A Quantum IP Predictor-Corrector Algorithm for Linear Programming

P. A. M. Casares$^{1}$ and M. A. Martin-Delgado$^{1}$

$^1$Departamento de Física Teórica, Universidad Complutense de Madrid.

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We introduce a new quantum optimization algorithm for Linear Programming (LP) problems based on Interior Point (IP) Predictor-Corrector (PC) methods whose (worst case) time complexity is $O(\sqrt{n}Ls^3k\epsilon^{-1}s^{-1})$. This represents a quantum speed-up in the number $n$ of variables in the cost function with respect to the comparable classical Interior Point (IP) algorithms that behave as $O((n+m)\sqrt{nKLs^3}\log(\epsilon^{-1})\epsilon^{-1})$ or $O(\sqrt{n(n+m)L})$ depending on the technique employed, where $m$ is the number of constraints and the rest of the variables are defined in the introduction. The average time complexity of our algorithm is $O(\sqrt{n}s^3k\epsilon^{-1}s^{-1})$, which equals the behaviour on $n$ of quantum Semidefinite Programming (SDP) algorithms based on multiplicative weight methods when restricted to LP problems and heavily improves on the precision $\epsilon^{-1}$ of the algorithm. Unlike the quantum SDP algorithm, the quantum PC algorithm does not depend on size parameters of the primal and dual LP problems $(R, r)$, and outputs a feasible and optimal solution whenever it exists.

Keywords: Linear Programming Problem, Quantum Algorithms, Quantum Linear Approximation, Interior Point Method, Iteration Complexity, Strong Polynomiality.

I. INTRODUCTION

Linear Programming (LP) problems are among the most basic optimization problems $\cite{1,2}$. Applications abound both at personal and professional fronts: improving a project delivery, scheduling of tasks, analyzing supply chain operations, shelf space optimization, designing better strategies and logistics and scheduling problems in general. LP is also used in Machine Learning where Supervised Learning works on the basics of linear programming. A system is trained to fit on a mathematical model of an objective (cost) function from the labeled input data that later can predict values from an unknown test data $\cite{4,5}$. More specifically, linear programming is a method to achieve the best outcome, such as maximum profit or lowest cost, in a mathematical model whose requirements are represented by linear relationships known as constraints. Semi-Definite Programming (SDP) is an extension of LP when the objective or cost function is formulated with a non-diagonal matrix and constraints contain more general inequalities $\cite{6,9}$.

We are in the time of small quantum computers with reduced computational capabilities due to noisy physical qubits $\cite{10,13}$. The challenge of surpassing the power of current and foreseeable classical computers is attracting a lot of attention in the academia and in technological companies $\cite{14,15}$. This motivates the endeavour of searching for new quantum algorithms beyond the standard ones that spurred the field of quantum computation in the mid 90s (Shor, Grover, etc.) $\cite{16,19}$. Only recently, SDP problems have been given a quantization procedure by Brandão and Svore providing us with the first quantum advantage for these optimization problems $\cite{20,21}$.

The development of methods to solve LP problems has a long tradition starting with the Simplex Method $\cite{3}$. Interior Point Methods (IP) stand out among the large variety of available methods $\cite{22}$. In turn, IP methods represent a whole class of solving strategies for LP optimization. Among them, the Predictor-Corrector Method $\cite{20,27}$ is arguably one of the best procedures to achieve an extremely well-behaved solution. Here we present a quantum algorithm that relies on the quantization of a Predictor-Corrector Method. One important feature of our quantum IP algorithm is that it is a hybrid algorithm: partially classical, partially quantum. This feature has become very common and a similar situation occurs with the Brandão-Svore algorithm in SDP or the Quantum Eigen-Solver for quantum chemistry $\cite{28,32}$, and many others. The core of this quantum IP algorithm relies on the quantization of a crucial step of the Predictor-Corrector method by means of the HHL quantum algorithm for solving linear systems of equations $\cite{33}$. More precisely, by means of a Quantum Linear System Algorithm (QLSA) $\cite{34}$ that has improved features w.r.t. the original HHL algorithm. In order to apply the QLSA in the context of LP programming, we have to solve several caveats since the straightforward application of it is doom to failure.

The quantum IP algorithm we propose benefits from several fundamental properties inherited from the classical Predictor-Corrector algorithm, and has a better performance than other classical IP algo-
rithms. In particular [27]:

1. The Predictor-Corrector method can solve the LP problem without assuming the existence of feasible or optimal solutions.

2. If LP has solution, the loop of this interior point algorithm approaches feasibility and optimality at the same time, and if the problem is infeasible or unbounded the algorithm detects infeasibility for either the primal or dual problem.

3. The algorithm can start from any point near the center of the positive orthant, and does not use any big-M penalty or lower bound (except in our case in the first iteration, as we will see).

The notions of feasible, optimal solutions etc. are defined in Sec. 11 where a self-contained review of the Predictor-Corrector method is presented.

