Balances of $m$-bonacci words

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December 11, 2013

Abstract

The $m$-bonacci word is a generalization of the Fibonacci word to the $m$-letter alphabet $A = \{0, \ldots, m-1\}$. It is the unique fixed point of the Pisot–type substitution $\varphi_m : 0 \to 01, 1 \to 02, \ldots, (m-2) \to 0(m-1),$ and $(m-1) \to 0$. A result of Adamczewski implies the existence of constants $c^{(m)}$ such that the $m$-bonacci word is $c^{(m)}$-balanced, i.e., numbers of letter $a$ occurring in two factors of the same length differ at most by $c^{(m)}$ for any letter $a \in A$. The constants $c^{(m)}$ have been already determined for $m = 2$ and $m = 3$. In this paper we study the bounds $c^{(m)}$ for a general $m \geq 2$. We show that the $m$-bonacci word is $(\lfloor \kappa m \rfloor + 12)$-balanced, where $\kappa \approx 0.58$. For $m \leq 12$, we improve the constant $c^{(m)}$ by a computer numerical calculation to the value $\lceil \frac{m+1}{2} \rceil$.

1 Introduction

The $m$-bonacci word is a generalization of the Fibonacci word to the $m$-letter alphabet $A = \{0, \ldots, m-1\}$. It is the unique fixed point of the substitution $\varphi = \varphi_m$ given by the prescription

$$0 \to 01, 1 \to 02, \ldots, (m-2) \to 0(m-1), \text{ and } (m-1) \to 0.$$ (1)

In particular, for $m = 3$, we obtain the substitution $0 \to 01, 1 \to 02, 2 \to 0$ with the fixed point

$$01020100102010102010010201020100102010102010010201020100102010102010010201020100\cdots,$$

usually called the Tribonacci word.

The aim of this article is to study a certain combinatorial property of the $m$-bonacci word for a general $m$. Namely, we examine the balance property, which describes a certain uniformity of occurrences of letters in an infinite word. In order to give its rigorous definition, let us precise the notation we will use in the sequel. A factor of an infinite word $u = u_0 u_1 u_2 \cdots \in A^\omega$ is any finite string in the form $w = u_i u_{i+1} \cdots u_{i+n-1}$ for certain $i \in \mathbb{N}_0$, $n \in \mathbb{N}$, where $|w| = n$ is the length of the factor $w$. The language of an infinite word $u$, denoted by $L(u)$, is the set of all its factors. The number of occurrences of a given letter $a \in A$ in a factor $w$ is denoted by $|w|_a$. Clearly, $\sum_{a \in A} |w|_a = |w|$. The balance property is related to the variability of $|w|_a$ within the meaning of the following definition.

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Definition 1. Let $c$ be a positive integer. An infinite word $u \in \mathcal{A}^\mathbb{N}$ is said to be $c$-balanced if
\[ |w|_a - |v|_a \leq c \]
for all factors $w, v \in \mathcal{L}(u)$ of the same length and for each letter $a \in \mathcal{A}$.

The notion of a 1-balanced word (originally referred to as “balanced word”) has been used by Morse and Hedlund already in 1940 [8] for a characterization of Sturmian sequences. Since the Fibonacci word (in our notation 2-bonacci word) is Sturmian, it is 1-balanced.

It was expected and announced in several papers since 2000 that the Tribonacci word is 2-balanced [5, 4, 13]. This statement has been proved in 2009 (in two different ways) by Richomme, Saari and Zamboni [11]. As for a general $m \geq 2$, in 2009 Glen and Justin [7] mentioned “the $k$-bonacci word is $(k-1)$-balanced”, but to the best of our knowledge, no proof of this proposition has ever been published.

The $m$-bonacci words belong to a broad class called Arnoux–Rauzy words. In the last ten years, balance properties of Arnoux–Rauzy words have been intensively studied. For the most recent results and a nice overview see [3].

The works of Adamczewski on discrepancy and balance properties of fixed points of primitive substitutions [1, 2] imply the existence of finite constants $c(m)$ such that the $m$-bonacci word is $c(m)$-balanced. Namely, Adamczewski proved that if all eigenvalues of the matrix of substitution except the dominant one are of modulus less than 1, then the fixed point of the primitive substitution is $c$-balanced for some $c$. It is well known (and explicitly shown in our text as well) that the substitution defined by (1) satisfies the Adamczewski condition.

In the present article, we approach the problem of determining $c(m)$ by refining the matrix method used by Adamczewski in [1, 2] (and also by Richomme, Saari, Zamboni in [11] in their Proof 2). Small values of $m$ can be treated numerically. We show that

- the 4-bonacci word and the 5-bonacci word are 3-balanced but not 2-balanced;
- for $m = 6, 7, \ldots, 12$ the $m$-bonacci word is $\lceil \frac{m+1}{2} \rceil$-balanced, Theorem 3.1.

The approach works for a general $m$ as well. We prove the following theorem.

Theorem. (Theorem 6.1.) The $m$-bonacci word is $c(m)$-balanced with
\[ c(m) = \lceil \kappa m \rceil + 12, \]
where $\kappa = \frac{2}{\pi} \int_0^{\pi} \frac{\cos x}{(1-\cos x)(\sin(5-\cos x))} \, dx \approx 0.58$.

Our results confirm the bound $c = m - 1$ proposed by Glen and Justin for all $m \leq 12$ and $m \geq 29$. Moreover, it turns out that the formerly proposed bound $c = m - 1$ is far from being optimal except for a few small values of $m$.

Our article is organized as follows: Section 2 explains relationship between balance and discrepancy and gives a formula estimating the balance constant using spectrum of the matrix $M$ of substitution (1). In Section 3 we present results obtained by computer evaluation of this formula. In Section 4 we show that for estimating the balance constant $c$ we can concentrate on the letter 0 only. Sections 5 and 6 are devoted to the proof of the main theorem. Our proof requires very detailed information about spectrum of the matrix $M$; in Appendix we use standard methods of calculus to describe this spectrum.

2 Balance property and discrepancy

This section describes the main idea that will be later applied to find for any letter $a \in \{0, \ldots, m-1\}$ upper bound on the letter balance constant
\[ c_a := \max\{|w|_a - |v|_a : v, w \in \mathcal{L}(u) \text{ and } |w| = |v|\}. \]

The derivation of these bounds uses the following two ingredients.
the $m$-bonacci sequence defined recursively

$$ T_0 = T_1 = \ldots = T_{m-2} = 0, \quad T_{m-1} = 1 $$

and

$$ T_n = T_{n-1} + T_{n-2} + \ldots + T_{n-m} \quad (2) $$

for any $n \geq m$;

• zeros $\beta \equiv \beta_0 > 1, \beta_1, \ldots, \beta_m$ of the polynomial

$$ p(x) = x^m - x^{m-1} - \ldots - x - 1. $$

It is well known that $p(x)$ is an irreducible polynomial, its root $\beta$ belongs to the interval $(1, 2)$, and the other roots (conjugates of $\beta$) are all of modulus less than 1. From now on, we order the roots $\beta_1, \ldots, \beta_m$ according to their arguments, i.e.,

$$ 0 \leq \arg(\beta_1) \leq \arg(\beta_2) \leq \cdots \leq \arg(\beta_m) < 2\pi. \quad (3) $$

The $m$-bonacci word is a fixed point of a primitive substitution. Therefore, density $\mu_a$ of any letter $a \in A$ is well defined and positive, i.e.,

$$ \mu_a = \lim_{n \to +\infty} \frac{|u[n]|_a}{n} > 0, $$

where $u[n]$ the prefix of $u$ of length $n$. We refer to [9], where the problem of letter densities is studied in detail.

The value $\mu_a$ can be interpreted in the way that the “expected” number of letters $a$ in the prefix $u[n]$ is $\mu_a n$. A simple consequence of the definition of $\mu_a$ is the following observation.

Observation 1. For any $\varepsilon > 0$ and for any positive integer $N$, there exist factors $v$ and $w$ in $L(u)$ such that

$$ |v| = |u| = N, \quad |w|_a \geq \mu_a N - \varepsilon \quad \text{and} \quad |v|_a \leq \mu_a N + \varepsilon. $$

Proof. Assume that there exist $\varepsilon > 0$ and $N \geq 1$ such that for any factor $w$ of length $N$, the inequality $|w|_a < \mu_a N - \varepsilon$ holds. It means that for the prefix of $u$ of length $n = kN$, we obtain $|u[n]| = |u[kN]|_a < (\mu_a N - \varepsilon)k$. This implies $\mu_a = \lim_{n \to +\infty} \frac{|u[n]|}{n} = \lim_{k \to +\infty} \frac{|u[kN]|}{kN} < \mu_a - \frac{\varepsilon}{N}$, which is a contradiction. The proof of existence of $v$ is analogous.

The difference between the expected and actual number of letters $a$ defines the discrepancy function $D_a : \mathbb{N} \to \mathbb{R}$:

$$ D_a(n) = |u[n]|_a - \mu_a n $$

for any $n \in \mathbb{N}$.

Lemma 2.1. For any letter $a$, denote

$$ \Delta_a := \sup_{n \in \mathbb{N}} D_a(n) - \inf_{n \in \mathbb{N}} D_a(n). $$

Then $\Delta_a \leq c_a \leq 2\Delta_a$.

Proof. Let $w, v \in L(u)$ be factors of the same length such that $c_a = |w|_a - |v|_a$. We can find prefixes $W$ and $V$ of $u$ such that $Ww$ and $Vv$ are prefixes of $u$ as well. Obviously

$$ |w|_a - |v|_a = |Ww|_a - |W|_a - |Vv|_a + |V|_a = D_a(|Ww|) - D_a(|W|) - D_a(|Vv|) + D_a(|V|) $$

$$ \leq 2 \sup_{n \in \mathbb{N}} D_a(n) - 2 \inf_{n \in \mathbb{N}} D_a(n) = 2\Delta_a. $$
To deduce the lower bound on $c_a$, let us choose $\varepsilon > 0$. There exist prefixes of $u$, say $u[n_1]$ and $u[n_2]$, such that $D_a(n_1) > \sup_{n \in \mathbb{N}} D_a(n) - \varepsilon$ and $D_a(n_2) < \inf_{n \in \mathbb{N}} D_a(n) + \varepsilon$, or equivalently

$$|u[n_1]|_a > \mu_a n_1 + \sup_{n \in \mathbb{N}} D_a(n) - \varepsilon,$$

$$|u[n_2]|_a < \mu_a n_2 + \inf_{n \in \mathbb{N}} D_a(n) + \varepsilon.$$

First suppose that $n_1 > n_2$ and put $N := n_1 - n_2$. Denote the suffix of $u[n_1]$ of length $N$ by $\hat{W}$. Then $\hat{W}$ contains at least $\mu_a N + \sup_{n \in \mathbb{N}} D_a(n) - \inf_{n \in \mathbb{N}} D_a(n) - 2\varepsilon$ letters $a$.

According to Observation 1 there exists a factor $W$ of length $N$ such that $|W|_a \leq \mu_a N + \varepsilon$. Hence $c_a \geq |\hat{W}|_a - |W|_a \geq \sup_{n \in \mathbb{N}} D_a(n) - \inf_{n \in \mathbb{N}} D_a(n) - 3\varepsilon = \Delta_a - 3\varepsilon$.

