ON THE SUM OF THE VORONOI POLYTOPE OF A LATTICE WITH A ZONOTOPE

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Abstract. A parallelotope $P$ is a polytope that admits a facet-to-facet tiling of space by translation copies of $P$ along a lattice. The Voronoi cell $P_V(L)$ of a lattice $L$ is an example of a parallelotope. A parallelotope can be uniquely decomposed as the Minkowski sum of a zone closed parallelotope $P$ and a zonotope $Z(U)$, where $U$ is the set of vectors used to generate the zonotope. In this paper we consider the related question: When is the Minkowski sum of a general parallelotope and a zonotope $P + Z(U)$ a parallelotope? We give two necessary conditions and show that the vectors $U$ have to be free. Given a set $U$ of free vectors, we give several methods for checking if $P + Z(U)$ is a parallelotope. Using this we classify such zonotopes for some highly symmetric lattices.

In the case of the root lattice $E_6$, it is possible to give a more geometric description of the admissible sets of vectors $U$. We found that the set of admissible vectors, called free vectors, is described by the well-known configuration of 27 lines in a cubic. Based on a detailed study of the geometry of $P_V(E_6)$, we give a simple characterization of the configurations of vectors $U$ such that $P_V(E_6) + Z(U)$ is a parallelotope. The enumeration yields 10 maximal families of vectors, which are presented by their description as regular matroids.

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1. Introduction

Let $L = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n$ be a lattice of rank $n$ in Euclidean space given by $n$ independent vectors $(v_i)_{1 \leq i \leq n}$ in $\mathbb{R}^n$. The Voronoi cell (or Voronoi polytope) of $L$ is the convex polytope

$$P_V(L) = \left\{ x \in \mathbb{R}^n : v^T x \leq \frac{1}{2} v^2 \text{ for all } v \in L - \{0\} \right\},$$

where $v^T x$ is scalar product of vectors $v, x \in \mathbb{R}^n$. By $R$ we denote the set of relevant vectors, i.e. vectors $v \in L$ that determine facet defining inequalities of $P_V(L)$. The quotient group $L/2L$ allows to determine the relevant vectors. A coset $v + 2L$ with $v \in L$ is called simple if it has a unique, up to sign, minimal vector. One can prove that $R$ is formed by the simple cosets. Above description of the Voronoi polytope $P_V(L)$ is given in an orthonormal basis of $\mathbb{R}^n$. This basis is self dual.

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If we consider vectors of the lattice $L$ in a basis $B$ of $L$, and points $x \in \mathbb{R}^n$ in the dual basis $B^*$, then we obtain the following affinely equivalent description of $P_V(L)$:

\[ P_V(L, f) = \{ x \in \mathbb{R}^n : z^T x \leq f(z) \text{ for all } z \in Z \}, \]

where $Z \subset \mathbb{Z}^n$ is a set of representations of minimal vectors vectors of $L$ in the basis $B$. Here $f(z) = \frac{1}{2} z^T D z$ is a quadratic form related to the chosen basis of $L$, and $D$ is Gram matrix of the basis.

For a root lattice $L$ the set $R$ is a set of all roots, i.e. of all vectors $v \in L$ of norm $v^2 = 2$. Root lattices $A_n$, $D_n$, for $n \geq 1$, and $E_n$, for $n = 6, 7, 8$, are described in many books and papers on lattices. See, for example, [CS91].

A parallelotope is a polytope whose translation copies under a lattice $L$ fill the space without gaps and intersections by inner points. For a given lattice $L$, \{ $P_V(L) + v$ \}_{v \in L}$ define a tiling of $\mathbb{R}^n$ and so $P_V(L)$ is a parallelotope.

We call a facet normal each vector $p$ orthogonal to a hyperplane supporting a facet $F(p)$, i.e., an $(n-1)$-dimensional face, of an $n$-parallelotope $P$. If $P = P_V(L)$ is a Voronoi polytope defined above, then every its facet normal $p$ is collinear to some lattice vector $s$ such that the facet $F(p)$ is shared by the parallelotopes $P_V(L)$ and $P_V(L) + s$. Such lattice vectors are called facet vectors of $F(p)$.

A parallelotope $P$ is necessarily centrally symmetric and its facets are necessarily centrally symmetric. A $k$-belt of an $n$-dimensional polytope $P$, whose facets are centrally symmetric, is a family of $k$ facets $F_1, F_2, \ldots, F_k$ such that $F_i \cap F_{i+1}$ and $F_i \cap F_{i-1}$ are antipodal $(n-2)$-dimensional faces in $F_i$ for $1 \leq i \leq k$, where the indexing $i$ in $F_i$ taken modulo $k$. A polytope $P$ is a parallelotope if and only if the following Venkov conditions hold [Ve54, Mu80, Zo96]:

- $P$ is centrally symmetric;
- The facets of $P$ are centrally symmetric;
- The facets of $P$ are organized into 4- and 6-belts.

A still open conjecture of Voronoi [Vo08] asserts that any parallelotope is affinely equivalent to a Voronoi polytope. Voronoi conjecture has been solved up to dimension 5 [En98]. For a given set $U$ of vectors, the zonotope $Z(U)$ is the Minkowski sum

\[ Z(U) = \sum_{u \in U} z(u), \]

where

\[ z(u) = [-u, u] = \{ x \in \mathbb{R}^n : x = \lambda u, \ -1 \leq \lambda \leq 1 \} \]

is a segment of a line spanned by the vector $u$.

Voronoi’s conjecture has been proved for zonotopes [ErRy94, Er99].

For a parallelotope $P$, a vector $v$ is called free if the Minkowski sum $P + z(v)$ is again a parallelotope. In $\mathbb{R}^4$ free vectors were characterized by the requirement that $z(v)$ is parallel to at least one facet of each 6-belt of $P$, but the argument that this condition was both necessary and sufficient was incomplete; a complete proof is provided below in Section 4.

A parallelotope is called free or nonfree depending upon whether or not it has free vectors; the first known nonfree parallelotope is $P_V(E_6^* )$ [En98] and it was later proved that the parallelotopes $P_V(D_4^{2m})$ are also nonfree for $m \geq 4$ [Gr06a]. A parallelotope $P$ is called zone-open if it can be represented as a Minkowski sum $P' + z(v)$, where $P'$ is a parallelotope and $v$ is a non-zero vector; otherwise the
parallelotope is called \textit{zone-closed}. Any parallelotope can be uniquely expressed as Minkowski sum of a zone-closed parallelotope and a zonotope.

A parallelotope \( P \) is called \textit{finitely free} if there exists a finite set \( \mathcal{F}(P) \) such that any free vector \( v \) is collinear to a vector in \( \mathcal{F}(P) \). The parallelotopes that we consider in this paper are finitely free Voronoi polytopes of highly symmetric lattices. But, for example, any vector \( v \) is free for the cube \([-1/2, 1/2]^n\), which is the Voronoi polytope for the lattice \( \mathbb{Z}^n \), because the cube has no 6-belts.

For a given zone-closed parallelotope \( P \), we want to find the vector systems \( U \) such that \( P + Z(U) \) is still a parallelotope. Of course, every vector in \( U \) has to be free. Another condition arising from the theory of matroid \cite{BLSWZ93, DG99, Gr04} is that \( U \) has to be \textit{unimodular}, i.e. in any basic subset \( B \subseteq U \), all vectors of \( U \) have integer coordinates in the basis \( B \).

The condition that elements of \( U \) are free for \( P \), and that \( U \) is unimodular are certainly necessary for \( P + Z(U) \) to be a parallelotope. But in general they are not sufficient, as shown below in the discussion of the Voronoi polytope \( P_V(\mathbb{E}_6) \) of the root lattice \( \mathbb{E}_6 \). If we want to check Venkov conditions, then we have to determine the faces and the \( k \)-belts of the sum \( P + Z(U) \).

If \( G \) is a face of \( P \), then we decompose \( U \) into \( U_1(G) \cup U_2(G) \cup U_3(G) \). Let \( x \in G \) be an inner point of \( G \) and let \( \lambda \) be a sufficiently small number. Then one of the following three cases are true.

\begin{itemize}
  \item \( u \in U_1(G) \) if \( x \pm \lambda u \notin G \) and one of the points \( x \pm \lambda u \) is in \( P \).
  \item \( u \in U_2(G) \) if \( x \pm \lambda u \in \text{Lin}(G) \).
  \item \( u \in U_3(G) \) if \( x \pm \lambda u \notin G \) and \( x \pm \lambda u \notin P \).
\end{itemize}

The vectors in \( U_1(G) \) translate \( G \) by some vector \( w = \sum_{u \in U_1(G)} \pm u \). The vectors \( u \in U_2(G) \) belong to the vector space \( \text{Lin}(G) \) defined by \( G \) and extend \( G \) to a larger face of the same dimension. The vectors \( u \in U_3(G) \) are \textit{strongly transversal} to \( G \), that is \( \dim(G + z(u)) = 1 + \dim G \) and \( G + z(u) \) is a new face of \( P \).

Hence, all faces of \( P + Z(U) \) are translated or extended facets of \( P \). So, if one computes all facets and \( (n - 2) \)-dimensional faces of \( P + Z(U) \), then one can determine if it is a parallelotope. The method is explained in Section \( \text{III} \) and applied in Section \( \text{IV} \) (with modifications) to the root lattice \( \mathbb{E}_6 \) and in Section \( \text{V} \) to other lattices.

The Voronoi polytope \( P_V(\mathbb{E}_6) \) of the root lattice \( \mathbb{E}_6 \) has only 6-belts. The edges of \( P_V(\mathbb{E}_6) \) are of two types: \( r \)-edges, which are parallel and equal by norm to minimal vectors of the lattice \( \mathbb{E}_6 \), and \( m \)-edges, which are parallel and equal by norm to minimal vectors of the dual lattice \( \mathbb{E}_6^* \). The set \( \mathcal{R} \) of minimal vectors of \( \mathbb{E}_6 \) is the root system \( \mathbb{E}_6 \). Roots of \( \mathcal{R} \) are also facet vectors of \( P_V(\mathbb{E}_6) \). One can choose a set \( \mathcal{M} \) of 27 vectors of \( \mathbb{E}_6 \) such that the pairwise scalar products \( q^T q' \) of \( q, q' \in \mathcal{M} \) take only two values \(-\frac{1}{2}\) and \(-\frac{1}{3}\) and the set of minimal vectors of \( \mathbb{E}_6^* \) is \( \mathcal{M} \cup -\mathcal{M} \).

For a lattice \( L \) we define an \textit{empty sphere} to be a ball containing no lattice point in its interior. The convex hull of the lattice points on the surface of the ball is called a \textit{Delaunay polytope}. The Delaunay polytopes define a tessellation, which is dual to the one by Voronoi polytopes \( P_V(\mathbb{E}_6) \), in particular the vertices of \( P_V(\mathbb{E}_6) \) are circumcenters of Delaunay polytopes (see, for example, \cite{Sch09}). It turns out that the Delaunay polytopes of \( \mathbb{E}_6 \) are Schläfli polytope \( P_{\text{Schl}} \) and its antipodal \( P_{\text{Schl}}^* \). The configuration of vectors \( \mathcal{M} \) is the vertex set of \( P_{\text{Schl}} \).

We prove that the set of free vectors of \( \mathbb{E}_6 \) is exactly \( \mathcal{M} \). Then we consider subsets \( U \subseteq \mathcal{M} \) and give a necessary and sufficient condition for \( P_V(\mathbb{E}_6) + Z(U) \)
to be a parallelotope. Namely, the subset $U$ should not contain a subset of five vectors $u_i \in M, 1 \leq i \leq 5$, such that $u_i^T u_j = \frac{1}{3}$ for $1 \leq i < j \leq 5$. We find all ten maximal by inclusion feasible subsets $U \subseteq M$ by a computer search. We give a detailed description of these subsets and regular matroids, represented by these subsets. Finally, we consider how the implemented methods can be extended to other lattices.

We use Coxeter’s notations $\alpha_n$ and $\beta_n$ for regular $n$-dimensional simplices and cross-polytopes respectively.

In Section 2 general results about free vectors are proved. In Section 3 the standard vectors of parallelotopes $P + z(v)$ are determined. In Section 4 enumeration algorithms for free-vectors are given. In Sections 5 the lattice $E_6$ is built. In Sections 6, 7 and 8 the Delaunay polytope, Voronoi polytope and free vectors of $E_6$ are determined. In Section 9 the criterion for $P_v(E_6) + Z(U)$ to be a parallelotope is given. In Sections 10 and 11 the enumeration of maximal feasible subsets is done for $E_6$ and some other lattices.

2. General results

Recall that a vector $x$ is called a facet vector of a parallelotope $P$ if it connects the center of $P$ with the center of a parallelotope $P + x$ such that $P + x$ and $P$ share a facet. If $P$ is a Voronoi parallelotope, then $x$ is orthogonal to the facet $P \cap (P + x)$.

In [Do09] Dolbilin introduced the notion of standard faces of parallelotopes. Namely, if $P$ and $P + x$ are parallelotopes of the face-to-face tiling by translates of $P$, and if $F = P \cap (P + x) \neq \emptyset$, then $F$ is called a standard face of $P$ and $x$ is called the standard vector of $F$. If $P$ is centered at the origin, then $F$ has a center of symmetry at $x/2$. In what follows the definition of standard faces will be used extensively.

The following Theorem gives a characterization of free vectors that can be used for a computer enumeration of all free vectors for a given parallelotope $P$, or in a case by case analysis as shown below in our discussion of $E_6$.

**Theorem 1.** ([Gr04]) Let $P$ be a parallelotope and $v$ be a vector. The following assertions are equivalent:

(i) $P$ is free along $v$ (i.e. $P + z(v)$ is a parallelotope);

(ii) $v^T p = 0$ for at least one facet normal $p$ of each 6-belt of $P$; equivalently, $p$ is parallel to at least one facet of each 6-belt.

