Nonassociativity in String Theory

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Abstract

I summarize some of the ideas and motivations behind a recently performed conformal field theory analysis of closed strings in both geometric and nongeometric three-form flux backgrounds. This suggests an underlying nonassociative structure for the coordinates.

\footnote{This article summarizes a blackboard talk given at the “Memorial Conference for Maximilian Kreuzer” held in Vienna, June 2011.}
1 Introduction

Max will remain in our memory for his contributions to the understanding of Calabi-Yau manifolds. The classification of reflexive polytopes in the framework of toric geometry is surely his most acknowledged contribution. However, his interests were much broader and together with M. Herbst and A. Kling he also wrote a couple of, I think, very nice papers on noncommutative geometry\[1\,\[2\]. In particular, they were analyzing open strings in the background of a non-constant two-form background, i.e. one with non-trivial three-form flux, and found that in this case the coordinates are not only noncommutative but also nonassociative. Mathematically, an important role in their analysis was played by the Rogers dilogarithm. Intrigued by their results and similar ones by Cornalba/Schiappa \[3\], my collaborators and myself wondered whether a non-trivial three-form flux background might also have similar effects on the closed string sector, which is the one governing gravity.

Let me pose three questions, which I believe have at least the potential to point to such new structures in closed string theory. It is expected that the answers to these three questions are related, and some first concrete computations make it conceivable that string theory at small scales is dual to a theory which involves nonassociative spaces, for which the Kalb-Ramond field has been traded for a nonassociative deformation of ordinary Riemannian geometry.

Closed string generalization of noncommutative geometry

It is a well established fact that the effective theory on a D-brane equipped with a non-trivial two-form magnetic flux $F = B + F$ becomes noncommutative. This can be deduced by studying the conformal field theory on a flat D-brane with a constant magnetic field. In this case, the two-point function of two open string coordinates $X^a(\tau)$ inserted on the boundary of a disk takes the form

$$\langle X^a(\tau_1) X^b(\tau_2) \rangle = -\alpha' G^{ab} \log(\tau_1 - \tau_2)^2 + i \theta^{ab} \epsilon (\tau_1 - \tau_2),$$  \hspace{1cm} (1)

where $\tau$ stands for the real part of the complex world-sheet coordinate $z$. The matrix $G^{ab}$ is symmetric and can be interpreted as the (inverse of the) effective metric seen by the open string. $\theta^{ab}$ is related to the two-form flux as $\theta^{ab} \sim \frac{F^{ab}}{4 + F}$ and thus is anti-symmetric. The reason for the appearance of noncommutativity is the second term in (1) which means that the flux
distinguishes between the order of the two-points on the boundary of the disk.

This has been made more precise by analyzing open string scattering amplitudes for open string vertex operators

\[ V = F(\partial X^\mu) e^{ipX}, \]  

where \( F \) is a function of \( \partial X \). Since the second term in (1) is locally constant, it only contributes to correlation functions involving the \( \exp(ipX) \) factor in the vertex operators. Its effect is that it introduces non-trivial momentum dependent phases, which can be described by the introduction of a noncommutative product on the space of functions

\[ f_1(x) \star f_2(x) = \exp \left( i \theta_{ab} \partial_a x^1 \partial_b x^2 \right) f_1(x_1) f_2(x_2) \bigg|_{x_1=x_2=x}. \]  

This is the Moyal-Weyl product, which implies \([x_a, x_b] = x_a \star x_b - x_b \star x_a = i \theta_{ab} \).

Thus, noncommutativity arises for open strings in a magnetic flux background leading to noncommutative gauge theories. One might have expected that noncommutative geometry should also play an important role for quantum gravity, but for closed strings a similar structure has not been identified. Thinking about this question, one realizes that the closed string analogue must clearly be different as here two vertex operators are inserted in the bulk of a two-sphere \( S^2 \) and no unambiguous ordering can be defined. Therefore, one does not expect the same kind of noncommutativity to arise. Moreover, for a closed string a constant \( B \)-field can be gauged away.

