ANALYSIS OF ODD/ODD VERTEX REMOVAL GAMES ON SPECIAL GRAPHS

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Abstract

In this paper we analyze the odd/odd vertex removal game introduced by P. Ot- taway. We prove that every bipartite graph has Grundy value 0 or 1 only depending on the parity of the number of edges in the graph. In addition we also reduce a conjecture proposed by K. Shelton to a seemingly simpler one, in order to be able to show that there are graphs in the odd/odd vertex removal game for every possible Grundy value. Only the proof of the latter is incomplete and depends on this new conjecture.
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1 Introduction

In this paper we will analyze some properties of a combinatorial game played on graphs. The game consists of two players removing the vertices of the graph depending on the parity of the degree of the vertices. We call these types of games *vertex removal games*, and the game we will study in particular we call the *odd/odd vertex removal game*. This game was introduced by Ottaway[5] and some results concerning the game when played on particular kinds of graphs (for example path graphs) are presented there. Furthermore, Shelton[6] studied a more general problem but conjectured some results concerning the odd/odd vertex removal game.

In this paper we will first make an introduction to the field of combinatorial game theory in general and Sprague-Grundy theory in particular to the extent that is needed to present the two main results of this article. There are quite a lot of games that are studied in the field of combinatorial game theory, many examples of which Conway and Berlekamp[2, 1], present and analyze (or even invent). We will however in this paper concentrate on one particular combinatorial game.

In Section 2 we prove two new results. The first of these is a more general case of a conjecture which was introduced by Shelton[6, Conjecture 17]. We prove that every bipartite graph has Grundy value 1 or 0 depending only on the number of edges in the graph. (For a description of what Grundy values are and their uses in analyzing a game, see Section 1.1.2.)

The second result we prove is incomplete; we would like to prove that there are graphs for every possible Grundy value. We can do this by proving a special case of another conjecture made by Shelton[6, Conjecture 16]. This is what we do except for a very particular, however necessary, case. We have to settle for just reducing the problem to another, hopefully and seemingly simpler conjecture. The presented conjecture has not been thoroughly investigated and might thus not be far from being solved.

In Section 3 we thus present some ideas of how one might attempt to prove the conjecture we left open and furthermore present some of the other open problems concerning the odd/odd vertex removal game.

1.1 Combinatorial game theory

This paper is concerned with what is known as combinatorial game theory which is quite different from what is most often meant by “game theory”. Combinatorial game theory deal with a specific type of games, namely those satisfying the following conditions:
1. There are two players, who alternate moves.

2. There is a set, usually finite, of possible positions of the game.

3. For each position the game specifies a set of legal moves for each player. If for all positions this set is the same for both players we call the game *impartial*. 

4. The game ends when a position is reached from which no moves are possible. The last player to move wins the game.

5. The game ends after a finite number of moves.

This paper will only deal with impartial games, since in particular the object of our study, the odd/odd vertex removal game, is an impartial combinatorial game. Henceforth in this paper when we talk about to a “game” we will always mean an impartial combinatorial game.

The positions which can be reached (in one move) from a position $P$ are called the *options* of $P$. If the set of options is empty we have reached the end of the game, positions with no options are said to be *terminal*. We will talk about positions of games as games since any position of a game may be considered as another game; just with the current position as the starting position.

1.1.1 Classes $\mathcal{N}$ and $\mathcal{P}$

There are several ways of classifying the possible positions of a game. The most trivial way is to divide the positions into two classes, $\mathcal{P}$ and $\mathcal{N}$. A position belongs to $\mathcal{P}$ if the previous player (last player to move) will win the game if that player makes optimal moves. The set $\mathcal{N}$ consists of the positions for which the next player to move wins when the game is played optimally. For impartial games every position either belongs to $\mathcal{P}$ or $\mathcal{N}$.

Clearly this means that a position belongs to $\mathcal{P}$ if and only if all options of that position belongs to $\mathcal{N}$, while a position lies in $\mathcal{N}$ if and only if there is at least one option which lies in $\mathcal{P}$.

1.1.2 Grundy value

Another way of classifying the positions of a game is by giving each position a numerical value, called its Grundy value. The Grundy value of a position $P$ is denoted $g(P)$ and is
defined recursively as
\[
g(P) = \begin{cases} 
0 & \text{if } P \text{ is terminal} \\
\text{mex}\{g(P') : P' \text{ option of } P\} & \text{otherwise}
\end{cases}
\]
where mex is the *minimal excludant* of a set of integers. The minimal excludant of a set \( S \) is the smallest nonnegative integer which does not lie in the set \( S \), e.g. \( \text{mex}\{0, 1, 2\} = 3, \) \( \text{mex}\{0, 2, 5, 7\} = 1 \).

It might not be immediately clear why this classification is more useful than the one of dividing the positions into \( \mathcal{N} \) and \( \mathcal{P} \), but we will later (in Section 1.1.4) present a result which shows that the Grundy value is extremely useful when considering sums of games. Here we will however state a result that the Grundy value classification is as least as powerful as the \( \mathcal{N}/\mathcal{P} \) one.

**Theorem 1.** A position \( P \) belongs to \( \mathcal{N} \) if and only if its Grundy value \( g(P) \) is positive.

**Proof.** \( P \in \mathcal{N} \) if and only if some option of \( P \) belongs to \( \mathcal{P} \) and \( P \in \mathcal{P} \) if and only if all options of \( P \) belongs to \( \mathcal{N} \).

Thus by the definition of Grundy value we may inductively claim that each position in \( \mathcal{P} \) has Grundy value 0 (since either it is terminal or all its options have positive Grundy value) and each position in \( \mathcal{N} \) has Grundy value > 0 (since at least one option has Grundy value 0).

\[\square\]

### 1.1.3 Nim-sum

An important concept for the analysis of impartial games is the *nim-sum*. The nim-sum of two nonnegative integers \( n \) and \( m \) is denoted \( n \oplus m \) and is defined as the bitwise XOR of the binary representation of the integers \( n \) and \( m \).

For example \( 21 \oplus 12 = (10101)_2 \) XOR \( (1100)_2 = (11001)_2 = 25 \).

The name nim-sum comes from that we use this sum when analyzing the classical game of Nim[1]. The game of Nim can be considered central in the theory of impartial games since all games can be shown to be equivalent to some Nim game.

### 1.1.4 Sums of impartial games

We define the sum of two games \( G \) and \( H \) as the game where \( G \) and \( H \) are played in parallel, and denote this game by \( G + H \). By being played in parallel we mean that a player can make a move in either \( G \) or \( H \), which corresponds to one move in \( G + H \).

More explicitly the set of options of \( G + H \) is \( \{G + H' : H' \text{ option of } H\} \cup \{G' + H : G' \text{ option of } G\} \). Naturally we are interested in the answer of the question: Does \( G + H \)
belong to $\mathcal{N}$ or $\mathcal{P}$? The answer to this question can be decided using the nim-sum of the Grundy value of each part, i.e:

**Theorem 2.** [2] Let $G$ and $H$ be two impartial games. The Grundy value of the sum $G + H$ is

\[ g(G + H) = g(G) \oplus g(H) \]

### 1.2 Vertex removal games

In this paper we are going to analyze a vertex removal game that was introduced by Ottaway[5, 4]. The vertex removal games are played on graphs and each move consists of removing a vertex and all the edges to and from that vertex. We decide what vertices we are allowed to remove by some rule concerning the parity of the degree of the vertices. For these games every position may be denoted by a graph and we will make no distinction between a graph and a position in these games.

For example one of the two players may only be allowed to remove edges of even degree and the other only edges of odd degree. This example is obviously a partial game since the two players have a different set of options. Games of the same type but played in digraphs is also possible, but will not be considered here.

