De Donder Construction for Higher Jets

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1 Introduction

In 1929, De Donder [5], [6], formulated an approach to study first order variational problems for several independent variables in terms of a differential form obtained from the Lagrangian by the Legendre transformation in all independent variables. His construction was generalized by Lepage [12] yielding a family of forms, each of which could be used in the same way as the De Donder form to reduce the original variational problem to a system of equations in exterior differential forms. A geometric formulation of the De Donder construction in terms of jets was given by Śniatycki in 1970 [16]. It showed that the De Donder form depended only on the original Lagrangian and the canonical structure of the appropriate jet bundle. De Donder form, also called Poincaré-Cartan form, facilitated an invariant multisymplectic formulation of field theories [10], [9], [2], [7]. The De Donder construction was generalized in 2017 by Kupferman, Olami and Segev [11] in the context of continuum mechanics of first order materials, to forms on the first jet bundle that need not be exact.

In 1977 Aldaya and Azcárraga [1] investigated generalization of Lepagean forms to higher order variational problems. For higher order Lagrangians, the natural generalization of the De Donder construction in terms of Ostrogradski’s Legendre transformation [14] in all independent variables leads to a form that depends on the adapted coordinate system used for its construction. This has lead to search for additional geometric structures, which would ensure global existence of Poincaré-Cartan forms, see [3] and references quoted there. In the context of continuum mechanics, the analysis for higher

\footnotesize{English transcription of the original Russian name is Ostrogradsky. However, Ostrogradski wrote in French and used the French transcription of his name.}
order jets has been replaced by an analogous, yet further underdetermined, analysis on iterated jet bundles [15].

In this paper, we generalize De Donder approach to construct boundary forms that depend on the adapted coordinate system used in the construction. In continuum mechanics, use of boundary forms leads to splitting of the total force acting on the body into body force and surface traction. Moreover, this splitting is independent of the choice of the boundary form used. In calculus of variations, use of boundary forms leads to equations in exterior differential forms that are equivalent to the Euler-Lagrange equations. Infinitesimal symmetries of the theory lead to conservation laws valid for any choice of the boundary form. In an example, we show that the boundary conditions lead to independence of constants of motion of the choice of the boundary form.

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2 Spaces of smooth sections

We are interested in geometric structure of calculus of variations with \( m > 1 \) independent variables and \( n \) dependent variables and its relation to continuum mechanics. Both theories deal with differentiation of functions on spaces of maps. There are several approaches to manifold structure of a space of maps. Here, we use the traditional approach of the calculus of variations flavoured by the insight from theory of differential spaces [17].

Consider a locally trivial fibration \( \pi : N \to M \) with \( \dim M = m \) and \( \dim N = m + n \). Let \( K \) be an open relatively compact sub-manifold of \( M \) with smooth boundary \( \partial K \). The closure \( \bar{K} = K \cup \partial K \) is a manifold with boundary. A map \( \sigma : \bar{K} \to N \) is a section of \( \pi \) if \( \pi \circ \sigma \) is the identity on \( \bar{K} \). In the spirit of theory of differential spaces, we say that a section \( \sigma : \bar{K} \to N \) is smooth if it extends to a smooth section of \( \pi \) defined on an open subset of \( M \) that contains \( \bar{K} \). We denote by \( S^\infty(\bar{K}, N) \) the space of smooth sections \( \sigma : \bar{K} \to N \) of \( \pi \). In the following, we assume that \( \bar{K} \) is contain in the domain of a chart on \( M \).

The next stage is to identify smooth functions on \( S^\infty(\bar{K}, N) \). We use terminology of jet bundles reviewed in the Appendix. Let \( \Lambda \) be an \( m \)-form
on the space $J^k(M, N)$ of k-jets of sections of $\pi$. We say that $\Lambda$ is semi-basic with respect to the source map $\pi^k : J^k(M, N) \to M$ if $X \downarrow \Lambda = 0$ for every vector field $X$ tangent to fibres of $\pi^k : J^k(M, N) \to M$, where $\downarrow$ denotes the left interior product (contraction) of vectors and forms. The form $\Lambda$ gives rise to the corresponding action functional

$$A : S^\infty(\bar{K}, N) \mapsto \mathbb{R} : \sigma \mapsto A(\sigma) = \int_K j^k \sigma^* \Lambda.$$  \hspace{1cm} (1)

Calculus of variations is concerned with study of critical points of action functionals. Let $A$ denote the space of all action functionals on $S^\infty(\bar{K}, N)$. In other words, a function $F : S^\infty(\bar{K}, N) \to \mathbb{R}$ is in $A$ if there exists an integer $k \geq 0$, and an $m$-form $\Lambda$ on $J^k(M, N)$, semi-basic with respect to the source map $\pi^k : J^k(M, N) \to M$, such that

$$F(\sigma) = \int_K j^k \sigma^* \Lambda \quad \forall \; \sigma \in S^\infty(\bar{K}, N).$$  \hspace{1cm} (2)

Here, for $k = 0$, we use identifications $J^0(M, N) = N$ and $\pi^0 = \pi$.

The tangent space $T_{\sigma}S^\infty(\bar{K}, N)$ is the space of smooth maps $Y_\sigma : \bar{K} \to TN$ such that, for each $x \in \bar{K}$, $Y_\sigma(x) \in T_{\sigma(x)}N$ is tangent to the fibre $\pi^{-1}(x)$. It should be noted, that every $Y_\sigma \in T_{\sigma}S^\infty(\bar{K}, N)$ can be extended to a vector field $Y$ on $N$ tangent to fibres of $\pi$ and such that $Y_\sigma(x) = Y(\sigma(x))$ for every $x \in \bar{K}$. For $Y_\sigma \in T_{\sigma}S^\infty(\bar{K}, N)$ and $F(\sigma)$ given by equation (2), the derivative of $F$ in direction $Y_\sigma$ is

$$D_{Y_\sigma}F = \int_K j^k \sigma^*(\mathcal{L}_{Y_\sigma} \Lambda),$$  \hspace{1cm} (3)

where $Y^k$ is the prolongation of an extension of $Y_\sigma$ to a vertical vector field $Y$ on $N$. It should be noted that the integral in (3) does not depend on the choice of the extension $Y$ of $Y_\sigma$.

The next step is to identify vector fields on $S^\infty(\bar{K}, N)$. On manifolds, vector fields play two roles: they are global derivations of the differential structure, and they generate local one-parameter local groups of diffeomorphisms. On manifolds with singularities, eg. stratified spaces, global derivations need not generate local diffeomorphisms [17]. In this paper, we consider only vector fields on $S^\infty(\bar{K}, N)$ that are generated by global vertical vector fields $Y$ on $N$ as follows. A vertical vector field $Y$ on $N$ gives rise to a section
\( Y : \bar{K} \to TS^\infty(\bar{K}, N) \) such that, \( Y(\sigma) = Y_\sigma \) for every \( \sigma \in S^\infty(\bar{K}, N) \). In other words, for every \( F \) given by equation (2)

\[
(YF)(\sigma) = D_{Y_\sigma} F = \int_{\bar{K}} j^k \sigma^*(\mathcal{L}_{Y_\sigma} \Lambda)
\]  

for every \( \sigma \in S^\infty(\bar{K}, N) \). We denote by \( \mathfrak{Y}(S^\infty(\bar{K}, N)) \) the space of vector fields on \( S^\infty(\bar{K}, N) \) defined above.

Now that we have vectors tangent to \( S^\infty(\bar{K}, N) \), we can consider forms on \( S^\infty(\bar{K}, N) \). Suppose that \( \Phi \) is an \((m + 1)\)-form on \( J^k(M, N) \) such that \( X \mathbf{\Phi} \) is semi-basic with respect to the source map \( \pi^k : J^k(M, N) \to M \) for every vector field \( X \) on \( J^k(M, N) \) tangent to fibres of \( \pi^k : J^k(M, N) \to M \). It gives rise to a 1-form \( \Phi \) on \( S^\infty(\bar{K}, N) \) defined as follows. For every \( Y \in \mathfrak{Y}(S^\infty(\bar{K}, N)) \) and \( \sigma \in S^\infty(\bar{K}, N) \),

\[
\langle \Phi | Y \rangle (\sigma) = \int_{\bar{K}} j^k \sigma^*(Y^k \mathbf{\Phi}) .
\]  

If \( Y_\sigma \in T_{\sigma}S^\infty(\bar{K}, N) \) is the restriction of \( Y \) to \( \sigma \), then the restriction to \( j^k \sigma(\bar{K}) \) of the prolongation \( Y^k \) of \( Y \) depends only on \( Y_\sigma \) and not on its extension off \( \sigma(\bar{K}) \). This shows that the 1-form \( \Phi \) restricts to a linear map \( \Phi_\sigma : T_{\sigma}S^\infty(\bar{K}, N) \to \mathbb{R} \) such that

\[
\langle \Phi_\sigma | Y_\sigma \rangle = \langle \Phi | Y \rangle (\sigma).
\]  

In applications to continuum mechanics, \( \bar{K} \) represents the body manifold, sections \( \sigma \in S^\infty(\bar{K}, N) \) are configurations of the body, vectors \( Y_\sigma \in T_{\sigma}S^\infty(\bar{K}, N) \) are virtual displacement fields. The form \( \Phi \) may be referred to as a force functional.

3 Boundary forms

Let \( \Phi \) be an \((m + 1)\)-form on \( J^k(M, N) \) such that \( X \mathbf{\Phi} \) is semi-basic with respect to the source map \( \pi^k : J^k(M, N) \to M \) for every vector field \( X \) on \( J^k(M, N) \) tangent to fibres of \( \pi^k : J^k(M, N) \to M \). Let \((x^i, y^a, \xi^a_1, ..., \xi^a_{i_1...i_k})\)
be local coordinates coordinates on $J^k(M,N)$. The corresponding local representation of $\Phi$ is

$$\Phi = \sum_{a=1}^{n} \left( \Phi_a dy^a + \sum_{i=1}^{m} \Phi_i^a dz_i^a + \ldots + \sum_{i_1 \leq \ldots \leq i_l} \Phi_{i_1}^{a_1 \ldots a_l} dz_{i_1}^{a_1} \right) \wedge dm x \quad (7)$$

$$+ \ldots + \sum_{a=1}^{n} \sum_{i_1 \leq \ldots \leq i_k} \Phi_{i_1}^{a_1 \ldots a_k} dz_{i_1}^{a_1} \ldots dz_{i_k}^{a_k} \wedge dm x.$$ 

Note that, for every $l = 2, \ldots, k$, coordinates $z_{i_1}^{a}, \ldots, i_l$ are symmetric in indices $i_1, \ldots, i_l$ and so are $\Phi_{i_1}^{a_1 \ldots a_l}$. Therefore, the sum in equation (7) is taken only over the independent components. In the following, we modify the summation convention by the requirement that the indices $i_1, \ldots, i_l$ occurring in $z_{i_1}^{a}, \ldots, i_l$ are taken in a non-decreasing order. This allows us to rewrite equation (7) as follows,

$$\Phi = (\Phi_a dy^a + \Phi_i^a dz_i^a + \ldots + \Phi_{i_1}^{a_1 \ldots a_k} dz_{i_1}^{a_1} \ldots dz_{i_k}^{a_k} + \ldots + \Phi_{i_1}^{a_1 \ldots a_k} dz_{i_1}^{a_1} \ldots dz_{i_k}^{a_k}) \wedge dm x. \quad (8)$$

An alternative approach would be use of multi-indices.

