Infinite-dimensional Lie algebras
and the period map for curves

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Abstract

We compute higher-order differentials of the period map for curves
and show how they factor through the corresponding higher Kodaira-
Spencer classes. Our approach is based on the infinitesimal equivari-
ance of the period map, due to Arbarello and De Concini [AD].

To the memory of Boris Moishezon

A holomorphic map between two complex manifolds \( \Phi : S \to M \) in-
duces, for every \( n \), a map between sheaves of differential operators of order
\( \leq n \) on \( S \) and \( M \), \( \mathcal{D}_S^{(n)} \to \mathcal{D}_M^{(n)} \). In Algebraic Geometry it is slightly more
convenient to replace \( \mathcal{D}_S^{(n)} \) with the sheaf of the \( n \)th order tangent vectors
\( T^{(n)}S := \mathcal{D}_S^{(n)}/\mathcal{O}_S \). The \( n \)th differential of \( \Phi \) is the corresponding map

\[
d^n\Phi : T^{(n)}S \to T^{(n)}M.
\]

In general, it is hard to describe \( T^{(n)}S \) or \( T^{(n)}M \), and little can be
said about \( d^n\Phi \). However, when \( \Phi \) is the period map arising from a family
of complex algebraic varieties or Kähler manifolds over \( S \), and \( M \) is an
appropriate period domain \( \mathcal{D} \), it turns out that despite the transcendental
nature of \( \Phi \), the differentials \( d^n\Phi \) admit an algebraic description.

In this paper we study higher differentials of the period map \( \Phi : S \to \mathcal{D} \)
associated to a miniversal deformation of a complete curve \( X \). Specializing

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to curves allows us to use constructions and facts so far unavailable in other cases. To simplify the exposition, we restrict our computations to $d^2\Phi$, which already presents all features found in higher-order cases. This also seems to be the most important case for potential applications. The results for higher-order cases are summarized in Theorem 8.2.

Our first main result is a description of the second differential of $\Phi$ at $0 \in S$,

$$d^2_0\Phi : T_0^{(2)}S \rightarrow T_{\Phi(0)}D,$$

in terms intrinsic to $X$.

An essential difficulty one encounters is the lack of a nice description for the second tangent space to the period domain, $T_F^{(2)}D$, comparable to the well-known identification $T_F D = Hom(s)(F, H/F)$. We bypass this problem by observing in Proposition 6.1 that there is a natural splitting

$$T_F^{(2)}D = T_F D \oplus S^2T_F D,$$

and that $d^2_0\Phi$ splits accordingly:

$$d^2_0\Phi = \ell \oplus \sigma.$$

Here $\sigma$ is the symbol part of $d^2_0\Phi$, which is simply $S^2d_0\Phi$ composed with the projection $T_0^{(2)}S \rightarrow S^2T_0S$.

The object of our attention is the linear part of $d^2_0\Phi$,

$$\ell : T_0^{(2)}S \rightarrow T_{\Phi(0)}D,$$

and in Theorem 6.4 we explain how to compute it. The result may be expressed in its simplest form as follows. Let $\zeta$ and $\xi$ be some Kodaira-Spencer representatives for two tangent vectors to $S$ at 0. Then $\zeta \otimes \xi$ represents, in a precise sense defined in the paper, a second-order tangent vector to $S$ at 0, and $\ell(\zeta \otimes \xi)$ is given by the map

$$\omega \mapsto \xi \triangledown \zeta \omega$$

regarded as an element of

$$T_{\Phi(0)}D = Hom(s)(H^0(X, \Omega^1_X), H^1(X, O_X)).$$

It was shown in [K1] that the same formula computes the second fundamental form of $\Phi$ (see [CGGH]),

$$I : S^2T_0S \rightarrow T_{\Phi(0)}D / image(d_0\Phi).$$
In fact, \( \Pi \equiv \ell \mod \text{image} (d_0 \Phi) \). The very important refinement here is that \( \Pi(\zeta \otimes \xi) \) is already determined by the symbol of \( \zeta \otimes \xi \) in \( S^2T_0S \), whereas \( \ell(\zeta \otimes \xi) \) (and, hence, \( d_0^2 \Phi(\zeta \otimes \xi) \)) involves the full second-order tangent vector \( [\zeta \otimes \xi] \in T_0^{(2)}S \).

The second main result of this paper is a cohomological interpretation of \( d_0^2 \Phi \). A well-known result of Griffiths \([Gr]\) states that the first differential of the period map is given by cup product with the Kodaira-Spencer class of the deformation. Recently there has appeared a series of papers \([BG, EV, R1, R2, HS]\) renewing the study of higher-order deformation theory and, in particular, introducing higher Kodaira-Spencer classes \( \kappa_n \). Pursuing an analogy with Griffiths’ result, we showed in \([K2]\) that the second fundamental form of \( \Phi \) depends only on \( \kappa_2 \) (more precisely, on \( \kappa_2 \) modulo the image of \( \kappa_1^2 \)). Here this fact receives a new proof as a corollary of Theorem 7.4, which states that \( \ell \), and hence \( d_0^2 \Phi \), factors through \( \kappa_2 \). In fact, Theorem 7.4 gives more: it brings \( d_0^2 \Phi \) in closer agreement with \( d_2^0 \Phi \) by explicitly displaying a kind of a cup product computing \( \ell \) on the cochain level.

In \([K1]\) and \([K2]\) the main technical tool was Archimedean cohomology — an infinite-dimensional replacement for the Hodge structure of \( X \). Here we use a different infinite-dimensional object: the “extended Hodge structure” of the curve \( X \), leading to the “extended period map” \( \hat{\Phi} \). Arbarello and De Concini introduced these notions in their paper \([AD]\), which serves as a point of departure for this work.

One starts with a curve \( X \), a point \( p \) on \( X \), and a formal parameter near \( p \), \( u : \hat{\mathcal{O}}_{X,p} \to \mathbb{C}[[z]] \). We may say that both the usual and the extended Hodge structures on \( X \) encode information about regular 1-forms on \( X - p \): one through their periods, the other through their Laurent expansions at \( p \).

The advantage of working with the extended period map is that it is easier to bring in a basic fact of moduli theory for curves that emerged in recent years — that the moduli of curves are (locally) infinitesimally uniformized by a very simple, though infinite-dimensional, Lie algebra. We are referring to the Witt Lie algebra of formal vector fields on a punctured disk, \( \mathfrak{d} = \mathbb{C}((z))d/dz \), whose central extension is the more famous Virasoro algebra.

The key observation of \([AD]\) used in this paper is that the extended Hodge structures are also infinitesimally uniformized by an infinite-dimensional Lie algebra, denoted \( \mathfrak{sp}(\mathcal{H}') \), and that there is a Lie algebra homomorphism \( \varphi : \mathfrak{d} \to \mathfrak{sp}(\mathcal{H}') \)
making the extended, and hence the usual period map \textit{infinitesimally equivariant}. This means that the differential of the period map \( \Phi \) associated to a miniversal deformation of \( X \) over a sufficiently small base \( S \) is induced by the (very simple) Lie-algebraic object \( \varphi \) (see diagram (20)). This sets the stage for computing higher differentials of \( \Phi \), which are evidently induced by filtered pieces of the enveloping algebra morphism corresponding to \( \varphi \).

The plan of the paper is as follows. Section 1 contains notation and definitions pertaining to infinitesimal uniformization. Section 2 is devoted to the proof of Theorem 2.5 on infinitesimal uniformization of moduli of curves. This result is due to [BMS], [Ko], [BS], cf. [N], [UY], but the author was unable to find a complete proof in the literature. Section 3 introduces extended Hodge structures and discusses infinitesimal uniformization for the period domain. Section 4 reviews the construction of the extended Hodge structure associated to a given curve and in doing so defines the extended period map. In section 5 all of this comes together in Theorem 5.2 on infinitesimal equivariance of the period map. The material in sections 3 through 5 is largely due to Arbarello and De Concini [AD], though we found it necessary to fill in a number of details.

Sections 6 and 7 develop our main results. Proposition 6.1 exploits the usual (i.e. finite-dimensional) uniformization of the period domain \( D \) under \( Sp(2g, \mathbb{C}) \) to obtain a splitting of the second tangent space to \( D \). Then lemmas 6.2 and 6.3 study the infinitesimal uniformization of \( D \) described in section 3 and compare the two uniformizations on the second-order level. As a corollary, we obtain in Theorem 6.4 a formula for the linear part of \( d_2^\Phi \), \( \ell \). The map \( \ell \) also determines the second fundamental form II of the VHS. We have computed II in [K1] in a different way, and in Theorem 6.6 we show that the two approaches agree.

Section 7 is devoted to showing that the second differential of the period map, as well as its linear part \( \ell \) introduced in section 6, factors through the second Kodaira-Spencer class \( \kappa_2 \). This requires an excursion into the recent description of the second tangent space at \([X]\) to the moduli of curves as a certain cohomology group on \( X \times X \). In fact, in Theorem 7.4 we give what amounts to a cohomological interpretation of the map \( \ell \). In Corollary 7.5 we obtain another proof that II factors through \( \kappa_2 \). This was already proved in [K2] by a more general but less explicit method.

Section 8 provides higher-order analogues of the statements in sections 6 and 7.
1 Notation and some preliminaries

The results discussed in this paper owe their explicitness to a very concrete object, the field of Laurent power series

\[ \mathcal{H} = \mathbb{C}((z)) = \mathbb{C}[z][z^{-1}] \, . \]

Most of the time we will regard it merely as an infinite-dimensional vector space. It has several distinguished subspaces:

\[ \mathcal{H}_+ = \mathbb{C}[z] \quad \text{and} \quad \mathcal{H}_- = \text{the span of negative powers of } z \, . \]

Thus \( \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \) . Also,

\[ \mathcal{H}'_+ = z\mathcal{H}_+ \, , \quad \text{and} \quad \mathcal{H}' = \mathcal{H}'_+ \oplus \mathcal{H}_- \, . \]

**Definition 1.1** \( < f, g > = \text{Res}_{z=0} f dg \, . \)

This is a symplectic form on \( \mathcal{H} \), non-degenerate on \( \mathcal{H}' \).

