JACOBI-TYPE ALGORITHM FOR LOW RANK ORTHOGONAL APPROXIMATION OF SYMMETRIC TENSORS AND ITS CONVERGENCE ANALYSIS

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Abstract. In this paper, we propose a Jacobi-type algorithm to solve the low rank orthogonal approximation problem of symmetric tensors. This algorithm includes as a special case the well-known Jacobi CoM2 algorithm for the approximate orthogonal diagonalization problem of symmetric tensors. We first prove the weak convergence of this algorithm, i.e. any accumulation point is a stationary point. Then we study the global convergence of this algorithm under a gradient based ordering for a special case: the best rank-2 orthogonal approximation of 3rd order symmetric tensors, and prove that an accumulation point is the unique limit point under some conditions. Numerical experiments are presented to show the efficiency of this algorithm.

1. Introduction

As the higher order analogue of vectors and matrices, in the last two decades, tensors have been attracting more and more attentions from various fields, including signal processing, numerical linear algebra and machine learning [6, 10, 12, 21, 30, 2]. One reason is that more and more real data are naturally represented in tensor form, e.g. hyperspectral image, brain fMRI image, or social networks. The other reason is that, compared with the matrix case, tensor based techniques can capture higher order and more complicated relationships, e.g. Independent Component Analysis (ICA) based on the cumulant tensor [8], and multilinear subspace learning methods [25].

Low rank approximation of higher order tensors is a very important problem and has been applied in various areas [12, 14, 31]. However, it is much more difficult than the matrix case, since it is ill-posed for many ranks, and this ill-posedness is not rare for 3rd order tensors [17].

Notation. Let $\mathbb{R}^{n_1 \times \cdots \times n_d}$ be the linear space of $d$th order real tensors and $\text{symm}(\mathbb{R}^{n \times \cdots \times n}) \subseteq \mathbb{R}^{n \times \cdots \times n}$ be the set of symmetric ones [11, 27], whose entries do not change under any permutation of indices. The identity matrix of size $n$ is denoted by
Let $\text{St}(p, n) \subseteq \mathbb{R}^{n \times p}$ be the Stiefel manifold with $1 \leq p \leq n$. Let $\mathcal{O}_n \subseteq \mathbb{R}^{n \times n}$ be the orthogonal group, i.e. $\mathcal{O}_n = \text{St}(n, n)$. Let $\text{SO}_n \subseteq \mathbb{R}^{n \times n}$ be the special orthogonal group, i.e. the set of orthogonal matrices with determinant 1. We denote by $\| \cdot \|$ the Frobenius norm of a tensor or a matrix, or the Euclidean norm of a vector. Tensor arrays, matrices, and vectors, will be respectively denoted by bold calligraphic letters, e.g. $\mathbf{A}$, with bold uppercase letters, e.g. $M$, and with bold lowercase letters, e.g. $u$; corresponding entries will be denoted by $A_{ijk}$, $M_{ij}$, and $u_i$. Operator $\bullet_p$ denotes contraction on the $p$th index of a tensor; when contracted with a matrix, it is understood that summation is always performed on the second index of the matrix. For instance, $(\mathbf{A} \bullet_1 M)_{ijk} = \sum_\ell A_{\elljk} M_{i\ell}$.

We denote $\mathbf{A}(M) \overset{\text{def}}{=} \mathbf{A} \bullet_1 M^T \bullet_2 \cdots \bullet_d M^T$ for convenience in this paper. For $\mathbf{A} \in \mathbb{R}^{n \times \cdots \times n}$ and a fixed set of indices $1 \leq k_1 < k_2 < \cdots < k_m \leq n$, we denote by $\mathbf{A}^{(k_1, k_2, \cdots, k_m)}$ the $m$-dimensional subtensor obtained from $\mathbf{A}$ by allowing its indices to vary in $\{k_1, k_2, \cdots, k_m\}$ only.

**Problem statement.** Let $\mathbf{A} \in \text{symm}(\mathbb{R}^{n \times \cdots \times n})$ and $1 \leq p \leq n$. In this paper, we study the best rank-$p$ orthogonal approximation problem, which is to find $C^* \overset{\text{def}}{=} \sum_{k=1}^p \sigma_k^* u_k^* \otimes \cdots \otimes u_k^* = \operatorname{argmin} \| \mathbf{A} - \sum_{k=1}^p \sigma_k u_k \otimes \cdots \otimes u_k \|$, (1)

where $[u_1, \cdots, u_p] \in \text{St}(p, n)$ and $\sigma_k \in \mathbb{R}$ for $1 \leq k \leq p$. If $p = 1$, then (1) is the best rank-1 approximation problem [16, 19, 22, 33, 13] of symmetric tensors, which is equivalent to the cubic spherical optimization problem [28, 34, 35]. If $p = n$, by [5, Proposition 5.1] and [24, Proposition 5.2], we see that (1) is closely related to the approximate orthogonal diagonalization problem for 3rd and 4th order cumulant tensors, which is in the core of **Independent Component Analysis** (ICA) [7, 8, 9], and finds many applications [12].

To our knowledge, the orthogonal tensor decomposition was first tackled in [7], but appeared more formally in [20], in which many examples were presented to illustrate the difficulties of this type of decomposition. In [5], the existence of $C^*$ in problem (1) was proved, and the low rank orthogonal approximation of tensors (LROAT) algorithm and symmetric LROAT (SLROAT) were developed to solve this problem based on the polar decomposition. These two algorithms boil down to the higher order power method (HOPM) and symmetric HOPM (SHOPM) algorithm [16, 19, 33] when $p = 1$. More recently, also based on the polar decomposition, a similar algorithm was developed in [26] to solve problem (1), and this algorithm was applied to the image reconstruction task.

**Contribution.** In this paper, we propose a Jacobi-type algorithm to solve problem (1). This algorithm is exactly the well-known Jacobi CoM2 algorithm [12] when $p = n$, and the same as the Jacobi-type algorithm in [18] when $p = 1$. We first prove the
weak convergence\(^1\) of this algorithm under the cyclic ordering based on a decomposition property of the identity matrix. Then, under the gradient based ordering defined in [18, 23, 32], we prove the global convergence\(^2\) of this algorithm for 3rd order tensors of rank \(p = 2\) under some conditions. By making some numerical experiments and comparisons, we show that the Jacobi-type algorithm proposed in this paper is efficient and stable.

**Organization.** The paper is organized as follows. In Section 2, we show that two optimization problems on Riemannian manifold are both equivalent to (1), and then calculate their Riemannian gradients. In Section 3, we propose a Jacobi-type algorithm to solve (1). This algorithm includes the well-known Jacobi CoM2 algorithm as a special case. In Section 4, we prove the weak convergence of this algorithm under the cyclic ordering. In Section 5, we study the global convergence of this algorithm under the gradient based ordering for the 3rd order tensor and \(p = 2\) case. In Section 6, we report some numerical experiments showing the efficiency of this algorithm.

## 2. Geometric Properties

### 2.1. Equivalent problems.

Let \(\mathcal{A} \in \text{symm}(\mathbb{R}^{n \times \cdots \times n})\) and \(1 \leq p \leq n\). Let \(X \in \text{St}(p, n)\) and \(\tilde{\mathcal{W}} = \mathcal{A}(X)\). One problem equivalent to (1) is to find

\[
X_* = \arg\max_{X \in \text{St}(p, n)} \tilde{f}(X),
\]

where

\[
\tilde{f}(X) = \sum_{i=1}^{p} \tilde{W}_{i \cdots i}^2.
\]

**Lemma 2.1.** ([5, Proposition 5.1]) Let \(C^*\) be as in (1). Then

\[
\langle \mathcal{A} - C^*, u_k^* \otimes \cdots \otimes u_k^* \rangle = 0 \quad \text{and} \quad \sigma_k^* = \langle \mathcal{A}, u_k^* \otimes \cdots \otimes u_k^* \rangle
\]

for \(1 \leq k \leq p\). Moreover, it holds that

\[
\|\mathcal{A} - C^*\|^2 = \|\mathcal{A}\|^2 - \|C^*\|^2 = \|\mathcal{A}\|^2 - \sum_{k=1}^{p} (\sigma_k^*)^2.
\]

**Remark 2.2.** (i) Let \(C^*\) be as in (1) and \(X_*\) be as in (2). We see from (4) that

\[
X_* = [u_1^*, \cdots, u_p^*] \quad \text{and} \quad \|\mathcal{A} - C^*\|^2 = \|\mathcal{A}\|^2 - \tilde{f}(X_*).
\]

In other words, to solve (1), it is enough for us to solve (2), which is an optimization problem on \(\text{St}(p, n)\).