The time complexity of the proposed quantum IP algorithm is \( O(\sqrt{n}Ls^3k\epsilon^{-1}\epsilon_r^{-1}) \) and the space complexity is \( O(n + m) \), where \( n \) is the number of variables of the cost function, \( m \) is the number of constraints, \( L \) is the size of the encoded data (see eq. [1]), \( s \) is the sparsity of the matrix of constraints, \( k \) is a precondition parameter to be specified later, \( \epsilon^{-1} \) is the precision of the algorithm and \( \epsilon_r^{-1} \) a precision coming from a sparsification procedure that we will explain afterwards. This represents a quantum speed-up in \( n \) with respect to the best classical IP comparable algorithm (preconditioned conjugate gradient descent) with efficiency \( O(\sqrt{n}(n + m)Ls^3\sqrt{k}\log(\epsilon^{-1})\epsilon_r^{-1}) \), or \( O(\sqrt{n}(n + m)L) \) and \( O((n + m)^2) \) space complexity, if we are using the customary Cholesky decomposition [25 33 36].

On the other hand the average time complexity of the quantum IP algorithm is \( O(\sqrt{ns^3k}\epsilon^{-1}\epsilon_r^{-1}) \), which equals the efficiency of the quantum SDP algorithm of Brandão-Svore when restricted to LP problems. A precise comparison among different classical and quantum algorithms can be found in Table I.

It is worth mentioning that our quantization approach to LP problems is radically different from the method of Brandão and Svore and this comes with several benefits. Namely, the problem of quantizing linear programming using multiplicative weight methods [37] as in Brandão-Svore is that they yield an efficiency depending on parameters \( R \) and \( r \) of the primal and dual problems. In fact, they might depend on the sizes \( n,m \) of the cost function, thereby the real time complexity of the algorithm remains hidden. Moreover and generically, unless specified, these \( R,r \) parameters cannot be computed before hand, but after running the algorithm. Thus, the real efficiency of the quantum algorithm is masqueraded by overhead refactors behaving badly on \( R \) and \( r \). This situation is clearly not satisfactory and quantization methods with a clean quantum advantage are desirable like the ones proposed here. The quantum IP algorithm has a better behaviour in the precision as \( \epsilon^{-1} \) compared to the strong power-like behaviour as \( \epsilon^{-5} \) in the most recent improvement of the Brandão-Svore algorithm [23] (although we should also take into account a precision coming from the sparsification process \( \epsilon_r^{-1} \)). On the contrary, the space complexity of the latter is \( O(1) \), whereas both the classical and quantum IP algorithms have space complexity \( O(n + m) \), or \( O((n + m)^2) \) if the classical algorithms uses Cholesky decomposition.

Next, we present a more detailed description of our main results and the structure of the paper.

## A. Results

This article combines the Predictor-Corrector algorithm [27] with the Quantum Linear System Algorithm (QLSA) [34] with the aim of obtaining an interior point hybrid (quantum-classical) algorithm for linear programming, that runs on average as \( O(\sqrt{n}) \), which is optimal as explained in [20]. The major price to pay is a \( O(n + m) \) space complexity, in the sense of classical parallelism.

The main constraint with using this algorithm is that the matrix \( A \) and vectors \( b \) and \( c \) from [2] and [3] must be sparse. This is because one way to control the condition number parameter of the Quantum Linear System Algorithm is using a preconditioner (Sparse Approximate Inverse [35]), which will have complexity \( O(s^3) \), for \( s \) the sparsity parameter (the number of non-zero terms).

The dependence on other parameters is comparatively rather good. We obtain \( O(\epsilon^{-1}) \) coming from the sparsification procedure and the Amplitude Estimation algorithm [39] for the readout procedure of QLSA. The sparsification may also lead to numerical precision problems in the first iteration of the algorithm, due to some terms of size \( \epsilon \). The dependence on \( r \) is worse than in comparable classical algorithms where one would get \( O(\log(\epsilon^{-1})) \), but other quantum algorithms do not reach this bound either and they are even worse.

Our algorithm inherits the nice properties of the Predictor-Corrector algorithm, since we have successfully implemented the QLSA algorithm at the core of solving the various linear systems of equations appearing in this classical Interior Point algorithm.
B. Structure of the paper.

The paper has two main sections. The first reviews the Predictor-Corrector algorithm from [27]. It is itself divided in subsections where we explain how to initialize and terminate the algorithm, and the main loop.

In the second section we explain the changes we carry out to be able to use the QLSA from [34]. In particular we start with two subsections discussing the condition number and sparsity. Then we focus on how to prepare the initial quantum states for the QLSA, following [40], and read out the results using Amplitude Estimation [39]. Finally we explain the QLSA, comment on the possibility of quantizing the termination of the algorithm, and devote one subsection to the complexity of the overall algorithm and its comparison with other alternatives.

We recommend the reader to look at the algorithms where we have described the precise process step by step. There are also informative figures depicting the flow of the algorithm. Since they can help understand the overall process, we recommend first to understand them before going to the details of each subsection.

