The tetralogy of Birkhoff theorems

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Abstract

We classify the existent Birkhoff-type theorems into four classes: First, in field theory, the theorem states the absence of helicity 0- and spin 0-parts of the gravitational field. Second, in relativistic astrophysics, it is the statement that the gravitational far-field of a spherically symmetric star carries, apart from its mass, no information about the star; therefore, a radially oscillating star has a static gravitational far-field. Third, in mathematical physics, Birkhoff’s theorem reads: up to singular exceptions of measure zero, the spherically symmetric solutions of Einstein’s vacuum field equation with $\Lambda = 0$ can be expressed by the Schwarzschild metric; for $\Lambda \neq 0$, it is the Schwarzschild-de Sitter metric instead. Fourth, in differential geometry, any statement of the type: every member of a family of pseudo-Riemannian space-times has more isometries than expected from the original metric ansatz, carries the name Birkhoff-type theorem. Within the fourth of these classes we present some new results with further values of dimension and signature of the related spaces; including them are some counterexamples: families of space-times where no Birkhoff-type theorem is valid. These counterexamples further confirm the conjecture, that the Birkhoff-type theorems have their origin in the property, that
the two eigenvalues of the Ricci tensor of two-dimensional pseudo-Riemannian spaces always coincide, a property not having an analogy in higher dimensions. Hence, Birkhoff-type theorems exist only for those physical situations which are reducible to two dimensions.

Keyword(s): Birkhoff theorem, Einstein space, isometry group

1 Introduction

Four different types of theorems carry the name Birkhoff theorem, all of them refer to the original Birkhoff result from 1923, see [1] for a presentation of the earlier papers about it. All of them are in a sense related to the spherically symmetric metric

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \frac{dr^2}{1 - 2m/r} + r^2 \left(d\psi^2 + \sin^2 \psi d\phi^2\right)$$

(1.1)

and its property that the $t$-translation $\partial/\partial t$ represents an isometry. It is one of the possibilities to write the metric of the Schwarzschild black hole and is valid for all points of the space-time except the horizon at $r = 2m$.

First, in field theory, the theorem states the absence of spin 0-parts of the gravitational field within Einstein’s theory following from the Einstein-Hilbert lagrangian $L_{EH} = R$. One of its possible counterparts is a theory, where the lagrangian $L_{FOG} = L_{EH} + l^2 R^2$ has a spin 0-part stemming from the $R^2$-term. These considerations may be restricted to the linearized field equations, where closed-form solutions are available. Accordingly, even in the linearized solutions, $\partial/\partial t$ fails to be an isometry in fourth-order gravity defined by $L_{FOG}$. The short-hand description of Birkhoff’s theorem with the words: spherically symmetric solutions of Einstein’s field equation are static is a little bit dangerous, as it may lead to misunderstandings about the validity of metric (1.1) in the region $0 < r < 2m$, where $t$ fails to be a timelike coordinate. The best version to circumvent this misunderstanding is to say, that spherically symmetric solutions of Einstein’s field equation possess a
fourth isometry, represented by the hypersurface-orthogonal Killing vector \( \frac{\partial}{\partial t} \). However, this formulation is not very convenient, so another possibility has been used in some places: There a space-time is defined to be spherically symmetric, if it has a SO(3)-group of isometries, whose orbits are isomorphic to a standard two-sphere \( S^2 \) with area \( 4\pi r^2 \), and the gradient of the scalar \( r \) represents a spacelike vector vanishing at the center of symmetry only. With this definition of spherical symmetry, the original Birkhoff formulation keeps valid. Another possibility to circumvent the region with \( r \leq 2m \) applied in field theory is to concentrate on the linearized field equation with smallness parameter being essentially \( m \), there a Fourier transform of the solutions is possible.

Second, in relativistic astrophysics, the main emphasis is on the formulation: the set of spherically symmetric vacuum solutions of Einstein’s field equation can be parametrized by one single parameter \( m \). Of course, for \( m = 0 \), metric (1.1) represents nothing but the flat Minkowski space-time of special relativity theory. Sometimes one can see formulations like: In all other cases, \( m > 0 \) represents the total mass of the central object. Again, such a formulation, though justified, is a little bit dangerous, as also for \( m < 0 \), metric (1.1) represents a vacuum solution. Formally, this case can be subsumed under the \( m > 0 \)-case, if one allows also negative values of \( r \) in that metric. But the real astrophysical reason for restricting \( m \) to values \( m \geq 0 \) is the fact, that all objects composed of normal matter have a positive total mass. The importance of the Birkhoff theorem is the following: the gravitational far-field of a spherically symmetric star carries, apart from its total mass, absolutely no information about the structure of the star, so e.g. a radially oscillating star has a static gravitational far-field. This property was already known to be valid in Newton’s theory of gravitation, but it came as a surprise that such a property will also be valid in Einstein’s theory, where the gravitational field, the metric \( g_{AB} \), carries 6 (namely 10 components of the metric minus 4 coordinate transformations) degrees of freedom in contrast to Newton’s with only one of them.
Third, in mathematical physics, especially in the search for exact solutions, Birkhoff’s theorem will be formulated like: up to singular exceptions of measure zero, the spherically symmetric solutions of Einstein’s vacuum field equation with Λ-term can be written by inserting the value $n = 2$ into the metric