**Proof.** If $v$ does not satisfy (ii) then Figure 1(i) shows that there is a 8-belt in $P + z(v)$. So, (ii) is necessary.
Assume that (ii) is true. We start by showing that the first two Venkov conditions must then hold for $P + z(v)$. Since $P$ and $z(v)$ are centrally symmetric, the Minkowski sum $P + z(v)$ is centrally symmetric as well. If $F$ is a facet of $P + z(v)$ that is a translate of a facet of $P$, then it is necessarily centrally symmetric. If the facet $F$ is not such a translate, then it has the form $F = G + z(v)$, where $G$ is either a $(n - 1)$ or $(n - 2)$-face of $P$. If $G$ is a facet of $P$, then it is centrally symmetric, and therefore the sum $G + z(v)$ is also symmetric. If $G$ is a $n - 2$-face, then condition (ii) implies that $G$ belongs to a 4-belt of $P$ and is symmetric. For this reason $G + z(v)$ is also symmetric.

We now show that $P + z(v)$ has only 4- and 6-belts. We establish this result using a local argument — we show that each $(n - 2)$-face $G$ of the polytope $P + z(v)$ belongs to either 3 or 4 distinct translates of $P + z(v)$ that fit facet-to-facet around $G$. From this local tiling property it immediately follows that all belts of $P + z(v)$ have length 4 or 6.

We need to consider the three possibilities for a $(n - 2)$-face, $G_v$, of $P + z(v)$:

1. $G_v = G \pm v$ (this notation means that $F$ is a translate of $G$ by $v$ or $-v$), where $G$ is a $(n - 2)$-face of $P$;
2. $G_v = G + z(v)$, where $G$ is a $(n - 2)$-face of $P$;
3. $G_v = G + z(v)$, where $G$ is a $(n - 3)$-face of $P$ (and the sum is direct);

By condition (ii), the local tiling property for $G_v$ follows from that for $G$ in cases (1) and (2). So we need only consider case (3).

Consider the face-to-face tiling by parallel copies of $P$. Let $G$ be an $(n - 3)$-face of the tiling and let $\pi_G$ be a projection along the linear space $\text{Lin}(G)$ onto the complementary space $(\text{Lin}(G))^\perp$. The projections of the tiles sharing $G$ split a sufficiently small neighborhood of $\pi_G(G)$ in the same way as some 3-dimensional complete polyhedral fan $\text{Fan}(G)$ does. The incidence type of the face $G$ is a combinatorial type of $\text{Fan}(G)$.

It is convenient to label the incidence type of a face (and therefore the combinatorial type of its fan) by its dual cell. By definition, the dual cell $D(G)$ of a face $G$ is the convex hull of centers of all paralleloptopes in the tiling that have the face $G$.

The five possible combinatorial types of $\text{Fan}(G)$, where $G$ is a $(n - 3)$-face, were classified by B.N. Delaunay in [De29]. We label these types by their corresponding dual cells: (a) tetrahedron, (b) octahedron, (c) pyramid with quadrangular base, (d) prism with triangular base and (e) cube. In the cases (b) and (e) the face $G$ is standard.

Since the face $G$ is $(n - 3)$-dimensional, and $G + z(v)$ is an $(n - 2)$-dimensional face of $P + z(v)$, the segment $\pi_G(z(v))$ is non-degenerate. Further, if a 1-dimensional face (ray) of $\text{Fan}(G)$ is trivalent, i.e. incident to exactly 3 two-dimensional cones, then these two-dimensional cones correspond to facets of the same 6-belt. Thus for every trivalent ray of $\text{Fan}(G)$ the segment $\pi_G(z(v))$ is parallel to at least one two-dimensional face containing that ray.

We call the segment $\pi_G(z(v))$ feasible if for every trivalent ray of $\text{Fan}(G)$ one of the two-dimensional cones incident to this ray is parallel to $\pi_G(z(v))$. Respectively, we will call the face $G + z(v)$ a feasible extension of the face $G$.

We consider each of the five types of local structure for the $(n - 3)$-faces $G$ of $P$, and determine how this structure can change when the individual tiles are replaced.
by their sum with a feasible $v$. Remarkably, in all cases these modified tiles fit back together as they initially did.

We enumerate all combinatorially distinct pairs of $(\text{Fan}, z)$ with Fan a 3-dimensional fan and $z$ a segment with the following properties.

1. Fan is combinatorially equivalent to a fan of some $(n-3)$-face in some tiling of $\mathbb{R}^n$ by parallelohedra.
2. $z$ is feasible for Fan.

The way we do the enumeration is as follows: given a 3-dimensional fan Fan and a subset of its 2-dimensional faces, we can determine if the subset covers all trivalent rays of Fan, and if there is a segment $z$ parallel to every 2-dimensional face from the chosen subset. If the answer is positive for both instances, $z$ is feasible for Fan. Of course, every possible pair (Fan, $z$) will be considered through this enumeration.

As a result, we obtain a total of 13 types of pairs (Fan, $z$). All these cases are listed and described explicitly in Figures 2 and 3. In fact, each of the cases d.1), e.1) and e.2) covers an infinite series of segments for a fixed fan Fan, however the treatment is the same within each single case.

We also emphasize that each of the 13 cases has its own subcases, depending on which 3-dimensional cone of Fan corresponds to $P$. Still in each case it is easy enough to treat all subcases simultaneously.

In each subcase one can prove that either there is no feasible extension of the face $G$, or the copies of $P + z(v)$ fit together around $G + z(v)$. The details are provided in Appendix A.

Therefore the third Venkov condition for $P + z(v)$ is fulfilled. Thus $P + z(v)$ is a parallelootope.

In [Gr06b] a graph $\Gamma(P)$ is built as follows for every parallelootope $P$. The vertices of $\Gamma(P)$ represent facets and two vertices are said to be adjacent if there exists a 6-belt containing the corresponding facets. If one contracts all pairs of vertices of $\Gamma(P)$ corresponding to pairs of antipodal facets, the result will be the so-called red Venkov graph $\Gamma_{RV}(P)$. $\Gamma_{RV}(P)$ is connected iff $\Gamma(P)$ is connected.

$P$ is a direct sum of two non-trivial paralleloptopes $P_1$ and $P_2$ if and only if the red Venkov graph of $P$ is not connected [Gr06b, Proposition 4] (see also [Or04, Or14, RyZa05]). The following is an extension of Theorem 1 to Voronoï polytopes:

**Theorem 2.** Let $P$ be a non-decomposable Voronoï parallelootope and $v$ be a vector. The following assertions are equivalent:

1. $P + z(v)$ is affinely equivalent to a Voronoï polytope.
2. There exists $a > 0$ such that for any 6-belt with facet vectors $\{p_1, \ldots, p_6\}$ either $v^T p_i = 0$ for $1 \leq i \leq 6$ or $v^T p_i = 0$ for two indices and $v^T p_i = \pm a$ for the other four indices.

**Proof.** See [Gr06b, Lemma 3].

**Corollary 1.** Let $P$ be a non-decomposable parallelootope and assume that Voronoï conjecture is true. Then $P$ is finitely free.

**Proof.** Since $P$ is non-decomposable, every facet vector $w$ belongs to at least one 6-belt. So, if $v$ is a free vector then we have $w^Tv \in \{-a, 0, a\}$. Since the facet vectors span $\mathbb{R}^n$ the linear system has a unique solution for a given choice of signs and zeros. The result follows by remarking that there is only a finite number of possible choices of zeros of signs.
The connection between Voronoi conjecture and finite freedom is noticed in [Ve06].

3. LATTICE AND STANDARD VECTORS OF $P + z(v)$

Let $\Lambda(P)$ be the lattice generated by facet vectors of a parallelotope $P$, i.e. the lattice such that

$$\{ P + x : x \in \Lambda(P) \}$$

is a face-to-face tiling. Let $v$ be a non-zero free vector of $P$. In this section we will prove several relations between the lattices $\Lambda(P + z(v))$ and $\Lambda(P)$, as well as between the sets of standard vectors of $P$ and $P + z(v)$. 

Figure 2. Possible arrangements of free segments and $(n - 3)$-faces
Lemma 1. $\Lambda(P + z(v)) = A_v \Lambda(P)$, where $A_v x = x + 2n_v(x)v$ and $n_v(x) = e_v^T x$ is a linear function from $\Lambda(P)$ to $\mathbb{Z}$.

Proof. Find all facet vectors of $P + z(v)$. As mentioned before, each facet $F$ of $P + z(v)$ is of one of the following types.

- $F = F' + z(v)$, where $F'$ is a facet of $P$, $F' \parallel z(v)$. Then if $s$ and $s'$ are the facet vectors of $F$ and $F'$ respectively, then $s = s'$.
- $F = F' + v$, where $F'$ is a facet of $P$. Then $s = s' + 2v$.
- $F = F' - v$, where $F'$ is a facet of $P$. Then $s = s' - 2v$.
- $F = F' + z(v)$, where $F'$ is a standard $(n-2)$-face of $P$ and $v$ is strongly transversal to $F'$. If $s'$ now denotes the standard vector of $F'$, then $s = s'$.  

From here it follows that every point of $\Lambda(P + z(v))$ is $x + 2nv$, where $x \in \Lambda(P)$ and $n \in \mathbb{Z}$. Moreover, for every $x \in \Lambda(P)$ there exists at least one point of $\Lambda(P + z(v))$ of that form.

Now suppose that $x + 2n_1v$ and $x + 2n_2v$ are points of $\Lambda(P + z(v))$. Then, obviously,

$$x + 2\lambda_n_1v \in \Lambda(P + \lambda z(v)) \quad \text{and} \quad x + 2\lambda_n_2v \in \Lambda(P + \lambda z(v))$$

for every $\lambda > 0$. Then there are two points of $\Lambda(P + \lambda z(v))$ which are arbitrarily close to each other as $\lambda \to 0$. This is impossible, so $n$ is a function of $x$.

Since we suppose that $P + z(v)$ is a parallelootope, it is of non-zero width along $v$. Venkov asserts and proves [Ve59, item (2) of Theorem 1] that facet vectors of all
facets parallel to \( v \) generate an \((n-1)\)-dimensional sublattice if a parallelootope has non-zero width in direction \( v \). In our case, denote this sublattice of \( \Lambda(P + z(v)) \) by \( L_v \). Obviously, we have also \( L_v \subset \Lambda(P) \). The lattice \( L_v \) determines a partition of \( \Lambda(P) \) into \((n-1)\)-dimensional layers.

The layers are equally spaced and cover the entire lattice \( \Lambda(P) \). Thus there exists a vector \( e_v \) such that the scalar product \( e_v^T x \) runs through all integers while \( x \) runs through \( \Lambda(P) \).

Put \( n_v(x) = e_v^T x \) and \( A_v x = x + 2 n_v(x) v \) as in the condition of the Lemma. Then every facet vector of the parallelootope \( P + z(v) \) has the form of \( A_v s \), where \( s \) is some standard vector of \( P \), and every facet vector of the parallelootope \( P \) has the form of \( A_v^{-1} s' \), where \( s \) is a facet vector of \( P + z(v) \). Hence \( A_v \) is an isomorphism between \( \Lambda(P) \) and \( \Lambda(P + z(v)) \). \( \square \)

Remark. It is not hard to see that \( n_v(x) = n_{\lambda v}(x) \) for every \( \lambda > 0 \).

**Lemma 2.** Let \( x_1, x_2, \ldots, x_k \) be points of \( \Lambda(P) \). Then the points

\[
A_v x_1, A_v x_2, \ldots, A_v x_k
\]

are affinely dependent if and only if \( x_1, x_2, \ldots, x_k \) are affinely dependent.

**Proof.** The matrix \( A_v \) is non-singular since \( \Lambda(P + z(v)) \) is full-dimensional. Hence the result follows. \( \square \)

**Lemma 3.** Let \( x \in \Lambda(P) \). Then the following conditions are equivalent.

(i) \( A_v x \) is a standard vector for \( P + z(v) \).

(ii) \( x \) is a standard vector for \( P \) and \( n_x(v) \in \{0, \pm 1\} \).

**Proof.** In this proof we write \( P + x \) for the translate of \( P \) centered at \( x \). This coincides with the standard notation under assumption that \( P \) is centered at the origin.

(ii) \( \Rightarrow \) (i). If \( x \) is a standard vector, then \( P + x \) and \( P \) share the point \( x/2 \). Under assumption \( n_v(x) \in \{0, \pm 1\} \) it immediately follows that

\[
(P + z(v)) + A_v x \quad \text{and} \quad P + z(v)
\]

share the point \( A_v x/2 \), which is equivalent to (i).

(i) \( \Rightarrow \) (ii). Suppose that the parallelootopes

\[
(P + z(v)) + A_v x \quad \text{and} \quad (P + z(v))
\]

share the point \( z = A_v x/2 \). Let \( \ell \) be the line passing through \( z \) and parallel to \( v \).

Since the tiling is face-to-face, then there are three options.

- For some \( \alpha \geq 1 \) we have

\[
\{(P + z(v)) + A_v x\} \cap \ell = (P + z(v)) \cap \ell = [v(z - \alpha v), v(z + \alpha v)]
\]

- For some \( \alpha \geq 1 \) we have

\[
\left\{
\begin{array}{ll}
\{(P + z(v)) + A_v x\} \cap \ell = [z - 2 \alpha v, z] \\
(P + z(v)) \cap \ell = [z, z + 2 \alpha v]
\end{array}
\right.
\]

- For some \( \alpha \geq 1 \) we have

\[
\left\{
\begin{array}{ll}
\{(P + z(v)) + A_v x\} \cap \ell = [z, z + 2 \alpha v] \\
(P + z(v)) \cap \ell = [z - 2 \alpha v, z]
\end{array}
\right.
\]
Now notice that for every $\lambda > 0$ there is a face-to-face tiling by translates of $P + \lambda z(v)$. Using this, consider all three cases separately.

