TESTS FOR COMPLETE $K$-SPECTRAL SETS

MICHAEL A. DRITSCHEL, DANIEL ESTÈVEZ, AND DMITRY YAKUBOVICH

Abstract. Let $\Phi$ be a family of functions analytic in some neighborhood of a complex domain $\Omega$, and let $T$ be a Hilbert space operator whose spectrum is contained in $\overline{\Omega}$. Our typical result shows that under some extra conditions, if the closed unit disc is complete $K'$-spectral for $\varphi(T)$ for every $\varphi \in \Phi$, then $\overline{\Omega}$ is complete $K$-spectral for $T$ for some constant $K$. In particular, we prove that under a geometric transversality condition, the intersection of finitely many $K'$-spectral sets for $T$ is again $K$-spectral for some $K \geq K'$. These theorems generalize and complement results by Mascioni, Stessin, Stampfli, Badea-Beckerman-Crouzeix and others. We also extend to non-convex domains a result by Putinar and Sandberg on the existence of a skew dilation of $T$ to a normal operator with spectrum in $\partial \Omega$. As a key tool, we use the results from our previous paper [8] on traces of analytic uniform algebras.

1. Introduction

Let $T$ be an operator on a Hilbert space $H$ and $\Omega$ a bounded subset of $\mathbb{C}$ containing the spectrum $\sigma(T)$. We recall that, given a constant $K \geq 1$, the closure $\overline{\Omega}$ of $\Omega$ is said to be a complete $K$-spectral set for $T$ if the matrix von Neumann inequality

\begin{equation}
\|p(T)\|_{B(H \otimes \mathbb{C}^s)} \leq K \max_{z \in \overline{\Omega}} \|p(z)\|_{B(\mathbb{C}^s)}
\end{equation}

holds for any square $s \times s$ rational matrix function $p(z)$ of any size $s$ and with poles off of $\overline{\Omega}$; here $B(H)$ denotes the space of linear operators on $H$. The set $\overline{\Omega}$ is called a $K$-spectral set for $T$ if (1) holds for $s = 1$. By a well-known theorem of Arveson [3], $\overline{\Omega}$ is a complete $K$-spectral set for $T$ for some $K \geq 1$ if and only if $T$ is similar to an operator, which has a normal dilation $N$ with $\sigma(N) \subset \partial \Omega$; the importance of complete $K$-spectral sets is due to this result.

We denote by $\hat{\mathbb{C}}$ the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. By a Jordan domain in $\hat{\mathbb{C}}$ we mean an open domain $\Omega \subset \hat{\mathbb{C}}$ whose boundary is a Jordan curve. A Jordan domain (or Jordan domain in $\mathbb{C}$) is just a bounded Jordan domain in $\hat{\mathbb{C}}$. A curve $\Gamma \subset \mathbb{C}$ is called Ahlfors regular if $|B(z, \varepsilon) \cap \Gamma| \leq C\varepsilon$, for every

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\( \varepsilon > 0 \) and every \( z \in \Gamma \), where \( C \) is a constant independent of \( \varepsilon \) and \( z \). Here \( | \cdot | \) denotes the arc-length measure and \( B(z, \varepsilon) \) is the open disk of radius \( \varepsilon \) and center \( z \).

By a circular sector with vertex \( z_0 \) we mean a set in \( \mathbb{C} \) of the form
\[
\{ z \in \mathbb{C} : 0 < |z - z_0| < r, \alpha < \arg z < \beta \},
\]
where \( r > 0 \) and \( \alpha, \beta \in \mathbb{R}, 0 < \beta - \alpha < 2\pi \). The aperture of such a circular sector is the number \( \beta - \alpha \).

If \( \Omega_1, \Omega_2 \subset \hat{\mathbb{C}} \) are two open sets, and \( \infty \neq z_0 \in \partial \Omega_1 \cap \partial \Omega_2 \) is a point in the intersection of their boundaries, we say that the boundaries of \( \Omega_1 \) and \( \Omega_2 \) intersect transversally at \( z_0 \) if one can find five pairwise disjoint circular sectors \( S_0, S_1^1, S_1^2, S_2^1, S_2^2 \) with vertex \( z_0 \), having the same aperture, and such that the following conditions are satisfied:

- \( S_0 \) does not intersect \( \overline{\Omega_1} \cup \overline{\Omega_2} \).
- \( B(z_0, \varepsilon) \cap \partial \Omega_j \subset S_j^1 \cup S_j^2 \cup \{ z_0 \} \) for \( j = 1, 2 \) and some \( \varepsilon > 0 \).
- For every \( \delta > 0 \), \( B(z_0, \delta) \cap \Omega_1 \cap \Omega_2 \) is not empty.

In the case when \( \infty \in \partial \Omega_1 \cap \partial \Omega_2 \), we say that the boundaries of \( \Omega_1 \) and \( \Omega_2 \) intersect transversally at \( \infty \) if the boundaries of \( \psi(\Omega_1) \) and \( \psi(\Omega_2) \) intersect transversally at 0, where \( \psi(z) = 1/z \). We say that the boundaries of \( \Omega_1 \) and \( \Omega_2 \) intersect transversally if they intersect transversally at every point of \( \partial \Omega_1 \cap \partial \Omega_2 \). Note that the third condition in the definition of a transversal intersection implies that \( \Omega_1 \cap \Omega_2 = \overline{\Omega_1} \cap \overline{\Omega_2} \).

By an analytic arc in \( \mathbb{C} \) we mean an image of the interval \([0, 1]\) under a function, analytic in its neighborhood. A piecewise analytic curve will mean a curve which can be subdivided into finitely many analytic arcs.

We can now state some of the main results of the paper.

**Theorem 1.** Let \( \Omega_1, \ldots, \Omega_n \) be open sets in \( \hat{\mathbb{C}} \) such that the boundary of each set \( \Omega_k \), \( k = 1, \ldots, n \), is a finite disjoint union of Jordan curves. We also assume that the boundaries of the sets \( \Omega_k \), \( k = 1, \ldots, n \), are Ahlfors regular and rectifiable, and intersect transversally. Put \( \Omega = \Omega_1 \cap \cdots \cap \Omega_n \).

Suppose that \( T \in \mathcal{B}(H) \), and \( \sigma(T) \subset \overline{\Omega} \). There is a constant \( K' \) such that

1. if each of the sets \( \overline{\Omega}_j \), \( j = 1, \ldots, n \), is \( K \)-spectral for \( T \), then \( \overline{\Omega} \) is also \( K' \)-spectral for \( T \); and
2. if each of the sets \( \overline{\Omega}_j \), \( j = 1, \ldots, n \), is complete \( K \)-spectral for \( T \), then \( \overline{\Omega} \) is a complete \( K' \)-spectral set for \( T \).

In both cases, \( K' \) depends only on the sets \( \Omega_1, \ldots, \Omega_n \) and the constant \( K \), but not on the operator \( T \).

As will be seen from the proof, the Ahlfors regularity condition can be weakened, by requiring that it hold only in some neighborhoods of the intersection points of the boundary curves \( \partial \Omega_j \).

The results of Theorem 1 can be viewed as a generalization of the so called surgery of \( K \)-spectral sets. The articles [21, 39, 40] are devoted to this topic.

In the case when the sets that one is dealing with are Jordan domains and their boundaries intersect transversally, the results of these articles can be obtained as a particular case of Theorem 1.

In [5], Badea, Beckermann and Crouzeix prove that the intersection of complete spectral sets which are disks on the Riemann sphere is a complete
Let $T$ be a bounded linear operator and $\Omega \subset \mathbb{C}$ an open set whose boundary is a finite disjoint union of Jordan curves. Assume that $\infty \notin \partial \Omega$, that $\Omega$ satisfies Condition A and that $\sigma(T) \subset \overline{\Omega}$. Furthermore, assume that for every $k = 1, \ldots, N$ and every $\lambda \in \gamma_k$ we have $\| (T - \mu_k(\lambda))^{-1} \| \leq R_k^{-1}$. Then $\overline{\Omega}$ is a complete $K$-spectral set for some $K > 0$.

It is easy to see that the hypotheses are satisfied (for any $R_k > 0$) if $\Omega$ is a convex Jordan domain and the numerical range of $T$ is contained in $\overline{\Omega}$. This
case was first proved by Delyon and Delyon in [14]. Theorem 2 will be deduced from this and from Theorem 1. Putinar and Sandberg gave a different proof of the Delyon-Delyon result in [35] by constructing a so called normal skew-dilation, and relate the constant in this result with C. Neumann’s “configuration constant” of a convex domain Ω, see [35], Proposition 1. These articles consider only K-spectral sets instead of complete K-spectral sets. However, the arguments used both in [14] and [35] imply the existence of a normal operator N on a larger Hilbert space K ⊃ H and having \( \sigma(N) \subset \partial\Omega \), and a bounded linear map \( \Xi : C(\partial\Omega) \to C(\partial\Omega) \) such that

\[
f(T) = P_H(\Xi(f))(N)|_H, \quad f \in \text{Rat}(\Omega).
\]

It follows from Lemma 6 below that the map \( \Xi \) is completely bounded (see also Crouzeix [10]). Therefore, (1) implies that \( \Omega \) is a complete K-spectral set for \( T \), and so that under the assumptions of the Delyon-Delyon theorem, \( T \) is similar to an operator having a normal dilation to \( \partial\Omega \).

It is also known that Theorem 2 is valid if \( \Omega \) is the unit disk. In fact, by results of Sz.-Nagy and Foias, if the hypotheses hold in this case, then \( T \) is a \( \rho \)-contraction for some \( \rho < \infty \) and hence is similar to a contraction. Therefore Theorem 2 can be considered as a generalization of both of the above mentioned results. We refer to Section 4 for a further discussion and some consequences of this result.

We will deduce the first part of Theorem 1 from results of Havin, Nersessian and Cerdà, where they give various geometric conditions on domains \( \Omega_1, \ldots, \Omega_s \) in \( \mathbb{C} \), guaranteeing that every function \( f \) in \( H^\infty(\cap_j \Omega_j) \) admits a representation

\[
f = f_1 + f_2 + \cdots + f_s, \quad f_j \in H^\infty(\Omega_j), \quad j = 1, 2, \ldots, s.
\]

In other words, the domains \( \Omega_1, \ldots, \Omega_s \) admit a separation of singularities.

The proof of the second part of Theorem 1 will also use Lemma 6. This lemma says, basically, that if the range of a bounded linear map is commutative, then the map is automatically completely bounded. The particular maps we will be considering have commutative ranges, so this lemma will be important in our proofs. The arguments by Havin, Nersessian and Cerdà will also be key here, because they allow us to deal with commutative algebras of functions (to which Lemma 6 can be applied) instead of noncommutative algebras of operators.

Suppose now that \( \Phi \) is a collection of functions mapping into \( \mathbb{D} \), such that each of them is analytic on (its own) neighborhood of \( \Omega \). The rest of the article is devoted to finding sufficient conditions for complete K-spectrality of the form

\[
\exists K', \forall \varphi \in \Phi \quad \overline{\mathbb{D}} \text{ is a complete } K'-\text{spectral set for } \varphi(T) \\
\implies \exists K : \overline{\Omega} \text{ is a complete } K\text{-spectral set for } T.
\]

Here \( \mathbb{D} \) stands for the open unit disk. Notice that for any \( \varphi \in \Phi \), \( \varphi(T) \) is defined by the Cauchy-Riesz functional calculus. Our conditions concern the set \( \Omega \) and the family \( \Phi \), but we do not impose extra conditions on \( T \).

In particular, a special case of (2) is that

\[
\forall \varphi \in \Phi \quad \|\varphi(T)\| \leq 1 \implies \exists K : \overline{\Omega} \text{ is a complete } K\text{-spectral set for } T.
\]
In this case, $\Phi$ will be called a test collection (a more precise definition of this notion will be given in the next section). As we will show, many known sufficient conditions for complete $K$-spectrality are easily formulated in the form (3) or in the form (2) for specific test collections. Indeed, Theorems 1 and 2 can also be given this form if one uses appropriate Riemann mappings for the test functions (see Section 3 below).

As we will show, implication (2) holds when we can solve the following:

**Algebra Generation Problem.** Suppose $\Phi$ is a finite family of functions in $A(\overline{\Omega})$, the algebra of functions in $C(\overline{\Omega})$ that are analytic on the interior of $\overline{\Omega}$. Find geometric conditions guaranteeing that $\Phi$ generates $A(\overline{\Omega})$ as an algebra.

A solution to this was given in our previous article [8] (which also was inspired by the techniques of Havin, Nersessian and Cerdà [18, 19]). In fact, we more generally prove that sometimes it is sufficient to show that the closed subalgebra of $A(\overline{\Omega})$ generated by $\Phi$ is of finite codimension.

The article is organized in the following manner. In Section 2, we introduce admissible test functions and we state the main results of this article concerning test collections. In Section 3, we interpret known criteria of complete $K$-spectrality in terms of test collections. Section 4 contains the proofs of Theorems 1 and 2, which were stated in the Introduction. We also formulate a question, related with Theorem 1. In Section 5, we will list the results of [8] that we will need to prove our theorems. Section 6 is devoted to the auxiliary lemmas that are needed to prove the main results. Section 7 contains the proofs of the main theorems. Finally, in Section 8, we treat weakly admissible test functions, a larger class of test functions for which we can also prove some results (see, in particular, Theorem 18).

2. Test collections

2.1. Preliminaries. We denote by $M_s$ the $C^*$-algebra of complex $s \times s$ matrices. If $S$ is a (not necessarily closed) linear subspace of a $C^*$-algebra $A$, we denote by $S \otimes M_s$ the tensor product equipped with the norm inherited from $A \otimes M_s$, which has a unique $C^*$ norm. One can view $S \otimes M_s$ as the space of $s \times s$ matrices with entries in $S$. The simplest way to norm this is to represent $A$ faithfully as a subspace of $B(H)$ and then to take the natural norm of $s \times s$ operator matrices. If $B$ is another $C^*$-algebra and $\varphi : S \to B$ is a linear map, we can form the map $\varphi \otimes \text{id}_s : S \otimes M_s \to B \otimes M_s$, which amounts to applying $\varphi$ entrywise to $s \times s$ matrices over $S$. The completely bounded norm of $\varphi$ is then defined as

$$\|\varphi\|_{cb} = \sup_{s \geq 1} \|\varphi \otimes \text{id}_s\|.$$  

If a compact set $X \subset \mathbb{C}$ is a complete $K$-spectral set for a bounded linear operator $T$ and $\text{Rat}(X)$, the algebra of rational functions with poles off of $X$, is dense in $A(X)$, then the functional calculus for $T$ extends continuously to $f \in A(X)$, and we say that such a $T$ admits a continuous $A(X)$-calculus. Note that there are various sorts of geometric conditions on $X$ guaranteeing that $\text{Rat}(X)$ is dense in $A(X)$ (see, for instance, [7, Chapter V, Theorem 19.2] for one such). In particular, it suffices for $X$ to be finitely connected...
(see [7, Chapter V, Corollary 19.3]). In what follows, we only consider finitely connected domains.

2.2. Different types of test collections. Here we give the definitions of the several kinds of test collections used throughout the paper. As a convenient notation, for $\lambda \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, define $p_\lambda(z) = (z - \lambda)^{-1}$ if $\lambda \neq \infty$, and $p_\infty(z) = z$.

Assume that $\Omega \subset \hat{\mathbb{C}}$ is some finitely connected set. A pole set for $\Omega$ is a finite set $\Lambda \subset \hat{\mathbb{C}} \setminus \overline{\Omega}$ that intersects each connected component of $\hat{\mathbb{C}} \setminus \overline{\Omega}$. If $T \in \mathcal{B}(H)$ and $\sigma(T) \subset \overline{\Omega}$, the $\Lambda$-pole size of $T$ is defined as $\max_{\lambda \in \Lambda} \|p_\lambda(T)\|$. We denote the $\Lambda$-pole size of $T$ by $S_\Lambda(T)$. In the setting of this article, $\Omega$ will be an (open or closed) $k$-connected domain and we usually choose pole sets of minimal cardinality; that is, having $k$ elements, one in each connected component of $\hat{\mathbb{C}} \setminus \overline{\Omega}$.

