Growth of Solutions of Second Order Linear Differential Equations

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August 23, 2022

Abstract

In this paper, we will prove that all non-trivial solutions of \( f'' + A(z)f' + B(z)f = 0 \) are of infinite order, where we have some restrictions on entire functions \( A(z) \) and \( B(z) \).
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1. Introduction and statement of main results

Consider the differential equation,

\[
  f'' + A(z)f' + B(z)f = 0
\]

where \( A(z) \) and \( B(z) \) are entire functions. It is well known result that solutions of (1) are also entire functions. All solutions of (1) are of finite order if and only if both coefficients are polynomials. If either of \( A(z) \) or \( B(z) \) is transcendental entire then almost all solutions are of infinite order. Gundersen\[4\] proved that any non-constant solution is of infinite order when \( \rho(A) < \rho(B) \). Hellerstein, Miles and Rossi\[5\] proved that any non-constant solution is of infinite order when \( \rho(B) < \rho(A) \leq 1/2 \). Kumar and Saini\[8\] consider \( \lambda(A) < \rho(A) \) and \( B(z) \) a transcendental entire function satisfying either \( \rho(B) \neq \rho(A) \) or \( B(z) \) having Fabry gap and proved that non-trival solutions of (1) are of infinite order. J. Wang and I. Laine\[13\] consider \( A(z) = h(z)e^{-z} \) and \( B(z) \) to satisfy

\[
  T(r,B) \sim \log M(r,B)
\]

in a set \( E \) satisfying \( \log \text{dens}(E) > 0 \), where the above notation means

\[
  \lim_{r \to \infty} \frac{T(r,B)}{\log M(r,B)} = 1.
\]

They have proved the following result.

2010 Mathematics Subject Classification. 34M10, 30D35.

Key words and phrases. entire function, order of growth, complex differential equation, Fabry gap and Fejer gap.

The research work of the first author is supported by research fellowship from Council of Scientific and Industrial Research (CSIR), New Delhi.
Theorem A[13] Suppose that $A(z)$ and $B(z)$ are entire functions where $A(z) = h(z)e^{-z}$ and $\rho(h) < \rho(B) = 1$, and that $B(z)$ satisfies (2) in a set of positive upper logarithmic density. Then every non-trivial solution $f$ of equation (1) is of infinite order.

They considered $\rho(A) = \rho(B) = 1$. We have considered $A(z) = h(z)e^{P(z)}$ satisfying $\lambda(A) < \rho(A) = n$, where $P(z)$ is a polynomial of degree $n$ and improved above result as follows.

**Theorem 1.** Let $A(z) = h(z)e^{P(z)}$ be an entire function with $\lambda(A) < \rho(A) = n$, where $P(z)$ is a polynomial of degree $n$ and $B(z)$ satisfies (2) in a set $E$ of positive upper logarithmic density. Then all non-trivial solutions of equation (1) are of infinite order.

For an entire function $f(z) = \sum_{n=0}^{\infty} a_{\lambda_n}z^{\lambda_n}$, we say that it has Fejér gaps if the infinite series $\sum_{n=0}^{\infty} \frac{1}{\lambda_n}$ converges. Murai [12] proved that function having Fejér gaps satisfies (2) in a set of positive upper logarithmic density and has no deficient value. Thus, we can state the corollary with $A(z)$ satisfying condition of previous theorem and $B(z)$ having Fejér gap.

**Corollary 1.** Let $A(z)$ satisfies the condition given in Theorem 1 and $B(z)$ has Fejér gaps in a set of positive upper logarithmic density. Then all solutions of (1) are of infinite order.

**Example 1.** $A(z) = e^z$ and $B(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n^2)!}$ satisfies the condition of Theorem 1, equation (1) has all non-trivial solutions of infinite order.

Kwon[6] considered the case of $A(z)$ to be of non-intergal order and proved that all non-trivial solutions of (1) are of infinite order when $0 < \rho(B) < 1/2$.

