Continuous and discontinuous Galerkin time stepping methods for nonlinear initial value problems with application to finite time blow-up

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Abstract We consider continuous and discontinuous Galerkin time stepping methods of arbitrary order as applied to first-order initial value ordinary differential equation problems in real Hilbert spaces. Our only assumption is that the nonlinearities are continuous; in particular, we include the case of unbounded nonlinear operators. Specifically, we develop new techniques to prove general Peano-type existence results for discrete solutions. In particular, our results show that the existence of solutions is independent of the local approximation order, and only requires the local time steps to be sufficiently small (independent of the polynomial degree). The uniqueness of (local) solutions is addressed as well. In addition, our theory is applied to finite time blow-up problems with nonlinearities of algebraic growth. For such problems we develop a time step selection algorithm for the purpose of numerically computing the blow-up time, and provide a convergence result.

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1 Introduction

In this paper we focus on continuous Galerkin (cG) as well as on discontinuous Galerkin (dG) time stepping discretizations (of any order) as applied to abstract first-order ordinary differential equation problems of the form

$$u'(t) = F(t, u(t)), \quad t \in (0, T), \quad u(0) = u_0.$$  \hspace{1cm} (1)

Here, $u : (0, T) \to H$, for some $T > 0$, is an unknown solution, with values in a real Hilbert space $H$ (with inner product denoted by $(\cdot, \cdot)_H$ and induced norm $\| \cdot \|_H$). The initial value $u_0 \in H$ prescribes the solution $u$ at the start, $t = 0$, and $F : [0, T] \times H \to H$ is a possibly nonlinear, continuous operator. We emphasize that we include, for instance, the case of $F$ being (continuous and nonlinear and) unbounded in the sense that

$$\frac{\|F(t, x)\|_H}{\|x\|_H} \to \infty \text{ as } \|x\|_H \to \infty, \quad 0 \leq t \leq T.$$ \hspace{1cm} (2)

In the sequel, we will usually omit to explicitly write the dependence on the first argument $t$.

For $H = \mathbb{R}^N$ and continuous nonlinearities $F$, the well-known Peano Theorem (see, e.g., [25]) guarantees the existence of $C^1$-solutions $u$ of (1) within some limited time range, $t \in (0, T_\infty)$, for some $T_\infty > 0$. Generalizations to problems in Banach spaces are available as well; see, e.g., [11]. Notice that the existence interval for solutions may be arbitrarily small even for smooth $F$: For instance, solutions of (1) may become unbounded in finite time, i.e.,

$$\|u(t)\|_H < \infty \text{ for } 0 < t < T_\infty, \quad \lim_{t \uparrow T_\infty} \|u(t)\|_H = \infty.$$  

This effect is commonly termed (finite-time) blow-up.

Galerkin time stepping

Galerkin-type time stepping methods for initial-value problems are based on weak formulations. For both the cG and the dG time stepping schemes, the test spaces consist of polynomials that are discontinuous at the time nodes. In this way, the discrete Galerkin formulations decouple into local problems on each time step, and the discretizations can therefore be understood as implicit one-step schemes. Results on a priori as well as a posteriori error estimates of Galerkin time stepping methods in the context of ordinary differential equations (ODEs), which is also the focus of the present work, can be found, e.g., in [5,7–10,12,16].

A key feature of Galerkin time stepping methods is their great flexibility with respect to the size of the time steps and the local approximation orders, thereby naturally leading to an $hp$-version Galerkin framework. The $hp$-versions of the cG and dG time stepping schemes were introduced and analyzed in the works [20,21,23,27]. In
particular, in the articles [20, 27], which focus on ordinary initial value problems with uniform Lipschitz nonlinearities, the use of the contraction mapping theorem made it possible to prove existence and uniqueness results for discrete Galerkin solutions, which are independent of the local approximation orders. We emphasize that the $hp$-approach is well-known for its ability to approximate smooth solutions with possible local singularities at high algebraic or even exponential rates of convergence; see, e.g., [6, 21, 22, 26] for the numerical approximation of problems with start-up singularities.

**Results**

The goal of the current paper is to extend the existence results on $hp$-type Galerkin time stepping schemes for initial value problems with Lipschitz-type nonlinearities in [20, 27] to abstract problems with nonlinearities which are merely continuous. We emphasize that this generalization is substantial; indeed, it covers, for example, the case of unbounded nonlinearities as in (2). Based on writing the weak Galerkin formulations in strong (pointwise) form along the lines of [3, 17, 23], we develop a new technique that allows to obtain existence results which are uniform with respect to the local polynomial degrees. As a first step, suitable fixed-point forms will be derived. In the context of the cG method, this is accomplished within an integral equation framework. For the dG scheme, matters are more sophisticated, and a careful investigation of the discrete time operator, which involves a lifting operator from [23], is required on the local polynomial approximation space; this operator turns out to be an isomorphism on the underlying polynomial spaces (with a continuity constant of the inverse operator that is independent of the local polynomial degrees) and allows to transform the strong dG form into a fixed point equation. Subsequently, for both the cG and the dG schemes the application of Brouwer’s fixed point theorem yields the existence of discrete solutions; see Theorem 1. In particular, as in the case of Lipschitz continuous nonlinearities [20, 27], the existence results do not depend on the local polynomial degrees, and only require the local time steps to be sufficiently small. In this sense, our theory constitutes a discrete version of Peano’s Theorem. Furthermore, employing a contraction argument along the lines of the approach presented in [4], we show that the local Galerkin formulations are uniquely solvable (within a certain range); cf. Theorem 2.

In addition, we apply our general theory to ordinary differential equation problems with nonlinearities of algebraic growth, i.e., $F(t, u) \sim \alpha \|u\|_H^\beta$, with $\alpha > 0$, $\beta > 1$, and for a given range of $t$; in this case, the initial value problem (1) features a solution that blows up at a finite time $T_\infty$. We will show that a careful selection of locally varying time steps in the cG and dG time stepping schemes results in discrete solutions that blow up as well; in this context, we mention the paper [24] which illustrates the importance of variable step size selection. More precisely, following some ideas from [18], we derive an analysis which allows to choose the local time steps a posteriori as the time marching process is moving forward. We develop a time step selection algorithm which guarantees the existence and uniqueness of local solutions, and provides a numerical approximation of the exact blow-up time. Moreover, again following the approach taken in [18], we prove a convergence result which shows that the blow-up time can be approximated arbitrarily well if the time steps are scaled sufficiently small.
Related work and extensions

The concepts and technical tools developed in our current work constitute an important stepping stone with regard to the numerical treatment of nonlinear parabolic partial differential equations (PDE) by fully discrete Galerkin methods in time and space. In this context, let us point to the work [2], where the cG and dG time-stepping methods for parabolic partial differential equations (PDE) with Lipschitz continuous nonlinearities have been studied. In addition, we mention the papers [13–15] on Galerkin time discretizations for nonlinear Schrödinger and wave type equations.

In principal, semi-discretizations of nonlinear parabolic problems in space, which result in a system of ordinary differential equations in time, could be treated by means of the Galerkin time stepping methods to be analyzed in this paper. We emphasize, however, that the Peano type approach taken here (guaranteeing local existence of Galerkin solutions for ODE type problems) will require time steps that will typically depend on the spatial stiffness matrix in an unfavourable way (i.e., on the mesh size within a finite difference or finite element framework); c.f., e.g. [4]. Indeed, the analysis of fully discrete Galerkin discretizations of semilinear PDE should exploit the parabolic evolution operator (including the linear elliptic operator) rather than treating the time discretization separately. In particular, like in the present paper, a $p$-robust existence (and local uniqueness) analysis of Galerkin time-stepping methods for nonlinear parabolic PDEs requires new ($p$-uniform) stability results for the fully discrete differential operators; this is subject of ongoing work [19].

Outline

Our article is organized as follows: Sect. 2 presents the cG and dG time stepping schemes. Furthermore, Sect. 3 centres on the development of existence proofs for discrete solutions. The question of uniqueness is addressed in Sect. 4. Moreover, the application of our results to algebraically growing nonlinearities causing finite time blow-ups will be worked out in Sect. 5. Finally, the article closes with a few concluding remarks in Sect. 6.

Notation

Throughout the paper, Bochner spaces will be used: For an interval $I = (a, b)$ and a real Hilbert space $H$ as before, the space $C^0(I; H)$ consists of all functions $u : I \rightarrow H$ that are continuous on $I$ with values in $H$. Moreover, introducing, for $1 \leq p \leq \infty$, the norm

$$
\|u\|_{L^p(I; H)} = \begin{cases} 
\left( \int_I \|u(t)\|^p_H \, dt \right)^{1/p}, & 1 \leq p < \infty, \\
\text{ess sup}_{t \in I} \|u(t)\|_H, & p = \infty,
\end{cases}
$$

we write $L^p(I; H)$ to signify the space of measurable functions $u : I \rightarrow H$ so that the corresponding norm is bounded. We notice that $L^2(I; H)$ is a Hilbert space with inner product and induced norm.
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\[(u, v)_{L^2(I; H)} = \int_I (u(t), v(t))_H \, dt, \quad \text{and} \quad \|u\|_{L^2(I; H)} = \left(\int_I \|u(t)\|_H^2 \, dt\right)^{1/2},\]

respectively.

2 Galerkin time discretizations

In this section we present the \(hp\)-cG and \(hp\)-dG time stepping methods as applied to (1).

2.1 \(hp\)-cG time stepping

On an interval \(I = [0, T]\), \(T > 0\), consider time nodes \(0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = T\) which introduce a time partition \(\mathcal{M} = \{I_m\}_{m=1}^M\) of \(I\) into \(M\) open time intervals \(I_m = (t_{m-1}, t_m), m = 1, \ldots, M\). The (possibly varying) length \(k_m = t_m - t_{m-1}\) of a time interval is called the \(m\)th time step. Furthermore, to each interval we associate a polynomial degree \(r_m \geq 0\) which takes the role of a local approximation order. Moreover, given a (real) Hilbert space \(X \subset H\), an integer \(r \in \mathbb{N}_0\), and an interval \(J \subset \mathbb{R}\), the set

\[P^r(J; X) = \left\{ p \in C^0(\bar{J}; X) : p(t) = \sum_{i=0}^r x_i t^i, x_i \in X \right\}\]

signifies the space of all polynomials of degree at most \(r\) on \(J\) with values in \(X\).

In practical computations, the Hilbert space \(H\), on which (1) is based, will typically be replaced by a finite-dimensional subspace \(H_m \subset H\), \(\dim(H_m) < \infty\), on each interval \(I_m, 1 \leq m \leq M\). The \(H\)-orthogonal projection from \(H\) to \(H_m\) is defined by

\[\pi_m : H \to H_m, \quad (x - \pi_m x, y)_H = 0 \quad \forall y \in H_m. \quad (3)\]

With these definitions, the (fully discrete) \(hp\)-cG time marching scheme is iteratively given as follows: For an initial value \(U_{m-1} := \lim_{t \to t_{m-1}} U|_{I_{m-1}}(t) \in H\) (with \(U_0 := u_0\), where \(u_0 \in H\) is the initial value from (1)), we find \(U|_{I_m} \in P^{r_m+1}(I_m; H_m)\) through the weak formulation

\[\int_{I_{m}} (U', V)_H \, dt = \int_{I_{m}} (\mathcal{F}(U), V)_H \, dt \quad \forall V \in P^{r_m}(I_m; H_m), \quad (4)\]

for any \(1 \leq m \leq M\). Notice that, in order to enforce the initial condition on each individual time step, the local trial space has one degree of freedom more than the local test space. Furthermore, if \(H_1 = H_2 = \cdots = H_M\), we remark that the continuous Galerkin solution \(U\) is globally continuous on \((0, T)\).
2.2 \( hp \)-dG time stepping

In order to define the discontinuous Galerkin scheme, some additional notation is required: We define the one-sided limits of a piecewise continuous function \( U \) at each time node \( t_m \) by

\[
U^+_m := \lim_{s \searrow 0} U(t_m + s), \quad U^-_m := \lim_{s \searrow 0} U(t_m - s).
\]

Then, the discontinuity jump of \( U \) at \( t_m \), \( 0 \leq m \leq M - 1 \), is defined by \( \|U\|_m := U^+_m - U^-_m \), where we let \( U^-_0 := u_0 \), with \( u_0 \) being the initial condition from (1). Then, the (fully discrete) \( hp \)-dG time stepping method for (1) reads: Find \( U|_{I_m} \in \mathcal{P}^{r_m}(I_m; H_m) \) such that

\[
\int_{I_m} (U', V)_H \, dt + (\|U\|_{m-1}, V^+_{m-1})_H = \int_{I_m} (\mathcal{F}(U), V)_H \, dt \quad \forall V \in \mathcal{P}^{r_m}(I_m; H_m), \tag{5}
\]

for any \( 1 \leq m \leq M \). We emphasize that, in contrast to the continuous Galerkin formulation, the trial and test spaces are the same for the discontinuous Galerkin scheme. This is due to the fact that the initial values are weakly imposed (by means of an upwind flux) on each time interval.

