Complete solution of an optimization problem in tropical semifield

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Abstract

We consider a multidimensional optimization problem that is formulated in the framework of tropical mathematics to minimize a function defined on vectors over a tropical semifield (a semiring with idempotent addition and invertible multiplication). The function, given by a matrix and calculated through a multiplicative conjugate transposition, is nonlinear in the tropical mathematics sense. We show that all solutions of the problem satisfy a vector inequality, and then use this inequality to establish characteristic properties of the solution set. We examine the problem when the matrix is irreducible. We derive the minimum value in the problem, and find a set of solutions. The results are then extended to the case of arbitrary matrices. Furthermore, we represent all solutions of the problem as a family of subsets, each defined by a matrix that is obtained by using a matrix sparsification technique. We describe a backtracking procedure that offers an economical way to obtain all subsets in the family. Finally, the characteristic properties of the solution set are used to provide a complete solution in a closed form.

Key-Words: tropical semifield, tropical optimization, matrix sparsification, complete solution, backtracking.

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1 Introduction

Tropical (idempotent) mathematics, which deals with the theory and applications of semirings with idempotent addition \([1, 2, 3, 4, 5, 6, 7, 8]\), offers a useful analytical framework to solve many actual problems in operations research, computer science and other fields. These problems can be formulated and solved as optimization problems in the tropical mathematics setting, referred to as the tropical optimization problems. Examples

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of the application areas of tropical optimization include project scheduling [9, 10, 11, 12, 13, 14], location analysis [15, 16, 17, 18], and decision making [19, 20, 21, 22, 23].

Many tropical optimization problems are formulated to minimize or maximize functions defined on vectors over idempotent semifields (semirings with multiplicative inverses). These problems may have functions to optimize (objective functions), which can be linear or non-linear in the tropical mathematics sense, and constraints, which can take the form of vector inequalities and equalities. Some problems have direct, explicit solutions obtained under general assumptions. For other problems, only algorithmic solutions under restrictive conditions are known, which apply iterative numerical procedures to find a solution if it exists, or to indicate infeasibility of the problem otherwise. A short overview of tropical optimization problems and their solutions can be found in [24].

In this paper, we consider the tropical optimization problem as to

\[
\text{minimize } (Ax)^- x,
\]

where \( A \) is a given square matrix, \( x \) is an unknown vector, and the minus sign in the superscript serves to specify conjugate transposition of vectors.

A partial solution of the problem was obtained in [25]. The main purpose of this paper is to continue the investigation of the problem to derive a complete solution. We follow an approach developed in [26] and based on a characterization of the solution set. We show that all solutions of the problem satisfy a vector inequality, and then use this inequality to establish characteristic properties of the solution set. The solutions are represented as a family of solution subsets, each defined by a matrix that is obtained by using a matrix sparsification technique. We describe a backtracking procedure that offers an economical way to obtain all subsets in the family. Finally, the characteristic properties of the solution set are applied to provide a complete solution in a closed form. The results obtained are illustrated with illuminating numerical examples.

The rest of the paper is organized as follows. In Section 2, we give a brief overview of basic definitions and preliminary results of tropical algebra. Section 3 formulates the tropical optimization problem under study, and performs a preliminary analysis of the problem. The analysis includes the evaluation of the minimum of the objective function, the derivation of a partial solution, and the investigation of the characteristic properties of the solutions. In Section 4, a complete solution to the problem is given as a family of subsets, and then represented in a compact closed vector form. Finally, Section 5 offers concluding remarks and suggestions for further research.
2 Preliminary Definitions and Results

We start with a brief overview of the preliminary definitions and results of tropical algebra to provide an appropriate formal background for the development of solutions for the tropical optimization problems in the subsequent sections. The overview is mainly based on the results in [27, 25, 12, 13, 14], which offer a useful framework to obtain solutions in a compact vector form, ready for further analysis and practical implementation. Additional details on tropical mathematics at both introductory and advanced levels can be found in many recent publications, including [1, 2, 3, 4, 5, 6, 7, 8].

2.1 Idempotent Semifield

An idempotent semifield is a system \((X, 0, 1, \oplus, \otimes)\), where \(X\) is a nonempty set endowed with associative and commutative operations, addition \(\oplus\) and multiplication \(\otimes\), which have as neutral elements the zero \(0\) and the one \(1\). Addition is idempotent, which implies \(x \oplus x = x\) for all \(x \in X\). Multiplication distributes over addition, has \(0\) as absorbing element, and is invertible, which gives any nonzero \(x\) its inverse \(x^{-1}\) such that \(x \otimes x^{-1} = 1\).

Idempotent addition induces on \(X\) a partial order such that \(x \leq y\) if and only if \(x \oplus y = y\). With respect to this order, both addition and multiplication are monotone, which means that, for all \(x, y, z \in X\), the inequality \(x \leq y\) entails that \(x \oplus z \leq y \oplus z\) and \(x \otimes z \leq y \otimes z\). Furthermore, inversion is antitone to take the inequality \(x \leq y\) into \(x^{-1} \geq y^{-1}\) for all nonzero \(x\) and \(y\). Finally, the inequality \(x \oplus y \leq z\) is equivalent to the pair of inequalities \(x \leq z\) and \(y \leq z\). The partial order is assumed to extend to a total order on the semifield.

The power notation with integer exponents is routinely defined to represent iterated products for all \(x \neq 0\) and integer \(p \geq 1\) in the form \(x^0 = 1\), \(x^p = x \otimes x^{p-1}\), \(x^{-p} = (x^{-1})^p\), and \(0^p = 0\). Moreover, the equation \(x^p = a\) is assumed to be solvable for any \(a\), which extends the notation to rational exponents. In what follows, the multiplication sign \(\otimes\) is, as usual, dropped to save writing.

