GEOMETRY OF MATROIDS AND HYPERPLANE ARRANGEMENTS

JAEHO SHIN

Abstract. There is a trichotomic relation between hyperplane arrangements, convex polytopes and matroids. We expand this theory while resolving the complexity issue expected by Mnëv’s universality theorem. In particular, we invent a combinatorial apparatus for the geometry of hyperplane arrangements in terms of semilattices and their operations, for instance, puzzle-pieces and the matroidal MMP. We also investigate matroid tilings and their extensions. As an algebro-geometric application, we answer Alexeev’s question.

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INTRODUCTION

Polynomials and polytopes are seemingly quite independent categories of objects, but relating them to each other is a classical idea, which goes back to Newton. For a Laurent polynomial over a field \(k\), say \(p(x) = \sum c_m x^m \in k[x]\) with \(m \in \mathbb{Z}^k\) for some positive integer \(k\), the convex hull of those \(m\) with \(c_m \neq 0\) is called the Newton polytope of the polynomial \(p(x)\). Throughout the paper, the underlying field \(k\) is assumed algebraically closed unless otherwise specified.

For Chow variety \(G(k, n, d)\), the projective variety of all the \((k - 1)\)-dimensional algebraic cycles in \(\mathbb{P}^{n-1}\) of degree \(d\), the relation between polynomials and polytopes becomes that between Chow forms and their weight polytopes, where the weight polytopes are the Newton polytopes of the Chow forms, also called Chow polytopes. When it comes to Grassmannians \(G(k, n)\), Chow varieties with \(d = 1\), the relation for generic Chow forms turns to a trichotomy:

- Arrangements of \(n\) hyperplanes in \(\mathbb{P}^{k-1}\) that the Chow forms induce
- Weight polytopes of the Chow forms, also called matroid polytopes
- Corresponding combinatorial structures, that is, matroids

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The association of arrangements goes as follows. Any Chow form recovers a \((k - 1)\)-dimensional projective subspace \(L \subseteq \mathbb{P}^{n-1}\). If \(L\) is not contained in a coordinate hyperplane of \(\mathbb{P}^{n-1}\), the transpose of the matrix form of \(L\) induces an arrangement of \(n\) hyperplanes \(B_i\) in \(\mathbb{P}^{k-1}\) and vice versa, such that those \(B_i\) are identified with the intersections of \(L\) and the \(n\) coordinate hyperplanes of \(\mathbb{P}^{n-1}\). Note that our arrangements are projective, which are central and essential in other contexts of hyperplane arrangements such as \([OT92, Sta07]\). Note also that any hyperplane arrangement in their sense can be reduced to a projective one.

The natural action of the algebraic torus \(G^n\) on \(\mathbb{P}^{n-1}\) extends to a torus action on the Grassmannian \(G(k, n)\). The weights of the Chow form of \(L \in G(k, n)\) are characters of the torus, which are identified with the incidence vectors of \(k\)-element subsets \(I\) of \([n] = \{1, \ldots, n\}\) with nonzero Plücker coordinates \(p_I(L)\). Then, those \(k\)-element subsets form a matroid, a combinatorial abstraction of a spanning set of a \(k\)-dimensional vector space with size \(n\) counted with multiplicity.

Consider a category whose objects are polytopes in the form of the convex hull of incidence vectors of \(k\)-element subsets of \([n]\). Matroid polytopes are polytopes in this category with the smallest possible edge length. Matroid polytopes are in a one-to-one correspondence with matroids.

The trichotomic relation goes further. The moduli of hyperplane arrangements \((\mathbb{P}^{k-1}, (B_1, \ldots, B_n))\) has a compactification due to Hacking-Keel-Tevelev \([HKT06]\). This situation can be generalized by assigning to the hyperplanes \(B_i\) numbers \(b_i \in \mathbb{R}\) with \(0 < b_i \leq 1\) and \(\sum b_i > k\), one for each, where \(\beta = (b_1, \ldots, b_n) \in \mathbb{R}^n\) is called a weight (vector) and \((\mathbb{P}^{k-1}, (b_1B_1, \ldots, b_nB_n))\) is called a \((\beta)\)-weighted hyperplane arrangement. The moduli \(M_\beta(k, n)\) of those weighted hyperplane arrangements also has a compactification \(\overline{M}_\beta(k, n)\) due to Alexeev \([Ale08]\), where then, the HKT’s space is \(\overline{M}_4(k, n)\) with \(\mathbf{1} = (1, \ldots, 1)\), unweighted version of Alexeev’s space.\(^1\)

The geometric fibers of \(\overline{M}_\beta(k, n)\) are called \((\beta)\)-weighted stable hyperplane arrangements, \((\beta)\)-weighted SHAs for short, and simply (unweighted) SHAs for \(\beta = \mathbf{1}\).

Fix \(\beta\). To any \(\beta\)-weighted SHA, there corresponds a matroid tiling, a polytopal complex whose cells are matroid polytopes. More explicitly, let \(X = \cup X_j\) be a \(\beta\)-weighted SHA with irreducible components \(X_j\). To each \(X_j\) there corresponds a full-dimensional matroid polytope \(P_j\) with \(P_j \cap \text{int } \Delta_\beta \neq \emptyset\)\(^2\) such that \(P_j\)’s generate with intersections a polytopal complex whose support covers \(\Delta_\beta\), cf. \([Ale15]\).

- Furthermore, each \(X_j\) is a normal toric variety. It comes from a \(\beta\)-weighted hyperplane arrangement whose log canonical model is \(X_j\).
- \(P_j\)’s are the matroid polytopes associated to the hyperplane arrangements. They glue to one another exactly the same way as the varieties \(X_j\)’s do.
- For any two weight vectors \(\beta, \beta’\) with \(\beta’ > \beta\), there exists a natural morphism \(\rho_{\beta’, \beta} : \overline{M}_{\beta’}(k, n) \to \overline{M}_\beta(k, n)\), called a reduction morphism. On the fibers, to \(X’ \to X\) there corresponds the extension of the matroid tiling associated to \(X\) to that associated to \(X’\).

Thus, matroid subdivisions directly helps our understanding of the weighted SHAs. Meanwhile, the notion of the secondary polytope of a marked polytope, the convex

\(^1\)However, their constructions of \(\overline{M}_1(k, n)\) are different from each other. In this paper, Alexeev’s construction is preferred since it reflects the gluing of the toric varieties along torus orbits and the gluing of the associated polytopes along faces; see also \([Ale15]\).

\(^2\)\(\Delta_\beta = \Delta_\beta(k, n) = \left[\prod_{i=1}^n [0, b_i] \cap \{ \sum x_i = k \}\right] \text{ and } \text{int } \Delta_\beta = (\text{the interior of } \Delta_\beta)\)
hull of the characteristic functions of triangulations of the marked polytope, gives a framework to study Chow varieties. In our context, the marked polytope consists of a matroid polytope and the collection of its vertices. The vertices of the secondary polytope are exactly the characteristic functions of all the coherent triangulations; the poset of its faces corresponds to that of the coherent polyhedral subdivisions of the marked polytope, ordered by refinement, cf. [GKZ94].

For all those known nice properties of secondary polytopes, however, they do not fully satisfy our expectations because it is not just triangulations of a certain matroid polytope we are interested in, but matroid tilings. Now, matroid polytopes are 0/1-polytopes, and the literature on them, e.g. [Zie00], tells that our intuition in general even for low dimensional cases can fail and there may be huge complexity. This complexity issue is confirmed by Mnëv’s universality theorem, [Laf03, Vak06].

The idea to break through is to convert the algebraic/polyhedral complexity to combinatorial, more precisely, matroidal complexity. Conceptually, this amounts to reducing exponential complexity to base level by taking logarithm. To implement the idea, we formulate the matroidal counterparts of objects of the other two realms and fill in missing parts between the trichotomic relation, while several new notions are developed. See the table below.

| Varieties                  | Polytopes                   | Matroids                   |
|---------------------------|-----------------------------|----------------------------|
| Hyperplane Arrangements   | Matroid polytopes           | Matroidal HAs              |
| (HAs, for short) of dim k − 1 | of dim n − 1               | of dim k − 1               |
| Sub-arrangements          | Faces                       | Sub-arrangements/Face matroids |
| Intersection of sub-HAs   | Intersection of faces       | ⊙ and ⊙                   |
| LUB (least upper bound)   | LUB of faces                | ⋄                           |
| of sub-HAs                |                             |                            |
| Flags                     | Flags                       | Flags/Flace sequences³     |
| Blowup/Contraction        |                             | Matroidal blowup⁴/Collapsing |
| Minimal Model Program (MMP) |                             | Matroidal MMP              |
| Log canonical models      |                             |                             |
| of (unweighted) HAs       |                             |                             |
| Tilings⁵                  |                             |                             |
| Semitilings⁶              |                             |                             |
| Local convexity           |                             |                             |
| SHAs                      | Complete tilings            | Matroidal SHAs/Complete puzzles |
| Weighted SHAs             | Weighted tilings            | Weighted objects           |
| A reduction morphism      | Extensions of weighted objects |                         |

³In the sense of a sequence of flats of a matroid, a flag is to a flace sequence what a normal series is to a subnormal series in the classical group theory.
⁴This notion is different from the combinatorial blowup of [FK04], see Subsection 4.6.
⁵Tilings in this paper are restrictedly defined as polytopal complexes in the hypersimplex.
⁶Any rank-3 full-dimensional (semi)tiling in the hypersimplex connected in codimension 1 has a natural quiver structure, cf. Theorem 3.21 and Definition 5.5.
where a minor, then, is defined as the matroid that a minor expression represents; this notion plays a crucial role thereafter. For matroids being structures on sets, pullback and pushforward are defined, and familiar concepts such as simplifications, restrictions, matroid unions, partition matroids, and possibly more can be redefined in terms of them. Base intersections and unions are defined in contrast to matroid intersections, which are related to the face intersection of a matroid polytope.

In Section 2, we interpret the faces of a matroid polytope and their intersections into matroidal terms. Some posets/semilattices of direct sums of minor expressions are considered, and operations $\circledcirc$, $\circledast$, and $\circledast$ on those posets are introduced with which the face computations are performed incredibly efficient way. A face can be expressed by two different notions of face sequences and flags.

In Section 3, matroid subdivisions/tilings are investigated.\(^7\) Tilings are restricted to those in the hypersimplices, and a semitiling is defined as a collection of convex polytopes that is locally a tiling. Then, any subdivision is a semitiling connected in codimension 1. Conversely, a matroid semitiling connected in codimension 1 with convex support is a matroid subdivision. It turns out that the local convexity is actually equivalent to the global convexity. Also, it turns out that the number of full-dimensional matroid polytopes in the hypersimplex face-fitting at a common codimension-2 face is at most 6. Weighted tilings and weights per se are studied.

In Section 4, a matroidal counterpart of a matroid polytope is defined, say a puzzle-piece, which has dimension $k - 1$ if the matroid polytope has full-dimension. Also, a matroidal hyperplane arrangement\(^8\) is defined, a combinatorial abstraction of a usual hyperplane arrangement over a field, whose dimension is $k - 1$. Then, any usual hyperplane arrangement over a field is a realization (over the given field) of a matroidal hyperplane arrangement, and its log canonical model is a realization of the corresponding puzzle-piece which has information on the cohomology of the associated toric pair.\(^9\) Those two combinatorial objects are related by the matroidal MMP as their algebro-geometric counterparts are by the MMP. Furthermore, by the straightforward correspondence between the toric varieties and the associated matroidal semilattices, the matroidal MMP tells that any hyperplane arrangement has a log canonical model, and even shows how to obtain it.

Finally, in Section 5, extension of tilings is discussed with a focus on the cases when $k = 2, 3$. It turns out that the case $k = 2$ is simple, but the case $k = 3$ gets drastically complicated, cf. Mnëv’s universality theorem. We develop an algorithm that extends a specific kind of semitilings to complete tilings, and then show that all the $(3, n \leq 9)$-tilings associated to weighted SHAs have complete extensions where the bound $n = 9$ is sharp. For realizable extensions, this is also true, which answers the question of whether or not the reduction morphisms between moduli spaces of weighted SHAs are surjective, proposed by Alexeev, cf. [Ale08].

All the computations are manually done with pen and paper.

\(^7\)The matroid subdivisions of particular interest to tropical geometers are coherent ones, and those give a partition of the Dressian $\text{Dr}(k, n)$. Note that Example 5.9 shows how efficient the computation becomes for $\text{Dr}(3, 6)$, cf. [HJJS09].

\(^8\)This notion of matroidal HA can be thought of as a generalization of a pseudoline arrangement [BVSWZ99], of an abstract tree arrangement [HJJS09], and of a projective geometry [Oxl92].

\(^9\)The 2-dimensional puzzle-pieces realizable over $\mathbb{C}$ seem closely related to the puzzle-pieces of [KT03], which are plane figures. A caveat is that their two triangles pointing in opposite directions can not happen in our sense, cf. Lemma 4.21 and Example 4.20(b).
1. Customization of Matroid Theory and More

1.1. Basic notions and matroid axioms. For any finite set \( S \), its power set \( 2^S := \{ A : A \subseteq S \} \) with the natural inclusion relation \( \subseteq \) is a poset. We consider an extra structure on \( 2^S \), and define a matroid.

Definition 1.1 (Matroid rank axioms). A submodular rank function \( r \) on \( 2^S \) is a \( \mathbb{Z}_{\geq 0} \)-valued function with the following properties:

1. \( 0 \leq r(A) \leq |A| \) for \( A \in 2^S \), (boundedness)
2. \( r(A) \leq r(B) \) for \( A, B \in 2^S \) with \( A \subseteq B \), (increasingness)
3. \( r(A \cup B) + r(A \cap B) \leq r(A) + r(B) \) for \( A, B \in 2^S \). (submodularity)

Definition 1.2. A (finite) matroid \( M \) on \( S \) is a poset \( 2^S \) equipped with a submodular rank function \( r \) on \( 2^S \), denoted by \( M := (r; S) \), where \( E(M) := S \) is called the ground set of \( M \). The function \( r \) is simply called the rank function of \( M \), and \( r(M) := r(S) \) is called the rank of \( M \). Two matroids \( (r; S) \) and \( (r'; S') \) are said to be isomorphic if there is a bijection \( f : S \to S' \) with \( r = r' \circ f \).

Let \( M = (r; S) \) be a matroid. We denote \( A + s := A \cup \{ s \} \) and \( A - s := A - \{ s \} \) for any subset \( A \subseteq S \) and any element \( s \in S \).

- The \( k \)-th graded piece \( \{ A \in 2^S : r(A) = k \} \) of \( 2^S \) is denoted by \( M^{(k)} \). For every \( A \in M^{(k)} \), the set \( T := \{ s \in S : r(A + s) = r(A) \} \) is a unique maximal member of \( M^{(k)} \) containing \( A \), called a flat. Denote by \( L(M) \) the collection of the flats in \( M \), then \( L(M) := \cup_{k=0}^{r(M)} M^{(k)} \) is called the (geometric) lattice of \( M \).
- If \( r(A) = |A| \) in (R1), then \( A \) is called an independent set. All the maximal independent sets, called the bases, have the same size \( r(M) \). We denote by \( \mathcal{B} = \mathcal{B}(M) \) the collection of the bases of \( M \).
- If \( r(A) < |A| \) in (R1), then \( A \) is called a dependent set. A minimal dependent set is called a circuit. The collection of the circuits of \( M \) is denoted by \( C = C(M) \).
- If \( r(A \cup B) + r(A \cap B) = r(A) + r(B) \) in (R3), the pair \( \{ A, B \} \) is called a modular pair (with respect to the rank function \( r \)).

Proposition 1.3 (Matroid flat axioms). A nonempty subcollection \( A \subseteq 2^S \) with \( S \in A \) is the lattice of a matroid if it satisfies the following axioms.

1. \( F, L \in A \), then \( F \cap L \in A \).
2. \( F \in A \) and \( s \in S - F \), by (F1), there exists the smallest member \( L \) of \( A \) containing \( F + s \). Then, there is no member of \( A \) between \( F \) and \( L \).

Proposition 1.4 (Matroid base axioms). A nonempty subcollection \( A \subseteq 2^S \) is the base collection of a matroid if it satisfies the base exchange property:

\( \text{(BEP)} \) For \( A, B \in A \), if \( x \in A - B \), then \( A - x + y \in A \) for some \( y \in B - A \).

Note that \( \text{(BEP)} \) implies that every member of \( A \) has the same size.

Proposition 1.5 (Matroid independent-set axioms). A nonempty subcollection \( A \subseteq 2^S \) with \( \emptyset \in A \) is the independent-set collection of a matroid if it satisfies the following axioms.

1. \( A \subseteq A \) and \( B \subseteq A \) implies that \( B \in A \).
2. \( A, B \in A \) with \( |A| < |B| \), there exists \( b \in B - A \) such that \( A + b \in A \).

The condition (I2) is called the exchange property for independent sets.
There is also a system of matroid axioms with respect to circuits, but we do not state it here. All these \( r, \mathcal{L}, \mathcal{I}, \mathcal{B}, \mathcal{C} \) are recovered from one another. Thus, we use the pair of some of those and \( S \) to denote the matroid, where \( S = \cup \mathcal{L} \supseteq \cup \mathcal{I} = \cup \mathcal{B} \).

1.2. **Realizable matroids.** Let \( V \) be a \( k \)-dimensional vector space over a field \( k \) with a spanning set \( \{v_1, \ldots, v_n\} \). Denote \([n] := \{1, \ldots, n\}\) and define a \( \mathbb{Z}_{\geq 0} \)-valued function \( r_V : 2^{[n]} \to \mathbb{Z}_{\geq 0} \) such that:

\[
r_V(A) = \dim \text{span}\{v_i : i \in A\}.
\]

Then, \( r_V \) satisfies \((R1)-(R3)\) and is the rank function of a matroid where \( A \in 2^{[n]} \) is an independent set if and only if \( \{v_i : i \in A\} \) is linearly independent over \( k \).

**Definition 1.6.** A matroid \( M = (r; [n]) \) is called realizable or representable over a field \( k \) if there is a collection of vectors \( \{v_i \in k^r(M) : i \in [n]\} \) such that \( r = r_V \) of (1.1). Any matroid isomorphic to a realizable matroid is also called realizable. A matroid that is realizable over every field is called a regular matroid.

**Example 1.7.** (a) The \((k, n)\)-uniform matroid \( U^k_n \) for \( 0 \leq k \leq n \) is defined by a rank function on \( 2^{[n]} \): \( A \mapsto \min(\{k, |A|\}) \). Its bases are all the \( k \)-sets of \([n]\), that is, \( k \)-element subsets of \([n]\). The matroid on \( S \) isomorphic to \( U^k_n \) is denoted by \( U^k_n[S] \), called the \((k, S)\)-uniform matroid or rank-\( k \) uniform matroid on \( S \).

(b) Let \( G \) be a graph with edges \( e_1, \ldots, e_n \) and \( A \) the collection of \( A \in 2^{[n]} \) such that \( \{e_i : i \in A\} \) is a cycle. Then, \( M(G) := (A, [n]) \) is a matroid with \( \mathcal{C}(M(G)) = A \), called a graphic matroid. This matroid is a regular matroid.

1.3. **Pullback and pushforward.** Regarding (finite) matroids as combinatorial structures on sets, we define pullback and pushforward of matroids. Let \( \hat{S} \) and \( S \) be finite sets with a map \( f : \hat{S} \to S \), and let \( \hat{M} = (\hat{r}; \hat{S}) \) and \( M = (r; S) \) be matroids. Now, let \( f^*(\mathcal{I}(M)) \) be the following nonempty subcollection of \( 2^{\hat{S}} \):

\[
f^*(\mathcal{I}(M)) := \{A \in 2^{\hat{S}} : f(A) \in \mathcal{I}(M), |A| = |f(A)| \neq 0\}.
\]

Then, \( f^*(\mathcal{I}(M)) \) satisfies (11)-(12), and \( f^*(M) := (f^*(\mathcal{I}(M)); \hat{S}) \) is a matroid with rank function \( r_{f^*(M)} = r_M \circ f \), which we call the pullback of \( M \) under the map \( f \). In particular, \( \mathcal{L}(f^*(M)) = f^{-1}(\mathcal{L}(M)) \). Also, consider the following collection:

\[
f_*(I(\hat{M})) := \{f(I) \in 2^{S} : I \in \mathcal{I}(\hat{M}) \neq \emptyset\}.
\]

Then, \( f_*(\hat{M}) := (f_*(I(\hat{M})); S) \) is a matroid with \( \mathcal{I}(f_*(\hat{M})) = f_*(I(\hat{M})) \), and we call this matroid the pushforward of \( \hat{M} \) under \( f \), cf. [Sch03, Theorem 42.1].

Note that if \( f \) is injective, \( \hat{M} \cong f_*(\hat{M})|_{\text{im} f} \). If \( f \) is surjective, \( M = f_*(f^*(M)) \).

1.4. **Operations on matroids.** Let \( M = (r; S) \) be a matroid.

- For an inclusion \( \iota : A \to S \), the matroid \( M|_A := \iota^*(M) \) is called the submatroid of \( M \) on \( A \), or the restriction of \( M \) to \( A \).
- The deletion \( M \setminus A \) of \( A \) from \( M \) is defined as the restriction of \( M \) to \( A^c = S \setminus A \).
- The contraction \( M/A \) of \( A \) in \( M \) is a matroid on \( A^c \) defined by a rank function \( J \mapsto r(J \cup A) - r(A) \). It is also called the contracted matroid of \( M \) over \( A \).
- The dual matroid \( M^* \) of \( M \) is a matroid on \( S \) defined by a rank function \( * \) given by \( A \mapsto |A| - r(S) + r(S - A) \). Note that \( (M^*)^* = M \).
- The \( k \)-level matroid \( M^{(\leq k)} \) of \( M \) is a matroid on \( S \) defined by \( A \mapsto \min(k, r(A)) \).
Let $M_1 = (r_1; S_1)$ and $M_2 = (r_2; S_2)$ be two matroids.

- Denote by $S_1 \oplus S_2$ the disjoint union of $S_1$ and $S_2$, and by $r_1 \oplus r_2$ the rank function $A_1 \oplus A_2 \mapsto r_1(A_1) + r_2(A_2)$ for $A_1 \in 2^{S_1}$ and $A_2 \in 2^{S_2}$. The matroid $M_1 \oplus M_2 := (r_1 \oplus r_2; S_1 \oplus S_2)$ is called the direct sum of $M_1$ and $M_2$.

- Let $f : S_1 \oplus S_2 \to S_1 \cup S_2$ be a natural surjection. Then, $M_1 \vee M_2 := f_*(M_1 \oplus M_2)$ is a matroid, called the matroid union\(^{10}\) of $M_1$ and $M_2$. It is a common upper bound matroid of $M_1$ and $M_2$ in the sense that $\mathcal{I}(M_1) \cup \mathcal{I}(M_2) \subseteq \mathcal{I}(M_1 \vee M_2)$. But, it is not the smallest nor a minimal such, and hence not a universal object.

- The pair $M_1 \wedge M_2 := (\mathcal{I}(M_1) \cap \mathcal{I}(M_2); S_1 \cap S_2)$ is called the matroid intersection of $M_1$ and $M_2$, which is in general not a matroid again, cf. [Sch03, Chapter 41]. Abusing notation, we mean by $M_1 \wedge M_2$ the collection $\mathcal{I}(M_1) \cap \mathcal{I}(M_2)$. Note that $\wedge$ is not compatible with $\oplus$: one has $(M_1 \oplus M_2) \wedge N \subset (M_1 \wedge N) \oplus (M_2 \wedge N)$ for any matroid $N$, but not the other way round in general.

