NEARGEODESICS IN JOHN DOMAINS IN BANACH SPACES

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Abstract. Let $E$ be a real Banach space with dimension at least 2. In this paper, we prove that if $D \subset E$ is a John domain which is homeomorphic to an inner uniform domain via a CQH map, then each neargeodesic in $D$ is a cone arc.

1. Introduction

Conformally invariant metrics, such as the hyperbolic metric, are some of the key tools of classical function theory of plane. Quasiconformal and quasiregular mapping [18, 22] generalize this theory to the Euclidean $n$-dimensional spaces. In the higher dimensions $n \geq 3$ there is no counterpart of the hyperbolic metric for a general subdomain of $\mathbb{R}^n$. In this case one can, however, introduce new metrics, hyperbolic type metrics, which still have some properties of the hyperbolic metrics. Also, it is useful to study domains where various hyperbolic type metrics compared. These ideas were presented for the first time in book form in [22]. Recently, many authors have studied this topic [7, 11, 12, 16, 17].

Hyperbolic type metrics generalize also to Banach spaces. In this case for instance quasiconformality is defined in terms of the quasihyperbolic metric introduced in [5]. The present paper deals with the hyperbolic type geometries in so called John domains of Banach spaces. For the statement of our main result we introduce some terminology and notation.

Throughout the paper, we always assume that $E$ denotes a real Banach space with dimension at least 2. The norm of a vector $z$ in $E$ is written as $|z|$, and for each pair of points $z_1$, $z_2$ in $E$, the distance between them is denoted by $|z_1 - z_2|$, the closed line segment with endpoints $z_1$ and $z_2$ by $[z_1, z_2]$. We always use $B(x_0, r)$ to denote the open ball $\{x \in E : |x - x_0| < r\}$ centered at $x_0$ with radius $r > 0$. Similarly, for the closed balls and spheres, we use the usual notations $\overline{B}(x_0, r)$ and $\mathbb{S}(x_0, r)$, respectively.

Definition 1. A domain $D$ in $E$ is called a $c$-John domain in the norm metric provided there exists a constant $c$ with the property that each pair of points $z_1, z_2$ in $D$ can be joined by a rectifiable arc $\alpha$ in $D$ such that for all $z \in \alpha$ the following holds:

\[
\min\{\ell(\alpha[z_1, z]), \ell(\alpha[z_2, z])\} \leq c d_D(z),
\]

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where \( d_D(z) \) denotes the distance from \( z \) to the boundary \( \partial D \) of \( D \), \( \ell(\alpha) \) denotes the length of \( \alpha \), \( \alpha[z_j, z] \) the part of \( \alpha \) between \( z_j \) and \( z \) (cf. [2, 13, 14, 15]). The arc \( \alpha \) is called to be a \( c \)-cone arc.

For \( z_1, z_2 \in D \), the inner length metric \( \lambda_D(z_1, z_2) \) between them is defined by
\[
\lambda_D(z_1, z_2) = \inf \{ \ell(\alpha) : \alpha \subset D \text{ is a rectifiable arc joining } z_1 \text{ and } z_2 \}.
\]

**Definition 2.** A domain \( D \) in \( E \) is called an inner \( c \)-uniform domain if there is a constant \( c \geq 1 \) such that each pair of points \( z_1, z_2 \in D \) can be joined by an arc \( \alpha \) satisfying (1.1) and
\[
\ell(\alpha) \leq c\lambda_D(z_1, z_2).
\]

Such an arc \( \alpha \) is called to be an inner \( c \)-uniform arc (cf. [21]).

From the Definition 2, we see that an inner \( c \)-uniform domain is \( c \)-John. If \( C \) is any compact subset of the line segment \([0, e_1] \subset \mathbb{R}^2\), then \( \mathbb{B}^2(0, 1) \setminus C \) is \( c \)-John with a universal \( c \), but it need not be inner uniform. This example is due to J. Heinonen and presented by Väisälä in [21]. See [3, 9, 10, 21] for more details on John domains and inner uniform domains.

In 1989, Gehring, Hag and Martio [6] discussed the following question.

**Question 1.2.** Suppose that \( D \subset \mathbb{R}^n \) is a \( c \)-John domain and that \( \gamma \) is a quasihyperbolic geodesic in \( D \). Is \( \gamma \) a \( b \)-cone arc for some \( b = b(c) \)?

And they proved the following result.

**Theorem A.** [6, Theorem 4.1] If \( D \subset \mathbb{R}^2 \) is a simply connected John domain, then every quasihyperbolic or hyperbolic geodesic in \( D \) is a cone arc.

Meanwhile, they construct several examples to show that a quasihyperbolic geodesic in a \( c \)-John domain need not be a \( b \)-cone arc with \( b = b(c) \) unless \( n = 2 \) and \( D \) is simply connected.

In 1989, Heinonen [8] proposed the following question.

**Question 1.3.** [8] Suppose that \( D \subset \mathbb{R}^n \) is a \( c \)-John domain which quasiconformally equivalent to the unit ball \( \mathbb{B} \) and that \( \gamma \) is a quasihyperbolic geodesic in \( D \). Is \( \gamma \) a \( b \)-cone arc for some constant \( b \)?

In 2001, Bonk, Heinonen and Koskela [3, Theorem 7.12] give an affirmative answer to Question 1.3. We remark that every ball is inner uniform.

**Theorem B.** [3, Theorem 7.12] If \( D \subset \mathbb{R}^n \) is a \( a \)-John domain which is homeomorphic to an inner \( c \)-uniform domain via a \( K \)-quasiconformal map, then each quasihyperbolic geodesic in \( D \) is a \( b \)-cone arc with \( b = b(a, c, K, n) \).

We note that the constant \( b \) in Theorem B depends on the dimensional \( n \) and the proof of Theorem B mainly used the modulus estimates of curves. As is known to all, the method of path families is useless in Banach spaces. Hence, it is natural to ask that if Theorem B could be dimensional free or not. In other words, if it holds in Banach spaces or not. The main aim of this paper is to consider this problem.
Our result shows that the answer to the problem is affirmative. Our main result is as follows.

