Normal Smoothings for Charney-Davis Strict Hyperbolizations

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Abstract

We prove that the Charney-Davis Strict Hyperbolization of a smoothly cubulated manifold admits a normal smooth structure. We also prove that this normal smooth structure is diffeomorphic to a smooth structure that has good tangential properties.

In his 1987 paper “Hyperbolic Groups” [4] M. Gromov introduced the process of hyperbolization. This process assigns to each simplicial complex $K$ a nonpositively curved (in the geodesic sense) complex. The hyperbolization process has a lego type flavor and it can roughly be described in simple terms: to construct the hyperbolization of a simplicial complex $K$ we replace its basic set of pieces (simplices) by another basic set of “hyperbolization pieces”. In other words, to construct the hyperbolization of $K$ we assemble the hyperbolization pieces using the same pattern as the one used to assemble $K$. The hyperbolization process was later studied and used by Davis and Januszkiewicz in [3].

An important property of hyperbolization is that if $K$ is a PL manifold then the hyperbolization of $K$ is also a PL manifold. Moreover Davis and Januszkiewicz [3] showed that the hyperbolization of a smoothly triangulated manifold is also a smooth manifold.

In [2] R. Charney and M. Davis built on hyperbolization previous versions and presented the strict hyperbolization process. In this case one begins with a cube complex $K$ (with large links) and obtains a negatively curved space $K_X$. In this process we replace the cubes by what we call Charney-Davis strict hyperbolization pieces. Again, the hyperbolization of a smoothly cubulated manifold is also a smooth manifold.

Notation: A smooth cubulation of a smooth manifold $M$ is a homeomorphism $f : K \to M$, where $K$ is a cube complex and $f$ is a smooth embedding when restricted to each cube of $K$. In this case, for simplicity, we will just say that $K$ is a smooth cube complex, or a smooth cube manifold. Therefore, if $K$ is a smooth cube complex, then $K_X$ is smoothable.

Let $K$ be a smooth cube complex. Then the Charney-Januszkiewicz-Davis smooth structure on $K_X$ is “good” from the Geometric Topology point of view because it has good tangential properties (see Section 4). But it is quite poor from the Geometry point of view because there is no a priori relationship between the smooth structure and the rich geometry of $K_X$. To correct this we introduce “normal smooth structures” in the paragraphs below. These normal structures

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are very natural and useful (see [7]). Of course, there is no guarantee that such structures exist on \(K_X\); actually there is no guarantee that \(K_X\) is smoothable at all. In this paper we prove the following.

**Main Theorem.** Let \(K\) be a cube complex. If \(K\) is smooth, then \(K_X\) admits a normal smooth structure.

**Addendum.** Furthermore the normal smooth structure has good tangential properties.

By “good tangential properties” we mean that \(K_X\), with its normal smooth structure, smoothly embeds in \(K \times X\) with trivial normal bundle (see Theorem 4.3 for more details). In [2] Charney and Davis ask the following:

**Question 7.4 in [2].** Is there a stably parallelizable Charney-Davis strict hyperbolization piece?

The addendum together with a positive answer to the question above would imply that if \(K\) is a smooth cube manifold then the natural map \(K_X \to K\) is covered by a map of stable tangent bundles, where we are considering \(K_X\) here with a normal smooth structure.

Before we explain what normal structures are we recall some features of the Charney-Davis strict hyperbolization process. For more details see Section 3.

We write \(\Box^n = [0, 1]^n\). A *Charney-Davis hyperbolization piece* \(X^n\) of dimension \(n\) is a compact connected orientable hyperbolic \(n\)-manifold with corners satisfying the properties stated in Lemma 6.2 of [2]. We state some relevant facts. There is a smooth map \(f : X^n \to \Box^n\), such that \(X^n\) and \(f\) satisfy the following.

1. For any \(k\)-face \(\Box^k\) of \(\Box^n\) we have that the \(k\)-face \(X_{\Box^k} = f^{-1}(\Box^k)\) of \(X^n\) is totally geodesic in \(X^n\), and it is a Charney-Davis hyperbolization piece of dimension \(k\).

2. The faces of \(X^n\) intersect orthogonally (unless one is a face of the other).

The interior \(X_{\Box^k} = f^{-1}(\Box^k)\) will be denoted by \(\hat{X}_{\Box^k}\).

Let \(X_{\Box^k}\) be a \(k\)-face. We denote by \(\text{Link}(X_{\Box^k}, X^n)\) the link of \(X_{\Box^k}\) in \(X^n\) (at \(p\)), that is, the set of inward vectors orthogonal to \(X_{\Box^k}\) at \(p\), for some \(p \in X_{\Box^k}\). The link can be identified with the canonical all-right \((n-k-1)\)-simplex \(\Delta_{S_{n-k-1}}\). In this sense we consider \(\Delta_{S_{n-k-1}} \subset T_p X\). Similarly we can consider the link \(\text{Link}(\Box^k, \Box^n)\) of \(\Box^k\) in \(\Box^n\). It can also be identified with \(\Delta_{S_{n-k-1}} \subset T_q \Box^n\), for \(q \in \Box^k\). We can identify the derivative of \(f\), \((Df_p|_{\Delta_{S_{n-k-1}}}) : \Delta_{S_{n-k-1}} \to \Delta_{S_{n-k-1}}\), with the identity map (see ).

Let \(K\) be a cube complex and \(\Box \in K\). Recall that the link \(\text{Link}(\Box, K)\) of \(\Box\) in \(K\) is the all-right spherical complex \(\{\text{Link}(\Box, \Box') : \Box \subset \Box' \in K\}\).

The strict hyperbolization process of Charney and Davis is done by gluing copies of \(X^n\) using the same pattern as the one used to obtain the cube complex \(K\) from its cubes (see Section 3.
for more details). This space is called $K_X$ in [2]. Note that we get a map $F : K_X \rightarrow K$, which restricted to each copy of $X^n$ is just the map $f : X^n \rightarrow □ n$. We will write $X_{□ k} = F^{-1}(□ k)$, for a $k$-cube $□ k$ of $K$.

The link $\text{Link}(X_□, X_K)$ of $X_□$ in $X_K$ is the all-right spherical complex $\{\text{Link}(X_□, X_{□'}) : □ \subset □' \in K\}$. We can use the derivative of the map $F : K_X \rightarrow K$ (in a piecewise fashion) to identify $\text{Link}(X_{□ k}, X_K)$ with $\text{Link}(□ k, K_X)$. Hence we write $\text{Link}(X_{□ k}, K_X) = \text{Link}(□ k, K_X)$, for a $k$-cube $□ k$ of $K$.

Let $□ \in K$. A link smoothing of $\text{Link}(□ k, K_X) = \text{Link}(X_{□ k}, K_X)$ is just a homeomorphism $h_{□ k} : S^{n-k-1} \rightarrow \text{Link}(X_{□ k}, K_X)$. A (complete) set of link smoothings is a set $A = \{h_□\}_□ \in K$.

We are now ready to define normal structures. Let $X_{□ k} \subset X_{□ n}$ be a $k$-face of $K_X$, contained in the copy $X_{□ n}$ of $X$ over $□ n$. For a non-zero vector $u$ normal to $X_{□ k}$ at $p \in X_{□ k}$, and pointing inside $X_{□ n}$, we have that $\exp_p(tu)$ is defined and contained in $X_{□ n}$, for $0 \leq t < t_0$, for certain fixed $t_0$ (see ). Let $A = \{h_□\}_□ \in K$ be a fixed set of link smoothings. We define the map $H_{□ k} : D^{n-k} \times X_{□ k} \rightarrow K_X$ given by

$$H_{□ k}(t v, p) = \exp_p\left(2rt h_{□ k}(v)\right)$$

where $v \in S^{n-k-1}$ and $t \in [0,1)$. We have that $H_{□ k}$ is a topological embedding (see ). The map $H_{□ k}$ is called a normal chart for the $k$-face $X_{□ k}$. The collection $\{H_{□ k}\}_{□ k \in K}$ of the normal charts is a normal atlas, and if this atlas is smooth (or $C^k$) the induced differentiable structure is called a normal smooth (or $C^k$) structure.

Note that the charts $H_□$ respect normal directions and radial distances. Note also that the normal atlas $\{H_{□ k}\}_{□ k \in K}$ depends only on the set $A$ of links smoothings.

Here is a description of the paper. In Section 1 we deal with smooth structures on cube complexes. In this section we recall and review some necessary concepts, definitions and results that appear in [3]. In Section 2 we study Charney-Davies strict hyperbolization pieces. In Section 3 we review the Charney-Davies strict hyperbolization process, compare two ways of doing this process. In this section we also introduce ways to construct smooth structures on Charney-Davies hyperbolizations with good tangential properties. In Section 4 we deal with normal smooth structures. We also treat the case of smooth manifolds with one point singularities. There are three appendices.

The results in this paper are key ingredients in problem of smoothing the metric of a strictly hyperbolized manifold (see [7]).
1. Smooth Structures on Cube Complexes and All-Right Spherical Complexes.

For the basic definitions and results about cube and spherical complexes see for instance [1]. Recall that a spherical complex is an all-right spherical complex if all of its edge lengths are equal to $\pi/2$. Given a (cube or all-right spherical) complex $K$ we use the same notation $K$ for the complex itself (the collection of all closed cubes or simplices) and its realization (the union of all cubes or simplices). For $\sigma \in K$ we denote its interior by $\dot{\sigma}$.

Let $M^n$ be a smooth manifold of dimension $n$. A smooth cubulation of $M$ is given by $(K, f)$, where $K$ a cube complex and $f : K \to M$ a non-degenerate $PD$ homeomorphism [3], that is, for all $\sigma \in K$ we have $f|_{\sigma}$ is a smooth embedding. Sometimes we will write $K$ instead of $(K, f)$. The smooth manifold $M$ together with a smooth cubulation is a smooth cube manifold or a smooth cube complex. A smooth all-right-spherical triangulation and a smooth all-right-spherical manifold (or complex) is defined analogously.

Note that if $K$ is a smooth cubulation (or all-right spherical triangulation) of $M$, then $K \cong_{PL} M$, that is, $K$ is $PL$-homeomorphic to the smooth manifold $M$.

The geometric link $\text{Link}(\sigma^j, K)$ of an open $j$-cube or $j$-all-right simplex $\sigma^j$ is the union of the end points of straight (geodesic) segments of small length $\epsilon > 0$ emanating perpendicularly (to $\dot{\sigma^j}$) from some point $x \in \dot{\sigma^j}$. We say that the link is based at $x$. And the star $\text{Star}(\sigma, K)$ as the union of such segments. We can identify the star with the cone of the link $C\text{Link}(\sigma, K)$ (or $\epsilon$-cone) defined as

$$C\text{Link}(\sigma, K) = \text{Link}(\sigma, K) \times [0, \epsilon) / \text{Link}(\sigma, K) \times \{0\}$$

We shall denote the cone point by $o$ or, more specifically, by $o_{C\text{Link}(\sigma, K)}$. Thus a point $x$ in $C\text{Link}(\sigma, K)$, different from the cone point $o$, can be written as $x = tu$, $t \in (0, \epsilon)$, $u \in \text{Link}(\sigma, K)$. For $s > 0$ we get the cone homothety $x \mapsto sx = (st)u$ (partially defined if $s > 1$). If we want to make explicit the dependence of the link or the cone on $\epsilon$ we shall write $\text{Link}_\epsilon(\sigma, K)$ or $C\text{Link}_\epsilon(\sigma, K)$ respectively. Also, we will always take $\epsilon < 1/2 (< \pi/4$ in the spherical case) and it can be verified that all results in this section (unless otherwise stated) are independent of the choice of the $\epsilon$'s. As usual we shall identify the $\epsilon$-neighborhood of $\dot{\sigma}$ in $K$ with $C_{\epsilon}\text{Link}(\sigma, K) \times \dot{\sigma}$ (or just $C\text{Link}(\sigma, K) \times \dot{\sigma}$). Hence a cone homothety induces a neighborhood homothety obtained by crossing it with the identity $1_{\dot{\sigma}}$. Note that $\text{Link}(\sigma, K)$ and $C\text{Link}(\sigma, K)$ are subsets of $K$.

In what follows we assume that $f : K \to M$ is a smooth cubulation (or all-right spherical triangulation) of the smooth manifold $M$. Recall that the link $\text{Link}(\sigma^i, K)$, $\sigma^i \in K$, has a natural all-right piecewise spherical structure, which induces a simplicial structure and thus a $PL$ structure on $\text{Link}(\sigma^i, K)$. Since the $PL$ structure on $M$ induced by $K$ is Whitehead compatible with $M$ we have that the link $\text{Link}(\sigma^i, K)$ is $PL$ homeomorphic to $S^{n-i-1}$. A link smoothing for $\dot{\sigma^i}$ (or $\sigma^i$) is just a homeomorphism $h_{\sigma^i} : S^{n-i-1} \to \text{Link}(\sigma^i, K)$. The cone of $h_{\sigma^i}$ is the map

$$C h_{\sigma^i} : D^{n-i} \to C\text{Link}(\sigma^i, K)$$

given by $tx = [x, t] \mapsto th_{q^i}(x) = [h_{q^i}(x), t]$, where we are canonically identifying the $\epsilon$-cone of
$\mathbb{S}^{n-i-1}$ with the disc $\mathbb{D}^{n-i}$. We remark that we are not assuming $h_{\sigma^i}$ to be smooth.

A link smoothing $h_{\sigma^i}$ induces the following smoothing of the normal neighborhood of $\dot{\sigma}^i$:

$$h_{\sigma^i}^* = f \circ \left( C \ h_{\sigma^i} \times 1_{\dot{\sigma}^i} \right) : \mathbb{D}^{n-i} \times \dot{\sigma}^i \rightarrow M$$

The pair $(h_{\sigma^i}^*, \mathbb{D}^{n-i} \times \dot{\sigma}^i)$, or simply $h_{\sigma^i}^*$, is a normal chart on $M$. Note that the collection $A = \{(h_{\sigma^i}^*, \mathbb{D}^{n-i} \times \dot{\sigma}^i)\}_{\sigma^i \in K}$ is a topological atlas for $M$. Sometimes will just write $A = \{h_{\sigma^i}^*\}_{\sigma^i \in K}$. The topological atlas $A$ is called a normal atlas. It depends uniquely on the the complex $K$, the map $f$ and the collection of link smoothings $\{h_{\sigma}\}_{\sigma \in K}$. To express the dependence of the atlas on the set of links smoothings we shall write $A = A(\{h_{\sigma}\}_{\sigma \in K})$ (this is different from $A = \{h_{\sigma}^*\}_{\sigma^i \in K}$, as written above).

The most important feature about these normal atlases is that they preserve the radial and sphere (link) structure given by $K$. These features make normal atlases very powerful tools for geometric constructions.

Note that not every collection of link smoothings induce a smooth atlas. But when the atlas is smooth we call $A$ a normal smooth atlas on $M$ with respect to $K$ and the corresponding smooth structure $S'$ a normal smooth structure on $M$ with respect to $K$.

Remarks.
1. If the normal atlas $A$ is smooth the maps $f|_{\dot{\sigma}^i} : \dot{\sigma}^i \rightarrow (M, S')$ and the link smoothings

$$h_{\sigma^i} : \mathbb{S}^{n-i-1} \rightarrow (M, S')$$

are, by construction, smooth embeddings. Here, as before, $S'$ is the normal smooth structure induced by the normal smooth atlas $A$.

2. The atlas $A$ is smooth if and only if there is a smooth structure $S'$ such that all normal charts $h_{\sigma^i}^* : \mathbb{D}^{n-i} \times \dot{\sigma}^i \rightarrow (M, S')$ are smooth embeddings. (This is true for any topological atlas)

Later in this section we show that the atlas $A(\{h_{\sigma}\})$ is smooth if and only if the set of link smoothings $\{h_{\sigma}\}$ is “smoothly compatible”. Here is the main result of [6].

**Theorem 1.1.** Let $M$ be a smooth cube manifold, with smooth structure $S$. Then $M$ admits a normal smooth structure $S'$ diffeomorphic to $S$.

Hence if $M^n$ is a smooth manifold with smooth structure $S$ and $K$ is a cubulation of $M$, then there are link smoothings $h_{\sigma^i}$, for all $\sigma^i \in K$, such that the atlas $A = A(\{h_{\sigma}\}_{\sigma \in K})$ is smooth. Moreover the normal smooth structure $S'$, induced by $A$, is diffeomorphic to $S$.

**Addendum Theorem 1.1.** The statement of Theorem 1.1 also holds for smooth all-right-spherical complexes.

The following is a corollary of the proof of Theorem 1.1 given in [6] (see Lemma 1.2 in [6]).
Corollary 1.1.2. Let \( f : K \to (M,S) \) be a smooth cubulation (or all-right spherical triangulation) of the smooth manifold \((M,S)\). Let \( S' \) as in Theorem 1.1 Then, for every \( \sigma \in K \) we have that \( f(\text{Link}(\sigma, K)) \) is a smooth submanifold of \((M,S')\).

Corollary 1.1.3. Let \( M, S \) and \( S' \) as in Theorem 1.1. (or its addendum). Then \( K \) is \( PL \)-homeomorphic to \((M,S')\).

Proof. Since \( K \) is a smooth cubulation of \((M,S)\) we have \( K \cong_{PL} (M,S) \). On the other hand, by Theorem 1.1 we get \((M,S) \cong_{DIFF} (M,S')\). Hence \( K \cong_{PL} (M,S')\). This proves the corollary.

Remark. Note that the image of the chart \( h_{\sigma}^* \) is the open normal neighborhood \( \mathcal{N}_{\epsilon} (\hat{\sigma}, K) \) of width \( \epsilon \) of \( \hat{\sigma} \) in \( K \). Even though we are assuming, for simplicity, that \( \epsilon < 1/2 \) (\( \epsilon < \pi/4 \) in the spherical case) it can be checked from the proof of the Theorem 1.1 in [6] that we can actually take \( \epsilon = 1 \) (\( \epsilon = \pi/2 \)) for the charts.

1.2. Induced Link Smoothings.

Let \( K \) be a cubical or all-right spherical complex. Then the links of \( \sigma \in K \) are all-right-spherical complexes. We explain here how to obtain from a given collection of link smoothings for \( K \) (and its corresponding normal atlas and structure) a collection of links smoothings for a link in \( K \) (and its corresponding normal atlas and structure).

