AFFINE BPS ALGEBRAS, W ALGEBRAS, AND THE COHOMOLOGICAL HALL ALGEBRA OF $\mathbb{A}^2$

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Abstract. We introduce affinizations and deformations of the BPS Lie algebra associated to a tripled quiver with its canonical cubic potential, and use them to precisely determine the $T$-equivariant cohomological Hall algebra $A^T_{\mathbb{A}^2}$ of compactly supported coherent sheaves on $\mathbb{A}^2$, acted on by a torus $T$. In particular we show that this algebra is spherically generated for all $T$.

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1. Introduction

1.1. Hall algebras. Let $\mathfrak{M}_d(\mathbb{C}[x,y])$ denote the moduli stack of coherent sheaves of length $d$ on $\mathbb{A}^2 := \mathbb{A}^2 \mathbb{C}$. We also consider the quotient $\mathfrak{M}_d^T(\mathbb{C}[x,y])$, by the induced action of various tori $T$ acting linearly on the two coordinates of $\mathbb{A}^2$. We consider the Borel–Moore homology of the stack

$$A^T_{\mathbb{A}^2} := \bigoplus_{d \in \mathbb{Z}_{\geq 0}} H_{BM}^*(\mathfrak{M}_d^T(\mathbb{C}[x,y]), \mathbb{Q}),$$

which carries the Hall algebra product defined in [SV13]. This algebra plays a key intermediate role in the geometric representation theory of $\text{Hilb}_d(\mathbb{A}^2)$ [Nak97, Gro96], $W$-algebras [SV13, Mik07], [AST13], [Neg16], Yangians and Hall algebras [RSYZ20a, RSYZ20b, LY20, GY20]. Setting $T = \{1\}$, the resulting algebra is important in the study of the cohomological Hall algebra of coherent sheaves with zero-dimensional support for an arbitrary smooth surface [KV19], as well as partially categorifying the degree zero motivic Donaldson–Thomas theory of $\mathbb{A}^3$, as studied in [BBS13]. In particular, via dimensional reduction [Dav17a] this algebra is isomorphic to a particular case of the critical cohomological Hall algebra introduced by Kontsevich and Soibelman [KS11].

In this paper we fully describe the algebra $A^T_{\mathbb{A}^2}$, for various $T$, via the introduction and study of deformed and affinized, BPS Lie algebras $\hat{g}^T_{Q,W}$ of tripled quivers with their canonical cubic potentials, extending the original BPS Lie algebra $\hat{g}_{Q,W}$ of [DM20] for this class of quivers with potential.

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1.2. Donaldson–Thomas theory. Donaldson–Thomas theory [Tho00], is a theory that assigns to a projective Calabi–Yau threefold $X$, a Chern class $\alpha \in H(X, \mathbb{Z})$, and a stability condition $\zeta$, a number $\omega_\zeta^\alpha$ “counting” $\zeta$-semistable sheaves on $X$ of Chern class $\alpha$. The inverted commas are due to the fact that the moduli space $\mathcal{M}_X^\alpha(\zeta)$ of such sheaves will generally have strictly positive dimension, i.e. there are infinitely many such sheaves, and so $\omega_\zeta^\alpha$ is defined by first constructing a virtual fundamental class of the right expected dimension (i.e. zero) and then taking its degree.

Subsequently it was realised by Kai Behrend [Beh09] that this virtual count could be computed as the Euler characteristic of $\mathcal{M}_X^\alpha(\zeta)$, weighted by a certain constructible function $\nu_X$. Furthermore, in the event that $X$ can be written as the critical locus of a function $f$ on a smooth ambient variety $Y$, this weighted Euler characteristic is equal to the Euler characteristic of the vanishing cycle cohomology $H(Y, p^* \phi_f \mathbb{Q})$. Firstly, these facts give us a way to define DT invariants for noncompact moduli spaces, and secondly, they suggest a natural (partial) “categorification” of the DT invariant: instead of considering the number $\chi(H(Y, p^* \phi_f \mathbb{Q})) = \sum_{i \in \mathbb{Z}} (-1)^i \dim(H^i(Y, p^* \phi_f \mathbb{Q}))$, we may study the cohomology $H(Y, p^* \phi_f \mathbb{Q})$ itself. In particular, being a vector space, and not a mere number, this cohomology is expected to have (and indeed does have, in cases of interest in this paper) a rich algebraic structure.

1.3. Noncommutative Donaldson–Thomas theory. Associated to a quiver with potential $Q, W \in \mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]_{\text{vect}}$, we may define the Jacobi algebra $\text{Jac}(Q, W)$ as in §3.2 below. This is a kind of noncommutative threefold, in the sense that the category of finite-dimensional modules over $\text{Jac}(Q, W)$ behaves in many ways like the category of compactly supported coherent sheaves on a 3CY variety. We refer to [Gin06] for background on this point of view. Furthermore, the moduli stack of finite-dimensional $\text{Jac}(Q, W)$-modules is realised as the critical locus of the function $\text{Tr}(W)$ on the smooth stack of $\mathbb{C}Q$-modules. From the above discussion, it is thus natural to study the vanishing cycle cohomology

$$A_{Q,W} := \bigoplus_{d \in \mathbb{N}_{\geq 0}} H(\mathcal{M}_d(Q), p^* \text{Tr}(W) \mathbb{Q}^\text{vir})$$

a categorification of the noncommutative DT theory associated to the pair $(Q, W)$, as initiated in [Sze08] in the case of the noncommutative conifold. Here and throughout the paper, the vir superscript denotes a shift in cohomological degree that will be specified below. The cohomology $A_{Q,W}$ carries an action of $H_{\mathbb{C}_*} := H(\mathbb{B}C^*, \mathbb{Q})$ by multiplying by tautological classes. Furthermore, the object $A_{Q,W}$ carries the cohomological Hall algebra structure introduced by Kontsevich and Soibelman [KS11].

The connection with the algebra $A_{\mathbb{A}^2}^T$ comes after setting $Q^{(3)}$ to be the quiver with one vertex and three loops $\alpha, a^*, \omega$, and considering the potential

$$\tilde{W} = \omega[a, a^*].$$

Then there is a natural isomorphism of Hall algebras (see §3.9)

$$A_{Q^{(3)}, \tilde{W}} \cong A_{\mathbb{A}^2}^T.$$

1For arbitrary moduli spaces of stable coherent sheaves on Calabi–Yau threefolds, this global critical locus description is too much to ask for, but can still be achieved locally: see [BBJ19, BBB17, BBB18].

2i.e. a smooth three-dimensional variety $X$ satisfying $\theta_X \cong \omega_X$. 
If \( T \) scales the \( x, y \) directions of \( \mathbb{A}^2 \) with weights \( w_1, w_2 \) respectively, we may consider the \( T \)-action on \( Q^{(3)} \) that scales the arrows \( a, a^*, \omega \) with weights \( w_1, w_2, -w_1-w_2 \) respectively, and there is a similar isomorphism of deformed CohAs \( \tilde{A}_{Q^{(3)},W}^{T} \cong \tilde{A}_{A^2}^{T} \).

Via a purity result of [Dav17b], the graded dimensions of \( A_{Q^{(3)},\tilde{W}} \) precisely encode the refined DT invariants of the Jacobi algebra \( \text{Jac}(Q^{(3)},\tilde{W}) \cong \mathbb{C}[x, y, z] \) (equivalently, the DT theory of compactly supported coherent sheaves on \( \mathbb{A}^3 \)) as defined in [KS08, KS11], and calculated in [BBS13]. This means that the DT theory of \( \text{Jac}(Q^{(3)},\tilde{W}) \) enables us to calculate the characteristic function encoding the graded dimensions of \( A_{A^2}^{T} \). From (2), the induced \( \mathbb{Z}\)-grading torus, with skew-symmetric form \( \omega((m,a),(n,b)) = an-bm \).

1.4. Results. The algebra \( W_{1+\infty} \) is defined to be the universal central extension of the Lie algebra of algebraic differential operators on the complex torus \( \mathbb{C}^* \). A basis for \( W_{1+\infty} \) is provided by the set

\[
\{z^m D^n \mid m \in \mathbb{Z}, n \in \mathbb{Z}_{\geq 0}\} \cup \{c\}
\]

where we set \( D = z(d/dz) \), and \( c \) accounts for the central extension, which will not feature in this paper. We define

\[
W_{1+\infty} = \text{Span}(z^m D^n \mid m \in \mathbb{Z}_{\geq 1}, n \in \mathbb{Z}_{\geq 0}),
\]

which is a Lie subalgebra of \( W_{1+\infty} \). Explicitly, the Lie bracket with respect to this basis is given by

\[
[z^m D^a, z^n D^b] = z^{m+n}((D + n)^a D^b - D^a(D + m)^b).
\]

The algebra \( W_{1+\infty} \) is \( \mathbb{Z} \)-graded, with \( n \)th graded piece spanned by the operators \( z^n D^a \) for \( a \geq 0 \). The Lie algebra \( W_{1+\infty}^+ \) has a filtration given by setting \( F_i W_{1+\infty}^+ = \text{Span}(z^m D^n \mid 0 \leq a \leq (i+2)/2, m \in \mathbb{Z}_{\geq 1}) \).

We denote by \( \text{Gr}_i W_{1+\infty}^+ \) the associated graded Lie algebra. From (2), the induced Lie bracket on \( \text{Gr}_i W_{1+\infty}^+ \) is nontrivial; it is given explicitly by

\[
[z^m D^a, z^n D^b] = (an-bm) z^{m+n} D^{a+b-1}.
\]

So the associated graded Lie algebra \( \text{Gr}_i W_{1+\infty}^+ \) is isomorphic to (part of) the quantum torus associated to the lattice \( \mathbb{Z}^2 \) with skew-symmetric form \( \omega((m,a),(n,b)) = an-bm \).

Theorem A (Theorem 5.4 and Corollary 5.5). There is an isomorphism of Lie algebras

\[
\tilde{A}_{Q^{(3)},\tilde{W}}^T \cong \text{Gr}_i W_{1+\infty}^+
\]

between the affine BPS Lie algebra for the pair \((Q^{(3)},\tilde{W})\) and the associated graded Lie algebra \( W_{1+\infty}^+ \) with respect to the filtration \( F_i \). As a result, there is an isomorphism of algebras

\[
A_{A^2}^T \cong U(\text{Gr}_i W_{1+\infty}^+).
\]

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3There is a doubling of degree in our definition, so that the associated graded object lives only in even degrees, in order to match a later, cohomological grading of Hall algebras.

4As defined in [KS11].
The algebras $A_{A^2}^T$ (for various $T$) have appeared in a great deal of work in geometric representation theory in the last decade. In particular, care is usually taken to restrict to the spherical subalgebra of $A_{A^2}^T$, i.e. the algebra generated by the subspace $A^T_{A^2,1}$, corresponding to coherent sheaves on $A^2$ of length one. One of the main consequences of Theorem A is the following

**Theorem B** (Theorem 5.3). Let $T$ act on the quiver $Q^{(3)}$, leaving the potential $\bar{W}$ invariant. Then the CoHA $A_{A^2}^T$ is spherically generated, i.e. it is generated by the piece $A^T_{A^2,1}$ corresponding to coherent sheaves of length one. In particular, $A^T_{A^2,1}$ is spherically generated.

It may come as a temporary disappointment to the reader that the associated graded algebra of the filtered algebra $W_{1+\infty}^+$ appears in Theorem A as opposed to $W_{1+\infty}$ itself. This is, however, somewhat inevitable: the algebra $A_{Q^{(3)},\bar{W}}$ is bigraded, with one grading coming from the decomposition of the stack of finite-dimensional $C[x,y]$-modules according to their dimension, and one grading coming from cohomological degree. The Lie algebra $W_{1+\infty}^+$, by contrast, is graded according to the exponent of $z$, but only filtered according to the exponent of $D$.

To improve upon this situation, we consider the action of $C^*$ on $A^2$ that acts by scaling the first coordinate, and leaving the second invariant. This induces a $C^*$-action on $\mathfrak{M}_{\sigma}(C[x,y])$ for all $d$, and we take the equivariant Borel–Moore homology of this stack with respect to this action, and define the Hall algebra

$$A_{A^2}^{C^*} := \bigoplus_{d \in \mathbb{N}} \mathbb{H}^{BM}(\mathfrak{M}_{\sigma}^C(C[x,y]), \mathbb{Q})$$

to be this equivariant Borel–Moore homology, equipped with the usual multiplication via correspondences. The algebra $\mathbb{Q}[t] \cong H_{C^*} := H(BC^*, \mathbb{Q})$ acts on $A_{A^2}^{C^*}$, with the action induced by the $C^*$-action on $A^2$, and the Hall algebra multiplication is $H_{C^*}$-linear.

We denote by

$$\mathbb{R}_F[W_{1+\infty}^+] := \sum_{i \geq -1} F_2(W_{1+\infty}^+)^t \mathbb{Q}[t] \subset W_{1+\infty}^+ \otimes \mathbb{Q}[t]$$

the Rees Lie algebra. This is a $\mathbb{Q}[t]$-linear Lie algebra, and so may be thought of as a family of Lie algebras over the affine line, deforming the algebra $\text{Gr}^F_{t} W_{1+\infty}^+$ to $W_{1+\infty}^+$ for $\epsilon \neq 0$ the specialisation at $t = \epsilon$ is isomorphic to $W_{1+\infty}^+$, and the specialisation at $t = 0$ is isomorphic to $\text{Gr}^F_{\epsilon} W_{1+\infty}^+$.

Let $A$ be a commutative ring. For $\mathfrak{g}$ a $A$-linear Lie algebra, we define the tensor algebra

$$T_A(\mathfrak{g}) := \bigoplus_{m \geq 0} \mathfrak{g} \otimes_A \cdots \otimes_A \mathfrak{g}$$

and define

$$U_A(\mathfrak{g}) = T_A(\mathfrak{g})/(\alpha, \beta - [\alpha, \beta]_\mathfrak{g} | \alpha, \beta \in \mathfrak{g}).$$

**Theorem C** (Corollary 5.10). Let $C^*$ act on $Q^{(3)}$ with weights $1, 0, -1$ on the arrows $a, a^*, \omega$ respectively. There is an isomorphism of $H_{C^*} = \mathbb{Q}[t]$-linear Lie algebras

$$\mathfrak{g}_{Q^{(3)}, \bar{W}} \cong \mathbb{R}_F[W_{1+\infty}^+]$$

between the deformed affine BPS Lie algebra for $(Q^{(3)}, \bar{W})$, introduced in §3.14, and the Rees Lie algebra of $W_{1+\infty}^+$. As a result, there is an isomorphism of $\mathbb{Q}[t]$-linear algebras:

$$A_{A^2}^{C^*} \cong U_{\mathbb{Q}[t]}(\mathbb{R}_F[W_{1+\infty}^+]).$$
We denote by \( \mathcal{Y}_{t_1,t_2,t_3}(\mathfrak{g}(1))^+ \) the strictly positive part of the affine Yangian; a presentation is recalled in §3.13. Let \( T = (\mathbb{C}^*)^2 \) act on \( \mathbb{A}^2 \), by scaling the two coordinates independently. Our final theorem is a strengthening of a result of Rapčák, Soibelman, Yau and Zhao [RSYZ20] (in fact it is an immediate consequence of their result, along with spherical generation):

**Theorem D** (Theorem 5.9). For \( T = (\mathbb{C}^*)^2 \) as above, there is an isomorphism of \( H_T \)-linear algebras

\[
A_T^{\mathbb{A}^2} \cong \mathcal{Y}_{t_1,t_2,t_3}(\mathfrak{g}(1))^+.
\]

1.5. **Notation and conventions.**

- Every cohomological Hall algebra that we consider throughout the paper will carry a \( \mathbb{H}(\mathbb{C}^*, \mathbb{Q}) \)-action arising from the morphism from the stack of finite-dimensional modules over an algebra to \( \mathbb{C}^* \) given by taking the determinant. We denote this copy of \( \mathbb{C}^* \) by \( \mathbb{H}_T \) to distinguish it from other instances of \( \mathbb{C}^* \) arising from e.g. torus actions on quivers. We write \( \mathbb{H}_{\mathbb{C}^*} := \mathbb{H}(\mathbb{C}^*, \mathbb{Q}) \cong \mathbb{Q}[u] \), with \( u \) in cohomological degree 2.

- More generally, if \( T = (\mathbb{C}^*)^j \) is a torus, we set \( \mathbb{H}_T := \mathbb{H}(BT, \mathbb{Q}) \).