**Definition 1.2**

\[ \text{sp}(\mathcal{H}') = \{ \alpha \in \text{End}(\mathcal{H}') \mid < \alpha(x), y > + < \alpha(y), x > = 0 \text{ for all } x, y \in \mathcal{H}' \} \, . \]

**Facts:** (a) \( \mathcal{H} \) (and, hence, \( \mathcal{H}' \)) is a topological vector space with the \( z \)-adic topology.

(b) \( \text{sp}(\mathcal{H}') \) is isomorphic to the completion \( \hat{S}^2(\mathcal{H}') \) of \( S^2(\mathcal{H}') \) — the symmetric square of \( \mathcal{H}' \), where \( S^2(\mathcal{H}') \) embeds in \( \text{sp}(\mathcal{H}') \) via

\[ hk \mapsto \{ x \mapsto \langle h, x \rangle k + \langle k, x \rangle h \} \, . \]

Finally, \( d = \mathcal{H}_+ \frac{d}{dz} \) will denote the Witt Lie algebra of formal vector fields on a punctured disc. Its central extension is the more famous Virasoro algebra. We will also use \( d_+ = \mathcal{H}_+ \frac{d}{dz} \, . \)

The Lie algebras \( d \) and \( \text{sp}(\mathcal{H}') \) will appear in the setting of the following

**Definition 1.3** A Lie algebra \( L \) acts by vector fields on a manifold \( M \) if there is a homomorphism (or anti-homomorphism) of Lie algebras

\[ L \rightarrow \Gamma(M, \Theta_M) \, . \]
If the composed map to the tangent space of $M$ at a point $x$

\[ L \mapsto \Gamma(M, \Theta_M) \mapsto T_xM \]

is surjective for each $x \in M$, $M$ is called \textit{infinitesimally homogeneous}, and one says that $L$ provides an \textit{infinitesimal uniformization} for $M$.

We will write $\Omega^1_X$ and $\omega_X$ interchangeably when $X$ is a curve. In turn, “curve” will mean a complex algebraic curve. We will consider the classical topology and the analytic structure on $X$ only when dealing with the cohomology of $X$ with coefficients in $\mathbb{Z}$ or $\mathbb{C}$, and when using the exponential sequence in the proof of Lemma 4.8, where we write $X^{an}$.

We will also use the following notation: if $g$ is a Lie algebra, then $Ug$ is its universal enveloping algebra, and $\overline{U}g := Ug/C$. $U^{(k)}g$ (respectively, $\overline{U}^{(k)}g$) will denote the elements of order $\leq k$ in the natural filtration of $Ug$ (respectively, $\overline{U}g$).

Finally, we will use the fact that Lie algebras of endomorphisms of Hodge structures, with or without polarization, carry a Hodge structure of their own, always of weight 0. E.g. if $H = \oplus H^{p,q}$ is a HS and $g = \text{End}(H)$, then the Hodge decomposition of $g$ is

\[ g = \oplus g^{-k,k}, \]

where $g^{-k,k} = \{ f \in \text{End}(H) \mid f(H^{p,q}) \subset H^{p-k,q+k} \forall p,q \}$.

2 \hspace{1cm} \textbf{Infinitesimal uniformization of moduli of curves}

Let $X$ be a complete curve of genus $g \geq 2$, $p$ — a point on $X$, and

\[ u : \mathcal{O}_{X,p} \overset{\cong}{\longrightarrow} \mathcal{H}_+\]

— a formal local coordinate at $p$; $u$ extends to an isomorphism of fields of fractions, also defining the obvious monomorphisms

\[ \Theta_{X,p} \hookrightarrow \mathcal{H}_+ \frac{d}{dz} \subset \mathcal{D}, \quad \Omega^1_{X,p} \hookrightarrow \mathcal{H}_+dz, \quad \Omega^1_X(*p)_p \hookrightarrow \mathcal{H}dz, \quad \text{etc.}, \]

all of which will also be denoted by $u$. As any point on a complete curve, $p$ is an ample divisor on $X$. Therefore, $X - p$ is an affine open set in $X$. Choose an affine neighborhood $V$ of $p$ in $X$. Then $\{(X - p), V\}$ is
an affine covering of $X$, suitable for computing the Čech cohomology of $X$ with coefficients in a coherent sheaf. Thus we have an exact sequence

$$0 \to \Gamma(X - p, \Theta_X) \oplus \Gamma(V, \Theta_X) \xrightarrow{\delta} \Gamma(V - p, \Theta_X) \xrightarrow{\pi} H^1(X, \Theta_X) \to 0.$$  

Exactness on the left is a consequence of $H^0(X, \Theta_X) = 0$, which, in turn, follows from the assumption $g \geq 2$.

The sheaf $\Theta_X$ is filtered by the subsheaves $\Theta_X(-ip)$ of vector fields vanishing at $p$ to an order $\geq i$ ($i \geq 0$). This induces a decreasing filtration $P_i$ on spaces of sections over $V$, and hence over $X - p$ and $V - p$. The Čech differential $\delta$ is strictly compatible with $P_i$, and so is the projection $\pi$, once $H^1(X, \Theta_X)$ receives the induced filtration $P_i$ from $\Gamma(V - p, \Theta_X)$. Therefore, the sequence (1) remains exact when reduced modulo $P_i$.

**Lemma 2.1** The maps 

$$u_i : \Gamma(V - p, \Theta_X)/P_i \to d/z^i$$  

induced via the identification $u$ are isomorphisms for each $i > 0$.

**Proof.** Suppose the points $q_1, \ldots, q_m$ constitute the complement of $V$ in $X$, and let $Q$ be the effective divisor $q_1 + \ldots + q_m$. Since $Q$ is ample, for $N$ sufficiently large $H^1(X, \Theta_X(NQ - ip))$ will vanish. We may also assume $\deg L \geq qg - 1$. With this choice of $N$, set $L = \Theta_X(NQ - ip)$. Then the Riemann-Roch Theorem gives

$$H^0(X, L(kp)) = \deg L + k + 1 - g$$  

for each $k \geq 1$, which means that for each $k$ there exists a section of $\Theta_V(-ip)$, regular on $V - p$ and with a pole of order exactly $k$ at $p$. This implies the surjectivity of $u_i$.

Now,  

$$u : \Gamma(V - p, \mathcal{O}_X) \to \mathcal{H}$$  

is injective, since any regular function on $V - p$ is completely determined by its Laurent expansion. And  

$$u^{-1}(z^i\mathcal{H}_+) = \Gamma(V - p, \mathcal{O}_X(-ip)),$$  

implying that the $u_i$’s are injective too. 

$\Box$
Corollary 2.2  \[ \lim_{i} \Gamma(V - p, \Theta_X)/P^i = d . \]

Lemma 2.3 Passing to the inverse limit in the exact sequence obtained from (2) by reduction mod\( P^i \) produces an exact sequence

\[ 0 \to u(\Gamma(X, \Theta_X(*p))) \oplus d_+ \to d \to H^1(X, \Theta_X) \to 0 \]

Proof. First we note that \( \Gamma(X - p, \Theta_X) = \Gamma(X, \Theta_X(*p)) \) by [Gro]. Also, \( P^i \Gamma(X, \Theta_X(*p)) = 0 \) for all \( i > 0 \), since \( X \) supports no non-zero global regular vector fields by virtue of the assumption \( g \geq 2 \). Hence \( \Gamma(X, \Theta_X(*p))^\wedge = \Gamma(X, \Theta_X(*p)) \). Second, \( \lim_{i} H^1(X, \Theta_X)/P^i = H^1(X, \Theta_X) \), because for all sufficiently large \( i \)

\[ H^1(X, \Theta_X)/P^i = H^1(X, \Theta_X) ; \]

this is simply a consequence of \( H^1(X, \Theta_X) \) being finite-dimensional. Finally, inverse limits preserve the exactness of \( \Gamma \) mod\( P^i \), because the directed system

\[ \{ A_i = (\Gamma(X - p, \Theta_X) \oplus \Gamma(V, \Theta_X))/P^i \} \]

satisfies the Mittag-Leffler condition (see [L], III, Prop. (9.3)):

For each \( n \), the decreasing sequence of images of natural maps \( \varphi_{mn} : A_m \to A_n \) stabilizes.
This is trivially so since all \( \varphi_{mn} \) are surjective in our situation. \( \square \)

Assume now that \( X \) moves in a flat family

\[ \begin{array}{ccc}
\mathcal{X} & \xrightarrow{\pi} & X_t \\
\downarrow & & \downarrow \\
S & \supset & t
\end{array} \]

(3)

with a section \( p : S \to \mathcal{X} \) and a local coordinate

\[ u : \mathcal{O}_{X_p} \longrightarrow \Gamma(S, \mathcal{O}_S) \otimes \mathcal{H}_+ \]

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on $X$ along $p$, so that the restriction of $u$ to $X_t$ provides a local formal coordinate $u_t$ near $p_t$.

For each $t \in S$ one has an analogue of (2). In particular, for each $t$ there is a surjection

$$d \to H^1(X_t, \Theta_{X_t});$$

these glue together into a map

$$d \to \Gamma(S, R^1\pi_*\Theta_{X/S}).$$

(4)

Assume further that $S$ is a disc centered at 0 in $\mathbb{C}^{3g-3}$, and the family

$$\begin{array}{c}
\mathcal{X} \\
\pi \\
S
\end{array} \supset \begin{array}{c} X \\
\downarrow \\
0
\end{array}$$

(5)

is a miniversal deformation of the curve $X$. Then the Kodaira-Spencer map of the family,

$$\kappa : \Theta_S \to R^1\pi_*\Theta_{X/S},$$

is an isomorphism. Composing its inverse with the map in (4) yields a linear map

$$\lambda : d \to \Gamma(S, \Theta_S).$$

(6)

Lemma 2.4 The map $\lambda$ in (6) is an anti-homomorphism of Lie algebras.

Proof. $\mathcal{X}$ admits an acyclic Stein covering $\mathcal{W} = \{W_0, W_1\}$ with $W_0 \cong S \times V$ and $W_1 = \mathcal{X} - p \cong S \times X - p$. It follows from the proof of the previous lemma that the map $\lambda$ fits in the commutative diagram

$$\begin{array}{c}
\Gamma(S, \mathcal{O}_S) \otimes d \\
j \uparrow \cong \\
\cong \downarrow \kappa
\end{array} \quad \begin{array}{c}
d \\
\lambda \quad \to \\
\Gamma(S, \Theta_S)
\end{array} \quad \begin{array}{c}
\Gamma(W_0 \cap W_1, \Theta_{X/S})^\wedge \\
\kappa \quad \to \\
\Gamma(S, R^1\pi_*\Theta_{X/S})
\end{array}$$

where $^\wedge$ indicates completion with respect to the filtration by the order of vanishing along $p$, and $j$ is the isomorphism given by taking Laurent expansionsof relative vector fields on $W_0 \cap W_1$ along $p$ via $u$. 

