ON THE NUMBER OF DISTINCT QUADRATIC FIELDS
GENERATED BY THE SHANKS SEQUENCE

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Abstract. Let \( g > 1 \) be an integer and \( f(X) \in \mathbb{Z}[X] \) a polynomial of positive degree with no multiple roots, and put \( u(n) = f(g^n) \). In this note, we study the sequence of quadratic fields \( \mathbb{Q}\left(\sqrt{u(n)}\right) \) as \( n \) varies over the consecutive integers \( M + 1, \ldots, M + N \). Fields of this type include Shanks fields and their generalizations. Using the square sieve together with new bounds on character sums, we improve an upper bound of Luca and Shparlinski (2009) on the number of \( n \in \{M + 1, \ldots, M + N\} \) with \( \mathbb{Q}\left(\sqrt{u(n)}\right) = \mathbb{Q}\left(\sqrt{s}\right) \) for a given squarefree integer \( s \).

1. Introduction

Let \( f(X) \in \mathbb{Z}[X] \) be a polynomial of degree \( d \geq 1 \) that has a positive leading coefficient and no multiple roots in its splitting field. Let \( g > 1 \) be a fixed integer, and put

\[
u(n) = f(g^n) \quad (n \geq 1).\tag{1.1}\]

Note that \( u(n) > 0 \) for all but finitely many \( n \). In this paper, we study those quadratic fields \( \mathbb{Q}\left(\sqrt{u(n)}\right) \) that arise as \( n \) varies over a sequence of consecutive integers.

Motivated by work of Shanks [17], which corresponds to the particular case in which \( u(n) = (2^n + 3)^2 - 8 \) and certain generalizations (see [15, 16, 18]), Luca and Shparlinski [12] have studied the distribution of the quadratic fields \( \mathbb{Q}\left(\sqrt{u(n)}\right) \) in the general setting of (1.1). To describe the results, let \( Q_u(M, N; s) \) be the number of integers \( n \in \{M + 1, \ldots, M + N\} \) for which \( \mathbb{Q}\left(\sqrt{u(n)}\right) = \mathbb{Q}\left(\sqrt{s}\right) \), where \( M, N, s \in \mathbb{Z} \) with \( N \geq 1 \) and \( s \) squarefree. Using the square sieve of Heath-Brown [7] along with some knowledge about prime divisors of shifted primes, in [12, Theorem 1.3] a nontrivial upper bound on \( Q_u(M, N; s) \) is given.

More precisely, let \( \tau_\ell \) denote the multiplicative order of \( g \) modulo a prime \( \ell \), that is, the smallest positive integer \( \tau \) for which \( g^\tau \equiv 1 \mod \ell \). Let \( \alpha_0 \) be a fixed real number for which one has a lower bound of the form

\[
\left|\{\ell \leq z : \ell \text{ is prime and } \tau_\ell \geq \ell^{\alpha_0}\}\right| \gg \frac{z}{\log z}\tag{1.2}
\]

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for all sufficiently large $z$, where the implied constant depends only on $\alpha_0$ (see §2.1 for the definitions of $\ll$, $\ll$ and other related symbols). We also let $\alpha$ a fixed real number for which one has a lower bound of the form

$$\left| \{ \ell \leq z : \ell \text{ is prime and } P^+(\ell - 1) \geq \ell^\alpha \} \right| \gg \frac{z}{\log z} \quad (1.3)$$

for all sufficiently large $z$, where $P^+(k)$ denotes the largest prime divisor of an integer $k \geq 2$, and the implied constant depends only on $\alpha$ Using a result of Baker and Harman [1] one can take

$$\alpha_0 \geq \alpha = 0.677, \quad (1.4)$$

and under the Extended Riemann Hypothesis (ERH) one can take any real number $\alpha_0 < 1$; see [4, 14]. It is straightforward to show that if $\alpha > 1/2$ is admissible for (1.3) then $\alpha_0 = \alpha$ is also admissible for (1.2); in fact, this is the only known approach for getting large values of $\alpha_0$ unconditionally. However, under the ERH every value $\alpha_0 < 1$ is admissible for (1.2) in a very strong sense (see Lemma 2.4 below), but the values of $\alpha$ are not improved under the ERH.

In the above notation, under natural conditions as in our Theorem 1.1 below, in [12, Theorem 1.3] it is shown that the bound

$$Q_u(M, N; s) \ll N^{\beta_0}(\log N)^{\gamma_0} \quad (1.5)$$

holds uniformly for all choices of $M, N, s$ as above, where

$$\beta_0 = \frac{3}{2(1 + \alpha_0)} \quad \text{and} \quad \gamma_0 = \frac{4 + \alpha_0}{1 + \alpha_0},$$

(in [12] the results are formulated in terms of $\alpha$, however the argument only depends on the parameter $\alpha_0$). Thus, using (1.4) we see that [12, Theorem 1.3] yields (1.5) unconditionally with

$$\beta_0 = 0.89445 \cdots \quad (1.6)$$

and conditionally (under ERH) with

$$\beta_0 = 0.75 + \varepsilon \quad (1.7)$$

for any fixed $\varepsilon > 0$.

**Theorem 1.1.** Let $g > 1$ be an integer, and let $f(X) \in \mathbb{Z}[X]$ be a polynomial of degree $\deg f \geq 3$ with a positive leading coefficient and no multiple roots in its splitting field. Put $u(n) = f(g^n)$. If $\alpha$ satisfies (1.3) then uniformly for $M, N, s \in \mathbb{Z}$ with $N \geq 1$ and $s$ squarefree, the upper bound

$$Q_u(M, N; s) \ll N^\beta(\log N)^\gamma$$

holds with

$$\beta = (2\alpha)^{-1} \quad \text{and} \quad \gamma = 2 - \alpha^{-1},$$

where the implied constant depends only on $f, g$ and $\alpha$. 
Consequently, in place of (1.6) we get the value
\[ \beta = 0.73855 \cdots, \]
which is unconditionally sharper than (1.7). Note that Theorem 1.1 immediately implies the lower bound \( cN^{1-\beta}(\log N)^{-\gamma} \) (with a constant \( c > 0 \) that depends on \( \alpha \)) for the number of distinct quadratic fields \( \mathbb{Q} \left( \sqrt{u(n)} \right) \) that arise as \( n \) varies over the interval \([M + 1, M + N] \).