II. THE PREDICTOR-CORRECTOR ALGORITHM

In this section we review the Predictor-Corrector algorithm of Mizuno, Todd and Ye for solving Linear Programming problems [27]. As stated in the original article, we will see that it performs $O(\sqrt{n}L)$ iterations of the main loop in the worst case scenario, where $n$ is the number of variables and $L$ the length of the encoding of the input data:

$$ L := \sum_{i=1}^{n} \sum_{j=1}^{m} \log_2(|a_{ij}| + 1) + 1. $$

(1)

Note that the smallest value $L$ can take is $2nm$. However, on the average case the number of iterations will not depend on $L$, rather $O(\sqrt{n} \log n)$ [31].

A. Initialization

The linear programming problem we want to solve is denoted as (LP): Given $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$, find $x \in \mathbb{R}^n$ such that:

$$ \text{minimizes } c^T x \quad (2a) $$

subject to $Ax \geq b$, $x \geq 0$. \hspace{1cm} (2b)

If the matrix $A$ fulfills some weak requirements (the duality gap being 0) its dual problem has the same solution. The dual problem is (DP): finding $y \in \mathbb{R}^m$ such that

$$ \text{maximizes } b^T y \quad (3a) $$

subject to $A^T y \geq c$. \hspace{1cm} (3b)

It is common to define $s = c - A^T y \in \mathbb{R}^n$ the slack (dual) variable to the constraint (3). However, in order to avoid confusion with the sparsity of $A$, we will call it $\zeta$ (zeta) in similarity with the notation used sometimes in semidefinite programming ($\zeta$).

To solve the previous problem, we set another which is artificial or auxiliary, homogeneous (in the sense that there is a single non-zero constraint), and self-dual (its dual problem is itself). This allows us to apply primal-dual interior-point algorithms without doubling the dimension of the linear system solved at each iteration. Therefore, given any $x^0 \geq 0 \in \mathbb{R}^n$, $z^0 \geq 0 \in \mathbb{R}^n$, and $y^0 \in \mathbb{R}^m$, formulate (HLP)

$$ \min((x^0)^T z^0 + 1) \theta \quad (4a) $$

such that $(\tau \in \mathbb{R})$:

$$ Ax - b\tau + \bar{b}\theta = 0 \quad (4b) $$
$$ -A^T y + c\tau - \bar{c}\theta \geq 0, \quad (4c) $$
$$ b^T y - c^T x + \bar{g}\theta \geq 0, \quad (4d) $$
$$ -\bar{b}^T y + \bar{c}^T x - \bar{g}\tau = -(x^0)^T z^0 - 1, \quad (4e) $$

with

$$ b := b - Ax^0, \quad \bar{b} := c - A^T y^0 - z^0, \quad \bar{g} := c^T x^0 + 1 - b^T y^0. \quad (5) $$

The constraint (4e) is used to impose self-duality. It is also important to remark that $\bar{b}$, $\bar{c}$ and $\bar{g}$ which in [27] is denoted by $\bar{z}$) indicate the infeasibility of the initial primal and dual points, and the dual gap, respectively.

Recall also that we use slack variables to convert inequality constraints into equality constraints. Those slack variables indicate the amount by which the original constraint deviates from an equality. As we have two inequality constraints, we define slack variables $z \in \mathbb{R}^n$ for (4c) and $\kappa \in \mathbb{R}$ for (4d):

$$ z := A^T y - c\tau + \bar{c}\theta \geq 0 \quad (6) $$
$$ \kappa := -b^T y + c^T x - \bar{g}\theta \geq 0 \quad (7) $$
This implies that we can rewrite (4e) as
\[(z^0)^T x + (x^0)^T z + \tau + \kappa - ((x^0)^T z^0 + 1)\theta = (x^0)^T z^0 + 1.
(8)

Once we have defined these variables, Theorem 2 of [27] indicates that

1. The dual (HLD) of (HLP) is again (HLP).
2. We can check that a feasible solution to this new system is:
   \[y = y^0, \ x = x^0 > 0, \ z = z^0 > 0, \ \tau = \kappa = \theta = 1.\]
(9)

3. There is an optimal solution such that
   \[x + z > 0; \ \tau + \kappa > 0,\]
   which we call strictly self-complementary. Self-complementary indicates that it solves (HLP) and strictly indicates that the inequalities are fulfilled strictly (with > rather than =).

Therefore, we can choose any point fulfilling (9) as a feasible starting point for our algorithm. A particularly simple one would be
\[y^0 = 0, \ x^0 = 1_{(n+1)\times1} = z^0.\]
(11)

**B. Main loop**

In this section we explain how to set up an iterative method that allows us to get close to the optimal point, following a path along the interior of the feasible region. The original references are [26, 27]. To define that path, theorem 5 of [27] indicates that

- For each \(\mu > 0\), there is a unique point \((y^0, x^0, \tau^0, z^0, \kappa^0) \in \mathcal{F}_h\), where \(\mathcal{F}_h\) is the set of feasible points of (HLP), such that
  \[
  \begin{pmatrix}
  Xz \\
  \tau\kappa
  \end{pmatrix} = \mu 1_{(n+1)\times1},
  \]
  (12)
  where \(X := \text{diag}(x)\). We will later see that we use the analog notation of \(Z := \text{diag}(z)\).