- If a torus \( T \) acts on a stack \( X \) with quotient \( Y \), we set \((\tau^i, \tau^j, \tau^k)\) to be the shift of the usual perverse t structure on the derived category of constructible complexes on \( Y \), so that \( F \) is in the heart if and only if \((X \to Y)^* \) is perverse (see §3.1).

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2. **Algebraic background**

2.1. **Heisenberg algebras.** We denote by \( \text{Heis} \) the Lie algebra over \( \mathbb{Q} \) having basis \( \{p, q, c\} \), subject to the relations

\[
[q, p] = c,
\]

\[
[c, p] = [c, q] = 0.
\]

We denote by \( \text{Heis} \)-mod the category of modules over \( \text{Heis} \). We give \( \text{Heis} \) a (cohomological) \( \mathbb{Z} \)-grading, putting \( p, c, q \) in degrees 2, 0, -2 respectively. A graded \( \text{Heis} \)-module is a \( \text{Heis} \)-module \( \rho \), together with a direct sum decomposition

\[
\rho = \bigoplus_{n \in \mathbb{Z}} \rho_n
\]

of the underlying vector space, such that \( p \) maps \( \rho_n \) to \( \rho_{n+2} \), \( q \) maps \( \rho_n \) to \( \rho_{n-2} \), and \( c \) prescribes the decomposition. We denote by \( \text{Heis}_{\mathbb{Z}} \)-mod the category of graded \( \text{Heis} \)-modules. For \( c \in \mathbb{Z} \) we say that a \( \text{Heis} \)-representation \( \rho \) has central charge \( c \) if \( c \) acts on \( \rho \) via scalar multiplication by \( c \).

For a quiver \( Q \) with vertex set \( Q_0 \), we define the Lie algebra

\[
\text{Heis}_{Q} := \bigoplus_{i \in Q_0} \text{Heis}.
\]

For \( i \in Q_0 \) we denote by \( p_i, q_i, c_i \) the basis elements in \( \text{Heis}_{Q} \) corresponding to the \( i \)th summand. We define the Lie subalgebra \( \text{Heis}_{Q} \subset \text{Heis}_{Q} \) to be the span of \( \{p_i \mid i \in Q_0\} \cup \{c_i \mid i \in Q_0\} \cup \{q_i := \sum_{i \in Q_0} q_i\} \).
For \( c \in \mathbb{Z}^{Q_0} \) we say that a Heis\(_{Q} \)-representation \( \rho \) has central charge \( c \) if \( c_i \) acts by multiplication by \( c_i \). There is a canonical inclusion Heis \( \hookrightarrow \) Heis\(_{Q} \) sending \( p \mapsto \sum_{i \in Q_0} p_i \), \( q \mapsto q \) and \( c \mapsto \sum_{i \in Q_0} c_i \). We will often consider a Heis\(_{Q} \)-module as a Heis-module via this inclusion. Placing all \( p_i, q_i, c_i \) in degrees \( 2, 0, -2 \) respectively, we give Heis\(_{Q} \) a \( \mathbb{Z} \)-grading, and define the category Heis\(_{Q} \)-mod\(_{Z} \) of graded modules as before.

The vector space \( Q_0 = \mathbb{Q} \) forms a Heis\(_{Q} \)-module of central charge \( 0 \) if we let all \( p_i, q_i, c_i \) act via the zero matrix. The categories Heis\(_{Q} \)-mod and Heis\(_{Q} \)-mod\(_{Z} \) are symmetric tensor categories with monoidal unit \( Q_0 \). If \( \mathcal{C} \) is a tensor category, a \( \mathcal{C} \)-algebra object is an object \( \mathcal{F} \) in \( \mathcal{C} \), along with morphisms \( 1_{\mathcal{C}} \rightarrow \mathcal{F} \) from the monoidal unit, and a multiplication morphism \( \mathcal{F} \otimes \mathcal{F} \rightarrow \mathcal{F} \), satisfying the standard properties. If \( \mathcal{C} \) is a symmetric tensor category, a \( \mathcal{C} \)-Lie algebra object is an object \( \mathcal{F} \) in \( \mathcal{C} \) along with an antisymmetric morphism \( \mathcal{F} \otimes \mathcal{F} \rightarrow \mathcal{F} \) satisfying the Jacobi identity. If \( \rho \) is a Lie algebra object in Heis\(_{Q} \), the universal enveloping algebra \( U(\rho) \) is an algebra object in Heis\(_{Q} \), with underlying algebra the universal enveloping algebra of the underlying Lie algebra of \( \rho \).

We say that a Heis\(_{Q} \)-module \( \rho \) is integrable if there is a decomposition

\[
\rho = \bigoplus_{d \in \mathbb{Z}^{Q_0}} \rho_{[d]} \tag{5}
\]

with each \( \rho_{[d]} \) a Heis\(_{Q} \)-module of central charge \( d \).

Let Heis\(_{Q} \) act on the \( \mathbb{N}^{Q_0} \)-graded algebra \( A \), where the \( \mathbb{N}^{Q_0} \)-grading is induced by a decomposition of \( A \) into sub-modules of fixed central charge, as in (5). Then \( A \) is an algebra object in Heis\(_{Q} \) if each \( p_i \) for \( i \in Q_0 \), and \( q_\Sigma \), act by derivations on \( A \). Similarly, if \( n \) is a \( \mathbb{N}^{Q_0} \)-graded Lie algebra, and also an integrable Heis\(_{Q} \)-module for which the \( \mathbb{N}^{Q_0} \)-graded decomposition is provided by the decomposition according to central charge, then \( n \) is a Lie algebra object in Heis\(_{Q} \) if and only if for every \( i \in Q_0 \), \( p_i \) and \( q_\Sigma \) act by Lie algebra derivations.

2.2. The Lie algebra \( W_{1+\infty}^{+} \). Recall from [1] the definition of \( W_{1+\infty}^{+} \subset W_{1+\infty} \). The Lie bracket on \( W_{1+\infty}^{+} \) is given by

\[
[z^m f(D), z^n g(D)] = z^{m+n} (f(D+n)g(D) - f(D)g(D+m)). \tag{6}
\]

**Proposition 2.1.** There is an action of Heis on \( W_{1+\infty}^{+} \) by derivations, given by sending \( p \) to \([D^2, -]\) and sending \( q \) to the endomorphism

\[
\partial_D: W_{1+\infty}^{+} \rightarrow W_{1+\infty}^{+}, \quad z^m f(D) \mapsto z^m f(D). \tag{7}
\]

For \( m \geq 1 \), the subspace \( \text{Span}(z^m D^n \mid n \in \mathbb{Z}_{\geq 0}) \) is preserved by this action, on which it has central charge \( 2m \).

**Proof.** The fact that \([D^2, -]\) is a derivation is obvious, while the fact that \( \partial_D \) is a derivation follows from the formula (7). Now we calculate

\[
[\partial_D, [D^2, -]](z^m D^n) = \partial_D[D^2, z^m D^n] - [D^2, n z^m D^{n-1}] = [2D, z^m D^n] + [D^2, n z^m D^{n-1}] - [D^2, n z^m D^{n-1}] = 2z^m ((D + m) D^n - D^{1+n}) = 2m z^m D^n
\]

as required. \( \Box \)
For even \( m \) we denote by \( V_m \subset W_{1+\infty}^+ \) the \textbf{Heis}-submodule with central charge \( m \). By the above proposition, it has a basis given by elements \( z^{m/2} D^a \). If we let \( \mathbb{Q}[u] \) act on \( V_m \) by letting \( u \) act via the raising operator \( p = [D^a,-] \), then all of the \( V_m \) are isomorphic to \( \mathbb{Q}[u] \), considered as \( \mathbb{Q}[u] \)-modules.

We endow \( V_{2m} \) with a filtration by setting

\[
F_i V_{2m} = \begin{cases} \text{span}(z^m, z^m D, \ldots, z^m D^{i/2} D^{i+1}) & \text{if } i \text{ is even} \\ F_{i-1} V_{2m} & \text{if } i \text{ is odd}. \end{cases}
\]

So if \( i \) is odd, the \( i \)-th piece of the associated graded object \( \text{Gr}_i^F V_{2m} \) is zero, and if \( i \in \mathbb{Z}_{\geq -1} \) then a basis of \( \text{Gr}_i^F V_{2m} \) is given by \( \{ z^m D^a \} \). We refer to the natural grading on \( \text{Gr}_i^F V_{2m} \) as the \textit{cohomological grading}. Taking the direct sum of the filtrations above, we obtain the \textit{order} filtration \( F_i \) on \( W_{1+\infty}^+ \) considered in the introduction. The Lie algebras \( \text{Gr}_i^F V_{2m} \) are generated by their respective \textbf{Heis}-subalgebras.

Lemma 2.2. \( \text{Gr}_i^F V_{2m} \) and \( \mathbb{R}[W_{1+\infty}^+] \) are \textit{spherically generated}; they are generated by their respective \textbf{Heis}-submodules of central charge 2.

Proof. Let \( g \subset \mathbb{R}[W_{1+\infty}^+] \) be the Lie subalgebra generated by \( \mathbb{R}[W_{1+\infty}^+] \) of \( z^m D^a \) \( a \geq 0 \) \( r \geq 0 \). We give \( z^m D^a t^r \) a \textit{cohomological degree} \( 2a + 2r \).

Let \( \mathbb{R}[W_{1+\infty}^+] \) be spanned by elements \( z^m D^a t^r \) with \( m \geq 1, a \geq 0, r \geq 0 \). We give \( z^m D^a t^r \) a \textit{cohomological degree} \( 2a + 2r \).

Lemma 2.3. The Lie algebras \( \text{Gr}_i^F V_{2m} \) and \( \mathbb{R}[W_{1+\infty}^+] \) are \textit{spherically generated}; they are generated by their respective \textbf{Heis}-submodules of central charge 2.

Proof. Let \( g \subset \mathbb{R}[W_{1+\infty}^+] \) be the Lie subalgebra generated by \( \mathbb{R}[W_{1+\infty}^+] \) of \( z^m D^a \) \( a \geq 0 \) \( r \geq 0 \). We give \( z^m D^a t^r \) a \textit{cohomological degree} \( 2a + 2r \).

Putting \( a = 1 \) and arguing inductively we deduce that \( z^{m} D^{((D + m)^a)} \) for every \( m \). Then fixing \( m \) and arguing by induction on \( a \), it also follows from \( \mathbb{Q}[t] \) that \( z^{m} D^{((D + m)^a)} \) for every \( m \geq 1 \) for every \( a \geq 0 \), and this algebra is spherically generated.

Spherical generation of \( \text{Gr}_i^F V_{2m} \) follows directly from \( \mathbb{Q}[t] \), or from the spherical generation of \( \mathbb{R}[W_{1+\infty}^+] \) and the isomorphism

\[
\mathbb{R}[W_{1+\infty}^+] \otimes_{\mathbb{Q}[t]} (\mathbb{Q}[t]/(t)) \cong \text{Gr}_i^F V_{2m}.
\]

We record a variant of the result, for later. It follows from \( \mathbb{Q}[t] \).

Lemma 2.4. The elements \( \text{ad}_{D^a} (z^m D^n) \) for \( m, n \geq 0 \) provide a basis for \( W_{1+\infty}^+ \).

The Lie algebra \( \text{Gr}_i^F V_{2m} \) carries two gradings, one by central charge, and one given by the cohomological grading, and \( \text{Gr}_i^F V_{2m} \) is a Lie algebra object in \( \textbf{Heis} \)-mod \( \mathbb{Z} \). It turns out that this object satisfies a simple universal property: the Lie algebra \( \text{Gr}_i^F V_{2m} \) is the universal Lie algebra object in \( \textbf{Heis} \)-mod \( \mathbb{Z} \) containing \( \text{Gr}_i^F V_{2m} \), and containing only representations \( \text{Gr}_i^F V_{2m} \) for various \( m \) as subrepresentations (i.e. no copies of \( \text{Gr}_i^F V_{2m} \)). More precisely,

Lemma 2.5. If \( H \) is a Lie algebra object in \( \textbf{Heis}_Q \)-mod \( \mathbb{Z} \) containing only unshifted representations \( \text{Gr}_i^F V_{2m} \) as \( \textbf{Heis}_Q \)-subrepresentations, and \( i \) : \( \text{Gr}_i^F V_{2m} \rightarrow H \) is an
inclusion in \( \text{Heis}_Q \text{-mod}_\mathbb{Z} \), then \( \iota \) extends uniquely to a morphism of \( \text{Heis}_Q \text{-Lie} \) algebra objects \( \text{Gr}^{\mathbb{F}}_{1+\infty} X \to \mathcal{H} \).

**Proof.** The lemma is equivalent to the claim that \( \text{Gr}^{\mathbb{F}}_{1+\infty} X \) is isomorphic to \( \mathcal{L}' \), the free \( \mathbb{Z} \text{-graded} \) Lie algebra \( \mathcal{L} \) generated by \( \text{Gr}^{\mathbb{F}}_{2} V_2 \), modulo the ideal containing all the \( \mathbb{Z} \text{-graded} \) \( \text{Heis}_Q \text{-submodules} \) of \( \mathcal{L} \) that are not isomorphic to unshifted copies of \( \text{Gr}^{\mathbb{F}}_{m} V_m \) for various \( m \).

The algebra \( \mathcal{L}' \) carries a grading by tensor degree, and we claim that \( \mathcal{L}'_m \cong \text{Gr}^{\mathbb{F}}_{m} V_{2m} \). This obviously implies the lemma. We prove the claim by induction. The base case \( m = 1 \) is trivial, and so we assume the claim is proved for \( i < m \), with \( m \geq 2 \). By the inductive hypothesis, the \( m \)-th piece of \( \mathcal{L}' \) is either a quotient of

\[
\text{Gr}^{\mathbb{F}}_{1} V_2 \otimes \text{Gr}^{\mathbb{F}}_{2m-2} \cong \bigoplus_{i \geq 0} \text{Gr}^{\mathbb{F}}_{2m}[2 - 2i]
\]

if \( m = 2 \), or

\[
\text{Gr}^{\mathbb{F}}_{1} V_2 \wedge \text{Gr}^{\mathbb{F}}_{4} V_4 \cong \bigoplus_{i \geq 0} \text{Gr}^{\mathbb{F}}_{4}[4i]
\]

if \( m \geq 3 \). In either case, at most a single unshifted copy of \( \text{Gr}^{\mathbb{F}}_{m} V_m \) survives after we quotient out by all of the shifted copies of \( \text{Gr}^{\mathbb{F}}_{m} V_m \). On the other hand, \( \mathcal{L}'_m \) contains at least one copy of \( \text{Gr}^{\mathbb{F}}_{2m} V_{2m} \), since by its universal property, and spherical generation of \( \text{Gr}^{\mathbb{F}}_{1+\infty} W_{1+\infty} \), there is a surjective morphism \( \mathcal{L}' \to \text{Gr}^{\mathbb{F}}_{1+\infty} W_{1+\infty} \). The inductive claim follows.

Given an integrable cohomologically-graded \( \text{Heis} \text{-module} \) \( \rho \), we define the characteristic function

\[
\chi(\rho) := \sum_{i,j \in \mathbb{Z}} \dim(\rho_{i,j}) v^i q^j.
\]

**Corollary 2.5.** Let \( \mathcal{H} \) be an integrable \( \mathbb{Z} \text{-graded} \) \( \text{Heis} \text{-module} \) generated by \( \text{Gr}^{\mathbb{F}}_{1} V_{2} \subset \mathcal{H} \) and satisfying

\[
\chi(\mathcal{H}) = \chi(\text{Gr}^{\mathbb{F}}_{1+\infty} W_{1+\infty}) = \sum_{i \geq 0} v^{2i} q^{2i} = v^2 q^{-2}(1 - v^2)^{-1}(1 - q^2)^{-1}.
\]

Then \( \mathcal{H} \cong \text{Gr}^{\mathbb{F}}_{1+\infty} W_{1+\infty} \).

**Proof.** By the universal property of \( \text{Gr}^{\mathbb{F}}_{1+\infty} W_{1+\infty} \) (Lemma 2.4), there is a morphism of Lie algebra objects in \( \text{Heis} \text{-mod}_\mathbb{Z} \)

\[
f : \text{Gr}^{\mathbb{F}}_{1+\infty} W_{1+\infty} \to \mathcal{H}
\]

sending the copy of \( \text{Gr}^{\mathbb{F}}_{1} V_{2} \) inside \( \text{Gr}^{\mathbb{F}}_{1+\infty} W_{1+\infty} \) to the copy inside \( \mathcal{H} \). By supposition, \( f \) is surjective. On the other hand, the source and the target have the same characteristic function, so \( f \) is an isomorphism.

