'^2 - 2\hat{\mathcal{C}}'
\]

\( \mathcal{W}_4^{(2)} \simeq \mathfrak{s}\ell(2)_{\text{def}} \oplus \hat{C}_2 \oplus \hat{C}_3. \)

\( \hat{C}_2 \) and \( \hat{C}_3 \) are two central elements.

\[
[\hat{U}, \hat{G}^\pm] = \pm\hat{G}^\pm \quad [\hat{G}^+, \hat{G}^-] = -\frac{1}{3}\hat{C}_3 - \hat{C}_2\hat{U} + \hat{U}^3 - \hat{U}
\]

\[
\hat{C}_4 \propto 2s_2(\hat{G}^+, \hat{G}^-) + \frac{1}{2}\hat{U}^4 - \hat{C}_2\hat{U}^2 - \frac{2}{3}\hat{C}_3\hat{U} - \frac{1}{2}\hat{U}^2
\]

\( \mathcal{W}_4^{(1)} \simeq \hat{C}_2 \oplus \hat{C}_3 \oplus \hat{C}_4. \)

This is the well-known Casimir algebra of \( \mathfrak{s}\ell(4) \): a commutative algebra.
7.2.4 Commutant of second order

It might be interesting to present, as already done in the previous paragraph for $W_3^{(1)}$ and $W_3^{(2)}$, some relations between $W_4$ algebras using the above mentioned secondary reduction scheme. Their above given C.R. allow to recognize natural gradings in each of these algebras. Working first at the classical level, it will be possible to realize some of them as the commutant of a part of another $W_4$ one. In particular we have:

$$W_4^{(3)} = \text{Com}\, W_4^{(3)}(\hat{H_1}, \hat{E})$$
$$W_4^{(2)} = \text{Com}\, W_4^{(3)}(\hat{H_2}, \hat{E}, \hat{G_1}, \hat{G_1}^+)$$
$$W_4^{(2)} = \text{Com}\, W_4^{(3)}(\hat{H}, \hat{J}^+)$$

One might remark that the finite quantum algebras $W_4^{(3)}$ and $W_4^{(4)}$ can be related in this framework, while their classical affine analogues cannot be linked by $W$-gauge transformations (see [4]).

8 Affine case

The generalization of the above results when $\mathcal{G}$ is now an affine Kac-Moody algebra is rather straightforward. Indeed recalling (classical case) the transformation eq. (1.4)

$$J \rightarrow J^g = g J g^{-1} + k\partial g . g^{-1}$$

one notes that the affine term $k\partial g . g^{-1}$ is in $\mathcal{G}_+$ with $g \in G_+$, which ensures that our symmetry fixing of sections [2] and [3] directly applies. As developed in section [6], one thus obtains the algebras $W(\mathcal{G}, \mu_{s\ell}(2))$ up to center part. The adjunction of Cartan elements to the $\mathcal{G}_+$ transformations -see section [5]- can again be performed without difficulties in order to get rid of this center part. However to complete the set of $W$ algebras, one will use the secondary reduction technics already mentioned in section [7]. Thus, as in the finite case, one obtains a $W$ algebra from the computation of commutants. For the quantum case, we will have first to symmetrize and regularize (i.e. normal order) the classical expressions of the $W$ elements. Then we adjust, in a unique way, the scalar $k$-functions which appear in the $W$ expressions, by requiring again $W$ to be the commutant of a special subalgebra, either of $\mathcal{G}$ or, if a secondary "reduction" is necessary, of another $W$ one.

Let us illustrate our methods in the simplest cases. All quantities depend in the variable $z$. We denote by $s_j$ the regularized (i.e. normal ordered) and symmetrized product of $j$ fields. To recover from the quantum expressions of the $W$ elements their classical counterpart, one has just, forgetting the $s_j$'s, to replace each $k$-polynomial by its highest degree monomial. Note also that, as expected, one gets for each $W$ algebra at the quantum level the same central charge as the one obtained by the BRS cohomology method.

