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Iwasawa Theory of Jacobians of Graphs

Sophia R. Gonet

Abstract

The Jacobian group (also known as the critical group or sandpile group) is an important invariant of a finite, connected graph $X$; it is a finite abelian group whose cardinality is equal to the number of spanning trees of $X$ (Kirchhoff’s Matrix Tree Theorem). A specific type of covering graph, called a derived graph, that is constructed from a voltage graph with voltage group $G$ is the object of interest in this paper.

Towers of derived graphs are studied by using aspects of classical Iwasawa Theory (from number theory). Formulas for the orders of the Sylow $p$-subgroups of Jacobians in an infinite voltage $p$-tower, for any prime $p$, are obtained in terms of classical $\mu$ and $\lambda$ invariants by using the decomposition of a finitely generated module over the Iwasawa Algebra.

1. Introduction

The Jacobian (or critical group, or sandpile group) is an algebraic invariant of a graph $X$ (in this paper the term graph will mean a simple graph with no loops or multiple edges, unless otherwise explicitly noted) which, for connected $X$, is a finite abelian group whose size is equal to the number of spanning trees of $X$ (this is well-known as the Matrix Tree Theorem). The study of Jacobians of graphs has a long history, and many applications, as described in [1, 2, 4, 5, 11, 20, 25]. Overall, there are relatively few graphs or families of graphs for which the Jacobian is exactly known: see [3, 6, 8, 12, 13, 19, 22]. In this paper we establish the “asymptotic structure” and orders of the Sylow $p$-subgroups of the Jacobians of certain covering graphs of a fixed base graph $X$, namely those that belong to a cyclic voltage $p$-tower cover of $X$.

More specifically, we adapt to voltage towers of graphs the classical work of Iwasawa for $\mathbb{Z}_p$-extensions—finite extensions $K_\infty$ of a number field $K$ with Galois group isomorphic to the additive $p$-adic integers, $\mathbb{Z}_p$, for some prime $p$. By using the general theory of $\mathbb{Z}_p[[\Gamma]]$-modules, where $\Gamma = \text{Gal}(K_\infty/K)$, Iwasawa was able to unravel the structure of the inverse limit of the $p$-Sylow subgroups of the class groups of the finite extension fields in his towers. This enabled him to prove the following theorem, which can be found in [18, 29]: Let $K_\infty/K$ be a $\mathbb{Z}_p$-extension. Let $p^e_m$ be the exact power of $p$ dividing the order of the class group of $K_m$, where $K_m$ is the fixed field of the subgroup $\Gamma_p^m$; then there exist nonnegative integers $\lambda, \mu$ and an integer $\nu$ such that $e_m = \mu p^m + \lambda m + \nu$ for all $m \geq m_0$ for some $m_0 \geq 0$.

The Main Theorem of this paper is the analog in the graph theory setting:
Theorem 1.1. Let
\[ X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_m \leftarrow \cdots \]
be a cyclic voltage \( p \)-tower (see Definition 3.2), where all \( X_m \) are connected. Let \( J_p(X_m) \) be the Sylow \( p \)-subgroup of the Jacobian of \( X_m \). Then there are nonnegative integers \( \mu \) and \( \lambda \) and an integer \( \nu \) such that
\[ |J_p(X_m)| = p^{e_m} \quad \text{where} \quad e_m = \mu p^m + \lambda m + \nu \]
for all \( m \geq m_0 \) for some \( m_0 \geq 0 \).

This theorem gives not only “asymptotic” order formulas for the \( p \)-Jacobians of the covering graphs \( X_m \), but also their “asymptotic” invariant factor decompositions, which, in particular give conditions under which the \( p \)-ranks grow without bound (which is also analogous to the classical number theoretic results of Iwasawa).

The ideas in [17, 23, 28] inspired the research that culminates in Section 3. However, the work is independent, contemporaneous, and by quite different methods.

The terminology and theory of Jacobians, voltage graphs and their derived covering graphs—including extending these results to infinite towers—is first summarized in Section 2. In Section 3, the theory of finitely generated modules over the Iwasawa algebra, \( \mathbb{Z}_p[[\Gamma]] \) is summarized. Consequences of the latter results, that form the essential underpinning of the Main Theorem, are also established. The Main Theorem is then proved using group-theoretic methods, that, in hindsight, illustrate how the decomposition theorem for Iwasawa modules plays the analogous role to the Smith Normal Form decomposition that describes ordinary Jacobians. The \( p \)-rank result mentioned above appears as a corollary to the Main Theorem.

This paper comprises the last part of the author’s dissertation [16], which contains significantly more details, examples, and an array of additional theoretical and computational material on voltage graphs and their associated derived graphs. We refer to it at points where its material expands on or expedites the development of this paper.

Added after refereeing: Just as this paper was submitted, Daniel Vallières and Kevin McGown circulated a manuscript [24] giving the generalization of Theorem 1.1 to multigraphs. Their work—which is completely independent—uses “analytic” methods (\( L \)-series etc.), and so provides a valuable complementary perspective on our result. It seems that the methods herein should also generalize to multigraphs, mutatis mutandis, since the main part of the proof, Section 3, essentially only involves cokernels of Laplacians, and these are well defined for multigraphs.

2. Preliminaries

In Section 2.1, we define the Picard and Jacobian groups. Then in Section 2.2 we define the Laplacian and reduced Laplacian. In Section 2.3 we describe a specific type of covering graph, called a derived graph, that arises from what is called a voltage graph—where elements from a group (which may be finite or infinite) are assigned to the edges of a fixed base graph \( X \). In Section 2.4, we give the definition of an intermediate covering graph. We then state the important result: given a voltage graph with derived graph \( Y \) such that \( Y \) is connected, \( Y/X \) is a normal (i.e. Galois) extension, and conversely, if \( Y/X \) is a normal extension with Galois group \( G \), then there exists a voltage assignment such that \((X, G, \alpha)\) is a voltage graph with derived graph \( Y \).
2.1. The Divisor, Picard, and Jacobian Groups. For more details on this section, refer to [10].

Definition 2.1. A divisor on a graph $X$ (possibly infinite) is an element of the free abelian group on the vertices $V = V(X)$:

$$\text{Div}(X) = \left\{ \sum_{v \in V(X)} a_v v \mid a_v \in \mathbb{Z} \right\}$$

where each $\sum_{v \in V(X)} a_v v$ is a formal linear combination of the vertices of $X$ with integer coefficients, where only finitely many $a_v$ are nonzero (in the case when $X$ is an infinite graph). The degree of a divisor is

$$\deg \left( \sum_{v \in V(X)} a_v v \right) = \sum_{v \in V(X)} a_v.$$  

When $V(X) = \{v_1, \ldots, v_n\}$, we may write the elements of $\text{Div}(X)$ as $a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$, where each $a_i \in \mathbb{Z}$, and its degree is $a_1 + a_2 + \cdots + a_n$. The degree map $\deg : \text{Div}(X) \to \mathbb{Z}$, is a surjective group homomorphism with kernel equal to the subgroup of $\text{Div}(X)$ of divisors of degree 0, denoted as $\text{Div}^0(X)$:

$$\text{Div}^0(X) = \{ D \in \text{Div}(X) \mid \deg D = 0 \}.$$  

Next let $X$ be a graph with vertices $\{v_1, \ldots, v_n\}$. For each fixed $v_i$ define the principal divisor, $p_i$, based at $v_i$ by

$$p_i = \deg(v_i) v_i - \sum_{j=1}^{n} \delta_{i,j} v_j$$

where $\delta_{i,j} = 1$ if $v_j$ is adjacent but not equal to $v_i$ and 0 otherwise (and here $\deg(v_i)$ is the valence of vertex $v_i$ in $X$). Define principal divisors to be elements of the $\mathbb{Z}$-submodule of $\text{Div}(X)$ spanned by the principal divisors based at the vertices:

$$\text{Pr}(X) = \text{Span}_\mathbb{Z}\{p_i \mid 1 \leq i \leq n\}.$$  

Evidently $\text{Pr}(X)$ is a submodule of $\text{Div}^0(X)$. From this we get the following groups.

Definition 2.2. The Picard group of $X$ is the quotient group

$$\text{Pic}(X) = \text{Div}(X)/\text{Pr}(X),$$

and the Jacobian group of $X$ is the subgroup of $\text{Pic}(X)$

$$\mathcal{J}(X) = \text{Div}^0(X)/\text{Pr}(X).$$

Theorem 2.3. If $X$ is connected, then $\mathcal{J}(X)$ is a finite abelian group.

2.2. The Laplacian and Reduced Laplacian.

Definition 2.4. Let $X$ be a graph with vertices $\{v_1, \ldots, v_n\}$. The graph Laplacian $L = L_X$ is the $n \times n$ matrix given by

$$L_{i,j} = \begin{cases} 
\deg(v_i) & \text{if } i = j \\
-1 & \text{if } v_i \text{ is adjacent to } v_j \\
0 & \text{if } i \neq j \text{ and } v_i \text{ is not adjacent to } v_j.
\end{cases}$$

The Laplacian is also the matrix representation of the following group homomorphism $\mathcal{L}$ defined as follows.

$$\mathcal{L} : \text{Div}(X) \to \text{Div}(X) \quad \text{where} \quad \mathcal{L}(v_i) = p_i.$$
When extended by $\mathbb{Z}$-linearity to all of $\text{Div}(X)$, this is a $\mathbb{Z}$-linear homomorphism from $\text{Div}(X)$ to itself, whose image is $\text{Pr}(X)$, the group of principal divisors. From this, we get the following important fact:

$$\text{Pic}(X) = \text{Div}(X)/\text{im}(\mathcal{L}) = \text{coker}(\mathcal{L}).$$

A reduced Laplacian $\tilde{L}$ is the $(n-1) \times (n-1)$ integer matrix obtained by removing the row and column corresponding to any vertex $v$ from the Laplacian matrix $L$. So the Jacobian group can be computed as the cokernel of the reduced Laplacian matrix

$$\mathcal{J}(X) \cong \mathbb{Z}^{n-1}/\text{im}(\mathcal{L}) = \text{coker}(\tilde{L}),$$

where $\mathbb{Z}^{n-1}$ denotes the free $\mathbb{Z}$-module on the set $V(X) - \{v\}$ of rank $n-1$.

