6+∞ new expressions
for the Euler-Mascheroni constant

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Abstract
In the first part we present results of four “experimental” determinations
of the Euler-Mascheroni constant \( \gamma \). Next we give new formulas expressing
the \( \gamma \) constant in terms of the Ramanujan–Soldner constant \( \mu \). Employing
the cosine integral we obtain the infinity of formulas for \( \gamma \).

1 Introduction
The Euler–Mascheroni constant is defined by the following limit:

\[
\gamma = \lim_{k \to \infty} \left( \sum_{n=1}^{k} \frac{1}{n} - \log(k) \right) = 0.57721566490153286 \ldots
\]

(1)
see [13], [14]. It is not known whether \( \gamma \) is irrational, see [19], [14]. The limit in (1)
is very slowly convergent (like \( n^{-1} \)) and in [8] it was shown that slight modification
of (1):

\[
\gamma = \lim_{k \to \infty} \left( \sum_{n=1}^{k} \frac{1}{n} - \log(k + \frac{1}{2}) \right)
\]

improves convergence to \( 1/n^2 \). Presently sequences converging to \( \gamma \) much faster are
known, see [16] where sequence which converge to \( \gamma \) like \( n^{-6} \) is presented. There is
a lot of formulas expressing \( \gamma \) as series, integrals or products, see [13] and e.g. [11],
[6], [14]. In particular there is infinity of formulas for \( \gamma \), we mention here [11 p. 4]:

\[
\gamma = \sum_{k=1}^{n} \frac{1}{k} - \log(n) - \sum_{k=2}^{\infty} \frac{\zeta(k, n+1)}{k}, \quad n = 2, 3, \ldots,
\]

(2)
where the Hurwitz zeta function:

$$\zeta(s, k) = \sum_{n=0}^{\infty} \frac{1}{(n + k)^s}, \quad \Re(s) > 1. \quad (3)$$

Another infinite set of formulas for \(\gamma\) we found in [3, eq.(9.3.10)]:

$$\gamma = \sum_{k=1}^{n} \frac{1}{k} - \log n - \int_{n}^{\infty} \frac{\{x\}}{x^2} dx, \quad n = 1, 2, 3, \ldots . \quad (4)$$

Here \(\{x\}\) is the fractional part of \(x\).

There are even uncountably many formulas for \(\gamma\), see e.g. [4]: for real \(r > 0\)

$$\gamma = \lim_{n \to \infty} \frac{\sum_{k=0}^{\infty} \left(\frac{n^k}{k!}\right) r \left(\sum_{j=1}^{k} \frac{1}{j} - \log(k)\right)}{\sum_{k=0}^{\infty} \left(\frac{n^k}{k!}\right) r}. \quad (5)$$

There are also doubly uncountably formulas for \(\gamma\), we present the formula (3.13) from [6]:

$$\gamma = r \int_{0}^{\infty} \left(\frac{1}{1 + x^q} - \exp(-x^r)\right) \frac{dx}{x}, \quad q > 0, \ r > 0. \quad (6)$$

The numerical value of the Euler–Mascheroni constant was calculated in the past many times see e.g. [4], the present day world record is 477,511,832,674 decimal digits of \(\gamma\) and belongs to Ron Watkins, see http://www.numberworld.org/digits/EulerGamma

The Euler–Mascheroni constant appears also in many places in number theory and in the theory of the Riemann zeta function, for example in the Nicolas’ and Robin’s criterions for the Riemann Hypothesis, see e.g. [5, vol.1, chapters 5 and 7]. One of the most amazing appearances of the \(\gamma\) constant is in the F. Merten’s two products over primes [12, p.351],, one of which involves constants \(\pi, e, \gamma\) ("holy trinity"):

$$\lim_{n \to \infty} \frac{1}{\log(n)} \prod_{p < n} \left(1 + \frac{1}{p}\right) = \frac{6e\gamma}{\pi^2}. \quad (7)$$

from which we obtain

$$\gamma = \log \left(\frac{\pi^2}{6} \lim_{n \to \infty} \frac{1}{\log(n)} \prod_{p < n} \left(1 + \frac{1}{p}\right)\right). \quad (8)$$