A typical example of the semifield is the system \((\mathbb{R} \cup \{-\infty\}, -\infty, 0, \max, +)\), which is usually referred to as the max-plus algebra. In this semifield, the addition \(\oplus\) is defined as \(\max\), and the multiplication \(\otimes\) is as arithmetic addition. The number \(-\infty\) is taken as the zero \(0\), and \(0\) is as the one \(1\). For each \(x \in \mathbb{R}\), the inverse \(x^{-1}\) coincides with the conventional opposite number \(-x\). For any \(x, y \in \mathbb{R}\), the power \(x^y\) corresponds to the arithmetic product \(xy\). The order induced by idempotent addition complies with the natural linear order on \(\mathbb{R}\).
2.2 Matrix and Vector Algebra

The set of matrices over \( \mathbb{X} \) with \( m \) rows and \( n \) columns is denoted by \( \mathbb{X}^{m \times n} \). A matrix with all entries equal to 0 is the zero matrix denoted by 0. A matrix without zero rows (columns) is called row- (column-) regular.

For any matrices \( A, B \in \mathbb{X}^{m \times n} \) and \( C \in \mathbb{X}^{n \times l} \), and scalar \( x \in \mathbb{X} \), matrix addition, matrix multiplication and scalar multiplication are routinely defined by the entry-wise formulas

\[
(A \oplus B)_{ij} = \{A\}_{ij} \oplus \{B\}_{ij}, \quad \{AC\}_{ij} = \bigoplus_{k=1}^{n} \{A\}_{ik} \{C\}_{kj}, \quad \{xA\}_{ij} = x \{A\}_{ij}.
\]

For any nonzero matrix \( A = (a_{ij}) \in \mathbb{X}^{m \times n} \), the conjugate transpose is the matrix \( A^\dagger = (a_{ji}) \in \mathbb{X}^{n \times m} \), where \( a_{ji}^{-1} = a_{ji}^{-1} \) if \( a_{ji} \neq 0 \), and \( a_{ji} = 0 \) otherwise.

The properties of the scalar addition, multiplication and inversion with respect to the order relations are extended entry-wise to the matrix operations.

Consider square matrices in the set \( \mathbb{X}^{n \times n} \). A matrix is diagonal, if its off-diagonal entries are all equal to 0. A diagonal matrix with all diagonal entries equal to 1 is the identity matrix denoted by \( I \). The power notation with non-negative integer exponents serves to represent repeated multiplication as \( A^0 = I \), \( A^p = AA^{p-1} \) and \( 0^p = 0 \) for any non-zero matrix \( A \) and integer \( p \geq 1 \).

If a row-regular matrix \( A \) has exactly one non-zero entry in each row, then the inequalities \( A^\dagger A \leq I \) and \( AA^\dagger \geq I \) hold (corresponding, in the context of relational algebra, to the univalent and total properties of a relation \( A \)).

The trace of any matrix \( A = (a_{ij}) \) is routinely defined as

\[
\text{tr} A = \bigoplus_{i=1}^{n} a_{ii},
\]

and retains the standard properties of traces with respect to matrix addition and to matrix and scalar multiplications.

To represent solutions proposed in the subsequent sections, we exploit the function, which takes any matrix \( A \in \mathbb{X}^{n \times n} \) to the scalar

\[
\text{Tr}(A) = \bigoplus_{m=1}^{n} \text{tr} A^m.
\]

Provided that the condition \( \text{Tr}(A) \leq 1 \) holds, the asterisk operator (also known as the Kleene star) maps \( A \) to the matrix

\[
A^* = \bigoplus_{m=0}^{n-1} A^m.
\]
If $\text{Tr}(A) \leq 1$, then the inequality $A^k \leq A^*$ holds for all integer $k \geq 0$.

The description of the solutions also involves the matrix $A^+$ which is obtained from $A$ as follows. First, we assume that $\text{Tr}(A) \leq 1$, and calculate the matrices $A^*$ and $AA^* = A \oplus \cdots \oplus A^n$. Then, the matrix $A^+$ is constructed by taking those columns in the matrix $AA^*$ which have the diagonal entries equal to 1.

Any matrix that consists of one row (column) is considered a row (column) vector. All vectors are assumed to be column vectors, unless otherwise specified. The set of column vectors of order $n$ is denoted $\mathbb{X}^n$. A vector with all zero elements is the zero vector $0$. A vector is regular if it has no zero elements.

For any non-zero vector $x = (x_j) \in \mathbb{X}^n$, the conjugate transpose is the row vector $x^- = (x_j)$, where $x_j^* = x_j^{-1}$ if $x_j \neq 0$, and $x_j^* = 0$ otherwise.

For any non-zero vector $x$, the equality $x^- x = 1$ is obviously valid.

For any regular vectors $x, y \in \mathbb{X}^n$, the matrix inequality $x y^- \geq (x^- y)^{-1} I$ holds and becomes $x x^- \geq I$ when $y = x$.

A vector $b$ is said to be linearly dependent on vectors $a_1, \ldots, a_n$ if the equality $b = x_1 a_1 \oplus \cdots \oplus x_n a_n$ holds for some scalars $x_1, \ldots, x_n$. The vector $b$ is linearly dependent on $a_1, \ldots, a_n$ if and only if the condition $(A(b^- A)^{-1}b = 1$ is valid, where $A$ is the matrix with the vectors $a_1, \ldots, a_n$ as its columns.

