On the ternary complex analysis and its applications

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February 2, 2008

Abstract

Previously a possible extension of the complex number, together with its connected trigonometry was introduced. In this paper we focus on the simplest case of ternary complex numbers. Then, some types of holomorphicity adapted to the ternary complex numbers and the corresponding results upon integration of differential forms are given. Several physical applications are given, and in particular one type of holomorphic function gives rise to a new form of stationary magnetic field. The movement of a monopole type object in this field is then studied and shown to be integrable. The monopole scattering in the ternary field is finally studied.

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1 Geometric origin of complex numbers.

The complex number theory is a seminal field in mathematics having many applications to geometry, group theory, algebra and also to the classical and quantum physics. Geometrically, it is based on the complexification of the $\mathbb{R}^2$ plane. The existence of similar structures in higher dimensional spaces is interesting for phenomenological applications.

As a first step toward an extension of the complex numbers, in a series of papers mathematicians studied a three dimensional space with linear arc defined by $d^3s = d^3x + d^3y + d^3z - 3dx dy dz$. This study was initiated by two papers of P. Appell \[1\], written in 1877, where he introduced some generalisation of the usual trigonometric functions which are closely related to the linear arc defined above. Then, the geometrical properties of these three-dimensional spaces was firstly studied by P. Humbert in a series of small papers (mostly in Comptes Rendus Académie des Sciences, Paris) and then by J. Devisme. One can see for instance \[2, 3, 4, 5, 6\] and references therein. In 1933 and 1940 J. Devisme \[5, 6\] wrote very interesting reviews on the subject summarizing all the results obtained at that time (especially in the first one). However, all the results of these papers were obtained without any reference to some extension of the complex number (except in a small remark in the first paper of Appell \[1\]).

It turns out, that there is a natural way to construct a generalized version of complex numbers in $\mathbb{R}^n$ spaces ($n = 3, 4, ...$). Such numbers were introduced in \[7, 8\]. In these papers, the authors suggested to use group theoretical methods to define these numbers, and in particular, the cyclic group $C_n$ was used for the “complexification” of $\mathbb{R}^n$. The simplest case beyond the complex numbers, corresponding to $n = 3$ is called ternary complex numbers, $T_n \subset \mathbb{C}$ is the algebra generated by one canonical generator $q$ satisfying $q^3 = 1$. The consideration of these numbers allows to rediscover in a more efficient way most of the results given in \[5\]. These numbers gave also a new insight on the achievements summarized in \[5\] together with some new relations. Very fundamental results obtained within the complex number approach (Virasoro and Kac-Moody algebras in (super)string theory and integrable models, etc.) give strong motivation to study with another point of view and more details the ternary complex numbers as well as their symmetric and geometric properties. The ternary complex numbers might be related to some new symmetries in high energy physics allowing to explain the three color- \[9\] and three family-\[10\] problems of the Standard Model.

In this paper we are going to investigate new aspects of the ternary complex analysis based on the “complexification” of $\mathbb{R}^3$ space \[7, 8\]. The use of the cyclic $C_3$ group for this purpose is a natural generalization of the similar application of the $C_2 = Z_2$ group in two dimensions. It is known that the complexification of $\mathbb{R}^2$ allows to introduce the new geometrical objects - the Riemannian surfaces. The Riemannian surfaces are defined as a pair $(M, \Omega)$, where $M$ is a connected two-dimensional manifold and $\Omega$ is a complex structure on $M$. Well-known examples of Riemann surfaces are the complex plane- $\mathbb{C}$, Riemann sphere - $CP^1 = \mathbb{C} \cup \{\infty\}$, complex tori- $T = \mathbb{C}/\Gamma$, $\Gamma := n\lambda_1 + m\lambda_2 : n, m \in Z$, $\lambda_1, 2 \in \mathbb{C}$, etc.

For the ternary complexification of the vector space, $\mathbb{R}^3$, one uses its cyclic symmetry subgroup $C_3 = R_3$. \[7, 8\]. In the physical context the elements of this subgroup are actually spatial rotations through a restricted set of angles, $0, 2\pi/3, 4\pi/3$ around, for example, the $x_0$-axis. After such rotations the coordinates, $x_0, x_1, x_2$, of the point in $\mathbb{R}^3$ are linearly related with the new coordinates, $x'_0, x'_1, x'_2$ which can be realized by the $3 \times 3$ matrices. The vector
representation $D^V$ is defined through the following three orthogonal matrices:

$$R^V(1) = O(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$R^V(q) = O(2\pi/3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/2 & \sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & -1/2 \end{pmatrix},$$

$$R^V(q^2) = O(4\pi/3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & -1/2 \end{pmatrix}.$$

These matrices realize the group multiplication rules due to the relations $R^V(q^2) = (R^V(q))^2$ and $(R^V(q))^3 = R^V(1)$. The representation is faithful because the kernel of its homomorphism consists only of identity: $\text{Ker}\, R = 1$.

Let us introduce the matrix

$$\hat{x} = \sum_{i=0}^{3} x_i R^V(q^i) = \begin{pmatrix} x_0 + x_1 + x_2 & 0 & 0 \\ 0 & x_0 - 1/2(x_1 + x_2) & -\sqrt{3}/2(x_1 - x_2) \\ 0 & \sqrt{3}/2(x_1 - x_2) & x_0 - 1/2(x_1 + x_2) \end{pmatrix}. \tag{1.1}$$

The determinant of this matrix is

$$\det(\hat{x}) = x_0^3 + x_1^3 + x_2^3 - 3x_0x_1x_2, \tag{1.2}$$

which is precisely the linear arc defined in [2, 3, 4, 5, 6].

The cyclic group $C_3$ has three conjugation classes, 1, $q$ and $q^2$, and, respectively, three one dimensional irreducible representations, $R^{(i)}$, $i = 1, 2, 3$. We write down the table of their characters [11]

| $C_3$ | $1$ | $q$ | $q^2$ |
|-------|-----|-----|-------|
| $R^{(1)}$ | 1  | 1   | 1     |
| $R^{(2)}$ | $j$ | $j^2$ | 1 , $j^3 = 1$ , |
| $R^{(3)}$ | $j^2$ | $j$  |       |

where $R^{(1)}$ is the trivial representation, whereby each elements is mapped onto unit, i.e. for $R^{(1)}$ the kernel is the whole group, $C_3$. For $R^{(2)}$ and $R^{(3)}$ the kernels can be identified with unit element, which means that they are faithful representations, isomorphic to $C_3$.

We remind that for the cyclic group $C_2$ there are two conjugation classes, 1 and $i$ and two one-dimensional irreducible representations:

| $C_2$ | 1  | $i$ |
|-------|-----|-----|
| $R^{(1)}$ | 1  | 1   | $z$ . |
| $R^{(2)}$ | 1  | $-1$ | $z$ |

2
Based on the character table and on (1.1) one can obtain
\[ \xi^V = \begin{pmatrix} \xi^V(1), \xi^V(q), \xi^V(q^2) \end{pmatrix} = \begin{pmatrix} \text{Tr}(R^V(1)), \text{Tr}(R^V(q)), \text{Tr}(R^V(q^2)) \end{pmatrix} = (3, 0, 0), \]
which demonstrates how the vector representation \( R^V \) decomposes in the irreducible representations \( R^{(i)} \):
\[ \xi^V = \xi^{(1)} + \xi^{(2)} + \xi^{(3)} \]
or
\[ R^V = R^{(1)} \oplus R^{(2)} \oplus R^{(3)}. \]
The combinations of coordinates on which \( R^V \) acts irreducible are given below
\[
\begin{pmatrix} z \\ \bar{z} \\ \bar{\bar{z}} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & j & j^2 \\ 1 & j^2 & j \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 q \\ x_2 q^2 \end{pmatrix}.
\]

The content of the paper is as follows. In the next section we remind the basic results on ternary complex numbers obtained in [7, 8] useful in the sequel, in particular we introduce some functions, which extend the usual sine and cosine functions, having generalized analytical properties. These functions coincide with those introduced by Appell [1]. Section 3 is devoted to the formulation of some types of holomorphicity adapted to the ternary complex numbers and to the integration of differential forms. In particular a generalization of the logarithm is introduced, and it turns out that corresponding functions are directly related to expressions extending the usual sine and cosine. In this section we give also a physical interpretation of one ternary differential form. It appears that this form can be interpreted as a magnetic field generated by a monopole string. In section 4 we construct a “ternary” Newton mechanics and study the movement of a monopole type object in the magnetic field constructed in the previous sub-section. It turns out, that the Newton equation is integrable, which allows us to investigate the monopole scattering in the ternary field.

2 Ternary complex numbers

In Refs. [7, 8] an \( n \)--dimensional commutative extension of complex numbers was introduced together with functions having generalized analytical properties (see also [11, 15]). The algebra of the multicomplex numbers is defined over the field of real numbers; it is generated by one element \( q \) satisfying the condition \( q^n = -1 \). Below we remind some results obtained in [7] (see also [5]) for the simplest non-trivial case \( n = 3 \). The class of these ternary complex numbers is denoted by \( T_3 \mathbb{C} \). Since 3 is an odd number the generator \( q \) can be equivalently normalized in such a way, that
\[ q^3 = 1, \]
which leads to some minor modifications in formulæ derived in Ref. [7].

A ternary complex number \( z \in T_3 \mathbb{C} \) is expressed in terms of \( x_0, x_1, x_1 \in \mathbb{R} \) as follows
\[ z = x_0 + x_1 q + x_2 q^2. \]

Formally \( T_3 \mathbb{C} \) can be defined by
\[ T_3 \mathbb{C} = \mathbb{R}[X]/(1 - X^3). \]

Since \( 1 - X^3 = (1 - X)(1 + X + X^2) \) we have some interesting algebraic properties for \( T_3 \mathbb{C} \):

1. \( T_3 \mathbb{C} \) is not simple: \( T_3 \mathbb{C} = I_1 \oplus I_2 \) with two ideals \( I_1, I_2 \);

2. \( \forall z_1 \in I_1, z_2 \in I_2, z_1 z_2 = 0; \)

3. \( I_1 \cong \mathbb{R}[X]/(1 - X) \cong \mathbb{R}, I_2 \cong \mathbb{R}[X]/(1 + X + X^2) \cong \mathbb{C}. \)

This can be proven easily. Indeed, let us introduce

\[
\begin{align*}
K_0 &= \frac{1}{3} (1 + q + q^2), \\
I &= \frac{1}{\sqrt{3}} \left( (1 - q) - (1 - q)^2 \right) = \frac{1}{\sqrt{3}} (q - q^2), \\
E_0 &= -I^2 = \frac{1}{3} (2 - q - q^2)
\end{align*}
\tag{2.1}
\]

and denote by \( \rho \) the projection from \( \mathbb{R}[X] \to T_3 \mathbb{C} \) and by \( q = \rho(X) \) the image of \( X \) in \( T_3 \mathbb{C} \).