**Theorem 1.** Suppose that $D \subset E$ is an $a$-John domain which is homeomorphic to an inner $c$-uniform domain via an $(M,C)$-CQH. Let $z_1, z_2 \in D$ and $\gamma$ be a $c_0$-neargeodesic joining $z_1$ and $z_2$ in $D$. Then $\gamma$ is a $b$-cone arc, where the positive constant $b$ depends only on $a, c, c_0, C$ and $M$.

The organization of this paper is as follows. In section 3, we prove several lemmas which is critical to the proof of our main result and in section 4, we will prove Theorem 1. In section 2, some preliminaries are stated.

2. Preliminaries

The quasihyperbolic length of a rectifiable arc or a path $\gamma$ in $D$ is the number (cf. [1, 4, 5, 19])

$$\ell_k(\gamma) = \int_\gamma \frac{1}{d_D(z)} |dz|.$$  

For each pair of points $z_1, z_2$ in $D$, the quasihyperbolic distance $k_D(z_1, z_2)$ between $z_1$ and $z_2$ is defined in the usual way:

$$k_D(z_1, z_2) = \inf \ell_k(\alpha),$$

where the infimum is taken over all rectifiable arcs $\alpha$ joining $z_1$ to $z_2$ in $D$.

For all $z_1, z_2$ in $D$, we have (cf. [19])

$$k_D(z_1, z_2) \geq \log \left(1 + \frac{\lambda_D(z_1, z_2)}{\min\{d_D(z_1), d_D(z_2)\}}\right) \geq \log \left|\frac{d_D(z_2)}{d_D(z_1)}\right|.$$  

Moreover, if $|z_1 - z_2| < d_D(z_1)$, we have [22, Lemma 3.7]

$$k_D(z_1, z_2) \leq \log \left(1 + \frac{|z_1 - z_2|}{d_D(z_1) - |z_1 - z_2|}\right).$$

Gehring and Palka [5] introduced the quasihyperbolic metric of a domain in $\mathbb{R}^n$, and it has been recently used by many authors in the study of quasiconformal mappings and related questions [4, 7, 12, 16] etc. Recall that an arc $\alpha$ from $z_1$ to $z_2$ is a quasihyperbolic geodesic if $\ell_k(\alpha) = k_D(z_1, z_2)$. Obviously, each subarc of a quasihyperbolic geodesic is a quasihyperbolic geodesic. It is known that a quasihyperbolic geodesic between every pair of points in $E$ exists if the dimension of $E$ is finite, see [4, Lemma 1]. This is not true in infinite dimensional Banach spaces (cf. [19, Example 2.9]). In order to remedy this shortage, Väisälä introduced the following concepts [20].

**Definition 3.** Let $D \neq E$ and $c \geq 1$. An arc $\alpha \subset D$ is a $c$-neargeodesic if $\ell_k(\alpha[x, y]) \leq c k_D(x, y)$ for all $x, y \in \alpha$.

In [20], Väisälä proved the following property concerning the existence of neargeodesics in $E$.

**Theorem C.** ([20, Theorem 3.3]) Let $\{z_1, z_2\} \subset D$ and $c > 1$. Then there is a $c$-neargeodesic in $D$ joining $z_1$ and $z_2$. 
Now let us recall the following characterization of inner uniform domains, which is due to Väisälä.

**Theorem D.** ([21, Theorem 2.33]) A domain $D \subset E$ is an inner $c$-uniform domain if and only if $k_D(x, y) \leq c' \log \left(1 + \frac{\lambda_D(x, y)}{\min\{d_D(x), d_D(y)\}}\right)$ for all $x, y \in D$, where the constants $c$ and $c'$ depend only on each other.

Generalizing quasiconformal, Väisälä introduced CQH homeomorphisms (cf. [18, 20]).

**Definition 4.** Suppose $f : D \to D'$ is a homeomorphism. Then $f$ is said to be $C$-coarsely $M$-quasihyperbolic, or briefly $(M, C)$-CQH, if it satisfies

$$\frac{k_D(x, y) - C}{M} \leq k_{D'}(f(x), f(y)) \leq M k_D(x, y) + C$$

for all $x, y \in D$.

### 3. Properties of cone arcs

In what follows, we always assume that $f : D \to D'$ is an $(M, C)$-CQH map, that $D$ is an $a$-John domain and that $D'$ is an inner $c$-uniform domain. Also we use $x, y, z, \cdots$ to denote the points in $D$, and $x', y', z', \cdots$ the images of $x, y, z, \cdots$ in $D'$, respectively, under $f$. For arcs $\alpha, \beta, \gamma, \cdots$ in $D$, we also use $\alpha', \beta', \gamma', \cdots$ to denote their images in $D'$.

For $x, y \in D$, let $\beta$ be an arc joining $x$ and $y$ in $D$. We are going to determine some special points on $\beta'$.

#### 3.1. Determination of special points on $\beta'$

Without loss of generality, we may assume that $d_{D'}(y') \geq d_{D'}(x')$. Then there must exist a point $w_0' \in \beta'$ which is the first point along the direction from $x'$ to $y'$ such that

$$d_{D'}(w_0') = \sup_{p' \in \beta'} d_{D'}(p').$$

It is possible that $w_0' = x'$ or $y'$. Obviously, there exists a nonnegative integer $m$ such that

$$2^m d_{D'}(x') \leq d_{D'}(w_0') < 2^{m+1} d_{D'}(x'),$$

and $x_0'$ the first point in $\beta'[x', w_0']$ from $x'$ to $w_0'$ with

$$d_{D'}(x_0') = 2^m d_{D'}(x').$$

Let $x_1' = x'$. If $x_0' = x_1'$, we let $x_2' = w_0'$. It is possible that $x_1' = x_2'$. If $x_0' \neq x_1'$, then we let $x_2', \ldots, x_{m+1}' \in \beta'[x', x_0']$ be the points such that for each $i \in \{2, \ldots, m + 1\}$, $x_i'$ denotes the first point from $x'$ to $x_0'$ with

$$d_{D'}(x_i') = 2^{i-1} d_{D'}(x_1').$$

Obviously, $x_{m+1}' = w_0'$. If $x_0' \neq w_0'$, then we use $x_{m+2}'$ to denote $w_0'$.

