PREDICATIVE WELL-ORDERING

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Abstract. Confusion over the predicativist conception of well-ordering pervades the literature and is responsible for widespread fundamental misconceptions about the nature of predicative reasoning. This short note aims to explain the core fallacy, first noted in [9], and some of its consequences.

1. Predicativism

Predicativism arose in the early 20th century as a response to the foundational crisis which resulted from the discovery of the classical paradoxes of naive set theory. It was initially developed in the writings of Poincaré, Russell, and Weyl.

Their central concern had to do with the avoidance of definitions they considered to be circular. Most importantly, they forbade any definition of a real number which involves quantification over all real numbers.

This version of predicativism is sometimes called "predicativism given the natural numbers" because there is no similar prohibition against defining a natural number by means of a condition which quantifies over all natural numbers. That is, one accepts \( \mathbb{N} \) as being "already there" in some sense which is sufficient to void any danger of vicious circularity. In effect, predicativists of this type consider uncountable collections to be proper classes. On the other hand, they regard countable sets and constructions as unproblematic.

2. Second order arithmetic

Second order arithmetic, in which one has distinct types of variables for natural numbers \((a, b, \ldots)\) and for sets of natural numbers \((A, B, \ldots)\), is thus a good setting for predicative reasoning — predicative given the natural numbers, but I will not keep repeating this. Here it becomes easier to frame the restriction mentioned above in terms of \(\mathcal{P}(\mathbb{N})\), the power set of \(\mathbb{N}\), rather than in terms of \(\mathbb{R}\). Thus we reject definitions of sets of numbers of the form

\[
\{a \in \mathbb{N} : P(a)\}
\]

when \(P\) is a formula which includes second order quantifiers, i.e., quantification over set variables.

The moral is that predicativists can accept only quite weak comprehension principles. They cannot reason about \(\{a : P(a)\}\) for general \(P\).

3. Countable well-orderings

Occasionally commentators have worried that predicativist scruples might forbid any reasoning about uncountable collections. This is wrong: it would be like saying that a finitist cannot make statements about all natural numbers, or a platonist
cannot make statements about all sets. They can, of course. In the same way that
a finitist can affirm that every natural number is even or odd, a predicativist can
affirm that every real number is rational or irrational. Indeed, for any particular
real number this question can be answered by a computation of countable length,
i.e., it is predicatively decidable.

An attractive position, expressed first in [9] and later, without credit, in [6],
is that in analogy with intuitionists, predicativists should assume the law of ex-
cluded middle only for formulas with no second order quantifiers. (A formal system
embodying this proposal appeared even earlier in [1].) This gives us the ability
to reason using such formulas while neatly explaining the obstruction to forming
\{a : P(a)\} as arising from the possibility that \(P(a)\) might not have a definite truth
value for some values of \(a\). However, I will not insist on this position here.

In any case, predicativists have a firm grasp of \(\mathbb{N}\) and can certainly affirm that
every nonempty set of natural numbers has a least element with respect to the
usual ordering. Given any set of natural numbers, a countable computation will
tell us either that it is empty or what its smallest element is, and predicativists
can see this. So there is nothing impredicative about defining a total ordering \(\preceq\)
of \(\mathbb{N}\) to be a well-ordering if every nonempty subset of \(\mathbb{N}\) has a \(\preceq\)-least element.
Yes, this definition involves quantification over \(\mathcal{P}(\mathbb{N})\), but there is no circularity
issue because it does not introduce any new objects. It does nothing more than fix
the term “well-ordered” as an abbreviation of the longer, predicatively intelligible
expression “every nonempty subset has a least element”.

4. Induction for sets

The following formulation of the well-ordering property is also useful. Say that
a subset \(A \subseteq \mathbb{N}\) is **progressive** for a total order \(\preceq\) on \(\mathbb{N}\) if \((\forall b < a)(b \in A)\) implies
\(a \in A\). That is, a number belongs to \(A\) whenever every preceding number belongs
to \(A\). Now consider the condition

\[(A): \text{If } A \subseteq \mathbb{N} \text{ is progressive for } \preceq \text{ then } A = \mathbb{N}.\]

This is equivalent to saying that \(\preceq\) is a well-order: in one direction, if \(\preceq\) is a well-
order and \(A\) is progressive, then \(\mathbb{N} \setminus A\) cannot have a least element by progressivity,
and hence it must be empty; conversely, if condition \((A)\) holds and \(A \subseteq \mathbb{N}\) is
nonempty, then \(\mathbb{N} \setminus A\) cannot be progressive, so there must exist \(a \in A\) such that
\((\forall b < a)(b \notin A)\), i.e., \(a\) is the least element of \(A\).