- The pair $M_1 \cap M_2 := (B(M_1) \cap B(M_2); S_1 \cap S_2)$ is called the base intersection of $M_1$ and $M_2$, and the pair $M_1 \cup M_2 := (B(M_1) \cup B(M_2); S_1 \cup S_2)$ is called the base union of $M_1$ and $M_2$. We follow the same notational convention as above.

- By definition, $M_1 \cap M_2 \subseteq M_1 \wedge M_2$ as collections. Note that if $M_1 \cap M_2 \neq 0$, the ranks of $M_1$ and $M_2$ are the same. In addition, if $M_1 \cap M_2 \neq 0$ is a matroid, so is $M_1 \wedge M_2$, and $M_1 \cap M_2 = M_1 \wedge M_2$ as matroids.

1.5. More terms and useful properties. Let $M = (r; S)$ be a matroid.

- The connectivity function $c_M$ of $M$ is a $\mathbb{Z}_{\geq 0}$-valued function on $2^S$ defined by $A \mapsto r(A) + r^*(A) - |A|$.

- A member $A \in 2^S$ is called a separator of $M$ if $c_M(A) = 0$. Then, $M$ and $M^*$ have the same collection of separators, which is closed under the set complement, intersection, and union. Note that $S$ and $\emptyset$ are always separators, and called the trivial separators. A nontrivial separator is called a proper separator.

- A matroid is called insep\_arable or connected if it has no proper separator.\(^{11}\)

- A 1-set separator with rank 0 is called a loop. The collection of loops is $S - \cup \mathcal{B}$ denoted by $\emptyset = \partial M$, and $M$ is called loopless if $\emptyset = \emptyset$.

- A 1-set separator with rank 1 is called a coloop. The collection of coloops is $\cap \mathcal{B}$ denoted by $\emptyset^*$. A coloop of $M$ is a loop of $M^*$, and $\emptyset^* = \partial M^*$. The matroid $M$ is said to be coloopless if $\emptyset^* = \emptyset$.

- $M\vert_{\emptyset}$ and $M\vert_{\emptyset^*}$ are uniform matroids of rank 0 and rank $|\emptyset^*|$, respectively.

- A loopless and coloopless matroid is said to be a relevant matroid.

- A simple matroid is a loopless matroid such that every rank-1 flat is a 1-set.

- We denote by $\lambda(M)$ the number of rank-1 flats of $M$.

- For a matroid $M$, consider a surjective map $f$ defined on $S - \emptyset$ by $f(i) = \bar{i}$. Then, $f_*(M\backslash\emptyset)$ is a simple matroid whose ground set size is $\lambda(M)$. This matroid is called the simplification of $M$. Note that $M\backslash\emptyset = f^*(f_*(M\backslash\emptyset))$. Any simple matroid is isomorphic to its simplification.

\(^{10}\)In contrast to this, a partition matroid is defined as a pullback of a certain uniform matroid.

\(^{11}\)"Inseparable" was used in [Sch03] to indicate a subset $A$ of $E(M)$ for a matroid $M$ such that the restriction matroid $M\vert_{\partial A}$ is connected. In this paper, we use inseparable (preferred) or connected for both inseparable subsets and connected matroids.
Let $A_1, \ldots, A_{\kappa(M)}$ be the nonempty minimal separators of $M$, then $M$ is written as $M = M|_{A_1} \oplus \cdots \oplus M|_{A_{\kappa(M)}}$ where $\kappa(M) \leq r(M) + |\emptyset|$. Each summand $M|_{A_i}$ is an inseparable matroid, called a **connected component** of $M$.

We denote by $\kappa(M)$ the **number of connected components** of $M$. Then, $\kappa$ is a $\mathbb{Z}_{\geq 0}$-valued function defined on the collection of (finite) matroids.

The independent sets of $M|_{A}$ are those independent sets of $M$ contained in $A$.

The independent sets of $M/A$ are those independent sets $I$ of $M|_{A'}$ such that $\{I, A\}$ is a modular pair, or equivalently, $M|_{I \cup A} = (M|_{I}) \oplus (M|_{A})$.

For a base $B$ of $M$, if $B \cap A$ is a base of $M|_{A}$, then $B - A$ is a base of $M/A$, and vice versa. Conversely, every base of $(M|_{A}) \oplus (M/A)$ is a base of $M$.

For $C,D \in 2^S$, one has $(M|_{C}) \cap (M|_{D}) = M|_{C \cap D}$, $(M|_{C}) \cap (M|_{D}) = M|_{C \cup D}/D$, and $(M|_{C}) \cap (M|_{D}) \supseteq I(M/(C \cup D))$.

### 1.6. Flats and non-degenerate subsets

The flats of a matroid behave like the closed sets in a topological space. Further, restriction to and contraction over a flat work like restriction and quotient morphisms, respectively. Let $M$ be a matroid.

- If $F$ is a flat of $M$, then $F \cap A$ is a flat of $M|_{A}$ for any $A \in 2^{E(M)}$.
- Conversely, if $F$ is a flat of $M|_{A}$, then $F = F \cap A$.
- A member $A \in 2^{E(M)}$ is a flat of $M$ if and only if $M/A$ is loopless.
- A member $F \in 2^{E(M)}$ is a flat of $M/A$ if and only if $F \cup A$ is a flat of $M$.

**Definition 1.8.** Let $M$ be an inseparable matroid. A member $A \in 2^{E(M)} \setminus \{\emptyset, E(M)\}$ is called **non-degenerate** if both $M/A$ and $M|_{A}$ are inseparable matroids.

- If $r(M) \leq 1$, there is no non-degenerate flat.
- If $r(M) = 2$, the non-degenerate flats are exactly the rank-1 flats. Then, $E(M)$ is written as the disjoint union of the non-degenerate flats.
- If $r(M) = 3$, a nonempty proper flat $A$ of $M$ is non-degenerate if and only if $\lambda(M/A) \geq 3$ or $\lambda(M|_{A}) \geq 3$ where $\lambda$ denotes the number of rank-1 flats.

### 1.7. Minors and expressions

Fix a matroid $M$. A finite sequence of restrictions and contractions is called a **minor expression** and denoted by concatenating $M$ and those operations in order from left to right. This notation is consistent with those of restriction and contraction. A **minor** of $M$ is defined as the matroid that a minor expression of $M$ represents.

Let $A,B,C,D \in 2^{E(M)}$ be such that $A \supseteq B$ and $C \cap D = \emptyset$, then the following are the **4 basic equations of minor expressions**, where $A,B,C,D$ are assumed to satisfy the implicit conditions imposed on each of those:

$$M|_{A \cup B} = M|_{A \cup B}, \quad M|_{A \cup B} = M/B|_{A \cup B}, \quad M|_{C \cup D} = M/C|_{C \cup D}, \quad M/C|_{D} = M/C \cup D$$

A minor expression of $M$ is said to be **empty** and denoted by $\emptyset$ if it is transformed into $M/F|_{\emptyset}$ for some $F \in E(M)$ using the 4 basic equations.

Consider a finite direct sum of minor expressions of matroids, say $X = \bigoplus_{i=1}^{t} X_i$. We call it a **matroidal expression** or simply an **expression**. We say that it is a **reduced** expression if none of $X_i$ is empty.

---

12The name comes from the torus action on Grassmannians $G(k,n)$, see [GS87]. This definition is generalized for any matroid later in Definition 2.18. Also, we show how to recover the lattice of a matroid from its collection of non-degenerate flats, Proposition 2.27.
Two reduced expressions \( X = \bigoplus_{i=1}^{\ell} X_i \) and \( Y = \bigoplus_{j=1}^{m} Y_j \) are said to be equivalent and denoted \( X = Y \) if \( \ell = m \) and there is a bijection \( \sigma : [\ell] \to [m] \) such that each \( X_i \) is transformed into \( Y_{\sigma(i)} \) using the 4 basic equations. Two expressions are said to be equivalent if their reduced expressions are. Any expression equivalent to \( \emptyset \) is also said to be the empty expression and denoted by \( \emptyset \).

Remark 1.9. Two non-equivalent expressions can represent the same matroid.

Let \( \phi \) denote the forgetful map that sends every expression \( X = \bigoplus_{i=1}^{\ell} X_i \) to the matroid that \( X \) represents. We often omit \( \phi \) if there is no confusion, for instance, if an operation such as \( r, I, B, \land, \cap \), etc. is written together. We say an expression is loopless or inseparable if the matroid it represents is. Also, if \( \phi(X) = \phi(Y) \) for two expressions \( X \) and \( Y \), we write \( X \sim Y \).

2. Incidence Geometry of Matroid Polytopes

A convex polytope is a convex hull of a finite number of points in \( \mathbb{R}^n \) for some nonnegative integer \( n \), denoted by \( \text{conv} \, A \) with \( A \) the collection of those points. Equivalently, a convex polytope is a bounded intersection of a finite number of half spaces. In this section, we study convex polytopes associated to matroids.

For any nonempty ground set \( S \), we denote by \( \mathbb{R}^S \) the product of \( |S| \) copies of \( \mathbb{R} \) labeled by the elements of \( S \), one for each, where \( \mathbb{R}^S \) is also understood as the space of all functions \( A \rightarrow \mathbb{R} \). For any \( A \in 2^S \) and a vector \( v = (v_i)_{i \in S} \in \mathbb{R}^S \), we denote:

\[
v(i) := v_i \quad \text{for} \quad i \in S \quad \text{and} \quad v(A) = v_A := \sum_{i \in A} v(i).
\]

**Definition 2.1.** The indicator or incidence vector of \( A \in 2^S \) is defined as a vector \( 1^A \in \mathbb{R}^S \) whose \( i \)-th entry is 1 if \( i \in A \), and 0 otherwise. An indicator vector is also called a 0/1-vector. In particular, \( 1 := 1^S \) is called the all-one vector.

For \( A, B \in 2^S \) with \( A \neq B \), the line segment \( \text{conv}(1^A, 1^B) \subset \mathbb{R}^S \) is denoted by \( 1^A 1^B \), and its length or the distance between \( 1^A \) and \( 1^B \) is defined as:

\[
d(1^A, 1^B) = d(A, B) := \frac{1}{2} |A \cup B - A \cap B|
\]

which is equal to the \( L^1 \)-norm of the vector \( 1^A - 1^B \) or \( 1^B - 1^A \) divided by 2. Thus, all matroids are discrete metric spaces with a metric \( d : 2^S \times 2^S \rightarrow \frac{1}{2} \mathbb{Z}_{\geq 0} \).

**Definition 2.2.** The moment polytope \( P_A \) of a nonempty subcollection \( A \subset 2^S \) is defined as \( P_A := \text{conv}(1^A : A \in A) \subset \mathbb{R}^S \), which is also called a 0/1-polytope. Denote \( 1 - P_A := \{ 1 - x : x \in P_A \} \), then one has \( 1 - P_A = \text{conv}(1 - 1^A : A \in A) \). This is said to be the dual moment polytope of \( P_A \).

**Definition 2.3.** A moment polytope \( P_A \) is said to be loopless if it is not contained in a coordinate hyperplane of \( \mathbb{R}^S \), coloopless if \( 1 - P_A \) is loopless, and relevant if it is both loopless and coloopless. Note that if \( A = B(M) \) for some matroid \( M \), then \( 1 - P_A = P_{A^*} \), where \( A^* = B(M^*) \); hence, \( P_A \) is (co)loopless if and only if the matroid \( M \) is (co)loopless, see also Lemma 2.12.

**Example 2.4.** The moment polytope of \( B(U^S_k) \) is called the \((k, S)\)-hypersimplex, denoted by \( \Delta_k^S = \Delta(k, S) \subset \mathbb{R}^S \), and a moment polytope in \( \Delta_k^S \) is said to be a \((k, S)\)-polytope. When \( S = [n] \), we often write the notations using \( n \) instead of \([n]\).

**Remark 2.5.** Two reduced expressions \( P_A \) and \( P_{A'} \) are the same if and only if two subcollections \( A \) and \( A' \) of \( 2^S \) are the same. By Definitions 2.1 and 2.2, we may identify subcollections of \( 2^S \) with their moment polytopes, and vice versa.
For a moment polytope, we call the following the \textbf{edge length property}:

\begin{equation}
\text{Every edge has length } \leq 1.
\end{equation}

Then, under the assumption of (I1), the edge length property is equivalent to the exchange property (I2), \cite[Theorem 40.6]{Sch03}. Furthermore, for a \((k, S)\)-polytope the edge length property alone is equivalent to the base exchange property (BEP). These facts lead us to the following definitions.

\textbf{Definition 2.6.} Suppose a moment polytope \(P_A\) satisfies the edge length property.

(a) If it satisfies (11) as well, it is called the \textbf{independent-set polytope} of the matroid \(M\) with \(I(M) = A\), denoted by \(IP_M\).

(b) If it is a \((k, S)\)-polytope, it is called the \textbf{base polytope} of the matroid \(M\) with \(B(M) = A\), denoted by \(BP_M\). It is also called a \textbf{matroid polytope}.

(c) For \(P_A = BP_M\) or \(P_A = IP_M\), we denote by \(MA_{P_A}\) the matroid \(M\).

\textbf{Proposition 2.7.} Every face of a base polytope is again a base polytope.

\textbf{Proposition 2.8} \cite[Corollaries 41.12b,d]{Sch03}. For any two matroids \(M\) and \(N\) with \(E(M) = E(N)\) one has \(IP_M \cap IP_N = P_{M \wedge N}\) and \(BP_M \cap BP_N = P_{M \cap N}\).

Let \(M\) be a loopless matroid on \(S\). Then, its independent-set polytope \(IP_M\) is the intersection of the following half spaces:

\begin{equation}
\{ x \in \mathbb{R}^S : x(i) \geq 0 \text{ } \forall i \in S \text{ and } \{ x \in \mathbb{R}^S : x(A) \leq r(A) \} \forall A \in 2^S. \tag{2.2}\end{equation}

The base polytope \(BP_M\) is the intersection of those half spaces and a hyperplane \{\(x \in \mathbb{R}^n : x(S) = r(M)\)\}. An inequality \(x(A) \leq r(A)\) with \(\emptyset \neq A \subset S\) is said to be \textbf{relevant} if \(0 < r(A) < |A|\). The system of the relevant inequalities of (2.2) reduces to a minimal one, see Corollary 2.20.

\textbf{Definition 2.9.} Let \(Q\) be a face of a moment polytope \(P_A\). We say that \(v \in \mathbb{R}^S\) is a \textbf{characterizer vector of} \(Q\) in \(P_A\) if there is a number \(c \in \mathbb{R}\) such that

\[ v(A) = \langle v, 1^A \rangle \geq c \text{ for all } A \in A \]

and equality holds precisely when \(1^A\) is a vertex of \(Q\), that is, \(Q\) is determined by the equation \(\langle v, x \rangle = c\) in \(P_A\) where \(\langle , \rangle\) denotes the usual inner product of \(\mathbb{R}^S\).

For a matroid \(M\) on \(S\) with \(|S| = n\), the \textbf{dimensions} of \(IP_M\) and \(BP_M\) are:

\[ \dim IP_M = n - |\emptyset| \] \[ \dim BP_M = n - \kappa(M) \]

where \(\kappa(M)\) is the number of connected components of \(M\). Then, \(M\) is inseparable if and only if \(BP_M\) is full-dimensional, i.e. \(BP_M\) is a facet of \(IP_M\), or equivalently, \(I\) is up to positive scalars a unique characterizer vector of \(BP_M\) in \(IP_M\).

\textbf{2.1. Faces of} \(IP_M\) \textbf{and} \(BP_M\). The edge length property describes the 1-dimensional faces of \(IP_M\) and \(BP_M\). Proposition 2.10 below describes their 2-dimensional faces, which generalizes \cite[Theorem 1.12.8]{Sch03}, without using representation theory.

\textbf{Proposition 2.10.} Let \(M\) be a rank-\(k\) matroid. Then, every 2-dimensional face of \(IP_M\) falls into the following 4 cases; see Figure 2.1, where the angles are Euclidean and the numbers in parentheses indicate the ranks of the vertices.

(a) \textit{A regular triangle with side length 1 contained in} \(P_{M^k}\).

(b) \textit{A square with side length 1 contained in} \(P_{M^k}\).

(c) \textit{An isosceles right triangle with lengths \(1/2, 1/2, 1\) connecting} \(P_{M^n}\) \textit{and} \(P_{M^{n+1}}\).

(d) \textit{A square with side length \(1/2\) connecting} \(P_{M^{n-1}}, P_{M^n}\) \textit{and} \(P_{M^{n+1}}\).
are the same.

be loopless, so is 

is loopless, so is

is a loopless matroid.

is written as follows:

is a base of 

is contained in a coordinate 

1

\( P = BP \)

Similarly, one shows

The following equivalent statements show 

Proof. Use the metric \( d \) of (2.1) and the edge length property.

Notation 2.11. Let \( M \) be a matroid with \( F \subset E(M) \) and let \( P = IP_M \) or \( P = BP_M \). Then, \( \{ x \in P : x(F) = r(F) \} \) is a nonempty face of \( P \) by the matroid axioms. Denote:

\[
P(F) = \{ x \in P : x(F) = r(F) \} \quad \text{and} \quad M(F) = \bigoplus (M/F).
\]

Then, Lemma 2.12 below tells \( BP_M(F) = BP_{M(F)} \). Further, for subsets \( F_1, \ldots, F_m \) of \( E(M) \), we simply write \( BP_M(F_1)(F_2) \cdots (F_m) \) for \( (\cdots ((BP_M(F_1))(F_2)) \cdots ) (F_m) \), and \( M(F_1)(F_2) \cdots (F_m) \) for \( (\cdots ((M(F_1))(F_2)) \cdots ) (F_m) \). Then, we have:

\[
BP_M(F_1)(F_2) \cdots (F_m) = BP_{M(F_1)(F_2) \cdots (F_m)}.
\]

Lemma 2.12. Assume the above setting and suppose that \( M \) is a loopless matroid. Then, \( P(F) \) is loopless if and only if \( F \) is a flat of \( M \). Furthermore, one has:

\[
IP_M(F) = IP_{M/F} \times BP_{M|F} \quad \text{and} \quad BP_M(F) = BP_{M/F} \times BP_{M|F}.
\]

Proof. The following equivalent statements show \( IP_M(F) = IP_{M/F} \times BP_{M|F} \):

- The indicator vector \( 1^A \) of \( A \subset [n] \) is a vertex of \( IP_M(F) \).
- \( A \) is an independent set of \( M \), and \( |A \cap F| = 1^A(F) = r(F) \).
- \( A \cap F \) is a base of \( M|F \), and \( A - F \) is an independent set of \( M/F \).
- \( 1^A \) is a vertex of \( IP_{M/F} \times BP_{M|F} \).

Similarly, one shows \( BP_M(F) = BP_{M/F} \times BP_{M|F} \).

Now, if \( M \) is loopless, so is \( M|F \). Hence, \( IP_M(F) \) is contained in a coordinate hyperplane if and only if \( IP_{M|F} \) is, that is, \( M/F \) has a loop, or equivalently, \( F \) is a non-flat of \( M \). The same argument is applied to \( BP_M(F) \).

Remark 2.13. Let \( M \) be a matroid and \( F, L \) subsets of \( E(M) \). Then, one has:

\[
M(F)(L) = M(F \cup L)(F \cap L) = M(F \cap L)(F - L)(L - F)
\]

where these three are the same as matroidal expressions. Further, let \( F_1, \ldots, F_m \) be subsets of \( E(M) \), and \( d_1 d_2 \cdots d_m \) a binary number converted from each decimal number \( i = 0, \ldots, 2^m - 1 \). Let \( L_i := F_1 \sqcap_{i_1} (\cdots (F_{m-1} \sqcap_{i_{m-1}} (F_m \sqcap_{i_m} \emptyset)) \cdots ) \) where \( \sqcap_{i_k} = \cup \) if \( d_{ik} = 0 \) and \( \sqcap_{i_k} = \cap \) if \( d_{ik} = 1 \). Then, \( \{ L_0, \ldots, L_{2^m-1} \} \) is a decreasing sequence of subsets with \( L_{2^m-1} = \emptyset \), and \( M(F_1) \cdots (F_m) \) is written as follows:

\[
M(F_1) \cdots (F_m) = M(L_0) \cdots (L_{2^m-2}). \tag{2.3}
\]

We provide a few corollaries to Lemma 2.12.

Lemma 2.14. Assume the above setting. Then, \( \{ F, L \} \) is a modular pair of \( M \) if and only if \( M(F) \cap M(L) \neq \emptyset \) if and only if (any) two of \( \phi(M(F \cup L)(F \cap L)) \), \( \phi(M(F)(L)) \), \( \phi(M(L)(F)) \) and \( M(F) \cap M(L) \) are the same.
Proof. Write $r = r_M$ as a matter of convenience. Suppose $\{F, L\}$ is a modular pair, then $r_{M(F)}(L) = r(L)$, and by Lemma 2.12 one has:

$$BP_{M(F)} \cap BP_{M(L)} = \{x \in BP_M : x(F) = r(F), x(L) = r(L)\} = \{x \in BP_{M(F)} : x(L) = r_{M(F)}(L)\} = BP_{M(F)}(L) \neq \emptyset.$$ Conversely, suppose $M(F) \cap M(L) \neq \emptyset$. Then, for any $B \in M(F) \cap M(L)$,

$$r(F) + r(L) = |B \cap F| + |B \cap L| = |B \cap F \cap L| + |B \cap (F \cup L)| \leq r(F \cap L) + r(F \cup L)$$

where equality holds by the submodularity of $r$, and $\{F, L\}$ is a modular pair. One can also check the remaining statement using the above arguments. \hfill \Box

Corollary 2.15. Let $M$ be a matroid with $F, L \subset E(M)$. Then, $\{F, L\}$ is a modular pair if and only if $F - L$ and $L - F$ are separators of $\phi(M/(F \cap L)\mid F \cup L)$. \hfill \Box

Definition 2.16. Let $M$ be a matroid, $(F_1, \ldots, F_c)$ a sequence of subsets of $E(M)$. Then, $\phi(M(F_1) \cdots (F_c))$ is called a face matroid of $M$.

The following theorem describes the facets of a base polytope of an inseparable matroid. We extend this theorem to general matroids, Lemma 2.19.

Corollary 2.17 ([GS87, Theorem 2.5.2]). Let $M$ be an inseparable matroid of positive rank. Then, every facet of the base polytope $BP_M$ is written as $BP_{M(F)}$ for a unique non-degenerate subset $F$ of $M$, and vice versa.

Definition 2.18. For a matroid $M$, a subset $F \subset E(M)$ is called non-degenerate if $\kappa(M(F)) = \kappa(M) + 1$, where $\phi(M(F))$ is called a facet matroid of $M$. There can be several different non-degenerate subsets $F_i$ with the same matroid $\phi(M(F_i))$, but there exists a unique inclusionwise minimal such. Note that if $F$ is a non-degenerate subset of $M$, then $E(M) - F$ is a non-degenerate subset of $M^*$.

Lemma 2.19. For a matroid $M$, there is a bijection between the facets $R$ of $BP_M$ and the matroids $\phi(M(F))$ for non-degenerate subsets $F$ of $M$ so that $R = BP_{M(F)}$. In particular, every loopless facet of $BP_M\setminus\emptyset$ is written as $BP_{M(F)\setminus\emptyset}$ for some non-degenerate flat $F$ of $M$, and vice versa.

