Application of the Shannon-Kotelnik theorem on the vortex structures identification

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Abstract. This paper presents a decomposition of unsteady vector fields, based on the principle of Shannon-Kotelnik theorem. The decomposition is derived from the Fourier transform of the Kotelnik series. The method can be used for the analysis of both forced and self-excited oscillation.

Nomenclature

- general time dependent vector
- \( F(\omega) \) – Fourier transform of function \( w(t) \)
- \( \Omega, \omega \) – angular velocity
- \( f \) – frequency
- \( \nu_i \) – i-th velocity vector coordinate
- \( x_i \) – independent variable
- \( t \) – time
- \( p \) – pressure function
- \( \nu \) – sound velocity
- \( \rho \) – density
- \( \eta \) – dynamic viscosity
- \( b \) – second viscosity
- \( g \) – gravity acceleration
- \( s_i \) – i-th eigenvalue

1. Introduction

The solution of systems of linear, time-dependent differential equations can be done for example by the expansion of eigenmodes. This gives not only an idea of its eigenfrequencies, but also of the possible causes of instability origins. This method also allows the study of the formation and disappearance of various vortex structures [1].

The method of eigenmode expansion cannot be used for the analysis of nonlinear problems. In this case other methods are used, such as singular decomposition on the principle of POD [2] - [7].

When the time dependence is expressed as the Kotelnik series, the resulting vector fields can again be decomposed into the characteristic mode shapes which are in linear systems consistent with eigenmode shapes [8].

2. Shannon-Kotelnik theorem [1], [2], [9], [11]

Let us consider the vector field given by a vector \( w = w(t) \). This time function may depend on a large number of frequencies (e.g. for turbulent flow of a liquid). Experience has shown that only a certain number of relatively low frequencies have a practical significance. Therefore, it is appropriate to filter out the higher frequencies from the vector \( w \) and delimit the solution by the frequency spectrum \( f \). This simplification is allowed by the Kotelnik series [9], [11], which approximates the function \( w(x, t) \) in the final interval \( j = n = 2fT \) by the vector \( u(t) \) as:

\[
 u(t) = \sum_{j=-n}^{n} w \left( j \frac{2f}{2f} \right) \sin[n(2ft - j)] \frac{1}{\pi(2ft - j)}. \quad (2.1)
\]
In points $t_j = \frac{j}{2f}$, this series regains the values:

$$\mathbf{u}(t = t_j) = \mathbf{w}(t = t_j).$$

(2.2)

The Fourier transform [9], [3] in the form below corresponds to the series (2.1):

$$F(\omega) = \frac{1}{2f} \sum_{j=-n}^{n} \mathbf{w}\left(\frac{j}{2f}\right) e^{-i\omega\frac{j}{2f}}.$$  

(2.3)

$$F(\omega > 2\pi f) = 0.$$  

(2.4)

3. Application on the flow of the liquid [11], [12]

Consider the Navier-Stokes equation for a compressible liquid in the form [10]:

$$\frac{\partial \mathbf{v}_i}{\partial t} + \frac{\partial \mathbf{v}_i}{\partial x_j} v_j - \frac{1}{\rho} \frac{\partial \Pi_{ij}}{\partial x_j} + \frac{\partial p}{\partial x_i} = \rho \mathbf{g}_i.$$  

(3.1)

$$\Pi_{ij} = 2\eta v_{ij} + b\delta_{ij}v_{kk}$$  

(3.2)

and the continuity equation:

$$\frac{\partial p}{\partial t} + \frac{\partial p}{\partial x_i} v_i + \rho \nu^2 \frac{\partial \mathbf{v}_i}{\partial x_i} = 0.$$  

(3.3)

For the eq. (3.1), (3.3), it is necessary to specify, according to the task type, the initial and boundary conditions.

In the finite-dimensional space, the following equations, given that they are non-linear, will have the form:

$$A \frac{\partial \mathbf{v}}{\partial t} + B(\mathbf{v})\mathbf{v} + R\mathbf{p} = \mathbf{f}$$  

(3.4)

$$H \frac{\partial \mathbf{p}}{\partial t} + D(\mathbf{p}, \mathbf{v})\mathbf{v} = \mathbf{n}.$$  

(3.5)

In case we are interested in the beginning of the steady flow instability, characterized by the stationary velocity vector $\mathbf{v}_s = \mathbf{v}_s(\mathbf{x})$, and the stationary pressure $\mathbf{p}_s$, the system (3.4), (3.5) can be linearized on the basis of decomposition.

$$\mathbf{v} = \mathbf{w}(t) + \mathbf{v}_s; \quad \mathbf{p} = \mathbf{p}_s + \mathbf{\sigma}(t)$$  

