Galactic dynamics in MOND—Existence of equilibria with finite mass and compact support

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Abstract

We consider a self-gravitating collisionless gas where the gravitational interaction is modeled according to MOND (modified Newtonian dynamics). For the resulting modified Vlasov-Poisson system we establish the existence of spherically symmetric equilibria with compact support and finite mass. In the standard situation where gravity is modeled by Newton’s law the latter properties only hold under suitable restrictions on the prescribed microscopic equation of state. Under the MOND regime no such restrictions are needed.

1 Introduction

One of the intriguing mysteries in current astrophysics is the possible existence and nature of dark matter around galaxies. Observed rotational velocities of stars and gas in typical spiral galaxies seem to be larger than the ones predicted from the gravitational potential of the directly observable matter. This is one manifestation of the so-called missing mass problem. The resolution of this problem which is currently favored by a majority in the astrophysics community is that a galaxy is typically surrounded by a spherical halo of dark matter which provides the missing mass. This dark component is supposed to outweigh the visible matter by a factor of the order 10. There are other indirect arguments for the existence of dark matter, but so far there is no direct observational evidence for its existence nor a consistent physical theory which predicts its existence.
Some 30 years ago M. Milgrom proposed MOND (modified Newtonian dynamics) which predicts the observed rotational velocities in galaxies from the visible matter without invoking a dark component. The basic MOND paradigm can be expressed as follows. If a particle (for example a star) would in Newtonian gravity experience an acceleration $g_N$, then according to MOND the particle experiences an acceleration $g$ which obeys the relation

$$\mu(|g|/a_0)g = g_N.$$  \hspace{1cm} (1.1)

Here $a_0 \approx 10^{-10} \text{m/s}^2$ is a new physical constant, an acceleration far below typical accelerations in the solar system, and $\mu$ is an interpolating function such that

$$\mu(\tau) \rightarrow 1 \text{ for } \tau >> 1 \text{ and } \mu(\tau) \rightarrow \tau \text{ for } \tau << 1.$$ \hspace{1cm} (1.2)

If $|g| << a_0$ then MOND predicts an acceleration $|g| \sim \sqrt{|g_N|}$ which is much larger than the Newtonian prediction, while the two coincide for $|g| >> a_0$.

It is a surprising empirical fact that a modification of Newtonian mechanics with one free parameter, namely $a_0$, seems to correctly predict the rotation curves for a large variety of galaxies. We refer to \cite{7, 8} for very readable introductions to MOND, and to \cite{3} for an in-depth discussion of MOND, where the observational support for dark matter and a very large part of the corresponding literature are reviewed. Background on dark matter is also found in \cite{2}.

In the present paper we investigate self-consistent mathematical models for galaxies or globular clusters where the gravitational interaction is to obey the MOND paradigm. A galaxy is often modeled as a large ensemble of particles which interact only via gravity. This results in the Vlasov or Collisionless Boltzmann equation, coupled to a suitable field equation for gravity. If $-\nabla_x U$ denotes the gravitational field then the Vlasov equation reads

$$\partial_t f + v \cdot \nabla_x f - \nabla_x U \cdot \nabla_v f = 0.$$ \hspace{1cm} (1.3)

Here $f = f(t, x, v) \geq 0$ is the density of the particle ensemble in phase space and $t \geq 0, x, v \in \mathbb{R}^3$ stand for time, position, and velocity. The density in phase space induces the spatial mass density

$$\rho(t, x) = \int f(t, x, v) dv.$$ \hspace{1cm} (1.4)

We close the system with the modified Poisson equation

$$\text{div} [\mu(|\nabla U|/a_0)\nabla U] = 4\pi \rho.$$ \hspace{1cm} (1.5)
which determines the gravitational potential $U$ in terms of $\rho$. If $\mu$ satisfies (1.2) this field equation implies the MOND paradigm (1.1), cf. [3, Eqn.(17)]; this is just one possible implementation of the MOND paradigm in terms of a consistent physical theory. The field equation (1.5) is non-linear while in the Newtonian situation the non-linearity in the resulting Vlasov-Poisson system arises through the coupling with the Vlasov equation. Astrophysical background on the Vlasov-Poisson system can be found in [2]. We refer to the system (1.3), (1.4), (1.5) as the MONDian Vlasov-Poisson system. In passing we note that the usual normalizing condition

$$\lim_{|x| \to \infty} U(t, x) = 0$$

(1.6)

does not work in the MOND case; we will come back to the issue of the behavior at infinity.

In the present paper we prove the existence of spherically symmetric steady states of this system which have finite mass and compact spatial support. In this context it is instructive to generalize the class of interpolating functions $\mu$ as follows. We require that $\mu(\tau) \to 1$, $\tau >> 1$ and $\mu(\tau) \to \tau^\alpha$, $\tau << 1$ for some $\alpha \in [0,1]$. Genuine MOND corresponds to the choice $\alpha = 1$, while the choice $\alpha = 0$ or more precisely $\mu = 1$ yields the standard Vlasov-Poisson system of Newtonian galactic dynamics. The MOND case $\alpha = 1$ and a case with $\alpha = 1 - \epsilon$ and $\epsilon > 0$ very small should be hard to distinguish observationally, but mathematically these two cases behave quite differently.