(ii) If \(p = 1\), then (2) is the **cubic spherical optimization problem** [28, 34, 35]. If \(p = n\), then (2) is the **approximate orthogonal tensor diagonalization problem** [8, 9, 12, 23].

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\(^1\)any accumulation point is a stationary point.

\(^2\)the iterations converge to a unique limit point for any starting point.
Let $Q \in \mathcal{O}_n$ and $\mathcal{W} = \mathcal{A}(Q)$. Another problem, equivalent to (2), is to find
\[
Q_* = \arg\max_{Q \in \mathcal{O}_n} f(Q),
\] where
\[
f(Q) \overset{\text{def}}{=} \sum_{i=1}^{p} W_{i...i}^2.
\]
In fact, if $X \in \text{St}(p, n)$ and $Q = [X, Y] \in \mathcal{O}_n$, then $W_{i_1...i_d} = \tilde{W}_{i_1...i_d}$ for any $1 \leq i_1, \cdots, i_d \leq p$. The equivalence between (2) and (5) follows from the fact that $f(Q) = \tilde{f}(X)$.

**Remark 2.3.** Let $\mathcal{W} \in \text{symm}(\mathbb{R}^{n \times n \times n})$ and $1 \leq p \leq n$. Let $\tilde{\mathcal{W}} = \mathcal{W}^{(1,2;\cdots,p)}$. Then the objective used in [18, (3.1)] is the sum of squares of all the elements in $\tilde{\mathcal{W}}$, while (6) is the sum of squares of the diagonal elements in $\tilde{\mathcal{W}}$. They are the same if $p = 1$.

### 2.2. Riemannian gradient.

**Definition 2.4.** Let $\mathcal{A} \in \text{symm}(\mathbb{R}^{n \times \cdots \times n})$ and $1 \leq i < j \leq n$. Define
\[
\sigma_{i,j}(\mathcal{A}) \overset{\text{def}}{=} A_{ii...i}A_{jj...j}, \quad d_{i,j}(\mathcal{A}) \overset{\text{def}}{=} \sigma_{i,j}(\mathcal{A}) - \sigma_{j,i}(\mathcal{A}) = A_{ii...i}A_{jj...j} - A_{ij...j}A_{ji...i}.
\]

**Theorem 2.5.** The Riemannian gradient of (6) at $Q$ is
\[
\text{Proj} \nabla f(Q) = Q \Lambda(Q),
\] where
\[
\Lambda(Q) \overset{\text{def}}{=} d \cdot \begin{bmatrix}
0 & -d_{1,2}(\mathcal{W}) & \cdots & -d_{1,p}(\mathcal{W}) & -\sigma_{1,p+1}(\mathcal{W}) & \cdots & -\sigma_{1,n}(\mathcal{W}) \\
d_{1,2}(\mathcal{W}) & 0 & \cdots & -d_{2,p}(\mathcal{W}) & -\sigma_{2,p+1}(\mathcal{W}) & \cdots & -\sigma_{2,n}(\mathcal{W}) \\
 \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
d_{1,p}(\mathcal{W}) & d_{2,p}(\mathcal{W}) & \cdots & 0 & -\sigma_{p,p+1}(\mathcal{W}) & \cdots & -\sigma_{p,n}(\mathcal{W}) \\
\sigma_{1,p+1}(\mathcal{W}) & \sigma_{2,p+1}(\mathcal{W}) & \cdots & \sigma_{p,p+1}(\mathcal{W}) & 0 & \cdots & 0 \\
 \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\sigma_{1,n}(\mathcal{W}) & \sigma_{2,n}(\mathcal{W}) & \cdots & \sigma_{p,n}(\mathcal{W}) & 0 & \cdots & 0
\end{bmatrix}.
\]

**Proof.** Note that
\[
f(Q) = \sum_{j=1}^{p} W_{jj...j}^2 = \sum_{j=1}^{p} \left( \sum_{i_1, i_2, \ldots, i_d} A_{i_1,i_2,\ldots,i_d} Q_{i_1,j} Q_{i_2,j} \cdots Q_{i_d,j} \right)^2.
\]
Let $\mathcal{V} = \mathcal{A} \cdot_2 Q^T \cdots \cdot_q Q^T$. Fix $1 \leq i \leq n$ and $1 \leq j \leq p$. Then
\[
\frac{\partial f}{\partial Q_{i,j}} = 2dW_{jj...j}V_{ij...j}
\]
by methods similar to [23, Section 4.1]. Note that $\mathbf{W} = \mathbf{V} \cdot \mathbf{Q}^T$. We get the Euclidean gradient of (6) at $\mathbf{Q}$ as follows:

$$\nabla f(\mathbf{Q}) = 2d\mathbf{Q} \begin{bmatrix} W_{11...1} & W_{12...2} & \ldots & W_{1p...p} \end{bmatrix} \begin{bmatrix} W_{11...1} & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & W_{p...p} & 0 \end{bmatrix}.$$  

By [1, (3.35)], we get that

$$\text{Proj}\nabla f(\mathbf{Q}) = \frac{1}{2} \mathbf{Q}(\mathbf{Q}^T\nabla f(\mathbf{Q}) - \nabla f(\mathbf{Q})^T \mathbf{Q}) = \mathbf{Q} \Lambda(\mathbf{Q}).$$  

Then the proof is complete. \hfill \Box

**Remark 2.6.** (i) If $p = 1$, we see that $\Lambda(\mathbf{Q}) = 0$ if and only if

$$W_{21...1} = W_{31...1} = \cdots = W_{n1...1} = 0,$$

which means that the first column of $\mathbf{Q}$ satisfies the condition in [28, (2)].

(ii) The definition of $\Lambda(\mathbf{Q})$ in (8) can be seen as an extension of [23, (12)].

**Theorem 2.7.** The Riemannian gradient of (3) at $\mathbf{X}$ satisfies

$$\mathbf{X}^T \text{Proj}\nabla \tilde{f}(\mathbf{X}) = d \begin{bmatrix} 0 & -d_{1,2}(\tilde{\mathbf{W}}) & \ldots & -d_{1,p}(\tilde{\mathbf{W}}) \\ d_{1,2}(\tilde{\mathbf{W}}) & 0 & \ldots & -d_{2,p}(\tilde{\mathbf{W}}) \\ \vdots & \ddots & \ddots & \vdots \\ d_{1,n}(\tilde{\mathbf{W}}) & d_{2,p}(\tilde{\mathbf{W}}) & \ldots & 0 \end{bmatrix}. \quad (10)$$

**Proof.** The proof goes along the same lines as for Theorem 2.5. Note that

$$\tilde{f}(\mathbf{X}) = \sum_{j=1}^p \tilde{W}_{jj...j}^2 = \sum_{j=1}^p \left( \sum_{i_1,i_2,...,i_d} A_{i_1,i_2,...,i_d} X_{i_1,j} X_{i_2,j} \cdots X_{i_d,j} \right)^2.$$  

Let $\tilde{\mathbf{V}} = \mathbf{A} \cdot_2 \mathbf{X}^T \cdots \cdot_d \mathbf{X}^T$. Fix $1 \leq i \leq n$ and $1 \leq j \leq p$. Then

$$\frac{\partial \tilde{f}}{\partial X_{i,j}} = 2d\tilde{W}_{jj...j} \tilde{V}_{ij...j}$$

by the similar methods in [23, Section 4.1]. Note that $\tilde{\mathbf{W}} = \tilde{\mathbf{V}} \cdot \mathbf{X}^T$. We get the Euclidean gradient of (6) at $\mathbf{X}$ as follows:

$$\nabla \tilde{f}(\mathbf{X}) = 2d \begin{bmatrix} \tilde{W}_{11...1} & \tilde{W}_{12...2} & \cdots & \tilde{W}_{1p...p} \\ \tilde{W}_{21...1} & \tilde{W}_{22...2} & \cdots & \tilde{W}_{2p...p} \\ \vdots & \ddots & \ddots & \vdots \\ \tilde{W}_{n1...1} & \tilde{W}_{n2...2} & \cdots & \tilde{W}_{np...p} \end{bmatrix} \begin{bmatrix} \tilde{W}_{11...1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{W}_{p...p} \end{bmatrix}.$$
It follows by [1, (3.35)] that

$$\text{Proj} \nabla \hat{f}(X) = (I_n - XX^T) \nabla \hat{f}(X) + dX \cdot \begin{bmatrix}
0 & -d_{1,2}(\hat{W}) & \cdots & -d_{1,p}(\hat{W}) \\
-d_{1,2}(\hat{W}) & 0 & \cdots & -d_{2,p}(\hat{W}) \\
\vdots & \vdots & \ddots & \vdots \\
d_{1,p}(\hat{W}) & d_{2,p}(\hat{W}) & \cdots & 0
\end{bmatrix}, \quad (11)$$

and the proof is completed.