In a similar way, let $s \geq 0$ be the integer such that
Let $x'_{1,0}$ the first point in $\beta'[y', x'_{1,0}]$ from $y'$ to $x'_{1,0}$ with

$$d_{D'}(x'_{1,0}) = 2^a d_{D'}(y').$$

Let $x'_{1,1} = y'$. If $x'_{1,0} = x'_{1,1}$, we let $x'_{1,2} = x'_{1,0}$. It is possible that $x'_{1,2} = x'_{1,1}$. If $x'_{1,0} \neq y'$, then we let $x'_{1,2}, \ldots, x'_{1,s+1}$ be the points in $\beta'[y', w'_0]$ such that for each $j \in \{2, \ldots, s+1\}$, $x'_{1,j}$ is the first point from $x'_{1,1}$ to $w'_0$ with

$$d_{D'}(x'_{1,j}) = 2^{j-1} d_{D'}(x'_{1,1}).$$

Then $x'_{1,s+1} = x'_{1,0}$. If $x'_{1,0} \neq w'_0$, we let $x'_{1,s+2} = w'_0$.

### 3.2. Elementary properties

In the following, we assume that for each $s_1, s_2 \in \beta$

$$\ell_k(\beta[s_1, s_2]) \leq 4a^2c_0 k_D(s_1, s_2) + 4a^2c_0,$$

where $a$ and $c_0$ are the same constants as in Theorem 1. Obviously, (3.1) is satisfied for each $c_0$-neargeodesic.

**Lemma 1.** For each $k \in \{1, \ldots, m\}$ and $z' \in \beta'[x'_k, x'_{k+1}]$

1. $d_{D'}(x'_{k+1}) \leq a_2 d_{D'}(z')$;
2. $\lambda_{D'}(x'_{k+1}, x'_k) \leq a_2 d_{D'}(z')$ and
3. $\max\{\lambda_{D'}(z', x'_k), \lambda_{D'}(x'_{k+1}, z')\} \leq a_2 d_{D'}(z'),$

where $a_2 = (1 + 2a_1)^{4a^2c_0cM^2 + 1}cC + 4a^2c_0M + 4a^2c_0CM$, $a_1 = e^{3(C+1)(ao+M)}$ and $a_0 = 2^4[c' + 4a^2c_0cM + C + 4a^2c_0]^4$. Here and in what follows, $[\cdot]$ always denotes the greatest integer part.

**Proof.** At first, we prove the following inequality: For any $k \in \{1, \ldots, m\}$,

$$\lambda_{D'}(x'_{k+1}, x'_k) < a_1 d_{D'}(x'_{k+1}).$$

We prove this inequality by contradiction. Suppose on the contrary that

$$\lambda_{D'}(x'_{k+1}, x'_k) \geq a_1 d_{D'}(x'_{k+1}).$$

Let $y'_{k,1}, y'_{k,2}, \ldots, y'_{k,a_0+1} \in \beta'[x'_k, x'_{k+1}]$ be $a_0 + 1$ points such that $y'_{k,1} = x'_k$, $y'_{k,a_0+1} = x'_{k+1}$ and $\lambda_{D'}(y'_{k,i+1}, y'_{k,i}) \geq \lambda_{D'}(x'_{k+1}, x'_k)$. Then for each $i \in \{1, 2, \ldots, a_0\}$,

$$k_{D'}(y'_{k,i}, y'_{k,i+1}) \geq \log \left(1 + \frac{\lambda_{D'}(y'_{k,i+1}, y'_{k,i})}{\min\{d_{D'}(y'_{k,i+1}, y'_{k,i})\}, d_{D'}(y'_{k,i})} \right) \geq \log \left(1 + \frac{\lambda_{D'}(x'_k, x'_{k+1})}{2a_0 d_{D'}(x'_k)} \right).$$
We see from (3.1) and Theorem D that
\[
a_0 \log \left(1 + \frac{\lambda_{D'}(x'_k, x'_{k+1})}{2a_0 d_{D'}(x'_k)}\right) \leq \sum_{i=1}^{a_0} k_{D'}(y'_k, y'_{k,i}) \leq M \sum_{i=1}^{a_0} k_D(y_k, y_{k,i}) + a_0 C
\]
\[
\leq M \ell_k(\beta[x_k, x_{k+1}]) + a_0 C
\]
\[
\leq 4a^2c_0 M k_D(x_k, x_{k+1}) + 4a^2c_0 M + a_0 C
\]
\[
\leq 4a^2c_0 M^2 k_{D'}(x'_k, x'_{k+1}) + (a_0 + 4a^2c_0 M)C + 4a^2c_0 M
\]
\[
\leq 4a^2c_0 c' M^2 \log \left(1 + \frac{\lambda_{D'}(x'_k, x'_{k+1})}{d_{D'}(x'_k)}\right)
\]
\[
\quad + (a_0 + 4a^2c_0 M)C + 4a^2c_0 M,
\]
whence
\[
a_0 \log \left(1 + \frac{\lambda_{D'}(x'_k, x'_{k+1})}{2a_0 d_{D'}(x'_k)}\right) \leq 8a^2c' M^2 \log \left(1 + \frac{\lambda_{D'}(x'_k, x'_{k+1})}{d_{D'}(x'_k)}\right),
\]
which contradicts with (3.3). Hence (3.2) holds.

