Every finite complex is the classifying space for proper bundles of a virtual Poincaré duality group.*

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Abstract

We prove that every finite connected simplicial complex is homotopy equivalent to the quotient of a contractible manifold by proper actions of a virtually torsion-free group. As a corollary, we obtain that every finite connected simplicial complex is homotopy equivalent to the classifying space for proper bundles of some virtual Poincaré duality group.

1 Introduction

Let $G$ be a discrete group. A $G$-CW-complex is, by definition, a CW-complex on which $G$ acts by permuting the cells and cell stabilizers act trivially on cells. A $G$-CW-complex $Y$ is said to be a model for $EG$ if every cell stabilizer is finite and, for every finite subgroup $H \leq G$, the fixed set $Y^H$ is contractible. The existence of such a model can be established by Milnor’s or Segal’s argument for the construction of the universal space for $G$. (See [12] for the general construction). Applying an equivariant obstruction theory proves that any two models for $EG$ are $G$-homotopy equivalent. We call $EG$ the classifying space for proper $G$-actions. We write the quotient $EG/G$ by $BG$ and call it the classifying space for proper $G$-bundles. Our main theorem can be stated as follows.

**Theorem 1** For any finite connected simplicial complex $X$, there exists a virtually torsion-free group $G$ with $EG$ a cocompact manifold such that $BG$ is homotopy equivalent to $X$.

A group $\Gamma$ is called a Poincaré duality group of dimension $n$ if $H^i(\Gamma; A) \cong H_{n-i}(\Gamma; A)$ for any $\mathbb{Z}\Gamma$-module $A$. Many interesting examples of Poincaré duality groups are manifold groups. More specifically, the fundamental group of a closed, aspherical $n$-dimensional manifold is a Poincaré duality group of dimension $n$. (The converse is false.) See [2], [4] for details about Poincaré duality groups. Finally, recall that a group virtually has some property if a finite index subgroup has the property.

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Let $T$ be a torsion-free finite index subgroup of $G$ in Theorem 1. Then one can take $EG$ as a model for $ET = ET$, where $ET$ is the universal space for $T$. Since $ET/T$ is a closed aspherical manifold, $T$ is a Poincaré duality group.

**Corollary 2** For any finite connected simplicial complex $X$, there is a virtual Poincaré duality group $G$ such that $BG$ is homotopy equivalent to $X$.

The statement of Corollary 2 is related to the theorem of Kan-Thurston, which says that every connected complex has the same homology as the classifying space for some group (See [9]). This theorem has been extended and generalized by a number of authors. For example, see [1], [13], [5], [11], [14], [12], [10]. Among many extensions and generalizations, Leary and Nucinkis proved in [12] that every connected CW-complex has the same homotopy type as the classifying space for proper bundles of some group. Corollary 2 says that the group can be taken as a virtual Poincaré duality group if the simplicial complex is finite.

The proof of Theorem 1 consists of three steps. In Section 2, we outline the embedding trick, due to Floyd [7], for equivariantly embedding a simplicial complex with an involution into some Euclidean space. In Section 3, we review the equivariant reflection group trick, due to Davis [6]. Finally, in Section 4, we use the two tricks to construct a contractible manifold, whose quotient by some group $G$ is homotopy equivalent to a given finite simplicial complex $X$. We also prove that the contractible manifold is the classifying space for proper $G$-actions and introduce the torsion-free subgroup of finite index to complete the proof of Theorem 1.

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### 2 The Equivariant Embedding Trick

Let $Z$ be a finite simplicial complex with a simplicial map of period $p$. In [7], Floyd introduced the embedding trick, namely, $Z$ can be embedded in some Euclidean space such that the restriction of specific coordinate changing map on $Z$ is the given simplicial map. We outline his construction in the case that $p = 2$. See [7] Section 2 for the full construction.

