Theta Functions for SL($n$) versus GL($n$)

Ron Donagi and Loring W. Tu

November 1, 1992

§1. Theta bundles
§2. A Galois covering
§3. Pullbacks
§4. Proof of Theorem 1
§5. A conjectural duality

Over a smooth complex projective curve $C$ of genus $g$ one may consider two types of moduli spaces of vector bundles, $\mathcal{M} := \mathcal{M}(n, d)$, the moduli space of semistable bundles of rank $n$ and degree $d$ on $C$, and $\mathcal{SM} := \mathcal{SM}(n, L)$, the moduli space of those bundles whose determinant is isomorphic to a fixed line bundle $L$ on $C$. We call the former a full moduli space and the latter a fixed-determinant moduli space. Since the spaces $\mathcal{SM}(n, L)$ are all isomorphic as $L$ varies in $\text{Pic}^d(C)$, we also write $\mathcal{SM}(n, d)$ to denote any one of them.

On both moduli spaces there are well-defined theta bundles, as we recall in Section §1. While the theta bundle $\theta$ on $\mathcal{SM}$ is uniquely defined, the theta bundles $\theta_F$ on $\mathcal{M}$ depend on the choice of complementary vector bundles $F$ of minimal rank over $C$. For any positive integer $k$, sections of $\theta_F^k$ generalize the classical theta functions of level $k$ on the Jacobian of a curve, and so we call sections of $\theta^k$ over $\mathcal{SM}$ and $\theta^k_F$ over $\mathcal{M}$ theta functions of level $k$ for SL($n$) and GL($n$) respectively.

Our goal is to study the relationship between these two spaces of theta functions. We prove a simple formula relating their dimensions, and then formulate a conjectural duality between these two spaces.

**Theorem 1** If $h = \gcd(n, d)$ is the greatest common divisor of $n$ and $d$, and $L \in \text{Pic}^d(C)$, then
\[
\dim H^0(\mathcal{SM}(n, L), \theta^k) \cdot k^g = \dim H^0(\mathcal{M}(n, d), \theta_F^k) \cdot h^g.
\]

Faltings [F] has proven the Verlinde formula for semisimple groups, which gives in particular the dimension of $H^0(\mathcal{SM}, \theta^k)$. The dimension of $H^0(\mathcal{M}, \theta_F^k)$ is thus determined by Theorem 1. When $k = 1$ and $d = 0$, [BNR] computes explicitly the two spaces in Theorem 1. Their result is a forerunner of Theorem 1.

Theorem 1 is consistent with and therefore lends credence to another, so far conjectural, relationship between these two types of theta functions. To explain this, start with integers $\bar{n}, \bar{d}, h, k$ such that $\bar{n}, h, k$ are positive and $\gcd(\bar{n}, \bar{d}) = 1$. Let $F \in \mathcal{M}(\bar{n}, \bar{d})$ and write
\[
\mathcal{SM}_1 = \mathcal{SM}(h\bar{n}, (\det F)^h) \quad \text{and} \quad \mathcal{M}_2 = \mathcal{M}(k\bar{n}, k(\bar{n}(g-1) - \bar{d})).
\]

The tensor product map $\tau$ sends $\mathcal{SM}_1 \times \mathcal{M}_2$ to $\mathcal{M}(hk\bar{n}^2, hk\bar{n}^2(g-1))$. 


Conjecture 2 The tensor product map induces a natural duality between $H^0(SM_1, \theta^k)$ and $H^0(\mathcal{M}_2, \theta_F^k)$.

For further discussion of this duality, including supporting evidence, see Section 5.