Proof. Suppose $M$ is loopless, and let $M = M_1 \oplus \cdots \oplus M_\ell$ be the decomposition of $M$ into its connected components. Since every face of $BP_M$ is written as $Q_1 \times \cdots \times Q_\ell$ where $Q_i$ are faces of $BP_{M_i}$, Corollary 2.17 and Lemma 2.12 prove the bijection and the remaining statement. For a general $M$, if $F$ is a non-degenerate flat of $M$, then $F \setminus \emptyset$ is a non-degenerate flat of $M\setminus\emptyset$. Further, $M(F)\setminus\emptyset = (M\setminus\emptyset)(F\setminus\emptyset)$ since $F \supset \emptyset$. Thus, apply the previous argument to $M\setminus\emptyset$. \hfill \Box

Corollary 2.20. Let $M$ be a rank-$k$ loopless matroid on $S$ with its decomposition $M = M_1 \oplus \cdots \oplus M_\kappa$ into $\kappa = \kappa(M)$ connected components. Then, its independent-set polytope $IP_M$ is determined by a minimal system of inequalities (2.4):

$$\begin{align*}
x(i) &\geq 0 \quad \text{for all } i \in S, \\
x(F) &\leq r(F) \quad \text{for all minimal non-degenerate flats } F \text{ of } M.
\end{align*}$$ (2.4)

Its base polytope $BP_M$ is determined by (2.4) and $\kappa$ equations $x(E(M_j)) = r(M_j)$, $j = 1, \ldots, \kappa$. Those inequalities $x(F) \leq r(F)$ are said to be essential.
2.2. The intersection of face matroids. Let $M$ be a matroid, and consider the collection of all the minor expressions. Define an operation $\circ$ on this collection:

$$(M/A|_{C}) \odot (M/B|_{D}) := M/(A \cup B)|_{C \cap D}$$

which is commutative and associative. Then, consider the collection of all finite direct sums of minor expressions, and extend $\circ$ by defining $N_0 \odot (\oplus_{i=1}^{m} N_i)$ for any sequence of minor expressions $N_0, N_1, \ldots, N_m$ such that:

$$N_0 \odot (\oplus_{i=1}^{m} N_i) := \oplus_{i=1}^{m} (N_0 \odot N_i). \quad (2.5)$$

Then, $\circ$ is even distributive over $\oplus$ by definition. For subsets $F_1, \ldots, F_m \subset E(M)$ and any $\sigma \subset [m]$, denote $F^-_\sigma = \cap_{j \notin \sigma} F_j$, $F^+_\sigma = \cup_{i \in \sigma} F_i$, and $M_\sigma = M/F^+_\sigma \setminus (F^-_\sigma - F^+_\sigma)$, where we set $F^-_\emptyset = E(M)$ by convention. Then, $E(M) = \cup_{\sigma \subset [m]} (F^-_\sigma - F^+_\sigma)$ and:

$$\cap_{i=1}^{m} M(F_i) = \oplus_{\sigma \subset [m]} M_\sigma.$$

The definitions of $\land$ and $\cap$ are expression-free while that of $\circ$ is not, and $\circ$ can produce different matroids for different expressions. However, for face matroids, $\circ$ gives an expression-free output as long as the output has full rank, Theorem 2.25.

Proposition 2.21. Let $M$ be a matroid and $F_1, \ldots, F_m$ subsets of $E(M)$, then:

$$\cap_{i=1}^{m} M(F_i) \subset B(\circ_{i=1}^{m} M(F_i)).$$

Proof. If $B$ is a common base of $\phi(M(F))$ and $\phi(M(L))$, then it is a common base of $\phi(M(F \cap L))$ and $\phi(M(F \cap L))$ by Lemma 2.14. Therefore, let $A$ be the Boolean algebra generated by $F_1, \ldots, F_m$ with unions and intersections, and $B$ a common base of all $\phi(M(F_i))$, then $r(B \cap A) = r(A)$ for all $A \in A$. Further, for any $\sigma \subset [m]$, $B_\sigma := B \cap (F^-_\sigma - F^+_\sigma)$ is a base of $\phi(M_\sigma)$ since:

$$r_{M_\sigma}(B_\sigma) \leq r(M_\sigma) = r(F^-_\sigma \cup F^+_\sigma) - r(F^+_\sigma) = r(B \cap (F^-_\sigma \cup F^+_\sigma)) - r(F^+_\sigma) \leq r((B \cap F^-_\sigma) \cup F^+_\sigma) - r(F^+_\sigma) = r_{M_\sigma}(B_\sigma).$$

Therefore, $B = \cup_{\sigma \subset [m]} B_\sigma$ is a base of $\phi(\oplus_{\sigma \subset [m]} M_\sigma) = \phi(\circ_{i=1}^{m} M(F_i))$. \qed

It also turns out that $\mathcal{I}(\circ_{i=1}^{m} M(F_i))$ is bounded above by $\cap_{i=1}^{m} M(F_i)$.

Proposition 2.22. Let $M$ be a matroid, $M'$ a minor of $M$, and $V = M/F \mid_A$ any minor expression of $M'$ with some $F, A \subset E(M)$. Then, for any $L \subset E(M)$:

$$\mathcal{I}(V \odot M(L)) \subset M' \land \phi(M(L)).$$

Proof. Let $N = M/F$, then $V \odot M(L)$ is written as $N|_{A\cap L} \oplus (N/L|_{A-L})$. Clearly, $\mathcal{I}(V \odot M(L)) \subset \mathcal{I}(M(L))$. Further, $\mathcal{I}(N/L|_{A-L}) \subset \mathcal{I}(N|_{A}/(A \cap L))$ and this implies $\mathcal{I}(V \odot M(L)) = \mathcal{I}(N|_{A\cap L} \oplus (N/L|_{A-L}) \subset \mathcal{I}(N|_{A}/(A \cap L)) \subset \mathcal{I}(M')$. \qed

Corollary 2.23. Assume the setting of Proposition 2.21, then one has:

$$\mathcal{I}(\circ_{i=1}^{m} M(F_i)) \subset \cap_{i=1}^{m} M(F_i).$$

Proof. One checks that $(\phi(M/F) \oplus M|_{F}) \land N = ((M/F) \land N) \oplus (M|_{F} \land N)$ for any matroid $N$. Then, recursively use Proposition 2.22. \qed

Lemma 2.24 (Squeeze lemma). If $\cap_{i=1}^{m} M(F_i)$ is nonempty, it is a matroid and:

$$\cap_{i=1}^{m} M(F_i) = \phi(\circ_{i=1}^{m} M(F_i)).$$

Proof. If $\cap_{i=1}^{m} M(F_i)$ is nonempty, it is a matroid by Lemma 2.12, Proposition 2.8, and Proposition 2.7; hence it is identical with $\phi(\circ_{i=1}^{m} M(F_i))$ by Proposition 2.21 and Corollary 2.23. \qed
Theorem 2.25. Denote by \( \mathcal{S}_m \) the group of permutations on \([m] \). The following are equivalent.

(a) \( \cap_{i=1}^m M(F_i) \neq \emptyset \)
(b) \( \phi(M(F_{\tau(1)}) \cdots (F_{\tau(m)})) \) are the same for all \( \tau \in \mathcal{S}_m \).
(c) \( \cap_{i=1}^m M(F_i) = \phi(M(F_{\tau(1)}) \cdots (F_{\tau(m)})) \) for some \( \tau \in \mathcal{S}_m \).
(d) \( \cap_{i=1}^m M(F_i) = \phi(\cap_{i=1}^m M(F_i)) \)
(e) \( \phi(\cap_{i=1}^m M(F_i)) = \phi(M(F_{\tau(1)}) \cdots (F_{\tau(m)})) \) for some \( \tau \in \mathcal{S}_m \).
(f) \( r(\cap_{i=1}^m M(F_i)) = r(M) \)

Proof. One shows (a) \( \Rightarrow \) (b) \( \Rightarrow \) (c) using Lemma 2.14. Also, (c) \( \Rightarrow \) (a) follows from \( r(M(F_{\tau(1)}) \cdots (F_{\tau(m)})) = r(M) \). Now, (a) \( \Rightarrow \) (d) is Lemma 2.24, and (a) \( \Rightarrow \) (e) follows from (a) \( \Rightarrow \) (c) and (a) \( \Rightarrow \) (d). Clearly, one has (d) \( \Rightarrow \) (f) and (e) \( \Rightarrow \) (f). And finally, Corollary 2.23 implies (f) \( \Rightarrow \) (a). The proof is done. \( \square \)

Corollary 2.26. If \( \cap_{i=1}^m M(F_i) \) is a nonempty loopless matroid, every member of the Boolean algebra generated by \( F_1, \ldots, F_m \) with unions and intersections is a flat.

Proof. It follows by Theorem 2.25 and formula (2.3). \( \square \)

Proposition 2.27. The flat collection of any loopless matroid can be set-theoretically recovered from the collection of its non-degenerate flats.

Proof. Let \( M \) be a loopless matroid of rank \( k \). We may assume \( M \) is inseparable. Let \( BP_{M_1}, \ldots, BP_{M_n} \) be all loopless codimension-\( k \) faces of \( BP_M \). By Theorem 2.25, each \( BP_{M_i} \) is the base polytope of \( \cap_{j=1}^i M(F_{i,j}) = M(F_{i1}) \cdots (F_{i\ell(i)}) \) for some distinct non-degenerate flats \( F_{i,j} \) of \( M \); let \( A_i \) be the Boolean algebra generated by those \( F_{i,j} \) with unions and intersections. Then, by Lemma 2.12 every nonempty proper flat of \( M \) is contained in some \( A_i \). Conversely, each \( A_i \) is contained in the lattice \( \mathcal{L} \) of \( M \) by Corollary 2.26, and hence \( \mathcal{L} = \cup_{i \in [n]} A_i \cup \{\emptyset, \mathcal{L} \} \). The proof is complete. \( \square \)

2.3. Loopless ridges. Let \( M \) be an inseparable matroid with \( r(M) \geq 3 \), and \( Q \) a codimension-2 loopless face of \( BP_M \). Then, \( Q = BP_{M(F)} \cap BP_{M(L)} \) for two distinct non-degenerate flats \( F \) and \( L \) of \( M \) by Lemma 2.19. Let \( A \) and \( T \) be the minimal non-degenerate flats of \( \phi(M(F)) \) and \( \phi(M(L)) \), respectively, such that:

\[
MA_Q = \phi(M(F)(A)) = \phi(M(L)(T)).
\]

Lemma 2.28. Assume the above setting. Then, up to symmetry, precisely one of the following three cases happens for the quadruple \((F, L, A, T)\).

| \( F \cap L = \emptyset \) | \( F \cup L = S \) | \( F \subset L \) | \( \) | \( \) | \( \) | \( \) | \( \) | \( \) |
|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| \( A \) | \( T \) | \( M/(F \cap L) \sim \) | \( M|_{F \cup L} \sim \) | \( M(F) \cap M(L) \sim \) |
| \( L \) | \( F \) | \( M \) | \( M|_F \oplus M|_L \) | \( M(F \cap L) \) |
| \( F \cap L \) | \( F \cap L \) | \( M/F \oplus M/L \) | \( M \) | \( M(F \cap L) \) |
| \( L\backslash F \) | \( F \) | \( M/F \) | \( M|_L \) | \( M/L \oplus M|_L/F \oplus M|_F \) |

Table 2.1. The classification of loopless ridge matroids.

Proof. By Theorem 2.25, one has \( MA_Q = M(F) \cap M(L) = \phi(M(F) \circ M(L)) \). Since \( \kappa(MA_Q) = 3 \), at least one of the four summands of the following vanishes:

\[
V := M(F) \circ M(L) = (M|_{F \cap L}) \oplus (M|_{F \cup L}/L) \oplus (M|_{F \cup L}/F) \oplus (M/(F \cup L))
\]
• If the 1st summand vanishes, that is, if $F \cap L = \emptyset$, the remaining three summands must represent inseparable matroids of positive ranks. In particular, $F \cup L \neq S$. Lemma 2.14 implies that $V \sim M(F \cup L)(F \cap L) = M(F \cup L)$. By Corollary 2.15, $M|_{F \cup L} \sim M|_F \oplus M|_L$ and $V \sim M|_F \oplus M|_L \oplus M/(F \cup L) \sim M(F)(L)$. Then, $L$ is the minimal $J$ with $V \sim M(F)(J)$ by the inseparability of $M|_L$; hence $A = L$. Similarly, one has $T = F$.

• If the 4th summand vanishes: $F \cup L = S$, then $F \cap L \neq \emptyset$ and $V \sim M(F \cap L)$. Also, $M/(F \cap L) \sim (M/F) \oplus (M/L)$ and $V \sim M(F)(F \cap L)$; hence $F \cap L$ is the minimal $J$ with $V \sim M(F)(J)$, and $A = F \cap L$. Similarly, $T = F \cap L$.

• If the 2nd summand vanishes: $F \subset L$, then $F \neq L$ and by similar argument as above, one has $V \sim M(L)(F) = M(F)(L/F)$, $A = L \setminus F$, and $T = F$.

• The case of the 3rd summand vanishing is symmetric to the above case.

Thus, there are up to symmetry three cases as in Table 2.1. □

**Corollary 2.29.** Let $M$ be an inseparable matroid of rank $\geq 3$ with a modular pair $\{F, L\}$ of non-degenerate flats. Then, $\text{BP}_{M(F)} \cap \text{BP}_{M(L)} \neq \emptyset$ is a codimension-2 loopless face of $\text{BP}_M$ if and only if:

(a) $F \cap L = \emptyset$ and $M/(F \cup L)$ is inseparable, or

(b) $F \cup L = S$ and $M|_{F \cap L}$ is inseparable, or

(c) $F \subset L$ and $M|_L/F$ is inseparable, or

(d) $F \supseteq L$ and $M|_F/L$ is inseparable.

**Proof.** By Theorem 2.25, one has $\text{BP}_{M(F)} \cap \text{BP}_{M(L)} \neq \emptyset$. By Lemma 2.28, it suffices to prove the if direction. Suppose (a) $F \cap L = \emptyset$ and $\phi(M/(F \cup L))$ is an inseparable matroid. Then, $r(F \cup L) < r(S)$ and $r(M/(F \cup L)) > 0$. By Corollary 2.15, one has $\phi(M|_{F \cup L}/L) = \phi(M|_F)$ and $\phi(M|_{F \cap L}/F) = \phi(M|_L)$ which are inseparable matroids. Thus, $\text{BP}_{M(F)} \cap \text{BP}_{M(L)}$ is a loopless codimension-2 face of $\text{BP}_M$ by Proposition 2.8. The other cases are similar. □

**2.4. Flags and face sequences.** For a fixed loopless matroid $M$, a **flag** of $M$ is defined as a sequence $(L_1, \ldots, L_c)$ of flats of $M$ such that $L_{i-1} \supseteq L_i$ (and hence $r(L_{i-1}) > r(L_i)$) for $i = 1, \ldots, c$ with $L_0 := E(M)$ and $L_{c+1} := \emptyset$. A flag is said to be **full** if $c = r(M) - r(L_1)$. A full flag is said to be **complete** if $c = r(M) - 1$.

On the other hand, a **face sequence** of $M$ is defined as a sequence $(F_1, \ldots, F_c)$ of subsets of $E(M)$ such that each $F_i$ is a flat of $M_{i-1} := M(F_1) \cdots (F_{i-1})$ and $\kappa(M_{i-1}) < \kappa(M_i)$ for $i = 1, \ldots, c$ with $M_0 := M$. A face sequence is said to be **full** if $c = \kappa(M_c) - \kappa(M)$, i.e. each $F_i$ is a non-degenerate flat of $M_{i-1}$. A full sequence is said to be **complete** if $c = r(M) - \kappa(M)$.

Proposition 2.30 below shows the relationship between flags and face sequences.

**Proposition 2.30.** For a full face sequence $(F_1, \ldots, F_c)$ of $M$, there exists a full flag $(L_1, \ldots, L_c)$ of $M$ with $M(F_1) \cdots (F_c) = M(L_1) \cdots (L_c)$, cf. Remark 2.13.

**Proof.** We use induction on $c$. The base case $c = 1$ is trivial. Assume $(L_1, \ldots, L_{c-1})$ is a full flag with $c \geq 2$ and $M(F_1) \cdots (F_{c-1}) = M(L_1) \cdots (L_{c-1})$. Let $L_0 := E(M)$, $L_c := \emptyset$, and $W := M(L_1) \cdots (L_{c-1})$, then $W = \oplus_{i=1}^c (M|_{L_{i-1}}/L_i)$. We may assume $F_c$ is a minimal non-degenerate flat of $\phi(W)$, then it is a flat of $\phi(M|_{L_{c-1}}/L_i)$ for

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13The etymology of “face” is flat + face, mimicking “flacet” of [FS05].
some $1 \leq i \leq c$. Let $T := F_i \sqcup L_i$, then $T$ is a flat of $M$ with $L_i \subseteq T \subseteq L_{i-1}$ such that $(M|_{L_{i-1}}/L_i)(F_i) = (M|_{L_{i-1}}/T) \oplus (M|T/L_i)$; hence one has:

$$M(L_1) \cdots (L_{c-1})(F_c) = M(L_1) \cdots (L_i)(T)(L_{i+1}) \cdots (L_{c-1}).$$

Renumbering the flats $L_1, \ldots, L_i, T, L_{i+1}, \ldots, L_{c-1}$ proves the statement. \hfill $\Box$

2.5. **Collections of expressions.** A poset $(K, \prec)$ is called a **join-semilattice** if any two elements have their **join** or **span,** i.e. their least common upper bound; a **meet-semilattice** if any two elements have their greatest common lower bound, say **meet** or **intersection:** a **lattice** if it is a semilattice with both join and meet.

In this paper all semilattices are finite and a lattice has both the greatest element $\hat{1}_K$, and the least element $\hat{0}_K$. The **rank function** $\rho$ of a lattice $(K, \prec)$ is:

$$K \mapsto \max \{ m : K_0, \ldots, K_m \in K, K = K_0 \nearrow \cdots \nearrow K_m = \hat{0} \}.$$

The number $\rho(K) := \rho(1)$ is called the **rank** of the lattice. If $K$ is the lattice of a matroid $M$, its rank function $\rho$ coincides with $r_M|K$.

**Collection $V(M)$ with 2 partial orders.** Fix a matroid $M$. For any two minor expressions $(M/A|C)$ and $(M/B|D)$, write:

$$(M/A|C) \oplus (M/B|D) \quad \text{if } A \supset B \text{ and } C \subset D.$$

Then, $\oplus$ defines a partial order on any collection of minor expressions of $M$. Further, let $V = V(M)$ be the collection of all the expressions $\oplus_{i=0}^m N_i$ such that $N_i$ are minor expressions of $M$ with **disjoint** ground sets. Define a partial order $\odot$ on $V$ such that for any two elements of $V$, say $V = \oplus_{i=0}^m N_i$ and $V' = \oplus_{j=0}^m N_j'$,

$$V \odot V' \quad \text{if } \forall i, \exists j = j(i) \text{ with } N_i \odot N_j'.$$

The poset $(V, \odot)$ is a meet-semilattice with meet $\odot$ where one has:

$$r(V \odot V') \leq \min\{r(V), r(V')\}.$$

Moreover, every fixed $V_0 \in V$ defines a map on $V$ assigning $V \odot V_0$ to $V \in V$ which preserves the partial order $\odot$.

We consider another partial order on $V$. For any $(M/A|C)$ and $(M/B|D)$, define:

$$(M/A|C) \odot (M/B|D) \quad \text{if } A \subset B \text{ and } C \supset D.$$

Then, extend $\odot$ onto $V$ in the same way as above.

Note that $\hat{0}_{(V, \odot)} = \hat{0}_{(V, \oplus)} = \emptyset$ and $\hat{1}_{(V, \odot)} = \hat{1}_{(V, \oplus)} = M$. Also, the following implications are worthy of attention.

$$V \odot V' \implies V \odot V' \implies \mathcal{I}(V) \subset \mathcal{I}(V').$$

**Posets $U(M)$ and $W(M)$.** Let $U = U(M)$ be the collection of loopless expressions $\oplus_{i=0}^m N_i \in V$ where $N_i$ are written as $N_i = M|F_i/L_i$ with $F_i, L_i \in E(M)$ such that $M|F_i$ are loopless and $L_i \in \mathcal{L}(M|F_i)$.

We may assume that $M$ is **loopless.** Consider any decreasing sequence of flats of $M$, say $(F_1, \ldots, F_{2m+1})$ for some $m \in \mathbb{Z}_{\geq 0}$, such that $F_{2i} \supseteq F_{2i+1}$ for all $0 \leq i \leq m$ where we set $F_0 := E(M)$ and $F_{2m+2} := \emptyset$. Then, $F_1 \neq E(M)$. Let $W = W(M)$ be the collection of the following expressions $W$ for all those sequences $(F_1, \ldots, F_{2m+1})$:

$$W = \oplus_{i=0}^m (M|F_{2i}/F_{2i+1}) \in U.$$

In particular, $M \in W$ but $\emptyset \notin W$. Note that $W = M$ if and only if $F_1 = \emptyset$. Denote

$$W^\perp = \oplus_{i=0}^m (M|_{F_{2i+1}/F_{2i+2}}).$$

Then, we have

$$W \ominus W^\perp = \emptyset.$$ 

By ignoring empty terms of $W \ominus W^\perp$ and arranging the remaining terms, we obtain a unique element of $W$:

$$\overline{W} = W \oplus W^\perp.$$ 

Henceforth, the above mentioned arrange process is a priori assumed for $W$.

**The poset $(W, \odot)$ is a join-semilattice.** Fix any two elements of $W$, say

$$W = \oplus_{i=0}^m (M|_{F_{2i}/F_{2i+1}}) \text{ and } W' = \oplus_{j=0}^{m'} (M|_{L_{2j}/L_{2j+1}}).$$

Let $A$ be the collection of all those terms $(M|_{F_{2i}/F_{2i+1}})$ and $(M|_{L_{2j}/L_{2j+1}})$ such that $(M|_{F_{2i}/F_{2i+1}}) \oplus W' \in W$ and $(M|_{L_{2j}/L_{2j+1}}) \oplus W \in W$, respectively. Note that $F_{2(m+1)} = \emptyset \subset L_{2m+1}$ and $L_{2(m'+1)} = \emptyset \subset F_{2m+1}$.

Let $i_0 = j_0 = 0$ and $T_0 = E(M)$.

- If $A = \emptyset$, let $i_1$ be the smallest $i > i_0$ such that $F_{2i} \subset L_{2j-1}$ and $L_{2j} \subset F_{2j-1}$ for some $j$, and let $J_i$ be the collection of all those $j$. Exchanging the roles of $i$ and $j$, construct $J_1$ and $I_1$ in the same way. Then, $I_1$ and $J_1$ are nonempty collections of consecutive numbers with $i_1 \in I_1$ and $j_1 \in J_1$, respectively. Assume $|I_1| \geq 2$, then $L_{2j-1} \supset F_{2i} \supset F_{2i+1} \supset L_{2j-1}$, and $(M|_{F_{2i}/F_{2i+1}}) \in A$, which is a contradiction; hence $|I_1| = |J_1| = 1$, that is, $I_1 = \{i_1\}$ and $J_1 = \{j_1\}$.