(3.6)

$$\mathbf{f} = \mathbf{f}_0 + \mathbf{\varphi}(t); \quad \mathbf{n} = \mathbf{n}_0 + \mathbf{\psi}(t).$$  

(3.7)

Using this decomposition and neglecting small nonlinear members, it is possible to write the mathematical model (3.4), (3.5) in the form [8]:

$$A \frac{\partial \mathbf{w}}{\partial t} + B(\mathbf{v}_s)\mathbf{w} + R\mathbf{\sigma} = \varphi$$  

(3.8)

$$H \frac{\partial \mathbf{\sigma}}{\partial t} + D(\mathbf{v}_s, \mathbf{p}_s)\mathbf{w} = \mathbf{\psi}.$$  

(3.9)
When a homogeneous system is associated with eq. (3.8), (3.9) and the modal matrices of the original, $X$, and associated, $Y$, systems are implemented, the solution of eq. (3.8), (3.9) can be written in the form:

$$
\mathbf{w} = X \Gamma(t) Y^* \mathbf{Na} + \int_{0}^{t} \Gamma(t - \tau) Y^* \mathbf{g}(\tau) \, d\tau.
$$

(3.10)

$\Gamma$ is a diagonal matrix with elements $e^{s_i t}, s_i = \alpha_i + i\omega_i$.

If at least one of $\alpha_i > 0$, the system is unsteady. Symbol $\omega_i$ represents the $i$-th eigen angular velocity.

If the following matrices and vectors are introduced [8]:

$$
\mathbf{N} = \begin{bmatrix} 0 & A \\ H & 0 \end{bmatrix} ; \quad \mathbf{P} = \begin{bmatrix} R & B \\ 0 & D \end{bmatrix}
$$

(3.11)

$$
\mathbf{g}^T = (\varphi^T, \Psi^T) ; \quad \mathbf{a}^T = \mathbf{w}(0).
$$

(3.12)

and for simplification, the excited function is assumed to be in the form:

$$
\mathbf{g} = g_0 e^{i\Omega t}
$$

(3.13)

considering that

$$
\mathbf{z}^T = \left( \begin{array}{c} \frac{v_1}{i\Omega - S_1} \\ \cdots \\ \frac{v_k}{i\Omega - S_k} \\ \cdots \\ \frac{v_N}{i\Omega - S_N} \end{array} \right)
$$

(3.14)

in the case of a stable system and after the damping of the transient process, eq. (3.10) can be written as:

$$
\mathbf{w} = X \mathbf{z} e^{i\Omega t} ; \quad X = (x_1, \ldots, x_k, \ldots, x_N).
$$

(3.15)

If $\alpha_k$ is in absolute value small and $\Omega = \omega_k$, therefore in resonance, it holds with acceptance of eq. (3.15), (3.14) that:

$$
\mathbf{w} = -\frac{v_k}{\alpha_k} x_k e^{i\Omega t}
$$

(3.16)

From here, it is visible that the system oscillates in resonance at the $k$-th eigen waveform. When $\Omega \neq \omega_k$, the forced waveform is different from the actual waveform. For the forced waveform due to (3.15), in general it holds:

$$
x = X \mathbf{z}.
$$

(3.17)

4. The spectral decomposition using the Fourier transform of the Kotelnik series [2], [8], [9]

In the previous chapter, the solution of a linear system excited by one harmonic function has been introduced.

The solution can be generalized to excitation by more harmonic functions with analogous results.
Another generalization can be also used for non-linear systems, where the self-excited oscillation happens at a particular frequency spectrum. This problem can be in a chosen time period described by the function in the form:

\[ w(t) = w_0 e^{i\Omega t}. \] (4.1)

Fourier transform of this function, expressed by the Kotelnik series is in the form [1], [9], [2]:

\[ F(\omega) = \frac{1}{2\pi} \sum_{j=0}^{n-1} w_0 \left( j \frac{\Omega}{2f} \right) e^{-i\omega \frac{j}{2f}} \]

and considering

\[ F(\omega) = \frac{1}{2\pi} \sum_{j=0}^{n-1} \bar{w}_0 e^{-i(\Omega-\omega) \frac{j}{2f}}. \] (4.2)

After an adjustment, summing the series in the previous expression, a simple relation applies:

\[ F(\omega) = \frac{1}{2\pi} \bar{w}_0 \left[ 1 + \frac{\sin \left( \frac{n-1}{2f} \right) e^{\Omega} - \omega \right] \sin \left( \frac{1}{2f} \right) e^{\frac{n-1}{2f} \omega} \right]. \] (4.3)

If we put \( \Omega = \omega \), from eq. (4.3) can be derived:

\[ F(\omega) = \frac{n}{2\pi} \bar{w}_0. \] (4.4)

Comparison of (4.1) and (3.15) shows that

\[ \bar{w}_0 = Xz = x \] (4.5)

From there it follows that the Fourier transform of the vector \( w \) defines the characteristic shape of the self-excited or forced oscillation that corresponds to the frequency of self-excited or forced oscillations.