2.3. Voltage Graphs. We first give the definition of a general covering graph.

**Definition 2.5.** An undirected graph $Y$ is a covering of an undirected graph $X$ if, after arbitrarily directing the edges of $X$, there is an assignment of directions to the edges of $Y$ and an onto graph homomorphism $\pi : Y \rightarrow X$ sending neighborhoods of $Y$ one-to-one onto neighborhoods of $X$ which preserve directions. We call such $\pi$ a covering map.

**Definition 2.6.** A $d$-sheeted covering means every fiber contains exactly $d$ elements, i.e.

$$|\pi^{-1}(x)| = d \forall x \in V(X).$$

**Definition 2.7.** Let $X$ be a graph whose edges have been oriented, and let $G$ be a group (finite or infinite). For a fixed orientation of the edges of $X$, let $E(X)^+$ denote the set of forward-directed edges of $X$; and let $E(X)^-$ denote the same edges but each with the reverse orientation (so each undirected edge of $X$ becomes two edges in the disjoint union of $E(X)^+$ and $E(X)^-$). An (ordinary) voltage assignment is a map

$$\alpha : E(X)^+ \cup E(X)^- \rightarrow G$$

such that if $e_{i,j} \in E(X)^+$ and $\alpha(e_{i,j}) = \alpha_{i,j} \in G$, then $e_{j,i} \in E(X)^-$ and $\alpha(e_{j,i}) = \alpha_{i,j}^{-1}$ (the inverse group element), where $e_{i,j}$ denotes the directed edge from $v_i$ to $v_j$.

The triple $(X,G,\alpha)$ is called an (ordinary) voltage graph. The values of $\alpha$ are called the voltages and $G$ is called the voltage group.

Note that a voltage assignment $\alpha$ is uniquely determined by its values on $E(X)^+$, so we will henceforth only specify $\alpha$ on the forward-directed edges of $X$.

The vertices of $X$ are labeled as $v_1, \ldots, v_n$. This imposes a natural lexicographic orientation on $X$, namely whenever there is an edge between $v_i$ and $v_j$, orient the edge $v_i \rightarrow v_j$ if $i < j$ (called the standard orientation). Note that results on derived graphs do not depend on the choice of orientation by [16], so without further mention, we adopt the standard orientation.

Any such voltage assignment can be codified by its $n \times n$ voltage adjacency matrix

$$A_\alpha = (A_{i,j})$$

with entries $A_{i,j} \in \mathbb{Z}[G]$ such that $A_{i,j} = 0$ if $i = j$ or there is no edge between $v_i$ and $v_j$, and $A_{i,j} = \alpha_{i,j}$ otherwise. (Note that the voltage adjacency matrix is also defined in [7, Definition 2.16].)

The purpose of assigning voltages to the graph $X$, called the base graph, is to obtain an object called the derived graph, called $Y$ here. To get the vertices of $Y$, make $d = |G|$ copies of each vertex $x \in V(X)$ labeling them as $x_{\tau_0}, x_{\tau_1}, x_{\tau_2}, \ldots, x_{\tau_{d-1}}$, where $G = \{\tau_0, \tau_1, \tau_2, \ldots, \tau_{d-1}\}$ has order $d$ (and the same formal construction works even if $|G|$ is uncountable). So there are $|G| \cdot |V(X)|$ vertices in $Y$. Now create the edges of $Y$ by the following rule: whenever there is an edge from $v_i$ to $v_j$ in the base...
graph $X$ with assigned voltage $\alpha_{i,j}$, create edges that go from $v_{i,g}$ to $v_{j,g\alpha_{i,j}}$ in $Y$, for every $g \in G$, where $g\alpha_{i,j}$ is the group-product of these two group elements in $G$. If $|G| = d$, then $\pi : Y \to X$ is a $d$-sheeted covering map (where again, $d$ may be any infinite cardinal too). Note that the degree (valence) of each vertex $v_\tau$ of $Y$ is the same as the degree of $v = \pi(v_\tau)$ in $X$. Also, since our base graph $X$ has no loops (i.e. $i \neq j$ here), no two vertices in the same fiber of $\pi$ are adjacent in $Y$.

Many examples as well as computational ways of constructing the ordinary adjacency matrix of $Y$ from the voltage adjacency matrix of $X$ by tensoring with matrices for the regular representation of $G$ appear in [16].

2.4. GALOIS THEORY OF COVERING GRAPHS AND VOLTAGE GRAPHS. Refer to [27] for Galois theory of (finite) Galois covers. For proof of the theorems presented below, see [16].

**Definition 2.8.** Suppose $Y$ is a covering of $X$ with projection map $\pi$. A graph $\tilde{X}$ is an intermediate covering to $Y/X$ if $Y/\tilde{X}$ is a covering, $\tilde{X}/X$ is a covering and the projection maps $\pi_1 : \tilde{X} \to X$ and $\pi_2 : Y \to \tilde{X}$ have the property that $\pi = \pi_1 \circ \pi_2$. If $Y/X$ is a $d$-sheeted covering with projection map $\pi : Y \to X$, then it is normal or Galois if there are exactly $d$ graph automorphisms $\sigma : Y \to Y$ such that $\pi \circ \sigma = \pi$. The Galois group is $G = \text{Gal}(Y/X) = \{ \sigma : Y \to Y \mid \pi \circ \sigma = \pi \}$.

**Theorem 2.9.** Suppose $Y/X$ is a normal covering with Galois group $G$ and $\tilde{X}$ an intermediate covering corresponding to the subgroup $H$ of $G$. Then $\tilde{X}$ itself is a normal covering of $X$ if and only if $H$ is a normal subgroup of $G$, in which case $\text{Gal}(\tilde{X}/X) \cong G/H$.

Now we put this in terms of voltage graphs.

**Theorem 2.10.** Let $(X, G, \alpha)$ be a voltage graph with $Y$ the derived graph. If $Y$ is connected, then $Y/X$ is a normal cover with $\text{Gal}(Y/X) \cong G$. Conversely, given a normal (Galois) cover $Y/X$, with $G = \text{Gal}(Y/X)$, then $Y/X$ is a voltage cover with the voltage group equal to $\text{Gal}(Y/X)$.

For any Galois cover $\pi : Y \to X$ of a connected base graph $X$, the graph $Y$ is necessarily connected except in the case where $G = \text{Gal}(Y/X)$ is the cyclic group of order 2 and $Y$ is two disjoint isomorphic copies of $X$ interchanged by $G$.

**Theorem 2.11.** Let $(X, G, \alpha)$ be a voltage graph with derived graph $Y$ such that $Y$ is connected. If $\tilde{X}$ is an intermediate cover of $Y/X$ corresponding to the normal subgroup $H$ of $G$, then $\tilde{X}/X$ is a voltage graph, whose voltage adjacency matrix is the voltage adjacency matrix of $Y/X$, but with nonzero entries reduced modulo $H$ (thus has entries in $\mathbb{Z}[G/H]$).

3. TOWERS OF VOLTAGE GRAPHS AND IWASAWA THEORY

We begin Section 3.1 by defining a cyclic $p$-tower of graphs. We then extend this definition to a cyclic voltage $p$-tower of graphs by using Theorems 2.10 and 2.11 from Section 2.4. From this, we get a “universal cover” of the tower by an infinite derived graph that we call $X_{\infty}$; it is the derived graph obtained from the voltage graph $(X, \mathbb{Z}_p, \alpha)$, where the voltage group is the additive $p$-adic integers and the voltage assignment $\alpha$ is determined by the cyclic voltage $p$-tower. We call $X_{\infty}$ the completion of the tower. In Section 3.2, we present important definitions and results pertaining to $A$-modules. Then in Subsection 3.2.1, we specify the $A$-modules be finitely generated. In Subsection 3.2.2 we construct a finitely generated torsion $A$-module, which we call $\text{Pic}_A$. Finally in Section 3.3 we prove the main theorem (Theorem 1.1) of this paper.
3.1. $p$-Tower Covering Graphs. We begin by defining a cyclic $p$-tower of graphs. We assume $p$ is a fixed prime.

**Definition 3.1.** A cyclic $p$-tower of graphs above a base graph $X$ is a sequence of covering graphs

$$X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_m \leftarrow \cdots$$

such that for $m \geq 0$, the cover $X_m/X$ is normal with $\text{Gal}(X_m/X) \cong \mathbb{Z}/p^m\mathbb{Z}$.

Note that for $m \geq 0$, this implies that the cover $X_{m+1}/X_m$ is normal with $\text{Gal}(X_{m+1}/X_m) \cong \mathbb{Z}/p\mathbb{Z}$ by the Fundamental Theorem of Galois Theory, along with the Third Isomorphism Theorem [14]. Now we specialize Definition 3.1 to voltage graphs.

**Definition 3.2.** A cyclic voltage $p$-tower of graphs above a base graph $X$ is a sequence of derived graphs

$$X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_m \leftarrow \cdots$$

such that for $m \geq 0$, $X_m/X$ is a derived graph with $\text{Gal}(X_m/X) \cong \mathbb{Z}/p^m\mathbb{Z}$.

Theorems 2.10 and 2.11 in Section 2.4 extend to towers, so we may choose notation that describes the vertices, edges, voltage assignments and Galois actions on the graphs $X_m$ in compatible ways that are determined by the covering maps. In short, each $X_m$ is a derived cover of $X$ defined by a voltage adjacency matrix of fixed degree $n$ whose $i,j$ entries are $\alpha_{m_{ij}}$ in the group ring $\mathbb{Z}[\mathbb{Z}/p^m\mathbb{Z}]$; and for each fixed $i,j$ the sequence of such entries has a limit as $m \to \infty$ in the $p$-adic integral group ring. The $n \times n$ matrix whose entries are these limits forms a voltage adjacency matrix for the “completion graph” that we now describe. This construction is straightforward, and the precise details are given explicitly in [16, Section 5.1]. This leads to a “universal cover” of the tower, by an infinite derived graph that we call $X_{p^\infty}$.