Shelton studies a more general problem than this[6], where one considers every vertex in a graph to be a coin, i.e. a binary value of either heads or tails. A player can remove any coin with heads up, and then flip any adjacent coins (adjacent vertices in the graph). This is more general than the problem we will study here. This type include the vertex removal game since if one starts with heads up on every vertex with the parity we want to be able to take this game is equivalent to the vertex removal game. Furthermore Shelton proves some results for the Grundy values of these types of games on particular graphs and that the problem played on directed graphs is PSPACE-Complete. He also makes two conjectures concerning the odd/odd vertex removal game which we will study. We will prove one of these conjectures and we will reduce the most interesting case of the other to another, simpler, conjecture.

We will be concerned with the impartial type of vertex removal games in undirected graphs and in particular we will analyze the game where both players only remove edges of odd degree, since for the other impartial game we have the following simple result:

**Theorem 3.** [5, Theorem 3.1.3] When both players can remove only even degree vertices, the game on graph $G = (V, E)$ is trivial and we have that:

\[ g(G) = \begin{cases} 
0 & \text{if } |V| \text{ is even} \\
1 & \text{if } |V| \text{ is odd} 
\end{cases} \]
This result follows from the fact that any graph with an odd number of vertices must have at least one even degree vertex. This means that no position with an odd number of vertices can be terminal. Since $|V|$ alternates between being odd and even the player who always moves to a position with an even number of vertices will win. Thus we get the theorem above.

1.2.1 Odd/odd vertex removal

The type of game where both players are only allowed to remove vertices of an odd degree is the main object of study in this paper. We will in this section present what results are known and in Section 2 we will show some new results.

\[ g(K_3) = 0 \quad g(K_4) = 1 \quad g(S_6) = 1 \quad g(K_{3,2}) = 0 \]

Figure 1: Examples of positions with known Grundy values. The vertices which are removable in the positions are filled in.

Theorem 4. [5] When both players can remove only odd degree vertices we have that the path on $n$ vertices $P_n$, the complete graph on $n$ vertices $K_n$, and the star on $n$ vertices $S_n$ have the same Grundy value

\[ g(P_n) = g(K_n) = g(S_n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \]

The complete bipartite graph $K_{n,m}$ has Grundy value

\[ g(K_{n,m}) = \begin{cases} 1 & \text{if } n \text{ and } m \text{ are both odd} \\ 0 & \text{otherwise} \end{cases} \]

Proof. This proof is presented here for illustrative purposes, so to exemplify the odd/odd vertex removal game.

Consider the path graph $P_n$, only the vertices in the two ends have odd degree and thus any move takes us to $P_{n-1}$. The only reachable terminal position is $P_1$, so $g(P_1) = 0$ and $g(P_n) = \text{mex} \{g(P_{n-1})\}$ recursively gives us precisely the assertion above.
For the complete graph $K_n$ we know that every vertex has degree $n - 1$ so for $K_n$ is
terminal when $n$ is even. If $n$ is odd every move takes us to $K_{n-1}$ which then is terminal,
here the assertion.

For the star $S_n$ of $n$ vertices if $n$ is odd you can only remove a leaf and you get $S_{n-1}$
while if $n$ is even you can either remove the center and reach a terminal position or by
a leaf $S_{n-1}$. Since the only terminal position that is also a star is $S_1$ we have recursively
$g(S_n) = \text{mex}\{g(S_{n-1})\}$, if $n$ is odd, and $g(S_n) = \text{mex}\{g(S_{n-1}), 0\}$, if $n$ is even, with base
case $g(S_1) = 0$ we get a simple induction to complete the assertion.

Lastly for the complete bipartite graph $K_{n,m}$ all moves takes us to another complete
bipartite graph. If there is a move then we reach either $K_{n-1,m}$ or $K_{n,m-1}$. If there is a
move from a position where $m$ or $n$ (inclusive) is even then it is to a complete bipartite
graph with both parts containing an odd number of vertices. There is always a move
when $m$ and $n$ are both odd, this move gives a bipartite graph where one part has an
even number of vertices. Since there is always a move from $K_{m,n}$ when $m$ and $n$ are odd
we have that all terminal positions are are such that $m$ or $n$ is even. Since the positions
alternate between “$m$ and $n$ are odd”, and “$m$ or $n$ is even” no matter how the game is
played. All $P$-positions are those where $m$ or $n$ is even and the others are $N$-positions.
Since all moves from a $N$-position takes us to a $P$-position we cannot get a Grundy value
larger than 1. This proves the last assertion.

The classes of path graphs, $P_n$, stars, $S_n$ and complete bipartite graphs $K_{n,m}$ are all
contained in the class of bipartite (not only complete but arbitrary) graphs. In Section
2 we prove a result that gives us a simple function for computing the Grundy value of
any bipartite graph. This will also prove a conjecture of Shelton[6] which states that all
grid graphs have Grundy value either 0 or 1, which we will prove is true for any bipartite
graph.

The ideas in the proof of Theorem 4 are somewhat similar to the ideas used in the
final part of the proof concerning bipartite graphs.

2 Odd/odd vertex removal analysis

In this section we will analyze the odd/odd vertex removal game in some particular graphs.
We start by proving a result for bipartite graphs. Bipartite graphs include such graph
classes as paths, grids, stars, trees and $k$-partite graphs. Recall that one way of defining a
bipartite graph is as a graph where every closed trail has even length[3, Proposition 1.6.1].
2.1 Bipartite positions

We will make use of the following classic theorem by Euler:

**Theorem 5.** [3, Theorem 1.8.1] A connected graph $G$ has a closed Eulerian trail (sometimes called Eulerian circuit) if and only if every vertex has even degree.

**Corollary 1.** A position $P$, in an odd/odd vertex removal game, is terminal if and only if $P$ is a graph where every connected component has a closed Eulerian trail (with the convention that the empty trail, in the single vertex component, is a closed Eulerian trail).

**Lemma 1.** Every terminal position, $P$, in an odd/odd vertex removal game played on a bipartite graph, $G$, has an even number of edges.

**Proof.** By Corollary 1 each edge in $P$ is part of some closed Eulerian trail. Since every subgraph of a bipartite graph clearly is bipartite (and by the definition of a bipartite graph as only having closed trails of even length) we know that each such trail must have even length by noting that a terminal position must be a subgraph of the initial graph $G$.

Thus each connected component of $P$ has an even number of edges. The total number of edges is the sum of these.

**Theorem 6.** Let $G = (V, E)$ be a bipartite graph, then

$$g(G) = \begin{cases} 
1 & \text{if } |E| \text{ odd} \\
0 & \text{if } |E| \text{ even}
\end{cases}$$

**Proof.** We need just use Lemma 1 to see that all terminal positions have an even number of edges. Thus any position with an odd number of edges cannot be terminal.

With each move an odd number of edges is removed so the number of edges remaining in the graph alternate between even and odd with every move. So one player always moves to a position with an even number of edges and the other always to a position with an odd number of edges. The player who moves to a position with an even number of edges must always win. This means that for $G = (V, E)$ we have $G \in \mathcal{P}$ if $|E|$ is even and $G \in \mathcal{N}$ if $|E|$ is odd.

Since every move from any $\mathcal{N}$-position takes you to a $\mathcal{P}$-position we cannot get any Grundy value larger than one. The assertion above follows.

**Corollary 2.** [6, Conjecture 17] The Grundy value of an odd/odd vertex removal game played on a grid graph is either 0 or 1.
2.2 Graphs for every Grundy value

Another interesting question concerning odd/odd vertex removal games is whether there are graphs for every possible Grundy value. This question was first posed by Ottaway[5]. Shelton[6] went even further and made a conjecture which would answer this question even for connected graphs.