However, if one considers three nearby points on the world-sheet \( S^2 \) of a closed string, one can very well decide whether the loop connecting the three points has positive or negative orientation. Thus, if there exists a background field which distinguishes these two orientations, one would expect a nonvanishing result not for the simple commutator, but for the cyclic double commutator

\[ [X^\mu, X^\nu, X^\rho] := \lim_{\sigma_i \to \sigma} \left[ [X^\mu(\sigma_1, \tau), X^\nu(\sigma_2, \tau)], X^\rho(\sigma_3, \tau) \right] + \text{cyclic}. \]  

Now, the question is whether there exists a three-form with this property?

**Nonlinear sigma models**

The usual approach to string theory is perturbative, i.e., one considers a string moving in a background with metric \( G_{\mu\nu} \), Kalb-Ramond field \( B_{\mu\nu} \) and...
dilaton $\Phi$, whose dynamics is governed by a two-dimensional non-linear sigma model. With $\Sigma$ denoting the world-sheet of the closed string, its action reads

$$S = \frac{1}{2\pi\alpha'} \int_{\Sigma} d^2z \left( G_{ab} + B_{ab} \right) \partial X^a \overline{\partial} X^b + \ldots ,$$ \hspace{1cm} (5)

where we suppressed the dilaton part. This is treated perturbatively in a dimensionless coupling $\sqrt{\alpha'}/R$, where $R$ is a characteristic length scale of the background. The guiding principle is conformal invariance. This means that the string equations of motion for the space-time fields $G_{\mu\nu}$, $B_{\mu\nu}$ and $\Phi$ are given by the vanishing beta-function equations. At leading order these equations read

\begin{align*}
0 &= \beta_{ab}^G = \alpha' \left( R_{ab} - \frac{1}{4} H_{a}^{\ \cdots d} H_{b c d} + 2 \nabla_a \nabla_b \Phi \right) + O(\alpha'^2) \\
0 &= \beta_{ab}^H = \alpha' \left( -\frac{1}{2} \nabla_c H_{a b}^{\ c} + \alpha' H_{a b}^{c} \nabla_c \Phi \right) + O(\alpha'^2) \\
0 &= \beta_{ab}^\Phi = \frac{1}{4} (d - d_{\text{crit}}) + \alpha' \left( (\nabla \Phi)^2 - \frac{1}{2} \nabla^2 \Phi - \frac{1}{24} H^2 \right) + O(\alpha'^2).
\end{align*} \hspace{1cm} (6)

The first equation, for instance, is nothing else than Einstein’s equation with sources. Clearly, in this approach one is assuming from the very beginning that the string is moving through a Riemannian geometry with additional smooth fields. However, it is well known that there exist conformal field theories which cannot be identified with such simple geometries. These are left-right asymmetric like for instance asymmetric orbifolds. The latter are asymmetric at some orbifold fixed points but one can imagine asymmetric CFTs which are not even locally geometric. What is the target space interpretation of such asymmetric CFTs?

In the non-linear sigma model one performs perturbation theory around the large volume limit with diluted fluxes. Can one also define a perturbation theory around the other limit, namely very small substringy sizes of the background? In view of double field theory, we have here in mind an effective field theory describing the dynamics of winding states in a $G_{\mu\nu}$, $B_{\mu\nu}$, $\Phi$ background.

**What is $R$-flux?**

In the past, applying T-duality to known configurations has led to new insights into string theory, where a prominent example is the discovery of
D-branes. Applying T-duality to the closed string background given by a flat space with constant non-vanishing three-form flux $H = dB$, results in a background with geometric flux. This so-called twisted torus is still a conventional string background, but a second T-duality leads to a non-geometric flux background. These are spaces in which the transition functions between two charts of a manifold are allowed to be T-duality transformations, hence they are also called T-folds. After formally applying a third T-duality, not along an isometry direction anymore, one obtains an $R$-flux background which does not admit a clear target-space interpretation. It was proposed not to correspond to an ordinary geometry even locally, but instead to give rise to a nonassociative geometry. In addition to involving a T-duality in a non-isotropic direction, another problem with this argument is that flat space with constant $H$-flux is not an exact solution to the string equations of motion. Therefore one should ask, whether nevertheless one can make this $R$-flux case more precise.