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• the Virasoro $\mathcal{W}_2$ algebra:

As usual we start with $\mathcal{G} = sl(2)^{(1)}$ with elements $\hat{J}_1(z)$ and $\hat{J}_{\pm\alpha}(z)$. We remark that the commutant of $\hat{J}_{-\alpha}$ contains $\hat{J}_{-\alpha}$ itself. To avoid this last field and determine uniquely $T$, we rather determine the commutant of $\hat{J}_{-\alpha}$ and $\hat{J}_1$.

$$T = \frac{1}{4(k+2)} \hat{J}_{-\alpha}^2 \left\{ 4s_4(\hat{J}_1, \hat{J}_1, \hat{J}_{-\alpha}, \hat{J}_{-\alpha}) + 4s_4(\hat{J}_{-\alpha}, \hat{J}_{-\alpha}, \hat{J}_{-\alpha}, \hat{J}_{-\alpha}) + 
 4(k+4) \left[ s_3(\hat{J}_1, \partial \hat{J}_{-\alpha}, \hat{J}_{-\alpha}) - s_3(\partial \hat{J}_1, \hat{J}_{-\alpha}, \hat{J}_{-\alpha}) \right] - 2\left( k^2 + \frac{7}{2} k + \frac{8}{3} \right) s_2(\partial^2 \hat{J}_{-\alpha}, \hat{J}_{-\alpha}) + 
 3(k^2 + 4k + \frac{14}{3}) s_2(\partial \hat{J}_{-\alpha}, \partial \hat{J}_{-\alpha}) \right\}$$

the central charge is equal to

$$c(k) = 1 - 6 \frac{(k+1)^2}{k+2}.$$ 

We recognize the central charge $c(p, q) = 1 - 6 \frac{(p-q)^2}{pq}$ of the Virasoro minimal models corresponding to $(p, q) = (k+2, 1)$.

Let us add that by taking the classical limit we recover exactly the usual Sugawara expression in the part of $T$ without derivatives.

• the Bershadsky $\mathcal{W}_3^{(2)}$ algebra:

Now we turn our attention to the $\mathcal{G} = sl(3)^{(1)}$ cases. In the following, we keep the notation of section 7.1. The commutant of the negatively graded part of $\mathcal{G}^*$ with respect to the principal gradation provides the classical $\mathcal{W}_3^{(2)}$ algebra plus a central term, which can be thrown away by enlarging $\mathcal{G}^*_-$ with the help of a Cartan generator (cf section 7.1).

As for the Virasoro algebra, the quantization reduces, after regularization and symmetrization, to a tuning of the k-coefficients obtained by imposing the OPE's between the $W$ generators and the extended $\mathcal{G}_-$ algebra, itself generated by $\hat{J}_1, \hat{J}_{-\alpha}, \hat{J}_{-\beta}, \hat{J}_{-\alpha-\beta}$, to vanish.

$$W_+ = \frac{1}{2\hat{J}_{-\alpha-\beta}^{\frac{3}{2}}} \left\{ 2s_3(\hat{J}_1, \hat{J}_{-\beta}, \hat{J}_{-\alpha-\beta}) + 3s_3(\hat{J}_2, \hat{J}_{-\beta}, \hat{J}_{-\alpha-\beta}) + 2s_3(\hat{J}_{-\alpha}, \hat{J}_{-\beta}, \hat{J}_{-\beta}) + 
 2s_3(\hat{J}_{-\alpha}, \hat{J}_{-\alpha-\beta}, \hat{J}_{-\alpha-\beta}) + 2(k + \frac{5}{2}) \left[ s_2(\hat{J}_{-\beta}, \partial \hat{J}_{-\alpha-\beta}) - s_2(\partial \hat{J}_{-\beta}, \hat{J}_{-\alpha-\beta}) \right] \right\}$$

$$W_- = \frac{1}{2\hat{J}_{-\alpha-\beta}^{\frac{3}{2}}} \left\{ 2s_3(\hat{J}_1, \hat{J}_{-\alpha}, \hat{J}_{-\alpha-\beta}) - 3s_3(\hat{J}_2, \hat{J}_{-\alpha}, \hat{J}_{-\alpha-\beta}) - 2s_3(\hat{J}_{-\alpha}, \hat{J}_{-\alpha}, \hat{J}_{-\beta}) + 
 2s_3(\hat{J}_{-\alpha}, \hat{J}_{-\alpha-\beta}, \hat{J}_{-\alpha-\beta}) + 2(k + \frac{5}{2}) \left[ s_2(\hat{J}_{-\alpha}, \partial \hat{J}_{-\alpha-\beta}) - s_2(\partial \hat{J}_{-\alpha}, \hat{J}_{-\alpha-\beta}) \right] \right\}$$
\[ Y = \frac{1}{J_{\alpha-\beta}} \left\{ s_2(J_2, J_{\alpha-\beta}) + s_2(J_{\alpha}, J_\beta) \right\} \]

