COORDINATE-FREE ACTION FOR $AdS_3$
HIGHER-SPIN-MATTER SYSTEMS

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Abstract

A coordinate-free action principle for the $\mathcal{N} = 2$ supersymmetric model of spin 0 and spin 1/2 matter fields interacting via Chern-Simons higher spin gauge fields in $AdS_3$ is formulated in terms of star-product algebra of two oscillators. Although describing proper relativistic dynamics the action does not contain space-time coordinates. It is given by a supertrace of a forth-order polynomial in the $3d$ higher spin superalgebra. The theory admits a free parameter characterizing the inequivalent vacua associated with the parameter of mass $m$ of the matter fields. For the case of the massless matter, the model has a form of the noncommutative $2d$ Yang-Mills theory with some infinite-dimensional gauge group. The limit $m \to \infty$ corresponds to the $SU(\infty) 2d$ Yang-Mills theory.

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1 Introduction

The pure gauge Chern-Simons higher spin action in $AdS_3$ was constructed in [1]. Since the Chern-Simons higher spin gauge fields do not propagate, this action does not describe any nontrivial local dynamics. So far, no attempt to analyze higher spin gauge interactions of 3d matter fields at the action level has been undertaken. The nonlinear equations of motion have been constructed however in all orders in interactions for 3d higher-spin-matter systems in [2] for massless matter and in [3] for an arbitrary mass of the matter fields. These equations are equivalent to the Lagrangian ones at least at the linearized level but do not have manifestly Lagrangian form because of the presence of an infinite set of auxiliary variables. This makes it not straightforward to derive the full nonlinear Lagrangian principle underlying the higher spin dynamics starting from this formulation of the higher spin equations.

The point is that, in this approach (referred to as “unfolded formulation” [4]; also see [3] and references therein) by virtue of introducing an infinite set of auxiliary variables the equations of motion acquire “unfolded” form of certain zero-curvature equations and covariant constancy conditions,

$$d\omega = \omega \wedge \omega, \quad dB^A = \omega_i^A B B^B,$$

supplemented with some gauge invariant constraints

$$\chi(B) = 0$$

that do not contain spacetime derivatives. Here $d = dx^a \frac{\partial}{\partial x^a}$ (spacetime base indices $a, b, \ldots = 0, \ldots, d - 1$ are underlined), $\omega(x) = dx^a \omega^a_i(x)T_i$ is a gauge field of some infinite-dimensional higher spin Lie superalgebra $l$ ($T_i \in l$), and $B^A(x)$ is a set of 0-forms that take values in the representation space of an appropriate infinite-dimensional representation $(t_i)^B_A$ of $l$. Note that in order to reformulate a relativistic dynamics in such a form one has to use an infinite-dimensional representation $t$ rich enough to allow $B^A(x)$ to contain dynamical fields along with all their on-mass-shell derivatives (for more detail see [3]). Such a form of the equations of motion is obviously non-Lagrangian because the number of equations exceeds greatly the number of variables. It is likely that to construct the full non-linear action leading to the equations (1), (2) one has to add more auxiliary fields. This problem is not yet solved.

Since the equations (1) have the form of zero-curvature and covariant constancy conditions, they admit an explicit (local) generic solution. The dynamical content is therefore hidden in the constraints (2). The role of the equations (2) is that any their solution describes, at any point of space-time, the complete set of all space-time derivatives of the dynamical fields, which are allowed to be non-zero by the field equations (for more detail see [3]). In this respect the unfolded formulation formulation is analogous to the coordinate-free approach of Penrose [4, 7].

The main observation of this paper is that for the 3d higher spin model [3] it is possible to build an action that leads to the constraints (2) as its field equations. This action principle is formulated not in the usual spacetime but in some auxiliary noncommutative spinor space that describes the infinite-dimensional representation space of $B^A$ as an appropriate functional space. Remarkably, the constructed action has a form of that of the
2-dimensional noncommutative Yang-Mills theory in this auxiliary space. Since the constraints (2) encode the full information about the dynamical system under investigation one can adopt an extreme point of view that the proposed action gives just the right variational principle for the model. As such, the proposed model provides the first example of the full nonlinear action principle for a nontrivial \( d = 3 \) model exhibiting higher spin gauge symmetries.

The paper is organized as follows. In sect. 2 we collect following some relevant facts about the nonlinear system of equations for massive matter fields of spin 0 and 1/2 interacting via the Chern-Simons higher spin gauge potentials in \( AdS_3 \). In sect. 3 we find the action functional for these equations and show that in the massless case it can be rewritten in a form analogous to the 2d noncommutative Yang-Mills action with some infinite-dimensional gauge group. In sect. 4 we discuss a generalization of this action to the case of an arbitrary mass. In sect. 5 we focus on the special cases of the parameter of mass at which the gauge algebra becomes finite-dimensional, as well as on the quasiclassical limit. In Conclusion we summarize the results and discuss some open problems.