**Theorem B[6]** Suppose that $A(z)$ is an entire function of finite non-integral order with $\rho(A) > 1$, and that all the zeros of $A(z)$ lie in the angular sector $\theta_1 < \arg z < \theta_2$ satisfying

$$\theta_2 - \theta_1 < \frac{\pi}{p+1}$$

if $p$ is odd, and

$$\theta_2 - \theta_1 < \frac{(2p-1)\pi}{2p(p+1)}$$

if $p$ is even, where $p$ is the genus of $A(z)$. Let $B(z)$ be an entire function with $0 < \rho(B) < 1/2$. Then every non-constant solution $f$ of (1) has infinite order.

It remains open what will be the behaviour of solution of (1) when $\rho(B) \geq \frac{1}{2}$. For this situation we proposed the following result. Before stating next result we are giving
definition of Fabry gap. For the entire function $f(z) = \sum_{n=0}^{\infty} a_{\lambda_n} z^{\lambda_n}$ we say it has Fabry gap if $\left\{ \frac{\lambda_n}{n} \right\}$ diverges when $n \to \infty$.

**Theorem 2.** Suppose that $A(z)$ is an entire function of finite non-integral order with $\rho(A) > 1$, and that all the zeros of $A(z)$ lie in the angular sector $\theta_1 < \arg z < \theta_2$ satisfying

$$\theta_2 - \theta_1 < \frac{\pi}{p + 1}$$

if $p$ is odd, and

$$\theta_2 - \theta_1 < \frac{(2p - 1)\pi}{2p(p + 1)}$$

if $p$ is even, where $p$ is the genus of $A(z)$. Let $B(z)$ be an entire function with Fabry gap. Then all non-trivial solutions $f$ of equation (1) are of infinite order.

In next section we list the preliminary results going to be used to prove the theorem and in section 3 we give proof of proposed theorem.

### 2. Preliminary lemmas

In this section we state some result which will be useful in proving our main results. We denote upper and lower logarithmic density of set $E$ by $\log dens(E)$ and $\log dens(E)$. Linear measure of a set $E$ is denoted by $m(E)$. Following two lemmas provide estimate of meromorphic function of finite order.

**Lemma 1.** [2] Let $f(z)$ be a meromorphic function of finite order $\rho$. Given $\xi > 0$ and $\delta$, $0 < \delta < 1/2$, there is a constant $K(\rho, \xi)$ such that for all $r$ in a set $E$ of lower logarithmic density greater than $1 - \xi$ and for every interval $J$ of length $\delta$

$$r \int_{J} \left\| \frac{f'\left(re^{i\theta}\right)}{f\left(re^{i\theta}\right)} \right\| d\theta < K(\rho, \xi)(\delta \log \frac{1}{\delta}) T(r, f).$$

**Lemma 2.** [3] Let $f$ be a transcendental meromorphic function with finite order and $(k, j)$ be a finite pair of integers that satisfies $k > j \geq 0$ and let $\epsilon > 0$ be a given constant. Then following statements holds:

(a) there exists a set $E_1 \subset [0, 2\pi]$ with linear measure zero such that for $\theta \in [0, 2\pi) \setminus E_1$ there exist $R(\theta) > 0$ such that

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\rho(f) - 1 + \epsilon)}$$

for all $k, j$; $|z| > R(\theta)$ and $\arg z = \theta$

(b) there exists a set $E_2 \subset (1, \infty)$ with finite logarithmic measure such that for all $|z| \not\in E_2 \cup [0, 1]$ such that inequality in (a) holds for all $k, j$ and $|z| \geq R(\theta)$.
(c) there exists a set $E_3 \subset [0, \infty)$ with finite linear measure such that for all $|z| \notin E_3$ such that

$$\frac{|f^{(k)}(z)|}{|f^{(j)}(z)|} \leq |z|^{(k-j)(\rho(f)+\epsilon)}$$

holds for all $k, j$.

Following lemma is due to Bank, Laine and Langley[1] that gives an estimate for an entire function with integral order.

