EXISTENCE AND NONEXISTENCE OF POSITIVE SOLUTIONS TO AN INTEGRAL SYSTEM INVOLVING WOLFF POTENTIAL

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Abstract. In this paper, we are concerned with the sufficient and necessary conditions for the existence and nonexistence of the positive solutions of the following system involving Wolff type potential:

\[
\begin{aligned}
  u(x) &= c_1(x)W_{\beta,\gamma}(v^q)(x), \\
  v(x) &= c_2(x)W_{\alpha,\tau}(u^p)(x).
\end{aligned}
\]  

(0.1)

Here \( x \in \mathbb{R}^n, 1 < \gamma, \tau \leq 2, \alpha, \beta > 0, 0 < \beta\gamma, \alpha\tau < n \), and the functions \( c_1(x), c_2(x) \) are double bounded. This system is helpful to well understand some nonlinear PDEs and other nonlinear problems. Different from the case of \( \alpha = \beta, \gamma = \tau \), it is more difficult to handle the critical condition. Fortunately, by applying the special iteration scheme and some critical asymptotic analysis, we establish the sharp criteria for existence and nonexistence of positive solutions to system (0.1). Then, we use the method of moving planes to prove the symmetry and monotonicity for the positive solutions of (0.1) when \( c_1(x) \equiv c_2(x) \equiv 1 \) in the case

\[
\frac{\gamma - 1}{p + \gamma - 1} + \frac{\tau - 1}{q + \tau - 1} = \frac{n - \alpha\tau}{2n - \alpha\tau + \beta\gamma} + \frac{n - \beta\gamma}{2n - \beta\gamma + \alpha\tau}.
\]

1. Introduction. The Wolff potential is defined for any non-negative Borel measure \( \mu \):

\[
W_{\beta,\gamma}\mu(x) = \int_0^\infty \left[ \frac{\mu(B_t(x))}{t^{n-\beta\gamma}} \right]^{\frac{1}{\gamma-1}} \frac{dt}{t},
\]

where \( 1 < \gamma < \infty, \beta > 0, \beta\gamma < n \), and \( B_t(x) \) is the ball of radius \( t \) centered at point \( x \).

If \( d\mu = f dx \) with \( f > 0 \) and \( f \in L_{loc}^1(\mathbb{R}^n) \), we write(cf. [11]):

\[
W_{\beta,\gamma}(f)(x) = \int_0^\infty \left[ \frac{\int_{B_t(x)} f(y)dy}{t^{n-\beta\gamma}} \right]^{\frac{1}{\gamma-1}} \frac{dt}{t}.
\]

It is easy to verify that \( W_{1,2}(\cdot) \) is the well-known Newton potential and \( W_{2,2}(\cdot) \) is the Riesz potential.

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There have been a series of studies on the relations between the Wolff potentials and the nonlinear PDEs (for example, see (cf. [10] and [17]). In [17], an integral estimate between the Wolff potential and Riesz potential was established as follows:

\[
C_1 \| W_{\beta,p}^{\alpha \mu} \|_{L^q(R^n)}^q \leq \| I_{\beta \mu} \|_{L^{p-1,q}(R^n)} \leq C_2 \| W_{\beta,p}^{\alpha \mu} \|_{L^q(R^n)}^q,
\]

where \( q > p - 1 \) and

\[
I_{\alpha \mu}(x) = \int_0^\infty \frac{\mu(B_t(x)) \, dt}{t^{n-\alpha}} = C \int_{R^n} \frac{d\mu(y)}{|x-y|^{n-\alpha}}
\]

is the Riesz potential. The result (1) plays a key role in the proof of Section 3 of this paper.

The Wolff potentials are helpful to well understand the nonlinear PDEs (cf.[10]-[13] and [17]) and other nonlinear problems (cf. [1] and [16]). For example, \( W_{1,\gamma}(w) \) can be used to estimate the solutions \( u \) of \( \gamma \)-Laplace equation

\[
-\text{div}(|\nabla u|^{\gamma-2} \nabla u) = w.
\]

If \( \inf_{R^n} u = 0 \), then there exist positive constants \( C_1 \) and \( C_2 \) such that (cf. [10])

\[
C_1 W_{1,\gamma}(w)(x) \leq u(x) \leq C_2 W_{1,\gamma}(w)(x), \quad x \in R^n.
\]

In this paper, we first consider the existence and nonexistence of positive solutions of the following system involving Wolff type

\[
\begin{cases}
  u(x) = c_1(x) W_{\beta,\gamma}(v^q)(x), \\
  v(x) = c_2(x) W_{\alpha,\tau}(u^p)(x).
\end{cases}
\]

Here \( x \in R^n, 1 < \gamma, \tau \leq 2, \alpha, \beta > 0, 0 < \beta \gamma, \alpha \tau < n \), and the functions \( c_1(x), c_2(x) \) are double bounded. Namely, there exist positive constants \( c \) and \( C \) such that

\[
0 < c \leq c_1(x), c_2(x) \leq C.
\]

When \( \alpha = \beta, \gamma = \tau \), system (3) becomes:

\[
\begin{cases}
  u(x) = c_1(x) W_{\beta,\gamma}(v^q)(x), \\
  v(x) = c_2(x) W_{\beta,\gamma}(u^p)(x).
\end{cases}
\]

The nonexistence result, known as Liouville type theorems, are fundamental in PDE theory and applications. Recently, by some new iteration technique and critical asymptotic analysis, Lei and Li [14] established the sharp criteria for nonexistence and nonexistence of positive solutions to system (5).

If \( c_1(x) \equiv c_2(x) \equiv 1 \), the system (5) becomes

\[
\begin{cases}
  u(x) = W_{\beta,\gamma}(v^q)(x), \\
  v(x) = W_{\beta,\gamma}(u^p)(x).
\end{cases}
\]

The critical case is

\[
\frac{\gamma - 1}{p + \gamma - 1} + \frac{\gamma - 1}{q + \gamma - 1} = \frac{n - \beta \gamma}{n}.
\]

For (6), Chen and Li [5] proved that the solutions \( u \) and \( v \) are radial symmetry and decreasing about some point \( x_0 \). Furthermore, Ma, Chen and Li [16] used the regularity lifting lemmas to derive the optimal integrability and the Lipschitz continuity. Based on these results, Lei [15] obtained the decay rates of the integrable solutions when \( |x| \to \infty \).
Theorem 1.1. For some double bounded functions decreasing about some planes, we show that the solutions of system (3) are radially symmetric and the existence of the positive solutions to (3). Then, by using the method of moving results above, in this paper we first establish sufficient and necessary conditions for ball. The inverse is also true by a similar procedure of their proof.

