Paradoxes in Sequential Voting

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Abstract

We analyse strategic, complete information, sequential voting with ordinal preferences over the alternatives. We consider several voting mechanisms: plurality voting and approval voting with deterministic or uniform tie-breaking rules. We show that strategic voting in these voting procedures may lead to a very undesirable outcome: Condorcet winning alternative might be rejected, Condorcet losing alternative might be elected, and Pareto dominated alternative might be elected. These undesirable phenomena occur already with four alternatives and a small number of voters. For the case of three alternatives we present positive and negative results.

1 Introduction

Traditionally, most game-theoretic models of voting study voting in a simultaneous setting. The reason is, no doubt, due to the fact that we perceive confidentiality as a necessary condition for the fairness of the election process. However, we do see many issues decided upon by means of sequential voting. In small committees and even in parliaments, sequential, open ballot is often the default method to make a decision.

When we speak of a possible voting outcome we mean the result when the voting is in an equilibrium. However, while simultaneous voting translates to a normal-form game and its solution relies on the notion of Nash equilibrium, the sequential voting translates to an extensive-form game for which there is the stronger notion of a subgame perfect Nash equilibrium (SPE). This equilibrium is guaranteed to always exist, and when the voters’ preferences are total orders, it also leads to a unique outcome.

While on the one hand a lot of work has been done in axiomatic comparison of simultaneous voting systems (e.g. [8]) and on the other hand there has been some interest in sequential voting with strategic voters ([7, 5]), the understanding of sequential voting, especially in the case of more than two alternatives, is very limited. As a first step toward the understanding of sequential voting, we analyse it in a complete information setting. Though in voting interactions with

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a large number of voters this assumption is unrealistic; in small committees it is more plausible. The goal of this note is to understand how good is the performance of sequential voting in scenarios with a small number of voters and a small number of alternatives (but not necessarily two).

We consider a setting of voters who have ordinal preferences over the alternatives. In such a setting it is not clear how to measure the quality of the elected alternative. However, there are some scenarios where a) it is clear which alternative is desirable, as for instance, if there is a Condorcet winner (2.1); b) it is clear which alternative is undesirable, as for instance, a Condorcet loser (2.2) or a Pareto dominated alternative (2.3, 2.4). We pose the questions: does sequential voting guarantee a choice of a desirable alternative whenever it exists? Does sequential voting guarantee that an undesirable alternative will not be elected?

Our results are mainly negative. Although in a two-alternatives setting the sequential voting works perfectly well; already with four alternatives the choice of (rejection of) a desirable (undesirable) alternative is not guaranteed by sequential voting. In case of three alternatives, we have both positive and negative results.

1.1 Related work

The field of computational social choice deals with mechanisms to collect agents’ preferences about different alternatives, and aggregate them to a single ranking (see, for instance, the recent handbook, [3]). Among the vast branches of this field is voting theory, which deals with axioms for ‘good’ voting rules (e.g. [9], [1]) and the analysis of voting mechanisms, scoring rules and the like (e.g. [8], [10]). In this paper we restrict our attention to voting mechanisms in which the voters vote sequentially.¹

In [7] Desmedt and Elkind analyse simultaneous and sequential voting systems employing the plurality voting rule. In their model a voter’s preference is a total ordering of all subsets of the alternatives such that it is consistent with a total ordering of the alternatives themselves; thus their model accepts draws as a legitimate result. They assume the voters are abstain-biased, i.e. a voter will not vote unless pivotal. They prove that in a plurality sequential voting with two alternatives, the winner is the most popular (if one exists), and the voters are truthful whenever they vote. They then show that these nice properties no longer hold when there are three alternatives. For instance, a voter may strategically abstain or vote for an alternative which will not be selected.

Conitzer and Xia ([5]) discuss sequential voting under a wide range of voting rules, classified by their domination-index (defined to be the smallest number such that any coalition of this size can make any alternative a winner). They prove a general necessary criterion for an alternative to be a winner, and show a voting scenario with an arbitrary number (but at least three) of alternatives such that the winner loses all pairwise comparisons but one, and is actually ranked in the bottom two positions in almost all voters’ preferences.