We infer from (3.2) that for each \(z' \in \beta'[x'_k, x'_{k+1}]\),
\[
(3.4) \quad \log \frac{d_{D'}(x'_{k+1})}{d_{D'}(z')} < k_{D'}(z', x'_{k+1}) \leq M k_D(z, x_{k+1}) + C
\]
\[
\leq M \ell_k(\beta[x_k, x_{k+1}]) + C
\]
\[
\leq 4a^2c_0 M k_D(x_k, x_{k+1}) + 4a^2c_0 M + 4a^2c_0 M
\]
\[
\leq 4a^2c_0 M^2 k_{D'}(x'_k, x'_{k+1}) + 4a^2c_0 C M + 4a^2c_0 M
\]
\[
\leq 4a^2c_0 c' M^2 \log \left(1 + \frac{\lambda_{D'}(x'_k, x'_{k+1})}{d_{D'}(x'_k)}\right)
\]
\[
\quad + C + 4a^2c_0 CM + 4a^2c_0 M
\]
\[
\leq 4a^2c_0 c' M^2 \log(1 + 2a_1) + C + 4a^2c_0 CM + 4a^2c_0 M,
\]
which implies that Lemma 1 (1) holds.

Hence, (3.2) and (3.4) yield that
\[
\lambda_{D'}(x'_k, x'_{k+1}) \leq (1 + 2a_1)4a^2c_0 c' M^2 + 1 e^{4a^2c_0 C M + 4a^2c_0 M + C} d_{D'}(z'),
\]
whence Lemma 1 (2) follows.

Obviously,
\[
\log \left(1 + \frac{\lambda_{D'}(x'_k, z')}{d_{D'}(z')}\right) \leq k_{D'}(x'_k, z') \leq M k_D(x_k, z) + C
\]
\[
\leq M \ell_k(\beta[x_k, x_{k+1}]) + C
\]
\[
\leq 4a^2c_0 M k_D(x_k, x_{k+1}) + 4a^2c_0 M + C
\]
\[
\leq 4a^2c_0 c^2 k_{D'}(x'_k, x'_{k+1}) + C + 4a^2c_0 M + 4a^2c_0 CM
\]
\[
\leq 4a^2c_0 c' M^2 \log \left(1 + \frac{\lambda_{D'}(x'_k, x'_{k+1})}{d_{D'}(x'_k)}\right)
\]
\[
\quad + C + 4a^2c_0 M + 4a^2c_0 CM,
\]
which, together with (3.2), yields

$$\lambda_{D'}(x_{k}', z') \leq (1 + 2a_1)^{4a^2c_0c' M^2} e^{C + 4a^2c_0M + 4a^2c_0CM} \ d_{D'}(z').$$

The similar discussion as in (3.5) shows that

$$\lambda_{D'}(z', x_{k+1}') \leq (1 + 2a_1)^{4a^2c_0c' M^2} e^{C + 4a^2c_0M + 4a^2c_0CM} \ d_{D'}(z').$$

The combination of (3.5) and (3.6) shows that Lemma 1 (3) holds.

The following two results easily follow from the similar reasoning as in the proof of Lemma 1.

**Corollary 1.** For each $k \in \{1, \cdots, s\}$ and $z' \in \beta'[x_{1,k}', x_{1,k+1}'],$

1. $d(x_{1,k+1}', z') \leq a_2 \ d_{D'}(z')$;
2. $\lambda_{D'}(x_{1,k+1}', z') \leq a_2 \ d_{D'}(z')$ and
3. $\max\{\lambda_{D'}(x_{1,k}', z'), \lambda_{D'}(x_{1,k+1}', z')\} \leq a_2 \ d_{D'}(z').$

**Corollary 2.** For each $z' \in \beta'[x_{m+1}', x_{1,s+1}'],$

1. $d_{D'}(w_0', z') \leq a_2 \ d_{D'}(z')$;
2. $\lambda_{D'}(x_{m+1}', z') \leq a_2 \ d_{D'}(z')$ and
3. $\max\{\lambda_{D'}(x_{m+1}', z'), \lambda_{D'}(x_{1,s+1}', z')\} \leq a_2 \ d_{D'}(z').$

**Lemma 2.** For each $z' \in \beta'[x', w_0'],$ $\lambda_{D'}(x', z') \leq a_3 \ d_{D'}(z')$, and for each $z' \in \beta'[y', w_0'],$ $\lambda_{D'}(y', z') \leq a_3 \ d_{D'}(z')$. Where $a_3 = a_2 + a_2^2$.

**Proof.** We only need to prove the first part of the Lemma as for the proof of the second part is similar.

If $z' \in \beta'[x', x_{m+1}]$, then there exists some $k \in \{1, \cdots, m\}$ such that $z' \in \beta'[x_{k}', x_{k+1}]$. If $k = 1$, then the result easily follows from Lemma 1. If $k > 1$, then by Lemma 1,

$$\lambda_{D'}(x', z') \leq \lambda_{D'}(x_{1}', z') + \cdots + \lambda_{D'}(x_{k-1}', x_{k}') + \lambda_{D'}(x_{k}', z')$$

$$\leq a_2 (d_{D'}(x_1') + \cdots + d_{D'}(x_{k-1}') + d_{D'}(z'))$$

$$\leq (a_2 + \frac{1}{2}a_2^2) d_{D'}(z').$$

Now we consider the case $z' \in \beta'[x_{m+1}', w_0'].$ Then we infer from Lemma 1 and Corollary 2 that

$$\lambda_{D'}(x', z') \leq a_2 (d_{D'}(x_1') + d_{D'}(x_2') + \cdots + d_{D'}(x_{m}') + d_{D'}(z'))$$

$$\leq a_2 (d_{D'}(x_{m+1}') + d_{D'}(z'))$$

$$\leq (a_2 + a_2^2) d_{D'}(z').$$

Hence the lemma holds.

Since $D$ is an $a$-John domain, then there exists an $a$-cone arc $\alpha$ in $D$ joining $x$ and $y$. Let $s_0$ bisect $\alpha$. 

Lemma 3. For each \( s_1, s_2 \in \alpha[x, s_0] \) (or \( \alpha[x, s_0] \)) with \( s_2 \in \alpha[s_1, s_0] \), we have

\[
k_D(s_1, s_2) \leq \ell_k(\alpha[s_1, s_2]) \leq 2a \log \left(1 + \frac{2\ell(\alpha[s_1, s_2])}{d_D(s_1)}\right)
\]

and

\[
\ell_k(\alpha[s_1, s_2]) \leq 4a^2 c_0 k_D(s_1, s_2) + 4a^2 c_0.
\]

That is, for each \( s_1, s_2 \in \alpha[x, s_0] \) (or \( \alpha[x, s_0] \)) (3.1) holds.

