Hydrodynamic electron transport near charge neutrality

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(Dated: April 28, 2020)

We develop theory of hydrodynamic electron transport in a long-range disorder potential for conductors in which the underlying electron liquid lacks Galilean invariance. For weak disorder we express the transport coefficients of the system in terms of the intrinsic kinetic coefficients of the electron liquid and the correlation function of the disorder potential. We apply these results to analyze the doping- and temperature-dependence of transport coefficients of graphene devices. Near charge neutrality the inhomogeneous hydrodynamic flow induced by the passage of electric current has a predominantly vortical character, and increases the conductivity of the system above the intrinsic value. The magnitude of enhancement is inversely proportional to the shear viscosity of the electron liquid and scales as the square of the disorder correlation radius. This is qualitatively different from the predominantly potential flow that arises away from charge neutrality, and produces negative viscous contributions to the conductivity that are proportional to the sum of shear and bulk viscosities and inversely proportional to the square of disorder correlation radius.

I. INTRODUCTION

There is mounting evidence that under certain conditions hydrodynamic effects predicted by Gurzhi [1] play an important role in electron transport properties of semiconductors [2–6], and monolayer or bilayer graphene-based systems including, for example, one-dimensional electron liquids in quantum wires [27–29], and electronic double-layers [30–33].

One of the most salient signatures of hydrodynamic electron transport is the violation of Matthiessen’s rule, according to which the resistivity should be proportional to the sum of momentum relaxation rates due to various scattering processes. Within this paradigm momentum conserving electron-electron (ee) collisions should not affect the resistivity.

In the hydrodynamic regime the resistivity depends on the rate of momentum conserving ee scattering via the viscosity and other dissipative characteristics of the electron liquid. This dependence arises from correlations between ee scattering and the underlying disorder and/or confining potential, which are ignored in the derivation of the Matthiessen’s rule.

Moreover, as the rate of ee scattering increases the resistance of the system often decreases. This effect, first pointed out by Gurzhi, occurs not only in finite geometries, such as the Poiseuille flow [2, 3, 34], or point contacts [10, 35], but also in the bulk [36–41].

On the other hand, disorder usually increases the resistivity. Indeed, away from charge neutrality acceleration of the liquid by the external electric field would result in vanishing resistivity in the absence of disorder. In particular, for Galilean invariant liquids the systems resistivity is proportional to the disorder strength [37]. In the more general case [40, 42–46] in which the electron liquid does not possess Galilean invariance, the system resistivity away from charge neutrality is still enhanced by disorder although the dependence of the resistivity on the disorder strength is more complicated.

For systems at charge neutrality, whose resistivity does not vanish even in the pristine disorder-free state [47, 48], the effect of disorder is not obvious. In previous studies of hydrodynamic electron transport [42–45] the system resistivity at charge neutrality was found to be independent of disorder and equal to the intrinsic resistivity of the electron liquid.

Here we show that in the hydrodynamic regime long range disorder increases the conductivity of the system at charge neutrality in comparison to the intrinsic conductivity σ0 of the electron liquid. Furthermore, the enhancement of the conductivity by disorder is inversely proportional to the shear viscosity of the liquid, and has extremely nonlocal character — its magnitude grows with increasing correlation radius of the disorder potential. This is in stark contrast to the situation sufficiently far away from charge neutrality [37, 40] where the viscous contributions enhance the resistivity, and their magnitude decreases as the disorder correlation radius grows.

This opposite effect on the conductivity is caused by a qualitative difference in the character of the flow caused by the passage of electric current near, and away from charge neutrality. We show below that near charge neutrality the flow has a predominantly vortical character
The vortical character of the induced flow and the reason for the enhancement of conductivity by long range disorder can be qualitatively understood as follows. Consider a two dimensional system in the $xy$ plane subject to the electric field $E_0$ along the $x$-axis. In a pristine system at charge neutrality, electron transport exerts a macroscopic convective electric current along the $x$ direction, which enhances the conductivity.

whereas away from charge neutrality the flow is predominantly potential.

The vortical character of the induced flow and the reason for the enhancement of conductivity by long range disorder can be qualitatively understood as follows. Consider a two dimensional system in the $xy$ plane subject to the electric field $E_0$ along the $x$-axis. In a pristine system at charge neutrality, electron transport is mediated by the intrinsic conductivity in a stationary liquid. In the presence of disorder charge neutrality is satisfied only on average, but locally the electron density $\delta n(r)$ is nonzero, see Fig. 1 for the illustration of a model. The resulting force density, $F(r) = eE_0\delta n(r)$, with $e$ being the electron charge, exerted on the fluid by the electric field must be compensated in a steady state by pressure gradients and viscous stresses arising in the fluid. It is important to note that only the potential part of the external force density, which is caused by density variations, can be compensated by the pressure gradients. In contrast, the vortical part of the external force that arises from density gradients perpendicular to $E_0$ must be compensated by viscous stresses. To understand the effect of the density modulations perpendicular to the electric field let us consider for simplicity a density modulation of the form, $\delta n(y) \sim \delta n_0 \cos(y/\xi)$. The force of the electric field $eE_0\delta n(y)$ induces an inhomogeneous flow along the $x$ axis, with velocity $u_x = u_0 \cos(y/\xi)$. The velocity amplitude may be estimated as $u_0 \sim \xi^2 eE_0 \delta n/\eta$ where $\eta$ is the shear viscosity. Due to correlations of the induced hydrodynamic velocity with disorder the hydrodynamic flow gives a nonvanishing contribution to the net current that enhances its magnitude. The corresponding enhancement of the conductivity may be estimated as

$$\delta \sigma \sim \frac{\epsilon^2 \xi^2}{\eta} \langle \delta n^2 \rangle,$$  \hspace{1cm} (1)

where $\xi$ is the correlation radius of the disorder potential, and $\langle \ldots \rangle$ denotes averaging over disorder. For a checkerboard pattern of density modulation that better mimics the isotropic long range disorder the vortical flow pattern is shown in Fig. 1. The contribution of the vortical velocity variations in the induced hydrodynamic flow to the transport properties of electron systems was ignored in previous considerations, resulting in disorder-independent conductivity near charge neutrality.

Below we develop a general theory of hydrodynamic transport in a long range disorder potential without assuming Galilean invariance of the underlying electron liquid. For weak disorder we obtain general expression for the transport coefficients of the system and apply our results to study thermal and electric transport in graphene devices near charge neutrality. Our considerations shows that the flow near charge neutrality is vorticity-dominated, in contrast to nearly potential flow that arises away from charge neutrality. This has a dramatic effect on the transport characteristics of the system. In particular, the estimate in Eq. (1) for the disorder-induced conductivity enhancement is borne out by the quantitative treatment presented in the subsequent sections.

The paper is organized as follows. In Sec. II we present the hydrodynamic description of electron transport in a long range disorder potential without assuming Galilean invariance of the underlying electron liquid. In Sec. III we apply this description to the study of electron transport in the linear response regime. In Sec. IV we obtain general expressions for transport coefficients of the system for a weakly disorder potential. We apply these results to obtain the transport coefficients of graphene near charge neutrality in Sec. V. The summary of our main results is given in Sec. VI.

