Integrability and solvability of polynomial Liénard differential systems

Maria V. Demina

Department of applied mathematics, HSE University, Moscow, Russian Federation

Correspondence
Maria V. Demina, HSE University, 34 Tallinskaya St, 123458 Moscow, Russian Federation.
Email: maria_dem@mail.ru

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Abstract
We provide the necessary and sufficient conditions of Liouvillian integrability for Liénard differential systems describing nonlinear oscillators with a polynomial damping and a polynomial restoring force. We prove that Liénard differential systems are not Darboux integrable excluding subfamilies with certain restrictions on the degrees of the polynomials arising in the systems. We demonstrate that if the degree of a polynomial responsible for the restoring force is greater than the degree of a polynomial producing the damping, then a generic Liénard differential system is not Liouvillian integrable with the exception of linear Liénard systems. However, for any fixed degrees of the polynomials describing the damping and the restoring force we present subfamilies possessing Liouvillian first integrals. As a by-product of our results, we find a number of novel Liouvillian integrable subfamilies. In addition, we study the existence of nonautonomous Darboux first integrals and nonautonomous Jacobi last multipliers with a time-dependent exponential factor.

KEYWORDS
Darboux integrability, invariant algebraic curves, Liénard differential systems, Liouvillian integrability, Puiseux series
INTRODUCTION

Performing classifications of integrable or solvable subfamilies for a given multiparameter system of ordinary differential equations is a very difficult problem. The aim of the present article is to solve the integrability problem for the following systems of first-order ordinary differential equations

\[ x_t = y, \quad y_t = -f(x)y - g(x). \] (1)

We suppose that \( f(x) \) and \( g(x) \) are polynomials

\[ f(x) = f_0 x^m + \cdots + f_m, \quad g(x) = g_0 x^n + \cdots + g_n, \quad f_0 g_0 \neq 0 \] (2)

with coefficients from the field \( \mathbb{C} \). Systems (1) are named in honor of the French physicist and engineer Liénard.\(^1\) These systems describe oscillators with a polynomial damping \( f(x) \) and a polynomial restoring force \( g(x) \). In addition, Liénard differential systems have a variety of other applications in physics, chemistry, biology, economics, and so forth. For example, these systems (1) arise as traveling wave reductions of the following general families of reaction–convection–diffusion equations

\[ u_t = D u_{xx} + A(u) u_x + B(u), \quad u = u(x, t), \] (3)

where \( D \) is a diffusion coefficient, \( A(u) \) describes a nonlinear convective flux, and \( B(u) \) is responsible for a reaction force.

Let the variable \( y \) be privileged with respect to the variable \( x \), then the function \( y(x) \) satisfies the following family of Abel differential equations of the second kind

\[ y y_x + f(x) y + g(x) = 0. \] (4)

The integrability properties of Liénard differential systems are investigated by various methods and in the framework of various theories. Let us enumerate the most important studies:

1. Local analysis\(^2\)–\(^6\) and formal first integrals\(^7\)–\(^9\);
2. Classical Lie symmetry analysis\(^10\),\(^11\) and \( \lambda \) symmetries\(^12\);
3. Local\(^13\)–\(^17\) and nonlocal transformations\(^18\)–\(^23\);
4. Differential Galois theory\(^24\);
5. Extended Prelle–Singer method\(^25\) and the Darboux theory of integrability.\(^19\),\(^22\),\(^26\)–\(^32\)

A collection of integrable and solvable subfamilies of Liénard differential systems is presented by Polyanin and Zaitsev.\(^33\) The transformation \( y(x) = 1/w(x) \) brings Abel differential equations of the second kind (4) to Abel differential equations of the first kind

\[ w_x = g(x) w^3 + f(x) w^2. \] (5)

Consequently, certain results available for such Abel differential equations can be transferred to Equation (4) and related Liénard differential systems.\(^13\)–\(^15\) Let us note that many scientific works
dealing with the global integrability problem present sufficient conditions of integrability. Thus, these works do not provide classifications of integrable Liénard differential systems and Abel differential equations. Meanwhile, it is an important scientific problem to find all integrable families for fixed degrees of the polynomials $f(x)$ and $g(x)$.

This article is devoted to the study of the general integrability properties of polynomial Liénard differential systems. We focus on the necessary and sufficient conditions of Darboux and Liouvillian integrability. We suppose that $f(x) \neq 0$. Any Liénard differential system is Hamiltonian with a polynomial first integral

$$I(x, y) = y^2 + 2 \int_0^x g(s) ds$$

whenever the relation $f(x) \equiv 0$ is valid. Llibre and Valls proved that Liénard differential systems (1) under the condition $\deg g \leq \deg f$ do not have Liouvillian first integrals excluding the trivial case $g(x) = \alpha f(x)$, $\alpha \in \mathbb{C}$. Consequently, we only need to study systems (1) satisfying the restriction $\deg g > \deg f$.

We demonstrate that the integrability properties of polynomial Liénard differential systems are substantially different in the following three cases:

(A) $\deg f < \deg g < 2\deg f + 1$;
(B) $\deg g = 2\deg f + 1$;
(C) $\deg g > 2\deg f + 1$.

Our goal is to prove that Liénard differential systems from families (A) and (C) are not Darboux integrable, while there are Liouvillian integrable subfamilies. In contrast, Liénard differential systems from family (B) exhibit a variety of rational, Darboux, and Liouvillian first integrals existing under certain restrictions on the parameters. This fact is also recognized by many scientists.10–13,17,20,25 Our main tools include the modern Darboux theory of integrability,34,35 the method of Puiseux series,36,37 and the local theory of invariants.38 We do not impose any nontrivial restrictions on the coefficients of the polynomials $f(x)$ and $g(x)$ with the exception of Liénard differential systems from family (B). We mainly study systems that are nonresonant near infinity provided that the following restriction $\deg g = 2\deg f + 1$ is valid. To be more precise, we say that a system (1) is resonant near infinity whenever the highest-degree coefficients $f_0$ and $g_0$ satisfy a resonance condition. This condition is explicitly given in Section 7 and arises only in the case $\deg g = 2\deg f + 1$. Let us note that the subset of resonant systems is of zero Lebesgue measure in the set of all polynomial systems (1) with fixed degrees of the polynomials $f(x)$ and $g(x)$. Our results are also valid in the resonant case, but they are not complete. For all other polynomial Liénard differential systems, we present a complete classification of Liouvillian integrable subfamilies. In addition, we classify polynomial Liénard differential systems possessing nonautonomous Darboux first integrals and nonautonomous Jacobi last multipliers with a time-dependent exponential factor.

This article is organized as follows. Sections 2, 3, and 4 contain a review of the known results and several preliminary observations on the methods we use in the subsequent part. In Section 2, we describe the Darboux theory of integrability and consider some related questions. Section 3 is devoted to the method of Puiseux series and to the local theory of invariants. In Section 4, the results on invariant algebraic curves of Liénard differential systems are described. In Section 5, we present some integrability properties valid for a generic polynomial Liénard differential system.
Sections 6, 7, and 8, we investigate the integrability and solvability of Liénard differential systems from families \((A), (B),\) and \((C),\) respectively. In Section 9, we consider an example: we study the Liénard differential systems satisfying the restrictions \(\text{deg } f = 2\) and \(\text{deg } g = 4\).

## 2 THE DARBOUX THEORY OF INTEGRABILITY

The main aim of the present section is to describe some basic aspects of the Darboux theory of integrability. We focus on the problem of finding Darboux and Liouvillian first integrals of polynomial differential systems in the plane

\[
x_t = P(x, y), \quad y_t = Q(x, y),
\]

where \(P(x, y)\) and \(Q(x, y)\) are relatively prime elements of the ring \(\mathbb{C}[x, y]\). By \(\mathbb{C}[x, y]\), we denote the ring of bivariate polynomials with complex-valued coefficients. The vector field related to system (7) is defined as

\[
\mathcal{X} = P(x, y) \frac{\partial}{\partial x} + Q(x, y) \frac{\partial}{\partial y}.
\]

**Definition 1.** A nonconstant function \(I(x, y): D \subset \mathbb{C}^2 \to \mathbb{C}\) is called a first integral of differential system (7) and the related vector field \(\mathcal{X}\) on an open subset \(D \subset \mathbb{C}^2\) if \(I(x(t), y(t)) = C\) with \(C\) being a constant for all values of \(t\) such that the solution \((x(t), y(t))\) of system (7) is defined in \(D\).

If \(I(x, y)\) is of a class at least \(C^1\) in \(D\), then \(I(x, y)\) is a first integral of differential system (7) if and only if \(\mathcal{X}I = 0\).

**Definition 2.** A nonconstant function \(M(x, y): D \subset \mathbb{C}^2 \to \mathbb{C}\) is called an integrating factor of differential system (7) and the related vector field \(\mathcal{X}\) in an open subset \(D \subset \mathbb{C}^2\) if the differential form \(M(x, y)(P(x, y)dy - Q(x, y)dx)\) is exact in \(D\). In other words, there exists a function \(I(x, y)\) of a class at least \(C^1\) in \(D\) such that the following relation is valid

\[
M(x, y)(P(x, y)dy - Q(x, y)dx) = dI(x, y).
\]

If an integrating factor \(M(x, y)\) is of a class at least \(C^1\) in \(D\), then it satisfies the following linear first-order partial differential equation \(\mathcal{X}M = -\text{div } \mathcal{X}M\), where \(\text{div } \mathcal{X} = P_x + Q_y\) is the divergence of the vector field \(\mathcal{X}\).

A function \(I(x, y)\) is referred to as a Liouvillian function of two variables \(x\) and \(y\) if it belongs to a Liouvillian extension of the field of rational functions \(\mathbb{C}(x, y)\) over \(\mathbb{C}\). Generally speaking, any Liouvillian function can be represented as a finite superposition of algebraic functions, antiderivatives, and exponentials. A function \(\Phi(x, y)\) is called a Darboux function of two variables \(x\) and \(y\), if it can be presented in the form

\[
\Phi(x, y) = F_1^{d_1}(x, y) \cdots F_K^{d_K}(x, y) \exp\{R(x, y)\},
\]

where \(F_1(x, y) \in \mathbb{C}[x, y], \ldots, F_K(x, y) \in \mathbb{C}[x, y], R(x, y) \in \mathbb{C}(x, y), d_1, \ldots, d_K \in \mathbb{C}.\) We see that any Darboux function is a Liouvillian function. The converse is not generally true.
A differential system \((7)\) is called *Darboux (Liouvillian) integrable* if it possesses a Darboux (Liouvillian) first integral. It is known that the problem of establishing Darboux or Liouvillian integrability of a differential system \((7)\) can be reduced to the problem of constructing all irreducible invariant algebraic curves of \((7)\) and all exponential invariants of \((7)\), for more details see Refs. 34, 35, 39.

**Definition 3.** The curve \(F(x, y) = 0\) with \(F(x, y) \in \mathbb{C}[x, y] \setminus \mathbb{C}\) is an invariant algebraic curve of a differential system \((7)\) whenever the following condition \(F_t|_{F=0} = (PF_x + QF_y)|_{F=0} = 0\) is valid.

If the polynomial \(F(x, y)\) producing the invariant algebraic curve \(F(x, y) = 0\) is irreducible in \(\mathbb{C}[x, y]\), then the ideal generated by \(F(x, y)\) is radical. Consequently, there exists an element \(\lambda(x, y)\) of the ring \(\mathbb{C}[x, y]\) such that \(F(x, y)\) satisfies the partial differential equation \(P(x, y)F_x + Q(x, y)F_y = \lambda(x, y)F\). The polynomial \(\lambda(x, y)\) is called the cofactor of the invariant algebraic curve \(F(x, y) = 0\). It is straightforward to show that the degree of \(\lambda(x, y)\) is at most \(L - 1\), where \(L\) is the maximum between the degrees of the polynomials \(P(x, y)\) and \(Q(x, y)\). We conclude that an invariant algebraic curve of differential system \((7)\) is formed from solutions of the latter. A solution of differential system \((7)\) has either empty intersection with the zero set of \(F(x, y)\) or it is entirely contained in \(F(x, y) = 0\). The generating polynomial \(F(x, y)\) of an invariant algebraic curve \(F(x, y) = 0\) is called a *Darboux polynomial* or an *algebraic invariant*.

**Definition 4.** A function \(E(x, y) = \exp\left[\frac{g(x, y)}{f(x, y)}\right]\) with the relatively prime polynomials \(g(x, y), f(x, y) \in \mathbb{C}[x, y]\) is called an exponential invariant of a differential system \((7)\) whenever the following condition \(\chi E = \varphi(x, y)E\) is valid, where \(\varphi(x, y) \in \mathbb{C}[x, y]\).

The polynomial \(\varphi(x, y)\) is referred to as the cofactor of the exponential invariant \(E(x, y)\). It is straightforward to show that the product of the exponential invariants \(E_1(x, y)\) and \(E_2(x, y)\) with the cofactors \(\varphi_1(x, y)\) and \(\varphi_2(x, y)\), respectively, is an exponential invariant possessing the cofactor \(\varphi(x, y) = \varphi_1(x, y) + \varphi_2(x, y)\). It is known that the polynomial \(f(x, y) \in \mathbb{C}[x, y] \setminus \mathbb{C}\) arising in an exponential invariant \(E(x, y) = \exp\left[\frac{g(x, y)}{f(x, y)}\right]\) produces an invariant algebraic curve \(f(x, y) = 0\) of the system under consideration.\(^{35}\)

It turns out that the study of autonomous first integrals and autonomous integrating factors is sometimes restrictive, even if an autonomous differential system is under consideration. In this article, we do not consider the Darboux theory of integrability for nonautonomous systems in the general case, for more details see Refs. 40–42.

A nonautonomous first integral \(I(x, y, t)\) and a nonautonomous integrating factor \(M(x, y, t)\) of a differential system \((7)\) are defined similarly to the autonomous case. They satisfy the following linear partial differential equations \(I_t + \chi I = 0\) and \(M_t + \chi M = -\div \chi M\), respectively, whenever \(I(x, y, t)\) and \(M(x, y, t)\) are function of a class at least \(C^1\) in \(D \subset \mathbb{C}^3\). Nonautonomous integrating factors are commonly referred to as Jacobi last multipliers or simply Jacobi multipliers.

The following theorems are the essence of the modern Darboux theory of integrability.

**Theorem 1.** A polynomial differential system \((7)\) is Darboux integrable if and only if it has a rational integrating factor.

The fact that a Darboux integrable differential system \((7)\) has a rational integrating factor was derived by Chavarriga et al.\(^{43}\) The converse statement was established by Christopher et al.\(^{44}\)
Theorem 2. A polynomial differential system (7) is Liouvillian integrable if and only if it has a Darboux integrating factor.

Theorem 2 was proved by Singer.34

Theorem 3. A polynomial differential system (7) possesses a first integral of the form

\[ I(x, y, t) = \prod_{j=1}^{K} F_j^{d_j}(x, y) \exp \left\{ \frac{S(x, y)}{R(x, y)} \right\} \exp(\omega t), \quad \omega, d_1, \ldots, d_K \in \mathbb{C}, \quad (10) \]

where \( F_1(x, y), \ldots, F_K(x, y) \) are pairwise relatively prime irreducible bivariate polynomials from the ring \( \mathbb{C}[x, y] \), \( S(x, y) \), and \( R(x, y) \) are relatively prime bivariate polynomials from the ring \( \mathbb{C}[x, y] \), if and only if \( F_1(x, y) = 0, \ldots, F_K(x, y) = 0, R(x, y) = 0 \) are invariant algebraic curves of the system and \( E(x, y) = \exp[S(x, y)/R(x, y)] \) is an exponential invariant of the system such that the following condition:

\[ \sum_{j=1}^{N} d_j \lambda_j(x, y) + \varphi(x, y) + \omega = 0 \quad (11) \]

holds identically. In this expression, \( \lambda_j(x, y) \) is the cofactor of the invariant algebraic curve \( F_j(x, y) = 0 \) and \( \varphi(x, y) \) is the cofactor of the exponential invariant \( E(x, y) \).

Theorem 3 follows from the classical theory of Darboux integrability (see Ref. 39). We name a first integral of Theorem 3 as a nonautonomous Darboux first integral provided that \( \omega \neq 0 \). If \( \omega = 0 \), then function (10) gives a Darboux first integral.

Theorem 4. Under the assumptions of Theorem 3, a polynomial differential system (7) possesses a Jacobi last multiplier of the form

\[ M(x, y, t) = \prod_{j=1}^{K} F_j^{d_j}(x, y) \exp \left\{ \frac{S(x, y)}{R(x, y)} \right\} \exp(\omega t), \quad \omega, d_1, \ldots, d_K \in \mathbb{C}, \quad (12) \]

if and only if \( F_1(x, y) = 0, \ldots, F_K(x, y) = 0, R(x, y) = 0 \) are invariant algebraic curves of the system and \( E(x, y) = \exp[S(x, y)/R(x, y)] \) is an exponential invariant of the system such that the following condition:

\[ \sum_{j=1}^{N} d_j \lambda_j(x, y) + \varphi(x, y) + \omega = -\text{div} \mathcal{X} \quad (13) \]

is identically valid. In this expression \( \lambda_j(x, y) \) is the cofactor of the invariant algebraic curve \( F_j(x, y) = 0 \) and \( \varphi(x, y) \) is the cofactor of the exponential invariant \( E(x, y) \).

Theorem 4 with the restriction \( \omega = 0 \) was derived by Christopher.35 The case \( \omega \neq 0 \) was considered in Ref. 42. A Jacobi last multiplier of Theorem 4 will be referred to as a nonautonomous Darboux–Jacobi last multiplier whenever \( \omega \neq 0 \).
These theorems suggest the following algorithm for searching autonomous and nonautonomous Darboux first integrals and Jacobi last multipliers:

1. Find all relatively prime irreducible invariant algebraic curves and all exponential invariants with linearly independent cofactors;
2. Find, or prove the nonexistence of, complex numbers \( d_1, \ldots, d_K, \omega \) such that condition (11) or (13) is identically satisfied; the polynomial \( \varphi(x, y) \) arising in conditions (11) and (13) equals the sum of the cofactors of exponential invariants found at the first step.

Let us note that there exist certain estimates of the number of pairwise distinct invariants which guarantees the existence of rational, Darboux or Liouvillian first integrals in the autonomous case. For more details, see books \(^{39,45}\) and the references therein.

The first step of this algorithm is extremely difficult. This is due to the absence of a priori upper bounds on the degrees of bivariate polynomials giving irreducible invariant algebraic curves. It is shown in Refs. \(^{36,37,46}\) that the method of Puiseux series, which is described in the next section, can facilitate the first step.

It is straightforward to see that integrating factors and Jacobi last multipliers are defined modulo to the multiplication by a nonzero constant. Two integrating factors or Jacobi last multipliers producing a constant ratio are supposed to be equivalent. We do not distinguish them. The uniqueness of integrating factors and Jacobi last multipliers is understood exactly in this sense.

In general, the existence of only one independent nonautonomous first integral is not sufficient for the complete integrability of the system under study in the framework of the Darboux theory. However, the knowledge of nonautonomous Darboux first integrals can be used to derive the general solutions. Several examples are given in Ref. \(^{32}\). Moreover, we demonstrate that some Liénard differential systems from family \((B)\) simultaneously have independent autonomous and nonautonomous Darboux first integrals. Eliminating the variable \( y = x_t \) from these first integrals, one can find explicit expressions of the general solutions; see also Ref. \(^{12}\), where some similar examples are presented. To conclude this section, let us note that trajectories lying in the zero set of an inverse Jacobi last multiplier are of importance in the qualitative theory of dynamical systems. \(^{39}\) Consequently, the classification of Liénard differential systems possessing Darboux–Jacobi last multipliers seem to be a significant problem.

### 3 THE METHOD OF PUISEUX SERIES AND THE LOCAL THEORY OF INVARIANTS

We start with a brief review of the theory of fractional-power (or Puiseux) series. A Puiseux series around a point \( x_0 \in \mathbb{C} \) can be presented as

\[
y(x) = \sum_{l=0}^{+\infty} c_l (x - x_0)^{l_0 + l}, \quad c_0 \neq 0, \tag{14}
\]

where \( l_0 \in \mathbb{Z} \) and \( n_0 \in \mathbb{N} \). It is without loss of generality to suppose that the number \( n_0 \) is relatively prime with the greatest common divisor of the numbers \( \{l_0 + l : c_l \neq 0, l \in \mathbb{N} \cup \{0\}\} \). While
a Puiseux series around the point \( x = \infty \) takes the form

\[
y(x) = \sum_{l=0}^{+\infty} b_l x^{\frac{l_0 - l}{n_0}}, \quad b_0 \neq 0,
\]

where \( l_0 \in \mathbb{Z} \) and \( n_0 \in \mathbb{N} \). Again we assume that the number \( n_0 \) is relatively prime with the greatest common divisor of the numbers \( \{l_0 - l : b_l \neq 0, l \in \mathbb{N} \cup \{0\}\} \).

Let us consider the algebraic equation \( F(x, y) = 0 \), where \( F(x, y) \in \mathbb{C}[x, y] \setminus \mathbb{C}[x] \). Giving preference to one of the variables with respect to another, a solution \( y \) of this equation viewed as a function of \( x \) can be locally expanded into convergent Puiseux series. This statement is known as the Newton–Puiseux theorem.

The set of all Puiseux series given by (14) or (15) forms an algebraically closed field, which we denote by \( \mathbb{C}_{x_0\{x\}} \) or \( \mathbb{C}_\infty\{x\} \), respectively. The ring of polynomials in one variable \( y \) with coefficients from the fields \( \mathbb{C}_{x_0\{x\}} \) or \( \mathbb{C}_\infty\{x\} \) is denoted as \( \mathbb{C}_{x_0\{x\}}[y] \) or \( \mathbb{C}_\infty\{x\}[y] \), respectively.

Suppose \( S(x, y) \) is an element of the ring \( \mathbb{C}_\infty\{x\}[y] \). Let us introduce two operators of projection acting on this ring. The first operator \( \{S(x, y)\}_+ \) gives the sum of the monomials of \( S(x, y) \) with nonnegative integer powers. In other words, \( \{S(x, y)\}_+ \) yields the polynomial part of \( S(x, y) \). Analogously, the projection \( \{S(x, y)\}_- = S(x, y) - \{S(x, y)\}_+ \) produces the nonpolynomial part of \( S(x, y) \). It is straightforward to show that these projections are linear operators. In addition, we see that the action of the projection operators can be extended to the ring of the Puiseux series in \( y \) near the point \( y = \infty \) with coefficients from the field \( \mathbb{C}_\infty\{x\} \). We denote this ring as \( \mathbb{C}_\infty\{x\}[y] \). Thus, we get the relation \( \{S(x, y)\}_+ \in \mathbb{C}[x, y] \), where \( S(x, y) \in \mathbb{C}_\infty\{x\}[y] \). We endow the fields \( \mathbb{C}_{x_0\{x\}} \) and \( \mathbb{C}_\infty\{x\} \) with the differential operator \( \partial_x \). Analogously, we endow the rings \( \mathbb{C}_{x_0\{x\}}[y] \) and \( \mathbb{C}_\infty\{x\}[y] \) with the differential operators \( \partial_x \) and \( \partial_y \). The action of these differential operators is standard.

The following basic statements are proved in Refs. 29,36,37.

**Lemma 1.** Let \( y(x) \) be a Puiseux series from one of the fields \( \mathbb{C}_{x_0\{x\}} \) or \( \mathbb{C}_\infty\{x\} \). Suppose that the series \( y(x) \) satisfies the equation \( F(x, y(x)) = 0 \), where \( F(x, y) \in \mathbb{C}[x, y] \setminus \mathbb{C}[x] \) gives an invariant algebraic curve \( F(x, y) = 0 \) of a system (7). Then the series \( y(x) \) solves the equation

\[
P(x, y)y_x - Q(x, y) = 0.
\]

Using algebraic closeness of the field of Puiseux series \( \mathbb{C}_\infty\{x\} \), it is straightforward to find the general structure of irreducible invariant algebraic curves and their cofactors.

**Theorem 5.** Let \( F(x, y) = 0 \), where \( F(x, y) \in \mathbb{C}[x, y] \setminus \mathbb{C}[x] \), be an irreducible invariant algebraic curve of a differential system (7). Then the polynomial \( F(x, y) \) and the cofactor \( \lambda(x, y) \in \mathbb{C}[x, y] \) of the curve \( F(x, y) = 0 \) take the form

\[
F(x, y) = \left\{ \mu(x) \prod_{j=1}^{N} \{y - y_{j, \infty}(x)\} \right\}_+,
\]

where \( \mu(x) \) is a polynomial in \( x \).
$$\lambda(x, y) = \left\{ \sum_{m=0}^{+\infty} \sum_{j=1}^{N} \frac{[Q(x, y) - P(x, y)(y_{j, \infty})_x](y_{j, \infty})^m}{y^{m+1}} + P(x, y) \sum_{m=0}^{+\infty} \sum_{l=1}^{L} \frac{v_l x^m}{x^{m+1}} \right\},$$  \hspace{1cm} (18)$$

where $y_{1, \infty}(x), \ldots, y_{N, \infty}(x)$ are pairwise distinct Puiseux series in a neighborhood of the point $x = \infty$ that satisfy Equation (16), $x_1, \ldots, x_L$ are pairwise distinct zeros of the polynomial $\mu(x) \in \mathbb{C}[x]$ with multiplicities $v_1, \ldots, v_L \in \mathbb{N}$ and $L \in \mathbb{N} \cup \{0\}$. Moreover, the degree of $F(x, y)$ with respect to $y$ does not exceed the number of distinct Puiseux series of the form (15) satisfying Equation (16) whenever this number is finite. If $\mu(x) = \mu_0$, where $\mu_0 \in \mathbb{C}$, then we suppose that $L = 0$ and the first series is absent in the expression of the cofactor $\lambda(x, y)$.

