Open string – string junction transitions

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Abstract

It is confirmed that geodesic string junctions are necessary to describe the
gauge vectors of symmetry groups that arise in the context of IIB superstrings
compactified in the presence of nonlocal 7-branes. By examining the moduli space
of 7-brane backgrounds for which the dilaton and axion fields are constant, we
are able to describe explicitly and geometrically how open string geodesics can fail
to be smooth, and how geodesic string junctions then become the relevant BPS
representatives of the gauge bosons. The mechanisms that guarantee the existence
and uniqueness of the BPS representative of any gauge vector are also shown to
generalize to the case where the dilaton and axion fields are not constant.

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1 Introduction

Type IIB superstring theory possesses BPS states \((p, q)\) which are bound states of \(p\) fundamental and \(q\) D-strings \([1]\), where \(p\) and \(q\) are relatively prime and represent the charges of the resulting string under the NS-NS and RR antisymmetric tensors of the theory, respectively. The theory also possesses a corresponding set of \([p, q]\) 7-branes. An elementary string, \(i.e.\) a \((1_0)\)-string, can end on a D-string \(i.e.\) a \((0_1)\)-string, and in this process part of the D-string turns into a \((1_1)\)-string in order to satisfy the charge conservation. This configuration is more appropriately thought of as a junction where three different strings join \([2, 3]\). The relevant conservation laws were explained by Schwarz who conjectured that string junctions would represent physically relevant BPS states \([3]\).

In \([4]\) we showed that there exists an interesting physical configuration where these string junctions seem to play an important role. This configuration is described by the compactification of type IIB on a two sphere in the presence of twenty-four 7-branes, not all of which are of the same \([p, q]\) type; this corresponds in F-theory to the compactification on an elliptically fibered K3 \([5]\). As the K3 develops an ADE-singularity, gauge bosons in the corresponding group become massless \([6]\). From the point of view of type IIB, these BPS states may be thought of as certain open strings that begin and end on various 7-branes \([7]\). However, as argued in \([4]\), open strings do not seem to suffice, and open string junctions seem necessary depending on the positions of the relevant 7-branes. Moreover (and this was the main motivation in \([4]\)), the pronged string states manifestly carry the correct gauge charges and this enables one to reproduce the multiplicative structure of the exceptional groups. This leads to a compelling brane picture for the gauge enhancement of exceptional symmetry.

Subsequently, several works have appeared giving detailed and convincing arguments that string junctions preserve supersymmetry \([8, 9, 10, 11, 12, 13]\). A second physical application for string junctions has also been proposed recently by Bergman \([14]\), who gave evidence that three string junctions should represent one-quarter-BPS states in a string picture of \(N = 4\), four-dimensional supersymmetric gauge theory with gauge groups of rank bigger than one. Type IIB theory has also \([p, q]\) five-branes and their junctions are relevant to five-dimensional gauge theory \([8]\). While the physics of five-branes is rather different from that of strings, the mathematical methods used to study five-brane junctions have relevance to string junctions.

In this paper we return to the subject of \([4]\). The 7-branes in question appear as points on the two-sphere and extend in the eight non-compact directions, and the background of the 7-branes induces a nontrivial metric on the two-sphere. Open strings that join the 7-branes and represent BPS states of the 8-dimensional field theory are naturally realized as smooth geodesics. A preliminary study of the metric on \(S^2\) suggested that it was unlikely that one could have smooth geodesics in all requisite homotopy classes \([4]\), and it was shown that open strings could become string junctions as the string crosses.
a 7-brane. In this process an extra prong is created, and this is U-dual to the Hanany-Witten effect [13]. This suggested a picture where a given BPS-state could be represented sometimes by a smooth geodesic string, and sometimes by a geodesic junction, where all prongs are smooth. The physical consistency of this proposal hinges on three properties:

(i) **Necessity of junctions.** As we change the positions of the 7-branes, the open string geodesic representing a gauge vector can fail to be smooth, and thus fail to be a valid representative.

(ii) **Existence.** In the situation of (i), a geodesic string junction of mass lower than that of the broken open string exists and represents the gauge vector.

(iii) **Uniqueness.** For any configuration of 7-branes the minimal mass object representing a gauge vector is unique; it is *either* a smooth geodesic open string *or* a geodesic string junction.

The main purpose of the present paper is to explain the mechanisms that guarantee that (i), (ii) and (iii) hold. This will also provide an indirect (and explicit) confirmation of the Hanany-Witten effect. We shall mainly consider particularly simple situations where the transitions can be understood very explicitly, and we shall demonstrate how the arguments generalize. Along the way we shall also learn much about the metric on the two-sphere.

The IIB backgrounds we are considering can also be viewed as an M-theory compactification on an elliptic K3 in the limit as the elliptic fibers, the tori, are of vanishing area [16, 3, 8, 12]. In this framework the relevant BPS states arise from membranes that wrap around supersymmetric cycles of genus zero, and in the limit of zero area, such cycles are expected to project to geodesics on the two-sphere base of the elliptic fibration. This viewpoint gives strong indirect evidence for the validity of (i)–(iii). Indeed, open strings and string junctions related by crossing operations should define K3 cycles belonging to the same homology class, and in a K3 cycles of genus zero in any given homology class have unique supersymmetric representatives. Moreover, we should expect the supersymmetric representatives to vary continuously as we vary the moduli of the K3. While this line of argumentation, familiar to experts, could conceivably be developed into a proof of (i)–(iii), we shall not attempt this here. We shall rather work directly with the IIB picture, and analyze the two-sphere with its nontrivial metric and 7-branes in detail. This picture has its merits: it is explicit, BPS states are easily visualized, and the role of string junctions is clearly exhibited.

In order to examine the various string geodesics and string junctions, it is often necessary to separate the different 7-branes (that define the singularity when they coincide), and we shall refer to this as resolving the singularity. When the branes are separated, however, the metric on the two-sphere is typically a rather complicated function of $\tau = a + ie^{-\phi}$
(where $a$ is the axion and $\phi$ the dilaton), which in itself is a nontrivial multivalued function on the sphere. We show that partial resolutions of the above singularities are possible while maintaining a constant value of $\tau$ throughout the sphere. For example the $so(8)$ singularity ($D_4$) defined by six coincident 7-branes can be resolved at constant $\tau = \exp(i\pi/3)$ into three separate singularities, each containing two coincident 7-branes. Analogous resolutions are also possible for the $E_6$, $E_7$ and $E_8$ singularities.

These partial resolutions fit nicely within the moduli space of compactifications at constant $\tau$ which consists of three branches: the branch where $\tau$ can take an arbitrary (constant) value which was described by Sen [17], and the two branches described by Dasgupta and Mukhi [18] that are consistent with particular values of constant $\tau$. For the former a description in terms of 7-branes was already given by Sen, and for the latter we shall provide such an explicit description in this paper. For example, on one of the two additional branches, the generic situation is described by twelve identical singularities, each defined by two mutually non-local 7-branes, and we show how collisions of these singularities generate all enhanced symmetry points on this branch. The main tool in this discussion is the understanding of how 7-branes are transformed as 7-brane branch cuts are moved across other 7-branes. This enables us to relate singularities defined by two types of 7-branes to singularities defined by three types of 7-branes; the latter are the conventional presentations for exceptional singularities presented by Johansen [7].

For the discussion of the transition between open string geodesics and string junctions, we shall mainly consider partial resolutions (at constant $\tau$) into three singularities. At constant $\tau$, all \binom{p}{q} strings feel the same metric (up to a constant factor that is due to the tension), and there exists a coordinate $w$, in which all \binom{p}{q} geodesics are straight lines. Moreover, the metric singularities are of conical type, and they can thus be represented (in the $w$-plane) by excising a wedge and identifying the edges of the wedge, with the proviso that strings change their $(p, q)$ character as they cross the seam. We can then examine the fate of a direct geodesic (between two of the singularities $P_1$ and $P_2$, say) as the position of the third singularity $Q$ changes. It will become apparent that the direct geodesic ceases to be smooth as the straight line between $P_1$ and $P_2$ crosses the wedge (see (i) above). In this case, as we shall demonstrate, a string junction is created whose overall mass is precisely described by the direct distance between $P_1$ and $P_2$, and this shows directly that the three string junction has strictly lower length than that of the broken direct geodesic ((iii) above) and that both representatives cannot simultaneously be smooth ((iii) above). Given that a three singularity resolution already exists for $so(8)$, this implies that string/junction transitions are not a phenomenon peculiar to exceptional groups; rather they are a generic non-perturbative phenomenon.

The above discussion can be generalized to backgrounds where $\tau$ is not necessarily constant. In particular, we can still find a simple expression for the overall length of the string junction as the distance between two points, and the main conclusions are therefore unaltered. We shall also show explicitly how for the $so(8)$ singularity, some of the
indirect geodesics fail to be smooth as the singularity (that corresponds to the orientifold plane in the corresponding type I description) is resolved.

