SOME MULTIPLICATIVE PRESERVERS ON $B(H)$

LAJOS MOLNÁR

Abstract. In this paper we describe the form of those continuous multiplicative maps on $B(H)$ ($H$ being a separable complex Hilbert space of dimension not less than 3) which preserve the rank, or the corank. Furthermore, we characterize those continuous *-semigroup endomorphisms of $B(H)$ which are spectrum non-increasing.

1. Introduction

The study of linear preserver problems has a long history. In fact, it is one of the most active research areas in matrix theory [10] (also see [3] for a survey on linear preservers on operator algebras). In a recent paper [7] Hochwald started to investigate multiplicative preserver problems. In his paper he described the form of those multiplicative selfmaps of a matrix algebra which preserve the spectrum (also see [1] for a result concerning Banach algebras). As a natural generalization, he also raised the question of spectrum-preserving multiplicative maps on operator algebras even under the possible additional condition of surjectivity. However, taking into account Martindale’s purely algebraic result [1, First Corollary], it follows that in the case of many operator algebras $\mathcal{A}$ (for example, if $\mathcal{A}$ is a standard operator algebra on a Banach space of dimension at least 2, i.e. a subalgebra of the whole operator algebra which contains the ideal of all finite rank operators), every multiplicative transformation on $\mathcal{A}$ which maps onto an arbitrary algebra is automatically additive. Since additivity is not so far from linearity, it seems a much more exciting problem to try to attack the problem if surjectivity is not assumed.

In the present paper we consider those multiplicative preservers on the operator algebra $B(H)$ which are the natural analogues of the most ‘popular’ linear preservers, that is, we try to describe the form of those multiplicative maps which preserve the rank, or the spectrum. Our main tool on the
way to obtain our results is the extensive theory of measures on lattices of projections of operator algebras.

Let us fix the notation and the concepts that we shall use throughout. Let $H$ be a Hilbert space. Denote by $B(H)$ the algebra of all bounded linear operators on $H$. An operator $P \in B(H)$ is called an idempotent if $P^2 = P$. Two idempotents $P, Q \in B(H)$ are said to be orthogonal if $PQ = QP = 0$. We denote $P \leq Q$ if $PQ = QP = P$. Any self-adjoint idempotent in $B(H)$ is called a projection. The set of all projections in $B(H)$ is denoted by $P(H)$. The notation $P_1(H)$ stands for the set of all rank-one projections on $H$. The ideal of all finite-rank elements in $B(H)$ is denoted by $F(H)$. If $x, y \in H$, then the operator $x \otimes y$ is defined by $(x \otimes y)(z) = \langle z, y \rangle x \quad (z \in H)$.

Clearly, every rank-one operator $A$ is of the form $A = x \otimes y$. Moreover, the rank-one projections are exactly the operators of the form $x \otimes x$ where $x$ is a unit vector.

A linear map $\phi : A \to B$ between the algebras $A$ and $B$ is called a Jordan homomorphism if $\phi(x)^2 = \phi(x^2)$ holds for every $x \in A$. Obviously, every homomorphism is a Jordan homomorphism and this is the case with every antihomomorphism as well, that is, with every linear map $\psi : A \to B$ for which $\psi(xy) = \psi(y)\psi(x)$ ($x, y \in A$).

2. Statement of the results

In what follows let $H$ be a separable complex Hilbert space of dimension at least 3.

Our first result describes the form of the continuous multiplicative rank preservers on $P(H)$. We emphasize that here we assign rank only to the elements of $F(H)$.

**Theorem 1.** Let $\phi : P(H) \to B(H)$ be a continuous multiplicative map which preserves the rank. Then $\phi$ is of the form

\[
\phi(P) = TPS + \phi'(P) \quad (P \in P(H))
\]

where $T, S : H \to H$ are either both bounded linear operators or both bounded conjugate-linear operators such that $ST = I$, $\phi' : P(H) \to B(H)$ is a continuous multiplicative map which vanishes on the set of all finite rank projections and $TPS\phi'(Q) = \phi'(Q)TPS = 0$ ($P, Q \in P(H)$).

**Remark 1.** To see that the 'singular' part $\phi'$ can really appear in (1) suppose that $H \cong H \oplus H$ (i.e., $H$ is infinite dimensional) and define a continuous
multiplicative map \( \phi : P(H) \to B(H) \) by

\[
\phi(P) = \begin{cases} 
\begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix}, & \text{if the corank of } P \text{ is infinite} \\
\begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix}, & \text{if the corank of } P \text{ is finite.}
\end{cases}
\]

The description of continuous multiplicative rank preservers on \( B(H) \) reads as follows.

**Theorem 2.** Let \( \phi : B(H) \to B(H) \) be a continuous multiplicative map which preserves the rank. Then \( \phi \) is of the form

\[
\phi(A) = TAS \quad (A \in B(H))
\]

where \( T, S : H \to H \) are either both bounded linear operators or both bounded conjugate-linear operators and \( ST = I \).