2 CFT analysis of $H$-flux

The remainder of this article is essentially a brief version of the more exhaustive analysis presented recently [8]. First, we note that the origin of T-duality lies in conformal field theory where it is nothing else than an asymmetric reflection $(X_L, X_R) \rightarrow (X_L, -X_R)$. Therefore, it is tempting to try to analyze $R$-flux from the CFT point of view. In order to see what is going on, let us first perform a perturbative analysis of the $H$-flux case and then apply a T-duality. We observe that a Ricci flat metric, vanishing dilaton and a constant $H$-flux solves the string equation of motion up to linear order in $H$ and arbitrary order in $\alpha'$. Therefore, the starting point is a flat metric and a constant $H$-flux specified by

$$ds^2 = \sum_{a=1}^{N} (dX^a)^2, \quad H = \frac{2}{\alpha' 2} \theta_{abc} dX^a \wedge dX^b \wedge dX^c,$$

where for simplicity we focus on $N = 3$. The expectation is that this background corresponds to a CFT up to linear order in $H$.

To proceed, we write the action (5) as the sum of a free part $S_0$ and a perturbation $S_1$. Choosing a gauge such that $B_{ab} = \frac{1}{3} H_{abc} X^c$, we have

$$S = S_0 + S_1 \quad \text{with} \quad S_1 = \frac{1}{2\pi \alpha'} \frac{H_{abc}}{3} \int_{\Sigma} d^2 z X^a \partial X^b \partial X^c.$$

5
We expect $S_1$ to be a marginal operator (only) up to linear order in $H$.

Now, one can apply conformal perturbation theory to compute the correction to the three-point functions of three currents $J^a = i\partial X^a$, $\overline{J}^a = i\partial \overline{X}^a$. It turns out that there are also non-vanishing correlators like $\langle J^a J^b J^c \rangle$, i.e. the currents are not holomorphic respectively anti-holomorphic. However, one can define new fields $J^a$ and $\overline{J}^a$:

\[
J^a(z, \overline{z}) = J^a(z) - \frac{1}{2} H^a_{bc} J^b(z) X^c(\overline{z}),
\]

\[
\overline{J}^a(z, \overline{z}) = \overline{J}^a(\overline{z}) - \frac{1}{2} H^a_{bc} X^b_L(z) \overline{J}^c(\overline{z})
\]

so that the three current correlators take the CFT form

\[
\langle J^a(z_1, \overline{z}_1) J^b(z_2, \overline{z}_2) J^c(z_3, \overline{z}_3) \rangle = -i \frac{\alpha'^2}{8} H^{abc} \frac{1}{z_{12} z_{23} z_{13}},
\]

\[
\langle \overline{J}^a(z_1, \overline{z}_1) \overline{J}^b(z_2, \overline{z}_2) \overline{J}^c(z_3, \overline{z}_3) \rangle = +i \frac{\alpha'^2}{8} H^{abc} \frac{1}{z_{12} z_{23} z_{13}}.
\]

The necessity of this redefinition can already be understood from the two-dimensional equation of motion $\partial \partial X^a = \frac{1}{2} H^a_{bc} \partial X^b \partial X^c$. Therefore, already at linear order the coordinate fields have to be adjusted to be consistent with a CFT description. However, the deformation is still marginal and nothing starts to run.