The all-right-spherical structure on \( \text{Link}(\sigma, K) \) induced by \( K \) has all-right-spherical simplices \( \{ \tau \cap \text{Link}(\sigma, K), \tau \in K \} \). Note that \( \tau \cap \text{Link}(\sigma, K) \) is non-empty only when \( \sigma \subseteq \tau \), hence we can write

\[
\text{Link}(\sigma, K) = \{ \tau \cap \text{Link}(\sigma, K), \sigma \subseteq \tau \in K \}
\]

Since \( \tau \cap \text{Link}(\sigma, K) \) is a simplex in the all-right spherical complex \( \text{Link}(\sigma, K) \) we can consider its link \( \text{Link} \left( \tau \cap \text{Link}(\sigma, K), \text{Link}(\sigma, K) \right) \). By definition we have:

\[
\text{Link} \left( \tau \cap \text{Link}(\sigma, K), \text{Link}(\sigma, K) \right) = \text{Link} \left( \tau, K \right)
\]

provided we choose the radii and bases of the links properly. In the formula above radii and bases are not specified but the radii are certainly not equal. The simple relationship between these radii is given by equation (1) in the proof of Lemma 1.2 [4] (or the corresponding one in the spherical case; see Remark 1 after the proof of Lemma 1.2 [6]). For the cone link we have a similar formula but it is not an equality, it is just an identification, which we call \( \mathfrak{R} \):

\[
(1.2.1) \quad C \text{Link} \left( \tau \cap \text{Link}(\sigma, K), \text{Link}(\sigma, K) \right) \xrightarrow{\mathfrak{R}} C \text{Link} \left( \tau, K \right)
\]

On the term in the right side the radial segments are “straight” (geodesic in each simplex of \( K \)) but on the left side the radial segments are “curved” (they lie in \( \text{Link}(\sigma, K) \)).
Remark 1.2.2. In cubical case the identification (1.2.1) above can be done in the following way. Let \( v \in \text{CLink}(\tau, K) \) = \( \text{CLink}_{\tau}(\tau, K) \) at some \( p \in \tau \) and consider \( \text{Link}(\sigma, K) = \text{Link}_{\sigma}(\sigma, K) \) at some point \( q \in \sigma \), with \( d_{K}(p, q) = s \) with the segment \([p, q] \) perpendicular to \( \sigma \). Then \( v \) corresponds to the point \( v' \) in the segment \([q, v] \) at a distance \( s \) from \( q \). The (angular) distance in \( \text{Link}(\sigma, K) \) from \( p \) to \( v' \) is \( \tan^{-1}\left(\frac{d_{K}(v, p)}{s}\right) \). We shall write \( \Re : \text{CLink}(\tau, K) \hookrightarrow \text{CLink}(\tau \cap \text{Link}(\sigma, K), \text{Link}(\sigma, K)) \subset \text{Link}(\sigma, K) \) for the radial projection described above. And we will write \( \Re = \Re_{p,q,r,s} \) if we want to make explicit the dependence of \( \Re \) on the choices above.

Remark 1.2.3. In the all-right spherical case the identification \( \Re \) is similar; we only need to replace the formula \( \tan^{-1}\left(\frac{d_{\Re}(v, p)}{s}\right) \) by \( \tan^{-1}\left(\frac{\tan d_{\Re}(v, p)}{\sin s}\right) \). The latter formula is obtained using spherical trigonometry.

Remark 1.2.4. Note that the representation of the radial projection \( \Re \) in the chart \( (h_{\tau}, \Re^{n-j} \times \tau) \), \( \tau \) a \( j \)-simplex, is smooth.

Now assume in addition that \( K \) is PL manifold, and let \( \{h_{\sigma}\}_{\sigma \in K} \) be a set of link smoothings for (the links of) \( K \). Since we have \( \text{Link}(\tau \cap \text{Link}(\sigma, K), \text{Link}(\sigma, K)) = \text{Link}(\tau, K) \) we can say that the set of link smoothings \( \{h_{\sigma}\}_{\sigma \in K} \) for \( K \) induces, just by restriction, a set of link smoothings for \( \text{Link}(\sigma, K), \sigma \in K \), given by \( \{h_{\tau}\}_{\tau \subseteq \tau} \). We have the following diagram:

\[
\begin{array}{ccc}
\mathbb{S}^{n-i-1} & \xrightarrow{h_{\tau \cap \text{Link}(\sigma, K)}} & \text{Link}(\tau \cap \text{Link}(\sigma, K), \text{Link}(\sigma, K)) \\
\| & & \| \\
\mathbb{S}^{n-i-1} & \xrightarrow{h_{\tau \cap \text{Link}(\sigma, K)}} & \text{Link}(\tau, K) \\
\end{array}
\]

The double vertical lines on the sides are equalities. This diagram is a triviality in the sense that the function on the top row \( h_{\tau \cap \text{Link}(\sigma, K)} \) is equal to the function on the bottom row \( h_{\tau \cap \text{Link}(\sigma, K)} \); but we shall use the notation \( h_{\tau \cap \text{Link}(\sigma, K)} \) when we consider the link smoothing for the link \( \text{Link}(\tau \cap \text{Link}(\sigma, K), \text{Link}(\sigma, K)) \), instead of the link smoothing for the link \( \text{Link}(\tau, K) \).

Note that the atlas \( A_{\sigma} = A_{\text{Link}(\sigma, K)} = \{h_{\tau \cap \text{Link}(\sigma, K)}\}_{\tau \subseteq \tau} \) is a (a priori just topological) normal atlas on \( \text{Link}(\sigma, K) \).

We now change the notation a little bit, to match the one we will use in 1.3: we replace \( \tau \) by \( \sigma^{i} \). So, let \( \sigma^{k} \subset \sigma^{i} \subset K \). We mentioned above that we have \( h_{\sigma^{i} \cap \text{Link}(\sigma^{k}, K)} = h_{\sigma^{i}} \); but because the map \( \Re \) of 1.2.1 is not an equality, we can not say that the neighborhood smoothings \( h_{\sigma^{i} \cap \text{Link}(\sigma^{k}, K)} \) and \( h_{\sigma^{i}} \) are equal (on their respective domains). But there is a relationship between \( h_{\sigma^{i} \cap \text{Link}(\sigma^{k}, K)} \) and \( h_{\sigma^{k}} \). The map \( h_{\sigma^{i} \cap \text{Link}(\sigma^{k}, K)} \) is given by the following composition:

\[
(1.2.5) \quad \mathbb{D}^{n-i} \times (\hat{\sigma}^{i} \cap \text{Link}(\sigma^{k}, K)) \xrightarrow{h_{\sigma^{i}}} \text{CLink}(\sigma^{i}, K) \times (\hat{\sigma}^{i} \cap \text{Link}(\sigma^{k}, K)) \xrightarrow{\Re \times 1} \text{Link}(\sigma^{k}, K)
\]

where \( 1 \) is the identity on \( \hat{\sigma}^{i} \cap \text{Link}(\sigma^{k}, K) \), \( \Re \) is the map in 1.2.2. Also the first arrow is really the map \( h_{\sigma^{i}} \) restricted to \( \mathbb{D}^{n-i} \times (\hat{\sigma}^{i} \cap \text{Link}(\sigma^{k}, K)) \) (the domain of \( h_{\sigma^{i}} \) is \( \mathbb{D}^{n-i} \times \hat{\sigma}^{i} \)).
1.3. Change of charts and smooth compatibility.

In this section we define what a “smoothly compatible set of link smoothings” is and prove that a normal atlas is smooth if and only if the set of link smoothings is smoothly compatible.

Let \( f : K \to M^n \) be a smooth cubulation (or all-right spherical triangulation). Since in what follows of this subsection the function \( f \) is not essential, to simplify our notation we identify \( K \) and \( M \) via \( f \). Let \( \{h_\sigma\} \) be a set of link smoothings on \( K \). Recall that this set determines a (not necessarily smooth) atlas \( A = \{h_\sigma^*\} \) on \( K \).

Let \( \sigma^k \subset \sigma^i \subset K \). We say that the link smoothings \( h_{\sigma^k}, h_{\sigma^i} \) are smoothly compatible if the neighborhood smoothing \( h^*_{\sigma^i \cap \text{Link}(\sigma^k,K)} \)

\[
\mathbb{D}^{n-i} \times (\hat{\sigma}^i \cap \text{Link}(\sigma^k,K)) \xrightarrow{h^*_{\sigma^i \cap \text{Link}(\sigma^k,K)}} \text{Link}(\sigma^k,K) \xrightarrow{h_{\sigma^k}^{-1}} S^{n-k-1}
\]

is a smooth embedding. Here we are considering \( \text{Link}(\sigma^k,K) \) with the smooth structure induced by the link smoothing \( h_{\sigma^k} : S^{n-k-1} \to \text{Link}(\sigma^k,K) \). That is, we consider \( \text{Link}(\sigma^k,K) \) with the smooth structure \( (h_{\sigma^k})_*(S_{S^{n-k-1}}) \), where \( S_{S^{n-k-1}} \) is the canonical smooth structure on \( S^{n-k-1} \). Equivalently, \( h_{\sigma^k}, h_{\sigma^i} \) are smoothly compatible if the composition

\[
\mathbb{D}^{n-i} \times (\hat{\sigma}^i \cap \text{Link}(\sigma^k,K)) \xrightarrow{h^*_{\sigma^i \cap \text{Link}(\sigma^k,K)}} \text{Link}(\sigma^k,K) \xrightarrow{h_{\sigma^k}^{-1}} S^{n-k-1}
\]

is a smooth embedding.

**Lemma 1.3.1.** Fix \( \sigma^k \in K \) and assume \( h_{\sigma^k} \) is smoothly compatible with \( h_{\sigma^i} \), for every \( \sigma^i \supset \sigma^k \). Then the atlas \( A_{\sigma^k} = A_{\text{Link}(\sigma^k,K)} = \{h^*_{\sigma^i \cap \text{Link}(\sigma^k,K)}\}_{\sigma^k \subset \sigma^i} \) is a smooth normal atlas on \( \text{Link}(\sigma^k,K) \). Moreover the link smoothing

\[
h_{\sigma^k} : S^{n-k-1} \to \left( \text{Link}(\sigma^k,K), S_{\sigma^k} \right)
\]

is a diffeomorphism. Here \( S_{\sigma^k} \) is the normal smooth structure induced by the normal atlas \( A_{\sigma^k} \).

**Proof.** It follows from the fact that the maps \( h_{\sigma^k}^{-1} \circ h^*_{\sigma^i \cap \text{Link}(\sigma^k,K)} \) are smooth embeddings. This proves the lemma.

The set of link smoothings \( \{h_\sigma\} \) is smoothly compatible if \( h_{\sigma^k}, h_{\sigma^i} \) are smoothly compatible whenever \( \sigma^k \subset \sigma^i \subset K \).

To simplify our notation write \( S = \text{Link}(\sigma^k,K) \), \( \sigma^i = \sigma^i \cap S \), and \( \hat{\sigma}^i = \hat{\sigma}^i \cap S \). Now using 1.2.5 we can say that the set of link smoothings is smoothly compatible if for every \( \sigma^k \subset \sigma^i \) the composition maps

\[
(1.3.2) \quad \mathbb{D}^{n-i} \times \hat{\sigma}^i \xrightarrow{h^*_{\sigma^i}} C \text{Link}(\sigma^i,K) \times \hat{\sigma}^i \xrightarrow{h_{\sigma^i}} \text{Link}(\sigma^k,K) \xrightarrow{(h_{\sigma^k})^{-1}} S^{n-k-1}
\]
are smooth embeddings. Notice that the image of the composition of the first two arrows is the normal neighborhood \( \text{CLink} \left( \sigma^i_S, \text{Link}(\sigma^k, K) \right) \times \hat{\sigma}^j_S \) of \( \hat{\sigma}^j_S \) in \( \text{Link}(\sigma^k, K) \). Note that the map given in (1.3.2) can also be written as

\[
(1.3.3) \quad \mathbb{D}^{n-i} \times \hat{\sigma}^j_S \xrightarrow{\mathcal{H}'_S} S' \xrightarrow{\mathcal{H}'_S} S = \text{Link}(\sigma^k, K) \xrightarrow{(h_{\sigma^k})^{-1}} \mathbb{S}^{n-k-1}
\]

where \( S' = (h_{\sigma^k})^{-1}(S) \) and \( \mathcal{H}' \) is the representation of \( \mathbb{R} \times 1_{\sigma^i_S} \) in the chart \( h_{\sigma^k}^* \) (which is always smooth, see 1.2.4). Note that the inverse of the map given in 1.3.2 (or 1.3.3) is just the change of charts \( (h_{\sigma^k})^{-1} \circ h_{\sigma^k}^* \) restricted to an open subset of \( \mathbb{S}^{n-k-1} \), plus the “straightening map” \( \mathbb{R}' \).

**Proposition 1.3.4.** The set of link smoothings \( \{h_{\sigma}\} \) is smoothly compatible if and only if the atlas \( \mathcal{A}(\{h_{\sigma}\}) \) is smooth.

**Proof.** If the atlas \( \mathcal{A} \) is smooth then all chart maps \( h_{\sigma}^* \) are embeddings (with respect to the smooth structure generated by \( \mathcal{A} \)). Therefore the composition given in 1.3.2 above is smooth.

Write \( k + j = n \). Suppose the set of smoothings \( \{h_{\sigma}\} \) is smoothly compatible. Denote by \( W_{\sigma^i} \) the image of \( h_{\sigma^i}^* \). Assume that we have proved that the atlas \( \mathcal{A}_j = \left\{(h_{\sigma^i}^*, \mathbb{D}^{n-i} \times \hat{\sigma}^i) \right\}_{k<i} \) is smooth, and we want to prove that the atlas \( \mathcal{A}_{j+1} = \left\{(h_{\sigma^i}^*, \mathbb{D}^{n-i} \times \hat{\sigma}^i) \right\}_{k\leq i} \) is smooth. Note that \( \mathcal{A}_j \) is a smooth atlas on the complement \( K - K_k \) of the \( k \)-skeleton \( K_k \). The difference between the atlas \( \mathcal{A}_j \) and the atlas \( \mathcal{A}_{j+1} \) are the charts with maps \( h_{\sigma^k}^* \), for all link smoothings \( h_{\sigma^k} \) of \( k \)-cubes or \( (k\text{-simplices}) \) \( \sigma^k \). We prove the proposition by proving that the following maps are smooth embeddings

\[
h_{\sigma^k}^*|_{\mathbb{D}^{k} \times \hat{\sigma}^k \cap \{0\} \times \hat{\sigma}^k} \to (M - M_j, \mathcal{A}_j)
\]

Fix \( \sigma^k \). The open sets \( U_{\sigma^i} = S \cap W_{\sigma^i} \), \( \sigma^i \supset \sigma^k \), form an open cover of \( S \). Note that \( U_{\sigma^i} \) is a normal neighborhood of \( \sigma^i = \sigma^k \cap \sigma^i \) in \( S \). Write \( V_{\sigma^i} = (h_{\sigma^k}^*)^{-1}(C^{+} U_{\sigma^i}) \) (here the vertex \( o \) of the open cone \( C^{+} U_{\sigma^i} \) is the center of \( S \)). Let \( u \in S \). Take \( i \) so that \( u \in U_{\sigma^i} \).

**Claim.** The map \( h_{\sigma^k}^*|_{V_{\sigma^i} \times \hat{\sigma}^k} : V_{\sigma^i} \times \hat{\sigma}^k \to W_{\sigma^i} \subset (K - K_k, \mathcal{A}_j) \) is an embedding.

**Proof of the Claim.** Since \( h_{\sigma^k}^* \) is already a smooth embedding it is enough to prove that the map \( h = (h_{\sigma^k}^*)^{-1} \circ h_{\sigma^k}^*|_{V_{\sigma^i}} \) is a smooth embedding. But we can consider \( V_{\sigma^i} \subset \mathbb{D}^{n-k} - \{0\} = \mathbb{S}^{n-k-1} \times (0, 1) \). Write \( \sigma^i = \sigma^i \times \sigma^k \), and let the link \( S = \text{Link}(\sigma^k, K) \) be based at \( p \in \hat{\sigma}^k \). For \( v = (t u, w) \in V_{\sigma^i} \times \hat{\sigma}^k \), \( u \in \mathbb{S}^{n-k-1} \), we can write

\[
h(v) = (\alpha_u(t), w) \in (\mathbb{D}^{n-i} \times \hat{\sigma}^i) \times \hat{\sigma}^k = \mathbb{D}^{n-i} \times \hat{\sigma}^i
\]

where \( \alpha_u \) is the segment \([p, g(u)]\) and \( g \) is the inverse of the map in 1.3.3. This proves the claim.
Since the open sets $V_{\sigma^i} \times \partial^k$ cover $(\mathbb{D}^{n-k} \times \partial^k) - (\{0\} \times \partial^k)$ we can conclude that $h^\bullet_{\sigma^k}$ is a smooth embedding away from $\{0\} \times \partial^k$, into the smooth manifold $(K - K_k, \mathcal{A}_{j+1})$. This proves the proposition.

**Corollary 1.3.5.** Let $\{h_\sigma\}$ be a set of link smoothings on $K$, and let $\sigma^k \in K$. If the atlas $\mathcal{A} = \mathcal{A}(\{h_\sigma\}_{\sigma \in K})$ is smooth, then:

1. the atlas $\mathcal{A}_{\sigma^k} = \mathcal{A}_{\text{Link}(\sigma^k, K)} = \{h^\bullet_{\sigma^i\cap\text{Link}(\sigma^k, K)}\}_{\sigma^k \subseteq \sigma^i}$ is a smooth normal atlas on $\text{Link}(\sigma^k, K)$.
2. The set of link smoothings $\{h_{\sigma^i\cap\text{Link}(\sigma, K)}\}_{\sigma^k \subseteq \sigma^i}$ for the links of $\text{Link}(\sigma^k, K)$ is smoothly compatible.
3. The link smoothing $h_{\sigma^k} : \mathbb{S}^{n-k-1} \to \left(\text{Link}(\sigma^k, K), S_{\sigma^k}\right)$ is a diffeomorphism.
4. Let $\mathcal{S}'$ be the normal smooth structure on $K$ induced by $\mathcal{A}$, and let $\mathcal{S}_{\sigma^k}$ be the normal smooth structure on $\text{Link}(\sigma^k, K)$ induced by $\mathcal{A}_{\sigma^k}$. We have that $\mathcal{S}'|_{\text{Link}(\sigma^k, K)} = \mathcal{S}_{\sigma^k}$.

