Gribov Copies And Other Gauge Fixing Beasties On The Lattice

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We study the nature of gauge fixing ambiguities in two dimensional gauge theories. We find that these ambiguities can be related to the associated spin model. They can be eliminated by means of a (multigrid) annealing algorithm.

Gauge fixing is by now a useful tool in lattice gauge calculations. Smooth gauges, like the Coulomb gauge and the Landau gauge, are commonly used in propagator calculations \[1\], as well as in wavefunction calculations \[2\]. In monopole studies, the so-called maximally Abelian gauge is frequently employed \[3\]. However, since the work of Gribov \[4\], it is known that 'fixing the gauge is not quite fixing the gauge'. In other words, it is possible to locally fix the gauge, but it need not always be possible to globally fix the gauge. Since one uses a slightly stronger condition on the lattice than in the continuum, it was long hoped that the Gribov ambiguity was absent on the lattice.

A few years ago, it was shown that these hopes were idle \[5\]. In order to get a better understanding of the nature of the Gribov ambiguity on the lattice, we shall study the occurrence of copies in 2-d U(1) and 2-d SU(2) when fixing to the Landau gauge.

The continuum version of the Landau gauge condition,
\[
\partial_{\mu}A_{\mu} = 0, \quad A^{g}_{\mu} = g^\dagger(\partial_{\mu} + A_{\mu})g,
\]
translates to the lattice as
\[
\text{Im} \sum_{\mu} [U^{g\dagger}_{\mu}(x) - U^{g\dagger}_{\mu}(x - \hat{\mu})]\text{traceless} = 0, \quad (1)
\]
\[
U^{g}_{\mu}(x) = g^\dagger(x)U_{\mu}(x)g(x + \hat{\mu}).
\]

As was mentioned above, usually a slightly stronger condition is used, namely minimizing a functional \(F\), where \(F\) is given by
\[
F(g, U) = \frac{1}{dVN} \sum_{x, \mu} \text{Re} \text{ Tr}[\mathbb{1} - U_{\mu}^{g}(x)], \quad (2)
\]
for SU(N) in \(d\) dimensions on a lattice containing \(V\) lattice sites. It is easy to see that if \(A\) is minimized for a given gauge configuration \(U\), \(A\) is automatically satisfied.

The Gribov problem, finding multiple solutions for the Landau gauge fixing condition
\[
\partial_{\mu}A_{\mu} + [D_{\mu}(A), (\partial_{\mu}g)g^\dagger] = 0,
\]
\(D_{\mu}(A)\) being the covariant derivative, is equivalent to finding local minima for the functional \(F\).

The search for Gribov copies on the lattice is done as follows. We start from an initial configuration \(C_1\), and fix that configuration to the Landau gauge, keeping track of the gauge transformations involved. The resulting configuration is called \(L_1\). We then perform a random gauge transformation on \(C_1\), to obtain \(C_2\), and fix \(C_2\) to the Landau gauge \(L_2\). We can repeat this as many times as we want. Since we have kept track of the gauge transformations \(g_i\) between \(C_1\) and \(L_i\), we can construct the gauge transformation \(g_{ij}\) between \(L_i\) and \(L_j\):
\[
g_{ij}(x) = g_{i}^\dagger(x)g_{j}(x). \quad (3)
\]
This procedure is displayed in figure 1.
One can look upon the gauge fixing procedure from another point of view. Minimizing the functional $F$ can be seen as finding the ground state of a spin system with pseudo-random couplings $U$, depending on both position and direction, with an action given by

$$S_U(g) = -\beta_s F(g, U). \quad (4)$$

We have put ‘pseudo-random couplings $U$', since only for gauge field coupling constant $\beta_g = 0$ are the couplings $U$ truly random. For finite $\beta_g$, the $U$ are constrained to form plaquettes $U_P$, which are to be taken from an equilibrium ensemble.

Making a random gauge transformation is, in some sense, creating a spin configuration at spin coupling constant $\beta_s = 0$. The standard procedure to find the minimum of (2) is employing a relaxation algorithm. Albeit impractical, this can be ‘simulated’ by performing Monte Carlo at $\beta_s = \infty$. We should therefore not be surprised if defects get trapped in the final, gauge fixed configuration: we don’t expect a perfect crystal to be formed by dropping liquid iron into liquid helium.

