CRYSTAL BASIS THEORY FOR A QUANTUM SYMMETRIC PAIR \((U, U')\)

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Abstract. We study the representation theory of a quantum symmetric pair \((U, U')\) with two parameters \(p, q\) of type \(\text{AIII}\), by using highest weight theory and a variant of Kashiwara’s crystal basis theory. Namely, we classify the irreducible \(U\)-modules in a suitable category and associate with each of them a basis at \(p = q = 0\), the \(\gamma\)-crystal basis. The \(\gamma\)-crystal basis of a finite-dimensional \(U\)-module is thought of as a “localization” of the \(\gamma\)-canonical basis, which was introduced by Huanchen Bao and Weiqiang Wang in 2013. Also, the \(\gamma\)-crystal bases have nice combinatorial properties as the ordinary crystal bases do.

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1. Introduction

1.1. Quantum Schur-Weyl duality. Jimbo [J86] established a quantum analog of the classical Schur-Weyl duality. Let $U_q(\mathfrak{gl}_n)$ denote the quantum enveloping algebra of $\mathfrak{gl}_n$, and $H(\mathfrak{S}_d)$ the Hecke algebra associated with the $d$-th symmetric group $\mathfrak{S}_d$, where $n \geq d$. Let $V$ denote the vector representation of $U_q(\mathfrak{gl}_n)$. Jimbo defined an $H(\mathfrak{S}_d)$-module structure on $V \otimes V$ by using the $R$-matrix for $V \otimes V$. Also, he proved that the actions of $U_q(\mathfrak{gl}_n)$ and $H(\mathfrak{S}_d)$ on $V \otimes V$ satisfy the double centralizer property, and hence, $V \otimes V$ decomposes as a $U_q(\mathfrak{gl}_n)$-$H(\mathfrak{S}_d)$-bimodule as:

$$V \otimes V = \bigoplus_{\lambda \in \Lambda} V(\lambda) \otimes S^\lambda,$$

where $\Lambda$ is an index set, and $\{V(\lambda) \mid \lambda \in \Lambda\}$ and $\{S^\lambda \mid \lambda \in \Lambda\}$ are families of nonisomorphic irreducible modules of $U_q(\mathfrak{gl}_n)$ and $H(\mathfrak{S}_d)$, respectively.

1.2. Quantum Schur-Weyl duality in type $B$. It has been known that there is no Schur-Weyl-type duality between the quantum enveloping algebra of type $B$ and the Hecke algebra $H(W_d)$ of type $B$. However, Bao and Wang discovered the double centralizer property between a quantum symmetric pair and $H(W_d)$ ([BW13]). More precisely, let $U^J = U^J_r$ be a coideal subalgebra of $U = U_{2r+1} = U_q(\mathfrak{sl}_{2r+1})$ such that $(U, U^J)$ forms a quantum analog of the symmetric pair of type $A_{3\{r\}}$ ([Le99], [Ko14]). In [BW13], Bao and Wang introduced the intertwiner $\Upsilon$, which played a central role when they defined the action of $H(W_d)$ on $V \otimes V$, and then, proved that the actions of $U^J$ and $H(W_d)$ on $V \otimes V$ satisfy the double centralizer property. A variant of this work, where $H(W_d)$ is replaced with the Hecke algebra of type $B_d$ with unequal parameters $(p, q)$, was done in [BWW16].

1.3. Representation theory of $U^J$. From the quantum Schur-Weyl duality in type $B$, we expect that there should exist a deep connection between the representation theory of $U^J$ and that of $H(W_d)$. However, here arises a problem: although the representation theory of $H(W_d)$ has been well-studied, little is known about that of $U^J$. This paper gives some fundamental results in the representation theory of $U^J$ by using analogs of highest weight theory and Kashiwara’s crystal basis theory.

In this paper, we treat the category $O^J_{\text{int}}$ consisting of all $U^J$-modules $M$ satisfying the following: $M$ is decomposed into its “weight spaces”, each of which is finite-dimensional; the set of weights of $M$ is bounded from above; $M$ is “integrable”.

References
We begin our study by decomposing $U^j$ into three parts. This is an analog of the triangular decomposition of $U$. Using this triangular decomposition of $U^j$, we define a “Verma module” associated with each weight. By its definition and the triangular decomposition of $U^j$, it possesses a unique irreducible quotient. Our first main result is

**Theorem A.** Every $U^j$-module in $O^j_{\text{int}}$ is completely reducible, and each irreducible $U^j$-module in $O^j_{\text{int}}$ is isomorphic to the irreducible quotient of a Verma module. Moreover, the isomorphism classes of irreducible $U^j$-modules in $O^j_{\text{int}}$ are parametrized by the pairs of partitions of length $r+1$ and $r$.

Following Kashiwara’s crystal basis theory, we introduce the notions of $j$-crystal basis and its $j$-crystal graph; this can be thought of as a basis at $p = q = 0$. The second main result of this paper is

**Theorem B.** Each irreducible $U^j$-module in $O^j_{\text{int}}$ admits a unique $j$-crystal basis whose $j$-crystal graph is connected with a single source.

This theorem and the complete reducibility of $U^j$-modules lead to the existence and uniqueness of $j$-crystal basis of a $U^j$-module. Also, as in the ordinary crystal basis theory, $j$-crystal bases have the tensor product rule.

**Theorem C.** Let $M$ be a $U^j$-module and $N$ a $U$-module. Suppose that $M$ admits a $j$-crystal basis $(L, B)$, and that $N$ has a crystal basis $(L', B')$. Then, $(L \otimes L', B \otimes B')$ is a $j$-crystal basis of $M \otimes N$. In particular, (by taking $M$ to be the trivial $U^j$-module) the crystal basis of a $U$-module $N$ is the $j$-crystal basis of $N$.

Here, let us recall a result in the representation theory of $U^j$ from [BW13]. In it, Bao and Wang introduced the notion of $j$-canonical basis for a finite-dimensional based $U$-module (in the sense of [Lu94 Chapter 27]). They proved that a finite-dimensional based $U$-module $(M, B)$ admits a unique $j$-canonical basis $B^j := \{ T_b \mid b \in B \}$ of the form

$$
T_b = b + \sum_{b' \in B, b' < b} t_{bb'}b', \quad t_{bb'} \in q\mathbb{Z}[q],
$$

where $<$ denotes a partial order on $B$ (see [BW13 Theorem 6.24] for details). By equation (1), the $\mathbb{Z}[q]$-span of $B^j$ coincides with that of $B$, and hence the set $\{ T_b + qB^j \mid b \in B \}$ is the crystal basis of $M$. Thus, the $j$-crystal basis of $M$ can be thought of as a “localization” of the $j$-canonical basis. Note that the category $O^j_{\text{int}}$ contains objects other than finite-dimensional based $U$-modules. For those objects, the notion of $j$-canonical basis has not been defined. We expect that we can “globalize” the $j$-crystal bases of such objects; namely, we expect that there exits a basis which we should call the $j$-canonical basis for each module in $O^j_{\text{int}}$.

Finally, we mention that $j$-crystal bases have rich combinatorial properties. In particular, the $j$-crystal basis of an irreducible $U^j$-module is realized as the set of pairs of semistandard Young tableaux of given shapes. As applications, we describe explicitly irreducible decompositions of $V_{2r+1}^{\otimes N}$ (Robinson-Shensted-type correspondence) and the tensor product of an irreducible $U^j$-module with an irreducible $U^j$-module (Littlewood-Richardson-type rule).

### 1.4. Organization of the paper.

This paper is organized as follows. Section 2 is devoted to introducing the quantum enveloping algebra $U = U_{2r+1} = \mathcal{U}(\mathfrak{sl}_{2r+1})$, its coideal subalgebra $U^j = U_{2r+1}^j$, and the category $O^j_{\text{int}}$.

We classify all the irreducible $U^j$-modules in $O^j_{\text{int}}$ and prove the complete reducibility of $U$-modules in $O^j_{\text{int}}$ for the case $r = 1$ in Section 3 and for a general $r$ in Section 4.

In Section 5, we introduce the notion of quasi-$j$-crystal basis of an integrable $U^j$-module in a naive way.

We study $U^j_2$-modules in Section 6. We associate with each irreducible $U^j_2$-module in $O^j_{\text{int}}$ a quasi-$j$-crystal basis in a systematic way.
In Section 2, we define \( \gamma \)-crystal bases by generalizing the quasi-\( \gamma \)-crystal bases constructed in Section 6 and state our main result: the existence and uniqueness theorem for \( \gamma \)-crystal bases of \( \mathcal{O}_\gamma \)-modules in \( \mathcal{O}_\gamma^{\text{int}} \). Its proof is given in Section 8 since we need some combinatorial tools, which we prepare in Section 3.

We end this paper by giving some applications of \( \gamma \)-crystal bases, such as Robinson-Schensted-type correspondence and Littlewood-Richardson-type rule, in Section 10.

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2. Basics of the Quantum Symmetric Pair \((\mathcal{U}, \mathcal{U}')\)

2.1. Definition of \(\mathcal{U}'\). Let \( r \geq 1 \), and set
\[
\mathbb{I} := \left\{ -\left( r - \frac{1}{2} \right), \ldots, -\left( r - \frac{3}{2} \right), \ldots, - \frac{1}{2} \right\}, \quad \mathbb{P} := \{1, 2, \ldots, r\}.
\]
Let \( \Phi \) denote the root system of type \( A_{2r} \) with simple roots \( \Pi = \{ \alpha_i := \epsilon_{i, \frac{1}{2}} - \epsilon_{i, \frac{1}{2}} | i \in \mathbb{I} \} \), where \( \{ \epsilon_i | i = -r, -r+1, \ldots, r \} \) is the standard basis of the Euclidean space \( \mathbb{R}^{2r+1} \) with the inner product \((\cdot, \cdot)\); the associated Dynkin diagram is
\[
\begin{array}{ccccccccc}
\bullet & \cdots & \bullet & \cdots & \bullet & \cdots & \bullet & - \frac{1}{2} & \frac{1}{2} \\
-\left( r - \frac{1}{2} \right) & \ldots & - \frac{1}{2} & \frac{1}{2} & - \frac{1}{2} & \frac{1}{2} & - \frac{1}{2} & \frac{1}{2} & - \frac{1}{2}
\end{array}
\]
We denote the set of positive roots by \( \Phi_+ \) and the weight lattice by \( \Lambda = \bigoplus_{i=-r}^{r} \mathbb{Z} \epsilon_i \).

Let \( \mathcal{U} = \mathcal{U}_{2r+1} \) denote the quantum group \( \mathcal{U}_q(\mathfrak{sl}_{2r+1}) \) of type \( A_{2r} \) over \( \mathbb{Q}(p, q) \) (with \( p \) and \( q \) indeterminates) with generators \( E_i, F_i, \) and \( K_i^{\pm 1}, i \in \mathbb{I} \), subject to the following relations:
\[
\begin{align*}
K_iK_i^{-1} &= K_i^{-1}K_i = 1, \\
K_iK_j &= K_jK_i, \\
K_iE_jK_i^{-1} &= q^{(\alpha_i, \alpha_j)}E_j, \\
K_iF_jK_i^{-1} &= q^{-(\alpha_i, \alpha_j)}F_j, \\
E_iF_j - F_jE_i &= \delta_{i,j} \frac{K_i - K_i^{-1}}{q - q^{-1}}, \\
E_i^2E_j - (q + q^{-1})E_iE_jE_i + E_jE_i^2 &= 0 \quad \text{if } |i - j| = 1, \\
F_i^2F_j - (q + q^{-1})F_iF_jF_i + F_jF_i^2 &= 0 \quad \text{if } |i - j| = 1, \\
E_iE_j - E_jE_i &= 0 \quad \text{if } |i - j| > 1, \\
F_iF_j - F_jF_i &= 0 \quad \text{if } |i - j| > 1.
\end{align*}
\]

Let \( \mathcal{U}^- \) denote the subalgebra of \( \mathcal{U} \) generated by \( F_i, i \in \mathbb{I} \).

We employ the comultiplication \( \Delta \) of \( \mathcal{U} \) given by:
\[
\Delta(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\pm 1}, \quad \Delta(E_i) = 1 \otimes E_i + E_i \otimes K_i^{-1}, \quad \Delta(F_i) = F_i \otimes 1 + K_i \otimes F_i \quad \text{for } i \in \mathbb{I}.
\]

Let \( \mathcal{U}' \) denote the quantum symmetric pair over \( \mathbb{Q}(p, q) \) of type \( A_{2r} \), that is, \( \mathcal{U}' \) is the subalgebra of \( \mathcal{U} \) generated by
\[
k_i^{\pm 1} := (K_i - \frac{1}{2} K_i^{-1})(\cdot)_{i, \frac{1}{2}}^{\pm 1},
\]
\[
e_i := E_i - \frac{1}{2} + p^{-\delta_{i,1}} F_{-i, \frac{1}{2}} K_{i, \frac{1}{2}},
\]
\[
f_i := E_{-i, \frac{1}{2}} + p^{\delta_{i,1}} K_{-i, \frac{1}{2}} F_{i, -\frac{1}{2}}, \quad i \in \mathbb{P}.
\]
When we want to emphasize the integer \( r \), we denote this subalgebra by \( \mathcal{U}'_r \) instead of \( \mathcal{U}' \).
The $U^j$ has the following defining relations: for $i, j \in \mathbb{P}$,
\[ k_ik_i^{-1} = k_i^{-1}k_i = 1, \]
\[ k_ik_j = k_jk_i, \]
\[ k_ie_ik_i^{-1} = q^{(a_i-a_j,a_j)}e_j, \]
\[ k_if_jk_i^{-1} = q^{-(a_i-a_j,a_j)}f_j, \]
\[ e_if_j - f_je_i = \delta_{ij}k_i - k_i^{-1} \quad \text{if } (i, j) \neq (1, 1), \]
\[ e_i^2 - (q + q^{-1})e_ie_j + e_j e_i^2 = 0 \quad \text{if } |i - j| = 1, \]
\[ f_i^2 - (q + q^{-1})f_if_j + f_jf_i^2 = 0 \quad \text{if } |i - j| = 1, \]
\[ e_if_j - e_je_i = 0 \quad \text{if } |i - j| > 1, \]
\[ f_if_j - f_j f_i = 0 \quad \text{if } |i - j| > 1, \]
\[ e_i^2 - (q + q^{-1})e_if_1 e_1 + f_1e_i^2 = -(q + q^{-1})e_1(qpk_1 + p - q^{-1}k_1^{-1}), \]
\[ f_i^2 - (q + q^{-1})f_if_1 e_1 + e_1f_i^2 = -(q + q^{-1})(qpk_1 + p - q^{-1}k_1^{-1})f_i. \]

Also, $U^j$ is a right coideal of $U$, that is, $\Delta(U^j) \subset U^j \otimes U$. Indeed, we have
\[ \Delta(k_i) = k_i^{\pm 1} \otimes k_i^{\pm 1}, \]
\[ \Delta(e_i) = e_i \otimes K_{-1}^{-\frac{1}{2}} + 1 \otimes E_i - \frac{1}{2} \otimes F_i - \frac{1}{2} \otimes K_i^{-1}, \]
\[ \Delta(f_i) = f_i \otimes K_{-1}^{\frac{1}{2}} - (i - \frac{1}{2}) + 1 \otimes E_i - (i - \frac{1}{2}) + p^{\delta_i,i}k_i \otimes K_{-1}^{\frac{1}{2}}F_i - \frac{1}{2} \quad \text{for } i \in \mathbb{P}. \]

This fact enables us to regard the tensor product $M \otimes N$ of a $U^j$-module $M$ and a $U$-module $N$ as a $U^j$-module. Thanks to the coassociativity of $\Delta$, we have a natural isomorphism $M \otimes (N_1 \otimes N_2) \simeq (M \otimes N_1) \otimes N_2$ of $U^j$-modules, where $N_1$ and $N_2$ are $U$-modules.

**Proposition 2.1.1.**

1. There exists a unique $\mathbb{Q}$-algebra automorphism $\psi^j$ of $U^j$ which maps $e_i, f_i, k_i, p, q$ to $e_i, f_i, k_i^{-1}, p^{-1}, q^{-1}$, respectively.
2. There exists a unique $\mathbb{Q}(p, q)$-algebra anti-automorphism $\sigma^j$ of $U^j$ which maps $e_i, f_i, k_i$ to $e_i, f_i, k_i$, respectively.

**Proof.** These assertions are easily verified by the defining relations (2) of $U^j$. \qed

For notational simplicity, we write $\pi$ instead of $\psi^j(x)$ for $x \in U^j$; it should be noted that $\psi^j$ is different from the restriction of the bar-involution of $U$, which we will not use in this paper.

### 2.2. Triangular decomposition of $U^j$.

Recall Lusztig’s braid group actions on $U$.

**Definition 2.2.1 (Lusztig 1994, Chapter 37).** Let $e \in \{1, -1\}$. For each $i \in \mathbb{I}$, define four automorphisms $T_{i,e}^j$ and $T_{i,-e}^j$ on $U$ by:

\[
T_{i,e}(E_j) = \begin{cases} 
-K_i^jF_i & \text{if } j = i, \\
E_j & \text{if } |i - j| > 1, \\
[E_j, E_i]_e & \text{if } |i - j| = 1,
\end{cases}
\]

\[
T_{i,e}(F_j) = \begin{cases} 
-E_iK_i^{-e} & \text{if } j = i, \\
F_j & \text{if } |i - j| > 1, \\
[F_i, F_j]_{-e} & \text{if } |i - j| = 1,
\end{cases}
\]

\[
T_{i,-e}(E_j) = \begin{cases} 
-F_iK_i^j & \text{if } j = i, \\
E_j & \text{if } |i - j| > 1, \\
[E_i, E_j]_e & \text{if } |i - j| = 1,
\end{cases}
\]

\[
T_{i,-e}(F_j) = \begin{cases} 
-K_i^jE_i & \text{if } j = i, \\
F_j & \text{if } |i - j| > 1, \\
[F_j, F_i]_{-e} & \text{if } |i - j| = 1,
\end{cases}
\]

\[
T_{i,e}(K_j) = T_{i,-e}(K_j) = \begin{cases} 
K_i^{-1} & \text{if } j = i, \\
K_j & \text{if } |i - j| > 1, \\
K_iK_j & \text{if } |i - j| = 1.
\end{cases}
\]
Definition 2.2.4. A total order \( \preceq \) on \( \Phi_+ \) is said to be a reflection order if it satisfies the following: for each \( \alpha, \beta \in \Phi_+ \) and \( a, b \in \mathbb{R}_{>0} \), if \( a \alpha + b \beta \in \Phi_+ \) and \( \alpha \prec \beta \), then \( \alpha \prec a \alpha + b \beta \prec \beta \).

Proposition 2.2.5 ([Lu94, Proposition 2.13]). Let \( i = (i_1, \ldots, i_N) \) be a reduced word for \( w_0 \in W(\Pi) \). Set \( \alpha_j(i) := s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j}) \). Then, the total order \( \preceq \) on \( \Phi_+ \) defined by \( \alpha_1(i) \prec \cdots \prec \alpha_N(i) \) is a reflection order. Moreover, this correspondence gives a bijection between the set of reduced words for \( w_0 \in W(\Pi) \) and the set of reflection orders on \( \Phi^+ \).
Lemma 2.2.6. There exists a reflection order $\preceq$ on $\Phi_+$ such that
\[
\Phi_{<0} \preceq \Phi_0 \preceq \Phi_{>0}.
\]
Here, for subsets $A, B \subset \Phi_+$, $A \prec B$ means that $\alpha \prec \beta$ for all $\alpha \in A$ and $\beta \in B$.

Proof. See the next example. □

Example 2.2.7. For simplicity, we write $(i, j)$ instead of $\epsilon_i - \epsilon_j$ for $i < j$. We decompose $\Phi_{<0}$ into $\Phi_{<0,-} := \{(i, j) \in \Phi_{<0} \mid j \leq 0\}$ and $\Phi_{<0,+} := \{(i, j) \in \Phi_{<0} \mid j > 0\}$. Similarly, we set $\Phi_{>0,-} := \{(i, j) \in \Phi_{>0} \mid i < 0\}$ and $\Phi_{>0,+} := \{(i, j) \in \Phi_{>0} \mid i \geq 0\}$. Let us define a total order $\preceq$ on $\Phi_+$ by:

1. $\Phi_{<0,-} \prec \Phi_{<0,+} \prec \Phi_0 \prec \Phi_{>0,-} \prec \Phi_{>0,+}$;
2. for $(i, j), (i', j') \in \Phi_{<0,-}$, $(i, j) \prec (i', j')$ if and only if $i < i'$ or $(i = i'$ and $j < j')$;
3. for $(i, j), (i', j') \in \Phi_{<0,+}$, $(i, j) \prec (i', j')$ if and only if $j < j'$ or $(j = j'$ and $i < i')$;
4. for $(i, j), (i', j') \in \Phi_0$, $(i, j) \prec (i', j')$ if and only if $j < j'$;
5. for $(i, j), (i', j') \in \Phi_{>0,-}$, $(i, j) \prec (i', j')$ if and only if $i < i'$ or $(i = i'$ and $j < j')$;
6. for $(i, j), (i', j') \in \Phi_{>0,+}$, $(i, j) \prec (i', j')$ if and only if $j < j'$ or $(j = j'$ and $i < i')$.

The $\preceq$ is a reflection order on $\Phi_+$ satisfying $\Phi_{<0} \prec \Phi_0 \prec \Phi_{>0}$; the proof is straightforward.

For example, when $r = 3$, this total order is given as follows:
\[
(-3, -2) \prec (-3, -1) \prec (-3, 0) \prec (-2, -1) \prec (-2, 0) \prec (-1, 0) \prec (-1, 1) \prec (-2, 1) \prec (-3, 2) \prec (-3, 3) \prec (-2, 3) \prec (-1, 2) \prec (-1, 3) \prec (0, 1) \prec (0, 2) \prec (1, 2) \prec (0, 3) \prec (1, 3) \prec (2, 3).
\]

Fix a reflection order $\preceq$ satisfying condition (1) in Lemma 2.2.6. Let $i$ be the reduced word for $w_0 \in W(\mathfrak{I})$ corresponding to $\preceq$ under the bijection of Proposition 2.2.5. We set $F_{i,j} := F_{i-\epsilon_j}(i)$ for $-r \leq i < j \leq r$. For each $i, j$, define $F'_{i,j} := \text{gr}^{-1}(F_{i,j})$, and set
\[
f_{-j,-i} := F'_{i,j} \quad \text{if } i + j < 0, \quad h_i := F'_{-i,i}, \quad e_{i,j} := F'_{i,j} \quad \text{if } i + j > 0.
\]

Let us compute some of these vectors. By [LS91] Lemma 1 (with a slight modification), we have
\[
F_{i-1,j} = [F_{i,j}, F_{i-\frac{1}{2}}]_1 \quad \text{if } (i - 1, i) \prec (i, j),
F_{i,j+1} = [F_{j+\frac{1}{2}}, F_{i,j}]_1 \quad \text{if } (i, j) \prec (j, j + 1).
\]

In particular, by condition (1),
\[
F_{-1,1} = [F_{\frac{1}{2}}, F_{\frac{1}{2}}]_1, \quad F_{-(i+1),i+1} = \left[F_{(i+\frac{1}{2}), F_{-i,i}]_1, F_{-(i+\frac{1}{2})}\right)_1 \quad \text{for } 1 \leq i \leq r - 1.
\]

Applying $\text{gr}^{-1}$, we obtain
\[
h'_i = [e_1, f_1]_1, \quad h'_{i+1} = [e_{i+1}, h_i]_1, f_{i+1} = [e_{i+1}, h_i]_1\cdot
\]

This shows that the $h'_i$’s are independent of the choice of a reflection order $\preceq$ satisfying condition (1) in Lemma 2.2.6.

Let $U_{<0}^J$ (resp., $U_0^J$, $U_{>0}^J$) denote the subspace of $U^J$ spanned by all ordered monomials in $f_{-j,-i}$ (resp., $h_i, e_{i,j}$). Then, we have an isomorphism of vector spaces
\[
U^J \cong U_{<0}^J \otimes \left(U_0^J \otimes U_{>0}^J\right) \otimes U_{>0}^J.
\]

We call this linear isomorphism the triangular decomposition of $U^J$ associated with the reflection order $\preceq$, and $U_{<0}^J$ (resp., $U_0^J \otimes U_{>0}^J$, $U_{>0}^J$) the negative part (resp., Cartan part, positive part) of $U^J$. The triangular decomposition enables us to establish an analog of highest weight theory for the representation theory of $U^J$. 
Remark 2.2.8. Unlike the ordinary triangular decomposition of a quantized enveloping algebra, the negative part, the Cartan part, and the positive part of $U^j$ are just subspaces, not subalgebras. In addition, the negative part and the positive part may depend on the choice of a reflection order.

2.3. Verma modules and their irreducible quotients. Recall that $\mathbb{R}^{2r+1} = \bigoplus_{i=-r}^r \mathbb{R} \epsilon_i$ is the Euclidean space with standard basis $\{ \epsilon_i \mid -r \leq i \leq r \}$ with respect to the inner product $(\cdot, \cdot)$, and $\alpha_i = \epsilon_{i-\frac{1}{2}} - \epsilon_{i+\frac{1}{2}}$, $i \in I$, are the simple roots. Set $\beta_i := \alpha_i - \frac{1}{2} - \alpha_{-(i-\frac{1}{2})} = \epsilon_{i-1} - \epsilon_{i-\frac{1}{2}} + \epsilon_{-(i-1)}$ for $i \in P$.

Definition 2.3.1. Let $J \subset \mathbb{R}^{2r+1} := \{ \lambda \in \mathbb{R}^{2r+1} \mid (\beta_i, \lambda) = 0 \text{ for all } i \in P \}$. Then the bilinear form $(\cdot, \cdot)$ on $\mathbb{R}^{2r+1} \times \mathbb{R}^{2r+1}$ induces a bilinear map $(\bigoplus_{i \in P} \mathbb{R} \beta_i) \times (\mathbb{R}^{2r+1}/J) \to \mathbb{R}$, which we also denote by $(\cdot, \cdot)$. For each $i \in P$, there exists a unique $\delta_i \in \mathbb{R}^{2r+1}/J$ such that

$$(\beta_i, \delta_i) = \delta_{i,j} \quad \text{for } i, j \in P.$$ 

Set $\Lambda := \sum_{i \in P} \mathbb{Z} \delta_i$ and $\Lambda_+ := \sum_{i \in P} \mathbb{Z}_{\geq 0} \delta_i$. Also, we set $\gamma_i := \epsilon_{i-1} - \epsilon_i + J \in \Lambda^j$.

By the definitions, we have

$$(\beta_i, \gamma_j) = (\alpha_i - \alpha_{-i}, \alpha_j) = \begin{cases} 3 & \text{if } i = j = 1, \\ 2 & \text{if } i = j \neq 1, \\ -1 & \text{if } |i - j| = 1, \\ 0 & \text{if } |i - j| > 1. \end{cases}$$

Define a partial order $\leq$ on $\Lambda^j$ by:

$$\mu \leq \lambda \text{ if and only if } \lambda - \mu \in \sum_{i \in P} \mathbb{Z}_{\geq 0} \gamma_i.$$ 

For a $U^j$-module $M$ and $m \in M$, we say that $m$ is of weight $\lambda \in \Lambda^j$ if it satisfies

$$k_i m = q^{(\beta_i, \lambda)} m$$ 

for all $i \in P$; we denote by $M_\lambda$ the subspace consisting of all $m \in M$ of weight $\lambda$.

Lemma 2.3.2. Let $M$ be a $U^j$-module and $\lambda \in \Lambda^j$. For each $i \in P$, we have

$$f_i(M_\lambda) \subset M_{\lambda - \gamma_i}, \quad e_i(M_\lambda) \subset M_{\lambda + \gamma_i}.$$ 

Proof. This follows immediately from the relations $k_i f_j k_i^{-1} = q^{(\beta_i, -\gamma_j)} f_j$ and $k_i e_j k_i^{-1} = q^{(\beta_i, \gamma_j)} e_j$.

Recall the triangular decomposition of $U^j$

$$U^j \simeq U^j_{<0} \bigotimes \left( U^j_0 \bigotimes U^j_{=0} \right) \bigotimes U^j_{\geq 0},$$

and the root vectors $f_{-\beta_i - \gamma_i}, h_i, e_{i,j}$ associated with a reflection order satisfying condition (H) in Lemma 2.2.6.

Definition 2.3.3. Let $\lambda \in \Lambda^j$ and $H_i \in \mathbb{Q}(p, q)$, $i = 1, 2, \ldots, r$. The Verma module $V(\lambda; H)$ over $U^j$ with highest weight $\lambda$ associated with $H := (H_1, \ldots, H_r) \in \mathbb{Q}(p, q)^r$ is defined to be

$$V(\lambda; H) := U^j/I(\lambda; H),$$

where $I(\lambda; H)$ denotes the left ideal of $U^j$ generated by $U^j_{>0}$ and $k_i - q^{(\beta_i, \lambda)}$, $h_i - H_i$ for $i \in P$.

