Note on lacunary Fourier series on nonabelian groups

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Abstract

We show that the classical equivalence between the BMO norm and the $L^2$ norm of a lacunary Fourier series has an analogue on any discrete group $G$ equipped with a conditionally negative function.

Introduction

Throughout this article, we consider a discrete group $G$ and a conditionally negative length $\psi$ on $G$. That is to say $\psi$ is a $\mathbb{R}_+^*$-valued function on $G$ satisfying

$$\psi(e) = 0, \psi(g) = \psi(g^{-1}),$$

and

$$\sum_{g,h} a_g a_h \psi(g^{-1} h) \leq 0$$

for any finite collection of coefficients $a_g \in \mathbb{C}$ with $\sum_g a_g = 0$. We say a sequence $h_k \in G$ is $\psi$-lacunary if there exists a constant $\delta > 0$ such that

$$\psi(h_{k+1}) \geq (1 + \delta) \psi(h_k)$$

and

$$\psi(h_k^{-1} h_{k'}) \geq \delta \max\{\psi(h_k), \psi(h_{k'})\}.$$

for any $k, k'$. Let $\lambda$ be the regular left representation of $G$. We say

$$x = \sum_k c_k \lambda_{h_k}$$

is a $\psi$-lacunary Fourier series if the sequence $h_k$ is $\psi$-lacunary.

When $G = \mathbb{Z}$, and $\psi(k) = |k|, k \in \mathbb{Z}$. Kochneff/Sagher/Zhou ([3]) prove that, for any $\psi$-lacunary Fourier series $x = \sum_k c_k \lambda_{h_k} \in L^2(\mathbb{T})$, we have

$$\|x\|_{BMO}^2 \simeq \sum_k |c_k|^2.$$  \hspace{1cm} (2)

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By interpolation, this implies that every lacunary Fourier series has an equivalent $L^p$ and $L^2$ norm, which is a fundamental theory in the classical Fourier analysis. We will show that Kochneff/Sagher/Zhou’ result extends to non-abelian discrete groups by considering semigroup BMO associated with $\psi$, while an analogue of Rudin’s theorem on the size of Sidon sets (thus Leliévre’s theorem on BMO) fails for $F_2$.

1 BMO estimate.

Given a discrete group $G$, we denote by $(\mathcal{L}(G), \tau)$ the group von Neumann algebra with its canonical trace $\tau$. Denote by $L^p(\hat{G})$ the associated noncommutative $L^p$ spaces, that is the closure of $L^p(G)$ w.r.t. the norm $\|x\|_p = (\tau|x|^p)^{\frac{1}{p}}$. If $G$ is abelian, then $L^p(\hat{G})$ is the canonical $L^p$ space of functions on the dual group $\hat{G}$. In particular, if $G = \mathbb{Z}$, then $\lambda_k = e^{ikt}, k \in \mathbb{Z}$ and $L^p(\hat{\mathbb{Z}}) = L^p(\mathbb{T})$, the space of all $p$-integrable functions on the unit circle. Please refer to [11] for details on noncommutative $L^p$ spaces.

Given a conditionally negative length $\psi$ on $G$, Schöenberg’s theorem says that $T_t : \gamma \mapsto e^{-t\psi(\gamma)}\lambda_\gamma$ extends to a symmetric Markov semigroups of operators on the group von Neumann algebra $\mathcal{L}(G)$. Following [5] and [8], let us set

$$\|x\|_{BMO(\psi)} = \sup_{0 < t < \infty} \|T_t|x - T_t x_i\|^\frac{1}{2}, \quad (3)$$

for $x \in L^2(\hat{G})$. Let $BMO(\psi)$ be the space of all $x \in L^2(\hat{G})$ such that

$$\|x\|_{BMO(\psi)}^2 = \max\{\|x\|_{BMO(\psi)}^2, \|x^\ast\|_{BMO(\psi)}^2\} < \infty. \quad (4)$$

**Lemma 1.** ([JM12]) We have the following interpolation result

$$[BMO(\psi), L^1(\hat{G})]_\frac{1}{p} = L^p(\hat{G})$$

for $1 < p < \infty$. Here $L^p_0(\hat{G}) = L^p(\hat{G})/\ker \psi$.