With present day computers we can check the accuracy of the above relation. In the Table I we present lhs and rhs of (7) as well as computed from the values of the finite products over primes values of the

$$\gamma(n) = \log \left(\frac{\pi^2}{6} \frac{1}{\log(n)} \prod_{p < n} \left(1 + \frac{1}{p}\right)\right). \quad (9)$$
The values of the product in (7) up to $n = 1000, 10000, \ldots, 10^{13}$ (second column) and values of its values following from the Mertens formula (third column), theirs ratio in fourth column and finite approximations to $\gamma$ in the last column. The fluctuations in the last digits of the values obtained from the computer are presumably caused by cumulation the floating-point errors.

| $n$  | $\prod_{p<n}(1 + 1/p_n)$ | $6e^\gamma \log(n)/\pi^2$ | ratio     | $\gamma(n)$   |
|------|------------------------|-----------------------------|-----------|---------------|
| $10^3$ | 7.5094464              | 7.4891425                   | 1.0027111 | 0.57992110    |
| $10^4$ | 9.9849904              | 9.9733461                   | 1.0011675 | 0.57838053    |
| $10^5$ | 12.4756558             | 12.4721158                  | 1.0002838 | 0.57749746    |
| $10^6$ | 14.9651229             | 14.9643917                  | 1.0000489 | 0.57726252    |
| $10^7$ | 17.4570890             | 17.4568441                  | 1.0000140 | 0.57722769    |
| $10^8$ | 19.9494269             | 19.9493052                  | 1.0000061 | 0.57721977    |
| $10^9$ | 22.4418428             | 22.4417674                  | 1.0000034 | 0.57721703    |
| $10^{10}$ | 24.9342956         | 24.9342295                  | 1.0000027 | 0.57721631    |
| $10^{11}$ | 27.4267504           | 27.4266917                  | 1.0000021 | 0.57721581    |
| $10^{12}$ | 29.9192150           | 29.9191539                  | 1.0000020 | 0.57721571    |
| $10^{13}$ | 32.4116846           | 32.4116161                  | 1.0000021 | 0.57721578    |

The average value of the divisor function $d(n)$ counting the number of divisors of $n$ including 1 and $n$ is given by the theorem proved by Dirichlet, see e.g. [12, Th.320]:

$$\frac{1}{n} \sum_{k=1}^{n} d(k) = \log n + 2\gamma - 1 + O\left(\frac{1}{\sqrt{n}}\right).$$  \hspace{2cm} (10)

In Table 2 values of $\gamma$ obtained from above formula for $n = 2^{15}, \ldots, 2^{23}$ are presented.

| $n$         | $\sum_{k=1}^{n} d(k)$ | $\gamma$ from (10) |
|-------------|-----------------------|---------------------|
| 32769       | 345785                | 0.5776565           |
| 65537       | 736974                | 0.5774880           |
| 131073      | 1564762               | 0.5773423           |
| 262145      | 3311206               | 0.5772996           |
| 524289      | 6985780               | 0.5772608           |
| 1048577     | 14698342              | 0.5772438           |
| 2097153     | 30850276              | 0.5772336           |
| 4194305     | 64607782              | 0.5772288           |
| 8388609     | 135030018             | 0.5772237           |

The sum of reciprocals of non–trivial zeros $\rho$ of the Riemann’s zeta function $\zeta(s)$ also involves $\gamma$ [9, p.67 and p. 159]:

$$\sum_{\rho} \frac{1}{\rho} = 2 + \gamma - \log(4\pi) = 0.046191417932 \ldots. \hspace{2cm} (11)$$
The above sum is real and convergent when zeros $\rho$ and complex conjugate $\overline{\rho}$ are paired together and summed according to increasing absolute values of the imaginary parts of $\rho$. Several years ago using the L-function calculator written by Michael Rubinstein (see http://doc.sagemath.org/html/en/reference/lfunctions/sage/lfunctions/lcalc.html) we have calculated 100,000,000 zeros of $\zeta(s)$; the last obtained zero has the value $\rho_{100000000} = \frac{1}{2} + i42653549.7609515$. In Table 2 we present approximations to $\gamma$ obtained from (11) after summing over 1000, 10,000, ..., 100,000,000 zeros of zeta function.

