On a theorem of Nosal

V. Nikiforov*

Abstract

Let \( G \) be a graph with \( m \) edges and spectral radius \( \lambda_1 \). Let \( bk(G) \) stand for the maximal number of triangles with a common edge in \( G \).

In 1970 Nosal proved that if \( \lambda_1^2 > m \), then \( G \) contains a triangle. In this paper we show that the same premise implies that

\[
bk(G) > \frac{1}{12} \sqrt{m}.
\]

This result settles a conjecture of Zhai, Lin, and Shu.

Write \( \lambda_2 \) for the second largest eigenvalue of \( G \). Recently, Lin, Ning, and Wu showed that if \( G \) is a triangle-free graph of order at least three, then

\[
\lambda_1^2 + \lambda_2^2 \leq m,
\]

thereby settling the simplest case of a conjecture of Bollobás and the author. We give a simpler proof of their result.

Keywords: triangle-free graph; spectral radius; graph booksize; second largest eigenvalue.

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1 Introduction

In 1970 Nosal showed that if \( G \) is a graph with \( m \) edges, and its largest adjacency eigenvalue \( \lambda_1 \) satisfies

\[
\lambda_1^2 > m,
\]

then \( G \) contains a triangle.

During the years this striking and elegant result has attracted significant attention (see, e.g., [8] and its references for some highlights.) In this note we discuss two recent developments of Nosal’s result.

The first one concerns the class of subgraphs that are present in \( G \) if \( \lambda_1^2 > m \) and \( m \) is sufficiently large.

As shown by Zhai, Lin, and Shu in the nice recent paper [8], this class contains graphs other than triangles. These authors studied similar problems in depth and surveyed some earlier research. In particular, they raised the following conjecture:

*Department of Mathematical Sciences, University of Memphis, Memphis TN 38152, USA. Email: vnikifrv@memphis.edu
Conjecture 1 For every natural number \( k \), there exists \( m(k) \) such that if \( m > m(k) \) and \( \lambda_2^2 \geq m \), then \( bk(G) > k \), unless \( G \) is complete bipartite graph with possibly some isolated vertices.

In the above conjecture, \( bk(G) \) stands for the booksize of \( G \), that is, the maximum number of triangles with a common edge in \( G \). Lower bounds on the booksize are known for Mantel’s theorem, but not for the context of Nosal’s inequality.

We confirm Conjecture 1 by the following theorem:

**Theorem 2** If \( G \) is a graph with \( m \) edges and \( \lambda_2^2 \geq m \), then
\[
bk(G) > \frac{1}{12} \sqrt[m]{m},
\]
unless \( G \) is a complete bipartite graph with possibly some isolated vertices.

The other main result of our note concerns the following conjecture of Bollobás and the author [1]:

**Conjecture 3** Let \( G \) be a graph with \( m \) edges, at least \( r + 1 \) vertices, and second largest eigenvalue \( \lambda_2 \). If \( G \) is \( K_{r+1} \)-free, then
\[
\lambda_1^2 + \lambda_2^2 \leq 2 \left( 1 - \frac{1}{r} \right) m. \quad (1)
\]

Recently, Lin, Ning, and Wu [4] settled the case \( r = 2 \) of Conjecture 3 by a clever argument using majorization theory.

Before stating their result, recall that a blow-up of a graph \( H \) is obtained by replacing each vertex \( v \) of \( H \) by an independent set \( B_v \) and replacing each edge \( \{u, v\} \) of \( H \) by a complete bipartite graph with vertex classes \( B_u \) and \( B_v \). Also, \( P_k \) stands for the path of order \( k \).

**Theorem 4 (Lin, Ning, Wu)** Let \( G \) be a graph with \( m \) edges, of order at least 3, and let \( \lambda_2 \) be its second largest adjacency eigenvalue. If \( G \) is triangle-free, then
\[
\lambda_1^2 + \lambda_2^2 \leq m.
\]
Equality holds if and only if \( G \) is a blow-up of \( P_2 \cup K_1, 2P_2 \cup K_1, P_4 \cup K_1, \) or \( P_5 \cup K_1 \).

In Section 5 we give a simple straightforward proof of Theorem 4. Theorem 2 is proved in Sections 3 and 4.

## 2 Notation and preliminary results

A \( k \)-walk stands for a walk on \( k \) vertices, that is, a walk of length \( k - 1 \).

