Universal 2-3 Symmetry

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ABSTRACT

Possible maximal mixing seen in the oscillations of the atmospheric neutrinos has led to postulate of a $\mu$-$\tau$ symmetry which interchanges $\nu_\mu$ and $\nu_\tau$. We argue that such symmetry need not be special to neutrinos but can be extended to all fermions. The assumption that all fermion mass matrices are approximately invariant under interchange of the second and the third generation fields is shown to be phenomenologically viable and has interesting consequences. In the quark sector, the smallness of $V_{ub}$ and $V_{cb}$ can be a consequences of this approximate 2-3 symmetry. The same approximate symmetry can simultaneously lead to large atmospheric mixing angle and can describe the leptonic mixing quite well provided the neutrino spectrum is quasi degenerate. We present this scenario, elaborate on its consequences and discuss its realization.

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The vastly different mixing patterns [1] of quarks and leptons have been used as an argument in favour of special leptonic symmetries such as $\mu-\tau$ interchange [2, 3, 4], $L_e - L_\mu - L_\tau$ [5], $D_4$ [6], $A_4$ [7] etc.. These symmetries lead to large or maximal mixing angles seen in the leptonic sector. Logically, such symmetries would then not be present in the quark sector which exhibits small mixing angles. This need not be so and it is possible to describe both the quark and leptonic mixing as a consequence of an approximately broken 2-3 symmetry which exchanges the second and the third generation fermionic fields. We argue that this symmetry manifests itself more forcefully in the quark sector than in the leptonic sector and results in the understanding of small values of $V_{ub}$ and $V_{cb}$ when it is broken at few percent level. The leptonic mass matrices can also be thought to be nearly invariant under the 2-3 symmetry if the neutrino mass spectrum is quasi degenerate.

Let us first elaborate on the well-known [2, 3] consequences of the $\mu-\tau$ symmetry. The light neutrino mass matrix $M_\nu$ is restricted to have the following form in the presence of this symmetry:

$$M_\nu = \begin{pmatrix} X_{\nu} & A_{\nu} & A_{\nu} \\ A_{\nu} & B_{\nu} & C_{\nu} \\ A_{\nu} & C_{\nu} & B_{\nu} \end{pmatrix}.$$  \hspace{1cm} (1)

This form leads to a maximal atmospheric mixing and zero $U_e3$ if it is assumed to be true in the flavour basis. In the same basis, the charged lepton mass matrix is diagonal and consequently, it is not invariant under the $\mu-\tau$ symmetry which would have implied $m_\mu = m_\tau$. It is possible to imagine a larger symmetry (e.g. $D_4$ [6]) which when broken leads to the above form for $M_\nu$ in the flavour basis. In this case, the $\mu-\tau$ symmetry appears to be only an effective neutrino symmetry.

It is important to stress that the $\mu-\tau$ symmetry by itself does not force equality of the muon and tau masses. To see this, let us simultaneously assume that both the charged lepton mass matrix $M_l$ and $M_\nu$ are $\mu-\tau$ symmetric and have the form\(^1\) given in eq. (1). In this case, the muon and tau masses are different but now the 23 mixing angle for the charged leptons is also maximal. As a consequence, the neutrino and the charged lepton mixing angles cancel and one

\(^1\)The 2-3 symmetry does not automatically imply the form given in eq. (1) for $M_l$ unless it is assumed to be symmetric. This assumption can easily be realized in the context of GUT such as $SO(10)$ which commutes with the 2-3 symmetry.
gets vanishing atmospheric mixing angle. In either case, the \( \mu - \tau \) symmetry does not appear to be an exact symmetry in the leptonic world.

In contrast to leptons, the 23 and the 13 mixing angles are indeed small for quarks. This suggests that a generalized \( \mu - \tau \) symmetry may be a good symmetry for quarks rather than for leptons. Let us then postulate that the quark mass matrices are symmetric and display an approximate 2-3 symmetry. Latter on we will show that this assumption can be extended to the leptonic masses as well. An approximate 2-3 symmetry dictates the following form for a symmetric fermion mass matrix \( M_f \):

\[
M_f = \begin{pmatrix}
X_f & A_f(1 - \epsilon_{1f}) & A_f(1 + \epsilon_{1f}) \\
A_f(1 - \epsilon_{1f}) & B_f(1 - \epsilon_{2f}) & C_f \\
A_f(1 + \epsilon_{1f}) & C_f & B_f(1 + \epsilon_{2f})
\end{pmatrix}.
\] (2)

The dimensionless parameters \( \epsilon_{1f,2f} \) break the 2-3 symmetry and are assumed to be \( \ll 1 \).