Our next result improves the bound of Theorem 1.1 on average over \( s \). Let
\[
\mathcal{Q}_u(M, N; S) = \sum_{\substack{s \leq S \text{ sqfree}}} Q_u(M, N; s). \tag{1.8}
\]
The uniformity with respect to \( s \) in the bound of Theorem 1.1 implies that
\[
\mathcal{Q}_u(M, N; S) \ll SN^{1/(2\alpha) + o(1)} \quad (N \to \infty).
\]
This can be strengthened as follows.

**Theorem 1.2.** In the notation of Theorem 1.1 and of (1.8), we have
\[
\mathcal{Q}_u(M, N; S) \leq \left( SN^{1/(1+\alpha)} + S^{1-1/(4\alpha)} N^{1/(2\alpha)} \right) N^{o(1)}
\]
as \( N \to \infty \), where the function of \( N \) implied by \( o(1) \) depends only on \( f \), \( g \) and \( \alpha \).

Furthermore, using a slightly different approach we show that

**Theorem 1.3.** In the notation of Theorem 1.1 and of (1.8), we have
\[
\mathcal{Q}_u(M, N; S) \leq \left( S^{3/(3+\alpha)} N^{(3-\alpha)/(3+\alpha)} + S^{1/2} N^{(3-\alpha)/(1+3\alpha)} \right) N^{o(1)} \tag{1.9}
\]
as \( N \to \infty \), where the function of \( N \) implied by \( o(1) \) depends only on \( f \), \( g \) and \( \alpha \).

One verifies that the second term in the bound of Theorem 1.2 dominates the first one for small values of \( S \) and the switching point at \( S = N^{2(1-\alpha)/(1+3\alpha)} \) is above the value of \( S = N^{2(1-\alpha)/(1+3\alpha)} \), where the bound of Theorem 1.3 become stronger. Thus, straightforward calculations show that Theorems 1.2 and 1.3 can be combined into the following statement:

**Corollary 1.4.** In the notation of Theorem 1.1 and of (1.8), we have
\[
\mathcal{Q}_u(M, N; S)
\leq \begin{cases} 
S^{1-1/(4\alpha)} N^{1/(2\alpha) + o(1)} & \text{if } S \leq N^{2(1-\alpha)/(1+3\alpha)}, \\
S^{1/2} N^{(3-\alpha)/(1+3\alpha) + o(1)} & \text{if } N^{2(1-\alpha)/(1+3\alpha)} < S \leq N^{4(1-\alpha)/(1+3\alpha)}, \\
S^{3/(3+\alpha)} N^{(3-\alpha)/(3+\alpha) + o(1)} & \text{if } N^{4(1-\alpha)/(1+3\alpha)} < S \leq N^{2\alpha/3},
\end{cases}
\]
as \( N \to \infty \), where the functions of \( N \) implied by \( o(1) \) depend only on \( f \), \( g \) and \( \alpha \).
We note that the end point in the third range of Corollary 1.4 is unnecessary; it is given to indicate the largest value of $S$ for which we have an improvement over the trivial bound $\mathcal{Q}_u(M, N; S) \leq N$.

The proofs of Theorems 1.2 and 1.3 involve sums with Jacobi symbols over integers $s \in [1, S]$. We use a result of Heath-Brown [8, Corollary 3] to estimate such sums “on-average,” but on several occasions we need “individual” estimates, and in those instances we use only the trivial bound $S$ to estimate the sums. This does not involve any substantial sacrifice, however, since in the most interesting ranges these sums are shorter than the range covered by the Polya-Vinogradov and Burgess bounds (see [9, Theorems 12.5 and 12.6]). On the other hand, under the ERH, not only can we use any $\alpha < 1$ in (1.3), but we can also exploit a square root cancellation in character sums (see Lemma 3.6 below), which leads to a much better estimate.

**Theorem 1.5.** Under the ERH, in the notation of Theorem 1.1 and of (1.8), we have

$$\mathcal{Q}_u(M, N; S) \leq S^{1/2} N^{1/(1+\alpha)+o(1)}$$

as $N \to \infty$, where the function of $N$ implied by $o(1)$ depends only on $f$, $g$ and $\alpha$.

Examining the proof of Theorem 1.5 we see that it also yields the bound

$$Q_u(M, N; s) \ll N^{1/(1+\alpha)+o(1)},$$

which with $\alpha$ as in (1.4) becomes

$$Q_u(M, N; s) \ll N^{1000/1677+o(1)};$$

this improves (1.7) and the unconditional bound of Theorem 1.1 (note that $1000/1677 = 0.596302\ldots$).

We remark that Cutter, Granville and Tucker [3, Theorems 1A and 1B] have obtained an asymptotic formula for the number of distinct fields of the form $\mathbb{Q}\left(\sqrt[\lambda(n)]{f(n)}\right)$ with $n = 1, \ldots, N$, where $f(X) \in \mathbb{Z}[X]$ is a given polynomial of degree at most two. For polynomials $f(X)$ of degree three or more, a conditional asymptotic formula based on the ABC-conjecture is given in [3, Theorem 1C]; see also [11].

2. Preliminaries

2.1. **General notation.** For an odd integer $m$, we use $\left(\frac{k}{m}\right)$ to denote the Jacobi symbol of $k$ modulo $m$.

We also use $\varphi(k)$ to denote the Euler function of an integer $k \geq 1$.

As usual, we write $e(t) = \exp(2\pi it)$ for all $t \in \mathbb{R}$.

For a fixed nonzero integer $\lambda$ and any integer $m$ that is coprime to $\lambda$, we use $\tau_m(\lambda)$ to denote the multiplicative order of $\lambda$ modulo $m$, that is, the smallest positive integer $\tau$ for which $g^\tau \equiv 1 \mod p$. 
Throughout the paper, we use the symbols $O$, $o$, $\ll$, $\gg$ and $\asymp$ along with their standard meanings; any constants or functions implied by these symbols may depend on the fixed polynomial $f(X) \in \mathbb{Z}[X]$ or the parameter $\alpha$ but are independent of other variables except where indicated.