Let $L$ be a finite connected simplicial complex with a simplicial involution $T$. Embed $L$ into $\mathbb{R}^n$ for some $n$ and suppose that $\mathbb{R}^n$ is triangulated so that $L$ is a subcomplex. Consider the following map.

$$\phi : L \to \mathbb{R}^n \times \mathbb{R}^n (= \mathbb{R}^{2n}), \quad x \mapsto (x, T(x)).$$

Note that a cell in the cellular decomposition of $\mathbb{R}^{2n}$ has the type of $s_1 \times s_2$, where each $s_i$ is a simplex in $\mathbb{R}^n$. We use the first barycentric subdivision of this cellular decomposition for the subdivision of $\mathbb{R}^{2n}$.

By passing to the barycentric subdivision $\text{Sd}(L)$ of $L$, we obtain that $\phi$ is a simplicial homeomorphism of $\text{Sd}(L)$ onto a subcomplex of $\mathbb{R}^{2n}$. Furthermore, the
map $S : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ defined by $S(x, y) = (y, x)$ is simplicial and satisfies $S \circ \phi = \phi \circ T$, hence $\phi$ is an equivalence between $(\operatorname{Sd}(L), T)$ and $(\phi(\operatorname{Sd}(L)), S)$. Hereafter, we suppose that $L$ is a subcomplex of $\mathbb{R}^{2n}$ and $S = T$ on $L$.

We may as well assume that $L$ is a full subcomplex of $\mathbb{R}^{2n}$. For if not, $\operatorname{Sd}(L)$ is a full subcomplex of $\operatorname{Sd}(\mathbb{R}^{2n})$. Let $U$ be the first regular neighborhood of $L$, i.e. the union of all open stars of vertices of $\operatorname{Sd}(L)$ relative to $\operatorname{Sd}(\mathbb{R}^{2n})$. Then $K = \overline{U}$ is a manifold with boundary of dimension $2n$. Denote the boundary of $K$ by $\partial K$.

Let $v_0, \cdots, v_k, v_{k+1}, \cdots$ be vertices of $\mathbb{R}^{2n}$, where $v_0, \cdots, v_k$ are vertices of $L$. Every point $x$ in $\mathbb{R}^{2n}$ has a unique barycentric representation $\sum t_i v_i$. Furthermore, $K$ consists of points $x$ for which $\max(t_0, \cdots, t_k) \geq \max(t_{k+1}, \cdots)$ and $\partial K$ consists of points $x$ for which $\max(t_0, \cdots, t_k) = \max(t_{k+1}, \cdots)$. Consider $f : K \to L$ defined by $f(x) = f(\sum t_i v_i) = \sum_{i=0}^k t_i v_i$. Then $\Phi : K \times I \to K$ defined by $\Phi(x, t) = (1-t)x + tf(x)$ is a deformation retract of $K$ onto $L$.

**Remark 3** By the unique barycentric representation of points in $\mathbb{R}^{2n}$, $\Phi(S(x), t) = S(\Phi(x), t)$ for any $x \in K$ and $t \in I$ so that $\Phi$ is $S$-equivariant. Therefore, $K$ equivariantly deformation retracts onto $L$. In particular, the fixed set $K^S$ is homotopy equivalent to the fixed set $L^T$ and $K/(\langle S \rangle)$ is homotopy equivalent to $L/(\langle T \rangle)$.

### 3 The Equivariant Reflection Group Trick

Suppose that we are given a space $M$ and a subspace $N \subset M$ such that $N$ is triangulated as a finite dimensional flag complex. Recall that a simplicial complex is a flag complex if any finite set of vertices, which are pairwise connected by edges, spans a simplex. Let a discrete group $G$ act on $M$ so that $G$ stabilizes the subspace $N$ and $G$ acts on $N$ by simplicial automorphisms. Following [6], we will associate a right-angled Coxeter group $W$ and construct a $(W \rtimes G)$-action on a space $\mathcal{U}(M, N, G)$.

Let $I$ be a vertex set of $N$. Define a right-angled Coxeter group $W$ as follows. There is one generator $s_i$ for each $i \in I$. Relations are given by $s_i^2 = 1$ and $(s_i s_j)^2 = 1$ if $\{i, j\}$ spans an edge in $N$. For $x \in N$, let $\sigma(x) = \{i \in I | x \in N_i\}$, where $N_i$ is the closed star of the vertex $i$ in the barycentric subdivision of $N$ and $W_x$ be the subgroup generated by $\{s_i | i \in \sigma(x)\}$.