The largest known prime numbers are of the form $M_p = 2^p - 1$ where in turn $p$ is also a prime and they are called Mersenne primes, see eg. https://www.mersenne.org/. In [17, p.101] (see also [22]) the Lenstra–Pomerance–Wagstaff conjecture was formulated: the number of $p < x$ with $2^p - 1$ prime grows like

$$\#\{p < x \text{ and } 2^p - 1 \text{ prime}\} \sim \frac{e^\gamma}{\log 2} \log x.$$  \hspace{1cm} (12)

The presence of $\gamma$ here comes from the Merten’s result (7). In the Fig. 1 we compare the Pomerance – Wagstaff conjecture with all 51 presently known Mersenne primes. From the fit of actual number of $p < x$ with $2^p - 1$ prime to the $\log x$ gives rather poor value $\gamma = 0.61$, thus it is rather not convenient way to compute the Euler–Mascheroni constant.

| TABLE 3 | The value of $\gamma$ obtained from (11) after summing over $n = 1000, 10000, \ldots, 100000000$ zeros of $\zeta(s)$. |
|---|---|
| $n$ | $\gamma$ |
| 1000 | 0.5757765 |
| 10000 | 0.5769463 |
| 100000 | 0.5771715 |
| 1000000 | 0.5772091 |
| 10000000 | 0.5772147 |
| 100000000 | 0.5772155 |

In this paper we will present some new formulas for $\gamma$ obtained by putting in the series for the logarithmic and cosine integrals special values for the argument. Similar idea appeared in [20], where the series for the exponential integral was used to calculate $\gamma$ up to 3566 decimal places. A few of these new expressions present the Euler-Mascheroni constant in the form of the difference of two numbers one of which is transcendental. It gives hopes for the proof not only of the irrationality of $\gamma$ but also its transcendentality.
2 Logarithmic integral

The logarithmic integral is defined for all positive real numbers \( x \neq 1 \) by the definite integral

\[
\text{li}(x) \equiv \begin{cases} 
\text{p.v.} \int_0^x \frac{du}{\log(u)}, & \text{for } x > 1; \\
\int_x^0 \frac{du}{\log(u)}, & \text{for } 0 < x < 1,
\end{cases}
\] (13)

where \( \text{p.v.} \) stands for Cauchy principal value around \( u = 1 \):

\[
\text{p.v.} \int_0^x \frac{du}{\log(u)} = \lim_{\epsilon \to 0} \left( \int_0^{1-\epsilon} \frac{du}{\log(u)} + \int_{1+\epsilon}^x \frac{du}{\log(u)} \right). 
\] (14)

There is the series giving logarithmic integral \( \text{li}(x) \) for all \( x > 1 \) and quickly convergent because it has \( n! \) in denominator and \( \log^n(x) \) in numerator (see [II §5.1])

\[
\text{li}(x) = \gamma + \log \log x + \sum_{n=1}^{\infty} \frac{\log^n x}{n \cdot n!} \quad \text{for } x > 1. 
\] (15)

The variant of the above series after some change of variable:

\[
\int_x^{\infty} \frac{e^{-t}}{t} dt = -\gamma - \log x + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n \cdot n!} 
\] (16)
was used in [20] for large \( x > 0 \), when lhs of above equation is practically zero (in fact it is \( O(e^{-x}/x) \)), to compute 3566 digits of \( \gamma \), see also [4].

The logarithmic integral takes a value 0 at only one number which is denoted by \( \mu \) and is called the Ramanujan–Soldner constant

\[
\int_0^\mu \frac{du}{\log u} = 0,
\]  

(17)

see e.g. [2, eq.(11.3)] and its numerical value is:

\[ \mu = 1.4513692348838105028396848589202745 \ldots . \]

Thus for \( x > \mu \) we have:

\[
\text{li}(x) = \int_\mu^x \frac{du}{\log(u)}.
\]  

(18)

Inserting in (15) \( x = \mu > 1 \) we obtain the first formula expressing the Euler-Mascheroni constant by the Ramanujan–Soldner constants:

\[
\gamma = -\log \log \mu - \sum_{n=1}^{\infty} \frac{\log^n \mu}{n \cdot n!}.
\]  

(19)

Using PARI [21] we checked that summing above to \( n = 20 \) reproduces 31 digits of \( \gamma \). In the Appendix we give the script to reproduce this result with whatever number of digits.