2.2. Auxiliary results. In §4 we use the following technical lemma; for a proof, see Graham and Kolesnik [5, Lemma 2.4].

**Lemma 2.1.** Let

$$B(z) = \sum_{j=1}^{J} A_j z^{B_j} + \sum_{k=1}^{K} C_k z^{-D_k},$$

where $A_j, B_j, C_k, D_k > 0$. For any $z_2 \geq z_1 > 0$ there exists $z \in [z_1, z_2]$ such that

$$B(z) \ll \sum_{j=1}^{J} \sum_{k=1}^{K} T_{jk} z^{B_j} + \sum_{j=1}^{J} A_j z^{B_j} + \sum_{k=1}^{K} C_k z^{-D_k},$$

where

$$T_{jk} = \left(\frac{A_j D_k C_j}{C_k}ight)^{1/(B_j + D_k)} \quad (1 \leq j \leq J, 1 \leq k \leq K).$$

As usual, we use $\pi(t; m, a)$ to denote the number of primes $p \leq t$ for which $p \equiv a \mod m$. We apply the Brun-Titchmarsh theorem in the following relaxed form (see [9, Theorem 6.6] for a much more precise statement).

**Lemma 2.2.** Fix $\eta > 0$. For any real number $t \geq 2$ and integer $m \leq t^{1-\eta}$, we have

$$\pi(t; m, a) \ll \frac{t}{\varphi(m) \log t},$$

where the implied constant depends only on $\eta$.

We also need the following estimate.

**Lemma 2.3.** For any real number $t \geq 2$, we have

$$\sum_{n \leq t} \frac{n}{\varphi(n)^2} \ll \log t.$$ 

**Proof.** If $f$ is any multiplicative function satisfying

$$0 \leq f(p^k) = 1 + \frac{c}{p} + O(p^{-2}) \quad (p \text{ prime, } k \geq 1),$$

(2.1) where the constant $c$ and that implied by the $O$-symbol depend only on $f$, then as a special case of the well known theorem of Wirsing [19] one sees that $\sum_{n \leq t} f(n) \ll t$. Applying this result with the function $f(n) = n^2/\varphi(n)^2$ (which verifies (2.1) with $c = 2$), we deduce the bound

$$\sum_{n \leq t} \frac{n^2}{\varphi(n)^2} \ll t.$$
The stated result follows from this by partial summation. \qed

One can easily obtain an asymptotic formula for the sum in Lemma 2.3, but the upper bound is quite sufficient for our purposes here.

2.3. **Multiplicative orders.** We recall the following result of Erdős and Murty [4, Theorem 4] in a slightly weakened form.

**Lemma 2.4.** Under the ERH, for any fixed integer \( g > 1 \) the inequality \( \tau_\ell(g) > \ell/\log \ell \) holds for all primes \( \ell \leq z \) with at most \( o(z/\log z) \) exceptions as \( z \to \infty \).

From this we immediately derive the next statement.

**Corollary 2.5.** Under the ERH, for any fixed integer \( g > 1 \) and any fixed \( \alpha > 1/2 \) which is admissible for (1.3) we have

\[
\left| \{ \ell \leq z : \ell \text{ is prime } \tau_\ell(g) > \ell/\log \ell \text{ and } P^+(\ell - 1) \geq \ell^\alpha \} \right| \gg \frac{z}{\log z} \quad (z \to \infty).
\]

3. **Character sums**

3.1. **Bounds on character sums with exponential functions.** In this section only, we write \( \tau_\ellm \) for \( \tau_m(\lambda) \) to simplify the notation.

For the proof of Theorem 1.1, we need some bounds for character sums.

We use the following variant of the result of Korobov [10, Theorem 3]. We present it here in a simplified form which is suited to our applications and can be extended in several directions.

**Lemma 3.1.** Let \( f(X) \in \mathbb{Z}[X] \), and let \( \lambda \in \mathbb{Z}, \lambda \neq 0 \). Let \( \ell, p \) be distinct primes with

\[
gcd(\ell p, \lambda) = gcd(\tau_\ell, \tau_p) = 1.
\]

For any integer \( a \) we define integers \( a_\ell \) and \( a_p \) by the conditions

\[
a_\ell \tau_\ell + a_p \tau_p \equiv a \mod \tau_\ell \tau_p, \quad 0 \leq a_\ell < \tau_\ell, \quad 0 \leq a_p < \tau_p.
\]

Then,

\[
\sum_{n=1}^{\tau_p} \left( \frac{f(\lambda^n)}{\ell p} \right) e(an/\tau_\ell \tau_p) = \sum_{x=1}^{\tau_\ell} \left( \frac{f(\lambda^x)}{\ell} \right) e(ax/\tau_\ell) \sum_{y=1}^{\tau_p} \left( \frac{f(\lambda^y)}{p} \right) e(a_p y/\tau_p).
\]

**Proof.** We follow closely the proof of [10, Theorem 3]. Using the coprimality condition \( gcd(\tau_\ell, \tau_p) = 1 \) we see that the integers

\[
x \tau_p + y \tau_\ell, \quad 0 \leq x < \tau_\ell, \quad 0 \leq y < \tau_p,
\]

run through the complete residue system modulo \( \tau_\ell \tau_p = \tau_\ell \tau_p \). Moreover,

\[
\lambda^{x \tau_\ell + y \tau_p} \equiv \lambda^{x \tau_\ell} \mod \ell, \quad \lambda^{x \tau_\ell + y \tau_p} \equiv \lambda^{y \tau_\ell} \mod p,
\]
and
\[ e(a(x\tau_p + y\tau_l)/\tau_{lp}) = e(ax/\tau_l) e(ay/\tau_p). \]

Hence, using the multiplicativity of the Jacobi symbol, we have
\[ \sum_{n=1}^{\tau_{lp}} \left( \frac{f(\lambda^n)}{\ell p} \right) e(an/\tau_{lp}) \]
\[ = \sum_{x=1}^{\tau_l} \sum_{y=1}^{\tau_p} \left( \frac{f(\lambda^{x\tau_p + y\tau_l})}{\ell} \right) \left( \frac{f(\lambda^{x\tau_p + y\tau_l})}{p} \right) e(a(x\tau_p + y\tau_l)/\tau_{lp}) \]
\[ = \sum_{x=1}^{\tau_l} \left( \frac{f(\lambda^{x\tau_p})}{\ell} \right) e(ax/\tau_l) \sum_{y=1}^{\tau_p} \left( \frac{f(\lambda^{y\tau_l})}{p} \right) e(ay/\tau_p). \]

Replacing \( x \) with \( x\tau_p^{-1} \mod \tau_l \) and \( y \) with \( y\tau_l^{-1} \mod \tau_p \), and taking into account that \( a\tau_p^{-1} \equiv a_l \mod \tau_l \) and \( a\tau_l^{-1} \equiv a_p \mod \tau_p \), the result follows.

