A note on one-loop soliton quantum mass corrections

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Abstract

We develop an alternative derivation of the renormalized expression for the one-loop soliton quantum mass corrections in \((1+1)\)-dimensional scalar field theories. We regularize implicitly such quantity by subtracting and adding its corresponding tadpole graph contribution and use the renormalization prescription that such added term vanishes with adequate counterterms. As a result, we get a finite unambiguous formula for the soliton quantum mass corrections up to one-loop order, which turns to be independent of the chosen regularization scheme.
1 Introduction

There has been a great deal of progress in studying one loop quantum corrections to the kink mass in (1+1)-dimensional field theories since the first appearance of these calculations in the bosonic $\phi^4$ and sine-Gordon models [1]–[3]. After that, several authors considered the corresponding supersymmetric extensions of those models and since then different approaches have been developed to calculate quantum corrections to the supersymmetric kink mass and central charge [4]–[13]. For some time, the two main concerns were if the bosonic and fermionic contributions in the quantum corrections to the supersymmetric kink mass cancel each other, and if the BPS saturation condition survives at the quantum level, which were exhaustively investigated in a series of publications [14]–[28].

Recently, these issues have been reconsidered by introducing new tools and methods to deal with renormalization, regularization and calculation of the one loop corrections to the kink mass in bosonic scalar and supersymmetric field theories [29]–[33], which bring up new interesting insights on this topic.

The purpose of this short note is to provide a simple derivation of the renormalized formula for the one loop soliton quantum mass correction in (1+1) dimensional bosonic field theory. As the result, we will get a formula, Eq. (2.23), which turns out to be independent of the regularization scheme used.

2 The method

Let us start with the bosonic Lagrangian,

$$\mathcal{L} = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - U(\phi) + \delta \mathcal{L},$$

where the potential $U(\phi)$ has at least two degenerate minima, and $\delta \mathcal{L}$ contains adequate counterterms in order to render finite the theory. By quantizing around the static kink configuration $\phi_c$, we can write the soliton mass at one loop order in the following form,

$$M = E[\phi_c] + \delta M + \frac{1}{2} \sum_n \omega_n - \frac{1}{2} \sum_k \omega(k),$$

where $E[\phi_c]$ is the energy of the static classical kink configuration, $\delta M$ are the counterterm contributions from $\delta \mathcal{L}$ term at one loop order and the eigenfrequencies $\omega_n$ and $\omega(k)$ are given by the eigenvalue equations

$$\left[ -\frac{d^2}{dx^2} + U''(\phi_c(x)) \right] \eta_n = \omega_n^2 \eta_n,$$  \hspace{1cm} (2.3)

and

$$\left[ -\frac{d^2}{dx^2} + m^2 \right] \eta_k = \omega^2(k) \eta_k,$$  \hspace{1cm} (2.4)

with $m$ being the mass of the quantum fluctuations around the trivial vacua. The last two terms in (2.2) usually are logarithmically divergent, which requires the use of certain regularization and renormalization schemes. There are several regularization techniques that have been used to deal with this problem (see for instance Refs. [1], [34]–[43], and references therein).

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In this note we consider a simple method to regularize the last two terms in Eq. (2.2). The method is based on the following formal identity [44],

\[
\frac{1}{2} \left( \sum_n \omega_n - \sum_k \omega(k) \right) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \text{Tr} \ln \left( 1 + \hat{A} \right), \tag{2.5}
\]

where

\[
\hat{A} = \frac{\mathcal{V}}{\omega^2 - \frac{d^2}{dx^2} + m^2}, \tag{2.6}
\]

with \( \mathcal{V} = U''[\phi_c(x)] - m^2 \). Equation (2.5) can be expanded formally in powers of \( \hat{A} \), and then, in terms of Feynman graphs, we have

\[
\frac{1}{2} \left( \sum_n \omega_n - \sum_k \omega(k) \right) = \ldots, \tag{2.7}
\]

from which we can identify the tadpole graph contribution to the one loop soliton mass, as the only ultraviolet divergent graph, and whose expression is given by

\[
\mathcal{V} - \bigcirc = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \text{Tr} \hat{A}. \tag{2.8}
\]