By the triangular decomposition of $U^j$, the Verma module $V(\lambda; H)$ has a unique maximal submodule, and hence, it has a unique irreducible quotient. We denote it by $L(\lambda; H)$ and call it the irreducible highest weight $U^j$-module with highest weight $\lambda$ associated with $H$, or simply, with highest weight $(\lambda; H)$. 
Definition 2.3.4. A nonzero $U^j$-module $M$ is called a highest weight module with highest weight $(\lambda; H)$ if there exists $m \in M_L$ such that $U^j_{\geq 0} m = 0$, $h_i m = H_i m$ for $i \in \mathcal{P}$, and $M = U^j m$. We call such an $m$ a highest weight vector of $M$ with highest weight $(\lambda; H)$.

Our definition of highest weight modules over $U^j$ depends on the choice of a reflection order satisfying condition (4) in Lemma 2.2.6. However, their $U^j$-module structure is independent of such a choice, as we explain below.

Let $M$ be a highest weight $U^j$-module with highest weight $(\lambda; H)$ associated with a reflection order $\preceq$. Take another reflection order $\preceq'$, and denote the corresponding root vectors by $f_{i,j}', h_i', e_{i,j}'$. Then, we see from equation (5) that $h_i' = h_i$. Also, by the triangular decomposition associated with $\preceq$, we have

\[ e_{i,j}' \in \sum_{\nu, \mu \in \Lambda^j_+} (U^j_{\geq 0})_{-\nu} \otimes (U^j_0 \otimes U^j_{\geq 0}) \otimes (U^j_{> 0})_{\mu}; \]

here, $(U^j_{\geq 0})_{-\nu} := \{ x \in U^j_{\geq 0} \mid k_i x k_i^{-1} = q^{(\beta_i, -\nu)} x \text{ for all } i \in \mathcal{P} \}$, and define $(U^j_{0})_{\mu}$ similarly. Therefore, it holds that $e_{i,j}' v = 0$ for all $i, j$. In addition, by expanding $f_{i,j}$ in ordered monomials in $f_{i,j}', h_i, e_{i,j}'$, we see that $f_{i,j}' v$ is a linear combination of $f_{i,j}' v$'s. From these, we conclude that $M$ is a highest weight module with highest weight vector $(\lambda; H)$ associated with $\preceq'$. In particular, if we denote Verma modules and their irreducible quotients associated with $\preceq'$ by $V'(\cdot; \cdot)$ and $L'(\cdot; \cdot)$, respectively, then we have

\[ V(\lambda; H) = V'(\lambda; H), \quad L(\lambda; H) = L'(\lambda; H). \]

Hence, in this paper, we use only the reflection order given in Example 2.2.7.

Let $O^j_{\text{int}}$ denote the category of all $U^j$-modules $M$ satisfying the following:

\begin{enumerate}[(M1)]
  \item $M$ is decomposed into weight spaces, i.e., $M = \bigoplus_{\lambda \in \Lambda^j} M_\lambda$.
  \item Each weight space is finite-dimensional.
  \item There exist finitely many weights $\mu_1, \ldots, \mu_n \in \Lambda^j$ such that each weight $\lambda \in \Lambda^j$ for which $M_\lambda \neq 0$ satisfies $\lambda \leq \mu_i$ for some $i = 1, \ldots, n$.
  \item $e_i$ and $f_i$ act on $M$ locally nilpotently, that is, for each $m \in M$, there exists $N \in \mathbb{N}$ such that $e_i^N m = 0 = f_i^N m$.
\end{enumerate}

Note that Verma modules and their irreducible quotients are not necessarily objects of $O^j_{\text{int}}$.

3. The case $r = 1$

3.1. Classification of the irreducible modules in $O^j_{\text{int}}$. We introduce some more notation.

Definition 3.1.1. \hspace{1em} (1) For $n \in \mathbb{Z}$, $[n] := \frac{q^n - q^{-n}}{q - q^{-1}}$.

(2) For $n \in \mathbb{Z}_{\geq 0}$, $[n]! := \prod_{i=1}^{n} [i]$; we set $[0]! := 1$.

(3) For $x \in U$ and $n \in \mathbb{Z}_{> 0}$, $x^{(n)} := \frac{x^n}{[n]!}$; we set $x^{(0)} := 1$, and $x^{(n)} := 0$ if $n < 0$.

(4) For $x, y \in U$ and $a \in \mathbb{Z}$, $[x, y]_a := xy - q^a yx$.

(5) For an invertible element $h$, $\{ h \} := h + h^{-1}$.

(6) For an integer $n \in \mathbb{Z}$, $\{ n \} := \{ pq^n \} = pq^n + p^{-1} q^{-n}$.

In the case $r = 1$, the root vectors are

\[ f_{0,1} = f_1, \quad h_1 = [e_1, f_1]_1, \quad e_{0,1} = e_1. \]

Lemma 3.1.2. In $U^1_j$, we have

\[ [h_1, f_1]_{-1} = -2\{ pqk \} f_1, \quad [e_1, h_1]_{-1} = -2\{ pqk \}. \]

Proof. By equation (2). \hfill \Box
Lemma 3.1.3. For each $n \in \mathbb{Z}_{\geq 0}$, we have
\[ e_1 f_1^{(n)} = f_1^{(n-1)} (h_1 - [n-1]_1 (pq^{-n}k_1)) + q^n f_1^{(n)} e_1. \]

Proof. We prove the assertion by induction on $n$. This is trivial when $n = 0$. Assume that the assertion holds for a fixed $n \in \mathbb{Z}_{\geq 0}$. Then, we compute as follows:
\[
e_1 f_1^{(n+1)} = \frac{1}{n+1} e_1 f_1^{(n)} f_1 \]
\[
= \frac{1}{n+1} \left( f_1^{(n-1)} (h_1 - [n-1]_1 (pq^{-n}k_1)) + q^n f_1^{(n)} e_1 \right) f_1 \]
\[
= \frac{1}{n+1} \left( f_1^{(n-1)} (q^{-1} f_1 h_1 - [2]_1 (pqk_1) f_1 - [n-1]_1 (pq^{-n}k_1) f_1) + q^n f_1^{(n)} (h_1 + q f_1 e_1) \right) \]
\[
= \frac{1}{n+1} \left( f_1^{(n-1)} (q^{-1} f_1 h_1 - [2]_1 f_1 (pq^{-2}k_1) - [n-1]_1 f_1 (pq^{-n-3}k_1)) + q^n f_1^{(n)} (h_1 + q f_1 e_1) \right) \]
\[
= \frac{1}{n+1} f_1^{(n)} (h_1 - [n]_1 (pq^{-n-1}k_1)) + q^{n+1} f_1^{(n+1)} e_1; \]
the second equality follows from our inductive hypothesis, the third from Lemma 3.1.2, and the rest is straightforward. This proves the lemma. \qed

Note that when $r = 1$, we have $\Lambda^1 = \mathbb{Z} \delta_1$ and $\gamma_1 = 3 \delta_1$. Let $M \in \mathcal{O}_\text{int}$. By the definition of $\mathcal{O}_\text{int}$, there exists $a \in \mathbb{Z}$ such that $M a \delta_1 \neq \{0\}$ and $M (a+3) \delta_1 = \{0\}$. Since the action of $h_1$ preserves weights, it defines a linear endomorphism of $M a \delta_1$. In order to consider the Jordan canonical form for the action of $h_1$ on $M a \delta_1$, we extend the base field $\mathbb{Q}(p, q)$ to its algebraic closure $\overline{\mathbb{Q}(p, q)}$ until the proof of Proposition 3.1.4. Let us write the Jordan canonical form as:
\[
\begin{pmatrix}
J_{d_1}(\mu_1) & & \\
& J_{d_2}(\mu_2) & \\
& & \ddots \\
& & & J_{d_m}(\mu_m)
\end{pmatrix},
\]
where $J_{d_i}(\mu_i)$ denotes the Jordan block of size $d_i$ whose eigenvalue is $\mu_i \in \overline{\mathbb{Q}(p, q)}$. We take a basis $\{v_{j,k} \mid j = 1, \ldots, m, k = 1, \ldots, d_j\}$ of $M a \delta_1$ in such a way that
\[ h_1 v_{j,k} = \mu_j v_{j,k} + v_{j,k-1} \]
for all $j = 1, \ldots, m, k = 1, \ldots, d_j$, where $v_{j,0} := 0$. By Lemma 3.1.3, we have
\[ e_1 f_1^{(n)} v_{j,k} = (\mu_j - [n-1]_1 (a-n)) f_1^{(n-1)} v_{j,k} + f_1^{(n-1)} v_{j,k-1}. \]

Proposition 3.1.4. We have $\mu_j = [N_j]_1 (a-N_j-1)$ for some $N_j \in \mathbb{Z}_{\geq 0}$. In particular, each $\mu_j$ belongs to $\mathbb{Q}(p, q)$.

Proof. Consider the case $k = 1$. By the local nilpotency of $f_1$, there exists a unique nonnegative integer $N_j$ such that
\[ f_1^{(N_j)} v_{j,1} \neq 0 \text{ and } f_1^{(N_j+1)} v_{j,1} = 0. \]

Then, by equation (7), we have
\[ 0 = e_1 f_1^{(N_j+1)} v_{j,1} = (\mu_j - [N_j]_1 (a-N_j-1)) f_1^{(N_j)} v_{j,1}. \]
Since $f_1^{(N_j)} v_{j,1} \neq 0$, we conclude that $\mu_j = [N_j]_1 (a-N_j-1)$, as desired. \qed

Proposition 3.1.5. Each $d_j$ is equal to 1, that is, $h_1$ is diagonalizable on $M a \delta_1$. 

Proof. We use the notation $N_j$ in the proof of Proposition 3.1.4. Assume, for a contradiction, that there exists $d_j > 1$. By equation (7), we have
\[ e_if_1^{(n)} v_{j,2} = (\mu_j - [n-1](a - n))f_1^{(n-1)} v_{j,2} + f_1^{(n-1)} v_{j,1} \]
for all $n \geq 0$. Let $N'_j$ denote the unique nonnegative integer such that
\[ f_1^{(N'_j)} v_{j,2} \neq 0, \text{ and } f_1^{(N'_j+1)} v_{j,2} = 0. \]
When $N'_j > N_j$, we have
\[ 0 = (\mu_j - [N'_j](a - N'_j - 1))f_1^{(N'_j)} v_{j,2} + f_1^{(N'_j)} v_{j,1} = (\mu_j - [N'_j](a - N'_j - 1))f_1^{(N'_j)} v_{j,2}. \]
This implies that $\mu_j = [N'_j](a - N'_j - 1) \neq [N_j](a - N_j - 1)$, which causes a contradiction. When $N'_j = N_j$, we have
\[ 0 = (\mu_j - [N_j](a - N_j - 1))f_1^{(N_j)} v_{j,2} + f_1^{(N_j)} v_{j,1}. \]
This contradicts the definition of $N_j$. When $N'_j < N_j$, we have
\[ 0 = (\mu_j - [N'_j](a - N'_j - 1))f_1^{(N'_j)} v_{j,2} + f_1^{(N'_j)} v_{j,1}. \]
Applying $e_i^{N'_j}$ on both sides, we obtain
\[ 0 = \prod_{l=1}^{N'_j+1} (\mu_j - [l-1](a - l))v_{j,2} + X v_{j,1} \text{ for some } X \in \mathbb{Q}(p, q). \]
Since the coefficient of $v_{j,2}$ is nonzero, this contradicts the linear independence of $v_{j,1}$ and $v_{j,2}$. This proves the proposition.

**Theorem 3.1.6.** For each $a \in \mathbb{Z}$ and $b \in \mathbb{Z}_{\geq 0}$, there exists a unique $(b + 1)$-dimensional irreducible $U'_1$-module $L(a; b) \in \mathcal{O}'_{\text{int}}$ such that
\[ L(a; b) = \bigoplus_{n=0}^{b} v_n, \]
\[ v_n = f_1^{(n)} v_0, \quad k_1 v_0 = q^n v_0, \quad h_1 v_0 = [b](a - b - 1) v_0. \]
Conversely, each irreducible $U'_1$-module in $\mathcal{O}'_{\text{int}}$ is isomorphic to $L(a; b)$ for some $a \in \mathbb{Z}$ and $b \in \mathbb{Z}_{\geq 0}$.

**Proof.** It is straightforward to show that $L(a; b)$ is a $(b + 1)$-dimensional irreducible $U'_1$-module, and so we omit the details. Let $V \in \mathcal{O}'_{\text{int}}$ be an irreducible $U'_1$-module. By the definition of $\mathcal{O}'_{\text{int}}$, there exists an integer $a \in \mathbb{Z}$ such that $V_{a \delta_1} \neq 0$ and $e_1 V_{a \delta_1} = 0$. Also, by Propositions 3.1.4 and 3.1.5, there exist $b \in \mathbb{Z}_{\geq 0}$ and $v \in V_{a \delta_1} \setminus \{0\}$ such that $f_1^{(b)} v \neq 0$, $f_1^{(b+1)} v = 0$, and $h_1 v = [b](a - b - 1) v$. Hence the $U'_1$-submodule generated by $v$ is identical to $\bigoplus_{n=0}^{b} f_1^{(n)} v$, which is isomorphic to $L(a; b)$ by the definitions of $v, a, b$. Since $V$ is irreducible, we have $V = U'_1 v \simeq L(a; b)$. This proves the theorem.

Note that $L(a; b)$ is the irreducible quotient $L(\lambda; H)$ of the Verma module $V(\lambda; H)$ with highest weight $(\lambda; H) = (a \delta_1; [b](a - b - 1))$. Hence, Theorem 3.1.6 gives a necessary and sufficient condition for $L(\lambda; H)$ to be an object of $\mathcal{O}'_{\text{int}}$.

**Corollary 3.1.7.** Let $a \in \mathbb{Z}$ and $H_1 \in \mathbb{Q}(p, q)$. Then, the irreducible highest weight module $L(a \delta_1; H_1)$ belongs to $\mathcal{O}'_{\text{int}}$ if and only if $H_1 = [b](a - b - 1)$ for some $b \in \mathbb{Z}_{\geq 0}$. Moreover, the assignment $(a, b) \mapsto [L(a; b)]$, where $[L(a; b)]$ denotes the isomorphism class of $L(a; b)$, gives a bijection from $\mathbb{Z} \times \mathbb{Z}_{\geq 0}$ to the set of isomorphism classes of irreducible $U'_1$-modules in $\mathcal{O}'_{\text{int}}$. 
3.2. Complete reducibility. Set $z_1 := h_1 + \frac{[2]pq}{1-q}k_1 + \frac{[2]p^{-1}q^{-1}}{1-q^{-1}}k_1^{-1} \in U_1^1$.

Lemma 3.2.1. In $U_1^1$, we have

$$z_1 f_1 = q^{-1} f_1 z_1, \quad z_1 e_1 = q e_1 z_1.$$ 

Proof. By Lemma 3.1.2 and the equalities

$$[k_1, f_1] = (1 - q^2) k_1 f_1, \quad [k_1^{-1}, f_1] = (1 - q^{-4}) k_1^{-1} f_1,$$

it follows that $z_1 f_1 = q^{-1} f_1 z_1$. Noting that $z_1$ is invariant under the anti-automorphism $\sigma^J$ defined in Proposition 2.1.1 (2), we obtain the other equality. \hfill $\square$

Let $a \in \mathbb{Z}$ and $b \in \mathbb{Z}_{\geq 0}$, and take a highest weight vector $v \in L(a; b)$. Then we have

$$z_1 f_1^{(n)} v = q^{-n} \left( [b] \{ a - b - 1 \} + \frac{[2]pq^{a+b+1}}{1-q} + \frac{[2]p^{-1}q^{-a-b-1}}{1-q^{-1}} \right) v.$$ 

Denoting by $z_1(a, b, n)$ the coefficient of $v$ on the right-hand side, one has

$$z_1(a, b, n) = \frac{-pq^{a-b-n}(q^{b+1} + q^{-b-1})}{q - q^{-1}} + \frac{p^{-1}q^{-a+2b-n+1}}{q - q^{-1}}.$$ 

Using this, one can verify that the function $\mathbb{Z}^3 \to \mathbb{Q}(p, q)$, $(a, b, n) \mapsto z_1(a, b, n)$, is injective.

Lemma 3.2.2. Let $M \in \mathcal{O}_{\text{int}}$, $a, a' \in \mathbb{Z}$, and $b, b' \in \mathbb{Z}_{\geq 0}$. Then, each short exact sequence of the form

$$0 \to L(a; b) \overset{\lambda}{\to} M \overset{\pi}{\to} L(a'; b') \to 0$$

splits.

Proof. Let $v \in L(a', b')$ be a highest weight vector, and take $u \in \pi^{-1}(v)$. Since $U_1^1$-module homomorphisms preserve generalized eigenspaces of $z_1$, we may assume that $u$ is a generalized eigenvector of $z_1$ with eigenvalue $z_1(a', b', 0)$. Then, $e_1 u$ is a generalized eigenvector of $z_1$ with eigenvalue $z_1(a', b', -1)$. Since $\pi(e_1 u) = e_1 \pi(u) = e_1 v = 0$, it follows that $e_1 u \in \nu(L(a', b'))$. However, the eigenvalues of $z_1$ on $L(a, b)$ are $z_1(a, b, n), 0 \leq n \leq b$. Therefore, $e_1 u = 0$, and hence we obtain a section $v \mapsto u$ of $\pi$. This proves the lemma. \hfill $\square$

Now, the complete reducibility of $U_1^1$-modules in $\mathcal{O}_{\text{int}}^1$ follows from a standard argument; see, for example, [HK02, Section 3.5].

Theorem 3.2.3. Every $U_1^1$-module in $\mathcal{O}_{\text{int}}^1$ is completely reducible.

Corollary 3.2.4. Let $M \in \mathcal{O}_{\text{int}}^1$. Then, $M$ is decomposed into a direct sum of $z_1$-eigenspaces with possible eigenvalues $z_1(a, b, n), a \in \mathbb{Z}, 0 \leq n \leq b$. In particular, if $z_1 m = z_1(a, b, 0)$, then $e_1 m = 0$.

4. Complete reducibility and the irreducible modules

Throughout this section, we fix $e \in \{1, -1\}$.

4.1. Braid group action on $U_1^1$. 

Definition 4.1.1. For \( i \in \mathbb{P} \setminus \{1\} \), define two automorphisms \( \tau'_{i,e} \) and \( \tau''_{i,e} \) of \( U^j \) by:

\[
\tau'_{i,e}(e_j) = \begin{cases} 
-k_i e_i & \text{if } j = i, \\
 e_j & \text{if } |i - j| > 1, \\
 [e_j, e_i] & \text{if } |i - j| = 1,
\end{cases}
\]

\[
\tau''_{i,e}(e_j) = \begin{cases} 
-f_i k_i e_i & \text{if } j = i, \\
 e_j & \text{if } |i - j| > 1, \\
 [e_j, e_i] & \text{if } |i - j| = 1,
\end{cases}
\]

\[
\tau'_{i,e}(k_j) = \tau''_{i,-e}(k_j) = \begin{cases} 
k_i^{-1} & \text{if } j = i, \\
k_j & \text{if } |i - j| > 1, \\
k_i k_j & \text{if } |i - j| = 1.
\end{cases}
\]

Proposition 4.1.2. The \( \tau'_{i,e} \) (resp., \( \tau''_{i,-e} \), \( i \in \mathbb{P} \setminus \{1\} \)), are indeed automorphisms of \( U^j \). Moreover, they satisfy the braid relation of type \( A_{r-1} \).

Proof. Set \( \tau_i := \tau'_{i,e} \) (resp., \( \tau''_{i,-e} \)), \( i \in \mathbb{P} \setminus \{1\} \). We need to verify that the relations in (2) hold if we replace \( e_i, f_i, k_i \) by \( \tau_j(e_i), \tau_j(f_i), \tau_j(k_i) \), respectively. By comparing Definition 4.1.1 with Definition 2.2.1, one immediately finds that the nontrivial assertions are

\[
\tau_2(e_1)^2 \tau_2(f_1) - (q + q^{-1}) \tau_2(e_1) \tau_2(f_1) + \tau_2(f_1) \tau_2(e_1)^2 = -q + q^{-1} \tau_2(e_1) + p q \tau_2(k_1) + p^{-1} q^{-1} \tau_2(k_1),
\]

\[
\tau_2(f_1)^2 \tau_2(e_1) - (q + q^{-1}) \tau_2(f_1) \tau_2(e_1) + \tau_2(e_1) \tau_2(f_1)^2 = -q + q^{-1} \tau_2(f_1) + p q \tau_2(k_1) + p^{-1} q^{-1} \tau_2(k_1)^{-1} \tau_2(f_1).
\]

These are checked by direct calculation, or by means of a computer program GAP [GAP16] with a package Quagroup (see [KP11, 4.5]). Also, one can verify the braid relation in the same way as for the braid group action on \( U \). This proves the proposition.

4.2. Braid group action on \( U^j \)-modules. In this subsection, we define a braid group action on \( U^j \)-modules in \( \mathcal{O}_{\text{int}}^j \). Since the proofs of the propositions in this subsection are almost the same as those in the ordinary quantum group theory, we omit the details.

Definition 4.2.1. Let \( M \in \mathcal{O}_{\text{int}}^j \). For each \( i \in \mathbb{P} \setminus \{1\} \), we define two automorphisms \( \tau'_{i,e} \) and \( \tau''_{i,e} \) on \( M \) by:

\[
\tau'_{i,e}(m) = \sum_{a, b, c \in \mathbb{Z} \geq 0} (-q)^b q^{e(ac+b)} f_i^{(a)} e_i^{(b)} f_j^{(c)} m,
\]

\[
\tau''_{i,e}(m) = \sum_{a, b, c \in \mathbb{Z} \geq 0} (-q)^b q^{e(ac+b)} e_i^{(a)} f_i^{(b)} e_j^{(c)} m,
\]

where \( n \in \mathbb{Z} \), and \( m \in M \) is such that \( k_i m = q^n m \).

Proposition 4.2.2 (see [Lu94] Proposition 5.2.2). Let \( M \in \mathcal{O}_{\text{int}}^j \), \( i \in \mathbb{P} \), and let \( \lambda \in \Lambda^j \) be such that \( (\beta_i, \lambda) \geq 0 \), \( j \in \{0, 1, \ldots, (\beta_i, \lambda)\} \); we set \( h := (\beta_i, \lambda) - j \).

1. If \( \eta \in M_{\lambda} \) is such that \( e_i \eta = 0 \), then \( \tau'_{i,e}(f_i^{(j)} \eta) = (-1)^j q^{e(2jh+j)} f_i^{(h)} \eta \).
2. If \( \xi \in M_{-\lambda} \) is such that \( f_i \xi = 0 \), then \( \tau''_{i,e}(e_i^{(j)} \xi) = (-1)^j q^{e(2jh+j)} e_i^{(h)} \xi \).

Proposition 4.2.3 (see [Lu94] Proposition 5.2.3). Let \( M \in \mathcal{O}_{\text{int}}^j \), \( i \in \mathbb{P} \), and \( m \in M_{\lambda} \).

1. We have \( \tau'_{i,e}^{-1} \tau''_{i,-e} = \text{id}_M = \tau''_{i,-e} \tau'_{i,e} \).
2. We have \( \tau''_{i,e}(m) = (-1)^{(\beta_i, \lambda)} q^{e(\beta_i, \lambda)} \tau'_{i,e}(m) \).
Proposition 4.2.4 (see [Lu94 Proposition 37.1.2]). Let $M \in \mathcal{O}_\text{int}^j$ and $i \in \mathbb{P} \setminus \{1\}$. Then, for each $m \in M$ and $x \in \mathcal{U}_i^j$, we have

$$\tau_i'(x_2m) = \tau_i'(x)m_2, \quad \tau_i''(x_2m) = \tau_i''(x)m_2.$$ 

In what follows, we write $\tau_i = \tau_i''$ for $i \in \mathbb{P} \setminus \{1\}$.

4.3. Classification of the irreducible modules in $\mathcal{O}_\text{int}^j$. Recall the triangular decomposition $\mathcal{U}^j = \mathcal{U}_{<0} \otimes (\mathcal{U}_0^j \otimes \mathcal{U}_{>0}^j)$ associated with the reflection order $\preceq$ defined in Example 2.2.7. Also, recall from [2] in Section 2.3 the explicit form of the root vectors $h_i = \text{gr}^{-1}(F_{-i}) \in \mathcal{U}_0^j$, $i \in \mathbb{P} = \{1, \ldots, r\}$. We remark that an irreducible highest weight module is determined by the eigenvalues of $k_i$'s and $h_i$'s for a highest weight vector. However, $h_i$'s are sometimes difficult to deal with.

Proposition 4.3.1. Let $V(\lambda; H)$ be the Verma module with highest weight $(\lambda; H)$. Then, $H$ is determined by the $\tau_i \cdots \tau_2(h_1)$-eigenvalue of $v$ for $i \in \mathbb{P}$.

Proof. For each $i \in \mathbb{P}$, set $\text{ef}(i) := e_i \cdots e_2 e_1 f_1 f_2 \cdots f_i$. By equation (3), the $h_i$ is of the form

$$h_i = \sum_{\sigma \in \mathcal{S}_{2i}} a_i(\sigma) x_{\sigma(1)} \cdots x_{\sigma(2i)},$$

where $\mathcal{S}_{2i}$ denotes the 2i-th symmetric group, $a_i(\sigma) \in \mathbb{Q}(q)$, $x_j = e_{i+1-j}$ for $1 \leq j \leq i$, and $x_j = f_{j-i}$ for $i+1 \leq j \leq 2i$. From this, noting that $v$ is a highest weight vector, we deduce that $h_i v$ is of the form

$$h_i v = \left(\text{ef}(i) + \sum_{1 \leq i_1 \leq \cdots \leq i_{2i}} f_{i_1 \cdots i_{2i}}(k_1, \ldots, k_i) \text{ef}(i_1) \cdots \text{ef}(i_i)\right) v,$$

where $f_{i_1 \cdots i_{2i}}(k_1, \ldots, k_i) \in \mathbb{Q}(q)[k_1, \ldots, k_i]$ for $i \geq 0$. Therefore, the $h_i$-eigenvalue $H_i$ of $v$ is determined by the ef(j)-eigenvalue of $v$ for $j \leq i$.

Also, $\tau_i \cdots \tau_2(h_1)$ is of the form

$$\tau_i \cdots \tau_2(h_1) = \sum_{\sigma \in \mathcal{S}_{2i}} b_i(\sigma) x_{\sigma(1)} \cdots x_{\sigma(2i)},$$

where $b_i(\sigma) \in \mathbb{Q}(q)$. In the same way as above, the $\tau_i \cdots \tau_2(h_1)$-eigenvalue of $v$ is determined by the ef(j)-eigenvalue of $v$ for $j \leq i$. Conversely, the $\tau_j \cdots \tau_2(h_1)$-eigenvalue of $v$ for $j \leq i$ altogether determine the ef(j)-eigenvalue of $v$ for $j \leq i$, which, in turn, determine the $h_i$-eigenvalue $H_i$ of $v$. This proves the proposition. 

This proposition enables us to replace $h_i$ with $\tau_i \cdots \tau_2(h_1)$ for $i \in \mathbb{P}$. Hence, from now on, we redefine $h_i$, $i \in \mathbb{P}$, as $h_i = [e_1, f_1]_1$ and $h_i = \tau_i \cdots \tau_2(h_1)$.

Let $L \in \mathcal{O}_\text{int}^j$ be an irreducible $\mathcal{U}^j$-module. By condition (M3), there exists $\lambda \in \Lambda^j$ such that $L_{\lambda} \neq 0$ and $L_{\lambda}^j = 0$ for all $\mu > \lambda$. By the case $r = 1$, $h_1$ acts on $L_{\lambda}$ semisimply.

Lemma 4.3.2. We have

$$[h_1, h_2]_0 = [h_1(q - q^{-1})(f_2 e_2 h_1) - p^{-1} q^2 f_2 e_2 k_1^{-1}] e_2 e_2 h_1 e_2 h_1^2,$$

where $\mathcal{U}^j(e_2, e_2 h_1, e_2 h_1^2)$ denotes the left ideal of $\mathcal{U}^j$ generated by $e_2, e_2 h_1, e_2 h_1^2$.

Proof. By direct calculation (or by using GAP). 

This lemma implies that $[h_1, h_2]_0 L_{\lambda} = 0$; namely, the actions of $h_1$ and $h_2$ commute with each other on $L_{\lambda}$.

Lemma 4.3.3. Let $i, j \in \mathbb{P}$. If $j \neq i, i + 1$, then we have $\tau_j(h_i) = h_i$. 

Proof. The assertion in the case \( j > i + 1 \) follows from the definitions of \( \tau_j \) and \( h_i \). When \( j < i \), by the braid relation for the \( \tau_j \)'s, we see that

\[
\tau_j(h_i) = \tau_j(\tau_i \tau_{i-1} \cdots \tau_2(h_1)) \\
= \tau_i \tau_j \tau_{i-1} \tau_{i-2} \cdots \tau_2(h_1) \\
= \tau_i \tau_j \tau_{i-1} \tau_{i-2} \cdots \tau_2(h_1) \\
= \tau_i \tau_j \tau_1 \tau_2 h_i.
\]

This proves the lemma.

\[\square\]

**Proposition 4.3.4.** Let \( L \in \mathcal{O}_\text{int}^\ell \) be an irreducible module. Take \( \lambda \in \Lambda^\ell \) such that \( L_\lambda \neq 0 \) and \( L_\mu = 0 \) for all \( \mu > \lambda \). Then, the actions of \( h_1, \ldots, h_r \) commute with each other on \( L_\lambda \).