The next statement follows immediately from the Weil bound on sums with multiplicative characters; see, for example, [9, Theorem 11.23].

**Lemma 3.2.** Let \( f(X) \in \mathbb{Z}[X] \) be monic of degree \( d \geq 1 \) with no multiple roots in its splitting field, and let \( \lambda \in \mathbb{Z} \), \( \lambda \neq 0 \). For any prime \( p \) coprime to \( \lambda f(0) \) and any integer \( a \), we have
\[ \sum_{x=1}^{\tau_p} \left( \frac{f(\lambda x)}{p} \right) e(ax/\tau_p) \ll p^{1/2}. \]

**Proof.** Denoting \( s = (p - 1)/\tau_p \), we can write \( \lambda = \vartheta^s \) with some primitive root \( \vartheta \) modulo \( p \). Then,
\[ \sum_{x=1}^{\tau_p} \left( \frac{f(\lambda x)}{p} \right) e(ax/\tau_p) = \frac{1}{\tau_p} \sum_{x=1}^{p-1} \left( \frac{f(\vartheta^{sx})}{p} \right) e(asx/(p-1)) \]
\[ = \frac{1}{s} \sum_{w=1}^{p-1} \left( \frac{f(w^s)}{p} \right) \chi(w), \]
where \( \chi \) is the multiplicative character modulo \( p \) defined by
\[ \chi(w) = e(asx/(p-1)) \quad (w \in \mathbb{Z}, \ p \nmid w), \]
where \( x \) is any integer for which \( w \equiv \vartheta^x \mod p \).

Let \( g(X) = f'(X) \) be the derivative of \( f \). Since \( f(X) \) has no multiple roots, \( f(0) \neq 0 \mod p \), and
\[ \frac{d}{dX} f(X^s) = sX^{s-1}g(X^s), \]
we see that \( f(X^s) \) has no multiple roots (and zero is not a root of \( f(X^s) \)). Thus, the Weil bound in the form given by [9, Theorem 11.23] shows that
\[ \sum_{w=1}^{p-1} \left( \frac{f(w^s)}{p} \right) \chi(w) \ll p^{1/2}. \]
Using (3.1) the result now follows. □

Combining Lemmas 3.1 and 3.2, we obtain the following statement.

**Lemma 3.3.** Let \( f(X) \in \mathbb{Z}[X] \) be monic of degree \( d \geq 1 \) with no multiple roots in its splitting field, and let \( \lambda \in \mathbb{Z}, \lambda \neq 0 \). Let \( \ell, p \) be distinct primes with
\[
\gcd(\ell p, \lambda f(0)) = \gcd(\tau_\ell, \tau_p) = 1.
\]
Then,
\[
\sum_{n=1}^{\tau_p} \left( \frac{f(\lambda^n)}{\ell p} \right) e(an/\tau_p) \ll (\ell p)^{1/2}.
\]

Using Lemma 3.3 we derive the following statement, which is our principal technical tool; this result improves upon [12, Lemma 4.1] but requires that the additional coprimality condition \( \gcd(\tau_\ell, \tau_p) = 1 \) is met; see also [2, 20] for some similar bounds with linear polynomials.

**Lemma 3.4.** Let \( f(X) \in \mathbb{Z}[X] \) be monic of degree \( d \geq 1 \) with no multiple roots in its splitting field, and let \( \lambda \in \mathbb{Z}, \lambda \neq 0 \). Let \( \ell, p \) be distinct primes with
\[
\gcd(\ell p, \lambda f(0)) = \gcd(\tau_\ell, \tau_p) = 1.
\]
Then, for any integer \( A \) with \( \gcd(A, \ell p) = 1 \) and \( K \geq 1 \), we have
\[
\sum_{n=1}^{K} \left( \frac{f(A\lambda^n)}{\ell p} \right) \ll \frac{K(\ell p)^{1/2}}{\tau_p} + (\ell p)^{1/2} \log(\ell p).
\]

The proof of Lemma 3.4 (which we omit) uses Lemma 3.3 in conjunction with the standard technique of deriving bounds on incomplete sums from bounds on complete sums; see, for example, [9, §12.2].

### 3.2. Sums with real characters

We need the following bound for character sums “on average” over squarefree moduli, which is due to Heath-Brown; see [8, Corollary 3].

**Lemma 3.5.** For positive integers \( R, S \) and a function \( \psi : \mathbb{R}_{>0} \to \mathbb{C} \) we have
\[
\sum_{m \leq R \atop m \text{ odd squarefree}} \left| \sum_{s \leq S} \psi(s) \left( \frac{s}{m} \right) \right|^2 \leq S(R + S)(RS)^{o(1)} \sum_{1 \leq s \leq S} |\psi(s)|^2
\]
as \( \max\{R, S\} \to \infty \), where the function of \( R, S \) implied by \( o(1) \) depends only on \( \psi \).

### 3.3. Character sums under the ERH

Under the ERH we have the following well-known estimate (see [13, §1]; it can also be derived from [6, Theorem 2]).
Lemma 3.6. For integers \( q > k \geq 1 \) and a primitive character \( \chi \) modulo \( q \), the bound
\[
\left| \sum_{n \leq k} \chi(n) \right| \leq k^{1/2} q^{o(1)}
\]
holds, where the function implied by \( o(1) \) depends only on \( k \).

4. The proofs

4.1. Proof of Theorem 1.1. Initially, we follow the proof of [12, Theorem 1.3]. For every \( \alpha \) satisfying (1.3), there are constants \( c > 0 \) and \( C > 1 \) depending only on \( \alpha \) with the following property. For every sufficiently large real number \( z > 1 \), there is a set \( \mathcal{L}_z \) containing at least \( cz/\log z \) primes \( \ell \in [z, Cz] \) for which
\[
\tau_\ell(g) \geq P^+(\ell - 1) \geq z^\alpha \quad (\ell \in \mathcal{L}_z);
\]
see [12, Lemma 5.1]. Let \( \alpha, c, C \) be fixed in what follows; we can assume that \( \alpha > \frac{1}{2} \), as even the value (1.4) is admissible. We also assume that \( z \) is large enough so that the aforementioned property holds.