This theorem introduces an algebraic tool of finding invariant algebraic curves and their cofactors in an explicit form, for more details see Ref. 47.

The method of Puiseux series can be also used whenever one wishes to find exponential invariants with nonpolynomial arguments. The following theorem giving necessary conditions for an exponential invariant related to an invariant algebraic curve to exist is proved in Ref. 29.

**Theorem 6.** Let the polynomial $f(x, y) \in \mathbb{C}[x, y] \setminus \mathbb{C}[x]$ give an invariant algebraic curve $f(x, y) = 0$ of a differential system (7). The cofactor of the invariant algebraic curve $f(x, y) = 0$ we denote by $\lambda(x, y) \in \mathbb{C}[x, y]$. Suppose that this system admits an exponential invariant $E = \exp(g/f)$ related to the algebraic curve $f(x, y) = 0$, then for each nonconstant Puiseux series $y_{j, \infty}(x)$ in a neighborhood of the point $x = \infty$ that satisfies the equation $f(x, y) = 0$ there exists a number $q \in \mathbb{Q}$ such that the Puiseux series for the function $\lambda(x, y_{j, \infty}(x))/P(x, y_{j, \infty}(x))$ in a neighborhood of the point $x = \infty$ is

$$\frac{\lambda(x, y_{j, \infty}(x))}{P(x, y_{j, \infty}(x))} = \sum_{k=0}^{+\infty} b_k x^{\frac{k+n}{n}}, \hspace{0.5cm} b_0 = q.$$  \hspace{1cm} (19)$$

Now our aim is to introduce a notion of local invariant curves that are given by elements from one of the rings $\mathbb{C}_{x_0}[x][y]$ and $\mathbb{C}_{\infty}[x][y]$. Let $x_0$ be a point of the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Elements of the ring $\mathbb{C}_{x_0}[x][y]$ are referred to as Puiseux polynomials. Note that the polynomials $P(x, y)$ and $Q(x, y)$ can be regarded as Puiseux polynomials for any $x_0 \in \overline{\mathbb{C}}$. Again giving preference to the variable $y$, we deal with formal Puiseux series solutions $y = y_{x_0}(x)$ of algebraic first-order ordinary differential equation (16). We can do the same analysis choosing the variable $x$ as depended.

**Definition 5.** The curve $F(x, y) = 0$, where $F(x, y) \in \mathbb{C}_{x_0}[x][y]$, is called a local invariant curve of a differential system (7) whenever the following condition $F_t|_{F=0} = (PF_x + QF_y)|_{F=0} = 0$ is valid.

If the element $F(x, y) \in \mathbb{C}_{x_0}[x][y]$ gives a local invariant curve $F(x, y) = 0$ of differential system (7), then there exists the Puiseux polynomial $\lambda(x, y) \in \mathbb{C}_{x_0}[x][y]$ such that the following equation $\lambda F(x, y) = \lambda(x, y) F(x, y)$ is satisfied.\(^ {38}\) By analogy with the algebraic case, the Puiseux polynomials $F(x, y)$ and $\lambda(x, y)$ will be called a local algebraic invariant and the cofactor of the related local algebraic invariant and local curve, respectively. The theory of local invariants is introduced in Ref. \(^ {38}\), where all the statements given below are proved.

In what follows we call local algebraic invariants and associated local invariant curves given by the polynomials $F(x, y) = h_{x_0}(x)$ and $F(x, y) = y - y_{x_0}(x)$, $h_{x_0}(x)$, $y_{x_0}(x) \in \mathbb{C}_{x_0}[x]$ elementary.
Any local algebraic invariant \( F(x, y) \in \mathbb{C}_{x_0}\{x\}[y] \) can be represented as a product of elementary local algebraic invariants and the cofactor of the local algebraic invariant \( F(x, y) \) is the sum of the cofactors of all the factors.

**Theorem 7.** Let \( y_{x_0}(x) \) be a Puiseux series near the point \( x = x_0 \). The element \( F(x, y) = y - y_{x_0}(x) \) of the ring \( \mathbb{C}_x\{x\}[y] \) gives a local invariant curve of differential system (7) if and only if the series \( y = y_{x_0}(x) \) satisfies Equation (16).

It is straightforward to obtain an explicit expression of the cofactor similar to that presented in Theorem 5. We introduce the operator of projection \( \{S(x, y)\}^+,y \) yielding the polynomial part with respect to \( y \) of the element \( S(x, y) \in \mathbb{C}_{x_0,\infty}\{x\}[[y]] \). The symbol \( \mathbb{C}_{x_0,\infty}\{x\}[[y]] \) denotes the ring of Puiseux series in \( y \) near the point \( y = \infty \) with coefficients from the field \( \mathbb{C}_{x_0}\{x\} \). In other words, we obtain the relation \( \{S(x, y)\}^+,y \in \mathbb{C}_{x_0}\{x\}[y] \).

**Theorem 8.** Let \( F(x, y) = y - y_{x_0}(x) \), \( y_{x_0}(x) \in \mathbb{C}_{x_0}\{x\} \) give an elementary local invariant curve \( F(x, y) = 0 \) of differential system (7), then its cofactor takes the form

\[
\lambda(x, y) = \left\{ \sum_{m=0}^{+}\frac{Q(x, y) - P(x, y)(y_{x_0}(x)x)^m}{y^{m+1}} y_{x_0}(x) \right\}^{+,y}_+. \tag{20}
\]

Similarly to the case of local invariant curves, we can introduce a concept of local exponential invariants.

**Definition 6.** A function \( E(x, y) = \exp[g(x, y)/f(x, y)] \) with relatively prime elements \( g(x, y) \) and \( f(x, y) \) of the ring \( \mathbb{C}_{x_0}\{x\}[y] \) is called a local exponential invariant of a differential system (7) if \( E(x, y) \) satisfies the partial differential equation \( \mathcal{X}E(x, y) = \varphi(x, y)E(x, y) \), where \( \varphi(x, y) \) is a Puiseux polynomial.

Again the Puiseux polynomial \( \varphi(x, y) \) will be referred to as the cofactor of the local exponential invariant \( E(x, y) \). Note that exponential invariants belong to a differential extension of the field of fractions of the ring \( \mathbb{C}_{x_0}\{x\}[y] \).

**Lemma 2.** Let \( E(x, y) = \exp[g(x, y)/f(x, y)] \), \( f(x, y) \in \mathbb{C}_{x_0}\{x\}[y] \setminus \mathbb{C}_{x_0}\{x\} \) be a local exponential invariant of a differential system (7), then \( f(x, y) = 0 \) is a local invariant curve of the system in question.

We say that local exponential invariants

\[
E(x, y) = \exp\left[ g_l(x)y^l \right], \quad g_l(x) \in \mathbb{C}_{x_0}\{x\}, \quad l \in \mathbb{N} \cup \{0\};
\]

\[
E(x, y) = \exp\left[ \frac{u_n(x, y)}{[y - y_{x_0}(x)]^n} \right], \quad y_{x_0}(x) \in \mathbb{C}_{x_0}\{x\}, \quad u_n(x, y) \in \mathbb{C}_{x_0}\{x\}[y], \quad n \in \mathbb{N}, \tag{21}
\]

where the degree with respect to \( y \) of \( u_n(x, y) \) is at most \( n - 1 \), are elementary local exponential invariants. Any local exponential invariant \( E(x, y) \) is as a product of elementary local exponential invariants and the cofactor of the local invariant \( E(x, y) \) is the sum of the cofactors of all the factors.
We conclude that if system (7) has a global invariant (algebraic or exponential), then for any $x_0 \in \mathbb{C}$ this invariant must be a product of local elementary invariants in such a way that the sum of their cofactors is a true, not Puiseux, polynomial. This observation is used in subsequent sections.

4 | INVARIANT ALGEBRAIC CURVES OF POLYNOMIAL LIÉNARD DIFFERENTIAL SYSTEMS

With the aim to study the integrability properties of polynomial Liénard differential systems, we need information on the existence of their invariant algebraic curves. The general structure of bivariate polynomials giving irreducible invariant algebraic curves is derived in Refs. 36, 47. Let us note that originally the problem of finding invariant algebraic curves was considered in Ref. 48. However, the results of Ref. 48 are not complete because the existence of an infinite number of Puiseux series from the field $\mathbb{C}_\infty\{x\}$ that solve Equation (4) is not taken into account. Some properties of the polynomials in $x$ arising in explicit representations of invariant algebraic curves are studied in Ref. 49.

Theorem 9. Let $F(x, y) = 0, F(x, y) \in \mathbb{C}[x, y] \setminus \mathbb{C}$ be an irreducible invariant algebraic curve of a Liénard differential system (1) from family (A). Then the polynomial $F(x, y)$ and the cofactor $\lambda(x, y)$ of the invariant algebraic curve $F(x, y) = 0$ take the form

$$F(x, y) = \left\{ \prod_{j=1}^{N-k} \left\{ y - y_{j,\infty}^{(1)}(x) \right\} \left\{ y - y_{N,\infty}^{(2)}(x) \right\} \right\}^k,$$

$$\lambda(x, y) = -Nf - (N-k)q_x - kp_x,$$

where $k = 0$ or $k = 1$, $N \in \mathbb{N}$, and $y_{1,\infty}^{(1)}(x), ..., y_{N-1,\infty}^{(1)}(x), y_{N,\infty}^{(2)}(x)$ are the series

(I) : $y_{j,\infty}^{(1)}(x) = q(x) + \sum_{l=0}^{+\infty} b^{(j)}_{m+1+l} x^{-l}$, $j = 1, ..., N - k$;

(II) : $y_{N,\infty}^{(2)}(x) = p(x) + \sum_{l=0}^{+\infty} b^{(N)}_{n-m+l} x^{-l}$.

The coefficients of the series of type (II) and of the polynomials

$$q(x) = -\frac{f_0}{m+1} x^{m+1} + \sum_{l=1}^{m} q_{m+1-l} x^l \in \mathbb{C}[x],$$

$$p(x) = -\frac{g_0}{m} x^n + \sum_{l=1}^{n-m} p_{n-m-l} x^l \in \mathbb{C}[x].$$
are uniquely determined. The coefficients $b^{(j)}_{m+1}$, $j = 1, ..., N - k$ are pairwise distinct. All other coefficients $b^{(j)}_{m+1+l}$, $l \in \mathbb{N}$ are expressible via $b^{(j)}_{m+1}$, where $j = 1, ..., N - k$. The corresponding product in (22) is unit whenever $k = 1$ and $N = 1$.

**Theorem 10.** A Liénard differential system (1) from family (A) with fixed coefficients of the polynomials $f(x)$ and $g(x)$ has at most two distinct irreducible invariant algebraic curves simultaneously. If the two distinct irreducible invariant algebraic curves exist, then the first has $k = 1$ in representation (22) and the second is given by a first-degree polynomial with respect to $y$ and takes the form $y - q(x) - z_0 = 0$.

**Corollary.** Liénard differential systems (1) from family (A) are not integrable with a rational first integral.

Theorems 9 and 10 are proved in Ref. 36.

The structure of polynomials producing invariant algebraic curves for Liénard differential systems from family (B) is in strong correlation with the properties of the following quadratic equation:

$$\eta^2 - \varphi \eta + (m + 1)\varphi = 0,$$

where we have introduced notation

$$\varphi = 4(m + 1) - \frac{f_0^2}{g_0}.$$  

The set of all positive rational numbers will be denoted as $\mathbb{Q}^+$. Let $\eta_1$ and $\eta_2$ be the roots of Equation (26).

**Theorem 11.** Suppose $F(x, y) = 0, F(x, y) \in \mathbb{C}[x, y] \setminus \mathbb{C}$ is an irreducible invariant algebraic curve of a Liénard differential system (1) from family (B) and Equation (26) has no solutions in $\mathbb{Q}^+$. One of the following statements holds:

1. If $\eta_1 \eta_2 \neq 0$, then the polynomial $F(x, y)$ is of degree at most two with respect to $y$ and

$$F(x, y) = \begin{cases} \{ y - y^{(1)}(x) \}^{s_1} \{ y - y^{(2)}(x) \}^{s_2} \text{,} \\ \lambda(x, y) = -(s_1 + s_2)f(x) - \left\{ s_1\left(y^{(1)}_\infty(x)\right)_x + s_2\left(y^{(2)}_\infty(x)\right)_x \right\}_+, \end{cases}$$

where $s_1$ and $s_2$ are either 0 or 1 independently, $s_1 + s_2 > 0$. The Puiseux series $y^{(k)}_\infty(x)$, $k = 1, 2$ are Laurent series and possess uniquely determined coefficients.
2. If $\eta_1 \eta_2 = 0$, then $\eta_1 = \eta_2 = 0$. The polynomial $F(x, y)$ and its cofactor $\lambda(x, y)$ take the form

\[
F(x, y) = y + \frac{f_0}{2(m + 1)} x^{m+1} - \sum_{l=1}^{m+1} b_l x^{m+1-l},
\]
\[
\lambda(x, y) = -f(x) + \frac{f_0}{2} x^m - \sum_{l=1}^{m} (m + 1 - l) b_l x^{m-l},
\]

(29)

where the coefficients $b_1, ..., b_{m+1}$ are uniquely determined. In addition, the following relation 

\[4(m + 1) g_0 - f_0^2 = 0\]

is valid.

This theorem is proved in Ref. 47. Note that the Liénard differential systems such that the related equation (26) has a positive rational solution are also studied in Ref. 47. We do not reproduce the related results here.

The structure of polynomials giving invariant algebraic curves and their cofactors for Liénard differential systems satisfying the condition $\deg g > 2 \deg f + 1$ is derived in Ref. 47. The following theorem is valid.

**Theorem 12.** Let $F(x, y) = 0, F(x, y) \in \mathbb{C}[x, y] \setminus \mathbb{C}$ be an irreducible invariant algebraic curve of a Liénard differential system (1) from family (C). Then the polynomial $F(x, y)$ and the cofactor $\lambda(x, y)$ of the invariant algebraic curve $F(x, y) = 0$ take the form

\[
F(x, y) = \left\{ \prod_{j=1}^{N_1} \left\{ y - y^{(1)}_{j, \infty}(x) \right\} \right\} \left\{ \prod_{j=1}^{N_2} \left\{ y - y^{(2)}_{j, \infty}(x) \right\} \right\} + ,
\]

(30)

\[
\lambda(x, y) = -(N_1 + N_2) f - \left\{ N_1 h^{(1)}_x + N_2 h^{(2)}_x \right\} +
\]

(31)

where the Puiseux series $y^{(1,2)}_{j, \infty}(x)$ are given by the relations

\[
y^{(1,2)}_{j, \infty}(x) = h^{(1,2)}(x) + \sum_{k=2(n+1)}^{+\infty} b^{(1,2)}_{k, j, x^{n+1} - \frac{k}{2}}, \quad h^{(1,2)}(x) = \sum_{k=0}^{2n+1} b^{(1,2)}_k x^{n+1} - \frac{k}{2},
\]

(32)

$N_1, N_2 \in \mathbb{N} \cup \{0\}$, and $N_1 + N_2 \geq 1$. The coefficients $b^{(1,2)}_{2(2n+1), j}$ with the same upper index are pairwise distinct and all the coefficients $b^{(1,2)}_{l, j}$ with $l > 2(n+1)$ are expressible via $b^{(1,2)}_{2(n+1), j}$. If $n$ is an odd number, then the corresponding Puiseux series are Laurent series and $b^{(1,2)}_{2l-1, j} = 0, b^{(1,2)}_{2l-1, j} = 0$, when $l \in \mathbb{N}$. In addition, $N_k = 1$ whenever $n$ is odd and $N_1 = 0$, where $k, l = 1, 2$ and $k \neq l$. If $n$ is an even number, then $N_1 = N_2$.

All the Puiseux series arising in Theorems 9, 11, and 12 solve Equation (4) related to the Liénard differential system under consideration. The following theorem provides the necessary and sufficient conditions for a Liénard differential system (1) to have an invariant algebraic curve.
Theorem 13. The polynomial \(F(x, y) \in \mathbb{C}[x, y] \setminus \mathbb{C}\) of degree \(N > 0\) with respect to \(y\) gives an invariant algebraic curve \(F(x, y) = 0\) of a Liénard differential system (1) if and only if there exists \(N\) Puiseux series \(y_{1, \infty}(x), \ldots, y_{N, \infty}(x)\) from the field \(\mathbb{C}_\infty[x]\) that solve Equation (4) and satisfy the conditions

\[
\left\{ \sum_{j=1}^{N} y_{j, \infty}(x) \right\} = 0. \tag{33}
\]

The theorems presented in this section essentially are based on the method of Puiseux series.

5 | INTEGRABILITY OF A GENERIC LIÉNARD DIFFERENTIAL SYSTEM

We begin this section by investigating the existence of exponential invariants with polynomial arguments.

Lemma 3. If a Liénard differential system (1) satisfying the condition \(\deg g > \deg f\) has an exponential invariant of the form \(E(x, y) = \exp[h(x, y)], h(x, y) \in \mathbb{C}[x, y]\) with the cofactor \(\varphi(x, y) \in \mathbb{C}[x, y]\) of degree at most \(\deg g - 1\), then the polynomial \(\varphi(x, y)\) is divisible by \(y\) in the ring \(\mathbb{C}[x, y]\).

Proof. It is straightforward to show that the polynomial \(h(x, y)\) satisfies the following linear inhomogeneous partial differential equation

\[
yh_x - \{f(x)y + g(x)\}h_y = \varphi(x, y). \tag{34}\]

Let us represent the functions \(h(x, y)\) and \(\varphi(x, y)\) as polynomials in \(y\) with polynomial in \(x\) coefficients. Substituting the expressions \(h(x, y) = h_0(x) + h_1(x)y + \cdots\) and \(\varphi(x, y) = \varphi_0(x) + \varphi_1(x)y + \cdots\) into Equation (34) and selecting the coefficients of \(y^0\), we get the relation \(g(x)h_1(x) = -\varphi_0(x)\). Since the degree of the polynomial \(\varphi_0(x)\) is at most \(\deg g - 1\), we conclude that \(h_1(x) \equiv 0\) and \(\varphi_0(x) \equiv 0\). As a result the polynomial \(\varphi(x, y)\) is given by the relation \(\varphi(x, y) = (\varphi_1(x) + \cdots)y\). This completes the proof. \(\blacksquare\)

As a direct consequence of this lemma, we establish that Darboux or Liouvillian integrable Liénard differential systems necessarily have invariant algebraic curves.

Theorem 14. If a Liénard differential system (1) satisfying the condition \(\deg g > \deg f\) has no invariant algebraic curves, then this system is not integrable with a Darboux or a Liouvillian first integral.

Proof. It is a simple result of the Darboux theory of integrability that a differential system (7) without invariant algebraic curves cannot have rational first integrals.

Suppose that a Liénard differential system without invariant algebraic curves possesses a Darboux first integral. Then this first integral is given by an exponential invariant with a zero cofactor. Consequently, the argument of the exponential function in the invariant provides a rational first integral. This is a contradiction.
If a Liénard differential system (1) without invariant algebraic curves is Liouvillian integrable, then there exists a Darboux integrating factor and this integrating factor equals some exponential invariant of the form $E(x, y) = \exp[h(x, y)]$, $h(x, y) \in \mathbb{C}[x, y]$. Calculating the divergence of the vector field

$$\mathcal{X} = y \frac{\partial}{\partial x} - (f(x)y + g(x)) \frac{\partial}{\partial y},$$

we get: $\text{div} \mathcal{X} = -f(x)$. The divergence is independent of $y$, while by Lemma 3 the cofactor of the exponential invariant $E(x, y) = \exp[h(x, y)]$ is divisible by $y$ provided that $\deg g > \deg f$. Consequently, condition (11) in the autonomous case is not valid. 

Thus, we conclude that Liouvillian integrable Liénard differential systems (1) with the restriction $\deg f < \deg g$ must have at least one invariant algebraic curve.

**Theorem 15.** A generic Liénard differential system (1) with fixed degrees of the polynomials $f(x)$ and $g(x)$ is neither Darboux nor Liouvillian integrable provided that the following restrictions $\deg g > \deg f$ and $(\deg f, \deg g) \neq (0, 1)$ are valid.

**Proof.** Let us denote the set of all Liénard differential systems with fixed degrees of the polynomials $f(x)$ and $g(x)$ as $L_{m,n}$. Any particular Liénard differential system can be identified with a point in $\mathbb{C}^{m+n} \times (\mathbb{C} \setminus \{0\})^2$. In view of Theorem 14, we need to prove that the subset of Liénard differential systems (1) without invariant algebraic curves is of full Lebesgue measure in the set $L_{m,n}$. Note that here deal with finite invariant algebraic curves. Extending systems (1) from the complex plane $\mathbb{C}^2$ to the complex projective plane $\mathbb{C}P^2$, we see that the infinite line is always invariant for Liénard differential systems.

We begin by considering systems from family $(A)$. The subset of Liénard differential systems such that the related equation (4) possesses a family of formal Puiseux series solutions $y^{(1)}_{\infty}(x)$ has Lebesgue measure zero. Indeed, there always arises a compatibility condition enabling this family of series to exist. We only need to show that the compatibility condition cannot be identically satisfied. For this aim, we track the appearance of the coefficient $g_{n-m}$ in the series $y^{(1)}_{\infty}(x)$. We use the following representation:

$$y^{(1)}_{\infty}(x) = \sum_{l=0}^{+\infty} v_l(x) (g_{n-m})^l,$$

where $\{v_l(x)\}$ are elements of the field $\mathbb{C}_\infty\{x\}$. The compatibility condition arises, when one tries to find the coefficient of $x^0$. Substituting representation (36) into Equation (4) and setting to zero the coefficients of $g_{n-m}^l$ for $l = 0, 1$, we find the ordinary differential equations

$$v_0 v_{0,x} + f(x)v_0 + \hat{g}(x) = 0, \quad v_0 v_{1,x} + (f(x) + v_{0,x})v_1 + x^m = 0,$$

where we use the designation $\hat{g}(x) = g(x) - g_{n-m}x^m$. The dominant behavior of the series $v_0(x)$ near the point $x = \infty$ is

$$v_0(x) = -\frac{f_0}{m+1} x^{m+1} + o(x^{m+1}), \quad x \to \infty.$$
Analyzing the ordinary differential equation for the series $v_1(x)$, we see that this equation should have a solution with the dominant behavior $v_1(x) = e_0 x^0$, $e_0 \in \mathbb{C} \setminus \{0\}$, $x \to \infty$, but it does not. In addition, the equations for $v_l(x)$, $l \geq 2$ have a zero solution whenever $v_1(x)$ is zero. Thus, the compatibility condition enabling the series $y^{(1)}_{\infty}(x)$ to exist is given by a first-degree polynomial with respect to $g_{n-m}$ and cannot hold identically. Consequently, it is necessary to consider systems with invariant algebraic curves given by bivariate polynomials not involving the series $y^{(1)}_{\infty}(x)$ into the factorization in the ring $\mathbb{C}_\infty[x][y]$. In view of Theorem 9, the corresponding irreducible invariant algebraic curve is given by the polynomial $F(x, y) = y - p(x) - z_1$, where $z_1 \in \mathbb{C}$ and the polynomial $p(x)$ of degree $n - m$ is described by expression (25). Now we consider the subset of Liénard differential systems with the invariant algebraic curve $y - p(x) - z_1 = 0$. Since the related equation (4) possesses the polynomial solution $y(x) = p(x) + z_1$, we find the polynomial $g(x)$. The result is $g(x) = -(p + z_1)(p_x + f)$. We see that the dimension of the subset of Liénard differential systems under consideration is less than the dimension of $L_{m,n}$.

We turn to systems from family (B). It follows from Theorem 11 that if a generic Liénard differential system from family (B) possesses irreducible invariant algebraic curves, then their generating polynomials are of degrees either 1 or 2 with respect to $y$. Let us suppose that there exists the irreducible invariant algebraic curve $y - q_l(x) = 0$. In this expression $q_l(x) = \{y^{(l)}_{\infty}(x)\}_+$, $l = 1, 2$ is a polynomial of degree $m + 1$. By analogy with systems from family (A), we can find the polynomial $g(x)$. We see from the relation $g(x) = -q_l(f + q_l,x)$ that the subset of Liénard differential systems with the invariant algebraic curve $y - q_l(x) = 0$ is of dimension $2m + 3$, while the dimension of $L_{m,n}$ is $m + n + 2 = 3(m + 1)$. Hence, the subset in question is of zero Lebesgue measure in $L_{m,n}$ provided that $m > 0$. Now we suppose that Liénard differential systems from family (B) have the hyperelliptic invariant algebraic curve $(y + u(x))^2 + w(x) = 0$. By Theorem 11, this curve is unique. In addition, we see that the polynomial $u(x)$ is of degree $m + 1$ and the polynomial $w(x)$ is of degree at most $2m + 2$. Following Żoładek,48 we substitute the bivariate polynomial $F(x, y) = (y + u(x))^2 + w(x)$ into the partial differential equation

$$yF_x - (f(x)y + g(x))F_y - \lambda(x, y)F = 0. \quad (39)$$

Furthermore, we set to zero the coefficients at different powers of $y$. Since the cofactor $\lambda(x, y)$ is independent of $y$, we express the polynomials $f(x)$ and $g(x)$ via $u(x)$ and $v(x)$. The result is

$$f(x) = u_x + \frac{uw_x}{2w}, \quad g(x) = \frac{w_x}{2} + \frac{u^2w_x}{2w}. \quad (40)$$

We see from these expressions that any zero of the polynomial $u(x)$ is also a zero of the polynomial $u(x)$. The polynomial $u(x)$ is parameterized by at most $m + 2$ parameters and the polynomial $w(x)$ adds only one new parameter. Thus, we conclude that the dimension of Liénard differential systems from family (B) with the hyperelliptic invariant algebraic curve $(y + u(x))^2 + w(x) = 0$ is at most $m + 3$. While the dimension of $L_{m,2m+1}$ is $3m + 3$.