$$ds^2 = - \left( 1 - \frac{2m}{r^{n-1}} - \frac{\Lambda r^2}{3} \right) dt^2 + \frac{dr^2}{1 - 2m/r^{n-1} - \Lambda r^2/3} + r^2 d\Omega^2. \quad (1.2)$$

These mentioned singular exceptions are not only the horizons where the component $g_{tt}$ of metric (1.2) vanishes, but also such solutions, where $r$ is constant, and therefore cannot be applied as coordinate; this can take place if the space-time $ds^2$ represents the direct product of two 2-spaces of constant curvature, and such spaces can all be generated as limits of portions of space-time metric (1.2), see [2]. The possible inclusion of several matter fields is possible, and can generally be transformed via the Einstein equation, or other theories of gravity under discussion, to properties of the Ricci tensor. The usually employed formulation reads: matter fields must be spherically symmetric. However, at least for those theories, where matter is coupled to gravity via the energy-momentum tensor $T_{AB}$ only, it suffices to require that the energy-momentum tensor must be spherically symmetric. This is, in some cases, a really weaker assumption.

And, at the end, fourth, in differential geometry, any statement of the type: Every member of a family of pseudo-Riemannian space-times has more isometries than expected from the original metric ansatz, carries the name Birkhoff-type theorem. It is this type of theorems we want to develop further; including them there will be some counterexamples: families of space-times where no Birkhoff-type theorem is valid. These counterexamples further confirm the conjecture, that the Birkhoff-type theorems have their origin in the property, that the two eigenvalues of the Ricci tensor of two-dimensional pseudo-Riemannian spaces always coincide, a property not having an analogy

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1Here, $d\Omega^2$ denotes the metric of the standard $n$-sphere $S^n$, the cases with $n > 2$ are covered here for later use.
in higher dimensions. Hence, Birkhoff-type theorems exist only for those physical situations which are reducible to two dimensions.

In the present paper, we shall try to put a further impetus to developing this fourth aspect of the Birkhoff theorem, namely to the question: under what circumstances, the solutions of a set of gravitational field equations have one more symmetry than should have been expected from the assumed metric ansatz?

Let us elucidate this by using Einstein’s theory without matter as example. Let $ds^2$ be an Einstein space, i.e. a vacuum solution of the Einstein field equation with arbitrary $\Lambda$-term, of differentiability class $C^2$ having the form

$$ds^2 = d\sigma^2 + r^2 d\Omega^2$$ (1.3)

where $d\sigma^2$ and $d\Omega^2$ are pseudo-Riemannian manifolds of dimension $k \geq 0$ and $n \geq 0$ respectively and arbitrary signature, and let $r \geq 0$ be a scalar function on $d\sigma^2$. Furthermore, let $d\Omega^2$ be a space of constant curvature with curvature scalar $R$, that means, the dimension of the isometry group of $d\Omega^2$ equals $n(n+1)/2$. In the most important application, $d\Omega^2$ is the metric on the unit sphere $S^n$, and then metric (1.3) is called to represent a spherically symmetric metric. If additionally $k = n = 2$ and $d\sigma^2$ has signature $(-, +)$, this metric is called to represent a spherically symmetric space-time.

Then the questions arise: For what values of $k$ and $n$ can we prove that $D \geq 1 + n(n+1)/2$, where $D$ is the dimension of the local isometry group of $ds^2$? That means, does the resulting $ds^2$ possess at least one non-expected isometry? Does the result depend on the signatures? Does the result depend on the sign of $R$? An affirmative answer is well-known for $k = n = 2$ and $R > 0$, for this case the result is called Birkhoff theorem. This fact motivates our notation:

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2 Of course, essentially $r > 0$, and only such isolated zeroes of the function $r$ are allowed which are compatible with the requested differentiability class.