These two parameters are sufficient to describe the most general 2-3 breaking \[3\] when fermion mass matrices are symmetric.

Let us first consider the symmetric limit assuming all parameters in eq. (2) to be real. All the eigenvalues of \( M_f \) are distinct and are given by

\[
m_{1f} = \frac{1}{2} \left[ B_f + C_f + X_f - \left( (B_f + C_f - X_f)^2 + 8A_f^2 \right)^{1/2} \right],
\]

\[
m_{2f} = \frac{1}{2} \left[ B_f + C_f + X_f + \left( (B_f + C_f - X_f)^2 + 8A_f^2 \right)^{1/2} \right],
\]

\[
m_{3f} = B_f - C_f.
\] (3)

We will assume the hierarchy \( |m_{1f}| < |m_{2f}| < |m_{3f}| \) and associate the fermionic states accordingly to these eigenvalues. The \( M_f \) can be diagonalized by a matrix \( V_f^0 \):

\[
V_f^0 = R_{23}(\pi/4)R_{12}(\theta_{12f}).
\] (4)

As a result, one gets in the symmetric limit,

\[
V_{CKM}^0 = V_u^{0\dagger}V_d^0 = R_{12}(\theta_c),
\] (5)

with

\[
\theta_c \approx \theta_{12d} - \theta_{12u}.
\]
It follows from eq. (5) that the 2-3 symmetry automatically leads to vanishing $V_{cb}$ and $V_{ub}$. This remains true even if $M_f$ is complex. The Cabibbo angle and the quark masses are not restricted by this symmetry. The Cabibbo angle can be constrained by imposing an additional discrete symmetry $D$ defined as:

$$ f_{1L} \to if_{1L} \quad f_{1R} \to -if_{1R}. $$

This symmetry forces $A_f$ and $X_f$ in eq. (2) to be zero. The $A_f$ term breaks this symmetry by one and $X_f$ by two units (of $i$). $B_f$ and $C_f$ are invariant. It is thus natural to assume that $D$-breaking (by some flavon field) can lead to a hierarchy $|B_f, C_f| >> |A_f| >> |X_f|$. This hierarchy leads to $A_f \sim O(\sqrt{m_1 m_2})$ and the celebrated relation

$$ \theta_c \sim \sqrt{\frac{m_d}{m_u}} - \sqrt{\frac{m_u}{m_c}}. $$

More precisely, one needs,

$$ |X_f| \ll |\sqrt{2}A_f| \ll |B_f + C_f| \ll |B_f - C_f|, $$

for $f = u, d$ in order to get eq. (7) and the hierarchical masses. It follows that an approximately broken $D$ and an exact 2-3 symmetry leads to eq. (7) and vanishing $V_{ub}, V_{cb}$. Subsequent breaking of the 2-3 symmetry can then induce the latter quantities.

While both $\epsilon_{1f}$ and $\epsilon_{2f}$ could be present in a model, we consider here one parameter breaking for all $M_f$ and assume that only $\epsilon_{2f}$ is non-zero. It is straightforward to add the effect of $\epsilon_{1f}$. We will also take all parameters to be real.

The non-zero $\epsilon_{2u}, \epsilon_{2d}$ are sufficient to generate the required values of $V_{ub}$ and $V_{cb}$. The $M_f$ can be diagonalized in the limit specified in eq. (8) as follows

$$ V_f^T M_f V_f = \text{Diag.}(m_{1f}, m_{2f}, m_{3f}), $$

with

$$ m_{3f} \approx B_f - C_f(1 + \frac{1}{2}\theta_{23f}^2), $$

$$ m_{2f} \approx B_f + C_f(1 + \frac{1}{2}\theta_{23f}^2) + \frac{2A_f^2}{m_{2f}}, $$