3. **DT theory background**

3.1. **Vanishing cycle toolkit.** If \( \mathfrak{X} \) is a smooth stack, we denote by \( \mathbb{Q}_\mathfrak{X}^{vir} = \mathbb{Q}_\mathfrak{X}[\dim(\mathfrak{X})] \) the constant perverse sheaf on \( \mathfrak{X} \). More generally, if an extra torus \( T \) acts on a smooth stack \( \mathfrak{X} \) with quotient \( \mathfrak{Y} \), we define \( \mathbb{Q}_\mathfrak{Y}^{vir} = \mathbb{Q}_\mathfrak{Y}[\dim(\mathfrak{X})] \).

---

5. The motivation for this slightly mixed convention is that we would like \( \mathbb{Q}_\mathfrak{X}^{vir} \) to correspond to a \( T \)-equivariant perverse sheaf on \( \mathfrak{Y} \), i.e. we would like the pullback \( (\mathfrak{X} \to \mathfrak{Y})^* \mathbb{Q}_\mathfrak{Y}^{vir} \) to be perverse.
Under the same conditions, we modify the usual perverse truncation functors for constructible complexes of sheaves on \( Y \) by setting \( p_T^{\leq -i} = p_T^{\leq -i - \dim(T)} \).

Given a function \( f \) on a smooth stack \( X \), we define the vanishing cycle functor \( \Phi_f \) as in [KS90]. It is an exact functor with respect to the perverse t structure, and in particular \( \Phi_f Q_X^\text{vir} \) is a perverse sheaf on \( X \), which we may refer to just as the sheaf of vanishing cycles on \( X \). We will often abuse notation by just writing \( H(X, \Phi_f Q_X^\text{vir}) \).

Let \( f, g \) be functions on smooth stacks \( X, Y \) respectively. The Thom–Sebastiani theorem states that there is a natural bifunctorial equivalence of constructible complexes of sheaves on \( X \times Y \)

\[
\text{TS}: \Phi_f \boxtimes g(F \boxtimes G)|_{f^{-1}(0) \times g^{-1}(0)} \cong \Phi_f F \boxtimes \Phi_g G.
\]

Assuming that \( \text{supp}((\Phi_f \boxtimes g(F \boxtimes G))) \subset f^{-1}(0) \times g^{-1}(0) \), and applying the derived global sections functor, yields the isomorphism

\[
H(X, \Phi_f F) \otimes H(Y, \Phi_g G) \cong H(X \times Y, \Phi_f \boxtimes g(F \boxtimes G)),
\]

the analogue in vanishing cycle cohomology of the Künneth isomorphism in ordinary cohomology.

Given a morphism \( q: X \to Y \) and a function \( f \) on \( Y \) there is a natural transformation \( \Theta: \Phi_f q_* \to q_* \Phi_f q^* \), which is an isomorphism if \( f \) is projective. The natural transformation \( \Theta \) induces the natural transformation \( \eta: \Phi_f \to q_* q^* \Phi_f \) by adjunction. Applying the derived global sections functor, \( \eta \) induces the pullback morphism

\[
q^*: H(Y, \Phi_f F) \to H(X, \Phi_f q^* F).
\]

Given a proper morphism \( p: X \to Y \) between smooth stacks and a function \( f \) on \( Y \), we obtain the pushforward morphism

\[
p_*: H(X, \Phi_f Q_X)[2 \dim(X)] \to H(Y, \Phi_f q_* Q)[2 \dim(Y)]
\]

by applying \( \Phi_f \) to \( p_* Q_X[2 \dim(X)] \to q_* Q_Y[2 \dim(Y)] \) (the Verdier dual of \( q_* Q_Y \to p_* Q_X \)), and composing with \( \Theta^{-1} \).

Let \( \mathfrak{X} \) be a smooth stack, \( f \) be a function on it, and \( F \in D^b_c(\mathfrak{X}) \) be a constructible complex. Let \( \Delta: \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X} \) be the diagonal embedding. Then \( f = (f \boxtimes 0) \circ \Delta \) and there is a natural isomorphism \( F \cong \Delta^*(F \boxtimes Q_X) \). The pullback morphism \( \Delta^* \), composed with the Thom-Sebastiani isomorphism, yields

\[
H(\mathfrak{X}, \Phi_f F) \otimes H(\mathfrak{X}, Q) \to H(\mathfrak{X}, \Phi_f F),
\]

and an action of the cohomology of \( \mathfrak{X} \) on \( H(\mathfrak{X}, \Phi_f F) \). If \( f = 0 \) this is just the usual action of \( H(\mathfrak{X}, Q) \) on \( H(\mathfrak{X}, F) \), and if in addition \( F = Q_X \) this is the usual algebra structure on \( H(\mathfrak{X}, Q) \). If \( q: \mathfrak{X} \to \mathfrak{Y} \) is a morphism of smooth stacks, from the commutativity of

\[
\begin{array}{ccc}
\mathfrak{X} & \xrightarrow{q} & \mathfrak{Y} \\
(id \times q) \circ \Delta & \downarrow & \\
\mathfrak{X} \times \mathfrak{Y} & \xrightarrow{\Delta} & \mathfrak{Y} \times \mathfrak{Y}
\end{array}
\]

we deduce that \( \Phi_f \) respects the \( H(\mathfrak{Y}, Q) \)-actions on the domain and target. If \( q \) is projective, one may show as in [Dav17a, Prop.2.14] that \( \Phi_f \) respects the \( H(\mathfrak{Y}, Q) \) actions on the domain and target.
3.2. Quivers. Throughout the paper, $Q$ will denote a finite quiver, i.e. a pair of finite sets $Q_1$ and $Q_0$ (the arrows and vertices, respectively) along with a pair of morphisms $s,t: Q_1 \to Q_0$. The free path algebra $kQ$ of $Q$ over the field $k$ is the algebra having as basis the paths in $Q$, including “lazy” paths $e_i$ of length zero at each of the vertices $i \in Q_0$.

We define the Euler form

$$\chi_Q: \mathbb{N}^{Q_0} \times \mathbb{N}^{Q_1} \to \mathbb{Z}$$

$$(d,e) \mapsto (d,e)_Q := \sum_{i \in Q_0} d_i e_i - \sum_{a \in Q_1} d_{(a)} e_{t(a)}.$$  

The double of $Q$ is obtained by setting $\overline{Q}_0 = Q_0$ and $\overline{Q}_1 = Q_1 \coprod Q_1$, where $Q_1^* := \{a^* \mid a \in Q_1\}$ and the orientation of $a^*$ is the opposite of the orientation of $a$. The preprojective algebra is defined by

$$\Pi_Q := C\overline{Q}/(\sum_{a \in Q_1} [a,a^*]).$$

The quiver $\tilde{Q}$ is defined by adding a loop $\omega_i$ to each vertex $i \in \overline{Q}_0$.

Example 3.1. If $Q = Q^{(1)}$ is the Jordan quiver with one vertex and one loop $a$, then $\tilde{Q} = Q^{(3)}$ is the three loop quiver, with one vertex and loops labelled $a,a^*,\omega$.

A potential $W \in \mathbb{C}Q/[CQ,CQ]_{\text{vect}}$ is a linear combination of cyclic words in $Q$. If $W = a_1 \ldots a_l$ is a single cyclic word then

$$\partial W/\partial a := \sum_{a_i = a} a_{i+1} \ldots a_1 \ldots a_{i-1}.$$  

We extend this definition to general $W$ by linearity. The Jacobi algebra associated to the pair $(Q,W)$ is defined by

$$\Jac(Q,W) := \mathbb{C}Q/\langle \partial W/\partial a \mid a \in Q_1 \rangle.$$  

If $Q$ is an arbitrary quiver, the tripled quiver $\tilde{Q}$ carries the canonical cubic potential

$$\tilde{W} := \bigoplus_{i \in Q_0} \omega_i (\sum_{a \in Q_1} [a,a^*]).$$

Let $\Pi_Q[\omega]$ denote the polynomial algebra in one variable $\omega$ with coefficients in $\Pi_Q$. There is an isomorphism

$$j: \Pi_Q[\omega] \to \Jac(\tilde{Q},\tilde{W})$$

where $j(a) = a$, $j(a^*) = a^*$, $j(\omega) = \sum_{i \in Q_0} \omega_i$.

3.3. Moduli spaces. Given a quiver $Q'$ and dimension vector $d \in \mathbb{N}^{Q_0}$ we define

$$\mathcal{H}_{Q',d} := \prod_{a \in Q_1'} \text{Hom}(\mathbb{C}^{d(s(a))},\mathbb{C}^{d(t(a))})$$

and the group $\text{GL}_d$ acts on $\mathcal{H}_{Q',d}$ by conjugation. The moduli stack $\mathcal{M}_d(Q')$, when evaluated on a scheme $X$, yields the groupoid of coherent sheaves on $X$, flat over $X$, equipped with an action of $\mathbb{C}Q'$, such that at every geometric $K$-point of $X$, with $K \supset \mathbb{C}$, the induced $KQ'$-module is $d$-dimensional. This stack is naturally isomorphic to the stack quotient $\mathcal{H}_{Q',d}/\text{GL}_d$.

Let $N = \mathbb{Z}^r$ be a lattice, and fix a function $\tau: Q_1' \to N$. We fix the torus $T = \text{Hom}_{\text{Grp}}(N,\mathbb{C}^*)$. The torus $T$ acts on $\mathcal{H}_{Q',d}$ by sending

$$(t,(\rho_a)_{a \in Q_1'}) \mapsto (t(\tau(a)) \cdot \rho_a)_{a \in Q_1'},$$

and this action commutes with the action of $\text{GL}_d$ by conjugation. We denote by

$$\mathcal{M}_d^T(Q') \cong \mathcal{H}_{Q',d}/(\text{GL}_d \times T)$$
the quotient of $\mathcal{M}_d(Q)$ by the $T$-action. Precisely, a homomorphism $X \to \mathcal{M}_d^T(Q')$ is provided by a $T$-torsor $E \to X$, along with a flat family $\mathcal{F}$ of $d$-dimensional $\mathbb{C}Q'$ modules on $E$, and a $T$-equivariant structure on $\mathcal{F}$, such that for every $a \in Q_1$ the following diagram of coherent sheaves on $G \times E$ commutes:

\[
\begin{array}{ccc}
I^*F & \xrightarrow{I\rho(a)} & I^*F \\
\downarrow \cong & & \downarrow \cong \\
p^*F & \xrightarrow{\prod_{i \in r(a)} z_i^{\rho(a)} \circ p^*\rho(a)} & p^*F.
\end{array}
\]

Here, $I: T \times E \to E$ is the action, $p: T \times E \to E$ is the projection, $\rho(a)$ is the action of $a$ on $\mathcal{F}$, $z_i$ are coordinates on $T$, and the vertical isomorphisms are defined by the $T$-equivariant structure on $\mathcal{F}$. We will use the morphism

\[
j: \mathcal{M}_d^T(Q') \times_{BT} \mathcal{M}_d^{T'}(Q') \to \mathcal{M}_d^T(Q') \times \mathcal{M}_d^{T'}(Q')
\]

induced by the structure map $BT \to \text{pt}$.

We denote by

\[\mathcal{M}_d(Q') = \text{Spec}(\Gamma(\mathcal{A}'Q',d)^{\text{GL}_d})\]

the coarse moduli space. For $K \supset \mathbb{C}$ a field, $K$-points of $\mathcal{M}_d(Q')$ correspond to isomorphism classes of semisimple $d$-dimensional $KQ'$-modules. Since the $T$-action commutes with the $\text{GL}_d$-action, $\mathcal{M}_d(Q')$ carries a natural $T$-action. We define the stack-theoretic quotient

\[\mathcal{M}_d^T(Q') := \mathcal{M}_d(Q')/T.\]

In the case of trivial $T$, there is a natural morphism

\[\mathcal{J}: \mathcal{M}_d(Q') \to \mathcal{M}_d(Q')\]

which, at the level of points, takes modules to their semisimplifications, and at the level of algebraic geometry is just the affinization morphism. This extends naturally to a morphism

\[\mathcal{J}^T: \mathcal{M}_d^T(Q') \to \mathcal{M}_d^T(Q').\] (12)

Precisely, if the morphism $f: X \to \mathcal{M}_d(Q')$ is represented by the $T$-torsor $E \to X$ and the morphism $f: E \to \mathcal{M}_d(Q')$, the morphism $\mathcal{J}^T \circ f$ is represented by the same torsor $E$, along with the morphism $\mathcal{J} \circ f$.

Given two dimension vectors $d', d'' \in \mathbb{N}_{Q_0}$ with $d = d' + d''$ we define $\mathcal{A}'Q',d',d'' \subset \mathcal{A}'Q',d \subset \mathcal{A}_Q,d$ and $\text{P}_{d',d''} \subset \text{GL}_d$ to be the space of representations, and changes of basis, respectively, preserving the flags $0 \subset \mathbb{C}^{d'(i)} \subset \mathbb{C}^{d(i)}$ for each $i \in Q_0$. Denoting by $\mathcal{M}_d',d''(Q')$ the stack of short exact sequences $0 \to \rho' \to \rho \to \rho'' \to 0$ of $\mathcal{C}Q'$-modules, for which the dimension vectors of $\rho'$ and $\rho''$ are $d',d''$ respectively, there is a natural isomorphism

\[\mathcal{M}_d',d''(Q') \cong \mathcal{A}_Q',\mathcal{A}',d'/\text{GL}_{d'},d''.\]

We define $\mathcal{M}_d'^T,d''(Q') := \mathcal{A}_Q',\mathcal{A}',d'/\text{GL}_{d'},d''/(\text{GL}_{d'},d'' \times T)$.

### 3.4. Cohomological Hall algebras

Let $Q'$ be a quiver. For the sake of exposition, we assume that $Q'$ is symmetric in this section, i.e. for every pair of vertices $i,j \in Q_0$ there are as many arrows from $i$ to $j$ as from $j$ to $i$. Let $W \in \mathbb{C}Q'/[\mathbb{C}Q',\mathbb{C}Q']_{\text{vect}}$ be a potential, which we assume to be $T$-invariant in the following sense: for every cyclic word $a_1 \ldots a_c$ appearing in $W$

\[\sum_{i=1}^c \tau(a_i) = 0.\] (13)
The potential \( W \) induces a function \( \text{Tr}(W) \) on \( A_{Q',d} \) which is GL\(_d\)-invariant by cyclic invariance of the trace map, and \( T \)-invariant by \([13]\). We define

\[
A_{Q',W}^T := \bigoplus_{d \in \mathbb{N}^Q} \text{H}(\mathcal{M}_d^T(Q'), \rho_{\text{Tr}(W)})
\]

(14)

the underlying \( \mathbb{N}^Q \)-graded vector space of the critical cohomological Hall algebra, defined by Kontsevich and Soibelman in \([KS11]\). Consider the convolution diagram

\[
\mathcal{M}_d^T(Q') \times_{B_T} \mathcal{M}_d^T(Q') \xrightarrow{\pi_1 \times \pi_3} \mathcal{M}_d^T(Q') \xrightarrow{\pi_3} \mathcal{M}_d^T(Q')
\]

where \( \pi_1, \pi_2, \pi_3 \) take a flag \( 0 \subset \mathcal{F}' \subset \mathcal{F} \) of flat families of \( CQ' \)-modules to the families \( \mathcal{F}', \mathcal{F}, \mathcal{F} / \mathcal{F}' \) respectively. We define

\[
m'_{d',d''} : \text{H}(\mathcal{M}_d^T(Q'), \rho_{\text{Tr}(W)} \otimes \text{Tr}(W)) \rightarrow \text{H}(\mathcal{M}_{d'}^T(Q'), \rho_{\text{Tr}(W)})
\]

\[
= \pi_{d',d''} \circ (\pi_1 \times \pi_3)
\]

and \( m_{d',d''} = m'_{d',d''} \circ j^* \otimes \text{TS} \) where \( j \) is as in \([14]\) and \( \text{TS} \) is the Thom–Sebastiani isomorphism. We then define the multiplication

\[
m : A_{Q',W}^T \otimes A_{Q',W}^T \rightarrow A_{Q',W}^T
\]

by summing \( m_{d',d''} \) over all pairs of dimension vectors.