**Theorem 1** There exists locally a smooth $m$-form $\Xi$ on $J^{2k-1}(M,N)$, which satisfies the following conditions.

1. (a) $\Xi$ is semi-basic with respect to the forgetful map $\pi_{k-1}^{2k-1} : J^{2k-1}(M,N) \to J^k(M,N)$. In other words, for any vector field $X$ tangent to fibres of $\pi_{k-1}^{2k-1} : J^{2k-1}(M,N) \to J^k(M,N)$,

$$X \cdot \Xi = 0.$$ 

(b) For every vector field $X$ on $J^{2k-1}(M,N)$ tangent to fibres of the source map $\pi^{2k-1} : J^{2k-1}(M,N) \to M$ the left interior product $X \cdot \Xi$ is semi-basic with respect to the source map. Thus, for every pair $X_1, X_2$ of vector fields on $J^{2k-1}(M,N)$ tangent to fibres of the source map $\pi^{2k-1} : J^{2k-1}(M,N) \to M$,

$$X_2 \cdot (X_1 \cdot \Xi) = 0.$$ 

2. For every section $\sigma$ of $\pi : N \to M$,

$$j^{2k-1} \sigma^* \Xi = 0,$$

where $j^{2k-1} \sigma^* \Xi = \Xi \circ \wedge^n T(j^{2k-1} \sigma)$ is the pull-back of $\Xi$ by $j^{2k-1} \sigma : M \to J^{2k-1}(M,N)$.
3. For every vector field \( X \) on \( J^{2k-1}(M, N) \) tangent to fibres of the target map \( \pi_0^{2k-1} : J^{2k-1}(M, N) \to N \), and every section \( \sigma \) of \( \pi : N \to M \),
\[
j^{2k-1}\sigma^* \left( X \left( \pi_k^{2k-1*}\Phi + d\Xi \right) \right) = 0.
\]

**Proof.** The first and the second condition imply that \( \Xi \) is a linear combination of contact forms up to order \( k \) with coefficients given by forms that are semi-basic with respect to the source map. In local coordinates,
\[
\Xi = p^i_a(dy^a - z_j^a dx^j) \wedge \left( \frac{\partial}{\partial x^i} d_m x \right)
\]
\[
+ p_a^{i_1i_2}(dz_{i_2}^a - z_{i_2j}^a dx^j) \wedge \left( \frac{\partial}{\partial x^{i_1}} d_m x \right)
\]
\[
+ ... + p_a^{i_1i_2...i_l}(dz_{i_2...i_l}^a - z_{i_2...i_lj}^a dx^j) \wedge \left( \frac{\partial}{\partial x^{i_1}} d_m x \right),
\]
where the coefficients \( p^a_{i_1i_2...i_l} \) are symmetric in indices \( i_2, ..., i_l \), for \( l = 3, ..., k \)
and the summation is taken over indices in non-decreasing order. Hence,
\[
\Xi = (p^i_a dy^a + p_a^{i_1i_2} dz_{i_2}^a + ... + p_a^{i_1i_2...i_k} dz_{i_2...i_k}^a) \wedge \left( \frac{\partial}{\partial x^{i_1}} d_m x \right)
\]
\[- (p_a^{i_1} z_{i_1}^a + p_a^{i_1i_2} z_{i_1i_2}^a + ... + p_a^{i_1i_2...i_k} z_{i_1i_2...i_k}^a) d_m x.
\]
Note that the symmetry of \( z_{i_1i_2...i_l}^a \) in \( i_1, ..., i_l \) implies in equation (10) that
only the fully symmetric parts of \( p_a^{i_1i_2...i_l} \) contribute to the sum in the second term.

In order to use the third condition, we need the exterior differential of \( \Xi \).
Equation (9) yields

\[ d\Xi = dp_a \wedge (dy^a - z^a_j dx^j) \wedge \left( \frac{\partial}{\partial x^i} \mathbf{j} m_x \right) + \] 
\[ dp_a^{i_1 i_2} \wedge (dz^a_{i_1} - z^a_{i_2} dx^j) \wedge \left( \frac{\partial}{\partial x^{i_1}} \mathbf{j} m_x \right) + \ldots + \] 
\[ + dp_a^{i_1 i_2 \ldots i_k} \wedge (dz^a_{i_2 \ldots i_k} - z^a_{i_2 \ldots i_k} dx^j) \wedge \left( \frac{\partial}{\partial x^{i_1}} \mathbf{j} m_x \right) + \] 
\[ - p^a_i dz^a_j \wedge dx^j \wedge \left( \frac{\partial}{\partial x^i} \mathbf{j} m_x \right) - \] 
\[ - p^a_{i_1 i_2} dz^a_{i_2 \ldots i_k} \wedge dx^j \wedge \left( \frac{\partial}{\partial x^{i_1}} \mathbf{j} m_x \right) + \ldots + \] 
\[ - p^a_{i_1 i_2 \ldots i_k} dz^a_{i_2 \ldots i_k} \wedge dx^j \wedge \left( \frac{\partial}{\partial x^{i_1}} \mathbf{j} m_x \right), \] 

which can be simplified to

\[ d\Xi = dp_a^i \wedge (dy^a - z^a_j dx^j) \wedge \left( \frac{\partial}{\partial x^i} \mathbf{j} m_x \right) + \] 
\[ dp_a^{i_1 i_2} \wedge (dz^a_{i_1} - z^a_{i_2} dx^j) \wedge \left( \frac{\partial}{\partial x^{i_1}} \mathbf{j} m_x \right) + \ldots + \] 
\[ + dp_a^{i_1 i_2 \ldots i_k} \wedge (dz^a_{i_2 \ldots i_k} - z^a_{i_2 \ldots i_k} dx^j) \wedge \left( \frac{\partial}{\partial x^{i_1}} \mathbf{j} m_x \right) + \] 
\[ - p^a_i dz^a_j \wedge dx^j \wedge \left( \frac{\partial}{\partial x^i} \mathbf{j} m_x \right) - \] 
\[ - p^a_{i_1 i_2} dz^a_{i_2 \ldots i_k} \wedge dx^j \wedge \left( \frac{\partial}{\partial x^{i_1}} \mathbf{j} m_x \right) + \ldots + \] 
\[ - p^a_{i_1 i_2 \ldots i_k} dz^a_{i_2 \ldots i_k} \wedge dx^j \wedge \left( \frac{\partial}{\partial x^{i_1}} \mathbf{j} m_x \right), \] 

Let \( X \) be a vector field on \( J^{2k-1}(M, N) \) tangent to fibres of of the target map \( \pi_0^{2k-1}: J^{2k-1}(M, N) \to N \). In local coordinates,

\[ X = X^a_{i_1} \frac{\partial}{\partial z^a_{i_1}} + \ldots + X^a_{i_1 \ldots i_{2k-1}} \frac{\partial}{\partial z^a_{i_1 \ldots i_{2k-1}}} \] 

(13)

Note that, for \( l = 2, \ldots, 2k - 1 \), \( X^a_{i_1 \ldots i_l} \) is symmetric in the indices \( i_1, \ldots, i_l \).
Then,

\[ X \cup d\Xi = (X p^i_a)(dy^a - z^a_j dx^j) \wedge \left( \frac{\partial}{\partial x^i} \cup d_m x \right) + \ldots + \]

\[ + (X p^{i_1 i_2 \ldots i_k}_a)(dz^a_{i_2 \ldots i_k} - z^a_{i_2 \ldots i_k} dx^j) \wedge \left( \frac{\partial}{\partial x^i} \cup d_m x \right) + \]

\[ - X^a_{i_2} dp^{i_1 i_2}_a \wedge \left( \frac{\partial}{\partial x^i} \cup d_m x \right) - \ldots - \]

\[ - X^a_{i_2 \ldots i_k} dp^{i_1 i_2 \ldots i_k}_a \wedge \left( \frac{\partial}{\partial x^i} \cup d_m x \right) + \]

\[ - (p^i_a X^a_i + p^{i_1 i_2}_a X^a_{i_1 i_2} + \ldots + p^{i_1 i_2 \ldots i_k}_a X^a_{i_1 i_2 \ldots i_k}) \cup d_m x, \]

where \((X p^i_a)\) is the derivation of \(p^i_a\) in direction \(X\), etc.

The first two lines of equation (14) do not contribute to \(j^{2k-1}\sigma^* (X \cup d\Xi)\), because they are linear combinations of contact forms. Hence, equation (14) implies that

\[ j^{2k-1}\sigma^* (X \cup d\Xi) = -X^a_{i_2} p^{i_1 i_2}_a \cup d_m x - \ldots - X^a_{i_2 \ldots i_k} p^{i_1 i_2 \ldots i_k}_a \cup d_m x \]

\[ - (p^i_a X^a_i + p^{i_1 i_2}_a X^a_{i_1 i_2} + \ldots + p^{i_1 i_2 \ldots i_k}_a X^a_{i_1 i_2 \ldots i_k}) \cup d_m x, \]

where

\[ p^{i_1}_a = j^{2k-1}\sigma^* p^{i_1}_a, \ldots, p^{i_1 i_2 \ldots i_k}_a = j^{2k-1}\sigma^* p^{i_1 i_2 \ldots i_k}_a, \]

and the components of \(X\) are evaluated on the range of \(j^{2k-1}\sigma\). Following the symmetry argument leading to equation (10), we can rewrite equation (15) in the form

\[ j^{2k-1}\sigma^* (X \cup d\Xi) = -X^a_{i_2} p^{i_1 i_2}_a \cup d_m x - \ldots - X^a_{i_2 \ldots i_k} p^{i_1 i_2 \ldots i_k}_a \cup d_m x \]

\[ - (p^i_a X^a_i + p^{i_1 i_2}_a X^a_{i_1 i_2} + \ldots + p^{i_1 i_2 \ldots i_k}_a X^a_{i_1 i_2 \ldots i_k}) \cup d_m x, \]

where \((i_1, \ldots, i_t)\) denotes symmetrization in indices \(i_1, \ldots, i_t\). Therefore, the third condition of the theorem yields

\[ 0 = \Phi^i_a X^a_i + \ldots + \Phi^{i_1 \ldots i_k}_a X^a_{i_1 \ldots i_k} - X^a_{i_2} p^{i_1 i_2}_a - \]

\[ - \ldots - X^a_{i_2 \ldots i_k} p^{i_1 i_2 \ldots i_k}_a - (p^i_a X^a_i + p^{i_1 i_2}_a X^a_{i_1 i_2} + \ldots + p^{i_1 i_2 \ldots i_k}_a X^a_{i_1 i_2 \ldots i_k}), \]
where $\Phi_{\alpha}^i, ..., \Phi_{\alpha}^{i_1...i_k}$ are evaluated on the range $\sigma^{j_k \sigma}$. Equation (18) is equivalent to

\begin{equation}
0 = (\Phi_{\alpha}^{i_1...i_k} - P^{(i_1i_2...i_k)}_{\alpha} X^{a}_{i_1 i_2...i_k} + (\Phi_{\alpha}^{i_2...i_k} - P^{(i_2...i_k)}_{\alpha} - P^{i_1 i_2...i_k}_{\alpha,i_1}) X^{a}_{i_2...i_k} + ... + (\Phi_{\alpha}^{i} - P^{i}_{\alpha} - P^{i_1i}_{\alpha,i_1}) X^{a}_{i_1}).
\end{equation}

Equation (19) is equivalent to

\begin{equation}
\begin{aligned}
\Phi_{\alpha}^{i_1...i_k} - P^{(i_1i_2...i_k)}_{\alpha} &= 0, \\
\Phi_{\alpha}^{i_2...i_k} - P^{(i_2...i_k)}_{\alpha} - P^{i_1i_2...i_k}_{\alpha,i_1} &= 0, \\
\Phi_{\alpha}^{i_1...i_k} - P^{(i_1...i_k)}_{\alpha} - P^{i_{l-1}i_1...i_k}_{\alpha,i_l} &= 0, \text{ for } l = 3, ..., k - 2, \\
\Phi_{\alpha}^{i} - P^{i}_{\alpha} - P^{i_1i}_{\alpha,i_1} &= 0.
\end{aligned}
\end{equation}

This shows that there is no unique local form $\Xi$ satisfying the conditions of our theorem. In order to prove existence, we are free to impose an additional condition that the coefficients $p^{i_1i_2...i_l}_{\alpha}$.