3 Existence of discrete solutions

In this Section our goal is to show existence of solutions to the discrete local problems (4) and (5):

**Theorem 1** Let \( m \geq 1 \). Then, if the local time step \( k_m > 0 \) is chosen sufficiently small (independent of the local polynomial degree \( r_m \)), then the continuous Galerkin method (4) and the discontinuous Galerkin method (5) on \( I_m \) both possess at least one solution \( U_{cG} \in \mathcal{P}^{r_m+1}(I_m; H_m) \) and \( U_{dG} \in \mathcal{P}^{r_m}(I_m; H_m) \), respectively.

Our general strategy of proof is to represent the Galerkin formulations in terms of strong (pointwise) equations, cf. [3,17,23], and then to derive suitable (integral equation) fixed-point formulations similarly as in the proof of the classical Peano existence theorem for ordinary differential equations. In the case of the dG method, this requires a stability result for the discrete time discretization operator which is uniform with respect to the local polynomial degree; see Proposition 1 below. For both time stepping schemes, the existence of discrete solutions will follow from the application of Brouwer’s fixed point theorem.
3.1 Existence of cG solutions

We begin by rewriting (4) as finding $U \in \mathcal{P}^{r_m+1}(I_m; H_m)$ such that

$$
\int_{I_m} (U' - \Pi_m^r \mathcal{F}(U), V)_H \, dt = 0 \quad \forall V \in \mathcal{P}^{r_m}(I_m; H_m),
$$

$$
U(t_{m-1}) = \pi_m U_{m-1}.
$$

Here, $\Pi_m^r : L^2(I_m; H) \rightarrow \mathcal{P}^{r_m}(I_m; H_m)$ denotes the $L^2$-projection onto the space $\mathcal{P}^{r_m}(I_m; H_m)$, which is uniquely defined by

$$
u \mapsto \Pi_m^r \nu : \int_{I_m} (u - \Pi_m^r u, V)_H \, dt = 0 \quad \forall V \in \mathcal{P}^{r_m}(I_m; H_m). \quad (6)
$$

Thence, noticing that $U' - \Pi_m^r \mathcal{F}(U) \in \mathcal{P}^{r_m}(I_m; H_m)$, we obtain the strong form

$$
U' - \Pi_m^r \mathcal{F}(U) = 0 \quad \text{on } I_m,
$$

$$
U(t_{m-1}) = \pi_m U_{m-1}.
$$

Integration results in

$$
U(t) = \pi_m U_{m-1} + \int_{t_{m-1}}^t \Pi_m^r \mathcal{F}(U) \, d\tau, \quad t \in I_m. \quad (7)
$$

We see that the operator

$$
\mathcal{T}^{cG}_m(U) := \pi_m U_{m-1} + \int_{t_{m-1}}^t \Pi_m^r \mathcal{F}(U) \, d\tau \quad (8)
$$

maps $\mathcal{P}^{r_m+1}(I_m; H_m)$ into itself, and hence, the integral equation (7) is a fixed point formulation,

$$
\mathcal{T}^{cG}_m(U) = U, \quad (9)
$$

on $\mathcal{P}^{r_m+1}(I_m; H_m)$. In particular, any solution of (7) will solve (4).

We are ready to prove Theorem 1 for the continuous Galerkin method (4): For some $\kappa_m, \theta_m > 0$ (with $t_{m-1} + \theta_m \leq T$) let us define the set

$$
Q_m = [t_{m-1}, t_{m-1} + \theta_m] \times B_m,
$$

where

$$
B_m = \{ y \in H_m : \| y - \pi_m U_{m-1} \|_H \leq \kappa_m \}. \quad (10)
$$

Since $\mathcal{F}$ is continuous, its maximum on the compact set $Q_m$,

$$
K_m := \max_{(t, y) \in Q_m} \| \mathcal{F}(t, y) \|_H. \quad (11)
$$
exists. We let

\[ 0 < k_m \leq \min(\theta_m, K_m^{-1} \kappa_m). \]

Then, we introduce

\[ M_{cG}^m := \{ Y \in \mathcal{P}_{r_m+1}(I_m; H_m) : Y(t) \in B_m \forall t \in T_m \}, \tag{12} \]

where \( I_m = (t_{m-1}, t_m) \), with \( t_m = t_{m-1} + k_m \).

Let \( U \in M_{cG}^m \) be arbitrary, and \( t^* \in T_m \) such that

\[ \| T_{m}^{cG}(U)(t^*) - \pi_m U_{m-1} \|_H = \| T_{m}^{cG}(U) - \pi_m U_{m-1} \|_{L^\infty(I_m; H)}. \]

Then, using Bochner’s Theorem as well as the Cauchy-Schwarz inequality, yields

\[
\| T_{m}^{cG}(U) - \pi_m U_{m-1} \|_{L^\infty(I_m; H)} \leq \left\| \int_{t_{m-1}}^{t^*} \Pi_{m}^r F(U) \, dt \right\|_H \\
\leq \int_{I_m} \| \Pi_{m}^r F(U) \|_H \, dt \\
\leq k_m^{1/2} \| \Pi_{m}^r F(U) \|_{L^2(I_m; H)}. 
\]

Taking into account the boundedness of the \( L^2 \)-projection on \( I_m \) (with constant 1) leads to

\[ \| T_{m}^{cG}(U) - \pi_m U_{m-1} \|_{L^\infty(I_m; H)} \leq k_m^{1/2} \| F(U) \|_{L^2(I_m; H)} \leq k_m \| F(U) \|_{L^\infty(I_m; H)}. \]

Therefore,

\[ \| T_{m}^{cG}(U) - \pi_m U_{m-1} \|_{L^\infty(I_m; H)} \leq K_m k_m \leq k_m. \]

Thus, we have \( T_{m}^{cG}(U) \in M_{cG}^m \), and more generally, it follows \( T_{m}^{cG}(M_{cG}^m) \subseteq M_{cG}^m \). Finally, since \( M_{cG}^m \) is convex and compact, and \( T_{m}^{cG} \) is continuous, Brouwer’s fixed point theorem implies that there exists at least one solution of (9) in \( M_{cG}^m \), and thus of (4).

### 3.2 Existence of dG solutions

The situation for the dG method is more involved for two reasons: Firstly, the dG formulation incorporates the initial value in a weak sense. Specifically, this leads to the appearance of a jump term in (5), which, in turn, prevents a direct derivation of a strong form. The problem can be resolved by applying a lifting operator technique as proposed in [17] (and discussed in the \( hp \)-context in [23]). This is the content of the following Sect. 3.2.1. Secondly, in contrast to the fixed point form (7) of the cG scheme, the dG method does not have an obvious representation in terms of an integral.
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Indeed, integrating the strong form of the dG method, see (13) below, in a straightforward way would lead to a fixed point operator that maps $\mathcal{P}^{r_m}(I_m; H_m)$ into $\mathcal{P}^{r_m+1}(I_m; H_m)$, and hence, would not be a self-map. This problem is addressed in Proposition 1 and Remark 3, where we investigate the invertibility and the stability of the discrete dG time derivate operator.

### 3.2.1 Discrete dG time operator

Following [23, Sect. 4.1] we define the lifting operator, for $1 \leq m \leq M$,

$$L_m^r : X \to \mathcal{P}^{r_m}(I_m; X),$$

by

$$\int_{I_m} (L_m^r(z), V)_X \, dt = (z, V_{m-1}^+) \quad \forall V \in \mathcal{P}^{r_m}(I_m; X), \ z \in X,$$

on a real Hilbert space $X$, with inner product $(\cdot, \cdot)_X$, and norm $\| \cdot \|_X$. In view of this definition with $X = H_m$, we have for the dG solution $U \in \mathcal{P}^{r_m}(I_m; H_m)$ from (5):

$$0 = \int_{I_m} \{ (U', V)_H - (F(U), V)_H \} \, dt + (\| U \|_{m-1}, V_{m-1}^+)$$

$$= \int_{I_m} \{ (U', V)_H - (\Pi_m^r F(U), V)_H \} \, dt + (\pi_m \| U \|_{m-1}, V_{m-1}^+)$$

$$= \int_{I_m} \{ (U' + L_m^r (\pi_m \| U \|_{m-1}) - \Pi_m^r F(U), V)_H \} \, dt,$$

for any $V \in \mathcal{P}^{r_m}(I_m; H_m)$. Here, $\pi_m$ is the $H$-orthogonal projection from (3), and $\Pi_m^r$ is the $L^2$-projection from (6).

Then, since $U', L_m^r (\pi_m \| U \|_{m-1}), \Pi_m^r F(U)$ all belong to $\mathcal{P}^{r_m}(I_m; H_m)$, we arrive at the strong formulation

$$U' + L_m^r (\pi_m \| U \|_{m-1}) = \Pi_m^r F(U)$$

(13) of (5). The term on the left-hand side of this equation is the $hp$-dG time discretization of the continuous derivative operator $u \mapsto u'$. This motivates the definition of a discrete operator

$$\chi : \mathcal{P}^{r_m}(I_m; H_m) \to \mathcal{P}^{r_m}(I_m; H_m)$$

given by

$$U \mapsto \chi(U) = U' + L_m^r (U^+_{m-1}).$$

(15)

For the proof of existence of solutions of (13) it is important to notice that the linear operator $\chi$ possesses an inverse that is uniformly stable with respect to the polynomial degree.
Proposition 1 Let $X$ be a real Hilbert space, and $1 \leq m \leq M$. Then, the operator $\chi$ from (14) and (15) is an isomorphism on $\mathcal{P}^{r_m}(I_m; X)$. In addition, there exists a constant $C_\chi > 0$ independent of the time step $k_m$ and the local approximation order $r_m$ such that, for any $p \in [1, \infty]$, there holds the bound

$$\| U \|_{L^\infty(I_m; X)} \leq C_\chi k_m^{1-1/p} \| \chi(U) \|_{L^p(I_m; X)},$$

for any $U \in \mathcal{P}^{r_m}(I_m; X)$.

In order to establish this estimate, we require two auxiliary results which will be proved first.

Lemma 1 Let $1 \leq m \leq M$, and $X$ a real Hilbert space. Then, there holds

$$\sup_{t \in I_m} \left\| z - \int_{t_m-1}^t L_m^r(z) \, d\tau \right\|_X = \| z \|_X,$$  \hspace{1cm} (17)

for any $z \in X$.