The case $n_1 < n_2$ is analogous. \hfill \square

To find the value $\Delta_a$, we apply the method of Adamczewski used in [1, 2]. Let us first recall the notation used in this method.

Let $M$ be a matrix of the substitution. Since entries of $M$ are defined as $M_{a,b} = |\varphi(b)|_a$ for $a, b \in \{0, 1, \ldots, m-1\}$, we have

$$M = \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 & 1 \\
1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 & 0
\end{pmatrix} \in \mathbb{R}^{m \times m}.$$

By $\Psi(w)$ we denote the Parikh vector of the word $w \in A^*$, i.e., $\Psi(w) = (|w|_0, |w|_1, \ldots, |w|_{m-1})^T$. The matrix of a substitution helps effectively calculate the Parikh vector of an image $w$ under $\varphi$.

It is easy to see that

$$\Psi(\varphi(w)) = M \Psi(w) \quad \text{for any } w \in A^*. \quad (4)$$

**Lemma 2.2.** For any prefix $u[n]$ of the $m$-bonacci word $u$, there exist $\ell \in \mathbb{N}$ and $\delta_0, \delta_1, \ldots, \delta_\ell \in \{0, 1\}$ such that

$$\Psi(u[n]) = \sum_{k=0}^\ell \delta_k M^k \Psi(0). \quad (5)$$

Moreover, for any choice of $\ell \in \{0, 1, 2, \ldots\}$ and $\delta_0, \ldots, \delta_\ell \in \{0, 1\}$, there exists a prefix $u[n]$ of $u$ such that $u[n]$ holds.

**Proof.** According to result [6], for any prefix there exist words $E_\ell \neq \epsilon, E_{\ell-1}, \ldots, E_1, E_0$ ($\epsilon$ is the empty word) such that

$$u[n] = \varphi(E_\ell) \varphi^{\ell-1}(E_{\ell-1}) \cdots \varphi(E_1) E_0 \quad (6)$$

and for any $k$, the word $E_k$ is a proper prefix of $\varphi(a)$ for some letter $a \in A$.

For our substitution $\varphi$, the only proper prefixes of $\varphi(a)$ are $E_k = \epsilon$ and $E_k = 0$. Since the Parikh vector of a concatenation of words is the sum of their Parikh vectors, we have

$$\Psi(u[n]) = \sum_{k=0}^\ell \delta_k \Psi(\varphi^k(0)),$$

where $\delta_k = 1$ if $E_k = 0$ and $\delta_k = 0$ if $E_k = \epsilon$. Applying formula (4) to $\Psi(\varphi^k(0))$, we get (5).

In general, not all sequences of $E_\ell, E_{\ell-1}, \ldots, E_1, E_0$ correspond to a prefix of $u$. The relevant sequences are described by paths in so called prefix graph of substitution. Nevertheless, since for our substitution the equality $\varphi^m(0) = \varphi^{m-1}(0) \varphi^{m-2}(0) \cdots \varphi(0) 0$ holds, any choice of $E_i \in \{\epsilon, 0\}$ gives a prefix of $u$. \hfill \square
Knowledge of the Parikh vector $\Psi(u[n])$ enables us to compute discrepancy $D_a(n)$. To make arithmetic manipulation more elegant, Adamczewski denotes row vectors

$$h^{(0)} = (1, 0, \ldots, 0) - \mu_0(1, 1, \ldots, 1),$$

$$h^{(1)} = (0, 1, \ldots, 0) - \mu_1(1, 1, \ldots, 1),$$

$$\vdots$$

$$h^{(m-1)} = (0, \ldots, 0, 1) - \mu_{m-1}(1, 1, \ldots, 1),$$

and expresses the discrepancy as the scalar product

$$D_a(n) = h^{(a)} \Psi(u[n]).$$

(7)

Verification of the formula is straightforward.

Now we can formulate the main tool for estimation of $c_a$.

**Proposition 2.3.** For any $a \in \{0, 1, \ldots, m - 1\}$ and $k \in \mathbb{N}$, denote

$$g(a, k) = |\varphi^k(0)|_a - \mu_a \cdot |\varphi^k(0)|,$$

(8)

where $\mu_a$ is the density of the letter $a$ in $u$. Then

$$g(a, k) = T_{k+m-a-1} - \frac{1}{\beta_{a+1}} T_{k+m}$$

(9)

and

$$\sup_{n \in \mathbb{N}} D_a(n) - \inf_{n \in \mathbb{N}} D_a(n) = \sum_{k=0}^{+\infty} |g(a, k)|$$

Proof. At first, since $g(a, k)$ is nothing but $D_a(|\varphi^k(0)|)$, equation (7) gives $g(a, k) = h^{(a)} \Psi(|\varphi^k(0)|)$. Using equation (4), we obtain $\Psi(|\varphi^k(0)|) = M^k \Psi(0)$, hence

$$g(a, k) = h^{(a)} M^k \Psi(0).$$

(10)

This expression combined with equations (5) and (7) gives $D_a(n) = \sum_{k=0}^{+\infty} \delta_k g(a, k)$, where $\delta_k \in \{0, 1\}$. Clearly, $\sup_{n \in \mathbb{N}} D_a(n) \leq \sum_{g(a,k)>0}^{+\infty} g(a,k)$ and $\inf_{n \in \mathbb{N}} D_a(n) \geq \sum_{g(a,k)<0}^{+\infty} g(a,k)$. According to Lemma 2.2 any choice of $\delta_k$'s corresponds to a prefix of $u[n]$, and, therefore, the equalities are reached in the previous inequalities. To sum up,

$$\sup_{n \in \mathbb{N}} D_a(n) - \inf_{n \in \mathbb{N}} D_a(n) = \sum_{k=0}^{+\infty} g(a,k) - \sum_{g(a,k)<0}^{+\infty} g(a,k) = \sum_{k=0}^{+\infty} |g(a,k)|.$$

In order to prove equation (10), let us observe that

$$
\begin{pmatrix}
T_n \\
T_{n-1} \\
\vdots \\
T_{n-m+1}
\end{pmatrix} =
\begin{pmatrix}
T_{n-1} \\
T_{n-2} \\
\vdots \\
T_{n-m}
\end{pmatrix}.
$$

Since $(T_{m-1}, T_{m-2}, \ldots, T_0) = (1, 0, 0, \ldots, 0) = (\Psi(0))^T$, we get using (10)

$$g(a, k) = h^{(a)} M^k \Psi(0) = h^{(a)} (T_{m+k-1}, T_{m+k-2}, \ldots, T_k)^T.$$

(11)
It is readily seen that the vector \( \vec{\beta} = (\beta^{-1}, \beta^{-2}, \ldots, \beta^{-m})^T \) is an eigenvector of \( M \) corresponding to the dominant eigenvalue \( \beta \). Moreover, sum of components of \( \vec{\beta} \) equals 1. It is well known that a vector \( \vec{\beta} \) with these properties is the vector of letter densities, see [9]. It means that for any letter \( a \in \{0, 1, \ldots, m-1\} \), the density of letter \( a \) is \( \mu_a = \beta^{-1-a} \). If we apply this fact to (11) and use the relation (2), we find

\[
g(a,k) = T_{m+k-a-1} - \beta^{-a-1}T_{m+k}.
\]

\[\square\]

**Corollary 2.4.** The balance constants of the \( m \)-bonacci word satisfy

\[
c_a \leq 2 \sum_{k=0}^{+\infty} |g(a,k)|
\]

for all \( a \in \mathcal{A} \).

**Proof.** The estimate follows easily from Lemma 2.1 and Proposition 2.3.

\[
c_a \leq 2\Delta_a = 2 \left( \sup_{n \in \mathbb{N}} D_a(n) - \inf_{n \in \mathbb{N}} D_a(n) \right) = 2 \sum_{k=0}^{+\infty} |g(a,k)|.
\]

\[\square\]

**Remark 1.** To estimate the sum \( \sum_{k=0}^{+\infty} |g(a,k)| \), we will use the explicit formula for elements \( T_n \) of the \( m \)-bonacci sequence. The characteristic equation of (9) is the polynomial \( p(x) \) with zeros \( \beta = \beta_0, \beta_1, \ldots, \beta_{m-1} \). Hence there exist constants \( a_0, a_1, \ldots, a_{m-1} \in \mathbb{C} \) such that

\[
T_n = a_0\beta_0^n + a_1\beta_1^n + \ldots + a_{m-1}\beta_{m-1}^n.
\]

The constants \( a_0, a_1, \ldots, a_{m-1} \) depend on the initial values \( T_0, T_1, \ldots, T_{m-1} \) only. A standard calculation provides \( T_n = \sum_{j=0}^{m-1} \frac{1}{\beta_j^{n+1}} + \frac{1}{\beta_j} \beta_j^{k+m} \), where \( p' \) denotes the derivative of the characteristic polynomial \( p \).

Using (9), we can conclude with

\[
g(a,k) = \sum_{j=1}^{m-1} \left( \frac{1}{\beta_j^{n+1}} - \frac{1}{\beta_j^{n+1}} \right) \frac{1}{p'(\beta_j)} \beta_j^{k+m}.
\]

3 **Numerical upper bounds on balance constant**

According to Corollary 2.4, the letter balance constants of the \( m \)-bonacci word \( u \) can be estimated by the formula

\[
c_a \leq 2 \sum_{k=0}^{+\infty} |g(a,k)|
\]

for any letter \( a \in \{0, 1, \ldots, m-1\} \) and for all \( m \geq 2 \).

In this section we estimate the expressions \( 2 \sum_{k=0}^{+\infty} |g(a,k)| \) using a computer calculation. The calculations are very time-consuming for \( m \) above 10, therefore, we confine ourselves to \( m \leq 12 \).