**Proposition 2.8.** Let $\mathcal{A} \in \text{symm}(\mathbb{R}^{n \times \cdots \times n})$ and $1 \leq p \leq n$. Let $X_* \in \text{St}(p, n)$ and $Q_* = [X_*, Y_*] \in \mathcal{O}_n$. Suppose that $\hat{f}$ is as in (3) and $f$ is as in (6). Then

$$\text{Proj} \nabla \hat{f}(X_*) = 0 \iff \text{Proj} \nabla f(Q_*) = 0.$$

**Proof.** Let $\tilde{W}_* = \mathcal{A}(X_*)$ and $W_* = \mathcal{A}(Q_*)$.

$(\Rightarrow)$. By (10), we see that $d_{i,j}(\tilde{W}_*) = d_{i,j}(W_*) = 0$ for any $1 \leq i < j \leq p$. It follows by (11) that

$$Y_*Y_*^T \nabla \hat{f}(X_*) = (I_n - X_*X_*^T) \nabla \hat{f}(X_*) = 0,$$

and thus

$$Y_*^T \nabla \hat{f}(X_*) = Y_*^T Y_* Y_*^T \nabla \hat{f}(X_*) = 0.$$  

Then $\sigma_{i,j}(W_*) = 0$ for any $1 \leq i \leq p < j \leq n$, and thus $\text{Proj} \nabla f(Q_*) = 0$ by (8).

$(\Leftarrow)$. By (8), we see that $d_{i,j}(\tilde{W}_*) = d_{i,j}(W_*) = 0$ for any $1 \leq i < j \leq p$. Note that $\sigma_{i,j}(W_*) = 0$ for any $1 \leq i \leq p < j \leq n$. It follows that $Y_*^T \nabla \hat{f}(X_*) = 0$, and thus

$$(I_n - X_*X_*^T) \nabla \hat{f}(X_*) = Y_* Y_*^T \nabla \hat{f}(X_*) = 0.$$  

Then $\text{Proj} \nabla \hat{f}(X_*) = 0$ by (11).

\[\square\]

3. **Jacobi low rank orthogonal approximation algorithm**

3.1. **Algorithm description.** Let $1 \leq p \leq n$ and $\mathcal{C} = \{(i, j), 1 \leq i < j \leq n, i \leq p\}$. We divide $\mathcal{C}$ to be two different subsets

$$\mathcal{C}_1 \overset{\text{def}}{=} \{(i, j), 1 \leq i < j \leq p\} \text{ and } \mathcal{C}_2 \overset{\text{def}}{=} \{(i, j), 1 \leq i \leq p < j \leq n\}.$$  

Denote by $G^{(i,j,\theta)}$ the Givens rotation matrix, as defined e.g. in [23, Section 2.2]. Now we formulate the Jacobi low rank orthogonal approximation (JLROA) algorithm for problem (5) as follows.

**Algorithm 1.** (JLROA algorithm)

**Input:** $\mathcal{A} \in \text{symm}(\mathbb{R}^{n \times \cdots \times n})$, $1 \leq p \leq n$, a starting point $Q_0$.

**Output:** Sequence of iterations $Q_k$.

- **For** $k = 1, 2, \ldots$ until a stopping criterion is satisfied do
Choose the pair \((i_k, j_k) \in \mathcal{C}\) in the following cyclic ordering:
\[
(1, 2) \rightarrow (1, 3) \rightarrow \cdots \rightarrow (1, n) \rightarrow \\
(2, 3) \rightarrow \cdots \rightarrow (2, n) \rightarrow \\
\cdots \rightarrow (p, p + 1) \rightarrow \cdots \rightarrow (p, n) \rightarrow \\
(1, 2) \rightarrow (1, 3) \rightarrow \cdots .
\] (12)

- Solve \(\theta_k^*\) that maximizes \(h_k(\theta) \overset{\text{def}}{=} f(Q_{k-1}G^{(i_k, j_k, \theta)})\).
- Set \(U_k \overset{\text{def}}{=} G^{(i_k, j_k, \theta_k^*)}\), and update \(Q_k = Q_{k-1}U_k\).
- End for

3.2. Elementary rotation. Let \(\mathcal{W} = \mathcal{A}(Q_{k-1})\) and \(\mathcal{T} = \mathcal{W}(G^{(i_k, j_k, \theta)})\). As in Algorithm 1, we define
\[
h_k : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow \mathbb{R}^+, \ \theta \mapsto f(Q_{k-1}G^{(i_k, j_k, \theta)}) = \sum_{i=1}^{p} T_{i-1}^2
\] (13)
where \(f\) is as in (6). Note that \(G^{(i_k, j_k, \theta)} = G^{(i_k, j_k, \theta + 2\pi)}\) and \(T_{i-1}^2(\theta) = T_{i-1}^2(\theta + \pi)\) for any \(\theta \in \mathbb{R}\) and \(1 \leq i \leq p\). We see that \(h_k\) has the same image with that defined on \(\mathbb{R}\). So it is sufficient to determine \(\theta_k^* \in [-\pi/2, \pi/2]\) such that \(h_k(\theta_k^*) = \max_{\theta} h_k(\theta)\), and we choose \(\theta_k^*\) with the smallest absolute value if there are more than one choices.

Denote by \(\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm \infty\}\). Define
\[
\tau_k : \overline{\mathbb{R}} \rightarrow \mathbb{R}^+, \ x \mapsto h_k(\arctan(x)).
\]
Let \(x = \tan(\theta) \in \overline{\mathbb{R}}\) and \(x_k^* = \tan(\theta_k^*)\). Then
\[
\tau_k(x) - \tau_k(0) = h_k(\theta) - h_k(0) = \sum_{i=1}^{p} T_{i-1}^2 - \sum_{i=1}^{p} W_i^2.
\]

**Lemma 3.1.** Let \(h_k\) be as in (13). Then \(h_k'(\theta) = -2\Lambda(Q_{k-1}G^{(i_k, j_k, \theta)})_{i_k, j_k}\).

**Proof.** We denote by \(G(\theta) = G^{(i_k, j_k, \theta)}\) for convenience. Then it follows from (9) and the methods similar to [23, Lemma 5.7] that
\[
h_k'(\theta) = \langle \text{Proj} \nabla f(Q_{k-1}G(\theta)), Q_{k-1}G'(\theta) \rangle = \langle Q_{k-1}G(\theta)\Lambda(Q_{k-1}G(\theta)), Q_{k-1}G'(\theta) \rangle = \langle \Lambda(Q_{k-1}G(\theta)), G(\theta)^T G'(\theta) \rangle = -2\Lambda(Q_{k-1}G(\theta))_{i_k, j_k}.
\]
\[
\square
\]

**Remark 3.2.** Let \((i_k, j_k) \in \mathcal{C}_1\). Then \(h_k(\theta)\) in (13) also has a period \(\pi/2\) by [23, Section 4.3]. In other words, we can choose \(\theta_k^* \in [-\pi/4, \pi/4]\) to maximize \(h_k(\theta)\). Equivalently, we can choose \(x_k^* \in [-1, 1]\) to maximize \(\tau_k(x)\).
3.3. Examples. Let $A \in \text{symm}(\mathbb{R}^{n \times \cdots \times n})$ be of 3rd or 4th order. Now we show the details of how to solve $\theta_k^*$ in Algorithm 1. In fact, the methods in Example 3.3(i) and Example 3.5(i) were first formulated in [9], and can also be found in [23, Section 6.2]. We present them here for convenience.

Example 3.3. (For 3rd order symmetric tensors)
(i) Case 1: $(i_k, j_k) \in C_1$. Take $p = 2$ and the pair $(1,2)$ for example. Let
\[
a = 6(W_{111}W_{112} - W_{122}W_{222}),
\]
\[
b = 6(W_{111}^2 + W_{122}^2 - 3W_{112}^2 - 3W_{121}^2 - 2W_{111}W_{122} - 2W_{112}W_{222}).
\]
Then we have that
\[
\tau_k(x) - \tau_k(0) = \frac{1}{(1 + x^2)^2} (a(x - x^3) - \frac{b}{2} x^2), \tag{14}
\]
\[
\tau_k'(x) = \frac{1}{(1 + x^2)^3} (a(1 - 6x^2 + x^4) - b(x - x^3)).
\]
Denote by $\xi = x - 1/x$. Then $\tau_k'(x) = 0$ if and only if $\Omega(\xi) \triangleq a\xi^2 + b\xi - 4a = 0$.

Solve $\Omega(\xi) = 0$ for all the real roots $\xi_\ell$. Then solve $x^2 - \xi_\ell x - 1 = 0$ for all $\ell$ and take the best real root as $x_k^*$.