3 A group $G$ is called to be a local isometry group of a pseudo-Riemannian manifold $M$, if each point of $M$ possesses an open neighbourhood $U$ which is isometric to an open subset of a pseudo-Riemannian manifold $V$ possessing $G$ as isometry group.
The type \((k, n)\)-Birkhoff theorem states the following: if metric (1.3) represents an Einstein space\(^4\), then the dimension of its local isometry group is larger than the dimension of the isometry group of the prescribed \(d\Omega^2\). In section 2 we shall outline, for which values of \(k\) and \(n\), a type\((k, n)\)-Birkhoff theorem is valid. Section 3 shall show in more details the necessary formulas, section 4 presents the coordinate-free proof of Birkhoff’s theorem for \(k = 2\), and the final section 5 shows how the metric of the generalized Schwarzschild-de Sitter black hole can be deduced and gives a summary of results and some further comments.

Before we continue with answering this type of questions in the next sections, here we present a very short overview about other papers on that theorem: The Birkhoff theorem for Einstein’s general relativity theory has been discussed e.g. in [3], [4]: here the original paper: G. D. Birkhoff, *Relativity and Modern Physics*, Harvard University Press, Cambridge (1923) is cited as follows: "The field outside of the spherical distribution of matter is static whether or not the matter is in a static or in a variable state . . . Thus the Schwarzschild solution is essentially the most general solution of the field equations with spherical symmetry.", [5]: here, a completely covariant proof is given without the necessity to introduce special adapted coordinates; that proof shows the geometric origin of the Birkhoff theorem: it rests on the property, that differently from all other dimensions \(k\) it holds for \(k = 2\): the Ricci tensor has no more than \(k - 1\) different eigenvalues, and this property has to be applied to the space perpendicular to the orbits of the spherical symmetry, [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19]: here a 5-dimensional exceptional case related to Birkhoff’s theorem is covered by a non-trivial limit of space-times, [20], [21], [22], [23], and [24].

The generalization of this theorem to fourth-order gravity is subject of the following references: [25]: here it is shown that the Birkhoff theorem is not valid in a fourth order theory of gravitation where \(L = R^2\), and that in this theory, the Newtonian limit is not well-behaved; only as a side-remark

\(^4\)An Einstein space is a space whose trace-free part of the Ricci tensor vanishes.
he mentions the possibility to use instead $L = R + l^2 \cdot R^2$ and comments this to be arbitrary and being only an unwarranted complication of the theory, [26]: here an example of a fourth order theory of gravitation is presented, where the Birkhoff theorem is valid, this is done by a Lagrangian, which coincides with $L = R$ in all those cases, where two of the eigenvalues of the Ricci tensor coincide, [27], [28], [29], [30], [31], [32]: here it is outlined that for the 3-dimensional case, i.e. for $k + n = 3$, the most general spherically symmetric metric cannot be presented in the form of metric (1.3); however, in the present paper we restrict to spaces of the form (1.3) from the beginning, [33]: here the Birkhoff theorem for Lovelock gravity is proven, and in [34], a minor error of that paper is corrected, [35], [36], [37], [38], [39], [40], [41], [42], [43], [44], [45], [46], [47], [48], [49], [50], [51], and [52]. The special case of the conformally invariant Weyl theory has been dealt with in [53], [54], [55], [56], [57], [58], [59], [60], [61], [62], [63], [64], [65], and [66].

The relation of the Birkhoff theorem to two-dimensional space-times is worked out in [67], [68], [69], [70], [71], [72], [73], [74], [75], [76], [77], [78], [79], [80], [81], [82], and [83]. For related work on black holes in Palatini gravity see e.g. [84]. Several variants of Birkhoff-type theorems including those in higher dimensions and those including many different types of matter fields are presented in [85]. The inverted Birkhoff theorem is the subject of references [86] and [87].

2 Arbitrary dimension of the spaces

Let us now return to the question posed in section 1, and discuss the different cases. The first two ones are trivial, we mention them only for completeness.

First case: $k = 0$, so with eq. (1.3) we have $ds^2 = r^2 d\Omega^2$ with a constant $r$, as $d\sigma^2$ represents a one-point set only. Hence, $D = n(n + 1)/2$, and so this case is not possible.