In the following result we describe the form of the continuous multiplicative maps on \( B(H) \) which preserve the corank. There are (at least) two possible definitions of the corank of an operator \( A \in B(H) \). Let \( n \) be a nonnegative integer. The first possibility is as follows. We say that the operator \( A \in B(H) \) has corank \( n \) if the algebraic dimension of the quotient space \( H/\text{rng } A \) is \( n \)-dimensional. The second possibility is when we say that \( A \) has corank \( n \) if the Hilbert dimension of \( \text{rng } A^\perp \) is \( n \). We shall see in the proof of our next result that we have the same description in both cases. In relation to the following theorem we also refer to [3, Theorem 3] and [13, Theorem 2].

**Theorem 3.** Let \( \phi : B(H) \to B(H) \) be a continuous multiplicative map which preserves the corank. Then \( \phi \) is of the form

\[
\phi(A) = TAT^{-1} \quad (A \in B(H))
\]

where \( T : H \to H \) is either a bounded linear operator or a bounded conjugate-linear operator. In particular, \( \phi \) is either a linear or a conjugate-linear algebra automorphism of \( B(H) \).

Finally, we consider multiplicative maps \( \phi \) on \( B(H) \) that are spectrum non-increasing which means that \( \sigma(\phi(A)) \subset \sigma(A) \) for every \( A \in B(H) \).

**Theorem 4.** Let \( \phi : B(H) \to B(H) \) be a continuous *-semigroup homomorphism (that is, a multiplicative map with the property that \( \phi(A)^* = \phi(A^*) \) \( (A \in B(H)) \)). If \( \phi \) is spectrum non-increasing, then \( \phi \) is a linear *-endomorphism of \( B(H) \). More precisely, there are linear isometries \( U_n : H \to H \ (n = 1, \ldots) \) with pairwise orthogonal ranges which generate \( H \) such that \( \phi \) is of the form

\[
\phi(A) = \sum_n U_n AU_n^* \quad (A \in B(H)).
\]
Notice that it is an obvious byproduct of the foregoing theorem that the spectrum non-increasing maps appearing there are necessarily spectrum preserving.

We remark that the form of our preservers in the low-dimensional cases (i.e. when \( \dim H \leq 2 \)) can be easily deduced from the result of Šemrl in [14] where the general form of the multiplicative selfmaps of a matrix algebra is given.

3. Proofs

Proof of Theorem 1. The idea of the proof is very simple. First extend \( \phi \) to a linear map on \( F(H) \) (this will be denoted by \( \tilde{\phi} \)) which preserves the rank and then apply a result on the form of linear rank preservers. So, the idea is obvious but we have to work hard to reach the desired conclusion.

Let \( P_1, \ldots, P_n \) be pairwise orthogonal rank-one projections. Let \( P = P_1 + \cdots + P_n \). By the properties of \( \phi \), \( \phi(P_1), \ldots, \phi(P_n) \) are pairwise orthogonal rank-one idempotents and \( \phi(P) \) is a rank-\( n \) idempotent. Since we have

\[
\phi(P_1) + \cdots + \phi(P_n) = \phi(P)(\phi(P_1) + \cdots + \phi(P_n))
\]

and

\[
\phi(P_1) + \cdots + \phi(P_n) = \phi(P_1)\phi(P) + \cdots + \phi(P_n)\phi(P) = (\phi(P_1) + \cdots + \phi(P_n))\phi(P),
\]

it follows that \( \phi(P_1) + \cdots + \phi(P_n) \leq \phi(P) \). But the idempotents on both sides of the latter inequality have rank \( n \) which implies that

\[
\phi(P_1) + \cdots + \phi(P_n) = \phi(P).
\]

Let \( H_d \) denote an arbitrary \( d \)-dimensional subspace of \( H \). Consider the natural embedding \( B(H_d) \hookrightarrow B(H) \) and for any \( h \in H \) let \( \phi_h \) be defined by \( \phi_h(P) = \langle \phi(P)h, h \rangle \). Taking (3) into account we easily obtain that \( \phi_h \) is a measure on \( P(H_d) \). We assert that \( \phi \) is bounded on \( P_1(H_d) \). Indeed, suppose on the contrary that there is a sequence \( (x_k) \) of unit vectors in \( H_d \) such that \( \|\phi(x_k \otimes x_k)\| \to \infty \). Since \( H_d \) is finite dimensional, \( (x_k) \) has a convergent subsequence. We can assume without any loss of generality that this subsequence is the original sequence \( (x_k) \). Let \( x = \lim_k x_k \). Then \( x \in H_d \) is a unit vector and by the continuity of \( \phi \) we have \( \|\phi(x \otimes x)\| = \infty \) which is an obvious contradiction. So, for any \( h \in H \), \( \phi_h \) is a so-called \( P_1 \)-bounded measure on \( P(H_d) \). By Gleason’s theorem [3, Theorem 3.2.16.] this implies that, in case \( d \geq 3 \), there exists a linear operator \( T_h \) on \( H_d \) such that

\[
\phi_h(P) = \text{tr} T_h P \quad (P \in P(H_d)).
\]

Our aim now is to extend \( \phi \) to a linear transformation of \( F(H) \). Let \( x_1, \ldots, x_n \in H \) be unit vectors (the pairwise orthogonality of the \( x_i \)’s is
not assumed) and let \( \lambda_1, \ldots, \lambda_n \) be real numbers. Define
\[
\psi\left( \sum_k \lambda_k x_k \otimes x_k \right) = \sum_k \lambda_k \phi(x_k \otimes x_k).
\]

