Characterization of entanglement in multiqubit systems via spin squeezing

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by

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Theoretical studies on the non-classicality of quantum state
DEDICATED TO MY

Parents,

Shri. M. Srinivas Rao
AND
Smt. Annapoorna
Theoretical studies on the non-classicality of quantum state
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# Contents

1 Introduction 1

2 Symmetric two qubit local invariants 7
   2.1 Introduction 7
   2.2 Arbitrary two qubit density matrix 10
   2.3 Local invariants of an arbitrary two qubit system 12
   2.4 Invariants for symmetric two-qubit states 14
   2.5 Special class of two qubit states 20
   2.6 Characterization of entanglement in symmetric two qubit states 22
   2.7 Necessary and sufficient criterion for a class of symmetric two qubit states 26
   2.8 Conclusions 29

3 Collective signatures of entanglement in symmetric multiqubit systems 30
   3.1 Introduction 30
   3.2 Collective spin observables in terms of two qubit variables 31
   3.3 Collective signatures of pairwise entanglement 34
      3.3.1 Spin squeezing in terms of the local invariant \( I_5 \) 36
      3.3.2 Collective signature in terms of \( I_4 \) 40
3.3.3 $I_4 - I_3^2$ in terms of the collective variables

3.3.4 Characterization of pairwise entanglement through $I_1$

3.4 Classification of pairwise entanglement

3.5 Conclusions

4 Dynamical models

4.1 Dicke State

4.1.1 Two qubit state parameters for Dicke state

4.1.2 Local invariants

4.2 Kitagawa-Ueda state generated by one axis twisting Hamiltonian

4.2.1 Two qubit state variables

4.2.2 Local invariants

4.3 Atomic spin squeezed states

4.3.1 Two qubit state parameters

4.3.2 Local invariants

4.4 Conclusions

5 Constraints on the variance matrix of entangled symmetric qubits

5.1 Peres-Horodecki inseparability criterion for CV states

5.2 Two qubit covariance matrix

5.3 Inseparability constraint on the covariance matrix

5.4 Complete characterization of inseparability in mixed two qubit symmetric states

5.5 Local invariant structure

5.6 Implications of $C < 0$ in symmetric $N$ qubit systems
5.7 Equivalence between the generalized spin squeezing inequalities and negativity of $C$ ................................................. 93

5.8 Conclusions ................................................. 95

6 Summary ....................................................... 96

Appendices ...................................................... 97

A Pure and mixed density operators .................................................. 98

B Peres PPT criterion .................................................. 104

C A Complete set of 18 invariants for an arbitrary two qubit state 107

Bibliography ....................................................... 112

List of Publications .................................................. 117
Chapter 1

Introduction

Quantum world opens up several puzzling aspects that are not amenable to classical intuition. Correlation exhibited by subsystems of a composite quantum state is one such striking feature and has been a source of philosophical debates - following the famous Einstein-Podolsky-Rosen discussion on the foundational aspects of quantum theory. It is now well established that entangled states play a crucial role in the modern quantum information science, including quantum cryptography [1], quantum communication and quantum computation [2-4].

Considerable interest has been evinced recently [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15] in producing, controlling and manipulating entangled multiqubit systems due to the possibility of applications in atomic interferometry [16, 17], high precession atomic clocks [18], quantum computation and quantum information processing [2]. Multiqubit systems, which are symmetric under permutation of the particles, allow for an elegant description in terms of collective variables of the system. Specifically, if we have \( N \) qubits, each qubit may be represented as a spin-\( \frac{1}{2} \) system and theoretical analysis in terms of collective spin operator

\[
\vec{J} = \frac{1}{2} \sum_{\alpha=1}^{N} \vec{\sigma}_\alpha
\]

(\( \vec{\sigma}_\alpha \) denote the Pauli spin operator of the \( \alpha^{th} \) qubit), leads to reduction of the dimension of the Hilbert space from \( 2^N \) to \( (N + 1) \), when the multiqubit system respects exchange symmetry. A large number of experimentally relevant multiqubit states
exhibit symmetry under interchange of qubits, facilitating a significant simplification in understanding the properties of the physical system. While complete characterization of multiqubit entanglement still remains a major task, collective behavior such as spin squeezing \[5, 6, 7, 8, 9, 10, 11, 12, 13, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30\], exhibited by multiqubit systems, has been proposed as a signature of quantum correlation between the atoms. A connection between spin squeezing and the nature of quantum entanglement has been explored \[29, 30\] and it is shown that the presence of spin squeezing essentially reflects pairwise entanglement. However, it is important to realize that spin squeezing serves only as a sufficient condition - not a necessary one - for pairwise entanglement. There will still be pairwise correlated states, which do not exhibit spin squeezing. In a class of symmetric multiqubit states it has been shown \[30\] that spin-squeezing and pairwise entanglement imply each other. Questions like “Are there any other collective signatures of pairwise entanglement?” are still being investigated. Recently, inequalities generalizing the concept of spin squeezing have been derived \[31\]. These inequalities are shown to provide necessary and sufficient conditions for pairwise entanglement and three-party entanglement in symmetric \(N\)-qubit states.

In this thesis, we have addressed the problem of characterizing pairwise entanglement in symmetric multiqubit systems in terms of two qubit local invariants \[32, 33, 34\]. This is important because quantum entanglement reflects itself through non-local correlations among the subsystems of a quantum system. These non-local properties remain unaltered by local manipulations on the subsystems and provide a characterization of quantum entanglement.

Two composite quantum states \(\rho_1\) and \(\rho_2\) are said to be equally entangled if they are related to each other through local unitary operations, which merely imply a choice of bases in the spaces of the subsystems. One may define a polynomial invariant, which is by definition any real valued function of density operators, taking the same value for equally entangled density operators \(\rho\).
Introduction

Basic issues of importance would then be

- to find complete set of polynomial entanglement invariants which assume identical values for density operators related to each other through local unitary operators.

- decide whether the set of separable states can be described in terms of a polynomial invariant \( f \), such that \( f(\rho) \geq 0 \) is equivalent to separability [35].

In this context, Y. Makhlin [36] has studied the entanglement invariants of an arbitrary mixed state of two-qubits and has identified a complete set of 18 local invariants characterizing the system. A set of 8 polynomial invariants has been identified in the case of pure three qubit states [37]. Linden et. al. [38] have outlined a general prescription to identify the invariants associated with a multi particle system [1]. Here, we focus on constructing a complete set of local invariants characterizing symmetric two qubit systems and analyzing the pairwise entanglement properties like collective spin squeezing - exhibited by multiqubits - in terms of two qubit entanglement invariants.

A brief Chapter wise summary of the thesis is given below.

Chapter 2: Symmetric two qubit local invariants

For an arbitrary two qubit mixed state, Makhlin [36] has proposed a complete set of 18 polynomial invariants. In this Chapter, we show that the number of invariants reduces from 18 to 6 in the case of symmetric two qubit states owing to the exchange symmetry. We quantify entanglement in symmetric two qubit states in terms of these complete set of six invariants. More specifically, we prove that the negative values of some of the invariants serve as signatures of quantum entanglement in symmetric two qubit states. This leads us to identify sufficient conditions for non-separability in terms of entanglement invariants [32]. Further, these conditions on invariants are shown here to be both necessary and sufficient for entanglement in a class of symmetric two qubit states.

[1] However, separability properties of two qubit states in terms of the local invariants, is not investigated in Ref. [36][37][38]
Chapter 3: Characterization of pairwise entanglement in symmetric multiqubit systems

As discussed in Chapter 2, some of the symmetric two qubit invariants reflect nonseparability [32]. In this Chapter, we focus on the characterization and classification of pairwise entanglement in symmetric multi-qubit systems, via local invariants associated with a random pair of qubits drawn from the collective systems. In other words, we investigate collective signatures of pairwise entanglement in symmetric N-qubit states as implied by the associated non-positive values of the two qubit invariants. More specifically, we identify here that a symmetric multi-qubit system is spin squeezed iff one of the entanglement invariant is negative. An explicit classification, based on the structure of local invariants for pairwise entanglement in symmetric N-qubit states is given [33]. We show that our characterization gets related to the generalized spin squeezing inequalities of Korbicz et al [31].

Chapter 4: Analysis of few dynamical models

In the light of our characterization of pairwise entanglement in symmetric multiqubit states discussed in Chapter 3, we analyze some of the experimentally relevant N qubit permutation symmetric states and explicitly demonstrate the non-separability of such states as exhibited through two qubit local invariants. In particular, we evaluate the two qubit local invariants and hence discuss the collective pairwise entanglement properties in the following multiqubit states:

1. Dicke states [39, 40]
2. Kitagawa-Ueda state generated by one axis twisting Hamiltonian [19]
3. Atomic squeezed states [20].
Introduction

Chapter 5: Necessary and sufficient criterion in symmetric two qubit states

Continuous variable systems (CV) \[41\] i.e., systems associated with infinite dimensional spaces are a focus of interest and attention due to their practical relevance in applications to quantum optics and quantum information science. Moreover two mode Gaussian states, a special class of CV systems provide a clean framework for the investigation of nonlocal correlations. Consequently, most of the results on CV entanglement have been obtained for Gaussian states. Entanglement for two-mode Gaussian states is \textit{completely} captured in its \textit{covariance matrix}. It is desirable to look for an analogous covariance matrix pattern in finite dimensional systems - in particular in multiqubits.

In this Chapter, we identify such a structural parallelism \[34\] between continuous variable states \[42\] and symmetric two qubit systems by constructing covariance matrix of the latter. Pairwise entanglement between any two qubits of a symmetric \(N\) qubit state is shown to be \textit{completely} characterized by the off-diagonal block of the two qubit covariance matrix. We establish the inseparability constraints satisfied by the covariance matrix \[34\] and identify that these are equivalent to the generalized spin squeezing inequalities \[31\] for pairwise entanglement. The interplay between two basic principles viz, the uncertainty principle and the nonseparability gets highlighted through the restriction on the covariance matrix of a quantum correlated two qubit symmetric state. So, the collective pairwise entanglement properties of symmetric multiqubit states depends entirely on the off diagonal block of the covariance matrix. We further establish an equivalence between the Peres-Horodecki \[43\] \[44\] criterion and the negativity of the covariance matrix \(\mathcal{C}\) showing that our condition is both necessary and sufficient for entanglement in symmetric two qubit states. We continue to identify the constraints satisfied by the collective correlation matrix \(V^{(N)}\) of pairwise entangled symmetric \(N\) qubit states.
In other words, the local invariant separability condition necessarily implies that

The symmetric $N$ qubit system is pairwise entangled iff the least eigen value of the real symmetric matrix $V^{(N)} + \frac{1}{N} SST$ is less than $N/4$.

($V^{(N)}$ denotes the collective covariance matrix and $S$ corresponds to the collective average spin of the symmetric $N$-qubit system.)

**Chapter 6: Summary**

In this Chapter, we briefly summarize the important results obtained in this thesis.
Chapter 2

Symmetric two qubit local invariants

2.1 Introduction

Initiated by the celebrated Einstein-Podolsky-Rosen criticism [45], counterintuitive features of quantum correlations have retained the focus for more than seven decades now, and quantum entanglement has emerged as an essential ingredient in the rapidly developing area of quantum computation and quantum information processing [2, 3, 4]. Characterization and quantification of entanglement has been one of the central tasks of quantum information theory. In simple terms, a bipartite quantum system is entangled, if it is not separable i.e., if the density matrix cannot be expressed as a convex mixture of product states,

\[ \rho = \sum_w p_w \rho^{(1)}_w \otimes \rho^{(2)}_w \text{ where } 0 \leq p_w \leq 1 \text{ and } \sum_w p_w = 1. \]  

Here, \( \{\rho^{(1)}_w\} \) and \( \{\rho^{(2)}_w\} \) denote a set of density operators associated with quantum systems 1 and 2. It is a non-trivial task to check whether a given state is expressible as a mixture of product states (see Eq. (2.1)) or not.

Peres [43] has identified that the partial transpose of a separable bipartite state \( \rho \) is positive definite (See Appendix [3]) and therefore negative eigenvalues of a partially transposed density matrix imply non-separability of a quantum state. Further, Horodecki et. al [44]
proved that negativity under partial transpose provides a necessary and sufficient condition for quantum entanglement in $2 \otimes 2$ and $2 \otimes 3$ systems only.

It is possible to quantify the amount of entanglement in a bipartite pure state $|\psi\rangle$ through the von Neumann entropy of either of the two subsystems [46]

$$E(\psi) = -\text{Tr}(\rho_A \log_2 \rho_A) = -\text{Tr}(\rho_B \log_2 \rho_B)$$  \hspace{1cm} (2.2)

where,

$$\rho_A = \text{Tr}_B (|\psi\rangle\langle\psi|)$$

and

$$\rho_B = \text{Tr}_A (|\psi\rangle\langle\psi|)$$

denote subsystem density matrices. $E(\psi)$ is referred to as Entropy of entanglement. $\lim_{n \to \infty} nE(\psi)$ gives the number of maximally entangled states that can be formed with $n$ copies of $|\psi\rangle$, in the asymptotic limit.

The entanglement of formation of a mixed bipartite state $\rho$ is defined as the minimum average entanglement

$$E(\rho) = \min \sum_i p_i E(\psi_i),$$  \hspace{1cm} (2.3)

where $|\psi_i\rangle$ corresponds to all possible decompositions of the state through

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|.$$  

Entanglement of formation $E(\rho)$ reduces to entropy of entanglement $E(\psi)$ in the case of pure states and is zero iff the state is separable.

An explicit analytical expression for the entanglement of formation has been derived for an arbitrary pair of qubits [47, 48] and is given by:

$$E(\rho) = h \left( \frac{1 + \sqrt{1 + C^2}}{2} \right),$$  \hspace{1cm} (2.4)
Symmetric two qubit local invariants

where

\[ h(x) = -x \log_2 x - (1 - x) \log_2 (1 - x). \]

In Eq. (2.4), \( C \), the Concurrence \(^{[48]}\) is given by

\[ C = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4), \]

with \( \lambda_1^2, \lambda_2^2, \lambda_3^2, \lambda_4^2 \) denoting the eigenvalues of

\[ \rho(\sigma_2 \otimes \sigma_2) \rho^*(\sigma_2 \otimes \sigma_2) \]

in the decreasing order. Here,

\[ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (2.5) \]

is the standard Pauli matrix and \( \rho \) is expressed in the standard two qubit basis set

\[ \{|0_1 0_2\}, |0_1 1_2\}, |1_1 0_2\}, |1_1 1_2\}. \]

The Concurrence \( C \) varies from zero to one and is monotonically related to entanglement of formation \( E(\rho) \), thus gaining the status of a measure of entanglement on its own \(^{[48]}\).

Entanglement properties of a quantum system remain unaltered when the subsystems are locally manipulated and two quantum states \( \rho_1 \) and \( \rho_2 \) are equally entangled if they are related to each other through local unitary transformations. Non-separability of a quantum state may thus be represented through a complete set of local invariants which contains functions of the quantum state that remain unchanged by local unitary operations on the subsystems. In this Chapter, we investigate a complete set of local invariants for arbitrary symmetric two qubit states.

We identify that a set of six invariants is sufficient to characterize a symmetric two qubit system, provided the average spin \( |\langle \vec{\sigma} \rangle| \) of the qubits is non-zero. If \( |\langle \vec{\sigma} \rangle| = 0 \), only two entanglement invariants represent the nonseparability of the system.
We further show that all the invariants associated with separable symmetric systems are positive. This allows us to identify criteria for non-separability in terms of invariants.

### 2.2 Arbitrary two qubit density matrix

Density matrix (See Appendix A) of an arbitrary two-qubit state in the Hilbert-Schmidt space \( \mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \) is given by

\[
\rho = \frac{1}{4} \left( I \otimes I + \sum_{i=1}^{3} s_i \sigma_{1i} + \sum_{i=1}^{3} \sigma_{2i} r_i + \sum_{i,j=1}^{3} t_{ij} \sigma_{1i} \sigma_{2j} \right),
\]

where \( I \) denotes the \( 2 \times 2 \) unit matrix. Here,

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

are the standard Pauli spin matrices and

\[
\sigma_{1i} = \sigma_i \otimes I \\
\sigma_{2i} = I \otimes \sigma_i.
\]

The average spins of the qubits \( \bar{s} = (s_1, s_2, s_3) \) and \( \bar{r} = (r_1, r_2, r_3) \) are given by

\[
s_i = \text{Tr} (\rho \sigma_{1i}) \\
r_i = \text{Tr} (\rho \sigma_{2i})
\]

and the two-qubit correlations are given by,

\[
t_{ij} = \text{Tr} [\rho (\sigma_{1i} \sigma_{2j})].
\]
Symmetric two qubit local invariants

It is convenient to express the two qubit correlations \( t_{ij}(i, j = 1, 2, 3) \), in the form of a \( 3 \times 3 \) matrix as follows:

\[
T = \begin{pmatrix}
  t_{11} & t_{12} & t_{13} \\
  t_{21} & t_{22} & t_{23} \\
  t_{31} & t_{32} & t_{33}
\end{pmatrix}
\] (2.11)

The correlation matrix \( T \) is real and in general, nonsymmetric.

**Transformation of state parameters under local unitary operations:**

The 15 parameters \( \{s_i, r_i, t_{ij}\}; i, j = 1, 2, 3 \), characterizing two qubit density matrix of Eq. (2.6) exhibit the following transformation properties under local unitary operations:

\[
s_i' = \sum_{j=1}^{3} O^{(1)}_{ij} s_j,
\]

\[
r_i' = \sum_{j=1}^{3} O^{(2)}_{ij} r_j,
\]

\[
t_{ij}' = \sum_{k,l=1}^{3} O^{(1)}_{ik} O^{(2)}_{jl} t_{kl} \quad \text{or} \quad T' = O^{(1)} T O^{(2)^T},
\]

(2.12)

where \( O^{(1)}, O^{(2)} \in SO(3, R) \) are the \( 3 \times 3 \) rotation matrices, uniquely corresponding to the \( 2 \times 2 \) unitary matrices \( U_i \in SU(2) \). The above transformation properties, facilitate the construction of polynomial functions of state parameters \( \{s_i, r_i, t_{ij}\} \) which remain invariant under local operations on individual qubits. We devote the next section for discussion of a complete set of local invariants associated with an arbitrary two qubit density matrix—which was proposed by Makhlin [36].
2.3 Local invariants of an arbitrary two qubit system

As has been emphasized earlier, genuine nonlocal properties should be described in terms of physical quantities that are invariant under local unitary operations. Makhlin [36] investigated such local invariant properties of mixed states of two-qubit system. Two density matrices \( \rho_1 \) and \( \rho_2 \) are called locally equivalent if one can be transformed into the other by local operations

\[
\rho_2 = (U_1 \otimes U_2) \rho_1 (U_1 \otimes U_2)^\dagger.
\]

A useful tool for verification of local equivalence of two states is a complete set of invariants that distinguishes all inequivalent states:

If each invariant from the set has equal values on two states \( \rho_1, \rho_2 \), their local equivalence is guaranteed.

A complete set of invariants for an arbitrary two qubit system as given by Makhlin [36] is listed in Table 2.1.

It is clear that all the invariants \( I_k, k = 1, 2, \ldots, 18 \), listed in Table 2.1, are invariant under local unitary transformations as can be verified by explicitly substituting Eqs. (2.12), (2.13) for transformed state parameters. For example consider the invariant \( I_4 \) which, under local unitary operation, transform as,

\[
I_4 = s'^T s' = s^T O^{(1)}^T O^{(1)} s
\]

\[
= s^T s \quad \text{since} \quad O^{(1)}^T O^{(1)} = 1. \quad (2.14)
\]

Similarly, it is easy to identify that

\[
I_{12} = s'^T T^r s' = s^T O^{(1)}^T O^{(1)} T O^{(2)}^T O^{(2)} r
\]

\[
= s^T T r. \quad (2.15)
\]
Symmetric two qubit local invariants

\[ I_1 = \det T \]
\[ I_2 = \text{Tr} (T^T T) \]
\[ I_3 = \text{Tr} (T^T T)^2 \]
\[ I_4 = s^T s \]
\[ I_5 = s^T T T^T s \]
\[ I_6 = s^T (T T^T)^2 s \]
\[ I_7 = r^T r \]
\[ I_8 = r^T T T^T r \]
\[ I_9 = r^T (T T^T)^2 r \]
\[ I_{10} = \epsilon_{ijk} s_i (T T^T s)_j ([T T^T]^2 s)_k \]
\[ I_{11} = \epsilon_{ijk} r_i (T^T T r)_j ([T T^T]^2 r)_k \]
\[ I_{12} = s^T T r \]
\[ I_{13} = s^T T T^T T r \]
\[ I_{14} = \epsilon_{ijk} \epsilon_{lmn} s_i r_l t_j m t_k n \]
\[ I_{15} = \epsilon_{ijk} s_i (T T^T s)_j (T r)_k \]
\[ I_{16} = \epsilon_{ijk} (T^T s)_i r_j (T T^T T r)_k \]
\[ I_{17} = \epsilon_{ijk} (T^T s)_i (T T T T^T s)_j r_k \]
\[ I_{18} = \epsilon_{ijk} s_i (T r)_j (T T^T T r)_k \]

Table 2.1: Complete set of 18 polynomial invariants for an arbitrary two qubit state.

Makhlin [36] has given an explicit procedure to find local unitary operations that transform any equivalent density matrices to a specific canonical form, uniquely determined by the set of 18 invariants given in Table 2.1. Further, it has been shown that when \(T^T T\) is nondegenerate, the entire set of invariants \(I_1-18\) is required to completely specify the canonical form of locally equivalent (See Appendix C) density matrices. However, when \(T^T T\) is degenerate, only a subset of 18 invariants would suffice for the complete specification of density matrices which are locally related to each other. In particular, (i) when two of the eigenvalues of \(T^T T\) are equal then only a subset of nine invariants \(\{I_{4-9}, I_{12-14}\}\) are required and (ii) when all the three eigenvalues of \(T^T T\) are equal, the subset \(\{I_{4-9}, I_{12}\}\) containing six invariants determines the canonical form of the density matrices.

In the next section, we identify that the number of invariants required to characterize an arbitrary symmetric two-qubit system reduces from 18 (as proposed by Makhlin [35]) to 6. Moreover, we consider a specific case of symmetric two-qubit system, and show that a subset of three independent invariants is sufficient to determine the non-local properties
Symmetric two qubit local invariants

completely.

2.4 Invariants for symmetric two-qubit states

Symmetric two qubit states $\rho_{\text{sym}}$, which obey exchange symmetry, are defined by,

$$\Pi_{12} \rho_{\text{sym}} = \rho_{\text{sym}} \Pi_{12} = \rho_{\text{sym}},$$

where $\Pi_{12}$ denotes the permutation operator.