Proof Let us first consider the lifting operator $\hat{\mathcal{L}}^m : X \to \mathcal{P}^{r_m}(\hat{I}; X)$ on the unit interval $\hat{I} = (-1, 1)$, defined by

$$\int_{-1}^1 (\hat{L}^m(z), \hat{V})_X \, d\hat{t} = (z, \hat{V}(-1))_X \hspace{1cm} \forall \hat{V} \in \mathcal{P}^{r_m}(\hat{I}; X), z \in X. \hspace{1cm} (18)$$

Referring to [23, Eq. (35) and Lemma 8] there holds the explicit formula

$$z - \int_{-1}^t \hat{L}^m(z) \, d\hat{\tau} = \frac{z}{2} \left( 1 - \hat{t} + \sum_{i=2}^{r_m+1} (-1)^i (2i - 1) \hat{Q}_i(\hat{t}) \right), \hspace{1cm} t \in \hat{I},$$

where

$$\hat{Q}_i(\hat{t}) = \int_{-1}^\hat{t} \hat{K}_{i-1}(\hat{\tau}) \, d\hat{\tau} = \frac{\hat{K}_i(\hat{t}) - \hat{K}_{i-2}(\hat{t})}{2i-1}, \hspace{1cm} i \geq 2,$$

with $\{\hat{K}_i\}_{i \geq 0}$ signifying the family of Legendre polynomials on $(-1, 1)$ (with degrees $\deg(\hat{K}_i) = i$), scaled such that $\hat{K}_i(-1) = (-1)^i$; cf. [23, Eq. (9) and Lemma 1]. Combining the above identities, we obtain

$$z - \int_{-1}^t \hat{L}^m(z) \, d\hat{\tau} = \frac{z}{2} \left( 1 - \hat{t} + \sum_{i=2}^{r_m+1} (-1)^i (\hat{K}_i(\hat{t}) - \hat{K}_{i-2}(\hat{t})) \right).$$

Noticing the telescope sum as well as the fact that $\hat{K}_0(\hat{t}) = 1$ and $\hat{K}_1(\hat{t}) = \hat{t}$, we arrive at

$$z - \int_{-1}^t \hat{L}^m(z) \, d\hat{\tau} = \frac{z}{2} (-1)^{r_m+1} \left( \hat{K}_{r_m+1}(\hat{t}) - \hat{K}_{r_m}(\hat{t}) \right).$$ \hspace{1cm} (19)
Then, employing the fact that
\[ |\tilde{K}_i(\hat{t})| \leq 1 \quad \forall \hat{t} \in [-1, 1], \forall i \geq 0, \] (20)
results in
\[ \|z - \int_{-1}^{\hat{t}} \tilde{L}_m(z) \, d\hat{\tau}\|_X \leq \|z\|_X \quad \forall \hat{t} \in \hat{I}. \]

Now we define the affine mapping
\[ F_m : \hat{I} \to I_m, \quad \hat{t} \mapsto \frac{1}{2} k_m \hat{t} + \frac{1}{2} (t_{m-1} + t_m). \] (21)
A scaling argument implies that
\[ \tilde{L}_m(z) \circ F_m = \frac{2}{k_m} \tilde{L}_m(z); \]
see [23, Lemma 7]. Hence, by a change of variables, \( \tau = F_m(\hat{\tau}), \, d\tau = \frac{k_m}{2} d\hat{\tau} \), we conclude that
\[ \|z - \int_{t_{m-1}}^{t} L_m(z) \, d\tau\|_X = \|z - \int_{-1}^{F_m^{-1}(t)} \tilde{L}_m(z) \, d\hat{\tau}\|_X \leq \|z\|_X \quad \forall t \in I_m. \]
Noticing that, for \( t = t_{m-1} \), there holds equality in the above bound, completes the proof. \( \square \)

**Remark 1** Using the identity (19), it is fairly elementary to verify that the lifting operator \( \tilde{L}_m \) from (18) can be written in the form
\[ \tilde{L}_m(z) = \frac{z}{2} (-1)^{r_m} (r_m + 1) J_{r_m}^{(0,1)}, \quad z \in X, \]
where \( \{J_{r}^{(\alpha, \beta)}\}_{r \in \mathbb{N}_0} \) denotes the class of Jacobi polynomials of order \((\alpha, \beta)\), with \(\alpha, \beta > -1\), on \( I \).

**Lemma 2** Let \( 1 \leq m \leq M \), and \( X \) a real Hilbert space. Then, the bound
\[ \|U_{m-1}^+\|_X \leq \|\chi(U)\|_{L^1(I_m; X)} \] (22)
holds true for any \( U \in \mathcal{P}^{r_m}(I_m; X) \).

**Proof** Let \( U \in \mathcal{P}^{r_m}(I_m; X) \). We define
\[ \mathcal{Y}^{r_m} := (-1)^{r_m} U_{m-1}^+ \left( \hat{K}_m \circ F_m^{-1} \right) \in \mathcal{P}^{r_m}(I_m; X), \]
where \( \hat{K}_{rm} \) is the \( rm \)-th Legendre polynomial on \((-1, 1)\), which we scale such that \( \hat{K}_{rm}(-1) = (-1)^{rm} \) (cf. the proof of Lemma 1), and \( F_m \) is the affine element mapping from (21). Then,

\[
\| \Upsilon_m^{rm} \|_{L^\infty(Im; X)} \leq \| U_{m-1}^+ \|_X \| \hat{K}_{rm} \circ F_m^{-1} \|_{L^\infty(Im)}
\leq \| U_{m-1}^+ \|_X \| \hat{K}_{rm} \|_{L^\infty(-1, 1)}.
\]

Involving (20) shows

\[
\| \Upsilon_m^{rm} \|_{L^\infty(Im; X)} \leq \| U_{m-1}^+ \|_X. \tag{23}
\]

Further, \( \Upsilon_m^{rm} \) is orthogonal to the space \( P_{rm}^{-1}(Im; X) \) (where \( P_{rm}^{-1}(Im; X) := \{0\} \subset X \)) with respect to the inner product in \( L^2(Im; X) \). In particular, since \( U' \in P_{rm}^{-1}(Im; X) \), we have

\[
\int_{Im} (U', \Upsilon_m^{rm})_X \, dt = 0.
\]

So, noticing that \( \Upsilon_m^{rm}(t_{m-1}^+) = U_{m-1}^+ \), it follows that

\[
\int_{Im} (\chi(U), \Upsilon_m^{rm})_X \, dt = \int_{Im} (L_{m}^{rm}(U_{m-1}^+), \Upsilon_m^{rm})_X \, dt
= (U_{m-1}^+, \Upsilon_m^{rm}(t_{m-1}^+))_X = \| U_{m-1}^+ \|_X^2.
\]

Therefore, using Hölder’s inequality and recalling (23), we conclude that

\[
\| U_{m-1}^+ \|_X^2 \leq \| \chi(U) \|_{L^1(Im; X)} \| \Upsilon_m^{rm} \|_{L^\infty(Im; X)} \leq \| \chi(U) \|_{L^1(Im; X)} \| U_{m-1}^+ \|_X.
\]

Dividing by \( \| U_{m-1}^+ \|_X \) shows the desired bound.

We are now ready to show Proposition 1.

**Proof (Proof of Proposition 1)** Consider \( U \in \mathcal{P}^{rm}(Im; X) \). We choose \( t^* \in \bar{I}_m \) such that \( \| U(t^*) \|_X = \| U \|_{L^\infty(Im; X)} \). It holds that

\[
U(t^*) = \int_{t_{m-1}}^{t^*} (U' + L_{m}^{rm}(U_{m-1}^+)) \, d\tau + U_{m-1}^+ - \int_{t_{m-1}}^{t^*} L_{m}^{rm}(U_{m-1}^+) \, d\tau.
\]

Applying the triangle inequality as well as Bochner’s Theorem, and recalling (17), this implies that

\[
\| U \|_{L^\infty(Im; X)} \leq \int_{t_{m-1}}^{t^*} \| \chi(U) \|_X \, d\tau + \| U_{m-1}^+ \|_X + \int_{t_{m-1}}^{t^*} L_{m}^{rm}(U_{m-1}^+) \, d\tau
\leq \| \chi(U) \|_{L^1(Im; X)} + \| U_{m-1}^+ \|_X.
\]
Inserting the bound (22) results in
\[ \| U \|_{L^\infty(I_m;X)} \leq 2 \| \chi(U) \|_{L^1(I_m;X)}, \]
and applying Hölder’s inequality completes the proof with \( C_\chi = 2 \).

**Remark 2** The proof of Proposition 1 reveals the upper bound \( C_\chi \leq 2 \). We emphasize, in particular, that the estimate (16) is uniform with respect to the local polynomial degree \( r_m \geq 0 \) as \( r_m \to \infty \).

**Remark 3** Upon setting \( U = \chi^{-1}(V) \) in (16), we obtain
\[ \| \chi^{-1}(V) \|_{L^\infty(I_m;X)} \leq C_\chi k_m^{1-1/p} \| V \|_{L^p(I_m;X)}, \]
for any \( V \in P_{r_m}(I_m;X) \).

### 3.2.2 Fixed point formulation and existence of discrete dG solutions

As for the cG method we prove the existence of solutions of (5) by means of a fixed point argument. For this purpose, we will derive a suitable fixed point formulation, and return to the case \( X = H_m \). Noticing the fact that \( \pi_m U_{m-1}^+ = U_{m-1}^+ \in H_m \), we observe that, on \( I_m \), there holds
\[
U' + L_m^r (\pi_m \| U \|_{m-1}) = (U - \pi_m U_{m-1}^-)' + L_m^r (U_{m-1}^+ - \pi_m U_{m-1}^-) \\
= \chi(U - \pi_m U_{m-1}^-),
\]
and recalling (13), we can write
\[ \chi(U - \pi_m U_{m-1}^-) = \Pi_m^r \mathcal{F}(U). \]

Applying Proposition 1 we infer that
\[ U = \pi_m U_{m-1}^- + \chi^{-1} \left( \Pi_m^r \mathcal{F}(U) \right); \]
this is the ‘dG-version’ of the integral equation (7) for the cG method. Now, for given \( U_{m-1}^- \) (where as before \( U_0^- := u_0 \)) we define the operator
\[ T_m^{dG} : P_{r_m}(I_m;H_m) \to P_{r_m}(I_m;H_m) \]
by
\[ T_m^{dG}(U) := \pi_m U_{m-1}^- + \chi^{-1} \left( \Pi_m^r \mathcal{F}(U) \right). \]
Then, \( U \in P_{r_m}(I_m;H_m) \) solves (13) if and only if \( U \) satisfies
\[ T_m^{dG}(U) = U. \]
We will now prove the existence of solutions to the local $hp$-dG time stepping scheme (5): Consider $\kappa_m, \theta_m > 0$ (with $t_{m-1} + \theta_m \leq T$), and define the set

$$ Q_m = [t_{m-1}, t_{m-1} + \theta_m] \times B_m, $$

where

$$ B_m = \{ y \in H_m : \| y - \pi_m u_{m-1} \|_H \leq \kappa_m \}. $$

Due to the continuity of $\mathcal{F}$, its maximum on the compact set $Q_m$,

$$ K_m := \max_{(t,y) \in Q_m} \| \mathcal{F}(t,y) \|_H, $$

exists. We choose

$$ 0 < k_m \leq \min(\theta_m, C^{-1}_\chi K^{-1}_m \kappa_m), $$

where $C_\chi$ is the constant from (16), and introduce

$$ M^dG_m := \{ Y \in \mathcal{P}^{r_m}(I_m; H_m) : Y(t) \in B_m \forall t \in I_m \}, $$

with $I_m = (t_{m-1}, t_m), t_m = t_{m-1} + k_m$.

