EXISTENCE OF A CLASS OF ROTOPULSATORS

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Abstract. We prove the existence of a class of rotopulsators for the n-body problem in spaces of constant curvature of dimension \( k \geq 2 \).

1. Introduction

By \( n \)-body problems, we mean problems where we want to find the dynamics of \( n \) point particles. If the space in which such a problem is defined is a space of zero curvature, then we call any solution to such a problem for which the point particles describe the vertices of a polytope that retains its shape over time (but not necessarily its size) a homographic orbit.

A rotopulsator, also known as a rotopulsating orbit, is a type of solution to an \( n \)-body problem for spaces of constant curvature \( \kappa \neq 0 \) that extends the definition of homographic orbits to spaces of constant curvature (see [7]).

Homographic orbits (and therefore rotopulsators) can be used to determine the geometry of the universe locally (see for example [4], [7]).

In this paper, we will prove the existence of a subclass of rotopulsators that form a natural generalisation of orbits found in [4] and [5].

While this paper mainly builds on results obtained in [4], [5] and [22], research on \( n \)-body problems for spaces of constant curvature goes back to Bolyai [1] and Lobachevsky [19], who independently proposed a curved 2-body problem in hyperbolic space \( \mathbb{H}^3 \) in the 1830s. In later years, \( n \)-body problems for spaces of constant curvature have been studied by mathematicians such as Dirichlet, Schering [20], Killing [12], [13], [14] and Liebmann [16], [17], [18]. More recent results were obtained by Kozlov, Harin [15], but the study of \( n \)-body problems in spaces of constant curvature for the case that \( n \geq 2 \) started with [9], [10], [11] by Diacu, Pérez-Chavela, Santoprete. Further results for the \( n \geq 2 \) case were then obtained by Cariñena, Rañada, Santander [2], Diacu [3], [4], [5], Diacu, Kordlou [7], Diacu, Pérez-Chavela [8]. For a more detailed historical overview, please see [4], [5], [6], [7], or [9].

In this paper, we will prove the following two theorems:

**Theorem 1.1.** For any rotopulsating solution of (2.2) formed by vectors \( \{q_i\}_{i=1}^{n} \) as defined in (2.3), the vectors \( \{Q_i\}_{i=1}^{n} \) have to form a regular polygon if \( \rho \) is non-constant.

**Theorem 1.2.** Rotopulsating orbits formed by vectors \( \{q_i\}_{i=1}^{n} \) as defined in (2.3) exist if the vectors \( \{Q_i\}_{i=1}^{n} \) form a regular polygon.

To prove these theorems, we will use a method strongly inspired by [4], [5] and [22]. Specifically, we will first deduce a necessary and sufficient criterion for the existence of rotopulsators. This will be done in section 2. We will then prove Theorem 1.1 and Theorem 1.2 in section 3 and section 4 respectively.
2. A criterion for the existence of rotopulsators

In this section, we will formulate a necessary and sufficient criterion for the existence of rotopulsating orbits of the type described in (2.3).

Consider the \( n \)-body problem in spaces of constant curvature \( \kappa \neq 0 \).

As has been shown in [6], we may assume that \( \kappa \) equals either \(-1\), or \(1\).

We will denote the masses of its \( n \) point particles to be \( m_1, m_2, ..., m_n > 0 \) and their positions by the \( k \)-dimensional vectors

\[
\mathbf{q}_i^T = (q_{i1}, q_{i2}, ..., q_{ik}) \in \mathbb{M}_{n-1}^k, \quad i = 1, n
\]

where

\[
\mathbb{M}_{n-1}^k = \{(x_1, x_2, ..., x_k) \in \mathbb{R}^k | \kappa(x_1^2 + x_2^2 + ... + x_{k-1}^2 + \sigma x_k^2) = 1\}, \quad k \in \mathbb{N}
\]

and

\[
\sigma = \begin{cases} 1 & \text{for } \kappa > 0 \\ -1 & \text{for } \kappa < 0 \end{cases}
\]