If Equation (2) holds, then $n_v(x) = 0$. Otherwise the tiling by $P + \lambda z(v)$ is not face-to face for $\lambda = 1 + \epsilon$ (and for $\lambda = 1 - \epsilon$) if $\epsilon$ is a sufficiently small positive number. Indeed, one can check that $(P + \lambda z(v)) + A_{\lambda x} \cap (P + \lambda z(v))$ is nonempty, but this intersection is not a face of any of the two parallelotopes.

If Equation (3) holds, then $n_v(x) = -1$. Indeed, if $n_v(x) < -1$, then the tiling by $P + \lambda z(v)$ is not face-to face for $\lambda = 1 - \epsilon$, where $\epsilon$ is a sufficiently small positive number. If $n_v(x) > -1$, then the tiling by $P + \lambda z(v)$ is not face-to face for $\lambda = 1 + \epsilon$.

The case of Equation (4) does not differ with the case of (3), and we get $n_v(x) = 1$. \hfill \Box

4. Algorithms for computing free vectors

For highly symmetric lattices, some efficient techniques for computing their Delaunay tessellation and so by duality Voronoi polytopes have been introduced in [DSV09]. These techniques use the quadratic form viewpoint for the actual computation and can be adapted to the enumeration of free vectors and strongly regular faces.

For the enumeration of free vectors, Theorem 1 gives implicitly a method for enumerating them. The first step is, given a lattice $L$, to use [DSV09] in order to get the Delaunay tessellation $D(L)$. The triangular faces of $D(L)$ enumerate all 6-belts of the parallelotope $P_V(L)$. By Theorem 1, every free vector of $P_V(L)$ must satisfy an orthogonality condition for each 6-belt. So, if we have $N$ 6-belts, then we have $3^N$ cases to consider, which can be large. The enumeration technique consider the 6-belts one by one by making choice at each step. We use symmetries to only keep non-isomorphic representatives of all choices. We also use the fact that any choice among the three possibilities implies a linear equality on the coefficients of the free vector $v$. Hence the choice made for some 6-belts might imply other choices for other 6-belts. So, the dimension decreases at each step and the number of choices is thus only $3^n$ at most, where $m = \min(n, N)$ and $n$ is dimension of $L$. At the end we have a number of vector spaces containing the free vectors. $P_V(L)$ is nonfree, respectively finitely free, if and only if all the vector spaces are 0-dimensional, respectively at most 1-dimensional.

The enumeration of strongly transversal faces can be done in the following way: If $G$ is a strongly transversal face of $P$, then any subface of $G$ is also strongly transversal. Thus starting from the vertices of $P$, which correspond to Delaunay polytopes of $P$, we can enumerate all strongly transversal faces of $P$ and hence describe the facet and belt structure of $P + Z(U)$ with $U$ a set of free vectors. By using Venkov’s condition this allows to determine whether or not $P + Z(U)$ is a parallelotope.

Another variant is to write $U = \{u_1, \ldots, u_p\}$ and write $P + Z(U) = P' + z(u_p)$ with $P' = P + Z(\{u_1, \ldots, u_{p-1}\})$.

If we know that $P'$ is a Voronoi polytope, then we can test whether $P + Z(U)$ is a parallelotope by testing whether $u_p$ is a free vector via Theorem 1. Furthermore, by using the sign condition from Theorem 2, we can test whether $P + Z(U)$ is a Voronoi polytope. The method relies on computing the Delaunay tessellation.
We choose to use the second method because it allows us to distinguish between parallelotope and Voronoi polytope. The process of enumeration is then done by considering all subsets $U$ of the set of free vectors and adding vectors one by one and testing whether they are feasible or not. By $F_{\text{min}}(L)$ we denote the set of minimal forbidden subsets of $L$. Similarly $F_{\text{max}}(L)$ denotes the maximal feasible subsets of $L$. Key information on those subsets are given in Table 4 for 12 lattices. The cost of computing the Delaunay tessellation is relatively expensive, hence we always use the list of already known forbidden subsets in order to avoid such computation whenever possible. We found out that in the case that we consider in Section 11, whenever a polytope $P_V(L) + Z(U)$ is a parallelotope then it is also a Voronoi polytope, thereby confirming Voronoi’s conjecture in those cases.

However, for $E_6$ we prefer to use the first enumeration method. Then we confirm the outcome by an explicit proof.

5. The root lattices $E_6$ and $E^*_6$

There are exactly 27 straight lines on any smooth non-degenerate cubic surface in the 3-dimensional projective space. This fact is very well known (see, for example, [Cox83], and other papers of Coxeter). It is proved in many textbooks on algebraic geometry (see, for example, [Re88, ch.3, §7], [Sha88, ch.IV, §2.5]). The combinatorial configuration of the set $L$ of these 27 lines is unique. For example, if two lines $l, l' \in L$ intersect, then there is a unique line $l'' \in L$ that intersects both lines $l, l'$. Every three mutually intersecting lines generate a tangent plane of the cubic surface. Each line intersects exactly 10 lines and belongs to exactly 5 of all 45 tangent planes.

Schläfli described his famous double six

\[
\begin{array}{cccccc}
  a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\
  b_1 & b_2 & b_3 & b_4 & b_5 & b_6
\end{array}
\]

which is a special arrangement of twelve lines from $L$ on the surface such that any two of them intersect if and only if they occur in different rows and different columns. Any two columns determine a pair of planes $a_i b_j$ and $a_j b_i$ whose intersection $c_{ij} = c_{ji}$ intersects all the four lines and therefore must lie entirely on the surface. In this way Schläfli obtained his notations $a_i, b_j, c_{ij}$ for all 27 lines.

Burnside [Burn11, pp.485–488], used these symbols of 27 lines as elements of an algebra (in fact as vectors in 6-dimensional space). This amounts to representing the lines by 27 points in an affine 6-space, such that the 45 triangles representing the tangent planes all have the same centroid. Using this centroid as the origin, he applies Schläfli’s symbols for the 27 lines to the positions of vectors of the 27 points, so that

(5) \[ a_i + b_j + c_{ij} = 0, \quad c_{ij} + c_{kl} + c_{mn} = 0, \text{ where } \{ijklmn\} = \{123456\} = I_6. \]

Choose a set $\mathcal{A}$ of six vectors $a_1, a_2, \ldots, a_6 \in \mathbb{R}^6$ so that $a_i^T a_i = 4/3$ and $a_i^T a_j = 1/3$ for all possible $i \neq j$. One can see that such set $\mathcal{A}$ exists and forms a basis of $\mathbb{R}^6$.

Define

\[ h = \frac{1}{3} \sum_{i \in I_6} a_i, \]

and let $b_i = a_i - h$, and $c_{ij} = h - a_i - a_j$ for all possible $i \neq j$. 

The lattice integrally generated by the basis $A$ and the vector $h$ is $E_6^*$ which is dual to the root lattice $E_6$. This representation of the lattice $E_6^*$ was used by Baranovskii in [Ba91] for a description of Delaunay polytopes of $E_6^*$, Barnes [Barn97] Formula (8.5)] uses the basis $A$ to describe minimal vectors of $E_6^*$. Vectors of $E_6^*$ have all coordinates in the basis $A$ equal to one third of an integer. The 27 vectors of the set

$$ \mathcal{M} = \{ a_i, b_i = a_i - h, c_{ij} = h - a_i - a_j \text{, where } 1 \leq i < j \leq 6 \}, $$

are, up to sign, the 27 minimal vectors of the lattice $E_6^*$. The lattice $E_6^*$ has an automorphism group equal to $W(E_6) \times \{ \pm Id_6 \}$ with $W(E_6)$ being the Weyl group of $E_6$ (see [Hu90] for more details), which is also the automorphism group of $\mathcal{M}$.

Let $A$ be the Gram matrix for the basis $A$. Then the inverse matrix $E = A^{-1}$, which is the Gram matrix of the dual basis $\mathcal{E} = \{ e_1, e_2, \ldots, e_6 \}$, has elements

$$ e_{ii} = e_i^2 = \frac{8}{9}, \quad i \in I_6, \quad e_{ij} = e_{ji} = e_i^T e_j = -\frac{1}{9}, \quad i \neq j. $$

It is easy to check that vectors of the dual lattice $E_6$ have in the basis $\mathcal{E}$ the following form

$$ \sum_{i \in I_6} z_i e_i, \text{ where } z_i \in \mathbb{Z} \text{ and } \sum_{i \in I_6} z_i \equiv 0 \pmod{3}. $$

6. The Schl"afli polytope $P_{\text{Schl}}$

The convex hull of end-points of all vectors of the set $\mathcal{M}$ is the Schl"afli polytope $P_{\text{Schl}}$, i.e., $P_{\text{Schl}} = \text{conv } \mathcal{M}$. Since $P_{\text{Schl}}$ is a Delaunay polytope of the lattice $E_6$ and the Voronoi polytope $P_v(E_6)$ of the lattice $E_6$ is the convex hull of $P_{\text{Schl}}$ and its centrally symmetric copy $P_{\text{Schl}}^*$ (see [CS91]), we have to study properties of $P_{\text{Schl}}$.

Two vertices $q, q' \in \mathcal{M}$ of $P_{\text{Schl}}$ are adjacent by an edge if and only if $q^T q' = \frac{1}{3}$, i.e., if the corresponding lines of $\mathcal{L}$ are skew. Let $X \subseteq \mathcal{M}$ be a subset of cardinality 6 such that $q^T q' = \frac{1}{3}$ for all $q, q' \in X, q \neq q'$. Then $\text{conv } X$ is a simplicial facet $\alpha_5(r)$, where $r = 1/3 \sum q$. Let $q \in \mathcal{M}$. Then there are 5 non-ordered pairs

$$ q'_1, q''_1 \in \mathcal{M}, \quad i = 1, 2, \ldots, 5 \quad \text{with } q + q'_1 + q''_1 = 0. $$

Denote by $T(q)$ the set $\{ q'_1, q''_1, q'_2, q''_2, \ldots, q'_5, q''_5 \}$. Then $\beta_5(q) = \text{conv } T(q)$ is a cross-polytopal facet of $P_{\text{Schl}}$.

The polytope $P_{\text{Schl}}$ has two types of facets: 36 pairs of simplicial facets $\alpha_5(r)$ and $\alpha_5(-r)$; and 27 cross-polytopal facets of form $\beta_5(q)$. Each facet $\beta_5(q)$ is orthogonal to the corresponding vector $q \in \mathcal{M}$ and is opposite to the vertex $q$. Both types of facets are regular polytopes with facets $\alpha_4$. Thus all faces of $P_{\text{Schl}}$ of dimension $k \leq 4$ are regular simplices $\alpha_k$.

Let $I_5 = \{ 1, 2, \ldots, 5 \}$, and let $q_1', q''_1, i \in I_5$, be the five pairs of opposite vertices of the facet $\beta_5(q)$. For every $J \subseteq I_5$ and $J' = I_5 \setminus J$ define

$$ T_J(q) = \{ q_i' : i \in J \} \cup \{ q_i'' : i \in J' \}. $$

Obviously, for each of the 32 subsets $J \subseteq I_5$ the polytope $\text{conv } T_J(q)$ is a 4-simplex which is a facet of $\beta_5(q)$, and vice versa: if $\alpha_4$ is a facet of $\beta_5(q)$, then $\alpha_4 = \text{conv } T_J(q)$ for some $J \subseteq I_5$.

**Lemma 4.** Any subset $X \subseteq \mathcal{M}$ of cardinality $|X| = 5$ such that $p^T p' = \frac{1}{3}$ for all $p, p' \in X, p \neq p'$, is a set $T_J(q)$ for some $q \in \mathcal{M}$ and $J \subseteq I_5$.
Table 1. Faces $G$ of the Schlafli polytope $P_{Schl}$.

| dim $G$ | 0    | 1    | 2    | 3    | 4    | 5    | 6    |
|--------|------|------|------|------|------|------|------|
| type of $G$ | vertex $\alpha_0$ | edge $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\alpha_4$ | $\alpha_5$ or $\beta_5$ | $P_{Schl}$ |
| $n(G)$ | 27   | 216  | 720  | 1080 | 432 + 216 | 72 + 27 | 1    |

**Proof.** It is not hard to see that $\text{conv } X = \alpha_4(X)$ is a 4-face of $P_{Schl}$. Therefore, $\alpha_4(X)$ is a 4-face either of a facet $\beta_5(q)$ for some $q \in M$, or of a facet $\alpha_5(r)$ for some $r \in R$. But each facet of type $\alpha_5(r)$ is contiguous in $P_{Schl}$ by a 4-face only to facets of type $\beta_5(q)$. This means that each 4-face of $P_{Schl}$ is a 4-face of some $\beta_5(q)$. The vertex set of each 4-dimensional subface of $\beta_5(q)$ equals $T_J(q)$ for some $J \subseteq I_5$. □

**7. The Voronoi polytope $P_{V}(E_6)$**

The convex hull of $P_{Schl} = \text{conv } M$ and its centrally symmetric copy $P_{Schl}^* = \text{conv}(\neg M)$ is the Voronoi polytope of the lattice $E_6$, i.e.,

$$P_{V}(E_6) = \text{conv}(P_{Schl} \cup P_{Schl}^*) = \text{diplo}(P_{Schl}),$$

(see [CS91], where the notation $\text{conv}(P \cup P^*) = \text{diplo}(P)$ is introduced). We also mention the notable fact that the intersection $P_{Schl} \cap P_{Schl}^*$ is the Voronoi polytope of the dual lattice $E_6^*$. $P_{V}(E_6)$ has 27 pairs of opposite vertices $q, -q$ for $q \in M$, and 36 pairs of parallel opposite facets

$$F(r) = \text{conv}((\alpha_5(r) \cup (\neg \alpha_5(-r)))) = \text{diplo}(\alpha_5).$$

The edges of $P_{V}(E_6)$ are of two types: $r$- and $m$-edges. Vertices $q$ and $q'$ are adjacent by an $r$-edge if and only if $q^Tq' = \frac{1}{3}$, and either $q, q' \in M$ or $q, q' \in \neg M$. Each $m$-edge connects a vertex $q \in M$ to a vertex $-q' \in \neg M$. The $m$-edge between $q$ and $q'$ exists if and only if $q^Tq' = -\frac{4}{3}$.