A system of vectors $a_1, \ldots, a_n$ is linearly dependent if at least one vector is linearly dependent on others, and linearly independent otherwise.

Suppose that the system $a_1, \ldots, a_n$ is linearly dependent. To construct a maximal linearly independent system, we use a procedure that sequentially reduces the system until it becomes linearly independent. The procedure applies the above condition to examine the vectors one by one to remove a vector if it is linearly dependent on others, or to leave the vector in the system otherwise.

A scalar $\lambda \in \mathbb{X}$ is an eigenvalue and a non-zero vector $x \in \mathbb{X}^n$ is a corresponding eigenvector of a square matrix $A \in \mathbb{X}^{n \times n}$ if they satisfy the equality

$$Ax = \lambda x.$$ 

2.3 Reducible and Irreducible Matrices

A matrix $A \in \mathbb{X}^{n \times n}$ is reducible if simultaneous permutations of its rows and columns can transform it into a block-triangular normal form, and irreducible otherwise. The lower block-triangular normal form of the matrix $A$ is given by

$$A = \begin{pmatrix}
A_{11} & 0 & \cdots & 0 \\
A_{21} & A_{22} & & 0 \\
& \ddots & \ddots & \ddots \\
A_{s1} & A_{s2} & \cdots & A_{ss}
\end{pmatrix},$$

(1)
where, in each block row $i = 1, \ldots, s$, the diagonal block $A_{ii}$ is either irreducible or the zero matrix of order $n_i$, the off-diagonal blocks $A_{ij}$ are arbitrary matrices of size $n_i \times n_j$ for all $j < i$, and $n_1 + \cdots + n_s = n$.

Any irreducible matrix $A$ has only one eigenvalue, which is calculated as

$$\lambda = \bigoplus_{m=1}^{n} \text{tr}^{1/m}(A^m). \quad (2)$$

From (2) it follows, in particular, that $\text{tr}(A^m) \leq \lambda^m$ for all $m = 1, \ldots, n$.

All eigenvectors of the irreducible matrix $A$ are regular, and given by

$$\mathbf{x} = (\lambda^{-1}A)^+ \mathbf{u},$$

where $\mathbf{u}$ is any regular vector of appropriate size.

Note that every irreducible matrix is both row- and column-regular.

Let $A$ be a matrix represented in the form (1). Denote by $\lambda_i$ the eigenvalue of the diagonal block $A_{ii}$ for $i = 1, \ldots, s$. Then, the scalar $\lambda = \lambda_1 \oplus \cdots \oplus \lambda_s$ is the maximum eigenvalue of the matrix $A$, which is referred to as the spectral radius of $A$ and calculated as (2). For any irreducible matrix, the spectral radius coincides with the unique eigenvalue of the matrix.

Without loss of generality, the normal form (1) can be assumed to order all block rows, which have non-zero blocks on the diagonal and zero blocks elsewhere, before the block rows with non-zero off-diagonal blocks. Moreover, the rows, which have non-zero blocks only on the diagonal, can be arranged in increasing order of the eigenvalues of diagonal blocks. Then, the normal form is refined as

$$A = \begin{pmatrix}
A_{11} & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & & \vdots \\
0 & A_{rr} & 0 & \cdots & 0 \\
A_{r+1,1} & \cdots & A_{r+1,r} & A_{r+1,r+1} & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
A_{s1} & \cdots & A_{sr} & A_{s,r+1} & \cdots & A_{ss}
\end{pmatrix}, \quad (3)$$

where the eigenvalues of $A_{11}, \ldots, A_{rr}$ satisfy the condition $\lambda_1 \leq \cdots \leq \lambda_r$, and each row $i = r + 1, \ldots, s$ has a block $A_{ij} \neq 0$ for some $j < i$.

### 2.4 Vector Inequalities and Equations

In this subsection, we present solutions to vector inequalities, which appear below in the analysis of the optimization problem under study.

Suppose that, given a matrix $A \in \mathbb{X}^{m \times n}$ and a vector $d \in \mathbb{X}^m$, we need to find vectors $x \in \mathbb{X}^n$ to satisfy the inequality

$$Ax \leq d. \quad (4)$$
A direct solution proposed in [12] can be obtained as follows.

**Lemma 1.** For any column-regular matrix $A$ and regular vector $d$, all solutions to inequality (4) are given by the inequality $x \leq (d^\top A)^\top$.

Next, we consider the following problem: given a matrix $A \in \mathbb{X}^{n \times n}$, find regular vectors $x \in \mathbb{X}^n$ to satisfy the inequality

$$Ax \leq x. \quad (5)$$

The following result [27, 13] offers a direct solution to inequality (5).

**Theorem 2.** For any matrix $A$, the following statements hold:

1. If $\text{Tr}(A) \leq 1$, then all regular solutions to (5) are given by $x = A^* u$, where $u$ is any regular vector.
2. If $\text{Tr}(A) > 1$, then there is no regular solution.

We conclude this subsection with a solution to a vector equation. Given a matrix $A \in \mathbb{X}^{n \times n}$ and a vector $b \in \mathbb{X}^n$, the problem is to find regular vectors $x \in \mathbb{X}^n$ that solve the equation

$$Ax \oplus b = x. \quad (6)$$

The next statement [27] offers a solution when the matrix $A$ is irreducible.