Let \( \mathcal{F} \) be a flat family of \( d \)-dimensional \( CQ' \)-modules on a scheme \( X \). For each \( i \in Q_0 \) we obtain a vector bundle \( \mathcal{F}_i = c_i \cdot \mathcal{F} \) of rank \( d_i \). There is a determinant line bundle \( \text{Det}(\mathcal{F}_i) \) on \( X \), which is defined to be the top exterior power of the underlying bundle of \( \mathcal{F}_i \). This defines a morphism of stacks

\[
\text{Det}_{(i)} : \mathcal{M}_d^T(Q') \rightarrow BC_u^*.
\]

Alternatively, we can construct this morphism as the composition

\[
\text{Det}_{(i)} = (\text{Det}_{GL_d} : BGL_d \rightarrow BC_u^*) \circ \left( \pi_{pt / GL_d} : pt / (GL_d \times T) \rightarrow BGL_d \right)
\]

\[
\circ (A_{Q',d} / (GL_d \times T) \rightarrow pt / (GL_d \times T)).
\]

We denote by \( u_{(i)} \in \text{H}(\mathcal{M}_d^T(Q'), Q) \) the image of \( u \in H_{\mathbb{C}^*_u} \) under \( \text{Det}^*_{(i)} \). Consider the morphism

\[
l_i = (\text{Det}_{(i)}, \text{id}) : \mathcal{M}_d^T(Q') \rightarrow BC_u^* \times \mathcal{M}_d^T(Q').
\]

Since the function \( \text{Tr}(W) \) is pulled back from the function \( 0 \otimes \text{Tr}(W) \) on the target of \( l_i \) we obtain the morphism in vanishing cycle cohomology

\[
\text{H}(BC_u^*, Q) \otimes \text{H}(\mathcal{M}_d^T(Q'), \rho_{\text{Tr}(W)}) \xrightarrow{l_i^*} \text{H}(\mathcal{M}_d^T(Q'), \rho_{\text{Tr}(W)})
\]

which defines an action of the algebra \( \mathbb{Q}[u] = H_{\mathbb{C}^*_u} \) on \( A_{Q',W}^T \):

\[
p(u) \bullet \alpha := l_i^*(p(u) \otimes \alpha).
\]

Similarly, we define the morphism

\[
l = (\bigotimes_{i \in Q_0} \text{Det}_{(i)}, \text{id}) : \mathcal{M}_d^T(Q) \rightarrow BC_u^* \times \mathcal{M}_d^T(Q).
\]

and an action of \( H_{\mathbb{C}^*_u} \) on \( A_{Q',W}^T \) by \( p(u) \bullet \alpha = l^*(p(u) \otimes \alpha) \).
3.5. PBW isomorphism. In this section we assume $T = \{1\}$ for simplicity, and continue to assume that $Q$ is symmetric. It follows that there exists a (non-unique) bilinear form
\[
\psi: \mathbb{Z}^Q \times \mathbb{Z}^Q \to \mathbb{Z}^Q
\]
satisfying
\[
\psi(d', d'') + \psi(d', d') = \chi_Q(d', d'') + \chi_Q(d', d'') \mod 2
\]
for all $d', d''$. We fix a choice of such $\psi$ in what follows, and denote by $A^{Q, W}$ the cohomological Hall algebra $A_{Q, W}$ with the multiplication twisted by setting
\[
m_{d', d''} = (-1)^{\psi(d', d'')} m_{d', d''}.
\]
where $m_{d', d''}$ and $m'_{d', d''}$ are the restrictions of the products in $A_{Q, W}$ (resp. $A^{Q, W}$) to the pieces of degree $d', d''$. In our main application, we will have that $Q' = Q(3)$, so that $\chi_Q$ is even, and we can (and will) set $\psi = 0$.

By [DM20] the direct image $j_* P_{\text{Tr}(W)}^! \mathbb{Q}_{\text{vir}}(Q)$ splits: there is an isomorphism in $D^+(\text{Perv}(\mathcal{M}_d(Q)))$:
\[
J_! P_{\text{Tr}(W)}^! \mathbb{Q}_{\text{vir}}(Q) \cong \bigoplus_{i \geq 1} \mathbb{Z}[H_! P_{\text{Tr}(W)}^! \mathbb{Q}_{\text{vir}}(Q)][-i].
\]

There is a natural morphism
\[
\mathcal{P}_i A_{Q, W, d} := \mathcal{H}(\mathcal{M}_d(Q), p^* \mathbb{P} \mathcal{V}) \to A_{Q, W, d}
\]
induced by the natural transformation $\mathcal{V} \to \text{id}$ and it follows from the splitting [13] that [15] is injective, and the subspaces $\mathcal{P}_i A_{Q, W, d}$ provide a filtration of $A_{Q, W, d}$, called the perverse filtration.

By [DM20 Sec.5.3] the multiplication on $A_{Q, W}^{(\psi)}$ respects the perverse filtration. There is vanishing
\[
\mathcal{P}_0 A_{Q, W} = 0,
\]
and furthermore the subspace
\[
\mathfrak{g}_{Q, W} := \mathcal{P}_1 A_{Q, W}^{(\psi)} \subset A_{Q, W}^{(\psi)}
\]
is preserved by the commutator Lie bracket $[\cdot, \cdot]$ by [DM20 Cor.6.11]. There are isomorphisms
\[
\mathfrak{g}_{Q, W, d} = \mathcal{H}(\mathcal{M}_d(Q), \mathcal{BPS}_{Q, W})[-1]
\]
where $\mathcal{BPS}_{Q, W, d}$ is the BPS sheaf defined in [DM20 Thm.A]. We call $\mathfrak{g}_{Q, W}$ the BPS Lie algebra associated to $(Q', W)$.

**Theorem 3.2.** [DM20 Thm.C] The PBW morphism
\[
\Psi: \text{Sym}(\mathfrak{g}_{Q, W} \otimes \mathcal{H}_{\mathbf{C}}) \to A_{Q, W}^{(\psi)}
\]
is an isomorphism of cohomologically graded $\mathbb{Z}^Q$-graded vector spaces.

The morphism $\Psi$ is constructed as follows. Firstly, we consider the composition
\[
\Psi': \mathcal{H}(\mathcal{B}_{C^u}, \mathbb{C}) \otimes \mathfrak{g}_{Q, W} \to \mathcal{H}(\mathcal{B}_{C^u}, \mathbb{C}) \otimes A_{Q, W}^{(\psi)} \to A_{Q, W}^{(\psi)}.
\]
Since the target carries an algebra structure, this extends uniquely to a morphism of algebras
\[
\Phi: \text{T}(\mathcal{H}(\mathcal{B}_{C^u}, \mathbb{C}) \otimes \mathfrak{g}_{Q, W}) \to A_{Q, W}^{(\psi)}
\]
from the free associative algebra generated by the domain of $\Psi'$. Then $\Psi$ is defined to be the restriction of $\Phi$ to the subspace of (supersymmetric) tensors. Note that the morphism $\Psi$ will typically not be a morphism of algebras; the CoHA $A_{Q, W}^{(\psi)}$ will in general have an interesting algebra structure, and not be a free (super)commutative algebra.
3.6. Deformed BPS Lie algebras. Let $Q'$ be a quiver, $T$ be a torus acting on the arrows of $Q'$, and let $W \in \mathbb{C}Q'/[\mathbb{C}Q', \mathbb{C}Q']_{\text{vec}}$ be a $T$-invariant potential. Recall the map $JH^T$ from [DM20]. We define the $H_T$-module
\[
\mathfrak{g}^T_{Q', W, d} = H(M^T_{\mathfrak{g}}(Q'), \nu_{\leq 1} JH^T \Phi_{\text{Tr}}(W) Q^\text{vir}_{20G}(Q')).
\]
Applying the derived global sections functor to the natural morphism of complexes
\[
\nu_{\leq 1} JH^T \Phi_{\text{Tr}}(W) Q^\text{vir}_{20G}(Q') \to JH^T \Phi_{\text{Tr}}(W) Q^\text{vir}_{20G}(Q'),
\]
we obtain the morphism
\[
\psi^T_{Q', W, d} : \mathfrak{g}^T_{Q', W, d} \to A^T_{Q', W, d}.
\]

**Theorem 3.3.** The morphism $\psi$ is injective, so that there is a natural inclusion of $\mathbb{N}G$-graded subspaces $\mathfrak{g}^T_{Q', W} \subset A^T_{Q', W}$. The subspace $\mathfrak{g}^T_{Q', W}$ is closed under the commutator Lie bracket on $A^T_{Q', W}$. Assume, furthermore, that $\mathfrak{g}^T_{Q', W}$ is pure, as a mixed Hodge structure. Then there is an isomorphism of vector spaces
\[
\mathfrak{g}^T_{Q', W} \cong \mathfrak{g}_{Q', W} \otimes H_T,
\]
and the PBW morphism
\[
\text{Sym}_{H_T} (\mathfrak{g}^T_{Q', W} \otimes H_T^*) \to A^T_{Q', W}
\]
constructed from the Hall algebra product on the target is an isomorphism.

**Proof.** The proof is a minor modification of the analogous statements from [DM20], to which we refer for more details. First, we claim that $JH^T$ is approximated by projective morphisms in the sense of [DM20, Sec.4.1]. For this the proof is as in [DM20], using the quotient of moduli spaces of framed $\mathbb{C}Q'$-modules by the $T$-action as auxiliary spaces. From this it follows as in [DM20] that there is a splitting
\[
JH^T \Phi_{\text{Tr}}(W) Q^\text{vir}_{20G}(Q') \cong \bigoplus_{i \in \mathbb{Z}} p^i \mathcal{H}^i \left( JH^T \Phi_{\text{Tr}}(W) Q^\text{vir}_{20G}(Q') \right) [-i]
\]
and so the morphism
\[
\nu_{\leq 1} JH^T \Phi_{\text{Tr}}(W) Q^\text{vir}_{20G}(Q') \to JH^T \Phi_{\text{Tr}}(W) Q^\text{vir}_{20G}(Q')
\]
has a left inverse, and the injectivity part of the theorem follows.

The monoid morphism $M(Q') \times M(Q') \to M(Q')$ taking a pair of semisimple $\mathbb{C}Q'$-modules to their direct sum is $T$-equivariant, so that we can define the monoid structure
\[
\oplus : M^T(Q') \times_{BT} M^T(Q') \to M^T(Q')
\]
by passing to the stack-theoretic quotient by the $T$-action. Given a pair of complexes of sheaves $\mathcal{F}$ and $\mathcal{G}$ on two stacks $\mathfrak{X}$ and $\mathfrak{Y}$ over a stack $\mathfrak{Z}$ we define the external product, a complex of sheaves on $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ in the usual way:
\[
\mathcal{F} \boxtimes \mathcal{G} := \pi^*_{\mathfrak{X}} \mathcal{F} \otimes \pi^*_{\mathfrak{Y}} \mathcal{G}.
\]
This defines a symmetric tensor product on complexes of sheaves on $M^T(Q')$: we define
\[
\mathcal{F} \boxtimes \mathcal{G} := \oplus_r (\mathcal{F} \boxtimes_{BT} \mathcal{G}).
\]
Let $p : M^T(Q') \to BT$ be the canonical map. This is a morphism of monoid objects over $BT$, where $BT$ is considered as a monoid over itself via the identity morphism. For arbitrary complexes of sheaves on $M^T(Q')$ it is too much to hope that there is an isomorphism
\[
H(M^T(Q'), \mathcal{F} \boxtimes \mathcal{G}) \cong H(M^T(Q'), \mathcal{F}) \otimes_{H_T} H(M^T(Q'), \mathcal{G}).
\]
However, let us assume that the complexes $p_\star \mathcal{F}$ and $p_\star \mathcal{G}$ lift to pure complexes of mixed Hodge modules. Then via the usual spectral sequence argument (see [Dav17b, Sec.9]), there are isomorphisms
\[
H(M^T(Q'), \mathcal{F}) \cong H(M(Q'), \mathcal{F}) \otimes H_T
\]
\[
H(M^T(Q'), \mathcal{G}) \cong H(M(Q'), \mathcal{G}) \otimes H_T
\]
where $\mathcal{F}$ and $\mathcal{G}$ denote the inverse images of $\mathcal{F}$ and $\mathcal{G}$ along the canonical morphism $M(Q') \to M^T(Q')$, and a natural isomorphism [21].

Via the same morphisms as in [DM20, Sec.5.1] (see also [3.4]), the complex $JH^T p \Phi_{Tr(W)} \mathcal{Q}_{2\mathcal{M}_d}(Q')$ carries a relative Hall algebra structure, i.e. for $d = d' + d''$ there is a morphism
\[
JH^T p \Phi_{Tr(W)} \mathcal{Q}_{2\mathcal{M}_d}(Q') \cong JH^T p \Phi_{Tr(W)} \mathcal{Q}_{3\mathcal{M}_d}(Q') \to JH^T p \Phi_{Tr(W)} \mathcal{Q}_{3\mathcal{M}_d}(Q')
\]
where $\mathcal{Q}_{2\mathcal{M}_d}(Q')$ is closed under the commutator Lie bracket. The first claim concerns the vanishing of certain perverse sheaves on $M^T(Q')$, and the second concerns the vanishing of certain morphisms $M^T(Q') \to M^T(Q')$, and a natural isomorphism [21].

Taking derived global sections, these two claims prove that $\mathcal{G}_Q^T_{d, W}$ is closed under the commutator Lie bracket. The first claim concerns the vanishing of certain perverse sheaves on $M^T(Q')$, and the second concerns the vanishing of certain morphisms of perverse sheaves on $M^T(Q')$. As such, both are local, and may be checked after pulling back along an open cover $U \to BT$, at which point they both reduce to the analogous claims on $M(Q)$, which are proved as [DM20, Thm.3.4], [DM20, Cor.6.11] respectively.

The isomorphism [21] follows from the usual spectral sequence argument (as in [Dav17b]). Since $p$ is a morphism of monoids over $BT$, the complex $p_* JH^T p \Phi_{Tr(W)} \mathcal{Q}_{3\mathcal{M}_d}(Q')$ inherits a $(\psi$-twisted) Hall algebra structure, along with an action of $H_{C_N}$. We claim that the morphism
\[
\text{Sym}_{3\mathcal{M}_d} \left( p_* JH^T p \Phi_{Tr(W)} \mathcal{Q}_{3\mathcal{M}_d}(Q') \otimes H_{C_N} \right) \to p_* JH^T p \Phi_{Tr(W)} \mathcal{Q}_{3\mathcal{M}_d}(Q')
\]
is an isomorphism. Again, this can be proved by passing to an open cover of $BT$, at which point it becomes the non-equivariant PBW theorem (Theorem 3.22). The equivariant PBW theorem then follows. \hfill \Box

We refer to $\mathcal{G}_Q^T_{d, W}$ as the deformed BPS Lie algebra. By the above theorem, as long as $\mathcal{G}_Q^T_{d, W}$ is pure, this deformed Lie algebra provides a family of Lie algebras over $\mathfrak{h} \cong \text{Spec}(H_T)$ specialising to the original BPS Lie algebra $\mathcal{G}_Q^T_{d, W}$ at the origin.

### 3.7. Kac polynomials and Kac–Moody Lie algebras.

Given a quiver $Q$ without loops we denote by $\mathfrak{n}_Q$ the negative half of the associated Kac–Moody Lie algebra. This is the free $\mathbb{N}^{Q_0}$-graded Lie algebra with generators $F_i$ in $\mathbb{N}^{Q_0}$-degree $1_i$, subject to the Serre relations
\[
[F_i, -]^{a_{ij}+1}(F_j) = 0
\]
for $i \neq j$. Here $a_{ij}$ denotes the number of edges joining $i$ and $j$ in the graph obtained from $Q$ by forgetting orientations of edges. If the underlying graph of $Q$ is a type ADE Dynkin diagram, $\mathfrak{n}_Q$ is (one half of) the associated simple Lie algebra.
Given a quiver $Q$ and a field $K$, a $KQ$-module $\rho$ is called indecomposable if it cannot be written as $\rho = \rho' \oplus \rho''$ with $\rho', \rho'' \neq 0$. It is called absolutely indecomposable if $\rho \otimes_K K$ is an indecomposable $KQ$-module. For $q = p^a$ a prime power, let $a_{Q,d}(q)$ denote the number of isomorphism classes of absolutely indecomposable $d$-dimensional $F_qQ$-modules. Victor Kac showed [Kac83] that $a_{Q,d}(q)$ is a polynomial in $q$, and moreover an element of $\mathbb{Z}[q]$.

If $Q$ is an orientation of a finite ADE graph, then
\begin{equation}
    a_{Q,d}(q) = \dim(n_{Q,d}).
\end{equation}
In particular, it is constant. Conversely, if $Q$ is any other type of quiver, $a_{Q,d}(q)$ is nonconstant for at least some values of $d$.