Consider any $U \in M^dG_m$. From the definition of $T^{dG}_m$ in (25), and from (24) with $p = 2$, we conclude that

$$ \left\| T^{dG}_m(U) - \pi_m u_{m-1} \right\|_{L^\infty(I_m; H)} \leq C_\chi k_m^{1/2} \left\| \Pi^{r_m}_m \mathcal{F}(U) \right\|_{L^2(I_m; H)}. $$

The boundedness of the $L^2$-projection on $I_m$ (with constant 1) implies that

$$ \left\| T^{dG}_m(U) - \pi_m u_{m-1} \right\|_{L^\infty(I_m; H)} \leq C_\chi k_m^{1/2} \left\| \mathcal{F}(U) \right\|_{L^2(I_m; H)}. $$

Then, we obtain

$$ \left\| T^{dG}_m(U) - \pi_m u_{m-1} \right\|_{L^\infty(I_m; H)} \leq C_\chi k_m \left\| \mathcal{F}(U) \right\|_{L^\infty(I_m; H)} \leq K_m C_\chi k_m \kappa_m \leq \kappa_m, $$

since $U \in M^dG_m$. This implies that $T^{dG}_m(M^dG_m) \subseteq M^dG_m$. Then, employing the fixed point theorem of Brouwer (based on the fact that $M^dG_m$ is convex and compact, and that $T^{dG}_m$ is continuous), there exists a solution of (26), and therefore of (13) and (5).

### 4 Uniqueness of Galerkin solutions

In order to obtain unique Galerkin solutions on each time step we apply a contraction argument following the approach presented in [4]. To this end, we make the assumption that the nonlinearity $\mathcal{F}$ is locally Lipschitz continuous. Then, if the local time step $k_m$ in the Galerkin time discretizations is chosen sufficiently small (again, independently of the local polynomial degree), we will show that the operators $T^{dG}_m$ and $T^{dG}_m$ from (8) and (25), respectively, are contractive. This will lead to the following uniqueness result.
Theorem 2 Let $m \geq 1$, and $\kappa_m, \theta_m > 0$ (with $t_{m-1} + \theta_m \leq T$). Furthermore, consider $B_m$ from (10), $K_m$ from (11), and $M_m^{cG}$ from (12) for the cG method (4), and the respective quantities for the dG scheme (5) from (27), (28), and (30). Moreover, for each of the two schemes, we suppose that there exists a constant $0 \leq L_{\mathcal{F}}(B_m) < \infty$ such that the local Lipschitz continuity condition,

$$
\|\mathcal{F}(t, u) - \mathcal{F}(t, v)\|_H \leq L_{\mathcal{F}}(B_m) \|u - v\|_H \quad \forall t \in \bar{T}_m, \forall u, v \in B_m,
$$

holds. In addition, for a parameter $\varrho \in (0, 1)$, suppose that

$$
k_m \leq \min \left(\theta_m, c^{-1} K_m^{-1} \kappa_m, \varrho c^{-1} L_{\mathcal{F}}(B_m)^{-1}\right),
$$

where

$$
c = \begin{cases} 1 & \text{for cG time stepping,} \\ C_X & \text{for dG time stepping,} \end{cases}
$$

with $C_X$ being the constant from (16). Then, the cG and dG methods on $I_m$ each possess unique solutions $U_{cG}$ and $U_{dG}$ in $M_m^{cG}$ and $M_m^{dG}$, respectively.

Proof We treat the cG and dG cases separately.

Uniqueness of cG solution: From Sect. 3.1 we recall the following fact: For given $\kappa_m, \theta_m > 0$ (with $t_{m-1} + \theta_m \leq T$), and for $K_m$ from (11), choosing the local time step $k_m$ to be bounded by $k_m \leq \min(\theta_m, K_m^{-1} \kappa_m)$ guarantees the self-mapping property $T_m^{cG}(M_m^{cG}) \subseteq M_m^{cG}$, where $T_m^{cG}$ is the cG operator from (8). Furthermore, for $U_1, U_2 \in M_m^{cG}$ we have

$$
\left\|T_m^{cG}(U_1) - T_m^{cG}(U_2)\right\|_{L^\infty(I_m; H)} = \left\|\int_{t_{m-1}}^t \Pi_{t_{m-1}}^t (\mathcal{F}(U_1) - \mathcal{F}(U_2)) \, dt\right\|_{L^\infty(I_m; H)}
\leq \int_{I_m} \left\|\Pi_{t_{m-1}}^t (\mathcal{F}(U_1) - \mathcal{F}(U_2))\right\|_H \, dt
\leq k_m^{1/2} \left\|\Pi_{t_{m-1}}^t (\mathcal{F}(U_1) - \mathcal{F}(U_2))\right\|_{L^2(I_m; H)}
\leq k_m^{1/2} \left\|\mathcal{F}(U_1) - \mathcal{F}(U_2)\right\|_{L^2(I_m; H)}
\leq k_m \left\|\mathcal{F}(U_1) - \mathcal{F}(U_2)\right\|_{L^\infty(I_m; H)}.
$$

Now involving the Lipschitz condition (31) on $B_m$ from (10), we infer that

$$
\left\|T_m^{cG}(U_1) - T_m^{cG}(U_2)\right\|_{L^\infty(I_m; H)} \leq L_{\mathcal{F}}(B_m) k_m \left\|U_1 - U_2\right\|_{L^\infty(I_m; H)},
$$

for all $U_1, U_2 \in M_m^{cG}$. This implies that, for $k_m < L_{\mathcal{F}}(B_m)^{-1}$, the operator $T_m^{cG}$ is a contraction on $M_m^{cG}$. Thus, by the Banach fixed point theorem, the Eq. (9) has a unique solution in $M_m^{cG}$.

Uniqueness of dG solution: In the case of the dG scheme we proceed in a similar way as for the cG time stepping method. For $\kappa_m, \theta_m > 0$ (with $t_{m-1} + \theta_m \leq T$),
and for $K_m$ from (28), choosing the local time step $k_m$ to be bounded by $k_m \leq \min(\theta_m, C^{-1}_{\chi}K_m^{-1}k_m)$ ensures that $\mathcal{T}^dG_m(M^dG_m) \subseteq M^dG_m$, where $C_{\chi}$ is the constant from (16), and $\mathcal{T}^dG$ is the operator defined in (25); cf. Sect. 3.2.2. In addition, for $U_1, U_2 \in M^dG_m$ there holds that

$$
\left\| \mathcal{T}^dG_m(U_1) - \mathcal{T}^dG_m(U_2) \right\|_{L^\infty(I_m; H)} = \left\| \chi^{-1}(\Pi^r_m\mathcal{F}(U_1) - \Pi^r_m\mathcal{F}(U_2)) \right\|_{L^\infty(I_m; H)}.
$$

Using (24), we deduce that

$$
\left\| \mathcal{T}^dG_m(U_1) - \mathcal{T}^dG_m(U_2) \right\|_{L^\infty(I_m; H)} \leq C_{\chi}k_m^{1/2} \left\| \Pi^r_m(\mathcal{F}(U_1) - \mathcal{F}(U_2)) \right\|_{L^2(I_m; H)}
$$

$$
\leq C_{\chi}k_m^{1/2} \left\| \mathcal{F}(U_1) - \mathcal{F}(U_2) \right\|_{L^2(I_m; H)}
$$

$$
\leq C_{\chi}k_m \left\| \mathcal{F}(U_1) - \mathcal{F}(U_2) \right\|_{L^\infty(I_m; H)}.
$$

By means of (31) we derive the bound

$$
\left\| \mathcal{T}^dG_m(U_1) - \mathcal{T}^dG_m(U_2) \right\|_{L^\infty(I_m; H)} \leq C_{\chi}L\mathcal{F}(B_m)k_m \left\| U_1 - U_2 \right\|_{L^\infty(I_m; H)},
$$

for all $U_1, U_2 \in M^dG_m$, where the ball $B_m$ is defined in (27). Hence, for $k_m < C_{\chi}^{-1}L\mathcal{F}(B_m)^{-1}$, the mapping $\mathcal{T}^dG_m : M^dG_m \rightarrow M^dG_m$ is a contraction. This implies that the Eq. (26) has a unique solution $U \in M^dG_m$. \( \square \)

**Remark 4** The above Theorem 2 shows that the cG and dG operators in (8) and (25) are contractions in each time step, and thus, have unique fixed points in $M^cG_m$ and $M^dG_m$, respectively. In particular, the corresponding fixed point iterations converge. For instance, in the case of the cG time stepping scheme, for $m \geq 1$, starting from an initial guess $U^{(0)} \in M^cG_m$ (which can be chosen, for example, to be the constant function $U^{(0)}(t) = \pi_m U_{m-1}$, $t \in I_m$), the iteration

$$
U^{(\ell+1)} = \mathcal{T}^cG_m(U^{(\ell)}), \quad \ell \geq 1,
$$

will tend to the unique solution $U|_{I_m} \in M^cG_m$ of (4). Similarly, for the dG scheme, for $m \geq 1$, and an initial guess $U^{(0)} \in M^dG_m$ (for example, $U^{(0)}(t) = \pi_m U_{m-1}$, $t \in I_m$), the iteration

$$
U^{(\ell+1)} = \mathcal{T}^dG_m(U^{(\ell)}), \quad \ell \geq 1,
$$

converges to the unique solution $U|_{I_m} \in M^dG_m$ of (5).

## 5 Application to finite-time blow-up problems

In this section we will discuss the existence and uniqueness Theorem 2 in the context of nonlinearities $\mathcal{F}$ that grow algebraically with respect to $u$, with a power strictly
larger than 1. We will show that both the exact solution $u$ of (1) as well as the cG and dG solutions blow up in finite time. In addition, we will provide a time step selection algorithm along the lines of [18], and prove a convergence result for the blow-up time. In order to keep the technical matters within a reasonable scope, we assume that $H$ is finite dimensional, and that $H = H_1 = H_2 = \cdots = H_m = \cdots$ holds for any $m \geq 1$.

5.1 Algebraic growth nonlinearities

We consider nonlinearities $F$ which feature the following algebraic growth condition:

Suppose that there exist constants $\alpha, \delta > 0$, $\beta > 1$, and $c_F \geq 0$ such that

$$\|F(t, u)\|_H \leq \alpha \|u\|_H^\beta$$

and

$$(F(t, u), u)_H \geq \delta \|u\|_H^{1+\beta},$$

(34)

for all $u \in H$ which satisfy $\|u\|_H \geq c_F$, and for any $t \in [0, \infty)$ (or for any $t \in [0, T]$, with sufficiently large $T > 0$). We note that such problems exhibit a blow-up in some finite time $T_\infty < \infty$. Indeed, let $u$ solve (1), and suppose that $\|u_0\|_H > c_F$ in (1). Then, under the conditions (34), it is easy to see that $\|u(t)\|_H$ is non-decreasing with respect to $t$, and thus,

$$\frac{d}{dt} \|u(t)\|_H^2 = 2(u'(t), u(t))_H = 2(F(t, u(t)), u(t))_H \geq 2\delta \left(\|u(t)\|_H^2\right)^{(1+\beta)/2}.$$

Hence,

$$\frac{1}{1 - \beta} \frac{d}{dt} \left[ \left(\|u(t)\|_H^2\right)^{(1-\beta)/2} \right] \geq \delta.$$

Integrating from 0 to some $t > 0$ shows that

$$\|u_0\|_H^{1-\beta} - \|u(t)\|_H^{1-\beta} \geq (\beta - 1)\delta t,$$

and therefore,

$$t \leq \frac{\|u_0\|_H^{1-\beta}}{(\beta - 1)\delta} =: \overline{T}_\infty.$$

It follows that $\overline{T}_\infty$ is an upper bound for the blow-up time.