9
We begin by reviewing the definition of $\kappa$. The Kodaira-Spencer map $\kappa$ is the connecting morphism in the direct-image sequence of the short exact sequence of $\mathcal{O}_X$ modules

\[(7) \quad 0 \to \Theta_{X/S} \to \Theta_X \to \pi^* \Theta_S \to 0 .\]

This contains an exact subsequence of $\pi^{-1}\mathcal{O}_S$-modules

\[(8) \quad 0 \to \Theta_{X/S} \to \tilde{\Theta}_X \to \pi^{-1}\Theta_S \to 0 .\]

whose direct-image sequence also has $\kappa$ as a connecting morphism (see [BS] and also [EV]). Furthermore, (8) is an exact sequence of sheaves of Lie algebras. The $\mathbb{C}$-linear brackets on $\tilde{\Theta}_X$ and $\pi^{-1}\Theta_S$ are inherited from $\Theta_X$ and $\Theta_S$, respectively. The bracket on $\Theta_{X/S}$ is even $\pi^{-1}\mathcal{O}_S$-linear.

We are ready to prove the lemma. Take any $\zeta, \xi \in d$, and let $Z = \lambda(\zeta)$, $\Xi = \lambda(\xi)$. We wish to show that $[\zeta, \xi] = -[Z, \Xi]$, where the first bracket is taken in the Witt Lie algebra $d$, and the second is in $\Gamma(S, \Theta_S)$. The elements $\zeta$ and $\xi$ of $d$, which we identify with their pre-images under $j$, may be taken as Kodaira-Spencer representatives of $Z$ and $\Xi$. Lift $Z$ to some sections of $\tilde{\Theta}_X$, $\zeta_0 \in \tilde{\Theta}_X$ on $W_0$ and $\zeta_1 \in \tilde{\Theta}_X$ on $W_1$, and similarly for $\Xi$: $\xi_0 \in \Gamma(W_0, \tilde{\Theta}_X)$, and $\xi_1 \in \Gamma(W_1, \tilde{\Theta}_X)$. Then $\zeta_1 - \zeta_0$ and $\xi_1 - \xi_0$, with all terms restricted to $W_{01} = W_0 \cap W_1$, also give KS representatives for $Z$ and $\Xi$. In particular,

\[\zeta = \zeta_1 - \zeta_0 + \delta \theta ,\]

and

\[\xi = \xi_1 - \xi_0 + \delta \eta ,\]

where $\theta$ and $\eta$ are some elements of $\delta \check{C}^0(W, \Theta_{X/S})$. Then $[Z, \Xi]$ admits as its KS representative the following expression, all terms of which are restricted to $W_{01}$:

\[(9) \quad [\zeta_1, \xi_1] - [\zeta_0, \xi_0] = [\zeta_1, \xi_1] - [\zeta_1 - \zeta + \delta \theta, \xi_1 - \xi + \delta \eta] = -[\zeta, \xi] + [\zeta_1, \xi] + [\zeta, \xi_1] + [\delta \theta, \xi_0] + [\zeta_0, \delta \eta] .\]

The Lie bracket of a section of $\tilde{\Theta}_X$ with that of $\Theta_{X/S}$ is again a section of $\Theta_{X/S}$, which implies that the last two terms in (9) are in $\delta \check{C}^0(W, \Theta_{X/S})$. We may assume that $u$ is induced by an isomorphism $u : \hat{\mathcal{O}}_{X,p} \to \mathcal{H}_+$ via the identification $W_{01} \cong S \times \{V - p\}$. The identification allows us to label some vector fields on $W_{01}$ as horizontal or vertical. By construction,
\( \zeta \) and \( \xi \) are vertical and constant in the horizontal direction. The fields \( \zeta_1 \) and \( \xi_1 \), on the other hand, may be chosen to be horizontal and constant in the vertical direction. Then \( [\zeta_1, \xi] = [\zeta, \xi_1] = 0 \). Collecting what is left of (9), we conclude that \( -[\zeta, \xi] \) is a Kodaira-Spencer representative for \([Z, \Xi]\), which proves the lemma. \( \Box \)

Recalling definition 1.3, we may summarize lemmas 2.4 and 2.3 in the following theorem, due to [BMS, Ko, BS], cf. [N, TUY].

**Theorem 2.5** For any curve \( X \) of genus \( g \geq 2 \) the Witt Lie algebra \( d \) acts by vector fields on the base \( S \) of a miniversal deformation of \( X \), making \( S \) infinitesimally homogeneous.

**Remark 2.6** The action above clearly depends on the choice of a point \( p_t \) on each curve \( X_t \), as well as on a formal parameter \( u_t \) at \( p_t \). For our purposes all these choices are equally good. More canonically, one may consider the moduli space of triples \( (X, p, u) \), encompassing all possible choices of \( p \) and \( u \) on each \( X \). The action of \( d \) extends to such “dressed” moduli spaces \( \hat{M}_g \), making them also infinitesimally homogeneous. We will not need these constructions, since the questions we study are local on \( \hat{M}_g \).
3 Infinitesimal uniformization of period domains of weight one

By definition, a Hodge structure of weight one consists of a lattice $\Lambda \cong \mathbb{Z}^{2g}$ and a decomposition of its complexification $H = \Lambda \otimes \mathbb{C}$, $H = H^{1,0} \oplus H^{0,1}$, such that $H^{1,0} = \overline{H^{0,1}}$. The Hodge filtration $F^\bullet$ on $H$ is given by $F^0 = H$, $F^1 = H^{1,0}$, $F^0 = 0$. The HS $(\Lambda, H, F^\bullet)$ is principally polarized if $\Lambda$ is equipped with a unimodular symplectic form $Q(\ , \ )$ such that $(u, v) = Q(\bar{u}, v)$ is a positive-definite Hermitian form on $H^{1,0}$ (and on $H^{0,1}$).

The data $(\Lambda, H, F^\bullet, Q)$ defines a principally-polarized abelian variety $A = H^{0,1}/i(\Lambda)$, where $i$ denotes the composition of the inclusion $\Lambda \to \Lambda \otimes \mathbb{C} = H$ with the projection $H = H^{1,0} \oplus H^{0,1} \to H^{0,1}$.

As is well-known, the space $\mathcal{D}$ of all Hodge structures $(H, F^\bullet)$ with a given lattice $\Lambda$ and polarization $Q$ (= the period domain) can be identified with the Siegel upper half-space $\mathbb{H}_g$ of complex symmetric $g \times g$ matrices whose imaginary parts are positive-definite. The moduli space of principally-polarized abelian varieties of dimension $g$, $\mathcal{A}_g$, is a quotient of $\mathbb{H}_g$ by the action of $Sp(2g, \mathbb{Z})$. Note that $\mathcal{D}$ is a homogeneous space for the group $Sp(2g, \mathbb{R})$.

We wish to present $\mathcal{D}$ locally as an infinitesimally homogeneous space for $\mathfrak{sp}(\mathcal{H}')$.

**Definition 3.1** An extended Hodge structure (of weight one) is a triple $(Z, K_0, \Lambda)$, where $Z$ is a maximal isotropic subspace of $\mathcal{H}'$ (with respect to the symplectic form $\langle \ , \rangle$), $K_0$ is a codimension $g$ subspace of $Z$, and $\Lambda$ is a rank $2g$ lattice in $K_0^\perp/K_0$, subject to several conditions.

First of all, $Z \cap \mathcal{H}'_+ = 0$. This implies the splittings $\mathcal{H}' = Z \oplus \mathcal{H}'_+$ and $H := K_0^\perp/K_0 = H^{1,0} \oplus H^{0,1}$, where $H^{0,1} = Z/K_0$, and $H^{1,0} = K_0^\perp \cap \mathcal{H}'_+$.

Let $Q$ be the bilinear form induced on $H$ by $\frac{1}{2\pi i} \langle \ , \rangle$ on $\mathcal{H}'$. The remaining conditions state that $H = \Lambda \otimes \mathbb{C}$, defining a real structure on $H$, that $H^{1,0} = \overline{H^{0,1}}$ with respect to this structure, and that $Q$ is unimodular on $\Lambda$. Thus $(\Lambda, H, H^{1,0}, H^{0,1}, Q)$ is a principally-polarized HS of weight one.

Arbarello and De Concini introduced an extended version of the Siegel upper half-space, $\tilde{\mathbb{H}}_g$, on which $Sp(2g, \mathbb{Z})$ acts transitively, and the quotient.
manifold $\hat{A}_g$ parameterizes extended Hodge structures. The latter may also be regarded as “extended abelian varieties” in view of the following commutative diagram:

$$
\begin{array}{ccc}
\hat{H}_g & \longrightarrow & \hat{A}_g \\
\downarrow & \triangleright & \downarrow \\
H_g & \longrightarrow & A_g
\end{array}
$$

(10)

The horizontal maps are quotients with respect to the $Sp(2g, \mathbb{Z})$-action. All spaces are manifolds (the top two are infinite dimensional), except $A_g$, which is a $V$-manifold. Note that all maps in the upper triangle are smooth.

**Proposition 3.2 ([AD])** $\hat{A}_g$ is an infinitesimally homogeneous space for $\text{sp}(\mathcal{H}')$.

Obviously, this also makes $D = H_g$ locally infinitesimally homogeneous for $\text{sp}(\mathcal{H}')$. Let us work out the surjection

$$
\text{sp}(\mathcal{H}') \longrightarrow T_H D
$$

explicitly. At any point $H = H^{1,0} \oplus H^{0,1}$ of $D$,

$$
T_H D = \text{Hom}(s)(H^{1,0}, H^{0,1}) = S^2 H^{0,1}.
$$

Suppose $H$ comes from an extended HS $(Z, K_0, \Lambda)$. Then any $\alpha \in \text{End}(\mathcal{H}')$ induces a map

$$
H^{1,0} = K_0^\perp \cap \mathcal{H}'_+ \longrightarrow \mathcal{H}'.
$$

(11)

We use the formulas $\mathcal{H}' = Z \oplus \mathcal{H}'_+$ and $K_0 \cap \mathcal{H}'_+ = 0$ to observe that

$$
H^{0,1} \cong Gr_1^0 H = Z/K_0 \cong \mathcal{H}'/K_0 + \mathcal{H}'_+.
$$

Then (11), composed with the natural projection

$$
\mathcal{H}' \longrightarrow \mathcal{H}'/K_0 + \mathcal{H}'_+,
$$

yields an element $a \in \text{Hom}(H^{1,0}, H^{0,1})$.

