Desingularization of singular hyperkähler varieties II.

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Abstract: We construct a natural hyperkähler desingularization for all singular hyperkähler varieties.

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1 Introduction

A hyperkähler manifold is a Riemannian manifold with an action of a quaternion algebra \( \mathbb{H} \) in its tangent bundle, such that for all \( I \in \mathbb{H} \), \( I^2 = -1 \), \( I \) establishes a complex, Kähler structure on \( M \) (see Definition 2.1 for details). We extend this definition to singular varieties. The notion of a singular hyperkähler variety has its origin in \( \text{V-bun} \) (see also \( \text{V3} \) and \( \text{V-des} \)). Examples of singular hyperkähler varieties are numerous, and come from several

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diverse sources (Remark 2.12, Theorem 2.13; for additional examples see [V3], Section 10). There is a weaker version of this definition: a notion of **hypercomplex variety**. Singular hypercomplex varieties is what this paper primarily deals with.

For a real analytic variety $M$, we say that $M$ is **hypercomplex**, if $M$ is equipped with the complex structures $I$, $J$ and $K$, such that $I \circ J = -J \circ I = K$, and certain integrability conditions are satisfied (for a precise statement, see Definition 2.9). The paper [V-des] dealt with the hypercomplex varieties with “locally homogeneous singularities” (LHS). A complex analytic variety $M$ is LHS if for each point $x \in M$, the completion $\mathcal{A}$ of the local ring $\mathcal{O}_x M$ can be represented as a quotient of the power series ring by a homogeneous ideal (Definition 3.2, Claim 3.3). In [V-des], we described explicitly the singularities of hypercomplex LHS varieties. We have shown that every such variety, considered as a complex variety with a complex structure induced from the quaternions, is locally isomorphic to a union of planes in $\mathbb{C}^n$ (Theorem 3.8). The normalization of such a variety is non-singular, which follows from this description of singularities. This gives a canonical, functorial way to desingularize hyperkähler and hypercomplex varieties (Theorem 3.9).

The purpose of the present paper is to show that all hypercomplex varieties have locally homogeneous singularities (Theorem 3.10). This is used to extend the desingularization results to all singular hyperkähler or hypercomplex varieties (Corollary 3.11).

In Sections 4–5, we prove that all hypercomplex varieties have locally homogeneous singularities. Section 4 is purely a commutative algebra. We work with a complete local Noetherian ring $\mathcal{A}$ over $\mathbb{C}$. By definition, an automorphism $e: \mathcal{A} \to \mathcal{A}$ is called **homogenizing** (Definition 4.1) if its differential acts as a dilatation on the Zariski tangent space of $\mathcal{A}$, with dilatation coefficient $|\lambda| < 1$. As usual, by the Zariski tangent space we understand the space $\mathfrak{m}_A/\mathfrak{m}_A^2$, where $\mathfrak{m}_A$ is a maximal ideal of $\mathcal{A}$.

The main result of Section 4 is the following. For a complete local Noetherian ring $\mathcal{A}$ over $\mathbb{C}$ equipped with a homogenizing automorphism $e: \mathcal{A} \to \mathcal{A}$, we show that $\mathcal{A}$ has locally homogeneous singularities.

In Section 5, we construct a natural homogenizing automorphism of the ring of germs of complex analytic functions on a hypercomplex variety $M$ (Proposition 5.1). Applying Section 4, we obtain that every hypercomplex variety has locally homogeneous singularities.
2 Preliminaries

2.1 Definitions

This subsection contains a compression of the basic definitions from hyperkähler geometry, found, for instance, in [Bes] or in [Beau].

Definition 2.1: (Bes) A hyperkähler manifold is a Riemannian manifold \( M \) endowed with three complex structures \( I, J \) and \( K \), such that the following holds.

(i) the metric on \( M \) is Kähler with respect to these complex structures and

(ii) \( I, J \) and \( K \), considered as endomorphisms of a real tangent bundle, satisfy the relation \( I \circ J = -J \circ I = K \).

The notion of a hyperkähler manifold was introduced by E. Calabi ([C]).

Clearly, a hyperkähler manifold has the natural action of quaternion algebra \( \mathbb{H} \) in its real tangent bundle \( TM \). Therefore its complex dimension is even. For each quaternion \( L \in \mathbb{H}, \ L^2 = -1 \), the corresponding automorphism of \( TM \) is an almost complex structure. It is easy to check that this almost complex structure is integrable ([Bes]).