**Definition 3.3.** Given a cyclic voltage $p$-tower as in Definition 3.2, with each $X_m$ the derived graph for the voltage assignment $\alpha_m: E(X)^+ \to \mathbb{Z}/p^n\mathbb{Z}$, let $X_{p^\infty}$ be the derived graph obtained from the voltage graph $(X,\mathbb{Z}_p,\alpha)$, where the voltage group is the (additive) $p$-adic integers and voltage assignment $\alpha$ is determined by the tower. We call $X_{p^\infty}$ the completion of the tower.

**Theorem 3.4.** Let $X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_m \leftarrow \cdots$ be a cyclic voltage $p$-tower, let $X_{p^\infty}$ be the completion of the tower, and for each $m \geq 0$ let $X_m$ be the associated intermediate graphs. Then for all $m \geq 0$ there are graph isomorphisms $\overline{X}_m \to X_m$, depicted as the horizontal maps in Figure 1, such that all the maps in that figure commute, and commute with the action of $\mathbb{Z}_p$ as automorphisms of each graph.
We will henceforth write the voltage group $\mathbb{Z}_p$ as the multiplicative profinite group $\Gamma$, where, as a profinite group, it is cyclic: it is the closure of an infinite (multiplicative) cyclic group $\langle \gamma \rangle$ under the $p$-adic metric topology, for some $\gamma$.

3.2. Iwasawa Modules. Fundamental to Iwasawa’s development of $p$-class groups in $\mathbb{Z}_p$-towers of number fields was his study of certain finitely generated modules over the $\mathbb{Z}_p$-algebra $\Lambda = \mathbb{Z}_p[[\Gamma]]$. Here $\Lambda$ is the compactification of $\mathbb{Z}_p[\Gamma]$, under the profinite topology defined by the open subgroups $\Gamma^p\Gamma$. In this section we list and establish some properties of $\Lambda$, as well as finitely generated modules over it, that are the underpinnings of our main theorem. The following two useful theorems about $\Lambda$ can be found in [29].

**Theorem 3.5.** For the indeterminate $T$, $\mathbb{Z}_p[[\Gamma]] \cong \mathbb{Z}_p[[T]]$ with the isomorphism being induced by $\gamma \mapsto T + 1$.

**Theorem 3.6.** $\Lambda = \mathbb{Z}_p[[\Gamma]]$ is a Noetherian local ring.

**Definition 3.7.** Two $\Lambda$-modules $M$ and $M'$ are said to be pseudo-isomorphic, written $M \sim M'$, if there is a homomorphism $M \rightarrow M'$ with finite kernel and co-kernel.

**Definition 3.8.** A nonconstant polynomial $P(T) \in \Lambda$

$$P(T) = T^n + a_{n-1}T^{n-1} + \cdots + a_0$$
is called distinguished if $p \mid a_i$ for all $0 \leq i \leq n-1$.

The following proposition can be obtained immediately from [29, Proposition 7.2 and Lemma 7.5].

**Proposition 3.9.** Let $F(T)$ be a distinguished polynomial in $\mathbb{Z}_p[T]$. Then

$$\mathbb{Z}_p[T]/F(T)\mathbb{Z}_p[T] \cong \mathbb{Z}_p[[T]]/F(T)\mathbb{Z}_p[[T]],$$

where the isomorphism is as $\mathbb{Z}_p[T]$-modules. The isomorphism is the natural one, namely for $r \in \mathbb{Z}_p[T]$, the coset $r + F(T)\mathbb{Z}_p[T]$ maps to $r + F(T)\mathbb{Z}_p[[T]]$.

Let $\Lambda = \mathbb{Z}_p[[T]]$ and fix a topological generator $\gamma$ for $\Gamma$; by Theorem 3.5 the map $\gamma \mapsto T + 1$ extends to an isomorphism from $\Lambda$ to $\mathbb{Z}_p[[T]]$. For $m \geq 0$ let $\omega_m = \gamma^p - 1$. 

**Lemma 3.10.** For $m \geq 0$ $\omega_m$ maps to a distinguished polynomial in $\mathbb{Z}_p[T]$.

**Proof.** By definition, $\omega_m$ maps to $(T + 1)^{p^m} - 1$. Thus

$$(T + 1)^{p^m} - 1 \equiv (T^{p^m} + 1^{p^m}) - 1 \equiv T^{p^m} \pmod{p\mathbb{Z}_p[T]},$$

which establishes claim. \qed

Let $R = \mathbb{Z}_p[[T]]$. Fix $m \geq 0$.

**Definition 3.11.** Let $D$ be any $\Lambda$-module and let $B$ be any subset of $D$. For every $m \geq 0$, define

$$\Omega^D_m(B) = B \cap \omega_m D.$$

Define $R_m = R/\Omega^A_m(R) = R/R \cap \omega_m \Lambda$.

In the special case when $B$ is an $R$-submodule of $D$ (where $D$ is considered as an $R$-module), we have that $\Omega^D_m(B)$ is an $R$-submodule of $B$ containing $\omega_m B$.

The sets $\Omega^D_m(B)$ define relatively open subsets of $B$ in the "$\omega$-adic topology" on $D$. They obey the appropriate transitive property: If $B$ and $C$ are subset of $D$ with $C \subseteq B$, then

$$\Omega^D_m(B) \cap C = \Omega^D_m(C).$$

It is not true in general that $\omega_m B \cap C = \omega_m C$ however.

**Proposition 3.12.** Let $D$ be a $\Lambda$-module, let $A$ be any $\Lambda$-submodule of $D$ and let $B$ be any $R$-submodule of $A$, where $A$ is considered as an $R$-module. Then the map

$$\phi : B/\Omega^D_m(B) \rightarrow A/\Omega^A_m(A) \quad \text{by} \quad \phi(x + \Omega^D_m(B)) = x + \Omega^A_m(A)$$

is a well-defined and injective $R$-module homomorphism. If $B$ contains a set of $\Lambda$-module generators for $A$, then $\phi$ is an isomorphism and $A = B + \Omega^D_m(A)$; and if additionally $\omega_m D \subseteq A$ then $A = B + \omega_m D$.

**Proof.** We first simplify notation by denoting $\Omega^D_m(C)$ by just $\Omega_m(C)$ for every subset $C$ of $D$ throughout the proof. The map $B \rightarrow A/\Omega_m(A)$ by $x \mapsto x + \Omega_m(A)$ is a well-defined $R$-module homomorphisms, and since $\Omega_m(B) \subseteq \Omega_m(A)$, its kernel clearly contains $\Omega_m(B)$. This map therefore factors through $B/\Omega_m(B)$, giving the homomorphism $\phi$. We also have

$$\ker \phi = \{ x + \Omega_m(B) \mid x \in B \text{ and } x + \Omega_m(A) = 0 + \Omega_m(A) \}$$

$$= \{ x + \Omega_m(B) \mid x \in B \text{ and } x \in \Omega_m(A) \}$$

$$= (B \cap \Omega_m(A))/\Omega_m(B)$$

$$= (B \cap (A \cap \omega_m D))/\Omega_m(B)$$

$$= (B \cap \omega_m D)/\Omega_m(B) = \Omega_m(B)/\Omega_m(B) = 1,$$
so \( \phi \) is injective. It remains to show if \( B \) contains a set of \( \Lambda \)-module generators for \( A \), then \( \phi \) is surjective. Assuming this hypothesis, every \( y \in A \) can be written as
\[
y = \alpha_1 b_1 + \cdots + \alpha_n b_n, \quad \text{for some } \alpha_1, \ldots, \alpha_n \in \Lambda \text{ and } b_1, \ldots, b_n \in B.
\]
By Proposition 3.9 and Lemma 3.10, for each \( \alpha_i \) there is some \( r_i \in \mathbb{Z}_p[\gamma] \leq R \) such that \( \alpha_i - r_i \in \omega_m \Lambda \). Let \( y' = r_1 b_1 + \cdots + r_n b_n \in B \). By construction,
\[
y - y' = (\alpha_1 - r_1) b_1 + \cdots + (\alpha_n - r_n) b_n \in A \cap \omega_m D = \Omega_m(A).
\]
Thus \( \phi(y' + \Omega_m(B)) = y' + \Omega_m(A) = y + \Omega_m(A) \), and so \( \phi \) is surjective, hence an isomorphism. Also, surjectivity of \( \phi \) implies that \( A = B + \Omega_m(A) \). If \( \omega_m D \leq A \), then \( \Omega_m(A) = \omega_m D \), so the last assertion holds too. \(\square\)

**Corollary 3.13.** For \( R = \mathbb{Z}_p[\Gamma] \), we have that \( \phi \) induces an \( R \)-module isomorphism
\[
R_m \cong \mathbb{Z}_p[\Gamma_m].
\]

**Proof.** As in Definition 3.11, we have \( R_m = R/\Omega_m(R) = R/(R \cap \omega_m \Lambda) \). Now \( \mathbb{Z}_p[T] \) corresponds to \( \mathbb{Z}_p[\gamma] \) in the isomorphism between \( \mathbb{Z}_p[[T]] \) and \( \Lambda \), where \( \gamma \) is a fixed topological generator for \( \Gamma \). So we have
\[
R_m = R/(R \cap \omega_m \Lambda) \quad \text{(by definition)}
\]
\[
\cong \Lambda/\omega_m \Lambda \quad \text{(by Proposition 3.12)}
\]
\[
\cong \mathbb{Z}_p[\gamma]/(\omega_m) \quad \text{(by Proposition 3.9)}
\]
\[
\cong \mathbb{Z}_p[\gamma]/(\gamma p^m - 1)
\]
\[
\cong \mathbb{Z}_p[\Gamma_m],
\]
where the last isomorphism follows since \( \mathbb{Z}_p[\gamma]/(\gamma p^m - 1) \) is isomorphic to the group ring of the cyclic group \( \mathbb{Z}/p^m \mathbb{Z} \cong \Gamma_m \). Hence \( R_m \cong \mathbb{Z}_p[\Gamma_m] \). \(\square\)

We now record an elementary lemma, which will be used in proving Claim 6(3) in Section 3.3.