- Let \((d_y, d_x, d_\tau, d_\theta, d_z, d_\kappa)\) be a point in the null space \(N\) of the constraint matrix of (HLP) with variables \(z\) and \(\kappa\), i.e.
  \[
  Ad_x - bd_\tau + bd_\theta = 0,
  \]
  \[
  - A^T d_y + cd_\tau - \bar{c}d_\theta - d_z = 0,
  \]
  \[
  b^T d_y - c^T d_x + \bar{g}d_\theta - d_\kappa = 0,
  \]
  \[
  - b^T d_y + \bar{c}^T d_x - \bar{g}d_\theta = 0.
  \]

  Then
  \[
  (d_z)^T d_z + d_\tau d_\kappa = 0.
  \]
(14)

This theorem defines the following path in (HLP)
\[
\mathcal{C} = \{(y, x, \tau, \theta, z, \kappa) \in \mathcal{F}_h^0 : \left|\begin{pmatrix}
  Xz \\
  \tau\kappa
  \end{pmatrix} - \mu 1_{(n+1)\times1}\right| = \beta\mu\}
\]

and its neighbourhood
\[
\mathcal{N}(\beta) = \{(y, x, \tau, \theta, z, \kappa) \in \mathcal{F}_h^0 : \left|\begin{pmatrix}
  Xz \\
  \tau\kappa
  \end{pmatrix} - \mu 1_{(n+1)\times1}\right| \leq \beta\mu\}.
\]
(16)

In consequence, the algorithm goes as follows: starting from an interior feasible point \((y^0, x^0, \tau^0, \theta^0, z^0, \kappa^0) \in \mathcal{F}_h^0\) and given the following system of equations for variables \((d_y, d_x, d_\tau, d_\theta, d_z, d_\kappa)\) and \(t = 0, 1, \ldots \in \mathbb{N}\):
\[
(d_y, d_x, d_\tau, d_\theta, d_z, d_\kappa) \in N,
\]
(17)
\[
\begin{pmatrix}
  X^t d_z + z^t d_x \\
  \tau^t d_\kappa + \kappa^t d_\tau
  \end{pmatrix} = \gamma \mu^t 1_{(n+1)\times1} - \begin{pmatrix}
  X^t z^t \\
  \tau^t \kappa^t
  \end{pmatrix},
\]
(18)
which can be written as \(M|d| = |f|\), i.e.
perform the following steps iteratively:

**Predictor step:** Solve (19) with $\gamma = 0$ for the previously calculated point $(y^t, x^t, \tau^t, z^t, \kappa^t) \in \mathcal{N}(1/4)$. Then find the biggest step length $\delta$ such that

$$
(y^{t+1}, x^{t+1}, \tau^{t+1}, z^{t+1}, \kappa^{t+1}) = (y^t, x^t, \tau^t, z^t, \kappa^t) + \delta(d_y, d_x, d_\tau, d_\theta, d_z, d_\kappa),
$$

(20)
is in $\mathcal{N}(1/2)$, and update the values accordingly.

**Corrector step:** Solve (19) with $\gamma = 1$ and set

$$
(y^{t+1}, x^{t+1}, \tau^{t+1}, z^{t+1}, \kappa^{t+1}) = (y^t, x^t, \tau^t, z^t, \kappa^t) + (d_y, d_x, d_\tau, d_\theta, d_z, d_\kappa),
$$

(21)
that will be in $\mathcal{N}(1/4)$.

### C. Termination

The loop from the previous section will run over $t$ until one of the following two criteria are fulfilled: For $\epsilon_1, \epsilon_2, \epsilon_3$ small numbers, either

$$(x^t/\tau^t)^T(z^t/\tau^t) \leq \epsilon_1 \quad \text{and} \quad (\theta^t/\tau^t)|||((\hat{b}, \hat{c})|||/\epsilon_2),
$$

(22)
or

$$
\tau^t \leq \epsilon_3.
$$

(23)
We will have to iterate up to $O(L\sqrt{n})$ times, with $t = \max\{\log((x^0)^T(z^0)/\epsilon_1\epsilon_2)\}, \log(|||((\hat{b}, \hat{c})|||/\epsilon_2\epsilon_3)\}$.

If the termination is due to condition (23), then we know that there is no solution fulfilling $|||(x, z)||| \leq 1/(2\epsilon_3) - 1$. We will then consider, following [22], that either (LP) or (LD) are infeasible or unbounded.

However, if termination is due to (22), denote by $z^t$ the index set \{\{0 \leq n: x^t_j \geq \frac{\epsilon_3}{2}\}\}. Let also $B$ the columns of $M$ such that their index is in $z^t$, and the rest by $C$.