## 2 3D Higher-Spin-Matter Equations of Motion

The full nonlinear system of 3d higher-spin-matter equations, which is a particular realization of the equations (1) and (2), is formulated in terms of the generating functions \( W(z,y;\psi_{1,2},k,\rho|x) \), \( B(z,y;\psi_{1,2},k,\rho|x) \), and \( S_\alpha(z,y;\psi_{1,2},k,\rho|x) \) that depend on the spacetime coordinates \( x^a \) \((a = 0,1,2)\), auxiliary commuting spinors \( z_\alpha, y_\alpha \) \((\alpha = 1,2)\), \([y_\alpha,y_\beta] = [z_\alpha,z_\beta] = [z_\alpha,y_\beta] = 0\), a pair of Clifford elements \( \{\psi_i,\psi_j\} = 2\delta_{ij} \) \((i = 1,2)\) that commute with all other generating elements, and another pair of Clifford-type elements \( k \) and \( \rho \) which have the following properties,

\[
k^2 = 1 \quad \rho^2 = 1 \quad k\rho + \rho k = 0 \quad ky_\alpha = -y_\alpha k \quad kz_\alpha = -z_\alpha k \quad \rho y_\alpha = y_\alpha \rho \quad \rho z_\alpha = z_\alpha \rho.
\]

(3)

The spacetime 1-form \( W = dx^2W_2(z,y;\psi_{1,2},k,\rho|x) \),

\[
W_2(z,y;\psi_{1,2},k,\rho|x) = \sum_{A,B,C,D=0}^{\infty} \frac{1}{m!n!} W_{ABCD}^{\alpha_1...\alpha_m\beta_1...\beta_n}(x) \times k^A \rho^B \psi_1^C \psi_2^D z^{\alpha_1} \ldots z^{\alpha_m} y^{\beta_1} \ldots y^{\beta_n},
\]

(4)

is the generating function for higher spin gauge fields. \( B = B(z,y;\psi_{1,2},k,\rho|x) \) is the generating function for the matter fields. It admits an analogous expansion with the component fields identified with the 3d matter fields and all their on-mass-shell nontrivial derivatives. The field \( S_\alpha(z,y;\psi_{1,2},k,\rho|x) \) is expressed in terms of \( B \) modulo gauge transformations by virtue of the field equations. As explained in sect. 2 (see also 3), \( S_\alpha \) and \( B \) can be interpreted as noncommutative connection and field strength in a two-dimensional noncommutative YM theory with some infinite-dimensional gauge group.

The multispinorial coefficients in the expansions like (1) of the functions \( W_2, B, \) and \( S_\alpha \) carry standard Grassmann parity: they are even (odd) if the number of spinor indices is even (odd), and are defined to commute with the generating elements \( z_\alpha, y_\alpha, k, \rho, \psi_{1,2} \).
The generating functions are treated as elements of the associative algebra with the product law
\[ (f * g)(z, y) = \frac{1}{(2\pi)^2} \int d^2u d^2v \exp(iu_\alpha v^\alpha) f(z + u, y + u)g(z - v, y + v) \]  
(5)
for any two functions \( f(z, y) \) and \( g(z, y) \). (The integration variables \( u \) and \( v \) are required to satisfy the commutation relations similar to those of \( y \) and \( z \) in (3). The spinorial indices are raised and lowered by the \( 2 \times 2 \) symplectic form \( \epsilon_{\alpha\beta} \) according to \( u^\alpha = \epsilon^{\alpha\beta} u_\beta \), \( u_\alpha = -\epsilon_{\alpha\beta} u^\beta \), \( \epsilon_{12} = e^{12} = 1 \).) For generic elements \( f(z, y; \psi_{1,2}, k, \rho) = F(z, y)\Phi(\psi_{1,2}, k, \rho) \) and \( g(z, y; \psi_{1,2}, k, \rho) = G(z, y)\Psi(\psi_{1,2}, k, \rho) \) the order of the product factors in the integrand of (3) is essential as the elements \( \psi_1, \psi_2, k, \rho \) do not commute. Due to the sign changes in (3) we have
\[ \Phi(\psi_{1,2}, k, \rho)G(z, y) = \tilde{G}(z, y)\Phi(\psi_{1,2}, k, \rho) \]  
(6)
with some \( \tilde{G} \). The full product is then defined as
\[ (f * g)(z, y; \psi_{1,2}, k, \rho) = (F * \tilde{G})(z, y) \Phi(\psi_{1,2}, k, \rho)\Psi(\psi_{1,2}, k, \rho). \]  
(7)
The product law (3) yields a particular realization of the Weyl algebra,
\[ [y_\alpha, y_\beta]_* = -[z_\alpha, z_\beta]_* = 2i\epsilon_{\alpha\beta}, \quad [y_\alpha, z_\beta]_* = 0 \]  
(8)
([\alpha, \beta]_* = a * b - b * a, \{a, b\}_* = a * b + b * a).

The generating functions therefore can be treated as elements of the associative algebra with the product law (3), (7).