**Lemma 3.14.** If \( A \cong \mathbb{Z}_p \) as a \( \mathbb{Z}_p \)-module and \( B \) is a \( \mathbb{Z}_p \)-submodule of \( A \), then either \( B = 0 \) or \( A/B \) is finite.

**Proof.** By hypothesis \( A \) is isomorphic to the ring \( \mathbb{Z}_p \) considered as a module over itself, so its submodules are ideals. If \( B \neq 0 \), then \( B = p^k A \), for some \( k \geq 0 \), and so \( A/B \cong \mathbb{Z}_p/p^k \mathbb{Z}_p \cong \mathbb{Z}/p^k \mathbb{Z} \), which is finite. \(\square\)

In the next subsection, we now specify our \( \Lambda \)-modules to be finitely generated. Theorem 3.22 will be used in Section 3.3 to ultimately prove Theorem 1.1.

### 3.2.1. Finitely Generated \( \Lambda \)-Modules

**Theorem 3.15 (Structure Theorem for Iwasawa modules).** For any finitely generated \( \Lambda \)-module \( M \), we get the following pseudo-isomorphism:
\[
M \sim \Lambda^r \oplus \left( \oplus_{i=1}^s \Lambda/(p^k_i) \right) \oplus \left( \oplus_{j=1}^t \Lambda/(g_j(T)^{m_j}) \right)
\]
where \( r = \text{rank}(M) \), \( s, t, k_i \) and \( m_j \in \mathbb{Z} \) and \( g_i \in \mathbb{Z}_p[T] \) is monic, distinguished and irreducible. This decomposition is uniquely determined by \( M \). If \( M \) is a torsion module, then \( r = 0 \).

The growth formula for the orders of the finite Jacobians in the conclusion of the main result, Theorem 1.1, ultimately comes from the orders of certain finite quotients of the cyclic factors in the Iwasawa Structure Theorem decomposition of a finitely generated, torsion Iwasawa module that we shall construct shortly. The general structure of finite quotients of cyclic \( \Lambda \)-modules is described in [29, Section 13.3].
Definition 3.16. As in the notation of Theorem 3.15, we define the Iwasawa invariants of $M$ by

$$
\mu = \sum_{i=1}^{s} k_{i} \quad \text{and} \quad \lambda = \sum_{j} m_{j} \deg g_{j}.
$$

Definition 3.17. Let $M$ be any finitely generated torsion $\Lambda$-module with $p^{k_{i}}$ and $g_{i}^{m_{j}}$, as in Theorem 3.15. The characteristic polynomial of $M$, denoted by $\text{Char}(M)$, is the product:

$$
\text{Char}(M) = p^{k_{1}+\cdots+k_{s}} g_{1}^{m_{1}} \cdots g_{t}^{m_{t}},
$$

where $\text{Char}(M) = 1$ if $M$ is finite.

We record some basic facts about finitely generated torsion $\Lambda$-modules. These may be found in Bence Forrás Master’s Thesis [15, Section 1.1]. Part (3) may also be found in [9].

Proposition 3.18. Let $P$ be any finitely generated torsion $\Lambda$-module.

1. The relation “pseudo-isomorphism” is an equivalence relation on any set of finitely generated torsion $\Lambda$-modules.
2. For any $\Lambda$-module $M$, the characteristic polynomial is an invariant of the pseudo-isomorphism equivalence class of $M$.
3. If $M$ is a submodule of $P$, then $\text{Char}(P) = \text{Char}(M) \text{Char}(P/M)$. In particular, $\text{Char}(M) \mid \text{Char}(P)$.

The following theorem can be found in Romyar Sharifi’s online notes [26, Theorem 2.4.7]. In it, $\mathcal{O}$ is a valuation ring of a $p$-adic field. We simplify his statement by taking $\mathcal{O} = \mathbb{Z}_{p}$. This can also be obtained from [29, Proposition 13.19 and Lemma 13.21].

Theorem 3.19. Let $M$ be a finitely generated, torsion, $\Lambda$-module, and let $n_{0} \geq 0$ be such that $\text{Char}(M)$ and $\omega_{m,n_{0}} = \omega_{m}/\omega_{n_{0}}$ are relatively prime for all $m \geq n_{0}$. Set $\lambda(M) = \lambda$ and $\mu(M) = \mu$. Then there exists an integer $\nu$ such that

$$
|M/\omega_{m,n_{0}}M| = q^{e_{m}} \quad \text{where} \quad e_{m} = \mu p^{m} + \lambda m + \nu
$$

for all sufficiently large $m \geq 0$.

This theorem is used to prove Theorem 3.22. First we present two lemmas.

Lemma 3.20. Let $U$ be any Unique Factorization Domain and let $d \in U$ with $d \neq 0$. Suppose $\{a_{m}\}_{m=0}^{\infty}$ is any sequence of nonzero elements of $U$ with $a_{m} \mid a_{m+1}$ for all $m \geq 0$. Then there exists some $n_{0} \geq 0$ such that

$$
gcd(a_{n_{0}}, d) = gcd(a_{m}, d) \quad \text{for all} \quad m \geq n_{0}, \quad \text{and} \quad gcd(a_{m}/a_{n_{0}}, d) = 1 \quad \text{for all} \quad m \geq n_{0}.
$$

Proof. This is an easy exercise. The key point is that $d$ has only finitely many divisors, so the chain of $gcd(a_{m}, d)$ must stabilize after finitely many steps. \qed

Lemma 3.21. Let $P$ be a finitely generated $\Lambda$-module. Let $P_{m} = \omega_{m}P$, for all $m \geq 0$. Assume there is a $\Lambda$-submodule $N$ of $P$ such that $P_{m} \subseteq N$ and $|N/P_{m}| < \infty$, for all $m \geq 0$. Assume also that $P/N \cong \mathbb{Z}_{p}$. Then $P$ is a torsion $\Lambda$-module.

Proof. By hypothesis $P/N$ is a projective (free) $\mathbb{Z}_{p}$-module, so as $\mathbb{Z}_{p}$-modules we have

1. $P/\omega_{m}P \cong (P/N) \times (N/\omega_{m}P) \cong \mathbb{Z}_{p} \times \text{finite}$ for all $m \geq 0$.

If $P$ is not a torsion $\Lambda$-module, then in Theorem 3.15 we have $r \geq 1$, so $P$ has a $\Lambda$-submodule $K$ such that $P/K$ is pseudo isomorphic to $\Lambda$ (where $K$ is the kernel of
the map \( P \sim \Lambda^* \oplus (\Lambda\text{-torsion}) \rightarrow \Lambda \). Let overbars denote passage to \( P/K \). Then (as in Claim 1 in Section 3.3 below)
\[
\overline{P}/\omega_m \overline{P} \cong \overline{P}/\omega_m \overline{P} \sim \Lambda/\omega_m \Lambda \cong \mathbb{Z}_p[\Gamma_m].
\]
However \( \mathbb{Z}_p[\Gamma_m] \) is a free \( \mathbb{Z}_p \)-module of rank \( p^m \), and so for any \( m \geq 1 \) it cannot be pseudo-isomorphic to a homomorphic image of \( P/\omega_m P \) by (1) and the characterization of finitely generated modules over the PID \( \mathbb{Z}_p \), a contradiction. \( \square \)

**Theorem 3.22.** Let \( P \) be a finitely generated \( \Lambda \)-module. Let \( P_m = \omega_m P \), for all \( m \geq 0 \). Assume there is a \( \Lambda \)-submodule \( N \) of \( P \) such that \( P_m \subseteq N \) and \( |N/P_m| < \infty \), for all \( m \geq 0 \). Assume also that \( P/N \cong \mathbb{Z}_p \). Then there are nonnegative integers \( \mu \) and \( \lambda \) and an integer \( \nu \) such that
\[
|N/P_m| = p^{e_m} \quad \text{where} \quad e_m = \mu p^m + \lambda m + \nu,
\]
for all \( m \geq m_0 \), for some constant \( m_0 \geq 0 \).

**Proof.** By the preceding lemma, \( P \) is a torsion \( \Lambda \)-module. Let \( d = \text{Char}(P) \) and apply Lemma 3.20 in \( U = \Lambda \) to \( a_m = \omega_m \), for all \( m \geq 0 \). Let \( n_0 \) be as provided by the conclusion of that lemma. For any \( m \geq n_0 \), define \( \omega_m/n_0 = \omega_m/\omega_{n_0} \in \Lambda \). Let \( M = P_{n_0} \).

Note that for all \( m \geq n_0 \) we have
\[
\omega_m/n_0 M = (\omega_m/\omega_{n_0})(\omega_m P) = \omega_m P = P_m.
\]
By hypotheses then, for all \( m \geq n_0 \),
\[
|M/\omega_m/n_0 M| = |P_{n_0}/P_m| = |N/P_m| < |N/P_m| < \infty.
\]
By Lemma 3.20 we have that \( \omega_m/n_0 = \omega_m/\omega_{n_0} \) is relatively prime to \( \text{Char}(P) = d \), for all \( m \geq n_0 \). By Proposition 3.18(3) we have that \( \omega_m/n_0 \) is relatively prime to \( \text{Char}(M) \) as well.

We now have the hypotheses of Theorem 3.19 above. This theorem proves that there are \( \mu, \lambda \), and some \( \nu' \) such that
\[
|M/\omega_m/n_0 M| = p^{e_m} \quad \text{where} \quad e_m = \mu p^m + \lambda m + \nu',
\]
for all \( m \) greater than or equal to some fixed \( m_0 \geq n_0 \).

