Critical Points of the Multiplier Map for the Quadratic Family

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\textbf{ABSTRACT}

The multiplier $\lambda_n$ of a periodic orbit of period $n$ can be viewed as a (multiple-valued) algebraic function on the space of all complex quadratic polynomials $p_c(z) = z^2 + c$. We provide a numerical algorithm for computing critical points of this function (i.e., points where the derivative of the multiplier with respect to the complex parameter $c$ vanishes). We use this algorithm to compute critical points of $\lambda_n$ up to period $n = 10$.

\textbf{KEYWORDS}

quadratic family; multipliers of periodic orbits; Mandelbrot set; critical points of the multipliers

1. Introduction

It has been known since the works of Fatou and Julia that multipliers of periodic orbits can carry not only local, but also global information about the holomorphic dynamical system at hand. In [Milnor 93] J. Milnor used the multipliers of the fixed points to parameterize the moduli space of degree 2 rational maps. Using this parameterization, he proved that this moduli space is isomorphic to $\mathbb{C}^2$. In the attempt to generalize this approach, it was observed by the second author [Gorbovickis 16] that the multipliers of any $m - 1$ distinct periodic orbits provide a local parameterization of the moduli space of degree $m$ polynomials in a neighborhood of its generic point. It is then a natural question to describe the set of polynomials at which this local parameterization fails, that is, to describe the set of all critical points of the multiplier map, defined as the map which assigns to each degree $m$ polynomial the $(m-1)$-tuple of multipliers at the chosen periodic orbits. The goal of the current article is to collect numerical data for this problem in the most basic case $m = 2$, i.e., the case of the quadratic family

$$p_c(z) = z^2 + c.$$

Even in this case the general problem seems to be quite complicated.

In the current article, we provide a numerical algorithm that computes critical points of the multiplier map on the space of quadratic polynomials $p_c$. More specifically, given $n \in \mathbb{N}$, the algorithm finds the values of the parameter $c$ for which the map $p_c$ has a periodic orbit of period $n$ whose multiplier has a vanishing derivative (when viewed as a locally analytic function of $c$). Using this algorithm, we compute critical points of the multiplier map together with the corresponding periodic orbits, for periods up to $n = 10$. In particular, we find a complete list of all critical points of the multiplier map, for periods up to $n = 8$.

Last but not least, let us mention another important motivation for the current study—the connection between the critical points of the multiplier map and the hyperbolic components of the famous Mandelbrot set. The argument of quasiconformal surgery implies that appropriate inverse branches of the multiplier map are Riemann mappings\textsuperscript{1} of the hyperbolic components [Milnor 12]. Possible existence of analytic extensions of these Riemann mappings to larger domains might allow to estimate the geometry of the hyperbolic components [Levin 09, 11] and, in turn, might shed light on one of the central questions in one-dimensional holomorphic dynamics: the question whether the Mandelbrot set is locally connected.

2. Notation and terminology

Let $p_c(z) = z^2 + c$ and denote its $n$th iteration by $p_c^n(z)$.

A point $z$ is a periodic point of $p_c$ if there exists a positive integer $n$ such that $p_c^n(z) = z$. The minimal such $n$ is called the period of $z$.

\textsuperscript{1}A Riemann mapping of a simply connected domain is a conformal diffeomorphism of the unit disk onto that domain.

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Given $n$, let the period $n$ curve $\text{Per}_n \subset \mathbb{C} \times \mathbb{C}$ be the closure of the locus of points $(c, z)$ such that $z$ is a periodic point of $p_c$ of period $n$ (see [Milnor 00] for more details). Observe that each pair $(c, z) \in \text{Per}_n$ determines a periodic orbit

$$z = z_0 \mapsto z_1 \mapsto \cdots \mapsto z_n = z_0.$$  

Let $\mathbb{Z}_n$ denote the cyclic group of order $n$. This group acts on $\text{Per}_n$ by cyclically permuting points of the same periodic orbits for each fixed value of $c$. Then the factor space $\text{Per}_n/\mathbb{Z}_n$ consists of pairs $(c, O)$ such that $O$ is a periodic orbit of $p_c$. Note that according to [Milnor 00], the space $\text{Per}_n/\mathbb{Z}_n$ (as well as $\text{Per}_n$) has a structure of a smooth algebraic curve. (Note that there is a natural projection from $\text{Per}_n$ to $\text{Per}_n/\mathbb{Z}_n$.)