Proof. It suffices to prove the case \( s_1, s_2 \in \alpha[x, s_0] \) since the proof for the other case is similar. For given \( s_2 \in \alpha[s_1, s_0] \), \( d_D(s_2) \geq \frac{\ell(\alpha[s_1, s_2])}{a} \). If \( \alpha[s_1, s_2] \subset B(s_1, \frac{d_D(s_1)}{2}) \), then \( d_D(z) \geq \frac{d_D(u)}{2} \). Otherwise, we have \( d_D(s_2) \geq \frac{d_D(a)}{2a} \). Hence \( d_D(s_2) \geq \frac{2\ell(\alpha[s_1, s_2]) + d_D(s_1)}{4a} \), which yields that

\[
k_D(s_1, s_2) \leq \ell_k(\alpha[s_1, s_2]) = \int_{\alpha[s_1, s_2]} \frac{|dz|}{d_D(z)} \leq 2a \log \left(1 + \frac{2\ell(\alpha[s_1, s_2])}{d_D(s_1)}\right) \leq 4a^2 \log \left(1 + \frac{d_D(s_2)}{d_D(s_1)}\right) \leq 4a^2 c_0 k_D(s_1, s_2) + 4a^2 c_0,
\]

from which the proof follows. \( \square \)

Let \( d_D'(v'_1) = \max\{d_D'(u') : u' \in \alpha'[x', s'_0]\} \) and \( d_D'(v'_2) = \max\{d_D'(u') : u' \in \alpha'[y', s'_0]\} \). Hence it follows from Lemma 2 that

Lemma 4. (1) For each \( z' \in \alpha'[x', v'_1] \), \( \lambda_D'(x', z') \leq a_3 \) \( d_D'(z') \) and for each \( z' \in \alpha'[v'_1, s'_0] \), \( \lambda_D'(s'_0, z') \leq a_3 d_D'(z') \).

(2) For each \( z' \in \alpha'[y', v'_2] \), \( \lambda_D'(y', z') \leq a_3 d_D'(z') \) and for each \( z' \in \alpha'[v'_2, s'_0] \), \( \lambda_D'(s'_0, z') \leq a_3 d_D'(z') \).

4. The proof of Theorem 1

Let \( z_1, z_2 \in D \) and \( \gamma \) be a \( c_0 \)-neargeodesic joining \( z_1, z_2 \) in \( D \). In the following, we prove that \( \gamma \) is a \( b \)-cone arc, that is, for each \( y \in \gamma \),

\[
(4.1) \quad \min\{\ell(\gamma[z_1, y]), \ell(\gamma[z_2, y])\} \leq b d_D(y),
\]

where \( b = 4a_1 c_0 e^{a_1 c_0} \), \( a_4 = a_5^2 e^{a_5 M} \), \( a_5 = a_6^{4 a_2 c_0 M + C} \) and \( a_6 = (8a_3)^{4 e^{a_4 c_0 M} a_2 e^{2C}} \). It is no loss of generality to assume that \( d_D(z_1) \leq d_D(z_2) \).

Let \( x_0 \in \gamma[z_1, z_2] \) be such that

\[
d_D(x_0) = \max_{z \in \gamma[z_1, z_2]} d_D(z).
\]

Then there exists an integer \( t_1 \geq 0 \) such that

\[
2^{t_1} d_D(z_1) \leq d_D(x_0) < 2^{t_1+1} d_D(z_1).
\]
Let $y_0$ be the first point in $\gamma[z_1, x_0]$ from $z_1$ to $x_0$ with
\[ d_D(y_0) = 2^{t_1} d_D(z_1). \]

Observe that if $d_D(x_0) = d_D(z_1)$, then $y_0 = z_1 = x_0$.

Let $y_1 = z_1$. If $z_1 = y_0$, we let $y_2 = x_0$. It is possible that $y_2 = y_1$. If $z_1 \neq y_0$, then we let $y_2, \ldots, y_{t_1+1}$ be the points such that for each $i \in \{2, \ldots, t_1+1\}$, $y_i$ denotes the first point in $\gamma[z_i, x_0]$ from $y_i$ to $x_0$ satisfying
\[ d_D(y_i) = 2^{i-1} d_D(y_1). \]

Then $y_{t_1+1} = y_0$. We let $y_{t_1+2} = x_0$. It is possible that $y_{t_1+2} = y_{t_1+1} = x_0 = y_0$. This possibility occurs once $x_0 = y_0$.

Now we are going to prove for each $i \in \{1, \ldots, t_1+1\}$,
\[ k_D(y_i, y_{i+1}) \leq a_4. \]

If $d_D(y_i) > \lambda_D(y_i, y_{i+1})$, then $k_D(y_i, y_{i+1}) \leq 2$. Inequality (4.2) obviously holds. Hence in the following, we assume that
\[ d_D(y_i) \leq \lambda_D(y_i, y_{i+1}). \]

To prove (4.2), we let $\alpha_i$ be an $a$-cone arc joining $y_i$ and $y_{i+1}$ in $D$ and let $v_i$ bisect $\alpha_i$. Without loss of generality, we may assume that $d_D(y_i') \leq d_D(y_{i+1}')$.

Hence Lemma 3 implies
\[ k_D(y_i, y_{i+1}) \leq k_D(y_i, v_i) + k_D(y_{i+1}, v_i) \leq 2a \left( \log \left( 1 + \frac{2\ell(\alpha_i[y_i, y_{i+1}])}{d_D(y_{i+1})} \right) \right. \]
\[ + \left. \log \left( 1 + \frac{2\ell(\alpha_i)}{d_D(y_i)} \right) \right) \leq 4a \log \left( 1 + \frac{\ell(\alpha_i)}{d_D(y_i)} \right). \]

Now we divide the rest proof of (4.2) into two cases.