Furthermore, consider for \( m \)-dimensional vectors \( \mathbf{a} = (a_1, a_2, ..., a_m) \), \( \mathbf{b} = (b_1, b_2, ..., b_m) \) the inner product

\[
\mathbf{a} \circ \mathbf{b} = a_1b_1 + a_2b_2 + ... + a_{m-1}b_{m-1} + \sigma a_mb_m.
\]

Then, following [3, 4, 5, 9, 10, 11] and the assumption that \( \kappa = \pm 1 \) from [6], we define the equations of motion for the curved \( n \)-body problem as the dynamical system described by

\[
\ddot{\mathbf{q}}_i = \sum_{j=1, j \neq i}^{n} \frac{m_j[\mathbf{q}_j - (\sigma \mathbf{q}_i \circ \mathbf{q}_j)\mathbf{q}_i]}{[\sigma - (\mathbf{q}_i \circ \mathbf{q}_j)^2]^\frac{1}{2}} - (\sigma \mathbf{q}_i \circ \mathbf{q}_i)\mathbf{q}_i, \quad i = 1, n.
\]

Let

\[
T(t) = \begin{pmatrix} \cos(\theta(t)) & -\sin(\theta(t)) \\ \sin(\theta(t)) & \cos(\theta(t)) \end{pmatrix}
\]

be a \( 2 \times 2 \) rotation matrix, where \( \theta(t) \) is some real valued, twice continuously differentiable, scalar function, for which \( \theta(0) = 0 \).

Furthermore, let \( \rho(t) \) be a nonnegative, twice continuously differentiable, scalar function.

We will consider rotopulsating orbit solutions of (2.3) of the form

\[
\mathbf{q}_i(t) = \begin{pmatrix} \rho(t)T(t)\mathbf{Q}_i \\ Z(t) \end{pmatrix}
\]

where \( \mathbf{Q}_i \in \mathbb{R}^2 \) is a constant vector and \( Z(t) \in \mathbb{R}^{k-2} \) is a twice differentiable, vector valued function.

Finally, before formulating our criterion, we need to introduce some notation and a lemma:

Let \( m \in \mathbb{N} \). Let \( \langle \cdot, \cdot \rangle_m \) be the Euclidean inner product on \( \mathbb{R}^m \) and let \( \| \cdot \|_m \) be the Euclidean norm on \( \mathbb{R}^m \). Let \( i, j \in \{1, ..., n\} \). By construction \( \|\mathbf{Q}_i\|_2 = \|\mathbf{Q}_j\|_2 \) for all \( i, j \in \{1, ..., n\} \) and we will assume that \( \|\mathbf{Q}_i\|_2 = 1 \). Let \( \beta_i \) be the angle between \( \mathbf{Q}_i \) and the first coordinate axis. The lemma we will need to prove our criterion is:

**Lemma 2.1.** The functions \( \rho \) and \( \theta \), are related through the following formula:

\[
\rho^2(t)\dot{\theta}(t) = \rho^2(0)\ddot{\theta}(0).
\]
Proof. In [5], using the wedge product, Diacu proved that
\[ \sum_{i=1}^{n} m_i \dot{q}_i \wedge q_i = c \]
where \( c \) is a constant bivector.
If \( \{e_i\}_{i=1}^{k} \) are the standard base vectors in \( \mathbb{R}^k \), then we can write \( c \) as
\[ c = \sum_{i=1}^{k} \sum_{j=1}^{k} c_{ij} e_i \wedge e_j. \]
(2.4)

where \( \{c_{ij}\}_{i=1,j=1}^{k} \) are constants. As \( e_i \wedge e_j = -e_j \wedge e_i \) and \( e_i \wedge e_i = 0 \) (see [5]), for \( i, j \in \{1, ..., n\} \), we can rewrite (2.4) as
\[ c = \sum_{i=1}^{k} \sum_{j=i+1}^{k} C_{ij} e_i \wedge e_j \]
(2.5)

where \( C_{ij} = c_{ij} - c_{ji} \).
Calculating \( C_{12} \), will give us our result:
Note that
\[ T^T = T^{-1} \text{ and } \dot{T} = \dot{\theta} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} T \]
(2.6)

and
\[ C_{12} = \sum_{i=1}^{n} m_i (q_{i1} \dot{q}_{i2} - q_{i2} \dot{q}_{i1}) \]
(2.7)

\[ = \sum_{i=1}^{n} m_i (q_{i1}, q_{i2}) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \dot{q}_{i1} \\ \dot{q}_{i2} \end{pmatrix}. \]

