The Topological $CP^1$ Model and the Large-$N$ Matrix Integral

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Abstract

We discuss the topological $CP^1$ model which consists of the holomorphic maps from Riemann surfaces onto $CP^1$. We construct a large-$N$ matrix model which reproduces precisely the partition function of the $CP^1$ model at all genera of Riemann surfaces. The action of our matrix model has the form $Tr V(M) = -2Tr M (\log M - 1) + 2 \sum t_{n,P} Tr M^n (\log M - c_n) + \sum 1/n \cdot t_{n-1,Q} Tr M^n (c_n = \sum 1/j)$ where $M$ is an $N \times N$ hermitian matrix and $t_{n,P}$ ($t_{n,Q}$), ($n = 0, 1, 2 \cdots$) are the coupling constants of the $n$-th descendant of the puncture (Kähler) operator.
The topological $CP^1$ model was introduced a few years ago by as a characteristic example of a topological $\sigma$ model coupled to the two-dimensional gravity. As is well-known, the topological $\sigma$-model (or topological A model) is a twisted version of the supersymmetric non-linear $\sigma$-model with a target space which has an almost complex structure. Its path-integral is given by a sum over contributions from instantons which are holomorphic maps from the (genus $g$) Riemann surface onto a target space $K$. When the target space $K$ is $CP^1$, the theory possesses two observables $P$ and $Q$ corresponding to the identity and the K"ahler class of $CP^1$ and thus the theory becomes a two-primary model. When the system is coupled to the two-dimensional gravity, additional excitations, e.g. gravitational descendants $\sigma_n(P), \sigma_n(Q)$ ($n = 1, 2, \cdots$) appear in the theory (by convention $\sigma_0(P) = P, \sigma_0(Q) = Q$). We denote the coupling constants associated with these observables as $t_{n,P}, t_{n,Q}$ ($n = 0, 1, 2, \cdots$).

In the following we construct a matrix model which reproduces exactly the partition function of the $CP^1$ model in the large phase space ($t_{n,P} \neq 0, t_{n,Q} \neq 0$) at all genera of the Riemann surface. The potential of our matrix model has the form

$$V(M) = -2M(\log M - 1) + 2 \sum_{n=1} t_{n,P} M^n (\log M - c_n)$$

$$+ \sum_{n=1} \frac{1}{n} t_{n-1,Q} M^n , \quad c_n = \sum_{j=1}^{n} \frac{1}{j} \quad (1)$$

and the partition function is given by

$$Z = \int dM \exp(N\text{Tr} V(M)) . \quad (2)$$

Here $M$ is an $N \times N$ hermitian matrix. The characteristic feature of our model (1) is the appearance of the logarithmic potentials indicative of the asymptotic freedom and the non-vanishing $\beta$ function. In the limit when all coupling constants vanish our action is reduced to $V(M) = -2M(\log M - 1)$ which is the form of the effective potential of the $CP^1$ model when $M$ is replaced by the chiral superfield $\Phi$ [4]. The extremum of the action $V'(\Phi) = 0$ occurs at $\Phi^2 = 1$ which is the basic relation of the $CP^1$ quantum cohomology [1, 5].

Let us first describe some basic features of the $CP^1$ model at genus $g = 0$. By denoting the free-energy of the system as $F$ basic two-point functions of
the theory are defined as
\[ \langle PP \rangle = \frac{\partial^2 F}{\partial t_{0,P}^2} \equiv u , \quad (3) \]
\[ \langle PQ \rangle = \frac{\partial^2 F}{\partial t_{0,Q} \partial t_{0,P}} \equiv v . \quad (4) \]

A crucial ingredient of the $CP^1$ model is the following (constitutive) relation
\[ \langle QQ \rangle = e^{\langle PP \rangle} = e^u \quad (5) \]
which comes from the instanton analysis [1]. (5) is the formula which distinguishes the $CP^1$ model from other two-primary models like the Ising model. From (3),(4),(5) we obtain the flow equations for the time variable $t_{0,Q}$
\[ \frac{\partial u}{\partial t_{0,Q}} = v' , \quad (6) \]
\[ \frac{\partial v}{\partial t_{0,Q}} = (e^u)' . \quad (7) \]