**Lemma 2.** For $a_k \in \mathbb{R}_+, c_k, b_k \in B(H)$, we have

$$\left\| \sum_k a_k c_k^* b_k \right\| \leq \sum_k |c_k|^2 a_k \| \| \sum_k |b_k|^2 a_k \|^\frac{1}{2}. \quad (5)$$

**Proof.** This is simply the Cauchy-Schwartz inequality. \hfill $\blacksquare$

**Theorem 1.** Assume $(h_k)$ is a $\psi$-lacunary sequence. Then, for any $x = \sum_k c_k \lambda_{h_k}$, we have

$$\|x\|_{BMO(\psi)}^{\psi} \lesssim_{\psi} \max\{\|\sum_k |c_k|^2\|, \|\sum_k |c_k|\|^2\}. \quad (6)$$
Proof. An easy calculation shows that

\[ T_t|x - T_t x|^2 = \sum_{k,j} a_{k,j} (c_k \lambda h_k)^* c_j \lambda h_j, \]

with

\[ a_{k,j} = e^{-t \psi(h_k^{-1} h_j)} (1 - e^{-t \psi(h_k)}) (1 - e^{-t \psi(h_j)}) \geq 0. \]

By the subadditivity of \( \psi \) we have that \( \psi(h_k^{-1} h_j) \geq |\psi(h_k) - \psi(h_j)| \). So

\[ \sum_k a_{k,j} \leq \sum_{t \psi(h_k) \leq 1} (1 - e^{-t \psi(h_k^{-1} h_j)}) + \sum_{t \psi(h_k) > 1} e^{-t \psi(h_k)} \]

\[ \leq 1 + \delta^{-1} + \frac{1}{1 - e^{-\delta t}} \leq c_\delta. \]

We then get \( \sup_j \sum_k a_{k,j} \leq c_\delta \). Similarly, \( \sup_k \sum_j a_{k,j} \leq c_\delta \). By Lemma, we have

\[ \| T_t|x - T_t x|^2 \| \leq \| \sum_k |c_k|^2 a_{k,j} \|^{\frac{1}{2}} \sum_k |c_j|^2 a_{k,j} \|^{\frac{1}{2}} \]

\[ \leq c_\delta \| \sum_k |c_k|^2 \|. \]

Taking supremum on \( t \), we get \( \| x \|_{B^{\infty}(\mathbb{G})}^2 \leq c_\delta \| \sum_k |c_k|^2 \|. \) Taking the adjoint, we prove the upper estimate. The lower estimate is obvious.

Corollary 1. Assume \( (h_k) \) is a \( \psi \)-lacunary sequence. We have that, for any \( x = \sum_k c_k \lambda h_k \),

\[ \| x \|_p^p \leq c_\delta^p p^2 \max\{ \| \sum_k |c_k|^2 \|_{\hat{\mathbb{G}}}^p, \| \sum_k |c_k^*|^2 \|_{\hat{\mathbb{G}}}^p \}. \]  

(7)

Remark 1. Corollary 1 is independently proved in \[6\] by using noncommutative Riesz transforms associated with semigroups. If, \( G = \mathbb{F}_n \), \( \psi \) is the reduced word length, it is also easy to verify that \( \psi \)-lacunary set is \( B(2) \) in the sense of W. Rudin, so it is a \( \Lambda_4 \) set by Harcharras’s work\[2\]. This does not seems clear for \( B(p) \) with \( p > 2 \).
2 Large $\Lambda_\infty$ sets on $\mathbb{F}_2$

We call a subset $A \in G$ is completely Sidon, if $\{\lambda_h, h \in A\}$ is completely unconditional in $L(\tilde{G})$, i.e. there exists a constant $C_A$ such that

$$\| \sum_{h_k \in A} \varepsilon_k c_k \lambda_{h_k} \| \leq C_A \max \{ \| \sum_{h_k \in A} |c_k|^2 \|^\frac{1}{2}, \| \sum_{h_k \in A} |c_k^*|^2 \|^\frac{1}{2} \},$$

(8)

for any choice $\varepsilon_k = \pm$, $c_k \in B(H)$. We call a subset $A \in G$ is completely $\Lambda_\infty$, if there exists a constant $C_A$ such that

$$\| \sum_{h_k \in A} c_k \lambda_{h_k} \| \leq C_A \max \{ \| \sum_{h_k \in A} |c_k|^2 \|^\frac{1}{2}, \| \sum_{h_k \in A} |c_k^*|^2 \|^\frac{1}{2} \},$$

for any choice of finite many $c_k \in B(H)$. We say $A$ is completely $\Lambda_{bmo,\psi}$ if we take the BMO($\psi$)-norm on the left hand side of (8). Obviously, a completely $\Lambda_\infty$ set is completely $\Lambda_{bmo,\psi}$ for any $\psi$, and is completely Sidon.