We have already seen graphs of Grundy value 1 and 0. Ottaway gives the following two examples of graphs with Grundy value 2:

![Graphs with Grundy value 2](image)

Figure 2: Odd/odd vertex removal positions with Grundy value 2.

To formulate his conjecture Shelton introduces a special type of graphs, these graphs are composed of the following kind of subgraphs, which we call W, V, P, and H.

![Component subgraphs](image)

Figure 3: The component subgraphs of the class of graphs we are concerned with.

To create the graph types we want to analyze we link the above listed parts together with an edge. Here are two examples of how this can look:

![Graph WWP](image)

Figure 4: The graph WWP, also denoted $W^2P$.

![Graph VHHHP](image)

Figure 5: The graph VHHHP, also denoted $VH^3P$. 
Note also that a string of H’s is just a path graph, which we already know the Grundy value of. More precisely \( g(H^n) = g(P_n) = [n + 1]_2 \), where \([p]_2\) gives the parity of \( p \), i.e. 1 if \( p \) is odd and 0 if \( p \) is even.

We will only be concerned with graphs where P’s only occur at the end of connected components pr by themselves because they only occur in that way in normal game play, unless they were introduced in the middle to start with. Now we restate the conjecture of Shelton in the following way:

**Conjecture 1.** [6, Conjecture 16]

\[
g(HV^mW^n) = 2n + [m]_2, \ m, n \geq 0
\]

We will reduce this conjecture, in the special case when \( m = 0 \), to a seemingly simpler one; Conjecture 2. Supposing that the conjecture holds we will answer the question of whether there are connected graphs for every even Grundy value, also since we know of graphs with Grundy value 1 we can easily make games of every possible Grundy value.

**Conjecture 2.**

\[
g(HW^nHH) = 0
\]

By computer runs it has been verified that this conjecture at least holds for \( 0 \leq n \leq 17 \). While by the simplicity of the statement one might be led to believe that there is some simple argument as proof, this might not be the case because the study of a Grundy values of strings inevitably concerns the options which in this case have more complicated structures.

For this special case when \( m = 0 \) in Conjecture 1 we will not need the V structure at all. This is because they never occur in any option of a position (graph) in any game unless they were there from the beginning. Here follows a list of the options that are available for every graph induced by each occurrence of the components H, W, and P.

Figure 6: The options induced by a graph that contains an H component. S is any nonempty graph of H, W and P components. If S were to be empty we would have a single vertex without edges, which is terminal and thus has Grundy value 0.
Figure 7: The options induced by a graph that contains an W component. S and Q are any nonempty graph of H, W and P components. Note that if either S or Q are empty graphs (no vertices) then the option of taking the vertex closest to that part is not available.

Figure 8: The options induced by a graph that contains a P component. S is any nonempty graph of H, W and P components. Note that if S is empty the single P is a terminal position and thus have Grundy value 0.

Note also that this means that a since every option of every graph consisting only of these H, W and P components can be described in terms of these same components these are all the component types we will need to analyze this particular type of games. We also see why we only are concerned with P’s at the end of connected components. This also gives us an opportunity to consider these graphs as just being strings of “H”’s, “W”’s and “P”’s. We will use this simplified way of describing the graphs throughout the rest of the paper, but it is always useful to keep in mind what graphs these strings actually represent.

When we have several components of a graph we represent these as the sum of the
strings which correspond to the individual components. For example: \( HWWP + H^2 \) represents the graph with one component \( HWWP \) and another consisting of two vertices which are connected. The Grundy value of two strings, \( S \) and \( Q \), when added together then, by Theorem 2, is

\[
g(S + Q) = g(S) \oplus g(Q)
\]

Note also that when considering a graph as a string we may write the same string backwards and still get the same graph. To exemplify this we have \( g(HHWWP) = g(PWWHH) \). This fact will be used extensively throughout without further explanation.

First, we will need to have a lemma that proves what we already know for path graphs (which is string of \( H \)'s in our new notation) with the additional option of appending a \( P \) at either end of this path.

**Lemma 2.** Here follows a collection of simple graphs and their corresponding Grundy values

1. \( g(\emptyset) = g(P) = g(PP) = g(H) = g(PHP) = 0 \) and \( g(HP) = 2 \).

2. \( g(PqH^kP^s) = \lfloor k + 1 \rfloor_2 \) for \( k \geq 2 \).

where \( [n]_m = n \mod m \), so \( [n]_2 \) is the parity of \( n \). Also \( q, s \in \{0, 1\} \) which just means that we can either have or not have a \( P \) at each end.

**Proof.** The first assertion (1) is easily proven by going through every option of each of these graphs and calculating the minimal excludant.

For the other assertion, (2), we use the first one to prove that it holds for the base case when \( k = 2 \):

\[
\begin{align*}
g(HH) &= \text{mex}\{g(H)\} = \text{mex}\{0\} = 1 \\
g(PPH) &= \text{mex}\{g(H^2 + H^2) = 0, g(PHP) = 2\} = 1 \\
g(PHPH) &= \text{mex}\{g(PHH + H^2) = 0\} = 1
\end{align*}
\]

Now suppose that the assertion in (2) holds for \( 2 \leq k \leq K \), then

\[
\begin{align*}
g(H^{K+1}) &= [K + 2]_2 \text{ which is known since this is just the path of } K + 1 \text{ vertices} \\
g(PHP^{K+1}) &= \text{mex}\{g(PHP^K) = [K + 1]_2, g(H^2 + H^{K+1}) = [K + 1]_2\} = [K + 2]_2 \\
g(PHP^{K+1}P) &= \text{mex}\{g(PHP^{K+1} + H^2) = [K + 2]_2 \oplus 1\} = \text{mex}\{[K + 1]_2\} = [K + 2]_2
\end{align*}
\]

Thus we have by induction proven that the assertion holds for all \( k \geq 2 \). \( \square \)
Lemma 3.

\[ g(\text{WH}^k) = \begin{cases} 
  3 & \text{if } k \text{ even} \\
  2 & \text{if } k \text{ odd}
\end{cases}, \text{ for } k \geq 1 \]

Proof. The base case \( g(\text{WH}) = 2 \) is easily verified.

Suppose that the assertion in the lemma statement holds for \( k = K \), we want to prove that then it holds for \( k = K + 1 \):

\[
g(\text{WH}^{K+1}) = \text{mex}\{g(P + H^{K+1}) = [K + 2]_2, \quad g(H^{K+4}) = [K + 5]_2, \quad g(\text{WH}^K) = \begin{cases} 
  3 & \text{if } K \text{ even} \\
  2 & \text{if } K \text{ odd}
\end{cases}\} =
\]

\[
= \begin{cases} 
  3 & \text{if } K + 1 \text{ even} \\
  2 & \text{if } K + 1 \text{ odd}
\end{cases}
\]

Thus the lemma is proved by induction. \(\square\)

Lemma 4.

\[ g(\text{PWH}^k) = \begin{cases} 
  3 & \text{if } k \text{ even} \\
  2 & \text{if } k \text{ odd}
\end{cases}, \text{ for } k \geq 2 \]

Proof. First we prove the base case \( g(\text{PWH}^2) = 3 \), we note that the Grundy values of options of \( \text{PWH}^2 \) are \( g(H^2 + \text{WH}^2) = 1 \oplus 3 = 2 \) by Lemma 3, \( g(P + PH^2) = 0 \oplus 1 = 1 \) using Lemma 2, \( g(PH^3) = 0, g(PP + H^2) = 0 \oplus 1 = 1 \) and \( g(\text{PWH}) = 4 \). These values have the minimal excludant 3.

