 ANALYTIC THEORY OF MULTICAVITY KLYSTRONS

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ABSTRACT. Multicavity Klystron (MCK) is a high power microwave (HPM) vacuum electronic device used to amplify radio-frequency (RF) signals. MCKs have numerous applications, including radar, radio navigation, space communication, television, radio repeaters, and charged particle accelerators. The microwave-generating interactions in klystrons take place mostly in coupled resonant cavities positioned periodically along the electron beam axis. Importantly, there is no electromagnetic coupling between cavities. The cavities are coupled only by the flow of bunched electrons drifting from one cavity to the next. We advance here a Lagrangian field theory theory of MCKs with the space being represented by one-dimensional continuum. The theory integrates into it the space-charge effects including the so-called debunching (electron-to-electron repulsion). The corresponding Euler-Lagrange equations are ODEs with coefficients varying periodically in the space. Utilizing the system periodicity we develop the instrumental features of the Floquet theory including the monodromy matrix and its Floquet multipliers. We use them to derive closed form expressions for a number of physically significant quantities. Those include in particular the dispersion relations and the frequency dependent gain foundational to the RF signal amplification. We assume that MCKs operate in voltage amplification mode associated with the maximal gain.

1. Introduction

A klystron is a specialized linear-beam vacuum tube, invented in 1935 by American electrical engineers Russell and Sigurd Varian. Klystron is used as an amplifier for high radio frequencies, from UHF up into the microwave range. It was the first genuine microwave electronic device to take full advantage of the principle of bunching and phasing, [Tsim, 7.1]. The original description by brothers Varians of the klystron concept is as follows, [VarVar]:

“A dc stream of cathode rays of constant current and speed is sent through a pair of grids between which is an oscillating electric field, parallel to the stream and of such strength as to change the speeds of the cathode rays by appreciable but not too large fractions of their initial speed. After passing these grids the electrons with increased speeds begin to overtake those with decreased speeds ahead of them. This motion groups the electrons into bunches separated by relatively empty spaces. At any points beyond the grids, therefore, the cathode ray current can be resolved into the original dc plus a nonsinusoidal ac. A considerable fraction of its power can then be converted into power of high frequency oscillations by running the stream through a second pair of grids between which is an ac electric field such as to take energy away from the electrons in bunches. These two ac fields are best obtained by making the grids form parts of the surfaces of resonators of type described in This journal by Hansen.”

Usage of cavity resonators in the klystron was a revolutionary idea of Hansen and the Varians, [Tsim, 7.1]. In the pursuit of higher power and efficiency the original design of Vairan klystrons evolve significantly over years featuring today multiple cavities and multiple electron beam, [Tsim]
The advantages of klystrons are their high power and efficiency, potentially wide bandwidth, phase and amplitude stability, [BenSweSch1a 9.1].

The distinct features of the klystron operation are as follows, [BenSweSch1a 9.1]:

"Klystrons have two distinguishing features. First, the microwave-generating interactions in these devices take place in resonant cavities at discrete locations along the beam. Second, the drift tube connecting the cavities is designed so that electromagnetic wave propagation at the operating frequency is cut off between the cavities; without electromagnetic coupling between cavities, they are coupled only by the bunched beam, which drifts from one cavity to the next. This latter feature of these devices, the lack of feedback between cavities, makes them perhaps the best-suited of HPM devices to operate as amplifiers."

Importantly, in klystrons the electron bunching is provided by cavity resonators (often of toroidal shape) acting as LC-circuit resonators. These cavities often utilize the lowest-frequency fundamental modes. For these modes the electric field energy is localized near the cavity gaps exposed to the e-beam whereas the magnetic field energy is stored in cavity toroidal tubes, [Tsim 7.1]. Cavities (resonator cavities) would interact with e-beam effectively if they satisfy the following conditions, [Shev 2.3]:

"In order to be used in an electron tube, a cavity resonator must have a region with a relatively strong high-frequency field which is polarized along the direction of electron flow. This region should, in the majority of cases, be so small that the electron transit time is less than the period of change of the field. Hollow toroidal resonators satisfy these conditions. Toroidal resonators consist of cylinders with a very prominent "bulge" in the middle."

For more information on klystrons and their operation we refer the reader to [BenSweSch1a 9], [ChoWes], [Grigo 7.2], [Nusi 3], [GewWat 10], [Gilm1 10, 11], [MAEAD], [Paol 4.3], [Tsim 7], [ValMid 16].

![Figure 1](attachment:figure1.png)

**Figure 1.** A schematic presentation of a multicavity klystron (also known as cascade amplifier) that exploits constructive interaction between the pencil-like electron beam and an array of electromagnetic cavities (often of toroidal shape). The interaction causes the electron bunching and consequent amplification of the RF signal.

The conceptual design of a multicavity klystron (MCK) also known as cascade amplifier, [Werne IIb], is shown in Fig. [I] and more detailed description of its operation is as follows, [BenSweSch1a 9.3], [Gilm1 8], [Tsim 7.7]. The e-beam enters the gap region of the first cavity (the buncher) where the electron velocities are modulated by the electric field in the gap driven by an RF signal.
The e-beam-cavity interaction through the cavity gap has the following features: (i) the input RF voltage in the cavity gap generates the electric field and that in turn initiates electron bunching (velocity modulation) and RF current in the e-beam; (ii) the RF current in the e-beam induces a current in the walls of the cavity and the induced current acts back on the e-beam enhancing the e-beam modulation. Exiting the gap region of the first cavity the velocity-modulated e-beam passes through the drift region and enters the gap region of the second cavity. When drifting between cavities the faster electrons “overtake” the slower electron resulting in charge wave bunches on the e-beam. Importantly, the drift tube separating the cavities is designed so that there is no electromagnetic communication between the cavities except for the bunched e-beam. Under properly designed conditions the e-beam charge wave interacts constructively in the gap of the second cavity achieving an amplified electron bunching upon its exit. This process of the e-beam charge wave amplification continues on as the e-beam electrons pass through the drift region and the consequent cavities. At the end of the process the e-beam enters the gap region of the last cavity (extraction cavity, catcher) where the power output signal is extracted. Actual multicavity klystron is a very complicated device with many independent parameters and with three important modes to be considered in choosing these parameters: the voltage amplifier, power amplifier, and bandwidth amplifier modes, \[T{\text{sim}}, 7.7\]. MCKs can be broadband exceeding 10% with reasonably flat power output across the band, \[Kreu\], \[Gilm1, 11.3\], and their efficiency can exceed 70%, \[Gilm1, 11.1\].

The subject of our work here is the construction of an analytic theory of multicavity klystrons operating in the voltage amplification mode associated with the maximal gain. This theory features in particular exact formulas for the MCK instability frequencies, its dispersion relations and optimal values of the MCK parameters providing for maximal gain.

The paper organized as follows. In Section 2 we concisely review our prior work on the analytic theory of traveling wave tubes (TWT) for its significant elements are utilized for the construction of the analytic theory of MCK's. In Section 3 we introduce the Lagrangian of the MCK system featuring a periodic array of cavity resonators and derive the corresponding Euler-Lagrange evolution equations. This Lagrangian has a term that integrates into it the space-charge effects including the so-called debunching (electron-to-electron repulsion). We show also in this section that the Euler-Lagrange equations have the Hamiltonian structure and develop all elements of the Floquet theory including the MCK monodromy matrix and its Floquet multipliers. In Section 4 we derive formulas for the Floquet multipliers that provide a basis for the evaluation of the MCK dispersion relations. In Section 5 we introduce and study the MCK instability parameter that determines the region of instability frequencies. In Section 6 we derive formulas for the gain as a function of frequency and its maximal value. In Section 7 we evaluate typical values of the MCK gain and its significant parameters. In Section 8 we derive explicit formulas for the MCK dispersion relations. In Section 9 we find the exceptional points of degeneracy of the MCK dispersion relations and study their properties. In Section 10 we construct the Lagrangian variational framework for the MCK system. In Appendices we provide information on a number of mathematical subjects relevant to the construction of the analytic theory of MCK's.

While quoting monographs we identify the relevant sections as follows. Reference \[X,Y\] refers to Section/Chapter “Y” of monograph (article) “X”, whereas \[X, p. Y\] refers to page “Y” of monograph (article) “X”. For instance, reference \[2, VI.3\] refers to monograph \[2\], Section VI.3; reference \[2, p. 131\] refers to page 131 of monograph \[2\].

2. Concise review of an analytic model of the traveling wave tube

When constructing an analytic model of the multicavity klystron we use some elements of an analytic model of the traveling wave tube (TWT) introduced and studied in our monograph \[FigTWTbk, 4, 24\]. We concisely review here this model of TWT. According to the simplest
version of the model an ideal TWT is represented by a single-stream e-beam interacting with single transmission line just as in the Pierce model [Pier51]. The main parameter describing the single-stream e-beam is e-beam intensity

\[ \beta = \frac{\sigma_B}{4\pi} R_{sc}^2 \omega_p^2 = \frac{e^2}{m} R_{sc}^2 \sigma_B \hat{n}, \quad \omega_p^2 = \frac{4\pi e^2}{m}, \tag{2.1} \]

where \( -e \) is electron charge with \( e > 0 \), \( m \) is the electron mass, \( \omega_p \) is the e-beam plasma frequency, \( \sigma_B \) is the area of the cross-section of the e-beam, \( s \hat{v} > 0 \) is stationary velocity of electrons in the e-beam and \( \hat{n} \) is the density of the number of electrons. The constant \( R_{sc} \) is the plasma frequency reduction factor that accounts phenomenologically for finite dimensions of the e-beam cylinder as well as geometric features of the slow-wave structure, [BraMih], [Gilm1, 9.2], [Nusi, 3.3.3]. The frequency

\[ \omega_{rp} = R_{sc} \omega_p \tag{2.2} \]

is known as reduced plasma frequency, [Gilm1, 9.2].

Assuming the Gaussian system of units the physical dimensions a complete set of the e-beam parameters as in Tables 1 and 2.

| Frequency | Plasma frequency | \( \omega_p = \sqrt{\frac{4\pi e^2}{m}} \) |
|-----------|-----------------|--------------------------------------|
| Velocity  | e-beam velocity | \( \hat{v} \) |
| Wavenumber| \( k_q = \frac{\omega_p}{v} = \frac{R_{sc} \omega_p}{\hat{v}} \) |
| Length    | Wavelength for \( k_q \) | \( \lambda_{rp} = \frac{2\pi \hat{v}}{\omega_p}, \omega_{rp} = R_{sc} \omega_p \) |
| Time      | Wave time period | \( \tau = \frac{2\pi}{\omega_p} \) |

**Table 1.** Natural units relevant to the e-beam.

We would like to point to an important spatial scale related to the e-beam, namely

\[ \lambda_{rp} = \frac{2\pi \hat{v}}{R_{sc} \omega_p}, \quad \omega_{rp} = R_{sc} \omega_p, \tag{2.3} \]

| \( i \) | current | [charge] \[time\] |
|--------|---------|-----------------|
| \( q \) | charge  | [charge] |
| \( \hat{n} \) | number of electrons p/u of volume | \[1\] \[length\] |
| \( \lambda_{rp} = \frac{2\pi \hat{v}}{\omega_{rp}}, \omega_{rp} = R_{sc} \omega_p \) | the electron plasma wavelength | [length] |
| \( g_B = \frac{\sigma_B}{4\lambda_{rp}} \) | the e-beam spatial scale | [length] |
| \( \beta = \frac{\sigma_B}{4\pi} R_{sc}^2 \omega_p^2 = \frac{e^2}{m} R_{sc}^2 \sigma_B \hat{n} \) | e-beam intensity | \[length\]^2 \[time\]^2 |
| \( \beta' = \frac{\beta}{\hat{v}^2} = \frac{\pi \sigma_B}{\lambda_{rp}} \frac{4\pi g_B}{\lambda_{rp}} \) | dimensionless e-beam intensity | [dim-less] |

**Table 2.** Physical dimensions of the e-beam parameters. Abbreviations: dimensionless – dim-less, p/u – per unit.
which is the distance passed by an electron for the time period \( \frac{2\pi}{\omega_{rp}} \) associated with the plasma oscillations at the reduced plasma frequency \( \omega_{rp} \). This scale is well known in the theory of klystrons and is referred to as the electron plasma wavelength, [Gilm1 9.2]. Another spatial scale related to the e-beam that arises in our analysis later on is

\[
g_B = \frac{\sigma_B}{4 \lambda_{rp}^2},
\]

and we will refer to it as e-beam spatial scale. Using these spatial scales we obtain the following representation for the dimensionless form \( \beta' \) of the e-beam intensity

\[
\beta' = \frac{\beta}{\bar{v}^2} = \frac{\pi \sigma_B}{\lambda_{tp}^2} = \frac{4\pi g_B}{\lambda_{tp}}.
\]

(2.5)

As for the single transmission line, its shunt capacitance per unit of length is a real number \( C > 0 \) and its inductance per unit of length is another real number \( L > 0 \). The coupling constant \( 0 < b \leq 1 \) is a number also, see [FigTWTbk, 3] for more details. The TL single characteristic velocity \( w \) and the single TL principal coefficient \( \theta \) defined by

\[
w = \frac{1}{\sqrt{CL}}, \quad \theta = \frac{b^2}{C}.
\]

(2.6)

Following to [FigTWTbk, 3] we assume that

\[
0 < \bar{v} < w.
\]

(2.7)

2.1. TWT system Lagrangian and evolution equations. Following to developments in [FigTWTbk] we introduce the TWT principal parameter \( \gamma = \theta \beta \). This parameter in view of equations (2.1) and (2.6) can be represented as follows

\[
\gamma = \theta \beta = \frac{b^2 \sigma_B}{C} R^2 \omega_p^2 = \frac{b^2 e^2}{C m} R^2 \sigma_B \hbar, \quad \theta = \frac{b^2}{C}, \quad \beta = \frac{e^2}{m} R^2 \sigma_B \hbar.
\]

(2.8)

The TWT-system Lagrangian \( \mathcal{L}_{TB} \) in the simplest case of a single transmission line and one stream e-beam is of the form, [FigTWTbk 4, 24]:

\[
\mathcal{L} (\{ Q \}, \{ q \}) = \mathcal{L}_{TB} (\{ Q \}, \{ q \}) + \mathcal{L}_B (\{ q \}),
\]

(2.9)

\[
\mathcal{L}_{TB} = \frac{L}{2} (\partial_t Q)^2 - \frac{1}{2C} (\partial_z Q + b \partial_t q)^2, \quad \mathcal{L}_B = \frac{1}{2\beta} (\partial_t q + \tilde{v} \partial_z q)^2 - \frac{2\pi}{\sigma_B} q^2,
\]

where

\[
\{ Q \} = Q, \partial_z Q, \partial_t Q, \quad Q = Q (z,t); \quad \{ q \} = q, \partial_z q, \partial_t q, \quad q = q (z,t),
\]

(2.10)

and \( q (z,t) \) and \( Q (z,t) \) are charges associated with the e-beam and the TL defined as time integrals of the corresponding e-beam currents \( i(z,t) \) and TL current \( I(z,t) \), that is

\[
q(z,t) = \int^t i(z,t') \, dt', \quad Q(z,t) = \int^t I(z,t') \, dt'.
\]

(2.11)

Note that term \( -\frac{2\pi}{\sigma_B} q^2 \) in the Lagrangian \( \mathcal{L}_B \) defined in equations (2.9) represents space-charge effects including the so-called debunching (electron-to-electron repulsion). The corresponding Euler-Lagrange equations is the following system of second-order differential equations

\[
\mathcal{L}_{\beta}^2 Q - \partial_z \left[ C^{-1} (\partial_z Q + b \partial_t q) \right] = 0,
\]

(2.12)

\[
\frac{1}{\beta} (\partial_t + \tilde{v} \partial_z)^2 q + \frac{4\pi}{\sigma_B} q - b \partial_z \left[ C^{-1} (\partial_z Q + b \partial_t q) \right] = 0,
\]

(2.13)

where \( \tilde{v} \) is the stationary velocity of electrons in the e-beam, \( \sigma_B \) is the area of the cross-section of the e-beam and \( \beta \) is the e-beam intensity defined by equations (2.8).
3. An analytic model of multicavity klystron

In the pursuit of powerful pulse microwave radiation the synchronization of multiple high-frequency sources emerged as a possible solution to the problem, [Tsim, 7.7]. High-power multicavity klystron (MCK) is a powerful amplifier that employs this kind of synchronization and it is the primary subject of our studies here. In particular, we advance the Lagrangian variational framework that includes: (i) the MCK system of evolution equations; (ii) closed form expressions for the MCK dispersion relations derived based on the Floquet theory; (iii) exact description of the frequency region of the MCK instability; (iv) exact formulas for the MCK gain as well as for the optimal values of the MCK parameters that yield the maximal gain. The proposed MCK model utilizes some of the elements of our analytic model of the traveling wave tube reviewed in Section 2. As to the features of electron bunching special to klystrons they are as follows, [Gilm1, 9.2]:

“A very important characteristic of the bunching process with space charge forces is that all electrons are either speeded up or slowed down to the same velocity (the dc beam velocity) at the same axial position \( \frac{\lambda}{4} \). In addition, even if the amplitude of the modulating field is changed so that initial electron velocities are changed, the axial position of the bunch remains the same. This result is extremely important to the klystron engineer because, unlike the situation when space charge forces are ignored, the cavity location for maximum RF beam current is not a function of signal level, of gap width, or of frequency of operation”.

As we already pointed out an actual multicavity klystron is a complicated device that can be designed to operate in one of three modes: the voltage amplifier, power amplifier, and bandwidth amplifier, [Tsim, 7.7]. We are interested here in the voltage amplifier mode for it yields the maximal gain, [Tsim, 7.7.1]. When in this mode the resonance frequencies of all cavities are identical and equal to the input operating frequency, a design known as the synchronous tuning regime with high amplification for a sufficiently small beam current.

When integrating into the mathematical model the identified significant features of MCKs we make a number of simplifying assumptions. In particular, we use the following basic assumptions of one-dimensional model of space-charge waves in velocity-modulated beams: (i) all quantities of interest depend only on a single space variable \( z \); (ii) the electric field has only an \( z \)-component; (iii) there are no transverse velocities of electrons; (iv) ac values are small compared with dc values; (v) electrons have a constant dc velocity which is much smaller than the speed of light; (vi) electron beams are nondense, [Tsim, 7.6.1].

Assumption 1. (ideal model of the e-beam and cavities interaction).

(i) \( E \)-beam is a flow of electrons confined effectively to \( z \)-axis (see Fig. 1) in consistency with the MCK operation when all significant energy transport is confined to \( z \)-axis.

(ii) The \( E \)-beam interacts with a periodic array of cavity resonators of toroidal shape through their electric field along \( z \)-axis in small cavity gaps. The cavity gap centers form a set of equidistant points on \( z \) axis which is a lattice:

\[ aZ : Z = \{ \ldots, -2, -1, 0, 1, 2, \ldots \}, \quad a > 0, \quad (3.1) \]

where \( a \) is the MCK period. The cavity resonators do not interact with each other directly but they interact only with the \( E \)-beam at the lattice points as in (3.1). This interaction feature is accomplished by designing the electron drift tube (drift space) so that its low cutoff frequency is above the klystron operating frequencies.

(iii) Each cavity interacts with the \( E \)-beam at the corresponding lattice points \( a\ell, \ell \in Z \) only by utilizing the single resonating cavity mode at frequency \( \omega_0 = \frac{1}{\sqrt{\ell_0 c_0}} \) where \( c_0 \) and \( l_0 \) are
respectively the capacitance and the inductance of each cavity resonator. This assumption enforces the voltage amplifier mode yielding the maximal gain.

An MCK state is described by charges \( q = q(z, t), \ z \in \mathbb{R} \) and \( Q = Q(z, t), \ z \in a\mathbb{Z} \) associated with respectively the e-beam charge-wave and the cavity resonators defined as the time integrals of the relevant currents

\[
Q = Q(z, t) = \int_{t}^{t'} I(z, t') \, dt', \quad q = q(z, t) = \int_{t}^{t'} i(z, t') \, dt'.
\]  

(3.2)

Since according to Assumptions 1 the interaction occurs only at the discrete set \( a\mathbb{Z} \) (lattice) of points embedded into one-dimensional continuum of real numbers \( \mathbb{R} \) some degree of singularity of function \( q(z, t) \) is expected. As the analysis shows it is appropriate to impose the following jump-continuity conditions on charge function \( q(z, t) \).

**Assumption 2.** (jump-continuity of charge functions).

(i) Functions \( q(z, t), \ z \in \mathbb{R} \) and their time derivatives \( \partial_{t}^{j}q(z, t) \) for \( j = 1, 2 \) are continuous for all real \( t \) and \( z \).

(ii) Functions \( Q(z, t), \ z \in a\mathbb{Z} \) and their time derivatives \( \partial_{t}^{j}Q(z, t) \) for \( j = 1, 2 \) are continuous for all real \( t \).

(iii) Derivatives \( \partial_{t}^{j}q(z, t), \ \partial_{z}^{j}q(z, t) \) for \( j = 1, 2 \), and the mixed derivatives \( \partial_{t}\partial_{z}q(z, t) = \partial_{t}^{2}\partial_{z}q(z, t) \) exist and continuous for all real real \( t \) and \( z \) except for the interaction points on the lattice \( a\mathbb{Z} \).

(iv) For a function \( F(z) \) and a real number \( b \) symbols \( F(b - 0) \) and \( F(b + 0) \) stand for its left and right limit at \( b \) assuming their existence, that is

\[
F(b \pm 0) = \lim_{z \to b \pm 0} F(z).
\]

We also denote by \([F](b)\) the jump of function \( F(z) \) at \( b \), that is

\[
[F](b) = F(b + 0) - F(b - 0).
\]

(3.3)

(3.4)

(v) The following right and left limits exist

\[
\partial_{z}^{j}q(a\ell \pm 0, t), \ j = 1, 2; \ \ell \in \mathbb{Z},
\]

and these limits are continuously differentiable functions of \( t \). The values \( \partial_{z}q(a\ell \pm 0, t) \) can be different and consequently the jumps \([\partial_{z}q](a\ell, t)\) can be nonzero.

