Mass of the bottom quark from Upsilon(1S) at NNNLO: an update

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Abstract. We update our perturbative determination of MS mass \( m_b^{\text{MS}} \), by including the recently obtained four-loop coefficient in the relation between the pole and MS mass. First the renormalon subtracted (RS or RS') mass is determined from the known mass of the \( \Upsilon(1S) \) meson, where we use the renormalon residue \( N_m \) obtained from the asymptotic behavior of the coefficient of the 3-loop static singlet potential. MS mass is then obtained using the 4-loop renormalon-free relation between the RS (RS') and MS mass. We argue that the effects of the charm quark mass are accounted for by effectively using \( N_f = 3 \) in the mass relations.

The extracted value is \( m_b^{\text{MS}} \) = 4222(40) MeV, where the uncertainty is dominated by the renormalization scale dependence.

1. Introduction
The (MS) mass of the bottom (b) quark, \( m_b \equiv m_b^{\text{MS}} \), is an important quantity in particle physics, free of renormalon ambiguities, and appears in many physical observables. Since it is relatively high, \( \sim 4 \) GeV, perturbative QCD methods are suitable for its extraction. The mass of the ground state of the \( b \bar{b} \) quarkonium, \( \Upsilon(1S) \), is one of the best quantities for such an extraction, \( M_{\Upsilon(1S)}^{\text{th}} = 2m_b + E_{\Upsilon(1S)} = 9.460 \) GeV, where \( m_b \) is the pole mass of the bottom quark, and \( E_{\Upsilon(1S)} \) is the binding energy. We use the available perturbative expansions of \( 2m_b/m_b \) and of \( E_{\Upsilon(1S)}/m_b \) in powers of QCD coupling \( a(\mu) \equiv \alpha_s(\mu)/\pi \) and thus extract the value of \( m_b^{\text{MS}} \).

In the extraction, we use the fact that the leading infrared (IR) renormalon ambiguity of \( 2m_b \) cancels out with that of \( E_{\Upsilon(1S)} \) [1, 2, 3]. These proceedings are a brief review of our previous work [4], which we update by including in the analysis the recently calculated [5] four-loop coefficient of the relation between the pole mass and the MS mass. Here we outline: (1) The correct treatment of charm quark mass effects in the perturbation expansion of \( m_b/m_b \); (2) Asymptotic expressions for the coefficients in the perturbation expansion of the ratio \( m_b/m_b \) and of the static singlet potential \( V(r) \), and the extraction of the renormalon residue \( N_m \); (3) The construction of the (modified) renormalon-subtracted mass \( m_{b,\text{RS}'} \) (using \( N_m \)), and the renormalon-free relation between \( m_{b,\text{RS}'} \), and \( m_b^{\text{MS}} \); (4) Renormalon-free perturbation expansion for \( M_{\Upsilon(1S)}^{\text{th}} \) in terms of \( m_{b,\text{RS}'} \), and extraction, from \( M_{\Upsilon(1S)}^{\text{th}} = 9.460 \) GeV, of the values of \( m_{b,\text{RS}'} \) (⇒ \( m_b^{\text{MS}} \)).
2. Charm mass effects in the bottom pole mass

The pole mass $m_b$ and the ${\overline{\text{MS}}}$ mass $\overline{m}_b$ are related:

$$m_b = \overline{m}_b (1 + S(N_f)) + \delta m_c ,$$  \hspace{1cm} (1)

where

$$S(N_f) = \frac{4}{3} a_+^{(0)} \left[ 1 + r_1^{(0)}(\mu) a_+^{(0)}(\mu) + r_2^{(0)}(\mu) a_+^2(\mu) + r_3^{(0)}(\mu) a_+^3(\mu) + \mathcal{O}(a_+^4) \right]$$  \hspace{1cm} (2)

and the evaluation is usually performed in QCD with $N_f = N_l + 1 = 4$ active flavors: $r_j^{(0)}(\mu) \equiv r_j(\mu; N_f)$, $a_+^{(0)}(\mu) = a(\mu; N_f)$. The coefficients $R_0 = 4/3$ and $r_j (j = 1, 2)$ were obtained in Refs. [6], [7], [8, 9], respectively. Recently, numerical values of the 4-loop coefficient $r_3$ were obtained [5], and we incorporate them here in the form given in [10].