**Theorem 3.** For any irreducible matrix $A$ and non-zero vector $b$, the following statements hold:

1. If $\text{Tr}(A) < 1$, then equation (6) has the unique regular solution $x = A^* b$.
2. If $\text{Tr}(A) = 1$, then all regular solutions to (6) are given by $x = A^* u \oplus A^* b$, where $u$ is any regular vector of appropriate size.
3. If $\text{Tr}(A) > 1$, then there is no regular solution.

### 3 Tropical Optimization Problem

We are now in a position to describe the optimization problem under study, and to provide some preliminary solution to the problem. The problem is formulated to minimize a function defined on vectors over a general idempotent semifield. Given a matrix $A \in \mathbb{X}^{n \times n}$, the problem is to find regular vectors $x \in \mathbb{X}^n$ that minimize

$$\text{minimize } (Ax)^\top x. \quad (7)$$
A partial solution to the problem for both irreducible and reducible matrices $A$ was given in [25]. The solutions offered below improve the previous results by adding new characterization properties of the solution set and by extending the partial solution given in the case of reducible matrices. The determination of the minimum value of the objective function, the derivation of the partial solution for irreducible matrices and the evaluation of the lower bound on the function for reducible matrices are taken from the previous proof, and presented here for the sake of completeness.

We start with a solution of problem (7) for an irreducible matrix.

**Lemma 4.** Let $A$ be an irreducible matrix with spectral radius $\lambda$. Then, the minimum value in problem (7) is equal to $\lambda - 1$, and all regular solutions are given by the inequality

$$x \leq \lambda - 1 Ax.$$  

Specifically, any eigenvector of the matrix $A$, given by $x = (\lambda - 1)A^+ u$, where $u$ is any regular vector of appropriate size, is a solution of the problem.

**Proof.** Let $x_0$ be an eigenvector of the matrix $A$. Since $A$ is irreducible, the vector $x_0$ is regular, and thus $x_0x_0^\top \geq I$. For any regular $x$, we obtain

$$(Ax)^- x \geq (A x_0 x_0^- x)^- x = (x_0^- x)^- (Ax_0)^- x = \lambda^{-1} (x_0^- x)^- x_0^- x = \lambda^{-1},$$

which means that $\lambda^{-1}$ is a lower bound for the objective function.

As the substitution $x = x_0$ yields $(Ax)^- x = (Ax_0)^- x_0 = \lambda^{-1} x_0^- x_0 = \lambda^{-1}$, the lower bound $\lambda^{-1}$ is strict, and thus presents the minimum in problem (7).

All regular vectors $x$ that solve the problem are determined by the equation $(Ax)^- x = \lambda^{-1}$. Since $\lambda^{-1}$ is the minimum, we can replace this equation by the inequality $(Ax)^- x \leq \lambda^{-1}$, where the matrix $A$ is irreducible and thus row-regular. Considering that $(Ax)^- x = \lambda^{-1}$ is then a column-regular matrix, we solve the last inequality by applying Lemma 1 in the form of (8).

**Example 1.** Let us examine problem (7), given in terms of the semifield $\mathbb{R}_{\max,+}$ by the irreducible matrix

$$A = \begin{pmatrix} 1 & -1 \\ 3 & -2 \end{pmatrix}.$$ 

First, we evaluate the minimum in the problem. We successively calculate

$$\text{tr } A = 1, \quad A^2 = \begin{pmatrix} 2 & 0 \\ 4 & 2 \end{pmatrix}, \quad \text{tr } A^2 = 2.$$ 

Since $\lambda = \text{tr } A \oplus \text{tr }^{1/2}(A^2) = 1$, we have the minimum $\lambda^{-1} = -1$. 

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Furthermore, we obtain the eigenvectors of the matrix $A$, which present a solution to the problem. We define $B = \lambda^{-1}A$ and calculate matrices

$$B = \begin{pmatrix} 0 & -2 \\ 2 & -3 \end{pmatrix}, \quad B^* = BB^* = B^+ = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}.$$

Considering that both columns in the matrix $B$ are collinear, we can take one of them to represent a solution to the problem in the form

$$x = \begin{pmatrix} -2 \\ 0 \end{pmatrix}u, \quad u \in \mathbb{R}.$$

We now extend this result to arbitrary matrices, and then obtain two useful consequences. For simplicity, we concentrate on the matrices in the block-triangular form (3), which have no zero rows. The case of matrices with zero rows follows the same arguments with minor technical modifications.

**Theorem 5.** Let $A$ be a matrix in the refined block-triangular normal form (3), where the diagonal blocks $A_{ii}$ for all $i = 1, \ldots, r$ have eigenvalues $\lambda_i > 0$.

Then, the minimum value in problem (7) is equal to $\lambda^{-1}_1$, and all regular solutions are characterized by the inequality

$$x \leq \lambda^{-1}_1Ax.$$  

Specifically, any block vector $x = (x^T_1, \ldots, x^T_s)^T$ with the blocks $x_i$ defined successively for each $i = 1, \ldots, s$ by the conditions

$$x_i = \begin{cases} (\lambda^{-1}_iA_{ii})^+u_i, & \text{if } \lambda_i \geq \lambda_1; \\ \lambda^{-1}_i(\lambda^{-1}_1A_{ii}) + \bigoplus_{j=1}^{i-1}A_{ij}x_j, & \text{if } \lambda_i < \lambda_1; \end{cases}$$

where $u_i$ are regular vectors of appropriate size, is a solution of the problem.