An obvious generalization of the De Donder construction corresponds to an additional conditions that all $p^{i_1i_2...i_l}_{\alpha}$ are fully symmetric in all indices $i_1, ..., i_l$. With this additional assumption, equation (20) yields

\begin{equation}
\begin{aligned}
\Phi_{\alpha}^{i_1...i_k} - P^{(i_1i_2...i_k)}_{\alpha} &= 0, \\
\Phi_{\alpha}^{i_2...i_k} - P^{(i_2...i_k)}_{\alpha} - P^{i_1i_2...i_k}_{\alpha,i_1} &= 0, \\
\Phi_{\alpha}^{i_1...i_k} - P^{(i_1...i_k)}_{\alpha} - P^{i_{l-1}i_1...i_k}_{\alpha,i_l} &= 0, \text{ for } l = 3, ..., k - 2, \\
\Phi_{\alpha}^{i} - P^{i}_{\alpha} - P^{i_1i}_{\alpha,i_1} &= 0.
\end{aligned}
\end{equation}

In equation (21) $\Phi$ depends on $\sigma^{j_k \sigma}(x)$. In local coordinates, the section $\sigma$ is given by $y^b = \sigma^b(x) = \sigma^b(x^1, ..., x^n)$ for $b = 1, ..m$, and $\sigma^{j_k \sigma}(x)$ is has coordinates

\begin{equation}
(x^i, y^b, z^b_{j_1}, ..., z^b_{j_1...j_k}) = (x^i, \sigma^b(x), \sigma^b_{j_1}(x), ..., \sigma^b_{j_1...j_k}(x)).
\end{equation}

Hence,

\begin{equation}
P^{(i_1i_2...i_k)}_{\alpha}(x) = \Phi_{\alpha}^{i_1i_2...i_k}(x, \sigma^b(x), \sigma^b_{j_1}(x), ..., \sigma^b_{j_1...j_k}(x)).
\end{equation}
Next,
\[ P_{a}^{i_1 \ldots i_k}(x) = P_{a}^{i_1 \ldots i_k}(x, \sigma^b(x), \sigma_{j_1}^b(x), \ldots, \sigma_{j_k}^b(x)) - P_{a,i_1}^{i_1 \ldots i_k}(x) \]  
(23)

Continuing, we get a complete solution of equations (21) in the form
\[ P_{a}^{i_1 \ldots i_k}(x) = \Phi_{a}^{i_1 \ldots i_k}(x, \sigma^b(x), \sigma_{j_1}^b(x), \ldots, \sigma_{j_k}^b(x)) + \]
\[ \sum_{j=1}^{l-1} (-1)^j \frac{\partial^j}{\partial x^{i-l} \ldots x^{i-1}} \left[ \Phi_{a}^{i_1 \ldots i_j \ldots i_{l+1} \ldots i_k}(x, \sigma^b(x), \sigma_{j_1}^b(x), \ldots, \sigma_{j_k}^b(x)) \right] \]
(24)

for \( l = 1, \ldots, k \). It shows that \( P_{a}^{i_1 \ldots i_k}(x) \) depends on \( j^{k+1} \sigma(x) \). In particular, \( P_{a}^{i_1 \ldots i_k}(x) \) depends on \( j^{2k-1} \sigma(x) \).

Recall that \( P_{a}^{i_1 \ldots i_k} = j^{2k-1} \sigma^* p_{a}^{i_1 \ldots i_k} \), for \( l = 1, \ldots, k-1 \), where \( p_{a}^{i_1 \ldots i_k} \) is a function on \( j^{2k-1}(M, N) \), see equation (16). Equation (24) gives \( P_{a}^{i_1 \ldots i_k} \) for all sections \( \sigma \) of \( \pi \). Hence, we can use it to get an explicit expression for \( p_{a}^{i_1 \ldots i_k} \) as a function of coordinates \((x^i, y^a, z_{i_1}^a, \ldots, z_{i_2k-1}^a)\). To this end we define a differential operator
\[ D_i = \frac{\partial}{\partial x^i} + z_{i_j}^a \frac{\partial}{\partial y^a} + z_{ij_1}^a \frac{\partial}{\partial z_{j_1}^a} + \ldots + z_{ij_1 \ldots j_{2k-2}}^a \frac{\partial}{\partial z_{j_1 \ldots j_{2k-2}}^a} + z_{ij_1 \ldots j_{2k-1}}^a \frac{\partial}{\partial z_{j_1 \ldots j_{2k-1}}^a} \]
(25)
acting on \( C^\infty(j^{2k-1}(M, N)) \).
It enables us to write
\[
p_a^{i_1 \ldots i_k}(x^i, y^b, z_{j_1}^b, \ldots, z_{j_{2k-1}}^b) = \Phi_a^{i_1 \ldots i_k}(x^i, y^b, z_{j_1}^b, \ldots, z_{j_{2k-1}}^b) + \sum_{j=1}^{l-1} (-1)^j D_{i_{j-1}} \ldots D_{i_{l-1}} \left[ \Phi_a^{j_{i_{j-1}+1 \ldots i_{l-1} i_{i_{j}} \ldots i_k}}(x^i, y^b, z_{j_1}^b, \ldots, z_{j_{2k-1}}^b) \right].
\]

Hence, in terms of local coordinates \((x^i, y^b, z_{j_1}^b, \ldots, z_{j_{2k-1}}^b)\) on \(J^{2k-1}(M, N)\), an obvious generalization of the De Donder construction yields form \(\Xi\) given by equation (14), where the coefficients \(p_a^{i_1}, \ldots, p_a^{i_{12} \ldots i_k}\) are given by equation (26).

**Definition 2** Local forms
\[
\Xi = p_a^i (dy^a - z_j^a dx^j) \wedge \left( \frac{\partial}{\partial x^i} j_{d_m x} \right) + p_a^{i_{12}} (dz_{i_2}^a - z_{i_2 j_3}^a dx^j) \wedge \left( \frac{\partial}{\partial x^{i_1}} j_{d_m x} \right) + \ldots + p_a^{i_{12} \ldots i_k} (dz_{i_2 \ldots i_k}^a - z_{i_2 i_3 \ldots i_{k+1}}^a dx^j) \wedge \left( \frac{\partial}{\partial x^{i_1}} j_{d_m x} \right),
\]
are called boundary forms. If \(\Xi\) satisfies Condition 3 of Theorem 1, we say that \(\Xi\) is a boundary form of \(\Phi\).

In the following we discuss some properties of boundary forms. This means, we do not make additional assumptions on the symmetry properties of coefficients \(p_a^{i_{1 \ldots i_j}}\), and do not specify the form \(\Phi\) explicitly.

**Lemma 3** For each vector field \(Y\) on \(N\), which projects to a vector field on \(M\), and every section \(\sigma\) of \(\pi : M \to N\),
\[
j^{2k-1} \sigma^* (\mathcal{L}_{Y^{2k-1}} \Xi) = 0,
\]
where \(Y^{2k-1}\) is the prolongation of \(Y\) to \(J^{2k-1}(M, N)\).

**Proof.** By definition,
\[
\mathcal{L}_{Y^{2k-1}} \Xi = \frac{d}{dt} (e^{Y^{2k-1}*\Xi})_{t=0} = \lim_{t \to 0} \left[ \frac{1}{t} \left( e^{Y^{2k-1}*\Xi} - \Xi \right) \right].
\]
Hence,
\[
\begin{align*}
    j^{2k-1} \sigma^* (L_{Y^{2k-1}} \Xi) &= j^{2k-1} \sigma^* \lim_{t \to 0} \left[ \frac{1}{t} (e^{t Y^{2k-1}} \Xi - \Xi) \right] \\
    &= \lim_{t \to 0} \left[ \frac{1}{t} \left( j^{2k-1} \sigma^* e^{t Y^{2k-1}} \Xi - j^{2k-1} \sigma^* \Xi \right) \right] \\
    &= \lim_{t \to 0} \left\{ \frac{1}{t} \left[ (e^{t Y} \circ j^{2k-1} \sigma)^* \Xi - j^{2k-1} \sigma^* \Xi \right] \right\} \\
    &= \lim_{t \to 0} \left\{ \frac{1}{t} \left[ (j^{2k-1} (e^{t Y} \sigma))^* \Xi - j^{2k-1} \sigma^* \Xi \right] \right\}.
\end{align*}
\]

Condition 2 of Theorem 1 ensures that \( j^{2k-1} \sigma^* \Xi \) and \( (j^{2k-1} (e^{t Y} \sigma))^* \Xi = 0 \) for every \( t \) in a neighbourhood of 0. Therefore, \( j^{2k-1} \sigma^* (L_{Y^{2k-1}} \Xi) = 0 \), which completes the proof. ■

**Proposition 4** For every \( Y_\sigma \in T_\sigma S^\infty(\bar{K}, N) \), a boundary form \( \Xi \) leads to a decomposition
\[
\int_K j^k \sigma^* (Y^k \mathcal{J} \Phi) = \int_K [j^k \sigma^* (\Phi_a) - P_a^i] Y_\sigma^a d_m x + \int_{j^{2k-1} \sigma(\partial K)} (Y_\sigma^{2k-1} \mathcal{J} \Xi),
\]
where \( P_a^i = j^{2k-1} \sigma^* p_a^i \), as in equation (16).

**Proof.** Let \( Y \) be an extension of \( Y_\sigma \) to a vertical vector field on \( N \). Clearly,
\[
\int_K j^k \sigma^* (Y^k \mathcal{J} \Phi) = \int_K j^{2k-1} \sigma^* (\pi_k^{2k-1} \sigma^* (Y^k \mathcal{J} \Phi))
\]
\[
= \int_K j^{2k-1} \sigma^* (Y^{2k-1} \mathcal{J} \pi_k^{2k-1} \sigma^* \Phi)
\]
\[
= \int_K j^{2k-1} \sigma^* (Y^{2k-1} \mathcal{J} (\pi_k^{2k-1} \Phi + d\Xi - d\Xi)).
\]
Thus,
\[
\int_K j^k \sigma^* (Y^k \mathcal{J} \Phi) = \int_K j^{2k-1} \sigma^* (Y^{2k-1} \mathcal{J} (\pi_k^{2k-1} \Phi + d\Xi)) + \quad (29)
\]
\[
- \int_K j^{2k-1} \sigma^* (Y^{2k-1} \mathcal{J} d\Xi).
\]
By Condition 3 in Theorem 1,

\[ j^{2k-1} \sigma^* \left( X \mathcal{J} \left( \pi_k^{2k-1} \Phi + d\Xi \right) \right) = 0. \]

for every vector field \( X \) tangent to fibres of the target map \( \pi_0^{2k-1} : J^{2k-1}(M, N) \rightarrow N \). On the other hand, the prolongation \( Y^{2k-1} \) of a vertical vector field \( Y = Y^a \frac{\partial}{\partial y^a} \) on \( N \) is \( \pi_0^{2k-1} \)-related to \( Y \). Therefore, in local coordinates, treating \( Y^a \frac{\partial}{\partial y^a} \) as a vector field on \( J^{2k-1}(M, N) \), the difference \( Y^{2k-1} - Y^a \frac{\partial}{\partial y^a} \) is tangent to fibres of the target map \( \pi_0^{2k-1} \), so that

\[
j^{2k-1} \sigma^* \left( Y^{2k-1} \mathcal{J} \left( \pi_k^{2k-1} \Phi + d\Xi \right) \right) \]

\[
= j^{2k-1} \sigma^* \left( [Y^{2k-1} - Y^a \frac{\partial}{\partial y^a}] \mathcal{J} \left( \pi_k^{2k-1} \Phi + d\Xi \right) \right)
\]

\[
= j^{2k-1} \sigma^* \left( Y^a \frac{\partial}{\partial y^a} \mathcal{J} \left( \pi_k^{2k-1} \Phi + d\Xi \right) \right).
\]

Taking into account equations (12) and (16), we get

\[ j^{2k-1} \sigma^* \left( Y^{2k-1} \mathcal{J} \left( \pi_k^{2k-1} \Phi + d\Xi \right) \right) = \left( j^k \sigma^* \Phi_a - P^i_{a,i} Y^a \right) Y^a d_m x. \quad (30) \]

Lemma 3 ensures that \( \mathcal{L}_{Y^{2k-1}} \Xi = 0 \). Hence,

\[ Y^{2k-1} \mathcal{J} d\Xi = -d(Y^{2k-1} \mathcal{J} \Xi). \]

Therefore, the second line in equation (29) can be rewritten in the form

\[
- \int_K j^{2k-1} \sigma^* \left( Y^{2k-1} \mathcal{J} d\Xi \right) = - \int_K j^{2k-1} \sigma^* \left( \mathcal{L}_{Y^{2k-1}} \Xi - d(Y^{2k-1} \mathcal{J} \Xi) \right)
\]

\[
= - \int_K j^{2k-1} \sigma^* \mathcal{L}_{Y^{2k-1}} \Xi + \int_K j^{2k-1} \sigma^* d(Y^{2k-1} \mathcal{J} \Xi)
\]

\[
= \int_K d \left( j^{2k-1} \sigma^* (Y^{2k-1} \mathcal{J} \Xi) \right)
\]

\[
= \int_{\partial K} j^{2k-1} \sigma^* (Y^{2k-1} \mathcal{J} \Xi)
\]

\[
= \int_{j^{2k-1} \sigma(\partial K)} Y^{2k-1} \mathcal{J} \Xi.
\]