**Case 1.** $\ell(\alpha_i) < a_5 \lambda_D(y_i, y_{i+1})$.

Then (4.4) yields
\[ \frac{\ell(\gamma[y_i, y_{i+1}])}{2d_D(y_i)} \leq \ell_k(\gamma[y_i, y_{i+1}]) \leq c_0 k_D(y_i, y_{i+1}) \leq 4ac_0 \log \left( 1 + \frac{\ell(\alpha_i)}{d_D(y_i)} \right) \]
\[ \leq 4ac_0 \log \left( 1 + \frac{a_5 \lambda_D(y_i, y_{i+1})}{d_D(y_i)} \right). \]

A necessary condition for (4.5) is
\[ \lambda_D(y_i, y_{i+1}) \leq a_5^2 d_D(y_i). \]

Hence (4.5) implies that $k_D(y_i, y_{i+1}) \leq a_4$.

**Case 2.** $\ell(\alpha_i) \geq a_5 \lambda_D(y_i, y_{i+1})$. 
We consider this case by a contradiction. Suppose on the contrary that
\[ k_D(y_i, y_{i+1}) > a_4, \]  
which implies that
\[ k_D'(y'_i, y'_{i+1}) > \frac{k_D(y_i, y_{i+1}) - C}{M} > 1. \]  
Then
\[ a_4 < k_D(y_i, y_{i+1}) \leq Mk_D'(y'_i, y'_{i+1}) + C \leq c'M \log \left(1 + \frac{\lambda_D'(y'_i, y'_{i+1})}{d_D'(y'_i)}\right) + C, \]
which implies that
\[ \lambda_D'(y'_i, y'_{i+1}) \geq a_5d_D'(y'_i). \]
Hence
\[ d_D(v_i) \geq \frac{\ell(\alpha_i)}{2a} \geq \frac{a_5\lambda_D(y_i, y_{i+1})}{2a} > a_6\lambda_D(y_i, y_{i+1}). \]
Then (4.3) guarantees that there exists \( v_{i,0} \in \alpha_i[y_i, v_i] \) such that
\[ d_D(v_{i,0}) = a_6\lambda_D(y_i, y_{i+1}). \]

**Claim 1.** \( k_D(y_i, v_{i,0}) \leq \frac{1}{a_5}k_D(y_i, y_{i+1}). \)

We prove this Claim also by a contradiction. Suppose that
\[ k_D(y_i, v_{i,0}) > \frac{1}{a_5}k_D(y_i, y_{i+1}). \]
Then Lemma 3 yields,
\[ \frac{\ell(\gamma[y_i, y_{i+1}])}{2d_D(y_i)} \leq \ell(\gamma[y_i, y_{i+1}]) \leq c_0k_D(y_i, y_{i+1}) \leq a_5c_0k_D(y_i, v_{i,0}) \]
\[ \leq 4aa_5c_0 \log \left(1 + \frac{\ell(\alpha_i[y_i, v_{i,0}])}{d_D(y_i)}\right) \leq 4aa_5c_0 \log \left(1 + \frac{ad(v_{i,0})}{d_D(y_i)}\right) \]
\[ \leq 4a^2a_5a_6 \log \left(1 + \frac{\lambda_D(y_i, y_{i+1})}{d_D(y_i)}\right), \]
whence
\[ \lambda_D(y_i, y_{i+1}) \leq a_5^2d_D(y_i), \]
which shows that \( k_D(y_i, y_{i+1}) \leq a_4 \) and this contradicts with (4.6). \[ \square \]

By (4.3) and (4.9), we get
\[ k_D(y_i, v_{i,0}) \geq \log \frac{d_D(v_{i,0})}{d_D(y_i)} \geq \log a_6 > C. \]
Thus Claim 1, Lemma 3, (4.7) and (4.8) imply that
\[
\log \left(1 + \frac{\lambda_{D'}(y_i', v_{i,0}')}{d_{D'}(y_i')}ight) \leq k_{D'}(y_i', v_{i,0}') \leq Mk_D(y_i, v_{i,0}) + C
\]
\[
< 2Mk_D(y_i, v_{i,0}) \leq \frac{2M}{a_5}k_D(y_i, y_{i+1})
\]
\[
\leq \frac{2M^2}{a_5}k_{D'}(y_i', y_{i+1}') + \frac{2CM}{a_5}
\]
\[
\leq \frac{4CM^2}{a_5}\log \left(1 + \frac{\lambda_{D'}(y_i', y_{i+1}')}{d_{D'}(y_i')}ight)
\]
\[
\leq \log \left(1 + \frac{\lambda_{D'}(y_i', y_{i+1}')}{a_5d_{D'}(y_i')}ight).
\]
Hence
\[
\lambda_{D'}(y_i', v_{i,0}') < \frac{1}{a_5}\lambda_{D'}(y_i', y_{i+1}'),
\]
which, together with (4.8), gives
\[
d_{D'}(v_{i,0}') \leq \lambda_{D'}(y_i', v_{i,0}') + d_{D'}(y_i') \leq \frac{2}{a_5}\lambda_{D'}(y_i', y_{i+1}).
\]
Claim 2. \(\lambda_{D'}(y_i', v_i') < \frac{\lambda_{D'}(y_i', y_{i+1}')}{2}\).

