A Formality Theorem for Hochschild Chains.

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Abstract

We prove Tsygan’s formality conjecture for Hochschild chains of the algebra of functions on an arbitrary smooth manifold \( M \) using the Fedosov resolutions proposed in [math.QA/0307212] and the formality quasi-isomorphism for Hochschild chains of \( \mathbb{R}[[y^1, \ldots, y^d]] \) proposed in paper [math.QA/0010321] by Shoikhet. This result allows us to describe traces on the quantum algebra of functions on an arbitrary Poisson manifold.

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1 Introduction

Proofs of Tsygan’s formality conjectures for chains [27], [28], [30] would unlock important algebraic tools which might lead to new generalizations of the famous index and Riemann-Roch-Hirzebruch theorems [1], [5], [12], [13], [17], [21], [22], [27]. Despite this pivot role in the traditional investigations and the efforts of various people [14], [24], [25], [27], [28] the most general version of Tsygan’s formality conjecture [27] has not yet been proved.

In this paper we prove Tsygan’s conjecture for Hochschild chains of the algebra of functions on an arbitrary smooth manifold \( M \) using the globalization technique proposed in [7] and [10] and the formality quasi-isomorphism for Hochschild chains of \( \mathbb{R}[[y^1, \ldots, y^d]] \) proposed in paper [24] by Shoikhet. This result allows us to prove Tsygan’s conjecture [30] about Hochschild homology of the quantum algebra of functions on an arbitrary Poisson manifold and to describe traces on this algebra.

The most general version of the formality theorem for chains says that a pair of spaces of Hochschild cochains and Hochschild chains of any associative algebra is endowed with the so-called \( T_\infty \)-structure and the \( T_\infty \)-algebra associated in this manner to the algebra of functions on a smooth manifold is formal. This statement was announced in [27] but the proof has not yet been formulated.

In this context we would like to mention paper [14], in which the authors prove a statement closely related to the cyclic formality theorem. In particular, this assertion allows them to prove a generalization of Connes-Flato-Sternheimer conjecture in the Poisson framework.

1On leave of absence from: ITEP (Moscow)
The structure of this paper is as follows. In the next section we recall basic notions related to $L_\infty$- or the so-called homotopy Lie algebras. We also describe a useful technical tool that allows us to utilize Maurer-Cartan elements of differential graded Lie algebras (DGLA). In the third section we recall algebraic structures on Hochschild complexes of associative algebra and introduce the respective versions of these complexes for the algebra of functions on a smooth manifold. In this section we formulate the main result of this paper (see theorem 1) and recall Kontsevich’s and Shoikhet’s formality theorems for $\mathbb{R}^d$. The bigger part of this paper is devoted to the construction of Fedosov resolutions of the algebras of polydifferential operators and polyvector fields, as well as the modules of Hochschild chains and exterior forms. Using these resolutions in section 5, we prove theorem 1. At the end of section 5 we also discuss applications of theorem 1 to a description of Hochschild homology for the quantum algebra of functions on a Poisson manifold $M$. In the concluding section we make a remark about an obvious version of theorem 1 in an algebraic geometric setting, mention an equivariant version, and raise some other questions.

Throughout the paper we assume the summation over repeated indices. Sometimes we omit the prefix “super-” referring to super-algebras, Lie super-brackets, and super(co)commutative (co)multiplications. We assume that $M$ is a smooth real manifold of dimension $d$. Our definition of antisymmetrization always comes with the standard $1/n!$-factor which makes the procedure idempotent. We omit symbol $\wedge$ referring to a local basis of exterior forms, as if we thought of $dx^i$’s as anti-commuting variables. The symbol $\circ$ always stands for a composition of morphisms. Finally, we always assume that a nilpotent linear operator is the one whose second power is vanishing.

## 2 $L_\infty$-structures

In this section we recall the notions of $L_\infty$-algebras, $L_\infty$-morphisms, $L_\infty$-modules and morphisms between $L_\infty$-modules. A more detailed discussion of the theory and its applications can be found in papers [16] and [20]. At the end of this section we introduce an important technical tool, which allows us to modify $L_\infty$-structures with the help of a Maurer-Cartan element.

### 2.1 $L_\infty$-algebras and $L_\infty$-morphisms

Let $\mathcal{L}$ be a $\mathbb{Z}$-graded vector space

$$\mathcal{L} = \bigoplus_{k \in \mathbb{Z}} \mathcal{L}^k. \quad (2.1)$$

We assume that there exists $N \geq 0$ such that $\mathcal{L}^k = 0$ for all $k < -N$. To the space $\mathcal{L}$ we associate a coassociative cocommutative coalgebra (without counit) $C(\mathcal{L}[1])$ cofreely cogenerated by $\mathcal{L}$ with a shifted parity.

The vector space of $C(\mathcal{L}[1])$ is the exterior algebra of $\mathcal{L}$

$$C(\mathcal{L}[1]) = \bigwedge \mathcal{L}, \quad (2.2)$$

where the antisymmetrization is graded, that is for any $\gamma_1 \in \mathcal{L}^{k_1}$ and $\gamma_2 \in \mathcal{L}^{k_2}$

$$\gamma_1 \wedge \gamma_2 = -(-)^{k_1 k_2} \gamma_2 \wedge \gamma_1.$$
The comultiplication

\[
\Delta : C(\mathcal{L}[1]) \mapsto \bigwedge C(\mathcal{L}[1])
\]  

is defined by the formulas \((n > 1)\)

\[
\Delta(\gamma_1) = 0,
\]

\[
\Delta(\gamma_1 \wedge \cdots \wedge \gamma_n) = \frac{1}{2} \sum_{k=1}^{n-1} \frac{1}{k!(n-k)!} \sum_{\varepsilon \in S_n} \pm \gamma_{\varepsilon(1)} \wedge \cdots \wedge \gamma_{\varepsilon(k)} \wedge \gamma_{\varepsilon(k+1)} \wedge \cdots \wedge \gamma_{\varepsilon(n)},
\]

where \(\gamma_1, \ldots, \gamma_n\) are homogeneous elements of \(\mathcal{L}\), \(S_n\) is the group of permutations of \(n\) elements and the sign in the latter formula depends naturally on the permutation \(\varepsilon\) and degrees of \(\gamma_k\).

We now give the definition of \(L_\infty\)-algebra.

**Definition 1** A graded vector space \(\mathcal{L}\) is said to be endowed with a structure of an \(L_\infty\)-algebra if the cocommutative coassociative coalgebra \(C(\mathcal{L}[1])\) cofreely cogenerated by the vector space \(\mathcal{L}\) with a shifted parity is equipped with a nilpotent coderivation \(Q\) of degree 1.

To unfold this definition we first mention that the kernel of \(\Delta\) coincides with the subspace \(\mathcal{L} \subset C(\mathcal{L}[1])\).

\[
ker \Delta = \mathcal{L}.
\]  

Next, we recall that a map \(Q\) is a coderivation of \(C(\mathcal{L}[1])\) if and only if for any \(X \in C(\mathcal{L}[1])\)

\[
\Delta Q X = (Q \otimes I + I \otimes Q) \Delta X.
\]  

Substituting \(X = \gamma_1 \wedge \cdots \wedge \gamma_n\) in (2.5), using (2.4), and performing the induction on \(n\) we get that equation (2.5) has the following general solution

\[
Q_{\gamma_1 \wedge \cdots \wedge \gamma_n} = Q_n(\gamma_1, \ldots, \gamma_n) + \sum_{k=1}^{n-1} \frac{1}{k!(n-k)!} \sum_{\varepsilon \in S_n} \pm Q_k(\gamma_{\varepsilon(1)}, \ldots, \gamma_{\varepsilon(k)}) \wedge \gamma_{\varepsilon(k+1)} \wedge \cdots \wedge \gamma_{\varepsilon(n)},
\]  

where \(\gamma_1 \ldots \gamma_n\) are homogeneous elements of \(\mathcal{L}\) and \(Q_n\) for \(n \geq 1\) are arbitrary polylinear antisymmetric graded maps

\[
Q_n : \wedge^n \mathcal{L} \mapsto \mathcal{L}[2 - n], \quad n \geq 1.
\]  

It not hard to see that \(Q\) can be expressed inductively in terms of the structure maps (2.7) and vice-versa.

Similarly, one can show that the nilpotency condition

\[
Q^2 = 0
\]  

is equivalent to a semi-infinite collection of quadratic relations on (2.7). The lowest of these relations are

\[
(Q_1)^2 = 0, \quad \forall \gamma \in \mathcal{L},
\]
\[ Q_1(Q_2(\gamma_1, \gamma_2)) - Q_2(Q_1(\gamma_1), \gamma_2) - (-)^{k_1}Q_2(\gamma_1, Q_1(\gamma_2)) = 0, \tag{2.10} \]

and

\[ (-)^{k_1+k_3}Q_2(Q_2(\gamma_1, \gamma_2), \gamma_3) + c.p.(1,2,3) = \]

\[ = Q_1Q_3(\gamma_1, \gamma_2, \gamma_3) + Q_3(Q_1\gamma_1, \gamma_2, \gamma_3) + (-)^{k_1}Q_3(\gamma_1, Q_1\gamma_2, \gamma_3) \]

\[ +(-)^{k_1+k_2}Q_3(\gamma_1, Q_2, \gamma_1\gamma_3), \tag{2.11} \]

where \( \gamma_i \in \mathcal{L}^{k_i} \).

Thus (2.9) says that \( Q_1 \) is a differential in \( \mathcal{L} \) (2.10) says that \( Q_2 \) satisfies Leibniz rule with respect to \( Q_1 \) and (2.11) implies that \( Q_2 \) satisfies Jacobi identity up to \( Q_1 \)-cohomologically trivial terms.

**Example.** Any differential graded Lie algebra (DGLA) \((\mathcal{L}, \partial, [\cdot,\cdot])\) is an example of an \(L_\infty\)-algebra with the only two nonvanishing structure maps

\[ Q_1 = \partial, \]

\[ Q_2 = [\cdot, \cdot]. \]

**Definition 2** An \(L_\infty\)-morphism \( F \) from the \(L_\infty\)-algebra \((\mathcal{L}, Q)\) to the \(L_\infty\)-algebra \((\mathcal{L}^\circ, Q^\circ)\) is a homomorphism of the cocommutative coassociative coalgebras

\[ F : C(\mathcal{L}[1]) \rightarrow C(\mathcal{L}^\circ[1]), \tag{2.12} \]

\[ \Delta F(X) = F \otimes F(\Delta X), \quad X \in C(\mathcal{L}[1]) \]

compatible with the nilpotent coderivations \( Q \) and \( Q^\circ \)

\[ Q^\circ F(X) = F(QX), \quad \forall X \in C(\mathcal{L}[1]). \tag{2.13} \]

In what follows the notation

\[ F : (\mathcal{L}, Q) \rightarrow (\mathcal{L}^\circ, Q^\circ) \]

means that \( F \) is an \(L_\infty\)-morphism from the \(L_\infty\)-algebra \((\mathcal{L}, Q)\) to the \(L_\infty\)-algebra \((\mathcal{L}^\circ, Q^\circ)\).

The compatibility of the map (2.12) with coproducts in \( C(\mathcal{L}[1]) \) and \( C(\mathcal{L}^\circ[1]) \) means that \( F \) is uniquely determined by the semi-infinite collection of polylinear graded maps

\[ F_n : \wedge^n \mathcal{L} \rightarrow \mathcal{L}^\circ[1-n], \quad n \geq 1 \tag{2.14} \]

via the equations \((n \geq 1)\)

\[ F(\gamma_1 \wedge \cdots \wedge \gamma_n) = F_n(\gamma_1, \ldots, \gamma_n) + \]

\[ \sum_{p \geq 1} \frac{1}{p!} \sum_{k_1, \ldots, k_p = n} \frac{1}{k_1! \cdots k_p!} \sum_{\epsilon \in S_n} \pm F_{k_1}(\gamma_{\epsilon(1)}, \ldots, \gamma_{\epsilon(k_1)}) \wedge \cdots \wedge F_{k_p}(\gamma_{\epsilon(n-k_p+1)}, \ldots, \gamma_{\epsilon(n)}), \tag{2.15} \]

where \( \gamma_1, \ldots, \gamma_n \) are homogeneous elements of \( \mathcal{L} \).
The compatibility of $F$ with coderivations (2.13) is a rather complicated condition for general $L_\infty$-algebras. However it is not hard to see that if (2.13) holds then
$$F_1(Q_1 \gamma) = Q_1^0 F_1(\gamma), \quad \forall \gamma \in \mathcal{L},$$
that is the first structure map $F_1$ is always a morphism of complexes $(\mathcal{L}, Q_1)$ and $(\mathcal{L}^\circ, Q_1^0)$.

This observation motivates the following natural

**Definition 3** An quasi-isomorphism $F$ from the $L_\infty$-algebra $(\mathcal{L}, Q)$ to the $L_\infty$-algebra $(\mathcal{L}^\circ, Q^\circ)$ is an $L_\infty$-morphism from $\mathcal{L}$ to $\mathcal{L}^\circ$, the first structure map $F_1$ of which induces a quasi-isomorphism of complexes $F_1 : (\mathcal{L}, Q_1) \mapsto (\mathcal{L}^\circ, Q_1^0)$.

Let us suppose for the next couple of paragraphs that our $L_\infty$-algebras $(\mathcal{L}, Q)$ and $(\mathcal{L}^\circ, Q^\circ)$ are just DG Lie algebras $(\mathcal{L}, d, [,])$ and $(\mathcal{L}^\circ, d^\circ, [,]^\circ)$. Then if $F$ is an $L_\infty$-morphism from $\mathcal{L}$ to $\mathcal{L}^\circ$ the compatibility of $F$ with the respective coderivations $Q$ and $Q^\circ$ is equivalent to the following semi-infinite collection of equations $(n \geq 1)$

$$d^\circ F_n(\gamma_1, \gamma_2, \ldots, \gamma_n) - \sum_{i=1}^{n} (-)^{k_1+\ldots+k_i-1+1-n} F_n(\gamma_1, \ldots, d\gamma_i, \ldots, \gamma_n) =$$

$$= \frac{1}{2} \sum_{k,l \geq 1, k+l=n} \frac{1}{k!l!} \sum_{\varepsilon \in S_n} \pm [F_k(\gamma_{\varepsilon_1}, \ldots, \gamma_{\varepsilon_k}), F_l(\gamma_{\varepsilon_{k+1}}, \ldots, \gamma_{\varepsilon_{k+l}})]^\circ -$$

$$- \sum_{i \neq j} \pm F_{n-1}(\hat{\gamma}_i, \gamma_1, \ldots, \hat{\gamma}_j, \gamma_i, \ldots, \gamma_{\hat{\gamma}_i}, \ldots, \gamma_n), \quad \gamma_i \in \mathcal{L}^{k_i},$$

where $\hat{\gamma}_i$ means that the polyvector $\gamma_i$ is missing.

**Remark.** Notice that in order to define the signs in formulas (2.17) one should use a rather complicated rule. For example, the signs that stand in front of the terms of the first sum at the right hand side depend on permutations $\varepsilon \in S_n$, on degrees of $\gamma_i$, and on the numbers $k$ and $l$. The simplest way to check that all the signs are correct is to show that the right hand side of equation (2.17) is closed with respect to the following differential acting on the space of graded polylinear maps

$$d_{Hom} : Hom(\wedge^n \mathcal{L}, \mathcal{L}^\circ[k]) \mapsto Hom(\wedge^n \mathcal{L}, \mathcal{L}^\circ[k + 1]),$$

$$d_{Hom} \Psi(\gamma_1, \gamma_2, \ldots, \gamma_n) = d^\circ \Psi(\gamma_1, \gamma_2, \ldots, \gamma_n) -$$

$$- \sum_{i=1}^{n} (-)^{k_1+\ldots+k_i-1+k} \Psi(\gamma_1, \ldots, d\gamma_i, \ldots, \gamma_n), \quad \gamma_i \in \mathcal{L}^{k_i},$$

where $\Psi \in Hom(\wedge^n \mathcal{L}, \mathcal{L}^\circ[k])$.

**Example.** An important example of a quasi-isomorphism from a DGLA $\mathcal{L}$ to a DGLA $\mathcal{L}^\circ$ is provided by a DGLA-homomorphism

$$\mathcal{H} : \mathcal{L} \mapsto \mathcal{L}^\circ,$$
which induces an isomorphism on the spaces of cohomology \( H^\bullet(L, \partial) \) and \( H^\bullet(L^\circ, \partial^\circ) \). In this case the quasi-isomorphism has the only nonvanishing structure map

\[ F_1 = \mathcal{H}. \]

### 2.2 \( L_\infty \)-modules and their morphisms

Another important object of the “\( L_\infty \)-world” we are going to deal with is an \( L_\infty \)-module over an \( L_\infty \)-algebra. Namely,

**Definition 4** Let \( L \) be an \( L_\infty \)-algebra. Then the graded vector space \( M \) is endowed with a structure of an \( L_\infty \)-module over \( L \) if the cofreely cogenerated comodule \( C(L[1]) \otimes M \) over the coalgebra \( C(L[1]) \) is endowed with a nilpotent coderivation \( \varphi \) of degree 1.