Note that $G$ acts on $N$ by permuting vertices, so we can form $W \rtimes G$.

Define the space $\mathcal{U}(M, N, G)$ by

$$\mathcal{U}(M, N, G) = W \rtimes M/\sim,$$

where $(w, x) \sim (w', x')$ if $x = x'$ and $w^{-1}w' \in W_x$. For $[w, x] \in \mathcal{U}(M, N, G)$ and $(v, g) \in W \rtimes G$, the action of $W \rtimes G$ on $\mathcal{U}(M, N, G)$ is defined by

$$(v, g)[w, x] = [vw^g, g.x].$$

**Remark 4** This construction enjoys the following properties. For details or proofs, see [6] Section 11.7, [3].
1. If $M$ is contractible, then so is $\mathcal{U}(M, N, G)$.

2. If $M$ is an $n$-dimensional manifold with boundary and $N = \partial M$, then $\mathcal{U}(M, N, G)$ is an $n$-dimensional manifold.

3. Let $C(N)$ be the cone on $N$. Then $\mathcal{U}(C(N), N, G)$ has a natural CAT(0) cubical structure so that the link of each vertex is isomorphic to $N$, and so that $W \rtimes G$ acts by a group of isometries. In particular, for any finite subgroup $F$ of $W \rtimes G$, the fixed point set $\mathcal{U}(C(N), N, G)^F$ is contractible.

A group action on a simplicial complex is said to be admissible if, for any simplex, setwise stabilizers are equal to pointwise stabilizers. If the $G$-action on $N$ is admissible, we have the following.

**Lemma 5** Let $H$ be a finite subgroup in $G$. Then

$$\mathcal{U}(M, N, G)^H = \mathcal{U}(M^H, N^H, V_H),$$

where $M^H$, and $N^H$ respectively, is the $H$-fixed set in $M$, and $N$ respectively, and $V_H = N_G(H)/H$.

**Remark 6** The above lemma is stated in [5, Proposition 11.7.1] without a proof. We provide the proof below. Note also that $N^H$ is a flag complex.

**Proof.** It is obvious that $\mathcal{U}(M^H, N^H, V_H)$ is a subspace of $\mathcal{U}(M, N, G)$.

Let $[w, m] \in \mathcal{U}(M, N, G)^H$ be given. In order to prove that $\mathcal{U}(M, N, G)^H$ is contained in $\mathcal{U}(M^H, N^H, V_H)$, it suffices to show that $m \in M^H$ and $w \in W_H$, where $W_H$ is the subgroup of $W$ generated by $\{s_i| i \text{ is a vertex in } N^H\}$. For any $h \in H$,

$$(1, h).[w, m] = [w^h, h.m] = [w, m]$$

$$\Rightarrow \quad h.m = m, \quad w^{-1}w^h \in W_m.$$

Since $h.m = m$ for all $h \in H$, it follows that $m \in M^H$. We prove that $w \in W_H$ by induction on the length $l(w)$ of $w$. To begin with, we point out that since the $G$-action on $M$ is admissible, every generator in $W_m$ is fixed by $H$. In particular, $W_m$ is a subgroup in $W_H$. Also note that $W_m$ is finite. (Consider the simplex containing $m$ of minimal dimension.)

Suppose that $l(w) = 1$, i.e. $w = w^{-1}$. Since $W_m$ is finite, $ww^h$ has finite order. But this happens only if two vertices corresponding to $w$ and $w^h$ are connected. Admissibility implies that $w$ is fixed by $h$, and hence, $w \in W_H$.

Suppose that $w = s_1 \cdots s_n$ is a reduced word (of length $n$). Let $t_i = s_i^h$.