The physical dimensions of quantities related to cavities are summarized in Table 3.
3.1. MCK Lagrangian and Euler-Lagrange equations. To simplify expressions of quantities of interest we use notations

\[
\{Q\} = Q, \quad \partial_t Q, \quad Q = Q(z, t), \quad z \in aZ;
\]
\[
\{q\} = q, \quad \partial_z q, \quad \partial_t q, \quad q = q(z, t), \quad z \in \mathbb{R},
\]
\[
\{x\} = Q, \quad \partial_t Q, \quad q, \quad \partial_t q.
\]

(3.6)

(3.7)

The dynamical properties of our MCK model are implemented through the Lagrangian variational formalism. Namely, the MCK system Lagrangian \(\mathcal{L}\) is defined as the sum of its two components: (i) \(\mathcal{L}_B\) is the e-beam Lagrangian; (ii) \(\mathcal{L}_{CB}\) the cavities and e-beam interaction Lagrangian. That is

\[
\mathcal{L}\left(\{x\}\right) = \mathcal{L}_B\left(\{q\}\right) + \mathcal{L}_{CB}\left(x\right),
\]

(3.8)

where we used notations (3.6) and (3.7). The expressions for \(\mathcal{L}_B\) are similar to the Lagrangian components in equations (2.9), (2.10), namely

\[
\mathcal{L}_B\left(\{q\}\right) = \frac{1}{2\beta} \left(\partial_t q + \dot{v} \partial_z q\right)^2 - \frac{2\pi}{\sigma_B} q^2,
\]

(3.9)

and the interaction Lagrangian \(\mathcal{L}_{CB}\) is defined by

\[
\mathcal{L}_{CB}\left(x\right) = \sum_{\ell=\infty}^{\infty} \delta(z - a\ell) \left\{ \frac{l_0}{2} \left(\partial_\ell Q(a\ell)\right)^2 - \frac{1}{2c_0} [Q(a\ell) + bq(a\ell)]^2 \right\}.
\]

(3.10)

Note that term \(-\frac{2\pi}{\sigma_B} q^2\) in the Lagrangian \(\mathcal{L}_B\) defined in equations (3.9) represents space-charge effects including the so-called debunching (electron-to-electron repulsion). Parameters \(\beta\) and \(\sigma_B\) are respectively the e-beam intensity and the area of the cross-section of the e-beam defined in Section 2.

We would like to point out that: (i) expression (3.10) for the interaction Lagrangian \(\mathcal{L}_{CB}\) limits the interaction by design to points \(a\ell\) as indicated by delta functions \(\delta(z - a\ell)\) and (ii) the factors before delta functions \(\delta(z - a\ell)\) are expressions similar to density \(L_{TB}\) in equations (2.9) adapted to set of discrete interaction points \(a\ell\); (iii) cavity capacitance \(c_0\) is of particular significance for the interaction between the cavities and the e-beam. Note that according to equations (3.8), (3.9) and (3.10) Lagrangian \(\mathcal{L}\) is a periodic function of \(z\) of the period \(a\).

As we derive in Section 10 the Euler-Lagrange (EL) equations for points \(z\) outside the lattice \(aZ\) are

\[
\frac{1}{\beta} (\partial_t + \dot{v} \partial_z)^2 q + \frac{4\pi}{\sigma_B} q = 0, \quad z \neq a\ell, \quad \ell \in \mathbb{Z},
\]

or equivalently

\[
\left(\frac{1}{v} \partial_t + \partial_z\right)^2 q + \frac{4\pi \beta}{\sigma_B v^2} q = 0, \quad z \neq a\ell, \quad \ell \in \mathbb{Z}.
\]

(3.11)

(3.12)

The EL equations at the interaction points \(a\ell\) (see equations (10.17)) are

\[
[q](a\ell) = 0,
\]

(3.13)

\[
\partial_t^2 Q(a\ell) + \omega_0^2 [Q(a\ell, t) + bq(a\ell, t)] = 0, \quad [\partial_z q](a\ell) = -\frac{b\beta_0}{v^2} [Q(a\ell) + bq(a\ell)],
\]

(3.14)

where we make use of parameters

\[
\omega_0 = \frac{1}{\sqrt{l_0 c_0}}, \quad \beta_0 = \frac{\beta}{c_0},
\]

(3.15)
and jumps $[q](a\ell)$ are defined by equation (3.4). We refer to $\beta_0$ as cavity e-beam interaction parameter and to $\omega_0$ as cavity resonance frequency. Note that equations (3.13) is just an acknowledgment of the continuity of charges $q(z,t)$ at the interaction points in consistency with Assumption 2. Equations (3.13), (3.14) can be viewed as the boundary conditions that are complementary to the differential equations (3.11) and (3.12).

In what follows the following parameters play an important role in the analysis

\[
f_B = \sqrt{\frac{4\pi \beta}{\sigma_B \beta^2}} = \frac{\omega_{rp}}{\tilde{\nu}}, \quad \lambda_{rp} = \frac{2\pi \tilde{\nu}}{\omega_{rp}}, \quad \omega_{rp} = R_{sc}\omega_p, \tag{3.16}
\]

where $\omega_{rp}$ and $\lambda_{rp}$ are respectively the reduced plasma frequency and the electron plasma wavelength.

The Fourier transform in $t$ (see Appendix A) of equations (3.12), (3.13) and (3.14) (see also equations (3.30) and (3.31)) yields the following ordinary differential equations in $z$

\[
\left( \partial_z - i\frac{\omega}{\tilde{\nu}} \right)^2 \tilde{q} + f_B^2 \tilde{q} = 0, \quad f_B = \sqrt{\frac{4\pi \beta}{\sigma_B \beta^2}} = \frac{\omega_{rp}}{\tilde{\nu}}, \quad z \neq a\ell, \quad \ell \in \mathbb{Z}, \tag{3.17}
\]

subjects to the boundary conditions at the interaction points

\[
[q](a\ell) = 0, \quad \ell \in \mathbb{Z}, \tag{3.18}
\]

\[
[\partial_z q](a\ell) = -\frac{b^2 \beta_0}{\tilde{\nu}^2} \frac{\omega^2}{\omega^2 - \omega_0^2} \tilde{q}(a\ell), \quad \omega_0 = \frac{1}{\sqrt{l_0 c_0}}, \quad \beta_0 = \frac{\beta}{c_0}, \tag{3.19}
\]

where $\tilde{q}$ is the time Fourier transform of $q$. Note also that

\[
\tilde{Q}(a\ell) = \frac{\omega_0^2}{\omega^2 - \omega_0^2} b\tilde{q}(a\ell), \quad \ell \in \mathbb{Z}, \tag{3.20}
\]

where $\tilde{Q}$ is the time Fourier transform of $Q$.

Hence, the EL differential equations (3.17) together with the boundary conditions (3.18) and (3.19) form the complete set of equation describing the MCK evolution. Boundary conditions (3.18) and (3.19) can be recast into matrix form as follows

\[
X(a\ell + 0) = S_b X(a\ell - 0), \quad S_b = \begin{bmatrix} 1 & 0 \\ \frac{-b^2 \beta_0}{\tilde{\nu}^2} \frac{\omega^2}{\omega^2 - \omega_0^2} & 1 \end{bmatrix}, \quad X = \begin{bmatrix} \tilde{q} \\ \partial_z \tilde{q} \end{bmatrix}. \tag{3.21}
\]

3.2. Euler-Lagrange equations in dimensionless variables. As to basic variables related to the e-beam and the klystron cavities we refer the reader to Section 2 and Tables 2, 3. The primary dimensionless variables of importance are

\[
z' = \frac{z}{a}, \quad \partial_z' = a \partial_z, \quad t' = \frac{\tilde{t}}{a}, \quad \partial_{t'} = a \partial_t, \quad \omega' = \frac{\omega}{\omega_a} = \frac{a}{2\pi \tilde{\nu}} \omega, \quad \omega_0' = \frac{\omega_0}{\omega_a} = \frac{a}{2\pi \tilde{\nu}} \omega_0, \tag{3.22}
\]

\[
\beta' = \frac{\beta}{\tilde{\nu}}, \quad \sigma_B' = \frac{\sigma_B}{a}, \quad c_0' = \frac{c_0}{a}, \quad l_0' = \frac{\tilde{l}_0}{a}, \tag{3.23}
\]

\[
\beta_0' = \frac{\beta_0}{c_0}, \quad f_B' = a f_B = \sqrt{\frac{4\pi \beta'}{\sigma_B' \tilde{\nu}}} = \frac{a R_{sc} \omega_p}{\omega_a} = \frac{2\pi a}{\lambda_{rp}}, \tag{3.24}
\]

\[
a\delta(z) = \delta(z'), \quad z = az'. \tag{3.25}
\]

For the reader convenience we collected in Table 4 all significant parameters associated with MCK.

The dimensionless form $L'$ of the Lagrangians is as follows:

\[
L' = L'_B + L'_{CB}, \quad L'_B = \frac{1}{2\beta'} (\partial_{t'} q + \partial_{z'} q)^2 - \frac{2\pi q^2}{\sigma_B'}, \quad \ell \in \mathbb{Z}, \tag{3.26}
\]
\( a \) | the MCK period | [length]  
\( \dot{v} \) | the e-beam stationary velocity | [length]  
\( \omega_a = \frac{2\pi \dot{v}}{a} \) | the period frequency | [1/time]  
\( \omega_p = \sqrt{\frac{4\pi \hbar e^2}{m}} \) | the plasma frequency | [1/time]  
\( \lambda_{tp} = \frac{2\pi \dot{v}}{\omega_p} \), \( \omega_{tp} = R_{sc}\omega_p \) | the electron plasma wavelength | [length]  
\( g_B = \frac{\sigma_B}{4\lambda_{tp}} \) | the e-beam spatial scale | [length]  
\( f' = \frac{2\pi \dot{v}}{\omega_p} = \frac{2\pi a}{\lambda_{tp}} \) | normalized period in units of \( \frac{\lambda_{tp}}{2\pi} \) | [dim-less]  
\( n \) | the number of electrons p/u of volume | [length]  
\( c_0, l_0 \) | the cavity capacitance, inductance | [length], [length]^2  
\( \omega_0 = \frac{1}{\sqrt{\omega_0 c_0}} \) | the cavity resonant frequency | [1/time]  
\( \beta = \frac{\sigma_B R_{sc}\omega_p^2}{4\pi} = \frac{e^2 R_{sc}\sigma_B}{m} = \frac{\pi \sigma_B e^2}{\lambda_{tp}^2} \) | the e-beam intensity | [length]^2 [time]^2  
\( \beta' = \frac{\beta}{c_0} = \frac{\varphi_B}{\lambda_{tp}} \) | dim-less e-beam intensity | [dim-less]  
\( \beta'_0 = \frac{\varphi_B}{c_0} = \frac{a_0}{c_0} \beta' \) | the first interaction par. | [dim-less]  
\( B(\omega) = B_0 \frac{\omega^2}{\omega^2 - \omega_0^2} \), \( B_0 = b^2 \beta'_0 \) | the second interaction par. | [dim-less]  
\( K_0 = \frac{B_0}{2f} = \frac{b^2 \beta'_0}{4\lambda_{tp} c_0} = \frac{2\beta_0}{c_0} \) | the MCK gain coefficient | [dim-less]  
\( K(\omega) = \frac{B(\omega)}{2f} = K_0 \frac{\omega^2}{\omega^2 - \omega_0^2} \) | the MCK gain par. | [dim-less]  

**Table 4.** MCK significant parameters. Abbreviations: dimensionless – dim-less, p/u – per unit, par. - parameter. For the sake of notation simplicity we often omit “prime” super-index indicating that the dimensionless version of the relevant parameter is involved when it is clear from the context.

\[
\mathcal{L}_{UB}^{c} = \sum_{\ell=\infty}^{\infty} \delta(z' - \ell) \left\{ \frac{f_0^2}{2} \left( \partial_{t}Q(a\ell) \right)^{2} - \frac{1}{2c_0} \left[ Q(a\ell) + bq(a\ell) \right]^{2} \right\}. \tag{3.27}
\]

The dimensionless form of the EL equations in between interaction points that corresponds to the Lagrangian \( \mathcal{L}' \) defined by equations (3.26) and (3.27) is

\[
(\partial_{t} + \partial_{z'})^{2} q + f_B^{2} q = 0, \quad z' \neq \ell, \quad \ell \in \mathbb{Z}; \quad f_{B}^{2} = \sqrt{\frac{4\pi \beta'}{\sigma_{B}}} = \frac{2\pi R_{sc} \omega_p}{\omega_a} = \frac{2\pi a}{\lambda_{tp}}, \tag{3.28}
\]

and the EL equations at the interaction points \( \ell \) are

\[
\partial_{z'}^{2} Q(a\ell) + \omega_{0}^{2} \left[ Q(a\ell) + bq(a\ell) \right] = 0, \quad \omega'_{0} = \frac{1}{\sqrt{\omega_0 c_0}}, \quad \beta'_0 = \frac{\beta'}{c_0}, \tag{3.29}
\]

\[
[\partial_{z'} q](a\ell) = -b\beta'_0 \left[ Q(a\ell) + bq(a\ell) \right], \quad \ell \in \mathbb{Z},
\]
where jumps \([q](a\ell)\) are defined by equation (3.4). Note that equations (3.29) can naturally be viewed as boundary (interface) conditions complimentary to the ordinary differential equations (3.28).

To simplify notations we will omit prime symbol in equations but rather will simply acknowledge their dimensionless form. So we will use from now on the following dimensionless form of the EL equations (3.28), (3.29)

\[
(\partial_t + \partial_z)^2 q + f^2 q = 0, \quad z \neq \ell, \quad \ell \in \mathbb{Z}; \quad f = \frac{2\pi R_{sc}\omega_p}{\omega_a} = \frac{2\pi a}{\lambda_{rp}}, \tag{3.30}
\]

\[
\partial_z^2 Q(a\ell) + \frac{\omega_0^2}{\omega^2} [Q(a\ell) + bq(a\ell)] = 0, \quad \omega_0 = \frac{1}{\sqrt{l_0 c_0}}, \quad \beta_0 = \frac{\beta}{c_0}, \tag{3.31}
\]

\[
[\partial_z q](a\ell) = -b\beta_0 [Q(a\ell) + bq(a\ell)], \quad \ell \in \mathbb{Z}. \tag{3.33}
\]

Note that in view of the definition of normalized frequency \(f = \frac{2\pi a}{\lambda_{rp}}\) in equation (3.30) we may view parameter \(\frac{f}{2\pi} = \frac{a}{\lambda_{rp}}\) in equations (3.30) as the MCK period measured in natural to the e-beam spatial unit \(\lambda_{rp}\).

The Fourier transform in \(t\) (see Appendix A) of equations (3.30), (3.31) yields

\[
(\partial_z - i\omega)^2 \tilde{q} + f^2 \tilde{q} = 0, \quad z \neq \ell, \tag{3.32}
\]

subjects to the boundary conditions at the interaction points

\[
[\tilde{q}](a\ell) = 0, \quad [\partial_z \tilde{q}](a\ell) = -B(\omega) \tilde{q}(a\ell), \quad \ell \in \mathbb{Z}, \tag{3.33}
\]

where \(\tilde{q}\) is the time Fourier transform of \(q\) and \(B(\omega)\) is a new important parameter defined by

\[
B = B(\omega) = \frac{b^2 \beta_0 \omega^2}{\omega^2 - \omega_0^2} = B_0 \frac{\omega^2}{\omega^2 - \omega_0^2}, \quad B_0 = b^2 \beta_0 = \frac{b^2 \beta}{c_0}, \tag{3.34}
\]

we refer to it as cavity e-beam interaction parameter. The representation of coefficient \(B_0\) by equations (3.34) suggests that it similar to the TWT principal parameter \(\gamma\) defined by equations (2.8). Note that according to equations (3.34) the following representations hold for the parameter \(B_0\)

\[
B_0 = b^2 \beta_0 = \lim_{\omega \to \infty} B(\omega) = B(\omega)|_{\omega_0 = 0}, \tag{3.35}
\]

indicating that \(B_0\) is the high-frequency limit of \(B(\omega)\) and at the same time it is the value of \(B(\omega)\) when the cavity resonant frequency \(\omega_0\) vanishes, that \(\omega_0 = 0\).

The Fourier transform in time of equation (3.31) yields

\[
\tilde{Q}(a\ell) = \frac{\omega_0^2}{\omega^2 - \omega_0^2} b\tilde{q}(a\ell), \quad \ell \in \mathbb{Z}, \tag{3.36}
\]

where \(\tilde{Q}\) is the time Fourier transform of \(Q\), and equation (3.36) was used to obtain the second equation in (3.33).

Boundary conditions (3.33) can be recast into matrix form as follows

\[
X(a\ell + 0) = S_b X(a\ell - 0), \quad S_b = \begin{bmatrix} 1 & 0 \\ -B(\omega) & 1 \end{bmatrix}, \quad X = \begin{bmatrix} \tilde{q} \\ \partial_z \tilde{q} \end{bmatrix}, \quad B(\omega) = \frac{b^2 \beta_0 \omega^2}{\omega^2 - \omega_0^2}. \tag{3.37}
\]

In order to use the standard form of the Floquet theory reviewed in Appendix F we recast the ordinary differential equations (3.32) with boundary (interface) conditions (3.33) as the following
single second-order ordinary differential equation with singular, frequency dependent, periodic potential:

\[
\partial^{2}_{z} \ddot{q} - 2i\omega \partial_{z} \dot{q} + (f^{2} - \omega^{2}) \dot{q} - B(\omega) p(z) \ddot{q} = 0, \quad p(z) = \sum_{\ell=-\infty}^{\infty} \delta(z - \ell), \quad \dot{q} = \dot{q}(z),
\]  

(3.38)

where the second interaction parameter \( B(\omega) \) is defined by equation (3.34).

Analysis of equations (3.38) based on the Floquet theory (see Appendix F) becomes now the primary subject of our studies. The second-order ordinary differential equation (3.38) can in turn be recast into the following matrix ordinary differential equation

\[
\partial_{z}X = A(z)X, \quad A(z) = A(z, \omega) = \begin{bmatrix} \omega^{2} - f^{2} + B(\omega) p(z) & 0 \\ 0 & 2i\omega \end{bmatrix}, \quad X = \begin{bmatrix} q \\ \partial_{z}q \end{bmatrix},
\]  

(3.39)

\[
B(\omega) = \frac{b^{2} \beta_{0}\omega^{2}}{\omega^{2} - \omega_{0}^{2}} = 2fK_{0} \frac{\omega^{2}}{\omega^{2} - \omega_{0}^{2}}, \quad p(z) = \sum_{\ell=-\infty}^{\infty} \delta(z - \ell).
\]

(3.40)

The eigenvalues are eigenvectors of metric matrix \( G \) are as follows:

\[
\omega + \sqrt{\omega^{2} + 1}, \begin{bmatrix} \frac{i}{\omega + \sqrt{\omega^{2} + 1}} \\ 1 \end{bmatrix}; \quad \omega - \sqrt{\omega^{2} + 1}, \begin{bmatrix} -\frac{i}{\omega - \sqrt{\omega^{2} + 1}} \\ 1 \end{bmatrix}.
\]  

(3.41)

Using expressions (3.39) and (3.40) for respectively matrices \( A(z) \) and \( G \) one can readily verify that \( A(z) \) is \( G \)-skew-Hermitian matrix, that is

\[
GA(z) + A^{*}(z)G = 0,
\]  

(3.42)

and that according to Appendix G implies that the system (3.39) is Hamiltonian. Consequently, according to Appendix G the matrizant \( \Phi(z) \) of the Hamiltonian system (3.39) is \( G \)-unitary and its spectrum \( \sigma \{ \Phi(z) \} \) is symmetric with respect to the unit circle, that is

\[
\Phi^{*}(z) G \Phi(z) = G, \quad \zeta \in \sigma \{ \Phi \} \Rightarrow \frac{1}{\zeta} \in \sigma \{ \Phi \}.
\]  

(3.43)

3.3. The monodromy matrix. We remind that according to the Floquet theory reviewed in Appendix F the monodromy matrix encodes significant information about the relevant first order periodic ODE related to the eigenmodes.

