The Complete Structure of the Cohomology Ring and Associated Symmetries in $D = 2$ String Theory

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ABSTRACT

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We determine explicitly all structure constants of the whole chiral BRST cohomology ring in $D = 2$ string theory including both the discrete states and tachyon states. This is made possible by establishing several identities for Schur polynomials with operator argument and exploring associativity. Furthermore we find that the (chiral) symmetry algebra of the charges obtained by using the descent equations can actually be read off from the cohomology ring structure by simple operation involving the ghost field $b$. We also determine the enlarged symmetry algebra which contains the charges having ghost number $-1$ and $1$. Finally the complete symmetry transformation rules are derived for closed string discrete states by carefully combining the left and right sectors. It turns out that the new states introduced recently by Witten and Zwiebach are naturally created when symmetries act on the old states.
1. Introduction

In the last three years or so we have witnessed a spectacular success in exactly solving certain $D \leq 2$ string theories by exploring first matrix models [1, 2, 3], and then integrable models [4] and topological field theory [5]. However, many of the results obtained in these approaches, though explicit and exact, can hardly be given a transparent physical interpretation. A consensus of opinion in this field has been that by understanding these results in the more conventional Liouville theory approach [6], we may be able to gain better understanding of the underlying physics, and to develop new methods and derive new results. Recent achievements in this regard include: 1) exact computation of some correlation functions in Liouville theory and explicit verification of the agreement with matrix models [7, 8, 9, 10]; 2) better understanding of the physical states and their symmetries in the Liouville theory for the $D \leq 2$ string theories (see below for references). In this paper we will report the progress we have made in the second direction.

Among the known soluble string theories the $D = 2$ model is interesting in many aspects: It has a simple two-dimensional space-time interpretation, incorporates the interactions between gravity and a massless scalar “tachyon” field, and embraces intriguing space-time physics such as the formation and evaporation of black holes. A by-now-well-known feature of the $D = 2$ string theory is the existence of the unusual “discrete states”, in addition to the familiar tachyons. Some discrete states were known [11] for some time. Recently they became popular because of their appearance first in the matrix-model calculations of Gross, Klebanov, and Newman [12] and then in the Liouville-theory analysis by Polyakov [13, 14]. The subsequent rigorous BRST analysis [15, 16, 17] (see also [18, 19]) has indicated a more complex pattern of the discrete states than one had expected. Besides the states of conventional ghost number one, there are also physical states of ghost number zero and two, appearing legitimately as non-trivial BRST cohomology classes. The study of the discrete states and the symmetries associated to them in $D = 2$ string theory could be a good starting point for discovering the fully fledged stringy symmetries and formulating a background independent string
theory. Moreover such a study, combined with relevant Ward identities, could lead
to systematic evaluation of correlation functions in string theory in the Liouville
theory approach.

It was Witten [20] who first pointed out that the ghost number zero states
generate a “ground ring”, which is characteristic of $D = 2$ string theory. By sim-
ple considerations involving the ground ring, he was able to explain many aspects
of the free fermion description that comes from the matrix model (see also [21]).
Further developments along this line of thought have been made in [22], where
a careful treatment of closed string BRST cohomology is given and a systematic
construction of symmetry charges from the discrete states via the “descent equa-
tions” is proposed. One of the main points in these works is that many of the
structures they found (at the $SU(2)$ radius) can be naturally described in terms
of the differential geometry of a certain three-dimensional cone, on which the ring of
functions is isomorphic to the ground ring of the ghost-number-zero states.

In this paper we want to study the same problems, the discrete states and
associated symmetries in $D = 2$ string theory, from somewhat different point of
view, purely algebraic and more explicit. We are motivated by the observation
that all the discrete states form a (graded) ring [22] in the following sense in terms
of BRST cohomology [20]. Let $V_1(z)$ and $V_2(z)$ be two (chiral) BRST invariant
operators. The operator product $V_1(z)V_2(w)$ is also BRST invariant, so all the
terms in its short distance expansion for $z \to w$ are BRST invariant. Negative
powers of $z - w$ may arise in this short distance expansion, but the operators
multiplying the negative powers of $z - w$ are of negative dimension and thus must be
BRST commutators (as there is no nontrivial BRST cohomology class at negative
dimensions). Therefore, modulo the BRST commutators, the short distance limit
of $V_1(z)V_2(w)$ is some BRST invariant operator $V_3(w)$ (which may be zero):

$$V_1(z)V_2(w) \sim V_3(w) + \{Q, \cdots \}. \quad (1)$$

This gives rise to the desired multiplication law, $V_1 \cdot V_2 = V_3$. This procedure
obviously defines an associative ring structure for the set of all BRST cohomology
classes, which is graded by ghost number. Clearly the ground ring is only a subring of this ring, which we will call the (chiral) BRST cohomology ring.

Instead of expressing the structure of the whole cohomology ring in terms of the ground (sub)ring in differential geometric language [20, 22], we want to directly attack the problem of explicitly calculating all the structure constants of the whole chiral BRST cohomology ring in \( D = 2 \) string theory, including both the discrete and tachyon states. This is possible, since an explicit representation for all discrete states is available in the literature [17], which involves Schur polynomials with operator argument. Starting from this representation and using several identities we have derived to rearrange Schur polynomials, we found that the BRST cohomology ring is computable and have obtained explicit results for the structure constants. We have been able to include the tachyon states in our computation and find that the product of two negative parity tachyon states can give rise to a discrete state of ghost number two, or \( P_{j,m} \) in the notation of [22]. This again shows the incompleteness of the physical states of standard ghost number one.

With the structure constants of the (chiral) BRST cohomology ring explicitly known, it is not too hard to derive the structure constants for the associated (chiral) symmetry charge algebra or the transformation rules for the discrete states. In doing so, the general procedure, proposed in ref. [22], of constructing conserved charges from the discrete states via descent equations is extremely powerful. Our results show that the (chiral) symmetry charge algebra can be read off from the cohomology ring structure by simple operation involving the ghost field \( b \). In combining the right and left (chiral) cohomology classes to obtain closed string BRST cohomology, we have to pay attention to the subtleties pointed out by Witten and Zwiebach [22], which allow the existence of more closed string discrete states and associated symmetries than what had been recognized before from naive combination of chiral BRST cohomologies. With these subtleties taken into account, we have determined the transformation rules in closed string theory. The rules obtained show that the new states introduced by Witten and Zwiebach [22] are naturally created when symmetry charges act on the old states. Since the coho-
ology ring structure implies the charge algebra but the converse is not true, we can say that the cohomology ring is more fundamental than the symmetry charge algebra.

This paper is organized as follows. In the next section we review the previous results about the classification of physical states in the $D = 2$ string theory at the $SU(2)$ point (with a vanishing cosmological constant). In particular we present an explicit representation of all the (chiral) discrete states in terms of BRST invariant local operators [20, 22]. A very useful rearrangement formula for $P_{j,m}$ is announced here and its proof will be deferred to sec. 3.

In sec. 3 we explicitly determine all structure constants of the chiral BRST cohomology ring. Part of the ring structure was already known [20, 23, 24]. The new results are derived by using several useful identities involving Schur polynomials with operator arguments, which are established by us and proved in appendix A, and by exploring associativity. In this section we will present some sample calculations and give the proof of the rearrangement formula for $P_{j,m}$. Our new results include the structure constants involving higher ghost number states, like $O_{j_1m_1}P_{j_2m_2}$ and $Y_{j_1m_1}^+P_{j_2m_2}$ in the notation of [22]. In particular, by explicit calculation we find $Y_{j_1m_1}^-Y_{j_2m_2}^-$ is zero.

The complete chiral cohomology ring involves also the tachyon states. The structure constants involving tachyon states are derived in sec. 4 and are compared with previous results [25] about the OPE and couplings of tachyons and discrete states in $D = 2$ string theory. Amazingly we find $T_p^{(-)}T_q^{(-)}$ is generally non-zero. Of particular interest is that this product gives rise to a term proportional to $P_{jm}$ in addition to the usual term $aY_{j+1,m}^-$. 

In sec. 5 we first explain how one can construct conserved charges from BRST invariant local operators. The procedure is just part of the general procedure of constructing conserved charges via descent equations (taking only one of the chiral sectors). Then we derive some general formulas of getting transformation rules from the ring multiplication laws. The complete enlarged (chiral) charge algebra
is explicitly computed.

In sec. 6 we derive the symmetry transformation rules for closed string discrete states by combining the left and right sectors and using the results of sec. 5. Here the relevant point is to show that the new closed string states introduced by Witten and Zwiebach [22] through the semi-relative cohomology are naturally created when symmetries act on the old states. The sample calculations presented in this section suggest some very simple relations involving the multiplication of the operator \((a + \bar{a})\). The proof of these relations and also some details about the calculations are presented in appendix C.

For the convenience of reader, this paper contains three appendices: Appendix A collects relevant formulas for Schur polynomials and presents the proof for our newly established identities which are useful in the text. Appendix B presents the complete chiral transformation rules and the chiral symmetry algebra by using the chiral BRST cohomology ring structure equations (sec. 3) and the general formulas derived in sec. 5. Appendix C gives some details for the calculations of the transformation rules in closed string theory and also the proof of two general results in this regard in sec. 6.
2. Representation of the Chiral Discrete States

Like in refs. [20, 22], we consider the $D = 2$ string theory at the $SU(2)$ radius with vanishing world-sheet cosmological constant. We will follow the notation of [20, 22] as much as possible. The basic fields appearing in the construction of the physical states are the world-sheet “matter” field $X$, the Liouville field $\phi$ and the ghost fields $b$ and $c$. At the $SU(2)$ radius and vanishing world-sheet cosmological constant, both the fields $X$ and $\phi$ are free fields, and the right and left movers are decoupled. Thus we may start with one (say, the right one) of the chiral sectors. The OPE’s for the basic fields are

\[
X(z)X(w) \sim -\ln(z - w),
\]

\[
\phi(z)\phi(w) \sim -\ln(z - w),
\]

\[
b(z)c(w) \sim \frac{1}{z - w}.
\]

The stress energy tensors are

\[
T_X = -\frac{1}{2}(\partial_z X)^2,
\]

\[
T_\phi = -\frac{1}{2}(\partial_z \phi)^2 + \sqrt{2}\partial_z^2 \phi,
\]

\[
T_{bc} = 2(\partial_z c)b + c\partial_z b.
\]

In the following we also use the light-cone fields $X^\pm = \frac{1}{\sqrt{2}}(X \pm i\phi)$. The combined matter-Liouville stress energy tensor is

\[
T = T_X + T_\phi = -\partial_z X^+ \partial_z X^- - i(\partial_z^2 X^+ - \partial_z^2 X^-).
\]

The central charge of $T$ is 26 and we have the following nilpotent BRST charge

\[
Q = \oint [dz] : c(z)(T(z) + \partial_z c(z)b(z)) :,
\]

where $[dz] \equiv \frac{dz}{2\pi i}$. Here and in what follows the normal ordering for the ghost fields
is such that the following is satisfied:

$$b(z)c(w) = \frac{1}{z-w} + :b(z)c(w):.$$  \hspace{1cm} (6)

According to the general principle of BRST quantization, the physical states are defined as $Q$-closed but not $Q$-exact states, i.e. $Q$-cohomology classes. By a one to one correspondence between states and fields, one can also define the states in terms of local field operators. We will use the local field operator representation exclusively in this paper so that a physical state is represented as a BRST invariant but not exact local operator.

Before presenting the explicit representation for all the discrete states, a technical remark is in order. It is easy to see that from the matter field $X(z)$ one can construct an $SU(2)$ Kac-Moody algebra:

$$J_{\pm}(z) = e^{\pm i\sqrt{2}X(z)}, \quad J_3(z) = \frac{i}{\sqrt{2}} \partial_z X(z),$$  \hspace{1cm} (7)

with the following OPE’s

$$J_+(z)J_-(w) \sim \frac{1}{(z-w)^2} + \frac{2}{z-w} J_3(w),$$

$$J_3(z)J_{\pm}(w) \sim \pm \frac{1}{z-w} J_{\pm}(w),$$

$$J_3(z)J_3(w) \sim \frac{1}{2} J_3(\frac{1}{z-w}^2).$$  \hspace{1cm} (8)