Writing the new currents as derivatives of corrected coordinates $X^a$, after three integrations the three-point function of these coordinates can be computed as

\[
\langle X^a(z_1, \overline{z}_1) X^b(z_2, \overline{z}_2) X^c(z_3, \overline{z}_3) \rangle^H = \theta^{abc} \left[ \mathcal{L} \left( \frac{z_{12}}{z_{13}} \right) - \mathcal{L} \left( \frac{z_{13}}{z_{12}} \right) \right]
\]

(11)

with $\theta^{abc} = \frac{\alpha'^2}{12} H^{abc}$ and

\[
\mathcal{L}(z) = L(z) + L \left( 1 - \frac{1}{x} \right) + L \left( \frac{1}{1 - x} \right),
\]

(12)

where the Rogers dilogarithm is defined as

\[
L(z) = \text{Li}_2(z) + \frac{1}{2} \log(z) \log(1-z).
\]

(13)

It satisfies the so-called fundamental identity $L(z) + L(1-z) = L(1)$. The three-point function should be considered as the closed string generalization of the second term in (I). However, there two essential differences:
• For the closed string it is the three- and not the two-point function which is corrected.

• For the closed string the Rogers dilogarithm gives rise to a non-trivial world-sheet dependence, whereas for the open string only the essentially constant step-function appeared.

One can also compute the correction to the two-point function of two coordinates. It reads

\[ \delta_2 \langle X^a(z_1, \bar{z}_1) X^b(z_2, \bar{z}_2) \rangle = \frac{\alpha'^2}{8} H^a_{pq} H^{bpq} \log |z_1 - z_2|^2 \log \epsilon , \quad (14) \]

where \( \epsilon \) is a cut-off. Therefore, we explicitly see that the perturbation \( S_1 \) ceases to be marginal at second order in the flux. The theory is no longer conformally invariant and starts to run according to the renormalization group flow equation for the inverse world-sheet metric \( G^{ab} \), which is of the form

\[ \mu \frac{\partial G^{ab}}{\partial \mu} = -\frac{\alpha'}{4} H^a_{pq} H^{bpq} . \quad (15) \]

This precisely agrees with equation (6) for constant space-time metric, \( H \)-flux and dilaton.

Up to linear order in the flux we can write the energy-momentum tensor as

\[ T(z) = \frac{1}{\alpha'} \delta_{ab} :J^a J^b:(z) , \quad \bar{T}(\bar{z}) = \frac{1}{\alpha'} \delta_{ab} :\bar{J}^a \bar{J}^b:(\bar{z}) . \quad (16) \]

They give rise to two copies of the Virasoro algebra with central charge \( c = 3 \) and \( J^a(\bar{J}^a) \) is indeed a (anti-)chiral primary field with \( h = 1(\bar{h} = 1) \).

The aim is to carry out a similar computation as for the open string case, i.e. to evaluate string scattering amplitudes for vertex operators and to see whether there is any sign of a new space-time noncommutative/nonassociative product. Recall that in the free theory the tachyon vertex operator is a primary field of conformal dimension \( (h, \bar{h}) = \left( \frac{\alpha'}{4} p^2, \frac{\alpha'}{4} p^2 \right) \), and in covariant quantization of the bosonic string physical states are given by primary fields of conformal dimension \( (h, \bar{h}) = (1, 1) \). The natural definition of the tachyon vertex operator for the perturbed theory is

\[ \mathcal{V}(z, \bar{z}) = :\exp(ip \cdot (X_L + X_R)):. \quad (17) \]
One can compute

\[ T(z_1) V(z_2, \bar{z}_2) = \frac{1}{(z_1 - z_2)^2} \frac{\alpha' p \cdot p}{4} V(z_2, \bar{z}_2) + \frac{1}{z_1 - z_2} \partial V(z_2, \bar{z}_2) + \text{reg.}, \]

and analogously for the anti-holomorphic part. This means that the vertex operator (17) is primary and has conformal dimension \((h, \bar{h}) = (\alpha' p^2, \alpha' \bar{p}^2) = (1, 1)\). It is therefore a physical quantum state of the deformed theory.