This paper is organized as follows. In section 2 the various conventions are explained in detail, and it is shown how the characterization of branes changes as branch cuts are moved across other branes. In section 3 we give a brane description of the moduli space of the constant \( \tau \) compactifications, and we show how to recover the \( D_4 \) and \( E_6, E_7, E_8 \) singularities by collision of simpler singularities. In section 4 we describe explicitly the transitions between direct strings, string junctions and indirect strings for partial resolutions compatible with constant \( \tau \). In section 5 we extend these results to non-constant \( \tau \) backgrounds. In section 6 we show, in the context of a familiar non-constant \( \tau \) background, the absence of smooth indirect geodesics for a certain class of strings. Finally, in section 7 we offer some concluding remarks and discuss open questions.

2 The description of the covering space

The main object of our discussion is type IIB string theory compactified on a two-sphere in the presence of twenty-four parallel 7-branes which extend along the eight uncompactified directions. In this section we shall explain our various conventions for the monodromies of the 7-branes (not all of which may be mutually local) on the compactifying sphere. In particular, we shall explain how to introduce branch cuts on the two-sphere, and how the description changes as branch cuts are moved across branes.

2.1 Seven-branes, monodromies and \((p,q)\)-strings

Let us denote by \( \tau = a + ie^{-\phi} \) the complex combination of the dilaton field \( \phi \) and the axion field \( a \) in type IIB. In the presence of 7-branes, \( \tau \) is not a single-valued function on the sphere, and the way in which it fails to be single-valued characterizes the 7-branes that are located on the sphere.\(^1\) In order to describe this in some detail, let us denote by \( z_i \) \((i = 1, \ldots, 24)\) the locations of the 7-branes on the sphere, and let \( S_0 \) be the punctured sphere where all twenty-four points \( z_i \) have been removed. We pick a reference point \( z_0 \neq z_i \) on the (punctured) sphere, and we introduce a basis of generators \( \gamma_i \) for the homotopy group \( \pi_1(S_0) \) of \( S_0 \) relative to \( z_0 \). (Here \( \gamma_i(t) \) is a path on \( S_0 \) that starts at \( \gamma_i(0) = z_0 \), encircles \( z_i \) anti-clockwise, and ends again at \( \gamma_i(1) = z_0 \).)

For the configuration we have in mind, the monodromy of \( \tau \) can be described by a collection of matrices \( M_i \in SL(2, \mathbb{Z}) \), where

\[
\tau(\gamma_i(1)) = M_i \tau(\gamma_i(0)),
\]

\(^1\)The following clarifies (and corrects) the conventions that were used in .
and $M \in \text{SL}(2, \mathbb{Z})$ acts on $\tau$ in the usual way

$$M \tau = \frac{a\tau + b}{c\tau + d}, \quad \text{where} \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \quad (2.2)$$

As $\tau$ is single-valued in every simply-connected subset of $S_0$, it follows from (2.1) that

the $M_i$ define a group homomorphism $M : \pi_1(S_0) \to \text{SL}(2, \mathbb{Z})$.

In order to represent a fundamental 7-brane (rather than a bound state of different 7-branes), $M_i$ must be of the form

$$M_{p,q} = g_{p,q} T g_{p,q}^{-1} = \begin{pmatrix} 1 - pq & p^2 \\ -q^2 & 1 + pq \end{pmatrix}, \quad (2.3)$$

where

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad g_{p,q} = \begin{pmatrix} p & r \\ q & s \end{pmatrix}, \quad ps - qr = 1. \quad (2.4)$$

In particular, the different monodromy matrices are thus SL(2, $\mathbb{Z}$) conjugates of $T$, the monodromy matrix of the [1, 0] D7-brane [19]. With respect to the chosen generators of the homotopy group of $S_0$, we say that the 7-brane at $z_i$ is of type $[p, q]$ if $M_i = M_{p,q}$.

This defines the $[p, q]$ label of a given 7-brane up to an overall sign, as $M_{-p,-q} = M_{p,q}$.

We should stress that this identification depends on the choice of $z_0$ and $\gamma_i$.

The system possesses a global SL(2, $\mathbb{Z}$) symmetry: for $g \in \text{SL}(2, \mathbb{Z})$, we can replace

$$\tau \mapsto g \tau, \quad \text{and then} \quad M_i \mapsto g M_i g^{-1}, \quad (2.5)$$

since $g\tau(\gamma_i(1)) = g M_i \tau(\gamma_i(0)) = (g M_i g^{-1}) g \tau(\gamma_i(0))$.

Given the multi-valued function $\tau$ on $S_0$, we can define a metric on $S_0$ by [20]

$$ds^2 = \tau_2 \eta(\tau)^2 \bar{\eta}(\bar{\tau})^2 \prod_i (z - z_i)^{1/12}(\bar{z} - \bar{z}_i)^{-1/12} dz d\bar{z}, \quad (2.6)$$

where $\tau_2 = \text{Im}(\tau)$, and

$$\eta^2(\tau) = q^{1/12} \prod_{n=1}^{\infty} (1 - q^n)^2, \quad q = \exp(2\pi i \tau), \quad (2.7)$$

is the square of the Dedekind eta-function which satisfies

$$\eta^2(-1/\tau) = -i \tau \eta^2(\tau), \quad \eta^2(\tau + 1) = e^{i\pi/6} \eta^2(\tau). \quad (2.8)$$

It is straightforward to show that this metric is invariant under the global SL(2, $\mathbb{Z}$) action on $\tau$, and this implies that the metric is single-valued on $S_0$. 

6
Type IIB theory possesses different strings which are labeled by \((p\ q)\), where \(p\) and \(q\) are coprime integers. The masses of the states associated to \((p\ q)\) strings must also take into account the string tension \(T_{p,q}\) of the \((p\ q)\) string [21],
\[
  T_{p,q} = \frac{1}{\sqrt{\tau_2}} |p - q\tau|.
\] (2.9)

It is then natural to introduce the length element \(ds_{p,q} = T_{p,q}ds\) which measures correctly the mass of the corresponding string, and the corresponding effective metric \(ds_{p,q}^2\)
\[
ds_{p,q} = |h_{p,q}(z)\ dz|, \quad \text{with} \quad h_{p,q}(z) = (p - q\tau)\eta^2(\tau)\prod_i(z - z_i)^{-1/12}.
\] (2.10)

For every fixed label \((p\ q)\), \(ds_{p,q}^2\) describes a continuous metric on a certain covering space \(\tilde{S}_{p,q}\) of \(S_0\). If we define the action of SL(2, \(\mathbb{Z}\)) on \((p\ q)\) by
\[
  \begin{pmatrix} p \\ q \end{pmatrix} \mapsto \begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} p \\ q \end{pmatrix}, \quad \text{where} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}),
\] (2.11)
then for \(\tau' = g\tau\), we have \(T_{p,q}(\tau) = T_{p',q'}(\tau')\). This implies that up to a global transformation of \(\tau\) by an element in \(\text{SL}(2, \mathbb{Z})\), the spaces \(\tilde{S}_{p,q}\) and \(\tilde{S}_{p',q'}\) are the same.

### 2.2 Introducing branch cuts

Different strings can end on different 7-branes, but there is no \(\tau\)-independent description of which string can end on which 7-brane. In order to discuss this issue it is therefore necessary to introduce branch cuts on \(S_0\) and to choose a fixed branch \(\tau_0\) of \(\tau\) on the space \(S_c = S_0 - \{\text{branch cuts}\}\). (The branch cuts are chosen so that \(S_c\) is simply connected.)

There is a large amount of freedom how to choose the branch cuts (and we shall later take some advantage of this fact) but there is a particularly simple choice: for each \(z_i\) we introduce a branch cut \(C_i\) from \(z_i\) to \(z_0\) so that the path \(\gamma_i\) does not cross \(C_i\) (and that it only touches the branch cut \(C_i\) at the common end point \(z_0\)).