II. HYDRODYNAMIC DESCRIPTION

The hydrodynamic equations express conservation of the number of particles, energy, and momentum of the electron liquid. Accordingly, the time evolution of the number density $n$, energy density $\epsilon$, and momentum density $p_i$, can be expressed to divergences of the corresponding conserved fluxes, $j$, $J$, and $\Pi_{ij}$. The particle number conservation is conventionally expressed by the continu-

![FIG. 1. Illustration of the vortical flow pattern induced by the electromotive force in the $x$-direction for a checkerboard density modulation near charge neutrality, $\delta n(r) = \delta n_0 \cos(2\pi(x + y)/\xi) + \cos(2\pi(x - y)/\xi)$. The flow velocity is indicated by arrows superimposed on the color plot of the density profile, with positive/negative density shown in yellow/blue respectively. The correlations of the flow velocity with the density profile produce a macroscopic convective electric current along the $x$ direction, which enhances the conductivity.](image-url)
The final hydrodynamic equation expresses energy conservation of the electron fluid. In addition to the density of particles and momentum of particles, the energy density depends on the entropy density of the liquid. Therefore, in hydrodynamics the energy conservation equation is traditionally replaced by an equivalent evolution equation for the entropy density \[ \frac{\partial \rho_s}{\partial t} = -\nabla \cdot \mathbf{j}_s + \varsigma, \] (4)
where \( \mathbf{j}_s \) is the entropy flux \[ \mathbf{j}_s = \frac{1}{T} \left[ \mathbf{j}_s - (\mu + e\phi + U) \mathbf{j} \right], \] (5)
and \( \varsigma \) denotes the local rate of entropy production due to electron-electron collisions.

A crucial ingredient of the hydrodynamic approach is the assumption of local thermal equilibrium of the electron liquid. Accordingly, the state of the liquid is characterized by the local equilibrium parameters: temperature \( T \), chemical potential \( \mu \), and the hydrodynamic velocity \( \mathbf{u} \), whose values are determined by the local densities of conserved quantities. Furthermore, the fluxes of conserved quantities may be described in terms of the gradients of the equilibrium parameters. To keep subsequent expressions more compact it is convenient to combine the particle and entropy fluxes into a two-component column vector \[ \mathbf{J} = \begin{pmatrix} \mathbf{j} \\ \mathbf{j}_s \end{pmatrix}. \] (10)

Throughout the paper we indicate two-component column vector quantities by arrows above them, and use bold face letters to denote the usual spatial vectors. Following the conventions of Ref. [50] we denote densities of thermodynamic variables by \( x_i \) and the corresponding thermodynamically conjugate quantities by \( X_i \). Introducing the column vector notations \[ \mathbf{x} = \begin{pmatrix} n \\ s \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} -e\mathbf{E} \\ \nabla T \end{pmatrix}, \] (11)
where
\[ e\mathbf{E} \equiv -\nabla (\mu + e\phi + U) \] (12)
is the electromotive force, we can write the constitutive relations for the particle and entropy currents in the form
\[ \mathbf{J} = \mathbf{J}_T - \mathbf{J}_r. \] (13)
Here \( \mathbf{J}_r \) is the matrix of kinetic coefficients that characterizes the dissipative properties of the electron liquid. It is given by
\[ \mathbf{J}_r = \left( \begin{array}{cc} \sigma/e^2 & \gamma/T \\ \gamma/T & \kappa/T \end{array} \right). \] (14)
where $\kappa$ is the thermal conductivity, $\sigma$ is the intrinsic conductivity, and $\gamma$ is the thermoelectric coefficient of the electron liquid. Throughout the paper we use units at which Boltzmann and Planck constants are set to unity, $k_B = \hbar = 1$.

The system of hydrodynamic equations and constitutive relations presented in this section does not assume Galilean invariance and provides a general description of the flow of electron liquid in an external potential at small velocities. For Galilean-invariant liquids $j = au$, and $\sigma = \gamma = 0$. In this case the second term in the numerator of the right hand side of Eq. (8) vanishes, and Eq. (8) reproduces the well known result for the energy dissipation rate in Galilean-invariant liquids [49].

Mathematically this problem is similar to the problem of finding a current distribution in an inhomogeneous conductor, where one needs to find two vector fields: one divergence-free, current density $j(r)$, and one purely potential, electric field $E(r)$, that are related by the position-dependent conductivity $j(r) = \sigma(r)E(r)$ [51].

Once the flow pattern is determined, the rate of entropy production in the system, $\dot{S}$, may be determined by integrating Eq. (8) over space. In the column vector notations we get

$$\dot{S} = \int dr \left[ \dot{X}^T \dot{Y} \dot{X} + \sigma'_{ij} \partial_i u_j \right].$$

The macroscopic transport coefficients of the system may be obtained by expressing the entropy production rate in terms of the net (macroscopic) particle and entropy current through the system. We combine the macroscopic densities of particle and entropy flux into a column vector current

$$\left\langle \vec{J} \right\rangle = \left\langle \vec{x}(r) u(r) - \vec{Y}(r) \dot{X}(r) \right\rangle,$$

where $\left\langle \ldots \right\rangle \equiv \frac{1}{V} \int dr \ldots$ denotes spatial average. We can then express the entropy production rate in the form

$$\frac{\dot{S}}{V} = \left\langle \vec{J} \rightangle^T \hat{\rho} \left\langle \vec{J} \right\rangle,$$

where $V$ is the volume of the system, and we introduced the matrix $\hat{\rho}$ of macroscopic kinetic coefficients of the system. This matrix must satisfy the Onsager symmetry principle. The matrix $\hat{\rho}$ defines a linear relation, $\left\langle \dot{X}(r) \right\rangle = \hat{\rho} \left\langle \vec{J} \right\rangle$, between the average currents $\left\langle j \right\rangle$, and the average forces $e\left\langle E \right\rangle$, $-\left\langle \nabla T \right\rangle$. Thus the problem of determination of transport properties of the system reduces to solving the linear flow problem, calculating the rate of entropy production (18), and expressing it in terms of the net particle and entropy fluxes (19) through the system. Below we show that this problem may be solved analytically for weak disorder.

### IV. WEAK DISORDER

In the framework of the present approach the disorder potential manifests itself via the spatial dependence of the equilibrium number and entropy densities, $n$ and $s$, the matrix of kinetic coefficients of the liquid, $\vec{Y}$, and the viscosities $\eta$ and $\zeta$. In this section we assume that these quantities are weakly inhomogeneous and study electron transport using perturbation theory in disorder. The presentation in this section is organized as follows. We begin by considering a uniform liquid in Sec. IV A. Then in Sec. IV B we develop perturbation theory about the
uniform solution and obtain general perturbative expressions for the transport coefficients of the system. Finally, in Sec. IV C we obtain simplified expressions for the transport coefficients in the regime of long range disorder.