I have detailed this simple argument because I want to make the point that this
equivalence is predicatively valid. That is, the argument which shows that condition
\((A)\) is equivalent to \(\preceq\) being a well-order is one which a predicativist could make.
It should be easy to convince oneself of this.

5. Induction for predicates

Say that a predicate \(P\) — a formula with one free number variable — is progressive for \(\preceq\) if \((\forall b \prec a)P(b)\) implies \(P(a)\). We have the following stronger version of
condition \((A)\).

\[(B): \text{If a predicate } P \text{ is progressive for } \preceq \text{ then } (\forall a \in \mathbb{N})P(a).\]

This condition is stronger in the sense that it reduces to condition \((A)\) when we
take \(P(a)\) to be the predicate “\(a \in A\)”. However, condition \((B)\) can also be inferred
from condition \((A)\) by the following simple argument:
(1) Assume condition (A) and suppose $P$ is progressive for $\preceq$.

(2) Let $A = \{a \in \mathbb{N} : P(a)\}$.

(3) Since $P$ is a progressive predicate, $A$ is a progressive subset. Thus condition (A) yields $A = \mathbb{N}$, which is just to say that $P(a)$ holds for all $a \in \mathbb{N}$.

But this is an argument predicativists cannot generally make. Step 2 invokes a comprehension principle which is impredicative unless $P$ has a special form.

Condition (B) does not predicatively follow from condition (A). This is what no one has understood.

6. But doesn’t it really?

Most working mathematicians have probably internalized the equivalence of conditions (A) and (B), making it seem so obvious as not to need any special argument. But the two are not a priori equivalent. Condition (B) is stronger. It is only by invoking a comprehension axiom that one gets from (A) to (B).

Some people I have spoken to have acknowledged that (A) and (B) are a priori inequivalent, but have maintained that a predicativist can somehow just intuit that (B) follows from (A). Again, I think this point of view arises from having internalized the equivalence. The two statements are not the same. To get from one to the other you need to invoke a comprehension principle, and predicativists cannot do this. If predicativism means anything, it entails weak comprehension axioms. We cannot simply postulate that predicativists can “just see” the validity of some argument that relies on impredicative principles. One could draw any conclusion whatever from that kind of reasoning.

A more subtle idea is that it may in fact be the case that whenever there is a predicatively valid argument which shows that $\preceq$ is well-ordered in the sense of condition (A), there will be some other predicatively valid argument which shows that it verifies condition (B), at least schematically. But I do not see how one could prove this without first delineating precisely which ordinals are predicatively available.

7. Feferman-Schütte

The celebrated Feferman-Schütte analysis [3, 8] of predicatively provable ordinals allegedly provides just such a delineation. But it is wrong.

The goal is to determine which ordinals are isomorphic to recursive total orderings of $\mathbb{N}$ which can be proved to be well-orderings by predicative means. The Feferman-Schütte analysis can be presented in several ways, but the most intuitive involves a system which allows proof trees of infinite (well-ordered) heights over a straightforwardly predicative base system. The idea is that one starts with proof trees of height $\epsilon_0$, say, but once one has proven that a notation for $\alpha$ is well-ordered — so that $\alpha$ is predicatively “recognized” to be an ordinal — one is allowed to use proof trees of height $\alpha$.

If we define $\gamma_1 = \epsilon_0$ and $\gamma_{n+1} = \phi_{\gamma_n}(0)$, where $\phi_{\alpha}$ are the Veblen functions, then using a proof tree of height $\gamma_n$ one can prove that a notation for $\gamma_{n+1}$ is well-ordered. Thus, iterating this process, every ordinal less than $\Gamma_0 = \sup \gamma_n$ is supposed to be predicatively provable. Conversely, one cannot prove that a notation for $\Gamma_0$ is well-ordered using a proof tree of any height less than $\Gamma_0$, and this allegedly shows that $\Gamma_0$ is the limit of predicative reasoning.
An obvious question is why the predicativist cannot recognize for himself that for all $n$ he can prove a notation for $\gamma_n$ is well-ordered, and infer from this that $\Gamma_0$ is well-ordered. It seems like just the sort of countable reasoning that predicativists are good at. This question was raised in [7], among other places. I will return to it in a moment.