Let \( \omega_z(k) \) be the number of distinct prime factors \( \ell \) of \( k \) that lie in \( \mathcal{L}_z \). Note that if \( k \geq 1 \) is a perfect square, then we always have
\[
\sum_{\ell \in \mathcal{L}_z} \left( \frac{k}{\ell} \right) = |\mathcal{L}_z| - \omega_z(k).
\]

Let \( \mathcal{M}_z \) be the set of integers \( n \in [M + 1, M + N] \) such that \( \omega_z(u(n)) \leq \frac{1}{2} |\mathcal{L}_z| \), and let \( \mathcal{E}_z \) be the set of remaining integers \( n \in [M + 1, M + N] \). The following bound is [12, Equation (6.1)]:
\[
|\mathcal{E}_z| \ll Nz^{-\alpha} + \log z. \tag{4.1}
\]

For a fixed squarefree \( s \geq 1 \), let \( \mathcal{N}_{s,z} \) denote the set of \( n \in \mathcal{M}_z \) for which \( Q\left(\sqrt{u(n)}\right) = Q(\sqrt{s}) \). For such \( n \), it is clear that \( su(n) \) is a perfect square, hence \( s \mid u(n) \), and \( \omega_z(su(n)) = \omega_z(u(n)) \leq \frac{1}{2} |\mathcal{L}_z| \). Thus, for every \( n \in \mathcal{N}_{s,z} \) we have
\[
\sum_{\ell \in \mathcal{L}_z} \left( \frac{su(n)}{\ell} \right) = |\mathcal{L}_z| - \omega_z(su(n)) \geq \frac{1}{2} |\mathcal{L}_z|.
\]

This implies that
\[
|\mathcal{N}_{s,z}| \leq \frac{2}{|\mathcal{L}_z|} \sum_{n \in \mathcal{M}_z} \left| \sum_{\ell \in \mathcal{L}_z} \left( \frac{su(n)}{\ell} \right) \right|^2. \tag{4.2}
\]

Using (4.1) and extending the summation in (4.2) to all \( n \in [M + 1, M + N] \) we deduce that
\[
Q_u(M, N; s) \ll Nz^{-\alpha} + \log z + \frac{\log z}{z^2} \sum_{n=M+1}^{M+N} \left| \sum_{\ell \in \mathcal{L}_z} \left( \frac{su(n)}{\ell} \right) \right|^2. \tag{4.3}
\]
Squaring out the right side and estimating the contribution from diagonal terms \( \ell = p \) trivially as \( N \), we have
\[
\sum_{n=M+1}^{M+N} \left| \sum_{\ell \in \mathcal{L}_z} \left( \frac{su(n)}{\ell} \right) \right|^2 = \sum_{\ell \neq p} \sum_{n=M+1}^{M+N} \left( \frac{su(n)}{\ell p} \right) \ll N|\mathcal{L}_z| + W,
\]
where
\[
W = \sum_{\ell \neq p} \sum_{n=M+1}^{M+N} \left( \frac{su(n)}{\ell p} \right).
\]
Since \( \alpha < 1 \), the contribution to (4.3) coming from diagonal terms, namely,
\[
N|\mathcal{L}_z| \frac{(\log z)^2}{z^2} \ll Nz^{-1} \log z,
\]
is dominated by the term \( Nz^{-\alpha} \) and so can be dropped. This yields the bound
\[
Q_u(M, N; s) \ll Nz^{-\alpha} + \log z + \frac{(\log z)^2}{z^2} W, \tag{4.4}
\]
which is a slightly simplified version of [12, Equation (6.4)].

Turning to the estimation of \( W \), we now write \( W = U + V \), where
\[
U = \sum_{\ell \neq p} \sum_{n=M+1}^{M+N} \left( \frac{su(n)}{\ell p} \right),
\]
\[
V = \sum_{\ell \neq p} \sum_{n=M+1}^{M+N} \left( \frac{su(n)}{\ell p} \right).
\]
To estimate \( U \), we use the trivial bound \( N \) on each inner sum, deriving that
\[
U \ll N \sum_{\ell \neq p} \sum_{n=M+1}^{M+N} 1 \ll N \sum_{r \geq z^{\alpha}} \sum_{\ell \equiv \ell_0 \mod r} 1 \ll N \sum_{r \geq z^{\alpha}} \frac{z^2}{r^2} \ll Nz^{2-\alpha}. \tag{4.5}
\]
To estimate \( V \), we first observe that the inequality \( \tau_p(g) \geq P^+(p-1) \geq p^\alpha \) implies (since \( \alpha > \frac{1}{2} \)) that \( P^+(\ell-1) \mid \tau_p(g) \) for every \( p \in \mathcal{L}_z \).

Fix a pair \( (\ell, p) \in \mathcal{L}_z \times \mathcal{L}_z \) with \( P^+(\ell-1) \neq P^+(p-1) \), and put
\[
h = \gcd(\tau_\ell(g), \tau_p(g)) \quad \text{and} \quad \lambda = g^h.
\]
It is easy to check that \( \lambda \) satisfies the conditions of Lemma 3.4 with
\[
\tau_\ell(\lambda) = \tau_\ell(g)/h \geq P^+(\ell-1) \quad \text{and} \quad \tau_p(\lambda) = \tau_p(g)/h \geq P^+(p-1).
\]
Furthermore, taking $K = \lfloor N/h \rfloor$ we obtain that
\[
\sum_{n=M+1}^{M+N} \left( \frac{su(n)}{lp} \right) = \left( \frac{s}{lp} \right) \sum_{j=1}^{h} \sum_{k=1}^{K} \left( \frac{u(M + k\ell + j)}{lp} \right) + O(h)
\]
\[
= \left( \frac{s}{lp} \right) \sum_{j=1}^{h} \sum_{k=1}^{K} \left( \frac{f(g^{M+j\ell}k)}{lp} \right) + O(h).
\]

Applying Lemma 3.4, we derive that
\[
\sum_{n=M+1}^{M+N} \left( \frac{su(n)}{lp} \right) \ll h \left( \frac{K(\ell p)^{1/2}}{\tau_{\ell p}(\lambda)} + (\ell p)^{1/2} \log(\ell p) \right) + h
\]
\[
\ll \frac{N(\ell p)^{1/2}}{\tau_{\ell p}(\lambda)} + h(\ell p)^{1/2} \log(\ell p).
\]

