Dynamic viscoelastic piezoelectric contact with adhesion

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Abstract. A model of a dynamic viscoelastic adhesive contact between a piezoelectric body and a deformable foundation is described. The model consists of a system of the hemivariational inequality of hyperbolic type for the displacement, the time dependent elliptic equation for the electric potential and the ordinary differential equation for the adhesion field. In the hemivariational inequality the normal and friction forces are derived from nonconvex superpotentials through the generalized Clarke subdifferential. Theorem on existence and regularity of a weak solution is provided. Several examples illustrate the applicability of our results.

1. Introduction
Recently, the analysis of various models in solid mechanics involving piezoelectric materials have attracted considerable attention. This is due to the fact that such materials are widely used to design smart structures, typically sensor-actuator systems. Most of the applications deal with plate- or rod-like bodies, cf. [1,9]. Piezoelectric materials can be either dielectrics or semiconductors, cf. [4].

The aim of this work is to obtain a mathematical model for dynamic and piezoviscoelectric adhesive contact between a body and a deformable foundation. We assume that the contact is modeled by subdifferential boundary conditions involving nonsmooth and nonconvex superpotentials. The behavior of the material is described by a modified Kelvin-Voigt constitutive law which takes into account the piezoelectric effect of the body. The evolution of the bonding field is described by an additional variable which is governed by an ordinary differential equation on the contact surface.

The present paper represents a continuation of [6] where a general method for the study of dynamic viscoelastic contact problems involving subdifferential boundary conditions was presented. Within the framework of evolutionary hemivariational inequalities, this method represents a new approach which unifies several methods used in the study of viscoelastic contact and allows to obtain new existence and uniqueness results. Note that the frictional contact models treated in [6] involve only viscoelastic materials. The novelty of the present paper arises in the extension of the results in [6] to materials with piezoelectric properties and taking into account the phenomenon of adhesion. Because of nonmonotone possibly multivalued boundary conditions, the mathematical problem is formulated as a system of the hemivariational inequality of hyperbolic type for the displacement, the time dependent elliptic boundary value problem for the electric potential and the ordinary differential equation for the bonding
field. In this context the paper opens the way to study further frictional contact problems involving additional effects which can be involved in contact.

We mention that analysis of various models for adhesive contact can be found in \[4,9,10\] and the references therein. We underline that the methods used in the aforementioned works can not be applied to the model under consideration since the subdifferential boundary conditions are nonmonotone and multivalued. Therefore, we use arguments as in \[5,6,8\] developed in the theory of hemivariational inequalities combined with results for elliptic problems and ordinary differential equations. We provide the existence and regularity of a weak solution to the model. The question of uniqueness of the solution is left open.

We recall the definitions of the generalized directional derivative and the generalized gradient of Clarke for a locally Lipschitz function \( h : X \to \mathbb{R} \), where \( X \) is a Banach space (see \[2\]). The generalized directional derivative of \( h \) at \( x \in X \) in the direction \( v \in X \) is defined by

\[
h^0(x;v) = \lim_{y \to x, y \neq x} \frac{h(y+tv) - h(y)}{t}.
\]

The generalized gradient of \( h \) at \( x \) is a subset of a dual space \( X^* \) given by

\[
\partial h(x) = \{ \zeta \in X^* : h(x) \geq \langle \zeta , v \rangle_{X^*} \quad \text{for all} \quad v \in X \},
\]

where \( \langle \cdot, \cdot \rangle_{X^* \times X} \) is the duality pairing of \( X \) and its dual \( X^* \). A locally Lipschitz function \( h \) is called regular (in the sense of Clarke) at \( x \) if for all \( v \in X \) the one-sided directional derivative \( h'(x;v) \) exists and satisfies

\[
h^0(x;v) = h'(x;v)
\]

for all \( v \in X \).

