On the (in)finiteness of the image of Reshetikhin-Turaev representations

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Abstract

We state a simple criterion to prove the infiniteness of the image of Reshetikhin-Turaev representations of the mapping class groups of surfaces at odd prime levels. We use it to study some of the Reshetikhin-Turaev representations associated to the one-holed torus and derive an alternative proof of the results of [6].

Keywords: Reshetikhin-Turaev representations, mapping class group, quantum representations, Topological Quantum Field Theory.

1 Introduction

At the end of the 80’ Witten [15] constructed a 2 + 1 dimensional TQFT using path integrals which gives a three-dimensional interpretation of the Jones polynomial and gives invariants for 3-dimensional closed oriented manifold equipped with some links and additional structure. Reshetikhin and Turaev ([12]) gave a rigorous construction using quantum groups and more recently Blanchet, Masbaum, Habegger and Vogel constructed these TQFTs by means of Kauffman bracket skein algebra ([1]) following the works of Lickorish ([9]). We will use the construction of the latter authors.

These TQFTs give rise to a tower of representations $\rho_{p,p} \geq 3$ of some central extensions of the mapping class group $\text{Mod}(\Sigma_g)$ of any closed oriented surface $\Sigma$ equipped with colored points, which act on finite rank modules $V_p(\Sigma)$. Here we will discuss whether these representations have finite image or not. Several studies have been made in that sense. The Reshetikhin-Turaev representations associated to a torus without marked points have finite image. This fact was known in the conformal field theory community (see [2] and references herein) and has been proved independently by Gilmer in [7]. In higher genus, the Reshetikhin-Turaev representations have finite image at level 3 and 6 (see [16]) and 4 (it is a character). In any other cases, they have infinite image. It was proved by Funar in [8] when $(g,p) \neq (2,10)$ (see also [10] for a proof that they contain an element of infinite order). The last remaining case $g = 2, p = 10$ was treated in [3]. Concerning the representation associated to a one holed torus, it was shown in [11] that they have infinite image for high enough level.

In this paper, we develop a simple criterion to check whether a given Reshetikhin-Turaev representation has infinite image or not. Unfortunately, this criterion only works when $p = r$ or $p = 2r$ for $r$ an odd prime. It permits us to recover the above results in these cases, and to study the finiteness of the representations associated to a one-holed torus, which could not be derived from previous papers.

Denote by $\rho_p^c$ the level $p$ Reshetikhin-Turaev representation associated to a torus $T^c$ equipped with a band colored by $2c$. The main theorem is the following:

**Theorem 1.1.** Let $r$ be an odd prime and $p = r$ or $p = 2r$. Then we have:
1. If \(2c = r - 3\), then \(\rho^{2r}_p\) has finite image.
2. If \(c \equiv 1\pmod{3}\) and \(r \neq 3, 5\), then \(\rho^{2r}_p\) has infinite image.
3. If \((c \equiv 3\pmod{5}\) and \(r \equiv 2\pmod{5}\) or \(r \equiv 3\pmod{5}\)) or \((c \equiv 1\pmod{5}\) and \(r \equiv 3\pmod{5}\)) or \((c \equiv 2\pmod{5}\) and \(r \equiv 2\pmod{5}\)), then \(\rho^{2r}_p\) has infinite image.
4. If \(c \leq \frac{r - 1}{3}\), then \(\rho^{2r}_p\) has infinite image.

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## 2 Complete positivity

Let \(\Sigma\) be a closed oriented surface with colored punctures and \(p \geq 3\). The associated representations \(\rho_p\) act on a finite rank \(k_p\)-module \(V_p(\Sigma)\) where \(k_p = \mathbb{Z}[A, \frac{1}{p}, \kappa]\) quotiented by the relations \(\phi_{2p}(A) = 0\), where \(\phi_{2p}\) represents the \(2p\)-th cyclotomic polynomial, and \(\kappa^6 = A^{-6} - \frac{2(p+1)}{p}\). A choice of a particular \(2p\)-th primitive root of unity \(A\) and a compatible complex \(\kappa \in \mathbb{C}\) give a complex vector space \(V_p^{A,\kappa}(\Sigma) = V_p(\Sigma) \otimes \mathbb{C}\) and a complex representation \(\rho_p^{A,\kappa}\) whose action is preserved by a non-degenerate Hermitian form \(\langle \cdot, \cdot \rangle^{A,\kappa}_p\). Note that in \([1]\), the invariant pairing is bilinear whereas in this paper we choose it Hermitian using the involution of \(k_p\) sending \(A\) to \(A^{-1}\).