(ii) Case 2: $(i_k, j_k) \in C_2$. Take $p = 2$ and the pair $(1,3)$ for example. It holds that
\[
\tau_k(x) - \tau_k(0) = T_{2111}^2 - W_{1111}^2 = \frac{1}{(1 + x^2)^3} [(W_{333}^2 - W_{1111}^2)x^6 + (6W_{133}W_{333})x^5
\]
\[
+ (-3W_{111}^2 + 9W_{133}^2 + 6W_{113}W_{333})x^4 + (18W_{113}W_{133} + 2W_{111}W_{333})x^3
\]
\[
+ (-3W_{111}^2 + 6W_{133}W_{111} + 9W_{113}^2)x^2 + (6W_{111}W_{113})x], \tag{15}
\]
\[
\tau_k'(x) = \frac{6T_{1111}(x)}{(1 + x^2)^{5/2}} [-W_{333}x^3 + (W_{333} - 2W_{113})x^2 + (2W_{133} - W_{111})x + W_{113}].
\]
Then we solve
\[
-W_{133}x^3 + (W_{333} - 2W_{113})x^2 + (2W_{133} - W_{111})x + W_{113} = 0, \tag{16}
\]
and take $x_k^*$ to be the best point among these real roots and $\pm \infty$.

Remark 3.4. (16) is similar to equations in [16, Section 3.5], which is for the best rank-1 approximation of a tensor in $\text{symm}(\mathbb{R}^{2\times 2\times 2})$.
Example 3.5. (For 4th order symmetric tensors)

(i) **Case 1**: \((i_k,j_k) \in C_1\). Take \(p = 2\) and the pair \((1,2)\) for example. It holds that
\[
\tau_k(x) - \tau_k(0) = T^2_{1111} + T^2_{2222} - W^2_{1111} - W^2_{2222} \\
= \frac{1}{(1 + x^2)^4}((8W_{1111}W_{1112} - 8W_{1122}W_{2222})(x - x^7) \\
+ (-4W^2_{1111} + 12W_{1122}W_{1111} + 16W^2_{1112} - 4W^2_{2222} + 12W_{1122}W_{2222})(x^2 + x^6) \\
+ (48W_{1112}W_{1122} + 8W_{1111}W_{2222} - 48W_{1122}W_{1122} - 8W_{1112}W_{2222})(x^3 - x^5) \\
+ (-6W^2_{1111} + 4W_{1111}W_{2222} + 72W^2_{1122} - 6W^2_{2222} + 64W_{1112}W_{1122})x^4).
\]

Denote by
\[
\begin{align*}
\alpha &= 8(W_{1111}W_{1112} - W_{1122}W_{2222}); \\
\beta &= 8(W^2_{1111} - 3W_{1122}W_{1111} - 4W^2_{1112} - 4W^2_{2222} + W^2_{2222} - 3W_{1122}W_{2222}); \\
\gamma &= 8(18W_{1112}W_{1122} - 7W_{1111}W_{1112} + 3W_{1111}W_{1122} \\
&\quad - 18W_{1122}W_{1122} - 3W_{1112}W_{2222} + 7W_{1122}W_{2222}); \\
\delta &= 8(9W_{1111}W_{1122} - 32W_{1112}W_{1122} - 2W_{1111}W_{2222} \\
&\quad + 9W_{1122}W_{2222} + 12W^2_{1112} - 36W^2_{1112} + 12W^2_{1122}); \\
\epsilon &= 80(6W_{1122}W_{1122} - W_{1111}W_{2222} - 6W_{1112}W_{1122} + W_{1112}W_{2222}).
\end{align*}
\]

Then
\[
\tau_k'(x) = \frac{1}{(1 + x^2)^4}[\alpha(1 + x^8) + \beta(x^7 - x) + \gamma(x^6 + x^2) + \delta(x^5 - x^3) + \epsilon x^4].
\]

Denote by \(\xi = x - 1/x\). It follows that \(\tau_k'(x) = 0\) if and only if
\[
\Omega(\xi) \overset{\text{def}}{=} a\xi^4 + b\xi^3 + (4a + c)\xi^2 + (3b + d)\xi + 2a + 2c + \epsilon = 0.
\]

Solve \(\Omega(\xi) = 0\) for all the real roots \(\xi_\ell\). Then solve \(x^2 - \xi_\ell x - 1 = 0\) for all \(\ell\) and take the best real root as \(x_\ell^k\).

(ii) **Case 2**: \((i_k,j_k) \in C_2\). Take \(p = 2\) and the pair \((1,3)\) for example. It holds that
\[
\tau_k(x) - \tau_k(0) = \frac{1}{(1 + x^2)^4}[(W^2_{3333} - W^2_{1111})x^8 + (8W_{1113}W_{3333})x^7 \\
+ (-4W^2_{1111} + 16W^2_{1333} + 12W_{1133}W_{3333})x^6 + (48W_{1133}W_{1333} + 8W_{1113}W_{3333})x^5 \\
+ (-6W^2_{1111} + 2W_{3333}W_{1111} + 36W^2_{1133} + 32W_{1113}W_{1333})x^4 \\
+ (48W_{1113}W_{1333} + 8W_{1111}W_{1333})x^3 \\
+ (-4W^2_{1111} + 12W_{1133}W_{1111} + 16W^2_{1113})x^2 + (8W_{1111}W_{1113})x],
\]

\[
\tau_k'(x) = \frac{-8T_{1111}}{(1 + x^2)^4}[W_{1333}x^4 + (3W_{1133} - W_{3333})x^3 + (3W_{1113} - 3W_{1333})x^2 \\
+ (W_{1111} - 3W_{1133})x - W_{1113}].
\]
Then we solve
\[ W_{1333}x^4 + (3W_{1113} - W_{3333})x^3 + (3W_{1113} - 3W_{1333})x^2 + (W_{1111} - 3W_{1333})x - W_{1113} = 0 \]
and take \( x_k^* \) to be the best point among these real roots and \( \pm \infty \).

4. Weak convergence to stationary points

Let \( N = p(2n - p - 1)/2 \) be the number of elements in \( \mathcal{C} \). We denote by \( \Sigma \) the set of all the ordered sets \( \mathcal{P} \) of index pairs in \( \mathcal{C} \), that is,

\[ \mathcal{P} = \{(i_1, j_1), (i_2, j_2), \ldots, (i_N, j_N)\} \in \Sigma. \]

We denote by \( \Sigma_0 \subseteq \Sigma \) the subset including

\[ \mathcal{P}^* = \{(i_1^*, j_1^*), (i_2^*, j_2^*), \ldots, (i_N^*, j_N^*)\}, \]

which satisfies that the first \( n-1 \) pairs \( \{(i_1^*, j_1^*), \ldots, (i_{n-1}^*, j_{n-1}^*)\} \} \) have one common index, the next \( n-2 \) pairs \( \{(i_n^*, j_n^*), \ldots, (i_{n-4}^*, j_{n-4}^*)\} \} \) have one common index, the next \( n-3 \) pairs \( \{(i_{n-2}^*, j_{n-2}^*), \ldots, (i_{n-6}^*, j_{n-6}^*)\} \} \) have one common index, until the last \( n-p \) pairs \( \{(i_{N-n+p+1}^*, j_{N-n+p+1}^*), \ldots, (i_N^*, j_N^*)\} \} \) have one common index.

**Definition 4.1.** Let \( \mathcal{P}_1, \mathcal{P}_2 \in \Sigma \). We say that \( \mathcal{P}_1 \) is equivalent to \( \mathcal{P}_2 \) if we can obtain \( \mathcal{P}_2 \) from \( \mathcal{P}_1 \) only by

(i) exchanging the positions of \((i_t, j_t)\) and \((i_{t+1}, j_{t+1})\) when \( \{i_t, j_t\} \cap \{i_{t+1}, j_{t+1}\} = \emptyset \);
(ii) moving the first element to the position after the last one;
(iii) moving the last element to the position before the first one;
(iv) reversing the positions of all the elements.

**Example 4.2.** (i) Let \( n = p = 4 \). Let \( \mathcal{P} = \{(1,3), (2,3), (2,4), (1,4), (3,4), (1,2)\} \). We can see that \( \mathcal{P} \) is equivalent to

\[ \{(2,4), (1,4), (3,4), (1,2), (1,3), (2,3)\} \] and \[ \{(3,4), (1,3), (2,3), (2,4), (1,4), (1,2)\}, \]

which are both in \( \Sigma_0 \). On the other hand, it is not difficult to see that

\[ \{(1,2), (1,4), (2,3), (2,4), (1,3), (3,4)\} \] (17)

is not equivalent to any \( \mathcal{P}^* \in \Sigma_0 \).