These coefficients have a specific dependence on the renormalization scale $\mu$, dictated by $\mu$-independence of $S(N_f)$.

$$r_1(\mu; N_f) = r_1(N_f) + \beta_0 L_m(\mu) , \text{ etc.}$$  \hspace{1cm} (3)

Finite-mass charm quark effects are incorporated in

$$\delta m_c^{(+)} = \delta m_c^{(1)} a_+^2(\overline{m}_b) + \delta m_c^{(2)} a_+^3(\overline{m}_b) + \mathcal{O}(a_+^4) ,$$  \hspace{1cm} (5)

which vanishes in the $m_c \to 0$ limit. We have

$$\delta m_c^{(1)} = \frac{4}{3} \overline{m}_b \Delta[\overline{m}_c/\overline{m}_b] = 1.9058 \text{ MeV} \ [7], \ \delta m_c^{(2)} = 48.6793 \text{ MeV} \ [11],$$  \hspace{1cm} (6)

$$\Rightarrow \delta m_c^{(1)} a_+^2(\overline{m}_b) = 9.3 \text{ MeV}, \ \delta m_c^{(2)} a_+^3(\overline{m}_b) = 18.1 \text{ MeV},$$  \hspace{1cm} (7)

so $\delta m_c^{(+)}$ is badly divergent. Why? At loop order $n$, the natural scale of the loop integral for $m_b$ is $m_b e^{-n}$ [12], which for $n$ large enough is: $m_b e^{-n} < m_c$. Therefore, for large $n$ (> 2) charm quark appears as very heavy (decoupled), leading to the effective number of flavors being $N_l = 3$ and not $N_f = N_l + 1 = 4$. Therefore, it is convenient to rewrite the relation between the pole and the ${\overline{\text{MS}}}$ mass in terms of $a_-^{(0)}(\mu) = a(\mu; N_l)$ and $r_j^{(-)}(\mu) \equiv r_j(\mu; N_l) [N_l = 3]$

$$m_b = \overline{m}_b (1 + S(N_l)) + \delta m_c ,$$  \hspace{1cm} (8)

where

$$S(N_l) = \frac{4}{3} a_-^{(0)} \left[ 1 + r_1^{(-)}(\mu) a_-^{(0)}(\mu) + r_2^{(-)}(\mu) a_-^2(\mu) + r_3^{(-)}(\mu) a_-^3(\mu) + \mathcal{O}(a_-^4) \right],$$  \hspace{1cm} (9)

and $r_j^{(-)}(\overline{m}_b) = 7.74, 87.2, 1265.3 \pm 16.1$, for $j = 1, 2, 3$. The effects of the decoupling of $S \ (N_f \to N_l = 3)$ are absorbed in the new $\delta m_c$

$$\delta m_c = \left[ \delta m_{c,+}^{(1)} + \delta m_{c,\text{dec.}}^{(1)} \right] a_+^2(\overline{m}_b) + \left[ \delta m_{c,+}^{(2)} + \delta m_{c,\text{dec.}}^{(2)} \right] a_+^3(\overline{m}_b) + \mathcal{O}(a_+^4) ,$$  \hspace{1cm} (10)
where \( \delta m_{(c, \text{dec.})}^{(j)} \) are generated by this decoupling and read

\[
\delta m_{(c, \text{dec.})}^{(1)} = \frac{2}{9} \overline{m}_b \left( \log \left( \frac{\overline{m}_b^2}{\mu_c^2} \right) - \frac{71}{32} - \frac{\pi^2}{4} \right) \tag{11}
\]

and \( \delta m_{(c, \text{dec.})}^{(2)} \) can be found in Ref. [4].