When $\alpha \in \text{sp}(\mathcal{H}')$,

$$
< \alpha(x), y > = - < x, \alpha(y) >,
$$
i.e. \(< x, \alpha(y) >\leq < y, \alpha(x) >\) for all \(x, y \in \mathcal{H}'\). Hence
\[
Q(x, a(y)) = Q(y, a(x))
\]
for all \(x, y \in H^{1,0}\), which means \(a\) is symmetric:
\[
a \in \text{Hom}^{(s)}(H^{1,0}, H^{0,1}) = S^2 H^{0,1}.
\]
For reasons that will be clear later, we prefer \(-a \in S^2 H^{0,1}\). Thus \(\alpha \mapsto -a\) indeed defines a map
\[
\rho : \text{sp}(\mathcal{H}') \to T_H D = \text{Hom}^{(s)}(H^{1,0}, H^{0,1}) = S^2 H^{0,1}.
\]

**Remark 3.3** For further use we record that the above construction presents the uniformizing map \((12)\) as a restriction of a more broadly defined map
\[
(13) \quad \text{End}(\mathcal{H}') \to \text{Hom}(H^{1,0}, H^{0,1}).
\]
Both maps will be denoted \(\rho\).

**Remark 3.4** In view of \((10)\), Proposition 3.2 implies that a sufficiently small open set \(U\) in \(D\) is an infinitesimally homogeneous space under the action of \(\text{sp}(\mathcal{H}')\). However, the action is not unique — it depends on the choice of a lift from \(U\) to \(\hat{A}_g\).

### 4 The extended period map

Let \(X\) be a complete smooth curve, \(p\) — a point on \(X\), and \(u : \mathcal{O}_{X,p} \cong \mathcal{H}_+\) — a formal local parameter at \(p\). In this section we review Arbarello and De Concini’s construction associating an extended HS \((Z, K_0, \Lambda)\) to the data \((X, p, u)\). When the triple \((X, p, u)\) varies in a flat family over some base \(S\), this construction defines “an extended period map”
\[
\hat{\Phi} : S \to \hat{A}_g,
\]
such that the usual period map \(\Phi : S \to D\) naturally factors through \(\hat{\Phi}\).

**Definition 4.1** \(K_0 := u(\Gamma(X - p, \mathcal{O}_X)) \cap \mathcal{H}'\).

This is the same as putting \(K_0 = u(\Gamma(X - p, \mathcal{O}_X))/\mathbb{C}\).
Note that $\Gamma(X - p, \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X(*p))$ by a theorem of Grothendieck \cite{Gro}, and that

$$u : \Gamma(X - p, \mathcal{O}_X) \to \mathcal{H}$$

is injective. There are no non-constant regular functions on $X$, and so $K_0 \cap \mathcal{H}'_+ = 0$.

**Lemma 4.2** $H^1(X, \mathcal{O}_X) \cong \mathcal{H}/\mathcal{H}_+ + K_0$.

**Proof.** This follows from the exact sequence

$$0 \to \Gamma(V, \mathcal{O}_X) \oplus \Gamma(X - p, \mathcal{O}_X) \to \Gamma(V - p, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X) \to 0$$

by completion with respect to the order-of-vanishing filtration $P^\bullet$ as in (2.3).

$\blacksquare$

Furthermore, $K_0$ is an isotropic subspace of $\mathcal{H}'$, i.e. $K_0$ is contained in $K_0^\perp$, the orthogonal complement of $K_0$ in $\mathcal{H}'$ with respect to the symplectic form $<,>$. We can be more specific about $K_0^\perp$.

**Definition 4.3** $\Omega := \{ f \in \mathcal{H}' \mid df \in u(\Gamma(X - p, \Omega^1_X(*p)))\}$.

We have $\Omega \cong \Gamma(X - p, \Omega^1_X) = \Gamma(X, \Omega^1_X(*p))$.

Now, Grothendieck’s Algebraic De Rham Theorem \cite{Gro}, coupled with the injectivity of the map $d : \mathcal{H}' \to \mathcal{H}dz$ and of $u$, gives

$$(14) \quad \Omega/K_0 \cong \frac{\Gamma(X, \Omega^1_X(*p))}{d\Gamma(X, \mathcal{O}_X(*p))} \cong H^1(X - p, \mathbb{C}) = H^1(X, \mathbb{C}).$$

**Lemma 4.4** $K_0^\perp = \Omega$.

**Proof.** If $f \in K_0$ and $g \in \Omega$, then $fdg$ is the Laurent expansion of a globally defined one-form on $X$ with poles only at $p$. Then $\text{Res}_0 fdg = 0$, i.e. $< K_0, \Omega > = 0$, and so $\Omega \subseteq K_0^\perp$. The well-known duality theorem of Serre \cite{S} implies that the residue pairing induces a duality between $H^0(X, \Omega^1_X)$ and $H^1(X, \mathcal{O}_X)$. The first of these groups is isomorphic to $\Omega \cap \mathcal{H}'_+$, the second — to

$$\frac{\mathcal{H}}{\mathcal{H}_+ + K_0} \cong \frac{\Omega}{\Omega \cap (\mathcal{H}_+ + K_0)} = \frac{\Omega}{\Omega \cap \mathcal{H}_+ + K_0}.$$
This implies that the residue pairing on $\Omega/K_0$ is non-degenerate. Coupled with the earlier statements that $\Omega \subseteq K_0^\perp$ and $K_0 \subset \Omega^\perp$, we have $\Omega^\perp = K_0$ and $(K_0^\perp)^\perp \subseteq \Omega^\perp$, which means that $(K_0^\perp)^\perp = K_0$.

This, in turn, says that $< , >$ is non-degenerate on $K_0^\perp/K_0$. However, the pairing is 0 on $K_0^\perp \cap H^+$ (since it is 0 on all of $H^+$), and on $K_0^\perp (K_0^\perp \cap H^+) + K_0 \cong H^1(X, \mathcal{O}_X)$.

Then $K_0^\perp \cap H^+$ must be dual to

$$\frac{K_0^\perp}{(K_0^\perp \cap H^+) + K_0} \cong H^1(X, \mathcal{O}_X)$$

under the residue pairing on $K_0^\perp/K_0$, which implies $K_0^\perp = \Omega$.  

**Corollary 4.5** $K_0^\perp/K_0 \cong H^1(X, \mathcal{C})$.

At this point we make the observation that the Laurent expansion via $u$ at $p$ can be made well-defined not only for regular functions on a punctured neighborhood of $p$, but also for sections of $\mathcal{O}_X/\mathbb{Z}$:

**Definition 4.6** $K := u(\Gamma(X - p, \mathcal{O}_X/\mathbb{Z})) \cap H'$.

Of course, by means of the exponential map, $\Gamma(X - p, \mathcal{O}_X/\mathbb{Z})$ may be regarded as a subspace of $\Gamma(X - p, \mathcal{O}_{Xan}^*)$. In other words, $K$ consists of those $f \in H'$ for which $e^f$ lies in $u(\Gamma(X - p, \mathcal{O}_{Xan}^*))$. Obviously, $K_0 \subset K$.

Since the exterior derivative $d$ of a constant function is 0, $d$ is well-defined on $\mathcal{O}_X/\mathbb{Z}$, and (4.4) implies that $K \subset K_0^\perp$.

**Definition 4.7** $\Lambda := K/K_0$.

**Lemma 4.8** The isomorphism (4.3): $K_0^\perp/K_0 \cong H^1(X, \mathcal{C})$ maps $\Lambda$ onto $H^1(X, \mathcal{Z})$; in particular, $K_0^\perp/K_0 \cong \Lambda \otimes \mathbb{C}$.

**Proof.** The starting point in identifying $H^1(X, \mathbb{Z})$ with $\Lambda$ is the exponential sequence (on $X^{an}$, of course)

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O}_{X^{an}} \rightarrow \mathcal{O}_{X^{an}}/\mathbb{Z} \rightarrow 1$$

(15)
and its cohomology sequence

\[(16)\]

\[
\begin{array}{ccc}
H^1(X_{an}, \mathcal{O}_{X_{an}}) & \overset{\sim}{\rightarrow} & \mathbb{Z} \\
H^1(X_{an}, \mathbb{Z}) & \hookrightarrow & H^1(X_{an}, \mathcal{O}_{X_{an}}) \rightarrow H^1(X_{an}, \mathcal{O}_{X_{an}}/\mathbb{Z}) \rightarrow H^2(X_{an}, \mathbb{Z})
\end{array}
\]

But we also have an algebraic partial analogue of (15) on \(X\):

\[
0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X/\mathbb{Z} \rightarrow 1,
\]

with the cohomology sequence

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
H^1(X, \mathbb{Z}) & \rightarrow & H^1(X, \mathcal{O}_X) \rightarrow H^1(X, \mathcal{O}_X/\mathbb{Z}) \rightarrow H^2(X, \mathbb{Z})
\end{array}
\]

mapping functorially to (16):

\[
\begin{array}{ccc}
0 & \rightarrow & H^1(X_{an}, \mathbb{Z}) \rightarrow H^1(X_{an}, \mathcal{O}_{X_{an}}) \rightarrow \text{Pic}^0(X) \rightarrow 0 \\
\cong & \overset{\sim}{\rightarrow} & H^1(X, \mathcal{O}_X) \cong H^1(X, \mathcal{O}_X/\mathbb{Z})
\end{array}
\]

The commutativity of the square implies that the right vertical arrow is surjective.