We have to check that \( \psi \) is well-defined. To see this, let \( y_1, \ldots, y_n \in H \) be unit vectors and \( \mu_1, \ldots, \mu_n \in \mathbb{R} \) such that
\[
\sum_k \lambda_k x_k \otimes x_k = \sum_k \mu_k y_k \otimes y_k.
\]

Let \( H_d \) be a finite dimensional subspace of \( H \) of dimension \( d \geq 3 \) which contains \( x_1, \ldots, x_n, y_1, \ldots, y_n \). Let \( h \in H \) be any vector. Let \( T_h \) denote the linear operator on \( H_d \) corresponding to \( \phi(h) \) (see (8)). We compute
\[
\begin{align*}
\langle \sum_k \lambda_k \phi(x_k \otimes x_k) h, h \rangle &= \sum_k \lambda_k \phi_h(x_k \otimes x_k) = \text{tr}_h(\sum_k \lambda_k x_k \otimes x_k) = \\
\text{tr}_h(\sum_k \mu_k y_k \otimes y_k) &= \sum_k \mu_k \phi_h(y_k \otimes y_k) = \langle \sum_k \mu_k \phi(y_k \otimes y_k) h, h \rangle.
\end{align*}
\]

Since this holds true for every \( h \in H \), we obtain that \( \psi \) is well-defined. The definition (8) now clearly implies that \( \psi \) is a real-linear operator on the set of all self-adjoint finite-rank operators. Clearly, \( \psi \) sends projections to idempotents. It is now a standard argument to verify that the extension \( \tilde{\psi} : F(H) \to B(H) \) of \( \psi \) defined by
\[
\tilde{\psi}(A + iB) = \psi(A) + i\psi(B)
\]
for any self-adjoint operators \( A, B \in F(H) \) is a Jordan homomorphism of \( F(H) \). See, for example, the proof of [12, Theorem 2].

Since \( F(H) \) is a locally matrix ring, it follows from a celebrated result of Jacobson and Rickart [9, Theorem 8] that \( \tilde{\psi} \) can be written as \( \tilde{\psi} = \psi_1 + \psi_2 \), where \( \psi_1 \) is a homomorphism and \( \psi_2 \) is an antihomomorphism. Let \( P \) be a rank-one projection. Since \( \psi(P) = \phi(P) \) is also rank-one, we obtain that one of the idempotents \( \psi_1(P), \psi_2(P) \) is zero. Since \( F(H) \) is a simple ring, it is now easy to see that this implies that either \( \psi_1 \) or \( \psi_2 \) is identically zero, that is, \( \tilde{\psi} \) is either a homomorphism or an antihomomorphism of \( F(H) \). In what follows we can assume without loss of generality that \( \tilde{\psi} \) is a homomorphism.

We show that \( \tilde{\psi} \) preserves the rank. Let \( A \) be a rank-\( n \) operator. Then there is a rank-\( n \) projection \( P \) such that \( PA = A \). Thus, \( \tilde{\psi}(A) = \tilde{\psi}(P)\tilde{\psi}(A) = \phi(P)\tilde{\psi}(A) \) which proves that \( \tilde{\psi}(A) \) is of rank at most \( n \). If \( Q \) is any rank-\( n \) projection, then there are finite rank operators \( U, V \) such that \( Q = UAV \). Since \( \phi(Q) = \tilde{\psi}(Q) = \tilde{\psi}(U)\tilde{\psi}(A)\tilde{\psi}(V) \) and the rank of \( \phi(Q) \) is \( n \), it follows that the rank of \( \tilde{\psi}(A) \) is at least \( n \). Therefore, \( \tilde{\psi} \) is rank preserving. We now refer to Hou’s work [8]. It follows from the argument leading to [8, Theorem 1.2] (which is in fact a standard ‘preserver-argument’ already) that there are linear operators \( T, S \) on \( H \) such that \( \tilde{\psi} \) is of the form
\[
\tilde{\psi}(x \otimes y) = (Tx) \otimes (Sy) \quad (x, y \in H)
\]
(recall that we have assumed that $\tilde{\psi}$ is a homomorphism). We claim that $T, S$ are bounded. This will follow from the following lemma.

**Lemma 1.** Let $T, S$ be linear operators on $H$ with the property that the map $x \mapsto (Tx) \otimes (Sx)$ is continuous on the unit ball of $H$. Then $T, S$ are bounded.

**Proof.** If $x_n \to x$ and $x \neq 0$, then we have

$$\frac{x_n}{\|x_n\|} \to \frac{x}{\|x\|}.$$ 

This implies that

$$\frac{(Tx_n) \otimes (Sx_n)}{\|x_n\|^2} \to \frac{(Tx) \otimes (Sx)}{\|x\|^2}.$$ 

which yields

$$(Tx_n) \otimes (Sx_n) \to (Tx) \otimes (Sx).$$

Consequently, the map $x \mapsto (Tx) \otimes (Sx)$ is continuous at any point different from 0. Now, let $x_n \to 0$ and pick a nonzero vector $y \in H$ for which $Sy \neq 0$ (observe that if $S = 0$, then there is nothing to prove). Using the polarization identity

$$(Tx_n) \otimes (Sy) = \frac{1}{4} \{ T(x_n + y) \otimes S(x_n + y) - T(x_n - y) \otimes S(x_n - y) + iT(x_n + iy) \otimes S(x_n + iy) - iT(x_n - iy) \otimes S(x_n - iy) \},$$

we see that $(Tx_n) \otimes (Sy) \to 0$ which gives us that $T$ is continuous at 0, that is, $T$ is bounded. The boundedness of $S$ is now obvious.