Second case: $k > 0$ and $n = 0$, with eq. (1.3) we have $ds^2 = d\sigma^2$, and the scalar $r$ does not enter the equations. For $k = 1$ we get $D = 1 > 0,$
so this case is possible, it expresses the well-known fact that 1-dimensional Riemannian spaces are always locally homogeneous Einstein spaces. For \( k = 3 \) we get \( D = 6 > 0 \), so this case is possible, too, it expresses the fact that 3-dimensional Einstein spaces are always locally of constant curvature. For all other values \( k \) however, and every signature, Einstein spaces without local isometries exist, so these cases are not possible. In sum up to now: If \( k \cdot n = 0 \), then exactly the type(1,0)-Birkhoff theorem and the type(3,0)-Birkhoff theorem are valid. From now on we will assume \( k > 0 \) and \( n > 0 \).

Third case: \( k = 1 \). For \( n = 1 \) the essential metric form is \( ds^2 = dx^2 + r^2(x)dy^2 \) which is always an Einstein space, but has typically only \( D = 1 \), so the type(1,1)-Birkhoff theorem is not valid. For \( n = 2 \), we again apply the fact that 3-dimensional Einstein spaces are locally of constant curvature, i.e. the type(1,2)-Birkhoff theorem is valid. For \( n \geq 3 \), there always exist Ricci-flat spaces of the required form with only one Killing vector, so no Birkhoff theorem of one of these types is valid.

Fourth case: \( k = 2 \). As is generally known, the type(2,\( n \))-Birkhoff theorems are valid for every \( n \geq 1 \). For a similar and in some respect more general approach in the context of multidimensional gravity see [85].

Fifth case: \( k \geq 3 \). This case cannot be adequately dealt by such general considerations, so we must go deeper into the details; this we will do in the next sections.

3 General warped product

We start with metric (1.3), representing a warped product with warping function \( r^2 \) equipped with coordinates \( x^A \), where \( A, B = 1, \ldots, N \)

\[
ds^2 = d\sigma^2 + r^2 d\Omega^2 = g_{AB} dx^A dx^B. \tag{3.1}
\]

With \( N = k + n \) and \( i, j = 1, \ldots, k \) we assume both \( r \) and \( g_{ij} \) to depend on the \( x^i \) only, and

\[
d\sigma^2 = g_{ij} dx^i dx^j, \quad r = e^\rho, \quad r \geq 0. \tag{3.2}
\]
Consequently, we get for the other part of the metric

\[ r^2d\Omega^2 = g_{\alpha\beta}dx^\alpha dx^\beta, \tag{3.3} \]

where \( \alpha, \beta = k + 1, \ldots, N \).

Now we perform a conformal transformation as follows

\[ ds^2 = e^{-2\varrho}ds^2 = h_{AB}dx^A dx^B. \tag{3.4} \]

Therefore,

\[ h_{AB} = e^{-2\varrho}g_{AB}, \quad d\Omega^2 = h_{\alpha\beta}dx^\alpha dx^\beta \tag{3.5} \]

and the \( h_{\alpha\beta} \) shall depend on the \( x^\alpha \) only. Consequently,

\[ d\sigma^2 = e^{2\varrho}h_{ij}dx^i dx^j, \tag{3.6} \]

and

\[ ds^2 = e^{-2\varrho}d\sigma^2 + d\Omega^2 \tag{3.7} \]

represents a direct product, so its Ricci tensor \( P_{AB} \) has a block structure composed from \( P_{ij} \) and \( P_{\alpha\beta} \), whereas all values \( P_{i\alpha} \) identically vanish. Let

\[ P = h^{AB}P_{AB} \quad \text{and} \quad Q = h^{\alpha\beta}P_{\alpha\beta} \tag{3.8} \]

that means, \( P \) is the curvature scalar for \( d\hat{s}^2 \) and \( Q \) is the curvature scalar for \( d\Omega^2 \). Let \( d \) be the dimension of the local isometry group of \( d\Omega^2 \). This implies that \( d \leq n(n+1)/2 \) with equality taking place only for spaces \( d\Omega^2 \) being locally of constant curvature.

Now, for a given \( d\Omega^2 \) but unspecified \( r \) and \( d\sigma^2 \) we request \( ds^2 \) to be an Einstein space. Let \( D \) be the dimension of the local isometry group of \( ds^2 \). For those cases where we get \( D > d \), we have the validity of a Birkhoff-type theorem.\(^5\)

The case \( N = 0 \) is trivial: for this case, we have \( D = d = 0 \), and no Birkhoff theorem is valid. So we assume \( N \geq 1 \) in the following. Let \( R_{AB} \)