Now suppose that the assertion in the lemma statement holds for \( 2 \leq k \leq K \). Then

\[
g(\text{PWH}^{K+1}) = \text{mex}\{g(H^2 + \text{WH}^{K+1}) = 1 \oplus \begin{cases} 
  3 & \text{if } K + 1 \text{ even} \\
  2 & \text{if } K + 1 \text{ odd}
\end{cases}\},
\]

\[
g(P + PH^{K+1}) = [K + 2]_2, 
\]

\[
g(PH^{K+4}) = [K + 5]_2, 
\]

\[
g(PP + H^{K+1}) = 0 \oplus [K + 2]_2,
\]

\[
g(\text{PWH}^K) = \begin{cases} 
  3 & \text{if } K \text{ even} \\
  2 & \text{if } K \text{ odd}
\end{cases}\} = \begin{cases} 
  3 & \text{if } K + 1 \text{ even} \\
  2 & \text{if } K + 1 \text{ odd}
\end{cases}
\]

This completes the proof by induction of the lemma. \(\square\)
Now we can prove the following theorem which gives the Grundy value for a larger collection of strings than the lemmas. The type of strings is \( P^sW^n(H^3)^kW^mP^q \), where \( k, m, n \geq 0 \) and \( s, q \in \{0, 1\} \). These types of strings occur as parts of options of \( HW^n \) strings frequently, that is why it is useful to know what they are. We will use induction to prove it for many cases of parities on the parameter on which it depends.

**Theorem 7.** Consider the position (graph/string) \( P^sW^n(H^3)^kW^mP^q \), with \( k, m, n \geq 0 \) and \( s, q \in \{0, 1\} \), then we have:

If \( k = 0 \):

\[
g(P^sW^n(H^3)^kW^mP^q) = P^sW^nW^mP^q = [n + m]_2
\]

If \( k \) even (nonzero):

\[
g(P^sW^n(H^3)^kW^mP^q) = \begin{cases} 
0 & \text{if } m, n \text{ both odd} \\
1 & \text{if } m, n \text{ both even} \\
3 & \text{if } m \text{ and } n \text{ have different parities}
\end{cases}
\]

If \( k \) odd:

\[
g(P^sW^n(H^3)^kW^mP^q) = \begin{cases} 
0 & \text{if } m \text{ and } n \text{ have the same parity} \\
2 & \text{otherwise}
\end{cases}
\]

**Proof.** We want to prove this using induction over the sum of \( m \) and \( n \), for all \( k \). Thus we first consider the base case when \( m + n = 0 \), i.e. we consider the string \( P^s(H^3)^kP^q \):

If \( k = 0 \): \( g(P^s(H^3)^kP^q) = [n + m]_2 = 0 \) by Lemma 2.

If \( k \) even (nonzero): \( g(P^s(H^3)^kP^q) = [3k + 1]_2 = 1 \) by Lemma 2.

If \( k \) odd: \( g(P^s(H^3)^kP^q) = [3k + 1]_2 = 0 \) by Lemma 2.

It is actually quite easy to even directly verifying the case when \( 0 \leq m + n \leq 1 \) using Lemmas 3 and 4 and observing that adding a P at the end of the string of H’s does not change the Grundy value since the additional options are such that they do not effect the minimal excludant operation.

Now we suppose (for induction) that the assertion in the theorem statement holds for \( 0 \leq m + n \leq N \) for all \( k \geq 0 \), we now need to show that it then also holds for \( m + n = N + 1 \) (it is actually sufficient to show that if \( m' + n' = N \) then it holds for \( m = m' \) and \( n = n' + 1 \)). Let \( m + n = N + 1 \), the Grundy values for each option of
\(P^sW^n(H^3)W^mPq\) then are

(A) \(g(P^sW^n + W^{n+m-r-1}Pq) = [r]_2 \oplus [n + m - r - 1]_2\)

where

\[
\begin{array}{ll}
1 \leq r \leq n + m - 2 & \text{if } s = q = 0 \\
1 \leq r \leq n + m - 1 & \text{if } s = 0, q = 1 \\
0 \leq r \leq n + m - 2 & \text{if } s = 1, q = 0 \\
0 \leq r \leq n + m - 1 & \text{if } s = q = 1
\end{array}
\]

(B) \(g(P^sW^n + PW^{n+m-r-1}Pq) = [r]_2 \oplus [n + m - r - 1]_2\)

which is the same as (A) by symmetry

(C) \(g(P^sW^nH^3W^{n+m-r-1}Pq) = \begin{cases} 
0 & \text{if } [n + m - 1]_2 = 0 \\
2 & \text{otherwise}
\end{cases}\)

If \(s \neq 0, q = 0\):

(D) \(g(H^2 + W^{n+m})\) and symmetry for reverse \(s, q\)

If \(s \neq 0, q \neq 0\):

(E) \(g(H^2 + W^{n+m}P)\)

Now what we want to show is that \(P^sW^n(H^3)^0W^mPq = [m + n]_2\), so suppose that \([m + n]_2 = 0, s = q = 0\) then

\[g(P^sW^n(H^3)^0W^mPq) = \operatorname{mex}\{(A),(B),(C)\}\]

\[= \operatorname{mex}\{[r]_2 \oplus [n + m - r - 1]_2 = [r]_2 \oplus [r - 1]_2, 2\}\]

\[= \operatorname{mex}\{1, 2\} = 0\]

suppose on the other hand that \([m + n]_2 = 1\) then

\[g(P^sW^n(H^3)^0W^mPq) = \operatorname{mex}\{(A),(B),(C)\}\]

\[= \operatorname{mex}\{[r]_2 \oplus [n + m - r - 1]_2 = [r]_2 \oplus [r - 1]_2, 0\}\]

\[= \operatorname{mex}\{0, 0\} = 1\]

note also that while (A),(B),(D) and (E) might not always exist we always have at least one option of type (C), since \(n + m \geq 1\) and thus always an option of Grundy value 0 in the latter case.

It remains to show that when having a \(P\) at either end this does not effect the Grundy value because the options (D) and (E) do not change the minimal excludant. We know that \((D) = g(H^2 + W^{n+m}) = 1 \oplus [n + m]_2\) which makes the Grundy value unchanged when having a \(P\) at one of the ends. Thus we can at last say that \((E) = g(H^2 + W^{n+m}P) = 1 \oplus [n + m]_2\) so for \(s = q = 1\) (having \(P\)'s at both ends) the assertion still holds.
This proves that the induction step holds at least for \( k = 0 \), we now need to prove the induction step for \( k > 0 \). We do this in a similar way, i.e. by analyzing each Grundy value of the options. We could have included the special case \( k = 0 \) in this induction, however the proof has many separate cases to analyze and it is easier to follow if one understands the proof of the \( k = 0 \) case.

