Safe reinforcement learning control for continuous-time nonlinear systems without a backup controller

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Abstract—This paper proposes an on-policy reinforcement learning (RL) control algorithm that solves the optimal regulation problem for a class of uncertain continuous-time nonlinear systems under user-defined state constraints. We formulate the safe RL problem as the minimization of the Hamiltonian subject to a constraint on the time-derivative of a barrier Lyapunov function (BLF). We subsequently use the analytical solution of the optimization problem to modify the Actor-Critic-Identifier architecture to learn the optimal control policy safely. The proposed method does not require the presence of external backup controllers, and the RL policy ensures safety for the entire duration. The efficacy of the proposed controller is demonstrated on a class of Euler-Lagrange systems.

I. INTRODUCTION

The reinforcement learning (RL) framework has seen reasonable success in solving optimal control problems under uncertain system dynamics. However, most RL-based methods need to explore the state-action spaces during the initial phases of training. Consequently, they tend to apply control inputs that may be detrimental to real-time safety-critical systems. Thus, the challenge of RL algorithms precludes their use in real-world systems lest they endanger the safety of humans and property. Therefore, researchers actively seek to bolster RL algorithms with provable safety guarantees. Formally, the notion of safety of dynamical systems is the certification of forward invariance [1] of state and actuation constraint sets. Under this definition of safety, the safe RL problem is the mathematical construct to solve optimal control problems under user-defined state and actuation constraints.

In literature, various methods are proposed to ensure the safety of RL algorithms. One school of thought is to exploit model predictive control (MPC) to buttress RL algorithms with safety guarantees [2]–[4]. While these algorithms provide a unified approach to handling state and actuation constraints, they solve an optimization routine at each time step of the controller run and thus are computationally expensive.

Another class of methods in the safe RL literature employs control barrier functions (CBF) [5], [6]. CBFs provide a Lyapunov-like analysis to ensure the safety of dynamical systems without the need to compute system trajectories. In literature, it is common to combine CBFs with control Lyapunov functions (CLFs) in the form of an optimization problem to trade-off safety and stability objectives [7]. However, these approaches are limited to discrete-time control problems.

The extension of the results of RL to uncertain continuous-time systems has been achieved by combining approximate dynamic programming (ADP) with adaptive control [8]–[11]. These approaches approximately solve the unconstrained optimal control problem for uncertain system dynamics, however, the constrained optimal control problems for continuous-time systems remain an active area of research.

The safety problem of continuous-time RL is primarily addressed by considering the continuous-time counterpart of CBFs, namely barrier Lyapunov functions (BLF) [12]. One research direction is to transform the constrained state dynamics into dynamics of an unconstrained state [13]–[15] and subsequently use ADP algorithms to solve the unconstrained problem. However, this approach typically handles rectangular state constraints (box constraints on individual components of states) and cannot be trivially extended to general convex state constraints. Additionally, these approaches modify the original cost function non-trivially.

Another approach involves adding BLF to the cost formulation [16], [17]. Such an addition often renders the system’s value function not continuously differentiable, which is typically needed to establish theoretical guarantees of the algorithms.

A common feature in both continuous-time and discrete-time RL algorithms is the use of the so-called “backup controllers” [15, Assm. 2] [18]. These are user-defined stabilizing controllers that step in place when RL algorithms generate control actions not in accordance with the safety requirements. Most literature assumes access to an initial policy that stabilizes the system under a wide range of epistemic uncertainties. The backup controllers are typically used as a fallback measure during the initial phase of the RL training when the agent has limited knowledge of the system under control. The assumption of the availability of such controllers is restrictive, and the formulation of backup controllers may be difficult for certain complex systems. Additionally, the act of switching to a backup controller deviates from the on-policy RL algorithm leading to suboptimal results.

In this paper, an on-policy RL algorithm is developed for the optimal control of continuous-time nonlinear systems that guarantee safety while obviating the need for a backup controller. Furthermore, the objective function of the optimal control problem remains unchanged. Inspired by [18], we fo-
cus our efforts on extending the Actor-Critic-Identifier (ACI) architecture [9] to solve the optimal regulation problem for a class of uncertain nonlinear systems under user-defined state constraints.