Dekel and Piccione ([6]), and later Battaglini ([2]), investigate the information cascade in sequential voting with two alternatives, and compare its equilibria to the symmetric equilibria of simultaneous voting.

¹Notice that the term ‘sequential voting’ is often used with respect to a completely different setting, where the voting procedure is divided into several stages; at each stage the voting is for a subset of the alternatives, e.g. [4], [11].
2 Model and Definitions

Denote by $V = \{v_1, \ldots, v_n\}$ the set of voters and by $A = \{a_1, \ldots, a_m\}$ the set of alternatives. We assume that for any $v_i \in V$, his preferences over the alternatives can be expressed as a total ordering $(A, \succ_{v_i})$. We denote by $\rho = \{\succ_{v_i}\}_{i=1}^n$ the collection of the voters’ preferences, and call the triplet $(V, A, \rho)$ a voting profile. We assume the set $V$ is totally ordered and the voters vote one by one according to this order. It is also assumed the voters have complete information, both about $\rho$ and about the voting order.

We consider two types of voting rules:

1.) **Sequential plurality voting:** each voter, in his turn, votes to at most one alternative (abstentions allowed).

2.) **Sequential approval voting:** each voter may vote to any subset of the alternatives.

In order for the voting system to be well defined we must also define a tie-breaking rule. We consider two settings:

1.) **Deterministic:** there is a predefined total ordering of the alternatives. The winner is the one which is positioned highest among all alternatives that received the highest number of votes.

2.) **Uniform:** randomly and uniformly pick a winner from all those who received the highest number of votes. Equivalently, we may think of the result as a ‘winning set’ instead of a winning alternative. Notice that when we use the uniform tie-breaking rule we must define the preferences of the voters over all possible subsets of alternatives. We therefore assume that each agent tries to maximize the probability of election of his most favourite alternative and then that of his second-most favourite and so on. Based on this assumption, we get the following relation on the power set of $A$. For any $S \subseteq A$ let $T_{v_i}(S)$ be the most favourite alternative of $v_i$ in $S$. Then for two non-empty subsets $S_k, S_\ell$,

$$S_k \succ_{v_i} S_\ell \iff$$

- $T_{v_i}(S_k) \succ_{v_i} T_{v_i}(S_\ell)$ or
- $T_{v_i}(S_k) = T_{v_i}(S_\ell)$ and $|S_k| < |S_\ell|$ or
- $T_{v_i}(S_k) = T_{v_i}(S_\ell)$ and $|S_k| = |S_\ell|$ and $(S_k - T_{v_i}(S_k)) \succ_{v_i} (S_\ell - T_{v_i}(S_k))$.

It’s not hard to verify that this relation defines a total ordering.

Thus, we have four combinations of voting rules and tie-breaking rules which define four different voting systems. We now present the paradoxes against which we examine our voting systems. Let $(A, \succ_{pw})$ be the binary relation which is defined by the pairwise comparison of alternatives. That is, for any $a_i, a_j \in A$

$$a_k \succ_{pw} a_\ell \iff$$

- $\# \{v_i \in V : a_k \succ_{v_i} a_\ell\} > \# \{v_i \in V : a_\ell \succ_{v_i} a_k\}$.

An alternative $a_k$ is called a Condorcet winner of $(V, A, \rho)$ if it beats pairwise all other alternatives, i.e. if $a_k \succ_{pw} a_\ell, \forall \ell \neq k$.

**Definition 2.1.** A Condorcet winner paradox is a voting scenario in which a Condorcet winner of $(V, A, \rho)$ is not part of the winning set.
An alternative $a_k$ is called a Condorcet loser of $(V, A, \rho)$ if it loses pairwise to all other alternatives, i.e. if $a_\ell \succ_{pw} a_k, \forall \ell \neq k$.

**Definition 2.2.** A Condorcet loser paradox is a voting scenario in which a Condorcet loser of $(V, A, \rho)$ is the lone winner.

An alternative $a_k$ is Pareto dominated by alternative $a_\ell$ in $(V, A, \rho)$ if for all $v_i \in V$, $a_\ell \succ_{v_i} a_k$.