$$ m_{1f} \approx -\frac{2A_f^2}{m_{2f}}. $$

(9)
where $f = u, d$. The mixing matrix is given as

$$V_f = R_{23}(\pi/4) R_{23}(\theta_{23f}) R_{13}(\theta_{13f}) R_{12}(\theta_{12f}),$$

with

$$\begin{align*}
\theta_{23f} & \approx \frac{\epsilon_{2f} B_f}{2 C_f} \approx -\frac{\epsilon_{2f}}{2}, \\
\theta_{12f} & \approx \sqrt{-\frac{m_{1f}}{m_{2f}}}, \\
\theta_{13f} & \approx \frac{m_{2f}}{m_{3f}} \theta_{12f} \theta_{23f}.
\end{align*}$$

This leads to

$$
\begin{align*}
V_{cb} & \approx \theta_{23d} - \theta_{23u}, \\
V_{ub} & \approx \theta_{13d} - \theta_{13u} + \theta_{12u}(\theta_{23d} - \theta_{23u}) \sim \theta_{12u} V_{cb}
\end{align*}
$$

and eq. (7) for $V_{us}$. Keeping a grand unified picture in mind, we assume that the $M_f$ in eq. (2) is defined at $M_{GUT} \sim 10^{16}$ GeV and require it to reproduce the parameters in quark sector at that scale. For definiteness, we choose the MSSM and quark masses corresponding to $\tan \beta = 10$ given in [8].

It follows from eq. (12) that a few percent breaking of the 2-3 symmetry can reproduce the observed mixing quite well for several choices of parameters in $M_f$. For illustration, we give one specific choice which is a typical phenomenologically consistent example:

$$\epsilon_{2u} = -\epsilon_{2d} \sim 0.045,$$

$$
M_d = \begin{pmatrix}
-0.003 & 0.0054 & 0.0054 \\
0.0054 & 0.49 & -0.54 \\
0.0054 & -0.54 & 0.54
\end{pmatrix};
M_u = \begin{pmatrix}
0 & 0.0084 & 0.0084 \\
0.0084 & 42.74 & -41.06 \\
0.0084 & -41.06 & 39.055
\end{pmatrix}.
$$

These mass matrices lead to the mixing angles $|V_{us}| \approx 0.221$, $|V_{cb}| \approx 0.044$ and $|V_{ub}| \approx 0.0026$. These values are in approximate agreement with the high scale estimates $|V_{us}| \sim 0.223 - 0.226$, $|V_{cb}| \sim 0.029 - 0.038$ and $V_{ub} \sim 0.0024 - 0.0038$ as given for example in Matsuda and Nishiura, ref. [4]. This agreement can be improved by switching on small $\epsilon_1$. The approximate 2-3 symmetry of quark mass matrices is apparent in eq. (13).
Let us now turn to the leptonic sector. Our discussion of the quark sector shows that the smallness of $V_{cb}$ is a natural consequence of the 2-3 symmetry. The corresponding mixing angle for leptons is known to be almost maximal. This has led to a view [2, 3] that the $\mu$-$\tau$ symmetry is an effective symmetry of neutrino mass matrix badly broken in the charged lepton sector. This need not always be the case as we argue now.

We assume that just as in case of the quarks, both $M_l$ and $M_\nu$ are having approximate 2-3 symmetric forms given in eq. (2). The $M_\nu$ here refers to the effective mass matrix of the light neutrinos. It can originate from an approximate 2-3 symmetric couplings to a triplet Higgs or may originate from the ordinary seesaw mechanism in which the Dirac neutrino mass matrix $m_D$ and the right handed neutrino mass matrix $M_R$ are approximately 2-3 symmetric with the form given in eq. (2).