Given a graph \( G \), we write:
- \( V(G) \) for the set of its vertices and \( E(G) \) for the set of its edges;
- \( e(G) \) for the number of its edges;
- \( t(G) \) for the number of its triangles;
For a vertex $u \in V(G)$, we write:

- $w_k(G)$ for the number of its $k$-walks;
- $\rho(G)$ for its largest adjacency eigenvalue.

Note, in particular, that

$$\sum_{u \in V(G)} t''(u) = t''(G).$$

For a proof of the following theorem of Wei [7] we refer the reader to [3], p. 182.

**Theorem 5 (Wei)** Let $G$ be a connected nonbipartite graph of order $n$ and let $(x_1, \ldots, x_n)$ be a positive eigenvector to $\rho(G)$. For every vertex $u \in V(G)$,

$$\lim_{k \to \infty} \frac{w_k(u)}{w_k(G)} = \frac{x_u}{x_1 + \cdots + x_n}.$$

The following inequality is the instance $r = 2$ of Theorem 1 in [1]:

**Lemma 6** If $G$ is a graph with $e(G) = m$, $\rho(G) = \rho$, and $t(G) = t$, then

$$3t \geq \rho^3 - \rho m.$$
for all edges \( \{i, j\} \in E(G') \), it turns out that \( G' \) may have only one nontrivial component and is nonbipartite. Under these premises for \( G' \), Theorem 9 implies that \( bk(G') \) is as large as needed.

Theorem 9 itself is based on two rather technical results–Theorem 7 and Lemma 8.

**Theorem 7** Let \( G \) be a connected nonbipartite graph of order \( n \) with

\[
e(G) = m, \quad \rho(G) = \rho, \quad t(G) = t, \quad t''(G) = t''.
\]

Suppose that \( (x_1, \ldots, x_n) \) is a positive unit eigenvector to \( \rho \) and let

\[
c = \frac{1}{x_1 + \cdots + x_n} \min \{x_1, \ldots, x_n\}.
\]

Then

\[
\rho^3 - \rho m + c\rho t'' \leq 3t.
\]

**Proof** For every edge \( \{u, v\} \in E(G) \), obviously

\[
w_k(G) = \sum_{w \in N(u) \cup N(v)} w_k(w) + \sum_{w \in N(u) \cap N(v)} w_k(w)
\]

\[
= \sum_{w \in N(u)} w_k(w) + \sum_{w \in N(v)} w_k(w) - \sum_{w \in N(u) \cap N(v)} w_k(w) + \sum_{w \in N(u) \cap N(v)} w_k(w)
\]

\[
= w_{k+1}(u) + w_{k+1}(v) - \sum_{w \in N(u) \cap N(v)} w_k(w) + \sum_{w \in N(u) \cap N(v)} w_k(w).
\]

Summing this identity over all edges \( \{u, v\} \in E(G) \), we get

\[
w_k(G)m = \sum_{u \in V(G)} w_{k+1}(u)d(u) - \sum_{u \in V(G)} t(u)w_k(u) + \sum_{u \in V(G)} t''(u)w_k(u)
\]

\[
= w_{k+2}(G) - \sum_{u \in V(G)} t(u)w_k(u) + t''(u)w_k(u).
\]

Since for any vertex \( u \) we have \( w_k(u) \leq w_{k-1}(G) \), it follows that

\[
\sum_{u \in V(G)} t(u)w_k(u) \leq \sum_{u \in V(G)} t(u)w_{k-1}(G) = 3tw_{k-1}(G).
\]

Hence,

\[
w_k(G)m \geq w_{k+2}(G) - 3tw_{k-1}(G) + \sum_{u \in V(G)} t''(u)w_k(u).
\]

Dividing both sides by \( w_{k-1}(G) \), we get

\[
\frac{w_k(G)}{w_{k-1}(G)}m \geq \frac{w_{k+2}(G)}{w_{k-1}(G)} - 3t + \frac{w_k(G)}{w_{k-1}(G)} \sum_{u \in V(G)} t''(u) \frac{w_k(u)}{w_k(G)}
\]