The calculation is based on the following strategy. We sum up the first \( n \) members of \( \langle |g(a,k)| \rangle_{k=0}^{+\infty} \) and estimate the rest of them;

\[
\sum_{k=0}^{+\infty} |g(a,k)| \leq \sum_{k=0}^{n-1} |g(a,k)| + E, \quad \text{where } E \text{ satisfies } E \geq \sum_{k=n}^{+\infty} |g(a,k)|.
\]
Table 1: 4-bonacci \( g(a, k) \) with quadruple of integer coefficients in linear combination of \( g(a, 0), \ldots, g(a, 3) \) and its signum.

| \( g(a, 0) \) | IC of \( (g(a, k))_k=0^3 \) | \( a = 0 \) | \( a = 1 \) | \( a = 2 \) | \( a = 3 \) |
|----------------|-------------------------------|--------|--------|--------|--------|
| (1, 0, 0, 0)   | +                             | −      | −      | −      | −      |
| (0, 1, 0, 0)   | −                             | +      | −      | −      | −      |
| (0, 0, 1, 0)   | −                             | −      | +      | −      | −      |
| (0, 0, 1, 0)   | −                             | −      | −      | +      | −      |
| (1, 1, 1, 1)   | +                             | −      | −      | −      | −      |
| (1, 2, 2, 2)   | −                             | +      | −      | −      | −      |
| (2, 3, 4, 4)   | −                             | −      | +      | −      | −      |
| (4, 6, 7, 8)   | −                             | −      | −      | +      | −      |
| (8, 12, 14, 15)| +                             | +      | −      | −      | −      |
| (15, 23, 27, 29)| −                             | +      | +      | −      | −      |
| (29, 44, 52, 56)| −                             | −      | +      | +      | −      |
| (56, 85, 100, 108)| +                             | −      | −      | +      | −      |
| (108, 164, 193, 208)| +                             | +      | −      | −      | −      |

Formula (13) provides setting

\[
E_{a,n} := |\beta_2^n| \sum_{j=1}^{m-1} \left( \frac{1}{\beta_j^{a+1}} - \frac{1}{\beta_j^{a+1}} \right) \frac{1}{\beta_j} \left( \frac{1}{1 - |\beta_j|} \right).
\]

To conclude, we have to find an \( n \) big enough to satisfy

\[
\left[ 2 \sum_{k=0}^{n-1} |g(a, k)| \right] = \left[ 2 \sum_{k=0}^{n-1} |g(a, k)| + E_{a,n} \right].
\]

Since we always compute on machines working in a finite precision, it is desirable to reduce the work with non-integer numbers. Therefore, we make use of the fact that, for a fixed letter \( a \) and the alphabet cardinality \( m \), the sequence of numbers \( g(a, k) \) satisfies the \( m \)-bonacci recurrence relation

\[
g(a, n + m) = g(a, n + m - 1) + \ldots + g(a, n),
\]

which follows from Proposition 2.3.

Let us demonstrate the method on the 4-bonacci word. The first step is calculating \( \text{sgn} g(a, k) \) from (9) for all \( k \in \{0, \ldots, m - 1\} \) (illustrated in Table 1). Then we express \( \sum_{k=0}^{n-1} |g(a, k)| \) as an integer combination (IC) of \( \left( \begin{array}{c} g(a, 0) \\ g(a, m - 1) \\ \vdots \end{array} \right) \), which can be rewritten in the form \( p + \frac{q}{m+1} \) for some \( p, q \in \mathbb{Z} \) (this follows from Proposition 2.3) and then evaluated\(^1\) (see Table 2). The final step is verification of the equality (14).

To make our procedure reliable with respect to possible rounding errors, we replace the estimated error \( E_{a,m} \) by a constant \( E > E_{a,m} \). If (14) holds, it is equal to the desired upper bound of \( c_a \) (but it may not be optimal). In the opposite case, we must increase \( n \) and repeat the procedure.

Our results obtained for \( m \in \{2, \ldots, 12\} \) are summarized in Table 3.

To find lower bounds on the constant \( c \), one needs to find two factors \( v, w \) of the \( m \)-bonacci word that are of the same length with \( |w| \_a - |v| \_a \) big enough. Computer searching in the set of all factors is very time-consuming. Nevertheless, for any given \( m \geq 4 \) and any \( a \in \{1, \ldots, m - 1\} \), a modification of the abelian co-decomposition method (12) allowed us to find a pair of factors \( v, w \) of the \( m \)-bonacci word such that \( |v| = |w| \) and \( |v|_a - |w|_a = 3 \). For instance, if \( m = 4 \), the words

\[
v = 1\varphi^2(0)\varphi^0(0)\varphi^5(0)\varphi^2(0),
\]

\(^1\)The calculation must be performed in an environment working in enough precision, e.g., Wolfram Mathematica.
Table 2: 4-bonacci – Estimates of $\sum_{k=0}^{+\infty} |g(a,k)|$ and the resulting upper bound on $c_a$.

| a   | $\sum_{k=0}^{12} |g(a,k)|$ as | $\sum_{k=0}^{12} |g(a,k)|$ numerical | $c_a$ upper bound |
|-----|-------------------------------|---------------------------------------|------------------|
| 0   | $(\frac{123}{125}, \frac{39}{76}, \frac{-133}{253}, \frac{-291}{253}, \frac{-47}{-85}, \frac{-254}{-86})$ | $(1.2778, 1.5157, 1.5611, 1.5776)$ | 2                |
| 1   | $(1664 - \frac{3205}{3}, 286 - \frac{1057}{3^2}, \frac{3499}{3^3} - 487, \frac{1209}{3^4} - 86)$ | $1.49844, 1.76006, 1.84618, 1.91919$ | 3                |
| 2   | $\sum_{k=0}^{12} |g(a,k)| + E$ | $c_a$ upper bound |

Table 3: Upper estimates of $c_a$ for $m \in \{2, \ldots, 12\}, a \in \{0, \ldots, m - 1\}$.

| m \ a | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|--------|---|---|---|---|---|---|---|---|---|---|----|----|
| 2      | 1 | 1 | x | x | x | x | x | x | x | x | x  | x  |
| 3      | 2 | 2 | 2 | x | x | x | x | x | x | x | x  | x  |
| 4      | 3 | 3 | 3 | 3 | x | x | x | x | x | x | x  | x  |
| 5      | 2 | 3 | 3 | 3 | 3 | x | x | x | x | x | x  | x  |
| 6      | 3 | 3 | 4 | 4 | 4 | x | x | x | x | x | x  | x  |
| 7      | 3 | 4 | 4 | 4 | 4 | 4 | x | x | x | x | x  | x  |
| 8      | 3 | 4 | 4 | 4 | 4 | 4 | 4 | x | x | x | x  | x  |
| 9      | 3 | 4 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5  | 5  | 5  |
| 10     | 3 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5  | 5  | 5  |
| 11     | 4 | 5 | 5 | 6 | 6 | 6 | 6 | 6 | 6 | 6  | 6  | 6  |
| 12     | 4 | 5 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6  | 6  | 6  |
\[ w = \left( \varphi^9(0)\varphi^8(0)\varphi^5(0)\varphi^2(0) \right)^{-1} \varphi^{11}(00)\varphi^{10}(0)\varphi^7(0)\varphi^6(0)\varphi^4(0)\varphi^3(0)\varphi^2(0) 0 \]

are factors of \( u \) such that \(|v| = |w| = 3305, |v|_1 = |w|_1 = 3\). Similarly, if \( m = 5 \), the words

\[ v = 1\varphi^{14}(0)\varphi^{11}(0)\varphi^6(0)\varphi^2(0), \]

\[ w = \left( \varphi^{11}(0)\varphi^{10}(0)\varphi^6(0)\varphi^2(0) \right)^{-1} \varphi^{13}(00)\varphi^{12}(0)\varphi^9(0)\varphi^8(0)\varphi^7(0)\varphi^5(0)\varphi^3(0)\varphi^2(0) 0 \]

are factors of \( u \) such that \(|v| = |w| = 15481, |v|_1 = |w|_1 = 3\).

Therefore, we can conclude with the following theorem.

**Theorem 3.1.** For \( m \in \{4, 5\} \), the \( m \)-bonacci word is \( c \)-balanced with \( c = 3 \) and this bound cannot be improved.

For \( m \in \{6, \ldots, 12\} \), the \( m \)-bonacci word is \( c \)-balanced for \( c = \lceil \frac{m+1}{2} \rceil \).

## 4 Balance property of letters in the \( m \)-bonacci word

The numerical calculation, performed in Section 3, is convenient only for small values of \( m \). In the rest of the paper we develop a technique to estimate the constant \( c \) for the balance property of the \( m \)-bonacci word for a general \( m \). The calculation will be again based on formula (12), but this time we bring in an improvement. Instead of estimating the sums \( \sum_{k=0}^{+\infty} |g(a,k)| \) for all letters \( a \in A \), we show that in case of the \( m \)-bonacci word, the balance constants \( c_a \) for \( a = 1, 2, \ldots, m-1 \) can be estimated by a simple formula in terms of \( c_0 \) providing that \( c_0 \) is small enough, see the following observation.

**Proposition 4.1.** Let \( m \geq 4 \). If \( c_0 \leq 2^{m-1} - 3 \), then

\[ c_j \leq \left( 2 - \frac{1}{2^j} \right) c_0 + 4 \left( 1 - \frac{1}{2^j} \right) \]  \hspace{1cm} (15)

for each \( j = 1,2,\ldots, m-1 \). In particular, the \( m \)-bonacci word is \( c \)-balanced with \( c = 2c_0 + 3 \).

With regard to this proposition, it will be sufficient to estimate \( \sum_{k=0}^{+\infty} |g(a,k)| \) and use formula (12) just once, for \( a = 0 \). All the remaining constants \( c_a \) for \( a = 1, \ldots, m-1 \) can be then easily estimated using formula (15).

Before we prove Proposition 4.1 we derive two simple observations.

**Observation 2.** For any factor \( f \) of \( u \) and for each \( j \in \{1, \ldots, m-1\} \), it holds

\[ |f|_0 = |\varphi^j(f)|_j \hspace{1cm} \text{and} \hspace{1cm} |f| = |\varphi^j(f)|_{j-1}. \]

**Proof.** From the form of the substitution \( \varphi \), we see \(|w|_{j-1} = |\varphi(w)|_j\) and \(|w| = |\varphi(w)|_0\) for any factor \( w \) and letter \( j \in 1,2,\ldots, m-1 \). Applying these relations on \( w = f, w = \varphi(f), \ldots, w = \varphi^{j-1}(f) \), we get the formulae in the observation. \( \square \)

**Observation 3.** If \( f \) is a factor of \( u \) such that \( |f| \leq 2^m \), then \(|f|_0 \leq \frac{1}{2}|f| + 1 \).

**Proof.** The form of the substitution \( \varphi \) implies that \( 00 \) is the longest block of zeros occurring in \( u \). Further, with exception of this block, the letter \( 0 \) is always sandwiched by nonzero letters. It is easy to see that the shortest factor \( w \neq 00 \), with the prefix \( 00 \) and the suffix \( 00 \) such that \( w \) has no other occurrences of \( 00 \), is the factor \( w = 0\varphi^m(0) \). Since \(|w| = 2^m + 1\), any factor \( f \) with \(|f| \leq 2^m \) contains at most one block \( 00 \). This implies the inequality for \(|f|_0 \) stated in the observation. \( \square \)

The following lemma is the combinatorial core for the proof of Proposition 4.1.

**Lemma 4.2.** Let \( j \in \{1, \ldots, m-1\} \). If \( c_{j-1} \leq 2^m - 2 \), then

\[ c_j \leq c_0 + 2 + \frac{c_{j-1}}{2}. \]  \hspace{1cm} (16)
Proof. With respect to the definition of \( c_j \), there exists a pair of factors \( v \) and \( w \) such that
\[
|v| = |w| \quad \text{and} \quad |v|_j - |w|_j = c_j. \tag{17}
\]

Without loss of generality, we can assume that \( v \) and \( w \) is the shortest possible pair satisfying (17). Then \( v \) and \( w \) are in the form \( v = j \cdots j \) and \( w = \ell \cdots \ell' \) for certain \( \ell, \ell' \neq j \). Moreover, we can assume that \( jw \) is a factor of \( u \) (otherwise we replace \( w = u_i \cdots u_{i+|w|-1} \) by \( w' = u_{i-1} \cdots u_{i+|w|-1-1} \)) without violating equations (17).