5.2 Discrete blow-up

Provided that the properties (34) hold true, the goal of this section is to show that the cG and dG time stepping methods yield discrete solutions which blow up in finite time. To this end, let us assume, in addition to (34), that the local Lipschitz property

$$\|F(t, u) - F(t, v)\|_H \leq \gamma \max(\|u\|_H, \|v\|_H)^{\beta-1} \|u - v\|_H$$

(36)
holds true whenever \( \|u\|_H, \|v\|_H \geq c_\mathcal{F} \), cf. (34), and for all \( t \in [0, \infty) \) (or for any \( t \in [0, T] \) with sufficiently large \( T > 0 \), with a uniform constant \( \gamma \geq 0 \). Under these conditions, following [18, Lemma 9], we choose the time step sizes \( k_m \) in such a way that the final values \( U_m^- \) of the cG and dG solutions on each time step, \( m \geq 1 \), can be proved to form an increasing sequence in the sense that

\[
\|U_m^-\| \geq C\|U_{m-1}^-\|_H,
\]

for a constant \( C > 1 \); see (43) below. This property, in turn, makes it possible to show that the discrete solutions blow up in finite time. In this context, let us mention the work [1], which is based on a related idea.

In the following elaborations, the function

\[
\Psi : [0, \gamma/\alpha) \to \mathbb{R}, \quad \varrho \mapsto \Psi(\varrho) = \delta(\gamma - \varrho\alpha)^\beta - \varrho\alpha\gamma^\beta - \varrho\gamma^\alpha (\gamma - \varrho\alpha)^{-1}
\]

will play an important role; here, \( \alpha, \beta, \delta, \) and \( \gamma \) are the constants from (34) and (36), respectively. We note that \( \Psi \) is decreasing, and that \( \Psi(0) = \delta\gamma^{\beta-1} > 0 \), and \( \lim_{\varrho \to \gamma/\alpha} \Psi(\varrho) = -\infty \). Hence, by continuity there exists exactly one zero \( \varrho \) of \( \Psi \) in the interval \([0, \gamma/\alpha)\).

**Proposition 2** Suppose that the conditions (34) and (36) hold, and that the initial value \( u_0 \in H \) from (1) satisfies \( \|u_0\|_H > c_\mathcal{F} \). Furthermore, let \( \varrho_0 \) be a fixed constant with \( 0 < \varrho_0 < \min(1, \varrho) \), where \( \varrho \) is the unique zero of \( \Psi \) from (37) in \([0, \gamma/\alpha)\). For any given \( \varrho \) with

\[
0 < \varrho \leq \min \left( \varrho_0, \frac{\alpha^{-1}\gamma}{1 + (1 - c_\mathcal{F}\|u_0\|_H^{-1})^{-1}} \right),
\]

choose the time steps to be

\[
k_m(\varrho) := c^{-1}\gamma^{-\beta}(\gamma - \varrho\alpha)^{\beta-1}\|U_m^-\|_H^{1-\beta}, \quad m = 1, 2, 3, \ldots,
\]

where \( U_{m-1}^- \), \( m \geq 1 \), signifies the left-sided value of the cG or dG solution \( U \) from (4) or (5), respectively, at the nodal point \( t_{m-1} \) (with \( U_{m-1}^- = U_{m-1} \) for the cG scheme, and \( U_0^- := u_0 \)). Then, there holds:

(i) For any \( m \geq 1 \), the cG and dG solutions resulting from (4) and (5) exist and are unique in \( M_{cG}^m \) from (12) and \( M_{dG}^m \) from (30), respectively, with \( \kappa_m = \varrho\alpha(\gamma - \varrho\alpha)^{-1}\|U_{m-1}^-\|_H^{-1} \), for any polynomial degree distribution.

(ii) Both the cG and the dG solutions blow up at finite times \( \tilde{T}_\infty^{cG}(\varrho) \) and \( \tilde{T}_\infty^{dG}(\varrho) \), respectively.

The constants \( \alpha, \beta, \delta, \) and \( \gamma \) were introduced in (34) and (36), respectively, and \( c \) is defined in (33).
Proof We focus on the dG method only; the proof for the cG method can be done verbatim. Let $m \geq 1$, and suppose that the dG solution on the first $m - 1$ time steps is well-defined, and that

$$
\|U_{m-1}^\pm\|_H \geq \|u_0\|_H > c_\mathcal{F} \geq 0. \tag{40}
$$

Then, with $\kappa_m = \eta_m \|U_{m-1}^-\|_H$, where

$$
\eta_m = \varrho \alpha (\gamma - \varrho \alpha)^{-1}, \tag{41}
$$

we see by means of (38) that $0 < \eta_m \leq 1 - c_\mathcal{F} \|u_0\|_H^{-1}$. Therefore, for any $y \in B_m := \{y \in H : \|y - U_{m-1}^-\|_H \leq \kappa_m\}$, it follows that

$$
\|y\|_H \geq \|U_{m-1}^-\|_H - \|y - U_{m-1}^-\|_H \geq \|U_{m-1}^-\|_H - \kappa_m \geq (1 - \eta_m) \|U_{m-1}^-\|_H \\
\geq (1 - \eta_m) \|u_0\|_H \geq c_\mathcal{F}.
$$

Consequently, in view of the growth condition (34), there holds

$$
K_m := \max_{(t,y) \in Q_m} \|\mathcal{F}(t,y)\|_H \\
\leq \alpha \|y\|_H^\beta \leq \alpha \left(\|U_{m-1}^-\|_H + \kappa_m\right)^\beta = \alpha (1 + \eta_m)^\beta \|U_{m-1}^-\|_H^\beta,
$$

with $Q_m = I_m \times B_m$, where $I_m = [t_{m-1}, t_{m-1} + \theta_m]$, and $\theta_m := k_m(\varrho)$. Hence,

$$
k_m(\varrho) = c^{-1} \alpha^{-1} \eta_m (1 + \eta_m)^{-\beta} \|U_{m-1}^-\|_H^{1-\beta} \leq c^{-1} K_m^{-1} \kappa_m,
$$

and revisiting the existence proof in Sect. 3.2.2 (in particular, see (29)), we infer that there is a dG solution in $M_m^{d\Gal}$. Furthermore, we bound the Lipschitz constant $L_\mathcal{F}(B_m)$ appearing in (31) by means of (36): For any $u, v \in B_m$ we have $\|u\|_H, \|v\|_H \geq c_\mathcal{F}$ as shown before, and, max($\|u\|_H, \|v\|_H$) $\leq \|U_{m-1}^-\|_H + \kappa_m$. Thus,

$$
L_\mathcal{F}(B_m) \leq \gamma (\kappa_m + \|U_{m-1}^-\|_H)^{\beta-1} \leq \gamma (1 + \eta_m)^{\beta-1} \|U_{m-1}^-\|_H^{1-\beta}, \tag{42}
$$

which implies that

$$
k_m(\varrho) = \varrho c^{-1} \gamma^{-1} (1 + \eta_m)^{1-\beta} \|U_{m-1}^-\|_H^{1-\beta} \leq \varrho c^{-1} L_\mathcal{F}(B_m)^{-1}.
$$

Then, with reference to (32), the uniqueness of a dG solution in $M_m^{d\Gal}$ follows immediately.

Next, consider the dG solution $U|_{I_m} \in \mathcal{P}_m^d(I_m; H)$. Using (5) with the constant test function $V(t) = U_{m-1}^-, t \in I_m$, we have that

$$
(U_{m-1}^-, U_{m-1}^-)_H = \|U_{m-1}^-\|_H^2 + \int_{I_m} (\mathcal{F}(U), U_{m-1}^-)_H \, dt.
$$
Recalling (40), and employing (34), we obtain

\[ \| U_m^- \|_H \| U_{m-1}^- \|_H \]
\[ \geq \| U_{m-1}^- \|_H^2 + k_m (\mathcal{F}(U_{m-1}^-), U_{m-1}^-)_H + \int_{I_m} (\mathcal{F}(U) - \mathcal{F}(U_{m-1}^-), U_{m-1}^-)_H \, dt \]
\[ \geq \| U_{m-1}^- \|_H^2 + k_m \delta \| U_{m-1}^- \|_H^{1+\beta} - \| U_{m-1}^- \|_H \int_{I_m} \| \mathcal{F}(U) - \mathcal{F}(U_{m-1}^-) \|_H \, dt. \]

Furthermore, dividing by \( \| U_{m-1}^- \|_H > 0 \), it holds that

\[ \| U_m^- \|_H \geq \| U_{m-1}^- \|_H + k_m \delta \| U_{m-1}^- \|_H^{\beta} - \int_{I_m} \| \mathcal{F}(U) - \mathcal{F}(U_{m-1}^-) \|_H \, dt. \]

Reviewing the proof of Theorem 1 in Sect. 3.2.2, we observe that \( U(t) \in B_m \) for all \( t \in I_m \). Therefore, using the local Lipschitz continuity (36) with the bound (42), it follows that

\[ \| U_m^- \|_H \geq \| U_{m-1}^- \|_H + k_m \delta (1 + \eta_m)^{\beta-1} \| U_{m-1}^- \|_H^{\beta}. \]

Inserting (41) yields

\[ \| U_m^- \|_H \geq \| U_{m-1}^- \|_H + k_m (\delta - \eta \eta_m (1 + \eta_m)^{\beta-1}) \| U_{m-1}^- \|_H^{\beta}. \]

Then, employing (39), and recalling that \( \Psi \) is monotone decreasing, leads to

\[ \| U_m^- \|_H \geq \left(1 + c^{-1} \gamma^{-\beta} \varrho \Psi(\varrho) \right) \| U_{m-1}^- \|_H \geq (1 + C_0 \varrho) \| U_{m-1}^- \|_H, \]

with

\[ C_0 = c^{-1} \gamma^{-\beta} \Psi(\varrho_0). \]  

The assumption (40) is trivially valid for \( m = 1 \). Furthermore, due to (43) we note the fact that \( \| U_1^- \|_H \geq \| u_0 \|_H \), and, thus, we conclude inductively that the previous derivations are applicable for any \( m \geq 2 \).

Moreover, from (43) we infer that

\[ \| U_{m-1}^- \|_H \geq (1 + C_0 \varrho)^{m-k} \| U_{k-1}^- \|_H \quad \forall m \geq k \geq 1, \]

which shows that \( \| U_{m-1}^- \|_H \to \infty \) as \( m \to \infty \). In addition, involving (39) it follows, for any \( m \geq i \geq 1 \), that

\[ \odot \text{ Springer} \]
\[ t_m = t_{i-1} + \sum_{j=i}^{m} k_j(\varrho) = t_{i-1} + \frac{\varrho(y - \alpha \varrho)\beta^{-1}}{c_y \beta} \sum_{j=i}^{m} \| U_j^- \|_H^{1-\beta} \]
\[ \leq t_{i-1} + \frac{\varrho(y - \alpha \varrho)\beta^{-1}}{c_y \beta} \| U_{i-1}^- \|_H^{1-\beta} \sum_{j=i}^{m} (1 + C_0 \varrho)^{(1-\beta)(j-i)} \]
\[ \leq t_{i-1} + \frac{\varrho(y - \alpha \varrho)\beta^{-1}}{c_y \beta} \| U_{i-1}^- \|_H^{1-\beta} \sum_{j=0}^{\infty} (1 + C_0 \varrho)^{(1-\beta)j}. \]

Therefore,
\[ t_m \leq t_{i-1} + \frac{\varrho(y - \alpha \varrho)\beta^{-1}}{c_y \beta} \frac{\| U_{i-1}^- \|_H^{1-\beta}}{1 - (1 + C_0 \varrho)^{1-\beta}}, \quad m \geq i \geq 1. \tag{46} \]

In particular, for \( i = 1 \) and \( m \to \infty \), we see that the discrete blow-up time \( \tilde{T}_{\infty}^{dG}(\varrho) \) for the dG method is bounded by
\[ \tilde{T}_{\infty}^{dG}(\varrho) \leq \frac{\varrho(y - \alpha \varrho)\beta^{-1}}{c_y \beta} \frac{\| u_0 \|_H^{1-\beta}}{1 - (1 + C_0 \varrho)^{1-\beta}} < \infty. \tag{47} \]

This concludes the proof. \( \square \)

**Remark 5** The above proof allows to establish an \( L^\infty \) bound on the cG and dG solution, again denoted by \( U \), on \((0, t_m)\), for \( m \geq 1 \). Indeed, for any \( 1 \leq i \leq m \), using (41), we have that
\[ \| U \|_{L^\infty(t_i; H)} \leq \| U_{i-1}^- \|_H + \kappa_i \leq (1 + \eta_i) \| U_{i-1}^- \|_H \leq \zeta \| U_{i-1}^- \|_H \leq \zeta \| U_{m-1}^- \|_H, \]
with \( \zeta = y(y - \varrho_0 \alpha)^{-1} \). Taking the maximum for all \( 1 \leq i \leq m \), we conclude that
\[ \| U \|_{L^\infty((0, t_m); H)} \leq \zeta \| U_{m-1}^- \|_H. \]

**Remark 6** We notice that
\[ \mu := \lim_{\varrho \to 0} \frac{\varrho(y - \alpha \varrho)\beta^{-1}}{c_y \beta (1 - (1 + C_0 \varrho)^{1-\beta})} = \frac{1}{c_y C_0 (\beta - 1)} > 0 \tag{48} \]
in (47). In particular, we see that the discrete blow-up times \( \tilde{T}_{\infty}^{cG}(\varrho) \) and \( \tilde{T}_{\infty}^{dG}(\varrho) \) for the cG and dG methods, respectively, are uniformly bounded for any \( \varrho \) satisfying (38).

