String Theory on Calabi–Yau Manifolds:

The Three Generations Case

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ABSTRACT

Recently, string theory on Calabi–Yau manifolds was constructed and was shown to be a fully consistent, space–time supersymmetric string theory. The physically interesting case is the case of three generations. Intriguingly, it appears at the present that there is a unique manifold which gives rise to three generations. We describe in this paper a full fledged string theory on this manifold in which the complete spectrum and all the Yukawa couplings can be computed exactly.

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String theory is a rarity among physical theories. For twenty years it has been developed without experimental input. Certainly, it is a beautifully consistent theory. Yet, it remained the theoretician playground.

The purpose of this note is to make a step towards the confrontation of string theory with experimental physics as a candidate theory for the unification of all natural forces. We do this by exploring the physically relevant case, the case of three generations, in the vast framework of the recently discovered [1, 2] string theory on Calabi–Yau manifolds.

String theories incorporate an elaborate and tightly woven structure with many consistency requirements. In particular, a superstring propagating in flat Minkowski space is a consistent theory only if this space is ten dimensional. The real world is however four dimensional.

A possible solution to this problem is to consider a world which is a manifold $M \times K$, where $M$ is the usual four dimensional Minkowski space and $K$ is some tiny, 'invisible', manifold, an idea first suggested in the late twenties by Kaluza and Klein [3]. The internal manifold $K$ would then give rise to the observable forces and the particular choice for the manifold has a profound influence on the physical predictions of the theory.

To implement the Kaluza–Klein idea in string theory, one needs to study string propagation on the manifold $M \times K$, where $K$ is some curved internal manifold. The first study of string propagation in curved space was described in [4], where a closed bosonic string theory on a manifold $K$, which is a Lie group manifold, was considered. As a result, string theory on a group manifold was shown to be a fully consistent, full fledged string theory, obeying all the severe constrains that string theory should obey. The constraints of the existence of sensible vertex operators, which are in one to one correspondence with the physical spectrum, unitarity at the tree level and at the one loop level, were all shown to be obeyed. Moreover, as a result of this work, it became clear that a full fledged string theory can be constructed, along the same lines, from any conformal field theory in two
For phenomenological reasons, one would actually like to study the case where the string theory has space–time supersymmetry. This symmetry between fermions and bosons enables the elimination of tachyons and the potential resolution of questions like the vanishing of the cosmological constant and the hierarchy problem.

The problem of the existence of supersymmetric string theory in curved space was open for a long time. Initial interest in the question came from the work of [5], in which the equations of classical ten dimensional supergravity were studied and were shown to have a solution, provided the manifold $K$ is a complex manifold of vanishing first Chern class (Calabi–Yau manifold). This gives initial indication that string theory on such a manifold might exist, since the same classical equations expresses the lowest order contribution to the conformal anomaly of the sigma model on the manifold [5], which describes string propagation on it. However, the tiniest conformal anomaly renders such a string theory inconsistent so one needs to study higher order contributions. Indeed, such a contribution in the four loop level was reported in [6], implying that the naive string theory on a Calabi Yau (CY) manifold is inconsistent. Arguments were suggested [7] that it may be possible to modify the metric and many workers in the field believed that, at least for large radius, a nearby string theory exists.

This is not the only problem. The existence of metrics of $SU(3)$ holonomy on Calabi Yau manifolds was conjectured by Calabi and proved by Yau [8]. However, no metric of a compact Calabi Yau manifold is known explicitly. The writing of such a metric is a hard mathematical problem, which even for the simplest $K3$ surface has been open for decades. Consequently, the propagation of a classical Newtonian particle on such a manifold is untractable, since one cannot compute distances. Thus, it appears to be entirely hopeless to compute the physical predictions from the seemingly much harder problem of the propagation of a quantized string on such a manifold.

In a recent series of papers [1, 2] the author has put forward the construction of
string theory on Calabi Yau manifolds and have shown them to be fully consistent, space–time supersymmetric string theories, where all the physical predictions can be computed exactly.

The new idea, which avoids all the aforementioned difficulties, is to proceed in two stages. At the first stage any possible geometrical interpretation was ignored and new space–time supersymmetric string theories were constructed from scratch [1], solving the stringent constraints that string theory must obey. At the second stage, by studying the massless spectrum of these new string theories and comparing it with the results of equivariant index theorems in particular Calabi Yau geometries, it was shown [2] that these theories correspond to string propagation on Calabi–Yau manifolds. The logic behind this procedure is that since string theory is a rarely constrained system, it should be possible to recover any such theory by simply considering string theory and its constraints, per se. This proves the consistency and existence of CY string theories, as well as giving their actual construction.