\[ T = \frac{1}{4(k+3)J_{\alpha-\beta}^2} \left\{ 4s_4(J_{\alpha+\beta}, J_{\alpha-\beta}, J_{\alpha-\beta}, J_{\alpha-\beta}) + 4s_4(J_{\beta}, J_{\beta}, J_{\alpha-\beta}, J_{\alpha-\beta}) + 4s_4(J_{\beta}, J_{\beta}, J_{\alpha-\beta}, J_{\alpha-\beta}) + 
+ 4s_4(J_\alpha, J_\alpha, J_{\alpha-\beta}, J_{\alpha-\beta}) + 3s_4(J_2, J_2, J_{\alpha-\beta}, J_{\alpha-\beta}) + 4s_4(J_1, J_1, J_{\alpha-\beta}, J_{\alpha-\beta}) + 
+ (k+5) \left[ 4s_3(J_1, J_{\alpha-\beta}, \partial J_{\alpha-\beta}) - 4s_3(\partial J_1, J_{\alpha-\beta}, J_{\alpha-\beta}) + 2s_3(J_\alpha, \partial J_{\beta}, J_{\alpha-\beta}) + 
+ 2s_3(\partial J_{\alpha}, \partial J_{\beta}, J_{\alpha-\beta}) \right] + 3(k^2 + \frac{31}{6}k + \frac{43}{6})s_2(\partial J_{\alpha-\beta}, \partial J_{\alpha-\beta}) + 
- 2(k^2 + \frac{9}{2}k + \frac{23}{6})s_2(\partial J_{\alpha-\beta}, \partial^2 J_{\alpha-\beta}) \right\} \]

the central charge is equal to

\[ c(k) = 1 - \frac{(k+1)^2}{k+3}. \]

which correspond to the series \( c(p, q) = 1 - 12\frac{(p-q)^2}{pq} \) presented in [8] for \((p, q) = (k+3, 2)\).

-the Zamolodchikov \( W_3^{(1)} \) algebra:

The classical \( W_3^{(1)} \) algebra can be obtained by a secondary "reduction" of the \( W_3^{(2)} \) algebra, or in other words by determining the commutant in the (closure of the) enveloping algebra of \( W_3^{(2)} \) of the \( W_- \) and \( Y \) generators. Once again, quantization is achieved, after introduction of the \( s_j \) products, by imposing the OPE’s between the \( W_3^{(1)} \) generators and the previous \( W_- \) and \( Y \) ones to be zero, determining in this way the \( k \)-dependent quantum corrections. Hereafter, \( W_2 \) and \( W_3 \) are written as functions of the \( W_3^{(2)} \) generators. Their expressions in terms of \( \text{sl}(3)^{(1)} \) generators are obviously obtained by using the above given \( W_3^{(2)} \) realization.

\[ W_3 = \sqrt{\frac{-3}{8(3k+4)(5k+12)}} \frac{1}{(k+3)W_3^2} \left\{ s_5(W_+, W_-, W_-, W_-, W_-) + 4s_6(Y, Y, Y, W_-, W_-) + 
- 4(k+3)s_5(T, Y, W_-, W_-, W_-) - 6(k+9) s_5(\partial Y, W_-, W_-, W_-) + 
+ 4(k+\frac{15}{2}) s_5(Y, Y, \partial W_-, W_-, W_-) + 2(k+6)(k+3) s_4(\partial T, W_-, W_-, W_-) + 
- \left( \frac{8k}{3} + 14 \right)(k+3) s_4(T, \partial W_-, W_-, W_-) + (k^2 + 2k + 25) s_4(\partial^2 Y, W_-, W_-, W_-) + 
- (2k^2 + 16k + 33) s_4(\partial Y, \partial W_-, W_-, W_-) - (2k + 5)(k+3) s_4(Y, \partial^2 W_-, W_-, W_-) + 
+ 2\left( \frac{5}{3}k^2 + 13k + 33 \right) s_4(Y, \partial W_-, \partial W_-, W_-) + \left( \frac{2}{3}k^3 + 5k^2 + \frac{103}{9}k + \frac{7}{3} \right) s_3(\partial^3 W_-, W_-, W_-) + 
- \frac{1}{3}(10k^3 + 76k^2 + 181k + 129) s_3(\partial^2 W_-, \partial W_-, W_-) + 
+ \frac{1}{3}\left( \frac{80}{9}k^3 + 73k^2 + 205k + 207 \right) s_3(\partial W_-, \partial W_-, \partial W_-) \right\} \]
\[ W_2 = \frac{1}{(k+3)W_2^2} \left\{ (k+3)s_3(T, W_-, W_-) + (k+3)(-\frac{3}{2}s_3(\partial Y, W_-, W_-) + s_3(Y, \partial W_-, W_-)) + \\
\quad + (\frac{4}{3}k^2 + \frac{15}{2}k + \frac{21}{2})s_2(\partial W_-, \partial W_-) - (k^2 + \frac{23}{4}k + \frac{33}{4})s_2(\partial^2 W_-, W_-) \right\} \]