Let $\lambda_n : \text{Per}_n \rightarrow \mathbb{C}$ be the map defined by

$$\lambda_n : (c, z) \mapsto \frac{\partial p_n}{\partial z}(z) = 2^n z_1 \cdots z_n.$$  

Observe that for all regular points of the projection $(c, z) \mapsto c$, the value $\lambda_n(c, z)$ is the multiplier of the periodic point $z$. Furthermore, if $z_1$ and $z_2$ belong to the same periodic orbit of $p_c$ then $\lambda_n(c, z_1) = \lambda_n(c, z_2)$; hence the map $\lambda_n$ projects to a well-defined map $\tilde{\lambda}_n : \text{Per}_n/\mathbb{Z}_n \rightarrow \mathbb{C}$ that assigns to each pair $(c, O)$ the multiplier of the periodic orbit $O$.

Both $\lambda_n$ and $\tilde{\lambda}_n$ are proper algebraic maps (c.f. [Milnor 00]). The goal of this work is to study (compute) critical points of the multiplier map $\lambda_n$.

### 3. Algorithm for computing critical points of the multiplier map

#### 3.1. Computing derivatives

Observe that all points $(c, z) \in \text{Per}_n$ satisfy the following equation

$$p_n^{\text{on}}(z) = z. \quad (3-1)$$  

Together with the Implicit Function Theorem this implies that the parameter $c$ can serve as a local chart on $\text{Per}_n$ at all points $(c, z) \in \text{Per}_n$ such that $\lambda_n(c, z) \neq 1$. Hence, in a neighborhood of any such point, one can implicitly define a map $z(c)$, so that $(c, z(c)) \in \text{Per}_n$ for all nearby values of $c$. Then one can express the multiplier map $\lambda_n$ in the above local chart as

$$\lambda_n(c) = \tilde{\lambda}_n(c, z(c)).$$  

According to Lemma 4.5 in [Milnor 00], if $\tilde{\lambda}_n(c, z) = 1$, then $(c, z)$ cannot be a critical point of the multiplier map $\lambda_n$. Thus, to study all critical points of this map, it is sufficient to work in local charts associated with the parameter $c$ (i.e., the critical points of the multiplier map $\lambda_n$ correspond to those points $(c, z) \in \text{Per}_n$ in a neighborhood of which the map $\lambda_n(c) = \lambda_n(c, z(c))$ is defined and $\lambda_n(c) = 0$).

We will use the following notation for the partial derivatives

$$\frac{\partial p_n^{\text{on}}}{\partial c} := \frac{\partial f_n}{\partial c},$$  

where $f_n(c, z) = p_n^{\text{on}}(z)$. By differentiating both left and right sides of (3–1) we get

$$\frac{dz}{dc} = \frac{\partial p_n^{\text{on}}}{\partial c}(z) + \frac{\partial p_n^{\text{on}}}{\partial z}(z) \frac{dz}{dc}.$$  

Therefore

$$z' = \frac{dz}{dc} = \frac{\partial p_n^{\text{on}}}{\partial c}(z) \left(1 - \frac{\partial p_n^{\text{on}}}{\partial z}(z)\right)^{-1} = \frac{\partial p_n^{\text{on}}}{\partial c}(z)(1 - \lambda_n(c))^{-1}. \quad (3-2)$$  

Observe that $\frac{\partial p_n^{\text{on}}}{\partial c}(z)$ satisfies the recurrence relation

$$\frac{\partial p_n^{\text{on}}}{\partial c}(z) = \frac{\partial}{\partial z} p_n(p_n^{\text{on}}(z)) = 1 + 2 \cdot p_n^{\text{on}}(z) \frac{\partial p_n^{\text{on}}}{\partial c}(z).$$

We therefore get an expression for the derivative of the multiplier map

$$\frac{dz}{dc} = d\lambda_n(c) = \frac{\partial p_n^{\text{on}}}{\partial c} + z' \frac{\partial p_n^{\text{on}}}{\partial z},$$  

where for $i = 1, \ldots, n - 1$, we denote

$$\frac{dp_i^{\text{on}}}{dc} = \frac{\partial p_i^{\text{on}}}{\partial c} + z' \frac{\partial p_i^{\text{on}}}{\partial z}.$$  

Finally, to find the critical points of the multiplier map $\tilde{\lambda}_n$, we combine (3–1), (3–2) and (3–3) into the following system of three algebraic equations

$$\begin{align*}
    p_n^{\text{on}}(z) - z &= 0, \\
    z' - \frac{\partial p_n^{\text{on}}}{\partial c}(z) \left(1 - \frac{\partial p_n^{\text{on}}}{\partial z}(z)\right)^{-1} &= 0, \\
    \frac{d\lambda_n}{dc} &= 0,
\end{align*} \quad (3-4)$$  

with three unknowns $c, z, z'$. Any critical point of the multiplier map $\tilde{\lambda}_n$ corresponds to a solution of the
above system. Thus, the problem of finding critical points of the map \( \hat{\lambda}_n \) can be reduced to the problem of solving the above system.