Finally, we are left with systems from family (C). Let us track the dependence of the Puiseux series $y^{(1)}_{\infty}(x)$ and $y^{(2)}_{\infty}(x)$ on the coefficient $f_m$. We introduce the representation

$$y^{(k)}_{\infty}(x) = \sum_{l=0}^{+\infty} u^{(k)}_l(x)(f_m)^l, \quad k = 1, 2, \quad (41)$$
where \(v^{(k)}(x)\) are elements of the field \(\mathbb{C}_\infty\{x\}\). Let us introduce the following designation: \(\hat{f}(x) = f(x) - f_m\). Substituting these representations into Equation (4) and setting to zero the coefficients of \((f_m)^l\) with \(l = 0, 1\), we find the ordinary differential equations

\[
u_0v_{0,x} + \hat{f}(x)v_0 + g(x) = 0, \quad v_0v_{1,x} + (\hat{f}(x) + v_{0,x})v_1 + v_0 = 0. \tag{42}\]

Note that we omit the upper index.

Let us suppose that \(n\) is even. Puiseux series from the field \(\mathbb{C}_\infty\{x\}\) that satisfy these equations are of the form

\[
v^{(k)}_0(x) = \sum_{j=0}^{+\infty} c^{(k)}_j x^{n+1-k} / 2, \quad k = 1, 2, \quad c^{(1,2)}_0 = \pm \sqrt{-2(n+1)g_0 / n+1};
\]

\[
v^{(k)}_1(x) = \sum_{j=0}^{+\infty} e^{(k)}_j x^{2-k} / 2, \quad k = 1, 2, \quad e^{(1,2)}_0 = -2 / (n+1). \tag{43}\]

Theorem 13 gives the following necessary condition for an invariant algebraic curve \(F(x, y) = 0\) to exist: \(N_1b^{(1)}_{n+3} + N_2b^{(2)}_{n+3} = 0\). Recall that \(b^{(k)}\) is a coefficient of the family of series \(y^{(k)}_\infty(x)\) and \(N_k\) is the number of times the family of series \(y^{(k)}_\infty(x)\) enters the factorization of \(F(x, y)\) in the ring \(\mathbb{C}_\infty\{x\}[y]\). Using Theorem 12, we get \(N_1 = N_2\). Calculating the first five coefficients of the series \(v^{(k)}_1(x)\), we see that \(e^{(1)}_4 + e^{(2)}_4\) is not identically zero for any nonzero polynomial \(f(x)\) of degree \(m < (n-1)/2\). Hence the expression \(N_1b^{(1)}_{n+3} + b^{(2)}_{n+3}\) is a polynomial with respect to \(f_m\) possessing a nonzero coefficient of \(f_m\) in the generic case.

Now we assume that \(n\) is odd. The Puiseux series \(v^{(k)}_0(x)\) and \(v^{(k)}_1(x)\) solving Equations (42) are of the form (43), where the coefficients \(\{c^{(k)}_j\}\) and \(\{e^{(k)}_j\}\) with odd lower indices equal zero. Furthermore, we again consider the necessary condition \(N_1b^{(1)}_{n+3} + N_2b^{(2)}_{n+3} = 0\), but now the numbers \(N_1\) and \(N_2\) may not be equal. Calculating the first three nontrivial coefficients of the series \(v^{(k)}_1(x)\), we see that the condition \(N_1e^{(1)}_4 + N_2e^{(2)}_4\) is not identically zero for any numbers \(N_1, N_2 \geq 0, N_1 + N_2 > 0\) and any nonzero polynomial \(f(x)\) of degree \(m < (n-1)/2\). Consequently, the expression \(N_1b^{(1)}_{n+3} + N_2b^{(2)}_{n+3}\), regarded as a polynomial with respect to \(f_m\), possesses a nonzero coefficient of \(f_m\) in the generic case. We conclude that a generic Liénard differential system from family (C) has no finite invariant algebraic curves. ■

It is straightforward to see that if \((\deg f, \deg g) = (0, 1)\), then the associated Liénard differential systems are linear. They always have invariant lines and are Darboux integrable. As far as the author is aware, the Liouvillian nonintegrability of a generic nonlinear polynomial Liénard differential system such that \(\deg g > \deg f\) is established in this article for the first time.

6 INTEGRABILITY OF LIÉNARD DIFFERENTIAL SYSTEMS FROM FAMILY (A)

We have proved in the previous section that if a polynomial Liénard differential system is Liouvillian integrable, then it has at least one invariant algebraic curve. Let us investigate the existence of exponential invariants with nonpolynomial arguments. Here and in what follows we use the
designations of Theorem 9. In particular, the polynomials \( q(x) \) and \( p(x) \) give the initial parts of the Puiseux series near the point \( x = \infty \) that solve Equation (4) whenever \( \deg f < \deg g < 2 \deg f + 1 \). These polynomials are presented in expression (25).

**Lemma 4.** Liénard differential systems (1) from family (A) do not have exponential invariants of the form, \( E(x, y) = \exp[h(x, y)/r(x, y)] \) where \( h(x, y) \in \mathbb{C}[x, y] \) and \( r(x, y) \in \mathbb{C}[x, y] \setminus \mathbb{C} \) are coprime polynomials.

**Proof.** The proof is by contradiction. Let \( E(x, y) = \exp[h(x, y)/r(x, y)] \) be an exponential invariant of a Liénard differential system (1) from family (A). Since the polynomial \( r(x, y) \) is not a constant, we see that \( r(x, y) = 0 \) is an invariant algebraic curve of the differential system under consideration. According to the results of Theorem 9 we can represent the polynomial \( r(x, y) \) in the form \( r(x, y) = F_1^{n_1}(x, y)F_2^{n_2}(x, y) \), where \( n_1, n_2 \in \mathbb{N}_0 \), \( n_1 + n_2 > 0 \), and the polynomials \( F_1(x, y) \) and \( F_2(x, y) \) produce invariant algebraic curves of the related differential system. These polynomials are given by expression (22) with \( N = 1, k = 0 \) (the invariant algebraic curve \( F_1(x, y) = 0 \)) and by expression (22) with \( N \in \mathbb{N}, k = 1 \) (the invariant algebraic curve \( F_2(x, y) = 0 \)). Relation (23) yields the explicit expressions of the cofactors

\[
\lambda_1(x, y) = -f(x) - q(x) = o(x^m), \quad x \to \infty,
\]
\[
\lambda_2(x, y) = -Nf(x) - (N - 1)q(x) - p(x) = -f_0x^m + o(x^m), \quad x \to \infty
\] (44)

related to these invariant algebraic curves, respectively. The cofactor of the invariant algebraic curve \( r(x, y) = 0 \) equals \( \lambda(x, y) = n_1\lambda_1(x, y) + n_2\lambda_2(x, y) \).

If the Liénard differential system under consideration possesses only one irreducible invariant algebraic curve, for example, \( F_1(x, y) = 0 \), then we assume that \( n_2 = 0 \) and vice versa.

Here and in what follows, \( O \) denotes the zero element of the field \( \mathbb{C}_\infty\{x\} \). Let us suppose that \( y^{(2)}_\infty(x) \) is a Puiseux series of type (II) such that the following condition \( F_2(x, y^{(2)}_\infty(x)) = O \) is valid. Substituting \( y = y^{(2)}_\infty(x) \) into the function \( \lambda(x, y)/y \), we can expand it into a Puiseux series near infinity. Supposing that \( n_2 > 0 \), we find the dominant behavior near the point \( x = \infty \) of this series. The result is

\[
\lambda\left(x, y^{(2)}_\infty(x)\right) \over y^{(2)}_\infty(x) = \frac{n_2f_0^2}{g_0}x^{2m-n} + o(x^{2m-n}), \quad x \to \infty.
\] (45)

The inequality \( \deg f < \deg g < 2 \deg f + 1 \) yields \( 2m - n > -1 \). Using Theorem 6, we come to a contradiction. Consequently, we should set \( n_2 = 0 \). The polynomial \( r(x, y) \) now takes the form \( r(x, y) = F_1^{n_1}(x, y) \) with \( F_1(x, y) = y - q(x) - z_0 \) and \( z_0 \in \mathbb{C} \).

Equation \( \lambda E = \varphi(x, y)E \) produces the following linear inhomogeneous partial differential equation:

\[
yh_x - (g(x)y + f(x))h_y = n_1\lambda_1(x, y)h + \varphi(x, y)F_1^{n_1}(x, y), \quad \varphi(x, y) \in \mathbb{C}[x, y]
\] (46)

satisfied by the polynomial \( h(x, y) \). Let us consider the truncated Puiseux series \( y(x) = q(x) + z_0 \) that solves Equation (4) whenever invariant algebraic curve \( F_1(x, y) = 0 \) exists. Recall that \( y(x) \) is a zero of the polynomial \( F_1(x, y) \). Considering the restriction \( H(x) = h(x, y)|_{y=y(x)} \), we obtain
the ordinary differential equation

\[(q(x) + z_0) \frac{dH}{dx} = n_1 \lambda_1(x, y) H, \quad \lambda_1(x, y) = -f(x) - q(x).\]  

(47)

Since the polynomials \(h(x, y)\) and \(F_1(x, y)\) are relatively prime, we conclude that \(H(x) \neq 0\). Indeed, assuming the converse and using the Bézout's theorem, we see that the bivariate polynomials \(h(x, y)\) and \(r(x, y)\) have a common factor.

It follows from the relations \(\deg q = m + 1\) and \(\deg \lambda_1 \leq m - 1\) that Equation (47) does not have nonzero polynomial solutions. This fact contradicts the existence of exponential invariants \(E(x, y) = \exp\{h(x, y)/r(x, y)\}\) with a nonconstant polynomial \(r(x, y)\).

Now let us establish that Liénard differential systems (1) do not have Darboux first integrals whenever \(\deg f < \deg g < 2 \deg f + 1\).

**Theorem 16.** Liénard differential systems (1) from family (A) are not Darboux integrable.

**Proof.** Using Theorem 10, we see that a Liénard differential system (1) with the restriction \(\deg f < \deg g < 2 \deg f + 1\) has at most two distinct irreducible invariant algebraic curves simultaneously. As in the proof of Lemma 4, we denote these curves as \(F_1(x, y) = 0\) and \(F_2(x, y) = 0\). Without loss of generality, we can suppose that the generating polynomials of these algebraic curves are given by expression (22) with \(N = 1, k = 0\), and \(N \in \mathbb{N}, k = 1\), respectively. If there exists a first integral being a Darboux function, then this first integral should be of the form

\[I(x, y) = F_1^{d_1}(x, y)F_2^{d_2}(x, y), \quad d_1, d_2 \in \mathbb{C}, \quad |d_1| + |d_2| > 0.\]  

(48)

Indeed, Lemma 4 forbids the existence of exponential factors given by invariants \(E(x, y) = \exp\{h(x, y)/r(x, y)\}\), where \(h(x, y) \in \mathbb{C}[x, y]\) and \(r(x, y) \in \mathbb{C}[x, y] \setminus \mathbb{C}\). By Lemma 3, exponential invariants \(E(x, y) = \exp\{h(x, y)\}\) that could arise in an expression of the first integral have divisible by \(y\) cofactors, while invariant algebraic curves \(F_1(x, y) = 0\) and \(F_2(x, y) = 0\) have cofactors that are independent of \(y\). Recall that there are no rational first integrals by the corollary to Theorem 10. Consequently, first integrals that are Darboux functions do not have exponential factors.

Let us note that if the Liénard differential system in question has only one irreducible invariant algebraic curve, for example, \(F_1(x, y) = 0\), then we suppose that \(d_2 = 0\) and vice versa.

The cofactors \(\lambda_1(x, y)\) and \(\lambda_2(x, y)\) of invariant algebraic curves \(F_1(x, y) = 0\) and \(F_2(x, y) = 0\) are given in relation (44). First, integral (48) exists provided that the following condition \(d_1 \lambda_1(x, y) + d_2 \lambda_2(x, y) = 0\) is satisfied. This condition can be rewritten as

\[d_1 [f(x) + q(x)] + d_2 [N f(x) + (N - 1) q(x)] + p(x) = 0.\]  

(49)

The highest-degree term in expression (48) is \(d_2 f_0 x^m\). Since \(f_0 \neq 0\), we get \(d_2 = 0\). Consequently, the first integral can be chosen as a rational function with \(d_1 = 1\). Again we recall the corollary to Theorem 10, which excludes the existence of rational first integrals.

Furthermore, our aim is to study the existence of nonautonomous Darboux first integrals with a time-dependent exponential factor.
Lemma 5. A Liénard differential system (1) from family (A) has a nonautonomous Darboux first integral with a time-dependent exponential factor (10) if and only if \( \text{deg } g = \text{deg } f + 1 \) and the following condition:

\[
g(x) = \omega \left( \int_{0}^{x} f(s) ds - \omega x - z_0 \right), \quad \omega, z_0 \in \mathbb{C}, \quad \omega \neq 0
\]  

(50)
is valid. A first integral takes the form

\[
I(x, y, t) = \left( y + \int_{0}^{x} f(s) ds - \omega x - z_0 \right) \exp(\omega t).
\]  

(51)

There are no other independent nonautonomous Darboux first integrals with a time-dependent exponential factor.

Proof. If the polynomial \( g(x) \) is given by relation (50), then it is straightforward to verify that function (51) is a nonautonomous first integral of the related differential system.

Let us establish the necessity of condition (50). We repeat the reasoning given in the proof of Theorem 16. By Theorems 3 and 10, a nonautonomous first integral (10) reads as

\[
I(x, y, t) = F_1^{d_1}(x, y) F_2^{d_2}(x, y) \exp(\omega t), \quad d_1, d_2, \omega \in \mathbb{C}, \quad |d_1| + |d_2| > 0, \quad \omega \neq 0.
\]  

(52)

Now we need to study the following condition:

\[
d_1[f(x) + q_x(x)] + d_2[Nf(x) + (N - 1)q_x(x) + p_x(x)] - \omega = 0.
\]  

(53)

Similarly to the case of Theorem 16, we obtain \( d_2 = 0 \). Furthermore, it is without loss of generality to set \( d_1 = 1 \). Considering relation (53) as an ordinary differential equation for the polynomial \( q(x) \) and performing the integration, we find its solution

\[
q(x) = -\int_{0}^{x} f(s) ds + \omega x
\]  

(54)
satisfying the condition \( q(0) = 0 \) presented in (25). Finally, we note that the invariant algebraic curve \( F_1(x, y) = 0 \) given by the polynomial \( F_1(x, y) = y - q(x) - z_0 \) exists whenever the series of type (I) defined in (24) terminates at the zero term. In this case, Equation (4) has a polynomial solution \( y(x) = q(x) + z_0 \). Substituting our results into Equation (4), we find expression (50). Consequently, we obtain the following equality \( \text{deg } g = \text{deg } f + 1 \). The absence of other independent nonautonomous Darboux first integrals (10) follows from the previous considerations and the uniqueness of the invariant algebraic curve \( F_1(x, y) = 0 \).

It follows from Lemma 5 that ordinary differential equation (4) related to a Liénard differential system from family (A) possessing a nonautonomous Darboux first integral has a polynomial solution \( y(x) = q(x) + z_0 \).

We recall that integrating factors and Jacobi last multipliers with a constant ratio belong to the same equivalence class. We do not distinguish between them.
Theorem 17. A Liénard differential system (1) from family (A) is Liouvillian integrable if and only if the following assertions are valid:

1. The system under consideration possesses two distinct irreducible invariant algebraic curves $F_1(x, y) = 0$ and $F_2(x, y) = 0$, where the polynomial $F_1(x, y) = y - q(x) - z_0$ is given by expression (22) with $N = 1$, $k = 0$ and the polynomial $F_2(x, y) \in \mathbb{C}[x, y]$ takes the form (22) with $N \in \mathbb{N}$, $k = 1$;

2. The polynomials $q(x)$ and $p(x)$ giving initial parts of the Puiseux series near the point $x = \infty$ that solve Equation (4) identically satisfy the condition

$$ (n - m)[f(x) + q_x(x)] + (m + 1)p_x(x) = 0. $$

(55)

The related Liénard differential system has the unique Darboux integrating factor

$$ M(x, y) = \left(\frac{y - q(x) - z_0}{F_2(x, y)}\right)^{\frac{N(m+1)-(n+1)}{m+1}}. $$

(56)

Proof. Using Theorem 2, we see that a Liouvillian integrable differential system should have a Darboux integrating factor. Arguing as in the proof of Theorem 16, we conclude that such an integrating factor does not involve exponential invariants and is of the form

$$ M(x, y) = F_1^{d_1}(x, y)F_2^{d_2}(x, y), \quad d_1, d_2 \in \mathbb{C}, \quad |d_1| + |d_2| > 0. $$

(57)

In this expression, the polynomials $F_1(x, y)$ and $F_2(x, y)$ define invariant algebraic curves. The polynomial $F_1(x, y)$ is given by expression (22) with $N = 1$ and $k = 0$. The polynomial $F_2(x, y)$ is produced by the same expression with $N \in \mathbb{N}$ and $k = 1$. Again we note that if the Liénard differential system in question has only one irreducible invariant algebraic curve, for example, $F_1(x, y) = 0$, then we set $d_2 = 0$ and vice versa. Calculating the divergence of vector field (35) yields $\operatorname{div} \mathbf{X} = -f(x)$. Hence, the necessary and sufficient condition for Darboux integrating factor (57) to exist takes the form $d_1\lambda_1(x, y) + d_2\lambda_2(x, y) = f(x)$, where $\lambda_1(x, y)$ and $\lambda_2(x, y)$ are the cofactors of invariant algebraic curves $F_1(x, y) = 0$ and $F_2(x, y) = 0$, respectively. These cofactors are given in expression (44). The condition enabling the existence of the Darboux integrating factor explicitly reads as

$$ d_1[f(x) + q_x(x)] + d_2[Nf(x) + (N - 1)q_x(x) + p_x(x)] = -f(x). $$

(58)

Balancing the highest-degree terms in this expression, we get $d_2 = -1$. Further, we recall that the invariant algebraic curve $F_1(x, y) = 0$ is given by the polynomial $F_1(x, y) = y - q(x) - z_0$. If $d_1 = 0$, then the curve $F_1(x, y) = 0$ either does not exist or does not enter the explicit expression (57) of the integrating factor.

Now let us consider the representation of the cofactors $\lambda_1(x, y)$ and $\lambda_2(x, y)$ in the ring $\mathbb{C}_\infty[x][y]$. They are of the form

$$ \lambda_1(x, y) = -f(x) - q_x(x), \quad \lambda_2(x, y) = -\sum_{j=1}^{N-1} \left[ f(x) + \left(y_j^{(1)}\right)_x \right] - \left[ f(x) + \left(y_{N, \infty}^{(2)}\right)_x \right]. $$

(59)
Substituting these expressions into the condition $d_1 \lambda_1(x, y) + d_2 \lambda_2(x, y) = f(x)$ with $d_2 = -1$, we find

$$d_1 [f(x) + q_x(x)] - \sum_{j=1}^{N-1} \left( f(x) + \left. y_{j, \infty}^{(1)} \right|_x \right) = \left( y_{N, \infty}^{(2)} \right|_x. \tag{60}$$

The Puiseux series $y(x) = y_{j, \infty}^{(1)}(x)$ and the polynomial $y(x) = q(x) + z_0$ solve Equation (4). Thus, we obtain

$$f(x) + q_x(x) = -\frac{g(x)}{q(x) + z_0}, \quad f(x) + \left. y_{j, \infty}^{(1)} \right|_x = -\frac{g(x)}{y_{j, \infty}^{(1)}. \tag{61}$$

Again we suppose that if there is no polynomial solution $y(x) = q(x) + z_0$, then $d_1 = 0$. Using expressions (61), we rewrite condition (60) in the form

$$g(x) \left[ \sum_{j=1}^{N-1} \frac{1}{y_{j, \infty}^{(1)}} - \frac{d_1}{q(x) + z_0} \right] = \left( y_{N, \infty}^{(2)} \right|_x. \tag{62}$$

Let us find the highest-order terms in a neighborhood of the point $x = \infty$. Using the asymptotic formulas

$$g(x) = g_0 x^n + o(x^n), \quad q(x) = -\frac{f_0}{m+1} x^{m+1} + o(x^{m+1}), \quad x \rightarrow \infty$$

$$y_{j, \infty}^{(1)}(x) = -\frac{f_0}{m+1} x^{m+1} + o(x^{m+1}), \quad y_{N, \infty}^{(2)}(x) = -\frac{g_0}{f_0} x^{n-m} + o(x^{n-m}), \quad x \rightarrow \infty, \tag{63}$$

we collect the coefficients of $x^{n-m-1}$ in expression (62). This yields

$$d_1 = N - 1 - \frac{n-m}{m+1}. \tag{64}$$

Using inequalities $m < n < 2m + 1$, we see that the parameter $d_1$ cannot be zero. Substituting relation (64) and $d_2 = -1$ into expressions (57) and (58), we find condition (55) and the Darboux integrating factor as given in (57).

\[\Box\]

Remark. Condition (55) is identically satisfied whenever $\deg g = \deg f + 1$. Indeed, if $n = m + 1$, then we get

$$f(x) + q_x(x) = \frac{(m+1)g_0}{f_0}, \quad p(x) = -\frac{g_0}{f_0} x. \tag{65}$$

Consequently, Liénard differential systems satisfying the restriction $\deg g = \deg f + 1$ and possessing two distinct irreducible invariant algebraic curves are always Liouvillian integrable. This fact is noted on examples in Refs. 28–30, 50.

\[\Box\]

We see from Theorem 17 that Equation (4) related to a Liouvillian integrable Liénard differential system from family (A) has a polynomial solution given by the expression $y(x) = q(x) + z_0$. In addition, using condition (55) and Equation (4), we can represent the polynomials $f(x)$ and $g(x)$
in the integrable cases as
\[ f(x) = -q_x(x) - \frac{m + 1}{n - m} p_x(x), \quad g(x) = \frac{m + 1}{n - m} [q(x) + z_0] p_x(x). \] (66)

Recall that the polynomial \( q(x) \) is of degree \( m + 1 \) and the polynomial \( p(x) \) is of degree \( n - m \). It follows from Theorem 17 that Liénard differential systems (1) from family \((A)\) are Liouvillian integrable if and only if the polynomials \( f(x) \) and \( g(x) \) are given by expressions (66) and the systems possess an invariant algebraic curve with the generating polynomial taking the form (22), where \( N \in \mathbb{N} \) and \( k = 1 \).

Now let us study the integrability of Liénard differential systems such that the related equation (4) has two distinct polynomial solutions. The following theorem is valid.

**Theorem 18.** A Liénard differential system (1) from family \((A)\) with two distinct invariant algebraic curves given by first-degree polynomials with respect to \( y \) is Liouvillian integrable if and only if the system is of the form
\[ x_t = y, \quad y_t = \left[ k \beta v^{k-1}(x) + (k + l) v^{l-1}(x) \right] v_x y - k \left[ \beta v^k(x) + v^l(x) \right] v^{l-1}(x) v_x, \] (67)
where \( \beta \in \mathbb{C} \setminus \{0\} \), \( v(x) \) is a polynomial of degree \( (n - m)/l \), \( k \) and \( l \) are relatively prime natural numbers such that the restriction \((m + 1)l = (n - m)k \) holds. The related Liénard differential system has the unique Darboux integrating factor
\[ M(x, y) = \left\{ y - \beta v^k(x) - v^l(x) \right\}^{-\frac{1}{k}} \left\{ v - v^l(x) \right\}^{-1} \] (68)
and the invariant algebraic curves \( y - \beta v^k(x) - v^l(x) = 0, y - v^l(x) = 0 \). A Liouvillian first integral is of the form
\[ I(x, y) = \frac{k \beta}{k - l} \left\{ y - \beta v^k(x) - v^l(x) \right\}^{\frac{k - l}{k}} + \sum_{j=0}^{k-1} \exp \left[ -\frac{\pi i (2j+1)l}{k} \right] \times \log \left\{ y - \beta v^k(x) - v^l(x) \right\}^{\frac{1}{l}} - \beta \exp \left[ -\frac{\pi i (2j+1)l}{k} \right] v(x), \] (69)

**Proof.** We begin the proof by noting that we use novel designations for the polynomial solutions of Equation (4). We set \( \tilde{q}(x) = q(x) + z_0 \) and \( \tilde{p}(x) = p(x) + z_1 \). Condition (55) and Equation (4) provide explicit expressions (66) for the polynomials \( f(x) \) and \( g(x) \). Substituting the relation \( y(x) = \tilde{p}(x) \) and expressions (66) into Equation (4), we integrate the resulting equation with respect to the polynomial \( \tilde{q}(x) \). This way we get
\[ \tilde{q}(x) = \beta \tilde{p}^{\frac{m+1}{n-m}}(x) + \tilde{p}(x), \] (70)
where \( \beta \in \mathbb{C} \) is a constant of integration. Recalling the fact that \( \tilde{q}(x) \) is a polynomial of degree \( m + 1 \) and \( \tilde{p}(x) \) is a polynomial of degree \( n - m \), we find that \( \beta \neq 0 \). Introducing relatively prime natural numbers \( k \) and \( l \) according to the rule \((m + 1)l = (n - m)k \), we represent the polynomials \( \tilde{p}(x) \) and \( \tilde{q}(x) \) as
\[ \tilde{p}(x) = v^l(x), \quad \tilde{q}(x) = \beta v^k(x) + v^l(x). \] (71)
In this expression, \( v(x) \) is an arbitrary polynomial of degree \((n - m)/l\). Substituting \( N = 1 \) into expression (56), we find the integrating factor as given in relation (68). Finally, we calculate the line integral

\[
I(x, y) = \int_{(x_0, y_0)}^{(x, y)} M(x, y)[ydy + \{f(x)y + g(x)\}dx]
\]

and obtain first integral (69), where \( i \) is the imaginary unit.