Likewise, recursively construct $i_\alpha$, $j_\alpha$, $I_\alpha = \{i_\alpha\}$, $J_\alpha = \{j_\alpha\}$ for each $\alpha \geq 1$, where $A = \emptyset$ assures $m + 1 = i_\ell$ and $m' + 1 = j_\ell$ for some $\ell \geq 1$. For $1 \leq \alpha \leq \ell$, let $T_{2\alpha-1}$ and $T_{2\alpha}$ be defined as follows, then $T_1 \subseteq E(M) = T_0$:

$$T_{2\alpha-1} = F_{2i_{\alpha-1}} \cap L_{2j_{\alpha-1}-1} \text{ and } T_{2\alpha} = F_{2i_{\alpha}} \cup L_{2j_{\alpha}}.$$ 

By dropping all $\alpha$ with $T_{2\alpha} = T_{2\alpha+1}$ and rearranging the indices if necessary, we may assume $T_{2\alpha-1} \supset T_{2\alpha} \supseteq T_{2\alpha+1}$ for all $\alpha$. Then, $\oplus_{\alpha=0}^{\alpha-1} (M|_{T_{2\alpha}/T_{2\alpha+1}})$ belongs to $W$, which is the least common upper bound of $W$ and $W'$, denoted by:

$$W \ominus W' = W' \ominus W = \ominus_{\alpha=0}^{\alpha-1} (M|_{T_{2\alpha}/T_{2\alpha+1}}).$$

- If $A \neq \emptyset$, let $W_0$ be the direct sum of $(M|_{F_{2i}/F_{2i+1}}) \notin A$, and $W'_0$ the direct sum of $(M|_{L_{2j}/L_{2j+1}}) \notin A$, then $W_0, W'_0 \in W$ since $M/F_1, M/L_1 \notin A$; hence $W_0 \odot W'_0$ exists. Denote by $W \odot W'$ the direct sum of $W_0 \odot W'_0$ and the elements of $A$, then $W \odot W' \in W$ and it is the least common upper bound of $W$ and $W'$.

**The poset $(W \cup \{\emptyset\}, \odot)$ is a meet-semilattice.** Since $(W \cup \{\emptyset\}, \odot)$ is a finite join-semilattice with $0_{W \cup \{\emptyset\}, \odot} = \emptyset$, it is a lattice with join $\odot$ and meet, say $\odot$, cf. [Bir67, Chapter 2]. The meet is explicitly found below. Fix any $W, W' \in W$ and write them as before. Note that $F_1 \subset E(M) = L_0$ and $L_1 \subset E(M) = F_0$.

Let $i_{-1} = j_{-1} = -1$ and $T_0 = E(M)$.

- For $\alpha \geq 0$, let $i_\alpha$ be the smallest $i > i_{\alpha-1}$ such that $F_{2i+1} \subset L_{2j}$ and $L_{2j+1} \subset F_{2i}$ for some $j > j_{\alpha-1}$, and $J_\alpha$ the collection of all those $j$. Exchanging $i$ and $j$, construct $j_\alpha$ and $I_\alpha$ in the same way. Then, $i_\alpha \in I_\alpha$ and $j_\alpha \in J_\alpha$, and there are precisely 3 cases for the pair $(|I_\alpha|, |J_\alpha|)$:

$$|I_\alpha| = |J_\alpha| = 2 \text{ or } |I_\alpha| = 1 \text{ or } |J_\alpha| = 1.$$
3. Matroid Semitilings and Weights

Unlike the convex polytope, the definition of a polytope depends on the context. In this paper, we mean by a polytope a finite union of convex polytopes.

3.1. Tilings and semitilings.

Definition 3.1. Let $\Sigma$ be a finite collection of convex polytopes contained in the $(k, S)$-hypersimplex $\Delta_S^k$. The support of $\Sigma$ is defined as $|\Sigma| := \cup_{P \in \Sigma} P$, and the dimension of $\Sigma$ is defined as $\dim \Sigma := \dim |\Sigma|$. We say that $\Sigma$ is equidimensional if all of its members have the same dimension. A face $Q$ of a member of $\Sigma$ is called a cell of $\Sigma$ with codimension $\text{codim}_\Sigma Q := |\Sigma| - \dim Q$. Also, the empty set $\emptyset$ is regarded as a cell of $\Sigma$ with $\dim \emptyset := -1$ and $\text{codim}_\Sigma \emptyset := \dim \Sigma + 1$.

When mentioning cells of $\Sigma$, we identify $\Sigma$ with the collection of all of its cells, which is a polytopal complex when $\Sigma$ is a tiling, see Definition 3.4.

Definition 3.2. Let $P$ and $P'$ be two distinct convex polytopes in $\Delta_S^k$. Then, we say that they are face-fitting if $P \cap P'$ is either empty or a common face of them. We also say that $P$ is face-fitting to $P'$ or vice versa.

Example 3.3. Two full-dimensional base polytopes $BP_M$ and $BP_N$ in $\Delta_S^k$ are face-fitting through a nonempty common facet $R$ if and only if $\text{MA}_R = M(F) = N(F^c)$ for a non-degenerate flat $F$ of $M$, cf. Lemma 2.12 and Corollary 2.20.

Definition 3.4. A $(k, S)$-tiling $\Sigma$ is a finite collection of convex polytopes of $\Delta_S^k$ that are pairwise face-fitting. We say $\Sigma$ is a subdivision of $|\Sigma|$.

Definition 3.5. A $(k, S)$-semitiling $\Sigma$ is a finite collection of convex polytopes of $\Delta_S^k$ such that $\Sigma_Q$ is a $(k, S)$-tiling for every codimension-2 cell $Q$ of $\Sigma$ where:

$$\Sigma_Q := \{ P \in \Sigma : Q \leq P, \text{ i.e. } Q \text{ is a face of } P \}.$$  

We omit $(k, S)$ if the context is clear. Note that a tiling is a semitiling by definition. We assume that $\Sigma$ is equidimensional and full-dimensional, i.e. $\dim \Sigma = \dim \Delta$, unless otherwise noted. The semitiling $\Sigma$ is said to be complete if $|\Sigma| = \Delta$. 

- If $|J_\alpha| = |J^\alpha| = 2$, then $F_{2\alpha+1} = F_{2(\alpha+1)} = L_{2\alpha+1} = L_{2(\alpha+1)}$, and let:

$$T_{2\alpha+1} = T_{2(\alpha+1)} = F_{2\alpha+1}.$$  

- If $|J_\alpha| = 1$, for each $j \in J_\alpha$ let:

$$T_{2\alpha+1,j} = T_{2(\alpha+1),j} = F_{2(\alpha+1)} \cup L_{2(\alpha+1)}.$$  

- Else if $|J_\alpha| = 1$, likewise construct $T_{2\alpha+1,i}$ and $T_{2(\alpha+1),i}$ for each $i \in J_\alpha$.

Note that in either case, the first constructed flat $T$ is $F_1 \cup L_1$.

\[ W \odot W' = W' \odot W = \begin{cases} \oplus_{\alpha=0}^{\ell} M |T_{2\alpha}/T_{2\alpha+1} if \ F_\alpha \cup L_\alpha \neq E(M), \\ \emptyset else. \end{cases} \tag{2.7} \]

- By rearranging these flats if necessary, we obtain a decreasing sequence of flats $T_0, T_1, \ldots, T_{2\ell}, T_{2\ell+1}$ for some $\ell \geq 0$ such that $T_{2\ell} \neq \emptyset$ and $T_{2\ell+2} = \emptyset$. We may assume $T_{2\alpha} \supseteq T_{2\alpha+1}$ for all $\alpha = 0, \ldots, \ell$. Define $W \odot W' \in W \cup \{\emptyset\}$ as follows, then it is the greatest common lower bound of $W$ and $W'$:

- Note that in either case, the first constructed flat $T$ is $F_1 \cup L_1$. 

- By rearranging these flats if necessary, we obtain a decreasing sequence of flats $T_0, T_1, \ldots, T_{2\ell}, T_{2\ell+1}$ for some $\ell \geq 0$ such that $T_{2\ell} \neq \emptyset$ and $T_{2\ell+2} = \emptyset$. We may assume $T_{2\alpha} \supseteq T_{2\alpha+1}$ for all $\alpha = 0, \ldots, \ell$. Define $W \odot W' \in W \cup \{\emptyset\}$ as follows, then it is the greatest common lower bound of $W$ and $W'$:

- If $|J_\alpha| = 1$, for each $j \in J_\alpha$ let:

$$T_{2\alpha+1,j} = T_{2(\alpha+1),j} = F_{2(\alpha+1)} \cup L_{2(\alpha+1)}.$$  

- Else if $|J_\alpha| = 1$, likewise construct $T_{2\alpha+1,i}$ and $T_{2(\alpha+1),i}$ for each $i \in J_\alpha$.

Note that in either case, the first constructed flat $T$ is $F_1 \cup L_1$. 

- By rearranging these flats if necessary, we obtain a decreasing sequence of flats $T_0, T_1, \ldots, T_{2\ell}, T_{2\ell+1}$ for some $\ell \geq 0$ such that $T_{2\ell} \neq \emptyset$ and $T_{2\ell+2} = \emptyset$. We may assume $T_{2\alpha} \supseteq T_{2\alpha+1}$ for all $\alpha = 0, \ldots, \ell$. Define $W \odot W' \in W \cup \{\emptyset\}$ as follows, then it is the greatest common lower bound of $W$ and $W'$: 

- $W \odot W' = W' \odot W = \begin{cases} \oplus_{\alpha=0}^{\ell} M |T_{2\alpha}/T_{2\alpha+1} if \ F_\alpha \cup L_\alpha \neq E(M), \\ \emptyset else. \end{cases} \tag{2.7} \]

3. Matroid Semitilings and Weights

Unlike the convex polytope, the definition of a polytope depends on the context. In this paper, we mean by a polytope a finite union of convex polytopes.
Definition 3.6. We say that a semitiling $\Sigma$ is connected in codimension $c$ if for any two distinct polytopes $P, P' \in \Sigma$ there is a sequence of distinct polytopes $P_1, \ldots, P_\ell \in \Sigma$ such that $\{P_i, P_{i-1}\}$ is a tiling with its support $P_i \cup P_{i-1}$ connected in codimension $c$ for each $i = 1, \ldots, \ell + 1$ where $P_0 = P$ and $P_{\ell+1} = P'$.

Definition 3.7. Let $\Sigma$ be a semitiling and $R$ a codimension-1 cell. Then, $R$ is said to be a facet of $\Sigma$ if there is a codimension-2 cell $Q$ of $\Sigma$ with $Q \leq R \subset \partial |\Sigma_Q|$, and denoted by $R \leq \Sigma$. Note that a facet of $\Sigma$ need not be a facet of $|\Sigma|$. However, if $\Sigma$ is a tiling, the facets of $\Sigma$ are the same as those of $|\Sigma|$. A cell of $\Sigma$ that is a face of a facet of $\Sigma$ is said to be a boundary cell of $\Sigma$.

Definition 3.8. Let $\Sigma$ be a semitiling. A subcollection $\Sigma'$ of $\Sigma$ is also a semitiling, and $\Sigma$ is called an extension of $\Sigma'$. The semitiling $\Sigma$ is said to be an extension of $\Sigma'$ at a cell $R$ of $\Sigma'$ if there is a (maximal) polytope $P \in \Sigma - \Sigma'$ such that $P \geq R$. In particular, a semitiling is called the trivial extension of itself.

3.2. Locally convex semitilings. From now on, we assume $\Sigma$ is a $(k,n)$-semitiling.

Definition 3.9. For a point $y \in |\Sigma|$, let $\Sigma_y$ be a maximal collection of pairwise face-fitting polytopes of $\Sigma$ containing $y$, which is a tiling. We say that $\Sigma$ is locally convex at $y$ if for any such $\Sigma_y$ there is a convex neighborhood of $y$ in $|\Sigma_y|$. If $\Sigma$ is locally convex at every point of $|\Sigma|$, it is said to be a locally convex semitiling.

For a cell $Q$ of $\Sigma$, we say $\Sigma$ is locally convex at $Q$ if $\Sigma$ is locally convex at every point of the relative interior $\text{relint}(Q)$ of $Q$.

Proposition 3.10. Let $\Sigma$ be a locally convex semitiling connected in codimension 1. Then, $\Sigma$ is a tiling and $|\Sigma|$ is a convex polytope.

Proof. Suppose that $P, P' \in \Sigma$ are two distinct polytopes that are not face-fitting. Then, $P \cap P'$ is nonempty, and take a point $y \in P \cap P'$. Since $\Sigma$ is connected in codimension 1, there is a sequence of polytopes of $\Sigma$, say $P = P_0, P_1, \ldots, P_\ell, P_{\ell+1} = P'$ with $\ell \geq 1$ such that $P_i \cap P_{i-1}$ for each $i = 1, \ldots, \ell + 1$ is the common facet of $P_i$ and $P_{i-1}$. Consider any path $\alpha(t), 0 \leq t \leq 1$, lying in $\cup_{i=0}^{\ell+1} P_i$ with a sequence of real numbers $0 = t_0 \leq t_1 \leq \cdots \leq t_{\ell+1} = t_{\ell+2} = 1$ such that $\alpha(0) = \alpha(1) = y$ and $\alpha(t) \in P_i$ for $t_{i-1} \leq t \leq t_i$. Among all those sequences of polytopes and all those paths, let $\alpha$ be with the shortest Euclidean length, and $\{P_i\}$ an associated sequence of polytopes of $\Sigma$ satisfying the above mentioned condition.

- Then, $\alpha$ is a piecewise linear closed curve with a finite number of vertices.
- By construction the number of vertices of $\alpha$ is at least 3, and let $y = y_0, y_1, y_2$ be three successive vertices of $\alpha$.
- Then, by the local convexity of $\Sigma$, there is a convex neighborhood $N(y_1)$ of $y_1$ in $\cup P_i$ that is small enough to choose $y_0' \in \overline{y_0y_1}$ and $y_2' \in \overline{y_1y_2}$ such that $\overline{y_0'y_2} \subset N(y_1)$. This contradicts the length minimality of $\alpha$.

Thus, the polytopes of $\Sigma$ are pairwise face-fitting, and $\Sigma$ is a tiling.

A similar argument shows that for any two distinct points $y'$ and $y''$ in $|\Sigma|$, the line segment $\overline{y'y''}$ is contained in $|\Sigma|$, and $|\Sigma|$ is a convex polytope. \(\square\)

By Proposition 3.10, we sometimes mean by a convex tiling a locally convex semitiling connected in codimension 1 if there is no confusion while some authors mean by convex subdivisions coherent subdivisions.
Corollary 3.11. Every complete semitiling connected in codimension 1 is a tiling.

Proposition 3.12. Let $\Sigma$ be a semitiling connected in codimension 1. Then, $\Sigma$ is locally convex if and only if $\Sigma$ is locally convex at every codimension-2 cell.

Proof. Let $Q$ be a codimension-2 boundary cell of $\Sigma$, then there are two unique facets $R_1, R_2$ of $\Sigma$ with $Q = R_1 \cap R_2$, and let $P_1, P_2 \in \Sigma$ be the full-dimensional polytopes with $P_1 \supseteq R_1$ and $P_2 \supseteq R_2$, respectively. Then, let $H_1, H_2$ be the half spaces with $H_1 \supseteq P_1$ and $H_2 \supseteq P_2$ whose boundaries are the affine hulls $\text{Aff}(R_1)$ and $\text{Aff}(R_2)$, respectively. Then, $\Sigma$ is locally convex at $Q$ if and only if $|\Sigma_Q| \subset H_1 \cap H_2$, which proves the proposition.

Lemma 3.13. Let $\Sigma$ be a semitiling connected in codimension 1. Then, $\Sigma$ is locally convex if and only if $\Sigma$ is locally convex at every relevant codimension-2 cell.

Proof. Every irrelevant codimension-2 cell $Q$ of $\Sigma$ is contained in a hyperplane with defining equation $x(i) = 0$ or $x(i) = 1$ for some $i \in [n]$, and $|\Sigma_Q|$ is contained in the half space defined by $x(i) \geq 0$ or $x(i) \leq 1$. Then, the local convexity of $\Sigma$ at $Q$ is equivalent to $\Sigma_Q$ being connected in codimension 1. So, $\Sigma$ is locally convex at every irrelevant codimension-2 cell. Applying Proposition 3.12 finishes the proof.

3.3. Matroid semitilings. We define matroid semitilings and henceforth assume that a semitiling is a matroid semitiling.

Definition 3.14. A semitiling $\Sigma$ whose members are base polytopes is called a matroid semitiling. This is well-defined due to Proposition 2.7. In particular, a matroid tiling whose support is a base polytope is called a matroid subdivision.

Lemma 3.15. Let $\Sigma = \{BP_{M_1}, \ldots, BP_{M_\ell}\}$ be a matroid semitiling connected in codimension 1 that is locally convex. Then, $\Sigma$ is a matroid tiling and $|\Sigma|$ is a base polytope with matroid structure $MA_{|\Sigma|}$ whose base collection is $\cup_{i=1}^\ell M_i$.

Proof. By Proposition 3.10, the semitiling $\Sigma$ is a tiling and $|\Sigma|$ is convex. Then, $|\Sigma|$ is a moment polytope of the base union $\cup_{i=1}^\ell M_i$ with the edge length property. Therefore, $\cup_{i=1}^\ell M_i$ is a matroid and $|\Sigma|$ is its base polytope.

Example 3.16. Let $\Sigma$ be a $(2,n)$-semitiling connected in codimension 1, and $Q$ a codimension-2 cell of $\Sigma$ with $MA_Q = M_X \oplus M_Y \oplus M_Z$ where $M_X, M_Y, M_Z$ are connected components of $MA_Q$. Then, since $MA_Q$ has rank 2, at least one of the 3 summands has rank 0; hence $Q$ is contained in a coordinate hyperplane. Therefore, $\Sigma$ is a convex tiling, and $|\Sigma|$ is a base polytope by Lemmas 3.13 and 3.15.14

Definition 3.17. Let $BP_M$ be a full-dimensional $(k,n)$-polytope with $k \geq 3$, and $Q$ a codimension-2 loopless face. Write $MA_Q = M_X \oplus M_Y \oplus M_Z$ where $M_X, M_Y, M_Z$ are inseparable matroids of positive ranks with ground sets $X, Y, Z$, respectively. If $R = BP_{M(J)}$ is a loopless facet of $BP_M$ with $Q < R$, then by Lemma 2.28, the number of $X, Y, Z$ contained in $J$ is 1 or 2, which is said to be the type of $R$ at $Q$ in $BP_M$, while $r_M(J)$ is said to be the rank of $R$ in $BP_M$.

The type of a facet is a relative notion depending on codimension-2 cells while the rank is not. When $k = 3$, however, the type of a facet equals its rank regardless of its codimension-2 cells $Q$, and we may omit the phrase “at $Q$”. Further, if two

---

14From an algebro-geometric perspective, this is one of the reasons that $\mathcal{M}_{0,n}$ and its weighted version $\mathcal{M}_{0}(2,n)$ are nice spaces, cf. [Kap93, Has03]. See also Example 4.31 and Theorem 5.1.
full-dimensional polytopes are face-fitting through their common facet $R$, the types of $R$ at $Q < R$ in them are complementary, that is, those types sum up to $k$.

**Definition 3.18.** Let $\Sigma$ be a $(k, n)$-semitiling with $k \geq 3$, $R$ a facet of $\Sigma$, and $Q$ a codimension-2 cell of $\Sigma$ with $Q < R$. Then, $R$ is a facet of a unique base polytope of $\Sigma$, say $BP_M$, and the type of $R$ at $Q$ in $\Sigma$ is defined as that of $R$ at $Q$ in $BP_M$. Also, the rank of $R$ in $\Sigma$ is defined as that of $R$ in $BP_M$.

**Definition 3.19 (Drawing Rule I).** Assume the setting of Definition 3.17. Then, the angle of $BP_M$ at $Q$, denoted by $\text{ang}_Q BP_M$, is defined as follows.

$$\text{ang}_Q BP_M = \begin{cases} 1 & \text{if } F \cap L = \emptyset \text{ or } F \cup L = E(M), \\ 2 & \text{otherwise.} \end{cases}$$

Denote by $B_\epsilon(x)$ the open ball centered at $x$ with radius $0 < \epsilon \ll 1$, and consider a neighborhood $C_\epsilon = C_{\epsilon, Q} := \cup_{x \in \text{Aff}(Q)} B_\epsilon(x)$ of the affine hull $\text{Aff}(Q)$. Then,

$$(\mathbb{R}^n / \mathbb{R} \text{ modulo } \text{Aff}(Q)) \cong \mathbb{R}^2$$

where $\mathbb{I}$ is perpendicular to all $(k, n)$-polytopes. We draw $C_\epsilon \cap BP_M$ modulo $\text{Aff}(Q)$ in $\mathbb{R}^2$ such that the facet $R = BP_M(J) > Q$ with $J = F, L$ is represented by a solid line segment if its type at $Q$ is 1 and by a dashed one otherwise, with the angle between these two line segments being $\frac{\pi}{3} \cdot \text{ang}_Q BP_M$.

**3.4. Polytopes with a common facet.** Let $\Sigma$ be a semitiling with a codimension-2 cell $Q$. For a neighborhood $C_\epsilon$ of $\text{Aff}(Q)$, define the local figure of $\Sigma$ at $Q$ as

$$\Sigma_Q \cap C_\epsilon := \{ P \cap C_\epsilon : P \in \Sigma_Q \}$$

which is a generalization of vertex figure. Then, $\Sigma$ is locally convex at $Q$ if and only if the support of the local figure of $\Sigma$ at $Q$ is convex.

**Lemma 3.20.** Let $\Sigma = \{BP_M, BP_N\}$ be a $(k, n)$-tiling connected in codimension 1 with $k \geq 3$. Let $R = BP_M \cap BP_N$ and $Q < R$ a loopless codimension-2 cell of $\Sigma$. Then, the local figure of $\Sigma$ at $Q$ is up to symmetry one of the 4 figures of Figure 3.1, where the middle black dots are $Q$ modulo $\text{Aff}(Q)$.

![Figure 3.1](image)

**Proof.** Write $\text{MA}_R = \phi(M(F)) = \phi(N(J))$ where $F$ and $J$ are non-degenerate flats of $M$ and $N$, respectively, with $J = F^c$, cf. Example 3.3. We may assume that $R$ at $Q$ is of type 1 in $BP_M$ and of type 2 in $BP_N$. Let $L$ and $D$ be the non-degenerate flats of $M$ and $N$, respectively, with $Q = R \cap BP_M(L) = R \cap BP_N(D)$. Further, as in Lemma 2.28, let $A$ be the minimal non-degenerate flat of $\phi(M(F))$ such that $Q = BP_M(F(A))$. Then, one has either $L = A$ or $L = F \cup A$. Also, either $D = A$ or $D = F \cup A$. Hence, there are only $2 \times 2 = 4$ cases, and Figure 3.1 depicts the local figure of $\Sigma$ at $Q$ for each case. \qed
3.5. Polytopes with a common ridge. Let $\Sigma$ be a $(k, n)$-tiling with $k \geq 3$ whose polytopes have a common loopless ridge $Q$ with $\text{MA}_Q = M_X \oplus M_Y \oplus M_Z$ where $M_X, M_Y, M_Z$ are inseparable matroids of positive ranks with ground sets $X, Y, Z$, respectively. Then, each base polytope $P \in \Sigma$ has exactly 2 facets containing $Q$. Up to positive scalars, there are at most 6 characterizer vectors of all such facets:

$$1^X, 1^Y, 1^Z, 1^{X \cup Y}, 1^{X \cup Z}, 1^{Y \cup Z}.$$ 

Note that $1^{Y \cup Z} \equiv -1^X$ modulo $\mathbb{R}$. There are up to symmetry two candidates for the counter-clockwise orientation: \{1$^X, 1^Y, 1^Z$\} and \{1$^X, 1^{X \cup Z}, 1^Z$\}, but the latter is incorrect, and we opt for the former; see Figure 3.2.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{characterizer_vectors.pdf}
\caption{Characterizer vectors in $\mathbb{R}^n / \mathbb{R}$ modulo $\text{Aff}(Q)$.}
\end{figure}

**Theorem 3.21** (Uniform finiteness of semitilings). Let $\Sigma$ be a $(k, n)$-semitiling with $k \geq 3$ (whether or not connected in codimension 1), and $Q$ a loopless codimension-2 cell of it. The local figure of $\Sigma$ at $Q$ is up to symmetry a subcollection of one of the figures either of Figure 3.3 if $Q$ is irrelevant, or of Figure 3.4 if $Q$ is relevant.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{local_figures.pdf}
\caption{The local figure of $\Sigma$ at irrelevant $Q$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{local_figures2.pdf}
\caption{The local figure of $\Sigma$ at relevant $Q$.}
\end{figure}

**Proof.** Since $\Sigma_Q$ is a tiling whose polytopes have a common ridge $Q$, use Figure 3.2 and Lemma 3.20. \qed

**Definition 3.22** (Drawing Rule II). Let $\Sigma$ be a $(k, n)$-semitiling with $k \geq 3$ and $Q$ a loopless codimension-2 cell. The **angle** and the **deficiency** of $\Sigma$ at $Q$, denoted by $\text{ang}_Q \Sigma$ and $\text{def}_Q \Sigma$, respectively, are defined to be integers:

$$\text{ang}_Q \Sigma := \sum_{P \in \Sigma_Q} \text{ang}_P$$

and

$$\text{def}_Q \Sigma := 6 - \text{ang}_Q \Sigma.$$ 

**Corollary 3.23.** Upon the assumption that $\Sigma$ is connected in codimension 1, the semitiling $\Sigma$ is locally convex at $Q$ if and only if $\text{def}_Q \Sigma \neq 1, 2$. 