(ii) Let \( n = p \). We can verify that there always exists such \( \mathcal{P} \in \Sigma \) as in (17) when \( n \) is odd and \( n \geq 5 \). In fact, in this case, we can construct a graph by setting the numbers as vertices and the index pairs in \( \mathcal{C} \) as edges. Then, by Euler’s Theorem, there always exists an Eulerian circuit, which is corresponding to a \( \mathcal{P} \in \Sigma \) not equivalent to any \( \mathcal{P}^* \in \Sigma_0 \). When \( n = 5 \), one such \( \mathcal{P} \) is

\[ \{(1,2), (2,3), (3,4), (4,5), (3,5), (1,3), (1,4), (2,4), (2,5), (1,5)\}. \]

**Algorithm 2.** (General algorithm)

**Input:** \( \mathcal{A} \in \text{symm}(\mathbb{R}^{n \times \cdots \times n}), 1 \leq p \leq n, \) a starting point \( Q_0 \), an ordered set \( \mathcal{P} \in \Sigma \).

**Output:** Sequence of iterations \( Q_k \).
• For $k = 1, 2, \ldots$ until a stopping criterion is satisfied do
• Choose the pair $(i_k, j_k) \in \mathcal{C}$ according to $\mathcal{P}$.
• Solve $\theta_k^*$ that maximizes $h_k(\theta)$ defined as in (13).
• Set $U_k \overset{\text{def}}{=} G^{(i_k, j_k, \theta_k^*)}$, and update $Q_k = Q_{k-1}U_k$.
• End for

Let $\mathcal{A} \in \text{symm}(\mathbb{R}^{n \times \cdots \times n})$ and $Q \in \mathcal{O}_n$. Let $\mathcal{W} = \mathcal{A}(Q)$ and $(i, j) \in \mathcal{C}$. Suppose that $\theta_*$ is the maximal point of the function
\[ h : \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \rightarrow \mathbb{R}_+, \ \ \theta \mapsto f(QG^{(i, j, \theta)}) \]
as in (13). We define the operators $\Phi_{i, j}$ by sending $Q$ to $QG^{(i, j, \theta_*)}$. Then the iterations in the $t$-th loop\(^3\) of Algorithm 2 are in fact generated as follows:
\[
\cdots \xrightarrow{\Phi_{i_3, j_3}} Q_{(t-1)N} \xrightarrow{\Phi_{i_2, j_2}} Q_{(t-1)N+1} \xrightarrow{\Phi_{i_1, j_1}} Q_{tN} \xrightarrow{\Phi_{i_2, j_2}} \cdots.
\]

We define $Q^{(t)} = Q_{tN}$ and $\Phi = \Phi_{i_N, j_N} \circ \cdots \circ \Phi_{i_2, j_2} \circ \Phi_{i_1, j_1}$. It is clear that $\Phi_{i, j}$ is continuous for all $(i, j) \in \mathcal{C}$. Therefore, $\Phi$ is also continuous. Now we rewrite [5, Lemma 5.5] as follows.

**Lemma 4.3.** Let $\Phi : \mathcal{O}_n \rightarrow \mathcal{O}_n$ be a continuous operator and the sequence $\{Q^{(t)}\}_{t=1}^{\infty} \subseteq \mathcal{O}_n$ satisfy $Q^{(t+1)} = \Phi(Q^{(t)})$. If a continuous function $f : \mathcal{O}_n \rightarrow \mathbb{R}$ satisfies

(i) the sequence $\{f(Q^{(t)})\}_{t=1}^{\infty}$ converges, and
(ii) if $f(\Phi(Q)) = f(Q)$, then $\Phi(Q) = Q$,

then every accumulation point $Q_*$ of $\{Q^{(t)}\}_{t=1}^{\infty}$ satisfies that $\Phi(Q_*) = Q_*$.  

**Lemma 4.4.** Suppose that $\mathcal{P}$ is equivalent to a $\mathcal{P}^* \in \Sigma_0$ and
\[ G^{(i_1, j_1, \theta_1)}G^{(i_2, j_2, \theta_2)} \cdots G^{(i_N, j_N, \theta_N)} = I_n, \]
where $\theta_k \in [-\pi/2, \pi/2]$. Then $G^{(i_k, j_k, \theta_k)} = I_n$ for $1 \leq k \leq N$.

**Proof.** Note that $\mathcal{P}$ is equivalent to $\mathcal{P}^* \in \Sigma_0$ and the position changes in Definition 4.1 preserve (18). After a finite number of such position changes, there exist $\theta_k^* \in [-\pi/2, \pi/2]$ such that
\[ G^{(i_1, j_1, \theta_1^*)}G^{(i_2, j_2^*, \theta_2^*)} \cdots G^{(i_N, j_N^*, \theta_N^*)} = I_n. \]

Without loss of generality, we can suppose that
\[ \mathcal{P}^* = \{(1, 2), (1, 3), \cdots, (1, n), (2, 3), \cdots, (2, n), (3, 4), \cdots, (p, n)\}, \]
as in (12). Then (19) is $G^{(1, 2, \theta_1^*)}G^{(1, 3, \theta_2^*)} \cdots G^{(p, n, \theta_N^*)} = I_n$. It follows that
\[ G^{(1, 3, \theta_2^*)} \cdots G^{(p, n, \theta_N^*)} = G^{(1, 2, -\theta_1^*)}. \]

\(3\)each loop contains $N$ successive iterations.
It is not difficult to verify that $(G^{(1,3,θ_2^*)} \cdots G^{(p,n,θ_N^*)})_{12} = 0$. Then $θ_1^* = 0$. Similarly, by $G^{(1,4,θ_2^*)} \cdots G^{(p,n,θ_N^*)} = G^{(1,3,−θ_2^*)}$, we get $θ_2^* = 0$. After repeating this process for $N−1$ times, we complete the proof. □

Remark 4.5. It may be interesting to ask whether Lemma 4.4 holds for any $P \in \Sigma$. In fact, when $p = n = 4$, a counterexample is

$$G^{(1,2,π/2)}G^{(1,4,π/2)}G^{(2,3,−π/2)}G^{(2,4,−π/2)}G^{(1,3,−π/2)}G^{(3,4,−π/2)} = I_4.$$ 

Theorem 4.6. In Algorithm 2, if $P$ is equivalent to a $P^* \in \Sigma_0$, then every accumulation point is a stationary point.

Proof. Suppose that $Q_*$ is an accumulation point of $\{Q_k, k \in \mathbb{N}\}$. Then there exists $1 ≤ ℓ_* ≤ N$ such that $Q_*$ is an accumulation point of $\{Q_{tN+ℓ_*}, t \in \mathbb{N}\}$.

Case I: If $ℓ_* = N$, by Lemma 4.3, we see that $Φ(Q_*) = Q_*$. It follows by Lemma 4.4 that $Φ_{i,j}(Q_*) = Q_*$ for all $(i, j) \in \mathcal{E}$. Then $Q_*$ is a stationary point by Theorem 2.5.

Case II: If $ℓ_* < N$, we can set the starting point as $Q_{ℓ_*}$. Let $P' \in \Sigma$ be obtained by doing the manipulation (ii) of Definition 4.1 on $P$ successively for $ℓ_*$ times. Let $Φ'$ be the composition corresponding to $P'$. Similar to Case I, we see that $Φ'(Q_*) = Q_*$. Note that $P'$ is also equivalent to $P^* \in \Sigma_0$. By the similar reduction as in Case I, we complete the proof. □

Corollary 4.7. (i) In Algorithm 1, every accumulation point is a stationary point.
(ii) In Jacobi CoM2 algorithm, every accumulation point is a stationary point.

5. JACOBI-G ALGORITHM AND ITS CONVERGENCE

5.1. Jacobi-G algorithm. Different from the cyclic ordering (12) in Algorithm 1 or the fixed ordering $P$ in Algorithm 2, another pair selection rule of Jacobi-type algorithm based on the Riemannian gradient was proposed in [18]. In this sense, the pair $(i_k, j_k)$ at each iteration is chosen such that

$$|h_k^{(i_k)}(0)| = 2|(Q_{k−1})_{i_k,j_k} \cdot ∇f(Q_{k−1})|_k| ≥ ε\| \text{Proj} ∇f(Q_{k−1})\|,$$ (20)

where $0 < ε ≤ 2/n$ is fixed. By [18, Lemma 5.2] and [23, Lemma 3.1], we see that it is always possible to find such a pair if $f$ is differentiable.