Even faster converging series was discovered by Ramanujan [2, p.130]:

\[
\int_\mu^x \frac{du}{\log u} = \gamma + \log \log x + \sqrt{x} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(\log x)^n}{n! 2^{n-1}} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{1}{2k+1} \quad \text{for } x > 1.
\]  

(20)

Putting here \( x = \mu \) we obtain second formula for the Euler-Mascheroni constant:

\[
\gamma = -\log \log \mu + \sqrt{\mu} \sum_{n=1}^{\infty} \frac{(-1)^n(\log \mu)^n}{n! 2^{n-1}} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{1}{2k+1}.
\]  

(21)

We checked using PARI that summing above to \( n = 20 \) reproduces correctly 37 digits of \( \gamma \).

Putting in (15) \( x = e \) simplifies series and we obtain third expression for the Euler-Mascheroni constant:

\[
\gamma = \int_\mu^e \frac{du}{\log u} - \sum_{n=1}^{\infty} \frac{1}{n \cdot n!} := \alpha - \beta,
\]  

(22)

where the numbers

\[
\alpha := \int_\mu^e \frac{du}{\log u} = 1.89511781635593675546652 \ldots ,
\]  

(23)
\[ \beta := \sum_{n=1}^{\infty} \frac{1}{n \cdot n!} = 1.317021514540389486 \ldots \quad (24) \]

The number \( \beta \) is irrational as the reasoning proving the irrationality of \( e = \sum_{n=0}^{\infty} \frac{1}{n!} \) (see e.g. [18, p.65]) can be repeated here *mutatis mutandis*. In fact from the Siegel–Shidlovsky theorem [10, see eq.5.2 for \( k = 1 \)] it follows that (24) is transcendental.

Putting in (20) \( x = e \) we get the fourth expression for Euler–Mascheroni constant

\[ \gamma = \int_{e}^{\infty} \frac{du}{\log u} + \sqrt{e} \sum_{n=1}^{\infty} \frac{(-1)^n \lfloor (n-1)/2 \rfloor}{n!} 2^{n-1} k=0 \frac{1}{2k+1}. \quad (25) \]

Finally let us notice that in [13] at several places (e.g. pp. 52, 104) we can read that Euler had hoped that \( \gamma \) is the logarithm of some important number. Above we have given for \( \gamma \) two series in logarithm of the Ramanujan–Soldner constant.

## 3 Cosine integral

Many special functions involve in theirs expansions the Euler-Mascheroni constant. The function cosine integral \( \text{Ci}(x) \) has the series expansion also containing \( \gamma \) (see e.g. [11, §5.2]):

\[ \text{Ci}(x) = - \int_{x}^{\infty} \frac{\cos u}{u} du = \gamma + \log x + \sum_{n=1}^{\infty} \frac{(-x^2)^n}{2n(2n)!} \quad (26) \]

\[ = \gamma + \log x + \sum_{n=1}^{\infty} \frac{(-x^2)^n}{2^{n+1} n! (2n-1)!!} \]

because \( (2n)! = 2^n n!(2n-1)!! \), where odd factorial \( (2n-1)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1) \). Putting above \( x = 1 \) we obtain the fifth expression for the Euler-Mascheroni constant:

\[ \gamma = - \int_{1}^{\infty} \frac{\cos u}{u} du + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n(2n)!}, \quad (27) \]

where:

\[ \int_{1}^{\infty} \frac{\cos u}{u} du = -0.3374039229009681346626\ldots \quad (28) \]

and

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n(2n)!} = 0.2398117420005647259439\ldots \quad (29) \]

The cosine integral \( \text{Ci}(x) \) has infinity of zeros that do not have their own names and are non–periodic; they are usually denoted by \( c_k \), see [15]. The first zeros are
\( c_0 = 0.61650548562, c_1 = 3.38418042255, c_2 = 6.42704774405, \ldots \). A.J. MacLeod in [15] gives the asymptotic expansion for these zeros:

\[
c_k \sim k\pi + \frac{1}{k\pi} - \frac{16}{3(k\pi)^3} + \frac{1673}{15(k\pi)^5} - \frac{507746}{105(k\pi)^7} + \ldots
\]

Putting zeros \( c_k \) into (26) we obtain an infinity of expressions for \( \gamma \)

\[
\gamma = -\sum_{n=1}^{\infty} \frac{(-c_k^2)^n}{2n(2n)!} - \log c_k, \quad k = 0, 1, 2, \ldots .
\]

In the Table 4 we present values for \( \gamma \) obtained from above formula when \( c_k \) are calculated from (30). In the last column the difference between values in third column and \( \gamma \) are presented.