3.6. Weighted tilings.

**Definition 3.24.** A vector \( \beta \in \mathbb{Q}^n \) of rational numbers is called a \((k, n)\)-weight or simply a weight if \( k < \beta([n]) \) and \( 0 < \beta(i) \leq 1 \) for all \( i \in [n] \). The weight domain \( D(k, n) \) is the set of all \((k, n)\)-weights. For a \((k, n)\)-weight \( \beta \), a \( \beta \)-weighted \((k, n)\)-hypersimplex or a \( \beta \)-cut hypersimplex is defined as:

\[
\Delta_\beta = \Delta_\beta(k, n) := \{ x \in \mathbb{Q}^n_k : x(i) \leq \beta(i) \text{ for all } i \in [n] \}.
\]

A \( \beta \)-weighted tiling or a \( \beta \)-tiling for short is a \((k, n)\)-tiling \( \Sigma \) such that \( \Delta_\beta \subset |\Sigma| \) and \( \text{int}(\Delta_\beta) \cap P \neq \emptyset \) for all \( P \in \Sigma \) where \( \text{int}(\Delta_\beta) = \Delta_\beta - \partial \Delta_\beta \).

Note that a \( \beta \)-tiling is connected in codimension 1. Let \( \Sigma \) be a \( \beta \)-tiling and \( Q \) a relevant codimension-2 cell. Write \( MA_Q = MA = M_B \oplus M_C \) where \( M_A, M_B, M_C \) are inseparable matroids of positive ranks whose ground sets are \( A, B, C \), respectively, with \( |A|, |B|, |C| \geq 2 \). Let \( x(A) = a, x(B) = b, x(C) = c \) be the \( \kappa(\text{MA}_Q) = 3 \) defining equations of \( Q \) where \( a, b, c \geq 1 \) and \( k = a + b + c \), cf. Corollary 2.20.

**Proposition 3.25.** Assume the above setting. If \( \text{def}_Q \Sigma = 0 \), then \( \beta(A) > a, \beta(B) > b \), and \( \beta(C) > c \). When \( k = 3 \), one has \( a = b = c = 1 \) and the converse is true.

**Proof.** Observe that if \( \beta(C) \leq c \), every polytope of \( \Sigma \) has empty intersection with open half space \( \{ x \in \Delta : x(C) > c \} \), and so \( \text{def}_Q \Sigma > 0 \) by Theorem 3.21. Hence, by symmetry, if \( \text{def}_Q \Sigma = 0 \), one has \( \beta(A) > a, \beta(B) > b \), and \( \beta(C) > c \).

Fix \( k = 3 \), then \( a = b = c = 1 \). Suppose \( \beta(A) > 1, \beta(B) > 1 \), and \( \beta(C) > 1 \), then there is a point \( v \in \mathbb{R}^n \) such that \( 0 < v(i) < \beta(i) \) for all \( i \in [n] \) and \( \sum_{i \in X} v(i) = 1 \) for all \( X = A, B, C \) since \( |X| \geq 2 \); hence \( v \in \text{int}(\Delta_\beta) \cap Q \). Consider 2 points:

\[
p_+ = \{ x \in \Delta : x(C) > c \} \quad \text{and} \quad p_- = \{ x \in \Delta : x(C) < c \}
\]

whose entries are \( p_+(i) = v(i) + \epsilon, \ i \in A \cup B \), and \( p_-(j) = v(j) \pm \frac{\epsilon(n-|C|)}{|C|}, \ j \in C \), for some \( \epsilon > 0 \). If \( \epsilon \) is sufficiently small, \( p_\pm \in \text{int}(\Delta_\beta) \). Likewise, we obtain 4 more points of \( \text{int}(\Delta_\beta) \) contained in \( \{ x \in \Delta : x(A) > a \} \), \( \{ x \in \Delta : x(A) < a \} \), \( \{ x \in \Delta : x(B) > b \} \), and \( \{ x \in \Delta : x(B) < b \} \), respectively. Then, the convexity of \( \Delta_\beta \) tells that \( \text{def}_Q \Sigma = 0 \). \( \square \)

**Proposition 3.26.** Let \( \Sigma \) be a weighted \((3, n)\)-tiling. Suppose that \( \Sigma \) is not locally convex at some relevant codimension-2 cell, say \( Q \). Then, the local figure of \( \Sigma \) at \( Q \) is up to symmetry one of the three of Figure 3.5, and \( \Sigma \) has at least one type-2 facet \( R \) that contains \( Q \). Moreover, for every loopless codimension-2 cell \( Q' \neq Q \) of \( \Sigma \) contained in \( R \), one has \( \text{def}_Q \Sigma' \geq 4 \).

![Figure 3.5. Non-convex local figures of weighted tilings.](image)

**Proof.** Assume the setting of Proposition 3.25, then \( \text{def}_Q \Sigma = 1, 2 \) by Corollary 3.23 and at least one of the 3 inequalities of Proposition 3.25 is violated. One checks that precisely one of them is violated using the observation of Proposition 3.25. Hence, we obtain up to symmetry the three of Figure 3.5 for the local figure of \( \Sigma \) at \( Q \). In either case, there exists a type-2 facet \( R \) of \( \Sigma \) with \( Q < R \). If \( Q' < R \) with \( Q' \neq Q \) is a loopless codimension-2 cell of \( \Sigma \), one has \( \text{def}_Q \Sigma' \geq 4 \) again by the observation, which is depicted as well in Figure 3.5. \( \square \)
4. Hyperplane Arrangements and Puzzle-pieces

Given a matroid, we construct two different geometric objects, say a matroidal hyperplane arrangement and a puzzle-piece, where the latter is obtained from the former via a sequence of meet-semilattice operations, say matroidal blowups and collapsings. This sequence is a lattice operation, called the matroidal MMP.

4.1. Puzzle-pieces and puzzles. For loopless matroids, \( r \) and \( \kappa \) are additive on direct sums, and \( r - \kappa \) is a dimension-like function.

**Definition 4.1.** The **dimension** of a nonempty matroid \( M \) is defined to be:

\[
\dim M := (r - \kappa)(M \setminus \emptyset).
\]

Every matroid \( N \) with \( BP_N \leq BP_M \) is a face matroid of \( M \), denoted by \( N \leq M \). For any nonempty such \( N \) with \( \bar{\emptyset}_N = \bar{\emptyset}_M \), its **codimension** in \( M \) is defined as:

\[
\text{codim}_M N := \dim M - \dim N = \dim BP_M - \dim BP_N \leq r(M).
\]

For empty matroid \( \emptyset \), we set \( \dim \emptyset = -1 \) and \( \text{codim}_\emptyset \emptyset = \dim \emptyset + 1 \) by convention.

For \( d \geq 1 \), denote by \( \lfloor M \rfloor_0 \) the direct sum of the connected components of \( M \) of dimension \( \geq d \). We write \( \lfloor M \rfloor_0 = M \setminus \emptyset \) by convention where, then, \( \lambda(M) = \lambda(\lfloor M \rfloor_0) \). When the subscript \( d \) is 1, we omit it and write \( \lfloor M \rfloor \) for \( \lfloor M \rfloor_1 \).

In light of Theorem 2.25, Corollaries 2.26 and 2.17, we define geometric objects from matroids corresponding to base polytopes.

**Definition 4.2.** For a matroid \( M \), let \( A \) be the collection of all the facet matroids \( \phi(M(F_i)) \) with non-degenerate flats \( F_i \) of \( M \). The **puzzle-piece** or simply a **piece** associated to \( M \) is defined to be a pair of \( M \) and \( A \):

\[
PZ_M = PZ_{BP_M} := (M, A)
\]

whose **dimension** and **rank** are defined as those of \( M \). In particular, \( PZ_M \) is said to be a **point-piece** if \( \dim M = 0 \), a **line-piece** if \( \dim M = 1 \), etc.

**Definition 4.3.** For two matroids \( M \) and \( N \) such that \( N \leq M \) and \( \bar{\emptyset}_N = \bar{\emptyset}_M \), we say \( PZ_N \) is a **sub-puzzle-piece** or simply a **subpiece** of \( PZ_M \), and write \( PZ_N \leq PZ_M \).

For a loopless matroid \( M \), let \( \mathcal{P}(PZ_M) \) denote the collection of its loopless face matroids, and let \( \emptyset \in \mathcal{P}(PZ_M) \) by convention. Then, \( \mathcal{P}(PZ_M) \) is a lattice isomorphic to that of loopless faces of \( BP_M \). The rank function \( \rho \) of \( \mathcal{P}(PZ_M) \) is given by:

\[
\rho(X) = \dim X + 1.
\]

For a general matroid \( M \), we define \( \mathcal{P}(PZ_M) \) by the following bijection:

\[
\mathcal{P}(PZ_{\lfloor M \rfloor_0}) \to \mathcal{P}(PZ_M), \quad X \mapsto X \oplus \bar{\emptyset}_M.
\]

In spite of the dimension difference, gluing of base polytopes is the same as that of puzzle-pieces in matroidal language. This leads us to the following definition.

**Definition 4.4.** A \((k, S)\)-semipuzzle \( \Psi \) is a collection of those puzzle-pieces whose base polytopes form a \((k, S)\)-semitiling \( \Sigma \). If \( \Sigma \) is a tiling, \( \Psi \) is said to be a **puzzle**.

We use the same terms/conventions of semitilings for semipuzzles. As is for semitilings, we often omit \((k, S)\) and assume that a semipuzzle is equidimensional and full-dimensional unless otherwise noted, and hence the associated matroids of the puzzle-pieces are assumed loopless and inseparable.
Definition 4.5. Let \( \Psi = \{ PZ_M^i : i \in \Lambda \} \) be a semipuzzle and \( \Sigma \) its corresponding semitiling. The boundary \( \partial \Psi \) of \( \Psi \) is defined as the collection of \( X \in \cup_{i \in \Lambda} \mathcal{P}(\text{PZ}_M^i) \) corresponding to the loopless facets of \( \Sigma \). If the members of \( \partial \Psi \) are all irrelevant, \( \Psi \) is said to be complete. The support \( |\Psi| \) of \( \Psi \) is defined later, see Definition 4.29.

The completeness of \( \Psi \) is well-defined since \( \Psi \) is complete if and only if \( \Sigma \) has no relevant facet, or equivalently, \( \Sigma \) is complete by Lemmas 3.13 and 3.15.

Notation 4.6. A \( k \)-partition of \( S \) is a partition of \( S \) into \( k \) nonempty subsets. For a \( k \)-partition \( \mathcal{A} := \cup_{i \in [k]} A_i \) of \( S \), denote \( U_{\mathcal{A}} := \oplus_{i \in [k]} U_{A_i} \). Then, any \((k, S)\)-point-piece is a matroid \( U_{\mathcal{A}} \) for some \( k \)-partition \( \mathcal{A} \), and hence is identified with \( \mathcal{A} \).

Definition 4.7. Let \( \Psi \) be a \((k, n)\)-semipuzzle. For any two \( k \)-partitions of \([n]\), say \( \mathcal{A} = \cup_{i \in [k]} A_i \) and \( \mathcal{B} = \cup_{i \in [k]} B_i \), the distance between point cells \( U_{\mathcal{A}} \) and \( U_{\mathcal{B}} \) is defined as follows, cf. formula (2.1):

\[
d(U_{\mathcal{A}}, U_{\mathcal{B}}) = \min_{\sigma \in \mathcal{E}_n} \left\{ \sum_{i=1}^k d(A_i, B_{\sigma(i)}) \right\}.
\]

(4.1)

Thus, the collection of point-pieces of \( \Psi \) is a metric space. Consider the convex hull of \( k \) points \((n-k+1, 1, \ldots, 1)\) for an \((n-k+1)\)-simplex with edge length \( n-k \). Then, the intersection of this simplex with \( \mathbb{Z}^k \) works as a local coordinate chart for \( \Psi \) with barycentric coordinates.

4.2. Hyperplane arrangements.

Definition 4.8. Fix a loopless matroid \( M \). A subspace of \( M \) is a matroid \( \phi(M/F) \) for a flat \( F \in \mathcal{L}(M) \). When considering \( \phi(M/F) \) as a subspace of \( M \) we often write \( \eta(M/F) \) instead. Let \( S = S(M) \) denote the collection of all subspaces \( \eta(M/F) \), then \( S \) is a lattice with rank function \( \rho \) such that \( \rho(\eta(M/F)) = r(M/F) \). The subspace dimension/codimension in \( M \) of \( \eta(M/F) \) are defined as:

\[
\text{sdim} \, \eta(M/F) := r(M/F) - 1 \quad \text{and} \quad \text{scodim}_M \, \eta(M/F) := r(M|_F).
\]

Then, \( \eta(M/F) \) said to be a point if \( \text{sdim} \, \eta(M/F) = 0 \), a line if \( \text{sdim} \, \eta(M/F) = 1 \), and a hyperplane with multiplicity \( |F| \) if \( \text{scodim}_M \, \eta(M/F) = 1 \).

Definition 4.9. Let \( M \) be a loopless matroid. Let \( \mathcal{T} = \mathcal{T}(M) \) be the collection of the expressions \( M/F \) for all \( F \in \mathcal{L} \). Then, \( \phi|_\mathcal{T} : \mathcal{T} \to S \) is a lattice isomorphism, and \( (T, \odot) \) is isomorphic to the dual lattice \( \mathcal{L}^\vee \) of \( (\mathcal{L}, \subseteq) \). Every \( \eta(M/F) \in S \) is the intersection of hyperplanes \( \eta(M/i) \), \( i \in F \), and as an intersection it is said to be trivial if \( r(M|_F) = 1 \), normal if \( \lambda(M|_F) = r(M|_F) \), and simple if \( M|_F \) is.

Remark 4.10. If \( \eta(M/F) \) is a nontrivial normal intersection, \( \phi(M|_F) \) is separable. When \( r(M) = 3 \), the converse is true, and further \( \eta(M/F) \) is a nontrivial and non-normal intersection if and only if \( F \) is a rank-2 non-degenerate flat, cf. Example 4.13.

Definition 4.11. Let \( M \) be loopless. For each \( i \in E(M) \) the pair \( B_i = (\eta(M/i), i) \) is called a labeled hyperplane with label \( i \). A hyperplane arrangement \( \text{HA}_M \) of \( M \) on \( E(M) \) is defined as the following pair:

\[
\text{HA}_M := (M, \{B_i\}_{i \in E(M)}).
\]

The hyperplane \( \eta(M/i) \) is said to be a hyperplane locus of \( \text{HA}_M \) where, then, \( \text{HA}_M \) has \( \lambda(M) \) hyperplane loci. We say that \( \text{HA}_M \) is a \((k, S)\)-arrangement if \( M \) is a \((k, S)\)-matroid, that is, a rank-\( k \) matroid on \( S \). For a general matroid \( M \), we understand the hyperplane arrangement \( \text{HA}_M \) as \( \text{HA}_{M|_a} \).
**Definition 4.12.** Let $M$ be a rank-$k$ matroid. For any $J \subseteq E([M]_0)$, we say that hyperplanes $\eta(M/J)$, $j \in J$, are in general position if $\phi(M|J) = U^\min_{j,J}(k,J)$.

**Example 4.13.** (a) The following are the classifications of point and line arrangements $HA_M$ with respect to the number of connected components of $[M]_0$.

\[
\begin{array}{ccc}
\text{\lambda = 2 point loci} & \text{\lambda = 3 point loci} & \text{\lambda (\geq 3) point loci} \\
\kappa([M]_0) = 2 & \kappa([M]_0) = 3 & \kappa([M]_0) = 1 \\
\end{array}
\]

**Figure 4.1.** The classification of point arrangements.

\[
\begin{array}{ccc}
\text{\lambda = 3 line loci with no common intersection} & \text{\lambda - 1 (\geq 3) line loci meeting at a point} & \text{\lambda 4 (\leq \lambda) line loci in general position} \\
\kappa([M]_0) = 3 & \kappa([M]_0) = 2 & \kappa([M]_0) = 1 \\
\end{array}
\]

**Figure 4.2.** The classification of line arrangements.

(b) Let $HA_M$ be a $(3,n)$-arrangement. Let us check in terms of flats that any two lines, say $\eta(M/T)$ and $\eta(M/\bar{T})$, passing through two distinct points $\eta(M/F)$ and $\eta(M/L)$ are the same: $F$ and $L$ are rank-2 flats, and $T \cup \bar{T}$ is contained in $F \cap L$ while $1 \leq r(T \cup \bar{T}) \leq r(F \cap L) = 1$; hence $T = \bar{T} = F \cap L$, and $\eta(M/T) = \eta(M/\bar{T})$.

**Lemma 4.14.** Let $M$ be a rank-$k$ loopless matroid. If $HA_M$ has $k + 1$ hyperplanes in general position, then $M$ is inseparable.

**Proof.** Let $\eta(M/J)$, $j \in J$, be $k + 1$ hyperplanes in general position. Suppose $M$ is separable, then there is a proper separator $T$ so $\phi(M|J) = \phi(M|J \setminus T) \oplus \phi(M|J \setminus T^c)$ and hence $J \cap T = \emptyset$ or $J \cap T^c = \emptyset$ by the inseparability of $\phi(M|J) \cong U^k_{k+1}$. Without loss of generality, assume $J \cap T^c = \emptyset$, i.e. $T \supset J$, then one has:

\[
k = r(T) + r(T^c) \geq r(J) + r(T^c) = k + r(T^c).
\]

Therefore, $r(T^c) = 0$ and $T^c = \emptyset$ since $M$ is loopless, which contradicts that $T$ is a proper separator of $M$. Thus, $M$ is inseparable. \enda

**Remark 4.15.** When $k = 2, 3$, the converse of Lemma 4.14 is true; Figures 4.1 and 4.2 show an easy check. However, when $k \geq 4$, the inseparability does not promise the existence of $k + 1$ hyperplanes in general position: Consider the graphic matroid $M(G)$ of graph $G$ given in Figure 4.3, which is inseparable since $G$ is 2-connected. But, its circuits all have size 4 while $U^5_5$ has a unique circuit $[5]$ of size 5. Therefore, there is no submatroid of $M(G)$ that is isomorphic to $U^5_5$. Another way to see this is to compute with the given matrix which is a regular realization of $M(G)$, that is, the same matrix over every field $k$ with $1 = 1_k$ represents the matroid $M(G)$.
Figure 4.3. A rank-4 inseparable graphic matroid without any 5 lines in general position and its regular realization.

**Definition 4.16.** Formula (2.7) describes intersections of hyperplanes of $HA_M$:

$$r(A) = \text{scodim}_M \otimes_{i \in A} (M/T)$$

which is equivalent to (1.1) if and only if $M$ is realizable. Therefore, we say that $HA_M$, $BP_M$, and $PZ_M$ are realizable if $M$ is. See Subsection 5.4 for the definitions of realizable SHAs/tilings/puzzles.

**Example 4.17.** (a) Every rank-2 matroid $M$ is realizable: Let $A_1 \sqcup \cdots \sqcup A_{|M|}$ be the partition of $E([M]_0)$ into the rank-1 flats of $[M]_0$ and consider $\mathbb{P}^1$ over a sufficiently large field, e.g. an infinite field. Pick $\lambda(M)$ distinct points $P_j$ on $\mathbb{P}^1$, one for each $A_j$, and let $j(i)$ for each $i \in E([M]_0)$ be the unique $j$ with $i \in A_j$. The point arrangement $(\mathbb{P}^1, \{(P_j(i), i)\}_{i \in E([M]_0)})$ has matroid structure $[M]_0$, and is a realization of $HA_{[M]_0}$.

(b) The uniform matroid $U_{k,n}^k$ for any $(k, n)$ is realizable since over a sufficiently large field one can pick $n$ hyperplanes on $\mathbb{P}^{k-1}$ in general position.

**Definition 4.18.** The characteristic polynomial of a loopless matroid $M$ in an indeterminate $x$ is written as follows where $\mu$ is the Möbius function:

$$p_M(x) = \sum_{A \in \mathcal{L}} \mu([M]_A) x^{r([M]_A)}.$$  

Since the structures of $HA_{[M]_A}$ and $HA_{[M]_A}$ both only depend on the structure of hyperplane arrangement $HA_M$, this polynomial $p_M$ is an invariant of $HA_M$ and thus we call it the characteristic or Poincaré polynomial of $HA_M$.

When $M$ is a simple matroid with a realization of $HA_M$ over some field $k$, the polynomial $p_M$ coincides with the usual characteristic polynomial of the realization, see [Sta11, Proposition 3.11.3] and [Whi87, Proposition 7.2.1].

**Example 4.19.** Let $HA_M$ be a line arrangement with $M = [M]_0$. We compute the coefficients of $p_M(x)$ using Boolean expansion formula, cf. [Whi87, Chapter 7].

- Since $M$ is loopless, $\emptyset = \emptyset_M$ is a unique rank-0 flat of $M$ and $\mu(\emptyset) = 1$.
- For every rank-1 flat $F$ of $M$, we have $\mu([M]_F) = -1$.
- If $L$ is a rank-2 flat of $M$, it is the disjoint union of rank-1 flats of $\phi([M]_L)$, say $L = \bigcup_{i=1}^{\lambda([M]_L)} F_i$. Then, $HA_{\phi([M]_L)}$ is a point arrangement with $\lambda([M]_L)$ points, and $\mu([M]_L)$ is a scaled product of multiplicities $|F_i|$ of all those points:

$$\mu([M]_L) = (\lambda([M]_L) - 1) \cdot \prod_{i=1}^{\lambda([M]_L)} |F_i|.$$  

- $E(M)$ is a unique rank-3 flat of $M$, and $\mu(M) = -\sum_{A \in \mathcal{L}} \mu([M]_A).$
4.3. Degenerations of hyperplanes. Let $M$ be a loopless matroid of rank $k \geq 2$. Let $F_1, \ldots, F_m$ be distinct nonempty proper subsets of $E(M)$ and $j_1, \ldots, j_m$ integers in $[k-1]$. Denote by $\delta^{[j_1, \ldots, j_m]}_{F_1, \ldots, F_m}(M)$ the set of vertices of the following polytope:

$$\delta^{[j_1, \ldots, j_m]}_{F_1, \ldots, F_m}(BP_M) := BP_M \cap \{x \in \mathbb{R}^{m} | x(F_i) \leq j_i\}.$$  

If this is a base polytope, abusing notation we denote by $\delta^{[j_1, \ldots, j_m]}_{F_1, \ldots, F_m}(M)$ its matroid and call $\delta^{[j_1, \ldots, j_m]}_{F_1, \ldots, F_m}$ a degeneration of hyperplanes of $HA_M$ where we also denote by $\delta^{[j_1, \ldots, j_m]}_{F_1, \ldots, F_m}(HA_M)$ its hyperplane arrangement.