The approach we use is known from the Newtonian case. The system under investigation is—by a suitable ansatz—reduced to the modified Poisson equation (1.5) where the right hand side becomes a function of the unknown potential $U$. The latter functional relation is determined by the assumed microscopic equation of state, i.e., by the functional dependence of $f$ on the particle energy $E = \frac{1}{2} |v|^2 + U(x)$ and possibly other local conserved quantities. In the Newtonian case the resulting steady state has the physically required properties of finite mass and compact support only under suitable restrictions on the equation of state. We refer to [9] for a quite general, sufficient such condition. The main result of the present paper is the following. For $\alpha \in [0,1]$ the same condition as in the Newtonian case guarantees finite mass and compact support, while for the genuine MOND case $\alpha = 1$ no such condition (beyond technical assumptions) is needed.

The paper proceeds as follows. In the next section we make precise the basic set-up for constructing spherically symmetric steady states of the MONDian Vlasov-Poisson system. In particular, we establish Jeans’ Theo-
rem, which says that in the spherically symmetric case the particle distribution function must be a function of the particle energy and modulus of angular momentum, we reduce the stationary system to a non-linear equation for the potential $U$, and we prove a corresponding existence result. In Section 3 we investigate the question of finite mass and compact support of the steady states. In a final section we discuss the asymptotics at spatial infinity and the question whether associated potential energies are finite. We consider steady states resulting from a Maxwellian ansatz. In the Newtonian situation they have infinite mass and extent, but in the genuine MOND regime $\alpha = 1$ their mass is finite. Finally, we extend our results to the MONDian Euler-Poisson system where matter is modeled as an ideal, compressible fluid.

To conclude this introduction we recall that for the Newtonian Vlasov-Poisson system there exists a satisfactory existence theory for the initial value problem including a global-in-time existence result for smooth initial data. Also the stability properties of steady states are by now quite well understood, and we refer to [4, 5, 6, 10] and the references there. Whether these results persist under the modification of the Poisson equation is an interesting open problem. The results of the present paper are mathematically fairly straightforward which is in part due to the flexibility of the analysis in [9], but our paper is intended only as a first step towards a better mathematical understanding of MONDian modifications of stellar dynamics.

2 Spherically symmetric steady states: The basic set-up

We first make precise our assumptions on the interpolating function $\mu$ which appears in the basic MOND paradigm (1.1) and in the modified Poisson equation (1.5).

**Assumptions on $\mu$.** Let $\mu \in C([0, \infty]) \cap C^1([0, \infty])$ be increasing with

$$\lim_{\tau \to \infty} \mu(\tau) = 1, \quad \lim_{\tau \to 0} \tau^{-\alpha} \mu(\tau) = 1$$

for some $\alpha \in [0, 1]$. Only the case $\alpha = 1$ corresponds to the genuine MOND theory. Furthermore we normalize the constant $a_0$ to unity, $a_0 = 1$, since its value does not affect the mathematical analysis.

We are interested in steady states of the system (1.3), (1.4), (1.5) which are spherically symmetric, i.e., $f(x, v) = f(Ax, Av)$ for all orthonormal matrices $A$ and $x, v \in \mathbb{R}^3$. A spherically symmetric distribution function de-
pends only on the variables
\[ r := |x|, \quad w := \frac{x \cdot v}{r}, \quad L := |x \times v|^2; \quad (2.2) \]

here \( w \) is the radial velocity and \( L \) the modulus of angular momentum squared. The induced spatial density and the potential are spherically symmetric in the sense that, by abuse of notation, \( \rho(x) = \rho(r) \) and \( U(x) = U(r) \). Under these symmetry assumptions the modified Poisson equation (1.5) takes the form
\[
\frac{1}{r^2} (r^2 \mu(|U'|) U')' = 4\pi \rho.
\]

We integrate once to find that
\[
\mu(|U'|) U' = \frac{m}{r^2},
\]

where
\[
m(r) := 4\pi \int_0^r s^2 \rho(s) \, ds \quad (2.3)
\]
is the mass within the ball of radius \( r \). Since \( \mu > 0 \) and \( m \geq 0 \), potentials solving the above equation are increasing so that the modulus inside \( \mu \) can be dropped. Now we observe that the mapping \( \tau \mapsto \tau \mu(\tau) \) is strictly increasing on \([0, \infty]\) and onto \([0, \infty]\). Let \( \zeta \) denote its inverse, i.e., \( \zeta : [0, \infty[ \to [0, \infty[ \) is one-to-one, onto, strictly increasing, \( \zeta \in C([0, \infty[) \cap C^1([0, \infty[) \), and
\[
\zeta(\tau \mu(\tau)) = \tau \text{ for } \tau \geq 0, \quad \lim_{\sigma \to 0} \sigma^{-1/(1+\alpha)} \zeta(\sigma) = 1, \quad \lim_{\sigma \to \infty} \sigma^{-1} \zeta(\sigma) = 1.
\]