Since by definition \( \ker \rho = (1 - X^3) \) it is clear that \( (1 - q)(1 + q + q^2) = 0 \). If we define \( I_1 \subseteq T_3 \mathbb{C} \) to be the one-dimensional sub-algebra generated by \( K_0 \), and \( I_2 \subseteq T_3 \mathbb{C} \) to be the two-dimensional sub-algebra generated by \( I \), it is easily to verify the relations

\[
K^n_0 = K_0, E^n_0 = E_0, n \geq 1, \quad I^2 = -E_0, K_0 E_0 = 0, E_0 I = I. \tag{2.2}
\]

This proves the above statements 1-2-3. Note also that we have the relation \( e^{2\pi i} = q \).

A ternary complex number can be presented in two different forms

\[
z = x_0 + x_1 q + x_2 q^2 = (x_0 + x_1 + x_2) K_0 + (x_0 - \frac{1}{2}(x_1 - x_2)) E_0 + \frac{\sqrt{3}}{2}(x_1 - x_2) I, \tag{2.3}
\]

with \( x = (x_0 + x_1 + x_2) K_0 \in I_1, w = (x_0 - \frac{1}{2}(x_1 - x_2)) E_0 + \frac{\sqrt{3}}{2}(x_1 - x_2) I \in I_2. \)

Furthermore, since \( T_3 \mathbb{C} = I_1 \oplus I_2 \) with \( I_1 \cong \mathbb{R} \) and \( I_2 \cong \mathbb{C} \), which means that there is only one automorphism of \( T_3 \mathbb{C} \) given by \( q \leftrightarrow q^2 \) (or \( I \to -I, E_0 \to E_0 \) and \( K_0 \to K_0 \) i.e. for \( w \in I_2 \) we have \( w = x E_0 + y I \to x E_0 - y I \)). This automorphism enables us to define an adapted “modulus” on \( z \in T_3 \mathbb{C} \). Namely, for any \( z \in T_3 \mathbb{C} \) such that \( z = x + w \), with \( x \in I_1, w \in I_2 \) we set

\[
\| z \|^2 = \| x \|_{I_1} \| w \|_{I_2}^2 = (x_0 + x_1 + x_2) \left( (x_0 - \frac{1}{2}(x_1 - x_2))^2 + \frac{3}{2}(x_1 - x_2)^2 \right) = (x_0 + x_1 + x_2)(x_0 + j x_1 + j^2 x_2)(x_0 + j^2 x_1 + j x_2) = x_0^3 + x_1^3 + x_2^3 - 3x_0 x_1 x_2 \tag{2.4}
\]
with \( j = \frac{\sqrt{-1}}{2} \). It is interesting that a different definition of \( \| z \| \) can be given. Indeed, let us consider three isomorphic copies of ternary complex numbers

\[
T_3 \C = \{ z = x_0 + x_1 q + x_2 q^2, \; x_0, x_1, x_2 \in \R \}
\]

\[
\tilde{T}_3 \C = \{ \tilde{z} = x_0 + x_1 j q + x_2 j q^2, \; x_0, x_1, x_2 \in \R \}
\]

\[
\tilde{\tilde{T}}_3 \C = \{ \tilde{\tilde{z}} = x_0 + x_1 j^2 q + x_2 j q^2, \; x_0, x_1, x_2 \in \R \}
\]

(2.5)

The ternary isomorphism \( \tilde{\cdot} \) is the mapping

\[
T_3 \C \rightarrow \tilde{T}_3 \C \rightarrow \tilde{\tilde{T}}_3 \C \rightarrow \tilde{T}_3 \C.
\]

For further use, note that for elements \( z, \tilde{z} \) and \( \tilde{\tilde{z}} \) of the algebras \( T_3 \C, \tilde{T}_3 \C \) and \( \tilde{\tilde{T}}_3 \C \) we have

\[
\tilde{z} + \tilde{\tilde{z}} = 2x_0 - x_1 q - x_2 q^2 \in T_3 \C, \quad \tilde{z} \tilde{\tilde{z}} = (x_0^2 - x_1 x_2) + (x_2^2 - x_0 x_1)q + (x_1^2 - x_2 x_0)q^2 \in T_3 \C.
\]

We also have

\[
x_0 = \frac{1}{3} (z + \tilde{z} + \tilde{\tilde{z}}), \quad x_1 = \frac{q^2}{3} (z + j^2 \tilde{z} + j \tilde{\tilde{z}}), \quad x_2 = \frac{q}{3} (z + j \tilde{z} + j^2 \tilde{\tilde{z}}).
\]

(2.6)

The ternary mapping allows to give an alternative definition of the pseudo-norm on \( T_3 \C \) by

\[
\| \|: \; T_3 \C \otimes \tilde{T}_3 \C \otimes \tilde{\tilde{T}}_3 \C \rightarrow \R, \quad z \otimes \tilde{z} \otimes \tilde{\tilde{z}} \mapsto \| z \|^3 = z \tilde{z} \tilde{\tilde{z}} = x_0^3 + x_1^3 + x_2^3 - 3x_0 x_1 x_2
\]

which coincide with (2.4). Thus, \( \| z \| = 0 \) if and only if \( z \) belongs to \( I_1 \) or to \( I_2 \). A ternary complex number is called non-singular if \( \| z \| \neq 0 \). From now on we also denote \( |z| \) the modulus of \( z \). The modulus introduced here coincide with the one considered in [2 3 4 5 6].

It was proven in [7] that any non-singular ternary complex number \( z \in T_3 \C \) can be written in the “polar form”:

\[
z = \rho e^{\varphi_1 q + \varphi_2 q^2} = \rho e^{\theta (q - q^2) + \varphi (q + q^2)}
\]

(2.7)

with \( \rho = |z| = \sqrt{x_0^3 + x_1^3 + x_2^3 - 3x_0 x_1 x_2} \in \R, \theta \in [0, 2\pi/\sqrt{3}], \varphi \in \R \). The combinations \( q - q^2 \) and \( q + q^2 \) generate in the ternary space respectively compact and non-compact directions. Using (2.2) and \( q + q^2 = 2K_0 + E_0 \), we can rewrite (2.7) in the form

\[
z = \rho [m_0(\varphi_1, \varphi_2) + m_1(\varphi_1, \varphi_2)q + m_2(\varphi_1, \varphi_2)q^2]
\]

(2.8)

where the multi-sine functions [7 12] are given below (in [4 5] these functions are denoted \( P, Q, R \)).

---

\footnote{If we complexify the ternary complex numbers \( T_3 \C^C = T_3 \C \otimes \R \C \), the three above copies become identical and \( \tilde{\cdot} \) is an automorphism.}
They satisfy the relations

\[
\frac{\partial}{\partial \varphi_\ell} m_k(\varphi_1, \varphi_2) = m_{k-\ell}(\varphi_1, \varphi_2),
\]

\[
m_k(\varphi_1 + \psi_1, \varphi_2 + \psi_2) = m_0(\varphi_1, \varphi_2)m_k(\psi_1, \psi_2) + m_1(\varphi_1, \varphi_2)m_{k-1}(\psi_1, \psi_2) + m_2(\varphi_1, \varphi_2)m_{k-2}(\psi_1, \psi_2),
\]

where the indices are defined modulo 3 and

\[
m_0^3(\varphi_1, \varphi_2) + m_1^3(\varphi_1, \varphi_2) + m_2^3(\varphi_1, \varphi_2) - 3m_0(\varphi_1, \varphi_2)m_1(\varphi_1, \varphi_2)m_2(\varphi_1, \varphi_2) = 1.
\]

Since for the product of two ternary complex numbers we have \( \|zw\| = \|z\|\|w\| \) the set of unimodular ternary complex numbers preserves the cubic form \[7, 12\]. The continuous group of symmetry of the cubic surface \( x_0^3 + x_1^3 + x_2^3 - 3x_0x_1x_2 = \rho^3 \) is isomorphic to \( SO(2) \times SO(1, 1) \). We denote the set of unimodular ternary complex numbers or the “ternary unit sphere” as \( \mathbb{T}U(1) = \left\{ e^{i(\theta + \varphi)}a^+(\varphi - \theta)a^2, \ 0 \leq \theta < 2\pi/\sqrt{3}, \varphi \in \mathbb{R} \right\} \sim \mathbb{T}S^1 \). This surface is also called the Appell sphere \[5\].

**Fig 1:** Cubic surface \( x_0^3 + x_1^3 + x_2^3 - 3x_0x_1x_2 = 1 \)
Looking at the above figure one can see, that the surface approaches asymptotically the plane \( x_0 + x_1 + x_2 = 0 \) and the line \( x_0 = x_1 = x_2 \) is orthogonal to it. In \( \mathbb{T}_3 \mathbb{C} \) they correspond to the ideals \( I_2 \) and \( I_1 \), respectively. The latter line will be called the “trisectrice”.

One can construct the inner metric of this surface for the general case \( \rho \neq 0 \). Let us introduce the new coordinate \( a = x_0 + x_1 + x_2 \) and parametrise a point on the circle around the trisectrice by its polar coordinates \((r, \theta)\). The surface in these coordinates has the simple equation \( ar^2 = \rho^3 \). It can be shown that a point \( M(x_0, x_1, x_2) \) on the cubic surface can be parametrised as follows

\[
x_0 = \frac{1}{3} \left( \frac{\rho^3}{r^2} - 2r \cos \theta \right), \quad x_1 = \frac{1}{3} \left( \frac{\rho^3}{r^2} + r \cos \theta + \sqrt{3r} \sin \theta \right), \quad x_2 = \frac{1}{3} \left( \frac{\rho^3}{r^2} + r \cos \theta - \sqrt{3r} \sin \theta \right).
\]

This gives

\[
ds^2 = \frac{1}{3} \left( 2 + 4\rho^6 \right) dr^2 + \frac{2}{3} r^2 d\theta^2
\]

and corresponds to the Gauss curvature

\[
K = -\frac{12a^4}{(4a^3 + \rho^3)^2}.
\]

The Gauss curvature is negative and tends to zero, when \( a \to 0 \) (\( r \to \infty \)). In this limit the metric becomes flat:

\[
ds^2 = \frac{2}{3} (dr^2 + r^2 d\theta^2).
\]

