On the number of extremal surfaces.

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Abstract

Let \( X \) be a compact Riemann surface of genus \( \geq 2 \) of constant negative curvature \(-1\). An extremal disk is an embedded (resp. covering) disk of maximal (resp. minimal) radius. A surface containing an extremal disk is an extremal surface.

This paper gives formulas enumerating extremal surfaces of genus \( \geq 4 \) up to isometry. We show also that the isometry group of an extremal surface is always cyclic of order 1, 2, 3 or 6.

Introduction

Let \( X \) be a compact Riemann surface of genus \( \geq 2 \) of constant negative curvature \(-1\). We consider the maximal radius of an embedded metric disk in \( X \) and the minimal radius of a disk covering \( X \).

An extremal disk is an embedded (resp. covering) disk of maximal (resp. minimal) radius. A surface containing an extremal disk is an extremal surface. C. Bavard [1] proved, that if a surface contains an embedded disk of maximal radius if and only if it contains a covering disk of minimal radius and that extremal surfaces are modular surfaces.

The radius \( R_g \) of an extremal embedded disk, as well as the radius \( C_g \) of an extremal covering disk were computed in [1]:

\[
R_g = \cosh^{-1}(1/2\sin\beta_g), \quad \beta_g = \pi/(12g - 6).
\]

\[
C_g = \cosh^{-1}(1/\sqrt{3}\tan\beta_g), \quad \beta_g = \pi/(12g - 6).
\]
C. Bavard [11] also proved, that discs of maximal radius occur in those surfaces which admit as Dirichlet domain a regular polygon with the largest possible number of sides $12g - 6$ and extremal surfaces occur in every genus.

We give an explicit construction of all extremal surfaces of genus $g \geq 4$. We show, that for genus $g \geq 4$ the isometry group of an extremal surface is always cyclic of order $1, 2, 3$ or $6$ and we give an explicit formula for the number of nonisometric extremal surfaces. Also we give explicit formulas of extremal surfaces having exactly $d$ automorphisms, where $d$ is $1, 2, 3$ or $6$. Form those formulas one can see that asymptotically almost all extremal surfaces have no isometries. In particular, for $d = 1$ we have a big family of explicitly constructed surfaces with no automorphisms. Let’s note, that another families of surfaces with no automorphisms were considered in [11], [18]. The questions of explicit construction, enumeration and description of isometries of genus 2 extremal surfaces were solved in [14]. We will show, that oriented maximal Wicks forms and extremal surfaces are in bijection for $g \geq 4$. For $g = 2$ this bijection was proved in [12] and for $g = 3$ the question is still open.

Section 1 formulates our main results and introduces oriented Wicks forms (cellular decompositions with only one face of oriented surfaces), our main tool. Wicks forms are canonical forms for products of commutators in free groups [19].

Section 2 contains a few facts concerning oriented maximal Wicks forms.

Section 3 contains the proof of our main results.

1 Main results

Definition 1.1. An oriented Wicks form is a cyclic word $w = w_1w_2\ldots w_{2l}$ (a cyclic word is the orbit of a linear word under cyclic permutations) in some alphabet $a_1^{\pm 1}, a_2^{\pm 1}, \ldots$ of letters $a_1, a_2, \ldots$ and their inverses $a_1^{-1}, a_2^{-1}, \ldots$ such that

(i) if $a_1^\epsilon$ appears in $w$ (for $\epsilon \in \{\pm 1\}$) then $a_1^{-\epsilon}$ appears exactly once in $w$,

(ii) the word $w$ contains no cyclic factor (subword of cyclically consecutive letters in $w$) of the form $a_ia_i^{-1}$ or $a_i^{-1}a_i$ (no cancellation),

(iii) if $a_i^\delta a_j^\delta$ is a cyclic factor of $w$ then $a_j^{-\delta}a_i^{-\epsilon}$ is not a cyclic factor of $w$ (substitutions of the form $a_i^\delta a_j^\delta \rightarrow x, a_j^{-\delta}a_i^{-\epsilon} \rightarrow x^{-1}$ are impossible).
An oriented Wicks form $w = w_1w_2\ldots$ in an alphabet $A$ is isomorphic to $w' = w'_1w'_2$ in an alphabet $A'$ if there exists a bijection $\phi : A \rightarrow A'$ with $\phi(a^{-1}) = \phi(a)^{-1}$ such that $w'$ and $\phi(w) = \phi(w_1)\phi(w_2)\ldots$ define the same cyclic word.