We also have the commutative ladder with exact columns

\[
\begin{array}{ccc}
0 & \rightarrow & H^1(X_{an}, \mathcal{O}_{X_{an}}) \rightarrow H^1(X_{an}, \mathcal{O}_{X_{an}}/\mathbb{Z}) \\
\overset{\cong}{\cong} & \rightarrow & \cong \\
\Gamma(V - p, \mathcal{O}_X) & \rightarrow & \Gamma(V - p, \mathcal{O}_X/\mathbb{Z}) \\
\Gamma(V, \mathcal{O}_X) \oplus \Gamma(V - p, \mathcal{O}_X) & \rightarrow & \Gamma(V, \mathcal{O}_X/\mathbb{Z}) \oplus \Gamma(V - p, \mathcal{O}_X/\mathbb{Z}) \\
\rightarrow & \rightarrow & \rightarrow \\
0 & \rightarrow & 0
\end{array}
\]

Again we note that the upper right vertical arrow must be surjective.

Splicing the two diagrams, and completing with respect to the order-of-
vanishing filtration $P^\bullet$ as in (23), we get

$$
\begin{array}{cccc}
0 & 0 \\
\uparrow & \uparrow & \uparrow & \\
0 \to H^1(X^\text{an}, \mathbb{Z}) & \to H^1(X^\text{an}, \mathcal{O}_{X^\text{an}}) & \to \text{Pic}^0(X) & \to 0 \\
\uparrow & \uparrow & \uparrow & \\
\mathcal{H} & = & \mathcal{H} & \\
\uparrow & \uparrow & \uparrow & \\
\mathcal{H}_+ + K_0 & \to \mathcal{H}_+ + K & \\
\uparrow & \uparrow & \uparrow & \\
0 & 0 & \\
\end{array}
$$

(17)

The lemma now follows by simple homological algebra. Consider the vertical ladder in the above diagram as a short exact sequence of three complexes

$$
0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0.
$$

Then $H^0(C^\bullet) = H^1(X^\text{an}, \mathbb{Z})$, $H^1(A^\bullet) = K/K_0$, and the connecting map in the corresponding cohomology sequence is precisely the sought-after isomorphism

$$
H^1(X^\text{an}, \mathbb{Z}) \xrightarrow{\cong} \Lambda = K/K_0.
$$

(18)

It remains to show that this isomorphism is induced by the one in (4.3). The map (18) factors through the monomorphism

$$
H^1(X^\text{an}, \mathbb{Z}) \to H^1(X^\text{an}, \mathcal{O}_{X^\text{an}}),
$$

which, in turn, factors through $H^1(X^\text{an}, \mathcal{C})$. And the vertical sequences in (17) may be amended as in the proof of Lemma 4.4. Then we arrive at the following variant of (17):

$$
\begin{array}{cccc}
H^1(X^\text{an}, \mathbb{C}) & 0 & 0 \\
\uparrow & \uparrow & \uparrow & \\
H^1(X^\text{an}, \mathbb{Z}) & \to H^1(X^\text{an}, \mathcal{O}_{X^\text{an}}) & \to \text{Pic}^0(X) & \to 0 \\
\uparrow & \uparrow & \uparrow & \\
K_0^\perp = K_0^\perp = K_0^\perp \\
\uparrow & \uparrow & \uparrow & \\
K_0 \to (K_0^\perp \cap \mathcal{H}_+) + K_0 & \to (K_0^\perp \cap \mathcal{H}_+) + K \to \Lambda \\
\uparrow & \uparrow & \uparrow & \\
0 & 0 & 0 \\
\end{array}
$$
With this diagram it is easy to trace the map $H^1(X^{an}, \mathbb{Z}) \to \Lambda$ and see that it fits in the commutative square

\[
\begin{array}{ccc}
H^1(X^{an}, \mathbb{C}) & \to & K_0^\perp / K_0 \\
\uparrow & & \uparrow \\
H^1(X^{an}, \mathbb{Z}) & \to & \Lambda
\end{array}
\]

with natural inclusions as the vertical arrows. 

\[\blacksquare\]

**Proposition 4.9** The isomorphism in (4.3) is symplectic, identifying $\frac{1}{2\pi} < , >$ on $\Lambda$ with the intersection form $Q( , )$ on $H^1(X, \mathbb{Z})$.

**Proof.** The above proposition is established in [AD], following [SW], by reasoning similar to that in the proof of the Riemann reciprocity laws. Alternatively, we can identify the residue pairing with the cup product

\[
H^0(X, \Omega^1_X) \otimes H^1(X, \mathcal{O}_X) \to H^1(X, \Omega^1_X) \cong H^2(X, \mathbb{C}) \cong \mathbb{C},
\]

as Serre suggests in [S], and then relate the cup product to the intersection pairing. 

\[\blacksquare\]

We now complete the identifications above to include the Hodge structure. First, $U := K_0^\perp \cap \mathcal{H}'_+$ is easily seen to be mapped onto

\[
H^0(X, \Omega^1_X) = H^{1,0}(X) = F^1 H^1(X, \mathbb{C})
\]

by the isomorphism (1.3). Let $\overline{U}$ be the complex conjugate of $U$ with respect to the real structure which $\Lambda$ defines on $K_0^\perp / K_0 = \Lambda \otimes \mathbb{C}$. Then (1.5) identifies $\overline{U}$ with $H^{0,1}(X)$. Finally, let $Z \subset K_0^\perp$ to be the pre-image of $\overline{U}$ with respect to the projection

\[
K_0^\perp \to K_0^\perp / K_0.
\]

It is easy to see that $Z \cap \mathcal{H}'_+ = 0$, $\mathcal{H}' = Z \oplus \mathcal{H}'_+$, and that $Z$ is a maximal isotropic subspace of $\mathcal{H}'$.

To summarize, we have constructed an extended HS $(Z, K_0, \Lambda)$ out of the data $(X, p, u)$. 

19
5 Infinitesimal equivariance of the period map

We will work with a miniversal deformation $\pi : X \to S$ of a complete smooth curve $X$ of genus $g \geq 2$, as in [3], with a sufficiently small contractible open Stein manifold $S$ as its base.

It was shown in Theorem 2.5 that $S$ is an infinitesimally homogeneous space for $d$. Consider the usual and the extended period maps on $S$:

\[
\begin{align*}
S & \xrightarrow{\Phi} \hat{A}_g \\
\Phi & \downarrow j \quad \downarrow \\
U & \subset D = H_g.
\end{align*}
\]

(19)

Let $U$ be a neighborhood of $\Phi(0)$ in $D$ containing the image of $S$; we assume that $U$ is small enough to admit lifts to $\hat{A}_g$. Choose the lift $j$ making the diagram commutative (i.e. $j \circ \Phi = \hat{\Phi}$ on $S$). This makes $U$ an infinitesimally homogeneous space for $\mathfrak{sp}(\mathcal{H}')$.

**Definition 5.1** $\varphi : d \rightarrow \mathfrak{sp}(\mathcal{H}')$ is the Lie-algebra homomorphism given by

\[
f \frac{d}{dz} \mapsto \{g \mapsto fg' \quad \forall g \in \mathcal{H}' \}.
\]

Using the identification $\mathfrak{sp}(\mathcal{H}') \cong \hat{S}^2(\mathcal{H}')$ (see Section [3]), the map $\varphi$ may also be written as

\[
\varphi : d \mapsto \hat{S}^2(\mathcal{H}')
\]

\[
z^{k+1} \frac{d}{dz} \mapsto \frac{1}{2} \sum_{j \in \mathbb{Z} \setminus \{0\}} z^{-j} z^{j+k}.
\]

We note that $\varphi$ is an irreducible representation of the Witt algebra on $\mathcal{H}'$, described in [4], (1.2), where it is denoted $V'_{0,0}$.

The following is an adaptation of a theorem of Arbarello and De Concini [3].

20
Theorem 5.2 The period map $\Phi : S \to U \subset D$ is infinitesimally equivariant, i.e. there exists a commutative diagram

$$
\begin{array}{ccc}
d & \xrightarrow{\varphi} & \text{sp}(H') \\
\downarrow & & \downarrow \\
\Gamma(S, \Theta_S) & \xrightarrow{d\Phi} & \Gamma(U, \Theta_D).
\end{array}
$$

The vertical arrows are Lie algebra anti-homomorphisms, while the horizontal ones are Lie algebra homomorphisms. The vertical arrows induce surjections onto $T_tS$ (respectively, $T_HD$) for any point $t \in S$ (respectively, $H \in U \subset D$).

Remark 5.3 The vertical arrows are not unique.
6 The second differential of the period map

We continue with a miniversal deformation of $X$. Theorem 5.2 allows one to calculate the various differentials of the period map. We begin by specializing diagram (20) to $0 \in S$:

\[
\begin{array}{ccc}
d & \xrightarrow{\varphi} & \mathbf{sp}(H') \\
\downarrow & & \downarrow \\
T_0S & \xrightarrow{d_0\Phi} & T_{\Phi(0)}D \\
\end{array}
\]

A well-known theorem of Griffiths factors $d_0\Phi$ as

\[
\begin{array}{ccc}
T_0S & \xrightarrow{d_0\Phi} & T_{\Phi(0)}D \\
\kappa \downarrow \cong & & \downarrow \\
H^1(\Theta_X) & \xrightarrow{\nu} & \text{Hom}^{(s)}(H^0(\omega_X), H^1(O_X)) ,
\end{array}
\]

where $\kappa$ is the Kodaira-Spencer isomorphism, and $\nu$ is the map defined by the cup-product pairing

\[
H^1(\Theta_X) \otimes H^0(\omega_X) \xrightarrow{\cup} H^1(O_X) ,
\]

itself induced by the contraction pairing of sheaves $\Theta_X \otimes \omega_X \xrightarrow{\cup} O_X$.