Now consider \( k > 0 \). Then the options of \( P^sW^n(H^3)^kW^mP^q \) are:

\[
\begin{align*}
(a) & \quad P^sW^rP + W^{n-r-1}(H^3)^kW^mP^q & \text{for } 0 \leq r \leq n - 1 \\
(a') & \quad P^sW^r + PW^{n-r-1}(H^3)^kW^mP^q & \text{for } 0 \leq r \leq n - 1 \\
(b) & \quad P^sW^n(H^3)^kW^{m-r-1} + PW^rP^q & \text{for } 0 \leq r \leq m - 1 \\
(b') & \quad P^sW^n(H^3)^kW^{m-r-1}P + W^rP^q & \text{for } 0 \leq r \leq m - 1 \\
(c) & \quad P^sW^rH^2W^{n-r-1}(H^3)^kW^mP^q & \text{for } 0 \leq r \leq n - 1 \\
(d) & \quad P^sW^n(H^3)^kW^{m-r-1}H^2W^rP^q & \text{for } 0 \leq r \leq m - 1 \\
\end{align*}
\]

If \( q = 0, m = 0 \):
\[
(e) \quad P^sW^n(H^3)^kH^2
\]

If \( s = 0, n = 0; \)
\[
(f) \quad H^2(H^3)^kW^mP^q
\]

If \( s = 1 \):
\[
(g) \quad H^2 + W^n(H^3)^kW^mP^q
\]

If \( s = 1, n > 0 \):
\[
(h) \quad P + PW^{n-1}(H^3)^kW^mP^q
\]

If \( q = 1 \):
\[
(i) \quad P^sW^n(H^3)^kW^m + H^2
\]

If \( q = 1, m > 0 \):
\[
(j) \quad P^sW^n(H^3)^kW^{m-1}P + P
\]

First off we note that by the induction assumption we have that \((a) = (a')\) and \((b) = (b')\) in terms of their Grundy values. This means that we may neglect the options \((a')\) and \((b')\) since they do not contribute anything to the Grundy value.

We will analyze each option individually first for all different cases of parities of the parameters.

**Option (a):**
\[
g(P^sW^rP + W^{n-r-1}(H^3)^kW^mP^q) = [r]_2 \oplus g(W^{n-r-1}(H^3)^kW^mP^q) \quad (1)
\]
Summary of the Grundy values of the options for different parities of $n, m$ and $k$:

| $[n]_2, [m]_2, [k]_2$ | 0,0,0 | 0,0,1 | 0,1,0 | 0,1,1 | 1,0,0 | 1,0,1 | 1,1,0 | 1,1,1 |
|-----------------------|-------|-------|-------|-------|-------|-------|-------|-------|
| $r$ even              | 3     | 2     | 0     | 0     | 1     | 0     | 3     | 2     |
| $r$ odd               | 0     | 1     | 2     | 3     | 2     | 3     | 1     | 1     |

These Grundy values are obtained in the following manner.

If $r$ is even then $[n - r - 1]_2 = [n + 1]_2$ and thus if $k$ is even we have

$$g(W^{n-r-1}(H^3)^kW^mP^q) = \begin{cases} 
0 & \text{if } n \text{ even and } m \text{ odd} \\
1 & \text{if } n \text{ odd and } m \text{ even} \\
3 & \text{if } [m + n]_2 = 0 
\end{cases}$$

while if $r$ is even and $k$ is odd we get

$$g(W^{n-r-1}(H^3)^kW^mP^q) = \begin{cases} 
0 & \text{if } [n + m]_2 = 1 \\
2 & \text{if } [n + m]_2 = 0 
\end{cases}$$

If $r$ is odd then $[n - r - 1]_2 = [n]_2$ and if $k$ is even:

$$g(W^{n-r-1}(H^3)^kW^mP^q) = \begin{cases} 
0 & \text{if both } n \text{ and } m \text{ odd} \\
1 & \text{if both } n \text{ and } m \text{ even} \\
3 & \text{if } [m + n]_2 = 1 
\end{cases}$$

while if $r$ is odd and $k$ is odd we get

$$g(W^{n-r-1}(H^3)^kW^mP^q) = \begin{cases} 
0 & \text{if } [n + m]_2 = 0 \\
2 & \text{if } [n + m]_2 = 1 
\end{cases}$$

We then get the Grundy values in the table by calculating the nim-sum with $[r]_2$ according to equation (1).

**Option (b):**

$$g(P^sW^n(H^3)^kW^{m-r-1} + PW^rP^q) = g(P^sW^n(H^3)^kW^{m-r-1}) \oplus [r]_2$$ (2)

Summary of the Grundy values of the options for different parities of $n, m$ and $k$:

| $[n]_2, [m]_2, [k]_2$ | 0,0,0 | 0,0,1 | 0,1,0 | 0,1,1 | 1,0,0 | 1,0,1 | 1,1,0 | 1,1,1 |
|-----------------------|-------|-------|-------|-------|-------|-------|-------|-------|
| $r$ even              | 3     | 2     | 1     | 0     | 0     | 0     | 3     | 2     |
| $r$ odd               | 0     | 1     | 2     | 3     | 2     | 3     | 1     | 1     |

These Grundy values are obtained in the following manner.

If $r$ is even then $[m - r - 1]_2 = [m + 1]_2$ and thus if $k$ is even we have

$$g(P^sW^n(H^3)^kW^{m-r-1}) = \begin{cases} 
0 & \text{if } n \text{ odd and } m \text{ even} \\
1 & \text{if } n \text{ even and } m \text{ odd} \\
3 & \text{if } [m + n]_2 = 0 
\end{cases}$$
while if $r$ is even and $k$ is odd we get

$$g(P^sW^n(H^3)^kW^{m-r-1}) = \begin{cases} 0 & \text{if } [n + m]_2 = 1 \\ 2 & \text{if } [n + m]_2 = 0 \end{cases}$$

If $r$ is odd then $[m - r - 1]_2 = [m]_2$ and if $k$ is even:

$$g(P^sW^n(H^3)^kW^{m-r-1}) = \begin{cases} 0 & \text{if both } n \text{ and } m \text{ odd} \\ 1 & \text{if both } n \text{ and } m \text{ even} \\ 3 & \text{if } [m + n]_2 = 1 \end{cases}$$

while if $r$ is odd and $k$ is also odd we get

$$g(P^sW^n(H^3)^kW^{m-r-1}) = \begin{cases} 0 & \text{if } [n + m]_2 = 0 \\ 2 & \text{if } [n + m]_2 = 1 \end{cases}$$

We then get the Grundy values in the table by calculating the nim-sum with $[r]_2$ according to equation (2).

**Option (c):**

$$g(P^sW^nH^3W^{m-r-1}(H^3)^kW^mP^q)$$

For this option and for option (d) we will use a different tactic, we will show that this Grundy value is not the value sought for $g(P^sW^n(H^3)^kW^mP^q)$ without necessarily specifying exactly what the Grundy value is.

In the study of all the options (c) through (f) we are concerned with proving that the Grundy value of these does not affect the Grundy value of the string for which these are options. Thus we want to prove them being different from the sought Grundy value. Thus it is useful to make the following definition

$$g_{n,m,k} := \begin{cases} 0 & \text{if both } n \text{ and } m \text{ odd} \\ 1 & \text{if both } n \text{ and } m \text{ even} \\ 3 & \text{if } [m + n]_2 = 1 \end{cases} \quad \text{if } k \text{ even}$$

$$g_{n,m,k} := \begin{cases} 0 & \text{if } [n + m]_2 = 0 \\ 2 & \text{if } [n + m]_2 = 1 \end{cases} \quad \text{if } k \text{ odd}$$

where everything is taken directly from the assertion in the Lemma we are proving. These values should be considered the sought value, which we want to prove (inductively) that the graph we are studying have.