Quantum states of symmetric two qubits get confined to a three-dimensional subspace of the Hilbert space. Explicitly the angular momentum states $\{\left| N/2 = 1, M = 1 \right>, \left| N/2 = 1, M = 0 \right>, \left| N/2 = 1, M = -1 \right>\}$ are related to the standard two qubit states $\{|0_1, 0_2\}, |0_1, 1_2\}, |1_1, 0_2\}, |1_1, 1_2\}$ as follows:

$$\left| N/2 = 1, M = 1 \right> = |0_1, 0_2\>,$$

$$\left| N/2 = 1, M = 0 \right> = \frac{1}{\sqrt{2}} (|0_1, 1_2\> + |1_1, 0_2\>),$$

$$\left| N/2 = 1, M = -1 \right> = |1_1, 1_2\>. \quad (2.16)$$

An arbitrary symmetric two qubit density matrix $\rho_{\text{sym}}$ has the form:

$$\rho_{\text{sym}} = \frac{1}{4} \left( I \otimes I + \sum_{i=1}^{3} s_i (\sigma_{1i} + \sigma_{2i}) + \sum_{i,j=1}^{3} t_{ij} \sigma_{1i} \sigma_{2j} \right), \quad (2.17)$$

where,

$$\text{Tr}(\rho_{\text{sym}} \sigma_{1i}) = \text{Tr}(\rho_{\text{sym}} \sigma_{2i}),$$

or $r_i = s_i, \quad (2.18)$

---

1The collective angular momentum basis states $\{\left| N/2, M \right>; M = -N \leq M \leq N\}$ span the Hilbert space $\mathcal{H}_{\text{sym}} = (C^2 \otimes C^2 \ldots \otimes C^2)_{\text{sym}}$ of symmetric $N$ qubit system. i.e., the dimension of the Hilbert space gets reduced from $2^N$ to $(N + 1)$ for symmetric $N$ qubit states.
Symmetric two qubit local invariants

and

\[ \text{Tr}[\rho_{\text{sym}}(\sigma_1;\sigma_2)] = \text{Tr}[\rho_{\text{sym}}(\sigma_2;\sigma_1)], \]
\[ \text{or } t_{ij} = t_{ji} \implies T^T = T. \] \hspace{1cm} (2.19)

(Please see Eqs. (2.6) - (2.10) for a comparison with the density matrix of an arbitrary two qubit system.)

Obviously, the number of state parameters for a symmetric two qubit system gets reduced from 15 (for an arbitrary two qubit state) to 9 owing to symmetry constraints given in Eqs. (2.18), (2.19). Moreover, there is one more constraint on correlation matrix \( T \) reducing the number of parameters required to characterize a symmetric two qubit system to 8. To see this, let us consider the collective angular momentum of two qubit system which is given by

\[ \vec{J} = \frac{1}{2}(\vec{\sigma}_1 + \vec{\sigma}_2). \] \hspace{1cm} (2.20)

We have,

\[ \text{Tr}[\rho_{\text{sym}}(\vec{J} \cdot \vec{J})] = 2, \] \hspace{1cm} (2.21)

for symmetric two qubit states. This is because, the symmetric two qubit system is confined to the maximum value of angular momentum \( j = 1 \) in the addition of two spin \( \frac{1}{2} \) (qubit) systems. (Note that in the addition of angular momentum of two spin \( \frac{1}{2} \) particles the total angular momentum can take values 0 and 1. The two qubit states with 0 net angular momentum are antisymmetric under interchange - whereas those with maximum angular momentum 1 are symmetric under interchange of particles.) Using Eq. (2.20) we obtain,

\[ \vec{J} \cdot \vec{J} = \frac{1}{4}[((\vec{\sigma}_1 + \vec{\sigma}_2) \cdot (\vec{\sigma}_1 + \vec{\sigma}_2)], \]
\[ = \frac{1}{4}[(\vec{\sigma}_1 \cdot \vec{\sigma}_1) + (\vec{\sigma}_2 \cdot \vec{\sigma}_2) + 2(\vec{\sigma}_1 \cdot \vec{\sigma}_2)]. \] \hspace{1cm} (2.22)
Since $$(\vec{\sigma}_1 \cdot \vec{\sigma}_1) = (\vec{\sigma}_2 \cdot \vec{\sigma}_2) = 3I$$, we can rewrite the above equation as,

$$\vec{J} \cdot \vec{J} = \frac{1}{4} [6I + 2(\vec{\sigma}_1 \cdot \vec{\sigma}_2)].$$

(2.23)

In other words, we have,

$$\text{Tr} [\rho_{\text{sym}}(\vec{J} \cdot \vec{J})] = \frac{1}{2} (3 \text{Tr}(I) + \text{Tr}[\rho_{\text{sym}}(\vec{\sigma}_1 \cdot \vec{\sigma}_2)])$$

$$= \frac{1}{2} [3 + \text{Tr}(T)].$$

(2.24)

From Eq. (2.21) we have,

$$\frac{1}{2} [3 + \text{Tr}(T)] = 2,$$

for symmetric two qubit systems. This in turn leads to the constraint

$$\text{Tr}(T) = 1.$$ 

(2.25)

Thus only eight real state parameters viz., three real parameters $s_i$ and five parameters $t_{ij}$, (completely specifying the $3 \times 3$ real symmetric two qubit correlation matrix $T$ with unit trace) determine a symmetric two qubit system.

We give below an explicit $4 \times 4$ matrix form (in the standard two-qubit basis $|0_1 \ 0_2\rangle, |0_1 \ 1_2\rangle, |1_1 \ 0_2\rangle, |1_1 \ 1_2\rangle$) of an arbitrary symmetric two qubit density matrix:

$$\rho_{\text{sym}} = \frac{1}{4} \begin{pmatrix}
1 + 2s_3 + t_{33} & A^* & A^* & (t_{11} - t_{22}) - 2it_{12} \\
A & (t_{11} + t_{22}) & (t_{11} + t_{22}) & B^* \\
A & (t_{11} + t_{22}) & (t_{11} + t_{22}) & B^* \\
(t_{11} - t_{22}) + 2it_{12} & B & B & 1 - 2s_3 + t_{33}
\end{pmatrix},$$

(2.26)

where $A = (s_1 + is_2) + (t_{13} + it_{23})$ and $B = (s_1 + is_2) - (t_{13} + it_{23})$.

In the following discussion, we show that the number of local invariants required to determine a symmetric two qubit density matrix also reduces, owing to the symmetry constraints.
on the state parameters Eqs. (2.18), (2.19).

**Symmetric two qubit local invariants:**

In the case of symmetric two qubit system, it is easy to see that the 18 polynomial invariants (for an arbitrary two qubit system) given in Table 2.1 in terms of the qubit average $\vec{s}$ and two qubit correlations $T$ reduces to twelve:

\[
\begin{align*}
I_1 &= \det T & I_4 &= I_7 = s^T s, \\
I_2 &= \text{Tr}(T^2) & I_5 &= I_8 = s^T T T^T s \\
I_3 &= \text{Tr}(T^2)^2 & I_6 &= I_9 = s^T (T T^T)^2 s \\
I_{12} &= s^T T s, & I_{10} &= I_{11} = \epsilon_{ijk} s_i (T T^T s)_j ([T T^T]^2 s)_k \\
I_{13} &= s^T T T^T T s & I_{15} &= I_{16} = \epsilon_{ijk} s_i (T T^T s)_j (T s)_k \\
I_{14} &= \epsilon_{ijk} \epsilon_{lmn} s_i s_l t_{jm} t_{kn} & I_{17} &= I_{18} = \epsilon_{ijk} s_i (T s)_j (T T^T T s)_k
\end{align*}
\]

Table 2.2: Invariants for a symmetric two qubit state.

We now proceed to identify that a set containing six local invariants \{I_{1–6}\} is sufficient to determine the canonical form of locally equivalent symmetric two qubit density matrices.
Symmetric two qubit local invariants

**Theorem 2.1** All equally entangled symmetric two-qubit states have identical values for the local invariants \( \{I_1 - I_6\} \) given below:

\[
I_1 = \det T, \quad I_2 = \text{Tr}(T^2), \\
I_3 = s^T s, \quad I_4 = s^T T s, \\
I_5 = \epsilon_{ijk} \epsilon_{lmn} s_i s_l t_{jm} t_{kn}, \\
I_6 = \epsilon_{ijk} s_i (T s)_j (T^2 s)_k, \quad (2.27)
\]

where \( \epsilon_{ijk} \) denotes Levi-Civita symbol; \( s \) (\( s^T \)) is a column (row) with \( s_1, s_2 \) and \( s_3 \) as elements.

**Proof:** Let us first note that the state parameters of a symmetric two-qubit density matrix transform under identical local unitary operation \( U \otimes U \) as follows:

\[
s'_i = \sum_{j=1}^{3} O_{ij} s_j \quad \text{or} \quad s' = O s, \\
t'_{ij} = \sum_{k,l=1}^{3} O_{ik} O_{jl} t_{kl} \quad \text{or} \quad T' = O T O^T, \quad (2.28)
\]

where \( O \in SO(3, R) \) denotes \( 3 \times 3 \) rotation matrix, corresponding uniquely to the \( 2 \times 2 \) unitary matrix \( U \in SU(2) \).

To find the minimum number of local invariants required to characterize a symmetric two qubit system, we refer to a canonical form of two qubit symmetric density matrix which is achieved by identical local unitary transformations \( U \otimes U \) such that the correlation matrix \( T \) is diagonal. This is possible because the real, symmetric correlation matrix \( T \) can be

---

\( ^2 \) A symmetric state transforms into another symmetric state under identical local unitary transformation on both the qubits.
diagonalized through identical local rotations:

\[ T^d = OT O^T = \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix}. \] (2.29)

It is clear that the invariants \( I_1 \) and \( I_2 \)

\[ I_1 = \det T = t_1 t_2 t_3, \]
\[ I_2 = \Tr(T^2) = t_1^2 + t_2^2 + t_3^2, \] (2.30)

along with the unit trace condition

\[ \Tr(T) = t_1 + t_2 + t_3 = 1, \] (2.31)

determine the eigenvalues \( t_1, t_2 \) and \( t_3 \) of the two-qubit correlation matrix \( T \). Further, the absolute values of the state variables \( s_1, s_2, s_3 \), can be evaluated using \( I_3, I_4 \) and \( I_5 \):

\[ I_3 = s^T s = s_1^2 + s_2^2 + s_3^2, \]
\[ I_4 = s^T T s = s_1^2 t_1 + s_2^2 t_2 + s_3^2 t_3, \]
\[ I_5 = \varepsilon_{ijk} \varepsilon_{lmn} s_i s_l t_j m t_k n = 2 (s_1^2 t_2 t_3 + s_2^2 t_1 t_3 + s_3^2 t_1 t_2). \] (2.32)

Having determined \( s_1^2, s_2^2, s_3^2 \) and thus fixing the absolute values of the components of the qubit orientation vector \( \vec{s} \) - the overall sign of the product \( s_1 s_2 s_3 \) is then fixed by \( I_6 \):

\[ I_6 = \varepsilon_{ijk} s_i (T T^T s)_j ([T T^T]^2 s)_k \]
\[ = s_1 s_2 s_3 \left[ t_1 t_2 (t_2 - t_1) + t_2 t_3 (t_3 - t_2) + t_3 t_1 (t_1 - t_3) \right]. \] (2.33)

Note that \( \Tr(T) \) is preserved by identical local operations \( U \otimes U \) i.e., we have \( t_1 + t_2 + t_3 = 1 \).

For an arbitrary two qubit state, the diagonal elements \( (t_1, t_2, t_3) \) of \( T^d \) are not the eigenvalues of the correlation matrix \( T \) (see Appendix C). Hence, to determine \( (t_1^2, t_2^2, t_3^2) \), (the eigenvalues of \( T T^T (T T^T) \)) local invariants \( I_{1-3} \) are required. However, in the case of a symmetric two qubit state, we have \( T^T = T \). Therefore only two polynomial invariants \( \det T, \Tr(T^2) \) suffice to determine the eigenvalues of \( T \).
It is important to realize that only the overall sign of \( s_1 s_2 s_3 \) - not the individual signs - is a local invariant. More explicitly, if \((+ , + , +)\) denote the signs of \( s_1, s_2, \) and \( s_3, \) identical local rotation through an angle \( \pi \) about the axes 1, 2 or 3 affect only the signs, not the magnitudes of \( s_1, s_2, s_3, \) leading to the possibilities \((+, - , -), (-, +, -), (-, - , +)\). All these combinations correspond to the ‘+’ sign for the product \( s_1 s_2 s_3 \). Similarly, the overall ‘-’ sign for the product \( s_1 s_2 s_3 \) arises from the combinations, \((- , - , -), (-, +, +), (+, - , +), (+, + , -)\), which are all related to each other by \( 180^\circ \) local rotations about the 1, 2 or 3 axes.

Thus we have shown that every symmetric two qubit density matrix can be transformed by identical local unitary transformation \( U \otimes U \) to a canonical form, specified completely by the set of invariants \( \{I_{1-6}\} \). In other words, symmetric two-qubit states are equally entangled iff \( \{I_{1-6}\} \) are same. In the next section, we consider a special class of symmetric density matrices and show that a subset of three independent invariants is sufficient to characterize the non-local properties completely.

### 2.5 Special class of two qubit states

Many physically interesting cases of symmetric two-qubit states like for e.g., even and odd spin states [49], Kitagawa - Ueda state generated by one-axis twisting Hamiltonian [19], atomic spin squeezed states [20], exhibit a particularly simple structure

\[
\rho_{\text{sym}} = \frac{1}{4} \begin{pmatrix}
  a & 0 & 0 & b \\
  0 & c & c & 0 \\
  0 & c & c & 0 \\
  b & 0 & 0 & d \\
\end{pmatrix},
\]

(2.34)

of the density matrix (in the standard two-qubit basis \( |0_1 0_2\rangle, |0_1 1_2\rangle, |1_1 0_2\rangle, |1_1 1_2\rangle \)) with

\[
\text{Tr}(\rho_{\text{sym}}) = a + 2c + d = 1.
\]

(2.35)
The qubit orientation vector \( \vec{s} \) for the density matrices of the form Eq. (2.34) has the following structure
\[
\vec{s} = (0, 0, (a - d)),
\]
and the real symmetric \( 3 \times 3 \) correlation matrix \( T \) for this special class of density matrix has the form:
\[
T = \begin{pmatrix}
2(c + b) & 0 & 0 \\
0 & 2(c - b) & 0 \\
0 & 0 & (a + d - 2c)
\end{pmatrix}.
\]
It will be interesting to analyze the non-local properties of such systems through local invariants. The specific structure \( \varrho_{\text{sym}} \) given by equation Eq. (2.34) of the two-qubit density matrix further reduces the number of parameters essential for the problem. Entanglement invariants associated with the symmetric two-qubit system \( \varrho_{\text{sym}} \), may now be identified through a simple calculation to be (see Eqs. (2.30), (2.32), (2.33))
\[
\begin{align*}
I_1 &= t_1 t_2 t_3 = (4 c^2 - 4 |b|^2) (1 - 4 c), \\
I_2 &= t_1^2 + t_2^2 + t_3^2 = (2 c + 2 |b|)^2 + (2 c - 2 |b|)^2 + (1 - 4 c)^2, \\
I_3 &= s_1^2 + s_2^2 + s_3^2 = (a - d)^2, \\
I_4 &= s_1^2 t_1 + s_2^2 t_2 + s_3^2 t_3 = (a - d)^2 (1 - 4 c), \\
I_5 &= 2 (s_1^2 t_2 t_3 + s_2^2 t_1 t_3 + s_3^2 t_1 t_2) = 8 (a - d)^2 (c^2 - |b|^2), \\
I_6 &= s_1 s_2 s_3 \left[ t_1 t_2 (t_2 - t_1) + t_2 t_3 (t_3 - t_2) + t_3 t_1 (t_1 - t_3) \right] = 0.
\end{align*}
\]
In this special case, we can express the invariants \( I_1 \) and \( I_2 \) in terms of \( (I_3, I_4, I_5) \) (provided \( I_3 \neq 0 \)):
\[
I_1 = \frac{I_5 I_4}{2 I_3^2}, \quad I_2 = \frac{(I_3 - I_4)^2 - I_3 I_5 + I_4^2}{I_3^2}.
\]
If \( I_3 = 0 \), then the set containing six invariants reduces to the subset of two non-zero invariants \( (I_1, I_2) \). Thus the non-local properties of symmetric two-qubit states - having a
Symmetric two qubit local invariants

specific structure $\rho_{\text{sym}}$ given by equation Eq. \((2.34)\) for the density matrix are characterized by

(i) subset of three invariants $(I_3, I_4, I_5)$ when $I_3 \neq 0$

(ii) subset of two invariants $(I_1, I_2)$ when $I_3 = 0$.

In the next section, we propose criteria, which provide a characterization of non-separability (entanglement) in symmetric two-qubit states in terms of the local invariants \{$I_{1-6}$\}.

### 2.6 Characterization of entanglement in symmetric two qubit states

A separable symmetric two-qubit density matrix is an arbitrary convex combination of direct product of identical single qubit states,

$$\rho_{\text{sym}} = \frac{1}{2} \left( I + \sum_{i=1}^{3} \sigma_i s_{wi} \right)$$  \hspace{1cm} (2.40)

and is given by

$$\rho_{(\text{sym-sep})} = \sum_{w} p_w \rho_w \otimes \rho_w,$$  \hspace{1cm} (2.41)

where $\sum_{w} p_w = 1$.

Separable symmetric system is a classically correlated system, which can be prepared through classical communications between two parties. A symmetric two qubit state which cannot be represented in the form Eq. \((2.41)\) is called **entangled**.

For a separable symmetric two qubit state, the components of the average spin of the
Symmetric two qubit local invariants

qubits are given by

\[ s_i = \text{Tr}[\rho_{(\text{sym} - \text{sep})} \sigma_{1i}] \]

\[ = \sum_{w} p_w \text{Tr}[\rho_w \sigma_{1i}] \]

\[ = \sum_{w} p_w \text{Tr}(\rho_w \sigma_{1i}) \]

\[ = \sum_{w} p_w s_{wi}, \quad (2.42) \]

and the elements of the correlation matrix \( T \) can be expressed as,

\[ t_{ij} = \text{Tr}[\rho_{(\text{sym} - \text{sep})} \sigma_{1i} \otimes \sigma_{2j}] \]

\[ = \sum_{w} p_w \text{Tr}[\rho_w \sigma_{1i}] \text{Tr}(\rho_w \sigma_{2j}) \]

\[ = \sum_{w} p_w s_{wi} s_{wj}. \quad (2.43) \]

One of the important goals of quantum information theory has been to identify and characterize inseparability. We look for such identifying criteria for separability, in terms of entanglement invariants, in the following theorem:

**Theorem 2.2** The invariants, \( I_4, I_5 \) and a combination \( I_4 - I_3^2 \) of the invariants, necessarily assume positive values for a symmetric separable two-qubit state, with \( I_3 \neq 0 \).

**Proof:** (i) The invariant \( I_4 \) has the following structure for a separable state:

\[ I_4 = s^T T s = \sum_{i,j=1}^{3} t_{ij} s_i s_j \]

\[ = \sum_{w} p_w \left( \sum_{i=1}^{3} s_i^{(w)} s_i \right) \left( \sum_{j=1}^{3} s_j^{(w)} s_j \right) \]

\[ = \sum_{w} p_w \left( \vec{s} \cdot \vec{s}^{(w)} \right)^2 \geq 0. \quad (2.44) \]
(ii) Now, consider the invariant $I_5$ for a separable symmetric system:

\[
I_5 = \epsilon_{ijk} \epsilon_{lmn} s_i s_l t_{jm} t_{kn} \nonumber
\]

\[
= \sum_{w,w'} p_w p_{w'} \left( \epsilon_{ijk} s_i s_j (w') \right) \left( \epsilon_{lmn} s_l s_m (w') \right) \nonumber
\]

\[
= \sum_{w,w'} p_w p_{w'} \left( \vec{s} \cdot \left( \vec{s} (w) \times \vec{s} (w') \right) \right)^2 \geq 0. \quad (2.45)
\]

(iii) For the combination $I_4 - I_3^2$ we obtain,

\[
I_4 - I_3^2 = \sum_w p_w \left( \vec{s} \cdot \vec{s} (w) \right)^2 - \left( \sum_w p_w \left( \vec{s} \cdot \vec{s} (w) \right) \right)^2, \quad (2.46)
\]

which has the structure $\langle A^2 \rangle - \langle A \rangle^2$ and is therefore, essentially non-negative. Negative value assumed by any of the invariants $I_4$, $I_5$ or the $I_3 - I_4^2$, is a signature of pairwise entanglement.

Further from the structure of the invariants in a symmetric separable state, it is clear that $I_3 = \sum_w p_w \left( \vec{s} \cdot \vec{s} (w) \right) = 0$ implies $\vec{s} (w) \equiv 0$ for all $w$, leading in turn to $I_4 = \sum_w p_w \left( \vec{s} \cdot \vec{s} (w) \right)^2 = 0$ and $I_5 = 0$. Thus when $I_3 = 0$, the two qubit local invariant $I_1$ characterizes pairwise entanglement as shown in the theorem given below:

**Theorem 2.3** For a symmetric separable two-qubit state, the invariant $I_1$ assumes positive value.

**Proof** : Consider,

\[
T = \text{diag}(t_1, t_2, t_3) \nonumber
\]

\[
= \text{diag} \left( \sum_w p_w \left( s_1 (w) \right)^2, \sum_w p_w \left( s_2 (w) \right)^2, \sum_w p_w \left( s_3 (w) \right)^2 \right). \quad (2.47)
\]

We therefore have,

\[
I_1 = \det T = t_1 t_2 t_3 = \prod_{i=1}^3 \left( \sum_w p_w \left( s_i (w) \right)^2 \right), \quad (2.48)
\]
which is obviously non-negative for all symmetric separable two qubit states. Thus when \( \mathcal{I}_3 = 0 \), \( \mathcal{I}_1 > 0 \) provides a sufficient criteria for separability.

A simple example illustrating our separability criterion in terms of two qubit local invariants is the two qubit bell state

\[
|\Phi\rangle = \frac{1}{\sqrt{2}}(|0\ 1\rangle + |1\ 0\rangle).
\]  

(2.49)

The density matrix for the two qubit bell state is written as,

\[
\rho = |\Phi\rangle \langle \Phi| = \frac{1}{2}
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]  

(2.50)

It is easy to identify that the above density matrix has a structure similar to \( \rho_{\text{sym}} \) (see Eq. (2.34)) with the matrix elements given by

\[
a = 0, \quad b = 0,
\]

\[
c = \frac{1}{2}, \quad d = 0.
\]  

(2.51)

The invariant \( \mathcal{I}_3 \) associated with the density matrix of Eq. (2.50) is given by,

\[
\mathcal{I}_3 = (a - d)^2 = 0,
\]

which, in turn implies that

\[
\mathcal{I}_4 = \mathcal{I}_3(1 - 4c) = 0,
\]

\[
\mathcal{I}_5 = \mathcal{I}_3(c^2 - |b|^2) = 0.
\]  

(2.52)
Symmetric two qubit local invariants

The invariant \( I_1 \) has the structure,

\[
I_1 = (4c^2 - 4|b|^2)(1 - 4c) = -1.
\]  
(2.53)

Since \( I_1 \leq 0 \), we can conclude from Theorem 2.3 that the given state Eq. (2.49) is entangled.

2.7 Necessary and sufficient criterion for a class of symmetric two qubit states

It would be interesting to explore how these constraints on the invariants, get related to the other well established criteria of entanglement. For two qubits states, it is well known that Peres’s PPT (positivity of partial transpose) criterion \[43\] is both necessary and sufficient for separability. We now proceed to show that in the case of symmetric states, given by Eq. (2.34), there exists a simple connection between the Peres’s PPT criterion and the non-separability constraints (see Eqs. (2.44) - (2.46)) on the invariants.