To carry out the first stage it was needed to understand how to construct space–time supersymmetric string theories. Previously, the only known method to get space–time supersymmetry in string theory was in the context of theories made entirely out of free fermions, along the lines of the GSO construction [9]. In order to get supersymmetry one needs some projection. The problem is that modular invariance almost always prevents one from projecting out any fields in a general conformal field theory. It is a kind of completeness condition. Thus it appears that supersymmetry cannot be achieved in general conformal field theory.

Surprisingly, I was able to show [1] that there is a very general supersymmetry projection, which works in consistency with modular invariance and can be implemented in any theory with $N = 2$ superconformal invariance, leading to space–time supersymmetry. The supersymmetry charge is given by $Q = \exp(i\phi)$, where $J = \partial_z \phi$ is the $U(1)$ current algebra part of the $N = 2$ superconformal algebra. In addition, one demands that the total $U(1)$ charge, in both the compactified
and uncompactified dimensions, should be an integer. The crucial point is that under the modular transformation $\tau \to -\frac{1}{\tau}$, these two conditions are exchanged and thus can be implemented simultaneously in any $N = 2$ superconformal field theory, without ruining modular invariance†. This new projection, the $G$ projection, then leads to space–time supersymmetry in any $N = 2$ superconformal field theory.

The next issue which needed to be addressed is how the left and right movers are correlated in the string theory. Again, the constraint of modular invariance is exceedingly restrictive. In general conformal field theory, in order to be able to achieve modular invariance, it is almost always required that the left movers and the right movers to be identical ‘half theories’. In a heterotic–like [12] string theory, where the left movers are fermionic and the right movers are bosonic, this presents a formidable problem, since the left and right movers, by definition, are completely different.

This problem is solved by a simple and completely general map which takes any superstring theory into a heterotic–like string theory [1]. A general superstring theory in $d + 2$ dimensions contains a $d$ dimensional flat superstring, described in the light–cone gauge by $d$ world sheet free fermions and $d$ world–sheet free bosons. The world–sheet fermions realize a level one $SO(d)$ current algebra, both in the right and in the left moving sectors. The map that takes a superstring into a heterotic string is then simply to replace the $SO(d)$ representations by $SO(24 + d)$ or $E_8 \times SO(8 + d)$ ones in the right moving sector, where one exchanges the vector by the singlet and changes the sign of the two spinors. The effect of this is to exchange the fermions that carry a space–time index with fermions that carry an internal index. This map preserves modular invariance and spin–statistics and thus sends any consistent superstring theory in $d + 2$ dimensions into a consistent heterotic string in $d + 2$ dimensions. In the case of space–time supersymmetric superstring

† The old GSO [9] projection is actually a particular case of the new G projection. The fact that the GSO projection can be written in terms of an $N = 2$ algebra is known for some time, and was used in orbifold calculations [10]. Preliminary observations that the $G$ projection might be possible were reported in [11].
compactification to four dimensions the resulting gauge groups are either \( SO(26) \) or \( E_8 \times E_6 \). The \( E_6 \) is obtained by combining \( SO(10) \) with the superconformal \( U(1) \). These gauge groups are the same as the ones obtained in the supergravity models on CY manifolds [5].

In [2] the following was demonstrated:

String theory on a Calabi–Yau manifold exists, it is a full fledged, fully consistent string theory, which is space–time supersymmetric. All string theories on a CY manifold have the structure described above. In addition, any string theory which has this structure is a string theory on some Calabi–Yau manifold.

More generally, this structure corresponds to a string compactification to \( 10 - 2k \) dimensions, with propagation on a manifold of \( SU(k) \) holonomy for the cases \( k = 1, 2, 3 \). Presently, the proof of the above statement is incomplete. A partial proof and additional conclusive evidence are presented in [13].

Our main tool in exploring this structure are the minimal \( N = 2 \) superconformal field theories. The reason is that these are the only presently known \( N = 2 \) superconformal field theories.\(^\dagger\) The minimal models have the trace anomaly

\[
c = \frac{3k}{k+2} \quad \text{for } k = 1, 2, \ldots, \infty.
\]  

The primary fields in the minimal models are labeled by three integers for the left movers, \( l, q, \) and \( s \), which obeys \( 0 \leq l \leq k \), \( q \) which is defined modulo \( 2(k+2) \) and \( s \) which labels the sector and is defined modulo \( 4 \). In addition, the right movers carry another set of such quantum numbers. We denote such a primary field by \( \Phi_{l,q,s,\bar{l},\bar{q},\bar{s}} \). The dimension and charge of this field are then

\[
\Delta = \frac{l(l+2) - q^2}{4(k+2)} + \frac{s^2}{8} + \text{integer}, \quad Q = -\frac{q}{k+2} + \frac{s}{2} + 2(\text{integer}).
\]  

One can get the correct total trace anomaly by an arbitrary tensoring of these models. The number of possibilities for consistent string theories is enormous, at

\(^\dagger\) Except of course for the trivial realization in terms of free fermions. This realization corresponds to the case of flat tori [14] and their discrete quotients (orbifolds) [15].
least several millions in the case of $c = 9$ which corresponds to a four dimensional string theory. It appears that by these possibilities alone one can get all Calabi–Yau manifolds up to diffeomorphisms [13].