An oriented Wicks form $w$ is an element of the commutator subgroup when considered as an element in the free group $G$ generated by $a_1, a_2, \ldots$.

We define the algebraic genus $g_a(w)$ of $w$ as the least positive integer $g_a$ such that $w$ is a product of $g_a$ commutators in $G$.

The topological genus $g_t(w)$ of an oriented Wicks form $w = w_1\ldots w_{2e-1}w_{2e}$ is defined as the topological genus of the oriented compact connected surface obtained by labeling and orienting the edges of a $2e$-gon (which we consider as a subset of the oriented plane) according to $w$ and by identifying the edges in the obvious way.

**Proposition 1.1.** The algebraic and the topological genus of an oriented Wicks form coincide (cf. [7, 9]).

We define the genus $g(w)$ of an oriented Wicks form $w$ by $g(w) = g_a(w) = g_t(w)$.

Consider the oriented compact surface $S$ associated to an oriented Wicks form $w = w_1\ldots w_{2e}$. This surface carries an immersed graph $\Gamma \subset S$ such that $S \setminus \Gamma$ is an open polygon with $2e$ sides (and hence connected and simply connected). Moreover, conditions (ii) and (iii) on Wicks form imply that $\Gamma$ contains no vertices of degree 1 or 2 (or equivalently that the dual graph of $\Gamma \subset S$ contains no faces which are 1-gones or 2-gones). This construction works also in the opposite direction: Given a graph $\Gamma \subset S$ with $e$ edges on an oriented compact connected surface $S$ of genus $g$ such that $S \setminus \Gamma$ is connected and simply connected, we get an oriented Wicks form of genus $g$ and length $2e$ by labeling and orienting the edges of $\Gamma$ and by cutting $S$ open along the graph $\Gamma$. The associated oriented Wicks form is defined as the word which appears in this way on the boundary of the resulting polygon with $2e$ sides. We identify henceforth oriented Wicks forms with the associated immersed graphs $\Gamma \subset S$, speaking of vertices and edges of oriented Wicks form.

The formula for the Euler characteristic

$$\chi(S) = 2 - 2g = v - e + 1$$

(where $v$ denotes the number of vertices and $e$ the number of edges in $\Gamma \subset S$) shows that an oriented Wicks form of genus $g$ has at least length $4g$ (the associated graph has then a unique vertex of degree $4g$ and $2g$ edges) and at
most length $6(2g - 1)$ (the associated graph has then $2(2g - 1)$ vertices of degree three and $3(2g - 1)$ edges).

We call an oriented Wicks form of genus $g$ maximal if it has length $6(2g - 1)$. Oriented maximal Wicks forms are dual to 1-vertex triangulations. This can be seen by cutting the oriented surface $S$ along $\Gamma$, hence obtaining a polygon $P$ with $2e$ sides. We draw a star $T$ on $P$ which joins an interior point of $P$ with the midpoints of all its sides. Regluing $P$ we recover $S$ which carries now a 1-vertex triangulation given by $T$ and each 1-vertex triangulation is of this form for some oriented maximal Wicks form (the immersed graphs $T \subset S$ and $\Gamma \subset S$ are dual to each other: faces of $T$ correspond to vertices of $\Gamma$ and vice-versa. Two faces of $T$ share a common edge if and only if the corresponding vertices of $\Gamma$ are adjacent). This construction shows that we can work indifferently with 1-vertex triangulations or with oriented maximal Wicks forms.

Similarly, cellular decompositions of oriented surfaces with one vertex and one face correspond to oriented minimal Wicks forms and were enumerated in [10]. The dual of an oriented minimal Wicks form is again a (generally non-equivalent) oriented minimal Wicks form and taking duals yields hence an involution on the set of oriented minimal Wicks forms.

A vertex $V$ (with oriented edges $a, b, c$ pointing toward $V$) is positive if

$$w = ab^{-1} \ldots bc^{-1} \ldots ca^{-1} \ldots \quad \text{or} \quad w = ac^{-1} \ldots cb^{-1} \ldots ba^{-1} \ldots$$

and $V$ is negative if

$$w = ab^{-1} \ldots ca^{-1} \ldots bc^{-1} \ldots \quad \text{or} \quad w = ac^{-1} \ldots ba^{-1} \ldots ab^{-1} \ldots$$

The automorphism group $\text{Aut}(w)$ of an oriented Wicks form

$$w = w_1w_2 \ldots w_{2e}$$

of length $2e$ is the group of all cyclic permutations $\mu$ of the linear word $w_1w_2 \ldots w_{2e}$ such that $w$ and $\mu(w)$ are isomorphic linear words (i.e. $\mu(w)$ is obtained from $w$ by permuting the letters of the alphabet). The group $\text{Aut}(w)$ is a subgroup of the cyclic group $\mathbb{Z}/2e\mathbb{Z}$ acting by cyclic permutations on linear words representing $w$.