Therefore, by adding and subtracting the tadpole graph above in (2.2), and using the renormalization prescription that the added tadpole graph cancels with \( \delta M \), we get the renormalized finite result for the one loop soliton mass,

\[
M = E[\phi_c] + \frac{1}{2} \sum n \omega_n - \frac{1}{2} \sum k \omega(k) - \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \text{Tr} \hat{A}. \tag{2.9}
\]

Of course, all the above manipulations are meaningful only if the divergent terms are taken as already regularized. Consequently, in order to get finite unambiguous results through (2.9), we have to consider the last three terms as a whole, i.e. we have to use same regularization procedure when dealing with each divergent term. Thus, considering such quantities as implicitly regularized, we have to take due care only in passing from formal sums to integrals for the continuous spectrum given by Eqs. (2.3) and (2.4). For this end we note that the free-soliton eigenfrequencies in Eq. (2.3) are continuous and given by \( \omega(k) = \sqrt{k^2 + m^2} \). On the other hand, the soliton eigenfrequencies in Eq. (2.3) fall in two sets, a finite discrete set denoted by \( \omega_i \), and a continuous set \( \omega(q) = \sqrt{q^2 + m^2} \). Both \( \omega(k) \) and \( \omega(q) \), range from \( m \) to \( \infty \), but they do not cancel each other in Eq. (2.2) because their corresponding densities of states are different. Splitting the second sum in (2.2), we have

\[
M = E[\phi_c] + \frac{1}{2} \sum_{i=1}^{N} \omega_i + \Delta, \tag{2.10}
\]

with \( \Delta \) given by

\[
\Delta = \frac{1}{2} \sum_{q=-\infty}^{\infty} \omega(q) - \frac{1}{2} \sum_{k=-\infty}^{\infty} \omega(k) - \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \text{Tr} \hat{A}. \tag{2.11}
\]
In order to go from formal sums to integrals we enclose the system in a box of size $L$, and then taking the limit $L \to \infty$ after applying periodic boundary conditions. For the free eigenfunctions $\eta_k \propto e^{ikx}$, we have the condition $e^{-ikL/2} = e^{ikL/2}$, and then

$$k_n = \frac{2\pi n}{L}, \quad n = 0, \pm 1, \pm 2, \ldots$$  \hspace{1cm} (2.12)

In this case we found that the free density of states is given by,

$$\frac{1}{(k_{n+1} - k_n)} = \frac{L}{(2\pi)}.$$  \hspace{1cm} (2.13)

For the soliton eigenfrequencies $\omega(q)$, let us consider first the simplest case in which $V(x)$ is a reflectionless potential, which is the case of the sine-Gordon and $\phi^4$ models. In this case, the asymptotic behaviour of $\eta_q(x)$ is given by,

$$\eta_q(x) = \begin{cases} e^{iqx} & x \to -\infty \\ e^{iqx+i\delta(q)} & x \to +\infty \end{cases},$$  \hspace{1cm} (2.14)

where $\delta(q)$ is the scattering phase shift. Imposing periodic boundary conditions, $\eta_q(-L/2) = \eta_q(L/2)$, we get

$$q_n = \frac{2\pi n}{L} - \frac{\delta(q_n)}{L}, \quad n = 0, \pm 1, \pm 2, \ldots$$  \hspace{1cm} (2.15)

Now, by using the free-soliton basis $\{\langle x|k_n \rangle = e^{ik_n x}/\sqrt{L}\}$ to compute the trace in the last term of Eq. (2.11) we get after integration in $\omega$,

$$\int_{-\infty}^{\infty} \frac{d\omega}{4\pi} \operatorname{Tr} \hat{A} = \int_{-\infty}^{\infty} \frac{d\omega}{4\pi} \sum_{n=-\infty}^{\infty} \langle k_n | \frac{V(x)}{\omega^2 - \frac{d}{dx}^2 + m^2} | k_n \rangle,$$

$$= \frac{\langle V \rangle}{4} \sum_{n=-\infty}^{\infty} \frac{1}{L \sqrt{k_n^2 + m^2}},$$  \hspace{1cm} (2.16)

with

$$\langle V \rangle = \int_{-\infty}^{\infty} V(x) dx.$$  \hspace{1cm} (2.17)