Assume that the $A_{\nu, l}$ are small parameters as in case of the quarks and concentrate first on the lower $2 \times 2$ block of eq. (2). Its diagonalization gives

$$\epsilon_{2f} = \left( \frac{m_{2f} - m_{3f}}{m_{2f} + m_{3f}} \right) \cos 2\tilde{\theta}_{23f}.$$  

(14)

$f = l, \nu$ above and $\tan 2\tilde{\theta}_{23f} \equiv \frac{C_l}{\epsilon_{2f}B_l}$ correspond to the 23 mixing angle for $f$. This equation gives a clue to obtaining approximate 23 symmetry simultaneously for $M_l$ and $M_\nu$ as well as large atmospheric mixing angle. The approximate 2-3 symmetry requires $\epsilon_{2\nu}, \epsilon_{2l} \ll 1$. For the charged leptons, small $\epsilon_{2l}$ necessarily means $\tilde{\theta}_{23l} \sim \frac{\pi}{4}$ in eq. (14) since $m_\mu$ substantially differs from $m_\tau$. In contrast, for neutrinos small $\epsilon_{2\nu}$ can be realized either with a large $\tilde{\theta}_{23\nu}$ or with $m_{2\nu} \sim m_{3\nu}$. The latter case will correspond to a large atmospheric mixing angle. It follows that in case of the quasi degeneracy, there exists ranges in parameters corresponding to approximately 23 symmetric $M_l$ and $M_\nu$ and large atmospheric mixing arising due to a small $\tilde{\theta}_{23\nu}$ and almost maximal $\tilde{\theta}_{23l}$. All three neutrinos are required to be quasi degenerate in order to obtain simultaneous explanation of the solar and atmospheric neutrino scales. In particular, the $m_{\nu_2}$ and $m_{\nu_3}$ would need to have the same sign to make $\epsilon_{2\nu}$ small.

The 2-3 symmetry can be exact in $M_l$ while it needs to be broken by $M_\nu$. The amount of the required breaking is quantified using eq. (14):

$$|\epsilon_{2\nu}| \approx \left| \frac{m_{\nu_2} - m_{\nu_3}}{m_{\nu_2} + m_{\nu_3}} \right| \approx \left| \frac{\Delta_A}{4m_0^2} \right| \sim 0.08$$

(15)
for the atmospheric scale
\[ \Delta_A \sim 3 \times 10^{-3} \text{ eV}^2 \]
and the quasi degenerate mass \( m_0 \sim 0.1 \text{ eV} \). This value is not very different from the symmetry breaking that was required in the quark sector.

In order to analyze the leptonic mixing in the full 3 \times 3 case, let us assume that \( M_l \) is 2-3 symmetric and go to the basis with a diagonal \( M_l \). In this basis, the neutrino mass matrix assumes the form
\[
M_{\nu f} \equiv R_{12}^T(\theta_{12l})R_{23}^T(\pi/4)M_{\nu}R_{23}(\pi/4)R_{12}(\theta_{12l}) .
\] (16)
The \( \theta_{12l} \) denotes the \( e-\mu \) mixing which in analogy with the quark case will be assumed to be small, \( \theta_{12l} \sim \sqrt{\frac{m_e}{m_\mu}} \). Neglecting its effect, the \( M_{\nu f} \) is approximately given by
\[
M_{\nu f} \approx \begin{pmatrix}
X_\nu & \sqrt{2}A_\nu & 0 \\
\sqrt{2}A_\nu & B_\nu + C_\nu & \epsilon_{2\nu}B_\nu \\
0 & \epsilon_{2\nu}B_\nu & B_\nu - C_\nu
\end{pmatrix} .
\] (17)
\( M_{\nu f} \) is diagonalized by the PMNS matrix [9] \( U \) as
\[
U^T M_{\nu f} U = \text{Dia.}(m_{\nu_1}, m_{\nu_2}, m_{\nu_3}) ,
\] (18)
with \( U = R_{23}(\theta_{23})R_{13}(\theta_{13})R_{12}(\theta_{12}) \) in the standard parameterization.

Consider the symmetric limit corresponding to \( \epsilon_{2\nu} = 0 \). The quasi degeneracy \( m_{\nu_2} \sim m_{\nu_3} \) is obtained for
\[
B_\nu \sim m_0 ; \quad C_\nu \sim \mathcal{O} \left( \frac{\Delta_A}{4m_0} \right) .
\] (19)
The atmospheric mixing is zero in this case but when \( \epsilon_{2\nu} \) is turned on, even a small value as given in eq. (15) can lead to a large atmospheric mixing due to smallness of \( C_\nu \). The smallness of \( C_\nu \), i.e. the quasi degeneracy does not follow from the underlying 2-3 symmetry but it is quite consistent with it.