\(^5\)Of course, \( D \geq d \) follows already from the assumptions, so the main point is, that for a special metric ansatz, the validity of Einstein’s vacuum equation with \( \Lambda \)-term shall imply the existence of at least one further isometry.
be the Ricci tensor of $ds^2$ and $R = g^{AB} R_{AB}$ the related scalar. According to our request we have

$$R_{AB} = \frac{R}{N} g_{AB}. \quad (3.9)$$

The case $N = 1$ can now be solved: eq. (3.9) is no additional requirement, as $R = 0$ and $R_{AB} = 0$ anyhow. This implies $D = 1$, because every 1-dimensional Riemannian space has a translational isometry, at least locally. Further, for $n = 1$ we have $d = 1$, and no Birkhoff theorem holds. For $n = 0$ we have $d = 0$, and a Birkhoff theorem is formally valid, but it carries no more information, than the well-known fact, that every one-dimensional Riemannian space has an isometry. The case $N = 2$ is like-wise trivial: every two-dimensional pseudo-Riemannian space is an Einstein space, so requesting it is an empty requirement, and cannot increase the dimension of the isometry group. So, for $N = 2$ no Birkhoff theorem holds.

Let us assume $N \geq 3$ in the following, this implies $R$ to be constant, see eq. (3.9). The conformal transformation eqs. (3.4)/(3.5) has the following consequence for the related Ricci tensors:

$$P_{AB} = R_{AB} + (N - 2) (\varrho_{;AB} + \varrho_{,A} \varrho_{,B}) + g_{AB} \left( \square \varrho - (N - 2) \varrho^{C} \varrho_{,C} \right), \quad (3.10)$$

where indices are moved and covariant derivatives $f_{;AB}$ are calculated with the metric $g_{AB}$, and the D’Alembertian $\square$ is defined by $\square f = g^{AB} f_{,AB}$ for any scalar $f$. Of course, $f_{,A}$ is identical to the partial derivative $f_{,A}$.

Transvecting eq. (3.10) with $g^{AB}$ we get

$$P \cdot e^{-2\varrho} = R + (N - 1) \left( 2 \varrho - (N - 2) \varrho^{C} \varrho_{,C} \right). \quad (3.11)$$

To calculate $\varrho_{,AB}$, we need the components of the Christoffel affinity. To this end we rewrite eq. (3.1) as follows:

$$ds^2 = g_{ij} dx^i dx^j + r^2 h_{\alpha\beta} dx^\alpha dx^\beta \quad (3.12)$$

with $g_{ij}$ and $r$ depending on $x^i$ only, whereas the $h_{\alpha\beta}$ depend on the $x^\alpha$ only. It is clear from the above, that we have also

$$r^2 h_{\alpha\beta} = e^{2\varrho} h_{\alpha\beta} = g_{\alpha\beta}.$$
The components $\Gamma^i_{jl}$ of the Christoffel affinity represent both the complete Christoffel affinity for the space $d\sigma^2$ as well as those components with indices all $\leq k$ of the Christoffel affinity for the space $ds^2$. In our case, both interpretations lead to the same values, so we need not distinguish the notation here.

The same takes place with the components $\Gamma^\alpha_{\beta\gamma}$ of the Christoffel affinity: in the following three spaces their value is always the same: for $d\Omega^2$, for $r^2d\Omega^2$, and for the components with indices all $> k$ of the Christoffel affinity for the space $ds^2$.

The only non-trivial influence of an allowed non-constancy of the warping factor $r^2 = e^{2\varrho}$ is via the following components of the Christoffel affinity for the space $ds^2$:

$$\Gamma^\alpha_{\beta i} = \delta^\alpha_\beta \varrho_i \quad \text{and} \quad \Gamma^i_{\alpha\beta} = -g_{\alpha\beta} \varrho^i .$$

(3.13)

Now we are ready to calculate the needed components of $\varrho_{,AB}$: all the mixed components $\varrho_{,ai}$ vanish, the components $\varrho_{,ij}$ can be calculated if they were simply within $d\sigma^2$, and the only non-trivial part is

$$\varrho_{,\alpha\beta} = g_{\alpha\beta} \varrho^i \varrho_{,i}.$$  

(3.14)

Denoting the D'Alembertian within $d\sigma^2$ by $\Delta$, i.e. $\Delta \varrho = g^{ij} \varrho_{,ij}$ we get

$$\Box \varrho = \Delta \varrho + n \cdot \varrho^i \varrho_{,i}.$$  

(3.15)