(2.4)
The asymptotic properties of \( \zeta \) follow from those of \( \mu \), cf. (2.1). Using this function we rewrite the spherically symmetric modified Poisson equation in the form in which it will be solved, namely
\[
U'(r) = \zeta \left( \frac{m(r)}{r^2} \right), \quad r > 0.
\]

(2.5)

In the new variables adapted to the spherical symmetry (1.4) becomes
\[
\rho(r) = \frac{\pi}{r^2} \int_{-\infty}^\infty \int_0^\infty f(r, w, L) \, dL \, dw.
\]

(2.6)
The characteristic system
\[
\dot{x} = v, \quad \dot{v} = -\nabla U(x)
\]
of the Vlasov equation (1.3) can be rewritten as

\[ \dot{r} = w, \quad \dot{w} = \frac{L}{r^3} - U'(r), \quad \dot{L} = 0. \]  

(2.7)

A spherically symmetric steady state of the MONDian Vlasov-Poisson system is by definition a triple \((f, \rho, U)\) such that \(f = f(r, w, L) \geq 0\) is measurable, \(\rho\) satisfies (2.6) and \(r^2 \rho\) is locally integrable on \([0, \infty[\), \(U\) is differentiable on \([0, \infty[\) and satisfies (2.5), and \(f\) is constant along characteristics (2.7) which must exist uniquely for initial data \(r > 0, w \in \mathbb{R}, \) and \(L > 0.\)

Since the potential \(U\) is time-independent the particle energy

\[ E = E(x, v) := \frac{1}{2} |v|^2 + U(x) = \frac{1}{2} w^2 + \frac{L}{2r^2} + U(r) \]  

(2.8)

is constant along characteristics, and due to spherical symmetry the same is true for \(L.\) Hence any function of the form

\[ f = \Phi(E, L) \]  

(2.9)

with a suitable, prescribed function \(\Phi \geq 0\) satisfies the Vlasov equation in the sense of being constant along characteristics. We show that by making the ansatz (2.9), no spherically symmetric steady states are lost, a fact which in the Newtonian case is known as Jeans' Theorem, cf. [1, Thm. 2.2].

**Proposition 2.1** Let \((f, \rho, U)\) be a stationary, spherically symmetric solution of (1.3), (1.4), (1.5). Then \(f\) is of the form (2.9).

**Proof.** For \(r > 0,\) any characteristic curve remains in a plane of constant \(L\) which we can take positive; \(L\) is now kept fixed. In such a plane the effective characteristic system can be rewritten as

\[ \dot{r} = w, \quad \dot{w} = - \Psi_L'(r) \]  

(2.10)

with effective potential

\[ \Psi_L(r) := \frac{L}{2r^2} + U(r), \quad r > 0, \]

in particular,

\[ E = E(r, w) = \frac{1}{2} w^2 + \Psi_L(r). \]

The level sets of the function \(E\) in the half plane \(r > 0, w \in \mathbb{R}\) are invariant under the flow of (2.10), and since \(f(\cdot, \cdot, L)\) is constant on the orbits of (2.10)
the assertion of the proposition follows if each level set of $E$ consists of a single orbit.

In order to see the latter we analyze the effective potential $\Psi_L$. We first show that $r^2 U(r) \to 0$ as $r \to 0$ so that $\Psi_L(r) \to \infty$ as $r \to 0$. Indeed, for $0 < r < 1$,

$$r^2 U(r) = r^2 U(1) - r^2 \int_r^1 \zeta \left( \frac{m(s)}{s^2} \right) ds.$$  

The first term vanishes as $r \to 0$, and the integral has a limit $I \geq 0$ as $r \to 0$. If $I < \infty$ we are done. If $I = \infty$, we apply l'Hôpital's rule, and

$$0 \leq \lim_{r \to 0} r^2 \int_r^1 \zeta \left( \frac{m(s)}{s^2} \right) ds = \frac{1}{2} \lim_{r \to 0} r^3 \zeta \left( \frac{m(r)}{r^2} \right) \leq \frac{1}{2} \lim_{r \to 0} r^3 \zeta \left( \frac{m(1)}{r^2} \right)$$

as desired. We can assume that the steady state is non-trivial so that $m(r) > m_0 > 0$ for $r$ large, and hence $\Psi_L$ is strictly increasing for $r$ large. Together with its behavior at $r = 0$ this implies that $\Psi_L$ attains a global minimum at some $r_L > 0$. We show that $\Psi'_L$ has at most one zero so that $r_L$ is unique and $\Psi_L$ is strictly decreasing on $]0, r_L[$ and strictly increasing on $]r_L, \infty[$. Altogether, these properties of $\Psi_L$ imply that each level curve for an energy value above $\Psi_L(r_L)$ is a smooth connected curve which does not contain the unique stationary point $(r_L, 0)$ of (2.10). Each such level curve therefore consists of a single orbit of (2.10) as desired.