Here $'$ denotes the derivative in $t_{0,P}$. The flow equations for the descendant times $t_{n,P}$, $t_{n,Q}$ $(n = 1, 2 \cdots)$ are obtained by using the genus $g = 0$ topological recursion relations [1]
\[ \langle \sigma_n(\Phi_\alpha)XY \rangle = n\langle \sigma_{n-1}(\Phi_\alpha)P \rangle \langle QXY \rangle + n\langle \sigma_{n-1}(\Phi_\alpha)Q \rangle \langle PXY \rangle , \quad (8) \]
where $\Phi_\alpha = P$ or $Q$. $\partial u/\partial t_{1,P}$ is, for instance, derived as
\[ \frac{\partial u}{\partial t_{1,P}} = \langle \sigma_1(P)PP \rangle = \langle PP \rangle \langle QPP \rangle + \langle PQ \rangle \langle PPP \rangle = u' + vu' = (uv)' . \quad (9) \]

Similarly
\[ \frac{\partial v}{\partial t_{1,P}} = \langle \sigma_1(P)PQ \rangle = \langle PP \rangle \langle QPQ \rangle + \langle PQ \rangle \langle PPQ \rangle = u(e^u)' + vu' = \left( \frac{1}{2}v^2 + (u - 1)e^u \right)' . \quad (10) \]
For the sake of illustration we also present the flow equations for the variables $t_{1,Q}$, $t_{2,P}$ and $t_{2,Q}$

\[
\frac{\partial u}{\partial t_{1,Q}} = \left(\frac{1}{2}v^2 + e^u\right)' , \quad \frac{\partial v}{\partial t_{1,Q}} = (ve^u)' , \quad (11)
\]

\[
\frac{\partial u}{\partial t_{2,P}} = (uv^2 + 2(u-2)e^u)' , \quad \frac{\partial v}{\partial t_{2,P}} = \left(\frac{1}{3}v^3 + 2v(u-1)e^u\right)' , \quad (12)
\]

\[
\frac{\partial u}{\partial t_{2,Q}} = \left(\frac{1}{3}v^3 + 2ve^u\right)' , \quad \frac{\partial v}{\partial t_{2,Q}} = \left(v^2e^u + e^2u\right)' . \quad (13)
\]

It is clear that one can write down flow equations for any of the variables $t_{n,P}$, $t_{n,Q}$ by using the recursion relations repeatedly. By construction these flows mutually commute with each other and thus they define an integrable hierarchy.

It is easy to see that the CP$^1$ system is closely related to the Toda lattice hierarchy [6]. In fact by combing (6) and (7) we obtain

\[
\frac{\partial^2 u}{\partial t_{0,Q}^2} = (e^u)''
\]

which may be regarded as the continuum (genus $g = 0$) limit of the Toda lattice equation

\[
\frac{\partial^2 u_n}{\partial z \partial \bar{z}} = e^{u_{n+1}} + e^{u_{n-1}} - 2e^{u_n}
\]

if we identify $t_{0,P} = n/N$ ($N$ is the size of the lattice). In (14) $t_{0,Q} \approx z = \bar{z}$ and thus we have the one-dimensional Toda theory. As we shall see in the following, our system can in fact be described by a suitable generalization of the one-dimensional Toda lattice hierarchy.

Let us now introduce the Lax formalism for the Toda lattice (at genus $g = 0$) and see how one can reproduce the evolution equations of the CP$^1$ model. We consider an operator

\[
L = p + v + e^u p^{-1} , \quad (16)
\]

where the momentum $p$ is the canonical conjugate of $t_{0,P}$. The Poisson bracket of the Toda system is defined as

\[
\{A, B\} = p \left(\frac{\partial A}{\partial p} \frac{\partial B}{\partial t_{0,P}} - \frac{\partial B}{\partial p} \frac{\partial A}{\partial t_{0,P}}\right) . \quad (17)
\]
Time-evolution is generated by the Hamiltonians \((L^n)_+, \ n = 1, 2 \cdots\), where + means taking terms with non-negative powers of \(p\). By introducing the Toda-times \(t_n\) we have

\[
\frac{\partial L}{\partial t_n} = \{(L^n)_+, L\}, \quad n = 1, 2, \cdots.
\] (18)

By comparing (18) with the flow equations of the \(CP^1\) model we find that

\[
\text{the } CP^1\text{-times } t_{n,Q}(n = 0, 1, 2 \cdots) \text{ are nothing but the Toda-times } t_{n+1}(n = 0, 1, 2 \cdots)
\]

\[
t_{n,Q} = (n+1)t_{n+1}, \quad n = 0, 1, 2, \cdots.
\] (19)