The automorphism group $\text{Aut}(w)$ of an oriented Wicks form can of course also be described in terms of permutations on the oriented edge set induced
by orientation-preserving homeomorphisms of $S$ leaving $\Gamma$ invariant. In particular an oriented maximal Wicks form and the associated dual 1-vertex triangulation have isomorphic automorphism groups.

We define the mass $m(W)$ of a finite set $W$ of oriented Wicks forms by

$$m(W) = \sum_{w \in W} \frac{1}{|\text{Aut}(w)|}.$$ 

Let us introduce the sets

$W^g_1$: all oriented maximal Wicks forms of genus $g$ (up to equivalence),

$W^g_2(r) \subset W^g_1$: all oriented maximal Wicks forms having an automorphism of order 2 leaving exactly $r$ edges of $w$ invariant by reversing their orientation. (This automorphism is the half-turn with respect to the “midpoints” of these edges and exchanges the two adjacent vertices of an invariant edge.)

$W^g_3(s, t) \subset W^g_1$: all oriented maximal Wicks forms having an automorphism of order 3 leaving exactly $s$ positive and $t$ negative vertices invariant (this automorphism permutes cyclically the edges around an invariant vertex).

$W^g_6(3r; 2s, 2t) = W^g_2(3r) \cap W^g_3(2s, 2t)$: all oriented maximal Wicks forms having an automorphism $\gamma$ of order 6 with $\gamma^3$ leaving $3r$ edges invariant and $\gamma^2$ leaving $2s$ positive and $2t$ negative vertices invariant (it is useless to consider the set $W^g_6(r'; s', t')$ defined analogously since 3 divides $r'$ and 2 divides $s', t'$ if $W^g_6(r'; s', t') \neq \emptyset$).

We define now the masses of these sets as

$$m^g_1 = \sum_{w \in W^g_1} \frac{1}{|\text{Aut}(w)|},$$

$$m^g_2(r) = \sum_{w \in W^g_2(r)} \frac{1}{|\text{Aut}(w)|},$$

$$m^g_3(s, t) = \sum_{w \in W^g_3(s, t)} \frac{1}{|\text{Aut}(w)|},$$

$$m^g_6(3r; 2s, 2t) = \sum_{w \in W^g_6(3r; 2s, 2t)} \frac{1}{|\text{Aut}(w)|}.$$ 

**Theorem 1.1.**

(i) The group $\text{Aut}(w)$ of automorphisms of an oriented maximal Wicks form $w$ is cyclic of order 1, 2, 3 or 6.

(ii) $m^g_1 = \frac{2}{12} \left( \frac{1}{12} \right)^g \frac{(6g - 5)!}{g!(3g - 3)!}$. 


and we have then

\[ m_2^g(r) = \frac{2}{12} \left( \frac{2^2}{2} \right) f \left( \frac{1}{r!} \right) \frac{(6f + 2r - 5)!}{(3f + r - 3)!} \]

and we have then

\[ m_3^g = \frac{2}{3} \left( \frac{3^2}{12} \right) f \left( \frac{1}{s!} \right) \frac{(6f + 2s + 2t - 5)!}{(3f + s + t - 3)!} \]

if \( g > 1 \) and \( m_3^g(0, 2) = \frac{1}{6} \).

(iv) \( m_4^g(s, t) > 0 \) if and only if \( f = \frac{2g + 1 - x - t}{3} \in \{0, 1, 2, \ldots\} \), \( s \equiv 2g + 1 \pmod{3} \) and \( t \equiv 2g \pmod{3} \) (which follows from the two previous conditions). We have then

\[ m_4^g(s, t) = \frac{2}{6} \left( \frac{6^2}{12} \right) f \left( \frac{1}{r!s!t!} \right) \frac{(6f + 2r + 2s + 2t - 5)!}{(3f + r + s + t - 3)!} \]

if \( g > 1 \) and \( m_4^g(3; 0, 2) = \frac{1}{6} \).