All analysis here is carried out for dimensionless variables. We begin with an observation that characteristic polynomial associated with equation (3.32) is

\[
A_{B}(s) = s^{2} - 2i\omega s + f^{2} - \omega^{2}, \quad s = \exp \{ ik \}, \quad k = k(\omega),
\]  

(3.44)

where \( s \) is the spectral parameter and \( k \) can be interpreted as complex-valued wave number. Note that in accordance with the general theory of differential equations (see Appendices C, D and E) the spectral parameter \( s \) in the expression (3.44) of the characteristic polynomial \( A_{B}(s) \) represents symbolically the differential operator \( \partial_{z} \).
The expression of $2 \times 2$ companion matrix $\mathcal{C}_B$ (see Appendix [D]) of the scalar characteristic polynomial $A_B(s)$ defined by equation (3.44) and the corresponding exponential $\exp\{z \mathcal{C}_B\}$ are

$$
\mathcal{C}_B = \begin{bmatrix}
\omega^2 - f^2 & 1 \\
2i\omega & 0
\end{bmatrix} = \mathcal{X}_B \begin{bmatrix}
i(\omega-f) & 0 \\
0 & i(\omega+f)
\end{bmatrix} \mathcal{X}_B^{-1}, \quad \mathcal{X}_B = \begin{bmatrix}
-\frac{i}{\omega-f} & -\frac{i}{\omega+f} \\
1 & 1
\end{bmatrix},
$$

(3.45)

$$
\exp\{z \mathcal{C}_B\} = \frac{1}{f} \exp\{i\omega z\} \begin{bmatrix}
f \cos(fz) - i\omega \sin(fz) & \sin(fz) \\
(\omega^2 - f^2) \sin(fz) & f \cos(fz) + i\omega \sin(fz)
\end{bmatrix} = \mathcal{X}_B \exp\begin{bmatrix}
i(\omega-f)z & 0 \\
0 & i(\omega+f)z
\end{bmatrix} \mathcal{X}_B^{-1}.
$$

(3.46)

Applying now the Floquet theory (see Appendix [F]) to the first-order ODE equivalent of the EL equations (3.32), (3.33) and (3.34) and their alternative form (3.38) we find that the corresponding $2 \times 2$ matrizant $\Phi(z)$ satisfies

$$
\Phi(z) = \exp\{(z-\ell) \mathcal{C}_B\} \Phi(\ell + 0), \quad \ell < z < \ell + 1,
$$

(3.47)

$$
\Phi(0 + 0) = I, \quad \Phi(\ell + 0) = S_b \Phi(\ell - 0), \quad \ell \in \mathbb{Z},
$$

where $\exp\{z \mathcal{C}_{TB}\}$ is defined by equation (3.46) and $2 \times 2$ matrix $S_b$ is defined by equations (3.37). Equations (3.47) imply in turn the following expression for the monodromy matrix $\mathcal{T} = \Phi(1 + 0)$:

$$
\mathcal{T} = \Phi(1 + 0) = S_b \exp\{\mathcal{C}_B\} = \exp\begin{bmatrix}
\cos(f) - i\omega \text{sinc}(f) & \text{sinc}(f) \\
\text{sinc}(f) \omega^2 + 2i\omega \text{ sinc}(f) + b_f - \frac{2b_f \cos(f) + \text{cos}^2(f) + 1}{\text{sinc}(f)} & i\omega \text{sinc}(f) - \cos(f) - 2b_f
\end{bmatrix},
$$

(3.48)

where

$$
b_f = \frac{B(\omega)}{2} \text{sinc}(f) - \cos(f), \quad \text{sinc}(f) = \frac{\sin(f)}{f}, \quad B(\omega) = \frac{b^2 \beta_0 \omega^2}{\omega^2 - \omega_0^2}.
$$

(3.49)

Note according to relations (3.43) the monodromy matrix $\mathcal{T} = \Phi(1 + 0)$ represented by equations (3.48) is $G$-unitary for metric matrix $G$ defined by equations (3.40) and its spectrum $\sigma\{\mathcal{T}\}$ is symmetric with respect to the unit circle, that is it satisfies relations (3.43).

We show in Section [4] that the parameter $b_f$ defined in equations (3.49) completely determines the Floquet multipliers (eigenvalues) of monodromy matrix $\mathcal{T}$ and consequently plays a key role in the analysis of the MCK instability.

4. The Floquet multipliers, the instability and the gain

The MCK instability is manifested by exponentially growing eigenmodes. In the case of periodic differential equation (3.38) according to the Floquet theory (see Appendix [F]) and particularly Remark [24] a Floquet eigenmode grows exponentially if and only if the absolute value of the corresponding Floquet multiplier $s$ (that is an eigenvalue of the MCK monodromy matrix) is greater than 1, that is $|s| > 1$. Consequently the MCK instability is reduced mathematically to the identification of conditions when the Floquet multipliers $s$ satisfy inequality $|s| > 1$. With that in mind we proceed as follows.

We introduce first new frequency dependent parameter

$$
K = K(\omega) = \frac{B(\omega)}{2f} = K_0 \frac{\omega^2}{\omega^2 - \omega_0^2}, \quad K_0 = \frac{b^2 \beta_0}{2f} = \frac{b^2 \sigma_B}{4\lambda_{rp} c_0} = \frac{b^2 g_B}{c_0}, \quad g_B = \frac{\sigma_B}{4\lambda_{rp}},
$$

(4.1)

where $\lambda_{rp}$ and $g_B$ are respectively the electron plasma wavelength and the e-beam spatial scale (see Table [4]). We refer to $K = K(\omega)$ as the gain parameter and to $K_0$ the coefficient of the gain.
parameter for they determine the MCK gain as we show below (see Definition 5). Note that gain parameter $K (\omega)$ defined by equations (4.11) is evidently a function of $\frac{\omega}{\omega_0}$:

$$K (\omega) = K_0 \frac{\omega^2}{\omega^2 - \omega_0^2} = K_0 \frac{\tilde{\omega}^2}{\tilde{\omega}^2 - 1}, \quad \tilde{\omega} = \frac{\omega}{\omega_0}, \quad K_0 = \frac{b^2 \beta_0}{2f} = \frac{b^2 g_B}{c_0}, \quad g_B = \frac{\sigma_B}{4\lambda_{rp}}. \quad (4.2)$$

As to the gain coefficient $K_0$ in view of equations (4.11) it satisfies the following identities

$$K_0 = \lim_{\omega \to -\infty} K (\omega) = K (\omega)\big|_{\omega_0=0} = \frac{b^2 \beta_0}{2f} = \frac{b^2 g_B}{c_0}, \quad g_B = \frac{\sigma_B}{4\lambda_{rp}}. \quad (4.3)$$

Using the gain parameter $K$ we recast parameter $b_f$ in equations (3.49) as follows:

$$b_f = b_f (\omega) = K (\omega) \sin (f) - \cos (f), \quad K (\omega) = K_0 \frac{\omega^2}{\omega^2 - \omega_0^2}, \quad K_0 = \frac{b^2 g_B}{c_0}. \quad (4.4)$$

and acknowledging its key role for the MCK instability we name $b_f$ instability parameter. It turns out that the high-frequency limit $b_f^\infty$ of instability parameter $b_f$ defined by

$$b_f^\infty = \lim_{\omega \to -\infty} b_f (\omega) = K_0 \sin (f) - \cos (f), \quad K_0 = \frac{b^2 g_B}{c_0}, \quad (4.5)$$

plays significant role in the analysis of the MCK instability and its gain.

We turn now to two Floquet multipliers $s_\pm$ which are the eigenvalues of the monodromy matrix $T$ defined by equations (3.48) and (3.49). Hence, $s_\pm$ are solutions to the characteristic equation $\det \{ T - s I \} = 0$ which is the following quadratic equation:

$$s = e^{ik} = e^{i\omega} S : \quad S^2 + 2b_f S + 1 = 0; \quad S = S_\pm = -b_f \pm \sqrt{b_f^2 - 1}, \quad (4.6)$$

readily implying

$$s_\pm = e^{ik_\pm} = e^{i\omega} S_\pm = e^{i\omega} \left( -b_f \pm \sqrt{b_f^2 - 1} \right), \quad b_f = b_f (\omega) = K (\omega) \sin (f) - \cos (f), \quad (4.7)$$

$$K (\omega) = K_0 \frac{\omega^2}{\omega^2 - \omega_0^2}, \quad K_0 = \frac{b^2 \beta_0}{2f} = \frac{b^2 g_B}{c_0}. \quad (4.4)$$

We will also use the following normalized form of the characteristic equation (4.6)

$$s^2 + 2b_f e^{i\omega} s + e^{2i\omega} = 0, \quad b_f = b_f (\omega) = K_0 \frac{\omega^2}{\omega^2 - \omega_0^2} \sin (f) - \cos (f). \quad (4.8)$$

Equations (4.6) show that parameter $b_f$ completely determines the two Floquet multipliers $s_\pm$ justifying its name the instability parameter. Importantly, the characteristic equation (4.6) can be viewed as an expression of the dispersion relations between the frequency $\omega$ and the wavenumber $k$ as we discuss in Section 8.

Note that quantities $S_\pm$ that solve quadratic equation (4.6) and $s_\pm = e^{i\omega} S_\pm$ satisfy the following elementary identities

$$S_+ S_- = 1; \quad s_+ s_- = e^{2i\omega}, \quad |s_+| |s_-| = 1, \quad (4.9)$$

implying that if the pairs $S_+$ and $S_-$ and consequently $s_+$ and $s_-$ are outside the unit circle then the both pairs are symmetric with respect to it as illustrated in Figures 2 (a) and 2 (a). Not also that expressions (4.7) for $s_\pm$ imply that $s_+$ and $s_-$ are equal if and only if $b_f^2 = 1$, that is

$$s_+ = s_- \iff b_f = \pm 1. \quad (4.10)$$

Figures 2 and 2 illustrate possible locations of the two Floquet multipliers $s_\pm$ in the complex plane $\mathbb{C}$. 
The instability parameter \( b_f = b_f(\omega) \) defined by equations (4.4) and its absolute value \(|b_f|\) are respectively 2\(\pi\)-periodic and \(\pi\)-periodic functions of \(f\), that is
\[
|b_{f+2\pi}| = b_f; \quad |b_{f+\pi}| = |b_f|, \quad b_f = b_f(\omega).
\]

(4.11)

Let also \( s_\pm \) be the two MCK Floquet multipliers solving quadratic equation (4.6). Then exactly one of the following three possibilities always occurs:

\[
|b_f| > 1: \quad s_\pm = -\text{sign} \left\{ b_f \right\} \left( |b_f| \pm \sqrt{|b_f|^2 - 1} \right) \exp \left\{ i\omega \right\}, \quad |s_-| < 1 < |s_+|;
\]

(4.12)

\[
|b_f| < 1: \quad s_\pm = -\text{sign} \left\{ b_f \right\} \exp \left\{ i \left[ \omega \pm \arccos \left( |b_f| \right) \right] \right\}, \quad |s| = 1;
\]

(4.13)

\[
|b_f| = 1: \quad s_\pm = -\text{sign} \left\{ b_f \right\} \exp \left\{ i\omega \right\}, \quad s_+ = s_-, \quad |s| = 1.
\]

(4.14)

where

\[
b_f = b_f(\omega) = \frac{\omega^2}{\omega_0^2 - \omega^2} \sin (f) - \cos (f), \quad K_0 = \frac{b_0^2 \beta_0}{2} = \frac{b^2 g_B}{c_0}, \quad g_B = \frac{\sigma_B}{4\lambda_{\text{rp}}}.
\]

(4.15)

Relations (4.11)-(4.15) imply

\[
s_\pm (f + 2\pi) = s_\pm (f); \quad s_\pm (f + \pi) = -s_\pm (f); \quad |s_\pm (f + \pi)| = |s_\pm (f)|.
\]

(4.16)

As it was already pointed out \(|s_\pm| \neq 1\) determines the onset of the MCK instability. According to Theorem 1 the absolute value of each of the Floquet multipliers \(|s_\pm (f)|\) is \(\pi\)-periodic functions of \(f\). Consequently, if we are interested in smaller values of normalized period \(f = \frac{2\pi a}{\lambda_{\text{rp}}} \), the parameter that effects the instability, we may impose the following assumption.
Figure 3. The two complex eigenvalues (the Floquet multipliers) $s_\pm = e^{i\omega} S_\pm$ of the monodromy matrix $S$ defined by equations (3.48) for $\beta_0 = 3.1$, $f = 0.2$, $b = 1$, $\omega_0 = 1.2$ and two values of $\omega$: (a) $\Omega_{0.2}^- < \omega = 0.2 < \omega_0 = 1.2$: two eigenvalues shown as solid (blue) dots reside outside the the unit circle; (b) $\omega = 0.02 < \Omega_{0.2}^-$: two eigenvalues shown as solid (blue) dots reside on the unit circle. The horizontal and vertical axes represent respectively $\Re\{s\}$ and $\Im\{s\}$. In both cases $\omega = 0.02, 0.2 < \omega_0 = 1.2$, that is the chosen frequencies $\omega$ are below the resonant frequency $\omega_0 = 1.2$. The dotted red circle represents the unit circle. See Fig. 5 showing plots of functions $\Omega^\pm_f$.

Assumption 3. (smaller MCK period). The MCK normalized period $f$ satisfies the following bounds:

$$0 < f = \frac{2\pi a}{\lambda_{vp}} < \pi.$$  \hspace{1cm} (4.17)

5. Instability parameter and instability frequencies

We assume here that Assumption 3 holds. As to the MCK instability according to Theorem 1 its presence is determined entirely by the instability parameter $b_f(\omega)$ defined by equations (4.4). More precisely, the instability occurs if and only if $|b_f(\omega)| > 1$ and we want to identify all points $(f,\omega)$ when it is the case and Figure 5 illustrates ultimate results of our analysis of the matter.

When searching for all points $(f,\omega)$ such that $|b_f(\omega)| > 1$ we want to identify first all points $(f,\omega)$ for which $b_f(\omega) = \pm 1$, that is

$$b_f(\omega) = K_0 \frac{\omega^2}{\omega^2 - \omega_0^2} \sin(f) - \cos(f) = \pm 1.$$ \hspace{1cm} (5.1)

To separate out variables $f$ and $\omega$ in equations (5.1) we recast them into

$$r(\omega) = \frac{\omega}{\omega_0^2} = \frac{1 + \cos(f)}{K_0 \sin(f)} = \frac{1}{K_0 \tan\left(\frac{f}{2}\right)}, \quad \omega > 0, \quad 0 < f < \pi,$$ \hspace{1cm} (5.2)

$$r(\omega) = \frac{\omega^2}{\omega^2 - \omega_0^2} = \frac{-1 + \cos(f)}{K_0 \sin(f)} = -\frac{\tan\left(\frac{f}{2}\right)}{K_0}, \quad \omega > 0, \quad 0 < f < \pi.$$ \hspace{1cm} (5.3)
Note that function \( r(\omega) \) in equations (5.2) and (5.3) has the following properties: (i) it is a monotonically decreasing function of \( \omega \geq 0 \) with a simple pole at \( \omega = \omega_0 \); (ii) it maps one-to-one interval \((\omega_0, +\infty)\) onto \((1, +\infty)\) and interval \((0, \omega_0)\) onto \((-\infty, 0)\). The monotonicity of \( r(\omega) \) and expressions for \( b_f(\omega) \) in equations (5.1) and \( b_f^\infty \) in equations (4.5) readily imply the following low bound

\[
b_f(\omega), b_f^\infty > -\cos(f) > -1, \quad 0 < f < \pi, \quad \omega > \omega_0.
\]  
Equations (4.5) for \( b_f^\infty \) imply also the following factorization

\[
b_f^\infty - 1 = K_0 \sin(f) - \cos(f) - 1 = K_0 \sin(f) \left[ 1 - \frac{1}{K_0 \tan \left( \frac{f}{2} \right)} \right].
\]  
The high-frequency limit \( b_f^\infty \) is of importance in the analysis of the MCK instabilities and its significant properties are collected in the following statement.

**Theorem 2** (the high-frequency limit of the instability parameter). Let the high-frequency limit \( b_f^\infty \) of the instability parameter be defined by equations (4.5). Then there exists a unique value \( f_{cr} \) on interval \((0, \pi)\) of the normalized period \( f \) such that

\[
b_f^\infty = K_0 \sin(f) - \cos(f) - 1 = 0, \quad 0 < f < \pi,
\]  
and we refer to it as the critical value. The following representation holds

\[
f_{cr} = 2 \arctan \left( \frac{1}{K_0} \right), \quad \text{where} \quad K_0 = \frac{b_f g_B}{c_0}, \quad g_B = \frac{\sigma_B}{4\lambda rp}, \quad |\arctan(\ast)| < \frac{\pi}{2}.
\]  
The following identities hold for \( f_{cr} \):

\[
\tan \left( \frac{f_{cr}}{2} \right) = \frac{1}{K_0}, \quad \sin \left( \frac{f_{cr}}{2} \right) = \frac{1}{\sqrt{1 + K_0^2}}, \quad \cos \left( \frac{f_{cr}}{2} \right) = \frac{K_0}{\sqrt{1 + K_0^2}}.
\]  

Figure 4 shows how \( f_{cr} = 2 \arctan \left( \frac{1}{K_0} \right) \) varies with \( K_0 \).

The high-frequency limit \( b_f^\infty \) can be alternatively represented by the following equations:

\[
b_f^\infty = \sqrt{1 + K_0^2} \sin \left( f - \frac{f_{cr}}{2} \right) = \sqrt{1 + K_0^2} \sin \left( f - \arctan \left( \frac{1}{K_0} \right) \right),
\]  

\[
b_f^\infty = -\sqrt{1 + K_0^2} \cos \left( f + \arctan(K_0) \right), \quad |\arctan(\ast)| < \frac{\pi}{2}.
\]  

In addition to that \( b_f^\infty \) satisfies the following relations

\[
b_f^\infty > -1, \quad 0 < f < \pi;
\]  

\[
b_f^\infty < 1, \quad 0 < f < f_{cr}, \quad b_f^\infty = 1; \quad b_f^\infty > 1, \quad f_{cr} < f < \pi.
\]  

Importantly, the MCK normalized period \( f_{cr} \) signifies the onset of the MCK instability for all frequencies \( \omega > \omega_0 \), that is for \( f_{cr} < f < \pi \) the MCK system is unstable for all \( \omega > \omega_0 \), see Fig. 7.

**Proof.** Note that equation (5.6) can be recast as

\[
b_f^\infty - 1 = K_0 \sin(f) c(f), \quad c(f) = \left[ 1 - \frac{1}{K_0 \tan \left( \frac{f}{2} \right)} \right], \quad 0 < f < \pi.
\]  
Since \( \sin(f) \neq 0 \) for \( 0 < f < \pi \) equation (5.13) is equivalent to

\[
c(f) = 1 - \frac{1}{K_0 \tan \left( \frac{f}{2} \right)} = 0, \quad 0 < f < \pi.
\]
Function \( c(f) \) is monotonically increasing on interval \((0, \pi)\) since
\[
\partial_f c(f) = \frac{1}{2K_0 \sin^2\left(\frac{f}{2}\right)} > 0, \quad 0 < f < \pi, \tag{5.15}
\]
and it varies from \(-\infty\) to 1 on this interval. Hence equation (5.14) and consequently equation (5.5) have a unique solution \( f_{cr} \) on interval \((0, \pi)\) satisfying equation (5.7).

The validity of equations (5.8)-(5.10) can be verified by carrying out elementary trigonometric transformations of involved expressions. Inequality (5.11) readily follows from the first equation in (5.5) \( \sin (f) > 0 \) for \( 0 < f < \pi \). Relations (5.11) follow straightforwardly from equations (5.13) and (5.14) and already established monotonicity of function \( c(f) \).

□

The following asymptotic formulas hold for \( f_{cr} \) defined by equations (5.7):
\[
f_{cr} \approx \pi - 2K_0 + \frac{2K_0^3}{3} + O(K_0^5), \quad K_0 \to 0, \tag{5.16}
\]
\[
f_{cr} \approx \frac{2}{K_0} - \frac{2}{3K_0^3} + O(K_0^5), \quad K_0 \to +\infty. \tag{5.17}
\]

(a) (b) (c)

**Figure 4.** Plots of \( f_{cr} = 2 \arctan \left( \frac{1}{K_0} \right) \) for different ranges of \( K_0 \): (a) \( 0 \leq K_0 \leq 1 \); (b) \( 1 \leq K_0 \leq 10 \); (c) \( 10 \leq K_0 \leq 20 \). In all plots the horizontal and vertical axes represent respectively \( K_0 \) and \( f_{cr} \).

The next statement specifies the set of points \((f, \omega)\) associated with the instability, namely the points for which \( |b_f(\omega)| > 1 \).

**Theorem 3** (instability frequencies). Let functions \( \Omega_f^\pm > 0 \) of \( f \) for \( 0 < f < \pi \) be defined by the following relations:
\[
\Omega_f^+ = \begin{cases} \omega_0 \sqrt{\frac{1}{1 - K_0 \tan \left( \frac{f}{2} \right)}} & > \omega_0 \quad \text{if} \quad 0 < f < f_{cr}, \\ +\infty & \text{if} \quad f_{cr} \leq f < \pi \end{cases}, \tag{5.18}
\]
\[
\Omega_f^- = \omega_0 \sqrt{\frac{\tan \left( \frac{f}{2} \right)}{\tan \left( \frac{f}{2} \right) + K_0}} < \omega_0, \quad 0 < f < \pi. \tag{5.19}
\]

Then the values of the instability parameter \( b_f(\omega) \) satisfy the following relations:
\[
\omega > \omega_0, \quad 0 < f < f_{cr}: \quad b_f(\Omega_f^+) = 1; \tag{5.20}
\]
\[
b_f(\omega) > 1, \quad \omega_0 < \omega < \Omega_f^-, \quad -1 < b_f(\omega) < 1, \quad \omega > \Omega_f^+; \tag{5.21}
\]
\[
\omega > \omega_0, \quad f_{cr} \leq f < \pi: \quad b_f(\omega) > b_f^\infty > 1; \tag{5.21}
\]
\[0 < \omega < \omega_0, \quad 0 < f < \pi: \quad b_f (\Omega_f^{-}) = -1;\]  
\[-1 < b_f (\omega) < 1, \quad 0 < \omega < \Omega_f^{-}; \quad b_f (\omega) < -1, \quad \Omega_f^{-} < \omega < \omega_0.\]  
Hence, for any \(0 < f < \pi\) interval \((\Omega_f^{-}, \Omega_f^{+})\) identifies frequencies of the MCK instability and gain, that is  
\[(\Omega_f^{-}, \Omega_f^{+}) = \{\omega: |b_f (\omega)| > 1\}, \quad 0 < f < \pi,\]  
and, in particular, for \(f_{cr} \leq f < \pi\) the frequency instability interval extends to infinity, that is  
\[(\Omega_f^{-}, \Omega_f^{+}) = (\Omega_f^{-}, +\infty), \quad f_{cr} \leq f < \pi.\]  

Figure 5 provides for graphical representation of functions \(\Omega_f^{\pm} > 0\) with shadowed area identifying points \((f, \omega)\) of instability where \(|b_f (\omega)| > 1\).