For instance,

\[
\frac{\partial L}{\partial t_1} = \{(L^2)_+, L\} = \{p + v, p + v + e^up^{-1}\}
= (e^u)' + v'e^up^{-1}.
\] (20)

Hence \(\partial v/\partial t_1 = (e^u)', \ \partial u/\partial t_1 = v'\) which coincide with eqs.(7),(6). Similarly

\[
\frac{\partial L}{\partial t_2} = \{(L^3)_+, L\} = \{p^2 + 2vp + v^2 + 2e^u, p + v + e^u p^{-1}\}
= 2(ve^u)' + (v^2 + 2e^u)'e^up^{-1}.
\] (21)

Hence \(\partial v/\partial t_2 = 2(ve^u)', \ \partial u/\partial t_2 = 2(v^2/2 + e^u)'\) which agree with eq.(11). Thus the generators of the \(t_{n,Q}\)-time evolutions are identified as simple powers of the Lax operator \(L\).

If one attempts to describe the \(t_{n,P}\)-time evolutions using the Lax formalism, one has to generalize the standard treatment of the Toda theory and introduce the logarithm of the Lax operator \(\log L\). We introduce a series of the operators \(L^n(\log L - c_n)\) with \(c_n = \sum_{j=1}^n 1/j\) and consider an additional set of evolution equations

\[
\frac{\partial L}{\partial s_n} = \{(L^n(\log L - c_n))_+, L\}, \quad n = 0, 1, 2, \cdots.
\] (22)

It is possible to show that only terms proportional to \(p^0\) and \(p^{-1}\) appear in the right-hand-side of eq.(22) as in the case of (18) (for the treatment of the logarithm see below, eq.(24)) and hence (22) gives rise to a set of differential
equations for the coefficient functions $u$ and $v$. Furthermore, the flows in the variables $s_n$ commute among themselves and also with those of the variables $t_m$.

Then by comparing (22) with the $CP^1$ $t_n,P$-flow equations we find

$$\frac{1}{2}s_n = t_{n,P}, \quad n = 0, 1, 2, \ldots \quad (23)$$

Let us check the $n = 0$ case. We first rewrite the log $L$ operator as

$$\log L = \log(p + v + e^u p^{-1})$$

$$= \frac{1}{2}\log p(1 + vp^{-1} + e^u p^{-2}) + \frac{1}{2}\log p^{-1}e^u(1 + ve^{-u}p + e^{-u}p^2)$$

$$= \frac{1}{2}u + \frac{1}{2}\log(1 + vp^{-1} + e^u p^{-2}) + \frac{1}{2}\log(1 + ve^{-u}p + e^{-u}p^2). \quad (24)$$

Taking the positive part simply drops the 2nd term in (24). Then by direct calculation,

$$\frac{\partial L}{\partial s_0} = \{(\log L)_+, L\} = \frac{1}{2}(v' + (e^u)' p^{-1}). \quad (25)$$

Hence

$$s_0 = 2t_{0,P}. \quad (26)$$

We can check other cases $n = 1, 2, \cdots$ of (23) with somewhat lengthy calculations. The coefficients $c_n$ are determined to eliminate the $\partial/\partial t_{n-1, Q}$ component in the action of the operator $(L^n \log L)_+$ on $L$. Our operators $L^n(\log L - c_n)$ have a special “scaling” property

$$\frac{d}{dL}L^n(\log L - c_n) = nL^{n-1}(\log L - c_{n-1}) \quad (27)$$

which plays an important role in what follows. We summarize our identification of the $CP^1$ evolution equations at the genus $g = 0$ level,

$$n\frac{\partial L}{\partial t_{n-1, Q}} = \{(L^n)_+, L\}, \quad n = 1, 2, \cdots \quad (28)$$

$$\frac{1}{2}\frac{\partial L}{\partial t_{n,P}} = \{(L^n(\log L - c_n))_+, L\}, \quad n = 1, 2, \cdots \quad (29)$$