It remains to show that $\Psi'_L$ has indeed at most one zero. By (2.5), $\Psi_L(r) = 0$ is equivalent to the equation

$$r^3 \zeta \left( \frac{m(r)}{r^2} \right) = L. \quad (2.11)$$

Since $L > 0$ and $\zeta(0) = 0$ no solutions exist with $m(r) = 0$. Since $m$ is increasing and non-trivial, $m(r) > 0$ on some interval $]r_0, \infty[$, and in order to see that there exists at most one solution of (2.11) we show that the derivative of the left hand side is positive on $]r_0, \infty[$. Clearly,

$$\frac{d}{dr} \left[ r^3 \zeta \left( \frac{m(r)}{r^2} \right) \right] = 3r^2 \zeta \left( \frac{m(r)}{r^2} \right) + r^3 \zeta' \left( \frac{m(r)}{r^2} \right) \left( -2 \frac{m(r)}{r^3} + 4\pi \rho(r) \right)$$

$$\geq 3r^2 \left( \zeta \left( \frac{m(r)}{r^2} \right) - \frac{2m(r)}{3} \zeta' \left( \frac{m(r)}{r^2} \right) \right).$$

We recall that $\zeta^{-1}(\tau) = \tau \mu(\tau)$ and that $\mu$ is increasing. Hence

$$\tau(\zeta^{-1})'(\tau) = \tau (\mu(\tau) + \tau \mu'(\tau)) \geq \tau \mu(\tau) > \frac{2}{3} \zeta^{-1}(\tau), \tau > 0.$$
In terms of $\zeta$ this is equivalent to the estimate
\[ \zeta(\sigma) - \frac{2}{3}\sigma \zeta'(\sigma) > 0, \quad \sigma > 0. \]

We substitute $\sigma = m(r)/r^2$ and conclude that
\[ \frac{d}{dr} \left[ r^3 \zeta \left( \frac{m(r)}{r^2} \right) \right] > 0, \quad r > r_0, \]
as desired. The proof is complete. \hfill \Box

In order to avoid mostly technical complications we restrict ourselves to the more specific ansatz
\[ f(x, v) = \Phi(E_0 - E)L^l \]
(2.12)
Here $l > -1/2$ and $E_0$ is a cut-off energy above which the distribution is to vanish, i.e., $\Phi$ has to vanish for negative arguments. Such a cut-off energy is necessary in order to obtain steady states with compact support. As we will see in the last section, a cut-off energy is also necessary in order to obtain finite mass for the case $\alpha < 1$, but not in the genuine MOND case $\alpha = 1$. We make the following technical assumptions on $\Phi$.

**Assumptions on $\Phi$.** $\Phi : \mathbb{R} \to [0, \infty]$ is measurable, $\Phi(\eta) = 0$ for $\eta < 0$, and $\Phi > 0$ a.e. on some interval $[0, \eta_1]$ with $\eta_1 > 0$. Moreover, there exists $\kappa > -1$ such that for every compact set $K \subset \mathbb{R}$ there exists a constant $C > 0$ such that
\[ \Phi(\eta) \leq C\eta^\kappa, \quad \eta \in K. \]

If we substitute the ansatz (2.12) into (2.6) then in terms of $y := E_0 - U$,
\[ \rho(r) = r^{2l} g(y(r)) \]
(2.13)
where
\[ g(y) := \begin{cases} c_l \int_0^y \Phi(\eta) (y - \eta)^{l+1/2} d\eta, & y > 0, \\ 0, & y \leq 0, \end{cases} \]
(2.14)
and $c_l > 0$ is a constant. The assumptions on $\Phi$ and Lebesgue’s dominated convergence theorem imply that $g \in C(\mathbb{R}) \cap C^1([0, \infty])$. In terms of $y$ we have to solve the equation
\[ y'(r) = -\zeta \left( \frac{m(r)}{r^2} \right) \]
(2.15)
where
\[ m(r) = m(r, y) = 4\pi \int_0^r s^{2l+2} g(y(s)) ds; \]
(2.16)
notice that the right hand side of (2.15) depends on \( y \) non-locally. A solution launched by a central value \( y(0) = \dot{y} > 0 \) gives a non-trivial steady state which has finite mass and compact support if \( y \), which is decreasing, has a zero. In order to define the potential \( U \) we have to make a choice for the cut-off energy \( E_0 \). One possibility is to let \( E_0 = 0 \) in which case \( U = -y \) will have some unspecified limit at infinity. If \( y_\infty := \lim_{r \to \infty} y(r) > -\infty \) which is the case if \( y \) has a zero and \( \alpha < 1 \), but not for the genuine MOND case, then we can define \( E_0 := y_\infty \) and \( U := E_0 - y \) to recover the standard boundary condition (1.6) at infinity. We now show that (2.15) does have a unique solution for every prescribed value of \( y(0) = \dot{y} > 0 \). The question when this solution has a zero so that the induced steady state has finite mass and compact support is considered in the next section.

