On the second Gegenbauer moment of $\rho$-meson distribution amplitude

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Using the soft pion theorem, crossing, and the dispersion relations for the two pion distribution amplitude (2πDA) we argue that the second Gegenbauer moment of the $\rho$-meson DA ($a_2^{(\rho)}$) most probably is negative. This result is at variance with the majority of the model calculations for $a_2^{(\rho)}$.

Using the instanton theory of the QCD vacuum, we computed $a_2^{(\rho)}$ at a low normalisation point and obtain for the ratio $a_2^{(\rho)}/M_\pi^3$ definitely negative value in the range of $a_2^{(\rho)}/M_\pi^3 \in [-2, -1]$. The range of values corresponds to a generous variation of the parameters of the instanton vacuum. The value of the second Gegenbauer moment of pion DA is positive in the whole range and is compatible with the most advanced lattice measurement. It seems that the topologically non-trivial field configurations in the QCD vacuum (instantons) lead to qualitatively different shapes of the pion and the $\rho$-meson DAs.

I. INTRODUCTION

The chiral-even leading-twist $\rho$-meson distribution amplitude (DA) $\phi_\rho$ is defined by the following matrix element

$$\langle p^0(P)|\bar{\psi}(x)\gamma_5\frac{n}{2}\psi(0)|0\rangle_{x^+_{\perp}=0} = (n \cdot \epsilon^\lambda)f_\rho m_\rho \sqrt{2} \int_0^1 dz e^{-iz(P \cdot x)} \phi_\rho(z).$$  

Here, $\epsilon^\lambda$ is the polarization vector of the $\rho$-meson and $n$ is a light-like vector. The dimensional constant $f_\rho$ is defined in a way that the DA follows the normalisation condition:

$$\int_0^1 dz \phi_\rho(z) = 1.$$

The $\rho$-meson DA can be expanded in terms of Gegenbauer polynomials

$$\phi_\rho(z) = 6z(1-z)\left(1 + \sum_{n=2,\text{even}}^{\infty} a_n^{(\rho)} C_{3/2}^n(2z-1)\right).$$

In the above expression, the first term in the parentheses $1 = a_0^{(\rho)}$ corresponds to the normalisation [2]. The expansion is related to the deviation from the perturbative DA $\phi_\rho(z) = 6z(1-z)$ and the leading coefficient $a_2^{(\rho)}$ provides information about the width of the DA.

Through the light-cone QCD sum-rules, the $\rho$-meson DA $\phi_\rho$ plays an important role to describe the processes such as exclusive semileptonic decays of the B-mesons ($B \to \rho l\bar{\nu}$) [11]. The resulting $B \to \rho$ transition form factors can be used to extract the CKM matrix element $|V_{ub}|$, as supplementary to the process $B \to \pi l\bar{\nu}$ [5, 6]. In there, the coefficient $a_2^{(\rho)}$ provide a nontrivial and prominent contribution, for detailed discussion, see [4].

Various non-perturbative methods have been applied to study of the $\rho$-meson DA: the QCD sum-rule [9, 12], lattice simulation of the QCD [13, 15], and other model approaches [16, 23]. In Table I we collected various model predictions for $a_2^{(\rho)}$. One sees that the majority of the predictions corresponds to positive value of the second Gegenbauer coefficient.

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In Ref. 22 it was shown that with help of the dispersion relations the Gegenbauer moments of the ρ-meson DA can be expressed in terms of the two pion distribution amplitudes (2πDAs). In particular, $a_2^{(ρ)}$ is related to the single pion observables thanks to the crossing symmetry and the low-energy theorem. In Ref. 22, using the instanton approach, the ratio of the ρ and pion DA Gegenbauer moments $a_2^{(ρ)}/a_2^{(π)} \simeq -2.3$ was obtained, whereas the majority of the other works predicts the opposite sign for the ratio. We argue here that the negative sign of the ratio $a_2^{(ρ)}/a_2^{(π)}$ is deeply rooted in chiral dynamics and general properties of quantum field theory such as unitarity, crossing and dispersion relations.