This completes the proof. \( \blacksquare \)
Next, we want to show that the decomposition (28) is independent of the choice of boundary form $\Xi$. Let $X$ be a vector field on $J^{2k-1}(M,N)$ tangent to fibres of the source map $\pi^{2k-1}: J^{2k-1}(M,N) \to M$. For any boundary form $\Xi$ of $\Phi$, equations (12) and (16) yield

$$j^{2k-1}\sigma^* (X \mathfrak{d}\Xi)$$

$$= j^{2k-1}\sigma^* \left[ (X^a d\xi_1 - X^a d\xi_2 - ... - X^a d\xi_2 - ... - X^a d\xi_{2k-1}) \wedge \left( \frac{\partial}{\partial x^{2k}} \mathfrak{d}m \right) \right]$$

$$- j^{2k-1}\sigma^* \left( P^a_{i_1} X^a_{i_1} + P^a_{i_2} X^a_{i_2} + P^a_{i_3} X^a_{i_3} \right) \mathfrak{d}m.$$

Hence

$$j^{2k-1}\sigma^* (X \mathfrak{d}\Xi) = - \left[ X^a P^a_{i_1} + X^a P^a_{i_2} + ... \right] \mathfrak{d}m \quad (31)$$

Let $\Xi'$ be another boundary form of $\Phi$ such that

$$j^{2k-1}\sigma^* (X \mathfrak{d}\Xi') = - \left[ X^a P^a_{i_1} + X^a P^a_{i_2} + ... \right] \mathfrak{d}m \quad (32)$$

where the coefficients $P^a_{i_1 i_2 ... i_k}$ are symmetric in the upper indices. For the sake of simplicity, we introduce the notation

$$Q^a_{i_1 i_2 ... i_l} = P^a_{i_1 i_2 ... i_l} - P^a_{i_2 i_1 ... i_l} \quad (33)$$

for $l = 1, ..., k$. Then

$$j^{2k-1}\sigma^* (X \mathfrak{d}(\Xi - \Xi')) = - \left[ X^a Q^a_{i_1 i_2 ... i_l} \right] \mathfrak{d}m \quad (34)$$

Since $\Xi$ and $\Xi'$ are boundary forms of the same form, and $X$ is an arbitrary vector field tangent to fibres of the source map, Condition 3 of Theorem 1 yields

$$Q^a_{i_1 i_2 ... i_k} = 0, \quad (35)$$

$$Q^a_{i_1 i_2 ... i_k} + Q^a_{i_2 i_1 ... i_k} = 0,$$

$$Q^a_{i_1 ... i_l} + Q^a_{i_2 ... i_l} = 0, \text{ for } l = 2, ..., k,$$

$$Q^a_{i_1 i_2} + Q^a_{i_2 i_1} = 0,$$
Note that, by construction, $Q_\alpha^{i_1i_2...i_l}$ is symmetric in the indices $i_2, ..., i_l$. Hence,

$$j^{2k-1} \sigma^\alpha (X \mathcal{J} d(\Xi - \Xi')) = -X^\alpha Q_\alpha^{i_1} d_m x.$$ (36)

**Lemma 5** For boundary forms $\Xi$ and $\Xi'$, given by equations (31) and (32), respectively,

$$Q_\alpha^{i_1} = (P_\alpha^i - P_\alpha^i)_{,i} = 0. \quad (37)$$

**Proof.** We begin with the case when the difference $Q_\alpha^i = (P_\alpha^i - P_\alpha^i)$ is generated at the highest differential level. In other words, we consider

$$0Q_\alpha^{i_1i_2...i_k} = P_\alpha^{i_1i_2...i_l} - P_\alpha^{i_1i_2...i_l} \neq 0 \quad \text{such that} \quad 0Q_\alpha^{i_1i_2...i_k} = 0, \quad (38)$$

and, the remaining differences are symmetric and satisfy the equations

$$0Q_\alpha^{i_1i_2...i_l} + 0Q_\alpha^{i_1i_2...i_l}_{,i} = 0, \quad \text{for } l = 3, ..., k - 1, \quad (39)$$

$$0Q_\alpha^{i_1i_2} + 10Q_\alpha^{i_1i_2} = 0.$$ 

Therefore,

$$0Q_\alpha^{i_1i_2} = (-1)^{k-1} 0Q_\alpha^{i_1i_2...i_{k-1}i_k}, \quad (40)$$

and

$$0Q_\alpha^{i_1i_2...i_k} = (-1)^{k-1} 0Q_\alpha^{i_1i_2...i_{k-1}i_k} = (-1)^{k-1} 0Q_\alpha^{i_1i_2...i_{k-1}i_k}_{,i} = 0. \quad (41)$$

because partial derivatives commute.

In the next step, we consider the situation when, $\Xi$ and $\Xi'$ agree on the highest differential, that is we assume that $iQ_\alpha^{i_1...i_k} = 0$. Moreover, we assume that

$$1Q_\alpha^{i_1...i_k} \neq 0,$$

$$1Q_\alpha^{(i_1...i_{k-1})} = 0, \quad (42)$$

$$1Q_\alpha^{i_2...i_l} + 1Q_\alpha^{i_1i_2...i_l} = 0, \quad \text{for } l = 3, ..., k - 1,$$

$$1Q_\alpha^{i_2} + 1Q_\alpha^{i_1i_2} = 0.$$ 

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The same arguments as above, lead to

\[ 1Q^i_k = (-1)^{k-2} 1Q^i_1...i_{k-1} \]

so that

\[ 1Q^i_{a,1k} = (-1)^{k-2} 1Q^i_1...i_{k-1} = 0 \]  

because \( 1Q^i_{1...i_k} = 0 \). Continuing this procedure, for every \( r = 2, \ldots, k-1 \), we consider \( rQ^i_{a,i} \) such that,

\[ rQ^i_{a,l...i_k} = 0 \]

for \( l < r \), \( rQ^i_{a,1...i_l} = 0 \), for \( l = 3, \ldots, k-r \), \( rQ^i_{a,k} + rQ^i_{a,k-1} = 0 \).

As before, for this choice of \( rQ^i_{a,i} \), we have

\[ rQ^i_{a,ik} = (-1)^{k-r} rQ^i_{a,ik-1} = 0. \]  

The general \( Q^i_{a,i} \) can be expressed as the sum of terms \( rQ^i_{a,i} \), for \( r = 0, \ldots, k-1 \). That is,

\[ Q^i_{a,i} = 0Q^i_{a,i} + 1Q^i_{a,i} + \ldots + k-1Q^i_{a,i}. \]  

The defining equations (44) for the terms \( rQ^i_{a,i} \) ensure that the decomposition (46) satisfies equations (37). Taking into account equations (41), (43) and (45) we get

\[ Q^i_{a,i} = (0Q^i_{a,i} + 1Q^i_{a,i} + \ldots + k-1Q^i_{a,i}) = 0Q^i_{a,i} + 1Q^i_{a,i} + \ldots + k-1Q^i_{a,i} = 0. \]  

We have shown that \( Q^i_{a,i} = P^i_{a,i} - P^i_{a,i} = 0 \), under the assumption that \( \Xi' \) is the obvious choice of boundary form with fully symmetric coefficients and no additional assumptions on \( \Xi \). Hence, equation (37) holds for any pair of boundary forms of the same form \( \Phi \). We have shown that \( P^i_{a,i} = P^i_{a,i} \) for any other boundary form \( \Xi' \). If \( \Xi'' \) is still another boundary form of \( \Phi \), then \( P^i_{a,i} = P^i_{a,i} \).

This implies the following result.
Corollary 6 1. If $\Xi$ and $\Xi'$ are boundary forms of the same form $\Phi$, then
\[ j^{2k-1} \sigma^* (X \mathcal{J} d (\Xi - \Xi')) = 0 \] (48)
for every vector field $X$ tangent to fibres of the source map $\pi^{2k-1}: J^{2k-1}(M, N) \to M$.

2. The decomposition (28) is independent of the choice of boundary form $\Xi$ for $\Phi$ such that $j^{2k-1} \sigma(K)$ is in the domain of definition of $\Xi$.

Proof. Equation (48) is the consequence of equations (36) and (37).
Equations (28), (36) and (37) yield
\[
\int_K \left[ j^k \sigma^* (\Phi_a) - P^i_{a,i} \right] Y^a \sigma d_m x = \int_K \left[ j^k \sigma^* (\Phi_a) - P^i_{a,i} - (P^i_{a,i} - P'^i_{a,i}) \right] Y^a \sigma d_m x
\]
\[ = \int_K \left[ j^k \sigma^* (\Phi_a) - P^i_{a,i} \right] Y^a \sigma d_m x. \]
because $P^i_{a,i} - P'^i_{a,i} = 0$. Therefore, decompositions (28) for the boundary forms $\Xi$ and $\Xi'$ yield
\[
\int_{j^{2k-1} \sigma(\partial K)} (Y^k \mathcal{J} \Xi) = \int_K j^k \sigma^* (Y^k \mathcal{J} \Phi) - \int_K \left[ j^k \sigma^* (\Phi_a) - P^i_{a,i} \right] Y^a \sigma d_m x
\]
\[ = \int_K j^k \sigma^* (Y^k \mathcal{J} \Phi) - \int_K \left[ j^k \sigma^* (\Phi_a) - P'^i_{a,i} \right] Y^a \sigma d_m x
\]
\[ = \int_{j^{2k-1} \sigma(\partial K)} (Y^k \mathcal{J} \Xi'). \]
This shows that the decomposition (28) is independent of the choice of $\Xi$. 

Since boundary forms are constructed in terms of adapted coordinate systems, non-uniqueness of the De Donder construction implies only local existence of the result. We see in Corollary 6 that decomposition (28) does not depend on the choice of boundary form with the same domain of definition. If boundary forms are globally defined, then decomposition (28) is unique and it holds for every section of $\pi$ and each relatively compact open submanifold $K$ of $M$ with piece-wise smooth boundary $\partial K$. The existence of global boundary forms is a topological condition on the fibration $\pi: N \to M$. It is satisfied if the fibration is trivial and $M$ and the stypical fibre of $\pi$ are diffeomorphic to open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. In particular, it is satisfied in many problems in continuum mechanics.
4 Application to variational problems

4.1 Critical points of action functionals

In this section, we consider the case when \( \Phi = d\Lambda \), where \( \Lambda \) is a semi-basic \( m \)-form on \( J^k(M, N) \). Let \( K \subseteq M \) be an open relatively compact submanifold of \( M \) with piece-wise smooth boundary \( \partial K \). As in Section 2, we consider the space \( S^\infty(\bar{K}, N) \) of smooth section \( \sigma : \bar{K} \to N \). The form \( \Lambda \) defines an action functional \( A \) on \( S^\infty(\bar{K}, N) \), given by

\[
A(\sigma) = \int_K j^k\sigma^*\Lambda = \int j^k_{\sigma(\bar{K})} \Lambda. \tag{49}
\]

**Definition 7** A section \( \sigma \in S^\infty(\bar{K}, N) \) is a critical point of \( A \) if \( D_\sigma A = 0 \) for every \( Y_\sigma \in T_\sigma S^\infty(\bar{K}, N) \), which vanishes on \( \partial K \) together with its partial derivatives up to order \( k - 1 \).

Taking into account equation (3), we see that \( \sigma \in S^\infty(\bar{K}, N) \) is a critical point of \( A \) if

\[
\int j^k_{\sigma(\bar{K})} \mathcal{L}_{Y_\sigma} \Lambda = 0 \tag{50}
\]

for every \( Y_\sigma \in T_\sigma S^\infty(\bar{K}, N) \), which vanishes on \( \partial K \) together with its partial derivatives up to order \( k - 1 \). Here, \( Y^k \) is the prolongation to \( J^k(M, N) \) of an extension of \( Y_\sigma \) to a vertical vector field \( Y \) on \( N \).

For every vector field \( Y \) on \( N \),

\[
\mathcal{L}_{Y^k} \Lambda = Y^k \cdot d\Lambda + d \left( Y^k \right) \Lambda. \tag{51}
\]

The identity (51) and Stokes’ Theorem yield

\[
\int j^k_{\sigma(\bar{K})} \mathcal{L}_{Y^k} \Lambda = \int j^k_{\sigma(\bar{K})} \left[ Y^k \cdot d\Lambda + d \left( Y^k \right) \Lambda \right]
= \int j^k_{\sigma(\bar{K})} Y^k \cdot d\Lambda + \int_{\partial j^k_{\sigma(\bar{K})}} Y^k \Lambda
= \int j^k_{\sigma(\bar{K})} Y^k \cdot d\Lambda
\]

because \( \Lambda \) is semi-basic with respect to the source map \( \pi^k : J^k(M, N) \to M \).