Before we turn to the subject of the matrix-model realization of the hierarchy (28),(29) we must discuss its higher-genus corrections. As we discussed
elsewhere [3], it is straightforward to determine corrections to the flow equations of the $CP^1$ model at each genus simply by demanding the commutativity of the flows. Up to the order of $g = 2$ the corrected flow equations (for $t_{0,Q}$, $t_{1,P}$ and $t_{1,Q}$) are given by

\[
\frac{\partial u}{\partial t_{0,Q}} = v',
\]

\[
\frac{\partial v}{\partial t_{0,Q}} = \left( e^u + \lambda^2 \left( \frac{1}{24} u'^2 + \frac{1}{12} u'' \right) e^u + \lambda^4 \left( \frac{1}{360} u'''' + \frac{1}{180} u' u''' + \frac{7}{1440} u'^2 u'' + \frac{1}{1920} u'^4 + \frac{1}{240} u''^2 e^u \right) \right)',
\]

\[
\frac{\partial u}{\partial t_{1,P}} = \left( \frac{1}{2} v^2 + (u-1)e^u + \lambda^2 \left( \frac{1}{24} (u+3) u'^2 + \frac{1}{12} (u+2) u'' \right) e^u + \lambda^4 \left( \frac{1}{360} (u+4) u'''' + \frac{1}{180} (u+5) u' u''' + \frac{7}{1440} (u+6) u'^2 u'' + \frac{1}{1920} (u+7) u'^4 + \frac{1}{240} (u+5) u''^2 e^u \right) \right)',
\]

\[
\frac{\partial v}{\partial t_{1,P}} = \left( \frac{1}{2} v^2 + e^u + \lambda^2 \left( \frac{1}{24} (3 u'^2 + 4 u'') \right) e^u + \lambda^4 \left( \frac{1}{120} u'''' + \frac{1}{48} u' u''' + \frac{1}{48} u'^2 u'' + \frac{1}{384} u'^4 + \frac{1}{72} u''^2 e^u \right) \right)',
\]

\[
\frac{\partial u}{\partial t_{1,Q}} = \left( \frac{1}{2} v^2 + e^u + \lambda^2 \left( \frac{1}{24} (4 u'^2 + 2 u' v + 2 u'' v + 4 u'') \right) e^u + \lambda^4 \left( \frac{1}{360} v u'''' + \frac{1}{180} v u' u''' + \frac{7}{1440} v u'^2 u'' + \frac{1}{1920} v u'^4 + \frac{1}{240} v u''^2 + \frac{1}{180} v' u''' + \frac{1}{480} v' u''^2 + \frac{1}{720} v'' u' + \frac{1}{80} v'' u' + \frac{1}{120} v'' u'' e^u \right) \right)',
\]

Here $\lambda^2$ is the genus expansion parameter. In deriving the above equations we first write down candidate correction terms in the right-hand-side of (31)-(35) with unknown coefficients and then fix them by demanding the commutativity of the flows, e.g. the vanishing of the cross-derivatives such as
\[ \partial/\partial t_1, P (\partial v/\partial t_0, Q) - \partial/\partial t_0, Q (\partial v/\partial t_1, P) = 0. \] The flow commutativity leads to an over-determined system for the unknown coefficients and gives rise to their unique solution.

We now turn to the construction of a matrix model which reproduces the \( CP^1 \) model. Let us first recall some basic formulas for the analysis of matrix models using orthogonal polynomials \[1\]. We consider a one-matrix model with an action \( \text{Tr} V(M) \) which depends only on the eigenvalues of the \( N \times N \) hermitian matrix \( M \). The orthogonal polynomials are defined by

\[
\int d\lambda e^{NV(\lambda)} \psi_n(\lambda) \psi_m(\lambda) = \delta_{n,m} h_m, \tag{36}
\]

\[
\psi_n(\lambda) = \lambda^n + \text{lower terms}. \tag{37}
\]

The multiplication by \( \lambda \) of the orthogonal polynomials leads to a three-term recursion relation

\[
\lambda \psi_n(\lambda) = \sum_m Q_{nm} \psi_m(\lambda) = \psi_{n+1}(\lambda) + v_n \psi_n(\lambda) + R_n \psi_{n-1}(\lambda), \tag{38}
\]

\[
R_n = h_n/h_{n-1} \equiv e^N(\phi_n - \phi_{n-1}). \tag{39}
\]

The free-energy of the system is given by 

\[
F = \log \prod_{n=0}^{N-1} h_n = N \sum_{n=0}^{N-1} \phi_n.
\]

The matrix \( Q \) has non-vanishing elements only on the diagonal and 1st off-diagonal lines and thus is a Jacobi matrix. In the continuum limit \( N(\phi_n - \phi_{n-1}) \approx F''/N^2 = u \) and the matrix \( Q \) turns into the Lax operator

\[
Q \approx L = p + v + e^u p^{-1}. \tag{40}
\]