We denote by \( S_d \) the linear space of second order symmetric tensors on \( \mathbb{R}^d \), or equivalently, the space of symmetric matrices of order \( d \). The inner products and the corresponding norms on \( \mathbb{R}^d \) are given by \( u \cdot v = u_i v_i \), \( \| v \|_{\mathbb{R}^d} = (v \cdot v)^{1/2} \) for all \( u, v \in \mathbb{R}^d \) and \( \sigma : \tau = \sigma_{ij} \tau_{ij} \), \( \| \tau \|_{S_d} = (\tau \cdot \tau)^{1/2} \) for all \( \sigma, \tau \in S_d \). In this paper the indices \( i,j,k,l \) run from 1 to \( d \). Summation convention over repeated indices is used and the index that follows a comma indicates a partial derivative. Let \( \Omega \subset \mathbb{R}^d \) be an open bounded set and let the boundary \( \Gamma \) of \( \Omega \) be Lipschitz continuous. Given regular \( v : \Gamma \to \mathbb{R}^d \) and \( \sigma : \Gamma \to S_d \), we denote by \( v_n, v_T, \sigma_N, \sigma_T \) the usual normal and tangential components of \( v \) and \( \sigma \) on the boundary, i.e. \( v_n = v \cdot n \), \( v_T = v - v_n n \), \( \sigma_N = (\sigma n) \cdot n \) and \( \sigma_T = \sigma n - \sigma_N n \).

2. Introduction

The physical setting of the problem under consideration is as follows. A viscoelastic piezoelectric body occupies, in its undeformed state, a bounded open set \( \Omega \) of \( \mathbb{R}^d \), \( d = 2, 3 \) with a regular boundary \( \Gamma \). The body is in adhesive contact with an insulator obstacle, the so-called foundation. We consider two partitions of \( \Gamma \). A first partition is given by three disjoint measurable parts \( \Gamma_D, \Gamma_N \) and \( \Gamma_C \) such that \( m(\Gamma_D) > 0 \) while a second one consists of two disjoint measurable parts \( \Gamma_1 \) and \( \Gamma_2 \) such that \( m(\Gamma_1) > 0 \) and \( \Gamma_1 \subset \Gamma_2 \). We suppose that the body is clamped on \( \Gamma_D \) and the displacement field vanishes there. Surface tractions of density \( g \) act on \( \Gamma_N \), \( \Gamma_C \) is a contacting surface and \( f_0 \) denotes the density of body forces. For simplicity, we assume also free electric charges. Due to the adhesive contact, we suppose two nonmonotone possibly multivalued laws between the normal stress \( \sigma_N \) and the normal displacement \( u_N \), and between the shear \( \sigma_T \) and the tangential displacement \( u_T \). These laws depend also on a bonding field, denoted by \( \beta \), which describes the pointwise fractional density of active bonds on \( \Gamma_C \).
The evolution of the bonding field is governed by an ordinary differential equation depending on the displacement and considered on contact surface (cf. [3]). When \( \beta = 1 \) at a point of the contact part, the adhesion is complete and all the bonds are active, when \( \beta = 0 \) all bonds are inactive and there is no adhesion, when \( 0 < \beta < 1 \) the adhesion is partial and a fracture \( \beta \) of the bonds is active. Let \( Q = \Omega \times (0,T) \) with \((0,T)\) being a finite time interval.

The dynamical model for the process is as follows: find a displacement field \( u : Q \rightarrow \mathbb{R}^d \), an electric potential \( \varphi : Q \rightarrow \mathbb{R} \) and a bonding field \( \beta : \Gamma_c \times (0,T) \rightarrow [0,1] \) such that

\[
\begin{aligned}
\sigma(t) &= A(\varepsilon(u(t))) + B(\varepsilon(u'(t))) - P^T E(\varphi(t)) \\
D(t) &= DE(\varphi(t)) + P \varepsilon(u(t)) \\
u(t) &= 0 \\
\sigma(t)n &= g(t) \\
\varphi(t) &= 0 \\
D(t) \cdot n &= 0 \\
-\sigma_n(t) &= \partial_j \sigma_j(u(t),u_n(t)) \\
-\sigma_t(t) &= \partial_j \sigma_j(u(t),u_t(t)) \\
\beta(t) &= F(t,u(t),\beta(t)) \\
\beta(0) &= \beta_0 \\
u(0) &= u_0, \quad u'(0) = u_1
\end{aligned}
\]