We may recall from Sec. 2.5 that the density matrix for the special class of symmetric two qubit states (see Eq. (2.34)) has the following structure:

\[
\rho_{\text{sym}} = \frac{1}{4} \begin{pmatrix}
    a & 0 & 0 & b \\
    0 & c & c & 0 \\
    0 & c & c & 0 \\
    b & 0 & 0 & d \\
\end{pmatrix}.
\]  
(2.54)

The partial transpose of the matrix \( \rho_{\text{sym}} \) has the form,

\[
(\rho_{\text{sym}})^{PT} = \frac{1}{4} \begin{pmatrix}
    a & 0 & 0 & c \\
    0 & c & b & 0 \\
    0 & b & c & 0 \\
    c & 0 & 0 & d \\
\end{pmatrix}.
\]  
(2.55)
The eigenvalues of the partially transposed density matrix \((\rho_{\text{sym}})^{PT}\) are given by

\[
\begin{align*}
\lambda_1 &= \frac{1}{2} \left( (a + d) - \sqrt{(a - d)^2 + 4c^2} \right), \\
\lambda_2 &= \frac{1}{2} \left( (a + d) + \sqrt{(a - d)^2 + 4c^2} \right), \\
\lambda_3 &= c - |b|, \\
\lambda_4 &= c + |b|,
\end{align*}
\]

(2.56)

of which \(\lambda_1\) and \(\lambda_3\) can assume negative values (\(a, c, d\) are positive quantities since they are the diagonal elements of the density matrix).

(i) If \(\lambda_1 < 0\), then we have,

\[(a + d)^2 < (a - d)^2 + 4c^2.\]  \hspace{1cm} (2.57)

The above inequality can be expressed as

\[(a + d + 2c)(a + d - 2c) < (a - d)^2,\]  \hspace{1cm} (2.58)

which on using Eq. (2.35), gets simplified to

\[(1 - 4c) < (a - d)^2.\]  \hspace{1cm} (2.59)

From Eq. (2.38), we have

\[I_4 - I_3^2 = (a - d)^2 \left( (1 - 4c) - (a - d)^2 \right),\]  \hspace{1cm} (2.60)

associated with the two qubit symmetric system \(\rho_{\text{sym}}\). It is obvious that \(I_4 - I_3^2 < 0\) when \((1 - 4c) < (a - d)^2\) i.e., when \(\lambda_1\) is negative.

(ii) When \(\lambda_3 < 0\), we can easily see that the invariant (see Eq. (2.38)),

\[I_5 = 8 (a - d)^2 (c + |b|) (c - |b|)\]  \hspace{1cm} (2.61)
Symmetric two qubit local invariants

associated with the special class of symmetric two qubit states (Eq. (2.34)) is also negative. Thus \( \lambda_3 < 0 \Rightarrow I_5 < 0 \).

We have thus established an equivalence between the Peres’s partial transpose criterion and the constraints on the invariants for two qubit symmetric state of the form \( \varrho_{(\text{sym})} \). In other words, the nonseparability conditions \( I_4 - I_3^2, I_5 < 0 \), are both necessary and sufficient for a class of two-qubit symmetric states given by Eq. (2.34).
2.8 Conclusions

We have shown that the number of 18 invariants as proposed by Makhlin [36] for an arbitrary two qubit system gets reduced to 12 due to symmetry constraints. Further, we realize that a canonical form of two qubit symmetric density matrix achieved by identical local unitary transformations $U \otimes U$ restricts the minimum number of local invariants to specify an arbitrary symmetric two qubit system to 6. In other words, we show that a subset of six invariants $\{I_{1-6}\}$ of a more general set of 18 invariants proposed by Makhlin [36], is sufficient to characterize the nonlocal properties of a symmetric two qubit states. For a special class of two qubit symmetric states, only 3 invariants are sufficient to characterize the system. The invariants $I_4$, $I_5$ and $I_4 - I_3^2$ of separable symmetric two-qubit states are shown to be non-negative. We have proposed sufficient conditions for identifying entanglement in symmetric two-qubit states, when the qubits have a non-zero value for the average spin. Moreover these conditions on the invariants are shown to be necessary and sufficient for a class of symmetric two qubit states.
Chapter 3

Collective signatures of entanglement in symmetric multiqubit systems

3.1 Introduction

Quantum correlated systems of macroscopic atomic ensembles [50] have been drawing considerable attention recently. This is especially in view of their possible applications in atomic interferometers [16, 17] and high precision atomic clocks [18] and also in quantum information and computation [2]. Spin squeezing [5, 19] has been established as a standard collective method to detect entanglement in these multiatom (multiqubit) systems.

Spin squeezing, is defined as the reduction of quantum fluctuations in one of the spin components orthogonal to the mean spin direction below the fundamental noise limit $N/4$. Spin squeezing of around $N \approx 10^7$ atoms is nowadays routinely achieved in laboratories. It has been shown in Ref. [30] that spin squeezing is directly related to pairwise entanglement in atomic ensembles - though it provides a sufficient condition for inseparability.

In the original sense, spin squeezing is defined for multiqubit states belonging to the maximum multiplicity subspace of the collective angular momentum operator $\vec{J}$. Multiqubit states with highest collective angular momentum value $J = \frac{N}{2}$ exhibit symmetry under
the interchange of particles. In other words, the concept of spin squeezing is defined for symmetric multiqubit systems and it is reflected through collective variables associated with the system.

In this Chapter, we concentrate on symmetric multiqubit systems and connect the average values of the collective first and second order spin observables in terms of the two qubit state parameters. We examine how collective signatures of pairwise entanglement, like spin squeezing, manifest themselves via negative values of two qubit local invariants. This leads to a classification of pairwise entanglement in symmetric multi-qubit states in terms of the associated two qubit local invariants.

### 3.2 Collective spin observables in terms of two qubit variables

The collective spin operator $\vec{J}$ for a $N$ qubit system is defined by,

$$\vec{J} = \frac{1}{2} \sum_{\alpha=1}^{N} \vec{\sigma}_{\alpha}. \quad (3.1)$$

Let us concentrate on symmetric multiqubit systems, which respect exchange symmetry:

$$\Pi_{\alpha \beta} \rho_{\text{sym}}^{(N)} = \rho_{\text{sym}}^{(N)} \Pi_{\alpha \beta} = \rho_{\text{sym}}^{(N)},$$

where $\Pi_{\alpha \beta}$ denotes the permutation operator interchanging $\alpha^{\text{th}}$ and $\beta^{\text{th}}$ qubits.

The expectation value of collective spin correlations $\langle J_i \rangle$ is given by

$$\langle J_i \rangle = \frac{1}{2} \sum_{\alpha=1}^{N} \langle \sigma_{\alpha i} \rangle; \quad i = 1, 2, 3. \quad (3.2)$$

---

1As a consequence of exchange symmetry, density matrices characterizing any random pair of qubits drawn from a symmetric multiqubit system are all identical. So, the average values of qubit observables are identical for any random pair of qubits belonging to a symmetric $N$-qubit system.
Collective signatures of entanglement in symmetric multiqubit systems

Here, we denote the average value \( \langle \ldots \rangle \) of any observable by

\[
\langle \ldots \rangle = \Tr [\rho_{\text{sym}} (\ldots)].
\]  

(3.3)

It may be noted that the density matrix of the \( \alpha \)th qubit extracted from a system of \( N \) qubits, which respect exchange symmetry, is given by

\[
\rho^{(\alpha)}_{\text{sym}} = \Tr_{1,2,\ldots,\alpha-1,\alpha+1,\ldots,N} \rho^{(N)}_{\text{sym}},
\]

\[
= \frac{1}{2} [1 + \sigma_i^{(\alpha)} s_i],
\]  

(3.4)

where,

\[
s_i = \Tr [\rho^{(\alpha)}_{\text{sym}} \sigma^{(\alpha)}_i]
\]

and

\[
\vec{\sigma}^{(\alpha)} = I \otimes I \otimes \ldots \otimes \vec{\sigma} \otimes I \otimes \ldots \otimes I
\]  

(3.5)

is the \( \alpha \)th qubit spin operator, with \( \vec{\sigma} \) appearing at \( \alpha \)th position. In Eq. (3.4), the density matrix \( \rho^{(\alpha)}_{\text{sym}} \) of the \( \alpha \)th qubit is obtained by tracing the multiqubit state \( \rho^{(N)}_{\text{sym}} \) over all the qubit indices, expect \( \alpha \).

We emphasize that the qubit averages are independent of the qubit index \( \alpha \):

\[
\langle \sigma_i^{(\alpha)} \rangle = s_i.
\]  

(3.6)

Therefore the first moments of the collective spin \( \langle J_i \rangle \) (see Eq. (3.2)) assume the form

\[
\langle J_i \rangle = \frac{1}{2} \sum_{\alpha=1}^{N} \langle \sigma_{\alpha i} \rangle = \frac{N}{2} s_i
\]  

(3.7)

in terms of the single qubit state parameter \( s_i \).

For our further discussion, we need to associate the second order moments of the collective spin variables i.e., \( \langle (J_i J_j + J_j J_i) \rangle \) with the two qubit correlation parameters.
Collective signatures of entanglement in symmetric multiqubit systems

Using Eq. (3.2), we obtain,

\[
\frac{1}{2} \langle (J_i J_j + J_j J_i) \rangle = \frac{1}{8} \sum_{\alpha, \beta = 1}^{N} \langle (\sigma_\alpha \sigma_\beta + \sigma_\beta \sigma_\alpha) \rangle
\]

\[
= \frac{N}{4} \delta_{ij} + \frac{1}{4} \sum_{\alpha \neq \beta = 1}^{N} \langle (\sigma_\alpha \sigma_\beta) \rangle.
\]  \tag{3.8}

Here, \( \langle (\sigma_\alpha \sigma_\beta) \rangle (\alpha \neq \beta) \) are the spin correlations of a pair of qubits \( \alpha, \beta \) drawn randomly from a symmetric multiqubit system.

The density matrix \( (\rho_{\text{sym}}^{(\alpha \beta)}) \) of such a pair of qubits obtained by taking a partial trace over the remaining \( (N - 2) \) qubits is given by

\[
(\rho_{\text{sym}}^{(\alpha \beta)}) = \text{Tr}_{1,2,\ldots,\text{except } (\alpha, \beta)} (\rho_{\text{sym}}^{(N)}).
\]

The general form of such a two qubit density matrix in terms of 8 (see Eq. (2.17)) state variables is given by,

\[
\rho_{\text{sym}}^{(\alpha \beta)} = \frac{1}{4} \left( I \otimes I + \sum_{i=1}^{N} s_i (\sigma_i^{(\alpha)} + \sigma_i^{(\beta)}) + \sum_{\alpha, \beta = 1}^{N} t_{ij} \sigma_i^{(\alpha)} \sigma_j^{(\beta)} \right),
\]  \tag{3.9}

where,

\[
t_{ij} = \text{Tr} [\rho_{\text{sym}}^{(\alpha \beta)} (\sigma_i^{(\alpha)} \sigma_j^{(\beta)})]
\]

\[
= \langle (\sigma_i^{(\alpha)} \sigma_j^{(\beta)}) \rangle,
\]  \tag{3.10}

irrespective of the qubit indices \( \alpha, \beta \).

Substituting Eq. (3.10) in Eq. (3.8), we obtain

\[
\frac{1}{2} \langle (J_i J_j + J_j J_i) \rangle = \frac{N}{4} \delta_{ij} + \frac{N(N - 1)}{4} \langle (\sigma_1 \sigma_2) \rangle
\]

\[
= \frac{N}{4} [\delta_{ij} + (N - 1) t_{ij}], \quad i, j = 1, 2, 3. \]  \tag{3.11}
Collective signatures of entanglement in symmetric multiqubit systems

From Eqs. (3.7), (3.11), it is evident that the collective spin observables (upto first and second order) can be expressed in terms of state parameters of a pair of qubits chosen arbitrarily from a symmetric $N$ qubit state. Thus the collective pairwise entanglement behavior in symmetric multiqubits results from the properties of the two qubit state parameters $\{s_i, t_{ij}\}$.

We now proceed to identify collective criteria of pairwise entanglement in a symmetric multi-qubit state, in terms of the two-qubit local invariants $\{I_1 - I_6\}$.

### 3.3 Collective signatures of pairwise entanglement

Collective phenomena, reflecting pairwise entanglement of qubits, can be expressed through two qubit local invariants as the first and second moments $\langle J_i \rangle$, $\langle (J_i J_j + J_j J_i) \rangle$ are related to two qubit state parameters $\{s_i, t_{ij}\}$ in symmetric multiqubit systems. Here, we show that spin squeezing- which is one of the collective signatures of pairwise entanglement in symmetric multiqubit systems-gets reflected through one of the separability criterion derived in Chapter 2.

For the sake of continuity, the main result of Chapter 2 is summarized in the following sentence:

*The non-positive values of the invariants $\{I_4$, $I_5$ or $I_4 - I_3^2\}$ serve as a signature of entanglement and hence provide sufficient conditions for non-separability of the quantum state.*

Let us now review spin squeezing criteria. Kitagawa and Ueda 19 pointed out that a definition of spin squeezing, 51 based only on the uncertainty relation,

$$ (\triangle J_1)^2 (\triangle J_2)^2 \geq \frac{|\langle J_3 \rangle|^2}{4} \tag{3.12} $$

exhibits co-ordinate frame dependence and does not arise from the quantum correlations...
Collective signatures of entanglement in symmetric multiqubit systems

among the elementary spins. They identified a mean spin direction

\[ \hat{n}_0 = \frac{\langle \vec{J} \rangle}{|\langle \vec{J} \rangle|}, \]  

(3.13)

where, \( |\langle \vec{J} \rangle| = \sqrt{\langle \vec{J} \rangle \cdot \langle \vec{J} \rangle} \) (The collective spin operator \( \vec{J} \) for an \( N \) qubit system is given by Eq. (3.2)).

Associating a mutually orthonormal set \( \{ \hat{n}_1, \hat{n}_2, \hat{n}_0 \} \), with the system, let us consider the following collective operators,

\[ J_{1\perp} = \vec{J} \cdot \hat{n}_{1\perp}, \quad J_{2\perp} = \vec{J} \cdot \hat{n}_{2\perp} \quad \text{and} \quad J_0 = \vec{J} \cdot \hat{n}_0 \]  

(3.14)

which satisfy the usual angular momentum commutation relations

\[ [J_{1\perp}, J_{2\perp}] = i J_0. \]  

(3.15)

Now, employing a collective spin component \( J_\perp \) orthogonal to the mean spin direction \( \hat{n}_0 \), given by,

\[ J_\perp = \vec{J} \cdot \hat{n}_\perp = J_{1\perp} \cos \theta + J_{2\perp} \sin \theta, \]  

(3.16)

minimization of the variance,

\[ (\Delta J_\perp)^2 = \langle J^2_\perp \rangle - \langle J_\perp \rangle^2 \]  

(3.17)

can be done over the angle \( \theta \). Kitagawa and Ueda [19] proposed that a multiqubit state can be regarded as spin squeezed if the minimum of \( \Delta J_\perp \) is smaller than the standard quantum limit \( \frac{\sqrt{N}}{2} \) of the spin coherent state.
A spin squeezing parameter incorporating this feature is defined by [19]

\[ \xi = \frac{2 (\Delta J_\perp)_{\text{min}}}{\sqrt{N}}. \] (3.18)

Symmetric multiqubit states with \( \xi < 1 \) are spin squeezed. We next proceed to show that the two qubit local invariant \( I_5 \) and the spin squeezing parameter \( \xi \) are related to each other.

### 3.3.1 Spin squeezing in terms of the local invariant \( I_5 \)

We now prove the following theorem.

**Theorem 3.1** *For all spin squeezed states, the local invariant \( I_5 \) is negative.*

**Proof.** It is useful to evaluate the invariant \( I_5 \) (see Eq. (2.27)), after subjecting the quantum state to an identical local rotation \( U \otimes U \otimes U \otimes \cdots \) on all the qubits, which is designed to align the average spin vector \( \langle \vec{J} \rangle \) along the 3-axis. *After this local rotation*, orientation of the qubits would be along 3 axis and the qubit orientation vector is given by

\[ \vec{s} \equiv (0, 0, s_0). \]

We may then express the local invariant \( I_5 \) as,

\[
I_5 = \epsilon_{ijk} \epsilon_{lmn} s_i s_l t_{jm} t_{kn},
\]

\[
= \epsilon_{3jk} \epsilon_{3mn} s_0^2 t_{jm} t_{kn},
\]

\[
= 2 s_0^2 \left( t'_{11} t'_{22} - (t'_{12})^2 \right),
\]

\[
= 2 s_0^2 \det T_\perp, \quad (3.19)
\]

where,

\[
T_\perp = \begin{pmatrix}
  t'_{11} & t'_{12} \\
  t'_{12} & t'_{22}
\end{pmatrix},
\] (3.20)
Collective signatures of entanglement in symmetric multiqubit systems

denotes the $2 \times 2$ block of the correlation matrix in the subspace orthogonal to the qubit orientation direction i.e., 3-axis.

Now, we can still exploit the freedom of local rotations $O_{12}$ in the $1-2$ plane, which leaves the average spin $\vec{s} = (0, 0, s_0)$ unaffected. We use this to diagonalize $T_\perp$:

$$O_{12}T_\perp O_{12}^T = T^d_\perp = \begin{pmatrix} t^{(+)}_\perp & 0 \\ 0 & t^{(-)}_\perp \end{pmatrix} \quad (3.21)$$

with the diagonal elements given by

$$t^{(\pm)}_\perp = \frac{1}{2} \left[ (t'_{11} + t'_{22}) \pm \sqrt{(t'_{11} - t'_{22})^2 + 4 (t'_{12})^2} \right].$$

We once again emphasize that local rotations on the qubits leave the invariants unaltered and here, we choose local operations to transform the two qubit state variables as,

$$\vec{s} = (0, 0, s_0), \quad (3.22)$$

and

$$T = \begin{pmatrix} t^{(+)}_\perp & 0 & t''_{13} \\ 0 & t^{(-)}_\perp & t''_{23} \\ t''_{13} & t''_{23} & t'_{33} \end{pmatrix}, \quad (3.23)$$

so that the two qubit invariant $I_5$ can be expressed as,

$$I_5 = 2 s_0^2 \det T_\perp, \quad (3.24)$$
We now express the spin squeezing parameter $\xi$, given by Eq. (3.18), in terms of the two-qubit state parameters

$$\xi^2 = \frac{4 (\Delta J_{\perp})^2_{\text{min}}}{N},$$

$$= \frac{4 (\langle \vec{J} \cdot \hat{n}_{\perp} \rangle^2 - \langle (\vec{J} \cdot \hat{n}_{\perp})^2 \rangle}{N}. \quad (3.25)$$

Writing the collective spin operator $\vec{J}$ for an $N$ qubit system in terms of the two qubit operators (see Eq. (3.1)), we obtain

$$\xi^2 = \frac{1}{N} \sum_{\alpha,\beta=1}^{N} \langle (\vec{\sigma}_{\alpha} \cdot \hat{n}_{\perp}) (\vec{\sigma}_{\beta} \cdot \hat{n}_{\perp}) \rangle_{\text{min}}$$

$$= 1 + \frac{1}{N} \sum_{\alpha=1}^{N} \sum_{\beta \neq \alpha=1}^{N} \langle (\vec{\sigma}_{\alpha} \cdot \hat{n}_{\perp}) (\vec{\sigma}_{\beta} \cdot \hat{n}_{\perp}) \rangle_{\text{min}}$$

$$= 1 + \frac{2}{N} \sum_{\alpha=1}^{N} \sum_{\beta > \alpha=1}^{N} \left( \sum_{i,j=1}^{3} \langle (\sigma_{\alpha i} \sigma_{\beta j}) \rangle_{\text{min}} n_{\perp i} n_{\perp j} \right). \quad (3.26)$$

Since for a symmetric system, we have $\langle \sigma_{\alpha i} \sigma_{\beta j} \rangle = t_{ij}$, we express the squeezing parameter as follows:

$$\xi^2 = 1 + (N - 1) \left( \sum_{i,j=1}^{3} t_{ij} n_{\perp i} n_{\perp j} \right)_{\text{min}}$$

$$= 1 + (N - 1) (n_{\perp}^T T n_{\perp})_{\text{min}}. \quad (3.27)$$

In Eq. (3.27), we have denoted the row vector $n_{\perp}^T = (n_{1\perp}, n_{2\perp}, 0) = (\cos \theta, \sin \theta, 0)$. The minimum value of the quadratic form $(n_{\perp}^T T n_{\perp})_{\text{min}}$ in Eq. (3.27) is fixed as follows:

$$(n_{\perp}^T T n_{\perp})_{\text{min}} = \left( \begin{array}{c} \min_{\theta} \end{array} \right) \left( t'_{11} \cos^2 \theta + t'_{22} \sin^2 \theta + t'_{12} \sin 2\theta \right)$$

$$= \frac{1}{2} \left[ (t'_{11} + t'_{22}) - \sqrt{(t'_{11} - t'_{22})^2 + 4 (t'_{12})^2} \right]$$

$$= t_{1\perp}^{(-)}. \quad (3.28)$$
Collective signatures of entanglement in symmetric multiqubit systems

where \( t^{(-)}_\perp \) is the least eigenvalue of \( T_\perp \) (see Eq. (3.20)).

We finally obtain,

\[
\xi^2 = \frac{4}{N} (\Delta J_\perp)^2_{\text{min}} = \left( 1 + (N - 1) t^{(-)}_\perp \right).
\] (3.29)

Following similar lines we can also show that

\[
\frac{4}{N} (\Delta J_\perp)^2_{\text{max}} = \left( 1 + (N - 1) t^{(+)}_\perp \right),
\] (3.30)

which relates the eigenvalue \( t^{(+)}_\perp \) of \( T_\perp \) to the maximum collective fluctuation \( (\Delta J_\perp)^2_{\text{max}} \) orthogonal to the mean spin direction. Substituting Eqs. (3.29), (3.30), and expressing \( s_0 = \frac{2}{N} |\langle J_3 \rangle| \) in Eq. (3.24), we get,

\[
I_5 = \frac{8 |\langle \vec{J} \rangle|^2}{(N(N - 1))^2} \left( \xi^2 - 1 \right) \left( \frac{4}{N} (\Delta J_\perp)^2_{\text{max}} - 1 \right).
\] (3.31)

Having related the local invariant \( I_5 \) to collective spin observables, we now proceed to show that \( I_5 < 0 \) iff \( \xi^2 < 1 \) i.e., iff the state is spin squeezed.

Note that the two qubit correlation parameters \( t_{ij} \) are bound by

\[-1 \leq t_{ij} \leq 1.\]

This bound, together with the unit trace condition \( \text{Tr} (T) = 1 \) on the correlation matrix of a symmetric two-qubit state, leads to the identification that only one of the diagonal elements of \( T \) can be negative. This in turn implies that if the diagonal element \( t^{(-)}_\perp \) is negative, then the other diagonal element \( t^{(+)}_\perp \) is necessarily positive. Thus, from Eq. (3.30), it is evident that

\[
t^{(+)}_\perp = \left( \frac{4}{N} (\Delta J_\perp)^2_{\text{max}} - 1 \right) \geq 0.
\]
Collective signatures of entanglement in symmetric multiqubit systems

whenever \( t^{(-)}_1 < 0 \). It is therefore clear (from Eq. (3.31)) that a symmetric multiqubit state is spin-squeezed \( \iff \mathcal{I}_5 < 0 \). In other words,

\[
\xi^2 < 1 \iff \mathcal{I}_5 < 0. \tag{3.32}
\]

Further, from the structure of the invariant \( \mathcal{I}_5 \) (Eq. (2.27)), it is clear that \( \mathcal{I}_5 < 0 \) implies

\[
(s_1^2 t_2 t_3 + s_2^2 t_1 t_3 + s_3^2 t_1 t_2) < 0,
\]

i.e., one of the eigenvalues \( t_1, t_2 \) or \( t_3 \) of the correlation matrix \( T \) must be negative. This in turn implies that the invariant

\[
\mathcal{I}_1 = t_1 t_2 t_3 < 0.
\]

In other words, when \( \mathcal{I}_5 < 0 \), the invariant \( \mathcal{I}_1 \) is also negative.