Hence equation (50) is equivalent to

\[
\int_K j^k\sigma^* \left( Y^k \cdot d\Lambda \right) = 0 \tag{52}
\]

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for every vertical vector field $Y$ on $N$. Therefore, $\sigma$ is a critical section of $A$ if, equation \((52)\) holds for every vertical vector field $Y$ on $N$ such that $Y^{k-1}$ vanishes on $\partial j^{k-1}(K) = j^{k-1}(\partial K)$.

### 4.2 Euler-Lagrange equations

The Euler-Lagrange equations are obtained by using the coordinate description of $\Lambda$,

$$\Lambda = L(x^i, y^a, z^a_{i_1, \ldots, i_k}) \, dx.$$

(53)

The usual rule that "variation of the derivative is the derivative of the variation" corresponds to the choice of extension of $Y_\sigma$ to a vertical vector field

$$Y = Y^a(x^i) \frac{\partial}{\partial y^a}$$

with components independent of $y^a$. Its prolongation to $J^k(M, N)$ is

$$Y^k(x^i, z^b_{j_1, \ldots, j_k}) = Y^a(x^i) \frac{\partial}{\partial y^a} \frac{\partial}{\partial z^a_{i_1, \ldots, i_k}} + \cdots + Y^a(x^i) \frac{\partial}{\partial z^a_{i_1, \ldots, i_k}}.$$  

(54)

Finally, the coordinate description of $j^k \sigma$ is

$$j^k \sigma : M \to J^k(M, N) : (x^i) \mapsto (x^i, y^a(x), z^a_{i_1, \ldots, i_k}(x)),$$

(55)

where

$$z^a_{i_1, \ldots, i_k}(x) = y^a(x), z^a_{j_1, \ldots, j_k}(x)$$

(56)

for every positive integer $l$. With this notation,

$$j^k \sigma^*(Y^k(\Lambda(\partial_m x))) = \left( \frac{\partial L}{\partial y^a} Y^a + \frac{\partial L}{\partial z^a_{i, j}} Y^a + \cdots + \frac{\partial L}{\partial z^a_{i_1, \ldots, i_k}} Y^a \right) \, dx.$$

(57)
where all quantities on the right hand side are expressed as functions of \((x^1, ..., x^n)\). Integrating this result over \(K\) and using Stokes Theorem yields

\[
\int_K j^k \sigma^*(Y^a \, d\Lambda) = \int_K \left( \frac{\partial L}{\partial y^a} Y^a + \frac{\partial L}{\partial z^a_{i_1}} Y^a_{i_1} + ... + \frac{\partial L}{\partial z^a_{i_1...i_k}} Y^a_{i_1...i_k} \right) \, d_m x
\]

\[
= \int_K \left[ \frac{\partial L}{\partial y^a} Y^a + \frac{\partial}{\partial x^{i_1}} \left( \frac{\partial L}{\partial z^a_{i_1}} Y^a_{i_1} \right) - \frac{\partial}{\partial x^{i_1}} \frac{\partial L}{\partial y^a} Y^a + \frac{\partial}{\partial x^{i_1}} \frac{\partial L}{\partial y^a} \right] \, d_m x +
\]

\[
+ \int_K \left[ \frac{\partial}{\partial x^{i_1}} \left( \frac{\partial L}{\partial z^a_{i_1...i_k}} Y^a_{i_1...i_k} \right) - \frac{\partial}{\partial x^{i_1}} \frac{\partial L}{\partial y^a} Y^a_{i_1...i_k} \right] \, d_m x,
\]

because \(Y^{k-1}\) vanishes on \(j^{k-1} \sigma(\partial K)\). Continuing integration by parts, we get

\[
\int_K j^k \sigma^*(Y^a \, d\Lambda) = \int_K \left( \frac{\partial L}{\partial y^a} - \frac{\partial}{\partial x^{i_1}} \frac{\partial L}{\partial z^a_{i_1}} + ... + (-1)^k \frac{\partial}{\partial x^{i_1...i_k}} \frac{\partial L}{\partial z^a_{i_1...i_k}} \right) Y^a \, d_m x
\]

\[
= \int_K \frac{\delta L}{\delta y^a} Y^a \, d_m x,
\]

(58)

where

\[
\frac{\delta L}{\delta y^a} = \frac{\partial L}{\partial y^a} - \frac{\partial}{\partial x^{i_1}} \frac{\partial L}{\partial z^a_{i_1}} + ... + (-1)^k \frac{\partial}{\partial x^{i_1...i_k}} \frac{\partial L}{\partial z^a_{i_1...i_k}}
\]

(59)

is called the Lagrange derivative of \(L\). Comparing equation (58) with equation (28) observe that, if \(\Phi = d(Ld_m x)\), then \(j^k \sigma^*(\Phi_a) - P^i_{a,j} = \frac{\delta L}{\delta y^a}\).

Taking into account the Fundamental Theorem in the Calculus of Variations, we conclude that \(\sigma\) is a critical section of \(A_K\) if and only if, for every \(a = 1, ..., n\),

\[
\left( \frac{\partial L}{\partial y^a} - \frac{\partial}{\partial x^{i_1}} \frac{\partial L}{\partial z^a_{i_1}} + ... + (-1)^k \frac{\partial}{\partial x^{i_1...i_k}} \frac{\partial L}{\partial z^a_{i_1...i_k}} \right) \bigg|_K = 0.
\]

(60)

Equations (60) are the Euler-Lagrange equations for critical points of the action functional corresponding to the Lagrangian \(L\).
4.3 De Donder equations

Let $\Xi$ be the boundary form of $d\Lambda$, and let

$$\Theta = \pi_k^{2k-1}\Lambda + \Xi.$$  \hfill (61)

Equation (61) generalizes the construction of De Donder [5] to $k > 1$. We refer to $\Theta$ as a De Donder form of $\Lambda$. It follows from Theorem 1 that $\Theta$ satisfies the following conditions.

**Corollary 8**  
1. $\Theta$ is semi-basic with respect basic to the forgetful map $\pi_{k-1}^{2k-1} : J^{2k-1}(M, N) \to J^k(M, N)$. In other words, for any vector field $X$ tangent to fibres of $\pi_{k-1}^{2k-1} : J^{2k-1}(M, N) \to J^k(M, N)$,

$$X \lrcorner \Theta = 0.$$  \hfill (62)

2. For every vector field $X$ on $J^{2k-1}(M, N)$ tangent to fibres of the source map $\pi_{k-1}^{2k-1} : J^{2k-1}(M, N) \to M$, the left interior product $X \lrcorner \Theta$ is semi-basic with respect to the source map. In other words, for every pair $X_1, X_2$ of vector fields on $J^{2k-1}(M, N)$ tangent to fibres of the source map $\pi_{k-1}^{2k-1} : J^{2k-1}(M, N) \to M$,

$$X_2 \lrcorner (X_1 \lrcorner \Theta) = 0.$$  \hfill (63)

3. For every section $\sigma$ of $\pi : N \to M$,

$$j^{2k-1}\sigma^* \Theta = j^k \sigma^* \Lambda.$$  \hfill (64)

4. For every vector field $X$ on $J^{2k-1}(M, N)$ tangent to fibres of the target map $\pi_{0}^{2k-1} : J^{2k-1}(M, N) \to N$, and every section $\sigma$ of $\pi : N \to M$,

$$j^{2k-1}\sigma^* (X \lrcorner d\Theta) = 0.$$  \hfill (65)

**Theorem 9** For $\sigma \in S^\infty(\bar{K}, N)$, suppose that $j^{2k-1}\sigma(\bar{K})$ is in the domain of a De Donder form $\Theta$. Then $\sigma$ is a critical section of the functional $A$, given by equation (49), if and only if

$$j^{2k-1}\sigma^* (X \lrcorner d\Theta) = 0$$  \hfill (66)

for every vector field $X$ on $J^{2k-1}(M, N)$ that is tangent to fibres of the source map $\pi_{k-1}^{2k-1} : J^{2k-1}(M, N) \to M$. 

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Proof. Equation (64) implies that replacing \( j^k \sigma^* \Lambda \) by \( j^{2k-1} \sigma^* \Theta \) in equation (49) does not change the action functional,

\[
A(\sigma) = \int_K j^k \sigma^* \Lambda = \int_{j^{2k-1} \sigma(K)} \Theta. \tag{67}
\]

Moreover, if \( Y \) is a vertical vector field on \( N \), then

\[
j^{2k-1} \sigma^* \mathcal{L}_{Y^{2k-1}} \Theta = j^k \sigma^* \mathcal{L}_{Y^k} \Lambda, \tag{68}
\]

where \( Y^{2k-1} \) is the prolongation of \( Y \) to \( J^{2k-1}(M, N) \). Hence, \( \sigma \) is a critical section of the functional \( A \) if and only if

\[
\int_K j^{2k-1} \sigma^* \mathcal{L}_{Y^{2k-1}} \Theta = 0 \tag{69}
\]

for every vertical vector field \( Y \) on \( N \) such that \( Y^{k-1} \) vanishes on \( j^{k-1} \sigma(\partial K) \).

The argument leading from equation (50) to equation (52) ensures that \( \sigma \) is a critical section of \( A \) if and only if

\[
\int_K j^{2k-1} \sigma^* (Y^{2k-1} \mathcal{J} d\Theta) = 0 \tag{70}
\]

for all vertical vector fields \( Y \) on \( N \), such that \( Y^{k-1} \) vanishes on \( j^{k-1} \sigma(\partial K) \).

Equation (65) ensures that in equation (70), we can replace \( Y^{2k-1} \) by an arbitrary vector field \( X \) on \( J^{2k-1}(M, N) \) that is tangent to fibres of the source map \( \pi^{2k-1} : J^{2k-1}(M, N) \to M \) and satisfies the condition \( T_{\pi^{2k-1}} \circ X \circ j^{k-1} \sigma(\partial K) = 0 \). In other words, we may omit the requirement that \( Y^{2k-1} \) is the prolongation of a vertical vector field \( Y \) on \( N \). This proves that that \( \sigma \) is a critical section of \( A_K \) if and only if

\[
\int_K j^{2k-1} \sigma^* (X \mathcal{J} d\Theta) = 0 \tag{71}
\]

for every vector field \( X \) on \( J^{2k-1}(M, N) \) that is tangent to fibres of the source map \( \pi^{2k-1} : J^{2k-1}(M, N) \to M \) and satisfies the condition \( T_{\pi^{2k-1}} \circ X \circ j^{k-1} \sigma(\partial K) = 0 \).

Suppose that \( \sigma \) is a critical section of \( A \). Equation (71) and the Fundamental Theorem in the Calculus of Variations, this condition ensure that

\[
j^{2k-1} \sigma^* (X \mathcal{J} d\Theta) = 0 \tag{72}
\]
for every vector field $X$ on $J^{2k-1}(M,N)$ that is tangent to fibres of the source map $\pi^{2k-1} : J^{2k-1}(M,N) \to M$.

Conversely, assume that equation (72) is satisfied for all vector fields on $J^{2k-1}(M,N)$ that are tangent to fibres of the source map $\pi^{2k-1} : J^{2k-1}(M,N) \to M$. Then, equation (71) is satisfied for every vector field $X$ on $J^{2k-1}(M,N)$ because the integrand is identically zero. In particular, equation (70) is satisfied for prolongations $Y^{2k-1}$ of vertical vector fields $Y$ on $N$ that vanish on $\partial K$ together with all derivatives up to order $k$. This ensures that $\sigma$ is a critical point of $A$.

We refer to (66) and (72) as De Donder equations. They are a system of equations in differential forms that is equivalent to Euler-Lagrange equations.

Note that Condition 4 in Corollary 8 on De Donder form $\Theta$, see equation (65), differs from equation (72) only by restriction on the range of the vector field $X$. We can combine these two conditions in the corollary below.

**Corollary 10** A section $\sigma \in S^\infty(K,\bar{N})$ is a critical section of the functional $A$, given by equation (49), if there exists a boundary form $\Xi$ such that

$$j^{2k-1}\sigma^*(X \star d(\pi^{2k-1}_k\Lambda + \Xi)) = 0$$

(73)

for every vector field $X$ on $J^{2k-1}(M,N)$ that is tangent to fibres of the source map $\pi^{2k-1} : J^{2k-1}(M,N) \to M$.