The expression for the atmospheric mixing angle follows from the diagonalization of the 23 block
\[
\tan 2\theta_{23} = \frac{\epsilon_{2\nu}B_\nu}{C_\nu} .
\] (20)
This gets a small correction when \( A_\nu \sim \mathcal{O}(\frac{\Delta_A}{4m_0}) \) is turned on.
While the small 2-3 breaking leads to a large atmospheric mixing, the $U_{e3}$ remains small. This follows because of the zero in eq. (17) at the (13) entry. Using $(\mathcal{M}_{\nu f})_{13} = (UD_{\nu}U^T)_{13} = 0$ and the quasi degeneracy, one finds

$$U_{e3} \sim \tan \theta_{23} \sin 2\theta_{12} \frac{\Delta_\odot}{2\Delta_A} \sim \pm 0.03,$$

where $\Delta_A \equiv m_{\nu_3}^2 - m_{\nu_1}^2$ and $\Delta_\odot \equiv m_{\nu_2}^2 - m_{\nu_1}^2$. Note that the normal and inverted neutrino mass hierarchies correspond to opposite signs for $U_{e3}$.

The above value for $U_{e3}$ would get corrected by (a) the $12$ mixing angle in the charged lepton sector and (b) the symmetry breaking parameter $\epsilon_{1\nu}$ which was also neglected here. The (a) gives a contribution [10] of

$$\mathcal{O} \left( \frac{1}{\sqrt{2}} \theta_{12l} \right) \sim 0.05,$$

which can add or subtract to the value $\sim 0.03$ given above depending upon the neutrino mass hierarchy. There can be a relative phase between these contribution in the presence of CP violation. As a consequence, one expects $U_{e3}$ in the present scheme to be typically $0.02 - 0.08$ if $\theta_{12l} \sim \mathcal{O} \left( \frac{1}{\sqrt{2}} \theta_{12l} \right)$. The $\epsilon_{1\nu}$ gives a very small $\sim \mathcal{O}(\frac{\Delta_\odot}{\Delta_A} \epsilon_{1\nu})$ contribution to $U_{e3}$ when $A_\nu \sim \mathcal{O}(\frac{\Delta_\odot}{m_0})$.

The quasi-degeneracy is an essential ingredient in this approach. One would therefore expect relatively large value for the effective neutrino mass $m_{ee}$ probed by the neutrinoless double beta decay experiments. This is quantified in in Figure 1. The parameters in $M_{\nu f}$ are determined in terms of the lightest neutrino mass $m_0$, the solar and the atmospheric scales and the corresponding mixing angles using eq. (17,18) after imposing eq. (21). These are then varied randomly in their allowed $2\sigma$ ranges [11] to generate the values of $m_{ee}$ and $\epsilon_{2\nu}$. The sum of neutrino masses is assumed to be $\leq 0.9$ eV as required by cosmology. One clearly sees that quite large values for $m_{ee}$ are possible which is understood from the fact that the scenario corresponds to quasi degeneracy with all the neutrinos having the same CP property. The $\epsilon_{2\nu}$ is restricted in the range $\sim 0.005 - 0.2$ with higher $m_0$ requiring smaller 23 breaking.

The atmospheric mixing angle can be large but it is not required to be maximal as would be the case if only $\mathcal{M}_{\nu f}$ was assumed to be $\mu-\tau$ symmetric. While strict maximality does
Figure 1: The allowed ranges of the 23-breaking parameter $\epsilon_{2\nu}$ and the neutrinoless double beta decay mass $m_{ee}$ obtained from eq.(17) with quasi degenerate spectrum. The solar and the atmospheric scales and mixing angles are randomly varied within their allowed 2$\sigma$ ranges.

not obtain, all values allowed by the present data are possible including close to the maximal mixing.