the central charge is equal to

\[ c(k) = 2(1 - 12\frac{(k+2)^2}{k+3}) \]

and we recognize the \( \mathcal{W}_3 \) minimal models

\[ c(p, q) = 2\left(1 - 12\frac{(p-q)^2}{pq}\right) \quad \text{with} \quad (p, q) = (k+3, 1) \]

Again, it is straightforward to obtain in the classical limit the Casimir expressions of \( W_2 \) and \( W_3 \) when restricting their expressions to terms without derivatives.

9 Conclusion

We have shown that each \( \mathcal{W} \) algebra, which arises from the usual Hamiltonian reduction, can explicitly be realized in terms of all the \( J \) components in the classical and quantum cases. The primary \( \mathcal{W} \) fields are no longer polynomials, but quotients of two polynomials. However the denominator quantities simply commute with all the numerator ones, allowing in particular to compute OPE without special difficulties in the quantum framework.

Actually, such an approach might be seen as a kind of generalized Sugawara construction. Considering, as an example, the Zamolodchikov \( \mathcal{W}_3^{(1)} \) algebra realization given in section 8, it seems natural to compare it with the Casimir algebra approach of \( \mathcal{W}_3 \). We remark that in our case, we are not restricted to the level value \( k = 1 \) of the \( \mathfrak{sl}(3)^{(1)} \) algebra in order to avoid the occurrence of extra fields and need not the help of the coset \( \mathfrak{G} \) \( \mathfrak{H} \) \( \mathfrak{g} \) \( \mathfrak{H} \). Moreover, let us emphasize that the developed technics in this paper do not limit to the Casimir-like \( \mathcal{W}_n \) algebras, but to any \( \mathcal{W}(\mathcal{G}, \mathcal{H}) \) algebra following the notation used in our text: see again in section 8 the realization of the Bershadsky \( \mathcal{W}_3^{(2)} \) algebra.

Let us also note that the generalization of such technics to (affine) superalgebras is a priori straightforward. Finally, it might be worthwhile to remind that in our construction the \( \mathcal{W} \) algebra appears as the commutant of a subalgebra. In this aspect, one can think in terms of a coset construction. Such a mathematical object, which is explicitly determined, can be of interest by itself. For example, in the finite case, it has been shown that \( \text{Com}(\mathcal{G}_-) \) with \( \mathcal{G}_- \) Abelian allows to construct general realizations of the algebra \( \mathcal{G} \) once a special \( \mathcal{G} \) representation is known—cf.\( \mathcal{W}_3 \).
A Appendix: Symmetry fixing for the principal gradation of $so(n)$ and $sp(2n)$ algebras

Before considering these different algebras, let us come back to the $s\ell(n)$ case developed in section \[3. We denote by $E_{-\psi_0}$ the smallest root generator, with $\psi_0 = \alpha_1 + \alpha_2 + \ldots + \alpha_{n-1}$, i.e. the sum of simple roots in $s\ell(n)$. From now on, we will work on the $n \times n$ matrix representation of $s\ell(n)$. Then, the generator $E_{-\psi_0}$ is represented by the matrix $e_{n,1}$.