Equation (73) is a relation in the space of pairs $(\Xi, \sigma)$. However, it is not in the form of the symplectic relation occurring in Tulczyjew triples. For a discussion of Tulczyjew triples in higher derivative field theory see reference [8].

Since boundary forms are defined only locally, the assumption in Theorem 9 appears to be quite restrictive. We show that this is not the case.

**Proposition 11** A section $\sigma \in S^\infty(K,\bar{N})$ is a critical section of the functional $A$ if there exists an open cover $\{U_\alpha\}$ of $\bar{K} \subset M$ such that $j^{2k-1}\sigma(U_\alpha)$ is in the domain of a De Donder form $\Theta_\alpha$, and

$$j^{2k-1}\sigma^*(X \star d\Theta_\alpha)|_{K \cap U_\alpha} = 0$$

(74)

for each $\alpha$, and every vector field $X$ on $J^{2k-1}(M,N)$.
Proof. Corollary 6 ensures that, if $\Xi$ and $\Xi'$ are boundary forms of $d\Lambda$ with the same domain and $\Theta$ and $\Theta'$ are the De Donder forms corresponding to $\Xi$ and $\Xi'$, respectively, then

$$j^{2k-1}\sigma^* (X \llcorner d\Theta') = j^{2k-1}\sigma^* (X \llcorner d\Theta)$$

for each section $\sigma$ of $\pi$ and every vector field $X$ on $J^{2k-1}(M, N)$. Hence, the choice of a De Donder form does not matter.

For each $\alpha$, equation (74) is equivalent to Euler-Lagrange equations for $\sigma$ in $K \cap U_\alpha$. Since Euler-Lagrange equations are local, the conditions of Proposition 11 imply that $\sigma$ satisfies Euler-Lagrange equations in $K$. $\blacksquare$

4.4 Symmetries and conservation laws

Definition 12 A vector field $Y$ on $N$ is an infinitesimal symmetry of the Lagrangian system with Lagrangian $\Lambda = Ld_mx$ of differential order $k$ if it projects to a vector field on $M$ and $\mathcal{L}_Y \Lambda = 0$.

Let $Y$ be an infinitesimal symmetry of $\Lambda$. For every boundary form $\Xi$ of $d\Lambda$, Lemma 3 ensures that $j^{2k-1}\sigma^*(\mathcal{L}_{Y^{2k-1}}\Xi) = 0$ for all sections $\sigma$ of $\pi : M \to N$. Since $\Theta = \pi^{2k-1}_*\Lambda + \Xi$ is the local De Donder form corresponding to $\Xi$, it follows that

$$j^{2k-1}\sigma^*(\mathcal{L}_{Y^{2k-1}}\Theta) = 0.$$ 

Hence,

$$j^{2k-1}\sigma^*(Y^{2k-1}\llcorner d\Theta) + j^{2k-1}\sigma^*[d(Y^{2k-1}\llcorner \Theta)] = 0.$$ 

If $\sigma$ satisfies De Donder equations, we get the conservation law

$$d \left[ j^{2k-1}\sigma^* (Y^{2k-1}\llcorner \Theta) \right] = 0.$$ 

If $K$ is an open, relatively compact submanifold of $M$ with boundary $\partial K$, which is contained in domain $\sigma$, then

$$\int_{\partial K} j^{2k-1}\sigma^* (Y^{2k-1}\llcorner \Theta) = 0.$$ 

In other words, if $\partial K = \Sigma_1 \cup \Sigma_2$, where $\Sigma_1$ and $\Sigma_2$ inherit outer orientation from $\partial K$, and $\Sigma_1 \cap \Sigma_2$ is smooth of dimension $n - 2$, then

$$\int_{\Sigma_1} j^{2k-1}\sigma^* (Y^{2k-1}\llcorner \Theta) = \int_{\Sigma_2} j^{2k-1}\sigma^* (Y^{2k-1}\llcorner \Theta).$$ 

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If the De Donder equations are hyperbolic, and $\Sigma_1$ and $\Sigma_2$ are Cauchy surfaces, then the integrals in equation (79) are conserved quantities corresponding to the infinitesimal symmetry $Y$.

A priori, the integrals on each side of equation (79) depend on the choice of the boundary form $\Xi$. However, the difference between the left and the right hand sides of equation (79) vanishes for every $\Xi$. In an example below, we show how boundary conditions lead to unique expressions for constants of motion.

5 Example

5.1 Cauchy problem

Consider $M = \mathbb{R}^2$ with coordinates $x = (x^1, x^2)$ and $N = T\mathbb{R}^2$ with coordinates $(x, y) = (x^1, x^2, y^1, y^2)$.

\begin{equation}
L = g_{ab}g^{ij}g^{kl}z^{a}_{ij}z^{b}_{kl},
\end{equation}

where $g_{ab}$ is the Minkowski metric.

\begin{equation}
\frac{\partial L}{\partial z^{a}_{ij}} = 2g_{ab}g^{ij}g^{kl}z^{b}_{kl},
\end{equation}

\begin{equation}
\frac{\partial L}{\partial z^{a}_{i}} = 0 \quad \text{and} \quad \frac{\partial L}{\partial y^{a}_{i}} = 0.
\end{equation}

Euler-Lagrange equations

\begin{equation}
\frac{\partial^2 L}{\partial x^i \partial x^j \partial y^{a}_{ij}} = 0
\end{equation}

\begin{equation}
2g_{ab}g^{ij}g^{kl}y^{a}_{klij} = 0.
\end{equation}

Writing $x^1 = t, x^2 = x$, we get

\begin{equation}
\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)y^a(t, x) = 0.
\end{equation}

\begin{equation}
\left(\frac{\partial^4}{\partial t^4} - 2\frac{\partial^2}{\partial t^2}\frac{\partial^2}{\partial x^2} + \frac{\partial^4}{\partial x^4}\right)y^a(t, x) = 0
\end{equation}
Set $y(t, x), \dot{y}(t, x), \ddot{y}(t, x)$, and $\dddot{y}(t, x)$ as the Cauchy data at $t$. Then
\[
\frac{\partial}{\partial t} y^a(t, x) = \dot{y}^a(t, x), \\
\frac{\partial}{\partial t} \dot{y}^a(t, x) = \ddot{y}^a(t, x), \\
\frac{\partial}{\partial t} \ddot{y}^a(t, x) = \dddot{y}^a(t, x), \\
\frac{\partial}{\partial t} \dddot{y}^a(t, x) = \frac{\partial^4}{\partial t^4} y^a(t, x) = 2 \frac{\partial^2}{\partial x^2} \dot{y}^a(t, x) - \frac{\partial^4}{\partial x^4} y^a(t, x).
\]

Therefore,
\[
\frac{\partial}{\partial t} \begin{pmatrix}
y^a(t, x) \\
\dot{y}^a(t, x) \\
\ddot{y}^a(t, x) \\
\dddot{y}^a(t, x)
\end{pmatrix} = \begin{pmatrix}
\dot{y}^a(t, x) \\
\ddot{y}^a(t, x) \\
\dddot{y}^a(t, x) \\
2 \frac{\partial^2}{\partial x^2} \dot{y}^a(t, x) - \frac{\partial^4}{\partial x^4} y^a(t, x)
\end{pmatrix} = A \begin{pmatrix}
y^a(t, x) \\
\dot{y}^a(t, x) \\
\ddot{y}^a(t, x) \\
\dddot{y}^a(t, x)
\end{pmatrix}
\]
where
\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-\frac{\partial^4}{\partial x^4} & 0 & 2 \frac{\partial^2}{\partial x^2} & 0
\end{pmatrix}.
\]

Since
\[
e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}
\]
is well defined and
\[
\begin{pmatrix}
y^a(t, x) \\
\dot{y}^a(t, x) \\
\ddot{y}^a(t, x) \\
\dddot{y}^a(t, x)
\end{pmatrix} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} \begin{pmatrix}
y^a(0, x) \\
\dot{y}^a(0, x) \\
\ddot{y}^a(0, x) \\
\dddot{y}^a(0, x)
\end{pmatrix}
\]
is a solution of the Cauchy problem at $t = 0$.

5.2 De Donder forms

De Donder forms are $\pi_2^3 \Lambda + \Xi$, where
\[
\Xi = p^i_a (dy^a - z^a_j dx^j) \wedge \left( \frac{\partial}{\partial x^i} d_2 x \right) + p^i_{a12} (dz^a_{i2} - z^a_{i2j} dx^j) \wedge \left( \frac{\partial}{\partial x^{a1}} d_2 x \right)
\]
is a local boundary form corresponding to $d\Lambda$. For a section $\sigma$ of $\pi$,

$$
\Theta_{\text{range } j^3} = n_{2}^{3s} \Lambda_{\text{range } j^3} + \Xi_{\text{range } j^3}
$$

(83)

$$
= Ld_2x + P_{a}^{i_{1}}\left(dy^{a} - z_{ij}^{a}dx^{j}\right) \wedge \left(\frac{\partial}{\partial x^{i_{1}}}d_2x\right) + P_{a}^{i_{1}i_{2}}\left(d_2x^{a} - z_{ij}^{a}dx^{j}\right) \wedge \left(\frac{\partial}{\partial x^{i_{1}}}d_2x\right)
$$

$$
= P_{a}^{i_{1}}dy^{a} \wedge \left(\frac{\partial}{\partial x^{i_{1}}}d_2x\right) + P_{a}^{i_{1}i_{2}}dz_{ij}^{a} \wedge \left(\frac{\partial}{\partial x^{i_{1}}}d_2x\right) - \left(P_{a}^{i}y_{i}^{a} + P_{a}^{ij}y_{ij} - L\right)d_2x.
$$

where the functions $P_{a}^{i_{1}}$ and $P_{a}^{i_{1}i_{2}}$ satisfy the equations

$$
\frac{\partial L}{\partial z_{ij}^{a}} - P_{a}^{(i_{1}i_{2})} = 0,
$$

(84)

$$
\frac{\partial L}{\partial z_{i}^{a}} - P_{a}^{i} - P_{a,i_{1}}^{i_{1}} = 0,
$$

that follow from equations (20).

Since

$$
d\Lambda = \frac{\partial L}{\partial z_{ij}^{a}}dz_{ij}^{a} \wedge d_2x + \frac{\partial L}{\partial z_{i}^{a}}dz_{i}^{a} \wedge d_2x + \frac{\partial L}{\partial y^{a}}dy^{a} \wedge d_2x = 2g_{ab}g^{ij}g^{kl}z_{kl} \wedge d_2x,
$$

it follows that

$$
\frac{\partial L}{\partial z_{ij}^{a}} = 2g_{ab}g^{ij}g^{kl}y_{kl},
$$

$$
\frac{\partial L}{\partial z_{i}^{a}} = 0.
$$

Hence, the symmetric solution is

$$
P_{a}^{i_{1}i_{2}} = \Phi_{a}^{i_{1}i_{2}} = 2g_{ab}g^{ij}g^{kl}y_{kl},
$$

$$
P_{a,i_{1}}^{i_{1}} = \Phi_{a,i_{1}}^{i_{1}} = 2g_{ab}g^{ij}g^{kl}y_{ijkl},
$$

$$
P_{a}^{i} = \Phi_{a}^{i} = -2g_{ab}g^{ij}g^{kl}y_{ijkl}.
$$

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In this case

\[ \Xi |_{\text{range } j^j} = P^{i_1 i_2}_a (dy^a - z^a_j dx^j) \wedge \left( \frac{\partial}{\partial x^i_2} \mathfrak{L} d_2 x \right) + P^{i_1 i_2}_a (dz^a_{i_2} - z^a_{i_2 j} dx^j) \wedge \left( \frac{\partial}{\partial x^{i_2}} \mathfrak{L} d_2 x \right) \]

\[ = -2 g_{ab} g^{ij} g^{kl} y_{,kli_1} (dy^a - z^a_j dx^j) \wedge \left( \frac{\partial}{\partial x^{i_2}} d_2 x \right) + 2 g_{ab} g^{ij} g^{kl} y_{,kl} (dz^a_{i_2} - z^a_{i_2 j} dx^j) \wedge \left( \frac{\partial}{\partial x^{i_2}} d_2 x \right), \]

and

\[ \Theta |_{\text{range } j^j} = \pi_{2,2}^3 \Lambda |_{\text{range } j^j} + \Xi |_{\text{range } j^j} \]

\[ = L d_2 x + P^{i_1 i_2}_a (dy^a - z^a_j dx^j) \wedge \left( \frac{\partial}{\partial x^i_2} \mathfrak{L} d_2 x \right) + P^{i_1 i_2}_a (dz^a_{i_2} - z^a_{i_2 j} dx^j) \wedge \left( \frac{\partial}{\partial x^{i_2}} \mathfrak{L} d_2 x \right) \]

\[ = g_{ab} g^{ij} g^{kl} z^a_{i_2} z^b_{kl} d_2 x - 2 g_{ab} g^{ij} g^{kl} y_{,kli_1} (dy^a - z^a_j dx^j) \wedge \left( \frac{\partial}{\partial x^{i_2}} \mathfrak{L} d_2 x \right) + 2 g_{ab} g^{ij} g^{kl} y_{,kl} (dz^a_{i_2} - z^a_{i_2 j} dx^j) \wedge \left( \frac{\partial}{\partial x^{i_2}} \mathfrak{L} d_2 x \right). \]