To unfold the definition we first mention that the total space of the comodule \( C(L[1]) \otimes M \) is

\[ C(L[1]) \otimes M = \bigwedge (L) \otimes M, \quad (2.19) \]

and the coaction

\[ a : C(L[1]) \otimes M \mapsto C(L[1]) \otimes (C(L[1]) \otimes M) \]

is defined on homogeneous elements as follows

\[ a(\gamma_1 \wedge \cdots \wedge \gamma_n \otimes v) = \]

\[ \sum_{k=1}^{n-1} \frac{1}{k!(n-k)!} \sum_{\varepsilon \in S_n} \pm \gamma_{\varepsilon(1)} \wedge \cdots \wedge \gamma_{\varepsilon(k)} \otimes \gamma_{\varepsilon(k+1)} \wedge \cdots \wedge \gamma_{\varepsilon(n)} \otimes v \]

\[ + \gamma_1 \wedge \cdots \wedge \gamma_n \otimes v, \quad (2.20) \]

where \( \gamma_1, \ldots \gamma_n \) are homogeneous elements of \( L \), \( v \in M \), \( S_n \) is the group of permutations of \( n \) elements and the signs in the equation depends naturally on the permutation \( \varepsilon \) and degrees of \( \gamma_k \). For example,

\[ a(v) = 0, \quad \forall v \in M, \]

\[ a(\gamma \otimes v) = \gamma \otimes v, \quad \forall v \in M, \gamma \in L, \]

and

\[ a(\gamma_1 \wedge \gamma_2 \otimes v) = \gamma_1 \wedge \gamma_2 \otimes v + \gamma_1 \otimes (\gamma_2 \otimes v) - (-)^{k_1 k_2} \gamma_2 \otimes (\gamma_1 \otimes v) \]

for any \( v \in M \) and for any pair \( \gamma_1 \in L^{k_1}, \gamma_2 \in L^{k_2} \).

Direct computation shows that the coaction \( (2.20) \) satisfies the required axiom

\[ (I \otimes a)a(X) = (\Delta \otimes I)a(X) \quad \forall X \in C(L[1]) \otimes M, \]

where \( \Delta \) is the comultiplication \( (2.3) \) in the coalgebra \( C(L[1]) \). It is also easily seen that

\[ \ker a = M \subset C(L[1]) \otimes M. \quad (2.21) \]
By definition \( \varphi \) is a coderivation of \( C(L[1]) \otimes M \). This means that for any \( X \in C(L[1]) \otimes M \)
\[
a \varphi X = I \otimes \varphi (aX) + Q \otimes I (aX),
\]
where \( Q \) is the \( L_\infty \)-algebra structure on \( L \) (that is the nilpotent coderivation of \( C(L[1]) \)).

Substituting \( X = \gamma_1 \wedge \cdots \wedge \gamma_n \) in (2.22), using (2.21), and performing the induction on \( n \) we get that equation (2.22) has the following general solution
\[
\varphi(\gamma_1 \wedge \cdots \wedge \gamma_n \otimes v) = \varphi_n(\gamma_1, \ldots, \gamma_n, v) +
\]
\[
\sum_{k=1}^{n-1} \frac{1}{k!(n-k)!} \sum_{\varepsilon \in S_n} \pm \gamma_{e(1)} \wedge \cdots \wedge \gamma_{e(k)} \wedge \varphi_{n-k}(\gamma_{e(k+1)}, \ldots, \gamma_{e(n)}, v) +
\]
\[
+ (-1)^{k_1 + \cdots + k_n} \gamma_1 \wedge \cdots \wedge \gamma_n \otimes \varphi_0(v) +
\]
\[
\sum_{k=1}^{n} \frac{1}{k!(n-k)!} \sum_{\varepsilon \in S_n} \pm Q_k(\gamma_{e(1)}, \ldots, \gamma_{e(k)}) \otimes \gamma_{e(k+1)} \wedge \cdots \wedge \gamma_{e(n)} \otimes v,
\]
where \( \gamma_i \in L^{k_i}, v \in M \), \( Q_k \)'s represent the \( L_\infty \)-algebra structure on \( L \) and \( \{ \varphi_n \} \) for \( n \geq 0 \) are arbitrary polylinear antisymmetric graded maps
\[
\varphi_n : \wedge^n L \otimes M \mapsto M[1-n].
\]

Equation (2.23) allows us to express \( \varphi \) inductively in terms of its structure maps (2.24) and vice-versa.

Similarly, one can show that the nilpotency condition
\[
\varphi^2 = 0
\]
is equivalent to the following semi-infinite collection of quadratic relations in \( \varphi_k \) and \( Q_l \) \( (n \geq 0) \)
\[
\varphi_0(\varphi_n(\gamma_1, \ldots, \gamma_n, v)) - (-1)^{1-n} \varphi_n(Q_1(\gamma_1), \ldots, \gamma_n, v) - (-1)^{k_1 + \cdots + k_{n-1} + 1-n} \varphi_n(\gamma_1, \ldots, Q_1(\gamma_n), v) =
\]
\[
- (-1)^{k_1 + \cdots + k_{n+1} - 1-n} \varphi_n(\gamma_1, \ldots, \gamma_n, \varphi_0(v)) =
\]
\[
\sum_{k=1}^{n-1} \frac{1}{2} \frac{1}{k!(n-k)!} \sum_{\varepsilon \in S_n} \pm \varphi_k(\gamma_{e(1)}, \ldots, \gamma_{e(k)}, \varphi_{n-k}(\gamma_{e(k+1)}, \ldots, \gamma_{e(n)}, v)) +
\]
\[
\sum_{k=1}^{n-1} \frac{1}{2} \frac{1}{k!(n-k)!} \sum_{\varepsilon \in S_n} \pm \varphi_{n-k}(Q_{k+1}(\gamma_{e(1)}, \ldots, \gamma_{e(k+1)}), \gamma_{e(k+2)}, \ldots, \gamma_{e(n)}, v),
\]
where \( \gamma_i \in L^{k_i}, v \in M \).

The signs in the above equations are defined similarly to those in (2.17) (see the remark after (2.17)). For \( n = 0 \) equation (2.25) says that \( \varphi_0 \) is a degree 1 differential on \( M \)
\[
(\varphi_0)^2 = 0
\]
and for \( n = 1 \) it says that \( \varphi_1 \) is closed with respect to the natural differential acting on the vector space \( \text{Hom}(\mathcal{L} \otimes \mathcal{M}, \mathcal{M}) \)

\[
\varphi_0 \varphi_1(\gamma, v) - \varphi_1(Q_1 \gamma, v) - (-)^{k} \varphi_1(\gamma, \varphi_0(v)) = 0
\]

\[\forall \gamma \in \mathcal{L}^k, \quad v \in \mathcal{M}.\]

For an \( L_\infty \)-module structure we reserve the following notation

\[
\mathcal{L} \\
\downarrow_{\varphi_{\text{mod}}} \\
(\mathcal{M}, \varphi_0)
\]

where \( \mathcal{L} \) stands for the \( L_\infty \)-algebra and \( \mathcal{M} \) stands for the respective graded vector space.

**Example.** The simplest example of an \( L_\infty \)-module is a DG module \((\mathcal{M}, b)\) over a DGLA \((\mathcal{L}, d, [\cdot, \cdot])\). In this case the only nonvanishing structure maps of \( \varphi \) are

\[
\varphi_0(v) = b(v), \quad v \in \mathcal{M},
\]

and

\[
\varphi_1(\gamma, v) = \rho(\gamma) v, \quad \gamma \in \mathcal{L}, \quad v \in \mathcal{M},
\]

where \( \rho \) is the action of \( \mathcal{L} \) on \( \mathcal{M} \). The axioms of DG module

\[
b^2 = 0,
\]

\[
b(\rho(\gamma)v) = \rho(d \gamma)v + (-)^{k}\rho(\gamma)b(v), \quad \gamma \in \mathcal{L}^k,
\]

\[
\rho(\gamma_1)\rho(\gamma_2)v - (-)^{k_1k_2}\rho(\gamma_2)\rho(\gamma_1)v = \rho([\gamma_1, \gamma_2])v,
\]

\[\gamma_1 \in \mathcal{L}^{k_1}, \quad \gamma_2 \in \mathcal{L}^{k_2}\]

are exactly the axioms of \( L_\infty \)-module.

**Definition 5** Let \( \mathcal{L} \) be an \( L_\infty \)-algebra and \((\mathcal{M}, \varphi^\mathcal{M})\), \((\mathcal{N}, \varphi^\mathcal{N})\) be \( L_\infty \)-modules over \( \mathcal{L} \). Then a morphism \( \kappa \) from the comodule \( C(\mathcal{L}[1]) \otimes \mathcal{M} \) to the comodule \( C(\mathcal{L}[1]) \otimes \mathcal{N} \) compatible with the coderivations \( \varphi^\mathcal{M} \) and \( \varphi^\mathcal{N} \)

\[
\kappa(\varphi^\mathcal{M} X) = \varphi^\mathcal{N}(\kappa X), \quad \forall X \in C(\mathcal{L}[1]) \otimes \mathcal{M}
\]

(2.26)

is called an morphism between \( L_\infty \)-modules \((\mathcal{M}, \varphi^\mathcal{M})\) and \((\mathcal{N}, \varphi^\mathcal{N})\).

Unfolding this definition one can easily show that the morphism \( \kappa \) is uniquely determined by its structure maps

\[
\kappa_n : \wedge^n \mathcal{L} \otimes \mathcal{M} \mapsto \mathcal{N}[-n], \quad n \geq 0
\]

(2.27)

via the following equations

\[
\kappa(\gamma_1 \wedge \cdots \wedge \gamma_n \otimes v) = \kappa_n(\gamma_1, \ldots, \gamma_n, v) +
\]
\[
\sum_{k=1}^{n-1} \frac{1}{k!(n-k)!} \sum_{\varepsilon \in S_n} \pm \gamma_{\varepsilon(1)} \wedge \cdots \wedge \gamma_{\varepsilon(k)} \otimes \kappa_{n-k}(\gamma_{\varepsilon(k+1)}, \ldots, \gamma_{\varepsilon(n)}, v)
\]
\[
+ \gamma_1 \wedge \cdots \wedge \gamma_n \otimes \kappa_0(v).
\] (2.28)

Relation (2.26) is equivalent to the following semi-infinite collection of equations \((n \geq 0)^2\)

\[
\varphi_0^N \kappa_n(\gamma_1, \ldots, \gamma_n, v) - (-)^n \kappa_n(Q_1 \gamma_1, \gamma_2, \ldots, \gamma_n) - \cdots - (-)^{k_1+\cdots+k_n+n} \kappa_n(\gamma_1, \ldots, \gamma_n, \varphi_0^M v) =
\]
\[
\sum_{p=0}^{n-1} \frac{1}{p!(n-p)!} \sum_{\varepsilon \in S_n} \pm \kappa_p(\gamma_{\varepsilon(1)}, \ldots, \gamma_{\varepsilon(p)}, \varphi_0^M(\gamma_{\varepsilon(p+1)}, \ldots, \gamma_{\varepsilon(n)}), v)
\]
\[
- \sum_{p=1}^{n} \frac{1}{p!(n-p)!} \sum_{\varepsilon \in S_n} \pm \varphi_0^N(\gamma_{\varepsilon(1)}, \ldots, \gamma_{\varepsilon(p)}, \kappa_{n-p}(\gamma_{\varepsilon(p+1)}, \ldots, \gamma_{\varepsilon(n)}), v)
\] (2.29)
\[
+ \sum_{p=2}^{n} \frac{1}{p!(n-p)!} \sum_{\varepsilon \in S_n} \pm \kappa_{n-p+1}(Q_p(\gamma_{\varepsilon(1)}, \ldots, \gamma_{\varepsilon(p)}), \gamma_{\varepsilon(p+1)}, \ldots, \gamma_{\varepsilon(n)}), v)
\]
\[
\gamma_i \in L^{k_i}, \quad v \in \mathcal{M}.
\]

It is not hard to check that the ordinary morphism of modules over an ordinary DGLA provides us with the simplest example of the morphism between \(L_\infty\)-modules.

For \(n = 0\) equation (2.29) reduces to

\[
\kappa_0(\varphi_0^M v) = \varphi_0^N \kappa_0(v), \quad v \in \mathcal{M}
\]

and hence the zero-th structure map of \(\kappa\) is always a morphism of complexes \((\mathcal{M}, \varphi_0^M)\) and \((\mathcal{N}, \varphi_0^N)\). This motivates the following

**Definition 6** A quasi-isomorphism \(\kappa\) of \(L_\infty\)-modules \((\mathcal{M}, \varphi_0^M)\) and \((\mathcal{N}, \varphi_0^N)\) is a morphism between these \(L_\infty\)-modules with the zero-th structure map \(\kappa_0\) being a quasi-isomorphism of complexes \((\mathcal{M}, \varphi_0^M)\) and \((\mathcal{N}, \varphi_0^N)\).

In what follows the notation \(\mathcal{M} \triangleright \kappa \triangleright \rightarrow \mathcal{N}\)

means that \(\kappa\) is a morphism from the \(L_\infty\)-module \(\mathcal{M}\) to the \(L_\infty\)-module \(\mathcal{N}\).

To this end we mention that there is another definition of an \(L_\infty\)-module over an \(L_\infty\)-algebra which is known \([30]\) to be equivalent to the definition we gave above.

**Definition 7** Let \(\mathcal{L}\) be an \(L_\infty\)-algebra. Then a complex \((\mathcal{M}, b)\) is called an \(L_\infty\)-module over \(\mathcal{L}\) if there is an \(L_\infty\)-morphism \(\chi\) from \(\mathcal{L}\) to \(\text{Hom}(\mathcal{M}, \mathcal{M})\), where \(\text{Hom}(\mathcal{M}, \mathcal{M})\) is naturally viewed as a DGLA with the differential induced by \(b\).

\(^2^{\text{see the remark after equation (2.17) about the signs in such formulas}}\)
The structure maps $\varphi_n$ of the respective nilpotent coderivation of the comodule $C(L[1]) \otimes M$ are related to $b$ and the structure maps of the $L_\infty$-morphism $\chi$ in the following simple way

$$b = \varphi_0, \quad \chi_n(\gamma_1, \ldots, \gamma_n)(v) = \varphi_n(\gamma_1, \ldots, \gamma_n, v) \quad (n \geq 1), \quad \gamma_i \in L, \ v \in M. \quad (2.30)$$

**REMARK.** Since all $L_\infty$-algebras we will consider will be DG Lie algebras in the rest of this paper we assume, for simplicity, that all our $L_\infty$-algebras are DG Lie algebras. “Weird” things we still borrow from the “$L_\infty$-world” are $L_\infty$-morphisms, $L_\infty$-modules, and morphisms between such modules.

### 2.3 Maurer-Cartan elements and twisting procedures

It would not be an exaggeration if we say that the deformation theory is full of Maurer-Cartan equations. The following definition is motivated by this common observation.

**Definition 8** Let $(L, d, [,])$ be a DGLA and $k$ be a local Artinian algebra (or pro-Artinian algebra) over $\mathbb{R}$ with a maximal ideal $m$. Then $\pi \in L^1 \otimes m$ is called a Maurer-Cartan element if

$$d\pi + \frac{1}{2}[\pi, \pi] = 0. \quad (2.31)$$

Let $G$ be a unipotent group corresponding to the nilpotent Lie algebra $L^0 \otimes m$. The group $G$ acts naturally on the Maurer-Cartan elements

$$\pi \mapsto P^{-1}dP + P^{-1}\pi P, \quad \forall P \in G \quad (2.32)$$

and the quotient space of the space of Maurer-Cartan elements with respect to $G$-action is called the *moduli space* of the DGLA $L$.

It turns out that a quasi-isomorphism (see definition 3) between DG Lie algebras gives a bijective correspondence between their moduli spaces. But for our purposes we need a weaker version of this statement. Namely, using (2.17) it is not hard to show that

**Proposition 1** If $F$ is an $L_\infty$-morphism from a DGLA $(L, d, [,])$ to a DGLA $(L^\circ, d^\circ, [,]^\circ)$ and $\pi \in L^1 \otimes m$ is a Maurer-Cartan element of $L$ then

$$S = \sum_{n \geq 1} \frac{1}{n!} F_n(\pi, \ldots, \pi) \quad (2.33)$$

is a Maurer-Cartan element in $L^\circ$. □

**Remark.** The infinite sum in (2.33) is well-defined because $k$ is local Artinian (pro-Artinian) algebra. All elements of this sum are of degree 1 since for any $n F_n$ shifts the degree by $1 - n$ (see 2.14).