**A. Uniform liquid**

In a uniform liquid with number density $n_0$ and entropy density $s_0$ the charge and energy transport can proceed in two ways: (i) via a hydrodynamic flow with a uniform hydrodynamic velocity $u_0$, and (ii) by transport relative to the liquid driven by the thermodynamic forces $\vec{X}_0$. The momentum balance condition (16) implies that for a uniform liquid a steady state can exist only if the uniform electric field $E_0$ and temperature gradient $\nabla T_0$ satisfy the relation

$$en_0 E_0 = s_0 \nabla T_0.$$  \hspace{1cm} (21)

Under this condition the force due to the external electric field is balanced by the pressure gradient, so that no acceleration of the liquid occurs. The corresponding column force $\vec{X}_0$ may be expressed in the form

$$\vec{X}_0 = \frac{\epsilon E_0}{s_0} \left( \begin{array}{c} -s_0 \\ n_0 \end{array} \right) = -\frac{i\hat{\gamma}_y E_0}{s_0} \epsilon E_0,$$  \hspace{1cm} (22)

where the column vector $\vec{x}_0$ describes the particle and entropy density in the uniform state, see Eq. (11), and $\hat{\gamma}_y$ is the Pauli matrix acting in the $2 \times 2$ vector space. Thus, the steady state current may be written in the form

$$\vec{J}_0 = \vec{x}_0 u_0 + \hat{Y}_0 \frac{i\hat{\gamma}_y \vec{x}_0}{s_0} \epsilon E_0,$$  \hspace{1cm} (23)

The first term describes the dissipationless transport in the hydrodynamic mode caused by the uniform flow of the liquid, and the second term describes charge and energy transport in the “relative” mode that occurs in the liquid at rest. The entropy production,

$$T \dot{S} = \int d\vec{r} \left( \frac{\epsilon E_0}{s_0} \right)^2 (i\hat{\gamma}_y \vec{x}_0)^T \hat{Y}_0 (i\hat{\gamma}_y \vec{x}_0),$$  \hspace{1cm} (24)

is entirely due to the transport in the relative mode.

The dissipationless transport mode exists only in the absence of disorder. At finite disorder the relative and the hydrodynamic modes mix. This mixing and the viscous stress arising in the inhomogeneous flow cause additional dissipation. At weak disorder this dissipation is expected to be especially significant for the hydrodynamic transport mode described by the first term in Eq. (23). However, as we will show below, transport in the relative mode (on average) can be significantly modified by the mixing between the relative and hydrodynamic modes. In particular, at charge neutrality the conductivity of the system may significantly exceed the intrinsic conductivity of the electron liquid. As shown in Sec. V this can occur even at weak disorder provided its correlation radius is sufficiently long.

**B. Perturbation theory in disorder**

We now consider the situation in which the parameters $\vec{x}(r)$ and $\hat{Y}(r)$ describing the equilibrium state of the liquid are weakly inhomogeneous functions of position $\vec{r}$ (as we will show, to second order in inhomogeneity the spatial variations of the viscosities of the electron liquid may be neglected). To this end, we write

$$\vec{x}(\vec{r}) = \vec{x}_0 + \delta\vec{x}(\vec{r}), \quad \hat{Y}(\vec{r}) = \hat{Y}_0 + \delta\hat{Y}(\vec{r}),$$  \hspace{1cm} (25)

where $\vec{x}_0$ and $\hat{Y}_0$ denote the uniform components of and $\vec{x}(\vec{r})$ and $\hat{Y}(\vec{r})$ respectively, while $\delta\vec{x}(\vec{r}) \ll \vec{x}_0$ and $\delta\hat{Y}(\vec{r}) \ll \hat{Y}_0$ denote their spatial variations. We assume that the resulting hydrodynamic velocity $u(\vec{r})$ and the thermodynamic force $\vec{X}(\vec{r})$ are also nearly homogeneous,

$$u(\vec{r}) = u_0 + \delta u(\vec{r}), \quad \vec{X}(\vec{r}) = \vec{X}_0 + \delta\vec{X}(\vec{r}),$$  \hspace{1cm} (26)

with $\delta u(\vec{r}) \ll u_0$ and $\delta\vec{X}(\vec{r}) \ll \vec{X}_0$.

The inhomogeneous components of the hydrodynamic velocity $\delta u(\vec{r})$ and the driving force $\delta\vec{X}(\vec{r})$ may be determined using perturbation theory about the uniform solution. The hydrodynamic equations to be solved consist of the continuity equation for the column current, Eq. (15), and the momentum balance equation, Eq. (17).

The solution strategy can be summarized as follows. We determine the inhomogeneous part of the flow velocity field $\delta u$ and forces $\delta\vec{X}$ in terms of their uniform counterparts to linear order in $\delta\vec{x}$ and $\delta\hat{Y}$. This enables us to express the spatial average of the currents (19) in terms of $\vec{X}_0$ and $u_0$ to second order accuracy in inhomogeneity. Furthermore, the spatial average of the momentum balance equation (17) imposes a linear relation between the inhomogeneous Fourier components of the column current $\vec{J}(\vec{q})$ and the driving force $\vec{X}(\vec{q})$ in the form

$$\langle \vec{J} \rangle = \hat{\gamma}_e \vec{X}_0.$$  \hspace{1cm} (27)

Here $\hat{\gamma}_e$ is the effective disorder-renormalized thermoelectric conductivity matrix of the system. Inverting the matrix one obtains the resistivity $\hat{q}$ defined in Eq. (20).

To implement this program we switch to the Fourier representation, defining the Fourier amplitudes of various quantities $O(\vec{r})$ in the standard way,

$$O_q = \int d\vec{r} O(\vec{r}) e^{-i\vec{q} \cdot \vec{r}}.$$  \hspace{1cm} (28)

To linear order accuracy in the perturbations we get for the inhomogeneous Fourier components of the column current

$$\vec{J}_q = \vec{x}_q u_0 + \vec{x}_0 u_q - \hat{Y}_q \vec{X}_0 - \hat{Y}_0 \vec{X}_q.$$  \hspace{1cm} (29)

The continuity equations for the heat and particle currents, Eq. (15), then become

$$\vec{x}_0 (q \cdot \vec{u}_q) - \hat{Y}_0 (q \cdot \vec{X}_q) = -\vec{x}_q (q \cdot \vec{u}_0) + \hat{Y}_q (q \cdot \vec{X}_0).$$  \hspace{1cm} (30)
The force balance equation (17) imposes the following relation on the Fourier components of velocity and thermodynamic forces with \( \mathbf{q} \neq 0 \),

\[
\eta q^2 \mathbf{u_q} + \zeta (\mathbf{q} \cdot \mathbf{u_q}) + \mathbf{x_0}^T \mathbf{X_q} + \mathbf{x_0}^T \mathbf{X_q} = 0.
\]  

(31)

The system of equations (30) and (31) determines the inhomogeneous components of the hydrodynamic velocity, \( \mathbf{u_q} \), and electric field/temperature gradient, \( \mathbf{X_q} \), in terms of the macroscopic hydrodynamic velocity, \( \mathbf{u_0} \), and macroscopic electric field/temperature gradient, \( \mathbf{X_0} \). However, since the macroscopic flow is characterized by only two macroscopic currents (particle and entropy flux, \( \langle \mathbf{j} \rangle \), \( \langle \mathbf{j_s} \rangle \)), the average velocity \( \mathbf{u_0} \) is not independent from \( \mathbf{X_0} \). The relation between them can be obtained by considering the zeroth Fourier component of the force balance equation (17), which can be written in the form

\[
-x_0^T \mathbf{X_0} - \mathbf{u_0} = 0.
\]  

(32)

Substituting the solutions of Eqs. (30) and (31) into (32) one can express average velocity \( \mathbf{u_0} \) in terms of \( \mathbf{X_0} \) to second order accuracy in disorder.