Proof. Let \( i, j \in \mathcal{P}^\ell \) be such that \( j < i \). By Lemma 4.3.3,

\[
[h_j, h_i]_0 = \tau_j \cdots \tau_2([h_1, h_i]_0) = \tau_j \cdots \tau_2 \tau_i \cdots \tau_3([h_1, h_2]_0).
\]

Also, by Lemma 4.3.2,

\[
\tau_j \cdots \tau_2 \tau_i \cdots \tau_3([h_1, h_2]_0) = U^j([\tau_j, i](e_2), \tau_j, i(e_2) h_j, \tau_j, i(e_2) h_j^2),
\]

where \( \tau_j, i \) denotes \( \tau_j \cdots \tau_2 \tau_i \cdots \tau_3 \). Since \( \tau_j, i(e_2) \in U^j \), the vectors \( \tau_j, i(e_2) h_j^l, l = 0, 1, 2 \), act on \( L_i \) by 0. This proves the proposition.

\[\square\]

As a corollary of this proposition, we can take a simultaneous eigenvector \( v \in L_\lambda \) for \( h_1, \ldots, h_r \).

Let \( H_i \in \mathbb{Q}(p,q) \) denote the eigenvalue of \( h_i \). Then the submodule generated by \( v \) is a highest weight module with highest weight \( (\lambda; H_1, \ldots, H_r) \). Since \( L \) is irreducible, we conclude that \( L \) is a highest weight module.

**Theorem 4.3.5.** Each irreducible module in \( \mathcal{O}_\text{int}^\ell \) is a highest weight module with highest weight \( (\lambda; H) \) for some \( \lambda \in \Lambda^\ell \) and \( H = (H_1, \ldots, H_r) \in \mathbb{Q}(p,q)^r \) satisfying the following:

1. \( a_i := (\beta_i, \lambda) \geq 0 \) for each \( i \in \mathcal{P}^\ell \setminus \{1\} \).
2. For each \( i \in \mathcal{P}^\ell \), there exists \( b_i \in \mathbb{Z}_{\geq 0} \) such that \( 0 \leq b_i \leq a_i \) and \( H_i = [b_1 + \cdots + b_i][a_1 + \cdots + a_i - (b_1 + \cdots + b_i) - 1] \); here, recall that \( \{n\} = pq^n + p^{-1}q^{-n} \in \mathbb{Q}(p,q) \) for \( n \in \mathbb{Z} \).

Proof. We have shown that each irreducible module in \( \mathcal{O}_\text{int}^\ell \) is a highest weight module with highest weight \( (\lambda; H) \) for some \( \lambda \in \Lambda^\ell \) and \( H = (H_1, \ldots, H_r) \in \mathbb{Q}(p,q)^r \). It is easy to verify that the irreducible highest weight module \( L(\lambda; H) \) belongs to \( \mathcal{O}^\ell \) if and only if \( f_1^r v = 0 \), \( i \in \mathcal{P}^\ell \), for a sufficiently large \( N \), where \( v \in L(\lambda; H) \) is a highest weight vector. By the case \( r = 1 \), the equality \( f_1^N v = 0 \) is equivalent to the existence of \( b_1 \in \mathbb{Z}_{\geq 0} \) satisfying the equality \( H_1 = [b_1][a_1 - b_1 - 1] \). Also, by the representation theory of \( U^q(\mathfrak{s}_2) \), the condition \( f_1^N v = 0, i \geq 2, \) is equivalent to \( a_i \geq 0 \).

It remains to determine the possible values of \( H_2, \ldots, H_r \). Let us assume the following:

(*) Let \( M \in \mathcal{O}_\text{int}^\ell \) be a \( U^2 \)-module, \( v \in M \) a highest weight vector with highest weight \( (a_1 b_1 + a_2 b_2; H_1, H_2) \). If \( H_1 = [b_1][a_1 - b_1 - 1] \) for some \( b_1 \in \mathbb{Z}_{\geq 0} \), then \( H_2 = [b_1 + b_2][a_1 + a_2 - (b_1 + b_2) - 1] \) for some \( 0 \leq b_2 \leq a_2 \).

In Section 6.1, we will prove that this assumption always holds (without assuming this theorem).

Let \( i \geq 3 \), and assume that for all \( j < i \), \( H_j = [b_1 + \cdots + b_j][a_1 + \cdots + a_j - (b_1 + \cdots + b_j) - 1] \) for some \( 0 \leq b_j \leq a_j \). Set \( T_i := ([\tau_i - 1 \tau_i] \cdots [\tau_1 \tau_2]) \subset U^1 \).

We have \( T_i(k_1) = k_1 \cdots k_{i-1}, T_i(k_2) = k_i, T_i(h_1) = h_{i-1}, \) and \( T_i(h_2) = h_i \). If we regard \( L \) as a \( U^2 \)-module via the algebra homomorphism \( T_1 : U^2 \rightarrow U_1^2 \), the \( v \) is a highest weight vector with highest weight \((a_1 + \cdots + a_{i-1}) b_1 + a_1 b_2; H_{i-1}, H_i) \). By assumption (*), \( H_i \) must be of the form \( [b_1 + \cdots + b_{i-1} + b_i][a_1 + \cdots + a_{i-1} + a_i - (b_1 + \cdots + b_{i-1} + b_i) - 1] \) for some \( 0 \leq b_i \leq a_i \). This proves the theorem.

\[\square\]

From now on, we write \( L(a; b) \) instead of \( L(\lambda; H) \), where \( a = (a_1, \ldots, a_r) \) and \( b = (b_1, \ldots, b_r) \) are such that \( a_i = (\beta_i, \lambda), H_i = [b_1 + \cdots + b_i][a_1 + \cdots + a_i] - (b_1 + \cdots + b_i) \).
Corollary 4.3.6. Let $\lambda \in \Lambda^\vee$ and $H \in \mathbb{Q}(p,q)^\vee$. Then, the irreducible highest weight module $L(\lambda; H)$ belongs to $\mathcal{O}_{\text{int}}^j$ if and only if $L(\lambda; H) = L(a,b)$ for some $(a,b) \in \mathbb{Z}^r \times \mathbb{Z}_{\geq 0}^r$ such that $a_i \geq b_i$, $i \in \mathbb{P} \setminus \{1\}$. Moreover, the assignment $(a,b) \mapsto [L(a,b)]$, where $[L(a,b)]$ denotes the isomorphism class of $L(a,b)$, gives a bijection from $\{(a,b) \in \mathbb{Z}^r \times \mathbb{Z}_{\geq 0}^r | a_i \geq b_i, \quad i \in \mathbb{P} \setminus \{1\}\}$ to the set of isomorphism classes of irreducible $\mathbb{U}^j$-modules in $\mathcal{O}_{\text{int}}^j$.

4.4. Complete reducibility. In this subsection only, we set $A := \mathbb{U}^j$, and write $B$ for $\mathbb{U}^j$ with $p$ replaced by $p^{-1}q$. Consider the anti-algebra homomorphism $S : A \rightarrow B$ over $\mathbb{Q}(q)$ defined by:

$$S(e_i) = -e_i, \quad S(f_i) = -k_i^{-1}f_i, \quad S(h_i) = k_i^{-1}, \quad S(p) = p^{-1}q.$$ 

It is easily checked that $S$ is an anti-algebra homomorphism. In addition, $S$ has the inverse:

$$S^{-1}(e_i) = -k_i e_i, \quad S^{-1}(f_i) = -f_i k_i^{-1}, \quad S^{-1}(h_i) = k_i, \quad S^{-1}(p) = p^{-1}q.$$ 

For an $A$-module $M$, define a $B$-module $S_*(M) := M^\vee$ by:

$$(x \cdot g)(m) = g(S^{-1}(x) \cdot g) \quad \text{for } x \in B, \ g \in S(M), \ m \in M,$$

where $M^\vee$ denotes the restricted dual of $M$, i.e., $M^\vee = \bigoplus_{\lambda \in \Lambda^\vee} \text{Hom}_{\mathbb{Q}(p,q)}(M, \mathbb{Q}(p,q))$. Similarly, we associate an $A$-module $S^*(N)$ with each $B$-module $N$.

Lemma 4.4.1. Let $L \in \mathcal{O}_{\text{int}}^j$ be the irreducible highest weight $A$-module with highest weight $(\lambda; H)$. Then, $S_*(L)$ is the irreducible lowest weight $A$-module with lowest weight $(-\lambda; H)$. 

Proof. Let $v \in L$ be a highest weight vector, and let $g \in S_*(L)$ be such that $g(v) = 1$ and $g(u) = 0$ for all $u \in L_\mu$, $\mu < \lambda$. Then, we have:

$$(k_\lambda g)(v) = g(k_i^{-1}v) = q^{-(\beta_i, \lambda)}g(v),$$

$$(h_i g)(v) = g(S^{-1}(h_i)v).$$

Since $S^{-1}(h_i)v \in L_\lambda = \mathbb{Q}(p,q)v$, we have $S^{-1}(h_i)v = H_i^\mu v$ for some $H_i^\mu \in \mathbb{Q}(p,q)$, and hence $h_ig = H_i^\mu g$. Therefore, $Bg$ is a lowest weight module with lowest weight $(-\lambda; H_i^\mu, \ldots, H_i^\mu)$.

Now, it remains to show that $S_*(L)$ is irreducible. Suppose that $N \subset S_*(L)$ is a submodule. Then $S^*(N)$ is a submodule of $S^*(S_*(L)) \simeq L$. Since $L$ is irreducible, $S^*(N)$ is identical either to 0 or to $L$, and hence $N$ is identical either to 0 or to $S_*(L)$. Thus, $S_*(L)$ is irreducible. This proves the lemma. □

Lemma 4.4.2. Let $M$ be an $A$-module. Suppose that $M$ contains an irreducible submodule $L \simeq L(\lambda; H)$ for some $\lambda \in \Lambda^\vee$ and $H \in \mathbb{Q}(p,q)^\vee$. Then, $M \simeq L \oplus (M/L)$. 

Proof. It suffices to show that the short exact sequence

$$0 \rightarrow L \rightarrow M \xrightarrow{\pi} M/L \rightarrow 0$$

splits. By the previous lemma, $S_*(M)$ has an irreducible submodule $S_*(L)$. Applying $S^*$ to the inclusion $S_*(L) \hookrightarrow S_*(M)$, we obtain a surjection $M \twoheadrightarrow L$ of $A$-modules. Since the composite map $L \xrightarrow{i} M \rightarrow L$ is nonzero, it follows from Schur’s lemma that this composite map is an isomorphism of $A$-modules. By composing the inverse of this isomorphism with the surjection $M \twoheadrightarrow L$, we obtain a retraction of $i$. This proves the lemma. □

Now, the complete reducibility of the $\mathbb{U}^j$-modules in $\mathcal{O}_{\text{int}}^j$ follows from a standard argument; see, for example, [HK02, Section 3.5].

Theorem 4.4.3. Every $\mathbb{U}^j$-module in $\mathcal{O}_{\text{int}}^j$ is completely reducible.

Corollary 4.4.4. Every highest weight module in $\mathcal{O}_{\text{int}}^j$ is irreducible.

Proof. Let $M$ be a highest weight vector in $\mathcal{O}_{\text{int}}^j$, and $v \in M$ a highest weight vector. By Theorem 4.4.3, we can decompose $M = \mathbb{U}^j v$ into the direct sum of irreducible submodules. Since the weight space of $\mathbb{U}^j v$ containing $v$ is one-dimensional, there exists a unique irreducible submodule $L \subset \mathbb{U}^j v$ containing $v$. This shows that $\mathbb{U}^j v = L$ is irreducible. This proves the corollary. □
Theorem 4.4.5. Let $M \in \mathcal{O}_{\text{int}}^1$. Irreducible decomposition of $M$ is unique in the following sense. If we have two irreducible decompositions $M = \bigoplus_{j \in J} L_j = \bigoplus_{k \in K} L_k$ for some index sets $J$ and $K$, then there exists a bijection $\phi : J \rightarrow K$ such that $L_j \simeq L_{\phi(j)}$ for all $j \in J$. Moreover, for each $j \in J$, the number of $j' \in J$ such that $L_{j'} \simeq L_j$ is finite.

Proof. Set $H := \{ v \in M \mid U_{\geq 0}^2 v = 0 \}$. Suppose that we have an irreducible decomposition $M = \bigoplus_{j \in J} L_j$ of $M$ for some index set $J$. Then, we obtain a highest weight vector $v_j \in L_j$ with highest weight $(\lambda_j, H_j)$ for each $j \in J$. Clearly, $\{ v_j \mid j \in J \}$ is a basis of $H$ consisting of simultaneous eigenvectors of $h_i$, $i \in \mathbb{P}$. This shows that each irreducible decomposition of $M$ depends only on the choice of a base of $H$ consisting of simultaneous eigenvectors of $h_i$, $i \in \mathbb{P}$. Since each weight space $M_\lambda$ is finite-dimensional, so is $H_\lambda$ for all $\lambda \in \Lambda$. Therefore, the number of $j \in J$ such that $L_j \simeq L(\lambda; H)$ is equal to $\dim H_\lambda$. This proves the theorem. \qed

5. Quasi-$\gamma$-Crystal Bases

5.1. Quasi-$\gamma$-Crystal Bases. The $\mathfrak{u}^\gamma = \mathfrak{u}^\gamma_\mathbb{Z}$ has $r - 1$ $\mathfrak{sl}_2$-triples: $(f_i, k_i, e_i)$ for $i = 2, \ldots, r$. Hence, one can define Kashiwara operators, $\tilde{f}_i$ and $\tilde{e}_i$, in the same way as in the crystal basis theory for quantum groups. Also, by the case $r = 1$, one can define Kashiwara operators, $\check{f}_1$ and $\check{e}_1$. Let us give the precise definition of these operators.

Definition 5.1.1. Let $M$ be a $\mathfrak{u}^\gamma$-module, $\lambda \in \Lambda^\gamma$, and $m \in M_\lambda$. For each $i \in \mathbb{P}$, there exist $m_j \in M_{\lambda + j \gamma}$, $j = 0, 1, \ldots, N$, uniquely for some $N$ such that

$$e_i m_j = 0, \quad e_i f_i m_j \in \mathbb{Q}(p, q) m_j, \quad m = \sum_{j=0}^N f_i^{(j)} m_j.$$ 

Using this expression, we define $\tilde{f}_i(m)$ and $\tilde{e}_i(m)$ by:

$$\tilde{f}_i(m) = \sum_{j=0}^N f_i^{(j+1)} m_j, \quad \tilde{e}_i(m) = \sum_{j=1}^N f_i^{(j-1)} m_j.$$ 

Set $\mathbf{A}_0 := \{ f/g \in \mathbb{Q}(p, q) \mid f \in p \mathbb{Q}[p, q, q^{-1}] + \mathbb{Q}[q], g \notin p \mathbb{Q}[p, q, q^{-1}] + q \mathbb{Q}[q] \}$; namely, $\mathbf{A}_0$ consists of all those $h \in \mathbb{Q}(p, q)$ for which $\lim_{q \to 0} \lim_{p \to 0} h$ exists.

Definition 5.1.2. Let $M$ be a $\mathfrak{u}^\gamma$-module and $\mathcal{L}$ an $\mathbf{A}_0$-submodule of $M$. We say that $\mathcal{L}$ is a quasi-$\gamma$-crystal lattice of $M$ if

($q\mathcal{L}1$) $\mathbb{Q}(p, q) \otimes_{\mathbf{A}_0} \mathcal{L} = M$,

($q\mathcal{L}2$) $\mathcal{L} = \bigoplus_{\lambda \in \Lambda^\gamma} \mathcal{L}_\lambda$, where $\mathcal{L}_\lambda := \mathcal{L} \cap M_\lambda$,

($q\mathcal{L}3$) $\tilde{f}_i(\mathcal{L}) \subset \mathcal{L}$ and $\tilde{e}_i(\mathcal{L}) \subset \mathcal{L}$ for all $i \in \mathbb{P}$.

If $\mathcal{L}$ is a quasi-$\gamma$-crystal lattice of $M$, then Kashiwara operators induce $\mathbb{Q}$-linear maps, denoted by the same symbols, on $\mathcal{L}/q\mathcal{L}$.

Definition 5.1.3. Let $M$ be a $\mathfrak{u}^\gamma$-module, $\mathcal{L}$ an $\mathbf{A}_0$-submodule of $M$, and $B$ a subset of $\mathcal{L}/q\mathcal{L}$. We say that $(\mathcal{L}, B)$ is a quasi-$\gamma$-crystal basis if

($qB1$) $\mathcal{L}$ is a quasi-$\gamma$-crystal lattice of $M$,

($qB2$) $B$ is a $\mathbb{Q}$-basis of $\mathcal{L}/q\mathcal{L}$,

($qB3$) $B = \bigsqcup_{\lambda \in \Lambda^\gamma} B_\lambda$, where $B_\lambda := B \cap (\mathcal{L}_\lambda/q\mathcal{L}_\lambda)$,

($qB4$) $\tilde{f}_i(B) \subset B \sqcup \{ 0 \}$ and $\tilde{e}_i(B) \subset B \sqcup \{ 0 \}$ for all $i \in \mathbb{P}$,

($qB5$) for each $b, b' \in B$ and $i \in \mathbb{P}$, one has $\tilde{f}_i(b) = b'$ if and only if $b = \tilde{e}_i(b')$. 

Definition 5.1.4. For a quasi-$\mathcal{J}$-crystal basis $(\mathcal{L}, \mathcal{B})$ and $i \in \mathbb{P}$, we define three maps $\varphi_i : \mathcal{B} \to \mathbb{Z}_{\geq 0}$, $\varepsilon_i : \mathcal{B} \to \mathbb{Z}_{\geq 0}$, and $\text{wt} : \mathcal{B} \to \Lambda^3$ by
\[
\varphi_i(b) := \max\{n \mid \tilde{f}_i^n(b) \neq 0\},
\varepsilon_i(b) := \max\{n \mid \tilde{e}_i^n(b) \neq 0\},
\text{wt}(b) := \lambda \text{ if } b \in \mathcal{B}_\lambda.
\]

Example 5.1.5. Let $r = 1$. For each $a \in \mathbb{Z}$ and $b \in \mathbb{Z}_{\geq 0}$, the irreducible $\mathbf{U}_1^r$-module $L(a;b)$ has the following quasi-$\mathcal{J}$-crystal basis. Let $\mathcal{L}(a;b)$ denote the $\mathbf{A}_0$-lattice spanned by $\{f_1^{(n)}v \mid 0 \leq n \leq b\}$, and set $\mathcal{B}(a;b) := \{f_1^{(n)}v + \mathcal{L}/\mathcal{L} \mid 0 \leq n \leq b\}$. Then, the Kashiwara operators $\tilde{f}_1$ and $\tilde{e}_1$ act on $\mathcal{B}(a;b)$ by:
\[
\tilde{f}_1(f_1^{(n)}v + q\mathcal{L}) = f_1^{(n+1)}v + q\mathcal{L}, \quad \tilde{e}_1(f_1^{(n)}v + q\mathcal{L}) = f_1^{(n-1)}v + q\mathcal{L}.
\]
In addition, one has $\varphi_1(f_1^{(n)}v + q\mathcal{L}) = b-n$, $\varepsilon_1(f_1^{(n)}v + q\mathcal{L}) = n$, and $\text{wt}(f_1^{(n)}v + q\mathcal{L}) = (a-3n)\delta_1$.

Definition 5.1.6. Let $M$ be a $\mathbf{U}_r^2$-module and $(\mathcal{L}, \mathcal{B})$ a quasi-$\mathcal{J}$-crystal basis of $M$. The quasi-$\mathcal{J}$-crystal graph associated with $(\mathcal{L}, \mathcal{B})$ is the colored directed graph with vertex set $\mathcal{B}$ and edges $b \xrightarrow{i} b'$, where $b, b' \in \mathcal{B}$, $i \in \mathbb{P}$ are such that $\tilde{f}_i b = b'$.

Note that a quasi-$\mathcal{J}$-crystal graph of an irreducible module is usually disconnected unless $r = 1$.

5.2. Tensor product rule. Recall that $\mathbf{U}^J$ is a right coideal of $\mathbf{U}$, i.e., $\Delta(\mathbf{U}^J) \subset \mathbf{U}^J \otimes \mathbf{U}$. Hence, we are interested in the $\mathbf{U}^J$-module structure of the tensor product of a $\mathbf{U}^J$-module and a $\mathbf{U}$-module. Let $V_r$ denote the vector representation of $\mathbf{U}$. It is spanned by $\{u_n \mid -r \leq n \leq r\}$, and is equipped with a $\mathbf{U}$-module structure by:
\[
F_j u_i = \delta_{j-\frac{1}{2}, \frac{1}{2}} u_{i+1}, \quad E_j u_i = \delta_{j+\frac{1}{2}, \frac{1}{2}} u_{i-1}, \quad K_j u_i = q^{(a_j, r_j)} u_i.
\]
If we set $\mathcal{L}_r := \bigoplus_{n=-r}^{r} \mathbf{A}_0 u_n$, $\mathcal{B}_r := \{u_n + q\mathcal{L}_r \mid -r \leq n \leq r\}$, then, $(\mathcal{L}_r, \mathcal{B}_r)$ is an ordinary crystal basis of $V_r$.

When we consider ordinary crystal bases, Kashiwara operators acting on them are denoted by capital letters $\tilde{E}_i$ and $\tilde{F}_i$, $i \in \mathbb{P}$, while those for quasi-$\mathcal{J}$-crystal bases are denoted by lowercase letters $\tilde{e}_i$ and $\tilde{f}_i$, $i \in \mathbb{P}$.

We first consider the case $r = 1$.

Proposition 5.2.1. Let $a \in \mathbb{Z}$, $b \in \mathbb{Z}_{\geq 0}$. Then we have an isomorphism
\[
L(a;b) \otimes V_1 \simeq L(a+2; b+1) \oplus L(a-1; b) \oplus L(a-1; b-1)
\]
of $\mathbf{U}_1^2$-modules. Moreover, $(\mathcal{L}(a;b) \otimes \mathcal{L}_1, \mathcal{B}(a;b) \otimes \mathcal{B}_1)$ is a quasi-$\mathcal{J}$-crystal basis of $L(a;b) \otimes V_1$.

Proof. Let $v \in L(a;b)$ be a highest weight vector, and set
\[
\begin{align*}
\mathbf{0} &:= v \otimes u_0, \\
\mathbf{1} &:= v \otimes u_1 - \frac{q^{-b+1}(q - q^{-1})}{(a - b - 1)} f_1 v \otimes u_0 - pq^{a-2b} v \otimes u_{-1}, \\
\mathbf{-1} &:= f_1 v \otimes u_0 - q^{b} [b] v \otimes u_{-1} - pq^{a-b-2}[b] v \otimes u_1.
\end{align*}
\]
Then, by direct calculation, we obtain
\[
\begin{align*}
h_1 \mathbf{0} &= [b+1] \{(a+2) - (b+1) - 1\} \mathbf{0}, \\
h_1 \mathbf{1} &= [b] \{(a-1) - (b-1) - 1\} \mathbf{1}, \\
h_1 \mathbf{-1} &= [b-1] \{(a-1) - (b-1) - 1\} \mathbf{-1}.
\end{align*}
\]
These equations, together with Corollary 3.2.3 and Theorem 3.1.6, show that $\mathbf{U}_1^2 \mathbf{0} \simeq L(a+2; b+1)$, $\mathbf{U}_1^2 \mathbf{1} \simeq L(a-1; b)$, and $\mathbf{U}_1^2 \mathbf{-1} \simeq L(a-1; b-1)$. Since $\dim(L(a; b) \otimes V_1) = 3b$,
\((b+1) + b + (b-1) = \sum_{k=-1}^{1} \dim U_1^k \) we see that \(L(a; b) \otimes V_1 = U_1^0 \oplus U_1^{-1} \oplus U_1^1 \).

Also, we calculate as:

\[
\begin{align*}
f_i^{(n)} (v_{k}) &= f_i^{(n-1)} v \otimes u_{-1} + q^n f_i^{(n)} v \otimes u_0 + pq^{n+1} f_i^{(n-1)} v \otimes u_1 \\
& \in \begin{cases} 
  v \otimes u_0 + q L(a; b) \otimes L_1 & \text{if } n = 0, \\
  f_i^{(n-1)} v \otimes u_{-1} + q L(a; b) \otimes L_1 & \text{if } 0 \leq n \leq b+1, \\
  f_i^{(n)} v \otimes u_{-1} + q L(a; b) \otimes L_1 & \text{if } 0 \leq n \leq b,
\end{cases} \\
f_i^{(n)} (u_{k}) &= q^{-n} \{a-b-n-1\} f_i^{(n)} v \otimes u_0 - \frac{q^{-b-n} + 1}{a-b-1} f_i^{(n+1)} v \otimes u_0 \\
& \quad - \frac{pq^{n-2}}{a-b-1} f_i^{(n)} v \otimes u_{-1} \\
& \in f_i^{(n)} v \otimes u_0 + q L(a; b) \otimes L_1 \quad \text{if } 0 \leq n \leq b-1.
\end{align*}
\]

Since \(\tilde{f}_i^{(n)} (v_{k}) = f_i^{(n)} (v_{k}) \), \(k \in \{0, \pm 1\}\), these equations imply that the \(A_0\)-span of \(\{\tilde{f}_i^{(n)} (v_{k}) \mid k \in \{0, \pm 1\}, \ n \in \mathbb{Z}_{\geq 0}\}\) coincides with \(L(a; b) \otimes L_1\), and that \(\{\tilde{f}_i^{(n)} (v_{k}) + q L(a; b) \otimes L_1 \mid k \in \{0, \pm 1\}, \ n \in \mathbb{Z}_{\geq 0}\} \setminus \{0\}\) is identical to \(B(a; b) \otimes B_1\). Now, it is easy to verify that \((L(a; b) \otimes L_1, B(a; b) \otimes B_1)\) is a quasi-\(j\)-crystal basis of \((L(a; b) \otimes V_1)\). This proves the proposition. \(\Box\)

We give the quasi-\(j\)-crystal graph of \(B(a; b) \otimes B_1\):

\[
\begin{array}{c}
  u_{-1} \xrightarrow{\frac{1}{2}} u_0 \xrightarrow{1} u_1 \\
  \vdots \\
  v \\
  \tilde{f}_i^{(1)} (v) \\
  \vdots \\
  \tilde{f}_i^{(v)} \\
\end{array}
\]

Let \(N \in \mathbb{N}\). Applying the above proposition repeatedly, we see that the tensor product module \(V_1^{\otimes N}\) has a quasi-\(j\)-crystal basis \((L_1^{\otimes N}, E_1^{\otimes N})\); we denote \(u_i \otimes \cdots \otimes u_N + q L_1^{\otimes N} \in E_1^{\otimes N}\), \(i_1, \ldots, i_N \in \{-1, 0, 1\}\), by \((i_1, \ldots, i_N)\). For each \(s = (s_1, \ldots, s_N) \in E_1^{\otimes N}\), let us describe \(\tilde{f}_i^{(s)}\) and \(\tilde{\epsilon}_i^{(s)}\) explicitly. First, ignore all the \(s_j\)'s such that \(s_j = 1\). Next, delete all the adjacent ordered pairs \((-1, 0)\), and repeat this until there are no such pairs. The resulting sequence \(s_{-\frac{1}{2}}\) (the subscript is for later use) is of the form \((0, \ldots, 0, -1, \ldots, -1)\). Then, \(\tilde{f}_i^{(s)}\) (resp., \(\tilde{\epsilon}_i^{(s)}\)) is obtained from \(s\) by replacing the rightmost \(0\) in \(s_{-\frac{1}{2}}\) with \(-1\) (resp., the leftmost \(-1\) in \(s_{-\frac{1}{2}}\) with \(0\)); if this is impossible, then \(\tilde{f}_i^{(s)}\) (resp., \(\tilde{\epsilon}_i^{(s)}\)) equals \(0\). Namely, \(\tilde{f}_i^{(s)} = \tilde{E}_{-\frac{1}{2}}^{(s)}\) and \(\tilde{\epsilon}_i^{(s)} = \tilde{F}_{-\frac{1}{2}}^{(s)}\).
Proposition 5.2.3. Let \( M \) be a \( \mathbf{U}_q^l \)-module having a quasi-\( l \)-crystal basis \( (\mathcal{L}, \mathcal{B}) \), and \( N \) a \( \mathbf{U}_q^3 \)-module having a crystal basis \( (\mathcal{L}', \mathcal{B}') \). Then, \( M \otimes N \) has a quasi-\( l \)-crystal basis \( (\mathcal{L} \otimes \mathcal{L}', \mathcal{B} \otimes \mathcal{B}') \), on which the Kashiwara operators act as follows:

\[
\tilde{f}_1(b \otimes b') = \begin{cases} 
 b \otimes \tilde{E}_{-\frac{1}{2}}(b') & \text{if } \varepsilon_1(b) < \varepsilon_{-\frac{1}{2}}(b'), \\
 \tilde{f}_1(b) \otimes b' & \text{if } \varepsilon_1(b) \geq \varepsilon_{-\frac{1}{2}}(b'), 
\end{cases}
\]

\[
\tilde{e}_1(b \otimes b') = \begin{cases} 
 b \otimes \tilde{F}_{-\frac{1}{2}}(b') & \text{if } \varepsilon_1(b) \leq \varepsilon_{-\frac{1}{2}}(b'), \\
 \tilde{e}_1(b) \otimes b' & \text{if } \varepsilon_1(b) > \varepsilon_{-\frac{1}{2}}(b'). 
\end{cases}
\]

More generally, we obtain the following theorem. As in the ordinary crystal basis theory, the proof is given by embedding the crystal basis of a \( \mathbf{U}_3 \)-module into \( (\mathcal{L}_1^N, \mathcal{B}_1^N) \) for a sufficiently large \( N \).