Proof. Let us start with points \((f, \omega)\) with \(\omega > \omega_0\). Note that in this case \(b_f (\omega)\) is a monotonically decreasing function of \(\omega\) on interval \((\omega_0, +\infty)\) for any \(0 < f < \pi\). Consequently the set of its values on interval \((\omega_0, +\infty)\) satisfies  
\[b_f ((\omega_0, +\infty)) = (b_f^\infty, +\infty).\]  
Combining relation (5.25) with the results of Theorem 2 we obtain relations (5.21).

Let us consider now points \((f, \omega)\) with \(0 < \omega < \omega_0\) and \(0 < f < \pi\). In this case \(b_f (\omega)\) is also a monotonically decreasing function of \(\omega\) on interval \((0, \omega_0)\) for any \(0 < f < \pi\). Consequently the set of its values on interval \((0, \omega_0)\) satisfies  
\[b_f ((0, \omega_0)) = (-\cos (f), -\infty).\]  
Combining relation (5.26) with the results of Theorem 2 we obtain relations (5.22). Representations (5.23) and (5.24) for the frequency instability interval \((\Omega_f^{-}, \Omega_f^{+})\) follow from equations (5.18) for \(\Omega_f^{\pm}\) and relations (5.21).

Note that function \(\Omega_f^{+}\) approaches \(+\infty\) as \(f \to f_{cr}\) and the following asymptotic formula can be obtained. Combing equation (5.7) defining \(f_{cr}\) and equations (5.18) defining \(\Omega_f^{+}\) we obtain the following alternative representation for function \(\Omega_f^{+}\):  
\[\Omega_f^{+} = \frac{\omega_0}{\sqrt{K_0}} \sqrt{\frac{1}{\tan \left(\frac{f_{cr}}{2}\right) - \tan \left(\frac{f}{2}\right)}} = \frac{\omega_0}{\sqrt{K_0}} \sqrt{\frac{\cos \left(\frac{f_{cr}}{2}\right) \cos \left(\frac{f}{2}\right)}{\sin \left(\frac{f_{cr} - f}{2}\right)}}, \quad 0 < f < f_{cr}.\]  
Then using representation (5.27) we find the following asymptotic expansion of \(\Omega_f^{+}\) at \(f_{cr}\):  
\[\Omega_f^{+} = \frac{\omega_0}{\sqrt{K_0}} \left|\frac{\cos \left(\frac{f_{cr}}{2}\right)}{\sqrt{\frac{f_{cr} - f}{2}}}\right| + \text{sign} \left\{\cos \left(\frac{f_{cr}}{2}\right)\right\} \sin \left(\frac{f_{cr}}{2}\right) \sqrt{\frac{f_{cr} - f}{2}} + O \left((f_{cr} - f)^{\frac{3}{2}}\right), \quad f \to f_{cr}.\]  
(5.28)

Combining statements of Theorems 3, 2 and 1, particularly equations (5.23), (5.23) and relations (5.12), (4.12), we obtain the following statement.

**Theorem 4** (Floquet multiplier and instability). Let Floquet multiplier \(s_+ = s_+ (f, \omega)\) be defined by equations (4.12). Then function \(|s_+ (f, \omega)| > 1\) if and only if \(\Omega_f^{-} < \omega < \Omega_f^{+}\) for \(0 < f < \pi\). The later relations describe all unstable states of the MCK.

For any \(0 < f < \pi\) function \(|s_+ (f, \omega)|\) is monotonically increasing for \(\Omega_f^{-} < \omega < \omega_0\) and monotonically decreasing for \(\omega_0 < \omega < \Omega_f^{+}\) and \(\lim_{\omega \to \omega_0} |s_+ (f, \omega)| = +\infty\). In addition to that, the following lower bound holds:  
\[|s_+ (f, \omega)| > b_f^\infty > 1, \quad f_{cr} < f < \pi, \quad \omega > \omega_0,\]  
(5.29)
ANALYTIC THEORY OF MULTICAVITY KLYSTRONS

Figure 5. Plots of two functions $\omega = \Omega_f^+ < \omega_0 < \Omega_f^-$ defined by equations (5.18) for the case when $K_0 = 0.3$, $\omega_0 = 1.2$, and $f_{cr} \approx 2.558679064$: $\Omega_f^+$ for $0 < f < f_{cr}$, $\Omega_f^-$ for $0 < f < \pi$. The horizontal and vertical axes represent respectively variables $f$ and $\omega$. The upper and lower solid (brown) curves represent respectively functions $\Omega_f^+$ and $\Omega_f^-$ and the dashed (green) line represent $\omega = \omega_0 = 1.2$. The diamond solid (brown) dot marks the value of $f_{cr}$. The shaded (light blue) region between the two curves for functions $\Omega_f^+$ and $\Omega_f^-$ identify points $(f, \omega)$ of instability, that is points for which $|b_f(\omega)| > 1$ and consequently the corresponding Floquet multipliers $s$ satisfy $|s_+| > 1 > |s_-|$ (see Theorem 1). For the points of instability the relevant Floquet modes either grow or decay exponentially. The remaining points correspond to the case when $|b_f(\omega)| \leq 1$. In the later case the Floquet multipliers $s$ satisfy $|s_\pm| = 1$ and the corresponding Floquet modes are bounded and oscillatory. The points $(f, \Omega_f^+)$ for $0 < f < f_{cr}$ correspond to $b_f = 1$ and $(f, \Omega_f^-)$ for $0 < f < \pi$ correspond to $b_f = -1$. Points $(f, \omega_0 \pm 0)$ for $0 < f < \pi$ laying on the dashed (green) line correspond respectively to $b_f = \pm \infty$.

where high-frequency limit instability $b_\infty^f$ of the instability parameter is defined by equations (4.5).

6. Gain and its dependence of the frequency

Based on the prior analysis we introduce the MCK gain $G$ in dB per one period as a the rate of the exponential growth of the MCK eigenmodes associated with Floquet multipliers $s_\pm$ defined by equations (4.12)-(4.14) (see Theorem 1). More precisely the definition is as follows.

Definition 5 (MCK gain per one period). Let $s_\pm$ be the MCK Floquet multipliers satisfying by equations (4.12)-(4.14). Then the corresponding to them gain $G$ in dB per one period is defined
by

\[ G = G(f, \omega, K_0) = \begin{cases} 
20 \left| \log (|s_+|) \right| = 20 \log \left( \left| b_f \right| + \sqrt{b_f^2 - 1} \right) & \text{if } |b_f| > 1 \\
0 & \text{if } |b_f| \leq 1
\end{cases}, \quad (6.1) \]

\[ b_f = b_f(\omega) = K(\omega) \sin (f) - \cos (f), \quad K(\omega) = K_0 \frac{\omega^2}{\omega_0^2 - \omega^2}, \quad K_0 = \frac{b^2 \beta_0}{2f} = \frac{b^2 g_B}{c_0}. \]

Fig. 6 shows the frequency dependence of the gain \( G \) per one period which is consistent with statements of Theorem 4 including that \( G(f, \omega, K_0) \) is a monotonically increasing and decreasing function of \( \omega \) on respective intervals \((\Omega_f, \omega_0)\) and \((\omega_0, \Omega_f^+)\). In view of Definition 5 a state of the

![Figure 6](image-url)

**Figure 6.** Plots of gain \( G \) as a function of frequency \( \omega \) for \( K_0 = 1, \omega_0 = 1 \) for which \( f_{\text{cr}} \approx 1.176005207, f_{\text{max}} \approx 2.158798931, G_{\text{max}} = 10.3775845 \) and for different values of \( f \): (a) \( f = 1.15 < f_{\text{cr}} \approx 1.176005207 \); (b) \( f = 1.17 < f_{\text{cr}} \approx 1.176005207 \).

In all plots the horizontal and vertical axes represent respectively frequency \( \omega \) and gain \( G \) in dB. The solid (brown) curves represent gain \( G \) as a function of frequency \( \omega \), the dashed (blue) line \( G = G_{\text{max}} \) represent the maximal value \( G_{\text{max}} \) of \( G \) in the high frequency limit. The diamond solid (green) dots mark the values of \( \Omega_f^- \) and \( \Omega_f^+ \) which are the frequency boundaries of the instability.

MCK is unstable and has positive gain \( G > 0 \) if and only if \( |b_f(\omega)| > 1 \).

### 6.1. Maximal values of the gain

In Sections 4 and 5 we carried out detailed studies of the MCK instability including its dependence on frequency \( \omega \) and dimensionless period \( f \), see Theorems 1, 3 and 4. In particular Theorem 4 implies the following sharp lower bound holds for the gain \( G \) per one period (see Fig. 7):

\[ G = G(f, \omega, K_0) > G^\infty(f) = \lim_{\omega \to +\infty} 20 \log (|s_+|) = 20 \log \left( \left| b_f^\infty + \sqrt{(b_f^\infty)^2 - 1} \right| \right) > 0, \quad (6.2) \]

\[ b_f^\infty = K_0 \sin (f) - \cos (f) > 1, \quad f_{\text{cr}} < f < \pi, \quad \omega > \omega_0. \]

Consequently, the maximal value of gain \( G \) is attained when \( b_f^\infty \) gets its maximal value for \( f = f_{\text{max}} \) such that \( f_{\text{cr}} < f_{\text{max}} < \pi \). To find \( f_{\text{max}} \) and the corresponding \( G_{\text{max}} \) we us representation (5.10)
for \( b_f^\infty \) which we copy here for the reader’s convenience

\[
b_f^\infty = -\sqrt{1 + K_0^2 \cos(f + \arctan(K_0))}, \quad f_c < f < \pi, \quad |\arctan(*)| < \frac{\pi}{2}.
\]  

(6.3)

An elementary trigonometric analysis of above representation shows \( b_f^\infty \) attains its maximal value of \( \sqrt{1 + K_0^2} \) at \( f_{\text{max}} \) which as follows

\[
f_{\text{max}} = \pi - \arctan(K_0), \quad b_f^{\text{max}} = \sqrt{K_0^2 + 1}.
\]  

(6.4)

Consequently, the desired maximum value \( G_{\text{max}} \) of \( G \) in view of relations (6.2) is

\[
G_{\text{max}} = 20 \left| \log \left( \frac{K_0 + \sqrt{K_0^2 + 1}}{\ln(10)} \right) \right| = 20 \frac{\ln \left( K_0 + \sqrt{K_0^2 + 1} \right)}{\ln(10)}.
\]  

(6.5)

Fig. 7 shows the dependence of the gain \( G \) on frequency \( \omega \) and its asymptotic behavior as \( \omega \to +\infty \). In particular, we have the following asymptotic formulas for \( f_{\text{max}} \) and \( G_{\text{max}} \):

\[
f_{\text{max}} = \pi - K_0 + \frac{K_0^3}{3} + O(K_0^5), \quad K_0 \to 0;
\]  

(6.6)

\[
f_{\text{max}} = \frac{\pi}{2} + \frac{1}{K_0^3} - \frac{3}{K_0^3} + O(K_0^5), \quad K_0 \to +\infty;
\]  

(6.7)

\[
G_{\text{max}} = \frac{20K_0}{\ln(10)} - \frac{10K_0^3}{3 \ln(10)} + O(K_0^5) \approx 8.685889638K_0, \quad K_0 \to +0
\]  

(6.8)

\[
G_{\text{max}} = \frac{20 \ln(2K_0)}{\ln(10)} + \frac{5}{\ln(10) K_0^5} + O\left( \frac{1}{K_0^4} \right), \quad K_0 \to +\infty.
\]  

(6.9)
Fig. 7 shows the dependence of the gain $G_{\text{max}}$ on $K_0$.

**Figure 8.** Plots of optimal gain $G_{\text{max}} = 20 \log \left( K_0 + \sqrt{K_0^2 + 1} \right)$ as solid (red) curves and its approximations by leading terms in asymptotic expansions (6.8) and (6.8) as dashed (blue) curves: (a) $0 \leq K_0 \leq 1$; (b) $0 \leq K_0 \leq 10$.

It is also an elementary exercise in trigonometry to verify that the following identity holds

$$f_{\text{max}} = \frac{f_{\text{cr}} + \pi}{2}, \text{ implying } \frac{\pi}{2} < f_{\text{max}} < \pi. \quad (6.10)$$

To find out how $f_{\text{max}} - f_{\text{cr}}$ depends on $K_0$ we use equations (5.7) and (6.4) that yield

$$f_{\text{max}} - f_{\text{cr}} = \pi - \arctan \left( \frac{1}{K_0} \right).$$

Equation (6.11) in turn readily implies the following relationships illustrated by Fig. 9:

$$\frac{d}{dK_0} (f_{\text{max}} - f_{\text{cr}}) = \frac{1}{1 + K_0^2} > 0; \quad \lim_{K_0 \to +0} (f_{\text{max}} - f_{\text{cr}}) = 0; \quad f_{\text{cr}} < f_{\text{max}} < \pi. \quad (6.12)$$

7. Typical values of the MCK gain and its significant parameters

It would instructive to make an assessment of typical values the MCK gain and its other parameters. There is the following rough empirical formula that shows the dependence of the maximum power gain $G_T(N)$ on the number $N$ of cavities in the klystron, \[Tsim, 7.7.1\], \[Grigo, 7.2.6\], \[ValMid, 16\]:

$$G_T(N) = 15 + 20 (N - 2) \text{ dB}. \quad (7.1)$$

Realistically achievable maximum amplification values though are smaller and are of the order of 50 dB to 70 dB. The main limiting factors are noise and self-excitation of the klystron because of parasitic feedback between cavities.

As to other universal values in the klystron theory the MCK theory features a fundamental scale $z_B$, called *bunching distance*, associated with one quarter a plasma oscillation cycle, \[BenSweScha, 9.3.4\], \[Gilm1, 9.2\]:

$$z_B = \frac{\lambda_{\text{rp}}}{4}, \quad \lambda_{\text{rp}} = \frac{2\pi v}{\omega_{\text{rp}}}. \quad (7.2)$$
Functions $f_{cr} = 2 \arctan \left( \frac{1}{K_0} \right)$, $f_{\text{max}} = \pi - \arctan \left( K_0 \right)$.

**Figure 9.** Plots of $f_{\text{max}} = \pi - \arctan \left( K_0 \right)$ as solid (brown) curve and $f_{cr} = 2 \arctan \left( \frac{1}{K_0} \right)$ as dashed (blue) curve. The horizontal and vertical axes represent respectively $K_0$ and $f$. Asymptotic formulas (5.16), (5.16) and (6.6), (6.6) describe respectively the behavior of $f_{cr}$ and $f_{\text{max}}$ for small and large $K_0$.

**Figure 10.** Plots of gain $G^\infty(f)$, that is the gain in high frequency limit, as a function of $f$ defined in relations (6.2) for $f_{cr} < f < \pi$ and (a) $K_0 = 3$, with $f_{cr} \cong 0.06435011088$, $f_{\text{max}} \cong 1.892546882$ and $G_{\text{max}} \cong 15.79482525$; (b) $K_0 = 4.95$, with $f_{cr} \cong 0.3986746100$, $f_{\text{max}} \cong 1.770133632$ and $G_{\text{max}} = 20$. In all plots the horizontal and vertical axes represent respectively frequency $f$ and gain $G$ in dB. The solid (brown) curves represent gain $G$ as a function of $f$. The diamond solid (green) dots mark the values of $f_{cr}$ and $f_{\text{max}}$.

The physical origin of the bunching distance scale $z_B$ defined by equations (7.2) is evidently related to the reduced plasma frequency $\lambda_p$ and it can be explained as follows, [Gilm1, 9.2] ($\lambda_p = \frac{2\pi}{\omega_p}$):

"At the axial position denoted by $\lambda_p/4$ fast electrons have been slowed to the dc velocity of the beam and slow electrons have been accelerated to the dc beam velocity. At $\lambda_p/4$, all electrons have the same velocity. Also, at $\lambda_p/4$, the RF electron density and the RF current reach maximum values. For small to medium RF signals, the RF current is nearly sinusoidal. A very important characteristic of the bunching process
with space charge forces is that all electrons are either speeded up or slowed down to the same velocity (the dc beam velocity) at the same axial position ($\lambda_{p}/4$). In addition, even if the amplitude of the modulating field is changed so that initial electron velocities are changed, the axial position of the bunch remains the same. This result is extremely important to the klystron engineer because, unlike the situation when space charge forces are ignored, the cavity location for maximum RF beam current is not a function of signal level, of gap width, or of frequency of operation.”

In other words the charge wave in the moving stream of electrons of stationary (dc) velocity $\dot{v}$ has the density that proportional to a sinusoidal traveling wave, namely

$$\sin \left[ \omega_{rp} \left( \frac{z}{\dot{v}} - t \right) \right] = \sin \left[ k_q z - \omega_{rp} t \right], \quad k_q = \frac{\omega_{rp}}{\dot{v}}, \quad \lambda_{rp} = \frac{2\pi}{k_q} = \frac{2\pi \dot{v}}{\omega_{rp}},$$

where $\lambda_{rp}$ is the electron plasma wavelength as in equations (7.2). In particular, equation (7.3) is consistent the formula (7.2) for bunching distance.

Comparing our formula (6.5) for the maximal gain $G_{\text{max}}$ for one period with Tsimring formula (7.1) and assuming their consistency we readily arrive with the following equation for a “typical” value $K_{0T}$ for the instability coefficient $K_0$:

$$G_{\text{max}} = 20 \ln \left( K_0 + \sqrt{K_0^2 + 1} \right) = \lim_{N \to \infty} \frac{G_T(N)}{N} = 20.$$  \hspace{1cm} (7.4)

The solution $K_0 = K_{0T}$ to equation (7.4) is

$$K_{0T} = 4.95.$$  \hspace{1cm} (7.5)

yielding the corresponding values of $f_{\text{cr}}$, $f_{\text{max}}$ and $G_{\text{max}}$:

$$f_{\text{crT}} \simeq 0.3986746100, \quad f_{\text{maxT}} \simeq 1.770133632, \quad G_{\text{maxT}} = 20.$$  \hspace{1cm} (7.6)

Based on typical values of the MCK parameters as in equations (7.5) and (7.6) we generate the gain $G$ as a function of frequency $\omega$ defined by equations (6.1) shown in Figs. 11 and 12.

![Figure 11. Plots of gain $G$ as a function of frequency $\omega$ defined by equations (6.1) for $\omega_0 = 1$, $K_0 = K_{0T} = 4.95$ and consequently $f_{\text{cr}} \simeq 0.3986746100$, $f_{\text{max}} \simeq 1.770133632$, $G_{\text{max}} = 20$ and: (a) $f = 0.35 < f_{\text{cr}}$; (b) $f = 0.5 > f_{\text{cr}}$. In all plots the horizontal and vertical axes represent respectively frequency $\omega$ and gain $G$ in dB.](attachment:image.png)

(a)  \hspace{1cm} (b)
Figure 12. Plot of gain $G$ as a function of frequency $\omega$ defined by equations (6.1) for $\omega_0 = 1$, $K_0 = K_{RT} = 4.95$ and consequently $f_{cr} \cong 0.3986746100$, $f_{max} \cong 1.7701336320$, $G_{max} = 20$ and $f = f_{max} \cong 1.7701336320$. The horizontal and vertical axes represent respectively frequency $\omega$ and gain $G$ in dB. The solid (brown) curve represents gain $G$ as a function of frequency $\omega$, the dashed (blue) line $G = G_{max}$ represent the maximal value $G_{max}$ of $G$ in the high frequency limit (see Section 6.1). The diamond solid (green) dot mark the value of $\Omega - f$ which is the frequency boundaries of the instability.

Note that according to equation (3.30) the normalized period $f = \frac{2\pi a}{\lambda_{rp}}$. Combining that with equations (5.7), (6.4) and (6.10) for $f_{cr}$ and $f_{max}$ we obtain the following expression for the corresponding values $a_{cr}$ and $a_{max}$ of the MCK period $a$:

$$a_{cr} = \frac{f_{cr}}{2\pi} \lambda_{rp} = \frac{\arctan \left( \frac{1}{K_0} \right)}{\pi} \lambda_{rp}, \quad a_{max} = \frac{f_{max}}{2\pi} \lambda_{rp} = \frac{\pi - \arctan (K_0)}{2\pi} \lambda_{rp} = \frac{a_{cr}}{2} + \frac{\lambda_{rp}}{4},$$

(7.7)

where $K_0 = \frac{b^2 g_B}{c_0}$, $g_B = \frac{\sigma_B}{4\lambda_{rp}}$, $|\arctan(*)| < \frac{\pi}{2}$.

The MCK period $a = a_{cr}$ signifies the onset of MCK instability for all frequencies $\omega > \omega_0$, that is for any $a_{cr} < a < \frac{\lambda_{rp}}{2}$ the MCK system is unstable for all frequencies $\omega > \omega_0$. The MCK period $a = a_{max}$ is the one at which the MCK system attains its maximal gain for all frequencies $\omega > \omega_0$, see Figs. 11 (b) and 12.

Equations (7.7) the following limit relations (see Fig. 9):

$$\lim_{K_0 \to \infty} a_{cr} = 0, \quad \lim_{K_0 \to 0} a_{cr} = \frac{\lambda_{rp}}{2},$$

(7.8)

$$\lim_{K_0 \to \infty} a_{max} = \frac{\lambda_{rp}}{4}, \quad \lim_{K_0 \to 0} a_{max} = \frac{\lambda_{rp}}{2}.$$  

(7.9)

Note the first limit relation in equations (7.9) yields the well known in the klystron theory bunching distance $z_B = \frac{\lambda_{rp}}{4}$ in equations (7.2) as the limit of $a_{max}$ for large values of $K_0$.