**Definition 2.1.** We say that \(V_p(\Sigma)\) is completely positive if for any choice of \(A \in \mathbb{C}\) as a primitive \(2p\)-th root of unity and compatible \(\kappa \in \mathbb{C}\), the Hermitian form \((V_p^{A,\kappa}(\Sigma), \langle \cdot, \cdot \rangle^{A,\kappa}_p)\) is positive definite or negative definite.

The only effect of changing \(\kappa\) is possibly to change the eigenvalues of the Hermitian form to its opposite so complete positivity only means that all eigenvalues have the same sign for a given \(\kappa\) and all \(A\).

**Proposition 2.2.** Let \(r\) be an odd prime number and \(p = r\) or \(p = 2r\). Then \((V_p(\Sigma), \langle \cdot, \cdot \rangle_p)\) is completely positive if and only if \(\rho_p\) has finite image.

If \(\rho_p(\text{Mod}(\Sigma))\) is finite, the following lemma, together with Roberts’ results ([13]), which states that \(\rho_p\) is irreducible at prime levels, imply that \(V_p(\Sigma)\) must be completely positive.

**Lemma 2.3.** Let \(\rho : G \to \text{GL}(V)\) be an irreducible group representation on a finite dimensional complex vector space \(V\) equipped with a non degenerate invariant Hermitian form \(\langle \cdot, \cdot \rangle_0\). If \(\rho(G)\) is finite, then \(\langle \cdot, \cdot \rangle_0\) is definite either positive or negative.

**Proof.** Let \(\langle \cdot, \cdot \rangle\) denotes an arbitrary scalar product on \(V\). If \(\rho(G)\) is finite, the Hermitian form on \(V\) defined by

\[
\langle u, v \rangle_1 := \frac{1}{\# \rho(G)} \sum_{\rho(g) \in \rho(G)} \langle \rho(g)u, \rho(g)v \rangle
\]

is definite positive and invariant under the action of \(G\).

If \(t \in [0, 1]\), we define the invariant Hermitian form

\[
\langle u, v \rangle_t := t \langle u, v \rangle_1 + (1 - t) \langle u, v \rangle_1
\]

Suppose there exists \(v_0 \in V\) so that \(\langle v_0, v_0 \rangle_0 < 0\). Since \(\langle v_0, v_0 \rangle_1 > 0\), there exists \(t_0 \in (0, 1)\) so that \(\langle v_0, v_0 \rangle_{t_0} = 0\). The invariance of \(\langle \cdot, \cdot \rangle_{t_0}\) under the action of \(G\), implies that \(\langle \rho(g)v_0, \rho(g)v_0 \rangle_{t_0} = 0\) for all \(g \in G\). Since \(\rho\) is irreducible, the vector \(v_0\) is cyclic so \(\langle \cdot, \cdot \rangle_{t_0} = 0\) and \(\langle \cdot, \cdot \rangle_0 = \frac{t_0 - 1}{t_0} \langle \cdot, \cdot \rangle_1\) is definite negative. \(\square\)
When \( p = r \) or \( p = 2r \) with \( r \) an odd prime, we write
\[
\alpha_p := \begin{cases}
p & \text{if } p \equiv 3 \pmod{4} \\
4r & \text{if } p \equiv 1, 2 \pmod{4}
\end{cases}
\]
and \( \mathcal{O}_p := \mathbb{Z}[A] / \phi_{\alpha_p}(A) \).