A non-symmetric solution of equation (84) is

\[ P^{i_1 i_2}_a = P^{i_1 i_2}_a + Q^{i_1 i_2}_a = 2 g_{ab} g^{ij} g^{kl} y_{,kli_1} + Q^{i_1 i_2}_a, \]

\[ P^{i_1 i_2}_a = P^{i_1 i_2}_a + Q^{i_1 i_2}_a = 2 g_{ab} g^{ij} g^{kl} y_{,kli_1} + Q^{i_1 i_2}_a, \]

\[ P^i_a = \Phi^i_a - P^{i_1 i_2}_a = -2 g_{ab} g^{ij} g^{kl} y_{,kli_1} - Q^{i_1 i_2}_a, \]

where \( Q^{i_1 i_2}_a \) is skew symmetric in \( i_1 \) and \( i_2 \),

\[ Q^{i_1 i_2}_a = -Q^{i_2 i_1}_a. \]

### 5.3 Symmetries

A vector field \( Y = Y^i \frac{\partial}{\partial x^i} + Y^a \frac{\partial}{\partial y^a} \) is a symmetry if

\[ \mathcal{L}_Y (L d_2 x) = 0, \]
where \( Y^2 = Y^i \frac{\partial}{\partial x^i} + Y^a \frac{\partial}{\partial y^a} + Y_i \frac{\partial}{\partial z_i^a} + Y_{ij} \frac{\partial}{\partial z_{ij}} \) of \( Y \) is the prolongation of \( Y \) to \( J^2(M, N) \) and

\[
Y^i_i = Y^a_i \frac{\partial}{\partial z_i^a} - z_i^a Y^a_i + Y^a_i,
\]

(89)

\[
Y^a_{ij} = Z(j) Y^a_i + Z(i) Y^a_j + Y^a_{ij},
\]

see equations (106) and (107) in the Appendix. The Lorentz metric \( g_{ij} dx^i dx^j = (dt)^2 - (dx)^2 \) occurring in the Lagrangian has Killing vector \( Y_T = \frac{\partial}{\partial x^i} \), \( Y_S = \frac{\partial}{\partial x^i} \)
corresponding to the time and the space translations, and the infinitesimal Lorentz transformation \( Y_L = x^2 \frac{\partial}{\partial x^i} + x^1 \frac{\partial}{\partial x^i} \).

We discuss here conservation of energy corresponding to the time translations \( Y_T \). Equations (89) show that the jet components of \( Y^2_T \) and \( Y^3_T \) vanish, so that \( Y^2_T = \frac{\partial}{\partial x^i} \) and \( Y^3_T = \frac{\partial}{\partial x^i} \). Hence,

\[
\mathcal{L}_{Y^2_T}(Ld_2 x) = \mathcal{L}_{Y^3_T}(Ld_2 x) = 0.
\]

Since

\[
(\frac{\partial}{\partial x^i} L d_2 x) = \delta^i_1,
\]

\[
\frac{\partial}{\partial x^i} L d_2 x = \frac{\partial}{\partial x^i} L d x^1 \wedge d x^2 = \delta^1_i d x^2 - \delta^2_i d x^1,
\]

and

\[
(\frac{\partial}{\partial x^i} L d x^j) = (\frac{\partial}{\partial x^i} L d x^j) (\delta^1_i d x^2 - \delta^2_i d x^1) = -\delta^i_1.
\]

Equation (83) yields

\[
J^3 \sigma^* (Y^3_T \Theta) = \int J^3 \sigma^* \left[ -P^a_i \, dy^a \wedge (Y^2_T \left( \frac{\partial}{\partial x^i} \right) L d_2 x) \right] + \int J^3 \sigma^* \left[ P^a_i z_i^a \wedge (Y^2_T \left( \frac{\partial}{\partial x^i} \right) L d_2 x) \right] - (P^a_i \, y^a_3 + P^a_i y^a_2) \, L \, (Y^2_T \left( \frac{\partial}{\partial x^i} \right) d_2 x) = P^2_a \, y^a_3 \, dx^3 + P^2_3 \, y^a_2 \, dx^3 - (P^a_i \, y^a_3 + P^a_i y^a_2) \, dx^2.
\]

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For an open relatively compact manifold $K$ with boundary $\partial K = \Sigma - \Sigma'$,

$$\int_{\Sigma} j^3 \sigma^e (Y^2_T \mathbf{j} \Theta) = \int_{\Sigma} [P^2 a y^a_j dx^j + P^{2i} a y^a_{ij} dx^j - (P^i a y^a_i + P^i j a y,ij - L) dx^2].$$

Let us decompose $P^i a$ into its symmetric and antisymmetric parts in the upper indices

$$P^i a = P^{(i)} a + P^{[i]} a.$$

Then,

$$P^i a = \frac{\partial L}{\partial z^a_i} - P_{a,i1}^i a = \frac{\partial L}{\partial z^a_i} - P^{(i)} a - P^{[i]} a i_1,$$

so that

$$\int_{\Sigma} j^3 \sigma^e (Y^2_T \mathbf{j} \Theta) = \int_{\Sigma} [P^2 a y^a_j dx^j + P^{2i} a y^a_{ij} dx^j - (P^i a y^a_i + P^i j a y,ij - L) dx^2].$$

The last integral in equations (92) involves only the odd terms $P^i a$.

It can
be rewritten as follows

\[
\int_\Sigma \left[ -P_{a,i_1}^{[i_2]} y_j^a d x^j + P_{a}^{[i_2]} y_{i_3 j}^a d x^j + P_{a,i_1}^{[i_2]} y_j^a d x^2 \right] = \tag{93}
\]

\[
= \int_\Sigma \left[ \left( -P_{a}^{[12]} - P_{a}^{[22]} \right) y_j^a d x^j + \left( P_{a}^{[21]} y_{1 j}^a + P_{a}^{[22]} y_{2 j}^a \right) d x^j \right] \\
+ \int_\Sigma \left( P_{a,1}^{[11]} y_1^a + P_{a,1}^{[12]} y_2^a + P_{a,2}^{[21]} y_1^a + P_{a,2}^{[22]} y_2^a \right) d x^2 \\
= \int_\Sigma \left[ -P_{a,1}^{[12]} \left( y_1^a d x^1 + y_2^a d x^2 \right) + P_{a}^{[21]} y_{1 j}^a d x^j + \left( P_{a,1}^{[12]} y_{2 j}^a + P_{a,1}^{[21]} y_1^a \right) d x^2 \right] \\
= \int_\Sigma \left[ -P_{a,1}^{[12]} y_1^a d x^1 + P_{a}^{[21]} y_{1 j}^a d x^j + P_{a,2}^{[21]} y_1^a d x^2 \right] \\
= \int_\Sigma \left[ P_{a}^{[21]} y_{1 j}^a d x^j - P_{a,1}^{[12]} y_1^a d x^1 + P_{a,2}^{[21]} y_1^a d x^2 \right] \\
= \int_\Sigma \left[ d \left( P_{a}^{[21]} y_1^a \right) - P_{a,1}^{[21]} y_1^a d x^j - P_{a,1}^{[12]} y_1^a d x^1 + P_{a,2}^{[21]} y_1^a d x^2 \right] \\
= \int_\Sigma \left[ d \left( P_{a}^{[21]} y_1^a \right) - P_{a,1}^{[21]} y_1^a d x^1 - P_{a,2}^{[21]} y_1^a d x^2 - P_{a,1}^{[12]} y_1^a d x^1 + P_{a,2}^{[21]} y_1^a d x^2 \right] \\
= \int_\Sigma \left[ d \left( P_{a}^{[21]} y_1^a \right) - \left( P_{a,1}^{[21]} y_1^a d x^1 + P_{a,1}^{[12]} y_1^a d x^1 \right) - P_{a,2}^{[21]} y_1^a d x^2 + P_{a,2}^{[21]} y_1^a d x^2 \right] \\
= \int_\Sigma d \left( P_{a}^{[21]} y_1^a \right)
\]

Since, we consider an evolution equation with non-compact Cauchy surfaces, replace \( K \) by a slice

\[
S = \{ (x^1, x^2) \in \mathbb{R}^2 \mid 0 < x^1 < t \}
\]

with boundary

\[
\partial S = \Sigma_t - \Sigma_0 = \{ (t, x_2) \in \mathbb{R}^2 \} - \{ (0, x_2) \in \mathbb{R}^2 \}.
\]

We assume that the fields \( y^a(x_1, x_2) \) vanish sufficiently fast as \( x_2 \to \pm \infty \), so that integrals over \( K, \Sigma_t \) and \( \Sigma_0 \) converge and permit integration by parts.
For \( \Sigma = \Sigma_t \), equations (92) and (93) yield

\[
\int_{\Sigma_t} j^3 \sigma^* (Y_T \Omega_\Theta) = \int_{\Sigma_t} \left[ \left( \frac{\partial L}{\partial z_a^2} - P_{a,i_1}^{(i_2)} \right) y_j^a dx^j + \left( P_{a}^{(2i_2)} \right) y_{i_2j}^a dx^j \right] \tag{94}
\]

\[
- \int_{\Sigma_t} \left( \left( \frac{\partial L}{\partial z_a^2} - P_{a,i_1}^{(i_2)} \right) y_i^a + P_{a}^{ij} y_{ij} - L \right) dx^2 + \int_{\Sigma_t} d \left( P_{a}^{[2]} y_1^a \right)
\]

\[
= \int_{\Sigma_t} \left[ \left( \frac{\partial L}{\partial z_a^2} - P_{a,i_1}^{(i_2)} \right) y_j^a dx^j + \left( P_{a}^{(2i_2)} \right) y_{i_2j}^a dx^j \right] +
\]

\[
- \int_{\Sigma_t} \left( \left( \frac{\partial L}{\partial z_a^2} - P_{a,i_1}^{(i_2)} \right) y_i^a + P_{a}^{ij} y_{ij} - L \right) dx^2 +
\]

\[
+ \lim_{x_2 \to \infty} (P_{a}^{[2]} y_1^a)(t, x_2) - \lim_{x_2 \to -\infty} (P_{a}^{[2]} y_1^a)(t, x_2)
\]

because our asymptotic conditions require that \( \lim_{x_2 \to \infty} (P_{a}^{[2]} y_1^a)(t, x_2) = 0 \) and \( \lim_{x_2 \to -\infty} (P_{a}^{[2]} y_1^a)(t, x_2) = 0 \). Hence the potential non-uniqueness of constants of motion is taken care of by the appropriate choice of boundary conditions.

6 Appendix

6.1 Jets

Let \( \pi : N \to M \) be a locally trivial fibration. A local section \( \sigma \) of \( \pi \) is a smooth map \( \sigma : M \to N \), defined on an open subset \( U \) of \( M \), such that \( \pi \circ \sigma(x) = x \) for every \( x \in U \). If \( U = M \), we say that \( \sigma : M \to N \) is a global section of \( \pi \). In the following, we say \( \sigma : M \to N \) is a section of \( \pi \) if \( \sigma \) is either local or global section.

