Some possible $q$-generalizations of harmonic numbers

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Abstract

We study three different $q$-analogue of the harmonic numbers. As applications, we present some generating functions involving numerical theoretical functions and give the $q$-generalization of Gospers’s exponential generating function of harmonic numbers. We involve also the $q$-gamma and $q$-digamma function.

Key words: $q$-harmonic number, harmonic number, divisor function, Gosper identity, Hockey Stick Theorem, $q$-gamma function, $q$-digamma function, $q$-Euler-Mascheroni constant

1991 MSC: 05A30

1 Introduction

The harmonic numbers are defined as

$$H_n = \sum_{k=1}^{n} \frac{1}{k}, \quad H_0 := 0.$$  \hspace{1cm} (1)

Our aim is to find some $q$-analogue of these numbers. We start from some basic identities satisfied by the harmonic numbers. For example,
\[ H_n = 1/n! s(n + 1, 2), \]  
\[ H_n = \sum_{k=1}^{n} \binom{n}{k} (-1)^{k+1}, \]  

(2)  
(3)

where \( s(n + 1, 2) \) is a Stirling number of the first kind \[6\].

Next we introduce the most basic notions of \( q \)-calculus. Let

\[ [n]_q = \frac{1 - q^n}{1 - q}, \]

and

\[ [n]_q! = [n]_q[n - 1]_q \cdots [1]_q. \]

Define also \((x; q)_n = (1 - x)(1 - qx) \cdots (1 - q^{n-1}x)\) and \((x; q)_\infty = \lim_{n \to \infty} (x; q)_n\).

Then the \( q \)-binomial coefficient with parameter \( n \) and \( k \) is

\[ \binom{n}{k}_q = \frac{[n]_q!}{[k]_q! [n - k]_q!} = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}} \quad (n \geq k \geq 0). \]

According to (1)-(3) we define three class of \( q \)-harmonic numbers as

\[ H^1_{n,q} = \sum_{k=1}^{n} \frac{1}{[k]_q}, \]  
\[ H^2_{n,q} = \frac{1}{[n]_q!} s_q(n + 1, 2), \]  
\[ H^3_{n,q} = \sum_{k=1}^{n} \binom{n}{k}_q q^{\binom{k}{2}} (-1)^{k+1} \]  
\[ H^4_{n,q} = \ln(q) \sum_{k=1}^{n} \frac{q^k}{q^k - 1}. \]  

(4)  
(5)  
(6)  
(7)

(Let \( H^i_{0,q} := 0 \) for \( i = 1, 2, 3, 4 \).) Here \( s_q(n + 1, 2) \) is a \( q \)-Stirling number of the first kind \[1\].

The first definition appears in \[4,11\], for example. We point out that the first and second definitions are the same but the third and the fourth are different from each other — although all of them tend to \( H_n \) as \( q \to 1 \). That \( H^4_{n,q} \) differs from the others is obvious because of the presence of \( \ln(q) \).

In what follows we deduce some identities and give applications involving these numbers.
2 Identities involving $H_{n,q}^1$ and $H_{n,q}^2$

2.1 Number theoretical results

In this section we point out that the $H_{n,q}^1 q$-harmonic numbers are connected to the divisor function. Moreover, the products of the Riemann zeta function and polylogarithms are Dirichlet generating functions of some interesting number theoretical “polynomials”.

By definitions,

$$H_{n,q}^1 = (1 - q) \sum_{k=1}^{n} \frac{1}{1 - q^k}.$$  

Cauchy’s product gives that

$$\sum_{n \geq 1} H_{n,q}^1 x^n = \frac{1 - q}{1 - x} \sum_{n \geq 1} x^n \frac{1}{1 - q^n}. \quad (8)$$

We need the notion of Lambert series \[7, p. 257\]. In general, a Lambert series has the form

$$F(q) = \sum_{n \geq 1} a_n \frac{q^n}{1 - q^n},$$

where $a_n$ is any suitable sequence. This connection implies some interesting results. We cite a useful theorem of \[7, Theorem 307\]: if

$$f(s) = \sum_{n \geq 1} \frac{a_n}{n^s}, \quad \text{and} \quad g(s) = \sum_{n \geq 1} \frac{b_n}{n^s},$$

then

$$F(q) = \sum_{n \geq 1} a_n \frac{q^n}{1 - q^n} = \sum_{n \geq 1} b_n q^n$$

holds if and only if

$$\zeta(s)f(s) = g(s),$$

where

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$$

is the Riemann zeta function.