**Theorem 5.2.2.** Let \( M \) be a \( \mathbf{U}_q^l \)-module having a quasi-\( l \)-crystal basis \( (\mathcal{L}, \mathcal{B}) \), and \( N \) a \( \mathbf{U}_3 \)-module having a crystal basis \( (\mathcal{L}', \mathcal{B}') \). Then, \( M \otimes N \) has a quasi-\( l \)-crystal basis \( (\mathcal{L} \otimes \mathcal{L}', \mathcal{B} \otimes \mathcal{B}') \), on which the Kashiwara operators act as follows:

For example, let \( s = (0, 0, -1, 1, 1, -1, 0, 1, 0, 0, -1) \). Then,

\[
s_{\frac{1}{2}} = (0, 0, \cdot, \cdot, \cdot, \cdot, \cdot, 0, 0, -1),
\]

\[
\tilde{f}_1(s) = (0, 0, -1, 1, 1, -1, 0, 1, 0, 0, -1),
\]

\[
\tilde{e}_1(s) = (0, 0, -1, 1, 1, -1, 0, 1, 0, 0, 0).
\]

Now, we turn to the case of a general \( r \). Recall that Kashiwara operators \( \tilde{f}_i \) and \( \tilde{e}_i \) for \( i \neq 1 \) are defined by means of the \( sl_l \)-triple \( (f_i, k_i, e_i) \). Therefore, the next proposition follows from a standard argument; see, for example, [HK02 Section 4.4].

**Proposition 5.2.3.** Let \( M \) be a \( \mathbf{U}_q^l \)-module having a quasi-\( l \)-crystal basis \( (\mathcal{L}, \mathcal{B}) \). Then \( (\mathcal{L} \otimes \mathcal{L}_r, \mathcal{B} \otimes \mathcal{B}_r) \) is a quasi-\( l \)-crystal basis of \( M \otimes V_r \), on which the Kashiwara operators act as follows: \( \tilde{f}_i \) and \( \tilde{e}_i \) acts as described in Theorem 5.2.2 for \( i \in \mathbb{Z} \setminus \{1\}, b \in \mathcal{B}, j \in \{-r, -r+1, \ldots, r\} \),

\[
\tilde{f}_i(b \otimes u_j) = \begin{cases} 
 0 & \text{if } j = i \text{ and } \tilde{f}_i^2(b) = 0, \\
 b \otimes u_i & \text{if } j = i - 1 \text{ and } \tilde{f}_i(b) = 0, \\
 b \otimes u_{-i} & \text{if } j = -(i - 1) \text{ and } \tilde{e}_i(b) = 0, \\
 \tilde{f}_i(b) \otimes u_j & \text{otherwise}, 
\end{cases}
\]

\[
\tilde{e}_i(b \otimes u_j) = \begin{cases} 
 b \otimes u_{i-1} & \text{if } j = i \text{ and } \tilde{f}_i(b) = 0, \\
 0 & \text{if } j = -(i - 1) \text{ and } \tilde{e}_i^2(b) = 0, \\
 b \otimes u_{-(i-1)} & \text{if } j = -i \text{ and } \tilde{e}_i(b) = 0, \\
 \tilde{e}_i(b) \otimes u_j & \text{otherwise}. 
\end{cases}
\]
The action of \( \tilde{f}_i \) for \( i \neq 1 \) is visualized as:

\[
\begin{array}{ccc}
\text{u}_{-i} & \overset{-(i - \frac{1}{2})}{\rightarrow} & \text{u}_{-(i-1)} \\
& b & i \\
\tilde{f}_i(b) & \rightarrow & i \\
\vdots & \vdots & \vdots \\
\tilde{f}_i^{\varphi_i^{-1}}(b) & \rightarrow & \varphi_i^{-1}(b) \\
\end{array}
\]

From Proposition 5.2.3, we see that \( V_r^\otimes N \) has a quasi-\( j \)-crystal basis \( (\mathcal{L}_r^\otimes N, \mathcal{B}_r^\otimes N) \); we denote \( u_1 \otimes \cdots \otimes u_N + q \mathcal{L}_r^\otimes N \in \mathcal{B}_r^\otimes N, i_1, \ldots, i_N \in \{-r, \ldots, r\}, \text{by} \ (i_1, \ldots, i_N) \). Before describing its quasi-\( j \)-crystal structure, recall the ordinary crystal structure of \( \mathcal{B}_r^\otimes N \). Let \( s = (s_1, \ldots, s_N) \in \mathcal{B}_r^\otimes N \) and \( i \in \mathbb{I} \). First, ignore all the \( s_j \)'s such that \( s_j \neq i \pm \frac{1}{2} \). Next, delete all the adjacent ordered pairs \( (i - \frac{1}{2}, i + \frac{1}{2}) \), and repeat this until there are no such pairs. The resulting sequence is of the form \( (i + \frac{1}{2}, i, i + \frac{1}{2}, i - \frac{1}{2}, \ldots, i - \frac{1}{2}) \); we denote this sequence by \( s_i \). Then, \( \tilde{F}_i(s) \) is obtained from \( s \) by replacing the leftmost \( i - \frac{1}{2} \) in \( s_i \) with \( i + \frac{1}{2} \); if this is impossible, then \( \tilde{F}_i(s) = 0 \). Also, \( \tilde{E}_i(s) \) is obtained from \( s \) by replacing the rightmost \( i + \frac{1}{2} \) in \( s_i \) with \( i - \frac{1}{2} \); if this is impossible, then \( \tilde{E}_i(s) = 0 \). Note that \( s_i \) consists of \( \varepsilon_i(s) \ (i + \frac{1}{2})'s \) and \( \varphi_i(s) \ (i - \frac{1}{2})'s:

\[
\begin{array}{c}
\varepsilon_i(s) \\
\varphi_i(s)
\end{array} = \begin{cases} 
(i + \frac{1}{2}, \ldots, i + \frac{1}{2}, i - \frac{1}{2}, \ldots, i - \frac{1}{2}) \\
\varepsilon_i(s) \quad \varphi_i(s)
\end{cases}
\]

From the consideration above, we can describe the quasi-\( j \)-crystal structure of \( \mathcal{B}_r^\otimes N \) as follows.

**Proposition 5.2.4.** Let \( s \) be as before and \( i \in \mathbb{I} \setminus \{1\} \). First, consider the concatenated sequence \( (s_{-(i - \frac{1}{2})}, s_{-(i - \frac{1}{2})}) \), where \( s_{-(i - \frac{1}{2})} \) is the sequence obtained by reversing the order of \( s_{-(i - \frac{1}{2})} \); we denote this sequence by \( s_i \), i.e.,

\[
\begin{array}{c}
\varepsilon_{-(i - \frac{1}{2})}(s) \\
\varphi_{-(i - \frac{1}{2})}(s)
\end{array} = \begin{cases} 
(i, \ldots, i, -(i - 1), \ldots, -(i - 1), i, \ldots, i, i - 1, \ldots, i - 1) \\
\varepsilon_{-(i - \frac{1}{2})}(s) \quad \varphi_{-(i - \frac{1}{2})}(s)
\end{cases}
\]

Next, delete all the adjacent ordered pairs \( -(i - 1), i \), and repeat this until there are no such pairs. Then, \( \tilde{f}_i(s) \) is obtained from \( s \) by replacing the leftmost \( -(i - 1) \) in \( s_i \) with \( -i \) (resp., the leftmost \( i - 1 \) in \( s_i \) with \( i \)) if \( -(i - 1) \notin s_i \) (resp., \( -(i - 1) \notin s_i \)); if this is impossible, then \( \tilde{f}_i(s) = 0 \). Also, \( \tilde{e}_i(s) \) is obtained from \( s \) by replacing the rightmost \( i \) in \( s_i \) with \( i - 1 \) (resp., the rightmost \( -i \) in \( s_i \) with \( -(i - 1) \)) if \( i \in s_i \) (resp., \( i \notin s_i \)); if this is impossible, then \( \tilde{e}_i(s) = 0 \).
Namely,
\[ f_1(s) = \tilde{E}_{-\frac{1}{2}}(s), \quad \tilde{e}_1(s) = \tilde{F}_{-\frac{1}{2}}(s), \]
\[ \tilde{f}_i(s) = \begin{cases} \tilde{E}_{-(i-\frac{1}{2})}(s) & \text{if } \varepsilon_{i-\frac{1}{2}}(s) < \varepsilon_{-(i-\frac{1}{2})}(s), \\ \tilde{F}_{-\frac{1}{2}}(s) & \text{if } \varepsilon_{i-\frac{1}{2}}(s) \geq \varepsilon_{-(i-\frac{1}{2})}(s), \end{cases} \]
\[ \tilde{e}_i(s) = \begin{cases} \tilde{F}_{-(i-\frac{1}{2})}(s) & \text{if } \varepsilon_{i-\frac{1}{2}}(s) \leq \varepsilon_{-(i-\frac{1}{2})}(s), \\ \tilde{E}_{-\frac{1}{2}}(s) & \text{if } \varepsilon_{i-\frac{1}{2}}(s) > \varepsilon_{-(i-\frac{1}{2})}(s). \end{cases} \]

Proof. The proof proceeds by induction on \( N \) by means of Proposition 5.2.3 \( \square \)

Now, we are ready to generalize Proposition 5.2.3. The following theorem describes the tensor product rule for the Kashiwara operators \( \tilde{f}'s \) and \( \tilde{e}'s \) in full generality. The proof is given by embedding the crystal basis of a \( U \)-module into \( (L^\Sigma_N, B^\Sigma_N) \) for a sufficiently large \( N \).

**Theorem 5.2.5.** Let \( M \) be a \( U_l \)-module having a quasi-\( \gamma \)-crystal basis \( (\mathcal{L}, \mathcal{B}) \), and \( N \) a \( U_{2r+1} \)-module having a crystal basis \( (\mathcal{L}', \mathcal{B}') \). Then, \( M \otimes N \) has a quasi-\( \gamma \)-crystal basis \( (\mathcal{L} \otimes \mathcal{L}', \mathcal{B} \otimes \mathcal{B}') \), on which the Kashiwara operators act as follows: for \( b \in \mathcal{B} \) and \( b' \in \mathcal{B}' \),

\[ f_1(b \otimes b') = \begin{cases} b \otimes \tilde{E}_{-\frac{1}{2}}(b') & \text{if } \varepsilon_1(b) < \varepsilon_{-\frac{1}{2}}(b'), \\ \tilde{f}_1(b) \otimes b' & \text{if } \varepsilon_1(b) \geq \varepsilon_{-\frac{1}{2}}(b'), \end{cases} \]
\[ \tilde{e}_1(b \otimes b') = \begin{cases} b \otimes \tilde{F}_{-\frac{1}{2}}(b') & \text{if } \varepsilon_1(b) \leq \varepsilon_{-\frac{1}{2}}(b'), \\ \tilde{e}_1(b) \otimes b' & \text{if } \varepsilon_1(b) \geq \varepsilon_{-\frac{1}{2}}(b'), \end{cases} \]

\[ \tilde{f}_i(b \otimes b') = \begin{cases} b \otimes \tilde{E}_{-(i-\frac{1}{2})}(b') & \text{if } \varepsilon_{i-\frac{1}{2}}(b') < \varphi_i(b) \text{ and } \varepsilon_i(b) \geq \varepsilon_{-(i-\frac{1}{2})}(b'), \\ \tilde{f}_i(b) \otimes b' & \text{if } \varepsilon_{i-\frac{1}{2}}(b') \geq \varphi_i(b) \text{ and } \varepsilon_i(b) < \varepsilon_{-(i-\frac{1}{2})}(b'), \end{cases} \]
\[ \tilde{e}_i(b \otimes b') = \begin{cases} b \otimes \tilde{F}_{-(i-\frac{1}{2})}(b') & \text{if } \varepsilon_{i-\frac{1}{2}}(b') \leq \varphi_i(b) \text{ and } \varepsilon_i(b) \leq \varepsilon_{-(i-\frac{1}{2})}(b'), \\ \tilde{e}_i(b) \otimes b' & \text{if } \varepsilon_{i-\frac{1}{2}}(b') > \varphi_i(b) \text{ and } \varepsilon_i(b) < \varphi_i(b). \end{cases} \]

6. The case \( r = 2 \)

6.1. **Quasi-\( \gamma \)-crystal bases of irreducible highest weight modules.** Throughout this subsection, we fix a \( U_2 \)-module \( M \in \mathcal{O}_\text{int}^l \). Recall from the case \( r = 1 \) that \( M \) is decomposed as:

\[ M = \bigoplus_{a \in \mathbb{Z}, \, b, n \in \mathbb{Z}_{>0}} M_{a, b, n}, \]
\[ M_{a, b, 0} = \{ u \in M \mid e_1 u = 0, \, k_1 u = q^a u, \, h_1 u = [b]\{a - b - 1\} u \}, \]
\[ M_{a, b, n} = f_1^{(n)}(M_{a, b, 0}). \]

While the representation theory of \( U_2 \) is similar to that of \( U_3 \), the representation theory of \( U_2^l \) is much more difficult than that of \( U_3 \). The main difficulty comes from the fact that \( f_2m \) is not necessarily an eigenvector of \( h_1 \) even if \( m \) is so. Hence, we need to investigate the action of \( f_2 \) on \( m \) carefully.
Proposition 6.1.1. Let \( a \in \mathbb{Z} \) and \( b, n \in \mathbb{Z}_{\geq 0} \). We define three linear maps \( f \) as follows. For each \( a \in \mathbb{Z} \) and \( b, n \in \mathbb{Z}_{\geq 0} \), we define \( f(a, b, n) = \mathbb{U} \), i.e.,

\[
\begin{align*}
  f_{2,1}(a, b, n) &= (pq^{a-b} - p^{-1}q^{-a-b+1})f_2 - q^{-b+n+1}f_2', \\
  f_{2,2}(a, b, n) &= (pq^{a-b-2}f_2 - (q^{b+1} + q^{-b-1})f_2 + p^{-1}q^{-a+b+n+2}f_2', \\
  f_{2,3}(a, b, n) &= q^{-n-2}f_2 + (pq^{a-2b-1} - p^{-1}q^{-a+2b+1})f_2 - q^{-n+2}f_2'.
\end{align*}
\]

Also, we define three linear maps \( f' \) as follows:

\[
\begin{align*}
  f'_{2,i}(m) &= f'(a, b, n)m \quad \text{for } m \in M_{a,b,n}.
\end{align*}
\]

Proposition 6.1.1. Let \( a \in \mathbb{Z} \), \( b, n \in \mathbb{Z}_{\geq 0} \), and \( m \in M_{a,b,n} \). Then, we have

\[
\begin{align*}
  f_{2,1}(m) &= M_{a+1,b+1,n}, \quad f_{2,2}(m) = M_{a+1,b,n}, \quad f_{2,3}(m) = M_{a-2,b-1,n-1}.
\end{align*}
\]

Proof. See Appendix \[3.1\]

Proposition 6.1.2. The linear maps \( f'_{2,i}, i = 1, 2, 3 \), commute with each other.

Proof. See Appendix \[3.2\]

We normalize \( f'_{2,i} \) as follows:

\[
\begin{align*}
  f_{2,1}(a, b, n) &= 1 \quad (q^{b+1} - q^{-b-1})(a - 2b - 1) f'_{2,1}(a, b, n), \\
  f_{2,2}(a, b, n) &= - \frac{1}{(a - b)(a - 2b - 1)} f'_{2,2}(a, b, n), \\
  f_{2,3}(a, b, n) &= - \frac{1}{(q^{b+1} - q^{-b-1})(a - b)} f'_{2,3}(a, b, n),
\end{align*}
\]

and define linear maps \( f_{2,i}, i = 1, 2, 3 \), by \( f_{2,i}(m) = f_{2,i}(a, b, n)m \) for \( m \in M_{a,b,n} \). Then, for each \( m \in M_{a,b,n} \), we have \( f_{2,m} = (f_{2,1} + f_{2,2} + f_{2,3})m \). Thanks to this equality and Proposition \[6.1.1\] in order to compute \( f_{2}(m) \), it is enough to decompose \( f_{2}m \) into three \( h_{1} \)-eigenvectors with distinct eigenvalues. The computation becomes easier when \( n = 0 \) since in this case, \( f_{2,3}(m) = 0 \). Also, it follows that \( f_{2}m \in M_{a+1,b+1,0} \oplus M_{a+1,b,0} \) for \( m \in M_{a,b,0} \). Repeating this, we have

(9)

\[
   f_{2}^{(l)}m \in \bigoplus_{k=0}^{l} M_{a+l,b+k,0} \quad \text{for } l \in \mathbb{Z}_{\geq 0}, \ m \in M_{a,b,0}.
\]

Let \( L \in \mathcal{O}_{\text{int}}^{\mu} \) be an irreducible \( \mathbb{U}_{2}^{\prime} \)-module. By the first half of the proof of Theorem \[4.3.5\] \( L \) is isomorphic to \( L(a_1 \delta_1 + a_2 \delta_2; H_1, H_2) \) for some \( a_1 \in \mathbb{Z} \), \( a_2 \in \mathbb{Z}_{\geq 0} \), and \( H_1, H_2 \in \mathbb{Q}(p, q) \). Moreover, \( H_1 = [b_1] \{ a_1 - b_1 - 1 \} \) for some \( b_1 \in \mathbb{Z}_{\geq 0} \). As we announced in the proof of Theorem \[4.3.5\] we show that \( H_2 = [b_1 + b_2] \{ a_1 + a_2 - (b_1 + b_2) - 1 \} \) for some \( b_2 \in \mathbb{Z}_{\geq 0} \). If \( v \in L \) is a highest weight vector, then \( \tau_{2}^{-1}(v) \) is a \( \mathbb{U}_{2}^{\prime} \)-highest weight vector. Hence, we have

\[
   h_{2}v = \tau_{2}(h_{2}^{-1}(v)) = b \{ a_1 + a_2 - b - 1 \} v
\]

for some \( b \in \mathbb{Z}_{\geq 0} \). This implies that \( f_{2}^{(a_1)}v = \tau_{2}^{-1}(v) \in L_{a_1 + a_2, b, 0} \). Also, we have \( f_{2}^{(a_2)}v \) by equation \[9\]. Therefore, we deduce that \( b_1 \leq b \leq b_1 + a_2 \), and hence \( b = b_1 + b_2 \) for \( 0 \leq b_2 \leq a_2 \). This completes the proof of Theorem \[4.3.5\] by Theorem \[4.3.5\] and \( 0 \leq b_2 \leq a_2 \). We define two linear operators \( \tilde{f}_{2}^{\prime} \) and \( \tilde{c}_{2}^{\prime} \) on \( L \) as follows. First, for \( c \in \mathbb{Z}_{\geq 0} \), we set

\[
   \tilde{f}_{2}^{\prime}(v) := q^{-cb}f_{2,1}^{(c)}v,
\]

and

\[
   \tilde{f}_{2}^{\prime}(\tilde{f}_{2}^{\prime}(v)) = \tilde{f}_{2}^{\prime+1}(v), \quad \tilde{c}_{2}^{\prime}(\tilde{f}_{2}^{\prime}(v)) = \tilde{f}_{2}^{\prime-1}(v) \quad \text{if } \tilde{f}_{2}^{\prime}(v) \neq 0.
\]
Next, note that \( L \) is decomposed as

\[
L = \bigoplus_{a \in \mathbb{Z}} \bigoplus_{b, n \in \mathbb{Z} \geq 0} (L_{a,b,n} \cap L_\lambda);
\]

here, recall that \( L_\lambda \) is the weight space of weight \( \lambda \), and \( L_{a,b,n} \) is defined as at the beginning of this subsection. Since \( D := \bigoplus_{c \in \mathbb{Z} \geq 0} \mathbb{Q}(p,q) f_2' (v) = \bigoplus_{c \in \mathbb{Z} \geq 0} (L_{a_1+c,b,0} \cap L_{a_1+b_1+a_2-b_2-c_0}) \), one can take the complementary subspace \( C \) to \( D \) with respect to the decomposition above of \( L \). We define \( f_2' \) and \( \tilde{c}_2 \) to be zero on \( C \).

**Theorem 6.1.3.** Let \( a_1 \in \mathbb{Z}, a_2, b_1, b_2 \in \mathbb{Z} \geq 0, b_2 \leq a_2 \), and let \( v \in L(a_1, a_2; b_1, b_2) \) be a highest weight vector. Set

\[
\mathcal{L}(a_1,a_2;b_1,b_2) := \sum_{i_1, \ldots, i_l \in \{1,2\}} A_0 f_{i_1} \cdots f_{i_l}(v) = \sum_{i_1, \ldots, i_l \in \{1,2\}} A_0 \tilde{f}_{i_1} \cdots \tilde{f}_{i_l} f_2'(v),
\]

\[
\mathcal{B}(a_1,a_2;b_1,b_2) := \{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_l}(v) + q \mathcal{L}(a_1,a_2;b_1,b_2) \mid l \in \mathbb{Z} \geq 0, i_1, \ldots, i_l \in \{1,2\} \} \setminus \{0\} = \{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_l} f_2'(v) + q \mathcal{L}(a_1,a_2;b_1,b_2) \mid l, c \in \mathbb{Z} \geq 0, i_1, \ldots, i_l \in \{1,2\} \} \setminus \{0\}.
\]

Then, \( (\mathcal{L}(a_1,a_2;b_1,b_2), \mathcal{B}(a_1,a_2;b_1,b_2)) \) is a quasi-crystal basis of \( L(a_1,a_2;b_1,b_2) \).

We will reformulate this theorem for a general \( r \) as Theorem 7.2.1 which is proved in Section 9. The next subsection is devoted to the preparation for the proof.

### 6.2. Preparation for the proof of Theorem 7.2.1

Let \( M \in \mathcal{O}_{\text{int}} \). For the computation in this subsection, it is important to obtain the commutation relations among \( f_{2,1}, f_{2,2}, \) and \( f_{2,3} \).

**Lemma 6.2.1.** For each \( m \in M_{a,b,n} \), we have

\[
f_{2,1} f_{2,2} m = \frac{\{a-b,2\}}{\{a-2b\}} f_{2,2} f_{2,1} m, \]

\[
f_{2,3} f_{2,1} m = \frac{\{b\}}{\{b+2\}} f_{2,1} f_{2,3} m, \]

\[
f_{2,3} f_{2,2} m = \frac{\{a-b,1\}}{\{a-b+1\}} f_{2,2} f_{2,3} m.
\]

**Proof.** This is an easy consequence of Proposition 6.1.2.

Recall that \( V_2 \) denotes the vector representation of \( U = U_q(\mathfrak{sl}_2) \). Let us consider \( M \otimes V_2 \), for which we know all the \( h_1 \)-eigenvectors and eigenvalues by the proof of Proposition 5.2.1.

For \( m \in M_{a,b,0} \), we define \( m \oplus 0 m \oplus 1 m \oplus -1 m \oplus 2 m \oplus -2 \in M \otimes V_2 \) by

\[
m \oplus 0 := m \otimes u_0,
\]

\[
m \oplus 1 := m \otimes u_1 - \frac{q^{-b+1}(q^{-1})}{\{a-b\}} f_1 m \otimes u_0 - p q^{a-2b} m \otimes u_{-1},
\]

\[
m \oplus -1 := f_1 m \otimes u_0 - q^{b}[b] m \otimes u_{-1} - p q^{a-b-2}[b] m \otimes u_1,
\]

\[
m \oplus 2 := m \otimes u_2,
\]

\[
m \oplus -2 := m \otimes u_{-2}.
\]

**Proposition 6.2.2.** We have

\[
m \oplus 0 \in (M \otimes V_2)_{a+2, b+1, 0}, \quad m \oplus 1 \in (M \otimes V_2)_{a-1, b, 0}, \quad m \oplus 2 \in (M \otimes V_2)_{a, b, 0},
\]

\[
m \oplus -1 \in (M \otimes V_2)_{a-1, b-1, 0}, \quad m \oplus -2 \in (M \otimes V_2)_{a, b, 0}.
\]
Proposition 6.2.8. Assume that $\tilde{f}^c_2(v) \neq 0$ if and only if $0 \leq c \leq a_2 - b_2$. Then we have

\[
\tilde{f}^c_2(v_0) \neq 0 \text{ if and only if } 0 \leq c \leq a_2 - b_2,
\]

\[
\tilde{f}^c_2(v_1) \neq 0 \text{ if and only if } 0 \leq c \leq (a_2 + 1) - b_2,
\]

\[
\tilde{f}^c_2(v_{-1}) \neq 0 \text{ if and only if } 0 \leq c \leq (a_2 + 1) - (b_2 + 1).
\]
Moreover, \( \tilde{f}_2^c(v_j) \in \mathcal{L} := L(a_1, a_2; b_1, b_2) \otimes \mathcal{L}_2 \), and \( \tilde{f}_2^c(v_j) + q\mathcal{L} \in \mathcal{B} := B(a_1, a_2; b_1, b_2) \otimes \mathcal{B}_2 \) for \( j = 0, \pm 1 \).

**Proof.** The assertions for \( v_0 \) and \( v_{-1} \) are clear from Lemma 6.2.5 and Lemma 6.2.6.

\[
\tilde{f}_2^c(v_0) = q^{-cb_2}f_{2,2}^{(c)}(v_0) = q^{-cb_2}f_{2,2}^{(c)}(v) \otimes u_0 = \tilde{f}_2^c(v) \otimes u_0,
\]

\[
\tilde{f}_2^c(v_{-1}) = q^{-c(b_2+1)}f_{2,2}^{(c)}(v_{-1}) = q^{-c(b_2+1)}\prod_{n=1}^{c} \left\{ \frac{a_1 - 2b_1 - 2a_1 + 2b_1 - (1+n)}{a_1 - 2b_1 - 1 + n} \right\} f_{2,2}^{(c)}(v_{-1}) = \frac{q^{-c}a_1 - 2b_1 - (1-c)}{a_1 - 2b_1 - 1} \tilde{f}_2^c(v_{-1}) \in (1 + p\mathfrak{A}_0)\tilde{f}_2^c(v_{-1}) \in 1.
\]

For \( j = 1 \), by Lemma 6.2.5 we have

\[
\tilde{f}_2^c(v_1) = q^{-cb_2}f_{2,2}^{(c)}(v_1) = q^{-cb_2} \left[ f_{2,2}^{(c)}(v) \right] + q^{a_2-c+1}f_{2,2}^{(c-1)}(v) \left[ 2 \right] - pq^{a_1-2b_1}f_{2,2}^{(c-1)}(v) \left[ 2 \right] - q^{a_1-2b_1-1}f_{2,2}^{(c-1)}(v) \left[ 0 \right]
\]

\[
= \tilde{f}_2^c(v) + q^{a_2-b_2-c+1}f_{2,2}^{c-1}(v) \left[ 2 \right] - pq^{a_1-2b_1-b_2}f_{2,2}^{c-1}(v) \left[ 2 \right] - q^{a_1-b_2-1}f_{2,2}^{(c-1)}(v) \left[ 0 \right]
\]

\[
= \tilde{f}_2^c(v) + \tilde{f}_2^c(v) \left[ 2 \right] - pq^{a_1-2b_1-b_2}f_{2,2}^{c-1}(v) \left[ 2 \right] - q^{a_1-b_2-1}f_{2,2}^{c-1}(v) \left[ 0 \right].
\]

Note that \( f_1\tilde{f}_2^{c-1}(v) = \tilde{f}_1f_{2}^{c-1}(v) \in \mathcal{L} \). Since \( \mathcal{L} \) is a quasi-\( j \)-crystal lattice, it follows that \( f_2(\mathcal{L}) \subset q^{-N}\mathcal{L} \) for sufficiently large \( N \geq 0 \). Also, by the complete reducibility of \( U_1^{j} \)-modules in \( \mathcal{O}_{\text{int}}^{j} \), \( \mathcal{L} \) is decomposed as \( \mathcal{L} = \bigoplus_{a,b,n} L(a, b, n) \), where \( L(a, b, n) := L(a, b, n) \cap \mathcal{L} \). Suppose that \( m \in L(a, b, n) \), and consider \( f_2m = f_{2,1}(m) + f_{2,2}(m) + f_{2,3}(m) \). Since \( f_{2,1}(m) \in L(a, b, n) \), for some \( a_1, b_1, n_1 \), it follows that \( f_{2,1}(m) \in q^{-N}\mathcal{L} \) for sufficiently large \( N \geq 0 \). In particular, \( \frac{q^{-a_1-b_2-1}(q-1)}{a_1 - b_1 - 2}f_{2,3}f_{2,2}^{c-1}(v) \) and \( \frac{q^{-a_1-b_2-1}(q-1)}{a_1 - b_1 - 2}f_{2,2}^{c-1}(v) \) belong to \( p\mathcal{L} \). Therefore, we deduce that \( \tilde{f}_2^c(v) \left[ 1 \right] \in \mathcal{L} \), and

\[
\tilde{f}_2^c(v) + q\mathcal{L} = \begin{cases} \tilde{f}_2^c(v) \otimes u_1 + q\mathcal{L} & \text{if } c \leq a_2 - b_2, \\ \tilde{f}_2^c(v) \otimes u_2 + q\mathcal{L} & \text{if } c = a_2 - b_2 + 1. \end{cases}
\]

This proves the proposition. \( \square \)

**Corollary 6.2.9.** We have \( \tilde{f}_2^c(v) \neq 0 \) if and only if \( 0 \leq c \leq a_2 - b_2 \)

**Proof.** It is clear that \( \tilde{f}_2^c(v) \neq 0 \) if and only if \( c = 0 \). Hence, the assertion follows inductively by Proposition 6.2.3. \( \square \)

The submodule generated by \( v_0, v_{-1} \), and \( v_1 \) may not be the whole of \( L \). We find the other highest weight vectors as follows.

