Resolving the $\tau$ vs. electroproduction discrepancy for the $I = 1$ vector spectral function and implications for the SM prediction for $a_\mu$

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Using only independent high-scale OPE input, we investigate QCD sum rule constraints on two currently incompatible versions of the isovector vector spectral function, one obtained from electroproduction (EM) data, the other from hadronic $\tau$ decay data. Sum rules involving weighted integrals over the spectral function, from threshold to a variable upper endpoint $s_0$, are employed. It is shown that both the normalization and slope with respect to $s_0$ of the EM spectral integrals disagree with the corresponding OPE expectations, while both normalization and slope are in good agreement when hadronic $\tau$ decay data is used instead. These results favor determinations of the leading hadronic vacuum polarization contribution to $a_\mu$ obtained using the $\tau$ decay data, and hence Standard Model predictions for $a_\mu$ compatible with the current experimental determination.

Keywords: muon, magnetic moment; vacuum polarization, hadronic; QCD sum rules

1. Introduction

The recent high-precision experimental determination of $a_\mu = (g - 2)_\mu$ has led to intense interest in the corresponding Standard Model (SM) prediction. The largest of the non-pure-QED SM contributions, that due to the leading order hadronic vacuum polarization, $[a_\mu]_{\text{LO}}^{\text{had}}$, can be obtained as an appropriately-weighted integral over the electromagnetic (EM) spectral function, $\rho_{EM}(s)$. The $I = 1$ component of $\rho_{EM}(s)$ is related to that of its charged current $I = 1$ partner, measured in hadronic $\tau$ decay, by CVC. Recent attempts to incorporate $\tau$ decay data into the evaluation of $[a_\mu]_{\text{LO}}^{\text{had}}$ using the isospin-breaking (IB) corrected version of this relation have encountered a consistency problem in the dominant ($\pi\pi$ final state) contribution to $[a_\mu]_{\text{LO}}^{\text{had}}$. Specifically, even after accounting for known sources of IB, the $\pi\pi$ component of the resulting $\tau$-based $I = 1$ vector spectral function is incompatible with that obtained from the high-precision CMD2 $e^+e^- \to \pi^+\pi^-$ data, with surprisingly large ($\sim 10\%$) discrepancies remaining in the region of $\pi\pi$ invariant masses from $\sim 0.85$ to $\sim 1$ GeV. SM predictions for $a_\mu$ based solely on electroproduction (EM) data indicate a $\sim 2.5\sigma$ deviation from the experimental result, while those based on the IB-corrected hadronic $\tau$ decay data indicate compatibility at the $\sim 1\sigma$ level. The discrepancy thus significantly impacts the question of whether or not the current experimental determination of $a_\mu$ shows evidence for beyond-the-SM contributions.
To resolve the above discrepancy, we have investigated sum rule constraints on the EM and \( \tau \)-decay-based data sets, working with sum rules of the form
\[
\int_{s_{th}}^{s_0} ds \, w(s) \, \rho(s) = -\frac{1}{2\pi i} \int_{|s|=s_0} ds \, w(s) \, \Pi(s),
\]
where \( s_{th} \) is the relevant threshold, \( \Pi(s) \) is the correlator with spectral function \( \rho(s) \), \( w(s) \) is a function analytic in the region of the contour, and the OPE representation of \( \Pi(s) \) is to be used on the RHS. At intermediate scales, weights satisfying \( w(s=s_0)=0 \) must be employed, in order to suppress duality violating contributions from the vicinity of the timelike point. This is most conveniently done by using the variable \( y = s/s_0 \) and weights \( w(y) \) satisfying \( w(1) = 0 \). Integrated OPE contributions of dimension \( D \) then scale as \( 1/s^{(D-2)/2} \), allowing reliable self-consistency checks for the absence of neglected higher \( D \) contributions.

In the current study, to be specific, we employ the ALEPH data and covariance matrix (with updated normalization) for the hadronic \( \tau \) decay version of the isovector vector spectral function. For the various exclusive components of the EM spectral function and their errors, we follow the assessments of Refs. More recent high-precision data is incorporated as discussed in Ref. In the region of the EM-\( \tau \) discrepancy, we focus on the CMD2 \( \pi\pi \) data. The weights, \( w(y) \), and scales, \( s_0 \), used in this study have been chosen in such a way that (i) the OPE is essentially entirely dominated by its \( D = 0 \) contribution, (ii) the convergence of the integrated \( D = 0 \) series, order-by-order in \( \alpha_s \), is excellent, and (iii) poorly known \( D = 6 \) contributions are absent. The OPE sides of the various sum rules are then determined, up to very small non-perturbative corrections, by the single input parameter, \( \alpha_s(M_Z) \), for which we employ an average of high-scale determinations independent of the EM and \( \tau \) data being tested. The values relevant to lower scales are obtained by standard four-loop running and matching.

For illustration, we present below results for three weight cases, \( w(y) = 1 - y, w_3(y), \) and \( w_6(y), \) where \( w_N(y) = 1 - \left( \frac{N-1}{N} \right) y + \frac{y^N}{N} \). The first weight has a zero of order 1 at \( y = 1 \), the remaining weights zeros of order 2. The weights are all non-negative in the integration region, simplifying the interpretation of the sum rule tests. A discussion of other advantages of these weights may be found in Ref.