**Proof.** Considering the refined block-triangular form of the matrix $A$ with $s$ rows, where all blocks above the diagonal are zero matrices, we write

$$(Ax)^-x = \bigoplus_{i=1}^{s} \left( \bigoplus_{j=1}^{i} A_{ij}x_j \right)^- x_i = \bigoplus_{i=1}^{r} (A_{ii}x_i)^-x_i + \bigoplus_{i=r+1}^{s} \left( \bigoplus_{j=1}^{i} A_{ij}x_j \right)^- x_i.$$

An application of Lemma 3 and the condition that $\lambda^{-1}_i \leq \lambda^{-1}_1$ for $i \leq r$ yield

$$(Ax)^-x \geq \bigoplus_{i=1}^{r} (A_{ii}x_i)^-x_i \geq \bigoplus_{i=1}^{r} \lambda^{-1}_i = \lambda^{-1}_1.$$
which means that $\lambda_1^{-1}$ is a lower bound for the objective function in the problem.

To verify that the bound $\lambda_1^{-1}$ is strict, and hence is the minimum value of the objective function, we need to present a vector $x$ that produces this bound.

We have to solve the inequality $(Ax)^-x \leq \lambda_1^{-1}$, which, due to the block-triangular form of $A$, is equivalent to the system of inequalities

$$\left( \bigoplus_{j=1}^{i} A_{ij}x_j \right)^- x_i \leq \lambda_1^{-1}, \quad i = 1, \ldots, s.$$  

We successively define a sequence of vectors $x_i$ for $i = 1, \ldots, s$. If $\lambda_i \geq \lambda_1$ we take $x_i$ to be an eigenvector of $A_{ii}$, given by the equation $\lambda_i^{-1}A_{ii}x_i = x_i$, which is solved as $x_i = (\lambda_i^{-1}A_{ii})^+u_i$, where $u_i$ is a regular vector of appropriate size.

Note that the condition $\lambda_i \geq \lambda_1$ is fulfilled if $i \leq r$. With this condition, we have

$$\bigoplus_{j=1}^{i} A_{ij}x_j = \bigoplus_{j=1}^{i-1} A_{ij}x_j \oplus A_{ii}x_i \geq A_{ii}x_i = \lambda_i x_i \geq \lambda_1 x_i$$

and therefore,

$$\left( \bigoplus_{j=1}^{i} A_{ij}x_j \right)^- x_i \leq \lambda_1^{-1}.$$  

If $\lambda_i < \lambda_1$ we define $x_i$ as the solution to the equation

$$\lambda_1^{-1} \bigoplus_{j=1}^{i-1} A_{ij}x_j \oplus \lambda_1^{-1} A_{ii}x_i = x_i.$$  

Since, in this case, $\text{Tr}(\lambda_1^{-1}A_{ii}) = \lambda_1^{-1} \text{tr} A_{ii} \oplus \cdots \oplus \lambda_1^{-n} \text{tr}(A_{ii}^n) < 1$, the equation is solved by Theorem 3 in the form

$$x_i = \lambda_1^{-1}(\lambda_1^{-1}A_{ii})^+ \bigoplus_{j=1}^{i-1} A_{ij}x_j.$$  

With the solution vector $x_i$, we write

$$\lambda_1 x_i = \bigoplus_{j=1}^{i-1} A_{ij}x_j \oplus A_{ii}x_i = \bigoplus_{j=1}^{i} A_{ij}x_j,$$

and then have

$$\left( \bigoplus_{j=1}^{i} A_{ij}x_j \right)^- x_i = \lambda_1^{-1}.$$  

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Combining all obtained vectors $x_i$ together yields

$$(Ax)^-x = \bigoplus_{i=1}^{s} \left( \bigoplus_{j=1}^{i} A_{ij}x_j \right)^- x_i \leq \lambda_1^{-1},$$

which shows that $\lambda_1^{-1}$ is the minimum value of the problem.

Finally, the application of Lemma 1 to solve the inequality $(Ax)^-x \leq \lambda_1^{-1}$ with respect to $x$ leads to inequality (9).

**Example 2.** Consider problem (7) defined in terms of $R_{\max,+}$ with the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 3 & -2 \end{pmatrix}. $$

Note that the matrix $A$ is reducible, and has the block-triangular form with the diagonal blocks given by $(1 \times 1)$-matrices. The eigenvalues of the diagonal blocks are easily found to be $\lambda_1 = 1$ and $\lambda_2 = -2$.

By Theorem 5 the minimum in the problem is equal to $\lambda_1^{-1} = -1$. The solution offered by the theorem is given by the vector $x = (x_1, x_2)^T$, where $x_1 = u$ for all $u \in \mathbb{R}$. The element $x_2$ is defined by the equation $x_2 = 2x_1 \oplus (-3)x_2$, which reduces to the equality $x_2 = 2x_1$. In vector form, the solution becomes

$$x = \begin{pmatrix} 0 \\ 2 \end{pmatrix} u, \quad u \in \mathbb{R}. $$

We now consider a special case of the problem, where the partial solution given by the previous theorem takes a more compact form.

**Corollary 6.** Under the conditions of Theorem 5, if $\lambda_1 \leq \lambda_i$ for all $i = 1, \ldots, s$, then the vector

$$x = Du, \quad D = \begin{pmatrix} (\lambda_1^{-1}A_{11})^+ & 0 & \cdots & 0 \\ \cdots & \ddots & \cdots & \cdots \\ 0 & \cdots & (\lambda_s^{-1}A_{ss})^+ \end{pmatrix},$$

where $u$ is any regular vector of appropriate size, is a solution of the problem.

**Proof.** It follows from Theorem 5 that the vector $x = (x_1^T, \ldots, x_s^T)^T$, which has, for all $i = 1, \ldots, s$, the blocks $x_i = (\lambda_i^{-1}A_{ii})^+u_i$, where $u_i$ are regular vectors of appropriate size, is a solution of the problem.