**Proposition 2.2** Let \( \dot{y} > 0 \). Then (2.15) has a unique solution \( y \in C^1([0, \infty]) \) with \( y(0) = \dot{y} \). This solution is strictly decreasing, \( y'(0) = 0 \), and \( y \in C^2([0, \infty]). \)

**Proof.** For \( \delta > 0 \) we define the set of functions

\[
Y := \{ y \in C([0, \delta]) \mid \dot{y}/2 \leq y(r) \leq \dot{y}, \ r \in [0, \delta]\},
\]

a bounded, closed, convex subset of the Banach space \( C([0, \delta]) \) equipped with the sup norm. The operator

\[
T(y)(r) := \dot{y} - \int_0^r \zeta \left( \frac{m(s, y)}{s^2} \right) \, ds
\]

is defined on \( Y \) and maps \( Y \) into itself, provided \( \delta \) is sufficiently small; note that for \( y \in Y \) and \( r \in [0, \delta]\),

\[
\frac{4\pi g(\dot{y})}{2l + 3} \frac{r^{2l+1}}{r^{2l+1}} \geq \frac{m(r, y)}{r^2} = \frac{4\pi}{r^2} \int_0^r s^{2+2l} g(y(s)) \, ds \geq \frac{4\pi g(\dot{y}/2)}{2l + 3} \frac{r^{2l+1}}{r^{2l+1}} \tag{2.17}
\]

and recall that \( 2l+1 > 0 \). Moreover, \( T \) is continuous, and \( T(Y) \) is a bounded and equicontinuous subset of \( C([0, \delta]) \). By the Arzela-Ascoli Theorem and Schauder’s Fixpoint Theorem there exists a fixed point \( y \in Y \) of \( T \) which is differentiable and satisfies (2.15) on \([0, \delta]\) together with the desired initial condition. The function \( \zeta \) is in general not Lipschitz near 0 so that the mapping \( Y \) need not be a contraction, but the solution is unique on \([0, \delta]\), provided \( \delta > 0 \) is sufficiently small. To see this we first observe that for \( \sigma \in [0, 1]\),

\[
0 \leq \frac{d}{d\sigma} \zeta^2(\sigma) = \frac{2\zeta(\sigma)}{\mu(\zeta(\sigma)) + \zeta(\sigma)\mu'(\zeta(\sigma))} \leq \frac{2\zeta(\sigma)}{\mu(\zeta(\sigma))} \leq C\zeta(\sigma)^{1-\alpha} \leq C\zeta(1)^{1-\alpha}
\]
so that $\zeta^2$ is Lipschitz on $[0, 1]$. Consider two solutions $y$ and $\tilde{y}$ of the above initial value problem. Then

$$ |y'(r) - \tilde{y}'(r)| = \left| \frac{\zeta^2 \left( \frac{m(r, y)}{r^2} \right) - \zeta^2 \left( \frac{m(r, \tilde{y})}{r^2} \right)}{\zeta \left( \frac{m(r, y)}{r^2} \right) + \zeta \left( \frac{m(r, \tilde{y})}{r^2} \right)} \right| \leq C \zeta \left( \frac{m(r, y)}{r^2} \right)^{-1} \left| \frac{m(r, y)}{r^2} - \frac{m(r, \tilde{y})}{r^2} \right|. $$

If $\delta$ is sufficiently small, (2.17) and the asymptotic behavior of $\zeta$ imply that

$$ \zeta \left( \frac{m(r, y)}{r^2} \right) \geq Cr^{(2l+1)/(1+\alpha)} $$

with a positive constant $C > 0$ which does not depend on $y, \tilde{y}, \delta$, or $r$. Since $g$ is Lipschitz on $[\tilde{y}/2, \tilde{y}]$,

$$ \left| \frac{m(r, y)}{r^2} - \frac{m(r, \tilde{y})}{r^2} \right| \leq Cr^{2l+1} \max_{0 \leq s \leq r} |y(s) - \tilde{y}(s)| $$

so that altogether,

$$ |y(r) - \tilde{y}(r)| \leq \int_0^r |y'(s) - \tilde{y}'(s)| ds \leq C \delta^{1+\alpha(2l+1)/(\alpha+1)} \max_{0 \leq s \leq \delta} |y(s) - \tilde{y}(s)| $$

where the constant $C > 0$ does not depend on $y, \tilde{y}$, or $\delta$. It follows that $y = \tilde{y}$ on $[0, \delta]$, provided $\delta$ is sufficiently small.

In order to extend this unique local solution we observe that in terms of the dependent variables $y$ and $m$ the equation (2.15) is recast into the non-autonomous first order system of ordinary differential equations

$$ y' = -\zeta \left( \frac{m(r, y)}{r^2} \right), \quad m' = 4\pi r^{2l+2} g(y). $$

Prescribing positive data for $y$ and $m$ at some positive radius $r > 0$ yields a unique local $C^1$ solution. The maximally extended solution is strictly decreasing. Either it is bounded from below by zero or it has a zero to the right of which $g(y) = 0$ and $m = \text{const}$, and in both cases the solution extends to $[0, \infty]$. The right hand side of (2.15) is continuously differentiable for $r > 0$, and it converges to zero for $r \to 0$. The proof is complete. \qed

Given a solution of (2.15) as obtained in Proposition 2.2

$$ f(x, v) = \Phi \left( y(r) - \frac{1}{2} |v|^2 \right) |x \times v|^{2l} $$

defines a spherically symmetric steady state of (1.3), (1.4), (1.5).
3 Compact support and finite mass

Theorem 3.1 Let \( f \) be a non-trivial, spherically symmetric steady state of the MONDian Vlasov-Poisson system (1.3), (1.4), (1.5) as obtained in the previous section.

