A FULL-TWISTING FORMULA FOR THE HOMFLY POLYNOMIAL

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Abstract. We introduce a certain class of link diagrams, which includes all closed braid diagrams. We show a generalized version of Kálmán’s full-twist formula for the HOMFLY polynomial in the class.

1. Introduction

The HOMFLY polynomial [2, 4, 11] \( P(L)(v, z) \) is an oriented link invariant, which is defined by the skein relation

\[ v^{-1}P(\emptyset) - vP(\emptyset) = zP(\emptyset), \]

and the normalization \( P(\text{trivial knot}) = 1 \).

The framed HOMFLY polynomial is a useful alternative to study the HOMFLY polynomial. With the blackboard-framing convention, it is defined for oriented link diagrams \( D \), and it is denoted by \( H(D)(v, z) \). The two versions are related by the formula \( H(D) = v^{-w(D)}P(D) \), where \( w(D) \) is the writhe of \( D \).

We can see that \( H(D) \) is invariant under the Reidemeister moves II and III. The skein relation and the Reidemeister move I relations for \( H(D) \) are as follows:

\[ H(\emptyset) - H(\emptyset) = zH(\emptyset), \]
\[ H(\emptyset) = v^{-1}H(\emptyset), \]
\[ H(\emptyset) = vH(\emptyset). \]

In this paper, we focus on certain extreme parts of \( H(D) \). They are specified by the Morton–Franks–Williams (MFW) bounds [8, 1].

Theorem 1.1 ([8, 1]). For an oriented link diagram \( D \), we have

\[ -s(D) + 1 \leq \mindeg_{v} H(D) \leq \maxdeg_{v} H(D) \leq s(D) - 1, \]

where \( s(D) \) is the number of Seifert circles of \( D \).

For a diagram \( D \), we denote the coefficient of \( v^{s(D)-1} \) (resp. \( v^{-s(D)+1} \)) in \( H(D) \) by \( H_{+}(D) \) (resp. \( H_{-}(D) \)). Here \( H_{+}(D) \) and \( H_{-}(D) \) are Laurent polynomials in \( z \). We note that they can be zero when the MFW bounds are not sharp for \( D \).

Now we recall the full-twist formula for braids by Kálmán [5].

Theorem 1.2 ([5]). Let \( B \) be an \( n \)-strand braid. Then we have

\[ H_{-}(\hat{B}) = (-1)^{n-1}H_{+}(\text{FTB}), \]

where FT is the positive full-twist braid and the hat \( \hat{\cdot} \) of a braid denotes its closure (see figure 1).

This formula is categorified in [10, 3], that is to say, it is generalized for the Khovanov–Rozansky HOMFLY homology [7]. In this paper, we will explore a different type of generalization, namely we will extend the formula to knitted diagrams, which include braid closures.
The definitions and the generalized formula are given in section 2. We prepare some terminology from the Hecke algebra in section 3, then we prove the formula in section 4.

2. KNITTED DIAGRAMS

We will define knitted diagrams as a generalization of closed braid diagrams. They are obtained by arranging several braid diagrams on the plane and connecting them appropriately.

Definition 2.1. Let $D \subset \mathbb{R}^2$ be an oriented link diagram. An $n$-strand braid box in $D$ is a rectangle $B \subset \mathbb{R}^2$ such that the restriction of $D$ in $B$ is an $n$-strand braid diagram.

We note that since all strands cross a braid box in the same direction, a Seifert circle of $D$ passes through each braid box in $D$ at most once.

The following abuse of notation will be useful: for a braid box $B$ in $D$, the symbol $B$ will also denote the braid represented by the diagram $D \cap B$.

Definition 2.2. Let $D$ be an oriented link diagram. A knitting pattern of $D$ is a finite set $\{B_i\}_{i=1}^m$ of braid boxes in $D$ such that:

• the braid boxes $B_i$ are disjoint,
• every crossing of $D$ is contained in some box $B_i$,
• there is no pair of Seifert circles of $D$ both of which pass through two common boxes in $\{B_i\}$.

A knitted diagram is a pair $D = (D, \{B_i\})$ of a diagram $D$ and its knitting pattern $\{B_i\}$.

Obviously, a closed braid diagram can be considered as a knitted diagram with a single braid box (as in figure 1).

Example 2.3. Let $G$ be a simple plane bipartite graph. By reversing Seifert’s algorithm, a special link diagram $D$ can be obtained from $G$, up to crossing signs and orientations, so that the Seifert graph of $D$ is $G$. By putting a 2-strand braid box on each crossing of $D$, we obtain a knitted diagram over $D$. (The simplicity of $G$ is needed for the third condition on knitting patterns in definition 2.2. See figure 2 for an example.)

