SOBOLEV EMBEDDING IMPLIES REGULARITY OF MEASURE IN METRIC MEASURE SPACES

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ABSTRACT. We prove that if the Sobolev embedding $M_1^{1,p}(X) \hookrightarrow L^q(X)$ holds for some $q > p \geq 1$ in a metric measure space $(X, d, \mu)$, then a constant $C$ exists such that $\mu(B(x,r)) \geq Cr^n$ for all $x \in X$ and all $0 < r \leq 1$, where $\frac{1}{p} - \frac{1}{q} = \frac{1}{n}$. This was proved in [3] assuming a doubling condition on the measure $\mu$.

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1. Introduction

It is well-known that, for an an open set $\Omega \subset \mathbb{R}^n$ and $1 \leq p < n$, the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ holds, where $p^* = \frac{np}{n-p}$, if the boundary of $\Omega$ is sufficiently regular, see e.g. [1]. On the other hand, it has been proved in [5] that if the embedding $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ holds, then $\Omega$ satisfies the so-called measure density condition, i.e. there exists a constant $c > 0$ such that for all $x \in X$ and all $0 < r \leq 1$

\begin{equation}
|B(x,r) \cap \Omega| \geq cr^n.
\end{equation}

Let $(X, d, \mu)$ be a metric measure space equipped with a metric $d$ and a Borel regular measure $\mu$. We assume throughout the note that the measure of every open nonempty set is positive and that the measure of every bounded set is finite. In a metric measure space $(X, d, \mu)$, Hajłasz [4] has shown that if the space $X$ is $n$-regular, then the embedding $M_1^{1,p}(X) \hookrightarrow L^{p^*}(X)$ holds, where $p^* = \frac{np}{n-p}$. Recall that a space $(X, d, \mu)$ is $n$-regular if there exists a constant $C$ such that

\begin{equation}
\mu(B(x,r)) \geq Cr^n
\end{equation}

for all $B(x,r) \subset X$ with $r < \text{diam} X$. Also recall that a $p$-integrable function $u$ belongs to the Hajłasz-Sobolev space $M_1^{1,p}(X)$ if there exists a non-negative $g \in L^p(X)$, called a generalized gradient, such that

$$|u(x) - u(y)| \leq d(x,y)(g(x) + g(y)) \quad \text{a.e. for } x, y \in X.$$
The space $M^{1,p}(X)$ is a Banach space with the norm
\[ \|u\|_{M^{1,p}(X)} = \|u\|_{L^p(X)} + \inf \|g\|_{L^p(X)}, \]
where the infimum is taken over all the generalized gradients.

In [3], it has been proved that if the embedding $M^{1,p}(X) \hookrightarrow L^q(X)$ holds for some $q > p$, then the measure $\mu$ satisfies (1.2) for all $x \in X$ and all $0 < r \leq 1$ provided that the space $(X, d, \mu)$ is doubling, i.e. there exists a constant $c_d$ such that for every ball $B(x, r)$,
\[ \mu(B(x, 2r)) \leq c_d \mu(B(x, r)). \]

In this note, we prove the same result but without assuming the doubling condition, as conjectured in [3] and the proof of the same is inspired by [2] and [6].

**Theorem 1.1.** Let $(X, d, \mu)$ be a metric measure space and $p \geq 1$. If $M^{1,p}(X) \hookrightarrow L^q(X)$, $q > p$, then there exists $C = C(p, q, C_e)$ such that
\[ \mu(B(x, r)) \geq Cr^n, \quad \text{for} \quad r \in (0, 1], \]
where \( \frac{1}{p} - \frac{1}{q} = \frac{1}{n} \) and $C_e$ is the constant of the embedding.

2. **Proof of Theorem 1.1**

For each $u \in M^{1,p}(X)$ and for any generalized gradient $g$ of $u$ we have, by the Sobolev embedding,
\[ \left( \int_X |u|^q \, d\mu \right)^{\frac{1}{q}} \leq C_e \left[ \left( \int_X |u|^p \, d\mu \right)^{\frac{1}{p}} + \left( \int_X g^p \, d\mu \right)^{\frac{1}{p}} \right]. \]

Fix $x \in X$ and $r \in (0, 1]$. For each fixed $j \in \mathbb{N}$, set $r_j = (2^{-j} + 2^{-1})r$, and $B_j = B(x, r_j)$. Note that, for all $j \in \mathbb{N}$,
\[ \frac{r}{4} < r_{j+1} < r_j \leq \frac{3r}{4}. \]