**Contributions:** The contributions of the present paper are three-fold. First, we formulate the safety problem as the minimization of the Hamiltonian subject to a constraint involving the time derivative of the BLF. We subsequently show that the proposed optimization problem is convex, and thus we compute the analytical solution for the optimal control policy by minimizing the Lagrangian. Second, we approximate the optimal control policy obtained from the proposed Lagrangian method and show that this approximate control law renders the system safe for each time step of the controller run without the help of a backup stabilizing controller. Third, we extend the ACI approach [9] to learn the controller run without the help of a backup stabilizing controller. Subsequently, we perform simulation studies on a class of Euler-Lagrange nonlinear systems to show the efficacy of our proposed methodology. We additionally compare our results with ACI approach to demonstrate the safety guarantees of the proposed method.

**Notations:** Let $\text{vec}(\cdot)$ denote the vectorization operator of a matrix yielding a column vector obtained by stacking the columns of the matrix on top of one another. We will use $\nabla$ to denote gradient operator with respect to (w.r.t.) $x$. We use $\|\cdot\|$ to denote the Euclidean norm for vectors and the corresponding induced norm for matrices. Let $\mathbb{L}_\infty$ denote the set of all bounded signals. $\lambda_{\min}(A)$ denotes the minimum eigenvalue of matrix $A$.

## II. Preliminaries

We consider the following control-affine nonlinear system

$$
\dot{x} = f(x) + g(x)u
$$

(1)

where $x(t) \in \mathbb{R}^n$ is the state vector and $u(t) \in \mathbb{R}^m$ is the control action. We assume that the state $x(t)$ is measurable. The functions $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $g(x) : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ are the drift dynamics and control matrix, respectively.

We define the notion of safety as the forward invariance of a compact set $C \subset \mathbb{R}^n$ w.r.t the state $x$. In other words, we deem the system to be safe if $x(0) \in C \implies x(t) \in C \forall t \in \mathbb{R}_{\geq 0}$. We assume that the origin is an element of the set $C$. Additionally, we define the sets $\partial C$ and $\text{Int}(C)$ to be the boundary and the interior of the set $C$, respectively.

**Assumption 1:** $f(x)$ and $g(x)$ are locally Lipschitz, second-order differentiable functions with $f(0) = 0$.

**Assumption 2:** The matrix $g(x)$ has full rank $\forall x \in C$.

**Assumption 3:** The matrix $g(x)$ is known and bounded as $\|g(x)\| < \bar{g}$, where $\underline{g}, \bar{g} \in \mathbb{R}_{>0}$.

We formulate the safe RL problem as the minimization of a cost functional w.r.t. the control policy $u(t)$, subject to the hard constraint on the state $x(t)$.

**Problem 1 (Constrained Optimal Control):**

$$
\min_{u(t) \forall t \in \mathbb{R}_{\geq 0}} \int_0^\infty r(x(s), u(s))ds
$$

s.t. \quad \dot{x} = f(x) + g(x)u \quad (2a)

$$
\begin{align*}
\text{s.t.} & \quad \dot{x} = f(x) + g(x)u \\
& \quad x(t) \in \mathbb{C} \quad \forall t \in \mathbb{R}_{\geq 0}
\end{align*}
$$

(2b)

(2c)

where $r : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}_{\geq 0}$ is the instantaneous cost function given by

$$
\begin{align*}
\forall \in \mathbb{C} \rightarrow \mathbb{R}_{\geq 0}
\end{align*}
$$

(3)

where $Q : \mathbb{C} \rightarrow \mathbb{R}_{\geq 0}$ is positive-definite cost function in $x$, and $R \in \mathbb{R}^{m \times m}$ is positive-definite.