### 3.2. The number of critical points of the multiplier map \( \lambda_n \)

A question which arises naturally when using numerical methods (such as Newton method) is how to ensure that all solutions are found. In this section we derive an upper bound for the number of critical points of the multiplier map which will be later used in the numerical algorithm.

Let \( \nu(n) \) be the number of periodic points of \( p_c \) of period \( n \) for a generic value of \( c \). One can observe that the numbers \( \nu(n) \) satisfy the recursive relation

\[
\nu(1) = 2 \quad \text{and} \quad \nu(n) = 2^n - \sum_{m \text{ divides } n, \ m \neq n} \nu(m).
\]

In particular, this implies that \( \nu(n) \sim 2^n \) as \( n \to \infty \).

It was shown in [Milnor 00] that \( c \) can be used as a local uniformizing parameter at any non-parabolic point of \( \text{Per}_n/\mathbb{Z}_n \) (i.e., where \( \lambda_n(c, \mathcal{O}) \neq 1 \)).

Moreover, the projection map \( \pi_n : \text{Per}_n/\mathbb{Z}_n \to \mathbb{C} \), defined as

\[ \pi_n : (c, \mathcal{O}) \mapsto c, \]

is proper of degree \( \deg \pi_n = \nu(n)/n \). Conversely, in a neighborhood of a point \((c, \mathcal{O})\) with \( \lambda_n(c, \mathcal{O}) = 1 \), the multiplier \( \lambda_n \) serves as a local uniformizing parameter for the curve \( \text{Per}_n/\mathbb{Z}_n \) and \( \lambda_n : \text{Per}_n/\mathbb{Z}_n \to \mathbb{C} \) is a proper map of degree \( \deg \lambda_n = \nu(n)/2 \).

To derive the number of critical points of \( \lambda_n \) recall the Riemann-Hurwitz formula (e.g., [Hamal Hubbard 06]). Let \( X, Y \) be Riemann surfaces, where \( X \) is connected with finite-dimensional homology, and \( \chi(X), \chi(Y) \) denote the corresponding Euler characteristics. Suppose \( f : Y \to X \) is a proper analytic map of degree \( \deg f \) with finitely many critical values. Then \( f \) has finitely many critical points and

\[
\chi(Y) = \deg f \cdot \chi(X) - \sum_{y \in Y} (\deg_y f - 1), \tag{3.5}
\]

where \( \deg_y f \geq 1 \) is called a ramification index (or a local degree) of \( f \) at \( y \) and is defined in the following way. There exists an open neighborhood \( U \subset Y \) of \( y \), such that \( x = f(y) \) has only one preimage in \( U \), i.e., \( f^{-1}(x) \cap U = \{y\} \), and for all other points \( \hat{x} \in f(U) \) the number of preimages is \( \deg_y f \). Observe that \( \deg_y f \neq 1 \) only at the critical points of \( f \) and hence the sum in (3.5) is finite.

Denote the number of (finite) critical points of \( \pi_n \) and \( \lambda_n \) by \( N_{\pi_n} \) and \( N_{\lambda_n} \) respectively. Let \( Y \) be the Riemann surface obtained from \( \text{Per}_n/\mathbb{Z}_n \) by smooth compactification (i.e., compactification in \( \mathbb{C}P^1 \), possibly followed by resolution of singularities at infinity).

Then \( Y = \text{Per}_n/\mathbb{Z}_n \cup Z \), where \( Z \) is a finite set of points at infinity.

We continuously extend \( \pi_n \) to the map \( \pi_n : Y \to \mathbb{C}P^1 \) of the whole surface \( Y \) by setting \( \pi_n(z) = \infty \), for all \( z \in Z \).

To continuously extend the multiplier map in the similar way we need the following proposition.