A. One-dimensional flow of a compressible liquid in a tube, induced by a pressure gradient [10]

Recordings of the pressure function and its Kotelnik series are shown in Fig. 1 for the selected parameters: \( f = 500Hz \), \( T = 5s \) [10].

The detail of the Kotelnik function compared to the original function is shown in Fig. 2; and its Fourier transform in Fig. 3 [11], [12].

![Fig. 1](image1.png)

A graph of the pressure function and the Kotelnik series
Fig 4 shows the eigenmode shapes of the oscillation that correspond to the eigenfrequencies by Fourier transform shown in Fig. 3.

A comparison of the actual course of the function $w$ and the Kotelnik series (detail)

A detailed view on the frequency spectrum of the pressure function

Fig 4 shows the eigenmode shapes of the oscillation that correspond to the eigenfrequencies by Fourier transform shown in Fig. 3.

The Eigenmode shapes
B. Two-dimensional flow

The time series corresponds to the runaround of the square by the incompressible liquid. The values of the pressure function behind the square were taken from the results of the computer simulation using software Fluent. The values of the pressure function were determined in nodes in accordance with mesh in Fig. 5 [5], [6], [7], [11], [12].

Fig 6 shows the values of the pressure function in a field behind the runaround body for four time steps.
The detail of Fourier transform of the Kotelnik series is shown in Fig. 7. In Fig 9, the prime waveform corresponding to Fourier transform for \( \omega = 0 \) is shown. The next Fig. 8 shows the first four mode shapes of pressure oscillation, corresponding to the Fourier transform according to Fig. 7.

**Fig. 7**
The Fourier transform of the Kotelnik series

**Fig. 8**
The first four mode shapes
From the eigenmodes of the pressure function, the formation of Karman vortex trail behind the body is obvious. Hence follows that the resonant oscillation of the body of the given shape may occur at multiple frequencies, which cannot be expected based on the Strouhal number.

C. Three dimensional flow – vortex rope in the draft tube.

Fig. 10 shows the first four time steps of the pressure function [5], [6], [7], [11], [12].
Detail of the Fourier transform of the Kotelnik function is shown in Fig. 11 and 12. The first four characteristic mode shapes are shown in Fig 13. The frequency of pressure pulsations corresponds to the points shown in Fig 12.

Fig. 11
The Fourier transform of the Kotelnik series

Fig. 12
A detail of the Fourier transform of the Kotelnik series

Fig. 13
The first four mode shapes (3rd – doubled spiral)
5. Conclusion

The Kotelnik series works with a much smaller set of time data than other conventional methods, such as methods of POD. A major advantage is that it is not necessary to solve the eigenvalue problem of large matrices. The simplicity of the proposed method therefore consists of the fact that higher frequencies than the given value \( f[Hz] \) are filtered out. This simplifies the calculation of the Fourier transform of the vector function \( w(t) \). The Fourier transform has a simple form:

\[
F(\omega) = \sum_{j=0}^{n-1} w\left( \frac{j}{T} \right) e^{-i\omega \frac{j}{2f}}; \quad F(\omega) = 0 \quad \text{pro} \quad \omega > 2\pi f. \tag{5.1}
\]

The peaks of the Fourier transform of the vector function \( w(t) \) for the selected point are determined by the characteristic frequency \( \omega_m \) of the eigen, forced or self-excited oscillation. These frequencies correspond to the eigen, forced or self-excited characteristic mode shapes \( w_m \). The average power \( S \) of the vector function \( w(t) \) can be defined in a simple relation based on the Kotelnik function in the form \([2], [8], [9]\).

\[
w_m = \sum_{j=0}^{n-1} w\left( \frac{j}{2f} \right) e^{-i\omega \frac{j}{2f}}. \tag{5.2}
\]

\[
w^T = (w_1, ... w_i, ... w_N) \tag{5.3}
\]

\[
S_i = \frac{1}{2fT} \sum_{k=1}^{2fT} w_i\left( \frac{k}{2f} \right) w_i\left( \frac{k}{2f} \right) \tag{5.4}
\]

\[
S = \frac{1}{2fT} \sum_{k=1}^{2fT} w^T\left( \frac{k}{2f} \right) w\left( \frac{k}{2f} \right) \tag{5.5}
\]

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