Therefore, by using that $\omega(-q) = \omega(q)$ (similarly for $\omega(k)$), Eq. (2.11) can be rewritten as follows,

$$\Delta = \sum_{q=0}^{\infty} \omega(q) - \sum_{k=0}^{\infty} \omega(k) - \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \operatorname{Tr} \hat{A},$$

$$= \sum_{n=n_0}^{\infty} \sqrt{q_n^2 + m^2} - \sum_{n=0}^{\infty} \sqrt{k_n^2 + m^2} - \frac{\langle V \rangle}{4} \sum_{n=-\infty}^{\infty} \frac{1}{L \sqrt{k_n^2 + m^2}},$$  \hspace{1cm} (2.18)

where $n_0$ in the first sum is not necessarily equal to zero, but must be chosen as the correct integer corresponding to $q = 0$ according to Eq. (2.15). So, by setting $q_{n_0} = 0^+$ in Eq. (2.15) and using the one dimensional Levinson theorem for potential with half-bound states (which is the case of the reflectionless potentials and the free case)$^{[45]}$, $\delta(0^+) = N\pi$, we have

$$2\pi n_0/L - N\pi/L = 0,$$  \hspace{1cm} (2.19)
from which we find that \( n_0 = N/2 \). It is worth noting that this solution makes sense only if \( N \) is even, which will be assumed at first. Now, by setting \( N = 2l \), we have that \( n_0 = l \), and Eq. (2.18) can be written as

\[
\Delta = \sum_{n=1}^{\infty} \sqrt{q_n^2 + m^2} - \sum_{n=0}^{\infty} \sqrt{k_n^2 + m^2} - \frac{\langle V \rangle}{4} \sum_{n=-\infty}^{\infty} \frac{1}{L\sqrt{k_n^2 + m^2}}
\]

(2.20)

From Eqs. (2.12) and (2.15), we obtain

\[
\sqrt{q_n^2 + m^2} = \sqrt{k_n^2 + m^2} - \frac{k_n\delta(k_n)}{L\sqrt{k_n^2 + m^2}} + \mathcal{O}(L^{-2}).
\]

(2.21)

Substituting in (2.20), we have

\[
\Delta = -\sum_{n=0}^{\infty} \frac{k_n\delta(k_n)}{L\sqrt{k_n^2 + m^2}} - \sum_{n=0}^{l-1} \sqrt{q_n^2 + m^2} - \frac{\langle V \rangle}{4} \sum_{n=-\infty}^{\infty} \frac{1}{L\sqrt{k_n^2 + m^2}} + \mathcal{O}(L^{-2}).
\]

(2.22)

Now, by replacing Eqs. (2.22) in Eq. (2.10), and taking the limit \( L \to \infty \), we get

\[
M = E[\phi_c] + \frac{1}{2} \sum_{i=1}^{N} (\omega_i - m) - \int_{0}^{\infty} \frac{dk}{4\pi} \frac{[2k\delta(k) + \langle V \rangle]}{\sqrt{k^2 + m^2}},
\]

(2.23)

where we have used that \( q_0, q_1, \ldots, q_{l-1} \), approach zero in the limit \( L \to \infty \) in the second sum in (2.22). Equation (2.23) corresponds to the renormalized formula for the one loop soliton quantum mass correction, see for example [36, 37].