3 T-duality, R-flux and tachyon amplitudes

Even though in the framework of the Buscher rules, applying three T-dualities on the \(H\)-flux background is questionable, on the level of the CFT, T-duality corresponds to a simple asymmetric transformation of the world-sheet theory. It is just a reflection of the right-moving coordinates. Since our corrected fields \(X^a(z, \bar{z})\) still admit a split into a holomorphic and an anti-holomorphic piece, we define T-duality on the world-sheet action along the direction \(X^a\) as

\[ X^a_L(z) \xrightarrow{T\text{-duality}} +X^a_L(z), \quad X^a_R(\bar{z}) \xrightarrow{T\text{-duality}} -X^a_R(\bar{z}). \]

Under a T-duality in all three directions, momentum modes in the \(H\)-flux background are mapped to winding modes in the \(R\)-flux background. We are now interested in momentum modes in the \(R\)-flux background which are related via T-duality to winding modes in the \(H\)-flux background. Therefore, the three-point function in the \(R\)-flux background should read

\[ \langle X^a(z_1, \bar{z}_1) X^b(z_2, \bar{z}_2) X^c(z_3, \bar{z}_3) \rangle^R = \theta^{abc} \left[ \mathcal{L} \left( \frac{\bar{z}_{12}}{z_{12}} \right) + \mathcal{L} \left( \frac{\bar{z}_{13}}{z_{13}} \right) \right], \]

which just has a different relative sign between the holomorphic and anti-holomorphic part. Here, we have the relation \(\theta^{abc} = \frac{\alpha'^2}{12} R^{abc}\).

For the correlator of three tachyon vertex operators one obtains

\[ \langle \mathcal{V}_1 \mathcal{V}_2 \mathcal{V}_3 \rangle^{H/R} = \frac{\delta(p_1 + p_2 + p_3)}{|z_{12} z_{13} z_{23}|^2} \times \exp \left[ -i \theta^{abc} p_{1,a} p_{2,b} p_{3,c} \left[ \mathcal{L} \left( \frac{\bar{z}_{12}}{z_{12}} \right) \pm \mathcal{L} \left( \frac{\bar{z}_{13}}{z_{13}} \right) \right] \right] \theta, \]
where [..]θ indicates that the result is valid only up to linear order in θ. The full string scattering amplitude of the integrated tachyon vertex operators then becomes

\[
\left\langle T_1 T_2 T_3 \right\rangle^{H/R} = \int \prod_{i=1}^{3} d^2 z_i \delta^{(2)}(z_i - z_i^0) \delta(p_1 + p_2 + p_3) \times \\
\exp \left[ -i \theta^{abc} p_{1,a} p_{2,b} p_{3,c} \left[ \mathcal{L}(\frac{z_1 z_2}{z_{12}}) \mp \mathcal{L}(\frac{z_2 z_3}{z_{12}}) \right] \right]_{\theta}.
\]

(22)

Let us now study the behavior of (21) under permutations of the vertex operators. Before applying momentum conservation, the three-tachyon amplitude for a permutation \(\sigma\) of the vertex operators can be computed using the relation \(L(z) + L(1-z) = L(1)\). With \(\epsilon = -1\) for the \(H\)-flux and \(\epsilon = +1\) for the \(R\)-flux, one finds

\[
\left\langle V_{\sigma(1)} V_{\sigma(2)} V_{\sigma(3)} \right\rangle^{H/R} = \\
\exp \left[ i \left( \frac{1+\epsilon}{2} \right) \eta_{\sigma} \pi^2 \theta^{abc} p_{1,a} p_{2,b} p_{3,c} \right] \left\langle V_1 V_2 V_3 \right\rangle^{H/R},
\]

where in addition \(\eta_{\sigma} = 1\) for an odd permutation and \(\eta_{\sigma} = 0\) for an even one. One observes that for \(H\)-flux the phase is always trivial while for \(R\)-flux a non-trivial phase may appear. Recall that our analysis is only reliable up to linear order in \(\theta^{abc}\).