Hence
\[
\sum_{n=M+1}^{M+N} \left( \frac{su(n)}{lp} \right) \ll \frac{N(\ell p)^{1/2}}{P^+(\ell - 1)P^+(p - 1)} + h(\ell p)^{1/2} \log(\ell p)
\]
\[
\ll Nz^{1-2\alpha} + hz \log z.
\]

Since
\[
h = \gcd(\tau_{\ell}(g), \tau_{p}(g)) \ll \gcd(\ell - 1, p - 1),
\]
we have therefore shown that
\[
\sum_{n=M+1}^{M+N} \left( \frac{su(n)}{lp} \right) \ll Nz^{1-2\alpha} + \gcd(\ell - 1, p - 1)z \log z. \tag{4.7}
\]

Combining the bound (4.7) with the definition of $V$, we have
\[
V \ll Nz^{3-2\alpha}(\log z)^{-2} + Tz \log z,
\]
where
\[
T = \sum_{\ell, p \in \mathbb{Z}_+ \atop P^+(\ell - 1) \neq P^+(p - 1)} \gcd(\ell - 1, p - 1).
\]

For each pair $(\ell, p)$ in this sum, the primes $\ell_0 = P^+(\ell - 1)$ and $p_0 = P^+(p - 1)$ are distinct and satisfy $\min\{\ell_0, p_0\} \geq z^\alpha$. Writing $\ell - 1 = \ell_0m_0$ and $p - 1 = p_0n_0$, it follows that
\[
\gcd(\ell - 1, p - 1) \leq \min\{m_0, n_0\} \leq Cz^{1-\alpha}. \tag{4.8}
\]

Thus, using Lemmas 2.2 and 2.3 we have
\[
T \ll \sum_{m \leq Cz^{1-\alpha}} m \sum_{\substack{\ell, p \in \mathbb{Z}_+ \\ \ell \equiv 1 \mod m \atop p \equiv 1 \mod m}} 1 \ll \sum_{m \leq Cz^{1-\alpha}} m \left( \frac{z}{\varphi(m) \log z} \right)^2 \ll z^2(\log z)^{-1}. \tag{4.9}
\]

Thus,
\[
V \ll Nz^{3-2\alpha}(\log z)^{-2} + z^3. \tag{4.10}
\]
Comparing (4.5) and (4.10), we see that the bound for $V$ always dominates (since $\alpha < 1$); hence, as $W = U + V$ we have

$$W \ll N z^{3-2\alpha} (\log z)^{-2} + z^3.$$ 

Inserting this bound into (4.4) and removing negligible terms, it follows that

$$Q_u(M, N; s) \ll N z^{1-2\alpha} + z (\log z)^2.$$ 

Choosing $z = N^{1/(2\alpha)} (\log N)^{-1/\alpha}$ we obtain the stated result.

4.2. Proof of Theorem 1.2. We proceed as in the proof of Theorem 1.1. In particular, since the sets $N_{s,z}$ are disjoint, using (4.1) and (4.2) we derive the following analogue of (4.3):

$$Q_u(M, N; S) \ll N z^{-\alpha} + \log z + \frac{(\log z)^2}{z^2} \sum_{s \in S} \sum_{n = M+1}^{M+N} \left| \sum_{\ell \in \mathbb{L}_z} \left( \frac{su(n)}{\ell p} \right) \right|^2.$$ 

Thus, extending the summation to all integer $s \in [1, S]$, we obtain the following analogue of (4.4):

$$Q_u(M, N; S) \ll N z^{-\alpha} + \log z + \frac{(\log z)^2}{z^2} \mathcal{W}.$$ 

Here, $\mathcal{W} = \mathcal{U} + \mathcal{V}$ with

$$\mathcal{U} = \sum_{\ell, p \in \mathbb{L}_z, \ell \neq p} \sum_{s \in S} \sum_{n = M+1}^{M+N} \left( \frac{su(n)}{\ell p} \right),$$

$$\mathcal{V} = \sum_{\ell, p \in \mathbb{L}_z} \sum_{s \in S} \sum_{n = M+1}^{M+N} \left( \frac{su(n)}{\ell p} \right).$$

We bound $\mathcal{U}$ trivially as in the proof of Theorem 1.1:

$$\mathcal{U} \ll S N z^{2-\alpha}.$$ 

Next, we estimate $\mathcal{V}$. Using (4.7) we have

$$\mathcal{V} \ll \sum_{\ell, p \in \mathbb{L}_z} \left| \sum_{s \in S} \left( \frac{s}{\ell p} \right) \right| \sum_{n = M+1}^{M+N} \left( \frac{u(n)}{\ell p} \right)$$

$$\ll N z^{1-2\alpha} \mathcal{V}_1 + z^{1+o(1)} \mathcal{V}_2 \quad (z \to \infty),$$

(4.13)
where
\[
\mathcal{V}_1 = \sum_{\ell \in \mathcal{L}_z} \left| \sum_{s \leq S} \left( \frac{s}{\ell p} \right) \right|,
\]
\[
\mathcal{V}_2 = \sum_{\ell \in \mathcal{L}_z} \gcd(\ell - 1, p - 1) \cdot \left| \sum_{s \leq S} \left( \frac{s}{\ell p} \right) \right|.
\]