The \( SO(2) \times SO(1,1) \) group of transformations is generated by the multi-sine functions. In particular, in the special case where \( \varphi = 0 \) the transformation in the compact direction is a rotation to the angle \( \sqrt{3} \theta \) and for \( \theta = 0 \) we have the dilatation in the non-compact direction

\[
\begin{align*}
\varphi = 0 : & \quad \begin{cases} 
 x_0 + x_1 + x_2 \to x_0 + x_1 + x_2 \\
 x_0 + jx_1 + j^2 x_2 \to e^{i\sqrt{3} \theta} (x_0 + jx_1 + j^2 x_2)
\end{cases} \\
\theta = 0 : & \quad \begin{cases} 
 x_0 + x_1 + x_2 \to e^{2\varphi} (x_0 + x_1 + x_2) \\
 x_0 + jx_1 + j^2 x_2 \to e^{-\varphi} (x_0 + jx_1 + j^2 x_2)
\end{cases}
\end{align*}
\]

Let us consider now the following discrete transformation preserving the modulus \( || z || \) of non-singular ternary complex numbers:

\[
z = \rho e^{\varphi_1 q - \varphi_2 q^2} \to \bar{z} = \frac{\bar{z} \bar{z}}{|| z ||} = \rho e^{-\varphi_1 q - \varphi_2 q^2}.
\]

Obviously we have \( \bar{\bar{z}} = z \). For multi-sine functions \( m_k(\varphi_1, \varphi_2) \) the symmetry (2.14) leads to the relations
\[ m_0(-\varphi_1, -\varphi_2) = m^2_0(\varphi_1, \varphi_2) - m_1(\varphi_1, \varphi_2) m_2(\varphi_1, \varphi_2), \]
\[ m_1(-\varphi_1, -\varphi_2) = m^2_2(\varphi_1, \varphi_2) - m_0(\varphi_1, \varphi_2) m_1(\varphi_1, \varphi_2), \]
\[ m_2(-\varphi_1, -\varphi_2) = m^2_1(\varphi_1, \varphi_2) - m_0(\varphi_1, \varphi_2) m_2(\varphi_1, \varphi_2). \]

These relations are also given in \[5\].

### 3 Differential and holomorphic forms

#### 3.1 Holomorphicity

In this section, we introduce notions of holomorphicity adapted to the ternary complex numbers, some partial results were previously given in \[8\]. For a point \(a\) of a Riemann manifold, the elements \(dx_0, dx_1, dx_2\) form a basis of the cotangent space \(T_a^{(1)}\). It should be relevant to consider also the alternative basis given by \((dz, d\tilde{z}, d\check{z})\). Thus, if one defines a differentiable function

\[
F(x_0, x_1, x_2) = f_0(x_0, x_1, x_2) + f_1(x_0, x_1, x_2) q + f_2(x_0, x_1, x_2) q^2
\]

(3.1)

\[
\equiv g_0(z, \tilde{z}, \check{z}) + g_1(z, \tilde{z}, \check{z}) q + g_2(z, \tilde{z}, \check{z}) q^2,
\]

it is possible to construct the 1-form

\[
dF = \frac{\partial F}{\partial x_0} dx_0 + \frac{\partial F}{\partial x_1} dx_1 + \frac{\partial F}{\partial x_2} dx_2 = \frac{\partial F}{\partial z} dz + \frac{\partial F}{\partial \tilde{z}} d\tilde{z} + \frac{\partial F}{\partial \check{z}} d\check{z}. \quad (3.2)
\]

Here we introduced

\[
\frac{\partial}{\partial z} = \frac{1}{3} \left( \frac{\partial}{\partial x_0} + q^2 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right),
\]

\[
\frac{\partial}{\partial \tilde{z}} = \frac{1}{3} \left( \frac{\partial}{\partial x_0} + j q^2 \frac{\partial}{\partial x_1} + j q \frac{\partial}{\partial x_2} \right), \quad (3.3)
\]

\[
\frac{\partial}{\partial \check{z}} = \frac{1}{3} \left( \frac{\partial}{\partial x_0} + j^2 q^2 \frac{\partial}{\partial x_1} + j^2 q \frac{\partial}{\partial x_2} \right).
\]

In a full analogy with the usual complex analysis, we are now able to define holomorphic functions. Thus, \(T_a^{(1)}\) is considered as a direct sum of three subspaces spanned by \(dz, d\tilde{z}\) and \(d\check{z}\):

\[
T_a^{(1)} = T_a^{(1,0,0)} \oplus T_a^{(0,1,0)} \oplus T_a^{(0,0,1)},
\]

or

\[
T_a^{(1)} = T_a^{(1,0,0)} \oplus T_a^{(0,1,1)}.
\]

We introduce now the following notations for one-forms

\[
\omega^{(1,0,0)} = g_0(z, \tilde{z}, \check{z}) \, dz, \quad \omega^{(0,1,1)} = g_1(z, \tilde{z}, \check{z}) \, d\tilde{z} + g_2(z, \tilde{z}, \check{z}) \, d\check{z},
\]

\[
\omega^{(1)} = g_0(z, \tilde{z}, \check{z}) \, dz + g_1(z, \tilde{z}, \check{z}) \, d\tilde{z} + g_2(z, \tilde{z}, \check{z}) \, d\check{z}. \quad (3.4)
\]
3.1.1 First type of holomorphic forms

The function \( F(z, \tilde{z}, \tilde{\tilde{z}}) \) is called holomorphic (or double analytic) if
\[
\frac{\partial F(z, \tilde{z}, \tilde{\tilde{z}})}{\partial \tilde{z}} = \frac{\partial F(z, \tilde{z}, \tilde{\tilde{z}})}{\partial \tilde{\tilde{z}}} = 0.
\]
(3.5)

It was proved in [8] that these conditions are equivalent to the equations of the Cauchy-Riemann type (see also [5] for analogous relations)
\[
\begin{align*}
\frac{\partial f_0}{\partial x_0} &= \frac{\partial f_1}{\partial x_1} = \frac{\partial f_2}{\partial x_2}, \\
\frac{\partial f_0}{\partial x_1} &= \frac{\partial f_1}{\partial x_2} = \frac{\partial f_2}{\partial x_0}, \\
\frac{\partial f_0}{\partial x_2} &= \frac{\partial f_1}{\partial x_0} = \frac{\partial f_2}{\partial x_1},
\end{align*}
\]
(3.6)
assuming that the function \( F \) is derivable. As a direct consequence, \( F \) is holomorphic if and only if the one-form \( \omega^{(1,0,0)} = F \partial z \) is closed i.e. \( d\omega^{(1,0,0)} = 0 \). Note, that in the polar form \( dz = z(\frac{d\rho}{\rho} + d\varphi_1 q + d\varphi_2 q^2) \) introducing the function \( zF(z) = h_0(\rho, \varphi_1, \varphi_2) + h_1(\rho, \varphi_1, \varphi_2)q + h_2(\rho, \varphi_1, \varphi_2)q^2 \) the condition \( d(F(z) \partial z) = 0 \) gives
\[
\begin{align*}
\frac{\partial h_1}{\partial \varphi_1} &= \frac{\partial h_2}{\partial \varphi_2}, & \rho \frac{\partial h_2}{\partial \rho} &= \frac{\partial h_0}{\partial \varphi_1}, & \frac{\partial h_0}{\partial \varphi_2} &= \rho \frac{\partial h_1}{\partial \rho}, \\
\frac{\partial h_0}{\partial \varphi_1} &= \frac{\partial h_2}{\partial \varphi_2}, & \rho \frac{\partial h_2}{\partial \rho} &= \frac{\partial h_0}{\partial \varphi_1}, & \frac{\partial h_0}{\partial \varphi_2} &= \rho \frac{\partial h_1}{\partial \rho}.
\end{align*}
\]
(3.7)

The above equations (3.6), (3.7) define ternary harmonical functions similar to the Cauchy-Riemann (Darbou-Euler) definition of the holomorphic functions in the binary case. From holomorphicity constraints one can derive that the three functions \( f_0, f_1, f_2 \) (see (3.1)) entering in the ternary double analytic function satisfy the ternary Laplace equation
\[
\nabla^3 f_i = \frac{\partial^3 f_i}{\partial z \partial \bar{z} \partial \tilde{z}} = \frac{\partial^3 f_i}{\partial x_0^3} + \frac{\partial^3 f_i}{\partial x_1^3} + \frac{\partial^3 f_i}{\partial x_2^3} - 3\frac{\partial^3 f_i}{\partial x_0 \partial x_1 \partial x_2} = 0, \quad i = 1, 2, 3.
\]
(3.8)

3.1.2 Second type of holomorphicity

The second type of holomorphicity, corresponds to the single analyticity that is to a function \( F(z, \tilde{z}, \tilde{\tilde{z}}) \) such that \( \frac{\partial F(z, \tilde{z}, \tilde{\tilde{z}})}{\partial z} = 0 \). A direct calculation gives the corresponding Cauchy-Riemann equations
\[
\begin{align*}
\frac{\partial f_0}{\partial x_0} + \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} &= 0, \\
\frac{\partial f_0}{\partial x_1} + \frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_0} &= 0, \\
\frac{\partial f_0}{\partial x_2} + \frac{\partial f_1}{\partial x_0} + \frac{\partial f_2}{\partial x_1} &= 0,
\end{align*}
\]
(3.9)
with the notations given in (3.1). Manipulating equations (3.9), we get

\[
\frac{\partial}{\partial x_i} \left\{ \frac{\partial^3}{\partial x_0^3} + \frac{\partial^3}{\partial x_1^3} + \frac{\partial^3}{\partial x_2^3} - 3 \frac{\partial^3}{\partial x_0 \partial x_1 \partial x_2} \right\} f_j = 0, i, j = 0, 1, 2.
\]

This means that

\[
\left\{ \frac{\partial^3}{\partial x_0^3} + \frac{\partial^3}{\partial x_1^3} + \frac{\partial^3}{\partial x_2^3} - 3 \frac{\partial^3}{\partial x_0 \partial x_1 \partial x_2} \right\} f_i = C_i, i = 0, 1, 2
\]

where \( C_i \) are a constant. The solution of the above equations is the sum of the general solution of the homogeneous equations and a particular solution of the inhomogeneous equations. The solution of the inhomogeneous equations can be searched as a linear combination of the functions \( x_3^0, x_3^1, x_3^2 \) and \( x_0 x_1 x_2 \), but it is easy to verify, that this combination does not satisfy the above set of equations for \( f_i \) linear in partial derivatives \( \partial/(\partial x_j) \). Therefore \( C_i = 0, i = 0, 1, 2 \).