The solution of the system of linear equations (30)–(31) is given by

\[
\begin{align*}
\mathbf{X_q} &= \frac{-\mathbf{q}}{q^2 \lambda_q} \mathbf{Y_0}^{-1} \left( \left[ \lambda_q - \mathbf{x_0} \otimes \mathbf{x_0}^T \mathbf{Y_0}^{-1} \right] \mathbf{x}(\mathbf{q} \cdot \mathbf{u_0}) + \left[ \left( \mathbf{x_0} \otimes \mathbf{x_0}^T \mathbf{Y_0}^{-1} - \lambda_q \right] \mathbf{Y_0} - \mathbf{x_0} \otimes \mathbf{x_0}^T \right] (\mathbf{q} \cdot \mathbf{X_0}) \right), \\
\mathbf{u_q}' &= -\frac{1}{\eta q^2} \mathbf{x_0}^T \left[ \mathbf{X_0} - \frac{\mathbf{q} (\mathbf{q} \cdot \mathbf{X_0})}{q^2} \right], \\
\mathbf{u_q}' &= -\frac{\mathbf{x_0}^T \mathbf{Y_0}^{-1} \mathbf{Y_0}^{-1} \mathbf{x_0}}{\lambda_q q^2} + \mathbf{x_0}^T \mathbf{Y_0}^{-1} \mathbf{Y_0}^{-1} \mathbf{x_0} \frac{\mathbf{q} (\mathbf{q} \cdot \mathbf{X_0})}{q^2}. \tag{33a, 33b, 33c}
\end{align*}
\]

Here \( \mathbf{u_q}' \) and \( \mathbf{u_q}' \) denote, respectively, the transverse and longitudinal components of the hydrodynamic velocity and \( \lambda_q \) is a function of momentum \( q \) given by

\[
\lambda_q = (\eta + \zeta) q^2 + \mathbf{x_0}^T \mathbf{Y_0}^{-1} \mathbf{x_0}.
\]  

(34)

We indicate transposition of column vectors by superscript \( T \) and use the standard notation for the direct product of two vectors \( \mathbf{a} \otimes \mathbf{b}^T \) that defines a corresponding matrix.

Substituting the result (33a) for \( \mathbf{X_q} \) into the macroscopic momentum balance equation (32) we obtain the following relation between \( \mathbf{u_0} \) and \( \mathbf{X_0} \)

\[
\begin{align*}
x_0^T \mathbf{X_0} &= -\frac{\mathbf{u_0}}{2} \int q \left[ \mathbf{x_0}^T \mathbf{Y_0}^{-1} \mathbf{x_0} - \frac{\mathbf{x_0}^T \mathbf{Y_0}^{-1} \mathbf{x_0}^2}{\lambda_q} \right] \\
&+ \frac{1}{2} \int q \left[ \mathbf{x_0}^T \mathbf{Y_0}^{-1} \mathbf{Y_0} - \mathbf{x_0}^T \mathbf{Y_0}^{-1} \mathbf{x_0} \right] \mathbf{X_0} + \frac{1}{2} \int q \left[ \mathbf{x_0}^T \mathbf{Y_0}^{-1} \mathbf{x_0} \right] \left[ \mathbf{x_0}^T \mathbf{Y_0}^{-1} \mathbf{x_0} \right] \mathbf{X_0}, \tag{35}
\end{align*}
\]

where the factor \( 1/2 \) arises from the projection into the longitudinal and transverse components that is specific to two dimensions, and consequently, \( \int q \ldots = \int q^2 (2\pi)^d \ldots \), denotes an integral over the wavevectors in \( d = 2 \).

Equation (35) expresses force balance on spatial scales large in comparison to the disorder correlation radius. Comparing this equation with Eq. (17) we see that the last two terms in the right-hand-side of Eq. (35) describe renormalization of the effective densities of particle number and entropy by disorder. In contrast, the first term in the right-hand-side of Eq. (35) describes the emergent friction force due to the macroscopic flow of the electron liquid, \( \mathbf{F_f} = -k \mathbf{u_0} \). Thus the spatially uniform part of the force balance equation, Eq. (35), can be written in the form

\[
-x_0^T \mathbf{X_0} - k \mathbf{u_0} = 0. \tag{36}
\]

Here the column-vector \( \mathbf{x_0} \) describing disorder-renormalized particle and entropy densities is given by

\[
\begin{align*}
\mathbf{x_0} &= \mathbf{x_0} - \frac{1}{2} \int q \left[ \mathbf{Y_0}^{-1} \mathbf{x_0} \right] \mathbf{x_0} \\
&- \frac{\mathbf{x_0}^T \mathbf{Y_0}^{-1} \mathbf{x_0}}{\lambda_q} \mathbf{Y_0}^{-1} \mathbf{Y_0}^{-1} \mathbf{x_0} \mathbf{Y_0}^{-1} \mathbf{x_0}, \tag{37}
\end{align*}
\]

and the friction coefficient \( k \) is given by

\[
k = \frac{1}{2} \int q \mathbf{x_0} \mathbf{K_q} \mathbf{x_0}, \tag{38}
\]

where we introduced the matrix

\[
\mathbf{K_q} = \mathbf{Y_0}^{-1} - \frac{1}{\lambda_q} \mathbf{Y_0}^{-1} \mathbf{x_0} \otimes \mathbf{x_0}^T \mathbf{Y_0}^{-1}. \tag{39}
\]

The friction coefficient \( k \) is positive definite, as we show below. Using the notations (37) and (38) the spatial average of the hydrodynamic velocity \( \mathbf{u_0} \) can be expressed in terms of the average electric field and temperature gradient \( \mathbf{X_0} \) from Eq. (36).

1. Macroscopic thermoelectric conductivity matrix

We are now in a position to evaluate the transport coefficients of the system. This can be done in two equivalent
ways: (i) by expressing the macroscopic particle and entropy current $\langle \mathbf{J} \rangle$ in terms of $\mathbf{X}_0$, or (ii) by evaluating the entropy production rate in terms of $\mathbf{X}_0$.

We begin by following the first route. To this end, we express the macroscopic particle and entropy currents, $\langle \mathbf{J} \rangle$, in terms of $\mathbf{X}_0$ and $u_0$. Rewriting next Eq. (19) in terms of the integrals in reciprocal space of wavenumbers

$$\langle \mathbf{J} \rangle = \bar{x}_0 u_0 - \bar{\Upsilon}_0 \bar{X}_0 + \int_q \left( \bar{x}_q u_q - \bar{\Upsilon}_{-q} \bar{X}_q \right),$$

(40)

and using the linear response solutions from Eq. (33), we can express the macroscopic particle and entropy currents in the systems in terms of $\bar{X}_0$ and $u_0$ in the form

$$\langle \mathbf{J} \rangle = \bar{x}_0 u_0 - \bar{\Upsilon}_0 \bar{X}_0 - \frac{1}{2} \int_q \left[ \frac{1}{\lambda_q} \bar{\Upsilon}_{-q} \bar{\Upsilon}_0^{-1} \left( \bar{x}_0 \otimes \bar{x}_0^T \bar{\Upsilon}_0^{-1} - \lambda_q \right) \bar{\Upsilon}_q 
- \frac{1}{\lambda_q} \left( \bar{x}_{-q} \otimes \bar{x}_0^T \bar{\Upsilon}_0^{-1} \bar{\Upsilon}_q + \bar{\Upsilon}_{-q} \bar{\Upsilon}_0^{-1} \bar{x}_0 \otimes \bar{x}_q^T \right) 
+ \left( \frac{1}{\eta q^2} + \frac{1}{\lambda_q} \right) \bar{x}_{-q} \otimes \bar{x}_q^T \right] \bar{X}_0. \quad (41)$$