Dolbilin in [Do09] shows that there is a one-to-one correspondence between minimal vectors of a coset of $L/2L$ and standard faces of the Voronoi polytope $P_{V}(L)$. A face $G$ of $P_{V}(L)$ is called standard if $G = G(t) = P_{V}(L) \cap P(t)$, where $P(t) = P_{V}(L) + t$ and $t$ is a minimal vector of a coset $L/2L$. If $G(t)$ is a facet of $P_{V}(L)$, then $t$ coincides with its facet vector.

Besides, there is a one-to-one correspondence between standard faces of $P_{V}(L)$ and centrally symmetric faces of the Delaunay tiling. Let $D(G)$ be a centrally symmetric Delaunay face related to a standard face $G(t)$. Then $t$ is a diagonal of the centrally symmetric Delaunay face $D(G)$. All other diagonals of $D(G)$ are all other minimal vectors of the coset containing $t$.

Translates and centrosymmetrical copies of $P_{Schl}$ form the Delaunay tiling for $E_6$. Hence all minimal vectors of $E_6$ are exactly all vectors connecting pairs of vertices of $P_{Schl}$. These vectors naturally split into two classes:

$$R = \{ p - p' : p, p' \in M, p^Tp' = 1/3 \} \quad \text{and}$$

$$T = \{ p - p' : p, p' \in M, p^Tp' = -2/3 \}.$$  

If $p^Tp' = 1/3$, then $[(v(p), v(p'))]$ is an edge of $P_{Schl}$, i.e. a 1-dimensional Delaunay cell of $E_6$. Thus $R$ is the set of facet vectors of $P_{V}(E_6)$. The notation $F(r)$ for the
facet of $P_V(E_6)$ with $r$ being its facet vector coincides with the definition of $F(r)$ above.

Two facets $F(r)$ and $F(r')$ of $P_V(E_6)$ intersect by a 4-face if and only if $r^T r' = 1$.
In this case $r'' = r - r' \in \mathcal{R}$, and $r^T r'' = -(r')^T r'' = 1$. Hence the six facets $F(\pm r), F(\pm r'), F(\pm r'')$ form a 6-belt. The Voronoi polytope $P_V(E_6)$ has 120 6-belts studied in Section 8.

If $p^T p' = -2/3$, then $[v(p), v(p')]$ is a diagonal of a facet of $P_{\mathcal{G}chl}$. Consequently, each vector $t \in \mathcal{T}$ is a standard vector of some $m$-edge of $P_V(E_6)$. Denote the endpoints of that edge by $a(t)$ and $-b(t)$ so that $a(t), b(t) \in \mathcal{M}$. Let $q(t) = -a(t) - b(t)$. It is easy to check that $q(t) \in \mathcal{M}$.

Since $\mathcal{R} \cup \mathcal{T}$ is the set of all minimal vectors of $E_6$, the Voronoi parallelotope $P_V(E_6)$ has no standard faces other than its facets and $m$-edges.

8. Freedom of $P_V(E_6)$

Recall that a vector $v$ is free for a parallelotope $P$ if the Minkowski sum $P + z(v)$ is also a parallelotope.

**Proposition 1.** There are the following 120 triples generating 6-belts of the Voronoi polytope $P_V(E_6)$, where facet vectors are given in the basis $\mathcal{E}$, $e(I_6) = \sum_{i \in I_6} e_i$,

$$e(S) = \sum_{i \in S} e_i$$

and $S$ is a subset of $I_6$ of cardinality 3.

(i) $e_i - e_j, e_j - e_k, e_k - e_i$, $i, j, k \in I_6$;
(ii) $e(S), e(S), e(I_6), \mathcal{S} = I_6 - S$;
(iii) $e(S), e_i - e_j, e(S) - e_i + e_j$, $i \in S, j \notin S$.

**Proof.** It is easy to verify that there are 20 triples of type (i), 10 triples of type (ii) and 90 triples of type (iii), total 120 triples. Each 6-belt is uniquely determined by each of its six 4-faces. It is known (see, for example [CS91]) that $P_V(E_6)$ has 720 4-faces. Since each 4-face belongs exactly to one 6-belt, $P_V(E_6)$ has $\frac{720}{6} = 120$ 6-belts.

Note that the above 6-belts form 1-vertex under the action of the automorphism group of $E_6$.

Call a vector free for a triple if it is orthogonal at least to one vector of the triple.

**Theorem 3.** The Voronoi polytope $P_V(E_6)$ is free along a line $I$ if and only if $l$ is spanned by a minimal vector $q \in \mathcal{M}$ of the dual lattice $E_6^*$, described in [4] and [5].

**Proof.** We seek a vector $a$ which is free for $P_V(E_6)$ in the basis $A = \{a_i : i \in I_6\}$ related to the lattice $E_6$ which is dual to the basis $\mathcal{E} = \{e_i : i \in I_6\}$ related to the lattice $E_6$. So, let $a = \sum_{i \in I_6} z_i a_i$ be a vector which is free for $P_V(E_6)$. We find conditions, when the vector $a$ is orthogonal to at least one vector of each triple of types (i), (ii) and (iii) of Proposition 1. We shall see that $a$ is, up to a multiple, one of the vectors $a_i, b_j, c_{kl}, i, j, k, l \in I_6$, of [4] and [5].

**Claim 1.** The coordinates $z_i, i \in I_6$ take only two values. Suppose that there are three pairwise distinct coordinates $z_i \neq z_j \neq z_k \neq z_l$. Then the vector $a$ is not free for a triple of type (i).

So, a free vector has the form $a_T = za(T) + z'a(T)$, where $T \subseteq I_6$, $\bar{T} = I_6 - T$ and $a(T) = \sum_{i \in T} a_i$.

**Claim 2.** If $z = 0$ or $z' = 0$, then $|\bar{T}| = 1$, or $|T| = 1$, respectively. In fact, if $z' = 0$ and $|T| \geq 2$, then the vector $a = za(T)$ is not free for each triple of type (ii) such that $S \cap T \neq \emptyset$ and $S \cap \bar{T} \neq \emptyset$. 

Note that Claim 2 implies that $T \neq \emptyset$ and $T \neq I_6$.

It is easy to verify that each of the six vectors $a_i$, $i \in I_6$, (which are minimal vectors of $E^*_6$) is free for all triples.

Now consider vectors $a_T = za(T) + z'a(T)$, with both non-zero coefficients $z$ and $z'$.

Claim 3. $|T| \neq 3$. In fact, let $|T| = 3$. Consider a triple of type (ii) for $S = T$. The vector $a_T$ is free for this triple only if $e(I_6)^T a_T = z + z' = 0$. Hence, $a_T$ should take the form $a_T = z(a(T) - a(T))$. But this vector is not free for a triple of type (iii) with $S = T$ and $i \in T$, $j \not\in T$.

So, without loss of generality we can consider vectors $a_T$ such that $|T| = 1, 2$.

Claim 4. $z = -2z'$. Let $|T| = 1$. For $a_T$ to be free for a triple of type (ii) with $S \supset T$, the coefficients $z$, $z'$ should satisfy one of the equalities $z + 2z' = 0$ or $z + 5z' = 0$. If $z = -5z'$, then $a_T$ is not free for a triple of type (iii) such that $S \subseteq T$. Hence, $z = -2z'$. It is easy to verify that, for all $i \in I_6$, the vector $a_T = -z'(2a_i - a(1_6 - \{i\})) = -3z'(a_i - \frac{1}{3}a(I_6)) = -3z'a_i$ is free for all triples.

Now, let $|T| = 2$. For $a_T$ to be free for a triple of type (ii) with $S \supset T$, the coefficients $z$ and $z'$ should satisfy one of the equalities $2z + z' = 0$ or $2z + 4z' = 0$. If $z' = -2z$, then the vector $a_T = z(a(T) - 2a(T))$ is not free for a triple of type (iii) such that $S \cap T = \{i\}$ and $j \not\in T$. One can verify that if $z = -2z'$ and $T = \{ij\}$, then, for $1 \leq i < j \leq 6$, the vector $a_T = 3z'(a_i + a_j - \frac{1}{3}a(I_6)) = 3z'c_{ij}$ is free for triples of all types. \hfill \square

9. A SUM OF $P_V(E_6)$ WITH A ZONOTOPE

Our main goal is to classify all subsets $U \subseteq M$ such that $P + Z(U)$ is a parallelotope. If $P + Z(U)$ is a parallelotope, we call $U$ feasible, otherwise $U$ is called forbidden.

We say that that $U_0 \subseteq M$ is a minimal forbidden set for $P_V(E_6)$, if $P_V(E_6) + Z(U_0)$ is not a parallelotope, but $P_V(E_6) + Z(U_1)$ is a parallelotope for every $U_1 \subseteq U_0$. Obviously, every forbidden set $U \subseteq M$ contains some minimal forbidden subset $U_0$.

Let $U_0 \subseteq M$ be a minimal forbidden set for $P_V(E_6)$. Choose an arbitrary element $p_0 \in U_0$ and let $U_1 = U_0 \setminus \{p_0\}$ be the pre-forbidden set. Notice that $p_0$ is not a free vector for the parallelotope $P_V(E_6) + Z(U_1)$. Hence by Theorem 4 $P_V(E_6) + Z(U_1)$ has a 6-belt such that $p_0$ is not parallel to any facet of that belt. This motivates us to study 6-belts of $P_V(E_6) + Z(U_1)$.

Lemma 3 implies that each standard vector $s$ of $P_V(E_6) + Z(U_1)$ can be represented in the form

$$s = s' + \sum_{p \in U_1} 2n_p(s')p,$$

where $s'$ is a standard vector of $P_V(E_6)$. In this situation we say that $s$ corresponds to $s'$ and vice versa.

Thus every facet vector of $P_V(E_6) + Z(U_1)$ arises from some standard vector of $P_V(E_6)$. Moreover, from Lemma 2 follows that the 3 pairs vectors of each 6-belt of $P_V(E_6) + Z(U_1)$ arise from 3 pairs of coplanar standard vectors of $P_V(E_6)$.

So, for what follows 2-dimensional subsets of the set $R \cup T$ are important. Each of these subsets generates a 2-plane $\alpha$, and it is the intersection $(R \cup T) \cap \alpha$. 

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Now we state several lemmas that are necessary to describe all minimal forbidden sets for \( P_V(E_6) \). Their proofs are technical, and therefore transferred to the Appendix B.

**Lemma 5.** Let \( \alpha \) be a two-dimensional plane such that the intersection 
\[
(R \cup T) \cap \alpha
\]
consists of at least 6 vectors, i.e., at least three pairs of antipodal vectors. Then this intersection is equivalent to one of the following 5 planar sets up to the action of \( W(E_6) \).

(a) \( \{ \pm (a_2 - a_1), \pm (b_1 - a_1), \pm (a_2 - b_1), \pm (b_2 - a_1) \} \);
(b) \( \{ \pm (a_2 - a_1), \pm (a_3 - a_1), \pm (a_3 - a_2) \} \);
(c) \( \{ \pm (b_2 - a_1), \pm (c_{12} - a_1), \pm (c_{12} - b_2) \} \);
(d) \( \{ \pm (b_2 - a_1), \pm (b_1 - a_3), \pm (c_{23} - c_{14}) \} \);
(e) \( \{ \pm (b_2 - a_1), \pm (b_3 - a_1), \pm (b_3 - b_2) \} \).

**Lemma 6.** Let \( p \in M \). Then the direct Minkowski sum \( [a_2, -b_1] \oplus z(p) \) is a standard 2-face of the parallelotope \( P_V(E_6) + z(p) \) iff
\[
p \in \{ a_1, b_2, c_{34}, c_{35}, c_{36}, c_{45}, c_{46}, c_{56} \}.
\]

**Lemma 7.** Let \( U \subset M \) and let \( P_V(E_6) + Z(U) \) be a parallelotope. Then

1. \( P_V(E_6) + Z(U) \) necessarily has a 6-belt with facet vectors corresponding to
\[
\{ \pm (a_2 - a_1), \pm (a_3 - a_1), \pm (a_3 - a_2) \}.
\]
2. \( P_V(E_6) + Z(U) \) has a 6-belt with facet vectors corresponding to
\[
\{ \pm (a_2 - a_1), \pm (b_1 - a_1), \pm (b_2 - a_1) \},
\]
iff \( U \) contains a subset \( U' = \{ p_1, p_2, p_3, p_4 \} \), where
\[
p_1 \in \{ c_{34}, c_{56} \}, \ p_2 \in \{ c_{35}, c_{46} \}, \ p_3 \in \{ c_{36}, c_{45} \}, \ p_4 \in \{ a_2, b_1 \}.
\]
3. \( P_V(E_6) + Z(U) \) has a 6-belt with facet vectors corresponding to
\[
\{ \pm (b_2 - a_1), \pm (b_3 - a_1), \pm (b_3 - b_2) \},
\]
iff \( U \) contains a subset \( \{ b_1, c_{45}, c_{46}, c_{56} \} \).
4. \( P_V(E_6) + Z(U) \) has no 6-belt with facet vectors corresponding to
\[
\{ \pm (a_2 - a_1), \pm (b_1 - a_2), \pm (b_2 - a_1) \}.
\]
5. \( P_V(E_6) + Z(U) \) has no 6-belt with facet vectors corresponding to
\[
\{ \pm (b_2 - a_1), \pm (c_{12} - a_1), \pm (c_{12} - b_2) \}.
\]
6. \( P_V(E_6) + Z(U) \) has no 6-belt with facet vectors corresponding to
\[
\{ \pm (b_2 - a_1), \pm (b_1 - a_3), \pm (c_{23} - c_{14}) \}.
\]

Now we are ready to state and prove the main result.