Degenerations commute by definition. Note that, however, some degenerations do not happen at the same time: consider the hypersimplex $\Delta^2_2$ and let $F_1 = \{1, 2\}$, $F_2 = \{1, 3\}$ and $j_1 = j_2 = 1$, then $\Delta^2_2 \cap (\cap_{i \in [2]} \{x(F_i) \leq 1\})$ is not a base polytope.

Example 4.20. (a) Let $M$ be the uniform matroid $U^k_n$ with $k < n$. For any subset $F \subseteq [n]$ with $1 \leq |F| \leq n - k$, define $f$ such that $f(i) = F$ if $i \in F$ and $f(i) = i$ otherwise. Then, $\delta^1_F = f$ for $M$ and so $\delta^k_{f}$ and $\delta^{1 - k}_{1 - f}$ are degenerations.

(b) Further, let $k = 3$. Suppose $F$ and $L$ are subsets of $[n]$ such that $F \cup L = [n]$, $F \cap L \neq \emptyset$, $|F - L| \geq 2$ and $|L - F| \geq 2$. Then, $\delta^{1,2,2}_{F \cap L, F, L}$ and $\delta^{2,2}_{F, L}$ are the same degeneration and $\delta^{1,2,2}_{F \cap L, F, L}(M) = \delta^{2,2}_{F, L}(M)$ is an inseparable matroid on $[n]$, cf. Lemmas 4.14 and 4.21; see Figure 4.4.

4.4. Line arrangements.

Lemma 4.21. Let $M$ be a rank-3 inseparable matroid on $S$. Then, $M$ has two proper (rank-2 non-degenerate) flats $F$ and $L$ with $F \cap L \neq \emptyset$ and $F \cup L = S$ if and only if $M$ has a rank-1 degenerate flat $T$. These flats are unique and $T = F \cap L$.

Proof. Let $F$ and $L$ be two proper flats of $M$ with $F \cap L \neq \emptyset$ and $F \cup L = S$. Since $M$ is inseparable, $r(F) = r(L) = 2$ and $r(F \cap L) = 1$, and moreover $F$ and $L$ are inseparable. Then, $F$ and $L$ are non-degenerate since $r(M) = 3$, and hence $F \cap L$ is degenerate by Lemma 2.28.

Conversely, let $T$ be a rank-1 degenerate flat, then $M/T = (M|_F/T) \oplus (M|_L/T)$ for some rank-2 flats $F$ and $L$ with $T = F \cap L$ and $S = F \cup L$. If $F$ or $L$ were separable, $M$ would be separable; therefore $F$ and $L$ are inseparable and hence non-degenerate since $r(M) = 3$. Further, if $A = (A \cap F) \cup (A \cap L)$ is an inseparable rank-2 flat that is different from $F$ and $L$, then it is separable. Therefore such $F$ and $L$ are unique, and the uniqueness of rank-1 degenerate flat also follows. $\square$

Remark 4.22. When $k \geq 4$, the uniqueness of a rank-1 degenerate flat fails. Indeed, consider the rank-4 graphic matroid $M(G)$ of Figure 4.5, which is inseparable since the graph $G$ is 2-connected. Now, contracting edge $e_6$ and contracting $e_7$ both produce loopless graphs $G'$ that are only 1-connected, and $M(G')$ are separable.
Remark 4.23. Lemma 4.21 provides a neat explanation why the log canonical model of an unweighted (realizable) line arrangement is either $\mathbb{P}^1 \times \mathbb{P}^1$ or $\text{Bl}_{\text{pts}} \mathbb{P}^2$, cf. [Ale15, Theorem 5.7.2]; see also Subsection 4.7.

4.5. Construction of line arrangements. In this subsection, we construct line arrangements using degenerations.

I. Suppose $R$ is a relevant $(3,n)$-polytope of codimension 1 (and of dimension $n-2$), or equivalently $[\text{MA}_R]$ is an inseparable rank-2 matroid with $m = \lambda([\text{MA}_R]) \geq 3$ on some subset $L \subset [n]$ such that $\text{int}(\Delta^3_{n}) \cap \{x(L) = 2\} \neq \emptyset$. Then, $\text{HA}_{[\text{MA}_R]}$ is a point arrangement with at least 3 point loci. Write $L$ as the disjoint union of the rank-1 flats $F_1, \ldots, F_m$ of $[\text{MA}_R]$, and consider the degeneration $M = \delta_{F_1, \ldots, F_m}^1(U_3^m)$ whose simplification is isomorphic to $U_{n+m-|L|}^3$. We construct two line arrangements $\text{HA}_N$ below whose visualizations are offered in Figure 4.6.

(L1) $N = \delta_{L^c}(M)$ where $L^c$ is a rank-1 non-degenerate flat with $N/L^c = [\text{MA}_R]$.

(L2) $N = \delta_{L}^2(M)$ where $L$ is a rank-2 non-degenerate flat with $N|_L = [\text{MA}_R]$.

II. Let $R_1$ and $R_2$ be face-fitting $(3,n)$-polytopes of dimension $n-2$ such that their affine hulls are not parallel and $Q = R_1 \cap R_2$ is a loopless polytope of dimension $n-3$. Then, $M_1 = [\text{MA}_{R_1}]$ and $M_2 = [\text{MA}_{R_2}]$ are rank-2 inseparable matroids.

Write $\text{MA}_Q = U_X^1 \oplus U_Y^1 \oplus U_Z^1$ with $[n] = X \cup Y \cup Z$, and let $E(M_1) = Y \sqcup Z$ and $E(M_2) = X \sqcup Z$ without loss of generality. Then, since $\text{MA}_Q = M_1(A) \oplus U_X^1$ for a non-degenerate flat $A$ of $M_1$, one has either $A = Y$ or $A = Z$. But, not both of $Y$ and $Z$ are flats of $M_1$ since otherwise $M_1$ should be separable, a contradiction. Then, up to symmetry, there are only three cases as follows.

| (L3) | $Z \notin \mathcal{L}(M_1)$ and $Z \notin \mathcal{L}(M_2)$ | $Y \in \mathcal{L}(M_1)$ and $X \in \mathcal{L}(M_2)$ |
| (L4) | $Z \in \mathcal{L}(M_1)$ and $Z \in \mathcal{L}(M_2)$ | $Y \notin \mathcal{L}(M_1)$ and $X \notin \mathcal{L}(M_2)$ |
| (L5) | $Z \in \mathcal{L}(M_1)$ and $Z \notin \mathcal{L}(M_2)$ | $Y \notin \mathcal{L}(M_1)$ and $X \in \mathcal{L}(M_2)$ |
For each of the above cases, we construct a line arrangement $HA_N$; see Figure 4.7 for the visualizations. Let $F_1, \ldots, F_A$ and $L_1, \ldots, L_A$ be the rank-1 flats of $M_1$ and $M_2$, respectively, with $\lambda_1 = \lambda(M_1) \geq 3$ and $\lambda_2 = \lambda(M_2) \geq 3$.

**(L3)** $N = \delta_{Y \cup L_1 \cup \ldots \cup L_A}^2(\delta_{X \cup F_1 \cup \ldots \cup F_A}^2(\delta_{X \cup Y}^1(U_n^3)))$ where $Y = F_i$ and $X = L_j$ for some $i$ and $j$ are rank-1 non-degenerate flats of $N$ with $N/X = M_1$ and $N/Y = M_2$, respectively. Then, $N(S - E(M_1) \cap E(M_2)) = N(X \cup Y) = MA_Q$.

**(L4)** $N = \delta_{E(M_1), E(M_2)}^2(\delta_{F_1 \cup \ldots \cup F_A}^1(\delta_{L_1 \cup \ldots \cup L_A}^1(U_n^3)))$ where $Z = F_i = L_j$ for some $i$ and $j$ is a unique degenerate rank-1 flat of $N$; $E(M_1)$ and $E(M_2)$ are rank-2 non-degenerate flats of $N$ with $N|_{E(M_1)} = M_1$ and $N|_{E(M_2)} = M_2$, respectively, cf. Example 4.20(b). Then, $N(E(M_1) \cap E(M_2)) = N(Z) = MA_Q$.

**(L5)** Note that $Z \in \mathcal{L}(M_1)$ is the disjoint union of at least two $L_j$’s, and let $Z = F_1$ and $L_1 \subseteq Z$ without loss of generality. Construct $N$ as follows:

$N = \delta_{E(M_2)}^2(\delta_{L_1 \cup F_2 \cup \ldots \cup F_A}^1(\delta_{F_1 \cup \ldots \cup F_A}^1(\delta_{L_1 \cup \ldots \cup L_A}^1(U_n^3))))$.

Then, $X$ and $E(M_2)$ are rank-1 and rank-2 non-degenerate flats, respectively, with $N/X = M_1$ and $N|_{E(M_2)} = M_2$, cf. Example 4.20(b).

![Figure 4.7. Construction of Line Arrangements II](image)

Note that if the underlying field $\mathbb{k}$ is large enough, all five line arrangements above can be constructed in $\mathbb{P}^2$ over $\mathbb{k}$, using the following simple facts:

- Given a point, there exist enough lines that pass through the point.
- For two distinct points, there exists a unique line passing through them.

Thus, the line arrangements of (L1)–(L5) are all **realizable**.

4.6. Matroidal semilattices and operations. Fix a nonempty loopless matroid $M$ and let $\mathcal{A}$ be a subcollection of $\mathcal{V} = \mathcal{V}(M)$, cf. Subsection 2.5.

- Denote $\phi(\mathcal{A}) := \{\phi(A) : A \in \mathcal{A}\}$ where $\emptyset$ is interpreted as the empty expression in $\mathcal{A}$ and as the empty matroid in $\phi(\mathcal{A})$, cf. Subsection 1.7.
- If $(\mathcal{A}, \preceq)$ is a poset, for any $V \in \mathcal{A}$, we denote:
  
  $\mathcal{A}|_V := \{A \in \mathcal{A} : A \preceq V\}$.

- A poset $(\mathcal{A}, \preceq)$ is said to be **matroidal** (on $M$) if $\hat{\mathcal{A}}(\mathcal{A}, \preceq) = M$, $\emptyset|_{\mathcal{A}, \preceq} = \emptyset$, and the partial order $\preceq$ is a restriction of $\circ$, that is, $V \preceq V'$ implies $V \circ V'$.
- Let $\xi(\mathcal{V})$ denote $V$ if $V \in \mathcal{A}$, and $\emptyset$ otherwise.

\footnote{For instance, if $\mathbb{k}$ is algebraically closed, its size is infinite and this suffices for the construction.}
For any $V = \bigoplus_{i=0}^n (M|_{D_i}/D_i) \in \mathcal{V}$, denote
$$\nabla := \xi_V (\bigoplus_{i=0}^n (M|_{\mathcal{D}_i}/\mathcal{D}_i)).$$

**Blowing up/down.** Let $M$ be a loopless matroid, and $(\mathcal{K}, \preceq, \alpha)$ a matroidal meet-semilattice with $\mathcal{K} \subset \mathcal{U} = \mathcal{U}(M)$, which is actually a lattice, cf. [Bir67, Chapter 2]. Fix $V \in \mathcal{K}$ with $\nabla \in \mathcal{W}(M) - \{\emptyset\}$.

- For any $K \in \mathcal{K}$, if $K \wedge V \neq \emptyset$ and $A \odot \nabla^\perp \in \mathcal{U}$ for all $A \in \mathcal{K}$ with $K \subset A$, let:
  $$K_V = (K \wedge V) \ominus (K \odot \nabla^\perp),$$
  and otherwise let $K_V = \emptyset$.

- The collection $\mathcal{E}_V$ with the partial order $\preceq_V$ defined below is a poset:
  $$\mathcal{E}_V := \{K_V \in \mathcal{U} : K \in \mathcal{K} - \mathcal{K}|_V \cup \{\emptyset\}\}$$
  - The empty expression $\emptyset$ is the $\preceq_V$-smallest member of $\mathcal{E}_V$.
  - For any two nonempty expressions $K_V, K'_V \in \mathcal{E}_V$,
    $$K_V \preceq_V K'_V$$
    if $(K \wedge V) \preceq (K' \wedge V)$ and $(K \odot \nabla^\perp) \odot (K' \odot \nabla^\perp)$.
  - In particular, $V \oplus \nabla^\perp$ is the $\preceq_V$-largest member of $\mathcal{E}_V$ since $M \in \mathcal{K}$.

- Moreover, $\mathcal{E}_V$ is a meet-semilattice with meet $\wedge_V$ defined by:
  $$K_V \wedge V K'_V := (K \wedge K')_V.$$

Now, the following collection $\text{Bl}_V \mathcal{K}$ is said to be the **blowup of $\mathcal{K}$ along $V$**:
$$\text{Bl}_V \mathcal{K} := (\mathcal{K} - \mathcal{K}|_V) \cup \mathcal{E}_V.$$ 

Obtaining $\text{Bl}_V \mathcal{K}$ is said to be **blowing up $\mathcal{K}$ along $V$** while **blowing down** is defined as the inverse operation of blowing up. The blowup $\text{Bl}_V \mathcal{K}$ is a poset with partial order $\preceq$ described as follows.

$$\begin{align*}
\preceq &\equiv \preceq_V \quad \text{on } (\mathcal{K} - \mathcal{K}|_V) \times (\mathcal{K} - \mathcal{K}|_V), \\
\preceq &\equiv \preceq_V \quad \text{on } \mathcal{E}_V \times \mathcal{E}_V, \\
K_V \preceq K' &\text{ if } K_V \preceq_K K'_V \text{ for } (K_V, K'_V) \in \mathcal{E}_V \times (\mathcal{K} - \mathcal{K}|_V). 
\end{align*}$$

Further, $\text{Bl}_V \mathcal{K}$ is a (matroidal) meet-semilattice with meet $\sqcap$ described as follows.

$$\begin{align*}
K \sqcap K' &= \xi_{\mathcal{K}-\mathcal{K}|_V} (K \wedge K') \quad \text{for } (K, K') \in (\mathcal{K} - \mathcal{K}|_V) \times (\mathcal{K} - \mathcal{K}|_V), \\
K_V \sqcap K'_V &= K_V \wedge_V K'_V \quad \text{for } (K_V, K'_V) \in \mathcal{E}_V \times \mathcal{E}_V, \\
K_V \sqcap K' &= K_V \sqcap K'_V \quad \text{for } (K_V, K') \in \mathcal{E}_V \times (\mathcal{K} - \mathcal{K}|_V). 
\end{align*}$$

**Remark 4.24.** In the algebro-geometric blowup, $\{K \oplus \nabla^\perp : K \in \mathcal{K}|_V\}$ corresponds to the collection of the dominant transforms of the subvarieties contained in the blowup center corresponding to $V$, cf. [Li09].

**Collapsing.** Let $(\mathcal{K}, \preceq, \alpha)$ be as above. For any fixed $V \in \mathcal{K}$, obtaining a collection
$$\mathcal{K}' = (\mathcal{K} - \mathcal{K}|_V) \cup \phi(\mathcal{K}|_V)$$
or the surjective map $\pi = \pi_{\mathcal{K}, V} : \mathcal{K} \to \mathcal{K}'$ is said to be **collapsing $\mathcal{K}$ over $V$** or **collapsing $V$ in $\mathcal{K}$**. Define a partial order $\preceq$ on $\mathcal{K}'$ as follows.

$$\begin{align*}
\preceq &\equiv \preceq_V \quad \text{on } (\mathcal{K} - \mathcal{K}|_V) \times (\mathcal{K} - \mathcal{K}|_V), \\
\preceq &\equiv \preceq_V \quad \text{on } \phi(\mathcal{K}|_V) \times \phi(\mathcal{K}|_V), \\
N \preceq K &\text{ if } N \preceq \phi(K \wedge V) \text{ for } (N, K) \in \phi(\mathcal{K}|_V) \times (\mathcal{K} - \mathcal{K}|_V). 
\end{align*}$$
If $K' = \pi(K)$ is a meet-semilattice, we say $V$ is collapsible (to $\phi(V)$) or $K$ is collapsible (over $V$) where the meet operation $\wedge$ is defined by:

$$\pi(K) \wedge \pi(K') = \phi(\xi_{K\cup K'}(K \wedge K')) \oplus \xi_{K-K'}(K \wedge K').$$

**Example 4.25.** (a) Let $M = U_{[3]}^3 \cong U_1^1 \oplus U_1^1 \oplus U_1^1$, then $M \in T$ is collapsible to a point-piece, see Figure 4.8(a), where $a_1 \cdots a_m$ denotes the set $\{a_1, \ldots, a_m\}$.

(b) Let $M = U_{[3]}^2 \oplus U_{[4]}^1$, and $K = \text{Bl}_{M/123}T$, then collapsing $K$ over $M$ produces $\mathcal{P}(PZ_M)$ where $PZ_M$ is a line-piece, see Figure 4.8(b).

![Figure 4.8. Matroidal semilattices and collapsing.](image)

4.7. Matroidal MMP. Let $M$ be a loopless matroid of rank $k \geq 2$. An expression $V = \bigoplus_{i=0}^m N_i \in \mathcal{V}(M)$ is called unstable if some $N_i$ is separable, that is:

$$
edim V := \sum_{i=0}^m (r(N_i) - 1) > \dim V$$

where edim is a generalization of sdim under the identification of $S(M)$ with $\mathcal{T}(M)$. This is a generalization of the GIT (un)stability of [KT04, Definition 1.25].

The following algorithm is said to be the matroidal MMP for the hyperplane arrangement $\text{HA}_M$. Let $c = 0$ and $K_0$ be the meet-semilattice $\mathcal{T}(M)$.

**MMP1** Starting with $K_c$, perform successive blowups along all those members $W$ contained in $\mathcal{V}(M) \cap \mathcal{T}(M)$ such that $\dim W = c$ and $\dim W > 0$, cf. (2.6); let $K_{c+1}$ be the resulting meet-semilattice; increase $c$ by 1.

**MMP2** Repeat the process (MMP1) until $c = k - 1$, and end up with $K_{k-1}$.

**MMP3** For each $c = 1, \ldots, k - 2$, collapse all the unstable elements $V \in K_{k-1}$ with edim $V = c$. Let $K$ be the final meet-semilattice.

Then, $\phi(K) = \mathcal{P}(PZ_M)$. To see this, consider the collection $K_{k-1}$ at (MMP2). Let $F$ and $L$ be flats of $M$ with rank $\geq 2$ such that $F \nsubseteq L$ and $L \nsubseteq F$, then precisely one of the following 4 cases happens:

$M(F), M(L) \in K_{k-1}, \ M/F, M/L \in K_{k-1}, \ M/F, M(L) \in K_{k-1}, \ M(F), M/L \in K_{k-1}.$

Note that the two members in each case do not intersect in $K_{k-1}$ by construction. Then, one checks that $K_c \in \mathcal{W} \cup \{\emptyset\}$ for any fixed $c = 1, \ldots, k - 1$, which is uniquely obtained, and also that $(K_c, \oplus, \ominus, \oslash)$ is a matroidal lattice, cf. Remark 4.26.

Now, let $K_k := \{\widetilde{W} \in \mathcal{W} : \emptyset \neq W \in K_{k-1}\} \cup \{\emptyset\}$, then it is a lattice isomorphic to $K_{k-1}$ via the bijective map $K_{k-1} \rightarrow K_k$ sending $W \mapsto \widetilde{W}$ for $W \neq \emptyset$ and $\emptyset \mapsto \emptyset$. By Proposition 2.30, any loopless face matroid of $M$ is expressed as $\phi(W)$ for some $W \in K_k$. Conversely, Theorem 2.25 tells that $\phi(W)$ for all $\emptyset \neq W \in K_k$ are loopless face matroids of $M$. Hence, collapsing $K_k$ over $M$ produces $\mathcal{P}(PZ_M)$. Here, every
nonempty intersection of \( \phi(V), \phi(V') \in \mathcal{P}(\mathbb{P}Z_M) \) is \( \phi(V \sqcup V') \). Moreover, observe that \( \text{(MMP3)} \) can be replaced by a single operation of collapsing \( K_{k-1} \) over \( M \). Therefore, we conclude that \( \phi(K) = \mathcal{P}(\mathbb{P}Z_M) \).

**Remark 4.26.** For a nonempty loopless face matroid \( \text{MA}_Q \) of \( M \), consider a maximal collection \( \mathcal{A} \) of nonempty proper flats \( F \) of \( M \) such that \( \text{MA}_Q = \phi(\circ_{F \in \mathcal{A}} M(F)) \). By Theorem 2.25, this collection is unique, and \( V_Q := \circ_{F \in \mathcal{A}} M(F) \) is the \( \circ \)-smallest among the expressions \( V \in \mathcal{V} \) with \( \text{MA}_Q = \phi(V) \). Thus, the expression \( V_Q \) can be used as a representative of \( \text{MA}_Q \).

### 4.8. Matroidal arrangements

Let \( (\mathcal{K}, \ll) \) be a matroidal lattice on \( M \) with rank function \( \rho \), cf. Subsection 2.5. Let \( \mathcal{A} \) be the collection of all those members of \( \mathcal{K} \) with rank \( \rho(K) - 1 \) equipped with an integer-valued function \( \mathcal{A} \rightarrow \mathbb{Z} \) which we call the multiplicity function. Then, the pair \( (M, \phi(\mathcal{A})) \) is said to be a matroidal arrangement if \( \mathcal{K} \) is a coatomic lattice, that is, if for every element \( K \) of \( \mathcal{K} \) such that \( K \ll A \), there is an element \( A \) of \( \mathcal{A} \) such that \( K \ll A \). For instance, a hyperplane arrangement and a puzzle-piece are matroidal arrangements.

Let \( (M_1, \phi(A_1)) \) and \( (M_2, \phi(A_2)) \) be two matroidal arrangements. Let \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \) be the intersection lattices of \( A_1 \) and \( A_2 \) with rank functions \( \rho_1 \) and \( \rho_2 \), respectively, and consider the following collection:

\[
K_1 \times \text{int} \mathcal{K}_2 := ((K_1 - \{\emptyset\}) \times (K_2 - \{\emptyset\})) \cup \{\emptyset\}.
\]

By identifying \( \emptyset \) with \( (\emptyset, \emptyset) \), this collection has a lattice structure induced from that of the Cartesian product \( \mathcal{K}_1 \times \mathcal{K}_2 \) such that \( \hat{1} = (M_1, M_2), \hat{0} = \emptyset \), and the rank of any member \( (K_1, K_2) \in \mathcal{K}_1 \times \text{int} \mathcal{K}_2 \) is:

\[
\rho_1(K_1) + \rho_2(K_2).
\]

Now, the product of the two matroidal arrangements \( (M_1, \phi(A_1)) \) and \( (M_2, \phi(A_2)) \) is defined as:

\[
(M_1, \phi(A_1)) \times (M_2, \phi(A_2)) := (M_1 \oplus M_2, (\phi(A_1) \times \{M_2\}) \cup (\{M_1\} \times \phi(A_2)))
\]

where the multiplicity of any member \( (\phi(A_1), M_2) \) or \( (M_1, \phi(A_2)) \) of the product is defined as that of \( A_1 \) or \( A_2 \), respectively. The intersection lattice of the product is defined as \( \mathcal{K}_1 \times \text{int} \mathcal{K}_2 \) where if the two arrangements \( (M_1, \phi(A_1)) \) and \( (M_2, \phi(A_2)) \) are equipped with dimension functions, we define the dimension of \( (K_1, K_2) \in \mathcal{K}_1 \times \text{int} \mathcal{K}_2 \) to be the sum of the dimensions of \( K_1 \) and \( K_2 \) as is for the rank function. Since its intersection lattice is coatomic, the product is a matroidal arrangement.