Algorithm 3. (Jacobi-G algorithm)
Input: $A \in \text{symm}(\mathbb{R}^{n×n})$, $1 ≤ p ≤ n$, $0 < ε ≤ 2/n$, a starting point $Q_0$.
Output: Sequence of iterations $\{Q_k\}_{k≥1}$.

- For $k = 1, 2, \ldots$ until a stopping criterion is satisfied do
  - Choose a pair $(i_k, j_k)$ satisfying (20) at $Q_{k−1}$.
  - Solve $θ_k^*$ that maximizes $h_k(θ)$ defined as in (13).
  - Set $U_k \overset{\text{def}}{=} G^{(i_k,j_k,θ_k^*)}$, and update $Q_k = Q_{k−1}U_k$.
- End for
Remark 5.1. (i) By [18, Theorem 5.4] and [23, Theorem 3.3], we see that every accumulation point of the iterations in Algorithm 3 is a stationary point of $f$.
(ii) Let $\mathcal{A} \in \text{symm}(\mathbb{R}^{n \times n \times n})$ and $p = 1$. Then Algorithm 3 is the same with the Jacobi-type algorithm in [18], which was developed to find the best low multilinear rank approximation of symmetric tensors.

In this section, we mainly prove the following result for Algorithm 3. The proof is postponed to Section 5.3.

Theorem 5.2. Let $\mathcal{A} \in \text{symm}(\mathbb{R}^{n \times n \times n})$ with $n \geq 3$. Suppose that $p = 2$ and $Q_*$ is an accumulation point of Algorithm 3 satisfying

\begin{align}
\mathcal{A}(Q_*)^2_{112} + \mathcal{A}(Q_*)^2_{122} &\neq 0, \quad (21) \\
\mathcal{A}(Q_*)_{333} \mathcal{A}(Q_*)_{444} \cdots \mathcal{A}(Q_*)_{nnn} &\neq 0. \quad (22)
\end{align}

Then either $Q_*$ is the unique limit point, or there exist an infinite number of accumulation points.

5.2. Some lemmas.

Lemma 5.3. Let $\mathcal{W} \in \text{symm}(\mathbb{R}^{2 \times 2 \times 2})$ and $\mathcal{T} = \mathcal{W}(G^{(1,2,\arctan x)})$ with $x \in \mathbb{R}$. Define $\tau : \mathbb{R} \to \mathbb{R}^+$ sending $x$ to $\mathcal{T}^2_{111}$. Suppose that $\mathcal{W}_{222} \neq 0$ and $\tau(0) = \max_{x \in \mathbb{R}} \tau(x)$. Then

(i) $\mathcal{W}_{111} \neq 0$, $\mathcal{W}_{112} = 0$,
(ii) $\mathcal{W}_{111}(2\mathcal{W}_{122} - \mathcal{W}_{111}) < 0$.

Proof. (i) It is clear that $|\mathcal{W}_{222}| \leq |\mathcal{W}_{111}|$ since $\tau(0) \geq \tau(\pm \infty)$. Then $\mathcal{W}_{111} \neq 0$. Let $\theta = \arctan x$. We have that

$$
d\mathcal{T}_{111} = 3\mathcal{T}_{112}, \quad d\mathcal{T}_{112} = 2\mathcal{T}_{122} - \mathcal{T}_{111}
$$

by straightforward differentiation [23, Page 10]. It follows that

$$
\tau'(x) = 2\mathcal{T}_{111} \frac{d\mathcal{T}_{111}}{d\theta} \frac{d\theta}{dx} = 6\mathcal{T}_{111} \mathcal{T}_{112} \frac{1}{1 + x^2},
$$

$$
\tau''(x) = \frac{6}{(1 + x^2)^2} (3\mathcal{T}_{112}^2 + 2\mathcal{T}_{111} \mathcal{T}_{122} - \mathcal{T}_{111}^2 - 2\mathcal{T}_{111} \mathcal{T}_{112} x).
$$

Note that $\tau'(0) = 0$. We have $\mathcal{W}_{112} = 0$ by (23).

(ii) Note that $\tau''(0) \leq 0$. We have $2\mathcal{W}_{111} \mathcal{W}_{122} - \mathcal{W}_{111}^2 \leq 0$ by (24). To complete the proof, we only need to prove that $\tau(0) < \max_{x \in \mathbb{R}} \tau(x)$ if $\mathcal{W}_{111} = 1$, $\mathcal{W}_{122} = 1/2$ and $\mathcal{W}_{222} = \beta \neq 0$ without loss of generality. In fact, it can be verified that

$$
\tau(x) = \frac{(1 + \frac{3}{2} x^2 + \beta x^3)^2}{(1 + x^2)^3}
$$

for $x \in \mathbb{R}$. Then $\tau(x)$ has a unique minimum at $x = 0$.
in this case, and
\[
\max_{x \in \mathbb{R}} \tau(x) \geq \tau(2\beta) = \frac{(1 + 6\beta^2 + 8\beta^4)^2}{(1 + 4\beta^2)^3} > \tau(0) = 1.
\]

**Definition 5.4.** ([24, Definition 3.11]) Let \( A \in \text{symm}(\mathbb{R}^{n \times n}) \) and \( 1 \leq i < j \leq n \). Suppose that \( A_{iii}A_{ijj} = A_{ijj}A_{jjj} \). The stationary diagonal ratio, denoted by \( \gamma_{ij}(A) \), is defined as follows.

\[
\gamma_{ij}(A) \begin{dcases}
0, & \text{if } A^{(i,j)} = 0; \\
\infty, & \text{if } A_{iii} = A_{jjj} = 0 \text{ and } A_{ijj}^2 + A_{ii}^2 \neq 0; \\
\end{dcases}
\]

otherwise, \( \gamma_{ij}(A) \) is the (unique) number such that

\[
\left( \begin{array}{c} A_{ijj} \\ A_{ii} \end{array} \right) = \gamma_{ij}(A) \left( \begin{array}{c} A_{iii} \\ A_{jjj} \end{array} \right).
\]

**Lemma 5.5.** Let \( W \in \text{symm}(\mathbb{R}^{2 \times 2}) \) and \( T = W(G(1,2,\arctan x)) \) with \( x \in \mathbb{R} \) and \( x \neq 0 \). Suppose that \( \| \text{diag}(W) \| = \| \text{diag}(T) \| \neq 0 \) and

\[
W_{111}W_{112} = W_{122}W_{222}, \quad T_{111}T_{112} = T_{122}T_{222}.
\]

Then \( \gamma_{12}(W) = \gamma_{12}(T) = -1 \) or \( 1/3 \).

**Proof.** Note that \( \| \text{diag}(W) \| = \| \text{diag}(T) \| \) and \( \| W \| = \| T \| \). We see that \( |\gamma_{12}(W)| = |\gamma_{12}(T)| \). Let \( T = W(G(1,2,\arctan x)) \). Define

\[
\tau : \mathbb{R} \to \mathbb{R}^+, \quad x \mapsto \| \text{diag}(T) \|^2 = T_{111}^2 + T_{222}^2.
\]

Then \( \tau(x) = \tau(0) \) by the condition. It follows by (14) that

\[
W_{111}^2 + W_{222}^2 - 3W_{112}^2 - 3W_{122}^2 - 2W_{111}W_{112} - 2W_{112}W_{222} = 0. \tag{25}
\]

After the substitution of \( W_{112} = \gamma_{12}(W)W_{111} \) and \( W_{112} = \gamma_{12}(W)W_{222} \) to (25), we get that \( \gamma_{12}(W) = -1 \) or \( 1/3 \). Note that \( W = T(G(1,2,\arctan x))^T \). We can similarly get that \( \gamma_{12}(T) = -1 \) or \( 1/3 \). \( \square \)

**Lemma 5.6.** Let \( W \in \text{symm}(\mathbb{R}^{3 \times 3}) \) and \( T = W(G(1,3,\arctan x)) \) with \( x \in \mathbb{R} \) and \( x \neq 0 \). Suppose that \( |W_{111}| = |T_{111}| > 0 \) and

\[
W_{111}W_{112} = W_{122}W_{222}, \quad T_{111}T_{112} = T_{122}T_{222}, \quad W_{113} = W_{223} = T_{113} = T_{223} = 0.
\]

Then \( W_{112} = W_{122} = T_{112} = T_{122} = 0 \).

**Proof.** It can be verified that

\[
-\frac{x}{\sqrt{1 + x^2}}W_{122} = T_{223} = 0,
\]

and thus \( W_{122} = 0 \). It follows by the condition that \( W_{112} = 0 \). Note that \( W = T(G(1,3,\arctan x))^T \). We can similarly get that \( T_{112} = T_{122} = 0 \). \( \square \)
5.3. Proof of Theorem 5.2.