Now, as noted above, \( \omega_m/n_0 M = P_m \), and so for all \( m \geq m_0 \), by Lagrange we have
\[
|N/P_m| = |N/P_{n_0}| \cdot |P_{n_0}/P_m| = |N/M| \cdot |M/\omega_m/n_0 M| = p^k \cdot p^{e_m} \quad \text{where} \quad p^k = |N/M| \quad \text{and} \quad e_m = \mu p^m + \lambda m + \nu'.
\]
Finally, let \( \nu = k + \nu' \) to obtain the conclusion to the theorem. \( \square \)

The goal of the next subsection is to construct a finitely generated torsion \( \Lambda \)-module, \( P = \text{Pic}_\Lambda \). We will then apply Theorem 3.22 to \( \text{Pic}_\Lambda \) in Section 3.3.

### 3.2.2. Constructing a Finitely Generated \( \Lambda \)-Module
Let \( R = \mathbb{Z}_p[\Gamma] \) be the usual group ring of \( \Gamma \) with coefficients from \( \mathbb{Z}_p \). For the given voltage \( p \)-tower let \( X_{p^\infty} \) be its completion, so by Theorem 3.4 we may henceforth identify the intermediate graphs of \( X_{p^\infty}/X \) with corresponding graphs in the tower.

Fix the following subset of \( X_{p^\infty} \):
\[
\mathcal{B} = \{ v_{i,0} \mid 1 \leq i \leq n \},
\]
where 0 is the additive identity of $\mathbb{Z}_p$, so these vertices are taken from the “zeroth sheet”. We fix the identification of $X$ and $X_0$ with $\mathcal{B}$ by $v_i$ is identified with $v_{i,0}$.

We first take the free $\mathbb{Z}$-module on basis $\mathcal{B}$, $\text{Div}_\mathbb{Z}(X)$, and extend scalars (see [14, Section 10.4, Corollary 18]) to the free $\mathbb{Z}_p$-module with the same basis, now viewed over $\mathbb{Z}_p$. Denote this module by $\text{Div}_{\mathbb{Z}_p}(X_0)$. We can do likewise for each of the graphs $X_m$ and for $X_{p^\infty}$ too. We obtain the free $\mathbb{Z}_p$-modules of divisors

$$\text{Div}_{\mathbb{Z}_p}(X_m) = \mathbb{Z}_p \otimes \mathbb{Z}_p \text{Div}_\mathbb{Z}(X_m), \quad m \geq 0$$
$$\text{Div}_{\mathbb{Z}_p}(X_{p^\infty}) = \mathbb{Z}_p \otimes \mathbb{Z}_p \text{Div}_\mathbb{Z}(X_{p^\infty}).$$

Now for every $m \geq 0$, each $\text{Div}_\mathbb{Z}(X_m)$ is a free $\mathbb{Z}[\Gamma_m]$-module on the set $\mathcal{B}$ too, once we consider the group indices for vertices in $X_m$ to be $p$-adic indices reduced to $\mathbb{Z}_p/p^m\mathbb{Z}_p \cong \mathbb{Z}/p^m\mathbb{Z}$; and so $\text{Div}_{\mathbb{Z}_p}(X_m)$ is a free module of rank $n$ over $\mathbb{Z}_p[\Gamma_m]$. We may do likewise for $X_{p^\infty}$ to obtain that $\text{Div}_{\mathbb{Z}_p}(X_{p^\infty})$ is a free $R$-module, also of rank $n$ (on basis $\mathcal{B}$). In order to emphasize the free, rank $n$ nature of these respective modules, we adopt the following notation:

$$\text{Div}_{R_m} = \text{Div}_{\mathbb{Z}_p}(X_m) \quad \text{and} \quad \text{Div}_R = \text{Div}_{\mathbb{Z}_p}(X_{p^\infty}).$$

Since $\text{Div}_{R_m} = \text{Div}_{\mathbb{Z}_p}(X_m)$ is a free $\mathbb{Z}_p$-module on the basis of vertices of $X_m$, \(\{v_{i,g} \mid 1 \leq i \leq n, \, g \in \Gamma_m\}\), we may define the usual degree zero divisors with respect to this $\mathbb{Z}_p$-basis, and denote this by

$$\text{Div}^0_{\mathbb{Z}_p}(X_m) = \left\{ \sum_{i,g} a_{i,g}v_{i,g} \mid a_{i,g} \in \mathbb{Z}_p \text{ and } \sum_{i,g} a_{i,g} = 0 \right\}$$

where these sums are for $1 \leq i \leq n$ and $g \in \Gamma_m$.

Next we extend scalars from $R$ to $\Lambda$. Since $\text{Div}_R$ is a free $R$-module of rank $n$, its extension is a free $\Lambda$-module of rank $n$, denoted by

$$\text{Div}_\Lambda = \Lambda \otimes_R \text{Div}_R.$$ 

Since $R$ is a subring of $\Lambda$ we may simply view the elements of $\text{Div}_\Lambda$ as $R$-linear combinations of $\mathcal{B}$ and $\text{Div}_R$ as the subset of these consisting of $R$-linear combinations of $\mathcal{B}$.

Next we define the Laplacian endomorphism:

$$\mathcal{L}_{p^\infty} : \text{Div}_R \rightarrow \text{Div}_R \quad \text{by} \quad \mathcal{L}_{p^\infty}(v_i,0) = p_{i,0} \quad 1 \leq i \leq n,$$

where $p_{i,0}$, the principal divisor “based at $v_{i,0}$” is, by definition,

$$p_{i,0} = n_iv_{i,0} - \sum_{j=1}^n v_{j,0 + a_{i,j}}$$

where $n_i$ is the degree of $v_i$ in $X$. This is extended by $R$-linearity to all of $\text{Div}_R$. Because $\Gamma$ acts transitively on vertices in each fiber of $X_{p^\infty}/X$, as usual we have that the image of $\mathcal{L}_{p^\infty}$ is the $\mathbb{Z}_p$-span of the set of all principal divisors. We encapsulate this by the following notation (definition):

$$\text{Pr}_R = \mathcal{L}_{p^\infty}(\text{Div}_R).$$

By taking the “same map”, but defined on the basis $\mathcal{B}$ of the free $\Lambda$-module $\text{Div}_\Lambda$ we denote this by

$$\hat{\mathcal{L}}_{p^\infty} : \text{Div}_\Lambda \rightarrow \text{Div}_\Lambda \quad \text{by} \quad \hat{\mathcal{L}}_{p^\infty}(v_i,0) = p_{i,0} \quad 1 \leq i \leq n,$$

extended now by $\Lambda$-linearity. (Formally, $\hat{\mathcal{L}}_{p^\infty} = 1 \otimes \mathcal{L}_{p^\infty}$. ) Now we just define

$$\text{Pr}_\Lambda = \hat{\mathcal{L}}_{p^\infty}(\text{Div}_\Lambda).$$
Likewise, because $\Gamma_m$ acts transitively on the vertices of $X_m$, using the same $L_{p^\infty}$, but instead reading the vertices $v_{i,0}$ as lying in $\text{Div}_{R_m}$ (i.e. with the vertex indices reduced to $\mathbb{Z}_p/p^m\mathbb{Z}_p$), and extended by $R_m$-linearity—call this map $L_m$—defines the usual Laplacian endomorphism of $\text{Div}_{R_m}$. Its image is the $R_m$-module of principal divisors of $\text{Div}_{R_m}$, denoted as

$$\text{Pr}_{R_m} = L_m(\text{Div}_{R_m}).$$

We now define the appropriate Picard groups as follows:

$$\text{Pic}_{R_m} = \text{Div}_{R_m}/\text{Pr}_{R_m} \quad \text{(an $R_m$-module)}$$
$$\text{Pic}_R = \text{Div}_R/\text{Pr}_R \quad \text{(an $R$-module)}$$
$$\text{Pic}_\Lambda = \text{Div}_\Lambda/\text{Pr}_\Lambda \quad \text{(a $\Lambda$-module)}.$$

So these modules are cokernels of the respective module endomorphisms.

Next, we identify the $\Lambda$-submodule that plays the role of “degree zero divisors” in the proof of Theorem 1.1.

**Definition 3.23.** Let

$$S_1 = \{v_{i,0} - v_{j,0} \mid 1 \leq j < i \leq n\} \quad \text{and} \quad S_2 = \{p_{i,0} \mid 1 \leq i \leq n\}$$

Let $M_\Lambda$ be the $\Lambda$-submodule of $\text{Div}_\Lambda$ generated by $S_1, S_2$ and $(\gamma - 1)\text{Div}_\Lambda$, and let $M_R = \text{Div}_R \cap M_\Lambda$ and $N_\Lambda = M_\Lambda/\text{Pr}_\Lambda$.

It turns out that $M_\Lambda$ is actually generated by just $S_1$ and $(\gamma - 1)\text{Div}_\Lambda$ (see Claim 5(1) in the next subsection).

Since $\text{Div}_\Lambda$ is a finitely generated $\Lambda$-module and $\Lambda$ is Noetherian, all of its submodules are finitely generated, and so it follows that the quotient modules $\text{Div}_\Lambda/\text{Pr}_\Lambda = \text{Pic}_\Lambda$ and $M_\Lambda/\text{Pr}_\Lambda = N_\Lambda$ are also finitely generated as $\Lambda$-modules.

**Theorem 3.24.** Let $\text{Div}_\Lambda$, $\text{Pr}_\Lambda$, $\text{Pic}_\Lambda$ and $N_\Lambda$ be as above. Then $\text{Pic}_\Lambda$ is a finitely generated module over the Iwasawa Algebra $\Lambda = \mathbb{Z}_p[[\Gamma]]$ and therefore so is its submodule $N_\Lambda$.

### 3.3. The Main Theorem

We now go on to prove Theorem 1.1.