Because of the form of \( v \), there exists a factor \( V = 0V' \in \mathcal{L}(u) \) such that \( v = j\varphi'(V') \). Clearly, \( v \) is a suffix of \( \varphi'(0V') = \varphi'(V) \).

Let \( wzj \) be a factor of \( u \) such that \( |z|_j = 0 \) (we extend the factor \( w \) to the right up to the next letter \( j \)). As \( jwzj \in \mathcal{L}(u) \) by assumption, there exists a factor \( W \) such that \( wzj = \varphi'(W) \).

Observation 2 implies
- \(|V|_0 = 1 + |V'|_0 = 1 + |\varphi'(V')|_j = |j\varphi'(V')|_j = |v|_j\)
- \(|W|_0 = |W|_0 - 1 = |\varphi'(W)|_j - 1 = |wzj|_j - 1 = |w|_j\)
- \(|V| = 1 + |V'| = 1 + |\varphi'(V')|_{j-1} = 1 + |v|_{j-1}\)
- \(|W| = |W|_0 - 1 = |\varphi(W)|_{j-1} - 1 = |wzj|_{j-1} - 1\)

Together, we have deduced
\[
|V|_0 - |W|_0 = c_j \quad \text{and} \quad |V| - |W| = |v|_{j-1} - |w|_{j-1} - |z|_{j-1} + 2. \tag{18}
\]

We distinguish two cases:

- **Case** \( |V| \leq |W| \). Let \( \tilde{V} = Vx \) be a factor of \( u \) such that \( |\tilde{V}| = |W| \). From the definition of \( c_0 \) and (18) we get \( c_0 \geq |\tilde{V}|_0 - |W|_0 = |V|_0 - |W|_0 = c_j \). Thus \( c_j \leq c_0 + 2 + \frac{c_{j-1}}{2} \) holds trivially.

- **Case** \( |V| > |W| \). Let \( \tilde{W} = Wy \) be a factor of \( u \) such that \( |\tilde{W}| = |V| \). Then \( c_0 \geq |\tilde{W}|_0 - |\tilde{W}|_0 = c_j \). To bound length of \( y \), we apply Equation (18). It gives \(|y| = |V| - |W| = |v|_{j-1} - |w|_{j-1} - |z|_{j-1} + 2 \leq |v|_{j-1} - |w|_{j-1} + 2 \leq c_{j-1} + 2 \). With regard to the assumption \( c_{j-1} \leq 2^{m-2} - 2 \), we have \(|y| \leq 2^{m-2} - 2 \). Therefore, \( |y| \leq \frac{1}{2}y + 1 \) due to Observation 3. To sum up, \( c_j \geq c_j - \left( \frac{1}{2} (c_{j-1} + 2) + 1 \right) \geq c_j - 2 - \frac{1}{2} c_{j-1} \).

\[\square\]

**Proof of Proposition 4.1**. Let us assume that \( c_0 \leq 2^{m-1} - 3 \). We prove equation (18) by induction on \( j \).

I. Let \( j = 1 \). It holds \( c_0 \leq 2^{m-1} - 3 \leq 2^m - 2 \) by assumption, therefore, we can use Lemma 1.2.

II. Let \( j > 1 \) and equation (18) hold for \( j - 1 \). Inequality \( c_0 \leq 2^{m-1} - 3 \) implies
\[
c_{j-1} \leq \left( 2 - \frac{1}{2^{j-1}} \right) c_0 + 4 \left( 1 - \frac{1}{2^{j-1}} \right) < 2c_0 + 4 \leq 2(2^{m-1} - 3) + 4 = 2^m - 2. \]

It allows us to apply Lemma 1.2. Equation (18) gives
\[
c_j \leq c_0 + 2 + \frac{1}{2} c_{j-1} \leq c_0 + 2 + \frac{1}{2} \left( 2 - \frac{1}{2^{j-1}} \right) c_0 + 4 \left( 1 - \frac{1}{2^{j-1}} \right) = \left( 2 - \frac{1}{2^j} \right) c_0 + 4 \left( 1 - \frac{1}{2^j} \right).
\]

In particular, (18) yields \( c_j < 2c_0 + 4 \). As \( c = \max\{c_j : j = 0, 1, \ldots, m-1\} \) and \( c \) and \( c_0 \) are integers, necessarily \( c \leq 2c_0 + 3 \). \[\square\]
Estimate of $\sum_{k=0}^{+\infty} |g(0, k)|$

As anticipated in Section 4, the balance constant $c_0$ will be obtained using formula 12. Therefore, we need to estimate the sum $\sum_{k=0}^{+\infty} |g(0, k)|$. This is the topic of this section; since we deal with the letter $a = 0$ only, we abbreviate the symbol $g(0, k)$ to $g(k)$.

The sum $\sum_{k=0}^{+\infty} |g(0, k)|$ will be estimated by splitting it into two parts, $\sum_{k=0}^{2m-1} |g(k)|$ and $\sum_{k=2m}^{+\infty} |g(k)|$, and estimating each of them separately. In Sections 5.1 and 5.2 we show that

$$\sum_{k=0}^{2m-1} |g(k)| < \frac{5}{4}$$

and

$$\sum_{k=2m}^{+\infty} |g(k)| < \frac{A}{2\pi} m + 1 \quad \text{for all } m \geq 4.$$

To get these estimates we will exploit bounds on absolute values and arguments of zeros of polynomials $p(x)$, derived in Appendix A.

5.1 An upper bound on the sum $\sum_{k=0}^{2m-1} |g(k)|$

At first we express $g(k)$’s for all $k = 0, 1, \ldots, 2m - 1$ and determine their signs. Recall that $\mu_0 = 1/\beta$, therefore, due to equation (8), it holds

$$g(k) = |\varphi^k(0)|_0 - \frac{1}{\beta} |\varphi^k(0)|.$$

In the sequel we use the following formula to calculate $g(k)$ for all $k \leq 2m - 1$.

**Proposition 5.1.** It holds

$$|\varphi^k(0)| = \begin{cases} 2^k & \text{for } k = 0, \ldots, m - 1; \\ 2^k - 2^k - (k - m)2^k - m - 1 & \text{for } k = m, \ldots, 2m - 1. \end{cases}$$

**Proof.** The identity $\varphi^k(0) = \varphi(\varphi^{k-1}(0))$ together with the substitution (11) implies

$$|\varphi^k(0)| = 2|\varphi^{k-1}(0)| - |\varphi^{k-1}(0)|_{m-1}.$$ (21)

Let us distinguish two cases.

- **Case $k \leq m - 1$.** It holds $\varphi^0(0) = 0$ and $|\varphi^k(0)|_{m-1} = 0$ for all $k \leq m - 2$, hence $|\varphi^k(0)| = 2|\varphi^{k-1}(0)|$ for all $k \leq m - 1$.
- **Case $k \geq m$.** We prove equation (20) for $k \in \{m, m + 1, \ldots, 2m - 1\}$ by induction on $k$.

I. $k = m$. We have $|\varphi^{m-1}(0)|_{m-1} = 1$, hence $|\varphi^m(0)| = 2|\varphi^{m-1}(0)| - 1 = 2^m - 1$. Since $2^m - 1 = 2^m - 2^m - (m - m)2^m - m - 1$, the statement holds true for $k = m$.

II. $k \geq m + 1$. Let $|\varphi^{k-1}(0)| = 2^k - 2^k - (k - 1 - m)2^k - m - 1$. The identity $|\varphi^{k-1}(0)|_{m-1} = |\varphi^{k-1-m}(0)|$, valid for every $k \geq m + 1$, allows us to use the formula (21) in the form

$$|\varphi^k(0)| = 2|\varphi^{k-1}(0)| - |\varphi^{k-1-m}(0)|.$$

Since $k - 1 - m < m - 1$, we can apply the results obtained above $k \leq m - 1$, whence we get

$$|\varphi^k(0)| = 2 \left(2^k - 2^k - (k - 1 - m)2^k - m - 1\right) - 2^k - 2^k - (k - m)2^k - m - 1.$$  

To determine signs of $g(k)$’s defined by (19), we need a fine estimate on $\beta$. Let us recall that $\beta$ is the dominant eigenvalue of the matrix of substitution $M$ and thus a zero of its characteristic polynomial $p(x) = x^m - x^{m-1} - x^{m-2} - \cdots - x - 1$.  

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Proposition 5.2. It holds
\[ g(0) = 1 - \frac{1}{\beta} > 0; \]
\[ g(k) = 2^{k-1} \left( 1 - \frac{2}{\beta} \right) < 0 \quad \text{for } k = 1, \ldots, m - 1; \]
\[ g(m) = 2^{m-1} \left( 1 - \frac{2}{\beta} \right) + \frac{1}{\beta} > 0; \]
\[ g(k) = \left( 1 - \frac{2}{\beta} \right) \left( 2^{k-1} - (k + 1 - m)2^{k-m-2} \right) + \frac{1}{\beta} \cdot 2^{k-m-1} < 0 \quad \text{for } k = m + 1, \ldots, 2m - 1. \]

Proof. The formula for \( g(0) \) follows immediately from equation (19).

For every \( k \geq 1 \), it holds \(|\varphi^k(0)| = |\varphi^{k-1}(0)|\), hence
\[ g(k) = |\varphi^{k-1}(0)| - \frac{1}{\beta} \cdot |\varphi^k(0)|, \]
cf. equation (19). All the formulae for \( g(k) \) listed in Proposition 5.2 then follow easily from equation (20).

In the rest of the proof we show that \( g(0) > 0, g(m) > 0, \) and \( g(k) < 0 \) for all \( k \in \{1, \ldots, m - 1\} \cup \{m + 1, \ldots, 2m - 1\} \).

At first, \( \beta \in (1, 2) \) immediately implies \( g(0) > 0 \) and \( g(k) < 0 \) for all \( k \in \{1, \ldots, m - 1\} \).

As for \( k = m \), we shall show that
\[ 2^{m-1} \left( 1 - \frac{2}{\beta} \right) + \frac{1}{\beta} > 0. \]
This inequality is equivalent to
\[ 2 - \beta < \frac{1}{2^{m-1}}, \]
which is valid due to (13) from Appendix, because \( 1/2^{m-1} > 1/(2^m - (m + 1)/2) \) for all \( m \geq 2 \).