Here $\mathcal{S}'|_{\text{Link}(\sigma^k, K)}$ denotes the restriction of $\mathcal{S}'$ to $\text{Link}(\sigma^k, K)$. (Recall that, by Corollary 1.1.2 $\text{Link}(\sigma^k, K)$ is a smooth submanifold of $(K, \mathcal{S}')$.)

**Proof.** Since $\mathcal{A}$ is smooth, by 1.3.4, the set of link smoothings $\{h_\sigma\}_{\sigma \in K}$ is smoothly compatible. This together with Lemma 1.3.1 imply (1). Item (2) follows from 1(1) and 1.3.4 (applied to the complex $\text{Link}(\sigma^k, K)$). Item (3) also follows from Lemma 1.3.1 and (2) (i.e. the fact that $\{h_{\sigma^i\cap\text{Link}(\sigma, K)}\}_{\sigma^k \subseteq \sigma^i}$ is smoothly compatible). Item (4) follows from (3) and the fact that $h_{\sigma^k} : \mathbb{S}^{n-k-1} \to (K, \mathcal{S}')$ is a smooth embedding (see Remark 1 before Theorem 1.1). This proves the corollary.

### 1.4. A few technical results.

This is a technical subsection. We present some results that will be needed later.

Let $f : K \to M$ be a smooth cubulation of $M$ and $\mathcal{A}$ be a normal atlas for $K$, inducing the smooth structure $\mathcal{S}'$ on $M$. In general for a (closed) cube (or simplex) $\sigma$ the inclusion $\sigma \hookrightarrow (M, \mathcal{S}')$ is (almost always) not a smooth embedding (see [6]). But we prove in the next lemma that a weaker regularity condition holds. Consider $\sigma^j = \sigma^i \times \sigma^k \in K$. As before we identify a normal neighborhood of $\sigma^k$ in $\sigma^j$ with $\text{CLink}(\sigma^k, \sigma^j) \times \sigma^k$ and write an element in $\text{CLink}(\sigma^k, \sigma^j)$ in the form $tu$, $u \in \text{Link}(\sigma^k, \sigma^j)$. We have an inclusion $\text{Link}(\sigma^k, \sigma^j) \times \sigma^k \subset \text{CLink}(\sigma^k, \sigma^j) \times \sigma^k$. Also
denote the inclusion $\sigma^j \hookrightarrow (M, S')$ by $\iota$.

**Lemma 1.4.1.** Let $(u_0, p_0), (u_n, p_n) \in \text{Link}(\sigma^k, \sigma^j) \times \hat{\sigma}^k \subset \sigma^j$ with $(u_n, p_n) \to (u_0, p_0)$. Also let $\tau_n \to 0 \in [0, \infty)$, and let $(0, v_0), (0, v_n) \in T_{(u_n, p_n)} \left(C \text{Link}(\sigma^k, \sigma^j) \times \sigma^k\right)$ (hence they are “parallel to $\sigma^k$), with $v_n \to v_0$. Then

(i) we have that $D_{\iota(u_0, p_0)}(u_n, 0) \to D_{\iota(0, p_0)}(u_0, 0)$,

(ii) we have that $D_{\iota(u_0, p_0)}(0, v_n) \to D_{\iota(0, p_0)}(0, v_0)$.

**Proof.** Just take the chart $h_{\sigma^j}$ and recall that $h_{\sigma^j}$ is a product map and respects the radial structure. This proves the lemma.

Let $U$ be a bounded open set of $\mathbb{R}^n$. A smooth map $f : U \to \mathbb{R}^N$ is polynomially bounded with respect to a subset $B \subset \mathbb{R}^n$ if for every $p \in B$ and every partial derivative $\partial^a f$ of $f$ we have constants $C, m \in \mathbb{R}$ such that $|\partial^a f(p)| \leq Cd_{\sigma}(p, B)^m$. The map is $C^k$ well-bounded if the norm of the first derivative $|Df|$ is bounded and bounded away from zero. For $U \subset \mathbb{S}^n$ we write $C^+ U = CU - \{0\}$, where $CU$ is the cone $CU = U \times [0, 1]/U \times \{0\}$ of $U$.

**Lemma 1.4.2.** Let $U \subset \mathbb{S}^n$ be open and $f : U \to f(U) \subset \mathbb{R}^N$ be smooth. Let $V$ open with $V \subset U$. Then the cone map $C(f|_V) : C^+ V \to C f(V) \subset \mathbb{R}^N$ is polynomially bounded at 0. Furthermore $C(f|_V)$ is also $C^1$ well-bounded.

**Proof.** We have $(C f)(x) = |x| f(x/|x|)$, and the lemma follows from differentiating this equation. This proves the lemma.

**Remark.** Actually since $C f$ is a cone map then $D(C f)_x = D(C f)_{|x|^{-1}}$, for $x \neq 0$. In particular if $u \in V$, and $t > 0$ then $D(C f)_{tu} f = f(u)$.

Let $\mathcal{A} = \left\{(h_{\sigma^i}, \mathbb{D}^{n-i} \times \hat{\sigma}^i)\right\}$ be a normal atlas. Let $S = f(\text{Link}(\sigma^k, K))$ and $U' \subset S \cap W_{\sigma^i}$, where $\sigma^i > \sigma^k$ and $W_{\sigma^i}$ is the image of $h_{\sigma^i}$. Write $U = (h_{\sigma^i})^{-1}(U')$ and let $V$ open with $V \subset U$. We have the following corollary about the change of charts $(h_{\sigma^i})^{-1} \circ h_{\sigma^k} : C^+ V \to \mathbb{D}^{n-i} \times \hat{\sigma}^i \subset \mathbb{R}^n$, restricted to the cone of $V$.

**Corollary 1.4.3.** The change of charts $(h_{\sigma^i})^{-1} \circ h_{\sigma^k}$, restricted to $C^+ V$, is polynomially bounded at 0, and $C^1$ well-bounded.

1.5. The case of manifolds with codimension zero singularities.

Here we treat the case of manifolds with a one point singularity. The case of manifolds with many (isolated) point singularities is similar.
Let $Q$ be a smooth manifold with a one point singularity $q$, that is $Q - \{q\}$ is a smooth manifold and there is a topological embedding $C_1 N \to Q$, with $o_{C,N} \to q$, that is a smooth embedding outside the vertex $o_{C,N}$. Here $N = (N, S_N)$ is a closed smooth manifold (with smooth structure $S_N$). Also $C_1 N$ is the (closed) cone of width 1 and we identify $C_1 N - \{o_{C,N}\}$ with $N \times (0,1]$. We write $C_1 N \subset Q$. We say that the singularity $q$ of $Q$ is modeled on $C N$.

Assume $(K, f)$ is a smooth cubulation of $Q$, that is

(i) $K$ is a cubical complex.

(ii) $f : K \to Q$ is a homeomorphism. Write $f(p) = q$ and $L = \text{Link}(p, K)$.

(iii) $f|_{\sigma}$ is a smooth embedding for every cube $\sigma$ not containing $p$.

(iv) $f|_{\sigma - \{p\}}$ is a smooth embedding for every cube $\sigma$ containing $p$.

(v) $L$ is $PL$ homeomorphic to $(N, S_N)$.

Many of the definitions and results given before for smooth cube manifolds still hold (with minor changes) in the case of manifolds with a one point singularity:

(1) A link smoothing for $L = \text{Link}(p, K)$ (or $p$) is just a homeomorphism $h_p : N \to L$. Since all but one of the links of $K$ are spheres, sets of link smoothings for $K$ are defined, that is they are sets of link smoothings for the sphere links plus a link smoothing for $L$.

(2) Given a set of link smoothings for $K$ we get a set of normal charts as before. For the vertex $p$ we mean the cone map $h_p^* = f \circ h : C N \to Q$. We will also denote the restriction of $h_p^*$ to $C N - \{o_{C,N}\}$ by the same notation $h_p^*$. As before $\{h_p^*\}_{p \in K}$ is a (topological) normal atlas on $Q$ with respect to $K$. The atlas on $Q$ is smooth if all transition functions are smooth, where for the case $h_p^* : C N - \{o_{C,N}\} \to Q - \{q\}$ we are identifying $C N - \{o_{C,N}\}$ with $N \times (0,1]$ with the product smooth structure obtained from some smooth structure $\tilde{S}_N$ on $N$. A smooth normal atlas on $Q$ with respect to $K$ induces, by restriction, a smooth normal structure on $Q - \{q\}$ with respect to $K - \{p\}$ (this makes sense even though $K - \{p\}$ is not, strictly speaking, a cube complex).

(3) We say that the set $\{h_{\sigma}\}$ is smooth compatible if condition 1.3.2 holds. It is straightforward to verify that Proposition 1.3.4 holds: $\{h_{\sigma}\}$ is smooth compatible if and only if $\{h_p^*\}$ is a smooth atlas on $K$. In this case we say that the smooth atlas $\{h_p^*\}$ (or the induced smooth structure, or the set $\{h_{\sigma}\}$) is correct with respect to $N$ if $S_N$ and $\tilde{S}_N$ are diffeomorphic.

(4) Also it is straightforward to verify that Corollary 1.3.5 holds in our present case.

(5) Theorem 1.1 also holds in this context:

**Theorem 1.5.1.** Let $Q$ be a smooth manifold with one point singularity $q$ modeled on $C N$, where $N$ is a closed smooth manifold. Let $(K, f)$ be a smooth cubulation of $Q$. Then $Q$ admits a normal smooth structure with respect to $K$, which restricted to $Q - \{q\}$ is diffeomorphic to $Q - \{q\}$. Moreover this normal smooth structure is correct with respect to $N$ if
(a) $\dim N \leq 4$.

(b) $\dim N \geq 5$ and the Whitehead group $\text{Wh}(N)$ of $N$ vanishes.

This Theorem is proved in [6].

2. Charney-Davis Strict Hyperbolization Pieces.

We use some of the notation in [2]. In particular the canonical $n$-cube $[0,1]^n$ will be denoted by $\square^n$. (This differs with the notation used in Section 1, where an $n$-cube was denoted by $\sigma^n$.) Also $B_n$ is the isometry group of $\square^n$.

A Charney-Davis strict hyperbolization piece of dimension $n$ is a compact connected orientable hyperbolic $n$-manifold with corners satisfying the properties stated in Lemma 6.2 of [2]. The group $B_n$ acts by isometries on $X^n$ and there is a smooth map $f : X^n \to \square^n$ constructed in Section 5 of [2] with certain properties. We collect some facts from [2].

(1) For any $k$-face $\square^k$ of $\square^n$ we have that $f^{-1}(\square^k)$ is totally geodesic in $X^n$ and it is a Charney-Davis hyperbolization piece of dimension $k$. The totally geodesic submanifold (with corners) $f^{-1}(\square^k)$ is a $k$-face of $X^n$. Note that the intersection of faces is a face and every $k$-face is the intersection of exactly $n-k$ distinct $(n-1)$-faces.

(2) The map $f$ is $B_n$-equivariant.

(3) The faces of $X^n$ intersect orthogonally.

(4) The map $f$ is transversal to the $k$-faces of $\square^n$, $k < n$.

The $k$-face $f^{-1}(\square^k)$ of $X$ will be denoted by $X_{\square^k}$. The interior $f^{-1}(\text{int } \square^k)$ will be denoted by $\text{int } X_{\square^k}$.

Lemma 2.1. For every $n$ and $r > 0$ there is a Charney-Davis hyperbolization piece of dimension $n$ such that the widths of the normal neighborhoods of every $k$-face, $k = 0,...,n-1$, are larger than $r$.

Proof. A piece $X^n$ is constructed in Section 6 of [2] by cutting a closed hyperbolic $n$-manifold $M$ along a system $\{Y_i\}_{i=1}^n$ of codimension one totally geodesic submanifolds of $M$ that intersect orthogonally. The $Y_i$’s are orientable and two sided. The group $B_n$ acts by isometries on $M$, permuting the $Y_i$’s. In particular each $Y_i$ is contained in the fixed point set of a nontrivial isometric involution $r_i$. Therefore $r_i$ interchanges both sides of $Y_i$.

The $(n-1)$-faces of $X^n$ correspond to the $Y_i$’s and a $k$-face of $X^n$ corresponds to the (transverse) intersection of different $n-k$ $Y_i$’s. Therefore it is enough to show that the intersections of the $Y_i$’s have normal neighborhoods with large width.

Claim. It is enough to show that the $Y_i$’s have normal neighborhoods with large width.
**Proof of Claim.** Let $Z = Y_i \cap Y_j$, $Y_i \neq Y_j$, and assume both $Y_i$, $Y_j$ have normal neighborhoods of width larger than $r$. Let $\alpha$ be a path with end points in $Z$ of length $< 2r$. Then $\alpha$ lies in the normal neighborhood of $Y_i$. Using the distance decreasing normal geodesic deformation of $Y_i$ we can deform $\alpha$, rel end points, to a shorter path $\beta$ in $Y_i$. Repeat the same argument now with $\beta$, and using the fact that $Y_i$ and $Y_j$ intersect orthogonally, we get that the deformation of $\beta$ lies in $Z$. The proof for larger intersections is similar. This proves the claim.

We continue with the proof of Lemma 2.1. The claim above and the existence of the nontrivial isometric involutions $\iota_i$ imply that it is enough to have $M$ with large injectivity radius. (To see this let $\alpha : [0.1] \to M$, $\alpha(0), \alpha(1) \in Y_i$, of length $< 2r$, with $\alpha$ not homotopic, rel $\{0, 1\}$, to a path in $Y_i$. Then take $\beta = (\iota_i \circ \alpha)^{-1} \ast \alpha$, which is a non-nullhomotopic loop of length $< 4r$.)

To obtain $M$ with large injectivity radius recall that $M$ is given in Section 6 of [2] as $M = \mathbb{H}^n / \Gamma$, where $\Gamma = \Gamma(J)$ is a congruence subgroup given by the ideal $J = \mathbb{Q}(\sqrt{d})$ of $\mathbb{Q}$. The only conditions required for $J$ are that $\Gamma$ is torsion free and $\Gamma \subset SO_o(n, 1)$. Hence any deeper congruence subgroup $\Gamma'$ will serve as well. But, by taking deeper congruence subgroups we can increase (in a well-known way) the injectivity radius as much as we want: let $\gamma_1, \ldots, \gamma_k \in \Gamma$ correspond the closed geodesics of length $< r$, and take a deeper ideal that contains none of the $\gamma_i$'s. This proves the lemma.

We need some extra properties for the map $f$, so we give an explicit construction of it. Recall from the proof above that $X$ is obtained from the closed hyperbolic manifold $M$ by cutting along the system $\{Y_i\}$. Similarly the map $f$ is obtained in [2] from a map $\varphi : M \to \mathbb{T}^n$. And this map has coordinate maps $\varphi_i : M \to S^1$, which are constructed by applying the Pontryagin-Thom construction to the framed (two sided) codimension one submanifolds $Y_i$. Here we need a bit more details so we give an specific construction for $\varphi$.

Let $Y_i \times (-r, r) \subset M$ be the normal geodesic neighborhood of $Y_i$ of width $r > 2$. Hence for $p = (y, t) \in Y_i \times (-r, r)$, the smooth map $p \mapsto t(p) = t$ gives the signed distance to $Y_i$. Let $\eta : \mathbb{R} \to [-1, 1]$ be a non-decreasing smooth map such that $\eta(t) = t/r$ for $t \in (-r + 1, r - 1)$, $\eta(t) = 1$ for $t \geq r$, $\eta(t) = -1$ for $t \leq -r$. By identifying $(S^1, 1)$ with $([-1, 1]/-1 = 1, 0)$, the smooth map $\eta \circ t$ induces the smooth map $\varphi_i : Y_i \times (-r, r) \to S^1$ that can be extended to the whole of $M$. Note that $\varphi_i^{-1}(1) = Y_i$ and $\varphi^{-1}(\mathbb{T}^n = \{1\} \times \mathbb{T}^{n-1}) = Y_i$. After cutting along the $Y_i$’s we get the map $f : X \to \mathbb{R}^n$ and each $Y_i$ corresponds to two $(n - 1)$-faces $X_{\mathbb{R}^n-1, 0, 1}$, $X_{\mathbb{R}^n-1, 1, 1}$ (one for each side of $Y_i$), where $\mathbb{R}^{n-1} = \mathbb{R}^n \cap \{x_i = j\}$, $j = 0, 1$. Moreover the normal neighborhood $Y_i \times (-r, r)$ corresponds to the two one-sided normal neighborhoods $X_{\mathbb{R}^n-1, 0, 1} \times [0, r)$, $X_{\mathbb{R}^n-1, 1} \times [0, r)$. Write $f(p) = (f_1(p), \ldots, f_n(p)) \in \mathbb{R}^n \subset \mathbb{R}^n$. Then if $p \in X_{\mathbb{R}^n-1, 0, 1} \times [0, r)$, we have $f_i(p) = \frac{1}{2} \eta(t_i(p))$, where $t_i(p)$ is the distance to $X_{\mathbb{R}^n-1, 1}$. Similarly if $p \in X_{\mathbb{R}^n-1, 1, 1} \times [0, r)$ we have $f_i(p) = 1 - \frac{1}{2} \eta(t_i(p))$, where $t_i(p)$ is the distance to $X_{\mathbb{R}^n-1, 0, 1}$. And if $p \in X_{\mathbb{R}^n-1, 0, 1} \times [0, r - 1)$, we have $f_i(p) = \frac{1}{2r} t_i(p)$, and similarly for $X_{\mathbb{R}^n-1, 1, 1}$. In particular $p \in X_{\mathbb{R}^n-1, 0, 1} \times [0, a)$, $a \leq r - 1$, if and only if $f(p) \in \mathbb{R}^{n-1} \times [0, \frac{a}{r})$. In what follows of this paper we assume $\varphi$ and $f$ are constructed as above.

In what follows we will write $\mathbb{R}^{n-1} = \mathbb{R}^{n-1, i, 0},$ if the context is clear.
Proposition 2.2. The derivative of \( f \) sends normal vectors to \( X_{\square k} \) to normal vectors to \( \square^k \).