To continue the proof of Theorem 1, we infer from (8) that $\langle Tx, Sx \rangle = \langle x, x \rangle$ for every unit vector $x \in H$ ($\phi$ sends rank-one projections to idempotents). Clearly, this implies that $\langle Tx, Sy \rangle = \langle x, y \rangle$ ($x, y \in H$). Consequently $S^*T = I$. We have $\phi(P) = TPS^*$ for every rank-one projection $P \in P(H)$. By the additivity property of $\phi$ appearing in (6), it follows that $\phi(P) = TPS^*$ holds true for every finite-rank projection $P$ as well.

Denote $Q = TS^*$. Clearly, $Q^2 = TS^*TS^* = TIS^* = Q$. Let $P$ be an arbitrary projection. Choose a monotone increasing sequence $(P_n)$ of finite-rank projections which weakly converges to $I$. We compute

(9) \[ \phi(P)Q = \phi(P) \text{ w- lim}_n TP_n S^* = \text{ w- lim}_n \phi(P) TP_n S^* = \]

$$\text{ w- lim}_n \phi(P) \phi(P_n) = \text{ w- lim}_n \tilde{\psi}(PP_n) = \text{ w- lim}_n TPP_n S^* = TPS^*$$

and

(10) \[ Q \phi(P) = (\text{ w- lim}_n TP_n S^*) \phi(P) = \text{ w- lim}_n (TP_n S^* \phi(P)) = \]

$$\text{ w- lim}_n \phi(P_n) \phi(P) = \text{ w- lim}_n \tilde{\psi}(P_n P) = \text{ w- lim}_n TP_n PS^* = TPS^*.$$ 

So, $Q$ is an idempotent commuting with the range of $\phi$. Therefore, $\phi$ can be written as

$$\phi(P) = \phi(P)Q + \phi(P)(I - Q)$$
where the maps $\phi_1 : P \mapsto \phi(P)Q$ and $\phi_2 : P \mapsto \phi(P)(I - Q)$ are multiplicative. We see that $\phi_1(P) = TPS^*$ ($P \in P(H)$) and thus $\phi_2$ vanishes on the set of all finite-rank projections. This completes the proof of the theorem.

Proof of Theorem 2. If we consider $\phi$ only on $P(H)$, then by Theorem 1 it follows that

$$\phi(P) = TPS$$

for every finite rank projection $P$, where $T, S$ are either both bounded linear operators or both bounded conjugate-linear operators with $ST = I$. In what follows we can suppose without loss of generality that $T, S$ are linear.

Let $A \in B(H)$ be a rank-one operator. Then there is another rank-one operator $R$ such that $A = ARA$. Since $\phi$ preserves the rank, it follows from the equality

$$\phi(\lambda A) = \phi(A)\phi(\lambda R)\phi(A)$$

that $\phi(\lambda A) = f_A(\lambda)\phi(A)$ with some scalar function $f_A$. If $B$ is a rank-one operator with $BA \neq 0$ and $f_B$ is the scalar function corresponding to $B$, then we have

$$f_A(\lambda)\phi(B)\phi(A) = \phi(B)\phi(\lambda A) = \phi(\lambda B)\phi(A) = f_B(\lambda)\phi(B)\phi(A)$$

which implies that $f_A(\lambda) = f_B(\lambda)$ ($\lambda \in \mathbb{C}$). If $C$ is a rank-one operator and $CA = 0$, then we can choose a rank-one operator $B$ such that $CB \neq 0$ and $BA \neq 0$. This gives us that $f_C = f_B = f_A$. Therefore, the scalar function $f_A$ does not depend on the rank-one operator $A$. In what follows this function will be denoted by $f$. It follows from the equality

$$f(\lambda \mu)\phi(A) = \phi(\lambda \mu A) = f(\lambda)\phi(\mu A) = f(\lambda)f(\mu)\phi(A)$$

that $f$ is a continuous multiplicative function. We show that it is additive as well. Let $x, y \in H$ be orthogonal unit vectors. Since $\phi$ is additive on the set of finite rank projections, we compute

$$\phi((\lambda x + \mu y) \otimes y) = \phi(x \otimes x + y \otimes y)\phi((\lambda x + \mu y) \otimes y) =$$
$$= (\phi(x \otimes x) + \phi(y \otimes y))\phi((\lambda x + \mu y) \otimes y) =$$
$$\phi(x \otimes x)\phi((\lambda x + \mu y) \otimes y) + \phi(y \otimes y)\phi((\lambda x + \mu y) \otimes y) =$$
$$\phi(\lambda x \otimes y) + \phi(\mu y \otimes y) = f(\lambda)\phi(x \otimes y) + f(\mu)\phi(y \otimes y)$$