**Remark 7** Using (34), we can show that there exists a constant \( C_1 > 0 \), with \( C_1 \geq C_0 \) from (44), such that
\[ \| U_m^- \|_H \leq (1 + C_1 \varrho) \| U_{m-1}^- \|_H, \quad m \geq 1, \]
for both the cG and dG solutions. To see this, consider, for instance, the dG solution \( U|_{l_m} \in \mathcal{P}^{l_m}(l_m; H) \). Applying (5) with the constant test function \( V(t) = U_m^- \), \( t \in l_m \), yields
\[
\|U_m\|_H^2 = (U_{m-1}, U_m)_H + \int_{I_m} (\mathcal{F}(U), U_m)_H \, dt.
\]

Then, proceeding as in the proof of Proposition 2, there holds
\[
\|U_m\|_H^2 = (U_{m-1}, U_m)_H + k_m (\mathcal{F}(U_{m-1}), U_m)_H
+ \int_{I_m} (\mathcal{F}(U) - \mathcal{F}(U_{m-1}), U_m)_H \, dt
\leq \|U_{m-1}\|_H \|U_m\|_H + k_m \|\mathcal{F}(U_{m-1})\|_H \|U_m\|_H
+ \|U_m\|_H \int_{I_m} \|\mathcal{F}(U) - \mathcal{F}(U_{m-1})\|_H \, dt.
\]

Dividing by \(\|U_m\|_H\), involving (34), (36), and (42), and recalling the fact that \(\|U - U_{m-1}\|_{L^\infty(I_m; H)} \leq \kappa_m\), we infer
\[
\|U_m\|_H \leq \|U_{m-1}\|_H + k_m \|U_{m-1}\|_H^{\beta} + k_m \mathcal{F}(B_m) \kappa_m
\leq \left(1 + k_m \left(\alpha + \gamma \eta_m (1 + \eta_m)^{\beta - 1}\right)\right) \|U_{m-1}\|_H^{\beta - 1} \|U_{m-1}\|_H.
\]

Inserting (39) and (41) we arrive at
\[
\|U_m\|_H \leq \left(1 + \frac{C_1 \alpha (\gamma - \alpha \varrho)}{c \gamma \beta (\gamma - \alpha \varrho)}\right) \|U_{m-1}\|_H \leq (1 + C_1 \varrho) \|U_{m-1}\|_H,
\]
with
\[
C_1 = \alpha c^{-1}(\varrho_0 + 1)(\gamma - \varrho_0 \alpha)^{-1}.
\]

Proceeding analogously for the cG method, precisely the same bound can be proved. Moreover, in analogy to the derivation of (46), the bounds
\[
I_m \geq I_{i-1} + \frac{\varrho (\gamma - \alpha \varrho)}{c \gamma \beta (\gamma - \alpha \varrho)} \|U_{i-1}\|_H^{1-\beta} \sum_{j=i}^{m} (1 + C_1 \varrho)^{(1-\beta)(j-i)}
\geq I_{i-1} + \frac{\varrho (\gamma - \alpha \varrho)}{c \gamma \beta} \|U_{i-1}\|_H^{1-\beta} \sum_{j=0}^{m-i} (1 + C_1 \varrho)^{(1-\beta)j},
\]
for any \(m \geq i \geq 1\), are obtained.

### 5.3 Convergence to blow-up time

We will now show that the cG and dG time stepping schemes are able to approximate the exact blow-up time arbitrarily well as \(\varrho \searrow 0\) in (39). To this end, we will apply the approach proposed in [18, Proposition 2] to the Galerkin time stepping methods in the present paper. More precisely, we will first recall some \(hp\)-version approximation
results for the cG and dG schemes from [27] and [20], respectively; see Lemma 3 below. Then, by means of a (non-constructive) contradiction argument, our goal is to show that
\[
\lim_{\rho \searrow 0} \tilde{T}_\infty(\rho) \leq T_\infty \leq \lim_{\rho \searrow 0} \tilde{T}_\infty(\rho),
\]
where \(T_\infty\) and \(\tilde{T}_\infty(\rho)\) represent the exact and discrete blow-up times, respectively; see Theorem 3.

Lemma 3 Suppose that the assumptions of Proposition 2 are fulfilled; in particular choose \(\rho\) as in (38), and let the time steps \(\{k_m(\rho)\}_{m \geq 1}\) be given by (39). In addition to the local Lipschitz property (36), suppose that there exists a constant \(L_{cF}\) such that
\[
\|F(u, t) - F(v, t)\|_H \leq L_{cF} \|u - v\|_H \quad \forall t \in [0, \infty),
\]
whenever \(\|u\|_H, \|v\|_H \leq c_F\). Furthermore, let \(T_0 > 0\) be fixed with
\[
T_0 < \min(T_\infty(\rho), \tilde{T}_\infty(\rho)) < \infty,
\]
where \(T_\infty(\rho)\) and \(\tilde{T}_\infty(\rho)\) are the exact and the discrete blow-up times (i.e., either the cG or the dG blow-up time), respectively. Moreover, define
\[
M(\rho, T_0) := \sup \left\{ m : t_m(\rho) = \sum_{l=1}^{m} k_l(\rho) \leq T_0 \right\} < \infty,
\]
and
\[
\Xi(u, U, T_0) := \max \left( \|u\|_{L^\infty((0, T_0); H)}, \|U\|_{L^\infty((0, T_M(\rho, T_0)); H)} \right) < \infty,
\]
where \(u\) is the solution of (1), and \(U\) signifies either the cG or the dG solution. Then, there holds the a priori error estimate
\[
\|u - U\|_{L^\infty((0, T_M(\rho, T_0)); H)} \leq C(T_0, \Xi(u, U, T_0)) \sqrt{C_r} \sqrt{\rho},
\]
where \(C_r = \sup_{1 \leq m \leq M(\rho, T_0)} \max(3, \ln(r_m))\), and \(C(T_0, \Xi(u, U, T_0)) > 0\) only depends on the time \(T_0\), on \(L_{cF}\), on \(\Xi(u, U, T_0)\), and on the constants \(c, \alpha, \beta, c_F, \) and \(\gamma\) from (33), (34), and (36), respectively.

Proof Let us first suppose that \(\Xi(u, U, T_0) \geq c_F\). Then, the operator \(F\) is Lipschitz continuous on the annulus \(R_{T_0} := \{ v \in H : c_F \leq \|v\|_H \leq \Xi(u, U, T_0) \}\), with Lipschitz constant \(L(R_{T_0}) = \gamma \Xi(u, U, T_0)^{\beta - 1}\); cf. (36). Furthermore, by (52) we know that \(F\) is Lipschitz continuous on \(\{ v \in H : \|v\|_H \leq c_F \}\), with a Lipschitz constant \(L_{cF}\). Moreover, if \(\|u\|_H < c_F\) and \(\Xi(u, U, T_0) \geq \|v\|_H \geq c_F\), then we
choose $\omega \in [0, 1]$ uniquely such that $z_\omega := (1 - \omega)u + \omega v$ satisfies $\|z_\omega\|_H = cF$. We deduce that

$$\|\mathcal{F}(t, u) - \mathcal{F}(t, v)\|_H \leq \|\mathcal{F}(t, u) - \mathcal{F}(t, z_\omega)\|_H + \|\mathcal{F}(t, z_\omega) - \mathcal{F}(t, v)\|_H$$

$$\leq L_{\mathcal{F}}\|u - z_\omega\|_H + L(R_{T_0})\|z_\omega - v\|_H$$

$$\leq (\omega L_{\mathcal{F}} + (1 - \omega)L(R_{T_0}))\|u - v\|_H$$

$$\leq \max(L_{\mathcal{F}}, L(R_{T_0}))\|u - v\|_H.$$ 

In summary, we conclude that $\mathcal{F}$ is Lipschitz continuous on $\{v \in H : \|v\|_H \leq \mathcal{E}(u, U, T_0)\}$, with Lipschitz constant $L_{T_0} := \max(L_{\mathcal{F}}, L(R_{T_0}))$. Evidently, due to (52), this still holds when $\mathcal{E}(u, U, T_0) < cF$.

Now, we introduce the operator

$$\mathcal{G} : [0, T_0] \times H \to H, \quad x \mapsto \left\{ \begin{array}{ll} \mathcal{F}(t, x) & \|x\|_H \leq \mathcal{E}(u, U, T_0), \\ \mathcal{F} \left( t, \mathcal{E}(u, U, T_0) \frac{x}{\|x\|_H} \right) & \|x\|_H > \mathcal{E}(u, U, T_0), \end{array} \right.$$ 

which is \textit{globally} Lipschitz continuous on the domain $[0, T_0] \times H$ with Lipschitz constant $L_{T_0}$; see Lemma 6. Then, for the cG method, applying [27, Theorem 3.1], we obtain, for $l \in \{0, 1\}$, that

$$\left\| (u - U)^{(l)} \right\|_{L^2((0,t_M(\varrho,T_0));H)}^2 \leq C_{cG}(T_0, L_{T_0})\|u\|_{H^1((0,T_0);H)}^2 \max_{1 \leq m \leq M(\varrho,T_0)} k_m^{2(1-l)},$$

for a constant $C_{cG} > 0$. Therefore, choosing a time $t^* \in [0, t_M(\varrho,T_0)]$ such that $\|u - U\|_{L^\infty((0,t_M(\varrho,T_0));H)} = \|u(t^*) - U(t^*)\|_H$, and noticing that $U_0 = u_0 = u(0)$, we have

$$\left\| u - U \right\|_{L^\infty((0,t_M(\varrho,T_0));H)}^2 = \int_0^{t^*} \frac{d}{dt}\left\| u - U \right\|_H^2 dt = 2 \int_0^{t^*} (u - U, u' - U')_H dt$$

$$\leq 2\|u - U\|_{L^2((0,t_M(\varrho,T_0));H)}\|u' - U'\|_{L^2((0,t_M(\varrho,T_0));H)}$$

$$\leq 2C_{cG}(T_0, L_{T_0})\|u\|_{H^1((0,T_0);H)}^2 \max_{1 \leq m \leq M(\varrho,T_0)} k_m$$

$$\leq 2T_0C_{cG}(T_0, L_{T_0})\|u\|_{W^{1,\infty}((0,T_0);H)}^2 \max_{1 \leq m \leq M(\varrho,T_0)} k_m.$$ 