**Definition 2.3.** A weak Pareto dominated paradox is a voting scenario in which a Pareto-dominated alternative in $(V, A, \rho)$ is part of the winning set.

**Definition 2.4.** A strong Pareto dominated paradox is a voting scenario in which a Pareto-dominated alternative in $(V, A, \rho)$ is the lone winner.

### 3 The Condorcet Winner and the Condorcet Loser Paradoxes

When there are only two alternatives the Condorcet winner and Condorcet loser paradoxes actually describe the same situation in which an alternative which is the most favourite by a majority of the voters, is not elected. It is not hard to see that this is impossible in the voting systems we consider here (see Corollary 1 of [7] for plurality voting). Here we show that already when there are three alternatives the classical voting systems are no longer resistant to those paradoxes. In all the examples, we list the voters’ preferences from high to low. We circle the votes in an SPE.

**Claim 3.1.** Already with three alternatives, there are examples of the Condorcet winner paradox and the Condorcet loser paradox, in all of our voting systems.

**Proof.** Consider the following sequential plurality voting scenario, which can either be seen as a five-voter uniform tie-breaking scenario or a four-voter and deterministic tie-breaker (in which case the last voter is the tie-breaker). It is easy to verify that $A$ is a Condorcet winner in either case. If Voter 1 votes for anything other than $B$, then Voters 2, 3 and 4 will vote for $A$, and $A$ will be elected. However, the vote of Voter 1 for $B$ creates a threat on Voters 2 and 3 since together with Voter 4 and Voter 5 (or the Tie-breaker) $B$ can beat $A$; thus they are forced to vote for $C$, and $C$ is elected.²

The two scenarios below demonstrate a situation where a Condorcet loser wins the election under the plurality-deterministic and plurality-uniform voting systems. In both of them $C$ is a Condorcet loser, but is a sure winner after the first three voters vote for him. A short analysis can show that any change in Voter 1’s vote leads the election of $B$ and any change in Voter 3’s vote leads to the election of $A$; thus none of them has a better vote.

²Interestingly, $C$ is Voter 1’s most preferred alternative, but to get it elected he must vote for $B$ — his least preferred alternative.
Four scenarios which demonstrate the rest of the paradoxes are presented below without analysis.

| Approval-Uniform, Condorcet winner loses | Approval-Deterministic, Condorcet winner loses |
|----------------------------------------|---------------------------------------------|
| Voter 1 C A B | Voter 1 C A B |
| Voter 2 C A B | Voter 2 C A B |
| Voter 3 A C B | Voter 3 A C B |
| Voter 4 A C B | Voter 4 A C B |
| Voter 5 B A C | Voter 5 B A C |
| Voter 6 A B C | Voter 6 A B C |
| Tie-Break C B A | Tie-Break B C A |

| Approval-Deterministic, Condorcet loser wins | Approval-Uniform, Condorcet loser wins |
|---------------------------------------------|----------------------------------------|
| Voter 1 B C A | Voter 1 B C A |
| Voter 2 B C A | Voter 2 B C A |
| Voter 3 A C B | Voter 3 A C B |
| Voter 4 A C B | Voter 4 A C B |
| Voter 5 B A C | Voter 5 B A C |
| Voter 6 A B C | Voter 6 A B C |
| Tie-Break C A B | Tie-Break A C B |

4 Pareto-efficiency

Consider the following Pareto-dominated paradox scenario for the Plurality-Deterministic voting system. Suppose we have one voter with preference order $A \succ C \succ B$ and another voter with preference $B \succ A \succ C$. Furthermore, assume the tie-breaking order is $C \succ B \succ A$. The first voter should not vote for $A$ as that would lead to the election of $B$ which is his worse outcome. He therefore votes for $C$, which is elected. Notice that $A$ dominates $C$ in the voters’ preferences, but the tie-breaking order gives preference to $C$ over $A$. We claim that the Plurality-Deterministic voting system is the only one of our four voting systems which allows this paradox.

**Proposition 4.1.** Let $S$ be a voting system with three alternatives and at least two voters. If $S$ is approval voting or it is plurality voting with uniform tie-breaking then the weak Pareto-dominated paradox cannot happen.