The rule for computing the spectrum is simple: anything that can appear should appear. One starts from any modular invariant $N = 2$ theory, and implements the $G$ projection, along with the map into heterotic–like string theory. The massless states in a tensor product of $c < 3$ minimal theories can then be described as all the states obeying the following conditions,

(C1) The left and right states have a total $U(1)$ charge which is odd integral.

(C2) The states are either in the Ramond sector of all the sub-theories or all in the Neveu–Schwarz sector.

(C3) In each of the discrete models, the $l$ and $\bar{l}$ quantum numbers are correlated according to any of the $A_{1}^{1}$ invariants, which were classified in [16]. The left and right $q$ and $s$ quantum numbers are equal. These conditions guarantee modular invariance.

(C4) In addition, we add to the spectrum states which can be obtained by the action of $Q$, the supersymmetry charge. This condition implements space–time supersymmetry. We also add to the spectrum states related by $G_i G_j$ where the $G_i$ are the superconformal stress tensors in any of the sub-theories.

In order to identify these string theories as string propagation on Calabi–Yau manifolds we explored the massless spectrum. In the supergravity models the gauge group is $E_8 \times E_6$, the number of generations (27 of $E_6$) is equal to $h^{2,1}$ and the number of anti–generation is $h^{1,1}$ [5]. If, in addition, the manifold has some automorphism group (this is the physically interesting case) then, by equivariant index theorems, the generations and anti–generations must transform in as the forms. By comparing the automorphisms with the discrete symmetries of the string theory and the way the massless spectra transform, we then obtain a highly unambiguous, model by model, identification of string theories with the spectrum.
expected for particular manifolds. An example of this procedure will be described later in the context of the three generations case.

As a result we find that the massless spectrum of a string theory on a Calabi–Yau manifold is as following,

1) The gauge symmetry is $E_8 \times E_6$ or $SO(26)$ times a possible extra gauge symmetry.

2) The theories have $N = 1$ space–time supersymmetry.

3) The number of generations is $h^{2,1}$ and the number of anti–generations is $h^{1,1}$.

4) The $E_6$ singlets are divided according to: singlets that perturb the complex structure (their number is $h^{2,1}$), singlets that change the radii (their number is $h^{1,1}$), singlets coming from $H^1(\text{End} T)$ and a number of Higgs singlets equal in number to the dimension of the extra gauge symmetry.

5) The automorphisms of the surface appear as discrete symmetries of the string theories and the massless spectra transform as their corresponding forms.

The physically interesting case is when the number of generations is three, arising when the manifold has the Euler number $|\chi| = 6$. Models with more than three generations tend to have problems with fast proton decay, as well as the flow of coupling constants [17], and thus can probably be ruled out.

The first examples of CY manifolds with $\chi = 6$ were described by Tian and Yau [18]. In refs. [19, 20], a comprehensive computer search for all complete intersection manifolds with $|\chi| = 6$ was carried out, and it was shown that no additional such manifold exists. It is also known that there are no orbifolds [15], which corresponds to propagation on a flat singular limit of some CY manifold (e.g. the $Z$ manifold [5]), that have three generations. In addition, all the known manifolds [18, 21] with $\chi = -6$ are either diffeomorphic to one another, or ill defined [22].

Thus, strikingly, there appears to be a unique manifold with three generations, the Tian–Yau manifold. In this paper we describe a string theory on this manifold.
Our starting point is a CY manifold with Euler number $\chi = -54$. It can be described as the hypersurface, $S$, in $CP^2 \times CP^3$, described as the manifold of solutions of the polynomial equations

$$
P_1 = z_0^3 + z_1^3 + z_2^3 + z_3^3 = 0,
$$
$$
P_2 = z_1x_1^3 + z_2x_2^3 + z_3x_3^3 = 0,
$$
\tag{3}

where $[z_0, z_1, z_2, z_3] \in CP^3$ and $[x_1, x_2, x_3] \in CP^2$. This manifold has a vanishing first Chern class as follows from the existence of a holomorphic three form, which can be written as

$$
\mu = \oint \oint \epsilon_{ijkl} z_i dz_j \wedge dz_k \wedge dz_l \epsilon_{ijk} x_i dx_j \wedge dx_k / P_1 P_2,
$$
\tag{4}

where the integrals are taken around close contours surrounding the surfaces $P_1 = 0$ and $P_2 = 0$.