**Remark.** The family of systems (67) can be transformed to the following simple form

\[
s_\tau = z, \quad z_\tau = [k\beta s^{k-1} + (k + l)s^{l-1}]z - k[\beta s^k + s^l]s^{l-1}
\]

via the generalized Sundman transformation \( s(\tau) = v(x) \), \( z(\tau) = y \), \( d\tau = v_x(x)dt \). Substituting \( v(x) = s \), \( y = z \) into (69), we find a Liouvillian first integral for systems (73).

Let us demonstrate that for any fixed degrees of the polynomials \( f(x) \) and \( g(x) \) there exist Liénard differential systems (1) from family (A) that have Liouvillian first integrals. With this aim, we suppose that the polynomial \( \tilde{p}(x) \) takes the following form \( \tilde{p}(x) = x^{n-m} \). We find the polynomial \( \tilde{q}(x) \) using expression (70). The result is \( \tilde{q}(x) = \beta x^{m+1} + x^{n-m} \). Thus, we conclude that the following family of Liénard differential systems:

\[
x_t = y, \quad y_t = [(m + 1)\beta x^m + (n + 1)x^{n-m-1}] - (m + 1)(\beta x^n + x^{2n-2m-1})
\]

is Liouvillian integrable. The related Darboux integrating factor reads as

\[
M(x, y) = (y - x^{n-m})^{-1}(y - \beta x^{m+1} - x^{n-m})^{m-n/m+1}.
\]

In addition, if the numbers \( n \) and \( m \) are the following \( n = l(k + 1) - 1 \) and \( m = lk - 1 \), where \( l \), \( k \in \mathbb{N} \), \( k > 1 \), then relations (66) and (70) produce Liouvillian integrable systems

\[
x_t = y, \quad y_t = [k\beta \tilde{p}^{k-1}(x) + k + 1] \tilde{p}_x(x)y - k(\beta \tilde{p}^{k-1}(x) + 1)\tilde{p}(x)\tilde{p}_x(x),
\]

where \( \tilde{p}(x) \) is a polynomial of degree \( n - m = l \). Using expression (68), we see that systems (76) possess the Darboux integrating factor

\[
M(x, y) = \{y - \beta \tilde{p}^{k}(x) - \tilde{p}(x)\}^{-1}\{y - \tilde{p}(x)\}^{-1}.
\]

A related Liouvillian first integral is fairly simple in the case \( k = 2 \). Let us explicitly present another expression of a first integral:

\[
I(x, y) = \text{arctanh} \left\{ \sqrt{\frac{\beta p^2(x)}{F(x, y)}} \right\} + \sqrt{\beta F(x, y)}, \quad F(x, y) = \beta \tilde{p}^2(x) + \tilde{p}(x) - y.
\]

Finally, let us note that there exist Liouvillian integrable Liénard differential systems from family (A) such that the polynomial \( F_2(x, y) \) in expression (56) is of degree with respect to \( y \) greater
than 1. In fact, the degree with respect to $y$ of the polynomial $F_2(x, y)$ can be an arbitrary natural number. This fact was established in Ref. 30.

Our next step is to investigate the existence of nonautonomous Darboux–Jacobi last multipliers. The cases $\deg g = \deg f + 1$ and $\deg g \neq \deg f + 1$ will be considered separately.

**Lemma 6.** A Liénard differential system (1) satisfying the condition $\deg f + 1 < \deg g < 2 \deg f + 1$ has a nonautonomous Darboux–Jacobi last multiplier of the form (12) if and only if the following assertions are valid:

1. The system under consideration possesses two distinct irreducible invariant algebraic curves $F_1(x, y) = 0$ and $F_2(x, y) = 0$, where the polynomial $F_1(x, y) = y - q(x) - z_0$ is given by expression (22) with $N = 1$, $k = 0$ and the polynomial $F_2(x, y) \in \mathbb{C}[x, y]$ reads as (22) with $N \in \mathbb{N}$, $k = 1$;
2. The polynomials $q(x)$ and $p(x)$ giving initial parts of the Puiseux series near the point $x = \infty$ that solve Equation (4) identically satisfy the condition

   $$(n - m)[f(x) + q_x(x)] + (m + 1)[p_x(x) + \omega] = 0, \quad (79)$$

   where $\omega \in \mathbb{C} \setminus \{0\}$ is a constant.

**The nonautonomous Darboux–Jacobi last multiplier is unique and takes the form**

$$M(x, y, t) = \left(\frac{y - q(x) - z_0}{F_2(x, y)}\right)^{\frac{N(m+1)-(n+1)}{m+1}} \exp(\omega t). \quad (80)$$

**Proof.** We use Theorem 4 and repeat the proof of the previous theorem. The only difference is in condition (58). Now this condition reads as

$$d_1[f(x) + q_x(x)] + d_2[Nf(x) + (N - 1)q_x(x) + p_x(x)] - \omega = -f(x), \quad (81)$$

where $\omega \neq 0$. The related nonautonomous Darboux–Jacobi last multiplier is given by the expression

$$M(x, y, t) = F_1^{d_1}(x, y)F_2^{d_2}(x, y) \exp[\omega t]. \quad (82)$$

Similarly to the case of Theorem 17, we find the values of $d_1$ and $d_2$. They are $d_1 = N - 1 - (n - m)/(m + 1)$ and $d_2 = -1$. Both of them are nonzero. Consequently, a Liénard differential system (1) with a nonautonomous Darboux–Jacobi last multiplier has two distinct irreducible invariant algebraic curves $F_1(x, y) = 0$ and $F_2(x, y) = 0$. Substituting explicit values of the parameters $d_1$ and $d_2$ into condition (81) yields relation (79).

If there exist two distinct nonautonomous Darboux–Jacobi last multipliers (12), then their ratio is a Darboux first integral either autonomous or nonautonomous of the form (10). By Theorem 16 and Lemma 5, such a situation is impossible.

**Lemma 7.** A Liénard differential system (1) satisfying the condition $\deg g = \deg f + 1 (n = m + 1)$ has a nonautonomous Darboux–Jacobi last multiplier of the form (12) if and only if the system under
study possesses the irreducible invariant algebraic curve \( F_2(x, y) = 0 \) given by relation (22) with \( N \in \mathbb{N} \) and \( k = 1 \). A nonautonomous Darboux–Jacobi last multiplier is of the form

\[
M(x, y, t) = \frac{(y - q(x) - z_0)^{d_1} \exp \left[ \frac{g_0}{f_0} \{1 + \{d_1 - (N - 1)\}(m + 1)\}t \right]}{F_2(x, y)},
\]

where \( d_1 = 0 \) whenever the system does not have another invariant algebraic curve given by the expression \( y - q(x) - z_0 = 0 \) and \( d_1 \in \mathbb{C} \) otherwise. The nonautonomous Darboux–Jacobi last multiplier is unique provided that the curve \( y - q(x) - z_0 = 0 \) does not exist.

**Proof.** Similarly to the case of Lemma 6, we see that a nonautonomous Darboux–Jacobi last multiplier of the form (12) exists if and only if the system under consideration has at least one invariant algebraic curve and condition (81) is valid. Again we suppose that \( d_j = 0 \) whenever the invariant algebraic curve \( F_j(x, y) = 0 \) does not exist. The explicit expression of a nonautonomous Darboux–Jacobi last multiplier is given by (82). Balancing the highest-degree terms in condition (81) yields \( d_2 = -1 \). This fact proves the existence of the invariant algebraic curve \( F_2(x, y) = 0 \). Furthermore, we find that the polynomials \( f(x) + q_x(x) \) and \( p_x(x) \) are constants, see relation (65). Substituting equalities \( d_2 = -1, f(x) + q_x(x) = (m + 1)g_0/f_0 \) and \( p_x(x) = -g_0/f_0 \) into condition (81), we obtain

\[
(m + 1)g_0d_1 + \{1 - (N - 1)(m + 1)\}g_0 - f_0\omega = 0.
\]

This expression provides the value of \( \omega \) as given in relation (83). If the Liénard differential system in question does not have the invariant algebraic curve \( F_1(x, y) = 0 \), then nonautonomous Darboux–Jacobi last multiplier (83), where \( d_1 = 0 \), is unique. In the converse case, we obtain the family of distinct nonautonomous Darboux–Jacobi last multipliers (83). We note that the invariant algebraic curve \( F_1(x, y) = 0 \) is \( y - q(x) - z_0 = 0 \). In addition, recall that a Liénard system (1) satisfying the restriction \( \deg g = \deg f + 1 \) and possessing the invariant algebraic curve \( y - q(x) - z_0 = 0 \) has time-dependent Darboux first integral (51). It is straightforward to show that the product of a first integral and a nonconstant Jacobi last multiplier is another Jacobi last multiplier.

It is demonstrated in Ref. 31 that Liénard differential systems with nonautonomous Darboux–Jacobi last multipliers (80) indeed exist. In addition, let us note that the famous Duffing–van der Pol oscillators belong to family (A). The classification of Liouvillian integrable Duffing–van der Pol oscillators is performed in Ref. 29.

## 7 INTEGRABILITY OF LIÉNARD DIFFERENTIAL SYSTEMS FROM FAMILY (B)

Let us consider families of Liénard differential systems with fixed degrees of the polynomials \( f(x) \) and \( g(x) \) such that the following relation \( \deg g = 2 \deg f + 1 \) is satisfied. If we do not impose restrictions on the highest-degree coefficients \( f_0 \) and \( g_0 \) of the polynomials \( f(x) \) and \( g(x) \), then it has been established in Ref. 47 that the Fuchs indices of the Puiseux series near the point \( x = \infty \) that solve related equations (4) depend on the parameters \( f_0 \) and \( g_0 \). Consequently, performing
the classification of irreducible invariant algebraic curves and Darboux or Liouvillian first 
integrals is a very difficult problem whenever only the degrees of the polynomials \( f(x) \) and \( g(x) \) are fixed. The method of Puiseux series can deal with each case of a positive rational Fuchs index 
separately. Interestingly, such a degeneracy leads to a variety of distinct integrable cases arising in 
Liénard differential systems from family (B).

This section is mainly devoted to the nonresonant case. We say that a Liénard differential sys-
tem from family (B) is resonant near infinity if Equation (26) possesses a solution in \( \mathbb{Q}^+ \). For 
convenience, we introduce the parameter \( \delta \) according to the rule

\[
g_0 = \frac{f_0^2 - \delta^2}{4(m + 1)}, \quad \delta \in \mathbb{C}, \quad \delta \neq \pm f_0. \tag{85}
\]

Using this normalization, we solve Equation (26). As a result, we find the Fuchs indices. They 
take the form

\[
\eta_1 = \frac{2(m + 1)\delta}{\delta - f_0}, \quad \eta_2 = \frac{2(m + 1)\delta}{\delta + f_0}. \tag{86}
\]

There are no positive rational Fuchs indices if and only if the condition \( \delta/f_0 \notin \mathbb{Q} \setminus \{0\} \) is valid. In 
addition, we assume that the following inequality \( \deg f > 0 \) holds. The integrability problem for 
Liénard differential systems (1) under restrictions \( \deg f = 0 \) and \( \deg g = 1 \) is simple, see the end 
of Section 5.

In this section, we use the designations of Theorem 11. In particular, the Puiseux series near the 
point \( x = \infty \) that solve Equation (4) are denoted as \( y^{(1)}_{\infty}(x) \) and \( y^{(2)}_{\infty}(x) \). Introducing the variable \( \delta \) 
instead of the parameter \( g_0 \), we see that these series have the following dominant behavior:

\[
y^{(1)}_{\infty}(x) = \frac{\delta - f_0}{2(m + 1)} x^{m+1} + o(x^{m+1}), \quad x \to \infty;
\]

\[
y^{(2)}_{\infty}(x) = -\frac{\delta + f_0}{2(m + 1)} x^{m+1} + o(x^{m+1}), \quad x \to \infty. \tag{87}
\]

Let us denote the polynomial parts of the series \( y^{(1)}_{\infty}(x) \) and \( y^{(2)}_{\infty}(x) \) as \( q_1(x) \) and \( q_2(x) \), respectively. Thus, we have the equalities

\[
q_1(x) = \left\{ y^{(1)}_{\infty}(x) \right\}^+, \quad q_2(x) = \left\{ y^{(2)}_{\infty}(x) \right\}^+. \tag{88}
\]

If \( \delta = 0 \), then the series \( y^{(1)}_{\infty}(x) \) and \( y^{(2)}_{\infty}(x) \) coincide. Let us omit the indices and set \( q(x) = \{y_{\infty}(x)\}^+ \).

We begin by investigating the existence of exponential invariants related to invariant algebraic 
curves.

**Lemma 8.** Let \( h(x, y) \in \mathbb{C}[x, y] \) and \( r(x, y) \in \mathbb{C}[x, y] \setminus \mathbb{C} \) be relatively prime polynomials. A Liénard differential system (1) satisfying the conditions \( \deg g = 2\deg f + 1 \) and \( \delta/f_0 \notin \mathbb{Q} \setminus \{0\} \) has exponential invariants \( E(x, y) = \exp\{h(x, y)/r(x, y)\} \) if and only if the following statements are valid:

1. \( \delta = 0; \)
2. There exists the invariant algebraic curve \( F(x, y) = 0 \) with the generating polynomial \( F(x, y) = y - q(x) \);

3. The ordinary differential equation

\[
q(x)u_x(x) + [f(x) + q_x(x)]u(x) = 0
\]

(89)

has a nonzero polynomial solution \( u(x) \).

The related exponential invariant is of the form

\[
E(x, y) = \exp \left[ \frac{u(x)}{y - q(x)} \right]
\]

(90)

and possesses the cofactor \( g(x, y) = u_x(x) \).

**Proof.** It follows from Theorem 11 that any Liénard differential system (1) satisfying the conditions \( \deg g = 2 \deg f + 1 \) and \( \delta/f_0 \notin \mathbb{Q} \setminus \{0\} \) has at most two distinct irreducible invariant algebraic curves simultaneously. The degrees with respect to \( y \) of polynomials producing irreducible invariant algebraic curves are either 1 or 2. If there exists an irreducible invariant algebraic curves of degree 2 with respect to \( y \), then it is unique. Furthermore, there can arise at most two distinct irreducible invariant algebraic curves of degree 1 with respect to \( y \). Since \( E(x, y) = \exp\{h(x, y)/r(x, y)\} \) is an exponential invariant, we conclude that \( r(x, y) = 0 \) is an invariant algebraic curve of the related system. Let us represent exponential invariants in the form

\[
E(x, y) = \exp \left[ \frac{h(x, y)}{F_1^n(x, y)F_2^n(x, y)} \right], \quad n_1, n_2 \in \mathbb{N}_0, \quad n_1 + n_2 > 0
\]

(91)

where \( F_1(x, y) = 0 \) and \( F_2(x, y) = 0 \) are irreducible invariant algebraic curves. Without loss of generality, we set \( n_k = 0 \) whenever the invariant algebraic curve \( F_k(x, y) = 0 \) does not exist. Here, \( k = 1 \) or \( k = 2 \).

**Case 1.** Let us suppose that there exists only one irreducible invariant algebraic curve \( F_1(x, y) = 0 \) of degree 2 with respect to \( y \). Using relation (28), we find the cofactor \( \lambda_1(x, y) \). The result is

\[
\lambda(x, y) = -2f(x) - \left\{ \left( y_\infty^{(1)} \right)_x + \left( y_\infty^{(2)} \right)_x \right\}_+
\]

(92)

The dominant behavior of the cofactor near the point \( x = \infty \) is \( \lambda(x, y) = -f_0 x^m + o(x^m) \). Now let us take one of the Puiseux series, for example, the series \( y_\infty^{(1)}(x) \). We find the asymptotic relation

\[
\frac{\lambda\left(x, y_\infty^{(1)}(x)\right)}{y_\infty^{(1)}(x)} = -\frac{2(m + 1)f_0}{(\delta - f_0)x} + o\left(\frac{1}{x}\right), \quad x \to \infty.
\]

(93)

Since \( \delta/f_0 \notin \mathbb{Q} \setminus \{0\} \), we conclude that the conditions of Theorem 6 are not satisfied whenever \( \delta \neq 0 \). The case \( \delta = 0 \) will be considered separately. Exponential invariants do not exist provided that \( \delta \neq 0 \).
Case 2. Let us suppose that the differential system under consideration has two distinct irreducible invariant algebraic curves \( F_1(x, y) = 0 \) and \( F_2(x, y) = 0 \). The polynomials producing the curves are of degree 1 with respect to \( y \) and have the cofactors

\[
\lambda_m(x, y) = -f(x) - \left\{ \left( y^{(m)} \right)_x \right\}_+, \quad m = 1, 2.
\]

The cofactor of the invariant algebraic curve \( F_1^n(x, y)F_2^n(x, y) = 0 \) is given by the polynomial \( \lambda(x, y) = n_1\lambda_1(x, y) + n_2\lambda_2(x, y) \) and has the following dominant behavior near the point \( x = \infty \):

\[
\lambda(x, y) = \frac{1}{2} [n_2(\delta - f_0) - n_1(\delta + f_0)]x^m + o(x^m), \quad x \to \infty.
\]

Suppose that one of the numbers \( n_1 \) and \( n_2 \) is nonzero, for example, \( n_1 \). Furthermore, we consider the following asymptotic relation:

\[
\frac{\lambda(x, y)}{y^{(1)}_\infty(x)} = (m + 1)(n_2 - (\delta + f_0)n_1)\frac{1}{x} + o\left(\frac{1}{x}\right), \quad x \to \infty.
\]

Using Theorem 6 and expression \( \delta/f_0 \notin \mathbb{Q} \setminus \{0\} \), we see that there are no exponential invariants \( \lambda \) whenever \( \delta \neq 0 \).

Case 3. Let us suppose that the differential system in question has only one irreducible invariant algebraic curve: either \( F_1(x, y) = 0 \) or \( F_2(x, y) = 0 \). If \( \delta \neq 0 \), then arguing as in the previous case, we prove the nonexistence of exponential invariants.

Case 4. Now let us suppose that \( \delta = 0 \). Recall that the Puiseux series \( y^{(1)}_\infty(x) \) and \( y^{(2)}_\infty(x) \) merge in this case. The unique Puiseux series centered at the point \( x = \infty \) that satisfies Equation (4) is denoted as \( y_\infty(x) \). Let a Liénard differential system (1) satisfying the conditions \( \deg g = 2 \deg f + 1 \) and \( \delta = 0 \) have an irreducible invariant algebraic curve \( F(x, y) = 0 \), then we get \( F(x, y) = y - q(x) \), \( q(x) = \{y_\infty(x)\}_+ \). The cofactor \( \lambda(x, y) \) of this curve reads as \( \lambda(x, y) = -f(x) - q_\infty(x) \). Again the cofactor does not depend on \( y \). If in addition the system possesses an exponential invariant \( E(x, y) = \exp[h(x, y)/F^k(x, y)] \) for some \( k \in \mathbb{N} \), then it is without loss of generality to suppose that the degree of the polynomial \( h(x, y) \) with respect to \( y \) is at most \( k - 1 \). By \( \varphi(x, y) \in \mathbb{C}[x, y] \) we denote the cofactor of the invariant \( E(x, y) \). We introduce the variable \( z \) as \( z = y - q(x) \) and represent the polynomial \( h(x, y) \) in the following form:

\[
h(x, y) = \sum_{j=0}^{k-1} u_j(x)z^j, \quad z = y - q(x), \quad u_j(x) \in \mathbb{C}[x],
\]

where it is without loss of generality to suppose that \( u_0(x) \neq 0 \). Otherwise, we need to reduce \( k \) by one. Substituting representation (97) into the partial differential equation \( \lambda h - k\lambda h = \varphi(x, y)z^k \) resulting from the equation for the invariant \( E(x, y) \), we set to zero the coefficients of \( z^j \), \( 0 \leq j \leq k \). As a result, we see that the cofactor \( \varphi(x, y) \) does not depend on \( y \): \( \varphi(x, y) = u_{k-1,x} \). In addition, we find the system

\[
\begin{align*}
qu_{0,x} - k\lambda u_0 &= 0, \\
qu_{j,x} - (k - j)\lambda u_j + u_{j-1,x} &= 0, \quad 1 \leq j < k - 1.
\end{align*}
\]
Solving the first two equations, we obtain the dominant terms \( u_0(x) = C_0 x^{k(m+1)} \) and \( u_1(x) = 2k(m + 1)^2 C_0 x^{(k-1)(m+1)} \log x / f_0 \), where \( C_0 \in \mathbb{C} \setminus \{0\} \). Since \( u_1(x) \) is a polynomial, we get \( k = 1 \). Consequently, the exponential invariant \( E(x, y) = \exp\{h(x, y)/F(x, y)\} \) exists if and only if \( E(x, y) = \exp[u_0(x)/(y - q(x))] \). Omitting the index for \( u_0(x) \), we get expressions (89) and (90).

Remark 1. Balancing the highest-degree terms in Equation (89), it can be shown that the polynomial \( u(x) \) is of degree \( m + 1 \).

Our next step is to derive the necessary and sufficient conditions of Darboux integrability for nonresonant Liénard differential systems from family (B). The case \( \delta = 0 \) will be considered separately.

Theorem 19. A Liénard differential system (1) satisfying the conditions \( \deg g = 2 \deg f + 1 \) and \( \delta / f_0 \notin \mathbb{Q} \) is Darboux integrable if and only if the system is of the form

\[
x_t = y, \quad y_t = \frac{2f_0}{f_0 - \delta} q_1, xy - \frac{f_0 + \delta}{f_0 - \delta} q_1, x q_1,
\]

where \( q_1(x) \) is a polynomial of degree \( m + 1 \) with the highest-degree coefficient \( (\delta - f_0)/(2m + 1) \).

A related Darboux first integral reads as

\[
I(x, y) = [y - q_1(x)]^{\delta - f_0} \left[ y - \frac{(f_0 + \delta)}{(f_0 - \delta)} q_1(x) \right]^{\delta + f_0}.
\]

Proof. It is straightforward to verify that expression (100) gives a Darboux first integral whenever all other conditions of the theorem are valid.

Let us prove the converse statement. It follows from Lemmas 3 and 8 that Darboux first integrals of Liénard differential systems (1) satisfying the conditions \( \deg g = 2 \deg f + 1 \) and \( \delta / f_0 \notin \mathbb{Q} \) do not have exponential factors. By Theorem 11, the systems under consideration have at most two distinct irreducible invariant algebraic curves simultaneously.

First, we consider a Liénard differential system (1) that satisfies the conditions \( \deg g = 2 \deg f + 1, \delta / f_0 \notin \mathbb{Q} \) and possesses only one irreducible invariant algebraic curve. If there exists a Darboux first integral, then it can be chosen as a bivariate polynomial \( I(x, y) = F(x, y) \) producing the invariant algebraic curve \( F(x, y) = 0 \). Using Theorem 11, we see that the generating polynomial \( F(x, y) \) is of degree at most 2 with respect to \( y \). The Darboux first integral \( I(x, y) = F(x, y) \) exists if an only if the cofactor \( \lambda(x, y) \) of the invariant algebraic curve \( F(x, y) = 0 \) is identically zero. We need to consider three possibilities. Let us write down the related cofactors and their dominant behavior near the point \( x = \infty \). With the help of relations (87) and (88), we get

\[
N = 1 : \quad \lambda(x, y) = -f(x) - q_1, x, \lambda(x, y) = -\frac{1}{2}(\delta + f_0)x^m + o(x^m), \quad x \to \infty;
\]

\[
N = 1 : \quad \lambda(x, y) = -f(x) - q_2, x, \lambda(x, y) = \frac{1}{2}(\delta - f_0)x^m + o(x^m), \quad x \to \infty;
\]

\[
N = 2 : \quad \lambda(x, y) = -2f(x) - q_1, x - q_2, x, \lambda(x, y) = -f_0 x^m + o(x^m), \quad x \to \infty.
\]

In these expressions, \( N \) denotes the degree of the polynomial \( F(x, y) \) with respect to \( y \). We conclude that the cofactors are not identically zero. Thus, there are no Darboux first integrals.