**Proposition 3.1.** The following relation holds:

\[
\lim_{(c, \mathcal{O}) \in \text{Per}_n/\mathbb{Z}_n, \ c \to \infty} \lambda_n(c, \mathcal{O}) = \infty.
\]

**Proof.** Assume that \( c \neq 0 \) and denote by \( \mathbb{D}_R := \mathbb{D}_R(0) \) the disc of radius \( R = |c|/10 \). Then for any \( z \in \mathbb{C} \setminus \mathbb{D}_R \) we have

\[
|p_c(z)| = |z^2 + c| > R|z| - |c| > (R-10)|z|.
\]

If \( |c| \) is sufficiently large, then \( R-10 > 2 \), and the above inequality implies that the orbit of any point \( z \in \mathbb{C} \setminus \mathbb{D}_R \) converges to \( \infty \) under the dynamics of the map \( p_c \). In particular, this means that all periodic points of the map \( p_c \) lie in the disc \( \mathbb{D}_R \).

We now consider the disc \( \mathbb{D}_r := \mathbb{D}_r(0) \) of radius \( r = \frac{1}{2} \sqrt{|c|} \). Let \( z \in \mathbb{D}_r \), i.e., \( 0 \leq |z| \leq r \). Observe that

\[
|p_c(z)| = |z^2 + c| \geq |c| - |c|/4 > R,
\]

i.e., any point \( z \) from the disc \( \mathbb{D}_r \) is mapped outside of the disc \( \mathbb{D}_R \) under one iteration of the map \( p_c \) and tends to infinity under further iterations, provided that \( |c| \) is sufficiently large. Hence all periodic points of the map \( p_c \) lie inside the annulus \( \mathbb{D}_R \setminus \mathbb{D}_r \).

Since \( R \to \infty \) and \( r \to \infty \) as \( c \to \infty \), for all periodic points \( z \) of \( p_c \) it follows that \( |z| \to \infty \) as \( c \to \infty \).

Recall that the multiplier \( \lambda_n \) of the periodic point \( z \) satisfies

\[
\lambda_n = \frac{\partial p_c^n}{\partial z}(z) = 2^n z_1 \cdots z_n,
\]

where \( z_1, \ldots, z_n \) denotes the points of the orbit of \( z \) under the map \( p_c \). Combining the above observations, it follows that

\[
\lim_{(c, \mathcal{O}) \in \text{Per}_n/\mathbb{Z}_n, \ c \to \infty} \lambda_n(c, \mathcal{O}) = \infty. \quad \square
\]

According to **Proposition 3.1**, we continuously extend \( \lambda_n \) to the map \( \lambda_n : Y \to \mathbb{C}P^1 \) of the whole surface \( Y \) by setting \( \lambda_n(z) = \infty \) for all \( z \in Z \).

For further reference, let us state the following propositions:
**Proposition 3.2.** For any \( y \in \text{Per}_n/\mathbb{Z}_n \), we have \( \deg_y \pi_n \leq 2 \).

**Proof.** If \( y = (c, \mathcal{O}) \) and \( \deg_y \pi_n > 1 \), then by the Implicit Function Theorem we have \( \lambda_n(y) = 1 \), which means that \( \mathcal{O} \) is a parabolic periodic orbit. According to the Fatou-Shishikura inequality (c.f. [Milnor 06]), the ramification index \( \deg_y \pi_n \) is not greater than 1 plus the number of critical points of \( p_c \) lying in the basin of attraction of the orbit \( \mathcal{O} \). Since the polynomial \( p_c \) has only one critical point, the statement of the proposition follows. \( \square \)

**Proposition 3.3.** If \( \lambda_n(c, \mathcal{O}) = 1 \) for some point \((c, \mathcal{O}) \in \text{Per}_n/\mathbb{Z}_n \), then either \( (c, \mathcal{O}) \) is a critical point of \( \pi_n \) or \( \mathcal{O} \) is a periodic orbit of period \( p < n \), with \( n = pr \) for some integer \( r > 1 \), and \( \lambda_p(c, \mathcal{O}) \) is a primitive root of unity of degree \( r \).