Using a Maurer-Cartan element $\pi \in L \otimes t$ one can naturally modify the structure of the DGLA on $L \otimes t$ by adding the inner derivation $[\pi, \cdot]$ to the initial differential $d$. Thanks to Maurer-Cartan equation (2.31) this new differential $d + [\pi, \cdot]$ is nilpotent and by definition it satisfies the Leibniz rule. This modification can be described in terms of the respective
\[ L_\infty \text{-stricture}. \] Namely, the nilpotent coderivation \( Q^\pi \) on the coassociative cocommutative coalgebra \( C(\mathcal{L} \otimes \mathfrak{t}[1]) \) corresponding to the new DGLA structure \((\mathcal{L} \otimes \mathfrak{t}, \partial + [\pi, \cdot], [], [])\) is related to the initial coderivation \( Q \) by the equation
\[
Q^\pi(X) = \exp((-\pi) \wedge) Q(\exp(\pi \wedge) X), \quad X \in C(\mathcal{L} \otimes \mathfrak{t}[1]), \tag{2.34}
\]
where the sum
\[
\exp(\pi \wedge) = + \pi \wedge + \frac{1}{2!} \pi \wedge \pi \wedge + \ldots
\]
is well-defined since \( \pi \in \mathcal{L} \otimes \mathfrak{m} \).

We call this procedure of changing the initial DGLA structure on \( \mathcal{L} \otimes \mathfrak{t} \) twisting\(^3\) of the DGLA \( \mathcal{L} \) by the Maurer-Cartan element \( \pi \).

Similar twisting procedures by a Maurer-Cartan element can be defined for an \( L_\infty \)-morphism, for an \( L_\infty \)-module, and for a morphism of \( L_\infty \)-modules. In the following propositions we describe these procedures.

**Proposition 2** If \( F \) is an \( L_\infty \)-morphism

\[
F : \mathcal{L} \rightarrow \mathcal{L}^\circ
\]
of DG Lie algebras, \( \pi \in \mathcal{L}^1 \otimes \mathfrak{m} \) is a Maurer-Cartan element of \( \mathcal{L} \) and \( S \in (\mathcal{L}^\circ)^1 \otimes \mathfrak{m} \) is the corresponding Maurer-Cartan element \( \text{(2.33)} \) of \( \mathcal{L}^\circ \) then

1. For any \( X \in C(\mathcal{L} \otimes \mathfrak{t}[1]) \)

\[
\Delta(\exp(\pi \wedge) X) = \exp(\pi \wedge) \bigotimes \exp(\pi \wedge) (\Delta X) + \exp(\pi \wedge) X \bigotimes \exp(\pi \wedge) + \ldots
\]

where
\[
\exp(\pi \wedge) = \sum_{k=1}^{\infty} \frac{1}{k!} \pi \wedge \ldots \wedge \pi.
\]

3. \( F(\exp(\pi)) = \exp(S) \). \tag{2.37}

5. If \( \mathfrak{t} = \mathbb{R}[[h]] \) and \( F \) is a quasi-isomorphism then so is \( F^\pi \).

---

\(^3\)This terminology is borrowed from \( \text{[24]} \) (see App. B 5.3). However, the twisting by a Maurer-Cartan element we use here is different from the one in \( \text{[24]} \).
In what follows we refer to $F^\pi$ in (2.38) as an $L_\infty$-morphism (or a quasi-isomorphism) twisted by the Maurer-Cartan element $\pi$. It is not hard to see that the structure maps of the twisted $L_\infty$-morphism $F^\pi$ are given by

$$F^\pi_n(\gamma_1, \ldots, \gamma_n) = \sum_{k=0}^{\infty} \frac{1}{k!} F_{n+k}(\pi, \ldots, \pi, \gamma_1, \ldots, \gamma_n), \quad \gamma_i \in L.$$  

(2.39)

**Proof.** Statements 1-3 are proved by straightforward computations and statements 4 and 5 can be proved along the lines of [19] (see section 4.5). Here we would like to illustrate a proof of 4 using a technique, which is different from the one used in [19].

**Proof of 4.** While the compatibility of $F^\pi$ with the coderivations $Q^\pi$ and $Q^{\pi S}$ follows directly from the definitions the compatibility of $F^\pi$ with the coproducts in $C(L \otimes \mathfrak{k}[1])$ and $C(L^\circ \otimes \mathfrak{k}[1])$ requires some work. Using 1 and 3 we get that for any $X \in C(L \otimes \mathfrak{k}[1])$

$$\Delta \exp(-S \wedge) F \exp(\pi \wedge) X = \exp(-S \wedge) \bigotimes \exp(-S \wedge)(F \bigotimes F) \Delta \exp(\pi \wedge) X + \exp(-S \wedge) \bigotimes \exp(-S \wedge) F \exp(\pi \wedge) X \bigotimes \exp(-S) = \exp(-S \wedge) \bigotimes F^\pi X + F^\pi X \bigotimes \exp(-S) + (F^\pi \bigotimes F^\pi)(\Delta X) + \exp(-S \wedge) \bigotimes \exp(-S \wedge)(F \bigotimes F)(\exp(\pi \wedge) X \bigotimes \exp(\pi \wedge)) + \exp(-S \wedge) \bigotimes \exp(-S \wedge)(F \bigotimes F)(\exp(\pi \wedge) X \bigotimes \exp(\pi \wedge)).$$

The first and the second terms in the latter expression cancel with the third and the forth terms, respectively, due to 3 and the following obvious identity between Taylor series

$$e^{-S} \exp(S) = -\exp(-S).$$  

(2.40)

Thus, we get the desired relation

$$\Delta F^\pi(X) = (F^\pi \bigotimes F^\pi)(\Delta X).$$  

□

**Proposition 3** If $(\mathcal{L}, \partial, [\cdot, \cdot])$ is a DGLA, $(\mathcal{M}, \varphi)$ is an $L_\infty$-module over $\mathcal{L}$ and $\pi \in \mathcal{L}^1 \otimes \mathfrak{m}$ is a Maurer-Cartan element then

1. For any $X \in C(\mathcal{L} \otimes \mathfrak{k}[1]) \otimes \mathcal{M} \otimes \mathfrak{k}$

$$a(\exp(\pi \wedge) X) = \exp(\pi \wedge) \bigotimes \exp(\pi \wedge)(aX) + \exp(\pi \wedge) X \bigotimes \exp(\pi \wedge),$$

where $a$ is the coaction and $\exp(\pi \wedge)$ is defined in the previous proposition.

2. The following map

$$\varphi^\pi = \exp(-\pi \wedge) \varphi \exp(\pi \wedge) : C(\mathcal{L} \otimes \mathfrak{k}[1]) \otimes \mathcal{M} \otimes \mathfrak{k} \mapsto C(\mathcal{L} \otimes \mathfrak{k}[1]) \otimes \mathcal{M} \otimes \mathfrak{k}$$

(2.42)

is a nilpotent coderivation of the comodule $C(\mathcal{L} \otimes \mathfrak{k}[1]) \otimes \mathcal{M} \otimes \mathfrak{k}$.  

\textit{if $X = v \in \mathcal{M} \otimes \mathfrak{k}$ we set “$\pi \wedge X = \pi \otimes X$”}

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3. If $\tilde{\varphi} : \mathcal{L} \rightarrow (\text{Hom}(\mathcal{M}, \mathcal{M}), \varphi_0)$ is the $L_\infty$-morphism induced by the module structure $\varphi$ then the twisted $L_\infty$-morphism $\tilde{\varphi}^\pi$ defines the $L_\infty$-module structure given in (2.42).

4. If $\kappa : \mathcal{M} \rightarrow \mathcal{N}$ is an $L_\infty$-morphism of $L_\infty$-modules $(\mathcal{M}, \varphi)$ and $(\mathcal{N}, \psi)$ over $\mathcal{L}$ then the map

$$\kappa^\pi = \exp(-\pi \wedge) \kappa \exp(\pi \wedge) : C(\mathcal{L} \otimes \mathfrak{k}[1]) \otimes \mathcal{M} \otimes \mathfrak{k} \mapsto C(\mathcal{L} \otimes \mathfrak{k}[1]) \otimes \mathcal{N} \otimes \mathfrak{k} \quad (2.43)$$

is an $L_\infty$-morphism between $L_\infty$-modules $(\mathcal{M} \otimes \mathfrak{k}, \varphi^\pi)$ and $(\mathcal{N} \otimes \mathfrak{k}, \psi^\pi)$ over $(\mathcal{L} \otimes \mathfrak{k}, \mathfrak{d} + [\pi, \cdot], [, ,])$ (2.43)

5. If $\mathfrak{k} = \mathbb{R}[[\hbar]]$ and $\kappa$ is a quasi-isomorphism of modules $\mathcal{M}$ and $\mathcal{N}$ then so is $\kappa^\pi$.

In what follows we refer to $\varphi^\pi$ in (2.42) and $\kappa^\pi$ in (2.43), respectively, as an $L_\infty$-module structure and a morphism of $L_\infty$-modules twisted by the Maurer-Cartan element $\pi$. It is not hard to see that the structure maps of the twisted coderivation $\varphi^\pi$ and the twisted morphism $\kappa^\pi$ are given by

$$\varphi^\pi_n(\gamma_1, \ldots, \gamma_n, v) = \sum_{m=0}^\infty \frac{1}{m!} \varphi_{n+m}(\pi, \ldots, \pi, \gamma_1, \ldots, \gamma_n, v), \quad \gamma_i \in \mathcal{L}, \ v \in \mathcal{M}. \quad (2.44)$$

$$\kappa^\pi_n(\gamma_1, \ldots, \gamma_n, v) = \sum_{m=0}^\infty \frac{1}{m!} \kappa_{n+m}(\pi, \ldots, \pi, \gamma_1, \ldots, \gamma_n, v), \quad \gamma_i \in \mathcal{L}, \ v \in \mathcal{M}. \quad (2.45)$$

Proof. Statement 1 is proved by a straightforward computation. Statement 2 follows from statement 1 of this proposition and statement 2 of the previous proposition. The proof of statement 4 is similar to the proof of statement 4 in the previous proposition. Statement 3 is proved by comparing the corresponding structure maps and statement 5 is borrowed from [24] (see the first lemma in section 3.2).

From the definitions of the above twisting procedures, it is not hard to see that these procedures are functorial. Namely,

**Proposition 4** If $F : \mathcal{L} \rightarrow \mathcal{L}^\circ$ and $F^\circ : \mathcal{L}^\circ \rightarrow \mathcal{L}^\bullet$ are $L_\infty$-morphisms of DG Lie algebras, $\pi$ is a Maurer-Cartan element of $\mathcal{L}$ and $S$ is the corresponding Maurer-Cartan element (2.33) of $\mathcal{L}^\circ$ then

$$(F^\circ \circ F)^\pi = F^\circ S \circ F^\pi,$$

where $\circ$ stands for the composition of $L_\infty$-morphisms. Furthermore, the twisting procedure assigns to any Maurer-Cartan element of a DGLA $\mathcal{L}$ a functor from the category of $L_\infty$-modules to itself.

**Remark.** In all our examples the local Artinian algebra $\mathfrak{k}$ (or pro-Artinian algebra) over $\mathbb{R}$ will be the algebra $\mathbb{R}[[h]]$ of formal power series in one variable. However, we will also use
a Maurer-Cartan element $\pi$ which will belong to the initial DGLA $L$ over $\mathbb{R}$. This element will also be a one-form on some manifold and therefore the expression

$$\pi \wedge \cdots \wedge \pi$$

will vanish for big enough $N$. For this reason all the above constructions will be well-defined as well as the propositions we proved will hold.

### 3 Algebraic structures on Hochschild (co)chains

For a unital algebra $\mathfrak{A}$ (over $\mathbb{R}$) we denote by $C^\bullet(\mathfrak{A})$ the vector space of Hochschild cochains with a shifted grading

$$C^n(\mathfrak{A}) = \text{Hom}(\mathfrak{A}^{(n+1)}, \mathfrak{A}), \quad (n \geq 0) \quad C^{-1}(\mathfrak{A}) = \mathfrak{A}. \quad (3.1)$$

The space $C^\bullet(\mathfrak{A})$ can be endowed with the so-called Gerstenhaber bracket [15] which is defined between homogeneous elements $\Phi_1 \in C^{k_1}(\mathfrak{A})$ and $\Phi_2 \in C^{k_2}(\mathfrak{A})$ as follows

$$[\Phi_1, \Phi_2]_G(a_0, \ldots, a_{k_1+k_2}) =$$

$$\sum_{i=0}^{k_1} (-)^{ik_2} \Phi_1(a_0, \ldots, \Phi_2(a_i, \ldots, a_{i+k_2}), \ldots, a_{k_1+k_2})$$

$$-(-)^{k_1k_2}(1 \leftrightarrow 2), \quad a_j \in \mathfrak{A}. \quad (3.2)$$

Direct computation shows that (3.2) is a Lie (super)bracket and therefore $C^\bullet(\mathfrak{A})$ is Lie (super)algebra.

For the same unital algebra $\mathfrak{A}$ (over $\mathbb{R}$) we denote by $C_\bullet(\mathfrak{A})$ the vector space of Hochschild chains with a converted grading

$$C_{-n}(\mathfrak{A}) = \mathfrak{A} \otimes \mathfrak{A}^{\otimes n}, \quad (n \geq 1), \quad C_0(\mathfrak{A}) = \mathfrak{A}. \quad (3.3)$$

The space $C_\bullet(\mathfrak{A})$ can be endowed with the structure of a graded module over the Lie algebra $C^\bullet(\mathfrak{A})$ of Hochschild cochains. For homogeneous elements the action of $C^\bullet(\mathfrak{A})$ on $C_\bullet(\mathfrak{A})$ is defined as follows. If $\Phi \in C^k(\mathfrak{A})$ then

$$R_\Phi(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-k} (-)^{ki} a_0 \otimes \cdots \otimes \Phi(a_i, \ldots, a_{i+k}) \otimes \cdots \otimes a_n +$$

$$\sum_{j=n-k}^{n-1} (-)^{n(j+1)} \Phi(a_{j+1}, \ldots, a_n, a_0, \ldots, a_{k+j-n}) \otimes a_{k+j-n} \otimes \cdots \otimes a_j, \quad a_i \in \mathfrak{A}. \quad (3.4)$$

The required axiom of the Lie algebra module

$$R_{[\Phi_1, \Phi_2]} = R_{\Phi_1} R_{\Phi_2} - R_{\Phi_2} R_{\Phi_1} \quad (3.5)$$

can be checked by a straightforward computation.
The multiplication \( \mu_0 \) in the algebra \( \mathfrak{A} \) can be naturally viewed as an element of \( C^1(\mathfrak{A}) \) and the associativity condition for \( \mu_0 \) can be rewritten in terms of bracket (3.2) as

\[
[\mu_0, \mu_0]_G = 0.
\]

Thus, on the one hand \( \mu_0 \) defines a nilpotent interior derivation of the graded Lie algebra \( C^\bullet(\mathfrak{A}) \)

\[
\partial \Phi = [\mu_0, \Phi]_G : C^k(\mathfrak{A}) \to C^{k+1}(\mathfrak{A}), \quad \partial^2 = 0,
\]

and on the other hand \( \mu_0 \) endows the graded vector space \( C^\bullet(\mathfrak{A}) \) with the (nilpotent) differential

\[
b = R_{\mu_0} : C_k(\mathfrak{A}) \to C_{k+1}(\mathfrak{A}), \quad b^2 = 0.
\]

Equation (3.5) implies that \( R_{\partial \Phi} = b R_{\Phi} - (-)^k R_{\Phi} b \), \( \Phi \in C^k(\mathfrak{A}) \)

and therefore the vector spaces \( C^\bullet(\mathfrak{A}) \) and \( C^\bullet(\mathfrak{A}) \) become a pair of a DGLA and a DG module over this DGLA.

One can easily see that the bizarre gradings of \( C^\bullet(\mathfrak{A}) \) and \( C^\bullet(\mathfrak{A}) \) are chosen intentionally. It is these gradings for which both the Gerstenhaber bracket (3.2) and the action \( R \) (3.4) have the degree 0 and the differentials (3.7) (3.8) have degree +1.

Notice that the differentials (3.7) and (3.8) are exactly the Hochschild coboundary and boundary operators on \( C^\bullet(\mathfrak{A}) \) and \( C^\bullet(\mathfrak{A}) \), respectively.

We will be mainly interested in the algebra \( A_0 = C^\infty(M) \) where \( M \) is a smooth manifold of dimension \( d \). A natural analogue of the complex of Hochschild cochains for this algebra is the complex \( D_{\text{poly}}(M) \) of polydifferential operators with the same differential as in \( C^\bullet(A_0) \)

\[
D_{\text{poly}}(M) = \bigoplus_{k=-1}^{\infty} D^k_{\text{poly}}(M),
\]

where \( D^k_{\text{poly}}(M) \) consists of polydifferential operators of rank \( k + 1 \)

\[
\Phi : C^\infty(M)^\otimes(k+1) \to C^\infty(M).
\]

Similarly, instead of the complex \( C^\bullet(A_0) \) we consider three versions of the vector space \( C^\text{poly}(M) \) of Hochschild chains for \( A_0 \)

1. \( C^\text{poly}_\text{function}(M) = \bigoplus_{n \geq 0} C^\infty(M^{n+1}) \),

2. \( C^\text{poly}_{\text{germ}}(M) = \bigoplus_{n \geq 0} \text{germs}_{\Delta(M^{n+1})} C^\infty(M^{n+1}) \),

3. \( C^\text{poly}_{\text{jet}}(M) = \bigoplus_{n \geq 0} \text{jets}_{\Delta(M^{n+1})} C^\infty(M^{n+1}) \),
where $\Delta(M^{n+1})$ is the diagonal in $M^{n+1}$.