Now, we consider a Liénard differential system (1) that satisfies the conditions \( \deg g = 2 \deg f + 1, \delta / f_0 \notin \mathbb{Q} \) and possesses two distinct irreducible invariant algebraic curves \( F_1(x, y) = 0 \) and
$F_2(x, y) = 0$. By Theorem 11, the polynomials $F_1(x, y)$ and $F_2(x, y)$ are of the form $F_1(x, y) = y - q_1(x)$ and $F_2(x, y) = y - q_2(x)$. If there exists a Darboux first integral, then it can be represented in the form

$$I(x, y) = F_1^{d_1}(x, y)F_2^{d_2}(x, y), \quad d_1, d_2 \in \mathbb{C}, \quad |d_1| + |d_2| > 0.$$  

This first integral exists if and only if the following condition $d_1 \lambda_1(x, y) + d_2 \lambda_2(x, y) = 0$ is satisfied. Finding the dominant behavior near the point $x = \infty$ of the cofactors

$$\lambda_1(x, y) = -\frac{1}{2}(\delta + f_0)x^m + o(x^m), \quad \lambda_2(x, y) = \frac{1}{2}(\delta - f_0)x^m + o(x^m),$$  

we obtain $d_1 = \delta - f_0$, $d_2 = \delta + f_0$ and the expression

$$2\delta f(x) + (\delta - f_0)q_{1,x}(x) + (\delta + f_0)q_{2,x}(x) = 0.$$  

Recall that one of the parameters $d_1$ or $d_2$ can be chosen arbitrary. By Theorem 19, Equation (4) related to the Liénard differential system under consideration possesses two distinct polynomial solutions $y(x) = q_1(x)$ and $y(x) = q_2(x)$. Thus, we obtain the following relations:

$$f(x) + q_{1,x}(x) = -\frac{g(x)}{q_1(x)}, \quad f(x) + q_{2,x}(x) = -\frac{g(x)}{q_2(x)}.$$  

Substituting these relations and the values of $d_1$ and $d_2$ into the condition on the cofactors $d_1[f(x) + q_{1,x}(x)] + d_2[f(x) + q_{2,x}(x)] = 0$ yields the expression $q_2(x) = -d_2q_1(x)/d_1$. Finally, we use relations (104) and (105) to derive the explicit representations of the polynomials $f(x)$ and $g(x)$.

**Corollary.** The family of first-order ordinary differential equations

$$yy_x - \frac{2f_0}{f_0 - \delta}q_{1,x}y + \frac{f_0 + \delta}{f_0 - \delta}q_{1,x}q_1 = 0$$  

associated with systems (99) has two distinct polynomial solutions of the form $y(x) = q_1(x)$ and $y(x) = (f_0 + \delta)q_1(x)/(f_0 - \delta)$.  

**Remark 2.** Suppose we are in assumptions of Theorem 19 with the exception of the condition $\delta/f_0 \notin \mathbb{Q} \setminus \{0\}$. Then function (100) is still a Darboux first integral of the related Liénard differential system. In addition, the corollary to Theorem 19 is also valid. However, there may exist other resonant Liénard differential systems from family $(B)$ with Darboux first integrals.

**Remark 3.** Expression (99) provides a set of systems with rational first integrals (100) provided that $f_0/\delta$ is a rational number.

Next, let us study the Darboux integrability in the case $\delta = 0$.

**Theorem 20.** A Liénard differential system (1) satisfying the conditions $\deg g = 2 \deg f + 1$ and $\delta = 0$ is Darboux integrable if and only if the system can be represented in the form

$$x_t = y, \quad y_t = 2q_s(x)y - q(x)q_s(x),$$  

(107)
where $q(x)$ is a polynomial of degree $m + 1$. A related Darboux first integral reads as

$$I(x, y) = [y - q(x)] \exp \left[ -\frac{q(x)}{y - q(x)} \right]. \tag{108}$$

**Proof.** By direct computations, we verify that expression (108) gives a Darboux first integral of a Liénard differential system (1) with the polynomials $f(x)$ and $g(x)$ satisfying relations (107) and $\delta = 0$. Thus, we have established sufficiency of conditions presented in the theorem.

Let us prove their necessity. Suppose that a Liénard differential system from family $(B)$ with $\delta = 0$ possesses a Darboux first integral. A Darboux integrable differential system (7) has at least one invariant algebraic curve. It follows from Theorem 11 that the Liénard differential system in question possesses at most one irreducible invariant algebraic curve. This curve is of the form $y - q(x) = 0$ and has the cofactor $\lambda(x, y) = -f(x) - q_x(x)$. Thus, we have established the existence of the invariant algebraic curve $y - q(x) = 0$. According to Lemma 8, the Liénard differential system under consideration may have exponential invariants associated with the invariant algebraic curve $y - q(x) = 0$. These invariants take the form (90) and possess the cofactor $\varphi(x, y) = u_x(x)$, where the polynomial $u(x)$ satisfies Equation (89). Using Lemma 3, we conclude that exponential invariants with polynomial arguments cannot enter explicit expressions of Darboux first integrals. Consequently, a Darboux first integral can be represented in the form

$$I(x, y) = [y - q(x)]^d \exp \left[ \frac{u(x)}{y - q(x)} \right], \quad d \in \mathbb{C}, \tag{109}$$

where we suppose that $u(x) \equiv 0$ whenever exponential invariants (90) do not exist.

If $d = 0$, then the related Liénard differential system has an invariant algebraic curve $u(x) = 0$ independent of $y$. It is impossible due to Theorem 11. Thus, it is without loss of generality to set $d = 1$. The cofactors of all the invariants identically satisfy the relation $\lambda(x, y) + \varphi(x, y) = 0$ provided that first integral (109) exists. As a result, we get the expression $f(x) = u_x(x) - q_x(x)$. Substituting this expression into Equation (89) yields $u(x) = -q(x)$. This relation proves existence of exponential invariants (90). Since $y = q(x)$ is a polynomial solution of Equation (4) and $f(x) = -2q_x(x)$, we find the polynomial $g(x)$. The result is $g(x) = q(x)q_x(x)$.

Interestingly, Liénard differential systems (99) and (107) are those characterized by the so-called Chiellini integrability condition

$$\frac{d}{dx} \left[ \frac{g(x)}{f(x)} \right] = \alpha f(x), \quad \alpha \in \mathbb{C} \setminus \{0\}. \tag{110}$$

This condition was originally introduced by Chiellini. Chiellini integrable Liénard differential systems can be transformed to linear systems $s_\tau = z, z_\tau = -z - \alpha s$ via the generalized Sundman transformation $s(\tau) = \int f(x)dx, z(\tau) = y, d\tau = f(x)dt$ (see Ref. 18). Some other properties of Chiellini integrable Liénard differential systems are presented in Ref. 17.

Our next step is to study the existence of nonautonomous Darboux first integrals with a time-dependent exponential factor (10).

**Lemma 9.** A Liénard differential system (1) satisfying the conditions $\deg g = 2 \deg f + 1, \deg f \neq 0$, and $\delta/f_0 \notin Q$ possesses a nonautonomous Darboux first integral (10) if and only if the system reads
as
\[
x_t = y, \quad y_t = -\left\{ f_0(x - x_0)^m + \frac{(m+2)\omega}{2(m+1)\delta} y - \frac{f_0^2 - \delta^2}{4(m+1)}(x - x_0)^{2m+1} \right\} y - \frac{f_0\omega}{2(m+1)\delta}(x - x_0)^{m+1} - \frac{\omega^2}{4(m+1)\delta^2}(x - x_0),
\]
(111)
where \( x_0 \in \mathbb{C} \) and \( \omega \in \mathbb{C} \setminus \{0\} \). A related nonautonomous Darboux first integral takes the form
\[
I(x, y, t) = \left[ y + \frac{f_0 - \delta}{2(m+1)}(x - x_0)^{m+1} + \frac{\omega}{2(m+1)\delta}(x - x_0) \right]^{\delta - f_0} \times \left[ y + \frac{\delta + f_0}{2(m+1)}(x - x_0)^{m+1} + \frac{\omega}{2(m+1)\delta}(x - x_0) \right]^{\delta + f_0} \exp(\omega t).
\]
(112)

Proof. It is straightforward to derive that expression (112) is a nonautonomous Darboux first integral of systems (111). We only need to prove the converse statement. Supposing that a nonresonant Liénard differential system from family \((B)\) possesses a nonautonomous Darboux first integral (10), we use the arguments given in the proof of Theorem 19 to represent this first integral as
\[
I(x, y, t) = [y - q_1(x)]^{\delta - f_0} [y - q_2(x)]^{\delta + f_0} \exp(\omega t), \quad \omega \neq 0,
\]
(113)
where \( q_1(x) \) and \( q_2(x) \) are distinct polynomial solutions of Equation (4). In addition, we get the following condition:
\[
2\delta f(x) + (\delta - f_0)q_{1,x}(x) + (\delta + f_0)q_{2,x}(x) - \omega = 0.
\]
(114)
Furthermore, we express the polynomial \( f(x) \) from this condition. Substituting \( y(x) = q_1(x) \) into Equation (4), we obtain the polynomial \( g(x) \). Let us introduce the polynomial \( v(x) \) according to the rule \( q_2(x) - q_1(x) = v(x) \). Requiring that the function \( q_2(x) = q_1(x) + v(x) \) is a solution of Equation (4), we get the relation
\[
[(\delta - f_0)v(x) + 2\delta q_1(x)]v_x(x) + \omega v(x) = 0.
\]
(115)
Since \( q_1(x) \) is a polynomial, we obtain the ordinary differential equation \( \beta(x - x_0)v_x = v \), where \( \beta, x_0 \in \mathbb{C} \). Integrating this equation yields \( v(x) = v_0(x - x_0)^{1/\beta} \) with \( v_0 \in \mathbb{C} \) being a constant of integration. It follows from expression \( q_2(x) - q_1(x) = v(x) \) that \( v(x) \) is a polynomial of degree \( m + 1 \). Thus, we obtain \( \beta = 1/(m + 1) \). As a result the polynomials \( q_1(x) \) and \( q_2(x) \) can be represented in the form
\[
q_1(x) = \frac{\delta - f_0}{2(m+1)}(x - x_0)^{m+1} - \frac{\omega}{2(m+1)\delta}(x - x_0),
\]
\[
q_2(x) = -\frac{\delta + f_0}{2(m+1)}(x - x_0)^{m+1} - \frac{\omega}{2(m+1)\delta}(x - x_0),
\]
(116)
Finally, we find the polynomials \( f(x) \) and \( g(x) \) from condition (114) and Equation (4) recalling the fact that, for example, \( y(x) = q_1(x) \) is a solution of the latter. The uniqueness of independent nonautonomous Darboux first integral (112) follows from the uniqueness of the polynomials \( q_1(x) \), \( q_2(x) \) and the dominant behavior of the cofactors given by expression (103).
Remark. Suppose we are in assumptions of Lemma 9 with the exception of the condition $\delta / f_0 \not\in \mathbb{Q} \setminus \{0\}$. Then function (112) is still a nonautonomous Darboux first integral of the related Liénard differential system. However, there may exist other resonant Liénard differential systems from family $(B)$ with nonautonomous Darboux first integrals of the form (10).

Let us note that if $\delta = \pm mf_0 / (m + 2)$, then any system (111) has not only a nonautonomous Darboux first integral (112), but also an independent Darboux first integral (100), where $q_1(x)$ is given by the relation

$$q_1(x) = -\frac{f_0 x^{m+1}}{(m+1)(m+2)} - \frac{(m+2)\omega x}{2m(m+1)f_0}.$$  

(117)

Let $I_1(x, y)$ be Darboux first integral (100) and $I_2(x, y, t)$ be nonautonomous Darboux first integral (112). Eliminating $y$ from the relations $I_1(x, y) = C_1$ and $I_2(x, y, t) = C_2$, we can find the general solution of a system (111) under the condition $\delta = \pm mf_0 / (m + 2)$. Note that such a system is resonant near $x = \infty$. The general solution in the case $m = 2$ previously appeared in Ref. 12, see also Ref. 22.

Lemma 10. A Liénard differential system (1) satisfying the conditions $\deg g = 2 \deg f + 1$, $\deg f \neq 0$, and $\delta = 0$ possesses a nonautonomous Darboux first integral (10) if and only if the system is of the form

$$x_t = y, \quad y_t = -\left\{f_0(x - x_0)^m + \frac{(m+2)\omega}{m+1}\right\}y - \frac{(x - x_0)}{4(m+1)}f_0(x - x_0)^m + 2\omega,$$

(118)

where $x_0 \in \mathbb{C}$ and $\omega \in \mathbb{C} \setminus \{0\}$. A related nonautonomous Darboux first integral reads as

$$I(x, y, t) = \exp \frac{f_0(x - x_0)^{m+1}}{2(m+1)y + f_0(x - x_0)^{m+1} + 2\omega(x - x_0)} \left[y + \frac{f_0(x - x_0)^{m+1}}{2(m+1)} + \frac{\omega(x - x_0)}{m+1}\right] \exp(\omega t).$$

(119)

Proof. By direct computations, we verify that expression (119) is a time-dependent first integral of system (118).

Let us prove the converse statement. We suppose that a Liénard differential system from family $(B)$ satisfies the restriction $\delta = 0$ and possesses a nonautonomous Darboux first integral (10). Repeating the arguments used in the proof of Theorem 20, we represent such a first integral in the form

$$I(x, y, t) = [y - q(x)]^d \exp \left[\frac{u(x)}{y - q(x)}\right] \exp(\omega t), \quad d \in \mathbb{C}, \quad \omega \in \mathbb{C} \setminus \{0\}.$$  

(120)

In addition, we note that the related system possesses the invariant algebraic curve $y - q(x) = 0$ with the cofactor $\lambda(x, y) = -f(x) - q_x(x)$. If $u(x) \equiv 0$ in expression (120), then the related system either has no exponential invariants or they do not take part in the representation of the first integral. By Lemma 8, the cofactor $\varphi(x, y)$ of the exponential invariant $E(x, y) = \exp[u(x)/(y - q(x))]$ reads as $\varphi(x, y) = u_x(x)$. Condition (11) relating the cofactors $\lambda(x, y), \varphi(x, y)$ and the parameter $\omega$ takes the form

$$d[f(x) + q_x(x)] - u_x(x) - \omega = 0, \quad d \in \mathbb{C}.$$  

(121)
Let us begin with the case $d = 0$. We conclude from condition (121) that $u(x)$ is a first-degree polynomial. Using the remark to Lemma 8, we find the value of $m$. The result is $m = 0$. As it is mentioned at the beginning of this section, we do not consider Liénard differential systems with the restriction $m = 0$ ($\deg f = 0$).

We turn to the case $d \neq 0$. Without loss of generality, we set $d = 1$. Now let us suppose that the restriction $u(x) \equiv 0$ is valid. The related system may not have exponential invariants. Finding the dominant behavior of the cofactor $\lambda(x, y) = -f(x) - q(x, x)$ near the point $x = \infty$, we obtain

$$\lambda(x, y) = -\frac{f_0}{2} x^m + o(x^m), \quad x \to \infty. \quad (122)$$

Recalling the inequality $m > 0$, we see that condition (121) is not satisfied. Thus, we conclude that the exponential invariant exists. Furthermore, we eliminate from relations (89) and (121) the polynomial $f(x)$. As a result, we get the following expression:

$$q(x) = -u(x) - \omega \frac{u(x)}{u_x(x)}. \quad (123)$$

Consequently, the ratio $u(x)/u_x(x)$ is a polynomial. It is straightforward to see that this polynomial is of the first degree and can be represented as $\beta(x - x_0)$, where $\beta, x_0 \in \mathbb{C}$. Integrating the ordinary differential equation $\{\beta(x - x_0)\} u_x(x) = u(x)$, we obtain $u(x) = u_0(x - x_0)^{1/\beta}$, where $u_0 \in \mathbb{C}$ is a constant of integration. We recall that $u(x)$ is a polynomial of degree $m + 1$. As a result, we get $\beta = 1/(m + 1)$. Substituting the relation $u(x) = u_0(x - x_0)^{m+1}$ into expressions (123) and (121), we find the polynomials $q(x)$ and $f(x)$. In addition, we choose the parameterization $u_0 = f_0/(2(m + 1))$. We find the polynomial $g(x)$ recalling the fact that $y(x) = q(x)$ is the polynomial solution of the related equation (4).

Thus, we see that if the Liénard differential system in question has a nonautonomous Darboux first integral (10), then there exist the invariant algebraic curve $y - q(x) = 0$ and the exponential invariant $E(x, y) = \exp[\alpha u(x)/(y - q(x))]$, where $\alpha \in \mathbb{C}$, the polynomial $q(x)$ is given by expression (123), and the polynomial $u(x)$ is $u(x) = f_0(x - x_0)^{m+1}/(2m + 1)$.

Below we prove that Liénard differential systems (111) and (118) are Liouvillian integrable.

Let us study the Liouvillian integrability of Liénard differential systems from family $(B)$. We begin with some partial case characterized by Darboux integrating factors of a special form. Note that in Theorem 21 we use novel designations for polynomial solutions of Equation (4) related to Liénard differential systems. We need novel designations because polynomials $p_1(x)$ and $p_2(x)$ may have coinciding dominant terms. After considering this special case, we turn to nonresonant systems.

**Theorem 21.** A Liénard differential system (1) from family (B) possesses the Darboux integrating factor

$$M(x, y) = [y - p_1(x)]^{d_1} [y - p_2(x)]^{d_2}, \quad d_1, d_2 \in \mathbb{C} \setminus \{0\}, \quad (124)$$

where $p_1(x)$ and $p_2(x)$ are distinct polynomials, if and only if one of the following assertions is valid:
1. The system is of the form (99) and the polynomials \( p_1(x) \) and \( p_2(x) \) are linearly dependent: 
\[ p_2(x) = \frac{f_0 + \delta}{f_0 - \delta} p_1(x) \]. In this case, the parameters \( d_1 \) and \( d_2 \) can be chosen as \( d_1 = d_2 = -1 \) and the following relations \( p_1(x) = q_1(x) \), \( p_2(x) = q_2(x) \) are valid. In fact, there exists a family of Darboux integrating factors (124) that are products of the integrating factor \( M_0(x, y) = \left[ y - q_1(x) \right]^{-1} \left[ y - q_2(x) \right]^{-1} \) and the Darboux first integrals \( I^x(x, y) \), where the function \( I(x, y) \) is given by expression (100) and \( x \in \mathbb{C} \).

2. The system reads as
\[
\begin{align*}
 x_t &= y, \\
 y_t &= \left[ \beta(l + k)u^{k-1} + \frac{(2d_1+1)l+k}{k-l} u^{l-1} \right] u_x y \\
 &\quad - \left[ l^2 u^{2k-1} + \frac{(2d_1+1)l+k}{k-l} u^{k+l-1} + \frac{(ld_1+k)(d_1+1)^2}{(k-l)^2} u^{2l-1} \right] u_x,
\end{align*}
\]  

(125)

where \( k \) and \( l \) are relatively prime both nonunit natural numbers such that \( (m+1)/\max\{k,l\} \) is a natural number, \( u(x) \) is a polynomial of degree \( (m+1)/\max\{k,l\} \), and \( \beta \in \mathbb{C} \setminus \{0\} \). The polynomials \( p_1(x) \) and \( p_2(x) \) can be represented as
\[
\begin{align*}
p_1(x) &= \beta u^k(x) + \frac{(d_1+1)}{k-l} u^l(x), \\
p_2(x) &= \beta u^k(x) + \frac{(ld_1+k)}{k-l} u^l(x)
\end{align*}
\]  

(126)

and the parameter \( d_2 \) is given by the relation \( d_2 = -(d_1 + 1 + k/l) \).

**Proof.** Expression (124) gives a Darboux integrating factor of a Liénard differential system if and only if the system possesses the invariant algebraic curves \( y - p_j(x) = 0 \) such that the following condition \( \lambda_j(x, y) + d_2 \lambda_2(x, y) - f(x) = 0 \) is identically satisfied. It is straightforward to find the cofactor \( \lambda_j(x, y) \) of the invariant algebraic curve \( y - p_j(x) = 0 \). The result is \( \lambda_j(x, y) = -f(x) - \frac{p_j(x)}{p_j, x(x)} \), \( j = 1 \), \( 2 \). Thus, we arrive at the condition
\[
(d_1 + d_2 + 1) f(x) + d_1 p_{1, x}(x) + d_2 p_{2, x}(x) = 0.
\]  

(127)

If the following restriction \( d_2 = -1 - d_1 \) is valid, then integrating Equation (127) with respect to the polynomial \( p_1(x) \), we obtain \( p_1(x) = (d_1 + 1) p_2(x)/d_1 + \beta \), where \( \beta \in \mathbb{C} \) is a constant of integration. Recalling the fact that \( y = p_1(x) \) and \( y = p_2(x) \) are polynomial solutions of Equation (4), we find the polynomials \( f(x) \) and \( g(x) \). The polynomial \( f(x) \) can be represented in the form
\[
f(x) = -\frac{(2d_1+1) p_2(x)+d_1(d_1+1)\beta}{d_1(p_2(x)+d_1\beta)} p_{2, x}(x).
\]  

(128)

Analyzing this expression, we conclude that the function on the right-hand side is a polynomial whenever either \( p_2(x) \) is a constant or \( \beta = 0 \). The case \( \beta = 0 \) produces Darboux integrable family (99).

Let us consider the case \( d_2 \neq -1 - d_1 \). We find the polynomials \( f(x) \) and \( g(x) \) from condition (127) and Equation (4), where we set \( y(x) = p_1(x) \). Substituting the resulting expressions into Equation (4) and recalling the fact that \( y(x) = p_2(x) \) is a solution of the latter, we obtain the equation
\[
\{(d_2 + 1)p_1(x) + d_1 p_2(x)\}p_{1, x}(x) - \{d_2 p_1(x) + (d_1 + 1)p_2(x)\}p_{2, x}(x) = 0.
\]  

(129)
Introducing the polynomial \( v(x) \) according to the rule \( v(x) = p_2(x) - p_1(x) \), we substitute the relation \( p_2(x) = p_1(x) + v(x) \) into Equation (129). Integrating the result with respect to the polynomial \( p_1(x) \), we obtain

\[
\begin{align*}
  d_2 &= -2 - d_1 : \\
  p_1(x) &= -(d_1 + 1)v(x) \log v(x) + \beta v(x); \\
  d_2 &\neq -2 - d_1 : \\
  p_1(x) &= \beta v^{-d_2 - d_1 - 1}(x) - \frac{d_1 + 1}{d_2 + d_1 + 2} v(x),
\end{align*}
\]

where \( \beta \in \mathbb{C} \) is a constant of integration. Analyzing the first possibility, we need to set \( d_1 = -1 \). As a result, we get Darboux integrable family (99) of Liénard differential systems. The parameter \( \beta \) can be derived with the help of the dominant behavior of the polynomials \( p_1(x) \) and \( p_2(x) \), which now coincide with \( q_1(x) \) and \( q_2(x) \), respectively. Recalling the fact that the product of an integrating factor and a first integral is again an integrating factor, we obtain the family of integrating factors \( M_0(x,y)I^x(x,y) \), where \( M_0(x,y) = [y - q_1(x)]^{-1}[y - q_2(x)]^{-1}, x \in \mathbb{C} \), and the Darboux first integral \( I(x,y) \) is given by expression (100).

Now we turn to the case \( d_2 \neq -2 - d_1 \). If \( \beta = 0 \), then we get the equality \( p_2(x) = -(d_2 + 1)p_1(x)/(d_1 + 1) \). In addition, we obtain the following representations of the polynomials \( f(x) \) and \( g(x) \):

\[
\begin{align*}
  f(x) &= (d_2 - d_1)p_1,\chi(x)/(d_1 + 1) \\
  g(x) &= -(d_1 + 1)p_1(x)p_1,\chi(x)/(d_1 + 1).
\end{align*}
\]

Considering these expressions, we again arrive at Darboux integrable systems (99).

Thus, it is without loss of generality to set \( \beta \neq 0 \). Recalling the restriction \( d_2 \neq -1 - d_1 \), we introduce relatively prime both nonunit natural numbers \( k \) and \( l \) satisfying the condition \( d_2 + d_1 + 1 = -k/l \). Analyzing expression (130), we conclude that there exists a polynomial \( u(x) \) such that the following relation \( v(x) = u'(x) \) holds. In this way, we express the polynomials \( p_1(x) \) and \( p_2(x) \) via the polynomial \( u(x) \). The result is given in expression (126). By construction, the degree of the polynomial \( u(x) \) equals \( (m + 1)/\max\{k, l\} \).

Using integrating factor (124), we find the following expression of a Liouvillian first integral:

\[
I(x,y) = \frac{p_2(x)B\left(\frac{y-p_1(x)}{p_2(x)-p_1(x)};1+d_1,-d_1-1\right)}{\left\{p_2(x)-p_1(x)\right\}^{\frac{k}{T}}} - \frac{B\left(\frac{y-p_1(x)}{p_2(x)-p_1(x)};1+d_1,1-d_1-\frac{k}{l}\right)}{\left\{p_2(x)-p_1(x)\right\}^{\frac{k}{T}-1}}
\]

of systems (125). The polynomials \( p_1(x) \) and \( p_2(x) \) are given by relation (126). Symbol \( B(s;\alpha,\delta) \) denotes the incomplete beta function

\[
B(s;\alpha,\delta) = \int_0^s z^{\alpha-1}(1-z)^{\delta-1}dz.
\]

The family of systems (125) can be transformed to the following simple form:

\[
\begin{align*}
  s_\tau &= z, \\
  z_\tau &= \left[\beta(l + k)s^{k-1} + \frac{(2d_1 + 1)(l + k)}{k-l}s^{l-1}\right]z \\
  &\quad - \left[l\beta^2s^{2k-1} + \frac{(2d_1 + 1)(l + k)}{k-l}s^{k+l-1} + \frac{(l + k)(d_1 + 1)^2}{(k-l)^2}s^{2l-1}\right]
\end{align*}
\]

via the generalized Sundman transformation \( s(\tau) = u(x), z(\tau) = y, d\tau = u,\chi(x)dt \). Substituting \( u(x) = s, y = z \) into (131) and (126), we find a Liouvillian first integral for systems (133).