By construction, see eq. (3.8), $Q$ depends on the $x^\alpha$ only, and $P - Q$, representing the curvature scalar for $e^{-2\varrho}d\sigma^2$, depends on the $x^i$ only. Now we are ready to evaluate eq. (3.10) in more details: Inspection of the mixed components implies that $R_{ai} = 0$ identically. So, we may split eq. (3.10) in the $\alpha\beta$-block and the $ij$-block. So we get

$$P_{\alpha\beta} = R_{\alpha\beta} + g_{\alpha\beta} \Box \varrho.$$  

(3.16)

and

$$P_{ij} = R_{ij} + (N - 2) (\varrho_{,ij} + \varrho_{,i} \varrho_{,j}) + g_{ij} \left( \Box \varrho - (N - 2) \varrho^l \varrho_{,l} \right).$$

(3.17)
Inserting eq. (3.9) into these two equations we get
\[ P_{\alpha\beta} = g_{\alpha\beta} \left( \Box \varrho + \frac{R}{N} \right) \]  \hfill (3.18)
and
\[ P_{ij} = (N - 2) (\varrho_{;ij} + \varrho_{;i} \varrho_{;j}) + g_{ij} \left( \Box \varrho + \frac{R}{N} - (N - 2) \varrho^{l} \varrho_{;l} \right). \]  \hfill (3.19)
Transvecting eq. (3.18) with \( h^{\alpha\beta} \) we get with eq. (3.8)
\[ Q = ne^{2\varrho} \left( \Box \varrho + \frac{R}{N} \right). \]  \hfill (3.20)
Transvecting eq. (3.19) with \( h^{ij} \) we get with eq. (3.8)
\[ P - Q = e^{2\varrho} \left( (2N - n - 2) \Box \varrho + \frac{R}{N} (N - n) - (N - 1) (N - 2) \varrho^{i} \varrho_{;i} \right). \]  \hfill (3.21)
Cross-checking of eqs. (3.20)/(3.21) with eq. (3.11) shows that these 3 equations are compatible.

By construction, the following terms depend on the \( x^{i} \) only: \( \varrho, \varrho^{i} \varrho_{;i}, \Delta \varrho, \Box \varrho, P - Q, g_{ij}, h_{ij}, P_{ij}, \text{ and } R_{ij} \). Likewise by construction, the following terms depend on the \( x^{\alpha} \) only: \( Q, P_{\alpha\beta}, \text{ and } h_{\alpha\beta} \).

We get the result: the l.h.s. of eq. (3.20) depends on the \( x^{\alpha} \) only, and its r.h.s. depends on the \( x^{i} \) only. Consequently, \( Q \) is a constant.

### 4 Coordinate-free proof of Birkhoff’s theorem

Now we restrict to the main case, \( k = 2 \), i.e. \( N = n + 2 \). The relevant equations from section 3 then lead to the following simplifications: From eq. (3.9) we get
\[ R_{AB} = \frac{R}{n + 2} g_{AB} \quad R = \text{const.} \]  \hfill (4.1)
Eq. (3.18) now reads
\[ P_{\alpha\beta} = g_{\alpha\beta} \left( \Box \varrho + \frac{R}{n + 2} \right). \]  \hfill (4.2)
from eq. (3.19) we get

\[ P_{ij} = n (\varrho_{ij} + \varrho_{i} \varrho_{j}) + g_{ij} \left( \Box \varrho + \frac{R}{n + 2} - n \varrho^{i} \varrho_{i} \right). \quad (4.3) \]

From eq. (3.20) we get

\[ Q = n e^{2\varrho} \left( \Box \varrho + \frac{R}{n + 2} \right) = \text{const.} \quad (4.4) \]

From eq. (3.21) we get

\[ P - Q = e^{2\varrho} \left( (n + 2) \Box \varrho + \frac{2R}{n + 2} - n(n + 1) \varrho^{i} \varrho_{i} \right). \quad (4.5) \]

At this place it proves useful to re-insert \( r = e^\varrho \) instead of \( \varrho \) into the equations: \( \varrho = \ln r \), and similarly we get with eq. (3.15)

\[ \varrho^{i} \varrho_{i} = \frac{1}{r^2} \cdot r^{i} r_{i}; \quad \Box \varrho = \frac{\Delta r}{r} + \frac{n - 1}{r^2} \cdot r^{i} r_{i}. \]

then eqs. (4.2) - (4.5) read

\[ P_{\alpha\beta} = g_{\alpha\beta} \left( \frac{\Delta r}{r} + \frac{n - 1}{r^2} \cdot r^{i} r_{i} + \frac{R}{n + 2} \right), \quad (4.6) \]

\[ P_{ij} = \frac{n}{r} \cdot r_{;ij} + g_{ij} \left( \frac{\Delta r}{r} - \frac{1}{r^2} \cdot r^{i} r_{;i} + \frac{R}{n + 2} \right). \quad (4.7) \]

\[ Q = n \left( r \Delta r + (n - 1) \cdot r^{i} r_{;i} + \frac{r^2 \cdot R}{n + 2} \right) = \text{const.} \quad (4.8) \]

\[ P - Q = (n + 2) r \Delta r + \frac{2r^2 R}{n + 2} - 2r^{i} r_{;i}. \quad (4.9) \]