Let us apply this and (8):

$$\sum_{n \geq 1} H_{n,q}(qx)^n = \frac{1 - q}{1 - qx} \sum_{n \geq 1} x^n \frac{q^n}{1 - q^n} = \frac{1 - q}{1 - qx} F(q) \quad (9)$$
with \( a_n = x^n \). A simple transformation shows that

\[
F(q) = \sum_{n \geq 1} q^n \left( \sum_{d|n} x^d \right).
\]  

(10)

Let us introduce the notation

\[
A(x, n) = \sum_{d|n} x^d.
\]

In special,

\[
A(1, n) = d(n) = \sigma_0(n),
\]

where \( d(n) \) is the divisor function and

\[
\sigma_x(n) = \sum_{d|n} d^x.
\]

It is worth to realize the symmetry between the power and the base in \( A(x, n) \) and \( \sigma_x(n) \).

Substituting \( x = 1 \) in (9) we have

**Proposition 1**

\[
\sum_{n \geq 1} H_{n,q} q^n = \sum_{n \geq 1} q^n d(n).
\]

So the generating function of the divisor function involves the \( q \)-harmonic numbers. More generally, with the help of (9) and (10) and the definition of \( A(x, n) \) one can write

\[
\sum_{n \geq 1} H_{n,q}(qx)^n = \frac{1-q}{1-qx} \sum_{n \geq 1} q^n A(x, n).
\]

Again, by (9)

\[
\sum_{n \geq 1} x^n \frac{q^n}{1-q^n} = \sum_{n \geq 1} q^n A(x, n).
\]

Hence we choose \( a_n = x^n \) and \( b_n = A(x, n) \) and apply the theorem cited above:

\[
\text{Li}_s(x) = \sum_{n \geq 1} \frac{x^n}{n^s}, \quad \text{and} \quad g(s) = \sum_{n \geq 1} \frac{A(x, n)}{n^s},
\]

then

**Proposition 2**

\[
\zeta(s) \text{Li}_s(x) = \sum_{n \geq 1} \frac{A(x, n)}{n^s}.
\]
Here \( \text{Li}_s(x) \) is the well-known polylogarithm function. In special \( \text{Li}_s(1) = \zeta(s) \), so we get the known sum

\[
\zeta^2(s) = \sum_{n \geq 1} \frac{d(n)}{n^s}.
\]

### 2.2 A recursion for \( H_{n,q}^1 \)

Since

\[
H_n = \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k+1}}{k},
\]

we may think that

\[
H_{n,q}^1 = \sum_{k=1}^{n} \binom{n}{k} a_k
\]

holds for some sequence \( a_k \). This is really true but, sadly, \( a_k \) does not have a simple form. See the table below.

We shall need the notion of the \( q \)-Seidel matrix [2]. Given a sequence \( a_n \), the \( q \)-Seidel matrix is associated to the double sequence \( a^n_k \) given by the recurrence

\[
a^n_0 = a_n \quad (n \geq 0),
\]

\[
a^n_k = q^n a^{k-1}_n + a^{k-1}_{n+1} \quad (n \geq 0, k \geq 1).
\]

In addition, \( a^n_0 \) is called the initial sequence and \( a^n_0 \) the final sequence of the \( q \)-Seidel matrix. Then the identity

\[
a^n_0 = \sum_{k=0}^{n} \binom{n}{k} q^k a^n_0.
\]

connects the initial and the final sequence.

Define the generating functions of \( a^n_0 \) and \( a^n_0 \):

\[
a(x) = \sum_{n \geq 0} a^n_0 x^n, \quad \overline{a}(x) = \sum_{n \geq 0} a^n_0 x^n,
\]

and

\[
A(x) = \sum_{n \geq 0} a^n_0 \frac{x^n}{[n]_q!}, \quad \overline{A}(x) = \sum_{n \geq 0} a^n_0 \frac{x^n}{[n]_q!}.
\]

A proposition given in [2] states that these functions are related by the following equations:
\[ \overline{x}(x) = \sum_{n \geq 0} a_n^0 \overline{x}^n, \quad (13) \]
\[ \overline{A}(x) = e_q(x) A(x), \quad (14) \]

where

\[ e_q(x) = \sum_{n \geq 0} \frac{x^n}{[n]_q!} \]

is the \( q \)-analogue of the exponential function \([5]\). We introduce the notation \( \text{Egf}(a_n) \) and \( \text{Gf}(a_n) \) for the exponential and ordinary generating function of \( a_n \), respectively.