2. Results and Discussion

We find that the \( \tau \)-decay-based spectral integrals are compatible with the high-scale OPE input for a wide range of \( s_0 \), but that the EM data yields spectral integrals consistently lower than those predicted by the OPE. The \( s_0 \) dependence of the OPE and corresponding spectral integrals is also in good agreement for the \( \tau \) case, but in significant disagreement for the EM case. Figures may be found in Ref.

Regarding the normalization of the EM spectral integrals: to quantify the normalization discrepancy, we have used the spectral integral values at the highest accessible common scale for the EM and \( \tau \) cases (\( s_0 = m_\tau^2 \)) to obtain an effective value for \( \alpha_s(M_Z) \). This is to be compared to the independent high-scale average, \( \alpha_s(M_Z) = 0.1200 \pm 0.0020 \). The results are shown in Table 1. While the values im-
plied by the EM data are low by only $\sim 2\sigma$, typical fluctuations in the data which would bring them into better agreement with OPE expectations would also serve to increase $[a_{\mu}]_{\text{had}}^{LO}$. The $\tau$ data, in contrast, yields $\alpha_s(M_Z)$ values in good agreement with the high-scale average. Somewhat lower values of $\alpha_s(M_Z)$ are also easily accommodated since the overall normalization uncertainty for the $\tau$ data corresponds to an uncertainty of $\pm 0.0010$ in the fitted $\alpha_s(M_Z)$.

Table 1. Values of $\alpha_s(M_Z)$ obtained by fitting to the $s_0 = m_{\tau}^2$ experimental EM and $\tau$ spectral integrals with central values for the $D = 2, 4$ OPE input.

| Weight | EM or $\tau$ | $\alpha_s(M_Z)$ |
|--------|--------------|-----------------|
| $1 - y$ | EM          | $0.1138 \pm 0.0035$ |
| $w_3$  | EM          | $0.1152 \pm 0.0021$ |
| $w_6$  | EM          | $0.1150 \pm 0.0022$ |
| $1 - y$ | $\tau$      | $0.1218 \pm 0.0022$ |
| $w_3$  | $\tau$     | $0.1195 \pm 0.0021$ |
| $w_6$  | $\tau$     | $0.1201 \pm 0.0022$ |

The problem of the $s_0$ dependence of the EM spectral integrals is similarly quantified in Table 2, for $w(y) = 1 - y$, $w_6(y)$. $S_{\text{exp}}$ and $S_{\text{OPE}}$ are the slopes with respect to $s_0$ of the spectral and OPE integrals, respectively. OPE entries labelled "indep" correspond to the high-scale average $\alpha_s(M_Z)$ input above, those labelled "fit" to the fitted values given in Table 1. The uncertainty in $S_{\text{OPE}}$ is seen to be very small. The very weak dependence on $\alpha_s$ means that even the change from "indep" to "fit" input on the OPE side has only marginal impact, the OPE versus spectral integral slope discrepancy being reduced from 2.6 to 2.3 $\sigma$ for $w(y) = 1 - y$ and 2.5 to 2.2 $\sigma$ for $w(y) = w_6(y)$. The source of the slope problem thus lies entirely on the data side. The problem can be cured only by a change in the shape of the EM spectral distribution, such as would occur if the IB-corrected $\tau$ data, rather than the EM data, represented the correct version of $\rho_{\text{EM}}(s)$. 

We conclude that either (i) non-one-photon physics effects, as yet unidentified, are contaminating the EM data or (ii) there remain experimental problems with the (pre-2005) EM data. The latter conclusion is favored by the just-released SND $e^+e^- \rightarrow \pi^+\pi^-$ results \cite{12}, which are compatible with the IB-corrected hadronic $\tau$ decay data. In either case it follows that determinations of $[a_{\mu}]_{\text{had}}^{LO}$ incorporating $\tau$ decay data are favored over those based solely on EM cross-sections and hence that
the SM prediction for $a_\mu$ is in good agreement with the current experimental result.

A final comment concerns the IB correction associated with $\rho$-$\omega$ “mixing”. The current standard evaluation is based on a chirally-constrained model (the “GP/CEN model”) fitted to a since-corrected form of the CMD2 data. Because of strong cancellations, the model dependence of the integrated $[a_\mu]^{LO}_{had}$ interference contribution turns out to be significantly larger than any one model’s fitting-induced uncertainty. Table 3 shows the results of fits to the CMD2 data for the Gounaris-Sakurai (GS), hidden local symmetry (HLS), and Kuhn-Santamaria (KS) models, as well as two versions of the GP/CEN model, one a refitting to the current version of the CMD2 data (GP/CEN$^\dagger$), the other a refitting which incorporates an additional phase, such as arises when direct IB $\omega \to \pi\pi$ decay contributions are taken into account (GP/CEN$^\ast$). Smaller values of $[\delta(a_\mu)]_{\rho-\omega}$ also serve to reduce the difference between the EM and $\tau$-based determinations of $[a_\mu]^{LO}_{had}$.

| Model       | $\chi^2$/dof | $[\delta(a_\mu)]_{\rho-\omega} \times 10^{10}$ |
|-------------|--------------|---------------------------------------------|
| GS          | 35.9/38      | $2.0 \pm 0.5$                               |
| HLS         | 36.6/38      | $4.0 \pm 0.6$                               |
| KS          | 37.1/38      | $3.8 \pm 0.6$                               |
| GP/CEN$^\dagger$ | 40.6/39 | $2.0 \pm 0.5$                               |
| GP/CEN$^\ast$ | 61.5/40   | $3.7 \pm 0.7$                               |

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