**Case 1:** If $\tau^t \geq \kappa$ solve for $y, x_B, \tau$

$$
\min ||y^t - y||^2 + ||x_B^t - x_B||^2 + (\tau^t - \tau)^2 
$$

(24a)
such that

$$
Bx_B - \tau = 0; \quad -B^Ty + c_B\tau = 0; \quad b^Ty - c_B^Tx_B = 0.
$$

(24b)

**Case 2:** If $\tau^t < \kappa^t$ and we solve for $y, x_B$, and $\kappa$ from

$$
\min ||y^t - y||^2 + ||x_B^t - x_B||^2 + (\kappa^t - \kappa)^2
$$

(25a)
such that

$$
Bx_B = 0; \quad -B^Ty = 0; \quad b^Ty - c_B^Tx_B - \kappa = 0
$$

(25b)
The result of either of these two calculations will be the output of our algorithm, and the estimate of the solution of the (HLP) problem. In particular, $x$ will be the calculated $x_B$ in the least square projection together with $x_C$, and $y$ will be the calculated $y$ again in the least square projection.

## III. THE QUANTUM ALGORITHM

The aim of this section is to explain how the Quantum Linear System Algorithm (QLSA) can help us efficiently run this algorithm, in the same spirit of, for example, [42] solving the problem of the Finite Element Method. This is due to the fact that solving (19) is clearly the most computationally expensive part of each step. We will use the following result (algorithm):

**Theorem 1** [34]: Let $M$ be an $n' \times n'$ Hermitian matrix (if the matrix is not hermitian it can be included as a submatrix of a Hermitian one) with $|||M||| |||M^{-1}||| \leq k$ (that is, the condition number: the ratio between the highest and smallest eigenvalue for positive definite matrices), and $M$ having an sparsity $s$ (at most $s$ nonzero entries in each row). The usual notation for the condition number is $\kappa$, but since we want to avoid confusion with the $\kappa$ slack variable from the Predictor-Corrector algorithm, we will call the former $k$.

Let $f$ be an $N$-dimensional unit vector, and assume that there is an oracle $P_f$ which produces the state $|f\rangle$, and another $P_M$ which, taking $(r, i)$ as input, outputs the location and value of the ith nonzero entry in row $r$ of $M$. Let

$$
d = M^{-1}f, \quad |d\rangle = \frac{d}{|||d|||}.
$$

(26)
Then, [34] construct an algorithm relying on Hamiltonian simulation that outputs the state $|d\rangle$ up to accuracy $\epsilon$, with bounded probability of failure, and makes

$$
o(sk\log^{2.5}(k/\epsilon))
$$

(27)
uses of $P_M$ and

$$
o(k\sqrt{\log(k/\epsilon)})
$$

(28)
of $P_f$; and has overall time complexity

$$
o(sk\poly(\log(n'sk/\epsilon))).
$$

(29)
In our case the variable $n'$ is the size of the matrix of (19), that is $n' = m + 2n + 3$, so the time complexity of running their proposed algorithm is $O(\poly\log n)$, considering that $m = O(n)$. 
Using the QLSA is not straightforward, so let us look first into the parameters involved: the condition number and the sparsity. The sparsity will play an important role on the Hamiltonian simulation necessary in the QLSA, while the condition number will appear when we try to implement a linear combination that calculates \( M^{-1} |f| \) as we shall see.

### A. The condition number \( k \)

We have seen that the QLSA is linear in \( k \). We also know, from [33], that this dependence is optimal. Therefore it is important to check that \( k \) does not depend (except maybe polylogarithmically) on \( n \).

The main problem, however, is that we should be able to control \( k \) on each of the iterations, in order to be able to calculate the overall complexity beforehand. This is not easy since the matrix of [19] will be updated each time we solve the system. For that reason we will use the Sparse Approximate Inverse (SPAI) algorithm [38] as suggested in [40], to precondition the system and control \( k \). Other preconditioning methods not discussed here might also be possible.

Let us follow [38]. The procedure consists in finding a matrix \( L \) that minimizes \( ||ML - I|| \) in the Frobenius norm.

\[
||ML - I||_F^2 = \sum_{j=1}^{n} ||(ML - I)e_j||_2^2.
\]

This problem can be separated into \( n' = m + 2n + 3 \) least squares problems

\[
\min_{\hat{L}_j} ||\hat{L}_j e_j - \hat{e}_j||_2,
\]

(31)

to be solved in parallel, where the hat means that columns and rows with all zeros have been removed and \( \hat{e}_j = (\delta_{ji})_1 \) being \( \delta_{ji} \) a Kronecker delta between the row index \( i \) and \( j \).

The key of the algorithm relies on choosing the sparsity pattern of \( L \). SPAI is a recursive algorithm itself, so this can be done in first place taking \( L \) with the same sparsity pattern as \( M \). In subsequent applications of the preconditioning algorithm, when we manage to find out the best pattern for \( L \), we can use that pattern in the rest of the recursions of the main loop of the predictor corrector algorithm, which will make it run faster.

This means solving \( n' \) independent \( s_r \times s_c \) least squares problems in parallel, with an iteration procedure \( O(s) \) times, with \( s_r \) and \( s_c \) the row and column sparsity respectively. Note that it is customary to refer to row sparsity as sparsity, but due to the (anti)symmetry of the matrix \( M \), \( s = s_r = s_c \). This can be done in \( O(s^3) \) operations with \( O(s^2) \) calls to the oracle of matrix \( M, \mathcal{P}_M \), and the additional \( O(s) \) coming from the iteration procedure of SPAI.