Suppose on the contrary that
\[
\lambda_{D'}(y_i', v_i') \geq \frac{\lambda_{D'}(y_i', y_{i+1}')}{2}.
\]
Let \(u_{0,i} \in \gamma'[y_i', y_{i+1}']\) be a point satisfying
\[
d_{D'}(u_{0,i}') = \max\{d_{D'}(w') : w' \in \gamma'[y_i', y_{i+1}']\}.
\]
Obviously,
\[
\max\{\lambda_{D'}(y_{i+1}', u_{0,i}'), \lambda_{D'}(u_{0,i}', y_i')\} \geq \frac{\lambda_{D'}(y_i', y_{i+1}')}{2}.
\]
Then we know from Lemma 2 that
\[
d_{D'}(u_{0,i}') \geq \frac{\lambda_{D'}(y_i', y_{i+1}')}{2a_3}.
\]
Hence by Lemma 2 and (4.8), there must exist some point \(y_{0,i}' \in \gamma'[y_i', u_{0,i}']\) satisfying
\[
d_{D'}(y_{0,i}') = \frac{\lambda_{D'}(y_i', y_{i+1}')}{2a_3} \text{ and } \lambda_{D'}(y_i', y_{0,i}') \leq a_3d_{D'}(y_{0,i}).
\]
Let \(u'_0 \in \alpha'[y_i', v_i']\) satisfy \(d_{D'}(u_0') = \max\{d_{D'}(u') : u' \in \alpha'[y_i', v_i']\}\). Then Lemma 4 shows that for each \(z' \in \alpha'[v_0', v_i']\),
\[
\lambda_{D'}(v_i', z') \leq a_3d_{D'}(z').
\]
By (4.10) and (4.11) we have
\[
\lambda_{D'}(v_i', v_{i,0}') \geq \lambda_{D'}(v_i', y_i') - \lambda_{D'}(v_{i,0}', y_i') \\
\geq \left( \frac{1}{2} - \frac{1}{a_5} \right) \lambda_{D'}(y_i', y_{i+1}') \\
\geq \left( a_5 - \frac{1}{2} \right) d_{D'}(v_{i,0}'),
\]
which together with (4.14) shows that \( v_0' \in \alpha'[v_i', v_i'] \).

Obviously, \( \max\{\lambda_{D'}(v_i', v_0'), \lambda_{D'}(v_i', y_i')\} \geq \frac{\lambda_{D'}(y_i', y_{i+1}')}{4a_3} \). We know from Lemma 4 that \( d_{D'}(v_0') \geq \frac{\lambda_{D'}(y_i', y_{i+1})}{4a_3} \). By (4.11) and Lemma 4, we see that there exists some point \( u_0' \in \alpha'[v_i', v_i'] \) such that
\[
\tag{4.15} d_{D'}(u_0') = \frac{\lambda_{D'}(y_i', y_{i+1}')}{4a_3}
\text{ and } \lambda_{D'}(v_i', u_0') \leq a_3 d_{D'}(u_0').
\]
Hence (4.13) shows that
\[
\log \frac{d_D(u_0)}{d_D(y_{0,i})} \leq k_D(y_{0,i}, u_0) \leq M k_{D'}(y_{0,i}, u_0') + C \\
\leq Mc' \log \left(1 + \frac{\lambda_{D'}(u_0', y_{0,i})}{\min\{d_{D'}(u_0'), d_{D'}(y_{0,i})\}}\right) + C \\
\leq Mc' \log \left(1 + \frac{\lambda_{D'}(u_0', y_{0,i}) + \lambda_{D'}(y_i', y_{0,i})}{\min\{d_{D'}(u_0'), d_{D'}(y_{0,i})\}}\right) + C \\
\leq Mc' \log \left(1 + \frac{a_3 d(u_0') + \lambda_{D'}(y_i', y_{0,i})}{\min\{d_{D'}(u_0'), d_{D'}(y_{0,i})\}}\right) + C \\
< Mc' \log (1 + 3a_3) + C,
\]
which yields that
\[
\tag{4.16} d_D(u_0) \leq (1 + 3a_3)^{Mc'} e^C d_D(y_{0,i}).
\]
Lemma 3, (4.11) and (4.15) imply that
\[
4a^2 M \log \left(1 + \frac{d_D(u_0)}{d_{D'}(v_{i,0})}\right) + C \geq M \ell_k(\alpha'[v_i,0], u_0) + C \geq M k_{D'}(v_{i,0}, u_0) + C \\
\geq k_{D'}(v_{i,0}', u_0') \geq \log \frac{d_{D'}(u_0')}{d_{D'}(v_{i,0}')} \geq \log \frac{a_5}{8a_3},
\]
whence \( d_D(u_0) \geq a_6 d_{D'}(v_{i,0}) \). So we infer from (4.3) and (4.9) that
\[
d_D(u_0) \geq a_6 d_{D'}(v_i,0) = a_6^2 \lambda_D(y_i, y_{i+1}) \geq \frac{a_6^2}{2} d_{D'}(y_{i+1}) \geq \frac{a_6^2}{2} d_D(y_{0,i}),
\]
which contradicts with (4.16). Hence Claim 2 holds. \(\square\)

It is obvious from Claim 2 that \( \lambda_{D'}(y_{i+1}', v_i') > \frac{\lambda_{D'}(y_i', y_{i+1}')}{2} \). Let \( q_0' \in \alpha'[y_i', v_i'] \) with
\[
\tag{4.17} \frac{\lambda_{D'}(y_i', v_i')}{2a_3} \geq \lambda_{D'}(q_0', v_i') \geq \frac{\lambda_{D'}(y_i', v_i')}{4a_3},
\]
and \( u'_1 \in \alpha'[y'_{i+1}, v'_i] \) with

\[
\frac{\lambda_{D'}(y'_i, v'_i)}{2a_3} \geq \frac{\lambda_{D'}(u'_1, v'_i)}{4a_3} \geq \frac{\lambda_{D'}(y'_i, v'_i)}{4a_3}.
\]

By Lemma 4, we get

\[
d_{D'}(q'_0) \geq \frac{\lambda_{D'}(y'_i, v'_i)}{4a_3} \quad \text{and} \quad d_{D'}(u'_1) \geq \frac{\lambda_{D'}(y'_i, v'_i)}{4a_3}.
\]