To reach our aim posed in (11), our approach is as follows. Let the final sequence \( b_n = H^1_{n,q} \). We determine the initial sequence \( a_0^0 = a_n \). Then \( \text{Egf}(b_n) = \text{Egf}(H^1_{n,q}) = e_q \text{Egf}(a_n) \). And, to get \( \text{Egf}(a_n) \) we determine \( a_n \) by using (8) and (13):

\[ \text{Gf}(b_n) \equiv \text{Gf}(H^1_{n,q}) = \sum_{n \geq 1} \frac{x^n}{(x; q)_n} \]

(15)

From this equation \( a_n \) can be determined. (Note that \( a_0 = 0 \).)

**Proposition 3** We have

\[ H^1_{n,q} = \sum_{k=1}^{n} \binom{n}{k}_q a_k, \]

where the sequence \( a_k \) is determined recursively by

\[ \sum_{k=1}^{n} a_k q^{n-k} \binom{n-1}{k-1}_q = \frac{1}{[n]_q} = \frac{1-q}{1-q^n} \quad (a_0 := 0). \]

**Proof.** The denominator of the right hand side of (13) is

\[ \frac{1}{(x; q)_n+1} = \frac{1}{1-x} \frac{1}{(q^n x; q)_n} = \frac{1}{1-x} (q^n q x; q)_{\infty}. \quad (16) \]

The \( q \)-binomial theorem \([5]\), Section 1.3] states that

\[ \frac{(az; q)_{\infty}}{(z; q)_{\infty}} = \sum_{k \geq 0} \frac{(a; q)_k}{(q; q)_k} z^k. \]

Applying this to (16),

\[ \frac{1}{1-x} (q^n q x; q)_{\infty} = \frac{1}{1-x} \sum_{k \geq 0} \frac{(q^n; q)_k}{(q; q)_k} (q x)^k. \]
Thus (15) becomes

\[(1 - q) \sum_{n \geq 1} \frac{x^n}{1 - q^n} = \sum_{n \geq 0} a_n x^n \left( \sum_{k \geq 0} \frac{(q^n; q)_k (q x)^k}{(q; q)_k} \right).\]

Let

\[B_{k,n} = \frac{(q^n; q)_k}{(q; q)_k} q^k,\]

for short. Then

\[B_{k,n} = q^k \binom{n + k - 1}{k},\]

for all \(n\) and \(k\). Moreover,

\[(1 - q) \sum_{n \geq 1} \frac{x^n}{1 - q^n} = \sum_{n \geq 0} a_n x^n \left( \sum_{k \geq 0} B_{k,n} x^k \right). \tag{17}\]

If we write the sums term by term, we get

\[a_0(B_{0,0} + B_{1,0}x + B_{2,0}x^2 + \cdots) + a_1 x(B_{0,1} + B_{1,1}x + B_{2,1}x^2 + \cdots) + \cdots =\]

\[a_0B_{0,0} + x(a_0B_{1,0} + a_1 B_{1,1}) + x^2(a_0B_{2,0} + a_1 B_{1,1} + a_2 B_{0,2}) + \cdots\]

Comparing the coefficients here and the left hand side of (17), we have

\[\sum_{k=0}^{n} a_k B_{n-k,k} = \frac{1 - q}{1 - q^n}.\]

Note that – because of (15) – \(a_0\) must be zero. Remember also that \(a_k\) is the initial sequence of our \(q\)-Seidel matrix, so (12) gives

\[H_{n,q}^1 = \sum_{k=1}^{n} \binom{n}{k} q^k. \tag{18}\]

This is our proposition.