Now define the residues

\[
r_j = \hat{M} \hat{L}_j - \hat{e}_j.
\]

(32)

Making \( r_j = 0 \) can be done inverting the matrix so that \( \hat{L}_j = \hat{M}^{-1} \hat{e}_j \). The classical complexity of inverting a \( s \times s \) matrix is \( O(s^3) \) (it is another way of obtaining that complexity bound) or we can fix a more modest objective of attaining

\[
\epsilon_c = \max_j ||r_j||_2.
\]

(33)

If so, we have a theorem [38] indicating that if \( \sqrt{s}\epsilon_c < 1 \), then the condition number is bounded as

\[
k \leq \frac{1 + \sqrt{s}\epsilon_c}{1 - \sqrt{s}\epsilon_c}.
\]

(34)

Since in [34] the complexity dependence on \( k \) and \( s \) is the same, we want to find out which \( \epsilon_c \) is necessary in order to have

\[
k \leq \frac{1 + \sqrt{s}\epsilon_c}{1 - \sqrt{s}\epsilon_c} \leq s.
\]

(35)

That means

\[
\epsilon_c \leq \frac{1}{\sqrt{s}} \frac{s - 1}{s + 1}.
\]

(36)

so if \( s \gg 1 \) then we can take \( \epsilon_c \approx s^{-1/2} \).

The preconditioning will be carried out classically. Overall the time complexity of SPAI is \( O(s^3) \) and the space complexity is \( O(n + m) \), since we need to solve \( O(n + m) \) systems of equations in parallel.

### B. The sparsity \( s \)

In the previous section we have seen that the sparsity plays an important role, and in the QLSA the time complexity of this parameter is linear, whereas in the preconditioning it is cubic.

Luckily, the sparsity of the system does not vary when we update [19]. However, we need to ensure that from the start it is a sparse system. Obviously, \( A \) needs to be sparse. Otherwise we might be able to employ the QLSA for dense matrices [44] but not the one that we have been using, the QLSA for sparse systems [34]. For the time being let us suppose that \( A \) is in fact sparse. What else do we need? Since \( b, c, \bar{b} \) and \( \bar{c} \) also appear in matrix \( M \), it is important that those vectors are sparse too.
Let us assume then that both $b$ and $c$ are sparse. Otherwise we might want to sparsify our matrix, using for example the sparsifier given in [45], but it would be of little help since it would take the sparsity to $s = O(\sqrt{n}/\epsilon)$, for $||M - \overline{M}||_2 \leq \epsilon$. However, the problem is that we want $s = O(\log(n))$, so it would not be a good procedure. If $m \geq n$, we can make $\overline{c} = 0_{n \times 1}$ by solving the linear system of equations $c - z^0 = A^T y^0$ for the variable $y^0$ (see [3]). This trick will not be available for $b$ though, because $x^0 > 0$, so we will abandon this procedure in favour of a more powerful one.

The strategy we use is setting $x^0 = z^0 = \epsilon$. We can take $y^0 = 0$ to make things simpler. The idea is dropping all those factors (because they are small) in $\overline{b}$ and $\overline{c}$ (check (3)). In particular, $\overline{b} \approx b$ and $\overline{c} \approx c$. Let us now calculate the error we incur when we effectively make $\overline{b} = b$ and $\overline{c} = c$.

First recall that we define

$$||M||_2 := \max_{ ||x||_2 = 1} ||Mx||_2.$$  (37)

The difference in this case between the sparsed matrix $\overline{M}$ and the original matrix $M$ is $O(n + m)$ terms in a column and those same terms in the corresponding row. It can be calculated that expression (37) for our matrix will be maximized for a vector with all (relevant) entries equal $(1/\sqrt{n})$ except the last one, that can take any value. $\overline{n}$ is defined such that there are $\overline{n} = n + m$ of those entries in the vector, and $2(n + m)$ of non-zero entries concentrated in one column and one row of $||M - \overline{M}||_2$.

It is then easy to see that $||\overline{M} - M||_2 = \sqrt{(\epsilon/\sqrt{n})^2 \overline{n} = \epsilon}$. In brief, we can choose $x^0 = z^0 = \epsilon$, $y^0 = 0$, and sparsify our matrix setting $\overline{b} = b$ and $\overline{c} = c$. We have seen that then $||\overline{M} - M||_2 = \epsilon$.

Let us finally see that this causes an error up to $\epsilon$ when solving the linear system of equations. Suppose that for any $x$, we have a $y$ such that $Mx = y$. If instead of $M$, we use $\overline{M}$, then, by (37),

$$||\overline{M} - M||_2 \leq \epsilon ||y||_2.$$  (38)

Applying $\overline{M}^{-1}$,

$$||\overline{M}^{-1}y - x||_2 = ||\overline{M}^{-1}(\overline{M} - M)x||_2 \leq \epsilon ||\overline{M}^{-1}y||_2,$$  (39)

and we obtain

$$\frac{||\overline{M}^{-1}y - x||_2}{||\overline{M}^{-1}y||_2} \leq \epsilon =: \epsilon_s,$$  (40)

where the subindex $s$ will be used to indicate that the error is associated with the sparsifying procedure. Therefore, we can see that the complexity on the precision of calculating the inverse is also bounded by $O(\epsilon_s^{-1})$.