In (1)-(13), in order to simplify the notation, we do not indicate explicitly the dependence of various functions and tensors on the spatial variable \( x \in \Omega \cup \Gamma \). Equations (1) and (2) are the equation of motion for the stress field and the equilibrium equation for the electric displacement field, respectively. Recall that \( \text{Div} \) is the divergence operator for tensor valued functions given by

\[
\text{Div} \sigma = (\sigma_{ij})
\]

and \( \text{div} \) stands for the divergence operator for vector valued functions, i.e. \( \text{div} D = (D_{ij}) \). The relations (3) and (4) represent the electroviscoelastic constitutive law of the material in which \( A, B, D \) and \( P \) are respectively the (fourth order) elasticity tensor, the (fourth order) viscosity tensor, the (second order) electric permittivity tensor and the (third order) piezoelectric tensor. The equation (3) describes the converse effect and (4) models the direct effect of piezoelectricity. Furthermore, \( \sigma : Q \rightarrow S_d, \sigma = (\sigma_j) \) and \( D : Q \rightarrow R^d, D = (D_j) \) denote the stress tensor and the electric displacement field, respectively and \( P^T \) is the tensor transposed to \( P \). We use here the notation \( p^T \) to denote the transpose of the tensor \( p \) given by \( p \tau \cdot v = \tau : p^Tv \) for \( \tau \in S_d \) and \( v \in \mathbb{R}^d \). The elastic strain-displacement and electric field-potential relations are given by

\[
\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T), \quad E(\varphi) = -\nabla \varphi \text{ in } Q, \quad \text{where } \varepsilon(u) = (\varepsilon_{ij}(u)) \text{ and } E(\varphi) = (E_i(\varphi)) \text{ denote the linear strain tensor and the electric vector field, respectively. Here } u : Q \rightarrow \mathbb{R}^d, \quad u = (u_i) \text{ and } \varphi : Q \rightarrow \mathbb{R} \text{ are the displacement vector field and the electric potential (scalar field), respectively. The equations (5) and (6) are the displacement and traction boundary conditions,}
\]

\[
\begin{aligned}
\text{in } Q \\
\text{on } \Gamma_D \times (0,T) \\
\text{on } \Gamma_N \times (0,T) \\
\text{on } \Gamma_1 \times (0,T) \\
\text{on } \Gamma_2 \times (0,T) \\
\text{on } \Gamma_c \times (0,T) \\
\text{on } \Gamma_c \times (0,T) \\
\text{on } \Gamma_c \\
in \Omega
\end{aligned}
\]
respectively while (7) and (8) represent the electric boundary conditions. Conditions (9) and (10) are the subdifferential boundary conditions with nonconvex nonsmooth superpotentials \( j_N \) and \( j_T \). These relations say that the normal (resp. tangential) traction \( \sigma_N \) (resp. \( \sigma_T \)) depends on the intensity of adhesion \( \beta \) and the normal (resp. tangential) displacement \( u_N \) (resp. \( u_T \)). The evolution of the adhesion (bonding) field \( \beta \) is governed (cf. [3]) by the ordinary differential equation (11) on the contact surface and (12) represents a given initial bonding field. The adhesive evolution rate function \( F \) depends on both the bonding field and the displacement and may change sign. This allows for rebonding after debonding took place, and it allows for possible cycles of debonding and rebonding (cf. [9] for examples of \( F \)). Finally, the initial values for the displacement and the velocity are prescribed in (13).

We now turn to the variational formulation of the problem (1)–(13). We introduce the spaces for the displacement and the electric potential: \( V = \{ v \in H^1(\Omega; R^d) : v = 0 \text{ on } \Gamma_D \} \) and \( \Phi = \{ \phi \in H^1(\Omega) : \phi = 0 \text{ on } \Gamma_N \} \). Let \( H = L^2(\Omega; R^d) \) and \( Z = H^1(\Omega; R^\nu) \) with a fixed \( \delta \in (1/2,1) \).