Numerical evaluation gives for \( \left[ \delta m_{(c,+)\text{,dec.}}^{(1)} + \delta m_{(c,\text{dec.})}^{(1)} \right] a_2^2 (\overline{m}_b) = -1.6 \) MeV and \( \left[ \delta m_{(c,+)\text{,dec.}}^{(2)} + \delta m_{(c,\text{dec.})}^{(2)} \right] a_2^2 (\overline{m}_b) = -0.3 \) MeV. This means that the previous divergent series (in QCD) \( \delta m_{(c,+)\text{,dec.}}^{(0)} = (9.3 + 18.1 + \ldots) \) MeV [Eq. (7)] now transforms (in QCD) to

\[
\delta m_c = (-1.6 - 0.3 + \ldots) \text{ MeV.} \tag{12}
\]

The series for \( \delta m_c \) in QCD formulation is convergent, strong cancellation takes place between \( \delta m_{(c,+)\text{,dec.}}^{(j)} \) and \( \delta m_{(c,\text{dec.})}^{(j)} \), as expected.

3. Leading renormalon of the pole mass

The asymptotic behaviour of \( r_N \) is determined by the leading IR renormalon:

\[
\frac{4}{3} \frac{\overline{m}}{m} (\mu) \approx \pi N_m \frac{\mu}{\overline{m}_b} (2\beta_0)^N \Gamma(\nu + N + 1) \left[ 1 + \sum_{s=1}^{3} \frac{\nu \cdots (\nu - s + 1)}{(N + \nu) \cdots (N + \nu - s + 1)} \tilde{c}_s + \mathcal{O} (N^{-4}) \right]. \tag{13}
\]

\[
\frac{4}{3} r_N (\mu) = \pi N_m \frac{\mu}{\overline{m}_b} (2\beta_0)^N \Gamma(\nu + N + 1) \left[ 1 + \sum_{s=0}^{\nu} \frac{\nu \cdots (\nu - s + 1)}{(N + \nu) \cdots (N + \nu - s + 1)} \tilde{c}_s \right] + h_N (\mu), \tag{14}
\]

where \( h_N \) is dominated by subleading renormalons, and the coefficients \( \tilde{c}_s \) \((s = 1, 2, 3)\) are given in [13, 14, 15, 4] \((\overline{c}_0 = 1)\) by convention).

Determining the pole mass from \( \Upsilon(1S) \) mass has large uncertainties due to the pole mass renormalon ambiguity \( \delta m_b \sim \Lambda_{\text{QCD}} [13] \). In order to avoid this problem, we work with the renormalon-subtracted (RS) bottom mass \( m_{b, \text{RS}} \) instead [14]. Then, \( \overline{m}_b \) is obtained from its stable (renormalon-free) relation with the \( m_{\text{RS}} \) mass.

The use of \( m_{\text{RS}} \) in the theoretical evaluation of the \( \Upsilon(1S) \) mass is convenient because it has no leading IR renormalon ambiguity, and the renormalon cancellation in the quarkonium mass \( M_{\Upsilon(1S)} = 2 m_b + E_{\Upsilon(1S)} \) is implemented automatically and explicitly.

4. Determination of the renormalon residue \( N_m \) and \( N_V \)

The asymptotic behavior of the coefficients \( v_N (\mu) \) of the static singlet potential,

\[
V (r) = -\frac{4 \pi}{3} \frac{1}{r} a_- (\mu) \left[ 1 + v_1 (\mu) a_- (\mu) + v_2 a_- (\mu)^2 + v_3 a_- (\mu)^3 + \ldots \right], \tag{15}
\]

can be determined in complete analogy with those of \( r_N \)

\[
-\frac{4}{3} v_N (\mu) = N_V \mu r (2\beta_0)^N \sum_{s \geq 0} \frac{\Gamma (\nu + N + 1 - s)}{\Gamma (\nu + 1 - s)} \tilde{c}_s \Rightarrow \tag{16}
\]