In this work, we first explain briefly how the $a_2^{(ρ)}$ is expressed in terms of the single pion observables owing to low-energy theorems, crossing symmetry, and the dispersion relations. After that, we adopt the pion observables from a recent lattice calculation and global data analysis to obtain the model independent relations for the ratio $a_2^{(ρ)}/a_2^{(π)}$. Finally we present the results from the instanton model of the QCD vacuum.

## II. RELATION BETWEEN $a_2^{(ρ)}$ MOMENT AND SINGLE PION OBSERVABLES

The twist-2 chiral-even two-pion distribution amplitude (2πDA) is defined as follows 25:

$$\Phi^{a b}(z, \zeta, W^2) = \frac{1}{4 \pi} \int dx^- \exp(-izP^+x^-/2) (\pi^a(p_1)\pi^b(p_2))\bar{\psi}(x)\gamma^0 T\psi(0)\bigg|_{x^+=x_-=0}. \quad (4)$$

Here we introduce the light-cone coordinate for a vector $v^\pm = n \cdot v = v^0 \pm v^3$, with the light-like vector $n^2 = 0$, represented as $n \rightarrow (1, 0, 0, 1)$. $T$ is the isospin matrix and in our case of interest, the isovector, $T = 3^F/2$. The 2πDA has three independent variables: $z$, the quark momentum fraction with respect to $P = p_1 + p_2$, $\zeta = p_1^\perp / P^+$, the longitudinal momentum distribution of two pions, and $W^2 = P^2 = (p_1 + p_2)^2$, the invariant mass. The isovector ($I = 1$) part of the 2πDA can be projected out and expanded in terms of the Gegenbauer polynomials $C_n^I(x)$ in the following form

$$\Phi_{I=1}(z, \zeta, W^2) = 6z(1-z) \sum_{n=0}^{\infty} \sum_{l=0}^{n+1} B_{nl}(W^2) C_n^{3/2}(2z-1) C_l^{1/2}(2\zeta-1). \quad (5)$$

In Ref. 22 the dispersion relations for the generalised Gegenbauer moments $B_{nl}(W^2)$ were derived.
Moreover the solution of these dispersion relations for $W^2 \leq 16m^2_\pi$ was found:

$$B_{nl}(W^2) = B_{nl}(0) \exp \left[ N \sum_{k=1}^{N-1} a_k^{(nl)} W^{2k} + \frac{W^{2N}}{2} \int_{4m^2_\pi}^{\infty} ds \frac{\delta_I^{l=1}(s)}{s^N(s-W^2-4m^2_\pi)} \right].$$

(6)

Here $\delta_I^{l=1}(s)$ is the isospin one $\pi\pi$ scattering phase shift with the orbital momentum $l$, $N$ is the number of the subtractions in the dispersion relations, and $a_k^{(nl)}$ are the corresponding low-energy subtraction constants. For the $\rho$-meson channel ($I = 1$ and $l = 1$) it is known [26] that the dispersion relations with two subtraction gives excellent description of the pion form factor for invariant mass till $W^2 \simeq 2.5$ GeV$^2$. Therefore we restrict ourselves to $N = 2$ in Eq. (6) for the studies of generalised Gegenbauer moments at $W$ around the $\rho$-meson mass.

The $\rho$-meson is a resonance in the $\pi\pi$ scattering amplitude in the channel $l = 1, I = 1$, the scattering phase shift $\delta_1^I(s)$ in Eq. (6) crosses rapidly the value of $\pi/2$ near $s = m^2_\rho$. As it was shown in Ref. [22] such behaviour of the scattering phase leads to the appearance of the pole in $B_{nl}(W^2)$ at $W = m_\rho - i\Gamma_\rho/2$, the residue in this pole corresponds to the Gegenbauer moment of the resonance DA, see detailed derivation in Ref. [22]. The $\rho$-meson DA second Gegenbauer moment can be obtained as:

$$a_2^{(\rho)} = B_{21}(0) \exp(c_1^{(21)} m^2_\rho),$$

(7)

where $c_1^{(21)} = a_1^{(21)} - a_0^{(101)}$ is the subtraction constant. Its value is not known a priori, however it can be estimated in the low-energy models or determined from the shape of $\pi\pi$ mass spectrum in hard exclusive processes, see detailed discussion and fits to experimental data in Ref. [24]. The expression (7) can be further reduced to the single pion observables by the soft pion theorem and the crossing symmetry [22].