Using the previously explained procedure may lead to numerical representation problems in the first iteration, but will be solved once $x'$, $z' \neq \epsilon$. We must nevertheless take into account that this problem does not only affect the quantum algorithm since any classical algorithm will have dependence $O(\sqrt{K})$, and the preconditioning will need $O(s^3)$ as well.

Finally, it might be possible to bypass the sparsity problem using [44], although we have not explored it here, and would worsen the dependence on $n$ of the overall algorithm. It would also require a different preconditioning procedure. Using our procedure requires $A$, $b$ and $c$ to be sparse.

### C. Quantum state preparation

If we want to use the QLSA [34] we need to be able to prepare the quantum state $|f\rangle$ and after the procedure, read out the solution. We will follow the procedure of [40] to carry out the former procedure, and we will explain it in this section.

The main problem that preparing a quantum state has is that no arbitrary quantum state preparation procedure is known [10]. However, it is possible to prepare a quantum state of the form

$$|f_T\rangle = \cos \phi_f |f^\perp\rangle |0\rangle_a + \sin \phi_f |f\rangle |1\rangle_a,$$  (41)

where $|f^\perp\rangle$ is a notation for a linear combination such that $\langle f^\perp | f \rangle = 0$.

For preparing it, we will use algorithm 1 from [40] which has a query complexity $O(1)$, and the only condition it gives is that we need an oracle that efficiently takes $|0\rangle \rightarrow |f_j\rangle$ for each $j$, and similarly another one taking $|0\rangle \rightarrow |\phi_j\rangle$ for each $j$ too, in superposition.

Once we have this state we could in principle prepare several copies, and postselect the result on the ancilla being measured on state $|1\rangle$. However, this last step will not be necessary. We will instead carry out the computations in the state $|f_T\rangle$ and we will see how to take care of the ancilla being in the right state during the readout procedure.

The explicit process is explained in algorithm 1 and depicted in figure 1.

### D. Readout of the solution of QLSA: Amplitude Estimation

In the same way that we need some procedure to prepare the quantum state that feeds in the QLSA, we need some way to read out the information in $|d\rangle$, defined as in equation (26). For this, first remember that we are only interested in the amplitude of those
Algorithm 1 State preparation.
1: procedure QUANTUM STATE PREPARATION
2: Direct preparation of an arbitrary quantum state $|b\rangle$ is not usually possible. Instead consider creating the state
\[ |f_T\rangle = \cos \phi_f |f^\perp\rangle |0\rangle_a + \sin \phi_f |f\rangle |1\rangle_a. \]  
This can be done as follows \[40\].
3: Initialize
\[ |\Psi\rangle = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i\rangle |0\rangle |0\rangle_a. \]
4: Query the oracle that calculates the amplitude $f_i$ and phase $\phi_i$ of the vector
\[ |f\rangle = \sum_{i=0}^{N-1} f_i e^{i\phi_i} |i\rangle, \]
controlled by the first register. Then
\[ |\Psi\rangle \rightarrow \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} e^{i\phi_i} |f_i\rangle |\phi_i\rangle |0\rangle_a. \]
5: Apply a controlled phase to $|0\rangle_a$ controlled by $|\phi_i\rangle$.
6: Apply a rotation to the same ancilla qubit, controlled by $|f_i\rangle$. The state is now
\[ |\Psi\rangle \rightarrow \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} e^{i\phi_i} |f_i\rangle |\phi_i\rangle \times \left( \sqrt{1 - C_f^2 f_i^2} |0\rangle_a + C_f f_i |1\rangle_a \right), \]
where $C_f \leq 1/\max f_i$.
7: Uncompute registers 2 and 3 with the inverse oracles. The result is
\[ |\Psi\rangle \rightarrow \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} e^{i\phi_i} |0\rangle |0\rangle \times \left( \sqrt{1 - C_f^2 f_i^2} |0\rangle_a + C_f f_i |1\rangle_a \right). \]
This means that
\[ \sin^2 \phi_f = \frac{C_f^2}{N} \sum_{i=0}^{N-1} f_i^2 = \frac{C_f^2}{N}. \]

FIG. 1. Quantum state preparation procedure. This procedure allows to create the state $|f_T\rangle := \cos \phi_f |f^\perp\rangle |0\rangle_a + \sin \phi_f |f\rangle |1\rangle_a$.

The reader might be surprised since in order to perform the Predictor-Corrector algorithm we need the full solution $|d\rangle$, not just an element of the basis. The solution to this problem consists in classically paralleling the entire procedure, so that the time complexity remains $O(\sqrt{n})$, whereas the space complexity scales to $O(n + m)$, the same as for the pre-conditioning procedure. Or put in another words, in parallel we will solve the same system of equations $O(n + m)$ times (specifically $m + 2n + 3$) and read out one element of the solution vector at each copy of the solution.