In addition to these Cauchy-Riemann equations (3.9) the holomorphic functions has to be real. This means that \( F(\tilde{z}, \bar{\tilde{z}}) = F(\bar{\tilde{z}}, \tilde{z}) \) and gives the relation

\[
\tilde{z} \frac{\partial}{\partial \tilde{z}} F(\tilde{z}, \bar{\tilde{z}}) = \bar{\tilde{z}} \frac{\partial}{\partial \bar{\tilde{z}}} F(\tilde{z}, \bar{\tilde{z}}).
\]

In components this gives the additional relations

\[
\begin{align*}
\left( x_0 \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} \right) f_2 &= \left( x_0 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_0} + x_2 \frac{\partial}{\partial x_2} \right) f_1, \\
\left( x_0 \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} \right) f_0 &= \left( x_0 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_0} + x_2 \frac{\partial}{\partial x_2} \right) f_2, \\
\left( x_0 \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} \right) f_1 &= \left( x_0 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_0} + x_2 \frac{\partial}{\partial x_2} \right) f_0.
\end{align*}
\]

(3.10)

As an example one sees that the function \( F(\tilde{z}, \bar{\tilde{z}}) = (\tilde{z} \bar{\tilde{z}})^m \) satisfies the conditions (3.9)-(3.10), although the function \( F(\tilde{z}, \bar{\tilde{z}}) = (\tilde{z})^n(\bar{\tilde{z}})^m, n \neq m \) only satisfies (3.9).

### 3.2 Ternary conformal transformations

The transformations of a domain in the complex plane \( \mathbb{C} \) into the other domain in \( \mathbb{C} \) can be performed with the use of two functions,

\[
u = u(x, y), \quad v = v(x, y),
\]

which are respectively the real and imaginary parts of the holomorphic function \( w = u + iv = f(z) = f(x + iy) \). The Jacobian of this transformation is:

\[
\frac{\partial(u, v)}{\partial(x, y)} = |f'(z)|^2.
\]
If $f'(z) \neq 0$, the corresponding transformation $w = f(z)$ is conformal. For the double holomorphic ternary function $F(z) = f_0 + qf_1 + f_2q^2$, $\bar{\partial} F = \partial F = 0$, $F$ defines a conformal transformation. Using the Cauchy-Riemann equations with $F'(z) = \frac{dF(z)}{dz}$, we get

$$\frac{\partial (f_0, f_1, f_2)}{\partial (x_0, x_1, x_2)} = |F'(z)|^3.$$  

3.3 The Integration of the Differential Forms

The ternary differential 1-forms can be integrated along some curves as it was in the ordinary holomorphic case. If the form is closed the result depends only on the homotopy class of the curve $L$. In particular on the simply connected manifold (surface) such integral $\int_L \omega$ is a well defined function, the integral from the exact differential form depends only on the initial and final points of the curve $L$. Moreover, locally for each closed form one can build the exact form.

3.3.1 Integral of holomorphic functions of the first type

Now we want to consider the integral from the ternary holomorphic function of the first type:

$$\int L \omega = \int L F(z)dz = \int L (\omega_0 + \omega_1q + \omega_2q^2),$$

where three associated 1-forms are given below (see (3.11))

$$\omega_0 = (f_0dx_0 + f_1dx_2 + f_2dx_1),$$
$$\omega_1 = (f_1dx_0 + f_0dx_1 + f_2dx_2),$$
$$\omega_2 = (f_2dx_0 + f_1dx_1 + f_0dx_2).$$

(3.12)

It is easy to prove that this 1-form $\omega = Fdz$ is exact for an holomorphic function $F(z)$ (we have already proved that is closed). Indeed, introduce a zero-form $V = V_0 + V_1q + V_2q^2$, for which we assume $dV = \omega$. This gives

$$f_0 = \frac{\partial V_0}{\partial x_0}, \quad f_2 = \frac{\partial V_0}{\partial x_1}, \quad f_1 = \frac{\partial V_0}{\partial x_2},$$
$$f_1 = \frac{\partial V_1}{\partial x_0}, \quad f_0 = \frac{\partial V_1}{\partial x_1}, \quad f_2 = \frac{\partial V_1}{\partial x_2},$$
$$f_2 = \frac{\partial V_2}{\partial x_0}, \quad f_1 = \frac{\partial V_2}{\partial x_1}, \quad f_0 = \frac{\partial V_2}{\partial x_2},$$

(3.13)

these equations admit a solution if and only if the function $V$ is holomophic because (3.13) are just the Cauchy-Riemann equations (3.6) for $V$. Therefore, if $V$ is holomorphic, we have

$$\int L F(z)dz = V(z_1) - V(z_2)$$

(3.14)
with \( z_1 \) and \( z_2 \) being the end points of the curve \( L \).

Now we consider an important example of ternary holomorphic functions

\[
\ln z = \int^z \frac{dz'}{z'} = (\ln z)_0 + (\ln z)_1 q + (\ln z)_2 q^2 ,
\]

Of course the curve along which we perform the integration is such that \( 1/z \) is always defined, \textit{i.e.} there is no singular ternary complex numbers. In the polar form this integral is easily calculated

\[
\ln z = \int^z \frac{dz'}{z'} = \int^z \left( \frac{d\rho'}{\rho'} + d\varphi' q + d\varphi' q^2 \right) = \ln \rho + \varphi q + \varphi^2 q^2 . \quad (3.15)
\]

This result corresponds to the obvious relation

\[
e^{\ln z} = z .
\]

It is useful to perform the calculation of the integral directly in the “cartesian coordinates”. One can take easily the integral for \((\ln z)_0\)

\[
\ln \rho = (\ln z)_0 = \int \frac{1}{|z|^3}[(x_0^2 - x_1 x_2)dx_0 + (x_0^2 - x_0 x_2)dx_1 + (x_2^2 - x_0 x_1)dx_2] = \frac{1}{3} \ln(x_0^3 + x_1^3 + x_2^3 - 3x_0 x_1 x_2). \quad (3.16)
\]

However, the second and third integrals are more involved

\[
(\ln z)_1 = \int \frac{1}{|z|^3}[(x_1^2 - x_0 x_2)dx_0 + (x_0^2 - x_1 x_2)dx_1 + (x_2^2 - x_0 x_1)dx_2],
\]

\[
(\ln z)_2 = \int \frac{1}{|z|^3}[(x_2^2 - x_0 x_2)dx_0 + (x_2^2 - x_0 x_1)dx_1 + (x_0^2 - x_1 x_2)dx_2].
\]

If we denote \( \varphi_1 = (\ln z)_1 = \int \frac{1}{|z|^3}(adx_0 + bdx_1 + cdx_2) \) a direct verification gives \( \frac{\partial \varphi_1}{\partial x_0} = \frac{\partial \ln}{\partial x_1} \) plus similar relations for the other terms (this was expected see \(3.15\)). Thus, we have

\[
\frac{\partial \varphi_1}{\partial x_0} = \frac{x_2^2 - x_0 x_1}{|z|^3}, \quad \frac{\partial \varphi_1}{\partial x_1} = \frac{x_1^2 - x_1 x_2}{|z|^3}, \quad \frac{\partial \varphi_1}{\partial x_2} = \frac{x_1^2 - x_0 x_2}{|z|^3},
\]

and

\[
\varphi_1 = (\ln z)_1 = \int \frac{dx_0(x_0^2 - x_0 x_1)}{(x_0 + x_1 + x_2)(x_0 + j x_1 + j^2 x_2)(x_0 + j^2 x_1 + j x_2)} \quad (3.17)
\]

\[
= \frac{1}{3}[\ln(x_0 + x_1 + x_2) + j^2 \ln(x_0 + j x_1 + j^2 x_2) + j \ln(x_0 + j^2 x_1 + j x_2)].
\]

Similarly,

\[
\varphi_2 = (\ln z)_2 = \frac{1}{3}[\ln(x_0 + x_1 + x_2) + j \ln(x_0 + j x_1 + j^2 x_2) + j^2 \ln(x_0 + j^2 x_1 + j x_2)]. \quad (3.18)
\]
A priori these functions seem to be ill-defined (multi-valued) and cuts in the complex plane should be taken, however it is not the case if $\rho > 0$ (if $\rho \leq 0$ the formulæ above are not defined). Indeed, one can write

$$
\varphi_1 = -\frac{1}{6} \ln[(x_0 + jx_1 + j^2 x_2)(x_0 + j^2 x_1 + jx_2)] + \frac{2}{\sqrt{3}} \psi
$$

with

$$
\psi = \arctg\left[\frac{\sqrt{3}(x_1 - x_2)}{2x_0 - x_1 - x_2}\right],
$$

and similar relations for $\varphi_2$. Using above relations we obtain finally

$$
(\ln z) = \frac{1}{3} \ln |z|^3 + q\frac{1}{3} \ln(x_0 + x_1 + x_2) + j^2 \ln(x_0 + jx_1 + j^2 x_2) + j \ln(x_0 + j^2 x_1 + jx_2)]
$$

$$
+ q^2 \frac{1}{3} \ln(x_0 + x_1 + x_2) + j \ln(x_0 + jx_1 + j^2 x_2) + j^2 \ln(x_0 + j^2 x_1 + jx_2)]
$$

It is interesting to notice that adding $\ln z$ with its ternary conjugations one gets

$$
\ln z + \ln \tilde{z} + \ln \tilde{\tilde{z}} = \ln |z|^3.
$$

Let us now consider $z$ in the “polar” coordinates (see (2.1))

$$
z = \exp\{\ln z\} = \rho \exp\{\varphi_1 q + \varphi_2 q^2\} = \rho \exp\{(2K_0 - E_0)\varphi + i\sqrt{3}\theta\},
$$

where

$$
\ln \rho = \frac{1}{3} \ln[(x_0 + x_1 q + x_2 q^2)(x_0 + jx_1 + j^2 x_2)(x_0 + j^2 x_1 + jx_2)],
$$

$$
\varphi = \frac{\varphi_1 + \varphi_2}{2} = \frac{(\ln z)_1 + (\ln z)_2}{2} = \frac{1}{\rho} \ln[x_0 + x_1 + x_2],
$$

$$
\theta = \frac{\varphi_1 - \varphi_2}{2} = \frac{(\ln z)_1 - (\ln z)_2}{2} = \frac{i}{2\sqrt{3}} \ln[x_0 + jx_1 + j^2 x_2] = \frac{1}{\sqrt{3}} \psi.
$$

The relations, (3.16), (3.17) and (3.18) which give the polar coordinates as functions of the cartesian ones, can be considered as inverse formulæ for the multisine-functions (2.9), which present the cartesian coordinates as functions of polar ones (these are well defined since by definition the angle in the compact direction is restricted to $[0, 2\pi/\sqrt{3}]$ see (2.7)). As a matter of a formal exercise, one can check that the expressions (3.20) inserted in the multisine-functions (2.9) give $x_k = \rho m_k(\varphi_1, \varphi_2)$.