Using (2.3) with (2.7) gives
\[ C_{12} = \sum_{i=1}^{n} m_i \rho^2 (Q_{i1}, Q_{i2}) T^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \dot{T} \begin{pmatrix} Q_{i1} \\ Q_{i2} \end{pmatrix} \]
(2.8)

\[ + \sum_{i=1}^{n} m_i \dot{\rho} (Q_{i1}, Q_{i2}) T^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} T \begin{pmatrix} Q_{i1} \\ Q_{i2} \end{pmatrix}. \]

Note that
\[ \dot{\rho} (Q_{i1}, Q_{i2}) T^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} T \begin{pmatrix} q_{i1} \\ q_{i2} \end{pmatrix} \]
\[ = \frac{\dot{\rho}}{\rho} (q_{i1}, q_{i2}) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} q_{i1} \\ q_{i2} \end{pmatrix} = 0. \]
So, using (2.6) repeatedly, we get that
\[ C_{12} = \sum_{i=1}^{n} m_i \rho^2 \dot{\theta} (Q_{i1}, Q_{i2}) T^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} T \begin{pmatrix} Q_{i1} \\ Q_{i2} \end{pmatrix} + 0 \]
\[ = \sum_{i=1}^{n} m_i \rho^2 \dot{\theta} (Q_{i1}, Q_{i2}) T^T (Q_{i1}, Q_{i2}) \]
\[ = \sum_{i=1}^{n} m_i \rho^2 \dot{\theta} (Q_{i1}, Q_{i2}) \left( Q_{i1}^2 + Q_{i2}^2 \right) + 0 \]
which means that
\[ C_{12} = \rho^2 \dot{\theta} \sum_{i=1}^{n} m_i \left( Q_{i1}^2 + Q_{i2}^2 \right). \]

As, by construction
\[ \sum_{i=1}^{n} m_i \left( Q_{i1}^2 + Q_{i2}^2 \right) > 0, \]
we may divide both sides of (2.9) by
\[ \sum_{i=1}^{n} m_i \left( Q_{i1}^2 + Q_{i2}^2 \right), \]
which gives that
\[ \rho^2 \dot{\theta} = \frac{C_{12}}{\sum_{i=1}^{n} m_i \left( Q_{i1}^2 + Q_{i2}^2 \right)}, \]
which is constant, so \( \rho^2 \dot{\theta} = \rho^2(0)\dot{\theta}(0). \) □

We now have the following necessary and sufficient criterion for the existence of a rotopulsating orbit, as described in (2.3):

**Criterion 1.** Let
\[ b_i = \sum_{j=1, j \neq i}^{n} \frac{m_j (1 - \cos (\beta_i - \beta_j))^{- \frac{1}{2}}}{2 - \sigma \rho^2 (1 - \cos (\beta_i - \beta_j))}, \]
Then necessary and sufficient conditions for the existence of a rotopulsating orbit of non-constant size are that \( b_1 = b_2 = ... = b_n \) and
\[ 0 = \sum_{j=1, j \neq i}^{n} \frac{m_j \sin (\beta_i - \beta_j)}{(1 - \cos (\beta_i - \beta_j))^{\frac{1}{2}} (2 - \sigma \rho^2 (1 - \cos (\beta_i - \beta_j)))^{\frac{1}{2}}} \]
for all \( i \in \{1, ..., n\}. \)

*Proof.* Note that
\[ \hat{T} = \dot{\theta} T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \]
and consequently
\[ \hat{T} = \dot{\theta} T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \dot{\theta}^2 T. \]
Inserting (2.3) into (2.2) and using (2.12) and (2.13) gives for the first and second lines of (2.2) that

\[
T \left( \dot{Q}_2 + 2\dot{\rho} \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) + \rho \left( \tilde{\rho} \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) - \rho^2 I_2 \right) \right) Q_i
\]

(2.14) 

\[
= \rho T \left( \sum_{j=1, j \neq i}^n m_{ij} \left[ \frac{Q_j - \sigma q_i \circ_k q_j}{[\sigma - (q_i \circ_k q_j)^2]^{\frac{1}{2}}} \right] - (\sigma \dot{q}_i \circ_k \dot{q}_i) Q_i \right)
\]

where \( I_2 \) is the \( 2 \times 2 \) identity matrix.