3.8. Preprojective CoHAs. We consider the class of special cases of quiver with potential $(Q', W) = (\tilde{Q}, \tilde{W})$ for a finite quiver $Q$. Although we do not assume that $Q$ is symmetric, $\tilde{Q}$ obviously is. We call these CoHAs preprojective CoHAs, for reasons that will become evident in [Dav17b, Thm. 4.7] By [Dav17b, Thm. 4.7] there is an equality of generating series
\begin{equation}
    \sum_{i \in \mathbb{Z}} \dim(g^i_{Q,\tilde{W},d})q^{i/2} = a_{Q,d}(q^{-1}).
\end{equation}
where $a_{Q,d}(q)$ is the Kac polynomial. So in particular, $g_{Q,\tilde{W}}$ is concentrated in even, negative cohomological degrees.

**Proposition 3.4.** [Dav20, Thm. 6.6] Let $Q$ be a finite quiver, and denote by $Q^{in}$ the quiver obtained by removing all vertices that support a loop, along with any arrows incident to at least one such vertex. There is an isomorphism of $\mathbb{N}^{Q^0}$-graded Lie algebras
\begin{equation}
    g^0_{Q,\tilde{W}} \cong n_{Q^{in}}^{\infty}.
\end{equation}

Choose a torus $T = (C^*)^l$, and a weighting function $\tau: \tilde{Q}_1 \to \mathbb{Z}^l$ defining an action of $T$ on the category of $CQ$-modules, for which $\tilde{W}$ is invariant. We denote by $\mathfrak{m}_T$ the maximal homogeneous ideal in $H_T$. The following theorem is a consequence of the purity of the natural mixed Hodge structure on $A_{\tilde{Q},\tilde{W}}$, see [Dav17b] for details.

**Proposition 3.5.** [Dav17b, Thm. 9.6] Let $l: \mathbb{Z}^r \to \mathbb{Z}^r'$ be a split surjection of lattices, with complement $\mathbb{Z}^{r''}$, inducing the split inclusion of tori
\begin{equation}
    T' = \text{Hom}_{\text{Grp}}(\mathbb{Z}^{r''}, C^*) \hookrightarrow T = \text{Hom}_{\text{Grp}}(\mathbb{Z}^{r'}, C^*)
\end{equation}
with complement $T'' = \text{Hom}_{\text{Grp}}(\mathbb{Z}^{r''}, C^*)$. There is an isomorphism of $\mathbb{N}^{Q^0}$-graded, cohomologically graded $H_T$-modules
\begin{equation}
    A^T_{Q,\tilde{W}} \cong A^{T'}_{Q,\tilde{W}} \otimes H_{T''}
\end{equation}
and an isomorphism of algebras
\begin{equation}
    A^{T',(\psi)}_{Q,\tilde{W}} \cong A^{T,(\psi)}_{Q,\tilde{W}} \otimes_{H_T} H_{T''}.
\end{equation}

For the special case $r' = 0$, the above proposition yields an isomorphism of $H_T$-modules
\begin{equation}
    A^T_{Q,\tilde{W}} \cong A^T_{Q,\tilde{W}} \otimes H_T
\end{equation}
and an isomorphism of algebras
\begin{equation}
    A^{(\psi)}_{Q,\tilde{W}} \cong A^{T,(\psi)}_{Q,\tilde{W}} / \mathfrak{m}_T : A^{T,(\psi)}_{Q,\tilde{W}}.
\end{equation}
We will be particularly interested in the action of the torus $T = (\mathbb{C}^*)^2 = \text{Hom}_{\text{Grp}}(\mathbb{Z}^2, \mathbb{C}^*)$ acting via the weighting function

$$\tilde{\tau}: Q_1 \rightarrow \mathbb{Z}^2$$

$$a \mapsto (1, 0) \quad a^* \mapsto (0, 1) \quad \omega_i \mapsto (-1, -1).$$

3.9. Dimensional reduction. We recall from [Dav17a, Appendix.A] the following result.

**Theorem 3.6.** [Dav17a, Thm.A.1] Let $X = Y \times \mathbb{A}^n$ be a smooth variety, let $f$ be a function on $X$ of weight one for the scaling action on $\mathbb{A}^n$, so that we can write

$$f = \sum_{i=1}^n x_i f_i$$

for $x_1, \ldots, x_n$ a system of coordinates on $\mathbb{A}^n$ and $f_1, \ldots, f_n$ functions on $Y$. Set $Z = Z(f_1, \ldots, f_n)$, let $i: Z \hookrightarrow Y$ be the inclusion, and let $\pi: X \rightarrow Y$ be the projection. Then there is a natural isomorphism

$$i_* f^* \rightarrow \pi_* f^*.$$

Applying the result to $DQ_Y$ and passing to global sections yields the isomorphism

$$H^B_{\mathbb{R}M}(Z, \mathbb{Q}) \rightarrow H(X, R\phi f \mathbb{Q})[2 \dim(Y)].$$

The theorem extends in the obvious way to affine fibrations over stacks [Dav17a, Cor.A.9]. In particular, for the affine fibration

$${\mathfrak{M}}_d(\bar{Q}) \rightarrow {\mathfrak{M}}_d(Q)$$

the function $\text{Tr}(\tilde{W})$ has weight one with respect to the action that scales the fibres (i.e. the space of choices of matrices for each of the loops $\omega_i$). As such, we obtain an isomorphism

$$\Psi_d: H^B_{\mathbb{R}M}(\mathfrak{M}_d(\Pi Q), \mathbb{Q})[-2\chi_Q(d, d)] \cong H(\mathfrak{M}_d(Q), R\phi \text{Tr}(\tilde{W})).$$

We slightly twist this isomorphism by setting $\Psi'_d = (-1)^{\binom{d}{2}} \Psi_d$. We use these sign-twisted isomorphisms, and the algebra structure on $A_{Q,W}^T$, to induce an algebra structure on

$$A_{Q,d}^T := \bigoplus_{d \in \mathbb{N}^n} H^B_{\mathbb{R}M}(\mathfrak{M}_d(\Pi Q), \mathbb{Q}^{vir}).$$

Here, in the definition of the sign twist in the definition of $A_{Q,W}^{T,\chi}$, we have chosen $\psi = \chi_Q$, noting that

$$\chi_Q(d', d'') + \chi_Q(d'', d') = \chi_Q(d', d'') + \chi_Q(d', d'') \mod 2.$$

Recall that we have defined the complex $Q^{vir}$ on the quotient of a smooth stack by a $T$-action to be the constant sheaf, shifted on each component of the stack by the dimension of the stack. Since $\mathfrak{M}_d(\Pi Q)$ is not smooth, we need a new convention to make sense of the above definition: on $\mathfrak{M}_d(\Pi Q)$ we define $Q^{vir} := \mathbb{Q}[-2\chi(d, d)]$.

The algebra structure on $A_{Q,d}^T$ induced by the algebra structure on $A_{Q,W}^{T,\chi}$ and the isomorphism $\Psi'$ is the same as the one defined by Schiffmann and Vasserot in [SV13], see [RS17, Appendix], [YZ16] for proofs. Our algebra structure is thus isomorphic to theirs, with the added sign twist provided by $\chi_Q$.

---

7See [RS17, Lem.4.1] for the origin of this sign.
3.10. *Affinizing BPS Lie algebras.* For pairs $(\tilde{Q}, \tilde{W})$ of a tripled quiver with potential there is extra structure on the CoHA $\mathcal{A}_{\tilde{Q}, \tilde{W}}^X$, ultimately derived from a factorization structure on the stack of representations of the Jacobi algebra. We briefly describe this here; full details for more general 3-Calabi–Yau completions of 2-Calabi–Yau categories will appear in joint work with Tasuki Kinjo. In the special case in which $Q$ is the one-loop quiver, i.e. the case we need for the main results of this paper, this structure can be extracted from [KV19] via dimensional reduction.

Under the isomorphism $j$ from [10] the element $j(\omega)$ is central in Jac$(\tilde{Q}, \tilde{W})$, so that every Jac$(\tilde{Q}, \tilde{W})$-module $\rho$ admits a canonical decomposition as a Jac$(\tilde{Q}, \tilde{W})$-module

$$\rho \equiv \bigoplus_{\lambda \in \mathbb{C}} \rho_\lambda$$

where $j(\omega)$ acts on $\rho_\lambda$ with unique generalised eigenvalue $\lambda$. From this one deduces (see [Dav17b, Lem.4.1]) that for $U \subset \mathbb{C}$ an open ball, if one denotes by $M_\mathfrak{d}^\omega(\tilde{Q})$ the open substack satisfying the condition that each $\omega_i$ acts with generalised eigenvalues inside $U$, the restriction

$$H(M_\mathfrak{d}^\omega(\tilde{Q}), r_\phi Tr(\tilde{W})) \to H(M_\mathfrak{d}^\omega(\tilde{Q}), r_\phi Tr(\tilde{W}))$$

is an isomorphism. Let $U'$ and $U''$ be disjoint open balls in $\mathbb{C}$, then via the Thom–Sebastiani isomorphism one constructs the composition of morphisms

$$\Delta: H(M(\tilde{Q}), r_\phi Tr(\tilde{W})(Q^{vir})) \to H(M(U') \coprod U''(\tilde{Q}), r_\phi Tr(\tilde{W})(Q^{vir}))$$

$$\cong H(M(U')(\tilde{Q}), r_\phi Tr(\tilde{W})(Q^{vir})) \otimes H(M(U'')(\tilde{Q}), r_\phi Tr(\tilde{W})(Q^{vir}))$$

$$\cong A_{\tilde{Q}, \tilde{W}}^X \otimes A_{\tilde{Q}, \tilde{W}}^X$$

defining a coproduct on $A_{\tilde{Q}, \tilde{W}}^X$. The second isomorphism is not quite canonical: there is a choice of sign, with the geometric origin explained in [BS17, Lem.4.1]. It is almost cocommutative since the space of choices of $U'$ and $U''$ is connected, so that we can continuously swap them. Because of the signs appearing in dimensional reduction, in order for this coproduct to be actually cocommutative, we again multiply the component $\Delta_{\mathfrak{d}', \mathfrak{d}''}$ by $(-1)^{\mathfrak{d}' \cdot \mathfrak{d}''}$. The resulting coproduct is compatible with $m'$, since $m'$ lifts to an algebra structure on the object

$$(M(\tilde{Q}) \xrightarrow{\lambda} \text{Sym}(A^1)) \cdot r_\phi Tr(\tilde{W})(Q^{vir})$$

where $\lambda$ is the morphism recording the generalised eigenvalues of $j(\omega)$. It follows by the Milnor–Moore theorem that $A_{\tilde{Q}, \tilde{W}}^X$ is a universal enveloping algebra. Furthermore, by the support lemma [Dav17b, Lem.4.1] $\lambda_{BPS_{\tilde{Q}, \tilde{W}}} \otimes \mathbb{H}_{\mathfrak{c}_L}$ is supported on the subspace of $M(\tilde{Q})$ for which all of the generalised eigenvalues of $j(\omega)$ are identical, and so

$$\hat{\mathfrak{g}}_{\tilde{Q}, \tilde{W}} := \mathfrak{g}_{\tilde{Q}, \tilde{W}} \otimes \mathbb{H}_{\mathfrak{c}_L} \subset A_{\tilde{Q}, \tilde{W}}^X$$

is primitive for the coproduct $\Delta$, i.e. for $\alpha \in \hat{\mathfrak{g}}_{\tilde{Q}, \tilde{W}}$, $\Delta(\alpha) = 1 \otimes \alpha + \alpha \otimes 1$. By Theorem [KZ2] and the PBW theorem for $A_{\tilde{Q}, \tilde{W}}^X$ the subspace $\hat{\mathfrak{g}}_{\tilde{Q}, \tilde{W}}$ has the same graded dimensions as the space of primitive elements for $A_{\tilde{Q}, \tilde{W}}^X$. Since all elements of $\hat{\mathfrak{g}}_{\tilde{Q}, \tilde{W}}$ are primitive, we find

**Proposition 3.7.** The subspace $\hat{\mathfrak{g}}_{\tilde{Q}, \tilde{W}} \subset A_{\tilde{Q}, \tilde{W}}^X$ is closed under the commutator Lie bracket, and $A_{\tilde{Q}, \tilde{W}}^X = U(\hat{\mathfrak{g}}_{\tilde{Q}, \tilde{W}})$.

**Definition 3.8.** We call $\hat{\mathfrak{g}}_{\tilde{Q}, \tilde{W}}$ the affine BPS Lie algebra for $(\tilde{Q}, \tilde{W})$. 
Remark 3.9. There are no known examples of quivers with potential \( A_Q^{c}, W \) for which \( B_Q, W \otimes H_C \) can be shown not to be closed under the commutator Lie bracket. It would be interesting to know what class of BPS Lie algebras can be affinized this way, and in particular whether the “partially fermionized” cousins [Dav22] of the BPS Lie algebras \( B_Q, W \) belong to this class.

Let \( C^* \) act on \( C \tilde{Q} \)-modules via the weighting function \( \tau: \tilde{Q} \rightarrow \mathbb{Z} \) which sends \( a \) to 1 and \( a^* \) to \(-1\) for all \( a \in Q_1 \), and \( \omega_i \) to 0 for all \( i \in Q_0 \). Then the morphism \( \lambda \) is \( C^* \)-invariant, so that we can repeat the construction of the commutative coproduct above for the CoHA

\[
\Lambda^{C^*} Q, \tilde{W}^*,
\]

considered as an algebra object in the category of \( H_C \)-modules. Via the symmetry that permutes the arrows \( a, a^*, \omega \), the algebra \( A^{C^*} Q, \tilde{W}^* \) carries a cocommutative coproduct in the same category, i.e. a morphism

\[
A^{C^*} Q, \tilde{W}^* \rightarrow A^{C^*} Q, \tilde{W}^* \otimes H_C, A^{C^*} Q, \tilde{W}^*.
\]

and is a universal enveloping algebra, as long as \( C^* \) acts on one of \( a, a^*, \omega \) with weight zero. Since the coproduct is \( H_C \)-linear by construction, we obtain a \( H_C \)-linear Lie algebra of primitive elements \( \tilde{H}^{C^*} Q, \tilde{W}^* \):

**Definition 3.10.** We denote by \( \tilde{H}^{C^*} Q, \tilde{W} \subset A^{C^*} Q, \tilde{W}^* \) the Lie algebra of primitives, which we call the deformed affinized BPS Lie algebra for \( \tilde{Q}, \tilde{W} \).

The following is a basic application of the Milnor–Moore theorem:

**Proposition 3.11.** There is an isomorphism of \( H_C \)-algebra objects:

\[
U_{H_C}(\tilde{H}^{C^*} Q, \tilde{W}) \rightarrow A^{C^*} Q, \tilde{W}^*.
\]  

(28)

The “deformed” in the definition is justified by the following

**Proposition 3.12.** There is an isomorphism of \( \mathbb{N}Q_0 \)-graded, cohomologically graded \( H_C \)-modules.

\[
\tilde{H}^{C^*} Q, \tilde{W} \cong \tilde{H}^{C^*} Q, \tilde{W} \otimes H_C.
\]

(29)

as well as an isomorphism of Lie algebras

\[
\tilde{H}^{C^*} Q, \tilde{W} \cong \tilde{H}^{C^*} Q, \tilde{W} / m_{C^*} \tilde{H}^{C^*} Q, \tilde{W}.
\]

**Proof.** Before we start, we remind the reader that \( H_C \neq H_C^* \): the first algebra has to do with the extra \( C^* \)-action on \( C \tilde{Q} \), the second has to do with the action studied at the end of \[3.3\].

Since the coproduct is \( H_C \)-linear, and \( A^{C^*} Q, \tilde{W}^* \) is free as a \( H_C \)-module by Proposition 3.5 it follows that \( \alpha \) is primitive if and only if \( t \cdot \alpha \) is primitive, for \( t \) a generator of \( m_{C^*} \). Therefore, the space of primitives for the coproduct (i.e. \( \tilde{H}^{C^*} Q, \tilde{W} \)) is free as a \( H_C \)-module. The graded dimensions of this module are determined by the existence of the isomorphisms (28) and (25), yielding (29).