**Theorem 4.** Let \( U \subset M \). The following assertions are equivalent.

(i) There exists a subset \( U' \subseteq U \) such that \( |U'| = 5 \) and \( p^T p' = 1/3 \) for every \( p, p' \in U' \), \( p \neq p' \).

(ii) The polytope \( P_V(E_6) + Z(U) \) is not a parallelotope.
Then the facets of this belt are parallel to the hyperplanes that correspond exactly to the 6-tuple $c_{12}$. Thus $U = \{ p_1, p_2, p_3, p_4 \}$, where

$$p_1 \in \{ c_{34}, c_{56} \}, \quad p_2 \in \{ c_{35}, c_{46} \}, \quad p_3 \in \{ c_{36}, c_{45} \}, \quad p_4 \in \{ a_2, b_1 \}, b_5 \in \{ a_1, b_2 \}.$$ 

Let $U_1 = \{ p_1, p_2, p_3, p_4 \}, U_2 = \{ p_1, p_2, p_3, p_5 \}$. Since $P_V(E_6) + Z(U_0)$ is a parallelotope, so are $P_V(E_6) + Z(U_1)$ and $P_V(E_6) + Z(U_2)$.

By Lemma 2 and Assertion 2, the parallelotope $P_V(E_6) + Z(U_1)$ has a 6-belt with facet vectors corresponding to

$$\{ \pm (a_2 - a_1), \pm (b_1 - a_1), \pm (b_2 - a_1) \}.$$ 

On the other hand, the parallelotope $P_V(E_6) + Z(U_2)$ has a 6-belt with facet vectors corresponding to

$$\{ \pm (a_2 - a_1), \pm (b_1 - a_1), \pm (b_1 - a_2) \}.$$ 

Indeed, this statement is analogous to Lemma 4 and Assertion 2 up to interchange of $a_1$ and $a_2$.

Hence $P_V(E_6) + Z(U)$ has two 6-belts with 4 common facets (these belts are inherited from $P_V(E_6) + Z(U_1)$ and $P_V(E_6) + Z(U_2)$). But 2 different belts can share only 2 facets. The contradiction shows that $P_V(E_6) + Z(U)$ is not a parallelotope.

(ii)$\Rightarrow$(i) Suppose that $P_V(E_6) + Z(U)$ is not a parallelotope. Then $U$ contains some minimal forbidden subset, say $U_0 \subseteq U$. Choose arbitrarily $p \in U_0$ and let $U_1 = U_0 \setminus \{ p \}$. Then $P_V(E_6) + Z(U_1)$ is a parallelotope and $P_V(E_6) + Z(U_0) + z(p)$ is not. Thus $P_V(E_6) + Z(U_0)$ has a 6-belt with no facet of this belt parallel to $p$.

According to Lemmas 3 and 4 the 6-belt is of one of the three types.

1. The facet vectors of the 6-belt correspond to a configuration of vectors from $\mathcal{R} \cup \mathcal{T}$ that is equivalent to

$$\{ \pm (a_2 - a_1), \pm (a_3 - a_1), \pm (a_3 - a_2) \}$$

up to an isometry of $P_V(E_6)$.

2. The facet vectors of the 6-belt correspond to a configuration of vectors from $\mathcal{R} \cup \mathcal{T}$ that is equivalent to

$$\{ \pm (a_2 - a_1), \pm (b_1 - a_1), \pm (b_2 - a_1) \}$$

up to an isometry of $P_V(E_6)$.

3. The facet vectors of the 6-belt correspond to a configuration of vectors from $\mathcal{R} \cup \mathcal{T}$ that is equivalent to

$$\{ \pm (b_2 - a_1), \pm (b_3 - a_1), \pm (b_3 - b_2) \}$$

up to an isometry of $P_V(E_6)$.

Consider these three cases separately.

**Case 1.** Without loss of generality, assume that the facet vectors of the 6-belt correspond exactly to the 6-tuple

$$\{ \pm (a_2 - a_1), \pm (a_3 - a_1), \pm (a_3 - a_2) \}.$$ 

Then the facets of this belt are parallel to the hyperplanes

$$\langle a_2 - a_1 \rangle \perp, \langle a_3 - a_1 \rangle \perp, \langle a_3 - a_2 \rangle \perp.$$
These are exactly the facet directions of one particular 6-belt of $P_V(E_6)$. Since $p$ is free for $P_V(E_6)$, it is parallel to one of those facets. This contradicts the assumption on the vector $p$, so this case is impossible.

**Case 2.** Without loss of generality, assume that the facet vectors of the 6-belt correspond exactly to the 6-tuple $$\{\pm(a_2 - a_1), \pm(b_1 - a_1), \pm(b_2 - a_1)\}.$$ Then the 4-dimensional direction of this 6-belt is $$\langle a_2 - a_1, b_1 - a_1 \rangle^\perp,$$ because two pairs of facets of the belt are extended facets of $P_V(E_6)$ and have the same direction with those facets. Thus the orthogonal projection of $P_V(E_6) + Z(U_0)$ onto the 2-plane $\langle a_2 - a_1, b_1 - a_1 \rangle$ should be an 8-gon. Similarly to the proof of Assertion 2 of Lemma 7, we conclude that $U_0$ should contain a vector $p_5 \in \{a_1, b_2\}$ (as well as $a_2$ or $b_1$). By the same Assertion 2 of Lemma 7, $U_1$ also contains the set $\{p_1, p_2, p_3, p_4\}$, where $$p_1 \in \{c_{34}, c_{56}\}, \quad p_1 \in \{c_{35}, c_{46}\}, \quad p_3 \in \{c_{36}, c_{45}\}, \quad p_4 \in \{a_2, b_1\}.$$ Setting $U' = \{p_1, p_2, p_3, p_4, p_5\}$ finishes the proof of this case, because $p_i^T p_j = 1/3$ for every $1 \leq i < j \leq 5$. The last identity follows, for instance, from the fact that $U'$ is of the form $T_j(a_2 - a_1)$ for every possible choice of its elements.

**Case 3.** Without loss of generality, assume that the facet vectors of the 6-belt correspond exactly to the 6-tuple $$\{\pm(b_2 - a_1), \pm(b_3 - a_1), \pm(b_3 - b_2)\}.$$ By Assertion 3 of Lemma 7, the set $U_0$ contains a subset $\{b_1, c_{45}, c_{46}, c_{56}\}$. The proof of the same assertion also shows that the facets of the 6-belt are orthogonal to the vectors $b_2 - a_1$, $b_3 - a_1$, $b_3 - b_2$. Therefore the vector $p$, which is not parallel to any of those facets, is not orthogonal to any of the vectors $b_2 - a_1$, $b_3 - a_1$, $b_3 - b_2$. This is possible only for $p = b_2$ or $p = b_3$.

Setting $U' = \{b_1, b_2, c_{45}, c_{46}, c_{56}\}$ or $U' = \{b_1, b_2, c_{45}, c_{46}, c_{56}\}$ correspondingly, we finish the proof of this case.

Since every possible case is considered, the implication (ii)⇒(i) is proved. \qed

Recall that a subset $U \subset \mathcal{M}$ is feasible if $P_V(E_6) + Z(U)$ is a parallelotope. As mentioned before, all feasible subsets are unimodular. Hence the following corollary.

It contains no subset $U' \subset U$ such that $|U'| = 5$ and $p^T p' = 1/3$ for every $p, p' \in U'$, $p \neq p'$. If $U$ is feasible, then $P_V(E_6) + Z(U)$ is a parallelotope. This immediately gives the following corollary.

**Corollary 2.** Suppose that $U \subset \mathcal{M}$, and there is no subset $U' \subset U$ such that $|U'| = 5$ and $p^T p' = 1/3$ for every $p, p' \in U'$, $p \neq p'$. Then $U$ is unimodular.

10. **Maximal feasible subsets in $\mathcal{M}$**

It is convenient to denote a unimodular system $U$ by a regular matroid $M_U$, which is represented by $U$. There are many definitions of a matroid, see, for example, [Aig79]. In particular, a matroid on a ground set $X$ is a family $\mathcal{C}$ of circuits $C \subseteq X$ satisfying the following axioms:

- If $C_1, C_2 \in \mathcal{C}$, then $C_1 \nsubseteq C_2, C_2 \nsubseteq C_1$
- and if $x \in C_1 \cap C_2$, then there is $C_3 \in \mathcal{C}$ such that $C_3 \subseteq C_1 \cup C_2 - \{x\}$.
Note that linear dependencies between vectors of any family of vectors satisfy these axioms. A matroid $M$ on a set $X$ is represented by a set of vectors $U$ if there is a one-to-one map $f : X \to U$ such that, for all $C \in \mathcal{C}$, the set $f(C)$ is a minimal by inclusion linearly dependent subset of $U$.

Each unimodular set of vectors represents a regular matroid. Special cases of a regular matroids are graphic $M(G)$ and cographic $M^*(G)$ matroids whose ground sets are the set of edges of a graph $G$. The families of circuits of these matroids are cycles and cuts of the graph $G$, respectively.

Seymour proved in [Se80] that a regular matroid is a 1-, 2- and 3-sum of graphic, cographic matroids and a special matroids $R_{10}$, which is neither graphic nor cographic (see also [Tr98]). Using this work of Seymour, the authors of [DG99] described maximal by inclusion unimodular systems of a given dimension. Their description is similar to that of Seymour, but they denote a $k$-sum of Seymour by $(k - 1)$-sum, and slightly changed the definition of the $k$-sum. Namely, $k$-sum of Seymour gives the symmetric difference of the summing sets, but the corresponding $(k - 1)$-sum of [DG99] gives the union of the summing sets. Below, we use the second $k$-sum from [DG99].

The graphic matroid $M(K_{n+1})$ of the complete graph $K_{n+1}$ on $n + 1$ vertices is represented by the root system $A_n$, which is one of maximal by inclusion unimodular $n$-dimensional systems. The matroid $R_{10}$ is represented by a ten-element unimodular system of dimension 5, which is denoted in [DG99] by $E_5$. Each maximal by inclusion cographic unimodular system of dimension $n$ represents the cographic matroid $M^*(G)$ of a 3-connected cubic non-planar graph $G$ on $2(n - 1)$ vertices and $3(n - 1)$ edges.

All ten maximal by inclusion unimodular feasible subsets $U \subseteq \mathcal{M}$ were found by computer. The method is to take an unimodular subset, to consider all possible ways to extend it in an acceptable way and to reduce by isomorphism. The results are presented on Table 2. We give vectors of these sets in the denotations (6). Let Aut($\mathcal{M}$) be the group of automorphisms of the set $\mathcal{M}$. We denote by Stab($U$) $\subseteq$ Aut($\mathcal{M}$) the stabilizer subgroup of the set $U$. Among these 10 sets there are two absolutely maximal 6-dimensional unimodular sets $U_3$ and $U_{10}$ (see [DG99]). The matroids which are neither graphic nor cographic are given in Table 3 and the rest in Figure 4.

| nr | $|U|$ | dim $U$ | $|\text{Stab } U|$ | status | ref |
|----|-----|--------|----------------|--------|-----|
| 1  | 9   | 5      | 384            | graphic | $M(G_1)$ |
| 2  | 12  | 5      | 96             | cographic | $M^*(G_2)$ |
| 3  | 12  | 6      | 12             | special | $R_{12}$ |
| 4  | 12  | 6      | 12             | special | $R_{10} \oplus_1 C_3$ |
| 5  | 13  | 6      | 4              | cographic | $M^*(G_5)$ |
| 6  | 13  | 6      | 4              | cographic | $M^*(G_6)$ |
| 7  | 13  | 6      | 2              | cographic | $M^*(G_7)$ |
| 8  | 14  | 6      | 8              | graphic | $M(G_8)$ |
| 9  | 14  | 6      | 24             | graphic | $M(G_9)$ |
| 10 | 15  | 6      | 24             | cographic | $M^*(G_{10})$ |

Table 2. All ten maximal admissible unimodular system of $M$
The algorithms explained in Section 4 have been implemented in [Du10] and applied to several highly symmetric lattices. We thus proved that the polytopes $P_V(L)$ are nonfree for $L = E_7^*, \kappa_9^*, \kappa_{16}^*$ or $K_{12}$. For the lattice $BW_{16}$ this direct approach does not work, because there are too many 6-belts. However, by selecting a subset of the 6-belts, one can prove that this lattice is nonfree as well.

For a general lattice $L$, we cannot expect a simple criterion for determining the forbidden and feasible subsets of $P_V(L)$. This is because the lattice $E_6$ is very symmetric and other cases are necessarily more complicated.

In Table 4 we give detailed information on 12 lattices of the computed data. For the lattice $ER_7$, we found out that there exist some minimal forbidden subsets $U$ such that $P_V(ER_7)$ has some faces $G$ such that $\dim G_U = 7$. Hence, for this case, one cannot limit oneself to the quasi 4-belt in the analysis.