It then follows that as \(\tau\) crosses \(C_i\) anti-clockwise, \(\tau\) jumps to \(K_i\tau\) where
\[
  K_i = M_i^{-1}.
\] (2.12)

We should stress that this relation only holds for this particular choice of cuts, and that for a different choice of cuts, \(K_i\) has to be replaced by \(K_i = D_i M_i^{-1} D_i^{-1}\), where \(D_i\)

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2 The different choices for \(z_0\) and \(\gamma_i\), and the different possible branch cuts \(C_i\) are in one-to-one correspondence: every choice of \(z_0\) and \(\gamma_i\) determines a family of cuts \(C_i\) by the above description, and conversely, every such family of cuts determines \(z_0\) (the common endpoint of the \(C_i\)) and \(\gamma_i\) up to homotopy.
is some matrix in $\text{SL}(2, \mathbb{Z})$; this will be explained in more detail in the next subsection. For this choice of cuts we can then say that a 7-brane is of type $[p, q]$ if $K_i = M_{-1}^{p,q}$. Given $\binom{p}{q}$ and $\tau_0$ on $S_c$, we can define a metric on $S_c$ by formula (2.10). This metric is not continuous across the branch cut, and in order for it to become continuous, we have to choose the convention that the labels of the string $\binom{p}{q}$ change to $K_i(\binom{p}{q})$ as we cross the branch cut anti-clockwise; the metric is then continuous because of (2.11). As $\binom{p}{q}$ is locally constant, it follows that the $K_i$ induce a group homomorphism $K : \pi_1(S_0) \to \text{SL}(2, \mathbb{Z})$; this can be easily understood from Fig. 1. In this example the monodromy corresponding to the curve $\gamma_1 \circ \gamma_2 \circ \gamma_3$ is $M_1M_2M_3 = 1$ (see figure); on the other hand, following the curve $\gamma$, one crosses first the cut $C_1$, then $C_2$ and finally $C_3$ which corresponds to

$$K_3K_2K_1 = M_3^{-1}M_2^{-1}M_1^{-1} = (M_1M_2M_3)^{-1} = 1.$$  

![Figure 1: The branes can be characterized by either the monodromies or the discontinuity at their branch cuts. The matrices $M$ and $K$ both induce a group homomorphism $\pi_1(S_0) \to \text{SL}(2, \mathbb{Z})$.](image)

We can thus summarize our convention as (see Fig. 3)

Upon anticlockwise crossing of the branch cut of a 7-brane with matrix $K$, $\tau$ transforms to $\tau \to K\tau$ and $\binom{p}{q}$ transforms to $\binom{p}{q} \to K\binom{p}{q}$. 

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Figure 2: As the branch cut is crossed anticlockwise both $\tau$ and $\binom{p}{q}$ are transformed by $K$.

We shall now say that a $\binom{p}{q}$ string can end on a brane described by $K$ if and only if $K = M_{p,q}^{-1}$. If $K$ is defined with respect to the specific choice of cuts described above, then this rule amounts to saying that a $\binom{p}{q}$-string can only end on a $[p, q]$ brane.

In this paper we shall mainly deal with three types of branes $A$, $B$ and $C$, whose corresponding $[p, q]$ labels (with respect to some choice of $z_0$ and $\gamma_i$) are $[1, 0]$, $[1, -1]$ and $[1, 1]$ respectively. Across the corresponding cuts $C_i$, the labels $\binom{p}{q}$ and $\tau$ then change according to

\[
\begin{align*}
A &= [1, 0]: K_A = M_{1,0}^{-1} = T^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \\
B &= [1, -1]: K_B = M_{1,-1}^{-1} = ST^2 = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}, \\
C &= [1, 1]: K_C = M_{1,1}^{-1} = T^2S = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix},
\end{align*}
\]

(2.13)

where $S$ is the matrix

\[
S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

We shall adopt the convention that for a given arrangement of the branes and their cuts, we shall write the corresponding matrices $K_i$ in the order (from right to left) in which we would cross the branch cuts as we encircle the relevant branes in a large anticlockwise circle, and we shall use the same convention also for the labels $A$, $B$ and $C$. For example, the list of branes

\[
A_1A_2 \cdots A_n
\]

(2.14)

represents a situation where the branes lie below the real axis, and all the branch cuts go upwards along the imaginary axis after crossing the real axis at points $x_{A_1} < x_{A_2} < \cdots < x_{A_n}$ \footnote{In this configuration, $z_0 = \infty$.} The corresponding $K$-matrices then read $K_{A_1}K_{A_2} \cdots K_{A_n}$. 

\[\text{In this configuration, } z_0 = \infty.\]
2.3 Moving branch cuts across branes

Let us now explain how the $K$-matrices of the branes change as we change the location of the cuts. First of all, it is clear that the $K_i$ are locally independent of the choice of the cuts, and that the matrix $K_i$ associated to the $i$-brane only changes as the cut of another 7-brane is moved through the position $z_i$ of this 7-brane.

For definiteness, let us consider the case where we move the cut $C_j$ of the $j$-brane clockwise through the position $z_i$ of an $i$-brane. This operation can be implemented by performing a physically immaterial local $\text{SL}(2,\mathbb{Z})$ transformation where we let $\tau \to K_j \tau$ in the region enclosed by $C_j$ and $C'_j$. (Indeed, this transformation makes $\tau$ continuous across $C_j$ and creates a new discontinuity across $C'_j$.)

Let us consider the situation where the cut $C_i$ that emerges from $z_i$ is inside the region that is defined by $C_j$ and $C'_j$. Before we moved the cut, $\tau$ transformed to $K_i \tau$ as we crossed $C_i$ anti-clockwise, and after we have moved $C_j$ to $C'_j$, we get $K_j \tau \to K_j K_i \tau = K_j K_i (K_j K_i)^{-1}$, implying that the $K_i \to K'_i \equiv K_j K_i K_j^{-1}$ as a consequence of the (clockwise) motion of $C_j$ through the $i$-brane; this is illustrated in Fig. 3. The shaded region denotes the region defined by the old and new cuts.

![Figure 3: The local SL(2, Z)-transformation induced by $K_j$ in the shaded area relocates the cut of the $j$-brane and changes the $i$-brane to an $i'$-brane. In the standard presentation the right picture reads $i'j$ as $C'_j$ is to the right of $C_i$ and thus we have $ji \to i'j$.](image)

The whole effect can be described as

$$j \ i \to i' \ j, \ \text{with} \ \ K'_i = K_j K_i K_j^{-1}. \hspace{1cm} (2.15)$$

This transformation respects, as expected, the product of the $K$-matrices that is associated with the transformation of $\tau$ along a large circle surrounding the pair of branes

$$K_j K_i = (K_j K_i K_j^{-1}) K_j = K'_i K_j.$$
Figure 4: The effect of a local $SL(2, \mathbb{Z})$-transformation induced by $K_A$ in the light and by $K_A^2$ in the dark shaded area results in the relocation of both $A$-cuts and the change of the $C$-brane into a $B$-brane: $AAC \rightarrow BAA$.

As an illustration of this, let us consider the configuration $AAC$ depicted in Fig. 4, and move successively the cuts of the $A$ branes across the $C$ brane. In this process we get $K_C \rightarrow K_A^2 K_C K_A^{-2} = K_B$, as is readily verified using (2.13). We can therefore write

$$AAC \rightarrow BAA.$$  \hspace{1cm} (2.16)

Similarly we have

$$ACB \rightarrow CA'B \rightarrow CBA,$$  \hspace{1cm} (2.17)

where $K_{A'} = K_C^{-1} K_A K_C$ as follows from moving the cut of $C$ anti-clockwise across $A$. The final step arises by moving the cut of $B$ anti-clockwise across $A'$ on account of $K_B^{-1} K_A' K_B = K_A$. We thus learn that the $A$ brane “commutes” with the $BC$ sequence.

In general, given a basepoint $z_0$ and contours $\gamma_i$, we can first choose the cuts $C_i$ as described in the previous subsection so that $K_i = M_i^{-1}$. We can then obtain any other choice of cuts by successively moving the cuts through the singularities. It is then clear that the matrices $K_i$ that are associated to a general family of cuts are of the form $K_i = D_i M_i^{-1} D_i^{-1}$ where $D_i \in SL(2, \mathbb{Z})$.

It is now also clear that the prescription that a $(p \ q)$ string can end on a brane with $K = M_{p,q}^{-1}$ is independent of the choice of the cuts. As we move the cut corresponding to $K_j$ clockwise through the brane, $K$ becomes $K' = K_j K K_j^{-1} = M_{p',q'}^{-1}$, where $(p' \ q') = K_j (p \ q)$. On the other hand, $(p \ q)$ becomes $(p' \ q')$, and it follows that the prescription is invariant as we change the cuts.

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4In the figures of the present paper we denote the $A$-branes by filled circles, the $B$-branes by empty circles and the $C$-branes by empty squares.
3 Resolutions, Singularities and the Moduli space of constant $\tau$

The dynamics of $(p, q)$-strings in a general axion-dilaton background is quite nontrivial. Although the metric is locally flat, the coordinate in which geodesics are straight lines is different for mutually nonlocal strings, and the transition functions between these coordinates are rather complicated. In section 5 we shall examine the general case but before doing so we want to analyze the problem for the special branches of the moduli space where $\tau$ is constant over $S^2$. As we shall see, all the important features of the problem will already be visible in this case.