Of particular importance is the fact that the zero modes of $J_i$,

$$\hat{J}_{\pm,3} = \oint \left[ dz \right] J_{\pm,3}(z),$$

commutes with $Q$: $[Q, \hat{J}_{\pm,3}] = 0$. Thus all the physical states should form $SU(2)$ multiplets. In fact all the discrete physical states do fall into finite dimensional
multiplets. If there exists a physical state $V_{j,j}(w)$ such that

$$[\hat{J}_+, V_{j,j}(w)] = 0,$$
$$[\hat{J}_3, V_{j,j}(w)] = j V_{j,j}(w),$$

one can get the remaining states in the multiplet by using the $\hat{J}_-$ operator

$$V_{j,m}(w) = \left( \frac{(j+m)!}{(2j)!(j-m)!} \right)^{1/2} \left[ \hat{J}_-, [\hat{J}_-, \cdots, [\hat{J}_-, V_{j,j}(w)] \cdots] \right]_{j-m}^{j-m}$$

$$\equiv \left( \frac{(j+m)!}{(2j)!(j-m)!} \right)^{1/2} (\hat{J}_-)^{j-m} V_{j,j}(w),$$

where

$$\hat{J}_- V_{j,m-1} = \oint \, [dz] J_-(z) V_{j,m}(w).$$

After all these preliminaries we can now summarize the results of [15, 16, 17] in compact form. All the physical states are divided into two categories: relative physical states that are annihilated by $b_0$ and absolute physical states that are not annihilated by $b_0$. Apart from the tachyon states (see sec. 4), the other relative physical states have discrete momenta $(p_X, p_\phi)$ and are created by following local vertex operators:

$$O_{j,m} = \left( \frac{(j+m)!}{(2j)!(j-m)!} \right)^{1/2} (\hat{J}_-)^{j-m} O_{j,j},$$

$$O_{j,j} = \left( -S_{2j}(-iX^-) + \sum_{q=1}^{2j} S_{2j-q}(-iX^-) \frac{c \partial^{q-1} b}{(q-1)!} \right) e^{2ijX^+} ;,$$

$$Y_{j,m}^\pm = c : \left( \frac{(j+m)!}{(2j)!(j-m)!} \right)^{1/2} (\hat{J}_-)^{j-m} \left( c e^{i(jX+(1\mp j)\phi)} \right)^{\sqrt{2}} ;,$$

$$P_{j,m} = \left( \frac{(j+m)!}{(2j)!(j-m)!} \right)^{1/2} (\hat{J}_-)^{j-m} P_{j,j},$$

$$P_{j,j} = \sum_{q=1}^{2j+1} : S_{2j+1-q}(-iX^+ - iX^-) \frac{\partial^{q+1} c}{(q+1)!} e^{-2iX^+ + 2i(j+1)X^-} ;,$$

where $m = -j, -j + 1, \cdots, j$ and $j = 0, 1/2, 1, 3/2, \cdots$. In the above equation
$S_k(-iX^-)$ is a Schur polynomial of degree $k$ with the operator $-iX^-$ as its argument. We refer the reader to appendix A for our notation and relevant properties of the Schur polynomials. In Fig. 1 we show the $(p_X, p_\phi)$ momenta of these discrete states. According to the type of Liouville dressing, $O_{j,m}$ and $Y^+_{j,m}$ are called the “plus” states, and $Y^-_{j,m}$ and $P_{j,m}$ are called the “minus” states. Note that the states $O_{j,m}$, $Y^\pm_{j,m}$ and $P_{j,m}$ have ghost number 0, 1 and 2 respectively.

A remark is in order about the representation of $P_{j,j}$ (and all the $P_{j,m}$’s by $SU(2)$ action). According to [16], the non-triviality of $P_{j,j}$ could be proved by changing the prefactor to a representation in terms of only the light-cone variable $X^+$ by adding some BRST exact terms. We find that it is quite difficult to prove the non-triviality of $P_{j,j}$ with the representation given in (13). In the process of seeking a general procedure of handling this rearrangement problem we have found the following general representation for $P_{j,j}$:

$$P_{j,j} \equiv \sum_{q=1}^{2j+1} S_{2j+1-q} (-iX^+ - iX^-) \frac{\partial^{q+1} c}{(q+1)!} e^{-2iX^+ + 2i(j+1)X^-}$$

with $\delta$ an arbitrary real number. Later we will see the usefulness of this formula. It will be proved in the next section.

As for the absolute physical states, they can be obtained from the relative physical states by “multiplying” the following local operator [22, 24]

$$a = \frac{1}{\sqrt{2}} [Q, \phi] = \partial c + \frac{1}{\sqrt{2}} c \partial \phi,$$

(15)

In the usual sense $\phi$ is not a conformal field, but $a$ is. Obviously $a$ is BRST invariant and is not BRST exact in the usual space of conformal fields. This being
so, we can get absolute physical states from the relative ones by forming the local product with \( a \), i.e.

\[
(aV_{jm})(w) = \oint_w \frac{1}{z-w}a(z)V_{jm}(w).
\]

(16)

The local multiplication with \( a \) commutes with the \( SU(2) \) action. Thus, \((aV_{j,m})\),
or simply \( aV_{j,m} \) when there is no confusion, has the same \( SU(2) \) quantum numbers,
as well as the same \((p_X, p_\phi)\) momenta, as \( V_{j,m} \). This way of forming an absolute
physical state can be viewed as the multiplication law for the BRST invariant
operator \( a \) and a relative physical state in the ring. This makes it easy to derive the
ring multiplication involving absolute physical states from that of relative physical
operators, as exemplified in the following:

\[
O_{j_1m_1}O_{j_2m_2} = O_{j_1+j_2, m_1+m_2} \rightarrow \\
aO_{j_1m_1}O_{j_2m_2} = O_{j_1m_1}aO_{j_2m_2} = aO_{j_1+j_2, m_1+m_2}.
\]

(17)

Note that the multiplication of \( a \) with itself gives zero. So only the ring structure
of the relative physical states will be computed.
3. Cohomology Ring of the Discrete Chiral States

First let us summarize the known results on the explicit ring structure of the discrete chiral states:

1) Witten [20] derived the chiral ground ring as the ring of polynomial functions in

\[ x \equiv O_{1/2,1/2}, \quad y \equiv O_{1/2,-1/2}. \]

This implies that the operators \( O_{jm} \) are closed under multiplication:

\[
O_{j_1m_1}O_{j_2m_2} = \langle j_1m_1j_2m_2 \mid j_1 + j_2, m_1 + m_2 \rangle O_{j_1+j_2,m_1+m_2},
\]

up to normalization. We will prove this result by explicit calculation and find the rescaling of \( O_{jm} \) such that all the structure constants here become unity. (Another, perhaps more natural, normalization will be presented at the end of this section.)

2) Polyakov and Klebanov [13] derived the following OPE (\( Y_{jm}^{\pm} = c\psi_{jm}^{(\pm)} \)):

\[
\begin{align*}
\Psi_{j_1m_1}^{(+)}(z)\Psi_{j_2m_2}^{(+)}(w) & \sim \cdots + \frac{1}{z-w}(j_2m_1 - j_1m_2)\Psi_{j_1+j_2-1,m_1+m_2}^{(+)}(w), \\
\Psi_{j_1m_1}^{(-)}(z)\Psi_{j_2m_2}^{(-)}(w) & \sim \cdots + \frac{1}{z-w} \times 0, \\
\Psi_{j_1m_1}^{(+)}(z)\Psi_{j_2m_2}^{(-)}(w) & \sim \cdots + \frac{1}{z-w} \times 0, \quad j_1 \geq j_2 + 1, \\
\Psi_{j_1m_1}^{(-)}(z)\Psi_{j_2m_2}^{(-)}(w) & \sim \cdots - \frac{1}{z-w}(j_2m_1 - j_1m_2)\Psi_{j_2,-m_2}^{(-)}(w).
\end{align*}
\]

This gives a term \( aY_{j_2+(j_1-1),m_1+m_2}^{\pm} \) in the product \( Y_{j_1m_1}^{\pm}Y_{j_2m_2}^{\pm} \). However, this is only part of the multiplication law. As we will see later, in addition, there is another term proportional to \( P_{j_2-j_1,m_1+m_2} \) in the product of \( Y_{j_1m_1}^{+}Y_{j_2m_2}^{-} \). This is due to the higher pole terms in (19) which were not computed explicitly.

3) Li [24] obtained the following multiplication law:

\[
O_{j_1m_1}Y_{j_2m_2}^{+} = \frac{j_2}{j_1 + j_2} Y_{j_1+j_2,m_1+m_2}^{+} - \frac{1}{j_1 + j_2}(j_2m_1 - j_1m_2)\alpha O_{j_1+j_2-1,m_1+m_2}. \quad (20)
\]

It was obtained partly by explicit calculation and partly by using the fact that \( Y_{jm}^{+} \) acts on the ground ring as a differential operator [20]. According to our experience
the second term in (20) can be derived from the associativity of the ring. We will explore associativity in several places either to check our results or to derive the ring structure constants which are not explicitly calculable.

To begin with, let us first compute \( O_{j_1 m_1} O_{j_2 m_2} \). Counting the momenta and ghost number, we know that this product must be proportional to \( O_{j_1 + j_2, m_1 + m_2} \):

\[
O_{j_1 m_1} O_{j_2 m_2} = g(j_1, j_2) \langle j_1 m_1 j_2 m_2 \mid j_1 + j_2, m_1 + m_2 \rangle O_{j_1 + j_2, m_1 + m_2},
\]

where \( \langle j_1 m_1 j_2 m_2 \mid j_1 + j_2, m_1 + m_2 \rangle \) are the Clebsch-Gordan coefficients. To determine the unknown function \( g(j_1, j_2) \), one may consider some special values of \( m_1 \) and \( m_2 \) such that a direct calculation is possible. Let us take \( m_1 = j_1, m_2 = j_2 \) and \( j_1 = \frac{1}{2} \). The general case can be reached by induction. We have

\[
O_{1/2,1/2} O_{j,j}(z) = \oint \frac{[dz]}{z - w} O_{1/2,1/2}(z) O_{j,j}(w),
\]

with

\[
O_{1/2,1/2}(z) O_{j,j}(w) = (c(z) b(z) + i \partial_z X^-(z)) e^{iX^+(z)} \times (-S_{2j}(-iX^-(w))) + \sum_{q=1}^{2j} S_{2j-q}(-iX^-(w)) \frac{c(w) \partial_w^{q-1} b(w)}{(q - 1)!} e^{2ijX^+(w)}.
\]

\[
= - \left[ (c(z) b(z) + i \partial_z X^-(z) + 2j \frac{1}{z - w}) \sum_{k=0}^{2j} S_{2j-k}(-iX^-(w)) \frac{1}{(z - w)^k} \right.
\]

\[
+ \sum_{q=1}^{2j} \left[ (c(z) b(z) + i \partial_z X^-(z) + 2j \frac{1}{z - w}) \frac{c(w) \partial_w^{q-1} b(w)}{(q - 1)!} \right.
\]

\[
+ \frac{c(z) \partial_w^{q-1} b(w)}{(q - 1)!} \frac{1}{z - w} + b(z) c(w) \frac{1}{(z - w)^q} + \frac{1}{(z - w)^{q+1}} \right] \times \sum_{k=0}^{2j-q} S_{2j-q-k}(-iX^-(w)) \frac{1}{(z - w)^k} \right]
\]

\[
\times e^{iX^+(z) + 2ijX^+(w)}.
\]

Let us compute the various terms appearing in (23). First one can show that the

\[\text{§ Some useful formulas about the OPEs involving Schur polynomials are given in the appendix A.}\]
term containing four ghost fields is zero:

\[
\sum_{q=1}^{2j} c(z)b(z)c(w)\partial_w^{q-1}b(w) \sum_{k=0}^{2j-q} S_{2j-k}(-iX-(w)) \frac{1}{(z-w)^k} : \\
= \sum_{k=1}^{2j} S_{2j-k}(-iX-(w)) \frac{1}{(z-w)^{k-1}} c(z)b(z)c(w) \sum_{q=1}^{k} (z-w)^{q-1} \frac{\partial_w^{q-1}b(w)}{(q-1)!} : \\
= \sum_{k=1}^{2j} S_{2j-k}(-iX-(w)) \frac{1}{(z-w)^{k-1}} c(z)b(z)c(w)(b(z) + o((z-w)^k)) : \\
= o(z-w). \tag{24}
\]

The integration over \(z\) then gives vanishing contribution. For the terms containing no ghost field, we have

\[
\left[ -(i\partial_z X-(z) + 2j \frac{1}{z-w}) \sum_{k=0}^{2j} S_{2j-k}(-iX-(w)) \frac{1}{(z-w)^k} \\
+ \sum_{q=1}^{2j} \sum_{k=0}^{2j-q} S_{2j-q-k}(-iX-(w)) \frac{1}{(z-w)^{q+k-1}} \right] \\
= \sum_{k=0}^{2j} (k-2j) S_{2j-k}(-iX-(w)) \frac{1}{(z-w)^{k+1}} + \sum_{k=0}^{2j} \frac{1}{(z-w)^k} P(k) + o(z-w) \tag{25}
\]

where

\[
P(k) = \sum_{l=k}^{2j} S_{2j-l}(-iX-(w)) \left( -i \frac{\partial_w^{l-k+1} X-(w)}{(l-k)!} \right). \tag{26}
\]

By using the last equation in appendix A we find that all the singular terms in (25) are zero and the regular term \(P(0)\) is \((2j+1) S_{2j+1}(-iX-(w))\). A similar but a bit more tedious calculation of the remaining terms (i.e. those containing two

\* The exponential factor \(e^{iX^+(z)+2jX^+(w)}\) is suppressed.
ghost fields) gives

\[-(2j + 1) \sum_{q=1}^{2j+1} : S_{2j+1-q}(-iX^-(w)) \frac{c(w) \partial_q b^{-1}(w)}{(q - 1)!} e^{iX^+(z) + 2i j X^+(w)} : + o(z - w).\]  

It is remarkable that all singular terms in the OPE of $O_{1/2,1/2}(z)$ and $O_{j,j}(w)$ are zero exactly without any BRST exact terms.