**Definitions.** Let $\Phi$ be a collection of functions mapping $\Omega$ into $\mathbb{D}$ and analytic in neighborhoods of $\Omega$. Fix a pole set $\Lambda$ for $\Omega$. We say that $\Phi$ is a

(i) uniform test collection over $\Omega$ if the implication (3) holds, where the constant $K$ depends only on $\Omega$ and $\Phi$ (and not on $T$);

(ii) quasi-uniform test collection over $\Omega$ if (3) holds, where $K$ depends on $\Omega$, $\Phi$ and $S_\Lambda(T)$;

(iii) non-uniform test collection over $\Omega$ if (3) holds, where $K$ can depend on $\Omega$, $\Phi$ and the operator $T$;

(iv) uniform strong test collection over $\Omega$ if (2) holds, where $K$ depends only on $\Omega$, $\Phi$ and $K'$ (but not on $T$);

(v) quasi-uniform strong test collection over $\Omega$ if (2) holds, where $K$ depends on $\Omega$, $\Phi$, $K'$ and $S_\Lambda(T)$;

(vi) non-uniform strong test collection over $\Omega$ if (2) holds, where $K$ depends on $\Omega$, $\Phi$, $K'$, and also may depend on $T$.

To summarize, there is the basic notion of a test collection, which roughly means that whenever $\varphi(T)$ is a contraction for every $\varphi$ in the collection, then $T$ has $\overline{\Omega}$ as a $K$-spectral set. To this, one can add the adjectives uniform, quasi-uniform and non-uniform, which mean respectively that $K$ does not depend on $T$, that $K$ depends only on $S_\Lambda(T)$, and that $K$ may depend on $T$. Finally, the term strong indicates that we can replace the condition $\|\varphi(T)\| \leq 1$ by the weaker condition that $\overline{\mathbb{D}}$ is a complete $K'$-spectral set for $\varphi(T)$ for all $\varphi \in \Phi$.

An operator $R$ has $\overline{\mathbb{D}}$ as a complete 1-spectral set if and only if $R$ is a contraction. In this case, $R$ has $\overline{\mathbb{D}}$ as a complete $K'$-spectral set for all $K' > 1$. Therefore, each strong test collection is a test collection.

Also note that when $\Phi = \{\varphi\}$ consists of a single element, the strong part comes for free, since if $\varphi(T)$ has $\overline{\mathbb{D}}$ as a complete $K$-spectral set for some $K$, then there is some invertible operator $S$ such that $S\varphi(T)S^{-1} = \varphi(STS^{-1})$ is a contraction, and so we can reason with $STS^{-1}$ instead of $T$.

In most cases, $\Omega$ will be an open domain or the closure of an open domain. Given a domain $\Omega$, the notions of a test collection over $\Omega$ and over $\overline{\Omega}$ might seem very similar, but as we will see below, the condition that $\sigma(T) \subset \overline{\Omega}$, as opposed to the stronger condition $\sigma(T) \subset \Omega$, represents an additional technical challenge in some arguments.
Finally, the notion of a non-uniform test collection over an open set $\Omega$ is trivial, since if $\sigma(T) \subset \Omega$, then $\Omega$ is a complete $K$-spectral set for $T$, where $K$ depends on $\Omega$ and $T$. This was first proved for $\Omega = \mathbb{D}$ by Rota [36], and follows in general from the Herrero-Voiculescu theorem (see [29, Theorem 9.13]).

2.3. Admissible function families. Let us recall the definition of an admissible function from [8].

Definition. Let $\Omega \subset \mathbb{C}$ be a domain whose boundary is a disjoint finite union of piecewise analytic Jordan curves such that the interior angles of the “corners” of $\partial \Omega$ are in $(0, \pi]$. We will say that an analytic function $\Phi = (\varphi_1, \ldots, \varphi_n) : \overline{\Omega} \to \overline{\mathbb{D}^n}$ is admissible if $\varphi_k \in A(\Omega)$, for $k = 1, \ldots, n$, and there is a collection of closed analytic arcs $\{J_k\}_{k=1}^n$ of $\partial \Omega$ and a constant $\alpha$, $0 < \alpha \leq 1$, such that the following conditions are satisfied:

(a) The arcs $J_k$ cover all $\partial \Omega$.
(b) $|\varphi_k| = 1$ in $J_k$, for $k = 1, \ldots, n$.
(c) For each $k = 1, \ldots, n$, there exists an open set $\Omega_k \supset \Omega$ such that the interior of $J_k$ relative to $\partial \Omega$ is contained in $\Omega_k$, $\varphi_k$ is defined in $\overline{\Omega_k}$, $\varphi_k \in A(\overline{\Omega_k})$, and $\varphi_k'$ is of class Hölder $\alpha$ in $\Omega_k$, i.e.,

$$|\varphi_k'(\zeta) - \varphi_k'(z)| \leq C|\zeta - z|^{\alpha}, \quad \zeta, z \in \Omega_k.$$

(d) If $z_0$ is an endpoint of $J_k$, then there exists an open sector $S_k(z_0)$ with vertex on $z_0$ and such that $S_k(z_0) \subset \Omega_k$ and $J_k \cap B(z_0, \varepsilon) \subset S_k(z_0) \cup \{z_0\}$, for some $\varepsilon > 0$. Here, $B(z_0, \varepsilon)$ denotes the open disk of center $z_0$ and radius $\varepsilon$. If $z_0$ is a common endpoint of both $J_k$ and $J_l$, $k \neq l$, then we require that $(S_k(z_0) \cap S_l(z_0)) \setminus \overline{\Omega}$ be nonempty.

(e) $|\varphi_k'| \geq C > 0$ in $J_k$, for $k = 1, \ldots, n$.

(f) For each $k = 1, \ldots, n$, $\varphi_k(\zeta) \neq \varphi_k(z)$ if $\zeta \in J_k$ and $z \in \overline{\Omega}$, $z \neq \zeta$.

We recall from [8] that there is no loss of generality in assuming in this definition that the arcs $J_k$ intersect only at their endpoints.
Given an admissible function \( \Phi = (\varphi_1, \ldots, \varphi_n) : \overline{\Omega} \to \mathbb{D}^n \), we will denote the set of functions \( \{\varphi_1, \ldots, \varphi_n\} \) by the same letter \( \Phi \).

**Theorem 3.** Assume that \( \Phi : \overline{\Omega} \to \mathbb{D}^n \) is admissible and analytic in an open neighborhood of \( \overline{\Omega} \), where \( \Omega \) is a Jordan domain. Then \( \Phi \) is a quasi-uniform strong test collection in \( \overline{\Omega} \). If, moreover, \( \Phi \) is injective and \( \Phi \) does not vanish on \( \Omega \), then \( \Phi \) is a uniform strong test collection over \( \overline{\Omega} \).

This means that if \( T \in \mathcal{B}(H) \) satisfies \( \sigma(T) \subset \overline{\Omega} \), \( S_\lambda(T) \) is an arbitrary fixed pole set for \( \Omega \), and \( \varphi_k(T) \) have \( \mathbb{D} \) as a complete \( K' \)-spectral set for \( k = 1, \ldots, n \), then \( T \) has \( \overline{\Omega} \) as a complete \( K \)-spectral set, with \( K = K(\Omega, \Phi, K', S_\lambda(T)) \). If \( \Phi \) is injective and \( \Phi' \) does not vanish on \( \Omega \), then one can even choose \( K \) independently of \( T \).

**Theorem 4.** Let \( \Phi : \overline{\Omega} \to \mathbb{D}^n \) be admissible and \( \Lambda \) an arbitrary fixed pole set for \( \Omega \). Given \( T \in \mathcal{B}(H) \), assume that there are operators \( C_1, \ldots, C_n \in \mathcal{B}(H) \) such that \( \mathbb{D} \) is a complete \( K' \)-spectral set for every \( C_k \), \( k = 1, \ldots, n \). Furthermore, assume that whenever \( f \in \text{Rat}(\overline{\Omega}) \) can be written as

\[
(4) \quad f(z) = \sum_{k=1}^{n} f_k(\varphi_k(z)), \quad f_k \in A(\mathbb{D}),
\]

we have

\[
(5) \quad f(T) = \sum_{k=1}^{n} f_k(C_k).
\]

Then \( \overline{\Omega} \) is a complete \( K \)-spectral set for \( T \) with \( K \) depending only on \( \Omega \), \( \Phi \) and \( S_\lambda(T) \). If moreover, \( \Phi \) is injective and \( \Phi' \) does not vanish on \( \Omega \), then one can choose \( K \) independently of \( T \).

A posteriori, since \( \overline{\Omega} \) is a complete \( K \)-spectral for \( T \), the operators \( \varphi_k(T) \) are defined by the \( A(\overline{\Omega}) \) calculus for \( T \). The hypotheses of the theorem imply that \( C_k = \varphi_k(T) \), so the operators \( C_k \) are uniquely defined. However, a priori, the operators \( \varphi_k(T) \) are not defined by any reasonable functional calculus, so the theorem cannot be stated in terms of these operators.

If \( \sigma(T) \subset \Omega \), then it is an easy consequence of the Cauchy-Riesz functional calculus that \( C_k = \varphi_k(T) \) satisfy the hypotheses of this theorem. Therefore, this proves the following corollary.

**Corollary 5.** If \( \Phi : \overline{\Omega} \to \mathbb{D}^n \) is admissible, then \( \Phi \) is a quasi-uniform strong test-collection over \( \Omega \). If moreover \( \Phi \) is injective and \( \Phi' \) does not vanish on \( \Omega \), then \( \Phi \) is a uniform strong test collection over \( \Omega \).

**Remark.** The main differences between Theorems 3 and 4 is that Theorem 3 assumes that \( \Omega \) is simply connected and Theorem 4 does not. On the other hand, Theorem 4 requires the existence of some operators \( C_k \) which behave in an informal sense like \( \varphi_k(T) \) (the formal condition is that (4) implies (5)). As it will be clear from the proofs of these theorems, the case when \( \sigma(T) \subset \Omega \) is easy to handle, while the case when \( \sigma(T) \) contains part of the boundary of \( \Omega \) presents some technical difficulties. Theorems 3 and 4 represent two different ways of sorting out these difficulties. In Theorem 3, we will use the existence of a certain family \( \{\psi_k\}_{0 \leq k \leq \delta} \) of univalent functions on \( \Omega \) to
pass from the operator $T$ to operators $\psi_\varepsilon(T)$, whose spectra are contained over $\Omega$. In Theorem 4 we postulate some kind of functional calculus for $T$. Ultimately, it would be desirable to extend Theorem 3 to multiply connected domains.

We remark that it follows from the proofs of our theorems that similar results hold if one replaces complete $K$-spectral sets by (not necessarily complete) $K$-spectral sets. For instance, in Theorem 4, if $C_k$ have $\overline{D}$ as a $K'$-spectral set, then $\overline{\Omega}$ is $K$-spectral for $T$.

Theorems 1 and 2, which were stated in the Introduction, can be reformulated in terms of test collections. Theorem 1 shows that if $\varphi_k : \Omega_k \to \mathbb{D}$ are Riemann conformal maps, then \{\varphi_1, \ldots, \varphi_N\} is a uniform strong test collection for $\overline{\Omega}$. In Theorem 2, we can put $\varphi_{k,\lambda}(z) = R(z - \mu_k(\lambda))^{-1}$. Then \{\varphi_{k,\lambda} : k = 1, \ldots, N, \lambda \in \gamma_k\} is a uniform test collection over $\Omega$.

In Theorem 3, it is easy to see that when $\Phi$ is not injective or $\Phi'$ has zeros, then $\Phi$ can only be a non-uniform strong test collection in $\overline{\Omega}$ (i.e., one cannot remove the adjective “non-uniform”). For instance, if $\Phi(z_1) = \Phi(z_2)$ for distinct points $z_1, z_2 \in \Omega$, then we can take an operator $T$ acting on $\mathbb{C}^2$ and having $z_1$ and $z_2$ as eigenvalues, with associated eigenvectors $v_1$ and $v_2$. For every $k$, we have $\varphi_k(T) = \varphi_k(z_1)I$, which is a contraction. If the angle between $v_1$ and $v_2$ is very small, then $\|T\|$ will be very large, so there is no constant $K$ independent of $T$ such that $\overline{\Omega}$ is $K$-spectral for $T$.

Similarly, if $\Phi'(z_0) = 0$ for some $z_0 \in \Omega$, we can take an operator $T$ such that $T \neq z_0I$ and $(T - z_0I)^2 = 0$. For every $n \geq 1$, we put $T_n = n(T - z_0I) + z_0I$. Then it is easy to check that for every $n$ and every $k$, we have $\varphi_k(T_n) = \varphi_k(z_0)I$, which is a contraction. However, $\|T_n\| \to \infty$ as $n \to \infty$. This implies that there is no constant $K$ independent of $n$ such that $\overline{\Omega}$ is $K$-spectral for $T_n$, for every $n$.

To illustrate the phenomenon described in the last paragraph, construct a domain $\Omega$ and an admissible function $\Phi : \overline{\Omega} \to \mathbb{D}^3$ such that $\Phi'$ vanishes at some point $z_0 \in \Omega$. Choose a small $\varepsilon > 0$ and put $z_1 = 0$, $z_2 = \varepsilon$, $z_3 = \varepsilon/2 + i\sqrt{3}\varepsilon/2$, so that $z_1, z_2, z_3$ are the vertices of a equilateral triangle of side length $\varepsilon$. Let $z_0$ be the center of this triangle.

Let $D_j$ be the disk of radius 1 and center $z_j$. We put $\Omega = D_1 \cap D_2 \cap D_3$. We can divide the boundary of $\Omega$ in three arcs $J_k$ by putting $J_k = (\partial \Omega) \cap (\partial D_k)$, for $k = 1, 2, 3$. Since $\varepsilon$ is small, it is easy to see that the length of each arc $J_k$ is close to $2\pi/3$.

Let $\varphi_1(z) = (z - z_0)^2/(1 - \overline{z_0}z)^2$. Then $\varphi_1$ maps $D_1$ onto $\mathbb{D}$, and it maps $J_1$ bijectively onto some arc of $T$. For $k = 2, 3$, let $\eta_k$ be the orientation-preserving rigid motion taking $z_k$ to $z_1$ and $J_k$ to $J_1$ (so that it maps $\overline{D_k}$ onto $\overline{D_1}$). Note that $\eta_k(z_0) = z_0$. We define $\varphi_k = \varphi_1 \circ \eta_k$, for $k = 2, 3$. We see that $\varphi_k'(z_0) = 0$ for $k = 1, 2, 3$. It is easy to check that $\Phi = (\varphi_1, \varphi_2, \varphi_3)$ is admissible in $\overline{\Omega}$, because, for every $k$, $\varphi_k$ is analytic on a neighborhood of $\overline{\Omega}$ and takes $J_k$ bijectively onto some arc of $T$.

It is worthwhile mentioning that the condition in the definition of an admissible family of functions requiring the interior angle of a corner of the domain $\Omega$ to be in $(0, \pi]$ can be relaxed in the results stated above if one instead requires that the corner is not in the spectrum of the operator $T$ under consideration. This is seen by altering $\Omega$, removing the intersection
with a small enough disk about the corner. The complement of the disk will be a complete spectral set for \( T \) and the new corners created will satisfy the condition that the interior angles are in \((0, \pi]\). Since the disk can be made arbitrarily small, it essentially has no effect on the statements given above, other than that there is now a dependence on the choice of \( T \) through this additional requirement on the spectrum.

Part of the inspiration for our definition of test collections comes from \([13]\). There, such a notion is defined abstractly as a (possibly infinite) collection of complex valued functions on a set with the property that at any given point in the set, the supremum over the test functions evaluated at the point is strictly less than 1 and functions separate the points of the set. In such cases as when the set \( X \) is contained in \( \mathbb{C}^n \), the boundary of \( X \) corresponds to points where some test function is equal to 1. A test collection in this context is used to define the dual notion of admissible kernels, and from these a normed function algebra is constructed, with the functions in the test collection in the unit ball of the algebra. The realization theorem then states that unital representations of the algebra which send the functions in the test collection to strict contractions are (completely) contractive. In the case that the set where we define the test collection is a bounded set \( \Omega \subset \mathbb{C} \), this is reminiscent of the test collection being a uniform test collection. In the general setting of \([13]\), the algebra obtained may not be equal to \( A(\Omega) \), which is the issue being addressed in this paper.