One can also verify the relation

$$
\int^z \frac{dz}{z} = \int \frac{d\rho}{\rho} + (2K_0 - E_0) \int d\varphi + \sqrt{3}I \int d\theta.
$$
In particular on the cubic surfaces, where $\rho = \text{const}, \varphi = \text{const}$, i.e. if we integrate over the closed contour around the “trisectrice”, the integral equals $2\pi I$. This result can also be obtained using the parametrisation given in (2.12).

### 3.3.2 Integral of holomorphic functions of the second type

Let us consider now the integral of holomorphic functions of the second type, which is related to the single ternary holomorphicity. We perform the integration of the two-form over a surface $S$ with some boundary $\partial S$

$$\int_S \Omega = \int_S \Phi(\bar{z} \tilde{z}) \frac{d\bar{z} \wedge d\tilde{z}}{j^2 - j} = \int_S (\Omega_0 + \Omega_1 q + \Omega_2 q^2).$$

(3.21)

Here from $\Phi(\bar{z} \tilde{z}) = \Phi_0 + \Phi_1 q + \Phi_2 q^2$, we defined three associated 2-forms

$$\Omega_0 = \Phi_0 dx_1 \wedge dx_2 + \Phi_1 dx_2 \wedge dx_0 + \Phi_2 dx_0 \wedge dx_1,$$

$$\Omega_1 = \Phi_1 dx_1 \wedge dx_2 + \Phi_2 dx_2 \wedge dx_0 + \Phi_0 dx_0 \wedge dx_1,$$

$$\Omega_2 = \Phi_2 dx_1 \wedge dx_2 + \Phi_0 dx_2 \wedge dx_0 + \Phi_1 dx_0 \wedge dx_1.$$  

(3.22)

This integral vanishes on a closed surface if $\Omega = d\omega$. If $\omega = \omega_z dz + \omega_{\bar{z}} d\bar{z} + \omega_{\tilde{z}} d\tilde{z}$, with (i) $\omega_z$ being a holomorphic function of the first type and (ii) $\omega_{\bar{z}}$ and $\omega_{\tilde{z}}$ being holomorphic functions of the second type, satisfying the following constraints $\partial_{\bar{z}} \omega_{\tilde{z}} - \partial_{\tilde{z}} \omega_z = \frac{1}{j^2 - j} \Phi(\bar{z} \tilde{z})$, we have $\Omega = d\omega$.

Let us consider the following example, where the function $\Phi$ has singularities

$$\int \Omega = \frac{1}{j^2 - j} \int \frac{d\bar{z} \wedge d\tilde{z}}{(\bar{z} \tilde{z})} = \frac{1}{j^2 - j} \int \frac{zd\bar{z} \wedge d\tilde{z}}{\rho^3},$$

(3.23)

$$\Omega = \frac{1}{j^2 - j} \frac{z}{\rho^3} d\bar{z} \wedge d\tilde{z} = \frac{1}{\rho^3} (x_0 dx_1 \wedge dx_2 + x_1 dx_2 \wedge dx_0 + x_2 dx_0 \wedge dx_1)$$

$$+ \frac{1}{\rho^3} (x_0 dx_0 \wedge dx_1 + x_1 dx_1 \wedge dx_2 + x_2 dx_2 \wedge dx_0)q$$

$$+ \frac{1}{\rho^3} (x_0 dx_2 \wedge dx_0 + x_1 dx_0 \wedge dx_1 + x_2 dx_1 \wedge dx_2)q^2.$$  

(3.24)

To integrate $\Omega$ over a piece $\Sigma$ of the three-dimensional surface, $z\bar{z}\tilde{z} = \rho^3$, one should choose a parametrization, $g: \mathcal{R} \to \Sigma$, where $\mathcal{R}$ is a subset of $\mathbb{R}^2$:

$$g(u, v) = (x_0(u, v), x_1(u, v), x_2(u, v)),$$

which gives $dx_i \wedge dx_j = J_{ij} du dv$, where the Jacobians are

$$J_{ij} = \begin{vmatrix} \frac{\partial x_i}{\partial u} & \frac{\partial x_i}{\partial v} \\ \frac{\partial x_j}{\partial u} & \frac{\partial x_j}{\partial v} \end{vmatrix}, \quad i, j = 0, 1, 2.$$
If we choose the parametrisation (2.12), with \( a = x_0 + x_1 + x_2 \) and \( ar^2 = \rho^3 \) we get (considering here the case where \( a > 0 \) corresponding to \( \rho > 0 \)).

\[
\begin{align*}
    x_0 &= \frac{a}{3} - \frac{2}{3} \frac{\rho^{3/2}}{\sqrt{a}} \cos \theta, \\
    x_1 &= \frac{a}{3} + \frac{1}{3} \frac{\rho^{3/2}}{\sqrt{a}} (\cos \theta + \sqrt{3} \sin \theta), \\
    x_2 &= \frac{a}{3} + \frac{1}{3} \frac{\rho^{3/2}}{\sqrt{a}} (\cos \theta - \sqrt{3} \sin \theta).
\end{align*}
\]

(3.25)

The region of integration is given below

\[
0 < a_1 \leq a \leq a_2, \quad 0 \leq \theta \leq 2\pi.
\]

We obtain also the relations

\[
\begin{align*}
    x_0^2 + x_1^2 + x_2^2 &= \frac{a^2}{3} \left( 1 + 2 \frac{\rho^3}{a^3} \right), \\
    x_0 x_1 + x_1 x_2 + x_2 x_0 &= \frac{a^2}{3} \left( 1 - \frac{\rho^3}{a^3} \right).
\end{align*}
\]

(3.26)

Based on the above parametrisation one can calculate the Jacobians:

\[
\begin{align*}
    J_{12} &= \frac{\sqrt{3}}{9} \frac{\rho^3}{a^2} - \frac{2}{9} \frac{\sqrt{3} \rho^{3/2}}{\sqrt{a}} \left( \cos \theta \right), \\
    J_{20} &= \frac{\sqrt{3}}{9} \frac{\rho^3}{a^2} + \frac{\sqrt{3}}{9} \frac{\rho^{3/2}}{\sqrt{a}} \left( \cos \theta + \sqrt{3} \sin \theta \right), \\
    J_{01} &= \frac{\sqrt{3}}{9} \frac{\rho^3}{a^2} + \frac{\sqrt{3}}{9} \frac{\rho^{3/2}}{\sqrt{a}} \left( \cos \theta - \sqrt{3} \sin \theta \right).
\end{align*}
\]

(3.27)

Note that the geometrical meaning of \( J_{01}, J_{12}, J_{20} \) is the surface on the planes \( \{01\}, \{12\}, \{20\} \), respectively. Therefore one can consider the following cubic relation for \( J_{jk} \) as a ternary analog of the binary Pythagor theorem

\[
J_{01}^3 + J_{12}^3 + J_{20}^3 - 3J_{01}J_{12}J_{20} = \frac{1}{3\sqrt{3}} \frac{\rho^6}{a^3}.
\]

Whereas the binary Pythagor theorem relates the lengths of the sides in a triangle, the ternary Pythagor theorem gives a relation among the squares of the faces, \( A, B, C, D \), in the tetrahedron:

\[
S_A^3 + S_B^3 + S_C^3 - 3S_A S_B S_C = S_D^3.
\]
where $S_i$ is the square of the corresponding face.

To obtain the final result of integration one can take into account the relations

\[
x_0 J_{12} + x_1 J_{20} + x_2 J_{12} = \frac{1}{\sqrt{3}} \frac{\rho^3}{a},
\]
\[
x_0 J_{01} + x_1 J_{12} + x_2 J_{20} = 0,
\]
\[
x_0 J_{20} + x_1 J_{01} + x_2 J_{12} = 0,
\]

\[(3.28)\]

\[
\int \Omega = \int (x_0 J_{12} + x_1 J_{20} + x_2 J_{01}) \frac{a}{\rho^3} d\theta da = \frac{2\pi}{\sqrt{3}} \ln \frac{a_2}{a_1}.
\]

\[(3.29)\]

We can perform the calculation in the polar coordinates. In this case one obtains

\[
\Omega = \frac{1}{j^2 - j} \frac{d\hat{z} \wedge d\hat{z}}{\hat{z}} = d\varphi_1 \wedge d\varphi_2 + \frac{d\rho}{\rho} \wedge d\varphi_1 q + d\varphi_2 \wedge \frac{d\rho}{\rho} q^2
\]

for the cubic surface $\Sigma$ parametrized by $\varphi' \leq \varphi \leq \varphi'', 0 \leq \theta \leq 2\pi/\sqrt{3}$. Using $\varphi_1 = \varphi + \theta, \varphi_2 = \varphi - \theta$, we get

\[
\int_{\Sigma} \Omega = \int_{\Sigma} d\varphi_1 \wedge d\varphi_2 = 2 \int_{0}^{2\pi/\sqrt{3}} d\theta \int_{\varphi'}^{\varphi''} d\varphi = \frac{4\pi}{\sqrt{3}} (\varphi'' - \varphi').
\]

This relation coincides with the above result since $a = \rho e^{2\varphi}$ (see (2.9)).

Let us consider now a generic two-form $\Omega$. It will be exact if there exists a 1-form $\omega$ (not necessarily holomorphic having $d\omega = \Omega$). In particular for the 1-form given below

\[
\omega = \varphi_1 d\varphi_2 + \ln \rho d\varphi_1 q + \varphi_2 \frac{d\rho}{\rho} q^2
\]

we obtain

\[
d\omega = \left( \frac{d\rho}{\rho} \frac{\partial}{\partial \rho} + d\varphi_1 \frac{\partial}{\partial \varphi_1} + d\varphi_2 \frac{\partial}{\partial \varphi_2} \right) \wedge \omega = \frac{1}{j^2 - j} \frac{d\hat{z} \wedge d\hat{z}}{\hat{z}}.
\]

Thus, $\oint_{\Sigma} \frac{1}{j^2 - j} \frac{d\hat{z} \wedge d\hat{z}}{\hat{z}} = 0$ for any closed surface without inner singularity points.