Notation and Conventions.

\[
j^0(\ ) = \dim H^0(\ )
\]

\[
J_d = \text{Pic}^d(C) = \{\text{isomorphism classes of line bundles of degree } d \text{ on } C\}
\]

\[
J = J_0 = \text{Pic}^0(C)
\]

\[
L_1 \boxtimes L_2 = \pi_1^*L_1 \otimes \pi_2^*L_2 \text{ if } L_i \text{ is a line bundle on } X_i \text{ and } \pi_i : X_1 \times X_2 \to X_i \text{ is the } i\text{-th projection}
\]

\[
S^hC = \text{the } h\text{th symmetric product of } C
\]

\[
T_n = \text{the group of } n\text{-torsion bundles on } C
\]

1 Theta bundles

We recall here the definitions of the theta bundles on a fixed-determinant moduli space and on a full moduli space. Our definitions are slightly different from but equivalent to those in [DN].

For $L \in \text{Pic}^d(C)$, the Picard group of $SM := SM(n, L)$ is $\mathbb{Z}$ and the theta bundle $\theta$ on $SM$ is the positive generator of $\text{Pic}(SM)$.

When $n$ and $d$ are such that $\chi(E) = 0$ for $E \in \mathcal{M}(n, d)$, i.e., when $d = (g - 1)n$, there is a natural divisor $\Theta \subset \mathcal{M}(n, n(g - 1))$:

\[
\Theta = \text{closure of } \{E \text{ stable in } \mathcal{M}(n, n(g - 1)) \mid h^0(E) \neq 0\}.
\]

The theta bundle $\theta$ over $\mathcal{M}(n, n(g - 1))$ is the line bundle corresponding to this divisor.

We say that a semistable bundle $F$ is complementary to another bundle $E$ if $\chi(E \otimes F) = 0$. We also say that $F$ is complementary to $\mathcal{M}(n, d)$ if $\chi(E \otimes F) = 0$ for any $E \in \mathcal{M}(n, d)$. It follows easily from the Riemann-Roch theorem that if $E \in \mathcal{M}(n, d)$, $h = \gcd(n, d)$, $n = h\tilde{n}$, and $d = h\tilde{d}$, then $F$ has rank $n_F$ and degree $d_F$, where

\[
n_F = k\tilde{n} \quad \text{and} \quad d_F = k(\tilde{n}(g - 1) - \tilde{d})
\]

for some positive integer $k$.

If $F$ is complementary to $\mathcal{M}(n, d)$, let

\[
\tau_F : \mathcal{M}(n, d) \to \mathcal{M}(nn_F, nn_F(g - 1))
\]

be the map

\[
E \mapsto E \otimes F.
\]

Pulling back the theta bundle $\theta$ from $\mathcal{M}(nn_F, nn_F(g - 1))$ via $\tau_F$ gives a line bundle $\theta_F := \tau_F^*\theta$ over $\mathcal{M}(n, d)$. (This bundle may or may not correspond to a divisor in $\mathcal{M}(n, d)$.) Let $\det : \mathcal{M}(n, d) \to J_d(C)$ be the determinant map. When $\text{rk } F$ is the minimal possible: $\text{rk } F = \tilde{n} = n/h$, then $\theta_F$ is called a theta bundle over $\mathcal{M}(n, d)$; otherwise, it is a multiple of a theta bundle. Indeed, we extract from [DN] the formula:
Proposition 3 Let $F$ and $F_0$ be two bundles complementary to $\mathcal{M}(n,d)$. If $\text{rk } F = \text{rk } F_0$, then
\[
\theta_F \simeq \theta_{F_0}^a \otimes \det^*(\det F \otimes (\det F_0)^{-a}),
\]
where we employ the usual identification of $\text{Pic}^0(C)$ with $\text{Pic}^0(J_0)$.

In particular, $\theta_F$ depends only on $\text{rk } F$ and $\det F$.

If $\theta_F$ is a theta bundle on $\mathcal{M}(n,d)$, then for any $L \in \text{Pic}^d(C)$, $\theta_F$ restricts to the theta bundle on $\mathcal{S}\mathcal{M}(n, L)$.

2 A Galois covering

Let $\tau: Y \to X$ be a covering of varieties, by which we mean a finite étale morphism. A deck transformation of the covering is an automorphism $\phi: Y \to Y$ that commutes with $\tau$. The covering is said to be Galois if the group of deck transformations acts transitively (hence simply transitively) on a general fiber of the covering.