Definition 2.2: Let \( M \) be a hyperkähler manifold, \( L \) a quaternion satisfying \( L^2 = -1 \). The corresponding complex structure on \( M \) is called an induced complex structure. The \( M \) considered as a complex manifold is denoted by \( (M, L) \).

Let \( M \) be a hyperkähler manifold. We identify the group \( SU(2) \) with the group of unitary quaternions. This gives a canonical action of \( SU(2) \) on the tangent bundle and all its tensor powers. In particular, we obtain a natural action of \( SU(2) \) on the bundle of differential forms.

Lemma 2.3: The action of \( SU(2) \) on differential forms commutes with the Laplacian.

Proof: This is Proposition 1.1 of [V-bun]. 

Thus, for compact \( M \), we may speak of the natural action of \( SU(2) \) in cohomology.
2.2 Trianalytic subvarieties in compact hyperkähler manifolds.

In this subsection, we give a definition and a few basic properties of trianalytic subvarieties of hyperkähler manifolds. We follow [V2].

Let $M$ be a compact hyperkähler manifold, $\dim_{\mathbb{R}} M = 2m$.

**Definition 2.4:** Let $N \subset M$ be a closed subset of $M$. Then $N$ is called **trianalytic** if $N$ is a complex analytic subset of $(M, L)$ for any induced complex structure $L$.

Let $I$ be an induced complex structure on $M$, and $N \subset (M, I)$ be a closed analytic subvariety of $(M, I)$, $\dim_{\mathbb{C}} N = n$. Denote by $[N] \in H_{2n}(M)$ the homology class represented by $N$. Let $\langle N \rangle \in H^{2m-2n}(M)$ denote the Poincare dual cohomology class. Recall that the hyperkähler structure induces the action of the group $SU(2)$ on the space $H^{2m-2n}(M)$.

**Theorem 2.5:** Assume that $\langle N \rangle \in H^{2m-2n}(M)$ is invariant with respect to the action of $SU(2)$ on $H^{2m-2n}(M)$. Then $N$ is trianalytic.

**Proof:** This is Theorem 4.1 of [V2].

**Remark 2.6:** Trianalytic subvarieties have an action of quaternion algebra in the tangent bundle. In particular, the real dimension of such subvarieties is divisible by 4. The non-singular part of a trianalytic subvariety is hyperkähler.

2.3 Hypercomplex varieties

This subsection is based on the results and definitions from [V-des].

Let $X$ be a complex variety, $X_{\mathbb{R}}$ the underlying real analytic variety. In [V-des], Section 2, we constructed a natural automorphism of the sheaf of Kähler differentials on $X_{\mathbb{R}}$

$$I : \Omega^1_{X_{\mathbb{R}}} \longrightarrow \Omega^1_{X_{\mathbb{R}}}, \quad I^2 = -1.$$ 

This endomorphism is a generalization of the usual notion of a complex structure operator, and its construction is straightforward. We called $I$ the **complex structure operator** on $X_{\mathbb{R}}$. The operator $I$ is functorial: for a morphism $f : X \longrightarrow Y$ of complex varieties, the natural pullback map
$df : f^*\Omega^1_{Y_\mathbb{R}} \to \Omega^1_{X_\mathbb{R}}$ commutes with the complex structure operators (see [V-des] for details). The converse statement is also true:

**Theorem 2.7:** Let $X, Y$ be complex analytic varieties, and

$$f_\mathbb{R} : X_\mathbb{R} \to Y_\mathbb{R}$$

be a morphism of underlying real analytic varieties which commutes with the complex structure. Then there exist a unique morphism $f : X \to Y$ of complex analytic varieties, such that $f_\mathbb{R}$ is its underlying morphism.

**Proof:** This is [V-des], Theorem 2.1. □

**Definition 2.8:** Let $M$ be a real analytic variety, and

$$I : \Omega^1(O_M) \to \Omega^1(O_M)$$

be an endomorphism satisfying $I^2 = -1$. Then $I$ is called an **almost complex structure on** $M$. If there exist a complex analytic structure $\mathcal{C}$ on $M$ such that $I$ appears as the complex structure operator associated with $\mathcal{C}$, we say that $I$ is **integrable**. Theorem 2.7 implies that this complex structure is unique if it exists.