Proof. For simplicity we assume that \( k = n - 2 \). The proof for general \( k \) is similar. We can also assume that \( \square^{n-2} = \square_1^{n-1} \cap \square_2^n \), where \( \square_1^{n-1} = \{ x_i = 0 \} \cap \square^n \). Write \( U_{i,j} = X_{\square_{i,j}^{n-1}} \), \( U_i = U_{i,0} \), and \( W = X_{\square_{i}^{n-2}} = U_1 \cap U_2 \). Let \( p \in W \). We certainly have that \( p \in U_1 \times [0,r) \) and \( p \in U_2 \times [0,r) \). We have to prove that \( (Df_i)_p u = 0 \) for \( i \geq 3 \). For each \( i \geq 3 \) we have two cases. Case 1: \( p \notin U_{i,j} \times [0,r) \) for \( j = 0 \) and \( j = 1 \). In this case it follows that \( (Df_i)_p = 0 \), hence \( (Df_i)_p = 0 \). Case 2: \( p \in U_{i,j} \times [0,r) \) for \( j = 0 \) or \( j = 1 \). Say \( p \in U_3 \times [0,r) \). We want to prove that \( (Df_3)_p u = 0 \). Since \( p \in U_3 \times [0,r) \) there is a geodesic \( \beta \) in \( U_3 \times [0,r) \), beginning at \( U_3 \), normal to \( U_3 \) and ending in \( p \). Also \( t_3(p) \) is equal to the length of \( \beta \). Since \( \alpha \) and \( \beta \) are perpendicular at \( p \) the function \( t \mapsto t_3(\alpha(t)) \) has a minimum at 0. Hence \( D(t_3)_p u = 0 \), therefore \( D(f_3)_p u = \eta'(t_3(p))D(t_3)_p u = 0 \). This proves the proposition.

For a \( k \)-face \( X_{\square k} \) and \( p \in X_{\square k} \), the set of inward normal vectors to \( X_{\square k} \) at \( p \) can be identified with the canonical all-right \((n-k-1)\)-simplex \( \Delta_{\square_{n-k-1}} \). In this sense we consider \( \Delta_{\square_{n-k-1}} \subset T_p X \). Similarly we can consider \( \Delta_{\square_{n-k-1}} \subset T_q \square^n \), for \( q \in \square^k \). We make the convention that the two identifications above are done with respect to an ordering of the \((n-1)\)-faces \( X_{\square_{i-1}^n} \) of \( X \) and the corresponding ordering for \( \square^n \). For instance the vectors in \( \Delta_{\square_{n-k-1}} \subset T_p X \) tangent to some \( X_{\square_{i-1}^n} \) correspond to the same \((n-1)\)-face of \( \Delta_{\square_{n-k-1}} \) as the vectors in \( \Delta_{\square_{n-k-1}} \subset T_{f(p)} \square^n \) tangent to \( \square_{i-1}^n \). With these identifications we get coordinates on \( \Delta_{\square_{n-k-1}} \): we write \( (u_1, ..., u_n) = u \in \Delta_{\square_{n-k-1}} \) where \( u_i \) is the angle between \( u \) and \( X_{\square_{i-1}^n} \) (or \( \square_{i-1}^n \)).

Proposition 2.3. For \( p \in X_{\square k} \), we have that

(i) \( Df_p \) sends non-zero normal vectors to non-zero normal vectors.

(ii) \( \eta \circ (Df_p|_{\Delta_{\square_{n-k-1}}}) : \Delta_{\square_{n-k-1}} \to \Delta_{\square_{n-k-1}} \) is the identity, where \( \eta(x) = \frac{x}{|x|} \) is the normalization map.

Remark. In (iii) we are using the coordinates on \( \Delta_{\square_{n-k-1}} \) mentioned above to identify the normal tangent spaces of \( X_{\square k} \) and \( \square^k \).

Proof. For simplicity we assume that \( k = n - 3 \). The proof for general \( k \) is similar. As in the proof of Proposition 2.2 write \( \square_{i-1}^n = \{ x_i = 0 \} \cap \square^n \) and \( U_i = X_{\square_{i-1}^n} \). For simplicity take \( W = X_{\square_{i-3}} = U_1 \cap U_2 \cap U_3 \). Let \( u = (u_1, u_2, u_3) \in \Delta_{\square_2} \subset T_p X \). Then \( u_i \) is the spherical distance from \( u \) to \( T_p U_i \), \( i = 1, 2, 3 \) (or angle between \( u \) and \( U_i \)). Let \( \alpha \) be the geodesic with \( \alpha(0) = p \) and \( \alpha'(0) = u \). Then \( u_i \) is the angle, at \( p \), between \( \alpha \) and \( U_i \). We have to prove that \( D(f_i)_p u = \frac{1}{2^r} u_i \) for \( i = 1, 2, 3 \). Since the length of \( \alpha|_{[0,t]} \) is \( t \) and the distance from \( \alpha(t) \) to \( U_i \), \( i = 1, 2, 3 \), is \( t_i(\alpha(t)) \), we get a right hyperbolic triangle with hypotenuse of length \( t \) and side equal to \( t_i(\alpha(t)) \) with opposite angle \( u_i \). Hence, the hyperbolic law of sines implies

\[
t_i(\alpha(t)) = \sinh^{-1} \left( \frac{\sin(u_i) \sinh(t)}{t_i(\alpha(t))} \right)
\]
for $i = 1, 2, 3$. But for $t$ small we have $f_i(\alpha(t)) = \frac{1}{2\eta} t_i(\alpha(t))$ and a simple differentiation (1), evaluated at 0, shows $D(f_i) \, u = \frac{1}{2\eta} u_i$, $i = 1, 2, 3$. This proves (i), (ii) and (iii) and completes the proof of the proposition.

Choose $r > 0$. In what follows we assume the width of the normal neighborhoods of the $X_{\Box}$ to be much larger than the number $r > 0$. Lemma 2.1 asserts this is always possible. Fix a point $p \in \bar{X}_{\Box}$ and consider $\Delta_{S_{n-k-1}} \subset T_p X$. The cone $C_r \Delta_{S_{n-k-1}}$ is the set $\{ t u, 0 \leq t < r, u \in \Delta_{S_{n-k-1}} \}$. We have the exponential map $E : C_r \Delta_{S_{n-k-1}} \rightarrow X$, given by $E(u,t) = \exp(t \, u)$. Write $\Box^k = \Box^k \times \Box^i \subset \mathbb{R}^k \times \mathbb{R}^i = \mathbb{R}^n$ and denote by $p_i : \mathbb{R}^n \rightarrow \mathbb{R}^i$, the projections. Also, as in Section 6, we write $\bar{R}_+ = [0, \infty)^n$.

**Lemma 2.4.** We have the following properties.

(i) The map $E$ respects faces, that is $E \left( (C_r \Delta_{S_{n-k-1}}) \cap T_p X_{\Box} \right) \subset X_{\Box}^j$.

(ii) The map $f \circ E$ respects faces, that is $(f \circ E) \left( (C_r \Delta_{S_{n-k-1}}) \cap T_p X_{\Box} \right) \subset \Box^j$. Hence $(C_r \Delta_{S_{n-k-1}}) \cap T_p X_{\Box} = (C_r \Delta_{S_{n-k-1}}) \cap T_{f(p)} \Box^j$.

(iii) The map $p_i \circ f \circ E$ does not depend on the point $p$.

(iv) The values of the map $p_i \circ f \circ E$ do not depend on the variable $u \in \Delta_{S_{n-k-1}}$ (but they do depend on $t$ and $p$).

(v) Write $T = (t_1, \ldots, t_n)$. The map $p_i \circ T \circ E : C_r \Delta_{S_{n-k-1}} \rightarrow \mathbb{R}^{n-k}$ is an embedding, provided $s < r - 1$.

**Remark.** In the second statement of (ii) we are considering both $C_r \Delta_{S_{n-k-1}} \subset T_p X$ and $C_r \Delta_{S_{n-k-1}} \subset T_{f(p)} \Box^n$.

**Proof.** Statement (i) follows from the fact that each $X_{\Box}$ is totally geodesic in $X$. Statement (ii) also follows because $f$ respects faces. Differentiating, using 2.3 and the fact that the derivative of $E$ (at 0) is the identity we obtain $(C_r \Delta_{S_{n-k-1}}) \cap T_p X_{\Box} \subset T_{f(p)} \Box^j$. To get the other inclusion use 2.3 again and count dimensions.

To prove (iii) and (iv) we assume for simplicity, as in the proof of 2.3, that $k = n - 3$ and $\Box^{n-3} = \Box^{n-1} \cap \Box^{n-1} \cap \Box^{n-1}$, where $\Box^{n-1} = \{ x_i = 0 \} \cap \Box^n$. Also $l = 3$ and $\Box^3 = \Box^2 \cap \ldots \cap \Box^2$. Let $u = (u_1, u_2, u_3) \in \Delta_{S_2}$. Since $f = (f_1, \ldots, f_n)$ we have

$$p_i \circ f \circ E(u,t) = \left( f_1(E(u,t)) , f_2(E(u,t)) , f_3(E(u,t)) \right)$$

But for $q \in X_{\Box_{n-1}} \times [0, r)$ we have that $f_i(q) = \frac{1}{2\eta} \eta(t_i(q))$ (see paragraph before 2.2). This together with (1) in the proof of 2.3 imply

$$f_i(E(u,t)) = \frac{1}{2} \eta \left( \sinh^{-1}\left( \sin(u_i) \sinh(t) \right) \right)$$

which depend only on $u$ and $t$. Here $u = (u_1, \ldots, u_{n-k})$. This proves (iii).
Note that \( p_k \circ f \circ E(u, t) = \left( f_1(E(u, t)), \ldots, f_n(E(u, t)) \right) \). We prove that \( f_1(E(u, t)) \) is independent of \( u \). The proof is similar for \( i > 4 \). Write \( U = X_{\square n-1}, V = X_{\square k}, W = U \cap V \) and let \( p \in \hat{V} \). Note \( U, V, W \) are totally geodesic. Let \( q \in W \) be such that the segment \([q, p]\) has length equal to the distance \( d_X(p, W) \) between \( p \) and \( W \). If \( d_X(p, W) \geq r \) we are done because then \( f_1(E(u, t)) \) is constant. We assume \( L = d_X(p, W) < r \).

Let \( B \) be the union of the images of all geodesics of length \( r \) in \( U \) beginning at \( q \) and perpendicular to \( W \). Then \( B \) is isometric to the \( r \)-cone of the canonical all-right simplex \( \Delta_{S^2} \), and it is totally geodesic. Let \( C \) be the union of the images of all geodesics of length \( r \) in \( X \) beginning at some point in \( B \) and perpendicular to \( U \). Then \( C \) is isometric to \( B \times [0, r) \) with the usual \( \cosh \)-warped product metric. We write \( C = B \times [0, r) \). Therefore \( C \) is also totally geodesic, and \( C, V \) intersect perpendicularly at \( C \cap V = [q, p] \). Now, if \( \alpha \) is a geodesic of length \( < r \) beginning at \( p \) and perpendicular to \( V \) then \( \alpha \) in contained in \( C \). Moreover, \( \alpha \) is contained in \( \ell \times [0, r) \subset C \) for some ray \( \ell \subset B \) beginning at \( q \). (To see this note that we can consider \( C \) convex in \( \mathbb{H}^n \), and the statement is true in \( \mathbb{H}^n \).) Note that \( \ell \times [0, r) \subset C \) is isometric to a convex set in \( \mathbb{H}^2 \). Finally we get that \( d_X(\alpha(t), U) \) can then be computed in \( \mathbb{H}^2 \) as the length of the side \( a \) of a quadrilateral with consecutive sides \( a, b, c, d \), angles \( \angle ab = \angle bc = \angle cd = \pi/2 \) and \( \text{length}(a) = t, \text{length}(b) = L \). This calculation only depends on \( t \) and \( L = \text{length}[q, p] \) (hence on the choice of \( p \)) but not on the “direction” \( u \).

To prove (v) note that from equation (1) in the proof of 2.3 and the fact that \( s < r - 1 \) we get that for \( u = (u_1, \ldots, u_{n-k}), 0 \leq t < s \), we have \( t_i(tu) = \sinh^{-1}\left( \sin(u_i) \sinh(t) \right) \). This equation together with \( \Sigma_{i=1}^{n-k} \sin^2(u_i) = 1 \) imply

\[
t = \sinh^{-1}\left( \left( \Sigma_{i=1}^{n-k} \sinh^2(t_i) \right)^{1/2} \right)
\]

Since we also get \( u_i = \sin^{-1}\left( \frac{\sinh(t_i)}{\sinh(t)} \right) \), the map \( p_i \circ T \circ E = (t_1 \circ E, \ldots, t_{n-k} \circ E) \) has a continuous inverse. Moreover this inverse is clearly smooth when \( t \neq 0 \) and all \( t_i < t \). But for \( t = 0 \) the derivative of \( p_i \circ T \circ E \) can be shown to be injective. This proves the lemma.

**Remark 2.5.** Using the method in the proof above together with hyperbolic trigonometry we can find an explicit formula for the coordinate functions of the function in (iv). It can be checked that these maps are even on the variable \( t \). In particular, \( \frac{d}{dt}(p_k \circ f \circ E)|_{t=0} = 0 \).

### 3. The Charney-Davis Hyperbolization Process

The strict hyperbolization process of Charney and Davis is done by gluing copies of \( X^n \) using the same pattern as the one used to obtain the cube complex \( K \) from its cubes. This space is called \( K_X \) in [2]. We call this space the piece-by-piece strict hyperbolization of \( K \). Note that we get a map \( F : K_X \to K \), which restricted to each copy of \( X \) is just the map \( f : X^n \to \square^n \) in Section 2. We will write \( X_{\square k} = F^{-1}(\square^n) \), for a \( k \)-cube \( \square^k \) of \( K \).
But to obtain good differential and tangential properties the process described above is not enough. Therefore in [3] and [2] an alternative method is given. We describe this next. As before let \( X^n \) be a strict hyperbolization piece and \( K \) be cube complex. We assume there is projection \( p : K \to \square^n \) (see 7.2 of [2]). Now consider \( K_X \) given as the fiber product

\[
\begin{array}{ccc}
K_X & \overset{q_X}{\longrightarrow} & X \\
q_K & \downarrow & \downarrow f \\
K & \overset{p}{\longrightarrow} & \square^n
\end{array}
\]

that is \( K_X = \{(y, x) : p(y) = f(x)\} \subset K \times X \). Here \( q_K, q_X \) are projections. We call this space the fiber-product strict hyperbolization of \( K \). We denoted both hyperbolizations by \( K_X \) but we shall write \( K_X \) piece-by-piece and \( K_X \) fiber-product if we need to. We shall write \( X_{\square^k} = q_K^{-1}(\square^k) \), for a \( k \)-cube \( \square^k \) of \( K \).

**Remark.** The space \( K_X \) fiber-product does depend on the projection map \( p \). For instance, if \( p \) is a cube map, that is \( p|_{\square^n} \) is an isometry for every \( \square^n \in K \), then \( K_X \) piece-by-piece and \( K_X \) fiber-product coincide. But in general these two hyperbolizations are homeomorphic but the obvious homeomorphism (see below) does not preserve the natural piecewise differentiable structures. If needed we shall write \( K_X \) fiber-product\((p) \) to show explicitly the dependence on \( p \).

We now assume that \( K \) has a smooth structure \( S \) compatible with the cube structure of \( K \) (hence \( 1_K \) is a smooth cubulation of the smooth manifold \( K \)). We assume further that the projection \( p : K \to \square^n \) is smooth. Using this and item (4) at the beginning of Section 2 it is argued in [2] that 0 is a regular value of the smooth map \( (k, x) \mapsto p(k) - f(x) \). Therefore \( K_X \) is a smooth submanifold of \( K \times X \) (with trivial normal bundle). Hence if \( K^n \) has a smooth structure (compatible with the cube structure \( K \)) then \( K_X \) has a natural smooth structure. This is an important point for us, so we need to analyze this with a bit more detail. First we remark two facts:

(i) For the regular value argument to work it is assumed (implicitly) in [3], 1C.5, that the restriction \( p|_{\square^n} \) of \( p \) to every open cube \( \square \) of \( K \) is an embedding. (In [3] simplices are used instead of cubes.)

(ii) Let \( \square \) be a \( k \)-cube of \( K \). Then \( p(\square) = \square^k \), for some \( k \)-face \( \square^k \) of \( \square^n \). If \( K \) has no boundary the smoothness of \( p \) implies that for every \( y \in \square \) we have \( Dp_y(T_y K) = T_{p(y)} \square^k \), where \( Dp \) is the derivative of \( p \). In particular, the image of \( Dp_y \) is \( k \)-dimensional, thus \( Dp_y \) is not an isomorphism; hence \( p|_{\square} \) is not an embedding. What is happening here is that whenever we “fold” two \( n \)-cubes into one (this is what \( p \) does), and we want this folding \( p \) to be smooth, then \( p \) has to “slow down to 0” at the place of the folding (which is an \((n-1)\)-cube). Of course none of this has to happen if, for instance, the complex \( K \) is equal to \( \square^n \).

It is important for us here to work with both hyperbolization constructions: \( K_X \) piece-by-piece and \( K_X \) fiber-product. Therefore we need a good way to identify them. We deal with this issue next. Note
that $K^\text{piece-by-piece}_X$ has a natural piecewise smooth structure: the inclusion of each copy of $X$ in $K^\text{piece-by-piece}_X$ is, by definition, a smooth embedding. But this is not true for $K^\text{fiber-product}_X$. We explain this next.

The copy of $X$ in $K^\text{fiber-product}_X$ over an $n$-cube $\square^n$ of $K$ is $X_{\square^n} = q_X^{-1}(\square^n)$. It is a “copy” of $X$ because the projection $q_X|_{X_{\square^n}} : X_{\square^n} \to X$ is a homeomorphism, whose inverse is given by

$$x \mapsto (\rho|_{\square^n})^{-1}(f(x), x)$$

But this map is not smooth because, as mentioned before, $p|_{\square}$ is not an embedding (we have $Dp|_{\square}.v = 0$, for some $v \neq 0$). Therefore, even though $q_X$ is smooth, the map $q_X|_{X_{\square^n}}$ is not a diffeomorphism because the natural (topological) embedding $(q_X|_{X_{\square^n}})^{-1} : X \to K_X$ is not smooth.

The price we paid for slowing down the cubes at the boundary (via $p$) was that we sped up the copies of $X$ at the boundaries. Therefore the natural piecewise hyperbolic (and piecewise differentiable) structure of $K^\text{piece-by-piece}_X$ does not directly give one in $K^\text{fiber-product}_X$.