Combining the above results, we have

\[O_{1/2,1/2} O_{j,j} = -(2j + 1) O_{j+1/2,j+1/2}.\]  

(28)

Using it recursively one obtains

\[O_{j_1,j_1} O_{j_2,j_2} = \frac{(2(j_1 + j_2))!}{(2j_1)!(2j_2)!} O_{j_1+j_2,j_1+j_2}.\]  

(29)

Recalling the explicit expression for the relevant Clebsch-Gordan coefficients here:

\[
\langle j_1 m_1 j_2 m_2 | j m \rangle |_{j=j_1+j_2, m=m_1+m_2} = \left[ \frac{(2j_1)!}{(j_1 + m_1)!(j_1 - m_1)!} \frac{(2j_2)!}{(j_2 + m_2)!(j_2 - m_2)!} \frac{(j + m)!(j - m)!}{(2j)!} \right]^{1/2},
\]

(30)

and rescaling $O_{j,m}$ to

\[-(2j)! \left[ \frac{(j + m)!(j - m)!}{(2j)!} \right]^{1/2} O_{j,m},\]  

(31)

we get the following ring multiplication law:

\[O_{j_1,m_1} O_{j_2,m_2} = O_{j_1+j_2,m_1+m_2}.\]  

(32)

As the second example, we derive the multiplication law for $Y_{j_1,m_1}^+ Y_{j_2,m_2}^+$ from the first equation of (19) as follows. First by momentum and ghost number counting

\* It is not quite natural to scale the vertex operator $O_{j,m}$'s (also $Y_{j,m}^\pm$ and $P_{j,m}$'s) by an $m$ dependent factor which spoil the $SU(2)$ structure among them. This normalization (for $Y_{j,m}^\pm$) was used in [23] and is quite useful as to make all the formulas explicit and simple. We will give the structure equations in their $SU(2)$ covariant form at the end of this section.
we conclude that

\[ Y_{j_1 m_1}^+ Y_{j_2 m_2}^+ = g(j_1, j_2) (j_1 m_1 j_2 m_2 \mid j_1 + j_2 - 1, m_1 + m_2) a Y_{j_1 + j_2 - 1, m_1 + m_2}^+. \]  

(33)

In order to determine the function \( g(j_1, j_2) \) we consider the special value \( m_1 = j_1 \) and \( m_2 = j_2 - 1 \). In this special case the higher order pole terms in the first equation of (19) vanish: Because \( Y_{jm}^+ (z) = c(z) \Psi_{jm}^+(z) \), and \( : c(z)c(w) : \mid z = w = 0 \), the single pole term is the only term contributing to \( Y_{j_1 m_1}^+ Y_{j_2 m_2}^+ \). Noticing that \( aY_{jm}^+ = j : \partial_z c(z)Y_{jm}^+(z) : \) and rescaling \( Y_{jm}^+ \) as \( \Psi_{jm}^+ \) in [13] to

\[
\sqrt{j/2(2j - 1)} \left[ \frac{(j + m)! (j - m)!}{(2j - 1)!} \right]^{1/2} Y_{jm}^+,
\]

(34)

we get the following ring multiplication law by making use of (19):

\[
Y_{j_1 m_1}^+ Y_{j_2 m_2}^+ = \frac{1}{j_1 + j_2 - 1} (m_1 j_2 - m_2 j_1) a Y_{j_1 + j_2 - 1, m_1 + m_2}^+.
\]

(35)

The other ring multiplication law involving the plus states only is \( O_{j_1 m_1} Y_{j_2 m_2}^+ \). By momentum and ghost number counting we have

\[
O_{j_1 m_1} Y_{j_2 m_2}^+ = g_1(j_1, j_2) (j_1 m_1 j_2 m_2 \mid j_1 + j_2, m_1 + m_2) Y_{j_1 + j_2, m_1 + m_2}^+
+ g_2(j_1, j_2) (j_1 m_1 j_2 m_2 \mid j_1 + j_2 - 1, m_1 + m_2) a O_{j_1 + j_2 - 1, m_1 + m_2}.
\]

(36)

The unknown function \( g_1(j_1, j_2) \) can easily be computed by setting \( m_1 = j_1 \) and \( m_2 = j_2 \) because the second term then vanishes. After appropriate rescaling as in (31) and (34) we find

\[
O_{j_1 m_1} Y_{j_2 m_2}^+ = \frac{j_2}{j_1 + j_2} Y_{j_1 + j_2, m_1 + m_2}^+ \tilde{g}_2(j_1, j_2) (m_1 j_2 - m_2 j_1) a O_{j_1 + j_2 - 1, m_1 + m_2}.
\]

(37)

To compute \( \tilde{g}_2(j_1, j_2) \), we explore the associativity of the ring multiplication as
follows. Multiplying both sides of (37) by $Y_{j_2m_3}^+$ from the right,

$$
(O_{j_1m_1} Y_{j_2m_2}^+) Y_{j_3m_3}^+ = \frac{j_2}{j_1 + j_2} Y_{j_1 + j_2, m_1 + m_2}^+ Y_{j_3m_3}^+ \\
+ \tilde{g}_2(j_1, j_2)(m_1j_2 - m_2j_1)aO_{j_1 + j_2 - 1, m_1 + m_2} Y_{j_3m_3}^+
$$

(38)

On one hand, we have

$$
aO_{j_1 + j_2 - 1, m_1 + m_2} Y_{j_3m_3}^+ = a(O_{j_1 + j_2 - 1, m_1 + m_2} Y_{j_3m_3}^+)
$$

(39)

because of $a^2 = 0$. On the other hand, for the LHS of (38) we have

$$
(O_{j_1m_1} Y_{j_2m_2}^+) Y_{j_3m_3}^+ = O_{j_1m_1}(Y_{j_2m_2}^+ Y_{j_3m_3}^+)
$$

$$
= O_{j_1m_1}(m_2j_3 - m_3j_2) \frac{1}{j_2 + j_3 - 1} aY_{j_2 + j_3 - 1, m_2 + m_3}^+
$$

$$
=(m_2j_3 - m_3j_2) \frac{1}{j_2 + j_3 - 1} a(O_{j_1m_1} Y_{j_2 + j_3 - 1, m_2 + m_3}^+)
$$

$$
=(m_2j_3 - m_3j_2) \frac{1}{j_1 + j_2 + j_3 - 1} aY_{j_1 + j_2 + j_3 - 1, m_1 + m_2 + m_3}^+
$$

(40)

Substituting (39) and (40) into (38) we get $\tilde{g}_2(j_1, j_2) = -1/(j_1 + j_2)$.

To compute the ring structure constants involving the minus states like $Y_{j_1m_1}^-$, one can not simply use (19) because of the higher order pole terms. Their existence can be seen from the product $Y_{j_1m_1}^- Y_{j_2m_2}^-$. By momentum and ghost number counting we have

$$
Y_{j_1m_1}^- Y_{j_2m_2}^- = g(j_1, j_2) \langle j_1m_1j_2m_2 | j_1 + j_2, m_1 + m_2 \rangle P_{j_1 + j_2, m_1 + m_2}.
$$

(41)

Setting $m_1 = j_1$ and $m_2 = j_2$ we get

$$
Y_{j_1j_1}^- Y_{j_2j_2}^-(w) = \oint [dz] : c(z) e^{-iX^+(z)+i(2j_1+1)X^-(z)} : c(w) e^{-iX^-(w)+i(2j_2+1)X^+(w)} : \\
= \sum_{q=0}^{2(j_1+j_2)+1} \frac{(q+1)!}{(q+1)!} cS_{2(j_1+j_2)+1-q} (-iX^+ + i(2j_1 + 1)X^-) \\
\times e^{-2iX^+ + 2i(j_1+j_2+1)X^-} :.
$$

(42)
From (41) we see that there should be no \(aY_{j_1+j_1+1,j_2+1}^-\) term in (42). So the \(q = 0\) term in (42) must be BRST equivalent to \(P_{j_1+j_2,j_1+j_2}\) (up to a proportional constant). In fact one can prove that \(cS_{2j_1+1}(-iX^+ - i\delta X^-)e^{-2iX^+ + 2i(j+1)X^-}\) is BRST invariant for any \(\delta\). For \(\delta = -1, -2, \cdots, -(2j+1)\), it is actually BRST exact. We have (modulo BRST exact terms)

\[
cS_{2j+1}(-iX^+ - i\delta X^-)e^{-2iX^+ + 2i(j+1)X^-} = \frac{\Gamma(2j + 2 + \delta)}{(2j + 2)!(\delta + 1)}\sqrt{2(2j + 1)}Y_{j_1+j_2}^+.
\]

This can easily be proved by using the raising operator \(\hat{T}_+\). The action of \(a\) on \(cS_{2j+1}(-iX^+ - i\delta X^-)e^{-2iX^+ + 2i(j+1)X^-}\) gives the following:

\[
a\left(cS_{2j+1}(-iX^+ - i\delta X^-)e^{-2iX^+ + 2i(j+1)X^-}\right)
\]

\[
= \left\{ \frac{1}{2}(\delta - 1) \sum_{q=1}^{2j+1} \frac{\partial^{q+1}c}{(q + 1)!} cS_{2j+1-q}(-iX^+ - i\delta X^-) \right\} e^{-2iX^+ + 2i(j+1)X^-}.
\]

By using the above formula in (42) \((\delta = -(2j_1 + 1)\) and \(j = j_1 + j_2\) we get then

\[
Y_{j_1,j_2,j_2}^- = -\frac{j_2}{j_1 + j_2 + 1} \sum_{q=1}^{2(j_1+j_2)+1} \frac{\partial^{q+1}c}{(q + 1)!} c
\]

\[
\times S_{2(j_1+j_2)+1-q}(-iX^+ - i(2j_1 + 1)X^-)e^{-2iX^+ + 2i(j+1)X^-}.
\]

This vanishes by making use of (14). Recalling (41), we have

\[
Y_{j_1,m_1}^- Y_{j_2,m_2}^- = 0,
\]

There is little difficulty to extend the above calculation to determine the ring
multiplication law for $Y_{j_{1m_{1}}}^{+}Y_{j_{2m_{2}}}^{-}$. The result is

$$
Y_{j_{1m_{1}}}^{+}Y_{j_{2m_{2}}}^{-} = \frac{j_{1}}{j_{2} - j_{1} + 1} P_{j_{2-j_{1}, m_{1}+m_{2}}} + \frac{1}{j_{2} - j_{1} + 1} (m_{1}j_{2} + m_{2}j_{1} + m_{1}) aY_{j_{2-j_{1}+1, m_{1}+m_{2}}}^{-},
$$

(47)

if we rescale $P_{jm}$ to

$$
(-1)^{j+m} \frac{1}{(2j + 1)!} \left[ \frac{(2j)!(2j-1)!}{(j + m)! (j - m)!} \right]^{1/2} P_{jm},
$$

(48)

and $Y_{jm}^{-}$ to

$$
(-1)^{j+m} \frac{1}{\sqrt{2j}(2j - 1)!} \left[ \frac{(2j - 1)!(2j-1)!}{(j + m)! (j - m)!} \right]^{1/2} Y_{jm}^{-}.
$$

(49)

Finally let us calculate the product $O_{j_{1m_{1}}} P_{j_{2m_{2}}}$. In the course of this calculation we will find a proof of (14). Generally we should have

$$
O_{j_{1m_{1}}} P_{j_{2m_{2}}} = g_{1}(j_{1}, j_{2}) (j_{1}m_{1}j_{2}m_{2} | j_{2} - j_{1}, m_{1} + m_{2}) P_{j_{2-j_{1}, m_{1}+m_{2}}} + g_{2}(j_{1}, j_{2}) (j_{1}m_{1}j_{2}m_{2} | j_{2} - j_{1} + 1, m_{1} + m_{2}) aY_{j_{2-j_{1}+1, m_{1}+m_{2}}}^{-}.
$$

(50)

We will compute $g_{1}(j_{1}, j_{2})$ only, which can be deduced from $g_{1}(1/2, j_{2})$. The other unknown function can be derived by using associativity. For $j_{1} = -m_{1} = 1/2$ and $m_{2} = j_{2} = j$ we have

$$
O_{1/2,-1/2} P_{j,j}(\delta) = \left\{ \begin{array}{c}
(2j + \delta) \sum_{q=1}^{2j} \frac{\partial^{q+1} cS_{2j-q}(-iX^{+} - i\delta X^{-})}{(q + 1)!} \\
+ \sum_{q=0}^{2j} \frac{\partial^{q+1} cS_{2j-q}(-iX^{+} - i\delta X^{-})}{(q + 1)!} e^{-2iX^{+}+i(2j-1)X^{-}} \\
= \frac{(2j + 1)(\delta + 2j)}{(2j + 1)} P_{j-1/2,j-1/2}(\delta) + \cdots,
\end{array} \right.
$$

(51)

where $\cdots$ denotes the term proportional to $aY_{j+1/2,j-1/2}^{-}$. Setting $P_{j,j}(\delta) = F_{j}(\delta) P_{j,j}$, one knows that $F_{j}(\delta)$ is a polynomial function with the highest degree $2j$ and $F_{j}(1)$
is 1. From (51) we get

\[
F_j(\delta) = \frac{\delta + 2j}{2j + 1} F_{j-1/2}(\delta) = \frac{\delta + 2j}{2j + 1} \frac{\delta + 2j - 1}{2j} \ldots \frac{\delta + 1}{2} F_0(\delta) = \frac{(\delta + 2j)!}{(2j + 1)! \delta!}.
\]

(52)

This proves (14). Substituting this result back into (51) one has

\[
O_{1/2,-1/2} P_{j,j} = (2j + 2) P_{j-1/2,j-1/2} + \ldots.
\]

(53)

For \(O_{j_1,-j_1} = (-1)^{2j_1} (O_{1/2,-1/2})^{2j_1}/(2j_1)^2\), we have then

\[
O_{j_1,-j_1} P_{j_2,j_2} = (-1)^{2j_1} \frac{(2(j_2 + 1))!}{(2j_1)!(2(j_2 - j_1 + 1))!} P_{j_2,j_2,j_1,j_1} + \ldots.
\]

(54)

From this result it follows

\[
O_{j_1,m_1} P_{j_2,m_2} = \frac{j_2 + 1}{j_2 - j_1 + 1} P_{j_2,j_1,m_1 + m_2}
\]

\[+
\frac{\tilde{g}_2(j_1,j_2)}{j_2 - j_1 + 1} (m_1 j_2 + m_2 j_1 + m_1) a Y_{j_2,j_1 + 1,m_1 + m_2}.
\]

(55)

after rescaling \(O_{j_1,m_1}\) and \(P_{j_2,m_2}\) as in (31) and (48). The unknown function 
\(\tilde{g}_2(j_1,j_2)\) will be determined in a moment.