The subset of the moduli space where $\tau$ is constant has been analyzed before by Sen [17], who considered only the case of the $D_4$ singularity, and by Dasgupta and Mukhi [18], who showed that there are two additional branches. We will briefly review their work, and then give brane descriptions of the relevant singularities that are necessary to describe points of enhanced gauge symmetry. It will turn out that two types of branes are sufficient to describe the general situation. We shall also see that there exist interesting partial resolutions of the $D_4$, $E_6$, $E_7$ and $E_8$ singularities at constant $\tau$; these will prove very useful for our explicit analysis of transitions.

The most efficient way to describe $\tau$ as a function of $z$, is to consider the torus bundle over $S^2$ that defines the K3 surface and that is determined by the function

$$y^2 = x^3 + f(z) x + g(z). \quad (3.1)$$

Here $f$ and $g$ are polynomials in $z$ of degree eight and twelve, respectively, and for each fixed $z$, (3.1) defines a torus whose modular parameter $\tau$ can be defined implicitly as

$$j(\tau(z)) = \frac{(24f(z))^3}{(4f(z)^3 + 27g(z))^2}, \quad (3.2)$$

where $j(\tau)$ is the standard $j$ function.

It is not difficult to see that $\tau$ is constant if $f^3 \sim g^2$ [17], or if $g = 0$ (branch I) and $f = 0$ (branch II) [18]. On the former branch, $g(z) \equiv 0$, and $f(z)$ is an arbitrary polynomial of degree eight whose zeros coincide with the zeros of the discriminant $\Delta(z) = 4f(z)^3 + 27g(z)^2$

$$f(z) = \prod_{i=1}^8 (z - z_i), \quad \Delta(z) = 4 \prod_{i=1}^8 (z - z_i)^3. \quad (3.3)$$

On this branch $\tau \equiv i$, and each of the eight zeros corresponds to a singularity of type $A_1$ with fiber type III [6]; this gives rise to the enhanced symmetry $SU(2)^8$ at a generic point. In the type IIB picture there are 24 7-branes altogether, and thus we have to

---

5Here and in the following, we do not write the relevant $U(1)$ factors.
have three branes at each of the eight singularities. Since the gauge algebra is $A_1$ (rather than $A_2$) we have to conclude that two of the three branes are of the same type, and that the third is relatively nonlocal to the former two; one solution is given as

$$\text{AAC} : \quad K_A K_A K_C = S ,$$

which is indeed compatible with $\tau = i$, since this is a fixed point of $S$. The resulting metric is flat except at the singularities, which are in this case conical, each with a defect angle of $\pi/2$.

On branch II, $f(z) \equiv 0$ and $g(z)$ is an arbitrary polynomial of degree twelve whose zeros coincide again with those of the discriminant

$$g(z) = \prod_{i=1}^{12} (z - z_i), \quad \Delta(z) = 27 \prod_{i=1}^{12} (z - z_i)^2 .$$

On this branch $\tau \equiv \exp(i\pi/3)$, and each of the twelve zeros corresponds to a fiber type II $[6]$, which does not give rise to an enhanced gauge symmetry. In the type IIB picture, each of the twelve singularities are therefore realized by two relatively non-local 7-branes; a solution can be given as

$$\text{AC} : \quad K_A K_C = TS ,$$

which is consistent with constant $\tau \equiv \exp(i\pi/3)$ since this is a fixed point for the transformation $TS$. In this case the singularities define conical singularities with a defect angle of $\pi/3$.

We can also understand from this point of view how the various other enhanced symmetries arise as some of the singularities coincide. Let us first consider branch I: as two of the AAC collapse, the singularity gets enhanced from $A_1 \times A_1$ to $D_4$. Indeed, in terms of the branes,

$$\text{AAC} \rightarrow \text{AACBAA} \rightarrow \text{CBAAAA} ,$$

which is recognized as the standard brane description of the $D_4$ singularity. In the first and second steps we used (2.16) and (2.17), respectively.

When three bunches collide, the singularity becomes of $E_7$ type $[4]$. Using (3.7) this can be represented as

$$\text{AAC AAC} \rightarrow \text{AAC CBAAAA} \rightarrow C'' C'' B'' AAAAAA ,$$

where $K_{C''} = K_A^2 K_C K_A^{-2}$ and $K_{B''} = K_A^2 K_B K_A^{-2}$. We thus recover (up to an irrelevant global $SL(2, \mathbb{Z})$ transformation taking $C'' \rightarrow C$, $B'' \rightarrow B$ and $A \rightarrow A$) the picture that was used in $[4]$. The coincidence of four bunches leads to $\text{ord}(f) = 4$, which according to $[5]$ destroys the triviality of the canonical bundle. It is not difficult to list all the
possible enhanced gauge symmetries that can be found in this way on branch I

\[ SU(2)^8 \quad E_7 \times SU(2)^5 \quad E_7 \times E_7 \times SU(2)^2 \]
\[ SO(8) \times SU(2)^6 \quad E_7 \times SO(8) \times SU(2)^3 \quad E_7 \times E_7 \times SO(8) \]
\[ SO(8)^2 \times SU(2)^4 \quad E_7 \times SO(8)^2 \times SU(2) \]
\[ SO(8)^3 \times SU(2)^2 \]
\[ SO(8)^4 \]

On branch II, the coincidence of two singularities gives rise to a fiber type IV, giving an \( A_2 \) singularity \([6]\). From the point of view of the explicit branes this corresponds to

\[ \text{ACAC} \rightarrow \text{AACA} \rightarrow \text{BAAA} . \]  

(3.9)

The first step corresponds to moving the cuts of the left \( \text{ACA} \) block to the right of the last \( \text{C} \) brane in three steps, one at a time. This turns the \( \text{C} \) brane into an \( \text{A} \) brane by virtue of \( K_A K_C K_A K_C^{-1} K_A^{-1} = K_A \). In the second step we simply used \((2.16)\).

Three coinciding singularities again give rise to a \( D_4 \) singularity whose usual representation can now be recovered as

\[ \text{AC ACAC} \rightarrow \text{AC BAAA} \rightarrow \text{CBAAAA} , \]

where we have used \((3.9)\) in the first step, and \((2.17)\) in the second step. Four coinciding singularities result in an \( E_6 \) singularity. Using \((3.10)\) this can be understood as

\[ \text{AC ACACAC} \rightarrow \text{AC CBAAAA} \rightarrow \text{C'C'B'AAAAA} , \]

with \( K_{C'} = K_A K_C K_A^{-1} \), and \( K_{B'} = K_A K_B K_A^{-1} \). This can be recognized as the description used in \([4, 4]\). Finally, when five singularities coincide, we get an \( E_8 \) singularity. Using \((3.9)\) and \((3.10)\) this corresponds to

\[ \text{ACAC ACACAC} \rightarrow \text{AACA CBAAAA} \rightarrow \text{AA CCB AAAAA} \rightarrow \text{C''C''B'' AA AAAAA} . \]  

(3.11)

This reproduces the description of \([4, 4]\). The allowed symmetry enhancements on branch II are therefore

\[
\begin{array}{cccc}
0 & SO(8) & SO(8)^2 & SO(8)^3 \\
SU(3) & SO(8) \times SU(3) & SO(8)^2 \times SU(3) & SO(8)^3 \times SU(3) \\
SU(3)^2 & SO(8) \times SU(3)^2 & SO(8)^2 \times SU(3)^2 & \\
SU(3)^3 & SO(8) \times SU(3)^3 & SO(8)^2 \times SU(3)^3 & \\
SU(3)^4 & SO(8) \times SU(3)^4 & & \\
SU(3)^5 & SO(8) \times SU(3)^5 & & \\
SU(3)^6 & SO(8) \times SU(3)^6 & & \\
\end{array}
\]
From the point of view of F-theory, the relevant BPS states correspond to supersymmetric cycles in an elliptically fibered K3 whose self-intersection numbers are \(-2\). In type IIB theory these states are described by smooth geodesics or possibly string junctions on the base space of the elliptic fibration (a two-sphere) whose metric is given by \((2.10)\). \(h_{p,q}(z)\) is an analytic function of \(z\) save for possible poles at the locations of the 7-branes, and the metric is therefore flat except at the locations of the poles.

The relevant smooth geodesics in type IIB begin and end at the position of a 7-brane, and, as they are smooth, must avoid the singularities of the other 7-branes. Each such geodesic defines therefore an element of a homotopy class. The property to be geodesic means that its length is minimal in the homotopy class of curves it defines. The computation of the length may require the use of several metrics, as the \((p, q)\) labels of the string can change whenever it crosses a cut. Given a fixed set of branch cuts, we can distinguish between homotopy classes associated to direct strings, containing representatives that do not cross branch cuts, and homotopy classes associated to indirect strings, where all representatives must necessarily cross branch cuts.