$$a_2^{(\rho)} = B_{21}(0) \exp(c_1^{(21)} m^2_\rho) = \left( a_2^{(\pi)} - \frac{7}{6} M_3^{(\pi)} \right) \exp(c_1^{(21)} m^2_\rho),$$

(8)

where $a_2^{(\pi)}$ is the second Gegenbauer moment of the $\pi$-meson light-cone DA and $M_3^{(\pi)}$ is the second moment of the pion quark distribution function,

$$M_3^{(\pi)} = \int_0^1 dx x^2 (q_\pi(x) - q_\bar{\pi}(x)).$$

(9)

In order to calculate the second Gegenbauer moment of the $\rho$-meson DA with help of Eq. (8) we use

- the results for $a_2^{(\pi)}$ from the lattice calculation,
- $M_3^{(\pi)}$ from global phenomenological analysis,
- the value of the subtraction constant $c_1^{(21)}$ from low-energy effective theory derived from instanton model of QCD vacuum. We note that the precise value of the subtraction constant $c_1^{(21)}$ in (8) does not influence the sign of $a_2^{(\rho)}$.

The value of $a_2^{(\pi)}$ we take from the lattice simulation of Bali et al. (RQCD) [27]:

$$a_2^{(\pi)}(\mu = 2 \text{ GeV}) = 0.101 \pm 0.024.$$

(10)

The central value is smaller by a factor of $\sim 1/2$ compared to the central values of older lattice results [13, 14, 28, 29] evaluated at the same renormalisation point or higher. We note that the lattice simulations [27] are the most advanced in terms of the extrapolation to the chiral and continuum limits. For the $M_3^{(\pi)}$, we consider the results of the recent phenomenological analysis by I. Novikov et al. (xFitter) [31].

* Eq. (8) has a correction of order $\sim m^2_\pi$, there are no corrections enhanced by a chiral logarithms [26]. Numerically such corrections are of order $\sim 2\%$. In what follows we shall consider consistently the chiral limit.
which supersedes the old analysis by M. Gluck et al. \cite{42}. At the renormalisation scale $\mu = 2$ GeV they obtained:

$$M^3(\pi) (\mu = 2 \text{ GeV}) = 0.114 \pm 0.020 \text{ (xFitter).} \quad (11)$$

In Table II the ratios $B_{21}(0)/M_{3}^{(\pi)}$ and $a_{2}^{(\rho)}/a_{2}^{(\pi)}$ obtained with help of Eq. (8) are shown. We give the results for ratios, instead of individual observables, as the former stay stable under the scale evolution at the one-loop order. Note that the positive factor $\exp(c_{1}^{(21)} m_{R}^{2})$ should be multiplied to $B_{21}(0)$ to obtain the $a_{2}^{(\rho)}$, as seen in Eq. (8). As the factor is not yet determined phenomenologically\footnote{The subtraction constant $c_{1}^{(21)}$ can be measured in hard exclusive two pion production, see discussion in Ref. \cite{24}} we use the model value calculated in this work, $c_{1}^{(21)} \in [0.7, 0.9]$ GeV$^{-2}$. The detailed discussion of this constant will be given in the following section. Note that the smaller value of $a_{2}^{(\pi)}$ results in a cancellation for $B_{21}(0)$. Even though it is difficult at this point to draw the conclusion on the sign of $a_{2}^{(\rho)}$ decisively due to relatively large error of the studies, the results still hints strongly to negative value of $a_{2}^{(\rho)}$.