**Lemma 3.** [1] Let $A(z) = h(z)e^{P(z)}$ be an entire function with $\lambda(A) < \rho(A) = n$, where $P(z)$ is a polynomial of degree $n$. Then for every $\epsilon > 0$ there exists $E \subset [0, 2\pi)$ of linear measure zero satisfying

(i) for $\theta \in [0, 2\pi) \setminus E$ with $\delta(P, \theta) > 0$, there exists $R > 1$ such that

$$\exp((1 - \epsilon)\delta(P, \theta)r^n) \leq |A(re^{i\theta})|$$

for $r > R$;

(ii) for $\theta \in [0, 2\pi) \setminus E$ with $\delta(P, \theta) < 0$, there exists $R > 1$ such that

$$|A(re^{i\theta})| \leq \exp((1 - \epsilon)\delta(P, \theta)r^n)$$

for $r > R$.

The next lemma gives estimates of function with non-integral order at particular value of $z$.

**Lemma 4.** [6] Let $f(z)$ be an entire function of finite non-integral order $\rho$ and of genus $p > 1$. Suppose that for any given $\epsilon > 0$, all the zeros of $f(z)$ have their arguments in the following set of real numbers:

$$S(p, \epsilon) = \{\theta : |\theta| \leq \frac{\pi}{2(p + 1)} - \epsilon\}$$

if $p$ is odd, and

$$S(p, \epsilon) = \{\theta : \frac{\pi}{2p} + \epsilon \leq |\theta| \leq \frac{3\pi}{2(p + 1)} - \epsilon\}$$

if $p$ is even. Then for any $c > 1$, there exists a real number $R > 0$ such that

$$|f(-r)| \leq \exp(-cr^p)$$

for all $r \geq R$.

The following lemma gives the property of an entire function with Fabry gap and can be found in [10] and [14].

**Lemma 5.** Let $g(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ be an entire function of finite order with Fabry gap, and $f(z)$ be an entire function with $\rho(f) \in (0, \infty)$. Then for any given $\epsilon \in (0, \rho(f))$, there exists a set $H \subset (1, +\infty)$ satisfying $\log\text{dense}(H) \geq \xi$, where $\xi \in (0, 1)$ is a constant such that for all $|z| = r \in H$, one has

$$\log M(r, h) > r^{\rho(f) - \epsilon}, \log m(r, g) > (1 - \xi) \log M(r, g),$$
where \( M(r,h) = \max \{|h(z)| : |z| = r \} \), \( m(r,g) = \min \{|g(z)| : |z| = r \} \) and \( M(r,g) = \max \{|g(z)| : |z| = r \} \).

**Lemma 6.** [11] Let \( f(z) \) be an entire function satisfying \( T(r,f) \sim \log M(r,f) \) in a set \( E \) of positive upper logarithmic density. For given \( 0 < c < \frac{1}{4} \) and \( r \in E \), the set

\[
I_r = \{ \theta \in [0,2\pi) : \log |f(re^{\theta})| \leq (1-c) \log M(r,f) \}
\]

has linear measure zero.

### 3. Proof of main theorems

**Proof of Theorem 1:**

**Proof.** Let us suppose that solution \( f \) of (1) is of finite order. For given \( 0 < c < \frac{1}{4} \), let

\[
I_r = \{ \theta \in [0,2\pi) : \log |B(re^{\theta})| \leq (1-c) \log M(r,B) \}.
\]

Since, \( B(z) \) satisfies \( T(r,B) \sim \log M(r,B) \) in a set \( E \) satisfying \( \log \text{dens}(E) > 0 \), using Lemma 6 we have \( m(I_r) \to 0 \) as \( r \to \infty \) and \( r \in E \). Using Lemma 1, for given \( \xi \), \( 0 < \delta < 1/2 \) and choosing \( \delta \) so small that \( K(\rho,\xi)(\delta \log \frac{1}{\delta}) < c \), we have

\[
r \int_{J} \left| \frac{B'(re^{\theta})}{B(re^{\theta})} \right| d\theta < cT(r,B)
\]