Similarly, if \( k \geq m + 1 \), we need to prove that
\[ \left( 1 - \frac{2}{\beta} \right) \left( 2^{k-1} - (k + 1 - m)2^{k-m-2} \right) + \frac{1}{\beta} \cdot 2^{k-m-1} < 0 \quad \text{for } k = m + 1, \ldots, 2m - 1; \]
i.e.,
\[ 2 - \beta > \frac{1}{2^{m-k+1}} \quad \text{for all } k = m + 1, \ldots, 2m - 1. \]
Since \( k + 1 - m \leq m \), the validity immediately follows from inequalities (13). \( \square \)

Proposition 5.3. It holds
\[ \sum_{k=0}^{2m-1} |g(k)| = 1 + \left( \frac{2}{\beta} - 1 \right) \left( 2^m \left( 2^{m-1} - 1 \right) - (m - 1)2^{m-2} \right) - \frac{1}{\beta} \left( 2^{m-1} - 1 \right) < 1 + \frac{1}{4}. \quad (22) \]

Proof. Proposition 5.2 implies
\[ \sum_{k=0}^{2m-1} |g(k)| = g(0) - \sum_{k=1}^{m-1} g(k) + g(m) - \sum_{k=m+1}^{2m-1} g(k). \]

When we substitute for \( g(k) \) from Proposition 5.2, we obtain
\[ g(0) + g(m) = 1 + 2^{m-1} \left( 1 - \frac{2}{\beta} \right). \]
Proposition 5.4. For any \(p\)

\[
- \sum_{k=1}^{m-1} g(k) = - \sum_{k=1}^{m-1} 2^{k-1} \left(1 - \frac{2}{\beta}\right) = -(2^{m-1} - 1) \left(1 - \frac{2}{\beta}\right),
\]

and, in a similar way, we get

\[
- \sum_{k=m+1}^{2m-1} g(k) = - \left(1 - \frac{2}{\beta}\right) \left[2^m (2^{m-1} - 1) - (m-1)2^{m-2}\right] - \frac{1}{\beta} (2^{m-1} - 1).
\]

Summing up these expressions, we get formula [22].

In the rest of the proof we show that

\[
(2 - \beta) \left[2^m (2^{m-1} - 1) - (m-1)2^{m-2}\right] - 2^{m-1} + 1 < \frac{\beta}{4},
\]

and also to

\[
(2 - \beta) \left[2^m (2^{m-1} - 1) - (m-1)2^{m-2} + \frac{1}{4}\right] - 2^{m-1} + 1 < \frac{1}{2}.
\]

Using inequality [13], we obtain

\[
(2 - \beta) \left[2^m (2^{m-1} - 1) - (m-1)2^{m-2} + \frac{1}{4}\right] - 2^{m-1} + 1
\]

\[
\leq \frac{1}{2m - m + 1} \left[2^m (2^{m-1} - 1) - (m-1)2^{m-2} + \frac{1}{4}\right] - 2^{m-1} + 1
\]

\[
= \frac{2^{m-1} - 1 - \frac{m-1}{2m + m}}{1 - \frac{m+1}{2m + m}} - 2^{m-1} + 1 = \frac{-m+1 + \frac{m+1}{2m+1} - \frac{m+1}{2m+1}}{1 - \frac{m+1}{2m+1}} = \frac{1}{2} \left(1 - \frac{m+1}{2m+1}\right) < \frac{1}{2}.
\]

5.2 An upper bound on the sum \(\sum_{k=2m}^{+\infty} |g(k)|\)

Proposition 5.4. For any \(k \in \mathbb{N}\) we have

\[
|g(k)| \leq \frac{1}{2(m-1)} \sum_{j=1}^{m-1} |\Re(\beta_j) - 1| \cdot |\beta_j|^k.
\]

Proof. With regard to equation [12] from Appendix,

\[
p'(x) = \frac{(m+1)x^m - 2mx^{m-1}}{x - 1} - \frac{x^m + 2x^m + 1}{(x - 1)^2} = \frac{(m+1)x^m - 2mx^{m-1}}{x - 1} - \frac{p(x)}{x - 1}.
\]

Since \(p(\beta_j) = 0\) for every eigenvalue of \(M\), we have

\[
p'(\beta_j) = \frac{(m+1)\beta_j^m - 2m\beta_j^{m-1}}{\beta_j - 1} = \frac{(m+1)\beta_j^m - 2m\beta_j^{m-1}}{\beta_j - 1}.
\]

Therefore, due to [13],

\[
g(k) = \sum_{j=1}^{m-1} \left(\frac{1}{\beta_j} - \frac{1}{\beta}\right) \frac{\beta_j^{k+m}}{(m+1)\beta_j^m - 2m\beta_j^{m-1}} = \sum_{j=1}^{m-1} \frac{\beta - \beta_j}{\beta} \cdot \frac{\beta_j - 1}{(m+1)\beta_j - 2m\beta_j^k}.
\]

As \(g(k)\) is real, we can write

\[
g(k) = \sum_{j=1}^{m-1} \frac{1}{\beta} \Re \left(\frac{\beta - \beta_j}{(m+1)\beta_j - 2m(\beta_j - 1)}\right).
\]
We have Lemma A.1 implies \(2\), i.e., (30) holds true. Equation (27) together with inequalities (29) and (30) implies 
\[
\Re \beta k - 1 \cdot |\beta_k|.
\]

To finish our proof we will deduce for all \(j = 1, \ldots, m - 1,\)
\[
\frac{1}{\beta} \left| \frac{\beta - \beta_j}{(m + 1)\beta_j - 2m} \right| \leq \frac{1}{2(m - 1)}.
\]  
(25)

Since
\[
\frac{1}{\beta} \left| \frac{\beta - \beta_j}{(m + 1)\beta_j - 2m} \right| = \frac{1}{2(m - 1)} \left| \frac{m - 1 - (m - 1)\beta_j}{m - (m + 1)\frac{m}{2}} \right|,
\]  
(26)

it suffices to prove that
\[
\left| \frac{m - 1 - (m - 1)\beta_j}{m - (m + 1)\frac{m}{2}} \right| \leq 1.
\]

We have
\[
\left| \frac{m - 1 - (m - 1)\beta_j}{m - (m + 1)\frac{m}{2}} \right|^2 = \left[ \frac{m - 1 - \frac{m+1}{2} \Re \beta_j}{m - \frac{m+1}{2} \Re \beta_j} \right]^2 + \left[ \frac{m - 1 - \frac{m+1}{2} \Im \beta_j}{m - \frac{m+1}{2} \Im \beta_j} \right]^2.
\]  
(27)

Lemma A.1 implies \(2 - \beta < \frac{2}{m+1} < \frac{4}{m+1}\); hence
\[
\frac{m - 1}{\beta} < \frac{m + 1}{2}.
\]  
(28)

Therefore
\[
\left[ \frac{m - 1 - \frac{m+1}{2} \Re \beta_j}{m - \frac{m+1}{2} \Re \beta_j} \right]^2 < \left[ \frac{m + 1}{2} \Im \beta_j \right]^2.
\]  
(29)

In what follows we demonstrate that
\[
\left| m - 1 - \frac{m - 1}{\beta} \Re \beta_j \right| < \left| m - \frac{m + 1}{2} \Re \beta_j \right|.
\]  
(30)

Since \(\beta \in (1, 2)\) and \(|\beta_j| < 1\), we have
\[
0 < m - 1 - \frac{m - 1}{\beta} \Re \beta_j = m - \frac{m + 1}{2} \Re \beta_j - 1 + \left( \frac{m + 1}{2} - \frac{m - 1}{\beta} \right) \Re \beta_j.
\]

It holds \(\Re \beta_j < 1\), and the expression \(\frac{m + 1}{2} - \frac{m - 1}{\beta}\) is positive due to equation (28) therefore
\[
-1 + \left( \frac{m + 1}{2} - \frac{m - 1}{\beta} \right) \Re \beta_j < -1 + \frac{m + 1}{2} - \frac{m - 1}{\beta} = -(m - 1) \left( \frac{1}{\beta} - \frac{1}{2} \right) < 0.
\]

Hence
\[
0 < m - 1 - \frac{m - 1}{\beta} \Re \beta_j < m - \frac{m + 1}{2} \Re \beta_j,
\]
i.e., (30) holds true. Equation (27) together with inequalities (29) and (30) implies
\[
\left| \frac{m - 1 - (m - 1)\beta_j}{m - (m + 1)\frac{m}{2}} \right|^2 < \left[ \frac{2m - (m + 1) \Re \beta_j}{2m - (m + 1) \Re \beta_j} \right]^2 + \left[ \frac{m + 1} \Im \beta_j \right]^2,
\]

\[= 1.\]
Corollary 5.5.
\[
\sum_{k=2m}^{+\infty} |g(k)| \leq \frac{1}{2(m-1)} \sum_{j=1}^{m-1} \left| \Re(\beta_j) - 1 \right| \frac{1}{1-|\beta_j|} \left| \frac{1}{2-\beta_j} \right|^2.
\]  
(31)

Proof. Using (23), we can estimate
\[
\sum_{k=2m}^{+\infty} |g(k)| \leq \frac{1}{2(m-1)} \sum_{j=1}^{m-1} |\Re(\beta_j) - 1| \sum_{k=2m}^{+\infty} |\beta_j^k| = \frac{1}{2(m-1)} \sum_{j=1}^{m-1} |\Re(\beta_j) - 1| |\beta_j|^{2m} \left| \frac{1}{1-|\beta_j|} \right|.
\]

Finally, we use Observation 4 to rewrite |β_j|^{2m} = 1/(2 - β_j)^2.

At this stage we apply the information on |β_j| for j = 1, ..., m - 1, derived in Lemma A.2.

Proposition 5.6. It holds
\[
\sum_{k=2m}^{+\infty} |g(k)| < \frac{1}{2(m-1)} \frac{1}{1-2\ln \frac{3}{m}} \left( \frac{2m}{1-\ln \frac{3}{m}} \sum_{j=1}^{m-1} \frac{1-\cos \gamma_j}{5-4\cos \gamma_j} \ln(5-4\cos \gamma_j) + \sum_{j=1}^{m-1} \frac{\cos \gamma_j}{5-4\cos \gamma_j} \right).
\]  
(32)

Proof. We will estimate summands from inequality (31). In the notation \( \beta_j = B_j e^{i\gamma_j} \), we have
\[
\left| \frac{\Re(\beta_j) - 1}{1-|\beta_j|} \right| = \frac{1-B_j \cos \gamma_j}{1-B_j} = \frac{1-\cos \gamma_j}{1-B_j} + \cos \gamma_j,
\]
thus equation (31) from Appendix implies
\[
\frac{1-\cos \gamma_j}{1-B_j} + \cos \gamma_j \leq \frac{1-\cos \gamma_j}{\ln(5-4\cos \gamma_j)} \cdot \frac{2m}{1-\ln \frac{3}{m}} + \cos \gamma_j.
\]  
(33)

Concerning the term 1/(2 - β_j)^2, it holds
\[
\frac{1}{(2 - \beta_j)^2} = \frac{1}{4 - 4B_j \cos \gamma_j + B_j^2} = \frac{1}{5 - 4\cos \gamma_j + 4(1 - B_j) \cos \gamma_j - 2(1 - B_j) + (1 - B_j)^2}.
\]

\[
< \frac{1}{5 - 4\cos \gamma_j} \cdot \frac{1}{1 - (1 - B_j) \frac{2 - 4\cos \gamma_j}{5 - 4\cos \gamma_j}}.
\]

It is easy to see that \( \frac{2 - 4\cos \gamma_j}{5 - 4\cos \gamma_j} \leq \frac{2}{3} \), therefore, it suffices to estimate 1 - B_j from above. Since
\[
B_j = \frac{1}{2m \sqrt{4 - 4B_j \cos \gamma_j + B_j^2}} > \frac{1}{\sqrt{2} \sqrt{9}} = \frac{1}{\sqrt{3}}
\]
and
\[
\sqrt{3} = e^{\frac{\ln 3}{m}} < \left[ 1 + \frac{1}{\ln \frac{3}{m} - 1} \right]^{\frac{\ln 3}{m}} = \frac{m}{\ln \frac{3}{m} - 1},
\]

it holds \( B_j > \frac{\ln 3}{m} \). Hence 1 - B_j < \( \frac{\ln 3}{m} \) for all \( j = 1, ..., m - 1 \). Consequently,
\[
\frac{1}{(2 - \beta_j)^2} < \frac{1}{5 - 4\cos \gamma_j} \cdot \frac{1}{1 - \frac{2}{3} \cdot \frac{\ln 3}{m}}.
\]  
(34)