Expressions (7.7) for $a_{cr}$ and $a_{max}$ imply also the following inequalities:

$$a_{cr} < a_{max}, \quad 0 < a_{cr} < \frac{\lambda_{rp}}{2}, \quad \frac{\lambda_{rp}}{4} < a_{max} = \frac{a_{cr}}{2} + \frac{\lambda_{rp}}{4} < \frac{\lambda_{rp}}{2}.$$  

(7.10)
8. Dispersion relations

We start with an observation that in view of the relation $s = \exp\{ik\}$ between the Floquet multiplier $s$ and the wave number $k$ (see Section [F] and Remark [24]) the characteristic equation (4.6) can be viewed as an expression of the dispersion relations between the frequency $\omega$ and the wavenumber $k$ and we will refer to it as the MCK dispersion relations or just the dispersion relations. Dispersion relation (4.6) can be readily recast as

$$S + 2bf + S^{-1} = 0, \quad S = s \exp\{-i\omega\} = \exp\{i(k - \omega)\}, \quad s = \exp\{ik\}, \quad (8.1)$$

or, equivalently, as

$$\cos(k - \omega) + bf(\omega) = 0, \quad bf(\omega) = K(\omega)\sin(f) - \cos(f), \quad (8.2)$$

$$K(\omega) = K_0 \frac{\omega^2}{\omega^2 - \omega_0^2}, \quad K_0 = \frac{b^2\beta_0}{2f} = \frac{b^2g_B}{c_0}. \quad (8.3)$$

Equation (8.2) in turn is equivalent to

$$k_{\pm}(\omega) = \omega \pm \arccos(-bf(\omega)) + 2\pi m, \quad bf(\omega) = K(\omega)\sin(f) - \cos(f), \quad m \in \mathbb{Z}, \quad (8.3)$$

$$K(\omega) = K_0 \frac{\omega^2}{\omega^2 - \omega_0^2}, \quad K_0 = \frac{b^2\beta_0}{2f} = \frac{b^2g_B}{c_0}. \quad (8.3)$$

When constructing the MCK dispersion relations we follow to the general approach reviewed in Section [F] (see Remark [24]) for finding the dispersion relations for periodic systems. Using Theorem [1] we obtain the following statement relating the frequency $\omega$ to the wavenumber $k = k_{\pm}(\omega)$.

**Theorem 6** (MCK dispersion relations). Let $s_{\pm}$ be the MCK Floquet multipliers and let $k_{\pm}(\omega)$ be the corresponding complex-valued wave numbers satisfying

$$s_{\pm} = s_{\pm}(\omega) = \exp\{ik_{\pm}(\omega)\}, \quad (8.4)$$

Then statements of Theorem [1] imply the following representation for $k_{\pm}(\omega)$:

$$k_{\pm}(\omega) = \left\{ \begin{array}{ll}
\frac{-1 + \text{sign}\{bf(\omega)\}}{2} \pi + \omega + 2\pi m \pm i \ln \left[ \left( |bf(\omega)| + \sqrt{b^2f(\omega) - 1} \right) \right] & \text{if } b^2f > 1, \\
\frac{1 + \text{sign}\{bf(\omega)\}}{2} \pi + \omega + 2\pi m \pm \arccos(|bf(\omega)|) & \text{if } b^2f \leq 1
\end{array} \right., \quad m \in \mathbb{Z}, \quad (8.5)$$

where $0 < f < \pi$ and

$$bf(\omega) = K_0 \frac{\omega^2}{\omega^2 - \omega_0^2} \sin(f) - \cos(f), \quad K_0 = \frac{b^2\beta_0}{2} = \frac{b^2g_B}{c_0}, \quad g_B = \frac{\sigma_B}{4\lambda_{\text{rp}}}. \quad (8.6)$$

Requirement for $\Re\{k_{\pm}(\omega)\}$ to be in the first (main) Brillouin zone $(-\pi, \pi]$ effectively selects the band number $m$ that depend on $\omega$ as follows. For any given $\omega > 0$ and $0 < f < \pi$ the band number $m \in \mathbb{Z}$ is determined by the requirement to satisfy the following inequalities:

$$-\pi < \frac{-1 + \text{sign}\{bf(\omega)\}}{2} \pi + \omega + 2\pi m \leq \pi, \quad \text{if } b^2f(\omega) > 1$$

$$-\pi < \frac{1 + \text{sign}\{bf(\omega)\}}{2} \pi \pm \arccos(-bf(\omega)) + \omega + 2\pi m \leq \pi, \quad \text{if } b^2f(\omega) < 1. \quad (8.7)$$

The equations (8.5) for the complex-valued wave numbers $k_{\pm}(\omega)$ represent the dispersion relations of the MCK.

**Remark 7** (real part of the wave number). Note that according to expression (8.5) in Theorem 6 and relations (5.20) in Theorem 3 (see also Figure 5) we have

$$\Re\{k_{\pm}(\omega)\} = \pi + \omega + 2\pi m, \quad \omega_0 < \omega < \Omega_f^+. \quad (8.8)$$
Figures 13, 14 and 15 illustrate graphically equation (8.8) by perfect straight lines parallel to $\Re \{k\} = \omega$ in the shadowed area.

There is yet another form of the dispersion relation (8.2) and (8.3) which is the high-frequency form:

$$D_K(\omega, k) = D_K^{(0)}(\omega, k) + \frac{K_0 \omega_0^2}{\omega^2 - \omega_0^2} = 0,$$

$$D_K^{(0)}(\omega, k) = \cos(\omega - k) + b_f^\infty, \quad b_f^\infty = K_0 \sin(f) - \cos(f).$$

This form readily yields the following high-frequency approximation to the MCK dispersion relations

$$\cos(\omega - k) + b_f^\infty = 0, \quad b_f^\infty = K_0 \sin(f) - \cos(f), \quad |b_f^\infty| \leq 1,$$

or, equivalently

$$\omega = k \pm \arccos(-b_f^\infty) + 2\pi m, \quad b_f^\infty = K_0 \sin(f) - \cos(f), \quad |b_f^\infty| \leq 1, \quad m \in \mathbb{Z},$$

where inequality $|b_f^\infty| \leq 1$ is necessary and sufficient for the existence of real-valued $\omega$ and $k$ satisfying the dispersion relation.

8.1. **Plotting the dispersion relations.** The conventional dispersion relations are defined as the relations between real-valued frequency $\omega$ and real-valued wavenumber $k$ associated with the relevant eigenmodes. In the case of interest $k$ can be complex-valued and to represent all system modes geometrically we follow to [FigTW1b, 7]. First, we parametrize every mode of the system uniquely by the pair $(k(\omega), \omega)$ where $\omega$ is its frequency and $k(\omega)$ is its wavenumber. If $k(\omega)$ is degenerate, it is counted a number of times according to its multiplicity. In view of the importance to us of the mode instability, that is, when $\Im \{k(\omega)\} \neq 0$, we partition all the system modes represented by pairs $(k(\omega), \omega)$ into two distinct classes – oscillatory modes and unstable ones – based on whether the wavenumber $k(\omega)$ is real- or complex-valued with $\Im \{k(\omega)\} \neq 0$. We refer to a mode (eigenmode) of the system as an oscillatory mode if its wavenumber $k(\omega)$ is real-valued. We associate with such an oscillatory mode point $(k(\omega), \omega)$ in the $k\omega$-plane with $k$ being the horizontal axis and $\omega$ being the vertical one. Similarly, we refer to a mode (eigenmode) of the system as a (convective) unstable mode if its wavenumber $k(\omega)$ is complex-valued. We associate with such an unstable mode point $(\Re \{k(\omega)\}, \omega)$ in the $k\omega$-plane. Since we consider here only convective unstable modes, we refer to them shortly as unstable modes. Notice that every point $(\Re \{k(\omega)\}, \omega)$ is in fact associated with two complex conjugate system modes with $\pm \Im \{k(\omega)\}$.

Based on the above discussion, we represent the set of all oscillatory and unstable modes of the system geometrically by the set of the corresponding modal points $(k(\omega), \omega)$ and $(\Re \{k(\omega)\}, \omega)$ in the $k\omega$-plane. We name this set the dispersion-instability graph. To distinguish graphically points $(k(\omega), \omega)$ associated oscillatory modes when $k(\omega)$ is real-valued from points $(\Re \{k(\omega)\}, \omega)$ associated unstable modes when $k(\omega)$ is complex-valued with $\Im \{k(\omega)\} \neq 0$ we mark by a shadow the region occupied by points with $\Im \{k(\omega)\} \neq 0$. We remind once again that every point $(\omega, \Re \{k(\omega)\})$ with $\Im \{k(\omega)\} \neq 0$ represents exactly two complex conjugate unstable modes associated with $\pm \Im \{k(\omega)\}$.

Figures 13, 14 and 15 illustrate graphically the dispersion relations $k_\pm(\omega)$ described by equations (8.5).
Figure 13. The MCK dispersion-instability plots (solid brown curves and lines) over main Brillouin zone $[-\pi, \pi]$ for $K_0 = 3$, $\omega_0 = 1$ for which $f_{\text{cr}} \cong 0.6435011088$, $f_{\text{max}} \cong 1.892546882$: (a) $f = 0.5 < f_{\text{cr}} \cong 0.6435011088$; (b) $f = 0.62 < f_{\text{cr}} \cong 0.6435011088$. In all plots the horizontal and vertical axes represent respectively $\Re\{k\}$ and $\omega/\omega_0$. Two solid (green) diamond dots identify the values of $\Omega_f^-$ and $\Omega_f^+$ which are the frequency boundaries of the instability. Two solid (brown) disk dots identify points of the transition from the instability to the stability which are also EPD points. Two (brown) dash-dot lines $\omega = \Omega_f^\pm$ identify the frequency boundaries of the instability and the shaded (light blue) region between the lines identify points $(\Re\{k\}, \omega)$ of instability. Dashed (green) line $\omega = \omega_0$ identifies the resonance frequency $\omega_0$. Note the plots have jump-discontinuity along the dashed (green) line, namely $\Re\{k(\omega)\}$ jumps by $\pi$ according to equations (8.5) as the frequency $\omega$ passes through the resonance frequency $\omega_0$ and the sign of $b_f(\omega)$ changes. The shadowed area marks points $(\Re\{k\}, \omega)$ associated with the instability. The dashed (blue) straight lines lines correspond to the high frequency approximation defined by equations (8.12).

9. Exceptional points of degeneracy

The concept of an exceptional point of degeneracy (EPD), [Kato II.1], refers to a system evolution matrix degeneracy when not only some eigenvalues of the matrix coincide but the corresponding eigenvectors coincide also. An important class of applications of EPDs is sensing, [CheN], [PeLiXu], [Wie], [Wie1], [KNAC], [OGC]. In our prior work in FigSynbJ and FigPert we advanced and studied simple circuits exhibiting EPDs and their applications to sensing. Our studies of traveling wave tubes (TWT) in FigTWTbk [4, 7, 13, 14, 54, 55] demonstrate that TWTs always have EPDs. In FigTWTDEPD we developed applications of EPDs to sensing based on TWTs. For more applications of EPDs to TWTs see [OTC], [OVFC], [OVFC1], [VPFC].
Figure 14. The MCK dispersion-instability plot (solid brown curves and lines) over 3 Brillouin zones $[-\pi, \pi]$ for $K_0 = 3$, $\omega_0 = 1$ for which $f_{cr} \approx 0.6435011088$, $f_{max} \approx 1.892546882$ and $f = 0.5 < f_{cr} \approx 0.6435011088$. The horizontal and vertical axes represent respectively $\Re\{k\}$ and $\omega/\omega_0$. Two solid (green) diamond dots identify the values of $\Omega_f^−$ and $\Omega_f^+$ which are the frequency boundaries of the instability. Solid (brown) disk dots identify points of the transition from the instability to the stability which are also EPD points. Two (brown) dash-dot lines $\omega = \Omega_f^\pm$ identify the frequency boundaries of the instability and the shaded (light blue) region between the lines identify points $(\Re\{k\}, \omega)$ of instability. Dashed (green) line $\omega = \omega_0$ identifies the resonance frequency $\omega_0$. Note the plot has a jump-discontinuity along the dashed (green) line, namely $\Re\{k(\pm)\}$ jumps by $\pi$ according to equations (8.5) as the frequency $\omega$ passes through the resonance frequency $\omega_0$ and the sign of $b_f(\omega)$ changes. The shadowed area marks points $(\Re\{k\}, \omega)$ associated with the instability. The dashed (blue) straight lines lines correspond to the high frequency approximation defined by equations (8.12).

In this section we study EPDs in the MCK system using the properties of the MCK Floquet multipliers established in Sections 4 and 5. In particular, it follows from equation (4.7) that the degeneracy of the Floquet multipliers

$$s_{\pm} = e^{i\omega}S_{\pm}, \quad S_{\pm} = -b_f \pm \sqrt{b_f^2 - 1}, \quad b_f = b_f(\omega),$$

occurs if and only if $b_f(\omega) = \pm 1$ where $b_f(\omega)$ is defined by equations (8.6). In this case the degenerate Floquet multiplier is a single number for each of the values of $b_f$ which is

$$s = -e^{i\omega}b_f(\omega) = \mp e^{i\omega}, \quad b_f(\omega) = K_0\frac{\omega^2}{\omega^2 - \omega_0^2}\sin(f) - \cos(f) = \pm 1.$$
The Floquet multiplier $s$ in equation (9.2) is associated with monodromy matrix $\mathcal{T}$ defined by equations (3.48) that for $b_f = \pm 1$ takes the following form

$$
\mathcal{T}_\pm = e^{i\omega} \left[ \frac{\cos(f) - i\omega \sin(f)}{f} \frac{\sin(f)}{f} - \frac{\sin(f)}{f^2} \right], \quad b_f = \pm 1. \quad (9.3)
$$

Note that for the each value $b_f = 1$ and $b_f = -1$ each of the corresponding monodromy matrices $\mathcal{T}_+$ and $\mathcal{T}_-$ has a single and hence degenerate eigenvalue respectively $s = -e^{i\omega}$ and $s = e^{i\omega}$ and the index $\pm$ for matrix $\mathcal{T}$ corresponds to the sign of $b_f = \pm 1$.

Using elementary identity $\sin^2(f) = 1 - \cos^2(f)$ we can recast representation (9.3) as

$$
\mathcal{T}_\pm = e^{i\omega} \left[ \frac{\cos(f) - i\omega \sin(f)}{f} \frac{\sin(f)}{f^2} + \frac{\sin(f)}{f} \right], \quad b_f = \pm 1. \quad (9.4)
$$
The spectral analysis of the monodromy matrices $\mathcal{J}_\pm$ shows that their Jordan canonical forms $\mathcal{J}_\pm$ are
\[
\mathcal{J}_\pm = \begin{bmatrix} \mp e^{i\omega} & 1 \\ 0 & \mp e^{i\omega} \end{bmatrix}, \quad \mathcal{J}_\pm = \mathcal{J}_\pm \mathcal{J}_\pm^{-1},
\]
where matrices $\mathcal{J}_\pm$ are defined by
\[
\mathcal{J}_\pm = \begin{bmatrix} e^{i\omega} \left[ \cos (f) \pm 1 - i\omega \sin(f) \right] / \cos(f) & 1 \\ e^{i\omega} \left[ \sin(f) + 2i\omega \cos(f) \pm 1 \right] / \cos(f) & 0 \end{bmatrix}.
\]
Note that the first column $c_\pm$ of the corresponding matrix $\mathcal{J}_\pm$ is the single eigenvector of $\mathcal{J}_\pm$ and its second column $r_\pm$ is the relevant root vector of $\mathcal{J}_\pm$, that is
\[
c_\pm = \begin{bmatrix} e^{i\omega} \left[ \cos (f) \pm 1 - i\omega \sin(f) \right] / \cos(f) \\ e^{i\omega} \left[ \sin(f) + 2i\omega \cos(f) \pm 1 \right] / \cos(f) \end{bmatrix}, \quad r_\pm = \begin{bmatrix} 1 \\ 0 \end{bmatrix},
\]
\[
c_\pm = (\mathcal{J}_\pm \pm e^{i\pi}) r_\pm, \quad (\mathcal{J}_\pm \pm e^{i\pi}) c_\pm = 0.
\]

According to the above analysis all EPD points of the MCK can be found by solving equations $b_f(\omega) = \pm 1$ for $\omega$. Then based on Theorem 1, particularly equation (4.14), and Theorem 3, particularly relations (5.18)-(5.22), we obtain the following statement on EPDs of the MCK.

**Theorem 8** (EPD points, their frequencies and wavenumbers). If $0 < f < f_{cr}$ then there are exactly two EPD points with the corresponding frequencies $\Omega_f^\pm$ satisfying
\[
b_f(\Omega_f^\pm) = \pm 1; \quad \Omega_f^+ = \omega_0 \sqrt{-\frac{1}{1 - K_0 \tan(\frac{f}{2})}} > \omega_0, \quad \Omega_f^- = \omega_0 \sqrt{\frac{\tan(\frac{f}{2})}{\tan(\frac{f}{2}) + K_0}} < \omega_0.
\]

where $b_f(\omega)$ is defined by equations (8.6). If $f_{cr} \leq f < \pi$ then there is exactly one EPD point with the corresponding frequency $\Omega_f^-$ satisfying
\[
b_f(\Omega_f^-) = -1; \quad \Omega_f^- = \omega_0 \sqrt{\frac{\tan(\frac{f}{2})}{\tan(\frac{f}{2}) + K_0}} < \omega_0.
\]

The expressions of the corresponding wavenumbers $k(\omega)$ with $\omega = \Omega_f^\pm$ are provided by relations (8.5) and (8.6) in Theorem 6.

The monodromy matrix $\mathcal{J}$ and its Floquet multipliers $s$ at the EPD points satisfy equations (9.2), (9.3) and (9.4) with the corresponding Jordan form $\mathcal{J}$ of $\mathcal{J}$ satisfying equations (9.5) and (9.6). Note that matrix $\mathcal{J}$ is the Jordan block of dimension 2 as expected for EPD points.

We would like to derive asymptotic formulas for wave numbers $k_\pm(\omega)$ defined by equations (8.5) when frequency $\omega$ is in a vicinity of EPD frequencies $\Omega_f^\pm$. In order to do that we set $\omega = \Omega_f^\pm + \delta$ assuming that $\delta$ is small and introduce the power series expansion for $b_f(\Omega_f^\pm + \delta)$, that is
\[
b_f(\Omega_f^\pm + \delta) = \pm 1 + \sum_{n=1}^{\infty} b_n^\pm \delta^n, \quad \delta = \omega - \Omega_f^\pm \to 0, \quad b_n^\pm = \partial_\omega [b_f(\omega)]_{\omega=\Omega_f^\pm}, \quad n = 1, 2, \ldots
\]

Using expression (8.6) for $b_f(\omega)$ and expressions (9.9) for $\Omega_f^\pm$ we obtain the following representations for $b_1^\pm$:
\[
b_1^+ = -\frac{2\omega_0^2 \sin(f)}{K_0 \tan^2(\frac{f}{2}) (\Omega_f^+)^3} < 0, \quad \Omega_f^+ = \omega_0 \sqrt{-\frac{1}{1 - K_0 \tan(\frac{f}{2})}} > \omega_0, \quad 0 < f < f_{cr},
\]
Figure 16. The dispersion-instability plot (solid brown curves and lines) for the MCK with $K_0 = 1$, $\omega_0 = 1$, $f = 1.3$ for which $f_{cr} \cong 1.57096327$, $f_{\text{max}} \cong 2.356194491$. The horizontal and vertical axes represent respectively $\Re\{k\}$ and $\omega/\omega_0$. Two solid (green) diamond dots identify the values of $\Omega^{-} - f$ and $\Omega^{+} + f$ which are the frequency boundaries of the instability. Solid (brown) disk dots identify points of the transition from the instability to the stability which are also EPD points. Two (brown) dash-dot lines $\omega = \Omega^{\pm}$ identify the frequency boundaries of the instability and the shaded (light blue) region between the lines identify points $(\Re\{k\},\omega)$ of instability. Dashed (green) line $\omega = \omega_0$ identifies the resonance frequency $\omega_0$. The two dashed (black) curves represent the approximations to the dispersion relations described by equations (9.17) and (9.19). Note the plot has a jump-discontinuity along the dashed (green) line, namely $\Re\{k^{\pm}(\omega)\}$ jumps by $\pi$ according to equations (8.5) as the frequency $\omega$ passes through the resonance frequency $\omega_0$ and the sign of $b_f(\omega)$ changes.