3. Some examples of test collections from the literature

Here we interpret the known criteria for being a complete \( K \)-spectral set in terms of our notion of a test collection and its variants. For a good recent review of different aspects of \( K \)-spectral sets and complete \( K \)-spectral sets, the reader is referred to \([4]\).

3.1. Intersection of disks. A set \( D \subset \mathbb{C} \) will be called a closed disk in the Riemann sphere \( \hat{\mathbb{C}} \) if it has of one of the following three forms:
\[
\{ z \in \hat{\mathbb{C}} : |z - a| \leq r \}, \quad \{ z \in \hat{\mathbb{C}} : |z - a| \geq r \}, \quad \{ z \in \hat{\mathbb{C}} : \text{Re} \alpha(z - a) \geq 0 \},
\]
i.e., it is either the interior of a disk in \( \mathbb{C} \), the exterior of a disk, or a half-plane.

**Theorem A** (Badea, Beckermann, Crouzeix \([5]\)). Let \( \{D_k\}_{k=1}^n \) be closed disks in \( \hat{\mathbb{C}} \) and \( \{\varphi_k\}_{k=1}^n \) be fractional linear transformations taking \( D_k \) onto \( \mathbb{D} \). Then \( \{\varphi_k\}_{k=1}^n \) is a uniform test collection for \( \bigcap_{k=1}^n D_k \).

3.2. Nice \( n \)-holed domains. We say that an open bounded set \( \Omega \subset \hat{\mathbb{C}} \) is an \( n \)-holed domain if its boundary \( \partial \Omega \) consists of \( n+1 \) disjoint Jordan curves. Given an \( n \)-holed domain \( \Omega \), we will denote by \( \{U_k\}_{k=0}^n \) the connected components of \( \hat{\mathbb{C}} \setminus \Omega \), with \( U_0 \) the unbounded component. Let \( X_k = \hat{\mathbb{C}} \setminus U_k \).

**Theorem B** (Douglas, Paulsen \([15]\)). Let \( \Omega \) be an \( n \)-holed domain, and define \( \{X_k\}_{k=0}^n \) as above. Assume that each \( X_k \) has an analytic boundary, so that there exist analytic homeomorphisms \( \varphi_k : X_k \to \mathbb{D} \), for \( k = 0, \ldots, n \). Then \( \{\varphi_k\}_{k=0}^n \) is a uniform strong test collection in \( \Omega \).

This theorem can also be found in Paulsen’s book \([29, \text{Chapter 11}]\).
3.3. Convex domains and the numerical range. For $T \in \mathcal{B}(H)$, the numerical range is defined as the set

$$W(T) = \{(Tx, x) : \|x\| = 1\}.$$ 

It is well-know that this set is convex, and so its closure can be written as the intersection of a (generally infinite) collection of closed half planes $\{H_\alpha\}$. Let $\varphi_\alpha$ be a linear fractional transformation taking $H_\alpha$ onto $\overline{D}$. It is easy to check that $W(T) \subset H_\alpha$ if and only if $\|\varphi_\alpha(T)\| \leq 1$. As we have already commented, it follows from the arguments in [14] and [35] that every compact convex set containing $W(T)$ is a complete $K$-spectral set for $T$. This result can be rewritten in terms of test collections as follows.

**Theorem C.** Let $\Omega$ be a convex domain in $\mathbb{C}$ and let $\{H_\alpha\}$ be a collection of closed half-planes such that $\overline{\Omega} = \bigcap H_\alpha$. Let $\varphi_\alpha$ be a fractional linear transform taking $H_\alpha$ onto $\overline{D}$. Then $\{\varphi_\alpha\}$ is a uniform test collection in $\overline{\Omega}$.

**Remark.** If $\Omega$ is a smooth bounded convex set, we denote by $C_\Omega$ the optimal constant $K$ such that $\overline{\Omega}$ is a (complete) $K$-spectral set for $T$ whenever $W(T) \subset \Omega$. The constant $Q = \sup_\Omega C_\Omega$ is know as Crouzeix constant. Crouzeix has conjectured that $Q = 2$. The best result so far is $Q \leq 1 + \sqrt{2}$, as shown by Crouzeix and Palencia in their recent preprint [11].

We also mention that in [25], a certain analogue of the Delyon-Delyon result [14] about a normal skew-dilation to the numerical range is given for a (possibly non-commuting) tuple of operators in the context of the symmetrized functional calculus.

3.4. $\rho$-contractions. If $\rho > 0$, we say that an operator $T \in \mathcal{B}(H)$ is a $\rho$-contraction if $T$ has an unitary $\rho$-dilation. This is a unitary operator $U$ acting on a larger Hilbert space $K \supset H$ and such that

$$T^n = \rho P_H U^n |H, \quad n \geq 1.$$ 

Alternatively, one can ask that $\sigma(T) \subset \overline{D}$ and that the operator-valued Poisson kernel of $T$

$$K_{r,t}(T) = (I - re^{it}T^*)^{-1} + (I - re^{-it}T)^{-1} - I \quad 0 < r < 1, \quad t \in \mathbb{R}$$

satisfies

$$(6) \quad K_{r,t}(T) + (\rho - 1)I \geq 0, \quad 0 < r < 1, \quad t \in \mathbb{R}.$$ 

The class of $\rho$ contractions becomes larger as $\rho$ increases, as (6) clearly shows, and $\rho = 1$ corresponds to the usual contractions, while $\rho = 2$ corresponds to $W(T) \subset \overline{D}$.

If $1 < \rho < 2$, then $T$ being a $\rho$-contraction is also equivalent to the condition that

$$(7) \quad \|\mu I - T\| \leq |\mu| + 1, \quad \frac{\rho - 1}{2 - \rho} \leq |\mu| < \infty.$$ 

(See, for instance, [26, Chapter I,]) If $a \in \mathbb{T}$, we denote by $D_a(\rho)$ the closed disk of radius $1 + (\rho - 1)/(2 - \rho)$ whose boundary is tangent to $\mathbb{T}$ at $a$ and which contains $\overline{D}$. Let $\varphi_{a,\rho}$ be a linear fractional transformation taking $D_a(\rho)$ onto $\overline{D}$. Then (7) is equivalent to the condition that $\varphi_{a,\rho}(T)$ is a contraction for every $a \in \mathbb{T}$. 
Similarly, if $\rho > 2$, $T$ is a $\rho$ contraction if and only if
\begin{equation}
\|(\mu I - T)^{-1}\| \leq \frac{1}{|\mu| - 1}, \quad 1 \leq |\mu| \leq \frac{\rho - 1}{\rho - 2}.
\end{equation}
For these values of $\rho$, denote by $D_a(\rho)$ the complement of the open disk of radius $(\rho - 1)/(\rho - 2) - 1$, which is tangent to $T$ at $a \in T$ and does not contain $D$. Let $\varphi_{a,\rho}$ be a linear fractional transformation which takes $D_a(\rho)$ onto $D$. Then (8) is equivalent to the condition that $\varphi_{a,\rho}(T)$ is a contraction for every $a \in T$.

For the case $\rho = 2$, let $D_a(2)$ be the closed half-plane which is tangent to $T$ at $a \in T$ and which contains $D$ and let $\varphi_{a,2}$ be a fractional linear transformation taking $D_a(2)$ onto $D$. Then it follows from the above comments regarding the numerical range that $T$ is a 2-contraction if and only if $\varphi_{a,2}(T)$ is a contraction for every $a \in T$.

It is also known [26] that every $\rho$-contraction is similar to a contraction. We summarize in terms of test collections as follows.

**Theorem D.** For $\rho > 1$, let $\Phi_\rho = \{\varphi_{a,\rho}\}_{a \in T}$, where $\varphi_{a,\rho}$ is defined as above. Then $\Phi_\rho$ is a uniform test collection over $D$.

### 3.5. Inner functions.

Recall that a Blaschke product is a function of the form
\[ B(z) = e^{i\theta} z^k \prod_{j=1}^{N} b_{\lambda_j}(z), \]
where
\[ b_{\lambda}(z) = \frac{\lambda - z}{|\lambda|^2 - 1}, \]
is a disk automorphism, $N$ may be either a finite number or $\infty$ (in which case its zeros $\lambda_j \in D$ satisfy the Blaschke condition $\sum_{j=1}^{\infty} (1 - |\lambda_j|) < \infty$). The Blaschke product is called finite if $N$ is finite.

**Theorem E** (Mascioni, [24]). Let $\varphi$ be a finite Blaschke product. Then the one element set $\{\varphi\}$ is a non-uniform strong test collection over $D$.

We cannot say that the one element set $\{\varphi\}$ is a uniform test collection in $D$. For example, take $\varphi(z) = z^2$, which is a finite Blaschke product. Then the operators $T_n$ on $C^2$ defined by the matrices $T_n = \begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix}$ satisfy $\varphi(T_n) = 0$, but we have $\|T_n\| = n$. Hence, $D$ can be a $K$-spectral set for $T_n$ only if $K \geq n$.

In some of the theorems given above, the conclusion is that some family of functions is a strong test collection, whereas in others the conclusion is just that the family is a test collection. Indeed, we do not know whether one can replace “test collection” by “strong test collection” in Theorems A and C. The proofs of these theorems involve some kind of operator valued Poisson kernel which turn out to be positive when $\varphi(T)$ is a contraction, but they do not seem to work well if $\varphi(T)$ simply has $D$ as a $K'$-spectral set. Similarly, we do not know whether one can replace “test collection” by “strong test collection” in Theorem D.
Theorem E has been generalized by Stessin [42] and Kazas and Kelley [20] to several classes of infinite Blaschke products. These generalizations give examples of test functions on a set $\Omega$ which is neither an open domain nor its closure. We restate here Stessin’s theorem in the language of test collections.

**Theorem F** (Stessin, [42]). Let $\varphi$ be a Blaschke product whose zeros $\{\lambda_j\}_{j=1}^{\infty}$ satisfy $\sum(1-|\lambda_j|^2)^{1/2} < \infty$. Let $\mathcal{P}$ be the set of poles of $\varphi$ and put $\Omega = \overline{\mathbb{D}} \setminus \mathcal{P}$. Then, the one element set $\{\varphi\}$ is a non-uniform strong test collection over $\Omega$.

Another recent result that can be put into the terminology of test collections is that of concerning lemniscates.

**Theorem G** (Nevanlinna, [27]). Let $p$ be a monic polynomial, $R > 0$, and denote by $\gamma_R$ the set $\{z \in \mathbb{C} : |p(z)| = R\}$. Assume that no critical point of $p$ lies on $\gamma_R$. Let $\Omega = \{z \in \mathbb{C} : |p(z)| < R\}$ and $\varphi = p/R$. Then the single-element set $\{\varphi\}$ is a non-uniform test collection over $\Omega$.

Remark. If $\Omega$ is a complete $K$-spectral set for an operator $T$, one can speak about constructing a concrete Sz.-Nagy-Foias like model of $T$ in $\Omega$. In the simplest case, this model will be the compression of the multiplication operator $f \mapsto zf$ on the space $H^2(\Omega, U) \oplus \theta H^2(\Omega, Y)$, where $U, Y$ are auxiliary Hilbert spaces and $\theta \in H^\infty(\Omega, L(Y, U))$ is an analogue of the characteristic function. As it was shown in [46], there are important cases when the function $\theta$ can be calculated explicitly. This is also true for some of the above examples. If $T$ is a $\rho$-contraction, there is an explicit formula for its similarity to a contraction. See, for instance, [28].

Such an explicit similarity transform is also available when $\|B(T)\| \leq 1$ for a finite Blaschke product $B$. Indeed, let

$$B(z) = \prod_{k=1}^{n} b_k(z),$$

where $b_k(z) = (z - \lambda_k)/(1 - \bar{\lambda}_k z)$ are Blaschke factors, $|\lambda_k| < 1$. The functions

$$s_k(z) = \frac{(1-|\lambda_k|^2)^{1/2}}{1-\bar{\lambda}_k z} \prod_{j=1}^{k-1} b_j(z), \quad k = 1, \ldots, n,$$

form an orthonormal basis of the model space $H^2 \oplus BH^2$, whose reproducing kernel is $(1 - B(w)B(z))/(1 - wz)$. For $z, w$ in a neighborhood of the closed unit disk, this gives

$$1 - B(w)B(z) = (1 - wz) \sum_{k=1}^{n} s_k(w)s_k(z),$$

This then implies that for any $h \in H$,

$$\sum_{k=1}^{n} \|s_k(T)h\|^2 - \sum_{k=1}^{n} \|s_k(T)Th\|^2 = \|h\|^2 - \|B(T)h\|^2 \geq 0,$$

and therefore $\|h\|^2 := \sum_{k=1}^{n} \|s_k(T)h\|^2$ defines a Hilbert space norm on $H$ for which $T$ is a contraction. Since $s_1(T)$ is an invertible operator, this norm is equivalent to the original.
As shown in [46], there are ways of calculating the characteristic function \( \theta \) of an operator \( T \) explicitly without knowing an explicit form of the similarity transform converting \( T \) into an operator for which \( \Omega \) is a complete spectral set with constant 1. As also explained in the paper, one obtains additional cases where explicit formulas are available by admitting a larger class of characteristic functions (which are then no longer unique). Apart from these examples, we do not know either an explicit form of the similarity transform of the above type, nor explicit characteristic functions.

4. Proofs of Theorems 1 and 2

In what follows, we will use the following lemma. It is a special case of a well-known principle in the theory of \( C^\ast \)-algebras that says that whenever the range of a linear map is commutative, the complete boundedness of this map comes for free.

**Lemma 6.** Let \( T : A(\overline{\Omega}_1) \rightarrow A(\overline{\Omega}_2) \) be a bounded operator, and \( \alpha \in A(\overline{\Omega})^\ast \) a bounded linear functional. Then \( T \) and \( \alpha \) are completely bounded, and \( \| T \|_{cb} = \| T \| \) and \( \| \alpha \|_{cb} = \| \alpha \| \).

**Proof.** If \( A \) and \( B \) are \( C^\ast \)-algebras, with \( B \) commutative, \( S \) a (not necessarily closed) linear subspace of \( A \) and \( \phi : S \rightarrow B \) is a bounded linear map, then it is well known that \( \phi \) is completely bounded and \( \| \phi \|_{cb} = \| \phi \| \) (see, for instance, [29, Theorem 3.9] or [22, Lemma 1]). The lemma then follows from the fact that \( A(\overline{\Omega}) \) is a subspace of the commutative \( C^\ast \)-algebra \( C(\partial \Omega) \) and the norm in \( A(\overline{\Omega}) \) coincides with the norm that it inherits as a subspace of \( C(\partial \Omega) \).

**Proof of Theorem 1.** We start by proving (ii) with \( n = 2 \).

Following the steps of [18, Example 4.1], we see that there are bounded operators \( G_k : A(\overline{\Omega}) \rightarrow A(\overline{\Omega}_k) \), \( k = 1, 2 \), such that \( f = G_1(f) + G_2(f) \), for every \( f \in A(\overline{\Omega}) \). Indeed, although [18, Example 4.1] is formulated for two simply connected domains whose boundaries intersect in only two points, the arguments used there are local at each point in \( \partial \Omega_1 \cap \partial \Omega_2 \) and can be applied in our setting. A key remark that we need to use here is that \( X = \partial \Omega_1 \cap \partial \Omega_2 \) is a finite set, a consequence of the transversality condition, which implies that every point \( z \in X \) has an open neighborhood in \( \hat{C} \) which contains no other points from \( X \).

Take \( f \in \text{Rat}(\overline{\Omega}_1 \cap \overline{\Omega}_2) \otimes M_s \), an \( s \times s \) matrix-valued rational function with poles off \( \overline{\Omega}_1 \cap \overline{\Omega}_2 \). We first want to check that

\[
(9) \quad f(T) = [(G_1 \otimes \text{id}_s)(f)](T) + [(G_2 \otimes \text{id}_s)(f)](T).
\]

Note that the operators \( [(G_k \otimes \text{id}_s)(f)](T) \) are defined by the \( A(\overline{\Omega}_k) \) calculus for \( T \), which is well defined because each \( \overline{\Omega}_k \) is a complete \( K \)-spectral set for \( T \).