### 3.4 Physical interpretation of the singular ternary potential

As we have seen, the integral (3.23) over the closed surface is zero if the singularity at the trisectrice $x_1 = x_2 = x_0$ is not inside this surface. We interprete the integrand as a magnetic field

\[
\vec{H} = \frac{\vec{x}}{|\vec{z}|^3},
\]

(3.30)
because due to the ternary holomorphicity of the second type (the single analyticity) the relation
\[ \nabla \vec{H} = 0 \]  
(3.31)
is valid everywhere apart from the singularity. Note, that the transmutation of field components \( \vec{H} = (H_1, H_2, H_0) \rightarrow \vec{H}' = (H_2, H_0, H_1) \) is equivalent to the rotation of the vector \( \vec{H} \) around the trisectrice on the angle \( 2\pi/3 \). The new vector \( \vec{H}' \) satisfies also the condition \( \nabla \vec{H}' = 0 \) in the region outside the trisectrice. It is convenient to rotate the coordinate system
\[ \vec{n}_0 = \frac{1}{\sqrt{3}} (1,1,1), \, \vec{n}_1 = \frac{1}{\sqrt{2}} (1,-1,0), \, \vec{n}_2 = \frac{1}{\sqrt{6}} (1,1,-2). \]  
(3.32)
in such way, that the new coordinates are
\[ r_0 = \frac{x_1 + x_2 + x_0}{\sqrt{3}}, \quad r_1 = \frac{x_1 - x_2}{\sqrt{2}}, \quad r_2 = \frac{x_1 + x_2 - 2x_0}{\sqrt{6}} \]  
(3.33)
and the new components of the magnetic field are
\[ \vec{h} = \frac{3\sqrt{3}}{2} \vec{H} = \frac{\vec{R}}{|\vec{r}|^2}, \quad R_0 = l, \, R_{1,2} = r_{1,2}, \, |\vec{r}|^2 = r_1^2 + r_2^2. \]  
(3.34)

Let us integrate \( \nabla \vec{h} \) over the inner part of the cylinder of the radius \( a \) with the coordinate axes \( l \) situated in its center. We use the Stokes theorem to express this volume integral through the integral over the cylinder surface surrounding the trisectrice
\[ \int d^2 r \, dl \, \nabla \vec{h} = \int_{-\infty}^{\infty} \frac{dl}{l} 2\pi \]  
(3.35)
independently of its radius \( a \). It means, that the correct equation for \( \vec{h} \) is
\[ \nabla \vec{h} = 4\pi \rho, \quad \rho = \frac{1}{2} \frac{1}{l} \delta^2(r), \]  
(3.36)
where \( \rho \) is the magnetic monopole density and \( \vec{r} \) is the coordinate transversal to the vector \( \vec{n}_0 \). The monopole string with an infinitesimal small radius is stretched along the vector \( \vec{n}_0 \).

We can calculate the density \( \rho \) directly from the expression for \( \vec{h} \) using the relations
\[ \nabla_3^2 \ln |r|^2 = \nabla_3^2 \frac{2\vec{r}}{|r|^2} = 4\pi \delta^2(r). \]  
(3.37)

Let us find the scalar potential corresponding to this magnetic density. It satisfies the equation
\[ \nabla_3^2 \varphi = \rho. \]  
(3.38)
Using the well known relation
\[ \nabla_3^2 \frac{1}{|\vec{R}|} = -4\pi \delta^3(R), \quad \vec{R} = l\vec{n}_0 + \vec{r}, \quad |\vec{R}|^2 = l^2 + |r|^2 \]  
(3.39)
we obtain for the scalar potential

$$
\varphi(l, \overrightarrow{r}) = -\frac{1}{2} \int_{-\infty}^{\infty} \frac{dl'}{l'} \frac{1}{\sqrt{(l-l')^2 + |\overrightarrow{r}|^2}} = \frac{1}{2} \ln \frac{|\overrightarrow{R}| - l}{|\overrightarrow{R}| + l}.
$$

One can easily calculate the magnetic field which corresponds to this density

$$
\overrightarrow{h}_0^{\text{rot}} = \partial_0 \varphi = -\frac{l}{2 |\overrightarrow{R}|^3} \ln \frac{|\overrightarrow{R}| - l}{|\overrightarrow{R}| + l} - \frac{1}{2 |\overrightarrow{R}|^2} + \frac{l \overrightarrow{r}}{2 |\overrightarrow{R}|^3 |r|^2 |\overrightarrow{R}|^2}.
$$

This is only a potential part of the field $h_\mu$, which can be written as follows

$$
h_\mu = h_\mu^{\text{pot}} + h_\mu^{\text{rot}}, \quad h_\mu^{\text{pot}} = \frac{\partial_\mu \varphi}{|\overrightarrow{R}|^2}, \quad h_\mu^{\text{rot}} = -\frac{1}{\partial^2} \left[ \nabla, \left[ \nabla, h \right] \right]
$$

and $h_\mu^{\text{rot}}$ is related to the existence of closed electric currents. By subtracting $h_\mu^{\text{pot}}$ from $h_\mu$ we obtain

$$
h_\mu^{\text{rot}} = -\frac{l}{2 |\overrightarrow{R}|^3} \ln \frac{|\overrightarrow{R}| - l}{|\overrightarrow{R}| + l} + \frac{1}{2 |\overrightarrow{R}|^4} \ln \frac{|\overrightarrow{R}| - l}{|\overrightarrow{R}| + l} + \frac{l \overrightarrow{r}}{2 |\overrightarrow{R}|^3 |r|^2 |\overrightarrow{R}|^2}.
$$

The divergency of this vector does not contain the singularity at $\overrightarrow{r} = 0$

$$
\partial_i h_i^{\text{rot}} = \left( \frac{|r|^2 - 2l^2}{2 |\overrightarrow{R}|^5} + \frac{2l^2 - |r|^2}{2 |\overrightarrow{R}|^5} \right) \ln \frac{|\overrightarrow{R}| - l}{|\overrightarrow{R}| + l} + \frac{l \overrightarrow{r}}{2 |\overrightarrow{R}|^4} - \frac{l \overrightarrow{r}}{4 |\overrightarrow{R}|^4} + 2 \frac{l}{|\overrightarrow{R}|^4} = 0.
$$

Note, however, that the vector $\overrightarrow{h}_\mu^{\text{rot}}$ does not have this property after the rotation around the axes $l$ on the angle $2\pi/3$ and therefore it can not be expressed in terms of the ternary number with the single analyticity.

The electromagnetic current according to the Maxwell equations can be found in terms of $\overrightarrow{h}_\mu^{\text{rot}}$ as follows

$$
h_0 = \frac{1}{|r|^2}, \quad h_i = \frac{r_i}{l |r|^2} (i = 1, 2),
$$

$$
\overrightarrow{j} = \left[ \nabla, h \right] = \left[ \nabla, \overrightarrow{h}_\mu^{\text{rot}} \right],
$$

because the potential part does not give to it any contribution.

In an explicit form we have for our case

$$
j_0 = 0, \quad j_k = \epsilon_{kl} r_l \left( -\frac{2}{|r|^4} + \frac{1}{l^2 |r|^2} \right),
$$

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where $\epsilon_{kl}$ is the two-dimensional anti-symmetric tensor with $\epsilon_{12} = 1$. Because $\vec{j} \cdot \vec{r} = 0$, the current $\vec{j}$ is tangential to the circles enclosing the trisectrice $\vec{n}_0$ and have the same value on them

$$|j| = |r| \left| -\frac{2}{|r|^4} + \frac{1}{|l|^2|r|^2} \right|. \quad (3.48)$$

Their direction is changed to opposite one at the distance

$$|r| = \sqrt{2}l. \quad (3.49)$$

We have the following representation for $h^{rot}_\mu$ in terms of the vector-potential $\vec{A}$

$$h^{rot}_\mu = [\vec{\nabla}, \vec{A}]. \quad (3.50)$$

It is valid everywhere including the trisectrice. Instead of this equation for $\vec{A}$ we consider a simpler equation valid in the region of analyticity of $\vec{h}$

$$\vec{h} = \frac{\vec{r}}{|l|\vec{r}|^2} = \text{rot}\vec{A},$$

Choosing the gauge

$$A_0 = 0, \quad (3.51)$$

we find its solution in the form

$$A_0 = 0,$$

$$A_1 = \frac{r_2}{r_1^2 + r_2^2} \ln \frac{l}{\sqrt{r_1^2 + r_2^2}},$$

$$A_2 = -\frac{r_1}{r_1^2 + r_2^2} \ln \frac{l}{\sqrt{r_1^2 + r_2^2}}. \quad (3.52)$$

For $W = \varphi_0 + \varphi_1 q + \varphi_2 q^2$, note, that one can consider the integration of the three form

$$\int_M W d\vec{z} \wedge d\vec{z} \wedge d\vec{z} = \int_M (\omega_0^{(3)} + \omega_2^{(3)} q^2 + \omega_1^{(3)} q), \quad (3.53)$$

$$\omega_i^{(3)} = \varphi_i dx_0 \wedge dx_1 \wedge dx_2. \quad (3.54)$$

In this case there is also a correspondence between the integrals written in the vector calculus and in terms of differential forms:
\[
\int_M W dV \leftrightarrow \int_M \Omega_W^{(3)} \\
\int_M (\nabla \times \vec{H}) dV \leftrightarrow \int_M d\Omega_W^{(3)}
\]

(3.55)

If \( V \) is a volume in \( \mathbb{R}^3 \), then we have:

\[
\int_{\partial V} (\vec{H} d\vec{S}) = \int_{\partial V} \Omega_W^{(2)} = \int_V \Omega_W^{(3)} = \int_V (\nabla \cdot \vec{H}) dV
\]

(3.56)

4 Ternary mechanics and monopole dynamics

4.1 Newton equation for a particle in a ternary field

Let us consider the ternary generalization of the classical Newton equation

\[
\frac{d^2 z}{(dt)^2} = -G \frac{1}{z \bar{z}}.
\]

(4.1)

where \( G \) is a coupling constant. We can write the Newton equation in the vector form

\[
\frac{d^2 \vec{x}}{(dt)^2} = -G \frac{\vec{x}}{Z^3},
\]

(4.2)

where \( \vec{x} = (x_1, x_2, x_0) \) and

\[
Z^3 \equiv z \bar{z} \tilde{z} = x_1^3 + x_2^3 + x_0^3 - 3x_1 x_2 x_0.
\]

(4.3)

We interpret the ternary Newton equation as the equation for a monopole moving in the magnetic field \( \vec{H} \) of the type

\[
\vec{H} = \frac{\vec{x}}{Z^3},
\]

(4.4)

because due to the analyticity condition for the ternary function \( H = H_0 + qH_1 + q^2H_2 \) in an accordance with the Maxwell equations one obtains \( \nabla \cdot \vec{H} = 0 \) everywhere apart the singularities on the trisectrice \( x_1 = x_2 = x_0 \). After the change of coordinates (3.32)-(3.33) the Newton equation becomes

\[
\frac{d^2 \vec{r}}{(dt)^2} = -g \frac{\vec{h}}{|r|^2 l^2}, \quad \vec{h} = \frac{\vec{r}}{|r|^2 l}, \quad g = G \frac{2}{3 \sqrt{3}}.
\]

(4.5)

Thus, we consider in fact the movement of a monopole-type object in the magnetic field of a simple form, created by the monopoles situated on the line \( r_1 = r_2 = 0 \)

\[
4\pi \rho = \partial_0 h_0 + \partial_1 h_1 + \partial_2 h_2 = \frac{2\pi}{|l|} \delta^2(r)
\]

(4.6)
and by electromagnetic currents
\[ j_0 = 0, \quad j_k = \epsilon_{kl} r_t \left( -\frac{2}{|r|^4} + \frac{1}{l^2 |r|^2} \right), \quad (4.7) \]
circulated around the trisectrice. We show below, that the above ternary equation is integrable similar to the case of the Newton equation in the central gravitational field.