Denote by $J = \text{Pic}^0(C)$ the group of isomorphism classes of line bundles of degree 0 on the curve $C$, and $G = T_n$, the subgroup of torsion points of order $n$. Fix $L \in \text{Pic}^d(C)$ and let $\mathcal{S}\mathcal{M} = \mathcal{S}\mathcal{M}(n,L)$, $J = J_0(C)$, and $\mathcal{M} = \mathcal{M}(n,d)$. Recall that the tensor product map
\[
\tau: \mathcal{S}\mathcal{M} \times J \to \mathcal{M}
\]
\[
(E,M) \mapsto E \otimes M
\]
gives an $n^{2g}$-sheeted étale map ([TT], Prop. 8). The group $G = T_n$ acts on $\mathcal{S}\mathcal{M} \times J$ by
\[
N.(E,M) = (E \otimes N^{-1}, N \otimes M).
\]

It is easy to see that $G$ is the group of deck transformations of the covering $\tau$ and that it acts transitively on every fiber. Therefore, $\tau: \mathcal{S}\mathcal{M} \times J \to \mathcal{M}$ is a Galois covering.

Proposition 4 If $\tau: Y \to X$ is a Galois covering with finite abelian Galois group $G$, then $\tau_*\mathcal{O}_Y$ is a vector bundle on $X$ which decomposes into a direct sum of line bundles indexed by the characters of $G$:
\[
\tau_*\mathcal{O}_Y = \sum_{\lambda \in \hat{G}} L_\lambda,
\]
where $\hat{G}$ is the character group of $G$.

Proof. Write $\mathcal{O} = \mathcal{O}_Y$. The fiber of $\tau_*\mathcal{O}$ at a point $x \in X$ is naturally a complex vector space with basis $\tau^{-1}(x)$. Hence, $\tau_*\mathcal{O}$ is a vector bundle over $X$. The action of $G$ on $\tau^{-1}(x)$ induces a representation of $G$ on $\tau_*\mathcal{O}(x)$ equivalent to the regular representation. Because $G$ is a finite abelian group, this representation of $G$ decomposes into a direct sum of one-dimensional representations indexed by the characters of $G$:
\[
(\tau_*\mathcal{O})(x) = \sum_{\lambda \in \hat{G}} L_\lambda(x).
\]

Thus, for every $\lambda \in \hat{G}$, we obtain a line bundle $L_\lambda$ on $X$ such such $\tau_*\mathcal{O} = \sum_{\lambda} L_\lambda$. \qed
3 Pullbacks

We consider the tensor product map
\[ \tau : \mathcal{SM}(n_1, L_1) \times \mathcal{M}(n_2, d_2) \to \mathcal{M}(n_1 n_2, n_1 d_2 + n_2 d_1) \]
\[ (E_1, E_2) \mapsto E_1 \otimes E_2, \]
where \( d_1 = \deg L_1 \). For simplicity, in this section we write \( \mathcal{SM}_1 = \mathcal{SM}(n_1, L_1) \), \( \mathcal{M}_2 = \mathcal{M}(n_2, d_2) \), and \( \mathcal{M}_{12} = \mathcal{M}(n_1 n_2, n_1 d_2 + n_2 d_1) \).

**Proposition 5** Let \( F = F_{12} \) be a bundle on \( C \) complementary to \( \mathcal{M}_{12} \). Then
\[ \tau^* \theta_F \simeq \theta^c \otimes \theta_{E_1 \otimes F} \]
for any \( E_1 \in \mathcal{SM}(n_1, L_1) \), where
\[ c := \frac{n_2 \rk F}{\rk F_1} = \frac{n_2 \rk F}{n_1 / \gcd(n_1, d_1)} \]
and \( F_1 \) is a minimal complementary bundle to \( E_1 \).