Lemma 5.7. Let $\mathcal{A} \in \text{symm}(\mathbb{R}^{n \times n \times n})$. Let $h_k(\theta)$ be as in (13) for $k \in \mathbb{N}$. Then there exists $\delta > 0$ such that
\[ h_k(\theta^*_k) - h_k(0) \geq \delta |h'_k(0)|^2 \]  
for any $k \in \mathbb{N}$ with $(i_k, j_k) \in C_2$ in Algorithm 3.

Proof. Let $\mathcal{W} = \mathcal{A}(Q_{k-1})$ and $\mathcal{T} = \mathcal{W}(\mathcal{G}^{(i_k, j_k, \theta)})$. Let $(i, j) = (i_k, j_k)$. It is clear that $\mathcal{T}_{ii}(\theta)$ is a trigonometric polynomial with a finite degree $n_0$ for all the iterations in $C_2$. By [3, Theorem 1], we see that
\[ \mathcal{T}_{ii}(0)^2 \leq n_0^2(\|\mathcal{T}_{ii}\|_{\infty} - \mathcal{T}_{ii}^2(0)) = n_0^2(h_k(\theta^*_k) - h_k(0)), \]
when $\theta = 0$. Note that $h'_k(0) = 2\mathcal{T}_{ii}(0)\mathcal{T}_{ii}'(0)$. Let $M > 0$ such that $|4n_0^2\mathcal{T}_{ii}^2(0)| < M$ for all the iterations in $C_2$. Then
\[ |h'_k(0)|^2 \leq 4n_0^2\mathcal{T}_{ii}^2(0)(h_k(\theta^*_k) - h_k(0)) < M(h_k(\theta^*_k) - h_k(0)). \]
The proof is completed if we set $\delta = 1/M$. \hfill \Box

Remark 5.8. Let $\mathcal{A} \in \text{symm}(\mathbb{R}^{n \times n \times n \times n})$ be of 4th order. By the similar methods, we can also prove (26) for pairs in $C_1$, or pairs in $C_2$.

Now we need a result in [23], which is the direct consequence of [29, Theorem 2.3].

Theorem 5.9. ([23, Corollary 5.4]) Let $f$ be a real analytic function from $\mathcal{O}_n$ to $\mathbb{R}$. Suppose that $\{Q_k : k \in \mathbb{N}\} \subseteq \mathcal{O}_n$ and, for large enough $k$,
(i) there exists $\sigma > 0$ such that
\[ |f(Q_k) - f(Q_{k-1})| \geq \sigma \|\text{Proj} \nabla f(Q_{k-1})\| \|Q_k - Q_{k-1}\|, \]
(ii) $\text{Proj} \nabla f(Q_{k-1}) = 0$ implies that $Q_k = Q_{k-1}$.
Then the iterations $\{Q_k : k \in \mathbb{N}\}$ converge to a point $Q_\ast \in \mathcal{O}_n$.

Proof of Theorem 5.2. Assume that there exist a finite number of accumulation points, denoted by $Q^{(\ell)}(1 \leq \ell \leq N)$. Then any accumulation point is a stationary point by Remark 5.1(i). In other words, it holds that $\Lambda(Q^{(\ell)}) = 0$ for all $1 \leq \ell \leq N$ by (9). Let $Q_\ast = Q^{(1)}$. Now we prove that $Q_\ast$ is the unique limit point.

Step 1. We first prove that all the accumulation points satisfy (21) and (22) if $Q_\ast$ satisfies them. Note that the number of accumulation points is finite. We can see that any two different accumulation points can be connected by finite combination of the following two possible paths.
(a) Take the pair $(1, 2) \in C_1$. If $\{x_k^*, (i_k, j_k) = (1, 2)\}$ is finite or converges to 0, this path doesn’t appear and we skip it. Otherwise, this set has a nonzero accumulation point $\zeta$ and a subsequence converges to it. We assume that
\[ \{x_k^*, (i_k, j_k) = (1, 2)\} \to \zeta \neq 0 \]
without loss of generality. Note that \( \{Q_{k-1}, (i_k, j_k) = (1, 2)\} \) has an accumulation point. We assume that
\[
\{Q_{k-1}, (i_k, j_k) = (1, 2)\} \rightarrow Q^{(\ell_1)}
\]
without loss of generality. Then \( Q^{(\ell_2)} = Q^{(\ell_1)} G^{(1,2,\arctan \zeta)} \) is another different accumulation point. It is clear that \( A(Q^{(\ell_1)})_{iii} = A(Q^{(\ell_2)})_{iii} \) for \( 3 \leq i \leq n \). Note that \( A(Q^{(\ell_1)})^{(1,2)} \) and \( A(Q^{(\ell_2)})^{(1,2)} \) satisfy the conditions in Lemma 5.5. We see that
\[
A(Q^{(\ell_1)})^2_{112} + A(Q^{(\ell_1)})^2_{122} \neq 0, \quad A(Q^{(\ell_2)})^2_{112} + A(Q^{(\ell_2)})^2_{122} \neq 0.
\]
(b) Take the pair \( (1, 3) \in C_2 \) for example. Other pairs in \( C_2 \) are similar. If \( x^*_k, (i_k, j_k) = (1, 3) \) is finite or converges to 0, this path doesn’t appear and we skip it. Otherwise, this set has a nonzero accumulation point \( \zeta \) and a subsequence converges to it. We assume that
\[
\{x^*_k, (i_k, j_k) = (1, 3)\} \rightarrow \zeta \neq 0
\]
without loss of generality. Note that \( \{Q_{k-1}, (i_k, j_k) = (1, 3)\} \) has an accumulation point. We assume that
\[
\{Q_{k-1}, (i_k, j_k) = (1, 3)\} \rightarrow Q^{(\ell_1)}
\]
without loss of generality. Then \( Q^{(\ell_2)} = Q^{(\ell_1)} G^{(1,3,\arctan \zeta)} \) is another different accumulation point. Note that \( A(Q^{(\ell_1)})^{(1,2,3)} \) and \( A(Q^{(\ell_2)})^{(1,2,3)} \) satisfy the conditions in Lemma 5.6. We see that
\[
A(Q^{(\ell_1)})_{112} = A(Q^{(\ell_1)})_{122} = A(Q^{(\ell_2)})_{112} = A(Q^{(\ell_2)})_{122} = 0.
\]
Since \( Q_\ast \) satisfies (21), we see that path (a) is the only possible path. Then all the accumulation points satisfy (21). Note that \( Q_\ast \) satisfies (22) and \( A(Q^{(\ell_1)})_{iii} = A(Q^{(\ell_2)})_{iii} \) for \( 3 \leq i \leq n \) in path (a). All the accumulation points satisfy (22).

**Step 2.** Since path (b) in Step 1 doesn’t appear, we get that
\[
\{x^*_k, (i_k, j_k) \in C_2\} \rightarrow 0 \tag{27}
\]
in Algorithm 3. Let \( N(Q_\ast, \eta) \) be the neighborhood of \( Q_\ast = Q^{(1)} \) in \( \Omega_n \) with radius \( \eta > 0 \) such that there exist no other accumulation points in this neighborhood. If pair \( (i, j) \in C_2 \) satisfies that
\[
\{Q_{k-1} \in N(Q_\ast, \eta), (i_k, j_k) = (i, j)\} \text{ is infinite,} \tag{28}
\]
then \( A(Q_\ast)^{(i,j)} \) satisfies the conditions in Lemma 5.3(iii) by condition (22). Then \( A(Q_\ast)_{iii} A(Q_\ast)_{iii} - 2A(Q_\ast)_{iij} \neq 0 \). Let
\[
\rho_1 \overset{\text{def}}{=} \min |A(Q_\ast)_{iii} A(Q_\ast)_{iii} - 2A(Q_\ast)_{iij}|
\]
for all pairs \( (i, j) \in C_2 \) satisfying (28). Then \( \rho_1 > 0 \). For other accumulation points, we can similarly get \( \rho_\ell \) for \( 1 < \ell \leq N \). Then
\[
\rho \overset{\text{def}}{=} \min \rho_\ell > 0. \tag{29}
\]
Step 3. Now we show that there exists $\kappa > 0$ such that
\[
|h_k(\theta_k^*) - h_k(0)| \geq \kappa|\tilde{h}_k(0)||\theta_k^*|
\]
for all $(i_k, j_k) \in \mathcal{C}_2$. Let $\mathcal{W} = \mathcal{A}(Q_{k-1})$. Denote $(i, j) = (i_k, j_k)$. Note that $|x_k^*| < +\infty$ when $k$ is large enough by (27). Then by (16) and (27), we have that
\[
\frac{h'_k(0)}{x_k^*} = \frac{6\mathcal{W}_{ii}\mathcal{W}_{jj}}{x_k^*} = -6\mathcal{W}_{ii}[(2\mathcal{W}_{ij} - \mathcal{W}_{ii}) + (\mathcal{W}_{jj} - 2\mathcal{W}_{ij})x_k^* - \mathcal{W}_{ij}x_k^2]
\]
have accumulation points in the set
\[
\{-6\mathcal{A}(Q^{(i)})_{ii}(2\mathcal{A}(Q^{(i)})_{ij} - \mathcal{A}(Q^{(i)})_{ii}), \text{ pair } (i, j) \text{ satisfies } (28), 1 \leq \ell \leq N\}
\]
when $k \in \mathbb{N}$ with $(i_k, j_k) \in \mathcal{C}_2$. It follows from (29) that there exists $\nu > 0$ such that $|h'_k(0)| \geq \nu|x_k^*|$ when $k$ is large enough with $(i_k, j_k) \in \mathcal{C}_2$. Then we get (30) by Lemma 5.7.