Hong, Li [9] investigated system of integral equations involving Wolff potential on PDE:

\[
\begin{aligned}
\left\{ \begin{array}{l}
u(x) = \int_{R^n} \frac{v(y)}{|x-y|^{n-\alpha}}dy, \\
u(x) = \int_{R^n} \frac{u(y)}{|x-y|^{n-\alpha}}dy.
\end{array} \right. 
\tag{8}
\end{aligned}
\]

(8) is associated with the study of the sharp constant of the well-known classical Hardy-Littlewood-Sobolev (HLS) inequality,

\[
\left| \int_{R^n} \int_{R^n} \frac{f(x)g(y)}{|x-y|^{n-\alpha}}dxdy \right| \leq C_{s,a,n} \| f \|_{r} \| g \|_{s}
\]

for any \( f \in L^r(R^n) \) and \( g \in L^s(R^n) \). Here \( 1 < r, s < \infty \) and \( 0 < \alpha < n \) such that \( \frac{1}{r} + \frac{1}{s} + \frac{n-\alpha}{n} = 2 \).

It was shown in [6] that the integral system (8) is equivalent to the system of PDE:

\[
\begin{aligned}
\left\{ \begin{array}{l}
(-\Delta)^{\frac{\sigma}{2}} u = v^q \quad \text{in} \quad R^n, \\
(-\Delta)^{\frac{\tau}{2}} v = u^p \quad \text{in} \quad R^n.
\end{array} \right. 
\tag{9}
\end{aligned}
\]

Several authors studied the special cases about (3) and (5). For example, Huang, Hong, Li [9] investigated system of integral equations involving Wolff potential on a bounded domain such as

\[
\begin{aligned}
u(x) &= \int_{0}^{\infty} \left[ \int_{B_t(x) \cap \Omega} u^{a} v^{b}(y)dy \right] \frac{dt}{t}, \\
u(x) &= \int_{0}^{\infty} \left[ \int_{B_t(x) \cap \Omega} u^{c} v^{d}(y)dy \right] \frac{dt}{t}, \quad x \in \Omega.
\end{aligned}
\]

They showed that if \( u \) and \( v \) are constants on \( \partial \Omega \), then \( \Omega \) is a ball. Furthermore, \( u \) and \( v \) are radially symmetric and monotone decreasing about the center of the ball. The inverse is also true by a similar procedure of their proof.

Intuitively, it would be interesting to study the system (3). Based on the results above, in this paper we first establish sufficient and necessary conditions for the existence of the positive solutions to (3). Then, by using the method of moving planes, we show that the solutions of system (3) are radially symmetric and decreasing about some \( x_0 \in R^n \).

**Theorem 1.1.** For some double bounded functions \( c_1(x), c_2(x) \), there exist positive solutions \( u, v \) of the integral system (3), if and only if

\[
pq > (\gamma - 1)(\tau - 1)
\]

and

\[
pq - \frac{(\gamma - 1)(\tau - 1)}{(\gamma - 1)(\tau - 1)} > \max \left\{ \frac{q\alpha \tau + \beta \gamma (\tau - 1)}{(n - \beta \gamma)(\tau - 1)}, \frac{p\beta \gamma + \alpha \tau (\gamma - 1)}{(n - \alpha \tau)(\gamma - 1)} \right\}. \tag{10}
\]

When \( \gamma = \tau = 2, \beta = \frac{\pi}{2}, \alpha = \frac{4}{2} \), system (3) becomes the following integral system involving Riesz potential

\[
\begin{aligned}
u(x) &= c_1(x) \int_{R^n} \frac{v^q(y)}{|x-y|^{n-\alpha}}dy, \\
u(x) &= c_2(x) \int_{R^n} \frac{u^p(y)}{|x-y|^{n-\alpha}}dy.
\end{aligned}
\tag{11}
\]
Therefore, we can easily obtain the following corollary.

**Corollary 1.** Let $0 < s, t < n$. The HLS system (11) has positive solutions $u, v$ for some double bounded functions $c_1(x), c_2(x)$, if and only if $pq > 1$ and

$$pq - 1 > \max \left\{ \frac{s + qt}{n - s}, \frac{t + ps}{n - t} \right\}.$$

According to the Theorem 3 in [6], the property can be extended to the following semilinear Lane-Emden type system

$$
\begin{cases}
(\Delta)^{k_1} u(x) = c_1(x)v^q(x), \\
(\Delta)^{k_2} v(x) = c_2(x)u^p(x).
\end{cases}
$$

(12)

**Corollary 2.** Let $k_1$ and $k_2$ be positive integers less than $\frac{n}{2}$, and assume $p, q > 1$. There exist positive $u, v$ of the system (12) for some double bounded functions $c_1(x), c_2(x)$, if and only if

$$pq > \max \left\{ \frac{n + 2k_2 q}{n - 2k_1}, \frac{n + 2k_1 p}{n - 2k_2} \right\}.$$

According to Theorem 1.1, we assume hereafter

$$pq > (\gamma - 1)(\tau - 1)$$

and

$$pq - (\gamma - 1)(\tau - 1) > \max \left\{ \frac{q\alpha\tau + \beta\gamma(\tau - 1)}{(n - \beta\gamma)(\tau - 1)}, \frac{p\beta\gamma + \alpha\tau(\gamma - 1)}{(n - \alpha\tau)(\gamma - 1)} \right\}.$$

Since $c_1(x)$ and $c_2(x)$ are not constants, the solutions of (3) have no radial structure. Then we consider another system for $c_1(x) \equiv c_2(x) \equiv 1$, that is:

$$
\begin{cases}
u(x) = W_{\beta, \gamma}(v^q), \\
v(x) = W_{\alpha, \tau}(v^p).
\end{cases}
$$

(13)