The formula for the number of \( k \)-walks by Cveković [2] (see also [3], p. 15) implies that

\[
\lim_{k \to \infty} \frac{w_k(G)}{w_{k-1}(G)} = \rho \quad \text{and} \quad \lim_{k \to \infty} \frac{w_{k+2}(G)}{w_{k-1}(G)} = \rho^3.
\]
Hence, in view of Theorem 5, we see that

\[ m\rho \geq \rho^3 - 3t + \rho \sum_{u \in V(G)} t''(u) \frac{x_u}{x_1 + \cdots + x_n} \]
\[ \geq \rho^3 - 3t + cpt''. \]

Theorem 7 is proved. \(\square\)

**Lemma 8** If \(G\) is a graph of order \(n\) with \(b_k(G) = \beta\), then

\[(n - 3\beta) t(G) \leq \beta t''(G).\]

**Proof** For \(0 \leq j \leq 2\), write \(k_4^{(j)}(G)\) for the number of induced subgraphs of \(G\) that are isomorphic to a triangle together with an additional vertex joined to precisely \(j\) vertices of the triangle. E.g., \(k_4^{(0)}(G)\) is the number of induced subgraphs of \(G\) that are isomorphic to a triangle with an isolated vertex. In addition, we write \(k_4\) for the number of 4-cliques of \(G\).

It is not hard to check the following three relations:

\[(n - 3) t = k_4^{(0)} + k_4^{(1)} + 2k_4^{(2)} + 4k_4. \quad (2)\]

\[3(\beta - 1) t \geq \sum_{\{u,v\} \in E(G)} |N(u) \cap N(v)| (|N(u) \cap N(v)| - 1) = 2k_4^{(2)} + 12k_4. \quad (3)\]

\[\beta t'' \geq \sum_{\{u,v\} \in E(G)} |N(u) \cap N(v)| |\overline{N}(u) \cap \overline{N}(v)| = k_4^{(1)} + 3k_4^{(0)}. \quad (4)\]

Now, subtracting (3) from (2), in view of (4), we find that

\[nt - 3\beta t \leq k_4^{(0)} + k_4^{(1)} + 2k_4^{(2)} + 4k_4 - 2k_4^{(2)} - 12k_4 \]
\[= k_4^{(0)} + k_4^{(1)} - 8k_4 \]
\[\leq \beta t''. \]

\(\square\)

Having Theorem 7 and Lemma 8 in hand, we are ready to prove a statement similar to Theorem 2 under some extra assumptions.

**Theorem 9** Let \(G\) be a connected graph with \(m\) edges such that \(\rho(G) > \sqrt{m}\). Suppose that \((x_1, \ldots, x_n)\) is a positive unit eigenvector to \(\rho(G)\). If

\[x_i x_j \geq \frac{1}{8\sqrt{m}}\]

for every edge \(\{i, j\} \in E(G)\), then

\[b_k(G) > \frac{1}{12} \sqrt{2m}.\]
Proof Let $\rho = \rho (G)$, let $i$ be a vertex, and suppose that $\{i, j\} \in E (G)$. We have
\[
\rho x_i^2 \geq x_j > \frac{1}{8\sqrt{m}} > \frac{1}{8\rho}.
\]
Now, letting
\[
c = \frac{1}{x_1 + \cdots + x_n} \min \{x_1, \ldots, x_n\},
\]
we see that
\[
c \rho > \frac{1}{\sqrt{8}} \cdot \frac{1}{x_1 + \cdots + x_n} \geq \frac{1}{\sqrt{8n}}.
\]
Note that $G$ is nonbipartite as $\rho > \sqrt{m}$. Hence, Theorem 7 implies that
\[
t'' < \sqrt{8n} \left(3t + \rho m - \rho^3\right).
\]
Combining this inequality with Lemma 8 and letting $\beta = bk (G)$, we find that
\[
(n - 3\beta) t \leq \beta t'' < \beta \sqrt{8n} \left(3t + \rho m - \rho^3\right)
\]
\[
\left(n - 3\beta - 3\beta\sqrt{8n}\right) t < \beta \sqrt{8n} \left(m - \rho^2\right) \rho \leq 0.
\]
Therefore, $n - 3\beta - 3\beta\sqrt{8n} < 0$, and we see that
\[
\beta > \frac{n}{3 \left(1 + \sqrt{8n}\right)} > \frac{n}{3 \left(4\sqrt{n}\right)} = \frac{\sqrt{n}}{12} \geq \frac{1}{12} \sqrt[4]{2m}.
\]
The proof of Theorem 9 is completed. \qed

4 Proof of Theorem 2

We may suppose that $G$ has no isolated vertices.