To determine whether we have reached convergence, we would like to use $F$. However, for non-trivial gauge configurations we do not know the minimal value(s) of $F$. Therefore, we define

$$f_i = F_{i-1} - F_i.$$

$F_i$ is the value of $F$ after $i$ iterations. We can assign a relaxation time $\tau$ to the minimization process by defining $\tau$ by

$$f_i = C \exp(-i/\tau). \quad (5)$$

For 2-d U(1), the associated spin model is the well known XY-model. It possesses a Kosterlitz-Thouless phase transition at a finite coupling $\beta_c$. The system is described by vortices and antivortices for couplings $\beta < \beta_c$, and by spin waves for couplings $\beta > \beta_c$.

We created pure gauge configurations by taking $U_\mu(x) = g^\dagger(x)g(x + \hat{\mu})$, and fixed these to the Landau gauge. We used a standard relaxation algorithm, combined with multigrid \cite{6}. We experienced, even on a $16^2$ lattice, difficulties in reaching the true Landau gauge, $U_\mu(x) = 1$. On larger lattices it became virtually impossible to reach the true Landau gauge. Figure 2 displays a typical gauge fixed configuration for a pure gauge configuration on a $32^2$ lattice.

Although the inclusion of overrelaxation in the minimization algorithm improves on the relaxation time $\tau$ and is able to remove all the vortex-antivortex pairs on the smaller lattices, the combined algorithm is not able to eradicate all pairs on the larger lattices. Nevertheless, in the following we shall make use of the combined algorithm. Turning to gauge fields taken out of an equilibrium ensemble, we find basically the same results. We found that the $\tau$ values depended slightly on the $\beta$ value of the gauge fields, as well as on the lattice size, but not on the number of (anti)vortices or spin waves left in the gauge fixed configuration.

For 2-d SU(2), the associated spin model is the O(4) non-linear sigma model. This model is asymptotically free and does not contain any topological objects such as instantons or (anti)vortices. Therefore, we do not expect to find any obstructions in fixing pure gauge to the Landau gauge. This is confirmed by figure 3.

Taking gauge field configurations from an equilibrium ensemble, and fixing these to the Landau...
We would like to find an observable \( f \) for the gauge transformation \( g_{ij} \) to distinguish between the different Gribov copies. This observable should be invariant under global spin rotations, i.e. \( f(g_{ij}) = f(g_{ij}h) \), \( h \in \text{SU}(2) \), since global spin rotations correspond to global gauge transformations \( U_\mu(x) \rightarrow U'_\mu(x) = h_\mu U_\mu(x)h \).

gauge, we at times do find Gribov copies.

These copies behave differently from the ones we found for 2-d U(1). The different copies relax with different autocorrelation times \( \tau \) to their minima. This is depicted in figure 4.

At fixed lattice size, it is observed that the relaxation time \( \tau \) on average becomes larger when \( \beta_g \) is increased. Furthermore, the relative occurrence of these copies becomes smaller; see table 1.

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The first observable to satisfy this criterion is the most obvious one: the (spin model) energy density
\[
e(x) = \sum_\mu \text{Tr} [g^\dagger(x)g(x + \hat{\mu}) + g^\dagger(x - \hat{\mu})g(x)].
\]
We use the parametrization
\[
g(x) = [g_4 \mathbb{I} - i \vec{\sigma} \cdot \vec{g}] (x),
\]
since it allows us to change conveniently from the O(4) or vector notation, to the SU(2) or matrix notation, and vise versa. The second observable is given by
\[
J_i(x) = \sum_i j_i^2(x),
\]
\[
j_i(x) = [\epsilon_{\mu\nu\epsilon_{ijkl}}g_j \partial_\mu g_k \partial_\nu g_l] (x).
\]

We have plotted \( 1 - \frac{1}{4}e(x) \) and \( J(x) \) for a gauge transformation taking one copy to the absolute minimum; see figure 5. Surprisingly, the two figures are almost identical in shape! The reason for this similarity is not (yet) understood and is being studied.