**Proof.** The proposition follows from the Fatou-Shishikura inequality in a similar way as Proposition 3.2.

We recall that \( Z \subset Y \) is the inverse image of infinity under the map \( \pi_n \). Applying the Riemann-Hurwitz formula (3–5) to the projection \( \pi_n \) and using Proposition 3.2, we get

\[
\chi(Y) = \deg \pi_n \cdot \chi(\mathbb{CP}^1) - N_{\pi_n} - \sum_{y \in Z} (\deg_y \pi_n - 1).
\]

Therefore, assuming that \( Z \) consists of \( \kappa \) points, we have

\[
\sum_{y \in Z} (\deg_y \pi_n - 1) = \deg \pi_n - \kappa,
\]

and

\[
\chi(Y) = \deg \pi_n - N_{\pi_n} + \kappa.
\]

Observe that if \( (c, \mathcal{O}) \in \text{Per}_n/\mathbb{Z}_n \) is a critical point of the projection \( \pi_n \), then \( \lambda_n(c, \mathcal{O}) = 1 \). Since all points in the closure of the unit disc \( \mathbb{D} \) are regular values of \( \lambda_k \) for all \( k \in \mathbb{N} \), (c.f. [Epstein]), Proposition 3.3 implies the following formula for the number of critical points of \( \pi_n \):

\[
N_{\lambda_n} = \deg \lambda_n - \sum_{\forall r, p \text{ s.t. } n = rp} \deg \lambda_p \cdot \varphi(r),
\]

where \( \varphi(r) \) is the Euler’s function that counts the positive integers up to \( r \) that are relatively prime with \( r \).

Analogous computations for \( \lambda_n \) using the Riemann–Hurwitz formula (3–5) show that

\[
\chi'(Y) \leq \deg \lambda_n \cdot \chi(\mathbb{CP}^1) - N_{\lambda_n} - \sum_{y \in Z} (\deg_y \lambda_n - 1)
\]

\[
= \deg \lambda_n - N_{\lambda_n} + \kappa.
\]

Finally, expressing \( \deg \pi_n \) and \( \deg \lambda_n \) as \( \nu(n)/n \) and \( \nu(n)/2 \), respectively, we get

\[
N_{\lambda_n} \leq \left( \frac{\nu(n)}{n} \right) - \frac{1}{2} \sum_{\forall r, p \text{ s.t. } n = rp} \nu(p) \cdot \varphi(r).
\]

**Remark 3.1.** We note that the above inequality turns into an equality if the critical points of the multiplier map \( \lambda_n \) are counted with their multiplicities.

### 3.3. Algorithm

To solve the system of equations (3–4) we use the Newton method. In addition to the critical points \( c \) of the multiplier map \( \lambda_n \), this allows us to determine the corresponding periodic points \( z \) and its derivatives \( z' \).

We fix the period \( n \). All initial guesses for the Newton method will be randomly chosen on the complex curve defined by the first two equation of (3–4). Since every solution to the system (3–4) lies on this curve, we hope that such initial conditions are more likely to belong to the domains of attraction of the solutions.

More specifically, initial guesses for the Newton method are chosen as follows: we first generate a random guess for the parameter \( c \) and compute all periodic points \( z \) of period \( n \) for the polynomial \( p_c \). To do this, we apply the algorithm of Hubbard, Schleicher, and Sutherland, developed in [Hubbard et al. 01]. For every choice of a periodic point \( z \) obtained this way, we compute the corresponding initial value for \( z' \).
plex conjugation, each solution \( \sigma \) once a solution \( \sigma \).

Remark 3.3. Since system (3–4) from each periodic orbit.

\[ \sigma \]

Remark 3.2. Input: the period \( n \).

0: Set the counter of the critical points of the multiplier map \( k = 0 \), compute the upper bound on \( N_n \) using (3–11).

1: Generate \( c \) randomly, find all \( z \) using the method described in [Hubbard et al. 01], select one \( z \) from each orbit, compute \( z' \). Store triplets of the initial guesses \( c, z, z' \) in the set \( \Sigma_0 \).

2: If \( \Sigma_0 \neq \emptyset \), then take an initial guess from the set \( \Sigma_0 \), remove it from \( \Sigma_0 \) and proceed to Step 3. If the set \( \Sigma_0 \) was empty, return to Step 1.

3: Iterate the 3-dimensional Newton operator applied to the system (3–4) at the initial guess for maximum 50 times. If the Newton method does not converge after 50 iterations with the desired tolerance, return to Step 2.

4: Let \( (c, z, z') \) be the triple obtained by the Newton method at Step 3. Test if the point \( z \) is of period \( n \). If yes, then check that a representative of its orbit is not already stored in the set \( \Sigma_n \) of solutions of (3–4). If at least one of these conditions is not fulfilled, then return to Step 2.