Again by \( H_C \)-linearity of the coproduct, the composition

\[
\tilde{H}^{C^*} Q, \tilde{W} / m_{C^*} \tilde{H}^{C^*} Q, \tilde{W} \hookrightarrow A^{C^*} Q, \tilde{W}^* / m_{C^*} A^{C^*} Q, \tilde{W}^* \cong A^X Q, \tilde{W}^*
\]

realises \( \tilde{H}^{C^*} Q, \tilde{W} / m_{C^*} \tilde{H}^{C^*} Q, \tilde{W} \) as a subspace of primitive elements in \( A^X Q, \tilde{W}^* \). But the graded dimensions of \( \tilde{H}^{C^*} Q, \tilde{W} / m_{C^*} \tilde{H}^{C^*} Q, \tilde{W} \) match those of \( \tilde{H}^{C^*} Q, \tilde{W} \), by the first statement, and the second statement follows.

\[\Box\]
3.11. Affinized finite type BPS algebras. Let $Q$ be an orientation of a finite type, ADE Dynkin diagram. We calculate the (undeformed) affine BPS Lie algebra $\tilde{g}_{Q,W}$. We start with an easy lemma

**Lemma 3.13.** Let $\mathcal{H}$ be an algebra object in $\text{Heis}_Q$-mod, such that there is an isomorphism of underlying $\text{Heis}$-modules

$$\mathcal{H} \cong \bigoplus_{m \in \mathbb{Z}_{\geq 1}} \text{Gr}^F_m(V_m)[-2]^\oplus_{a_m}$$

for integers $a_m$. Then up to isomorphism, $\mathcal{H}$ is determined by the sub-algebra $\mathcal{H}^0$. Similarly, if $g$ is a Lie algebra object in $\text{Heis}$-mod$_Z$ such that there is an isomorphism of underlying $\text{Heis}$-modules

$$g \cong \bigoplus_{m \in \mathbb{Z}_{\geq 1}} \text{Gr}^F_m(V_m)[-2]^\oplus_{b_m}$$

for integers $b_m$, then $g$ is determined up to isomorphism by the Lie sub-algebra $g^0$.

**Proof.** Under the conditions of the Lemma, $\mathcal{H}$ is concentrated in cohomological degrees indexed by $2\mathbb{Z}_{\geq 0}$, and the lowering operator $q_i : \mathcal{H}^i \to \mathcal{H}^{i-2}$ is injective for $i > 0$. As such, if $\alpha, \beta \in \mathcal{H}$ are not both of cohomological degree zero, $\alpha \star \beta$ is determined by $q(\alpha \star \beta) = q(\alpha) \star \beta + \alpha \star q(\beta)$, and the result follows by induction. The proof for $g$ is identical. \qed

Given a $\mathbb{N}^Q$-graded Lie algebra $g$ we form the Lie algebra $g \otimes_{\mathbb{Q}} \mathbb{Q}[D]$ by $\mathbb{Q}[D]$-linear extension. We make this into a $\text{Heis}_Q$-module by setting $q_i(g \otimes D^m) = mg \otimes D^{m-1}$ and $p_i(g \otimes D^m) = d_i g \otimes D^{m+1}$, for $g$ of homogeneous $\mathbb{N}^Q$-degree $d$.

In particular, for $Q$ a finite type ADE Dynkin diagram with associated Lie algebra $g$, the Lie algebra object $g \otimes_{\mathbb{Q}} \mathbb{Q}[D]$ is a Lie algebra object in $\text{Heis}$-mod$_Z$ satisfying the assumptions of Lemma 3.13

**Proposition 3.14.** Let $Q$ be a quiver obtained by orienting a finite type Dynkin diagram. There is an isomorphism of Lie algebras

$$\tilde{g}_{Q,W} \cong n^Q_{W} \otimes_{\mathbb{Q}} \mathbb{Q}[D]$$

where $n^Q_{W}$ is half of the simple Lie algebra associated to the underlying Dynkin diagram of $Q$.

**Proof.** By Proposition 3.4 there is an isomorphism $\tilde{g}^0_{Q,W} \cong n^Q_{W}$, and then the result follows from Lemma 3.13. \qed

We leave it to the enthusiastic reader to calculate the deformed affine BPS Lie algebra $\tilde{g}^*_W$.

3.12. Nakajima quiver varieties. Fix a finite quiver $Q$, and a framing vector $f \in \mathbb{N}^Q$. We define the quiver $Q_f$ to be the quiver obtained by adding a single vertex $\infty$ to the set $Q_0$, and for each $i \in Q_0$, adding $f_i$ arrows from $\infty$ to $i$, which we label $a_{i,m}$ for $m = 1, \ldots, f_i$. We define $Q_f$ to be the quiver obtained by tripling $Q_f$, and set $Q^+_f$ to be the quiver obtained by removing the loop $\omega_\infty$ from $Q_f$. We define

$$W^+ = \left( \sum_{a \in Q_1} [a, a^*] + \sum_{i \in Q_0} \sum_{m=1}^{f_i} a_{i,m}a^*_{i,m} \right) \left( \sum_{i \in Q_0} \omega_i \right).$$

Given a dimension vector $d \in \mathbb{N}^Q$ we extend $d$ to a dimension vector $d^+$ for $Q^+_f$ by setting $d^+_{\infty} = 1$. We define $A^-_{Q_f,d^+} \subset A^+_{Q_f,d^+}$ to be the subspace of representations
\( F \) such that the one-dimensional vector space \( e_\infty \cdot F \) generates \( F \) under the action of \( \mathbb{C}Q_\triangleright^F \). We call such modules \( \zeta \)-stable. Then we define the scheme
\[
\mathcal{M}_{f,d}(Q) = \mathcal{M}_{\zeta, \mathbb{C}Q_\triangleright^F, d+} / GL_d.
\]
This is the moduli scheme parameterising pairs of a \( d+ \)-dimensional \( \zeta \)-stable \( \mathbb{C}Q_\triangleright^F \)-module \( F \), along with a trivialisation \( e_\infty \cdot F \cong \mathbb{C} \).

We define \( \mathcal{M}_{\zeta, \mathbb{C}Q_\triangleright^F, d+} \subset \mathcal{M}_{\zeta, \mathbb{C}Q_\triangleright^F, d} \) to be the subspace of representations \( F \) of the doubled framed quiver such that the one-dimensional space \( e_\infty \cdot F \) generates \( F \) under the action of \( \mathbb{C}Q_\triangleright^F \). We consider the function
\[
\mu_d : \mathcal{M}_{\zeta, \mathbb{C}Q_\triangleright^F, d+} \to \mathfrak{gl}_d
\]
\[
((A_a, A_a^*)_{a \in Q_1}, (I_{i,m}, J_{i,m})_{i \in \mathbb{Q}_0, m \leq t_i}) \mapsto \sum_{a \in Q_1} [A_a, A_a^*] + \sum_{i \in \mathbb{Q}_0, m \leq t_i} I_{i,m}J_{i,m}.
\]
The group \( GL_d \) acts freely on \( \mathcal{M}_{\zeta, \mathbb{C}Q_\triangleright^F, d} \), and thus on \( \mu_d^{-1}(0) \), and we define the Nakajima quiver variety \( \mathcal{N}_{f,d}(Q) = \mu_d^{-1}(0) / GL_d \) \cite{nak94}. Via dimensional reduction, one may show that there is a natural isomorphism (see \cite{dav20} Prop.6.3)).
\[
\mathcal{H}(\mathcal{N}_{f,d}(Q), \mathbb{Q}^{\text{vir}}) \cong \mathcal{H}(\mathcal{M}_{f,d}(Q), r_{\Phi_{\text{T}(W)}}).
\]
(30)

Let \( d', d'' \in \mathbb{N}_0 \) be a pair of dimension vectors satisfying \( d' + d'' = d \). We define \( \mathcal{M}_{f,d',d''}(Q) \) to be the moduli scheme of pairs of a short exact sequence
\[
0 \to F' \to F \to F'' \to 0
\]
along with a trivialisation \( e_\infty \cdot F \cong \mathbb{C} \), where \( F \) is a \( d+ \)-dimensional \( \zeta \)-stable \( \mathbb{C}Q_\triangleright^F \)-module, and \( F' \) is a \( d' \)-dimensional \( \mathbb{C}Q \)-module, considered as a \( \mathbb{C}Q_\triangleright^F \)-module with dimension vector zero at the vertex \( \infty \). The module \( F'' \) appearing in such a short exact sequence is a \( d'' \)-dimensional \( \zeta \)-stable \( \mathbb{C}Q_\triangleright^F \)-module. We consider the standard correspondence diagram
\[
\mathfrak{M}_d(\tilde{Q}) \times \mathcal{M}_{f,d''}(Q) \xrightarrow{\pi_1 \times \pi_3} \mathcal{M}_{f,d',d''}(Q) \xrightarrow{\pi_2} \mathcal{M}_{f,d}(Q).
\]

Composing the Thom–Sebastiani isomorphism with the pushforward \( \pi_{2,*} \), and pullback \((\pi_1 \times \pi_3)^*\) we define the operation
\[
*: \mathcal{H}(\mathfrak{M}_d(\tilde{Q}), r_{\Phi_{\text{T}(W)}}) \otimes \mathcal{H}(\mathcal{M}_{f,d''}(Q), r_{\Phi_{\text{T}(W)}}) \to \mathcal{H}(\mathcal{M}_{f,d}(Q), r_{\Phi_{\text{T}(W)}}).
\]

**Proposition 3.15.** Let \( Q \) be a finite quiver.

- Via the isomorphism (30), the operation \( * \) defines an action of the CoHA \( \mathcal{A}_{\tilde{Q}, \tilde{W}}^\times \) on the vector space
\[
\mathcal{M}_f(Q) = \bigoplus_{d \in \mathbb{N}_0} \mathcal{H}(\mathcal{N}_{f,d}(Q), \mathbb{Q}^{\text{vir}})
\]
- The operation is \( \mathcal{H}_{\mathbb{C}_u} \)-linear: there are equalities \( u \cdot (\alpha \ast \beta) = (u \cdot \alpha) \ast \beta + \alpha \ast (u \cdot \beta) \).

**Proof.** For the first part, see e.g. \cite{so16} \cite{yz18}. The second part is proved as in Proposition 3.1 below. □

For \( Q \) the one loop quiver, and framing vector \( f = 1 \), we have \( \mathcal{A}_{\tilde{Q}, \tilde{W}, 1} \cong \mathcal{H}_{\mathbb{C}^*} \cong \mathbb{Q}[x] \). One shows as in \cite{dav20} that for \( 1 \) the copy of \( 1 \in \mathbb{Q}[x] \cong \mathcal{A}_{\tilde{Q}, \tilde{W}, 1} \) we have
\[
1 \ast = p_1
\]
(31)

where \( p_1 \) is the Nakajima raising operator \cite{nak97} on \( \text{Hilb}_n(\mathbb{A}^2) \cong \mathcal{M}_{1,n}(Q) \). From the second part of Proposition 3.15, it follows that
\[
(u \cdot \alpha) \ast (-) = [u \cdot \alpha](-)
\]
(32)
as operators on \(\text{Hilb}_n(\mathbb{A}^2)\). We recall the following theorem of Manfred Lehn:

**Theorem 3.16** (Leh99 Thm.3.10). Write \(p_n\) for Nakajima’s \(n\)th raising operator on \(\text{Hilb}(\mathbb{A}^2)\). Write \(p_n' = [u \cdot p_n]\). Then
\[
[p'_n, p_n] = -np_{n+1}.
\]

We refer to \([Nak97, Leh99]\) for the definition of the operators \(p_n\). Alternatively, via \([32, 31]\) and Theorem 3.16 these operators may be defined in terms of the action of \(1 \in \mathcal{A}_{\mathcal{Q}W, 1}\) and \(u\) on \(\text{Hilb}(\mathbb{A}^2)\).

### 3.13. Affine Yangian and shuffle algebra embedding

Let \(T\) be a torus, acting on the quiver \(Q'\). The superpotential \(W = 0\) is obviously \(T\)-invariant, and the vanishing cycle sheaf \(\mathcal{R}_{\mathcal{T}(W)}\mathcal{Q}^\text{vir}_{\mathcal{W}}(Q')\) on \(\mathcal{M}^T_d(Q')\) is isomorphic to the constant (shifted) perverse sheaf \(\mathcal{Q}^\text{vir}_{\mathcal{W}}(Q')\). It is thus straightforward to describe the vanishing cycle cohomology (leaving out the overall cohomological shifts for now):
\[
H(\mathcal{M}^T_d(Q'), \mathcal{R}_{\mathcal{T}(W)}\mathcal{Q}^\text{vir}_{\mathcal{W}}(Q')) \cong H(pt/(\text{GL}_d \times T), \mathcal{Q})
\cong \mathbb{Q}[x_1, \ldots, x_d, t_1, \ldots, t_i]^{\mathfrak{g}_d} \otimes \mathbb{Q}[t_1, \ldots, t_i]
\]
where \(Q' = \{1, \ldots, r\}\) and \(\mathfrak{g}_d = \prod_{i \in Q'_0} \mathfrak{g}_d\) is the product of symmetric groups generated by permutations of the variables preserving their first subscript.

**Proposition 3.17.** Let the torus \(T = \text{Hom}_{\text{Grp}}(\mathbb{Z}, (\mathbb{C}^*)^2)\) act on \(\tilde{T}\) via the function \(\tau\) from \([27]\). The natural morphism \(\Phi: \mathcal{A}^T_{\mathcal{Q}W} \to \mathcal{A}^T_{\tilde{T}}\) of algebras is an embedding of algebras.

The definition of \(\phi\) and proof that \(\Phi\) preserves the algebra product is given in \([SV13]\). The fact that it is an embedding of algebras is proved in \([Dav17b, Thm.10.2]\), slightly modifying and fixing a gap the proof in \([SV20]\).

By \([20]\) there is a (non-canonical) isomorphism of \(\text{H}_{T^*}\)-modules
\[
\mathcal{A}^T_{\mathcal{Q}W} \cong \mathcal{A}_{\mathcal{A}^2} \otimes \text{H}_{T^*},
\]
and we set \(\tilde{\mathcal{X}}^{(0)}_1 = \mathcal{X}^{(0)}_1 \otimes 1\) via this isomorphism, and \(\tilde{\mathcal{X}}^{(m)}_1 = u^m \cdot \mathcal{X}^{(0)}_1\). Although \([33]\) is not canonical, \(\tilde{\mathcal{X}}^{(0)}_1\) is uniquely defined, up to multiplication by a scalar, by the condition that it lies in cohomological degree \(-2\) and lies in \(\mathfrak{g}_d^T \subset \mathcal{A}^T_{\mathcal{A}^2}\).

For what follows, we introduce (the strictly positive part of) the affine Yangian \(\mathcal{Y}_{\mathcal{W}}\), \(\mathcal{W}\) be formal parameters satisfying \(t_1 + t_2 + t_3 = 0\). Then \(\mathcal{Y}_{\mathcal{W}}(\mathfrak{g}(1))\) is the algebra over \(\mathbb{Q}[t_1, t_2, t_3]/(t_1 + t_2 + t_3)\) generated by symbols \(e_i\) with \(i \in \mathbb{Z}_{\geq 0}\) subject to the two relations
\[
\begin{align*}
[e_{i+3}, e_j] - 3[e_{i+2}, e_{j+1}] + 3[e_{i+1}, e_{j+2}] - [e_i, e_{j+3}] & \quad (34) \\
+ \sigma_2(e_{i+1}, e_{j}) - [e_i, e_{j+1}] & = -\sigma_3(e_{i}, e_{j} + e_{j}, e_{i}) \\
\text{Sym}_{e_i}(e_{i+1}, [e_{i+2}, e_{i+1}]) & = 0 \\
\end{align*}
\]
where \(\sigma_2 = t_1 t_2 + t_1 t_3 + t_2 t_3\) and \(\sigma_3 = t_1 t_2 t_3\). The shuffle algebra embedding \(\Phi\) is used in \([RSYZ20b]\) to prove that the relations \([33]\) and \([35]\) hold in \(\mathcal{A}^T_{\mathcal{A}^2}\):

**Theorem 3.18.** \([RSYZ20b, Thm.7.1.1]\) There is an algebra morphism
\[
\iota: \mathcal{Y}_{\mathcal{W}}(\mathfrak{g}(1)) \to \mathcal{A}^T_{\mathcal{A}^2}
\]
sending \(e_i\) to \(\tilde{\mathcal{X}}^{(i)}_1\). Furthermore, this morphism is injective, so that \(\mathcal{Y}_{\mathcal{W}}(\mathfrak{g}(1))\) may be identified with the spherical subalgebra \(\mathcal{S}\mathcal{A}^T_{\mathcal{A}^2} \subset \mathcal{A}^T_{\mathcal{A}^2}\) under \(\iota\), i.e. the subalgebra generated by \(\mathcal{A}^T_{\mathcal{A}^2, 1}\).
We include here the explicit description of the shuffle algebra product on $A^T_{Q(3)}$. Since there is only one vertex, we can simplify the subscripts and write
\[
H(\mathfrak{H}^T_{d}(Q(3)), \mathbb{Q}) \cong \mathbb{Q}[t_1, t_2] \otimes \mathbb{Q}[x_1, \ldots, x_d]^\otimes.
\]
We define $\zeta(z) := (z + t_1)(z + t_2)(z - t_1 - t_2)z^{-1}$. Given two functions $f(x_1, \ldots, x_d) \in H(\mathfrak{H}^T_{d}(Q(3)), \mathbb{Q})$ and $g(x_1, \ldots, x_e) \in H(\mathfrak{H}^T_{d}(Q(3)), \mathbb{Q})$ we have
\[
f \ast g(x_1, \ldots, x_{d+e}) = \sum_{\pi \in \text{Sh}(d,e)} \pi \left( f(x_1, \ldots, x_d)g(x_{d+1}, \ldots, x_{d+e}) \prod_{1 \leq i \leq d \leq j \leq d+e} \zeta(x_i - x_j) \right)
\]
where $\text{Sh}(d,e) \subset S_{d+e}$ is the subset of the permutation group containing those $\pi$ such that $\pi(i) \leq \pi(i+1)$ for $i \neq d$.