Let $P_d$ ($P_{\leq d}$) be the collection of all reduced words of $\mathbb{F}_n$ with length $= d$ ($\leq d$). When $G = \mathbb{Z}$, a classical theory of Rudin says that, for any Sidon set $A$ of $\mathbb{F}_1$, we have $\#(A \cap P_{\leq d}) \lesssim \log \#P_{\leq d}$, and Lelièvre ([4]) prove that every $\Lambda_{bmo,\psi}$ is a finite combination of Hadamard lacunary sets, thus a Sidon set.

Fix a generating set $S = \{g_k, k \in \mathbb{Z}_+\}$ of $\mathbb{F}_\infty$, with the convention that $g_k^{-1} = g_{-k}$. Let $Q_n \subset \mathbb{F}_\infty$ be the collection of symmetric words of length $2n$,

$$Q_n = \{g_{k_1}g_{k_2}\cdots g_{k_n}g_{k_n}\cdots g_{k_2}g_{k_1}; |g_{k_j}| = 1, g_{k_j} \neq g_{k_{j+1}}^{-1}, k_j \in \mathbb{Z}_+\}.$$

The following Proposition is the key observation for our example. We include a proof although this maybe obvious for experts.

**Proposition 1.** $Q_n$ is a free subset of $\mathbb{F}_\infty$.

**Proof.** Let us first introduce a few notations. Given a reduced word $h \in \mathbb{F}_\infty$, we denote by $L_h$ the subset of all reduced words $g$ that start with $h$, that is $L_h = \{ g \in \mathbb{F}_\infty; |g| \geq |h|, |h^{-1}g| = |g| - |h| \}$. Suppose $h \in P_{2n}$, denote by $h^l, h^r$ the left half and the right half of $h$, i.e. the reduced words in $P_n$ such that $h = h^l h^r$. We will use the fact, that the condition $|hg| > |g|$ holds iff $g \notin L(h^r)^{-r}$ and implies that $hg \in L_{h^l}$.

Given any $m$ elements $x_j \in Q_n, 1 \leq j \leq m$ such that $x_k^{-1} \neq x_{k+1}$ for any $1 \leq k < m$, it is obvious that $|x_2 x_1| > |x_1|$. Assume $|x_j \cdots x_2 x_1| > |x_j \cdots x_2 x_1|$. That is $|x_j g| > |g|$ for $g = x_j \cdots x_2 x_1$. Then $x_j g \in L_{x_j}$. So $g' = x_j g \notin L_{(x_j)^{-1}}$ since $x_j \neq x_j^{-1}$. Then $|x_{j+1} g'| > |g'|$. We then get $|x_j \cdots x_2 x_1| > |x_j \cdots x_2 x_1|$ for all $1 < j \leq m$ by induction. Therefore, $x_m \cdots x_2 x_1 \neq e$. We then conclude that $Q_n$ is a free set. \hfill $\square$

Now, for $x = \sum_{h \in Q_n} c_h \lambda_h$ with $h \in Q_n$ and $c_h \in B(H)$, we have by Haagerup and Pisier’s inequality ([1]) that

$$\|x\| \leq 2 \max\{\|\tau x^2\|^\frac{1}{2}, \|\tau x^* x\|^\frac{1}{2}\}.$$

(9)
Example. Let $\pi$ be the group homomorphism from $\mathbb{F}_\infty$ into $\mathbb{F}_2$ with free generators $a, b$, such that

$$\pi(g_k) = a^k ba^{-k}, k \in \mathbb{N}.$$ 

By (9), $\pi(Q_n)$ is a complete $\Lambda_\infty$ set of $\mathbb{F}_2$ for each $n \in \mathbb{N}$. Therefore, it is completely Sidon, and is completely $\Lambda_{bmo(\cdot | \cdot)}$ with $| \cdot |$ the word length on $\mathbb{F}_2$. However, $\pi(Q_n)$ is not a finite union of $| \cdot |$-lacunary set, contrary to Lelièvre’s theorem for $\mathbb{F}_1$. In fact, it is easy to see that $\#(\pi(Q_n) \cap \mathcal{P}_{\leq 2nm}) \simeq mn$ while $\log \#\mathcal{P}_{\leq 2nm} \simeq nm$ as $m \to \infty$.

Remark 2. Let $\phi$ be an injection from $\mathbb{N}$ to $\mathbb{N}$. Let $\phi(k) = -\phi(-k)$ for $k < 0$. (9) holds for

$$Q_n = \{g_{k_1}g_{k_2} \cdots g_{k_n}g_{\phi(k_n)} \cdots g_{\phi(k_2)}g_{\phi(k_1)}; |g_{k_i}| = 1, g_{k_i} \neq g_{-k_i+1}\}$$

as well.

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