\[
-\frac{4}{3} v_N^{\text{asym}} (\mu) \approx N_V \mu r (2\beta_0)^N \Gamma (\nu + N + 1) \left[ 1 + \sum_{s=1}^{3} \frac{\nu \cdots (\nu - s + 1)}{(N + \nu) \cdots (N + \nu - s + 1)} \tilde{c}_s \right], \tag{17}
\]
where in Eq. (17) $d_N = 0$ was taken. We can determine the “strength”, $N_V$, of the leading IR renormalon by approximating the asymptotic $v_N^{\text{asym}}(\mu)$ with the exact $v_N(\mu)$ ($N = 0, 1, 2, 3$): $v_N^{\text{asym}}(\mu) \approx v_N(\mu) \Rightarrow$

$$N_V \approx -\frac{4}{3} v_N(\mu) / \left\{ \mu r(2/3) \frac{\Gamma(\nu + N + 1)}{\Gamma(\nu + 1)} \left[ 1 + \sum_{s=1}^{3} \frac{\nu \cdots (\nu - s + 1)}{(N + \nu) \cdots (N + \nu - s + 1)} \bar{c}_s \right] \right\}.$$  \hspace{1cm} (18)

The result for $N_V$ should be the best for the highest available $N$ ($N = 3$) and should also have reduced spurious $\mu$-dependence. At present, the $v_j$ are known up to $N^3\text{LO}$ ($v_3$) [16, 17, 18, 19, 20, 21, 22, 23, 24, 25].

In the sum $2m_b + V(r)$ the leading IR renormalon gets cancelled. $N_V$ is then related with $N_m$ by the renormalon cancellation of the sum $2m_b + V(r)$: $2N_m + N_V = 0$. Determining $N_m$ via $N_V$ gives us the value that we use [4]

$$N_m = -N_V / 2 = 0.56255(260) \ (N_l = 3).$$ \hspace{1cm} (19)

**5. Renormalon-subtracted \text{(RS, RS')} mass of bottom**

The RS mass is defined by subtracting the leading IR renormalon singularity from the pole mass [14]:

$$m_{\text{b,RS}}(\nu_f) = m_b - N_m \pi \nu_f \sum_{N=0}^{\infty} a_{-}^{N+1}(\nu_f) \frac{(2/3)^N}{\nu_f} \frac{c_N}{\nu_f} \frac{\Gamma(\nu + N + 1 - s)}{\Gamma(\nu + 1 - s)}$$  \hspace{1cm} (20)

Equation (20) is still formal. In practice, one rewrites $m$ in terms of $\overline{m}$ using Eqs. (8)-(9)

$$m_b = \overline{m}_b (1 + (4/3) a_{-}(\nu_f) + \ldots),$$ \hspace{1cm} (21)

and reexpands the perturbation series in Eq. (20) around the same coupling $a_{-}(\mu)$, at fixed but otherwise arbitrary scale $\mu$:

$$m_{\text{b,RS}}(\nu_f) = \overline{m}_b \left[ 1 + \sum_{N=0}^{\infty} h_N(\nu_f) a_{-}^{N+1}(\nu_f) \right] \Rightarrow m_{\text{b,RS}}(\nu_f) = \overline{m}_b \left[ 1 + \sum_{N=0}^{\infty} \tilde{h}_N(\nu_f; \mu) a_{-}^{N+1}(\mu) \right],$$ \hspace{1cm} (22)

where $h_N(\nu_f)$ is determined from Eq. (14) (with $\mu = \nu_f$ and with the sum truncated at $\bar{c}_3$) for $N = 0, 1, 2, 3$. For $N \geq 4$ we take $h_N(\overline{m}_b) = 0$. The coefficients $\tilde{h}_N(\nu_f; \mu)$ in Eq. (22) are obtained by expanding $a_{-}(\nu_f)$ in the expansion in powers of $a_{-}(\mu)$. Note that $m_{\text{b,RS}}(\nu_f)$ will only marginally depend on $\mu$ when we truncate the infinite sum in Eq. (22). On the other hand, the coefficients $h_N$ are functions of $\nu_f$, $\mu$, and $\overline{m}_b$, and are much smaller than $r_N(\mu)$.