It was showed in \([8, 11]\) that \( V_p(\Sigma) \) contains a free \( \mathcal{O}_p \) lattice of maximal rank preserved by \( \hat{\text{Mod}}(\Sigma_g) \).

We denote by \( \mu(\alpha_p) = \{q_1, \ldots, q_{\phi(\alpha_p)}\} \) the set of primitive \( \alpha_p \)-th roots of unity and by \( \phi(\alpha_p) = \# \mu(\alpha_p) \) the Euler totient function.

**Lemma 2.4.** Let \( p = r \) or \( p = 2r \) with \( r \) an odd prime. Then the injective linear map
\[
\Psi : \mathcal{O}_p \to \mathbb{C}^{\phi(\alpha_p)}
\]
sending \( A^n \) to \( (q_1^i, \ldots, q_{\phi(\alpha_p)}^i) \), has a discrete image.

**Proof.** We endow \( \mathbb{C}^{\phi(\alpha_p)} \) with the norm:
\[
\|(z_1, \ldots, z_{\phi(\alpha_p)})\|^2 := \frac{1}{\phi(\alpha_p)} \sum_i |z_i|^2
\]
where \( |z|^2 \) represents the Euclidean norm in \( \mathbb{C} \).

If \( P(X) = \sum_n n_i X^i \in \mathcal{O}_p \), then:
\[
\|\Psi(P)\|^2 = \frac{1}{\phi(\alpha_p)} \sum_{q \in \mu(\alpha_p)} \sum_{i,j} n_i n_j q^{i-j} = \sum_i n_i^2 + \sum_{i\neq j} n_i n_j \left( \frac{1}{\phi(\alpha_p)} \sum_{q \in \mu(\alpha_p)} q^{i-j} \right) = \begin{cases}
\sum_i n_i^2 & , \text{if } p \text{ is prime.} \\
\sum_i n_i^2 - 2 \sum_{|i-j|=2r} n_i n_j & , \text{if } p = 2r.
\end{cases}
\]

Thus we have \( \|\Psi(\mathcal{O}_p)\|^2 \in \mathbb{N} \), so \( \Psi(\mathcal{O}_p) \) is discrete in \( \mathbb{C}^{\phi(\alpha_p)} \). \( \square \)

**Proof of Proposition 2.3** If \( \rho_p(\hat{\text{Mod}}(\Sigma_g)) \) is finite, then the irreducibility of \( \rho_p \) \([13]\) together with Lemma 2.3 imply that every invariant forms \( \langle \cdot, \cdot \rangle_p^A, \kappa \) must be either positive or negative, so \( V_p(\Sigma) \) is completely positive.

Conversely, if \( V_p(\Sigma) \) is completely positive, using the \( \mathcal{O}_p \) lattice of \([8, 11]\), we have an injective homomorphism from \( \rho_p(G) \) to the group of matrices with coefficients in \( \mathcal{O}_p \). Once composed with the map \( \Psi \) of Lemma 2.4, we get an injective group morphism \( \hat{\Psi} : \rho(G) \to \text{GL}_d(\mathbb{C}) \times \cdots \times \text{GL}_d(\mathbb{C}) \), where \( d := \text{dim}(V_p(\Sigma)) \). Lemma 2.4 implies that \( \hat{\Psi} \) has a discrete image, and the complete positivity of \( V_p(\Sigma) \) implies that the image lies in the compact product of \( r-1 \) unitary groups. This implies the finiteness of \( \rho_p(\hat{\text{Mod}}(\Sigma_g)) \). \( \square \)
3 (In)finiteness of the one holed torus representations

Using Proposition 2.2, we can prove the main theorem of this paper:

(Proof of Theorem 1.1). For $0 \leq i \leq r - 2 - 2c$, denote by $w^i \in V_p(T^i)$ the vector associated to a lollipop graph whose stick is colored by $2c$ and loop colored by $i + c$, that is:

$$w_i^c := \frac{2c}{i+c}$$

Using Theorem 4.11 of [1], and using the usual quantum numbers defined by $[n] := \frac{4^n - 4^{-n}}{A^2 - A^{-2}} \in k_p$, we have:

$$\frac{\langle u^c_{i+1}, u^c_{i+1} \rangle}{\langle u^c_i, u^c_i \rangle} = \frac{2c + i + 2}{[c + i + 2][c + i + 1]}$$

$$\frac{\langle u^c_{i+2}, u^c_{i+2} \rangle}{\langle u^c_i, u^c_i \rangle} = \frac{2c + i + 3[2c + i + 2][i + 1]}{[c + i + 2][c + i + 3][c + i + 2]^2}$$