**Proposition 3.25.** The following diagram holds.
Proposition 3.25 is proved by combining the following six claims 1-6 concerning the columns of Figure 2.

First consider the reduction map

$$\pi_m : \text{Div}_R \to \text{Div}_{R_m} \quad \text{by} \quad v_{i,g} \mapsto v_{i,\gamma}$$

where $g \in \Gamma$ and $\gamma \in \Gamma_m$ is the reduction of $g$ to $\Gamma/\Gamma^m \cong \mathbb{Z}_p/p^m \mathbb{Z}_p$, (and recall $\text{Div}_{R_m} = \text{Div}_{\mathbb{Z}_p}(X_m)$). Here we are really defining $\pi_m$ on the free $R$-basis vectors on the zeroth sheet, and then extending by $R$-linearity to all of $\text{Div}_R$. It is helpful to keep in mind that for all $m \geq 0$, by the above map and by the previous subsection we have

$$\text{Div}_R \text{ is an } R\text{-submodule of } \text{Div}_\Lambda, \text{ and } \text{Div}_{R_m} \text{ is an } R\text{-quotient module of } \text{Div}_R.$$
Proof. By Proposition 3.12 and Corollary 3.13, we get the following isomorphisms, where the composition of these isomorphisms is the induced map on \( \Div_R \) mod \( \ker \pi_m \):

\[
\Div_R / \Omega_m(\Div_R) \cong \Div_\Lambda / \omega_m \Div_\Lambda \\
\cong (\Lambda \oplus \Lambda \oplus \cdots \oplus \Lambda) / (\omega_m (\Lambda \oplus \Lambda \oplus \cdots \oplus \Lambda)) \\
\cong (\Lambda / (\omega_m)) \oplus \cdots \oplus (\Lambda / (\omega_m)) \\
\cong R_m \oplus \cdots \oplus R_m \\
\cong \mathbb{Z}_p[\Gamma_m] \oplus \cdots \oplus \mathbb{Z}_p[\Gamma_m] \\
\cong \Div_{R_m}
\]

the free \( \mathbb{Z}_p[\Gamma_m] \)-module of rank \( n \). Thus, the kernel of \( \pi_m \) is \( \Omega_m(\Div_R) \).

Now let

\[ K_m = \ker \pi_m \quad \text{and} \quad Q_R = \Pr_\Lambda \cap \Div_R. \]

**Claim 2:**

Columns 1 and 2 have the following intersections:

1. \( \Div_1 \subseteq \Div_\Lambda \),
2. \( \Div_1 \cap M_\Lambda = M_R \),
3. \( \Pr_\Lambda \cap \Div_R = Q_R \),
4. \( (\omega_m \Div_\Lambda + \Pr_\Lambda) \cap \Div_R = K_m + Q_R = K_m + \Pr_R \).

Proof. (1) holds by Subsection 3.2.2, while (2) and (3) are by definition of \( M_R \) and \( Q_R \), respectively. By Proposition 3.12 applied with \( D = \Div_\Lambda \), \( A = \Pr_\Lambda + \omega_m \Div_\Lambda \) and \( B = \Pr_R \), since \( \Pr_R \) and \( \Pr_\Lambda \) are both generated (as \( R \)- and \( \Lambda \)-modules, respectively) by the same generators, they both have the same image in \( \Div_\Lambda / \omega_m \Div_\Lambda \) as in Claim 1. So, by the last sentence of Proposition 3.12,

\[ (\omega_m \Div_\Lambda + \Pr_\Lambda) \cap \Div_R = K_m + Q_R = K_m + \Pr_R. \]

Then since

\[ \Pr_R \subseteq Q_R \subseteq \Pr_\Lambda, \]

by Equation (2), we get

\[ \Pr_R + \omega_m \Div_\Lambda = \Pr_\Lambda + \omega_m \Div_\Lambda. \]

Now because \( \Pr_R \) and \( Q_R \) are contained in \( \Div_R \), intersecting the subgroups in Equation (3) with \( \Div_R \) gives

\[
(\Pr_R + \omega_m \Div_\Lambda) \cap \Div_R = \Pr_R + (\omega_m \Div_\Lambda \cap \Div_R) = \Pr_R + K_m \\
= Q_R + (\omega_m \Div_\Lambda \cap \Div_R) = Q_R + K_m \\
= (\Pr_\Lambda + \omega_m \Div_\Lambda) \cap \Div_R,
\]

which gives (4).

**Claim 3:**

Columns 1 and 2 have the following containments:

1. \( \Pr_\Lambda \subseteq \omega_m \Div_\Lambda + \Pr_\Lambda \subseteq M_\Lambda \subseteq \Div_\Lambda \),
2. \( \Pr_R \subseteq Q_R \subseteq K_m + Q_R \subseteq M_R \subseteq \Div_R \).

Proof. (1) is clear. From Claim 1, we have that \( \ker \pi_m = K_m = \Div_R \cap \omega_m \Div_\Lambda \). Then since \( \omega_m = (\gamma - 1)(1 + \gamma + \cdots + \gamma^{p^{m-1}}) \), we have

\[ \omega_m \Div_\Lambda = (\gamma - 1)(1 + \gamma + \cdots + \gamma^{p^{m-1}}) \Div_\Lambda \subseteq (\gamma - 1) \Div_\Lambda \subseteq M_\Lambda. \]

Thus we have \( \ker \pi_m \subseteq \Div_R \cap M_\Lambda = M_R \). So all containments in (2) are clear.
Claim 4:
Columns 1 and 2 have the following joins:
(1) $M_A + \text{Div}_R = \text{Div}_A$,
(2) $(\omega_m \text{Div}_A + \text{Pr}_A) + M_R = M_A$.

Proof. Apply Proposition 3.12 to $D = A = \text{Div}_A$ and $B = \text{Div}_R$. Its final assertion gives that $\omega_m \text{Div}_A + \text{Div}_R = \text{Div}_A$. Then (1) is immediate because $\omega_m \text{Div}_A \subseteq M_A$ by definition. Finally, since $\omega_m \text{Div}_A + \text{Div}_R = \text{Div}_A$ and by the latter observation, we have

$$M_A = M_A \cap (\omega_m \text{Div}_A + \text{Div}_R)$$
$$= \omega_m \text{Div}_A + (M_A \cap \text{Div}_R)$$
$$= \omega_m \text{Div}_A + M_R$$

as needed for (2). □

Claim 5:
As $R$-modules we have the following:
(1) $\text{Div}_A/M_A \cong \mathbb{Z}_p$ and $M_A$ is the $\Lambda$-submodule of $\text{Div}_A$ generated by $S_1 \cup (\gamma - 1)\text{Div}_A$,
where $S_1$ is as in Definition 3.23,
(2) $\text{Div}_R/M_R \cong \mathbb{Z}_p$.

Proof. We first prove (1). We may obtain $\text{Div}_A/M_A$ as follows: first factor $\text{Div}_A$ by $(\gamma - 1)\text{Div}_A$. By Corollary 3.13 applied with $m = 0$, and as in the proof of Claim 1, we have that

$$\text{Div}_A/(\gamma - 1)\text{Div}_A \cong (\Lambda/\omega_0 \Lambda) \oplus \cdots \oplus (\Lambda/\omega_0 \Lambda) \cong \mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p,$$

where the divisors $v_{1,0}, \ldots, v_{n,0}$ map to a basis of this free $\mathbb{Z}_p$-module of rank $n$. Now factor out the submodule generated by the images of all $v_{i,0} - v_{j,0}$ for all $i, j$ from the quotient $\text{Div}_A/(\gamma - 1)\text{Div}_A$. By doing this, we are simply identifying all the basis vectors with each other, leaving the rank-1 $\mathbb{Z}_p$-module quotient. The latter process is the same as modding $\text{Div}_\mathbb{Z}_p(X_0)$ by the degree zero divisors in $\text{Div}_\mathbb{Z}_p(X_0)$. So this tells us that $\text{Pr}_0$ for $1 \leq i \leq n$ must already be contained in the $\Lambda$-submodule generated by just $S_1 \cup (\gamma - 1)\text{Div}_A$. This proves (1).

To prove (2): First note that by definition $M_R = M_A \cap \text{Div}_R$. Then by Claim 4(1) we have that

$$\text{Div}_A/M_A = (\text{Div}_R + M_A)/M_A.$$ 

Then by the Diamond Isomorphism Theorem (which says that $(\text{Div}_R + M_A)/M_A \cong \text{Div}_R/M_R$) and the previous part we see that

$$\text{Div}_R/M_R \cong \mathbb{Z}_p.$$

as desired. □

By the proof of Claim 3(2), it follows that the kernel of the map $\pi_m$ restricted to $M_R$ (which we simply denote by $\pi_m$ too) is also equal to $K_m = \text{Div}_R \cap \omega_m \text{Div}_A$. From the definition in Section 3.2.2 we introduce the new notation:

$$M_{R_m} = \text{Div}_\mathbb{Z}_p^2(X_m).$$

Claim 6:
For Columns 2 and 3 the following holds:
(1) $\text{Pr}_{R_m} \subseteq M_{R_m} \subseteq \text{Div}_{R_m}$
(2) \( \pi_m : \text{Div}_R \rightarrow \text{Div}_{R_m} \) is well-defined and surjective,
(3) \( \pi_m(M_R) = M_{R_m} \) and \( M_R = \pi_m^{-1}(M_{R_m}) \),
(4) \( \pi_m(K_m + Q_R) = \text{Pr}_{R_m} \) and \( \pi_m^{-1}(\text{Pr}_{R_m}) = K_m + Q_R \).