We first describe a simple procedure that constructs a sequence of graphs
\[
G_0 \supset G_1 \supset \cdots \supset G_l
\]
such that $V (G_i) = V (G)$ and $e (G_i) = m - i$ for every $i = 1, \ldots, l$.

Step 1 Set $l := 0$ and $G_0 := G$.

Step 2 If $l = \lceil m/2 \rceil$, stop.

Step 3 Let $x_l = (x_1, \ldots, x_n)$ be a nonnegative unit eigenvector to $\rho (G_l)$.

Step 4 If there is an edge $\{i, j\} \in E (G_l)$ with
\[
x_i x_j < \frac{1}{8\sqrt{m - l}}.
\]
set
\[ G_{l+1} := G_l - \{i, j\} \]
\[ l := l + 1 \]

and iterate the procedure from step 2.

**Step 5** If there is no such edge, stop.

Let \( k \) be the number of the last graph constructed by the procedure. Note that for every \( s = 1, \ldots, k \)
\[ \rho (G_s) > \rho (G_{s-1}) - \frac{1}{4\sqrt{m - s + 1}}. \]

Indeed, let \( x_{s-1} = (x_1, \ldots, x_n) \) be the unit eigenvector to \( \rho_{s-1} \) and let \( \{i, j\} \in E(G_{s-1}) \) be the edge such that
\[ G_s = G_{s-1} - \{i, j\} , \]
which entails
\[ x_i x_j < \frac{1}{8\sqrt{m - s + 1}}. \]

The Rayleigh principle implies that
\[ \rho (G_s) \geq \sum_{\{u,v\} \in E(G_s)} x_u x_v = -2x_i x_j + 2 \sum_{\{u,v\} \in E(G_s)} x_u x_v \]
\[ = \rho (G_{s-1}) - 2x_i x_j > \rho_{s-1} - \frac{1}{4\sqrt{m - s + 1}}. \]

Now, adding inequalities (5) for \( s = 1, \ldots, k \), we get
\[ \rho (G_k) \geq \rho (G_0) - \frac{1}{4\sqrt{m}} - \cdots - \frac{1}{4\sqrt{m - k + 1}} \]
\[ \geq \sqrt{m - k} + \sqrt{m} - \sqrt{m - k} - \frac{k}{4\sqrt{m - k + 1}} \]
\[ > \sqrt{m - k} + \frac{k}{2\sqrt{m}} - \frac{k}{4\sqrt{m/2}} \]
\[ = \sqrt{m - k} + \frac{k}{4\sqrt{m}} (2 - \sqrt{2}) . \]

Suppose that the procedure stops because \( k = \lceil m/2 \rceil \). Then we have
\[ \rho (G_k) > \sqrt{m - k} + \frac{1}{4} (\sqrt{2} - 1) \sqrt{m - k} , \]
and in view of
\[ b_k (G_k) (m - k) \geq 3t (G_k) , \]
Lemma 6 implies that
\[
bk(G_k) (m-k) \geq \rho(G_k) \left( \rho^2(G_k) - (m-k) \right)
\]
\[
\geq (m-k) \sqrt{m-k} \left( \left( 1 + \frac{1}{4} (\sqrt{2} - 1) \right)^2 - 1 \right)
\]
\[
> \frac{1}{5} (m-k) \sqrt{m-k}.
\]

Now, we see that
\[
bk(G) \geq bk(G_k) > \frac{1}{5} \sqrt{\lceil m/2 \rceil} > \frac{1}{12} \sqrt{m},
\]
completing the proof of Theorem 2 in the case \( k = \lfloor m/2 \rfloor \).

Next, suppose that the procedure stops because
\[
x_i x_j \geq \frac{1}{8 \sqrt{m-k}}
\]
for every edge \( \{i,j\} \in E(G_k) \).

Let us drop all isolated vertices that \( G_k \) may have and write \( G'_k \) for the resulting graph. Let \( (x_1, \ldots, x_p) \) be the restriction of \( x_k = (x_1, \ldots, x_n) \) to the vertices of \( G'_k \).