Let us now consider the case when \( N \) is odd. In this case \( n_0 = N/2 \) will be a half integer, and then it will be necessary to disregard this value since it does not satisfy the condition (2.15). In fact, it turns to be that such half integer \( n_0 \) satisfies anti-periodic boundary conditions. Now, disregarding \( n_0 \) is totally equivalent to disregard \( q = 0 \), and this implies now that the Eq. (2.18) is no longer correct in the box. Then, by introducing a proper modification, we get a new form of the expression (2.18) for the odd case, namely

\[
\Delta = \frac{1}{2} \sum_{q=-\infty}^{\infty} \omega(q) - \frac{1}{2} \sum_{k=-\infty}^{\infty} \omega(k) - \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \text{Tr} \hat{A},
\]

\[
\Delta = \sum_{q>0} \sqrt{q^2 + m^2} - \sum_{k>0} \sqrt{k^2 + m^2} - \frac{m}{2} - \frac{\langle V \rangle}{4} \sum_{n=-\infty}^{\infty} \frac{1}{L\sqrt{k_n^2 + m^2}},
\]

\[
\Delta = \sum_{n=n_0}^{\infty} \sqrt{q_n^2 + m^2} - \sum_{n=1}^{\infty} \sqrt{k_n^2 + m^2} - \frac{m}{2} - \frac{\langle V \rangle}{4} \sum_{n=-\infty}^{\infty} \frac{1}{L\sqrt{k_n^2 + m^2}},
\]

(2.24)

where we have used again the symmetry \( \omega(q) = \omega(-q) \), \( \omega(k) = \omega(-k) \), and that \( \omega(0) = m \), by passing from the first to the second line. The first sum in Eq (2.24) must begin at the next integer greater than \( n_0 \). So, by setting \( N = 2l + 1 \), we get

\[
\Delta = \sum_{n=l+1}^{\infty} \sqrt{q_n^2 + m^2} - \sum_{n=1}^{\infty} \sqrt{k_n^2 + m^2} - \frac{m}{2} - \frac{\langle V \rangle}{4} \sum_{n=-\infty}^{\infty} \frac{1}{L\sqrt{k_n^2 + m^2}},
\]

\[
\Delta = \sum_{n=1}^{\infty} \left( \sqrt{q_n^2 + m^2} - \sqrt{k_n^2 + m^2} \right) - \sum_{n=1}^{l} \sqrt{q_n^2 + m^2} - \frac{m}{2} - \frac{\langle V \rangle}{4} \sum_{n=-\infty}^{\infty} \frac{1}{L\sqrt{k_n^2 + m^2}}.
\]

(2.25)
Now, by using Eq. (2.21), and taking the limit \( L \to \infty \), we find
\[
\Delta = -(2l + 1) \frac{m}{2} - \int_0^\infty \frac{dk}{4\pi} \frac{[2k\delta(k) + \langle \mathcal{V} \rangle]}{\sqrt{k^2 + m^2}},
\]  
(2.26)
and replacing this expression in (2.10), we get the same formula (2.23) of the even case.

At this point it is worth highlighting that in its more general form, the Levinson’s theorem in one dimension includes a one-half contribution, i.e \( \delta(0) = (N - \frac{1}{2})\pi \). However, we can show that the results are not altered, and that the same procedure can be applied. Note that, in our method the Levinson’s theorem is used to determine the lowest integer \( n_0 \), from where the sum over the soliton eigenfrequencies starts. In that case, we can see that the relation (19) is modified by,
\[
\frac{2\pi n_0}{L} - \frac{\pi}{L} \left( N - \frac{1}{2} \right) = 0, \quad \rightarrow \quad n_0 = \frac{N}{2} - \frac{1}{4}.
\]

Then, whether \( N \) being even or odd, we should disregard that value (which is equivalent to disregard \( q = 0 \) from the spectrum) since it does not satisfy the condition (15), and then the sum should start from the next integer. In the even case \( N = 2l \), we should start from \( n_0 = l \), while in the odd case, \( N = 2l + 1 \), from \( n_0 = l + 1 \). After doing that, the derivation follows similar steps as the odd case.