Splicing (21) and (22) yields

\[
\begin{array}{ccc}
d & \xrightarrow{\varphi} & \mathbf{sp}(H') \\
\downarrow & & \downarrow \\
H^1(\Theta_X) & \xrightarrow{\nu} & \text{Hom}^{(s)}(H^0(\omega_X), H^1(O_X)) ,
\end{array}
\]

which we want to work out explicitly. As before, let $V$ be an affine open set in $X$ containing $p$, so that $X - p$ and $V$ form an affine covering of $X$. Let $\xi$ be a vector field on $V - p$ with $u(\xi) = f(z)\frac{d}{dz}$ in $d$. Let $\omega$ be a global holomorphic 1-form on $X$ with $u(\omega|_{V-p}) = dg$ for some $g \in H'$. Then the cup-product pairing (23) gives

\[
[\xi] \cup [\omega] = [-\xi \cup (\omega|_{V-p})] \in H^1(O_X) .
\]

We observe that the minus sign is built into $\rho$ (see (3.3)), and that

\[
u(\xi \cup (\omega|_{V-p})) = f \frac{d}{dz} \omega = fg' = \varphi(\frac{d}{dz}g) ,
\]

which is how (24) and, indeed, the theorem of Arbarello and De Concini (5.2) is proved.
We would like to work out an equally explicit realization of the second differential of \( \Phi \) (higher-order cases are similar). Our starting point is again Theorem 5.2. We simply pass from Lie algebras to their (reduced) enveloping algebras to obtain a commutative diagram

\[
\begin{array}{ccc}
\mathcal{U}^{(2)} d & \xrightarrow{\varphi^{(2)}} & \mathcal{U}^{(2)} \mathfrak{sp}(\mathcal{H}') \\
\Gamma(S, \Theta_{S}^{(2)}) & \xrightarrow{d_2 \Phi} & \Gamma(U, \Theta_{D}^{(2)})
\end{array}
\]

where \( \Theta^{(2)} = \mathcal{D}^{(2)}/\mathcal{O} \) stands for the second-order tangent sheaf, and \( \mathcal{U}^{(2)} \) is the notation introduced in Section [6].

Again, to be precise, the maps emanating from the upper-left corner reverse the order of products, while the remaining maps are the second-degree parts of filtered ring homomorphisms.

Restricting to \( 0 \in S \), we obtain

\[
\begin{array}{ccc}
\mathcal{U}^{(2)} d & \xrightarrow{\varphi^{(2)}} & \mathcal{U}^{(2)} \mathfrak{sp}(\mathcal{H}') \\
\lambda^{(2)} & \xrightarrow{\lambda^{(2)}} & \Gamma(S, \Theta_{S}^{(2)}) \\
\rho^{(2)} & \xrightarrow{\rho^{(2)}} & \Gamma(U, \Theta_{D}^{(2)})
\end{array}
\]

Proposition 6.1 The second tangent space of the period domain \( D \) at the point corresponding to a HS \( (H, F^\bullet) \) admits a canonical splitting

\[
T_{F}^{(2)} D = T_{F} D \oplus S^2 T_{F} D.
\]

Proof. Let \( g = \text{End}(H) \ (= \mathfrak{gl}(2g, \mathbb{C})) \), and \( s = \mathfrak{sp}(H) \) (symplectic with respect to the polarization on \( H \)). Then \( D \) is infinitesimally homogeneous under the action of \( s \), and

\[
T_{F} D \cong s^{-1,1} \cong \text{Hom}^{(s)}(F^1, H/F^1).
\]

We also have a natural surjection \( \mathcal{U}^{(2)} s \rightarrow T_{F}^{(2)} D \). Its restriction to \( \mathcal{U}^{(2)} s^{-1,1} \) is an isomorphism by reason of dimension. But \( s^{-1,1} \) is an abelian Lie algebra, i.e.

\[
\mathcal{U}^{(2)} s^{-1,1} = s^{-1,1} \oplus S^2 s^{-1,1}.
\]

\( \Box \)
In view of Proposition 6.1, $d^2_0 \Phi : T^{(2)}_0 S \to T^{(2)}_{\Phi(0)} D$ breaks up into a direct sum of two components:

**the symbol map**

$$T^{(2)}_0 S \xrightarrow{\sigma} S^2 s^{-1,1}$$

$$(\Upsilon + \sum_i Z_i \Xi_i)|_0 \mapsto d_0\Phi(Z|_0) \otimes d_0\Phi(\Xi|_0) \quad \text{(order does not matter here)},$$

where $\Upsilon, Z_i, \Xi_i \in \Gamma(S, \Theta S)$, and

**the linear part**

$$\ell : T^{(2)}_0 S \to s^{-1,1}.$$  

It is the linear part that is really interesting. A typical second-order tangent vector to $S$ at $0$, $(\Upsilon + \sum Z_i \Xi_i)|_0$, is sent by $\ell$ to

$$d_0\Phi(\Upsilon) + \sum_i \ell((Z_i \Xi_i)|_0).$$

Thus, it suffices to understand $\ell((Z\Xi)|_0)$ for $Z, \Xi \in \Gamma(S, \Theta S)$.

By surjectivity of $\lambda^{(2)}$ in (26), we may assume that the vector fields $Z$ and $\Xi$ on $S$ are the images, respectively, of some $f_1 \frac{d}{dz}$ and $f_2 \frac{d}{dz}$ in $d$, under the map $d \to \Gamma(S, \Theta S)$, whose restriction $\lambda$ is. Then (26) implies that

$$d^2_0\Phi((Z\Xi)|_0) = \rho(2) \circ \varphi(2)(f_1 \frac{d}{dz}, f_2 \frac{d}{dz}) = \rho(2)(vw)$$

where $v = \varphi(f_1 \frac{d}{dz})$ and $w = \varphi(f_2 \frac{d}{dz})$.

Now, the map $\rho(2)$ in (26) is not induced by $\rho : \text{sp}(\mathcal{H}') \to s^{-1,1}$, which was a restriction of the map, also denoted $\rho$ in (3.3).

$$\text{End}(\mathcal{H}') \to g^{-1,1}.$$  

In fact, the maps $\rho$ are not even Lie algebra morphisms. Nevertheless, there is a way to reduce $\rho(2)$ to $\rho$. This will require a more detailed understanding of the infinitesimal action of $\text{sp}(\mathcal{H}')$; in fact, we need to work out how the group $Sp(\mathcal{H}') \subset \text{Aut}(\mathcal{H}')$ acts on a neighborhood of a point in $\widehat{A}_g$.

Since any element in the group $\text{Aut}(\mathcal{H}')$ may be written as $I + \alpha$, where $\alpha \in \text{End}(\mathcal{H}')$, we have the following map from $\text{Aut}(\mathcal{H}')$ to $\text{Aut}(\mathcal{H})$:

$$A \mapsto I - \rho(\alpha), \text{ where } I + \alpha = A^{-1}.$$  

This map will be denoted $R$. So

$$R(A) = I - \rho(A^{-1} - I).$$  

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Caution: $R$ is not a group homomorphism.

Let $(H, F^\bullet_t)$ be the Hodge structure corresponding to a point in $U$ near $\Phi(0)$. The HS $(H, F^\bullet_t)$ comes from an extended HS $(\mathbb{Z}, K_0^0, K_0^1, \Lambda_t)$. The assumption that $U$ is small and infinitesimally homogeneous under $\mathfrak{sp}(\mathcal{H}^\prime)$ implies that there exists $A_t \in \mathfrak{sp}(\mathcal{H}^\prime)$ such that $K_0^0$ and $Z_t$ are images under $A_t$ of $K_0$ and $Z$, respectively (we refer to the components of the extended HS corresponding to $\Phi(0)$).

**Lemma 6.2** In this situation $F^1_t = R(A_t)F^1$.

**Proof.** We wish to compare the Hodge structures

\[(H = K_0^\perp / K_0, F^1 = K_0^\perp \cap \mathcal{H}^\prime_+)\]

and

\[(H_t = K_0^\perp / K_0^t, F^1_t = K_0^\perp \cap \mathcal{H}^\prime_+).\]

To do so, we identify $H_t$ with $H$ by $A_{-1}t$. Then the comparison involves two subspaces of $H$: $F^1 = K_0^\perp \cap \mathcal{H}^\prime_+$ and

\[A_{-1}t^{-1}F^1_t = A_{-1}t^{-1}(K_0^\perp) \cap A_{-1}t^{-1}(\mathcal{H}^\prime_+) = K_0^\perp \cap A_{-1}t^{-1}(\mathcal{H}^\prime_+).\]

We regard $U$ as a subset of the Grassmannian

\[Grass(F^1, H) = \text{Aut}(H)/\{A | A(F^1) \subseteq F^1\}\]

Any element of $\text{Aut}(H)$ may be written as $I + T$ for some $T \in \mathfrak{g}$, and if $I + T \in \{A | A(F^1) \subseteq F^1\}$, then $T \in \mathfrak{g}^{0,0}$. If some $I + T \in \text{Aut}(H)$ moves $F^1$ to $A_{-1}t^{-1}F^1_t$, then so does $I + T^{-1,1}$, where the subscript refers to the $(-1, 1)$-component of $T$ under the direct sum decomposition $\mathfrak{g} = \mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{0,0} \oplus \mathfrak{g}^{1,-1}$. Thus we only need to find the map

\[T^{-1,1} : H^{-1,0} = \mathcal{H}^\prime_+ \cap K_0^\perp \rightarrow \mathcal{H}^\prime/\mathcal{K}^\prime_+ + K_0 \cong H^{0,1}\]

which measures deviation of $A_{-1}t^{-1}F^1_t$ from $F^1$. The above formulas for $F^1$ and $A_{-1}t^{-1}F^1_t$ show that $T^{-1,1}$ is induced by

\[A_{-1}^{-1} : \mathcal{H}^\prime_+ \rightarrow \mathcal{H}^\prime/\mathcal{H}^\prime_+\]

But if $A_{-1}^{-1} = I + \alpha$, then

\[\alpha : \mathcal{H}^\prime_+ \rightarrow \mathcal{H}^\prime/\mathcal{H}^\prime_+\]
induces the same $T^{-1,1}$. Recalling the definition of $\rho$ (3.3), this says that $T^{-1,1} = -\rho(\alpha)$. It remains to consult the definition of $R$ (28) and to abuse notation by putting $F^1_t = A_t^{-1}F^1_t$. \hfill \Box

We are now able to establish the principal formula relating the two infinitesimal uniformizations of $U$ on the second-order level.

**Lemma 6.3** Let $v, w \in \text{End}(\mathcal{H}')$. Then

$$\rho^{(2)}(vw) = \rho(v)\rho(w) - \rho(w \circ v),$$

where $w \circ v$ denotes the composition law in $\text{End}(\mathcal{H}')$.