It is not hard to see that the Gerstenhaber bracket (3.2), the action (3.4), and the differentials (3.7), (3.8) still make sense if we replace $C^\bullet(A_0)$ by $D_{\text{poly}}(M)$ and $C^\bullet(A_0)$ by either of versions (3.10), (3.11), (3.12) of $C_{\text{poly}}(M)$. Thus, $D_{\text{poly}}(M)$ and $C_{\text{poly}}(M)$ are DGLA and a DG module over this DGLA, respectively. We use the same notations for all the operations $[,]_G$, $R\Phi$, $\partial$, and $b$ when we speak of $D_{\text{poly}}(M)$ and $C_{\text{poly}}(M)$.

The cohomology of $D_{\text{poly}}(M)$ and $C_{\text{poly}}(M)$ are described by the Hochschild-Kostant-Rosenberg type theorems. The original version of the theorem [18] by Hochschild, Kostant, and Rosenberg says that the module of Hochschild homology of a smooth affine algebra is isomorphic to the module of exterior differential forms on the respective affine algebraic variety. A dual version of this theorem was originally proved by Vey.

**Proposition 5 (Vey, [31])** Let

$$T_{\text{poly}}(M) = \bigoplus_{k=-1}^{\infty} T_{\text{poly}}^k(M), \quad T_{\text{poly}}^k(M) = \Gamma(\wedge^{k+1}TM) \quad (3.13)$$

be a vector space of the polyvector fields on $M$ with shifted grading. If we regard $T_{\text{poly}}(M)$ as a complex with a vanishing differential $d = 0$ then the natural map

$$\mathcal{V}(\gamma)(a_0, \ldots, a_k) = \iota_\gamma(da_0 \wedge \cdots \wedge da_k) : T_{\text{poly}}^k(M) \mapsto D_{\text{poly}}^k(M), \quad k \geq -1 \quad (3.14)$$

defines a quasi-isomorphism of complexes $(T_{\text{poly}}(M), 0)$ and $(D_{\text{poly}}(M), \partial)$. Here $d$ stands for the De Rham differential and $\iota_\gamma$ denotes the contraction of the polyvector field with an exterior form.

One can easily check that the Lie algebra structure induced on cohomology $H^\bullet(D_{\text{poly}}(M)) = T_{\text{poly}}(M)$ coincides with the one given by the so-called Schouten-Nijenhuis bracket $[\cdot, \cdot]_{SN} : T_{\text{poly}}(M) \wedge T_{\text{poly}}(M) \mapsto T_{\text{poly}}(M)$.

This bracket is defined as an ordinary Lie bracket between vector fields and then extended by Leibniz rule with respect to the $\wedge$-product to an arbitrary pair of polyvector fields.

The most general $C^\infty$-manifold version of the Hochschild-Kostant-Rosenberg theorem is due to N. Teleman [29].

**Proposition 6 (Teleman, [29])** Let

$$\mathcal{A}^\bullet(M) = \bigoplus_{k \leq 0} \mathcal{A}^k(M), \quad \mathcal{A}^k(M) = \Gamma(\wedge^{-k}T^*M) \quad (3.15)$$

be a vector space of the exterior forms on $M$ with a converted grading. If we regard $\mathcal{A}^\bullet(M)$ as a complex with a vanishing differential $b = 0$ then the natural map

$$\mathcal{C}(a_0 \otimes \cdots \otimes a_k) = a_0 da_1 \wedge \cdots \wedge da_k : C_{\text{poly}}^{-k}(M) \mapsto A^{-k}(M), \quad k \geq 0 \quad (3.16)$$

defines a quasi-isomorphism of complexes $(C_{\text{poly}}^{-k}(M), b)$ and $(\mathcal{A}^\bullet(M), 0)$ for either of versions (3.10), (3.11), (3.12) of $C_{\text{poly}}(M)$.

---

5See also [8], in which this result was proven for any compact smooth manifold.
Remark. In what follows we will restrict ourselves to the third version (3.12) of $C^{poly}(M)$ and since all $D_{poly}(M)$-modules (3.10), (3.11), (3.12) are naturally quasi-isomorphic our further results will hold for versions (3.10), (3.11) as well.

DG $D_{poly}(M)$-module structure on $C^{poly}(M)$ induces a DG $T_{poly}(M)$-module structure on the vector space $\mathcal{A}^\bullet(M)$ which coincides with the one defined by the action of a polyvector field on exterior forms via the Lie derivative

$$L_\gamma = d\ i_\gamma + (-)^k i_\gamma d, \quad \gamma \in T^k_{poly}(M),$$

where as above $d$ stands for the De Rham differential and $i_\gamma$ denotes the contraction of the polyvector field $\gamma$ with an exterior form.

Unfortunately, the maps (3.14) and (3.16) are not compatible with the Lie brackets on $T_{poly}(M)$ and $D_{poly}(M)$ and with the respective actions (3.4) and (3.17). In particular, the equation

$$\mathcal{E} \circ R_{V(\gamma)} \overset{?}{=} L_\gamma \mathcal{E}$$

does not hold in general. This defect can be cured by

**Theorem 1** The DG modules $(T_{poly}(M), \mathcal{A}^\bullet(M))$ and $(D_{poly}(M), C^{poly}(M))$ are quasi-isomorphic. More precisely, for any smooth manifold $M$ one can construct the following commutative diagram

$$
\begin{array}{cccccc}
T_{poly}(M) & \xrightarrow{U} & (\mathcal{L}, [\cdot, \cdot]) & \xleftarrow{\tau_1} & D_{poly}(M) \\
\downarrow{\overset{L}{\text{mod}}} & & \downarrow{\overset{\varphi}{\text{mod}}} & & \downarrow{\overset{R}{\text{mod}}} \\
\mathcal{A}^\bullet(M) & \xrightarrow{\tau_2} & (\mathfrak{N}, \bar{b}) & \xleftarrow{\bar{R}} & (\mathfrak{N}, b) & \xleftarrow{\tau_3} & C^{poly}(M)
\end{array}
$$

where the horizontal arrows in the upper row are quasi-isomorphisms of the DG Lie algebras $T_{poly}(M)$, $\mathcal{L}$, and $D_{poly}(M)$. The inclined arrow $L$, and the vertical arrows $\bar{L}$, $\varphi$, and $R$ denote DGLA module structures on the terms of the lower row. $\tau_2$ and $\tau_3$ are embeddings and also quasi-isomorphisms of the respective DGLA modules. And $\bar{R}$ is a quasi-isomorphism of $L_\infty$-modules $(\mathfrak{N}, \bar{b})$ and $(\mathfrak{N}, b)$ over the DGLA $T_{poly}(M)$, where the $L_\infty$-modules structure on $(\mathfrak{N}, b)$ over $T_{poly}(M)$ is obtained by composing the quasi-isomorphism $U$ with the DGLA module structure $\varphi$.

Although the statement of the theorem looks a little bit complicated the objects that enter the diagram are simply vector spaces of smooth sections of some vector bundles on $M$. The construction of the quasi-isomorphism $\bar{R}$ is explicit and in section 5 we show how it allows us to prove Tsygan’s conjecture (see the first part of corollary 4.0.3 in [30]) about Hochschild homology of the quantum algebra of functions on an arbitrary Poisson manifold, and in particular, to describe the space of traces on this algebra.

The proof of the theorem occupies the rest of this paper. The bigger part of the proof is devoted to the construction of Fedosov resolutions of the DGLA modules $(T_{poly}(M), \mathcal{A}^\bullet(M))$ and $(D_{poly}(M), C^{poly}(M))$. After completing this stage it will only remain to use Kontsevich’s [19] and Shoikhet’s [24] formality theorems for $\mathbb{R}^d_{\text{formal}}$ and apply the twisting procedures we developed in the previous section.

Let us now recall the theorems we just mentioned.
Theorem 2 (Kontsevich, [19]) \[ \text{There exists a quasi-isomorphism } U : T_{\text{poly}}(\mathbb{R}^d) \to D_{\text{poly}}(\mathbb{R}^d) \] (3.20)
from the DGLA \( T_{\text{poly}}(\mathbb{R}^d) \) of polyvector fields to the DGLA \( D_{\text{poly}}(\mathbb{R}^d) \) of polydifferential operators on the space \( \mathbb{R}^d \) such that

1. One can replace \( \mathbb{R}^d \) in (3.20) by its formal completion \( \mathbb{R}^d_{\text{formal}} \) at the origin.
2. The quasi-isomorphism \( U \) is equivariant with respect to linear transformations of the coordinates on \( \mathbb{R}^d_{\text{formal}} \).
3. If \( n > 1 \) then
\[
U_n(v_1, v_2, \ldots, v_n) = 0
\] (3.21)
for any set of vector fields \( v_1, v_2, \ldots, v_n \in T_0^{\text{poly}}(\mathbb{R}^d_{\text{formal}}) \).
4. If \( n \geq 2 \) and \( \chi \in T_0^{\text{poly}}(\mathbb{R}^d_{\text{formal}}) \) is linear in the coordinates on \( \mathbb{R}^d_{\text{formal}} \) then for any set of polyvector fields \( \gamma_2, \ldots, \gamma_n \in T_{\text{poly}}(\mathbb{R}^d_{\text{formal}}) \)
\[
U_n(\chi, \gamma_2, \ldots, \gamma_n) = 0.
\] (3.22)

Composing the quasi-isomorphism \( U \) with the action (3.4) of \( D_{\text{poly}}(M) \) on \( C_{\text{poly}}(M) \) we get an \( L_{\infty} \)-module structure \( \psi \) on \( C_{\text{poly}}(M) \) over the DGLA \( T_{\text{poly}}(M) \). For this module structure \( \psi \) we have the following

Theorem 3 (Shoikhet, [24]) \[ \text{There exists a quasi-isomorphism } K : \left( (C_{\text{poly}}(\mathbb{R}^d), \psi) \right) \to \left( (A^*(\mathbb{R}^d), L) \right) \] (3.23)
of \( L_{\infty} \)-modules over \( T_{\text{poly}}(\mathbb{R}^d) \), the zeroth structure map \( K_0 \) of which is the map (3.16) of Connes and such that

1. One can replace \( \mathbb{R}^d \) in (3.23) by its formal completion \( \mathbb{R}^d_{\text{formal}} \) at the origin.
2. The quasi-isomorphism \( K \) is equivariant with respect to linear transformations of the coordinates on \( \mathbb{R}^d_{\text{formal}} \).
3. If \( n \geq 1 \) and \( \chi \in T_0^{\text{poly}}(\mathbb{R}^d_{\text{formal}}) \) is linear in the coordinates on \( \mathbb{R}^d_{\text{formal}} \) then for any set of polyvector fields \( \gamma_2, \ldots, \gamma_n \in T_{\text{poly}}(\mathbb{R}^d_{\text{formal}}) \) and any Hochschild chain \( a \in C_{\text{poly}}(\mathbb{R}^d_{\text{formal}}) \)
\[
K_n(\chi, \gamma_2, \ldots, \gamma_n; a) = 0.
\] (3.24)
Proof. The first two properties are obvious from the construction [24] of the structure maps $K_n$ and we are left with the third property.

As for Kontsevich’s quasi-isomorphism the proof of the third property for Shoikhet’s quasi-isomorphism reduces to calculation of integrals entering the construction of the structure maps $K_n$ (see section 2.2 of [24]). To do this calculation we first transform the unit disk $\{|w| \leq 1\}$ into the upper half plane $H^+ = \{z, \text{Im}(z) \geq 0\}$ via the standard fractional linear transformation

$$z = -i\frac{w+1}{w-1}.$$  \hspace{3cm} (3.25)

The origin of the unit disk goes to $z = i$ and the point $w = 1$ goes to $z = \infty$. The angle function corresponding to an edge of the first type [24] (see figure 1) connecting $p \neq i$ and $q \neq i$ looks as follows

$$\alpha^{Sh}(p,q) = \text{Arg}(p - q) - \text{Arg}(\bar{p} - q) - \text{Arg}(p - i) + \text{Arg}(\bar{p} - i).$$ \hspace{3cm} (3.26)

If we fix the rotation symmetry by placing the first function of the Hochschild chain at the point $z = \infty$ then the angle function corresponding to an edge of the second type (see figure 2) connecting $p = i$ and $q$ takes the form

$$\beta^{Sh}(q) = \text{Arg}(i - q) - \text{Arg}(-i - q).$$ \hspace{3cm} (3.27)

Let us suppose that $\chi$ is a vector linear in coordinates on $\mathbb{R}^d_{\text{formal}}$. Then there are two types of the diagrams corresponding to $K_n(\chi, \ldots n \geq 2$. In the diagram of the first type (see figure 3) there are no edges ending at the vertex $z$ corresponding to the vector $\chi$ and in the diagrams of the second type (see figures 4 and 5) there is exactly one edge ending at the vertex $z$.

The coefficient corresponding to a diagram of the first type vanishes because the angle functions entering the integrand form turn out to be dependent. The coefficients corresponding to diagrams of the second type vanish since so do the following integrals

$$\int_{z \in H^+ \setminus \{w,v,i\}} d\alpha^{Sh}(w,z)d\alpha^{Sh}(z,v) = 0, \quad \int_{z \in H^+ \setminus \{v,i\}} d\beta^{Sh}(z)d\alpha^{Sh}(z,v) = 0.$$ \hspace{3cm} (3.28)

Equations (3.28) follow immediately from lemmas 7.3, 7.4, and 7.5 in [19]. □
Hopefully, quasi-isomorphisms (3.20) and (3.23) satisfying the listed properties could also
be obtained along the lines of Tamarkin and Tsygan [26, 27, 28].

4 Fedosov resolutions of the DGLA modules \((T_{poly}(M), A^\bullet(M))\) and \((D_{poly}(M), C_{poly}(M))\)

In this section we extend the definition of resolutions for the DG Lie algebras \(T_{poly}(M)\) and \(D_{poly}(M)\) proposed in [10] and construct resolutions of DGLA modules \((T_{poly}(M), A^\bullet(M))\) and \((D_{poly}(M), C_{poly}(M))\).

First, we recall a definition of a bundle \(S_M\) of the formally completed symmetric algebra of the cotangent bundle \(T^*M\) used in paper [10]. This bundle is a classical analogue of the Weyl algebra bundle used in paper [11] by Fedosov.

**Definition 9** The bundle \(S_M\) of formally completed symmetric algebra of the cotangent bundle \(T^*M\) is defined as a bundle over the manifold \(M\) whose sections are infinite collections of symmetric covariant tensors \(a_{i_1...i_p}(x)\), where \(x^i\) are local coordinates, \(p\) runs from 0 to \(\infty\), and the indices \(i_1, \ldots, i_p\) run from 1 to \(d\).

It is convenient to introduce auxiliary variables \(y^i\), which transform as contravariant vectors. These variables allow us to rewrite any section \(a \in \Gamma(S_M)\) in the form of the formal power series

\[
a = a(x, y) = \sum_{p=0}^{\infty} a_{i_1...i_p}(x)y^{i_1} \cdots y^{i_p}.
\]

It is easy to see that the vector space \(\Gamma(S_M)\) is naturally endowed with the commutative product which is induced by a fiberwise multiplication of formal power series in \(y^i\). This product makes \(\Gamma(S_M)\) into a commutative algebra with a unit.

Now we recall from [10] definitions of formal fiberwise polyvector fields and formal fiberwise polydifferential operators on \(S_M\).

**Definition 10** A bundle \(T_{poly}^k\) of formal fiberwise polyvector fields of degree \(k\) is a bundle over \(M\) whose sections are \(C^\infty(M)\)-linear operators \(v : \wedge^{k+1} \Gamma(S_M) \mapsto \Gamma(S_M)\) of the form

\[
v = \sum_{p=0}^{\infty} v_{i_1...i_p}^{j_0...j_k}(x)y^{i_1} \cdots y^{i_p} \frac{\partial}{\partial y^{j_0}} \wedge \cdots \wedge \frac{\partial}{\partial y^{j_k}},
\]

where we assume that the infinite sum in \(y\)'s is formal and \(v_{i_1...i_p}^{j_0...j_k}(x)\) are tensors symmetric in indices \(i_1, \ldots, i_p\) and antisymmetric in indices \(j_0, \ldots, j_k\).