### A. Approximate Dynamic Programming

In the theory of Dynamic Programming, the optimal value function is defined as

$$
V^*(x(t)) = \min_{u(t)} \int_t^\infty r(x(s), u(s))ds
$$

(4)

The Hamiltonian of the system is defined as follows

$$
H(x, u, \nabla V) \triangleq r(x, u) + \nabla V^T(f(x) + g(x)u)
$$

(5)

We obtain the optimal control law $u^*(x)$ for the unconstrained optimal control problem in (2a) by minimizing the Hamiltonian w.r.t. the control action $u$

$$
\begin{align*}
\forall \in \mathbb{C} \rightarrow \mathbb{R}_{\geq 0}
\end{align*}
$$

(10)

where $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^p$ denotes the basis function chosen to approximate the value function, satisfying $\phi(0) = 0$. The parameter $W \in \mathbb{R}^{p}$ denotes the true NN weight and $\epsilon_v : \mathbb{R}^n \rightarrow \mathbb{R}$ denotes the function approximation error.

**Assumption 4:** The value function approximation error $\epsilon_v$ and its derivative w.r.t. state are bounded as $\|\epsilon_v(x)\| \leq \bar{\epsilon}_v$, $\|\nabla \epsilon_v(x)\| \leq \tau_d$. Additionally, these bounds approach 0 as the number of neurons approaches infinity.

Since the NN weight $W$ is unknown in (10), we maintain two estimates $\hat{W}_a \in \mathbb{R}^p$ and $\hat{W}_c \in \mathbb{R}^p$ for the control law and the value function estimate, respectively.

### B. BLF-based Constrained Optimal Control Problem

A positive-definite differentiable function $B_f : \mathbb{C} \rightarrow \mathbb{R}$ satisfying the following properties is called a Barrier Lya-
punov function (BLF) if its time derivative along the system trajectories is negative semi-definite, i.e. $\dot{B}_f(x) \leq 0$

\[ B_f(0) = 0, \quad B_f(x) > 0 \quad \forall x \in C/\{0\}, \quad \lim_{x \to 0^+} B_f(x) = \infty \]

The existence of a BLF over $C$ implies the forward invariance of $C$ [12, Lemma 1].

**Construction 1:** $B_f(x)$ is constructed in a way such that there exists $\gamma \in \mathbb{R}_{>0}$ satisfying $\gamma \|
abla B_f\| \geq B_f \quad \forall x \in C$.

**Example 1:** For $x \in \mathbb{R}$ and $C = [-1, 1]$ a candidate BLF $B_f(x) = \log(\frac{1}{1-x^2})$ with $\gamma = 0.5$ satisfies the condition in Construction 1.

**Remark 1:** The constant $\gamma$ would be used to compute the largest attracting subset of $C$.

Problem 1 can be reformulated in terms of BLF as

\[
\begin{align*}
\min_{u(s)} & \quad H(x, u, \nabla V^*) \\
\text{s.t.} & \quad \frac{d B_f}{dt}|_{\dot{x} = f(x) + g(x)u} \leq 0 \\
& \quad B_f(x(0)) < \infty
\end{align*}
\]

The constraint in (11b) can be rewritten as

\[ \nabla B_f(x)^T [f(x) + g(x)u] \leq 0 \] (12)

We observe that the constraint in (12) is affine in the decision variable $u$. This, combined with the fact that the Hamiltonian in (5) is convex in $u$, makes Problem 2 a convex optimization problem. To find an analytical solution, we define the Lagrangian as

\[ L(x, u, \nabla V^*, \lambda) = H(x, u, \nabla V^*) + \lambda \nabla B_f^T(f(x) + g(x)u) \] (13)

where $\lambda \in \mathbb{R}_{\geq 0}$ is the Lagrange multiplier. The control law can be obtained by minimizing the Lagrangian

\[ u^*_{safe}(x, \lambda) = -\frac{1}{2} R^{-1} g^T(x) [\nabla V^*(x) + \lambda \nabla B_f(x)] \] (14)

**Remark 2:** The Lagrange multiplier $\lambda$ provides a way to reformulate a constrained optimization problem into a weighted unconstrained optimization problem. Typically, the expression for Lagrange multipliers are obtained from the KKT conditions [18]. For simplification of analysis, we approximate the optimal Lagrange multiplier with a user-defined constant $\lambda$, resulting in a suboptimal solution.