To bound \( \mathcal{V}_1 \) we apply the Cauchy inequality and Lemma 3.5, deriving that
\[
\mathcal{V}_1 \leq \sqrt{z^{2+o(1)} \cdot S(z^2 + S)(Sz)\alpha(1)} = (S^{1/2}z^2 + Sz)(S\alpha) \ (z \to \infty). \tag{4.14}
\]
For \( \mathcal{V}_2 \) we bound the sum over \( s \) trivially as \( O(S) \) and use (4.9), obtaining
\[
\mathcal{V}_2 \leq S^{2+o(1)} \ (z \to \infty). \tag{4.15}
\]
Inserting the bounds (4.14) and (4.15) into (4.13) we have
\[
\mathcal{V} \leq (S^{1/2}z^{3-2\alpha} + SNz^{2-2\alpha} + Sz^3)(S\alpha) \ (z \to \infty). \tag{4.16}
\]
The right side of (4.12) dominates the second term on the right side of (4.16); hence, as \( \mathcal{W} = \mathcal{U} + \mathcal{V} \), we have
\[
\mathcal{W} \leq (S^{1/2}Nz^{3-2\alpha} + SNz^{2-\alpha} + Sz^3)(S\alpha) \ (z \to \infty). \tag{4.17}
\]
After inserting this bound in (4.11) we derive that
\[
\mathcal{Q}_u(M, N; S) \leq (Sz + S^{1/2}Nz^{1-2\alpha} + SNz^{-\alpha})(S\alpha) \ (z \to \infty) \tag{4.17}
\]
(here we have changed the order of terms to make (4.17) readily available for an application of Lemma 2.1 with \( J = 1 \) and \( K = 2 \)). Noting that (4.17) is trivial for \( z \leq \log N \), we now apply Lemma 2.1 with \( z_1 = \log N \) and with a very large value of \( z_2 \) (for example \( z_2 = (SN)^{100} \) so that the single sums are always dominated by the double sum. Since \( z \to \infty \) as \( N \to \infty \), this yields the bound
\[
\mathcal{Q}_u(M, N; S) \leq (T_{11} + T_{12})(SN)\alpha(1) \ (N \to \infty),
\]
where
\[
T_{11} = (S^{2\alpha-1}(S^{1/2}N))^{1/(2\alpha)} = S^{1-1/(4\alpha)}N^{1/(2\alpha)},
\]
\[
T_{12} = (S^\alpha(SN))^{1/(1+\alpha)} = SN^{1/(1+\alpha)}.
\]
We also remark that for \( S > N \) the result is trivial, so we can replace \((SN)^\alpha(1)\) with \(N^{o(1)}\), and the proof is complete.

4.3. Proof of Theorem 1.3. We proceed as in the proof of Theorem 1.2, but we estimate the sum \( \mathcal{V}_2 \) in a different way. To simplify the notation, we now denote \( g(\ell, p) = \gcd(\ell - 1, p - 1) \) for all \( \ell, p \in \mathcal{L}_z \). As in (4.8) we have the bound
\[
g(\ell, p) \leq Cz^{1-\alpha} \quad (\ell, p \in \mathcal{L}_z, \ P^+(\ell - 1) \neq P^+(p - 1)).
\]
Hence, setting \( J = \lceil \log(Cz^{1-\alpha}) \rceil \) we have

\[
\mathcal{V}_2 \leq \sum_{\nu=0}^{J} \sum_{\ell, p \leq z, \ell \neq p, e^\nu < g(\ell, p) \leq e^{\nu+1}} g(\ell, p) \left| \sum_{s \leq S \text{ sqfree}} \left( \frac{s}{\ell p} \right) \right| \ll \sum_{\nu=0}^{J} e^{\nu} \mathcal{V}_{2,\nu},
\]

(4.18)

where

\[
\mathcal{V}_{2,\nu} = \sum_{\ell, p \leq z, \ell \neq p, g(\ell, p) > e^\nu} \left| \sum_{s \leq S} \left( \frac{s}{\ell p} \right) \right|.
\]

Trivially,

\[
\sum_{\ell, p \leq z, g(\ell, p) > e^\nu} 1 \ll \sum_{m > e^\nu} \sum_{\ell, p \leq z} 1 \ll \sum_{m > e^\nu} \left( \frac{z}{m} \right)^2 \ll z^2 e^{-\nu}.
\]

Hence, using the Cauchy inequality and Lemma 3.5 we see that

\[
\mathcal{V}_{2,\nu}^2 \ll z^2 e^{-\nu} \left| \sum_{\ell, p \leq z} \sum_{s \leq S} \left( \frac{s}{\ell p} \right) \right|^2 \ll z^2 e^{-\nu} \cdot (S^2 z^2 + S^2) (S z)^{o(1)} \quad (z \to \infty),
\]

and therefore

\[
\mathcal{V}_{2,\nu} \ll e^{-\nu/2} (S^{1/2} z^2 + S z) (S z)^{o(1)} \quad (z \to \infty).
\]

Substitution in (4.18) gives

\[
\mathcal{V} \ll e^{J/2} (S^{1/2} z^2 + S z) (S z)^{o(1)} \ll z^{1/2 - \alpha/2} (S^{1/2} z^2 + S z) (S z)^{o(1)}.
\]

(4.19)

Inserting the bounds (4.14) and (4.19) into (4.13) we derive that

\[
\mathcal{V} \leq (S^{1/2} N z^{-3/2} + SN z^{-2} - S z^{1/2} z^{-7/2} - S z^{5/2} - S z^{2} z^{-\alpha/2} + S z^{5/2} - S z^{2} z^{-\alpha/2}) (S z)^{o(1)}.
\]

(4.20)

The right side of (4.12) dominates the second term in the bound (4.20); hence, recalling that \( \mathcal{W} = \mathcal{U} + \mathcal{V} \), we see that

\[
\mathcal{W} \leq (S^{1/2} N z^{-3/2} + SN z^{-2} - S z^{1/2} z^{-7/2} - S z^{5/2} - S z^{2} z^{-\alpha/2} + S z^{5/2} - S z^{2} z^{-\alpha/2}) (S z)^{o(1)}.
\]

Inserting this bound into (4.11) we find that

\[
\mathcal{Q}_u(M, N; S) \leq (S^{1/2} z^{-3/2} - S z^{1/2} - S z^{-\alpha/2} + SN z^{-\alpha} + S z^{1/2} N z^{-2} z) (S z)^{o(1)}
\]

(4.21)

as \( z \to \infty \) (we have rearranged terms to make (4.21) readily available for an application of Lemma 2.1 with \( J = K = 2 \)).