Multiplying by $\phi(x \otimes (x + y))$ from the left we can compute

$$f(\lambda + \mu)\phi(x \otimes y) = \phi((\lambda + \mu)(x \otimes y)) =$$
$$= \phi((x \otimes (x + y))(\lambda x + \mu y) \otimes y) = \phi(x \otimes (x + y))\phi((\lambda x + \mu y) \otimes y) =$$
$$\phi(x \otimes (x + y))(f(\lambda)\phi(x \otimes y) + f(\mu)\phi(y \otimes y)) =$$
$$f(\lambda)\phi((x \otimes (x + y))(x \otimes y)) + f(\mu)\phi((x \otimes (x + y))(y \otimes y)) =$$
\[ f(\lambda) \phi(x \otimes y) + f(\mu) \phi(x \otimes y). \]

It follows that \( f(\lambda + \mu) = f(\lambda) + f(\mu) \), that is, \( f \) is additive. Therefore, \( f \) is a continuous ring endomorphism of \( \mathbb{C} \) with \( f(1) = 1 \). It is well-known that this implies that \( f \) is either the identity or the conjugation on \( \mathbb{C} \). We show that in our case \( f \) is the identity. Suppose on the contrary that \( f(\lambda) = \bar{\lambda} \) (\( \lambda \in \mathbb{C} \)). Let \( x, y \) be non-orthogonal unit vectors. Since \( ST = I \), we compute

\[
\langle x, y \rangle \phi(x \otimes y) = \phi(\langle y, x \rangle x \otimes y) = \phi(x \otimes x \cdot y \otimes y) =\]

\[
\phi(x \otimes x) \phi(y \otimes y) = Tx \otimes S^* x \cdot Ty \otimes S^* y =\]

\[
\langle Ty, S^* x \rangle Tx \otimes S^* y = \langle y, x \rangle Tx \otimes S^* y.
\]

So, we have

\[
(12) \quad \phi(x \otimes y) = \frac{\langle y, x \rangle}{\langle x, y \rangle} Tx \otimes S^* y.
\]

Now, let \( x, y, u, v \) be unit vectors for which \( \langle x, y \rangle, \langle x, v \rangle, \langle u, v \rangle \neq 0 \). We then have

\[
\phi(x \otimes y) \phi(u \otimes v) = \phi(\langle u, y \rangle x \otimes v) = \langle y, u \rangle \phi(x \otimes v) = \langle y, u \rangle \langle v, x \rangle Tx \otimes S^* v =\]

\[
\frac{\langle y, x \rangle}{\langle x, y \rangle} \langle u, y \rangle \langle u, v \rangle \langle v, x \rangle Tx \otimes S^* v.\]

On the other hand,

\[
\phi(x \otimes y) \cdot u \otimes v = \phi(\langle u, y \rangle x \otimes v) = \langle y, u \rangle \phi(x \otimes v) = \langle y, u \rangle \langle v, x \rangle Tx \otimes S^* v.
\]

Comparing these two equalities we arrive at

\[
\langle y, x \rangle \langle u, v \rangle \langle v, x \rangle = \langle x, y \rangle \langle y, u \rangle \langle u, v \rangle \langle v, x \rangle.
\]

Since this equality obviously does not hold true for every possible choice of \( x, y, u, v \in H \), we obtain that \( f \) is really the identity.

Now, the same argument that has led to (12) shows that

\[
\phi(x \otimes y) = Tx \otimes S^* y
\]

if \( x, y \in H \) are non-orthogonal unit vectors. If \( x, y \) are orthogonal, then choosing a unit vector \( z \in H \) such that \( \langle x, z \rangle \neq 0 \) and \( \langle z, y \rangle \neq 0 \) we have

\[
\phi(x \otimes y) = \phi(x \otimes z) \phi(z \otimes y) = Tx \otimes S^* z \cdot Tz \otimes S^* y =\]

\[
\langle Tz, S^* z \rangle Tx \otimes S^* y = \langle z, z \rangle Tx \otimes S^* y = Tx \otimes S^* y.
\]

Since \( f \) is the identity, we thus obtain \( \phi(A) = TAS \) for every rank-one operator \( A \). If \( A \in F(H) \) and \( P \) is a finite-rank projection such that \( A = PA \) and \( P_1, \ldots, P_n \) are pairwise orthogonal rank-one projections such that \( P = P_1 + \cdots + P_n \), then it follows that

\[
\phi(A) = \phi(PA) = \phi(P) \phi(A) =
\]
\[
\sum_{i} \phi(P_i) \phi(A) = \sum_{i} \phi(P_i) A = \sum_{i} T(P_i) A S = TAS.
\]