The full system of equations of motion for the system of 3d massive spin-0 and spin-1/2 matter interacting via higher spin gauge fields has the form (3)
\[ dW = W * \wedge W, \]  
(9)
\[ dB = W * B - B * W, \]  
(10)
\[ dS_\alpha = W * S_\alpha - S_\alpha * W, \]  
(11)
\[ S_\alpha * S^\alpha = -2i(1 + B), \]  
(12)
\[ S_\alpha * B + B * S_\alpha = 0, \]  
(13)
where \( W \) and \( B \) are independent of \( \rho \) while \( S_\alpha \) is linear in \( \rho \),
\[ W(z, y; \psi_{1,2}, k, -\rho|x) = W(z, y; \psi_{1,2}, k, \rho|x), \quad B(z, y; \psi_{1,2}, k, -\rho|x) = B(z, y; \psi_{1,2}, k, \rho|x), \]  
(14)
\[ S_\alpha(z, y; \psi_{1,2}, k, -\rho|x) = -S_\alpha(z, y; \psi_{1,2}, k, \rho|x). \]  
(15)
Eqs.(11)-(13) are general coordinate invariant (because of using the exterior algebra formalism) and invariant under the infinitesimal higher spin gauge transformations
\[ \delta W = d\varepsilon - W * \varepsilon + \varepsilon * W, \quad \delta B = \varepsilon * B - B * \varepsilon, \quad \delta S_\alpha = \varepsilon * S_\alpha - S_\alpha * \varepsilon, \]  
(16)
where \( \varepsilon = \varepsilon(z, y; \psi_{1,2}, k|x) \) is an arbitrary gauge parameter.
A particular vacuum solution of the system (9)-(13) can be chosen in the form [3]

\[ B_0 = \nu K, \quad S_{0\alpha} = s_{0\alpha}, \quad W_0 = \frac{1}{8i} (\omega^{\alpha\beta} + \lambda h^{\alpha\beta} \psi_1) \{ \tilde{y}_\alpha, \tilde{y}_\beta \}, \quad (17) \]

where \( \nu \) is a constant parameter,

\[ K = ke^{i(z_y)}, \quad (z_y) = z_\alpha y^\alpha, \quad (18) \]

\[ s_{0\alpha}(z, y) = \rho \left( z_\alpha + \nu (z_\alpha + y_\alpha) \int_0^1 dt e^{it(z_y)} k \right), \quad (19) \]

\[ \tilde{y}_\alpha(z, y) = y_\alpha + \nu (z_\alpha + y_\alpha) \int_0^1 dt (t - 1) e^{it(z_y)} k, \quad (20) \]

and \( \omega^{\alpha\beta} \) and \( h^{\alpha\beta} \) are some \( \alpha\beta \)-symmetric 1-forms on \( AdS_3 \) satisfying the equations

\[ d\omega^{\alpha\beta} = \omega^{\alpha\gamma} \wedge \omega^{\beta\gamma} + \lambda^2 h^{\alpha\gamma} \wedge h^{\beta\gamma}, \quad dh^{\alpha\beta} = \omega^{\alpha\gamma} \wedge h^{\beta\gamma} + \omega^{\beta\gamma} \wedge h^{\alpha\gamma}. \quad (21) \]

Provided that \( h^{\alpha\beta} \) is a non-degenerate \( 3 \times 3 \) matrix, \( \omega^{\alpha\beta} \) and \( h^{\alpha\beta} \) are identified with the background \( AdS_3 \) Lorentz connection and frame field, respectively. The parameter \( \lambda \) is identified with the inverse radius of \( AdS_3 \).

As shown in [3], the elements \( s_{0\alpha}(z, y) \) and \( \tilde{y}_\alpha(z, y) \) commute to each other,

\[ [s_{0\alpha}, \tilde{y}_\beta]_* = 0, \quad (22) \]

and obey the commutation relations of the “deformed oscillator” algebra [5, 6],

\[ [\tilde{y}_\alpha, \tilde{y}_\beta]_* = 2i \epsilon_{\alpha\beta} (1 + \nu k), \quad \tilde{y}_\alpha k = -k \tilde{y}_\alpha; \quad (23) \]

\[ [s_{0\alpha}, s_{0\beta}]_* = -2i \epsilon_{\alpha\beta} (1 + \nu K), \quad s_{0\alpha} * K = -K * s_{0\alpha}; \quad (24) \]

\[ k^2 = K * K = 1. \quad (25) \]

These properties guarantee that the substitution of the ansatz (17) into (9) leads to the \( AdS_3 \) equations (21).

Fixation of the vacuum solution (17)-(21) breaks the local symmetry (16) down to some global symmetry, the symmetry of the vacuum. It is generated by a parameter \( \varepsilon_{gl}(x) \) obeying the conditions

\[ d\varepsilon_{gl} = [W_0, \varepsilon_{gl}]_* = 0, \quad (26) \]

\[ [\varepsilon_{gl}, S_{0\alpha}]_* = 0, \quad (27) \]