Here the contributions proportional to $u_0$ were expressed in terms of the column vector of disorder-renormalized densities defined in Eq. (37). Finally, substituting $u_0$ from Eq. (36) into last equation we obtain Eq. (27) with the matrix of effective kinetic coefficients of the medium $\hat{\Upsilon}_e$ given by

$$\hat{\Upsilon}_e = \frac{1}{k} (\bar{x}_e \otimes \bar{x}_e^T) + \bar{\Upsilon}_0$$
$$+ \frac{1}{2} \int_q \left[ \frac{1}{\lambda_q} \bar{\Upsilon}_{-q} \bar{\Upsilon}_0^{-1} \left( \bar{x}_0 \otimes \bar{x}_0^T \bar{\Upsilon}_0^{-1} - \lambda_q \right) \bar{\Upsilon}_q 
- \frac{1}{\lambda_q} \left( \bar{x}_{-q} \otimes \bar{x}_0^T \bar{\Upsilon}_0^{-1} \bar{\Upsilon}_q + \bar{\Upsilon}_{-q} \bar{\Upsilon}_0^{-1} \bar{x}_0 \otimes \bar{x}_q^T \right) 
+ \left( \frac{1}{\eta q^2} + \frac{1}{\lambda_q} \right) \bar{x}_{-q} \otimes \bar{x}_q^T \right]. \quad (42)$$

The matrix in the above expression is symmetric in agreement with the Onsager symmetry principle of kinetic coefficients.

2. Entropy production rate

Equivalently, we can obtain the matrix of kinetic coefficients $\hat{\Upsilon}_e$ by evaluating the entropy production rate and setting $T S = V \mathbf{X}_0^T \hat{\Upsilon}_e \mathbf{X}_0$. Using Eq. (18) we get

$$\bar{X}_0^T \hat{\Upsilon}_e \mathbf{X}_0 = \bar{X}_0^T \bar{\Upsilon}_0 \bar{X}_0 + \int_q \left[ \bar{X}_{-q}^T \bar{\Upsilon}_0 \bar{X}_q + 2 \bar{X}_0^T \bar{\Upsilon}_{-q} \bar{X}_q \right] \eta q^2 (u_q^T \cdot u_{-q}^T) + \eta \zeta q^2 (u_q^T \cdot u_{-q}^T). \quad (43)$$

Substituting Eqs. (33) into this equation, and using Eqs. (37)–(39) we reproduce the result of Eq. (42) for $\hat{\Upsilon}_e$.

Let us now summarize the results of this subsection. Equation (42), together with Eqs. (34), (37), (38), and (39), express the thermoelectric conductivity matrix of the system in terms of the position-dependent matrix of kinetic coefficients of the liquid, $\bar{\Upsilon}(r)$, and densities of particles and entropy, $\bar{x}(r)$, in the equilibrium state. Note that the conductivity of the system cannot be expressed in terms of local fluctuations of $\bar{x}(r)$ and $\bar{\Upsilon}(r)$.

C. Long range disorder

We now apply the general perturbative results of Sec. IV B to an experimentally relevant case of long range disorder. This situation can be realized in high mobility semiconductor quantum wells with modulation doping and boron nitride encapsulated graphene devices [6–13].

In this case the expressions for the transport coefficients of the system obtained in the previous section may be simplified significantly by selecting the terms that scale as leading powers of the correlation radius $\xi$ of the disorder potential. More specifically, we assume that the correlation radius satisfies the condition

$$\varepsilon \equiv \frac{1}{\xi^2} \frac{\eta + \zeta}{\bar{x}_0^T \bar{\Upsilon}_0^{-1} \bar{x}_0} \ll 1 \quad (44)$$

and obtain the transport coefficients of the system to leading order in $\varepsilon$.

It is important to note that when the condition (44) is satisfied the transverse component of the velocity in Eq. (33b), which corresponds to the vortical flow, exceeds the last term in Eq. (33c). The latter corresponds to the potential component of the flow caused by the thermodynamic forces $\bar{X}_0$. This implies that in transport measurements dominated by the relative mode, for which the macroscopic flow velocity $u_0$ is small, the inhomogeneous part of the flow induced on spatial scale of order of the correlation radius $\xi$ is primarily vortical. In particular, this situation is realized in charge transport near the neutrality point, as was qualitatively discussed in the introduction. For transport measurements in which the macroscopic hydrodynamic flow characterized by the velocity $u_0$ plays a substantial role, the first term in Eq. (33c) may exceed the vortical contribution (33b) rendering the flow mostly potential. This situation is realized for charge transport sufficiently far away from charge neutrality [37, 40].

Below we obtain the transport coefficients that are valid in the entire crossover region between these two regimes. We work to leading order accuracy in the parameter $\varepsilon$ in Eq. (44). In this approximation we may neglect the wave number dependence of $\lambda_q$ in Eq. (34), setting $\lambda_q \rightarrow \lambda_0$ in the subsequent expressions of the previous section.

In particular, the matrix $\hat{K}_q$ in Eq. (39) can be reduced
to a more compact expression
\[\hat{K}_0 = \frac{1}{(i\gamma x_0)\mathbf{T}_0(i\gamma x_0)} \mathbf{T}_0(i\gamma x_0) \otimes (i\gamma x_0)\mathbf{T}^+\]
\[= \frac{1}{n_0^2/s_0^2 - 2\gamma_0 n_0 s_0 + n_0^2 n_0^2} \begin{pmatrix} s_0^2 & -n_0 s_0 \\ -n_0 s_0 & n_0^2 \end{pmatrix}, \quad (45)\]
As a result, Eq. (38) for the friction coefficient simplifies to
\[k = \frac{\langle (s_0 \delta n - n_0 \delta s)^2 \rangle}{2(n_0^2/s_0^2 - 2\gamma_0 n_0 s_0 + n_0^2 n_0^2)}. \quad (46)\]
The friction coefficient \(k\) here is positive definite because the denominator in Eq. (46) is positive. The latter statement follows from the fact that the matrix \(\hat{Y}\) of kinetic coefficients of the liquid is positive definite. Furthermore, from the positivity of \(k\) in Eq. (46) it follows that the friction coefficient in the general expression of Eq. (38) is also positive definite [52].