**Definition 2.9:** (Hypercomplex variety) Let $M$ be a real analytic variety equipped with almost complex structures $I, J$ and $K$, such that $I \circ J = -J \circ I = K$. Assume that for all $a, b, c \in \mathbb{R}$, such that $a^2 + b^2 + c^2 = 1$, the almost complex structure $aI + bJ + cK$ is integrable. Then $M$ is called a **hypercomplex variety**.

**Remark 2.10:** As follows from [V-des], Claim 2.7, every hyperkähler manifold is hypercomplex, in a natural way. The proof is straightforward.

### 2.4 Singular hyperkähler varieties

Throughout this paper, we never use the notion of hyperkähler variety. For our present purposes, the hypercomplex varieties suffice. However, for the reader’s benefit, we give a definition and a list of examples of hyperkähler varieties. All hyperkähler varieties are hypercomplex, and the converse is (most likely) false. However, it is difficult to construct examples of hypercomplex varieties which are not hyperkähler, and all “naturally” occuring hypercomplex varieties come equipped with a singular hyperkähler structure.
This subsection is based on the results and definitions from \[V_3\] and \[V\text{-des}\]. For a more detailed exposition, the reader is referred to \[V_3\], Section 10.

**Definition 2.11:** (\[V\text{-bun}\], Definition 6.5) Let \(M\) be a hypercomplex variety (Definition 2.9). The following data define a structure of a hyperkähler variety on \(M\).

(i) For every \(x \in M\), we have an \(\mathbb{R}\)-linear symmetric positively defined bilinear form \(s_x : T_xM \times T_xM \to \mathbb{R}\) on the corresponding real Zariski tangent space.

(ii) For each triple of induced complex structures \(I, J, K\), such that \(I \circ J = K\), we have a holomorphic differential 2-form \(\Omega \in \Omega^2(M, I)\).

(iii) Fix a triple of induced complex structure \(I, J, K\), such that \(I \circ J = K\). Consider the corresponding differential 2-form \(\Omega\) of (ii). Let \(J : T_xM \to T_xM\) be an endomorphism of the real Zariski tangent spaces defined by \(J\), and \(\text{Re}\Omega\big|_x\) the real part of \(\Omega\), considered as a bilinear form on \(T_xM\). Let \(r_x\) be a bilinear form \(r_x : T_xM \times T_xM \to \mathbb{R}\) defined by \(r_x(a, b) = -\text{Re}\Omega\big|_x(a, J(b))\). Then \(r_x\) is equal to the form \(s_x\) of (i). In particular, \(r_x\) is independent from the choice of \(I, J, K\).

**Remark 2.12:**

(a) It is clear how to define a morphism of hyperkähler varieties.

(b) For \(M\) non-singular, Definition 2.11 is equivalent to the usual one (Definition 2.1). If \(M\) is non-singular, the form \(s_x\) becomes the usual Riemann form, and \(\Omega\) becomes the standard holomorphically symplectic form.

(c) It is easy to check the following. Let \(X\) be a hypercomplex subvariety of a hyperkähler variety \(M\). Then, restricting the forms \(s_x\) and \(\Omega\) to \(X\), we obtain a hyperkähler structure on \(X\). In particular, trianalytic subvarieties of hyperkähler manifolds are always hyperkähler, in the sense of Definition 2.11.

**Caution:** Not everything which is seemingly hyperkähler satisfies the conditions of Definition 2.11. Take a quotient \(M/G\) of a hyperkähler manifold by an action of finite group \(G\), acting in accordance with hyperkähler
structure. Then $M/G$ is, generally speaking, not hyperkähler (see \cite{V3}, Section 10 for details).

The following theorem, proven in \cite{V-bun} (Theorem 6.3), gives a convenient way to construct examples of hyperkähler varieties.

**Theorem 2.13:** Let $M$ be a compact hyperkähler manifold, $I$ an induced complex structure and $B$ a stable holomorphic bundle over $(M,I)$. Let $\text{Def}(B)$ be the reduction of the deformation space of stable holomorphic structures on $B$. Assume that $c_1(B)$, $c_2(B)$ are $SU(2)$-invariant, with respect to the standard action of $SU(2)$ on $H^*(M)$. Then $\text{Def}(B)$ has a natural structure of a hyperkähler variety.