(a) In the genuine MOND case \( \alpha = 1 \) the steady state has finite mass and compact support.

(b) In the general case the steady state has finite mass and compact support, provided that \( \Phi \) satisfies the additional assumption

\[
\Phi(\eta) \geq c\eta^k \quad \text{for} \quad \eta \in [0, \eta_0[ \\
\text{with parameters} \quad c > 0, \eta_0 > 0, \text{and} -1 < k < l + 3/2.
\]

Proof. Since the steady state is non-trivial with \( g(y(0)) > 0 \) and since the mass function \( m \) is increasing it follows that in the general case,

\[
m(r) \geq m(1) > 0, \quad r \geq 1,
\]

and since \( \zeta \) is increasing,

\[
y'(r) \leq -\zeta \left( \frac{m(1)}{r^2} \right), \quad r \geq 1. \tag{3.1}
\]

In the genuine MOND case \( \alpha = 1 \) the asymptotic behavior (2.4) of \( \zeta \) implies that there exists some constant \( \sigma_0 > 0 \) such that

\[
\zeta(\sigma) \geq \frac{1}{2} \sigma^{1/2}, \quad 0 \leq \sigma \leq \sigma_0.
\]

We can choose \( r_0 \geq 1 \) such that

\[
\frac{m(1)}{r^2} \leq \sigma_0, \quad r \geq r_0,
\]

and hence

\[
y'(r) \leq -\frac{1}{2} \frac{m(1)^{1/2}}{r}, \quad r \geq r_0
\]

This estimate implies that \( \lim_{r \to \infty} y(r) = -\infty \), in particular \( y(R) = 0 \) for some \( R > 0 \) and \( y(r) < 0 \) for \( r > R \). By (2.13), \( \rho(r) = 0 \) for \( r > R \), and the proof of part (a) is complete.
Let us consider the general case; in what follows $C > 0$ denotes a constant which can change from line to line and which does not depend on $r$. Clearly, (2.14) implies that $\zeta(\sigma) \geq C\sigma$, $\sigma \geq 0$, and hence

$$y'(r) \leq -C \frac{m(r, y)}{r^2}, \quad r > 0.$$  

We can now rely on the argument in [9, Lemma 3.1] to conclude that $y$ again has a zero. We include the short argument for the sake of completeness. The monotonicity of $g$ and $y$ imply that

$$y'(r) \geq -Cr^{2l+1} g(y(r)).$$

Hence for all $r > 0$,

$$\int_{y(r)}^{\eta_0} \frac{d\eta}{g(\eta)} = - \int_0^r \frac{y'(s)}{g(y(s))} ds \geq C \int_0^r s^{2l+1} ds = Cr^{2l+2}. \quad (3.2)$$

We need to show that $y_\infty := \lim_{r \to \infty} y(r) < 0$; the limit exists by monotonicity. If $y_\infty > 0$ then (3.2) gives a contradiction since the left hand side is then bounded by $\dot{y}/g(y_\infty)$. It remains to derive a contradiction from the assumption that $y_\infty = 0$. The assumption in (b) and (2.14) imply that

$$g(y) \geq Cy^{n+l}, \quad 0 \leq y \leq \eta_0, \quad (3.3)$$

where $0 < n = k + 3/2 < 3 + l$ by the assumption on $k$. By the assumption that $y_\infty = 0$, $0 < y(r) < \eta_0$ for $r$ sufficiently large. We split the left hand side in (3.2) accordingly, and by (3.3),

$$C r^{2l+2} \leq \int_{y(r)}^{\eta_0} \frac{d\eta}{\eta^{n+l}} + 1.$$  

We compute the integral and multiply by $y(r)^{2l+2}$ to find that

$$C \left( ry(r) \right)^{2l+2} \leq \left( \eta_0 \right)^{1-l-n} - y(r)^{1-l-n} \left| y(r)^{2l+2} + y(r)^{2l+2} \right|;$$

assume for the moment that $n + l \neq 1$. Since $n < l + 3$, the right hand side of this estimate goes to zero as $r \to \infty$. The same is true if $n + l = 1$, since then the integral yields $\ln y(r)$ and $2l + 2 > 0$. But (3.1), the fact that $\zeta(\sigma) \geq C\sigma$, and the assumption $y_\infty = 0$ imply that the left hand side is bounded from below by a positive constant. This contradiction completes the proof. \hfill \Box
Remark. In the Newtonian case it is known that an ansatz of the form
\[ f(x, v) = (E_0 - E)^k L^l; \] (3.4)
where the subscript + denotes the positive part leads to finite mass and compact support, provided \( k, l > -1 \) with \( k + l + 3/2 \geq 0 \) and \( k < 3l + 7/2 \) while the corresponding steady states have infinite mass for \( k > 3l + 7/2 \). For \( l + 3/2 < k < 3l + 7/2 \) these polytropic states have finite mass and radius, but are structurally unstable in the sense of [11, p. 378]. The robust finite radius proof based on [9, Lemma 3.1] breaks down for \( k > l + 3/2 \), and it is questionable whether part (b) of the theorem persists for polytropes with \( k > l + 3/2 \).