As in the proof of Theorem 1.3, we note that (4.21) is trivial for \( z \leq \log N \). We now apply Lemma 2.1 with \( z_1 = \log N \), and a very large value of \( z_2 \), say \( z_2 = (SN)^{100} \), so that the single sums are dominated by the double sum. This gives

\[
\mathcal{Q}_u(M, N; S) \leq (T_{11} + T_{12} + T_{21} + T_{22}) (SN)^{o(1)} \quad (N \to \infty),
\]

(4.22)
where
\[
T_{11} = \left((S^{1/2})^\alpha (SN)^{(3-\alpha)/2}\right)^{2/(3+\alpha)} = S^{3/(3+\alpha)} N^{(3-\alpha)/(3+\alpha)},
\]
\[
T_{12} = \left((S^{1/2})^{2\alpha-1} (S^{1/2} N)^{(3-\alpha)/2}\right)^{2/(1+3\alpha)} = S^{1/2} N^{(3-\alpha)/(1+3\alpha)},
\]
\[
T_{21} = \left(S^\alpha (SN)^{(1-\alpha)/2}\right)^{2/(1+\alpha)} = SN^{(1-\alpha)/(1+\alpha)},
\]
\[
T_{22} = \left(S^{2\alpha-1} (S^{1/2} N)^{(3-\alpha-1)}\right)^{2/(3\alpha-1)} = S^{(7\alpha-3)/(6\alpha-2)} N^{(1-\alpha)/(3\alpha-1)}.
\]

Denoting
\[
\vartheta = \frac{(1-\alpha)^2 (1+3\alpha)}{(1+\alpha^2)(3\alpha - 1)},
\]
it is straightforward to check that \( \vartheta \in (0, 1) \) for any \( \alpha \in \left(\frac{1}{2}, 1\right) \), and also
\[
1 - \frac{\vartheta}{2} = \frac{-3 + 5\alpha + 3\alpha^2 + 3\alpha^3}{(6\alpha - 2)(1 + \alpha^2)} \geq \frac{7\alpha - 3}{6\alpha - 2}.
\]

Thus,
\[
T_{22} \leq S^{1-\vartheta/2} N^{(1-\alpha)/(3\alpha-1)} = T_{12}^\vartheta T_{21}^{1-\vartheta} \leq \max\{T_{12}, T_{21}\}.
\]

Hence, the term \( T_{22} \) can be omitted from (4.22).

Elementary calculations reveal that the inequality \( T_{21} \geq T_{11} \) cannot hold unless \( S \geq N \) (in fact, \( S \geq N^{4/(1+\alpha)} \)), in which case \( T_{11} \geq N^{(6-\alpha)/(3+\alpha)} \geq N \); but then our stated bound (1.9) is weaker than the trivial bound \( \mathcal{D}_u(M, N; S) \leq N \). Consequently, we can assume \( T_{21} \leq T_{11} \), and so the term \( T_{21} \) can be omitted from (4.22). Since we can also assume that \( S \leq N \) we can replace \( (SN)^{o(1)} \) with \( N^{o(1)} \) in (4.22), and this completes the proof.

4.4. Proof of Theorem 1.5. We proceed as in the proof of Theorem 1.2, but in place of \( \mathcal{L}_z \) we use the set \( \mathcal{Z}_z \) consisting of primes \( \ell \in [z, Cz] \) for which
\[
\tau_\ell(g) \geq \ell / \log \ell \quad \text{and} \quad P^+ (\ell - 1) \geq z^\alpha.
\]
By Corollary 2.5, we can assume that \( |\mathcal{Z}_z| \geq cz / \log z \). Another difference is that we estimate the sums over \( s \) directly via Lemma 3.6.

Thus, instead of the bound (4.12) we obtain
\[
\mathcal{U} \leq S^{1/2} N z^{2-\alpha + o(1)} \quad (z \to \infty).
\]

Next, we estimate \( \mathcal{V} \). First we apply Lemma 3.6 and derive
\[
\mathcal{V} \leq S^{1/2} z^{o(1)} \sum_{\ell, p \in \mathcal{Z}_z \atop P^+ (\ell - 1) \neq P^+ (p - 1)} \left| \sum_{n = M + 1}^{M+N} \left( \frac{u(n)}{\ell p} \right) \right| \quad (z \to \infty).
\]

Using the equation
\[
\tau_{\ell p}(\lambda) = \frac{\tau_\ell(g) \tau_p(g)}{h^2},
\]
where (as in the proof of Theorem 1.1)

\[ h = \gcd(\tau_\ell(g), \tau_p(g)) \leq \gcd(\ell - 1, p - 1), \]

and bearing in mind our choice of \( \mathcal{L}_z \), we can rewrite (4.6) as

\[
\sum_{n=M+1}^{M+N} \left( \frac{su(n)}{\ell p} \right) \ll \frac{N((\ell p)^{1/2}}{\tau_\ell(p)} + h((\ell p)^{1/2}\log(\ell p)) \\
\ll \frac{N((\ell p)^{1/2}h^2}{\tau_\ell(g)\tau_p(g)} + h((\ell p)^{1/2}\log(\ell p)) \\
= Nz^{-1+\alpha(1)} \gcd(\ell - 1, p - 1)^2 + z^{1+\alpha(1)} \gcd(\ell - 1, p - 1). \]

Hence, we obtain

\[ \mathcal{V} \ll S^{1/2} \left( NQz^{-1+\alpha(1)} + Tz^{1+\alpha(1)} \right), \quad (4.23) \]

where

\[
Q = \sum_{\ell, p \in \mathcal{L}_z} \gcd(\ell - 1, p - 1) \\
T = \sum_{\ell, p \in \mathcal{L}_z} \gcd(\ell - 1, p - 1). 
\]

Clearly, we can still use (4.8) with \( \ell, p \in \mathcal{L}_z \) with \( P^+(\ell - 1) \neq P^+(p - 1) \), and we have (4.9) as before. Also,

\[ Q \ll \sum_{m \leq Cz^{1-\alpha}} m^2 \sum_{\ell, p \in \mathcal{L}_z} 1 \ll \sum_{m \leq Cz^{1-\alpha}} m^2 \left( \frac{z}{m} \right)^2 \ll z^{3-\alpha}. \quad (4.24) \]

Substituting (4.9) and (4.24) in (4.23), we obtain that

\[ \mathcal{V} \ll S^{1/2} \left( Nz^{-2-\alpha+\alpha(1)} + z^{3+\alpha(1)} \right). \]

Hence

\[
\mathcal{Q}_h(M, N; S) \ll Nz^{-\alpha} + S^{1/2}Nz^{-\alpha+\alpha(1)} + S^{1/2}z^{1+\alpha(1)} \\
\ll S^{1/2}Nz^{-\alpha+\alpha(1)} + S^{1/2}z^{1+\alpha(1)}. 
\]

Taking \( z = N^{1/(1+\alpha)} \) to balance the two terms, we conclude the proof.

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