Suppose that \( m = \dim M \) and \( m + n = \dim N \). We use local coordinates \((x^i)\) on \( M \), where \( i = 1, ..., m \), and \((x^i, y^a)\) on \( N \), where \( a = 1, ..., n \). The local coordinate description of a section \( \sigma : M \to N \) is given by \( y^a = \sigma^a(x^1, ..., x^m) \) for \( a = 1, ..., n \).
For each \( x \in M \) and \( k = 1, 2, \ldots \), sections \( \sigma \) and \( \tilde{\sigma} \) of \( \pi \) are \( k \)-equivalent at \( x \) if \( \sigma(x) = \tilde{\sigma}(x) \) and, in local coordinates,

\[
\sigma^a_{,i_1\ldots i_l}(x) = \tilde{\sigma}^a_{,i_1\ldots i_l}(x),
\]

where

\[
\sigma^a_{,i_1\ldots i_l}(x) = \frac{\partial^l \sigma^a}{\partial x^{i_1} \cdots \partial x^{i_l}}(x^1(x), \ldots, x^n(x)),
\]

for all \( l = 1, \ldots, k \).

The \( k \)-equivalence class at \( x \) of a section \( \sigma \) is called the \( k \)-jet of \( \sigma \) at \( x \) and denoted \( j^k \sigma(x) \). The space of \( k \)-equivalence classes at \( x \) of all section \( \sigma \) is denoted \( J^k_x(M, N) \) and

\[
J^k(M, N) = \bigcup_{x \in M} J^k_x(M, N)
\]

is called the space of \( k \)-jets of sections of \( \pi \). In terms of local coordinates, \( j^k \sigma(x) \) has coordinates \( (x^i, y^a, z^a_{i_1}, \ldots, z^a_{i_1\ldots i_l}) \), where

\[
z^a_{i_1\ldots i_l} = \sigma^a_{,i_1\ldots i_l}(x),
\]

for \( l = 1, \ldots, k \), \( i_1, \ldots, i_l = 1, \ldots, n \), and \( a = 1, \ldots, m \). Since partial derivatives of a smooth function commute, the variables \( z^a_{i_1\ldots i_l} \) cannot be considered as independent coordinates. In the case when it matters, we use an independent collection

\[
\{z^a_{i_1\ldots i_l} | a = 1, \ldots, m, \text{ and } 1 \leq i_1 \leq i_2 \leq \ldots \leq i_l\}; \tag{97}
\]

see equation (7). However, in general, we use symmetry of variables \( z^a_{i_1\ldots i_l} \) in the indices \( i_1, \ldots, i_l \).

There are several maps defined on \( J^k(M, N) \):

- **the source map**
  \[
  \pi^k : J^k(M, N) \to M : j^k \sigma(x) \mapsto x,
  \]

- **the target map**
  \[
  \pi^k_0 : J^k(M, N) \to N : j^k \sigma(x) \mapsto \sigma(x),
  \]

- **the \((k, l)\)-forgetful**
  \[
  \pi^k_l : J^k(M, N) \to J^l(M, N) : j^k \sigma(x) \mapsto j^l \sigma(x) \quad \text{for } k > 1 > 0.
  \]

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Each of these maps defines a fibre bundle structure in $J^k(M, N)$. For this reason, $J^k(M, N)$ is also called the k-jet bundle of sections of $\pi$.

Let $\sigma : M \to N$ be a section of $\pi : N \to M$. We denote the k-jet extension of $\sigma$ by

$$j^k\sigma : M \to J^k(M, N) : x \mapsto j^k\sigma(x).$$

For every integer $k > 0$,

$$\pi^k_0 \circ j^k\sigma = \sigma.$$ (98)

Similarly, for each $k > l > 0$,

$$\pi^k_l \circ j^k\sigma = j^l\sigma.$$

A section $\rho : M \to J^k(M, N)$ of the source map $\pi^k : J^k(M, N) \to M$ is called holonomic if there exists a section $\sigma$ of $\pi$ such that

$$\rho = j^k(\sigma).$$

It follows from equation (98) that $\rho$ is holonomic if and only if

$$\rho = j^k(\pi^k_0 \circ \rho).$$

Each local chart $M, \ldots$ gives rise to local contact forms on $J^k(M, N)$ given by

$$\vartheta^a = dy^a - \sum_{i=1}^{m} z_i^a dx^i, \quad \vartheta^a_i = dz_i^a - \sum_{j=1}^{m} z_{ij}^a dx^j, \quad \ldots$$ (99)

$$\vartheta^{a_{i_1 \ldots i_{k-1}}} = dz^{a_{i_1 \ldots i_{k-1}}} - \sum_{i_k=1}^{m} z^{a_{i_1 \ldots i_{k-1} i_k}} dx^{i_k}.$$ 

A section $\rho : M \to J^k(M, N)$ of the source map $\pi^k$ is holonomic if the tangent space of its range is annihilated by the contact forms $\vartheta^a_{i_1 \ldots i_l}$ for all $l = 0, \ldots, k$ and all indices $i_1, \ldots, i_l = 1, \ldots, n$ and any collection of coordinate charts covering $M$.

6.2 Prolongations

Let $Y$ be a vector field on $N$ which projects to a vector field $Y^0$ on $M$. In other words, $Y$ is $\pi$-related to a vector field $Y^0$, that is

$$T\pi \circ Y = Y^0 \circ \pi.$$ (100)
This implies that $\pi : N \rightarrow M$ intertwines the actions of local-one-parameter local groups $e^{tY}$ and $e^{tY_0}$ generated by $Y$ and $Y_0$, respectively.

\[
\begin{array}{ccl}
N & \rightarrow & N \\
\pi & \downarrow & \downarrow \pi \\
M & \rightarrow & M \\
e^{tY_0}
\end{array}
\]  

Hence, for every section $\sigma$ of $\pi$,

\[e^{tY_0} \sigma = e^{tY} \circ \sigma \circ e^{-tY_0}\]  

is a local section of $\pi$. For every integer $k$, the map $\sigma \mapsto e^{tY_0} \sigma$ induces a local-one parameter local group

\[e^{tY_k} : J^k(M, N) \rightarrow J^k(M, N) : j^k \sigma(x) \mapsto [j^k(e^{tY_0} \sigma)](e^{tY_0} x).\]  

of diffeomorphisms of $J^k(M, N)$ to itself, generated by a vector field $Y_k$ on $J^k(M, N)$, called the prolongation of $Y$ to $J^k(M, N)$. In other words,

\[e^{tY_k} \circ j^k \sigma = j^k(e^{tY_0} \sigma).\]  

For every $0 < l < k$, $Y_k$ is $\pi_l^k$-related to $Y_l$,

\[T_{\pi_l^k} Y_k = Y_l \circ \pi_l^k,\]  

where $\pi_l^k : J^k(M, N) \rightarrow J^l(M, N)$ is the forgetful map.

Following reference [13], we show how to find the prolongation $Y^k$ of a vector field $Y = Y^i(x) \frac{\partial}{\partial x^i} + Y^a(x, y) \frac{\partial}{\partial y^a}$ on $N$ that is $\pi$-related to $Y^0 = Y^i(x) \frac{\partial}{\partial x^i}$ on $M$ using the condition that, for every local contact form $\vartheta$ on $J^k(M, N)$, the Lie derivative $\mathcal{L}_{Y^k} \vartheta$ of $\vartheta$ with respect to $Y^k$ is a linear combination of local contact forms. Let

\[Y^k = Y^i \frac{\partial}{\partial x^i} + Y^a \frac{\partial}{\partial y^a} + Y_a^i \frac{\partial}{\partial z_a^i} + ... + Y_a^{i_1 ... i_k} \frac{\partial}{\partial z_a^{i_1 ... i_k}}\]
be the prolongation of $Y$ to $J^k(M,N)$. Then,

$$
\mathcal{L}_{Y^k}[dy^a - z_i^a dx^i]
= Y^k \left(a[dz_i^a - z_i^a dx^i]\right) + d \left(Y^k \left(a[dy^a - z_i^a dx^i]\right)\right)
= -Y^k \left(a[dz_i^a \wedge dx^i]\right) + d \left(Y^a - z_i^a Y^i\right)
= -Y^a_i dx^i + Y^i dz_i^a + Y^a_{ib} dy^b + Y^a_{ij} dx^j - Y^i_i dx^a - z_i^a Y^i_j dx^j
= Y^a_{ib} \left(dy^b - z_i^a dx^i\right) + Y^a_{ij} dz_i^a + Y^a_{ij} dx^j + Y^a_{ij} dx^i - Y^a_{ij} dz_i^a - z_i^a Y^a_{ij} dx^j
= Y^a_{ib} \left(dy^b - z_i^a dx^i\right) + \left(\left(Y^a_{ib} - z_i^a Y^a_{ij} + Y^a_{ij}\right) - Y^a_{ij}\right) dx^j,
$$

which implies that

$$
Y^a_i = Y^a_{ib} z_i^b - z_j^a Y^j_i + Y^a_i.
$$

Similarly,

$$
\mathcal{L}_{Y^k}[dz_i^a - z_{ij}^a dx^j]
= Y^k \left(a[dz_i^a - z_{ij}^a dx^j]\right) + d \left(Y^k \left(a[dz_i^a - z_{ij}^a dx^j]\right)\right)
= -Y^k \left(a[dz_i^a \wedge dx^j]\right) + d \left(Y^i - z_{ij}^a Y^j\right)
= -Y^a_{ij} dx^j + Y^j dz_i^a + Y^a_{ib} dy^b + Y^a_{ij} dx^j - Y^i_i dx^a - z_{ij}^a Y^i_k dx^k
= Y^a_{ib} \left(dy^b - z_i^a dx^i\right) + Y^a_{ij} dz_i^a + Y^a_{ij} dx^j + Y^a_{ij} dx^i - Y^a_{ij} dz_i^a - z_i^a Y^a_{ij} dx^j
= Y^a_{ib} \left(dy^b - z_i^a dx^i\right) + \left(\left(Y^a_{ib} - z_i^a Y^a_{ij} + Y^a_{ij}\right) - Y^a_{ij}\right) dx^j,
$$

so that, for $i \leq j$,

$$
Y^a_{ij} = Y^a_{ib} z_i^b - z_k^a Y^k_{ij} + Y^a_{ij}.
$$

Symmetrizing, we get

$$
Y^a_{ij} = z_i^b Y^a_{ib} - z_{k(i,i)}^a Y^k_{ij} + Y^a_{i,j}.
$$

(107)
In general,
\[
\mathcal{L}_{Y^k}[dz^{a_{i_1...i_l}}_i - z^{a_{i_1...i_l}}_i dx^j] = 
\]
\[
Y^k \mathcal{L} \left( d[z^{a_{i_1...i_l}}_i - z^{a_{i_1...i_l}}_i dx^j] \right) + d \left( Y^k \right)
\]
\[
= -Y^k \mathcal{L} \left( dz^{a_{i_1...i_l}}_i dx^j \right) + d \left( Y^k_{i_1...i_l} z^{a_{i_1...i_l}}_i dx^j \right)
\]
\[
= -Y^a_{i_1...i_l} dx^j + Y^b_{i_1...i_l, j} dy^b + Y^a_{i_1...i_l} dx^j
\]
\[
- Y^k_{i_1...i_l} z^{a_{i_1...i_l}}_i dx^j - z^{a_{i_1...i_l}}_i dx^k
\]
\[
= -Y^a_{i_1...i_l} dx^j + Y^a_{i_1...i_l, i} dx^j - z^{a_{i_1...i_l}}_i dx^j
\]
\[
= Y^a_{i_1...i_l} \left( dy^b - z^{b}_{j} dx^j \right) + Y^a_{i_1...i_l, i} z^{b}_{j} dx^j - Y^a_{i_1...i_l} dx^j
\]
\[
+ Y^a_{i_1...i_l, i} \left( dy^b - z^{b}_{j} dx^j \right) + \left( Y^a_{i_1...i_l, i} z^{b}_{j} - z^{a_{i_1...i_l}}_i \right) dx^j
\]
\[
= Y^a_{i_1...i_l} dx^j - z^{a_{i_1...i_l}}_i dx^k
\]
\[
= Y^a_{i_1...i_l, i} dx^j - z^{a_{i_1...i_l}}_i dx^k
\]
\[
= Y^a_{i_1...i_l} dx^j - z^{a_{i_1...i_l}}_i dx^k
\]
Therefore
\[
Y^a_{i_1...i_l, i} dx^j = z^{b}_{j} \left( Y^a_{i_1...i_l} - z^{a_{i_1...i_l}}_i \right) + Y^a_{i_1...i_l} dx^j. \tag{108}
\]

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