It is obvious, that there is an integral of motion - the angular moment
\[ \vec{M} = [\vec{r}, \partial_t \vec{r}], \quad \frac{d}{dt} \vec{M} = 0 \quad (4.8) \]
or in the components
\[ M_0 = r_1 \partial_t r_2 - r_2 \partial_t r_1, \quad M_1 = r_2 \partial_t l - l \partial_t r_2, \quad M_2 = l \partial_t r_1 - r_1 \partial_t l. \quad (4.9) \]

To begin with, let us consider the particular case, in which the particle moves in the plane to which the trisectrice belongs. Without the loss of generality we can put
\[ r_2 = M_0 = M_1 = 0. \quad (4.10) \]
In this case the only non-trivial integral of motion is
\[ M_2 = l \partial_t r_1 - r_1 \partial_t l. \quad (4.11) \]
This constraint should be considered together with the Newton equation for \( r_1 \)
\[ \partial_t^2 r_1 = -\frac{g}{r_1} \frac{1}{l}. \quad (4.12) \]

It is convenient to introduce the new variable
\[ z = \frac{l}{r_1}. \quad (4.13) \]
In the variables \((r_1, z)\) we can rewrite as follows the integral of motion
\[ M_2 = -r_1^2 \partial_t z. \quad (4.14) \]
and the Newton equation
\[ \partial_t^2 r_1 = -\frac{g}{r_1^2} \frac{1}{z} = \frac{g}{M_2} \partial_t z \quad (4.15) \]
One can integrate the last equation
\[ \partial_t r_1 = \frac{g}{M_2} \ln \left| \frac{z}{z_0} \right|, \quad (4.16) \]
where \(z_0\) is the value of the ratio \(l/r_1\) on the trajectory for which \(\partial_t r_1 = 0\).

With the use of the conserved momentum \(M_2\) we rewrite the above integral of motion as follows
\[ \partial_t \sqrt{-\frac{M_2}{\partial_t z}} = -\frac{1}{2\partial_t z} \sqrt{-\frac{M_2}{\partial_t z}} \partial_t^2 z = \frac{g}{M_2} \ln \left| \frac{z}{z_0} \right| \quad (4.17) \]
and integrate it again finding the monopole trajectory

$$-\sqrt{-M_2}\partial_t z = \frac{M_2}{r_1} = \frac{g}{M_2} \left( z \ln \left| \frac{z}{e^{z_0}} \right| - z_1 \ln \left| \frac{z_1}{e^{z_0}} \right| \right), \quad (4.18)$$

where $z_1$ is the value of the ratio $l/r_1$, for which $\partial_t z = 0$.

Integrating the last relation, we obtain the relation

$$t = -\frac{M_2^3}{g^2} \int^z dz' \left( z' \ln \left| \frac{z'}{e^{z_0}} \right| - z_1 \ln \left| \frac{z_1}{e^{z_0}} \right| \right)^{-2} \quad (4.19)$$

determining the dependence of $z$ from $t$.

Because the integral over $z'$ is divergent at $z' = z_1$, we conclude, that at $t \to \pm \infty$ the coordinates $l$ and $r_1$ grow linearly with $t$, but their ration $z$ tends to the constants $z_1$ or $\tilde{z}_1$, where $\tilde{z}_1$ is the second solution of the equation

$$\tilde{z}_1 \ln \left| \frac{\tilde{z}_1}{e^{z_0}} \right| = z_1 \ln \left| \frac{z_1}{e^{z_0}} \right| \quad (4.20)$$

appearing if $|z_1| < |ez_0|$. Really the particle does not perform any oscillation before going to infinity. We show below, that this instability takes place in the general case.

Using the above expression for the angular momenta one can express $l$ in terms of the coordinates $r_1$ and $r_2$

$$l = -\frac{M_1}{M_0} r_1 - \frac{M_2}{M_0} r_2 \quad (4.21)$$

Because $M_0$ is conserved, it is enough to consider only the first vector component of the Newton equation

$$\partial^2_t r_1 = \frac{g}{r_1^2} \frac{1}{(1 + y^2) \left( \frac{M_1}{M_0} + \frac{M_2}{M_0} \right)} \quad (4.22)$$

where we introduced the new variable $y$

$$y = \frac{r_2}{r_1} \quad (4.23)$$

In the variables $r_1$ and $y$ the integral of motion $M_0$ has the form

$$M_0 = r_1^2 \partial_t y \quad (4.24)$$

and therefore the equation for $r_1$ can be written as follows

$$\partial^2_t r_1 = \frac{g}{M_0} \frac{\partial_t y}{(1 + y^2) \left( \frac{M_1}{M_0} + \frac{M_2}{M_0} \right)} \quad (4.25)$$

By integrating it we obtain the fourth integral of motion

$$\partial_t r_1 = \frac{g}{M_0} \int^{y_0}_y \frac{d\tilde{y}}{(1 + \tilde{y}^2) \left( \frac{M_1}{M_0} + \frac{M_2}{M_0} \right)} \quad (4.26)$$
where the constant \( y_0 \) is equal to the value of \( r_2/r_1 \) for which \( \partial_t r_1 = 0 \).

Now it is convenient to use again the integral of motion \( M_0 \) to express \( r_1 \) in terms of \( y \)

\[
- \frac{1}{2} \partial_t y \sqrt{M_0 \partial_t y \partial_t y} = \frac{g}{M_0} \int_{y_0}^{y} \frac{d\tilde{y}}{(1 + \tilde{y}^2) \left( \frac{M_1}{M_0} + \frac{M_2}{M_0} \tilde{y} \right)}.
\]

This operation allows one to find the fifth integral of motion

\[
\sqrt{M_0 \partial_t y} = -\frac{g}{M_0} \int_{y_1}^{y} dz \int_{y_0}^{z} \frac{d\tilde{y}}{(1 + \tilde{y}^2) \left( \frac{M_1}{M_0} + \frac{M_2}{M_0} \tilde{y} \right)},
\]

where the constant \( y_1 \) is the value of the ratio \( r_2/r_1 \) for which \( \partial_t y = 0 \). We can write this integral of motion in the form

\[
\frac{M_0}{r_1} = \sqrt{M_0 \partial_t y} = -\frac{g}{M_0} \int_{y_0}^{y} \frac{d\tilde{y}}{(1 + \tilde{y}^2) \left( \frac{M_1}{M_0} + \frac{M_2}{M_0} \tilde{y} \right)}. \tag{4.29}
\]

The last equation gives a possibility to calculate the particle trajectory parametrized by \( y \)

\[
r_1 = \frac{r_2}{y} = -\frac{l}{\frac{M_1}{M_0} + \frac{M_2}{M_0} y} = -\frac{|M_0|}{g} \left( \int_{y_0}^{y} \frac{d\tilde{y}}{(1 + \tilde{y}^2) \left( M_1 + M_2 \tilde{y} \right)} \right)^{-1}. \tag{4.30}
\]

To find the coordinate dependence from time one should invert the equation

\[
t = \frac{M_0}{g^2} \int_{y_0}^{y(t)} dz \left( \int_{y_0}^{z} \frac{d\tilde{y}}{(1 + \tilde{y}^2) \left( M_1 + M_2 \tilde{y} \right)} \right)^{-2}. \tag{4.31}
\]

Thus, in the general case we see, that due to the divergency of the integral over \( z \) at \( z \to y_1 \) the particle in the ternary potential at \( t \to \pm \infty \) goes to infinity along the line with a fixed ratio of coordinates

\[
r_1 = \frac{r_2}{y_1} = -\frac{l}{\frac{M_1}{M_0} + \frac{M_2}{M_0} y_1} \sim t. \tag{4.32}
\]

The reason for this instability is the reflective character of the ternary force at \( l < 0 \). In principle a particle or a large cosmic object with the magnetic field of the ternary type can exist in nature. Therefore it is interesting to calculate the differential cross-section for the particle scattering off such field.

### 4.2 Monopole scattering in the ternary field

Here we investigate the scattering of the monopole in the magnetic field \( h \) introduced above. Initially we consider the symmetric case, where the trisectrice lies in the scattering plane \((r_1, l)\) fixed by the angular momentum \( \bar{M} = (0, M_2, 0) \). The monopole trajectory approaches the critical slope \( z = l/r_1 \to z_1 \) only asymptotically

\[
r_1 \to \frac{M_2^2}{g} \frac{1}{(z - z_1) \ln \left| \frac{z_1}{z_0} \right|}, \quad \bar{r}_1 \to \frac{g}{M_2} \ln \left| \frac{z_1}{z_0} \right|, \quad \frac{l}{r_1} \to z_1. \tag{4.33}
\]
Apart from $M_2$ one can fix for the colliding monopole also the asymptotic slope $z_1$ and its velocity

$$v_1^{-\infty} = \lim_{t \to -\infty} \partial_t r_1. \quad (4.34)$$

The velocity component along the axis $r_0 = l$ is calculated in terms of $v_1^{-\infty}$ and $z_1$

$$v_0^{-\infty} = v_1^{-\infty} z_1. \quad (4.35)$$

Together with $M_2$ the total velocity determines the impact parameter $b = M_2/|v^{-\infty}|$. Note, that the velocity vector at $t \to -\infty$ should be directed to the center

$$\frac{\vec{v}}{|v|} = - \frac{\vec{R}}{|R|}. \quad (4.36)$$

The initial conditions fix also the parameter $z_0$

$$v_1^{-\infty} = \frac{g}{M_2} \ln |\frac{z_1}{z_0}|. \quad (4.37)$$

Due to the invariance of the equations under the time inversion $t \to -t$ and the reflection $r_1 \to -r_1$ it is enough to investigate only the positive values of $M_2$ and $z_1$

$$M_2 > 0, \; z_1 > 0. \quad (4.38)$$

Let us consider initially the case $z_0 < z_1$.