Note that it is non-trivial that this phase is independent of the worldsheet coordinates, which can be traced back to the form of the fundamental identity of \(L(z)\). For this reason, it can be thought of as a property of the underlying target space. Indeed, the phase in (23) can be recovered from a new three-product on the space of functions \(V_{p_n}(x) = \exp(i p_n \cdot x)\) which is defined as

\[
V_{p_1}(x) \Delta V_{p_2}(x) \Delta V_{p_3}(x) \overset{\text{def}}{=} \\
\exp \left( -i \frac{\pi^2}{2} \theta^{abc} p_{1,a} p_{2,b} p_{3,c} \right) V_{p_1 + p_2 + p_3}(x).
\]

(24)

However, in CFT correlation functions operators are understood to be radially ordered and so changing the order of operators should not change the form of the amplitude. This is known as crossing symmetry. In the case of the \(R\)-flux background, this is reconciled by applying momentum conservation leading to

\[
p_{1,a} p_{2,b} p_{3,c} \theta^{abc} = 0 \quad \text{for} \quad p_3 = -p_1 - p_2.
\]

(25)
Therefore, scattering amplitudes of three tachyons do not receive any corrections at linear order in $\theta$ both for the $H$- and $R$-flux.

The three-product (24) can be generalized to more generic functions as

$$f_1(x) \Delta f_2(x) \Delta f_3(x) \overset{\text{def}}{=} \exp\left(\frac{\pi^2}{2} \theta^{abc} \partial_a^{x_1} \partial_b^{x_2} \partial_c^{x_3}\right) f_1(x_1) f_2(x_2) f_3(x_3) \bigg|_x,$$

where we used the notation $(\ )|_x = (\ )|_{x_1=x_2=x_3=x}$. This is to be compared with the $\star$-product (3) and can be thought of as a possible closed string generalization of the open string noncommutative structure. Note that (26) is precisely the three-product anticipated in an analysis of the $SU(2)$ WZW model\cite{9}. Indeed, the three-bracket for the coordinates $x^a$ can then be rederived as the completely antisymmetrized sum of three-products

$$[x^a, x^b, x^c] = \sum_{\sigma \in P^3} \text{sign}(\sigma) x^{\sigma(a)} \Delta x^{\sigma(b)} \Delta x^{\sigma(c)} = 3\pi^2 \theta^{abc},$$

where $P^3$ denotes the permutation group of three elements. For the WZW model\cite{9}, this three-bracket was defined as the Jacobi-identity (4) of the coordinates, which can only be non-zero if the space is noncommutative and nonassociative (see also the similar paper by Lüst\cite{10}).

This result generalizes to the $N$-tachyon amplitude, where the relative phase can be described by the following deformed product

$$f_1(x) \triangle_N f_2(x) \triangle_N \ldots \triangle_N f_N(x) \overset{\text{def}}{=} \exp\left[\frac{\pi^2}{2} \theta^{abc} \sum_{1 \leq i < j < k \leq N} \partial_a^{x_i} \partial_b^{x_j} \partial_c^{x_k}\right] f_1(x_1) f_2(x_2) \ldots f_N(x_N) \bigg|_x,$$

which is the closed string generalization of the open string noncommutative product (3). The phase becomes trivial after taking momentum conservation into account or equivalently

$$\int d^n x f_1(x) \triangle_N f_2(x) \triangle_N \ldots \triangle_N f_N(x) = \int d^n x f_1(x) f_2(x) \ldots f_N(x).$$

4 Comments

We have used conformal perturbation theory to analyze the bosonic string moving in an $H$- respectively $R$-flux background, at least up to linear order.
in the flux. In the $R$-flux case, the application of T-duality to the string scattering amplitudes of tachyon vertex operators revealed a non-trivial three-product structure which was visible, however, only prior to implementing momentum conversation. At first sight this might be puzzling, but it actually makes sense. If we had found a non-vanishing phase factor for a closed string scattering amplitude, it would have been in clear conflict with crossing symmetry of CFT amplitudes. Another way of saying this is: The deformation of space-time as implied by a non-vanishing three-bracket for the coordinates is consistent with the structure of two-dimensional CFT. In view of the fact that asymmetric CFTs are known to not admit a usual geometric target-space interpretation, this is an interesting observation.