The critical case is

$$\gamma - 1 \quad \frac{p + \gamma - 1}{\rho + \gamma - 1} + \frac{\gamma - 1}{\tau + \gamma - 1} > \max \left\{ \frac{n - \alpha\tau}{2n - \alpha\tau + \beta\gamma}, \frac{n - \beta\gamma}{2n - \beta\gamma + \alpha\tau} \right\}. \quad (14)$$

**Remark 1.** As $\beta\gamma = \alpha\tau$, (10) and (14) becomes

$$\frac{pq - (\gamma - 1)(\tau - 1)}{(\gamma - 1)(\tau - 1)} > \max \left\{ \frac{\alpha\gamma(q + \gamma - 1)}{(n - \alpha\tau)(\gamma - 1)}, \frac{\alpha\gamma(p + \gamma - 1)}{(n - \alpha\gamma)(\gamma - 1)} \right\} \quad (15)$$

and

$$\frac{\gamma - 1}{\rho + \gamma - 1} + \frac{\tau - 1}{\rho + \gamma - 1} = \frac{n - \alpha\gamma}{n}, \quad (16)$$

respectively. Clearly, (15) is weaker than (16). But as $\beta\gamma \neq \alpha\tau$, we can’t be sure that (10) is weaker than (14).

For (13), using the method of moving planes, we get

**Theorem 1.2.** Let $p, q > 1$, and assume $(u, v)$ is a pair of positive solutions of (13). If $u \in L^s(R^n)$, $v \in L^t(R^n)$,

$$s = \frac{n(pq - (\gamma - 1)(\tau - 1))}{q\alpha\tau + \beta\gamma(\tau - 1)}, \quad r = \frac{n(pq - (\gamma - 1)(\tau - 1))}{p\beta\gamma + \alpha\tau(\gamma - 1)}.$$ 

Then $u$ and $v$ are radially symmetric and monotonic about some points in $R^n$. 
Remark 2. When $\gamma = \tau = 2$, $\beta = \frac{4}{7}$, $\alpha = \frac{1}{7}$, system (13) becomes

\[
\begin{align*}
u(x) &= \int_{R^n} \frac{u^q(y)}{|x-y|^{n-s}} dy, \\
v(x) &= \int_{R^n} \frac{u^p(y)}{|x-y|^{n-t}} dy,
\end{align*}
\]

and (14) becomes

\[
\frac{1}{p+1} + \frac{1}{q+1} = \frac{n-t}{2n-t+s} + \frac{n-s}{2n-s+t}.
\]

Huang, Li, and Wang [8] proved the symmetry and monotonicity of its solution in such case.

2. Existence and nonexistence for system (3). In this section, we prove Theorem 1.1 by applying the special iteration scheme and some critical asymptotic analysis (cf. [14]).

Proof of Theorem 1.1. Step 1. Existence.

Similar to the argument in the proof of Theorem 3.3 in [14], we can find three pairs solutions.

Let

\[
u(x) = \frac{1}{(1 + |x|^2)^{n_1}}, \quad v(x) = \frac{1}{(1 + |x|^2)^{n_2}}
\]

where $\theta_1, \theta_2 > 0$ will be determined later.

Inserting the form of $v$ into $W_{\beta,\gamma}(v^q)(x)$, we obtain

\[
W_{\beta,\gamma}(v^q)(x) = \left[ \int_{\mathbb{R}^n} \left[ \int_{|y|=r} \frac{dy}{(1 + |y|^2)^{q\theta_2}} \right] \frac{ln}{t} \right] dt = I_1 + I_2,
\]

When $|x| \leq R$ for some $R > 0$, then $u$ is proportional to $W_{\beta,\gamma}(v^q)(x)$. So we only consider suitably large $|x|$.

Clearly,

\[
I_1 = \int_{0}^{\frac{1}{4}} \left[ \int_{B_{r}(x)} \frac{dy}{(1 + |y|^2)^{\theta_2}} \right] \frac{ln}{t} dt
\]

\[
= c(x)(1 + |x|^2)^{-\frac{\theta_2}{2}} \int_{0}^{\frac{1}{4}} t^{\frac{\beta_2}{\gamma_2}} dt = c(x)(1 + |x|^2)^{\frac{\beta_2 - 2\theta_2}{2(\gamma_2 - 1)}}.
\]

Case 1. Take the slow rates

\[
2\theta_1 = \frac{\alpha n + \beta \gamma (\tau - 1)}{pq - (\gamma - 1)(\tau - 1)}, \quad 2\theta_2 = \frac{p\beta \gamma + \alpha \gamma (\tau - 1)}{pq - (\gamma - 1)(\tau - 1)}.
\]

Then $pq > (\gamma - 1)(\tau - 1)$, (10) lead to $\alpha \tau < 2p\theta_1 < n$ and $\beta \gamma < 2q\theta_2 < n$. Hence,

\[
I_2 = c(x) \int_{\frac{1}{4}}^{\infty} \left[ \int_{B_{r}(x)} \frac{dy}{(1 + |y|^2)^{\theta_2}} \right] \frac{ln}{t} dt
\]

\[
= c(x) \int_{\frac{1}{4}}^{\infty} t^{\frac{\beta_2 - 2\theta_2}{\gamma_2 - 1}} dt = c(x)(1 + |x|^2)^{\frac{\beta_2 - 2\theta_2}{2(\gamma_2 - 1)}}.
\]

Thus, for some double bounded functions $c_1(x), c_2(x)$, we have

\[
W_{\beta,\gamma}(v^q)(x) = c(1 + |x|^2)^{\frac{\beta_2 - 2\theta_2}{\gamma_2 - 1}} + c(x)(1 + |x|^2)^{\frac{\beta_2 - 2\theta_2}{2(\gamma_2 - 1)}} = c_2(x)u(x).
\]
Similarly, we also obtain
\[ W_{\alpha,\tau}(u^p)(x) = \frac{c_1(x)}{(1 + |x|^2)\frac{n - \alpha}{n - \beta}} = c_1(x)v(x). \]

This implies that (3) has a pair of radial solutions \((u, v)\) as (17).