Summary of the Grundy values of the options for different parities of $n,m$ and $k$: 20
\[ [n]_2, [m]_2, [k]_2 \rightarrow \begin{array}{cccccccc}
\text{r even} & 0,0,0 & 0,0,1 & 0,1,0 & 0,1,1 & 1,0,0 & 1,0,1 & 1,1,0 & 1,1,1 \\
\text{r odd} & \neq 1 & \neq 0 & \neq 3 & \neq 2(1) & \neq 3 & \neq 2(1) & \neq 0 & \neq 0 \\
\end{array} \]

By the numbers in the table we mean by \( \neq i(j) \) that we will prove below that \( g(P^sW^rH^3W^{n-r-1}(H^3)^kW^mP^q) \) cannot have Grundy value \( i \) and that at least one \( r \) gives the Grundy value \( j \). We will need this to ensure that the strings with Grundy value 2 has an option of Grundy value 1. We get these differences from the following. Consider the case when \( n - r - 1 = 0 \), then we have

\[
P^sW^rH^3W^{n-r-1}(H^3)^kW^mP^q = P^sW^{n-1}(H^3)^{k+1}W^mP^q
\]

and we have an induction assumption for

\[
g(P^sW^{n-1}(H^3)^{k+1}W^mP^q) = \begin{cases} 0 & \text{if } n \text{ even and } m \text{ odd} \\ 1 & \text{if } n \text{ odd and } m \text{ even} \\ 3 & \text{if } [m + n]_2 = 0 \end{cases} \text{ if } k \text{ odd}
\]

\[
g(P^sW^{n-1}(H^3)^{k+1}W^mP^q) = \begin{cases} 0 & \text{if } [n + m]_2 = 1 \\ 2 & \text{if } [n + m]_2 = 0 \end{cases} \text{ if } k \text{ even}
\]

We see that the only common value is 0 for each parity of \( k \) and it does not coincide in these parities of \( m \) and \( n \). Thus we have \( g(P^sW^{n-1}(H^3)^{k+1}W^mP^q) \neq g_{n,m,k} \).

Furthermore we can observe that either \( P^sW^{n-1}(H^3)^{k+1}W^mP^q \) or \( P^sW^n(H^3)^{k+1}W^{m-1}P^q \) has Grundy value 1 when \( k \) is odd and \( [m]_2 \neq [n]_2 \), thus the 1’s in column four and six in the table above are justified.

Now consider the case when \( n - r - 1 > 0 \) then:

\[
P^sW^rH^3W^{n-r-1}(H^3)^kW^mP^q \text{ has option } P^sW^rH^3 + P^{n-r-2}(H^3)^kW^mP^q
\]

and first let \( r \) be even, then

\[
g(P^sW^rH^3 + P^{n-r-2}(H^3)^kW^mP^q) = 0 \oplus g(P^{n-r-2}(H^3)^kW^mP^q) = g_{n,m,k}
\]

since \([n - r - 2]_2 = [n]_2 \) and the definition of \( g_{n,m,k} \) is taken from the assertion in the theorem which only depends on the parity of the parameters. Thus we must have that

\[
g(P^sW^rH^3W^{n-r-1}(H^3)^kW^mP^q) \neq g_{n,m,k}
\]

because a position cannot have the same Grundy value as any of its options.
Now, let $r$ be odd and $k$ be even, then consider the option $P^rW^3W^{n-r-2}P + (H^3)^kW^mP^q$ of $P^rW^3W^{n-r-1}(H^3)^kW^mP^q$:

$$g(P^rW^3W^{n-r-2}P + (H^3)^kW^mP^q) = g(P^rW^3W^{n-r-2}P) \oplus g((H^3)^kW^mP^q) =$$

$$= \begin{cases}
0 \oplus 1 = 1 & \text{if } [n]_2 = [m]_2 = 0 \\
0 \oplus 3 = 3 & \text{if } [n]_2 = 0, [m]_2 = 1 \\
2 \oplus 1 = 3 & \text{if } [n]_2 = 1, [m]_2 = 0 \\
2 \oplus 3 = 1 & \text{if } [n]_2 = [m]_2 = 1 
\end{cases} = g_{n,m,k}$$

$$\Rightarrow g(P^rW^3W^{n-r-1}(H^3)^kW^mP^q) \neq g_{n,m,k}$$

Lastly consider the case when both $r$ and $k$ is odd, then

$$g(P^rW^3W^{n-r}P + PW^{n-r-2}(H^3)^kW^mP^q) = 2 \oplus g(PW^{n-r-2}(H^3)^kW^mP^q) =$$

$$= \begin{cases}
2 \oplus 2 = 0 & \text{if } [n]_2 = [m]_2 \\
2 \oplus 0 = 2 & \text{if } [n]_2 \neq [m]_2 
\end{cases} = g_{n,m,k}$$

$$\Rightarrow g(P^rW^3W^{n-r-1}(H^3)^kW^mP^q) \neq g_{n,m,k}$$

This completes the proof that no option of type (c) have the sought values $g_{n,m,k}$.

**Option (d):**

We use symmetry and the fact that $g_{n,m,k} = g_{m,n,k}$ to give us that the option (d) is in fact no different in the analysis than option (c). Thus we may just reverse the string in option (d) to get an option of type (c) and use the result we just proved above.

**Option (e):**

We want to prove that

$$g(P^rW^m(H^3)^kH^2) \neq g_{n,0,k}$$

Summary of the differences we prove below:

| $[n]_2$, $[m]_2$, $[k]_2$ | 0,0,0 | 0,0,1 | 0,1,0 | 0,1,1 | 1,0,0 | 1,0,1 | 1,1,0 | 1,1,1 |
|--------------------------|--------|--------|--------|--------|--------|--------|--------|--------|
|                          | $\neq 1$ | $\neq 0$ | N/A | N/A | $\neq 3$ | $\neq 2$ | N/A | N/A |

We start with the case $n = 1$ (since $m = 0$ $n$ cannot be less than 1). By Lemmas 2 and 3 we have that

$$g(P^rW^n(H^3)^kH^2) = \begin{cases}
2 & \text{if } k \text{ even} \\
3 & \text{if } k \text{ odd} 
\end{cases} \neq g_{1,0,k} = \begin{cases}
1 & \text{if } k \text{ even} \\
2 & \text{if } k \text{ odd} 
\end{cases}$$
Now suppose that \( n > 1 \), then if \( k \) is even we want to prove \( g(\text{P}^*W^n(H^3)^{k-1}H^2) \neq 1 \) when \( n \) even, and \( g(\text{P}^*W^n(H^3)^{k-1}H^2) \neq 3 \) when \( n \) is odd. Let both \( k \) and \( n \) be even, then \( \text{P}^*W^n(H^3)^{k-1}H^2 \) has option \( \text{P}^*W^{n-1} + \text{P}(H^3)^{k-1}H^2 \) with Grundy value
\[
g(\text{P}^*W^{n-1} + \text{P}(H^3)^{k-1}H^2) = 1 \oplus 0 = 1
\]
While when \( k \) even and \( n \) odd we have that \( \text{P}^*W^n(H^3)^{k-1}H^2 \) has option \( \text{P}^*W^{n-2} + \text{PW}(H^3)^{k-1}H^2 \) with Grundy value
\[
g(\text{P}^*W^{n-2} + \text{PW}(H^3)^{k-1}H^2) = 1 \oplus 2 = 3
\]
using Lemma 4.

On the other hand if \( k \) is odd we want to prove that \( g(\text{P}^*W^n(H^3)^{k-1}H^2) \neq 0 \) when \( n \) is even and \( g(\text{P}^*W^n(H^3)^{k-1}H^2) \neq 2 \) when \( n \) is odd. Let \( k \) be odd and \( n \) be even, then
\[
g(\text{P}^*W^{n-1} + \text{P}(H^3)^{k-1}H^2) = 1 \oplus 1 = 0
\]
While when \( k \) and \( n \) both are odd we have that \( \text{P}^*W^n(H^3)^{k-1}H^2 \) has option \( \text{P}^*W^{n-2} + \text{PW}(H^3)^{k-1}H^2 \) with Grundy value
\[
g(\text{P}^*W^{n-2} + \text{PW}(H^3)^{k-1}H^2) = 1 \oplus 3 = 2
\]

**Option (f):** This is the same as option (e) if we only reverse the string first and apply the exact same argument we may derive the same inequalities.