Suppose $s_1 \neq t_1$. Again, $w^{-1}w^h$ has finite order.

$$(w^{-1}w^h)^n = 1 \text{ for some } n$$

$$\Rightarrow \quad (s_n \cdots s_1 \cdot t_1 \cdots t_n)(s_n \cdots s_1 \cdot t_1 \cdots t_n) \cdots (s_n \cdots s_1 \cdot t_1 \cdots t_n) = 1$$

In order for the left hand side to be reduced to the identity, in particular, there exists $t_i$ for $2 \leq i \leq n$ such that $t_i$ commutes with $t_1$ and cancels with $s_1$, i.e.
\[ t_i = s_1. \] But this is impossible, because \( s_1 \) and \( t_1 \) do not commute. Therefore, \( s_1 = t_1. \)

\[ w^{-1}w^h = s_n \cdots s_1 \cdot t_1 \cdots t_n = s_n \cdots s_2 \cdot t_2 \cdots t_n \in W_m \]

By the induction hypothesis, \( s_1w \in W_H \), so \( w \in W_H. \)

### 4 The proof of Theorem 1

The proof of Theorem 1 consists of three steps. First, we use the equivariant embedding trick to embed a given finite simplicial complex \( X \) into the manifold \( M \) with an involution \( \tau \) such that \( M/\langle \tau \rangle \) is homotopy equivalent to \( X \). Then we apply the equivariant reflection group trick on \( M \) with boundary to obtain a contractible manifold on which some group \( G \) acts. The quotient of the contractible manifold by \( G \) will be homotopy equivalent to \( X \). Finally, we show that the contractible manifold is the classifying space for proper \( G \)-actions. Additionally, we introduce a finite index torsion-free subgroup of \( G \), which proves that \( G \) is a virtual Poincaré duality group.

Let \( X \) be a finite connected simplicial complex. Note that the equivariant embedding trick requires a simplicial complex with a periodic simplicial map. In this paper, we use the construction appearing in [11]. Applying [11] Theorem A, we obtain a finite connected locally CAT(0) cubical complex \( Y \) with a cubical involution \( \tau \) such that \( Y/\langle \tau \rangle \) is homotopy equivalent to \( X \). By passing to the barycentric subdivision, we may assume that \( Y \) is a finite connected simplicial complex and \( \tau \) is a simplicial involution on \( Y \). Now we apply the equivariant embedding trick introduced in Section 2 to obtain a manifold \( M \) with a boundary \( N \) and a simplicial involution \( \omega \) on \( M \) such that \( M \) equivariantly deformation retracts onto \( Y \). By passing to the barycentric subdivision, we can assume that \( N \) is a flag complex and a cyclic group of order two, \( C_2 = \langle \omega \rangle \), acts on \( N \) admissibly. Note that \( Y \) is locally CAT(0). Therefore, \( M \) is aspherical and \( M^\omega \) is homotopy equivalent to \( Y^\tau \).

Next we apply the equivariant reflection group trick from Section 3 to obtain a space \( \mathcal{U}(M, N, C_2) \) on which \( W \rtimes C_2 \) acts, where \( W \) is a right-angled Coxeter group associated to \( N \). Let \( \tilde{M} \) be the universal cover of \( M \), \( \tilde{N} \) be the inverse image of \( N \) in \( M \) and \( \tilde{W} \) be the associated right-angled Coxeter group to \( \tilde{N} \). Repeat the equivariant reflection group trick to obtain a space \( \mathcal{U}(\tilde{M}, \tilde{N}, \Gamma) \) on which \( \tilde{W} \rtimes \Gamma \) acts, where \( \Gamma \) is the group of liftings of the \( C_2 \)-action on \( M \) to \( \tilde{M} \). Note that every torsion element in \( \Gamma \) has order at most two and every finite subgroup of \( \Gamma \) is cyclic of order two.

**Proposition 7** Let \( \mathcal{U}(M, N, C_2) \) and \( \mathcal{U}(\tilde{M}, \tilde{N}, \Gamma) \) be the spaces constructed above. Then

1. \( \mathcal{U}(\tilde{M}, \tilde{N}, \Gamma) \) and \( \mathcal{U}(M, N, C_2) \) are manifolds.
2. \( \mathcal{U}(\tilde{M}, \tilde{N}, \Gamma) \) is the universal cover of \( \mathcal{U}(M, N, C_2) \).
3. $\mathcal{U}(\tilde{M}, \tilde{N}, \Gamma)/(\tilde{W} \times \Gamma)$ is homotopy equivalent to $X$.