Given a homotopy class with two fixed endpoints, we can analyze whether a smooth geodesic exists in this class. If this is not the case, we can ask whether there exists a geodesic string junction or some other geodesic that corresponds to the same cycle in K3. (In particular, the configurations that can be obtained by brane crossings always define the same cycle in K3.) Here a geodesic three-string junction is a trivalent graph, where the three prongs are smooth and end on 7-branes. The total length of the geodesic junction is defined as the sum of the lengths of the prongs. It is not difficult to show that in order for the string junction to be geodesic (i.e. of minimal length), the familiar force balance condition [3, 10] must be satisfied at the junction vertex.

We shall find (at least in the examples we consider) that for the cycles with self-intersection number \(-2\), precisely one of the possible geodesics objects is smooth and

\[
\begin{align*}
E_6 & \quad E_6 \times SO(8) & \quad E_6 \times E_6 & \quad E_6 \times E_6 \times E_6 \\
E_6 \times SU(3) & \quad E_6 \times SO(8) \times SU(3) & \quad E_6 \times E_6 \times SU(3) & \quad E_6 \times E_6 \times SU(3) \\
E_6 \times SU(3)^2 & \quad E_6 \times SO(8) \times SU(3)^2 & \quad E_6 \times E_6 \times SU(3)^2 & \quad E_6 \times E_6 \times SO(8) \\
E_6 \times SU(3)^3 & \quad E_6 \times SO(8)^2 & \quad E_6 \times E_6 \times SO(8) \\
E_6 \times SU(3)^4 & \quad E_6 \times SO(8)^2 \times SU(3) & \quad E_6 \times E_6 \times SU(3)^3 & \quad E_6 \times E_6 \times SO(8)^2 \\
E_8 & \quad E_8 \times SO(8) & \quad E_8 \times E_6 & \quad E_8 \times E_8 \\
E_8 \times SU(3) & \quad E_8 \times SO(8) \times SU(3) & \quad E_8 \times E_6 \times SU(3) & \quad E_8 \times E_8 \times SU(3) \\
E_8 \times SU(3)^2 & \quad E_8 \times SO(8) \times SU(3)^2 & \quad E_8 \times E_6 \times SO(8) & \quad E_8 \times E_8 \times SO(8) \\
E_8 \times SU(3)^3 & \quad E_8 \times SO(8) \times SO(8)
\end{align*}
\]
thus represents the BPS state in this configuration. This confirms the proposal of [4] that the conventional geodesics are not the relevant BPS states in the whole of the moduli space; it also reaffirms indirectly, via a string of dualities, the Hanany-Witten effect [15] (see also [22]), and the string creation effect in the D0-D8 brane system [23].

In this section we use the resolutions of the various gauge enhancement points (that have been discussed in the previous section) for which $\tau$ is constant. This will simplify the analysis considerably, as there exists then a coordinate in which all $(p, q)$ geodesic strings are straight lines. We shall first consider the resolution of $so(8)$ into two bunches of 7-branes, and we shall see that an indirect $A-A$ geodesic is in fact realized as a degenerate three string junction. Much of the technology shall be developed through this example. We shall then turn to the case of a three singularity resolution of $so(8)$ where the transitions between a direct $A-A$ geodesic, the corresponding $A-A-C$ geodesic junction, and the associated indirect $A-C$ strings can be seen very explicitly. We also discuss briefly other examples. The main lesson of this section is that in the appropriate coordinate the total length of a string junction is given by the length of a mathematical straight line joining the positions of the initial and final 7-branes.

4.1 Indirect strings in a two-singularity resolution of $so(8)$

Let us first consider the $so(8)$ singularity which we resolve at constant $\tau = i$ as $(AAC BAA)$. For definiteness, let us place the $AAC$ group at $z_0$ and the $BAA$ at $z_1$, where both $z_0 = x_0$ and $z_1 = x_1$ are real and $x_0 < x_1$ (Fig. 5(a)).

![Figure 5: Two-singularity resolution of so(8). (a) The groups of three branes are located at the real points $z_0$ and $z_1$. (b) The singularities are mapped to 0 and $w_1$ while the whole $z$-plane is mapped to the non-shaded region of the $w$-plane.](image)

The metric is defined as

$$h_{p,q}(z) = C(p - iq)(z - z_0)^{-\frac{1}{4}}(z - z_1)^{-\frac{1}{4}},$$

16
where $C$ is some constant that does not depend on $(p, q)$. All strings feel the same identical metric up to an overall $(p, q)$ dependent constant, and thus all geodesics are necessarily along the same trajectories. The metric is flat and has conical type singularities of deficit angle $\frac{\pi}{2}$ at $z_0$ and $z_1$.

Let us take $(p, q) = (1, 0), C = \exp(i\pi/4)$, and let us draw a branch cut running horizontally to the left from $z_0$ with angles defined by $-\pi \leq \theta_0 \leq \pi$ on the sheet, and a second cut running horizontally to the right from $z_1$ with angles defined by $0 \leq \theta_1 \leq 2\pi$. Let us introduce a new coordinate $w$ via

$$w(z) = \int_{z_0}^{z} \frac{e^{i\pi/4} \, dz'}{[(z' - x_0)(z' - x_1)]^{1/4}}. \quad (4.1)$$

By construction it is clear that in the Euclidean $w$-plane, geodesics are represented by straight lines. The singularity at $z_0$ maps to a singularity at $w = 0$, and the two sides of the branch cut emanating from $z_0$ are mapped to rays with arguments $\pm 3\pi/4$ in the $w$-plane (see Fig. 5(b)). The other singularity at $z_1$ is mapped to $w_1$ which is real because

$$w_1 = \int_{x_0}^{x_1} \frac{dx}{[(x - x_0)(x_1 - x)]^{1/4}}.$$

The two sides of the cut originating at $z_1$ are mapped to the rays departing from $w_1$ with arguments $\pi/4$ and $2\pi - \pi/4$. It is then clear that the image of the whole cut $z$-plane is the part of the $w$-plane, where appropriate wedges with $|\theta_0| > 3\pi/4$ at $w = 0$, and $|\theta_{w_1}| \leq \pi/4$ at $w_1$ have been removed; this is depicted in Fig. 5(b). There is an implicit identification of the rays representing the boundaries of the shaded regions, but this should not be taken to mean that the metric is simply an ordinary conical metric; in fact, the type of string changes as it crosses the seam described by the identification.

The structure of the $w$-plane could have been guessed directly from inspection of (4.1), and the only non-trivial information that the map carries is a determination of $w_1$ in terms of $x_1$ and $x_0$. As we consider $w_1$ to be an adjustable parameter, its relation to the $z$-plane parameters is not important for us and we can work directly in the $w$ plane. This will also be the case for metrics with three singular points.

We can consider resolving the two bunches of branes further into individual branes (see Fig. 6(b)), so that we can distinguish more clearly the various types of geodesics, but this is not compatible with the assumption of constant $\tau = i$. We shall therefore use this further resolution only as a formal tool, with the implicit understanding that we have to take the limit in which the individual branes collapse to the above bunches. We want to show that (in this limit) the indirect $A$-$A$ geodesic is not smooth and that the associated three-string geodesic junction $A$-$A$-$C$ has total length which is smaller than that of the indirect $A$-$A$ geodesic by a finite amount proportional to the distance between the two bunches. This should be sufficient to guarantee that the three-string junction remains the actual BPS state for a small resolution of the bunches into individual branes.
Figure 6: (a) The two-singularity resolution of $\text{so}(8)$. (b) An indirect $\text{A-A}$ string. (c) The corresponding three-pronged junction. When the branes eventually collapse into two punctures the $(0_1^1)$ and the $(1_0^1)$ prongs degenerate.

The indirect $\text{A-A}$ string departs from each of two $\text{A}$-branes in the $\text{AAB}$ bunch as a $(1_0^1)$ string, and it is a $(1^1_0)$ and $(0^1_0)$ string, respectively in between the bunches (see Fig. 6(b)). As the bunches collapse, the top and bottom geodesic strings are straight lines from one bunch to the other, and the total length is thus the sum of the two contributions

$$ |\text{A-A}| = l_{1,0} + l_{0,1} = 2w_1. $$

This path is not a smooth geodesic: as the path goes through the singularity at $z_0$ it reverses direction.