Let us summarize the ring structures we have determined. First we have

\[
O_{j_1,m_1} O_{j_2,m_2} = O_{j_1 + j_2,m_1 + m_2},
\]

\[
O_{j_1,m_1} Y_{j_2,m_2}^+ = \frac{j_2}{j_1 + j_2} Y_{j_1 + j_2,m_1 + m_2}^+ - \frac{1}{j_1 + j_2} (m_1 j_2 - m_2 j_1) a O_{j_1 + j_2 - 1,m_1 + m_2},
\]

\[
Y_{j_1,m_1}^+ Y_{j_2,m_2}^+ = \frac{j_1}{j_1 + j_2 - 1} (m_1 j_2 - m_2 j_1) a Y_{j_1 + j_2 - 1,m_1 + m_2}^+.
\]

\[
Y_{j_1,m_1}^+ Y_{j_2,m_2}^- = \frac{j_1}{j_2 - j_1 + 1} P_{j_2,j_1,m_1 + m_2}
\]

\[+
\frac{1}{j_2 - j_1 + 1} (m_1 j_2 + m_2 j_1 + m_1) a Y_{j_2,j_1 + 1,m_1 + m_2}^-.
\]

(56)
The other non-vanishing products are

\[
O_{j_1m_1}Y_{j_2m_2}^- = g_1(j_1, j_2)Y_{j_2-j_1,m_1+m_2}^{-},
\]

\[
Y_{j_1m_1}^+ P_{j_2m_2} = \frac{g_2(j_1, j_2)}{j_2 - j_1 + 2}(m_1j_2 + m_2j_1 + m_1)aP_{j_2-j_1+1,m_1+m_2},
\]

\[
O_{j_1m_1} P_{j_2m_2} = \frac{j_2 + 1}{j_2 - j_1 + 1}P_{j_2-j_1,m_1+m_2}
+ \frac{\tilde{g}_2(j_1, j_2)}{j_2 - j_1 + 1}(m_1j_2 + m_2j_1 + m_1)aY_{j_2-j_1+1,m_1+m_2}^-,
\]

with

\[
g_1(j_1, j_2) = g_2(j_1, j_2) = \tilde{g}_2(j_1, j_2) = 1. \tag{58}
\]

while the vanishing products are

\[
Y_{j_1m_1}^+ Y_{j_2m_2}^- = 0,
\]

\[
Y_{j_1m_1}^- P_{j_2m_2} = 0,
\]

\[
P_{j_1m_1} P_{j_2m_2} = 0. \tag{59}
\]

The three unknown functions in (57) are determined from associativity. Multiplying both sides of (35) from the right by \(Y_{j_3m_3}^-\) and using the known result (47) and the second equation of (57), we get \(g_2(j_1, j_3 - j_2) = 1\) or \(g_2(j_1, j_2) = 1\). To determine \(\tilde{g}_2(j_1, j_2)\) we multiply the third equation of (57) by \(Y_{j_3m_3}^+\) and using (37) and the second equation of (57). The result is \(\tilde{g}_2(j_1, j_2) = 1\). Finally \(g_1(j_1, j_2)\) is determined by multiplying the first equation of (57) by \(aY_{j_3m_3}^+\), resulting in \(g_1(j_1, j_2) = 1\). This result is also confirmed by explicit calculation.

To conclude this section let us note that we have chosen an \(m\) dependent normalization for all the vertex operator \(O_{jm}, Y_{jm}^\pm\) and \(P_{jm}\)’s. Here we also give the structure equations in their \(SU(2)\) covariant form, which are obtained from
(56) and (57) by the following replacement:

\[
\begin{align*}
O_{jm} & \rightarrow C(jm)O_{jm}, \\
Y_{jm}^+ & \rightarrow N(jm)Y_{jm}^+, \\
Y_{jm}^- & \rightarrow (-1)^{j+m}\sqrt{j(j+1)}Y_{jm}^-, \\
P_{jm} & \rightarrow (-1)^{j+m}\sqrt{j(j+1)}P_{jm},
\end{align*}
\]

where \(C(jm) = \left[\frac{(j+m)!(j-m)!}{(2j)!}\right]^{1/2}\) and \(N(jm) = \left[\frac{(j+m)!(j-m)!}{(2j-1)!}\right]^{1/2}\). We have then

\[
\begin{align*}
O_{j_1m_1}O_{j_2m_2} = & \langle j_1 m_1 j_2 m_2 | j_1 + j_2, m_1 + m_2 \rangle O_{j_1+j_2, m_1+m_2}, \\
O_{j_1m_1}Y_{j_2m_2}^+ = & \sqrt{\frac{j_2}{j_1 + j_2}} \langle j_1 m_1 j_2 m_2 | j_1 + j_2, m_1 + m_2 \rangle Y_{j_1+j_2, m_1+m_2}^+ \\
& - \sqrt{\frac{j_1}{j_1 + j_2}} \langle j_1 m_1 j_2 m_2 | j_1 + j_2 - 1, m_1 + m_2 \rangle aO_{j_1+j_2-1, m_1+m_2}, \\
Y_{j_1m_1}Y_{j_2m_2}^+ = & \sqrt{\frac{j_1 + j_2}{j_1 + j_2 - 1}} \langle j_1 m_1 j_2 m_2 | j_1 + j_2 - 1, m_1 + m_2 \rangle aY_{j_1+j_2-1, m_1+m_2}^+, \\
Y_{j_1m_1}Y_{j_2m_2}^- = & \sqrt{\frac{j_2}{j_2 - j_1 + 1}} \langle j_1 m_1 j_2 m_2 | j_2 - j_1, m_1 + m_2 \rangle P_{j_2-j_1, m_1+m_2}^- \\
& - \sqrt{\frac{j_2 + 1}{j_2 - j_1 + 1}} \langle j_1 m_1 j_2 m_2 | j_2 - j_1 + 1, m_1 + m_2 \rangle aY_{j_2-j_1+1, m_1+m_2}^-, \\
O_{j_1m_1}Y_{j_2m_2}^- = & \langle j_1 m_1 j_2 m_2 | j_2 - j_1, m_1 + m_2 \rangle Y_{j_2-j_1, m_1+m_2}^- \\
Y_{j_1m_1}P_{j_2m_2}^+ = & - \sqrt{\frac{j_2 - j_1 + 1}{j_2 - j_1 + 2}} \langle j_1 m_1 j_2 m_2 | j_2 - j_1 + 1, m_1 + m_2 \rangle aP_{j_2-j_1+1, m_1+m_2}, \\
O_{j_1m_1}P_{j_2m_2} = & \sqrt{\frac{j_2 + 1}{j_2 - j_1 + 1}} \langle j_1 m_1 j_2 m_2 | j_2 - j_1, m_1 + m_2 \rangle P_{j_2-j_1, m_1+m_2} \\
& - \sqrt{\frac{j_1}{j_2 - j_1 + 1}} \langle j_1 m_1 j_2 m_2 | j_2 - j_1 + 1, m_1 + m_2 \rangle aY_{j_2-j_1+1, m_1+m_2}^-.  
\end{align*}
\]

The other three vanishing products remain unchanged.
4. Chiral Ring Structure Including the Tachyon States

In this section we incorporate the tachyon states into the chiral cohomology ring. The tachyon states are labelled by a continuous momentum $p$ and there are two chiralities:

$$T_p^{(\pm)} = ce^{ipX + (\sqrt{2}p)}\phi.$$ (62)

As in the case of discrete states, there also exist absolute tachyon states

$$aT_p^{(\pm)} = \pm\frac{p}{\sqrt{2}}\partial ce^{ipX + (\sqrt{2}p)}\phi.$$ (63)

To simplify our presentation we will consider only those tachyon states which are not included in the discrete states, i.e. $p \neq j\sqrt{2}$, where $j$ is an integer or half integer. Let us first consider the product of discrete states with tachyon states. By momentum addition this product must give rise to another tachyon state. Since any tachyon states must be of the form given by either (62) or (63), there are only four non-vanishing products of this type:

$$O_{j,\pm j}T_p^{(\pm)} \sim T_p^{(\pm)};$$

$$Y^+_{j,\pm (j-1)}T_p^{(\pm)} \sim aT_p^{(\pm)}.$$ (64)

The explicit calculation of these products is easy and we get, in agreement with [22],

$$O_{j,j}T_p^{(+)} = \frac{p}{p + \sqrt{2}j}T_p^{(+)};$$

$$O_{j,-j}T_p^{(-)} = T_p^{(-)};$$

$$Y^+_{j,(j-1)}T_p^{(+)} = -\frac{p}{p + \sqrt{2}(j-1)}aT_p^{(+)};$$

$$Y^+_{j,-(j-1)}T_p^{(-)} = -aT_p^{(-)}.$$ (65)

if we rescale $T^{(\pm)}$ to

$$(\Gamma(1 + \sqrt{2}p))^{\pm 1}T_p^{(\pm)}.$$ (66)

Now let us consider the product of two tachyons. For the product $T_p^{(-)}T_q^{(-)}$
the exponential factor is $e^{i(p+q)X+(2\sqrt{2}+(p+q))\phi}$ which can not be the exponential factor of any tachyon state. To be able to obtain a discrete state we must have

$$p + q = j \sqrt{2},$$  \hspace{1cm} (67)

where $j$ is an integer or half integer. For $j \geq -1/2$ we have

$$T_p^{(-)} T_q^{(-)} = d_1 a Y_{j+1,1}^- + d_2 P_{j,j},$$

where $d_1$ and $d_2$ are two constants. For $j \leq -1$ the product is zero, for there is no $a Y_{-j-1,j}^+$. By explicit calculation we have

$$T_p^{(-)} T_q^{(-)} = \sum_{k=0}^{2j+1} \frac{\partial^{k+1} c}{(k+1)!} c S_{2j+1-k} (-iX^+ + i(\sqrt{2}p + 1)X^-) e^{ijX+(2j+\phi)\sqrt{2}}$$

$$= \Gamma(2j+1-\sqrt{2}p) \frac{\Gamma(2j+1) - \sqrt{2}p}{2(2j+1)\Gamma(\sqrt{2}p)} (P_{j,j} + a Y_{j+1,j}^-),$$

or

$$T_p^{(-)} T_q^{(-)} = \frac{\sin \sqrt{2} \pi p}{2(j+1)\pi} (P_{j,j} + a Y_{j+1,j}^-),$$

after making rescaling (66). Similarly we have, with $p + q = -j \sqrt{2}$,

$$T_p^{(+)} T_q^{(+)} = \frac{\pi p q}{(j+1)\sin \sqrt{2} \pi q} (P_{j,-j} - a Y_{j+1,-1}^-).$$

The product $T_p^{(+)} T_q^{(-)}$ gives no new result. It either reduces to (65) or to $Y_{j_1,\pm j_1}^+ Y_{j_2,\pm j_2}^-$, which is a special case of (47).

Finally let us point out that part of the above results was also obtained in [25]. There they computed the single pole term in the OPE of $\Psi_p^{(\pm)} \Psi_q^{(\pm)}$, where $T_p^{(\pm)} = c \Psi_p^{(\pm)}$. This only gives the term $a Y_{j+1,\pm j}^-$ in (70) or (71). The other term $P_{j,\pm j}$ comes from the higher order pole terms in the OPE of $\Psi_p^{(\pm)} \Psi_q^{(\pm)}$. 

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5. Chiral Transformation Rules and the Chiral Charge Algebra

In this section we will use the above results to study the (chiral) symmetry charge algebra associated to the chiral states. We will follow the general strategy of constructing BRST invariant charges from BRST invariant operators \[22\]. We consider only the holomorphic part in this section. In the next section we will use the results to derive the transformation rules for the discrete closed-string states by combining the left and right sectors.