**Case II.** If the stronger condition \( p > \frac{n(\gamma - 1)}{n - \beta\gamma}, q > \frac{n(\tau - 1)}{n - \alpha\tau} \) holds, we take the fast rates
\[
2\theta_1 = \frac{n - \beta\gamma}{\gamma - 1}, \quad 2\theta_2 = \frac{n - \alpha\tau}{\tau - 1}.
\]

Then \( 2\theta_1p > n \) and \( 2\theta_2q > n \), and hence
\[
I_2 = \int_{\mathbb{R}^n} \left[ \frac{\int_{B_1(x)} (1 + |y|^2)^{-p\theta_2} dy + \int_{B_1(x) \setminus B_1(0)} (1 + |y|^2)^{-q\theta_2} dy}{t^{n - \beta\gamma}} \right]^{\frac{1}{p - \gamma}} dt = c(x)(1 + |x|^2)^{\frac{\beta\gamma - n}{\gamma - 1}}.
\]

Therefore, for some double bounded functions \( c_1(x), c_2(x) \), we have
\[ W_{\beta,\gamma}(v^q)(x) = c(x)(1 + |x|^2)^{\frac{\beta\gamma - 2\theta_2 q}{\gamma - 1}} + c(x)(1 + |x|^2)^{\frac{\beta\gamma - n}{\gamma - 1}} = c_2(x)u(x). \]

Similarly, we also obtain
\[ W_{\alpha,\tau}(u^p)(x) = \frac{c_1(x)}{(1 + |x|^2)\frac{n - \alpha}{n - \beta}} = c_1(x)v(x). \]

This implies that (3) has a pair of radial solutions \((u, v)\) as (17) with fast decay rates.

**Case III.** If another stronger condition \( pq > (\gamma - 1)(\tau - 1) \) as well as
\[
\frac{\alpha\tau}{n - \beta\gamma} < p < \frac{n(\gamma - 1)}{n - \beta\gamma}, \quad \frac{\alpha\tau}{n - \beta\gamma} < q < \frac{n(\tau - 1)}{n - \alpha\tau},
\]

holds, we can find a pair of solutions \( u, v \). Now, \( u, v \) have two different fast decay rates.

We claim that if \( pq > (\gamma - 1)(\tau - 1) \), the condition (18) together (19) are stronger than (10). In fact,
\[
\frac{pq\alpha\tau + p\beta\gamma(\tau - 1)}{pq - (\gamma - 1)(\tau - 1)} < \frac{n - \beta\gamma}{\gamma - 1} < \frac{n(\gamma - 1)}{n - \beta\gamma} \cdot \frac{n - \beta\gamma}{\gamma - 1} = n.
\]

Now, (20) implies
\[
\frac{p\beta\gamma + \alpha\tau(\gamma - 1)}{pq - (\gamma - 1)(\tau - 1)} < \frac{n - \alpha\tau}{\tau - 1}.
\]

Combing (19) and (21) yields (10).

Take
\[
2\theta_1 = \frac{n - \beta\gamma}{\gamma - 1}, \quad 2\theta_2 = \frac{2p\theta_1 - \alpha\tau}{\tau - 1} = \frac{p(n - \beta\gamma)}{(\gamma - 1)(\tau - 1)} - \frac{\alpha\tau}{\tau - 1}.
\]
Therefore, $\alpha \tau < 2p\theta_1 < n$ and $2q\theta_2 > n$, and hence
\[
W_{\alpha,\tau}(u^q)(x) = c(1 + |x|^2)^{\frac{\beta\gamma-2q\theta_1}{n-\alpha\tau}} + c(x)(1 + |x|^2)^{\frac{\beta\gamma-n}{2(n-1)}}
= \frac{c_2(x)}{(1 + |x|^2)^{\frac{n-\beta\gamma}{n-\alpha\tau}}} = c_2(x)u(x).
\]

Similarly, we obtain
\[
W_{\alpha,\tau}(u^p)(x) = \frac{c_1(x)}{(1 + |x|^2)^{\frac{p\theta_1-\beta\gamma}{n-\alpha\tau}}} = c_1(x)v(x).
\]

This shows (3) has radial solution as (17).

Similar to the argument above, if another stronger condition $pq > (\gamma - 1)(\tau - 1)$ as well as
\[
\frac{\beta\gamma}{n-\alpha\tau} < q < \frac{n(\tau - 1)}{n-\alpha\tau}, \tag{22}
\]
\[
p\beta\gamma + \alpha\tau(\gamma - 1) < pq - (\gamma - 1)(\tau - 1) < \frac{n-\alpha\tau}{\tau - 1}, \tag{23}
\]
holds, (3) also has radial solutions as (19) with two different fast rates $2\theta_2 = \frac{n-\alpha\tau}{\tau-1}$ and
\[
2\theta_1 = \frac{2q\theta_2 - \beta\gamma}{\gamma - 1} = \frac{q(n-\alpha\tau)}{(\gamma - 1)(\tau - 1)} - \frac{\beta\gamma}{\gamma - 1}.
\]

Step 2. Nonexistence.
Substep 2.1. Suppose either $0 < pq \leq (\gamma - 1)(\tau - 1)$ or
\[
\frac{pq - (\gamma - 1)(\tau - 1)}{(\gamma - 1)(\tau - 1)} < \max \left\{ \frac{q\alpha\tau + \beta\gamma(\tau - 1)}{(n-\beta\gamma)(\tau - 1)}, \frac{p\beta\gamma + \alpha\tau(\gamma - 1)}{(n-\alpha\tau)(\gamma - 1)} \right\}.
\]