Similarly as in \cite{[12]}, \cite{[14]} in the proof of Theorem \cite{[11]} we see that the operator \( Q = TS \) is an idempotent commuting with the range of \( \phi \) and \( \phi(A)Q = TAS \) (\( A \in B(H) \)). Therefore, \( \phi \) can be written as

\[
\phi(A) = \phi(A)Q + \phi(A)(I - Q)
\]

where the maps \( \phi_1 : A \mapsto \phi(A)Q \) and \( \phi_2 : A \mapsto \phi(A)(I - Q) \) are multiplicative and \( \phi_2 \) vanishes on the set of all finite rank operators. We claim that \( \phi_2 \) is identically 0. Indeed, if \( \phi_2 \) is not zero, then \( \phi_2(I) \neq 0 \). If \( P \) is a projection of infinite rank, then due to the fact that in that case there is a coisometry \( U \) such that \( UPU^* = I \), it follows that \( \phi_2(P) \neq 0 \). Choosing an uncountable set of infinite rank projections in \( B(H) \) with the property that the product of any two of them has finite rank (see the first part of the proof of \cite[Theorem 1]{[12]}) and taking the values of those projections under \( \phi_2 \), we would obtain uncountably many pairwise orthogonal nonzero idempotents in \( B(H) \) which contradicts the separability of \( H \). This shows that \( \phi_2 = 0 \). So, \( \phi(A) = \phi(A)Q = TAS \) for every \( A \in B(H) \). This completes the proof.\( \square \)

Proof of Theorem \cite{[3]}. We prove that \( \phi \) preserves the rank of projections. First suppose that \( \phi(P) = 0 \) for every finite rank projection \( P \in B(H) \). Since \( \phi(I) = I \), just as in the proof of Theorem \cite{[4]} we see that \( \phi(P) \neq 0 \) for every infinite rank projection \( P \) and then we arrive at a contradiction in the same way as there. So, let \( n \) be the smallest positive integer with the property that \( \phi(P) \neq 0 \) whenever \( P \in P(H) \) is of rank \( n \) (observe that by the multiplicativity of \( \phi \), we have rank \( \phi(Q) = \text{rank} \phi(Q') \) if rank \( Q = \text{rank} Q' \).

We claim that the rank of \( \phi(P) \) is 1 for every such \( P \). Indeed, let \( Q \) be a rank-one projection and \( P \) be a rank-\( n \) projection such that \( (I - Q)P = P(I - Q) \) is of rank \( n - 1 \). Then \( \phi(I - Q) \) and \( \phi(P) \) are orthogonal and we have \( \phi(I - Q) + \phi(P) \leq I \). Since the corank of \( \phi(I - Q) \) is 1, this gives us that the rank of \( \phi(P) \) is 1. We show that rank \( P = 1 \). Suppose on the contrary that rank \( P = n > 1 \). Let \( P \leq R \) be a projection of rank \( n + 1 \). Similarly as just before, we can verify that the rank of \( \phi(R) \) is at most 2. On the other hand there are rank-\( n \) projections \( P_1, \ldots, P_{n+1} \leq R \) such that the product of any two of them is a rank-\( (n - 1) \) projection. Consequently, \( \phi(P_1), \ldots, \phi(P_{n+1}) \) are orthogonal and \( \phi(P_1) + \cdots + \phi(P_{n+1}) \leq \phi(R) \). Therefore, we have \( n + 1 \leq 2 \). This gives us that \( n = 1 \) and hence \( \phi \) sends rank-one projections to rank-one idempotents.

Let now \( P \) be a rank-\( n \) projection. Since \( \phi(I - P), \phi(P) \) are orthogonal idempotents and \( \phi(I - P) \) has corank \( n \), we obtain that \( \phi(P) \) has rank at most \( n \). Now, if \( P_1, \ldots, P_n \) are pairwise orthogonal rank-1 projections, then like in the proof of Theorem \cite{[4]} we see that

\[
\phi(P_1) + \cdots + \phi(P_n) \leq \phi(P_1 + \cdots + P_n).
\]
Since the idempotent appearing on the left-hand side of this inequality has rank \(n\) and the one on the right-hand side has rank at most \(n\), we infer that
\[
\phi(P_1) + \cdots + \phi(P_n) = \phi(P_1 + \cdots + P_n).
\]

Therefore, \(\phi\) preserves the rank of projections. Similarly to the argument in the proof of Theorem \(\text{I} 1\) before Lemma \(\text{I} 2\) we get that \(\phi\) is rank-preserving. By Theorem \(\text{I} 3\) we have the form \(\text{(I)}\) of \(\phi\). Since \(\phi(I) = I\), we also obtain \(TS = I\).

**Proof of Theorem \(\text{I} 4\).** Let \(\lambda \in \mathbb{C}\). By the properties of \(\phi\), \(\phi(\lambda I)\) is a normal operator whose spectrum does not contain any scalar different from \(\lambda\). Therefore, \(\phi(\lambda I) = \lambda I\). This gives us that \(\phi\) is homogenous.

We prove that for any orthogonal projections \(P, Q\) we have \(\phi(P + Q) = \phi(P) + \phi(Q)\). Let \(P, Q\) be of infinite rank such that \(P + Q = I\). Pick a scalar \(0 < \mu < 1\). We have \(\sigma(\phi(P + \mu Q)) \subset \{1, \mu\}\). We distinguish three cases. First suppose that \(\sigma(\phi(P + \mu Q)) = \{1\}\). Since \(\phi\) preserves normality, this yields \(\phi(P + \mu Q) = I\). Taking powers, we obtain
\[
\phi(P + \mu^n Q) = (\phi(P + \mu Q))^n = I \quad (n \in \mathbb{N}).
\]

Using the continuity of \(\phi\) we have \(\phi(P) = I\). Since \(\phi(P) + \phi(Q) \leq I\), we get \(\phi(Q) = 0\). On the other hand, \(P, Q\) are equivalent projections and it follows that \(\phi(P) = 0\) which is a contradiction. Next suppose that \(\sigma(\phi(P + \mu Q)) = \{\mu\}\). Then we have \(\phi(P + \mu Q) = \mu I\). Taking powers again, we have
\[
\phi(P + \mu^n Q) = (\phi(P + \mu Q))^n = \mu^n I \longrightarrow 0 \quad (n \in \mathbb{N}).
\]

Thus, we infer \(\phi(P) = 0\) which gives us that \(\phi(I) = 0\), a contradiction. Consequently, we have \(\sigma(\phi(P + \mu Q)) = \{1, \mu\}\). This implies that \(\phi(P + \mu Q) = P' + \mu Q'\) where \(P', Q'\) are nonzero projections such that \(P' + Q' = I\).