The estimated safe control law is given by

\[ \hat{u}(x, \lambda) = -\frac{1}{2} R^{-1} g^T(x) [\nabla \phi(x)^T \hat{W}_a + \lambda \nabla B_f(x)] \] (15)

**Theorem 1:** Under the control law in (15) and provided Assumptions 1-4 hold, the set $C$ is forward invariant for the system in (1) if $x(0) \in C$.

**Proof:** Consider the candidate Lyapunov function as $B_f(x) : C \to \mathbb{R}$. The time derivative of $B_f(x)$ along the trajectories of $\dot{x} = f(x) + g(x)u$ is given by

\[ \dot{B}_f = \nabla B_f^T(f(x) + g(x)u) \] (16)

Substituting the control law from (15), we have

\[ \dot{B}_f = \nabla B_f^T (f(x) - \frac{1}{2} \nabla B_f^T R_g \phi^T \hat{W}_a - \frac{1}{2} \nabla B_f^T R_g \nabla B_f) \] (17)

where we define $R_g(x) \triangleq g(x) R^{-1} g^T(x)$ and $R_g(x) \triangleq \nabla \phi(x) R_g(x) \nabla \phi^T(x)$. Under Assumption 2, $R_g(x)$ is positive-definite. Additionally, $R_g$ is bounded as $\| R_g(x) \| \leq \bar{R}_g \quad \forall x \in C$. Since $f(x)$ and $\nabla \phi(x)$ are continuous functions over compact set $C$, $\| f(x) \| \leq \bar{F}, \| \nabla \phi(x) \| \leq \bar{\phi}_d \quad \forall x \in C$. We can upper bound the right hand side of (17) by

\[ \dot{B}_f \leq (\bar{F} + \frac{1}{2} \bar{\phi}_d \| \hat{W}_a \|) \| \nabla B_f \| - \frac{\lambda}{2} \min(R_g) \| \nabla B_f \|^2 \] (18)

where $\hat{W}_a \in \mathbb{R}_{>0}$ is the bound on the true NN weight $W$ which is subsequently enforced on $\hat{W}_a$ via a projection operator [19]. We observe that the $\dot{B}_f$ is negative outside the compact set $\Omega = \{ x \in \mathbb{R}^n : \| \nabla B_f \| \leq \bar{B}_d \}$, where $\bar{B}_d \triangleq \frac{\bar{F} + \frac{1}{2} \bar{\phi}_d \| \hat{W}_a \|}{\frac{1}{2} \min(R_g)}$ is a computable finite positive constant. Under the condition in Construction 1 we can upper bound the value of Barrier function as

\[ B_f(x(t)) \leq \max (B_f(x(0)), \bar{B}_d) \] (19)

Since $x(0) \in C$, the $B_f(x(0))$ is finite. Thus, $B_f(t) \in \mathcal{L}_\infty$. Since the value of the Barrier function along the system trajectory is bounded, then by the definition of $B_f(x)$, at no point in time, the state trajectory intersects the boundary of the safe set $\partial C$ [12, Lemma 1]. Thus the state $x(t) \in C \ \forall t \in \mathbb{R}_{\geq 0}$ and the system is forward invariant. Since the BLF is continuously differentiable in $x$, the $\nabla B_f(x)$ is a continuous function over the compact set $\Omega$. Thus, $\| \nabla B_f \| \in \mathcal{L}_\infty$. Since all constituents of the control law in (15) are bounded, we can conclude that $\hat{u}(t) \in \mathcal{L}_\infty$.

**Remark 3:** Theorem 1 proves that the control policy in (15) guarantees safety for all time. Further, the control policy doesn’t switch between a stabilizing backup policy and the RL policy, which is a distinct advantage over approaches that rely on an elusive backup policy.