Next, we note that in the long wavelength approximation (44) the column vector of renormalized densities \(\bar{x}_0\) in Eq. (37) can be expressed using the matrix \(\hat{K}_0\) in Eq. (45) in terms of the local fluctuations of densities, \(\delta\bar{x}\) and kinetic coefficients, \(\delta\hat{Y}\), in the form
\[\bar{x}_0 = \bar{x}_0 - \frac{1}{2} \delta\hat{K}_0 \delta\bar{x} - \frac{1}{2\lambda_0} \delta\bar{x} \otimes \delta\bar{x}^T \hat{Y}_0^{-1} \bar{x}_0. \quad (47)\]

Let us now turn to the thermoelectric conductivity matrix. Note that since the friction coefficient \(k\) is quadratic in the disorder amplitude the first two terms in Eq. (42) do not vanish in the limit of weak disorder. In contrast, the remaining terms in (42) are proportional to the variance of the disorder potential and are generally small in comparison to the first two. However, while most of these terms can be expressed in terms of local correlators, and are independent of disorder correlation radius \(\xi\), the first term in the last line of Eq. (42), which is inversely proportional to the shear viscosity \(\eta\), scales as \(\xi^2\). Therefore, for systems where the correlation radius of disorder satisfies the condition (44) we may neglect all terms in Eq. (42) that are proportional to the disorder variance except for the first term in the last line. Doing so we obtain
\[\hat{\Gamma}_e = \frac{1}{k} (\bar{x}_e \otimes \bar{x}_e^T) + \hat{\Gamma}_0 + \frac{1}{2\gamma_0} (\delta\bar{x} \otimes \delta\bar{x}^T) \hat{Y}_0^{-1}. \quad (48)\]
Here \(k\) and \(\bar{x}_e\) are given, respectively, by Eqs. (46) and (47), and within the accuracy of our approximation only terms of zeroth and second order in the fluctuations should be retained in \(\bar{x}_e \otimes \bar{x}_e^T\). The matrix in (48) is obviously positive definite.

The electrical resistivity of the system can be expressed in terms of the 11 matrix element of \(\hat{\Gamma}_e\) in Eq. (48) as \(\rho = 1/e^2[\hat{\Gamma}_e]_{11}\). To leading accuracy in the disorder strength and in the long wavelength limit this yields \(\rho \approx k/(e^2 n_0^2)\). Substituting here \(k\) from Eq. (46) we get
\[\rho = \frac{1}{2e^2} \frac{T \langle (\delta s/n)^2 \rangle}{\kappa_0 - 2\gamma_0 n_0 + T \frac{\alpha_0}{n_0} (\frac{\alpha_0}{n_0})^2}. \quad (49)\]
This expression generalizes the result of Ref. [37] in the long wavelength limit. Setting \((\sigma_0, \gamma_0) = 0\) in Eq. (49) we reproduce the Galilean-invariant result [37] in the long wavelength limit.

The expressions for the effective friction coefficient, Eq. (46), disorder-renormalized densities, Eq. (47), and thermoelectric conductivity matrix, (48), are the main results of this subsection. They are applicable to systems with weak, \((\delta n, \delta s) \ll \max\{n_0, s_0\}\), long range disorder whose correlation length \(\xi\) satisfies the condition (44).

Note that the first term in Eq. (48) has the form of a projection operator on the column vector of effective densities of particles and entropy in Eq. (47). This term may be identified with disorder-renormalized hydrodynamic transport mode. The corresponding conductivity is inversely proportional to the friction coefficient \(k\) in Eq. (46), and diverges at vanishing disorder. The remaining terms in Eq. (48) are associated with disorder-renormalized transport mode relative to the liquid. The first of these terms arises from the intrinsic transport relative to the liquid. The second term represents the contribution of convective vortical flow of particles and entropy induced on spatial scales of order of the correlation radius of disorder. Although this contribution is proportional to the disorder strength it grows as \(\xi^2\) as the correlation radius increases. Therefore, the perturbative smallness is compensated in this term by the large parameter \(1/\varepsilon\).

The conditions \((\delta n, \delta s) \ll \max\{n_0, s_0\}\) and Eq. (44) that define applicability of Eqs. (47)–(48) ensure that the neglected nonlocal corrections to the hydrodynamic mode and to the relative mode separately, are relatively small. However, in a particular transport setup both of these two modes may provide a contribution. In this case our approximation, which involves retaining the last two terms in Eq. (48) while neglecting the nonlocal corrections to the first term, requires further justification. Let us consider the electrical conductivity of the system as an example. The nonlocal corrections to the main term in Eq. (48) come from two sources: (i) corrections to the friction coefficient, and (ii) corrections to the effective density. Using the form of the friction matrix in Eq. (39) the nonlocal correction to the friction coefficient in Eq. (38) can be estimated as \(\delta k \sim (\delta n^2)(\eta + \zeta)/\xi^2\). This yields the correction to the conductivity \(\delta \sigma_\perp \sim n_0^2 \delta k/k^2\). The modification of conductivity due to nonlocal corrections to the effective density are of the same order of magnitude. On the other hand, the conductivity enhancement due to vortical viscous flow that is determined by the third term in Eq. (48) is estimated as \(\delta \sigma_\perp \approx \xi^2 (\delta n^2)/\eta\). Thus the last term in Eq. (48) exceeds the nonlocal corrections to the first term provided the following condition is satisfied,
\[n_0^2 < \frac{\xi^2 k \lambda_0}{\sqrt{\eta(\eta + \zeta)}}. \quad (50)\]
Note here that to the leading order \(k \sim (\delta n^2)\) per Eq. (46). We will consider the implications of the above con-
tion in greater detail in the next section where we analyze electron transport in graphene near charge neutrality.

V. GRAPHENE NEAR CHARGE NEUTRALITY

We now apply the results of Sec. IV to study the thermoelectric properties of graphene near the charge neutrality point. Our perturbative approach assumes that the variations of density are small, \( \langle \delta n^2 \rangle \ll s_0^2 + n_0^2 \). This condition can be satisfied in boron nitride encapsulated graphene devices. Furthermore, it is easy to see that for graphene near charge neutrality the condition (44) is satisfied in the regime of applicability of the hydrodynamic approximation, \( v/(T \xi) \ll 1 \). Therefore transport properties of graphene devices can be investigated using the long range approximation of Sec. IV C. Indeed, near charge neutrality we can estimate \( \eta \sim T^2/(v \xi)^2 \), where \( v \) is the band velocity in graphene, and \( \xi \) is the correlation length of the disorder potential. Here we took \( \eta \sim T^2/v^2 \) [53] and suppressed all logarithmic renormalizations in \( \eta \) which are beyond the accuracy of these estimates.

One can further simplify the expressions of Sec. IV C by neglecting small terms of order \( n_0/s_0 \), which is valid in the regime near charge neutrality, \( n_0 \ll s_0 \). In this approximation the friction coefficient, \( k_0 = k(n_0 \to 0) \), in Eq. (38) can be expressed in terms of the intrinsic conductivity and the electron density variance as follows

\[
k_0 = \frac{e^2 \langle \delta n^2 \rangle}{\sigma_0}.
\]

Furthermore, the expression (47) for the vector of renormalized densities simplifies to

\[
\vec{x}_e = \left( s_0 - \frac{n_0}{e^2 \langle \delta n^2 \rangle} \right) \cdot \langle \delta n \delta \gamma \rangle.
\]

Substituting the expressions (51) and (52) for the friction coefficient and disorder-renormalized densities of graphene into Eq. (48) we can express the thermoelectric conductivity matrix in the form

\[
\hat{\chi} \approx \frac{2\sigma_0}{e^2 \langle \delta n^2 \rangle} \left( \begin{array}{cc} n_0^2 & n_0 s_0 \\ n_0 s_0 & s_0^2 \end{array} \right) - \frac{\langle \delta n \delta \gamma \rangle}{T \langle \delta n^2 \rangle} \left( \begin{array}{cc} n_0 & 0 \\ 0 & 2s_0 \end{array} \right) + \left( \begin{array}{cc} \sigma_0/e^2 + \chi & \gamma_0/T \\ \gamma_0/T & \kappa_0/T \end{array} \right),
\]

where we introduced a dimensionless quantity

\[
\chi = \frac{1}{2\eta} \int \frac{d^2q}{(2\pi)^2} |n_q|^2.
\]

The electrical conductivity of the system is defined by the 11 element of the matrix \( \hat{\chi} \) in Eq. (53), and is given by

\[
\sigma = \sigma_0 + e^2 \chi + \sigma_0 \frac{2n_0^2}{\langle \delta n^2 \rangle}.
\]

Note that long range correlated disorder enhances the conductivity of the system at charge neutrality.