3 Desingularization of hyperkähler varieties

In this section, we recall the desingularization theorem for the hypercomplex varieties with locally homogeneous singularities, as it was proven in \cite{V-des}. In the last subsection, we state the main result of this paper, which is proven in Sections 4–5.

3.1 Spaces with locally homogeneous singularities.

**Definition 3.1:** (local rings with LHS) Let $A$ be a local ring. Denote by $m$ its maximal ideal. Let $A_{\text{gr}}$ be the corresponding associated graded ring. Let $\widehat{A}$, $\widehat{A}_{\text{gr}}$ be the $m$-adic completion of $A$, $A_{\text{gr}}$. Let $(\widehat{A})_{\text{gr}}, (\widehat{A}_{\text{gr}})_{\text{gr}}$ be the associated graded rings, which are naturally isomorphic to $A_{\text{gr}}$. We say that $A$ has locally homogeneous singularities (LHS) if there exists an isomorphism $\rho : \widehat{A} \rightarrow (\widehat{A}_{\text{gr}})_{\text{gr}}$ which induces the standard isomorphism $i : \widehat{A}_{\text{gr}} \rightarrow (\widehat{A}_{\text{gr}})_{\text{gr}}$ on associated graded rings.

**Definition 3.2:** (SLHS) Let $X$ be a complex or real analytic space. Then $X$ is called a space with locally homogeneous singularities (SLHS) if for each $x \in M$, the local ring $\mathcal{O}_x$ has locally homogeneous singularities.

The following claim might shed a light on the origin of the term “locally homogeneous singularities”.

\footnote{The deformation space might have nilpotents in the structure sheaf. We take its reduction to avoid this.}
**Claim 3.3:** Let $A$ be a complete local Noetherian ring over $\mathbb{C}$. Then the following statements are equivalent

(i) $A$ has locally homogeneous singularities

(ii) There exist a surjective ring homomorphism $\rho : \mathbb{C}[[x_1,\ldots,x_n]] \rightarrow A$, where $\mathbb{C}[[x_1,\ldots,x_n]]$ is the ring of power series, and the ideal $\ker \rho$ is homogeneous in $\mathbb{C}[[x_1,\ldots,x_n]]$.

**Proof:** Clear. ■

### 3.2 Hyperkähler varieties with locally homogeneous singularities

**Proposition 3.4:** Let $M$ be a complex variety, $M_\mathbb{R}$ the underlying real analytic variety. Then $M_\mathbb{R}$ is a space with locally homogeneous singularities (SLHS) if and only if $M$ is a space with locally homogeneous singularities.

**Proof:** This is [V-des], Proposition 4.6. ■

**Corollary 3.5:** [V-des] Let $M$ be a hyperkähler variety, $I_1, I_2$ induced complex structures. Then $(M, I_1)$ is a space with locally homogeneous singularities if and only if $(M, I_2)$ is SLHS.

**Proof:** The real analytic variety underlying $(M, I_1)$ coinsides with that underlying $(M, I_2)$. Applying Proposition 3.4, we immediately obtain Corollary 3.5. ■

**Definition 3.6:** Let $M$ be a hyperkähler or hypercomplex variety. Then $M$ is called a space with locally homogeneous singularities (SLHS) if the underlying real analytic variety is SLHS or, equivalently, for some induced complex structure $I$ the $(M, I)$ is SLHS.

Some of the canonical examples of hyperkähler varieties are spaces with locally homogeneous singularities *per se*. For instance, it is easy to prove the following theorem:

**Theorem 3.7:** ([V-des], Theorem 4.9) Let $M$ be a compact hyperkähler manifold, $I$ an induced complex structure and $B$ a stable holomorphic bundle over $(M, I)$. Assume that $c_1(B), c_2(B)$ are $SU(2)$-invariant, with respect to the standard action of $SU(2)$ on $H^*(M)$. Let Def($B$) be a reduction of a deformation space of stable holomorphic structures on $B$, which is a
hyperkähler variety by Theorem 2.13. Then Def(\(B\)) is a space with locally homogeneous singularities (SLHS).

However, for the other examples of hyperkähler varieties, there is no easy ad hoc way to show that they are SLHS. The main aim of this paper, however, is to prove that every hypercomplex variety is SLHS (see Subsection 3.4).