which follow from the requirements that \( \delta W_0 = 0 \) and \( \delta S_{0\alpha} = 0 \) (\( \delta B_0 = 0 \) holds trivially), i.e. \( \varepsilon_{gl} \) belongs to the stability subalgebra of the vacuum solution. The condition (24) implies that \( \varepsilon_{gl} \) belongs to the centralizer subalgebra for the element \( s_{0\alpha} \) (19), an infinite dimensional algebra \( l^g \) spanned by the elements \( y_\alpha, k, \psi_1 \) [3]. The equation (26) fixes a dependence of \( \varepsilon_{gl} \) on the space-time coordinates \( x^a \) in terms of the initial data \( \varepsilon_{gl}(x_0) = \varepsilon_{gl}^0 \) at any space-time point \( x_0 \) (in some neighborhood of \( x_0 \)). \( l^g \) contains the 3d \( N2 \) SUSY.
superalgebra $osp(2,2) \oplus osp(2,2)$ as a finite dimensional subalgebra spanned by generators $\Pi_\pm T^A$, where
\[ \Pi_\pm = \frac{1 \pm \psi_1}{2} \] (28)
and $T^A = \{ T_{\alpha\beta}, Q^{(1)}_{\alpha}, Q^{(2)}_\alpha, J \}$ with
\[ T_{\alpha\beta} = \frac{1}{4i} \{ \tilde{y}_\alpha, \tilde{y}_\beta \}, \quad Q^{(1)}_\alpha = \tilde{y}_\alpha, \quad Q^{(2)}_\alpha = \tilde{y}_\alpha k, \quad J = k + \nu. \] (29)
The fact that these generators form $osp(2,2)$ is a simple consequence \[10\] of the properties of the deformed oscillator algebra (23). Thus, the system (9)-(13) possesses $N=2$ global SUSY for arbitrary $\nu$. Perturbations near the vacuum (17) describe a massive scalar $AdS_3$ hypermultiplet interacting via Chern-Simons type higher spin gauge fields \[3\]. The values of mass are related to the parameter $\nu$ as follows: $m^2_{\pm} = \lambda^2 \nu (\nu \mp 2)/2$ for bosons and $m^2_{\pm} = \lambda^2 \nu^2/2$ for fermions.

Since Eq. (9) and Eqs. (10), (11) have, respectively, a form of zero-curvature equation and covariant constancy conditions they admit a (local) generic solution of the form
\[ W = dg(x) \ast g^{-1}(x), \quad B(x) = g(x) \ast b \ast g^{-1}(x), \quad S_\alpha(x) = g(x) \ast s_\alpha \ast g^{-1}(x), \] (30)
where $g(z, y; \psi_{1,2}, k|x)$ is an arbitrary invertible element, $g \ast g^{-1} = g^{-1} \ast g = 1$, while $b(z, y; \psi_{1,2}, k)$ and $s_\alpha(z, y; \psi_{1,2}, k, \rho)$ are arbitrary $x$-independent elements.

Plugging these expressions into (12) and (13) one obtains, as a consequence of the gauge invariance,
\[ s_\alpha \ast s_\alpha = -2i(1 + b), \] (31)
\[ s_\alpha \ast b + b \ast s_\alpha = 0. \] (32)
Since these equations contain no dependence on the spacetime coordinates they describe the dynamical system under consideration turns out to be entirely reformulated in terms of the auxiliary space of spinor variables, noncommutative spinor space. Recall that the role of the equations (31) and (32) is that any their solution describes, at any point of space-time, the complete set of all space-time derivatives of the dynamical fields, which are allowed to be non-zero by the field equations (for more detail see \[3\]).

### 3 Coordinate-Free Action for the Massless Case and Non-Commutative Yang-Mills Theory

The main observation of this paper is that the system of equations (31), (32) admits a Lagrangian formulation with the action of the form
\[ S = \text{str} \ L, \] (33)
where $L$ is some star-product Lagrangian function of $b$ and $s_\alpha$ specified below and str is the supertrace operation \[11\] which for the particular star-product (5) has the form \[3\]
\[ \text{str} \ f(z, y) = \frac{1}{(2\pi)^2} \int d^2z \ d^2y \ \exp(-iz_\alpha y^\alpha) f(z, y). \] (34)
Since the algebra of functions $f(z, y; k, \rho, \psi_{1,2})$ is in fact a tensor product of the Weyl algebra with some matrix algebra, this definition admits a unique extension to the elements dependent on $k, \rho$ and $\psi_{1,2}$ with $\text{str} f = 0$ for any $f$ proportional to the first power of any of the variables $k, \rho$ or $\psi_{1,2}$.

The defining property of the supertrace (34) that fixes it uniquely \cite{9} for the class of polynomial functions $f(z, y; k, \rho, \psi_{1,2})$ is

$$\text{str} (f * g) = (-1)^{\pi(f)} \text{str} (g * f) = (-1)^{\pi(g)} \text{str} (g * f),$$

(35)

where $\pi(f)$ is the parity of $f$, i.e. $f(-z, -y) = (-1)^{\pi(f)} f(z, y)$. This definition of parity is in agreement with the standard spin-statistics relationship, so that

$$\text{str} [f, g]_* = 0$$

(36)

for any two elements $f$ and $g$ provided that fermions carry an additional Grassmann grading.

The definition (34) works for polynomial elements of the star-product algebra but not necessarily makes sense beyond this class. For example,

$$\text{str} \exp i(zy) = \infty.$$  \hspace{1cm} (37)

As a result, the supertrace (34) is not well-defined for the algebra of $\tilde{y}$ (20) because the integrals in the definition of the supertrace may diverge. This is the only point where the definition of the action is sensitive to the value of the parameter $\nu$. In sect. 4 we discuss the possibility of generalization of (34) to the case of arbitrary $\nu$. The content of this section is therefore only applicable to the particular case of $\nu = 0$. Note that, as shown in \cite{3}, the equations (31), (32) are well-defined for any value of $\nu$.