\( Q \) replaces \( L \) in the discussion of the \( CP^1 \) model at higher genera. Time evolution equations are now given by the commutation relations

\[
\frac{n}{N} \frac{\partial Q}{\partial t_{n-1, Q}} = [(Q^n)_,, Q], \quad n = 1, 2, \cdots \tag{41}
\]

\[
\frac{1}{2N} \frac{\partial Q}{\partial t_{n, P}} = [(Q^n(\log Q - c_n))_+, Q], \quad n = 0, 1, 2, \cdots, \tag{42}
\]

where + means to take the upper-triangular part of a matrix including the diagonal line. It is easy to check that the matrix commutators reduce to the Poisson bracket (17) in the continuum limit and thus (41) (42) reduce to (28) (29) at genus \( g = 0. \)
The derivative operator $\frac{d}{d\lambda}$, on the other hand, becomes a lower-triangular matrix when acting on the orthogonal polynomials

$$\frac{d}{d\lambda} \psi_n(\lambda) = n\psi_{n-1}(\lambda) + \cdots = \sum_m P_{nm} \psi_m(\lambda).$$

(43)

By partial integration we have

$$\int d\lambda e^{NV(\lambda)} \frac{d}{d\lambda} \psi_n(\lambda) \psi_m(\lambda) = -\int d\lambda e^{NV(\lambda)} NV'(\lambda) \psi_n(\lambda) \psi_m(\lambda) = -NV'(Q)_{nm} h_m, \quad n > m.\quad (44)$$

Hence

$$P = -N V'(Q)_-,\quad (45)$$

where $-$ means to take the lower-triangular part of the matrix. The string equation then reads

$$[Q, P] = -N [Q, V'(Q)_-] = 1.\quad (46)$$

Let us now introduce our matrix model defined by an action

$$\text{Tr} \, V(M) = -2 \text{Tr} \, M \log M - 1 + 2 t_{n,P} \text{Tr} \, M^n \log M - c_n$$

$$+ \sum_{n=1}^{\infty} \frac{1}{n} t_{n-1,Q} \text{Tr} \, M^n.\quad (47)$$

We then obtain

$$N [V'(Q)_-, Q]$$

$$= -2N[(\log Q)_-, Q] + N \sum_{n=1}^{\infty} 2 nt_{n,P} [(Q^{n-1} \log(Q - c_{n-1}))_-, Q]$$

$$+ N \sum t_{n-1,Q} [(Q^{n-1})_-, Q]$$

$$= \frac{\partial Q}{\partial t_{0,P}} - \sum_{n=1}^{\infty} nt_{n,P} \frac{\partial Q}{\partial t_{n-1,P}} - \sum_{n=2}^{\infty} (n-1)t_{n-1,Q} \frac{\partial Q}{\partial t_{n-2,Q}},\quad (48)$$

where we used (41),(42) and the “scaling” property of our logarithmic potentials (27). (46) is now compactly expressed as

$$1 + \sum_{n=1}^{\infty} \sum_{\alpha} nt_{n,\alpha} \frac{\partial Q}{\partial t_{n-1,\alpha}} = \frac{\partial Q}{\partial t_{0,P}},\quad (\alpha = P, Q).\quad (49)$$
The diagonal element of \((49)\) reads
\[
\frac{\partial (PQ)}{\partial t_{0,P}} = 1 + \sum_n \sum_\alpha n t_{n,\alpha} \langle \sigma_{n-1}(\Phi_\alpha) PQ \rangle \quad (50)
\]
in terms of the correlation functions. By integrating \((50)\) we obtain
\[
u = t_{0,Q} + \sum_n \sum_\alpha n t_{n,\alpha} \langle \sigma_{n-1}(\Phi_\alpha) P \rangle + \sum_n \sum_\alpha n t_{n,\alpha} \langle \sigma_{n-1}(\Phi_\alpha) Q \rangle . \quad (51)
\]
These are the string equations of the \(CP^1\) model \([2]\).