3.3 Desingularization of hypercomplex varieties which are SLHS

For hypercomplex varieties which are SLHS, we have a complete list of possible singularities ([V-des]; see also Theorem 3.8). This makes it possible to desingularize every hypercomplex (or hyperkähler) variety in a natural way. The present paper shows that every hypercomplex variety is SLHS, thus extending the results of [V-des] to all hypercomplex varieties. For the benefit of the reader, we relate in this Subsection the main results of [V-des]. We don’t use these results in the rest of the article, so the reader is free to skip this Subsection.

Here is the theorem describing the shape of singularities.

**Theorem 3.8:** Let \(M\) be a hypercomplex variety, and \(I\) an induced complex structure. Assume that \(M\) is SLHS. Then, for each point \(x \in M\), there exists a neighbourhood \(U\) of \(x \in (M, I)\), which is isomorphic to \(B \cap (\bigcup L_i)\), where \(B\) is an open ball in \(\mathbb{C}^n\) and \(\bigcup L_i\) is a union of planes \(L_i \in \mathbb{C}^n\) passing through \(0 \in \mathbb{C}^n\). In particular, the normalization of \((M, I)\) is smooth.

**Proof:** See Corollary 5.3 of [V-des].

Here is the desingularization theorem.

**Theorem 3.9:** ([V-des], Theorem 6.1) Let \(M\) be a hyperkähler or a hypercomplex variety, \(I\) an induced complex structure. Assume that \(M\) is a space with locally homogeneous singularities. Let

\[
\widehat{(M, I)} \xrightarrow{n} (M, I)
\]

be the normalization of \((M, I)\). Then \(\widehat{(M, I)}\) is smooth and has a natural hyperkähler (respectively, hypercomplex) structure \(\mathcal{H}\), such that the associ-
ated map \( n : (\tilde{M}, I) \to (M, I) \) agrees with \( \mathcal{H} \). Moreover, the hyperkähler (hypercomplex) manifold \( \tilde{M} := (M, I) \) is independent from the choice of induced complex structure \( I \).

3.4 The main result: every hypercomplex variety is SLHS

The proof of the following theorem is given in Sections 4–5.

**Theorem 3.10:** (the main result of this paper) Let \( M \) be a hypercomplex variety. Then \( M \) is a space with locally homogeneous singularities (SLHS).

Theorem 3.10 has the following immediate corollary.

**Corollary 3.11:** Let \( M \) be a hypercomplex or a hyperkähler variety. Then Theorem 3.8 (a theorem describing the shape of the singularities of \( M \)) and Theorem 3.9 (desingularization theorem) hold.

4 Complete rings with automorphisms

**Definition 4.1:** Let \( A \) be a local Noetherian ring over \( \mathbb{C} \), equipped with an automorphism \( e : A \to A \). Let \( m \) be a maximal ideal of \( A \). Assume that \( e \) acts on \( m/m^2 \) as a multiplication by \( \lambda \in \mathbb{C}, |\lambda| < 1 \). Then \( e \) is called a homogenizing automorphism of \( A \).

The aim of the present section is to prove the following statement.

**Proposition 4.2:** Let \( A \) be a complete Noetherian ring over \( \mathbb{C} \), equipped with a homogenizing automorphism \( e : A \to A \). Then there exist a surjective ring homomorphism \( \rho : \mathbb{C}[x_1, \ldots, x_n] \to A \), such that the ideal \( \ker \rho \) is homogeneous in \( \mathbb{C}[x_1, \ldots, x_n] \). In particular, \( A \) has locally homogeneous singularities (Claim 3.3).

This statement is well known. A reader who knows its proof should skip the rest of this section.

**Proposition 4.3:** Let \( A \) be a complete Noetherian ring over \( \mathbb{C} \), equipped with a homogenizing automorphism \( e : A \to A \). Then there exist a system
of ring elements
\[ f_1, \ldots, f_m \in \mathfrak{m}, \quad m = \dim_\mathbb{C} \mathfrak{m}/\mathfrak{m}^2, \]
which generate \( \mathfrak{m}/\mathfrak{m}^2 \), and such that \( e(f_i) = \lambda f_i \).

**Proof:** Let \( a \in \mathfrak{m}/\mathfrak{m}^2 \). Let \( a \in \mathfrak{m} \) be a representative of \( a \) in \( \mathfrak{m} \). To prove Proposition 4.3 it suffices to find \( c \in \mathfrak{m}^2 \), such that \( e(a - c) = \lambda a - \lambda c \).