**Proof.** For \( E_2 \in \mathcal{M}(n_2, d_2) \), let \( \tau_{E_2} : \mathcal{SM}_1 \to \mathcal{M}_{12} \) be tensoring with \( E_2 \). Then
\[ \left( \tau^* \theta_F \right)|_{\mathcal{SM} \times \{E_2\}} = \tau_{E_2}^* \theta_F = \tau_{E_2}^* \tau_F^* \theta = \tau_{E_1 \otimes F}^* \theta = \theta^c, \]
where by Proposition 5
\[ c = \frac{\rk (E_2 \otimes F)}{\rk F_1} = \frac{n_2 \rk F}{n_1 / \gcd(n_1, d_1)}. \]
Similarly,
\[ \left( \tau^* \theta_F \right)|_{\{E_1\} \times \mathcal{M}_2} = \tau_{E_1}^* \theta_F = \tau_{E_1}^* \tau_F^* \theta = \tau_{E_1 \otimes F}^* \theta. \]
Note that the bundle \( \theta_{E_1 \otimes F} \) depends only on \( \rk (E_1 \otimes F) = n_1 \rk F \) and \( \det(E_1 \otimes F) = L_1^\rk F \otimes (\det F)^n_1 \). Hence, both \( (\tau^* \theta_F)|_{\mathcal{SM}_1 \times \{E_2\}} \) and \( (\tau^* \theta_F)|_{\{E_1\} \times \mathcal{M}_2} \) are independent of \( E_1 \) and \( E_2 \). By the seesaw theorem,
\[ \tau^* \theta_F \simeq \theta^c \otimes \theta_{E_1 \otimes F}. \]
\[ \Box \]

**Corollary 6** Let \( L \in \text{Pic}^d(C) \) and
\[ \tau : \mathcal{SM}(n, L) \times J_0 \to \mathcal{M}(n, d) \]
be the tensor product map. Suppose \( F \) is a minimal complementary bundle to \( \mathcal{M}(n, d) \). Choose \( N \in \text{Pic}^{g-1}(C) \) to be a line bundle such that \( N^n = L \otimes (\det F)^h \), where \( h = \gcd(n, d) \). Then
\[ \tau^* \theta_F = \theta \otimes \theta_N^{n^2/h}. \]
Proof. Apply the Proposition with \( \text{rk} F = n/h \) and \( n_1 = n, d_1 = d, n_2 = 1, d_2 = 0 \). Then \( c = 1 \).

By Proposition 3,
\[
\theta_{E_1 \otimes F} = \theta_{N}^{n_2/h} \otimes \text{det}^*(\text{det}(E_1 \otimes F) \otimes N^{-n^2/h}) = \theta_{N}^{n_2/h}.
\]

\[\square\]

4 Proof of Theorem 1

We apply the Leray spectral sequence to compute the cohomology of \( \tau^* \theta^k_F \) on the total space of the covering \( \tau : \mathcal{SM} \times J \to \mathcal{M} \) of Section 2. Recall that \( \mathcal{SM} = \mathcal{SM}(n, d) \), \( J = J_0 \), and \( \mathcal{M} = \mathcal{M}(n, d) \).

Because the fibers of \( \tau \) are 0-dimensional, the spectral sequence degenerates at the \( E_2 \)-term and we have
\[
H^0(\mathcal{SM} \times J, \tau^* \theta^k_F) = H^0(\mathcal{M}, \tau_* \tau^* \theta^k_F).
\]

By Cor. 3 and the Künneth formula, the left-hand side of (1) is
\[
H^0(\mathcal{SM} \times J, \tau^* \theta^k_F) = H^0(\mathcal{SM} \times J, \theta_{E_1} \otimes \theta_{N}^{n_2/h}) = H^0(\mathcal{SM}, \theta^k) \otimes H^0(J, \theta_{N}^{n_2/h}).
\]

By the Riemann-Roch theorem for an abelian variety,
\[
h^0(J, \theta_{N}^{n_2/h}) = (kn^2/h)^g.
\]

So the left-hand side of (1) has dimension
\[
h^0(\mathcal{SM}, \theta^k) \cdot (kn^2/h)^g. \tag{2}
\]