Proposition 2.2 states that $\rho^c_A$ has infinite image if and only if there exists a primitive $2p$-th root of unity $A$ such that one of the above numbers is negative.

When $2c = r - 3$, choose $A = \exp\left(\frac{i\pi k}{2^p}\right)$ with $g.c.d.(k, 2r) = 1$. The vectors $\{u^c_0, u^c_1\}$ form a basis of $V_p(T^c)$ and we have:

$$\frac{\langle u^c_1, u^c_1 \rangle}{\langle u^c_0, u^c_0 \rangle} = \frac{\sin\left(\frac{1}{2^p - 1} \pi k\right)}{\sin\left(\frac{1}{2^p - 1} \pi k\right)}$$

This product appears to be always positive for every $k$: indeed $\sin\left(\frac{1}{2^p - 1} \pi k\right)$ and $\sin\left(\frac{1}{2^{p-1}} \pi k\right)$ are both positive if $k < r$ and both negative elsewhere whereas $\sin\left(\frac{1}{2^p - 1} \pi k\right)$ and $\sin\left(\frac{1}{2^{p-1}} \pi k\right)$ are both positive if $\frac{1}{2^p - 1}$ is even and both negative elsewhere. Thus $\rho^c_A$ has finite image.

When $c \equiv 1 \pmod{3}$, we take $k = \left\{\frac{2r+1}{2^{p-1}}, \frac{2r-1}{2^{p-1}}\right\}$, if $r \equiv 1 \pmod{3}$ and we have $\frac{\langle u^c_2, u^c_3 \rangle}{\langle u^c_0, u^c_0 \rangle} < 0$.

If $r \equiv 2 \pmod{5}$, we put $k = \frac{2r+1}{2}$ and we have $\frac{\langle u^c_5, u^c_3 \rangle}{\langle u^c_0, u^c_0 \rangle} < 0$.

If $r \equiv 3 \pmod{5}$, we put $k = \frac{2r-1}{2}$ and we have $\frac{\langle u^c_5, u^c_3 \rangle}{\langle u^c_0, u^c_0 \rangle} < 0$ when $c \equiv 1, 3 \pmod{5}$.

When $c = 1$, then $\frac{\langle u^c_5, u^c_3 \rangle}{\langle u^c_0, u^c_0 \rangle} = \frac{6}{2^l} = \frac{\sin\left(\frac{1+2}{2^p - 1} \pi k\right)}{\sin\left(\frac{2}{2^p - 1} \pi k\right)}$ is negative if $\frac{r}{12} < k < \frac{r}{6}$. Such a $k$ exists when $r \geq 12$.

For $r = 1$ or 7, we take $k = 1$ and for $r = 13, 17$, we take $k = 2$, and find again $\frac{\langle u^c_5, u^c_3 \rangle}{\langle u^c_0, u^c_0 \rangle} < 0$.