A variant of the RS mass is the modified renormalon-subtracted (RS') mass $m_{\text{b,RS'}}$, where subtractions start at $\sim a^2$ [14]. Specifically in this case, Eqs. (20) and (22) are repeated, with the replacements $m_{\text{b,RS}}(\nu_f) \rightarrow m_{\text{b,RS'}}(\nu_f)$ and $\sum_{N=0}^{\infty} \rightarrow \sum_{N=1}^{\infty}$.

**6. Bottom mass from heavy quarkonium**

The perturbation expansion of $M_{\text{T(1S)}}^{(\text{th})}$ is presently known up to $O(m_b a^5)$ [19, 20, 21, 22]:

$$M_{\text{T(1S)}}^{(\text{th})} = 2m_b - \frac{4\pi^2}{9} m_b a^2(\mu) \left\{ 1 + a_{-}(\mu) [K_{1,0} + K_{1,1} L_p(\mu)] + a^2(\mu) \frac{2}{j=0} K_{2,j} L_p(\mu) j^j + a^3(\mu) [K_{3,0,0} + K_{3,0,1} \ln a_{-}(\mu) + \sum_{j=1}^{3} K_{3,j} L_p(\mu) j^j] + O(a^4) \right\},$$  \hspace{1cm} (23)
μ is the renormalization scale, \( L_p(μ) = \ln(μ/μ_b) \) where \( μ_b = (4π/3) m_b a_−(μ) \). \( K_{i,j}(N_f) \) and \( K_{3,0,j} \) are given, e.g., in [4]. We then rewrite \( m_b \) in terms of \( m_{b,RS} \) to implement the leading IR renormalon cancellation. This gives

\[
\frac{M_{\gamma(1S)}^{(th)}}{m_{b,RS}(\nu_f)} = 2 + \left[ 2π N_m ba K_0 - \frac{4π²}{9} a² \right] + \left[ 2π N_m ba² (K_1 + z_1 K_0) - \frac{4π²}{9} a³ (K_{1,0} + K_{1,1} L_{RS}) \right] + 2π N_m ba³ (K_2 + 2z_1 K_1 + z_2 K_0) - \frac{4π²}{9} a⁴ \left( \sum_{j=0}^{2} K_{2,j} L_{RS}^j + ba³ π N_m K_0 \right) + \mathcal{O}(ba⁴,a⁵). \tag{24}
\]

The terms \( \mathcal{O}(ba⁴,a⁵) \) have a similar structure and were written in [4]. The notations are

\[
a = a_−(μ) = a(μ, N_f = 3); \quad b = b(ν_f) = ν_f/m_{b,RS}(ν_f), \quad N_m = N_m(N_l = 3), \quad \tag{25a}
\]

\[
L_{RS} \equiv L_{RS}(μ) = \ln \left( \frac{μ}{(4π/3)m_{b,RS}(ν_f) a_−(μ)} \right), \quad K_N = (2β_0)^N \sum_{s=0}^{3} \tilde{c}_s \Gamma(ν + N + 1 - s) \Gamma(ν + 1 - s). \tag{25b}
\]

In the expression (24) for \( M_{\gamma(1S)} \), the terms of the same order \( (ν_f/m_{b,RS}) a^n \) and \( a^{n+1} \) were combined in common brackets \([…]\), in order to account for the renormalon cancellation.

If using the RS’ mass in our approach instead, the above expressions are valid without changes, except that \( m_{b,RS} \to m_{b,RS'} \) and \( K_0 \to 0 \) (and: \( h_0(μ) \to 4/3 \)).

We note that we take \( N_l = 3 \) active flavours, as the charm quark mass effects in the binding energy \( E_{\gamma(1S)} \) are negligible [26].