Let \( \lambda \in \Lambda^j \) be the highest weight of \( L(a_1, a_2; b_1, b_2) \), i.e., \( \lambda = a_1\delta_1 + a_2\delta_2 \). Note that every weight of \( L \) is less than or equal to \( \lambda + 2b_1 \) with respect to the partial order defined by equation (6) in Section 2.3.
Lemma 6.2.10. The subspace $L_{\lambda+2\delta_1-\gamma_1-\gamma_2} \cap \text{Ker}(e_1) \cap \text{Ker}(e_2) \cap \text{Ker}(h_1 - [b_1]|a_1-b_1-1|)$ is at most two-dimensional; it is spanned by

$$m'_1 := f_{2,2}(v) \square - q^{-1} yv \otimes u_2 + \frac{q^{-b_1+1}(q-q^{-1})}{q^{a_1+a_2-2b_1+1}y} (f_{2,3}f_1)(v) \square 0 + p q^{a_1+a_2-2b_1+1} yv \otimes u_{-2},$$

$$m'_2 := f_{2,1}(v) \square -1 + p q^{-b_1+1} v [b_1+1] xv \otimes u_2 - \frac{x}{z} (f_{2,3}f_1)(v) \square 0 + q^{a_2+b_1+1} [b_1+1] xv \otimes u_{-2},$$

where we set

$$x := \frac{[b_2]|a_1+a_2-2b_1-b_2-1|}{[a_1-2b_1-1]}, \quad y := \frac{[a_2-b_2]|a_1-2b_1-b_2-1|}{[a_1-2b_1-1]},$$

$$z := \frac{[b_1+b_2+1]|a_1+a_2-b_2-b_2|}{[b_1+1]|a_1-b_1|}.$$

Proof. See Appendix [3.4].

We would like to determine all the $U_2$-highest weight vectors (or equivalently, $h_2$-eigenvectors) in $L_{\lambda+2\delta_1-\gamma_1-\gamma_2} \cap \text{Ker}(e_1) \cap \text{Ker}(e_2) \cap \text{Ker}(h_1 - [b_1]|a_1-b_1-1|$); this is completed in Proposition [6.2.11]. If one of $m'_1$ and $m'_2$ is equal to zero, then we are done. Hence we assume that $m'_1 \neq 0$ and $m'_2 \neq 0$. Note that $m'_1 \neq 0$ if and only if $a_2-b_2 \neq 0$, and that $m'_2 \neq 0$ if and only if $b_2 \neq 0$. For notational simplicity, we write $u \sim u'$ to indicate that $u - u' \in qL'$ for $u, u' \in L'$, where $L'$ is either $L(a_1, a_2; b_1, b_2)$ or $L$, and write $u \preceq u'$ to indicate that $u' \in (1+qA_0)u$. Also, we write $u \preceq u'$ to indicate that $q^n(u - u') \in pL'$ for a sufficiently large $N \geq 0$. By setting

$$m_1 := \frac{q}{y} m'_1, \quad m_2 := \frac{q}{y} m'_2,$$

we see that $m_1 \preceq v \otimes u_2$ and $m_2 \preceq f_{1,2}(v) \otimes u_0$.

**Proposition 6.2.11.** Assume that $a_2 - b_2 > 0$ and $b_2 > 0$.

1. There exists $v_2 \in A_0 m_1 \oplus A_0 m_2$ such that $v_2 - m_1 \in q(A_0 m_1 \oplus A_0 m_2)$ and $h_2 v_2 = [b_1+b_2]|a_1+(a_2-1) - (b_1+b_2)-1| v_2$. In particular, we have $U_2 v_2 \simeq L(a_1, a_2-1; b_1, b_2)$.

2. There exists $v_{-2} \in A_0 m_1 \oplus A_0 m_2$ such that $v_{-2} - m_2 \in q(A_0 m_1 \oplus A_0 m_2)$ and $h_2 v_{-2} = [b_1+b_2-1] |a_1+(a_2-1) - (b_1+b_2)-1| v_{-2}$. In particular, we have $U_2 v_{-2} \simeq L(a_1, a_2-1; b_1, b_2-1)$.

Proof. See Appendix [3.5].

As the last step of the preparation for the proof of Theorem [7.2.1], we compute $f_{2}(v \pm 2) + qL$. Since $m_1 \preceq v \otimes u_2 - \frac{q}{y} f_{2,2}(v)$, we have $q^{-cb_2} f_{2,2}(v) \preceq q^{-cb_2} f_{2,2}(v) \otimes u_2 - \frac{q}{y} f_{2,2}(v)$, and

$$q^{-cb_2} f_{2,2}(v) \otimes u_2 = \frac{q}{y} f_{2,2}(v) \otimes u_2,$$

$$q^{-cb_2} f_{2,2}(v) \otimes u_2 = \frac{q}{y} f_{2,2}(v) \otimes u_2 - \frac{q}{y} f_{2,2}(v) \square 1 - q^{2(a_2-b_2-c)} \tilde{f}_{2}^c(v) \square 2.$$
Similarly, we have
\[ m_2 \overset{p}{\sim} f_{2,1}(v|-1) - \frac{x}{z} (f_{2,3} f_{1})_0 0 + q^{a_2+b_1+1} [b_1 + 1] x v - 2, \]
and
\[ q^{-c(b_2-1)} f^{(c)}_{2,2}(f_{2,1}(v)[-1] - \frac{x}{z} (f_{2,3} f_{1})_0 0 + q^{a_2+b_1+1} [b_1 + 1] x v - 2) \]
\[ = q^{-c(b_2-1)} \left( \prod_{i=1}^c \frac{\{ a_1 - 2b_1 - 3 + i \}}{\{ a_1 - 2b_1 - 2 + i \}} f^{(c)}_{2,2} f_{2,1}(v)[-1] \right) \]
\[ - \frac{x}{z} \prod_{i=1}^c \frac{\{ a_1 - b_1 + i \}}{\{ a_1 - b_1 + i - 1 \}} (f_{2,3} f_{1})_0 0 \]
\[ + q^{a_2+b_1+1-c} [b_1 + 1] x f^{(c)}_{2,2}(v)[-2] \]
\[ = q^{-c(b_2-1)} \left( \frac{q^{c_2} a_1 - 2b_1 + c - 1}{a_1 - 2b_1 - 1} f^{(c)}_{2,1} f^{(c)}_{2,2}(v)[-1] \right) \]
\[ - \frac{x}{z} \frac{q^{c_2} a_1 - b_1 + c}{a_1 - b_1} (f_{2,3} f_{1})_0 0 \]
\[ + q^{a_2+b_1+1-c+c_2} [b_1 + 1] x f^{(c)}_{2,2}(v)[-2] \].

Therefore, we obtain
\[ \tilde{f}^{(c)}_{2,2}(m_2) \overset{q}{\sim} f_{1,2} f_{2,1} f^{(c)}_{2,2}(v) \otimes u_0 \quad \text{if } 0 \leq c \leq (a_2 - 1) - (b_2 - 1). \]

**Proposition 6.2.12.** Assume that \( a_2 - b_2 > 0 \) and \( b_2 > 0 \). For \( c \in \mathbb{Z}_{\geq 0} \), we have
\[ \tilde{f}^{(c)}_{2,2}(v) + q \mathcal{L} = \begin{cases} \tilde{f}^{(c)}_{2,2}(v) \otimes u_0 & \text{if } 0 \leq c \leq (a_2 - 1) - b_2, \\ 0 & \text{if } c > (a_2 - 1) - b_2, \end{cases} \]
\[ \tilde{f}^{(c)}_{2,2}(v-2) + q \mathcal{L} = \begin{cases} \tilde{f}^{(c)}_{1,2} \tilde{f}^{(c)}_{2,2}(v) \otimes u_0 & \text{if } 0 \leq c \leq (a_2 - 1) - (b_2 - 1), \\ 0 & \text{if } c > (a_2 - 1) - (b_2 - 1). \end{cases} \]

**Proof.** We prove the assertion for \( v_2 \). By Corollary 6.2.9, we have \( \tilde{f}^{(c)}_{2,2}(v_2) = 0 \) for \( c > (a_2 - 1) - b_2 \). Also, by equality (11) and (12), we have
\[ \tilde{f}^{(c)}_{2,2}(v_2) + q \mathcal{L} = \tilde{f}^{(c)}_{2,2}(m_2) + q \mathcal{L} = \tilde{f}^{(c)}_{2,2}(v_2) + q \mathcal{L} \]
for \( 0 \leq c \leq (a_2 - 1) - b_2 \), as desired. The proof of the assertion for \( v_{-2} \) is similar. \( \square \)

### 7. Crystal basis theory for \( U_2^\lambda \)

#### 7.1. \( \lambda \)-crystal bases.

**Definition 7.1.1.** Let \( A \) be an associative algebra over \( \mathbb{Q}(p, q) \). We call \((f_1^A, f_2^A, k_1^A, k_2^A, e_1^A, e_2^A) \in A^6 \) a \( U_2^\lambda \)-sextuple if there is an injective algebra homomorphism \( U_2^\lambda \to A \) which sends \( f_1, f_2, k_1, k_2, e_1, e_2 \) to \( f_1^A, f_2^A, k_1^A, k_2^A, e_1^A, e_2^A \), respectively.

**Example 7.1.2.** Consider \( A = U_2^\lambda \). For each \( i \in \{2, \ldots, r\} \), the sextuple
\[ (T_i, f_i, T_i(k_1), T_i(k_2), T_i(e_1), T_i(e_2)) \in (U_2^\lambda)^6 \]
is a \( U_2^\lambda \)-sextuple, where \( T_i := (\tau_{i-1} \tau_i) \cdots (\tau_3 \tau_4)(\tau_2 \tau_3) (i > 2) \), and \( T_2 := \text{id} \). Note that \( T_i(x_2) = x_i \) for \( x \in \{ c, k, f \} \), \( T_i(k_1) = k_1 \cdots k_{i-1}, T_i(h_1) = h_{i-1}, \) and \( T_i(h_2) = h_i \).

Let us define linear maps \( \tilde{f}_i \) and \( \tilde{e}_i \), \( i \in \mathbb{N} \setminus \{1\} \), on each \( M \in \mathcal{O}_\lambda^\text{int} \) as follows. Let \( i \in \mathbb{N} \setminus \{1\} \), and \( L \) an irreducible \( U_2^\lambda \)-module with highest weight vector \( v \in L \). Let \( L^{(i)} \) denote the module \( L \) over the \( U_2^\lambda \)-sextuple \((T_i(f_1), T_i(f_2), T_i(k_1), T_i(k_2), T_i(e_1), T_i(e_2)) \), that is, \( L^{(i)} \) is the vector space \( L \) equipped with a \( U_2^\lambda \)-module structure via the homomorphism \( T_i : U_2^\lambda \to U_2^\lambda \). Then, \( \tilde{f}_i \) (resp., \( \tilde{e}_i \)) is defined to be \( \tilde{f}_i \) (resp., \( \tilde{e}_i \)) on the irreducible component of the \( U_2^\lambda \)-module.
Definition 7.1.3. Let $M$ be a $U^1$-module and $\mathcal{L}$ an $A_0$-submodule of $M$. We say that $\mathcal{L}$ is a $\tau$-crystal lattice of $M$ if

(L1) $\mathcal{L}$ is a quasi-$\tau$-crystal lattice of $M$,
(L2) $\tilde{f}_\tau(\mathcal{L}) \subset \mathcal{L}$ and $\tilde{e}_\tau(\mathcal{L}) \subset \mathcal{L}$ for all $i \in \mathbb{P} \setminus \{1\}$.

If $\mathcal{L}$ is a $\tau$-crystal lattice of $M$, then the Kashiwara operators $\tilde{f}_\tau$ induce $\mathbb{Q}$-linear maps, denoted by the same symbols, on $\mathcal{L}/q\mathcal{L}$.

Definition 7.1.4. Let $M$ be a $U^1$-module, $\mathcal{L}$ an $A_0$-submodule of $M$, and $B$ a subset of $\mathcal{L}/q\mathcal{L}$. We say that $(\mathcal{L}, B)$ is a $\tau$-crystal basis if

(B1) $\mathcal{L}$ is a $\tau$-crystal lattice of $M$,
(B2) $(\mathcal{L}, B)$ is a quasi-$\tau$-crystal basis of $M$,
(B3) for each $b, b' \in B$ and $i \in \mathbb{P} \setminus \{1\}$, one has $\tilde{f}_\tau(b) = b'$ if and only if $b = e_i(b')$.

In order to describe $\tilde{f}_\tau$, we use a symmetry of quasi-$\tau$-crystal bases. Let $S_{r-1}$ denote the $(r-1)$-st symmetric group with simple reflections $\{s_i \mid i = 2, \ldots, r\}$.

Lemma 7.1.5. Let $(\mathcal{L}, B)$ be a quasi-$\tau$-crystal basis of a $U^1$-module $M$. Then $S_{r-1}$ acts on $B$ as follows:

$$s_i(b) := \begin{cases} \tilde{e}_1^{-(\beta_i, \text{wt}(b))}(b) & \text{if } (\beta_i, \text{wt}(b)) \geq 0, \\ \tilde{e}_1^{(\beta_i, \text{wt}(b))}(b) & \text{if } (\beta_i, \text{wt}(b)) < 0. \end{cases}$$

Proof. Since the subalgebra of $U^1$ generated by $\{f_i, k_i, e_i \mid i \in \mathbb{P} \setminus \{1\}\}$ is isomorphic to $U_q(\mathfrak{sl}_r)$, the $(\mathcal{L}, B)$ is equipped with a $U_q(\mathfrak{sl}_r)$-crystal structure by ignoring the actions of $\tilde{f}_1$ and $\tilde{e}_1$. Hence, the assertion follows from the ordinary crystal basis theory for quantum groups. \hfill \Box

For convenience, we introduce operators $\tilde{f}_i^{\max}$ and $\tilde{e}_i^{\max}$, $i \in \mathbb{P}$, acting on $U^1$-modules. Let $M \in \mathcal{O}_{\text{int}}^1$ and $m \in M$. By the definition of $\mathcal{O}_{\text{int}}^1$, there exists a unique integer $N_i$ such that $\tilde{f}_i^{N_i}(m) \neq 0$ and $\tilde{f}_i^{N_i+1}(m) = 0$. Then, $\tilde{f}_i^{\max}(m)$ is defined to be $\tilde{f}_i^{N_i}(m)$. The $\tilde{e}_i^{\max}$ is defined in a similar way.

Lemma 7.1.6. Let $M$ be a $U^1$-module and $m \in M$ a $U^1$-highest weight vector. For each $c \in \mathbb{Z}_{\geq 0}$, we have $\tilde{e}_i^{c}(m) = (\tilde{f}_i^{c})^{\max}(m)$.

Proof. We prove the assertion by induction on $i \geq 2$. The case $i = 2$ is trivial. Hence let $i \geq 3$, and assume that the assertion holds for $i - 1$. Since $m$ is a $U^1$-highest weight vector, $(\tau_1^{-1})^{-1}(m) = \tilde{f}_i^{\max}(m)$ is a $U^1$-highest weight vector with highest weight $(a_1, \ldots, a_{i-3}, a_{i-2} + 1, a_i, b_1, \ldots, b_{i-3}, b_{i-2} + 1, b_i)$. Therefore, we have

$$q^{ch} \tilde{f}_i^{c}(m) = f_{i-1}^{c}(m) = (\tau_{i-1})^{T_{i-1}}(\tau_{i-1})^{-1}(m).$$

Let $\lambda \in \Lambda^1$ be the weight of $m$. Then, the weight of $\tilde{f}_i^{c}(m)$ is equal to $\delta_i s_{i-1}(\lambda) - c_{i-1}$. Since $m$ is a $U^1$ highest weight vector, every weight of the submodule $U^1 m$ is less than
or equal to \( \lambda \). Noting this, we can show that \( f_i f_{(i-1)}^c (\tau_{i-1})^{-1}(m) = 0 \). Indeed, if not, then 
\[
s_{i} s_{i-1}(\lambda) - c_{i-1} - \gamma_i \text{ is a weight of } U^J m, \text{ and hence so is } s_{i-1} s_{i} s_{i-1}(\lambda) - c_{i-1} - \gamma_i = \lambda + \gamma_i - (c - 1) \gamma_i \not\subseteq \lambda, \text{ which is a contradiction. Similarly, } f_{i-1} \tau_i f_{(i-1)}^c (\tau_{i-1})^{-1}(m) = 0.
\]
Hence, we compute as follows:
\[
(\tau_{i-1}) (\tau_{i-1})^{-1}(m) = (\tau_{i-1}^\max \tau_{i-1}^\max) \cdots (\tau_{2}^\max \tau_{2}^\max) (\tau_{i-1}^\max \tau_{i-1}^\max) = (\tau_{i-1}^\max \tau_{i-1}^\max) \cdots (\tau_{i-1}^\max \tau_{i-1}^\max) (\tau_{i-1}^\max \tau_{i-1}^\max) \cdots (\tau_{i-1}^\max \tau_{i-1}^\max) (\tau_{i-1}^\max \tau_{i-1}^\max) (m).
\]
This proves the lemma.

Set \( \overline{P} := P \cup \{r' \mid 2 \leq i \leq r\} \).

**Definition 7.1.7.** Let \( M \) be a \( U^J \)-module and \( (L, B) \) a \( J \)-crystal basis of \( M \). The \( J \)-crystal graph associated with \( (L, B) \) is a colored directed graph with vertex set \( B \) and edges \( b \xrightarrow{i} b' \), where \( b, b' \in B, i \in \overline{P} \) are such that \( f_i b = b' \).

**7.2. Existence and uniqueness theorem.** We are ready to state one of our main results in this paper; this gives the existence and uniqueness of a \( J \)-crystal basis of \( L(a;b) \).

**Theorem 7.2.1.** Consider the irreducible highest weight module \( L(a;b) \) and take a highest weight vector \( v \in L(a;b) \). Set
\[
L(a;b) := \sum_{i \in \mathbb{Z}_{\geq 0}} \mathbf{A}_0 \tilde{f}_{i_1} \cdots \tilde{f}_{i_1}(v) = \sum_{l \in \mathbb{Z}_{\geq 0}} \mathbf{A}_0 \tilde{f}_{i_1} \cdots \tilde{f}_{i_1} f_{l}(v),
\]
\[
B(a;b) := \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_1}(v) + q L(a;b) \mid i \in \mathbb{Z}_{\geq 0}, i_1, \ldots, i_l \in \overline{P} \} \setminus \{0\} = \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_1} f_{l}(v) + q L(a;b) \mid i, c \in \mathbb{Z}_{\geq 0}, i_1, \ldots, i_l \in \overline{P} \setminus \{r'\} \} \setminus \{0\}.
\]

Then the following hold:
1. \( (L(a;b), B(a;b)) \) is a \( J \)-crystal basis of \( L(a;b) \).
2. Let \( M \) be a \( U^J \)-module having a \( J \)-crystal basis \( (L, B) \). Suppose that \( M \cong \bigoplus_{t \in T} L(a_t;b_t) \), where \( T \) is an index set. Then, there exists an isomorphism \( \phi : M \to \bigoplus_{t \in T} L(a_t;b_t) \) of \( U^J \)-modules which induces an isomorphism \( (L, B) \to \bigoplus_{t \in T} L(a_t;b_t) \bigcup_{t \in T} B(a_t;b_t) \) of \( J \)-crystal bases.

We prove this theorem in Section 8 after introducing some combinatorial tools in the next section.

**8. Explicit description of \( J \)-crystal bases.**

**8.1. Double partitions and double Young tableaux.**

**Definition 8.1.1.** Let \( N \) be a nonnegative integer. A partition \( \alpha = (\alpha_1, \ldots, \alpha_l) \) of \( N \) is a nonincreasing sequence of nonnegative integers \( \alpha_1 \geq \cdots \geq \alpha_l \geq 0 \) such that \( \sum_{i=1}^{l} \alpha_i = N \). We call \( |\alpha| := N \) the size of \( \alpha \), and \( \ell(\alpha) := l \) the length of \( \alpha \).

**Definition 8.1.2.** Let \( \alpha \) be a partition of \( N \). The Young diagram \( D(\alpha) \) associated with \( \alpha \) is the set \( \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \mid 1 \leq i \leq l, 1 \leq j \leq \alpha_i\} \). Note that the Young diagram \( D(0,0,\ldots,0) \) is the empty set.

We often identify a partition \( \alpha \) with its Young diagram \( D(\alpha) \).

**Definition 8.1.3.** Let \( \alpha \) be a partition of \( N \). A Young tableau \( T \) of shape \( \alpha \) is a map form \( D(\alpha) \) to a totally ordered set. A Young tableau \( T \) is said to be semistandard if it satisfies \( T(i,j) \leq T(i,l) \) and \( T(i,j) < T(k,j) \) for all \((i,j), (i,l), (k,j) \in D(\alpha) \) such that \( j < l, i < k \). A semistandard Young tableau \( T \) is said to be standard if it satisfies \( T(i,j) < T(i,l) \) for all \((i,j), (i,l) \in D(\alpha) \) such that \( j < l \), and if \( T(D(\alpha)) = \{1, 2, \ldots, |\alpha|\} \).
Definition 8.1.4. Let \( N \) be a nonnegative integer. A double partition \((\alpha; \beta)\) of \( N \) is an ordered pair of partitions such that \(|\alpha| + |\beta| = N\). We call \( N \) the size of \((\alpha; \beta)\), and \((\ell(\alpha); \ell(\beta))\) the length of \((\alpha; \beta)\); we denote the size of \((\alpha; \beta)\) by \(|\alpha; \beta|\).

Note that we distinguish between \((\alpha; \beta)\) and \((\beta; \alpha)\); in particular, for a partition \( \alpha \), the pairs \((\alpha; \emptyset)\) and \((\emptyset; \alpha)\) are distinct double partitions.

Let \( L(\alpha; \beta) \) be an irreducible highest weight \( U^j \)-module; namely, \( \alpha = (a_1, \ldots, a_r) \), \( \beta = (b_1, \ldots, b_r) \), with \( a_1 \in \mathbb{Z}, a_2, \ldots, a_r, b_1, \ldots, b_r \in \mathbb{Z}_{\geq 0} \), and \( 0 \leq b_i \leq a_i \) for \( i = 2, \ldots, r \). Set

\[
\alpha := \left( \sum_{i=1}^{r} b_i, \sum_{i=2}^{r} b_i, \ldots, b_r, 0 \right) + a_1^+ \rho_{r+1},
\]

\[
\beta := \left( \sum_{i=2}^{r} (a_i - b_i), \sum_{i=3}^{r} (a_i - b_i), \ldots, a_r - b_r, 0 \right) - a_1^- \rho_r,
\]

where \( a_1^+ := \max\{a_1 - (2 \sum_{i=1}^{r} b_i - \sum_{i=2}^{r} a_i), 0\} \), \( a_1^- := \min\{a_1 - (2 \sum_{i=1}^{r} b_i - \sum_{i=2}^{r} a_i), 0\} \), \( \rho_n := (1, 1, \ldots, 1) \) \((n\ \text{components})\), and the addition is defined componentwise. The assignment \((\alpha; \beta) \mapsto (\alpha; \beta)\) gives a bijection from \(\{(\alpha; \beta) \mid a_1 \in \mathbb{Z}, b_1 \in \mathbb{Z}_{\geq 0}, 0 \leq b_i \leq a_i, \ i \geq 2\}\) to the set of double partitions of length \((r+1; r)\) containing at least one 0; the inverse map \( \pi \) is given by

\[
(13) \quad a_1 = 2a_1 - \alpha_2 - \beta_1, \ a_1 = a_i - a_{i+1} + \beta_i - \beta_{i-1}, \ b_1 = \alpha_i - a_{i+1}.
\]

We write \( L(\alpha; \beta) = L(\pi(\alpha; \beta)) \). If we define \( \pi(\alpha; \beta) \) by equation (13) for a double partition \((\alpha; \beta)\) of length \((r+1; r)\), then \( \pi(\alpha; \beta) = \pi(\alpha'; \beta') \) if and only if \((\alpha'; \beta') = (\alpha + n\rho_{r+1}; \beta + n\rho_r)\) for some \( n \in \mathbb{Z} \). We denote this condition by \((\alpha; \beta) \sim_{\pi} (\alpha'; \beta')\), and define \( L(\alpha; \beta) \) to be \( L(\alpha; \beta) \), where \((\alpha; \beta)\) is the unique double partition containing at least one 0 such that \((\alpha; \beta) \sim_{\pi} (\alpha'; \beta')\). From these observations, we obtain the following.

Proposition 8.1.5. The isomorphism classes of irreducible \( U^j \)-modules in \( \mathcal{O}^1_{\text{int}} \) are parametrized by the double partitions of length \((r+1; r)\) modulo \( \sim_{\pi} \).

Definition 8.1.6. Let \((\alpha; \beta)\) be a double partition of \( N \). The double Young diagram \( D(\alpha; \beta) \) associated with \((\alpha; \beta)\) is the ordered pair \((D(\alpha); D(\beta))\); we often identify a double partition with its double Young diagram.

Example 8.1.7. Let \( r = 3 \), \( \alpha = (2, 2, 3) \), \( \beta = (2, 0, 1) \). Then the corresponding double partition is \((4, 2, 2, 1; 4, 2, 0)\), and the associated double Young diagram is

\[
\begin{array}{|c|c|c|}
\hline
\hline
& 1 & 2 \\
\hline
3 & & 4 \\
\hline
\end{array}
\]

Definition 8.1.8. For \( s \in \{ -r, -r+1, \ldots, r \} \), a double partition \((\alpha; \beta)\) is said to be \( s \)-addable if \( a_{|s|+1} < a_{|s|} \) when \( s \leq 0 \), and \( \beta_s < \beta_{s-1} \) when \( s > 0 \). Here we understand that \( \alpha_0 = \beta_0 = \infty \) by convention.

Example 8.1.9. A double partition \((4, 2, 2, 1; 4, 2, 0)\) is \( s \)-addable for \( s = -3, -1, 0, 1, 2, 3 \).

Definition 8.1.10. Let \((\alpha; \beta)\) be a double partition of \( N \). A double Young tableau \((T_1; T_2)\) of shape \((\alpha; \beta)\) is an ordered pair of a Young tableau \( T_1 \) of shape \( \alpha \) and a Young tableau \( T_2 \) of shape \( \beta \). A double Young tableau is said to be semistandard if \( T_1 \) and \( T_2 \) are both semistandard.

Definition 8.1.11. We denote by \( \text{SST}_r(\alpha; \beta) \) the set of double Young tableaux \((T_1; T_2)\) of shape \((\alpha; \beta)\) such that \( T_1(i, j) \in \{0, -1, \ldots, -r\} \) and \( T_2(i, j) \in \{1, \ldots, r\} \). Here, we equip \( \{0, -1, \ldots, -r\} \) with a total order \( 0 < -1 < \cdots < -r \).