### Appendix A. Feasible extensions of $(n - 3)$-faces

Hereby we consider feasible extensions of $(n - 3)$-faces of the parallelotope $P$ and construct local matchings around these extensions. The classification of feasible arrangements of $\text{Fan}(G)$ and $\pi_G(z(v))$ is given in Figures 2 and 3.

We will use the notation $v \in U_i(G)$ ($i = 1, 2, 3$) introduced in Section 1. Recall the meaning of the notation. If $\lambda > 0$ is sufficiently small, and $x$ is an arbitrary point in the relative interior of $G$, then we write

- $v \in U_1(G)$ if $x \pm \lambda v \notin G$ and one of the points $x \pm \lambda v$ is in $P$.
- $v \in U_2(G)$ if $x \pm \lambda v \in \text{Lin}(G)$.
- $v \in U_3(G)$ if $x \pm \lambda v \notin G$ and $x \pm \lambda v \notin P$.

A feasible extension of the face $G$ exists iff $v \in U_3(G)$.

If $\text{Fan}(G)$ and $\pi_G(z(v))$ are arranged together as in one of the cases a.1), a.2), b.2), c.2), c.3), d.2), d.3), e.3), then $v \in U_1(G)$, regardless of the position of $\pi_G(P)$ relative to $\text{Fan}(G)$. Hence $P + z(v)$ has no $(n - 2)$-face of the form $G + z(v)$.

Consider the remaining 5 cases.

### Table 3. Unimodular systems which are neither graphic nor cographic

| $b_2$ | 1 0 0 0 0 0 | $b_2$ | 1 0 0 0 0 0 |
|------|-------------|------|-------------|
| $c_{26}$ | 0 1 0 0 0 0 | $c_{26}$ | 0 1 0 0 0 0 |
| $a_2$ | 0 0 1 0 0 0 | $a_2$ | 0 0 1 0 0 0 |
| $a_3$ | 0 0 0 1 0 0 | $a_3$ | 0 0 0 1 0 0 |
| $b_1$ | 0 0 0 0 1 0 | $b_1$ | 0 0 0 0 1 0 |
| $b_5$ | 0 0 0 0 0 1 | $b_5$ | 0 0 0 0 0 1 |
| $b_3$ | 1 0 −1 1 0 0 | $b_3$ | 1 0 −1 1 0 0 |
| $c_{36}$ | 0 1 1 −1 0 0 | $c_{36}$ | 0 1 1 −1 0 0 |
| $a_4$ | 0 1 0 −1 −1 −1 | $a_4$ | 0 1 0 −1 −1 −1 |
| $a_1$ | −1 0 1 0 1 0 | $a_1$ | −1 0 1 0 1 0 |
| $c_{14}$ | 0 −1 0 1 0 1 | $c_{14}$ | 0 −1 0 1 0 1 |
| $c_{15}$ | 1 0 −1 0 −1 −1 | $c_{15}$ | 1 0 −1 0 −1 −1 |

The unimodular system $R_{12}$ The unimodular system $R_{10} \oplus_1 C_3$

11. Results for other lattices
b.1) Suppose that Fan(G) and πG(z(v)) are arranged together as in case b.1). Then, if πG(P) lies in one of the cones O(−x₁)(−x₂)x₃x₄ or O(−x₁)x₄(−x₂)(−x₃), we have v ∈ U₁(G). In all four other subcases we have v ∈ U₃(G).

Let P, P₁, P₂, and P₃ be the translates of P that share the face G so that their projections along G lie in the four cones O₁, O₃, O₂, and O₄, respectively.

Figure 4. Occurring graphs
Table 4. For 12 lattices, we give the following information on their free structure: \( |\mathcal{F}(L)| \) is the size of \( \mathcal{F}(L) \), \( |O(\mathcal{F}_{\text{min}}(L))| \) is the number of orbits of minimal forbidden subsets, \( |O(\mathcal{F}_{\text{max}}(L))| \) is the number of orbits of maximal unimodular subsets, \( \dimmax(L) \) is the maximum dimension of admissible subsets of \( \mathcal{F}(L) \), and \( S_{\text{max}}(L) \) is the maximum size of an admissible subsets of \( \mathcal{F}(L) \). The lattices are given in [NeSo12], and the lattice \( ER_7 \) is given in [ErRy02].

| \( L \) | \( |\mathcal{F}(L)| \) | \( |O(\mathcal{F}_{\text{min}}(L))| \) | \( |O(\mathcal{F}_{\text{max}}(L))| \) | \( \dimmax(L) \) | \( S_{\text{max}}(L) \) |
|---|---|---|---|---|---|
| \( E_6 \) | 27 | 2 | 10 | 6 | 15 |
| \( E_7 \) | 28 | 2 | 4 | 7 | 14 |
| \( ER_7 \) | 28 | 9 | 49 | 7 | 15 |
| \( ER_7 \) | 6 | 0 | 1 | 6 | 6 |
| \( \kappa_7 \) | 11 | 2 | 2 | 6 | 9 |
| \( \kappa_7 \) | 10 | 2 | 1 | 1 | 1 |
| \( \kappa_8 \) | 6 | 1 | 2 | 3 | 3 |
| \( \kappa_9 \) | 3 | 1 | 1 | 2 | 2 |
| \( \Lambda_9 \) | 1 | 0 | 1 | 1 | 1 |
| \( \Lambda_9 \) | 16 | 2 | 1 | 1 | 1 |
| \( \Lambda_{10} \) | 3 | 0 | 1 | 2 | 3 |
| \( O_{10} \) | 40 | 2 | 2 | 5 | 10 |

For \( 12 \) lattices, we give the following information on their free structure: \( |\mathcal{F}(L)| \) is the size of \( \mathcal{F}(L) \), \( |O(\mathcal{F}_{\text{min}}(L))| \) is the number of orbits of minimal forbidden subsets, \( |O(\mathcal{F}_{\text{max}}(L))| \) is the number of orbits of maximal unimodular subsets, \( \dimmax(L) \) is the maximum dimension of admissible subsets of \( \mathcal{F}(L) \), and \( S_{\text{max}}(L) \) is the maximum size of an admissible subsets of \( \mathcal{F}(L) \). The lattices are given in [NeSo12], and the lattice \( ER_7 \) is given in [ErRy02].

\( O(\pm x_1)(\pm x_2)(\pm x_3)(\pm x_4) \), and \( O(\pm x_1 \pm y)(\pm x_2 \pm y) \). Then \( P + z(v) \), \( P + z(v) \), \( P + z(v) \), and \( P + z(v) \) match around their common face \( G + z(v) \). This matching can be presented as splitting \( D(G) \) into two quadrangular pyramids by a parallelogram, which is the “dual cell” of \( G + z(v) \). We do not assume that the tiling by translates \( P + z(v) \) exists, however, the local matching proves that \( G + z(v) \) determines a 4-belt.

c.1 Suppose that \( \text{Fan}(G) \) and \( \pi_G(z(v)) \) are arranged together as in case c.1). Then, if \( \pi_G(P) \) lies in one of the cones \( O(\pm x_1 \pm y) \), \( O(\pm x_2 \pm y) \), or \( O(\pm x_3 \pm y) \), we have \( v \in U_1(G) \). For the other three subcases we have \( v \in U_3(G) \).

Let \( P_1 \), \( P_2 \), and \( P_3 \) be the translates of \( P \) that share the face \( G \) so that their projections along \( G \) lie in the three cones \( O(\pm x_1 \pm x_3) \), \( O(\pm x_2 \pm x_4) \), and \( O(\pm x_3 \pm x_4) \). Then \( P + z(v) \), \( P + z(v) \), \( P + z(v) \) match around their common face \( G + z(v) \). This matching can be presented as splitting the dual cell \( D(G) \) into two tetrahedra by a triangle, which is the “dual cell” of \( G + z(v) \). Thus \( G + z(v) \) determines a 6-belt.

d.1 Suppose that \( \text{Fan}(G) \) and \( \pi_G(z(v)) \) are arranged together as in case d.1). Then, if \( \pi_G(P) \) lies in one of the three cones \( O(\pm x_1 \pm y) \), \( O(\pm x_2 \pm y) \), or \( O(\pm x_3 \pm y) \), then we have \( v \in U_1(G) \). For the other three subcases we have \( v \in U_3(G) \).

Let \( P_1 \), \( P_2 \), and \( P_3 \) be the translates of \( P \) that share the face \( G \) so that their projections along \( G \) lie in the three cones \( O(\pm x_1 \pm y) \), \( O(\pm x_2 \pm y) \), or \( O(\pm x_3 \pm y) \). Then \( P + z(v) \), \( P + z(v) \), \( P + z(v) \) match around their common face \( G + z(v) \). This matching can be presented as splitting \( D(G) \) into a tetrahedron and a quadrangular pyramid by a triangle, which is the “dual cell” of \( G + z(v) \).

e.2 Suppose that \( \text{Fan}(G) \) and \( \pi_G(z(v)) \) are arranged together as in case e.2). Then, if \( \pi_G(P) \) lies in one of the four cones \( O(\pm x_1 \pm x_3) \), \( O(\pm x_2 \pm x_4) \), \( O(\pm x_3 \pm x_4) \), or \( O(\pm x_4 \pm x_3) \), then we have \( v \in U_3(G) \). For the other three subcases we have \( v \in U_3(G) \).
Let $P$, $P_1$, $P_2$, and $P_3$ be the translates of $P$ that share the face $G$ so that their projections along $G$ lie in the four cones $Ox_1(-x_2)x_3$, $Ox_1(-x_2)(-x_3)$, $O(-x_1)x_2x_3$, and $O(-x_1)x_2(-x_3)$. Then $P + z(v)$, $P_1 + z(v)$, $P_2 + z(v)$, and $P_3 + z(v)$ match around their common face $G + z(v)$. This matching can be presented as splitting $D(G)$ into two triangular prisms by a parallelogram, which is the “dual cell" of $G + z(v).

**e.1)** Suppose that $\pi_G(z(v))$ and $\pi_G(z(v))$ are arranged together as in case e.1). Then, if $\pi_G(P)$ lies in one of the cones $O(-x_1)x_2x_3$ or $Ox_1(-x_2)(-x_3)$, we have $v \in U_1(G)$. For the other three subcases we have $v \in U_3(G)$.

Let $P$, $P_1$, $P_2$, $P_3$ be the translates of $P$ that share the face $G$ so that their projections along $G$ lie in the six cones of $\pi_G(P)$ not mentioned in the previous paragraph. Let $P_6$ and $P_7$ be the other two translates of $P$ the face $G$.

Without loss of generality, we can assume that $P_6$ shares a facet with $P$, $P_1$ and $P_2$. Then $P_7$ shares a facet with $P_3$, $P_4$ and $P_5$. We can also assume that the vector $\pi_G(v)$ is directed from $\pi_G(P_6)$ to $\pi_G(P_7)$. Then $P + z(v)$, $P_1 + z(v)$, and $P_2 + z(v)$ match around their common face $G + z(v)$, as well as $P_3 + 2v + z(v)$, $P_4 + 2v + z(v)$, $P_5 + 2v + z(v)$ match around $G + 2v + z(v)$ and share common facets with the three polytopes mentioned just before. Thus $G$ produces two $(n-2)$-faces of the local matching. This can be presented as splitting $D(G)$ into two tetrahedra and an octahedron by two triangles, being the “dual cells" for $G + z(v)$ and $G + 2v + z(v)$.

### Appendix B. 6-belts of parallelohedra $PV(E_6) + Z(U)$

**Proof of Lemma** Notice that for every $r \in \mathcal{R}$ and $t \in \mathcal{T}$ one has $|r| = \sqrt{2}$ and $|t| = 2$.

Since every vector of $\mathcal{R} \cup \mathcal{T}$ is minimal for the lattice $E_6$, then every vector from the set $(\mathcal{R} \cup \mathcal{T}) \cap \alpha$ is minimal for the two-dimensional lattice $E_6 \cap \alpha$.

Now we have two possibilities — either the Delaunay tessellation $D(E_6 \cap \alpha)$ for the lattice $E_6 \cap \alpha$ is triangular, or it is rectangular.

**Case 1.** $D(E_6 \cap \alpha)$ is triangular. Then there is no vector in $(\mathcal{R} \cup \mathcal{T}) \cap \alpha$, except of the edge vectors of $D(E_6 \cap \alpha)$. But, since the intersection $(\mathcal{R} \cup \mathcal{T}) \cap \alpha$ consists of at least 3 pairs of antipodal vectors, it should contain every edge vector of $D(E_6 \cap \alpha)$. As a consequence, $(\mathcal{R} \cup \mathcal{T}) \cap \alpha$ contains three vectors that form a Delaunay triangle for $E_6 \cap \alpha$.

Every Delaunay triangle is acute-angled. Among all triangles that have all side-lengths equal to $\sqrt{2}$ or 2, the acute-angled are the following:

- equilateral with side-length $\sqrt{2}$;
- equilateral with side-length 2;
- isosceles with sides equal to 2, 2, and $\sqrt{2}$.

This forces us to restrict the search to the following subcases:

1.1. $r, r' \in \mathcal{R}$ with $r^T r' = 1$ and $r - r' \in \mathcal{R}$;
1.2. $t, t' \in \mathcal{T}$ with $t^T t' = 2$ and $t - t' \in \mathcal{T}$;
1.3. $t, t' \in \mathcal{T}$ with $t^T t' = 3$ and $t - t' \in \mathcal{R}$.

Consider these subcases separately.

**Subcase 1.1.** Consider a lattice triangle $\Delta$ in $E_6$ with edge vectors $r$, $r'$ and $r - r'$. This triangle is equilateral with side-length $\sqrt{2}$. We prove that $\Delta$ is a Delaunay triangle for $E_6$. 