**Remark.** It is worth to present the first terms of the sequence \(a_n\):

\[a_0 = 0,\]
\[a_1 = 1,\]
\[a_2 = -\frac{q^2 + q - 1}{q + 1},\]
\[a_3 = \frac{q^5 + q^4 - q^2 - q + 1}{q^2 + q + 1},\]
\[a_4 = -\frac{q^9 + q^8 - 2q^5 + q^2 + q - 1}{q^3 + q^2 + q + 1},\]
\[a_5 = \frac{q^{14} + q^{13} - q^{10} - q^9 - q^8 + q^7 + q^6 + q^5 - q^2 - q + 1}{q^4 + q^3 + q^2 + q + 1},\]
\[ a_6 = \frac{-q^{20} + q^{19} - q^{16} - 2q^{14} + q^{12} + q^{11} + q^{10} + q^9 - 2q^7 - q^5 + q^2 + q - 1}{q^6 + q^4 + q^3 + q^2 + q + 1}. \]

It seemed to be interesting to give a simple formula for the nominator. However, one can easily see that

\[ a_k \to \frac{(-1)^{k+1}}{k} \quad \text{as} \quad q \to 1 \quad (k = 1, 2, 3, 4, 5, 6). \]

According to (3), this is plausible for all \( a_k \).

As a consequence of (14) and (18), we have the next connection:

\[ \text{Egf}(H_{n,q}^1) = e_q \text{Egf}(a_n). \]

This is a curious version of Gosper’s identity (21) involving the exponential generating function of the harmonics.

\[ \text{(19)} \]

### 2.3 The case \( H_{n,q}^2 \)

To close the case of \( H_{n,q}^1 \), we remark also that

\[ H_{n,q}^1 = H_{n,q}^2 \]

(see (5) for the definitions). The \( q \)-Stirling numbers of the first kind are defined recursively by [1, p. 155]

\[ s_q(n + 1, k) = s_q(n, k - 1) + [n]_q s_q(n, k), \quad (20) \]

and \( s_q(0, 0) = 1, \ s_q(n, 0) = 0 \) when \( n > 0 \).

These relations imply

\[ H_{n,q}^1 = \frac{1}{[n]_q} s_q(n + 1, 2) = \frac{1}{[n]_q} s_q(n, 1) + \frac{1}{[n - 1]_q} s_q(n, 2), \]

hence

\[ H_{n,q}^1 = H_{n-1,q}^1 + \frac{1}{[n]_q!} s_q(n, 1) = H_{n-1,q}^1 + \frac{[n - 1]_q!}{[n]_q!}. \]

Then (20) gives (19).
3 Identities involving $H_{n,q}^3$

3.1 A $q$-analogue of Gosper’s result

The exponential generating function of the harmonic numbers is deduced by Gosper [3,9]:

$$\sum_{n \geq 1} H_n \frac{x^n}{n!} = x e^{x} \sum_{n \geq 0} \binom{a_1}{n} \binom{a_2}{n} \cdots \binom{a_r}{n} x^n$$

(21)

Here

$$\sum_{n \geq 0} \binom{a_1}{n} \binom{a_2}{n} \cdots \binom{a_r}{n} x^n (a_1 q^n \cdots (a_r q^n x^n)$$

is the hypergeometric function with parameters $a_i$ and $b_j$, and

$$(a)_k = a(a + 1)(a + 2) \cdots (a + k - 1)$$

under the agreement $(a)_0 = 1$.

The $q$-version of the hypergeometric function is called basic hypergeometric function and defined as [5]

$$\sum_{n \geq 0} \binom{a_1}{n} \binom{a_2}{n} \cdots \binom{a_r}{n} x^n$$

(22)

One may see from (22) that

$$\frac{(k + 1)\text{th term}}{k\text{th term}} = \frac{x(-q^k)^{1+s-r}}{1 - q^{k+1}} (1 - a_1 q^k) \cdots (1 - a_r q^k)$$

$$\cdots (1 - b_1 q^k) \cdots (1 - b_s q^k).$$

Now we derive the $q$-analogue of Gosper’s result (21) with respect to $H_{n,q}^3$.

Equations (6) and (14) give that

$$\sum_{k \geq 1} q^{(k)} \binom{a}{[k]} x^k \frac{(-1)^{k+1}}{[k]_q}.$$

Hence it is enough to determine the sum

$$\sum_{k \geq 1} q^{(k)} \binom{a}{[k]} x^k \frac{(-1)^{k+1}}{[k]_q}.$$
We would like to express this function as a basic hypergeometric series. Thus \( k \) should run from 0, so we write
\[
x \sum_{k \geq 0} q^{(k+1)}\frac{(-1)^{k+2}}{[k+1]_q [k+1]_q!} x^k
\]
whence
\[
\frac{(k+1)\text{th term}}{k\text{th term}} = (1 - q)(-xq)^k \frac{(1 - q^2)^k}{1 - q^{k+1} (1 - q^2q^k)^2}.
\]
Since \( s = r = 2 \) in our case, the place of \((-1)^n\) is indifferent. Finally we can state the following