The asymptotic slope $z = z_1$ can be reached at $t = -\infty$ or $t = \infty$. At the first sub-case the monopole at large negative $t$ moves to the center along the line $l/r_1 = z_1 - 0$ from the negative values of $l$ and $r_1$ with the initial velocity $(v_1^{-\infty}, v_0^{-\infty})$. The velocity $v_1$ changes its sign at $z = z_0$. Providing that $z_1 < ez_0$ corresponding to the condition $v_1^{-\infty} < g/M_2$, the monopole turns before reaching the axes $r_1 = 0$, $l = 0$ and moves backward along the asymptotic line $z = \tilde{z}_1 < z_0$ being another solution of the equation

$$\tilde{z}_1 \ln \left|\frac{\tilde{z}_1}{ez_0}\right| = z_1 \ln \left|\frac{z_1}{ez_0}\right|. \quad (4.39)$$

In particular, we have $\tilde{z}_1 \to z_0 - \epsilon$ for $z_1 \to z_0 + \epsilon$ and $\tilde{z}_1 \to eez_0/\ln(1/\epsilon)$ for $z_1 \to eez_0 - \epsilon$. Note, that for $M_2 < 0$ the monopole goes along the same trajectory but in the opposite direction if one would interchange $z_1 \leftrightarrow \tilde{z}_1$.

If $z_1 > ez_0$ the particle approaches at $z = 0$ the line $l = 0$ where its velocity $v_1$ tends to $-\infty$. In this moment we have

$$\lim_{l \to 0} r_1 = - \frac{M_2^2}{g} \frac{1}{z_1 \ln \left|\frac{z_1}{ez_0}\right|} < 0. \quad (4.40)$$

After that the velocity $|v_1|$ decreases and vanishes at $z = -z_0$. At $z \to -\infty$ the monopole trajectory approaches the point $r_1 = -0$, $l = +0$

$$\lim_{z \to -\infty} r_1 \to \frac{M_2^2}{g} \frac{1}{z \ln \left|\frac{z}{ez_0}\right|}, \quad \lim_{z \to -\infty} l \to \frac{M_2^2}{g} \frac{1}{\ln \left|\frac{z}{ez_0}\right|}. \quad (4.41)$$
Simultaneously the velocities grow rapidly

\[
\lim_{z \to -\infty} v_1 = \frac{g}{M_2} \ln \left| \frac{z}{z_0} \right|, \quad \lim_{z \to -\infty} v_0 = \frac{g}{M_2} z \ln \left| \frac{z}{z_0} \right| \tag{4.42}
\]

and the monopole reaches the coordinate center with an infinite energy for a finite period of time. One can prolong the trajectory in the sector \( l < 0, r_1 > 0 \) where after the change of the velocity sign \( v_1/v_1 \) at \( z = -z_0 \) the particle goes asymptotically to \(-\infty\) along the axes \( l \). In the second sub-case the monopole moves to \( r_1 = l = \infty \) along the line \( z = z_1 + 0 \) from the coordinate center \( r_1 = +0, l = +0, l/r_1 = \infty \).

In the case \( z_0 < z_1 \) we have \( v_1 < 0 \) for \( z \to z_1 \) and therefore the monopole can arrive from \( r_1 = l/z_1 = \infty \) or depart to \( r_1 = l/z_1 = -\infty \). In the first sub-case it reaches the line \( l = 0 \) at \( z = 0 \) with the velocity \( v_1 = -\infty \) in the point

\[
\lim_{l \to 0} r_1 = -\frac{M_2^2}{g} \frac{1}{z_1 \ln \left| \frac{z_1}{ez_0} \right|}. \tag{4.43}
\]

Then in the point \( z = -z_0 \) its velocity \( v_1 \) changes its sign and the monopole goes to \( r_1 = \infty \) at \( t \to \infty \) for the value \( z' < 0 \) which is a solution of the equation

\[
z' \ln \left| \frac{z'}{ez_0} \right| = z_1 \ln \left| \frac{z_1}{ez_0} \right|. \tag{4.44}
\]

In the second sub-case the monopole arises at \( r_1 = -0, l = -\infty \) at \( z = -\infty \) and goes to \(-\infty\) at \( z \to z_1 - 0 \).

Let us consider now the movement of the monopole in the general case, when the scattering plane does not contain the trisectrice. The initial data include the three components of angular momentum \( \overrightarrow{M} \), and velocity components

\[
v_1^{-\infty} = \frac{v_2^{-\infty}}{y_1} = -\frac{v_0^{-\infty}}{\frac{M_1}{M_0} + \frac{M_2}{M_0} y_1}, \tag{4.45}
\]

where \( y_1 \) is the integral of motion. It means, that the angular momenta \( M_i \) satisfy the relation

\[
M_1 v_1^{-\infty} + M_2 v_2^{-\infty} + M_0 v_0^{-\infty} = 0 \tag{4.46}
\]

and only two of them \((M_1, M_2)\) are independent.

Note, that due to the symmetry to the rotations around the axes \( l \) without any restriction of generality one can put

\[
v_2^{-\infty} = y_1 = 0.
\]

In this case the introduced above parameter \( z_1 \) is

\[
z_1 = -\lim_{y_1, M_0 \to 0} \left( \frac{M_1}{M_0} + \frac{M_2}{M_0} y_1 \right).
\]

The other integral of motion \( y_0 \) is fixed by the equation

\[
v_1^{-\infty} = \frac{g}{M_0} \int_{y_0}^{y_1} \frac{d \tilde{y}}{1 + \tilde{y}^2} \left( \frac{M_1}{M_0} + \frac{M_2}{M_0} \tilde{y} \right).
\]

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\[\begin{align*}
&= g \left( \frac{i/2}{M_1 - iM_2} \ln \frac{y_1 - i}{y_0 - i} - \frac{i/2}{M_1 + iM_2} \ln \frac{y_1 + i}{y_0 + i} + \frac{M_2}{M_1^2 + M_2^2} \ln \frac{M_1 + y_1M_2}{M_1 + y_0M_2} \right). 
\end{align*}\] (4.47)

It allows one to calculate the velocity \(v_1\) as function of the parameter \(y = r_2/r_1\)

\[v_1(y) = g \left( \frac{i/2}{M_1 - iM_2} \ln \frac{y - i}{y_0 - i} - \frac{i/2}{M_1 + iM_2} \ln \frac{y + i}{y_0 + i} + \frac{M_2}{M_1^2 + M_2^2} \ln \frac{M_1 + yM_2}{M_1 + y_0M_2} \right). \] (4.48)

and the trajectory

\[r_1 = \frac{r_2}{y} = -\frac{l}{M_1/M_0 + M_2/M_0} = -\frac{M_0}{g} \left( \int_{y_1}^y dy' \frac{v_1(y')}{g} \right)^{-1}. \] (4.49)

Here

\[\int_{y_1}^y dy' \frac{v_1(y')}{g} = \frac{i(y - i)/2}{M_1 - iM_2} \ln \frac{y - i}{e(y_0 - i)} - \frac{(y + i)/2}{M_1 + iM_2} \ln \frac{y + i}{e(y_0 + i)} + \frac{M_1 + yM_2}{M_1^2 + M_2^2} \ln \frac{M_1 + yM_2}{M_1 + y_0M_2}. \]

The above formulas give a possibility to calculate the differential cross-section for the scattering of the monopole moving at \(t \to -\infty\) with the velocity \(\bar{v}^{-}\) to the coordinate center

\[d\sigma = d^2\rho = \rho d\rho d\varphi, \] (4.50)

where \(\rho\) is impact parameter and \(\varphi\) is its azimuthal angle around the velocity \(\bar{v}^{-}\). We have the following relation between \(\vec{M}\) and \(\vec{\rho}\)

\[\vec{M} = [\vec{\rho}, \bar{v}^{-}] , \quad \vec{\rho} = \frac{1}{|\bar{v}^{-}|^2} [\bar{v}^{-}, \vec{M}] , \] (4.51)

where we took into account, that \((\vec{\rho}, \bar{v}^{-}) = 0\). As it was argued above, one can put \(y = v_2^{-} = 0\) without the generality loss. In this case the above relations are simplified

\[\rho^d|\bar{v}^{-}| = M_2 , \quad \rho^\perp|\bar{v}^{-}|^2 = M_1v_1^{-} - M_0v_1^{-} = M_1 \frac{|\bar{v}^{-}|^2}{v_0^{-}}, \] (4.52)

where we introduced the components \(\rho^d\) and \(\rho^\perp\) belonging to the plane \((r_1, l)\) and orthogonal to it, respectively. These expressions give a possibility to calculate the Jacobian of the transition from the vector \(\vec{\rho}\) to the variables \(M_{1,2}\)

\[d^2\rho = \frac{dM_1 dM_2}{|v_0^{-}| |\bar{v}^{-}|}. \] (4.53)

Further, the parameters of the final states can be expressed in terms of angular momenta \(M_1, M_2\). For example, if the monopole moves from \(r_1 = r_2 = l = -\infty\) with such small velocity, that the equation

\[\int_{y_1}^y dy' \frac{v_1(y')}{g} = 0 \] (4.54)
has the other solution \( y = \tilde{y}_1(M_1, M_2) \) apart from trivial one \( y = y_1 \), than the momenta of the particle at \( t \to \infty \) are related by the formulas
\[
v_1^\infty = \frac{v_2^\infty}{y_1} = -\frac{v_0^\infty}{M_1}\frac{M_2}{M_0} \tilde{y}_1, \quad (4.55)
\]
The energy is not conserved in the ternary field and therefore the final kinetic energy \( E \) is also a function of the angular momenta \( M_1, M_2 \)
\[
E(M_1, M_2) = \frac{(\vec{v}^\infty)^2}{2} \quad (4.56)
\]
and can be obtained from the previous formulas. Therefore we can calculate the Jacobian of the transition from the momenta \((M_1, M_2)\) to the variables \((\tilde{y}_1, E)\)
\[
d\tilde{y}_1 \, dE = J \, dM_1 \, dM_2, \quad J = \det \left( \begin{array}{cc} \frac{\partial \tilde{y}_1}{\partial M_1} & \frac{\partial \tilde{y}_1}{\partial M_2} \\ \frac{\partial E}{\partial M_1} & \frac{\partial E}{\partial M_2} \end{array} \right). \quad (4.57)
\]
Thus, the differential cross-section for small momenta of the monopole moving to the center from negative values of \( r_1, r_2, l \) is
\[
d\sigma = \frac{d\tilde{y}_1 \, dE}{J |\vec{v}_0^\infty| |\vec{v}^\infty|}. \quad (4.58)
\]

**Acknowledgments** One of us (GGV) acknowledges Laboratoire de Physique Théorique, Strasbourg and LAPTH Annecy-Le-Vieux, where part of the work was done, for hospitality. He also thanks for discussions and support L. Alvarez-Gaume, U. Aglietti, I. Antoniadis, G. Belanger, G. Costa, I. Crotty, B. Dubrovin, A. Dubrovskiy, J. Ellis, T. Faberge, L. Fellin, G. Girardi, M. Goze, R. Kerner, A. Masiero, J. Richert, A. Sabio-Vera, P. Sorba, A. Uranga and J.-B. Zuber.

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