In [Neg16] a related shuffle algebra is studied, in reference to the AGT relation\footnote{Many thanks to Andrei Neguţ for patiently explaining the following to me.}. Firstly, define $\zeta'(z) := (z + t_1)(z + t_2)(z - t_1 + t_2)^{-1}z^{-1}$. Then we fix the coefficient field $\mathbb{F} = \mathbb{Q}(t_1, t_2)$ and for each $d$ define $A^T_{A^2,d,\text{loc}} := \mathbb{F}(x_1, \ldots, x_d)^\otimes$ the field of symmetric rational functions. Then $A^T_{A^2,\text{loc}} := \bigoplus_{d \geq 0} A^T_{A^2,d,\text{loc}}$ carries a shuffle algebra structure defined as in [36] but with $\zeta'(z)$ replacing $\zeta(z)$. We define the algebra isomorphism
\[
J : A^T_{A^2,\text{loc}} \to A^T_{A^2,\text{loc}} \quad f(x_1, \ldots, x_d) \mapsto \prod_{1 \leq i < j \leq d} (z^2 - (t_1 + t_2)^2)^{-1} f(x_1, \ldots, x_d)(t_1t_2)^{-\deg(f)}.
\]
Composing with the injections
\[
A^T_{A^2} \xrightarrow{i_1} \mathbb{Q}[t_1, t_2] \otimes \mathbb{Q}[x_1, \ldots, x_d]^\otimes \xrightarrow{i_2} A^T_{A^2,\text{loc}}
\]
we obtain the injection $\Gamma = J \circ i_2 \circ i_1$. In [Neg16], explicit elements $B_d$ and $\tilde{L}_d$ of a Heisenberg–Virasoro algebra are given: it is easy to see that $\tilde{B}_1 = \Gamma(1)$ and $\tilde{L}_1 = \Gamma(x_1)$. Then $\tilde{B}_d$ is equal to a generator of the 1-dimensional vector space $\Gamma(\mathfrak{g}_{A^2,d})$ by Equation (2.43) of [Neg16] and Proposition 5.2 below.

4. Heisenberg actions on $A^T_{Q,W}$

It turns out that Heis$Q$-actions on cohomological Hall algebras exist at a very high level of generality\footnote{Note that the Heisenberg algebra appearing in this section has a different origin to the half Heisenberg algebra occurring inside $\mathfrak{g}_{A^2}$, studied in [4].}. We explain this first, before zooming in on the CoHAs that this paper is most concerned with, i.e. $A^T_{A^2}$ for various choices of $T$.

4.1. Raising operators.

**Proposition 4.1.** Let $Q'$ be an arbitrary quiver, with a $T$-action for some torus $T = \text{Hom}_\text{Grph}(N, \mathbb{C}^*)$, and with $T$-invariant potential $W \in \mathbb{C}Q'/\mathbb{C}Q'\mathbb{C}Q'\mathbb{C}Q'$, and for each $i \in Q_0$ the morphism $\alpha \mapsto u_i \bullet \alpha \quad \alpha := u_i(\cdot) \alpha$ defines a derivation of $A^T_{Q',W}$.
Proof. Fix $d', d'' \in \mathbb{N}^{q_0}$ and set $d = d' + d''$. Consider the following commutative diagram

$$
\begin{array}{c}
\mathcal{M}_d^T(Q') \times_{BT} \mathcal{M}_{d''}^T(Q') \xrightarrow{\pi_1 \times \pi_3} \mathcal{M}_{d'}^T(Q') \times_{BT} \mathcal{M}_{d'''}^T(Q') \xrightarrow{\pi_2} \mathcal{M}_d^T(Q') \\
\mathcal{M}_d^T(Q') \xrightarrow{\text{Det}(1)} \mathcal{M}_d^T(Q') \xrightarrow{\otimes} \mathcal{M}_d^T(Q') \\
BC_u \times BC_u \xrightarrow{\otimes} BC_u.
\end{array}
$$

The morphisms $(\pi_1 \times \pi_3)^*$ and $\pi_2^*$ respect the $H_{C^*}$-actions induced by the morphism to $BC_u^*$ in the above diagram. On the other hand, under the morphism

$$Q[u] = H(BC_u^*, Q) \xrightarrow{\otimes^*} Q[u_1, u_2] = H(BC_u^* \times BC_u^*, Q)$$

we have

$$\otimes^*(u) = u_1 + u_2.$$ (37)

So

$$u \cdot_i (m(\alpha \otimes \beta)) = m(((u_1 \otimes 1) + (1 \otimes u_2)) \cdot_i (\alpha \otimes \beta))$$

$$= m(u \cdot_i (\alpha \otimes \beta)).$$

4.2. Lowering operators. The morphism $\otimes^*$ in the proof of Proposition 4.1 makes $H_{C^*}$ into a cocommutative coalgebra. Let $\mathcal{L}$ be a line bundle on a scheme $X$, and let $\mathcal{F}$ be a flat family of $d$-dimensional $\mathbb{C}Q'$-modules. Then $\mathcal{L} \otimes \mathcal{F}$ is a flat family of $d$-dimensional $\mathbb{C}Q'$-modules, where the $\mathbb{C}Q'$-action is induced by the given action on $\mathcal{F}$ and the natural isomorphism $\text{End}_X(\mathcal{L} \otimes \mathcal{F}, \mathcal{L} \otimes \mathcal{F}) \cong \text{End}_X(\mathcal{F}, \mathcal{F})$. Recall that an $X$-point of $\mathcal{M}_d^T(Q')$ is defined by a $T$-torsor $p: E \to X$ along with a flat family $\mathcal{F}$ of $d$-dimensional $\mathbb{C}Q'$-modules on $E$ compatible with the $T$-action. We define a morphism

$$A: BC_u^* \times \mathcal{M}_d^T(Q') \to \mathcal{M}_d^T(Q')$$

$$(\mathcal{L}, (p: E \to X, \mathcal{F})) \mapsto (E, p^* \mathcal{L} \otimes \mathcal{F})$$

which makes $\mathcal{M}_d^T(Q')$ into a module over the algebra object $BC_u^*$ in stacks. We obtain an induced morphism in vanishing cycle cohomology

$$A^*: H(\mathcal{M}_d^T(Q'), p^* \mathcal{P}_{\mathcal{M}(W)}) \to H(BC_u^*, \mathcal{M}_d^T(Q'), p^* \mathcal{P}_{\mathcal{M}(W)})$$

making $A_{Q', W, d}$ into a comodule for the coalgebra $H_{C^*}$. Via our fixed identification $H_{C^*} = Q[u]$, for $\alpha \in A_{Q', W}$ we define coefficients $\alpha_n$ via the expansion

$$A^*(\alpha) = \sum_{n \geq 0} u^n \otimes \alpha_n.$$ (38)

Proposition 4.2. The morphism

$$\partial_u: A_{Q', W}^T \to A_{Q', W}^T$$

$$\alpha \mapsto \alpha_1$$

defines a derivation of $A_{Q', W}^T$, where $\alpha_1$ is as defined in 3.5.

Proof. We abbreviate $\mathcal{M}_d^T = \mathcal{M}_d^T(Q')$ etc. to remove clutter. Consider the commutative diagram

$$
\begin{array}{c}
\mathcal{M}_d^T(Q') \times_{BT} \mathcal{M}_{d''}^T(Q') \xrightarrow{\text{Det}(1)} \mathcal{M}_d^T(Q') \xrightarrow{\otimes} \mathcal{M}_d^T(Q') \\
\mathcal{M}_d^T(Q') \xrightarrow{\text{Det}(1)} \mathcal{M}_d^T(Q') \xrightarrow{\otimes} \mathcal{M}_d^T(Q') \\
BC_u \times BC_u \xrightarrow{\otimes} BC_u.
\end{array}
$$

□
with morphisms of points over a scheme $X$ defined by
\[ a: (\mathcal{L}, (E \xrightarrow{d} X, F', F'')) \mapsto (E \to X, p^* \mathcal{L} \otimes F', p^* \mathcal{L} \otimes F''), \]
and so since $\Delta \star$ from which we deduce that
\[ \text{Proposition 4.5.} \quad \text{Heis}_{[Q^*]} \text{according to central charge of the} \]
\[ \text{Proposition 4.3.} \quad \text{For each} \ d \in \mathbb{N}^{Q_0} \ \text{there is an action of} \ \text{Heis}_{Q'} \ \text{on} \ \mathcal{A}_{Q',W,d}^T \ \text{defined by letting} \ p_i \ \text{act by} \ u \bullet, \ \text{and letting} \ \partial_{Q_0} \ \text{act by} \ \partial_u. \ \text{This action has central charge} \ d. \]
\[ \text{Proof.} \quad \text{Consider the composition defined on} \ X\text{-points by} \]
\[ BC_u \times \mathfrak{M}_d(Q') \xrightarrow{(\mathcal{L}, (E \xrightarrow{d} X, F)) \mapsto (E \xrightarrow{d} X, p^* \mathcal{L} \otimes F).} \]
\[ \text{The induced endomorphism} \ F = (l_i \circ A)^* \ \text{of} \ \text{Heis}_{[Q^*]} \ \text{satisfies} \]
\[ F(u \otimes \alpha) = (d_i (u \otimes 1)) \cdot A^*(\alpha) \]
\[ \partial_{u \bullet} = (u_{(i)} \cdot \alpha_1) = d_i \alpha_0 + u_{(i)} \cdot \alpha_1 = d_i \alpha + u \bullet (\partial_u \alpha) \]
\[ \text{as required.} \]
\[ \text{Combining Propositions 4.1, 4.2, and 4.3 yields} \]
\[ \text{Theorem 4.4.} \quad \text{The algebra} \ \mathcal{A}_{Q',W,d}^T \ \text{is an algebra object in the category} \ \text{Heis}_{Q'}\text{-mod}, \]
\[ \text{for which the underlying} \ \text{Heis}_{Q'}\text{-module is integrable. The decomposition} \]
\[ \mathcal{A}_{Q',W,d}^T \cong \bigoplus_{d \in \mathbb{N}^{Q_0}} (\mathcal{A}_{Q',W,d}^T)[ad] \]
\[ \text{according to central charge of the} \ \text{Heis}_{Q'}\text{-action is given by the standard decomposition} \]
\[ \text{Proposition 4.5.} \quad \text{Let} \ Q' \ \text{be an arbitrary quiver with potential} \ W \in \mathbb{C}Q'/[\mathbb{C}Q', \mathbb{C}Q']_{\text{vect}}. \]
\[ \text{The inclusion} \ \text{Heis}_{Q',W} \hookrightarrow \mathcal{A}_{Q',W}^T \ \text{is an inclusion of graded} \ \text{Heis-modules}. \]
Proof. Since the image of $\mathfrak{g}_{Q',\mathbb{W}}^T \otimes \mathbb{C}_z$ is just the closure of the image of $\mathfrak{g}_{Q',\mathbb{W}}^T \subset A_{Q',\mathbb{W}}^T$ under the raising operator $\omega$, the proposition follows from the claim that the lowering operator $\partial_u$ annihilates $\mathfrak{g}_{Q',\mathbb{W}}^T$. Consider the following diagram

$$
\begin{array}{ccc}
BC_u^* \times \mathcal{M}_d^T(Q') & \xrightarrow{A} & \mathcal{M}_d^T(Q') \\
\downarrow & & \downarrow \mathbb{H}^T \\
\mathcal{M}^T_d(Q'). & & 
\end{array}
$$

We claim that the diagram commutes: this is easy to see at the level of geometric points, from which the claim follows. Thus there is a natural isomorphism

$$
\mathbb{H}^T_\ast A_\ast (Q \otimes \mathbb{F}_{\mathbb{T}(\mathbb{W})} Q^{\text{vir}}) \cong H(BC_u^*, \mathbb{Q}) \otimes \mathbb{H}^T_\ast \mathbb{F}_{\mathbb{T}(\mathbb{W})} Q^{\text{vir}}
$$

and so passing to the first perverse truncation and using the perverse vanishing result \[\text{(39)}\]

$$
\mathfrak{f}_T^{\leq 1} \mathbb{H}^T_\ast A_\ast (Q \otimes \mathbb{F}_{\mathbb{T}(\mathbb{W})} Q^{\text{vir}}) \cong \mathfrak{f}_T^{\leq 1} \mathbb{H}^T_\ast \mathbb{F}_{\mathbb{T}(\mathbb{W})} Q^{\text{vir}}.
$$

It follows that the morphism

$$
\mathfrak{f}_T^{\leq 1} \mathbb{H}^T_\ast (\mathbb{F}_{\mathbb{T}(\mathbb{W})} Q^{\text{vir}}) \rightarrow A_\ast (Q_{BC_u^*} \otimes \mathbb{F}_{\mathbb{T}(\mathbb{W})} Q^{\text{vir}})
$$

factors through the morphism

$$
\mathfrak{f}_T(\mathbb{B}_u^* \otimes \mathbb{F}_{\mathbb{T}(\mathbb{W})} Q^{\text{vir}}) \rightarrow H(\mathbb{B}_u^*, \mathbb{Q}) \otimes \mathfrak{f}_T^{\leq 1} \mathbb{H}^T_\ast \mathbb{F}_{\mathbb{T}(\mathbb{W})} Q^{\text{vir}}
$$

and $\mathfrak{f}_T(\mathbb{Q}^\ast) = 1 \otimes \mathfrak{f}_T(\mathbb{Q}^\ast)$ for $\mathfrak{f}_T(\mathbb{Q}^\ast)$, as such, $\mathfrak{f}_T(\mathbb{Q}^\ast) = 0$ as required.

5. PROOFS OF MAIN RESULTS

Let $Q = Q^{(1)}$ be the Jordan quiver, with one vertex and one loop, so $\tilde{Q} = Q^{(3)}$ is the quiver with three loops. Then for every $d \geq 1$ there are $q$ absolutely indecomposable $d$-dimensional $\mathbb{F}_q Q$-modules, i.e. $\mathfrak{a}_{Q,d}(q) = q$, and so

$$
\mathfrak{g}_{Q,\mathbb{W},d} \cong \mathbb{Q}[2]
$$

by \[\text{(39)}\]. We define

$$
\mathfrak{g}_{\mathbb{A}_2} := \mathfrak{g}_{Q^{(3)},\mathbb{W}}, \quad \mathfrak{g}_{\mathbb{A}_2}^T := \mathfrak{g}_{Q^{(3)},\mathbb{W}}^T
$$

the deformed and the deformed affinized BPS Lie algebras for the pair $(Q^{(3)}, \mathbb{W})$, respectively.

Proposition 5.1. Let the torus $T = \text{Hom}_{\text{Grp}}(\mathbb{Z}, (\mathbb{C}^\ast)^r)$ act on the quiver $Q^{(3)}$, with $\mathbb{W}$ $T$-invariant. Then the Lie bracket on $\mathfrak{g}_{\mathbb{A}_2}$ vanishes.