Proof. The first statement follows from the fact that $M$ and $\tilde{M}$ are manifolds with boundary. It is clear that $\mathcal{U}(M, N, \Gamma)$ is a cover of $\mathcal{U}(M, N, C_2)$. Since $M$ is aspherical, $\tilde{M}$ is contractible and so is $\mathcal{U}(\tilde{M}, \tilde{N}, \Gamma)$. This proves the second statement. By construction,

$$\mathcal{U}(\tilde{M}, \tilde{N}, \Gamma)/(\tilde{W} \times \Gamma) \simeq \mathcal{U}(M, N, C_2)/(W \times C_2) \simeq M/C_2 \simeq X,$$

where $\simeq$ is a homotopy equivalence.

We compare the $(\tilde{W} \times \Gamma)$-action on the space $\mathcal{U}(\tilde{M}, \tilde{N}, \Gamma)$ with the same action on $\mathcal{U}(C(\tilde{N}), \tilde{N}, \Gamma)$, and prove $\mathcal{U}(\tilde{M}, \tilde{N}, \Gamma) = E(\tilde{W} \times \Gamma)$. Denote the image of $\{\tilde{w}\} \times \tilde{M}$ in $\mathcal{U}(\tilde{M}, \tilde{N}, \Gamma)$ by $\tilde{w}\tilde{M}$ and the image of $\{\tilde{w}\} \times (\tilde{M} \setminus \tilde{N})$ by $int(\tilde{w}\tilde{M})$.

First, we prove that all stabilizers are finite.

**Proposition 8** Let $H$ be a subgroup of $\tilde{W} \times \Gamma$ fixing some point in $\mathcal{U}(\tilde{M}, \tilde{N}, \Gamma)$. Then $H$ is finite.

Proof. Suppose that $H$ fixes some point in $int(\tilde{w}\tilde{M})$ for some $\tilde{w}$.

Then $H' = (\tilde{w}, 1)^{-1}H(\tilde{w}, 1)$ fixes some point in $int(\tilde{M})$. Denote this point by $[1, x]$. For any $(\tilde{v}, \gamma) \in H'$,

$$(\tilde{v}, \gamma)[1, x] = [\tilde{v}, \gamma.x] = [1, x] \Rightarrow \gamma.x = x, \quad \tilde{v} \in \tilde{W}_x$$

Since $x \in \tilde{M} \setminus \tilde{N}$ (recall that $\tilde{N} = \partial \tilde{M}$, and $x \in int(\tilde{M})$), $\tilde{W}_x$ is trivial. It follows that $H'$ is a subgroup of $\Gamma$. Recall that $\Gamma$ is the group of liftings of the $C_2$-action on $M$ to $\tilde{M}$. Therefore, if a nontrivial element $\gamma$ fixes some point $x$, $\gamma$ is the only nontrivial element in $\Gamma$ fixing $x$. This tells us that $H'$ must be finite of order 2.

Suppose that the fixed point is not contained in $int(\tilde{w}\tilde{M})$ for any $\tilde{w}$. As in the previous case, choose some $\tilde{w}'$ so that $H'' = (\tilde{w}', 1)^{-1}H(\tilde{w}', 1)$ fixes some point in the image of $\tilde{N}$ in $\mathcal{U}(M, N, \Gamma)$. Denote such a point by $[1, y]$. For any $(\tilde{v'}, \gamma') \in H''$,

$$(\tilde{v'}, \gamma')[1, y] = [\tilde{v'}, \gamma'.y] = [1, y] \Rightarrow \gamma'.y = y, \tilde{v}' \in \tilde{W}_y$$

As before, we have at most two possibilities for $\gamma'$. Furthermore, $\tilde{W}_y$ is finite. (Consider the simplex containing $y$ of minimal dimension.) Therefore, $H''$ is finite, and hence, $H$ is finite.