Starting from a generic BRST invariant operator \( V_{jm} \), one defines \( V^{(1)}_{jm} \) by the OPE

\[
b(z)V_{jm}(w) \sim \cdots + \frac{1}{z-w}V^{(1)}_{jm},
\]

(72)
or

\[
V^{(1)}_{jm}(w) = \oint_0 [dz]b(z)V_{jm}(w),
\]

(73)
up to total derivatives and BRST exact terms. Then one can prove

\[
\partial_w V_{jm}(w) = \{Q, V^{(1)}_{jm}(w)\}.
\]

(74)

Indeed,

\[
\{Q, V^{(1)}_{jm}(w)\} = \oint_w [dz]c(z)(T(z) + \partial_z c(z)b(z)) : \oint_w [d\bar{z}]b(\bar{z})V_{jm}(w)
\]

\[
= \left( \oint_w [d\bar{z}] \oint_w [dz] + \oint_w [d\bar{z}] \oint [dz] \right) : c(z)(T(z) + \partial_z c(z)b(z)) : b(\bar{z})V_{jm}(w)
\]

\[
= \oint_w [dz] \left( T(z) + 2\partial_z c(z)b(z) + c(z)\partial_z b(z) \right) V_{jm}(w)
\]

\[
= \partial_w V_{jm}(w).
\]

(75)

In terms of \( V^{(1)}_{jm} \) one can define a BRST invariant charge

\[
\hat{V}_{jm} = \oint_0 [dz]V^{(1)}_{jm}(z).
\]

(76)
The (anti-)commutator of $\hat{V}_{jm}$ with any BRST invariant local operator gives another BRST invariant operator, i.e.

$$[\hat{V}_{jm_1}, V_{jm_2}(z)] = \sum_{jm} g(j_1, j_2; j) \langle j_1 m_1 j_2 m_2 \mid jm \rangle V_{jm}(z).$$  \hfill (77)$$

Let us see how one can compute the RHS of (77) explicitly from the knowledge of the cohomology ring structure. Let us consider first the relative physical states. For them the double pole term in (72) actually vanishes, because there is no $\partial c$ term in $V_{jm}$. By definition, the ring multiplication law is given by

$$(V_{jm_1} V_{jm_2})(w) = \oint_w [dz] \frac{1}{z - w} V_{jm_1}(z) V_{jm_2}(w).$$  \hfill (78)$$

Applying $\oint_w [d\tilde{z}](\tilde{z} - w)b(\tilde{z})$ on both sides from the left, we get

$$\text{RHS} = \left( \oint_w [dz] \oint_w [d\tilde{z}] + \oint_w [dz] \oint_w [d\tilde{z}] \oint_w [dz] \right) \frac{\tilde{z} - w}{z - w} b(\tilde{z}) V_{jm_1}(z) V_{jm_2}(w)$$

$$= \oint_w [dz] V^{(1)}_{jm_1}(z) V_{jm_2}(w) = [\hat{V}_{jm_1}, V_{jm_2}(w)];$$  \hfill (79)$$

Thus we have derived the important formula

$$[\hat{V}_{jm_1}, V_{jm_2}] = \oint_w [dz](z - w)b(z)(V_{jm_1} V_{jm_2})(w).$$  \hfill (80)$$

This formula allows us to compute the action of a charge on a physical states from the knowledge of the cohomology ring structure, exhibited by

$$(V_{jm_1} V_{jm_2})(w) = \sum_{jm} \tilde{g}(j_1, j_2; j) \langle j_1 m_1 j_2 m_2 \mid jm \rangle V_{jm}(w).$$  \hfill (81)$$

In fact, by using (81) the RHS of (80) is simply given by

$$\text{RHS of } (80) = \sum_{jm} \tilde{g}(j_1, j_2; j) \langle j_1 m_1 j_2 m_2 \mid jm \rangle \oint_w [dz](z - w)b(z)V_{jm}(w).$$  \hfill (82)$$

We note that the last integration selects only terms with a factor $\partial c$ in $V_{jm}$. In this way we see that the transformation rules for the discrete charges acting on
the discrete states can be simply read off from the structure equation (i.e. the multiplication law) of the cohomology ring.

Now let us apply the above results to the discrete states \( O_{jm}, Y_{jm}^\pm \), and \( P_{jm} \). From these discrete states one can construct BRST invariant charges of ghost number \(-1, 0\) and \(1\). We denote these charges as \( \hat{X}_{jm}, Q_{jm}^\pm \) and \( \hat{Z}_{jm} \):

\[
\hat{X}_{jm} \equiv \oint_0 [dw] \oint \frac{dz}{b(z)} O_{jm}(w) \equiv \oint [dw] X_{jm}(w),
\]

\[
Q_{jm}^\pm \equiv \oint_0 [dw] \oint \frac{dz}{b(z)} Y_{jm}^\pm(w) \equiv \oint [dw] W_{jm}^\pm(w),
\]

\[
\hat{Z}_{jm} \equiv \oint_0 [dw] \oint \frac{dz}{b(z)} P_{jm}(w) \equiv \oint [dw] Z_{jm}(w).
\]

Their action on discrete states can be directly read off from the ring structure by using (80). We have

\[
[\hat{X}_{j1m1}, O_{j2m2}] = 0, \quad \{\hat{X}_{j1m1}, Y_{j2m2}^-\} = 0,
\]

\[
\{\hat{X}_{j1m1}, Y_{j2m2}^+\} = -(m_1j_2 - m_2j_1)O_{j_1+j_2-1,m_1+m_2},
\]

\[
[\hat{X}_{j1m1}, P_{j2m2}] = -(m_1j_2 + m_2j_1 + m_1)Y_{j_2-j_1+1,m_1+m_2},
\]

\[
[Q_{j1m1}^+, Y_{j2m2}^+] = (m_1j_2 - m_2j_1)Y_{j_1+j_2-1,m_1+m_2},
\]

\[
[Q_{j1m1}^+, Y_{j2m2}^-] = -(m_1j_2 + m_2j_1 + m_1)Y_{j_2-j_1+1,m_1+m_2},
\]

\[
[Q_{j1m1}^+, P_{j2m2}] = -(m_1j_2 + m_2j_1 + m_1)P_{j_2-j_1+1,m_1+m_2},
\]

\[
\{Q_{j1m1}^-, Y_{j2m2}^-\} = 0, \quad [Q_{j1m1}^-, P_{j2m2}] = 0, \quad [\hat{Z}_{j1m1}, P_{j2m2}] = 0.
\]

Applying once more \( \oint_0 [dw] \oint dz \) from the left on these equations*, we get the

* This operation changes all the discrete states into the corresponding charges.
following chiral charge algebra

\[ [\hat{X}_{j_1 m_1}, Q^+_{j_2 m_2}] = (m_1 j_2 - m_2 j_1)\hat{X}_{j_1 + j_2 - 1, m_1 + m_2}, \]
\[ \{\hat{X}_{j_1 m_1}, \hat{Z}_{j_2 m_2}\} = (m_1 j_2 + m_2 j_1 + m_1)Q^-_{j_2 - j_1 + 1, m_1 + m_2}, \]
\[ [Q^+_{j_1 m_1}, Q^+_{j_2 m_2}] = (m_1 j_2 - m_2 j_1)Q^+_{j_1 + j_2 - 1, m_1 + m_2}; \]
\[ [Q^+_{j_1 m_1}, Q^-_{j_2 m_2}] = -(m_1 j_2 + m_2 j_1 + m_1)Q^-_{j_2 - j_1 + 1, m_1 + m_2}, \]
\[ [Q^-_{j_1 m_1}, \hat{Z}_{j_2 m_2}] = -(m_1 j_2 + m_2 j_1 + m_1)\hat{Z}_{j_2 - j_1 + 1, m_1 + m_2}. \]  

The remaining (anti-)commutators are zero.

Similarly one can also construct BRST invariant charges from absolute physical states. We simply denote these absolute charges as \(a\hat{X}_{jm}, aQ^\pm_{jm}\) and \(a\hat{Z}_{jm}**\). Then one easily proves the following general relations

\[ [\hat{V}_{j_1 m_1}, aV_{j_2 m_2}(w)] = -(-1)^{|V_{j_1}|}gV(j_2)(V_{j_1 m_1}V_{j_2 m_2})(w) \]
\[ + \oint w [dz](z - w)b(z)(V_{j_1 m_1}aV_{j_2 m_2})(w), \]
\[ [a\hat{V}_{j_1 m_1}, V_{j_2 m_2}(w)] = -gV(j_1)(V_{j_1 m_1}V_{j_2 m_2})(w) \]
\[ + \oint w [dz](z - w)b(z)(aV_{j_1 m_1}V_{j_2 m_2})(w), \]
\[ [a\hat{V}_{j_1 m_1}, aV_{j_2 m_2}(w)] = -(-1)^{|V_{j_1}|}(gV(j_1) - gV(j_2))(aV_{j_1 m_1}V_{j_2 m_2})(w), \]

just by deforming the integration contour as in (79). Here in the first and third equations \([V_{j_1}]\) denotes the ghost number of \(V_{j_1 m_1}\). The function \(gV(j_a) (a = 1, 2)\) is defined as

\[ gO(j_a) = -gP(j_a) = j_a + 1, \quad gY^\pm(j_a) = \pm j_a, \]  

depending on the type of the operator \(V_{j_a m_a}\).

** ** \(a\hat{X}_{jm} \equiv \oint [d\omega]\oint [dz]b(z)(aO_{jm})(w) \neq a \cdot \hat{X}_{jm}, \) etc.
Now we present some examples to see how the above formulas can be used to derive the chiral transformation rules and chiral charge algebra involving absolute physical states or absolute charges. We have, e.g.,

\[
\hat{X}_{j_1m_1}, aY_{j_2m_2}^+ = \frac{j_1j_2}{j_1 + j_2} Y_{j_1+j_2,m_1+m_2}^+ + \frac{j_2}{j_1 + j_2} (m_1j_2 - m_2j_1) aO_{j_1+j_2-1,m_1+m_2},
\]

\[
[aQ_{j_1m_1}^+, O_{j_2m_2}^-] = \frac{j_1j_2}{j_1 + j_2} Y_{j_1+j_2,m_1+m_2}^+ - \frac{j_1}{j_1 + j_2} (m_1j_2 - m_2j_1) aO_{j_1+j_2-1,m_1+m_2},
\]

\[
[Q_{j_1m_1}, aO_{j_2m_2}^-] = -\frac{j_1(j_1 - 1)}{j_1 + j_2} Y_{j_1+j_2,m_1+m_2}^+ + \frac{j_2 + 1}{j_1 + j_2} (m_1j_2 - m_2j_1) aO_{j_1+j_2-1,m_1+m_2},
\]

\[
[a\hat{X}_{j_1m_1}^+, Y_{j_2m_2}^+] = \frac{j_2(j_2 - 1)}{j_1 + j_2} Y_{j_1+j_2,m_1+m_2}^+ + \frac{j_1 + 1}{j_1 + j_2} (m_1j_2 - m_2j_1) aO_{j_1+j_2-1,m_1+m_2}.
\]

Applying \( \oint_{\gamma} [dw] \oint_{\gamma} [dz] b(z) \) from the left to the above equations, we get the following (chiral) (anti-)commutation relations

\[
\{ \hat{X}_{j_1m_1}, aO_{j_2m_2}^+ \} = -\frac{j_1j_2}{j_1 + j_2} Q_{j_1+j_2,m_1+m_2}^+ + \frac{j_1}{j_1 + j - 2} (m_1j_2 - m_2j_1) a\hat{X}_{j_1+j_2-1,m_1+m_2},
\]

\[
[Q_{j_1m_1}^+, a\hat{X}_{j_2m_2}^-] = -\frac{j_1(j_1 - 1)}{j_1 + j_2} Q_{j_1+j_2,m_1+m_2}^+ + \frac{j_2}{j_1 + j - 2} (m_1j_2 - m_2j_1) a\hat{X}_{j_1+j_2-1,m_1+m_2}.
\]

We have derived the complete set of the chiral transformation rules and the chiral charge algebra and collect them in appendix B.
6. Symmetry Transformation of the Discrete String States

All of our discussions up to now deal with the right-moving sector only. In this section we will combine the left and right sectors to derive the symmetry transformation rules for the discrete states in closed string theory. Before doing this let us recall briefly the physical states in \( D = 2 \) closed string theory.

As discussed in [22], the physical closed string states in \( D = 2 \) should satisfy an extra condition \( (b_0 - \bar{b}_0)|\text{phys.}\rangle = 0 \). We will not go into any details for solving this condition but simply quote their results. By matching the Liouville momenta, the physical closed string discrete states that obey the \( b_0 - \bar{b}_0 \) condition are as follows (\( G \) is the total ghost number):

1) the plus states

\[
\begin{align*}
G = 0 &: \quad O_{jm} \tilde{O}_{jm'}; \\
G = 1 &: \quad O_{jm} \tilde{Y}_{j+1,m}' \tilde{O}_{jm'}, \quad (a + \bar{a})O_{jm} \tilde{O}_{jm'}; \\
G = 2 &: \quad Y_{jm}^+ \tilde{Y}_{jm'}^+ \quad (a + \bar{a})O_{jm} \tilde{Y}_{j+1,m}'^+ \quad (a + \bar{a})Y_{j+1,m}'^+ \tilde{O}_{jm'}; \\
G = 3 &: \quad (a + \bar{a})Y_{jm}^+ \tilde{Y}_{jm'}^+ \\
\end{align*}
\]

2) the minus states

\[
\begin{align*}
G = 2 &: \quad Y_{jm}^- \tilde{Y}_{jm'}^-; \\
G = 3 &: \quad P_{jm} \tilde{Y}_{j+1,m}'^- \quad Y_{j+1,m}' \tilde{P}_{jm'} \quad (a + \bar{a})Y_{jm}^- \tilde{Y}_{jm'}^-; \\
G = 4 &: \quad P_{jm} \tilde{P}_{jm'} \quad (a + \bar{a})P_{jm} \tilde{Y}_{j+1,m}'^- \quad (a + \bar{a})Y_{j+1,m}'^- \tilde{P}_{jm'}; \\
G = 5 &: \quad (a + \bar{a})P_{jm} \tilde{P}_{jm'} \\
\end{align*}
\]