To obtain the thermoelectric properties of the systems we reexpress the linear relation (27) with the matrix coefficients (53) in a more familiar form [54],

\[
\langle \mathcal{E} \rangle = \rho_e \langle \mathbf{j} \rangle + Q_e \langle \nabla T \rangle,
\]

\[
T \langle \mathbf{j} \rangle = \Pi_e \langle \mathbf{j} \rangle - \kappa_e \langle \nabla T \rangle.
\]

Here \( \rho_e \) and \( \kappa_e \) are the electrical resistivity and the thermal conductivity of the system, and \( Q_e \) and \( \Pi_e \) are, respectively, the Seebeck coefficient (thermopower) and the Peltier coefficient. The latter are related by the Onsager symmetry relation \( \Pi_e = Q_e T \).

For the Peltier and Seebeck coefficients we obtain to
The density dependence of the Lorentz ratio in Eq. (59) (scaled with the Wiedemann-Franz value $L_{WF} = \pi^2/3k^2$) for three different values of the disorder-induced density fluctuations as quantified by the parameter $\Gamma = \sqrt{\frac{1}{2} \langle \delta n^2 \rangle (1 + \frac{\sigma_n}{\sigma_0})}$.

The Seebeck coefficient $Q_e$ is given by the entropy per unit charge that is transported by the current. At relatively large doping, $n_0^2 \gg \langle \delta n^2 \rangle$, it approaches the value in the pristine electron liquid, $Q_0 = \frac{1}{T} \frac{\sigma_n}{\sigma_0}$, but is always reduced from it. This reduction is especially strong, near charge neutrality. The doping dependence of the Seebeck coefficient is illustrated in Fig. 2.

Finally, for the thermal conductivity we obtain

$$\kappa_e = \frac{\Pi_e}{T} = \frac{1}{e^2} \frac{2n_0 s_0}{\langle \delta n^2 \rangle} \left[ 1 + \frac{\sigma_n}{\sigma_0} + \frac{2n_0^2}{\langle \delta n^2 \rangle} \right]. \quad (57)$$

This yields the Lorentz ratio $L_e = \kappa_e \Pi_e / T$ in the form

$$L_e = \frac{2s_0}{e^2 \langle \delta n^2 \rangle} \left[ 1 + \frac{\sigma_n}{\sigma_0} + \frac{2n_0^2}{\langle \delta n^2 \rangle} \right]^2. \quad (59)$$

The density dependence of the Lorentz ratio in Eq. (59) is illustrated in Fig. 3.

Equations (53), (55), and (57)-(59) for the transport coefficients of graphene near charge neutrality represent the main results of this section. Note that the disorder affects the transport coefficients of the system in two qualitatively different ways: (i) via the friction coefficient $k$ in Eq. (51), and (ii) via the parameter $\chi$ in Eq. (54).

(i) The first part of the dependence can be obtained by introducing friction force in the hydrodynamic equations [42–44, 55–58]. This results in viscosity-independent transport coefficients that correspond to setting $\chi \to 0$ in our equations. In particular, the conductivity of the system at charge neutrality is unaffected by disorder and is equal to the intrinsic conductivity of the electron liquid [42–44]. The reason is that within such an approach at charge neutrality electron transport is decoupled from the hydrodynamic flow at charge neutrality.

(ii) The dependence of the transport coefficients on the shear viscosity of the electron liquid is described by the parameter $\chi$ in Eq. (54). Although $\chi$ is proportional to the disorder strength it scales with the correlation radius as $\xi^2$, and thus is inversely proportional to the small parameter $\varepsilon$ in Eq. (44). This shows that the effect of the long range disorder on hydrodynamic electron transport near charge neutrality is extremely nonlocal. This is qualitatively different from the situation away from charge neutrality, where the resistivity becomes independent of the correlation radius for long range disorder [37, 40]. The dependence of the transport coefficients on $\chi$ arises from the vortical component of the inhomogeneous flow described by Eq. (33b). This vortical flow gives a positive contribution to the conductivity of the system that is inversely proportional to the shear viscosity $\eta$.

Note that accounting for the dependence of $\lambda_q$ on the wavenumber $q$ in our general expressions would also produce viscosity-dependent corrections to the resistivity of the system. These corrections arise from the oscillatory component of the flow, and are scale as $(\eta + \zeta)/\xi^2$, c.f. Refs. [37, 40, 41]. Their contribution to the electrical conductivity is proportional to $n_0^2$ and becomes essential far away from charge neutrality. The condition that these corrections are small in comparison to the second term in Eq. (55) is expressed by Eq. (50). Using Eq. (51) and the estimates at the beginning of this section it is easy to see that the applicability condition (50) reduces to

$$|n_0| \lesssim \frac{\xi}{l_T} \sqrt{\langle \delta n^2 \rangle}, \quad l_T = \frac{v}{T}. \quad (60)$$

This condition is satisfied in a parametrically wide range of electron doping near charge neutrality, in which the viscous contribution to resistivity is dominated by vortical flow at spatial scales comparable to $\xi$. This flow produces a positive contribution to the electrical conductivity of the system, which is proportional to the square of the disorder correlation radius. In contrast, sufficiently far from charge neutrality, the viscous contribution to the conductivity arises predominantly from the nonlocal corrections to Eq. (53) due the momentum dependence of $\lambda_q$ in the general expression (42). They correspond to the longitudinal component of the hydrodynamic flow, and give a positive contribution to the resistivity that is proportional to the combination $(\eta + \zeta)/\xi^2$ [37, 40, 41].

It is important to note that the dimensionless parameter $\chi$ in Eq. (54), which can be estimated near charge neutrality as $\chi \sim (\xi l_T^2)/(\delta n^2)$, may become large even at weak, $\langle \delta n^2 \rangle \ll s_0^2$, but sufficiently long range disorder, $\xi \gg l_T$. This imposes additional constraints on the applicability of our results. They may be obtained by considering an “isotropic” extension of the toy model from the qualitative discussion in Sec. 1.
with a checkerboard pattern of inhomogeneous density, 
\[ n(r) = \delta n_0 [\cos(y/\xi) + \cos(x/\xi)]. \]
In this case, in addition to the viscous force an inhomogeneous flow will cause the friction force, whose magnitude can be estimated from Eq. (51) as \[ ku_0 \sim u_0 e^2 / \sigma_0 \delta n_0^2. \] Our results obtained in the long wavelength approximation of Sec. (IV C) apply as long as the friction force is smaller than the viscous force, \( e^2 \xi^2 \langle \delta n^2 \rangle / \eta \lesssim \sigma_0. \) This imposes the following constraint, 
\[ \chi \lesssim \sigma_0 / e^2 \tag{61} \]
on the applicability of our results. For Euclidean potentials, the constraint (61) on the applicability of our perturbative results can also be verified by considering higher order terms in the perturbation theory.