In order to compute the Hodge numbers for this manifold it is enough to find $h^{2,1}$, since $\chi = 2(h^{1,1} - h^{2,1})$. Now, the Hodge number $h^{2,1}$ counts the number of deformations of the complex structure in manifolds which admit a metric of $SU(3)$ holonomy. The reason is that these deformations are given, in general, by $(1,0)$ forms with values in the tangent bundle, $H^1(T)$ (e.g. see [23]), which can in turn be converted to $(1,2)$ forms using the holomorphic three form. The deformations of the complex structure for the particular surface $S$ may all be described as perturbations of the defining equation (3). We may perturb $P_1$ by adding any of the 20 polynomials which are cubic in $z$ and of zero order in $x$, or perturb $P_2$ by any of the 40 polynomials which are linear in $z$ and cubic in $x$. In total there are 60 possible polynomials. However, polynomials related by a linear redefinition of $z$ or $x$ correspond to the same complex structure. There are 25 such redefinitions and thus the net number of perturbations is $h^{2,1} = 35$. Since $\chi = -54$ we also find $h^{1,1} = 8$.

The surface $S$ enjoys a large global automorphism group. First, we can permute the indices 1, 2, 3 by an arbitrary permutation, $p \in S_3$, of these indices: $z_i \rightarrow z_{p(i)}$. 
simultaneously with \( x_i \to x_{p(i)} \). Next, we have a \( Z_3 \times Z_9^3 \) automorphism group given by different phases. Denoting by \( \{ r_0, r_1, r_2, r_3 \} \) an element of \( Z_3 \times Z_9^3 \), where \( r_0 \) is defined modulo 3 and the other \( r \) modulo 9, its action is given by

\[
z_i \rightarrow e^{2\pi i r_i/3} \quad \text{for } i = 0, 1, 2, 3 \tag{5}
\]

\[
x_i \rightarrow e^{-2\pi i r_i/9} x_i \quad \text{for } i = 1, 2, 3. \tag{6}
\]

Since an overall phase is irrelevant in \( CP^n \), the group element \( g = \{1, 1, 1, 1\} \) acts trivially. To summarize, the global automorphism group is \( G = S_3 \times (Z_3 \times Z_9^3)/(g) \). It is of order 1458.

Under the automorphism group \( G \) the deformations of the complex structure transform like their corresponding polynomial perturbations. We denote by a column vector the perturbations, where the up (down) component perturb \( P_1 \) (\( P_2 \)). Due to the freedom to linearly redefine \( z \), the perturbations of \( P_1 \) may all be assumed to be linear in any of the \( z_i \). Similarly, redefinitions of \( x \) allow us to write the perturbations of \( P_2 \) as \( z_i x_j^2 x_k \), \( z_i x_j^3 \), or \( z_i x_1 x_2 x_3 \), where \( i \neq j \). The possible perturbations then come in the patterns,

\[
\begin{pmatrix}
  z_0 z_1 z_2 \\
  0 \\
\end{pmatrix} \quad (1, 3, 3, 0) \tag{3}
\]

\[
\begin{pmatrix}
  z_1 z_2 z_3 \\
  0 \\
\end{pmatrix} \quad (0, 3, 3, 3) \tag{1}
\]

\[
\begin{pmatrix}
  0 \\
  z_0 x_3^3 \\
\end{pmatrix} \quad (1, 6, 0, 0) \tag{3}
\]

\[
\begin{pmatrix}
  0 \\
  z_0 x_1^2 x_2 \\
\end{pmatrix} \quad (1, -2, -1, 0) \tag{6}
\]

\[
\begin{pmatrix}
  0 \\
  z_0 x_1 x_2 x_3 \\
\end{pmatrix} \quad (1, -1, -1, -1) \tag{1}
\]

\[
\begin{pmatrix}
  0 \\
  z_1 x_2^3 \\
\end{pmatrix} \quad (0, 3, 6, 0) \tag{6}
\]
where we denote by \((m_0, m_1, m_2, m_3)\) the charge of a vector, \(v\), in any of the one dimensional irreducible representation of \(Z_3 \times Z_9^3\). A vector in this representation transforms as

\[
v \rightarrow e^{2 \pi i (r_0 m_0/3 + r_1 m_1/9 + r_2 m_2/9 + r_3 m_3/9)} v
\]

(7)

under \(\{r_0, r_1, r_2, r_3\} \in G\). Since \(g = (1, 1, 1, 1) \in G\) is equivalent to \(0 \in G\), it must act trivially in all the representations of \(G\) and thus

\[
3m_0 + m_1 + m_2 + m_3 = 0 \mod 9,
\]

(8)

for all the representations \((m_0, m_1, m_2, m_3)\).