From the above list we see that half of the states (the states without \((a + \bar{a})\)) are just the products of left and right relative physical states which satisfy the stronger condition: \( b_0|\text{phys.}\rangle = \bar{b}_0|\text{phys.}\rangle = 0 \). For later convenience we call these states relative physical (string) states. The other half of the states contain \((a + \bar{a})\) and do not satisfy either \( b_0|\text{phys.}\rangle = 0 \) or \( \bar{b}_0|\text{phys.}\rangle = 0 \). We will call them absolute physical (string) states. For closed string states, all the absolute physical states can be obtained from the relative physical states simply by multiplying \((a + \bar{a})\).
The general framework for constructing conserved charges in $D = 2$ closed string theory again has been given in sec. 4 of [22]. Summarizing their results, one starts with any BRST invariant zero form $\Omega^{(0)}$ and constructs the one and two forms $\Omega^{(1)}$ and $\Omega^{(2)}$ satisfying the descent equations

$$0 = \{Q_T, \Omega^{(0)}\},$$
$$d\Omega^{(0)} = \{Q_T, \Omega^{(1)}\},$$
$$d\Omega^{(1)} = \{Q_T, \Omega^{(2)}\}.$$  \hspace{1cm} (92)

These equations imply that the charge $A = \frac{1}{2\pi i} \oint_0 \Omega^{(1)}$ is a BRST invariant charge (the second equation) conserved up to BRST trivial operators (the third equation). $\Omega^{(1)}$ and $\Omega^{(2)}$ are given by

$$\Omega^{(1)}(w, \bar{w}) = \left( dw \oint_w [dz] b(z) - d\bar{w} \oint_w [d\bar{z}] \bar{b}(\bar{z}) \right) \Omega^{(0)}(w, \bar{w}),$$
$$\Omega^{(2)}(w, \bar{w}) = - dw \wedge d\bar{w} \oint_w [dz'] b(z') \oint_w [d\bar{z}] \bar{b}(\bar{z}) \Omega^{(0)}(w, \bar{w}).$$  \hspace{1cm} (93)

In what follows we take $\Omega^{(0)}$ to be $Y^+_{j+1,m} \tilde{O}_{jm'}$ as an example and derive the symmetry transformations for discrete string states. We have then

$$\Omega^{(1)}(w, \bar{w}) = dw W^+_{j+1,m} \tilde{O}_{jm'}(w, \bar{w}) - d\bar{w} Y^+_{j+1,m} \tilde{X}_{jm'}(w, \bar{w}),$$
$$\Omega^{(2)}(w, \bar{w}) = - dw \wedge d\bar{w} W^+_{j+1,m} \tilde{X}_{jm'}(w, \bar{w}).$$  \hspace{1cm} (94)

The conserved charge $A_{j:mm'} \equiv \frac{1}{2\pi i} \oint_0 \Omega^{(1)}(w, \bar{w})$ acting on the (plus) relative dis-
crete states gives rise to

\[
[A_{j_1;m_1',1}, O_{j_2;m_2} \bar{O}_{j_2;m_2}]
\]

\[
= (m_1 j_2 - m_2(j_1 + 1)) O_{j_1+j_2,m_1+m_2} \bar{O}_{j_1+j_2,m_1'+m_2'},
\]

\[
[A_{j_1;m_1',1}, O_{j_2;m_2} \bar{Y}^+_{j_2+1,m_2}]
\]

\[
= -\frac{1}{j_1 + j_2 + 1} (m_1 j_2 - m_2(j_1 + 1))(m_1'(j_2 + 1) - m_2' j_1)
\times \left(\bar{a} \bar{a} \right) O_{j_1+j_2,m_1+m_2} \bar{O}_{j_1+j_2,m_1'+m_2'}
\]

\[
+ \frac{j_2 + 1}{j_1 + j_2 + 1} (m_1 j_2 - m_2(j_1 + 1)) O_{j_1+j_2,m_1+m_2} \bar{Y}^+_{j_1+j_2+1,m_1'+m_2'}
\]

\[
- \frac{j_1 + 1}{j_1 + j_2 + 1} (m_1'(j_2 + 1) - m_2' j_1) Y^+_{j_1+j_2+1,m_1+m_2} \bar{O}_{j_1+j_2,m_1'+m_2'}, \quad (95)
\]

\[
[A_{j_1;m_1',1}, Y^+_{j_2+1,m_2} \bar{O}_{j_2;m_2}]
\]

\[
= (m_1(j_2 + 1) - m_2(j_1 + 1)) Y^+_{j_1+j_2+1,m_1+m_2} \bar{O}_{j_1+j_2,m_1'+m_2'},
\]

\[
[A_{j_1;m_1',1}, Y^+_{j_2;m_2} \bar{Y}^+_{j_2+1,m_2}]
\]

\[
= \frac{1}{j_1 + j_2} (m_1 j_2 - m_2(j_1 + 1))(m_1' j_2 - m_2' j_1)
\times \left(\bar{a} \bar{a} \right) Y^+_{j_1+j_2,m_1+m_2} \bar{O}_{j_1+j_2-1,m_1'+m_2'}
\]

\[
+ \frac{j_2}{j_1 + j_2} (m_1 j_2 - m_2(j_1 + 1)) Y^+_{j_1+j_2,m_1+m_2} \bar{Y}^+_{j_1+j_2,m_1'+m_2'}.
\]

Note that the action of the charge \(A_{j_1;m_1,m_1'}\) on the relative discrete states also give rise to absolute physical states. This feature was noted by Witten and Zwiebach [22] too. The example given by them is a special case of the last equation in (95). Here we are able to systematically derive the complete set of transformation rules with the help of the knowledge of the cohomology ring structure. In appendix C we will show that the action of any charge on the physical closed string discrete states gives only (a linear combination of) physical closed string discrete states, i.e., those listed in (90) and (91).

The action of the charge \(A_{j_1;m_1,m_1'}\) on the (plus) absolute discrete states can
also be derived. We have

\[ [A_{j_1;m_1m_1'}, (a + \bar{a})O_{j_2m_2} \hat{O}_{j_2m_2'}] \]
\[ = (m_1j_2 - m_2(j_1 + 1))(a + \bar{a})O_{j_1+j_2,m_1+m_2} \hat{O}_{j_1+j_2,m_1'+m_2'} , \]
\[ [A_{j_1;m_1m_1'}, (a + \bar{a})O_{j_2m_2} \hat{Y}^+_{j_2+1,m_2}] \]
\[ = - \frac{j_1 + 1}{j_1 + j_2 + 1}(m_1'(j_2 + 1) - m_2'(j_1))(a + \bar{a})Y^+_{j_1+j_2+1,m_1+m_2} \hat{O}_{j_1+j_2,m_1'+m_2'} \]
\[ + \frac{j_2 + 1}{j_1 + j_2 + 1}(m_1j_2 - m_2(j_1 + 1))(a + \bar{a})O_{j_1+j_2,m_1+m_2} \hat{Y}^+_{j_1+j_2+1,m_1'+m_2'} , \]
\[ [A_{j_1;m_1m_1'}, (a + \bar{a})O_{j_2m_2} \bar{Y}^+_{j_2m_2'}] \]
\[ = (m_1(j_2 + 1) - m_2(j_1 + 1))(a + \bar{a})Y^+_{j_1+j_2+1,m_1+m_2} \bar{O}_{j_1+j_2,m_1'+m_2'} , \]
\[ [A_{j_1;m_1m_1'}, (a + \bar{a})O_{j_2m_2} \bar{Y}^+_{j_2m_2'}] \]
\[ = \frac{j_2}{j_1 + j_2}(m_1j_2 - m_2(j_1 + 1))(a + \bar{a})Y^+_{j_1+j_2+1,m_1+m_2} \bar{Y}^+_{j_1+j_2,m_1'+m_2'} . \]  

(96)

Comparing (95) and (96), we observe that the left sides of all the equations in (96) can be obtained from (95) simply by the multiplication of \((a + \bar{a})\). (Note that \((a + \bar{a})^2 = 0\).) This suggests the following general formula:

\[ [\hat{V}_{j_1;m_1m_1'}, (a + \bar{a})V_{j_2;m_2m_2'}] = \pm (a + \bar{a})[\hat{V}_{j_1;m_1m_1'}, V_{j_2;m_2m_2'}] , \]  

(97)

where \(\hat{V}_{j_1;m_1m_1'}\) is the charge derived from the relative physical state \(V_{j_1;m_1m_1'}\) and \(V_{j_2;m_2m_2'}\) is a relative physical state. Here one takes the “−” sign if \(\hat{V}_{j_1;m_1m_1'}\) has odd (total) ghost number. The general validity of (97) will be proved in Appendix C.

Quite similarly one can construct conserved charges from absolute discrete states and study their action on discrete states. The corresponding charges will simply be denoted as \((a + \bar{a})\hat{V}_{j:mm'}\). Now take \(\Omega^{(0)}\), for example, to be \((a +
By explicit calculation we have

\[
\left[(a + \bar{a})A_{j_1;m_1}, O_{j_2;m_2} \right] \tilde{o} = - (m_1 j_2 - m_2 (j_1 + 1))(a + \bar{a})O_{j_1 + j_2, m_1 + m_2} \tilde{o}_{j_1 + j_2, m_1 + m_2},
\]

\[
\left[(a + \bar{a})A_{j_1;m_1}, O_{j_2;m_2} Y^+_{j_2+1,m_2} \right] \tilde{o} = - \frac{j_2 + 1}{j_1 + j_2 + 1} (m_1 j_2 - m_2 (j_1 + 1))(a + \bar{a})O_{j_1 + j_2 + 1, m_1 + m_2} Y^+_{j_1 + j_2 + 1, m_1 + m_2}
\]

\[
+ \frac{j_1 + 1}{j_1 + j_2 + 1} (m_1' (j_2 + 1) - m_2' j_1)(a + \bar{a})Y^+_{j_1 + j_2 + 1, m_1 + m_2} \hat{o}_{j_1 + j_2, m_1' + m_2},
\]

\[
\left[(a + \bar{a})A_{j_1;m_1}, Y^+_{j_2+1,m_2} \right] \tilde{o} = - (m_1 (j_2 + 1) - m_2 (j_1 + 1))(a + \bar{a})Y^+_{j_1 + j_2 + 1, m_1 + m_2} \hat{o}_{j_1 + j_2, m_1' + m_2},
\]

\[
\left[(a + \bar{a})A_{j_1;m_1}, Y^+_{j_2+1,m_2} \right] \tilde{o} = - \frac{j_2}{j_1 + j_2} (m_1 j_2 - m_2 (j_1 + 1))(a + \bar{a})Y^+_{j_1 + j_2 + 1, m_1 + m_2} \hat{o}_{j_1 + j_2, m_1' + m_2},
\]

(98)

and

\[
\{ (a + \bar{a})A_{j_1;m_1}, (a + \bar{a})O_{j_2;m_2} \hat{o}_{j_2,m_2} \} = 0,
\]

\[
\left[(a + \bar{a})A_{j_1;m_1}, (a + \bar{a})O_{j_2,m_2} \hat{o}_{j_2+1,m_2} \right] = 0
\]

(99)

\[
\left[(a + \bar{a})A_{j_1;m_1}, (a + \bar{a})Y^+_{j_2+1,m_2} \hat{o}_{j_2,m_2} \right] = 0,
\]

\[
\{ (a + \bar{a})A_{j_1;m_1}, (a + \bar{a})Y^+_{j_2,m_2} \hat{o}_{j_2+1,m_2} \} = 0.
\]

From the above results we would like to guess the following general formulas

\[
\left[(a + \bar{a})\hat{V}_{j_1;m_1}, V_{j_2;m_2} \right] = -(a + \bar{a})[\hat{V}_{j_1;m_1}, V_{j_2;m_2}],
\]

\[
\left[(a + \bar{a})\hat{V}_{j_1;m_1}, (a + \bar{a})V_{j_2;m_2} \right] = 0.
\]

(100)

Again the general validity of these equations will be proved in Appendix C.

A remark is in order about the transformation rules in closed string theory. For the chiral transformation rules we see quite clearly from the general formulas (80) and (86) and also the explicit results in appendix B that the multiplication by \(a\) is not (anti-)commutative with the operation of taking (anti-) commutator
between charge and discrete state, and also the operation of getting the charges from the states, i.e. in general

\[ [\hat{V}_{j_1m_1}, aV_{j_2m_2}] \neq -(-1)^{[V_{j_1}]} a[\hat{V}_{j_1m_1}, V_{j_2m_2}], \]

\[ [a\hat{V}_{j_1m_1}, V_{j_2m_2}] \neq -a[\hat{V}_{j_1m_1}, V_{j_2m_2}], \]

\[ [a\hat{V}_{j_1m_1}, aV_{j_2m_2}] \neq 0. \]

This is in sharp contrast with the transformation rules for closed string theory. In closed string theory, the \((a + \bar{a})\) plays the role of \(a\) played in chiral theory. The eqs. (97) and (100) shows clearly that the multiplication by \((a + \bar{a})\) can be moved freely and can be taken out of the commutator. At the moment we do not have a deep understanding of this result.

7. Discussion and Conclusions

In this paper we have explicitly determined the complete set of structure constants of the chiral BRST cohomology ring for \(D = 2\) string theory. As we demonstrated in sec. 5, this ring structure encodes the full chiral charge algebra and the corresponding symmetry transformation rules. The latter in turn can be used to derive the full symmetry algebra and transformation rules for closed string states, as shown in sec. 6. We also found that the operator \((a + \bar{a})\) in closed string theory obeys a quite simple rule as given in eqs. (97) and (100).