Note that there exits a natural bijection \( \text{SST}_r(\alpha; \beta) \to \text{SST}_r(\alpha + n\rho_{r+1}; \beta + n\rho_r) \) for all \( n \in \mathbb{N} \).
8.2. $\rho$-crystal structure on $B_r^{\otimes N}$. Recall that $B_r^{\otimes N} = \{-r, \ldots, r\}^N$ is equipped with a quasi-$\rho$-crystal structure described in Section 5.2. There, we showed that for $s \in B_r^{\otimes N}$ and $i \in \mathbb{P}$, the $\tilde{x}_i(s)$, $x \in \{e, f\}$, is of the form $\tilde{X}_{x(i-\frac{1}{2})}(s)$ or $\tilde{X}_{x(i+\frac{1}{2})}(s)$, $X \in \{E, F\}$. Now, for $i \in \mathbb{P} \setminus \{1\}$, we define maps $\tilde{f}_i, \tilde{e}_i : B_r^{\otimes N} \to B_r^{\otimes N} \sqcup \{0\}$ as follows. Let $s = (s_1, \ldots, s_N) \in B_r^{\otimes N}$, and recall the definition of $s_{i-\frac{1}{2}}$ from Section 5.2. Then, $\tilde{f}_i(s)$ (resp., $\tilde{e}_i(s)$) is defined to be the element obtained from $s$ by replacing the leftmost $i - 1$ (resp., the rightmost $i$) in $s_{i-\frac{1}{2}}$ with $(i, \text{ resp., } i - 1)$ if $i - 1 \in s_{i-\frac{1}{2}}$ (resp., $i \in s_{i-\frac{1}{2}}$) and $\tilde{e}_i(s) = 0$ for all $j = 1, \ldots, i, i', \ldots, (i - 1)'$; otherwise, $\tilde{f}_i(s)$ (resp., $\tilde{e}_i(s')$) is defined to be 0. Note that $\tilde{f}_i(s)$ equals either $\tilde{E}_{i-\frac{1}{2}}(s)$ or 0, and that $\tilde{e}_i(s)$ equals either $\tilde{E}_{i-\frac{1}{2}}(s)$ or 0.

Remark 8.2.1. In Section 5 we prove that $V_r^{\otimes N}$ has a $\rho$-crystal basis $(L_r^{\otimes N}, B_r^{\otimes N})$, on which the Kashiwara operators $\tilde{f}_i$ and $\tilde{e}_i$, $i \in \mathbb{P}$, act as we described above.

Let $(\alpha; \beta)$ be a double partition of size $N$ of length $(r + 1, r)$. Consider the map $\text{SST}_r(\alpha; \beta) \to B_r^N$ given by the assignment $(T_1; T_2) \mapsto (\text{EM}(T_1), \text{ME}(T_2))$, where $\text{ME}(T_2)$ means the Middle-Eastern reading of $T_2$, and $\text{EM}(T_1)$ is obtained by reversing $\text{ME}(T_1)$. For example,

$$
\begin{bmatrix}
0 & 0 & -2 & -3 \\
-1 & -1 & 1 & 2 \\
-2 & -3 & 2 & 1 \\
-4 & & & \\
\end{bmatrix}
\mapsto (-4, -2, -3, -1, -1, 0, 0, -2, -3, 2, 1, 1, 4, 2).
$$

For $(T_1; T_2) \in \text{SST}_r(\alpha; \beta)$, we define $\tilde{f}_i(T_1; T_2)$ to be the unique (not necessarily semistandard) double Young tableau $(T'_i; T'_2)$ of shape $(\alpha; \beta)$ such that $(\text{EM}(T'_i), \text{ME}(T'_2)) = \tilde{f}_i(\text{EM}(T_1), \text{ME}(T_2))$ for $i \in \mathbb{P}$. The double Young tableau $\tilde{e}_i(T_1; T_2)$ is defined similarly. By the first paragraph of this subsection, for each $i \in \mathbb{P}$, the $\tilde{x}_i(T_1; T_2)$, $x \in \{e, f\}$, is of the form $\left(\tilde{X}_{x(i-\frac{1}{2})}(T_1; T_2)\right)$ or $\left(T_1; \tilde{X}_{x(i-\frac{1}{2})}(T_2)\right)$, $X \in \{E, F\}$. Therefore, $\text{SST}_r(\alpha; \beta) \sqcup \{0\}$ is stable under the operators $\tilde{f}_i$ and $\tilde{e}_i$, $i \in \mathbb{P}$. Clearly, there exists an isomorphism $\text{SST}_r(\alpha; \beta) \to \text{SST}_r(\alpha + n\rho_{r-1}; \beta + n\rho_r)$ of quasi-$\rho$-crystal graphs which is compatible with $\tilde{f}_i$ and $\tilde{e}_i$, $i \in \mathbb{P} \setminus \{1\}$.

Now, we define $(T_\alpha; T_\beta) \in \text{SST}_r(\alpha; \beta)$ by $T_\alpha(i, j) = -(i - 1)$, and $T_\beta(i, j) = i$. For example, when $\alpha = (4, 2, 2, 1)$ and $\beta = (4, 2, 0)$,

$$(T_\alpha; T_\beta) = \begin{bmatrix}
0 & 0 & 0 \\
-1 & -1 & 1 \\
-2 & -2 & 2 \\
-3 & & & \\
\end{bmatrix}.$$

Proposition 8.2.2. For each $(T_1; T_2) \in \text{SST}_r(\alpha; \beta)$, there exists a sequence $i_1, \ldots, i_l \in \mathbb{P}$ such that $\tilde{f}_{i_1} \cdots \tilde{f}_{i_l}(T_1; T_2) = (T_\alpha; T_\beta)$.

Proof. Let $(T_1; T_2) \in \text{SST}_r(\alpha; \beta)$. Suppose that $\tilde{e}_i(T_1; T_2) = 0$ for all $i \in \mathbb{P}$. By the tensor product rule, this implies that $\tilde{F}_{-(i-\frac{1}{2})}(T_1) = 0$ for all $i \in \mathbb{P}$, or equivalently, $T_1 = T_\alpha$. Set $d(T_2) := \sum_{i,s}(T_2(i, j) - T_\beta(i, j))$. This measures the distance between $T_2$ and $T_\beta$, that is, one has $d(T_2) \geq 0$, and the equality holds if and only if $T_2 = T_\beta$. Suppose that $d(T_2) > 0$. Then there exists a minimal $i_l \in \mathbb{P} \setminus \{1\}$ such that $\tilde{E}_{i_l-\frac{1}{2}}(T_2) \neq 0$. For such $i_l$, we have $\tilde{e}_{i_l}(T_\alpha; T_2) = (T_\alpha, \tilde{E}_{i_l-\frac{1}{2}}(T_2))$ and $d(\tilde{E}_{i_l-\frac{1}{2}}(T_2)) = d(T_2) - 1$. Repeating this, we obtain a sequence $(i_1, \ldots, i_l)$ such that $\tilde{e}_{i_1} \cdots \tilde{e}_{i_l}(T_\alpha; T_2) = (T_\alpha; T_\beta)$. This proves the proposition. □
Example 8.2.3. Let $r = 3$, $\alpha = (1,1,1,0)$, $\beta = (1,1,0)$. Then, the $\beta$-crystal graph of $\text{SST}_3(\alpha;\beta)$ is as follows:

\[
\begin{array}{c|cc}
\begin{array}{cccc}
0 & 1 & 2 \\
-1 & -1 & -2 \\
-2 & -3 & -2 \\
\end{array} & \begin{array}{cccc}
0 & 1 & 2 \\
-1 & -1 & -2 \\
-2 & -3 & -3 \\
\end{array} & \begin{array}{cccc}
0 & 1 & 2 \\
-1 & -1 & -2 \\
-2 & -3 & -3 \\
\end{array}
\end{array}
\]

Note that the quasi-$\beta$-crystal graph of $\text{SST}_3(\alpha;\beta)$, which is obtained by removing the directed edges colored by $2', 3'$, is not connected.

9. Proof of Theorem 7.2.1

Let $N \in \mathbb{N}$, and consider $V_r^{\otimes N}$. Following the proof of the existence of crystal bases in [Ka91], we prove the existence of $\beta$-crystal bases by decomposing $V_r^{\otimes N}$ into irreducible modules. We can write $V_r^{\otimes N} \simeq \bigoplus_{t \in T} L(\alpha_t;\beta_t)$, where $T$ is an index set and $(\alpha_t;\beta_t)$ is a double partition of length $(r+1; r)$. In this section, we prove $A_r(N)$: $V_r^{\otimes N}$ has a $\beta$-crystal basis $(\mathcal{L}_r^{\otimes N}, \mathcal{B}_r^{\otimes N})$, on which Kashiwara operators act as in the first paragraph of Section 8.2 and that there exists an irreducible decomposition $V_r^{\otimes N} = \bigoplus_{t \in T} L_t$ of $V_r^{\otimes N}$ such that $L_t \simeq L(\alpha_t;\beta_t)$ for all $t \in T$, and satisfies the following conditions $B_r(N) = D_r(N)$.

$B_r(N)$: For $t \in T$, set $\mathcal{L}_t := \mathcal{L}_r^{\otimes N} \cap L_t$, $\mathcal{B}_t := \mathcal{B}_r^{\otimes N} \cap (\mathcal{L}_t/q\mathcal{L}_t)$. Then, we have $\mathcal{L}_r^{\otimes N} = \bigoplus_{t \in T} \mathcal{L}_t$, $\mathcal{B}_r^{\otimes N} = \bigsqcup_{t \in T} \mathcal{B}_t$. Moreover, for each $t \in T$, $(\mathcal{L}_t, \mathcal{B}_t)$ is a $\beta$-crystal basis of $L_t$, on which Kashiwara operators act as the restriction of those acting on $(\mathcal{L}_r^{\otimes N}, \mathcal{B}_r^{\otimes N})$.

$C_r(N)$: For each double partition $(\alpha;\beta)$ of $N$, there exists a unique $t \in T$ such that $(\alpha;\beta) = (\alpha_t;\beta_t)$, and the $\beta$-crystal graph of $(\mathcal{L}_t, \mathcal{B}_t)$ is connected with a single source $(\text{EM}(T_\alpha), \text{ME}(T_\beta))$.

$D_r(N)$: $|\alpha_t;\beta_t| = N$ for all $t \in T$.

If we are done, then Theorem 7.2.1 is proved as follows. Let $(\alpha;\beta)$ be a double partition of $N$ of length $(r+1; r)$, and $v \in L(\alpha;\beta)$ a highest weight vector. By assumption $C_r(N)$, we may assume that $L(\alpha;\beta) = L_t$ for some $t \in T$ and $v + q\mathcal{L}_t = (\text{EM}(T_\alpha), \text{ME}(T_\beta))$. Also, by assertions $B_r(N)$ and $C_r(N)$, we have

$$
\mathcal{B}_t = \{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_l}(v) + q\mathcal{L}_t \mid l \in \mathbb{Z}_{\geq 0}, i_1, \ldots, i_l \in \mathcal{V} \} \setminus \{0\},
$$
and hence,
\[ \mathcal{L}_t = \sum_{i \in \mathbb{Z}_{\geq 0}, i_1, \ldots, i_r} A_0 f_{i_1} \cdots f_{i_r}(v). \]

This implies that \((\mathcal{L}(\alpha; \beta), \mathcal{B}(\alpha; \beta))\) equals \((\mathcal{L}_1, \mathcal{B}_1)\), which is a \(\gamma\)-crystal basis of \(L(\alpha; \beta)\). Moreover, we obtain an isomorphism \(\mathcal{B}(\alpha; \beta) \simeq \text{SST}_r(\alpha; \beta)\) of \(\gamma\)-crystal graphs. The uniqueness of \(\gamma\)-crystal basis of a \(U^\rho\)-module follows in the same way as that of the ordinary crystal basis of a \(U\)-module. This proves Theorem 7.2.1.

In addition, by assertion \(A_r(N)\), we obtain a rule for writing the \(\gamma\)-crystal graph of \((\mathcal{L}^{\otimes N}_r, \mathcal{B}^{\otimes N}_r)\). For example, the \(\gamma\)-crystal graph of \(V_2 \otimes V_2\) is as follows:

\[ u_{-2} \quad u_{-1} \quad u_0 \quad u_1 \quad u_2. \]

9.1. The case \(r = 2\). Let us prove assertions \(A_2(N) - D_2(N)\) by induction on \(N\). The \(U^2\)-module structure of \(V_2\) (more generally, the \(U^2\)-modules structure of \(V_r\)) can be found in [BWW16]. From it, one can easily verify that \(V_2 \simeq L(\emptyset; \emptyset) \oplus L(\emptyset; \emptyset)\), and that \((\mathcal{L}_2, \mathcal{B}_2)\) is a \(\gamma\)-crystal basis of \(V_2\) whose \(\gamma\)-crystal graph is

\[ u_{-2} \quad 2 \quad u_{-1} \quad 1 \quad u_0 \quad u_1 \quad 2' \quad u_2. \]

Thus, assertions \(A_2(1) - D_2(1)\) are obvious.

Let \(N \geq 1\), and assume that assertions \(A_2(N) - D_2(N)\) hold. Fix \(t \in T\), and write \((\alpha; \beta) = (\alpha_t; \beta_t)\). By \(D_2(N)\), we have \(|\alpha; \beta| = N\). Let \(v \in L(\alpha; \beta)\) be a highest weight vector. In Section 5 we considered \(L := L(\alpha; \beta) \otimes V_2 \subset V_2^{\otimes (N+1)}\) and defined five vectors (some of which are equal to 0) \(v_0, v_{\pm 1}, v_{\pm 2} \in L\); see that for \(s \in \{0, \pm 1, \pm 2\}, v_s\) is nonzero if and only if \((\alpha; \beta)\) is \(s\)-addable. Set \(S = S(\alpha; \beta) := \{s \in \{0, \pm 1, \pm 2\} \mid (\alpha; \beta)\) is \(s\)-addable\}. For each \(s \in S\), we denote by \((\alpha^s, \beta^s) = (\alpha^s_t, \beta^s_t)\) the double partition obtained from \((\alpha; \beta)\) by adding a box to the \((|s| + 1)\)-st row of \(\alpha\) if \(s \leq 0\), and to the \(s\)-th row of \(\beta\) if \(s > 0\).

Proposition 9.1.1. For each \(s \in S\), we have \(U_2^2v_s \simeq L(\alpha^s; \beta^s)\). In particular, assertion \(D_2(N + 1)\) holds.

Proof. By Propositions 6.2.7 and 6.2.11 \(\square\)

Corollary 9.1.2. For each double partition \((\alpha', \beta')\) of \(N + 1\), there exists \(v_{(\alpha', \beta')} \in L_2^{\otimes (N+1)}\) such that \(U_2^2v_{(\alpha', \beta')} \simeq L(\alpha'; \beta')\) and \(v_{(\alpha', \beta')} + qL_2^{\otimes (N+1)} = (\text{EM}(T_{\alpha'}), \text{ME}(T_{\beta'}))\).

Proof. This is easy by the construction of the \(v_s\)'s and Proposition 9.1.1 \(\square\)
Proposition 9.1.3. Let $s \in S$. Then, $\tilde{f}_2^c(v_s) + qL \in B \setminus \{0\}$ if $0 \leq c \leq (\beta^*)_1 - (\beta^*)_2$. Moreover, we have the explicit formula:

$$\tilde{f}_2^c(v_s) + qL = \begin{cases} 
\tilde{f}_2^c(v) \otimes u_0 + qL & \text{if } s = 0, \\
\tilde{f}_1^c\tilde{f}_2^c(v) \otimes u_0 + qL & \text{if } s = -1, \\
\tilde{f}_2^c(v) \otimes u_1 + qL & \text{if } s = 1 \text{ and } c \neq (\beta^*)_1 - (\beta^*)_2, \\
\tilde{f}_2^{c-1}(v) \otimes u_2 + qL & \text{if } s = 1 \text{ and } c = (\beta^*)_1 - (\beta^*)_2, \\
\tilde{f}_1^c\tilde{f}_2^c(v) \otimes u_0 + qL & \text{if } s = -2, \\
\tilde{f}_2^c(v) \otimes u_2 + qL & \text{if } s = 2 \text{ and } 0 \leq c \leq (\beta^*_2)_1 - (\beta^*_2)_2, \\
0 & \text{otherwise.}
\end{cases}$$

Proof. The assertions follow from Propositions 6.2.8, 6.2.12 and Corollary 6.2.9. □

Remark 9.1.4. If we regard $v_s + qL$, $s \in S$, as an element $s = (s_1, \ldots, s_{N+1}) \in B_2^{(N+1)}$, then for $0 \leq s \leq (\beta^*)_1 - (\beta^*)_2$, the $\tilde{f}_2^c(v_s) + qL$ described in Proposition 9.1.3 is identical to $\tilde{f}_2^c(s)$ defined in Section 8.2.

For $s \in S$, let $L_s = L_{t,s} := U_3^0 v_s$, $L_s = L_{t,s} := L \cap L_s$, and $B_s = B_{t,s} := B \cap (L_s/qL_s)$. We will show that $L = \bigoplus_{s \in S} L_s$, $\mathcal{L} = \bigoplus_{s \in S} L_s$, $\mathcal{B} = \bigcup_{s \in S} B_s$, and that $(\mathcal{L}, \mathcal{B})$ is a $j$-crystal basis of $L_s$.

Lemma 9.1.5. We have $\dim L = \sum_{s \in S} \sharp \text{SST}_2(\alpha^*; \beta^*)$.

Proof. For a partition $\lambda$, we denote by $\text{SST}_1(\lambda)$ the set of semistandard Young tableaux of shape $\lambda$ in letters $1, \ldots, l$. Clearly, we have

$$\dim L = \sharp \text{SST}_3(\alpha^*) \cdot \sharp \text{SST}_2(\beta^*) \cdot 5.$$ 

By the Pieri rule for $U$, it follows that

$$\sharp \text{SST}_3(\alpha^*) \cdot 3 = \sum_{s \in S \cap \mathbb{Z} \leq 0} \sharp \text{SST}_3(\alpha^*),$$
$$\sharp \text{SST}_2(\beta^*) \cdot 2 = \sum_{s \in S \cap \mathbb{Z} > 0} \sharp \text{SST}_2(\beta^*).$$

Therefore, we see that

$$\dim L = \sharp \text{SST}_3(\alpha^*) \cdot 3 \cdot \sharp \text{SST}_2(\beta) + \sharp \text{SST}_3(\alpha^*) \cdot \sharp \text{SST}_2(\beta) \cdot 2 = \sum_{s \in S \cap \mathbb{Z} \leq 0} \sharp \text{SST}_3(\alpha^*) \cdot \sharp \text{SST}_2(\beta) + \sharp \text{SST}_3(\alpha^*) \cdot \sum_{s \in S \cap \mathbb{Z} > 0} \sharp \text{SST}_2(\beta^*).$$

Here, notice that $(\alpha^*; \beta^*) = (\alpha^*; \beta)$ if $s \leq 0$, and $(\alpha^*; \beta^*) = (\alpha; \beta^*)$ if $s > 0$. This implies that

$$\dim L = \sum_{s \in S \cap \mathbb{Z} \leq 0} \sharp \text{SST}_3(\alpha^*) \cdot \sharp \text{SST}_2(\beta) + \sum_{s \in S \cap \mathbb{Z} > 0} \sharp \text{SST}_3(\alpha^*) \cdot \sharp \text{SST}_2(\beta^*) = \sum_{s \in S} \sharp \text{SST}_3(\alpha^*) \cdot \sharp \text{SST}_2(\beta^*) = \sum_{s \in S} \sharp \text{SST}_2(\alpha^*; \beta^*),$$

as desired. □

Proposition 9.1.6. The following hold.

1. For each $s \in S$, the $(\mathcal{L}_s, \mathcal{B}_s)$ is a $j$-crystal basis of $L_s$ whose $j$-crystal graph is isomorphic to $\text{SST}_2(\alpha^*; \beta^*)$.
2. We have $L = \bigoplus_{s \in S} L_s$, $\mathcal{L} = \bigoplus_{s \in S} \mathcal{L}_s$, and $\mathcal{B} = \bigcup_{s \in S} \mathcal{B}_s$. 

Proof. (1) Let $s \in S$. By Corollary 9.1.2 we may assume that $v_s \in \mathcal{F}(\mathbb{C}(T_{\alpha^*}^0), \mathcal{F}(T_{\beta^*}^0)).$

From Proposition 9.1.3, we see that $\tilde{f}_2^s(v_s) \in \mathcal{L}$, and $q \mathcal{L} \in \mathcal{B} \cup \{0\}$, and that $\tilde{f}_2^s(v_s) + q \mathcal{L} \neq 0$ if and only if $0 \leq c \leq (\beta^*)_1 - (\beta^*)_2$. Therefore, we have

$$\mathcal{L}_s \supset \bigoplus_{c=0}^{(\beta^*)_1 - (\beta^*)_2} \sum_{i_1, \ldots, i_l \in \{1, 2\}} \tilde{f}_{i_1} \cdots \tilde{f}_{i_l} \tilde{f}_2^s(v_s),$$

(14)

$$\mathcal{B}_s \supset \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_l} \tilde{f}_2^s(v_s) + q \mathcal{L}_s | c, l \in \mathbb{Z}_{\geq 0}, i_1, \ldots, i_l \in \{1, 2\} \} \setminus \{0\}.$$  

Also, it is easy to see that $\tilde{f}_2^s(T_{\alpha^*}; T_{\beta^*}) \neq 0$ if and only if $0 \leq c \leq (\beta^*)_1 - (\beta^*)_2$, and that $\mathcal{SST}_2(\alpha^*; \beta^*) = \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_l} \tilde{f}_2^s(T_{\alpha^*}; T_{\beta^*}) | c, l \in \mathbb{Z}_{\geq 0}, i_1, \ldots, i_l \in \{1, 2\} \} \setminus \{0\}$.

Therefore, the assignment $0 \neq \tilde{f}_{i_1} \cdots \tilde{f}_{i_l} \tilde{f}_2^s(T_{\alpha^*}; T_{\beta^*}) \mapsto \tilde{f}_{i_1} \cdots \tilde{f}_{i_l} \tilde{f}_2^s(v_s) + q \mathcal{L}_s$ gives an injection $\mathcal{SST}_2(\alpha^*; \beta^*) \hookrightarrow \mathcal{B}_s$, and hence, $\dim L_s \geq \mathcal{SST}_2(\alpha^*; \beta^*)$. However, by Lemma 9.1.3, this inequality is indeed an equality, and hence, so are the inclusions $\supset$ in equation (14). In addition, by Proposition 9.1.3, we see that $\mathcal{L}_s$ is closed under $\tilde{f}_2^s$ and $\tilde{e}_2^s$. Now, it is easy to check that $(\mathcal{L}_s, \mathcal{B}_s)$ is a $\mathcal{J}$-crystal basis of $L_s$ such that $\mathcal{B}_s \simeq \mathcal{SST}_2(\alpha^*; \beta^*)$. This proves assertion (1).

(2) Since $L \supset \bigoplus_{s \in S} L_s$ and $\dim L = \sum_{s \in S} \dim L_s$, we have $L = \bigoplus_{s \in S} L_s$. Next, We show that the $\mathcal{B}_s$’s are pairwise disjoint. Let $b \in \mathcal{B}_s \cap \mathcal{B}_{s'}$ for some $s, s' \in S$. Then, there exist $i_1, \ldots, i_k, j_1, \ldots, j_l \in \{1, 2\}$ and $c, c' \in \mathbb{Z}_{\geq 0}$ such that

$$\tilde{f}_{i_1} \cdots \tilde{f}_{i_k} \tilde{f}_2^s(v_s) + q \mathcal{L} = b = \tilde{f}_{j_1} \cdots \tilde{f}_{j_l} \tilde{f}_2^{s'}(v_{s'}) + q \mathcal{L}.$$  

Equivalently, we have

$$v_s' + q \mathcal{L} = \tilde{e}_2^{s'} \tilde{f}_{j_1} \cdots \tilde{f}_{j_l} \tilde{f}_2^{s'}(v_{s'}) + q \mathcal{L}.$$  

Since $\mathcal{B}_s \cup \{0\}$ is closed under the Kashiwara operators, we have $v_{s'} \in \mathcal{B}_s$, and hence, $s' = s$. Thus, $\mathcal{B} = \bigsqcup_{s \in S} \mathcal{B}_s$. Now, $\mathcal{L} = \bigoplus_{s \in S} \mathcal{L}_s$ is obvious. This completes the proof of the proposition.

\[ \square \]

Recall the irreducible decomposition $V_2^{(N+1)} = \bigoplus_{t \in T} L_t$. Since we took $(\alpha; \beta) = (\alpha_1; \beta_1)$ with $t \in T$ arbitrarily in the second paragraph of this subsection, this proposition implies the equalities

$$V_2^{(N+1)} = \bigoplus_{t \in T} (L_t \oplus V_2) = \bigoplus_{t \in T} \mathcal{L}_t,$$

$$L_2^{(N+1)} = \bigoplus_{t \in T} (L_t \otimes L_2) = \bigoplus_{t \in T} \mathcal{L}_t,$$

$$B_2^{(N+1)} = \bigcup_{t \in T} (B_t \otimes B_2) = \bigcup_{t \in T} B_t,$$

Moreover, $(\mathcal{L}_t, B_t)$ is a $\mathcal{J}$-crystal basis of $L_t$. This proves assertions $A_2(N + 1)$ and $B_2(N + 1)$. Now, assertion $C_2(N + 1)$ follows from Corollary 9.1.2.

9.2. The case $r \geq 3$. Let $r \geq 3$. We assume that assertions $A_r(N) - D_r(N)$, $N \geq 1$, hold for all $r' < r$, and prove assertions $A_r(N) - D_r(N)$, $N \geq 1$. We proceed by induction on $N$. The case $N = 1$ is easy (see [BW16]); indeed, $V_r$ is decomposed as $V_r = U'T_0 \oplus U'T_{u_1 - pu_0} \simeq L(\emptyset) \oplus L(\emptyset)$, and the $\mathcal{J}$-crystal graph is

$$u_r \leftarrow u_{r-1} \leftarrow \cdots \leftarrow u_0 \leftarrow 2u_1 \leftarrow 3u_2 \leftarrow \cdots \leftarrow (r-1)u_{r-1} \leftarrow ru_r.$$  

Let $N \geq 1$, and assume that assertions $A_r(N) - D_r(N)$ hold. Fix $t \in T$, and write $(\alpha; \beta) = (\alpha_t; \beta_t)$. By $D_r(N)$, we have $|\alpha; \beta| = N$. Let $v \in L(\alpha; \beta)$ be a highest weight vector. Set
$S = S(\alpha, \beta) := \{s \in \{-r, \ldots, r\} \mid (\alpha, \beta) \text{ is } s\text{-addable}\}$. For each $s \in S$, we denote by $(\alpha^s, \beta^s) = (\alpha^s_0, \beta^s_0)$ the double partition obtained from $(\alpha; \beta)$ by adding a box to the $(|s| + 1)$-st row of $\alpha$ if $s \leq 0$, and to the $s$-th row of $\beta$ if $s > 0$.

**Proposition 9.2.1.** Let $r \geq 2$. For each $s \in S$, there exists a highest weight vector $v_s \in \mathcal{L}$ such that $U^s v_s \simeq L(\alpha^s; \beta^s)$ and

$$v_s + q\mathcal{L} = \begin{cases} v \otimes u_s & \text{if } s \geq 0, \\ \tilde{f}_1 \tilde{f}_2 \cdots \tilde{f}_s(v) \otimes u_0 & \text{if } s < 0. \end{cases}$$

In particular, assertion $D_r(N + 1)$ holds.

**Proof.** See Appendix [3.6] □

**Corollary 9.2.2.** For each double partition $(\alpha; \beta)$ of $N$, there exists $v_{(\alpha; \beta)} \in \mathcal{L}^\otimes N$ such that $U^\otimes N v_{(\alpha; \beta)} \simeq L(\alpha; \beta)$, and $v_{(\alpha; \beta)} + q\mathcal{L}^\otimes N \in \mathcal{B}^\otimes N = (\text{EM}(T_\alpha), \text{ME}(T_\beta))$.

