Proton $\beta$ Decay in Large Magnetic Fields

Myron Bander$^*$

Department of Physics, University of California, Irvine, California 92717, USA

H. R. Rubinstein$^\dagger$

Department of Radiation Sciences, University of Uppsala, Uppsala, Sweden

(Received April 1992)

A delicate interplay between the anomalous magnetic moments of the proton and neutron makes, in magnetic fields $B \geq 2 \times 10^{14}$ T, the neutron stable and for fields $B \geq 5 \times 10^{14}$ T the proton becomes unstable to a decay into a neutron via $\beta$ emission. Limits on the field strengths for which these arguments hold are presented and are related to questions of vacuum stability in the presence of such fields. Possible astrophysical consequences are discussed.

$^*$e-mail: mbander@ucivmsa.bitnet; mbander@funth.ps.uci.edu

$^\dagger$e-mail: rub@vand.physto.se
I. INTRODUCTION

Very intense magnetic fields have been postulated to exist in connection with some astrophysical objects. Fields with strengths larger than $10^{14}$ T are associated with superconducting cosmic strings [1] and recently a proposal to explain extragalactic gamma ray bursts involves fields of around $10^{13}$ T [2]. We have been investigating several questions related to the existence of such fields and to the behavior of elementary and composite states in such environments. In a previous paper [3], we discussed the possible breakdown of constant magnetic fields with strengths beyond $10^{14}$ T. Other mechanisms that destabilize strong magnetic fields have been proposed [4], and here we add another intriguing phenomenon to this list. For fields greater than $5 \times 10^{15}$ T the proton becomes heavier than the neutron and decays into the latter by positron emission.

In Section II we study the behavior of an proton, neutron and electron in an intense magnetic field and find the aforementioned amusing result that the proton becomes unstable against neutron, positron and neutrino decay. The decay rates and spectrum are obtained in Section III. Whereas the discussion of the behavior of the electron is on firm footing, questions may be raised as to the validity of our treatment of the proton and neutron; these questions and the stability of the vacuum in the presence of strongly interacting particles with anomalous magnetic moments of non-electromagnetic origin are discussed in Section IV. Conclusions and experimental consequences are presented in the last section.

II. LOW LYING STATES FOR PARTICLES IN UNIFORM MAGNETIC FIELDS

The quantum mechanics of a Dirac particle with no anomalous magnetic moment in a uniform external magnetic field is straightforward. We shall present the results for the case where particles do have such anomalous moments. In reality, in fields so strong that the mass shifts induced by such fields are of the order of the mass itself one cannot define a magnetic moment as the energies are no longer linear in the external field. Schwinger [5]
calculated the self energy of an electron in an external field and we shall use his results subsequently. We cannot follow this procedure for the proton or neutron as we do not have a good field theory calculation of the magnetic moments of these particles, even for small magnetic fields; all we have at hand is a phenomenological anomalous magnetic moment. However, for fields that change the energies of these particles by only a few percent, we will consider these as point particle with the given anomalous moments. In the Section IV we will discuss possible limitations of this approach.

A. Protons in an External Field

The Dirac Hamiltonian for a proton with a uniform external magnetic field $B$ is

$$H = \alpha \cdot (p - eA(r)) + \beta M_p - \frac{e}{2M_p} \left( \frac{g_p}{2} - 1 \right) \beta \Sigma \cdot B.$$  \hspace{1cm} (1)

The vector potential $A(r)$ is related to the field by $A(r) = \frac{1}{2} \mathbf{r} \times \mathbf{B}$ and $g_p = 5.58$ is the proton’s Landé g factor. We first solve this equation for the case where the momentum along the magnetic field direction is zero and then boost along that direction till we obtain the desired momentum. For $B$ along the $z$ direction and $p_z = 0$ the energy levels are

$$E_{n,m,s} = \left[ 2eB(n + \frac{1}{2}) - eBs + M_p \right]^{\frac{1}{2}} - \frac{e}{2M_p} \left( \frac{g_p}{2} - 1 \right) Bs.$$  \hspace{1cm} (2)

In the above, $n$ denotes the Landau level, $m$ the orbital angular momentum about the magnetic field direction and $s = \pm 1$ indicates whether the spin is along or opposed to that direction; the levels are degenerate in $m$. $n = 0$ and $s = +1$ yield the lowest energy

$$E = \tilde{M}_p = M_p - \frac{e}{2M_p} \left( \frac{g_p}{2} - 1 \right) B.$$  \hspace{1cm} (3)