Therefore, in order to demonstrate the realization of the \(CP^1\) model by our matrix integral we finally have to examine our flow equations \((41)\),\((42)\) and check if they agree with the flow equations of the \(CP^1\) model at higher genera. Checking \((41)\) is straightforward. In the cases \(n = 1, 2\), for instance, \((41)\) gives
\[
\frac{\partial \phi_n}{\partial t_{0,Q}} = v_n , \quad (53)
\]
\[
\frac{\partial v_n}{\partial t_{0,Q}} = N(R_{n+1} - R_n) , \quad (54)
\]
\[
\frac{\partial \phi_n}{\partial t_{1,Q}} = \frac{1}{2}(R_{n+1} + R_n + v_n^2) , \quad (55)
\]
\[
\frac{\partial v_n}{\partial t_{1,Q}} = \frac{N}{2}((v_{n+1} + v_n)R_{n+1} - (v_n + v_{n-1})R_n) . \quad (56)
\]

If we use the definition \(t_{0,P} = n/N, u_n = \phi_n', R_n = \exp(N(\phi_n - \phi_{n-1}))\) and Taylor-expand \(R_{n\pm 1}(v_{n\pm 1})\) around \(R_n(v_n)\), we easily recover the \(CP^1\) equations. For instance, by expanding the right-hand-side of \((54)\) as
\[
N(R_{n+1} - R_n) = e^{u_n} \left( u_n' + \frac{u_n''}{12N^2} + \frac{u_n^3}{24N^2} + \frac{u_n' u_n''}{6N^2} + \frac{u_n''^2}{1920N^4} + \frac{u_n^3 u_n''}{144N^4} + \frac{u_n' u_n''^2}{72N^4} + \frac{u_n^2 u_n'''}{96N^4} + \frac{u_n'' u_n'''}{72N^4} + \frac{u_n' u_n'''}{120N^4} + \frac{u_n'''^2}{360N^4} + \cdots \right) \quad (57)
\]
we reproduce (31). Note that the genus expansion parameter $\lambda^2$ becomes $1/N^2$.

Checking (42), on the other hand, is rather non-trivial. We first have to compute the matrix elements of the logarithm of a Jacobi matrix $Q$. We use a formula,

$$ (\log Q)_{nm} = \frac{1}{2\pi i} \oint \frac{dz}{z} z^{n-m} \log \left( z + z^{-1} R(n + z \frac{d}{dz}) + v(n + z \frac{d}{dz}) \right) \cdot 1 $$ (58)

and expand the arguments of $R$ and $v$. We can then compute $\log Q$ perturbatively for each fixed order of differentiations of $R, v$. Up to the 2nd order in the derivatives, for instance, we obtain

$$ (\log Q)_{n,n} = \frac{1}{2} \log R_n + \frac{1}{4N} \frac{R'_n}{R_n} + \frac{1}{24N^2} \frac{R''_n}{R_n} - \frac{1}{24N^2} \frac{R'^2_n}{R_n^2}, \quad (59) $$

$$ (\log Q)_{n,n-1} = \frac{1}{2} v_n - \frac{1}{4N} v'_n + \frac{1}{24N^2} v''_n, \quad (60) $$

$$ (\log Q)_{n,n+1} = \frac{1}{2} \frac{v_n}{R_n} + \frac{1}{4N} \left( \frac{v'_n}{R_n} - \frac{2R'_n v_n}{R_n^2} \right) + \frac{1}{24N^2} \frac{v''_n}{R_n} - \frac{1}{4N^2} \frac{R''_n v_n}{R_n^2} + \frac{1}{2N^2} \frac{R'^2_n v_n}{R_n^3} - \frac{1}{4N^2} \frac{R'^2 v'_n}{R_n^2} \right). \quad (61) $$

Using the explicit expressions for $\log Q$ as above one can evaluate the right-hand-side of (42) and check its agreement with the $CP^1$ equations. We have calculated $\log Q$ up to the 3rd order in the derivatives and have confirmed the agreement of (42) with the $t_{n,P}$ flow equations up to $n = 2$ and $g = 1$.

We should note that in our construction we take the ordinary large-$N$ limit of the matrix model rather than the double scaling limit: we do not adjust the coupling constants to any specific set of values. This is in accord with the idea that the $CP^1$ model, due to its asymptotic freedom, has no phase transition points. In this sense our model is in a spirit closer to the Kontsevich [11] or Penner [11] model rather than the double-scaled matrix models describing the $c < 1$ minimal matter coupled to the two-dimensional gravity [12]. We think it is fortunate that the $CP^1$ model has a relatively simple representation in terms of a single matrix integral. Our model may prove useful in the study of the moduli space of holomorphic maps from Riemann surface to $CP^1$. It will be very interesting to see if our construction may be generalized to $CP^n$ or Grassmann manifolds.
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