The Lagrangian leading to the equations (31), (32) has the form

$$L = i s_\alpha * s^{\alpha} * b - 2b - b * b.$$ \hspace{1cm} (38)

The variation of the action (33), (38) w.r.t. $b$ gives rise to (31) while the variation w.r.t. $s_\alpha$ gives rise to (32). (It is the property of the supertrace (35) along with the fact that $s_\alpha$ carries odd parity that leads to the anticommutator on the left hand side of (32)).

The field $b$ is auxiliary. It is expressed in terms of $s_\alpha$ by virtue of its equation of motion (31),

$$b = \frac{i}{2} s_\alpha * s^{\alpha} - 1.$$ \hspace{1cm} (39)

Plugging this expression back into (33), (38) we obtain an equivalent “second-order” action,

$$S' = \frac{1}{4} \text{str} (i s_\alpha * s^{\alpha} - 2)^2.$$ \hspace{1cm} (40)

According to the discussion in sect. 2, this action functional encodes the dynamics of massless spin-0 and spin-1/2 fields in $AdS_3$, interacting via higher spin gauge fields. The way this dynamics is described by the action (33), (38), (40) is different as compared to the standard field theoretical language since it does not contain an explicit reference to the spacetime. Nevertheless, according to the general “unfolded formulation” technics
the equations (9)-(11) provide a map to the usual description of the spacetime fields in terms of appropriate partial differential equations [2, 3, 4].

Remarkably, the action functional (40) admits representation in the form of the two-dimensional noncommutative Yang-Mills theory [12, 13] with some infinite-dimensional gauge group. Indeed, let us write

\[ s_\alpha = \rho z_\alpha - 2i\sigma_\alpha \]

with the vacuum part \( s_0\alpha = \rho z_\alpha \) and dynamical perturbation \(-2i\sigma_\alpha\). Plugging (41) into (40) and using the formula

\[ [\rho z_\alpha, f]_\star = -2i\rho \frac{\partial f}{\partial z_\alpha}, \]

which is a simple consequence of (5) and (3), one obtains

\[ is_\alpha \star s^\alpha = 2 - 2i\rho \epsilon^{\alpha\beta} \left( \frac{\partial \sigma_\beta}{\partial z_\alpha} - \frac{\partial \sigma_\alpha}{\partial z_\beta} \right) - 2i\epsilon^{\alpha\beta} [\sigma_\alpha, \sigma_\beta]_\star. \]

Introducing auxiliary spinor differentials \( dz^\alpha \)

\[ dz^\alpha dz^\beta = -dz^\beta dz^\alpha, \]

an exterior differential \( d = \rho dz^\alpha \frac{\partial}{\partial z_\alpha} \) and a spinor 1-form \( \sigma = dz^\alpha \sigma_\alpha \), one defines the field strength

\[ F = dz \sigma + \sigma \star \sigma = dz^\alpha dz^\beta F_{\alpha\beta}. \]

The action (40) takes a form

\[ S'' = -8 \text{str} \left( F_{\alpha\beta} \star F^{\alpha\beta} \right). \]

Let us now interpret the elements \( z^\alpha \) as coordinates of a two-dimensional noncommutative spacetime, identifying \( y^\alpha, k, \rho, \psi_{1,2} \) with the generating elements of the infinite-dimensional associative algebra \( H \) – the universal enveloping algebra of the commutation relations for \( y^\alpha, k, \rho, \psi_{1,2} \) in (3) and (8). The 2-form (45) can be identified with the field strength in the noncommutative non-Abelian Yang-Mills theory. The action (40) has a form reminiscent of the noncommutative non-Abelian Yang-Mills action with the higher spin gauge algebra \( H \). Here the supertrace (34) serves both for the \( z \)-integration and supertrace for the gauge algebra in the sense that it gives rise to the gauge invariant functional of the noncommutative Yang-Mills curvatures. There is however some difference compared to the standard situation due to the fact that the integration measure in (34) contains the factor \( \exp \left( -i(yz) \right) \), thus mixing the “coordinate” and “color” degrees of freedom.

Let us note that despite the fact that the Weyl algebra with polynomial elements admits a basis in which the supertrace factorises in the \( z \) and \( y \) spaces (i.e. with the star-product algebra of polynomials in \( z \) and \( y \) interpreted as the tensor product of two individual algebras of polynomials in \( z \) and \( y \)) one can hardly use this construction for the problem under investigation because it admits no meaningful analog of the nonpolynomial operator \( \exp i(yz) \) (18) that comes out as a result of solution of the field equations in our
construction. In other words, the same action rewritten in the tensor product basis leads to the equations of motion that admit no solutions.