We have \( V \subset Z \subset H \subset Z^* \subset V^* \) with all embeddings being compact and \( (V,H,V^*) \) forms an evolution triple of spaces (cf. Chapter 3.4 of [2]). Define \( \mathcal{H} = L^2(\Omega; S^d) \), \( \mathcal{V} = L^2(0,T;V) \) and \( \mathcal{W} = \{ w \in L^2(0,T;V) : w \in L^2(0,T;V^*) \} \), where the time derivative involved in the definition is understood in the sense of vector valued distributions.

Assuming sufficient regularity of the functions and tensors involved in the problem (1)–(13), multiplying (1) by \( v \in V \), (2) by \( \phi \in \Phi \), using the Green formula and the boundary conditions (6), (9) and (10), we obtain (cf. [6,7] for details) the following variational formulation: find a displacement \( u : (0,T) \to V \), an electric potential \( \phi : (0,T) \to \Phi \) and a bonding field \( \beta : (0,T) \to L^2(\Gamma_C) \) such that

\[
\begin{align*}
\langle u''(t), v \rangle_{V',V} + \langle A \varepsilon(u(t)) + B \varepsilon(u'(t)) + P^T \nabla \phi(t), \varepsilon(v) \rangle + \\
\int_{\Gamma_C} \left( f_N^b(x, \beta(t), u_N(t), v_N) + f_T^b(x, \beta(t), u_T(t), v_T) \right) d\Gamma(x) \geq \langle f(t), v \rangle
\end{align*}

\text{for all } v \in V \text{ and a.e. } t \in (0,T) \tag{14}
\]

\[
\langle D\nabla \phi(t), D\psi \rangle - \langle P\varepsilon(u(t)), D\psi \rangle = 0 \quad \text{for all } v \in V \text{ and a.e. } t \in (0,T)
\]

\[
\beta'(t) = F(t,u(t),\beta(t)) \quad \text{on } \Gamma_C, \quad \text{for a.e. } t \in (0,T)
\]

\[
\beta(0) = \beta_0 \quad \text{on } \Gamma_C, \quad u(0) = u_0, \quad u'(0) = u_1 \quad \text{in } \Omega
\]

We impose the following hypotheses.

\( \overline{H(a)} \): The elasticity tensor field \( a = (a_{ijkl}) \) satisfies \( a_{ijkl} = a_{kl ij} = a_{ijlk} \in L^\nu(\Omega) \) and \( a_{ijkl}(x) \tau_{ij} \tau_{kl} \geq m_a \tau_{ij} \tau_{kl} \) for a.e. \( x \in \Omega \), all \( \tau = (\tau_{ij}) \in S_d \) with \( m_a > 0 \).

\( \overline{H(b)} \): The viscosity tensor field \( b = (b_{ijkl}) \) satisfies \( b_{ijkl} = b_{kl ij} = b_{ijlk} \in L^\nu(\Omega) \) and \( b_{ijkl}(x) \tau_{ij} \tau_{kl} \geq m_b \tau_{ij} \tau_{kl} \) for a.e. \( x \in \Omega \), all \( \tau = (\tau_{ij}) \in S_d \) with \( m_b > 0 \).

\( \overline{H(d)} \): The dielectric tensor field \( d = (d_{ij}) \) satisfies \( d_{ij} = d_{ji} \in L^\nu(\Omega) \) and \( d_{ij}(x) \xi_i \xi_j \geq m_d \xi_i \xi_j \) for a.e. \( x \in \Omega \), all \( \xi = (\xi_i) \in R^d \) with \( m_d > 0 \).

\( \overline{H(p)} \): The piezoelectric tensor field \( p = (p_{ijk}) \) satisfies \( p_{ijk} = p_{kj i} \in L^\nu(\Omega) \).