Assume $u, v$ are positive solutions of (3). Noting $\int_{B_R(0)} v^q(y)dy \geq c$, we obtain that for $|x| > R$,
\[
u(x) \geq \int_{|x|+R}^{\infty} \left[ \int_{B_{R+|x|}(0)} v^q(y)dy \right]^{\frac{1}{q}} \geq c \int_{|x|+R}^{\infty} \int_{B_{R+R}(0)} \frac{v^q}{t^{n-\beta\gamma}} \frac{dt}{t} \geq c \int_{2|x|}^{\infty} \frac{a_0^{\frac{\beta\gamma}{n-\alpha\tau}}}{t^{\frac{n-\beta\gamma}{n-\alpha\tau}}} dt.
\]
Here $a_0 = \frac{n-\beta\gamma}{\gamma-1}$. By this estimate, for $|x| > R$, there holds
\[
v(x) \geq c \int_{2|x|}^{\infty} \left[ \int_{B_{2|x|}(0)} \frac{dt}{|y|^{p\alpha\tau-n}} \right]^{\frac{1}{\alpha\tau-n}} dt \geq c \int_{2|x|}^{\infty} \frac{a_1^{\frac{\alpha\tau-p\theta_0}{n-\alpha\tau}}}{t^{\frac{\alpha\tau-p\theta_0}{n-\alpha\tau}}} dt.
\]
When $\alpha\tau - p\theta_0 \geq 0$, we see $v(x) = \infty$ for $|x| > R$. This implies the nonexistence of positive solutions of (3) since $R$ is an arbitrary positive number. When $\alpha\tau - p\theta_0 < 0$, then
\[
v(x) \geq \frac{c}{|x|^{b_0}}, \text{ for } |x| > R,
\]
where $b_0 = \frac{p\theta_0 - \alpha\tau}{n-1}$. Similarly, using this estimate, we also obtain that if $\beta\gamma - qb_0 \geq 0$, then $u(x) = \infty$; if $\beta\gamma - qb_0 < 0$, then
\[
u(x) \geq \frac{c}{|x|^{a_1}}, \text{ for } |x| > R,
\]
where $a_1 = \frac{qb_0 - \beta\gamma}{\gamma-1}$.
For \( k = 1, 2, \cdots \), write
\[
a_0 = \frac{n - \beta \gamma}{\gamma - 1}, \quad b_0 = \frac{p a_0 - \alpha \tau}{\tau - 1}, \quad a_k = \frac{q b_{k-1} - \beta \gamma}{\gamma - 1}, \quad b_k = \frac{p a_k - \alpha \tau}{\tau - 1}.
\]

By induction, we can obtain the following conclusions:

(i) If \( a_k < 0 \), then \( u(x) = \infty \). This leads to the nonexistence. If \( a_k \geq 0 \), then
\[
\frac{c}{|x|^\alpha} \implies u(x) \geq \frac{c}{|x|^\alpha}.
\]

(ii) If \( b_k < 0 \), then \( v(x) = \infty \). This also leads to the nonexistence. If \( b_k \geq 0 \), then
\[
v(x) \geq \frac{c}{|x|^\alpha} \implies u(x) \geq \frac{c}{|x|^\alpha}.
\]

In view of
\[
a_k = \frac{q}{\gamma - 1} b_{k-1} - \frac{\beta \gamma}{\gamma - 1} = \frac{pq}{(\gamma - 1)(\tau - 1)} a_{k-1} - \frac{q \alpha \tau + \beta \gamma (\tau - 1)}{(\gamma - 1)(\tau - 1)},
\]
we deduce that
\[
a_j = \frac{pq}{(\gamma - 1)(\tau - 1)} a_{j-1} - \frac{q \alpha \tau + \beta \gamma (\tau - 1)}{(\gamma - 1)(\tau - 1)} = \left(\frac{pq}{(\gamma - 1)(\tau - 1)}\right)^{j} a_0 - \frac{q \alpha \tau + \beta \gamma (\tau - 1)}{(\gamma - 1)(\tau - 1)} \sum_{k=0}^{j-1} \left(\frac{pq}{(\gamma - 1)(\tau - 1)}\right)^{k}.
\]

When \( \frac{pq}{(\gamma - 1)(\tau - 1)} = 1 \), then for some large \( j_0 \),
\[
a_{j_0} = a_0 = \frac{q \alpha \tau + \beta \gamma (\tau - 1)}{(\gamma - 1)(\tau - 1)} < 0.
\]
This implies \( u(x) = \infty \).

When \( 0 < \frac{pq}{(\gamma - 1)(\tau - 1)} < 1 \), letting \( j \to \infty \), we get
\[
a_j = \left(\frac{pq}{(\gamma - 1)(\tau - 1)}\right)^{j} a_0 - \frac{q \alpha \tau + \beta \gamma (\tau - 1)}{(\gamma - 1)(\tau - 1)} \cdot \frac{1 - (\frac{pq}{(\gamma - 1)(\tau - 1)})^j}{1 - \frac{pq}{(\gamma - 1)(\tau - 1)}} \to - \frac{q \alpha \tau + \beta \gamma (\tau - 1)}{(\gamma - 1)(\tau - 1) - pq} < 0.
\]
Therefore, we can find \( j_0 \) such that \( a_{j_0} < 0 \). This implies \( u(x) = \infty \).

When \( \frac{pq}{(\gamma - 1)(\tau - 1)} > 1 \) and \( \frac{pq}{\gamma - 1} < \frac{q \alpha \tau + \beta \gamma (\tau - 1)}{n - \beta \gamma} \), there holds
\[
a_0 < \frac{q \alpha \tau + \beta \gamma (\tau - 1)}{pq - (\gamma - 1)(\tau - 1)}.
\]

We deduce that
\[
a_{j_0} = \left(\frac{pq}{(\gamma - 1)(\tau - 1)}\right)^{j_0} a_0 - \frac{q \alpha \tau + \beta \gamma (\tau - 1)}{(\gamma - 1)(\tau - 1)} \left(\frac{pq}{(\gamma - 1)(\tau - 1)}\right)^{j_0} - \frac{pq}{(\gamma - 1)(\tau - 1)}
\]
\[
= \left(\frac{pq}{(\gamma - 1)(\tau - 1)}\right)^{j_0} [a_0 - \frac{q \alpha \tau + \beta \gamma (\tau - 1)}{pq - (\gamma - 1)(\tau - 1)}] + \frac{q \alpha \tau + \beta \gamma (\tau - 1)}{pq - (\gamma - 1)(\tau - 1)} < 0.
\]
Therefore, exchanging the order of variables yields
\[
\frac{pq - (\gamma - 1)(\tau - 1)}{\tau - 1} < \frac{p\beta\gamma + \alpha\tau(\gamma - 1)}{n - \alpha\tau},
\]
there holds \(\frac{n - \alpha\tau}{\tau - 1} < \frac{p\beta\gamma + \alpha\tau(\gamma - 1)}{pq - (\gamma - 1)(\tau - 1)}\). By the same argument above, we handle \(b_k\) instead of \(a_k\), we can also find some large \(k_0\) such that \(b_{k_0} < 0\). This implies \(v(x) = \infty\).