Another important difference is that the construction we use is essentially quantum admitting no quasiclassical (i.e. deformation) interpretation. This is again a consequence of the relevance of the operator \( \exp i(z\gamma) \) that has the form \( \exp (i\hbar^{-1}(z\gamma)) \) if one reintroduces \( \hbar \neq 1 \) into the right hand side of the equations (3). In other words, the system under investigation admits no “classical” solution at \( \hbar \to 0 \) (see however sect. 3).

Another relevant issue is that the supertrace operation on the Weyl algebra differs from the trace used in the deformation quantization approach. This is a consequence of the fact that the Weyl algebra operates with formal power series while the deformation quantization technics assumes integrable functions of the space-time variables that tend to zero at infinity.

4 Generic \( \nu \)

As shown in [3], the one-parametric family of vacua (17), (19), (21) describes matter systems with different masses parametrized by \( \nu \). The equations of motion (31, 32) are well defined for any value of \( \nu \). However, to define the consistent action functional for the case of general \( \nu \), one should find an appropriate generalization of the supertrace operation. In this section we mainly set up the problem rather then solve it explicitly. A possible interpretation of this analysis is that the functional space describing the fluctuations near the different vacua seem to be inequivalent. The theories with different values of \( \nu \) are somewhat reminiscent of different ”topological” sectors, like, e.g., instantons in the 4d Yang-Mills.

To see what happens, one has to analyze the action perturbatively. Plugging the perturbative expansions \( s_\alpha = s_{0\alpha} + s_{1\alpha}, \ b = \nu K + b_1 \) into (38) one finds

\[
L = \nu^2 + i(s_{1\alpha} \ast s_{\alpha}^0 + s_{0\alpha} \ast s_{1\alpha}) \ast b_1 + i\nu s_{1\alpha} \ast s_{1\alpha}^\alpha \ast K - b_1 \ast b_1,
\]

where we neglected the terms linear in \( s_{1\alpha} \) and \( b_1 \) that do not contribute to the action as the vacuum fields satisfy the equations of motion. Because of the cancellation of the exponential factors contained in the vacuum fields (17)-(19) with the exponential factor in the definition of the supertrace, the bilinear form in the Gaussian integral with respect to the spinorial variables in (34) may degenerate so that the resulting expressions are not necessarily well-defined. In particular, after the Gaussian integration in the spinorial integration variables in (34) is completed, some terms, containing the integrals in the variable \( t \) arising from (19), diverge in (47) at \( t \to 1 \). The conclusion is that the naive extrapolation of the action to the case of arbitrary \( \nu \) is not working.

There are two main alternatives. One is that the vacuum solution should be somehow modified (as shown in [3] there exists a broad class of the vacuum solutions equivalent at the level of equations of motion). Another alternative is that one should find a proper definition of the supertrace operation for every value of \( \nu \). We believe this latter alternative is most promising. One argument is due to the fact that the on-mass-shell higher spin algebras and the corresponding supertrace operations are essentially different for different values of \( \nu \).
Indeed, plugging the perturbative expansions into the equations of motion (31), (32) one gets in the linear approximation

\[ s_{0\alpha} * b_1 = -b_1 * s_{0\alpha} . \]  

(48)

As a result,

\[ b_1 = K * c , \]  

(49)

where \( c \) takes values in the subalgebra of elements commuting to \( s_{0\alpha} (19) \), i.e.

\[ c(z, y; \psi_1, \psi_2, k) = \sum_{n=0}^{\infty} \frac{1}{n!} c_{\alpha_1...\alpha_n}(k; \psi_1, \psi_2) \tilde{y}^{\alpha_1} * ... * \tilde{y}^{\alpha_n} , \]  

(50)

where \( c_{\alpha_1...\alpha_n} \) are totally symmetric multispinors (i.e., we choose the Weyl ordering). This formula is a simple consequence \([3]\) of (22) and of the properties of the elements \( k, \psi_1 \) and \( \psi_2 \).

The important fact shown in \([4]\) is that the universal enveloping algebra of the commutation relations (23) \( Aq(2, \nu) \) (in the notation of \([4]\)) with a generic element of the form

\[ f(\tilde{y}, k) = \sum_{A=0}^{1} \sum_{n=0}^{\infty} \frac{1}{n!} f^A_{\alpha_1...\alpha_n} k^A \tilde{y}^{\alpha_1} * ... * \tilde{y}^{\alpha_n} , \]  

(51)

where \( f^A_{\alpha_1...\alpha_n} \) are totally symmetric multispinors, admits a unique supertrace operation which for the element (51) has the form

\[ \text{str}_\nu f(\tilde{y}, k) = f^0 - \nu f^1 . \]  

(52)

For \( \nu = 0 \), it reduces to the supertrace (34) for \( z \)-independent functions.

One way to see that this definition of supertrace cannot be derived from the supertrace (34) is to observe that the following formal computation

\[ \text{str} \{ k\tilde{y}_\alpha, \tilde{y}_\beta \} = 2i\epsilon_{\alpha\beta} \text{str} (k + \nu) = 2i\epsilon_{\alpha\beta} \nu \]  

(53)

is in contradiction with the defining property of the supertrace (35) requiring the left hand side to be zero. A more careful analysis shows that this result can be interpreted as certain anomaly due to divergences for non-polynomial functions under the supertrace operation (34). (Note, that, as expected, \( \text{str}_\nu \{ k\tilde{y}_\alpha, \tilde{y}_\beta \} = 0 \).)