**Theorem 9** $\mathcal{U}(\tilde{M}, \tilde{N}, \Gamma) = E(\tilde{W} \times \Gamma)$.

Proof. It suffices to prove that the fixed point set by a finite subgroup is contractible. As mentioned before, we consider the $(\tilde{W} \times \Gamma)$-action on $\mathcal{U}(C(\tilde{N}), \tilde{N}, \Gamma)$.

Let $F$ be a finite subgroup of $\tilde{W} \times \Gamma$. Recall that $\mathcal{U}(C(\tilde{N}), \tilde{N}, \Gamma)^F$ is contractible. (See Remark 4.) In particular, it is nonempty.
Suppose that $F$ does not fix any cone point in $\mathcal{U}(C(\tilde{N}), \tilde{N}, \Gamma)$. Then $F$ fixes no point in $\text{int}(\tilde{w}\tilde{M})$ for any $\tilde{w}$. In other words,

$$\mathcal{U}(\tilde{M}, \tilde{N}, \Gamma)^F \subset \mathcal{U}(\tilde{M}, \tilde{N}, \Gamma) \setminus \bigcup_{\tilde{w} \in \tilde{W}} \text{int}(\tilde{w}\tilde{M}).$$

Therefore,

$$\mathcal{U}(\tilde{M}, \tilde{N}, \Gamma)^F = \mathcal{U}(C(\tilde{N}), \tilde{N}, \Gamma)^F$$

and $\mathcal{U}(\tilde{M}, \tilde{N}, \Gamma)^F$ is contractible.

Suppose that $F$ fixes some cone point in $\mathcal{U}(C(\tilde{N}), \tilde{N}, \Gamma)$. Choose some $\tilde{w}''$ so that $F' = (\tilde{w}'', 1)^{-1}F(\tilde{w}'', 1)$ fixes the cone point of $\text{int}(C\tilde{N})$. Denote the cone point by $c$. For any $(\tilde{v}'', \gamma'') \in F'$,

$$(\tilde{v}'', \gamma''), [1, c] = [\tilde{v}'', \gamma''.c] = [1, c] \Rightarrow \gamma''.c = c, \quad \tilde{v}'' \in \tilde{W}_c.$$  

Since $c$ is a cone point, $\tilde{v}''$ is trivial. It follows that $F'$ is a finite subgroup of $\Gamma$, and hence, cyclic of order 2. By Lemma 5, it follows that $\mathcal{U}(\tilde{M}, \tilde{N}, \Gamma)^{F'}$ is $\mathcal{U}(\tilde{M}^{F'}, \tilde{N}^{F'}, V_{F'})$. Recall $M$ is equivariantly homotopy equivalent to a locally CAT(0) space $Y$. Therefore, $\tilde{M}^{F'}$ is homotopy equivalent to the fixed set of $CAT(0)$ space by a cyclic group of order 2. It follows that $\tilde{M}^{F'}$ is contractible, hence so is $\mathcal{U}(\tilde{M}^{F'}, \tilde{N}^{F'}, V_{F'})$. Finally, $\mathcal{U}(\tilde{M}, \tilde{N}, \Gamma)^F = (\tilde{w}'', 1)\mathcal{U}(\tilde{M}, \tilde{N}, \Gamma)^{F''}$ is contractible.

\begin{remark}
Consider the commutator subgroup $T$ of $W$ and its inverse image $\tilde{T}$ in $\tilde{W}$. Since $\tilde{T}$ is torsion-free and of finite index in $\tilde{W}$, $\tilde{T} \rtimes \pi_1(M)$ is a torsion-free finite index subgroup of $\tilde{W} \rtimes \Gamma$. Since $\mathcal{U}(\tilde{M}, \tilde{N}, \Gamma)/(\tilde{T} \rtimes \pi_1(M))$ is an aspherical closed manifold, $\tilde{T} \rtimes \pi_1(M)$ is a Poincaré duality group. This verifies that $\tilde{W} \rtimes \Gamma$ is a virtual Poincaré duality group.
\end{remark}

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