But first, let us ask how the predicativist is to infer the soundness of proof trees of height $\alpha$ from the fact that $\alpha$ is well-ordered. Why should it follow that such trees prove true theorems? It is easy to see that soundness is progressive — if every proof tree of height less than $\alpha$ is sound, then every proof tree of height $\alpha$ is sound — so one wants to use an induction argument in the form of condition (B) with $P(\alpha) =$ “proof trees of height $\alpha$ are sound”, where $\alpha$ is a notation for $\alpha$.

But all we prove with a tree of height $\gamma_n$ is that a notation for $\gamma_{n+1}$ is well-ordered in the sense of condition (A). Lacking an impredicative comprehension principle, we cannot infer the soundness of proof trees of height $\gamma_{n+1}$.

The Feferman-Schütte analysis relies essentially on an impredicative inference of condition (B) from condition (A).

8. INDUCTION VS. RECURSION

Now Feferman-Schütte does not really need the soundness of all proof trees of height $\alpha$. Closer examination of the argument reveals that what is essential is to be able to carry out recursive constructions of length $\alpha$. That is, given that for any $\beta \prec \alpha$ and any indexed family $(A_\gamma)_{\gamma \prec \beta}$ there exists a unique $A_\beta \subseteq \mathbb{N}$ which satisfies some condition relative to the previous $A_\gamma$’s, we need to conclude that there is a family $(A_\gamma)_{\gamma \prec \alpha}$ which satisfies the condition at each stage. That is all we really need. But this still requires an instance of condition (B) with $P$ containing one second order quantifier.

Thus, even stripped to its bare minimum, the argument which gets one from $\gamma_n$ to $\gamma_{n+1}$ requires impredicative comprehension.

A prominent logician recently told me that “everyone agrees” with the Feferman-Schütte analysis. I think this is basically true. Because no one has appreciated the distinction between induction for sets and induction for predicates.

One possible exception is Feferman himself: buried in his little-known paper [4] is a comment that “the well-ordering statement . . . on the face of it only impredicatively justifies the transfinite iteration of accepted principles up to $\alpha$,” which looks a lot like a direct acknowledgement of the defect we have been discussing. But then in the later paper [5] he refers to “prima facie impredicative notions such as those of ordinals or well-orderings,” revealing a basic misunderstanding of predicativism (cf. Section 3 above), and in [5] he also affirms that a version of the inference from condition (A) to condition (B) is predicative because it “accords with ordinary informal reasoning”.

In a personal communication to me, Feferman firmly denied that he had not repudiated the autonomous systems analysis, which I suppose means that he felt he had in fact repudiated it. His position, I think, was that later papers of his such as [4] or [5] remedied this defect by analyzing the problem in a different way. However, these later systems are shown in [9] to suffer from similar defects (and indeed, were obviously deliberately engineered to get the answer $\Gamma_0$).
9. A wild hope

It may be hard for anyone who has already bought into the Feferman-Schütte analysis to accept that it is wrong, no matter how plain the error is. But I do not expect anyone to attempt the futile task of trying to find some predicatively valid argument that infers recursion up to \( \alpha \) from induction up to \( \alpha \). The lazy response, the only one I have personally encountered, is to simply postulate that a predicativist can directly intuit this inference without needing to prove it.

Besides being absurd on its face, this response runs into the obvious question mentioned in Section 7, about the predicativist being able to recognize that for all \( n \) he can prove a notation for \( \gamma_n \) is well-ordered. He would then be able to get beyond \( \Gamma_0 \). So if one really wants to stop exactly at \( \Gamma_0 \), one has to postulate that the predicativist is able to magically intuit the inference of recursion from induction in any particular case, but he cannot recognize that he has this general ability, as this would get him past \( \Gamma_0 \). For every \( n \), when he proves that a notation for \( \gamma_n \) is well-ordered, the revelatory intuition that it also supports recursion has to come as a surprise. That is the depth of irrationality to which one must sink in order to preserve \( \Gamma_0 \) as the limit of predicative reasoning.

Let me return to the more subtle idea mentioned in Section 6. Could it be that whenever a predicativist can prove that a total order is a well-order, he can also prove in some different way that it supports transfinite recursion? As I said before, I do not see how one could prove this in general, but proving it for all ordinals less than \( \Gamma_0 \), say, does not seem out of the question.

In fact, I showed in [9] that using hierarchies of Tarskian truth predicates one can prove stronger forms of condition (B) at lower ordinals which allow one to prove weaker forms of (B) at higher ordinals, sufficiently to get up to \( \Gamma_0 \). In my opinion these techniques are predicatively legitimate. The problem for Feferman-Schütte is that using them one can continue on well past \( \Gamma_0 \), as was shown in detail in [9].