5: If possible, use the methods of [Giusti and Yakoubsohn 13] to determine the multiplicity \( m_r \) of the root/cluster of roots at \( (c, z, z') \). If not possible (see Remark 3.4), then assume that \( m_r = 1 \). Store the triplet \( (c, z, z') \) and its multiplicity \( m_r \) in the set \( \Sigma_n \) of solutions of (3–4), and set \( k = k + m_r \). Next, if \( c \notin \mathbb{R} \) or \( c \in \mathbb{R} \), but the points \( z \) and \( z \) belong to different periodic orbits of \( p_c \), then store the complex conjugate \( (\bar{c}, \bar{z}, \bar{z}') \) and its multiplicity \( m_r \) in the set of solutions \( \Sigma_n \), and increase \( k \) by \( m_r \), again.

6: If \( k < N_n \), return to Step 2.

- Output: the set \( \Sigma_n \).

Remark 3.4. The coefficients of the polynomials in the left-hand side of (3–4) are never computed explicitly in practice as they tend to grow super-exponentially with \( n \). Instead, the values of the left-hand side of (3–4) at particular triplets \( (c, z, z') \) can be computed by evaluating certain expressions along the orbit of \( z \) (see the formulas preceding formula (3–4)). Even though the Jacobian matrix of (3–4) can also be computed at particular triplets \( (c, z, z') \) in a similar way, the formulas for it are rather complicated, so instead of using those formulas, we were approximating the partial derivatives with finite divided differences. Finally, due to the super-exponential growth of the coefficients, it is rather difficult to rigorously determine the multiplicity of a cluster of a root. Because of that, instead of a rigorous approach, we were simply computing the smallest eigenvalue of the Jacobian as a measure of how close it is to being singular. No obvious signs of existence of multiple roots/clusters of roots have been numerically detected in our computations.

### 4. Results of the numerical experiments

The algorithm described above has been implemented in a C++ program. In this section we present the outcome of the numerical experiments. The complete list of critical points of \( \lambda_n \) found by the program can be downloaded from https://www.dropbox.com/sh/nr5847qnhapd8zc/AADCdqv2rOxghrBLGQO47zbcma?dl=0. The tolerance for the Newton’s method has been set up at \( 10^{-10} \).

Note that the multiplier map \( \lambda_n \) does not have critical points for periods \( n = 1, 2 \). We ran the algorithm for several periods \( n = 3, \ldots, 10 \). Table 1 displays the upper bound for the number of critical points \( N_n \) of the multiplier map for each period and the number of critical points computed by the implemented algorithm, i.e., \( \#\Sigma_n \). It is likely that not all critical points have been detected for periods \( n = 9 \) and \( n = 10 \), which can be seen in Table 1. One of the reasons for the missing points might be that the critical points are lying too close to each other and cannot be distinguished using the standard double precision in the computations. However, increasing the precision can

| \( n \) | \( N_n \) | \( \#\Sigma_n \) | \( \text{Outside the Mandelbrot set} \) | \( \text{Outside the Mandelbrot set} \) |
|---|---|---|---|---|
| 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| Upper bound for \( N_n \) | 2 | 6 | 20 | 38 | 102 | 198 | 436 | 686 |
| \( \#\Sigma_n \) | 2 | 6 | 20 | 38 | 102 | 198 | 436 | 686 |
| Outside the Mandelbrot set (%) | 0 | 0 | 20 | 10 | 15 | 14 | 14 | 9 |
| Outside the Mandelbrot set (%) | 100 | 100 | 80 | 90 | 85 | 86 | 86 | 91 |
significantly change the running time. It could also be that the basins of attraction of some of the points for the Newton’s method are very small and are easily missed by the initial guesses. The computations for each period up to $n = 7$ took less than 30 s in double precision while for period $n = 8$ it took almost 2 h. Due to the randomness of the initial guesses the running time might be slightly different for the same period for different shots though our experiments showed that this difference is minor.

5. Discussion of the results

In this section we give a basic discussion of the results of our computations and state some questions and conjectures.

Figure 1 represents critical points of the multiplier map $\lambda_n(c, z)$ on the parameter space. (a) $n = 6$, (b) $n = 7$, (c) $n = 8$, (d) $n = 9$.