We show that in this case \(\phi(P + \epsilon Q) = P' + \epsilon Q'\) holds for every \(0 < \epsilon < 1\). By what we have just proved, for an arbitrary \(0 < \epsilon < 1\) we can write \(\phi(P + \epsilon Q)\) as
\[
\phi(P + \epsilon Q) = P'' + \epsilon Q''
\]

where \(P'', Q''\) are orthogonal nonzero projections with \(P'' + Q'' = I\). Since \(\phi\) clearly preserves the commutativity, we get that \(P' + \mu Q'\) and \(P'' + \epsilon Q''\) commute. Referring to the spectral theorem we obtain that \(P', Q', P'', Q''\) are pairwise commuting. Furthermore, since
\[
\phi(P + \mu Q)\phi(P + \epsilon Q) = \phi(P + \mu \epsilon Q),
\]

it follows that \((P' + \mu Q')(P'' + \epsilon Q'')\) is of the form \(P'' + \mu Q''\). Because of the equality
\[
(P' + \mu Q')(P'' + \epsilon Q'') = P' P'' + \epsilon P' Q'' + \mu Q' P'' + \epsilon \mu Q' Q''
\]

and the fact that the spectrum of \(\phi(P + \mu \epsilon Q)\) is \(\{1, \mu\}\) we obtain that \(P' Q'' = Q' P'' = 0\). Therefore, \(P' \leq P''\) and \(Q' \leq Q''\). Since \(P' + Q' = P'' + Q'' = I\), it follows that \(P' = P''\) and \(Q' = Q''\). So, we have
\[
\phi(P + \mu Q) = P' + \mu Q'.
\]
for every $0 < \mu < 1$. Sending $\mu$ to 0, we get $\phi(P) = P'$. Since $\phi(I) = I$, $\phi$ preserves the inverse operation. This yields that

$$\phi(P + (1/\mu)Q) = P' + (1/\mu)Q'.$$

By the homogeneity of $\phi$ we infer that

$$\phi(\mu P + Q) = \mu P' + Q'.$$

If $\mu \to 0$, we arrive at $\phi(Q) = Q'$. Consequently, we have

(13) \quad \phi(P) + \phi(Q) = I.

If $R, R'$ are projection such that $R \leq P$ and $R' \leq Q$, then multiplying (13) by $\phi(R + R')$ we arrive at

(14) \quad \phi(R) + \phi(R') = \phi(R + R').

Therefore, we have (14) whenever $R, R'$ are orthogonal, either both infinite or both finite rank projections. If $R$ is of finite rank and $R'$ is of infinite rank, then we can write $R' = P + Q$ where $P, Q$ are orthogonal and they are of infinite rank. The argument leading to (14) gives us that $\phi(R') = \phi(P) + \phi(Q)$ and $\phi(R + P) = \phi(R) + \phi(P)$. We then have

$$\phi(R + R') = \phi(R + P + Q) = \phi(R + P) + \phi(Q) = \phi(R) + \phi(P) + \phi(Q) = \phi(R) + \phi(R').$$

Hence, $\phi$ is additive on the set of all projections. Since $\phi$ sends projections to projections, $\phi$ is bounded on $P(H)$. By a deep result due to Bunce and Wright [4, Theorem A] it follows that $\phi|_{P(H)}$ can be extended to a bounded linear transformation $\psi$ on $B(H)$. Since $\psi$ sends projections to projections and $\psi$ is continuous, it is a standard argument to verify that $\psi$ is a Jordan *-homomorphism (once again, see the proof of [14, Theorem 2]).

We next refer to the proof of [4, Theorem 3]. Similarly to the argument followed there, we obtain that there is a central projection $Q$ in the $C^*$-algebra generated by the range of $\psi$ such that $\psi_1(.) = \psi(.)Q$ is a *-homomorphism and $\psi_2(.) = \psi(.)((I - Q)$ is a *-antihomomorphism. This gives us that $\psi$ is the direct sum of the maps

(15) \quad \psi_1 : A \mapsto \sum_n U_n A U_n^*

and

$$\psi_2 : A \mapsto \sum_n V_n A \text{tr} V_n^*,$$

where $U_n, V_n : H \to H$ are isometries with pairwise orthogonal ranges and $\text{tr}$ denotes the transpose with respect to a fixed orthonormal basis in $H$. 
Consequently, $\psi$ can be represented as

$$
\psi(A) = \begin{bmatrix}
A & 0 & \ldots & 0 & 0 & \ldots \\
0 & A & \ldots & 0 & 0 & \ldots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
0 & 0 & \ldots & A^\text{tr} & 0 & \ldots \\
0 & 0 & \ldots & 0 & A^\text{tr} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
$$