As we shall be interested in these states only we will drop the $n$ and $s$ quantum numbers. The Dirac wave function for this state is

$$\psi_{m,p_z=0}(\mathbf{r}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \phi_m(x,y).$$  \hspace{1cm} (4)
φ_m’s are the standard wave functions of the lowest Landau level;

\[ \phi_m(x, y) = \left[ \frac{\sqrt{\pi}}{eB} \right]^{\frac{m+1}{2}} [x + i y]^m \exp \left[ -\frac{1}{4} |eB|(x^2 + y^2) \right]. \]  

(5)

Boosting to a finite value of \( p_z \) is straightforward; we obtain

\[ E_m(p_z) = \sqrt{p_z^2 + \tilde{M}^2}, \]  

(6)

with a wave function

\[ \psi_{m,p_z}(r) = \begin{pmatrix} \cosh \theta \\ 0 \\ \sinh \theta \\ 0 \end{pmatrix} \frac{e^{ip_z z}}{\sqrt{2\pi}} \phi_m(x, y), \]  

(7)

where \( 2\theta \), the rapidity, is obtained from \( \tanh 2\theta = \frac{p_z}{E_m(p_z)}. \)

In the non-relativistic limit the energy becomes

\[ E_m(p_z) = \tilde{M} + \frac{p_z^2}{2M}, \]  

(8)

and the wave function reduces to

\[ \psi_{m,p_z}(r) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \frac{e^{ip_z z}}{\sqrt{2\pi}} \phi_m(x, y). \]  

(9)

**B. Neutrons in an External Field**

For a neutron the Dirac Hamiltonian is somewhat simpler

\[ H = \alpha \cdot p + \beta M_n - \frac{e}{2M_n} \left( \frac{g_n}{2} \right) \beta \Sigma \cdot B. \]  

(10)

with \( g_n = -3.82 \). Again for \( p_z = 0 \) the states of lowest energy, the ones we shall be interested in, have energies
\[ E(p_\perp, p_z = 0) = \frac{e}{2M_n} \left( \frac{g_n}{2} \right) B + \sqrt{p_\perp^2 + M_n^2} \] (11)

Boosting to a finite \( p_z \) we obtain

\[ E(p) = \sqrt{E(p_\perp, p_z = 0)^2 + p_z^2}. \] (12)

The wave functions corresponding to this energy are

\[ \psi_p(r) = \frac{e^{ip\cdot r}}{(2\pi)^{3/2}} u(p, s = -1), \] (13)

where \( u(p, s = -1) \) is the standard spinor for a particle with momentum \( p \), energy \( \sqrt{p^2 + M_n^2} \) (not the energy of Eq. (12)) and spin down.

In the non-relativistic limit

\[ E(p) = M_n + \frac{e}{2M_n} \left( \frac{g_n}{2} \right) B + \frac{p^2}{2M_n} \] (14)

and the wave functions are

\[ \psi_p(r) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \frac{e^{ip\cdot r}}{(2\pi)^{3/2}}. \] (15)

### C. Electrons in an External Field

We might be tempted to use, for the electron, the formalism used for the proton with the Landé factor replaced by \( g_e = 2 + \alpha/\pi \). However as we shall see for magnetic fields sufficiently strong as to make the proton heavier than the neutron the change in energy of the electron would appear to be larger than the mass of the electron itself. The point particle formalism breaks down and we have solve QED, to one loop, in a strong magnetic field; fortunately this problem was treated by Schwinger \[ 5 \]. The energy of an electron with \( p_z = 0 \), spin up and in the lowest Landau level is
\[ E_{m,p_z=0} = M_e \left[ 1 + \frac{\alpha}{2\pi} \ln \left( \frac{2eB}{M_e^2} \right) \right]. \]  \hspace{1cm} (16)

For field strengths of subsequent interest this correction is negligible; the energy of an electron in the lowest Landau level, with spin down and a momentum of \( p_z z \) is

\[ E_{m,p_z} = \sqrt{p_z^2 + M_e^2}. \]  \hspace{1cm} (17)

and with wave function similar to those of the proton

\[ \psi_{m,p_z}(r) = \begin{pmatrix} 0 \\ \cosh \theta \\ 0 \\ \sinh \theta \end{pmatrix} \frac{e^{ip_z z}}{\sqrt{2\pi}} \phi_m^*(x, y), \]  \hspace{1cm} (18)

where the boost rapidity, \( 2\theta \), is defined below Eq. (7) while the Landau level wave function is defined in Eq. (5). The reason the complex conjugate wave function appears is that the electron charge is opposite to that of the proton.