Extending the definition of the formal fiberwise polyvector field by allowing the fields to be inhomogeneous we define the total bundle \(T_{poly}\) of formal fiberwise polyvector fields

\[
T_{poly} = \bigoplus_{k=-1}^{\infty} T_{poly}^k, \quad T_{poly}^{-1} = S_M.
\]

The fibers of the bundle \(T_{poly}\) are endowed with the DGLA-structure \(T_{poly}(\mathbb{R}^d_{\text{formal}})\) of polyvector fields on the formal completion \(\mathbb{R}^d_{\text{formal}}\) of \(\mathbb{R}^d\) at the origin.
Definition 11 A bundle $\mathcal{D}_k^{\text{poly}}$ of formal fiberwise polydifferential operator of degree $k$ is a bundle over $M$ whose sections are $C^\infty(M)$-polylinear maps $\mathfrak{P} : \bigotimes^{k+1} \Gamma(TM) \mapsto \Gamma(TM)$ of the form

$$\mathfrak{P} = \sum_{\alpha_0 \ldots \alpha_k} \sum_{p=0}^{\infty} \mathfrak{P}_{i_1 \ldots i_p}^{\alpha_0 \ldots \alpha_k}(x)y^{i_1} \ldots y^{i_p} \frac{\partial}{\partial y^{\alpha_0}} \otimes \cdots \otimes \frac{\partial}{\partial y^{\alpha_k}}, \quad (4.4)$$

where $\alpha$’s are multi-indices $\alpha = j_1 \ldots j_l$ and

$$\frac{\partial}{\partial y^\alpha} = \frac{\partial}{\partial y^{j_1}} \cdots \frac{\partial}{\partial y^{j_l}}.$$

The infinite sum in $y$’s is formal, and the sum in the orders of derivatives $\partial/\partial y$ is finite.

Notice that the tensors $\mathfrak{P}_{i_1 \ldots i_p}^{\alpha_0 \ldots \alpha_k}(x)$ are symmetric in covariant indices $i_1, \ldots, i_p$.

As well as for polyvector fields we define the total bundle $\mathcal{D}_{\text{poly}}$ of formal fiberwise polydifferential operators as the direct sum

$$\mathcal{D}_{\text{poly}} = \bigoplus_{k=-1}^{\infty} \mathcal{D}_k^{\text{poly}}, \quad \mathcal{D}_{\text{poly}}^{-1} = SM. \quad (4.5)$$

The fibers of the bundle $\mathcal{D}_{\text{poly}}$ are endowed with the DGLA-structure $D_{\text{poly}}(\mathbb{R}^d_{\text{formal}})$ of polydifferential operators on $\mathbb{R}^d_{\text{formal}}$.

Definition 12 A bundle $\mathcal{C}_{-k}^{\text{poly}}$ of formal fiberwise Hochschild chains of degree $-k$ ($k \geq 0$) is a bundle over $M$ whose sections are formal power series in $k + 1$ fiber variables $y^0, \ldots, y_k$ of the tangent bundle

$$a(x, y_0, \ldots, y_k) = \sum_{\alpha_0 \ldots \alpha_k} a_{\alpha_0 \ldots \alpha_k}(x)y_0^{\alpha_0} \cdots y_k^{\alpha_k}, \quad (4.6)$$

where $\alpha$’s are multi-indices $\alpha = j_1 \ldots j_l$ and

$$y^\alpha = y^{j_1}y^{j_2} \cdots y^{j_l}.$$

The total bundle $\mathcal{C}_{\text{poly}}$ of formal fiberwise Hochschild chains is the direct sum

$$\mathcal{C}_{\text{poly}} = \bigoplus_{k=0}^{\infty} \mathcal{C}_{-k}^{\text{poly}}, \quad \mathcal{C}_{\text{poly}}^0 = SM. \quad (4.7)$$

The operations $R$ (3.4) and $b$ (3.8) turn each fiber of $\mathcal{C}_{\text{poly}}$ into a DG $D_{\text{poly}}(\mathbb{R}^d_{\text{formal}})$-module.

Now we want to introduce an additional copy of the local basis $dx^i$ of exterior forms on $M$. Namely, along with the basis $\{dx^i\}$ we let $\{C^i\}$ be another set of anticommuting coordinates on the fibers of the tangent bundle $TM$. For the relations between $C^i$ and $dx^i$ we accept the following convention

$$C^i dx^j = -dx^j C^i. \quad (4.8)$$
We think of the space of polynomials in $C^i$ whose coefficients are smooth covariant tensors on $M$ as the space $A^*(M)$ of exterior forms

$$A^*(M) = \bigoplus_{k=0}^{\infty} A^{-k}(M), \quad A^{-k}(M) = \{ a = a_{i_1...i_k}(x)C^{i_1}...C^{i_k} \}. \quad (4.9)$$

Although in physical literature the notation $C^i$ is usually reserved for the so-called “ghosts” (auxiliary variables) in our constructions the role of these auxiliary variables is played by $dx^i$’s.

In order to have a fiberwise analogue of exterior forms we give the following

**Definition 13** The bundle $E$ is a bundle over $M$ whose sections are formal power series in $y^i$ taking values in polynomials in $C^i$

$$a(x, y, C) = \sum_{p, q \geq 0} a_{i_1...i_p; j_1...j_q}(x)y^{i_1}...y^{i_p}dx^{j_1}...dx^{j_q}, \quad (4.10)$$

where $a_{i_1...i_p; j_1...j_q}(x)$ are covariant tensors symmetric in indices $i_1, \ldots, i_p$ and antisymmetric in indices $j_1, \ldots, j_q$.

One can say that $E$ is a bundle of exterior forms with values in $SM$. However, this definition would be confusing because in this paper we do use the doubled basis of exterior forms. In particular, we consider exterior forms with values in $S\ M$ which we want to distinguish from the series (4.10). For this purpose we reserve the notation $\Omega(M, S\ M)$ for the space of exterior forms whose homogeneous elements are the following formal series in $y^i$’s

$$a(x, y) = \sum_{p \geq 0} a_{i_1...i_p; j_1...j_q}(x)y^{i_1}...y^{i_p}dx^{j_1}...dx^{j_q}, \quad (4.11)$$

where $a_{i_1...i_p; j_1...j_q}(x)$ are covariant tensors symmetric in indices $i_1, \ldots, i_p$ and antisymmetric in indices $j_1, \ldots, j_q$.

Similarly, homogeneous elements of the graded vector spaces $\Omega(M, T_{poly})$ and $\Omega(M, D_{poly})$ are the following formal series in $y$’s

$$v = \sum_{p \geq 0} dx^{j_1}...dx^{j_k}v_{i_1...i_q; i_1...i_p}(x)y^{i_1}...y^{i_p} \frac{\partial}{\partial y^{j_0}} \wedge \ldots \wedge \frac{\partial}{\partial y^{j_k}}, \quad (4.12)$$

and

$$\mathfrak{P} = \sum_{\alpha_0...\alpha_k} \sum_{p \geq 0} dx^{j_1}...dx^{j_k} \mathfrak{P}_{i_1...i_q; i_1...i_p}^{\alpha_0...\alpha_k}(x)y^{i_1}...y^{i_p} \frac{\partial}{\partial y^{\alpha_0}} \otimes \ldots \otimes \frac{\partial}{\partial y^{\alpha_k}}, \quad (4.13)$$

where as above $\alpha$’s are multi-indices $\alpha = j_1 \ldots j_l$ and

$$\frac{\partial}{\partial y^{\alpha}} = \frac{\partial}{\partial y^{j_1}} \ldots \frac{\partial}{\partial y^{j_l}}.$$

Finally, homogeneous elements of $\Omega(M, E)$ and $\Omega(M, C^{poly})$ are the following formal series

$$a(x, dx, y, C) = \sum_{p \geq 0} dx^{j_1}...dx^{j_k}a_{i_1...i_q; i_1...i_p j_1...j_k}(x)y^{i_1}...y^{i_p}C^{j_1}...C^{j_k}, \quad (4.14)$$
and
\[ b(x, dx, y_0, \ldots, y_k) = \sum_{\alpha_0, \ldots, \alpha_k} dx^{i_1} \cdots dx^{i_q} b_{i_1, \ldots, i_q ; \alpha_0, \ldots, \alpha_k}(x) y_0^{\alpha_0} \cdots y_k^{\alpha_k}, \quad (4.15) \]
where as above \( \alpha \)'s are multi-indices \( \alpha = j_1 \ldots j_t \) and
\[ y_\alpha = y^{j_1} y^{j_2} \cdots y^{j_t}. \]

The symmetries of tensor indices in formulas (4.12), (4.13), (4.14), and (4.15) are obvious.

The space \( \Omega(M, SM) \) is naturally endowed with the structure of (super)commutative algebra which is \( \mathbb{Z} \)-graded with respect to the ordinary exterior degree \( q \) and filtered with respect to the powers in \( y \)'s. The graded vector spaces \( \Omega(M, T_{\text{poly}}) \) and \( \Omega(M, D_{\text{poly}}) \) are, in turn, endowed with fiberwise DGILA structures induced by those on \( T_{\text{poly}}(\mathbb{R}^d) \) and \( D_{\text{poly}}(\mathbb{R}^d) \). Similarly, \( \Omega(M, E) \) and \( \Omega(M, C_{\text{poly}}) \) become fiberwise DGILA modules over \( \Omega(M, T_{\text{poly}}) \) and \( \Omega(M, D_{\text{poly}}) \), respectively. We denote the Lie bracket in \( \Omega(M, D_{\text{poly}}) \) by \([,]_G\) and the Lie bracket in \( \Omega(M, T_{\text{poly}}) \) by \([,]_{SN}\). For fiberwise Lie derivative on \( \Omega(M, E) \) and for the fiberwise action of \( \Omega(M, D_{\text{poly}}) \) on \( \Omega(M, C_{\text{poly}}) \) we also use the same notation \( L \) and \( R \), respectively. It is not hard to see that the formulas for the fiberwise differentials on \( \Omega(M, D_{\text{poly}}) \) and \( \Omega(M, C_{\text{poly}}) \) can be written similarly to (3.7) and (3.8)
\[ \partial = [\mu, \bullet]_G, \quad b = R_\mu, \]
where \( \mu \in \Gamma(D_{\text{poly}}^1) \) is the (commutative) multiplication in \( \Gamma(SM) \). Notice that, we regard \( \Omega(M, T_{\text{poly}}) \) and \( \Omega(M, E) \) as a DGILA and a DGILA module with vanishing differentials.

The parity of elements in the algebras \( \Omega(M, T_{\text{poly}}) \), \( \Omega(M, D_{\text{poly}}) \) and the modules \( \Omega(M, E) \) and \( \Omega(M, C_{\text{poly}}) \) is defined by the sum of the exterior degree and the degree in the respective fiberwise algebra or the respective fiberwise module.

Now we are going to exploit the DGILA modules \( (\Omega(M, D_{\text{poly}}), \Omega(M, C_{\text{poly}})) \) and \( (\Omega(M, T_{\text{poly}}), \Omega(M, E)) \) in order to construct Fedosov resolutions of the DGILA modules \( (D_{\text{poly}}(M), C_{\text{poly}}(M)) \) \( (T_{\text{poly}}(M), A^\bullet(M)) \). In doing this, we will proceed with the algebra of functions \( C^\infty(M) \), the DG Lie algebras \( D_{\text{poly}}(M) \) and \( T_{\text{poly}}(M) \) and their modules \( C_{\text{poly}}(M) \) and \( A^\bullet(M) \) simultaneously and denote the same operations on different vector spaces \( \Omega(M, SM) \), \( \Omega(M, T_{\text{poly}}) \), \( \Omega(M, D_{\text{poly}}) \), \( \Omega(M, E) \) and \( \Omega(M, C_{\text{poly}}) \) by the same letters. In what follows it does not lead to any confusion.

Following \[10\] we introduce the differential
\[ \delta = dx^i \frac{\partial}{\partial y^i} : \Omega^\bullet(M, SM) \mapsto \Omega^{\bullet+1}(M, SM), \quad \delta^2 = 0 \quad (4.16) \]
on the algebra \( \Omega(M, SM) \). This differential can be obviously extended to differentials on \( \Omega(M, T_{\text{poly}}) \), \( \Omega(M, D_{\text{poly}}) \), \( \Omega(M, E) \), and \( \Omega(M, C_{\text{poly}}) \). Namely,
\[ \delta = [dx^i \frac{\partial}{\partial y^i} ; \bullet]_{SN} : \Omega^\bullet(M, T_{\text{poly}}) \mapsto \Omega^{\bullet+1}(M, T_{\text{poly}}), \quad \delta^2 = 0, \quad (4.17) \]
\[ \delta = [dx^i \frac{\partial}{\partial y^i} ; \bullet]_G : \Omega^\bullet(M, D_{\text{poly}}) \mapsto \Omega^{\bullet+1}(M, D_{\text{poly}}), \quad \delta^2 = 0, \quad (4.18) \]
\[ \delta = L_{dx^i \frac{\partial}{\partial y^i}} : \Omega^\bullet(M, E) \mapsto \Omega^{\bullet+1}(M, E), \quad \delta^2 = 0, \quad (4.19) \]
and
\[
\delta = R_{dx^i}\frac{\partial}{\partial y^i} : \Omega^\bullet(M, C^{\text{poly}}) \mapsto \Omega^{\bullet+1}(M, C^{\text{poly}}), \quad \delta^2 = 0.
\]

By definition, \(\delta\) is a derivation of the Lie algebras \(\Omega(M, T_{\text{poly}})\) and \(\Omega(M, D_{\text{poly}})\) and Lie algebra module structures on \(\Omega(M, E)\) and \(\Omega(M, C^{\text{poly}})\). Moreover, since the multiplication \(\mu \in \Gamma(D_1^{\text{poly}})\) in \(\Gamma(SM)\) is \(\delta\)-closed
\[
\delta \mu = 0
\]
\(\delta\) (anti)commutes with the differentials \(\partial\) and \(b\). Thus \(\delta\) is compatible with DGLA structures on \(\Omega(M, T_{\text{poly}})\) and \(\Omega(M, D_{\text{poly}})\) and DGLA module structures on \(\Omega(M, E)\) and \(\Omega(M, C^{\text{poly}})\).

The subspaces \(\ker(\delta) \cap \Gamma(T_{\text{poly}})\) and \(\ker(\delta) \cap \Gamma(D_{\text{poly}})\) will subsequently play an important role in our construction. For this reason we reserve for them special notations
\[
\ker(\delta) \cap \Gamma(T_{\text{poly}}) = \Gamma_{\delta}(T_{\text{poly}}), \quad \ker(\delta) \cap \Gamma(D_{\text{poly}}) = \Gamma_{\delta}(D_{\text{poly}}).
\]
These subspaces can be described in the following way. \(\Gamma_{\delta}(T_{\text{poly}})\) is a subspace of \(\Gamma(T_{\text{poly}})\) whose elements are fiberwise polyvector fields (4.2)
\[
v = \sum_k v^{j_0...j_k}(x) \frac{\partial}{\partial y^{j_0}} \wedge \cdots \wedge \frac{\partial}{\partial y^{j_k}}
\]
whose components do not depend on \(y\)'s. \(\Gamma_{\delta}(D_{\text{poly}})\) is a subspace of \(\Gamma(D_{\text{poly}})\) whose elements are fiberwise polydifferential operators (4.4)
\[
\mathcal{P} = \sum_k \sum_{\alpha_0...\alpha_k} \mathcal{P}^{\alpha_0...\alpha_k}(x) \frac{\partial}{\partial y^{\alpha_0}} \otimes \cdots \otimes \frac{\partial}{\partial y^{\alpha_k}}
\]
whose coefficients do not depend on \(y\)'s.

In the following proposition we describe cohomology of the differential \(\delta\) in \(\Omega(M, SM)\), \(\Omega(M, T_{\text{poly}})\), \(\Omega(M, D_{\text{poly}})\), and \(\Omega(M, E)\)

**Proposition 7** The complexes \((\Omega(M, SM), \delta)\), \((\Omega(M, T_{\text{poly}}), \delta)\), \((\Omega(M, D_{\text{poly}}), \delta)\), and \((\Omega(M, E), \delta)\) are acyclic in all terms except the zeroth one. The zeroth cohomology of these complexes are
\[
H^0(\Omega(M, T_{\text{poly}}), \delta) = \Gamma_{\delta}(T_{\text{poly}}), \quad H^0(\Omega(M, D_{\text{poly}}), \delta) = \Gamma_{\delta}(D_{\text{poly}}),
\]
\[
H^0(\Omega(M, SM), \delta) = \mathcal{C}^{\infty}(M), \quad H^0(\Omega(M, E), \delta) = \mathcal{A}^\bullet(M).
\]

**Proof.** For either of the complexes it is not hard to guess a map \(\sigma\)
\[
\sigma : \Omega(M, SM) \mapsto \mathcal{C}^{\infty}(M) \subset \Omega(M, SM),
\]
\[
\sigma : \Omega(M, T_{\text{poly}}) \mapsto \Gamma_{\delta}(T_{\text{poly}}) \subset \Omega(M, T_{\text{poly}}),
\]
\[
\sigma : \Omega(M, D_{\text{poly}}) \mapsto \Gamma_{\delta}(D_{\text{poly}}) \subset \Omega(M, D_{\text{poly}}),
\]
\[
\sigma : \Omega(M, E) \mapsto \mathcal{A}^\bullet(M) \subset \Omega(M, E),
\]

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and a contracting homotopy $\delta^{-1}$

$$\delta^{-1} : \Omega^\bullet(M, SM) \mapsto \Omega^{\bullet-1}(M, SM), \quad \delta^{-1} : \Omega^\bullet(M, T_{poly}) \mapsto \Omega^{\bullet-1}(M, T_{poly}),$$

$$\delta^{-1} : \Omega^\bullet(M, D_{poly}) \mapsto \Omega^{\bullet-1}(M, D_{poly}), \quad \delta^{-1} : \Omega^\bullet(M, E) \mapsto \Omega^{\bullet-1}(M, E)$$

between the map $\sigma$ and the identity map. Namely,

$$\sigma a = a \big|_{y^i = dx^i = 0}, \quad (4.21)$$

and

$$\delta^{-1} a = y^k i \left( \frac{\partial}{\partial x^k} \right) \int_0^1 a(x, ty, tdx, ty^0, \ldots, y^k + (t-1)y^0) \frac{dt}{t}, \quad (4.22)$$

where $i(\partial/\partial x^k)$ denotes the contraction of an exterior form with the vector field $\partial/\partial x^k$, and $\delta^{-1}$ is extended to $\Gamma(SM)$ (resp. $\Gamma(T_{poly})$, resp. $\Gamma(D_{poly})$, resp. $\Gamma(E)$) by zero.