**C. Actor-Critic Design**

The actor NN weight $\hat{W}_a$ and the critic NN weight $\hat{W}_b$ are updated to minimize the norm of the estimation errors $\hat{W}_c \triangleq \hat{W} - \hat{W}_c$ and $\hat{W}_a \triangleq \hat{W} - \hat{W}_a$. A least-squares update law for the critic can be obtained from the consideration of the integral squared Bellman error [9] as follows

\[ E_c = \int_0^t \delta_{hjb}(\tau) d\tau \] (20)

Defining $\omega \triangleq \frac{\partial \delta_{hjb}}{\partial \hat{W}_c}$ the update law for critic is given as

\[ \hat{W}_c = \eta_c \Gamma \omega \frac{1}{1 + \nu \omega^T \Gamma \omega} \delta_{hjb} \] (21)

where the learning rate $\eta_c$ and normalizing factor $\nu$ are positive user-defined constants. The positive-definite covariance matrix $\Gamma \in \mathbb{R}^{p \times p}$ is updated via the update law

\[ \hat{\Gamma} = \beta \Gamma - \eta_c \Gamma \omega^T \frac{\omega \omega^T}{1 + \nu \omega^T \Gamma \omega} \] (22)
Under the aforementioned update law of the covariance matrix, the following bounds can be established
\[ \varphi_1 I_p \preceq \Gamma(t) \preceq \varphi_0 I_p \quad \forall t \geq 0 \] (23)
where \( \preceq \) denotes the semi-definite ordering and \( \varphi_0 > \varphi_1 \) are positive constants. The update law for the actor is obtained by the gradient descent of the cost function in (20)
\[ \dot{\tilde{W}}_a = \text{proj} \left[ - \frac{\eta_{a1}}{\sqrt{1 + \omega^T \omega}} R_s (\tilde{W}_a - \tilde{W}_c) \delta_{hjb} \right. \\
\left. - \eta_{a2} (\tilde{W}_a - \tilde{W}_c) - \frac{1}{2} \lambda \nabla \phi R_g \nabla B_f \right] \] (24)
where the projection operator, \( \text{proj}(. \right) \) [19] is used to keep the estimates of the actor parameter bounded. The positive constants \( \eta_{a1}, \eta_{a2} \in \mathbb{R} > 0 \) user defined gains. The last two terms in the argument of the projection operator are attributed to the subsequent Lyapunov analysis in Subsection II-E.

\[ \text{Fig. 1. Block diagram of the proposed Reinforcement Learning algorithm} \]

### D. Identifier Design

We represent the system drift dynamics \( f(x) \) in (1) via a two-layer NN parameterized by \( W_f \in \mathbb{R}^{l \times n} \) and \( V_f \in \mathbb{R}^{n \times l} \). We represent the activation function of the NN by \( \sigma : \mathbb{R}^l \to \mathbb{R}^l \). The dynamics of the system can be written as
\[ \dot{x} = W_f^T \sigma(V_f^T x) + \epsilon_f + g(x) \tau \] (25)

The following state estimator is designed by involving estimates of \( W_f \) and \( V_f \) in the form of \( \hat{W}_f \) and \( \hat{V}_f \) respectively
\[ \dot{\hat{x}} = \hat{W}_f^T \sigma(\hat{V}_f^T x) + g(x) \tau + k \hat{x} \] (26)
where \( \hat{x} \triangleq x - \tilde{x} \) denotes the state estimation error and \( k \in \mathbb{R} > 0 \) is a feedback gain.

**Assumption 5:** The parameters \( W_f, V_f \) are assumed to be bounded and \( ||\sigma(\cdot)|| < \sigma, ||\nabla \sigma(\cdot)|| < \sigma_d \quad \forall x \in \mathcal{C} \).