**Proof.** Let $V = \rho(v)$, $W = \rho(w)$. $V$ and $W$ are vectors in $T_{\Phi(0)}D \cong s^{-1,1}$. For any $Z \in s^{-1,1}$ we will write $\tilde{Z}$ to denote the vector field on $U$ corresponding to $Z$ under the Lie algebra homomorphism $s^{-1,1} \rightarrow \Gamma(U, \Theta_D)$.

In particular, $\tilde{V}|_{\Phi(0)} = V$, $\tilde{W}|_{\Phi(0)} = W$. Let $f$ be any smooth function on $U$. Then

$$\rho^{(2)}(vw)f =$$

$$= \frac{\partial^2}{\partial t \partial s}|_0 f\{R(\exp tv \circ \exp sw)\Phi(0)\}$$

$$= \frac{\partial^2}{\partial t \partial s}|_0 f\{[I - \rho(\exp(-sw) \circ \exp(-tv) - I)]\Phi(0)\}$$

$$= \frac{\partial^2}{\partial t \partial s}|_0 f\{[I - \rho(-sw - tv + ts(w \circ v + \ldots))]\Phi(0)\}$$

$$= \frac{\partial^2}{\partial t \partial s}|_0 f\{(I + tV + sW - tsp(w \circ v) + \ldots)\Phi(0)\}$$

$$= \frac{d}{dt}|_0 \left\{ \frac{\partial}{\partial s}|_0 f\{(I + tV + o(t)) + s(W - t\rho(w \circ v) + o(t)) + o(s)\Phi(0)\} \right\}$$

$$= \frac{d}{dt}|_0 \left\{ [\tilde{W} - t(\rho(w \circ v))^\sim + o(t)]f\{(I + tV + o(t))\Phi(0)\} \right\}$$

$$= \frac{d}{dt}|_0 \left\{ [\tilde{W}f]\{(I + tV + \ldots)\Phi(0)\} - t[(\rho(w \circ v))^\sim f]\{(I + tV + \ldots)\Phi(0)\} + o(t) \right\}$$

$$= V\tilde{W}f - \rho(w \circ v)f.$$
Observe that $\rho$ takes its values in $g^{-1,1}$, which is an abelian Lie algebra. Thus the lemma gives a splitting of

$$\rho^{(2)} : \mathcal{T}^{(2)} \text{End}(\mathcal{H}') \rightarrow \mathcal{T}^{(2)} g^{-1,1} = g^{-1,1} \oplus S^2 g^{-1,1} :$$

$V\tilde{W} = \rho(v)\rho(w)$ is purely quadratic (=the symbol part), and $-\rho(w \circ v)$ is the linear part. Going back to (27), this implies that $\ell((Z\Xi)|_0) = -\rho(w \circ v)$, which proves the following

**Theorem 6.4** If $Z, \Xi \in \Gamma(S, \Theta_S)$ lift to $f_1 \frac{d}{dz}, f_2 \frac{d}{dz} \in d$, then the linear part of $d_0^2 \Phi$,

$$\ell : T^{(2)}_0 S \longrightarrow T_{\Phi(0)} D = s^{-1,1},$$

sends $(Z\Xi)|_0$ to the negative of the image under $\rho : \text{End}(\mathcal{H}') \rightarrow g^{-1,1}$ of the composition in $\text{End}(\mathcal{H}')$ of $\varphi(f_1 \frac{d}{dz})$ and $\varphi(f_2 \frac{d}{dz})$, in reverse order:

$$g \mapsto f_2f'_1g' + f_1f_2g''.$$  

(29)

**Remark 6.5** A composition (in $\text{End}(\mathcal{H}')$) of two elements of $\mathfrak{sp}(\mathcal{H}')$ need not be in $\mathfrak{sp}(\mathcal{H}')$. In particular, it is not a priori obvious that the image of $\ell$ is in

$$\text{Hom}^{(s)}(H^0(\omega_X), H^1(\mathcal{O}_X)) \cong s^{-1,1}.$$  

There is a better-known object which carries part of the information contained in the linear part $\ell$ of the period map’s second differential. It is the second fundamental form of the VHS of [CGGH], the map

$$\Pi : T^{(2)}_0 S/T_0 S = S^2 T_0 S \longrightarrow T_{\Phi(0)} D/im(d_0 \Phi)$$

induced by $\ell$.

**Theorem 6.6** The prescription (29) for computing $\ell$ gives a formula for $\Pi$, which coincides with that in [K1], §6:

$$\Pi(Z \otimes \Xi) = \{\omega \mapsto \xi \rightharpoonup \zeta \omega \} \mod im(d_0 \Phi)$$

(30)

for any $Z, \Xi \in T_0 S$ with KS representatives $\zeta, \xi \in \Gamma(V - p, \Theta_X)$, and $\omega \in H^0(X, \omega_X)$.  

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Proof. Recall that a choice of a point \( p \) on the curve \( X \) and a local parameter \( z \) near \( p \) allows one to represent \( \omega \in H^0(X, \omega_X) \) by some \( g \in \mathcal{H}' \) with \( dg = \omega \) near \( p \). The vectors \( Z \) and \( \Xi \) are the images under \( \rho \) of some \( f_1 \frac{d}{dz} \) and \( f_2 \frac{d}{dz} \) in \( d \), i.e. \( f_1 \frac{d}{dz} \) and \( f_2 \frac{d}{dz} \) are the Laurent expansions at \( p \) of \( \zeta \) and \( \xi \), respectively. Working out (30) in terms of \( z \), using the formula for the Lie derivative

\[ \mathcal{L}_\zeta \omega = d\zeta \omega + \zeta d\omega , \]

easily yields \( (29) \pmod{\text{im}(d_0\Phi)} \), as was already done, in fact, in [K1], §6.

\[ \square \]

Remark 6.7 Thus the prescription for \( \Pi \) given in [K1] turns out to be well-defined for second-order differential operators on \( S \) and not just their symbols, and the values given by that prescription are not merely equivalence classes modulo \( \text{im}(d_0\Phi) \).
7 Relation with the second Kodaira-Spencer class

As explained at the beginning of the previous section, the first differential of the period map is given by cup product with the (first) Kodaira-Spencer class \( \kappa = \kappa_1 \) of the deformation. In [K2] we have shown that \( \Pi \) depends only on the second Kodaira-Spencer class \( \kappa_2 \) (more precisely, on \( \kappa_2 \mod \text{im}(\kappa_1) \)) introduced recently in [BG], [EV] and [R1]. In this section we will explain, in the case of curves, how the full second differential

\[
d^2 \Phi : T_0^{(2)} S \to T_{\Phi(0)}^{(2)} D
\]

factors through the second KS mapping

\[
\kappa_2 : T_0^{(2)} S \to T_X^{(2)}
\]

Let us recall first the construction of \( T_X^{(2)} \), the space of second-order deformations of \( X \). Our reference is [R1] or [K2].

Let \( X_2 \) denote the symmetric product of the curve \( X \) with itself; write

\[
g : X \times X \to X_2
\]

for the obvious projection map, and \( i : X \hookrightarrow X_2 \) for the inclusion of the diagonal. Then \( T_X^{(2)} = H^1(X_2, \mathcal{K}^\bullet) \), where \( \mathcal{K}^\bullet \) is the sheaf complex on \( X_2 \)

\[
\begin{array}{c}
-1 \\
0 \\
(g_*(\Theta_X^{(2)}))^- & \downarrow \\
i_*\Theta_X
\end{array}
\]

Here \( \boxtimes \) stands for the exterior tensor product on \( X \times X \), \( (\ )^- \) denotes anti-invariants of the \( \mathbb{Z}/2\mathbb{Z} \)-action, and the differential is the restriction to the diagonal followed by the Lie bracket of vector fields.

In practice it seems easier to do the following. Letting \( C^\bullet \) denote the Čech cochain complex \( C^\bullet(\mathcal{U}, \Theta_X) \) of \( \Theta_X \) with respect to an affine covering \( \mathcal{U} \) of \( X \), one may compute \( T_X^{(2)} \) as the cohomology of the simple complex associated to the double complex

\[
\begin{array}{cccc}
2 & (C^1 \otimes C^1)^{(s)} & \downarrow & C^1 \\
1 & (C^0 \otimes C^1 + C^1 \otimes C^0) & \uparrow +\delta & C^1 \\
0 & (C^0 \otimes C^0) & \uparrow -\delta & C^0 \\
\end{array}
\]

Here \( \delta \) is the exterior derivation on \( \Theta_X \).
The superscripts \((s)\) and \(−\) denote the invariants and the anti-invariants, respectively, of the \(\mathbb{Z}/2\)-action.

Working with \(\mathcal{U} = \{X − p, V\}\) and using Laurent expansions at \(p\), we may follow the proof of Lemma 2.3 and replace \(C^1\) with \(d\) and \(C^0\) with the completion of its image in \(d \oplus d\). The resulting bicomplex still computes \(T_X^{(2)}\). In particular, \(T_X^{(2)}\) is a quotient of \(d \oplus (d \otimes d)^{(s)}\).

**Lemma 7.1** Assume the vector fields \(Z\) and \(Ξ\) on \(S\) are the images of \(ζ, \xi \in d\) under the infinitesimal uniformization map \(\lambda: d \rightarrow Γ(S, Θ_S)\). Then
\[
\frac{1}{2}(\langle ξ, ζ \rangle + ζ \otimes ξ + ξ \otimes ζ) \in d \oplus (d \otimes d)^{(s)}
\]
is a representative for \(κ_2((ZΞ)|_0) \in T_X^{(2)}\).

**Proof.** The Kodaira-Spencer maps are compatible with the symbol map \(T_X^{(2)} \rightarrow S^2 T_X^1\) in the sense that there is a commutative diagram
\[
\begin{array}{ccc}
T_0^{(2)}S & \xrightarrow{κ_2} & T_X^{(2)} \\
\downarrow & & \downarrow \\
S^2T_0S & \xrightarrow{κ_2^1} & S^2 T_X^1.
\end{array}
\]
Thus it is natural to look for a representative of \(κ_2((ZΞ)|_0)\) of the form
\[
θ + \frac{1}{2}(\langle ξ, ζ \rangle + ξ \otimes ζ + ζ \otimes ξ) \in d \oplus (d \otimes d)^{(s)}
\]
for some \(θ \in d\). The construction of \(κ_2\) as the connecting morphism in a certain long exact sequence ([EV, R1, R2]), presented more explicitly in [K2], offers the following way to determine \(θ\). Working with a covering \(\mathcal{W} = \{W_0, W_1\}\) of \(X\) as in the proof of Lemma 2.4, and using the sub-sheaf \(Θ_X\) of \(Θ_X\) introduced in ([8]), let \(ζ_0, ζ_1\) be lifts of \(Z\) to \(Γ(W_0, \tilde{Θ}_X), Γ(W_1, \tilde{Θ}_X)\). Write \(\tilde{ζ}\) to denote \(ζ_0 + ζ_1\) viewed as a cochain in \(ˇC^1(\mathcal{W}, \tilde{Θ}_X)\).