In addition, the holomorphic three form \(\mu\) transforms in the representation \((1, 2, 2, 2)\) of \(G\). The subgroup of \(G\) which commutes with supersymmetry, \(H\), is given by the elements that act trivially on the holomorphic three form,

\[
H = \{\{r_0, r_1, r_2, r_3\} \in G \mid 3r_0 + 2r_1 + 2r_2 + 2r_3 = 0 \mod 9\}.
\]

(9)

All the permutations commute with supersymmetry. The other elements of \(G\), which are not in \(H\), are \(R\) symmetries.

The \((2, 1)\) forms are obtained from the deformations of the complex structure, which are elements of \(H^1(T)\), by multiplying with holomorphic three form. Thus, the \((2, 1)\) forms transform like the deformations times the holomorphic three form, i.e. they differ by the charge \((1, 2, 2, 2)\).
The transformation properties of the $(1,1)$ forms under $G$ may be computed using Lefshetz fixed point theorem. Let $f$ be some element of the automorphism group $G$. Then $f$ acts on the cohomology group as some matrix, $f^*$. Lefshetz fixed point theorem tells us that

$$
\sum_{p,q}(-1)^{p+q} \text{Tr}_{H^{(p,q)}} f^* = \chi(M_f), \quad (10)
$$

where $\chi(M_f)$ is the Euler character of the submanifold which is fixed by $f$, $M = \{ x \in M | f(x) = x \}$. By calculating all the Euler numbers in eq. (10), we find that the eight $(1,1)$ forms transform as

$$(0,0,0,0) \times 2, \quad (1,3,6,6), \quad (2,3,3,6). \quad (11)$$

Let us turn now to one forms with values in the endomorphism of the tangent bundle, $H^1(\text{End} T)$. In the field theory, such forms give rise to massless $E_6$ singlets [24]. These forms are in correspondence with deformations of the tangent bundle. Denote a tangent vector by $(U_a, V_b)$, where $U_a$ is a tangent vector of $CP^3$ ($a=0,1,2,3$), and $V_b$ is a tangent vector in $CP^2$, $b = 1,2,3$. The tangent vectors $U_a$ and $V_b$ are defined modulo the equivalence relation

$$
U_a \sim U_a + \lambda z_a, \quad V_b \sim V_b + \rho x_b, \quad (12)
$$

for any $\lambda$ and $\rho$. In addition, the vector $(U_a, V_b)$ must be tangent to the two surfaces $P_1$ and $P_2$. This implies,

$$
\frac{\partial P_1}{\partial z_a} U_a = \frac{\partial P_1}{\partial x_b} V_b = 0 \quad \frac{\partial P_2}{\partial z_a} U_a = \frac{\partial P_2}{\partial x_b} V_b = 0 \quad (13)
$$

A simple method to deform the tangent bundle is to change equation (13) by adding to it some small perturbation. We can perturb any of the partial derivatives in (13) by adding to it an arbitrary polynomial of the same bi-degree as the
corresponding partial derivative. Denote a perturbation by the matrix

\[ M = \begin{pmatrix} P_a & Q_b \\ L_a & R_b \end{pmatrix}. \]  

Eq (13) is then perturbed by \( \left( \frac{\partial P_a}{\partial z_a} + P_a \right) U_a = 0 \), etc. The bi-degrees of the polynomials \( P, Q, L \) and \( R \) are \((2, 0), (0, 0), (0, 3)\) and \((1, 2)\), respectively. In addition, the equivalence relation eq. (12) implies

\[ P^a z_a = Q^b x_b = L^a z_a = R^b x_b = 0. \]  

Any such set of polynomials defines a perturbation of the tangent bundle. All the perturbations come either from \( P \) or from \( R \). To perturb \( P \) we may take \( P_a = C_a P/z_a \), for an arbitrary \( P \) which is of bi-degree \((3, 0)\) and where the \( C_a \) are some constants which obey \( \sum C_a = 0 \). The possible \( P \) come in the patterns

\[
\begin{align*}
  &z_0^2 z_1^2 
  &z_0 z_1^2 
  &z_1^2 z_2 
  &z_0 z_1 z_2 
  &z_1 z_2 z_3 
\end{align*}
\]

where the numbers above denote the \( \mathbb{Z}_3 \mathbb{Z}_9 \) charges and multiplicities.

Similarly, the perturbations of \( R \) can be written as \( R_b = C_b R/x_b \), where the constants \( C_b \) obey, \( \sum C_b = 0 \), and \( R \) is any polynomial of bi-degree \((1, 3)\). The possible \( R \)'s fall into the patterns

\[
\begin{align*}
  &z_0 x_1^2 x_2 
  &z_1 x_1^2 x_2 
  &z_2 x_1^2 x_2
\end{align*}
\]
In total we find 52 elements of $H^4(\text{End } T)$. There can be more deformations which cannot be obtained in this way. Using spectral sequences it should be possible to compute the entire cohomology.