Substep 2.2. Suppose \(pq > (\gamma - 1)(\tau - 1)\) and
\[
\frac{pq - (\gamma - 1)(\tau - 1)}{(\gamma - 1)(\tau - 1)} = \max \left\{ \frac{q\alpha\tau + \beta\gamma(\tau - 1)}{(n - \beta\gamma)(\tau - 1)}, \frac{p\beta\gamma + \alpha\tau(\gamma - 1)}{(n - \alpha\tau)(\gamma - 1)} \right\}.
\]

First, write \(H := \int_{B_r(x)} v^q(y) dy\). By Hölder inequality,
\[
\int_0^R H dt \leq \left( \int_0^R H^{\frac{1}{p(n-\beta\gamma)} \frac{1}{n-\beta\gamma} - 1} dt \right)^{\frac{p}{n-\beta\gamma}} \left( \int_0^R t^{\frac{n-\beta\gamma-\gamma-1}{2-\gamma}} dt \right)^{\frac{2-\gamma}{\gamma - 1}} = CR^{\gamma - \beta\gamma + 1} \left( \int_0^R \frac{H}{\frac{1}{n-\beta\gamma}} \frac{1}{\gamma - 1} \frac{dt}{t} \right)^{\frac{1}{\gamma - 1}}.
\]

Therefore, exchanging the order of variables yields
\[
u(x) \geq cR^{-\frac{n-\beta\gamma}{\gamma - 1}} \left( \int_{B_{\frac{R}{4}}} v^q(y) dy \right)^{\frac{1}{\gamma - 1}}.
\]

Thus,
\[
u^p(x) \geq cR^{-\frac{p(n-\beta\gamma)}{\gamma - 1}} \left( \int_{B_{\frac{R}{4}}} v^q(y) dy \right)^{\frac{p}{\gamma - 1}}. \tag{24}
\]

Similarly,
\[
v^q(x) \geq cR^{-\frac{q(n-\alpha\tau)}{\gamma - 1}} \left( \int_{B_{\frac{R}{4}}} u^p(y) dy \right)^{\frac{q}{\gamma - 1}}. \tag{25}
\]

Without loss of generality, we suppose
\[
\frac{pq - (\gamma - 1)(\tau - 1)}{\gamma - 1} = \frac{q\alpha\tau + \beta\gamma(\tau - 1)}{n - \beta\gamma}.
\]

Inserting (24) into (25) yields
\[
v^q(x) \geq cR^{-\frac{q(n-\alpha\tau)}{\gamma - 1} - \frac{pq(n-\beta\gamma)}{(\gamma - 1)(\tau - 1)} + \frac{nq}{\gamma - 1} + \frac{pq(n-\beta\gamma)}{\gamma - 1} \left( \int_{B_{\frac{R}{4}}} v^q(y) dy \right)^{\frac{pq}{\gamma - 1} \frac{1}{(\gamma - 1)(\tau - 1)}}}. \tag{27}
\]

Integrating on \(B_{\frac{R}{4}}\), we get
\[
\int_{B_{\frac{R}{4}}} v^q(x) dx \geq cR^{-\frac{q(n-\alpha\tau)}{\gamma - 1} - \frac{pq(n-\beta\gamma)}{(\gamma - 1)(\tau - 1)} + \frac{nq}{\gamma - 1} + \frac{pq(n-\beta\gamma)}{\gamma - 1} \left( \int_{B_{\frac{R}{4}}} v^q(y) dy \right)^{\frac{pq}{\gamma - 1} \frac{1}{(\gamma - 1)(\tau - 1)}}}. \tag{28}
\]

We claim that the exponent of \(R\) is zero. In fact, (26) implies \(\frac{pq(n-\beta\gamma)}{\gamma - 1} = q\alpha\tau + n(\tau - 1)\).
Thus, we obtain
\[-\frac{q(n - \alpha\tau)}{\tau - 1} - \frac{pq(n - \beta\gamma)}{(\gamma - 1)(\tau - 1)} + \frac{nq}{\tau - 1} + n = 0.\]
The claim is proved.

Letting $R \to \infty$ in (28), we see that $v \in L^q(\mathbb{R}^n)$ in view of $pq > (\gamma - 1)(\tau - 1)$. Integrating (27) on $A_R := B_r \setminus B_{\frac{r}{2}}$ and letting $R \to \infty$, we also have $\int_{\mathbb{R}^n} v^q dy = 0$. It is impossible. Thus, we complete our proof. □

3. Radial symmetry and monotonicity of solutions. In this section, we use the method of moving planes in integral forms which was established by Chen et al. [6] to prove that the positive solutions $u$ and $v$ are radially symmetric and decreasing about some point in $\mathbb{R}^n$.

**Claim 1.** There holds $s > \max\{1, \frac{n(\gamma - 1)}{n - \beta\gamma}\}$ and $r > \max\{1, \frac{n(\tau - 1)}{n - \alpha\tau}\}$.

In fact, by the relationship of $p, q$ in (14), it is easy to see
\[s = \frac{n(pq - (\gamma - 1)(\tau - 1))}{q\alpha\tau + \beta\gamma(\tau - 1)} = \frac{2n^2(\alpha\tau + \beta\gamma)(p + \gamma - 1)(q + \tau - 1)}{(q\alpha\tau + \beta\gamma(\tau - 1))(2n - \alpha\tau + \beta\gamma)(2n - \beta\gamma + \alpha\tau)} > 1,\]
and
\[r = \frac{n(pq - (\gamma - 1)(\tau - 1))}{p\beta\gamma + \alpha\tau(\gamma - 1)} = \frac{2n^2(\alpha\tau + \beta\gamma)(p + \gamma - 1)(q + \tau - 1)}{(p\beta\gamma + \alpha\tau(\gamma - 1))(2n - \alpha\tau + \beta\gamma)(2n - \beta\gamma + \alpha\tau)} > 1,\]
since $p > 1 \geq \gamma - 1, q > 1 \geq \tau - 1$ and $0 < \alpha\tau, \beta\gamma < n$.

On the other hand, by (10), it is easy to discover that
\[s = \frac{n(pq - (\gamma - 1)(\tau - 1))}{q\alpha\tau + \beta\gamma(\tau - 1)} > \frac{n(\gamma - 1)}{n - \beta\gamma},\]
\[r = \frac{n(pq - (\gamma - 1)(\tau - 1))}{p\beta\gamma + \alpha\tau(\gamma - 1)} > \frac{n(\tau - 1)}{n - \alpha\tau}.\]

**Claim 2.** There holds $\frac{\gamma - 1}{s} = \frac{n - \beta\gamma}{n - \alpha\tau}$ and $\frac{\tau - 1}{r} = \frac{\alpha\tau}{n} + \frac{\beta\gamma}{n}$.