For the last \( k - 2 \) lines, we get

\[
\dot{Z} = \left( \sum_{j=1, j \neq i}^n m_{ij} \left[ \frac{1 - (\sigma q_i \circ_k q_j)}{[\sigma - (q_i \circ_k q_j)^2]^{\frac{1}{2}}} \right] - (\sigma \dot{q}_i \circ_k \dot{q}_i) \right) Z.
\]

(2.15)

Note that

\[
q_i \circ_k q_j = \rho^2 (Q_i, Q_j)_2 + Z \circ_{k-2} Z.
\]

(2.16)

As we have that \( (Q_i, Q_j)_2 = 1 \) and as by (2.16),

\[
\sigma^{-1} = q_i \circ_k q_i = \rho^2 (Q_i, Q_i)_2 + Z \circ_{k-2} Z,
\]

we may rewrite (2.16) as

\[
q_i \circ_k q_j = \sigma^{-1} + \rho^2 (Q_i, Q_j)_2 - \rho^2,
\]

which can, in turn, be written as

\[
q_i \circ_k q_j = \sigma^{-1} + \rho^2 (\cos (\beta_i - \beta_j) - 1).
\]

(2.17)

Furthermore,

\[
\dot{q}_i \circ_k \dot{q}_i = \langle \rho T Q_i, \rho T Q_i, \rho T Q_i, \rho T Q_i, \rho T Q_i, \rho T Q_i \rangle_2 + \dot{Z} \circ_{k-2} \dot{Z}.
\]

(2.18)

As \( T \) is a rotation in \( \mathbb{R}^2 \), it is a unitary map, meaning that for \( v, w \in \mathbb{R}^2 \),

\[
\langle T v, T w \rangle_2 = (v, w)_2,
\]

meaning that (2.18) can be written as

\[
\dot{q}_i \circ_k \dot{q}_i = \langle \rho T Q_i, \rho T^{-1} T Q_i, \rho T Q_i, \rho T^{-1} T Q_i, \rho T^{-1} T Q_i, \rho T^{-1} T Q_i \rangle_2 + \dot{Z} \circ_{k-2} \dot{Z}.
\]

(2.19)

Using (2.12) with (2.19) gives

\[
\dot{q}_i \circ_k \dot{q}_i = \rho^2 + 2\rho \dot{\rho} \left( Q_i, \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) Q_i \right)_2 + \rho^2 \dot{\rho}^2 |Q_i|^2 + \dot{Z} \circ_{k-2} \dot{Z}
\]

(2.20)

\[
= \rho^2 + 0 + \rho^2 \dot{\rho}^2 + \dot{Z} \circ_{k-2} \dot{Z}.
\]

(2.21)

Inserting (2.20) and (2.17) into (2.14) and multiplying both sides by \( T^{-1} \) provides us with

\[
\left( (\dot{\rho} - \rho \dot{\rho}) I_2 + (2\dot{\rho} + \rho \ddot{\rho}) \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \right) Q_i
\]

(2.21) 

\[
= \rho \sum_{j=1, j \neq i}^n m_{ij} \left[ \frac{Q_j - (1 - \sigma \rho^2 (1 - \cos (\beta_i - \beta_j))) Q_i}{[\rho^2 (1 - \cos (\beta_i - \beta_j)) (2 - \sigma \rho^2 (1 - \cos (\beta_i - \beta_j)))]^{\frac{1}{2}}} \right]
\]

\[
- (\sigma \rho \dot{\rho}^2 + \sigma \rho^3 \dot{\rho}^2 + \sigma \rho \dot{Z} \circ_{k-2} \dot{Z}) Q_i.
\]
Taking the Euclidean inner product with $Q_i$ on both sides of (2.21) and using that $\|Q_i\|_2 = \|Q_j\|_2 = 1$, provides us with

$$\ddot{\rho} - \rho \dot{\theta}^2 + \sigma \rho^2 \dot{\theta}^2 + \sigma \rho \dot{\theta} \circ_{k-2} \dot{Z} = \left(\sigma - \frac{1}{\rho^2}\right) \sum_{j=1,j \neq i}^n \frac{m_j [(1 - \cos (\beta_i - \beta_j))^{-\frac{1}{2}}]}{\sqrt{(2 - \sigma \rho^2 (1 - \cos (\beta_i - \beta_j)))^2}}.$$ 