We now explore other collective signatures of pairwise entanglement, which are manifestations of negative values of the invariants \( \mathcal{I}_4 \) and \( \mathcal{I}_4 - \mathcal{I}_3^2 \).

### 3.3.2 Collective signature in terms of \( \mathcal{I}_4 \)

When the average spin is aligned along the 3-axis through local rotations such that \( \vec{s} = (0, 0, s_0) \), the local invariant \( \mathcal{I}_4 \) (see Eq. (2.27)) assumes the form,

\[
\mathcal{I}_4 = s^T T s = \begin{pmatrix} 0 & 0 & s_0 \\ 0 & 0 & s_0 \\ s_0 & s_0 & 0 \end{pmatrix} \begin{pmatrix} t^{(+)}_1 & 0 & t''_1 \\ 0 & t^{(-)}_1 & t''_2 \\ t''_1 & t''_2 & t''_3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ s_0 \end{pmatrix} = s_0^2 t'_{33}. \tag{3.33}
\]

It is evident from the above equation that

\[
\mathcal{I}_4 < 0 \iff t'_{33} < 0.
\]
Collective signatures of entanglement in symmetric multiqubit systems

Simplifying Eq. (3.7) and Eq. (3.11), we express $t'_{33}$ and $s_0$ in terms of the collective spin observables i.e.,

$$s_0 = \frac{2}{N} \langle \vec{J} \cdot \hat{n}_0 \rangle = \frac{2}{N} \langle |\vec{J}| \rangle,$$

$$t'_{33} = \frac{1}{(N-1)} \left( \frac{4}{N} \langle (\vec{J} \cdot \hat{n}_0)^2 \rangle - 1 \right), \quad (3.34)$$

leading further to the following structure for the invariant $I_4$

$$I_4 = \frac{4}{N^2 (N-1)} \langle |\vec{J}| \rangle^2 \left( \frac{4}{N} \langle (\vec{J} \cdot \hat{n}_0)^2 \rangle - 1 \right), \quad (3.35)$$

where $\hat{n}_0$ denotes a unit vector along the direction of mean spin. We therefore read from Eq. (3.35), that the average of the squared spin component, along the mean spin direction, reduced below the value $N/4$, signifies pairwise entanglement in symmetric $N$-qubit system.

Further, we may note that when $I_4 \leq 0$, the invariant $I_5$, which reflects spin squeezing is not negative. This is because $t'_{33} \leq 0 \implies t_{\pm}^{(\pm)} \geq 0$ as,

(i) $t'_{\perp} + t'_{\perp} - t'_{33} = 1$, (unit trace condition), and

(ii) $-1 \leq t_{\perp}^{(\pm)}$, $t'_{33} \leq 1$.

Therefore, spin squeezing and $\langle (\vec{J} \cdot \hat{n}_0)^2 \rangle \leq \frac{N}{4}$ are two mutually exclusive criteria of pairwise entanglement.

However, from the structure of the invariant $I_4$, as given in Eq. (2.32), it is obvious that

$$I_4 = s_1^2 t_1 + s_2^2 t_2 + s_3^2 t_3 \leq 0 \quad (3.36)$$

implying that one of the eigenvalues $t_1$, $t_2$, $t_3$ of two qubit correlation matrix must be negative. This leads to the identification that

$$I_1 = t_1 t_2 t_3 \leq 0.$$

We now continue to relate the combination of invariants $I_4 - I_3^2$ to the collective spin
3.3.3 $\mathcal{I}_4 - \mathcal{I}_3^2$ in terms of the collective variables

In terms of the two qubit state parameters, expressed in a conveniently chosen local coordinate system (see Eqs. (3.22), (3.23)), the invariant combination $\mathcal{I}_4 - \mathcal{I}_3^2$ can be written as,

$$\mathcal{I}_4 - \mathcal{I}_3^2 = s^T T s - s^T s = s_0^2 (t_{33}^2 - s_0^2).$$

(3.37)

Since the state parameters can be expressed in terms of the expectation values of the collective spin variables Eq. (3.34), the invariant quantity $\mathcal{I}_4 - \mathcal{I}_3^2$ may be rewritten as

$$\mathcal{I}_4 - \mathcal{I}_3^2 = \frac{4}{N^2} |\langle \vec{J} \rangle|^2 \left[ \frac{4}{N(N-1)} \langle (\vec{J} \cdot \hat{n}_0)^2 \rangle - \frac{1}{(N-1)} - \frac{4}{N^2} |\langle \vec{J} \rangle|^2 \right],$$

$$= \frac{16}{N^3(N-1)} |\langle \vec{J} \rangle|^2 \left[ \langle (\vec{J} \cdot \hat{n}_0)^2 \rangle - \left( \frac{N}{4} + \frac{(N-1)}{N} |\langle \vec{J} \rangle|^2 \right) \right].$$

(3.38)

Negative value of the combination $\mathcal{I}_4 - \mathcal{I}_3^2$ manifests itself through

$$\langle (\vec{J} \cdot \hat{n}_0)^2 \rangle < \frac{N}{4} + \frac{(N-1)}{N} |\langle \vec{J} \rangle|^2.$$

From Eqs. (3.35) and (3.38), we conclude that pairwise entanglement resulting from

$$\mathcal{I}_3 \neq 0, \quad \mathcal{I}_4 > 0, \quad \text{but} \quad \mathcal{I}_4 - \mathcal{I}_3^2 < 0,$$

is realized, whenever

$$\frac{N}{4} < \langle (\vec{J} \cdot \hat{n}_0)^2 \rangle < \frac{N}{4} + \frac{(N-1)}{N} |\langle \vec{J} \rangle|^2.$$

All the cases discussed above are valid when the average spin vector $|\langle \vec{J} \rangle| \neq 0$ i.e., when $|\langle \vec{J} \rangle|$ is oriented along the 3-axis. In the special case when $|\langle \vec{J} \rangle| = 0$, we show that pairwise entanglement manifests itself through negative value of the local invariant $\mathcal{I}_1$. 

Collective signatures of entanglement in symmetric multiqubit systems
3.3.4 Characterization of pairwise entanglement through $I_1$

In the cases where the qubits have no preferred orientation, i.e., when $|⟨\vec{J}⟩| = 0$, it is evident from Eq. (2.27) that the local invariants $\{I_{3−6}\}$ are zero

$$I_3 = s^T s = 0,$$
$$I_4 = s^T T s = 0,$$
$$I_5 = \epsilon_{ijk} \epsilon_{lmn} s_i t_{jm} t_{kn} = 0,$$
$$I_6 = \epsilon_{ijk} s_i (T s)_j (T^2 s)_k = 0. \quad (3.39)$$

The remaining two nonzero invariants $I_1$ and $I_2$ are,

$$I_1 = \det T,$$
$$I_2 = \text{Tr} (T^2). \quad (3.40)$$

In such situations, i.e., when $|⟨\vec{J}⟩| = 0$, pairwise entanglement manifests itself through $I_1 < 0$.

Writing the invariant $I_1$ in terms of collective observables Eq. (3.11), we have,

$$I_1 = \det T = t_1 t_2 t_3 = \left(\frac{4}{N(N−1)}\right)^{\frac{3}{2}} \prod_{i=1}^{3} \left(⟨J_i^2⟩ - \frac{N}{4}\right). \quad (3.41)$$

Negative value of $I_1$ shows up through $⟨J_i^2⟩ < \frac{N}{4}$ along the axes $i = 1, 2$ or 3, which are fixed by verifying $⟨(J_i J_j + J_j J_i)⟩ = 0; \ i \neq j$, as $T$ is diagonal with such a choice of the axes.

Note that,

$$I_1 = \det T < 0 \implies I_2 = \text{Tr} (T^2) > 1$$

since $t_1 + t_2 + t_3 = 1$ (unit trace condition) and $−1 \leq t_1, t_2, t_3 \leq 1$. Therefore we have,

$$I_1 < 0, \ I_2 > 1, \quad (3.42)$$
both implying pairwise entanglement.

Thus, we have related the two qubit entanglement invariants Eq. (2.27) to the collective spin observables and shown that the collective signatures of pairwise entanglement are manifested through the negative values of the invariants $I_4$, $I_5$, $I_4 - I_3^2$.

In Sec. 3.4 (a) we relate the invariant criteria with the recently proposed generalized spin squeezing inequalities for two qubits [31] and (b) propose a classification scheme for pairwise entanglement in symmetric multiqubit systems.

### 3.4 Classification of pairwise entanglement

Recently, Korbicz et al. [31] proposed generalized spin squeezing inequalities for pairwise entanglement, which provide necessary and sufficient conditions for genuine 2-, or 3- qubit entanglement for symmetric states: These generalized spin squeezing inequalities are given by [31]

\[
\frac{4(\Delta J_k)^2}{N} < 1 - \frac{4(J_k)^2}{N^2} \tag{3.43}
\]

where $J_k = \vec{J} \cdot \hat{k}$; with $\hat{k}$ denoting an arbitrary unit vector.

We now show that that the generalized spin squeezing inequality given in Eq. (3.43) can be related to our invariant criteria.

We consider various situations as discussed below:

(i) Let $\hat{k} = \hat{n}_\perp$, a direction orthogonal to the mean spin vector $\langle \vec{J} \rangle$. The inequality given by Eq. (3.43) reduces to

\[
(\Delta J_{n_\perp})^2 < \frac{N}{4}.
\]

Minimizing the variance $\langle \Delta J_{n_\perp} \rangle^2$ we obtain

\[
\xi^2 = \frac{4(\Delta J_{n_\perp})^2_{\text{min}}}{N} < 1.
\]
Collective signatures of entanglement in symmetric multiqubit systems

This is nothing but the conventional spin squeezing condition $I_5 < 0$ in terms of the invariant.

(ii) If $\hat{k}$ is aligned along the mean spin direction i.e., $\hat{k} = \hat{n}_0$ with $\hat{n}_0 = \frac{\langle \vec{J} \rangle}{|\langle \vec{J} \rangle|}$, the generalized spin squeezing inequalities (see Eq. (3.43)) reduce to the form,

$$\langle (\vec{J} \cdot \hat{n}_0)^2 \rangle < \frac{N}{4} + \frac{(N - 1)}{N} |\langle \vec{J} \rangle|^2.$$  \hfill (3.44)

From Eq. (3.35) we have,

$$I_4 = \frac{4}{N^2 (N - 1)} |\langle \vec{J} \rangle|^2 \left( \frac{4}{N} \langle (\vec{J} \cdot \hat{n}_0)^2 \rangle - 1 \right).$$  \hfill (3.45)

Now the condition $I_4 < 0$ on the local invariant leads to the collective signature [see Table.1]

$$\langle (\vec{J} \cdot \hat{n}_0)^2 \rangle < \frac{N}{4},$$

which is a stronger restriction than that given by Eq. (3.44).

Further, if $I_4 > 0$ but $I_4 - I_3^2 < 0$ we obtain the inequality [see Table.1]

$$\frac{N}{4} < \langle (\vec{J} \cdot \hat{n}_0)^2 \rangle < \frac{N}{4} + \frac{(N - 1)}{N} |\langle \vec{J} \rangle|^2,$$

which covers the remaining range of possibilities contained in the generalized spin squeezing inequalities of Eq. (3.44) with $\hat{n}$ along the mean spin direction.

(iii) If the average spin is zero for a given state i.e., we have $\langle J_k \rangle = 0$ for all directions $\hat{k}$. The inequalities of Korbicz et al. [31] assume a simple form

$$\langle J_k \rangle^2 < \frac{N}{4}.$$

This case obviously corresponds to $I_3 = 0$ and $I_1 < 0$. 
In the following table we summarize the results and prescribe a classification of pairwise entanglement in symmetric multiqubit states.

| Criterion of pairwise entanglement | Collective behaviour to look for |
|------------------------------------|----------------------------------|
| $\mathcal{I}_3 \neq 0$             | \begin{align*}
\mathcal{I}_5 & \leq 0 \\
(\Delta J_\perp)^2_{\text{min}} & \leq \frac{N}{4}
\end{align*} |
| $\mathcal{I}_3 = 0$                | \begin{align*}
\mathcal{I}_1 & < 0 \\
\langle J_i^2 \rangle & < \frac{N}{4} \\
\text{for any direction } i &= 1, 2, 3, \text{ so that} \\
\langle (J_i J_j + J_j J_i) \rangle & = 0; \text{ for } i \neq j
\end{align*} |

Table 3.1: Classification of pairwise entanglement in symmetric multi-qubit states in terms of two-qubit local invariants.
3.5 Conclusions

In summary, we have shown that a set of six local invariants \( \{I_1 - I_6\} \), associated with the two-qubit partition of a symmetric multiqubit system, characterizes the pairwise entanglement properties of the collective state. We have proposed a detailed classification scheme, for pairwise entanglement in symmetric multiqubit system, based on negative values of the invariants \( I_1, I_4, I_5 \) and \( I_4 - I_3^2 \). Specifically, we have shown, collective spin squeezing in symmetric multi-qubit states is a manifestation of \( I_5 < 0 \). Moreover, we have related our criteria, which are essentially given in terms of invariants of the quantum state, to the recently proposed generalized spin squeezing inequalities [31] for two qubit entanglement.
Chapter 4

Dynamical models

In the previous chapters, we have proposed separability criteria for symmetric multiqubit states in terms of two qubit local invariants. In the light of our characterization for pairwise entanglement, we analyze few symmetric multi-qubit dynamical models like,

1. Dicke states [39, 40]
2. Kitagawa-Ueda state generated by one axis twisting Hamiltonian [19]
3. Atomic squeezed states [20].

4.1 Dicke State

Collective spontaneous emission from dense atomic systems has been of interest since the pioneering work of Dicke, who predicted that two-level atoms (or qubits) possess collective quantum states in which spontaneous emission is enhanced (superradiance) or suppressed (subradiance). Multiqubit Dicke states are of interest for quantum information processing because they are robust under qubit loss [52, 53] and stand as an example of decoherence-free subspaces. An N-qubit symmetric Dicke states $|J = \frac{N}{2}, M\rangle$; $-J \leq M \leq J$, with
Dynamical models

\((\frac{N}{2} - M)\) excitations (spin up) is defined as \[52\]

\[
|\frac{N}{2}, M\rangle = \left( \begin{array}{c} N \\ M \end{array} \right)^{-\frac{1}{2}} \sum_{k} P_{k} \left( |1_{1}, 1_{2}, \ldots 1_{m}, 0_{m+1}, \ldots 0_{N}\rangle \right)
\]

(4.1)

where \(m = \frac{N}{2} - M\) and \(\{P_{k}\}\) is the set of all distinct permutations of the spins. A well known example is the W-state given by,

\[
|\frac{N}{2}, M = \frac{N}{2} - 1\rangle = \frac{1}{\sqrt{N}} \left[ |1_{1}0_{2}0_{3} \cdots 0_{N}\rangle + |0_{1}1_{2}0_{3} \cdots 0_{N}\rangle + \cdots + |0_{1}0_{2}0_{3} \cdots 1_{N}\rangle \right]
\]

(4.2)

which is a collective spin state with one excitation.

Multiqubit Dicke states are symmetric under permutation of atoms and entanglement-robust against particle loss \[52\]. They exhibit unique entanglement properties \[54\] and are excellent candidates for experimental manipulation and characterization of genuine multipartite entanglement. It has been shown that a wide family of Dicke states can be generated in an ion chain by single global laser pulses \[55\]. Further, a selective technique that allow a collective manipulation of the ionic degrees of freedom inside the symmetric Dicke subspace has been proposed \[56\]. An experimental scheme to reconstruct the spin-excitation number distribution of the collective spin states i.e., tomographic reconstruction of the diagonal elements of the density matrix in the Dicke basis of macroscopic ensembles containing atoms, with low mean spin excitations, has also been put forth \[57\]. More recently, Thiel et. al. \[58\] proposed conditional detection of photons in a Lambda system, as a way to produce symmetric Dicke states.

In order to analyze the pairwise entanglement properties of N-qubit Dicke states in terms of two qubit local invariants, we need to evaluate the first and second order moments \(\langle J_{i}\rangle, \langle J_{i}J_{j} + J_{j}J_{i}\rangle\) of the collective spin observable. These moments in turn allow us to determine the two qubit state parameters associated with a random pair of qubits (atoms), drawn from a multiqubit Dicke state.
Average values of the collective spin observable $J$:

It is easy to see that

$$\langle J_1 \rangle = \langle J, M | J_1 | J, M \rangle = \frac{1}{2} \left[ \langle J, M | J_+ + J_- | J, M \rangle \right] = 0$$

$$\langle J_2 \rangle = \langle J, M | J_2 | J, M \rangle = \frac{1}{2i} \left[ \langle J, M | J_+ - J_- | J, M \rangle \right] = 0$$

$$\langle J_3 \rangle = \langle J, M | J_3 | J, M \rangle = M \langle J, M | J, M \rangle = M.$$  \hspace{1cm} \text{(4.3)}

The average values of second order collective spin correlations in N-qubit Dicke states are readily evaluated and are given by,

$$\langle J_1^2 \rangle = \langle J, M | J_1^2 | J, M \rangle$$

$$= \frac{1}{4} \langle J, M | (J_+ + J_-)^2 | J, M \rangle$$

$$= \frac{1}{8} \left[ (N^2 + 2N - 4M^2) \right]$$

$$= \frac{(N^2 + 2N - 4M^2)}{8},$$

$$\langle J_2^2 \rangle = \langle J, M | J_2^2 | J, M \rangle$$

$$= \frac{1}{4} \langle J, M | (J_+ - J_-)^2 | J, M \rangle$$

$$= \frac{1}{8} \left[ (N^2 + 2N - 4M^2) \right]$$

$$= \frac{(N^2 + 2N - 4M^2)}{8},$$

$$\langle J_3^2 \rangle = \langle J, M | J_3^2 | J, M \rangle$$

$$= M^2 \langle J, M | J, M \rangle$$

$$= M^2.$$  \hspace{1cm} \text{(4.4)}
\[ \langle J_1 J_2 + J_2 J_1 \rangle = \langle J, M | J_1 J_2 + J_2 J_1 | J, M \rangle \]
\[ = \frac{1}{4} \langle J, M | (J_+ + J_-)(J_+ - J_-) + (J_+ + J_-)(J_+ - J_-) | J, M \rangle \]
\[ = 0. \quad (4.5) \]

Similarly, we find that
\[ \langle J_i J_j + J_j J_i \rangle = 0 \quad \text{for} \quad i \neq j. \quad (4.6) \]

### 4.1.1 Two qubit state parameters for Dicke state

We may recall here that the components of the single qubit orientation vector \( s_i \) (Eq. (2.9)) are related to the first order moments of the collective spin observables (see Eq. (3.7)) through the relation \( \langle J_i \rangle = \frac{N}{2} s_i \). Thus it is clear from Eq. (4.3), that
\[
\begin{align*}
  s_1 & = \frac{2}{N} \langle J_1 \rangle = 0, \\
  s_2 & = \frac{2}{N} \langle J_2 \rangle = 0, \\
  s_3 & = \frac{2}{N} \langle J_3 \rangle = \frac{2M}{N}.
\end{align*}
\]

(4.7)

In other words, the qubit orientation vector of any random qubit drawn from a \( N \) qubit Dicke state has the form,
\[
\vec{s} \equiv \left( 0, 0, \frac{2M}{N} \right). \quad (4.8)
\]

As the qubit correlations \( t_{ij} \) are related to the collective second moments \( \langle J_i J_j + J_j J_i \rangle \) through (see Eq. (3.11))
\[
t_{ij} = \frac{1}{N - 1} \left[ \frac{2 \langle J_i J_j + J_j J_i \rangle}{N} - \delta_{i,j} \right],
\]

(4.9)
we can evaluate the matrix elements of $T$ by using Eq. (4.4)

$$
t_{11} = \frac{4\langle J_1^2 \rangle}{N(N-1)} - \frac{1}{N-1} \left[ \frac{4(N^2 + 2N - 4M^2)}{8} - N \right]$

$$= \frac{1}{N(N-1)} \left[ \frac{N^2 - 4M^2}{2N(N-1)} \right]$$

$$t_{22} = \frac{4\langle J_2^2 \rangle}{N(N-1)} - \frac{1}{N-1} \left[ \frac{4(N^2 + 2N - 4M^2)}{8} - N \right]$$

$$= \frac{1}{N(N-1)} \left[ \frac{N^2 - 4M^2}{2N(N-1)} \right]$$

$$t_{33} = \frac{4\langle J_3^2 \rangle}{N(N-1)} - \frac{1}{N-1} \left[ 4M^2 - N \right]$$

$$= \frac{4M^2 - N}{N(N-1)}. \quad (4.10)$$

Further from Eq. (4.6), it can be easily seen that the off diagonal elements of the correlation matrix corresponding to Dicke state are all zero i.e.,

$$t_{ij} = 0 \quad \text{with} \quad i \neq j. \quad (4.11)$$

We thus find that the $3 \times 3$ real symmetric two qubit correlation matrix $T$ associated for any random pair of qubits, drawn from a multiqubit Dicke state is explicitly given by,

$$T = \text{diag} \left( t_1, t_2, t_3 \right) = \begin{pmatrix}
\frac{N^2 - 4M^2}{2N(N-1)} & 0 & 0 \\
0 & \frac{N^2 - 4M^2}{2N(N-1)} & 0 \\
0 & 0 & \frac{4M^2 - N}{N(N-1)}
\end{pmatrix}. \quad (4.12)$$
We construct the density matrix characterizing a pair of qubits arbitrarily chosen from a multiqubit Dicke state

$$\rho_{\text{sym}} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & c & c & 0 \\ 0 & c & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}, \quad (4.13)$$

where,

$$a = \frac{(N + 2M)(N - 2 + 2M)}{4N(N - 1)},$$
$$c = \frac{N^2 - 4M^2}{4N(N - 1)},$$
$$d = \frac{(N - 2M)(N - 2 - 2M)}{4N(N - 1)}. \quad (4.14)$$

We may note here that the above density matrix belongs to the special class of density matrices discussed in Sec 2.5. Thus the non-local properties of symmetric N-qubit Dicke states are characterized either by subset of three invariants ($\mathcal{I}_3, \mathcal{I}_4, \mathcal{I}_5$), when $\mathcal{I}_3 \neq 0$ or a subset of two invariants ($\mathcal{I}_1, \mathcal{I}_2$), when $\mathcal{I}_3 = 0$. We now proceed to evaluate the two qubit local invariants associated with the two qubit density matrix Eq. (4.13).