![Figure 1. FTB (left) and $\hat{B}$ (right) for a braid $B$](image1)

![Figure 2. A knitted diagram constructed from a graph](image2)
We will often replace the braids $B_i$ in a knitted diagram $(D, \{B_i\})$. For a set of braids $\{\beta_i\}$ to replace $\{B_i\}$, the new knitted diagram is denoted by $D' = (D, \{B_i \to \beta_i\})$.

Now we can state our main result, which is a generalization of theorem 1.2.

**Theorem 2.4.** Let $D = (D, \{B_i\}_{i=1}^m)$ be a knitted diagram. By adding a positive full-twist to each $B_i$, we obtain a new knitted diagram $FTD = (D, \{B_i \to FTB_i\})$. Then we have $H_-(D) = (-1)^{s(D)}H_+(FTD)$, where $s(D)$ is the number of Seifert circles of $D$.

Figure 3 shows a knitted diagram $D$ of a knot and its full-twisted diagram $FTD$. The framed HOMFLY polynomial of them are shown as tables in figure 4. In each table, the coefficient of $v^s z^t$ is displayed at the point $(s, t)$ unless it is zero. The left table is for $D$, and its bottom-left coefficient is at $(-6, 0)$. The right table is for $FTD$, and its bottom-right coefficient is at $(6, 0)$. We emphasize that one step between adjacent coefficients in the table is by the vector $(2, 0)$ or $(0, 2)$. I.e., we read off from the table that $H(D) = (2+3z^2+z^4)v^6-(1+2z^2+3z^4+z^6)v^4-(1+2z^2+3z^4+z^6)v^2+(2+3z^2+z^4)v^0-v^2$. Since $H_-(D) = H_+(FTD) = 2 + 3z^2 + z^4$, theorem 2.4 is valid for this example.

**Figure 3.** A knitted diagram $D$ and $FTD$

### 3. Hecke algebra

We need some terminology from the (type $A$) Hecke algebra to prove our result.

**Definition 3.1.** The Hecke algebra $H_n$ is obtained from the group algebra of the braid group $Br_n$, over the ring $\mathbb{Z}[z, z^{-1}]$, by imposing the skein relation

$$\sigma_i - \sigma_i^{-1} = z \quad (i = 1, \ldots, n - 1),$$
where \( \sigma_1, \ldots, \sigma_{n-1} \) are the standard generators of \( \text{Br}_n \).

As a \( \mathbb{Z}[z, z^{-1}] \)-module, the Hecke algebra \( H_n \) has several well-known bases. The positive permutation braids (PPBs) \( \{ T_w \}_{w \in S_n} \) and the negative permutation braids (NPBs) \( \{ U_w \}_{w \in S_n} \) are used as bases here. They are indexed by the symmetric group \( S_n \), and the braid \( T_w \) (resp. \( U_w \)) is the positive (resp. negative) braid representing the permutation \( w \) with the fewest crossings possible in its diagram. We note that \( T_\delta \) is the positive half-twist braid \( HT \), where here \( \delta \) is the longest element in \( S_n \):

\[
\delta = \begin{pmatrix}
1 & 2 & \cdots & n - 1 & n \\
n & n - 1 & \cdots & 2 & 1
\end{pmatrix}.
\]

An important property of these bases is that multiplying by \( HT \) from the left (or right) yields a bijection from \( \{ U_w \}_{w \in S_n} \) to \( \{ T_w \}_{w \in S_n} \). The following fact is also essential for the full-twist phenomenon.

**Lemma 3.2** ([5, proposition 3.1]). Let \( x \) be an element in the Hecke algebra \( H_n \). We expand \( x \) in terms of PPBs and NPBs in \( H_n \) as follows:

\[
x = \sum_{w \in S_n} a_w T_w = \sum_{w \in S_n} b_w U_w \quad (a_w, b_w \in \mathbb{Z}[z, z^{-1}] ).
\]

Then we have \( a_\delta = b_\delta \), where \( \delta \) is the longest element in \( S_n \).

For more detail on the type A Hecke algebra, see [5, 6].

\[
\begin{array}{ccccccc}
1 & 2 & 1 \\
20 & 35 & 15 & -1 \\
177 & 257 & 86 & -16 \\
914 & 1013 & 216 & -101 & 1 \\
3055 & 2206 & 112 & -317 & 10 \\
6925 & 2052 & -508 & -509 & 33 \\
10833 & -1804 & -554 & -424 & 36 & 1 \\
11655 & -7747 & 1727 & -498 & 13 & 6 \\
-1 & -1 & 8416 & -10076 & 4817 & -1314 & 125 & 4
\end{array}
\]

\[
\begin{array}{ccccccc}
1 & -3 & -3 & 1 \\
3 & -2 & -2 & 3 \\
2 & -1 & -1 & 2 & -1
\end{array}
\begin{array}{ccccccc}
3864 & -6816 & 4921 & -1892 & 359 & -27 \\
1008 & -2384 & 2328 & -1212 & 344 & -49 \\
112 & -336 & 419 & -281 & 107 & -22
\end{array}
\]

**Figure 4.** The framed HOMFLY polynomials of \( D \) and \( FT D \)
4. Proof

We will prove theorem 2.4 in this section, by generalizing the original proof in [5]. A key tool here is the refined MFW bound by Murasugi and Przytycki [9].