where \( K(\rho,\xi) \) is a constant and \( r \in E' \) satisfying \( \log \text{dens}(E') > 1 - \xi \), for every interval \( J \) of length \( \delta \). If \( \theta, \theta' \in [0,2\pi)/I_r \), \( |\theta - \theta'| < \delta \) and for all sufficiently large \( r \in E \cap E' \), we have

\[
\log |B(re^{\theta})| = \log |B(re^{\theta'})| + \int_{\theta'}^{\theta} \frac{d}{d\theta} \log |B(re^{\theta})| d\theta
\]

\[
> (1 - c) \log M(r,B) - r \int_{\theta'}^{\theta} \left| \frac{B'(re^{\theta})}{B(re^{\theta})} \right| d\theta
\]

\[
> (1 - 2c) \log M(r,B).
\]

Using Lemma 2(a), we have

\[
(3) \quad \left| \frac{f^{(k)}(re^{\theta})}{f(re^{\theta})} \right| \leq r^{k\rho(f)}
\]

for \( \theta \in [0,2\pi)/E'' \), where \( E'' \) is a set with linear measure 0 and \( r > R(\theta) \). Choosing \( \theta \in [0,2\pi)/E'' \) such that \( \delta(P,\theta) < 0 \), where \( E'' \) is a set with linear measure 0 we have

\[
(4) \quad |A(re^{\theta})| \leq \exp((1-c)\delta(P,\theta)r^\mu)
\]

for \( r > R \).

From equation (1), (3) and (4) for \( r > R(\theta) \) such that \( r \in E \cap E', \theta \in [0,2\pi)/I_r \cup E'' \cup E''' \) and \( \delta(P,\theta) < 0 \), we have

\[
|B(z)| \leq \left| \frac{f''(re^{\theta})}{f(re^{\theta})} \right| + |A(z)| \left| \frac{f'(re^{\theta})}{f(re^{\theta})} \right|
\]
\[ M(r, B)^{1-2c} < |B(z)| \leq (1 + o(1))r^{2\rho(f)} \]
\[ M(r, B) < (1 + o(1))r^{4\rho(f)} \]

There is a contradiction for transcendental entire function \( B(z) \).

**Proof of Theorem 2:**

**Proof.** Let us suppose \( f \) be a solution of (1) of finite order. Then by Lemma 2 (c) for \( r > R \) and \( r \notin G \) where \( G \in [0, \infty) \) is a set of finite linear measure, we have

\[ \left| \frac{f^{(k)}(re^{i\theta})}{f(re^{i\theta})} \right| \leq r^{k\rho(f)} \]

Let us rotate the axes of the complex plane, assume that all the zeros of \( A(z) \) have their arguments in the set \( S(\rho, \epsilon) \) defined in Lemma 4 for some \( \epsilon > 0 \). Hence, by Lemma 4, there exists a positive real number \( R \) such that for all \( r > R \), we have

\[ \min_{|z|=r} |A(z)| \leq |A(-r)| \leq \exp(-cr^\rho) < 1. \]

Since \( B(z) \) has Fabry gap, by Lemma 5 there exist a set \( H \subset (1, +\infty) \) satisfying

\[ \log \text{dens}(H) \geq \xi, \text{ where } \xi \in (0, 1) \text{ is a constant such that for all } |z| = r \in H. \]

\[ (1 - \xi) \log M(r, B) < \log m(r, B) < \log |B(z)| \]

From equation (1), we get

\[ |B(z)| \leq \left| \frac{f''(re^{i\theta})}{f(re^{i\theta})} \right| + |A(z)| \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right| \]

From (5), (6) and (7), for \( r \in H/G, r > R \) and \( \theta \in \{ \theta : \min_{|z|=r} |A(z)| = |A(z)| \} \), we have

\[ M(r, B) \leq r^{4\rho(f)} \]

There is a contradiction for transcendental entire function \( B(z) \).

**Acknowledgement** We are thankful to Dr. Dinesh Kumar for reading the manuscript and suggesting some changes.

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