The function \( f \) can be decomposed as \( f = f_1 + f_2 \), with \( f_j \in \text{Rat}(\overline{\Omega}_j) \otimes M_s \), because any pole \( a \) of \( f \) satisfies \( a \in \hat{C} \setminus \overline{\Omega}_j \) for either \( j = 1 \) or \( j = 2 \). Put \( g_k = (G_k \otimes \text{id}_s)(f) \), \( k = 1, 2 \). We have

\[
f_1 - g_1 = g_2 - f_2, \quad \text{in} \quad \overline{\Omega}_1 \cap \overline{\Omega}_2.
\]
The left hand side of this equality belongs to $A(\overline{\Omega_1}) \otimes M_s$ and the right hand side belongs to $A(\overline{\Omega_2}) \otimes M_s$. Thus this equation defines a function $h$ in $A(\overline{\Omega_1} \cup \overline{\Omega_2}) \otimes M_s$ by $h = f_1 - g_1$ in $\overline{\Omega_1}$ and $h = g_2 - f_2$ in $\overline{\Omega_2}$. Let \( \{h_n\}_{n=1}^\infty \subset \text{Rat}(\overline{\Omega_1} \cap \overline{\Omega_2}) \otimes M_s \) be rational functions such that $h_n \to h$ uniformly in $\overline{\Omega_1} \cup \overline{\Omega_2}$. Since $h_n \to f_1 - g_1$ uniformly in $\overline{\Omega_1}$, we have that $h_n(T) \to f_1(T) - g_1(T)$ in operator norm. On the other hand, since $h_n \to g_2 - f_2$ uniformly in $\overline{\Omega_2}$, we have that $h_n(T) \to g_2(T) - f_2(T)$ in operator norm. Hence $f_1(T) - g_1(T) = g_2(T) - f_2(T)$. This proves (9), because $f_1(T) + f_2(T) = f(T)$ by the rational functional calculus.

Now we estimate
\[
\|f(T)\| \leq \sum_{k=1}^2 \|[(G_k \otimes 1_s)(f)](T)\| \leq K \left( \sum_{k=1}^2 \|G_k\|_{cb} \right) \|f\|_{A(\overline{\Omega_1}) \otimes M_s}.
\]

By Lemma 6, $\overline{\Omega}$ is a complete $K'$-spectral set for $T$, with $K' = K(||G_1|| + ||G_2||)$.

Now suppose that $n > 2$ and that the sets $\Omega_1, \ldots, \Omega_n$ satisfy the hypotheses of the theorem. Then the transversality conditions imply that $\Omega_1 \cap \Omega_2, \Omega_3, \ldots, \Omega_n$ also satisfy the hypotheses of the theorem. This enables us to apply induction in $n$.

The proof of (i) is by the same argument, and indeed is somewhat simpler, since one only has to deal with scalar analytic functions and there is no need to invoke Lemma 6.

\[\square\]

**Remark.** The proof relies essentially on the Havin-Nersessian separation of singularities: given some domains $\Omega$, $\Omega_1$, $\Omega_2$ with good geometry such that $\Omega = \Omega_1 \cap \Omega_2$, any function $f$ in $H^\infty(\Omega)$ admits a decomposition $f = f_1 + f_2$, $f_j = G_j(f) \in H^\infty(\Omega_j)$. However, there are cases when the Havin-Nersessian separation fails, but nevertheless the following assertion is still true: if $\overline{\Omega_j}$ is a (1-)spectral set for $T$, then $\overline{\Omega}$ is a complete $K$-spectral set for $T$. This holds, for instance, if $\Omega_1$ and $\Omega_2$ are open half-planes such that $\Omega_1 \cup \Omega_2 = \mathbb{C}$. In this case, $\Omega_2$ are simply connected, and so they are spectral sets for $T$ if and only if they are complete spectral sets.

This assertion follows, for instance, from [10]. However, there is no Havin-Nersessian separation in this case. We reproduce arguments similar to those in [18, Example 2.1] for the convenience of the reader. By applying a linear map, we can assume that $\Omega_1 = \{\text{Re} \, z < 1\}$ and $\Omega_2 = \{\text{Re} \, z > -1\}$. Then the function $f(z) = \log((z + 2)/(z - 2))$ is in $H^\infty(\Omega_1 \cap \Omega_2)$, but cannot be represented as $f = f_1 + f_2$, $f_j \in H^\infty(\Omega_j)$. Indeed, an easy application of a variant of the Liouville theorem shows that any representation $f = f_1 + f_2$ with analytic functions $f_j$ satisfying, say, $|f_j(z)| \leq C(|z| + 1)^{1/2}$ should satisfy $f_1(z) = S - \log(z - 2)$, $f_2(z) = -S + \log(z + 2)$, where $S$ is a constant, and in no case $f_j$ are bounded in $\Omega_j$.

It is easy to modify this example to likewise produce bounded simply connected domains $\Omega_1$ and $\Omega_2$ with the same properties.

**Question.** Consider Theorem 1 for the case of two bounded and convex domains $\Omega_1$ and $\Omega_2$. Can the constant $K'$ be chosen to depend only on $K$ and not on the geometry of $\Omega_1$ and $\Omega_2$? Several modifications of this question are possible, for instance, we can pose it for two general Jordan domains or
for domains, the boundaries of which have bounded curvature. Even when \( K = 1 \) the questions are still of interest.

Now we pass to the proof of Theorem 2. We need some preliminaries. Recall that we say that \( D \subset \hat{C} \) is a closed disk in \( \hat{C} \) if it has one of the following three forms:

\[
\{ z \in \hat{C} : |z - a| \leq r \}, \quad \{ z \in \hat{C} : |z - a| \geq r \}, \quad \{ z \in \hat{C} : \Re \alpha(z - a) \geq 0 \},
\]

i.e., it is either the interior of a disk in \( C \), the exterior of a disk, or a half-plane. We will refer to disks \( \{ z \in \hat{C} : |z - a| \leq r \} \) as “genuine” disks. Next, suppose \( T \) is a Hilbert space operator and \( D \) has one of the above three forms. We say that \( D \) is a good disk for \( T \) if the following condition holds (depending on the case):

- If \( D = \{ z \in \hat{C} : |z - a| \leq r \} \), we require that \( \|T - a\| \leq r \);
- If \( D = \{ z \in \hat{C} : |z - a| \geq r \} \), we require that \( a \notin \sigma(T) \) and \( \|(T - a)^{-1}\| \leq r^{-1} \);
- If \( D = \{ z \in \hat{C} : \Re \alpha(z - a) \geq 0 \} \), we require that \( \Re (\alpha(T - a)) \geq 0 \).

**Lemma 7.** Let \( T \) be a Hilbert space operator.

(i) Suppose \( \psi : C \to \hat{C} \) is a Möbius map, and that \( \sigma(T) \) does not contain the pole of \( \psi \), so that the operator \( \psi(T) \) is bounded. Then, given a closed disk \( D \) in the Riemann sphere, \( D \) is good for \( T \) if and only if its image \( \psi(D) \) is good for \( \psi(T) \).

(ii) Whenever \( D_1 \subset D_2 \) are two Riemann sphere disks such that \( D_1 \) is good for \( T \), the disk \( D_2 \) is also good for \( T \).

**Proof.** We will show that \( D \) is good for \( T \) if and only if \( \sigma(T) \subset D \) and \( \psi(T) \) is a contraction, where \( \psi \) is a Möbius transform taking \( D \) onto \( \overline{D} \).

Part (i) clearly follows from this property. First note that if \( \varphi \) is a disk automorphism, then \( T \) is a contraction if and only if \( \varphi(T) \) is a contraction. Since every two Möbius maps taking \( D \) onto \( \overline{D} \) differ by composition on the right with a disk automorphism, we see that \( \psi(T) \) is a contraction for every Möbius map \( \psi \) taking \( D \) onto \( \overline{D} \) if and only if \( \psi(T) \) is a contraction for some particular choice of such Möbius map.

Now we examine the three kinds of disks separately. If \( D = \{ z : |z - a| \leq r \} \), then we can take \( \psi(z) = (z - a)/r \) as a Möbius map taking \( D \) onto \( \overline{D} \). The disk \( D \) is good for \( T \) precisely when \( \|\psi(T)\| \leq 1 \). Similarly, a disk \( D = \{ z : |z - a| \geq r \} \) is good for \( T \) if and only if \( \psi(T) \) is well-defined and is a contraction, where now we put \( \psi(z) = r/(z - a) \), which is a Möbius map taking \( D \) onto \( \overline{D} \).

In the last case, when \( D = \{ z : \Re \alpha(z - \alpha) \geq 0 \} \) is a half-plane, \( D \) is good for \( T \) if and only if \( C_+ = \{ z : \Re z \geq 0 \} \) is good for \( \alpha(T - a) \). Hence it suffices to consider only the case when \( D = C_+ \). Using the standard fact that \( \Re T \geq 0 \) if and only if

\[
\| (I + T)x \|^2 \geq \| (I - T)x \|^2, \quad \forall x,
\]

we see that \( \Re T \geq 0 \) if and only if \( \psi(T) \) is a contraction, where now \( \psi \) is a Möbius map that takes \( C_+ \) onto \( \overline{D} \), given by \( \psi(z) = (1 - z)/(1 + z) \).

To prove (ii), we can use (i) to reduce first to the case when \( D_1 = \overline{D} \). In this case, \( T \) is a contraction. Let \( \psi \) be a Möbius transform taking \( D_2 \)
onto \( \overline{\mathbb{B}} \). Then \(|\psi| \leq 1\) in \( \overline{\mathbb{B}} \), so \( \psi(T) \) is a contraction by von Neumann’s inequality. It follows that \( D_2 \) is good for \( T \), since \( \sigma(T) \subset D_1 \subset D_2 \).

**Proof of Theorem 2.** By Condition A, there are closed arcs \( \gamma_1, \ldots, \gamma_N \) satisfying the hypotheses listed there. We are going to construct domains \( \Omega_1, \ldots, \Omega_N \), whose closures are complete \( K \)-spectral for \( T \), with \( \Omega \) their intersection. Then we will apply Theorem 1 to deduce that \( \Omega \) is also complete \( K' \)-spectral for \( T \), for some \( K' \).

Fix \( k \in \{1, \ldots, N\} \), and choose some point \( z_k \in \bigcap_{\lambda \in \gamma_k} B(\mu_k(\lambda), R_k) \). Put \( \varphi_k(z) = (z - z_k)^{-1} \). Now take some \( \lambda \in \gamma_k \). Since \( z_k \in B(\mu_k(\lambda), R_k) \), it follows that the closed disk

\[
D^k_\lambda = \varphi_k(\mathbb{C} \setminus B(\mu_k(\lambda), R_k))
\]

is genuine. Since \( \lambda \in \partial B(\mu_k(\lambda), R_k) \), we have \( \varphi_k(\lambda) \in \partial D^k_\lambda \). Let \( \ell^k_\lambda \) be the straight line tangent to \( \partial D^k_\lambda \) at \( \varphi_k(\lambda) \) and let \( \Pi^k_\lambda \) be the closed half plane bordered by \( \ell^k_\lambda \) that contains \( D^k_\lambda \). Consider the (possibly unbounded) closed convex sets

\[
G_k = \bigcap_{\lambda \in \gamma_k} \Pi^k_\lambda.
\]

Since \( \Omega \subset \mathbb{C} \setminus B(\mu_k(\lambda), R_k) \), we have \( \varphi_k(\Omega) \subset D^k_\lambda \subset \Pi^k_\lambda \), for any \( \lambda \in \gamma_k \). Therefore \( \varphi_k(\Omega) \subset G_k \). By Lemma 7, the disk \( D^k_\lambda \) and the half plane \( \Pi^k_\lambda \) are good for \( \varphi_k(T) \). It follows that \( W(\varphi_k(T)) \subset G_k \). By the Delyon-Delyon theorem [14], \( G_k \) is a complete \( K \)-spectral set for \( \varphi_k(T) \) (see the comments in the Introduction).

Next, we consider the Jordan domains \( \Omega_k = \text{int}(\varphi_k^{-1}(G_k)) \) in the Riemann sphere \( \hat{\mathbb{C}} \). Each \( \Omega_k \) contains \( \Omega \), and its closure is a complete \( K \)-spectral for \( T \). By construction, \( \varphi_k(\gamma_k) \subset \partial G_k \). Hence, \( \gamma_k \subset \partial \Omega_k \). We wish to apply Theorem 1 to the intersection of the sets \( \Omega_k, \ k = 1, \ldots, N \). It may happen however that the boundaries of these sets do not intersect transversally. Nevertheless, it is possible to choose larger Jordan domains \( \tilde{\Omega}_k \supset \Omega_k \) whose boundaries do intersect transversally, and such that \( \gamma_k \subset \partial \tilde{\Omega}_k \).

To prove this, it suffices to choose the sets \( \tilde{\Omega}_k \) in such a way that they intersect transversally at the endpoints of the arcs \( \gamma_k \), as it is otherwise easy to ensure transversality at any other intersection points. So suppose \( \lambda \) is a common endpoint of two arcs \( \gamma_k \) and \( \gamma_l \). By construction, the open disk \( D_k = \varphi_k^{-1}(\mathbb{C} \setminus \Pi^k_\lambda) \) has the point \( \lambda \) on its boundary and does not intersect \( \Omega_k \), and similarly for the disk \( \Delta_l \). Therefore, there is an open circular sector \( S \) with vertex \( \lambda \) that does not intersect \( \tilde{\Omega}_k \cup \tilde{\Omega}_l \). Since \( \gamma_k \) and \( \gamma_l \) intersect transversally, we can find disjoint open circular sectors \( S^+_k \) and \( S^+_l \) which are also disjoint with \( S \) and such that \( \gamma_k \cap B(\lambda, \varepsilon) \subset S^+_k \cup \{\lambda\} \) and \( \gamma_l \cap B(\lambda, \varepsilon) \subset S^+_l \cup \{\lambda\} \) for some \( \varepsilon > 0 \). Now observe that we can choose disjoint open circular sectors \( S^-_k \) and \( S^-_l \) which are also disjoint from \( S, S^+_k, S^+_l \), and then the larger sets \( \tilde{\Omega}_k \supset \Omega_k \) and \( \tilde{\Omega}_l \supset \Omega_l \) to satisfy \( (\partial \tilde{\Omega}_k \setminus \gamma_k) \cap B(\lambda, \varepsilon) \subset S^-_k \) and \( (\partial \tilde{\Omega}_l \setminus \gamma_l) \cap B(\lambda, \varepsilon) \subset S^-_l \). Consequently, \( \tilde{\Omega}_k \) and \( \tilde{\Omega}_l \) intersect transversally at \( \lambda \).

Put

\[
\tilde{\Omega} = \tilde{\Omega}_1 \cap \cdots \cap \tilde{\Omega}_N.
\]
By Theorem 1, the closure of $\hat{\Omega}$ is a complete $K'$-spectral set for $T$, for some $K'$. By construction, each point $\lambda$ of $\partial\Omega$ has a neighborhood $B(\lambda,\varepsilon)$ such that $B(\lambda,\varepsilon) \cap \partial\Omega = B(\lambda,\varepsilon) \cap \hat{\partial}\Omega$. Therefore $\hat{\Omega} \setminus \Omega$ is at a positive distance from $\Omega$. Since $\Omega \subset \hat{\Omega}$ and $\sigma(T) \subset \overline{\Omega}$, it follows that $\hat{\Omega}$ also is a complete $K''$-spectral set for $T$. 

We recall that a Hilbert space operator $T$ is hyponormal if $T^*T \geq TT^*$. In this case, the equality $\|(T - \lambda)^{-1}\| = 1/\text{dist}(\lambda, \sigma(T))$ holds for all $\lambda \notin \sigma(T)$; see, for instance the book [23] by Martin and Putinar, Proposition 1.2. Consequently, we get the following corollary to Theorem 2.

**Corollary 8.** Let $T$ be hyponormal and let $\Omega \subset \hat{C}$ be an open set satisfying the hypotheses of Theorem 2 (in particular, the exterior disc condition) and such that $\sigma(T) \subset \overline{\Omega}$. Then $\hat{\Omega}$ is a complete $K$-spectral set for $T$.