Next we look at the right-hand side of (1). By the projection formula and Prop. 4,
\[
\tau_* \tau^* \theta^k_F = \theta_{F}^k \otimes \tau_* \mathcal{O} = \theta_{F}^k \otimes \sum_{\lambda \in \hat{G}} L_{\lambda} = \sum_{\lambda \in \hat{G}} \theta_{F}^k \otimes L_{\lambda}.
\]

Our goal now is to show that for any character \( \lambda \in \hat{G} \),
\[
H^0(\mathcal{M}, \theta_{F}^k \otimes L_{\lambda}) \simeq H^0(\mathcal{M}, \theta_{F}^k). \tag{3}
\]

This will follow from two lemmas.

Lemma 7 The line bundle \( L_{\lambda} \) on \( \mathcal{M} \) is the pullback under \( \text{det} : \mathcal{M} \to J_d \) of some line bundle \( N_{\lambda} \) of degree 0 on \( J_d := \text{Pic}^d(C) \).
Lemma 8 For \( F \) a vector bundle as above, \( k \) a positive integer, and \( M \) a line bundle of degree 0 over \( C \),
\[
H^0(\mathcal{M}, \theta^k_F \otimes M) \simeq H^0(\mathcal{M}, \theta^k_F).
\]

Assuming these two lemmas, let's prove (3). By Proposition 3,
\[
\theta_{F \otimes M} = \theta_F \otimes \det^n \;
\]
hence,
\[
\theta^k_{F \otimes M} = \theta^k_F \otimes \det^n.
\]
If \( L = \det^* N \), and we choose a root \( M = N^{1/(nF)} \), then
\[
\theta^k_F \otimes L = \theta^k_F \otimes \det^* N = \theta^k_{F \otimes M}.
\]

Equation (3) then follows from Lemma 8.

Proof of Lemma 7. Define \( \alpha : SM \times J \to J \) to be the projection onto the second factor, \( \beta : M \to J \) to be the composite of \( \det : \mathcal{M} \to J_d \) followed by multiplication by \( L^{-1} : J_d \to J \), and \( \rho : J \to J \) to be the \( n \)-th tensor power map. Then there is a commutative diagram
\[
SM \times J \xrightarrow{\tau} \mathcal{M} \]
\[
\alpha \quad \downarrow \quad \downarrow \beta
\]
\[
J \xrightarrow{\rho} J.
\]

Furthermore, in the map \( \alpha : SM \times J \to J \) we let \( G = T \) act on \( J \) by
\[
N \cdot M = N \otimes M, \quad M \in J,
\]
and in the map \( \beta : \mathcal{M} \to J \) we let \( G \) act trivially on both \( \mathcal{M} \) and \( J \). Then all the maps in the commutative diagram above are \( G \)-morphisms.

By the push-pull formula ([H], Ch. III, Prop. 9.3, p. 255),
\[
\tau_\ast \alpha^* \mathcal{O}_J = \beta^* \rho_\ast \mathcal{O}_J.
\]

By Proposition 4, \( \rho_\ast \mathcal{O}_J \) is a direct sum of line bundles \( V_\lambda \) on \( J \), where \( \lambda \in \hat{G} \). In fact, these \( V_\lambda \) are precisely the \( n \)-torsion bundles in \( J \); in particular, their degrees are zero. If \( \tau_{L^{-1}} : J_d \to J \) is multiplication by the line bundle \( L^{-1} \), we set \( N_\lambda := \tau^*_{L^{-1}} V_\lambda \). Then
\[
\tau_\ast \mathcal{O}_{SM \times J} = \beta^* \sum_{\lambda \in \hat{G}} V_\lambda
\]
\[
= \det^* \tau^*_{L^{-1}} \sum V_\lambda
\]
\[
= \sum \det^* N_\lambda.
\]