Hence we have

\[
\left| \log \frac{d_D(u_1)}{d_D(q_0)} \right| \leq k_D(u_1, q_0) \\
\leq Mk_D(u'_1, q'_0) + C \\
\leq Mc' \log \left( 1 + \frac{\lambda_{D'}(u'_1, q'_0)}{\min\{d_{D'}(q'_0), d_{D'}(u'_1)\}} \right) + C \\
\leq Mc' \log \left( 1 + \frac{\lambda_{D'}(u'_1, v'_i) + \lambda_{D'}(v'_i, q'_0)}{\min\{d_{D'}(q'_0), d_{D'}(u'_1)\}} \right) + C \\
\leq Mc' \log(1 + 4a_3) + C,
\]

which implies that

\[
d_D(u_1) \leq (1 + 4a_3)^{MC} e^{C}d_D(u_1).
\]

Claim 3. \( d_D(q_0) \geq a_5d_D(v_{i,0}). \)

Otherwise, Lemma 3, (4.9), (4.20) and (4.21) show that

\[
\frac{\ell(\gamma[y_i, y_{i+1}])}{2d_D(y_i)} \leq \ell_k(\gamma[y_i, y_{i+1}]) \leq c_0k_D(y_i, y_{i+1}) \\
\leq c_0(k_D(y_i, q_0) + k_D(q_0, u_1) + k_D(u_1, y_{i+1})) \\
\leq 4a^2c_0 \log \left( 1 + \frac{d_D(q_0)}{d_D(y_i)} \right) + Mc'c_0 \log \left( 1 + 2a_3 \right) \\
+Cc_0 + 4a^2c_0 \log \left( 1 + \frac{d_D(u_1)}{d_D(y_{i+1})} \right) \\
\leq 9a^2a_5c_0 \log \left( 1 + \frac{\lambda_D(y_i, y_{i+1})}{d_D(y_i)} \right).
\]

A necessary condition for (4.22) is \( \lambda_D(y_i, y_{i+1}) \leq a_5d_D(y_i) \). Hence by (4.22), we know that

\[ k_D(y_i, y_{i+1}) \leq 9a^2a_5 \log(1 + a_5^2), \]

which contradicts with (4.6). We complete the proof of Claim 3. \( \square \)

By (4.8) and (4.12)

\[ \lambda_{D'}(u'_{0,i}, y'_i) \geq d_{D'}(u'_{0,i}) - d_{D'}(y'_i) \geq \frac{1}{3a_5} \lambda_{D'}(y'_{i+1}, y'_i). \]
Then Claim 2 guarantees that there exists $y'_0 \in \gamma'[y'_i, u'_{0,i}]$ such that

\[(4.23) \quad \frac{\lambda_{D'}(y'_i, v'_i)}{2a_3} \geq \lambda_{D'}(y'_0, y'_i) \geq \frac{\lambda_{D'}(y'_i, v'_i)}{3a_3}.\]

Hence Lemma 2 implies that

\[\frac{\lambda_{D'}(y'_i, v'_i)}{2a_3} = \lambda_{D'}(y'_0, y'_i) \leq a_3 d_{D'}(y'_0).\]

Hence (4.17), (4.18), (4.19) and (4.23) give

\[\log \frac{d_D(q_0)}{d_D(y_0)} \leq k_D(q_0, y_0) \leq M k_{D'}(q'_0, y'_0) + C' \leq M c' \log \left(1 + \frac{\lambda_{D'}(y'_0, q'_0)}{\min\{d_{D'}(q'_0), d_{D'}(y'_0)\}}\right) + C \leq M c' \log\left(1 + \frac{\lambda_{D'}(y'_i, v'_i) + \lambda_{D'}(v'_i, q'_0) + \lambda_{D'}(y'_i, y'_0)}{\min\{d_{D'}(q'_0), d_{D'}(y'_0)\}}\right) + C \leq M c' \log(1 + 4a_3 + 4a_3^2) + C.\]

We infer from (4.3) and (4.9) that

\[d_D(q_0) \leq (1 + 4a_3 + 4a_3^2) M e^C d_D(y_0) \leq 2(1 + 4a_3 + 4a_3^2) M e^C d_D(y_i) \leq 2(1 + 4a_3 + 4a_3^2) M c e^C \lambda_D(y_i, y_{i+1}) = \frac{2(1 + 4a_3 + 4a_3^2) M c e^C}{a_6} d_D(v_{i,0}),\]

which contradicts with Claim 3. We complete the proof of (4.2).

Then by (4.2) we have for all $i \in \{1, \cdots, t_1 + 1\}$,

\[(4.24) \quad \frac{\ell(\gamma[y_i, y_{i+1}])}{2d_D(y_i)} \leq \ell_k(\gamma[y_i, y_{i+1}]) \leq c_0 k_D(y_i, y_{i+1}) \leq a_4 c_0,\]

which implies that

\[(4.25) \quad \ell(\gamma[y_i, y_{i+1}]) \leq 2a_4 c_0 d_D(y_i).\]

Further, for each $y \in \gamma[y_i, y_{i+1}]$, it follows from (4.24) that

\[(4.26) \quad \log \frac{d_D(y_i)}{d_D(y)} \leq k_D(y, y_i) \leq c_0 k_D(y_i, y_{i+1}) \leq a_4 c_0,\]

whence

\[d_D(y_i) \leq e^{a_4 c_0} d_D(y).\]
For each \( y \in \gamma[y_1, x_0] \), there is some \( i \in \{1, \cdots, t_1 + 1\} \) such that \( y \in \gamma[y_i, y_{i+1}] \). It follows from (4.25) and (4.26) yield that
\[
\ell(\gamma[z_1, y]) = \ell(\gamma[y_1, y_2]) + \ell(\gamma[y_2, y_3]) + \cdots + \ell(\gamma[y_i, y]) \\
\leq 2a_4c_0(d_D(y_1) + d_D(y_2) + \cdots + d_D(y_i)) \\
\leq 4a_4c_0 d_D(y_i) \\
\leq 4a_4c_0 e^{a_4c_0} d_D(y).
\]

By replacing \( \gamma[z_1, x_0] \) by \( \gamma[z_2, x_0] \) and repeating the procedure as above, we also get that
\[
\ell(\gamma[z_2, y]) \leq 4a_4c_0 e^{a_4c_0} d_D(y).
\]

The combination of (4.27) and (4.28) conclude the proof of Theorem 1. \( \square \)

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