So in other words, in this situation, $T$ can be dilated to an operator $S$ which is similar to a normal operator and satisfies $\sigma(S) \subset \partial\Omega$. It is interesting to compare Corollary 8 with Putinar’s result [33] that every hyponormal operator $T$ is subscalar and can in fact be represented as a restriction of a scalar operator $L$ of order 2 (in the sense of Colojoara-Foias) to an invariant subspace. Thus, if $T = L|H$, where $H$ is an invariant subspace of $L \in B(K)$, then $L$ is a dilation of $T$ of a special kind. On the other hand, the spectrum of a scalar operator $L$, as constructed by Putinar, contains a neighborhood of $\sigma(T)$. By contrast, in Corollary 8, if $\sigma(T)$ is a closed Jordan domain satisfying the exterior disk condition, the dilation $S$ of $T$ is a scalar operator of order 0 and its spectrum is contained in the spectrum of $T$ (and even in its boundary).

Generally speaking, the conditions of Corollary 8 do not imply that $\hat{\Omega}$ is a (1-)spectral set for $T$; this is seen from any of the examples by Wadhwa [45] and Hartman [17], where one can put $\hat{\Omega} = \sigma(T)$ (it is an annulus for the Hartman’s example and a disjoint union of an annulus and a disc for Wadhwa’s example). On the other hand, consider the hyponormal operator from Clancey’s example [9]; let us call it $B$. Its spectrum is a compact subset of $\hat{C}$ of positive area, whose interior is empty. It is proved in [9] that $\sigma(B)$ is not a 1-spectral set for $B$. A modification of Clancey’s arguments also shows that it is not even $K$-spectral for any $K$. Indeed, by applying [29, Exercise 9.11], one gets that if $\sigma(B)$ were $K$-spectral for $B$, then $B$ would be similar to a normal operator. Since $B$ is hyponormal, [41, Corollary 1] would then give that $B$ is normal, which is not true. So the equality $\|(T - \lambda)^{-1}\| = 1/\text{dist}(\lambda, \sigma(T)) \ (\lambda \notin \sigma(T))$ in general does not imply that $\sigma(T)$ is a $K$-spectral set for $T$.

We also refer to [34, Theorem 4] for a result on subscalarity of operators with a power-like estimate for the resolvent.

5. Admissible functions and generators of $A(\overline{\Omega})$

In this section we state the results from [8] needed in this article.

**Theorem I (Theorem 1.5 in [8]).** Let $\Phi : \overline{\Omega} \rightarrow \overline{\mathbb{D}}^n$ be admissible. Then there exist bounded linear operators $F_k : A(\overline{\Omega}) \rightarrow A(\overline{\mathbb{D}})$, $k = 1, \ldots, n$ such
that the operator in $A(\overline{\Omega})$ defined by

$$f \mapsto f - \sum_{k=1}^{N} F_{k}(f) \circ \varphi_{k}, \quad f \in A(\overline{\Omega}),$$

is compact.

If $\Phi = (\varphi_{1}, \ldots, \varphi_{n}) : \overline{\Omega} \rightarrow \overline{\mathbb{D}}^{n}$, we define the algebra $\mathcal{A}_{\Phi}$ to be the (not necessarily closed) subalgebra of $A(\overline{\Omega})$ generated by the functions of the form $f \circ \varphi_{k}$, with $f \in A(\overline{\mathbb{D}})$ and $k = 1, \ldots, n$. Explicitly,

$$\mathcal{A}_{\Phi} = \left\{ \sum_{j=1}^{N} f_{j,1}(\varphi_{1}(z)) \cdot \ldots \cdot f_{j,n}(\varphi_{n}(z)) : f_{j,k} \in A(\overline{\mathbb{D}}), \; N \in \mathbb{N} \right\}.$$

**Theorem II** (Theorem 1.1 in [8]). Let $\Phi : \overline{\Omega} \rightarrow \overline{\mathbb{D}}^{n}$ be admissible and injective, and such that $\Phi'$ does not vanish on $\Omega$. Then $\mathcal{A}_{\Phi} = A(\overline{\Omega})$.

**Lemma III.** [Lemma 8.1 in [8]] Let $\Phi_{\varepsilon} = (\varphi_{1}^{\varepsilon}, \ldots, \varphi_{n}^{\varepsilon}) : \overline{\Omega} \rightarrow \overline{\mathbb{D}}^{n}$, $0 \leq \varepsilon \leq \varepsilon_{0}$ be a collection of functions. Assume that $\Psi_{\varepsilon}$ is admissible for every $\varepsilon$, and, moreover, that one can choose sets $\Omega_{k}$ in the definition of an admissible collection that do not depend on $\varepsilon$. Assume that $\varphi_{k}^{\varepsilon} \in C^{1+\alpha}(\Omega_{k})$, with $0 < \alpha < 1$, and that the mapping $\varepsilon \mapsto \varphi_{k}^{\varepsilon}$ is continuous from $[0, \varepsilon_{0}]$ to $C^{1+\alpha}(\Omega_{k})$.

Then for $0 \leq \varepsilon \leq \varepsilon_{0}$, there exist bounded linear operators $F_{k}^{\varepsilon} : A(\overline{\Omega}) \rightarrow A(\overline{\mathbb{D}})$, such that if

$$L_{\varepsilon}(f) = \sum_{k=1}^{N} F_{k}^{\varepsilon}(f) \circ \varphi_{k}^{\varepsilon}$$

then $L_{\varepsilon} - I$ is a compact operator on $A(\overline{\Omega})$ for all $\varepsilon$, the mapping $\varepsilon \mapsto L_{\varepsilon}$ is continuous in the norm topology, and $\|F_{k}^{\varepsilon}\| \leq C$ for $k = 1, \ldots, n$, where $C$ is some constant independent of $k$ and $\varepsilon$.

**6. Auxiliary lemmas**

In this section we state and prove the lemmas that are needed in the proof of the Theorems 3 and 4.

**Lemma 9.** Let $X$ be a closed subspace of finite codimension $r$ in a Banach space $V$ and $Y$ a (not necessarily closed) subspace of $V$ such that $X + Y = V$. Then there exist vectors $g_{1}, \ldots, g_{r} \in Y$ such that $Z = \text{span}\{g_{1}, \ldots, g_{r}\}$ is a complement of $X$; that is, $V = X + Z$, and there are functionals $\alpha_{1}, \ldots, \alpha_{r} \in V^*$ such that

$$G(f) \overset{\text{def}}{=} f - \sum_{k=1}^{r} \alpha_{k}(f)g_{k}$$

is the projection of $V$ onto $X$ parallel to $Z$.

**Proof.** Let $\pi : V \rightarrow V/X$ be the natural projection onto the quotient. Then $\pi(Y) = \pi(X + Y) = V/X$. We can therefore choose vectors $g_{1}, \ldots, g_{r} \in Y$ such that $\{\pi(g_{1}), \ldots, \pi(g_{r})\}$ is a basis of $V/X$. It follows that $Z = \text{span}\{g_{1}, \ldots, g_{r}\}$ is a complement of $X$ in $V$. The existence of the functionals $\alpha_{1}, \ldots, \alpha_{r}$ is now clear. \hfill $\Box$

Note that, since $X + Y$ is always closed, the hypotheses of the lemma in particular hold in the case when $Y$ is a dense subspace of $V$.

The next lemma roughly says that to prove von Neumann’s inequality with a constant, it is enough to prove it only for rational functions in a space of finite codimension.
Lemma 10. Let $T \in \mathcal{B}(H)$ be such that for all $s \geq 1$,
\begin{equation}
\|f(T)\| \leq C\|f\|_{A(\overline{\Omega}) \otimes M_s}, \quad \forall f \in (X \cap \text{Rat}(\overline{\Omega})) \otimes M_s,
\end{equation}
where $X$ is some closed subspace of finite codimension in $A(\overline{\Omega})$. Then $\overline{\Omega}$ is a complete $K$-spectral set for $T$, where $K$ depends only on $X$, $C$ and $S_\Lambda(T)$, where $\Lambda$ is an arbitrary pole set for $\Omega$.

Proof. Fix a pole set $\Lambda$ for $\Omega$. Denote by $\text{Rat}_\Lambda$ the set of rational functions with poles in $\Lambda$. Note that $\text{Rat}_\Lambda$ is dense in $A(\overline{\Omega})$. Hence, we apply Lemma 9 with $V = A(\overline{\Omega})$, $Y = \text{Rat}(\overline{\Omega})$ to obtain functions $g_1, \ldots, g_r \in \text{Rat}_\Lambda$, functionals $\alpha_1, \ldots, \alpha_r \in A(\overline{\Omega})^*$, and an operator $G : A(\overline{\Omega}) \to A(\overline{\Omega})$ as in the statement of that lemma.

We can write
\[ g_k(z) = c_0 + \sum_{\lambda \in \Lambda} \sum_{j=1}^N c_{\lambda,j,k} p_\lambda(z)^j \]
for suitable coefficients $c_{\lambda,j,k}$. (Recall that $p_\lambda(z) = (z - \lambda)^{-1}$ for $\lambda \neq \infty$, and $p_\infty(z) = z$.) This shows that for $k = 1, \ldots, r$, $\|g_k(T)\| \leq K'$, where $K'$ is a constant depending only on $g_1, \ldots, g_r$ and $S_\Lambda(T)$, but not on $T$.

Let $f \in \text{Rat}(\overline{\Omega}) \otimes M_s$. By Lemma 6, $G$ and $\alpha_1, \ldots, \alpha_r$ are completely bounded, so by (10),
\begin{align*}
\|f(T)\| &= \left\| \left( (G \otimes \text{id}_s)(f)\right)(T) + \sum_{k} g_k(T) \otimes \left[ (\alpha_k \otimes \text{id}_s)(f) \right] \right\| \\
&\leq \left\| (G \otimes \text{id}_s)(f)\right\| + \sum_{k=1}^r \|g_k(T)\otimes \left[ (\alpha_k \otimes \text{id}_s)(f) \right]\| \\
&\leq C\| (G \otimes \text{id}_s)(f)\|_{A(\overline{\Omega}) \otimes M_s} + \sum_{k=1}^r K'\|\alpha_k \otimes \text{id}_s\| \cdot \|f\|_{A(\overline{\Omega}) \otimes M_s} \\
&\leq C\|G\|_{cb} \|f\|_{A(\overline{\Omega}) \otimes M_s} + \sum_{k=1}^r K'\|\alpha_k\|_{cb} \|f\|_{A(\overline{\Omega}) \otimes M_s},
\end{align*}
and the result follows. \hfill \Box

Definition. Given a domain $\Omega \subset \mathbb{C}$, a shrinking for $\Omega$ is a collection $\{\psi_\varepsilon\}_{0 \leq \varepsilon \leq \varepsilon_0}$ of univalent analytic functions in some open set $U \supset \overline{\Omega}$, such that $\psi_0$ is the identity map on $U$, $\psi_\varepsilon(\overline{\Omega}) \subset \Omega$ for $\varepsilon > 0$, and the map $\varepsilon \mapsto \psi_\varepsilon$ is continuous in the topology of uniform convergence on compact subsets of $U$.

If $\Omega$ is star-shaped with respect to $a \in \mathbb{C}$, then it admits a shrinking; namely, $\psi_\varepsilon(z) = (1 - \varepsilon)(z - a) + a$. The next lemma says that any admissible Jordan domain admits a shrinking.

Lemma 11. Let $\Omega$ be a Jordan domain with piecewise $C^2$ smooth boundary composed of closed $C^2$ arcs $\{J_k\}_{k=1}^n$. If the angles between these arcs are non-zero, then $\Omega$ admits a shrinking.

Proof. Denote by $z_1, \ldots, z_n \in \partial \Omega$ the endpoints of the arcs $J_1, \ldots, J_n$, so that $z_k$ is a common endpoint of $J_{k-1}$ and $J_k$ (we assume that the numbering of these arcs is counterclockwise and cyclic modulo $n$). The points $z_1, \ldots, z_n$
will be referred to as the corners of $\partial \Omega$. First we construct a function $\mu \in A(\overline{\Omega})$ such that $\mu \neq 0$ on $\partial \Omega$ and for any $z \in \overline{\Omega}$, $\mu(z)$ points strictly inside $\Omega$, by which we mean that there is $\sigma = \sigma(z) > 0$ such that the interval $[z, z + \sigma \mu(z)]$ is contained in $\overline{\Omega}$ and is not tangent to $\partial \Omega$ at $z$. If $z = z_k$, we require this interval to be non-tangential to both $J_{k-1}$ and $J_k$ at $z$. We denote by $\rho(z) \in \mathbb{C}$ the unit inner normal vector to the boundary. It is defined for points $z \in \partial \Omega$ which are not corners.

Let $\eta : \overline{\Omega} \to \overline{D}$ be a Riemann conformal map, and put

$$\nu(z) = \frac{\eta(z)}{\eta'(z)}.$$ 

Then $\nu$ is continuous on $\overline{\Omega} \setminus \{z_1, \ldots, z_n\}$; moreover, $\nu(z_0)$ points strictly inside $\Omega$ for any non-corner point $z_0 \in \partial \Omega$ and, in fact, $\rho(z_0) = c(z_0)\nu(z_0)$ for some $c(z_0) > 0$. Indeed, if $z(t) (0 \leq t \leq T)$ is a counterclockwise parametrization of $\partial \Omega$, which is smooth at $z_0$, $z(t_0) = z_0$, $z'(t_0) = b$, then $\eta'(z_0)b = ic\eta(z_0)$ for some $c > 0$, so that $\rho(z_0) = ib = cv(z_0)$. The function $\mu(z)$ will be, in a sense, a small correction of $\nu(z)$, which mostly affects neighborhoods of the corner points.

Denote by $R_{z,\theta} = \{z \in \mathbb{C} : \arg(w - z) = \theta\}$ the ray starting at $z$ with angle $\theta$. We assume that the rays $R_{z_k, \theta_k - \delta_k}$, $R_{z_k, \theta_k + \delta_k}$ are correspondingly tangent to $\partial \Omega$ at $z_k$ to the arcs $J_{k-1}$, $J_k$, where $0 < \beta_k < \pi$, and $\theta_k \in [0, 2\pi)$ is such that the ray $R_{z_k, \theta_k}$ points strictly inside $\Omega$. Theorem 3.9 in [32] implies that for $z \in \overline{\Omega}$,

$$\eta(z) = \eta(z_k) + u_k(z)(z - z_k)^{a_k},$$  

$$\eta'(z) = v_k(z)a_k(z - z_k)^{a_k-1},$$

where $u_k(z)$, $v_k(z)$ have finite non-zero limits as $z \to z_k$, and $a_k = \frac{\pi}{2\beta_k} \in \left(\frac{1}{2}, +\infty\right)$. (We use the principal branch of the logarithm in the definition of powers.) For small $\sigma > 0$, put $\tau_{k,\sigma} := z_k - \sigma e^{i\theta_k} \notin \overline{\Omega}$, and set

$$\mu_\sigma(z) = \Pi_\sigma(z)\nu(z) = -\Pi_\sigma(z)\frac{\eta(z)}{\eta'(z)},$$

where

$$\Pi_\sigma(z) = \prod_{k=1}^{n} \left( \frac{z - \tau_{k,\sigma}}{z - z_k} \right)^{1-a_k}.$$ 

Since the intervals $[z_k, \tau_{k,\sigma}]$ are outside $\Omega$, the function $\Pi_\sigma$ is well-defined and analytic in $\Omega$.

We assert that for sufficiently small $\sigma > 0$, $\mu(z) = \mu_\sigma(z)$ will satisfy all the necessary requirements. To begin with, it follows from (11) and (12) that for any fixed (small) $\sigma > 0$ and any $k$, $\mu_\sigma(z)$ has a finite non-zero limit as $z \to z_k$, $z \in \overline{\Omega}$. Hence $\mu_\sigma$ continues to a function in $A(\overline{\Omega})$ such that $\mu_\sigma \neq 0$ on $\partial \Omega$.