**Theorem 1.2.** (i) The group \( \text{Aut}(S_g) \) of automorphisms of an extremal surface is cyclic of order 1, 2, 3 or 6 in every genus.

(ii) There is a bijection between the number of equivalence classes of oriented maximal genus \( g \) Wicks forms and extremal genus \( g \) surfaces for \( g \geq 4 \).

Set

\[ m_2^g = \sum_{r \in \mathbb{N}} \frac{(2g + 1 - r)}{4} \in \mathbb{N} \cup \{0\} m_2^g(r) \]
\[ m_3^g = \sum_{s, t \in \mathbb{N}, \ (g + 1 - s - t) \in \mathbb{N} \cup \{0\}, \ s \equiv 2g + 1 \pmod{3}} m_3^g(s, t) \]
\[ m_4^g = \sum_{r, s, t \in \mathbb{N}} \frac{(2g + 5 - 3r - 4s - 4t)}{12} \in \mathbb{N} \cup \{0\}, \ 2s \equiv 2g + 1 \pmod{3}} m_4^g(3r; 2s, 2t) \]

(all sums are finite) and denote by \( M_d^g \) the number of isometry classes of extremal genus \( g \) surfaces having an automorphism of order \( d \) (i.e. an automorphism group with order divisible by \( d \)).

**Theorem 1.3.** We have

\[ M_1^g = m_1^g + m_2^g + 2m_3^g + 2m_4^g \]
\[ M_2^2 = 2m_2^g + 4m_6^g \]
\[ M_3^3 = 3m_2^g + 3m_6^g \]
\[ M_4^6 = 6m_6^g \]

and \( M_d^g = 0 \) if \( d \) is not a divisor of 6.
The number $M_{g}^{1}$ of this Theorem is the number of extremal surfaces of genus $g$ for $g \geq 4$. The first 15 values $M_{1}^{1}, \ldots, M_{1}^{15}$ are displayed in the Table at the end of this paper.

The following result is an immediate consequence of Theorem 1.3.

**Corollary 1.1.** For $g \geq 4$ there are exactly
- $M_{g}^{6}$ nonisometric extremal surfaces with 6 automorphisms,
- $M_{g}^{3} - M_{g}^{6}$ nonisometric extremal surfaces with 3 automorphisms,
- $M_{g}^{2} - M_{g}^{3}$ nonisometric extremal surfaces with 2 automorphisms and
- $M_{g}^{1} - M_{g}^{2} - M_{g}^{3} + M_{g}^{6}$ nonisometric extremal surfaces without non-trivial automorphisms.

**Remark.** Computing masses amounts to enumerating pointed objects, i.e. linear words instead of cyclic words in Definition 1.1. Their number is $(12g - 6) m_{d}^{g}$, where $d$ is 1, 2, 3 or 6.

Let us remark that formula (ii) can be obtained from [20] (formula (9) on page 207 and the formula on the top of page 211) or from [15] (Theorem 2.1 with $\lambda = 2^{6g-3}$ and $\mu = 3^{4g-2}$). Related objects have also been considered in [16].

**2 Oriented Wicks forms**

To understand better the structure of extremal surfaces, we need to describe now some properties of Wicks forms (see [19], [6] for all the details).

Let $V$ be a negative vertex of an oriented maximal Wicks form of genus $g > 1$. There are three possibilities, denoted configurations of type $\alpha$, $\beta$ and $\gamma$ (see Figure 1) for the local configuration at $V$.

![Figure 1](image_url)

**Figure 1.**

**Type $\alpha$.** The vertex $V$ has only two neighbours which are adjacent to each other. This implies that $w$ is of the form

$$w = x_{1}abcdb^{-1}ec^{-1}d^{-1}e^{-1}a^{-1}x_{2}u_{1}x_{2}^{-1}x_{1}^{-1}u_{2}$$
(where $u_1, u_2$ are subfactors of $w$) and $w$ is obtained from the maximal oriented Wicks form

$$w' = xu_1x^{-1}u_2$$

of genus $g - 1$ by the substitution $x \mapsto x_1abcdb^{-1}ec^{-1}d^{-1}e^{-1}a^{-1}x_2$ and $x^{-1} \mapsto x_2^{-1}x_1^{-1}$ (this construction is called the $\alpha-$construction in [19]).