By Prop. 4, \( \tau_\ast \mathcal{O}_{SM \times J} = \sum L_\lambda \). Since both \( L_\lambda \) and \( \det^* N_\lambda \) are eigenbundles of \( \tau_\ast \mathcal{O}_{SM \times J} \) corresponding to the character \( \lambda \in \hat{G} \),
\[
L_\lambda = \det^* N_\lambda.
\]
Proof of Lemma 8. Tensoring with \( M \in J_0(C) \) gives an automorphism

\[
\tau_M : \mathcal{M} \to \mathcal{M} \\
E \mapsto E \otimes M,
\]
under which

\[
\theta_{F \otimes M} = \tau_M^* \theta_F.
\]
Hence,

\[
\theta_{F \otimes M}^k = \tau_M^*(\theta_F^k)
\]
and the lemma follows. \( \square \)

Returning now to Eq. (1), its right-hand side is

\[
H^0(\mathcal{M}, \tau_* \tau^* \theta_F^k) = \sum_{\lambda \in \hat{G}} H^0(\mathcal{M}, \theta_F^k \otimes L_{\lambda}) \\
\simeq \sum_{\lambda \in \hat{G}} H^0(\mathcal{M}, \theta_F^k), \quad \text{(by (3))}
\]
which has dimension

\[
h^0(\mathcal{M}, \theta_F^k) \cdot n^{2g}.
\]
By (2) the left-hand side of Eq. (1) has dimension

\[
h^0(\mathcal{S}\mathcal{M}, \theta^k) \cdot (kn^2/h)^g.
\]
Equating these two expressions gives

\[
h^0(\mathcal{M}, \theta_F^k) = h^0(\mathcal{S}\mathcal{M}, \theta^k) \cdot \left(\frac{k}{h}\right)^g.
\]
This completes the proof of Theorem 1.

5 A conjectural duality

As in the Introduction we start with integers \( \bar{n}, \bar{d}, h, k \) such that \( \bar{n}, h, k \) are positive and \( \gcd(\bar{n}, \bar{d}) = 1 \). Take

\[
n_1 = h\bar{n}, \quad d_1 = h\bar{d}, \quad n_2 = k\bar{n}, \quad d_2 = k(\bar{n}(g - 1) - \bar{d}), \quad \text{and} \quad L_1 \in \text{Pic}^{d_1}(C).
\]
The tensor product induces a map

\[
\tau : \mathcal{S}\mathcal{M}(n_1, L_1) \times \mathcal{M}(n_2, d_2) \to \mathcal{M}(n_1 n_2, n_1 n_2(g - 1)).
\]
As before, write \( \mathcal{S}\mathcal{M}_1 = \mathcal{S}\mathcal{M}(n_1, L_1), \mathcal{M}_2 = \mathcal{M}(n_2, d_2), \) and \( \mathcal{M}_{12} = \mathcal{M}(n_1 n_2, n_1 n_2(g - 1)) \). Let \( F_2 = F \) and \( F_{12} = \mathcal{O} \) be complementary to \( \mathcal{M}_2 \) and \( \mathcal{M}_{12} \) respectively.

By the pullback formula (Proposition 3)

\[
\tau^* \theta_{\mathcal{O}} = \theta_{n_2/\bar{n}} \boxtimes \theta_{E_1}.
\]
But by Proposition \[3\],

\[ \theta_{E_1} = \theta_F^h \otimes \det^*(L \otimes (\det F)^{-h}). \]

If \( L = (\det F)^h \), then \( \theta_{E_1} = \theta_F^h \) and

\[ \tau^* \theta_O = \theta^k \boxtimes \theta_F^h. \]

By the Künneth formula,

\[ H^0(SM_1 \times M_2, \tau^* \theta_O) = H^0(SM_1, \theta^k) \otimes H^0(M_2, \theta_F^h). \]

In \[BNR\] it is shown that up to a constant, \( \theta_O \) has a unique section \( s \) over \( M_{12} \). Then \( \tau^* s \) is a section of \( H^0(SM_1 \times M_2, \tau^* \theta_O) \) and therefore induces a natural map

\[ H^0(SM_1, \theta^k)^{\vee} \to H^0(M_2, \theta_F^h). \tag{4} \]

We conjecture that this natural map is an isomorphism.