The desired contracting property

$$a = \sigma(a) + \delta\delta^{-1} a + \delta^{-1}\delta a, \quad \forall \ a \in \Omega(M, B) \quad (4.23)$$

with $B$ being either of the bundles $SM$, $T_{poly}$, $D_{poly}$, or $E$ can be checked by straightforward computations. □

It is worth noting that the homotopy operator $\delta^{-1}$ is nilpotent for either of complexes

$$(\delta^{-1})^2 = 0.$$ 

For the cohomology of the complex $(\Omega^\bullet(M, C^{poly}), \delta)$ we need a less precise description. We claim that

**Proposition 8** The complex $(\Omega^\bullet(M, C^{poly}), \delta)$ has vanishing higher cohomology

$$H^{\geq 1}(\Omega^\bullet(M, C^{poly}), \delta) = 0.$$ 

**Proof.** To prove the statement we define an operator similar to $\delta^{-1}$

$$h : \Omega^q(M, C^{poly}) \mapsto \Omega^{q-1}(M, C^{poly}), \quad q \geq 1,$$

which though acts only on elements with nonzero exterior degree

$$(hb)(x, dx, y_0, \ldots, y_k) = y_0^k i \left( \frac{\partial}{\partial x^k} \right) \int_0^1 a(x, tdx, ty_0, y_1 + (t-1)y_0, \ldots, y_k + (t-1)y_0) \frac{dt}{t}, \quad (4.24)$$

where as above $i(\partial/\partial x^k)$ denotes the contraction of an exterior form with the vector field $\partial/\partial x^k$.

Direct computation shows that for any $a \in \Omega^1(M, C^{poly})$

$$(\delta h + h\delta)a = a. \quad (4.25)$$
Thus the proposition follows. □

For our purposes we introduce an affine torsion free connection $\nabla_i$ on $M$ and associate to it the following derivation of $\Omega^i(M, SM)$

$$\nabla = dx^i \frac{\partial}{\partial x^i} + \Gamma : \Omega^i(M, SM) \mapsto \Omega^{i+1}(M, SM),$$  \hspace{1cm} (4.26)

where

$$\Gamma = -dx^i \Gamma^k_{ij}(x)y^j \frac{\partial}{\partial y^k},$$  \hspace{1cm} (4.27)

with $\Gamma^k_{ij}(x)$ being Christoffel symbols of $\nabla_i$.

The derivation $\nabla$ obviously extends to derivations of the Lie algebras $\Omega(M, T_{poly})$ and $\Omega(M, D_{poly})$

$$\nabla = dx^i \frac{\partial}{\partial x^i} + [\Gamma, \bullet]_{SN} : \Omega^i(M, T_{poly}) \mapsto \Omega^{i+1}(M, T_{poly}),$$  \hspace{1cm} (4.28)

$$\nabla = dx^i \frac{\partial}{\partial x^i} + [\Gamma, \bullet]_{G} : \Omega^i(M, D_{poly}) \mapsto \Omega^{i+1}(M, D_{poly}).$$  \hspace{1cm} (4.29)

and to the derivations of Lie algebra modules $\Omega(M, E)$ and $\Omega(M, C_{poly})$

$$\nabla = dx^i \frac{\partial}{\partial x^i} + L_\Gamma : \Omega^i(M, E) \mapsto \Omega^{i+1}(M, E),$$  \hspace{1cm} (4.30)

$$\nabla = dx^i \frac{\partial}{\partial x^i} + R_\Gamma : \Omega^i(M, C_{poly}) \mapsto \Omega^{i+1}(M, C_{poly}).$$  \hspace{1cm} (4.31)

Direct computations show that $\nabla$ acts on the components of fiberwise polyvector fields, on the coefficients of the fiberwise operators, and on the components of exterior forms and fiberwise Hochschild chains as the usual covariant derivative. This proves that $\nabla$ is defined correctly. Moreover, the multiplication $\mu \in \Gamma(D^1_{poly})$ in $\Gamma(SM)$ is "covariantly constant" $d\mu + [\Gamma, \mu]_G = 0$ and hence the derivation $\nabla$ commutes with the differentials $\partial$ and $b$. Thus, $\nabla$ (4.28), (4.31) is a derivation of the DGLA structure on $\Omega^i(M, T_{poly})$ and DGLA module structure on $\Omega^i(M, E^\bullet)$.

Since the DGLA $\Omega(M, T_{poly})$ and DG $\Omega(M, T_{poly})$-module $\Omega(M, E^\bullet)$ have vanishing differentials the operator $\nabla$ (4.28), (4.31) also respects the DGLA structure on $\Omega^i(M, T_{poly})$ and DGLA module structure on $\Omega^i(M, E^\bullet)$.

In general derivation (4.26) is not nilpotent as $\delta$. Instead we have the following expression for $\nabla^2$

$$\nabla^2 a = \mathcal{R} a : \Omega^i(M, SM) \mapsto \Omega^{i+2}(M, SM),$$  \hspace{1cm} (4.32)

where

$$\mathcal{R} = -\frac{1}{2} dx^i dx^j (R_{ij})^k_l(x) y^l \frac{\partial}{\partial y^k},$$

and $(R_{ij})^k_l(x)$ is the standard Riemann curvature tensor of the connection $\nabla_i$.

Similarly, for $\Omega(M, T_{poly})$, $\Omega(M, D_{poly})$, $\Omega(M, E)$, and $\Omega(M, C_{poly})$ we have

$$\nabla^2 a = [\mathcal{R}, a]_{SN} : \Omega^i(M, T_{poly}) \mapsto \Omega^{i+2}(M, T_{poly}),$$  \hspace{1cm} (4.33)
\[ \nabla^2 a = [\mathcal{R}, a]_G : \Omega^\bullet(M, \mathcal{D}_{\text{poly}}) \mapsto \Omega^{\bullet+2}(M, \mathcal{D}_{\text{poly}}), \quad (4.34) \]

\[ \nabla^2 a = L_\mathcal{R} a : \Omega^\bullet(M, \mathcal{E}) \mapsto \Omega^{\bullet+2}(M, \mathcal{E}), \quad (4.35) \]

\[ \nabla^2 a = R_\mathcal{R} a : \Omega^\bullet(M, \mathcal{C}_{\text{poly}}) \mapsto \Omega^{\bullet+2}(M, \mathcal{C}_{\text{poly}}). \quad (4.36) \]

Notice that, since the connection \( \nabla_i \) is torsion free derivations \( \nabla \) and \( \delta \) (anti)commute

\[ \delta \nabla + \nabla \delta = 0. \quad (4.37) \]

Following [10] we use the derivation (4.26) in order to deform the nilpotent differential \( \delta \) on \( \Omega(M, \mathcal{S}M), \Omega(M, \mathcal{T}_{\text{poly}}), \Omega(M, \mathcal{D}_{\text{poly}}), \Omega(M, \mathcal{E}), \) and \( \Omega(M, \mathcal{C}_{\text{poly}}) \)

\[ D = \nabla - \delta + A : \Omega^\bullet(M, \mathcal{S}M) \mapsto \Omega^{\bullet+1}(M, \mathcal{S}M), \]
\[ D = \nabla - \delta + [A, \bullet]_{SN} : \Omega^\bullet(M, \mathcal{T}_{\text{poly}}) \mapsto \Omega^{\bullet+1}(M, \mathcal{T}_{\text{poly}}), \]
\[ D = \nabla - \delta + [A, \bullet]_G : \Omega^\bullet(M, \mathcal{D}_{\text{poly}}) \mapsto \Omega^{\bullet+1}(M, \mathcal{D}_{\text{poly}}), \quad (4.38) \]
\[ D = \nabla - \delta + L_A : \Omega^\bullet(M, \mathcal{E}) \mapsto \Omega^{\bullet+1}(M, \mathcal{E}), \]
\[ D = \nabla - \delta + R_A : \Omega^\bullet(M, \mathcal{C}_{\text{poly}}) \mapsto \Omega^{\bullet+1}(M, \mathcal{C}_{\text{poly}}), \]

where

\[ A = \sum_{p=2}^{\infty} dx^k A_{ki1\ldots ip}^j (x)y^{i_1} \ldots y^{i_p} \frac{\partial}{\partial y^j} \]

is simultaneously viewed as an element of \( \Omega^1(M, \mathcal{T}_{\text{poly}}^0) \) and an element of \( \Omega^1(M, \mathcal{D}_{\text{poly}}^0) \).

Due to the following theorem one can explicitly construct a nilpotent differential \( D \) in the framework of ansatz (4.38)

**Theorem 4 ([10] (Theorem 2))** Iterating the equation

\[ A = \delta^{-1} R + \delta^{-1}(\nabla A + \frac{1}{2}[A, A]) \quad (4.39) \]

in degrees in \( y \) one constructs \( A \in \Omega^1(M, \mathcal{T}_{\text{poly}}^0) \subset \Omega^1(M, \mathcal{D}_{\text{poly}}^0) \) such that \( \delta^{-1} A = 0 \) and the derivation \( D \) (4.38) is nilpotent

\[ D^2 = 0. \quad \square \]

In what follows we refer to the nilpotent differential \( D \) (4.38) as Fedosov differential.

A proof of the following statement can be essentially read off from [10] (see the proof of theorem 3).

**Theorem 5** Let \( a \) be an element in \( C^\infty(M) \) (resp. \( \Gamma_\delta(\mathcal{T}_{\text{poly}}), \) resp. \( \Gamma_\delta(\mathcal{D}_{\text{poly}}), \) resp. \( \mathcal{A}^\bullet(M) \)). Then iterating the following equation

\[ \tau(a) = a + \delta^{-1}(\nabla \tau(a) + [A, \tau(a)]) \quad (4.40) \]

in degrees in \( y \) we get an isomorphism \( \tau \) from \( C^\infty(M) \) (resp. \( \Gamma_\delta(\mathcal{T}_{\text{poly}}), \) resp. \( \Gamma_\delta(\mathcal{D}_{\text{poly}}), \) resp. \( \mathcal{A}^\bullet(M) \)) to \( Z^0(\Omega(M, \mathcal{S}M), D) \) (resp. \( Z^0(\Omega(M, \mathcal{T}_{\text{poly}}), D), \) resp. \( Z^0(\Omega(M, \mathcal{D}_{\text{poly}}), D), \) resp. \( Z^0(\Omega(M, \mathcal{C}_{\text{poly}}), D), \) resp. \( Z^0(\Omega(M, \mathcal{E}), D), \) resp. \( Z^0(\Omega(M, \mathcal{C}_{\text{poly}}), D) \) and so on...
$Z^0(\Omega(M, \mathcal{E}), D))$, where $Z^0(\Omega(M, \mathcal{E}), D) = \ker D \cap \Omega^0(M, \mathcal{E})$. For either of the bundles $SM$, $T_{\text{poly}}$, $D_{\text{poly}}$, $\mathcal{E}$ higher cohomology of the Fedosov differential (4.38) are vanishing

$$H^{\geq 1}(\Omega(M, \mathcal{E}), D) = 0. \quad \square$$

Thus, the map $\tau$ (4.40) induces an isomorphism of graded vector spaces

$$H^\bullet(\Omega(M, SM), D) \cong C^\infty(M), \quad H^\bullet(\Omega(M, T_{\text{poly}}), D) \cong \Gamma_\delta(T_{\text{poly}}),$$

$$H^\bullet(\Omega(M, D_{\text{poly}}), D) \cong \Gamma_\delta(D_{\text{poly}}), \quad H^\bullet(\Omega(M, \mathcal{E}), D) \cong A^\bullet(M).$$

It is also worth noting that the map $\sigma$ provides us with a natural section of $\tau$

$$\sigma \circ \tau = Id. \quad (4.41)$$

It turns out that the map $\tau$ is compatible with the natural algebraic structures on $C^\infty(M)$, $\Omega(M, SM)$, $A^\bullet(M)$, and $\Omega(M, \mathcal{E})$. Namely,

**Proposition 9** Let

$$d_C = C^i \frac{\partial}{\partial x^i} \quad (4.42)$$

be a De Rham differential on $A^\bullet(M)$ and

$$\delta^f = C^i \frac{\partial}{\partial y^i} \quad (4.43)$$

be a the fiberwise De Rham differential on $\Omega(M, \mathcal{E})$. Then the map

$$\tau : C^\infty(M) \mapsto \Omega(M, SM) \quad (4.44)$$

$$\tau : (A^\bullet(M), d_C) \mapsto (\Omega(M, \mathcal{E}), \delta^f) \quad (4.45)$$

is a morphism of commutative (resp. DG commutative) algebras.

**Proof.** Since the statement about the map (4.44) follows from the statement about the map (4.45), we focus on the map (4.45).

First, we mention that the Fedosov differential (4.38) is a derivation of (super)commutative multiplication in $\Omega(M, \mathcal{E})$. Hence, for any pair $a_1, a_2 \in A^\bullet(M)$

$$D(\tau(a_1)\tau(a_2)) = 0.$$

But $\sigma(\tau(a_1)\tau(a_2)) = a_1a_2$ and the map $\tau$ is an isomorphism of the vector spaces $A^\bullet(M)$ and $Z^0(\Omega(M, \mathcal{E}), D) = \ker D \cap \Omega^0(M, \mathcal{E})$. Therefore, $\tau(a_1)\tau(a_2) = \tau(a_1a_2)$ and $\tau$ is the morphism of algebras.

Second, by definition the differential $D$ on $\Omega(M, \mathcal{E})$ can be rewritten as

$$D = d + L_B,$$

where $d = dx^i \frac{\partial}{\partial x^i}$ is the ordinary De Rham differential and $B$ is a one-form taking values in fiberwise vector fields. Notice that $d$ and $B$ are defined only locally.
The following computation

\[ \delta^f L_B + L_B \delta^f = \delta^f (\delta^f i_B - i_B \delta^f) + (\delta^f i_B - i_B \delta^f) \delta^f = 0 \]

shows that \( \delta^f \) anticommutes with \( L_B \). But \( \delta^f \) also obviously anticommutes with \( d \) and therefore

\[ D \delta^f + \delta^f D = 0. \tag{4.46} \]

Now we use the same trick as for the multiplication. Due to \( (4.46) \) we have that for any \( a \in A^\bullet(M) \)

\[ D \delta^f \tau(a) = 0. \]

On the other hand since the connection \( \nabla_i \) we use is torsion free \( \sigma \delta^f \tau(a) = d_C a \). Thus, we get the desired equation

\[ \delta^f \tau(a) = \tau(d_C a) \]

because \( \tau \) is an isomorphism of the vector spaces \( A^\bullet(M) \) and \( Z^0(\Omega(M, E), D) = \ker D \cap \Omega^0(M, E) \). \( \square \)

Our next task is to establish an isomorphism of graded vector spaces \( H(\Omega(M, C^{poly}), D) \) and \( C^{poly}(M) \). For this we first observe that for any function \( a \in C^\infty(M) \) and for any integer \( p \geq 0 \)

\[ \frac{\partial}{\partial y^1} \cdots \frac{\partial}{\partial y^p} \tau(a) \bigg|_{y=0} = \partial_{x^1} \cdots \partial_{x^p} a(x) + \text{lower order derivatives of } a. \tag{4.47} \]

Thus, the map \( \tau \) allows us to identify the algebra \( jets^\infty_x(M) \) of the \( \infty \)-jets of functions on \( M \) at the point \( x \in M \) with the fiber \( \Gamma_x(SM) \). We denote the respective isomorphism attached to the point \( x \in M \) by \( \varrho_x \)

\[ \varrho_x : jets^\infty_x(M) \xrightarrow{\sim} \Gamma_x(SM). \tag{4.48} \]

This isomorphism induces a natural embedding

\[ \varrho : C^{poly}(M) \hookrightarrow Z^0(\Omega(M, C^{poly}), D), \]

which is defined on the homogeneous chains by

\[ \varrho(a)(x) = \varrho_x \otimes \cdots \otimes \varrho_x(a). \tag{4.49} \]

Now we are ready to describe cohomology of \( (\Omega(M, C^{poly}), D) \)

**Proposition 10** The complex \( (\Omega(M, C^{poly}), D) \) has vanishing higher cohomology

\[ H^{\geq 1}(\Omega(M, C^{poly}), D) = 0 \]

and the map \( \varrho \ (4.49) \) gives an isomorphism of graded vector spaces

\[ \varrho : C^{poly}(M) \xrightarrow{\sim} Z^0(\Omega(M, C^{poly}), D). \tag{4.50} \]
Proof. Let \( a \in \Omega^q(M, C^\text{poly}) \) for some \( q \geq 1 \) and \( Da = 0 \). Our purpose is to solve the equation

\[ a = Db. \tag{4.51} \]

We claim that iterating the following relation

\[ b = -ha + h(\nabla b + R_A b) \tag{4.52} \]

in degrees in \( y_0 \) we get an element \( b \in \Omega^{q-1}(M, C^\text{poly}) \) such that \( Db = a \). We denote by \( f \) the element

\[ f = a - Db \in \Omega^q(M, C^\text{poly}) \]

and observe that \( Df = 0 \) or equivalently \( \delta f = \nabla f + R_A f \). Using (4.25) it is not hard to show that

\[ b = h\delta f \]

Therefore \( f \) satisfies the equation

\[ hf = 0. \]

Furthermore, since \( f \in \Omega^{\geq 1}(M, C^\text{poly}) \) we can apply homotopy property (4.25) and get

\[ f = h(\nabla f + [A, f]). \]

The latter equation has a unique vanishing solution since \( h \) raises the degree in \( y_0 \). Thus the first statement of the proposition follows.