### 4.1.2 Local invariants

The two-qubit local invariants (Eq. (2.27)), associated with Dicke state can be readily evaluated and we obtain,

$$\mathcal{I}_1 = \det T = t_1 t_2 t_3$$
$$= \left( \frac{N^2 - 4M^2}{2N(N - 1)} \right)^2 \left( \frac{4M^2 - N}{N(N - 1)} \right), \quad (4.15)$$
\[ I_2 = \text{Tr} (T^2) = t_1^2 + t_2^2 + t_3^2 = 2 \left( \frac{N^2 - 4M^2}{2N(N - 1)} \right)^2 + \left( \frac{4M^2 - N}{N(N - 1)} \right)^2, \]

\[ I_3 = s^T s = s_1^2 + s_2^2 + s_3^2 = \frac{4M^2}{N^2}, \]

\[ I_4 = s^T T s = s_1^2 t_1 + s_2^2 t_2 + s_3^2 t_3 = I_3 \left( \frac{4M^2 - N}{N(N - 1)} \right), \]

\[ I_5 = \epsilon_{ijk} \epsilon_{lmn} s_i t_{jm} t_{kn} = 2 (s_1^2 t_2 t_3 + s_2^2 t_1 t_3 + s_3^2 t_1 t_2) = 8 I_3 \left( \frac{N^2 - 4M^2}{4N(N - 1)} \right)^2, \]

\[ I_6 = \epsilon_{ijk} s_i (T s)_j (T^2 s)_k = s_1 s_2 s_3 [t_1 t_2 (t_2 - t_1) + t_2 t_3 (t_3 - t_2) + t_3 t_1 (t_1 - t_3)] = 0. \quad (4.16) \]

Further, the combination \( I_4 - I_3^2 \) of invariants, is given by

\[ I_4 - I_3^2 = \left( \frac{4M^2 - N^2}{N^2(N - 1)} \right) I_3. \quad (4.17) \]

We now consider three different cases (for different values of \( M \)) and explicitly verify pairwise entanglement of Dicke states through two qubit local invariants.

(i) When \( M = \pm \frac{N}{2} \):

In this case, the multiqubit dicke state state has the form

\[ \left| \frac{N}{2}, \frac{N}{2} \right\rangle = |0_1 0_2 \cdots 0_N \rangle. \quad (4.18) \]
Dynamical models

This corresponds to a situation in which all the qubits are spin-up. The N-qubit Dicke state in which all the qubits are spin-down is given by.

\[ |\frac{N}{2}, -\frac{N}{2}\rangle = |1_11_2 \cdots 1_N\rangle. \]  \hspace{1cm} (4.19)

The collective state, corresponding to this case, is obviously a uncorrelated product state. The invariants in this case are given by

\[ I_1 = I_5 = 0, \]
\[ I_2 = I_3 = I_4 = 1, \]  \hspace{1cm} (4.20)

which are all non-negative indicating that \(|\frac{N}{2}, \pm \frac{N}{2}\rangle\) Dicke states are separable.

(ii) \(M=0\):

\(|\frac{N}{2}, M = 0\rangle\) Dicke states are written as

\[ |\frac{N}{2}, 0\rangle = \frac{1}{\sqrt{N}} |1_11_2 \cdots 1_N, 0_{\frac{N}{2}+1}0_{\frac{N}{2}+2} \cdots 0_N\rangle. \]  \hspace{1cm} (4.21)

The invariants in this case are given by

\[ I_3 = I_4 = I_5 = 0, \]  \hspace{1cm} (4.22)

while the non-zero invariant,

\[ I_1 = -\frac{1}{4} \left(\frac{N}{N-1}\right)^3 \]  \hspace{1cm} (4.23)

assumes negative value.

So, the Dicke state \(|\frac{N}{2}, 0\rangle\), (with even number of atoms), exhibits pairwise entanglement, which is signalled in terms of collective signature (see Table. (3.1)) \(\langle J_i^2 \rangle < \frac{N}{4}\).
(iii) \( M \neq \pm \frac{N}{2}, \ 0 \):

In this case, the invariant, \( I_4 \) is bound by

\[
-\frac{1}{N-1} < I_4 < 1,
\]

and the combination \( I_4 - I_3^2 \) is always negative, thus revealing pairwise entanglement in Dicke atoms in this case too. The corresponding collective signature is given by \( \frac{N}{4} < \langle (\vec{J} \cdot \hat{n}_0)^2 \rangle < \frac{N}{4} + \frac{(N-1)}{N} |\langle \vec{J} \rangle|^2 \) (see Table. (3.1)).

### 4.2 Kitagawa-Ueda state generated by one axis twisting Hamiltonian

In 1993, Kitagawa and Ueda [19] had proposed the generation of correlated \( N \)-qubit states, which are spin squeezed, through the nonlinear Hamiltonian evolution \( H = J^2 \chi \),

\[
|\Psi_{K-U}\rangle = e^{-iHt} |J, -J\rangle; \quad J = \frac{N}{2},
\]

(4.24)

referred to as one-axis twisting mechanism. The \( N \) qubit state \( |J, -J\rangle \) is the all spin down state

\[
|J, -J\rangle = |1_1, 1_2, 1_3, ...1_N\rangle.
\]

The one-axis twisting Hamiltonian has been realized in various quantum systems including quantum optical systems [59], ion traps [60], cavity quantum electro magnetic dynamics [61]. This effective Hamiltonian \( H = J^2 \chi \), has already been employed to produce entangled states of four qubit maximally entangled states in an ion trap [62]. Collisional interactions between atoms in two-component Bose-Einstein condensation are also modeled using this one-axis twisting Hamiltonian [13].

In order to investigate the entanglement properties for a random pair of qubits drawn from the Kitagawa-Ueda state, we evaluate the first and second order moments of the
collective spin operators.

**Expectation values of the spin operator $J$:**

To evaluate the expectation values $\langle J_i \rangle$ and $\langle J_i J_j + J_j J_i \rangle$, let us first consider the time dependent operators $J_3(t)$ under the Hamiltonian evolution:

$$J_3(t) = e^{iHt} J_3 e^{-iHt} = J_3 + [iHt, J_3] + \frac{1}{2!} [iHt, [iHt, J_3]] + \cdots \quad (4.25)$$

(Here we have used the Baker-Campbell-Hausdorff formula $e^A e^{-A} = B + [A, B] + \frac{1}{2!} [A, [A, B]] + \cdots$)

The commutators in Eq. (4.25) are given by:

$$[iHt, J_3] = [i\chi t J_1^2, J_3] = i\chi t \{ J_1 [J_1, J_3] + [J_1, J_3] J_1 \}$$

$$= i\chi t \{ J_1 (-iJ_2) + (-iJ_2) J_1 \}$$

$$= \chi t [J_1, J_2]_+,$$

where we have denoted $[O_1, O_2]_+ = O_1 O_2 + O_2 O_1$. We further obtain,

$$[iHt, [iHt, J_3]] = [i\chi t J_1^2, \chi t [J_1, J_2]_+]$$

$$= i\chi^2 t^2 \{ J_1 [J_1, [J_1, J_2]_+] + [J_1, [J_1, J_2]_+] J_1 \}$$

$$= -\chi^2 t^2 \{ J_1^2 J_3 + 2 J_1 J_3 J_1 + J_3 J_1^2 \}.$$ 

Therefore we get,

$$J_3(t) = J_3 + \chi t [J_1, J_2]_+ - \frac{1}{2!} \chi^2 t^2 \{ J_1^2 J_3 + 2 J_1 J_3 J_1 + J_3 J_1^2 \} + \cdots \quad (4.26)$$

Now we consider the time dependent operator $J_+ (t)$ under the Hamiltonian evolution $H$,

$$J_+(t) = e^{iHt} J_+ e^{-iHt} = J_+ + [iHt, J_+] + \frac{1}{2!} [iHt, [iHt, J_+]] + \cdots \quad (4.27)$$
The commutators can be computed as follows:

\[
[iHt, J_+] = [i\chi t J_1^2, J_+] = i\chi t \{ J_1 [J_1, J_+] + [J_1, J_+] J_1 \} \\
= i\chi t \{ J_1 (-J_3) + (-J_3)J_1 \} \\
= -i\chi t [J_1, J_3]^+ \\
\]

\[
[iHt, [iHt, J_+]] = [i\chi t J_1^2, -i\chi t [J_1, J_3]^+] \\
= \chi^2 t^2 \{ J_1 [J_1, [J_1, J_3]^+] + [J_1, [J_1, J_3]^+] J_1 \} \\
= -i\chi^2 t^2 \{ J_1^2 J_2 + 2 J_1 J_2 J_1 + J_2 J_1^2 \}. \\
\]

Hence,

\[
J_+(t) = J_+ - i\chi t [J_1, J_3]^+ - i\chi^2 t^2 \{ J_1^2 J_2 + 2 J_1 J_2 J_1 + J_2 J_1^2 \} + \cdots \quad (4.28)
\]

Similarly \( J_-(t) \) can be evaluated as

\[
J_-(t) = e^{iHt} J_- e^{-iHt} = J_- + [iHt, J_-] + \frac{1}{2!} [iHt, [iHt, J_-]] + \cdots \quad (4.29)
\]

\[
[iHt, J_-] = [i\chi t J_1^2, J_-] = i\chi t \{ J_1 [J_1, J_-] + [J_1, J_-] J_1 \} \\
= i\chi t \{ J_1 (J_3) + (J_3)J_1 \} \\
= i\chi t [J_1, J_3]^+ \\
\]

\[
[iHt, [iHt, J_-]] = [i\chi t J_1^2, i\chi t [J_1, J_3]^+] \\
= -\chi^2 t^2 \{ J_1 [J_1, [J_1, J_3]^+] + [J_1, [J_1, J_3]^+] J_1 \} \\
= i\chi^2 t^2 \{ J_1^2 J_2 + 2 J_1 J_2 J_1 + J_2 J_1^2 \}. \\
\]
Dynamical models

So we obtain,

\[ J_- (t) = J_- + i\chi [J_1, J_3]_+ + i\chi^2 t^2 \left\{ J_2^2 J_2 + 2 J_1 J_2 J_1 + J_2 J_3^2 \right\} + \cdots \quad (4.30) \]

Using Eqs. (4.26), (4.28), (4.30), the expectation values are evaluated as follows:

\[ \langle \Psi_{K-U} | J_3(t) | \Psi_{K-U} \rangle = \langle J - J | J_3(t) | J - J \rangle \]
\[ = \langle J - J | [J_3 + \chi t [J_1, J_2]_+] - \frac{1}{2!} \chi^2 t^2 \left\{ J_2^2 J_3 + 2 J_1 J_2 J_1 + J_3 J_2^2 \right\} + \cdots | J - J \rangle \]
\[ = -\frac{N}{2} \left[ 1 - \frac{1}{2!} \chi^2 t^2 (N - 1) \right] + \frac{1}{4!} \chi^4 t^4 (N - 1)^2 - \cdots \]
\[ = -\frac{N}{2} \cos^{N-1} (\chi t), \quad (4.31) \]

\[ \langle \Psi_{K-U} | J_+ (t) | \Psi_{K-U} \rangle = \langle J - J | J_+ (t) | J - J \rangle \]
\[ = \langle J - J | J_+ - i\chi [J_1, J_3]_+ - i\chi^2 t^2 \left\{ J_2^2 J_2 + 2 J_1 J_2 J_1 + J_2 J_3^2 \right\} + \cdots | J - J \rangle \]
\[ = 0 \quad (4.32) \]

\[ \langle \Psi_{K-U} | J_- (t) | \Psi_{K-U} \rangle = \langle J - J | J_- (t) | J - J \rangle \]
\[ = \langle J - J | J_- + i\chi [J_1, J_3]_+ + i\chi^2 t^2 \left\{ J_2^2 J_2 + 2 J_1 J_2 J_1 + J_2 J_3^2 \right\} + \cdots | J - J \rangle \]
\[ = 0 \quad (4.33) \]

\[ \langle \Psi_{K-U} | J_3^2 (t) | \Psi_{K-U} \rangle = \langle J - J | J_3^2 (t) | J - J \rangle \]
\[ = \langle J - J | \left( J_3 + \chi [J_1, J_2]_+ - \frac{1}{2!} \chi^2 t^2 \left\{ J_2^2 J_3 + 2 J_1 J_2 J_1 + J_3 J_2^2 \right\} + \cdots \right)^2 | J - J \rangle \]
\[ = \frac{1}{8} [N^2 + N + N(N - 1) \cos^{N-2} (2\chi t)]. \quad (4.34) \]

In a similar manner, the remaining first and second order expectation values can be obtained. The average values of the collective spin observables for Kitagawa-Ueda state are
Dynamical models

listed below:

\[
\begin{align*}
\langle J_1 \rangle &= \langle J_2 \rangle = 0, \\
\langle J_3 \rangle &= -\frac{N}{2} \cos^{N-1}(\chi t) \\
\langle J^2_1 \rangle &= \frac{N}{4}, \\
\langle J^2_2 \rangle &= \frac{1}{8} \left( N^2 + N - N(N - 1) \cos^{N-2}(2\chi t) \right), \\
\langle J^2_3 \rangle &= \frac{1}{8} \left( N^2 + N + N(N - 1) \cos^{N-2}(2\chi t) \right), \\
\langle [J_1, J_2] + \rangle &= \frac{1}{2} N(N - 1) \cos^{N-2}(\chi t) \sin(\chi t), \\
\langle [J_+, J_3] + \rangle &= 0.
\end{align*}
\]

(4.35)

4.2.1 Two qubit state variables

The qubit state parameter \( s_i \) associated with a random qubit chosen from a multiqubit Kitagawa-Ueda state can be written (from Eq. (4.35)) as,

\[
\begin{align*}
\ s_1 &= \frac{2}{N} \langle J_1 \rangle = 0, \\
\ s_2 &= \frac{2}{N} \langle J_2 \rangle = 0, \\
\ s_3 &= \frac{2}{N} \langle J_3 \rangle = -\cos^{(N-1)}(\chi t).
\end{align*}
\]

(4.36)

Therefore the orientation vector \( \vec{s} \) for qubit drawn randomly from the Kitagawa-Ueda state is given by

\[
\vec{s} = \left( 0, 0, -\cos^{(N-1)}(\chi t) \right).
\]
The two qubit correlation matrix elements are readily obtained from the second order moments of the collective spin observable Eq. (4.35),

\[ t_{ij} = \frac{1}{N-1} \left[ \frac{2 \langle J_i J_j + J_j J_i \rangle}{N} - \delta_{ij} \right]. \]

Thus we have,

\[ t_{11} = t_{13} = t_{23} = 0, \]
\[ t_{12} = \cos((N-2)(\chi t)) \sin(\chi t), \]
\[ t_{22} = \frac{1}{2} \left( 1 - \cos((N-2)(2\chi t)) \right), \]
\[ t_{33} = \frac{1}{2} \left( 1 + \cos((N-2)(2\chi t)) \right). \] (4.37)

The 3 \times 3 real two qubit correlation matrix \( T \) can be thus written as,

\[
T = \begin{pmatrix}
0 & \cos((N-2)(\chi t)) \sin(\chi t) & 0 \\
\cos((N-2)(\chi t)) \sin(\chi t) & \frac{1}{2} \left( 1 - \cos((N-2)(2\chi t)) \right) & 0 \\
0 & 0 & \frac{1}{2} \left( 1 + \cos((N-2)(2\chi t)) \right)
\end{pmatrix}.
\]

The two qubit density matrix for a pair of qubits arbitrarily chosen from a multiqubit Kitagawa-Ueda state is given by,

\[
\rho_{\text{sym}} = \begin{pmatrix}
a & 0 & 0 & b \\
0 & c & c & 0 \\
0 & c & c & 0 \\
b & 0 & 0 & d
\end{pmatrix}, \] (4.38)
Dynamical models

with the matrix elements given by,

\[ a = \frac{3 + \cos^{N-2}(2\chi t) - 2\cos^{N-1}(\chi t)}{8}, \]
\[ b = \frac{1 - \cos^{N-2}(2\chi t)}{8}, \]
\[ c = \frac{N^2 - 4M^2}{4N(N - 1)}, \]
\[ d = \frac{3 + \cos^{N-2}(2\chi t) + 2\cos^{N-1}(\chi t)}{8}. \]

4.2.2 Local invariants

The two qubit local invariants Eq. (\ref{eq:2.27}) associated with the \( N \) qubit Kitagawa-Ueda state are given by

\[ I_1 = \det T = -\frac{1}{2} \cos^2(N-2)(\chi t) \sin^2(\chi t) \left( 1 + \cos^{(N-2)}(2\chi t) \right), \]
\[ I_2 = \text{Tr} (T^2) = 2 \cos^2(N-2)(\chi t) \sin^2(\chi t) + \frac{1}{2} \left( 1 + \cos^{2(N-2)}(\chi t) \right), \]
\[ I_3 = s^T s = s_1^2 + s_2^2 + s_3^2 = \cos^2(N-1)(\chi t), \]
\[ I_4 = s^T T s = \frac{1}{2} I_3 \left( 1 + \cos^{(N-2)}(2\chi t) \right), \]
\[ I_5 = \epsilon_{ijk} \epsilon_{lmn} s_i s_l t_j m t_k n = -2 I_3 \cos^2(N-2)(\chi t) \sin^2(\chi t), \]
\[ I_6 \epsilon_{ijk} s_i (T s)_j (T^2 s)_k = 0. \]

We see that pairwise entanglement is manifest through the negative value of the invariant \( I_5 \). Further, from Eq. (\ref{eq:3.32}) it is evident that \( I_5 < 0 \) implies that the state is spin squeezed.
Figure 4.1: The invariants $I_4$, $I_4 - I_3^2$, and $I_5$, corresponding to a $N$-qubit Kitagawa-Ueda state. Curve $a: N = 4$, $b: N = 6$, and $c: N = 8$. 
In Fig. 4.1, we have plotted the two qubit local invariants $I_4$, $I_4 - I_3^2$, and $I_5$, associated with the multiqubit kitagawa-Ueda state for different values of $N$.

### 4.3 Atomic spin squeezed states

Consider a system of $N$ identical two-level atoms. The atomic squeezed states are defined by

$$\left| \Psi_M \right\rangle = A_0 \exp (\theta J_3) \exp \left( -i \frac{\pi}{2} J_2 \right) \left| J = \frac{N}{2}, M \right\rangle,$$  \hfill (4.40)

where $\theta$ is conveniently parameterized as $e^{2\theta} = \tanh 2\xi$.

The atomic spin squeezed states $\left| \Psi_M \right\rangle$ are shown [20] to be the eigenvectors of the nonhermitian operator $R_3$, given by

$$R_3 = \frac{J_- \cosh \xi + J_+ \sinh \xi}{\sqrt{2 \sinh 2\xi}},$$  \hfill (4.41)

where

$$R_3 \left| \Psi_M \right\rangle = M \left| \Psi_M \right\rangle.$$  \hfill (4.42)

Here, we concentrate on the state

$$\left| \Psi_0 \right\rangle = A_0 \exp (\theta J_3) \exp \left( -i \frac{\pi}{2} J_2 \right) \left| J = \frac{N}{2}, 0 \right\rangle.$$  \hfill (4.43)

In the interaction of a collection of even number of $N$ two-level atoms with squeezed radiation, it has been shown [20] that the steady state $\left| \Psi_0 \right\rangle$ of Eq. (4.43) is the pure atomic squeezed state for certain values of external field strength and detuning parameters.

It is convenient to express the atomic squeezed state $\left| \Psi_0 \right\rangle$ as,

$$\left| \Psi_0 \right\rangle = A_0 \sum_{M=-J}^{J} d^{J}_M 0 \left( \frac{\pi}{2} \right) e^{M\theta} \left| J = \frac{N}{2}, M \right\rangle,$$  \hfill (4.44)
where the coefficients \( d_{J M}^I \left( \frac{\pi}{2} \right) \) are given by [63]

\[
d_{J M}^I \left( \frac{\pi}{2} \right) = \frac{J! \sqrt{(J + M)! (J - M)!}}{2^J} \sum_{p=M}^{J-M} \frac{(-1)^p}{(J - M)! p! (p - M)! (J + M - p)!}.
\]

\[(4.45)\]

We now proceed to study the pairwise entanglement properties of atomic squeezed state \( |\Psi_0\rangle \).

**First and second order moments of the collective spin observable for the Atomic squeezed states:**

The first order expectation values of \( \langle J_i \rangle \) are evaluated as follows:

Consider the expectation value of the nonhermitian operator \( R_3 \) (see Eq. (4.41))

\[
\langle \Psi_0 | R_3 | \Psi_0 \rangle = \frac{1}{\sqrt{2 \sinh 2 \xi}} \langle \Psi_0 | J_- \cosh \xi + J_+ \sinh \xi | \Psi_0 \rangle.
\]

\[(4.46)\]

However, it is clear from Eq. (4.42) that

\[
\langle \Psi_0 | R_3 | \Psi_0 \rangle = 0.
\]

So, we obtain,

\[
\langle \Psi_0 | J_- \cosh \xi + J_+ \sinh \xi | \Psi_0 \rangle = 0.
\]

\[(4.47)\]

Further, it is evident that,

\[
\langle \Psi_0 | R_3^\dagger | \Psi_0 \rangle = 0,
\]

which in turn leads to,

\[
\langle \Psi_0 | J_+ \cosh \xi + J_- \sinh \xi | \Psi_0 \rangle = 0.
\]

\[(4.48)\]
Using Eq. (4.47) and Eq. (4.48), it is easy to see that

\[ \langle J_+ \rangle = 0, \quad \langle J_- \rangle = 0, \]

or \[ \langle J_1 \rangle = 0, \quad \langle J_2 \rangle = 0. \]

The first order moment \( \langle J_3 \rangle \) can be explicitly evaluated as follows: We have,

\[
J_3 |\Psi_0\rangle = A_0 \sum_{M=-J}^J M d_M^J \left( \frac{\pi}{2} \right) e^{(M\theta)} |M\rangle,
\]

and therefore

\[
\langle \Psi_0 | J_3 | \Psi_0 \rangle = A_0^2 \sum_{M=-J}^J M \left[ d_M^J \left( \frac{\pi}{2} \right) \right]^2 e^{(2M\theta)}. \tag{4.49}
\]

The second order moments \( \langle (J_i J_j + J_j J_i) \rangle \) of the collective spin operator, are conveniently evaluated in terms of the expectation values of the non-hermitian operator \( R_3^2 \) :

\[
R_3^2 = (J_\perp \cosh \xi + J_\parallel \sinh \xi) (J_\perp \cosh \xi + J_\parallel \sinh \xi)
\]

\[
\frac{2 \sinh \xi}{2 \sinh 2\xi}
\]

\[
= J_\perp^2 \cosh^2 \xi + J_\parallel^2 \sinh^2 \xi + (J_\perp J_\parallel + J_\parallel J_\perp) \sinh \xi \cos \xi
\]

\[
\frac{2 \sinh 2\xi}{2 \sinh 2\xi}
\]

\[
\langle \Psi_0 | R_3^2 | \Psi_0 \rangle = \langle J_1^2 - J_2^2 \rangle \coth 2\xi + \langle J_1^2 + J_2^2 \rangle - i \langle [J_1, J_2]_+ \rangle \coth 2\xi. \tag{4.50}
\]

Since \( \langle \Psi_0 | R_3^2 | \Psi_0 \rangle = 0 \), (see Eq. (4.42)) we obtain,

\[
\langle J_1^2 - J_2^2 \rangle \coth 2\xi + \langle J_1^2 + J_2^2 \rangle - i \langle [J_1, J_2]_+ \rangle \coth 2\xi = 0. \tag{4.51}
\]

In other words, we have,

\[
\text{Re} \left( \langle R_3^2 \rangle \right) = \langle J_1^2 - J_2^2 \rangle \coth 2\xi + \langle J_1^2 + J_2^2 \rangle = 0 \tag{4.52}
\]

and

\[
\text{Im} \left( \langle R_3^2 \rangle \right) = \langle [J_1, J_2]_+ \rangle \coth 2\xi = 0. \tag{4.53}
\]

---

1Since \( J_\parallel = \frac{J_1 + J_2}{2} \) and \( J_\perp = \frac{J_1 - J_2}{2i} \), \( \langle J_\parallel \rangle = 0 \), \( \langle J_\perp \rangle = 0 \) \( \Rightarrow \) \( \langle J_1 \rangle = \langle J_2 \rangle = 0 \).
We now evaluate the expectation value of $R_3^\dagger R_3$ given explicitly as,

\[
R_3^\dagger R_3 = \frac{(J_+ \cosh \xi + J_- \sinh \xi)(J_- \cosh \xi + J_+ \sinh \xi)}{2 \sinh 2\xi}
= \frac{J_+ J_- \cosh^2 \xi + J_- J_+ \sinh^2 \xi + (J_+^2 + J_-^2) \sinh \xi \cos \xi}{2 \sinh 2\xi}
\]

\[
\langle \Psi_0 | R_3^\dagger R_3 | \Psi_0 \rangle = \langle J_1^2 + J_2^2 \rangle \coth 2\xi + \langle J_1^2 - J_2^2 \rangle + \langle J_3 \rangle. \tag{4.54}
\]

From Eq. (4.42), it is clear that $\langle R_3^\dagger R_3 \rangle = 0$, and therefore we get,

\[
\langle J_1^2 + J_2^2 \rangle \coth 2\xi + \langle J_1^2 - J_2^2 \rangle + \langle J_3 \rangle = 0
\]

i.e.,

\[
- \langle J_1^2 - J_2^2 \rangle - \langle J_1^2 + J_2^2 \rangle \coth 2\xi = \langle J_3 \rangle. \tag{4.55}
\]

Simplifying the Eqs. (4.52), (4.55), we obtain the expectation values of $J_1^2$ and $J_2^2$ as,

\[
\langle J_1^2 \rangle = -\frac{1}{2} \langle J_3 \rangle e^{-2\xi}
\]

\[
\langle J_2^2 \rangle = -\frac{1}{2} \langle J_3 \rangle e^{2\xi}. \tag{4.56}
\]

Now, to compute the average value of $J_3^2$, we use

\[
\langle J^2 \rangle = \langle J_1^2 + J_2^2 + J_3^2 \rangle = J(J + 1) \tag{4.57}
\]

and obtain,

\[
\langle J_3^2 \rangle = \langle J^2 - J_1^2 - J_2^2 \rangle = J(J + 1) - \langle J_3 \rangle \cosh 2\xi. \tag{4.58}
\]

The second order expectation values $\langle \Psi_0 | [J_1, J_3]_+ | \Psi_0 \rangle$ and $\langle \Psi_0 | [J_2, J_3]_+ | \Psi_0 \rangle$ are determined as follows:

Let us consider $\langle [J_+, J_3]_+ \rangle = \langle \psi_0 | J_+ J_3 | \psi_0 \rangle + \langle \psi_0 | J_3 J_+ | \psi_0 \rangle$. By computing each
term separately,

\[
\langle \Psi_0 | J_+ J_3 | \Psi_0 \rangle = A_0 \sum_{M=-J}^{J} M d^J_M \left( \frac{\pi}{2} \right) d^{J'}_{M'} \left( \frac{\pi}{2} \right) e^{(M+M')\theta} \sqrt{(J+M')(J-M'+1)} \delta_{M',M+1}
\]

\[
= A_0 \sum_{M=-J}^{J} M d^J_M \left( \frac{\pi}{2} \right) d^{J}_{M+10} \left( \frac{\pi}{2} \right) e^{(2M+1)\theta} \sqrt{(J+M+1)(J-M)}
\]

\[
\langle J_+, J_3 \rangle = 0,
\]

(4.59)

since \(d^{J}_M d^{J}_{M+10} = 0\).