Type $\beta$. The vertex $V$ has two non-adjacent neighbours. The word $w$ is then of the form

$$w = x_1abca^{-1}x_2u_1y_1db^{-1}ec^{-1}d^{-1}y_2u_2$$

(where perhaps $x_2 = y_1$ or $x_1 = y_2$, see [19] for all the details). The word $w$ is then obtained by a $\beta-$construction from the word $w' = xu_1yu_2$ which is an oriented maximal Wicks form of genus $g - 1$.

Type $\gamma$. The vertex $V$ has three distinct neighbours. We have then

$$w = x_1ab^{-1}y_2u_1z_1ca^{-1}x_2u_2y_1bc^{-1}z_2u_3$$

(some identifications among $x_i$, $y_j$ and $z_k$ may occur, see [19] for all the details) and the word $w$ is obtained by a so-called $\gamma-$construction from the word $w' = xu_2y_1z_3$.

**Definition 2.1.** We call the application which associates to an oriented maximal Wicks form $w$ of genus $g$ with a chosen negative vertex $V$ the oriented maximal Wicks form $w'$ of genus $g - 1$ defined as above the reduction of $w$ with respect to the negative vertex $V$.

An inspection of figure 1 shows that reductions with respect to vertices of type $\alpha$ or $\beta$ are always paired since two doubly adjacent vertices are negative, of the same type ($\alpha$ or $\beta$) and yield the same reductions.

The above constructions of type $\alpha$, $\beta$ and $\gamma$ can be used for a recursive construction of all oriented maximal Wicks forms of genus $g > 1$.

**Definition 2.2.** Consider an oriented maximal Wicks form

$$w = w_1 \ldots w_{12g-6}.$$ 

To any edge $x$ of $w$ we associate a transformation of $w$ called the IH-transformation on the edge $x$. 

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Geometrically, an IH-transformation amounts to contracting the edge $x$ of the graph $\Gamma \subset S$ representing the oriented maximal Wicks form $w$. This creates a vertex of degree 4 which can be split in two different ways (preserving planarity of the graph on $S$) into two adjacent vertices of degree 3: The first way gives back the original Wicks form and the second way results in the IH-transformation. Graphically, an IH-transformation amounts hence to replace a (deformed) letter I (a topological neighbourhood of the edge $x \in \Gamma \subset S$) by a (deformed) letter H.

More formally, one considers the two subfactors $axb$ and $cx^{-1}d$ of the (cyclic) word $w$. Geometric considerations and Definition 1.1 show that $b \neq a^{-1}, c \neq b^{-1}, d \neq a^{-1}, d \neq c^{-1}$ and $(c, d) \neq (a^{-1}, b^{-1})$.

According to the remaining possibilities we consider now the following transformation:

Type 1. $c \neq a^{-1}$ and $d \neq b^{-1}$. This implies that $d^{-1}a^{-1}$ and $b^{-1}c^{-1}$ appear as subfactors in the cyclic word $w$. The IH-transformation on the edge $x$ is then defined by the substitutions

$$
\begin{align*}
axb & \mapsto ab \\
 cx^{-1}d & \mapsto cd \\
d^{-1}a^{-1} & \mapsto d^{-1}ya^{-1} \\
b^{-1}c^{-1} & \mapsto b^{-1}y^{-1}c^{-1}
\end{align*}
$$

in the cyclic word $w$.

Type 2a. Suppose $c^{-1} = a$. This implies that $b^{-1}axb$ and $d^{-1}a^{-1}x^{-1}d$ are subfactors of the cyclic word $w$. Define the IH-transformation on the edge $x$ by

$$
\begin{align*}
b^{-1}axb & \mapsto b^{-1}yab \\
d^{-1}a^{-1}x^{-1}d & \mapsto d^{-1}y^{-1}a^{-1}d
\end{align*}
$$

Type 2b. Suppose $d^{-1} = b$. Then $axba^{-1}$ and $cx^{-1}b^{-1}c^{-1}$ are subfactors of the cyclic word $w$ and we define the IH-transformation on the edge $x$ by

$$
\begin{align*}
axba^{-1} & \mapsto aby^{-1} \\
cx^{-1}b^{-1}c^{-1} & \mapsto cb^{-1}y^{-1}c^{-1}
\end{align*}
$$

**Lemma 2.1.** (i) IH-transformations preserve oriented maximal Wicks forms of genus $g$.

(ii) Two oriented maximal Wicks forms related by a IH-transformation of type 2 are equivalent.