In fact,
\[\frac{\gamma - 1}{s} + \frac{\beta\gamma}{n} = \frac{(\gamma - 1)[q\alpha\tau + \beta\gamma(\tau - 1)]}{n(pq - (\gamma - 1)(\tau - 1))} + \frac{\beta\gamma}{n} = \frac{n - \alpha\tau}{r}.\]

Similarly, we can get the second equality by a simple calculation.

**Proof of Theorem 1.2.** Let
\[\Sigma_\lambda = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n | x_1 < \lambda\},\]
and
\[T_\lambda = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n | x_1 = \lambda\}\]
be the boundary of $\Sigma_\lambda$, the hyper plane we will move with.

Let $x^\lambda = (2\lambda - x_1, x_2, \ldots, x_n)$ be the reflection point of $x$ about the plane $T_\lambda$ and write $u_\lambda(x) = u(x^\lambda)$.

Denotes $D_t(x)$ the intersection of the ball $B_t(x)$ with its mirror image $B_t(x_\lambda)$, and
\[\Omega_t(x) = B_t(x) \setminus D_t(x).\]

**Step 1.** We first show that for $\lambda$ sufficiently negative, we have
\[u_\lambda(x) \geq u(x)\quad\text{and}\quad v_\lambda(x) \geq v(x),\quad \forall x \in \Sigma_\lambda.\]

(29)
To this end, we introduced the sets
\[ \Sigma^u_\lambda = \{ x \in \Sigma_\lambda | u_\lambda (x) < u(x) \}, \quad \Sigma^v_\lambda = \{ x \in \Sigma_\lambda | v_\lambda (x) < v(x) \}. \]

We will show that \( \Sigma^u_\lambda \) and \( \Sigma^v_\lambda \) must have measure zero for \( \lambda \) sufficiently negative.

We have for \( x \in \Sigma^u_\lambda \),
\[ 0 < u(x) - u_\lambda (x) = \int_0^\infty \left\{ \int_{B_t(x)} v^q(y)dy \right\}^{1/t} \frac{1}{t} \frac{dt}{t} \]
\[ = \int_0^\infty \left\{ \int_{\Omega_t(x)} v^q(y)dy + A_t(x) \right\}^{1/t} \frac{1}{t} \frac{dt}{t} \]
\[ = \int_0^\infty \left\{ \int_{\Omega_t(x)} v^q(y)dy \frac{1}{t^{n-\beta \gamma}} - \int_{\Omega_t(x)} v^q_\lambda(y)dy \frac{1}{t^{n-\beta \gamma}} + A_t(x) \right\}^{1/t} \frac{dt}{t} \]

where by symmetry
\[ A_t(x) = \int_{D_t(x)} v^q(y)dy = \int_{D_t(x)} v^q_\lambda(y)dy. \]

Applying the Mean Value Theorem to the above, we derive
\[ 0 < u(x) - u_\lambda (x) = \frac{1}{\gamma - 1} \int_0^\infty \left\{ \int_{B_t(x)} v^q(y)dy \frac{1}{t^{n-\beta \gamma}} - \int_{B_t(x)} v^q_\lambda(y)dy \frac{1}{t^{n-\beta \gamma}} \right\}^{1/t} \frac{dt}{t}, \]
\[ \text{(30)} \]

where \( \xi_t(x) \) is valued between
\[ \int_{B_t(x)} v^q(y)dy \text{ and } \int_{B_t(x)} v^q_\lambda(y)dy \]
and is obviously positive and less than the sum of the two.

By the definition of \( \Omega_t(x) \) and \( \Sigma^u_\lambda \) and again by the Mean Value Theorem, we have
\[ \int_{\Omega_t(x)} (v^q(y) - v^q_\lambda(y))dy \leq \int_{B_t(x) \cap \Sigma^u_\lambda} (v^q(y) - v^q_\lambda(y))dy \]
\[ \leq q \int_{B_t(x)} v^{q-1}(y)(v^q(y) - v^q_\lambda(y))^+dy, \]
\[ \text{(31)} \]

where as usual \( f(x)^+ = \max\{0, f(x)\} \).

Applying Hölder inequality to (30) and taking account of (31), we deduce
\[ 0 < u(x) - u_\lambda (x) \]
\[ \leq C \left( \int_0^\infty \left[ \int_{B_t(x)} v^{q-1}(y)(v(y) - v_\lambda(y))^+ \right] \frac{1}{t^{n-\beta \gamma}} \frac{dt}{t} \right)^{\gamma - 1} \]
\[ \times \left( \int_0^\infty \left[ \frac{\int_{B_t(x)} v^q_\lambda(y) + v^q(y)dy}{t^{n-\beta \gamma}} \right] \frac{1}{t^{1-\beta \gamma}} \frac{dt}{t} \right)^{2-\gamma} \]
\[ \leq C [W_{\beta,\gamma}(v^{q-1}(v - v_\lambda)(x))]^{\gamma - 1} (u(x) + u_\lambda(x))^{2-\gamma} \]
\[ \leq C [W_{\beta,\gamma} v^{q-1}(v - v_\lambda)(x)]^{\gamma - 1} u(x)^{2-\gamma}. \]
\[ \text{(32)} \]
Then it follows from (36) that

$$ (A + B)^s \leq C(A^s + B^s) $$

and the equation

$$ u(x) = W_{\beta, \gamma}(v^q)(x). $$

Since \( u \in L^s(R^n) \), \( v \in L^r(R^n) \) by the Hardy-Littlewood-Sobolev inequality, we derive from (32) that

$$ \|u - u_\lambda\|_{L^r(\Sigma_\lambda^1)} \leq C\|W_{\beta, \gamma}(v^{q-1}(v - v_\lambda))^+\|_{L^r(\Sigma_\lambda^1)} \|u\|_{L^r(\Sigma_\lambda^1)}^{-\gamma} $$

$$ \leq C\|W_{\beta, \gamma}(v^{q-1}(v - v_\lambda))^+\|_{L^r(\Sigma_\lambda^1)} \|u\|_{L^r(\Sigma_\lambda^1)}^{-\gamma} $$

$$ \leq C\|I_{\beta, \gamma}(v^{q-1}(v - v_\lambda))^+\|_{L^r(\Sigma_\lambda^1)} \|u\|_{L^r(\Sigma_\lambda^1)}^{-\gamma} $$

$$ \leq C\int_{R^n} \frac{v^{q-1}(v - v_\lambda)^+}{|x - y|^{n+\beta\gamma}} \, dy \|_{L^r(\Sigma_\lambda^1)} \|u\|_{L^r(\Sigma_\lambda^1)}^{-\gamma}. \quad (33) $$

The last two inequalities above was deduced by the integral estimate between the Wolff potential and Riesz potential, for \( \frac{n}{\gamma} \geq 1 \).