Finally, note that if there exist $0 \leq i \leq r - 3$ and an odd $k < r$ so that $\frac{r}{2c+i+2} < k < \frac{r}{c+i+2}$ then $\frac{\langle u^c_{i+1}, u^c_{i+1} \rangle}{\langle u^c_i, u^c_i \rangle} < 0$ for $A = \exp\left(\frac{i\pi k}{r}\right)$. Then we can find such a $i$ for $k = 3$ whenever $3 \in \bigcup_{i=0}^{r-3-2c} \left[\frac{r}{2c+i+2}, \frac{r}{c+i+2}\right] = \left[\frac{r}{r-1}, \frac{r}{c+2}\right]$, i.e. when $c \leq \frac{r-1}{3}$. □
Remark. We could have recovered the finiteness in the case where $2v = r - 3$ by using a theorem of Formanek (see [3]) which states that the only 2-dimensional irreducible representations of $B_2 = \langle t, t' | tt' = t'tt' \rangle$ are conjugate to one of the $\chi(y) \otimes \beta_3(q)$, where $\chi(y)$ is the character which sends both generators $t$ and $t'$ to $y \in \mathbb{C}^*$ and $\beta_3(q)$ is the reduced Burau representation in $q$. Since the eigenvalues of $\rho_p(t)$ are $\mu_c = (-1)^c A^{(c+1)(c+3)}$ and $\mu_{c+1} = (-1)^c A^{(c+1)(c+3)}$ we must have $y = -\mu_c$ and $q = A'$ or $y = -\mu_{c+1}$ and $q = A'^{-1}$ so $q$ is a 4-th primitive root of unity and $\beta_3(q)$ appears to have finite image (see [5] for a detailed discussion of finiteness of reduced Burau representations at roots of unity).

3.1 (In)finiteness of Reshetikhin-Turaev representations associated to closed surfaces

The following theorem results from [7, 6, 16, 3]. We derive another proof from Proposition [4].

Theorem 3.1 ([7, 6, 16, 3]). Let $g \geq 1$ and $p = r$ or $p = 2r$ with $r$ an odd prime. Then $\rho_{p,g}$ has finite image if and only if $g = 1$ or $r = 3$.

Proof. If $g = 1$, then $\langle \mu_{c+1}, \mu_{c+1} \rangle_{p} = 1$ for all $i$, so $\rho_{p,1}$ is completely positive.

If $p = 3$ then $V_3(\Sigma_g)$ is one dimensional so is completely positive. If $p = 6$, we have $I_p = \{0, 1\}$ and $(1, 1, 1)$ is not 6-admissible so, if $\Gamma$ is a trivalent graph and $\sigma$ a coloration of $\Gamma$, then $u_\sigma$ is a union of disjoint circles, say $n$ ones, colored by 1. Thus $\langle u_\sigma, u_\sigma \rangle_6 = 1$ and $V_6(\Sigma_g)$ is completely positive.

Now consider the case when $r = 5$ and $g = 2$. Take $\Gamma = \begin{array}{c} \hline \hline \end{array}$ and consider the associated vectors $u_{a,b,c}$ for $(a, b, c) = (a, b, c)$ a $p$-admissible triple.

If $p = 5$, then $\langle u_{2,2,2}, u_{2,2,2} \rangle_{p} = \frac{2}{|U_5|}$ has the sign of [4] which is negative for $A = \exp \left( \frac{2i\pi}{5} \right)$.

If $p = 10$, then $\langle u_{2,1,1}, u_{2,1,1} \rangle_{10} = \frac{(2,1,1)}{(1,1,1)}$ has the sign of $\langle 2 \rangle = [3]$ which is negative for $A = \exp \left( \frac{2i\pi}{10} \right)$.

Now the fact that $V_p(\Sigma_2)$ is not completely positive for $r = 5$ implies that $V_p(\Sigma_g)$ is not completely positive when $g \geq 3$ as well, for we can embed the Theta graph into a genus $g \geq 3$ graph and define vectors $u_{a,b,c} \in V_p(\Sigma_g)$, with same norm as in genus 2, by coloring by 0 the complementary of the embedding.

Finally, if $r = 3, 5$, and $g = 2$, we isolate one handle of $\Sigma_g$ and write $\Sigma_g \cong \Sigma_1 \cup g \Sigma_{g-1}$. Then using Theorem 1.14 of [1], we have:

$$V_p(\Sigma_g) \cong \bigoplus_{c \in I_p} V_p(7^c) \otimes V_p(\Sigma_{g-1})$$

So the fact that $V_p(7^c)$ is not completely positive (Theorem [1]) implies that $V_p(\Sigma_g)$ is not completely positive either for $g \geq 2$. We conclude using Proposition [2,2]. □
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