Consider a ball $B \subset \mathbb{R}^6$ centered at the center of $\Delta$ and containing the vertices of $\Delta$ on its boundary. $B$ contains no points of $E_6$ other than the vertices of $\Delta$. Indeed, every point of $B$ is at least as close to the nearest vertex of $\Delta$ as $2/\sqrt{3}$.

But the minimum distance between points of $E_6$ is $\sqrt{2} > 2/\sqrt{3}$, a contradiction.

Since $\Delta$ is a Delaunay triangle for $E_6$, it is a face of $P_{Schl}$. All such faces are equivalent to $\text{conv}\{a_1, a_2, a_3\}$ up to some action of $W(E_6)$. Hence Subcase 1.1 gives exactly the planar sets of type (b).

**Subcase 1.2.** In this case we write $t = p_1 - p'_1$, $t' = p_2 - p'_2$, where $p_i, p'_i \in \mathcal{M}$, $p_i^T p'_i = -2/3$ ($i = 1, 2$). The condition $t^T t' = 2$ now can be written as

$$p_1^T p_2 + (p'_1)^T p'_2 - p_1^T p'_2 - (p'_1)^T p_2 = 2.$$ 

Each scalar product equals $4/3, 1/3, or -2/3$, with no more than one scalar product being equal to $4/3$ (i.e., no more than one pair of vectors with different indices coincide). This leaves the following four possibilities (up to an interchange of pairs $(p_1, p'_1)$ and $(p_2, p'_2)$ or swapping vectors within these two pairs simultaneously).

$$4/3 + (-2/3) - (-2/3) - (-2/3) = 2, \quad 4/3 + 1/3 - 1/3 - (-2/3) = 2, \quad$$

$$4/3 + 1/3 - (-2/3) - 1/3 = 2, \quad 1/3 + 1/3 - (-2/3) - (-2/3) = 2.$$

The first three identities imply $p_1 = p_2$. Also, without loss of generality we can assume that $p_1 = a_1$ and $p'_1 = b_2$, as all other cases are similar up to some action of $W(E_6)$. Then the distance from $p'_2$ to both $a_1$ and $b_2$ should equal $\sqrt{2}$, which leaves the only case $p'_2 = c_{12}$, giving the planar sets of type (c).

If the fourth identity emerges, we can also assume $p_1 = a_1$ and $p'_1 = b_2$. Then

$$p_2 \in \{a_3, a_4, a_5, a_6, c_{23}, c_{24}, c_{25}, c_{26}\};$$

$$p'_2 \in \{b_3, b_4, b_5, b_6, c_{13}, c_{14}, c_{15}, c_{16}\}.$$ 

Up to reassignment of indices, this results in the following pairs for ($p_2, p'_2$): ($a_3, b_4$), ($c_{23}, c_{14}$), ($b_4, c_{13}$), or ($a_3, c_{13}$). In all these cases we have a planar set of type (d).

**Subcase 1.3.** In this case we write $t = p_1 - p'_1$, $t' = p_2 - p'_2$, where $p_i, p'_i \in \mathcal{M}$, $p_i^T p'_i = -2/3$ ($i = 1, 2$). The condition $t^T t' = 3$ can be rewritten as

$$p_1^T p_2 + (p'_1)^T p'_2 - p_1^T p'_2 - (p'_1)^T p_2 = 3.$$ 

Again, no more than one scalar product is equal to $4/3$, which results in the only possible case

$$4/3 + 1/3 - (-2/3) - (-2/3) = 3.$$ 

This leads to $p_1 = p_2$. Similarly to Subcase 1.2, we can assume $p_1 = p_2 = a_1$ and $p'_1 = b_2$. Therefore

$$p'_2 \in \{b_3, b_4, b_5, b_6, c_{13}, c_{14}, c_{15}, c_{16}\}.$$ 

Each option results in a planar set of type (e).

**Case 2.** The Delaunay tessellation $D(E_6 \cap \alpha)$ is rectangular. Let $\Xi$ be one of the rectangles of $D(E_6 \cap \alpha)$.

Again, the set $(R \cup T) \cap \alpha$ contains at least 3 pairs of antipodal vectors. Therefore at least one of the diagonal vectors of $\Xi$ is a minimal vector for $E_6$.

Consequently, the sphere in $\mathbb{R}^6$, that has the diagonal of $\Xi$ as a diameter, is empty for $E_6$. Hence $\Xi$ is a planar section of a centrally symmetric Delaunay polytope from $D(E_6)$. This polytope has at least 4 vertices, so it is not a segment. Therefore it is a crosspolytope. But all planar sections of such crosspolytopes are equivalent.
up to automorphisms of $E_6$, so only one type of a planar set is possible, namely, the type (a).

\[\text{Proof of Lemma}\]

Define

\[F = \text{conv}\{a_1, a_2, a_3, a_4, a_5, a_6, -b_1, -b_2, -b_3, -b_4, -b_5, -b_6\},\]

\[F' = \text{conv}\{a_2, b_2, c_{13}, c_{14}, c_{15}, c_{16}, -a_1, -b_1, -c_{23}, -c_{24}, -c_{25}, -c_{26}\}.\]

$F$ and $F'$ are facets of $P_V(E_6)$, and their facet vectors are $q = a_1 - b_1$ and $q' = a_2 - a_1$ respectively. The edge $[a_2, -b_1]$ is standard for $P_V(E_6)$ with the standard vector $t = a_2 - b_1$. It is easy to see that $t = q + q'$.

Suppose that $[a_2, -b_1] \oplus z(p)$ is a standard 2-face of the parallelotope $P_V(E_6) + z(p)$.

Let $A_p$ be the operator moving the lattice $\Lambda(P_V(E_6))$ to the lattice $\Lambda(P_V(E_6) + z(p))$ as defined in Lemma 1. One can see that $t$ is a standard vector of the parallelotope $P_V(E_6) + z(p)$, passing through the center of the face $[a_2, -b_1] \oplus z(p)$. Therefore $A_p t = t$. Further, $A_p q$ and $A_p q'$ are facet vectors of the parallelotope $P_V(E_6) + z(p)$, because facet vectors of $P_V(E_6)$ always correspond to facet vectors of $P_V(E_6) + z(p)$. Hence

\[A_p q + A_p q' = t = q + q'.\]

Recall that the operator $A_p$ has the form $A_p = Id + 2pe_p^T$, where $e_p$ is a particularly chosen normal to the layer defined by $p$. Hence $e_p^T q = -e_p^T q'$.

If $e_p^T q = -e_p^T q' = 0$, then $p$ is parallel to both $F$ and $F'$, which results in $p \in \{c_{12}, c_{34}, c_{35}, c_{36}, c_{45}, c_{46}, c_{56}\}$.

However, $p = c_{12}$ is not a possible option because $z(c_{12})$ and $[a_2, -b_1]$ are parallel and do not form a direct Minkowski sum. But if $p \in \{c_{34}, c_{35}, c_{36}, c_{45}, c_{46}, c_{56}\}$, then $q = A_p q$ and $q' = A_p q'$ are lattice vectors of the lattice $\Lambda(P_V(E_6) + z(p))$. Therefore $t = q + q'$ is also a lattice vector of $\Lambda(P_V(E_6) + z(p))$. Consequently, $P_V(E_6) + t + z(p)$ belongs to the same tiling as $P_V(E_6) + z(p)$, and

\[(P_V(E_6) + z(p)) \cap (P_V(E_6) + t + z(p)) = [a_2, -b_1] + z(p).\]

The sum in the right-hand side is direct, because the segments $[a_2, -b_1]$ and $z(p)$ are not parallel. Hence $p$ is strongly transversal to $[a_2, -b_1]$.

Now suppose that $e_p^T q \neq 0$. Then the scalar products $e_p^T q$ and $e_p^T q'$ have opposite signs. Therefore $p$ is parallel neither to $F$ nor to $F'$. Moreover, if the vector $p$ intersects the facet $F$ inwards $P_V(E_6)$, then $p$ intersects the facet $F'$ outwards $P_V(E_6)$, and vice versa. Thus scalar products of $p$ with the facet normals of $F$ and $F'$, i.e., with $q$ and $q'$, should have opposite signs. Among all vectors of $M$ this property holds only for $a_1$ and $b_2$.

If $p = a_1$, then $p$ intersects $F$ in outer direction, therefore $A_p q = q + 2p$ and $A_p q' = q' - 2p$. Similarly, if $p = b_2$, $A_p q = q - 2p$ and $A_p q' = q' + 2p$. Thus $t = A_p q + A_p q'$ is a lattice vector of $\Lambda(P_V(E_6) + z(p))$ as well.

Again, this means that $P_V(E_6) + t + z(p)$ belongs to the same tiling as $P_V(E_6) + z(p)$, and

\[(P_V(E_6) + z(p)) \cap (P_V(E_6) + t + z(p)) = [a_2, -b_1] + z(p).\]

Similarly to the previous case, the sum in the right-hand side is direct, and thus $p$ is strongly transversal to $[a_2, -b_1]$. \qed
Proof of Lemma 7. We prove all the assertions independently.

1. Indeed, $P_6(E_6)$ already has a 6-belt with facet vectors \{\pm(a_2 - a_1), \pm(a_3 - a_1), \pm(a_4 - a_3)\}. This 6-belts turns into a 6-belt of $P_6(E_8) + Z(U)$ with the same direction of the generating $(n-2)$-face.

2. The vectors \pm(a_2 - a_1) and \pm(b_1 - a_1) are already facet vectors of $P_6(E_6)$. Their directions, $(a_2 - a_1)^\perp$ and $(b_1 - a_1)^\perp$ do not change after adding a zonotope $Z(U)$.

Now consider the 6-belt of $P_6(E_6) + Z(U)$ with facet vectors corresponding to \{\pm(a_2 - a_1), \pm(b_1 - a_1), \pm(b_2 - a_1)\}. Its generating face $G$ should be parallel to both $(a_2 - a_1)^\perp$ and $(b_1 - a_1)^\perp$, so

$$G \parallel \langle a_2 - a_1, b_1 - a_1 \rangle ^\perp.$$

We also notice that

$$\mathcal{M} \cap (a_2 - a_1, b_1 - a_1)^\perp = \{c_{12}, c_{34}, c_{56}, c_{35}, c_{46}, c_{36}, c_{45}\}.$$

Consider the facets $F$ and $F'$ of $P_6(E_8)$ as in formula 7. Recall that their facet vectors are $a_1 - b_1$ and $a_2 - a_1$ respectively. Let $F + Z(U_1)$ and $F' + Z(U_1')$ be the corresponding facets of $P_6(E_6) + Z(U)$. $F + Z(U_1)$ and $F' + Z(U_1')$ are adjacent, because in the set \{\pm(a_2 - a_1), \pm(b_1 - a_1), \pm(b_2 - a_1)\} the vectors $a_1 - b_1$ and $a_2 - a_1$ are neighbouring, i.e. no other vector from this 6-tuple lies is a positive combination of these two.

The direction $(a_2 - a_1, b_1 - a_1)^\perp$ generates a belt of $P_6(E_6) + Z(U)$. Therefore the intersection

$$F + Z(U_1) \cap F' + Z(U_1')$$

should be a 4-dimensional face parallel to $(a_2 - a_1, b_1 - a_1)^\perp$.

However, $F \cap F' = [a_2, -b_1]$, i.e., 1-dimensional. To expand the intersection to a 4-dimensional polytope, the set $U$ should contain 3 vectors, making together with $-b_1 - a_2 = c_{12}$ a 4-dimensional linearly independent set. This can only be achieved by taking one vector from each pair

$$(8) \quad (c_{34}, c_{56}), \quad (c_{35}, c_{46}), \quad (c_{36}, c_{45}).$$

On the other hand, let $U_2$ be a 3-element set obtained by an arbitrary choice of representatives from each pair in (8). Then $P_6(E_8) + Z(U_2)$ has a 4-belt generated by a 4-face parallel to $(a_2 - a_1, b_1 - a_1)^\perp$.

The parallelotope $P_6(E_6) + Z(U_2)$ has a 4-face $G = [b_2, -a_1] + Z(U_2)$ which is standard. The standard vector of $G$ corresponds to $b_2 - a_1$. Therefore extending $P_6(E_6) + Z(U_2)$ to $P_6(E_6) + Z(U)$ should turn $G$ into a facet. For this purpose it is necessary and sufficient for $U$ to contain a vector $p_4$ strongly transversal to $G$.

Therefore $p_4$ is strongly transversal to the edge $[b_2, -a_1]$ of $P$, but is not parallel to $G$. By Lemma 8, this is possible only for $p_4 = a_2$ or $p_4 = b_1$.

All the arguments are reversible. Indeed, if $U$ contains $U_2 = \{p_1, p_2, p_3\}$ as a subset, then $P_6(E_6) + Z(U_2)$ has a standard 4-face $G = [b_2, -a_1] + Z(U_2)$ whose standard vector is exactly $t = b_2 - a_1$. Similarly to the argument of Lemma 8, $t$ is also a lattice vector of $\Lambda(P_6(E_6) + Z(U_2) + z(p_4))$. Thus

$$(P_6(E_6) + Z(U_2) + z(p_4)) \cap (P_6(E_6) + Z(U_2) + z(p_4) + t) = G + z(p_4),$$

which is a 5-polytope.

Hence the 4-belt of $P_6(E_6) + Z(U_2)$ has turned into a 6-belt of $P_6(E_6) + Z(U_2) + z(p_4)$, which persists in $P + Z(U)$. Assertion 2 is proved completely.
3. Let the 4-face \( G \) generate the 6-belt of \( P_V(E_6) + Z(U) \), whose facet vectors correspond to
\[
\{ \pm(b_2 - a_1), \pm(b_3 - a_1), \pm(b_3 - b_2) \}.
\]

First we prove that the affine hull of \( G \) is spanned by 4 vectors of \( M \).