It remains to introduce the block vector $u = (u_1^T, \ldots, u_s^T)^T$ and the block-diagonal matrix $D = \text{diag}((\lambda_1^{-1}A_{11})^+, \ldots, (\lambda_s^{-1}A_{ss})^+)$ to finish the proof. 

The next result shows a useful property of the solutions of problem (7).
Corollary 7. Under the conditions of Theorem 5, the set of solution vectors of problem (7) is closed under vector addition and scalar multiplication.

Proof. Suppose that the vectors \( x \) and \( y \) are solutions of the problem, which implies, by Theorem 5, that \( x \leq \lambda_1^{-1}Ax \) and \( y \leq \lambda_1^{-1}Ay \). We take arbitrary scalars \( \alpha \) and \( \beta \), and consider the vector \( z = \alpha x \oplus \beta y \). Since

\[
z = \alpha x \oplus \beta y \leq \alpha \lambda_1^{-1}Ax \oplus \beta \lambda_1^{-1}Ay = \lambda_1^{-1}A(\alpha x \oplus \beta y) = \lambda_1^{-1}Az,
\]

the vector \( z \) is a solution of the problem, which proves the statement. \( \square \)

4 Derivation of Complete Solution

It follows from the results from the previous section that, under the assumptions of Theorem 5, all solutions of problem (7) are given by inequality (9). Below, we derive all solutions of the inequality, which we represent, without loss of generality, in a form without a scalar factor on the right-hand side.

Given a matrix \( A \in \mathbb{R}^{n \times n} \), we consider the inequality

\[
x \leq Ax.
\]

(10)

4.1 Solution via Matrix Sparsification

We start with the description of all solutions in the form of a family of solution sets, each defined by means of sparsification of the matrix \( A \).

Theorem 8. Let \( A \) be a matrix in the refined block-triangular normal form \( 3 \), where the diagonal block \( A_{11} \) has eigenvalue \( \lambda_1 > 1 \).

Denote by \( A \) the set of matrices \( A_1 \) that are obtained from \( A \) by fixing one non-zero entry in each row and by setting the others to \( 0 \), and that satisfy the condition \( \text{Tr}(A_1^{-1}(A \oplus I)) \leq 1 \).

Then, all regular solutions of inequality (10) are given by the conditions

\[
x = (A_1^{-1}(A \oplus I))^*u, \quad u > 0, \quad A_1 \in A.
\]

(11)

Proof. First we note that, under the conditions of the theorem, regular solutions to inequality (10) exist. Indeed, using similar arguments as in Theorem 5 one can see that the block vector \( x = (x_1^T, \ldots, x_s^T)^T \), where, for all \( i = 1, \ldots, s \), we take \( x_i \) to be an eigenvector of the matrix \( A_{ii} \) if \( \lambda_i \geq 1 \), or to be a solution of the equation \( A_{11}x_1 \oplus \cdots \oplus A_{ii}x_i = x_i \) otherwise, satisfies the inequality.

To prove the theorem, we show that any regular solution of inequality (10) can be represented as (11), and vice versa. Assume \( x = (x_j) \) to be a regular solution of (10) with a matrix \( A = (a_{ij}) \), and consider the scalar inequality

\[
x_k \leq a_{k1}x_1 \oplus \cdots \oplus a_{kn}x_n,
\]

(12)
which corresponds to row \( k \) in the matrix \( A \).

If this inequality holds for some \( x_1, \ldots, x_n \), then, as the order defined by the relation \( \leq \) is assumed linear, there is a term in the sum on the right-hand side that provides the maximum of the sum. Suppose that the maximum is attained at the \( p \)th term \( a_{kp}x_p \), and hence \( a_{kp} > 0 \). Under this condition, we can replace the above inequality by the two inequalities

\[
a_{kp}x_p \geq a_{k1}x_1 + \cdots + a_{kn}x_n \text{ and } a_{kp}x_p \geq x_k,
\]
or, equivalently, by one inequality

\[
a_{kp}x_p \geq a_{k1}x_1 + \cdots + (a_{kk} \oplus 1)x_k + \cdots + a_{kn}x_n.
\]

(13)

Now assume that we determine maximum terms in all scalar inequalities in (10). Similarly as above, we replace each inequality by an inequality with the maximum term isolated on the left side.

To represent the new inequalities in a vector form, we introduce a matrix \( A_1 \) that is obtained from \( A \) by fixing one entry, which corresponds to the maximum term, in each row, and by setting the other entries to 0. With the matrix \( A_1 \), the scalar inequalities are written in vector form as

\[
A_1x \geq (A \oplus I)x.
\]

Let us verify that this inequality is equivalent to the inequality

\[
x \geq A^{-1}_1(A \oplus I)x.
\]

We multiply the former inequality by \( A_1^{-1} \) on the left. Since \( A_1^{-1}A_1 \leq I \), we have \( x \geq A_1^{-1}A_1x \geq A_1^{-1}(A \oplus I)x \), which gives the latter one. At the same time, the multiplication of the latter inequality by \( A_1 \) on the left, and the condition \( A_1A_1^{-1} \geq I \) result in the former inequality as

\[
A_1^{-1}(A \oplus I)x \geq (A \oplus I)x.
\]

By Theorem 2, the last inequality has regular solutions if and only if the condition

\[
\text{Tr}(A_1^{-1}(A \oplus I)) \leq 1
\]

holds. All solutions are given by

\[
x = (A_1^{-1}(A \oplus I))^*u, \quad u > 0,
\]

which means that the vector \( x \) is represented in the form of (11).