Proof. By \[\text{(59)}\] and Theorem \[\text{[33]}\] there is an isomorphism of cohomologically graded vector spaces

$$
\mathfrak{g}_{\mathbb{A}_2,d} \cong \mathbb{H}_T[2]
$$

for every $d \geq 1$. Let $\alpha_{d'}^{(0)} \in \mathfrak{g}_{\mathbb{A}_2,d'}$ and $\alpha_{d''}^{(0)} \in \mathfrak{g}_{\mathbb{A}_2,d''}$ be the unique (up to scalar) nonzero elements of cohomological degree $-2$. Then

$$
[\alpha_{d'}^{(0)}, \alpha_{d''}^{(0)}] \in \mathfrak{g}_{\mathbb{A}_2,d'+d''}^T
$$

has cohomological degree $-4$, and so is zero. For arbitrary $p(t), q(t) \in \mathbb{H}_T$ it follows that

$$
[p(t) \cdot \alpha_{d'}^{(0)}, q(t) \cdot \alpha_{d''}^{(0)}] = p(t)q(t) \cdot [\alpha_{d'}^{(0)}, \alpha_{d''}^{(0)}] = 0.
$$

\[\square\]
By (39) we have the equality of generating series
\[ \chi(\hat{\mathcal{g}}_{\mathbb{A}^2}) := \sum_{i,j \in \mathbb{Z}} \dim(\hat{\mathcal{g}}^i_{\mathbb{A}^2}) v^i q^j = q^{-2} v (1 - v)^{-1} (1 - q^2)^{-1}. \] (40)
From the isomorphism \( \hat{\mathcal{g}}^m_{\mathbb{A}^2} \cong \hat{\mathcal{g}}_{\mathbb{A}^2} \otimes \mathbb{H} \) (Proposition 3.12) we deduce the equality
\[ \chi(\hat{\mathcal{g}}^m_{\mathbb{A}^2}) = q^{-2} v (1 - v)^{-1} (1 - q^2)^{-2}. \] (41)

We denote by \( \alpha_{d}^{(0)} \) a basis element of \( \mathcal{g}^m_{\mathbb{A}^2,d} \), and set \( \alpha_{d}^{(m)} := u^m \cdot \alpha_{d}^{(0)} \). Then the elements \( \alpha_{d}^{(m)} \) with \( d \geq 1 \) and \( m \geq 0 \) form a basis for \( \hat{\mathcal{g}}_{\mathbb{A}^2} \).

**Proposition 5.2.** For every \( d \geq 1 \) we have the inequality
\[ [\alpha_{1}^{(1)}, \alpha_{d}^{(0)}] = \lambda_{d} \alpha_{d+1}^{(0)} \] (42)
for some nonzero \( \lambda_{d} \in \mathbb{Q} \).

**Proof.** We consider the \( \mathcal{A}_{\mathbb{A}^2} \)-action on \( M_{1,d}(\mathbb{Q}) \cong \bigoplus_{d \geq 2} \mathcal{H}(\text{Hilb}_{d}(\mathbb{C}^2), \mathbb{Q}) \), recalled in (3.12). We denote by \( p : \hat{\mathcal{g}}_{\mathbb{A}^2} \rightarrow \text{End}(\text{Hilb}(\mathbb{A}^2)) \) the action restricted to \( \hat{\mathcal{g}}_{\mathbb{A}^2} \). Then \( \rho(\alpha_{0}^{(0)}) = p_{1} \) is the action of the Nakajima raising operator and from
\[ u \cdot (\alpha \ast m) = (u \ast \alpha) \ast m + \alpha \ast (u \ast m) \]
we deduce that
\[ \rho(\alpha_{1}^{(1)}) = [u \ast \rho(\alpha_{1}^{(0)}), p_{1}] \neq 0. \] (43)
From Theorem 3.16 we deduce inductively that
\[ \rho \left( \text{ad}_{\alpha_{1}^{(1)}}(\alpha_{d}^{(0)}) \right) \neq 0. \] (44)
Set \( \alpha = \text{ad}_{\alpha_{1}^{(1)}}(\alpha_{d}^{(0)}) \in \hat{\mathcal{g}}_{\mathbb{A}^2,d+1} \). By (43), \( \alpha \neq 0 \). On the other hand, \( \alpha_{d+1}^{(0)} \) is the unique nonzero element of \( \mathcal{g}^m_{\mathbb{A}^2,d+1} \) of cohomological degree \(-2\), which is the cohomological degree of \( \alpha \), and the result follows by induction. \( \square \)

**Corollary 5.3.** The affinized BPS Lie algebra \( \hat{\mathcal{g}}_{\mathbb{A}^2} \) is generated by \( \alpha_{1}^{(m)} \) for \( m \geq 0 \).

**Proof.** We assume, for an inductive argument, that the Lie subalgebra \( \mathcal{g}' \subset \hat{\mathcal{g}}_{\mathbb{A}^2} \) generated by \( \alpha_{i}^{(m)} \) for \( m \geq 0 \) contains all of the elements \( \alpha_{i}^{(m)} \) for \( i' < i \) and \( m \in \mathbb{Z}_{\geq 0} \). From Proposition 5.2 we deduce that, possibly after replacing \( \alpha_{d}^{(0)} \) by itself multiplied by some \( \lambda \in \mathbb{Q} \setminus \{0\} \), we have the equality
\[ [\alpha_{1}^{(1)}, \alpha_{d}^{(0)}] = \alpha_{d+1}^{(0)}. \]
Acting by \( u^m \ast \), we find
\[ \sum_{0 \leq r \leq m} \binom{m}{r} [\alpha_{1}^{(1+r)}, \alpha_{d}^{(m-r)}] = \alpha_{d+1}^{(m)}. \]
\( \square \)

**Theorem 5.4.** There is an isomorphism \( F : \hat{\mathcal{g}}_{\mathbb{A}^2} \cong \mathcal{G}_{\mathbb{F}^{1}_{\infty}} \).

**Proof.** This is a corollary of (40), Theorem 4.14, Corollary 5.3, which states that \( \hat{\mathcal{g}}_{\mathbb{A}^2} \) is generated by \( \hat{\mathcal{g}}_{\mathbb{A}^2,1} = \mathcal{G}_{\mathbb{F}^{1}_{1}} \), and Corollary 2.5. \( \square \)

**Corollary 5.5.** There is an isomorphism of algebras
\[ \mathcal{A}_{\mathbb{A}^2} \cong \mathcal{U}(\mathcal{G}_{\mathbb{F}^{1}_{\infty}}). \]
In particular, \( \mathcal{A}_{\mathbb{A}^2} \) is spherically generated (i.e. it is generated by \( \mathcal{A}_{\mathbb{A}^2,1} \)).

The second statement in the corollary follows from Lemma 2.2.
Now we reintroduce the torus $T = (\mathbb{C}^*)^2$, acting by independently rescaling the two coordinates of $\mathbb{A}^2$ (i.e. via the weighting $\tau$ of (27)), and the fully equivariant CoHA $\mathcal{A}_{\mathbb{A}^2}^T$.

**Theorem 5.6.** If $T'$ is a torus acting on $Q^{(3)}$, leaving $\bar{W}$ invariant, then $\mathcal{A}_{Q^{(3)},\bar{W}}^T$ is spherically generated. In particular, the algebra $\mathcal{A}_{\mathbb{A}^2}^T$ is spherically generated.

**Proof.** For $d \geq 1$ we define a morphism of $\mathbf{H}_{T'}$-modules
\[
\Psi_d : \mathbb{Q}[u] \otimes \mathbf{H}_{T'} \to \mathcal{A}_{\mathbb{A}^2}^T
\]
\[
u^n \otimes t^n \mapsto t^n \left( [\bar{\alpha}^{(1)}]_t, \ldots, [\bar{\alpha}^{(m)}]_t \right)
\]
and define
\[
\Psi = \bigoplus_{d \geq 1} \Psi_d : \bigoplus_{d \geq 1} \mathbb{Q}[u] \otimes \mathbf{H}_{T'} \to \mathcal{A}_{\mathbb{A}^2}^T
\]
Finally we define
\[
\Phi : \text{Sym}_{\mathbf{H}_{T'}} \left( \bigoplus_{d \geq 1} \mathbb{Q}[u] \otimes \mathbf{H}_{T'} \right) \to \mathcal{A}_{\mathbb{A}^2}^T
\]
via $\Psi$ and the CoHA product on the target. This is a morphism of free $\mathbf{H}_{T'}$-modules that (via Lemma 2.3 Corollary 5.5 and Proposition 3.5) becomes an isomorphism after applying $\otimes_{\mathbf{H}_{T'},\mathfrak{m}_{T'}}$. It follows that $\Phi$ is an isomorphism, and in particular, the algebra $\mathcal{A}_{\mathbb{A}^2}^T$ is spherically generated. \qed

There is a partial converse to Theorem 5.6 provided by the next two propositions:

**Proposition 5.7.** Let $Q$ be a quiver without loops that is not an orientation of a finite type ADE Dynkin quiver, let $T$ act on $Q$, leaving $\bar{W}$ invariant. Then $\mathcal{A}_{Q^{(3)},\bar{W}}^T$ is not spherically generated, i.e. it is not generated by the subspaces $\mathcal{A}_{Q^{(3)},\bar{W},1}^T$ for $i \in Q_0$.

**Proof.** This follows for cohomological degree reasons. The conditions on $Q$ imply that $\mathfrak{a}_{Q,d}(q)$ is not a constant for some $d \in \mathbb{N}Q_0$. From (26) and (23) we deduce that $\mathcal{A}_{Q^{(3)},\bar{W}}^T$ has a summand in strictly negative cohomological degree. On the other hand
\[
\mathcal{A}_{Q^{(3)},\bar{W},1}^T \cong \mathbf{H}_T
\]
as a cohomologically graded vector space, so the collection of such spaces for $i \in Q_0$ generates an algebra lying in positive cohomological degrees. \qed

The next proposition is in the same vein. Before we state it, we introduce some notation. We label the three loops of $Q^{(3)}$ by the symbols $x, y, z$. We let $\mathfrak{M}^{\mathbb{S}N}(Q^{(3)}) \subset \mathfrak{M}(Q^{(3)})$ be the reduced closed substack, the points of which correspond to representations for which $z$ acts via a nilpotent operator, and let $\mathfrak{M}^{\mathbb{N}}(Q^{(3)}) \subset \mathfrak{M}(Q^{(3)})$ be the reduced closed substack, the points of which correspond to representations for which both $y$ and $z$ act via nilpotent operators. Then
\[
\mathcal{A}_{Q^{(3)},\bar{W}}^\triangledown := \bigoplus_d \mathbf{H} \left( \mathfrak{M}_d^{\mathbb{S}N}(Q^{(3)}), \{ \mathfrak{H}_{\mathbf{H}}(\bar{W}) \mathfrak{Q}_{\mathfrak{M}_d^{\mathbb{N}N}(Q^{(3)})} \} \right)
\]
carry Hall algebra structures defined as in §3.4 for $\triangledown = \mathbb{S}N, \mathbb{N}$. If the torus $T' \cong (\mathbb{C}^*)^n$ acts on $\mathbb{A}^3$, preserving the 3-form $dx \wedge dy \wedge dz$ then there is again
an induced action on $A_{A^2}^{T,\triangledown} := A_{Q^{(3)},\bar{W}}^{T,\triangledown}$. By the PBW theorem, there exist Lie subalgebras $\mathfrak{g}_{A^2}^{T,\triangledown} \subset A_{A^2}^{T,\triangledown}$ such that the induced morphism

$$\text{Sym}_{H^*} \left( \mathfrak{g}_{A^2}^{T,\triangledown} \otimes H_{C^*} \right) \rightarrow A_{A^2}^{T,\triangledown}$$

is an isomorphism. The generating functions of these Lie algebras are given by

$$\chi(A_{Q^{(3)},\bar{W}}^{T,\triangledown}) = vq f(\triangledown)(1 - v)^{-1}(1 - q^2)^{-1-n}$$

where $f(SN) = 0$ and $f(N) = 2$; see [BSV17, Dav17b].

**Proposition 5.8.** The algebras $A_{A^2}^{T,\triangledown}$ for $\triangledown = SN, N$ are not spherically generated.

**Proof.** Both results follow by dimension counting. Firstly, both algebras are concentrated entirely in non-negative cohomological degrees. Then, (44) and (45) imply that the subspace $V \subset A_{A^2,SN}^{T,\triangledown}$ in cohomological degree zero is 2 dimensional, while the cohomological degree zero piece of $U \subset A_{A^2,1}^{T,SN}$ is 1-dimensional, so cannot generate $V$. The argument for $A_{A^2,N}^{T,\triangledown}$ is similar. 

It would be interesting to have some neat description of the images of any one of the injections $A_{A^2}^{T,\triangledown} \hookrightarrow \bigoplus_{d \geq 1} Q[t_1, t_2] \otimes Q[x_1, \ldots, x_d]^{E_8}$, for $\triangledown = 0, SN, N$ to compare with the main result of [Neg22] in K-theory for the case $\triangledown = SN$ — we leave this to future work.

Finally we can completely describe the fully equivariant CoHA:

**Theorem 5.9.** There is an isomorphism of algebras $Y_{t_1, t_2, t_3}(\mathfrak{gl}(1))^{+} \rightarrow A_{A^2}^{T,\triangledown}$ sending $e_i$ to $\tilde{\alpha}_1^{(i)}$.

**Proof.** This is an immediate corollary of Theorems 3.18 and 5.6. 

We can now give a more complete statement regarding the relation between BPS Lie algebras and $W_{1+\infty}^+$. 

**Corollary 5.10.** Let $C^*$ act with weights $(1,0)$ on the two coordinates of $A^2$. Then there is an $H_{C^*}$-linear isomorphism

$$F: \mathfrak{g}_{A^2}^{C^*} \cong R_F[W_{1+\infty}^{+}]$$

between the deformed affinized BPS Lie algebra for the pair $(Q^{(3)}, \bar{W})$ and the Rees Lie algebra of $W_{1+\infty}^+$, uniquely determined by setting $F(\tilde{\alpha}_1^{(m)}) = z^m$. Thus, there is an isomorphism of algebras

$$A_{A^2}^{C^*} \cong U_{Q[t]}(R_F[W_{1+\infty}^+]).$$

**Proof.** Let $T = \text{Hom}_{\mathbb{Z}}(\mathbb{Z}, (C^*)^2)$ act on $\bar{Q}$ via the weighting $\bar{\tau}: \bar{Q} \rightarrow \mathbb{Z}^2$. By Proposition 5.9 there is an isomorphism of algebras

$$A_{A^2}^{C^*} \cong A_{A^2}^{T} \otimes_{H^*} \bar{Q}[t_1, t_2]/(t_2).$$

By Theorem 5.9 and (46) there is an isomorphism of algebras

$$F: (Y_{t_1, t_2, t_3}(\mathfrak{gl}(1))^{+})_{t_2=0} \cong A_{A^2}^{C^*}$$

setting $t_1 = t$ and sending $e_i$ to $\tilde{\alpha}_1^{(i)}$. By Proposition 5.11 there is an isomorphism of algebras

$$G: A_{A^2}^{C^*} \cong U_{H_{C^*}} \left( \mathfrak{g}_{A^2}^{C^*} \right).$$

Composing $F$ and $G$, we obtain an isomorphism

$$U_{H_{C^*}} \left( \mathfrak{g}_{A^2}^{C^*} \right) \rightarrow (Y_{t_1, t_2, t_3}(\mathfrak{gl}(1))^{+})_{t_2=0}.$$
sending $\tilde{\alpha}_i^{(i)}$ to $e_i$. Note that, after setting $t_2 = 0$, we have $\sigma_3 = 0$, and both the identities (44) and (45) become Lie algebra identities. It is easy to check that the identities

$$[z\tilde{D}^{(i+3)}, z\tilde{D}^{(j)}] - 3[z\tilde{D}^{(i+2)}, z\tilde{D}^{(j+1)}] + 3[z\tilde{D}^{(i+1)}, z\tilde{D}^{(j+2)}] - [z\tilde{D}^{(i)}, z\tilde{D}^{(j+3)}]$$

(48)

$$+ t^2([z\tilde{D}^{(i+1)}, z\tilde{D}^{(j)}] - [z\tilde{D}^{(i)}, z\tilde{D}^{(j+1)}]) = 0$$

(49)

hold in $R[W^+_{1+\infty}]$, so that there is a well-defined morphism of Lie algebras

$$F: \hat{\mathfrak{g}}_{K_2}^+ \to R[W^+_{1+\infty}]$$

(50)

and induced morphism of algebras

$$U_{Q[t]}(\hat{\mathfrak{g}}_{K_2}^+) \cong A_{K_2}^+ \to U_{Q[t]}(R[W^+_{1+\infty}])$$

(51)

both sending $\tilde{\alpha}_i^{(i)}$ to $z\tilde{D}^{(i)}$. The targets of both morphisms (50) and (51) are spherically generated by Lemma 2.2 and so both morphisms are surjective. On the other hand, the domain and target have the same generating series by (41) and we are done. □

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