**D. Decay Kinematics**

From Eq. (3) and Eq. (14) we note that the neutron becomes stable against \( \beta \)-decay when the following inequality is satisfied

\[ -\frac{e}{2M_n} \left( \frac{g_n}{2} \right) B - \frac{e}{2M_p} \left( \frac{g_p}{2} - 1 \right) B \geq M_n - M_p - M_e, \]  \hspace{1cm} (19)

or \( B \geq 2 \times 10^{14} \) T. On the other hand the proton becomes unstable for decay into a neutron and a positron whenever

\[ \frac{e}{2M_n} \left( \frac{g_n}{2} \right) B - \frac{e}{2M_p} \left( \frac{g_p}{2} - 1 \right) B \sim 0.12\mu_N B \geq M_n + M_e - M_p, \]  \hspace{1cm} (20)

or for \( B \geq 5 \times 10^{14} \) T. We shall now turn to a calculation of the life time of the proton in fields satisfying this inequality.
III. PROTON LIFE TIME

A. Proton, Neutron and Electron Fields

With the wave functions of the various particles in the magnetic fields we may define field operators for these particles. For the proton and electron we shall restrict the summation over states to the lowest Landau levels with spin up, down respectively; for magnetic fields of interest the other states will not contribute to the calculation of decay properties. For the same reason, the neutron field will be restricted to spin down only. The proton and neutron kinematics will be taken as non-relativistic.

\[ \Psi_p(r) = \sum_m \int dp_z \left[ a_m(p_z) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \frac{e^{ip_z z}}{\sqrt{2\pi}} \phi_m(x, y) \\ + b_m^\dagger(p_z) \begin{pmatrix} 0 \\ 0 \quad 1 \\ 0 \end{pmatrix} \frac{e^{-ip_z z}}{\sqrt{2\pi}} \phi_m(x, y) \right], \quad (21) \]

with \( \phi_m(x, y) \) defined in Eq. (5) and the energy, \( E_m(p_z) \) in Eq. (8). \( a_m(p_z) \) is the annihilation operator for a proton with momentum \( p_z \mathbf{z} \) and angular momentum \( m \); \( b_m(p_z) \) is the same for the negative energy states. For the neutron the field is

\[ \Psi_n(r) = \int d^3p \left[ a(p) \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \frac{e^{ip \cdot r}}{(2\pi)^{3/2}} + b^\dagger(p) \begin{pmatrix} 0 \\ 0 \\ 0 \quad 1 \end{pmatrix} \frac{e^{-ip \cdot r}}{(2\pi)^{3/2}} \right], \quad (22) \]

with an obvious definition of the annihilation operators. For the electron we use fully relativistic kinematics and the field is
\[ \Psi_e(r) = \sum_m \int dp_z \sqrt{\frac{M_e}{E}} \begin{bmatrix} 0 \\ \cosh \theta \\ 0 \\ \sinh \theta \end{bmatrix} a_m(p_z) \begin{bmatrix} e^{ip_zz} \\ 0 \\ e^{-ip_zz} \end{bmatrix} \sqrt{2\pi} \phi^*_{m}(x, y) + b^\dagger_m(p_z) \begin{bmatrix} 0 \\ \cosh \theta \\ 0 \\ \sinh \theta \end{bmatrix} \sqrt{2\pi} \phi^*_{m}(x, y). \] (23)

**B. Decay Rates and Spectrum**

The part of the weak Hamiltonian responsible for the decay \( p \to n + e^+ + \nu_e \) is

\[ H = \frac{G_F}{\sqrt{2}} \int d^3x \bar{\Psi}_n \gamma_\mu (1 + \gamma_5) \Psi_p \bar{\Psi}_\nu \gamma^\mu (1 + \gamma_5) \Psi_e. \] (24)

For non-relativistic heavy particles the matrix element of this Hamiltonian between a proton with quantum numbers \( p_z = 0, m = m_i \), a neutron with momentum \( p_n \), a neutrino with momentum \( p_\nu \) and an electron in state \( m = m_f \) and with \( p_{z,e} \) is

\[
\langle H \rangle = \frac{2G_F}{(2\pi)^3} \left( \frac{E_e + p_{z,e}}{E_e - p_{z,e}} \right)^{1/2} \sin(\theta_\nu/2) \sqrt{\frac{M_e}{E_e}} \delta(p_{z,e} + p_{z,\nu} + p_{z,n}) \\
\int dx dy \phi^*_{m_f}(x, y) \phi_{m_i}(x, y) \exp[-i(p_{\perp,n} + p_{\perp,\nu}) \cdot r_{\perp}];
\] (25)

\( \theta_\nu \) is the azimuthal angle of the neutrino. The integral in the above expression can be evaluated in a multipole expansion. Note that the natural extent of the integral in the transverse direction is \( 1/\sqrt{eB} \) whereas the neutron momenta are, from Eq. (20), of the order of \( \sqrt{0.12eB} \); thus setting the exponential term in this integral equal to one will yield a good estimate for the rate and spectrum of this decay. The positron spectrum is given by

\[
\frac{d\Gamma}{dp_{z,e}} = \frac{4G_F^2 M_p E_e + p_{z,e}}{3 (2\pi)^6} (\Delta - E_e)^3;
\] (26)

where \( \Delta = 0.12\mu_N B - M_n + M_p \). For \( \Delta \gg M_e \) the total rate is easily obtained...
\[ \Gamma = \frac{2 G_F^2 M_p}{3} (2\pi)^6 \Delta^4. \quad (27) \]

The lifetime is \( \tau \sim 1.5 \times 10^2 (10^{15} T/B)^4 \) s.