The pictures also suggest that as $n$ increases, most of the critical points of the multiplier map tend to accumulate on the boundary of the Mandelbrot set. This leads to the following question: let $X_n \subset \mathbb{C}$ be the projection of the set of all critical points of $\lambda_n$ onto the coordinate $c$ (counted with multiplicity) and let $\nu_n$ be the probability measure

$$\nu_n = \frac{1}{\text{card}(X_n)} \sum_{c \in X_n} \delta_c.$$

Is it true that as $n \to +\infty$, the sequence of measures $\nu_n$ converges to a measure $\mu$ supported on the boundary of the Mandelbrot set? If yes, is $\mu$ the bifurcation measure? Positive answers to these questions (for weak convergence of measures) have been recently obtained by Firsova and the second author in [Firsova and Gorbovickis 19]. Similar results for various other classes of dynamically significant points have been previously proven, for example, in [Levin 89] and [Buff and Gauthier 15].

It was also shown in [Firsova and Gorbovickis 19] that the set of all accumulation points of the sets $X_n$, namely the set
Table 2. Examples of periodic points corresponding to the multiplier map \( \lambda_n \) with a critical point at \( c = 0 \).

| \( n \) | \( z_0 \) |
|-------|-------|
| 6     | \( \exp(2\pi i/9) \) |
| 12    | \( \exp(2\pi i/45) \) |
| 18    | \( \exp(2\pi i/27) \) |
| 20    | \( \exp(2\pi i/25) \) |
| 24    | \( \exp(2\pi i/49) \) |
| 30    | \( \exp(2\pi i/153) \) |

\[
X := \bigcap_{k=3}^{\infty} \left( \bigcup_{n=k}^{\infty} X_n \right),
\]

is strictly larger than the support of the limiting measure, i.e., the boundary of the Mandelbrot set. In particular, it was shown that every point \( c \) that belongs to some set \( X_n \) and lies in the complement of the Mandelbrot set, is an accumulation point.

Next, we can observe that while most of the elements of the sets \( X_n \) are strictly complex, for periods \( n = 6, 8 \) the sets \( X_n \) also contain purely real elements. Can we understand this phenomenon? Furthermore, for \( n = 6 \) one of these purely real critical points lies exactly at \( c = 0 \). The latter suggests the following: given a periodic point \( z_0 \neq 0 \) of the polynomial \( p_0(z) = z^2 \), one can compute the derivative of the multiplier map \( \frac{d\lambda}{dc}(0, z_0) \) using the formula

\[
\frac{d\lambda_n}{dc}(0, z_0) = -2^n \sum_{j=0}^{n-1} z_0^{-2^{j+1}}, \tag{5.1}
\]

which was obtained in [Gorbovickis 16]. Using this formula, we can check numerically whether \( c = 0 \) is a critical point of the multiplier map \( \lambda_n \) for periods \( n > 8 \). Due to the limited computational time, we performed computations up to period \( n = 30 \) and obtained that the multiplier map \( \lambda_n \) has a critical point at \( c = 0 \), for periods \( n = 6, 12, 18, 20, 21, 24 \) and \( 30 \) (Table 2). Furthermore, for each of these periods except \( n = 6 \), the value \( \frac{d\lambda_n}{dc} = 0 \) is obtained at more than one different periodic orbit.

As suggested to us by the anonymous referee, one can start with a periodic point \( z_0 \) rather than with the period \( n \). Indeed, for every odd integer \( m \geq 1 \), the point \( z_0 = \exp(2\pi i/m) \) is a periodic point of some period \( n = n(m) \) for the polynomial \( p_0 \). Then one can use formula (5.1) to check numerically, whether the point \( (0, z_0) \) is a critical point of the multiplier map \( \lambda_n \). We checked all odd integers \( m \) in the range \( 3 \leq m \leq 1699 \). The results are summarized in Table 3. We also prove the following lemma:

**Lemma 5.1.** For every \( k \in \mathbb{N} \), the point \( z_k = \exp(2\pi i/3^{k+1}) \) is a periodic point of period \( n_k = 2 \times 3^k \) for the polynomial \( p_0(z) = z^2 \). Furthermore, \((0, z_k)\) is a critical point of the multiplier map \( \lambda_{n_k} \).