Proof. Items (1) and (2) are clear by their respective definitions. It is clear that
\[ S_1 \cup (\gamma - 1)\text{Div}_R \subseteq \text{Div}_R \cap M_A = M_R. \]

Next we show that the image of \( S_1 \cup (\gamma - 1)\text{Div}_R \) under \( \pi_m \) generates \( \text{Div}_{Z_p}(X_m) \) as a \( Z_p \)-module. The \( Z_p \)-module \( \text{Div}_{Z_p}(X_m) \) is generated as a \( Z_p \)-module by differences of vertices, \( v_{i,r} - v_{j,s} \), where \( 1 \leq i, j \leq n \) and \( r, s \in Z/p^mZ \). Such differences can be written as
\[ v_{i,r} - v_{j,s} = (v_{i,r} - v_{i,0}) + (v_{i,0} - v_{j,0}) + (v_{j,0} - v_{j,s}), \]
where the middle term on the right is in \( S_1 \). For each \( i \) (and \( j \)) we may express the first (and third, resp.) differences on the right as telescoping sums of divisors of the form \( v_{i,t+\tau} - v_{i,t} \) for all \( t \), where \( \tau = \pi_m(\gamma) \) is any additive generator for \( Z/p^mZ \). The claim then follows since all of the latter differences are in the image of \( (\gamma - 1)\text{Div}_R \) under \( \pi_m \). This argument shows that
\[ M_{R_m} \subseteq \pi_m(M_R). \]

To show the reverse containment, let \( D = \pi_m^{-1}(M_{R_m}) \), (the complete preimage). By basic properties of homomorphisms (part of the Lattice Isomorphism Theorem) and since \( \ker \pi_m \subseteq M_R \) we have:
\[ \pi_m^{-1}(\pi_m(M_R)) = M_R + \ker \pi_m = M_R. \]

By applying \( \pi^{-1} \) to Equation (4) we get \( D \subseteq M_R \). By the Lattice Isomorphism Theorem we have that \( \pi_m \) induces an isomorphism
\[ \text{Div}_R/D \cong \text{Div}_{R_m}/\pi_m(D) = \text{Div}_{R_m}/M_{R_m} = \text{Div}_{Z_p}(X_m)/\text{Div}_{Z_p}(X_m) \cong Z_p, \]
where the latter follows from \( \deg : \text{Div}_{Z_p}(X_m) \rightarrow Z_p \).

Since \( D \subseteq M_R \) we get that \( \text{Div}_R/M_R \) is a quotient \( Z_p \)-module of the \( Z_p \)-module \( \text{Div}_R/D \). By Claim 5(2) we also have \( \text{Div}_R/M_R \cong Z_p \). This is illustrated in Figure 3.

\[
\begin{array}{c}
\text{Div}_R \\
\left\{ \begin{array}{c}
\mathbb{Z}_p \\
M_R \\
D \\
\end{array} \right\}
\end{array}
\]

Figure 3. \( \text{Div}_R/M_R \cong Z_p \) is a quotient \( Z_p \)-module of the \( Z_p \)-module \( \text{Div}_R/D \)

However, the only \( Z_p \)-module quotient of \( Z_p \) that is also isomorphic to \( Z_p \) is the quotient by the zero submodule (this follows by Lemma 3.14 in Section 3.2) i.e. we must have \( M_R = D \); and so \( \pi_m(M_R) = \pi_m(D) = M_{R_m} \). This gives (3).

Now \( \pi_m \) induces the surjective map
\[ \pi_m : M_R/\text{Pr}_R \rightarrow M_{R_m}/\text{Pr}_{R_m} \]
where, by definition, $M_R/\text{Pr}_R = N_R$ and $M_{R_m}/\text{Pr}_{R_m} = \mathcal{J}_p(X_m)$ (the isomorphism $M_{R_m}/\text{Pr}_{R_m} \cong \mathcal{J}_p(X_m)$ is obtained by taking $\otimes \mathbb{Z}_p$ to $\text{Div}^0(X_m)/\text{Pr}(X_m) = \mathcal{J}(X_m)$). This is defined by the following commutative diagram in Figure 4.

$$
\begin{array}{ccc}
M_R & \xrightarrow{\pi_m} & M_{R_m} & \xrightarrow{\text{proj}} & M_{R_m}/\text{Pr}_{R_m} = \mathcal{J}_p(X_m) \\
\downarrow{\text{proj}} & & \downarrow{\pi_m} & & \\
M_R/\text{Pr}_R = N_R & & & &
\end{array}
$$

**Figure 4.** The map $\pi_m : N_R \to \mathcal{J}_p(X_m)$ commutes with the natural projection map.

To prove (4), first invoke Claim 2(4) to obtain that $K_m + Q_R = K_m + \text{Pr}_R$. Since $K_m$ is the kernel of $\pi_m$, the subgroups $Q_R$ and $\text{Pr}_R$ have the same image under $\pi_m$. By definition, $\text{Pr}_R$ is generated as an $R$-module by the principal divisors based at the vertices in the zeroth sheet of $X_{p^\infty}$; and likewise $\text{Pr}_{R_m}$ is generated as an $R_m$-module by the images of these principal divisors in $\text{Div}_{R_m}$. Thus $\pi_m$ maps $\text{Pr}_R$, hence also $K_m + \text{Pr}_R$, surjectively onto $\text{Pr}_{R_m}$. This gives the first assertion of (4). Furthermore, since $K_m + \text{Pr}_R$ contains the full kernel of $\pi_m$ and $\text{Pr}_R$ maps onto $\text{Pr}_{R_m}$, the Lattice Isomorphism Theorem immediately gives the second assertion of (4).

Claims 1–6 prove Proposition 3.25.

Now for each subgroup $A$ of $\text{Div}_\Lambda$ let $\bar{A}$ denote the image of $A$ under the natural projection map

$$
\sim : \text{Div}_\Lambda \longrightarrow \text{Div}_\Lambda/\text{Pr}_\Lambda
$$

(which is both a $\Lambda$- and an $R$-module homomorphism). Since $\sim$ is a $\Lambda$-module homomorphism, the image of $\omega_m \text{Div}_\Lambda + \text{Pr}_\Lambda$ under it is $\omega_m \text{Pic}_\Lambda$. Since $\text{Div}_R$ is an $R$-submodule of $\text{Div}_\Lambda$, we may apply $\sim$ to it as well, and to its submodules.

By the Diamond Isomorphism Theorem, since we have checked all the appropriate intersections from column 1 to column 2 in Figure 2, this natural projection gives the first two columns in Figure 5 as well as all intersections (depicted, as usual, by horizontal lines) between their subgroups in column 2. To get the third column of Figure 5, factor the third column of Figure 2 by $\text{Pr}_{R_m}$. The horizontal lines—which are homomorphisms—relating column 2 to column 3 in Figure 5 are obtained by taking images of the subgroups in column 2 under $\pi_m$.

By the Lattice Isomorphism Theorem, the (already established) quotient groups (in red) are consequently also preserved when passing between columns (thus also transitively from column 1 to column 3). We only need these to be abelian group isomorphisms; but they are, in fact, $R$- and $\Lambda$-module isomorphisms.
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Figure 5. The natural projection homomorphism from $\text{Div}_\Lambda$ to $\text{Pic}_\Lambda$ indicated in the first two columns and the passage from $\pi_m$ to $\overline{\pi}_m$ indicated in the third column.

Now we apply Theorem 3.22 with $P = \text{Pic}_\Lambda$ and $N = N_\Lambda = M_\Lambda / \text{Pr}_\Lambda$ used as $P$ and $N$ in its hypotheses. Since $X_m$ is connected and $\omega_m \text{Pic}_\Lambda \subseteq N_\Lambda$ for all $m$ by Figure 5, we have

$$|N_\Lambda / \omega_m \text{Pic}_\Lambda| = |J_p(X_m)| < \infty.$$  

This leads immediately to the conclusion of Theorem 1.1.

An important invariant of any voltage graph with abelian voltage group is the reduced Stickelberger Element. With respect to the basis $B$ of both $\text{Div}_R$ (as an $R$-basis) and $\text{Div}_\Lambda$ (as a $\Lambda$-basis) (see Subsection 3.2.2), the two maps, $\mathcal{L}_p^\infty$ and $\mathcal{L}_p^\infty$, have the same matrix representation. Thus we define

$$\Theta_p^\infty = \det \mathcal{L}_p^\infty = \det \mathcal{\hat{L}}_p^\infty,$$

which is an element of $R = \mathbb{Z}_p[\Gamma]$ that plays the role of the reduced Stickelberger element.

Remark 3.26. $\Theta_p^\infty$ annihilates both $\text{Pic}_R$ and $\text{Pic}_\Lambda$— see [16] for discussion and additional uses of the reduced Stickelberger element.

Corollary 3.27. Under the hypothesis and notation of Theorem 1.1, the ranks of $J_p(X_m)$ are bounded as $m \to \infty$ if and only if $p$ does not divide $\Theta_p^\infty$ in $\Lambda$ (or in $\mathbb{Z}_p[\Gamma]$).

Proof. By definition, $\text{Pic}_\Lambda$ is the cokernel of the voltage Laplacian, $\mathcal{L}_p^\infty : \text{Div}_\Lambda \to \text{Div}_\Lambda$, where $\Theta_p^\infty = \det \mathcal{L}_p^\infty$. In the notation of Theorem 3.15, let $p^\mu$ be the product of the $p^{k_i}$. Then the characteristic polynomial, as in Definition 3.17, is equal to

$$p^\mu \prod_{j=1}^{t} g_j^{m_j} = \text{Char}(\text{Pic}_\Lambda).$$

Let $M = \omega_m \text{Pic}_\Lambda$ where $m_0 \geq 0$ is fixed. Then since $\text{Pic}_\Lambda / M$ has finite $p$-rank (fixed, independent of $m \to \infty$), the ranks of $\text{Pic}_\Lambda$ and $M$ differ by a constant, and one is bounded as $m \to \infty$ if and only if the other is bounded.
We now compare \( \mu \) invariants for \( \text{Pic}_A \) and \( M \), as follows. By Proposition 3.18(3), we have

\[
(6) \quad \text{Char}(M) = \frac{\text{Char}(\text{Pic}_A)}{\text{Char}(\text{Pic}_A/M)}.
\]

The \( \Lambda \)-module \( \text{Pic}_A/M \) is a quotient of the module \( \text{Div}_A/(\omega_{m_0}\text{Div}_A) \); and as in Claim 1,

\[
\text{Div}_A/(\omega_{m_0}\text{Div}_A) \cong (\Lambda/\omega_{m_0}\Lambda) \oplus \cdots \oplus (\Lambda/\omega_{m_0}\Lambda) \oplus \cdots.
\]

But by Lemma 3.10 we know \( \omega_{m_0} \) maps to a distinguished polynomial in \( \mathbb{Z}_p[[T]] \cong \Lambda \), so \( (\Lambda/\omega_{m_0})^n \) is already in Iwasawa decomposition form, and it clearly has characteristic polynomial \( \omega_{m_0}^n \) (again, under the identification \( \gamma \mapsto T + 1 \)). One more usage of Proposition 3.18(3) gives that

\[
\text{Char}(\text{Pic}_A/M)\mid \omega_{m_0}^n,
\]

so \( \text{Char}(\text{Pic}_A/M) \) is relatively prime to \( p \) (since the distinguished polynomial \( \omega_{m_0} \) is).