On the other hand, Eq. (2.23) can be also obtained for more general potentials \( \mathcal{V} \), which are not necessarily reflectionless. In that case, it can be shown that identical results are obtained once the phase shift is properly generalized in terms of the \( S \)-matrix of the associated one-dimensional scattering problem given by the continuous solutions of Eq. (2.3). In terms of the reflection and transmission coefficient amplitudes, the \( S \)-matrix can be parametrized in the following form,[45]
\[
S = \begin{pmatrix} T & R' \\ R & T \end{pmatrix},
\]  
(2.27)
where \( R \) denotes the reflection coefficient of scattering from left to right, and \( R' \) from right to the left [46]. From unitarity of the \( S \) matrix, we have \( T^* R' + TR^* = 0 \), and then we obtain
\[
S(k) = \begin{pmatrix} T(k) & -R^*(k)T(k)/T^*(k) \\ R(k) & T(k) \end{pmatrix},
\]  
(2.28)

Now, to decouple into the scattering channels, we can be performed an unitary transformation \( USU^\dagger \), where
\[
U = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{2i\gamma(k)} & -1 \\ e^{-2i\gamma(k)} & 1 \end{bmatrix}, \quad e^{2i\gamma(k)} = \sqrt{R'/R},
\]  
(2.29)
and
\[
USU^\dagger = \begin{bmatrix} e^{2i\delta_1(k)} & 0 \\ 0 & e^{2i\delta_2(k)} \end{bmatrix}.
\]  
(2.30)

Then, the scattering decoupled channels become
\[
\eta_1(x) = A_1 \sin (kx + \gamma(k) \pm \delta_1(k)), \quad x \to \pm \infty,
\]
\[
\eta_2(x) = A_2 \cos (kx + \gamma(k) \pm \delta_2(k)), \quad x \to \pm \infty.
\]
By applying periodic boundary conditions at \( x = \pm L/2 \), we will find
\[
k = \frac{2\pi n}{L} - \frac{2}{L} \delta_1(k), \quad k = \frac{2\pi n}{L} - \frac{2}{L} \delta_2(k),
\]
where each scattering phase shift channel follows similar equation as eq. (15). Therefore, repeating the same arguments above, we get the same formula (2.23) for the one loop renormalized soliton mass, in terms of the total phase shift defined as \( \delta(k) = \delta_1(k) + \delta_2(k) \). In addition, by taking the determinant of the \( S \) matrix, we find that the phase shift is given by,
\[
\delta(k) = \frac{1}{2i} \ln \det S(k),
\]
\[
= \frac{1}{2i} \ln \left[ \frac{T(k)}{T^*(k)} \right], 
\tag{2.31}
\]
where we have used that \( |T|^2 + |R|^2 = 1 \). It is also worth mentioning that unitarity of the \( S \)-matrix ensures that the phase shift \( \delta(k) \) in Eq. (2.31) is a real function of \( k \).

3 Discussion and conclusions

The formula (2.23) that we have obtained for the renormalized mass of the soliton at one loop order is totally equivalent to the one obtained some time ago in reference [36], where authors introduced the so called phase shift technique. At this point, we call attention to the fact that although we have used the scattering phase shift as a basic ingredient, our approach is different from Ref. [36]. First of all, in order to regularize the quantum mass corrections, the authors subtracted the first Born approximation from the phase shift, and in order to overcome the infrared divergence introduced by the first Born approximation, the one dimensional Levinson’s theorem is used. In our approach, we regularize implicitly the one loop soliton mass by subtracting the tadpole graph contribution to it, which is free of infrared divergences. Therefore no use is made of the one dimensional Levinson’s theorem with regard to the regularization issue. We get the same results, since we are using the same renormalization prescription.

From Eq. (2.20), we note that our method resembles somehow the so called mode number regularization method. However, we need to call attention again to the fact that the starting points and assumptions in both methods are quite different. On one hand, in the mode number regularization method the system is enclosed in a finite discrete box, which implies that the number of modes of Eqs. (2.3) and (2.4) become finite, and then they are assumed to be the same in both cases. On the other hand, in the method presented in this note we enclose the system in a non-discrete box of finite size, and therefore the number of modes is no longer finite. So, in our approach, nothing else about the number of modes is then assumed. We can proceed in this way since, as it was already mentioned, \( \Delta \) given by Eq. (2.20) is finite and unambiguous, if each separately divergent term is considered as implicitly regularized in the same scheme. Then, the only issue that we have to be careful about is in passing from formal sums to integrals for the continuum spectrum of Eqs. (2.3) and (2.4).
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