Inequality (6) implies that \( (x_1, \ldots, x_p) \) is positive. We shall show that \( G'_k \) is connected. Indeed, since \( (x_1, \ldots, x_p) \) is positive, the spectral radius of each component of \( G'_k \) is equal to \( \rho(G'_k) \). If \( G \) has more than one component, let \( C \) be a component of \( G'_k \) with smallest number of edges. We see that
\[
2e(C) \leq e(G'_k) \leq \rho(G'_k) = \rho(C),
\]
which is a contradiction. Hence, \( G'_k \) is connected.

If \( k \geq 1 \), we have
\[
\rho(G'_k) > \sqrt{m-k},
\]
so \( G'_k \) is nonbipartite. Now, Theorem 4 implies that
\[
bk(G) \geq bk(G'_k) > \frac{1}{12} \sqrt{2(m-k)}.
\]
In view of \( k \leq \lfloor m/2 \rfloor \), we get
\[
bk(G) \geq \frac{1}{12} \sqrt{m}.
\]

It remains the case \( k = 0 \), that is, \( G_k = G \). We assumed that \( G \) has no isolated vertices, and we showed above that \( \rho(G) \geq m \) implies that \( G \) is connected. Hence, if \( G \) is nonbipartite, then Theorem 9 implies that
\[
bk(G) > \frac{1}{12} \sqrt{2m} > \frac{1}{12} \sqrt{m}.
\]
Finally, if \( G \) is bipartite, then \( \rho(G) = \sqrt{m} \) implies that \( G \) is complete bipartite. Theorem 2 is proved.
5 Proof of Theorem 4

In this section we prove Theorem 4. Our proof is based on a simple analytic result:

**Lemma 10** Let \( k \geq 3 \), and \( a, b, x_1, \ldots, x_k \) be nonnegative numbers such that

\[
0 \leq x_1 \leq a, \ldots, x_k \leq a.
\]

If

\[
x_1^2 + \cdots + x_k^2 \leq a^2 + b^2,
\]

then for every real \( p > 2 \),

\[
x_1^p + \cdots + x_k^p < a^p + b^p,
\]

unless

\[
x_1 = a, x_2 = b, x_3 = \cdots = x_k = 0.
\]

**Proof** We may suppose that \( a > 0 \), as otherwise the assertion is trivially true.

Fix \( a > 0 \) and \( b \geq 0 \), and write \( X_{a,b} \) for the compact set of all vectors \((y_1, \ldots, y_k)\) satisfying

\[
0 \leq y_i \leq a, \quad (1 \leq i \leq k)
\]

\[
y_1^2 + \cdots + y_k^2 \leq a^2 + b^2.
\]

Let the continuous function \( y_1^p + \cdots + y_k^p \) attains maximum over \( X_{a,b} \) at \((x_1, \ldots, x_k)\), and suppose by symmetry that

\[
x_1 \leq x_2 \leq \cdots \leq x_k.
\]

We shall show that \( x_k = a \). Assume for contradiction that \( x_k < a \). Then we have

\[
x_1^2 + \cdots + x_k^2 = a^2 + b^2,
\]

as otherwise we can increase \( x_k \) by a tiny bit, keeping the resulting vector in \( X \) and increasing \( x_1^p + \cdots + x_k^p \), which contradicts the choice of \((x_1, \ldots, x_k)\).

Since \( x_k < a \), we find that \( x_{k-1} > 0 \), for otherwise

\[
x_{k-1} = \cdots = x_1 = 0,
\]

and so

\[
x_1^2 + \cdots + x_k^2 < a^2 + b^2.
\]

Since the function \( x^2 \) is a homeomorphism for \( x > 0 \), we can find \( \varepsilon > 0 \) and \( \delta > 0 \) such that

\[
x_{k-1} - \varepsilon > 0, \quad x_k + \delta < a,
\]

and

\[
(x_{k-1} - \varepsilon)^2 + (x_k + \delta)^2 = x_{k-1}^2 + x_k^2.
\]

Therefore, the \( k \)-vector

\[
(x_1, \ldots, x_{k-2}, x_{k-1} - \varepsilon, x_k + \delta)
\]
belongs to $X_{a,b}$.