Inequality (31) combined with estimates (33) and (34) leads to formula (32). }

The following lemma is an essential component of our calculation. It uses the information on \( \gamma_j \) obtained in Lemma A.3.
Lemma 5.7. It holds
\[
\sum_{j=1}^{m-1} \frac{1 - \cos \gamma_j}{(5 - 4 \cos \gamma_j) \ln(5 - 4 \cos \gamma_j)} \leq \frac{m}{2\pi} \int_0^{2\pi} \frac{1 - \cos x}{(5 - 4 \cos \gamma_j) \ln(5 - 4 \cos x)} \, dx - \frac{1}{6} \left( \frac{m - 1}{m} \frac{\pi}{16} \left( 1 + \frac{1}{36} \right) \right).
\]

Proof. Let us denote
\[ f(x) = \frac{1 - \cos x}{\ln(5 - 4 \cos x)} ; \]
then
\[
\sum_{j=1}^{m-1} \frac{1 - \cos \gamma_j}{(5 - 4 \cos \gamma_j) \ln(5 - 4 \cos \gamma_j)} = \frac{m}{2\pi} \sum_{j=1}^{m-1} \frac{2\pi}{m} f(\gamma_j).
\]

The estimate \( \frac{1}{m} \) implies \( \gamma_j \in \left( \frac{2\pi}{m} (j - \frac{1}{2}) , \frac{2\pi}{m} (j + \frac{1}{2}) \right) \). Therefore, the sum \( \frac{1}{36} \) is a Riemann sum of the function \( f \) with respect to the tagged partition
\[ \frac{\pi}{m} = x_0 < x_1 < \ldots < x_{m-1} = 2\pi - \frac{\pi}{m} ; \]
where \( x_j = \frac{2\pi}{m} \left( j + \frac{1}{2} \right) \), of interval \( \left[ \frac{\pi}{m} , 2\pi - \frac{\pi}{m} \right] \). Let us rewrite the summands of \( \frac{1}{36} \) using a trivial identity
\[ 2\pi m f(\gamma_j) = \int_{x_{j-1}}^{x_j} f(x) \, dx + \int_{x_{j-1}}^{x_j} (f(\gamma_j) - f(x)) \, dx. \]

Since
\[ f(\gamma_j) - f(x) \leq |x - \gamma_j| \cdot \max_{x \in (x_{j-1}, x_j)} \{|f'(x)|\} \leq |x - \gamma_j| \cdot \max_{x \in [0, 2\pi]} \{|f'(x)|\}, \]
we have
\[ 2\pi m f(\gamma_j) \leq \int_{x_{j-1}}^{x_j} f(x) \, dx + \max_{x \in [0, 2\pi]} \{|f'(x)|\} \int_{x_{j-1}}^{x_j} |x - \gamma_j| \, dx. \]

Now we apply another identity, valid for any \( \gamma_j \in [x_{j-1}, x_j], \)
\[ \int_{x_{j-1}}^{x_j} |x - \gamma_j| \, dx = \int_{x_{j-1}}^{x_j} (x - \gamma_j) \, dx + \int_{x_{j-1}}^{x_j} (\gamma_j - x) \, dx = \int_{0}^{\gamma_j - x_{j-1}} x \, dx + \int_{0}^{x_j - \gamma_j} x \, dx, \]
which provides us, using the estimate \( \frac{1}{36} \), the inequality
\[ \int_{x_{j-1}}^{x_j} |x - \gamma_j| \, dx \leq \int_{0}^{\gamma_j - x_{j-1}} x \, dx + \int_{0}^{x_j - \gamma_j} x \, dx = \frac{\pi^2}{m^2} \left( 1 + \frac{1}{36} \right). \]

Hence
\[ \frac{2\pi m f(\gamma_j)}{m} \leq \int_{x_{j-1}}^{x_j} f(x) \, dx + \max_{x \in [0, 2\pi]} \{|f'(x)|\} \frac{\pi^2}{m^2} \left( 1 + \frac{1}{36} \right). \]

Consequently,
\[ \sum_{j=1}^{m-1} f(\gamma_j) \leq \frac{m}{2\pi} \left( \int_{\pi}^{2\pi - \frac{\pi}{m}} f(x) \, dx + (m - 1) \max_{x \in [0, 2\pi]} \{|f'(x)|\} \frac{\pi^2}{m^2} \left( 1 + \frac{1}{36} \right) \right). \]

Furthermore, it can be checked that \( f(x) \geq 1/6 \) for all \( x \in (0, \pi/2) \cup (3\pi/2, 2\pi) \) and \( \lim_{x \to 0} f(x) = 1/4 > 1/6, \) hence
\[ \int_{\pi}^{2\pi - \frac{\pi}{m}} f(x) \, dx = \int_{0}^{2\pi} f(x) \, dx - \int_{0}^{\pi} f(x) \, dx - \int_{2\pi - \frac{\pi}{m}}^{2\pi} f(x) \, dx \leq \int_{0}^{2\pi} f(x) \, dx - \frac{2\pi}{m} \cdot \frac{1}{6}. \]
Finally, a numerical calculation gives \( \max_{x \in [0, 2 \pi]} \{|f'(x)|\} < \frac{1}{36} \). To sum up,

\[
\sum_{j=1}^{m-1} \frac{1 - \cos \gamma_j}{(5 - 4 \cos \gamma_j) \ln(5 - 4 \cos \gamma_j)} \leq \frac{m}{2 \pi} \left( \int_0^{2 \pi} \frac{1 - \cos x}{(5 - 4 \cos \gamma_j) \ln(5 - 4 \cos x)} \, dx - \frac{2 \pi}{m} \frac{1}{6} + (m - 1) \frac{1}{8} \frac{\pi^2}{m^2} \left( 1 + \frac{1}{36} \right) \right),
\]

whence we obtain the sought formula \( \delta \).

**Lemma 5.8.** It holds

\[
\sum_{j=1}^{m-1} \frac{\cos \gamma_j}{5 - 4 \cos \gamma_j} \leq \frac{m}{6} + \frac{5}{6}.
\]  

**Proof.** If we define \( \gamma_m := 2 \pi \), we can write

\[
\sum_{j=1}^{m-1} \frac{\cos \gamma_j}{5 - 4 \cos \gamma_j} = \frac{m}{2 \pi} \sum_{j=1}^{m} \frac{2 \pi}{m} \frac{\cos \gamma_j}{5 - 4 \cos \gamma_j} - 1.
\]

The sum \( \sum_{j=1}^{m} \frac{2 \pi}{m} f(\gamma_j) \) for \( f(\gamma) := \frac{\cos \gamma}{5 - 4 \cos \gamma} \) will be calculated in a similar way as in the proof of Lemma 5.7. Namely, it is a Riemann sum of the function \( f \) with respect to the tagged partition

\[
\frac{\pi}{m} = x_0 < x_1 < \ldots < x_{m-1} < x_m = 2 \pi + \frac{\pi}{m}, \quad \text{where} \quad x_j = \frac{2 \pi}{m} \left( j + \frac{1}{2} \right),
\]

of interval \( \left[ \frac{\pi}{m}, 2 \pi + \frac{\pi}{m} \right] \). Following the steps of the proof of Lemma 5.7, we obtain

\[
\sum_{j=1}^{m} f(\gamma_j) \leq \frac{m}{2 \pi} \left( \int_0^{2 \pi} f(x) \, dx + (m - 1) \max_{x \in [0, 2 \pi]} \{|f'(x)|\} \frac{\pi^2}{m^2} \left( 1 + \frac{1}{36} \right) \right)
\]

\[
< \frac{m}{2 \pi} \left( \int_0^{2 \pi} f(x) \, dx + m \cdot \max_{x \in [0, \pi]} \{|f'(x)|\} \frac{\pi^2}{m^2} \left( 1 + \frac{1}{36} \right) \right).
\]

With regard to the properties of \( \cos \), we find

\[
\int_0^{2 \pi} \frac{\cos x}{5 - 4 \cos x} \, dx = \int_0^{\pi} \frac{\cos x}{5 - 4 \cos x} \, dx = 2 \left[ \frac{x}{4} + \frac{5}{6} \arctan \left( 3 \tan \frac{x}{2} \right) \right]_0^\pi = \frac{\pi}{3}.
\]

Furthermore,

\[
\max_{x \in [0, \pi]} \{|f'(x)|\} = \frac{5}{2} \sqrt{\frac{153 - 11}{15 - 153}^2} < \frac{\pi}{8}.
\]

To sum up,

\[
\sum_{j=1}^{m-1} \frac{\cos \gamma_j}{5 - 4 \cos \gamma_j} \leq \frac{m}{2 \pi} \left( \frac{\pi}{3} + \frac{9}{8} \frac{\pi^2}{m} \left( 1 + \frac{1}{36} \right) \right) - 1 = \frac{m}{6} + \frac{\pi}{2} \left( 1 + \frac{1}{36} \right) \frac{9}{8} - 1 < \frac{m}{6} + \frac{5}{6}.
\]

**Proposition 5.9.** For all \( m \geq 4 \), it holds

\[
\sum_{k=2m}^{+\infty} |g(k)| < \frac{A}{2 \pi} m + 1,
\]

where

\[
A := \int_0^{2 \pi} \frac{1 - \cos x}{(5 - 4 \cos x) \ln(5 - 4 \cos x)} \, dx \approx 0.909.
\]
Proof. Recall that
\[ \sum_{k=2m}^{+\infty} |g(k)| < \frac{1}{2(m-1)} \left( \frac{2m}{1-2m} \sum_{j=1}^{m-1} \frac{1}{5 - 4 \cos \gamma_j \ln(5 - 4 \cos \gamma_j)} + \sum_{j=1}^{m-1} \cos \gamma_j \right). \]

cf. formula (32). If we estimate the sums using inequalities (35) and (37), we obtain
\[ \sum_{k=2m}^{+\infty} |g(k)| - A \frac{2\pi}{m} m - 1 < \frac{1}{2(m-1)} \left( \frac{1}{1-2m} \left( \frac{2m}{A - \frac{1}{6} + \frac{m-1}{m} \cdot \frac{\pi}{16} \left( 1 + \frac{1}{36} \right) + \frac{5}{6} + \frac{5}{6} \right) - A \frac{2\pi}{m} - 1. \right. \]

A numerical integration gives \( A \approx 0.909 \in (0.9, 0.91) \). For such value of \( A \), the expression above is negative for all \( m \geq 4 \); i.e.,
\[ \sum_{k=0}^{+\infty} |g(k)| - A \frac{2\pi}{m} m - 1 < 0 \quad \text{for all } m \geq 4. \]

\[ \Box \]