Moreover, for the dG time stepping scheme we employ [20, Theorem 3.12] to infer

$$\left\| u - U \right\|_{L^\infty((0,t_M(\varrho,T_0));H)}^2 \leq C_{dG}(T_0, L_{T_0})C_r\|u\|_{W^{1,\infty}((0,T_0);H)}^2 \max_{1 \leq m \leq M(\varrho,T_0)} k_m,$$
where \( C_{\text{dG}} > 0 \) is again a constant. Then, in view of (39) and (45), we observe that
\[
 k_m(\varrho) \leq \frac{c^{-1}\gamma^{-1}Q}{\|U_{m-1}\|_H^{\beta-1}} \leq \frac{c^{-1}\gamma^{-1}Q}{\|u_0\|_H^{\beta-1}} \leq \frac{c_{dG}^{-1}\gamma^{-1}Q}{c_{dG}^{\beta-1}}.
\]

Thus,
\[
\|u - U\|_{L^\infty((0,T_0);H)}^2 \leq \varrho c^{-1}\gamma^{-1}c_{dG}^{1-\beta} \max\left(2T_0C_{\text{cG}}(T_0, L_{T_0}), C_{\text{dG}}(T_0, L_{T_0})\right) C_r \|u\|_{W^{1,\infty}((0,T_0);H)},
\]
where \( U \) is either the cG or dG solution. Upon recalling (34), we conclude that
\[
\|u\|_{W^{1,\infty}((0,T_0);H)} \leq \|u\|_{L^\infty((0,T_0);H)} + \|\mathcal{F}(u)\|_{L^\infty((0,T_0);H)} \leq \mathcal{E}(u, U, T_0) + \alpha \mathcal{E}(u, U, T_0)^\beta,
\]
and the proof is complete. \( \Box \)

**Remark 8** We note that the error estimate (53) above is not optimal in terms of \( k_m \) and \( r_m \). It is, however, sufficient to establish the blow-up time convergence result in Theorem 3 below.

**Lemma 4** Suppose that the assumptions of Proposition 2 hold. Moreover, consider a time \( T_0 > 0 \) with \( T_{\sup} := \lim_{\varrho \searrow 0} \tilde{T}_{\infty}(\varrho) > T_0 \) (note that, by Remark 6, it holds that \( T_{\sup} < \infty \)). Furthermore, let \( \{Q_l\}_{l \geq 1} \) be a sequence with \( Q_l \overset{l \to \infty}{\to} 0^+ \) that satisfies the bound (38), and \( \lim_{l \to \infty} \tilde{T}_{\infty}(Q_l) = T_{\sup} \). Moreover, by Remark 6, we may suppose that the sequence \( \{Q_l\}_l \) satisfies
\[
\frac{Q_l(\gamma - \alpha Q_l)^{\beta-1}}{c\gamma^{\beta(1 - (1 + C_0Q_l)^{1-\beta})}} \leq 2\mu \quad \forall l \geq 1,
\]
where \( \mu > 0 \) is the constant from (48). Then, whenever \( l, m \in \mathbb{N} \) are such that
\[
t_m(Q_l) = \sum_{i=1}^m k_i(Q_l) \leq T_0,
\]
with \( k_m \) from (39), the dG and cG time stepping solutions are bounded by
\[
\|U_{m}^-(Q_l)\|_H \leq \left( \frac{T_{\sup} - T_0}{2\mu} \right)^{(1-\beta)}. \]

**Proof** Suppose that (55) holds. Then, applying (46) (with \( m \to \infty \)) it follows that
\[
0 < T_{\sup} - T_0 \leq T_{\sup} - t_m(Q_l) \leq \frac{Q_l(\gamma - \alpha Q_l)^{\beta-1}}{c\gamma^{\beta(1 - (1 + C_0Q_l)^{1-\beta})}} \|U_{m}^-(Q_l)\|_H^{1-\beta}.
\]
Using (54), we infer that \( T_{\sup} - T_0 \leq 2\mu \|U_{m}^-(Q_l)\|_H^{1-\beta} \), which shows the assertion. \( \Box \)
Before stating the next lemma, we recall, by Remark 7, that $C_0 \leq C_1$. Hence, for any $A \in \mathbb{R}$ with $A > \|u_0\|_H > 0$, there holds

$$A \leq \left( \frac{A}{\|u_0\|_H} \right)^{C_1/C_0} \|u_0\|_H < 2 \left( \frac{A}{\|u_0\|_H} \right)^{C_1/C_0} \|u_0\|_H.$$

**Lemma 5** Suppose that the assumptions of Proposition 2 are fulfilled. Furthermore, let $A > \|u_0\|_H$. Then, there exists a sequence $\rho_l \to \infty$ (satisfying the bound (38)) with

$$\lim_{l \to \infty} \tilde{T}_\infty(\rho_l) = T_{\text{inf}} := \lim_{\rho \to 0} \tilde{T}_\infty(\rho),$$

and a time $T_0 < T_{\text{inf}}$ (depending, in particular, on $A$ and on $\|u_0\|_H$) such that for any $l$ there is a time index $m_A(\rho_l) \geq 0$ with

$$t_{m_A(\rho_l)} \leq T_0, \quad A \leq \|U(\rho_l)_{m_A(\rho_l)}\|_H \leq 2 \left( \frac{A}{\|u_0\|_H} \right)^{C_1/C_0} \|u_0\|_H.$$

Here, we denote by $U(\rho_l)$ either the discrete $cG$ or $dG$ solution from Proposition 2, and by $\tilde{T}_\infty(\rho_l)$ the corresponding discrete blow-up time. Furthermore, $C_0$ and $C_1$ are the constants from (44) and (50), respectively.

**Proof** Due to (45), for $\|U_m^-\|_H \geq A$ to hold, it is sufficient that

$$m \geq m_A(\rho) := \left\lceil \frac{\ln(A\|u_0\|_H^{-1})}{\ln(1 + C_0\rho)} \right\rceil.$$

Then, using (51) with $M \geq m_A(\rho) + 1 \geq 1$, we have

$$t_M - t_{m_A(\rho)} \geq \frac{\rho(\gamma - \alpha\rho)\beta^{-1}}{c\gamma^\beta} \|U_{m_A(\rho)}^-\|_H^{1-\beta} \sum_{j=0}^{M-m_A(\rho)-1} (1 + C_1\rho)^{(1-\beta)j} \|U_{m_A(\rho)}^-\|_H^{1-\beta} \frac{1 - (1 + C_1\rho)^{(1-\beta)(M-m_A(\rho))}}{1 - (1 + C_1\rho)^{1-\beta}}.$$

Letting $M \to \infty$, leads to

$$\tilde{T}_\infty(\rho) - t_{m_A(\rho)} \geq \frac{\rho(\gamma - \alpha\rho)\beta^{-1}}{c\gamma^\beta} \|U_{m_A(\rho)}^-\|_H^{1-\beta} \frac{1 - (1 + C_1\rho)^{1-\beta}}{1 - (1 + C_1\rho)^{1-\beta}}.$$

Applying (49) $m_A(\rho)$-times, we note that

$$\|U_{m_A(\rho)}^-\|_H \leq (1 + C_1\rho)^{m_A(\rho)} \|u_0\|_H \leq (1 + C_1\rho)^{1 + \frac{\ln(A\|u_0\|_H^{-1})}{\ln(1 + C_0\rho)}} \|u_0\|_H.$$
We observe that
\[
\lim_{\varrho \to 0} (1 + C_1 \varrho)^{1 + \frac{\ln(A\|u_0\|_H^{-1})}{\ln(1 + C_0 \varrho)}} = (A\|u_0\|_H^{-1})^{C_1/C_0}.
\]
Furthermore,
\[
\tilde{T}_\infty(\varrho) - t_{m_A(\varrho)} \geq \frac{\varrho(\gamma - \alpha \varrho)^{\beta - 1}\|u_0\|_H^{-\beta}}{c \gamma^\beta (1 - (1 + C_1 \varrho)^{1-\beta})(1 + C_1 \varrho)^{(1-\beta)}} (1 + \frac{\ln(A\|u_0\|_H^{-1})}{\ln(1 + C_0 \varrho)}),
\]
and since the right-hand side of the above inequality tends to
\[
v := \frac{\|u_0\|_H^{-\beta}}{c (\beta - 1)^{\gamma} C_1} (A\|u_0\|_H^{-1})^{(1-\beta) C_1/C_0} > 0,
\]
as \varrho \searrow 0, we conclude that we can choose \(\varrho^*\) small enough (and satisfying (38)) so that
\[
\|U_{m_A(\varrho)}^-\|_H \leq 2 (A\|u_0\|_H^{-1})^{C_1/C_0} \|u_0\|_H, \quad \tilde{T}_\infty(\varrho) - t_{m_A(\varrho)} \geq \frac{v}{2},
\]
for any \(0 < \varrho \leq \varrho^*\). Now consider a sequence \(\varrho_l \xrightarrow{l \to \infty} 0^+\), with \(0 < \varrho_l \leq \varrho^*\) for all \(l\), that satisfies (56) as well as
\[
|\tilde{T}_\infty(\varrho_l) - T_{\text{inf}}| \leq \frac{v}{4} \quad \forall l.
\]
Then, upon defining \(T_0 := T_{\text{inf}} - v/4\), we see that
\[
T_0 = t_{m_A(\varrho)} + (\tilde{T}_\infty(\varrho_l) - t_{m_A(\varrho_l)}) - (\tilde{T}_\infty(\varrho_l) - T_{\text{inf}}) - \frac{v}{4} \geq t_{m_A(\varrho_l)},
\]
and thus, the proof is complete. \(\square\)

We are now ready to show the following result on the convergence of the Galerkin time stepping schemes to the exact blow-up time.

**Theorem 3** Let the assumptions of Proposition 2 and of Lemma 3 be satisfied, and suppose that \(\sup_{m \geq 1} t_m < \infty\). Then, there holds
\[
\lim_{\varrho \to 0} \tilde{T}_\infty(\varrho) = T_\infty,
\]
where \(\tilde{T}_\infty(\varrho)\) denotes either the discrete cG or dG blow-up time, and \(T_\infty < \infty\) is the blow-up time of (1) under the conditions (34) and (36).
Proof We establish the proof by contradiction. Suppose first that $T_{\sup} : = \lim_{\varrho \to 0} \tilde{T}_\infty (\varrho) > T_\infty$. Thence, $\infty > \Delta_\infty : = T_{\sup} - T_\infty > 0$. We can find a sequence $\{\varrho_l\} \subset \mathbb{R}_{>0}$ satisfying (38), with $\varrho_l \to 0^+$, such that, for all $l$, there holds
\[
\tilde{T}_\infty (\varrho_l) \geq T_\infty + \frac{1}{2} \Delta_\infty.
\] (57)
as well as (54). Since the exact solution $u$ of (1) blows up at $T_\infty$, there is a time $T_0$, $T_0 < T_\infty$, such that
\[
\|u(T_0)\|_H \geq 2 \left( \frac{4\mu}{\Delta_\infty} \right)^{1/(\beta - 1)}.
\] Furthermore, choosing $l'$ large enough, there exists a time node index $m(\varrho_{l'})$ with
\[
T_0 \leq t_{m(\varrho_{l'})} \leq \frac{1}{2} (T_0 + T_\infty),
\]and such that $\sqrt{\varrho_l}$ is sufficiently small. Referring to Lemma 4, we have
\[
\|U_{m(\varrho_{l'})}^-\|_H \leq \left( \frac{T_{\sup} - 1/2(T_0 + T_\infty)}{2\mu} \right)^{1/(\beta - 1)},
\]uniformly with respect to $\varrho_{l'}$. Hence, by virtue of Remark 5 and Lemma 3 (noting that $C_r < \infty$ in (53)), it is possible to establish the estimate
\[
\|u(t_{m(\varrho_{l'})}) - U_{m(\varrho_{l'})}^-\|_H \leq \left( \frac{4\mu}{\Delta_\infty} \right)^{1/(\beta - 1)}.
\]Since $t \mapsto \|u(t)\|_H$ is non-decreasing this implies that
\[
\|U_{m(\varrho_{l'})}^-\|_H \geq \|u(t_{m(\varrho_{l'})})\|_H - \|u(t_{m(\varrho_{l'})}) - U_{m(\varrho_{l'})}^-\|_H \\
\geq \|u(T_0)\|_H - \left( \frac{4\mu}{\Delta_\infty} \right)^{1/(\beta - 1)} \geq \left( \frac{4\mu}{\Delta_\infty} \right)^{1/(\beta - 1)}.
\]Then, recalling (46), leads to
\[
\tilde{T}_\infty (\varrho_{l'}) \leq t_{m(\varrho_{l'})} + \frac{\|U_{m(\varrho_{l'})}^-\|_H^{1 - \beta}}{c\gamma^\beta} \varrho_{l'}^{\gamma - \alpha\varrho_{l'}}^{\beta - 1} \\
\times \frac{1 - (1 + C_0 \varrho_{l'})^{1 - \beta}}{1 - (1 + C_0 \varrho_{l'})^{1 - \beta}} < T_\infty + 2\mu \|U_{m(\varrho_{l'})}^-\|_H^{1 - \beta} \leq T_\infty + \frac{1}{2} \Delta_\infty,
\]which is a contradiction to (57).
Next, let us assume that $T_{\text{inf}} := \lim_{\varrho \to 0} \tilde{T}_\infty (\varrho) < T_\infty$, and define $\Delta_\infty := T_\infty - T_{\text{inf}} > 0$. Furthermore, let