At the first step, the positive root generators acting on $E_{-\psi_0}$ and which constitute $G_+^{(0)}$ (see eq. \[2.2\]) are those with non-zero entries on the first row or on the last column of the $n \times n$ matrix, except the two end points on the diagonal. One can therefore put to zero all the entries of the first column and the last row, included the Cartan generator $H_{\psi_0} = e_{1,1} - e_{n,n}$ on the diagonal, and excepted $E_{-\psi_0} = e_{n,1}$ itself.

At the second step, one acts on $E_{-\psi_1}$ with $\psi_1 = \alpha_2 + \ldots + \alpha_{n-2}$, and then describes the square $(n-2) \times (n-2)$ with the point $(n-1, 2)$ as the low-left corner. Iterating the process, one is led to the nested boxes represented in figure \[1. Note that the general element $E_{-\psi_j}$, $0 \leq j \leq \left[\frac{n}{2}\right]$ corresponds to the root

\[
\begin{align*}
\psi_j &= \alpha_{j+1} + \alpha_{j+2} + \ldots + \alpha_{n-j-1} \quad \text{for} \quad 0 \leq j \leq \left[\frac{n}{2}\right] - 1 \\
\psi_0 &= \left\{ \begin{array}{ll}
\frac{\alpha_1}{2} & \text{if } n \text{ even} \\
\frac{\alpha_{n-1} + \alpha_{n+1}}{2} & \text{if } n \text{ odd}
\end{array} \right. \label{eq:A.1}
\end{align*}
\]

These elements are the only ones below the diagonal that have not been set to zero in the process of fixing the symmetry. On the diagonal, we have set to zero all the Cartan generators $H_{\psi_j} = e_{j,j} - e_{n+1-j,n+1-j}$ with $j = 1, \ldots, \mu = \left[\frac{n}{2}\right]$. Thus, it remains $n-1-\mu$ generators, which can be taken for instance of the form $Y_j = e_{1,1} + e_{n,n} - e_{j,j} - e_{n+1-j,n+1-j}$ with $j = 2, \ldots, \left[\frac{n+1}{2}\right]$.

A.1 $so(2n+1)$ case

It might be useful to remind that the $so(2n+1)$ algebra can be obtained by a folding of the $s\ell(2n+1)$ one. Indeed, considering the $(2n+1) \times (2n+1)$ matrix representation of $s\ell(2n+1)$:

\[
M = m^{ij}e_{ij} \quad \text{with} \quad m^{ij} \in \mathbb{R} \quad \text{and} \quad \sum_{i=1}^{2n+1} m^{ii} = 0, \quad \label{eq:A.2}
\]

the $so(2n+1)$ elements are $s\ell(2n+1)$ elements satisfying the condition

\[
m^{ij} = (-)^{i+j+1} m^{2n+2-j,2n+2-i} \quad \forall i,j
\]

which reflects a (graded) symmetry w.r.t. the anti-diagonal.

Now, the smallest root generator reads

\[
E_{-\psi_0} = e_{2n,1} + e_{2n+1,2} \quad \text{with} \quad \psi_0 = \alpha_1 + 2\alpha_2 + \ldots + 2\alpha_n \quad \label{eq:A.4}
\]

The $G_+^{(0)}$ subalgebra appears to be made of elements with non-zero entries on the two first rows and two last columns, but without the element $E_{\alpha_1} = e_{12} + e_{2n,2n+1}$ and the diagonal.
The corresponding parameters of symmetry transformations will be fixed by the condition \( J_{G(0)} = 0 \) where \( G(0) \) is formed by the elements with non-zero entries on the two first rows and two last columns, including the diagonal (Cartan generator \( H_\psi = e_{1,1} - e_{2n+1,2n+1} \)), but without the element \( E_{-\alpha_1} = e_{21} + e_{2n+1,2n} \) and the root generator \( E_{-\psi_0} \).