**Proof.** This is easily verified by the construction of $v_s$ and Proposition 9.2.1 □

**Lemma 9.2.3.** Let $k \in \{-r, \ldots, r\}$ and $i \in \mathcal{P}$. Then, for $b \otimes u_k \in \mathcal{B}$, we have

$$s_i(b \otimes u_k) = \begin{cases} s_i(b) \otimes u_k & \text{if } k \neq \pm i, \pm(i - 1), \\ s_i(\tilde{f}_i(b)) \otimes u_{-i} & \text{if } k = -i \text{ and } \tilde{f}_i(b) \neq 0, \\ s_i(b) \otimes u_{-(i-1)} & \text{if } k = -i \text{ and } \tilde{f}_i(b) = 0, \\ s_i(\tilde{c}_i(b)) \otimes u_{i-(i-1)} & \text{if } k = -(i-1) \text{ and } \tilde{c}_i(b) \neq 0, \\ s_i(b) \otimes u_i & \text{if } k = (i-1) \text{ and } \tilde{c}_i(b) = 0, \\ s_i(\tilde{c}_i(b)) \otimes u_{i-1} & \text{if } k = i-1 \text{ and } \tilde{c}_i(b) \neq 0, \\ s_i(b) \otimes u_i & \text{if } k = i \text{ and } \tilde{f}_i(b) \neq 0, \\ s_i(b) \otimes u_{i-1} & \text{if } k = i \text{ and } \tilde{f}_i(b) = 0. \end{cases}$$

**Proof.** This is straightforward by using Proposition 5.2.3 □

**Proposition 9.2.4.** Let $b \in \mathcal{B}$ and $c \in \mathbb{Z}_{\geq 0}$. If $\tilde{c}_r'(b) = 0$, then we have

$$\tilde{f}_r'(b) = \begin{cases} \tilde{f}_r'(v) \otimes u_s + q\mathcal{L} & \text{if } b = v_s \text{ and } s \in S \setminus \{\pm(r-1), \pm r\}, \\ \tilde{f}_1 \cdots \tilde{f}_{r-1} \tilde{f}_r'(v) \otimes u_{-(r-1)} + q\mathcal{L} & \text{if } b = v_{-(r-1)}, \\ \tilde{f}_r'(v) \otimes u_{r-1} + q\mathcal{L} & \text{if } b = v_{r-1} \text{ and } 0 \leq c < (\beta_{r-1})_{r-1} - (\beta_{r-1})_r, \\ \tilde{f}_r'(v) \otimes u_r + q\mathcal{L} & \text{if } b = v_{r-1} \text{ and } c = (\beta_{r-1})_{r-1} - (\beta_{r-1})_r, \\ f_1 \cdots f_r \tilde{f}_r'(v) \otimes u_0 + q\mathcal{L} & \text{if } b = v_r, \\ \tilde{f}_r'(v) \otimes u_r + q\mathcal{L} & \text{if } b = v_r \text{ and } 0 \leq c < (\beta_r)_{r-1} - (\beta_r)_r, \\ \tilde{f}_r'(v) \otimes u_r + q\mathcal{L} & \text{if } b = v_r \text{ and } 0 \leq c < (\beta_r)_{r-1} - (\beta_r)_r, \end{cases}$$

**Proof.** We prove the assertion by induction on $r$. The case $r = 2$ follows from Proposition 9.1.3

When $r \geq 3$, we use Lemma 7.1.6 we have

$$\tilde{f}_r'(b) = s_{r-1}s_r \tilde{f}_{(r-1)}(s_{r-1}(b)).$$

Since $\tilde{f}_r'(b) = 0$ unless $b = \tilde{f}_d'(v_s) + q\mathcal{L}$ for some $s \in S$ and $d \in \mathbb{Z}_{\geq 0}$, we may assume that $b = v_s$ for some $s \in S$. In particular, $b$ is identified with $(\text{EM}(T_{\alpha^s}), \text{ME}(T_{\beta^s}))$. Based on this fact, we can compute $s_{r-1}s_r \tilde{f}_{(r-1)}(s_{r-1}(b))$ in terms of semistandard Young tableaux of shape $(\alpha^s; \beta^s)$. Since this calculation is straightforward, we omit the details. □

**Lemma 9.2.5.** We have $\dim L = \sum_{s \in S} \mathcal{I}_{ST_r}(\alpha^s; \beta^s)$.

**Proof.** The proof is the same as the proof of Lemma 9.1.5 □
Proposition 9.2.6. We have $L = \bigoplus_{s \in S} L_s$, $\mathcal{L} = \bigoplus_{s \in S} \mathcal{L}_s$, $B = \bigcup_{s \in S} B_s$. For each $s \in S$, the $(L_s, B_s)$ is a $\gamma$-crystal basis of $L_s$ whose $\gamma$-crystal graph is isomorphic to $\text{SST}_r(\alpha_s; \beta_s)$.

Proof. The proof is the same as the proof of Proposition 9.1.6. \hfill \Box

Recall the irreducible decomposition $V_r^{\otimes N} = \bigoplus_{t \in T} L_t$. Since we took $(\alpha; \beta) = (\alpha_r; \beta_r)$ with $t \in T$ arbitrarily in the second paragraph of this subsection, this proposition implies the equalities

$$V_r^{\otimes (N+1)} = \bigoplus_{t \in T} (L_t \otimes V_r) = \bigoplus_{t \in T} \bigoplus_{s(\alpha_t; \beta_t)} L_{t,s},$$

$$\mathcal{L}_r^{\otimes (N+1)} = \bigoplus_{t \in T} (\mathcal{L}_t \otimes \mathcal{L}_r) = \bigoplus_{t \in T} \bigoplus_{s(\alpha_t; \beta_t)} \mathcal{L}_{t,s},$$

$$B_r^{\otimes (N+1)} = \bigcup_{t \in T} (B_t \otimes B_r) = \bigcup_{t \in T} \bigcup_{s(\alpha_t; \beta_t)} B_{t,s}.$$

Moreover, $(\mathcal{L}_s, B_t,s)$ is a $\gamma$-crystal basis of $L_{t,s} \simeq L(\alpha_t^s; \beta_t^s)$. This proves assertions $A_r(N+1)$ and $B_r(N+1)$. Now, assertion $C_r(N+1)$ follows from Corollary 9.2.2.

As a byproduct, we obtain the following.

Corollary 9.2.7. Let $M \in \mathcal{O}_\text{int}$ be a $U^J$-module having a $\gamma$-crystal basis $(\mathcal{L}, \mathcal{B})$, and let $N$ be a $U$-module having a crystal basis $(\mathcal{L}', \mathcal{B}')$. Then, $M \otimes N$ has a $\gamma$-crystal basis $(\mathcal{L} \otimes \mathcal{L}', \mathcal{B} \otimes \mathcal{B}')$.

Proof. By Proposition 9.2.6, the assertion holds for $M = L(\alpha; \beta)$ for some double partition $(\alpha; \beta)$, and $N = V_r$. In the general case, $M$ is a direct sum of various $L(\alpha; \beta)$’s, and $N$ is a direct summand of $V_r^{\otimes n}$ for some $n \geq 1$. Therefore, the assertion follows by applying Proposition 9.2.6 repeatedly. \hfill \Box

10. Applications

In this section, we consider how a given $U^J$-module decomposes into irreducible modules. By the existence and uniqueness of a $\gamma$-crystal basis, together with the connectedness (with a single source) of the $\gamma$-crystal basis of an irreducible $U^J$-module, the problem is reduced to determining the highest weight vectors in the $\gamma$-crystal basis of a given module. We will frequently use results in [Kw09] without mentioning it.

10.1. Irreducible decomposition of $V_r^{\otimes N}$. Let us consider $V_r^{\otimes N}$ and its $\gamma$-crystal basis $B_r^{\otimes N}$. The connected components of $B_r^{\otimes N}$ are in one-to-one correspondence with $s \in B_r^{\otimes N}$ such that $\tilde{c}_t(s) = 0$ for all $i \in \overline{t}$. Such $s$’s are characterized as follows.

Proposition 10.1.1. Let $s = (s_1, \ldots, s_N) \in B_r^{\otimes N}$. For each $-r \leq j \leq r$ and $1 \leq n \leq N$, set $c_j^\leq_n(s) := \sharp \{1 \leq m \leq N \mid s_m = j\}$ and $c_j^\geq_n(s) := \sharp \{n \leq m \leq N \mid s_m = j\}$. Then the following are equivalent:

(1) $\tilde{c}_t(s) = 0$ for all $i \in \overline{t}$.
(2) $c_0^\leq_n(s) \geq c_1^\leq_n(s)$, $c_{-1}^\leq_n(s)$, $c_{-2}^\leq_n(s)$, and $c_{-j}^\leq_n(s) \geq c_j^\leq_n(s)$ for all $1 \leq n \leq N$ and $j \in \mathbb{Z} \setminus \{1\}$.

Proof. This follows easily from the $\gamma$-crystal structure of $B_r^{\otimes N}$ described at the beginning of Section 8.2. \hfill \Box

We call an element $s \in B_r^{\otimes N}$ satisfying condition (2) of Proposition 10.1.1 a double Yamanouchi word, since $s$ is a Yamanouchi word when we read only letters 1, 2, ..., $r$ and so is $s^\text{rev}$ when we read only letters 0, $-1$, ..., $-r$ and then ignore negative signs.

Remark 10.1.2. What we call a Yamanouchi word is called a lattice permutation in [Kw09]. For a partition $\lambda$ of length $r$, we denote by $\text{Yam}(\lambda)$ the set of Yamanouchi words in letters 1, ..., $r$ of shape $\lambda$, that is, the number of appearances of $i$ in the word equals $\lambda_i$ for all $i \in \{1, \ldots, r\}$. By the Robinson-Schensted correspondence, one has $\sharp \text{Yam}(\lambda) = \sharp \text{ST}(\lambda)$. 

Proposition 10.1.3. Let $s \in B_T^{N_1}$ be a double Yamanouchi word. Then, the connected component of $B_T^{N_1}$ containing $s$ is isomorphic to $\ST_r(\alpha; \beta)$, where $\alpha = (\alpha_1, \ldots, \alpha_{r+1})$ and $\beta = (\beta_1, \ldots, \beta_r)$ are partitions given by
\begin{equation}
\alpha_i = \sharp\{m \mid s_m = -(i-1)\}, \quad \beta_i = \sharp\{m \mid s_m = i\}.
\end{equation}

Proof. By the complete reducibility of $V_r^{\otimes N}$, the connected component of $B_T^{N_1}$ is isomorphic to $\ST_r(\alpha; \beta)$ for some double partition $(\alpha; \beta)$ of size $N$. Since $\tilde{\tau}_i(s) = 0$ for all $i \in \mathbb{P}$, we may identify $s$ with $\EM(T_\alpha), \ME(T_\beta)$, which satisfies condition (15). This proves the proposition.

We denote by $\Yam(\alpha; \beta)$ the set of double Yamanouchi words in $B_T^{N_1}$ satisfying condition (15), and call each element in $\Yam(\alpha; \beta)$ a Yamanouchi word of shape $(\alpha; \beta)$.

Definition 10.1.4. (1) A semistandard Young tableau $T$ of shape $\alpha$ is said to be standard if $\Im T = \{1, \ldots, |\alpha|\}$. We denote by $\ST(\alpha)$ the set of standard Young tableaux of shape $\alpha$.

(2) A semistandard double Young tableau $(T_1, T_2) = (\{p_1, \ldots, p_{|\alpha|}\}, \{q_1, \ldots, q_{|\beta|}\})$, with $p_1 < \cdots < p_{|\alpha|}$, $q_1 < \cdots < q_{|\beta|}$. We denote by $\ST(\alpha; \beta)$ the set of standard double Young tableaux of shape $(\alpha; \beta)$.

Let $(\alpha; \beta)$ be a double partition and $(T_1, T_2) \in \ST(\alpha; \beta)$. We write $\Im T_1 = \{p_1, \ldots, p_{|\alpha|}\}$ and $\Im T_2 = \{q_1, \ldots, q_{|\beta|}\}$, with $p_1 < \cdots < p_{|\alpha|}$, $q_1 < \cdots < q_{|\beta|}$. Let $T'_1$ denote the standard Young tableau of shape $\alpha$ obtained from $T_1$ by replacing each $p_i$ with $i$. Define $T'_2$ similarly. Then, the map $\ST(\alpha; \beta) \rightarrow \ST(\alpha) \times \ST(\beta) \times \{(q_1, \ldots, q_{|\beta|}) \mid 1 \leq q_1 < \cdots < q_{|\beta|} \leq |\alpha; \beta| \}$ defined by $(T_1, T_2) \mapsto (T'_1, T'_2, (q_1, \ldots, q_{|\beta|}))$ is a bijection.

Theorem 10.1.5. Let $(\alpha; \beta)$ be a double partition of $N$ of length $(r+1; r)$. Then, the multiplicity of the irreducible component of $V_r^{\otimes N}$ isomorphic to $L(\alpha; \beta)$ is equal to $\sharp \ST(\alpha; \beta)$. Namely, we have an isomorphism
\begin{equation}
V_r^{\otimes N} \cong \bigoplus_{(\alpha; \beta)} L(\alpha; \beta)^{\otimes (\sharp \ST(\alpha; \beta))}
\end{equation}
of $U^J$-modules, where $(\alpha; \beta)$ runs over all double partitions of $N$ of length $(r+1; r)$.

Proof. By Proposition 10.1.3, the multiplicity of the irreducible component of $V_r^{\otimes N}$ isomorphic to $L(\alpha; \beta)$ is equal to $\sharp \Yam(\alpha; \beta)$. Here, the set $\Yam(\alpha; \beta)$ is in one-to-one correspondence with $\Yam(\alpha) \times \Yam(\beta) \times \{(q_1, \ldots, q_{|\beta|}) \mid 1 \leq q_1 \leq \cdots \leq q_{|\beta|} \leq N\}$ under the assignment
\begin{equation}
s = (s_1, \ldots, s_N) \mapsto \left( (s_{p_1}, \ldots, s_{p_{|\alpha|}}), (s_{q_1}, \ldots, s_{q_{|\beta|}}), (q_1, \ldots, q_{|\beta|}) \right),
\end{equation}
where $s_{p_1}, \ldots, s_{p_{|\alpha|}} \leq 0$ with $p_1 < \cdots < p_{|\alpha|}$, and $s_{q_1}, \ldots, s_{q_{|\beta|}} \geq 1$ with $q_1 < \cdots < q_{|\beta|}$. From this and the bijection $\ST(\alpha; \beta) \rightarrow \ST(\alpha) \times \ST(\beta) \times \{(q_1, \ldots, q_{|\beta|}) \mid 1 \leq q_1 \leq \cdots \leq q_{|\beta|} \leq |\alpha; \beta| \}$, we obtain
\begin{align*}
\sharp \Yam(\alpha; \beta) &= \sharp \Yam(\alpha) \cdot \sharp \Yam(\beta) \cdot \binom{N}{|\beta|} \\
&= \sharp \ST(\alpha) \cdot \sharp \ST(\beta) \cdot \binom{N}{|\beta|} \quad \text{(By Remark 10.1.2)} \\
&= \sharp \ST(\alpha; \beta),
\end{align*}
as desired. This proves the theorem.

By the double centralizer property for $U^J$ and the Hecke algebra $H(W_d)$ of type $B_d$ with unequal parameters $(p, q)$ on $V_r^{\otimes d}$ for $r \geq d$ (BWW16), we have an irreducible decomposition
\begin{equation}
V_r^{\otimes d} \cong \bigoplus_{(\alpha; \beta)} L(\alpha; \beta) \otimes V(\alpha; \beta)
\end{equation}
as a $U^r\cdot H(B_d)$-bimodule, where $(\alpha; \beta)$ runs over all double partitions of $N$ of length $r+1$, and $V(\alpha; \beta)$ is an irreducible $H(W_d)$-module. According to [47], the irreducible $H(W_d)$-modules are classified by the double partitions of size $d$. Hoefsmit constructed the irreducible modules by giving the representation matrices for the generators of $H(W_d)$ explicitly. Later, Dipper and James [49] realized the irreducible $H(W_d)$-modules $S^\alpha\cdot \beta$ as ideals of $H(W_d)$. In Appendix A we prove that $V(\alpha; \beta) \simeq S^\alpha\cdot \beta$.

10.2. Littlewood-Richardson rule for $U^r$. In this section, we consider the irreducible decomposition of $L(\alpha; \beta) \otimes L(\lambda)$, where $L(\lambda)$ denotes the irreducible highest weight $U$-module with highest weight $\lambda$. In terms of $\nu$-crystal bases, we will determine the double Yamanouchi words in $B(\alpha; \beta) \otimes B(\lambda) \subset B(\alpha; \beta) \otimes B(\lambda)$; here $B(\lambda)$ denotes the crystal basis of $L(\lambda)$ embedded in $B(\lambda)$ by the Middle-Eastern reading. Let $\text{LR}_{(\alpha; \beta), \lambda}(r)$ denote the multiplicity of $L(\alpha'; \beta')$ in $L(\alpha) \otimes L(\lambda)$; clearly, it is equal to the number of double Yamanouchi words in $B(\alpha; \beta) \otimes B(\lambda)$ of shape $(\alpha'; \beta')$.

Let us briefly recall the Littlewood-Richardson rule for ordinary crystal bases in type $A$. Let $\text{LR}_{\mu, \nu}(2r+1)$ denote the multiplicity of $L(\lambda)$ in $L(\mu) \otimes L(\nu)$. A semistandard tableau $T$ of shape $\lambda/\mu$ is called a Littlewood-Richardson tableau of shape $\lambda/\mu$ with content $\nu$ if $T$ contains $\nu_i$'s, and if $ME(T)$ is a Yamanouchi word $[1899]$. Hence $\text{LR}_{\mu, \nu}(2r+1)$ equals the number of Littlewood-Richardson tableaux of shape $\lambda/\mu$ with content $\nu$ in $2r+1$ letters. Also, it is known that the multiplicity of $B(\nu)$ in $B(\lambda/\mu)$ is equal to $\text{LR}_{\mu, \nu}(2r+1)$.

**Theorem 10.2.1.** Let $(\alpha; \beta), (\alpha'; \beta')$ be double partitions of length $(r+1; r)$, and $\lambda$ a partition of length $2r+1$. Then, we have

$$\text{LR}_{(\alpha; \beta), \lambda}(r) = \sum_{\mu \subset \lambda} \sum_{\ell(\mu) \leq r+1} \text{LR}_{\mu, \alpha}(r+1) \text{LR}_{\mu, \nu}(r) \text{LR}_{\beta', \nu}(r),$$

where $\nu$ runs over all partitions of size $|\lambda/\mu|$.

**Proof.** Let $(T_1; T_2) \in B(\alpha; \beta)$ and $T \in B(\lambda)$. If we read only letters $\leq 0$ in $T$, then it is also a semistandard tableau $T'$ of shape, say $\mu \subset \lambda$. Since there are $r+1$ kinds of letters $\leq 0$, we have $\ell(\mu) \leq r+1$. Suppose that $(T_1; T_2) \otimes T$ is a double Yamanouchi word of shape $(\alpha'; \beta')$. By the definition of double Yamanouchi words, $(\text{EM}(T_2), \text{ME}(T/T'))$ is a Yamanouchi word of shape $\beta'$ in letters $1, \ldots, r$, and $(\text{EM}(T_1), \text{ME}(T'))^{ev} = (\text{EM}(T'), \text{ME}(T_1))$ is a Yamanouchi word of shape $\alpha'$ in letters $0, 1, \ldots, r$ if we ignore negative signs. In addition, by Proposition 5.2.3 we have $\bar{F}_{(r-\frac{1}{2}, \frac{1}{2})}(T') = 0$ for all $t \in \mathbb{P}$. This implies that $\text{EM}(T')$ is a Yamanouchi word of shape $\mu$ if we ignore negative signs, and that $T'$ is determined uniquely by $\mu$ and this condition; hence, we write $T' = T$. With this notation, for an arbitrary partition $\mu \subset \lambda$ of length $\leq r+1$, let $Y(\mu)$ be the number of $(T_2; T)$ such that $(\text{ME}(T_2), \text{ME}(T/T')(\mu))$ is a Yamanouchi word of shape $\beta'$ in letters $1, \ldots, r$, and $Z(\mu)$ the number of $T_1$ such that $(\text{EM}(T'\mu), \text{ME}(T_1))$ is a Yamanouchi word of shape $\alpha'$ in letters $0, 1, \ldots, r$ if we ignore negative signs. Then, by the above, we obtain

$$\text{LR}_{(\alpha; \beta), \lambda}(r) = \sum_{\mu \subset \lambda} Y(\mu) \cdot Z(\mu);$$

here, $Y(\mu)$ is equal to the cardinality of $\text{Yam}(\beta') \cap (B(\beta) \otimes B_r(\lambda/\mu))$, where $B_r(\lambda/\mu)$ denotes the set of semistandard tableaux of shape $\lambda/\mu$ in letters $1, \ldots, r$. Therefore, we see that $Y(\mu) = \sum_{\nu \vdash |\lambda/\mu|} \text{LR}_{\beta', \nu}(r) \cdot \text{LR}_{\mu, \nu}(r)$ by the Littlewood-Richardson rule for ordinary crystal bases in type $A$.

In order to compute $Z(\mu)$, let us count the number $Z'(\mu)$ of Yamanouchi words in $B(\mu) \otimes B(\alpha)$ of shape $\alpha'$ in letters $0, 1, \ldots, r$. By the tensor product rule for ordinary crystal bases, if $T_3 \otimes T_4 \in B(\mu) \otimes B(\alpha)$ is a Yamanouchi word, then so is $T_3$. Since $\text{EM}(T'\mu))$ is a Yamanouchi word of
shape $\mu$ in letters $0, 1, \ldots, r$ if we ignore negative signs, $Z(\mu)$ is equal to $Z'(\mu)$, which, in turn, equals $LR^\alpha_{\mu, \alpha}(r + 1)$ by the Littlewood-Richardson rule for ordinary crystal basis in type $A$.

Summarizing, we conclude that

$$LR^{(\alpha'; \beta')}_{(\alpha; \beta), \lambda} = \sum_{\mu \leq \lambda} \sum_{\ell(\mu) \leq r + 1} LR^\gamma_{\mu, \nu}(r) \cdot LR^\lambda_{\mu, \nu}(r) \cdot LR^\alpha_{\mu, \alpha}(r + 1),$$

as desired. This proves the theorem. \hfill \Box

In particular, if we take $(\alpha; \beta)$ to be $(\emptyset; \emptyset)$, then the tensor product module $L(\emptyset; \emptyset) \otimes L(\lambda)$ is just $L(\lambda)$ regarded as a $U^\mu$-module. Hence, Theorem 10.2.1 gives the branching rule for $U$-modules restricted to $U^\mu$:

**Corollary 10.2.2.** The multiplicity of $L(\alpha'; \beta')$ in $L(\lambda)$ is equal to

$$LR^{(\alpha'; \beta')}_{(\emptyset; \emptyset), \lambda} = LR^\lambda_{\alpha', \beta'}.$$

**Proof.** By Theorem 10.2.1 we have

$$LR^{(\alpha'; \beta')}_{(\emptyset; \emptyset), \lambda} = \sum_{\mu, \nu} LR^\lambda_{\mu, \emptyset}(r + 1) \cdot LR^\gamma_{\mu, \nu}(r) \cdot LR^\beta_{\emptyset, \nu}(r).$$

However, $LR^\alpha_{\mu, \emptyset}(r + 1) = \delta_{\mu, \alpha}$ and $LR^\beta_{\emptyset, \nu}(r) = \delta_{\nu, \beta'}$. This proves the corollary. \hfill \Box

**APPENDIX A. IRREDUCIBLE DECOMPOSITION OF $V_r^{\otimes d}$ AS A $U^\mu$-$\mathcal{H}(W_d)$-BIMODULE**

**A.1. The action of $\mathcal{H}(W_d)$ on $V_r^{\otimes d}$.** Let $\mathcal{H}(W_d)$ be the $\mathbb{Q}(p, q)$-algebra generated by $H_i$, $0 \leq i \leq d - 1$, subject to the relations:

- $H_0^2 = (p^{-1} - p)H_0 + 1$,
- $H_i^2 = (q^{-1} - q)H_i + 1$ if $i \neq 0$,
- $H_0H_1H_0H_1 = H_1H_0H_1H_0$,
- $H_iH_{i+1}H_i = H_{i+1}H_iH_{i+1}$ if $1 \leq i \leq n - 2$,
- $H_iH_j = H_jH_i$ if $|i - j| > 1$.

Let $W_d$ denote the Weyl group of type $B_d$ with simple reflections $s_0, s_1, \ldots, s_{d-1}$. We denote by $\mathfrak{S}_d$ and $\mathfrak{S}_{a,d-a}$ the subgroup of $W_d$ generated by $s_i$, $i \neq 0$, and $s_i$, $i \neq 0, a$, respectively. Let $\mathcal{H}(\mathfrak{S}_d)$ and $\mathcal{H}(\mathfrak{S}_{a,d-a})$ be the subalgebra of $\mathcal{H}(W_d)$ generated by $H_i$, $i \neq 0$, and $H_i$, $i \neq 0, a$, respectively.

Following [BWY10], let us recall the action of $\mathcal{H}(W_d)$ on $V_r^{\otimes d}$. For a map $f : \{1, \ldots, d\} \to \{-r, -r + 1, \ldots, r\}$, we set $M_f := u_{f(1)} \otimes \cdots \otimes u_{f(d)} \in V_r^{\otimes d}$. The Weyl group $W_d$ acts on the set of maps from $\{1, \ldots, d\}$ to $\{-r, -r + 1, \ldots, r\}$ by:

$$(f \cdot s_j)(i) = \begin{cases} f(j + 1) & \text{if } i = j, \\ f(j) & \text{if } i = j + 1, \\ f(i) & \text{otherwise}, \end{cases}$$

$$(f \cdot s_0)(i) = \begin{cases} -f(1) & \text{if } i = 1, \\ f(i) & \text{otherwise}. \end{cases}$$
Then the Hecke algebra $\mathcal{H}(W_d)$ acts on $V_r^\otimes d$ by:

$$M_f \cdot H_j = \begin{cases} q^{-1} M_f & \text{if } f(i) = f(i+1), \\ M_{f,s_i} & \text{if } f(i) < f(i+1), \\ M_{f,s_i} + (q^{-1} - q) M_f & \text{if } f(i) > f(i+1), \end{cases}$$

$$M_f \cdot H_0 = \begin{cases} p^{-1} M_f & \text{if } f(1) = 0, \\ M_{f,s_0} & \text{if } f(1) > 0, \\ M_{f,s_0} + (p^{-1} - p) M_f & \text{if } f(1) < 0. \end{cases}$$

### A.2. Irreducible $\mathcal{H}(S_d)$-modules

Let us recall from [G86] how to construct the irreducible $\mathcal{H}(S_d)$-modules. Note that our normalization of the generators of the Hecke algebra differs from that in [G86], because of this, we construct right $\mathcal{H}(S_d)$-modules, while Gyoja treated left $\mathcal{H}(S_d)$-modules.

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0)$ be a partition of $d$ and $\lambda'$ its transposed partition. Let $T_+ (\lambda)$ be the standard tableau of shape $\lambda$ defined by $T_+(i,j) = \lambda_1 + \cdots + \lambda_i - 1 + j$, and $T_- (\lambda)$ the standard tableau of shape $\lambda$ defined by $T_-(i,j) = \lambda'_1 + \cdots + \lambda'_{i-1} + i$. Also, let $I_+$ be the set of those $s_i$ which preserves each row of $T_+$, i.e., the set of those $s_i$ which preserves each column of $T_-$. Then the Hecke algebra $\mathcal{H}(S_d)$ acts on $\mathcal{H}(S_d)$ corresponding to $\mathcal{H}(S_d)$.

Let $[+, -] \in S_d$ be the unique element such that $T_+ \cdot [+, -] = T_-$. Then, for each $x \in S_+$ and $y \in S_-$, one has $\ell(x[+, -]y) = \ell(x) + \ell([+, -]) + \ell(y)$. By [G86] Section 2, the following holds.

**Theorem A.2.1 (G86).** The right ideal $S^\lambda$ of $\mathcal{H}(S_d)$ generated by $e_+ H_{[+, -]} e_-$ is an irreducible $\mathcal{H}(S_d)$-module. Moreover, the set $\{S^\lambda \mid \lambda \vdash d\}$ provides a complete list of nonisomorphic irreducible $\mathcal{H}(S_d)$-modules.

By this theorem, we can realize each $S^\lambda$, $\lambda \vdash d$, as a submodule of $V_r^\otimes d$. We define a map $f_{\lambda}$ by: $f_{\lambda}(i) = j$ if $\lambda_{j-1} < i \leq \lambda_j$. It is easy to verify that the $\mathcal{H}(S_d)$-submodule generated by $M_{f_{\lambda}}$ is isomorphic to $e_+ \mathcal{H}(S_d)$. Therefore, the $\mathcal{H}(S_d)$-submodule generated by $M_{f_{\lambda}} := M_{f_{\lambda}} (H_{[+, -]} e_-)$ is isomorphic to $S^\lambda$. Since $\ell(x[+, -]y) = \ell(x) + \ell([+, -]) + \ell(y)$ for all $x \in S_+$ and $y \in S_-$, we see that

$$M_{f_{\lambda}} = \sum_{y \in S_-} (-q)^{\ell(y)} M_{f_{\lambda}, [+, -]y}.$$

Also, by the definitions of $f_{\lambda}$ and $[+, -]$, it follows that

$$M_{f_{\lambda}, [+, -]} = (u_1 \otimes u_2 \otimes \cdots \otimes u_{\lambda'_1}) \otimes (u_1 \otimes u_2 \otimes \cdots \otimes u_{\lambda'_2}) \otimes \cdots \otimes (u_1 \otimes u_2 \otimes \cdots \otimes u_{\lambda'_k}).$$

These imply that $M_{f_{\lambda}} \in M_{f_{\lambda}, [+, -]} + qL_r^\otimes d$.

By the quantum Schur-Weyl duality of type $A$, the irreducible $\mathcal{H}(S_d)$-module $M_{f_{\lambda}} \mathcal{H}(S_d) \simeq S^\lambda$ is contained in the direct sum of some copies of the irreducible highest weight $U$-module with highest weight corresponding to a partition, say $\mu$. Applying Kashiwara operators $\hat{E}_i$'s on $M_{f_{\lambda}},$ repeatedly, one can easily verify that $\mu = \lambda$.

Exchanging the roles of $H_i$ and $H_i^{-1}$, we obtain $M_{f_{\lambda}} \in V_r^\otimes d$ such that

$$M_{f_{\lambda}} \in \left( u_{-\lambda'_1} \otimes \cdots \otimes u_{-2} \otimes u_{-1} \right) \otimes \left( u_{-\lambda'_2} \otimes \cdots \otimes u_{-2} \otimes u_{-1} \right) \otimes \cdots \otimes \left( u_{-\lambda'_k} \otimes \cdots \otimes u_{-2} \otimes u_{-1} \right) + qL_r^\otimes d$$

and $M_{f_{\lambda}} \mathcal{H}(S_d) \simeq S^\lambda$. 