The search for the infinite symmetries in \(D = 2\) string theory has been done recently in various approaches, with apparently different outcomes. In the \(c = 1\) matrix model [26, 27, 28, 29], it is the linear \(W_\infty\) symmetry [30], which is realized through its well-known free-fermion realization. In the gauged WZW model approach [31], the first-quantized \(D = 2\) string theory is described by the world-sheet \(SL(2, R)/U(1)\) model at level \(k = 9/4\). Recently it has been shown [32, 33, 34] that for generic \(k\) there exists a hidden non-linear \(\hat{W}_\infty\) [32] current algebra, whose generators are higher-spin currents formed from the basic coset currents through an elegant generating function [32]. In this paper we are discussing the symmetries
in the Liouville approach, initiated in [20, 22], for physical string states. Through we have been able to work out explicit symmetry transformation rules, the relationship with the $W_\infty$ symmetries appearing in other approaches as mentioned above remains to be clarified.

As shown in several recent papers [36, 37, 38], the symmetry algebra and transformation rules can be used to derive Ward identities relating different correlation functions. In fact the sub-algebra of charges with zero ghost number can be exploited to completely determine the multi-tachyon amplitudes. Similarly one expects that the amplitudes containing more than one discrete states may be determined from amplitudes containing less number of discrete states, by exploiting Ward identities derived from charges with non-zero ghost number. Let us close this paper with some remarks about this problem.

First, if one tries to compute the four-point amplitudes containing the exotic discrete states ($O_{jm}$ or $P_{jm}$) directly, one always gets zero or $\infty$. This is because the cosmological constant has been taken to be $\mu = 0$ and the momenta been taken at discrete values. One can circumvent the problems by taking a generic $\mu$ and compute the amplitudes in the presence of a cosmological constant [24] and regularizing the $D = 2$ string theory by $D < 2$ string theory or equivalently the $c < 1$ matter coupled to gravity [39]. At least in certain cases, taking the limit $c \to 1$ could lead to some meaningful results. In principle there seems no problem to extend our approach to $D < 2$ string theory, but in practice this extension may be confronted with some serious technical problems. (See refs. [40, 41, 42].)

Second, while it is not too difficult to derive the Ward identities [43, 37], it is highly non-trivial to use them in computing correlation functions. The difficulty lies in choosing the proper charges to derive a proper set of Ward identities that is soluble to give more-point amplitude in terms of less-point amplitudes. There seems no guarantee that all the more-point amplitudes can be obtained from less-point amplitudes by Ward identities [43]. Combined with the regularization problem mentioned above, the practical use of Ward identities in computing
more correlation functions remains a topic to be attacked.

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Appendix A. Schur Polynomials

The elementary Schur polynomials \(S_k(x)\) are defined through the generating function

\[
\sum_{k \geq 0} S_k(x) z^k = \exp\{\sum_{k \geq 1} x_k z^k\}. \tag{102}
\]

where the \(x_k\)'s, with \(k = 1, 2, \cdots\), are the arguments of the polynomials. What we are using in this paper is the Schur polynomials having \(x_k\)'s to be

\[
x_k = -i\delta^- \frac{\partial^k X^+}{k!} - i\delta^+ \frac{\partial^k X^-}{k!}, \tag{103}
\]

where \(\delta^+\) and \(\delta^-\) are two real numbers. To simplify our notations we denote

\[
S_k(x) = S_k(-i\delta^- \frac{\partial^j X^+}{j!} - i\delta^+ \frac{\partial^j X^-}{j!}) \equiv S_k(-i\delta^- X^+ + i\delta^+ X^-). \tag{104}
\]

These special Schur polynomials are related with the exponential function \(\exp\{iX^\pm(z + w)\}\) as follows

\[
\sum_{k \geq 0} S_k(-i\delta^- X^+(w) - i\delta^+ X^-(w)) z^k
= \exp\{-i\delta^-(X^+(z + w) - X^+(w)) - i\delta^+(X^-(z + w) - X^-(w))\}, \tag{105}
\]

or

\[
S_k(-i\delta^- X^+ - i\delta^+ X^-) = \frac{1}{k!} \partial^k\left(\exp\{-i\delta^- X^+ - i\delta^+ X^-\}\right). \tag{106}
\]

We have been able to prove the following identities for OPE's containing Schur polynomials:

\[
\partial X^+(z) : S_k(-i\delta X^-(w)) : \sim i\delta \sum_{j=1}^{k} \frac{1}{(z-w)^{j+1}} : S_{k-j}(-i\delta X^-(w)) :, \tag{107}
\]
: \( S_k(-i\delta X^-(z)) : \partial X^+(w) \sim i\delta \sum_{j=1}^k \frac{(-1)^{j+1}}{(z-w)^j+1} : S_{k-j}(-i\delta X^-(z)) : \) \quad (108)

: \( S_k(-i\delta X^-(z)) : e^{isX^+(w)} : \)

\[
\sim \sum_{n=1}^k \frac{\Gamma(\delta s+n)}{n!\Gamma(\delta s)} \frac{(-1)^n}{(z-w)^n} : S_{k-n}(-i\delta X^-(z)) e^{isX^+(w)} : ,
\]

: \( e^{isX^+(z)} : S_k(-i\delta X^-(w)) : \)

\[
\sim \sum_{n=1}^k \frac{\Gamma(\delta s+n)}{n!\Gamma(\delta s)} \frac{1}{(z-w)^n} : S_{k-n}(-i\delta X^-(w)) e^{isX^+(z)} : .
\]

These identities are crucial in our explicit determination of the structure constants of the chiral BSRT cohomology ring. Let us prove eqs. (107) and (109). The other two equations can be proved similarly.

Writing the Schur polynomial \( S_k(-i\delta X^-(w)) \) in a contour integration form

\[
S_k(-i\delta X^-(w)) = \oint_{\gamma} \frac{1}{Z^{k+1}} e^{-i\delta \sum_{j=1}^\infty \frac{Z^j}{j!} \partial^j X^-(w)},
\]

we have

\[
\partial X^+(z) : S_k(-i\delta X^-(w)) := \oint_{\gamma} \frac{1}{Z^{k+1}} \partial X^+(z) : e^{-i\delta \sum_{j=1}^\infty \frac{Z^j}{j!} \partial^j X^-(w)} : \]

\[
\sim -i\delta \oint_{\gamma} \frac{1}{Z^{k+1}} \sum_{j=1}^\infty \frac{Z^j}{j!} (\partial X^+(z) \partial^j X^-(w)) : e^{-i\delta \sum_{j=1}^\infty \frac{Z^j}{j!} \partial^j X^-(w)} : \]

\[
= i\delta \oint_{\gamma} \frac{1}{Z^{k+1}} \sum_{j=1}^\infty \frac{Z^j}{(z-w)^j+1} : e^{-i\delta \sum_{j=1}^\infty \frac{Z^j}{j!} \partial^j X^-(w)} : \]

\[
= i\delta \sum_{j=1}^k \frac{1}{(z-w)^j+1} : S_{k-j}(-i\delta X^-(w)).
\]

This proves (107). To prove (109) we use the following equation for the normal
ordering of two exponential functions:

\[ : e^{-i\delta \partial_j X^-(z)} :: e^{isX^+(w)} : \sim e^{\delta s (\partial_j X^-(z)X^+(w))} : e^{-i\delta \partial_j X^-(z)+isX^+(w)} : . \]  

(113)

We have

\[ : S_k(-i\delta X^-(w)) :: e^{isX^+(w)} := \oint [dZ] \frac{1}{Z^{k+1}} e^{-i\delta \sum_{j=1}^\infty \frac{Z_j}{j!} \partial_j X^-(w)} : e^{isX^+(w)} : \]

\[ \sim \oint [dZ] \frac{1}{Z^{k+1}} e^{\delta s \sum_{j=1}^\infty \frac{Z_j}{j!} (\partial_j X^-(z)X^+(w))} : e^{-i\delta \partial_j X^-(z)+isX^+(w)} : \]

\[ = \oint [dZ] \frac{1}{Z^{k+1}} (1 + \frac{1}{z-w}) e^{\delta s} : e^{-i\delta \partial_j X^-(z)+isX^+(w)} : \]

\[ = \sum_{n=1}^k \frac{\Gamma(\delta s + n)}{n! \Gamma(\delta s)} \frac{(-1)^n}{(z-w)^n} : S_{k-n}(-i\delta X^-(z)) e^{isX^+(w)} : . \]  

(114)

The other formula we use in the text is [16]

\[ \sum_{m=j+1}^k (m-j)x_{m-j}S_{k-m}(x) = (k-j)S_{k-j}(x), \]  

(115)

which can easily be proved through the generating function:

\[ \sum_{k\geq j}^k (k-j)S_{k-j}z^{k-j} = z \frac{d}{dx} \left( \sum_{k\geq j}^k S_{k-j}(x)z^{k-j} \right) \]

\[ = z \frac{d}{dz} \left( \exp(\sum_k x_kz^k) \right) = \sum_{k\geq 1}^k x_kz^k \exp(\sum_k x_kz^k) \]

\[ = \sum_{k\geq 1} \sum_{l\geq 0}^k x_kz^l S_l(z)z^{k-l} = \sum_{k\geq j}^k \left( \sum_{m=j+1}^k (m-j)x_{m-j}S_{k-m}(x) \right) z^{k-j}. \]

Appendix B. The Chiral Transformation Rules and Chiral Charge Algebra
In this appendix we present the full set of chiral transformation rules. All of them are derived from (80) and (86) by using our results on the cohomology ring structure. We have

\[ \{ \hat{X}_{j_1m_1}, Y^+_{j_2m_2} \} = -(m_1j_2 - m_2j_1)O_{j_1 + j_2 - 1, m_1 + m_2}, \]
\[ [\hat{X}_{j_1m_1}, P_{j_2m_2}] = -(m_1j_2 + m_2j_1 + m_1)Y^-_{j_2 - j_1 + 1, m_1 + m_2}, \]

\[ \{ \hat{X}_{j_1m_1}, aO_{j_2m_2} \} = j_1O_{j_1 + j_2, m_1 + m_2}, \]
\[ [\hat{X}_{j_1m_1}, aY^+_{j_2m_2}] = \frac{j_1j_2}{j_1 + j_2} Y^+_{j_1 + j_2, m_1 + m_2} + \frac{j_2}{j_1 + j_2} (m_1j_2 - m_2j_1) aO_{j_1 + j_2 - 1, m_1 + m_2}, \]
\[ [\hat{X}_{j_1m_1}, aY^-_{j_2m_2}] = j_1 Y^-_{j_2 - j_1, m_1 + m_2}, \]
\[ \{ \hat{X}_{j_1m_1}, aP_{j_2m_2} \} = \frac{j_1(j_2 + 1)}{j_2 - j_1 + 1} P_{j_2 - j_1, m_1 + m_2} 
+ \frac{j_2 + 1}{j_2 - j_1 + 1} (m_1j_2 + m_2j_1 + m_1) aY^-_{j_2 - j_1 + 1, m_1 + m_2}, \]

\[ [Q^+_{j_1m_1}, O_{j_2m_2}] = (m_1j_2 - m_2j_1)O_{j_1 + j_2 - 1, m_1 + m_2}, \]
\[ [Q^+_{j_1m_1}, Y^+_{j_2m_2}] = (m_1j_2 - m_2j_1) Y^+_{j_1 + j_2 - 1, m_1 + m_2}, \]
\[ [Q^-_{j_1m_1}, Y^-_{j_2m_2}] = -(m_1j_2 + m_2j_1 + m_1) Y^-_{j_2 - j_1 + 1, m_1 + m_2}, \]
\[ [Q^+_{j_1m_1}, P_{j_2m_2}] = -(m_1j_2 + m_2j_1 + m_1) P_{j_2 - j_1 + 1, m_1 + m_2}, \]

\[ [Q^+_{j_1m_1}, aO_{j_2m_2}] = -\frac{j_1(j_1 - 1)}{j_1 + j_2} Y^+_{j_1 + j_2, m_1 + m_2} \]
+ \frac{j_2 + 1}{j_1 + j_2} (m_1j_2 - m_2j_1) aO_{j_1 + j_2 - 1, m_1 + m_2}, \]
\[ [Q^+_{j_1m_1}, aY^+_{j_2m_2}] = \frac{j_2}{j_1 + j_2} (m_1j_2 - m_2j_1) aY^+_{j_1 + j_2 - 1, m_1 + m_2}, \]
\[ [Q^+_{j_1m_1}, aY^-_{j_2m_2}] = -\frac{j_1(j_1 - 1)}{j_2 - j_1 + 1} P_{j_2 - j_1, m_1 + m_2} \]
- \frac{j_2}{j_2 - j_1 + 1} (m_1j_2 + m_2j_1 + m_1) aY^-_{j_2 - j_1 + 1, m_1 + m_2}, \]
\[ [Q^+_{j_1m_1}, aP_{j_2m_2}] = -\frac{j_2 + 1}{j_2 - j_1 + 2} (m_1j_2 + m_2j_1 + m_1) aP_{j_2 - j_1 + 1, m_1 + m_2}, \]