Now suppose that a vector \( x \) is defined by the conditions at (11). To verify that \( x \) satisfies (10), we first use the condition \( \text{Tr}(A_1^{-1}(A \oplus I)) \leq 1 \) to see that \((A_1^{-1}(A \oplus I))^* \geq (A_1^{-1}(A \oplus I))^m\). Then, we write

\[
A(A_1^{-1}(A \oplus I))^* = A \bigoplus_{m=0}^{n-1}(A_1^{-1}(A \oplus I))^m = \bigoplus_{m=0}^{n-1}A(A_1^{-1}(A \oplus I))^m.
\]

Considering that \( A \geq A_1 \), we have \( AA_1^{-1} \geq A_1A_1^{-1} \geq I \). For each \( m \geq 1 \), we obtain

\[
A(A_1^{-1}(A \oplus I))^m \geq (A \oplus I)(A_1^{-1}(A \oplus I))^{m-1} \geq (A_1^{-1}(A \oplus I))^{m-1},
\]

13
from which it follows that

\[ A(A_i^-(A \oplus I))^* = A \oplus \bigoplus_{m=1}^{n} A(A_i^-(A \oplus I))^m \geq \bigoplus_{m=1}^{n} (A_i^-(A \oplus I))^{m-1} = \bigoplus_{m=0}^{n-1} (A_i^-(A \oplus I))^m = (A_i^-(A \oplus I))^*. \]

Since, in this case, \( Ax = A(A_i^-(A \oplus I))^*u \geq (A_i^-(A \oplus I))^*u = x \), we conclude that \( x \) satisfies inequality (10).

### 4.2 Backtracking Procedure of Generating Solution Sets

Note that, although the generation of the sparsified matrices according to the solution described above is a quite simple task, the number of the matrices in practical problems may be excessively large. Below, we propose a backtracking procedure that allows to reduce the number of matrices under examination.

The procedure successively checks rows \( i = 1, \ldots, n \) of the matrix \( A \) to find and fix one non-zero entry \( a_{ij} \) for \( j = 1, \ldots, n \), and to set the other entries to zero. On selection of an entry in a row, we examine the remaining rows to modify their non-zero entries by setting to 0, provided that these entries do not affect the current solution. One step of the procedure is completed when a non-zero entry is fixed in the last row, and hence a sparsified matrix is fully defined.

To prepare the next step, we take the next non-zero entry in the row, provided that such an entry exists. If there is no non-zero entries left in the row, the procedure has to go back to the previous row. It cancels the last selection of non-zero entry, and rolls back the modifications made to the matrix in accordance with the selection. Then, the procedure fixes the next non-zero entry in this row if it exists, or continues back to the previous rows until a new unexplored non-zero entry is found, otherwise. If the new entry is fixed in a row, the procedure continues forward to fix non-zero entries in the next rows, and to modify the remaining rows. The procedure is completed when no more non-zero entries can be selected in the first row.

To describe the modification routine implemented in the procedure, assume that there are non-zero entries fixed in rows \( i = 1, \ldots, k - 1 \), and we now select the entry \( a_{kp} \) in row \( k \). Since this selection implies that \( a_{kp}x_p \) is considered the maximum term in the right-hand side of inequality (12), it follows from (13) that \( x_p \geq a_{kp}^{-1}a_{kj}x_j \) for all \( j \neq k \), and \( x_p \geq a_{kp}^{-1}(a_{kk} \oplus 1)x_k \) for \( j = k \).

Let us examine the inequality \( x_i \leq a_{i1}x_1 \oplus \cdots \oplus a_{in}x_n \) for \( i = k+1, \ldots, n \). If the condition \( a_{ip}a_{kp}^{-1}a_{ki} \geq 1 \) holds, then the inequality is fulfilled at the expense of its \( p \)th term alone, because \( a_{ip}x_p \geq a_{ip}a_{kp}^{-1}a_{ki}x_i \geq x_i \). Since, in
this case, the contribution of the other terms is of no concern, we can set the entries \( a_{ij} \) for all \( j \neq p \) to 0 without changing the solution set under construction.

Suppose that the above condition is not satisfied. Then, we can verify the conditions \( a_{ip}a_{kp}^{-1}a_{kj} \geq a_{ij} \) for all \( j \neq p, k \), and \( a_{ip}a_{kp}^{-1}(a_{kk} \oplus 1) \geq a_{ik} \) for \( j = k \). If these conditions are satisfied for some \( j \neq k \) or \( j = k \), then we have \( a_{ip}x_p \geq a_{ip}a_{kp}^{-1}a_{kj}x_j \geq a_{ij}x_j \) or \( a_{ip}x_p \geq a_{ip}a_{kp}^{-1}(a_{kk} \oplus 1)x_k \geq a_{ik}x_k \). This means that term \( p \) dominates over term \( j \). As before, considering that the last term does not affect the right-hand side of the inequality, we put \( a_{ij} = 0 \).

4.3 Closed-Form Representation of Complete Solution

We conclude with the representation of the solution to the entire optimization problem under investigation, in a compact closed form.

**Theorem 9.** Let \( A \) be a matrix in the refined block-triangular normal form \( (3) \), where the diagonal blocks \( A_{ii} \) for all \( i = 1, \ldots, r \) have eigenvalues \( \lambda_i = 0 \).

Define \( B = \lambda_1^{-1}A \), and denote by \( \mathcal{B} \) the set of matrices \( B_1 \) that are obtained from \( B \) by fixing one non-zero entry in each row and by setting the other entries to 0, and that satisfy the condition \( \text{Tr}(B_1^{-1}(B \oplus I)) \leq 1 \).