Of course, there is no formal contradiction with the uniqueness theorem for the supertraces (34) and (52) because they were proved for polynomials (or formal power series). The formulae like (19) drive us away from this class. In \([3]\), it was shown that the star-product (3) is well-defined for such nonpolynomial functions. This is not true for the supertrace (34) however.

The question therefore is whether there exists a consistent generalization \( \text{str}_\nu \) of the supertrace (34) to the whole algebra, such that it reduces to (52) for the functions depending only on \( \tilde{y} \) and to (34) at \( \nu = 0 \). One complication is that the polynomials in \( s_{0\alpha}, \tilde{y}_\alpha, k, \psi_{1,2} \) do not form a closed algebra as the commutation relations (24) contain explicitly a nonpolynomial function \( K = k \exp i(zy) \) for \( \nu \neq 0 \). This makes it impossible to work with the Weyl-ordered polynomials in \( s_{0\alpha} \) and \( \tilde{y}_\alpha \) as a starting point towards a
proper generalization of (34). However, such an ordering prescription is not relevant for the problem under consideration because the star-product \[5\] is itself defined via certain normal ordering \[3, 5\]. The best we can do at this stage is to conjecture that at \(\nu \neq 0\) there exists some operation \(\text{str}_\nu\) that reduces to (52), when restricted to the subalgebra of functions of \(\tilde{y}_\alpha\) and \(k\), and to the supertrace operation (34) in the limit \(\nu \to 0\). In that case, the models with different parameters \(\nu\) admit the action functionals with the same Lagrangian (38) but with \(\text{str}_\nu\) instead of \(\text{str}\) in the action (33).

Let us note that if the corresponding actions exist they may also be well-defined on-mass-shell what is \textit{a priori} neither necessary nor guaranteed. Note that the difference between the off-mass-shell and on-mass-shell setting in our approach is that in the former case the fluctuational parts of the fields are arbitrary (i.e., formal) functions of the spinor variables \(y_\alpha\) and \(z_\alpha\), while in the latter case they depend on these variables in a specific way governed by the field equations (31) and (32) (e.g., via the dependence on \(\tilde{y}\) as in (50)). By analogy with the standard space-time action formulations one can admit that the action functional should not necessarily be on-mass-shell finite (for example because of a volume divergence). Nevertheless, the insertion of the solution (49), (50) into the action defined with \(\text{str}_\nu\) leads to some well-defined functional by virtue of (52) at least at the quadratic level. Note that even at \(\nu = 0\) the variable \(b\) contains via (41) a factor of \(\exp(i(z\tilde{y}))\) on which the supertrace diverges. Fortunately, this factor falls out of the Lagrangian (38) and \(\text{str}_\nu L\) turns out to be well-defined at least in the quadratic approximation in \(b\). It is an interesting question even for the case of \(\nu = 0\) whether this is true beyond the quadratic approximation (i.e., at the interaction level). In fact, the requirement that the nonlinear action is well-defined on-mass-shell may give a useful criterion for the selection of appropriate vacuum solutions in connection of the locality problem discussed in [3].

5 Special \(\nu\)

It was shown in [3] that for special values \(\nu = 2l + 1, l \in \mathbb{Z}\), the algebra \(Aq(2, \nu)\) acquires infinite-dimensional ideals \(I_\nu\) identified with the null spaces w.r.t. the invariant bilinear form constructed from the supertrace (32), i.e. \(\text{str}_\nu (ab) = 0, \forall a \in I_\nu, b \in Aq(2, \nu)\). As a result, all elements that belong to \(I_\nu\) do not contribute under \(\text{str}_\nu\) and therefore fall out of the action \(S = \text{str}_\nu L\). The factor algebra \(Aq(2, 2l + 1)/I_{2l+1}\) is isomorphic [3] to the associative algebra of \((2|l| + 1) \times (2|l| + 1)\) matrices treated as supermatrices from \(\text{Mat}(|l|, |l| + 1)\). The supertrace \(\text{str}_\nu\) for polynomials \(P(\tilde{y})\) amounts for this case to the usual matrix supertrace.

As a result, in the \(\tilde{y}_\alpha\) sector (i.e., at least for the on-mass-shell action) for the odd integer values of \(\nu\) the action acquires the form of the supertrace of some forth-order matrix polynomial. The analysis of the off-mass-shell action requires a better understanding of the properties of \(\text{str}_\nu\) in \(s_{0\alpha}\) sector. One comment is that because of the linear dependence of the field \(s_\alpha\) on \(\rho\) (cf. (15)) and nontrivial commutation relations of \(\rho\) with \(s_{0\alpha}\), the analysis in this sector does not amount to the algebraic analysis of \(Aq(2; \nu)\) despite the

\[1\]For example, one can see the difficulty from the formal computation of \(\text{str}_\nu \{kK * s_{0\alpha}, s_0^n\}\) \(= -4i\text{str}_\nu (\exp(i(z\tilde{y}) - i\nu k)\) that leads to a finite expression for \(\text{str}_\nu \exp(i(z\tilde{y})\) by virtue of (52) that is in contradiction with (33).
fact that the variables $s_{0\alpha}$ form themselves the deformed oscillator algebra (24). As a result there is no reason to expect any peculiarity in the $s_{0\alpha}$ sector for the special values of $\nu$. We therefore conclude that for odd integer values of $\nu$ the action of the model becomes analogous to the noncommutative version of Matrix String Theory action [14].