6 Main result

Theorem 6.1. For every \( m \geq 5 \), the \( m \)-bonacci word is \( c \)-balanced with
\[ c = \lceil \kappa m \rceil + 12, \]
where \( \kappa = \frac{2}{\pi} \int_0^{2\pi} \frac{1 - \cos x}{(5 - 4 \cos x) \ln(5 - 4 \cos x)} \, dx \approx 0.58. \)

Proof. In Propositions 5.3 and 5.9 we showed
\[ \sum_{k=0}^{2m-1} |g(k)| < \frac{5}{4} \quad \text{and} \quad \sum_{k=2m}^{+\infty} |g(k)| < A \frac{2\pi}{m} m + 1 \quad \text{for all } m \geq 4; \]
therefore,
\[ \sum_{k=0}^{+\infty} |g(0, k)| < \frac{9}{4} + A \frac{2\pi}{m}, \quad (40) \]
where \( A = \int_0^{2\pi} \frac{1 - \cos x}{(5 - 4 \cos x) \ln(5 - 4 \cos x)} \, dx \approx 0.909. \)

Having this bound in hand, we can use Corollary 2.4 to estimate the balance constant of letter 0 by
\[ c_0 \leq 2 \sum_{k=0}^{+\infty} |g(0, k)| \leq \frac{9}{4} + A \frac{2\pi}{m}. \]

Since \( \frac{9}{4} + \frac{A}{2} m \leq 2^{m-1} - 3 \) for any \( m \geq 5 \), the assumption of Proposition 4.1 is fulfilled and thus the \( m \)-bonacci word is \( c \)-balanced with
\[ c = 2c_0 + 3 \leq 3 + 4 \sum_{k=0}^{+\infty} |g(0, k)| \leq 12 + \frac{2A}{\pi} m, \]
which proves the theorem. \( \Box \)
7 Acknowledgement

We acknowledge financial support by the Czech Science Foundation grant GAČR 201/09/0584, by the Grant Agency of the Czech Technical University in Prague, grant SGS11/162/OHK4/3T/14, and by the Foundation "Nadání Josefa, Marie a Zdeňka Hlávkových".

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A On eigenvalues of $M$

In this section we examine the eigenvalues of the matrix of substitution. In particular, we estimate their absolute values and arguments. Such information is essential for estimating the sums $\sum_{k=0}^{2^m-1} |g(0, k)|$ and $\sum_{k=2^m}^{+\infty} |g(0, k)|$ in Section 5.

Let us recall that the eigenvalues of the matrix of substitution $M$ are zeros of its characteristic polynomial $p(x) = x^m - x^{m-1} - x^{m-2} - \ldots - x - 1$. The following observation will make further calculations substantially simpler.
Observation 4. Every zero of the polynomial \( p(x) \) is a root of the equation
\[
x^m (2 - x) = 1. \tag{41}
\]

Proof. For every \( x \neq 1 \), we can write
\[
p(x) = x^m - \frac{x^m - 1}{x - 1} = \frac{x^{m+1} - 2x^m + 1}{x - 1}. \tag{42}
\]

In particular, \( p(\beta_j) = 0 \) implies \( \beta_j^{m+1} - 2\beta_j^m + 1 = 0 \), whence \( \beta_j \) is a root of equation (41). \( \square \)

At first we derive a fine estimate on \( \beta \), which is needed for calculating the sum \( \sum_{k=0}^{2m-1} |g(0,k)| \).

Lemma A.1. The dominant eigenvalue \( \beta > 1 \) of the matrix of substitution \( M \) obeys the inequalities
\[
\frac{1}{2m - \frac{m}{2}} < 2 - \beta < \frac{1}{2m - \frac{m+1}{2}}. \tag{43}
\]

Proof. Observation 4 implies \( \beta^m (2 - \beta) = 1 \), hence \( \beta < 2 \). Let us set \( x_0 := 2 - \beta \). Obviously, \( x_0 \) is a root of the polynomial
\[
q(x) = (2 - x)^m \cdot x - 1.
\]

Since \( \beta \in (1,2) \), necessarily \( x_0 \in (0,1) \). It holds \( q'(x) = (2 - x)^{m-1} (2 - mx) \), therefore, \( q \) grows in \([0, 2/(m+1)]\) and decreases in \([2/(m+1), 1]\). Since \( q(0) = -1 \) and \( q(1) = 0 \), the root \( x_0 \) belongs to the interval \((0, 2/(m+1)]\), in which \( q \) grows. Consequently, proving inequalities (43) consists in showing that
\[
q \left( \frac{1}{2m - \frac{m+1}{2}} \right) < 0 < q \left( \frac{1}{2m - \frac{m+1}{2}} \right).
\]

Let us start with the estimate of \( 2 - \beta \) from above. We have
\[
q \left( \frac{1}{2m - \frac{m+1}{2}} \right) = \left( 2 - \frac{1}{2m - \frac{m+1}{2}} \right)^m \frac{1}{2m} - 1 = \left( 1 - \frac{1}{2m+1 - (m+1)} \right)^m \frac{1}{1 - \frac{m}{2m+1}} - 1.
\]

Since \((1+x)^m > 1 + mx \) for all \( x \in (-1,1) \), it holds
\[
q \left( \frac{1}{2m - \frac{m+1}{2}} \right) > \frac{1}{2m+1 - (m+1)} - 1 = \frac{1}{2m+1 - (m+1)} \left[ \left( 1 - \frac{1}{2m+1 - m} \right)^m - \left( 1 - \frac{m}{2m+1} \right) \right].
\]

For all \( m \geq 3 \), hence, \( q \left( 1/(2^m - \frac{m+1}{2}) \right) > 0 \) for all \( m \geq 3 \). If \( m = 2 \), the statement can be proved in the same way, just we use the exact expression \((1+x)^2 = 1 + 2x + x^2\) instead of the estimate \((1+x)^m > 1 + mx \).

Let us proceed to the estimate of \( 2 - \beta \) from below.
\[
q \left( \frac{1}{2m - \frac{m+1}{2}} \right) = \left( 2 - \frac{1}{2m - \frac{m+1}{2}} \right)^m \frac{1}{2m} - 1 = \frac{1}{2m} \left[ \left( 1 - \frac{1}{2m+1 - m} \right)^m - \left( 1 - \frac{m}{2m+1} \right) \right].
\]

For all \( x \in (-1,0) \), it holds \((1+x)^m < 1 + mx + \binom{m}{2} x^2\); therefore,
\[
\left( 1 - \frac{1}{2m+1 - m} \right)^m - \left( 1 - \frac{m}{2m+1} \right) < 1 - \frac{m}{2m+1} + \frac{m(m-1)}{2(2m+1 - m)^2} = 1 + \frac{m}{2m+1} - \frac{m}{2(2m+1 - m)^2} \left( -2m^2 + 2m + m - 1 + 2m^2 - 4m + \frac{m^2}{2m} \right)
\]
\[
= \frac{m}{2(2m+1 - m)^2} \left( -1 - m + \frac{m^2}{2m} \right) < 0
\]
for all \( m \geq 2 \). Hence \( q \left( 1/(2^m - \frac{m+1}{2}) \right) < 0 \). \( \square \)
Now we proceed to the eigenvalues $\beta_j$ for $j = 1, \ldots, m - 1$. For the sake of convenience let us set $B_j := |\beta_j|$ and $\gamma_j := \arg(\beta_j)$, i.e.,

$$\beta_j = B_j e^{i\gamma_j} \quad \text{for all } j = 1, \ldots, m - 1.$$  

**Lemma A.2.** It holds

$$|\beta_j| < 1 - \frac{\ln(5 - 4 \cos \gamma_j)}{2m} \left(1 - \frac{\ln 3}{m}\right)$$

for all $j = 1, \ldots, m - 1$.

**Proof.** Since the value $\beta_j = B_j e^{i\gamma_j}$ is a solution of equation (44), necessarily

$$|B_j^m e^{i m \gamma_j} (2 - B_j e^{i \gamma_j})|^2 = 1.$$  

Hence

$$B_j^m (4 - 4 B_j \cos \gamma_j + B_j^2) = 1.$$  

(45)

Note that if $m \gg 1$, then obviously $B_j \approx 1$. Therefore, equation (45) can be expressed approximately as

$$B_j^m (4 - 4 \cos \gamma_j + 1) \approx 1 \quad \text{for } m \gg 1.$$  

Consequently, for $m \gg 1$ we have

$$B_j \approx \frac{1}{\sqrt{5 - 4 \cos \gamma_j}} = e^{-\frac{\ln(5 - 4 \cos \gamma_j)}{2m}} \approx \left[1 + \frac{1}{2m}\right]^{-\frac{\ln(5 - 4 \cos \gamma_j)}{2m}}$$

$$= \left(1 + \frac{1}{2m}\right)^{-\ln(5 - 4 \cos \gamma_j)} \approx 1 - \frac{\ln(5 - 4 \cos \gamma_j)}{2m}. \quad (46)$$

With regard to this approximation, let us set

$$B_j = 1 - \frac{\ln(5 - 4 \cos \gamma_j)}{2m} (1 + \delta_j), \quad (47)$$

for all $m$, where $\delta_j$ compensates the error of the approximation (46). Comparing the statement (44) with the definition of $\delta_j$, we shall prove that

$$\delta_j > -\frac{\ln 3}{m} \quad \text{for all } j = 1, \ldots, m - 1.$$  

We proceed by contradiction. Let there be a $j \in \{1, \ldots, m - 1\}$ such that $\delta_j \leq -\frac{\ln 3}{m}$. (Note that necessarily $\delta_j > -1$, because $\beta_j$’s are of moduli less than one.) For all $x > \alpha > 1$, it holds

$$\frac{1}{(1 - \frac{\alpha}{x})^x} = \left(1 + \frac{\alpha}{x - \alpha}\right)^x \left[1 + \frac{1}{\frac{\alpha}{x} - 1}\right]^{\frac{x - 1}{\frac{\alpha}{x} - 1}} < e^{\frac{x - 1}{\frac{\alpha}{x} - 1}} = (e^{\alpha})^{1 + \frac{x - 1}{\frac{\alpha}{x} - 1}}.$$  

Since $B_j = 1 - \frac{\alpha}{x}$ for $x = 2m$ and $\alpha = (1 + \delta_j) \ln(5 - 4 \cos \gamma_j)$, we have

$$\frac{1}{B_j^{2m}} < \left(\frac{(1 + \delta_j) \ln(5 - 4 \cos \gamma_j)}{2m - (1 + \delta_j) \ln(5 - 4 \cos \gamma_j)}\right)^{1 + \frac{(1 + \delta_j) \ln(5 - 4 \cos \gamma_j)}{2m - (1 + \delta_j) \ln(5 - 4 \cos \gamma_j)}} = (5 - 4 \cos \gamma_j)^{(1 + \delta_j)} \left(\frac{(1 + \delta_j) \ln(5 - 4 \cos \gamma_j)}{2m - (1 + \delta_j) \ln(5 - 4 \cos \gamma_j)}\right).$$