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\[ [Q^-_{j_1m_1}, Y^+_{j_2m_2}] = (m_1j_2 + m_2j_1 + m_2)Y^-_{j_1-j_2+1,m_1+m_2}, \]
\[ [Q^-_{j_1m_1}, aO_{j_2m_2}] = (j_1 + 1)Y^-_{j_1-j_2,m_1+m_2}, \]
\[ [Q^-_{j_1m_1}, aY^+_{j_2m_2}] = -\frac{j_2(j_1 + 1)}{j_1 - j_2 + 1} P_{j_1-j_2,m_1+m_2} - \frac{j_2}{j_1 - j_2 + 1}(m_1j_2 + m_2j_1 + m_2)aY^-_{j_1-j_2+1,m_1+m_2}, \]
\[ [\hat{Z}_{j_1m_1}, O_{j_2m_2}] = -(m_1j_2 + m_2j_1 + m_2)Y^-_{j_1-j_2+1,m_1+m_2}, \]
\[ [\hat{Z}_{j_1m_1}, Y^+_{j_2m_2}] = -(m_1j_2 + m_2j_1 + m_2)P_{j_1-j_2+1,m_1+m_2}, \]
\[ [\hat{Z}_{j_1m_1}, aO_{j_2m_2}] = -\frac{(j_1 + 1)(j_1 + 2)}{j_1 - j_2 + 1} P_{j_1-j_2,m_1+m_2} - \frac{j_2 + 1}{j_1 - j_2 + 1}(m_1j_2 + m_2j_1 + m_2)aY^-_{j_1-j_2+1,m_1+m_2}, \]
\[ [\hat{Z}_{j_1m_1}, aY^+_{j_2m_2}] = -\frac{j_2}{j_1 - j_2 + 2}(m_1j_2 + m_2j_1 + m_2)aP_{j_1-j_2+1,m_1+m_2}, \]
\[ [a\hat{X}_{j_1m_1}, O_{j_2m_2}] = j_2O_{j_1+j_2,m_1+m_2}, \]
\[ [a\hat{X}_{j_1m_1}, Y^+_{j_2m_2}] = \frac{j_2(j_2 - 1)}{j_1 + j_2} Y^+_{j_1+j_2,m_1+m_2} + \frac{j_1 + 1}{j_1 + j_2}(m_1j_2 - m_2j_1)aO_{j_1+j_2-1,m_1+m_2}, \]
\[ [a\hat{X}_{j_1m_1}, Y^-_{j_2m_2}] = -(j_2 + 1)Y^-_{j_2-j_1,m_1+m_2}, \]
\[ [a\hat{X}_{j_1m_1}, P_{j_2m_2}] = -\frac{(j_2 + 1)(j_2 + 2)}{j_2 - j_1 + 1} P_{j_2-j_1,m_1+m_2} - \frac{j_1 + 1}{j_2 - j_1 + 1}(m_1j_2 + m_2j_1 + m_1)aY^-_{j_2-j_1+1,m_1+m_2}, \]
\[ [a\hat{X}_{j_1m_1}, aO_{j_2m_2}] = -(j_1 - j_2)aO_{j_1+j_2,m_1+m_2}, \]
\[ [a\hat{X}_{j_1m_1}, aY^+_{j_2m_2}] = -\frac{j_2(j_1 - j_2 + 1)}{j_1 + j_2} aY^+_{j_1+j_2,m_1+m_2}, \]
\[ [a\hat{X}_{j_1m_1}, aY^-_{j_2m_2}] = -(j_1 + j_2 + 1)aY^-_{j_1+j_2,m_1+m_2}, \]
\[ [a\hat{X}_{j_1m_1}, aP_{j_2m_2}] = -\frac{(j_2 + 1)(j_1 + j_2 + 2)}{j_2 - j_1 + 1} aP_{j_2-j_1,m_1+m_2}, \]
\begin{align*}
[aQ^+_{j_1,m_1}, O_{j_2,m_2}] &= \frac{j_1 j_2}{j_1 + j_2} Y^+_{j_1 + j_2, m_1 + m_2} - \frac{j_1}{j_1 + j_2} (m_1 j_2 - m_2 j_1) aO_{j_1 + j_2 - 1, m_1 + m_2}, \\
\{aQ^+_{j_1,m_1}, Y^+_{j_2,m_2}\} &= - \frac{j_1}{j_1 + j_2 - 1} (m_1 j_2 - m_2 j_1) aY^+_{j_1 + j_2 - 1, m_1 + m_2}, \\
\{aQ^+_{j_1,m_1}, Y^-_{j_2,m_2}\} &= - \frac{j_1 (j_2 + 1)}{j_2 - j_1 + 1} P_{j_2 - j_1, m_1 + m_2} \\
& \quad - \frac{j_1}{j_2 - j_1 + 1} (m_1 j_2 + m_2 j_1 + m_1) aY^-_{j_2 - j_1 + 1, m_1 + m_2}, \\
[aQ^+_{j_1,m_1}, P_{j_2,m_2}] &= - \frac{j_1}{j_2 - j_1 + 2} (m_1 j_2 + m_2 j_1 + m_1) aP_{j_2 - j_1 + 1, m_1 + m_2}, \\

\{aQ^+_{j_1,m_1}, aO_{j_2,m_2}\} &= \frac{j_1 (j_1 - j_2 - 1)}{j_1 + j_2} aY^+_{j_1 + j_2, m_1 + m_2}, \\
[aQ^+_{j_1,m_1}, aY^-_{j_2,m_2}] &= \frac{j_1 (j_1 + j_2)}{j_2 - j_1 + 1} aP_{j_2 - j_1, m_1 + m_2}, \\

[aQ^-_{j_1,m_1}, O_{j_2,m_2}] &= j_2 Y^-_{j_1 - j_2, m_1 + m_2}, \\
\{aQ^-_{j_1,m_1}, Y^+_{j_2,m_2}\} &= - \frac{j_2 (j_2 - 1)}{j_1 - j_2 + 1} P_{j_1 - j_2, m_1 + m_2} \\
& \quad - \frac{j_1}{j_1 - j_2 + 1} (m_1 j_2 + m_2 j_1 + m_2) aY^-_{j_1 - j_2 + 1, m_1 + m_2}, \\
\{aQ^-_{j_1,m_1}, aO_{j_2,m_2}\} &= - (j_1 + j_2 + 1) aY^-_{j_1 - j_2, m_1 + m_2}, \\
[aQ^-_{j_1,m_1}, aY^+_{j_2,m_2}] &= \frac{j_2 (j_1 + j_2)}{j_1 - j_2 + 1} aP_{j_1 - j_2, m_1 + m_2}, \\

[a\hat{Z}_{j_1,m_1}, O_{j_2,m_2}] &= \frac{j_2 (j_1 + 1)}{j_1 - j_2 + 1} P_{j_1 - j_2, m_1 + m_2} \\
& \quad + \frac{j_1 + 1}{j_1 - j_2 + 1} (m_1 j_2 + m_2 j_1 + m_2) aY^-_{j_1 - j_2 + 1, m_1 + m_2}, \\
[a\hat{Z}_{j_1,m_1}, Y^+_{j_2,m_2}] &= \frac{j_1 + 1}{j_1 - j_2 + 1} (m_1 j_2 + m_2 j_1 + m_2) aP_{j_1 - j_2 + 1, m_1 + m_2}, \\
[a\hat{Z}_{j_1,m_1}, aO_{j_2,m_2}] &= \frac{(j_1 + 1) (j_1 + j_2 + 2)}{j_1 - j_2 + 1} aP_{j_1 - j_2, m_1 + m_2},
\end{align*}
The chiral charge algebra can be obtained from the above equations by acting with corresponding charges, as shown in sec. 5.

To simplify the presentation we omit the “magnetic” index of the (chiral) discrete formation rules in closed string theory and also the proof of eqs. (97) and (100).

In this appendix we will give some details about the computation of the trans-

}\]

\[\{\hat{Z}_{j_1 m_1}, aP_{j_2 m_2}\} =0, \quad [a\hat{Z}_{j_1 m_1}, aP_{j_2 m_2}] =0, \quad [a\hat{Z}_{j_1 m_1}, aP_{j_2 m_2}] =0.

The chiral charge algebra can be obtained from the above equations by acting with \(f_0[\mathrm{d}w] f_w[\mathrm{d}z] b(z)\) from the left. This action transforms all the local fields into the corresponding charges, as shown in sec. 5.

**Appendix C. Proof of Eqs. (97) and (100)**

In this appendix we will give some details about the computation of the transformation rules in closed string theory and also the proof of eqs. (97) and (100). To simplify the presentation we omit the “magnetic” index of the (chiral) discrete states and write the chiral cohomology ring structure equation as follows:

\[V_{j_1} V_{j_2} = A_j + aB_{j'},\]  

(116)

where \(V_{j_1}, V_{j_2}, A_j\) and \(B_{j'}\) are all chiral relative discrete states. By using (80) and (86) we can write the chiral transformation rules as follows:

\[[\hat{V}_{j_1}, V_{j_2}] = g_B(j')B_{j'};\]

\[[\hat{V}_{j_1}, aV_{j_2}] = (-1)^{|V_{j_1}|}(g_{V_j(j_1)} - 1)A_j - g_{V_j(j_2)}aB_{j'},\]

\[[a\hat{V}_{j_1}, V_{j_2}] = (g_{V_j(j_2)} - 1)A_j - g_{V_j(j_1)}aB_{j'},\]

\[[a\hat{V}_{j_1}, aV_{j_2}] = -(-1)^{|V_{j_1}|}(g_{V_j(j_1)} - g_{V_j(j_2)})aA_j.\]  

(117)
Note that the relation $g_A(j) = g_B(j') = g_V(j_1) + g_V(j_2) - 1$ has been used in our derivation of the above rules.

A relative closed string discrete state is just the product a left relative discrete state and a right relative discrete states: $V_{j_1,j_1} = V_{j_1}V_{j_1}$. A necessary and sufficient condition for the pairing (at the $SU(2)$ point, of course) is $g_V(j_1) = g_V(j_1)$. The charge $\hat{V}_{j_1} V_{j_1}$ derived from the discrete state $V_{j_1,j_1}$ can be written symbolically (for the purpose of doing calculation) as follows

$$\hat{V}_{j_1} V_{j_1} = \hat{V}_{j_1} V_{j_1} - (-1)^{|V_{j_1}|} V_{j_1} \hat{V}_{j_1}. \quad (118)$$

The action of $\hat{V}_{j_1,j_1}$ on $V_{j_2,j_2} = V_{j_2} V_{j_2}$ is computed as follows:

$$[\hat{V}_{j_1,j_1}, V_{j_2,j_2}] = [\hat{V}_{j_1} V_{j_1} - (-1)^{|V_{j_1}|} V_{j_1} \hat{V}_{j_1}, V_{j_2} V_{j_2}]$$

$$= (-1)^{|V_{j_1}|} V_{j_2} \left[ \hat{V}_{j_1} V_{j_2} - (-1)^{|V_{j_1}|} V_{j_2} \hat{V}_{j_1} \hat{V}_{j_2} \right]$$

$$= (-1)^{|V_{j_1}|} V_{j_2} \left[ g_B(j') \left( B_{j'} \hat{A}_{j'} - (-1)^{|V_{j_1}|} V_{j_1} + |V_{j_2}| A_j \hat{B}_{j'} \right) \right] \quad \quad (119)$$

From the above result we see that the action of any charge on the closed string discrete state gives only physical closed string states. No states other than those listed in (90) and (91) are created.

The action of $\hat{V}_{j_1,j_1}$ on the absolute state $(a + \bar{a}) V_{j_2,j_2}$ can also be computed. We have

$$[\hat{V}_{j_1,j_1}, (a + \bar{a}) V_{j_2,j_2}] = [\hat{V}_{j_1} V_{j_1} - (-1)^{|V_{j_1}|} V_{j_1} \hat{V}_{j_1}, a V_{j_2} \hat{V}_{j_2} + (-1)^{|V_{j_2}|} V_{j_2} \hat{V}_{j_2}]$$

$$= (-1)^{|V_{j_1}|} V_{j_2} \left[ g_B(j') (a + \bar{a}) \left( B_{j'} \hat{A}_{j'} - (-1)^{|V_{j_1}|} V_{j_1} + |V_{j_2}| A_j \hat{B}_{j'} \right) \right] \quad \quad (120)$$

by using (119). This proves (97).
Eq. (100) can be proved similarly. We have

\[
\begin{align*}
&\left[(a + \bar{a}) \hat{V}_{j_1\bar{j}_1}, V_{j_2\bar{j}_2}\right] \\
&= \left[\hat{V}_{j_1}\bar{V}_{\bar{j}_1} - V_{j_1} \bar{a} \hat{V}_{\bar{j}_1} + (-1)^{|V_{j_1}|} a V_{j_1} \hat{V}_{\bar{j}_1} + (-1)^{|V_{j_1}|} \hat{V}_{j_1} \bar{a} \hat{V}_{\bar{j}_1}, V_{j_2} \bar{V}_{\bar{j}_2}\right] \\
&= (-1)^{|V_{j_1}|}|V_{j_2}| g_A(j)(a + \bar{a}) \left( - B_{j'} \bar{A}_{\bar{j}} + (-1)^{|V_{j_1}|} + |V_{j_2}| A_j \bar{B}_{\bar{j}} \right) \\
&= - (a + \bar{a}) \left(\hat{V}_{j_1\bar{j}_1}, V_{j_2\bar{j}_2}\right),
\end{align*}
\]

(121)

and

\[
\begin{align*}
&\left[(a + \bar{a}) \hat{V}_{j_1\bar{j}_1}, (a + \bar{a}) V_{j_2\bar{j}_2}\right] = \left[\hat{V}_{j_1}\bar{V}_{\bar{j}_1} - V_{j_1} \bar{a} \hat{V}_{\bar{j}_1} + (-1)^{|V_{j_1}|} a V_{j_1} \hat{V}_{\bar{j}_1} \\
&\quad + (-1)^{|V_{j_1}|} \hat{V}_{j_1} \bar{a} \hat{V}_{\bar{j}_1}, a V_{j_2} \bar{V}_{\bar{j}_2} + (-1)^{|V_{j_2}|} V_{j_2} \bar{a} \hat{V}_{\bar{j}_2}\right] \\
&= 0.
\end{align*}
\]

(122)

This concludes the proof of (100).
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