To have a chance to solve this problem we need a more concrete expression for $p$. We will consider maps $p$ of the form $p = \bar{\rho} \circ c$, where $c$ is a cube map $c : K \to \square^n$ (i.e. $c|_{\square^n}$ is an isometry for every $\square^n$) and $\bar{\rho}$ is a slow-down-at-the-boundary map $\bar{\rho} : \square^n \to \square^n$ given by $\bar{\rho}(x_1, ..., x_n) = (\rho(x_1), ..., \rho(x_n))$, with $\rho : I \to I$ a smooth homeomorphism that is a smooth diffeomorphism on $(0,1)$ and $\frac{dk}{dt}\rho(0) = \frac{dk}{dt}\rho(1) = 0$, $k > 0$. In what follows we write $K^\text{fiber-product}_X = K^\text{fiber-product}_X(\bar{\rho} \circ c)$.

**Remark.** Let $c : K \to \square^n$ be a cube map. For any $\square^n \in K$ the map $(c|_{\square^n})^{-1} : \square^n \to \square^n \subset K$ can be identified with the inclusion $\square^n \hookrightarrow K$. In particular if $K$ has a smooth structure compatible with the cube structure of $K$, the map $(c|_{\square^n})^{-1}$ is an embedding. Also, note that cube maps $c : K \to \square^n$ are not smooth (unless $K = \square^n$).

The following proposition that says that $\bar{\rho} : \square^n \to \square^n$ can be covered by a homeomorphism $X \to X$.

**Proposition 3.1.** We can choose $\rho : I \to I$ so that there is a smooth homeomorphism $P : X^n \to X^n$ such that $f \circ P = \bar{\rho} \circ f$, i.e. the following diagram commutes

$$
\begin{array}{ccc}
X & \xrightarrow{P} & X \\
\downarrow f & & \downarrow f \\
\square^n & \xrightarrow{\bar{\rho}} & \square^n
\end{array}
$$

Moreover we can choose $P$ so that its restriction to every open face $X_{\square^k}$ is an embedding.

**Remark.** With a bit extra work we can get $P$ to be $B_n$-invariant, but this fact will not be needed.

Note that from the construction of the map $\bar{\rho}$ we have that $D\bar{\rho}|_{\square}w = 0$ for every $\square \in \square^n$, $q \in \square$ and $w$ perpendicular to $\square$. We have the following addition to Lemma 3.1.
Addendum to Proposition 3.1. We can choose $P$ in Proposition 3.1 so that for any $\square \in \square^n$ we have that $DP|_{\mu,v} = 0$ for every $p \in \square$ and $v$ perpendicular to $X_\square$.

The proposition and its addendum are proved in Appendix A.

Now we get a new embedding $X \to X_{\square^n} \subset K_{X_{\text{fiber-product}}}$ given by $(q_{X_{\square^n}})^{-1} \circ P$. This is the “correct” embedding, as the next proposition shows.

Proposition 3.2. Suppose $p = \bar{\rho} \circ c$ is smooth. Then $(q_{X_{\square^n}})^{-1} \circ P : X \to X_{\square^n} \subset K_{X_{\text{fiber-product}}}$ is a smooth embedding. Moreover the following diagram commutes for every $n$-cube $\square^n$ of $K$.

\[
\begin{array}{cccc}
X & \xrightarrow{(q_{X_{\square^n}})^{-1} \circ P} & X_{\square^n} & \xrightarrow{\text{inclusion}} & K_X \\
\downarrow f & & \downarrow q_K & & \downarrow q_K \\
\square^n & \xrightarrow{\text{inclusion}} & \square^n & \xrightarrow{\text{inclusion}} & K
\end{array}
\]

Proof. From the definition of $K_{X_{\text{fiber-product}}}$ and Proposition 3.1 we get that the following diagram commutes

\[
\begin{array}{cccc}
K_X & \xrightarrow{q_X} & X & \xleftarrow{P} & X \\
q_K & \downarrow & f & \downarrow & f \\
K & \xrightarrow{p} & \square^n & \xleftarrow{\bar{\rho}} & \square^n
\end{array}
\]  \hspace{1cm} (1)

The commutativity of (1) together with $p = \bar{\rho} \circ c$, and the fact that $(c|\square)^{-1}$ is the inclusion $\square \to K$ imply that the left square of the diagram in the statement of Proposition 3.2 commutes. The right square commutes by definition.

Write $g = (q_{X_{\square^n}})^{-1} \circ P$. We have that the map $g : X \to K_X \subset K \times X$ is smooth if and only if the coordinate maps $q_K \circ g$, $q_X \circ g$ are smooth. First we have $q_X \circ g = q_X \circ (q_{X_{\square^n}})^{-1} \circ P = P$, which is smooth.

From diagram (1) and $p = \bar{\rho} \circ c$ we get

\[q_K \circ g = q_K \circ (q_{X_{\square^n}})^{-1} \circ P = (p|\square^n)^{-1} \circ \bar{\rho} \circ f = (c|\square^n)^{-1} \circ \bar{\rho} \circ f = (c|\square^n)^{-1} \circ f\]

Since $(c|\square^n)^{-1}$ is just the inclusion $\square^n \to K$ we get that $q_K \circ g$ is smooth. It remains to prove that $g$ is a smooth embedding. Since $q_X$ is smooth and $P^{-1}$ is smooth on $\hat{X}$ we get that $g^{-1}$ is smooth on $X_{\square^n}$. Note that the same argument shows that the restriction of $g$ to any $\hat{X}_{\square^k}$ is an
embedding. Therefore if \(u\) is a non-zero vector tangent to some \(\dot{X}_{\Box^k}\) then \(Dg.u\) is non-zero and tangent to the corresponding \(k\)-face \(g(X_{\Box^k})\). If \(u\) is a non-zero vector normal to \(\dot{X}_{\Box^k}\) then, by Lemma 2.3(i), \(Df.u\) is non-zero and normal to \(\Box^k \subset \Box^n\) and certainly \(D((e|_{\Box^n})^{-1} \circ f).u\) is also non-zero and normal to \(\Box^k\). But from diagram (1) we have that \(q_K \circ g = (e|_{\Box^n})^{-1} \circ f\), hence \(Dg.u\) is non-zero. Moreover, \(q_K\) sends \(g(X_{\Box^k})\) to \(\Box^k\), therefore \(Dg.u\) is not tangent to \(g(X_{\Box^k})\). This proves that \(Dg\) is injective on every point of \(\dot{X}_{\Box^k}\). This proves the proposition.

We can now use \((q_X|_{\Box^n}^{-1} \circ P)\) on each copy of \(X\) in \(K_X^{\text{piece-by-piece}}\) and get a map \(\Phi : K_X^{\text{piece-by-piece}} \to K_X^{\text{fiber-product}}\) that is a smooth embedding on each copy of \(X\). Hence we can consider \(K_X^{\text{piece-by-piece}}\) as \(K_X^{\text{piece-by-piece}}\) with the pulled back (by \(\Phi\)) differentiable structure, or \(K_X^{\text{fiber-product}}\) with the pushed forward piecewise hyperbolic structure.

**Corollary 3.3.** The following diagram commutes.

\[
\begin{array}{ccc}
K_X^{\text{piece-by-piece}} & \xrightarrow{\Phi} & K_X^{\text{fiber-product}} \\
F \downarrow & & \downarrow q_K \\
K & \xrightarrow{1_K} & K
\end{array}
\]

Moreover \(\Phi\) is a smooth embedding on each copy of \(X\) in \(K_X^{\text{piece-by-piece}}\).

Here is an important caveat. We showed how to identify \(K_X^{\text{piece-by-piece}}\) and \(K_X^{\text{fiber-product}}\) in a good way, so that we can benefit from the different properties of both constructions. But a key piece was missing: this identification was done under the assumption that \(p = \bar{\rho} \circ c\) is smooth (see statement of Proposition 3.2). We do not know how to prove that \(p\) is smooth, because the smooth structure on \(K\), though \(PL\)-compatible with \(K\), could be quite arbitrary. (If \(p\) is not of the form \(p = \bar{\rho} \circ c\) the problem of finding \(\Phi\) seems to be even harder.) But in our case this does not matter because we will work with normal smooth structures. The next result shows that \(p = \bar{\rho} \circ c\) is smooth on \((K, S')\), where \(S'\) is a normal smooth structure on \(K\) for \(K\) (see Section 1).

**Proposition 3.4.** Let \(S'\) be a normal smooth structure on \(K\) for \(K\). Then \(p : (K, S') \to \Box^n\) is smooth.

The proof is presented in appendix B.

Note that we also have that the restriction \(p|_{\Box^i}\) on every open cube is an embedding, because the inclusions \(\Box^i \to (K, S')\) are also embeddings (see remarks before Theorem 1.1). Therefore the regular value argument in \([2]\) (see items (i) and (ii) at the beginning of Section 3) goes through and we get the following result.

**Corollary 3.5.** We have that \(K_X^{\text{fiber-product}}\) is a smooth submanifold of \((K, S') \times X\), with trivial normal bundle.
We denote by $K'_X$ the submanifold $K'_X \subset (K, S') \times X$ with its induced smooth differentiable structure.

The proof of Proposition 3.2 also works if we replace $K'_X$ by $K'_X$, but with one change: we have to substitute $X_{\square^n}$ by $\hat{X}_{\square^n}$ (these are open faces), that is, the map $\hat{X} \to \hat{X}_{\square^n} \subset K'_X$ is an embedding (the key point in the proof is that the inclusion $(c|_{\square^n})^{-1} : \square^n \to (K, S')$ in not an embedding, but its restriction to $\square^n$ is). It follows that $\hat{X}$ is a submanifold of $K'_X$, for every $\square \in K$. Therefore we obtain the following corollary.

**Corollary 3.6.** The map

$$\Phi : K_{X \text{piece-by-piece}} \to K'_X$$

is a smooth embedding on each copy of $\hat{X}$ in $K_{X \text{piece-by-piece}}$.

4. Normal neighborhoods on Charney-Davis Hyperbolizations.

Theorem 1.1 gives a normal smooth atlas and a normal smooth structure for a given smooth cubulation of a smooth manifold. In this section we will construct a similar atlas on the Charney-Davis strict hyperbolization of $K$. In what follows of this section we will use the notation $K_X$ for $K_{X \text{piece-by-piece}}$. Recall that we are denoting by $K'_X$ the submanifold $K_X \subset (K, S') \times X$ with its induced smooth structure. Recall we have a map $\Phi : K_X \to K'_X$ (see Section 3). The normal smooth structure $S'$ on $K$ has a normal atlas $A = A(\mathcal{L})$, where $\mathcal{L} = \{ h_{\square^k} \}_{\square^k \in K}$ is a smoothly compatible set of link smoothings for $K$. We assume that the normal bundle of any face of the hyperbolization piece $X$ has width larger than $s_0 > 0$. Choose $r$, with $3r < s_0$. By Lemma 2.1 the number $s_o$ can be taken as large as we want.

By Lemma 2.3 we can use the derivative of the map $F : K_X \to K$ (in a piecewise fashion) to identify $\text{Link}(X_{\square^k}, K_X)$ with $\text{Link}(\square^k, K)$, where in both cases we consider the “direction” definition of link, that is, the link $\text{Link}(X_{\square^k}, K_X)$ (at $p \in X_{\square^k}$) is the set of normal vectors to $X_{\square^k}$ (at $p$) and the link $\text{Link}(\square^k, K)$ (at $q \in \square^k$) is the set of normal vectors to $\square^k$ (at $q$). Hence we write $\text{Link}(X_{\square^k}, K_X) = \text{Link}(\square^k, K)$; thus the set of links for $K$ coincides with the set of links for $K_X$.

Let $X_{\square^k} \subset X_{\square^n}$ be a $k$-face of $K_X$, contained in the copy $X_{\square^n}$ of $X$ over $\square^n$. For a non-zero vector $u$ normal to $X_{\square^k}$ at $p \in X_{\square^k}$, and pointing inside $X_{\square^n}$, we have that $\exp_p(tu)$ is defined and contained in $X_{\square^n}$, for $0 \leq t < s_0/|u|$. Recall that $h_{\square^k} : S^{n-k-1} \to \text{Link}(\square^k, K) = \text{Link}(X_{\square^k}, K_X)$ is the smoothing of the link corresponding to $\square^k$. In the Introduction we defined the map

$$H_{\square^k} : \mathbb{D}^{n-k} \times X_{\square^k} \to K_X$$

given by

$$H_{\square^k}(tv, p) = \exp_p\left( 2rt\ h_{\square^k}(v) \right)$$
where \( v \in \mathbb{S}^{n-k-1} \) and \( t \in [0,1) \). For \( k = n \) we have that \( H_{\triangle^n} \) is the inclusion \( \hat{X}_{\triangle^n} \subset K_X \) (or we can take this as a definition). Note that \( H_{\square^k} \) is a topological embedding because we are assuming the width of the normal neighborhood of \( X_{\square} \) to be larger than \( s_0 > 2r \). We called a chart of the form of \( H_{\square^k} \) (for some link smoothing \( h_{\square^k} \)) a normal chart for the \( k \)-face \( X_{\square^k} \). A collection \( \{ H_{\square^k} \}_{\square^k \in K} \) of normal charts is a normal atlas, and if this atlas is smooth (or \( C^k \)) the induced differentiable structure is called a normal smooth (or \( C^k \)) structure. The following is the Main Theorem in the Introduction.

**Proposition 4.1.** The normal atlas \( \{ H_{\square^k} \}_{\square^k \in K} \) on \( K_X \) is smooth.

**Proof.** Since we are assuming \( \mathcal{A} = \mathcal{A}(\{ h_{\square} \}) \) smooth we get from Proposition 1.3.4 that the set of smoothings \( \{ h_{\square} \} \) is smoothly compatible, that is, the maps in 1.3.2 (or 1.3.3) are smooth embeddings. For \( \square^k \subset \square^j \in K \) these maps have domains \( \mathbb{D}^{n-j} \times (\square^j \cap \text{Link}(\square^k, K)) \) (the second factor is denoted by \( \hat{\sigma}^j \) in 1.3) and target space \( \mathbb{S}^{n-k-1} \). We remark that in this definition (and in Section 1.3) we use the “geometric” definition of link, while here in Section 4 we are using the “direction” definition of link. But using (piecewise euclidean or piecewise hyperbolic) exponential maps in \( K \) or \( K_X \) we can identify these definitions. Therefore we can identify \( \square^j \cap \text{Link}(\square^k, K) \) with \( \hat{X}_{\square^j} \cap \text{Link}(X_{\square^k}, K) \) (the links here are geometric). This together with the fact that \( \text{Link}(\square, K) = \text{Link}(X_{\square}, K_X) \) imply that we can obtain maps in the \( K_X \) case similar to the maps in 1.3.2, and these maps have the same domains and target spaces. Moreover they coincide modulo a slight smooth change (see remark below). Therefore \( K_X \) versions of 1.3.2 are also “smoothly compatible”. Now, the proof that \( \{ H_{\square^k} \} \) is smooth is similar to the proof that \( \mathcal{A} = \{ h_{\square^k} \} \) is smooth (assuming \( \{ h_{\square} \} \) is smoothly compatible) given in the proof of Proposition 1.3.4. This proves Proposition 4.1.

**Remark.** There is only one adjustment that has to be made in the proof given in Proposition 1.3.4 to be applied to the case of Proposition 4.1. In Section 1.2 (see 1.2.1) we identified \( \mathcal{C}\text{Link}(\square^k, K) \) as a subset of \( \text{Link}(\square^j, K), \square^k \subset \square^j \) using the radial projection \( \mathcal{R} \) described in Remark 1.2.2. In the hyperbolic case (for Proposition 4.1) hyperbolic radial projection give a similar identification (call it \( \mathcal{R}_H \)). Moreover, since ray structures are preserved these two projections coincide in directions and just differ on the length. This length in the cube case is given in Remark 1.2.2. Using hyperbolic trigonometry the analogous formula (using the same setting as in 1.2.2) is given by: 

\[
\tan^{-1}\left( \frac{\tanh(d_K(v,p))}{\sinh(2r)} \right),
\]

which is a smooth function. Therefore if \( \{ h_{\square} \} \) is smoothly compatible using the identifications \( \mathcal{R} \), then \( \{ h_{\square} \} \) is also smoothly compatible using the identifications \( \mathcal{R}_H \).

We will denote by \( S_{K_X} = S_{K_X}(\{ h_{\square} \}) \) the smooth structure on \( K_X = K_X^{\text{piece-by-piece}} \) induced by the smooth atlas \( \mathcal{A}_{K_X} = \{ H_{\square^k} \}_{\square^k \in K} \). Note that \( \mathcal{A}_{K_X} \) depends uniquely on the smoothly compatible set of link smoothings \( \mathcal{L} = \{ h_{\square} \}_{\square \in K} \) for \( K \) (hence for \( K_X \)), and to express this dependence we will sometimes write \( \mathcal{A}_{K_X} = \mathcal{A}_{K_X}(\mathcal{L}) \).

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Proposition 4.2. The map $\Phi : (K_X, S_{K_X}) \to K'_X$ is a $C^1$-diffeomorphism.

The proof is a bit technical and it is given in appendix C.

Hence the atlas $\{H^k\}$ is a normal $C^1$-atlas for the smooth manifold $K'_X$. The following is a more detailed version of the addendum to the Main Theorem given in the Introduction.

Proposition 4.3. The smooth manifolds $(K_X, S_{K_X})$ and $K'_X$ are smoothly diffeomorphic.

Proof. Just approximate the $C^1$-diffeomorphism $\Phi$ by a smooth diffeomorphism.

5. Normal structures for Hyperbolized Manifolds with Codimension Zero Singularities.

In this section we treat the case of manifolds with a one point singularity. The case of manifolds with many (isolated) point singularities is similar.

We assume the setting and notation of Section 1.5. Let $K_X$ be the Charney-Davis strict hyperbolization of $K$. Denote also by $p$ the singularity of $K_X$. Many of the definitions and results given in Sections 2, 3, 4 still hold (with minor changes) in the case of manifolds with a one point singularity:

1. Given a set of link smoothings for $K$ (hence for $K_X$) we also get a set of charts $H_{\square}$ as in Section 4. For the vertex $p$ we mean the cone map $H_p = Ch_p : CN \to CL \subset K_X$. We will also denote the restriction of $H_p$ to $CN - \{o_{CN}\}$ by the same notation $H_p$. As in item (3) of 1.5 here we are identifying $CN - \{o_{CN}\}$ with $N \times (0, 1]$ with the product smooth structure obtained from some smooth structure $\tilde{S}_N$ on $N$. As before $\{H_{\square}\}_{\square \in K}$ is a normal atlas for $K_X$ (or $K_X - \{p\}$). A normal atlas for $K - \{p\}$ induces a normal smooth structure on $K_X - \{p\}$.