Similarly,

\[
J_3 J_+ | \Psi_0 \rangle = A_0 \sum_{M=-J}^{J} M d^J_M \left( \frac{\pi}{2} \right) d^{J}_{M'} \left( \frac{\pi}{2} \right) e^{(M+M')\theta} \sqrt{(J-M')(J+M'+1)} \delta_{M',M-1}
\]

\[
= A_0 \sum_{M=-J}^{J} M d^J_M \left( \frac{\pi}{2} \right) d^{J}_{M-10} \left( \frac{\pi}{2} \right) e^{(2M+1)\theta} \sqrt{(J-M+1)(J+M)}
\]

\[
\langle J_3 J_+ \rangle = 0.
\]

(4.60)

From Eqs. (4.59, 4.60), we obtain

\[
\langle [J_+, J_3]_+ \rangle = 0.
\]

(4.61)

So, we obtain,

\[
\langle [J_1, J_3]_+ \rangle = 0, \quad \langle [J_2, J_3]_+ \rangle = 0.
\]

(4.62)

We next determine the two qubit state parameters associated with the atomic squeezed systems.

\[\text{The coefficient} \quad d^J_M \left( \frac{\pi}{2} \right) = \sqrt{(J+M)(J-M+1)} \left( \frac{1}{2} \right)^{J-M} \left( \frac{1}{2} \right)^{J-M} (-1)^{J-M} \left( \frac{J-M}{2} \right) \text{ for } J + M = \text{ even and}
\]

\[d^J_M \left( \frac{\pi}{2} \right) = 0 \text{ for } J + M = \text{ odd. Therefore, we obviously have } d^J_M d^{J}_{M+10} = 0.\]
4.3.1 Two qubit state parameters

The components of the single qubit orientation vector $\vec{s}$ drawn from a collective atomic system, are given by (see Eqs. (4.49), (4.49))

\[ s_1 = \frac{2}{N} \langle J_1 \rangle = 0, \]

\[ s_2 = \frac{2}{N} \langle J_2 \rangle = 0, \]

\[ s_3 = \frac{2}{N} \langle J_3 \rangle = \frac{2}{N} A_0^2 \sum_{M=-J}^{J} M \left[ d_{M0} \left( \frac{\pi}{2} \right) \right]^2 e^{(2M\theta)}. \quad (4.63) \]

Thus the average spin vector $\vec{s}$ for atomic squeezed states of Eq. (4.43) assumes the form

\[ \vec{s} = \left( 0, 0, \frac{2 \langle J_3 \rangle}{N} \right). \]

The elements of the two qubit correlation matrix which are expressed in terms of the second order moments (see Eq. (3.11)) are given by,

\[ t_{ij} = \frac{1}{N-1} \left[ \frac{2 \langle J_i J_j + J_j J_i \rangle}{N} - \delta_{i,j} \right]. \]
The diagonal elements of the correlation matrix $T$ are obtained using Eqs. (4.56), (4.58) and are given by,

\[
t_{11} = \frac{4\langle J_1^2 \rangle}{N(N-1)} - \frac{1}{N-1} = \frac{1}{N(N-1)} \left[ -\frac{1}{2} \langle J_3 \rangle e^{-2\xi} - N \right] = -2 \langle J_3 \rangle e^{-2\xi} - N \frac{1}{N(N-1)},
\]

\[
t_{22} = \frac{4\langle J_2^2 \rangle}{N(N-1)} - \frac{1}{N-1} = \frac{1}{N(N-1)} \left[ -\frac{1}{2} \langle J_3 \rangle e^{2\xi} - N \right] = -2 \langle J_3 \rangle e^{2\xi} - N \frac{1}{N(N-1)},
\]

\[
t_{33} = \frac{4\langle J_3^2 \rangle}{N(N-1)} - \frac{1}{N-1} = \frac{1}{N(N-1)} \left[ J(J+1) - \langle J_3 \rangle \cosh 2\xi - N \right] = 4 \langle J_3 \rangle \cosh(2\xi) + N^2 + N \frac{1}{N(N-1)}. \tag{4.64}
\]

Further, from Eqs. (4.62), (4.53), it is easy to see that the off-diagonal elements of $T$ are all zero

\[
t_{12} = t_{21} = 0,
\]

\[
t_{13} = t_{31} = 0,
\]

\[
t_{23} = t_{32} = 0. \tag{4.65}
\]
Thus, the correlation matrix $T$ has the following structure,

$$
T = \text{diag}(t_1, t_2, t_3) = \begin{pmatrix}
-\frac{2 \langle J_3 \rangle e^{-2\xi} - N}{N(N-1)} & 0 & 0 \\
0 & -\frac{2 \langle J_3 \rangle e^{2\xi} - N}{N(N-1)} & 0 \\
0 & 0 & 4 \langle J_3 \rangle \cosh(2\xi) + N^2 + N \\
\end{pmatrix}.
$$

### 4.3.2 Local invariants

The two qubit local invariants (see Eq. (2.27)) associated with the atomic spin squeezed states are listed below:

$$
\mathcal{I}_1 = t_1 t_2 t_3,
$$

$$
\mathcal{I}_2 = t_1^2 + t_2^2 + t_3^2,
$$

$$
\mathcal{I}_3 = s^T s = s_1^2 + s_2^2 + s_3^2 = \frac{4 \langle J_3 \rangle^2}{N^2},
$$

$$
\mathcal{I}_4 = s^T T s = s_1^2 t_1 + s_2^2 t_2 + s_3^2 t_3 = \mathcal{I}_3 \left[ \frac{4 \langle J_3 \rangle \cosh(2|\xi|) + N^2 + N}{N(N-1)} \right],
$$

$$
\mathcal{I}_5 = \epsilon_{ijk} \epsilon_{lmn} s_i s_l t_j m n
$$

$$
= -\frac{2 \mathcal{I}_3}{N^2(N-1)^2} \left( 2 \langle J_3 \rangle e^{-2|\xi|} + N \right) \left( 2 \langle J_3 \rangle e^{2|\xi|} + N \right),
$$

$$
\mathcal{I}_6 = \epsilon_{ijk} s_i (T s)_j (T^2 s)_k
$$

$$
= 0. \quad (4.66)
$$

In Fig. 4.2, we have plotted the invariants $\mathcal{I}_4$, $\mathcal{I}_4 - \mathcal{I}_3^2$ and $\mathcal{I}_5$, as a function of the parameter $x = e^{2\theta}$, for different values of $N$. These plots demonstrate that the invariant $\mathcal{I}_5$ is negative, highlighting the pairwise entanglement (spin squeezing) of the atomic state.
Figure 4.2: The invariants $I_4$, $I_4 - I_5^2$ and $I_5$, associated with the atomic squeezed state of $N$ two level atoms interacting with squeezed radiation. Curve $a$: $N = 4$, $b$: $N = 6$, $c$: $N = 8$, and $d$: $N = 20$. 
4.4 Conclusions

In this chapter, we have considered few interesting symmetric multi-qubit dynamical models like Dicke states, Kitagawa-Ueda state generated by one axis twisting Hamiltonian and Atomic squeezed state $|\Psi_0\rangle$. The density matrix of these states have a specific structure and belong to the special class of symmetric states Eq. (2.34) discussed in Chapter. We have evaluated the two qubit local invariants and investigated the nonlocal properties associated with the states. In each case, the entanglement properties are reflected through the negative values of some of the two qubit invariants thus highlighting our separability criteria.
Chapter 5

Constraints on the variance matrix of entangled symmetric qubits

The nonseparability constraints on the two qubit local invariants $I_4 < 0$, $I_5 < 0$ and $I_4 - I_3^2 < 0$ derived, in Chapters 2 and 3, serve only as sufficient condition for pairwise entanglement in symmetric qubits. In this Chapter, we derive necessary and sufficient condition for entanglement in symmetric two qubit states by establishing an equivalence between the Peres-Horodecki criterion [43, 44] and the negativity of the two qubit covariance matrix. Pairwise entangled symmetric multiqubit states necessarily obey these constraints. We also bring out a local invariant structure exhibited by these constraints.

5.1 Peres-Horodecki inseparability criterion for CV states

Peres-Horodecki inseparability criterion [43, 44] viz., positivity under partial transpose (PPT) has been extremely fruitful in characterizing entanglement for finite dimensional systems. It provides necessary and sufficient conditions for $2 \times 2$ and $2 \times 3$ dimensional systems. It is found that the PPT criterion is significant in the case of infinite dimensional bipartite Continuous Variable (CV) states too. An important advance came about through an identification of how Peres-Horodecki criterion gets translated elegantly into the properties of the
Constraints on the variance matrix of entangled symmetric qubits

second moments (uncertainties) of CV states [42]. This results in restrictions [42, 64] on
the covariance matrix of an entangled bipartite CV state. In the special case of two-mode
Gaussian states, where the basic entanglement properties are imbibed in the structure of its
covariance matrix, the restrictions on the covariance matrix are found to be necessary and
sufficient for inseparability [42, 64].

Here, we construct a two qubit variance matrix (analogous to that of CV states) for
two qubits and derive corresponding inseparability constraints imposed on it.

Let us first recapitulate succinctly the approach employed by Simon [42] for bipartite
CV states: The basic variables of bi-partite CV states are the conjugate quadratures of two
field modes,

\[ \hat{\xi} = (\hat{q}_1, \hat{p}_1, \hat{q}_2, \hat{p}_2), \]  

which satisfy the canonical commutation relations

\[ [\hat{\xi}_\alpha, \hat{\xi}_\beta] = i \Omega_{\alpha\beta}, \quad \alpha, \beta = 1, 2, 3, 4 \]  

where,

\[ \Omega = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}, \]

and \[ J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \]  

The matrix \( \Omega \) is known as a symplectic matrix. All real linear transformations applied to
the operators \( \xi_{\alpha, \beta} \) that obey the commutation realtions Eq. [5.2], form a group known as the
symplectic group.

The second moments are embodied in the real symmetric \( 4 \times 4 \) covariance matrix of
Constraints on the variance matrix of entangled symmetric qubits

A bipartite CV state, which is defined through its elements:

\[ V_{\alpha\beta} = \frac{1}{2} \langle \{ \Delta \hat{\xi}_\alpha, \Delta \hat{\xi}_\beta \} \rangle, \]  

where,

\[ \Delta \hat{\xi} = \hat{\xi}_\alpha - \langle \hat{\xi}_\alpha \rangle, \]  

and

\[ \{ \Delta \hat{\xi}_\alpha, \Delta \hat{\xi}_\beta \} = \Delta \hat{\xi}_\alpha \Delta \hat{\xi}_\beta + \Delta \hat{\xi}_\beta \Delta \hat{\xi}_\alpha. \]  

Under canonical transformations, the variables of the two-mode system transform as

\[ \hat{\xi} \rightarrow \hat{\xi}' = S \hat{\xi}, \]

where \( S \in Sp(4, R) \) corresponds to a real symplectic \( 4 \times 4 \) matrix. Under such transformations, the covariance matrix goes as

\[ V \rightarrow V' = SVS^T. \]

It is convenient to cast the covariance matrix in a \( 2 \times 2 \) block form:

\[ V = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}. \]  

The entanglement properties hidden in the covariance matrix \( V \) remain unaltered under a local \( Sp(2, R) \otimes Sp(2, R) \) transformation. Such a local operation transforms the blocks \( A, B, C \) of the variance matrix Eq. (5.7) as

\[ A \rightarrow A' = S_1 A S_1^T, \]

\[ B \rightarrow B' = S_2 B S_2^T, \]

\[ C \rightarrow C' = S_1 C S_2^T. \]  

(5.8)
Constraints on the variance matrix of entangled symmetric qubits

There are four local invariants associated with $V$ given in terms of the blocks $A$, $B$, $C$:

\begin{align*}
I_1' &= \det A, \\
I_2' &= \det B, \\
I_3' &= \det C, \\
I_4' &= \text{Tr}(AJC BC^T J). \quad (5.9)
\end{align*}

The Peres-Horodecki criterion imposes the restriction \cite{42} on the second moments of every separable CV state.

\begin{equation}
I_1' I_2' + \left(\frac{1}{4} - |I_3'|\right)^2 - I_4' \geq \frac{1}{4} (I_1' + I_2') \quad (5.10)
\end{equation}

The signature of the invariant $I_3' = \det C$ has an important consequence: Gaussian states with $I_3' \geq 0$ are necessarily separable, while those with $I_3' < 0$ and violating Eq. (5.10) are entangled.

In other words, for Gaussian states violation of the condition Eq. (5.10) is both necessary and sufficient for entanglement.

In the next section, we explain a similar formalism for symmetric two qubits by constructing a covariance matrix and analyzing its inseparability behaviour.

### 5.2 Two qubit covariance matrix

The basic variables of a two qubit system are expressed as an operator column(row) as

\[ \hat{\zeta}^T = (\sigma_{1i}, \sigma_{2j}), \quad i, j = 1, 2, 3. \]

The $6 \times 6$ real symmetric covariance matrix $\mathcal{V}$ of a two qubit system may be defined through,

\begin{equation}
\mathcal{V}_{\alpha i; \beta j} = \frac{1}{2} \langle \{ \Delta \hat{\zeta}_{\alpha i}, \Delta \hat{\zeta}_{\beta j} \} \rangle, \quad (5.11)
\end{equation}
with $\alpha, \beta = 1, 2; \ i, j = 1, 2, 3$. The variance matrix $\mathcal{V}$ can be conveniently written in the $3 \times 3$ block form as

$$\mathcal{V} = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix},$$

(5.12)

where,

$$A_{ij} = \frac{1}{2} [\langle \sigma_{1i} \sigma_{1j} \rangle - \langle \sigma_{1i} \rangle \langle \sigma_{1j} \rangle] = \delta_{ij} - \langle \sigma_{1i} \rangle \langle \sigma_{1j} \rangle = \delta_{ij} - s_i s_j,$$

$$B_{ij} = \frac{1}{2} [\langle \sigma_{2i} \sigma_{2j} \rangle - \langle \sigma_{2i} \rangle \langle \sigma_{2j} \rangle] = \delta_{ij} - \langle \sigma_{2i} \rangle \langle \sigma_{2j} \rangle = \delta_{ij} - r_i r_j,$$

$$C_{ij} = \frac{1}{2} [\langle \sigma_{1i} \sigma_{2j} \rangle - \langle \sigma_{1i} \rangle \langle \sigma_{2j} \rangle] = t_{ij} - s_i r_j.$$  

(5.13)

In other words, we have,

$$A = \mathcal{I} - s s^T,$$

$$B = \mathcal{I} - r r^T,$$

$$C = T - s r^T.$$  

(5.14)

Here $\mathcal{I}$ denotes a $3 \times 3$ identity matrix and $s_i, r_i$ and $t_{ij}$ are the state parameters of an arbitrary two qubit state (see Eqs. (2.6) - (2.10)).

In the case of symmetric states, considerable simplicity ensues as a result of Eqs. (2.18), (2.19).
and the covariance matrix of Eq. (5.12) assumes the form:

\[
\mathcal{V} = \begin{pmatrix} \mathcal{A} & \mathcal{C} \\ \mathcal{C}^T & \mathcal{A} \end{pmatrix},
\]

(5.15)

where,

\[
\mathcal{A} = \mathcal{B} = I - ss^T,
\]

and \( \mathcal{C} = T - ss^T \).

Explicitly,

\[
\mathcal{A} = \begin{pmatrix} 1 - s_1^2 & -s_1s_2 & -s_1s_3 \\
-s_1s_2 & 1 - s_2^2 & -s_2s_3 \\
-s_1s_3 & s_2s_3 & 1 - s_3^2 \end{pmatrix},
\]

(5.16)

and

\[
\mathcal{C} = \begin{pmatrix} t_{11} - s_1^2 & t_{12} - s_1s_2 & t_{13} - s_1s_3 \\
t_{12} - s_1s_2 & t_{22} - s_2^2 & t_{23} - s_2s_3 \\
t_{13} - s_1s_3 & t_{23} - s_2s_3 & t_{33} - s_3^2 \end{pmatrix}.
\]

(5.17)

We now establish an important property exhibited by the off-diagonal block \( \mathcal{C} \) of the covariance matrix of a symmetric two qubit state.

### 5.3 Inseparability constraint on the covariance matrix

**Lemma:** For every separable symmetric state, \( \mathcal{C} = T - ss^T \) is a positive semidefinite matrix.
Proof: Consider a separable symmetric state of two qubits

$$\rho_{(\text{sym-sep})} = \sum_w p_w \rho_w \otimes \rho_w, \quad \sum_w p_w = 1; \quad 0 \leq p_w \leq 1. \quad (5.18)$$

The state variables $s_i$ and $t_{ij}$ associated with a separable symmetric state have the following structure:

$$s_i = \text{Tr} \left( \rho_{(\text{sym-sep})} \sigma_{\alpha i} \right) = \sum_w p_w s_{wi},$$

$$t_{ij} = \text{Tr} \left( \rho_{(\text{sym-sep})} \sigma_{1i} \sigma_{2j} \right) = \sum_w p_w s_{wi} s_{wj}. \quad (5.19)$$

Let us now evaluate the quadratic form $n^T (T - ss^T) n$ where $n (n^T)$ denotes any arbitrary real three componental column (row), in a separable symmetric state:

$$n^T (T - ss^T) n = \sum_{i,j} (t_{ij} - s_i s_j) n_i n_j$$

$$= \sum_{i,j} \left[ \sum_w p_w s_{wi} s_{wj} - \sum_w p_w s_{wi} \sum_{w'} p_{w'} s_{w'j} \right] n_i n_j$$

$$= \sum_w p_w (\vec{s} \cdot \hat{n})^2 - \left( \sum_w p_w (\vec{s} \cdot \hat{n}) \right)^2, \quad (5.20)$$

which has the structure $\langle A^2 \rangle - \langle A \rangle^2$ and is therefore a positive semi-definite quantity.$\square$

This lemma establishes the fact that the off diagonal block $C$ of the covariance matrix is necessarily positive semidefinite for separable symmetric states. And therefore, $C < 0$ serves as a sufficient condition for inseparability in two-qubit symmetric states.

We now investigate the inseparability constraint $T - ss^T < 0$ in the case of a pure entangled two qubit state.
Constraints on the variance matrix of entangled symmetric qubits

An arbitrary pure two qubit state can be written in a Schmidt decomposed form

\[ |\Phi\rangle = \kappa_1 |0_1 0_2\rangle + \kappa_2 |1_1 1_2\rangle, \quad \kappa_1^2 + \kappa_2^2 = 1 \] (5.21)

where,

\[ 0 < \kappa_2 \leq \kappa_1 < 1, \]

are the Schmidt coefficients. The two qubit state can be written in the 4 × 4 matrix form using the basis \{ |0_1 0_2\rangle, |0_1 1_2\rangle, |1_1 0_2\rangle, |1_1 1_2\rangle \}:

\[ \rho = |\Phi\rangle \langle \Phi| = \begin{pmatrix} \kappa_1^2 & 0 & 0 & 2\kappa_1 \kappa_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2\kappa_1 \kappa_2 & 0 & 0 & \kappa_2^2 \end{pmatrix}. \] (5.22)

The 3 × 3 real symmetric correlation matrix \( T \) may be readily obtained using Eq. (2.10) as,

\[ T = \begin{pmatrix} 2\kappa_1 \kappa_2, & 0 & 0 \\ 0 & -2\kappa_1 \kappa_2, & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{Tr} T = 1. \] (5.23)

The average qubit orientation (Eq. (2.9)) has the form

\[ s = (0, 0, \kappa_1^2 - \kappa_2^2) = r. \] (5.24)

From Eq. (5.23) and Eq. (5.24), it is clear that an arbitrary two qubit pure state is symmetric in the Schmidt basis.
The $3 \times 3$ matrix $C = T - ss^T$ takes the form

$$C = T - ss^T = \begin{pmatrix} 2\kappa_1\kappa_2, & 0 & 0 \\ 0 & -2\kappa_1\kappa_2, & 0 \\ 0 & 0 & 4\kappa_1^2\kappa_2^2 \end{pmatrix}.$$

(5.25)

It can be clearly seen that $C < 0$, for all entangled pure two-qubit states.

In other words, the condition $C < 0$ is both necessary and sufficient for pure entangled two-qubit states.