**Proposition 4** We have
\[
\sum_{n \geq 1} H^3_{n,q} x^n \frac{x^n}{[n]_q!} = (1 - q)x e_q(x) _2\phi_2 \left( \begin{array}{c} q, q \\ q^2, q^2 \end{array} \right| q; -qx \right).
\]

3.2 The Hockey Stick Theorem – a \( q \)-analogue

It is natural to ask, what is the recursion satisfied by \( H^3_{n,q} \)? If \( H^3_{n,q} = H^3_{n-1,q} + \frac{1}{[n]_q} \) would be true, we knew that \( H^3_{n,q} = H^1_{n,q} \). But this relation does not hold. In fact, the next recursion is valid.

**Proposition 5** For \( H^3_{n,q} \)
\[
H^3_{n,q} = H^3_{n-1,q} + q - \frac{[n-1]_q}{[n]_q}.
\]

We see that the limit of \( q - \frac{[n-1]_q}{[n]_q} \) is \( 1 - \frac{n-1}{n} = \frac{1}{n} \). So the standard recursion for harmonic numbers holds just asymptotically. To prove the proposition, we need the following statement. The standard (not \( q \)) version can be found at the webpage [http://binomial.csueastbay.edu/IdentitiesNamed.html](http://binomial.csueastbay.edu/IdentitiesNamed.html), Catalog #: 3400001.

**Proposition 6 (Hockey Stick Theorem)**
\[
\sum_{k=1}^{n-j} (-1)^{k+1} \binom{n}{j+k} = \binom{n-1}{j}.
\]

The name comes from the fact that the summands and the sum has a shape in the Pascal triangle like a hockey stick.
We prove the $q$-analogue version.

**Proposition 7 (Hockey Stick Theorem – a $q$-analogue)**

$$
\sum_{k=1}^{n-j} (-1)^{k+1} q^{(j+k)} \binom{n}{j+k}_q = \binom{n-1}{j}_q.
$$

**Proof.** Write the sum term by term:

$$
q^{(j+1)} \binom{n}{j+1}_q - q^{(j+2)} \binom{n}{j+2}_q + q^{(j+3)} \binom{n}{j+3}_q - \cdots + 

(-1)^{n-j} q^{(n-1)} \binom{n}{n-1}_q + (-1)^{n-j+1} q^{(n)} \binom{n}{n}_q.
$$

The binomial coefficients are rewritten with the recursion [5]

$$
\binom{n}{k}_q = q^k \binom{n-1}{k}_q + \binom{n-1}{k-1}_q:
$$

$$
q^{(j+1)} \left[ q^{j+1} \binom{n-1}{j+1}_q + \binom{n-1}{j}_q \right] - q^{(j+2)} \left[ q^{j+2} \binom{n-1}{j+2}_q + \binom{n-1}{j+1}_q \right] +

q^{(j+3)} \left[ q^{j+3} \binom{n-1}{j+3}_q + \binom{n-1}{j+2}_q \right] - \cdots +

(-1)^{n-j} q^{(n-1)} \left[ q^{n-1} \binom{n-1}{n-1}_q + \binom{n-1}{n-2}_q \right] + (-1)^{n-j+1} q^{(n)} \left[ 0 + \binom{n-1}{n-1}_q \right].
$$

Realize that the first members in the square brackets with the $q$-power coefficients are cancelled by the second member in the next square bracket:

$$
q^{(j+k)} q^{j+k} \binom{n-1}{j+k}_q - q^{(j+k+1)} \binom{n-1}{j+k}_q = 0.
$$

This is true for all $k = 1, 2, \ldots, n-j-1$.