Then by Claim 1 and Claim 2, we can use the Hardy-Littlewood-Sobolev inequality and the Hölder inequality on (33) to obtain

$$ \|u - u_\lambda\|_{L^r(\Sigma_\lambda^1)} \leq C\|v^{q-1}(v - v_\lambda)^+\|_{L^r(\Sigma_\lambda^1)} \|v\|_{L^r(\Sigma_\lambda^1)}^{-\gamma} \|u\|_{L^r(\Sigma_\lambda^1)} \|u\|_{L^r(\Sigma_\lambda^1)}^{-\gamma} $$

$$ \leq C\|v\|_{L^r(\Sigma_\lambda^1)} \|v - v_\lambda\|_{L^r(\Sigma_\lambda^1)} \|v\|_{L^r(\Sigma_\lambda^1)}^{-\gamma}. \quad (34) $$

Similarly, we have

$$ \|v - v_\lambda\|_{L^r(\Sigma_\lambda^1)} \leq C\|u\|_{L^r(\Sigma_\lambda^1)} \|v - u_\lambda\|_{L^r(\Sigma_\lambda^1)} \|v\|_{L^r(\Sigma_\lambda^1)}^{-\gamma}. \quad (35) $$

Combining (34) and (35), we arrive at

$$ \|u - u_\lambda\|_{L^r(\Sigma_\lambda^1)} \leq C\|v\|_{L^r(\Sigma_\lambda^1)}^{\gamma-\gamma+1} \|u\|_{L^r(\Sigma_\lambda^1)}^{\gamma-\gamma+1} \|u - u_\lambda\|_{L^r(\Sigma_\lambda^1)}. \quad (36) $$

Since \( u \in L^s(R^n) \) and \( v \in L^r(R^n) \), for sufficiently negative \( \lambda \), we have

$$ C\|v\|_{L^r(\Sigma_\lambda^1)}^{\gamma-\gamma+1} \|u\|_{L^r(\Sigma_\lambda^1)}^{\gamma-\gamma+1} \leq \frac{1}{2}. $$

Then it follows from (36) that

$$ \|u - u_\lambda\|_{L^s(\Sigma_\lambda^1)} = 0, $$

and hence \( \Sigma_\lambda^1 \) must be zero. Then from (35), \( \Sigma_\lambda^1 \) must also be measure zero. This verifies (29).

**Step 2.** Inequality (29) provides a starting point to move the plane \( T_\lambda \). Now we start from this neighborhood of \( x_1 = -\infty \) and move the plane to the right as long as symmetric about the limiting plane. More precisely, define

$$ \lambda_0 = \sup\{\mu \mid (29) \text{ holds for any } \lambda \leq \mu\}. $$

One can see that \( \lambda_0 < +\infty \) by using the similar argument in Step 1 and starting the plane \( T_\lambda \) near \( x_1 = +\infty \).

We will show that \((u, v)\) is symmetric about the plane \( T_{\lambda_0} \):

$$ u(x) \equiv u_{\lambda_0}(x) \text{ and } v(x) \equiv v_{\lambda_0}(x), \text{ a.e. } x \in \Sigma_{\lambda_0}. \quad (37) $$

Suppose for such a \( \lambda_0 \), we have, on \( \Sigma_{\lambda_0} \),

$$ v(x) \leq v_{\lambda_0}(x) \text{ but } v(x) \not\equiv v_{\lambda_0}(x), \text{ a.e. } x \in \Sigma_{\lambda_0}. \quad (38) $$
we show that the plane can be moved further to the right. More precisely, there exists an $\epsilon > 0$ such that
\[ u(x) \leq u_{\lambda}(x) \text{ and } v(x) \leq v_{\lambda}(x) \text{ a.e. } \forall x \in \Sigma_\lambda, \forall \lambda \in [\lambda_0, \lambda_0 + \epsilon). \tag{39} \]
This would contradict with the definition of $\lambda_0$.

By (30) and (39), we have in fact $u(x) \leq u_{\lambda_0}(x)$ in the interior of $\Sigma_{\lambda_0}$. Let
\[ \Sigma^u_{\lambda_0} = \{ x \in \Sigma_{\lambda_0} | u(x) \geq u_{\lambda_0}(x) \}, \quad \Sigma^v_{\lambda_0} = \{ x \in \Sigma_{\lambda_0} | v(x) \geq v_{\lambda_0}(x) \}. \]
Then obviously, $\Sigma^u_{\lambda_0}$ has measure zero, and $\lim_{\lambda \to \lambda_0} \Sigma^u_{\lambda} \subset \Sigma^u_{\lambda_0}$. The same is true for that of $v$.

Again the integrability conditions $u \in L^1(R^n)$ and $v \in L^1(R^n)$ ensure that one can choose $\epsilon$ sufficiently small, so that for all $\lambda$ in $[\lambda_0, \lambda_0 + \epsilon)$,
\[ C \|v\|_{L^r(\Sigma^v_{\lambda})} \|u\|_{L^r(\Sigma^v_{\lambda})} \leq \frac{1}{2}. \]

Now by (36), we have
\[ \|u - u_{\lambda}\|_{L^r(\Sigma^u_{\lambda})} = 0, \]
and hence $\Sigma^u_{\lambda}$ must be zero. Similarly, $\Sigma^v_{\lambda}$ must also be measure zero. This verifies (39) and hence (37).

Since $x_1$ direction can be chosen arbitrarily, we deduce that $(u, v)$ must be radially symmetric and monotone decreasing about some point in $R^n$. This completes the proof of Theorem 1.2. \hfill \qed

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