Based on a Lyapunov analysis (omitted here in the interest of space), we design the following adaptive laws for the NN parameters
\[ \dot{\hat{W}}_f = \text{proj}(\Gamma_w \hat{x} \hat{x}^T), \quad \dot{\hat{V}}_f = \text{proj}(\Gamma_v \hat{x} \hat{x}^T \hat{W}_f^T \nabla \sigma) \] (27)
where \( \Gamma_w \in \mathbb{R}^{l \times l} \) and \( \Gamma_v \in \mathbb{R}^{n \times n} \) are positive definite gain matrices. We define \( \hat{W}_f \triangleq W_f - \hat{W}_f \) and \( \hat{V}_f \triangleq V_f - \hat{V}_f \).

**Theorem 2:** Under the identifier update laws given by (26), (27) and Assumption 5, the state identification error \( (\tilde{x}(t)) \), the error in NN parameters \( (\hat{W}_f(t) \text{ and } \hat{V}_f(t)) \) are ultimately Upclosely (UUB)

**Proof:** (Sketch) We define an auxiliary state \( \zeta \triangleq \left[ \tilde{x}^T, \text{vec}(\hat{W}_f)^T, \text{vec}(\hat{V}_f)^T \right]^T \). Considering the following Lyapunov function
\[ V_1(\zeta) = \frac{1}{2} \tilde{x}^T \tilde{x} + \frac{1}{2} tr(\hat{W}_f^T \Gamma_w^{-1} \hat{W}_f) + \frac{1}{2} tr(\hat{V}_f^T \Gamma_v^{-1} \hat{V}_f) \]
One can show that \( V_1(\cdot) \) is negative whenever \( \zeta \) lies outside the compact set \( \Omega_1 \triangleq \{ \zeta : ||\tilde{x}|| \leq \sqrt{\frac{\|
abla \psi\|}{4k_1\omega^2}} + \frac{1}{\sqrt{2}} \} \), where \( \gamma \) is the computable upper bound of \( \sigma \) and higher order terms originating from Taylor’s approximation of \( \sigma \). Hence the state \( \zeta \) is UUB.

**Block diagram of the resulting system is shown in Fig. 1**

### E. Stability analysis

The Bellman estimation error can be written in its unmeasurable form as
\[ \delta_{hjb} = \nabla \hat{V}_f R_f x + \nabla V^T F_u x - r(x, u^*) \] (28)
where \( F_u^* \triangleq f(x) + g(x) u^* \) and \( F_u \triangleq \hat{f}(x) + g(x) \hat{u} \). Additionally, we define \( F_{\hat{u}} \triangleq F_u - F_{\hat{u}} \).

Substituting the instantaneous cost from (3), and the NN approximations of \( V^* \) from (10) and its estimate \( V \) we have \( \delta_{hjb} = \hat{W}_c^T \omega - [W^T \nabla \phi + \nabla \epsilon_f] F_u + u^T R\hat{u} - u^* T R u^* \) (29)
Substituting optimal control \( u^* \) from (14) and its estimate \( \hat{u} \) from (15) in (29) and simplifying, we have
\[ \delta_{hjb} = - \hat{W}_c^T \omega + T_1 \] (30)
where
\[ T_1 \triangleq -W^T \nabla \phi F_u - \nabla \epsilon_f F_u^* + \frac{1}{4} W_f^T R_s \hat{W}_a - \frac{1}{4} W^T R_s W \]
\[ - \frac{1}{4} \nabla \epsilon_f R_s \nabla \epsilon_f - \frac{1}{2} \nabla B_f^T R_s (\nabla \phi^T \hat{W}_a + \nabla \epsilon_f) \]
\[ - \frac{1}{2} W^T \nabla \phi R_g \nabla \epsilon_f \]
Substituting (30) into the dynamics of the critic estimation error \( \dot{\hat{W}}_c \), we obtain two components, a nominal dynamics term \( (\Omega_{nom}) \) and a perturbation term \( (\Delta) \)
\[ \dot{\hat{W}}_c = -\kappa \hat{W} \psi \psi^T + \eta \hat{W} \frac{\omega}{1 + \omega^T \Omega_{nom} T_1} \] (31)
where \( \psi(t) \triangleq \frac{\omega(t)}{\sqrt{1 + \omega^T \Omega_{nom} T_1 \omega(t)}} \in \mathbb{R}^n \) is the normalized gradient vector for the update law of the critic. The regressor \( \psi(t) \) is bounded as
\[ ||\psi(t)|| \leq \frac{1}{\sqrt{2} \sqrt{\varphi_1}} \quad \forall t \geq 0 \] (32)