$$A := \max \left( 2 \left( \delta (\beta - 1) \Delta_\infty \right)^{1/(1-\beta)}, 1 + \|u_0\|_H \right) > \|u_0\|_H.$$

Due to Lemma 5 we can find a sequence $\{\varrho_l\}_{l \in \mathbb{N}} \subset \mathbb{R} > 0$, with $\varrho_l \to 0^+$, and a time $T_0 < T_{\text{inf}}$ so that, for all $l$, there exists $m_A(\varrho_l)$ with $t_{m_A(\varrho_l)} \leq T_0$, and

$$A \leq \left\| U_{m_A(\varrho_l)}^- \right\|_H \leq 2 \left( \frac{A}{\|u_0\|_H} \right)^{C_1/C_0} \|u_0\|_H.$$

In particular, $\left\| U_{m_A(\varrho_l)}^- \right\|_H$ is bounded independently of $\varrho_l$; evidently, since $T_0 < T_{\text{inf}} < T_\infty$, it follows that $\|u(t_{m_A(\varrho_l)})\|_H$ is bounded as well. Thus, as before, recalling Remark 5, and using Lemma 3, we may find a sufficiently large index $l'$ such that

$$\|u(t_{m_A(\varrho_l')}) - U_{m_A(\varrho_l')}^-\|_H \leq \frac{1}{2} A.$$

Then,

$$\|u(t_{m_A(\varrho_l')})\|_H \geq \|U_{m_A(\varrho_l')}^-\|_H - \|u(t_{m_A(\varrho_l')}) - U_{m_A(\varrho_l')}^-\|_H \geq \frac{1}{2} A \geq (\delta (\beta - 1) \Delta_\infty)^{1/(1-\beta)}.$$

Integrating (35) from $t_{m_A(\varrho_l')}$ to $T_\infty$, we arrive at

$$T_\infty \leq t_{m_A(\varrho_l')} + \frac{\|u(t_{m_A(\varrho_l')})\|_H^{1-\beta}}{\delta (\beta - 1)} \leq T_0 + \Delta_\infty < T_{\text{inf}} + \Delta_\infty = T_\infty,$$

which constitutes a contradiction.

In summary, we have shown that

$$\lim_{\varrho \to 0} \tilde{T}_\infty (\varrho) \leq T_\infty \leq \lim_{\varrho \to 0} \tilde{T}_\infty (\varrho),$$

which concludes the proof.  

\[ \square \]

5.4 A time step selection algorithm

The theory in the previous sections suggests the following algorithm for computing a numerical approximation of the exact blow-up time of (1) under the conditions (34) and (36).

Algorithm 1 Suppose that the assumptions of Proposition 2 are satisfied. Choose a parameter $\varrho$ as in (38), and a tolerance $\tau > 0$. Then:
1: Set $m = 0$; $\tilde{T}_\infty = 0$;
2: loop
3: $m \leftarrow m + 1$;
4: Compute $k_m(\varrho)$ using (39);
5: $\tilde{T}_\infty \leftarrow \tilde{T}_\infty + k_m(\varrho)$;
6: if $k_m(\varrho) > \tau$ then
7: Compute the cG (4) or the dG (5) solution on $I_m$
8: (in $M_{m}^{cG}$ from (12) or $M_{m}^{dG}$ from (30), respectively);
9: else
10: return $\tilde{T}_\infty$;
11: end if
12: end loop

The result, $\tilde{T}_\infty$, is an approximation of the exact blow-up time.

Remark 9 A practical (and platform-independent) implementation of the stopping criterion in the if-statement in line 6 of the above Algorithm 1 is to run the time marching process until

$$\tilde{T}_\infty + k_m(\varrho) == \tilde{T}_\infty$$

is true, where we make use of the equality operator “==”. Note that this also eliminates the need of specifying the tolerance parameter $\tau$.

In order to provide an illustrating example for Algorithm 1, let us consider the initial value problem of finding a function $u = u(t)$, $t \geq 0$, such that

$$u'(t) = \frac{|u(t)| + 1}{1 + e^{-t}} =: F(t, u(t)), \quad u(0) = 3.$$ 

It has an exact solution $u(t) = 3(e^t + 1)(5 - 3e^t)^{-1}$, and, thus, features a blow-up at $T_\infty = \ln(5/3)$. Here, $H = \mathbb{R}$, and $cF < 3$ in (34) in alignment with Proposition 2.

Choosing $cF = 2$, a few elementary calculations show that $\alpha = 3/2$, $\beta = 2$, and $\delta = 1/2$ in (34). Furthermore, $\gamma = 5/2$ in (36), and (52) holds with $L_F = 5$. The unique positive root of $\Psi$ in (37) is given by $\varrho \approx 0.243163$. We see that the assumptions of Theorem 3 hold, although, in our computations, we select larger values of $\varrho$ and of $k_m(\varrho)$ than would be mandated by (38) and (39), respectively. More precisely, we consider

$$k_m(\varrho) = \varrho|U_{m-1}^-|^{-1},$$

for $\varrho \in \{2^{-p/2}, p = 4, \ldots, 10\}$. We run Algorithm 1 based on the stopping criterion mentioned in Remark 9. The polynomial degree $r = r_m$ is kept fixed for all time steps. The results are displayed in Fig. 1 (left) for both the cG and the dG time stepping methods for different values of $\varrho$, and for various choices of $r \in \{0, \ldots, 3\}$. The numerical solution on each time step is obtained with the aid of a fixed point iteration as described in Remark 4 (we note that solving the nonlinear problems by means of an adaptive Newton method is potentially more efficient from a computational view.
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point, however, we remark that this approach seems more fragile close to the blow-up due to the large magnitude of the numerical solution. Even though Theorem 3 does not provide any theoretical evidence on convergence rates, the results suggest a convergence to blow-up time of order $O(q^{2(r+1)})$ for the cG method (based on a local polynomial degree $r + 1$), and $O(q^{2r+1})$ for the dG scheme (based on a local polynomial degree $r$), for $r = 0, 1, 2$; for $r \geq 3$ the errors are too small to allow for a precise identification of the convergence behaviour. The number of time steps was found to be independent of the polynomial degrees (however, strongly dependent on $q$) and of whether the cG or dG method was employed; see Fig. 1 (right).

6 Conclusions

In this paper we have investigated the $hp$-version continuous and discontinuous Galerkin time stepping methods for the numerical approximation of general ordinary differential equation initial value problems with continuous (and possibly unbounded) nonlinearities in real Hilbert spaces. Our main findings include Peano-type existence results for the discrete systems, and a blow-up time step selection algorithm, together with a convergence (to blow-up time) result, for problems with algebraically growing nonlinearities. We have shown that discrete solutions exist (and are unique within suitable ranges) provided that the local time steps are chosen sufficiently small (depending on the numerical solutions themselves, however, independent of the local polynomial degrees). The key ingredients in the existence and uniqueness proofs include the derivation of strong forms of the Galerkin discretizations, the transformation into suitable fixed point equations, and the application of fixed point theory. The application of the techniques derived in this article to nonlinear parabolic PDE, and the development of a posteriori error estimates for the blow-up time (in conjunction with the $hp$-framework) are subjects of ongoing research.

An auxiliary result

**Lemma 6** Let $\mathcal{F} : H \rightarrow H$ be a continuous function on a (real) Hilbert space $H$, and $M > 0$ a constant such that the Lipschitz condition

\[ |\mathcal{F}(x) - \mathcal{F}(y)| \leq M |x - y| \]
\[ \| \mathcal{F}(x) - \mathcal{F}(y) \|_H \leq L_M \| x - y \|_H \]  

(58)

holds for any \( x, y \in H \) with \( \| x \|_H, \| y \|_H \leq M \); here \( L_M > 0 \) is a constant. Then, the function

\[ \mathcal{G} : H \to H, \quad x \mapsto \begin{cases} \mathcal{F}(x) & \| x \|_H \leq M, \\ \mathcal{F} \left( \frac{M}{\| x \|_H} x \right) & \| x \|_H > M \end{cases} \]

is (globally) Lipschitz continuous on \( H \) with Lipschitz constant \( L_M \).

**Proof** For \( \| x \|_H, \| y \|_H \leq M \) the claim follows immediately from the definition of \( \mathcal{G} \) and from (58). If \( \| x \|_H, \| y \|_H > M \), we have

\[ \| \mathcal{G}(x) - \mathcal{G}(y) \|_H \leq L_M \| x \|_H - \| y \|_H \|_H, \]

where we notice that

\[
\left\| \frac{x}{\| x \|_H} - \frac{y}{\| y \|_H} \right\|^2_H = \frac{2}{\| x \|_H \| y \|_H} (\| x \|_H \| y \|_H - (x, y)_H) \\
\leq \frac{1}{\| x \|_H \| y \|_H} \left( \| x \|_H^2 + \| y \|_H^2 - 2(x, y)_H \right) < \frac{1}{M^2} \| x - y \|_H^2.
\]

Thus, \( \| \mathcal{G}(x) - \mathcal{G}(y) \|_H < L_M \| x - y \|_H \). Moreover, if \( \| x \|_H \leq M < \| y \|_H \), then it holds that

\[ \| \mathcal{G}(x) - \mathcal{G}(y) \|_H \leq L_M \| x \|_H - \frac{y}{\| y \|_H}, \]

where

\[
\left\| \frac{x}{M} - \frac{y}{\| y \|_H} \right\|^2_H = \frac{1}{M \| y \|_H} \| x - y \|_H^2 - \frac{1}{M \| y \|_H} \left( \frac{\| y \|_H}{M} - 1 \right) \left( \frac{\| y \|_H M - \| x \|_H^2}{M \| y \|_H} \right) < \frac{1}{M \| y \|_H} \| x - y \|_H^2 < \frac{1}{M^2} \| x - y \|_H^2.
\]

Therefore, again \( \| \mathcal{G}(x) - \mathcal{G}(y) \|_H < L_M \| x - y \|_H \). The proof for \( \| x \|_H > M \geq \| y \|_H \) follows from symmetry. \( \square \)

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