This geodesic should be compared with the junction that corresponds to the same cycle in K3. This junction arises as the $\text{A-A}$ string cuts through the $\text{C}$ brane in the left singularity producing a $\text{C}$ prong, i.e. a $(1_1^1)$ string (see Fig. 6(c)). As the resolved branes recollapse into the two separate bunches we get $(1_0^1)$ and $(0_1^1)$ prongs of negligible length, but a long $(1_1^1)$ prong connecting the two bunches, and thus

$$ |\text{A-A-C}| = \sqrt{2} w_1, $$

where the $\sqrt{2}$ arises as the absolute value of the $(1 - i)$ factor in the metric. It thus follows that

$$ \frac{|\text{A-A-C}|}{|\text{A-A}|} = \frac{1}{\sqrt{2}}, $$

and we have verified that the geodesic junction has lower length than the non-smooth open string geodesic. Because of the above arguments this result should also hold when the two bunches are resolved infinitesimally.
4.2 Transitions for a three singularity resolution of so(8)

In the above example, the 3-pronged string was somewhat degenerate in that two of the prongs had vanishing length in the limit where the branes collapsed to bunches. We shall now consider examples where this does not happen. To this end we have to consider resolutions (at constant $\tau$) into at least three bunches of branes. The simplest case already occurs (perhaps slightly surprisingly) for so(8) which can be resolved as

$$\begin{array}{ccc}
AC & AC & AC \\
\ P_1 & Q & P_2 \\
\end{array}$$

where $\tau = \exp(i\pi/3)$ (see Fig. 7). We expect that as long as the middle singularity is above the line between the two other singularities, there exists a smooth geodesic representing a direct $A$-$A$ string connecting the right and left bunches of branes. As the middle singularity is moved downwards, we expect that the smooth (direct) geodesic eventually develops a corner, and that there is a transition to a geodesic three-pronged junction whose different prongs end on the three singularities.

![Figure 7: The direct $A$-$A$ string and the corresponding three-pronged $A$-$A$-$C$ junction in the three-singularity resolution of so(8).](image)

It is again useful to resolve the three bunches further in order to identify the types of strings (Fig. 8). The direct string is a $(1,0)$ string that avoids any branch cut, and the three-pronged string starts as a $(1,0)$ string on the right $A$-brane, emits a $(1,1)$-string (that ends on the middle $C$ brane), and thus becomes a $(0,-1)$ string. It then crosses the $AC$-cut thereby becoming a $(1,0)$ string that can end on the $A$-brane of the left singularity (avoiding the leftmost $C$-cut). The configuration in the $z$-plane is depicted in Fig. 8(a) where the location of the three bunches are labeled as $P_1$, $Q$ and $P_2$, the position of the junction is denoted by $S$, and $B$ and $A$ are the points to the right and left of the branch cut emanating from $Q$.

As before we introduce the coordinate

$$w(z) = \int_{z_{P_2}}^{z} h_{1,0}(z')dz', \quad h_{1,0}(z) = C(z - z_{P_1})^{-\frac{1}{3}}(z - z_{P_2})^{-\frac{1}{3}}(z - z_{Q})^{-\frac{1}{3}}, \quad (4.2)$$

that is appropriate for $(1,0)$ strings. The two sides of the cut emanating from $Q$ become rays that enclose an angle of $60^\circ$, and the image of the cut $z$-plane is the $w$-plane where
the shaded region has been removed (see Fig. 8(b)). As \( \tau = e^{i\pi/3} \) we have that \( T_{1,0} = T_{1,1} = T_{0, -1} \), and thus the appropriate length for the corresponding strings coincides with their actual length in the \( w \)-plane (up to an overall immaterial factor).

\[
\begin{align*}
\text{(a)} & \quad & \text{(b)} \\
& & \\
& & \\
& & \\
& & \\
& & \\
\end{align*}
\]

Figure 8: (a) Three-pronged junction in the \( z \)-plane. (b) Three-pronged junction in the \( w \)-plane. Since \( \tau = e^{i\pi/3} \), the \( w \)-plane represents faithfully the effective length of the prongs. The total mass of the junction is given by the length of the straight segment \( P_1P_2 \).

As long as the \( P_1P_2 \) line segment in the \( w \)-plane avoids the shaded region, the corresponding geodesic is smooth. If this is not the case (as shown in Fig. 8(b)), the shortest path that avoids the shaded region (and thus corresponds to an element in the same homotopy class as before) goes through the singularity at \( Q \), and thus does not represent a smooth geodesic. On the other hand, the three-string junction that is depicted in Fig. 8(b) is a smooth geodesic and its overall length is strictly smaller than the length of the above path. Let us justify this claim in some detail.

First, in order to see that the diagram represents a geodesic junction we observe that (i) the angles at the junction \( S \) are all 120°, and (ii) all prongs are straight lines. (This is obvious for the \( SQ \) and \( SP_2 \) prongs; as the defect angle at \( Q \) is 60°, the \( SP_1 \) is prong is a straight line provided that the \( AP_1 \) line when rotated by 60° to \( BP_1' \) forms a straight line with the first portion \( SB \).)

In fact, given \( Q, P_2 \) and \( P_1' \), there exists at most one point \( S \) for which these conditions are satisfied, and this point can be found as follows. The points \( X_1 \) for which \( \angle QX_1P_1' = 120^\circ \) form an arc (which goes through \( Q \) and \( P_1' \)), and so do the points \( X_2 \) for which \( \angle QX_2P_2 = 120^\circ \). \( S \) is therefore the (unique) intersection of the two arcs. (The intersection point may not exist, in which case the junction does not exist, see below.) The line \( P_1'S \)

---

The precise form of the shaded region depends on the choice of the branch cut that emanates from \( Q \) in the \( z \)-plane.
determines then the point $B$, and thus also $A$.

Next, we want to determine the overall length of this (unique) junction. As the triangle $QP_1P'_1$ is equilateral, $P_1$ is on the arc over $QP'_1$ with angle $60^\circ$, and $S$ is on the arc over $QP'_1$ with angle $120^\circ$, and thus the two arcs form together an actual circle. This implies that both $Q$ and $S$ lie on the same arc over $P_1P'_1$ with angle $60^\circ$, and therefore that the angles $\angle P_1SP'_1 = \angle P_1QP'_1 = 60^\circ$. It then follows that the points $P_1$, $S$ and $P_2$ define a line, and thus that the $P_1A$ and $P_2S$ prongs lie on the $P_1P_2$ line segment.

Let us denote by $S'$ the image of $S$ after an counterclockwise rotation by $60^\circ$ around $Q$. As this rotation maps $B$ to $A$, it then follows that $AS'Q$ and $BSQ$ are identical triangles. Finally, since $QSS'$ is equilateral, the distance $SQ$ equals $SS'$, and we therefore find that the length of the $P_1P_2$ line segment is equal to the true length of the junction. In particular, this implies that the length of the junction (if it exists) is strictly lower than that of the broken direct geodesic. Conversely, if the direct geodesic is smooth, $Q$ must lie above the $P_1P_2$ segment, and the above construction leads to two arcs that do not intersect, and the geodesic junction does not exist.

It is actually possible to establish the existence of the junction directly by choosing the shaded region to lie symmetrically with respect to the line passing through $Q$ and perpendicular to $P_1P_2$. (This choice corresponds to a certain choice for the location of the branch cut that emanates from $Q$.) In this case, the length of the $(0,-1)$ string is zero, and there are no broken segments in the picture (Fig. 9). It is then very easy to visualize the open string/junction transition. Let us fix the points $P_1$ and $P_2$ on the $w$-plane and suppose that $Q$ is somewhere above the $P_1P_2$ segment as in Fig. 10(a). As $Q$ is moved downwards and crosses the $P_1P_2$ line, the would-be geodesic string is transformed into a geodesic junction (Fig. 10(b) and (c)). It should be stressed that the

![Figure 9](image-url)
mass of the representative of this BPS state is unchanged in the process, as it is given by $\mathcal{T}_1\mathcal{T}_2$.

Figure 10: For fixed $P_1$ and $P_2$, a prong is created as the third puncture $Q$ crosses the $P_1P_2$ line segment.

This argument covers most of the possible configurations, but there exists yet another region where $Q$ is below the $P_1P_2$ segment (so that the direct geodesic has a corner), but yet the angle $\angle P_1'P_2Q > 120^\circ$, and thus we cannot find a satisfactory junction $S$ as the relevant arcs do not intersect (Fig. 8(b)). This configuration can be reached from the three-string configuration discussed above as $P_2$ approaches $S$, and the length of one of the prongs of the junction vanishes. We want to demonstrate next that as $P_2$ crosses $S$, the junctions turns into an indirect C-A string that runs from $Q$ to $P_1$ (Fig. 11).

Figure 11: The indirect A-C string. The junction of Fig. 7 can be obtained by moving the string across the rightmost singularity.

In order to discuss this configuration, it is necessary to draw also the branch cut that emanates from $P_2$, which in the $w$-plane again corresponds to a wedge of angle $60^\circ$ that is drawn as a shaded region in Fig. 12. Let $P_1'$ be the image of $P_1$ after a rotation about $Q$ by $60^\circ$, and let $P_1''$ and $Q'$ be the images of $P_1'$ and $Q$ after a rotation about $P_2$ by $60^\circ$ and $-60^\circ$, respectively. The relevant (indirect) geodesic is depicted in heavy lines, and it is manifest that it does not have any corners. By rotating the line segments about $P_2$ and $Q$, it is easy to see that the total length of the string equals $P_1Q'$. Finally, since the
two triangles $P_1Q_2P_2$ and $P_1'Q_2Q'$ are identical (as the triangle $P_2QQ'$ is equilateral), it follows that the overall length is again equal to $P_1P_2$.