Among the evidence for the duality \( \tag{4} \), we cite the following.

i) (Rank 1 bundles) The results of \[BNR\] that

\[ H^0(SM(n, O), \theta)^{\vee} \simeq H^0(M(1, g - 1), \theta_O^k) \quad \text{and} \quad H^0(M(n, n(g - 1)), \theta_O) = \mathbb{C}, \]

are special cases of \( \tag{4} \), for \((n_2, d_2) = (1, g - 1)\) and \((n_1, d_1) = (1, 0)\) respectively.

ii) (Consistency with Theorem \[3\]) Given a triple of integers \((n_1, d_1, k)\), we define \( h, \bar{n}, \bar{d} \) by

\[ h = \gcd(n_1, d_1), \quad n_1 = h\bar{n}, \quad d_1 = h\bar{d} \]

and let \( n_2, d_2 \) be as before:

\[ n_2 = k\bar{n}, \quad d_2 = k(\bar{n}(g - 1) - \bar{d}). \]

Assuming \( n_1 \) and \( k \) to be positive, it is easy to check that the function

\[ (n_1, d_1, k) \mapsto (n_2, d_2, h) \]

is an involution. Write \( v(n, d, k) = h^0(M(n, d), \theta_F^k) \) and \( s(n, d, k) = h^0(SM(n, d), \theta^k) \). Then Theorem \[3\] assumes the form

\[ v(n, d, k) \cdot h^g = s(n, d, k) \cdot k^g. \tag{5} \]

The duality \( \tag{4} \) implies that there is an equality of dimensions

\[ s(n_1, d_1, k) = v(n_2, d_2, h). \tag{6} \]

Because \((n_1, d_1, k) \mapsto (n_2, d_2, h)\) is an involution, it follows that

\[ s(n_2, d_2, h) = v(n_1, d_1, k). \tag{7} \]

Putting \( \tag{3}, \tag{5}, \) and \( \tag{7} \) together, we get

\[ v(n_2, d_2, h)k^g = s(n_2, d_2, h)h^g, \]

which is Theorem \[3\] again.
iii) (Elliptic curves) We keep the notation above, specialized to the case of a curve $C$ of genus $g = 1$:

$$n_1 = h\bar{n}, d_1 = h\bar{d}, n_2 = k\bar{n}, d_2 = -k\bar{d}.$$ 

Set $C' := \text{Pic}^d(C)$.

The map sending a line bundle to its dual gives an isomorphism $C' \simeq \text{Pic}^{-d}(C)$. If $L \in \text{Pic}^d(C)$, viewed as a line bundle on $C$, we let $\ell$ be the corresponding point in $C'$, and $\mathcal{O}_{C'}(\ell)$ the associated line bundle of degree 1 on the curve $C'$. There is a natural map

$$\gamma : \text{Pic}^{h\bar{d}}(C) \to \text{Pic}^{h}(C')$$

which sends $L := L_1 \otimes \cdots \otimes L_h \in \text{Pic}^{h\bar{d}}(C)$ to $L' := \mathcal{O}_{C'}(\ell_1 + \cdots + \ell_h)$, where $L_i \in \text{Pic}^{\bar{d}}(C)$ corresponds to the point $\ell_i \in C'$.

From [A] and [T] we see that there are natural identifications

$$\mathcal{M}(h\bar{n}, h\bar{d}) \simeq S^h\mathcal{M}(\bar{n}, \bar{d}) \simeq S^h\text{Pic}^d(C) = S^hC'$$

and

$$\mathcal{M}(k\bar{n}, -k\bar{d}) \simeq S^k\mathcal{M}(\bar{n}, -\bar{d}) \simeq S^k\text{Pic}^{-d}(C) \simeq S^kC'.$$

Furthermore, there is a commutative diagram

$$\begin{array}{ccc}
\mathcal{M}(h\bar{n}, h\bar{d}) & \supseteq & S^hC' \\
\text{det} & \downarrow & \downarrow \alpha \\
\text{Pic}^{h\bar{d}}(C) & \xrightarrow{\gamma} & \text{Pic}^{h}(C').
\end{array}$$