Let \( S \) be the matrix, which is constituted by the maximal linear independent system of columns in the matrices \( S_1 = (B_1^{-1}(B \oplus I))^t \) for all \( B_1 \in \mathcal{B} \).

Then, the minimum value in problem \( (7) \) is equal to \( \lambda_1^{-1} \), and all regular solutions are given by \( x = Sv, \quad v > 0 \).

**Proof.** By Theorem 5, we find the minimum in the problem to be \( \lambda_1^{-1} \), and characterize all solutions by the inequality \( x \leq Bx \) with the matrix \( B = \lambda_1^{-1}A \).

Application of Theorem 8 involves defining a set \( \mathcal{B} \) of matrices \( B_1 \) that are obtained from \( B \) by fixing one non-zero entry in each row together with setting the others to 0, and such that \( \text{Tr}(B_1^{-1}(B \oplus I)) \leq 1 \). The theorem yields a family of solutions \( x = S_1u \) with \( S_1 = (B_1^{-1}(B \oplus I))^t \) for all \( u > 0 \) and \( B_1 \in \mathcal{B} \).

Considering that each solution \( x = S_1u \) defines a subset of vectors generated by the columns of the matrix \( S_1 \), we apply Corollary 7 to represent all solutions as the linear span of the columns in the matrices \( S_1 \), corresponding to all \( B_1 \in \mathcal{B} \).

Finally, we reduce the set of all columns by eliminating those, which are linearly dependent on others. We take the remaining columns to form a matrix \( S \), and then write the solution as \( x = Sv \), where \( v \) is any regular vector.
Example 3. We now apply the results offered by Theorem 8 to derive all solutions of the problem considered in Example 1. We take the matrix $B$ and replace one element in each row of $B$ by $0 = -\infty$ to produce the sparsified matrices

$$B_1 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 0 \\ 0 & -3 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 0 & -2 \\ 0 & -3 \end{pmatrix}.$$ 

To find those sparsified matrices, which satisfy the conditions of the theorem, we need to calculate the matrices

$$B_1^-(B \oplus I) = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}, \quad B_2^-(B \oplus I) = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix},$$

$$B_3^-(B \oplus I) = \begin{pmatrix} 0 & -2 \\ 5 & 3 \end{pmatrix}, \quad B_4^-(B \oplus I) = \begin{pmatrix} 0 & 0 \\ 5 & 3 \end{pmatrix}.$$ 

Furthermore, we obtain $\text{Tr}(B_1^-(B \oplus I)) = \text{Tr}(B_2^-(B \oplus I)) = 0 = 1$. Since the matrices $B_1$ and $B_2$ satisfy the conditions, they are accepted. Considering that $\text{Tr}(B_3^-(B \oplus I)) = \text{Tr}(B_4^-(B \oplus I)) = 3 > 1$, the last two matrices are rejected.

To represent all solutions of the problem, we calculate the matrices

$$S_1 = (B_1^-(B \oplus I))^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad S_2 = (B_2^-(B \oplus I))^* = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix},$$

and then combine their columns to form a matrix that generates the solutions.

Taking into account that both columns of the matrix $S_2$ are collinear to the second column of $S_1$, we drop these columns, and define $S = S_1$. As a result, we represent the solution of the problem as

$$x = Su, \quad S = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}, \quad u \in \mathbb{R}^2.$$

Example 4. Finally, we apply Theorem 8 to the problem in Example 2. We first calculate the matrix

$$B = \lambda_1^{-1} A = \begin{pmatrix} 0 & 0 \\ 2 & -3 \end{pmatrix}.$$ 

We can derive two sparsified matrices $B_1$ and $B_2$ from $B$, and write

$$B_1 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \quad B_1^-(B \oplus I) = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}, \quad \text{Tr}(B_1^-(B \oplus I)) = 0;$$

$$B_2 = \begin{pmatrix} 0 & 0 \\ 0 & -3 \end{pmatrix}, \quad B_2^-(B \oplus I) = \begin{pmatrix} 0 & 0 \\ 5 & 3 \end{pmatrix}, \quad \text{Tr}(B_2^-(B \oplus I)) = 3.$$
The matrix $B_2$ does not satisfy the condition of the theorem, and thus is rejected. Note that the matrix $B_1$ coincides with the corresponding matrix in Example 3. Since the matrix $B_1$ completely determines the computations, the solution to the problem has the same form as that obtained in this example.

5 Conclusions

The paper focused on the development of methods and techniques for the complete solution of an optimization problem, formulated in the framework of tropical mathematics to minimize a nonlinear function defined by a matrix on vectors over idempotent semifield. As the starting point, we have taken our previous result, which offers a partial solution to the problem with both irreducible and reducible matrices. To further extend this result, we have first obtained a new partial solution of the problem in the case of a reducible matrix, and derived a characterization of the solutions in the form of a vector inequality. We have developed an approach to describe all solutions of the problem as a family of solution subsets by using a matrix sparsification technique. To generate all members of the family in a reasonable way, we have proposed a backtracking procedure. Finally, we have offered a representation for the complete solution of the problem in a compact vector form, ready for further analysis and calculation.

The results obtained were illustrated with illuminating numerical examples.

The directions of future research will include the development of real-world applications of the solutions proposed. A detailed analysis of the computational complexity of the backtracking procedure is of particular interest. Various extensions of the solution to handle other classes of optimization problems with different objective functions and constraints, and in different algebraic settings are also considered promising lines of future investigation.

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