What can we directly see here is the following: if we insert \( g_{\alpha\beta} = r^2 \cdot h_{\alpha\beta} \) into eqs. (4.6) and (4.8) we get

\[ P_{\alpha\beta} = \frac{Q}{n} \cdot h_{\alpha\beta}, \quad Q = \text{const.} \quad (4.10) \]

That means, \( d\Omega^2 \) is an Einstein space with constant curvature scalar. It is essential to point out that we have not assumed \( d\Omega^2 \) to be an Einstein space, or even a space of constant curvature, but moreover, it follows from the other assumptions; of course, the constancy of \( Q \) is a non-trivial extra property for \( n = 2 \) only.
The antisymmetric Levi-Civita pseudo-tensor $\varepsilon_{ij}$ in $d\sigma^2$ is completely defined by $\varepsilon_{12} = \sqrt{|\det g_{ij}|}$. It is covariantly constant. We now define the pseudo-vector $\xi^i$ via

$$\xi_i = \varepsilon_{ij} r^j.$$  \hspace{1cm} (4.11)

Here is the most relevant point of the deduction: In two-dimensional spaces, the two eigenvalues of the Ricci tensor coincide.\(^6\) Therefore, $P_{ij}$ is proportional to $g_{ij}$, and with eq. (4.7) we see that this also takes place for $r_{;ij}$. So we insert $r_{;ij} = c \cdot g_{ij}$ with a scalar $c$ into eq. (4.11) and get finally $\xi_{;ij} + \xi_{;ji} = 0$. Hence, $\xi_i$ is a Killing vector.\(^7\)

## 5 Discussion

Now we can summarize the results in the following

**Generalized Birkhoff Theorem:** Let the warped product

$$ds^2 = d\sigma^2 + r^2 d\Omega^2$$  \hspace{1cm} (5.1)

be an Einstein space, where $d\sigma^2$ is two-dimensional with coordinates $x^i$, and $d\Omega^2$ is $n$-dimensional with $n \geq 1$. The warping factor $r^2$ depends on the $x^i$ only. Then it holds: $\xi_i = \varepsilon_{ij} r^j$ represents a hypersurface-orthogonal Killing vector for $ds^2$. Hence, the dimension of the isometry group of $ds^2$ is larger than the dimension of the isometry group of $d\Omega^2$.

**Proof:** That $\xi_i$ is a Killing vector in $d\sigma^2$ was already deduced earlier, and in 2 dimensions, every vector is hypersurface-orthogonal anyhow. That both properties are maintained if $\xi_i$ is lifted to $ds^2$ becomes clear from the construction. If $r_{;i}$ vanishes on a hypersurface only, then so does $\xi_i$, but this does not prevent $\xi_i$ to induce an isometry, as it remains non-zero in a dense subset of the manifold. If $r_{;i}$ is a non-vanishing light-like vector in a whole region, then $d\sigma^2$ is flat, see the first paper in [67], sc. V A, so 3 Killing

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\(^6\)And, by the way, just equal the Gaussian curvature of the surface.

\(^7\)Of course, formally it is a pseudo-vector only, but replacement of $\xi_i$ by $-\xi_i$ does not alter the Killing equation, so we may keep the word Killing vector.
vectors appear. It still remains to look for the case, that \( r_{ij} = 0 \) in a whole region. Then with eq. (4.7) we get \( P_{ij} = g_{ij} R / (n + 2) \) with constant \( R \), so \( d\sigma^2 \) must be a space of constant curvature, it possesses 3 independent Killing vectors. q.e.d.