Step 4. If $\{x_k^*, (i, j_k) = (1, 2) \in \mathcal{C}_1\}$ is finite, we skip it. Otherwise, by [23, (27)], we know that
\[
|h_k(\theta_k^*) - h_k(0)| = \frac{|x_k^*h'_k(0)|}{2(1 - x_k^2)} \geq \frac{1}{2}|h'_k(0)||\theta_k^*|
\]
for all $(i, j_k) \in \mathcal{C}_1$. Let $\omega = \min\{\kappa, 1/2\} > 0$. By (30) and (31), we get that
\[
|h_k(\theta_k^*) - h_k(0)| \geq \omega|h'_k(0)||\theta_k^*| \geq \frac{\sqrt{2}}{2}\omega\varepsilon\|\text{Proj}_f(Q_{k-1})\|\|Q_k - Q_{k-1}\|,
\]
for all $k \in \mathbb{N}$. Then $Q_*$ is the unique limit point by Theorem 5.9. \hfill \Box

6. Numerical experiments

In this section, we make some experiments to compare the performance of JLROA algorithm with the LROAT and SLROAT algorithms in [5], and Trust region algorithm by Manopt Toolbox in [4]. When $p = 1$, LROAT and SLROAT are exactly the HOPM and SHOPM algorithms in [16, 19], respectively. We use the cyclic ordering of JLROA algorithm in Algorithm 1 for simplicity except Example 6.4 and Example 6.5. The LROAT and SLROAT algorithms are both initialized via HOSVD [15], because we find they generally have better performance in this case.

Example 6.1. We randomly generate 1000 tensors in symm($\mathbb{R}^{10\times10\times10}$), and run JLROA and SLROAT algorithms for them. Denote by JVAL and SVAL the final value of (3) obtained by JLROA and SLROAT, respectively. Set the following notations.
(i) NUMG : the number of cases that JVAL is greater than SVAL;
(ii) NUMS : the number of cases that JVAL is smaller than SVAL;
(iii) NUME : the number of cases that JVAL is equal\footnote{the difference is smaller than 0.0001.} to SVAL;
(iv) RATIOG : the average of JVAL/SVAL when JVAL is greater than SVAL;
(v) RATIOS : the average of JVAL/SVAL when JVAL is smaller than SVAL.

Example 6.2. We randomly generate 1000 tensors in symm($\mathbb{R}^{10\times10\times10}$), and run JLROA and SLROAT algorithms for them. Denote by JVAL and SVAL the final value of (3) obtained by JLROA and SLROAT, respectively. Set the following notations.
(i) NUMG : the number of cases that JVAL is greater than SVAL;
(ii) NUMS : the number of cases that JVAL is smaller than SVAL;
(iii) NUME : the number of cases that JVAL is equal\footnote{the difference is smaller than 0.0001.} to SVAL;
(iv) RATIOG : the average of JVAL/SVAL when JVAL is greater than SVAL;
(v) RATIOS : the average of JVAL/SVAL when JVAL is smaller than SVAL.
The results are shown in Table 1 and Figure 1. It can be seen that JLROA algorithm has better performance when \( p > 2 \). They always get the same result when \( p = 1 \).

\[\begin{array}{cccccc}
\text{NumG} & \text{NumS} & \text{NumE} & \text{RatioG} & \text{RatioS} \\
\hline
p = 1 & 0 & 0 & 1000 & — & — \\
p = 2 & 328 & 441 & 231 & 1.0023 & 0.9982 \\
p = 5 & 747 & 246 & 7 & 1.0042 & 0.9985 \\
p = 8 & 900 & 99 & 1 & 1.0044 & 0.9992 \\
p = 10 & 815 & 180 & 5 & 1.0039 & 0.9996 \\
\end{array}\]

Example 6.2. Let \( \mathcal{A} \in \text{symm}(\mathbb{R}^{3 \times 3 \times 3 \times 3}) \) such that

\[
\begin{align*}
A_{1111} &= 0.2883, & A_{1122} &= -0.2485, & A_{1222} &= 0.2972, & A_{1333} &= -0.3619, \\
A_{2223} &= 0.2127, & A_{1112} &= -0.0031, & A_{1123} &= -0.2939, & A_{1223} &= 0.1862, \\
A_{2222} &= 0.1241, & A_{2333} &= 0.2727, & A_{1113} &= 0.1973, & A_{1133} &= 0.3847, \\
A_{1233} &= 0.0919, & A_{2223} &= -0.3420, & A_{3333} &= -0.3054,
\end{align*}
\]
as in [19, Example 1] and [5, Section 6.1]. It has been shown in [19, 5] that SHOPM \((p = 1)\) and SLROAT \((p = 2)\) fail to converge for \( \mathcal{A} \). We now see the convergence behaviour of JLROA algorithm. The results of JLROA, SLROAT and LROAT algorithms are shown in Figure 2. It can be seen that JLROA performances are always better than or equal to those of SLROAT and LROAT.

Example 6.3. We randomly generate 1000 tensors in \( \text{symm}(\mathbb{R}^{10 \times 10 \times 10}) \), and run JLROA and Trust region algorithms for them. Denote by JVal and TVal the final value of (3) obtained by JLROA and Trust region, respectively. Set the following notations.

(i) \( \text{NumG} \) : the number of cases that JVal is greater than TVal;
(ii) \( \text{NumS} \) : the number of cases that JVal is smaller than TVal;
(iii) \( \text{NumE} \) : the number of cases that JVal is equal to TVal;
(iv) \( \text{RatioG} \) : the average of JVal/TVal when JVal is greater than TVal;
(v) \( \text{RatioS} \) : the average of JVal/TVal when JVal is smaller than TVal.

The results are shown in Table 2 and Figure 3. It can be seen that RatioG is very large when \( p = 1, 2 \), which means that Trust region is not so stable as JLROA in these two cases. Correspondingly, Trust region algorithm has generally better performance when \( p > 2 \).

\(^{5}\)the difference is smaller than 0.0001.
Example 6.4. In this example, we show the influence of choice of $P \in \Sigma$ on the final results. Fix $1 \leq p \leq 10$ and randomly generate a $A \in \text{symm}(\mathbb{R}^{10 \times 10 \times 10})$. We first choose the cyclic ordering (12), and then randomly choose $P \in \Sigma$ for 200 times to run Algorithm 2. The results are shown in Figure 4. It can be seen that all the $P \in \Sigma$ have almost the same result when $p = 1$. However, when $p = 2$, these $P \in \Sigma$ are separated into different groups corresponding to different results. It may be interesting to study how to determine the $P \in \Sigma$ with the best result.

Example 6.5. Let $A \in \text{symm}(\mathbb{R}^{10 \times 10 \times 10})$ and $p = 2$. Suppose that $Q_*$ is an accumulation point of Algorithm 3. To check the frequency of conditions (21) and (22) being satisfied, we define

$$\omega = \min\{|W_{112}|, |W_{122}|, |W_{333}|, \ldots, |W_{nnn}|\},$$

where $W = A(Q_*)$. We choose the iteration $Q_K$ as the approximation of an accumulation point when $K$ is large enough ($K = 500$ in this experiment). We randomly generate $A \in \text{symm}(\mathbb{R}^{10 \times 10 \times 10})$ for 1000 times, and run Algorithm 3 to see the frequency that $\omega > 0$ (greater than 0.0001). The results are shown in Figure 5, where $\omega > 0$ for 991 times. It can be seen that the conditions (21) and (22) are satisfied in most cases.

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Figure 4. Results of Example 6.4. The unique green point is for cyclic ordering (12). Red points mean higher results than (12), while blue points mean lower results than (12).

Figure 5. Results of Example 6.5. Blue points mean that $\omega > 0$, while red points mean that $\omega = 0$.

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