**Proof.** Assume that we have a voting scenario with alternatives $\{A, B, C\}$ and $C$ is Pareto-dominated by $A$. For $X \in \{A, B, C\}$, we call a voter which has $X$ at the top of his preference order an ‘$X$-type’ voter. Since $C$ is Pareto-dominated, there are only $A$-type and $B$-type voters.
Now, in any voting system, if there is a strict majority of $X$-type voters then they all just vote for $X$ in an SPE (since then $X$ is elected alone and that's the best possible result for them). We assume then that there is exactly the same number of $A$-type and $B$-type voters. If the tie-breaking is uniform then when all voters vote for their top alternative we get an SPE with the result $\{A, B\}$, and no voter can get a better result. The remaining case is when $S$ is approval-deterministic. Assume for contradiction that $C$ is selected. Since all the $B$-type voters rank $C$ last, any other outcome is better for them. Let $i$ be the first $B$-type voter. It must be that all the voters before $i$ vote for $C$, otherwise all the $B$-type voters may vote for $B$ and get a better outcome (since we assume that half of the voters are $B$-type). Now, suppose that all the voters before $i$ vote for $\{A, C\}$. Then still $B$ will not win in an SPE of the remaining voting; but on the other hand, voter $i$ can now vote for just $A$ which would allow $A$ to be elected since now the $A$-type voters together with voter $i$ outnumber the remaining $B$-type voters. This is a better result for all the voters, which is a contradiction to the assumption that $C$ is elected in an SPE.

The next proposition, then, settles the question of Pareto-efficiency for the systems with deterministic tie-breaking.

**Claim 4.2.** If there are at least four alternatives, then there are examples of the strong Pareto-dominated paradox for both plurality and approval voting systems with deterministic tie-breaking.

**Proof.** Consider first the plurality scenario. Here $C$ is Pareto-dominated by $A$. If the first two voters vote for $A$ then Voters 3, 4 and 5 will vote for $B$, who will be elected. However, if the first two voters vote for $C$ then if Voter 3 will now vote for $B$ then Voters 4 and 5 will get $D$ elected (notice that the tie-breaking prefers $A$ to $D$ to $C$), therefore Voter 3 would rather vote for $C$ in this case and $C$ gets elected.

### Plurality-Deterministic, Pareto-dominated wins

| Voter 1 | A | C | B | D |
| Voter 2 | A | C | B | D |
| Voter 3 | B | A | C | D |
| Voter 4 | D | B | A | C |
| Voter 5 | D | B | A | C |
| Tie-Break | B | A | D | C |

### Approval-Deterministic, Pareto-dominated wins

| Voter 1 | A | C | B | D |
| Voter 2 | A | C | B | D |
| Voter 3 | D | A | C | B |
| Voter 4 | D | A | C | B |
| Voter 5 | D | A | C | B |
| Tie-Break | D | A | B | C |

The scenario for the approval voting is a bit different. Again $C$ is Pareto-dominated by $A$. The first two voters vote for $\{C, B\}$ thus creating a threat on Voters 3 and 4 that if they don’t vote for $C$ as well then $B$ will be elected. If Voters 1 and 2 change their vote (for example, for $\{A, B\}$) then Voter 3 will vote for $\{D, A\}$, Voter 2 will vote for $D$ and Voter 5 will be forced to vote for $D$ as well, making $D$ the winner.

**Claim 4.3.** If there are at least four alternatives, then there are examples of the weak Pareto-dominated paradox for both plurality and approval voting systems with uniform tie-breaking.

**Proof.** We give here without analysis two examples in which a Pareto-dominated alternative is in the winning set (but not alone).
Plurality-Uniform, Pareto-dominated wins
Voter 1: A B D C
Voter 2: D A C B
Voter 3: D C B A
Voter 4: A B D C
Voter 5: B A D C
Voter 6: B D A C

Approval-Uniform, Pareto-dominated wins
Voter 1: A B D C
Voter 2: D C B A
Voter 3: D C A B
Voter 4: A B D C
Voter 5: A B D C
Voter 6: B D A C

The question of whether systems with uniform tie-breaking are weakly Pareto-efficient remains open.

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