(2.22)

Taking the Euclidean inner product of (2.21) with $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Q_i$ and using that $\|Q_i\|_2 = \|Q_j\|_2 = 1$ gives that

$$2 \dot{\rho} \dot{\theta} + \rho \ddot{\theta} = \sum_{j=1,j \neq i}^n \frac{m_j \sin (\beta_i - \beta_j)}{\sqrt{(1 - \cos (\beta_i - \beta_j))(2 - \sigma \rho^2 (1 - \cos (\beta_i - \beta_j)))^2}}.$$ 

(2.23)

Let

$$b_i := \sum_{j=1,j \neq i}^n \frac{m_j [(1 - \cos (\beta_i - \beta_j))^{-\frac{1}{2}}]}{\sqrt{(2 - \sigma \rho^2 (1 - \cos (\beta_i - \beta_j)))^2}}$$

and

$$c_i := \sum_{j=1,j \neq i}^n \frac{-m_j \sin (\beta_i - \beta_j)}{\sqrt{(1 - \cos (\beta_i - \beta_j))(2 - \sigma \rho^2 (1 - \cos (\beta_i - \beta_j)))^2}}.$$ 

Inserting (2.20) and (2.17) into (2.15), combined with (2.22) and (2.23) gives the following system of differential equations:

$$\begin{cases}
\ddot{\rho} = \rho \dot{\theta}^2 - \sigma \rho^2 \dot{\theta}^2 - \sigma \rho \dot{\theta} \circ_{k-2} \dot{Z} + \left(\sigma - \frac{1}{\rho^2}\right) b_i \\
\dot{\theta} = \frac{\alpha_i}{\rho} - 2 \dot{\rho} \dot{\theta} \\
\dot{Z} = \left(b_i - \sigma \rho^2 \dot{\theta}^2 - \sigma \dot{\theta} \circ_{k-2} \dot{Z}\right) Z
\end{cases}$$

(2.24)

For (2.24) to make sense, we need that

$$b_1 = ... = b_n \text{ and } c_1 = ... = c_n$$

(2.25)

which shows the necessity of (2.24).

Furthermore, that (2.24) has a global solution holds by the same argument as the argument used in the proof of Criterion 1 in [4] to prove global existence of a solution of (15) and (17). By the uniqueness of solutions to ordinary differential equations given suitable initial conditions, the solution to (2.24) must be a rotopulsating orbit, as every step from (2.14) and (2.15) to (2.24) is invertible.

Thus (2.25) is both necessary and sufficient. Finally, as by Lemma 2.1 $\rho^2 \dot{\theta} = \rho^2 (0) \dot{\theta}(0)$, we have that $\frac{d}{dt}(\rho^2 \dot{\theta}) = 0$, which means that the left hand side of (2.23) equals zero, which means that $c_i = 0$. This completes the proof.

□

3. PROOF OF THEOREM 1.1

In Criterion 1 let $r := \rho$, $\alpha_i := \beta_i$, $b_i := b_i$ and $\gamma_i := c_i$. Then the conditions of Criterion 1 become exactly the conditions of Criterion 1 in [4] with the added bonus that $\gamma_i = 0$. The proof of Theorem 1.1 in [22] is therefore a proof for Theorem 1.1 as well.
4. Proof of Theorem 1.2

Let again \( r := \rho \), \( \alpha_i := \beta_i \), \( \delta_i := b_i \) and \( \gamma_i := c_i \) in Criterion 1. Then the conditions of Criterion 1 become exactly the conditions of Criterion 1 in [4] with the added bonus that \( \gamma_i = 0 \). Theorem 1.2 now follows directly from the proofs of Theorem 1 and Theorem 2 in [4].

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