A slight modification of the proof of Prop. 2 in [K2] shows that \(θ\) should be cohomologous (in \(ˇC^1(\mathcal{W}, Θ_X/S)\)) to
\[
\frac{1}{2}(\tilde{ζ}, ξ + [ξ, \tilde{ζ}]) = \frac{1}{2}(\langle ξ_0, ξ \rangle + [ξ, ζ_1]) = \frac{1}{2}[ξ, ζ_1 − ζ_0].
\]

But \(ζ_1 − ζ_0\) is cohomologous to \(ζ\). Hence we can take \(θ = \frac{1}{2}[ξ, ζ]\). \(\Box\)
In (6.1) we explained how $T^{(2)}_{\Phi(0)}D$ splits into $T_{\Phi(0)}D \oplus S^2T_{\Phi(0)}D$, with the second differential of the period map breaking up accordingly:

$$d^2_0\Phi = \ell \oplus \sigma.$$

The symbol part factors through the square of the first KS class:

$$\begin{array}{ccc}
T^{(2)}_0S & \xrightarrow{\sigma} & S^2T_{\Phi(0)}D \\
\downarrow & & \uparrow (d_0\Phi)^2 \\
S^2T_0S & \xrightarrow{\kappa^2} & S^2T_X = S^2\text{Hom}^{(s)}(H^0(\omega_X), H^1(O_X)) .
\end{array}$$

This diagram may be directly obtained from (22) and carries no additional information.

Now to the linear part $\ell$ of $d^2_0\Phi$. “Recall” the canonical bijection $b$ given by the composition of the obvious maps

$$b : d \oplus (d \otimes d)^{(s)} \leftrightarrow d \oplus (d \otimes d) \longrightarrow \mathcal{T}^{(2)}d .$$

**Lemma 7.2** The canonical bijection $b$ fits in the commutative square

$$\begin{array}{ccc}
\mathcal{T}^{(2)}d & \xleftarrow{b} & d \oplus (d \otimes d)^{(s)} \\
\downarrow & & \downarrow \\
T^{(2)}_0S & \xrightarrow{\kappa^2} & T^{(2)}X
\end{array}$$

with bijective horizontal arrows, and surjective vertical ones.

**Proof.** It suffices to show that if $\zeta, \xi \in d$ lift the vector fields $Z$ and $\Xi$ on $S$, then $b^{-1}(\zeta \xi)$ lifts $\kappa_2((\Xi Z)|_0)$ under the projection

$$d \oplus (d \otimes d)^{(s)} \longrightarrow T^{(2)}X .$$

In other words, we must verify that $b^{-1}(\zeta \xi)$ is a KS representative for $(\Xi Z)|_0 \in T^{(2)}_0 S$. From the definition of $b$ it easily follows that

$$b^{-1}(\zeta \xi) = \frac{1}{2}([\xi, \zeta] + \zeta \otimes \xi + \xi \otimes \zeta) .$$
This, together with Lemma 7.1, implies our statement. □

**Definition 7.3** We define \( \nu_2 : T_X^{(2)} \to T_{\Phi(0)}D = \text{Hom}^{(s)}(H^0(\omega_X), H^1(\mathcal{O}_X)) \) as the composition

\[
T_X^{(2)} \xrightarrow{\kappa_2^{-1}} T_0^{(2)} S \xrightarrow{\ell} T_{\Phi(0)}D.
\]

Thus we have a commutative triangle

\[
\begin{array}{ccc}
T_0^{(2)} S & \xrightarrow[\nu_2]{\ell} & T_{\Phi(0)}D \\
\searrow & & \nearrow \\
& T_X^{(2)} & \\
\end{array}
\]

(31)

**Theorem 7.4** \( \nu_2 : T_X^{(2)} = H^1(K^\bullet) \to \text{Hom}(H^0(\omega_X), H^1(\mathcal{O}_X)) \) is induced by the pairing

\[
(\tilde{C}^1(\Theta_X) \oplus \tilde{C}^1(\Theta_X)) \times (\tilde{C}^0(\omega_X) \times \omega) \to \tilde{C}^1(\mathcal{O}_X),
\]

(32)

defined on the Čech cochain level by the coupling

\[
(\zeta \otimes \xi + \nu)\otimes \omega \mapsto -\zeta \otimes \xi \otimes \omega - \nu \otimes \omega.
\]

(33)

**Proof.** It suffices to study the effect of \( \nu_2 = \ell \circ \kappa_2^{-1} \) on an element \( x \) of \( T_X^{(2)} \) represented by

\[
\frac{1}{2}(\zeta \otimes \xi + \xi \otimes \zeta) + \nu \in (\mathbf{d} \otimes \mathbf{d})^{(s)} \oplus \mathbf{d}.
\]

According to Lemma 7.1,

\[
\kappa_2^{-1}(x) = (Z \Xi + \Upsilon - \frac{1}{2}[Z, \Xi])|_0,
\]

where \( Z, \Xi \) and \( \Upsilon \) are the images of \( \zeta, \xi \) and \( \nu \), respectively, under the uniformization map \( \lambda : \mathbf{d} \to \Gamma(S, \Theta_S) \). Note that \( \lambda([\zeta, \xi]) = -[Z, \Xi] \).

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As explained in the previous section, \( \ell ((Z \Xi + Y - \frac{1}{2}[Z, \Xi])|_0) \) is a map \( H^0(\omega_X) \rightarrow H^1(O_X) \) given by

\[
\omega \mapsto \xi \llcorner \mathcal{L}_\zeta \omega - (v - \frac{1}{2} [\zeta, \xi]) \llcorner \omega .
\]

However, Cartan’s identity gives

\[
\mathcal{L}_\zeta (\xi \llcorner \omega) - \xi \llcorner \mathcal{L}_\zeta \omega = [\zeta, \xi] \llcorner \omega ,
\]

and on a curve \( \mathcal{L}_\zeta (\xi \llcorner \omega) = \zeta \llcorner \mathcal{L}_\zeta \omega . \) Hence the right-hand side of \( (34) \) equals

\[
\frac{1}{2} (\xi \llcorner \mathcal{L}_\zeta \omega + \zeta \llcorner \mathcal{L}_\zeta \omega) - v \llcorner \omega .
\]

\[
\square
\]

**Corollary 7.5** The second fundamental form of the VHS, II, factors through \( S^2T_X \).

This fact was already proved in complete generality (for a deformation of any compact Kähler manifold) in [K2], using Archimedean cohomology.

We conclude with a diagram summarizing the relationships between some of the maps discussed in this section:

\[
\begin{array}{cccccc}
T_0^{(2)}S & \xrightarrow{\kappa_2} & T_X^{(2)} & \xrightarrow{\nu_2} & T_{\Phi(0)}D & \\
\downarrow & & \downarrow & & \downarrow & \\
S^2T_0S & \xrightarrow{\kappa_1^2} & S^2T_X & \xrightarrow{\nu_2/\text{im} \nu_1} & T_{\Phi(0)D/\text{im} \nu_1} & \\
\end{array}
\]

\[
\text{II}
\]

**8 The higher-order case**

We have the following analogues of the results in sections 6 and 7. The proofs, which are notationally cumbersome transcriptions of the \( n = 2 \) case, are omitted.

**Proposition 8.1** The \( n \)th tangent space of the period domain \( D \) at a point corresponding to a HS \((H, F^*)\) admits a canonical splitting

\[
T_F^{(n)}D = T_FD \oplus S^2T_FD \oplus \ldots \oplus S^nT_FD .
\]
The \( n^{th} \) differential of the period map splits accordingly:

\[
d_n^0 \Phi = \ell_1^{(n)} + \ldots + \ell_n^{(n)}.
\]

E.g. what we called \( \ell \) and \( \sigma \) earlier are \( \ell_1^{(2)} \) and \( \ell_2^{(2)} \), respectively.

Thus it suffices to describe the \( k^{th} \) component of \( d_n^0 \Phi \),

\[
\ell_k^{(n)} : T_0^{(n)} D \to S^k T_{\Phi(0)} D.
\]

**Theorem 8.2** If \( Z_1, \ldots, Z_n \in \Gamma(S, \Theta_S) \) lift to \( \zeta_1, \ldots, \zeta_n \in d \), then

a) \( \ell_1^{(n)} \) sends \( (Z_1 \ldots Z_n)_0 \) to \( (-1)^{n-1} \) times the image under \( \rho : \text{End}(H') \to g^{-1,1} \) of the composition in \( \text{End}(H') \) of \( \varphi(\zeta_1), \ldots, \varphi(\zeta_n) \) in reverse order;

b) \( \ell_k^{(n)} \) is the sum, over all partitions of \( k \), of the symmetrized tensor products

\[
\bigotimes \sum_{i, p_i = k} \ell_i^{(p_i)};
\]

c) \( d_n^0 \Phi \), as well as each \( \ell_k^{(n)} \), factors through \( T_X^{(n)} \).

We may add to (a) that in terms of the covering \( \{V, X - p\} \) of \( X \) as above, \( \ell_1^{(n)}((Z_1 \ldots Z_n)_0) \) can be also described as follows: it is a map

\[
H^0(X, \omega_X) \to H^1(X, \mathcal{O}_X)
\]

sending the class represented by a form \( \omega \) on \( V \) to the class represented by the function

\[
(-1)^n \zeta_n \rightarrow \mathcal{L}_{\zeta_{n-1}} \ldots \mathcal{L}_{\zeta_1} \omega
\]

on \( V - p \). Here we assume that the lifts \( \zeta_i \in d \) of \( Z_i \in \Gamma(S, \Theta_S) \) converge and define regular vector fields on \( V - p \).
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