**Definition 4.27.** Two matroidal arrangements \( (M_1, \phi(A_1)) \) and \( (M_2, \phi(A_2)) \) are said to be isomorphic if there is a lattice isomorphism between their intersection lattices that preserves their multiplicity functions.

**Remark 4.28.** There are two isomorphic puzzle-pieces such that their associated base polytopes are not affinely isomorphic. Let \( M \) be a matroid on \([5]\) of Lemma 4.21 with \( F = \{1, 2, 3\} \) and \( L = \{3, 4, 5\} \), and let \( N = U^a_2 \oplus U^a_2 \). Then, \( \mathbb{P}Z_M \) and \( \mathbb{P}Z_N \) are all isomorphic to \( \text{HA}_{U^a_2} \times \text{HA}_{U^a_2} \), but \( \text{BP}_M \) and \( \text{BP}_N \) are not affinely isomorphic since the number of bases of \( M \) is 8 while that of \( N \) is 9.

**Definition 4.29.** Let \( \Psi = \{(M_i, \phi(A_i))\}_{i \in \Lambda} \) be a collection of matroidal arrangements. Gluing the arrangements of \( \Psi \) is the operation of obtaining:

\[
\cup \Psi := (\cup_{i \in \Lambda} M_i, \cup_{i \in \Lambda} \phi(A_i)).
\]
Let $\mathcal{A}^{\text{com}}$ denote the collection of those $A \in \bigcup_{i \in \Lambda} \mathcal{A}_i$ such that $\phi(A) \in \phi(\mathcal{A}_j) \cap \phi(\mathcal{A}_l)$ for some $\mathcal{A}_j \neq \mathcal{A}_l$. Merging the arrangements of $\Psi$ is the operation of obtaining the following collection $|\Psi|$ which is called the support of $\Psi$:

$$|\Psi| := (\bigcup_{i \in \Lambda} M_i, \bigcup_{i \in \Lambda} \phi(A_i) - \phi(\mathcal{A}^{\text{com}})).$$

**Definition 4.30.** Let $\Psi$ be a semipuzzle connected in codimension 1, and $\Phi$ the associated collection of hyperplane arrangements:

$$\Psi = \{(M_i, \phi(A_i))\}_{i \in \Lambda} \quad \text{and} \quad \Phi = \{HA_{M_i} = (M_i, L_i)\}_{i \in \Lambda}.$$  

For each $i \in \Lambda$, let all $M(F) \in \mathcal{A}_i \cap \mathcal{A}^{\text{com}}$ replace $M/F \in L_i$ and obtain a new collection, say $L'_i$. Then, let $\Phi' = \{(M_i, \phi(L'_i))\}_{i \in \Lambda}$ and denote:

$$GA_\Phi := \bigcup_{i \in \Lambda} \Phi'.$$

If $\Psi$ is a puzzle, we say that the hyperplane arrangements of $\Phi$ are compatible and $GA_\Phi$ is a matroidal stable hyperplane arrangement, or matroidal SHA for short. Further, if $\Psi$ is a locally convex puzzle, the support of $\Phi'$ is a hyperplane arrangement:

$$|GA_\Phi| := HA_{\bigcup_{i \in \Lambda} M_i}.$$  

**Example 4.31.** Let $M$ and $N$ be two inseparable $(2, 4)$-matroids whose collections of rank-1 flats are $\{1, 2, 34\}$ and $\{12, 3, 4\}$, respectively. Let $\Psi = \{PZ_M, PZ_N\}$ and $\Phi = \{HA_M, HA_N\}$. Then, $\Psi$ is a complete puzzle and $|GA_\Phi| = HA_{U/2}$, cf. Example 3.16. Figure 4.9 shows how to glue or merge the arrangements of $\Phi$. Note that $GA_\Phi$ corresponds to a unique (up to symmetry) nontrivial matroid subdivision of $\Delta_2^4$.

![Figure 4.9. Gluing and merging of two compatible (2, 4)-arrangements.](image)

5. **Extensions of Matroid Tilings and Reduction Morphisms**

In this section, we study extensions of $(k, n)$-semitilings with a focus on $k = 2, 3$. For rank-2 semitilings connected in codimension 1, completion is always possible. When $k = 3$, however, this is not the case, and we consider a special class of rank-3 semitilings generalizing weighted tilings, say regular semitilings, Definition 5.2. The generalization is justified by Propositions 3.26 and 5.3.

Basically, a completion of a semitiling is achieved by repeated saturations at codimension-2 cells where a saturation of a semitiling $\Sigma$ at a codimension-2 cell $Q$ means a process of extending $\Sigma$ at $Q$ to a semitiling $\hat{\Sigma}$ such that $\text{def}_Q \hat{\Sigma} = 3$ if $Q$ is contained in an irrelevant facet, that is, if $Q$ is irrelevant, and $\text{def}_Q \hat{\Sigma} = 0$ otherwise, that is, if $Q$ is relevant. All (loopless) codimension-2 cells are saturated if and only if there is no relevant facet, that is, the semitiling is complete.

Throughout this section, semitilings are assumed connected in codimension 1 unless otherwise noted.
5.1. Extensions of $(2, n)$-semitilings.

**Theorem 5.1.** Let $\Sigma$ be a $(2, n)$-semitiling (connected in codimension $1$). Then, $\Sigma$ extends to a complete tiling connected in codimension $1$.

**Proof.** Note that $\Sigma$ is a convex tiling, cf. Example 3.16. If $\Sigma$ has no relevant facet, it is already a complete tiling. So, suppose that $\Sigma$ has a relevant facet, say $R$, then $R = BP_{M(A)}$ for some $BP_M \in \Sigma$ where $M$ is an inseparable rank-$2$ matroid and $A$ is a rank-$1$ flat of $M$ with $|A| \geq 2$. Let $N$ be a rank-$2$ matroid whose rank-$1$ flats are $A^c$ and singletons $\{i\}$ for all $i \in A$, cf. Example 4.17(a). Then, $N$ is inseparable since $\lambda(N) = |A| + 1 \geq 3$, and $MA_R = M(A) = N(A^c)$. Hence, $\{BP_M, BP_N\}$ is a full-dimensional convex tiling, and so is $\Sigma \cup \{BP_N\}$. Since any matroid semitiling connected in codimension $1$ is finite, repeating this extension process ends up with a full-dimensional convex tiling without any relevant facet, i.e. a complete tiling. □

5.2. Extensions of $(3, n)$-semitilings. In the rest of this section, we assume that a semitiling is a $(3, n)$-semitiling connected in codimension $1$. Also, the letters $R$ and $Q$ are reserved for $(3, n)$-polytopes of codimensions $1$ and $2$, respectively. Moreover, every cell of a semitiling to be mentioned is assumed loopless.

Fix a $(3, n)$-semitiling $\Sigma$. For any integer $m \geq 2$, a **chain of facets** of $\Sigma$ with **length** $m - 1$ is a sequence $(R_1, \ldots, R_{m-1})$ of facets of $\Sigma$ such that **intermediate vertices** $Q_i := R_i \cap R_{i-1}$ for $2 \leq i \leq m - 1$ are codimension-$2$ cells of $\Sigma$. Then, $\cup_{i=1}^{m-1} R_i$ is connected in codimension $2$ in $\Delta$. Further, any two codimension-$2$ cells $Q_1 < R_1$ and $Q_m < R_{m-1}$ that are non-intermediate vertices are said to be **initial** and **final vertices**, respectively. Note that there may be different choices for initial and final vertices while the intermediate ones are unique.

A chain $(R_1, \ldots, R_{m-1})$ of facets of $\Sigma$ is said to be **parallel** if $\text{def}_Q \Sigma = 3$ for all intermediate vertices $Q$ where their affine hulls $\text{Aff}(R_i)$ are indeed parallel and all $R_i$ have the same type by Theorem 3.21. If $Q_1$ and $Q_m$ are initial and final vertices of the parallel chain such that $\text{def}_Q \Sigma \neq 3 \neq \text{def}_Q \Sigma$, the chain together with these two vertices is said to be a **maximal parallel chain**.

Let $R$ be a facet of $|\Sigma|$ containing two distinct codimension-$2$ cells at which $\Sigma$ has deficiency $1$ or $2$, and consider the facets of $|\Sigma|$ whose intersections with $R$ are codimension-$2$ cells. We call the union of $R$ and those facets an **alcove** of $\Sigma$.

We consider a class of semitilings with nice enough alcoves.

**Definition 5.2.** Let $\Sigma$ be a $(3, n)$-semitiling (connected in codimension $1$).

(a) $\Sigma$ is said to be **regular at** a loopless codimension-$2$ cell $Q$ if $\text{def}_Q \Sigma \neq 1$.

(b) $\Sigma$ is said to be **regular at** a loopless facet $R$ if $R$ has type $1$, or if $R$ has type $2$ and the support of any maximal parallel chain of type-$2$ facets of $\Sigma$ involving $R$ contains at most one loopless codimension-$2$ cell $Q$ with $\text{def}_Q \Sigma = 2$.

(c) $\Sigma$ is said to be a **regular semitiling** if it is regular at all of its loopless facets and all of its loopless codimension-$2$ cells.

Fix a regular semitiling $\Sigma$. Suppose $R$ and $R'$ are two relevant facets of $\Sigma$ such that $Q = R \cap R'$ is a codimension-$2$ cell of $\Sigma$ with $\text{def}_Q \Sigma = 2$. One can construct

\begin{footnotesize}
\textsuperscript{16}We do not use “regular” to indicate a coherent subdivision of a polytope which is a subdivision induced by a convex or concave function, cf. [GKZ94].
\end{footnotesize}
a full-dimensional \((3,n)\)-polytope \(BP_N\) along the constructions \((L3)-(L5)\) of line arrangements such that \((\Sigma \cup \{BP_N\})_Q\), cf. \((3.1)\), is a tiling with 0-deficiency at \(Q\). Then, the issue of extendability occurs only when \(BP_N\) is inserted into an alcove of \(\Sigma\) that meets \(BP_N\) at more than two facets. Therefore we elaborate the extension process as eluding this situation as far as we can hereafter.

Recall that the drawing rule is a local property and that for a rank-3 matroid \(M\), the type and the rank of a loopless facet of \(BP_M\) coincide. Thus, the local convexity and the regularity of a \((3,n)\)-semitiling can be checked by drawing puzzle-pieces of its associated \((3,n)\)-semitiling assuming local coordinate charts of Definition 4.7. Loopless facets will be depicted as line segments whose lengths are measured by the metric of \((4.1)\) if necessary. But, specifying the lengths is not important in this paper (while it may be in a tropical sense) and we skip it.

We begin by showing that the regularity is general enough for our interests.

**Proposition 5.3.** Every weighted \((3,n)\)-tiling \(\Sigma\) extends to a regular semitiling.

**Proof.** Let \(Q\) be a codimension-2 cell of \(\Sigma\). Then, there are unique two facets \(R_1\) and \(R_2\) of \(\Sigma\) with \(Q = R_1 \cap R_2\). If \(\text{def}_Q \Sigma \geq 2\), then \(\Sigma\) is regular at all of \(Q\), \(R_1\) and \(R_2\) by Proposition 3.26. Suppose \(\text{def}_Q \Sigma = 1\), then \(R_1\) and \(R_2\) are non-parallel type-2 facets of \(\Sigma\). One constructs a base polytope \(BP_N\) of \((L3)\) such that \(R_1\) and \(R_2\) are its type-1 facets and \(\text{ang}_{BP_N} = 1\). Then, \(\Sigma' := \Sigma \cup \{BP_N\}\) is a semitiling with \(\text{def}_Q \Sigma_1 = 0\), see Figure 5.1. Further, \(\Sigma_1\) is regular at every \(Q' \neq Q\) containing \(R_1 \cup R_2\) since \(\text{ang}_{BP_N} = 2\) and \(\text{def}_{Q'} \Sigma \geq 4\) by Proposition 3.26, and hence:

\[
2 \leq \text{def}_{Q'} \Sigma_1 = \text{def}_{Q'} \Sigma - \text{ang}_{Q'} BP_N \leq 4.
\]

Now, let \(R_3 < BP_N\) and \(R_4 < \Sigma\) be the unique facets of \(\Sigma_1\) with \(Q' = R_3 \cap R_4\), then \(R_3 \not\in \Sigma\) and \(R_4 \not\in BP_N\).

- If \(\text{def}_{Q'} \Sigma_1 = 4\), then \(\Sigma_1\) is regular at \(R_3\).
- If \(\text{def}_{Q'} \Sigma_1 = 2\), then \(R_3\) and \(R_4\) are type-2 and type-1 facets of \(\Sigma_1\), respectively. Then, there is only one maximal parallel chain of type-2 facets of \(\Sigma_1\) involving \(R_3\), which is \((R_3)\). If \(Q'' \neq Q'\) is a loopless codimension-2 cell of the chain, then \(\text{def}_{Q''} \Sigma_1 = \text{def}_{Q''} BP_{N_i} \geq 4\). Thus, \(\Sigma_1\) is regular at \(R_3\).
- If \(\text{def}_{Q'} \Sigma_1 = 3\) and \(R_3\) is a type-1 facet of \(\Sigma_1\), then \(\Sigma_1\) is regular at \(R_3\).
- Else if \(\text{def}_{Q'} \Sigma_1 = 3\) and \(R_3\) is a type-2 facet of \(\Sigma_1\), then \(\Sigma_1\) might not be regular at \(R_3\), but \(\Sigma_1\) has the shape described in Proposition 3.26.

\[\text{Figure 5.1. Regular extensions of weighted tilings.}\]

\[\text{\textsuperscript{17}This computation is a 2-dimensional process that can be manually done.}\]

\[\text{\textsuperscript{18}Note that a polygon, broken or not, depicted for a base polytope or a puzzle-piece is not the whole picture, but just a partial collection of facets or line-pieces that is connected in codimension 1. For an efficient procedure, we may add more information inside, e.g. Figures 5.5 and 5.6.}\]
In either case, $\Sigma_1$ has the shape described in Proposition 3.26. Therefore, the above process can be repeatedly performed until one obtains an extension $\bar{\Sigma}$ of $\Sigma$ without any codimension-2 cell $Q$ with $\text{def}_Q \bar{\Sigma} = 1$, which is regular by construction. \hfill $\square$

We consider two kinds of regular semitilings in Lemma 5.4 and Algorithm 5.8, respectively, that extend to complete tilings.

**Lemma 5.4.** Let $\Sigma$ be a $(3,n)$-semitiling whose boundary codimension-2 cells are all irrelevant. Then, there is a full-dimensional base polytope $\text{BP}_N$ such that $\Sigma \cup \{\text{BP}_N\}$ is a convex tiling whose boundary codimension-2 cells are all irrelevant again. Hence, $\Sigma$ extends to a complete tiling.

**Proof.** Let $R$ be any relevant facet of $\Sigma$, then $R$ is written as $R = \text{BP}_{M(F)}$ for some $\text{BP}_M \in \Sigma$ and $F \in \mathcal{L}(M)$. Then, $|M(F)| = U^2_{E(M(F))}$ since all codimension-2 cells of $\Sigma$ are irrelevant. Construct a line arrangement $\mathcal{HA}_N$ of $(L_1)$ if $R$ is of type 2, and of $(L_2)$ if $R$ is of type 1, such that $|N(F^c)| = |M(F)|$ and its points away from $\eta(N/F^c)$ are all simple and normal. Then, $N(F^c) = M(F)$ and $R = \text{BP}_M \cap \text{BP}_N$.

Further, every point-piece of $\mathcal{PZ}_N$ is irrelevant and so is every codimension-2 cell of $\text{BP}_N$. So, $\Sigma \cup \{\text{BP}_N\}$ is a semitiling connected in codimension 1 whose boundary codimension-2 cells are all irrelevant, which is a convex tiling by Lemmas 3.13 and 3.15. This extension process can be performed recursively until one obtains a convex tiling without any relevant facet, which is a complete extension of $\Sigma$. \hfill $\square$

**Definition 5.5.** Let $\Sigma$ be a semitiling and $\Psi$ its associated semipuzzle. The **dual graph** of $\Sigma$ is a graph that has a vertex corresponding to each puzzle-piece of $\Psi$ and an edge joining two distinct puzzle-pieces with a common facet.

**Corollary 5.6.** Let $\Sigma$ be a $(3,n)$-semitiling such that its codimension-2 cells are all irrelevant. Then, its dual graph is a tree such that any two adjacent vertices collapse to a vertex along the edge and produce the dual graph of another semitiling.

**Definition 5.7.** Let $N$ be a rank-3 inseparable matroid. A nonempty proper flat $F$ of $N$ is said to be a **branch flat** if $\text{BP}_{N(F)}$ has at least 3 relevant codimension-2 cells, that is, $2 \leq |F| \leq n - 2$ and $|N(F)|$ has at least 3 rank-1 flats of size $\geq 2$. Then, $F$ is a non-degenerate flat, and $\text{BP}_{N(F)}$ is said to be a **branch facet** of $\text{BP}_N$.

Let $\Sigma$ be a regular semitiling without any branch facet. Algorithm 5.8 below, consisting of 3 steps, produces a complete extension of $\Sigma$. In the first two steps, the algorithm finds a nontrivial extension of $\Sigma$ without any type-2 facet, and in the last step finds one without any type-1 relevant facet, which is a complete tiling. This also means that $\Sigma$ was a tiling in the first place.

**Algorithm 5.8.** Let $\Sigma$ be a regular semitiling (connected in codimension 1) without any branch facet. Let $\nu = 0$, $\Sigma_0 = \Sigma$, and go to **Step 1**.

**Step 1** If there is no codimension-2 cell $Q < \Sigma_\nu$ such that $\text{def}_Q \Sigma_\nu = 2$, go to **Step 2**. Otherwise, let $Q_1$ be one of such. Let $R'_1$ and $R_1$ be the type-2 and -1 facets of $\Sigma_\nu$, respectively, with $Q_1 = R'_1 \cap R_1$, both of which are relevant. Let $(R_1, \ldots, R_{m-1})$ be a maximal parallel chain of type-1 facets with an initial vertex $Q_1$ and a final vertex $Q_m < R_{m-1}$. Then, $Q_i := R_i \cap R_{i-1}$ for $2 \leq i \leq m - 1$ are relevant. Let $R_m$ be the facet of $\Sigma_\nu$ with $Q_m = R_m \cap R_{m-1}$, see Figure 5.2. Since $\Sigma_\nu$ has no branch facet, $R_1, \ldots, R_{m-1}$ are non-branch facets, and so every $Q \subset \bigcup_{j=1}^{m-1} R_j$ other than $Q_1$ with $1 \leq i \leq m$ is irrelevant; hence $\text{ang}_Q \Sigma_\nu = 1$. Three sub-steps follow.
(a) Construct a base polytope $BP_{Q_1}$ of (L5) such that $R_i'$ and $R_1$ are type-1 and -2 facets, respectively, and $\text{ang}_{Q_1}(BP_{Q_1}) = 3 - i$ for $i = 1, 2$. Then, $\Sigma_{\nu + 1} := \Sigma_{\nu} \cup \{BP_{Q_1}\}$ is a semitiling (connected in codimension 1) with $\text{def}_{Q_1, \Sigma_{\nu + 1}} = 0$ and $\text{def}_{Q_2, \Sigma_{\nu + 1}} \geq 1$ since $\text{ang}_{Q_2, \Sigma_{\nu}} \leq 4$. Note that $BP_{Q_1}$ has no branch facet, and so does the semitiling $\Sigma_{\nu + 1}$. Every $Q < R_1$ other than $Q_i$ satisfies that:

$$\text{def}_{Q, \Sigma_{\nu + 1}} = 6 - \text{ang}_{Q, \Sigma_{\nu}} - \text{ang}_{Q, BP_{Q_1}} = 3$$

so $\Sigma_{\nu + 1}$ is regular at $Q$. Similarly, $\text{def}_{Q, \Sigma_{\nu + 1}} \geq 3$ for every $Q < R_i'$ other than $Q_i$, and $\Sigma_{\nu + 1}$ is regular at $Q$. Let $R_2'$ be the facet of $BP_{Q_1}$ with $Q_2' = R_2' \cap R_2$, which is a unique newly added type-2 facet of $\Sigma_{\nu + 1}$ by Lemma 4.21; therefore if $(R, R_i')$ was a chain of parallel facets of $\Sigma_{\nu}$, then $R$ is a type-2 facet of $\Sigma_{\nu + 1}$ at which $\Sigma_{\nu + 1}$ is regular. Now, unless $\text{ang}_{Q_2, \Sigma_{\nu}} = 4$, the regularity of $\Sigma_{\nu + 1}$ follows from that of $\Sigma_{\nu}$, and increase $\nu$ by 1 and go to (the beginning of) Step 1.

(b) Now, $\text{ang}_{Q_2, \Sigma_{\nu}} = 4$, so $m = 2$ and $\text{def}_{Q_2, \Sigma_{\nu + 1}} = 1$. Construct $BP_{N_1}$ of (L3) such that $R_i'$ and $R_2$ are type-1 facets with $\text{ang}_{Q_2}(BP_{N_1}) = 1$. Then, $\Sigma_{\nu + 2} := \Sigma_{\nu + 1} \cup \{BP_{N_1}\}$ is a semitiling with $\text{def}_{Q_2, \Sigma_{\nu + 2}} = 0$. Let $(R_2, \ldots, R_l)$, $l \geq 2$, be a maximal parallel chain of type-2 facets of $\Sigma_{\nu + 1}$ with an initial vertex $Q_2$ and a final vertex $Q_{l+1}$; then $\text{ang}_{Q_{l+1}, \Sigma_{\nu + 1}} \leq 2$. If $l = 2$, or if $l \geq 3$ and $\text{ang}_{Q_2}(BP_{N_2}) = 1$, then $\Sigma_{\nu + 2}$ is regular. Since both $R_2'$ and $R_2$ are non-branch facets of $\Sigma_{\nu + 1}$, we may assume $BP_{N_2}$ and $\Sigma_{\nu + 2}$ have no branch facet. Increase $\nu$ by 2 and go to Step 1.

(c) Else if $l \geq 3$ and $\text{ang}_{Q_1}(BP_{N_2}) = 2$, recursively construct $BP_{N_3}, \ldots, BP_{N_l}$ of (L3) until $\text{ang}_{Q_{l+1}, \Sigma_{\nu + 1}}(BP_{N_l}) = 1$ or $j = l$ such that $\Sigma_{\nu + j} := \Sigma_{\nu + 2} \cup \{BP_{N_3}, \ldots, BP_{N_l}\}$ is a semitiling. This process terminates with a regular semitiling $\Sigma_{\nu + j}$. We may assume that $\Sigma_{\nu + j}$ has no branch facet as before. Increase $\nu$ by $j$ and go to Step 1.