10. Inductive definitions

The Feferman-Schütte analysis is not the only place where the fallacy of conflating conditions (A) and (B) appears. For instance, consider the theory \( ID_1 \) of one inductive definition. It has been called “generalized predicative”, although according to the Stanford Encyclopedia of Philosophy it is simply “considered predicative in today’s foundations of constructive mathematics” [2]. Its proof theoretic ordinal is much larger than \( \Gamma_0 \).

In \( ID_1 \) one wants to introduce a class as the smallest class contained in \( \mathbb{N} \) which respects some closure operation. One formalizes this idea by introducing a constant symbol \( C \) to represent the class and expressing its minimality by an axiom scheme which states, for any predicate \( P \), that if \( P \) respects the closure condition then every element of \( C \) satisfies \( P \).

Classically \( C \) exists as the intersection of all classes which respect the closure condition, but this is not a predicatively valid definition: it defines a set of numbers by a condition which quantifies over all sets of numbers. However, it seems like one ought to be able to predicatively regard \( C \) in roughly the same way one predicatively regards \( \mathbb{R} \), as a proper class which can be built up in stages but never reaches a finished state.

This is not right. The suggestion that \( ID_1 \) is predicative in any sense is wrong, because the minimality axioms are impredicative. To see this, imagine how one
would establish that $P(a)$ holds for every $a \in C$, given that $P$ respects the closure operation. This hypothesis tells us that if every element appearing at some stage of the construction of $C$ satisfies $P$, then every element at the next stage will too. And at limit stages one merely collects together all the preceding stages, so nothing new appears and thus the minimality axiom is still verified, provided it was satisfied at all previous stages.

In other words, the condition “every element in the construction of $C$ at stage $\alpha$ satisfies $P$” is progressive in $\alpha$. So if one imagines constructing $C$ in a well-ordered series of stages, then the minimality axiom for $P$ should be satisfied — provided we can infer condition (B) from condition (A). Which we cannot predicatively do. It is the same error that invalidates Feferman-Schütte.

Another idea is to specify that $C$ is to be constructed in stages which are well-ordered not in the sense of condition (A), but in the stronger sense of supporting induction for any predicate of the language. But no, this is badly impredicative because the language contains the constant symbol “$C$”, which has no meaning until one has specified what $C$ is, so that one would be explaining how $C$ is to be constructed in terms of conditions which refer to $C$.

11. Kripke-Platek and CZF

Kripke-Platek set theory (KP) and Constructive Zermelo-Fraenkel set theory (CZF) are two set theoretic systems which are also routinely claimed to be predicative. (According to Wikipedia, KP is “roughly the predicative part of ZFC” and CZF has “a fairly direct constructive and predicative justification”.)

In fact, both are impredicative for the same reason $ID_1$ is: yet again, the fallacy involves a confusion between conditions (A) and (B). In both cases the problematic axioms are the set induction scheme, which states, for any formula $P$,

$$\forall y (\forall x (x \in y \rightarrow P(x)) \rightarrow P(y)) \rightarrow (\forall y)P(y).$$

Informally, if a predicate holds of a set $y$ whenever it holds of all the elements of $y$, that predicate must hold of all sets.

The informal justification for this scheme hinges on the premise that the universe of sets is built up in a well-ordered series of stages. One then applies progressivity of $P$ to infer, inductively, that it holds of all sets in the universe.

Just as with $ID_1$, this justification fails because being well-ordered in the sense of condition (A) does not predicatively entail the instances of condition (B) which would be needed to make the induction argument. And also as in that case, there is no option of strengthening the premise to say that the universe of sets is built up in a series of stages which are well-ordered in some stronger way which affirms condition (B). The instances of condition (B) which we would need in order to justify set induction involve all predicates expressible in the language of set theory, but the latter does not have an interpretation until we specify how the universe of sets is to be built up. So this would be circular.

12. Future directions

All this was explained in great detail in \[9\]. As I mentioned above, an attempt was also made there to rehabilitate the argument that gets us up to $\Gamma_0$ by means of iterated truth predicates. But when one does this there is no particular reason to stop at $\Gamma_0$; the iterated truth technique gets us up to at least the small Veblen
ordinal $\phi_{\omega^2}(0)$. However, the farther one goes the more complicated the analysis becomes.

Very likely the argument given in [9] can be simplified. I would still expect it to become more complicated the higher one goes. In my opinion predicativists can get past $\Gamma_0$, but not up to the Bachmann-Howard ordinal — the evidence for this is that, as discussed above, all known systems which get that far are clearly impredicative. The exact ordinal limit of predicative reasoning might not be a well-defined concept.

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