The careful examination of expression (126) shows that systems (125) are resonant near infinity whenever the following inequality \( k > l \) is valid. Indeed, two distinct polynomial solutions
of Equation (4) have the coinciding dominant behavior near the point \( x = \infty \) only in a resonant case. Let us find the necessary and sufficient conditions of the Liouvillian integrability in the nonresonant case.

**Theorem 22.** A Liénard differential system (1) satisfying the conditions \( \deg g = 2 \deg f + 1 \) and \( \delta / f_0 \not\in \mathbb{Q} \) is Liouvillian integrable if and only if the system is either Darboux integrable and reads as (99) or takes the form (125), where \( k < l \) and the following normalization

\[
 d_1 = \frac{(k - l)f_0}{2l\delta} - \frac{l + k}{2l}, \quad u_0 = \left( -\frac{\delta}{m + 1} \right)^{\frac{1}{7}}
\]

is introduced. By \( u_0 \), we denote the highest-degree coefficient of the polynomial \( u(x) \). In the case of systems (125), the Darboux integrating factor is the following

\[
 M(x, y) = \{y - q_1(x)\}^{\frac{(k-1)f_0}{2l\delta} - \frac{l+k}{2l}} \{y - q_2(x)\}^{\frac{(l-k)f_0}{2l\delta} - \frac{l+k}{2l}}
\]

with the polynomials \( q_j(x) \equiv p_j(x), j = 1, 2 \) given by expression (126).

**Proof.** By direct computations, we verify that systems (99) and (125) are Liouvillian integrable provided that all other conditions of the theorem are satisfied. This observation proves the sufficiency of these conditions.

Let us prove their necessity. It follows from Lemmas 3 and 8 that exponential invariants cannot arise in a Darboux integrating factor. Consequently, a Darboux integrating factor is constructed from generating polynomials of invariant algebraic curves. It is straightforward to see that if there are no invariant algebraic curves, then Darboux integrating factors do not exist. In view of Theorem 11, we need to consider three distinct cases.

**Case 1.** Let us suppose that a Liouvillian integrable Liénard differential system (1) satisfying the conditions \( \deg g = 2 \deg f + 1 \) and \( \delta / f_0 \not\in \mathbb{Q} \) has only one irreducible invariant algebraic curve with a generating polynomial of the first degree with respect to \( y \). This curve reads as \( y - q_k(x) = 0, k = 1 \) or \( k = 2 \), and has the cofactor \( \lambda(x, y) = -f(x) - q_k, x(x) \). A Darboux integrating factor can be represented in the form

\[
 M(x, y) = [y - q_k(x)]^{dk}, \quad d_k \in \mathbb{C} \setminus \{0\}.
\]

This integrating factor exists if and only if the following condition:

\[
 d_k \{f(x) + q_k, x(x)\} + f(x) = 0
\]

is identically valid. Balancing the terms at \( x^m \) in this relation, we obtain

\[
 k = 1 : \quad d_1 = -\frac{2f_0}{\delta + f_0}, \quad k = 2 : \quad d_2 = \frac{2f_0}{\delta - f_0}.
\]
Expressing $f(x)$ and $g(x)$ from relation (137) and Equation (4) with $y(x) = q_k(x)$, we see that our Liénard differential system is of the form (99) and possesses two distinct irreducible invariant algebraic curves. It is a contradiction.

Case 2. Let us suppose that a Liouvillian integrable Liénard differential system (1) satisfying the conditions $\deg g = 2 \deg f + 1$ and $\delta / f_0 \notin \mathbb{Q}$ has an irreducible invariant algebraic curve with a generating polynomial of the second degree with respect to $y$. This curve is given by the expression $\{[y - y_{\infty}^{(1)}(x)][y - y_{\infty}^{(2)}(x)]\} = 0$. Its cofactor reads as $\lambda(x, y) = -2f(x) - q_{1,x}(x) - q_{2,x}(x)$. A Darboux integrating factor can be represented in the form

$$
M(x, y) = F^d(x, y), \quad F(x, y) = \{[y - y_{\infty}^{(1)}(x)][y - y_{\infty}^{(2)}(x)]\} = 0, \quad d \in \mathbb{C} \setminus \{0\}
$$

and exists if and only if the following condition:

$$
d\{2f(x) + q_{1,x}(x) + q_{2,x}(x)\} + f(x) = 0
$$

is identically satisfied. Considering the coefficients of $x^m$ in this condition yields the value of $d$: $d = -1$. Furthermore, we represent the polynomial $F(x, y)$ in the form $F(x, y) = y^2 + u(x)y + w(x)$, where $u(x), w(x) \in \mathbb{C}[x]$. Substituting expression $M(x, y) = F^{-1}(x, y)$ into the partial differential equation

$$
yM_x - [f(x)y + g(x)]M_y - f(x)M = 0,
$$
we get rid of the denominator. Setting to zero the coefficients of different powers of $y$, we get $w(x) = 2\beta u(x)^2$, where $\beta \in \mathbb{C} \setminus \{0\}$. This equality contradicts irreducibility of the polynomial $F(x, y)$.

Case 3. Now we assume that a Liouvillian integrable Liénard differential system (1) satisfying the conditions $\deg g = 2 \deg f + 1$ and $\delta / f_0 \notin \mathbb{Q}$ has two distinct irreducible invariant algebraic curves $y - q_1(x) = 0$ and $y - q_2(x) = 0$. Their cofactors are the following $\lambda_1(x, y) = -f(x) - q_{1,x}(x)$ and $\lambda_2(x, y) = -f(x) - q_{2,x}(x)$, respectively. Condition (13) enabling the existence of a Darboux integrating factor

$$
M(x, y) = \{y - q_1(x)\}^{d_1}\{y - q_2(x)\}^{d_2}, \quad d_1, d_2 \in \mathbb{C}, \quad |d_1| + |d_2| > 0
$$

is of the form

$$
d_1\{f(x) + q_{1,x}(x)\} + d_2\{f(x) + q_{2,x}(x)\} + f(x) = 0.
$$

All the Liénard differential systems from family $(B)$ with integrating factor of the form (142) have been identified in Theorem 21. We need to extract nonresonant systems from those given by expression (125). Thus, the polynomials $q_1(x)$ and $q_2(x)$ necessarily have distinct dominant terms. This fact yields the inequality $k < l$. Finally, we need to introduce the normalization adopted at the beginning of this section. Using relations (87) and (88), we obtain expression (134).
Corollary. Liénard differential systems \((111)\) with a nonautonomous Darboux first integral \((112)\) are Liouvillian integrable. These systems have the Darboux integrating factor

\[
M(x, y) = \begin{vmatrix}
\frac{\omega + f_0(x-x_0)^{m+1}}{2(m+1)} + \frac{\omega}{2(m+1)\delta}(x-x_0) & \frac{m f_0 - (m+2)\delta}{2(m+1)\delta} \\
\frac{\omega + f_0(x-x_0)^{m+1}}{2(m+1)} + \frac{\omega}{2(m+1)\delta}(x-x_0) & \frac{m f_0 + (m+2)\delta}{2(m+1)\delta}
\end{vmatrix}
\] (144)

Proof. We establish the validity of the statement substituting relations

\[
k = 1, l = m + 1, u(x) = u_0(x - x_0), \ beta = -\frac{\omega}{2(m+1)\delta u_0}, u_0 = \left\{ -\frac{\delta}{m+1} \right\}^{\frac{1}{m+1}}
\] (145)

into expressions \((125)\) and \((135)\). In addition, we recall that the parameter \(d_1\) reads as \((134)\). □

Theorem 23. A Liénard differential system \((1)\) satisfying the conditions \(\deg g = 2\deg f + 1\) and \(\delta = 0\) is Liouvillian integrable if and only if the system has the irreducible invariant algebraic curve \(y - q(x) = 0\) and an exponential invariant \(E(x, y) = \exp[u(x)/(y - q(x))]\) such that one of the following assertions is valid:

1. The system is Darboux integrable and takes the form \((107)\). A related Darboux integrating factor reads as

\[
M(x, y) = \frac{1}{(y - q(x))^2}.
\] (146)

The polynomial \(u(x)\) arising in the exponential invariant is \(u(x) = \alpha q(z), \alpha \in \mathbb{C} \setminus \{0\}\).

2. The system is of the form

\[
x_t = y, \quad y_t = -\left[\frac{2l^2}{l-k} v^{l-1} - (l+k)\beta v^{k-1}\right] v_x y + \left[\frac{2l^2}{l-k} v^{l+k-1} \right] v_x,
\] (147)

where \(\beta \in \mathbb{C} \setminus \{0\}, v(x)\) is a nonconstant polynomial, \(k\) and \(l\) are relatively prime natural numbers satisfying the inequality \(k < l\). The associated Darboux integrating factor reads as

\[
M(x, y) = [y - q(x)]^{-\frac{l+k}{l-k}} \exp \left[\frac{v'(x)}{y - q(x)}\right], \quad q(x) = -\frac{l}{l-k} v^l + \beta v^k.
\] (148)

In addition, the following relation \(m + 1 = l\deg v\) is valid.

Proof. It is straightforward to verify that expressions \((146)\) and \((148)\) are Darboux integrating factors of Liénard differential systems \((1)\) satisfying the restrictions \(\deg g = 2\deg f + 1\) and \(\delta = 0\) whenever all other conditions of the theorem are satisfied.

Let us prove the converse statement. Suppose we consider a Liouvillian integrable Liénard differential system \((1)\) such that \(\deg g = 2\deg f + 1\) and \(\delta = 0\). By Theorems 2, 11 and Lemmas 3, 8, the system has the irreducible invariant algebraic curve \(y - q(x) = 0\) and a Darboux integrating
factor that can be represented in the form

\[ M(x, y) = (y - q(x))^d \exp \left[ \frac{u(x)}{y - q(x)} \right], \quad d \in \mathbb{C}, \quad (149) \]

where we suppose that \( u(x) \equiv 0 \) whenever the exponential invariant \( E(x, y) = \exp[u(x)/(y - q(x))] \) related to the invariant algebraic curve \( y - q(x) = 0 \) either does not exist or is not involved into an explicit expression of the integrating factor. Condition (13) with \( \omega = 0 \) now takes the form

\[ d[f(x) + q_x(x)] - u_x(x) + f(x) = 0. \quad (150) \]

Furthermore, we consider several distinct cases separately.

**Case 1.** Let us begin with the case \( u(x) \equiv 0 \). Substituting the asymptotic relations \( f(x) = f_0 x^m + o(x^m) \) and \( q(x) = -f_0 x^{m+1} / (2m + 1) + o(x^{m+1}), \ x \to \infty \) into condition (150) and setting to zero the coefficient of \( x^m \), we obtain the equalities \( d = -2 \) and \( f(x) = -2q_x(x) \). Recalling the fact that \( y = q(x) \) is the polynomial solution of Equation (4), we find the polynomial \( g(x) \) as given in (107). Using Theorem 20, we conclude that the related Liénard differential system possesses a Darboux first integral (108) and the exponential invariants \( E(x, y) = \exp\left[\alpha q(x)/(y - q(x))\right], \alpha \in \mathbb{C} \setminus \{0\} \). Note that integrating the differential form

\[ \frac{y dy + (qq_x - 2q_x y)dx}{\{y - q(x)\}^2} \quad (151) \]

yields a first integral \( \log I(x, y) \) with the function \( I(x, y) \) given by expression (108).

**Case 2.** Now let us suppose that the polynomial \( u(x) \) is not identically zero. We see that the system under consideration has the exponential invariant \( E(x, y) = \exp[u(x)/(y - q(x))] \) with the polynomial \( u(x) \) satisfying Equation (89). Expressing \( f(x) \) from the latter equation and substituting the result into condition (150) yields the relation

\[ (d + 1)q u_x + q_x u + uu_x = 0. \quad (152) \]

This relation viewed as an ordinary differential equation with respect to the polynomial \( q(x) \) can be integrated. Thus, we find the expressions

\[ d \neq -2 : \quad q(x) = -\frac{1}{d+2} u + \beta u^{-(d+1)}; \quad d = -2 : \quad q(x) = (\beta - \log u)u, \quad (153) \]

where \( \beta \in \mathbb{C} \) is a constant of integration. If \( d = -2 \), then \( q(x) \) is not a polynomial. Furthermore, we set \( d \neq -2 \). The case \( \beta = 0 \) again leads to a Darboux integrable family of Liénard differential systems given in Theorem 20. Thus, we suppose that the constant \( \beta \) is nonzero. Recalling the fact that \( q(x) \) and \( u(x) \) are polynomials with the dominant behavior \( q(x) = -f_0 x^{m+1} / (2m + 1) + o(x^{m+1}) \) and \( u(x) = u_0 x^{m+1} + o(x^{m+1}), \ u_0 \in \mathbb{C} \setminus \{0\} \) near the point \( x = \infty \), we find two possibilities

\[ d = -1 : \quad u(x) = \beta - q(x); \quad d = -\frac{i+1}{l} : \quad u(x) = v^l(x). \quad (154) \]
In these expressions, \( l \) and \( k \) are relatively prime natural numbers satisfying the restriction \( k < l \) and \( v(x) \) is a nonconstant polynomial. Analyzing the possibility \( d = -1 \), we substitute the equality \( u(x) = \beta - q(x) \) into relation (89) and find the polynomial \( f(x) \). The result is

\[
f(x) = \frac{(2q(x) - \beta)q_x(x)}{\beta - q(x)}.
\] (155)

Let \( x_0 \) be a zero of the polynomial \( \beta - q(x) \). Considering the behavior near \( x_0 \) of the rational function on the right-hand side of expression (155), we see that \( f(x) \) is not a polynomial whenever \( \beta \neq 0 \).

Finally, we suppose that the following relations \( d = -(l + k)/l \) and \( u(x) = v^l(x) \) are valid. We use expressions (153), (150), and (4) to find the polynomials \( f(x) \), \( g(x) \), and \( q(x) \) as given in relations (147) and (148). In addition, we verify that Equation (89) is identically satisfied. The proof is completed.

**Corollary 1.** Liénard differential systems (118) possessing a nonautonomous Darboux first integral (119) are Liouvillian integrable with the Darboux integrating factor

\[
M(x, y) = \exp \left[ mf_0(x-x_0)^{m+1}(m+1)[2(m+1)y + f_0(x-x_0)^{m+1} + 2\omega(x-x_0)] \right] \\
\times \left[ y + \frac{f_0(x-x_0)^{m+1}}{2(m+1)} + \frac{\omega(x-x_0)}{m+1} \right].
\] (156)

**Proof.** We prove the validity of the statement substituting relations

\[
k = 1, \ l = m + 1, \ v(x) = v_0(x-x_0), \ \beta = -\frac{\omega}{(m+1)v_0}, \ v_0 = \left\{ \frac{mf_0}{2(m+1)^2} \right\}^{\frac{1}{m+1}}
\] (157)

into expressions (147) and (148).

**Corollary 2.** If the following inequality \( k > l \) holds, then systems (147) are also Liouvillian integrable Liénard differential systems from family (B). However, these systems are resonant near infinity. The related Darboux integrating factor again is given by expression (148).

A Liouvillian first integral produced by integrating factor (148) reads as

\[
I(x, y) = v^{l-k}(x)y \left( -\frac{l-k}{l}, \frac{v^l(x)}{q(x)-y} \right) - \frac{q(x)}{v^k(x)}y \left( \frac{k}{l}, \frac{v^l(x)}{q(x)-y} \right),
\] (158)

where the polynomial \( q(x) \) is given in expression (147) and \( \gamma(\delta, s) \) is the lower incomplete Gamma function

\[
\gamma(\delta, s) = \int_0^s t^{\delta-1} \exp(-t)dt.
\] (159)
Note that we need to consider the analytic continuation of this integral for complex or real non-positive values of $s$. If $k = 1$ and $l = 2$, then we obtain another representation of a Liouvillian first integral

$$ I(x, y) = 2\sqrt{\beta}v(x) - 2v^2(x) - y \exp \left[ \frac{v^2(x)}{y + 2v^2(x) - \beta v(x)} \right] - \sqrt{\pi \beta} \text{erfc} \left[ \frac{v(x)}{\sqrt{\beta v(x) - 2v^2(x) - y}} \right], $$

(160)

where erfc(s) is the complementary error function

$$ \text{erfc}(s) = \frac{2}{\sqrt{\pi}} \int_{s}^{\infty} \exp(-t^2) dt. $$

(161)

The family of systems (147) can be transformed to the following simple form:

$$ s_\tau = z, z_\tau = -\left[ \frac{2l^2}{l - k} s^{l-1} - (l + k)\beta s^{k-1} \right] z + \frac{2l^2 \beta}{l - k} s^{l+k-1} - \frac{l^3}{(l - k)^2} s^{2l-1} - l\beta^2 s^{2k-1} $$

(162)

via the generalized Sundman transformation $s(\tau) = v(x), z(\tau) = y, d\tau = u_\nu(x) dt$. Substituting $u(x) = s, y = z$ into (158), we find a Liouvillian first integral for systems (162).

It seems that Liouvillian integrable families of Liénard differential systems given in Theorems 21, 22, and 23 are new with the exception of systems (118). The latter are presented by Stachowiak.28

Now let us investigate the existence of Jacobi last multipliers with a time-dependent exponential factor.

**Lemma 11.** A Liénard differential system (1) satisfying the conditions $\deg g = 2 \deg f + 1$ and $\delta / f_0 \not\in \mathbb{Q}$ has a nonautonomous Darboux–Jacobi last multiplier of the form (12) if and only if one the following assertions is valid:

1. The system under consideration possesses one irreducible invariant algebraic curve $y - q_k(x) = 0$, where $k = 1$ or $k = 2$, such that the polynomials $f(x)$ and $q_k(x)$ identically satisfy the condition

$$ k = 1 : \ (f_0 - \delta) f(x) + 2f_0 q_1,x(x) + (f_0 + \delta) \omega = 0, \omega \in \mathbb{C} \setminus \{0\}, $$

$$ k = 2 : \ (f_0 + \delta) f(x) + 2f_0 q_2,x(x) + (f_0 - \delta) \omega = 0, \omega \in \mathbb{C} \setminus \{0\}. $$

(163)

A related Darboux–Jacobi last multiplier reads as

$$ k = 1 : \ M(x, y, t) = [y - q_1(x)]^{\frac{2f_0}{\delta + f_0}} \exp(\omega t), $$

$$ k = 2 : \ M(x, y, t) = [y - q_2(x)]^{\frac{2f_0}{\delta - f_0}} \exp(\omega t). $$

(164)

2. The system under consideration possesses one irreducible invariant algebraic curve $F(x, y) = 0$ with $F(x, y) = \{[y - y_1^{(1)}(x)][y - y_2^{(2)}(x)]\}^+$ such that the polynomials $f(x), q_1(x) = \{y_1^{(1)}(x)\}^+$,
and \(q_2(x) = \{y^{(2)}_\infty(x)\}_+\) identically satisfy the condition

\[
f(x) + q_{1,x}(x) + q_{2,x}(x) + \omega = 0, \quad \omega \in \mathbb{C} \setminus \{0\}.
\] (165)

A related Darboux–Jacobi last multiplier reads as

\[
M(x, y, t) = \frac{\exp(\omega t)}{F(x,y)}, \quad F(x,y) = \left\{ \left[ y - y^{(1)}_\infty(x) \right] \left[ y - y^{(2)}_\infty(x) \right] \right\}_+.
\] (166)

3. The system under consideration possesses two distinct irreducible invariant algebraic curves \(y - q_1(x) = 0\) and \(y - q_2(x) = 0\) such that the polynomials \(f(x), q_1(x),\) and \(q_2(x)\) identically satisfy the condition

\[
[(2d_2 + 1)\delta - f_0]f(x) + [(\delta - f_0)d_2 - 2f_0]q_{1,x}(x) + (\delta + f_0)d_2q_{2,x}(x)
\]

\[
-(\delta + f_0)\omega = 0, \quad d_2 \in \mathbb{C}, \quad \omega \in \mathbb{C} \setminus \{0\}.
\] (167)

A related Darboux–Jacobi last multiplier reads as

\[
M(x, y, t) = [y - q_1(x)]^\frac{(\delta - f_0)d_2 - 2f_0}{\delta + f_0} [y - q_2(x)]^{d_2} \exp(\omega t).
\] (168)

This lemma is proved similarly to Theorem 22.

**Lemma 12.** A Liénard differential system (1) satisfying the conditions \(\deg g = 2 \deg f + 1\) and \(\delta = 0\) has a nonautonomous Darboux–Jacobi last multiplier of the form (12) if and only if one of the following assertions is valid:

1. The system under consideration possesses the irreducible invariant algebraic curve \(y - q(x) = 0\) such that the polynomials \(f(x)\) and \(g(x)\) can be represented as

\[
f(x) = -2q_x(x) - \omega, \quad g(x) = q(x)(q_x(x) + \omega), \quad \omega \in \mathbb{C} \setminus \{0\}.
\] (169)

A related Darboux–Jacobi last multiplier reads as

\[
M(x, y, t) = \frac{\exp(\omega t)}{|y - q(x)|^2}.
\] (170)

2. The system under consideration possesses the irreducible invariant algebraic curve \(y - q(x) = 0\) and the exponential invariant \(E(x, y) = \exp[u(x)/(y - q(x))]\) such that the polynomials \(f(x), q(x),\) and \(u(x)\) identically satisfy the condition

\[
(d + 1)f(x) + dq_x(x) = u_x(x) + \omega, \quad d \in \mathbb{C}, \quad \omega \in \mathbb{C} \setminus \{0\}.
\] (171)

A related Darboux–Jacobi last multiplier reads as

\[
M(x, y, t) = [y - q(x)]^d \exp \left[ \frac{u(x)}{y - q(x)} \right] \exp(\omega t).
\] (172)
The proof of this lemma is analogous to the proof of Theorem 23.

Concluding this section let us note that autonomous and nonautonomous Darboux first integrals of nonresonant Liénard differential systems (1) satisfying the conditions \( \deg f = 1 \), \( \deg g = 3 \) and \( \deg f = 2 \), \( \deg g = 5 \) are classified in Refs. 19, 22.

8 | INTEGRABILITY OF LIÉNARD DIFFERENTIAL SYSTEMS FROM FAMILY \((C)\)

We start investigating integrability properties of Liénard differential systems from family \((C)\) by proving the absence of exponential invariants related to invariant algebraic curves. We use designations of Theorem 12. In particular, by \( h^{(1)}(x) \) and \( h^{(2)}(x) \) we denote the initial parts of the Puiseux series \( y_\infty^{(1)}(x) \) and \( y_\infty^{(2)}(x) \), accordingly. These initial parts involve monomials with exponents exceeding \(-(n + 1)/2\). Recall that we use the designations \( \deg g = n \) and \( \deg f = m \). Thus, the following inequality \( n > 2m + 1 \) is valid.

**Lemma 13.** Suppose a Liénard differential system (1) from family \((C)\) is not integrable with a rational first integral. Then this system does not have exponential invariants of the form \( E(x, y) = \exp\{h(x, y)/r(x, y)\} \), where \( h(x, y) \in \mathbb{C}[x, y] \) and \( r(x, y) \in \mathbb{C}[x, y] \setminus \mathbb{C} \) are relatively prime polynomials.