**Summary of results:** For the metric (1.3), reading \( ds^2 = d\sigma^2 + r^2 d\Omega^2 \), where \( d\sigma^2 \) and \( d\Omega^2 \) are pseudo-Riemannian manifolds of dimension \( k \geq 0 \) and \( n \geq 0 \) respectively and arbitrary signature, and where \( r \) lives on \( d\sigma^2 \), we prescribe \( d\Omega^2 \) with a \( d \)-dimensional isometry group. Concerning \( r \) and \( d\sigma^2 \) we only require that \( ds^2 \) is an Einstein space. Let \( D \) be the dimension of the isometry group of \( ds^2 \). Then the type\((k,n)\)-Birkhoff theorem states that \( D \geq d + 1 \).

\[
(1,0), \ (3,0), \ (1,2), \ (2,n), \ n \geq 1
\]

**Table 1:** Values of \((k,n)\), where the type\((k,n)\)-Birkhoff theorem is valid

\[
(0,n), \ n \geq 0, \ (2,0), \ (k,0), \ k \geq 4, \ (1,1), \ (1,n), \ n \geq 3, \ (k,n), \ k \geq 3
\]

**Table 2:** Values of \((k,n)\), where the type\((k,n)\)-Birkhoff theorem is *not* valid

**Comments:** 1. As a byproduct we have shown that under the conditions of the theorem, \( d\Omega^2 \) turns out to be an Einstein space with constant curvature scalar, a property, which is presupposed in many other presentations.

2. No similar consideration is possible for dimensions \( k \geq 3 \), as for spaces of dimension \( \geq 3 \) the Ricci tensor may have \( k \) different eigenvalues. Therefore, no generalization of the Birkhoff theorem into this direction is to be expected.
3. Differently from other proofs, we did not introduce any coordinates. Besides aesthetic reasons, this approach has the great advantage, that no special care is needed to deal with the horizon. Let us make this point more detailed: In regions, where \( r; i \) is different from zero, one could be tempted to use \( r \) as one coordinate, and to define the other coordinate, denoted by \( t \), by the condition: the \( t \)-lines shall always be perpendicular to the \( r \)-lines. But then immediately it becomes clear, that for light-like values \( r; i \), the coordinate \( t \) is not well-defined.\(^8\)

4. To check the applicability of the presented formulas\(^9\), let us try to deduce the higher-dimensional Schwarzschild-de Sitter solution eq. (1.2) using the presented approach. Now it is indeed worthwhile to introduce the coordinates \( r \) and \( t \) for \( ds^2 \) as described in the previous comment, knowing that we now do not cover those points of the manifold, where \( r; i \) changes from space-like to time-like.\(^10\) By construction, \( g_{12} = 0 \), and the other components \( g_{ij} \) depend on \( r \) only. Restricting now to time-like \( x^1 = t \) and space-like \( x^2 = r \) only, we can now write

\[
d\sigma^2 = -A(r)dt^2 + \frac{dr^2}{B(r)}
\]

with positive functions \( A(r) \) and \( B(r) \). Now we skip the standard argument that shows that putting \( A(r) = B(r) \) does not restrict generality in this context. So we use eq. (5.1) with

\[
d\sigma^2 = -A(r)dt^2 + \frac{dr^2}{A(r)} \tag{5.2}
\]

and \( d\Omega^2 \) being the metric of the standard sphere \( S^n \), and we restrict to the case \( n \geq 2 \). With a dash denoting the derivative with respect to \( r \) we get

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\(^8\)In more details: That the \( t \)-lines are perpendicular to the \( r \)-lines can be expressed by the condition \( t^i r; i = 0 \), but if \( r; i \) is a non-vanishing light-like vector, then \( t^i \) must be parallel to \( r^i \), so the coordinates \( r, t \) fail to be independent ones.

\(^9\)and also to increase the confidence in their correctness

\(^10\)Of course, these are the same points of the manifold, where \( \xi_i \) changes from time-like to space-like, i.e., the points of the horizon.
\[ \Delta r = A'(r) \text{ and } r_i r^{;i} = A(r). \]

With eq. (4.8) we get then

\[ \frac{Q}{n} = r A'(r) + (n - 1) A(r) + \frac{r^2 R}{n + 2}. \]

To solve this equation it proves useful to define the function \( F(r) = r^{n-1} A(r) \).

We simply get

\[ F'(r) = \frac{Q}{n} \cdot r^{n-2} - \frac{r^n R}{n + 2} \]

which can be integrated to

\[ F(r) = c_1 + Q_1 r^{n-1} - R_1 r^{n+1} \]

with constants \( c_1, Q_1 \) and \( R_1 \), where \( Q \) and \( Q_1 \) have the same sign, and \( R \) and \( R_1 \) have the same sign. Finally we get

\[ A(r) = Q_1 + \frac{c_1}{r^{n-1}} - R_1 r^2. \quad (5.3) \]

Comparison with eq. (1.2) clearly shows the physical interpretation of the three constants in metric (5.1) with (5.2) and (5.3).

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