Then, one will have to consider, at the second step, the \( \mathfrak{so}(2n-3) \) subalgebra made with the \( (2n-3) \times (2n-3) \) submatrix of low-left corner \( (2n-1,3) \). One will consider separately (in the last step) the \( \mathfrak{sl}(2) \) subalgebra, which commutes with \( \mathfrak{so}(2n-3) \) and is constructed from \( E_{\alpha_1} \), with negative root generator \( E_{-\alpha_1} = e_{21} + e_{2n+1,2n} \). In the \( \mathfrak{so}(2n-3) \) subalgebra, the lowest root generator is

\[
E_{-\psi_1} = e_{2n-2,3} + e_{2n-1,4} \quad \text{with} \quad \psi_1 = \alpha_3 + 2\alpha_4 + \ldots + 2\alpha_n \quad (A.5)
\]

Iterating the process leads to the chain of embeddings \( \mathfrak{so}(2n+1) \supset \mathfrak{so}(2n-3) \oplus \mathfrak{sl}(2), \mathfrak{so}(2n-3) \supset \mathfrak{so}(2n-7) \oplus \mathfrak{sl}(2), \) and so on: see figure 3. At the last step, we will deal with a direct sum of \( \mathfrak{sl}(2) \) subalgebras, and the parameters of the corresponding root generators \( E_{\alpha_1}, E_{\alpha_3}, \ldots \) will be fixed by annihilating the associated Cartan generators \( H_{\alpha_1}, H_{\alpha_3}, \ldots \).

It follows that the generators of non-positive grade which cannot be eliminated in the process are the elements \( E_{-\psi_j} \) with \( \psi_j \) in the set

\[
\begin{align*}
\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_n & ; \quad \alpha_1 \quad ; \quad \alpha_3 + 2\alpha_4 + \ldots + 2\alpha_n \\
\alpha_3 & ; \quad \alpha_5 + 2\alpha_6 + \ldots + 2\alpha_n & ; \quad \ldots ; \quad \left\{ \begin{array}{ll}
& \text{If } n \text{ odd} \quad \alpha_{n-2} \quad ; \quad \alpha_n \\
& \text{If } n \text{ even} \quad \alpha_{n-1}
\end{array} \right.
\end{align*}
\]

\[
\text{Figure 3: Remaining non-zero entries on and below the diagonal, after fixing of the symmetry}
\]
A.2 \textit{so}(2n) case

Deleting the \((n + 1)\)th row and the \((n + 1)\)th column in the \(so(2n + 1)\) matrix leads to a matrix realization of the \(so(2n)\) algebra. It is a simple exercise to check that the same kind of nesting can be obtained, \textit{i.e.} \(so(2n) \supset so(2n - 4) \oplus sl(2)\), \(so(2n - 4) \supset so(2n - 8) \oplus sl(2)\), and so on. The list of roots corresponding to the remaining (negatively graded) generators is now:

\[
\begin{align*}
\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n & ; \quad \alpha_1 ; \quad \alpha_3 + 2\alpha_4 + \ldots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n \\
\alpha_3 ; \quad \alpha_5 + 2\alpha_6 + \ldots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n & ; \ldots ; \quad \left\{ \begin{array}{l}
\alpha_{n-4} ; \quad \alpha_{n-2} + \alpha_{n-1} + \alpha_n ; \quad \alpha_{n-2} ; \quad \alpha_n & \text{if } n \text{ odd} \\
\alpha_{n-3} ; \quad \alpha_{n-1} + \alpha_n ; \quad \alpha_{n-2} ; \quad \alpha_n & \text{if } n \text{ even}
\end{array} \right\
\end{align*}
\]

We get the same picture as in figure 3 for the non-zero generators of non-positive grade that remain after fixing of the symmetry.

A.3 \textit{sp}(2n) case

The \(sp(2n)\) algebra can be obtained from the folding of the \(sl(2n)\) algebra. Again, we will impose to the \(sl(2n)\) elements a graded symmetry as we did above with \(sl(2n + 1)\) elements in the \(so(2n + 1)\) case. Then, as in the \(sl(n)\) case, the \(G_+^{(0)}\) subalgebra will be constituted by the first row and last column with exception of the diagonal. We will have a situation analogous to the one of figure 4 for \(sl(n)\), with the embeddings \(sp(2n) \supset sp(2n - 2) \supset sp(2n - 4) \supset \ldots\). Now, the set of roots corresponding to the negatively graded remaining generators stand as follows:

\[
\begin{align*}
2\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{n-1} + \alpha_n & ; \quad 2\alpha_2 + \ldots + 2\alpha_{n-1} + \alpha_n \\
2\alpha_3 + \ldots + 2\alpha_{n-1} + \alpha_n & ; \quad \ldots ; \quad 2\alpha_{n-1} + \alpha_n ; \quad \alpha_n
\end{align*}
\] (A.7)
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