In the field theory limit the number of generations (27 of $E_6$) is 35, corresponding to the 35 harmonic $(2, 1)$ forms, and the number of anti–generations (27 of $E_6$) is 8, corresponding to the harmonic $(1, 1)$ forms. The net number of generations is $\frac{1}{2}|\chi| = 27$.

Consider now the theory made by gluing one copy of the $k = 1$ model with three copies of the $k = 16$ model. In addition, in condition (C3) we use the sporadic modular invariant at level 16 [16]. The resulting spectrum may be easily computed from (C1–C4). The theory, denoted by $1^116^3$, contains 35 generations (27 of $E_6$), 8 anti–generations (27 of $E_6$) and 197 massless $E_6$ singlets.

As will be seen the theory $1^116^3$ corresponds to a string theory on the manifold $S$. The number of generations and anti–generations are indeed the same as $h^{2,1}$ and $h^{1,1}$ for this manifold.

What are the discrete symmetries of the theory $1^116^3$? The $k$’th minimal model has a $Z_{k+2}$ discrete symmetry. Thus, the theory $1^116^3$ has a $Z_3 \times Z_{18}^3$ symmetry. In addition, we can permute the three identical $k = 16$ sub–theories. However, by examining the massless spectrum, it becomes clear that each of the $Z_2$ subgroups of $Z_{18}$ acts trivially on the spectrum and thus may be ignored. Hence the symmetry group of the theory $1^116^3$ is $G = (Z_3 \times S_3 \times Z_9^3)/Z_9$. Denote an element of $Z_3 \times Z_9^3$ by $\{r_0, r_1, r_2, r_3\}$. The quotient by $Z_9$ corresponds to the fact that the total superconformal $U(1)$ charge of all the fields is an odd integer, implying that the element $\{1, 1, 1, 1\}$ acts trivially on all the fields in the theory, so the actual symmetry group is a quotient by the $Z_9$ subgroup generated by this element.
We see that the theory $1^116^3$ has a symmetry group which is isomorphic to the automorphism group of the hypersurface $S$.

Under the $Z_{k+2}$ charge of the $k$ minimal model a field in the theory, $\Phi_{t,q,s,\overline{t},\overline{q},\overline{s}}$, has a charge which is

$$Q = (q + \overline{q})/2 \mod (k + 2).$$

We assume in this definition that $q = \overline{q} \mod 2$. This does not create any problem for the non $R$ symmetries. For some of the $R$ symmetries, however, eq. (16) may imply that the charges are ill defined, suggesting that these group elements are bad symmetries that should be ignored. In the case at hand, though, no such problem arises. The $R$ symmetries, since they do not commute with space–time supersymmetry, are very tricky. Different supersymmetry partners transform differently under them, and similarly, the different representations of $SO(10)$, which make the $27$ or $\overline{27}$ of $E_6$, transform differently.

We would like to compare the transformation properties of the various massless fields in the spectrum of the $1^116^3$ theory, with those that are predicted in the field theory.

The first thing we note is that the plus and minus chirality components of the $E_6$ gluino field come from the $H^{0,0}$ and $H^{0,3}$ Dolbeault cohomology groups or the positive chirality gluino corresponds to the unique constant $(0,0)$ form and the negative chirality gluino corresponds to the antiholomorphic $(0,3)$ form. The adjoint representation of $E_6$ decomposes into $SO(10)$ as $78 = 1 + 16 + \overline{16} + 45$. The different $SO(10)$ representations always transform differently under the $R$ symmetries. Now, for a fixed $SO(10)$ representation, say the singlet, the different $Z_{k+2}$ charges of the positive and negative chirality gluinos will always differ by 1. This is simply because these two modes are related by the square of the supersymmetry charge, $Q^2$, which, in turn, carries the $Z_{k+2}$ charge which is 1 for all the sub–theories.

On the other hand, this ratio corresponds to the holomorphic $(3,0)$ form. Thus, the holomorphic $(3,0)$ form always carry the $Z_{k+2}$ charge 1, when this charge is defined as in eq. (16).
From eq. (4) we see that the \((3,0)\) form has the \(Z_3 \times Z_9^3\) charge which is \((1,2,2,2)\). On the other hand, the positive and negative chirality gluinos differ by a \(Z_3 \times Z_{18}^3\) charge which is \((1,1,1,1)\). Using this correspondence we can ‘fix the normalization’ of the discrete charges. The discrete charges of the automorphisms of the manifold \((m_0, m_1, m_2, m_3)\), which are elements of \(Z_3 \times Z_9^3\), and the conformal field theory charges \((Q_0, Q_1, Q_2, Q_3)\), which are defined according to (16) and are elements of \(Z_3 \times Z_{18}^3\), are then seen to be related as

\[
m_0 = Q_0 \mod 3, \quad m_i = 2Q_i \mod 9, \quad \text{for } i = 1, 2, 3.
\] (17)