Let us now turn to the second statement. From the construction of (4.49) it is obvious that \( \varrho \) is injective. Thus we are left with a proof of surjectivity.

We claim that it suffices to prove the surjectivity of the map

\[ \varrho : C^\text{poly}(V) \mapsto Z^0(\Omega(V, C^\text{poly}), D), \tag{4.53} \]

for any coordinate chart \( V \subset M \). Indeed, if \( a \) is a global section in \( Z^0(\Omega(M, C^\text{poly}), D) \) and for each coordinate chart \( V \subset M \) we have \( b_V \in C^\text{poly}(V) \) such that \( a \big|_V = \varrho(b_V) \). Then on any intersection \( V \cap V' \) \( b_V = b'_{V'} \) since the map \( \varrho \) is injective and hence we have \( b \in C^\text{poly}(M) \) such that \( \varrho(b) = a \).

To prove surjectivity of (4.53) we observe that if the Fedosov differential had the simplest possible form

\[ D_0 = d - \delta \]

with \( d = dx^i \partial / \partial x^i \) being the De Rham differential then the problem would reduce to a simple task of the theory of partial differential equations.

Thus it suffices to prove that on any coordinate chart the Fedosov differential \( D \) can be conjugated to \( D_0 \) by some invertible formal fiberwise operator \( \mathfrak{P} \in \Gamma(V, D^0_{\text{poly}}) \).

To prove this we rewrite \( D \) in the form

\[ D = d - \delta + T, \]

where \( T \) is a formal fiberwise vector field

\[ T = \sum_{k=1}^{\infty} dx^i T_{i;j_1...j_k} y^{j_1} \ldots y^{j_k} \frac{\partial}{\partial y^j} \in \Omega^1(V, T_{\text{poly}}). \]
Next, we claim that iterating the equation
\[ \mathfrak{P} = I + \delta^{-1}(d(\mathfrak{P}) - \mathfrak{P} \circ T) \] (4.54)
in degrees in \( y \) we get a formal fiberwise invertible operator \( \mathfrak{P} \) such that
\[ \mathfrak{P}^{-1} \circ D^0 \circ \mathfrak{P} = D, \] (4.55)
Let us rewrite equation (4.55) as
\[ D_0 \circ \mathfrak{P} - \mathfrak{P} D = 0 \] (4.56)
and denote the left hand side of (4.56) by \( J \). Applying equation (4.23) to the operator \( \mathfrak{P} \) we get that \( \mathfrak{P} \) satisfies the consequence of (4.56)
\[ \delta^{-1} J = 0. \] (4.57)
On the other hand both \( D_0 \) and \( D \) are nilpotent. Therefore,
\[ D_0 \circ J + J \circ D = 0. \] (4.58)
Applying equation (4.23) to \( J \) and using (4.57) and (4.58) we get
\[ J = \delta^{-1}(d(J) + J \circ T). \]
The latter equation has a unique vanishing solution because \( \delta^{-1} \) raises the degree in \( y \).

Thus the Fedosov differential \( D \) can be always locally conjugated to \( D_0 = d - \delta \) and the desired surjectivity of the map \( \varrho \) follows. □

The map (4.49) provides us with the isomorphisms of graded vector spaces
\[ \nu : \Gamma_\delta(T_{poly}) \simarrow T_{poly}(M), \quad \nu : \Gamma_\delta(D_{poly}) \simarrow D_{poly}(M), \] (4.59)
which are defined by
\[ (\nu v)(a_0, \ldots, a_k)(x) = \nu(v(g_x a_0, \ldots, g_x a_k)), \] (4.60)
where \( v \) is either an element of \( \Gamma_\delta(T_{poly}) \) or an element of \( \Gamma_\delta(D_{poly}) \).

Collecting the results obtained so far we conclude that we have the following isomorphisms of graded vectors spaces
\[ \tau \circ \nu^{-1} : T_{poly}(M) \simarrow H^\bullet(\Omega(M, T_{poly}), D), \]
\[ \tau \circ \nu^{-1} : D_{poly}(M) \simarrow H^\bullet(\Omega(M, D_{poly}), D), \]
\[ \tau : A^\bullet(M) \simarrow H^\bullet(\Omega(M, E), D), \]
\[ \varrho : C_{poly}(M) \simarrow H^\bullet(\Omega(M, C_{poly}), D). \] (4.61)
But since the Fedosov differential (4.38) is compatible with the fiberwise operations \( \partial, [,],_G, [,],_SN, L_\bullet \) and \( R_\bullet \) the vector spaces \( H^\bullet(\Omega(M, T_{poly}), D) \) and \( H^\bullet(\Omega(M, D_{poly}), D) \) acquire a DGLA structure, as well as the vector spaces \( H^\bullet(\Omega(M, E), D) \) and \( H^\bullet(\Omega(M, C_{poly}), D) \) become DGLA modules over \( H^\bullet(\Omega(M, T_{poly}), D) \) and \( H^\bullet(\Omega(M, D_{poly}), D) \), respectively. This naturally raises the question as to whether the maps (4.61) are isomorphisms of DG Lie algebra and DGLA modules, respectively. The following proposition gives a positive answer to this question.
Proposition 11 The maps
\[ \tau \circ \nu^{-1} : T_{\text{poly}}(M) \xrightarrow{\sim} H^\bullet(\Omega(M, T_{\text{poly}}), D), \]  
(4.62)
\[ \tau \circ \nu^{-1} : D_{\text{poly}}(M) \xrightarrow{\sim} H^\bullet(\Omega(M, D_{\text{poly}}), D) \]  
(4.63)
are isomorphisms of DG Lie algebras and the maps
\[ \tau : \mathcal{A}^\bullet(M) \xrightarrow{\sim} H^\bullet(\Omega(M, \mathcal{E}), D), \]  
(4.64)
\[ \theta : C^\text{poly}(M) \xrightarrow{\sim} H^\bullet(\Omega(M, C^\text{poly}), D) \]  
(4.65)
are isomorphisms of DGLA modules.

Proof. The part of this proposition concerning the maps (4.62) and (4.63) has been proved in [10] (see proposition 2). Thus we are left with the maps (4.64) and (4.65).

Concerning the map (4.64) we have to prove that for any exterior form \( a = a_{i_1...i_q}(x)C^{i_1}...C^{i_q} \) and any polyvector field \( \gamma = \gamma^{i_0...i_k}(x)\partial_{y^{i_0}} \wedge ... \wedge \partial_{y^{i_k}} \)
\[ \tau(L_\gamma(a)) = L_{\tau(\nu^{-1}(\gamma))(\tau(a))}. \]  
(4.66)
Since Fedosov differential \( D \) is compatible with the fiberwise Lie derivative \( L \) the form \( L_{\tau(\gamma)}(\tau(a)) \) is \( D \)-closed. Therefore it suffices to show that
\[ L_{\tau(\nu^{-1}(\gamma))}(\tau(a)) \bigg|_{y=0} = L_\gamma(a). \]  
(4.67)
To prove (4.67) we need the expressions for \( \tau(\nu^{-1}(\gamma)) \) and \( \tau(a) \) only up to the second order terms in \( y \). They are
\[ \tau(\nu^{-1}(\gamma)) = \nu^{-1}(\gamma) + y^i \frac{\partial \nu^{-1}(\gamma)}{\partial x^i} - y^i [\Gamma_i(x), \nu^{-1}(\gamma)]_{SN} \mod (y)^2, \]  
(4.68)
\[ \tau(a) = a + y^i \frac{\partial a}{\partial x^i} - y^i L_{\Gamma_i(a)}(a) \mod (y)^2, \]  
(4.69)
where \( \Gamma^k_{ij}(x) \) are Christoffel symbols and
\[ \nu^{-1}(\gamma) = \gamma^{i_0...i_k}(x)\partial_{y^{i_0}} \wedge ... \wedge \partial_{y^{i_k}}. \]
Using symmetry of indices for the Christoffel symbols \( \Gamma^k_{ij} = \Gamma^k_{ji} \) we can rewrite (4.68) and (4.69) in the form
\[ \tau(\nu^{-1}(\gamma)) = \nu^{-1}(\gamma) + y^i \frac{\partial \nu^{-1}(\gamma)}{\partial x^i} - [\bar{\Gamma}(x), \nu^{-1}(\gamma)]_{SN} \mod (y)^2, \]  
(4.70)
\[ \tau(a) = a + y^i \frac{\partial a}{\partial x^i} - L_{\bar{\Gamma}(a)}(a) \mod (y)^2, \]  
(4.71)
where $\tilde{\Gamma} = \frac{1}{2} \Gamma_{ijy}^k \frac{\partial}{\partial y^k}$. Using these formulas it is not hard to show that equation (4.67) is equivalent to

$$ L_{\nu^{-1}(\gamma)} L_{\tilde{\Gamma}}(a) + L_{[\tilde{\Gamma}, \nu^{-1}(\gamma)]_{SN}}(a) = 0, $$

which obviously holds because $L_{\nu^{-1}(\gamma)}(a) = 0$.

The compatibility of the map (4.65) with the action $R$ essentially follows from definitions. Indeed, the action $R$ is defined in terms of the action of a $k$-cochain on a $k$-chain. But for any $\Psi \in Z^0(\Omega(M, D^k_{poly}), D)$ and any chain $a \in C^k_{poly}(M)$

$$ (\nu \Psi)(a) = \sigma \Psi(\rho(a)) $$

by definition of the map $\nu$.

Finally, since the differentials on $\Omega(M, D_{poly})$ and $\Omega(M, C_{poly})$ are defined via the multiplication $\mu \in \Gamma(D^1_{poly})$ in $\Gamma(SM)$ and the differentials on $D_{poly}(M)$ and on $C_{poly}(M)$ are defined via the multiplication $\mu_0 \in D^1_{poly}(M)$ in $C^\infty(M)$ the map (4.65) is compatible with the differentials because by proposition 9 the map $\tau : C^\infty(M) \mapsto \Gamma(SM)$ preserves the multiplication. □

5 Formality theorem for $C_{poly}(M)$ and its applications

5.1 Proof of formality theorem for $C_{poly}(M)$ via Fedosov resolutions

The results of the previous section can be represented in the form of the following commutative diagrams of DG Lie algebras, their modules, and morphisms

$$
\begin{array}{ccc}
T_{poly}(M) & \xrightarrow{\tau_{ob^{-1}}} & (\Omega(M, T_{poly}), D, [,]_{SN}) \\
\downarrow \mod & & \downarrow \mod \\
A^\bullet(M) & \xrightarrow{\tau} & (\Omega(M, E), D),
\end{array}
$$

(5.1)

$$
\begin{array}{ccc}
(\Omega(M, D_{poly}), D + \partial, [,]_G) & \xleftarrow{\tau_{ob^{-1}}} & D_{poly}(M) \\
\downarrow \mod & & \downarrow \mod \\
(\Omega(M, C_{poly}), D + b) & \xleftarrow{\theta} & C_{poly}(M),
\end{array}
$$

where the horizontal arrows correspond to embeddings of DG Lie algebras (resp. DGLA modules) which are also quasi-isomorphisms by proposition 11.

Next, due to properties 1 and 2 in theorem 2 we have a fiberwise quasi-isomorphism

$$
U : (\Omega(M, T_{poly}), 0, [,]_{SN}) \mapsto (\Omega(M, D_{poly}), \partial, [,]_G). 
$$

(5.2)

from the DGLA $(\Omega(M, T_{poly}), 0, [,]_{SN})$ to the DGLA $(\Omega(M, D_{poly}), \partial, [,]_G)$. Composing quasi-isomorphism (5.2) with the action of $\Omega(M, D_{poly})$ on $\Omega(M, C_{poly})$ we get an $L_\infty$-module structure on $\Omega(M, C_{poly})$ over $\Omega(M, T_{poly})$. We denote this structure by $\tilde{\Psi}$. 34
Due to properties 1 and 2 in theorem 3, we have a fiberwise quasi-isomorphism

\[ K : (\Omega(M, C^{\text{poly}}), b, \Upsilon) \xrightarrow{\sim} (\Omega(M, \mathcal{E}), 0, L) \]  

from the \(L_\infty\)-module \(\Omega(M, C^{\text{poly}})\) to the DGLA module \(\Omega(M, \mathcal{E})\) over \(\Omega(M, T^{\text{poly}})\). The zeroth structure map \(K_0\) of (5.3) is the map of Connes

\[ K_0(a(x, y_0, \ldots, y_k)) = \left( \frac{\partial}{\partial y_0} \ldots \frac{\partial}{\partial y_k} a(x, y_0, \ldots, y_k) \right) \bigg|_{y_0 = y_1 = \cdots = y_k} . \]  

Thus we get the following commutative diagram

\[ \begin{array}{ccc}
(\Omega(M, T^{\text{poly}}), 0, [\cdot], [\cdot]_{SN}) & \xrightarrow{U} & (\Omega(M, D_{\text{poly}}), \partial, [\cdot]_{G}) \\
\downarrow^{L \ mod} & & \downarrow^{R \ mod} \\
(\Omega(M, \mathcal{E}), 0) & \xleftarrow{K} & (\Omega(M, C^{\text{poly}}), b),
\end{array} \]  

where by commutativity we mean that \(K\) is a morphism of the \(L_\infty\)-modules \((\Omega(M, C^{\text{poly}}), b)\) and \((\Omega(M, \mathcal{E}), 0)\) over the DGLA \((\Omega(M, T^{\text{poly}}), 0, [\cdot]_{SN})\) where the \(L_\infty\)-module structure on \((\Omega(M, C^{\text{poly}}), b)\) over \((\Omega(M, T^{\text{poly}}), 0, [\cdot]_{SN})\) is obtained by composing the quasi-isomorphism \(U\) with the action \(R\) of \((\Omega(M, D_{\text{poly}}), \partial, [\cdot]_{G})\) on \((\Omega(M, C^{\text{poly}}), b)\).

Let us now restrict ourselves to a contractible coordinate chart \(V \subset M\). Since the quasi-isomorphisms (5.2) and (5.3) are fiberwise we can add to all the differentials in diagram (5.5) the ordinary De Rham differential \(d = dx^i \frac{\partial}{\partial x^i}\). In this new commutative diagram

\[ \begin{array}{ccc}
(\Omega(V, T^{\text{poly}}), d, [\cdot]_{SN}) & \xrightarrow{U} & (\Omega(V, D_{\text{poly}}), d + \partial, [\cdot]_{G}) \\
\downarrow^{L \ mod} & & \downarrow^{R \ mod} \\
(\Omega(M, \mathcal{E}), d) & \xleftarrow{K} & (\Omega(M, C^{\text{poly}}), d + b),
\end{array} \]  

\(U\) and \(K\) are still quasi-isomorphisms since the chart \(V\) is contractible. On the chart \(V\) we can represent the Fedosov differential (4.38) in the following (non-covariant) form

\[ D = d + B, \]  

\[ B = \sum_{p=0}^{\infty} dx^i B_{i; j_1 \ldots j_p}^k (x) y^{j_1} \ldots y^{j_p} \frac{\partial}{\partial y^k}. \]

If we regard \(B\) as an element of \(\Omega^1(V, T^{\text{poly}}_{\text{poly}})\) then the nilpotency condition \(D^2 = 0\) says that \(B\) is a Maurer-Cartan element in the DGLA \((\Omega(V, T^{\text{poly}}_{\text{poly}}), d, [\cdot]_{SN})\). Applying\(^6\) the technique developed in section 2 to the element \(B\) we see that the DGLA \((\Omega(V, T^{\text{poly}}_{\text{poly}}), D, [\cdot]_{SN})\) is obtained from \((\Omega(V, T^{\text{poly}}_{\text{poly}}), d, [\cdot]_{SN})\) by twisting via \(B\).