This completes the analysis of options (a) through (f) we may now summarize the result in the following way:

| option | \([n]_2, [m]_2, [k]_2 \rightarrow\) |
|--------|----------------------------------|
| (a)    | \( r \) even                      |
|        | \( 3 \) 2 0 0 1 0 3 2             |
| (a)    | \( r \) odd                       |
|        | \( 0 \) 1 2 3 2 3 1 1             |
| (b)    | \( r \) even                      |
|        | \( 3 \) 2 1 0 0 0 3 2             |
| (b)    | \( r \) odd                       |
|        | \( 0 \) 1 2 3 2 3 1 1             |
| (c)/(d) | \( r \) even                     |
|        | \( \neq 1 \) \( \neq 0 \) \( \neq 3 \) \( \neq 2(1) \) \( \neq 3 \) \( \neq 2(1) \) \( \neq 0 \) \( \neq 0 \) |
| (c)/(d) | \( r \) odd                      |
|        | \( \neq 1 \) \( \neq 0 \) \( \neq 3 \) \( \neq 2(1) \) \( \neq 3 \) \( \neq 2(1) \) \( \neq 0 \) \( \neq 0 \) |
| (e)/(f) |                                |
|        | \( \neq 1 \) \( \neq 0 \) \( \neq 3 \) \( \neq 2 \) \( \neq 3 \) \( \neq 2 \) \( N/A \) \( N/A \) |
| column mex |                                    |
|          | \( 1 \) 0 3 2 3 2 0 0            |
To complete the proof all we have to do is to verify that option $(g)$ through $(j)$ gives no new Grundy values. It is easy to see that $(g)$ and $(i)$ does not change the Grundy value since
\[ g(H^2 + S) = 1 \oplus g(S) \neq g(S) \]
for all strings $S$.

The options $(h)$ and $(j)$ cannot give any new values either since they are just additional cases (one more value for $r$) of $(a')$ and $(b')$ respectively whose values we have already obtained. Extending these for an additional value of $r$ does not make any difference because their value only depend on the parity of $r$.

This completes the proof by induction.

The preceding theorem is the one that requires the most calculations however does not yield any immediately interesting result. However we will use it to prove our main theorem in this section. This will in turn give us graphs of all possible Grundy values, supposing Conjecture 2 holds.

First we will need the following lemma however, it gives the fact that for the strings which occur in normal game play, starting from a $H/W$-position, we can append a $P$ without changing the Grundy value, unless the position belongs to a special class. The a special class may actually contain more than what is necessary, however this does not concern us.

**Lemma 5.** Let $K_N$ be the set of reachable sums of strings $\sum_{n=0}^{\infty} S_n$ from $W^N H$ ($S_n$‘s may be empty), and let $M$ denote the subset of sums of strings for which the $S_0$ is either $W^n H$ or $W^n H^2$, for some $n \geq 0$.

Then

\[ S_0 + \sum_{n=1}^{\infty} S_n \in K_N \setminus M \Rightarrow g(S_0 + \sum_{n=1}^{\infty} S_n) = g(PS_0 + \sum_{n=1}^{\infty} S_n) \]

**Proof.** We will use induction over the positions of the game. As a base case we have that the assertion is true for the terminal positions of $K_N$ since $g(P) = g(PP) = 0$, $g$ (the empty string) $= g(P) = 0$ and $H \in M$.

Now suppose $T \in K_N$ such that the assertion holds for all the options of $T$. The only $T \in K_N$ which has options in $M$ are such that $T = \sum_{n=0}^{\infty} S_n$ and $S_0 = W^n H^3$, some $n \geq 0$. But we already know (by Theorem 7) that $PW^n H^3 = W^n H^3$ so that case is trivial.

We thus only have to consider any other $\sum_{n=0}^{\infty} S_n \in K_N \setminus M$, then the set of Grundy values of options are equal (since they must all lie in $K_N \setminus M$ as well) for $\sum_{n=0}^{\infty} S_n$ and $PS_0 + \sum_{1}^{\infty} S_n$ with the exception of the following additional options of $PS_0$:
1. \( g(H^2 + S_0) \neq S_0 \), so it does not change the Grundy value alone (there must exist some other Grundy value that \( S_0 \) does not have for this to have effect).

2. If \( S_0 = WS'_0 : P + PS'_0 \), but for \( T \notin \mathbb{M} \) we have \( S'_0 + \sum_{i=1}^{\infty} S_n \notin \mathbb{M} \) and thus 
\[
g(PS'_0 + \sum_{i=1}^{\infty} S_n) = g(S'_0 + \sum_{i=1}^{\infty} S_n) = g(P + PS'_0 + \sum_{i=1}^{\infty} S_n) \quad \text{unless} \quad S'_0 \quad \text{is the empty string, but then we may just exclude it from the sum and the assertion holds trivially.}
\]

By induction we have shown that the assertion in the lemma statement holds. 

\[\square\]

**Theorem 8.** If Conjecture 2 holds, then

\[
g(HW^n) = 2n \quad \text{and} \quad g(HW^nP) = 2(n + 1)
\]

for \( n \geq 0 \).

To prove this final theorem we will need another lemma for which we will have the induction assumption of the theorem and then use a counting argument. We find an upper bound on the number of unique Grundy values of all options, since the maximal excludant must have all numbers less than it in the set on which it is calculated we can then also bound the Grundy value for the graph.

**Lemma 6.** If the assertion in Theorem 8 holds for \( 0 \leq n \leq N \), then for \( 0 \leq k + l \leq N - 1 \) we have

\[
g(HW^kH^lW^l) < 2(k + l + 1)
\]

\[
g(HW^kH^lW^lP) < 2(k + l + 2)
\]

given that Conjecture 2 holds.

**Proof.** (Of Lemma 6) We want to count the unique Grundy values of the options of
HW\(^k\)H\(^3\)W\(^l\), the options are listed below.

\(\text{(i) } W^kH^3W^l\)

\(\text{(ii) } HW^r + PW^{k-r-1}H^3W^l\) for \(0 \leq r \leq k - 1\)

\(\text{(iii) } HW^rP + W^{k-r-1}H^3W^l\) for \(0 \leq r \leq k - 1\)

\(\text{(iv) } HW^kH^3W^{l-r-1} + PW^r\) for \(0 \leq r \leq l - 1\)

\(\text{(v) } HW^kH^3W^{l-r-1}P + W^r\) for \(1 \leq r \leq l - 1\)

\(\text{(vi) } HW^rH^3W^{k-r-1}H^3W^l\) for \(0 \leq r \leq k - 1\)

\(\text{(vii) } HW^kH^3W^{l-r-1}H^3W^r\) for \(0 \leq r \leq l - 1\)

If \(l = 0\):

\(\text{(viii) } HW^nH^2\)

First we suppose that \(l \neq 0\), and \([k + l]_2 = 0\) then we have:

\(\text{(i) } = 0\)

\(\text{(ii) } = \left\{2r \oplus \begin{cases} 2 & \text{if } r \text{ even} \\ 0 & \text{if } r \text{ odd} \end{cases} \right\} = \{2, 2, 6, 6, 10, 10, \ldots \} \leq 2(k - 1)\)

\(\text{(iii) } = \left\{2(r + 1) \oplus \begin{cases} 2 & \text{if } r \text{ even} \\ 0 & \text{if } r \text{ odd} \end{cases} \right\} = \{0, 4, 4, \ldots \} \leq 2n\)

If \(k\) is even then \((ii)\) gives \(\frac{k}{2}\) values and \((iii)\) gives \(\frac{k-2}{2} + 2 = \frac{k}{2} + 1\). While if \(k\) is odd we get from \((ii)\) and from \((iii)\) \(\frac{k-1}{2} + 1\) values each. Thus the total number of values of 

\((i), (ii)\) and \((iii)\) becomes:

\[\#(i) + \#(ii) + \#(iii) = k + 1\]

since \((i)\) does not give anything because 0 also occurs in option \((iii)\).