\( \overline{H(j_N)} \): The superpotential \( j_N : \Gamma_C \times R \times R \to R \) is a function such that
\( j_N(\cdot,r,s) \) is measurable for all \( r,s \in R \) and \( j_N(\cdot,0) \in L^1(\Gamma_C) \);

(ii) \( j_N(x,r,r) \) is locally Lipschitz for a.e. \( x \in \Gamma_C \) and all \( r \in R \);

(iii) \( |\tilde{\partial}_N(x,r,s)| \leq c_N(1 + |s|) \) for all \( (x,r,s) \in \Gamma_C \times R \times R \) with \( c_N > 0 \);

(iv) either \( j_N(x,r,r) \) or \( -j_N(x,r,r) \) is regular for a.e. \( x \in \Gamma_C \) and all \( r \in R \);

(v) limsup \( j_N^0(x,r_n,s_n;\eta) \leq j_N^0(x,r,s;\eta) \) for all \( r_n \to r \), \( s_n \to s \) for all \( \eta \in R \).

**H(\( j_T \)):** The superpotential \( j_T : \Gamma_C \times R \times R^d \to R \) is a function such that

(i) \( j_T(\cdot,r,\xi) \) is measurable for all \( (r,\xi) \in R \times R^d \) and \( j_T(\cdot,0) \in L^1(\Gamma_C) \);

(ii) \( j_T(x,r,r) \) is locally Lipschitz for a.e. \( x \in \Gamma_C \) and all \( r \in R \);

(iii) \( |\tilde{\partial}_T(x,r,\xi)| \leq c_T(1 + |\xi|) \) for all \( (x,r,\xi) \in \Gamma_C \times R \times R^d \) with \( c_T > 0 \);

(iv) either \( j_T(x,r,r) \) or \( -j_T(x,r,r) \) is regular for a.e. \( x \in \Gamma_C \) and all \( r \in R \);

(v) limsup \( j_T^0(x,r_n,\xi_n;\eta) \leq j_T^0(x,r,\xi;\eta) \) for all \( r_n \to r \), \( \xi_n \to \xi \) for all \( \eta \in R^d \).

where the Clarke subdifferentials and the generalized directional derivatives are taken with respect to the last variables of \( j_N \) and \( j_T \).

**H(\( F \)):** The adhesive evolution rate function \( F : \Gamma_C \times (0,T) \times R^d \times R \to R \) satisfies

(i) \( F(\cdot,r,\xi,r) \) is measurable on \( \Gamma_C \times (0,T) \) and \( F(x,t,\cdot,\cdot) \) is continuous on \( R^d \times R \);

(ii) \( |F(x,t,\xi,\eta) - F(x,t,\xi',\eta) - L_F| \leq r_{\xi} \) for all \( x \in R^d \), \( \xi,\xi' \in R^d \) with \( L_F > 0 \);

(iii) \( |F(x,t,\xi_i,\eta_i) - F(x,t,\xi_i,\eta_i) - L_F| \leq |\xi_1 - \xi_2| + |\eta_i - \eta_j| \) for all \( \xi_i,\xi_j \in R^d \), \( \eta_i,\eta_j \in [0,1] \);

(iv) \( F(x,t,\xi,0) = 0 \), \( F(x,t,\xi,r) \geq 0 \) for \( r \leq 0 \) and \( F(x,t,\xi,r) \leq 0 \) for \( r \geq 1 \).

We can now state our main result on the existence and regularity of weak solutions to the contact model.

**Theorem.** Assume that the hypotheses \( H(a) \), \( H(b) \), \( H(d) \), \( H(p) \), \( H(j_N) \), \( H(j_T) \), \( (H_0) \) and \( H(F) \) hold and \( m_a > 2 \max\{|c_N,c_T|T\} M^2 \| \gamma \|^2 \), where \( \| \gamma \| \) denotes the norm of the trace operator \( \gamma : Z \to L^2(\Gamma;R^d) \), \( M > 0 \) is the embedding constant of \( V \) into \( Z \), \( m_a \), \( c_N \), and \( c_T \) are given by \( H(a) \), \( H(j_N)(iii) \) and \( H(j_T)(iii) \), respectively. Then the problem (14) has a solution \( (u,\varphi,\beta) \) such that \( u \in \mathcal{V} \) with \( u' \in \mathcal{W} \), \( \varphi \in C(0,T;\Phi) \) and \( \beta \in W^{1,\infty}(0,T;L^2(\Gamma_C)) \) with \( 0 \leq \beta(x) \leq 1 \) a.e. \( x \in \Gamma_C \).