From the MacLeod formula (30) we see that large zeros of \( \text{Ci}(x) \) approach just zeros of \( \sin(x) = \int \cos(x) dx \): \( c_k \sim k\pi \) for large \( k \). Thus we have the sixth formula for the Euler-Mascheroni constant:

\[
\gamma = \lim_{k \to \infty} \left( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(k^2\pi^2)^n}{2n(2n)!} - \log(k\pi) \right).
\]

This formula in some sense resembles the original definition (1). We checked that for \( k = 8000 \) the expression in above big parentheses gives 0.577215664867633997998 i.e. it reproduces correctly first 8 digits of \( \gamma \).

| \( k \) | \( c_k \) from eq.(30) | eq.(31) for this \( c_k \) | \( \text{eq.(31) for this } c_k - \gamma \) |
|---|---|---|---|
| 10 | 31.447589011629313 | 0.5772156649004098 | 1.123 \times 10^{-12} |
| 20 | 62.847747177749027 | 0.5772156649015328 | 1.953 \times 10^{-17} |
| 30 | 94.258383581485718 | 0.5772156649015328 | 2.888 \times 10^{-20} |
| 40 | 125.67166120666795 | 0.5772156649015328 | 2.657 \times 10^{-22} |
| 50 | 157.08599750231211 | 0.5772156649015328 | 6.519 \times 10^{-24} |
| 60 | 188.50086358429127 | 0.5772156649015328 | 2.871 \times 10^{-25} |
| 70 | 219.91603253410894 | 0.5772156649015328 | 1.771 \times 10^{-26} |
| 80 | 251.33139082491842 | 0.5772156649015328 | 1.180 \times 10^{-27} |
| 90 | 282.74687536370536 | 0.5772156649015328 | 2.181 \times 10^{-29} |
| 100 | 314.16244828586940 | 0.5772156649015328 | 2.861 \times 10^{-29} |

Appendix: Below is a simple PARI/GP script checking (19) to arbitrary accuracy declared by \( \text{\textbackslash p precision, below it is 2222. The output gives agreement between lhs and rhs of (19) up to the number of digits given by precision. It takes a fraction of a second to get results. For really big precision the allocated memory 300 MB maybe not sufficient.}
In the above script we used the fact that logarithmic integral is related to the exponential integral $Ei(x)$, see e.g. [1, chapt. 5]:

\[
\text{li}(x) = Ei(\log x),
\]

where

\[
Ei(x) = \begin{cases} 
-p.v. \int_{-x}^{\infty} \frac{e^{-t}}{t} dt, & \text{for } x > 0. \\
-\int_{-x}^{\infty} \frac{e^{-t}}{t} dt, & \text{for } x < 0
\end{cases}
\]

and principal value is needed to avoid singularity of the integrand at $t = 0$. The logarithmic integral is not implemented in Pari while exponential integral is implemented as $\text{eint1}(x)$. We have obtained as the result the number $4.27 \times 10^{-2235}$. To check (21) change last lines to

\[
\text{ss} = \text{suminf}(n=1, (-1)^n \text{tmp}^n/(2^n(n-1.0)*n!)) \times \text{sum}(k=0, \text{floor}((n-1)*0.5), 1.0/(2.0*k+1.0)))
\]

write("EMRS.txt", Euler+log(tmp)-sqrt(Soldner)*ss);

As the output this time we obtained $2.7328 \times 10^{-2233}$.

The equation (27) can be checked in Pari using the following commands:

```
allocatemem(5000000000)
\p 2222

tmp=\text{sumalt}(n=1, (-1)^{n-1}/(2*n*(2*n)!));
\text{oo}=[1];
c_i=\text{intnum}(u=1, [\text{oo}, 1], \cos(u)/u);
\text{print}(\text{Euler}+c_i-\text{tmp});
```

The explanations are needed: PARI contains the numerical routine \text{sumalt} for summing infinite alternating series in which extremely efficient algorithm of Cohen,
Villegas and Zagier [7] is implemented; oo=[1] denotes in Pari infinity; intnum( ) is the function for numerical integration and flag k*I (I= i, i.e. i^2 = −1) tells the procedure that the integrand is an oscillating function of the type cos(kx), here k = 1. After a few minutes we obtained 1.42335 × 10^{-2235}. This result shows the power of Pari’s procedures: the value of the cosine integral at 1 is indeed calculated numerically without using the expansion [26] and the value of γ to avoid a vicious circle (tautology).

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