|                      | RQCD/xFitter |
|----------------------|--------------|
| $B_{21}(0)/M_{3}^{(\pi)}$ | $-0.28 \pm 0.26$ |
| $a_{2}^{(\rho)}/a_{2}^{(\pi)}$ | $(-0.51 \pm 0.53) (1.0 \pm 0.1)$ |

TABLE II. Phenomenologically determined ratios $B_{21}(0)/M_{3}^{(\pi)}$ and $a_{2}^{(\rho)}/a_{2}^{(\pi)}$ at QCD normalisation point $\mu = 2$ GeV. The ratios are stable under scale evolution at the one-loop order. For calculation of the ratio $a_{2}^{(\rho)}/a_{2}^{(\pi)}$ the subtraction constant is taken in the range $c_{1}^{(21)} \in [0.7, 0.9]$ GeV$^{-2}$, this range is calculated within the instanton model in the present work. The uncertainty related to the variation of $c_{1}^{(21)}$ is shown in the last bracket in the low row (note that this bracket is always positive).

III. $a_{2}^{(\rho)}$ FROM INSTANTONS

The low-energy effective action of quarks interacting with (pseudo)Goldstone bosons derived from the theory of the instanton vacuum \cite{33} \cite{34} has the following form in the Euclidean space:

$$S_{\text{eff}} = \int d^{4}x \Psi(x) \left[ i \partial + i \sqrt{M(i\partial)} U^{\gamma^5}(x) \sqrt{M(i\partial)} \right] \Psi(x). \quad (12)$$

$U^{\gamma^5}$ is the non-linear chiral $SU(2)_{f}$ field defined by

$$U^{\gamma^5}(x) = U(x) \frac{1 + \gamma^5}{2} + U^t(x) \frac{1 - \gamma^5}{2}, \quad (13)$$

with

$$U(x) = \exp \left[ i \frac{\pi}{F_{\pi}} \tau^{a} \right]. \quad (14)$$

The momentum dependence of the dynamical quark mass arises from the Fourier transform of the quark zero-mode and has the following representation in the momentum space \cite{34}:

$$M(k) = M_{0}F^{2}(k), \quad (15)$$

$$F(k) = 2t \left[ I_{0}(t)K_{1}(t) - I_{1}(t)K_{0}(t) - \frac{1}{t} I_{1}(t)K_{1}(t) \right] \bigg|_{t = \frac{\mu^{2}}{2}}. \quad (16)$$

\footnote{In what follows we shall implicitly assume that $a_{2}^{(\pi)} \geq 0$, for a negative $a_{2}^{(\pi)}$ the second Gegenbauer moment of $\rho$-meson DA is obviously negative, see Eq. (8).}
In the above expression $K$ and $I$ are modified Bessel functions and $\rho$ is the average instanton size. The zero-momentum quark mass $M_0 = M(k = 0)$ generated by the spontaneous breakdown of the chiral symmetry is calculated by using the gap equation:

$$N/V = 4N_c \int \frac{d^4k}{(2\pi)^4} \frac{M^2(k)}{k^2 + M^2(k)},$$

(17)

where $N$ is the average instanton number and $V$ is the 4-Euclidean volume. In the above expression, the average instanton density $N/V$ can be re-expressed as $N/V = 1/R^4 = (\rho^4/R^4)/\rho^4$, where $R$ is the average instanton inter-distance. Hence, $M_0$ is obtained with given average instanton packing fraction and average instanton size. Typically the set of values $\rho/R \approx 1/3$ and $R \approx 1$ fm was obtained from Feynman variational principle for QCD partition function [33, 35]. In principle, the values of $\rho$ and $R$ are related to the $\Lambda_{\text{QCD}}$ and can be matched to perturbative QCD to provide the model renormalisation point, see Ref. [35].