The idea of the proof is as follows. We first observe that in the elliptic equation in (14), the time appears as a parameter and the dependence on time is induced by the displacement field. We study the dependence of the electric potential on the displacement in this equation. Next, we formulate the hemivariational inequality in terms of the displacement field only and establish the existence by formulating this inequality in an equivalent form of a second order evolution inclusion. The existence of a solution to the associated evolution inclusion is proved by applying the surjectivity result for L-pseudomonotone operators in Banach spaces (cf. Theorem 1.3.73 of [2]). Finally, using the Cauchy-Lipschitz type theorem for ordinary differential equation (11) we obtain the desired regularity of the bonding field.

2.1. Examples
We comment on examples of functions $j_T$ and $j_r$ which are met in the literature and satisfy our hypotheses.

**Example 1.** Consider the function $j_T: \Gamma_c \times R \times R^d \to R$ given by

$$j_T(x,r,\xi) = g(x,r)h(\xi)$$

for $(x,r,\xi) \in \Gamma_c \times R \times R^d$, where $g$ and $h$ satisfy suitable assumptions (cf. [6,7]). This choice leads to the boundary condition (10) of the form $-\sigma_T(t) \in g(x,\beta(t))\partial h(u_T(t))$ on $\Gamma_c \times (0,T)$. The case when $g$ is independent of $x$, Lipschitz continuous, nonnegative and bounded, and $h: R^d \to R$ is a convex (and thus regular) function given by $h(\xi) = 0.5 \| \xi \|^2$ if $\| \xi \| \leq L_0$ and $h(\xi) = L_0 \| \xi \| - 0.5L_0^2\| \xi \|^2$ if $\| \xi \| > L_0$ was considered in Chapter 11.4 of [9]. An example of nonconvex superpotential $h: R^d \to R$ is given by $h(\xi) = \| \xi \|^2$ if $\| \xi \| \leq L_0$ and $h(\xi) = L_0^2\| \xi \|^2$ if $\| \xi \| > L_0$, where a positive constant $L_0$ represents the bond length.

**Example 2.** We remark that the laws (9) and (10) are extensions of conditions considered in [6]. The contact with nonmonotone normal compliance is obtained when $h$ depends only on its last variable and $j_T(r) = \int_0^r p_N(\tau) d\tau$, $r \in R$, where $p_N \in L^\infty_{\text{loc}}(R)$ is such that $|p_N(s)| \leq p_1(1+|s|)$ for $s \in R$ with $p_1 > 0$. This model example can be modified to obtain nonmonotone zig-zag relations which describe the adhesive contact problems and contact laws for a granular material and a reinforced concrete (see Sections 2.4 and 7.2 of [8]). We remark that this example covers the nonmonotone Winkler law. If $j_T = 0$, we are lead to frictionless contact. If $j_T(x,t,r,\xi) = p(t)g(x,r)\| \xi \|$, where $p \in L^\infty(0,T)$ is positive and $g$ is as before, then we obtain a version of Tresca’s friction law. The multivalued condition (10) appears in several mechanical problems. We mention the reaction-displacement diagrams which are the nonmonotone variants of the friction law of Coulomb. Analogous situations arise in geomechanics and rock interface analysis as well as in friction laws between reinforcement and concrete in concrete structures. The sawtooth laws generated by nonconvex superpotentials $j_T$ (see Chapter 2.4 of [8]) describe the partial cracking and crushing of the adhesive bonding material.

For examples of the adhesive evolution rate function we refer to [9,10,7].

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