The reason for copies to occur is that once we have come close to a local minimum, there is no chance to get away from this minimum, due to the minimization algorithm. This immediately implies a way out: using an annealing algorithm. The implementation for gauge fixing is clear from (4): we perform Monte Carlo on the spins \( g \) using the action \( S_U(g) \) at some value of \( \beta_s \). We should start with a (very) low value of \( \beta_s \), and gradually increase \( \beta_s \) by small amounts \( \Delta \beta_s \). After increasing \( \beta_s \), we should let the system reach equilibrium before increasing \( \beta_s \) once more. When \( \beta_s \) is large

Table 1. The number of times a gauge configuration yielded \# copies for several combinations of the lattice size \( N \) and the gauge coupling \( \beta_g \). The \( \beta_g \) values are chosen to keep the ratio \( N/\xi_s \) roughly constant. \( \xi_s \) is the string tension correlation length.
enough, we can turn to ordinary relaxation, with or without overrelaxation. The starting value of $\beta_S$ and $\Delta \beta_S$, as well as the value of $\beta_S$ where we can turn to the normal minimization must be determined in practice, and may depend on $\beta_g$.

The interpretation of this process is straightforward. Making a random gauge transformation is (in some sense) taking a spin configuration at infinite temperature ($\beta \equiv T^{-1}$). Performing Monte Carlo at low $\beta_S$ is nothing more than letting the (spin) system evolve in time, deep into the high temperature phase. Increasing $\beta_S$ is then lowering the temperature, which should be done gradually especially near a phase transition. If we have been careful enough, we can throw the system into liquid helium once we are deep into the low temperature phase.

We used this annealing algorithm, using a multigrid algorithm as described in [6] for the Monte Carlo part, for both U(1) and SU(2) gauge fields. After tuning the aforementioned parameters, we found no copies, even for very rough gauge fields; $\beta = 24.0$ on a 128$^2$ lattice for SU(2), where $\xi_\sigma \simeq 4$.

We can draw several conclusions from this finding. Firstly, the copies for SU(2) are probably a relic from the spin model.

Secondly, the analogy with spin glasses, which sometimes is made to explain gauge fixing ambiguities, may be misleading. To be sure, it strictly holds for $\beta_g = 0$ only, since for $\beta_g \neq 0$ the couplings $U$ are constrained, as was mentioned before. Especially for larger values of $\beta_g$, we should expect that the local minima thin out, and become well separated. This is indeed observed, but surprisingly, for $\beta_g$ values in the range of $\beta_g = 6.0 - 192.0$ on 32$^2$ lattices, we find $\Delta F_\infty / F_\infty$ to be in the order of a few percent. $F_\infty$ denotes the final value of $F$ for a given configuration. This suggests that the deviation from a spin glass starts at lower values of $\beta_g$, and therefore, that the problem of finding the true Landau gauge is not as severe as finding the ground state of a spin glass. This fact encourages us to apply annealing algorithms to 4-d gauge fixing problems.

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REFERENCES
[1] C. Bernard, D. Murphy, A. Soni and K. Yee, Nucl. Phys. B (Proc. Suppl.) 17 (1990) 593.
[2] See, e.g. the contributions of Ph. de Forcrand and Th. DeGrand to these proceedings.
[3] T. Suzuki, these proceedings.
[4] V. N. Gribov, Nucl. Phys. B139 (1978) 1.
[5] Ph. de Forcrand, J. E. Hetrick, A. Nakamura and M. Plewnia, Nucl. Phys. B (Proc. Suppl.) 20 (1991) 194; A. Hulsebos, M. L. Laursen, J. Smit and A. J. van der Sijs, ibid., 20 (1991) 98; E. Marinari, R. Ricci and C. Parrinello, ibid., 20 (1991) 199.
[6] A. Hulsebos, J. Smit and J. C. Vink, Nucl. Phys. B356 (1991) 775; A. Hulsebos, M. L. Laursen and J. Smit, Phys. Lett. B291 (1992) 431.