The nominal dynamics \( \dot{\hat{W}}_c = \Omega_{nom} \) is globally exponentially stable (GES), provided that the bounded signal \( \psi(t) \) is persistently exciting (PE) [9]. Consequently, there exists a positive-definite scalar-valued function \( V_c(\hat{W}_c, t) \) such that the following conditions are satisfied
\[ c_1 ||\hat{W}_c||^2 \leq V_c(\hat{W}_c, t) \leq c_2 ||\hat{W}_c||^2 \]
\[ \frac{\partial V_c}{\partial t} + \frac{\partial V_c}{\partial \hat{W}_c} \Omega_{nom} \leq -c_3 ||\hat{W}_c||^2 \]
\[ \frac{\partial V_c}{\partial \hat{W}_c} \leq c_4 ||\hat{W}_c|| \] (33)
where $c_1, c_2, c_3, c_4$ are positive scalar constants. Additionally, we define the following term that would appear in the subsequent Lyapunov analysis

$$
T_2 \triangleq \frac{1}{4} \nabla c_T R_g \nabla c_v - \frac{1}{2} \lambda \nabla B_f^T R_g \nabla c_v + \frac{1}{2} \nabla T_a \nabla \phi R_g \nabla c_v
- \frac{1}{4} \hat{W}_a^T R_a \hat{W}_a + \lambda \nabla B_f f(x) - \lambda \nabla B_f^T R_g \nabla \phi^T \hat{W}_a
$$

Under the Assumptions 3-4, Theorems 1-2, we can obtain the subsequent Lyapunov analysis where $c$ is the actor weight estimation error.

Provided Assumptions 1-5 hold, the regressor $L \leq -\frac{1}{\lambda \min(R_g)} \|\nabla B_f\|^2 \leq \alpha_2(\|z\|)$

(41)

The derivative of Lyapunov function is upper-bounded by

$$
\dot{V}_L \leq -\alpha_1(\|z\|) + \frac{T_3^2}{40(\varsigma_3 - \eta_1 k_a k_s)} + k_2 + \eta_1 k_a^2 k_s k_1
$$

(42)

we observe that $\dot{V}_L(z, t)$ is negative whenever $z(t)$ lies outside the compact set $\Omega_1 \triangleq \{z: \|z\| \leq \alpha_1^{-1}(\frac{T_3^2}{40(\varsigma_3 - \eta_1 k_a k_s)} + k_2 + \eta_1 k_a^2 k_s k_1)\}$. We can thus conclude that the norm of the auxiliary state $\|z(t)\|$ is UUB.

### III. Simulation Results

To test the efficacy of the proposed control law, we perform a simulation study on a class of nonlinear Euler-Lagrange systems

$$
M(q)\ddot{q} + C_m(q, \dot{q})\dot{q} + G(q) + F_d(q) = \tau(t)
$$

(43)

Specifically, we consider the safe, optimal control problem for a two-link robot manipulator system

$$
M(q) = \begin{bmatrix}
p_1 + 2p_3 c_2 & p_2 + p_3 c_2 \\
p_2 & p_2
\end{bmatrix}, 
F_m(q) = \begin{bmatrix} f_1 q_1 \\ f_2 q_2 \end{bmatrix}, 
C_m(q, \dot{q}) = \begin{bmatrix}
-p_1 s_2 q_2 & -p_3 s_2 (q_1 + q_2) \\
p_1 s_2 q_1 & 0
\end{bmatrix}, 
G(q) = 0_{2 \times 1}
$$

where the signals $q_1(t), q_2(t) \in \mathbb{R}$ denote the angular position of the two link joints in radians. The parameters used for the simulation are $p_1 = 3.473$ kg m, $p_2 = 0.196$ kg m, $p_3 = 0.242$ kg m, $f_d = 5.3$ N, $f_d = 1.1$ N.