VI. SUMMARY

In this paper we presented a hydrodynamic theory of electron transport in the conductors with a long range disorder potential. It generalizes the approach of Ref. [37] to conductors in which the underlying electron liquid lacks Galilean invariance.

For weak disorder the matrix of kinetic coefficients of the system defined by Eq. (27) can be expressed in terms of the position-dependent densities of particles and entropy Eq. (11) and intrinsic kinetic coefficients Eq. (14) of the liquid in the form of Eq. (42).

For a long-range disorder potential, whose correlation radius satisfies the condition Eq. (44), the transport coefficients may be expressed in a simplified form Eq. (48) in terms of the friction coefficient Eq. (46), disorder-renormalized particle- and entropy-densities, and an additional viscosity dependent parameter \( \chi \) defined in Eq. (54).

For graphene devices subjected to long range disorder the transport coefficients are described by Eqs. (55)–(59). They show strong violation of Wiedemann-Franz law and the breakdown of the Mott formula indicative of hydrodynamic regime. Our results for the transport coefficients depend on the shear viscosity \( \eta \) of the electron liquid, and differ from those of the previous studies of hydrodynamic electron transport in graphene. In particular, the conductivity at charge neutrality, \( n_0 \rightarrow 0 \) in Eq. (55), is enhanced in comparison to the intrinsic conductivity of the electron liquid. The conductivity enhancement arises from the convective charge transport by the vortical component of the hydrodynamic flow.

Acknowledgments

We gratefully acknowledge illuminating discussions with I. Aleiner, K. C. Fong, P. Kim, S. Kivelson, L. Levitov, K. A. Matveev, A. Principi, and B. Spivak of various physical phenomena relevant to this work. This work was supported by the U.S. Department of Energy Office of Science, Basic Energy Sciences under Award No. DE-FG02-07ER46452 and by the National Science Foundation Grant MRSEC DMR-1719797 (A. V. A). S. L. and A. L. acknowledge support by the National Science Foundation CAREER Grant DMR-1653661 and the Ray MacDonald Endowment Award. This work was performed in part at Aspen Center for Physics, which is supported by National Science Foundation Grant PHY-1607611.

Appendix: entropy production rate

In this appendix we derive the expression for the local entropy production rate in Eq. (8). Assuming local thermal equilibrium we can characterize the state of the liquid by the densities of conserved quantities: particle number \( n \), energy \( \epsilon \), and momentum \( p \). The entropy density \( s \) is a function of the conserved quantities, and its differential is given by the thermodynamic relation
\[ ds = \frac{dc}{T} - \frac{(\mu + e\phi + U)dn}{T} - \frac{u \cdot dp}{T}. \tag{62} \]

Where \( U \) is the external potential, and the local equilibrium parameters \( T, \mu, \) and \( u \) are functions of \( n, s \) and \( p \) which depend on the band structure. The electric potential \( \phi \) is determined by the density of electrons and external charges,
\[ e\phi(r) = \int dr' e^2 \left[ n(r') + n_{\text{ext}}(r') \right] / |r - r'|. \tag{63} \]

Using the thermodynamic relation Eq. (62), and the evolution equations (2) and (3) for the conserved quantities, we get
\[ \partial_t s = -\frac{1}{T} \left[ \nabla \cdot j_e - (\mu + e\phi + U) \nabla \cdot j \right. \]
\[ \left. - u_i \partial_j \Pi_{ij} - n u \cdot \nabla (e\phi + U) \right]. \tag{64} \]

Using Eqs. (5) and (6c) we can rewrite the evolution equation (64) in the form of Eq. (4) where the expression for the local entropy production rate \( \varsigma \) is given by
\[ \varsigma = \nabla \cdot \frac{1}{T} \left[ j_e - j \right] \cdot \nabla \left( \frac{\mu + e\phi + U}{T} \right) \]
\[ + u \cdot \nabla P + nu \cdot \nabla (e\phi + U) - u_i \partial_j \sigma'_{ij}. \tag{65} \]

As the next step, using the thermodynamic identity \( \nabla P = n \nabla \mu + s \nabla T \) we can rewrite Eq. (65) in the form
\[ \varsigma = -\frac{1}{T} \left[ (j_e - su) \cdot \nabla T + (j - nu) \cdot \nabla (\mu + e\phi + U) \right] \]
\[ - u_i \partial_j \sigma'_{ij} / T. \tag{66} \]
Substituting the definitions (6a) and (9) of the dissipative particle and entropy fluxes we obtain the local entropy production rate in the form of Eq. (8).
[42] S. A. Hartnoll, P. K. Kovtun, M. Müller, and S. Sachdev, Phys. Rev. B 76, 144502 (2007).
[43] M. Müller, L. Fritz, and S. Sachdev, Phys. Rev. B 78, 115406 (2008).
[44] M. S. Foster and I. L. Aleiner, Phys. Rev. B 79, 085415 (2009).
[45] B. N. Narozhny, I. V. Gornyi, M. Titov, M. Schütt, A. D. Mirlin, Phys. Rev. B 91, 035414 (2015).
[46] A. A. Patel, R. A. Davison, and A. Levchenko, Phys. Rev. B 96, 205417 (2017).
[47] A. Kashuba, Phys. Rev. B 78, 085415 (2008).
[48] L. Fritz, J. Schmalian, M. Müller, and S. Sachdev, Phys. Rev. B 78, 085416 (2008).
[49] L. D. Landau and E. M. Lifshitz, Fluid Mechanics: Volume 6 Course of Theoretical Physics Series (Butterworth-Heinemann, 2 edition, 1987).
[50] L. D. Landau and E. M. Lifshitz, Statistical Physics: Volume 5 Course of Theoretical Physics Series (Butterworth-Heinemann, 3 edition, 2013).
[51] A. M. Dykhne, Zh. Eksp. Teor. Fiz. 59, 110 (1970). [Sov. Phys. JETP 32, 3263 (1971)].
[52] Indeed, the long wavelength approximation underestimates the denominator in the second term in the first line of Eq. (38). Thus, the long wavelength approximation (46) provides a positive lower bound for the friction coefficient in (38).
[53] M. Müller, J. Schmalian, and L. Fritz, Phys. Rev. Lett. 103, 025301 (2009).
[54] A. A. Abrikosov, Fundamentals of the Theory of Metals, (Dover Publications; Reprint edition 2017).
[55] H.-Y. Xie and M. S. Foster, Phys. Rev. B 93, 195103 (2016).
[56] H.-Y. Xie and A. Levchenko, Phys. Rev. B 99, 045434 (2019).
[57] M. Zarenia, A. Principi, G. Vignale, 2D Mater. 6, 035024 (2019).
[58] Sven Danz and Boris N. Narozhny, preprint arXiv:1910.14473 [cond-mat.mes-hall].