2. Again we say that the smooth atlas $\{H_{\square}\}$ (or the induced smooth structure, or the set $\{h_{\sigma}\}$) is correct with respect to $N$ if $S_N$ is diffeomorphic to $\tilde{S}_N$.

3. Let the set $L = \{h_{\square}\}_{\square \in K}$ induce a smooth structure on $K - \{p\}$, hence $L$ is smoothly compatible (see item (3) of 1.5). As Proposition 4.1 we get that $\{H_{\square}\}_{\square \in K}$ is a smooth atlas on $K_X - \{p\}$ that induces a normal smooth structure $S_{K_X}$ on $K_X - \{p\}$. Moreover, from Theorem 1.5.1 we get that $S_{K_X}$ is correct with respect to $S_N$ when $\dim N \leq 4$ (always) or when $\dim N > 4$, provided $W h(N) = 0$. Note that in this case we can take the domain $CN - \{o_{CN}\} = N \times (0, 1]$ of $H_p$ with smooth product structure $S_N \times S_{[0,1]}$.

4. It can be verified that a version of Proposition 4.3 also holds in this case: $(K_X - \{p\}, S_{K_X})$ smoothly embeds in $(K - \{p\}, S') \times X$ with trivial normal bundle.
Appendix A. Proof of Proposition 3.1 and its Addendum.

As always we write $I = [0, 1]$. Recall that the function $\tilde{\rho}$ is defined as $\tilde{\rho}(x_1, \ldots, x_n) = (\rho(x_1), \ldots, \rho(x_n))$, where $\rho: I \to I$ is as in Section 3. We will assume the following extra condition on $\rho$:

\[\rho(x) = x \text{ for } \delta \leq x \leq 1 - \delta\]  
\[\text{(A.1.)}\]

for some small $\delta > 0$. Let $\Box^{n-1}$ be an $(n - 1)$-face of $\Box^n = \{(x_1, \ldots, x_n), 0 \leq x_i \leq 1\}$. For simplicity write $\Box^n = \Box^{n-1} \times I$, and consider the vector field on $\Box^n$, depending on $\Box^{n-1}$, given by $V_{\Box^{n-1}}(x) = e_n = (0, \ldots, 0, 1)$. This vector field is perpendicular to $\Box^{n-1}$ and generates the collar $\eta_{\Box^{n-1}}: \Box^{n-1} \times I \to \Box^n$, of $\Box^{n-1}$ in $\Box^n$ (which for the decomposition $\Box^n = \Box^{n-1} \times I$ is just the identity).

Let $\hat{\rho}$ be the smooth self-homeomorphism on $\Box^{n-1} \times [0, \delta]$ given by $\hat{\rho}(x, t) = (x, \rho(t))$. Let $\Lambda_{\Box^{n-1}}$ be the smooth self-homeomorphism on $\Box^n \to \Box^n$ that is the identity outside $\eta_{\Box^{n-1}}(\Box^{n-1} \times [0, \delta])$ and on the image of $\eta_{\Box^{n-1}}$ it is equal to $\eta_{\Box^{n-1}} \circ \hat{\rho} \circ \eta_{\Box^{n-1}}^{-1}$. Hence we can write

\[\hat{\rho} = \Lambda_{\Box^{n-1}} \circ \ldots \circ \Lambda_{\Box^{n-1}}\]  
\[\text{(A.2.)}\]

for any ordering $\Box_{1}^{n-1}, \ldots, \Box_{2n}^{n-1}$ of all the $(n - 1)$-faces of $\Box^n$.

We will assume that the width of the normal neighborhoods of the $X_\Box$ in $X$ are larger than $3r$ (see Section 2).

Lemma A.3. For each $\Box^{n-1}$ the vector field $V_{\Box^{n-1}}$ has a lifting $W_{\Box^{n-1}}$ to $X$ near $X_{\Box^{n-1}}$. Moreover $W_{\Box^{n-1}}$ is perpendicular to $X_{\Box^{n-1}}$.

Remark. By $W_{\Box^{n-1}}$ being a lifting of $V_{\Box^{n-1}}$ near $X_{\Box^{n-1}}$ we mean that $W_{\Box^{n-1}}$ is defined on a normal neighborhood of $X_{\Box^{n-1}}$ of width $\leq r$, and $Df \cdot W_{\Box^{n-1}} = V_{\Box^{n-1}}$.

Before we present the proof of Lemma A.3 we show how it implies Proposition 3.1. The addendum to 3.1 will be proved later, at the end of this appendix. There is an $s'$ such that all $W_{\Box^{n-1}}$ are defined on the normal neighborhood of $X_{\Box^{n-1}}$ of width $s'$. Using the vector fields $W_{\Box^{n-1}}$ we get collars $\tau_{\Box^{n-1}}: X_{\Box^{n-1}} \times [0, a] \to X$, for some fixed $a > 0$. Since $W_{\Box^{n-1}}$ is a lifting of $V_{\Box^{n-1}}$ we get

\[f(\tau_{\Box^{n-1}}(x, t)) = \eta_{\Box^{n-1}}(f(x), t)\]  
\[\text{(A.4.)}\]

For instance, in the special case of the trivial decomposition $\Box^n = \Box^{n-1} \times I$ we get $f(\tau_{\Box^{n-1}}(x, t)) = (f(x), t)$ because, in this case $\eta_{\Box^{n-1}}$ is just the identity. Let now $\theta_{\Box^{n-1}}$ be the smooth self-homeomorphism on $X_{\Box^{n-1}} \times [0, a]$ given by

\[\theta_{\Box^{n-1}}(x, t) = (x, \rho(t))\]  
\[\text{(A.5.)}\]
Assuming $\delta > 0$ in (A.1) such that $\delta < a$, we get that $\theta_{\square^{n-1}}$ is the identity outside $X_{\square^{n-1}} \times [0, \delta] \subset X_{\square^{n-1}} \times [0, a)$. Finally define $\Theta_{\square^{n-1}}$ to be the the smooth self-homeomorphism on $X$ that is the identity outside $\tau_{\square^{n-1}} \left( X_{\square^{n-1}} \times [0, \delta) \right)$ and on the image of $\tau_{\square^{n-1}}$ is equal to $\tau_{\square^{n-1}} \circ \theta_{\square^{n-1}} \circ \tau_{\square^{n-1}}^{-1}$.

Claim A.6. For every $\square^{n-1}$ we have that $f \circ \Theta_{\square^{n-1}} = \Lambda_{\square^{n-1}} \circ f$.

Proof of Claim. By (A.4) we have that $f(\tau_{\square^{n-1}} (X_{\square^{n-1}} \times [0, \delta)) = \eta_{\square^{n-1}} (\square^{n-1} \times [0, \delta))$. Hence a point $p \in X$ is in $\tau_{\square^{n-1}} (X_{\square^{n-1}} \times [0, \delta))$ if and only if its image $f(p)$ is in $\eta_{\square^{n-1}} (\square^{n-1} \times [0, \delta))$. If $p$ is not in $\tau_{\square^{n-1}} (X_{\square^{n-1}} \times [0, \delta))$ we get that $\Theta_{\square^{n-1}}(p) = p$ and $\Lambda_{\square^{n-1}}(f(p)) = f(p)$ and the claim is true in this case. Assume now that $p$ is in $\tau_{\square^{n-1}} (X_{\square^{n-1}} \times [0, \delta))$. Write $\tau_{\square^{n-1}}(x, t) = p$, thus $f(p) = \eta_{\square^{n-1}}(f(x), t)$. By applying (A.4) and (A.5) several times we get

\[
\begin{align*}
  f \circ \Theta_{\square^{n-1}}(p) &= f \circ \tau_{\square^{n-1}} \circ \theta_{\square^{n-1}} \circ \tau_{\square^{n-1}}^{-1}(p) \\
  &= f \circ \tau_{\square^{n-1}} \circ \theta_{\square^{n-1}}(x, t) \\
  &= f \circ \tau_{\square^{n-1}}(x, \rho(t)) \\
  &= \eta_{\square^{n-1}}(f(x), \rho(t)) \\
  &= \eta_{\square^{n-1}} \circ \hat{\rho} (f(x), t) \\
  &= \eta_{\square^{n-1}} \circ \hat{\rho} \circ \eta_{\square^{n-1}}^{-1} \circ f(p) \\
  &= \Lambda_{\square^{n-1}} \circ f(p)
\end{align*}
\]

This proves the claim.

To finish the proof of Proposition 3.1 just define $P = \Theta_{\square^{n-1}} \circ \ldots \circ \Theta_{\square^{n}}$. The fact that $f \circ P = \hat{\rho} \circ f$ follows from (A.2) and Claim A.6. This proves Proposition 3.1.

It remains to prove Lemma A.3.

Proof of Lemma A.3. Fix $\square^{n-1}$. Without loss of generality we assume $\square^{n-1} = \square^{n-1}_1$, where $\square^{n-1}_i = \{ x_i = 0 \} \cap \square^n$. We have $\square^n = I \times \square^{n-1}_1$. Write $V = V_{\square^{n-1}_1}$ and $W = W_{\square^{n-1}_1}$. Now, since the condition $Df.W = V$ is linear we have, using a partition of unity and taking $\delta$ small in (A.1), that it is enough to find a lift of $V$ just locally, that is:

(A.7.) for every $p \in \square^{n-1}_1$ there is a neighborhood $U$ of $p$ in $X$ and vector field $W$ on $U$ such that $Df.W = V$.

Let $\square^k \subset \square^n$. Write $D^j(s) = C_s \Delta_{\square^j} = \{ tu \in \mathbb{R}^{j+1}, t \in [0, s], u \in \Delta_{\square^j} \}$. We identify the closed normal neighborhood $N_s(\square^k)$ of $\square^k$ of width $s$ with $\square^k \times D^{n-k}(s)$ (here $s < 1$). Similarly we identify the closed normal neighborhood $N_s(X_{\square^k})$ of $X_{\square^k}$ of width $s$ (via the exponential map) with $X_{\square^k} \times D^{n-k}(s)$. Note that for $\square^k \subset \square^n$ we can write $\square^k = \bigcap_{\square^k \subset \square^n-1} \square^{n-1}_1$. Define

\[
\begin{align*}
  A_s(\square^k) &= \bigcap_{\square^k \subset \square^n-1} N_s\left( \square^{n-1}_1 \right) \\
  A_s(X_{\square^k}) &= \bigcap_{\square^k \subset \square^n-1} N_s\left( X_{\square^{n-1}} \right)
\end{align*}
\]

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and for \( k < n \)

\[
L(X_{\square^k}) = A_{3r}(X_{\square^k}) - \bigcup_{\square^k \subset \square^n} N_{2r}(X_{\square^{n-1}})
\]

\[
= \bigcap_{\square^k \subset \square^n} N_{3r}(X_{\square^{n-1}}) - \bigcup_{\square^k \subset \square^n} N_{2r}(X_{\square^{n-1}})
\]

Note that \( A_s(X_{\square^k}) \subset N_{s'}(X_{\square^k}) \) for large \( s' \) (how large \( s' \) should be with respect to \( s \) can be calculated using hyperbolic trigonometry). Hence \( L(X_{\square^k}) \subset N_s(X_{\square^k}) \) for large \( s \).

**Claim A.8.** We have \( X_{\square_1^{n-1}} \subset \bigcup_{\square^k \subset \square_1^{n-1}} L(X_{\square^k}) \).

Let \( p \in X_{\square_1^{n-1}} \). If \( p \notin L(X_{\square_1^{n-1}}) \) then \( p \in N_{2r}(X_{\square_1^{n-1}}) \) for some \( \square^{n-1} \). Hence \( p \in A_{2r}(X_{\square_1^{n-1}} \cap \square_1^{n-1}) \). Therefore we either have \( p \in L_{2r}(X_{\square_1^{n-1}}) \), or \( p \in N_{2r}(X_{\square_1^{n-1}}) \), for some \( \square_2^{n-1} \) different from \( \square_1^{n-1} \) and \( \square^{n-1} \). Arguing in the same way by induction we get that if \( p \notin L(X_{\square^k}) \) for all \( \square^k \subset \square_1^{n-1} \) with \( k > 0 \) then \( p \in A_{2r}(X_{\square^0}) \subset L(X_{\square^0}) \), for some vertex \( \square^0 \). This proves Claim A.8.

We now prove statement (A.7). We use the construction of the map \( f \) given in Section 2. Let \( p \in \square_1^{n-1} \). From Claim A.8 we can assume that \( p \in L(X_{\square^k}) \), for some \( \square^k \subset \square_1^{n-1} \). Write \( l = n - k \). Note that \( L(X_{\square^k}) \subset N_s(X_{\square^k}) = X_{\square^k} \times D^l(s) \) (for large \( s \)), hence we will sometimes write \( p = (p,0) \in X_{\square^k} \times D^l(s) = N_s(X_{\square^k}) \).

For simplicity we assume \( \square^k = \square_1^{n-1} \cap ... \cap \square_l^{n-1} \), \( l = n - k \). Hence, using the notation in Lemma 2.4, we have that \( p_i \circ f = (f_1, ..., f_l) \) and \( p_i \circ T = (t_1, ..., t_l) \).

**Claim A.9.** We have

(a) if \( i > l \) then \( f_i(p) = 1/2 \),

(b) if \( (q,u) \in L(X_{\square^k}) \) is close to \( p = (p,0) \), then \( f_i(q,u) = 1/2, i > l \),

(c) let \( U = U' \times D \subset X_{\square^k} \times D^l(s) \) be a product neighborhood where (b) holds for every \( (q,u) \in U \). Then \( p_i \circ T \) is an embedding on \( \{q\} \times D \), for every \( q \in U' \).

Since \( p \in L(\square^k) \) we have that \( p \notin N_{2r}(X_{\square_1^{n-1}}) \), for \( i > l \). Therefore \( t_i(p) = d_x(p,X_{\square_1^{n-1}}) > 2r > r \), \( i > l \), and (a) follows. Item (b) follows from (a), continuity and the fact that the sets \( N_s \) are closed. Item (c) follows from Lemma 2.4 (v). This proves Claim A.9.

To finish the proof of (A.7) on \( U \) just take \( W(q,u) = \frac{1}{2r} \left((p_i \circ T)|_{(q) \times D}\right)^*(e_1) \), where \( e_1 \) is the constant vector field \((1,0,...,0)\) on \( \mathbb{R}^l \). (Note that \( W \) is different from the gradient, with respect to the hyperbolic metric on \( X \), of the distance to \( X_{\square_1^{n-1}} \) function \( t_1 \).) It follows now from (b) of Claim A.9 and the fact that \( f_1(x) = \frac{1}{2r} \rho(t_1(x)) = \frac{t_1(x)}{2r} \), if \( x \) is close to \( X_{\square_1^{n-1}} \), that \( Df_iW = e_1 = V \). This proves (A.7). It can be verified from the construction that the second statement of A.3 holds. This proves Lemma A.3 and completes the proof of Proposition 3.1.
Proof of the Addendum to Proposition 3.1. Since for $p \in \Box^{n-1}$ we have that $W_{\Box^{n-1}}(p)$ is perpendicular to $X_{\Box^{n-1}}$, it is enough to prove that $DP_p.W_{\Box^{n-1}}(p) = 0$, for every $\Box^{n-1}$ and $p \in \Box^{n-1}$. To make this happen we need to modify our construction of $P$ a little bit.

Note that from the second statement in Lemma A.3, (A.5) and the definition of $\Theta_{\Box^{n-1}}$ we get

\[(A.10.)\]  
\[D\Theta_{\Box^{n-1}}.W_{\Box^{n-1}} = 0\]

We need a lemma, which is is essentially an initial value version of Lemma A.3.

Lemma A.11. Let $U$ be a (not necessarily tangent) vector field on $X_{\Box^{n-1}}$. Suppose that $Df.U = V_{\Box^{n-1}}$. Then there is a self-diffeomorphism $g$ on $X$ covering the identity $1_{\Box^n} : \Box^n \rightarrow \Box^n$ (see diagram) with $Dg.U = W_{\Box^{n-1}}$.

\[\begin{array}{ccc}
X & \xrightarrow{\theta} & X \\
\downarrow f & & \downarrow f \\
\Box^n & \xrightarrow{1_{\Box^n}} & \Box^n
\end{array}\]

Proof. Using collars and integral curves the problem is reduced to finding an extension $U$ of $U$ to a neighborhood of $X_{\Box^{n-1}}$, with $Df.U = V_{\Box^{n-1}}$ and $U = W_{\Box^{n-1}}$ outside an even smaller neighborhood of $X_{\Box^{n-1}}$ (the argument uses the integral curves of $-U$). The proof that such an extension exists is similar to that of Lemma A.3. (without the perpendicularly condition). The only change needed is at the very end of the proof of (A.7) (after the proof of (A.9)). In our present case we have that $U(q) = U(q, 0) = W(q, 0) + T(q)$, where $T(q)$ is tangent to $X_{\Box^{n-1}}$ (this is because $Df.U = V_{\Box^{n-1}}$ and $q \in L(X_{\Box^n})$). Now take $U(q, v) = W(q, u) + \rho(|v|)T(q)$, where $\rho(t)$ is equal to 1 near $t = 0$ and equal to 0 for $t \geq \mu$, for some small $\mu > 0$. This proves the lemma.

We now prove the addendum. Recall that at the beginning of Appendix A we ordered the $(n - 1)$-cubes: $\Box^{n-1}_1, ..., \Box^{n-1}_2$, and we constructed the corresponding $\Lambda_{\Box^{n-1}}, \Theta_i = \Theta_{\Box^{n-1}}$. Write $V_i = V_{\Box^{n-1}}$ and $W_i = W_{\Box^{n-1}}$. We will need the following statement which follows from the definition of the $\Lambda_{\Box^{n-1}}$.

\[(A.12.)\]  
\[D\Lambda_{\Box^{n-1}}.V_j = V_j, \text{ for } i \neq j.\]

Take now $P = \Theta_{2n} \circ g_{2n-1} \circ ... \circ g_1 \circ \Theta_1$, where the $g_i$ are obtained in the following way. From Claim A.6. (A.12) and the fact that $Df.W_i = V_i$ we get that $Df.(D\Theta_1.W_2) = V_2$, hence we can apply Lemma A.11 to get a self-diffeomorphism $g_1 : X \rightarrow X$ lifting the identity and satisfying $Dg_1.(D\Theta_1.W_2) = W_2$. Next note that from A.6. (A.12), Lemma A.11 and the fact that $Df.W_i = V_i$ we get that $Df.(Dg_1 \circ \Theta_1).W_3) = V_3$ and we can apply Lemma A.11 to get a self-diffeomorphism $g_2 : X \rightarrow X$ lifting the identity and satisfying $Dg_2.(D(g_1 \circ \Theta_1).W_3) = W_3$, and so on. From the choice of the $g_i$ and (A.10) we get that $DP_i.W_i = 0$. Also from Claim A.6 and the fact that $g_i$ lifts the identity we get $f \circ P = \rho \circ f$. This proves the addendum to Proposition 3.1.
Appendix B. Proof of Proposition 3.4.