By Lemma 5 we have \((iv) = (v)\) for each choice of \(r \in \{1, 2, \ldots, l - 1\}\) and thus \(\#(iv) + \#(v) \leq \frac{2(l-1)}{2} + 1 = m\). Lastly we can easily see that \(\#(vii) + \#(viii) \leq k + l\).

Thus we get the following bound on the total:

\[\#(i)+\#(ii)+\#(iii)+\#(iv)+\#(v)+\#(vi)+\#(vii)+\#(viii) \leq k+1+l+l+k < 2(k+l+1)\]

But since for a position to have Grundy value \(2(k + l + 1)\) it needs to have at least \(2(k + l + 1)\) options \((0 \text{ through } 2(k + l + 1) - 1)\) we get that \(g(HW^kH^3W^l) < 2(k + l + 1)\).
Now suppose instead that \([k + l]_2 = 1\), but still \(l \neq 0\). Then we have:

\[
(i) = 2
\]

\[
(ii) = \left\{ \begin{array}{ll} 
2r & \text{if } r \text{ even} \\
0 & \text{if } r \text{ odd} 
\end{array} \right\} = \{0, 0, 4, 4, \ldots\} \leq 2k
\]

\[
(iii) = \left\{ \begin{array}{ll} 
2(r + 1) & \text{if } r \text{ even} \\
2 & \text{if } r \text{ odd} 
\end{array} \right\} = \{2, 6, 6, 10, 10, \ldots\} \leq 2k
\]

As before we count and add the number of unique values. If \(k\) is even \(#(ii) + #(iii) = \frac{k}{2} + \frac{k}{2} + 1\) and if \(k\) is odd \(#(ii) + #(iii) = \frac{k-1}{2} + 1 + \frac{k-1}{2} + 1\). Note also that since \((a) = 2\) which also occur in \((iii)\) we do not get anything new from \((i)\), i.e.

\[#(i) + #(ii) + #(iii) = k + 1\]

For the remaining options the same argument as above for \([k + l]_2 = 0\) still holds, and we get the same total:

\[#(i) + #(ii) + #(iii) + #(iv) + #(v) + #(vi) + #(vii) + #(viii) \leq k + 1 + l + l + k < 2(k + l + 1)\]

which implies \(g(HW^kH^3W^l) < 2(k + l + 1)\).

The only remaining case is for when \(l = 0\). Same argument as above holds for \((i), (ii), (iii)\) gives \#(i) + #(ii) + #(iii) = k + 1 unique values. \((iv), (v), (vii)\) are not applicable and \((vi)\) gives \(k\) values, total of \(2k + 1\) options. Lastly here we apply Conjecture 2 to see that \((viii)\) does not give any new options.

We complete the proof by observing that since we know that a P at either end does not make any difference for the Grundy value in Lemma 6 we see that a completely analogous argument works for the case where the string ends in a P.

\[\square\]

**Proof.** (Of Theorem 8) That the assertion holds for small \(n\) is easily verified, so we have a clear base case when \(n = 0\). Suppose that the assertion in the theorem holds for \(0 \leq n \leq N\), we want to prove that then it also holds for \(N + 1\).
The options of \( H^N + P^s \) where \( s \in \{0, 1\} \) are

\[
\begin{align*}
(I) & \quad W^{N+1}P^s \\
(II) & \quad HW^r + PW^{N-r}P^s \quad , 0 \leq r \leq N \\
(III) & \quad HW^rP + W^{N-r}P^s \quad , 0 \leq r \leq N \\
(IV) & \quad HW^rH^3W^{N-r}P^s \quad , 0 \leq r \leq N
\end{align*}
\]

If \( s = 1 \):

\[
\begin{align*}
(V) & \quad HW^NP + P \\
(VI) & \quad HW^{N+1} + H^2
\end{align*}
\]

Now we just have to observe that for \((I)\) we have \( g(W^{N+1}P^s) = [N + 1]_2 \), for \((II)\) : 
\( g(H^{N+1}P + PW^{N-r}P^s) = 2r \oplus [N-r]_2 \) and for \((III)\) : 
\( g(HW^rP + W^{N-r}P^s) = 2(r+1) \oplus [N-r]_2 \).

Lastly by Lemma 6 we get that the options of type \((IV)\) cannot effect the Grundy value since they are bounded by the sought Grundy value and is thus obtained by some other option type, \((I)\) through \((III)\).

The two additional options when \( s = 1 \) gives Grundy values \( 2(N+1) \) and \( 2(N+1) \oplus 1 = 2(N + 1) + 1 \) which increases the mex by two.

Thus the assertion in the theorem holds for \( n = N+1 \) and we have shown by induction that the assertion holds for any \( n \geq 0 \).

\[ \square \]

**Corollary 3.** For every \( n \geq 0 \) there exist some graph \( G \) such that \( g(G) = n \), given that Conjecture 2 holds.

### 3 Conclusions and future work

We have successfully proved one conjecture proposed by Shelton[6, Conjecture 17], while on the other conjecture we have (hopefully) made some useful progress. It is however hard to assess how difficult it will be to prove Conjecture 2, or even if this method is the right one.

It should be mentioned that even though we have looked at the surface of the problem and seen that we cannot find any obvious direct argument to prove this conjecture it has not been studied thoroughly enough, and some of the methods and results presented in this paper might be useful for further studies. In particular we might attempt to solve the problem in the following ways:
The method used in the proof of Theorem 7 of finding a simple class of graphs, which includes $HW^nH^2$ or $HW^nH^3$, for which every option that lies outside this specified class we can make some argument to show that these does not affect the Grundy value. Then it would be easy to construct a proof by induction.

Using the specific structure of the string we want to study one might be able to make some argument of how to play the game optimally (for example a tit-for-tat argument). Then since what the conjecture says is that the Grundy value should be 0 we will have to prove that the first player to move always loses if the other player uses some optimal strategy. One example of how this could be done is by making use of some symmetry in the structure of the graph.

The general structure of a string which is reachable from $HW^nH^2$ or even $HW^n$ is not impossible to specify. If one is able to make some assumption concerning the Grundy values of all of these one might be able to construct an induction proof, similar to the proof of Theorem 7 where we are only concerned with parities of the parameters. However this method probably requires a computer to verify the actual induction since the proof would be even more monotonous and lengthy than the proof of Theorem 7 in this paper.

These are just suggestions for what might help finish of the proof of existence of graphs of any Grundy value, however they might also lead nowhere. In personal experience however it seems at least feasible that one method such as these would work.

### 3.1 Open problems

In addition to the problem of proving Conjecture 2 we have still some interesting open problems concerning the game which we study in this paper: odd/odd vertex removal.

The main problem of constructing a function for computing the Grundy value of any graph, i.e. finding $g(G)$ for general graphs and not just simple classes of graphs such as bipartite which is done in this paper, is still an open problem. However the neither the result or the method from this paper will be of much use for proving that. This is because we use specific properties of the cycles in bipartite graphs and prove that these games are much simpler than the one played on arbitrary graphs (it does not matter what move you make). Hence, when solving this problem one will have to use some other method than the one for bipartite graphs.

Since Conjecture 2 remains, the problem of whether there are graphs of any Grundy value is still open. Furthermore the conjecture of Shelton[6, Conjecture 16] which implies
the existence of connected graphs of every Grundy value is still open. It seems that what has been done in this paper gives us an angle at which one might attack this problem as well. However, it does not seem any easier to prove this without having to prove the specific case when $m = 0$ (no V type components), which we study in this paper.

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