For each $j \in \mathbb{N}$, let us define $u_j : X \to \mathbb{R}$ as follows:
\[ u_j(y) = \begin{cases} 
1 & \text{if } y \in B_{j+1}, \\
\frac{r_j - d(x, y)}{r_j - r_{j+1}} & \text{if } y \in B_j \setminus B_{j+1}, \\
0 & \text{if } y \in X \setminus B_j.
\end{cases} \]

It is easy to see that, for each $j \in \mathbb{N}$, $u_j$ is a $(r_j - r_{j+1})^{-1}$-Lipschitz function on $X$ and the function $g_j := (r_j - r_{j+1})^{-1} \chi_{B_j}$ is a generalized gradient of $u_j$. In particular, $u_j \in M^{1,p}(X)$ and
hence the functions \( u_j \) and \( g_j \) satisfy (2.1). Noting that \((r_j - r_{j+1})^{-1} = 2^{j+2}r^{-1}\) we have, for each \( j \in \mathbb{N} \),

\[
\int_X g_j^p \, d\mu = \frac{2^{p(j+2)}}{r^p} \mu(B_j) \quad \text{and} \quad \int_X |u_j|^p \, d\mu \leq \mu(B_j).
\]

Moreover, for each \( j \in \mathbb{N} \),

\[
\int_X |u_j|^p \, d\mu \geq \mu(B_{j+1}).
\]

Use these estimates while applying (2.1) for the pair \((u_j, g_j)\), for every \( j \in \mathbb{N} \), to obtain

\[
\mu(B_{j+1})^{1/q} \leq C_e \left(1 + \frac{2^{j+2}}{r}\right) \mu(B_j)^{1/p}
\]

\[
\leq \frac{C_e}{r} 2^{j+3} \mu(B_j)^{1/p},
\]

where in the last inequality we have used the fact that \( r \leq 1 \). Raising both sides of the inequality (2.2) to the power \( p/\alpha^j-1 \), where \( \alpha = q/p \in (1, \infty) \), yields

\[
\mu(B_{j+1})^{1/\alpha^j} \leq \left(\frac{C_e}{r}\right)^{p/\alpha^j-1} 2^{p(j+3)/\alpha^j-1} \mu(B_j)^{1/\alpha^j-1}.
\]

Letting \( P_j = \mu(B_j)^{1/\alpha^j-1} \), we rewrite the above inequality as

\[
P_{j+1} \leq \left(\frac{C_e}{r}\right)^{p/\alpha^j-1} 2^{p(j+3)/\alpha^j-1} P_j \quad \forall j \in \mathbb{N}.
\]

After iteration, we obtain, for every \( j \in \mathbb{N} \),

\[
P_{j+1} \leq P_1 \prod_{k=1}^{j} 2^{p(k+3)/\alpha^k-1} \left(\frac{C_e}{r}\right)^{p/\alpha^k-1}.
\]

Observe that

\[
\prod_{k=1}^{\infty} \left(\frac{C_e}{r}\right)^{p/\alpha^k-1} = \left(\frac{C_e}{r}\right)^{p \sum_{k=1}^{\infty} \alpha^{-k}} = \left(\frac{C_e}{r}\right)^{\frac{p\alpha}{\alpha-1}}
\]

and

\[
\prod_{k=1}^{\infty} 2^{p(k+3)/\alpha^k-1} = 2^p \sum_{k=1}^{\infty} (k+3)\alpha^{-k} = 2^{\frac{p\alpha^2}{(\alpha-1)^2} + \frac{3p\alpha}{\alpha-1}}.
\]

On the other hand, from the construction of \( B_j \)'s, we have

\[
\mu(B(x, r/2))^{1/\alpha^j-1} \leq P_j = \mu(B_j)^{1/\alpha^j-1} \leq \mu(B(x, r))^{1/\alpha^j-1}
\]

and therefore \( \lim_{j \to \infty} P_j = 1 \). Consequently, passing to the limit in (2.3) and using \( P_1 \leq \mu(B(x, r)) \), we obtain

\[
1 \leq 2^{\frac{p\alpha^2}{(\alpha-1)^2} + \frac{3p\alpha}{\alpha-1}} \left(\frac{C_e}{r}\right)^{\frac{p\alpha}{\alpha-1}} \mu(B(x, r)).
\]

Therefore

\[
\mu(B(x, r)) \geq Cr^{\frac{p\alpha}{3}},
\]
where
\[ \frac{1}{C} = 2^{\frac{n^2}{2(\alpha-1)^2}} \left( \frac{\alpha}{\alpha-1} \right)^{\frac{n\alpha}{\alpha-1}}. \]

Finally, we use \( q = np/(n - p) \) and \( \alpha = q/p \) to get the desired result.

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