**Proof.** We use the local theory presented in Section 3. If a Liénard differential system (1) has an exponential invariant \( E(x, y) = \exp\{h(x, y)/r(x, y)\} \) with \( r(x, y) \in \mathbb{C}[x, y] \setminus \mathbb{C} \), then \( r(x, y) \in \mathbb{C}[x, y] \setminus \mathbb{C}[x] \) and it is without loss of generality to assume that the degree of the polynomial \( h(x, y) \) with respect to \( y \) is less than the degree of the polynomial \( r(x, y) \) with respect to \( y \). There exists a finite number of local elementary exponential invariants

\[
E_j(x, y) = \exp\left[\frac{u_j(x,y)}{(y-Y_{j,\infty}(x))^n_j}\right], \quad u_j(x, y) \in \mathbb{C}_\infty\{x\}[y], \quad Y_{j,\infty}(x) \in \mathbb{C}_\infty\{x\}, \quad n_j \in \mathbb{N}, \quad j = 1, \ldots, K, \quad K \in \mathbb{N}
\]

(173)

such that the exponential invariant \( E(x, y) \) equals the product \( E_1(x, y) \times \cdots \times E_K(x, y) \). It follows from Theorem 7 and Lemma 2 that each series \( Y_{j,\infty}(x) \) satisfies Equation (4). In fact, the series \( Y_{j,\infty}(x) \) coincides with one of the series \( y^{(1,2)}_j(x) \) presented in Theorem 12. Let us denote the cofactor of the local elementary exponential invariant \( E_j(x, y) \) by \( \varphi_j(x, y) \in \mathbb{C}_\infty\{x\}[y] \). We see that the following expression \( \varphi_1(x, y) + \cdots + \varphi_K(x, y) \) equals the cofactor \( \varphi(x, y) \) of the invariant \( E(x, y) \). Recall that the cofactor \( \varphi(x, y) \) is an element of the ring \( \mathbb{C}[x, y] \). Furthermore, we represent the numerator in (173) as

\[
u_j(x, y) = \sum_{k=0}^{n_j-1} \nu_{(j,k)}(x)z^k, \quad z = y - Y_{j,\infty}(x), \quad \nu_{(j,k)}(x) \in \mathbb{C}_\infty\{x\}
\]

(174)

where it is without loss of generality to suppose that \( \nu_{j,0}(x) \neq O_\infty \). Otherwise, we need to reduce \( n_j \) by one. By \( O_\infty \), we denote the zero element of the field \( \mathbb{C}_\infty\{x\} \). Substituting the explicit representation of \( E_j(x, y) \) into the partial differential equation \( \mathcal{L}E_j(x, y) = \varphi_j(x, y)E_j(x, y) \), we set to zero the coefficients of different powers of \( z \). As a result, we see that the cofactor \( \varphi_j(x, y) \) does not
depend on $y$: $\varphi_j(x, y) = u_{(j, n_j - 1), x}$. In addition, we find the system of linear ordinary differential equations

$$
\begin{align*}
Y_{j, \infty}v_{(j, 0), x} - n_j \lambda_j v_{(j, 0)} &= 0, \\
Y_{j, \infty}v_{(j, k), x} - (n_j - k) \lambda_j v_{(j, k)} + u_{(j, k-1), x} &= 0, \quad 1 \leq k < n_j - 1.
\end{align*}
$$

(175)

In this expression, $\lambda_j(x, y) \in C_\infty[x][y]$ is the cofactor of the local elementary invariant $F_j(x, y) = y - Y_{j, \infty}(x)$. Using Theorem 8, we obtain the cofactor $\lambda_j(x, y)$. The result is

$$
\lambda_j(x, y) = -f(x) - \{Y_{j, \infty}(x)\}_x.
$$

(176)

Solving system (175), we find the dominant terms $v_{(j, k)}(x) = C_{(j, k)}x^{-(n_j+k)(n+1)/2}$, where $C_{(j, k)} \in \mathbb{C} \setminus \{0\}$. Notice that we need to consider only partial solutions of linear inhomogeneous equations provided that $0 < k < n_j - 1$. Indeed, solutions of linear homogeneous equations in (175) are included into exponential invariants with lower values of $n_j$. Hence, the cofactor $\varphi_j(x, y) = u_{(j, n_j - 1), x}$ does not have monomials with nonnegative exponents. Consequently, the polynomial $\varphi(x, y) = \varphi_1(x, y) + \cdots + \varphi_k(x, y)$ should be identically zero. We conclude that the argument $h(x, y)/r(x, y)$ of the exponential invariant $E(x, y) = \exp\{h(x, y)/r(x, y)\}$ is a rational first integral of the differential system in question. It is a contradiction.

\[\square\]

**Remark.** It will be shown below that Liénard differential systems (1) satisfying the condition $\deg g > 2 \deg f + 1$ do not have rational first integrals. Thus, this condition can be removed from the statement of Lemma 13.

Now our aim is to prove that Liénard differential systems (1) do not have Darboux first integrals provided that $\deg g > 2 \deg f + 1$.

\[\square\]

**Theorem 24.** Liénard differential systems (1) from family $(C)$ are not Darboux integrable.

**Proof.** Let us suppose that a Liénard differential system (1) with $\deg g > 2 \deg f + 1$ has a Darboux first integral. In view of Theorem 1, the system possesses a rational integrating factor. Consequently, there exists $K \in \mathbb{N}$ pairwise distinct irreducible invariant algebraic curves $F_1(x, y) = 0, \ldots, F_K(x, y) = 0$ and $K$ nonzero integer numbers $d_1, \ldots, d_K$ such that the following condition:

$$
\sum_{j=1}^{K} d_j \lambda_j(x, y) = f(x), \quad d_1, \ldots, d_K \in \mathbb{Z} \setminus \{0\}
$$

(177)

is valid. In this expression, $\lambda_j(x, y)$ is the cofactor of the invariant algebraic curve $F_j(x, y) = 0$. Suppose the family of Puiseux series $y^{(l)}_\infty(x)$ arises $N_{l,j}$ times in the factorization of the polynomial $F_j(x, y)$ in the ring $C_\infty[x][y]$. Here, $l = 1, 2$ and we use the designations of Theorem 12. The cofactor $\lambda_j(x, y)$ reads as

$$
\begin{align*}
\lambda_j(x, y) &= -(N_{1,j} + N_{2,j})f(x) - \left\{N_{1,j}h^{(1)}_x(x) + N_{2,j}h^{(2)}_x(x)\right\}, \\
N_{1,j}, N_{2,j} &\in \mathbb{N}_0, \quad N_{1,j} + N_{2,j} > 0.
\end{align*}
$$

(178)
where $h^{(1)}(x)$ is the initial part of Puiseux series $y_{\infty}^{(1)}(x)$ introduced in Theorem 12. Substituting expression (178) into relation (177) yields

$$
\sum_{j=1}^{K} d_j N_{1,j} \left( f(x) + \left\{ h_x^{(1)}(x) \right\}_+ \right) + \sum_{j=1}^{K} d_j N_{2,j} \left( f(x) + \left\{ h_x^{(2)}(x) \right\}_+ \right) = -f(x).
$$

(179)

Let us suppose that $n$ is odd. The dominant behavior of the truncated series $h^{(1)}(x)$ and $h^{(2)}(x)$ is $b_0 x^{(n+1)/2}$ and $-b_0 x^{(n+1)/2}$, respectively. Setting to zero the coefficient of $x^{(n-1)/2}$ in expression (179), we obtain

$$
\sum_{j=1}^{K} d_j N_{1,j} = \sum_{j=1}^{K} d_j N_{2,j}.
$$

(180)

If $n$ is even, then by Theorem 12 we get $N_{1,j} = N_{2,j}$. Consequently, relation (180) is identically satisfied. As a result, condition (179) takes the form

$$
B \left( 2f(x) + \left\{ h_x^{(1)}(x) + h_x^{(2)}(x) \right\}_+ \right) = -f(x), \quad B = \sum_{j=1}^{K} d_j N_{1,j}.
$$

(181)

Obviously, $B$ is a nonzero integer number. Now let us consider the equation

$$
yy_x + \varepsilon f(x)y + g(x) = 0.
$$

(182)

Puiseux series near the point $x = \infty$ that satisfy this equation coincide with the Puiseux series $y_{\infty}^{(1,2)}(x)$ presented in Theorem 12 provided that $\varepsilon = 1$. Analogously to the case $\varepsilon = 1$, we find two families of Puiseux series near the point $x = \infty$ solving Equation (182). We denote these families as $Y_{\infty}^{(1)}(x)$ and $Y_{\infty}^{(2)}(x)$. It is straightforward to see that these series can be represented as

$$
Y_{\infty}^{(1,2)}(x) = \sum_{k=0}^{\infty} \varepsilon^k v_{k}^{(1,2)}(x),
$$

(183)

where the coefficients $v_{k}^{(1,2)}(x)$ are Puiseux series near the point $x = \infty$. Substituting expression (183) into Equation (182) and setting to zero the coefficients of different powers of $\varepsilon$, we find ordinary differential equations for the series $v_{k}^{(1,2)}(x)$. Two first equations take the form

$$
v_0 v_{0,x} + g(x) = 0, \quad v_0 v_{1,x} + v_{0,x} v_1 + f(x)v_0 = 0,
$$

(184)

where the upper index is omitted. Thus, we get

$$
v_0^{(2)}(x) = -v_0^{(1)}(x), \quad v_1^{(1,2)}(x) = -\frac{2f_0}{2m + n + 3} x^{m+1} + o(x^{m+1}), \quad x \to \infty.
$$

(185)

Analyzing other ordinary differential equations for the series $v_{k}^{(1,2)}(x)$ with $k \geq 2$, we find $v_{k}^{(1,2)}(x) = o(x^{m+1}), \quad x \to \infty, \quad k \geq 2$. We recall that the Puiseux series $y_{\infty}^{(1,2)}(x)$ with the same upper index have coinciding initial parts involving monomials with exponents exceeding $-(n + 1)/2$. 
These initial parts are given by $h^{(1)}(x)$ and $h^{(2)}(x)$. Substituting our results into condition (181) and considering the coefficients of the leading term $x^m$, we come to the equation

$$B\left(2f_0 - \frac{4(m+1)f_0}{2m+n+3}\right) = -f_0.$$  

Solving this equation, we find the expression

$$B = -\frac{m+1}{n+1} - \frac{1}{2}.$$  

The inequality $n > 2m + 1$ shows that $B$ is not an integer. It is a contradiction.

**Corollary 1.** A Liénard differential system (1) from family (C) has at most one invariant algebraic curve $F(x, y) = 0$ with the property $N_1 = N_2$.

**Proof.** Recall that the variables $N_1$ and $N_2$ give the number of distinct Puiseux series $y^{(1)}_\infty(x)$ and $y^{(2)}_\infty(x)$ from the field $\mathbb{C}_\infty\{x\}$ arising in the factorization of the polynomial producing the algebraic curve $F(x, y) = 0$, respectively. For more details, see Theorem 12. Note that we do not require the polynomial $F(x, y)$ to be irreducible. The cofactor of such an algebraic curve takes the form

$$\lambda(x, y) = -2N_1 f(x) - N_1 \left\{h^{(1)}_x(x) + h^{(2)}_x(x)\right\}_+. $$  

If there exists another invariant algebraic curve with the property $N_1 = N_2$, then the system under consideration possesses a rational first integral. This fact contradicts Theorem 24.

**Corollary 2.** A Liénard differential system (1) from family (C) cannot have two distinct irreducible invariant algebraic curves $F_1(x, y) = 0$ and $F_2(x, y) = 0$ with the property $N_{1,1}N_{2,2} - N_{1,2}N_{2,1} = 0$, where $N_{l,j}$ is the number of times the family of Puiseux series $y^{(l)}_\infty(x)$ enters the factorization of the polynomial $F_j(x, y)$ in the ring $\mathbb{C}_\infty\{x\}[y]$.

**Proof.** The proof is by contradiction. Suppose a Liénard differential system (1) from family (C) possesses two distinct irreducible invariant algebraic curves $F_1(x, y) = 0$ and $F_2(x, y) = 0$ with the property $N_{1,1}N_{2,2} - N_{1,2}N_{2,1} = 0$.

First of all, let us assume that one of the numbers $N_{l,j}$, where $l, j = 1, 2$, is zero. Without loss of generality, we choose $N_{2,1} = 0$. This gives $N_{1,1}N_{2,2} = 0$. The polynomial $F_j(x, y)$, $j = 1, 2$ should have at least one Puiseux series near the point $x = \infty$ in its factorization. Thus, we get $N_{1,1} > 0$, $N_{2,2} = 0$, and $N_{1,2} > 0$. By Theorem 12, there exists at most one irreducible invariant algebraic curve possessing only one family of Puiseux series near the point $x = \infty$ in the factorization. This yields a contradiction.

Now suppose that all the numbers $N_{l,j}$, where $l, j = 1, 2$, are nonzero. Introducing the variable

$$\chi = \frac{N_{1,1}}{N_{2,1}} = \frac{N_{1,2}}{N_{2,2}}, $$  

(189)
we represent the cofactors $\lambda_1(x, y)$ and $\lambda_2(x, y)$ of the invariant algebraic curves $F_1(x, y) = 0$ and $F_2(x, y) = 0$ in the form

$$
\lambda_j(x, y) = -N_{2,j} \left[(x + 1)f(x) + \left\{xh^{(1)}(x) + h^{(2)}(x)\right\}_+\right], \quad j = 1, 2.
$$

(190)

We conclude that the cofactors are dependent over the ring $\mathbb{Z}$. Consequently, the Liénard differential system under study has a rational first integral. This is a contradiction. ■

We see from Corollary 1 and Theorem 12 that if $\deg g$ denoted as $n$ is an even number, then a Liénard differential system (1) from family $(C)$ cannot simultaneously have more than one irreducible invariant algebraic curve.

Next, let us investigate the existence of non-autonomous Darboux first integrals. The following lemma is valid.

**Lemma 14.** A Liénard differential system (1) from family $(C)$ has a nonautonomous Darboux first integral with a time-dependent exponential factor (10) if and only if $\deg f = 0$ and one of the following assertions is valid:

1. There exists an irreducible invariant algebraic curve $F(x, y) = 0$ such that the family of Puiseux series $y_\infty^{(1)}(x)$ arises in the factorization of the polynomial $F(x, y)$ in the ring $\mathbb{C}_\infty[x][y]$ as many times as so does the family $y_\infty^{(2)}(x)$, that is, $N_1 = N_2$. A first integral takes the form

$$
I(x, y, t) = F(x, y) \exp \left[\frac{2(n+1)f_0N_1t}{n+3}\right].
$$

(191)

2. There exist two distinct irreducible invariant algebraic curves $F_1(x, y) = 0$ and $F_2(x, y) = 0$ such that the following relation $N_{1,j} \neq N_{2,j}, j = 1, 2$ is valid, where $N_{i,j}$ is the number of times the family of Puiseux series $y_\infty^{(i)}(x)$ enters the factorization of the polynomial $F_j(x, y)$ in the ring $\mathbb{C}_\infty[x][y]$. A first integral reads as

$$
I(x, y, t) = \frac{[F_1(x, y)]^{N_{1,2} - N_{2,2}}}{[F_2(x, y)]^{N_{1,1} - N_{2,1}}} \exp \left[\frac{2(n+1)f_0\Omega t}{n+3}\right]
$$

(192)

with the parameter $\Omega$ given by the relation $\Omega = N_{1,2}N_{2,1} - N_{1,1}N_{2,2}$.

There are no other independent nonautonomous Darboux first integrals with a time-dependent exponential factor (10).

**Proof.** By Lemmas 3 and 13, exponential factors cannot enter an explicit expression of a nonautonomous Darboux first integral (10). It follows from Theorem 3 that a Liénard differential system (1) satisfying the condition $\deg g > 2\deg f + 1$ has a first integral (10) if and only if there exists $K \in \mathbb{N}$ pairwise distinct irreducible invariant algebraic curves $F_1(x, y) = 0, ..., F_K(x, y) = 0$ and $K + 1$ nonzero complex numbers $d_1, ..., d_K, \omega$ such that the following condition:

$$
\sum_{j=1}^K d_j\lambda_j(x, y) + \omega = 0, \quad d_1, ..., d_K, \omega \in \mathbb{C} \setminus \{0\}
$$

(193)
is valid. In this expression, \( \lambda_j(x, y) \) is the cofactor of the invariant algebraic curve \( F_j(x, y) = 0 \). The related first integral reads as

\[
I(x, y, t) = \prod_{j=1}^{K} F_j^d(x, y) \exp[\omega t].
\]

(194)

Suppose the family of Puiseux series \( y^{(l)}_{\infty}(x) \) arises \( N_{l,j} \) times in the factorization of the polynomial \( F_j(x, y) \) in the ring \( \mathbb{C}_\infty\{x\}[y] \). Here, \( l = 1, 2 \) and again we use the designations of Theorem 12. The cofactor \( \lambda_j(x, y) \) is given by relation (178). The necessary and sufficient condition for first integral (194) to exist reads as

\[
\sum_{j=1}^{K} d_j N_{1,j} \left( f(x) + \left\{ h_x^{(1)}(x) \right\}_+ \right) + \sum_{j=1}^{K} d_j N_{2,j} \left( f(x) + \left\{ h_x^{(2)}(x) \right\}_+ \right) = \omega.
\]

(195)

This condition is not satisfied whenever relation (180) is not valid. Furthermore, we rewrite condition (195) in the form

\[
B \left( 2f(x) + \left\{ h_x^{(1)}(x) + h_x^{(2)}(x) \right\}_+ \right) = \omega, \quad B = \sum_{j=1}^{K} d_j N_{1,j},
\]

(196)

where unlike the case of Theorem 24 the parameter \( B \) may be complex-valued. If \( m > 0 \), then we arrive at the expression

\[
B \left( 2f_0 - \frac{4(m+1)f_0}{2m+n+3} \right) = 0,
\]

(197)

which is not valid. Consequently, we should set \( m = 0 \). Relations (180) and (196) now become

\[
\sum_{j=1}^{K} d_j N_{1,j} = \sum_{j=1}^{K} d_j N_{2,j}, \quad 2f_0(n+1) \sum_{j=1}^{K} d_j N_{1,j} = (n+3)\omega.
\]

(198)

If the original Liénard differential system has only one irreducible invariant algebraic curve \( (K = 1) \), then algebraic system (198) is satisfied if and only if the following relation \( N_{1,1} = N_{2,1} \) is valid. We note that the parameter \( d_1 \neq 0 \) can be chosen arbitrarily. Thus, we set \( d_1 = 1 \). The second equation in (198) produces the value of \( \omega \). As a result, we obtain nonautonomous Darboux first integral (191), where the index \( j = 1 \) is omitted.

Now let us suppose that the Liénard differential system under consideration has at least two distinct irreducible invariant algebraic curves \( F_1(x, y) = 0 \) and \( F_2(x, y) = 0 \). By Corollary 1 to Theorem 24, these curves satisfy the restriction \( N_{1,j_0} \neq N_{2,j_0} \) for some \( j_0 = 1, 2 \). Without loss of generality, let us first suppose that \( N_{1,1} = N_{2,1} \) and \( N_{1,2} \neq N_{2,2} \). Then, \( d_2 = 0 \) and again we find nonautonomous Darboux first integral (191), where the index \( j = 1 \) is omitted. Now we assume that \( N_{1,j} \neq N_{2,j} \) for \( j = 1, 2 \). We see that algebraic system (198) is always satisfied. Indeed, setting \( K = 2 \) and recalling the fact that one of the exponents \( d_1 \) and \( d_2 \) can be chosen arbitrary, we obtain a solution

\[
d_1 = N_{1,2} - N_{2,2}, \quad d_2 = N_{2,1} - N_{1,1}, \quad \omega = \frac{2(n+1)f_0\Omega}{n+3},
\]

(199)
where we use the designation \( \Omega = N_{1,2} N_{2,1} - N_{1,1} N_{2,2} \). Hence we have found time-dependent Darboux first integral (192). By Corollary 2 to Theorem 24, the following relation \( \Omega \neq 0 \) is valid.

Suppose a Liénard differential system has two independent nonautonomous Darboux first integrals with a time-dependent exponential factor (10), then this system is Darboux integrable. This fact contradicts Theorem 24.

\[ \square \]

**Remark.** Liénard differential systems (1) satisfying the conditions of item 2 of Lemma 14 can only arise when \( n = \deg g \) is an odd number.

\[ \square \]

There exist Liénard differential systems (1) with nonautonomous Darboux first integrals (191) and (192). An example is given in Ref. 32. Finally, we turn to the Liouvillian integrability.

**Theorem 25.** A Liénard differential system (1) from family (C) is Liouvillian integrable if and only if the relation

\[ 4(m + 1)f(x) + (2m + n + 3)\left\{ h_{x}^{(1)}(x) + h_{x}^{(2)}(x) \right\} = 0 \] 

is identically satisfied and one of the following assertions is valid:

1. There exists an irreducible invariant algebraic curve \( F(x, y) = 0 \) such that the family of Puiseux series \( y_{\infty}^{(1)}(x) \) arises in the factorization of the polynomial \( F(x, y) \) in the ring \( \mathbb{C}_{\infty}[x][y] \) as many times as so does the family \( y_{\infty}^{(2)}(x) \), that is, \( N_1 = N_2 \). In this case, the system has the unique Darboux integrating factor

\[ M(x, y) = \left\{ F(x, y) \right\}^{\frac{2m+n+3}{2(n+1)N_1}}. \] 

(201)

2. There exist two distinct irreducible invariant algebraic curves \( F_1(x, y) = 0 \) and \( F_2(x, y) = 0 \) such that the following relation \( N_{1,j} \neq N_{2,j} \), \( j = 1, 2 \) is valid, where \( N_{1,j} \) is the number of times the family of Puiseux series \( y_{\infty}^{(j)}(x) \) enters the factorization of the polynomial \( F_j(x, y) \) in the ring \( \mathbb{C}_{\infty}[x][y] \). In this case, the system has the unique Darboux integrating factor

\[ M(x, y) = \frac{\left\{ F_1(x, y) \right\}^{\frac{(2m+n+3)(N_{2,2}-N_{1,2})}{2(n+1)\Omega}}}{\left\{ F_2(x, y) \right\}^{\frac{(2m+n+3)(N_{2,1}-N_{1,1})}{2(n+1)\Omega}}}. \] 

(202)

The following designation \( \Omega = N_{1,2} N_{2,1} - N_{1,1} N_{2,2} \) is introduced in expression (202).

**Proof.** It follows from Theorem 2 that a Liouvillian integrable differential system (7) has a Darboux integrating factor. By Lemmas 3 and 13, exponential invariants cannot enter an explicit expression of a Darboux integrating factor. Thus, the Darboux integrating factor reads as

\[ M(x, y) = \prod_{j=1}^{K} F_j^{d_j}(x, y), \quad d_1, \ldots, d_k \in \mathbb{C}, \quad K \in \mathbb{N}, \] 

(203)

where the polynomials \( F_1(x, y), \ldots, F_K(x, y) \) give pairwise distinct irreducible invariant algebraic curves \( F_1(x, y) = 0, \ldots, F_K(x, y) = 0 \) of a Liénard differential system. Without loss of generality, we
Suppose that the numbers $d_1, ..., d_K$ are all nonzero. The cofactor $\lambda_j(x, y)$ of the invariant algebraic curve $F_j(x, y) = 0$ is given by relation (178). The necessary and sufficient condition $d_1\lambda_1(x, y) + \cdots + d_K\lambda_K(x, y) = -\text{div} \mathbf{X}$ for Darboux integrating factor (203) to exist now takes the form

$$\sum_{j=1}^{K} d_j N_{1,j} \left( f(x) + \left\{ h_x^{(1)}(x) \right\}_+ \right) + \sum_{j=1}^{K} d_j N_{2,j} \left( f(x) + \left\{ h_x^{(2)}(x) \right\}_+ \right) = -f(x). \quad (204)$$

This condition is not satisfied provided that relation (180) is not valid. Using relation (180), we simplify condition (204). Thus, we get

$$B \left( 2f(x) + \left\{ h_x^{(1)}(x) + h_x^{(2)}(x) \right\}_+ \right) = -f(x), \quad B = \sum_{j=1}^{K} d_j N_{1,j}, \quad (205)$$

where unlike the case of Theorem 24 the parameter $B$ may be complex-valued. Doing the same as in the proof of Theorem 24, we find the following equality:

$$B \left( 2f_0 - 4(m+1)f_0 \frac{2m+n+3}{2(n+1)} \right) = -f_0, \quad (206)$$

which gives the value of $B$. The result is

$$B = -\frac{2m+n+3}{2(n+1)}. \quad (207)$$

Substituting this equality into condition (205) yields relation (200). Finally, we are left with the following algebraic system:

$$\sum_{j=1}^{K} d_j N_{1,j} = \sum_{j=1}^{K} d_j N_{2,j}, \quad \sum_{j=1}^{K} d_j N_{1,j} = -\frac{2m+n+3}{2(n+1)} \quad (208)$$

with respect to the unknowns $d_1, ..., d_K$.

Suppose the Liénard differential system under study has only one irreducible invariant algebraic curve ($K = 1$). Algebraic system (208) is satisfied if and only if $N_{1,1} = N_{2,1}$. Omitting the index $j$, we find the value of $d$ and Darboux integrating factor (201).

Now we assume that the Liénard differential system in question possesses at least two distinct irreducible invariant algebraic curves. Setting $K = 2$, we see that the determinant of algebraic system (208) equals $\Omega = N_{1,2}N_{2,1} - N_{1,1}N_{2,2}$. By Corollary 2 to Theorem 24, we get $\Omega \neq 0$. Consequently, algebraic system (208) has the unique solution. As a result, we obtain Darboux integrating factor (202). Note that (202) becomes (201) whenever $N_{1,j} = N_{2,j}$ for $j = 1$ or $j = 2$.

If there are no invariant algebraic curves or there exists the unique irreducible invariant algebraic curve satisfying the condition $N_{1} \neq N_{2}$, then system (208) is inconsistent.

Suppose the Liénard differential system has two distinct Darboux integrating factors. Then their ratio is a Darboux first integral. This fact contradicts Theorem 24.

**Corollary.** A Liouvillian integrable Liénard differential system (1) from family \((C)\) has at most two distinct irreducible invariant algebraic curves simultaneously provided that $n = \text{deg} g$ is an odd number.
Proof. Assuming that a Liouvillian integrable Liénard differential system has three or more pairwise distinct irreducible invariant algebraic curves, we use Theorem 25 to find at least two distinct Darboux integrating factors. Consequently, the system possesses a Darboux first integral. It is a contradiction.

Remark 1. Item 2 of Theorem 25 can only arise if the number \( n = \deg g \) is odd. Moreover, integrating factor (202) transforms into integrating factor (201) whenever the following condition \( N_{1,1} + N_{1,2} = N_{2,1} + N_{2,2} \) holds. In this case, the polynomial \( F(x, y) \) in (201) is reducible: \( F(x, y) = F_1(x, y)F_2(x, y) \).

Remark 2. Equality (200) is identically satisfied whenever \( m = 0 \) (\( \deg f = 0 \)). This statement follows from relations (183) and (184).

Let us obtain all Liouvillian integrable Liénard differential systems (1) from family (C) with a hyperelliptic invariant algebraic curve \( y^2 + u(x)y + v(x) = 0 \), where \( u(x), v(x) \in \mathbb{C}[x] \).