Indeed, \( P_V(E_6) \) has the \( m \)-edge \([b_2, -a_1]\) with standard vector \( t = b_2 - a_1 \). The 6-belt from the condition of Assertion 3 has a facet \([b_2, -a_1] + Z(U_1)\) for some \( U_1 \subseteq U \). All zone vectors of the zonotope \([b_2, -a_1] + Z(U_1)\) belong to \( M \), therefore every subface of \([b_2, -a_1] + Z(U_1)\) is also a zonotope with all zone vectors in \( M \). In particular, this holds for \( G \).

Put
\[
F = \text{conv}\{b_3, a_1, c_{12}, c_{24}, c_{25}, c_{26}, -b_2, -a_2, -c_{13}, -c_{34}, -c_{35}, -c_{36}\},
\]
i.e., \( F \) is the facet of \( P_V(E_6) \) with the facet vector \( b_3 - b_2 \). Let \( U_2 \subseteq U_1 \cup \{c_{12}\} \) be the 4-element vector set spanning the linear hull of \( G \). \( G \) is parallel to \( F \), so \( q(t) \notin U_2 \). Thus \( U_2 \subseteq U_1 \) and \( P_V(E_6) + Z(U_2) \) is a parallelotope.

But \( P_V(E_6) + Z(U_2) \) has nonzero width in the direction \( (U_2) \). Thus it has a belt parallel to \( (U_2) \). The facet vectors of this belt cannot correspond to other vectors than \( \{\pm(b_2 - a_1), \pm(b_3 - a_1), \pm(b_3 - b_2)\} \), because they persist in the belt of \( P_V(E_6) + Z(U) \) parallel to \( G \).

The belt of \( P_V(E_6) + Z(U_2) \) parallel to \( (U_2) \) should be a 6-belt, because every 4-belt requires 8 coplanar standard vectors, but the plane
\[
\langle b_2 - a_1, b_3 - a_1, b_3 - b_2 \rangle
\]
contains only six. Hence \( P_V(E_6) + Z(U_2) \) has facets that have expanded from the edges \([b_2, -a_1]\) and \([b_3, -a_1]\). The only candidates for these facets are \([b_2, -a_1] + Z(U_2)\) and \([b_3, -a_1] + Z(U_2)\) respectively, because a 5-dimensional zonotope has at least 5 zone vectors. By Lemma 3, if a standard face expanded to a standard face, all intermediate expansions are also standard. As a result, for every \( p \in U_2 \) the 2-faces \([b_2, -a_1] + z(p)\) and \([b_3, -a_1] + z(p)\) are standard for \( P_V(E_6) + z(p) \). Applying Lemma 4 and transforms from \( W(E_6) \), that send the segment \([a_2, -b_1] \) to \([b_2, -a_1]\) and \([b_3, -a_1]\) respectively, gives
\[
\begin{align*}
U_2 \subset \{b_1, a_2, c_{34}, c_{35}, c_{36}, c_{45}, c_{46}, c_{56}\} \cap \\
\{b_1, a_3, c_{24}, c_{25}, c_{26}, c_{45}, c_{46}, c_{56}\} = \{b_1, c_{45}, c_{46}, c_{56}\}.
\end{align*}
\]

Since \(|U_2| = 4\), \( U_2 = \{b_1, c_{45}, c_{46}, c_{56}\} \).

However, the polytope \( P_V(E_6) + Z(U_2) \) has a belt consisting of facets \( \pm(F + Z(U_2)) \), \( \pm([b_2, -a_1] + Z(U_2)) \), and \( \pm([b_2, -a_1] + Z(U_2)) \), and this belt remains in all polytopes \( P_V(E_6) + Z(U) \), where \( U \supseteq U_2 \), if only \( P_V(E_6) + Z(U) \) is a parallelotope.

4. The facet, whose facet vectors corresponds to the vector \( a_2 - a_1 \), is orthogonal to \( a_2 - a_1 \). Further, the facet, whose facet vectors corresponds to the vector \( b_1 - a_2 \), is orthogonal to \( b_1 - a_2 \). Indeed, if a segment \( z(p) \), \( p \in M \), is strongly transversal to the edge \([b_1, -a_2]\), then, by Lemma 6 \( p \) is orthogonal to \( b_1 - a_2 \). Therefore the facet obtained by extensions of the edge edge \([b_1, -a_2]\) is orthogonal to \( b_1 - a_2 \).

Assume that the 6-belt with facet vectors corresponding to
\[
\{\pm(a_2 - a_1), \pm(b_1 - a_2), \pm(b_2 - a_1)\}
\]
exists. Then it is generated by a 4-face parallel to
\[
\langle a_2 - a_1, b_1 - a_2 \rangle^\perp = \langle a_2 - a_1, b_1 - a_1 \rangle^\perp.
\]
But a belt with this direction of the generating 4-face should contain two facets parallel to \((b_1 - a_1)^\perp\), because such facets are already in \(P_V(E_6)\). The facet vectors of these facets correspond to \(\pm(b_1 - a_1)\). But by assumption, there are no such facet vectors in the 6-belt we consider. The contradiction finishes the proof of Assertion 4.

5. Similarly to the proof of Assertion 3 we can establish that there is a 4-element set \(U_2 \subset U\) such that \(P_V(E_6) + Z(U_2)\) has a 6-belt consisting of facets \(\pm([a_1, -b_2] + Z(U_2)), \pm([a_1, -c_12] + Z(U_2))\) and \(\pm([b_2, -c_12] + Z(U_2))\). This means that for every \(p \in U_2\) the 2-faces \([a_1, -b_2] \oplus z(p), [a_1, -c_12] \oplus z(p), \) and \([b_2, -c_12] \oplus z(p)\) are standard for \(P_V(E_6) + z(p)\). As a result,

\[
U_2 \subset \{a_2, b_1, c_{34}, c_{35}, c_{36}, c_{45}, c_{46}, c_{56}\} \cap \{a_3, a_4, a_5, a_6, c_{23}, c_{24}, c_{25}, c_{26}\} \cap \{b_3, b_4, b_5, b_6, c_{13}, c_{14}, c_{15}, c_{16}\} = \emptyset.
\]

A contradiction proves that no 6-belt can appear as in the condition of Assertion 5.

6. Assume that a parallelootope \(P_V(E_6) + Z(U)\) has a 6-belt with facet vectors corresponding to \(\{\pm(b_2 - a_1), \pm(b_4 - a_3), \pm(c_{23} - c_{14})\}\). Then the facets of such a 6-belt are extensions of edges \(\pm[a_1, -b_2], \pm[a_3 - b_4], \) and \(\pm[c_{23}, -c_{14}]\). But these edges are pairwise disjoint, then so are their extensions.

Indeed, let \(G_1\) and \(G_2\) be faces of \(P_V(E_6)\) such that \(G_1 + Z(U_1)\) and \(G_2 + Z(U_2)\) are faces of \(P_V(E_6) + Z(U)\) satisfying

\[
(G_1 + Z(U_1)) \cap (G_2 + Z(U_2)) \neq \emptyset.
\]

Then one can see that the intersection is of the form

\[
(G_1 + Z(U_1)) \cap (G_2 + Z(U_2)) = G_3 + Z(U_3)
\]

with \(G_3 \subset G_1 \cap G_2\).

Hence a contradiction, because facets making together a belt cannot be pairwise disjoint.

\[\square\]

References

[Aig79] M. Aigner, Combinatorial Theory, Springer-Verlag, 1979.
[Ba91] E.P. Baranovskii, Partition of Euclidean spaces into L-polytopes of some perfect lattices, Proc. Steklov Inst. Math. 196 (1991) 29–51.
[Barn57] E.S. Barnes, The complete enumeration of extreme senary forms, Phil. Trans. Roy. Soc., London A249 (1957) 461–506.
[BLSWZ93] A. Björner, M. LasVernas, B. Sturmfels, N. White, G. Ziegler, Oriented Matroids, Encyclopedia of Math. and its Appl. 46, Cambridge Univ. Press, 1993.
[Burn11] W. Burnside, Theory of groups of finite order, Cambridge Univ. Press, 1911.
[CS91] J.H. Conway, N.J.A. Sloane, The cell structures of certain lattices, in: Miscellanea Mathematica, Springer-Verlag, (1991) 71–107.
[Cox83] H.S.M. Coxeter, The twenty-seven lines on the cubic surface, in: Convexity and its Applications, Birkhäuser Verlag, (1983) 111–119.
[DG99] V.I. Danilov, V.P. Grishukhin, Maximal unimodular systems of vectors, Eur. J. Combin. 20 (1999) 507–526.
[De29] B.N. Delaunay, Sur la partition régulière de l’espace à 4 dimensions, Izvestia Acad. Sci. URSS, Ser VII, Sect. phys.-math. sci. 1-2 (1929) 79–100, 147–164.
[DeGr04] M. Deza, V. Grishukhin, Voronoi’s conjecture and space tiling zonotopes, Mathematika 51 (2004) 1–10.
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[Do09] N.P. Dolbilin, Properties of faces of parallelohedra, Proc. Steklov Inst. of Math. 266 (2009) 112–126.
[Du10] M. Dutour, polyhedral, http://drobilica.irb.hr/~mathieu/Polyhedral/index.html
[DSV09] M. Dutour Sikirić, A. Schürmann, F. Vallentin, Complexity and algorithms for computing Voronoi cells of lattices, Math. Comput. 78 (2009) 1713–1731.
[DSV08] M. Dutour Sikirić, A. Schürmann, F. Vallentin, A generalization of Voronoi’s reduction theory and applications, Duke Math. J. 142 (2008) 127–164.
[En98] P. Engel, Investigations of parallelohedra in $\mathbb{R}^d$, in: Voronoi’s Impact on Modern Science, vol. 2, eds. P. Engel and H. Syta, Institute of Math., Kyiv, (1998) 22–60.
[ErRy94] R.M. Erdahl, S.S. Ryshkov, On lattice dicings, Eur. J. Combin. 15 (1994) 459–481.
[Er99] R.M. Erdahl, Zonotopes, Dicings, and Voronoi’s conjecture on Parallelohedra, Eur. J. Combin. 20 (1999) 527–549.
[ErRy02] R.M. Erdahl, K. Rybnikov, Voronoi-Dickson Hypothesis on Perfect Forms and L-types, Peter Gruber Festschrift: Rendiconti del Circolo Matemati ko di Palermo, Serie II, Tomo LII, part I (2002) 279–296.
[Gr04] V.P. Grishukhin, Parallelopetopes of non-zero width, Math. Sbornik 195 (2004) 59–78, translated in: Sb. Math 195 (2004) 669–686.
[Gr06a] V.P. Grishukhin, Free and Nonfree Voronoi Polyhedra (in Russian), Matematicheskie Zametki 80 (2006) 367–378, translated in Math. Notes 80 (2006) 355–365.
[Gr06b] V.P. Grishukhin, The Minkowski sum of a parallelope and a segment (in Russian), Math. Sbornik 197 (2006) 15–32, translated in: Sb. Math. 197 (2006) 1417–1433.
[Hu90] J.E. Humphreys, Reflection groups and Coxeter groups, Cambridge University Press, 1990.
[Mu80] P. McMullen, Convex bodies which tile space by translation, Mathematika 27 (1980) 113–121.
[NeSo12] G. Nebe, N.J.A. Sloane, A catalog of lattices, http://www.math.rwth-aachen.de/~Gabriele.Nebe/LATTICES/
[Or05] A. Ordine, Proof of the Voronoi conjecture on parallelopetopes in a new special case, Ph.D. Thesis, Queen’s University, Ontario, 2005.
[Or04] A. Ordin, A criterion for the nondecomposability of a parallelohedron into the Minkowski direct sum. (Russian) Chebyshevskii Sb. 5 (2004) 160–164.
[Re88] M. Reid, Undergraduate algebraic geometry, Lond. Math. Soc. student texts, Cambridge Univ. Press, 1988.
[RyZa05] K. Rybnikov, T. Zaslavsky, Criteria for balance in abelian gain graphs, with applications to piecewise-linear geometry, Disc. Comput. Geom. 34 (2005) 251–268.
[Sch09] A. Schürmann, Computational geometry of positive definite quadratic forms, University Lecture Notes, AMS, 2009.
[Se80] P.D. Seymour, Decomposition of regular matroids, J. Comb. Theory ser. B 28 (1980) 305–359.
[Sha88] I.R. Shafarevich, Foundations of algebraic geometry Vol. 1 (in Russian), Nauka, Moscow, 1988, translated in: Springer Verlag, 1994.
[Tr98] K. Treumer, Matroid decomposition, revised edition, Leibniz, Plano, Texas, 1998.
[Ve54] B.A. Venkov, On a class of Euclidean polytopes (in Russian), Vestnik Leningradskogo Univ., Ser. Math. Phys. Chem. 9 (1954) 11–31.
[Ve59] B.A. Venkov, On projection of parallelohedra (in Russian), Math. Sbornik 49 (1959) 207–224.
[Ve06] A. Végh, Rácsek, kör- és gömbelrendezések, PhD thesis (in Hungarian), Budapest University of Technology and Economics, Budapest, 2006.
[Vo08] G.F. Voronoi, Nouvelles applications des paramètres continus à la théorie des formes quadratiques, Deuxième Mémoire, Recherches sur les paralleloedres primitifs, J. Reine Angew. Math. 134 (1908) 198–287 and 136 (1909) 67–181.
[Zo96] C. Zong, Strange phenomena in convex and discrete geometry, Universitext, Springer-Verlag, New York, 1996.