Thus, we need to solve an equation
\[ \lambda c - e(c) = e(a) - \lambda(a). \tag{4.1} \]
Let \( r := e(a) - \lambda a \). Clearly, \( r \in \mathfrak{m}^2 \). A solution of (4.1) is provided by the following lemma.

**Lemma 4.4:** In assumptions of Proposition 4.3, let \( r \in \mathfrak{m}^2 \). Then, the equation
\[ e(c) - \lambda c = r \tag{4.2} \]
has a unique solution \( c \in \mathfrak{m}^2 \).

**Proof:** We need to show that the operator \( P := (e - \lambda)|_{\mathfrak{m}^2} \) is invertible.
Consider the adic filtration \( \mathfrak{m}^2 \subset \mathfrak{m}^3 \subset \ldots \) on \( \mathfrak{m}^2 \). Clearly, \( P \) preserves this filtration. Since \( \mathfrak{m}^2 \) is complete with respect to the adic filtration, it suffices to show that \( P \) is invertible on the successive quotients. The quotient \( \mathfrak{m}^2/\mathfrak{m}^i \) is finite-dimensional, so to show that \( P \) is invertible it suffices to calculate the eigenvalues. Since \( e \) is an automorphism, restriction of \( e \) to \( \mathfrak{m}^i/\mathfrak{m}^{i-1} \) is a multiplication by \( \lambda^i \). Thus, the eigenvalues of \( e \) on \( \mathfrak{m}^2/\mathfrak{m}^i \) range from \( \lambda^2 \) to \( \lambda^{i-1} \). Since \( |\lambda| > |\lambda|^2 \), all eigenvalues of \( P|_{\mathfrak{m}^2/\mathfrak{m}^i} \) are non-zero and the restriction of \( P \) to \( \mathfrak{m}^2/\mathfrak{m}^i \) is invertible. This proves Lemma 4.4.

**The proof of Proposition 4.2.** Consider the map
\[ \rho : \mathbb{C}[[x_1, \ldots x_m]] \to A, \quad \rho(x_i) = f_i, \]
where \( f_1, \ldots, f_m \) is the system of functions constructed in Proposition 4.3.
By Nakayama, \( \rho \) is surjective.

Let \( e_\lambda : \mathbb{C}[[x_1, \ldots x_m]] \to \mathbb{C}[[x_1, \ldots x_m]] \) be the automorphism mapping \( x_i \) to \( \lambda x_i \). Then, the diagram
\[
\begin{array}{ccc}
\mathbb{C}[[x_1, \ldots x_m]] & \xrightarrow{\rho} & A \\
e_\lambda \downarrow & & \downarrow e \\
\mathbb{C}[[x_1, \ldots x_m]] & \xrightarrow{\rho} & A
\end{array}
\]
is by construction commutative. Therefore, the ideal $I = \ker \rho$ is preserved by $e_\lambda$. Clearly, every $e_\lambda$-preserved ideal $I \subset \mathbb{C}[[x_1, \ldots, x_m]]$ is homogeneous. **Proposition 4.2** is proven. ◼

5 Authomorphisms of local rings of holomorphic functions on hyperkähler varieties

Let $M$ be a hypercomplex variety, $x \in M$ a point, $I$ an induced complex structure. Let $A_I := \hat{O}_x(M, I)$ be the adic completion of the local ring $O_x(M, I)$ of $x$-germs of holomorphic functions on the complex variety $(M, I)$. Clearly, the sheaf ring of the antiholomorphic functions on $(M, I)$ coincides with $O_x(M, -I)$. Thus, the corresponding completion ring is $A_{-I}$. As in [V-des], Claim 2.1, we have the natural isomorphism of completions:

$$A_I \otimes_{\mathbb{C}} A_{-I} = \hat{O}_x(\hat{M_R}) \otimes_{\mathbb{R}} \mathbb{C}, \quad (5.1)$$

where

$$\hat{O}_x(\hat{M_R}) \otimes_{\mathbb{R}} \mathbb{C}$$

is the $x$-completion of the ring of germs of real analytic complex-valued functions on $M$. Consider the natural quotient map

$$p : A_{-I} \rightarrow \mathbb{C}.$$  

Denote the ring

$$\hat{O}_x(\hat{M_R}) \otimes_{\mathbb{R}} \mathbb{C}$$

by $A_{\mathbb{R}}$. Let $i_I : A_I \hookrightarrow A_{\mathbb{R}}$ be the natural embedding