Since the fiber of the Abel-Jacobi map $\alpha : S^hC' \to \text{Pic}^{h}(C')$ above $L'$ is the projective space $\mathbb{P}H^0(C', L')$, it follows that there is a natural identification

$$S\mathcal{M}(h\bar{n}, L) \simeq \mathbb{P}H^0(C', L').$$

Since the theta bundle is the positive generator of $S\mathcal{M}(h\bar{n}, L)$, it is the hyperplane bundle. For $F \in \mathcal{M}(\bar{n}, \bar{d})$, let $q \in C'$ be the point corresponding to the line bundle $Q := \text{det} F \in \text{Pic}^d(C)$. Then

$$H^0(S\mathcal{M}(h\bar{n}, (\det F)^h), \theta^k) \simeq H^0(\mathbb{P}H^0(C', \mathcal{O}_{C'}(hq)), \mathcal{O}(k)) = S^kH^0(C', \mathcal{O}_{C'}(hq))^\vee.$$ 

Recall that each point $q \in C'$ determines a divisor $X_q$ on the symmetric product $S^kC'$:

$$X_q := \{ q + D \mid D \in S^{k-1}C' \}.$$ 

The proof of Theorem 6 in [T] actually shows that if $F \in \mathcal{M}(\bar{n}, -\bar{d})$, then under the identification $\mathcal{M}(k\bar{n}, -k\bar{d}) \simeq S^kC'$, the theta bundle $\theta_F$ corresponds to the bundle associated to the divisor $X_q$ on $S^kC'$, where $q$ is the point corresponding to $\text{det} F \in \text{Pic}^\bar{d}$. Therefore, by the calculation of the cohomology of a symmetric product in [T]

$$H^0(\mathcal{M}(k\bar{n}, -k\bar{d}), \theta_F^h) = H^0(S^kC', \mathcal{O}(hX_q)) = S^kH^0(C', \mathcal{O}(hq)).$$

So the two spaces $H^0(S\mathcal{M}(h\bar{n}, (\det F)^h), \theta^k)$ and $H^0(\mathcal{M}(k\bar{n}, -k\bar{d}), \theta_F^h)$ are naturally dual to each other.
iv) (Degree 0 bundles) Consider the moduli space $\mathcal{S}\mathcal{M}(n,0)$ of rank $n$ and degree 0 bundles. In this case,

$$n_1 = n, \quad d_1 = 0, \quad h = \gcd(n,0) = n, \quad n_2 = k, \quad d_2 = k(g - 1).$$

So the conjectural duality is

$$H^0(\mathcal{S}\mathcal{M}(n,\mathcal{O}), \theta^k) \cong H^0(\mathcal{M}(k, k(g-1)), \theta^n_{\mathcal{O}}).$$

Because $\mathcal{M}(k, k(g-1))$ is isomorphic to $\mathcal{M}(k,0)$ (though noncanonically), it follows that in the notation of ii)

$$s(n, 0, k) = v(k, 0, n).$$

According to R. Bott and A. Szenes, this equality follows from Verlinde’s formula.

References

[A] M. Atiyah, Vector bundles over an elliptic curve, *Proc. London Math. Soc.* 7 (1957), 414-452.

[BNR] A. Beauville, M. S. Narasimhan, and S. Ramanan, Spectral curves and the generalized theta divisor, *J. reine angew. Math.* 398 (1989), 169-179.

[DN] J.-M. Drezet and M. S. Narasimhan, Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques, *Invent. Math.* 97 (1989), 53-94.

[F] G. Faltings, A proof of the Verlinde formula, preprint.

[H] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York, 1977.

[TT] M. Teixidor and L. W. Tu, Theta divisors for vector bundles, in *Curves, Jacobians, and Abelian Varieties*, Contemporary Mathematics 136 (1992), 327-342.

[T] L. W. Tu, Semistable bundles over an elliptic curve, to appear in *Advances in Mathematics.*