**IV. FIELD STABILITY IN THE PRESENCE OF COMPOSITE HADRONS**

In Refs. [3,4] it was noted that vector particles with anomalous magnetic moments induce instabilities for large magnetic fields. Such a mechanism is not induced by spin one-half fields, even with anomalous moments. It is however clear that the discussion of the previous section regarding protons and neutrons has to break down at a sufficiently strong field. Let us look specifically at the neutron case. From Eq. (12) we see that for \( M_n - \mu B \leq 0 \) positive and negative energy levels cross and pair creation becomes possible. Following the analysis of Ref. [3] we can show that the vacuum does not decay into neutron-antineutron pair. (We cannot rule out the possibility that the strong interactions responsible for the anomalous moment could induce, for example, pion pair creation.) What occurs, is that the vacuum becomes a linear combination of the old vacuum and two fermion-antifermion pairs; a mixing with a single fermion-antifermion pair is ruled out on parity grounds.

It is amusing to look at this problem in two dimensions\(^3\). A Lagrangian for a charged Fermion with an anomalous moment is

\[ \mathcal{L} = i \overline{\psi} \gamma_\mu (\partial^\mu - e A^\mu) \psi + m \overline{\psi} \psi + \mu \overline{\psi} \sigma_{\mu\nu} \psi F^{\mu\nu}. \quad (28) \]

The bosonized version of this Lagrangian is [7]

\[ \mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + m \left[ 1 - \cos(2\sqrt{\pi} \phi) \right] + e E \phi + \frac{1}{2} \mu E \sin(2\sqrt{\pi} \phi). \quad (29) \]

In two dimensions there is only one field component, \( F^{0,1} = E \). We note that for non-zero charge, \( e \), a true instability develops, whereas for \( e = 0 \) and non-zero \( \mu \) the field \( \phi \) acquires a finite expectation value, corresponding to fermion pairs in the original model.

\(^3\)This two dimensional argument is due to A. Schwimmer.
From the above we expect the arguments of the previous sections to break down for critical fields, $B_c \sim M_n^2/e \sim 10^{16}$ T. However, the composite nature of the nucleons will cause a breakdown for lower fields. A reasonable estimate would be a critical field for the constituent quarks, $B \sim M_q^2/e \sim 10^{15}$ T. Thus our analysis should be valid for fields below this value and that leaves a window, $5 \times 10^{14}$ T $\leq B \leq 10^{15}$ T where we expect the proton to be heavier than the neutron and to decay into it.

V. CONCLUSIONS AND EXPERIMENTAL CONSEQUENCES

We studied the mass evolution of protons, neutrons and electrons in strong magnetic field and concluded that the proton will decay $\beta$ decay, $p \rightarrow n + e^+ + \nu_e$, in a sufficiently strong external magnetic field. The new mechanism for positron generation might have astrophysical interest. There is indeed an overabundance of positrons as compared with that accounted for by existing mechanisms [8].

VI. ACKNOWLEDGMENTS

We thank Miriam Leurer and Adam Schwimmer as well as other members of the Weizmann Institute for interesting discussions.
REFERENCES

[1] V. Berezinsky and H.R. Rubinstein, Nuclear Physics B323 (1989) 95

[2] R. Narayan, B. Paczyński and T. Piran, Harvard-Smithsonian Ctr. Astrophys. preprint CFA-3372, submitted to Astrophys. J. Letters

[3] M. Bander and H. R. Rubinstein, to appear in Phys. Letters

[4] See for example J. Ambjorn and P. Olesen, Nuclear Physics B310 (1988) 625

[5] J. Schwinger, *Particles, Sources, and Fields*, Vol. 3, (Addison-Wesley Pub. Co., Advanced Book Program, Redwood City, California, 1988), p. 164

[6] C. Itzykson and J.-B. Zuber, *Quantum Field Theory*, (McGraw Hill, New York, 1980), p. 67

[7] S. Coleman, Phys. Rev. D 11 (1975) 2088; J. Kogut and L. Susskind, Phys. Rev. D 11 (1975) 3594; S. Mandelstam, Phys. Rev. D 11 (1975) 3026; M. Bander, Phys. Rev. D 13 (1976) 1566

[8] C. J. Cesarsky, private communication; W. R. Webber in *20th International Cosmic Ray Conference*, Vol. 8, (Nauka, Moscow, 1988), p. 65