Our assumption on $\delta_j$ implies $\delta_j < 0$, therefore

$$\frac{(1 + \delta_j) \ln(5 - 4 \cos \gamma_j)}{2m - (1 + \delta_j) \ln(5 - 4 \cos \gamma_j)} \leq \frac{\ln(5 - 4 \cos \gamma_j)}{2m - \ln(5 - 4 \cos \gamma_j)} :$$
hence
\[ \frac{1}{B_j} < (5 - 4 \cos \gamma_j)^{(1 + \delta_j)\left(1 + \frac{\ln (5 - 4 \cos \gamma_j)}{2m - \ln (5 - 4 \cos \gamma_j)}\right)}. \]  \hfill (48)

At the same time we have from equation (45)
\[ \frac{1}{B_j} = 4 - 4B_j \cos \gamma_j + B_j^2 = 5 - 4 \cos \gamma_j + (1 - B_j)(4 \cos \gamma_j - 2) + (1 - B_j)^2 \]
\[ > 5 - 4 \cos \gamma_j + (1 - B_j)(4 \cos \gamma_j - 2). \]  \hfill (49)

Putting inequalities (48) and (49) together, we get
\[ (5 - 4 \cos \gamma_j)^{(1 + \delta_j)\left(1 + \frac{\ln (5 - 4 \cos \gamma_j)}{2m - \ln (5 - 4 \cos \gamma_j)}\right)} > 5 - 4 \cos \gamma_j + (1 - B_j)(4 \cos \gamma_j - 2); \]
\[ \text{hence} \]
\[ (5 - 4 \cos \gamma_j)^{\delta_j + (1 + \delta_j)\left(1 + \frac{\ln (5 - 4 \cos \gamma_j)}{2m - \ln (5 - 4 \cos \gamma_j)}\right)} > 1 + (1 - B_j)^2 \frac{2}{5 - 4 \cos \gamma_j}. \]

This gives, with regard to equation (47),
\[ e^{\left(\delta_j + (1 + \delta_j)\frac{\ln (5 - 4 \cos \gamma_j)}{2m - \ln (5 - 4 \cos \gamma_j)}\right)} \ln (5 - 4 \cos \gamma_j) - 1 > \frac{\ln (5 - 4 \cos \gamma_j)}{2m - \ln (5 - 4 \cos \gamma_j)} \left(1 + \delta_j\right) \frac{4 \cos \gamma_j - 2}{5 - 4 \cos \gamma_j}. \]  \hfill (50)

Since \( \delta_j \leq \frac{\ln 9}{2m} \leq -\frac{\ln (5 - 4 \cos \gamma_j)}{2m} \) by assumption, it holds
\[ \delta_j + (1 + \delta_j)\frac{\ln (5 - 4 \cos \gamma_j)}{2m - \ln (5 - 4 \cos \gamma_j)} \leq 0, \]
therefore, the exponent in (50) is negative (or zero). Moreover, a simple analysis of the exponent, using the fact \( \delta_j > -1 \), leads to the inequality
\[ \left(\delta_j + (1 + \delta_j)\frac{\ln (5 - 4 \cos \gamma_j)}{2m - \ln (5 - 4 \cos \gamma_j)}\right) \ln (5 - 4 \cos \gamma_j) \geq - \ln 9 \quad \text{for all} \ \gamma_j \in \mathbb{R}. \]

The convexity of the exponential function implies
\[ e^x - 1 < \frac{b - 1}{b} x \]
for all \( b < x \leq 0 \). Therefore, the left hand side of (50) obeys
\[ e^{\left(\delta_j + (1 + \delta_j)\frac{\ln (5 - 4 \cos \gamma_j)}{2m - \ln (5 - 4 \cos \gamma_j)}\right)} \ln (5 - 4 \cos \gamma_j) - 1 \]
\[ < \frac{1 - e^{-\ln 9}}{\ln 9} \left(\delta_j + (1 + \delta_j)\frac{\ln (5 - 4 \cos \gamma_j)}{2m - \ln (5 - 4 \cos \gamma_j)}\right) \ln (5 - 4 \cos \gamma_j) \]
\[ = \frac{8}{9 \ln 9} \left(\delta_j + (1 + \delta_j)\frac{\ln (5 - 4 \cos \gamma_j)}{2m - \ln (5 - 4 \cos \gamma_j)}\right) \ln (5 - 4 \cos \gamma_j). \]

Inequality (50) together with this estimate imply
\[ \frac{8}{9 \ln 9} \left(\delta_j + (1 + \delta_j)\frac{\ln (5 - 4 \cos \gamma_j)}{2m - \ln (5 - 4 \cos \gamma_j)}\right) \ln (5 - 4 \cos \gamma_j) > \frac{\ln (5 - 4 \cos \gamma_j)}{2m} \left(1 + \delta_j\right) \frac{4 \cos \gamma_j - 2}{5 - 4 \cos \gamma_j}. \]

We divide both sides by \( \ln (5 - 4 \cos \gamma_j) \), which is allowed due to \( \gamma_j \neq 0 \) (recall that \( \beta_j \notin (0, +\infty) \) for all \( j = 1, \ldots, m - 1 \); hence
\[ \delta_j + (1 + \delta_j)\frac{\ln (5 - 4 \cos \gamma_j)}{2m - \ln (5 - 4 \cos \gamma_j)} > \frac{9 \ln 9}{8} \cdot \frac{1 + \delta_j}{2m} \cdot \frac{4 \cos \gamma_j - 2}{5 - 4 \cos \gamma_j}. \]  \hfill (51)
For all $\gamma_j \in \mathbb{R}$, $\ln(5 - 4\cos \gamma_j) \leq \ln 9$ and
$$\frac{4 \cos \gamma_j - 2}{5 - 4\cos \gamma_j} = -1 + \frac{3}{5 - 4\cos \gamma_j} \geq -1 + \frac{3}{9} = -\frac{2}{3}.$$  
therefore, with regard to inequality (51),
$$\delta_j + (1 + \delta_j) \frac{\ln 9}{2m - \ln 9} > \frac{9 \ln 9}{8} \cdot \frac{1 + \delta_j}{2m} \cdot \frac{-2}{3} = -\frac{3 \ln 9}{8m} (1 + \delta_j).$$
Consequently,
$$\left(1 + \frac{1}{2m - \ln 9} + \frac{3}{8m}\right) \delta_j > -\frac{1}{2m - \ln 9} - \frac{3}{8m};$$
\hspace{1cm} hence
$$\delta_j > \frac{1}{1 + \frac{1}{2m - \ln 9} + \frac{3}{8m}}.$$  
This is a contradiction with the assumption $\delta_j \leq -\frac{\ln 3}{m}$, because
$$-\frac{\ln 3}{m} < -\frac{1}{2m - \ln 9} + \frac{3}{8m} \quad \text{for all } m \geq 2. $$
\hfill \Box

**Lemma A.3.** The arguments of $\beta_j$ satisfy
$$\gamma_j \in \left(\frac{2\pi}{m} - \frac{\pi}{6m}, j\frac{2\pi}{m} + \frac{\pi}{6m}\right)$$
\hspace{1cm} (52)
for all $j = 1, \ldots, m - 1$.

**Proof.** Equation (41) has $m + 1$ solutions, namely $1$, $\beta$ and $\beta_1, \ldots, \beta_{m-1}$. Therefore, it suffices to show that every sector
$$S_j := \{Be^{i\gamma} \mid B > 0, \gamma \in \left(\frac{2\pi}{m} - \frac{\pi}{6m}, j\frac{2\pi}{m} + \frac{\pi}{6m}\right)\}$$
\hspace{1cm} for $j = 1, \ldots, m - 1$
contains exactly one solution of equation (41).

Let
$$\beta = Be^{i\gamma}$$
\hspace{1cm} be a solution of (41), i.e.,
$$B^m e^{im\gamma} (2 - Be^{i\gamma}) = 1.$$  
Hence
$$m\gamma = -\arg (2 - Be^{i\gamma}) + 2j\pi \quad \text{for a certain } j \in \mathbb{Z}.$$  
(53)
We can obviously assume $j \in \{0, 1, \ldots, m-1\}$ without loss of generality. Since the solutions $1$ and $\beta$ of equation (53) are obtained for $\gamma = 0$, and, therefore, for $j = 0$, we prove the statement in two steps: 1. We demonstrate that equation (53) has exactly one solution for every $j = 1, \ldots, m-1$. 2. We show that the solution corresponding to $j$ belongs to the sector $S_j$ for every $j = 1, \ldots, m-1$.

It holds
$$2 - Be^{i\gamma} = 2 - B \cos \gamma - iB \sin \gamma,$$
hence
$$\tan (\arg (2 - Be^{i\gamma})) = \frac{-B \sin \gamma}{2 - B \cos \gamma} = \frac{-\sin \gamma}{\frac{2}{\pi} - \cos \gamma}. $$
Furthermore, $B < 1$ implies $2 - B \cos \gamma > 0$, hence
\[
\arg(2 - Be^\gamma) \in (-\pi/2, \pi/2),
\] (54)
i.e., we can write
\[
\arg \left(2 - Be^\gamma\right) = \arctan \frac{-\sin \gamma}{\frac{2}{B} - \cos \gamma}.
\]
To sum up, equation (53) is equivalent to
\[
m\gamma - \arctan \sin \gamma \frac{2}{B} - \cos \gamma = 2j\pi.
\] (55)
For every $j = 1, \ldots, m - 1$, the left hand side $L(\gamma) = m\gamma - \arctan \frac{\sin \gamma}{\frac{2}{B} - \cos \gamma}$ of equation (55), regarded as a function of $\gamma$ with a fixed $B < 1$, is continuous and satisfies
\[
0 = L(0) < 2j\pi < 2m\pi = L(2\pi).
\]
Also, a simple calculation gives
\[
L'(\gamma) = m - \frac{\frac{2}{B} \cos \gamma - 1}{\left(\frac{2}{B}\right)^2 - 2 \cdot \frac{2}{B} \cos \gamma + 1} > m - \frac{1}{\frac{2}{B} - 1} > m - 1 > 0.
\]
Consequently, equation (55) has indeed exactly one solution for every $j = 1, \ldots, m - 1$. The solution satisfies $m\gamma - 2j\pi \in (-\pi/2, \pi/2)$. With regard to the numbering (3), we conclude that
\[
\gamma_j \in \left(\frac{2j\pi}{m} - \frac{\pi}{2m}, \frac{2j\pi}{m} + \frac{\pi}{2m}\right).
\]
Now we improve this estimate in order to prove $\gamma_j \in S_j$. Since $2/B_j > 2$ for all $j = 1, \ldots, m - 1$, we have
\[
\left|\frac{-\sin \gamma_j}{\frac{2}{B_j} - \cos \gamma_j}\right| \leq \left|\frac{\sin \gamma_j}{2 - \cos \gamma_j}\right|.
\]
It is easy to show that
\[
\left|\frac{\sin \gamma}{2 - \cos \gamma}\right| \leq \frac{1}{\sqrt{3}} \quad \text{for all } \gamma \in \mathbb{R},
\]
hence
\[
\left|\arctan \frac{\sin \gamma_j}{\frac{2}{B_j} - \cos \gamma_j}\right| \leq \arctan \frac{1}{\sqrt{3}} = \frac{\pi}{6}.
\] (56)
By substituting estimate (56) into equation (55), we obtain statement (52). \qed