Note that for $z > 0$ the function $z^{p/2}$ is strictly convex, as its second derivative

$$\frac{p(p-2)}{4} z^{p/2-2}$$

is positive. Hence, if $0 < \alpha < z_1 \leq z_2$, we have

$$(z_1 - \alpha)^{p/2} + (z_2 + \alpha)^{p/2} > z_1^{p/2} + z_2^{p/2}.$$

Now setting

$$z_1 = x_{k-1}^2,$$

$$z_2 = x_k^2,$$

$$\alpha = x_{k-1}^2 - (x_{k-1} - \delta)^2 = (x_k + \delta)^2 - x_k^2,$$

we see that

$$(x_{k-1} - \delta)^p + (x_k + \delta)^p = (z_1 - \alpha)^{p/2} + (z_2 + \alpha)^{p/2}$$

$$> z_1^{p/2} + z_2^{p/2}$$

$$= x_{k-1}^p + x_k^p.$$ 

This inequality contradicts the assumption that $y_1^p + \cdots + y_k^p$ attains maximum over $X_{a,b}$ at $(x_1, \ldots, x_k)$, and therefore $x_k = a$.

Further, we see that

$$x_1^2 + \cdots + x_{k-1}^2 \leq b^2,$$

which yields

$$x_1 \leq b, \ldots, x_{k-1} \leq b.$$

Therefore,

$$b^p \geq x_1^2 b^{p-2} + \cdots + x_{k-1}^2 b^{p-2}$$

$$\geq x_1^p + \cdots + x_{k-1}^p.$$ 

Equality may hold only if

$$x_{k-1} = b \quad \text{and} \quad x_{k-2} = \cdots = x_1 = 0.$$

Hence,

$$a^p + b^p \geq x_1^p + \cdots + x_{k-1}^p + a^p$$

$$= x_1^p + \cdots + x_k^p,$$

completing the proof of the lemma. \hfill \Box

**Proof of Theorem 4** Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the adjacency eigenvalues of $G$. 

10
We first prove the contrapositive of the statement of the theorem: if
\[ \lambda_1^2 + \lambda_2^2 > m, \]
then \( G \) has a triangle.

Clearly, we may assume that \( G \) is a noncomplete graph and therefore \( \lambda_2 \geq 0 \). Since
\[ \lambda_1^2 + \cdots + \lambda_n^2 = 2m, \]
we see that
\[ \lambda_1^2 + \lambda_2^2 > m > \lambda_3^2 + \cdots + \lambda_n^2. \] (7)

Let
\[ k = n - 2, \]
\[ a = \lambda_1, \]
\[ b = \lambda_2, \]
\[ x_i = |\lambda_{i+2}|, \quad i = 1, \ldots, k, \]
\[ p = 3. \]

Since \( a \geq b \) and \( a \geq x_i \) for \( i = 1, \ldots, k \), Lemma 10 implies that
\[ \lambda_1^3 + \lambda_2^3 \geq |\lambda_3|^3 + \cdots + |\lambda_n|^3. \]

Note that equality cannot hold above, for otherwise Lemma 10 implies that
\[ \lambda_1^2 + \lambda_2^2 = \lambda_3^2 + \cdots + \lambda_n^2, \]
contradicting (3).

Hence,
\[ 6t(G) = \lambda_1^3 + \lambda_2^3 + \lambda_3^3 + \cdots + \lambda_n^3 \geq \lambda_1^3 + \lambda_2^3 - |\lambda_3|^3 - \cdots - |\lambda_n|^3 > 0. \]

Therefore, \( G \) contains a triangle, proving the inequality of Theorem 4.

If \( G \) is triangle-free and
\[ \lambda_1^2 + \lambda_2^2 = m, \]
setting \( k, a, b, x_i, p \) as above, Lemma 10 implies that
\[ \lambda_1^3 + \lambda_2^3 = |\lambda_3|^3 + \cdots + |\lambda_n|^3, \]
and therefore, the condition for equality in Lemma 10 implies that
\[ \lambda_1^2 = \lambda_n^2, \]
\[ \lambda_2^2 = \lambda_{n-1}^2, \]
\[ \lambda_3 = \cdots = \lambda_{n-2} = 0. \]

Now the condition for equality in Theorem 4 follows from a result of Oboudi [6], exactly as in [4].
6 Concluding remarks

The bound on $bk(G)$ given by Theorem 2 seems far from optimal, as the multiplicative constant and perhaps the exponent $1/4$ can be improved.

It seems unlikely that Lemma 10 can be extended to support the proof of Conjecture 3 for $r \geq 3$.

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