Interestingly, non-positivity of $C$ completely characterizes inseparability in an arbitrary symmetric two qubit state, which will be proved in the Sec. (5.4).

### 5.4 Complete characterization of inseparability in mixed two qubit symmetric states

We prove the following theorem:

**Theorem:** The off-diagonal block $C$ of covariance matrix of an entangled two qubit mixed state is necessarily non-positive.

**Proof:** An arbitrary two qubit symmetric state, characterized by the density matrix Eq. (2.17), with the state parameters obeying the permutation symmetry requirements Eqs. (2.18), (2.19) has the following matrix form:

$$\rho_{\text{sym}} = \frac{1}{4} \begin{pmatrix} 1 + 2s_3 + t_{33} & A^{*} & A^{*} & (t_{11} - t_{22}) - 2it_{12} \\ A & (t_{11} + t_{22}) & (t_{11} + t_{22}) & B^{*} \\ A & (t_{11} + t_{22}) & (t_{11} + t_{22}) & B^{*} \\ (t_{11} - t_{22}) + 2it_{12} & B & B & 1 - 2s_3 + t_{33} \end{pmatrix}.$$

(5.26)
in the standard two-qubit basis \{ |0_1 0_2 \rangle, |0_1 1_2 \rangle, |1_1 0_2 \rangle, |1_1 1_2 \rangle \}.

Here, we have denoted,

\[ A = (s_1 + i s_2) + (t_{13} + i t_{23}) \]
\[ B = (s_1 + i s_2) - (t_{13} + i t_{23}). \]

Note that the two qubit basis is related to the total angular momentum basis \(|J, M\rangle\) with \(J = 1, 0; -J \leq M \leq J\) as follows:

\[
\begin{align*}
|0_1 0_2 \rangle &= |1, 1\rangle, \\
|0_1 1_2 \rangle &= |1, -1\rangle, \\
|1_1 0_2 \rangle &= \frac{1}{\sqrt{2}}(|1, 0\rangle + |0, 0\rangle), \\
|1_1 1_2 \rangle &= \frac{1}{\sqrt{2}}(|1, 0\rangle - |0, 0\rangle),
\end{align*}
\]

and the following unitary matrix,

\[
U = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 1 \\
0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0
\end{pmatrix},
\]

transforms the two qubit density matrix \(\rho_{\text{sym}}\) of Eq. (5.26) to the angular momentum basis Eq. (5.27):

\[
U \rho_{\text{sym}} U^\dagger = \begin{pmatrix}
\rho_S & 0 \\
0 & 0
\end{pmatrix},
\]

where,

\[
\rho_S = \frac{1}{4} \begin{pmatrix}
1 + 2 s_3 + t_{33} & \sqrt{2} A^* & (t_{11} - t_{22}) - 2i t_{12} \\
\sqrt{2} A & 2(t_{11} + t_{22}) & \sqrt{2} B^* \\
(t_{11} - t_{22}) + 2i t_{12} & \sqrt{2} B & 1 - 2 s_3 + t_{33}
\end{pmatrix}.
\]
So, an arbitrary two qubit symmetric state always gets restricted to the 3 dimensional maximal angular momentum subspace spanned by \(|J_{\text{max}} = 1, M\rangle − 1 \leq M \leq 1\). However, the partial transpose of \(\rho_{\text{sym}}\), does not get restricted to the symmetric subspace with \(J_{\text{max}} = 1\), when transformed to the total angular momentum basis Eq. (5.27).

Under the Partial transpose (PT) operation (say, on the second qubit), the Pauli spin matrices of second qubit change as

\[
\sigma_{21} \rightarrow \sigma_{21}, \quad \sigma_{22} \rightarrow -\sigma_{22}, \quad \sigma_{23} \rightarrow \sigma_{23}.
\]

When this PT operation, is followed by a local rotation about the 2-axis by an angle \(\pi\), the spin operators of the second qubit completely reverse their signs:

\[
\sigma_{2i} \rightarrow -\sigma_{2i}.
\]

Thus, PT map on the symmetric density operator Eq. (2.17)

\[
\rho_{\text{sym}} = \frac{1}{4} \left( I \otimes I + \sum_{i=1}^{3} s_i (\sigma_{1i} + \sigma_{2i}) + \sum_{i,j=1}^{3} \sigma_{1i} \sigma_{2j} t_{ij} \right), \tag{5.30}
\]

leads to

\[
\rho_{\text{sym}}^{T_2} = \frac{1}{4} \left( I \otimes I + \sum_{i=1}^{3} (\sigma_{1i} s_i - \sigma_{2i} s_i) - \sum_{i=1}^{3} \sigma_{1i} \sigma_{2j} t_{ij} \right). \tag{5.31}
\]

(Here \(T_2\) corresponds to partial transpose map on the second qubit).

We thus obtain,

\[
\rho_{\text{sym}}^{T_2} = \frac{1}{4} \begin{pmatrix}
(t_{11} + t_{22}) & a^* & b^* & -(t_{11} - t_{22}) + 2it_{12} \\
1 + 2s_3 + t_{33} & t_{33} - 1 & -a^*
\end{pmatrix}
\]

\[
\begin{pmatrix}
t_{33} - 1 & 1 - 2s_3 + t_{33} & -b \\
-(t_{11} - t_{22}) - 2it_{12} & -a & -b^* \\
-a & -(t_{11} + t_{22}) & (t_{11} + t_{22})
\end{pmatrix}. \tag{5.32}
\]
in the basis \( \{ |0_1 0_2\rangle, |0_1 1_2\rangle, |1_1 0_2\rangle, |1_1 1_2\rangle \} \). Here we have denoted,

\[
a = -(s_1 + is_2) - (t_{13} + i t_{23}), \quad b = (s_1 + is_2) - (t_{13} - i t_{23}).
\]

Now a unitary transformation Eq. (5.27) which corresponds to a basis change Eq. (5.28) gives:

\[
\bar{\rho}^\text{T}_{\text{sym}} = U \rho^\text{T}_2 U^\dagger
\]

\[
= \frac{1}{4} \begin{pmatrix}
(t_{11} + t_{22}) & -\sqrt{2} t_{13} & -(t_{11} - t_{22}) + 2i t_{12} & \bar{a} \\
-\sqrt{2} t_{13} & 2 t_{33} & \bar{b} & 2 s_3 \\
-(t_{11} - t_{22}) - 2i t_{12} & \bar{b} & (t_{11} + t_{22}) & \sqrt{2} s_1 \\
\bar{a}^* & 2 s_3 & \sqrt{2} s_1 & 4
\end{pmatrix}, \tag{5.33}
\]

where,

\[
\bar{a} = \sqrt{2}(-s_1 + is_2 + i t_{23}) \quad \bar{b} = \sqrt{2}(t_{13} + is_2 + i t_{23}).
\]

It may therefore be seen that 3 dimensional subspace spanned by \( J_{\max} = 1 \) of total angular momentum of a symmetric two qubit state does not restrict itself to a \( 3 \times 3 \) block form.

Interestingly, a further change of basis defined by,

\[
|X\rangle = \frac{-1}{\sqrt{2}} (|1, 1\rangle - |1, -1\rangle), \quad |Y\rangle = \frac{-i}{\sqrt{2}} (|1, 1\rangle + |1, -1\rangle), \quad |Z\rangle = |1, 0\rangle,
\]

(5.34)
Constraints on the variance matrix of entangled symmetric qubits

which corresponds to a unitary transformation

\[
U' = \frac{1}{\sqrt{2}} \begin{pmatrix}
-1 & 0 & 1 & 0 \\
-i & 0 & -i & 0 \\
0 & \sqrt{2} & 0 & 0 \\
0 & 0 & 0 & \sqrt{2}
\end{pmatrix}
\]

(5.35)

on \( \bar{\rho}_{\text{sym}}^T \), leads to the following elegant structure

\[
\bar{\rho}_{\text{sym}}^{T'} = U' \bar{\rho}_{\text{sym}}^T U'^\dagger
\]

\[
= \frac{1}{2} \begin{pmatrix}
t_{11} & t_{12} & t_{13} & s_1 \\
t_{12} & t_{22} & t_{23} & s_2 \\
t_{13} & t_{23} & t_{33} & s_3 \\
s_1 & s_2 & s_3 & 1
\end{pmatrix}
\]

= \frac{1}{2} \begin{pmatrix}
T & s \\
s^T & 1
\end{pmatrix}
\]

(5.36)

Now a congruence \[ operation \]

\[
L \bar{\rho}_{\text{sym}}^{T'} L^\dagger
\]

with

\[
L = \begin{pmatrix}
I & -s \\
0 & 1
\end{pmatrix}
\]

gives,

\[
L \bar{\rho}_{\text{sym}}^{T'} L^\dagger = \frac{1}{2} \begin{pmatrix}
t_{11} - s_1^2 & t_{12} & t_{13} & 0 \\
t_{12} & t_{22} - s_2^2 & t_{23} & 0 \\
t_{13} & t_{23} & t_{33} - s_3^2 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[1\]Note that the congruence operation does not alter the positivity (negativity) of the eigenvalue structure of the matrix.
or

\[ L \tilde{\rho}_{\text{sym}}^{T_2} L^\dagger = \frac{1}{2} \begin{pmatrix} T - ss^T & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.37} \]

It is therefore evident that \textit{negativity of } \( T - ss^T \) \textit{necessarily implies negativity of the partially transposed arbitrary two qubit symmetric density matrix} as,

\[ \rho_{\text{sym}}^{T_2} < 0 \iff \tilde{\rho}_{\text{sym}}^{T_2} < 0 \iff L \tilde{\rho}_{\text{sym}}^{T_2} L^\dagger < 0 \iff \mathcal{C} = T - ss^T < 0; \]

i.e., non-positivity of \( \mathcal{C} = T - ss^T \) necessarily implies that partially transposed two qubit symmetric density matrix \( \rho_{\text{sym}}^{T_2} \) is negative. In other words, \( \mathcal{C} < 0 \) captures Peres’s inseparability criterion on symmetric two qubit state completely. Hence the theorem. □

In the next section, we explore how negativity of the matrix \( \mathcal{C} \) reflects itself on the structure of the local invariants associated with the two qubit state.

### 5.5 Local invariant structure

The off-diagonal block \( \mathcal{C} \) of the covariance matrix is a real \( 3 \times 3 \) symmetric matrix and so, can be diagonalized by an orthogonal matrix \( O \) i.e.,

\[ OCO^\dagger = \mathcal{C}^d = (c_1, c_2, c_3). \]

The orthogonal transformation corresponds to identical unitary transformation \( U \otimes U \) on the qubits.

We denote the eigenvalues of the off-diagonal block \( \mathcal{C} \) of the covariance matrix Eq. (5.12) by \( c_1, c_2, \text{ and } c_3 \). Restricting ourselves to identical local unitary transformations, we define
three local invariants, which completely determine the eigenvalues $c_1, c_2, c_3$ of $C = T - ss^T$:

$$
\bar{I}_1 = \det (C) = c_1c_2c_3,
\bar{I}_2 = \text{Tr} (C) = c_1 + c_2 + c_3,
\bar{I}_3 = \text{Tr} (C^2) = c_1^2 + c_2^2 + c_3^2.\quad (5.38)
$$

The invariant $\bar{I}_2$ may be rewritten as

$$
\bar{I}_2 = \text{Tr} (T - ss^T) = 1 - s_0^2,\quad (5.39)
$$

since $\text{Tr} (T) = 1$ for a symmetric state. Here, we have denoted

$$
\text{Tr} (ss^T) = s_1^2 + s_2^2 + s_3^2 = s_0^2.
$$

Another useful invariant, which is a combination of the invariants defined through Eq. (5.38), may be constructed as

$$
\bar{I}_4 = \frac{\bar{I}_2^2 - \bar{I}_3}{2} = c_1c_2 + c_2c_3 + c_1c_3.\quad (5.40)
$$

Positivity of the single qubit reduced density operator demands $s_0^2 \leq 1$ and leads in turn to the observation that the invariant $\bar{I}_2$ is positive for all symmetric states. Thus, all the three eigen values $c_1$, $c_2$, $c_3$ of $C$ can never assume negative values for symmetric qubits and at most two of them can be negative.

We consider three distinct cases encompassing all pairwise entangled symmetric states.

Case (i): Let one of the eigenvalues $c_1 = 0$ and of the remaining two, let $c_2 < 0$ and $c_3 > 0$.

Clearly, the invariant $\bar{I}_1 = 0$ in this case. But we have

$$
\bar{I}_4 = c_2c_3 < 0,\quad (5.41)
$$
which leads to a local invariant condition for two-qubit entanglement.

Case (ii): Suppose any two eigenvalues say, $c_1, c_2$, are negative and the third one $c_3$ is positive.

Obviously, $\tilde{I}_1 > 0$ in this case. But the invariant $\tilde{I}_4$ assumes negative value:

$$\tilde{I}_4 = c_1 \tilde{I}_2 - c_1^2 + c_2 c_3 < 0$$  \hspace{1cm} (5.42)

as each term in the right hand side is negative. In other words, $\tilde{I}_4 < 0$ gives the criterion for bipartite entanglement in this case too.

Case (iii): Let $c_1 < 0, c_2$ and $c_3$ be positive.

In this case we have

$$\tilde{I}_1 < 0,$$  \hspace{1cm} (5.43)

giving the inseparability criterion in terms of a local invariant.

The new set of local invariants (see Eqs. (5.38), (5.40)) associated with the off-diagonal block $C$ of the covariance matrix can be related to the symmetric two qubit local invariants given by Eq. (2.27). In the following discussion, we restrict ourselves to the identical local unitary transformations $U \otimes U$ (Eq. (3.23)) which transform the state vectors to the following form:

$$\vec{s} = (0, 0, s_0),$$

and

$$T = \begin{pmatrix} t_1^{(+)} & 0 & t_1^\prime \\ t_1^\prime & t_1^{(-)} & t_2^\prime \\ t_1 & t_2 & t_3 \end{pmatrix}.$$  \hspace{1cm} (5.44)
Constraints on the variance matrix of entangled symmetric qubits

We note that

\[
\det(C) = \det(T - ss^T)
\]

\[
= t_1^+ t_1^- (t_3^3 - s_0^2) - (t_{23}^I)^2 t_1^+ - (t_{13}^I)^2 t_1^-.
\]

\[
= \det T - s_0^2 t_1^+ t_1^-.
\]

\[
= \mathcal{I}_1 - \frac{\mathcal{I}_5}{2}, \quad (5.45)
\]

where we have used Eqs. (2.30), (3.24). We thus have,

\[
\bar{\mathcal{I}}_1 = \mathcal{I}_1 - \frac{\mathcal{I}_5}{2}, \quad (5.46)
\]

We further find that,

\[
\bar{\mathcal{I}}_2 = \text{Tr}(C) = \text{Tr}(T - ss^T)
\]

\[
= \text{Tr}(T) - \text{Tr}(ss^T)
\]

\[
= 1 - s^T s
\]

\[
= 1 - \mathcal{I}_3. \quad (5.47)
\]

The invariant \(\bar{\mathcal{I}}_3\), can be written as

\[
\bar{\mathcal{I}}_3 = \text{Tr}(C^2) = \text{Tr}[(T - ss^T)^2]
\]

\[
= \text{Tr}[(T^2) + (ss^T ss^T) - Tss^T - ss^T T]
\]

\[
= \text{Tr}(T^2) + (s^T s)^2 - 2 s^T Ts.
\]

From Eq. (2.27), we have \(\bar{\mathcal{I}}_3\) given by,

\[
\bar{\mathcal{I}}_3 = \mathcal{I}_2 + \mathcal{I}_3^2 - 2 \mathcal{I}_4. \quad (5.48)
\]

Now, we proceed to explore how this basic structure \(C < 0\) reflects itself via collective
Constraints on the variance matrix of entangled symmetric qubits

second moments of a symmetric $N$ qubit system.

5.6 Implications of $C < 0$ in symmetric $N$ qubit systems

Collective observables are expressible in terms of total angular momentum operator as

$$\vec{J} = \sum_{\alpha=1}^{N} \frac{1}{2} \vec{\sigma}_{\alpha}$$

(5.49)

where $\vec{\sigma}_{\alpha}$ denote the Pauli spin operator of the $\alpha^{th}$ qubit.

The collective correlation matrix involving first and second moments of $\vec{J}$ may be defined as,

$$V^{(N)}_{ij} = \frac{1}{2} \langle J_i J_j + J_j J_i \rangle - \langle J_i \rangle \langle J_j \rangle.$$

(5.50)

Using Eq. (3.7) and Eq. (3.8), we can express the first and second order moments $\langle J_i \rangle$, $\langle J_i J_j + J_j J_i \rangle$ in terms of the two qubits state parameters. After simplification, we obtain,

$$V^{(N)} = \frac{N}{4} \begin{pmatrix}
1 - t_{11} + N (t_{11} - s_1^2) & (N - 1) t_{12} & (N - 1) t_{13} \\
(N - 1) t_{12} & 1 - t_{22} + N (t_{22} - s_2^2) & (N - 1) t_{23} \\
(N - 1) t_{13} & (N - 1) t_{23} & 1 - t_{33} + N (t_{33} - s_3^2)
\end{pmatrix}.$$

(5.51)

We can now express $V^{(N)}$ in the following compact form,

$$V^{(N)} = \frac{N}{4} \left( \mathcal{I} - ss^T + (N - 1) (T - ss^T) \right).$$

(5.52)

where $\mathcal{I}$ is a $3 \times 3$ identity matrix; $s^T = (s_1, s_2, s_3)$ and $T$ denotes the two qubit correlation matrix (see Eq. (2.11)). We simplify Eq. (5.52) further.

By shifting the second term i.e., $\frac{N}{4} ss^T$ to the left hand side, we obtain,

$$V^{(N)} + \frac{N}{4} ss^T = \frac{N}{4} (\mathcal{I} + (N - 1) C), \quad C = T - ss^T$$

(5.53)
Constraints on the variance matrix of entangled symmetric qubits

Expressing $s_i$ in terms of the collective observables, we have,

$$s_i = \frac{2}{N} \langle J_i \rangle = \frac{2}{N} S_i. \quad (5.54)$$

Now, substituting Eq. (5.54) in Eq. (5.53), we obtain,

$$V^{(N)} + \frac{1}{N} SS^T = \frac{N}{4} (I + (N - 1) C). \quad (5.55)$$

For all symmetric separable states we have established that $C \geq 0$ (see our theorem in Sec. 5.4). We thus obtain the following constraint on the collective correlation matrix $V^{(N)}$:

$$V^{(N)} + \frac{1}{N} SS^T < \frac{N}{4} I. \quad (5.56)$$

Pairwise entangled symmetric multiqubit states necessarily satisfy the above condition.

Note that under identical local unitary transformations $U \otimes U \otimes \ldots \otimes U$ on the qubits, the variance matrix $V^{(N)}$ and the average spin $S$ transform as

$$V^{(N)'} = O V^{(N)} O^T,$$

and

$$S' = O S, \quad (5.57)$$

where $O$ is a $3 \times 3$ real orthogonal rotation matrix corresponding to the local unitary transformation $U$ on all the qubits. Thus, the $3 \times 3$ real symmetric matrix $V^{(N)} + \frac{1}{N} SS^T$ can always be diagonalized by a suitable identical local unitary transformation on all the qubits.

In other words, (5.56) is a local invariant condition and it essentially implies:

*The symmetric $N$ qubit system is pairwise entangled iff the least eigen value of the real symmetric matrix $V^{(N)} + \frac{1}{N} SS^T$ is less than $N/4$.***
5.7 Equivalence between the generalized spin squeezing inequalities and negativity of $C$

Let us consider the generalized spin squeezing inequalities of Ref. [31]

$$\frac{4\langle \Delta J^2_k \rangle}{N} < 1 - \frac{4\langle J_k \rangle^2}{N^2},$$  \hspace{1cm} (5.58)

where $J_k = \vec{J} \cdot \hat{k}$, with $\hat{k}$ denoting an arbitrary unit vector, and

$$\langle \Delta J^2_k \rangle = \langle J^2_k \rangle - \langle J_k \rangle^2.$$  

Expressing $\langle J_k \rangle$ and $\langle J^2_k \rangle$ in terms of the two qubit state variables $s_i$ and $t_{ij}$ we have,

$$\langle J_k \rangle = \frac{1}{2} \sum_{\alpha=1}^{N} \sum_{i=1}^{3} \langle \sigma_{\alpha i} \rangle k_i = \frac{N}{2} (\vec{s} \cdot \hat{k}) \hspace{1cm} (5.59)$$

$$\langle J^2_k \rangle = \frac{N}{4} + \frac{1}{4} \sum_{i,j} \sum_{\alpha, \beta \neq \alpha} \langle \sigma_{\alpha i} \sigma_{\beta j} \rangle k_i k_j$$

$$= \frac{N}{4} + \frac{1}{2} \sum_{i,j} \sum_{\alpha=1}^{N} \sum_{\beta > \alpha} \langle \sigma_{\alpha i} \sigma_{\beta j} \rangle k_i k_j$$

$$= \frac{N}{4} + \frac{1}{2} \sum_{i,j} \sum_{\alpha, \beta > \alpha} t_{ij} k_i k_j$$

$$= \frac{N}{4} + \frac{N(N-1)}{4} \sum_{i,j} t_{ij} k_i k_j$$

i.e., $\langle J^2_k \rangle = \frac{N}{4} (1 + (N - 1) k^T \hat{k})$.  \hspace{1cm} (5.60)
Using Eqs. (5.59), (5.60) in the generalized spin squeezing inequalities given in Eq. (5.58), we obtain,

\[
\frac{4}{N} \left[ (J_k^2) - \langle J_k \rangle^2 \right] < 1 - \frac{4\langle J_k^2 \rangle}{N^2}
\]

\[
\frac{4}{N} \left[ \frac{N}{4} (1 + (N - 1) k^T T k) - \frac{N^2}{4} (\vec{s} \cdot \hat{k})^2 \right] < 1 - \frac{4}{N^2} \left( \frac{N^2 (\vec{s} \cdot \hat{k})^2}{4} \right)
\]

\[
\left[ 1 + (N - 1) k^T T k \right] - N (\vec{s} \cdot \hat{k})^2 < 1 - (\vec{s} \cdot \hat{k})^2
\]

\[
(N - 1) k^T T k < (N - 1) (\vec{s} \cdot \hat{k})^2
\]

\[
k^T (T - ss^T) k < 0,
\]

or \( C < 0. \) (5.61)

Thus we find that the generalized spin squeezing inequality is equivalent to the condition 

\( C = T - ss^T < 0. \)
5.8 Conclusions

We have constructed a two qubit variance matrix and have shown here that the off-diagonal block of the variance matrix $C$ of a separable symmetric two qubit state is a positive semidefinite quantity. An equivalence between the Peres-Horodecki criterion and the negativity of the covariance matrix $C$ is established, showing that the covariance matrix criterion is both necessary and sufficient for entanglement in symmetric two qubit states. Thus symmetric two-qubit states satisfying the condition $C < 0$ are identified as inseparable. Further, the inseparability constraint $C < 0$ is shown to be equivalent to the recently proposed generalized spin squeezing inequalities for pairwise entanglement in symmetric $N$-qubit states. An elegant local invariant structure exhibited by these constraints on the two qubit covariance matrix has also been discussed.
Chapter 6

Summary

Quantum correlated multiqubit states offer promising possibilities in low-noise spectroscopy [18], high precision interferometry [16, 17, 65] and in the implementation of quantum information protocols [66]. Multiqubit states which are symmetric under interchange of particles (qubits) form an important class due to their experimental significance [13, 62, 67] as well as the mathematical simplicity and elegance associated with them.