The effective action (12) can be used to compute many low-energy observables for the pions. For example, the pion decay constant which is well established experimentally $F_\pi \approx 93$ MeV ($F_\pi \approx 88$ MeV in the chiral limit [36]), can be computed as the loop integral in 4D-Euclidean momentum space which has the following parametrical dependence:

$$F^2_\pi = \frac{C}{\rho^2} \left(\frac{\rho}{R}\right)^4 \ln \left(\frac{R^2}{\rho^2}\right),$$

(18)

with the constant $C \sim 1$. We see that in the theory of the instanton vacuum the pion decay constant has a natural suppression by the small instanton packing fraction, hence the instanton mechanism of the spontaneous chiral symmetry breaking explains the “accidental” smallness of the pion decay constant. With the typical choice of $\rho/R \approx 1/3$ with $R \approx 1$ fm, we obtain $F_\pi \approx 100$ MeV. The best values of the instanton vacuum parameters were determined with help of Feynman variational principle in Refs. [33, 34], the values are $\rho/R \approx 1/3$ and $R \approx 0.33$ fm. In this paper we vary these parameters in the range which corresponds to pion decay constant (in the chiral limit) $F_\pi = 88 \pm 15$ MeV.

The calculations of the pion and two-pion DAs were pioneered in Ref. [37] and Ref. [38] respectively. Main finding in these papers was that the momentum dependence of the quark mass is very important to determine the shape of (two)pion DAs. Using the simplifying assumption about the momentum dependence of the quark mass the first estimates of the DAs were performed. The technique was refined in Refs. [39–42], and we refer to these papers for details of calculations of pion DAs. In all these previous works a pole-type quark-mass momentum dependence was used to simplify the calculation and to obtain some analytic results. In this work, we will use directly (16) to explore the instanton parameter dependences of the observables as it follows from the theory of the instanton vacuum.

In the limit $\rho \to 0$ with fixed $\rho/R \ll 1$ one obtains that:

$$a^{(\pi)}_2 = \frac{7}{18}, \quad M^{(\pi)}_3 = \frac{1}{3}, \quad \text{[limit } \rho \to 0].$$

(19)

Therefore, owing to the soft pion theorem and crossing relations, we have in this limit:

$$B_{21}(0) = 0 \quad \text{[limit } \rho \to 0].$$

(20)

For small $W^2$ one can also easily obtain in the limit of small $\rho$ [22].

$$B_{21}(W^2) = -\frac{7N_c}{1440\pi^2F^2_\pi}W^2 + O(W^4) \quad \text{[limit } \rho \to 0].$$

(21)

The result indicates that $B_{21}(W^2)$ is getting negative with increasing of the two-pion invariant mass.

The above limiting results correspond to flat pion DA and to flat quark distribution function in the pion. Also these limiting results implies that $a^{(\rho)}_2 \to 0$ for $\rho \to 0$, see Eq. (16). Furthermore we made an important observation that beyond the limit of zero instanton size $B_{21}(0)$ is always negative. Therefore, independently of the details of the dynamics (e.g. the shape of the form factor $F(k)$) the second Gegenbauer moment for $\rho$-meson DA is always negative.

In Fig. 1 we show the results of our calculations of the ratio $B_{21}(0)/M^{(\pi)}_3$ for various values of instanton parameters $\rho$ and $\rho/R$. We also show on this figure by the vertical shaded band the range of instanton...
parameters which leads to reasonable values of  $F_\pi = 88 \pm 15$ MeV, about 20% around the phenomenological value of the decay constant in the chiral limit for $\rho = 0.33$ fm. From Fig. 1 we clearly see that the ratio $B_{21}(0)/M_3^{(\pi)}$ in the instanton model is definitely negative and is compatible (within 2$\sigma$) with the phenomenological analysis from previous section. The latter is shown by the horizontal shaded 1$\sigma$ band.