We shall demand the following condition on $\rho$: that the derivatives of $\rho$ approach zero exponentially fast, at 0 and 1. That is

\[(B.1.)\quad \text{for every } k \text{ there are positive } a \text{ and } b \text{ such that } \left| \frac{d^k}{dt^k}\rho(t) \right| \leq ae^{-\frac{b}{1-t}}.\]

Let $A = \{(h^i, \mathbb{D}^{n-i} \times \hat{\Box}^i)\}$ be a normal atlas inducing $S'$. We write $W_{\hat{\Box}^i}$ for the image of $h^i$. Note that $W_{\hat{\Box}^i}$ is a normal neighborhood $C\text{Link}(\Box, K) \times \hat{\Box}^i$ of $\hat{\Box}^i$. Write $c = (c_1, ..., c_n)$. We will prove that $\mu = \rho \circ c_1$ is smooth. The prove for $\rho \circ c_i$ is the same.

We state three facts about the map $c_1$, which can be verified by inspecting each of them cube by cube.

(1) There are three possibilities for a cube $\Box \in K$: First $c_1(\Box) = \{0\}$ and we say $\Box$ is a 0-valued-cube, second $c_1(\Box) = \{1\}$ and we say $\Box$ is a 1-valued-cube and finally $c_1|\Box$ is onto $I = [0, 1]$ and we say in this last case that $\Box$ is an I-cube. In what follows everything we do for 0-valued-cubes can be done for 1-valued-cubes, so we will just ignore 1-valued-cubes.

(2) For a 0-valued-cube $\Box^i$ the map $c_1$ (and hence $\mu$ and all its derivatives) is a product map on a neighborhood of $\Box^i$. Specifically $c_1$ factors through a composition

\[W_{\Box^i} = C\text{Link}(\hat{\Box}^i, K) \times \Box^i \xrightarrow{\text{projection}} C\text{Link}(\hat{\Box}^i, K) \longrightarrow I\]

(3) For a I-cube $\Box^i$ the map $c_1$ (and hence $\mu$ and all its derivatives) is a product map on a neighborhood of $\Box^i$. Specifically $c_1$ factors through a composition

\[W_{\Box^i} = C\text{Link}(\hat{\Box}^i, K) \times \Box^i \xrightarrow{\text{projection}} \Box^i \longrightarrow I\]

where the last arrow is also a projection: $(x_1, ..., x_n) \mapsto x_1$.

We prove that $\mu = \rho \circ c_1$ is smooth by showing that its representative $\mu_{\Box} = \mu \circ (h^i)^{-1}$ on each chart is smooth. We prove this by induction on the decreasing dimension of the cubes. Consider first the following two statements that depend on the $i$-cube $\Box^i$:

A($\Box^i$): We have that $\mu_{\Box^i}$ is smooth on $\mathbb{D}^{n-i} \times \hat{\Box}^i$. (Hence $\mu$ is smooth on $W_{\Box^i}$.)

B($\Box^i$): For every $\Box^j < \Box^i$, $\Box^j$ a 0-valued-cube, the map $\mu_{\Box^j}$ and all its derivatives approach zero exponentially fast with respect to the distance to $\Box^j$. That is, for every $k$ there are positive $a$ and $b$ such that

\[\left| \frac{d^k}{dt^k}\mu_{\Box^j}(p) \right| \leq ae^{-\frac{b}{1-t}}, \text{ where } t = d_{\mathbb{D}^{n-i} \times \Box^i}(p, \Box^j), p \in \mathbb{D}^{n-i} \times \hat{\Box}^i.\]
Recall that the chart maps $h^\bullet_{\square^i}$ respect the product $\mathbb{D}^{n-i} \times \square^i$ and the inclusion maps of cubes $\square \to (M, S')$ are embeddings. Therefore item (3) above implies that for an $I$-cube $\square^i$ the map $\mu_{\square^i}$ is just projection given by the composition

\[
\mathbb{D}^{n-i} \times \square^i \xrightarrow{\text{projection}} \square^i \xrightarrow{\text{projection}} I \xrightarrow{\rho} I
\]

where the last projection is projection to the $x_1$ coordinate. Since $\mu_{\square^i} = \rho \circ \pi$, where $\pi$ is linear, we have that $A(\square)$ is true for every $I$-cube $\square$. Also, if $\square^i$ is an $I$-cube and $\square^j < \square^i$ is a 0-valued-cube, we can write $\square^i = \square^{i-1} \times \square^1$, $\square^j < \square^{i-1}$, where the projection on to the $x_1$ coordinate is $\square^{i-1} \times \square^1 \to \square^1 = I$. But for $p \in \mathbb{D}^{n-i} \times \square^1$ we have

\[
\pi(p) \leq d_{\mathbb{D}^{n-i}\times \square^1}(p, \square^{i-1}) \leq d_{\mathbb{D}^{n-i}\times \square^1}(p, \square^j)
\]

This together with $\mu_{\square^i} = \rho \circ \pi$, B1, and the fact that $\pi$ is linear imply that $B(\square^i)$ is also true for every $I$-cube $\square^i$.

For a 0-valued $(n-1)$-cube $\square^{n-1}$ it is straightforward to verify that $A(\square^{n-1})$ and $B(\square^{n-1})$ hold true. Assume now that $A(\square^i)$ and $B(\square^i)$ hold true for every 0-valued-cube $\square^i$, $i > k$. We prove the same is true for 0-valued-cubes $\square^k$.

Let $\square^k$ be a cube of dimension $k$. Since $W_{\square^k} - \hat{\square}^k \subset \bigcup_{j > k} W_{\square}$, it follows from the inductive hypothesis $A(\square^i)$, $i > k$, that $\mu_{\square^k}$ is smooth on $\mathbb{D}^{n-k} \times \hat{\square}^k - \hat{\square}^k$, where we are writing $\hat{\square}^k = \{0\} \times \hat{\square}^k$. By item (2) above $\mu_{\square^k}$ is a product on $\mathbb{D}^{n-k} \times \hat{\square}^k$, hence, it is enough to prove that the restriction $\nu = \mu_{\square^k} |_{\mathbb{D}^{n-k}} : \mathbb{D}^{n-k} \to I$ is smooth at $0 \in \mathbb{D}^{n-i}$. And by the Mean Value Theorem we only need to prove that all partial derivatives of $\nu : \mathbb{D}^{n-k} \to I$ tend to zero as a point $p$ tends to $0 \in \mathbb{D}^{n-1}$.

Let $v_m = t_m u_m \in \mathbb{D}^{n-k}$, $u \in S^{n-k-1} = \partial \mathbb{D}^{n-k}$, $t_m \in (0, 1)$, $t_m \to 0$. We want to prove that all partial derivatives of $\nu$ at $v_m$ tend to zero as $m \to \infty$. We can assume (arguing by contradiction) that $u_m \to u \in S^{n-k-1}$.

Corollary 1.1.2 says that the link $S = \text{Link}(\square^k, K)$ is a submanifold of $(M, S')$. The open sets $U_{\square^i} = S \cap W_{\square}$, $\square^i > \square^k$, form an open cover of $S$. Note that $U_{\square^i}$ is a normal neighborhood of $\square^i \cap S$ in $S$. Write $x_m = h^\bullet_{\square^k}(v_m)$ and $y_m = h^\bullet_{\square^k}(u_m) \in S$. (Rigorously $h^\bullet_{\square^k}$ is not defined on $\partial \mathbb{D}^{n-k}$ but, after rescaling, we can assume this does happen). Since $h^\bullet_{\square^k}$ restricted to $\mathbb{D}^{n-k}$ is, by definition, a cone map we can write $x_m = t_m y_m$, where this last product is realized on the cone link $C \text{Link}(\square^k, K)$ of $\square^k$. And we also get $y_m \to z = h^\bullet_{\square^k}(u)$.

We have that $z \in \hat{\square}^i \cap S$, for some $\square^i > \square^k$. Let $V$ be a small neighborhood of $z$ in $S$ with $\hat{V} \subset U_{\square^i}$ and we assume $y_m \in V$ for all $m$. Write

\[
\nu = \mu \circ (h^\bullet_{\square^k} |_{\mathbb{D}^{n-k}}) = (\mu \circ h^\bullet_{\square^i}) \circ ((h^\bullet_{\square^i})^{-1}(h^\bullet_{\square^k} |_{\mathbb{D}^{n-k}}))
\]

By $B(\square^i)$ all partial derivatives of the first term $\mu \circ h^\bullet_{\square^i}$ approach zero exponentially fast as a point get close to $\square^i$. Likewise, by Corollary 1.4.3 the derivatives of the second term $(h^\bullet_{\square^i})^{-1}(h^\bullet_{\square^k} |_{\mathbb{D}^{n-k}})$ grow at most polynomially fast. Therefore, by applying the chain rule the
composition above we get that all partial derivatives of \( \nu \) tend to zero as \( v_m \to 0 \in \mathbb{D}^{n-i} \). This proves \( A(\square^k) \).

Note that the convergence of the derivatives of \( \nu \) to zero shown above is exponentially fast. This together with the fact that (see item (2) above) the map \( \mu_{\square^k} \) is a product on \( \mathbb{D}^{n-i} \times \square^k \) imply \( B(\square^k) \). This proves the proposition.

Appendix C. Proof of Proposition 4.2.

Recall that \( A = \{(h_{\square^k}, \mathbb{D}^{n-k} \times \square^k)\} \) is a normal atlas on \( K \), that generates the normal smooth structure \( S_{X}' \). Also \( \{H_{x}\} \) is a normal atlas for \( K_X = K_{X}^{\text{piece-by-piece}} \), generating the smooth structure \( S_{K_X} \). We will assume that the charts \( H_{\square^k} : \mathbb{D}^{n-k} \times \square^k \to K_X \) are defined on the larger sets \( \mathbb{D}^{n-k}(1+\delta) \times \square^k \) (here \( \mathbb{D}(1+\delta) \) is the open disc of radius 1+\( \delta \)). We can obtain this using 2.1.

Write \( H_{\square^k}' = \Phi \circ H_{\square^k} \). It is enough to prove that the maps \( H_{\square^k}' : \mathbb{D}^{n-k} \times \hat{X}_{\square^k} \to K_X \) are \( C^1 \)-embeddings. To prove this we need to prove that the following coordinate maps are both \( C^1 \)

\[
q_X \circ H_{\square^k}' : \mathbb{D}^{n-k} \times \hat{X}_{\square^k} \to (K, S')
\]

\[
q_X \circ H_{\square^k}' : \mathbb{D}^{n-k} \times \hat{X}_{\square^k} \to X
\]

We prove this by induction down the dimension of the skeleta. First for \( k = n \) recall that \( H_{\square^n} : \hat{X}_{\square^n} \to K_X \) is just the inclusion. Hence Proposition 3.2 implies that \( q_X \circ H_{\square^n}' : \hat{X}_{\square^n} \to \hat{\square^n} \) is the map \( \iota \circ f \), where \( \iota : \hat{\square^n} \to (K, S') \) is the inclusion, which is smooth. (Recall that the inclusion \( \hat{\square^n} \to (K, S') \) is not necessarily differentiable but its restriction \( \iota \) to \( \hat{\square^n} \) is smooth; see Remark 1 before Theorem 1.1). Therefore \( q_X \circ H_{\square^n}' \) is smooth. Also, by the definition of the map \( \Phi \), we have \( q_X \circ H_{\square^n}' = P \), which is also smooth. Moreover, by Proposition 3.1, \( P|_{\square^n} \) is an embedding. Therefore \( H_{\square^n}' \) is a smooth embedding for every \( n \)-cube \( \square^n \in K \).

Assume we have proved that \( H_{\square^j}' \) is a \( C^1 \)-embedding for every \( j \)-cube \( \square^j \in K \), \( j > k \). We have to prove that the same is true for all \( k \)-cubes. We prove this in three parts. In the first part we prove that \( q_X \circ H_{\square^k}' \) is \( C^1 \). In the second part we prove that \( q_X \circ H_{\square^k}' \) is \( C^1 \). This two parts imply that \( H_{\square^k}' \) is \( C^1 \). Finally, in the third part we prove that \( H_{\square^k}' \) is an embedding. Fix a \( k \)-cube \( \square^k \).

**FIRST PART.** The map \( q_X \circ H_{\square^k}' \) is \( C^1 \).

**Proof.** Denote by \( V_{\square} \) the image of \( H_{\square} \). For each \( \square^i \) with \( \square^k < \square^i \) we have that on \( U_{\square^i} = (H_{\square^k})^{-1}(V_{\square^i}) \) we can write

\[
q_X \circ H_{\square^k}' = (q_X \circ \Phi \circ H_{\square}) \circ (H_{\square}^{-1} \circ H_{\square^k}) = (q_X \circ H_{\square}' \circ (H_{\square}^{-1} \circ H_{\square^k})
\]

which is \( C^1 \) by inductive hypothesis and proposition 4.1. Since \( (\mathbb{D}^{n-k} - \{0\}) \times \hat{X}_{\square^k} \) is contained
in the union of the $U_{□}$, $i > k$, we have that $q_{X} \circ H'_{□}^{k}$ is $C^{1}$ outside $\hat{X}_{□} = \{0\} \times \hat{X}_{□}$. 

Since the map $q_{X} \circ \Phi|_{X_{□}^{n}}$ can be identified with the map $P : X \to X$ for a $n$-cube $□_{n}$ (recall $X_{□}^{n}$ is a copy of $X$), Proposition 3.1 implies that the derivatives at a point $(0,p) \in \{0\} \times \hat{X}_{□}^{k}$ in the $X$ directions $(0,v)$ exist because $q_{X} \circ H'_{□}^{n}$ on $\hat{X}_{□}^{k} = \{0\} \times \hat{X}_{□}^{k}$ is an embedding. We next show that the derivatives in the radial directions also exist and vanish. For this take a ray $\alpha(t) = (tu,p) \in \mathbb{D}^{n-k} \times \hat{X}_{□}^{k}$ and write $\beta(t) = H'_{□}^{k}(\alpha(t))$. Note that $\beta'(0) = DH_{□}^{k}u$ is normal to $X_{□}^{k}$. We have that the image of $\beta$ is contained in some $X_{□}^{n}$. As mentioned above the map $q_{X} \circ \Phi$ on $X_{□}^{n}$ can be identified with the map $P : X \to X$. The fact that the radial derivative in the direction $u$ exits and vanishes now follows from the addendum to 3.1.

Finally we need to prove that the first derivatives are continuous. But this follows from a result analogous to Lemma 1.4.1 with $X_{□}$ replacing $i$-cubes, which can easily be verified. This concludes the proof of the first part.

SECOND PART. The map $q_{K} \circ H'_{□}^{k} : \mathbb{D}^{n-k} \times \hat{X}_{□}^{k} \to (K,S')$ is $C^{1}$.

Proof. This proof will take the next five pages. First note that, by Corollary 3.3 and the definition of $H_{□}$ we have

$$q_{K} \circ H'_{□}^{k} (tv,p) = F \left( \exp_{p} \left( 2rt \ h_{□}(v) \right) \right)$$

(1)

Write $G_{□} = (h_{□}^{*})^{-1} \circ q_{K} \circ H'_{□}^{k} : \mathbb{D}^{n-i} \times \hat{X}_{□}^{k} \to \mathbb{D}^{n-i} \times □^{i}$. Since $\{h_{□}^{*}\}$ is an atlas for $(K,S')$, by inductive hypothesis we have that $G_{□}$ is $C^{1}$, for $i > k$, and

(C.1) to prove that $q_{K} \circ H'_{□}^{k}$ is $C^{1}$ it is enough to prove that $G_{□}^{k}$ is $C^{1}$.

Write also $G_{□} = (R_{□},T_{□})$. For $u = tv \in \mathbb{D}^{n-i}$, $t = |u|$, we have

$$G_{□}(tv,p) = \left( R_{□}(tv,p), T_{□}(tv,p) \right) = (h_{□}^{*})^{-1} \circ F \left( \exp_{p} \left( 2rt \ h_{□}(v) \right) \right)$$

(2)

It follows from (2) and Lemma 2.4 (iii), (iv), that we can write

$$G_{□}(u,p) = \left( R_{□}(u), T_{□}( |u|, p) \right)$$

(3)

that is, $R$ does not depend on $p$ and $T$ depends on $p$ and the length $|u|$ of $u$ (not on the direction of $u$). Also it can be checked from 2.4 (iv) and 2.5 that $(u,p) \mapsto T_{□}( |u|, p)$ is smooth. This together with (C.1) and (3) imply that

(C.2) to prove that $q_{K} \circ H'_{□}^{k}$ is $C^{1}$ it is enough to prove that $R_{□}^{k} : \mathbb{D}^{n-k} \to \mathbb{D}^{n-k}$ is $C^{1}$.

Claim C.3. The map $R_{□}^{k}$ is $C^{1}$ on $\mathbb{D}^{n-k} - \{0\}$. 32
Proof of Claim C.3. Recall that by inductive hypothesis we have that $G_{\square^i}$, $R_{\square^i}$ and $q_K \circ H'_{\square^i}$ are $C^1$, for all $\square^i$, $i > k$. Denote by $V_\square$ the image of $H_{\square}$. For each $\square^i$ with $\square^k < \square^i$ we have that on $U_{\square^i} = (H_{\square^k})^{-1}(V_{\square^i})$ we can write

$$q_K \circ H'_{\square^k} = (q_K \circ \Phi_{\square^k}) \circ (H^{-1}_{\square^k} \circ H_{\square^k}) = (q_K \circ H'_{\square^k}) \circ (H^{-1}_{\square^k} \circ H_{\square^k})$$

which is $C^1$ by inductive hypothesis and Proposition 4.1. Since $\mathbb{D}^{n-k} - \{0\} = (\mathbb{D}^{n-k} - \{0\}) \times \{p\}$ (for any $p \in X_{\square^k}$) is contained in the union of the $U_{\square^i}$, $i > k$, we have that $q_K \circ H'_{\square^k}$ (hence $G_{\square^k}$ and $R_{\square^k}$) is $C^1$ outside 0. This proves Claim C.3.