Let us make a few comments on the limit $\nu \to \infty$ that leads to one more interesting interpretation of the model. As shown in [9], the Lie superalgebra constructed from $Aq(2,\nu)$ via (anti)commutators tends in this limit to the superextension of the Lie algebra of two-dimensional volume preserving diffeomorphisms with the even part isomorphic to the direct sum of two volume preserving diffeomorphism algebras of a sphere $S^2$, $w_\infty \sim su(\infty)$. Choosing odd integer $\nu$ for the limiting sequence $\nu \to \infty$, one recovers the approximation of 2d volume-preserving diffeomorphism algebra by matrix algebras discovered by Hoppe [15].

The simplest way to see this correspondence is to rewrite the commutation relations (23) in terms of appropriately rescaled variables as

$$\left[\tilde{y}_\alpha, \tilde{y}_\beta\right]_* = 2i\epsilon_{\alpha\beta}(\nu^{-1} + k), \quad \tilde{y}_\alpha k = -k\tilde{y}_\alpha.$$  

(54)

The identification $\hbar = \nu^{-1}$ leads to the quasiclassical interpretation of the limit $\nu \to \infty$ in terms of two-dimensional sphere phase space. The Poisson brackets of $S^2$ coordinates identified with the symmetric bilinear combinations of oscillators $\tilde{y}_\alpha$ do not depend on $k$ and describe $so(3)$ Lie algebra, while the role of the $k$-dependent term in (54) is that it leads to the unit radius-squared of the sphere $S^2$ given by the appropriately rescaled quadratic Casimir operator of $so(3)$ (for more detail see [9]).

Another important point is that, for large $\nu$, $m \sim \nu \lambda$ where $\lambda$ is the inverse radius of $AdS_3$. Therefore $m$ tends to infinity in the limit $\nu \to \infty$. In other words, the sector of matter fields decouples from the spectrum in this limit. In terms of the coordinate-free formulation developed in this paper this fact manifests itself as follows. The part of the field $b$ describing the matter fields is proportional to the generating element $\psi_2$ which effectively replaces some of the commutators in the algebra by anticommutators due to anticommutativity with $\psi_1$. As a result, the limit $\nu \to \infty$ turns out to be ill-defined in this sector and the only possibility to have $\nu = \infty$ is to set the part of $b$ proportional to $\psi_2$ equal to zero. This conclusion is indeed natural in the context of analysis of [3] where it was shown that the corresponding relativistic matter fields acquire infinite masses in the limit $\nu \to \infty$. The fact that the large $N$ limit in two-dimensional theory is equivalent to the limit $m \to \infty$ in $AdS_3$ for certain matter fields looks quite interesting.

An interesting question is whether the limiting procedure in the $s_{0\alpha}$ sector can be performed in such a way that the coordinates $z^\alpha$ would admit interpretation of commuting coordinates of 2d spacetime in the limit $\nu \to \infty$. This is not obvious because of the appearance of the terms containing $\exp i(zy)$ (e.g. via $K$ in (14)). If nevertheless such a procedure is possible, the resulting model may provide us with some sort of manifest realization of the bulk/boundary correspondence [16]. Indeed, the original 3d space-time (bulk) theory containing gravity will be equivalent in the limit $\nu \to \infty$ to a 2d YM theory with an infinite-dimensional gauge group.
6 Conclusion

The action proposed in this paper for $AdS_3$ higher-spin-matter system is very different from the usual field-theoretical actions since it is formulated in the space of auxiliary spinor variables rather than in the usual space-time. Nevertheless, it reproduces correctly the dynamics of the $3d$ system under consideration. An interesting question for the future is to use this action for quantization of the model.

For special values of the parameter of mass the model acquires a form analogous to that of the matrix models. It is tempting to speculate that the relationship of our action with the space-time higher spin actions may be analogous to the relationship between relativistic $M$ theory and its Matrix formulation [17].

Note that consistent field-theoretical higher spin actions in $AdS_4$ are known not only at the free field level [18] but also to the cubic order in interactions [19]. The full $d \geq 4$ nonlinear higher spin gauge actions are not yet known however. Hopefully, the approach proposed in this paper will make it possible to construct full nonlinear “matrix” actions for the higher spin theories in $d \geq 4$.

The proposed construction exhibits a number of striking similarities with the current ideas of the theory of fundamental interactions associated with $M$ theory. The $\nu \to \infty$ limit is analogous to the $SU(\infty)$ $2d$ Yang-Mills theory thus indicating the formal connection to Matrix String Theory [14]. On the other hand, $\nu = 0$ case is proved to have a form of the $2d$ noncommutative Yang-Mills with an infinite-dimensional gauge group. An interesting fact is that the large $N$ limit in our model is equivalent to the large mass limit for certain $AdS_3$ matter fields.

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