Fix some small positive $\delta$ such that for all $k$, $2\delta < \beta_k < 2\pi - 2\delta$. Easy geometric arguments show that there is some $\sigma_0 > 0$ such that for any $k$, any $z \in J_{k-1}$ such that $|z - z_k| < \sigma_0$ and any $\sigma \in (0, \sigma_0)$, either $\arg \Pi_\sigma(z) \in (-\delta, \beta_k - \frac{\pi}{2} + \delta)$ if $\beta_k < \frac{\pi}{2}$, or $\arg \Pi_\sigma(z) \in (\beta_k - \frac{\pi}{2} + \delta, \delta)$ if $\beta_k \in (0, \frac{\pi}{2})$. One has symmetric estimates for $\arg \Pi_\sigma(z)$ if $z \in J_k$, $|z - z_k| < \sigma_0$. Since
\( \Pi_\sigma(z) \to 1 \) uniformly on \( \partial \Omega \setminus \bigcup_k B_{\sigma_0}(z_k) \), it follows that for any \( z \in \partial \Omega \), \( z \neq z_1, \ldots, z_n \) when \( \sigma \in (0, \sigma_0) \) is sufficiently small,
\[
-\frac{\pi}{2} + \delta \leq \arg \frac{\mu_\sigma(z)}{\rho(z)} = \arg \Pi_\sigma(z) \leq \frac{\pi}{2} - \delta.
\]

For such fixed \( \sigma, \mu(z) := \mu_\sigma(z) \) satisfies all the requirements.

By Mergelyan’s theorem [38], there is a sequence of polynomials \( \mu_m \), such that \( \mu_m \to \mu \) uniformly on \( \overline{\Omega} \). For a sufficiently large \( m \), put \( \tilde{\mu}(z) = \mu_m(z) \). Then the polynomial \( \tilde{\mu} \) also satisfies all the requirements on \( \mu \).

We assert that \( \psi_\varepsilon(z) = z + \varepsilon \mu(z) \) defines a shrinking of \( \Omega \). Indeed, for small \( \varepsilon > 0 \), \( \psi_\varepsilon(\partial \Omega) \) is a Jordan curve, contained in \( \Omega \). An application of the argument principle shows that for these values of \( \varepsilon \), \( \psi_\varepsilon \) maps \( \Omega \) univalently onto the interior of the curve \( \psi_\varepsilon(\partial \Omega) \). There exists a Jordan domain \( \Omega' \) such that \( \overline{\Omega} \subset \Omega' \) and the boundary of \( \Omega' \) consists of \( C^2 \) smooth arcs \( J_1, \ldots, J_n \), which are close to the arcs \( J_1, \ldots, J_n \) in \( C^1 \) metric. The domain \( \Omega' \) can be chosen in such a way that \( \tilde{\mu} \neq 0 \) on \( \partial \Omega' \) and \( \tilde{\mu}(z) \) points strictly inside \( \Omega' \) for all \( z \in \partial \Omega' \). By the same argument, the functions \( \psi_\varepsilon \) are univalent on \( \Omega' \) for \( 0 \leq \varepsilon \leq \varepsilon_0 \), where \( \varepsilon_0 > 0 \). Therefore the family \( \{ \psi_\varepsilon \} \) of functions, defined on \( \Omega' \), is a shrinking of \( \Omega \).

The following lemma improves upon the results of Lemma 10 by imposing certain constraints on \( \varphi_k(T) \) and \( A_\Phi \).

**Lemma 12.** Let \( \Phi \subset A(\overline{\Omega}) \) be a collection of functions taking \( \Omega \) into \( \mathbb{D} \). If, in addition to the hypotheses of Lemma 10, we have that for every \( \varphi \in \Phi, \overline{\mathbb{D}} \) is a (not necessarily complete) \( K' \)-spectral set for \( \varphi(T) \), then for all \( s \geq 1 \),
\[
\|f(T)\| \leq K \|f\|_{A(\overline{\Omega}) \otimes M_s}, \quad \forall f \in (X + A_\Phi) \otimes M_s,
\]
where \( K \) depends only on \( X, \Phi, C \) and \( K' \), but not on \( T \). In the case when \( X + A_\Phi = A(\overline{\Omega}) \), then \( \overline{\Omega} \) is a complete \( K \)-spectral set for \( T \), with \( K = K(X, \Phi, C, K') \).

Note that the operators \( \varphi(T) \) and \( f(T) \) are defined by the \( A(\overline{\Omega}) \)-functional calculus for \( T \) because by Lemma 10, \( \overline{\Omega} \) is a complete \( K \)-spectral set for \( T \) for some \( K \). On the other hand, and in contrast to the situation in most of this paper, here the complete \( K' \)-spectrality of \( \overline{\mathbb{D}} \) for \( \varphi(T) \) is not needed — \( K' \)-spectrality suffices. The reason for this is that all the functions that appear in the proof of this lemma are scalar-valued rather than matrix-valued.

**Proof of Lemma 12.** First we apply Lemma 9 with \( V = X + A_\Phi \), and \( Y = A_\Phi \) to obtain functions \( g_1, \ldots, g_r \in A_\Phi \), functionals \( \alpha_1, \ldots, \alpha_r \in A(\overline{\Omega})^* \), and an operator \( G \) as in the statement of that lemma.

By Lemma 10, \( \overline{\Omega} \) is a complete \( K \)-spectral set for \( T \) (with \( K \) depending on \( T \)). It follows that \( T \) has a continuous \( A(\overline{\Omega}) \)-functional calculus, and so the operators \( g_k(T) \) are well defined. Let us show that there is some constant \( C' \) depending only on \( g_1, \ldots, g_r \) (and not on \( T \)) such that \( \|g_k(T)\| \leq C' \), for \( k = 1, \ldots, r \). Since \( g_k \in A_\Phi \), we can write
\[
g_k(z) = \sum_{j=1}^N f_k^{j,1}(\varphi_1(z)) \cdots f_k^{j,n}(\varphi_n(z)),
\]
where $f^k_{j,l} \in A(\overline{\Omega})$. (Because there are a finite number of functions $g_k$, the same $N$ will do every $k$.) By the properties of the $A(\Omega)$-functional calculus for $T$ we see that for $k = 1, \ldots, r,$

$$g_k(T) = \sum_{j=1}^{N} f^k_{j,1}(\varphi_1(T)) \cdots f^k_{j,n}(\varphi_n(T)).$$

Using the fact that $\overline{\Omega}$ is a $K'$-spectral set for $\varphi_k(T)$, we get

$$\|g_k(T)\| \leq \sum_{j=1}^{N} \|f^k_{j,1}(\varphi_1(T))\| \cdots \|f^k_{j,n}(\varphi_n(T))\|$$

$$\leq \sum_{j=1}^{N} (K')^n \|f^k_{j,1}\|_{A(\overline{\Omega})} \cdots \|f^k_{j,n}\|_{A(\overline{\Omega})}.$$  

This shows that for $k = 1, \ldots, n$, $\|g_k(T)\| \leq C'$, with $C'$ independent of $T$.

Finally, we proceed as in the proof of Lemma 10. Take $f \in (X + A_{\Phi}) \otimes M_s$ and estimate

$$\|f(T)\| \leq \|((G \otimes \text{id}_s)(f))(T)\| + \sum_{k=1}^{r} \|g_k(T) \otimes [(\alpha_k \otimes \text{id}_s)(f)]\|$$

$$\leq C\|G\|_{\text{cb}} \|f\|_{A(\overline{\Omega}) \otimes M_s} + \sum_{k=1}^{r} C'\|\alpha_k\|_{\text{cb}} \|f\|_{A(\overline{\Omega}) \otimes M_s}.$$  

Apply Lemma 6 to get (13). The remaining part of the lemma now follows. 

We will also need a lemma that allows one to pass to the limit in a family of inequalities of the form (10) depending on some parameter $\varepsilon$. The subspaces which play the role of $X$ will be given by the kernels of finite rank operators $\Sigma_\varepsilon$.

**Lemma 13.** Let $\{T_\varepsilon\}_{0 \leq \varepsilon \leq \varepsilon_0} \subset B(H)$, with $\sigma(T_\varepsilon) \subset \overline{\Omega}$ for $0 \leq \varepsilon \leq \varepsilon_0$, and $\{\Sigma_\varepsilon\}_{0 \leq \varepsilon \leq \varepsilon_0} \subset B(A(\overline{\Omega}), C^*)$. Assume that the maps $\varepsilon \mapsto T_\varepsilon$ and $\varepsilon \mapsto \Sigma_\varepsilon$ are continuous in the norm topology. Assume also that $\Sigma_0$ is surjective and that for all $s \geq 1$ and for all $\varepsilon \in (0, \varepsilon_0]$,

$$(14) \quad \|f(T_\varepsilon)\| \leq C\|f\|_{A(\overline{\Omega}) \otimes M_s}, \quad \forall f \in (\ker \Sigma_\varepsilon \cap \text{Rat}(\overline{\Omega})) \otimes M_s,$$

where $C$ is a constant independent of $\varepsilon$. Then (14) also holds with $\varepsilon = 0$.

**Proof.** Since $\Sigma_0$ is surjective, $X = \ker \Sigma_0$ has codimension $r$ in $A(\overline{\Omega})$. We apply Lemma 9 with $Y = \text{Rat}(\overline{\Omega})$ to obtain functions $g_1, \ldots, g_r \in \text{Rat}(\overline{\Omega})$, a subspace $Z = \text{span}\{g_1, \ldots, g_r\}$, functional $\alpha_1, \ldots, \alpha_r \in A(\overline{\Omega})^*$ and an operator $G$ as in the statement of that lemma.

Consider the restrictions $\Sigma_\varepsilon|Z : Z \to C^*$. The operator $\Sigma_0|Z$ is invertible, therefore, $\Sigma_\varepsilon|Z$ is invertible for $\varepsilon$ sufficiently small. Put $P_\varepsilon = (\Sigma_\varepsilon|Z)^{-1}\Sigma_\varepsilon$. Thus $P_\varepsilon : A(\overline{\Omega}) \to Z$ and $P_\varepsilon^2 = P_\varepsilon$. Indeed, $P_\varepsilon$ is the projection onto $Z$ parallel to $\ker \Sigma_\varepsilon$. Define $\alpha_\varepsilon^k \in (A(\overline{\Omega}))^*$ by $\alpha_\varepsilon^k(f) = \alpha_k(P_\varepsilon f)$, and check that
$G_\varepsilon(f) \overset{\text{def}}{=} f - \sum \alpha_k^\varepsilon(f)g_k$ is in ker $\Sigma_\varepsilon$ for every $f \in A(\overline{\Omega})$. We compute

$$P_\varepsilon G_\varepsilon(f) = P_\varepsilon f - \sum_{k=1}^r \alpha_k^\varepsilon(f)P_\varepsilon g_k = P_\varepsilon^2 f - \sum_{k=1}^r \alpha_k(P_\varepsilon f)P_\varepsilon g_k = P_\varepsilon G(P_\varepsilon f) = 0,$$

because $P_\varepsilon f \in Z$ and ker $G = Z$. It follows that $G_\varepsilon(f)$ is in ker $P_\varepsilon = \ker \Sigma_\varepsilon$.

Since $T_\varepsilon$ depends continuously on $\varepsilon$, there is some constant $K$ independent of $\varepsilon$ such that $\|g_k(T_\varepsilon)\| \leq K$ for small $\varepsilon$ and $k = 1, \ldots, r$. Take $f \in \text{Rat}(\overline{\Omega}) \otimes M_s$ and estimate

$$\|f(T_\varepsilon)\| = \left\| \left( (G \otimes \text{id}_s)(f) \right) (T_\varepsilon) + \sum_{k=1}^r g_k(T_\varepsilon) \otimes \left[ (\alpha_k^\varepsilon \otimes \text{id}_s)(f) \right] \right\|$$

$$\leq C\|G \otimes \text{id}_s(f)\|_{A(\overline{\Omega}) \otimes M_s} + \sum_{k=1}^r K\|\alpha_k^\varepsilon \otimes \text{id}_s(f)\|_{A(\overline{\Omega}) \otimes M_s}.$$

Since $G_\varepsilon$ and $\alpha_k^\varepsilon$ depend continuously on $\varepsilon$, we can let $\varepsilon \to 0$ to obtain

$$\|f(T_0)\| \leq C\|G \otimes \text{id}_s(f)\|_{A(\overline{\Omega}) \otimes M_s} + \sum_{k=1}^r K\|\alpha_k \otimes \text{id}_s(f)\|_{A(\overline{\Omega}) \otimes M_s}.$$

The proof concludes by noting that if $f \in \ker \Sigma_0 \otimes M_s$, then $(G \otimes \text{id}_s)(f) = f$ and $(\alpha_k \otimes \text{id}_s)(f) = 0$ for $k = 1, \ldots, r$. \hfill $\Box$

The next lemma constructs a family of admissible functions $\Phi_\varepsilon$ which work well with the operators $\psi_\varepsilon(T)$, where $\{\psi_\varepsilon\}$ is a shrinking for $\Omega$.

**Lemma 14.** Let $\Omega$ be a Jordan domain with a shrinking $\{\psi_\varepsilon\}_{0 \leq \varepsilon \leq \varepsilon_0}$, and let $\Phi : \overline{\Omega} \to \overline{\Omega}^n$ be admissible and analytic in a neighborhood of $\Omega$. Let $T \in \mathcal{B}(H)$ with $\sigma(T) \subset \overline{\Omega}$ and such that $\overline{\Omega}$ is a complete $K$-spectral set for $\varphi_k(T)$, for $k = 1, \ldots, n$. Then there is some $0 < \delta \leq \varepsilon_0$ and a family of admissible functions $\{\Phi_\varepsilon\}_{0 \leq \varepsilon \leq \delta}$ over $\overline{\Omega}$ with $\Phi_\varepsilon = (\varphi_1^\varepsilon, \ldots, \varphi_n^\varepsilon)$ and $\Phi_0 = \Phi$, such that each $\varphi_k^\varepsilon$ is analytic in some neighborhood $U_k \subset \overline{\Omega}$ of $1$ in $\overline{\Omega}$, the map $\varepsilon \mapsto \varphi_k^\varepsilon$ is continuous from $[0, \delta]$ to $C^\infty(U_k)$, and $\overline{\Omega}$ is a complete $K$-spectral set for $\varphi_k^\varepsilon(\psi_\varepsilon(T))$.

**Proof.** We construct admissible functions $\Phi_\varepsilon = (\varphi_1^\varepsilon, \ldots, \varphi_n^\varepsilon)$ satisfying the statement of the lemma by choosing $\varphi_k^\varepsilon$ to have the form $\varphi_k^\varepsilon = \eta_k^\varepsilon \circ \varphi_k \circ \psi_\varepsilon^{-1}$, where $\eta_k \in A(\overline{\Omega})$ and $\|\eta_k\|_{A(\overline{\Omega})} \leq 1$. Because $\varphi_k^\varepsilon(\psi_\varepsilon(T)) = \eta_k^\varepsilon(\varphi_k(T))$, this will guarantee that $\varphi_k^\varepsilon(\psi_\varepsilon(T))$ has $\overline{\Omega}$ as a complete $K$-spectral set. The construction of $\eta_k^\varepsilon$ is geometric.

First, continue analytically the arcs $J_k \subset \partial \Omega$ to larger arcs $\tilde{J}_k$ such that $\varphi_k$ and $\psi_\varepsilon$ are analytic in a neighborhood of $\tilde{J}_k$ (recall that $\varphi_k$ and $\psi_\varepsilon$ are analytic in a neighborhood of $\partial \Omega$). In this proof, we only deal with closed arcs. Assume that $\tilde{J}_k$ are small enough that each $\varphi_k(\tilde{J}_k)$ is still one to one. Put $\Gamma_k^\varepsilon = \varphi_k(\psi_\varepsilon^{-1}(J_k))$ and $\overline{\Gamma}_k^\varepsilon = \varphi_k(\psi_\varepsilon^{-1}(J_k))$. Since $\Gamma_k^0 = \varphi_k(J_k)$ is an arc of $T$, it follows by continuity that for small $\varepsilon$, there exists $\tilde{I}_k^\varepsilon$ an arc of $T$, and a function $a_k^\varepsilon : I_k^\varepsilon \to \mathbb{R}_+$ such that $\overline{\Gamma}_k^\varepsilon = \{a_k^\varepsilon(\zeta) \zeta : \zeta \in I_k^\varepsilon\}$. Also, $a_k^\varepsilon \geq 1$
in $\tilde{T}_k$ and $a_k^0 = 1$ in $\tilde{D}_k$. The functions $a_k^\varepsilon$ are assumed to be defined for $0 \leq \varepsilon \leq \delta$. Let $I_k^\varepsilon$ be the sub-arc of $\tilde{T}_k$ such that $\Gamma_k^\varepsilon = \{a_k^\varepsilon(\zeta) : \zeta \in I_k^\varepsilon\}$.