From Lemma 2.3 (ii) and the fact that the derivative of the exponential (at 0) is the identity we get

$$\frac{\partial}{\partial v} R_{\square^k}(0) = v \tag{4}$$

That is, all directional derivatives at 0 of $R_{\square^k}$ exist and if $R_{\square^k}$ were differentiable its derivative at 0 would be the identity matrix $I$. It follows from (4), (C.2) and Claim C.3 that

(C.4.) to prove that $q_K \circ H'_{\square^k}$ is $C^1$ it suffices to prove $DR_{\square^k}|_q \rightarrow 1$ (the identity matrix) as $q \rightarrow 0$.

Write $S = \text{Link}(\square^k, K) = \text{Link}(X_{\square^k}, K_X)$ (at some point $F(p) \in \mathcal{X}'^k$ and $p \in X_{\square^k}$, respectively, and recall we are using “direction” links). Also write $\mathbb{D}^{n-k} = \mathbb{D}^{n-k} \times \{p\} \subset \mathbb{D}^{n-k} \times \mathcal{X}'_{\square^k}$. For $\square^i$, $\square^k < \square^i$ set $\sigma_{\square^i} = \mathbb{D}^{n-k} \cap (H_{\square^k})^{-1}(X_{\square^i})$ and $\dot{\sigma}_{\square^i} = \mathbb{D}^{n-k} \cap (H_{\square^k})^{-1}(\dot{X}_{\square^i})$. Note that the sets $\sigma_{\square^i}$ and $\dot{\sigma}_{\square^i}$ are cone sets. That is, if $u \in \sigma_{\square^i}$ then $tu \in \sigma_{\square^i}$, $t \in [0,1]$. Similarly for $\dot{\sigma}_{\square^i}$ (see 2.4).

Claim C.5. We have $R_{\square^k}(\sigma_{\square^i}) = \sigma_{\square^i}$ and $R_{\square^k}(\dot{\sigma}_{\square^i}) = \dot{\sigma}_{\square^i}$.

Proof of Claim C.5. We prove the first identity, the second one is similar. Let $u = tv \in \dot{\sigma}_{\square^i}$, $|v| = 1$. We assume $t > 0$. Then $\exp_p (2\pi t h_{\square^k}(v)) = H_{\square^k}(u,p) \in X_{\square^i}$. By (1) and Lemma 2.4 (ii) we have that $h^*_{\square^k} \circ G_{\square^k}(u,p) \in \square^i$. By the definition of $h^*_{\square^k}$ we get $(a h_{\square^k}(\ell(a)v)) \in \square^i$, where $a$ is the length of $R_{\square^k}(u)$. By 2.4 (ii) we get $h_{\square^k}(\ell(a)R_{\square^k}(u)) \in T_p X_{\square^i}$, which implies $\ell(a)R_{\square^k}(u) \in \sigma_{\square^i}$. Since this set is a cone set it follows that $R_{\square^k}(u) \in \sigma_{\square^i}$. This proves Claim C.5.

Now, let $q_n \rightarrow 0$ in $\mathbb{D}^{n-k}$. We can assume (arguing by contradiction) that $q_n = t_n u_n$, with $(t_n, u_n) \in \mathbb{R}^+ \times S^{n-k-1}$, $t_n \rightarrow 0$, $u_n \rightarrow u \in \dot{\sigma}_{\square^i}$ for some $\square^i \in K$ containing $\square^k$. Hence:

(C.6.) to prove that $q_K \circ H'_{\square^k}$ is $C^1$ it suffices to prove that $DR_{\square^k}|_{(t_n u_n)} \rightarrow 1$, as $n \rightarrow \infty$.

where $u_n \rightarrow u \in \dot{\sigma}_{\square^i}$ and $t_n \rightarrow 0$.

Claim C.7. Statement (C.6) holds for $j = n$.

Proof of Claim C.7. We have that $u \in \dot{\sigma}_{\square^n}$ for some $\square^n \in K$. Therefore there is a small compact neighborhood $V$ of $u \in S^{n-k-1} \cap \sigma_{\square^n} \subset \mathbb{D}^{n-k}$ such that (we can assume that) all $q_n$
and \(tu, t \in (0,1]\), lie in the interior of the cone \(CV\). Denote by \(h : \mathbb{D}^{n-k} \to C S\) the map \(Ch_{\square^k}\), where \(h_{\square^k}\) is the link smoothing of \(S = \text{Link}(\square^k, K)\). Since \(F|_{X_{\square^k}} = f\), on \(CV\) we can write \(R_{\square^k} = h^{-1} \circ (\pi \circ f \circ e) \circ h\), where \(e\) is the exponential map given by \(e(v) = exp_p(2rv)\), and \(\pi : CS \times \square^k \to CS\) is the projection. (Note that \(e(tu) = E(2rt, v)\), where \(E\) is as in 2.4.) Hence \((DR_{\square^k})|_{qn} = (Dh|_{qn})^{-1}D(\pi \circ f \circ e)|_{bn(qn)} Dh|_{qn}\), where \(y_n = h^{-1}(\pi \circ f \circ e)(h(q_n))\). By Lemma 2.3 (ii) and the fact that the derivative of the exponential at 0 is the identity we have that \(D(\pi \circ f \circ e)|_{bn(qn)} \to 1\), as \(q_n \to 0\). On the other hand, since \(h\) is a cone map, by Lemma 1.4.2 we get that \(Dh\) and \(Dh^{-1}\) are both bounded on \(CV\). Moreover \(Dh|_{qn} = Dh|_{un}\) (see remark after 1.4.2) and \(Dh|_{qn} = Dh|_{\text{un}}\). But, since \(\pi \circ f \circ e\) is smooth and \(D(\pi \circ f \circ e)|_{p} = 1\) we have that for any \(v\) we get
\[
\frac{\pi \circ f \circ e(t_n v)}{t_n} \to v
\]
(5)

From the fact that \(h^{-1}\) is a cone map and (5) we have
\[
\frac{t_n}{|h^{-1}(\pi \circ f \circ e(t_n h(u_n)))|} = \left[|h^{-1}(\pi \circ f \circ e(t_n h(u_n)))|\right]^{-1} \to \lim_{n \to \infty} |u_n|^{-1} = 1
\]
This together with (5) and the fact that \(h\) and \(h^{-1}\) are cone maps imply
\[
\frac{\frac{y_n}{|y_n|}} = \frac{h^{-1}(\pi \circ f \circ e(t_n u_n))}{|h^{-1}(\pi \circ f \circ e(t_n u_n))|} = \frac{h^{-1}(\pi \circ f \circ e(t_n u_n))}{|h^{-1}(\pi \circ f \circ e(t_n u_n))|}
\]
\[
\to \quad u
\]
Consequently \(\frac{y_n}{|y_n|} \to u\). Therefore \(DR_{\square^k}|_{qn} \to 1\). This proves Claim C.7.

We will prove statement (C.6) by decreasing induction on \(j\). Claim C.7 was the first step of this induction. We assume statement C.6 holds for all \(\square^j, j < i\).

**Claim C.8.** **Statement (C.6) holds for \(j\).**

**Proof of Claim C.8.** The proof has two steps.

**Step 1. It is enough to assume that \(u_n = u\).**

As in the proof of Claim C.7 let \(V\) be a small compact neighborhood of \(u \in S^{n-k-1} \subset \mathbb{D}^{n-k}\) such that all \(q_n\) and \(tu, t \in (0,1]\), lie in the interior of the cone \(CV\). From the definition of \(G_{\square}\) we have that on \(CV\) we can write
\[
G_{\square^k} = \left((h_{\square^k}^\bullet)^{-1} \circ h_{\square^k}^\bullet\right)^{-1} \circ G_{\square^j} \circ \left((H_{\square^k})^{-1} \circ H_{\square^k}\right)
\]
To simplify the notation write \(h = (h_{\square^k}^\bullet)^{-1} \circ h_{\square^k}^\bullet\) and \(H = (H_{\square^k})^{-1} \circ H_{\square^k}\). Hence
We next compare $DG_{\square}^k |_{(t_n u_n, p)}$ and $DG_{\square}^k |_{(t_n u, p)}$. We analyze the three terms $DH$, $DG_{\square}^j$, $Dh$ in (6).

**First term:** $DH$. Since $H$ is a cone map we get that $DH |_{t_n u_n} - DH |_{t_n u} \to 0$.

**Remark.** The map $H$ is an euclidean-to-hyperbolic cone map, and it is not an euclidean cone map but it is an euclidean cone map up to a smooth change of coordinates on a compact set.

**Second term:** $DG_{\square}^j$. Differentiating (3) we get

$$DG_{\square}^j |_{(u,y)} (v, w) = \left( DR_{\square}^j |_u . v , \frac{\partial}{\partial t} T_{\square}^j |_{(t,y)} \frac{u.v}{|u|} + \frac{\partial}{\partial y} T_{\square}^j |_{(t,y)},w \right)$$

where $t = |u|$. It can be checked from 2.4(iv) (see also (3)) that $T_{\square}$ can be extended to a smooth map on $\mathbb{D}^{n-k} \times X_{\square}$ (which is compact) the second term (i.e the $T_{\square}$ term) is Lipschitz on the variables $t$ and $y$. Since the distance between $H(t_n u_n, p)$ and $H(t_n u, p)$ goes to zero it follows that the $T_{\square}$ terms in the right hand side of the equation above evaluated at $H(t_n u_n, p)$ and $H(t_n u, p)$ get close as $n \to \infty$. Also, by inductive hypothesis, the first terms tend both to 1. Therefore we get

$$DG_{\square}^j |_{H(t_n u_n)} - DG_{\square}^j |_{H(t_n u)} \to 0$$

as $n \to \infty$.

**Third term:** $Dh$. Since $h$ is a cone map to prove that $Dh_{\square}^k (H(t_n u_n))$ and $Dh_{\square}^k (H(t_n u))$ are close we need to prove that the directions of $G_{\square}^k (H(t_n u_n))$ and $G_{\square}^k (H(t_n u))$ are close. This is equivalent to proving that the directions of their images by $h$ are close. Since $G_{\square}^k = h \circ G_{\square}^k \circ H$ this means proving that the directions of $G_{\square}^k (t_n u_n)$ and $G_{\square}^k (t_n u)$ are close. Let $E$ and $p_l$ be the maps in Lemma 2.4. We can assume (arguing by contradiction) that $t_n u_n$, $t_n u$ lie on $X_{\square}$, for some $\square$. Since $h_{\square}^k$ is also a cone map (on the first variable) it is enough to prove that the directions of $p_l \circ f \circ E(2rt_n, h_{\square}^k (u_n))$ and $p_l \circ f \circ E(2rt_n, h_{\square}^k (u))$ are close. But this is true because $p_l \circ f \circ E$ is smooth. This concludes step 1.

**Step 2.** We prove that $DF_{\square}^k |_{(t_n u)} \to 1$, as $n \to \infty$, where $u \in \hat{\sigma}_{\square}$ and $t_n \to 0$.

Note that every inclusion $\hat{\sigma}_{\square} \hookrightarrow \mathbb{D}^{n-k}$ is a smooth embedding (see Section 1). We have two cases.

**First case.** We have that $DF_{\square}^k |_{(t_n u)} v \to v$, as $n \to \infty$, when $v$ is tangent to $\sigma_{\square}$.

This follows from an argument similar to the one given in the proof of C.7 (recall that from C.5 we have $R_{\square}^k (\sigma_{\square}) = \sigma_{\square}$). This proves the first case.
Recall that $H$ is the change of variables $H = (H_{ij})^{-1} \circ H_{\square k}$. Let $u \in \hat{\sigma}_{\square}$ and $v \in \mathbb{R}^{n-k} = T_u \mathbb{D}^{n-k}$. We say that $v$ is a $X$-fiber vector at $u$ if $DH|_u v = (z, 0) \in \mathbb{R}^{n-j} \times T_H(u) \hat{X}_{\square j} = T_H(u)(\mathbb{D}^{n-j} \times \hat{X}_{\square j})$, for some $z \in \mathbb{R}^{n-j}$. We write $v = v^X_z$ (the reason for the upper index $X$ will be clear in a moment). Fixing $z$ we obtain a constant vector field $(z, 0)$, hence we obtain the corresponding vector field $v^X_z$ of $X$-fiber vectors on $\hat{\sigma}_{\square j}$. Thus $v^X_z$ is characterized by $DH|_u v^X_z(u) = (z, 0)$. Equivalently $v^X_z(u) = (DH|_u)^{-1}(z, 0)$.

Similarly, we can work on $K$ instead of $K_X$, and $h$ instead of $H$ and obtain vector fields of $\square$-fiber vectors $v^\square_z$ on $\hat{\sigma}_{\square j}$ with the property that $Dh|_u v^\square_z(u) = (z, 0) \in \mathbb{R}^{n-j} \times T_h(u) \square j = T_h(u)(\mathbb{D}^{n-j} \times \square j)$. Equivalently

$$v^\square_z(u) = (Dh|_u)^{-1}(z, 0) \quad (8)$$

**Claim C.9.** We have $v^X_z = v^\square_z$.

**Proof of Claim C.9.** Fix $u$. Then the (hyperbolic) geodesic $t \mapsto H_{\square k}(tu)$, and the straight segment $t \mapsto h^\bullet_{\square k}(tu)$ are both contained in $X_{\square k}$ and $\square^n$, respectively, for some $\square^n$. This together with the fact that both $h^\bullet_{\square k}$ and $H_{\square k}$ use the same link smoothing $h_{\square k}$ in their definition imply that we can reduce our problem to the following setting. Consider $\mathbb{R}^{n} = \mathbb{R}^{n-k} \times \mathbb{R}^k, \mathbb{R}^k \subset \mathbb{R}^j$, with canonical metrics $\sigma_{\mathbb{R}^n} = \sigma_{\mathbb{R}^{n-k}} + \cosh^2(r)\sigma_{\mathbb{R}^k}$, and $z$ (a constant vector field) perpendicular to $\mathbb{R}^j$. Here $r$ is the distance to $\mathbb{H}^{k}$. In this case $h^\bullet_{\square k}$ corresponds to the perpendicular (to $\mathbb{R}^k$) exponential map from a point $p \in \mathbb{R}^k$ and $H_{\square k}$ corresponds to the perpendicular (to $\mathbb{H}^{k} = (\mathbb{R}^k, \sigma_{\mathbb{R}^k})$) exponential map from $p$. The former exponential is just the inclusion and the latter exponential is done with respect to $\sigma_{\mathbb{R}^n}$. But in this setting these two exponentials coincide, hence the preimage of $z$ by them also coincide. This proves the claim.

Given $z$ as above we write $v_z$ to denote $v^X_z$ and $v^\square_z$ and we say that $v = v_z$ is a fiber vector.

The following statement can be easily verified in the euclidean case (i.e for $v^\square_z$).

$$v_z(tu) = v_z(u) \quad (9)$$

To prove step 2 it is enough to prove the following.

**Second case.** If $v$ is fiber vector at $u$, then $D R_{\square k} |_{(t_nu)} v \rightarrow v$, as $n \rightarrow \infty$.

We have $v = v_z(u)$, hence

$$DH|_{(t_nu)}v = (z, 0) \quad (10)$$

Using formula (7), the inductive hypothesis and the fact that $(0, q_i) = H(tu) \in \{0\} \times \hat{X}_{\square}$ we see that for any $z' \in \mathbb{R}^{n-j}$ we have

$$DG_{\square j}|_{H(t_nu)}(z', 0) = (z', 0)$$

This together with the equation (6) imply

$$DG_{\square k}|_{(t_nu)}(v, 0) = Dh_{\square j}^{-1}(H(t_nu)) \cdot DH_{t_nu}v \quad (11)$$
Therefore, using (8), (10) and Claim 9 in (11) we get

\[ DG_{\Box}^k|_{(t_nu)}(v, 0) = DG_{\Box}^k|_{(t_nu)}(v_z(u), 0) = v_z(G_{\Box}^k(t_nu)) \tag{12} \]

Note that, by (7) and Remark 2.5, \( DG_{\Box}^k|_{(t_nu)}(v, 0) = (DR_{\Box}^k|_{(t_nu)}.v, 0). \) This together with (12) imply that to prove case 2 we have to prove that \( v_z(G_{\Box}^k(t_nu)) = (v, 0) = v_z(u). \) Consequently, since \( h \) is a cone map it is enough to prove that the directions of \( G_{\Box}^k(t_nu) = h^{-1} \circ G_{\Box}^j \circ H(t_nu) \) tend to \( u, \) as \( n \to \infty. \) But this last statement is implied by

\[
\lim_{n \to \infty} \frac{G_{\Box}^k(t_nu)}{t_n} = DG_{\Box}^k|_{(0)} \cdot u = (DR_{\Box}^k|_{0}.u, 0) = u
\]

which follows from equation (4). This proves the second case, step 2, Claim C.8 and concludes the second part.

THIRD PART. The maps \( H'_{\Box} \) are \( C^1 \)-embeddings.

Proof. Again by induction. This is true for \( H'_{\Box}^n : \tilde{X}_{\Box}^n \hookrightarrow K'_{\Box}, \) which can be identified with \( P|_{\tilde{X}_{\Box}^n} \) (see 3.1). Assume that the \( H'_{\Box}^j \) are \( C^1 \)-embeddings for \( j > k. \) Fix a \( \Box^k. \) Using the argument used in the first and second parts we get that \( H'_{\Box}^k \) is a \( C^1 \)-embedding outside \( X_{\Box}^k = \{0\} \times \tilde{X}_{\Box}^k. \) From the second part we see that the derivative \( DH'_{\Box}^k \) maps non-zero vectors \( v \) at \( \tilde{X}_{\Box}^k \) in the \( \mathbb{R}^{n-k} \) direction to non-zero vectors (see equation (4)). As mentioned in the first part the map \( q_{\Box} \circ H'_{\Box}^k|_{\tilde{X}_{\Box}^k} \) can be identified with \( P|_{\tilde{X}_{\Box}^k}, \) which is a diffeomorphism (see 3.1). Hence \( DH'_{\Box}^k \) maps non-zero vectors \( w \) in the \( X_{\Box}^k \) direction to non-zero vectors. Moreover \( DH'_{\Box}^k.w \) is perpendicular to \( DH'_{\Box}^k.w. \) It follows that \( H'_{\Box}^k \) is an embedding on \( \{0\} \times \tilde{X}_{\Box}^k. \) This concludes the third part and completes the proof of Proposition 4.2.

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