In the original paper [8], we also computed the complete four-tachyon scattering amplitude. It could be written in the $SL(2, \mathbb{C})$ invariant and explicitly crossing symmetric form

$$\langle T_1 T_2 T_3 T_4 \rangle^{H/R} = \int d^2 X \exp \left[ -i \theta^{\mu \nu} p_1^\mu p_2^\nu \left( (-\frac{3}{2} L(1) + \mathcal{L}(X)) \mp (-\frac{3}{2} L(1) + \mathcal{L}(\bar{X})) \right) \right] \theta,$$

where $X$ denotes the cross-ratio $X = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}$ and

$$a = \frac{\alpha'}{4} (p_1 + p_4)^2 - 1, \quad b = \frac{\alpha'}{4} (p_1 + p_3)^2 - 1, \quad c = \frac{\alpha'}{4} (p_1 + p_2)^2 - 1. \quad (31)$$

The three corresponding Mandelstam variables read $u = -(p_1 + p_4)^2$, $t = -(p_1 + p_3)^2$ and $s = -(p_1 + p_2)^2$, and the on-shell external tachyons satisfy $\alpha' p_i^2 = 4$ so that $a + b + c = 1$. Equation (30) is the fluxed version of the Virasoro-Shapiro amplitude. The analysis of the pole-structure revealed that the spectrum of states changes at linear order in the flux. For instance, some of the former massless modes, like the graviton, seem to become massive and some even tachyonic. This is conceptually consistent with the observation that the vertex operator of the “graviton”

$$\mathcal{V}_G(z, \bar{z}) = G_{ab} \cdot \mathcal{J}^a \bar{\mathcal{J}}^b \exp(ip \cdot \mathcal{X})$$

is generically not any longer a primary field of conformal dimension one and therefore not a physical state.

So far, only correlation functions involving tachyons were analyzed. In contrast to the open string case, one expects that vertex operators of the
form (32) will also contain new contractions between the $H/R$-flux and the polarizations. Recall that for the open string they were absent, as $\theta^{ab}$ was multiplied by a piecewise constant function (see eq. (1)). This is clearly not true for the Rogers dilogarithm.

Let me close with another observation. Ignoring for the moment that the graviton $G$ and the analogously defined two-form $B$ vertex operators (32) are not physical, the computation of the $\langle GGB \rangle, \langle BGB \rangle$ scattering amplitudes contain $SL(2, \mathbb{Z})$ invariant contributions of the schematic form

$$H\text{-flux : } \langle GGB \rangle \simeq \theta^{abc} G_{am} p_{3}^{m} G_{bn} B_{cn} + \ldots ,$$

$$R\text{-flux : } \langle BGB \rangle \simeq \theta^{abc} B_{am} p_{3}^{m} G_{bn} B_{cn} + \ldots .$$

For $H$-flux the second amplitude is vanishing and for $R$-flux the first. In the first case, such a term arises from the term $H^{abc} H_{abc}$ in the effective action with $H = dB$. The second amplitude, however, rather suggests that the effective action contains a term $R^{abc} R_{abc}$ with $R^{abc} = B^{am} \partial_{m} B^{bc} + \ldots$. This is very similar to the form of the non-geometric $R$-flux as it appears for instance in double field theory [11]. In this context, $B$ is usually denoted as $\beta$ and is rather a bi-vector than a two-form. Thus, the question arises whether one can formulate something like an effective action for these non-geometric fluxes where the structures presented in this talk might play an important role.

I deeply regret that all these exciting questions cannot be discussed with Max anymore.

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