Next find functions $b_k^\varepsilon : \mathbb{T} \to \mathbb{R}_+\setminus \{0\}$, $0 \leq \varepsilon \leq \delta$, such that $b_k^\varepsilon \in C^\infty(\mathbb{T})$ for each $\varepsilon$, the map $\varepsilon \mapsto b_k^\varepsilon$ is continuous from $[0, \delta]$ to $C^\infty(\mathbb{T})$, $b_k^\varepsilon = a_k^\varepsilon$ in $I_k^\varepsilon$, $b_k^\varepsilon \geq 1$ in $\mathbb{T}$, and if $D_k^\varepsilon$ is the interior domain of the Jordan curve $\{b_k^\varepsilon(\zeta) : \zeta \in \mathbb{T}\}$, then $\varphi_k(\psi_\varepsilon^{-1}(\Omega)) \subset \overline{D_k^\varepsilon}$. These are first constructed in a local manner and then a partition of unity argument is employed. This construction is done as follows.

For each $k$, define the following closed subsets of $\mathbb{T} \times [0, \delta]$:

$$V_k = \bigcup_{0 \leq \varepsilon \leq \delta} (I_k^\varepsilon \times \{\varepsilon\}), \quad \tilde{V}_k = \bigcup_{0 \leq \varepsilon \leq \delta} (\tilde{I}_k^\varepsilon \times \{\varepsilon\}).$$

(These are closed because $I_k^\varepsilon$ and $\tilde{I}_k^\varepsilon$ depend continuously on $\varepsilon$.) Next, for every point $p = (\zeta, \varepsilon) \in \mathbb{T} \times [0, \delta]$ and every $k$, construct a function $c_{p_k}^\varepsilon : W_p \to \mathbb{R}_+$, where $W_p$ is some neighborhood of $p$ in $\mathbb{T} \times [0, \delta]$. If $\zeta \in I_k^\varepsilon$, choose $W_p$ small enough so that $W_p \subset \tilde{V}_k$ and put $c_{p_k}^\varepsilon(\zeta', \varepsilon') = a_k^\varepsilon(\zeta')$. Note that if $(\zeta', \varepsilon') \in W_p$ and $r\gamma' \in \partial \varphi_k(\psi_\varepsilon^{-1}(\Omega))$, then $r = c_{p_k}^\varepsilon(\zeta', \varepsilon')$. If $\zeta \notin I_k^\varepsilon$, then choose $W_p$ small enough so that $W_p$ does not intersect $V_k$, and then choose as $c_{p_k}^\varepsilon$ some $C^\infty$ function satisfying the property that if $(\zeta', \varepsilon') \in W_p$ and $r\gamma' \in \partial \varphi_k(\psi_\varepsilon^{-1}(\Omega))$, then $r \leq c_{p_k}^\varepsilon(\zeta', \varepsilon')$. We also require $c_{p_k}^\varepsilon \geq 1$ in all $W_p$.

By compactness, choose a finite subfamily $\{W_{p_j}\}$ of $\{W_p\}$, which still covers $\mathbb{T} \times [0, \delta]$. Let $\{\tau_{p_j}\}$ be a $C^\infty$ partition of unity in $\mathbb{T} \times [0, \delta]$ subordinate to the cover $\{W_{p_j}\}$ and put

$$b_k^\varepsilon(\zeta) = \sum_{p_j} \tau_{p_j}(\zeta, \varepsilon)c_{p_k}^\varepsilon(\zeta, \varepsilon).$$

It is easy to see that $b_k^\varepsilon$ satisfies the required conditions because the functions $c_{p_k}^\varepsilon$ satisfy them in a local manner.

Let $T_k^\varepsilon$ be defined as above and let $\eta_k^\varepsilon$ be the Riemann map from $D_k^\varepsilon$ onto $\mathbb{D}$ such that $\eta_k^\varepsilon(0) = 0$ and $(\eta_k^\varepsilon)'(0) > 0$. This exists since $\mathbb{D} \subset D_k^\varepsilon$. Clearly, $\eta_k^\varepsilon \in A(\mathbb{D})$ and $\|\eta_k^\varepsilon\|_{A(\mathbb{D})} \leq 1$.

We prove that $\varphi_k^\varepsilon = \eta_k^\varepsilon \circ \varphi_k \circ \psi_\varepsilon^{-1}$ depend continuously on $\varepsilon$. Put $\beta = \max_{k, \varepsilon, \zeta} b_k^\varepsilon(\zeta)$, which is greater than 1. Let $\gamma : \mathbb{R} \to \mathbb{R}$ be a $C^\infty$ function such that $\gamma(r) = 0$ in a neighborhood of 0, $\gamma(r) = r$ on $(\sigma, \infty)$ for some $\sigma \in (0, 1)$ and $\gamma'(r) < \beta/(\beta - 1)$ for all $r$. For each $\varepsilon \in (0, \delta)$, put

$$h_k^\varepsilon(r\zeta) = \rho_k^\varepsilon(r)\zeta, \quad \rho_k^\varepsilon(r) = r - \left(1 - \frac{1}{b_k^\varepsilon(\zeta)}\right) \gamma(r), \quad r \geq 0, \zeta \in \mathbb{T}.$$

The condition $\gamma'(r) < \beta/(\beta - 1)$ implies that $(\rho_k^\varepsilon)' > 0$. Thus, (15) defines maps $h_k^\varepsilon : \mathbb{C} \to \mathbb{C}$ which are diffeomorphisms from $\overline{D_k^\varepsilon}$ to $\overline{\mathbb{D}}$ and depend continuously on $\varepsilon$. By [6, Corollary 9.4], the maps $\varepsilon \mapsto \eta_k^\varepsilon \circ (h_k^\varepsilon)^{-1}$ are continuous from $[0, \delta]$ to $C^\infty(\mathbb{D})$. Hence, the maps $\varepsilon \mapsto \varphi_k^\varepsilon$ are continuous from $[0, \delta]$ to $C^\infty(\overline{\Omega})$.

Since by construction $|\varphi_k^\varepsilon| = 1$ in $\tilde{J}_k$, the Schwartz reflection principle implies that each $\varphi_k^\varepsilon$ is analytic in some neighborhood $U_k$ of $\Omega \cup J_k$ and that the map $\varepsilon \mapsto \varphi_k^\varepsilon$ is continuous from $[0, \delta]$ to $C^\infty(U_k)$. As $\Phi_0 = \Phi$ is
admissible, by continuity the maps $\Phi_\varepsilon$ must also be admissible for sufficiently small $\varepsilon$. This finishes the proof. \hfill $\Box$

The following is a continuous ($\varepsilon$-dependent) version of the right regularization for Fredholm operators of index 0.

Lemma 15. Let $V$ be a Banach space, and $\{L_\varepsilon\}_{0 \leq \varepsilon \leq \varepsilon_0} \subset \mathcal{B}(V)$ be such that the map $\varepsilon \mapsto L_\varepsilon$ is continuous in the norm topology and $L_0 - I$ is compact. Then there is a finite rank operator $P \in \mathcal{B}(V)$, some $0 < \delta \leq \varepsilon_0$, and operators $\{R_\varepsilon\}_{0 \leq \varepsilon \leq \delta}, \{S_\varepsilon\}_{0 \leq \varepsilon \leq \delta} \subset \mathcal{B}(V)$ such that the maps $\varepsilon \mapsto R_\varepsilon$ and $\varepsilon \mapsto S_\varepsilon$, $S_0 = I$, are continuous in the norm topology, and

$$L_\varepsilon R_\varepsilon = I + PS_\varepsilon$$

holds for $0 \leq \varepsilon \leq \delta$.

Proof. Since $L_0 - I$ is compact, it is well know that there is a finite rank operator $P$ and an operator $R_0$ such that $LR_0 = I + P$. Let $B_\varepsilon = I + (L_\varepsilon - L_0)R_0$. Then there is some $\delta > 0$ such that $B_\varepsilon$ is invertible for $0 \leq \varepsilon \leq \delta$. We have $L_\varepsilon R_0 B_\varepsilon^{-1} = I + PB_\varepsilon^{-1}$, so the lemma holds with $R_\varepsilon = R_0 B_\varepsilon^{-1}$ and $S_\varepsilon = B_\varepsilon^{-1}$. \hfill $\Box$

Lemma 16. Let $\Phi : \Omega \to \overline{\mathcal{D}}$ be admissible. Assume that there are operators $T \in \mathcal{B}(H)$ and $C_1, \ldots, C_n \in \mathcal{B}(H)$ such that $\overline{\mathcal{D}}$ is a complete $K'$-spectral set for every $C_k, k = 1, \ldots, n$. Assume that if $f \in \text{Rat}(\Omega)$ can be written as in (4), then (5) holds (see the statement of Theorem 4). Then $\overline{\Omega}$ is a complete $K$-spectral set for $T$ for some $K$ depending on $\Omega$, $\Phi$, $K'$ and $S_A(T)$. Furthermore,

$$\|f(T)\| \leq C\|f\|_{A(\overline{\Omega}) \otimes M_s}, \quad \forall f \in (A_\Phi \cap \text{Rat}(\Omega)) \otimes M_s,$$

where $C$ is a constant depending only on $\Omega$, $\Phi$ and $K'$, and not on $T$.

The main point of (16) is that, under the hypotheses of this Lemma, $A_\Phi$ is a closed subspace of finite codimension in $A(\overline{\Omega})$. Thus, (16) shows that, in a space of finite codimension, the inequality $\|f(T)\| \leq C\|f\|$ holds with a constant independent of $T$.

Proof of Lemma 16. Use Theorem 1 to obtain operators $F_k$ as in the statement of the theorem. Denote by $L \in \mathcal{B}(A(\overline{\Omega}))$ the operator defined by $L(f) = \sum F_k(f) \circ \varphi_k$. Since $I - L$ is compact, there exist an operator $R$ and a finite rank operator $P$ such that $LR = I + P$. The space $X = \ker P$ has finite codimension in $A(\overline{\Omega})$ and does not depend on $T$. We will now check that (10) holds for some constant $C$ independent of $T$.

Take $f \in (X \cap \text{Rat}(\Omega)) \otimes M_s$ and put $g = (R \otimes \text{id}_s)f$. Then $(L \otimes \text{id}_s)g = f$, and so by (5),

$$f(T) = \sum_{k=1}^n [(F_k \otimes \text{id}_s)(g)](C_k).$$
Since $\overline{D}$ is complete $K'$-spectral for $C_k$,
\[
\|f(T)\| \leq \sum_{k=1}^{n} K' \|F_k\| cb \|g\| A(\overline{\Omega}) \otimes M_k \leq \sum_{k=1}^{n} K' \|F_k\| cb \|R\| cb \|f\| A(\overline{\Omega}) \otimes M_k
\]
\[
= \sum_{k=1}^{n} K' \|F_k\| \cdot \|R\| \cdot \|f\| A(\overline{\Omega}) \otimes M_k
\]
where the last equality uses Lemma 6. Thus (10) holds with $C = \sum K' \|F_k\| \cdot \|R\| < \infty$. Apply Lemma 12 to get (16). The remaining part of the lemma follows from Lemma 10.

7. Proofs of Theorems 3 and 4

We first give the proof of Theorem 4, as it is simpler than that of Theorem 3 and both proofs follow the same general idea.

Proof of Theorem 4. The first part of Theorem 4 is already contained in Lemma 16. For the case when $\Phi$ is injective and $\Phi'$ does not vanish, use Theorem IV (see Section 5). Then (16) implies that $\Omega$ is a complete $K$-spectral set for $T$, with $K$ independent of $T$.

To prove Theorem 3, in the case when $\sigma(T) \subset \Omega$, one can argue as in the proof of Theorem 4, putting $C_k = \varphi_k(T)$ and using the Cauchy-Riesz functional calculus for $T$ to get (5). However, such a direct proof will not work in the general case. The idea then is to apply a shrinking $\{\psi_\varepsilon\}$ for $\Omega$ to obtain operators $T_\varepsilon = \psi_\varepsilon(T)$ which have $\sigma(T_\varepsilon) \subset \Omega$, so that the above argument is again valid. The difficulties reside in constructing admissible functions $\Phi_\varepsilon = (\varphi_1^\varepsilon, \ldots, \varphi_n^\varepsilon)$ adapted to $T_\varepsilon$, in the sense that each $\varphi_k^\varepsilon(T_\varepsilon)$ has $\overline{D}$ as a complete $K'$-spectral set, as well as in passing to the limit as $\varepsilon$ tends to 0.

Proof of Theorem 3. Let $T \in B(H)$ with $\sigma(T) \subset \overline{\Omega}$ and such that for $k = 1, \ldots, n$, $\overline{D}$ is a complete $K'$-spectral set for $\varphi_k(T)$. We must prove that $\Omega$ is a complete $K$-spectral set for $T$ with $K$ depending on $\overline{\Omega}$, $\Phi$, $K'$ and $S_A(T)$.

By Lemma 11, there is a shrinking $\{\psi_\varepsilon\}$ for $\Omega$. Apply Lemma 14 to obtain a collection of admissible functions $\{\Phi_\varepsilon\}_{0 \leq \varepsilon \leq \varepsilon_0}$ such that $\overline{D}$ is a complete $K'$-spectral set for $\varphi_k^\varepsilon(\psi_\varepsilon(T))$, and such that the maps $\varepsilon \mapsto \varphi_k^\varepsilon$ are continuous from $[0, \varepsilon_0]$ to $C^\infty(U_k)$, where $U_k$ is a neighborhood of $J_k$. Then use Lemma III to get operators $L_\varepsilon$, where
\[
L_\varepsilon(f) = \sum F_k^\varepsilon(f) \circ \varphi_k^\varepsilon.
\]
Since $L_0 - I$ is compact, Lemma 15 (with $V = A(\overline{\Omega})$) yields operators $P, R_\varepsilon, S_\varepsilon : A(\overline{\Omega}) \to A(\overline{\Omega})$, where $\varepsilon \in [0, \delta]$, with the properties stated in the lemma.

Next we wish to apply Lemma 13. To this end, fix $Q : \text{ran } P \to \mathbb{C}^r$ an isomorphism, where $r$ is the rank of $P$, put $\Sigma_\varepsilon = QPS_\varepsilon$, so that $\Sigma_\varepsilon : A(\overline{\Omega}) \to \mathbb{C}^r$, $\Sigma_\varepsilon$ depends continuously on $\varepsilon$ in the norm topology and $\Sigma_0 = QPS_0 = QP$ is surjective, and set $T_\varepsilon = \psi_\varepsilon(T)$. Note that $T_\varepsilon$ depends continuously on $\varepsilon$ in the norm topology because $\psi_\varepsilon$ depends continuously on $\varepsilon$ in the
topology of uniform convergence on compact subsets of $U$, where $U$ is some open neighborhood of $\sigma(T)$.

It is necessary to check that (14) holds. For this, take $f \in (\ker \Sigma_\varepsilon \cap \text{Rat}(\overline{\Omega})) \otimes M_s$, put $g = (R_\varepsilon \otimes \text{id}_s)f$ and note that $f = (L_\varepsilon \otimes \text{id}_s)g$. Since $\sigma(T_\varepsilon) \subset \Omega$, an application of the Cauchy-Riesz functional calculus gives

$$f(T_\varepsilon) = \sum_{k=1}^n [(F_k^\varepsilon \otimes \text{id}_s)(g)](\varphi_k^\varepsilon(T_\varepsilon)).$$

Therefore, by Lemma 6, and since $\|F_k^\varepsilon\| \leq C$ (coming from Lemma III),

$$\|f(T_\varepsilon)\| \leq \sum_{k=1}^n K'\|F_k^\varepsilon\|_C\|g\|_{A(\overline{\Omega}) \otimes M_s} \leq \sum_{k=1}^n K'C\|R_\varepsilon\|_C\|f\|_{A(\overline{\Omega}) \otimes M_s}.$$  

Since $R_\varepsilon$ depends continuously on $\varepsilon$, (14) holds, as desired.