$\frac{b_1}{K_0 (\Omega^{-})^3} < 0, \quad \Omega^{-} = \omega_0 \sqrt{\frac{\tan(\frac{\pi}{2})}{\tan(\frac{\pi}{2}) + K_0}} < \omega_0 < 0, \quad 0 < f < \pi. \quad (9.13)$

Then based on equation (9.2) for $S$, that is $S = -b_f \pm \sqrt{b_f^2 - 1}$, and relations (9.11)-(9.13) we obtain

$S (\Omega^{\pm} + \delta) = \mp 1 + \sqrt{\pm 2b_1^\mp \delta + b_1^\pm \delta - \left[\left(b_1^\mp 2 \pm 2b_2^\pm\right)\right]^{\frac{3}{2}} + O (\delta^2)}, \quad \delta = \omega - \Omega_f^{\pm} \to 0. \quad (9.14)$

Using equations (8.5) for $k^{\pm}(\omega)$ in the case of the primary Brillouin zone with $m = 0$ we get

$k_{\text{sign}(\sigma)} (\Omega_f^{\pm}) = \Omega_f^{\pm} - \frac{\mp 1}{2} \pi, \quad \sigma = \pm 1, \quad 0 < f < \pi, \quad \sigma = \pm 1, \quad (9.15)$

$k_{\text{sign}(\sigma)} (\Omega_f^{\pm} + \delta) = \Omega_f^{\pm} - \frac{1}{2} \pi \mp i \sigma \sqrt{\pm 2b_1^\pm \delta + \delta - O (|\delta|^2)}, \quad \delta = \omega - \Omega_f^{\pm} \to 0. \quad (9.16)$
As to the real and imaginary parts of $k_{\text{sign}}(\sigma) \left( \Omega_f^\pm + \delta \right)$ equations (8.5) and (9.16) imply

\[
\Re \left\{ k_{\text{sign}}(\sigma) \left( \Omega_f^+ + \delta \right) \right\} = \begin{cases} 
\Omega_f^+-\pi+\sigma\sqrt{2|b_\perp^+|}\sqrt{\delta}+\delta+O\left(\delta^{3/2}\right) & \text{if } \delta > 0, \\
\Omega_f^+-\pi-\delta & \text{if } \delta < 0,
\end{cases}
\]
\quad (9.17)

\[
\Im \left\{ k_{\text{sign}}(\sigma) \left( \Omega_f^+ + \delta \right) \right\} = \begin{cases} 
\sigma\sqrt{2|b_\perp^+|}\sqrt{-\delta}+O\left(\delta^{3/2}\right) & \text{if } \delta < 0, \\
0 & \text{if } \delta > 0
\end{cases}
\]
\quad (9.18)

\[
\Re \left\{ k_{\text{sign}}(\sigma) \left( \Omega_f^- + \delta \right) \right\} = \begin{cases} 
\Omega_f^-+\sigma\sqrt{2|b_\perp^-|}\sqrt{-\delta}+\delta+O\left(\delta^{3/2}\right) & \text{if } \delta < 0, \\
\Omega_f^-+\delta & \text{if } \delta > 0
\end{cases}
\]
\quad (9.19)

\[
\Im \left\{ k_{\text{sign}}(\sigma) \left( \Omega_f^- + \delta \right) \right\} = \begin{cases} 
\sigma\sqrt{2|b_\perp^-|}\sqrt{\delta}+O\left(\delta^{3/2}\right) & \text{if } \delta > 0, \\
0 & \text{if } \delta < 0
\end{cases}
\]
\quad (9.20)

Figure 17. Plot of $|\Im \{k_{\pm}(\omega)\}|$ as a function of frequency $\omega$ for $K_0 = 1$, $\omega_0 = 1$ and $f = 1.15 < f_{cr} \approx 1.176005207$. The horizontal and vertical axes represent respectively frequency $\omega$ and $\Im \{k\}$. The solid (brown) curves represent function $|\Im \{k_{\pm}(\omega)\}|$. The diamond solid (green) dots mark the values of $\Omega_f^-$ and $\Omega_f^+$ which are the frequencies of the MCK EPDs and also are frequency boundaries of the instability. The two dashed (black) curves represent the approximations of $|\Im \{k_{\pm}(\omega)\}|$ described by equations (9.18) and (9.20).

10. **LAGRANGIAN VARIATIONAL FRAMEWORK**

We construct here the Lagrangian variational framework for our model of the MCK. According to Assumption [1] the model integrates into it quantities associated with continuum of real numbers on one hand and features associated with discrete points on the another hand. The continuum features are represented by Lagrangian densities $\mathcal{L}_B$ in equations (3.9) whereas discrete features are represented by Lagrangian $\mathcal{L}_{CB}$ in equations (3.10) with energies concentrated in a set of discrete points $aZ$. One possibility for constructing the desired Lagrangian variational framework is to apply the general approach developed in [FigRey2] when the “rigidity” condition holds. Another possibility is to directly construct the Lagrangian variational framework using some ideas from [FigRey2] and that is what we actually pursue here.
Following to the standard procedures of the Least Action principle [ArnMech II.3], [GantM 3], [GelFom 7], [GoldM 8.6] we start with setting up the action integral \( S \) based on the Lagrangian \( \mathcal{L} \) defined by equations (3.8), (3.9) and (3.10). Using notations (3.6) and (3.7) we define the action integral \( S \) as follows:

\[
S \{ \{ x \} \} = \int_{t_0}^{t_1} dt \int_{z_1}^{z_2} \mathcal{L} \{ \{ x \} \} dz = S_B \{ \{ q \} \} + S_{CB} \{ x \}, \quad t_0 < t_1, \quad z_0 < z_1,
\]

where

\[
S_B \{ \{ q \} \} = \int_{t_0}^{t_1} dt \int_{z_1}^{z_2} \mathcal{L}_B \{ \{ q \} \} dz = \int_{t_0}^{t_1} dt \int_{z_1}^{z_2} \left[ \frac{1}{2\beta} (\partial_t q + \tilde{v}\partial_z q) - \frac{2\pi}{\sigma_B} q^2 \right] dz,
\]

\[
S_{CB} \{ \{ Q \} \} = \int_{t_0}^{t_1} dt \int_{z_1}^{z_2} \mathcal{L}_{CB} \{ Q, q \} dz = \sum_{\xi < a \ell < \delta z} \int_{z_1}^{z_2} dt \left[ \frac{l_0}{2} (\partial_t (a\ell))^2 - \frac{1}{2c_0} (Q (a\ell) + bq (a\ell)) \right] dz.
\]

To make expressions of the action integrals less cluttered we suppress notationally their dependence on intervals \((z_0, z_1)\) and \((t_0, t_1)\) that can be chosen arbitrarily. We consider then variation \( \delta S \) of action \( S \) assuming that variation \( \delta q \) of charge \( q = q(z, t) \) vanishes outside intervals \((z_0, z_1)\) and \((t_0, t_1)\), that is

\[
\delta q (z, t) = 0, \quad (z, t) \notin (z_0, z_1) \times (t_0, t_1),
\]

implying, in particular, that \( \delta q \) vanishes on the boundary of the rectangle \((z_0, z_1) \times (t_0, t_1)\), that is

\[
\delta q (z, t) = 0, \quad \text{if } z = z_0, z_1 \text{ or if } t = t_0, t_1.
\]

We refer to variations \( \delta Q \) and \( \delta q \) satisfying equations (10.4) and hence (10.4) for a rectangle \((z_0, z_1) \times (t_0, t_1)\) as admissible.

Following to the least action principle we introduce the functional differential \( \delta S \) of the action by the following formula [GelFom 7(35)]

\[
\delta S = \lim_{\varepsilon \to 0} \frac{S \{ \{ x + \varepsilon \delta x \} \} - S \{ \{ x \} \}}{\varepsilon}.
\]

Then the system configurations \( x = x(z, t) \) that actually can occur must satisfy

\[
\delta S = \lim_{\varepsilon \to 0} \frac{S \{ \{ x + \varepsilon \delta x \} \} - S \{ \{ x \} \}}{\varepsilon} = 0 \quad \text{for all admissible variations.}
\]

Let us choose now any \( z \) outside lattice \( a\mathbb{Z} \). Then there always exist a sufficiently small \( \varepsilon > 0 \) and an integer \( \ell_0 \) such that

\[
a\ell_0 < z_0 = z - \xi < z < z_1 = z + \xi < a(\ell_0 + 1).
\]

If we apply now the variational principle (10.7) for all admissible variations \( \delta Q \) and \( \delta q \) such that space interval \((z_0, z_1)\) is compliant with inequalities (10.8) we readily find that

\[
\delta S = \delta S_B = 0,
\]

where \( S_B \) is defined by expression (10.2). Using equations (10.5) and carrying out in the standard way the integration by parts transformations we arrive at

\[
\delta S_B = - \int_{t_0}^{t_1} dt \int_{z_1}^{z_2} \left[ \frac{1}{\beta} (\partial_t + \tilde{v}\partial_z) q + \frac{4\pi}{\sigma_B} q \right] \delta q dz.
\]

Combining equations (10.9) and (10.10) we arrive at the following EL equations
\[
\frac{1}{\beta} (\partial_t + \hat{v} \partial_z)^2 q + \frac{4\pi}{\sigma_B} q = 0, \quad z \neq a\ell, \quad \ell \in \mathbb{Z}.
\] (10.11)

Consider now the case when \( z = a\ell_0 \) for an integer \( \ell_0 \) and select space interval \((z_0, z_1)\) as follows
\[
a (\ell_0 - 1) < z_0 = a \left( \ell_0 - \frac{1}{2} \right) < z = a\ell_0 < z_1 = a \left( \ell_0 + \frac{1}{2} \right) < a (\ell_0 + 1).
\] (10.12)

Notice that in this case both actions \( S_B \) and \( S_{CB} \) contribute to the variation \( \delta S \). In particular, as consequence of the presence of delta functions \( \delta (z - a\ell) \) in the expression of the Lagrangian \( \mathcal{L}_{CB} \) defined by equation (3.10), the space derivatives \( \partial_z q \) can have jumps at \( z = a\ell_0 \) as it was already acknowledged by Assumption [2]. Based on this circumstance we proceed as follows: (i) we split the integral with respect to the space variable \( z \) into two integrals:
\[
\int_{z_0}^{z_1} = \int_{a\ell_0}^{a(\ell_0 - \frac{1}{2})} + \int_{a\ell_0}^{a(\ell_0 + \frac{1}{2})};
\] (10.13)

(ii) we carry out the integration by parts for each of the two integrals in the right-hand side of equation (10.13); (iii) we use already established EL equations (10.11) to simplify the integral expressions. When that is all done we arrive at the following:
\[
\delta S_B = - \int_{t_0}^{t_1} \frac{\bar{\psi}^2}{\beta} [\partial_z q] (a\ell_0, t) \delta q (a\ell_0, t) \, dt,
\] (10.14)

where jumps \([\partial_z q] (a\ell)\) are defined by equation (3.4), and
\[
\delta S_{CB} = - \int_{t_0}^{t_1} \left\{ \frac{l_0 \partial_t^2 Q (a\ell) + 1}{c_0} [Q (a\ell_0, t) + bq (a\ell_0, t)] \right\} \delta Q (a\ell_0, t) \, dt - \int_{t_0}^{t_1} \left\{ \frac{b}{c_0} [Q (a\ell_0, t) + bq (a\ell_0, t)] \right\} \delta q (a\ell_0, t) \, dt.
\] (10.15)

Using the variational principle (10.7), that is
\[
\delta S_B + \delta S_{CB} = 0,
\] (10.16)

and the fact that variations \( \delta Q \) and \( \delta q \) can be chosen arbitrarily we arrive at the following
\[
l_0 \partial_t^2 Q (a\ell) + \frac{1}{c_0} [Q (a\ell_0, t) + bq (a\ell_0, t)] = 0,
\] (10.17)

\[
\frac{\bar{\psi}^2}{\beta} [\partial_z q] (a\ell_0, t) = - \frac{b}{c_0} [Q (a\ell_0) + bq (a\ell_0, t)],
\]

where jumps \([\partial_z q] (a\ell)\) are defined by equation (3.4). We remind also that as consequence of continuity of \( q \) we also have
\[
[q] (a\ell_0, t) = 0.
\] (10.18)

Hence equations (10.17) and (10.18) can be viewed as the EL equations at point \( a\ell_0 \).

Equations (10.17) at interaction point \( a\ell_0 \) are perfectly consistent with boundary conditions (2.12) of the general treatment in [Fig,Rev2], which are
\[
- \frac{\partial L_D}{\partial \partial_t \psi^0_D} (b_1, t) + \frac{\partial L_B}{\partial \psi^0_D (b_1, t)} - \partial_0 \left( \frac{\partial L_B}{\partial \partial_0 \psi^0_D (b_1, t)} \right) = 0, \quad \partial_0 = \partial_t, \quad \partial_1 = \partial_z; \quad (10.19)
\]

\[
\frac{\partial L_D}{\partial \partial_t \psi^0_D} (b_2, t) + \frac{\partial L_B}{\partial \psi^0_D (b_2, t)} - \partial_0 \left( \frac{\partial L_B}{\partial \partial_0 \psi^0_D (b_2, t)} \right) = 0.
\]
where (i) $b_1 = a\ell_0 - 0$ and $b_2 = a\ell_0 + 0$; (ii) $L_D$ corresponds to $L_B + L_{CB}$; (iii) $L_B$ corresponds to $L_{CB}$; (iv) fields $\psi^D_\ell$ correspond to charges $Q$ and $q$; (v) boundary fields $\psi^B_\ell$ correspond to $Q (a\ell_0, t)$ and $q (a\ell_0, t)$. We remind the reader that boundary conditions (2.12) in [FigRey2] is an implementation of the “rigidity” requirement which is appropriate for Lagrangian $L_{CB}$ defined by equation (3.10). If fact, the signs of the terms containing $L_D$ in equations (10.19) are altered compare to original equations (2.12) in [FigRey2] to correct an unfortunate typo there.

Thus equations (10.11), (10.17) and (10.18) form a complete set of the EL equations.

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**NOMENCLATURE:**

- EL stands for the Euler-Lagrange (equations)
- HF stands for high-frequency
- MCK stands for multi-cavity klystron
- $\mathbb{C}$ is a set of complex number.
- $\bar{s}$ is complex-conjugate to complex number $s$
- $\mathbb{C}^n$ is a set of $n$ dimensional column vectors with complex complex-valued entries.
- $\mathbb{C}^{n \times m}$ is a set of $n \times m$ matrices with complex-valued entries.
- $\mathbb{R}^{n \times m}$ is a set of $n \times m$ matrices with real-valued entries.
- $\text{diag} (A_1, A_2, \ldots, A_r)$ is block diagonal matrix with indicated blocks.
- $\text{dim} (W)$ is the dimension of the vector space $W$.
- $\ker (A)$ is the kernel of matrix $A$, that is the vector space of vector $x$ such that $Ax = 0$.
- $\det \{A\}$ is the determinant of matrix $A$.
- $\sigma \{A\}$ is the spectrum of matrix $A$.
- $\chi_A (s) = \det \{sI_\nu - A\}$ is the characteristic polynomial of a $\nu \times \nu$ matrix $A$.
- $I_\nu$ is $\nu \times \nu$ identity matrix.
- $M^T$ is a matrix transposed to matrix $M$.
- EL stands for the Euler-Lagrange (equations).
- ODE stands for ordinary differential equations.

**Appendix A. Fourier Transform**

Our preferred form of the Fourier transforms as in [Foll 7.2, 7.5], [ArfWeb 20.2]:

$$f (t) = \int_{-\infty}^{\infty} \hat{f} (\omega) e^{-i\omega t} \, d\omega, \quad \hat{f} (\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f (t) e^{i\omega t} \, dt,$$  \hspace{1cm} (A.1)

$$f (z, t) = \int_{-\infty}^{\infty} \hat{f} (k, \omega) e^{-i(kz - \omega t)} \, dk \, d\omega, \quad \hat{f} (k, \omega) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} f (z, t) e^{i(kz - \omega t)} \, dz \, dt.$$  \hspace{1cm} (A.2)

This preference was motivated by the fact that the so-defined Fourier transform of the convolution of two functions has its simplest form. Namely, the convolution $f \ast g$ of two functions $f$ and $g$ is
defined by \cite{Foll,7.2,7.5},
\[
[f \ast g](t) = [g \ast f](t) = \int_{-\infty}^{\infty} f(t-t')g(t') \, dt',
\]
(A.3)
\[
[f \ast g](z,t) = [g \ast f](z,t) = \int_{-\infty}^{\infty} f(z-z',t-t')g(z',t') \, dz' \, dt'.
\]
(A.4)

Then its Fourier transform as defined by equations (A.1) and (A.2) satisfies the following properties:
\[
\hat{f} \ast \hat{g} (\omega) = \hat{f}(\omega) \hat{g}(\omega),
\]
(A.5)
\[
\hat{f} \ast \hat{g}(k,\omega) = \hat{f}(k,\omega) \hat{g}(k,\omega).
\]
(A.6)

**Appendix B. Jordan canonical form**

We provide here very concise review of Jordan canonical forms following mostly to \cite{Hale,III.4}, \cite{HorJohn,3.1,3.2}. As to a demonstration of how Jordan block arises in the case of a single $n$-th order differential equation we refer to \cite{ArnODE,25.4}.

Let $A$ be an $n \times n$ matrix and $\lambda$ be its eigenvalue, and let $r(\lambda)$ be the least integer $k$ such that $N[(A - \lambda I)^k] = N[(A - \lambda I)^{k+1}]$, where $N[C]$ is a null space of a matrix $C$. Then we refer to $M_\lambda = N[(A - \lambda I)^{\rho(\lambda)}]$ is the *generalized eigenspace* of matrix $A$ corresponding to eigenvalue $\lambda$. Then the following statements hold, \cite{Hale,III.4}.

**Proposition 9 (generalized eigenspaces).** Let $A$ be an $n \times n$ matrix and $\lambda_1, \ldots, \lambda_p$ be its distinct eigenvalues. Then generalized eigenspaces $M_{\lambda_1}, \ldots, M_{\lambda_p}$ are linearly independent, invariant under the matrix $A$ and
\[
\mathbb{C}^n = M_{\lambda_1} \oplus \ldots \oplus M_{\lambda_p}.
\]
(B.1)

Consequently, any vector $x_0$ in $\mathbb{C}^n$ can be represented uniquely as
\[
x_0 = \sum_{j=1}^{p} x_{0,j}, \quad x_{0,j} \in M_{\lambda_j},
\]
(B.2)

and
\[
\exp \{At\} x_0 = \sum_{j=1}^{p} e^{\lambda_j t} p_j(t),
\]
(B.3)

where column-vector polynomials $p_j(t)$ satisfy
\[
p_j(t) = \sum_{k=0}^{r(\lambda_j)-1} \frac{(A - \lambda_j I)^k t^k}{k!} x_{0,j}, \quad x_{0,j} \in M_{\lambda_j}, \quad 1 \leq j \leq p.
\]
(B.4)

For a complex number $\lambda$ a Jordan block $J_r(\lambda)$ of size $r \geq 1$ is a $r \times r$ upper triangular matrix of the form
\[
J_r(\lambda) = \lambda I_r + K_r = \begin{bmatrix}
\lambda & 1 & \cdots & 0 & 0 \\
0 & \lambda & 1 & \cdots & 0 \\
0 & 0 & \ddots & \ddots & \ddots \\
\vdots & \vdots & \ddots & \ddots & \lambda \\
0 & 0 & \cdots & 0 & \lambda
\end{bmatrix}, \quad J_1(\lambda) = [\lambda], \quad J_2(\lambda) = \begin{bmatrix}
\lambda & 1 \\
0 & \lambda
\end{bmatrix},
\]
(B.5)
The special Jordan block \( K_r = J_r(0) \) defined by equation (B.6) is a nilpotent matrix that satisfies the following identities:

\[
K_r^2 = \begin{bmatrix}
0 & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 0 & 0
\end{bmatrix}, \quad K_r^{r-1} = \begin{bmatrix}
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & \vdots & 0 \\
\vdots & \ddots & \ddots & \vdots & 0 \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0
\end{bmatrix}, \quad K_r^r = 0.
\] (B.7)

A general Jordan \( n \times n \) matrix \( J \) is defined as a direct sum of Jordan blocks, that is

\[
J = \begin{bmatrix}
J_{n_1}(\lambda_1) & 0 & \cdots & 0 & 0 \\
0 & J_{n_2}(\lambda_2) & \cdots & 0 & 0 \\
0 & 0 & \ddots & \cdots & \vdots \\
\vdots & \vdots & \ddots & J_{n_{q-1}}(\lambda_{n_{q-1}}) & 0 \\
0 & 0 & \cdots & 0 & J_{n_q}(\lambda_q)
\end{bmatrix}, \quad n_1 + n_2 + \cdots + n_q = n, \quad (B.8)
\]

where \( \lambda_j \) need not be distinct. Any square matrix \( A \) is similar to a Jordan matrix as in equation (B.8) which is called the Jordan canonical form of \( A \). Namely, the following statement holds, [HorJohn, 3.1].

**Proposition 10** (Jordan canonical form). Let \( A \) be an \( n \times n \) matrix. Then there exists a non-singular \( n \times n \) matrix \( Q \) such that the following block-diagonal representation holds

\[
Q^{-1}AQ = J
\] (B.9)

where \( J \) is the Jordan matrix defined by equation (B.8) and \( \lambda_j, 1 \leq j \leq q \) are not necessarily different eigenvalues of matrix \( A \). Representation (B.9) is known as the Jordan canonical form of matrix \( A \), and matrices \( J_j \) are called Jordan blocks. The columns of the \( n \times n \) matrix \( Q \) constitute the Jordan basis providing for the Jordan canonical form (B.9) of matrix \( A \).

A function \( f(J_r(s)) \) of a Jordan block \( J_r(s) \) is represented by the following equation [MeyCD 7.9, BernM 10.5]

\[
f(J_r(s)) = \begin{bmatrix}
f(s) & \partial f(s) & \frac{\partial^2 f(s)}{2} & \cdots & \frac{\partial^{r-1} f(s)}{(r-1)!}
0 & f(s) & \partial f(s) & \cdots & \frac{\partial^{r-2} f(s)}{(r-2)!}
\vdots & \vdots & \ddots & \ddots & \vdots 
0 & 0 & \cdots & f(s) & \partial f(s)
0 & 0 & \cdots & 0 & f(s)
\end{bmatrix}.
\] (B.10)

Notice that any function \( f(J_r(s)) \) of the Jordan block \( J_r(s) \) is evidently an upper triangular Toeplitz matrix.
There are two particular cases of formula (B.10) which can be also derived straightforwardly using equations (B.7):

\[ \exp \{ K_r t \} = \sum_{k=0}^{r-1} \frac{t^k}{k!} K_r^k = \begin{bmatrix} 1 & \frac{t}{2!} & \cdots & \frac{t^{r-1}}{(r-1)!} \\ 0 & 1 & \cdots & \frac{t^{r-2}}{(r-2)!} \\ 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \] (B.11)

\[ [J_r (s)]^{-1} = \sum_{k=0}^{r-1} s^{-k-1} (-K_r)^k = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} & \frac{1}{s^3} & \cdots & \frac{(-1)^{r-1}}{s^{r-1}} \\ 0 & \frac{1}{s} & \frac{1}{s^2} & \cdots & \frac{(-1)^{r-2}}{s^{r-2}} \\ 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix}. \] (B.12)

**Appendix C. Companion matrix and cyclicity condition**

The companion matrix \( C(a) \) for monic polynomial

\[ a(s) = s^\nu + \sum_{1 \leq k \leq \nu} a_{\nu-k} s^{\nu-k} \] (C.1)

where coefficients \( a_k \) are complex numbers is defined by [BernM 5.2]

\[ C(a) = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ a_0 & a_1 & \cdots & a_{\nu-2} & a_{\nu-1} \end{bmatrix}. \] (C.2)

Notice that

\[ \det \{ C(a) \} = (-1)^\nu a_0. \] (C.3)

An eigenvalue is called cyclic (nonderogatory) if its geometric multiplicity is 1. A square matrix is called cyclic (nonderogatory) if all its eigenvalues are cyclic [BernM 5.5]. The following statement provides different equivalent descriptions of a cyclic matrix [BernM 5.5].