**Theorem 26.** A Liénard differential system (1) from family (C) with a hyperelliptic invariant algebraic curve \( y^2 + u(x)y + v(x) = 0 \), where \( u(x), v(x) \in \mathbb{C}[x] \), is Liouvillian integrable if and only the system is of the form

\[
\begin{align*}
x_t &= y, \\
y_t &= -(k + 2l)/4 \cdot w^{l-1} w_x y - k/8 \left( w^{2l-1} + 4\beta w^{k-1} \right) w_x,
\end{align*}
\]

where \( \beta \in \mathbb{C} \setminus \{0\} \), \( w(x) \) is a polynomial of degree \( (m + 1)/l \), \( k \) and \( l \) are relatively prime natural numbers such that the following relation \( (m + 1)k = (n + 1)l \) is valid. The associated Liénard differential system has the unique Darboux integrating factor

\[
M(x, y) = \left\{ y^2 + w^l y + \frac{1}{4} w^{2l} + \beta w^k \right\}^{-\left(\frac{1}{2} + \frac{1}{l}\right)}
\]

and the hyperelliptic invariant algebraic curve reads as \( 4y^2 + 4w^l y + w^{2l} + 4\beta w^k = 0 \). A Liouvillian first integral is of the form

\[
I(x, y) = \frac{(2l-k)(2y+w^l)}{4kw^\frac{7}{2} \beta^{\frac{7}{2} + \frac{k}{4}}} _2\!F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{3}{k}; \frac{(2y+w^l)^2}{4\beta w^k}\right) + \left\{ y^2 + w^l y + \frac{1}{4} w^{2l} + \beta w^k \right\}^{\frac{1}{2} - \frac{1}{k}},
\]

where \( _2\!F_1(\alpha, \delta; \sigma; s) \) is the hypergeometric function.

Proof. It is straightforward to verify that system (209) is Liouvillian integrable with an integrating factor and a first integral given by expressions (210) and (211), respectively. Since \( \beta \neq 0 \) and \( w(x) \) is a polynomial of degree \( (m + 1)/l \), we conclude that system (209) is from family (C).

Now our goal is to prove the converse statement. Let us suppose that a Liénard differential system (1) from family (C) is Liouvillian integrable and possesses a hyperelliptic invariant algebraic curve \( y^2 + u(x)y + v(x) = 0 \). By Theorem 25, condition (200) is identically satisfied and the system has integrating factor given by expression (201), where the polynomial \( F(x, y) \) can be chosen in the form \( F(x, y) = y^2 + u(x)y + v(x) \). In addition, we set \( N_1 = 1 \). Substituting integrating factor (201) into the partial differential equation \( yM_x - [f(x)y + g(x)]M_y - f(x)M = 0 \) and equating to
zero the coefficients of different powers of \( y \) yields the relations

\[
f(x) = \frac{2m+n+3}{4(m+1)} u_x, \quad g(x) = \frac{1}{2}v_x + \frac{n-2m-1}{8(m+1)} uu_x
\]  

(212)

and the following equation

\[
u v_x - \frac{n+1}{m+1} u_x v + \frac{n-2m-1}{4(m+1)} u^2 u_x = 0.
\]  

(213)

Let us note that condition (200) produces an explicit expression of the polynomial \( f(x) \) similar to that given in relations (212). Integrating Equation (213) with respect to the function \( v(x) \), we obtain

\[
v(x) = \beta u \frac{n+1}{m+1} + \frac{1}{4} u^2
\]  

(214)

where \( \beta \in \mathbb{C} \) is a constant of integration. Using Theorem 12 and the arguments given in the proof of Theorem 24, we conclude that \( u(x) \) is a polynomial of degree \( m + 1 \) and \( v(x) \) is a polynomial of degree \( n + 1 \). Thus, we see that \( \beta \) is nonzero. Furthermore, we introduce relatively prime natural numbers \( k \) and \( l \) satisfying the relation \( (m+1)k = (n+1)l \). It follows from expression (214) that there exists a polynomial \( w(x) \) of degree \( (m+1)/l \) such that the polynomial \( u(x) \) can be represented in the form \( u(x) = w^l(x) \). Hence, we obtain the equality \( v(x) = \beta w^k(x) + w^{2l}(x) \). Substituting the explicit representations of the polynomials \( u(x) \) and \( v(x) \) into relations (212), we find the polynomials \( f(x) \) and \( g(x) \) as given in (209). Expressing the number \( n \) from the relation \( (m+1)k = (n+1)l \), we find Darboux integrating factor (210) giving Liouvillian first integral (211).

Remark 1. We do not require that the polynomial \( y^2 + u(x)y + v(x) \) is irreducible. See also Remark 1 to Theorem 25.

Remark 2. The family of systems (209) can be transformed to the following simple form

\[
s(\tau) = z, \quad z(\tau) = -\frac{k+2l}{4} s^{l-1} z - \frac{k}{8} (s^{2l-1} + 4\beta s^{k-1})
\]  

(215)

via the generalized Sundman transformation \( s(\tau) = w(x), z(\tau) = y, d\tau = w_x(x) dt \). Substituting \( w(x) = s, y = z \) into (211), we find a Liouvillian first integral for systems (215).

It follows from Theorem 12 that Equation (4) related to a Liénard differential system (1) from family (C) may have a polynomial solution only if \( n = \deg g(x) \) is an odd number. Such a polynomial solution gives rise to an invariant algebraic curve with the generating polynomial of the first degree with respect to \( y \). Let us study the Liouvillian integrability of Liénard differential systems (1) from family (C) possessing invariant algebraic curves with generating polynomials of the first degree with respect to \( y \). Since arbitrary coefficients arise in the nonpolynomial part of the series \( y_{\infty}^{(l)}(x) \), \( l = 1, 2 \), we conclude that Equation (4) has at most two distinct polynomial solutions simultaneously provided that the inequality \( \deg g > 2 \deg f + 1 \) holds. In what follows, we denote these polynomial solutions as \( y = p_1(x) \) and \( y = p_2(x) \). Note that the following relations \( p_l(x) = \{h_l^{(l)}(x)\}_+, l = 1, 2 \) are valid, where \( h_l^{(l)}(x) \) is the initial part of the series \( y_{\infty}^{(l)}(x) \).
Theorem 27. A Liénard differential system (1) from family (C) with two distinct invariant algebraic curves given by first-degree polynomials with respect to \( y \) is Liouvillian integrable if and only if \( n = \deg g(x) \) is an odd number, the system is of the form (209) and other conditions of Theorem 26 are satisfied with the additional restriction: either \( k \) is an even number or otherwise \((m+1)/l\) is an even number and the polynomial \( w(x) \) has only double roots. The polynomials \( p_1(x) \) and \( p_2(x) \) producing the invariant algebraic curves \( y - p_1(x) = 0 \) and \( y - p_2(x) = 0 \) can be represented in the form

\[
p_1(x) = \sqrt{\beta w(x)^k} + \frac{1}{2} w(x), \quad p_2(x) = -\sqrt{\beta w(x)^k} + \frac{1}{2} w(x), \quad \beta \in \mathbb{C} \setminus \{0\}. \tag{216}
\]

Proof. We use item 2 of Theorem 25 and the arguments given in the proof of Theorem 26. Let us note that the hyperelliptic invariant algebraic curve of Theorem 26 with the generating polynomial

\[
y^2 + w(x)y + \frac{1}{4} w^2 + \beta w^k = (y + w(x)/2)^2 + \beta w^k
\]

splits into two distinct invariant algebraic curves \( y - p_1(x) = 0 \) and \( y - p_2(x) = 0 \) if and only if \( n \) is an odd number and either \( k \) is an even number or otherwise \((m+1)/l\) is an even number and \( w(x) \) is a polynomial with double roots. In addition, recall that the degree of the polynomial \( w(x) \) equals \((m+1)/l\).

Remark. This theorem can also be proved directly without using Theorem 26. As an example, see Theorem 18.

Furthermore, our goal is to demonstrate that there exist Liouvillian integrable Liénard differential systems (1) from family (C) for any choice of the numbers \( m = \deg f(x) \) and \( n = \deg g(x) \).

Setting \( u(x) = x^{m+1} \) in expression (214), we find the following Liouvillian integrable Liénard differential systems from family (C):

\[
x_t = y, \quad y_t = -\frac{2m+n+3}{4} x^m y - \frac{n+1}{8} (4\beta x^n + x^{2m+1}). \tag{217}
\]

The related Darboux integrating factor reads as

\[
M(x, y) = \left( y^2 + x^m y + \beta x^{n+1} + \frac{1}{4} x^{2(m+1)} \right)^{\frac{2m+n+3}{2(n+1)}}. \tag{218}
\]

The numbers \( n = \deg g \) and \( m = \deg f \) can be chosen arbitrarily.

Now let us study the existence of nonautonomous Darboux–Jacobi last multipliers. The case \( \deg f = 0 \) is simple. There are families of distinct Jacobi last multipliers arising as products of integrating factors (201), (202) and nonautonomous first integrals \( I_\varphi(x, y, t) \), where \( \varphi \in \mathbb{C} \) and the function \( I(x, y, t) \) is given by relations (191) and (192).

Lemma 15. A Liénard differential system (1) satisfying the conditions \( \deg g > 2 \deg f + 1 \) and \( \deg f > 0 \) has a nonautonomous Darboux–Jacobi last multiplier of the form (12) if and only if there exists a nonzero complex number \( \omega \) such that the relation

\[
4(m + 1)f(x) + (2m + n + 3) \left\{ h_1^{(1)}(x) + h_1^{(2)}(x) \right\}_+ + 2(n + 1)\omega = 0 \tag{219}
\]

is identically satisfied and one of the following assertions is valid:
1. There exists an irreducible invariant algebraic curve $F(x, y) = 0$ such that the family of Puiseux series $y^{(1)}_\infty(x)$ arises in the factorization of the polynomial $F(x, y)$ in the ring $\mathbb{C}_\infty\{x\}[y]$ as many times as so does the family $y^{(2)}_\infty(x)$, that is, $N_1 = N_2$. In this case, the system has the unique Darboux–Jacobi last multiplier

$$M(x, y, t) = \{F(x, y)\}^{\frac{2m+n+3}{2(n+1)\Omega}} \exp[\omega t]. \quad (220)$$

2. There exist two distinct irreducible invariant algebraic curves $F_1(x, y) = 0$ and $F_2(x, y) = 0$ such that the following relation $N_{1,j} \neq N_{2,j}$, $j = 1, 2$ is valid, where $N_{l,j}$ is the number of times the family of Puiseux series $y^{(l)}_\infty(x)$ enters the factorization of the polynomial $F_j(x, y)$ in the ring $\mathbb{C}_\infty\{x\}[y]$. In this case, the system has the unique Darboux–Jacobi last multiplier

$$M(x, y, t) = \frac{\{F_1(x, y)\}^{(2m+n+3)N_{1,2}}}{\{F_2(x, y)\}^{(2m+n+3)N_{1,1}}} \exp[\omega t], \quad (221)$$

where the parameter $\Omega$ is given by the relation $\Omega = N_{1,2}N_{2,1} - N_{1,1}N_{2,2}$.

**Proof.** We repeat the proof of Theorem 25. The only difference is in condition (204). In the nonautonomous case, this condition takes the form

$$\sum_{j=1}^{K} d_j N_{1,j} \left( f(x) + \left\{ h^{(1)}_x(x) \right\}_+ \right) + \sum_{j=1}^{K} d_j N_{2,j} \left( f(x) + \left\{ h^{(2)}_x(x) \right\}_+ \right) = \omega - f(x), \quad (222)$$

where $\omega$ is a nonzero complex constant. ■

We have established that Liénard differential systems (1) from family (C) have neither rational nor Darboux first integrals. Let us note that the famous Duffing oscillators belong to family (C). These oscillators are studied in Refs. 29, 32 in detail.

### 9 QUARTIC LIÉNARD DIFFERENTIAL SYSTEMS WITH A QUADRATIC DAMPING FUNCTION

The aim of the present section is to demonstrate that the necessary and sufficient conditions of Liouvillian integrability presented in the previous sections can be used to find all Liouvillian integrable subfamilies of Liénard differential systems without performing the classification of irreducible invariant algebraic curves. As an example, we consider Liénard differential systems with the restrictions $\deg f = 2$ and $\deg g = 4$:

$$x_t = y, \quad y_t = -(\xi x^2 + \beta x + \alpha)y - (\epsilon x^4 + \xi x^3 + \sigma x + \delta), \quad \xi \notin 0. \quad (223)$$

Introducing suitable rescalings and shifts, it is without loss of generality to set $\xi = 3$, $\epsilon = -3$, and $\beta = 0$. In what follows, we work with the following systems:

$$x_t = y, \quad y_t = -(3x^2 + \alpha)y + 3x^4 - \xi x^3 - \epsilon x^2 - \sigma x - \delta. \quad (224)$$
Let us solve the integrability problem for systems (224).

**Theorem 28.** Quartic Liénard differential systems with a quadratic damping function (224) are Liouvillian integrable if and only if the tuple of the parameters \((\alpha, \xi, \delta, \sigma, e)\) equals

\[
I : (\alpha, \xi, \delta, \sigma, e) = \left( -\frac{25}{12}, -7, \frac{125}{36}, -\frac{25}{36}, -5 \right); \\
II : (\alpha, \xi, \delta, \sigma, e) = \left( -\frac{61}{12}, -7, \frac{3905}{432}, -\frac{25}{36}, 4 \right).
\]  

The related Darboux integrating factors can be represented as

\[
I : M(x, y) = \frac{1}{\left( y + x^3 + \frac{3}{2} x^2 + \frac{5}{12} x - \frac{25}{36} \right)^{\frac{7}{10}}} \left( y - x^2 - \frac{5}{3} x + \frac{25}{36} \right)^{\frac{2}{7}}; \\
II : M(x, y) = \frac{1}{y^2 + \frac{(6x+5)(6x-13)(6x+11)y}{216} - \frac{(6x-13)(6x+11)^2(6x+5)^2}{7776}}.
\]

**Proof.** Our proof is based on the results of Theorems 9 and 17. The Puiseux series given in relation (24) are now the following

\[
y^{(1)}(x) = -x^3 - \frac{3}{2} x^2 + \left( \xi - \alpha + \frac{9}{2} \right) x + b_3 + \sum_{l=1}^{\infty} b_{l+3} x^{-l}; \\
y^{(2)}(x) = x^2 - \frac{1}{3} (\xi + 2) x + \frac{1}{3} (\xi + 2 - \alpha - e) + \sum_{l=1}^{\infty} a_{l+2} x^{-l}.
\]  

The Puiseux series \(y^{(1)}(x)\) has an arbitrary coefficient \(b_3\) and exists whenever the restriction \(e = \frac{3(27 + 6\xi - 4\alpha)}{4}\) holds. The Puiseux series \(y^{(2)}(x)\) possesses uniquely determined coefficients. Note that we use novel designations for the coefficients of the Puiseux series \(y^{(2)}(x)\). The series \(y^{(1)}(x)\) terminates at the zero term under the condition

\[
\delta = \frac{1}{24} (2\xi + 9)(18\alpha + 4\xi\alpha - 4\sigma - 4\xi^2 - 36\xi - 81).
\]

Thus, we see that systems (224) possess the invariant algebraic curve \(F_1(x, y) = 0\) of Theorem 17 whenever \(e = \frac{3(27 + 6\xi - 4\alpha)}{4}\) and \(\delta\) is of the form (228). The related polynomial and the cofactor can be represented as

\[
F_1(x, y) = y + x^3 + \frac{3}{2} x^2 - \left( \xi - \alpha + \frac{9}{2} \right) x - \frac{1}{3} (\sigma + \xi^2 - \xi\alpha) - \frac{1}{4} (27 + 12\xi - 6\alpha), \quad \lambda_1(x, y) = 3x - \xi - \frac{9}{2}.
\]

Condition (55) gives the following restriction: \(\xi = -7\). Finally, we use Theorem 13 to find an irreducible invariant algebraic curve that exists simultaneously with \(F_1(x, y) = 0\) and is given by expression (22) where \(k = 1\) and \(N \in \mathbb{N}\). As a result, we obtain the values of the parameters as presented in relation (225). The related invariant algebraic curves are given by the polynomials

\[
(I) : \quad F_2(x, y) = y - x^2 - \frac{5}{3} x - \frac{25}{36}, \quad \lambda_2(x, y) = -3x^2 - 2x + \frac{5}{12}; \\
(II) : \quad F_2(x, y) = y^2 + \frac{(6x+5)(6x-13)(6x+11)y}{216} - \frac{(6x-13)}{7776} \frac{71}{12}, \\
\times (6x+11)^2(6x+5)^2, \quad \lambda_2(x, y) = -3x^2 + x + \frac{71}{12}.
\]
We calculate explicit expressions of Darboux integrating factors with the help of expression (56).

In case (I), a Liouvillian first integral is given by expression (69), where one sets

\[ l = 2, \quad k = 3, \quad \beta = -1, \quad v(x) = x + \frac{5}{6}. \] (231)

In case (II), a Liouvillian first integral reads as

\[
I(x, y) = \frac{\sqrt{w(x)}}{w(x)} \sum_{l=0}^{2} \left( \frac{\sqrt{w(x)} - v(x)}{2} \right) U(x) \log \left( z^\frac{1}{3} - U(x) \exp \left( \frac{2\pi li}{3} \right) \right) \\
+ \left( \sqrt{w(x)} + v(x) \right) V(x) \log \left( z^\frac{1}{3} - V(x) \exp \left( \frac{2\pi li}{3} \right) \right) \exp \left( \frac{2\pi li}{3} \right) + 6z^\frac{1}{3},
\] (232)

where we have introduced the notation

\[
U(x) = \left\{ u(x) - v(x) + \sqrt{w(x)} \right\}^\frac{1}{3}, V(x) = \left\{ u(x) - v(x) - \sqrt{w(x)} \right\}^\frac{1}{3}, z = y + u(x).
\] (233)

The polynomials \( u(x), v(x), w(x) \) take the form

\[
u(x) = \frac{(6x + 5)(6x - 13)(6x + 11)}{432}, \quad \begin{align*}
w(x) &= \frac{(6x - 13)(6x + 11)^3(6x + 5)^2}{186624}. \end{align*}
\] (234)

The integrability in case (II) of Theorem 28 is related to the existence of an invariant algebraic curve of degree 3 with respect to \( y \). This curve is reducible. Some examples of integrable Liénard differential systems with irreducible invariant algebraic curves of degree 3 and greater with respect to \( y \) are given in Refs. 30, 49.

Concluding this section we note that the method of Puiseux series and the explicit expression (18) of the cofactor of an invariant algebraic curve greatly facilitate the classification of integrable multiparameter planar differential systems.

10 CONCLUSION

This work completely solves the Liouvillian integrability problem for polynomial Liénard differential systems (1) satisfying the condition \( \deg g \neq 2 \deg f + 1 \). In the case \( \deg g = 2 \deg f + 1 \), our results are complete for the nonresonant systems. We say that a Liénard differential system with the restriction \( \deg g = 2 \deg f + 1 \) is resonant near infinity if Equation (26) possesses a positive rational solution. The resonance condition introduces a restriction on the highest-degree coefficients \( f_0 \) and \( g_0 \) of the polynomials \( f(x) \) and \( g(x) \).

We have established that a generic nonlinear polynomial Liénard differential system (1) with fixed degrees of the polynomials \( f(x) \) and \( g(x) \) is not Liouvillian integrable provided that the following restriction \( \deg g > \deg f \) is valid. However, as we have demonstrated, Liouvillian integrable subfamilies exist for any degrees of the polynomials \( f(x) \) and \( g(x) \) whenever \( \deg g > \deg f \). Besides that, we have classified polynomial Liénard differential systems possessing nonautonomous Darboux first integrals and nonautonomous Jacobi last multipliers with a
| Family | \( f(x), \ g(x) \) | \( M(x, y) \) | \( I(x, y), \ type \ of \ first \ integral \) |
|--------|----------------|----------------|----------------------------------|
| \((A)_1\) | \( f(x) = -[k \beta v^{k-1} + (k + l)u^{l-1}]v_x \) | \( z^\frac{j}{\tau} \) | \( I(x, y) = \frac{k \beta}{\kappa}v_{k-1}z^\frac{j}{\tau} + \sum_{j=0}^{k-1} \exp[-\frac{\pi(2j+1)\gamma}{k}]u \) |
| \( Th \ 18\) | \( g(x) = k[\beta u^{k-1} + v^l]u^{l-1}v_x \) | \( z = y - \beta v^k - v^l \) | \( \Delta L \) |
| \((B)_i\) | \( f(x) = -\frac{2f_0}{f_0-\delta}q_1, g(x) = f_0+\delta q_1 \) | \( y - q(x) \) | \( \Delta L \) |
| \( Th \ 21\) | \( u(\tau) = u_x \) | | |
| \( (B)_3\) | \( f(x) = -[\frac{(2d_1+1)k + kl}{k-l}]v^{l-1} - kl \beta v^{k-1} \) | \( \exp[\frac{j}{\tau}] \) | \( \Delta L \) |
| \( Th \ 26\) | \( g(x) = \frac{1}{8}(w^{2l-1} + 4\beta w^{k-1})w_x \) | | |
| \( Th \ 27\) | \( z = [y + \frac{w}{2}]^2 + \beta w^k \) | | |

Remark to Table 1

1. Natural numbers \( l \) and \( k \) are both nonunit.
2. Symbols \( D \), \( E \), and \( L \) mean Darboux, elementary, and Liouvillian, respectively. Symbols \( \Delta L \) and \( \Delta E \) mean non-Darboux and nonelementary.
3. Family \((A)_1\) gives all Liouvillian integrable families of Liénard differential systems (1) such that the related equation (4) possesses two distinct polynomial solutions and the following inequalities \( \deg f < \deg g < 2 \deg f + 1 \) are valid.

The explicit Liouvillian integrable families of Liénard differential systems are gathered in Table 1.
4. Families \((B)_1\) and \((B)_2\) produce all nonresonant Darboux integrable Liénard differential systems (1) satisfying the restriction \(\deg g = 2 \deg f + 1\).

5. Families \((B)_1, (B)_2, (B)_3,\) and \((B)_4\) include all nonresonant Liouvillian integrable Liénard differential systems (1) satisfying the restriction \(\deg g = 2 \deg f + 1\). Note that families \((B)_1, (B)_3,\) and \((B)_4\) also involve integrable resonant systems. Consequently, additional restrictions should be imposed if one is interested only in the nonresonant case. For family \((B)_1\), these restrictions take the form \(\delta / f_0 \not\in \mathbb{Q}\). For families \((B)_3\) and \((B)_4\), these restrictions are described in Theorems 22 and 23.

6. Family \((C)_1\) gives all Liouvillian integrable families of Liénard differential systems (1) with \(\deg g > 2 \deg f + 1\) possessing either a hyperelliptic invariant algebraic curve or two distinct invariant algebraic curves with generating polynomials of the first degree with respect to \(y\).

Let us note that families \((B)_1\) and \((B)_2\) are those given by the Chiellini integrability condition \(\{f(x)/g(x)\}_x = \alpha f(x)\) (see Ref. 13). Remarkably, Chiellini integrable Liénard differential systems can be linearized via generalized Sundman transformations.\(^\text{18}\) Other integrable families from Table 1 can also be transformed to a more simple form via generalized Sundman transformations, see remarks and comments to Theorems 18, 21, 23, and 26. These systems with the exception of a number of partial cases that appear in Refs. 10, 11, 25, 28, 33 seem to be new.

Let us enumerate some unsolved problems related to the integrability and solvability of Liénard differential systems. Despite the fact that the subset of resonant Liénard differential systems is of Lebesgue measure zero in the set of all polynomial Liénard differential systems satisfying the condition \(\deg g = 2 \deg f + 1\), it is an interesting open problem to perform a classification of invariant algebraic curves and integrable subfamilies of particular resonant polynomial Liénard differential systems provided that only a resonant condition is imposed on the parameters of the systems. The method of Puiseux series\(^\text{37,47}\) can deal with each family of resonant Liénard differential systems characterized by a fixed positive rational Fuchs index separately. Note that several novel families of Liouvillian integrable resonant systems are presented in Theorems 21 and 23, see also Corollary 2 after the latter theorem. In addition, a number of integrable resonant families with \(\deg f = 2\) and \(\deg g = 5\) are found via \(\lambda\) symmetries in Ref. 12. These families possess an exciting property: they simultaneously have autonomous and nonautonomous Darboux first integrals given by expression (10) with \(\omega = 0\) and \(\omega \neq 0\), respectively. This fact allowed Ruiz and Muriel to obtain nice expressions of the general solutions. As established in Section 7, systems (11) with \(\delta = \pm m f_0 / (m + 2)\) have the same property.

Along with this, it is a difficult open problem to perform a classification of integrable polynomial Liénard differential systems with non-Liouvillian first integrals. At the moment, only particular examples that can be transformed to linear equations are available, for more details see Refs. 16,24,38,51.

Another important problem is to study rational Liénard differential systems. If the functions \(f(x)\) and \(g(x)\) in expression (1) are rational, then systems (1) give rise to the following polynomial differential systems in the plane:

\[
x_t = h(x)y, \quad y_t = -\tilde{f}(x)y - \tilde{g}(x), \quad h(x), \tilde{f}(x), \tilde{g}(x) \in \mathbb{C}[x].
\]

(235)

Investigating the analytic and qualitative properties of these systems with respect to the degrees of polynomials \(h(x), \tilde{f}(x),\) and \(\tilde{g}(x)\) is a future challenge.
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DATA AVAILABILITY STATEMENT
The data that support the findings of this study are available from the corresponding author upon reasonable request.

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