It should be noted that $\angle P_1'P_2Q > 120^\circ$ implies that $Q'P_1'$ crosses the excised region emanating from $P_2$, which is necessary for the charge conservation of the indirect string. (The indirect string has to cross the cuts from both $Q$ and $P_2$, as is immediate from Fig. 11.) Completely analogous arguments hold also for the indirect $A-C$ string running from $Q$ to $P_2$.

![Figure 12: The indirect $A-C$ string is represented by three broken line segments in the $w$-plane. Charge conservation requires that it crosses both the $P_2$ and the $Q$-cut.](image)

We have thus found that depending on the configuration of the 7-branes, the actual BPS state is realized either as a direct $A-A$ string, as one of two indirect $C-A$ strings, or as an $A-A-C$ junction. Only one realization is smooth at any point of the moduli space (see Fig. 13), and its mass is always given in terms of the length of the straight line joining the two $A$-branes in the $w$-plane. As we shall demonstrate below (see section 5), similar results also hold for the case where $\tau$ is not constant.

### 4.3 Other three-singularity resolutions

We can also consider other resolution of singularities into three bunches of branes. For example, we can resolve the $E_6$ singularity at $\tau = \exp(i\pi/3)$ as

\[
\begin{array}{ccc}
AC & BAAA & AC \\
P_1 & Q & P_2
\end{array}
\]

The direct $A-A$ string and its junction counterpart are shown in Fig. 14(a). The defect angle associated with $Q$ is $120^\circ$ in this case, and the junction is shown in Fig. 14(b).
Figure 13: For fixed $P_1$ and $P_2$ we have depicted the different regions for $Q$ in the $w$-plane where different configurations are BPS. The shaded regions are the excised wedges associated to $P_1$ and $P_2$, and the arc through $P_1$ and $P_2$ is the boundary of the region where both $P_1$ and $P_2$ lie outside the wedge emanating from $Q$.

The angles at $S$ are $150^\circ$, $150^\circ$ and $60^\circ$, and the effective length of the $QS$ $(−1, 1)$-prong is indeed equal to the length of the portion of the line segment $P_1P_2$ inside the excised region, by virtue of $T_{-1,1} = \sqrt{3} T_{1,0}$. Again we see that the effective length of the junction equals the length of the straight mathematical line joining $P_1$ and $P_2$.

There is also a three singularity resolution of the ten 7-branes that make up the $E_8$ singularity as $\text{AC ACBAAA AC}$, where $\tau = \exp(i\pi/3)$. The middle singularity has a defect angle of $180^\circ$, and thus the direct strings always remain geodesics.

On the branch $\tau = i$, there exists a resolution of the $E_7$ singularity as (see Fig. 13(a))

\[
\begin{align*}
\text{AAC} & \quad \text{AAC} & \quad \text{AAC} \\
P_1 & \quad Q & \quad P_2
\end{align*}
\]

The conical singularity at $Q$ now carries a defect angle of $90^\circ$, and the effective length of the junction is given again by the distance between $P_1$ and $P_2$, as $T_{1,1} = \sqrt{2} T_{1,0}$ for $\tau = i$ (Fig. 13(b)).

5 Geodesic junctions in general $\tau$-backgrounds

In the previous section we gave a geometrical interpretation of the length of geodesic prongs that are created as the corresponding open string geodesic develops corners. In
Figure 14: (a) Constant $\tau$ three-singularity resolution of $E_6$. (b) The conical singularity at $Q$ ($\text{BAAA}$) has defect angle $120^\circ$. The physical length of the $SQ$ prong of the geodesic junction is $\sqrt{3}(SQ)$ which is equal to the length of the portion of $P_1P_2$ inside the shaded area.

Figure 15: (a) Constant $\tau$ three-singularity resolution of $E_7$. (b) The conical singularity at $Q$ ($\text{AAC}$) has defect angle $90^\circ$. The physical length of the $SQ$ prong of the geodesic junction is $\sqrt{2}(SQ)$ which is equal to the length of the portion of $P_1P_2$ inside the shaded area.
particular, we considered open string geodesics between singularities $P_1$ and $P_2$ that fail to be smooth when a third singularity $Q$ crosses the line $P_1P_2$. Our main observation was that in the $w$-plane, the length of the $Q$-prong was represented as that part of the straight line segment connecting $P_1$ and $P_2$ which falls into the wedge that has been removed from the $w$-plane on account of the cut originating at $Q$ in the $z$-plane. We want to explain now that effectively the same mechanism also works in the case where $\tau$ is not constant.

Let us consider an $(r)^{th}$-string in the vicinity of a $[p,q]$-brane, to which it is compatible, say, $ps - qr = 1^\text{7}$ (see Fig. 16(a)): a $(p)^{th}$-string and a $(r)^{th}$-string meet at the junction, whose third prong has quantum numbers $(r+p)^{s+q}$. This prong crosses the cut of the $[p,q]$-brane counterclockwise and is transformed back to an $(r)^{th}$-string. Charge conservation requires

$$M_{p,q} \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} r + p \\ s + q \end{pmatrix},$$

which holds if $ps - qr = 1$.  

\footnote{This compatibility condition is equivalent to demanding that the associated process of brane crossing creates a single $(r)^{th}$ prong.}
Although all \((p, q)\) metrics are flat away from branes, when \(\tau\) is not constant there is no coordinate in which all \((p)\) strings are straight lines. Given that two of the prongs ultimately are \((r)\) strings it is useful to introduce the \(w\)-coordinate where such strings are straight lines

\[
    w(z) = \int z h(z) dz .
\]  

(5.2)

In general, the geodesic of the \((p)\)-prong is then a curved line in this \(w\)-plane, whose explicit form may be difficult to determine. However, as we shall show, its length can be easily visualized in the \(w\)-plane.

The \((r)\) metric and the corresponding \(w\) coordinate has a cut emanating from the \([p, q]\)-brane (whose position we denote by \(z_0\) in the \(z\)-plane and \(w_0\) in the \(w\)-plane). The precise position of the cut is not physical, and we are free to choose it when defining \(w\). Suppose we are given already the physical junction. Then we can define the \([p, q]\)-cut to run along the geodesic \((p)\)-prong (Fig. 16(b)). Let the coordinate of the junction point on the \(z\)-plane be indicated by \(z_1\) when approached from below the cut, and by \(z_2\) when approached from above the cut \((z_1 = z_2)\) and let their images under (5.2) be \(w_1\) and \(w_2\), respectively (see Fig. 16(c)). Our claim is that the straight distance between \(w_1\) and \(w_2\) in the \(w\)-plane is in fact identical to the length of the \((p)\)-prong, as measured with the required \((p, q)\) metric. This is the generalized version of the result mentioned at the beginning of this section.

By definition we have

\[
    w_2 - w_1 = \int_{z_1}^{z_2} h(z) dz ,
\]  

(5.3)

where the path of the integration surrounds \(z_0\) in a clockwise direction. Let us choose this path so that it runs along the prong just below the cut from \(z_1\) to \(z_0\) and just above it from \(z_0\) to \(z_2\). Then (5.3) can be written as

\[
    w_2 - w_1 = \int_{z_1}^{z_2} h^-(z) dz - \int_{z_1}^{z_0} h^+(z) dz ,
\]  

(5.4)

where we parameterized both halves of the integration curve by the same variable (that runs from \(z_1\) to \(z_0\)), and the values of the metric on the two sides have been distinguished by the \(\pm\) superscripts. The discontinuity of the metric can be expressed in terms of the discontinuity of \(\tau\), which is given by the SL(2, \(\mathbb{Z}\))-matrix defining the \([p, q]\)-brane

\[
    w_2 - w_1 = \int_{z_1}^{z_0} \left( h^-(\tau, z) - h^-(M^{-1} p, q \tau, z) \right) dz .
\]  

(5.5)

From the SL(2, \(\mathbb{Z}\))-invariance of the metric (\(M\) is any SL(2, \(\mathbb{Z}\))-matrix),

\[
    h_M(\tau, z) = h_M(\tau, z)
\]  

(5.6)
it then follows that
\[ w_2 - w_1 = \int_{z_1}^{z_0} \left( h_{r s}^-(\tau(z), z) - h_{M_{p,q}}^-(\tau(z), z) \right) dz. \] (5.7)

Because of (5.1) and the linear dependence of the metric on the \( p, q \) labels, we have
\[ h_{M_{p,q}}^-(\tau_{s}) = h_{(r+s)}^- = h_{(r)}^- + h_{(q)}^- \] (5.8)
which leads to the final formula
\[ w_2 - w_1 = -\int_{z_1}^{z_0} h_{(q)}^-(\tau(z), z) dz. \] (5.9)

We have thus been able to relate the \( w \)-coordinate (which is tied to the \((r)\)-metric and is the convenient arena for the \((r'\))-strings) to the \((p)\)-metric which one uses to determine the length of the \((p)\)-prong. In particular the length of the geodesic \((p)\)-string between the junction point and \( z_0 \) is given as
\[ l_{(p)}^- = |\int_{z_1}^{z_0} h_{(q)}^-(\tau(z), z) dz| = |w_2 - w_1|, \] (5.10)
which is the true length of the line segment between \( w_1 \) and \( w_2 \) in the \( w \)-plane inside the shaded wedge. This generalizes our previous statement about the mass of the string junctions, and in effect guarantees that for those situations where a string junction exists, the corresponding direct string will be longer and will not be smooth.