**Proposition 11** (criteria for a matrix to be cyclic). Let \( A \in \mathbb{C}^{n \times n} \) be \( n \times n \) matrix with complex-valued entries. Let \( \text{spec} (A) = \{ \zeta_1, \zeta_2, \ldots, \zeta_r \} \) be the set of all distinct eigenvalues and \( k_j = \text{ind}_A (\zeta_j) \) is the largest size of Jordan block associated with \( \zeta_j \). Then the minimal polynomial \( \mu_A (s) \) of the matrix \( A \), that is a monic polynomial of the smallest degree such that \( \mu_A (A) = 0 \), satisfies

\[ \mu_A (s) = \prod_{j=1}^r (s - \zeta_j)^{k_j}. \] (C.4)

Furthermore, and following statements are equivalent:

(i) \( \mu_A (s) = \chi_A (s) = \det \{ sI - A \} \).
(ii) \( A \) is cyclic.
(iii) For every \( \zeta_j \) the Jordan form of \( A \) contains exactly one block associated with \( \zeta_j \).
(iv) \( A \) is similar to the companion matrix \( C(\chi_A) \).
Proposition 12 (companion matrix factorization). Let \( a(s) \) be a monic polynomial having degree \( \nu \) and \( C(a) \) is its \( \nu \times \nu \) companion matrix. Then, there exist unimodular \( \nu \times \nu \) matrices \( S_1(s) \) and \( S_2(s) \), that is \( \det \{ S_m \} = \pm 1, m = 1, 2, \) such that
\[
sI_\nu - C(a) = S_1(s) \begin{bmatrix} I_{\nu-1} & 0_{(\nu-1) \times 1} \\ 0_{1 \times (\nu-1)} & a(s) \end{bmatrix} S_2(s).
\] (C.5)

Consequently, \( C(a) \) is cyclic and
\[
\chi_{C(a)}(s) = \mu_{C(a)}(s) = a(s).
\] (C.6)

The following statement summarizes important information on the Jordan form of the companion matrix and the generalized Vandermonde matrix, [BernM, 5.16], [LanTsi, 2.11], [MeyCD, 7.9].

Proposition 13 (Jordan form of the companion matrix). Let \( C(a) \) be an \( n \times n \) a companion matrix of the monic polynomial \( a(s) \) defined by equation (C.1). Suppose that the set of distinct roots of polynomial \( a(s) \) is \( \{ \zeta_1, \zeta_2, \ldots, \zeta_r \} \) and \( \{ n_1, n_2, \ldots, n_r \} \) is the corresponding set of the root multiplicities such that
\[
n_1 + n_2 + \cdots + n_r = n.
\] (C.7)

Then
\[
C(a) = RJR^{-1},
\] (C.8)

where
\[
J = \text{diag} \{ J_{n_1}(\zeta_1), J_{n_2}(\zeta_2), \ldots, J_{n_r}(\zeta_r) \}
\] (C.9)
is the the Jordan form of companion matrix \( C(a) \) and \( n \times n \) matrix \( R \) is the so-called generalized Vandermonde matrix defined by
\[
R = [R_1 | R_2 | \cdots | R_r],
\] (C.10)

where \( R_j \) is \( n \times n_j \) matrix of the form
\[
R_j = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
\zeta_j & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\zeta_j^{n-2} & \zeta_j^{n-3} & \cdots & (n-2)_{j}^{\frac{1}{n_j-1} \zeta_j^{n-1-j}} \\
\zeta_j^{n-1} & \zeta_j^{n-2} & \cdots & (n-1)_{j}^{\frac{1}{n_j-1} \zeta_j^{n-2}}
\end{bmatrix}.
\] (C.11)

As a consequence of representation (C.9) \( C(a) \) is a cyclic matrix.

As to the structure of matrix \( R_j \) in equation (C.11), if we denote by \( Y(\zeta_j) \) its first column then it can be expressed as follows [LanTsi, 2.11]:
\[
R_j = [Y^{(0)} | Y^{(1)} | \cdots | Y^{(n_j-1)}], \quad Y^{(m)} = \frac{1}{m!} \partial_s^m Y(\zeta_j), \quad 0 \leq m \leq n_j - 1.
\] (C.12)

In the case when all eigenvalues of a cyclic matrix are distinct then the generalized Vandermonde matrix turns into the standard Vandermonde matrix
\[
V = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
\zeta_1 & \zeta_2 & \cdots & \zeta_n \\
\vdots & \vdots & \ddots & \vdots \\
\zeta_1^{n-2} & \zeta_2^{n-2} & \cdots & \zeta_n^{n-2} \\
\zeta_1^{n-1} & \zeta_2^{n-1} & \cdots & \zeta_n^{n-1}
\end{bmatrix}.
\] (C.13)
Appendix D. Matrix polynomials

An important incentive for considering matrix polynomials is that they are relevant to the spectral theory of the differential equations of the order higher than 1, particularly the Euler-Lagrange equations which are the second-order differential equations in time. We provide here selected elements of the theory of matrix polynomials following mostly to [GoLaRo II.7, II.8], [Baum 9]. General matrix polynomial eigenvalue problem reads

\[ A(s)x = 0, \quad A(s) = \sum_{j=0}^{\nu} A_j s^j, \quad x \neq 0, \quad (D.1) \]

where \( s \) is complex number, \( A_k \) are constant \( m \times m \) matrices and \( x \in \mathbb{C}^m \) is \( m \)-dimensional column-vector. We refer to problem (D.1) of finding complex-valued \( s \) and non-zero vector \( x \in \mathbb{C}^m \) as polynomial eigenvalue problem.

If a pair of a complex \( s \) and non-zero vector \( x \) solves problem (D.1) we refer to \( s \) as an eigenvalue or as a characteristic value and to \( x \) as the corresponding to \( s \) eigenvector. Evidently the characteristic values of problem (D.1) can be found from polynomial characteristic equation

\[ \det \{ A(s) \} = 0. \quad (D.2) \]

We refer to matrix polynomial \( A(s) \) as regular if \( \det \{ A(s) \} \) is not identically zero. We denote by \( m(s_0) \) the multiplicity (called also algebraic multiplicity) of eigenvalue \( s_0 \) as a root of polynomial \( \det \{ A(s) \} \). In contrast, the geometric multiplicity of eigenvalue \( s_0 \) is defined as \( \dim \{ \ker \{ A(s_0) \} \} \), where \( \ker \{ A \} \) defined for any square matrix \( A \) stands for the subspace of solutions \( x \) to equation \( Ax = 0 \). Evidently, the geometric multiplicity of eigenvalue does not exceed its algebraic one, see Corollary 16.

It turns out that the matrix polynomial eigenvalue problem (D.1) can be always recast as the standard “linear” eigenvalue problem, namely

\[ (sB - A)x = 0, \quad (D.3) \]

where \( m\nu \times m\nu \) matrices \( A \) and \( B \) are defined by

\[
B = \begin{bmatrix}
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 0 & A_\nu \\
\end{bmatrix}, \quad A = \begin{bmatrix}
0 & 1 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-A_0 & -A_1 & \cdots & -A_{\nu-2} & -A_{\nu-1} \\
\end{bmatrix}, \quad (D.4)
\]

with \( I \) being \( m \times m \) identity matrix. Matrix \( A \), particularly in monic case, is often referred to as companion matrix. In the case of monic polynomial \( A(\lambda) \), when \( A_\nu = I \) is \( m \times m \) identity matrix, matrix \( B = I \) is \( m\nu \times m\nu \) identity matrix. The reduction of original polynomial problem (D.1) to an equivalent linear problem (D.3) is called linearization.

The linearization is not unique, and one way to accomplish is by introducing the so-called known “companion polynomial” which is \( m\nu \times m\nu \) matrix

\[
C_A(s) = sB - A = \begin{bmatrix}
\lambda I & -\lambda I & \cdots & 0 & 0 \\
0 & \lambda I & -\lambda I & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & A_\nu \\
\end{bmatrix}, \quad (D.5)
\]
Notice that in the case of the EL equations the linearization can be accomplished by the relevant Hamilton equations.

To demonstrate the equivalency between the eigenvalue problems for \( m \nu \times m \nu \) companion polynomial \( C_A(s) \) and the original \( m \times m \) matrix polynomial \( A(s) \) we introduce two \( m \nu \times m \nu \) matrix polynomials \( E(s) \) and \( F(s) \). Namely,

\[
E(s) = \begin{bmatrix}
E_1(s) & E_2(s) & \cdots & E_{\nu-1}(s) & \mathbb{I}

-\mathbb{I} & 0 & 0 & \cdots & 0

0 & -\mathbb{I} & \ddots & \cdots & : \\
\vdots & \vdots & \ddots & 0 & 0 \\
0 & 0 & \cdots & -\mathbb{I} & 0
\end{bmatrix},
\]

\[
\det \{E(s)\} = 1,
\]

where \( m \times m \) matrix polynomials \( E_j(s) \) are defined by the following recursive formulas

\[
E_{\nu}(s) = A_{\nu}, \quad E_{j-1}(s) = A_{j-1} + sE_j(s), \quad j = \nu, \ldots, 2.
\]

Matrix polynomial \( F(s) \) is defined by

\[
F(s) = \begin{bmatrix}
\mathbb{I} & 0 & \cdots & 0 & 0 \\
-s\mathbb{I} & \mathbb{I} & 0 & \cdots & 0 \\
0 & -s\mathbb{I} & \ddots & \cdots & : \\
\vdots & \vdots & \ddots & \mathbb{I} & 0 \\
0 & 0 & \cdots & -s\mathbb{I} & \mathbb{I}
\end{bmatrix}, \quad \det \{F(s)\} = 1.
\]

Notice, that both matrix polynomials \( E(s) \) and \( F(s) \) have constant determinants readily implying that their inverses \( E^{-1}(s) \) and \( F^{-1}(s) \) are also matrix polynomials. Then it is straightforward to verify that

\[
E(s) C_A(s) F^{-1}(s) = E(s) (sB - A) F^{-1}(s) = \begin{bmatrix}
A(s) & 0 & \cdots & 0 & 0 \\
0 & \mathbb{I} & 0 & \cdots & 0 \\
0 & 0 & \ddots & \cdots & : \\
\vdots & \vdots & \ddots & \mathbb{I} & 0 \\
0 & 0 & \cdots & 0 & \mathbb{I}
\end{bmatrix}. \quad (D.9)
\]

The identity \((D.9)\) where matrix polynomials \( E(s) \) and \( F(s) \) have constant determinants can be viewed as the definition of equivalency between matrix polynomial \( A(s) \) and its companion polynomial \( C_A(s) \).

Let us take a look at the eigenvalue problem for eigenvalue \( s \) and eigenvector \( x \in \mathbb{C}^{m\nu} \) associated with companion polynomial \( C_A(s) \), that is

\[
(sB - A)x = 0, \quad x = \begin{bmatrix}
x_0 \\
x_1 \\
x_2 \\
\vdots \\
x_{\nu-1}
\end{bmatrix} \in \mathbb{C}^{m\nu}, \quad x_j \in \mathbb{C}^m, \quad 0 \leq j \leq \nu - 1, \quad (D.10)
\]
ANALYTIC THEORY OF MULTICAVITY KLYSTRONS

where

\[(sB - A)x = \begin{bmatrix}
    sx_0 - x_1 \\
    sx_1 - x_2 \\
    \vdots \\
    \sum_{j=0}^{\nu-2} A_j x_j + (sA_\nu + A_{\nu - 1}) x_{\nu - 1}
\end{bmatrix}.
\] (D.11)

With equations (D.10) and (D.11) in mind we introduce the following vector polynomial

\[x_s = \begin{bmatrix}
    x_0 \\
    sx_0 \\
    \vdots \\
    s^{\nu-2} x_0 \\
    s^{\nu-1} x_0
\end{bmatrix}, \quad x_0 \in \mathbb{C}^m.\] (D.12)

Not accidentally, the components of the vector \(x_s\) in its representation (D.12) are in evident relation with the derivatives \(\partial^j_t (x_0 e^{st}) = s^j x_0 e^{st}\). That is just another sign of the intimate relations between the matrix polynomial theory and the theory of systems of ordinary differential equations, see Appendix E.

**Theorem 14** (eigenvectors). Let \(A(s)\) as in equations (D.1) be regular, that \(\det \{A(s)\}\) is not identically zero, and let \(m\nu \times m\nu\) matrices \(A\) and \(B\) be defined by equations (D.2). Then the following identities hold

\[(sB - A)x_s = \begin{bmatrix}
    0 \\
    0 \\
    \vdots \\
    A(s) x_0
\end{bmatrix}, \quad x_s = \begin{bmatrix}
    x_0 \\
    sx_0 \\
    \vdots \\
    s^{\nu-2} x_0 \\
    s^{\nu-1} x_0
\end{bmatrix},\] (D.13)

\[\det \{A(s)\} = \det \{sB - A\}, \quad \det \{B\} = \det \{A_\nu\},\] (D.14)

where \(\det \{A(s)\} = \det \{sB - A\}\) is a polynomial of the degree \(m\nu\) if \(\det \{B\} = \det \{A_\nu\} \neq 0\). There is one-to-one correspondence between solutions of equations \(A(s) x = 0\) and \((sB - A) x = 0\). Namely, a pair \(s, x\) solves eigenvalue problem \((sB - A) x = 0\) if and only if the following equalitieshold

\[x = x_s = \begin{bmatrix}
    x_0 \\
    sx_0 \\
    \vdots \\
    s^{\nu-2} x_0 \\
    s^{\nu-1} x_0
\end{bmatrix}, \quad A(s) x_0 = 0, \quad x_0 \neq 0; \quad \det \{A(s)\} = 0.\] (D.15)

**Proof.** Polynomial vector identity (D.13) readily follows from equations (D.11) and (D.12). Identities (D.14) for the determinants follow straightforwardly from equations (D.12), (D.15) and (D.9). If \(\det \{B\} = \det \{A_\nu\} \neq 0\) then the degree of the polynomial \(\det \{sB - A\}\) has to be \(m\nu\) since \(A\) and \(B\) are \(m\nu \times m\nu\) matrices.

Suppose that equations (D.15) hold. Then combining them with proven identity (D.13) we get \((sB - A) x_s = 0\) proving that expressions (D.15) define an eigenvalue \(s\) and an eigenvector \(x = x_s\).

Suppose now that \((sB - A) x = 0\) where \(x \neq 0\). Combing that with equations (D.11) we obtain

\[x_1 = sx_0, \quad x_2 = sx_1 = s^2 x_0, \cdots, \quad x_{\nu - 1} = s^{\nu - 1} x_0,\] (D.16)
implying that
\[
x = x_s = \begin{bmatrix} x_0 \\ sx_0 \\ \vdots \\ s^{\nu-2}x_0 \\ s^{\nu-1}x_0 \end{bmatrix}, \quad x_0 \neq 0,
\]
(D.17)
and
\[
\sum_{j=0}^{\nu-2} A_j x_j + (sA_{\nu} + A_{\nu-1}) x_{\nu-1} = A(s) x_0.
\]
(D.18)

Using equations (D.17) and identity (D.13) we obtain
\[
0 = (sB - A) x = (sB - A) x_s = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ A(s) x_0 \end{bmatrix}.
\]
(D.19)

Equations (D.19) readily imply \( A(s) x_0 = 0 \) and \( \det \{ A(s) \} = 0 \) since \( x_0 \neq 0 \). That completes the proof. \( \square \)

Remark 15 (characteristic polynomial degree). Notice that according to Theorem 14 the characteristic polynomial \( \det \{ A(s) \} \) for \( m \times m \) matrix polynomial \( A(s) \) has the degree \( m\nu \), whereas in linear case \( sI - A_0 \) for \( m \times m \) identity matrix \( I \) and \( m \times m \) matrix \( A_0 \) the characteristic polynomial \( \det \{ sI - A_0 \} \) is of the degree \( m \). This can be explained by observing that in the non-linear case of \( m \times m \) matrix polynomial \( A(s) \) we are dealing effectively with many more \( m \times m \) matrices \( A \) than just a single matrix \( A_0 \).

Another problem of our particular interest related to the theory of matrix polynomials is eigenvalues and eigenvectors degeneracy and consequently the existence of non-trivial Jordan blocks, that is Jordan blocks of dimensions higher or equal to 2. The general theory addresses this problem by introducing so-called “Jordan chains” which are intimately related to the theory of system of differential equations expressed as \( A(\partial_t) x(t) = 0 \) and their solutions of the form \( x(t) = p(t) e^{st} \) where \( p(t) \) is a vector polynomial, see Appendix E and [GoLaRo, I, II], [Baum, 9]. Avoiding the details of Jordan chains developments we simply notice that an important to us point of Theorem 14 is that there is one-to-one correspondence between solutions of equations \( A(s) x = 0 \) and \( (sB - A) x = 0 \), and it has the following immediate implication.

Corollary 16 (equality of the dimensions of eigenspaces). Under the conditions of Theorem 14 for any eigenvalue \( s_0 \), that is \( \det \{ A(s_0) \} = 0 \), we have
\[
\dim \{ \ker \{ s_0B - A \} \} = \dim \{ \ker \{ A(s_0) \} \}.
\]
(D.20)
In other words, the geometric multiplicities of the eigenvalue \( s_0 \) associated with matrices \( A(s_0) \) and \( s_0B - A \) are equal. In view of identity (D.20) the following inequality holds for the (algebraic) multiplicity \( m(s_0) \)
\[
m(s_0) \geq \dim \{ \ker \{ A(s_0) \} \}.
\]
(D.21)

The next statement shows that if the geometric multiplicity of an eigenvalue is strictly less than its algebraic one than there exist non-trivial Jordan blocks, that is Jordan blocks of dimensions higher or equal to 2.
Theorem 17 (non-trivial Jordan block). Assuming notations introduced in Theorem 14 let us suppose that the multiplicity $m(s_0)$ of eigenvalue $s_0$ satisfies

$$m(s_0) > \dim \{ \ker \{ A(s_0) \} \}.$$  \hfill (D.22)

Then the Jordan canonical form of companion polynomial $C_A(s) = sB - A$ has a least one nontrivial Jordan block of the dimension exceeding 2.

In particular, if

$$\dim \{ \ker \{ s_0B - A \} \} = \dim \{ \ker \{ A(s_0) \} \} = 1,$$  \hfill (D.23)

and $m(s_0) \geq 2$ then the Jordan canonical form of companion polynomial $C_A(s) = sB - A$ has exactly one Jordan block associated with eigenvalue $s_0$ and its dimension is $m(s_0)$.

The proof of Theorem 17 follows straightforwardly from the definition of the Jordan canonical form and its basic properties. Notice that if equations (D.23) hold that implies that the eigenvalue 0 is cyclic (nonderogatory) for matrix $A(s_0)$ and eigenvalue $s_0$ is cyclic (nonderogatory) for matrix $B^{-1}A$ provided $B^{-1}$ exists, see Appendix C.

APPENDIX E. VECTOR DIFFERENTIAL EQUATIONS AND THE JORDAN CANONICAL FORM

In this section we relate the vector ordinary differential equations to the matrix polynomials reviewed in Appendix D following to [GoLaRo2, 5.1, 5.7], [GoLaRo, II.8.3], [Hale, III.4], [MeyCD, 7.9].

Equation $A(s)x = 0$ with polynomial matrix $A(s)$ defined by equations (D.1) corresponds to the following $m\nu$-vector $\nu$-th order ordinary differential

$$A(\partial_t)x(t) = 0,$$  \hfill (E.1)

where $A_j = A_j(t)$ are $m \times m$ matrices. Introducing $m\nu$-column-vector function

$$Y(t) = \begin{bmatrix} x(t) \\ \partial_t x(t) \\ \vdots \\ \partial_t^{\nu-2} x(t) \\ \partial_t^{\nu-1} x(t) \end{bmatrix}$$  \hfill (E.2)

and under the assumption that matrix $A_\nu(t)$ is the identity matrix the differential equation (E.1) can be recast and the first order differential equation

$$\partial_t Y(t) = AY(t),$$  \hfill (E.3)

where $A$ is $m\nu \times m\nu$ matrix defined by

$$A = A(t) = \begin{bmatrix} 0 & \mathbb{I} & \cdots & 0 & 0 \\ 0 & 0 & \mathbb{I} & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 & \mathbb{I} \\ -A_0(t) & -A_1(t) & \cdots & -A_{\nu-2}(t) & -A_{\nu-1}(t) \end{bmatrix},$$  \hfill (E.4)

and $A_\nu(t) = \mathbb{I}$. The proof of Theorem 17 follows straightforwardly from the definition of the Jordan canonical form and its basic properties. Notice that if equations (D.23) hold that implies that the eigenvalue 0 is cyclic (nonderogatory) for matrix $A(s_0)$ and eigenvalue $s_0$ is cyclic (nonderogatory) for matrix $B^{-1}A$ provided $B^{-1}$ exists, see Appendix C.
E.1. **Constant coefficients case.** Let us consider an important special case of equation (E.1) when matrices $A_j$ are $m \times m$ that do not depended on $t$. Then equation (E.1) can be recast as

$$B \partial_t Y (t) = A Y (t), \quad \text{(E.5)}$$

where $A$ and $B$ are $m \nu \times m \nu$ companion matrices defined by equations (D.4) and

In the case when $A_\nu$ is an invertible $m \times m$ matrix equation (E.5) can be recast further as

$$\partial_t Y (t) = \dot{A} Y (t), \quad \text{(E.6)}$$

where

$$\dot{A} = \begin{bmatrix} 0 & \mathbb{I} & \cdots & 0 & 0 \\ 0 & 0 & \mathbb{I} & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 & \mathbb{I} \\ -A_0 & -A_1 & \cdots & -A_{\nu-2} & -A_{\nu-1} \end{bmatrix}, \quad \dot{A}_j = A_\nu^{-1} A_j, \quad 0 \leq \nu - 1. \quad \text{(E.7)}$$

Notice one can interpret equation (E.6) as particular case of equation (E.5) where matrices $A_\nu$ and $B$ are identity matrices of the respective dimensions $m \times m$ and $m \nu \times m \nu$, and that polynomial matrix $A (s)$ defined by equations (D.1) becomes monic matrix polynomial $\dot{A} (s)$, that is

$$\dot{A} (s) = \mathbb{I} s^\nu + \sum_{j=0}^{\nu-1} \dot{A}_j s^j, \quad \dot{A}_j = A_\nu^{-1} A_j, \quad 0 \leq \nu - 1. \quad \text{(E.8)}$$

Notice that in view of equation (E.2) one recovers $x (t)$ from $Y (t)$ by the following formula

$$x (t) = P_1 Y (t), \quad P_1 = \begin{bmatrix} \mathbb{I} & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad \text{(E.9)}$$

where $P_1$ evidently is $m \times m \nu$ matrix.