Individual qubits (two-level atoms) in a multiqubit system are not accessible in the macroscopic ensemble and therefore only collective measurements are feasible. Any characterization of entanglement requiring individual control of qubits cannot be experimentally implemented. For example, spin squeezing [19], i.e., reduction of quantum fluctuations in one of the spin components orthogonal to the mean spin direction below the fundamental noise limit N/4 is an important collective signature of entanglement in symmetric N qubit systems and is a consequence of two-qubit pairwise entanglement [13, 30, 32].

Spin squeezing is one of the important quantifying signatures of quantum correlations in multiqubit systems. Spin squeezed atomic states are produced routinely in several laboratories [6, 7] today. However, it is important to realize that spin squeezing does not capture quantum correlations completely and it serves only as a sufficient condition for
Summary

pairwise entanglement in symmetric N qubit states. Investigations on other collective signatures \[31, 32, 33\] of entanglement gain their significance in this context.

In this thesis, we have investigated the pairwise entanglement properties of symmetric multiqubits obeying permutation symmetry by employing two qubit local invariants. We have shown that a subset of 6 invariants \(\{I_1 - I_6\}\), of a more general set of 18 invariants proposed by Makhlin \[36\], completely characterizes pairwise entanglement of the collective state. For a specific case of symmetric two-qubit system, which is realized in several physically interesting examples like, even and odd spin states \[49\], Kitagawa - Ueda state generated by one-axis twisting Hamiltonian \[19\], Atomic spin squeezed states \[20\] etc, a subset of three independent invariants is sufficient to characterize the non-local properties completely. For symmetric separable states, we have proved that the entanglement invariants \(I_1, I_4, I_5\) and \(I_4 - I_5^2\) assume non-negative values.

Based on negative values of the invariants \(I_1, I_4, I_5\) and \(I_4 - I_5^2\), we have proposed a detailed classification scheme, for pairwise entanglement in symmetric multiqubit system, Our scheme also relates appropriate collective non-classical features, which can be identified in each case of pairwise entanglement. Further, we have expressed collective features of entanglement, such as spin squeezing, in terms of these invariants. More specifically, we have shown that a symmetric multi-qubit system is spin squeezed iff one of the entanglement invariant is negative. Moreover, our invariant criteria are shown to be related to the family of generalized spin squeezing inequalities \[31\] (two qubit entanglement) involving collective first and second order moments of total angular momentum operator.

Further, we have established an equivalence between the Peres-Horodecki criterion and the negativity of the off diagonal block \(C\) of the two qubit covariance matrix thereby showing that our condition is both necessary and sufficient for entanglement in symmetric two qubit states \[34\]. Pairwise entangled symmetric multiqubit states necessarily obey these constraints. We have also brought out an elegant local invariant structure exhibited by our constraints.
Appendix A

Pure and mixed density operators

Consider a quantum system characterized by a state $|\psi\rangle$ in the Hilbert space $\mathcal{H}$. Using a complete orthonormal basis $\{|u_n\rangle\}$ satisfying

- orthonormality: $\langle u_m|u_n \rangle = \delta_{mn}$
- completeness: $\sum_n |u_n\rangle\langle u_n| = I,$

where $I$ is the unit operator.

We can expand $|\psi\rangle$ as follows:

$$|\psi\rangle = \sum_n c_n |u_n\rangle, \quad c_n = \langle u_n|\psi\rangle. \quad (A.1)$$

The expansion coefficients $c_n$ satisfy the normalization condition

$$\sum_n |c_n|^2 = \sum_n |\langle u_n|\psi\rangle|^2 = 1.$$
Let us evaluate the expectation value of an observable $A$ in the state $|\psi\rangle$

$$
\langle A \rangle = \langle \psi | A | \psi \rangle \\
= \sum_m \sum_n \langle \psi | u_m \rangle \langle u_m | A | u_n \rangle \langle u_n | \psi \rangle \\
= \sum_m \sum_n \langle u_n | \rho | u_m \rangle \langle u_m | A | u_n \rangle \\
= \sum_m \sum_n \rho_{nm} A_{mn}
$$

or

$$
\langle A \rangle = \text{Tr} (\rho A), 
$$

(A.2)

where,

$$
\rho = |\psi\rangle \langle \psi|
$$

(A.3)

is the density operator associated with the quantum system (elements of which are denoted by $\langle u_n | \rho | u_m \rangle = \rho_{nm}$ in the given basis $\{ |u_n\rangle \}$).

The density operator satisfies the following properties:

1. $\rho$ is a hermitian matrix

$$
\rho^*_{mn} = \rho_{nm}. 
$$

(A.4)

2. $\rho$ is positive semi definite,

$$
\rho \geq 0.
$$

(A.5)

3. $\rho$ has unit trace,

$$
\text{Tr} \rho = 1.
$$

(A.6)

From the structure of $\rho = |\psi\rangle \langle \psi|$, (see Eq. (A.3)) it is clear that,

$$
\rho^2 = \rho
$$

(A.7)

i.e., $\rho$ is idempotent operator.
Density operators satisfying Eq. (A.3) form a subclass of the more general class of density operators and are termed as pure density operators in contrast to mixed states.

Mixed density operators are, in general, a convex mixture of pure states $|\psi_i\rangle$:

$$\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|, \quad 0 \leq p_i \leq 1, \sum_i p_i = 1.$$  \hspace{1cm} (A.8)

Number of real independent parameters characterizing the density operators:

Writing a pure density operator $\rho$ (A.3) in the basis $\{|u_n\rangle\}$ explicitly, we obtain,

$$\rho = \begin{pmatrix} |c_1|^2 & c_1 c_2^* & \ldots & c_1 c_n^* \\ c_2 c_1^* & |c_2|^2 & \ldots & c_2 c_n^* \\ \vdots & \vdots & \ddots & \vdots \\ c_n c_1^* & c_n c_2^* & \ldots & |c_n|^2 \end{pmatrix},$$  \hspace{1cm} (A.9)

i.e., the density matrix given above is specified completely by the complex coefficients $c_n$, which are constrained by the normalization condition $\sum_n |c_n|^2 = 1$. In other words, the density matrix of Eq. (A.3) is completely characterized by $(2n - 1)$ real parameters.

However, the more general class of mixed density operators satisfying the properties (Eqs. (A.4) - (A.6)) are characterized by $n^2 - 1$ real parameters as shown below:
The density operator $\rho$ of a mixed quantum state may be explicitly written in the matrix form (in a suitable complete orthonormal basis $\{|u_n\}\}$ as,

$$
\rho = \begin{pmatrix}
\rho_{11} & \rho_{12} & \cdots & \rho_{1n} \\
\rho^*_{12} & \rho_{22} & \cdots & \rho_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\rho^*_{n1} & \rho^*_{n2} & \cdots & \rho_{nn}
\end{pmatrix}, \quad \text{Tr}(\rho) = \sum_n \rho_{nn} = 1. \tag{A.10}
$$

We count the number of parameters as follows:

- diagonal elements of $\rho$ are real and are constrained by the unit trace condition. So, we have $(n - 1)$ real parameters specifying the diagonal elements.

- There are $\frac{n(n-1)}{2}$ independent complex parameters $\rho_{nm} = \rho^*_{mn}, n \neq m$, which fix the off-diagonal elements of $\rho$. This leads to $n(n - 1)$ real parameters.

So, the total number of independent real parameters characterizing a mixed quantum state $\rho$ are given by $(n - 1) + n(n - 1) = n^2 - 1$.

**Single qubit density matrices:**

There are three real parameters specifying a mixed density operator of single qubit system which is represented by a $2 \times 2$ hermitian matrix of unit trace as,

$$
\rho = \begin{pmatrix}
\rho_{11} & \rho_{12} \\
\rho^*_{12} & 1 - \rho_{11}
\end{pmatrix}. \tag{A.11}
$$

The Pauli spin matrices,

$$
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.12}
$$
Pure and mixed density operators

together with the identity matrix
\[
I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{A.13}
\]
provide a matrix basis for any $2 \times 2$ matrices.

In the case of spin-$\frac{1}{2}$ quantum states, we can always express
\[
\rho = \frac{1}{2} [I + \sigma_1 s_1 + \sigma_2 s_2 + \sigma_3 s_3], \tag{A.14}
\]
where, it is readily seen that
\[
s_1 = \langle \sigma_1 \rangle = \text{Tr} (\rho \sigma_1), \\
s_2 = \langle \sigma_2 \rangle = \text{Tr} (\rho \sigma_2), \\
s_3 = \langle \sigma_3 \rangle = \text{Tr} (\rho \sigma_3). \tag{A.15}
\]

Two qubit quantum system:

There are $n^2 - 1 = 15$ parameters characterizing a two qubit system. A natural operator basis \{ $I \otimes I$, $\sigma_{1i}$, $\sigma_{2i}$, $\sigma_{1i} \sigma_{2j}$ \} is employed generally to expand the two qubit density matrix:
\[
\rho = \frac{1}{4} \left( I \otimes I + \sum_{i=1}^{3} s_i \sigma_{1i} + \sum_{i=1}^{3} \sigma_{2i} r_i + \sum_{i,j=1}^{3} t_{ij} \sigma_{1i} \sigma_{2j} \right), \tag{A.16}
\]
with
\[
\sigma_{1i} = \sigma_i \otimes I \\
\sigma_{2i} = I \otimes \sigma_i. \tag{A.17}
\]

The single qubit state parameters $s_i$ and $r_i$ are given by
\[
s_i = \text{Tr} (\rho \sigma_{1i}) \\
r_i = \text{Tr} (\rho \sigma_{2i}), \quad i = 1, 2, 3. \tag{A.18}
\]
and the remaining two qubit correlation parameters are evaluated as follows:

$$t_{ij} = \text{Tr} (\rho \sigma_1 \sigma_2).$$  \hspace{1cm} (A.19)
Appendix B

Peres PPT criterion

Any bipartite state $\rho$ defined on the Hilbert space $H$ is separable if it can be expressed in the convex product form

$$\rho_{\text{sep}} = \sum_w p_w \rho_w^{(1)} \otimes \rho_w^{(2)} \text{ where } 0 \leq p_w \leq 1 \text{ and } \sum_w p_w = 1.$$  \hspace{1cm} (B.1)

Here, $\rho_w^{(1)}$ and $\rho_w^{(2)}$ are the density operators defining the systems 1 and 2 respectively. Quantum systems which cannot be written in the form Eq. (B.1) are called entangled states. The fundamental question in entanglement theory is the following: Given a composite quantum state how do we test if the state is entangled?

In this context, Peres [43] has shown that the partially transposed density matrix of a separable state is a positive definite matrix. In other words, negative eigenvalues of a partially transposed density matrix necessarily imply entanglement in a quantum state.

To demonstrate this explicitly, let us write the matrix elements of the separable state given in Eq. (B.1) in the standard basis.

$$\langle m\mu | \rho_{\text{sep}} | n\nu \rangle = \sum_w p_w \langle m \rho_w^{(1)} | n \rangle \langle \mu \rho_w^{(2)} | \nu \rangle$$

or $$\langle \rho_{\text{sep}} \rangle_{m\mu;n\nu} = \sum_w p_w (\rho_w^{(1)})_{mn} (\rho_w^{(2)})_{\mu\nu}.$$  \hspace{1cm} (B.2)
Here, Latin indices refer to the system 1 and Greek indices correspond to the system 2.

The partial transposition (PT) of any density matrix $\rho$ with respect to one of the subsystems, (say system 2) is defined as,

$$\rho_{T_2}^{m\mu;n\nu} = \rho_{m\nu;n\mu}.$$  

Thus, the partial transpose of $\rho_{\text{sep}}$ (Eq. (B.2)) is given by

$$\rho_{\text{sep}}^{T_2} = \sum_w p_w (\rho_w^{(1)})_{mn} (\rho_w^{(2)})_{\nu\mu}^*.$$  

Using the hermiticity property, $\rho_i^\dagger = \rho_i$ (see Eq. (A.4)) of the density matrices, we obtain,

$$\rho_{\text{sep}}^{(2)} = (\rho_{\text{sep}}^{(2)})^*.$$  

and thus,

$$\rho_{\text{sep}}^{T_2} = \sum_w p_w (\rho_w^{(1)})_{mn} (\rho_w^{(2)})_{\nu\mu}^*.$$  

or

$$\rho_{\text{sep}}^{T_2} = \sum_w p_w (\rho_w^{(1)}) \otimes (\rho_w^{(2)})^*.$$  

Since $\{(\rho_w^{(2)})^*\}$ correspond to physical density matrices, $\rho_{\text{sep}}^{T_2}$ is again a physically valid separable density matrix.

Therefore, the PT operation preserves the trace, hermiticity and also the positive semi definiteness of a separable state whereas the last property need not be respected by an entangled state.

Horodecki et al. [44] showed that Peres’ positivity under partial transpose (PPT) is both necessary and sufficient for entanglement in $2 \times 2$ and $2 \times 3$ systems. However, for higher dimensions, there exist bound entangled states which are PPT states, though
they are inseparable. Negative under partial transpose (NPT) is a sufficient condition for entanglement in higher dimensional composite quantum systems.
Appendix C

A Complete set of 18 invariants for an arbitrary two qubit state

An arbitrary two qubit density matrix (see Eq. (2.6)), specified by 15 real parameters \( \{s_i, r_i, t_{ij}\} \), is characterized by a complete set of 18 polynomial invariants [36]. These local invariants are given in terms of the state parameters associated with an arbitrary two qubit system. Any two density matrices \( \rho_1 \) and \( \rho_2 \) are said to be locally equivalent if and only if all the 18 invariants (see Table.1) have identical values for these states. To illustrate that these 18 invariants form a complete set, we may chose to work in a basis in which the \( 3 \times 3 \) real correlation matrix \( T \) (which is nonsymmetric, in general) is diagonal. Such a singular value decomposition of \( T \) can be achieved by proper rotations \( O^{(1)}, O^{(2)} \in SO(3, R) \):

\[
T^d = O^{(1)} T O^{(2)\top} = \text{diag}(t_1, t_2, t_3). \quad (C.1)
\]
We may note here that the diagonal elements of the correlation matrix $T^d$ are not the eigenvalues of $T$ as $O^{(1)} T O^{(2)^\dagger}$ is not similarity transformation. However,

\[
T^d T^{d^\dagger} = O^{(1)} T O^{(2)^\dagger} O^{(2)} T^{\dagger} O^{(1)^\dagger},
\]

\[
= O^{(1)} T T^{\dagger} O^{(1)^\dagger},
\]

\[
= \begin{pmatrix}
 t_1^2 & 0 & 0 \\
 0 & t_2^2 & 0 \\
 0 & 0 & t_3^2
\end{pmatrix},
\]

(C.2)

and

\[
T^{d^\dagger} T^d = O^{(2)} T^{\dagger} T O^{(2)^\dagger}.
\]

In other words, $(t_1^2, t_2^2, t_3^2)$ are the eigenvalues of real symmetric matrix $T T^{\dagger}$ as well as $T^{\dagger} T$. In order to determine the eigenvalues $(t_1^2, t_2^2, t_3^2)$ the following polynomial quantities may be employed:

\[
\det (T T^{\dagger}) = t_1^2 t_2^2 t_3^2,
\]

\[
\text{Tr} (T^{\dagger} T) = t_1^2 + t_2^2 + t_3^2,
\]

\[
\text{Tr} [(T^{\dagger} T)^2] = t_1^4 + t_2^4 + t_3^4.
\]

(C.3)

Note that $\det (T^{\dagger} T)$, $\text{Tr} (T^{\dagger} T)$, $\text{Tr} (T^{\dagger} T)^2$ are invariant under local unitary transformations on the two qubit state. It is easy to see that

\[
\det (T T^{\dagger}) = \det (T) \det (T^{\dagger}) = (\det (T))^2,
\]

\[
= \det (O^{(1)} T^d O^{(2)^\dagger}) \det (O^{(2)} T^{d^\dagger}, O^{(1)^\dagger}),
\]

\[
= \det (T^d) \det (T^{d^\dagger}),
\]

\[
= (\det (T^d))^2.
\]

(C.4)
Thus $|\det(T)|$ itself may be used in the first line of Eq. (C.3) instead of $\det(T T^T)$. So, Makhlin chooses the first three elements of his set of local invariants as,

$$I_1 = \det(T) = t_1 t_2 t_3, \quad (C.5)$$
$$I_2 = \text{Tr}(T^T T) = t_1^2 + t_2^2 + t_3^2, \quad (C.6)$$
$$I_3 = \text{Tr}[(T^T T)^2] = t_1^4 + t_2^4 + t_3^4 \quad (C.7)$$

The diagonal form of $T$ viz., $(t_1, t_2, t_3)$ can be determined using the invariants $I_{1-3}$ up to a simultaneous sign change for any two of them. Now, a local rotation on the first qubit $R(\pi)^i \otimes I$ (where $R(\pi)^i$ is the $\pi$ rotation about the axis $i = 1, 2, 3$), may be used to fix the signs of $t_1, t_2, t_3$. It is then convenient to adopt the convention, (i) if $I_1 \geq 0$, elements of $T^d$ are all positive and (ii) $t_1, t_2, t_3$, are all negative, when $I_1 < 0$.

Let us restrict to two qubit states with a fixed diagonal correlation matrix $T$ (which is achieved through appropriate local unitary operations on the qubits).

To determine the absolute values of the state parameters, $s_1, s_2, s_3$, the invariants $I_{4-6}$ of Table 2.1 are used:

$$I_4 = s^T s = s_1^2 + s_2^2 + s_3^2,$$
$$I_5 = s^T T T^T s = s_1^2 t_1^2 + s_2^2 t_2^2 + s_3^2 t_3^2,$$
$$I_6 = s^T (T T^T)^2 s = s_1^2 t_1^4 + s_2^2 t_2^4 + s_3^2 t_3^4 \quad (C.8)$$

(Here, $T$ is considered to be nondegenerate, i.e., $t_1 \neq t_2 \neq t_3$). The absolute values of state parameters $r_1, r_2, r_3$, can be determined via the invariants $I_{7-9}$ of Table 2.1:

$$I_7 = r^T r = r_1^2 + r_2^2 + r_3^2,$$
$$I_8 = r^T T T^T r = r_1^2 t_1^2 + r_2^2 t_2^2 + r_3^2 t_3^2,$$
$$I_9 = r^T (T T^T)^2 r = r_1^2 t_1^4 + r_2^2 t_2^4 + r_3^2 t_3^4. \quad (C.9)$$
Now, the invariant, $I_{10}$ ($I_{11}$) is useful to fix the overall sign of $s_1$, $s_2$, and $s_3$ ($r_1$, $r_2$, and $r_3$):

$$I_{10} = \epsilon_{ijk} s_i (T T^T s)_j ([T T^T]^2 s)_k$$

$$= (t_1^4(t_3^2 - t_2^2) + t_2^4(t_1^2 - t_3^2) + t_3^4(t_2^2 - t_1^2)) s_1 s_2 s_3,$$

$$I_{11} = \epsilon_{ijk} r_i (T^T T r)_j ([T^T T]^2 r)_k$$

$$= (t_1^4(t_3^2 - t_2^2) + t_2^4(t_1^2 - t_3^2) + t_3^4(t_2^2 - t_1^2)) r_1 r_2 r_3. \tag{C.10}$$

Furthermore, the relative signs between $s_ir_i$ are determined using the invariants $I_{12-14}$,

$$I_{12} = s^T T r = s_1 r_1 t_1 + s_2 r_2 t_2 + s_3 r_3 t_3,$$

$$I_{13} = s^T T T^T T r = s_1 r_1 t_1^3 + s_2 r_2 t_2^3 + s_3 r_3 t_3^3,$$

$$I_{14} = \epsilon_{ijk} \epsilon_{lmn} s_i r_i t_j m t_k n = s_1 r_1 t_2 t_3 + s_2 r_2 t_1 t_3 + s_3 r_3 t_1 t_2, \tag{C.11}$$

which provide three linear constraints on $s_1r_1$, $s_2r_2$ and $s_3r_3$.

Next, the individual signs of $(s_1$, $s_2$, $s_3)$ and $(r_1$, $r_2$, $r_3)$ can also be determined when, atleast two components, say, $s_1$, $s_2$ are nonzero, and $s_3 = 0$. In this case, the signs of $s_1$ and $s_2$ can be made positive with the help of local rotations $\frac{1}{2}(R(\pi)^1 \otimes R(\pi)^1)$, $(R(\pi)^2 \otimes R(\pi)^2)$. So, with $s_1$, $s_2 > 0$ and $s_3 = 0$ the invariant $I_{15}$ of Table 2.1 has the form

$$I_{15} = s_1 s_2 t_3 r_3 [t_2^2 - t_1^2], \tag{C.12}$$

thus fixing the sign of $r_3$, provided $t_3 \neq 0$. If $t_3 = 0$, the invariant $I_{17}$ (which is evaluated for $s_1$, $s_2 > 0$ and $s_3 = t_3 = 0$)

$$I_{17} = s_1 s_2 t_2 t_3 r_3 [t_2^2 - t_1^2] \tag{C.13}$$

is utilized for determining the sign of $r_3$.

\footnote{Note that the diagonal form of $T$ remains unchanged under $R(\pi)^i \otimes R(\pi)^i$ with $i = 1, 2, 3$.}
A similar argument is applicable using the invariants $I_{11,16,18}$ for fixing the sign of $s_3 (s_3 \neq 0)$, when $r_1$ and $r_2$ are nonzero and $r_3 = 0$. 
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List of Publications

Journals:

1. Non-local properties of a symmetric two-qubit system

A. R. Usha Devi, M. S. Uma, R. Prabhu and Sudha

*J. Opt. B: Quantum Semiclass. Opt.* 7 (2005) S740-S744.

2. Local invariants and pairwise entanglement in symmetric multi-qubit system

A. R. Usha Devi, M. S. Uma, R. Prabhu and Sudha

*Int. J. Mod. Phys. B.* 20 (2006) 1917-1933.

3. Non-classicality of photon added coherent and thermal radiations

A. R. Usha Devi, R. Prabhu and M. S. Uma

*Eur. Phys. J. D* 40 (2006) 133-138.

4. Constraints on the uncertainties of entangled symmetric qubits

A. R. Usha Devi, M. S. Uma, R. Prabhu and A. K. Rajagopal

*Phys. Lett. A* 364 (2007) 203-207.
**Conferences/Symposia/Workshop**

1. **Pairwise entanglement properties of a symmetric multi-qubit system**

   A. R. Usha Devi, **M. S. Uma**, R. Prabhu and Sudha

   XVI-DAE-BRNS High Energy Physics Symposium held at Saha Institute of Nuclear Physics, Kolkata, India, during 29th November to 3rd December 2004.

2. **Non-local properties of a symmetric two-qubit system**

   A. R. Usha Devi, **M. S. Uma**, R. Prabhu and Sudha

   Seventh International Conference on Photoelectronics, Fiber Optics and Photonics held at International School of Photonics, Cochin University of Science and Technology, Kochi, India, during 9-11 December 2004.

3. **Separability, negativity, concurrence and local invariants of symmetric two qubit states**

   A. R. Usha Devi, **M. S. Uma**, R. Prabhu and Sudha

   International Conference on Squeezed States and Uncertainty Relations (ICSSUR’05) held at Besancon, France, during 2-6 May 2005.

4. **Nonclassicality of photon added Gaussian light fields**

   A. R. Usha Devi, R. Prabhu and **M. S. Uma**

   Second International Conference on Current Developments in Atomic, Molecular and Optical Physics with Applications (CDAMOP’06) held at Delhi University, New Delhi, during 21-23 March 2006.