Proof. This results easily by considering the effect of an IH-transformation on the graph $\Gamma \subset S$. QED
**Proposition 2.1.** An oriented maximal Wicks form of genus $g$ has exactly $2(g - 1)$ positive and $2g$ negative vertices.

**Lemma 2.2.** An $\alpha$ or a $\beta$ construction increases the number of positive and negative vertices by 2.

The proof is easy.

**Lemma 2.3.** The number of positive or negative vertices is constant under IH-transformations.

Proof of Lemma 2.3. The Lemma holds for IH-transformations of type 2 by Lemma 2.1 (ii). Let hence $w, w'$ be two oriented maximal Wicks forms related by an IH-transformation of type 1 with respect to the edge $x$ of $w$ respectively $y$ of $w'$. This implies that $w$ contains the four subfactors

- $axb$, $cx^{-1}d$, $d^{-1}a^{-1}$, $b^{-1}c^{-1}$

and $w'$ contains the subfactors

- $ab$, $cd$, $d^{-1}ya^{-1}$, $b^{-1}y^{-1}c^{-1}$

in the same cyclic order and they agree everywhere else. It is hence enough to check the lemma for the six possible cyclic orders of the above subfactors.

One case is

- $w = axbu \ldots cx^{-1}d \ldots d^{-1}a^{-1} \ldots b^{-1}c^{-1} \ldots$
- $w' = abu \ldots cd \ldots d^{-1}ya^{-1} \ldots b^{-1}y^{-1}c^{-1} \ldots$

In this case the two vertices of $w$ incident in $x$ and the two vertices of $w'$ incident in $y$ have opposite signs. All other vertices are not involved in the IH-transformation and keep their sign and the Lemma holds hence in this case.

The five remaining cases are similar and left to the reader. QED

Proof of Proposition 2.1. The result is true in genus 1 by inspection (the cyclic word $a_1a_2a_3a_1^{-1}a_2^{-1}a_3^{-1}$ is the unique oriented maximal Wicks form of genus 1 and has two negative vertices.)

Consider now an oriented maximal Wicks form $w$ of genus $g + 1$. Choose an oriented embedded loop $\lambda$ of minimal (combinatorial) length in $\Gamma$.

First case. If $\lambda$ is of length 2 there are two vertices related by a double edge in $\Gamma$. This implies that they are negative and of type $\alpha$ or $\beta$. The assertion of Proposition 2.1 holds hence for $w$ by Lemma 2.2 and by induction on $g$. 
Second case. We suppose now that $\lambda$ is of length $\geq 3$. The oriented loop $\lambda$ turns either left or right at each vertex. If it turns on the same side at two consecutive vertices $V_i$ and $V_{i+1}$ the IH-transformation with respect to the edge joining $V_i$ and $V_{i+1}$ transforms $w$ into a form $w'$ containing a shorter loop. By Lemma 2.2, the oriented maximal Wicks forms $w$ and $w'$ have the same number of positive (respectively negative) vertices.

If $\lambda$ does not contain two consecutive vertices $V_i$ and $V_{i+1}$ with the above property (ie. if $\lambda$ turns first left, then right, then left etc.) choose any edge $\{V_i, V_{i+1}\}$ in $\lambda$ and make an IH-transformation with respect to this edge. This produces a form $w'$ which contains a loop $\lambda'$ of the same length as $\lambda$ but turning on the same side at the two consecutive vertices $V_{i-1}, V_i$ or $V_{i+1}, V_{i+2}$. By induction on the length of $\lambda$ we can hence relate $w$ by a sequence of IH-transformation to an oriented maximal Wicks form $\tilde{w}$ of genus $g + 1$ containing a loop of length 2 for which the result holds by the first case. The Wicks forms $w$ and $\tilde{w}$ have of course the same number of positive (respectively negative) vertices by Lemma 2.2. QED

3 Proof of Theorem 1.2

Proof of Theorem 1.2.

Proof of (i).

Let $w$ be an oriented maximal Wicks form with an automorphism $\mu$ of order $d$. Let $p$ be a prime dividing $d$. The automorphism $\mu' = \mu^{d/p}$ is hence of order $p$. If $p \neq 3$ then $\mu'$ acts without fixed vertices on $w$ and Proposition 2.1 shows that $p$ divides the integers $2(g - 1)$ and $2g$ which implies $p = 2$. The order $d$ of $\mu$ is hence of the form $d = 2^a3^b$. Repeating the above argument with the prime power $p = 4$ shows that $a \leq 1$.