Observe also that, [GoLaRo2, Prop. 5.1.2], [LanTsi, 14]

$$\left[ \dot{A} (s) \right]^{-1} = P_1 \left[ \mathbb{I} s - \dot{A} \right]^{-1} R_1, \quad P_1 = \begin{bmatrix} \mathbb{I} & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \mathbb{I} \end{bmatrix}, \quad \text{(E.10)}$$

where $P_1$ and $R_1$ evidently respectively $m \times m \nu$ and $m \nu \times m$ matrices.

The general form for the solution to vector differential equation (E.6) is

$$Y (t) = \exp \left\{ \dot{A} t \right\} Y_0, \quad Y_0 \in \mathbb{C}^{m \nu}. \quad \text{(E.11)}$$

Then using the formulas (E.9), (E.11) and Proposition 9 we arrive the following statement.

**Proposition 18** (solution to the vector differential equation). Let $\dot{A}$ be $m \nu \times m \nu$ companion matrix defined by equations (E.7), $\zeta_1, \ldots, \zeta_p$ be its distinct eigenvalues, and $M_{\zeta_1}, \ldots, M_{\zeta_p}$ be the corresponding generalized eigenspaces of the corresponding dimensions $r (\zeta_j), \; 1 \leq j \leq p$. Then the $m \nu$ column-vector solution $Y (t)$ to differential equation (E.6) is of the form

$$Y (t) = \exp \left\{ \dot{A} t \right\} Y_0 = \sum_{j=1}^{p} e^{\zeta_j t} p_j (t), \quad Y_0 = \sum_{j=1}^{p} Y_{0,j}, \quad Y_{0,j} \in M_{\zeta_j}, \quad \text{(E.12)}$$
where \( m \)-column-vector polynomials \( p_j(t) \) satisfy
\[
p_j(t) = \sum_{k=0}^{r(\zeta_j) - 1} \frac{t^k}{k!} (\hat{A} - \zeta_j \mathbb{I})^k Y_{0,j}, \quad 1 \leq j \leq p.
\] (E.13)

Consequently, the general \( m \)-column-vector solution \( x(t) \) to differential equation (E.1) is of the form
\[
x(t) = \sum_{j=1}^{p} e^{\zeta_j t} P_1 p_j(t), \quad P_1 = \begin{bmatrix} \mathbb{I} & 0 & \cdots & 0 \end{bmatrix}.
\] (E.14)

Notice that \( \chi_{\hat{A}}(s) = \det \{ s\mathbb{I} - \hat{A} \} \) is the characteristic function of the matrix \( \hat{A} \) then using notations of Proposition 18 we obtain
\[
\chi_{\hat{A}}(s) = \prod_{j=1}^{p} (s - \zeta_j)^{r(\zeta_j)}.
\] (E.15)

Notice also that for any values of complex-valued coefficients \( b_k \) we have
\[
(\partial_t - \zeta_j)^{r(\zeta_j)} [e^{\zeta_j t} p_j(t)] = 0, \quad p_j(t) = \sum_{k=0}^{r(\zeta_j) - 1} b_k t^k,
\] (E.16)

implying together with representation (E.15)
\[
\chi_{\hat{A}}(\partial_t) [e^{\zeta_j t} p_j(t)] = 0, \quad p_j(t) = \sum_{k=0}^{r(\zeta_j) - 1} b_k t^k.
\] (E.17)

Combing now Proposition 18 with equation (E.17) we obtain the following statement.

**Corollary 19** (property of a solution to the vector differential equation). Let \( x(t) \) be the general \( m \)-column-vector solution \( x(t) \) to differential equation (E.1). Then \( x(t) \) satisfies
\[
\chi_{\hat{A}}(\partial_t) x(t) = 0.
\] (E.18)

**APPENDIX F. Floquet theory**

We provide here a concise review of the Floquet theory following to [DalKre III], [Hale III.7] and [YakStar II.2]. The primary subject of the Floquet theory is the general form of solutions to the ordinary differential equations with periodic coefficients. With that in mind suppose that: (i) \( z \) is real valued variable, (ii) \( x(z) \) is \( n \)-vector valued function of \( z \), (iii) \( A(z) \) is \( n \times n \) matrix valued \( \zeta \)-periodic function of \( z \) and consider the following homogeneous linear periodic system:
\[
\partial_z x(z) = A(z) x(z), \quad A(z + \zeta) = A(z), \quad \zeta > 0.
\] (F.1)

We would like to give a complete characterization of the general structure of the solutions to equation (F.1). We start with the following statement showing how to define the logarithm \( B \) of a matrix \( C \) so that \( C = e^B \).

**Lemma 20** (logarithm of a matrix). Let \( C \) be is \( n \times n \) matrix with \( \det \{ C \} \neq 0 \). Suppose \( C = Z^{-1} J Z \) where \( J \) is Jordan canonical form of \( C \) as described in Proposition 10. Then using the block representation (B.8) for \( J \), that is
\[
J = J = \text{diag} \{ J_{n_1}(\zeta_1), J_{n_2}(\zeta_2), \ldots, J_{n_r}(\zeta_r) \}, \quad n_1 + n_2 + \cdots + n_q = n,
\] (F.2)

we decompose \( J \) into its diagonal and nilpotent components:
\[
J = \text{diag} \{ \lambda_1 \mathbb{I}_{n_1}, \lambda_2 \mathbb{I}_{n_2}, \ldots, \lambda_q \mathbb{I}_{n_q} \} + K
\] (F.3)
where
\[ D = \text{diag} \{ \lambda_1 \mathbb{I}_{n_1}, \lambda_2 \mathbb{I}_{n_2}, \ldots, \lambda_q \mathbb{I}_{n_q} \}, \quad K = \text{diag} \{ K_{n_1}, K_{n_2}, \ldots, K_{n_q} \}, \] \hspace{1cm} (F.4)

Then let \( \ln(*) \) be a branch of the logarithm and let
\[ H = \ln J = \text{diag} \{ \ln(\lambda_1) \mathbb{I}_{n_1}, \ln(\lambda_2) \mathbb{I}_{n_2}, \ldots, \ln(\lambda_q) \mathbb{I}_{n_q} \} + S \] \hspace{1cm} (F.5)

where \( \mathbb{I}_{n_j} \) are identity matrices of identified dimensions and
\[ S = \text{diag} \{ S_{n_1}, S_{n_2}, \ldots, S_{n_q} \}, \quad S_{n_j} = \sum_{m=1}^{n_j-1} \frac{1}{m \lambda_j^m} K_{n_j}^m, \quad 1 \leq j \leq q. \] \hspace{1cm} (F.6)

Then
\[ C = e^B, \quad B = \ln C = Z^{-1} H Z, \] \hspace{1cm} (F.7)

where matrix \( H \) is defined by equation \((F.5)\).

Note that matrix \( S \) in equations \((F.5)\) and \((F.6)\) is associated with the nilpotent part of Jordan canonical form \( J \). The expression for \( S_{n_j} \) originates in the series
\[ \ln(1+s) = \sum_{m=1}^{\infty} (-1)^{m-1} \frac{1}{m} s^m = s - \frac{s^2}{2} + \frac{s^3}{3} + \cdots, \] \hspace{1cm} (F.8)

and it is a finite sum since \( K_{n_j} \) is a nilpotent matrix such that
\[ K_{n_j}^m = 0, \quad m \geq n_j, \quad 1 \leq j \leq q. \] \hspace{1cm} (F.9)

An \( n \times n \) matrix \( \Phi(z) \) is called matrizant (matriciant) of equation \((F.1)\) if it satisfies the following equation:
\[ \partial_z \Phi(z) = A(z) \Phi(z), \quad \Phi(0) = \mathbb{I}, \quad A(z + \varsigma) = A(z), \quad \varsigma > 0, \] \hspace{1cm} (F.10)

where \( \mathbb{I} \) is the \( n \times n \) identity matrix. Matrix \( \Phi(z) \) is also called principal fundamental matrix solution to equation \((F.1)\). Evidently \( x(z) = \Phi(z) x_0 \) is the a solution to equation \((F.1)\) with the initial condition \( x(0) = x_0 \). Using the fundamental solution \( \Phi(z) \) we can represent any matrix solution \( \Psi(z) \) to equation \((F.1)\) based on its initial values as follows
\[ \partial_z \Psi(z) = A(z) \Psi(z), \quad \Psi(0) = \Phi(z) \Psi(0). \] \hspace{1cm} (F.11)

In the case of \( \varsigma \)-periodic matrix function \( A(z) \) the matrix function \( \Psi(z) = \Phi(z + \varsigma) \) is evidently a solution to equation \((F.11)\) and consequently
\[ \Phi(z + \varsigma) = \Phi(z) \Phi(\varsigma). \] \hspace{1cm} (F.12)

It turns out that matrix \( M_\varsigma = \Phi(\varsigma) \) called the monodromy matrix is of particular importance for the analysis of solutions to equation \((F.10)\) with \( \varsigma \)-periodic matrix function \( A(z) \).

The monodromy matrix is integrated into the formulation of the main statement of the Floquet theory describing the structure of solutions to equation \((F.10)\) for \( \varsigma \)-periodic matrix function \( A(z) \).

**Theorem 21** (Floquet). **Suppose that** \( A(z) \) is a \( \varsigma \)-periodic continuous function of \( z \). **Let** \( \Phi(z) \) be the matrizant of equation \((F.10)\) and let \( M_\varsigma = \Phi(\varsigma) \) be the corresponding monodromy matrix. **Using the statement of Lemma 20**, we introduce matrix \( \Gamma \) defined by
\[ \Gamma = \frac{1}{\varsigma} \ln M_\varsigma = \frac{1}{\varsigma} \ln \Phi(\varsigma), \quad \text{implying} \quad M_\varsigma = \Phi(\varsigma) = e^{F_\varsigma}. \] \hspace{1cm} (F.13)

Then matrizant \( \Phi(z) \) satisfies the following equation called Floquet representation
\[ \Phi(z) = P(z) e^{F_\varsigma}, \quad P(z + \varsigma) = P(z), \quad P(0) = \mathbb{I}, \] \hspace{1cm} (F.14)

where \( P(z) \) is a differentiable \( \varsigma \)-periodic matrix function of \( z \).
Proof. Let us define matrix $P(z)$ by the following equation

$$P(z) = \Phi(z) e^{-\Gamma z}.$$  \hfill (F.15)

Then combining representation (F.15) for $P(z)$ with equations (F.12) and (F.13) we obtain

$$P(z + \varsigma) = \Phi(z + \varsigma) e^{-\Gamma(z+\varsigma)} = \Phi(z) \Phi(\varsigma) e^{-\Gamma \varsigma} e^{-\Gamma z} = \Phi(z) e^{-\Gamma z} = P(z),$$ \hfill (F.16)

that is $P(z)$ is a differentiable $\varsigma$-periodic matrix function of $z$. Equality $P(0) = I$ readily follows from equation (F.15) and equality $\Phi(0) = I$. \hfill $\square$

The eigenvalues of the monodromy matrix $\Phi(\varsigma) = e^{\Gamma \varsigma}$ are called Floquet (characteristic) multipliers and their logarithms (not uniquely defined) are called characteristic exponents.

**Definition 22** (Floquet multipliers, characteristic exponents and eigenmodes). Using notation of Theorem 21 let us consider complex numbers $\kappa$, $s_\kappa$ and vector $y_\kappa$ satisfying the following equations

$$\Gamma y_\kappa = \kappa y_\kappa, \quad \Phi(\varsigma) y_\kappa = e^{-\Gamma \varsigma} y_\kappa = s_\kappa y_\kappa, \quad s_\kappa = e^{\kappa \varsigma},$$ \hfill (F.17)

where evidently $\kappa$ and $y_\kappa$ are respectively an eigenvalue and the corresponding eigenvector of matrix $\Gamma$. We refer to $\kappa$ and $s_\kappa$ respectively as the Floquet characteristic exponent and the Floquet (characteristic) multiplier.

Using $\kappa$ and $y_\kappa$ defined above we introduce the following special solution to the original differential equation (F.1):

$$\psi_\kappa(z) = p_\kappa(z) e^{\kappa z} = \Phi(z) y_\kappa = P(z) e^{\Gamma z} y_\kappa, \quad p_\kappa(z) = P(z) y_\kappa,$$ \hfill (F.18)

and refer to it as the Floquet eigenmode. Note that $p_\kappa(z)$ in equations (F.18) is $\varsigma$-periodic vector-function of $z$.

**Remark 23** (Floquet eigenmodes). If $\psi_\kappa(z)$ is the Floquet eigenmode defined by equations (F.18) and $\Re \{\kappa\} > 0$ or equivalently $|s_\kappa| > 1$ then $\psi_\kappa(z)$ grows exponentially as $z \to +\infty$ and we refer to such $\psi_\kappa(z)$ as exponentially growing Floquet eigenmode. In the case when $\Re \{\kappa\} = 0$ or equivalently $|s_\kappa| = 1$ function $\psi_\kappa(z)$ is bounded and we refer to such $\psi_\kappa(z)$ as an oscillatory Floquet eigenmode.

**Remark 24** (dispersion relations). In physical applications of the Floquet theory $\varsigma$-periodic matrix valued function $A(z)$ in differential equation (F.1) depends on the frequency $\omega$, that is $A(z) = A(z, \omega)$. In this case we also have $\kappa = \kappa(\omega)$. If we naturally introduce the wave number $k$ by

$$k = k(\omega) = -i \kappa(\omega),$$ \hfill (F.19)

then the relation between $\omega$ and $k$ provided by equation (F.19) is called the dispersion relation.

**Appendix G. Hamiltonian systems of linear differential equations**

We follow here to [DalKre, I.8, V.1] and [YakStar, III]. We introduce first indefinite scalar product $\langle x, y \rangle$ on the vector space $\mathbb{C}^n$ associated with a nonsingular Hermitian $n \times n$ matrix $G$, namely

$$\langle x, y \rangle = \langle y, x \rangle = x^* G y, \quad G^* = G, \quad \det \{G\} \neq 0, \quad x, y \in \mathbb{C}^n.$$ \hfill (G.1)

We refer to matrix $G$ metric matrix. We define then for any $n \times n$ matrix $A$ another matrix $A^\dagger$ called adjoint by the following relations:

$$\langle A z, y \rangle = \langle x, A^\dagger y \rangle \quad \text{or equivalently} \quad A^\dagger = G^{-1} A^* G.$$ \hfill (G.2)

Notice that relations (G.2) readily imply

$$(AB)^\dagger = B^\dagger A^\dagger.$$ \hfill (G.3)
Let \( G \) and \( H(t) \) be Hermitian \( n \times n \) matrices and suppose that matrix \( G \) is nonsingular. We defined Hamiltonian system of equations to be a system of the form,

\[
- iG \partial_t x(t) = H(t) x(t), \quad H^*(t) = H(t).
\]

If based on matrices \( G \) and \( H(t) \) we introduce G-skew-Hermitian matrix

\[
A(t) = iG^{-1}H(t),
\]

we can recast the Hamiltonian system (G.4) in the following equivalent form,

\[
\partial_t x(t) = A(t) x(t), \quad A^\dagger(t) = -A(t).
\]

It turns out that the matrizant \( \Phi(t) \) of equation (G.6) with G-skew-Hermitian matrix \( A(t) \) is a G-unitary matrix for each value of \( t \). Indeed, using equation (G.6) together with equations (G.2), (G.3) we obtain

\[
\partial_t \left[ \Phi^\dagger(t) \Phi(t) \right] = \left\{ \partial_t \left[ \Phi(t) \right] \right\}^\dagger \Phi(t) + \Phi^\dagger(t) \partial_t \left[ \Phi(t) \right] = -\Phi^\dagger(t) A(t) \Phi(t) + \Phi^\dagger(t) A(t) \Phi(t) = 0,
\]

implying that matrizant \( \Phi(t) \) satisfies

\[
\Phi^\dagger(t) \Phi(t) = I, \text{ or equivalently } \Phi^*(t) G \Phi(t) = G,
\]

implying that \( \Phi(t) \) is a G-unitary matrix for each value of \( t \). Identity (G.8) implies in turn that for any two solutions \( x(t) \) and \( y(t) \) to the Hamiltonian system (G.4) we always have

\[
\langle x(t), y(t) \rangle = x^*(t) G y(t) = x^*(0) \Phi^*(t) G \Phi(t) y(0) = \langle x(0), y(0) \rangle,
\]

that is \( \langle x(t), y(t) \rangle \) does not depend on \( t \).

G.1. **Symmetry of the spectra.** G-unitary, G-skew-Hermitian and G-Hermitian matrices have special properties described in Table 5. These properties can viewed as symmetries and not surprising they imply consequent symmetries of the spectra of the matrices. Let \( \sigma\{A\} \) denote the spectrum of matrix \( A \). It is a straightforward exercise to verify based on matrix properties described in Table 5 that the following statements hold.

**Theorem 25** (spectral symmetries). Suppose that matrix \( A \) is either G-unitary or G-skew-Hermitian or G-Hermitian. Then the following statements hold:

(i) If \( A \) is G-unitary then \( \sigma\{A\} \) is symmetric with respect to the unit circle, that is

\[
\zeta \in \sigma\{\Phi\} \Rightarrow \frac{1}{\zeta} \in \sigma\{\Phi\}.
\]

(ii) If \( A \) is G-skew-Hermitian then \( \sigma\{A\} \) is symmetric with respect the imaginary axis, that is

\[
\zeta \in \sigma\{\Phi\} \Rightarrow -\zeta \in \sigma\{\Phi\}.
\]

(iii) If \( A \) is G-Hermitian then \( \sigma\{A\} \) is symmetric with respect to real axis, that is

\[
\zeta \in \sigma\{\Phi\} \Rightarrow \zeta \in \sigma\{\Phi\}.
\]
The following statement describes $G$-orthogonality of invariant subspaces of $G$-unitary, $G$-skew-Hermitian and $G$-Hermitian matrices, [DalKre, 1.8].

**Theorem 26** (eigenspaces). Suppose that matrix $A$ is either $G$-unitary or $G$-skew-Hermitian or $G$-Hermitian. Then the following statements hold. Let $\Lambda \subset \sigma\{A\}$ be a subset of the spectrum $\sigma\{A\}$ of the matrix $A$, and let $\tilde{\Lambda}$ be the relevant symmetric image of $\Lambda$ defined by

$$
\tilde{\Lambda} = \begin{cases} 
\{ \zeta : \zeta \in \Lambda \} & \text{if } A \text{ is } G\text{-unitary} \\
\{-\bar{\zeta} : \zeta \in \Lambda \} & \text{if } A \text{ is } G\text{-skew-Hermitian} \\
\{\bar{\zeta} : \zeta \in \Lambda \} & \text{if } A \text{ is } G\text{-Hermitian}
\end{cases}
$$

Let $\Lambda_1, \Lambda_2 \subset \sigma\{A\}$ be two subsets of the spectrum $\sigma\{A\}$ so that $\tilde{\Lambda}_1$ and $\Lambda_2$ are separated from each other by non-intersecting contours $\tilde{\Gamma}_1$ and $\Gamma_2$. Then the invariant subspaces $E_1$ and $E_1$ of the matrix $A$ corresponding to $\Lambda_1$ and $\Lambda_2$ are $G$-orthogonal.

The statement below describes a special property of eigenvectors of a $G$-unitary matrix.

**Lemma 27** (isotropic eigenvector). Let $A$ be a $G$-unitary matrix and $\zeta$ be its eigenvalue that does not lie on the unit circuit, that $|\zeta| \neq 1$. Then if $x$ is the eigenvector corresponding to $\zeta$ it is isotropic, that is

$$
\langle x, x \rangle = x^*Gx = 0, \quad Ax = \zeta x, \quad |\zeta| \neq 1.
$$

**Proof.** Since $Ax = \zeta x$ and $A$ is a $G$-unitary we have

$$
\langle Ax, Ax \rangle = \langle \zeta x, \zeta x \rangle = |\zeta|^2 \langle x, x \rangle, \quad \langle Ax, Ax \rangle = \langle x, x \rangle.
$$

Combining the two equation above with $|\zeta| \neq 1$ we conclude that $\langle x, x \rangle = 0$ which is the desired equation (G.10). $\square$

**DATA AVAILABILITY:** The data that supports the findings of this study are available within the article.

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