4 Additional results

4.1 Asymptotic behavior and potential energy

We define the energy of the gravitational field with potential \( U \) by
\[ S(U) := \frac{1}{2} a_0^2 \int F(|\nabla U|^2/a_0^2) dx, \] (4.1)
where we have restored the threshold parameter \( a_0 \) for easier comparison with (1.5), and
\[ F(\tau) := \int_0^\tau \mu(\sqrt{s}) ds, \quad \tau \geq 0. \]
Formally, the modified Poisson equation (1.5) is the Euler-Lagrange equation of \( S(U) + 4\pi \int U \rho \), and \( S(U) \) is part of the formally conserved total energy of the MONDian Vlasov-Poisson system. Hence it is desirable that \( S(U) \) be defined for steady states with compact support and finite mass, and this issue is related to the asymptotic behavior of the potential at infinity and the possibility of restoring the standard boundary condition (1.6).

Proposition 4.1 Consider a non-trivial, spherically symmetric steady state of the MONDian Vlasov-Poisson system.

(a) If \( \alpha < 1 \) and the steady state has finite mass, then \( \lim_{r \to \infty} U(r) < \infty \) and this limit can be taken to vanish so that (1.6) holds. The integral \( S(U) \) converges.

(b) In the genuine MOND case \( \alpha = 1 \), \( \lim_{r \to \infty} U(r) = \infty \) and \( S(U) = \infty \) even if the steady state has finite mass and compact support.
Proof. Since the steady state has positive, finite mass,
\[
\lim_{r \to \infty} \frac{m(r)}{r^2} = 0
\]
and by (2.4) there exist constants \(C_1, C_2 > 0\) such that
\[
C_1 r^{-2/(1+\alpha)} \leq \zeta \left( \frac{m(r)}{r^2} \right) \leq C_2 r^{-2/(1+\alpha)}
\]
(4.2)
for \(r\) sufficiently large. Now we observe that the assumption (2.1) on \(\mu\) implies that \(F \in C^1([0, \infty[)\) and
\[
\lim_{\tau \to 0} F(\tau^2 \tau^{-2-\alpha} = \frac{2}{2+\alpha}.
\]
Hence \(S(U)\) converges if and only if the function \(r^{2-2(2+\alpha)/(1+\alpha)} = r^{-2/(1+\alpha)}\)
is integrable on some interval \([R, \infty[\) with \(R > 0\). This is the case if and only if \(\alpha \in [0, 1[\) which proves the assertions on \(S(U)\). Moreover, (4.2) and (2.5) imply that
\[
\lim_{r \to \infty} U(r) = U(R) + \int_R^\infty U'(s) \, ds
\]
is finite for \(\alpha < 1\) and infinite for \(\alpha = 1\). The proof is complete.

Remark. The above proposition shows that the potential \(U\) is always confining in the genuine MOND case \(\alpha = 1\). This is consistent with the fact that in that case no restriction on the ansatz (2.12) is needed to guarantee finite mass and compact support of the resulting steady state. We illustrate this fact further by considering Maxwellian distributions.

4.2 Maxwellians
We assume that the distribution function \(f\) of a steady state is of the form
\[
f(x, v) = e^{-E} = e^{-\frac{1}{2}v^2 - U(x)}.
\]
(4.3)
Then the relation between the mass density and the potential becomes
\[
\rho(x) = (2\pi)^{3/2} e^{-U(x)},
\]
in particular, this relation has the regularity and monotonicity properties required for Proposition (2.2) so that the ansatz (1.3) leads to corresponding steady states of the MONDian Vlasov-Poisson system. It turns out that in the genuine MOND case such a Maxwellian steady state has finite mass.
Proposition 4.2  In the genuine MOND case $\alpha = 1$ the ansatz (4.3) leads to spherically symmetric steady states of the system (1.3), (1.4), (1.5) with finite mass.

Proof. We recall that there exists some constant $\sigma_0 > 0$ such that
\[ \zeta(\sigma) \geq \frac{1}{2} \sigma^{1/2}, \quad 0 \leq \sigma \leq \sigma_0. \]

Let us assume that a steady state of Maxwellian type has infinite mass. Then there exists some $r_0 > 0$ such that $m(r_0) > 36$. We can choose $R > r_0$ such that
\[ \frac{m(r_0)}{r^2} < \sigma_0, \quad r \geq R, \]
and hence by the monotonicity of $m$ and $\zeta$,
\[ U'(r) = \zeta \left( \frac{m(r)}{r^2} \right) \geq \zeta \left( \frac{m(r_0)}{r^2} \right) \geq \frac{m(r_0)^{1/2}}{2r} =: \frac{c_0}{r}, \quad r \geq R, \]
where $c_0 > 3$. Integrating this implies that
\[ U(r) \geq U(R) - c_0 \ln R + c_0 \ln r, \quad r \geq R, \]
and therefore
\[ m(r) = m(R) + (2\pi)^{3/2} A_{\pi} \int_R^\infty s^2 e^{-U(s)} ds \]
\[ \leq m(R) + (2\pi)^{3/2} A_{\pi} c_0 \ln R - U(R) \int_R^\infty s^2 - c_0 ds, \quad r \geq R. \]
Since $c_0 > 3$, the latter integral converges, and this contradicts the assumption that the total mass of the steady state is infinite.