Next, we can check whether the generations and anti–generations transform as they are supposed to, in the field theory limit. Under the non R symmetries (i.e. the ones which commute with SUSY) the generations and anti–generations must transform like their corresponding \((2,1)\) and \((1,1)\) forms. The R symmetries are trickier since, as discussed earlier, different supersymmetry components transform under them differently. Which component, then, should we compare with the forms? The answer is that the correct component for supersymmetry multiplets in the 27 or the \(\bar{27}\) of \(E_6\) is the scalar (helicity zero) which is a vector of \(SO(10)\). The reason is that such scalars are related to the \(E_6\) singlets which perturb the radius (in the \(\bar{27}\) case) or deform the complex structure (in the 27 case) \[13\] and thus must transform in precisely the same way as the forms of \(H^1(T)\) (for 27) or the \((1,1)\) forms (for \(\bar{27}\)) do.

The following is an enumeration of the 35 generations in the \(1^{16^3}\) theory,

\[
\begin{align*}
(3) & \quad \Phi_{1,2,1,1,3,2} \Theta_{12,13,1,4,32,0} \Theta_{0,1,1,16,20,0} & (1,6,0,0) \\
(6) & \quad \Phi_{1,2,1,1,3,2} \Theta_{8,9,1,8,28,0} \Theta_{4,5,1,12,24,0} \Theta_{0,1,1,16,20,0} & (1,-2,-1,0) \\
(3) & \quad \Phi_{1,2,1,1,3,2} \Theta_{0,1,1,16,20,0} & (1,3,3,0) \\
(1) & \quad \Phi_{1,2,1,1,3,2} \Theta_{0,1,1,16,20,0} & (1,-1,-1,-1) \\
(6) & \quad \Phi_{0,1,1,0,2,2} \Theta_{12,13,1,4,32,0} \Theta_{6,7,1,10,26,0} \Theta_{0,1,1,16,20,0} & (0,6,3,0)
\end{align*}
\]
The fields in the list correspond to anti–spinors which are singlets of $SO(10)$. We denoted by $\Phi$ and $\Theta$ the fields from the $k = 1$ and $k = 16$ theories. The six indices on each field correspond to the three left quantum numbers $(l, q, s)$ and the three right quantum numbers. The numbers on the right are the $Z_3 Z_9^3$ charges of each of the fields. The $Z_9$ charges are computed according to $m_i = q_i + \bar{q}_i - 3 \mod 9$, for $i = 1, 2, 3$; the $Z_3$ charge is $m_0 = -q_0 - \bar{q}_0 \mod 3$. We see that the 35 generations transform in precisely the same representation of $G$ as the deformations of the complex structure do (p. 10).

We can now check the anti–generations. The 8 anti–generations, along with their corresponding $Z_3 \times Z_9^3$ charges, are enumerated below,

$$
\begin{align*}
(6) & \quad \Phi_{0,1,0,2,2} \Theta_{10,1,1,6,30,0} \Theta_{8,9,1,8,28,0} \Theta_{0,1,1,16,20,0} & (0, 2, -2, 0) \\
(3) & \quad \Phi_{0,1,1,0,2,2} \Theta_{10,1,1,6,30,0} \Theta_{2,5,1,12,24,0} & (0, 2, -1, -1) \\
(6) & \quad \Phi_{0,1,1,0,2,2} \Theta_{8,9,1,8,28,0} \Theta_{6,7,1,10,26,0} \Theta_{4,5,1,12,24,0} & (0, -2, 3, -1) \\
(1) & \quad \Phi_{0,1,1,0,2,2} \Theta_{6,7,1,10,26,0} & (0, 3, 3, 3)
\end{align*}
$$

The fields above are also anti–spinors which are $SO(10)$ singlets. The $Z_3 \times Z_9$ charge, $(m_0, m_1, m_2, m_3)$ is computed in this case by $m_0 = -q_0 - \bar{q}_0 - 1 \mod 3$, $m_i = q_i + \bar{q}_i + 1 \mod 9$. Again, we see that the anti–generations transform in precisely the same way as the (1, 1) forms do, eq. (11). This completes the identification of the $1^{16} 3^3$ theory as a string theory on the hypersurface $S$.

We can further compare the $E_6$ singlets. The 197 singlets can be seen to contain, in a completely unambiguous way, 35 modes which transform like $H^1(T)$, these are the singlets related to deformations of the complex structure, 8 modes
transforming like (1, 1) forms, these are singlets related to change of radii and 52 singlets transforming like the modes of $H^1(\text{End } T)$ described earlier (p. 13). The remaining 102 singlets may correspond to additional perturbations of the tangent bundle that we have not computed, or less likely, to accidental massless particles. The resolution of this question must await the complete calculation of $H^1(\text{End } T)$ for this manifold.