Now we are ready to prove Proposition 5. A consequence of (6):

$$
H^3_{n,q} - H^3_{n-1,q} = \left( \binom{n}{1}_q - \binom{n-1}{1}_q \right) q^{(3)}[1]_q + \left( \binom{n}{2}_q - \binom{n-1}{2}_q \right) q^{(3)}[2]_q +

\left( \binom{n}{3}_q - \binom{n-1}{3}_q \right) q^{(3)}[3]_q + \cdots =

\left( \binom{n-1}{0}_q - (1-q) \binom{n-1}{1}_q \right) q^{(3)}[1]_q + \left( \binom{n-1}{1}_q - (1-q^2) \binom{n-1}{2}_q \right) q^{(3)}[2]_q +
$$
\[
\left(\binom{n-1}{2}_q - (1-q^3)\binom{n-1}{3}_q\right) \frac{q^{(3)}}{[3]_q} + \cdots = 
1-(1-q)\binom{n-1}{1}_q + \sum_{k=1}^{n-1} \binom{n-1}{k}_q \frac{(-1)^k}{[k+1]_q} q^{(k+1)}_q + \sum_{k=2}^{n-1} \binom{n-1}{k}_q \frac{(-1)^k(1-q^k)}{[k]_q} q^{(k)}_q = 
1 + \sum_{k=1}^{n-1} \binom{n-1}{k}_q \frac{(-1)^k}{[k+1]_q} q^{(k+1)}_q + \sum_{k=1}^{n-1} \binom{n-1}{k}_q \frac{(-1)^k(1-q)^k}{[k]_q} q^{(k)}_q = 
1 + \frac{1}{[n]_q} \sum_{k=1}^{n-1} \binom{n}{k+1}_q (-1)^k q^{(k+1)}_q + (1-q) \sum_{k=1}^{n-1} \binom{n-1}{k}_q (-1)^k q^{(k)}_q.
\]

The first sum is exactly the left hand side of the \(q\)-Hockey Stick Theorem with \(j = 1\), up to a minus sign. The second sum equals \(-1\) \([1, \text{p. 153}]\). Thus

\[
H^3_{n,q} - H^3_{n-1,q} = 1 - \frac{1}{[n]_q} \binom{n-1}{1}_q + q - 1.
\]

So we have the desired result.

4 Identity involving \(H^4_{n,q}\)

The digamma function is defined as

\[
\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)},
\]

where \(\Gamma(x)\) is the Euler gamma function. Latter satisfies the functional equation

\[
\Gamma(x + 1) = x\Gamma(x).
\]

(23)

The harmonic numbers are connected to the digamma function (as one immediately sees by the logarithmic derivative of (23)):

\[
H_n = \psi(n + 1) + \gamma.
\]

Here \(\gamma = -\psi(1)\) is the Euler-Mascheroni constant. Our goal is to find the \(q\)-analogue of this formula.

Let us start from the definition of the \(q\)-gamma function (see [5]):

\[
\Gamma_q(x) = \frac{(q^x; q)_\infty}{(q^x q^{1-x}; q)_\infty} (1 - q)^{1-x}.
\]

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The $q$-digamma function is simply the logarithmic derivative of $\Gamma_q(x)$ as in the ordinary case \( [5,8,10] \):

$$
\psi_q(x) = -\ln(1 - q) + \ln(q) \sum_{n \geq 0} \frac{q^{n+x}}{1 - q^n}.
$$

It is known that $\Gamma_q(x)$ satisfies the next $q$-version of \( [28] \):

$$
\Gamma_q(x + 1) = [x]_q \Gamma_q(x).
$$

Then taking logarithm and then the derivative, we get

$$
\psi_q(x + 1) = \frac{d}{dx} \ln([x]_q) + \psi_q(x).
$$

Since

$$
\frac{d}{dx} \ln([x]_q) = \frac{\ln(q)q^x}{q^x - 1},
$$

the previous equation is rewritten recursively as

$$
\psi_q(x + 1) = \frac{\ln(q)q^x}{q^x - 1} + \frac{\ln(q)q^{x-1}}{q^{x-1} - 1} + \cdots + \frac{\ln(q)q}{q - 1} + \psi_q(1).
$$

So definition \( [7] \) seems to be correct.

On the other hand, we may define the $q$-analogue of the Euler-Mascheroni constant as

$$
\gamma_q = -\psi_q(1) = \ln(1 - q) - \ln(q) \sum_{n \geq 1} \frac{q^n}{1 - q^n},
$$

or more simply,

$$
\gamma_q = \ln(1 - q) + \frac{\ln(q)}{q - 1} \sum_{n \geq 1} \frac{q^n}{[n]_q}.
$$

With these we have

$$
H_{n,q}^4 = \psi_q(n + 1) + \gamma_q.
$$

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