If $\alpha < 1$ then an ansatz of the form (4.3) necessarily leads to steady states with infinite mass. This is a corollary to the following more general result which in turn is just an extension of the corresponding Newtonian result \[11,\] Theorem 2.1 (a)] to the present modified situation; for the sake of completeness we include the corresponding argument.

Proposition 4.3  Let $\alpha < 1$ and assume that an ansatz of the general form (2.7) leads to a spherically symmetric steady state with finite mass. Then there exists a cut-off energy $E_0$ such that $\Phi(E, L) = 0$ for almost all $(E, L)$ with $E > E_0$. 

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Proof. As shown in Proposition 4.1, \( U \) is increasing with a finite limit \( U_\infty \) at infinity. A simple change of variables shows that the total mass \( M \) of the steady state is given by

\[
M = 8\pi^2 \int_0^\infty \int_0^{U(r)} \int_0^{2r^2(E - U(r))} \Phi(E, L) \frac{dL dE dr}{\sqrt{2(E - U(r) - L/2r^2)}}
\]

\[
\geq 8\pi^2 \int_0^{U_\infty} \int_0^{\infty} \int_0^{\infty} \Phi(E, L) \frac{dr}{\sqrt{L/2(E - U(r))}} \frac{dL dE}{\sqrt{2(E - U(r))}}
\]

For \( E > U_\infty \) and \( L > 0 \) the integral with respect to \( r \) in the latter expression is infinite, and hence \( \Phi \) must vanish for such arguments. \( \square \)

4.3 The MONDian Euler-Poisson system

The results which we have discussed to far have counterparts if matter is described as an ideal, compressible fluid instead of a collisionless gas. We replace the Vlasov equation by the compressible Euler equations and from the start restrict ourselves to the spherically symmetric, time-independent case. The pressure \( p = p(r) \) is given in terms of the mass density \( \rho = \rho(r) \) via an equation of state

\[
p = P(\rho), \tag{4.4}
\]

the velocity field vanishes, and the static, spherically symmetric Euler equation reads

\[
P'(\rho) \rho' + \rho U' = 0. \tag{4.5}
\]

Supplemented with \( (2.5) \) these equations constitute the stationary, spherically symmetric case of the MONDian Euler-Poisson system; we refer to \([9]\) for its Newtonian analogue. We make the following technical assumptions on the equation of state.

Assumptions on \( P \). Let \( P \in C^1([0, \infty[) \) be such that \( P' > 0 \) on \( ]0, \infty[ \),

\[
\int_0^1 \frac{P'(s)}{s} ds < \infty, \quad \text{and} \quad \int_0^\infty \frac{P'(s)}{s} ds = \infty.
\]

We define

\[
Q(\rho) := \int_0^\rho \frac{P'(s)}{s} ds, \quad \rho \geq 0.
\]

Then \( Q : [0, \infty[ \to [0, \infty[ \) is one-to-one and onto, \( Q \in C([0, \infty[) \cap C^1([0, \infty[) \) with \( Q(0) = 0 \) and \( Q'(\rho) = P'(\rho)/\rho \) for \( \rho > 0 \). The Euler equation \( (4.5) \) holds provided

\[
Q(\rho(r)) = c - U(r), \quad r \geq 0, \tag{4.6}
\]
with some integration constant $c$. If we define

$$g(y) := \begin{cases} Q^{-1}(y) , & y > 0, \\ 0 , & y \leq 0, \end{cases} \quad (4.7)$$

then $g \in C(\mathbb{R}) \cap C^1([0, \infty[)$, and writing $y = c - U$ we invert the relation (4.6) to read as in (2.13) with $l = 0$ there. Hence the static Euler-Poisson system is reduced to the same equation (2.15) with mass function defined by (2.16) with $l = 0$ and with $g$ defined by (4.7) instead of (2.14). We are therefore back to the same situation as for the MONDian Vlasov-Poisson system. As in Proposition 2.2 we obtain for any choice of $y(0) = y_0 > 0$ a unique solution of (2.15). It induces a stationary, spherically symmetric solution of the MONDian Euler-Poisson system. In the genuine MOND case $\alpha = 1$ this steady state always has compact support and finite mass, while for the general case $\alpha \in [0, 1]$ these properties hold provided

$$P'(\rho) \leq c \rho^{1/n}$$

for $\rho > 0$ and small and with $0 < n < 3$. These assertions follow exactly as in Theorem 3.1 notice that under the above assumption $Q(\rho) \leq C \rho^{1/n}$ for small values of $\rho$ which in turn implies that $g$ defined in (4.7) satisfies the estimate (3.3) with $l = 0$, and the proof can proceed as for the Vlasov case.

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