By Equation (6), this shows

\[
p \mid \text{Char}(\text{Pic}_A) \iff p \mid \text{Char}(M).
\]

If \( \mu(\text{Pic}_A) = 0 \), then \( \text{Char}(\text{Pic}_A) = \Theta_{p^\infty} \) by [21, Proposition 10.23]. In this case, \( p \) does not divide \( \Theta_{p^\infty} \) by definition of \( \text{Char}(\text{Pic}_A) \). By [29, Lemma 13.20], we have that the ranks of the finite \( \Lambda \)-module quotients of a finitely generated torsion \( \Lambda \)-module stay bounded if and only if the \( \mu \) invariant of the Iwasawa decomposition is zero. So if the ranks of \( J_p(X_m) \) stay bounded as \( m \to \infty \), then \( p \) does not divide \( \Theta_{p^\infty} \).

Conversely, we show that if the ranks of \( J_p(X_m) \) don’t stay bounded as \( m \to \infty \), then \( p \) does divide \( \Theta_{p^\infty} \) in \( \Lambda \). So if the rank of \( J_p(X_m) \to \infty \) as \( m \to \infty \), then the \( \mu \)-invariant of the submodule \( M \), and hence also of \( \text{Pic}_A \), must be nonzero. i.e. the Iwasawa decomposition of \( \text{Pic}_A \) must have at least one factor of the form \( \Lambda/(p^a) \), for some \( a \geq 1 \). This forces \( p \) to divide \( \Theta_{p^\infty} \) as follows. By [16, Corollary 9] or [29, p. 297], \( \Theta_{p^\infty} \) annihilates \( \text{Pic}_A \). It follows from the definition of pseudo-isomorphism that there is some submodule of fixed finite index in the Iwasawa Decomposition factor \( \Lambda/(p^a) \) that is also annihilated by \( \Theta_{p^\infty} \); and hence \( \Theta_{p^\infty} \) annihilates a submodule of fixed finite index in every quotient of \( \Lambda/(p^a) \). But the latter module has quotient modules, \( \Lambda/(p^a T^k) \), of order \( p^{ak} \), for every \( k \geq 1 \), and none of these possess a nonzero submodule annihilated by an element of \( \Lambda \) that is prime to \( p \). Thus \( p \) must divide \( \Theta_{p^\infty} \).

**Example 3.28.** Let \( X = X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_m \rightarrow \cdots \) be a cyclic voltage \( p \)-tower of derived graphs over a base graph \( X \) whose vertices are \( v_1, \ldots, v_m \), with \( v_1 \) adjacent to \( v_2 \). Assume that the voltage adjacency matrix for \( X_m/X \) (with the standard orientation on \( X \)) has a generator for the cyclic group of order \( p^m \) in entry 1, 2; its inverse in entry 2, 1; and has a 1 (the identity of the voltage group) in entry \( i, j \) whenever \( v_i \) is adjacent to \( v_j \) for \( \{i, j\} \neq \{1, 2\} \); and has zeros elsewhere (including on the diagonal). Then

\[
|J_p(X_m)| = p^{c_m}, \quad \text{where} \quad c_m = mp^m + \lambda m + \nu \quad \text{for all} \quad m \geq 0,
\]

where \( p^h \) is the largest power of \( p \) dividing the reduced Stickelberger element \( (p^h \text{ can be shown to be independent of } m) \), and \( \lambda = 1 \). In particular, when \( X \) is the complete graph on \( n \) vertices, \( p^h \) is the largest power of \( p \) dividing \((n - 2)n^{n-3}\), for \( n \geq 3 \).

Starting just with a base graph \( X \) as above, we can choose generators for voltage groups \( \mathbb{Z}/p^m \mathbb{Z} \) so that the sequence of 1, 2 entries in such voltage adjacency matrices converges in \( \mathbb{Z}_p \). This constructs derived graphs \( X_m/X \) that form a cyclic voltage.
$p$-tower (and if $X_1$ is connected, then all $X_m$ are too) — see [16] for further details on these “single voltage” derived covers.

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References

[1] Roland Bacher, Pierre de la Harpe, and Tatiana Nagnibeda, The lattice of integral flows and the lattice of integral cuts on a finite graph, Bull. Soc. Math. France 125 (1995), 167–198.
[2] Spencer Backman, Matthew Bakek, and Chi Ho Yuen, Geometric bijections for regular matroids, zonotopes, and Ehrhart theory, Forum Math. Sigma 7 (2019), article no. E45 (37 pages).
[3] Hua Bai, On the critical group of the $n$-cube, Linear Algebra Appl. 369 (2003), 251–261.
[4] Norman Biggs, Chip-firing and the critical group of a graph, J. Algebraic Combin. 9 (1999), 25–45.
[5] , The critical group from a cryptographic perspective, Bull. Lond. Math. Soc. 39 (2007), 829–836.
[6] David B. Chandler, Peter Sin, and Qing Xiang, The Smith and critical groups of Paley graphs, J. Algebraic Combin. 41 (2015), 1013–1022.
[7] Sunita Chepuri, Christopher J. Dowd, Andrew Hardt, Gregory Michel, Sylvester W. Zhang, and Valerie Zhang, Arborescences of covering graphs, Algebr. Comb. 5 (2022), no. 2, 319–346.
[8] Hans Christianson and Victor Reiner, The critical group of a threshold graph, Linear Algebra Appl. 394 (2002), 233–244.
[9] John Coates and Ramadorai Sujatha, Cyclotomic fields and zeta values, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2006.
[10] Scott Corry and David Perkinson, Divisors and sandpiles. An introduction to chip-firing, American Mathematical Society, Providence, RI, 2018.
[11] Deepak Dhar, Self-organized critical state of sandpile automaton models, Phys. Rev. Lett. 64 (1990), 1613–1616.
[12] Joshua Ducey, Ian Hill, and Peter Sin, The critical group of the Kneser graph on 2-subsets of an $n$-element set, Linear Algebra Appl. 516 (2018), 154–168.
[13] Joshua E. Ducey and Deelan M. Jalil, Integer invariants of abelian Cayley graphs, Linear Algebra Appl. 445 (2014), 316–325.
[14] David S. Dummit and Richard M. Foote, Abstract algebra, third ed., John Wiley & Sons, Inc., Hoboken, NJ, 2004.
[15] Bence Forrás, Iwasawa theory, 2020, http://bforras.eu/docs/Forrás_Iwasawa_Theory_blue.pdf.
[16] Sophia Gonet, Jacobians of finite and infinite voltage covers of graphs, Ph.D. thesis, University of Vermont, 2021.
[17] Kyle Hammer, Thomas W. Mattman, Jonathan W. Sands, and Daniel Valliès, The special value $u = 1$ of Artin-Ihara L-functions, 2019, https://arxiv.org/abs/1907.04910.
[18] Kenkichi Iwasawa, On $\Gamma$-extensions of algebraic number fields, Bull. Amer. Math. Soc. 65 (1959), 183–226.
[19] Brian Jacobson, Andrew Niedermaier, and Victor Reiner, Critical groups for complete multipartite graphs and Cartesian products of complete graphs, J. Graph Theory 44 (2003), 231–250.
[20] Caroline K. Kivans, The mathematics of chip-firing, Discrete Mathematics and its Applications (Boca Raton), CRC Press, Boca Raton, FL, 2019.
[21] Nobushige Kurokawa, Masato Kurihara, and Takeshi Saito, Number theory. 3. Iwasawa theory and modular forms, translated from the Japanese by Masato Kuwata, Iwanami Series in Modern Mathematics, Translations of Mathematical Monographs, vol. 242, American Mathematical Society, Providence, RI, 2012.
[22] Young Soo Kwon, Aleksandr D. Mednykh, and Ilya A. Mednykh, On Jacobian group and complexity of the generalized Petersen graph $gp(n,k)$ through Chebyshev polynomials, Linear Algebra Appl. 529 (2017), 355–373.
[23] Kevin J. McGown and Daniel Valliès, On abelian $\ell$-towers of multigraphs II, 2021, https://arxiv.org/abs/2105.08661.
[24] Kevin J. McGown and Daniel Valliès, On abelian $\ell$-towers of multigraphs III, 2021, https://arxiv.org/abs/2107.07639.
[25] Victor Reiner and Dennis Tseng, Critical groups of coverings, voltage and signed graphs, Discrete Math. 318 (2014), 10–40.
[26] Romyar Sharifi, Iwasawa theory, https://www.math.ucla.edu/~sharifi/ivasawa.pdf.
[27] Audrey Terras, Zeta functions of graphs, 1st ed., Cambridge University Press, 2011.
[28] Daniel Vallières, On abelian $\ell$-towers of multigraphs, Ann. Math. Qué. 45 (2021), no. 2, 433–452.
[29] Lawrence C. Washington, Introduction to cyclotomic fields, Graduate Texts in Mathematics, vol. 83, Springer-Verlag, New York, 1982.

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