\(^6\)see the last remark in subsection 2.3
Due to property 3 in theorem 2 the Maurer-Cartan element in \((\Omega(V, D_{poly}), d + \partial, [,]_G)\)
\[ B_D = \sum_{k=1}^{\infty} \frac{1}{k!} U_k(B, \ldots, B) \]
corresponding to the Maurer-Cartan element \(B\) in \((\Omega(V, T_{poly}), d, [,]_{SN})\) coincides with \(B\) viewed as an element of \(\Omega^1(V, D_{poly})\). Thus twisting of the quasi-isomorphism \(U\) via the Maurer-Cartan element \(B\) we get the quasi-isomorphism
\[ \Upsilon : (\Omega(V, T_{poly}), D, [,]_{SN}) \rightarrowto (\Omega(V, D_{poly}), D + \partial, [,]_G) \]

Next, using (2.44) it is not hard to show that the DGLA module structure on \(\Omega(V, \mathcal{E})\) and \(\Omega(V, C_{poly})\) over \((\Omega(V, T_{poly}), D, [,]_{SN})\) and \((\Omega(V, D_{poly}), D + \partial, [,]_G)\), respectively, obtained by twisting via \(B\) coincides with those defined by the fiberwise structures \(L\) and \(R\)
\[ \begin{array}{c}
(\Omega(V, T_{poly}), D, [,]_{SN}) \\
\downarrow L_{mod} \quad \uparrow R_{mod}
\end{array} \quad \begin{array}{c}
(\Omega(V, D_{poly}), D + \partial, [,]_G)
\end{array} \tag{5.8}
\]
Hence, by virtue of propositions 3 and 4 twisting procedure turns diagram (5.6) into the commutative diagram
\[ \begin{array}{c}
(\Omega(V, T_{poly}), D, [,]_{SN}) \\
\downarrow L_{mod} \quad \uparrow R_{mod}
\end{array} \quad \begin{array}{c}
(\Omega(V, D_{poly}), D + \partial, [,]_G)
\end{array} \tag{5.9}
\]
where \(K\) is a quasi-isomorphism obtained from \(K\) by twisting via the Maurer-Cartan element \(B \in (\Omega(V, T_{poly}), d, [,]_{SN})\).

Surprisingly, due to property 4 in theorem 2 and property 3 in theorem 3 the maps \(U\) and \(K\) are defined globally. Indeed, using (2.39) and (2.45) we get the structure maps of \(U\) and \(K\)
\[ \Upsilon_n(\gamma_1, \ldots, \gamma_n) = \sum_{k=0}^{\infty} \frac{1}{k!} U_{n+k}(B, \ldots, B, \gamma_1, \ldots, \gamma_n), \]
\[ \Phi_n(\gamma_1, \ldots, \gamma_n, a) = \sum_{k=0}^{\infty} \frac{1}{k!} K_{n+k}(B, \ldots, B, \gamma_1, \ldots, \gamma_n, a), \]
in terms of structure maps of \(U\) and \(K\). But the only term in \(B\) that transforms not as a tensor is
\[ \Gamma = -d x^i \Gamma^k_{ij} y^j \frac{\partial}{\partial y^k}, \]
and this term contributes neither to \(U_n\) nor to \(K_n\) since it is linear in \(y\)'s.
Thus the quasi-isomorphisms $\mathfrak{U}$ and $\mathfrak{K}$ are defined globally and we arrive at the following commutative diagram

$$
\begin{array}{ccc}
(\Omega(M, T_{\text{poly}}), D, [\cdot]_{SN}) & \xrightarrow{\mathfrak{U}} & (\Omega(M, D_{\text{poly}}), D + \partial, [\cdot]_G) \\
\downarrow^{L_{\text{mod}}} & & \downarrow^{R_{\text{mod}}} \\
(\Omega(M, E), D) & \xleftarrow{\mathfrak{K}} & (\Omega(M, C_{\text{poly}}), D + b).
\end{array}
$$

Assembling (5.10) with (5.11) we get the desired commutative diagram

$$
\begin{array}{ccc}
T_{\text{poly}}(M) & \xrightarrow{\mathfrak{U}} & \mathfrak{L} \\
\downarrow^{L_{\text{mod}}} & & \downarrow^{R_{\text{mod}}} & & \\
\mathfrak{A}^\bullet(M) & \xrightarrow{\tau \circ \nu^{-1}} & (\Omega(M, E), D) \xleftarrow{\mathfrak{R}} \mathfrak{N} \xleftarrow{\theta} C_{\text{poly}}(M)
\end{array}
$$

where $\mathfrak{L}$ is the DGLA $(\Omega^\bullet(M, D_{\text{poly}}), D + \partial, [\cdot]_G)$, $\mathfrak{N}$ is the DGLA module $(\Omega^\bullet(M, C_{\bullet}), D + b)$, $\mathfrak{R}$ is the action of $T_{\text{poly}}(M)$ on $(\Omega(M, E), D)$ obtained by composing the embedding

$$
\tau \circ \nu^{-1} : T_{\text{poly}}(M) \hookrightarrow \Omega(M, T_{\text{poly}})
$$

with fiberwise Lie derivative and $\mathfrak{U} = \mathfrak{U} \circ \tau \circ \nu^{-1}$ is a composition of quasi-isomorphisms hence is also a quasi-isomorphism.

Theorem $\blacksquare$ is proved. $\Box$

### 5.2 Applications of theorem $\blacksquare$

The first obvious applications of the formality theorem for $C_{\text{poly}}(M)$ are related to computation of Hochschild homology for the quantum algebra of functions on a Poisson manifold and to description of traces on this algebra. These applications were suggested in Tsygan’s paper [30] (see the first part of corollary 4.0.3 and corollary 4.0.5) as immediate corollaries of the conjectural formality theorem (conjecture 3.3.1 in [30]).

Although theorem $\blacksquare$ implies the existence of the desired quasi-isomorphism in conjecture 3.3.1 in [30] we decided to give direct proofs of the first part of corollary 4.0.3 and corollary 4.0.5 in [30] without making use of the fact that quasi-isomorphisms of $L_\infty$-algebras and $L_\infty$-modules are invertible.

Let $M$ be a smooth manifold endowed with a Poisson structure $\alpha_1 \in T^1_{\text{poly}}(M)$ and $\Pi$ be a star-product, which quantizes $\alpha_1$ in the sense of deformation quantization [3, 4]. Let

$$
\alpha = \sum_{k=1}^{\infty} \hbar^k \alpha_k, \quad \alpha_k \in T^1_{\text{poly}}(M)
$$

represent Kontsevich’s class of the star-product $\Pi$.

Then we claim that$^7$

$^7$see the first part of corollary 4.0.3 in [30]
Corollary 1 The complex of Hochschild homology

\[(C_{\text{poly}}^\ast(M)[[\hbar]], R_{\Pi})\]  \hspace{1cm} (5.13)

is quasi-isomorphic to the complex of exterior forms

\[(\mathcal{A}^\ast(M)[[\hbar]], L_\alpha)\]  \hspace{1cm} (5.14)

with the differential \(L_\alpha\).

Proof. Due to associativity of \(\Pi\) the difference

\[\Psi = \Pi - \mu_0\]

of \(\Pi\) and the ordinary (commutative) multiplication \(\mu_0 \in C^\infty(M)\) is a Maurer-Cartan element in \(D_{\text{poly}}(M)[[\hbar]]\). The fact that \(\alpha\) \hspace{1cm} (5.12)\ represents Kontsevich’s class of the star-product \(\Pi\) means the Maurer-Cartan elements

\[\mathfrak{B} = \sum_{k=1}^\infty \frac{1}{k!} U_k(\alpha, \ldots, \alpha)\]  \hspace{1cm} (5.15)

obtained from \(\alpha\) via the quasi-isomorphism

\[\mathfrak{U} = \mathfrak{U} \circ \tau \circ \nu^{-1}(\Psi)\]

are equivalent. In other words, there exists an invertible element \(F\) in the prounipotent group \(\mathfrak{H}\) corresponding the Lie algebra

\[\mathfrak{h} = (\Omega^0(M, \mathcal{D}^0_{\text{poly}}) \oplus \Omega^1(M, \mathcal{D}^{-1}_{\text{poly}})) \otimes \hbar \mathbb{R}[[\hbar]]\]

such that

\[\mathfrak{F}^{-1} \mathfrak{F} = \mathfrak{B}^\circ,\]  \hspace{1cm} (5.16)

where \(\mathfrak{B}^\circ = \tau \circ \nu^{-1}(\Psi)\).

Twisting the terms in the second diagram in \hspace{5cm} (5.1) via the Maurer-Cartan element \(\Psi\) we get the following commutative diagram

\[\Downarrow^R_{\text{mod}}\]

Thus \(\varphi\) gives a quasi-isomorphism of complexes \((C_{\text{poly}}^\ast(M)[[\hbar]], R_{\Pi})\) and \((\Omega(M, C_{\text{poly}}^\ast)[[\hbar]], D + b + R_{\mathfrak{B}}\) \hspace{2cm} (5.11)\). But the latter complex is quasi-isomorphic to \((\Omega(M, C_{\text{poly}}^\ast)[[\hbar]], D + b + R_{\mathfrak{B}})\) since \(\mathfrak{B}\) is obtained from \(\mathfrak{B}^\circ\) via conjugation \hspace{5cm} (5.16).

On the other hand twisting the terms in the left part of diagram \hspace{5cm} (5.11)\ by the Maurer-Cartan element \(\alpha \in T_{\text{poly}}(M)[[\hbar]]\) \hspace{1cm} (5.12)\ we get the following commutative diagram

\[\Downarrow^L_{\text{mod}}\]

\[\Downarrow^L_{\text{mod}}\]

\[\Downarrow^R_{\text{mod}}\]

\[\Downarrow^R_{\text{mod}}\]

\[\Downarrow^R_{\text{mod}}\]

Thus \(\varphi\) gives a quasi-isomorphism of complexes \((C_{\text{poly}}^\ast(M)[[\hbar]], R_{\Pi})\) and \((\Omega(M, C_{\text{poly}}^\ast)[[\hbar]], D + b + R_{\mathfrak{B}}\) \hspace{2cm} (5.11)\). But the latter complex is quasi-isomorphic to \((\Omega(M, C_{\text{poly}}^\ast)[[\hbar]], D + b + R_{\mathfrak{B}})\) since \(\mathfrak{B}\) is obtained from \(\mathfrak{B}^\circ\) via conjugation \hspace{5cm} (5.16).

On the other hand twisting the terms in the left part of diagram \hspace{5cm} (5.11)\ by the Maurer-Cartan element \(\alpha \in T_{\text{poly}}(M)[[\hbar]]\) \hspace{1cm} (5.12)\ we get the following commutative diagram

\[\Downarrow^L_{\text{mod}}\]

\[\Downarrow^L_{\text{mod}}\]

\[\Downarrow^R_{\text{mod}}\]

\[\Downarrow^R_{\text{mod}}\]

Thus \(\varphi\) gives a quasi-isomorphism of complexes \((C_{\text{poly}}^\ast(M)[[\hbar]], R_{\Pi})\) and \((\Omega(M, C_{\text{poly}}^\ast)[[\hbar]], D + b + R_{\mathfrak{B}}\) \hspace{2cm} (5.11)\). But the latter complex is quasi-isomorphic to \((\Omega(M, C_{\text{poly}}^\ast)[[\hbar]], D + b + R_{\mathfrak{B}})\) since \(\mathfrak{B}\) is obtained from \(\mathfrak{B}^\circ\) via conjugation \hspace{5cm} (5.16).

On the other hand twisting the terms in the left part of diagram \hspace{5cm} (5.11)\ by the Maurer-Cartan element \(\alpha \in T_{\text{poly}}(M)[[\hbar]]\) \hspace{1cm} (5.12)\ we get the following commutative diagram

\[\Downarrow^L_{\text{mod}}\]

\[\Downarrow^L_{\text{mod}}\]

\[\Downarrow^R_{\text{mod}}\]

\[\Downarrow^R_{\text{mod}}\]

Thus \(\varphi\) gives a quasi-isomorphism of complexes \((C_{\text{poly}}^\ast(M)[[\hbar]], R_{\Pi})\) and \((\Omega(M, C_{\text{poly}}^\ast)[[\hbar]], D + b + R_{\mathfrak{B}}\) \hspace{2cm} (5.11)\). But the latter complex is quasi-isomorphic to \((\Omega(M, C_{\text{poly}}^\ast)[[\hbar]], D + b + R_{\mathfrak{B}})\) since \(\mathfrak{B}\) is obtained from \(\mathfrak{B}^\circ\) via conjugation \hspace{5cm} (5.16).
where $\mathfrak{L}_h$ is the DGLA $(\Omega(M, D_{poly})[[h]], D + \partial + [\mathfrak{B}, , ]_G, [ , ]_G)$, $\mathfrak{M}_h$ is the DGLA module $(\Omega(M, C^{poly})[[h]], D + b + R_3)$, and $U^\alpha$ and $\mathfrak{R}^\alpha$ are the quasi-isomorphisms twisted by the Maurer-Cartan element $\alpha$.

Thus the zero-th structure map of $\mathfrak{R}^\alpha$ gives the quasi-isomorphism from the complex $(\Omega(M, C^{poly})[[h]], D + b + R_3)$ to the complex $(\Omega(M, E)[[h]], D + L_3)$ while $\tau$ provides us with the quasi-isomorphism from $\mathcal{A}^*(M)[[h]], L_0)$ to $(\Omega(M, E)[[h]], D + L_3)$ and the desired statement follows. $\Box$

Another application of theorem 1 is related to description of traces on the algebra $(C_c^\infty(M)[[h]], \Pi)$, where by $C_c^\infty(M)$ we denote the vector space of smooth functions with a compact support.

By definition trace is a continuous $\mathbb{R}[[h]]$-linear $\mathbb{R}[[h]]$-valued$^8$ functional $tr$ on $C_c^\infty(M)[[h]]$ vanishing on commutators

$$tr(\Pi(a) - \Pi(\lambda(a))) = 0,$$

where $a = a(x_0, x_1, h)$ is a function in $C^\infty(M \times M)$ with a compact support in its first argument and $\lambda$ denotes permutation of arguments $\lambda(a)(x_0, x_1) = a(x_1, x_0)$.

One can easily verify that our constructions still make sense if we replace the first version $^{39}$ of $C^{poly}(M)$ by

$$C^{poly-com}(M) = \bigoplus_{n \geq 0} C^\infty_{com}(M^{n+1}),$$

and the vector space of exterior forms $\mathcal{A}^*(M)$ by the vector space $\mathcal{A}^*(M)$ of exterior forms with a compact support. Here by $C^\infty_{com}(M^{n+1})$ we denote the vector space of smooth functions on $M^{n+1}$ with a compact support in the first argument.

Then the corresponding version of the above corollary implies that

**Corollary 2** (\cite{30}, Corollary 4.0.5) *The vector space of traces on the algebra $(C_c^\infty(M)[[h]], \Pi)$ is isomorphic to the vector space of continuous $\mathbb{R}[[h]]$-linear $\mathbb{R}[[h]]$-valued functionals on $C_c^\infty(M)[[h]]$ vanishing on all Poisson brackets $\alpha(a, b)$ for $a, b \in C_c^\infty(M)[[h]]*."

For a symplectic manifold this statement has been proved in \cite{9}, \cite{12}, and \cite{21}.

### 6 Concluding remarks.

We would like to mention that a natural algebraic geometric version of theorem \cite{1} holds for a smooth affine algebraic variety $X$ (over a $\mathbb{C}$). To formulate the theorem one has to replace $D_{poly}(X)$ by the DGLA of algebraic polydifferential operators, $C^{poly}(X)$ by the DGLA module of algebraic differential operators on products of $X, T_{poly}(X)$ by the Lie algebra of algebraic polyvector fields, and $\mathcal{A}^*(X)$ by the module of algebraic differential forms. This version of the theorem immediately follows from the fact that any smooth affine algebraic variety admits an algebraic connection on a tangent bundle to $X$.

It is perhaps worth examining whether the quasi-isomorphism $\mathfrak{R}$ is compatible in some sense with the cup product on $T_{poly}(M)$. More generally, we hope that the technique developed in this paper will allow us to prove the formality of the much richer algebraic structure

---

$^8$The result about traces will still hold if one replaces real valued functions (resp. traces) by smooth complex valued functions (resp. complex valued traces), as well as the ring $\mathbb{R}[[h]]$ by the field $\mathbb{C}[[h, h^{-1}]]$.  

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on the space of Hochschild (co)chains of $C^\infty(M)$, the existence of which was announced in \cite{27}.

Note also that theorem 1 admits an equivariant version similar to the one proposed in \cite{10} for $D_{\text{poly}}(M)$. We are going to discuss this theorem and its applications in a separate paper.

Finally, we mention that it would be interesting to derive the formality quasi-isomorphism of $C^\text{poly}(M)$ for a general manifold along the lines of the path integral approach \cite{2}, \cite{6}.

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