We extract the bottom masses from the condition \( M_{\gamma(1S)}^{(th)} = M_{\gamma(1S)}^{(exp)} (= 9.460 \text{ GeV}) \). The error estimates are made assuming \( μ = 2.5^{+1.5}_{-1.0} \text{ GeV} \) [we varied \( μ \) in Eq. (24) but not in Eq. (22)], \( ν_f = 2 ± 1 \text{ GeV}, \quad α_s(M_2) = 0.1184(7) \) (and decoupling at \( \bar{m}_b = 4.2 \text{ GeV} \) and at \( \bar{m}_c = 1.27 \text{ GeV} \), \( N_m = 0.56255(260) \), and \( (4/3)r_3(\bar{m}_b); N_l = 1687.1 ± 21.5 [10] \).

In RS and RS’ approaches we extract, in MeV, respectively

\[
m_{b,RS}(2 \text{ GeV}) = 4437^{-11}_{+14}(μ)^3/3(ν_f)^2/3(α_s)^{-1/41}(N_m)^{-0/1}(r_3); \quad \tag{26a}
\]

\[
⇒ \bar{m}_b = 4217^{-10}_{+39}(μ)^3/3(ν_f)^2/3(α_s)^{-1/41}(N_m)^{-0/1}(r_3); \quad \tag{26b}
\]

\[
m_{b,RS'}(2 \text{ GeV}) = 4761^{-10}_{+14}(μ)^3/3(ν_f)^2/3(α_s)^{-1/41}(N_m)^{-0/1}(r_3); \quad \tag{26c}
\]

\[
⇒ \bar{m}_b = 4223^{-14}_{+36}(μ)^2/4(ν_f)^2/4(α_s)^{-1/41}(N_m)^{-0/1}(r_3); \quad \tag{26d}
\]

The uncertainties in \( \bar{m}_b \) are dominated by the variation of the renormalization scale \( μ \).

The renormalon cancellations are reflected numerically in Eq. (24) [we take \( μ = 2.5 \text{ GeV} \)]:

\[
\text{RS : } M_{\gamma(1S)} = (8874 + 431 + 167 + 18 - 30) \text{ MeV} , \quad \tag{27a}
\]

\[
\text{RS' : } M_{\gamma(1S)} = (9521 - 150 + 112 + 8 - 31) \text{ MeV} , \quad \tag{27b}
\]

The convergence is good; except for the last (NNNLO) term \( \mathcal{O}(a^5,ba^4) \), where the factorization scale dependence becomes stronger, which may signal the importance of ultrasoft effects.

The relations between RS (RS’) mass and \( \bar{M}_S \) mass are reasonably convergent:

\[
m_{b,RS}(2 \text{ GeV}) = (4217 + 191 + 36 + 12 - 19) \text{ MeV} , \quad \tag{28a}
\]

\[
m_{b,RS'}(2 \text{ GeV}) = (4223 + 478 + 60 + 18 - 17) \text{ MeV} , \quad \tag{28b}
\]

where the expansion parameter is taken to be \( a(2.5 \text{ GeV}) \). A bigger value for the renormalization scale, closer to the bottom quark mass, makes the last term smaller.
Until now we have approximated $\delta m_c = 0$ in Eq. (20). However, $\delta m_c \approx -2$ MeV, Eq.(12). Hence, we have to add 2 MeV to the values of $\bar{m}_b$ obtained in Eqs. (26b) and (26b) (in Ref. [4] it was incorrectly subtracted), leading to the final average of the RS and RS’ extractions

$$\bar{m}_b = 4222(40) \text{ MeV} \ .$$  \hspace{1cm} (29)

where we have rounded the $\pm$ variation of each parameter to the maximum and added them in quadrature.

7. Conclusions

(i) We presented strong numerical indications that the charm quark decouples in the relation between $m_b$ and $m_b$ ($\Rightarrow N_l = 3$).

(ii) An improved determination of the residue of the leading renormalon for the bottom pole mass (and static potential with $N_l = 3$) was performed: $N_m = 0.56255(260)$.

(iii) Use of the 3-loop ($\sim a^3 \bar{m}_b$) corection to the $\Upsilon(1S)$ binding energy, and 4-loop relation between $m_b$ and $m_b$, allowed us to perform extraction of $m_b, \text{RS}'$ and $m_b$ to NNNLO, with the resulting values Eq. (29). The uncertainties are dominated by the variation of the renormalization scale.

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