The system is then reformulated to the control affine form given in (1) by defining the system state as $x = [q_1, q_2, \dot{q}_1, \dot{q}_2]^T$ and the control action as $u = \tau$. We seek to solve the optimal control problem (Problem 1) considering the following the cost function components as $Q(x) = x^T x$ and $R = I_{2 \times 2}$ and the state constraint set $C = \{x \in \mathbb{R}^4: |x_i| < a_i, \forall i \in \{1, 2, 3, 4\}\}$. We consider the following candidate Barrier Lyapunov Function: $B_f = \sum_{i=1}^{n} \log \frac{\sigma_i^2}{a_i^2}$.

For the given two-link robot manipulator system, we have considered $a_i = 5, \forall i \in \{1, 2, 3, 4\}$. We observe that there exists a $\gamma = 5$ that satisfies the condition $\gamma \|\nabla B_f\| > B_f$.

The Critic NN and the Identifier NN were considered to be two-layer NNs with sigmoidal activation function and hidden layer consisting of 30 and 5 neurons respectively. The gains for the actor-critic components were chosen as $\eta_v = 2, \eta_a = 1$ and $\eta_c = 50$. The forgetting factor $\beta = 0.001$ and the multiplier $\nu = 5$. The Lagrangian multiplier $\lambda$ was set to 100 to ensure that the value of the bound $\beta_d$ is of a reasonable magnitude. For identifier, we chose the gains $\Gamma_w \parallel f = 10 I_{d \times l}$, $\Gamma_v \parallel f = 10 I_{n \times n}$. The identifier feedback gain

1We consider rectangular constraints for the ease of visualization. The proposed method can be easily extended to consider other types of state constraints.
Finally, we demonstrate the effectiveness of our controller policy and demonstrate that all closed-loop signals are UUB. The forward invariance of the constraint set without the need for supervision using a constraint-admissible set, in American Control Conference (ACC), 2018, pp. 6390–6395.

We develop an online Actor-Critic-Identifier architecture-based safe RL algorithm to solve the optimal regulation problem for a class of uncertain nonlinear systems while adhering to user-defined state constraints. We formulate the safety problem as a convex optimization problem involving the minimization of the Hamiltonian subject to the negative semi-definiteness of a candidate BLF. We derive an optimal control law for the constrained system by solving the Lagrange multiplier obtained from KKT conditions.

IV. CONCLUSIONS AND FUTURE WORK

We develop an online Actor-Critic-Identifier architecture-based safe RL algorithm to solve the optimal regulation problem for a class of uncertain nonlinear systems while adhering to user-defined state constraints. We formulate the safety problem as a convex optimization problem involving the minimization of the Hamiltonian subject to the negative semi-definiteness of a candidate BLF. We derive an optimal control law for the constrained system by solving the Lagrange multiplier obtained from KKT conditions.

Fig. 2a shows the state trajectory of the system under the influence of the proposed control law. We observe that all of the states are inside the prescribed limit shown in red dotted lines. Additionally, the state remains uniformly ultimately bounded. Fig. 2b shows the control effort imposed by the controller. Fig. 2c shows the estimation error of the identifier. It can be seen that the estimation error converges very close to zero. In the Fig. 2d we observe a comparison of performance in the initial 2 seconds of training of the proposed method with the ACI method [9]. The hyper-parameters for the algorithm outlined in [9], were taken in similar orders of magnitude as detailed in that article to enable a juxtaposition of the two results for better comparison. We observe that while the ACI method initially violates the safety criterion, the proposed method manages to keep the states well within the boundaries of the safe set, highlighting the transient safety guarantees of the proposed method.

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