A.3. Irreducible $\mathcal{H}(W_d)$-modules. In this subsection, we construct the irreducible $\mathcal{H}(W_d)$-modules following [DJ92]. For $1 \leq i < j \leq d - 1$, we set

$$s_{i,j} := s_{i} s_{i+1} \cdots s_{j-1}, \quad s_{j,i} := s_{i,j}^{-1}.$$ 

Fix two nonnegative integers $a, b$ such that $a + b = d$, and set $w_{a,b} := (s_{d,1})^b \in \mathfrak{S}_d$. Also, we define $v_{a,b} \in \mathcal{H}(W_d)$ by

$$v_{a,b} := \sum_{i=1}^{a} (p + H_{s_{i,1}} H_0 H_{s_{1,i}} H_{w_{a,b}})^b \prod_{j=1}^{b} (1 - p H_{s_{j,1}} H_0 H_{s_{1,j}}).$$

Let $\lambda \vdash a$ and $\mu \vdash b$. By Appendix A.2, one can construct the irreducible $\mathcal{H}(\mathfrak{S}_a)$-module $S^\lambda$ in the subalgebra of $\mathcal{H}(W_d)$ generated by $H_1, \ldots, H_{a-1}$, and the irreducible $\mathcal{H}(\mathfrak{S}_b)$-module $S^\mu$ in the subalgebra generated by $H_{a+1}, \ldots, H_{d-1}$. It follows that $S^\lambda \cdot S^\mu \subset \mathcal{H}(\mathfrak{S}_{a,b})$. Set

$$S^{\lambda,\mu} := S^\lambda \cdot S^\mu : v_{a,b} \mathcal{H}(W_d) = v_{a,b} S^\mu \cdot S^\lambda \mathcal{H}(W_d).$$

**Theorem A.3.1** ([DJ92]). The set $\{S^{\lambda,\mu} \mid 0 \leq a \leq d, \lambda \vdash a, \mu \vdash d - a\}$ provides a complete list of nonisomorphic irreducible $\mathcal{H}(W_d)$-modules.

Let us find a good generator of $S^{\lambda,\mu}$ in $V_r^{\otimes d}$. Define a map $f_{\lambda,\mu}$ by:

$$f_{\lambda,\mu}(i) = \begin{cases} f_{\lambda}(i) & \text{if } 1 \leq i \leq a, \\ f_{\mu}(i - a) & \text{if } a + 1 \leq i \leq d. \end{cases}$$

By Appendix A.2 we have

$$S^\lambda \cdot S^\mu \simeq M_{f_{\lambda,\mu}} S^\lambda \cdot S^\mu \subset M_{f_{\lambda,\mu}} \mathcal{H}(\mathfrak{S}_d),$$

and hence,

$$S^{\lambda,\mu} \simeq M_{f_{\lambda,\mu}} S^\lambda \cdot S^\mu v_{a,b} \mathcal{H}(W_d) = M_{f_{\lambda,\mu}} S^\mu \cdot S^\lambda \mathcal{H}(W_d).$$

Also, we see that

$$M_{f_{\lambda,\mu}} v_{a,b} \subset u_{f_{\mu}(1)} \otimes \cdots \otimes u_{f_{\mu}(b)} \otimes u_{-f_{\lambda}(1)} \otimes \cdots \otimes u_{-f_{\lambda}(a)} + p \mathcal{L}_r^{\otimes d}.$$ 

Therefore, $M_{f_{\lambda,\mu}} v_{a,b} S^\mu \cdot S^\lambda \mathcal{H}(W_d)$ is generated by $M_{\mu,+) \otimes M_{\lambda,-}$, which is of the form

$$M_{\mu,+) \otimes M_{\lambda,-} = (u_1 \otimes \cdots \otimes u_{\mu_1}) \otimes (u_1 \otimes \cdots \otimes u_{\mu_2}) \otimes \cdots \otimes (u_1 \otimes \cdots \otimes u_{\mu_d}) \otimes \cdots \otimes (u_{-\lambda_1} \otimes \cdots \otimes u_{-1}) \otimes \cdots \otimes (u_{-\lambda_k} \otimes \cdots \otimes u_{-1}) + q \mathcal{L}_r^{\otimes d}.$$ 

By the quantum Schur-Weyl duality of type $B$, the irreducible $\mathcal{H}(W_d)$-modules $M_{\mu,+) \otimes M_{\lambda,-} \mathcal{H}(W_d)$ is contained in the direct sum of some copies of the irreducible highest weight $U^+\mathfrak{u}$-module $L(\alpha; \beta)$ for some double partition $(\alpha; \beta)$. By the descriptions of $M_{\mu,+) + q \mathcal{L}_r^{\otimes a}$ and $M_{\lambda,-} + q \mathcal{L}_r^{\otimes b}$, it is clear that

$$\varphi_{\lambda}^{(d)} \cdots \varphi_{\lambda}^{(1)} (M_{\mu,+) \otimes M_{\lambda,-} + q \mathcal{L}_r^{\otimes d}) \in B_r^{\otimes d}$$

is a double Yamanouchi word of shape $(\lambda; \mu)$, and hence, we conclude that $L(\alpha; \beta) \simeq L(\lambda; \mu)$.

**Theorem A.3.2**. Let $r \geq d$. As a $U^+\mathcal{H}(W_d)$-bimodule, $V_r^{\otimes d}$ is decomposed as follows:

$$V_r^{\otimes d} \simeq \bigoplus_{(\alpha; \beta)} L(\alpha; \beta) \otimes S^\alpha \otimes S^\beta,$$

where $(\alpha; \beta)$ runs over all the double partitions of $d$. 
This theorem implies that the functor $F$, from the category of $\mathcal{H}(W_d)$-modules to $\mathcal{O}_\text{int}'$, defined by $F(M) := V_{\text{red}} \otimes \mathcal{H}(W_d) M$ gives a category equivalence which maps $(S^{\alpha, \beta})^{op}$ to $L(\alpha; \beta)$ for all double partitions $(\alpha; \beta)$ of $d$.

**Appendix B.**

**B.1. Proof of Proposition 6.1.1.** Let $M$ be a $U^j$-module and $m \in M_{a,b,n} \setminus \{0\}$. Set $h'_1 := h_1 + \frac{p^{-1}q_{a+3n}^{-1}}{q-q^{-1}}$.

**Lemma B.1.1.** We have the following:

1. $[h'_1, f_2]_1 = q^2 f'_2$.
2. $[h'_1, f'_2]_1 = q^2 f'_2$.
3. $[h'_1, f'_2]_{-1} = -p \left( q^{-3}f'_2 - [2]f'_2 - f'_2 \left( q^{-1}(q - q^{-1})h'_1(a,b,n) + [2]p^{-1}q^{-1}q^{-a+3n} \right) \right) k_1$.

**Proof.** This is easy and straightforward. \(\square\)

Since $h_1$ and $k_1$ act on $m$ as scalar multiplication, so does $h'_1$; explicitly, we have $h'_1 m = h'_1(a,b,n)m$, where

$$h'_1(a,b,n) := [n+1][b-n](a-b-n-1) - q[n][b-n+1](a-b-n) + \frac{p^{-1}q^{-a+3n}}{q-q^{-1}}.$$  

By Lemma [B.1.1] we have

$$h'_1 f'_2 m = q h'_1(a,b,n) f'_2 m + q^2 f_2 m,$$

$$h'_1 f_2 m = q h'_1(a,b,n) f_2 m + q^2 f'_2 m,$$

$$h'_1 f'_2 m = q^{-1} h'_1(a,b,n) f'_2 m - p \left( q^{-3}f'_2 - [2]f'_2 - f'_2 \left( q^{-1}(q - q^{-1})h'_1(a,b,n) + [2]p^{-1}q^{-1}q^{-a+3n} \right) \right) q^{a-3n}$$

$$= -pq^{a-3n-3}f'_2 m + pq^{a-3n} \left( q^{-1}(q - q^{-1})h'_1(a,b,n) + [2]p^{-1}q^{-a+3n} \right) f_2 m$$

$$+ (q^{-1}h'_1(a,b,n) + pq^{a-3n}[2]) f'_2 m.$$  

Therefore, $h'_1$ defines a linear endomorphism on the vector space spanned by $\{f'_2 m, f_2 m, f'_2 m\}$ whose representation matrix is

$$\begin{pmatrix} q h'_1(a,b,n) & 0 & -pq^{a-3n-3} \\ q^2 & q h'_1(a,b,n) & pq^{a-3n-1}(q - q^{-1})h'_1(a,b,n) + q^{-1}[2] \\ 0 & q^2 & q^{-1}h'_1(a,b,n) + pq^{a-3n}[2] \end{pmatrix}.$$  

Hence, in order to prove Proposition 6.1.1, it suffices to show that the following three vectors

$$\begin{pmatrix} p q^{a-b-1} - p^{-1}q^{-a+b} \\ -q^{-b+n+1} \end{pmatrix}, \begin{pmatrix} -p q^{-a-b-n-2} \\ -q^{b+1} + q^{-b-1} \end{pmatrix}, \begin{pmatrix} p q^{a-2b-1} - p^{-1}q^{-a+2b+1} \\ -q^{-n+2} \end{pmatrix}$$

are eigenvectors of the matrix (17) with eigenvalues $h'_1(a + 1, b + 1, n)$, $h'_1(a + 1, b, n)$, and $h'_1(a - 2, b - 1, n - 1)$, respectively. This can be checked by using a computer, or possibly by direct calculation.

**B.2. Proof of Proposition 6.1.2.**

**Lemma B.2.1.** It holds that

$$f'_2 f_2 = q^{-1} f_2 f'_2, \quad f_2 f'_2 = q^{-1} f'_2 f_2, \quad f'_2 f'_2 = f_2 f_2 - (q - q^{-1}) f'_2.$$  

**Proof.** By direct calculation. \(\square\)
In order to prove Proposition 6.1.2, it suffices to prove the following three equalities for all \( a \in \mathbb{Z}, \ b, n \in \mathbb{Z}_{\geq 0} \):

\[
\begin{align*}
&f_{2,2}(a+1, b+1, n)f_{2,1}(a, b, n) = f_{2,1}(a+1, b, n)f_{2,2}(a, b, n), \\
&f_{2,2}(a+1, b, n)f_{2,2}(a, b, n) = f_{2,2}(a-2, b-1, n-1)f_{2,3}(a, b, n), \\
&f_{2,1}(a-2, b-1, n-1)f_{2,3}(a, b, n) = f_{2,3}(a+1, b+1, n)f_{2,1}(a, b, n).
\end{align*}
\]

This is straightforward by Lemma 3.2.1 and the definitions of \( f_{2,1}, f_{2,2}, \) and \( f_{2,3} \).

### B.3. Proof of Proposition 6.2.7

Let \( a_1 \in \mathbb{Z}, \ a_2, b_1, b_2 \in \mathbb{Z}_{\geq 0} \), and let \( v \in L(a_1, a_2; b_1, b_2) \) be a highest weight vector. Then, we have

\[
h_1v = [b_1\{a_1 - b_1 - 1\}v, \quad h_2v = [b_1 + b_2\{a_1 + a_2 - (b_1 + b_2) - 1\}v, \quad \tau_2^{-1}(v) = f_2^{(a_2)}v.
\]

We need to prove that

\[
\begin{align*}
h_2(v \sqrt{0}) &= [b_1 + b_2 + 1\{a_1 + a_2 - (b_1 + b_2) + 1\}v \sqrt{0}, \\
h_2(v \sqrt{1}) &= [b_1 + b_2\{a_1 + a_2 - (b_1 + b_2) - 1\}v \sqrt{1}, \\
h_2(v \sqrt{-1}) &= [b_1 + b_2\{a_1 + a_2 - (b_1 + b_2) - 1\}v \sqrt{-1}.
\end{align*}
\]

We compute as follows:

\[
h_2(v \sqrt{0}) = \tau_2(h_1 \cdot \tau_2^{-1}(v \sqrt{0})) = \tau_2(h_1(v \tau_2^{-1}(v \sqrt{0}))),
\]

Since \( \tau_2^{-1}(v) \in L(a_1, a_2; b_1, b_2)_{a_1 + a_2, b_1 + b_2} \), we have \( \tau_2^{-1}(v) \otimes u_0 \in (L(a_1, a_2; b_1, b_2) \otimes V_2)_{a_1 + a_2 + 2, b_1 + b_2} \). This proves equation (18).

Next, we prove equation (19). We have

\[
\begin{align*}
h_2(v \sqrt{1}) &= \tau_2(h_1 \cdot \tau_2^{-1}(v \sqrt{1})), \\
\tau_2^{-1}(v \sqrt{1}) &= \tau_2(v) \otimes u_2 - \frac{q^{b+1}(q-q^{-1})}{a-b-1}\tau_2^{-1}(f_1v) \otimes u_0 - pq^{a-2b}\tau_2^{-1}(v) \otimes u_{-2}.
\end{align*}
\]

Since \( \tau_2(v) \otimes u_2, \tau_2^{-1}(v) \otimes u_{-2} \in (L(a_1, a_2; b_1, b_2) \otimes V_2)_{a_1 + a_2 + b_1 + b_2} \), it remains to show that \( \tau_2^{-1}(f_1v) \otimes u_0 \in (L(a_1, a_2; b_1, b_2) \otimes V_2)_{a_1 + a_2 + b_1 + b_2} \). Indeed, we have

\[
h_1\tau_2^{-1}(f_1v) = h_1f_2^{(a_2 + 1)}f_1v
\]

as desired. Hence, equation (19) holds.

Finally, we prove equation (20). We have

\[
\begin{align*}
h_2(v \sqrt{-1}) &= \tau_2(h_1 \cdot \tau_2^{-1}(v \sqrt{-1})), \\
\tau_2^{-1}(v \sqrt{-1}) &= \tau_2^{-1}(f_1v) \otimes u_0 - q^{b_1}[b_1]\tau_2^{-1}(v) \otimes u_{-2} - pq^{a_1-b_1-2}[b_1]\tau_2^{-1}(v) \otimes u_2.
\end{align*}
\]

Since \( \tau_2^{-1}(f_1v) \otimes u_0, \tau_2^{-1}(v) \otimes u_{-2}, \) and \( \tau_2^{-1}(v) \otimes u_2 \) belong to \( (L(a_1, a_2; b_1, b_2) \otimes V_2)_{a_1 + a_2, b_1 + b_2} \), so does \( \tau_2^{-1}(v \sqrt{-1}) \). This proves equation (20). This completes the proof of the proposition.
B.4. Proof of Lemma 6.2.10. Let $x, y,$ and $z$ be as in the statement of Proposition 6.2.10. Recall that $v \in L = L(a_1, a_2; b_1, b_2)$ is a highest weight vector, and that $h_1 v = [b_1]\{a_1 - b_1 - 1\}v$, $h_2 v = [b_1 + b_2]\{a_1 + a_2 - (b_1 + b_2) - 1\}v$, where $h_1 = [e_1, f_1], h_2 = \tau_2(h_1) = \{e_2, e_1\} - 1, [f_1, f_2]_1$.

Lemma B.4.1. The following hold.

\[ e_2 f_{2,1} v = xv, \quad e_2 f_{2,2} v = yv, \quad e_2 (f_{2,3} f_1) v = z f_1 v. \]

Proof. Let $\lambda = a_1 \delta_1 + a_2 \delta_2$. Since $L_\lambda = \mathbb{Q}(p, q)v$ and $L_{\lambda - \gamma_1} = \mathbb{Q}(p, q)f_1 v$, there exist $X, Y, Z \in \mathbb{Q}(p, q)$ such that $e_2 f_{2,1} v = X v$, $e_2 f_{2,2} v = Y v$, and $e_2 (f_{2,3} f_1) v = Z f_1 v$. By the definition of $f_{2, i}$’s, we have $f_{2,2} v = f_{2,1} v + f_{2,2,2} v$. Applying $e_2$ to this equation, we obtain

\[ (21) \quad [a_2] = X + Y. \]

In addition, since $f_{2,1} v \in L_{a_1 +, b_1 +, 1, 0}$ and $f_{2,2} v \in L_{a_1 +, b_1, 0}$, it follows that

\[ e_2 e_1 f_1 (f_{2,1} v + f_{2,2} v) = e_2 ([b_1 + 1]\{a_1 - b_1 - 1\}f_{2,1} v + [b_1]\{a_1 - b_1\}f_{2,2} v) \]

\[ = ([b_1 + 1]\{a_1 - b_1 - 1\}X + [b_1]\{a_1 - b_1\}Y)v. \]

Also, we have

\[ e_2 e_1 f_1 f_2 v = (h_2 + q^{-1}e_1 e_2 f_1 f_2 + q e_2 e_1 f_2 f_1 - e_1 e_2 f_2 f_1) v \]

\[ = (h_2 + q^{-1}e_1 f_1 e_2 f_2 + q e_2 f_1 e_2 f_1 - e_1 f_2 e_2 + \frac{k_2 - k_2^{-1}}{q - q^{-1}})f_1 v \]

\[ = (h_2 + q^{-1}e_1 f_1 e_2 f_2 + q e_2 f_1 e_2 f_1 - e_1 f_1 q k_2 - q^{-1}k_2^{-1})v \]

\[ = ([b_1 + b_2]\{a_1 + a_2 - (b_1 + b_2) - 1\} \]

\[ + (q - q^{-1})[a_2]\{a_1 - b_1 - 1\} - [a_2 + 1][b_1]\{a_1 - b_1 - 1\})v \]

\[ = ([b_1 + b_2]\{a_1 + a_2 - (b_1 + b_2) - 1\} + [a_2 - 1][b_1]\{a_1 - b_1 - 1\})v. \]

Combining these two equalities, we have

\[ [b_1 + b_2]\{a_1 + a_2 - (b_1 + b_2) - 1\} + [a_2 - 1][b_1]\{a_1 - b_1 - 1\} \]

\[ = [b_1 + 1]\{a_1 - b_1 - 1\}X + [b_1]\{a_1 - b_1\}Y. \]

Solving the system of equations (21), (22), we obtain $X = x$ and $Y = y$.

Let us determine $Z$. By the definition of $U_{<0}$, we see that $\dim(U_{<0})_{\lambda_{\gamma_1} - \gamma_2} = 3$, and hence, $L_{\lambda - \gamma_2} = \text{Span}_{\mathbb{Q}(p, q)}\{f_1 f_{2,1} v, f_1 f_{2,2} v, f_{2,3} f_1 v\}$. Therefore, there exist $s, t \in \mathbb{Q}(p, q)$ such that $f_2 f_1 v = s f_1 f_{2,1} v + t f_1 f_{2,2} v + f_{2,3} f_1 v$. applying $e_1$ to this equation, we deduce that

\[ [b_1]\{a_1 - b_1 - 1\} f_2 v = s [b_1 + 1]\{a_1 - b_1 - 1\} f_{2,1} v + t [b_1]\{a_1 - b_1\} f_{2,2} v. \]

Since $f_2 v = f_{2,1} v + f_{2,2} v$, we obtain $s = \frac{[b_1]}{[b_1 + 1]}$ and $t = \frac{\{a_1 - b_1 - 1\}}{\{a_1 - b_1\}}$. In addition, we have

\[ [a_2 + 1] f_1 v = e_2 f_2 f_1 v = (sx + ty + Z)f_1 v, \]

and hence,

\[ Z = [a_2 + 1] - sx - ty = z, \]

as desired. \qed

Now, we can complete the proof of Lemma 6.2.10 by direct calculation using the previous lemma; we omit the details.
B.5. Proof of Proposition 6.2.11

Since we have assumed that \((L(a_1, a_2 ; b_1, b_2), B(a_1, a_2 ; b_1, b_2))\) is a \(\gamma\)-crystal basis of \(L(a_1, a_2 ; b_1, b_2)\), we may identify \(v + qL(a_1, a_2 ; b_1, b_2) \in B(a_1, a_2 ; b_1, b_2)\) with the semistandard double tableau

\[
\begin{pmatrix}
0 & 0 & 0 & 1 \\
-1 & -1 & -1 & b_1 \\
\end{pmatrix}
\]

Recall that \(M := L_{\lambda + 2\delta_1 - \gamma_1 - \gamma_2} \cap \text{Ker}(e_1) \cap \text{Ker}(e_2) \cap \text{Ker}(h_1 - [b_1]_1 \{a_1 - b_1 - 1\})\) is spanned by \(m_1\) and \(m_2\). Since \(h_2 m = \gamma_2(h_1 \gamma_2^{-1}(m))\) for \(m \in M\), if \(m \in M\) is an \(h_2\)-eigenvector, then \(\gamma_2^{-1}(m)\) is an \(h_1\)-eigenvector. So, let us consider the vector space \(\gamma_2^{-1}(M) = \tilde{f}^{-1}_2(M)\). Since \(m + qL = v \otimes u_2 + qL\), we have \(\tilde{f}^{-1}_2(m) + qL = \tilde{f}^{-1}_2(v) \otimes u_2 + qL\). Also, since \(m_2 + qL = \tilde{f}_1 \tilde{f}_2(v) \otimes u_0 + qL\), we have \(\tilde{f}^{-1}_2(m_2) + L = \tilde{f}^{-1}_2 \tilde{f}_1 \tilde{f}_2(v) \otimes u_0 + qL\). By identifying \(v + qL(a_1, a_2 ; b_1, b_2)\) with the semistandard double tableau above, it is easy to see that \(\tilde{f}^{-1}_2 \tilde{f}_1 \tilde{f}_2(v) + qL = \tilde{f}^{-1}_2 \tilde{f}_1 \tilde{f}_2(v) + qL\).

Indeed, \(\tilde{f}^{-1}_2(v) + qL(a_1, a_2 ; b_1, b_2)\) and \(\tilde{f}_1 \tilde{f}^{-1}_2(v) + qL(a_1, a_2 ; b_1, b_2)\) are identified with

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 2 & 2 \\
-2 & -2 & -2 & -2 & 2 & 2 & 2 \\
\end{pmatrix}
\]

respectively. Hence, \(\tilde{f}^{-1}_2(M)\) is spanned by an element in \(\tilde{f}^{-1}_2(v) \otimes u_2 + qL\) and an element in \(\tilde{f}_1 \tilde{f}^{-1}_2(v) \otimes u_0 + qL\). By the representation theory of \(U_1^\gamma\), this implies that \(\tilde{f}^{-1}_2(M)\) is spanned by an \(h_1\)-eigenvector in \(\tilde{f}^{-1}_2(v) \otimes u_2 + qL\) with eigenvalue \([b_1 + b_2]_1 \{a_1 + (a_2 - 1) - (b_1 + b_2) - 1\}\) and an \(h_1\)-eigenvector in \(\tilde{f}_1 \tilde{f}^{-1}_2(v) \otimes u_0 + qL\) with eigenvalue \([b_1 + b_2 - 1]_1 \{a_1 + (a_2 - 1) - (b_1 + b_2 - 1)\}\). Applying \(\gamma_2^{-1}\) on these vectors, we conclude that there exist an \(h_2\)-eigenvector \(v_2 \in M\) with eigenvalue \([b_1 + b_2]_1 \{a_1 + (a_2 - 1) - (b_1 + b_2) - 1\}\) such that \(v_2 \in v \otimes u_2 + qL\) and an \(h_2\)-eigenvector \(v_- \in M\) with eigenvalue \([b_1 + b_2 - 1]_1 \{a_1 + (a_2 - 1) - (b_1 + b_2 - 1)\}\) such that \(v_- \in \tilde{f}_1 \tilde{f}_2(v) \otimes u_0\).

This proves the proposition.

B.6. Proof of Proposition 9.2.1

We prove the assertion by induction on \(r \geq 2\). The case \(r = 2\) is already proved by Proposition 6.2.11. Let \(r \geq 3\) and assume that the assertion holds for \(r' < r\). Then, there exist \(v_s \in L\) satisfying the assertion for \(s \in S \setminus \{\pm r\}\). We set \(L(r - 1) := \bigoplus_{s \in S \setminus \{\pm r\}} U^\gamma v_s, L(r - 1) := L \cap L(r - 1),\) and \(B(r - 1) := B \cap (L(r - 1)/qL(r - 1))\). Let \(\lambda \in \Lambda_1\) denote the weight of \(v_s\), and set \(\mu := \lambda + 2\delta_1 - \sum_{n=1}^r \gamma_n\). It is straightforward to verify that \(B_\mu \cap B(r - 1) = B_\mu\) for all \(\mu > \mu\), and that

\[
B_\mu \cap B(r - 1) = \begin{cases}
B_\mu \setminus \{v \otimes u_r + qL, \tilde{f}_1 \cdots \tilde{f}_r(v) \otimes u_0 + qL\} & \text{if } \pm r \in S, \\
B_\mu \setminus \{v \otimes u_r + qL\} & \text{if } r \in S \text{ and } -r \notin S, \\
B_\mu \setminus \{\tilde{f}_1 \cdots \tilde{f}_r(v) \otimes u_0 + qL\} & \text{if } r \notin S \text{ and } -r \in S, \\
B_\mu & \text{if } r \notin S. 
\end{cases}
\]

This implies that the quotient module \(L/L(r - 1)\) has no weights greater than \(\mu\), and the weight space \((L/L(r - 1))_\mu\) is spanned by \(v \otimes u_r + L(r - 1)\) and \(\tilde{f}_1 \cdots \tilde{f}_r(v) \otimes u_0 + L(r - 1)\). By the representation theory of \(U_{r - 1}^\gamma\), the space \((L/L(r - 1))_\mu\) is spanned by at most two \(U_{r - 1}^\gamma\)-highest weight vectors. In the same way in the proof of Proposition 6.2.11, we see that there exist \(h_r\)-eigenvectors \(v_r' \in (L/L(r - 1))_\mu\) such that \(v_r' \in v \otimes u_r + q(L/L(r - 1))\) if \(r \in S\), and \(v_{-r} \in (L/L(r - 1))_\mu\) such that \(v_r' \in \tilde{f}_1 \cdots \tilde{f}_r \otimes u_0 + q(L/L(r - 1))\) if \(-r \in S\). Therefore, we obtain a \(U^\gamma\)-highest weight vector \(v_r \in L_\mu\) such that \(v_r + L(r - 1) = v_r'\) if \(r \in S\), and \(v_{-r} \in L_\mu\) such that \(v_{-r} + L(r - 1) = v_{-r}'\) if \(-r \in S\).
Next, we show that $v_{\pm r} \in \mathcal{L}$. If $v_r \notin \mathcal{L}$, then there exists $(k, l) \in (\mathbb{Z}_{>0} \times \mathbb{Z}) \cup (\{0\} \times \mathbb{Z}_{>0})$ such that $p^kq^l v_r \in \mathcal{L} \setminus \mathcal{L}$. Since $v_r + L(r - 1) \in v \otimes u_r + q\mathcal{A}_0 v \otimes u_r + q\mathcal{A}_0 f_1 \cdots f_r(v) \otimes u_r + L(r - 1)$, the vector $p^kq^l v_r + q\mathcal{L}$ is a linear combination of elements in $\mathcal{B}_q \setminus \{v \otimes u_r + q\mathcal{L}, f_1 \cdots f_r(v) \otimes u_0 + q\mathcal{L}\}$, and hence, $\tilde{c}_i(p^kq^l v_r + q\mathcal{L}) \neq 0$ for some $i \in \mathbb{P}$. However, since $p^kq^l v_r$ is a $U^q_l$-highest weight vector, we have $\tilde{c}_i(p^kq^l v_r + q\mathcal{L}) = 0$ for all $i \in \mathbb{P}$. This causes a contradiction. Thus, we obtain $v_r \in \mathcal{L}$. Similarly, we can prove that $v_{-r} \in \mathcal{L}$.

It remains to show that $U^q v_{\pm r} \simeq L(\alpha^{2r}; \beta^{2r})$. This is done by determining the eigenvalues of $v_{\pm r}$ for $k_i$ and $h_i, i \in \mathbb{P}$. To do this, we identify $v + q\mathcal{L}$ with the semistandard double Young tableau $(T_{\alpha'}; T_{\beta'})$, where $\alpha' = (\alpha_1, \ldots, \alpha_r)$ and $\beta' = (\beta_1, \ldots, \beta_{r-1})$. By the representation theory of $U^q_{-1}$, the $h_i$-eigenvalues of $v_{\pm r}$ for $i \in \mathbb{P} \setminus \{r\}$ are determined by regarding $v_{\pm r} + q\mathcal{L}$ as an element of $B^N_{q-1}$ and then ignoring all the $\pm r$'s. For example, since $v_r + q\mathcal{L} = v \otimes u_r + q\mathcal{L} = (EM(T_{\alpha'}), ME(T_{\beta'}), r)$, the $k_r$- and $h_i$-eigenvalues of $v_r, i \neq r$, are the same as those of $v$. Similarly, $s_r(v_{\pm r} + q\mathcal{L})$ determines the $k_r$- and $h_i$-eigenvalues. This proves the proposition.

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