$$a \mapsto a \times 1 \in A_I \otimes_{\mathbb{C}} A_{-I},$$

and $e_I : A_{\mathbb{R}} \rightarrow A_I$ be the natural epimorphism associated with the surjective map

$$A_I \otimes_{\mathbb{C}} A_{-I} \rightarrow A_I, \quad a \otimes b \mapsto a \otimes p(b),$$

where $a \in A_I$, $b \in A_{-I}$, and

$$a \otimes b \in A_I \otimes_{\mathbb{C}} A_{-I} \subset A_{\mathbb{R}}.$$  

For an induced complex structure $J$, we define $A_J$, $A_{-J}$, $i_J$, $e_J$ likewise. Let $\Psi_{I,J} : A_I \rightarrow A_J$ be the composition

$$A_I \xrightarrow{i_I} A_{\mathbb{R}} \xrightarrow{e_J} A_J \xrightarrow{i_j} A_{\mathbb{R}} \xrightarrow{e_I} A_I.$$
Clearly, for \( I = J \), the ring morphism \( \Psi_{I,J} \) is identity, and for \( I = -J \), \( \Psi_{I,J} \) is an augmentation map.

**Proposition 5.1:** Let \( M \) be a singular hyperkähler variety, \( x \in M \) a point, \( I, J \) be induced complex structures, such that \( I \neq J \) and \( I \neq -J \). Consider the map \( \Psi_{I,J} : A_I \rightarrow A_I \) defined as above. Then \( \Psi_{I,J} \) is a homogenizing automorphism of \( A_I \).

**Proof:** Let \( d\Psi \) be differential of \( \Psi_{I,J} \), that is, the restriction of \( \Psi_{I,J} \) to \( m/m^2 \), where \( m \) is the maximal ideal of \( A_I \). By Nakayama, to prove that \( \Psi_{I,J} \) is an automorphism it suffices to show that \( d\Psi \) is invertible. To prove that \( \Psi_{I,J} \) is homogenizing, we have to show that \( d\Psi \) is a multiplication by a complex number \( \lambda \), \( |\lambda| < 1 \). As usually, we denote the real analytic variety underlying \( M \) by \( M_R \). Let \( T_I, T_J, T_R \) be the Zariski tangent spaces to \((M, I), (M, J) \) and \( M_R \), respectively, in \( x \in M \). Consider the complexification \( T_R := T_R \otimes \mathbb{C} \), which is a Zariski tangent space to the local ring \( A_R \). To compute \( d\Psi : T_I \rightarrow T_I \), we need to compute the differentials of \( e_I, e_J, i_I, i_J \), i.e., the restrictions of the homomorphisms \( e_I, e_J, i_I, i_J \) to the Zariski tangent spaces \( T_I, T_J, T_R \). Denote these differentials by \( de_I, de_J, di_I, di_J \).

**Lemma 5.2:** Let \( M \) be a hyperkähler variety, \( M_R \) the associated real analytic variety, \( x \in M \) a point. Consider the space \( T_R := T_x(M_R) \otimes \mathbb{C} \). For an induced complex structure \( I \), consider the Hodge decomposition \( T_R = T_I^{1,0} \oplus T_I^{0,1} \). In our previous notation, \( T_I^{1,0} \) is \( T_I \). Then, \( di_I \) is the natural embedding of \( T_I^{1,0} \) to \( T_R \), and \( de_I \) is the natural projection of \( T_R = T_I^{1,0} \oplus T_I^{0,1} \) to \( T_I^{1,0} = T_I \).

**Proof:** Clear.

We are able now to describe the map \( d\Psi : T_I \rightarrow T_I \) in terms of the quaternion action. Recall that the space \( T_I \) is equipped with a natural \( \mathbb{R} \)-linear quaternionic action. For each quaternionic linear space \( V \) and each quaternion \( I, I^2 = -1, I \) defines a complex structure in \( V \). Such a complex structure is called **induced by the quaternionic structure**.