6 On the absence of smooth indirect geodesics

In the previous sections we showed through several examples that one may arrange the 7-branes in the IIB moduli space such that the would-be BPS open strings fail to be smooth geodesics and are replaced by pronged objects. The purpose of this section is to show explicitly that open strings can even fail to be smooth geodesics in the familiar resolution of the \( so(8) \) singularity \( (D_4) \) which is related to the \( su(2) \) Seiberg-Witten theory with \( N_f = 4 \) \cite{24}. This result confirms once more that the necessity of pronged objects is not limited to the case of exceptional groups.

Consider the resolved \( so(8) \) singularity, but for the sake of simplicity let us restrict ourselves to the case where the four \( A \)-branes coincide. There are then three singular points on the sphere (the \( B \) brane, the \( C \) brane, and the four \( A \) branes), and the moduli space is that of the \( su(2) \) Seiberg-Witten theory with four flavors of equal masses \cite{24}.

\footnote{Our arguments here are related to and inspired by those of \cite{28} where it is shown that in \( su(2) \) SW-theory, realized as the world-volume theory of a three brane in the vicinity of two seven-branes, one needs a pronged object to represent the W-boson.}
Let \( a(z) \) and \( a_D(z) \) be the usual multivalued functions of the sphere which transform as an SL(2,Z) doublet, \( (a^D_a) \) under the monodromies of \( \tau \), where

\[
\frac{da_D(z)}{da(z)} = \tau(z).
\]  

(6.1)

In the conventions of section 2.2, \( (a^D_a) \) changes to \( (a^D_a) \to K(a^D_a) \) as we change a branch cut anti-clockwise.

The case when the B and C brane are on top of each other and the A’s are separated was studied in detail in [17, 4], and the BPS states were shown to be direct and indirect strings between the A-branes. The latter are those which start on one of the four A-branes and go to one of the other A-branes after encircling the conical singularity of the BC singularity. Since we are considering the limiting case where all four A-branes coincide, all indirect strings hit the apex, but the strings that begin and end on different A-branes become smooth geodesics as we resolve the four A branes infinitesimally, whereas the strings that begin and end on the same A brane continue to go through the apex. (From an F-theory point of view, these last geodesics do not correspond to cycles with self-intersection number \((-2)\); they have self-intersection number \((-4)\) and thus do not correspond to simple spheres.)

As the B and C branes are resolved, we may expect (and this was conjectured in [4]) that the indirect A-A strings that begin and end on different nearby A-branes do not necessarily remain smooth geodesics, but that some become string junctions. In the following we shall demonstrate that this seems indeed to be the case: when the two A branes coincide, there is no smooth geodesic in the homotopy class of the indirect string, irrespective of whether B and C coincide or not. As a consequence, when B and C are finitely separated we expect the indirect string to fail to be smooth for small separation of the two A branes.

Figure 17: The indirect A-A string can be described as two \((1_0)\)-strings departing \( z_0 \) which meet smoothly on both sides of the unified BC-cut.

In the homotopy class of the indirect A-A string, the \((1_0)\)-string crosses the branch cuts
of both the B and the C brane before returning as a \((-1)^{-1}\)-string. To simplify the analysis, let us choose the cuts for the B and the C so that they run on top of each other where possible as in Fig. [17]. The point \(P\) with coordinate \(z_0\) represents the location of the A branes, and the two oriented curves \(C_-\) and \(C_+\) that extend from \(P\) to the point \(Q\) from the bottom and top of the branch cuts, respectively, each represent part of the indirect path where the string is described as a \((-1)^0\)-string. In order for the complete loop \(C_+ - C_-\) to be a smooth geodesic, both \(C_+\) and \(C_-\) must be geodesic segments extending from \(P\) to \(Q\), and the two segments must join smoothly at \(Q\) (where \(Q\) should not coincide with any of the branes if the geodesic is to be smooth).

According to [24], and taking account of our conventions, in the coordinate

\[
dw_{p,q} = pda - qda_D ,
\]

\((-p)\) geodesic strings are straight lines. The curves \(C_+\) and \(C_-\), both being \((-1)_0\)-strings, are therefore straight lines in the coordinate \(w\), where \(w\) is defined as

\[
dw(z) = da(z).
\]

Let us now introduce parameters \(t_+, t_- \in [0, 1]\) for the lines \(w_+(t_+)\) and \(w_-(t_-)\) describing \(C_+\) and \(C_-\), respectively, and write

\[
w_+(t_+) = w_0 + t_+(w_+(1) - w_0) ,
\]

\[
w_-(t_-) = w_0 + t_-(w_- (1) - w_0) ,
\]

where \(w_0 = w(z_0)\) is the starting point of both segments. As the monodromy around the CB-system is

\[
K_C K_B = -T^4 = \begin{pmatrix} -1 & -4 \\ 0 & -1 \end{pmatrix} ,
\]

the values of \(a\) above and below the cut are related to each other as \(a_+(z) = -a_-(z)\). The condition that the two segments join smoothly at \(Q\) requires then that

\[
\left( \frac{dz}{dt} \right)_+ (Q) = -\alpha_0 \left( \frac{dz}{dt} \right)_- (Q) ,
\]

where the minus sign is due to the parameterizations used, and \(\alpha_0\) is a real positive constant. On the other hand using (6.3) and (6.4) we find

\[
\left( \frac{dz}{dt} \right)_+ = \frac{dw}{dt}_+ + \left( \frac{dx}{dz} \right)_+ = \frac{w_+(1) - w_0}{w_+(1) - w_0} \cdot \frac{dz}{dt}_+ - \left( \frac{da}{dz} \right)_- = \frac{w_+(1) - w_0}{w_- (1) - w_0} \cdot \frac{dz}{dt}_- .
\]

Comparing with (6.7) this implies that

\[
\text{Arg} \left( w_+(1) - w_0 \right) = \text{Arg} \left( w_- (1) - w_0 \right) ,
\]

30
and the two curves $C_+$ and $C_-$ must have exactly the same slope at their coincident origin $z_0$. A smooth geodesic, however, is uniquely determined by a initial point and its slope at this point, and thus the $C_+$ and $C_-$ curves in the $z$ plane coincide, unless they encounter a singularity. If the geodesics are to be smooth, they do not encounter any singularity, and it follows that these indirect geodesics cannot be smooth.

7 Conclusions and open questions

In this paper we have examined type IIB superstring compactified on a two-sphere in the background of parallel 7-branes. We have shown that the open string geodesics which represent BPS states in a given background arrangement of 7-branes may fail to be smooth as the background is changed. The corresponding state then does not disappear from the spectrum, but instead it is represented by a different object, a geodesic three-string junction or another open string. This property guarantees that the number of BPS states that correspond to gauge vectors in the limit of the enhanced symmetry is independent of the position in moduli space. It should also be emphasized that these multipronged objects are in general necessary to describe the BPS spectrum, and this phenomenon is relevant not only to exceptional gauge groups.

The mechanism by means of which the different geodesic objects transform into one another is most easily understood in the subspace of the moduli space where the modular parameter $\tau$ is constant. It is then possible to analyze the transitions by means of elementary geometry, and the existence and uniqueness of the representative is manifest. This gives an explicit and concrete illustration of (one of the versions of) the Hanany-Witten effect.

We also demonstrated that all essential features survive in the general case where $\tau$ is not constant and the system is much less tractable. It would be interesting to analyze this case further: in particular, the discontinuities in the BPS spectrum of states in four dimensional supersymmetric gauge theories that are described using 3-brane probes in the backgrounds of 7-branes \[26, 25\] may be understood using similar techniques. It would also be of interest to try to understand the spectrum of massive BPS states. Such states should presumably be represented by geodesics that remain of finite length as the branes collapse to form the singularities with enhanced gauge symmetry.

At a more conceptual level, our work suggests that in type IIB superstring theory, string junctions are not only necessary ingredients for the description of the theory, but are, in fact, on the same footing as the $(p,q)$ strings, as there exist transitions between them. A good understanding of the physics of the junctions may therefore guide the way to a proper formulation of type IIB superstring theory.
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