The automorphism group of $S$ has a certain $Z_3 \times Z_3$ subgroup, $H$, which will be very important for us. The first $Z_3$ is generated by the permutation: $z_1 \to z_2 \to z_3 \to z_1$ along with $x_1 \to x_2 \to x_3 \to x_1$, denoted by $h$. This can be seen to be a freely acting automorphism. The second $Z_3$ is generated by the $Z_3 \times Z_3^3$ group element $g = \{0, 3, 6, 0\}$. This $Z_3$ is not freely acting. Thus, the quotient manifold $S/H$ is a singular manifold. However, these singularities can be resolved, as discussed in ref. [21] and the resulting manifold is a CY manifold with Euler number $\chi = -6$. Thus, a string theory on $S/H$ should have three generations.

The spectrum of a heterotic string theory propagating on the manifold $S/H$ can be computed as a quotient of the theory $1^{16}3^3$. The partition function of the $k$’th minimal model, twisted in the space and time directions by the $Z_{k+2}$ elements $x$ and $y$, respectively, is given by [1, 2]

$$Z(x, y) = \frac{1}{2} e^{2\pi i x y/(k+2)} \sum_{l, q, s} e^{2\pi i q y/(k+2)} \lambda_{q+2y}^l \chi_{l, q, s},$$

(18)

where $\chi_{l, q, s}^l$ is the partition function of the $N = 2$ conformal block with the quantum numbers $(l, q, s)$. Implementing eq. (18) in the string theory amounts to a simple modification of the conditions (C1–C4) and enables us to compute the spectrum of string propagation on the manifold $S/H$.

Consider first string theory on the quotient manifold $S/(g)$. By a straightforward enumeration of states we find that this theory has 23 generations, 14 anti–generations and 173 singlets. Of these, 17 generations, 8 anti–generations and 85 singlets come from the untwisted sector.
The Hodge numbers $h^{2,1} = 23$ and $h^{1,1} = 14$ are the same as those of a well known CY manifold, namely the one constructed by Tian and Yau [18]. This manifold can be described as the intersection of three hypersurfaces of bi-degrees $(3, 0), (0, 3)$ and $(1, 1)$ in the product space $CP^3 \times CP^3$. Its most symmetric shape is

$$
\sum_{i=0}^{3} x_i^3 = 0, \quad \sum_{i=0}^{3} y_i^3 = 0, \quad \sum_{i=0}^{3} x_i y_i = 0.
$$

(.19)

This manifold has the Hodge numbers $h^{2,1} = 23$ and $h^{1,1} = 14$. Thus, taking a quotient of it by the freely acting $Z_3$ automorphism group, which is generated by

$$(x_0, x_1, x_2, x_3) \times (y_0, y_1, y_2, y_3) \rightarrow (x_0, \alpha^2 x_1, \alpha x_2, \alpha x_3) \times (y_0, \alpha y_1, \alpha^2 y_2, \alpha^2 y_3), \quad (.20)$$

where $\alpha = \exp(2\pi i/3)$, we get to a three generation manifold. The manifold $S/H$ is indeed diffeomorphic to the Tian–Yau manifold [22].

The supergravity model on the Tian–Yau manifold was studied by a number of authors [25] and the indications are that the discrete symmetries of the manifold may well be rich enough to prevent fast proton decay.

Returning to the string theory on $S$, the next step after twisting the theory by $g$, is to further twist it by the permutation $h$. By projecting the spectrum of the string theory on $Q/(g)$ onto the $h$ invariant subspace, it is easy to write the spectrum of the closed string (untwisted) sector. We find in this sector 9 generations, 6 anti–generations and 62 singlets. In addition, one needs to take into account the winding sectors. Since $h$ acts freely, these sectors do not contribute any generations or anti–generations. Thus, all together, in this string theory we find 9 generations and 6 anti–generations, or a net number of three generations.

The Yukawa couplings in this three generation string theory can all be computed exactly since they are given as products of the structure constants of the $N = 2$ minimal models. These, in turn, are related to the structure constants of the $SU(2)$ WZW models which have been studied by several authors [26]. Using
the isomorphism of states and vertex operators that create them out of the vacuum [4], one can express the vertex operators in this string theory in terms of WZW fields and free bosons, and thus to calculate the Yukawa couplings exactly (see [1, 2] for more explanation).

In conclusion, we have presented a new string theory which corresponds to string propagation on a three generation Calabi–Yau manifold. It is a full fledged string theory in which all the physical predictions can be computed exactly. This string theory appears to be the unique viable candidate in its class. In addition, it comes intriguingly close to a realistic description of nature, which is moreover a consistent unification of gravity.

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