All orbits of $\mu'^{2^a}$ on the set of positive (respectively negative) vertices have either $3^b$ or $3^{b-1}$ elements and this leads to a contradiction if $b \geq 2$. This shows that $d$ divides 6 and proves that the automorphism groups of oriented maximal Wicks forms are always cyclic of order 1,2,3 or 6.

Let’s consider an extremal genus $g$ surface $S_g$. It was proved in [1], that surface is extremal if and only if it can be obtained from a regular hyperbolic 12$g - 6$-gon with angles $2\pi/3$ such that the image of the boundary of the polygon after identification of corresponded sides is a geodesic graph with $4g - 2$ vertices of valence 3 and $6g - 3$ edges of equal length. It was shown in [14], that any isometry of an extremal surface of genus $g > 3$ is realized
by a rotation of the $12g - 6$-gon.

Let $P$ be a regular geodesic hyperbolic polygon with $12g - 6$ equal sides and all angles equal to $2\pi/3$, equipped with a oriented maximal genus $g$ Wicks form $W$ on its boundary. Consider the surface $S_g$ obtained from $P$ by identification of sides with the same labels. Since we made the identification using an oriented maximal Wicks form of length $12g - 6$, then the boundary of $D$ becomes a graph $G$ with $4g - 2$ vertices of valence 3 and $6g - 3$ edges (see section 2). We started from a regular geodesic hyperbolic polygon with angles $2\pi/3$, so $G$ is a geodesic graph with edges of equal length. By a result of C. Bavard, mentioned above, the surface $S_g$ is extremal.

So, the surface is extremal if and only if it can be obtained from a regular hyperbolic polygon with $12g - 6$ equal sides and all angles equal to $2\pi/3$, equipped with an oriented maximal genus $g$ oriented maximal Wicks form $W$ on its boundary. The isometry of $S_g$ must be realized by a rotation of the $12g - 6$-gon \[14\], so the isometry must be an automorphism of the Wicks form. Since the automorphism groups of oriented maximal Wicks forms are always cyclic of order 1, 2, 3 or 6 then the automorphism groups of genus $g \geq 4$ extremal surfaces are also cyclic of order 1, 2, 3 or 6.

Proof of (ii).

Every oriented maximal Wicks form defines exactly one extremal surface, namely the surface obtained from a regular hyperbolic polygon with $12g - 6$ equal sides and all angles equal to $2\pi/3$ with an oriented maximal genus $g$ oriented maximal Wicks form $W$ on its boundary.

So, to prove the bijection between the number of equivalence classes of oriented maximal genus $g$ Wicks forms and extremal genus $g$ surfaces for $g \geq 4$ we need to show, that for every extremal surface $S_g$ there is only one oriented maximal Wicks $W$ form such that $S_g$ can be obtained from a regular hyperbolic $12g - 6$-gon with $W$ on its boundary. It was proved in \[14\], that for $g \geq 4$ the extremal disk $D$ of radius $R_g = cosh^{-1}(1/sin\beta_g)$, $\beta_g = \pi/(12g - 6)$, embedded in $S_g$ is unique. Consider the center $c$ of the disk $D$. The discs of radius $R_g$ with the centers in the images of $c$ in the universal covering of $S_g$ form a packing of the hyperbolic plane by discs. To this packing one can classically associate a tesselation $T$ of the hyperbolic plane by regular $12 - 6$-gons, which are Dirichlet domains for $S_g$. And such a tesselation is unique because of negative curvature (see \[1\], for example). But each $T$ defines a Wicks form of length $12g - 6$ in a unique way.

Theorem 1.2 is proved.

QED
Table. The number of extremal surfaces in genus 1, 2, 4 . . . , 15:

| Genus | Number          |
|-------|----------------|
| 1     | 1              |
| 2     | 9              |
| 4     | 1349005        |
| 5     | 2169056374     |
| 6     | 584968696988   |
| 7     | 23808202021448662 |
| 8     | 136415042681045401661 |
| 9     | 1047212810636411989605202 |
| 10    | 10378926166167927379808819918 |
| 11    | 129040245485216017874985276329588 |
| 12    | 196689594180840390142132270340417352 |
| 13    | 36072568973390464496963227953956789552404 |
| 14    | 783676560946907841153290887110277871996495020 |
| 15    | 19903817294929565349602352185144632327980494486370 |

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