**Proof.** Let \( G \) be the cyclic multiplicative group, generated by \( z_k \), and let \( \psi : G \to (\mathbb{Z}/(3^{k+1}\mathbb{Z}), +) \) be the group homomorphism sending \( z_0 \) to 1. Then \( \psi \) sends the orbit of \( z_k \) under the map \( p_0 \) bijectively to the orbit of 1 under multiplication by 2. Since 2 and \( 3^{k+1} \) are relatively prime, this orbit eventually returns to 1, so \( z_k \) is a periodic point of \( p_0 \). The period of \( z_k \) is equal to the order of 2 in the multiplicative group \((\mathbb{Z}/3^{k+1}\mathbb{Z})^\times \) of integers modulo \( 3^{k+1} \). We show that this order is equal to \( n_k \) by proving a stronger statement that for any \( k \in \mathbb{N} \), the following holds simultaneously:

\[
2^{n_k} \equiv 1 \pmod{3^{k+1}}, \quad 2^{n_k} \not\equiv 1 \pmod{3^{k+2}}, \tag{5.2}
\]

\[
2^{2^k} \equiv -1 \pmod{3^{k+1}}, \tag{5.3}
\]

and \( n_k \) is the smallest positive integer satisfying the first identity from (5-2). The proof goes by induction.

**Base case:** \( k = 1 \) is obvious.

**Induction step:** Assume that the statement holds for \( k = s-1 \in \mathbb{N} \). Then

\[
2^{n_s} - 1 = (2^{n_{s-1}} - 1)(2^{3^{s-1}} + 2^{3^{s-1}} + 1),
\]

and since \( 2^{3^{s-1}} + 2^{3^{s-1}} + 1 \equiv 3 \pmod{9} \), the condition (5.2) for \( k = s-1 \) implies (5.2) for \( k = s \). Similarly,

\[
2^{2^s} + 1 = (2^{2^{s-1}} + 1)(2^{2^{s-1}} - 2^{2^{s-1}} + 1),
\]

and since \( 2^{2^{s-1}} - 2^{2^{s-1}} + 1 \equiv 0 \pmod{3} \), the condition (5.3) for \( k = s-1 \) implies (5.3) for \( k = s \). Finally, if \( a \in \mathbb{N} \) is the smallest positive integer such that \( 2^a \equiv 1 \pmod{3^{s+1}} \), then by the inductive assumption, \( a \geq n_{s-1} = 2 \cdot 3^{s-1} \). Since \( a \) must divide the order of the group \((\mathbb{Z}/3^{s+1}\mathbb{Z})^\times \), and \(|(\mathbb{Z}/3^{s+1}\mathbb{Z})^\times| = 2 \cdot 3^s \), we have three options: \( a = 2 \cdot 3^{s-1}, a = 3^s \) and \( a = 2 \cdot 3^s \). The first two options are ruled out by the second condition of (5.2) for \( k = s-1 \) and (5.3) for \( k = s \) respectively. This completes the inductive proof.

Since \(|(\mathbb{Z}/3^{k+1}\mathbb{Z})^\times| = 2 \cdot 3^k = n_k \) for any \( k \in \mathbb{N} \), we conclude from the above statement that the group \((\mathbb{Z}/3^{k+1}\mathbb{Z})^\times \) is cyclic and the number 2 is its generator. This implies that the orbit of \( z_k \) under the map \( p_0 \) coincides with \( S = \psi^{-1}((\mathbb{Z}/3^{k+1}\mathbb{Z})^\times) \) as a set. Since \( G \) and \( G \setminus S \) considered as subsets of \( \mathbb{C} \), are the sets of all vertices of a regular \( 3^{k+1} \)-gon and a regular \( 3^k \)-gon, respectively, centered at zero, we conclude that the sum of all points in \( S \) is zero as well. Then the statement of the lemma follows after combining this result with (5.1). \( \square \)

As an immediate corollary from Lemma 5.1, we obtain the following:
There exist infinitely many different periods \( n \in \mathbb{N} \), for which the map \( \lambda_n \) has a critical point at \( c = 0 \).

We note that Lemma 5.1 is a partial case of more general results that have been communicated to us by Bernhard Reinke.

Another important problem is to study the critical values of the multiplier maps \( \lambda_n \). As it was mentioned in the introduction, the inverse branches of \( \lambda_n \) projected onto the \( c \)-coordinate are Riemann mappings of the corresponding hyperbolic components of the Mandelbrot set. In particular, this implies that all critical values of the multiplier maps \( \lambda_n \) lie outside of the open unit disk. The question is: How close can they get to the unit disk? Are the critical values of \( \lambda_n \) bounded away from the unit disk uniformly in \( n \)? If the answer to this question is positive, then one might use the Koebe Distortion Theorem to get uniform bounds on the geometric shape of the hyperbolic components. The results of our computations, summarized in Figure 2, cannot obviously give a definite answer to the stated question. Nevertheless, Figure 2 suggests that the answer might be positive.

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