(16)

We show that the $*$-antihomomorphic part of $\psi$ is in fact missing, and hence $\psi$ is a $*$-homomorphism. Let $P_1, \ldots, P_n$ be pairwise orthogonal projections and let $\lambda_1, \ldots, \lambda_n$ be scalars. We compute

$$
\phi(\sum_i \lambda_i P_i) = \phi((\sum_i \lambda_i P_i) (\sum_i P_i)) = \phi(\sum_i \lambda_i P_i) \phi(\sum_i P_i) = \\
\phi(\sum_i \lambda_i P_i) \sum_i \phi(P_i) = \sum_k \phi((\sum_i \lambda_i P_i) P_k) = \\
\sum_k \phi(\lambda_k P_k) = \sum_k \lambda_k \phi(P_k) = \psi(\sum_k \lambda_k P_k).
$$

By the continuity of $\phi, \psi$ and the spectral theorem we get that $\phi(N) = \psi(N)$ holds for every normal operator $N \in B(H)$. Suppose that $\text{tr}$ do appear in (16). If $S_1, \ldots, S_n$ are self-adjoint operators such that $N = S_1 \cdot \ldots \cdot S_n$ is normal, then by the multiplicativity of $\phi$ we have

$$
N^\text{tr} = S_1^\text{tr} \cdot \ldots \cdot S_n^\text{tr} = (S_n \cdot \ldots \cdot S_1)^\text{tr} = N^*\text{tr}
$$

which yields that $N = N^\ast$. Let $x, y$ be orthogonal unit vectors and $S_1 = x \otimes y + y \otimes x$, $S_2 = x \otimes x - y \otimes y$. It is trivial to check that $N = S_1 S_2$ is normal but not self-adjoint. Therefore, we obtain that $\psi_2 = 0$, that is, $\psi$ is a $*$-homomorphism. It is easy to see that every rank-one operator is the scalar multiple of the product of at most three rank-one projections. This gives us that $\psi$ and $\phi$ coincide on the rank-one operators. To complete the proof, let $A \in B(H)$ be arbitrary. Choose a maximal set $(P_n)$ of pairwise orthogonal rank-one projections in $B(H)$. We compute

$$
\phi(A) = \phi(A) \phi(I) = \phi(A) \psi(I) = \phi(A) \sum_n \psi(P_n) = \phi(A) \sum_n \phi(P_n) = \\
\sum_n \phi(A) \phi(P_n) = \sum_n \phi(AP_n) = \sum_n \psi(AP_n) = \psi(A),
$$

where we have used the weak continuity of $\psi$ which clearly holds by (15). Finally, since $\phi(I) = I$, we have $\sum_n U_n U_n^* = I$. This completes the proof. □
1. B. Aupetit, *Sur les transformations qui conservent la spectre*, in Banach Algebras’97, Walter de Gruyter, 1998, 55–78.
2. C.J.K. Batty and L. Molnár, *On topological reflexivity of the groups of *-automorphisms and surjective isometries of $B(H)$*, Arch. Math. 67 (1996), 415–421.
3. M. Bresar and P. Šemrl, *Linear preservers on $B(X)$*, Banach Cent. Publ. 38 (1997), 49–58.
4. L.J. Bunce and D.M. Wright, *The Mackey-Gleason problem*, Bull. Amer. Math. Soc. 26 (1992), 288–293.
5. A. Dvurečenskij, *Gleason’s Theorem and Its Applications*, Kluwer Academic Publishers, 1993.
6. M. Győry, L. Molnár and P. Šemrl, *Linear rank and corank preserving maps on $B(H)$ and an application to *-semigroup isomorphisms of operator ideals*, Linear Algebra Appl. 280 (1998), 253–266.
7. S.H. Hochwald, *Multiplicative maps on matrices that preserve the spectrum*, Linear Algebra Appl. 212/213 (1994), 339–351.
8. J.C. Hou, *Rank-preserving linear maps on $B(X)$*, Sci. China Ser. A 32 (1989), 929–940.
9. N. Jacobson and C. Rickart, *Jordan homomorphisms of rings*, Trans. Amer. Math. Soc. 69 (1950), 479–502.
10. C.K. Li and N.K. Tsing, *Linear preserver problems: A brief introduction and some special techniques*, Linear Algebra Appl. 162-164 (1992), 217–235.
11. W.S. Martindale III, *When are multiplicative mappings additive?*, Proc. Amer. Math. Soc. 21 (1969), 695–698.
12. L. Molnár, *The set of automorphisms of $B(H)$ is topologically reflexive in $B(B(H))$*, Studia Math. 122 (1997), 183–193.
13. L. Molnár, *Some linear preserver problems on $B(H)$ concerning rank and corank*, Linear Algebra Appl. 286 (1999), 311–321.
14. P. Šemrl, *Endomorphisms of the multiplicative semigroup of matrices*, preprint.

Institute of Mathematics, Lajos Kossuth University, 4010 Debrecen, P.O.Box 12, Hungary
*E-mail address:* molnarl@math.klte.hu