**Lemma 5.3:** Let \( V \) be a space with quaternion action, and \( V := V \otimes \mathbb{C} \) its complexification. For each induced complex structure \( I \in \mathbb{H} \), consider the Hodge decomposition \( V := V_I^{1,0} \oplus V_I^{0,1} \). For an induced complex structures \( I, J \in \mathbb{H} \), let \( \Phi_{I,J}(V) \) be a composition of the natural embeddings and

\footnote{For the definition of a homogenizing automorphism, see Definition 4.1.}
projections
\[ V_I^{1,0} \to V \to V_J^{1,0} \to V \to V_I^{1,0}. \]

Using the natural identification \( V \cong V_I^{1,0} \), we consider \( \Phi_{I,J}(V) \) as an \( \mathbb{R} \)-linear automorphism of the space \( V \). Then, applying the operator \( \Phi_{I,J}(V) \) to the quaternionic space \( T_I \), we obtain the operator \( d\Psi \) defined above.

**Proof:** Follows from Lemma 5.2

As we have seen, to prove Proposition 5.1 it suffices to show that \( d\Psi \) is a multiplication by a non-zero complex number \( \lambda \), \( |\lambda| < 1 \). Thus, the proof of Proposition 5.1 is finished with the following lemma.

**Lemma 5.4:** In assumptions of Lemma 5.3, consider the map \( \Phi_{I,J}(V) : V_I^{1,0} \to V_I^{1,0} \).

Then \( \Phi_{I,J}(V) \) is a multiplication by a complex number \( \lambda \). Moreover, \( \lambda \) is a non-zero number unless \( I = -J \), and \( |\lambda| < 1 \) unless \( I = J \).

**Proof:** Let \( V = \oplus V_i \) be a decomposition of \( V \) into a direct sum of \( \mathbb{H} \)-linear spaces. Then, the operator \( \Phi_{I,J}(V) \) can also be decomposed: \( \Phi_{I,J}(V) = \oplus \Phi_{I,J}(V_i) \). Thus, to prove Lemma 5.4 it suffices to assume that \( \dim_{\mathbb{H}} V = 1 \). Therefore, we may identify \( V \) with the space \( \mathbb{H} \), equipped with the right action of quaternion algebra on itself.

Consider the left action of \( \mathbb{H} \) on \( V = \mathbb{H} \). This action commutes with the right action of \( \mathbb{H} \) on \( V \). Consider the corresponding action \( \rho : SU(2) \to \text{End}(V) \) of the group of unitary quaternions \( \mathbb{H}^{un} = SU(2) \subset \mathbb{H} \) on \( V \). Since \( \rho \) commutes with the quaternion action, \( \rho \) preserves \( V_I^{1,0} \subset V \), for every induced complex structure \( I \). By the same token, for each \( g \in SU(2) \), the endomorphism \( \rho(g) \in \text{End}(V_I^{1,0}) \) commutes with \( \Phi_{I,J}(V) \).

Consider the 2-dimensional \( \mathbb{C} \)-vector space \( V_I^{1,0} \) as a representation of \( SU(2) \). Clearly, \( V_I^{1,0} \) is an irreducible representation. Thus, by Schur’s lemma, the automorphism \( \Phi_{I,J}(V) \in \text{End}(V_I^{1,0}) \) is a multiplication by a complex constant \( \lambda \). The estimation \( 0 < |\lambda| < 1 \) follows from the following elementary argument. The composition \( i_I \circ e_J \) applied to a vector \( v \in V_I^{1,0} \) is a projection of \( v \) to \( V_I^{1,0} \) along \( V_J^{0,1} \). Consider the natural Euclidean metric on \( V = \mathbb{H} \). Clearly, the decomposition \( V = V_I^{1,0} \oplus V_J^{0,1} \) is orthogonal. Thus, the composition \( i_I \circ e_J \) is an orthogonal projection of \( v \in V_I^{1,0} \) to \( V_J^{1,0} \).
Similarly, the composition \( i_J \circ e_I \) is an orthogonal projection of \( v \in V^{1,0}_I \) to \( V^{1,0}_{J} \). Thus, the map \( \Phi_{I,J}(V) \) is an orthogonal projection from \( V^{1,0}_I \) to \( V^{1,0}_{J} \) and back to \( V^{1,0}_I \). Such a composition always decreases a length of vectors, unless \( V^{1,0}_I \) coincides with \( V^{1,0}_{J} \), in which case \( I = J \). Also, unless \( V^{1,0}_I = V^{1,0}_{J} \), \( \Phi_{I,J}(V) \) is non-zero; in the later case, \( I = -J \). **Proposition 5.1** is proven. This finishes the proof of **Theorem 3.10**.

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