Apply Lemma 13 to obtain for all $s \geq 1$,

$$\|f(T)\| \leq C'\|f\|_{A(\overline{\Omega}) \otimes M_s}, \quad \forall f \in (\ker \Sigma_\varepsilon \cap \text{Rat}(\overline{\Omega})) \otimes M_s.$$  

By Lemma 10, this yields that $\overline{\Omega}$ is a complete $K$-spectral set for $T$, with $K$ depending on $\Omega$, $\Phi$ and $S_A(T)$. Therefore, $\Phi$ is a quasi-uniform strong test collection.

In the case that $\Phi$ is injective and $\Phi'$ does not vanish on $\Omega$, Theorem II and Lemma 12 together imply that $\overline{\Omega}$ is a complete $K$-spectral set for $T$, with $K$ independent of $T$.  

\[ \Box \]

8. Weakly admissible functions

In this section we will expand the class of functions under consideration to a wider class that we call weakly admissible functions. The main goal of this class is to replace condition (f) in the definition of an admissible function (see Section 2.3) by a weaker separation condition. In particular, a collection of functions which includes inner functions (i.e., functions with modulus 1 in all $\partial \Omega$) may be weakly admissible, though not admissible, except in trivial cases.

Let $\zeta \in \partial \Omega$. A right neighborhood of $\zeta$ in $\partial \Omega$ is understood to be the image $\gamma(\{0, \varepsilon\})$, where the function $\gamma : [0, \varepsilon) \to \partial \Omega$ is continuous and injective, $\gamma(0) = \zeta$, and as $t$ increases $\gamma(t)$ follows the positive orientation of $\partial \Omega$. Define the left neighborhoods of $\zeta$ in a similar manner.

If $\Psi \subset A(\overline{\Omega})$ is a collection of functions taking $\Omega$ into $\mathbb{D}$ and $\zeta \in \partial \Omega$, set

$$\Psi^+_{\zeta} = \{\psi \in \Psi : |\psi| = 1 \text{ in some right neighborhood of } \zeta\},$$

and

$$\Psi^-_{\zeta} = \{\psi \in \Psi : |\psi| = 1 \text{ in some left neighborhood of } \zeta\}.$$

**Definition.** Let $\Omega$ be a domain whose boundary is a disjoint finite union of piecewise analytic Jordan curves such that the interior angles of the corners of $\partial \Omega$ are in $(0, \pi]$. Then $\Psi = (\psi_1, \ldots, \psi_n) : \overline{\Omega} \to \mathbb{D}^n$, $\psi_k \in A(\overline{\Omega})$ for $k = 1, \ldots, n$, is weakly admissible if for $\Gamma_k = \{\zeta \in \partial \Omega : |\psi_k(\zeta)| = 1\}$ in place of $J_k$ and a constant $\alpha$, $0 < \alpha \leq 1$, it is the case that conditions (a)–(e) for an admissible function hold, and additionally:

(f) \quad $\forall \zeta \in \partial \Omega$, $\forall z \in \partial \Omega$, $z \neq \zeta$, $\exists \psi \in \Psi^+_{\zeta}$ : $\psi(\zeta) \neq \psi(z)$.
\((g')\quad \forall \zeta \in \partial \Omega, \forall z \in \partial \Omega, z \neq \zeta, \exists \psi \in \Psi^- \zeta : \psi(\zeta) \neq \psi(z).\)

In fact, it is easy to see that conditions (a) and (b) follow formally from conditions (c)–(e), (f) and \((g').\)

**Lemma 17.** Let \(\Psi : \overline{\Omega} \to \overline{\Omega}^\alpha\) be a weakly admissible function. Then there is an admissible function \(\Phi : \overline{\Omega} \to \overline{\Omega}^\alpha, \Phi = (\varphi_1, \ldots, \varphi_m),\) such that its components \(\varphi_k\) are of the form

\(\varphi_k = (h_{1,k} \circ \psi_1) \cdot \ldots \cdot (h_{n,k} \circ \psi_n),\)

where \(h_{j,k} : \overline{\Omega} \to \overline{\Omega}\) and \(h_{j,k} \in A(\overline{\Omega}).\)

**Proof.** First, fix some \(\zeta \in \partial \Omega.\) For each \(\psi \in \Psi_+ \zeta,\) put \(P_\psi = \psi^{-1}(\{\psi(\zeta)\}),\) which is a finite set of points of \(\partial \Omega.\) By condition (f) in the definition of a weakly admissible function, \(\bigcap_{\psi \in \Psi_+ \zeta} P_\psi = \{\zeta\}.\) Let \(J_\zeta^+\) be the closure of a sufficiently small right neighborhood of \(\zeta.\) For \(\psi \in \Psi_+ \zeta,\) put \(Q_\psi = \psi^{-1}(\psi(\zeta)).\) If \(J_\zeta^+\) is small enough, then each set \(Q_\psi\) is a union of disjoint right neighborhoods of each of the points in \(P_\psi.\) Since \(\bigcap_{\psi \in \Psi_+ \zeta} P_\psi = \{\zeta\},\) it can then be assumed that \(\bigcap_{\psi \in \Psi_+ \zeta} Q_\psi = J_\zeta^+.\)

Next, for each \(\psi \in \Psi_+ \zeta,\) construct a function \(h_\psi^+ \in A(\overline{\Omega})\) such that the function

\(\psi_\zeta^+ = \prod_{\psi \in \Psi_+ \zeta} h_\psi^+ \circ \psi\)

associated with \(J_\zeta^+\) satisfies \(|\psi_\zeta^+| = 1 in J_\zeta^+\) and \(|\psi_\zeta^+| < 1 in \partial \Omega \setminus J_\zeta^+.\) This is done as follows.

Take \(\psi \in \Psi_+ \zeta.\) Choose a function \(h_\psi^+\) satisfying the following conditions:

- \(|h_\psi^+| = 1 in \psi(J_\zeta^+)\) and \(|h_\psi^+| < 1 in \partial \Omega \setminus \psi(J_\zeta^+);\)
- \(h_\psi^+ map \psi(J_\zeta^+) bijectively onto a small arc of \(\mathbb{T};\)
- \(h_\psi^+\) is analytic on some open set \(U \supset \mathbb{D}\) such that the interior of \(\psi(J_\zeta^+);\)
  relative to \(\mathbb{T}\) is contained in \(U,\) and \((h_\psi^+)'\) is Hölder \(\alpha\) in \(U;\)
- \(|(h_\psi^+)'| \geq C > 0 in \psi(J_\zeta^+);\)
- \(\zeta\) is an endpoint of the set \(\{w \in \partial \Omega : |\psi(w)| = 1\}\) and \(S(\zeta)\) is the sector that appears on condition (d) in the definition of an admissible function (for \(\varphi_k = \psi),\) then \(\psi(S_k(\zeta)) \subset U.\)

Then \(|h_\psi^+ \circ \psi| = 1 in Q_\psi,\) and \(|h_\psi^+ \circ \psi| < 1 in \partial \Omega \setminus Q_\psi.\) Since \(|\psi_\zeta^+(z)| = 1 only when |h_\psi^+(\psi(z))| = 1 for every \(\psi \in \Psi_+ \zeta,\) that is, when \(z \in \bigcap_{\psi \in \Psi_+ \zeta} Q_\psi = J_\zeta^+,\) we get that \(|\psi_\zeta^+| = 1 in J_\zeta^+\) and \(|\psi_\zeta^+| < 1 in \partial \Omega \setminus J_\zeta^+.\) Also, since \(h_\psi^+(\psi(J_\zeta^+));\) is a small arc of \(\mathbb{T},\) it follows that \(\psi_\zeta^+\) maps \(J_\zeta^+\) bijectively onto some arc of \(\mathbb{T}.

Similarly, construct an arc \(J_\zeta^-\) which is the closure of a small left neighborhood of \(\zeta,\) and a corresponding function \(\psi_\zeta^-\). By compactness, we can choose a finite set of points \(\zeta_1, \ldots, \zeta_r\) such that \(J_{\zeta_k^+} \cup J_{\zeta_k^-}, k = 1, \ldots, r,\) cover all \(\partial \Omega.\) Rename the functions \(\psi_{\zeta_1}, \psi_{\zeta_1}^+, \psi_{\zeta_1}^-, \psi_{\zeta_2}, \psi_{\zeta_2}^+, \psi_{\zeta_2}^-\) as \(\varphi_1, \ldots, \varphi_m\) and the corresponding arcs \(J_{\zeta_1^+}, J_{\zeta_1^-}, \ldots, J_{\zeta_r^-}, J_{\zeta_r^+}\) as \(J_1, \ldots, J_m.\) Functions \(\varphi_1, \ldots, \varphi_m\) now satisfy condition (f) in the definition of an admissible family, because
if, for instance, \( J_k = J_\zeta^+ \), then \( \varphi_k = \psi_\zeta^+ \) sends \( J_\zeta^+ \) bijectively onto an arc of \( T \) and \( |\varphi_k| < 1 \) on \( \partial \Omega \setminus J_\zeta^+ \).

The functions \( \varphi_k \) satisfy conditions (c)–(e) because the functions \( \psi_k \) satisfy these conditions, and the functions \( h_\psi^+, h_\psi^- \) satisfy similar regularity conditions which have been given above. It follows that \( \Phi = (\varphi_1, \ldots, \varphi_m) \) is admissible.

**Theorem 18.** Let \( \Psi : \overline{\Omega} \to \overline{\mathbb{D}}^m \) be a weakly admissible function. Then \( \Psi \) is a quasi-uniform test collection over \( \Omega \). Moreover, if \( \Psi_0 \subseteq A(\overline{\Omega}) \) is any collection of functions taking \( \Omega \) into \( \mathbb{D} \) with the property that \( \Psi \subseteq \Psi_0 \), \( \Psi_0 : \overline{\Omega} \to \overline{\mathbb{D}}^m \) is injective and \( \Psi_0^{'} \) does not vanish on \( \Omega \), then \( \Psi_0 \) is a uniform test collection over \( \Omega \).

**Proof.** Suppose that \( T \in \mathcal{B}(H) \), \( \sigma(T) \subseteq \Omega \) and \( \psi_k(T) \) are contractions for \( k = 1, \ldots, n \). Let \( \Phi \) be the admissible function obtained from \( \Psi \) using Lemma 17. Put \( C_k = \varphi_k(T) \), \( k = 1, \ldots, m \). Check that \( C_k \) is a contraction for all \( k \).

Since \( \varphi_k = (h_{1,k} \circ \psi_1) \cdots (h_{n,k} \circ \psi_n) \),

\[
C_k = \varphi_k(T) = h_{1,k}(\psi_1(T)) \cdots h_{n,k}(\psi_n(T)).
\]

Each \( \psi_j(T) \) is a contraction, and \( \|h_{j,k}\|_{A(\overline{\mathbb{D}})} \leq 1 \), so \( h_{j,k}(\varphi_j(T)) \) is also a contraction for all \( j \) and \( k \). It follows that, being a product of contractions, \( \psi_k(T) \) is a contraction.

Because \( \sigma(T) \subseteq \Omega \), the hypotheses of Theorem 4 are satisfied by the Cauchy-Riesz functional calculus for \( T \). Therefore, Lemma 16 applies, and so \( \overline{\Psi} \) is a compact K-spectral set for \( T \) with \( K = K(\Omega, \Psi, S_\Lambda(T)) \), for an arbitrary pole set \( \Lambda \) for \( \Omega \). In other words, \( \Psi \) is a quasi-uniform test collection over \( \Omega \).

Now assume that \( \Psi_0 \) is as in the statement of the theorem. Form an admissible function \( \Phi_0 \) from \( \Phi \) by adding to \( \Phi \) all the functions in \( \Psi_0 \setminus \Phi \).

The arcs \( J_k \) corresponding to functions in \( \Psi_0 \setminus \Phi \) are defined to be equal to the empty set. Then \( \Phi_0 \) is injective and \( \Phi_0^{'} \) does not vanish, because \( \Psi_0 \) already had these properties. Therefore, \( \Phi_0 \) is a uniform strong test collection over \( \Omega \) by Corollary 5. If \( \psi(T) \) is a contraction for every \( \psi \in \Psi_0 \), then \( \varphi(T) \) is also a contraction for every \( \varphi \in \Phi_0 \). Hence, \( \Phi_0 \) is a uniform test collection over \( \Omega \). \( \square \)

Unfortunately, the methods of the above proof cannot be used to show that \( \Psi \) is a strong test collection over \( \Omega \), and we do not know whether the hypotheses imply it. If the operators \( \psi_k(T) \), \( k = 1, \ldots, n \), are contractions, then it follows that \( \varphi_k(T) \), \( k = 1, \ldots, m \), is a product of contractions and therefore a contraction. However, if \( \psi_k(T) \), \( k = 1, \ldots, n \), just have \( \overline{\mathbb{D}} \) as a complete K-spectral set for some \( K \), then we only get that \( \varphi_k(T) \) is a product of operators which have \( \overline{\mathbb{D}} \) as a complete K-spectral set. In general, an operator which is the product of two commuting operators both similar to contractions need not itself be similar to a contraction; see [31]. Therefore, one cannot prove by this method that \( \varphi_k(T) \) has \( \overline{\mathbb{D}} \) as a complete K'-spectral set for some \( K' \).
Corollary 19. Let \( \Omega \) be a finitely connected domain with analytic boundary and let \( \psi_1, \ldots, \psi_n : \overline{\Omega} \to \overline{D} \) be inner (i.e., \( |\psi_j| = 1 \) in \( \partial \Omega \) for \( j = 1, \ldots, n \)). Assume that the restriction of the map \( \Psi = (\psi_1, \ldots, \psi_n) : \overline{\Omega} \to \overline{D}^n \) to \( \partial \Omega \) is injective. Then \( \Psi \) is a quasi-uniform test-collection over \( \Omega \). If, moreover, \( \Psi \) is injective in \( \Omega \) and \( \Psi' \) does not vanish in \( \Omega \), then \( \Psi \) is a uniform test collection over \( \Omega \).

Proof. Since \( \psi_1, \ldots, \psi_n \) are inner, \( \Psi - \zeta = \Psi + \zeta \) for all \( \zeta \in \partial \Omega \). Therefore, the conditions (f) and (g') in the definition of a weakly admissible are equivalent to the condition that \( \Psi|\partial \Omega \) is injective. Since \( \psi_1, \ldots, \psi_n \) are inner and \( \partial \Omega \) is analytic, \( \psi_1, \ldots, \psi_n \) can be extended analytically across \( \partial \Omega \). Hence, \( \Psi \) is a weakly admissible function. To finish the proof, apply Theorem 18 with \( \Psi_0 = \Psi \).

On a general finitely connected domain \( \Omega \) with analytic boundary, one can always choose three inner functions \( \psi_1, \psi_2, \psi_3 \) such that the map \( \Psi = (\psi_1, \psi_2, \psi_3) : \overline{\Omega} \to \overline{D}^3 \) is injective and \( \Psi' \) does not vanish in \( \Omega \). Hence, such \( \Psi \) is a uniform test collection according to Corollary 19. See [43, Theorem IV.1] and [16, §3] for two different proofs of the existence of such a \( \Psi \). It is also known that when \( \Omega \) is doubly connected then the same can be done using only two inner functions \( \psi_1, \psi_2 \). However, for a domain \( \Omega \) of connectivity greater or equal than 3, a pair of inner functions \( \psi_1, \psi_2 \) will never be enough under the constraint that \( \Psi \) is injective (see [16, 37]).

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School of Mathematics and Statistics, Herschel Building, University of Newcastle, Newcastle upon Tyne, NE1 7RU, UK

E-mail address: michael.dritschel@newcastle.ac.uk

Departamento de Matemáticas, Universidad Autónoma de Madrid, Cantoblanco 28049 (Madrid), Spain

E-mail address: daniel.estevez@uam.es

Departamento de Matemáticas, Universidad Autónoma de Madrid, Cantoblanco 28049 (Madrid), Spain, and Instituto de Ciencias Matemáticas (CSIC - UAM - UC3M - UCM)

E-mail address: dmitry.yakubovich@uam.es