We now turn to a concrete realization of our basic ansatz $M_f$ given in eq. (2). This can be derived in a straightforward manner within the standard two double model by imposing a 2-3 symmetry on the Yukawa couplings. One of the doublets ($\phi_1$) is assumed to be invariant while the other ($\phi_2$) is odd under the 2-3 symmetry. The Yukawa couplings for a fermion $f$ are then given by

$$-\mathcal{L}_Y = \bar{f}_L (\Gamma_1 \phi_1 + \Gamma_2 \phi_2) f_R + \text{H.c.} \ .$$

(22)

The (assumed) symmetry of $\Gamma_{1,2}$ and the 2-3 symmetry together lead to the matrix $M_f$. The $\Gamma_1$ generates the parameters $A_f, B_f, C_f$ in eq. (2) and $\Gamma_2$ generates $\epsilon_{1f,2f}$ terms. The smallness of $\epsilon_{1f,2f}$ compared to the leading elements can be obtained by assuming corresponding elements of $\Gamma_{1,2}$ to be similar but taking $\langle \phi_2 \rangle / \langle \phi_1 \rangle$ to be small $\leq 0.1$
The neutrino mass matrix also follows in a straightforward manner, e. g. consider a model with right handed neutrinos which obtain mass from a standard model singlet (or 126 in $SO(10)$) assumed to be invariant under the 2-3 symmetry. The mass matrix $M_R$ would then be 2-3 symmetric. This together with the Dirac mass matrix obtained from eq. (22) would lead to a neutrino mass matrix having the form of eq. (2).

Simplicity of the above scheme is to be contrasted with other models [6] which try to obtain a $\mu$-$\tau$ symmetric neutrino mass matrix in the flavour basis.

In summary, let us recapitulate the salient features of the scheme and open problems.

- We showed that the $\mu$-$\tau$ symmetry can be extended to all fermions with interesting consequences. Many earlier studies [2, 3] postulated this only for the neutrino mass matrix in flavour basis and its extension to other fermions was found problematic. As shown here, approximate 23 symmetry is quite consistent with observations if neutrinos have quasi degenerate spectrum. We quantified the amount of breaking of the 2-3 symmetry needed for successful phenomenology.

- In the quark sector, the 2-3 symmetry provides explanation of the smallness of $V_{cb}, V_{ub}$ compared to the Cabibbo angle. The latter can be naturally explained if an additional symmetry $D$ as defined in eq. (6) is imposed. This needs to be broken badly by the effective neutrino mass matrix in order to get the quasi-degenerate spectrum.

- The $\mu$-$\tau$ symmetry has been extended to the quark sector in some of the earlier works [4]. These relied on breaking it through complex phases in the mass matrix. In contrast, the 2-3 symmetry breaking here occurs even when the phases are turned off but requires quasi degenerate neutrino spectrum. This feature can be tested through the neutrinoless double beta decay and direct neutrino mass measurements in future experiments.

- The maximal atmospheric mixing is one of the predictions of the unbroken $\mu$-$\tau$ symmetry of the neutrino mass matrix. This maximality is not obtained here but values very close to the maximal are possible. While realization of the effective $\mu$-$\tau$ symmetry for neutrino requires complicated models [6], the present scenario gets realized in the standard two Higgs doublet model.
The Yukawa couplings in eq. (22) generate the flavour changing neutral currents (FCNC). One finds that the specific structures of the Yukawa couplings $\Gamma_{1,2}$ lead to hierarchical strengths ($|F_{12}| \ll |F_{13}| \ll |F_{23}|$) for the FCNC current couplings $F_{ij}$ between flavours $i$ and $j$ to Higgs in case with one parameter symmetry breaking, i.e., with $\epsilon_2 \neq 0$. Rough estimates give for the down quarks, $F_{12} \sim \frac{m_b}{v}\lambda^5, F_{13} \sim \frac{m_b}{v}\lambda^3$ and $F_{23} \sim \frac{m_b}{v}\lambda^2$, where $\lambda \sim 0.22$ and $v$ is the weak scale. Because of this suppression, the FCNC do not require unusually large Higgs mass. Similar hierarchy in FCNC is found in some Higgs doublets models [12] and in $Z$ couplings [13] in models with additional quarks.

As in case of the $\mu$-$\tau$ symmetry, $U_{e3}$ remains small but can be close to the measurable values in future.

We assumed CP conservation and identical CP properties for the neutrinos. The latter is required to make symmetry breaking parameter small.

The entire scenario is compatible with grand unification and can be embedded in theories such as SO(10).

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