**Lemma 4.1** (Corollary of [9, theorem 8.3]). Let \( D \) be an oriented link diagram. If there is a pair of Seifert circles of \( D \) with exactly one crossing between them, which is positive (resp. negative), then we have \( H_+(D) = 0 \) (resp. \( H_-(D) = 0 \)).

Indeed, with the condition in the above lemma, the index [9, definition 2.1] of the Seifert graph of \( D \) becomes positive. It then follows from [9, theorem 8.3] that the MFW bound cannot be sharp.

**Proof of Theorem 2.4.** We denote the framed HOMFLY polynomial of the knitted diagram \( (D, \{B_i \rightarrow \beta_i\})_{i=1}^m \) by \( H(\beta_i)_{i=1}^m \), for a replacing set \( \{\beta_i\} \) of braids.

Let \( n_i \) be the number of strands in the braid \( B_i \). We expand each \( B_i \) in terms of PPBs in \( H_{n_i} \), and write \( B_i = \sum w_ia_i T_w \), where \( w \) runs over the symmetric group \( S_{n_i} \). Note that this expansion only requires the skein relation and braid isotopies. Hence, by the same computation, the framed HOMFLY polynomial \( H(D) \) can be expanded into a linear sum

\[
H(D) = \sum_{a_1, \ldots, a_m} a_{w_1}^1 \cdots a_{w_m}^m H(T_{w_i})_{i=1}^m,
\]

where each \( w_i \) runs over \( S_{n_i} \). (We stress that \( w_i \) is not determined by \( i \), rather it indicates the \( i \)-th set of subscripts.) We expand \( H(D) \) also in terms of NPBs, and write

\[
H(D) = \sum_{a_1, \ldots, a_m} b_{w_1}^1 \cdots b_{w_m}^m H(U_{w_i})_{i=1}^m.
\]

Let us apply negative half-twists \( HT^{-1} \) to (1) and apply positive half-twists \( HT \) to (2). By restricting the equations to \( H_\mp \), we have

\[
H_-(HT^{-1}D) = \sum_{a_1, \ldots, a_m} a_{w_1}^1 \cdots a_{w_m}^m H_-(HT^{-1}T_{w_i}),
\]

\[
H_+(HTD) = \sum_{a_1, \ldots, a_m} b_{w_1}^1 \cdots b_{w_m}^m H_+(HTU_{w_i}).
\]

We recall that the set \( \{HT^{-1}T_{w_i}\} \) is equal to \( \{U_{w_i}\} \). Unless \( w_i = 1 \), a reduced diagram of the braid \( U_{w_i} \) has a pair of adjacent strands with exactly one negative crossing between them. Hence by the definition of knitting patterns and lemma 4.1, we have \( H_-(U_{w_i}) = 0 \) unless \( w_i = 1 \) for all \( i \). Therefore, in the right hand side of (3), only one term can be non-zero; it is given by the longest elements \( \delta_i \in S_{n_i} \). Now we have

\[
H_-(HT^{-1}D) = a_{\delta_1}^1 \cdots a_{\delta_m}^m H_-(1),
\]

where \( 1 \) is the trivial braid \( HT^{-1}T_{\beta_i} = U_1 \). In a similar way, we have

\[
H_+(HTD) = b_{\delta_1}^1 \cdots b_{\delta_m}^m H_+(1).
\]

from (4). Since \( (D, \{B_i \rightarrow 1\}) \) is a trivial diagram, we have \( H(1) = \{(v^{-1} - v)/z\}^{s(D)-1} \). With lemma 3.2, it follows that

\[
H_-(HT^{-1}D) = a_{\delta_1}^1 \cdots a_{\delta_m}^m z^{1-s(D)} = b_{\delta_1}^1 \cdots b_{\delta_m}^m z^{1-s(D)} = (-1)^{s(D)-1} H_+(HTD).
\]

The claimed formula is obtained by replacing \( D \) with \( HTD \).
We end with an open question regarding categorification. Unfortunately, a direct way to compute the Khovanov–Rozansky HOMFLY homology of a link from an arbitrary diagram is not known; rather, we need a braid representation of the link [7]. However, in light of the full-twist formulas of [10, 3] it is still reasonable to ask the following.

**Question 4.2.** Can theorem 2.4 be generalized for the HOMFLY homology?

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