**Step 2** If $\Sigma_{\nu}$ has no type-2 facet, go to Step 3. Otherwise, let $(R_1, \ldots, R_{m-1})$ be a maximal parallel chain of type-2 facets of $\Sigma_{\nu}$ with vertices $Q_1, \ldots, Q_m$. Now, after Step 1, one has $\text{ang}_{Q, \Sigma_{\nu}} = 1, 2$ for $Q = Q_1, Q_m$. Construct $BP_N$ of (L1) without any branch facet such that $R_1$ is its type-1 facet and $\text{ang}_{Q, BP_N} = 1$ for every $Q < BP_N$. Then, $\Sigma_{\nu + 1} := \Sigma_{\nu} \cup \{BP_N\}$ is a regular semitiling without any branch facet; see Figure 5.3. Increase $\nu$ by 1 and go to Step 1.

**Figure 5.3.** $\Sigma_{\nu}$ has a type-2 facet.
Step 3 The regular semitiling $\Sigma_\nu$ has no type-2 facet, and $\text{def}_Q \Sigma_\nu \geq 3$ for every boundary codimension-2 cell $Q$; hence it is a convex tiling by Corollary 3.23 and Lemmas 3.13 and 3.15. Two sub-steps follow.

(a) If the convex tiling $\Sigma_\nu$ has no type-1 relevant facet, it is a complete tiling, and hence terminate the algorithm.

(b) Otherwise, let $(R_1, \ldots, R_{m-1})$ be a maximal parallel chain of type-1 relevant facets of $\Sigma_\nu$ with vertices $Q_1, \ldots, Q_m$. Now, after Step 2, one has $\text{ang}_Q \Sigma_\nu = 1$ for $Q = Q_1, Q_m$. Then, construct $\text{BP}_N$ of (L2) such that $R_1$ is its type-2 facet and $\text{ang}_Q \text{BP}_N = 1$ for all $Q < \text{BP}_N$ with $Q \not\sim R_1$ where $R_1$ is its unique type-2 facet, and hence no type-2 facet is added to the semitiling $\Sigma_{\nu+1} := \Sigma_\nu \cup \{\text{BP}_N\}$, see Figure 5.4.

If $m > 2$, then $\text{def}_Q \Sigma_{\nu+1} = 1$, and recursively construct $\text{BP}_N, \ldots, \text{BP}_{N_{m-1}}$ of (L4) such that $\Sigma_{\nu+m-1} := \Sigma_{\nu+1} \cup \{\text{BP}_N, \ldots, \text{BP}_{N_{m-1}}\}$ is a semitiling without any type-2 facet. Now, $\text{def}_Q \Sigma_{\nu+m-1} = 1, 2$ for all $Q \subset \bigcup_{j=1}^{m-1} \text{BP}_N$, and thus $\Sigma_{\nu+m-1}$ is regular. Since $\Sigma_\nu$ has no branch facet, we may assume $\text{BP}_N$ were constructed without any branch facet, and so is $\Sigma_{\nu+m-1}$. Increase $\nu$ by $m - 1$ and go to Step 3.

![Figure 5.4. $\Sigma_\nu$ has no type-2 facet and $\text{def}_Q \Sigma_\nu \neq 1, 2$ for all $Q < \Sigma_\nu$.](image)

Example 5.9. Let $\Sigma$ be a $(3, 6)$-semitiling connected in codimension 2, and $\Psi$ its associated semipuzzle. Note that the number of relevant point-pieces of $\Psi$ is at most 1, and therefore $\Sigma$ has no branch facet.

- If $\Psi$ is connected in codimension 1 and has no relevant point-piece, $\Sigma$ extends to a complete tiling $\tilde{\Sigma}$ by Lemma 5.4. If $\tilde{\Sigma}$ is maximally split, the dual graph of $\tilde{\Sigma}$ is a cross shaped tree with 4 leaves and 1 vertex of degree 4, cf. Corollary 5.6.

- If $\Sigma$ is connected in codimension 1 and regular, it extends to a complete tiling by Algorithm 5.8. The condition for complete extension of $\Sigma$ even can be weakened: If $\Psi$ is connected in codimension 2 and has a relevant point-piece, say $\text{PZ}_Q$, there are up to symmetry only three puzzle-pieces containing $\text{PZ}_Q$, see Figure 5.5, in which the last two are only two nontrivial splits of the first. Then, the puzzle $\Psi$ is obtained from a subcollection of one of the complete puzzles of Figure 5.6 by a sequence of merging operations if necessary.

![Figure 5.5. Up to symmetry all splits of any $(3, 6)$-puzzle-piece containing a relevant point-piece.](image)
Remark 5.10. The 7 dual graphs of the semitilings in Example 5.9 are precisely the 7 types of generic tropical planes in $\mathbb{TP}^5$; see [HJJS09, Figure 1].

5.3. Complete extensions of regular semitilings. In this subsection, we show that every regular or weighted $(3,n)$-semitiling with $n \leq 9$ extends to a complete tiling and the bound $n = 9$ is sharp.

Theorem 5.11. Every regular $(3,n \leq 9)$-semitiling extends to a complete tiling.

Proof. It suffices to prove for regular $(3,9)$-semitilings $\Sigma$. We make a variation of Algorithm 5.8 for regular semitilings with branch facets.

We modify Step 1 first. Let $Q_1$ be a codimension-2 cell of $\Sigma$ with $\text{def}_{Q_1}\Sigma = 2$, and assume the remaining setting of Step 1. Since $R_1$ is a relevant type-1 facet of $\Sigma$, it is written as $R_1 = \text{BP}_{M(F)}$ for some $\text{BP}_M \in \Sigma$ and some rank-1 flat $F$ of $M$ with $2 \leq |F| \leq 5$ where $|M(F)| = M/F$. On the other hand, $R_1'$ cannot be a branch facet of $\Sigma$ since otherwise $|F| \geq 6$, a contradiction. Therefore we assume that $R_1$ is a branch facet of $\Sigma$. Then, since $2 \cdot \lambda(M/F) \geq 6$, we have:

$$|F| = 2 \text{ or } |F| = 3.$$ 

Further, $M/F$ has exactly 3 rank-1 flats of size $\geq 2$, say $A_1$, $A_2$ and $A_3$. The sizes of $A_i$ are bounded above by 3, and hence 2 or 3. But, at most one of $A_i$ has size 3 which happens only when $|F| = 2$. Thus, there are precisely 4 cases:

$$|A_1| = 3 \text{ or } |A_2| = 3 \text{ or } |A_3| = 3 \text{ or } |A_1| = |A_2| = |A_3| = 2.$$ 

We may assume $Q_i = \text{BP}_{M(F)(A_i)}$ for $i = 1,2$. Let $Q_3'' = \text{BP}_{M(F)(A_3)}$, and let $R_3''$ be the (unique) facet of $\Sigma$ with $Q_3'' = R_3'' \cap R_1$, then $2 \leq \text{def}_{Q_3''}\Sigma \leq 5$ since $\Sigma$ is regular.

Observe that for any $Q < R_1$, one has $\text{def}_Q\Sigma \geq 2$. Moreover, let $R$ be the facet of $\Sigma$ with $Q = R \cap R_1$, and note the following.

- If $\text{def}_Q\Sigma \leq 3$, then $R$ is a non-branch facet of $\Sigma$.
- For any parallel chain of facets of $\Sigma$ containing $R$ or $R_1$, its facets other than $R$ and $R_1$ are non-branch facets of $\Sigma$. 

Figure 5.6. Up to symmetry all maximally split complete $(3,6)$-puzzles containing a relevant point-piece.
Suppose $|F| = 2$, and write $F = 12$. Consider the case $|A_1| = 3$ and $|A_2| = |A_3| = 2$. As pretending that $A_1$ is a singleton and $R_i$ is a non-branch facet, run Step 1(a) and obtain $\Sigma_1 = \Sigma \cup \{BP_{N_i}\}$, a semitiling or not. Then, $\text{def}_{Q^*}\Sigma_1 \leq 3$ and $\text{def}_{Q^*}\Sigma_1 \geq 1$ since $\text{ang}_{Q^*}BP_{N_1} = 2$ and $\text{ang}_{Q^*}BP_{N_1} = 1$, respectively.

- If $\text{def}_{Q^*}\Sigma_1 \geq 1$, then $\Sigma_1$ is a semitiling.
- Else if $\text{def}_{Q^*}\Sigma_1 = 0$, i.e. $\text{def}_{Q^*}\Sigma = 2$, both $|N_1(A_3)|$ and $|\text{MA}_{R_3^*}|$ are rank-2 matroids with the same collection of rank-1 flats which is $\{1, 2, [9] \mod F - A_3\}$; therefore $|N_1(A_3)| = |\text{MA}_{R_3^*}|$ and $\text{BP}_{N_1(A_3)} = R_3^*$. Thus, $\Sigma_1$ is a regular extension of $\Sigma_1$.

Moreover, there is a regular extension of $\Sigma_1$:

- If $\text{def}_{Q^*}\Sigma_1 = 0$, then $\Sigma_1$ is regular and is its own trivial extension.
- If $\text{def}_{Q^*}\Sigma_1 = 2$, i.e. $\text{def}_{Q^*}\Sigma = 4$, the rank-1 flats of $|\text{MA}_{R_3^*}|$ aside from $F$ are precisely the two singletons of $A_3$, and for every $Q < R_3^*$ with $Q \neq Q_3^*$ one has $\text{def}_{Q}\Sigma_1 = \text{def}_{Q}\Sigma = 4$; hence $\Sigma_1$ is regular.
- Else if $\text{def}_{Q^*}\Sigma_1 = 1$, i.e. $\text{def}_{Q^*}\Sigma = 3$, consider a parallel chain $(R_1, \ldots)$ of $\Sigma$ of length $m' - 1 \geq 2$ with $Q_3^*$ being an intermediate vertex of it. Apply to $\Sigma_1$ the argument of Step 3(b) with $m' > 2$ and $Q_3^*$ replacing $m$ and $Q_2$, respectively, and get a regular extension of $\Sigma_1$.

The other cases are similar.

Suppose $|F| = 3$, then $|A_1| = |A_2| = |A_3| = 2$. By similar argument, it suffices to consider the alcove-fitting case only, that is, $\text{def}_{Q^*}\Sigma = \text{def}_{Q^*}\Sigma = \text{def}_{Q^*}\Sigma = 2$. Write $F = 123$ and let $L_1$, $L_2$, and $L_2$ be the partitions of $F$ into the rank-1 flats of $|\text{MA}_{R_3^*}|_F$, $|\text{MA}_{R_3^*}|_F$, and $|\text{MA}_{R_3^*}|_F$, respectively. Then, $\lambda(|\text{MA}_R|_F) = 2, 3$ for $R = R_1, R_3$, and $R_2$, and there are up to symmetry 5 cases for $(L_1, L_2)$:

| (a) $\{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 3\}$ | (b) $\{1, 23\}, \{1, 2, 3\}, \{1, 2, 3\}$ |
|--------------------------------------------------|--------------------------------|
| (c) $\{1, 23\}, \{1, 2, 3\}, \{3, 12\}$ | (d) $\{1, 23\}, \{2, 13\}, \{3, 12\}$ |
| (e) $\{1, 23\}, \{1, 23\}, *$ | with $* = \text{any } L_2$ |

Table 5.1. Up to symmetry all cases for the alcove of $\Sigma$ containing $R_1$.

- For (e), apply the argument for $|F| = 2$ and obtain a regular extension of $\Sigma$, see Figure 5.7. The number of constructed puzzle-pieces (or base polytopes) is at most 3. When the number is 3, the two matroids $N_1$ and $N_2$ have simplifications isomorphic to each other and $N_3$ has simplification isomorphic to $\mathcal{U}_4^3$.

Figure 5.7. Step 1 adapted for the alcove-fitting case (e)

---

21The pictures are presented in terms of hyperplane arrangements and puzzle-pieces. The numbers are labels for lines or line-pieces, and $A_1$, $A_3$, and $A_2$ are sets of labels.
• For each of (a)–(d), there is a base polytope $BP_{N_1}$ such that $R'_1, R''_3$ and $R_2$ are type-1 facets of it and $R_1$ is a type-2 facet of it, see Figure 5.8. 22 One checks that $\Sigma \cup \{BP_{N_1}\}$ is a regular semitiling.

Thus, Step 1 is modified for regular $(3,9)$-semitilings.

For Step 2, if $R_1$ is a type-2 branch facet of $\Sigma_\nu$, then pretend $R_1$ is a non-branch facet and construct $BP_N$ of (L1) such that $R_1$ is its type-1 facet and $\text{ang}_{Q}BP_N = 1$ for every $Q < BP_N$ so that the semitiling $\Sigma_{\nu+1} := \Sigma_\nu \cup \{BP_N\}$ is a regular extension of $\Sigma$. This is just a copy of Step 2 except the branch facet condition.

For Step 3, similarly by pretending facets are non-branch facets, construct $BP_{N_1}$ of (L1), and if $m > 2$ also construct $BP_{N_2}, \ldots, BP_{N_{m-1}}$ of (L4) so that the semitiling $\Sigma_{\nu+m-1} := \Sigma_{\nu+1} \cup \{BP_{N_2}, \ldots, BP_{N_{m-1}}\}$ is a regular extension of $\Sigma$.

The proof is complete. $\square$

The following example by construction tells that for any $(k,n)$ with $k \geq 3$ and $n \geq 10$, there exists a counterexample to the complete extension of a $(k,n)$-tiling. This proves that the bound $n = 9$ of Theorem 5.11 is sharp.

Example 5.12. Fix $n = 10$ and write $[10] = \{1, \ldots, 9, 0\}$. Consider the realizable line arrangements $HA_{M_i}, i = 0, 1, 2, 3$, of Figure 5.9 with the relevant and essential describing inequalities of the base polytopes $BP_{M_i}$. Let $\Sigma := \{BP_{M_i} : i = 0, 1, 2, 3\}$, then $\Sigma$ is a regular tiling. Write $F_1 = 12, F_2 = 34, F_3 = 56$, and $F_0 = 7890$. Then,

$$
\begin{array}{l}
BP_{M_0} & x_{7890} \leq 1, x_{127890} \leq 2, x_{347890} \leq 2, x_{567890} \leq 2 \\
BP_{M_1} & x_{78} \leq 1, x_{3456} \leq 1, x_{34567890} \leq 2 \\
BP_{M_2} & x_{78} \leq 1, x_{90} \leq 1, x_{1290} \leq 1, x_{12907890} \leq 2 \\
BP_{M_3} & x_{90} \leq 1, x_{1234} \leq 1, x_{12347890} \leq 2
\end{array}
$$

The black dots indicate non-normal points of $HA_{N_1}$ which correspond to all the rank-2 facets of $BP_{N_1}$, cf. Remark 4.10.
$R_0 := BP_{M_0(F_0)}$ is a type-1 facet of $\Sigma$, and $R_i := BP_{M_i(F_i)}$, $i = 1, 2, 3$, are type-2 facets of $\Sigma$. Further, $Q_i := R_i \cap R_0$, $i = 1, 2, 3$, are boundary relevant codimension-2 cells of $\Sigma$ with $\def_{Q_i} \Sigma = 2$.

Now, suppose $\Sigma$ is a complete extension of $\Sigma$. Then, since $R_0$ is relevant, there is a base polytope $BP_N \in \Sigma$ such that $R_0$ is its type-2 facet. By Lemma 4.21, there is at most one $Q_1$ with $\ang_{Q_1} BP_N = 1$. If $Q_1$ is such, $\ang_{Q_1} BP_N = \ang_{Q_2} BP_N = 2$, and $BP_N$ is face-fitting to both $BP_{M_2}$ and $BP_{M_3}$. Then, $R_2$ and $R_3$ are type-1 facets of $BP_N$ with $[MA_{R_2}]|_{F_0} = [MA_{R_3}]|_{F_0}$, but this is a contradiction since:

$$[MA_{R_2}]|_{F_0} = [MA_{R_2}]|_{F_0} \neq [MA_{R_3}]|_{F_0}.$$ 

Similarly, $\ang_{Q_i} BP_N \neq 1$, that is, $\ang_{Q_i} BP_N = 2$ for all $i = 1, 2, 3$, and $R_i$ are facets of $BP_N$ so that $\eta(N/F_i)$ are lines of $HA_N$ with $N/F_i = M_i|_{F_i}$. Then, two lines $\eta(N/9)$ and $\eta(N/0)$ of $HA_N$ both pass through two distinct points $\eta(N/3490)$ and $\eta(N/5690)$, and hence those lines are the same. But, this is a contradiction since the line $\eta(N/9)$ does not pass through the point $\eta(N/120)$ while $\eta(N/0)$ does. Therefore, we conclude that there exists no complete extension of $\Sigma$.

The regular tiling $\Sigma$ is a tiling weighted by a weight $\beta = (1, 1, 1, 1, 1, 1, \frac{1}{3}, \frac{1}{4}, \frac{1}{4})$, cf. Definition 3.24, as follows:

- Write $v = (v_1, \ldots, v_9, v_0)$. For a point $v = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}) \in BP_{M_0}$, one can decrease entries $v_7, v_8, v_9, v_0$ by a sufficiently small positive number $\epsilon$ and increase $v_1, \ldots, v_6$ by $\frac{4\epsilon}{9}$ so that the new point is still contained in $BP_{M_0}$ and int$(\Delta_{\beta})$ both, and hence $BP_{M_0} \cap \text{int}(\Delta_{\beta}) \neq \emptyset$. Similarly, for all $i = 1, 2, 3$, one has $BP_{M_i} \cap \text{int}(\Delta_{\beta}) \neq \emptyset$.

- Moreover, $|\Sigma|$ covers $\Delta_{\beta}$. Suppose not, i.e. $\Delta_{\beta} - |\Sigma| \neq \emptyset$, then there is a point $v \in \Delta_{\beta}$ that violates at least one of the describing inequalities of $BP_{M_i}$ for all $i = 0, 1, 2, 3$. Since $v_{7890} \leq \beta_{7890} = 1$, the point $v$ must violate all 3 inequalities $x_{3456} \leq 1$, $x_{1256} \leq 1$ and $x_{1234} \leq 1$ of $BP_{M_1}$, $BP_{M_2}$ and $BP_{M_3}$, respectively. Also, $v$ must violate at least one of the inequalities of $BP_{M_0}$ except $x_{7890} \leq 1$. Whichever is violated, one reaches a contradiction because:

$$3 = v_{127890} + v_{3456} > 2 + 1 = 3 \quad \text{if } x_{127890} \leq 2 \text{ is violated,}$$

$$3 = v_{347890} + v_{1256} > 2 + 1 = 3 \quad \text{if } x_{347890} \leq 2 \text{ is violated,}$$

$$3 = v_{567890} + v_{1234} > 2 + 1 = 3 \quad \text{if } x_{567890} \leq 2 \text{ is violated.}$$

**Theorem 5.13.** Every weighted $(3, n)$-tiling with $n \leq 9$ extends to a complete tiling and the bound $n = 9$ is sharp.

**Proof.** It immediately follows by simply combining Proposition 5.3, Theorem 5.11, and Example 5.12. □

### 5.4. Extensions of realizable tilings

A matroidal SHA is said to be realizable if it corresponds to a weighted SHA, cf. Subsection 4.8. Accordingly, a tiling or a puzzle is said to be realizable if its associated matroidal SHA is. This definition is not redundant because even if all components of the matroidal SHA are realizable over a field $k$, realizations of them not necessarily glue to one another.

We prove the realization version of Theorem 5.13. This answers the question of whether or not the reduction morphisms between moduli spaces of weighted SHAs are surjective, which was proposed by Alexeev, cf. [Ale08].
Theorem 5.14. Fix an algebraically closed field \(\mathbb{k}\). When \(n \leq 9\), every reduction morphism \(\rho_{\beta', \beta} : \overline{M}_{\beta'}(3, n) \to \overline{M}_{\beta}(3, n)\) is surjective and the bound \(n = 9\) is sharp.

Proof. Since the sharpness of the bound \(n = 9\) follows from Example 5.12, we only prove the surjectivity of reduction morphism when \(n = 9\). For a fixed weight \(\beta\), let \(X = \sqcup X_i\) be a \(\beta\)-weighted SHA and \(\Sigma\) its corresponding matroid tiling. We extend \(X\) to an unweighted SHA by gluing irreducible components \(X_i\)'s, which is equivalent to showing that every geometric point of \(\overline{M}_{\beta}(3, 9)\) whose fiber is a \(\beta\)-weighted SHA has a preimage via the reduction morphism \(\rho_{\beta, \beta} : \overline{M}_{\beta}(3, 9) \to \overline{M}_{\beta}(3, 9)\). Then, for any weight \(\beta'\) with \(\beta' > \beta\), the reduction morphism \(\rho_{\beta', \beta} : \overline{M}_{\beta'}(3, 9) \to \overline{M}_{\beta}(3, 9)\) is surjective by the following commutative diagram:

\[
\begin{array}{ccc}
\overline{M}_{\beta'}(3, 9) & \xrightarrow{\rho_{\beta', \beta}} & \overline{M}_{\beta}(3, 9) \\
\text{\Large \(\rho_{\beta, \beta}\)} & & \text{\Large \(\rho_{\beta, \beta}\)} \\
\overline{M}_{\beta}(3, 9) & \xrightarrow{\rho_{\beta, \beta}} & \overline{M}_{\beta}(3, 9)
\end{array}
\]

We adapt the proof of Theorem 5.11 for realizations. It suffices to consider the alcove-fitting cases with \(|F| = 3\) only. Then, \(R_1\) is a type-1 branch facet, and \(R_1', R_2', R_2\) are type-2 non-branch facets of \(\Sigma\). For each \(R = R_1, R_1', R_2', R_2\) let \(Y_R\) be the corresponding subvariety of \(X\) (and hence of a unique \(X_i\)) which is a realization (over \(\mathbb{k}\)) of the point arrangement \(HA_{[MA_R]}\). In particular, \([MA_{R_1}] = M/F\) has simplification isomorphic to \(U_2^2\), so every realization of \(HA_{M/F}\) is isomorphic to one another, that is, there is an element of \(PGL_3(\mathbb{k})\) between any two realizations of \(HA_{M/F}\). Now, see Table 5.1.

For (a)–(d), each line arrangement \(HA_{N_i}\) of Figure 5.8 has a realization, say \(X'\), such that for each \(R = R_1', R_2', R_2\) the subvariety of \(X'\) corresponding to \(HA_{[MA_R]}\) is isomorphic to \(Y_R\). Indeed, for (a), take a realization \(X''\) of the line arrangement \(HA_{N_i}|_{[9] - A_2}\) and take a generic line in \(X''\) passing through the point corresponding to \(\eta((N_1|_{[9] - A_2} / (A_1 \sqcup A_3)))\) with induced point arrangement structure by pre-existing lines, which is a realization of \(HA_{[MA_{R_2}]}\) and hence a whole realization \(X'\) of \(HA_{N_i}\).

This process can be performed such that for each \(R = R_1', R_2', R_2\) the subvariety of \(X'\) corresponding to \(HA_{[MA_R]}\) is isomorphic to \(Y_R\) so that after appropriate blowups and contractions \(X'\) glues to \(X\). The other cases are similar.

For (e), see Figure 5.7. One can construct realizations \(X_{N_1}\) and \(X_{N_2}\) of \(HA_{N_1}\) and \(HA_{N_2}\), respectively, and \(X_{N_3}\) of \(HA_{N_3}\) if necessary, such that after appropriate blowups and contractions they glue to \(X\) in order.

Thus, we adapted Algorithm 5.8 for \(\beta\)-weighted SHAs for any weight \(\beta\) so that running this algorithm for \(X\) produces an output of an unweighted SHA as desired.

The proof is complete. \(\square\)

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