In order to obtain the second Gegenbauer moment of the $\rho$-meson DA from Eq. (8), we need to calculate the subtraction constant $c_{(21)}$. In Fig. 2 the result of the calculation of the dimensionless combination $c_{1(21)} F_\pi^2$ is shown for various values of instanton vacuum parameters $\rho$ and $\rho/R$. We see that

the subtraction constant has rather strong dependence on $\rho/R$, this prevents us from precise determination of this constant. Therefore for the phenomenological analysis presented in the previous section we assumed generously that $c_{1(21)} \in [0.7, 0.9]$ GeV$^{-2}$ which follows from results presented in Fig. 2. We note that the sharp dependence of the subtraction constant on $\rho/R$ can be used for the phenomenological determination.
of this instanton parameter from a measurement of $c_{1}^{(21)}$.

Eventually, combining the calculations of $B_{21}(0)$ and the subtraction constant, we present in Fig. 3 the ratio $a_{2}^{(\rho)}/M_{3}^{(\pi)}$ at various values of the instanton parameters. The value of the ratio $a_{2}^{(\rho)}/M_{3}^{(\pi)}$ has a rather wide range, roughly from $-2$ to $-1$, depending on the given values of the packing fraction $\rho/R$ and the average instanton size $\rho$. Important observation is that the ratio $a_{2}^{(\rho)}/M_{3}^{(\pi)}$, and hence $a_{2}^{(\rho)}$, is always negative in the instanton model. This is at variance with the majority of the results for this quantity in various models, see Table I. It is also very interesting that the recent calculation of the $\rho$-meson DA in an approach which emphasises the role of topologically (instanton) induced quark interactions [23] also predicts the negative $a_{2}^{(\rho)}$.

**Figure 3**. The ratio $a_{2}^{(\rho)}/M_{3}^{(\pi)}$ as a function of $\rho/R$ is plotted for various $\rho$ values. The green dotted horizontal line indicates the result $a_{2}^{(\rho)}/M_{3}^{(\pi)} = -0.63$ from Ref. [22]. Presented range of $\rho/R$ corresponds to $F_{\pi} = 88 \pm 15$ MeV for $\rho = 0.33$ fm.

**IV. SUMMARY**

Main findings of the present work are:

- Using the soft pion theorem, crossing, and the dispersion relations for two pion distribution amplitude ($2\pi$DA) we performed phenomenological analysis for the ratio of the second Gegenbauer moments of the pion and the $\rho$ meson DAs with the result at $\mu = 2$ GeV:

$$a_{2}^{(\rho)}/a_{2}^{(\pi)} = (-0.51 \pm 0.53)(1.0 \pm 0.1).$$

(22)

As the input for our analysis we used the value of $a_{2}^{(\pi)}$ from the most advanced to date lattice simulation in Ref. [27] and the most recent phenomenological analysis of the pion PDFs [31]. Eventually the yet experimentally unknown subtraction constant for the dispersion relations is calculated here using the effective low-energy theory derived from the instanton theory of the QCD vacuum. The uncertainty of the result [22] related to the model uncertainty of the model calculations of the subtraction constant is given in the last brackets of Eq. [22]. Note that this bracket is always positive – uncertainty in the subtraction constant does not influence the sign in Eq. [22].

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5 Note that the ratio $a_{2}^{(\rho)}/a_{2}^{(\pi)}$ is renormalisation scale independent at one-loop order.
We computed the ratio \( a_2^{(\rho)} / M_3^{(\pi)} \) in the instanton model of the QCD vacuum:

\[
a_2^{(\rho)} / M_3^{(\pi)} \in [-1, -2].
\]  

(23)

Here the range of the ratio reflects rather generous range of variation for the parameters of the instanton vacuum. The model calculations are in qualitative disagreement, especially what concerns the sign, with the majority of calculations of \( a_2^{(\rho)} \) in the literature (see summary of results in Table I).

From these results we may conjecture that the topologically non-trivial field configurations in the QCD vacuum (instantons) lead to qualitatively different shapes of the pion and the \( \rho \)-meson distribution amplitudes. Similar picture was obtained, although from different perspective, by E. Shuryak in Ref. [23].

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