The only negative side of using this procedure is that Amplitude Estimation has a time-complexity of $O(\epsilon^{-1})$ instead of the $O(\log(\epsilon^{-1}))$ we would have wished for. Unfortunately, we are not aware of any procedure that could allow us to readout the state faster, and in principle this procedure for Amplitude Estimation is optimal \[39\]. Since Amplitude Estimation requires $O(\epsilon^{-1})$ iterations and in each of those we get an error of sparsification of $M$ (i.e. making $b = b$ and $c = c$) proportional to $\epsilon$, due to the sparsification of $M$, that means that the total complexity on the procedure is $O(\epsilon^{-1} e^s^{-1})$.

There is one more thing we should do: find out the sign of each term in the vector, since amplitude estimation only estimates the absolute value of the amplitude. We propose to use the following method: we are going to check the relative sign of every amplitude to each other, and later on calculate the global sign correction classically checking if one entry ful-

Terms of the result who have the preparation ancilla at state $|1\rangle$, and the ancillas of QLSA in state $|0^\perp\rangle$, where $I$ is defined in the step 6 of the algorithm 3.
Let us now explain the heart of our construction: the QLSA as in [41]. In particular, we will use the Fourier version since even if it has slightly worse scaling in polylogarithmic terms, its Hamiltonian simulation procedure can be used as a black-box and is more generally applicable. Needless to say, it can be substituted by the Chebyshev polynomial approach whenever useful.

The standard procedure at the end of the QLSA algorithm explained here, would be to perform amplitude amplification to amplify the amplitude $\alpha^{-1}$ appearing in front of the term we are interested in, in step 12 in algorithm 3. Note, from steps 4 and 5, that the complexity would be $O(\alpha) = O(k)$. This complexity would be multiplied by the one of preparing $U$ in step 14 of algorithm 3 ($O(sk \text{ polylog}(n + m, e^{-1}))$), which is much bigger than that of preparing $V$ in the same algorithm but in the step 13 [34]. However, in order to avoid having to take into account this additional $k$ factor without making the algorithm more complicated with the trick of Variable-time Amplitude Amplification [33], we pass on the result without amplification as it is done in [10] with the original HHL algorithm [33]. Note also that we cannot use Variable-time Amplitude Amplification entries of the solution vector (and therefore the solution up to a global sign) with the previous procedure in classical parallelism, which amounts to a total space overhead $O(n + m)$ that we already had, and the same time complexity we already incurred in when calculating the amplitudes.
Algorithm 3 QLSA.

1: procedure QLSA
2: Here we explain the algorithm of [34] using the Fourier approach, which is more generally applicable since it uses Hamiltonian simulation as an oracle subroutine.
3: Fourier decomposition of $M^{-1}$
4: Define $\Delta_y = \Theta(\epsilon/\sqrt{\log(k/\epsilon)})$ and $\Delta_z = \Theta(k/\sqrt{\log(k/\epsilon)})^{-1}$ fixed, positive and arbitrarily small.
5: Define $j \in \{0, ..., J - 1\}$ and $l \in \{-L, ..., L\}$, for large $J = \Theta((k/\epsilon) \log(k/\epsilon))$ and $L = \Theta(k \log(k/\epsilon))$.
6: Define $i = j + (l + L) J \in \{0, ..., I\}$.
7: Define $y_l = j \Delta_y$ and $z_l = l \Delta_z$.
8: Define, for sparse operator $M$
   $$T_i(M) = i \sgn(z_i) e^{-i M y_i} z_i.$$ (53)
9: Define
   $$\alpha_i = \frac{1}{2 \pi} \Delta_y \Delta_z |z_i| e^{-i z_i^2 / 2}.$$ (54)
10: One may prove (Lemma 11 on [34]) that $\sum_i \alpha_i T_i$ is $\epsilon$-close to $M^{-1}$ in the domain $D_i := [-1, -1/k] \cup [1/k, 1]$.
11: Implementing the combination of unitaries.
   Lemmas 6, 7, and corollary 10 of [34].
12: Define $\alpha = \sum_i \alpha_i$.
13: Define the operator $V$ such that $V |0^I\rangle = \frac{1}{\sqrt{\pi}} \sum_i \sqrt{\alpha_i} |i\rangle$.
14: Define $U = \sum_i |i\rangle \langle i| \otimes U_i$, where $U_i$ is such that
   $$U_i |0^I\rangle \langle f| = |0^I\rangle \langle f| T_i |f\rangle + |\Psi^+\rangle.$$ (55)
   Whenever $T_i$ is unitary, then $T_i = U_i$.
15: Define $W = V^1 U V$.
16: When we apply $W$ we get
   $$W |0^I\rangle \langle f| \approx \frac{1}{\alpha} |0^I\rangle \langle M^{-1} f| + |\Psi^+\rangle$$ (56)
such that $(|0^I\rangle \langle 0^I| \otimes 1) |\Psi^+\rangle = 0$. Additionally, the first term (the one we are interested in) is $O(\epsilon)$ close to $\frac{1}{\alpha} |0^I\rangle \langle M^{-1} f|$. 

because there is a step involving measurements, incompatible with Amplitude Estimation. In brief: there is no need to perform amplitude amplification, nor a way to do it efficiently (Variable-time Amplitude Amplification would not be unitary as required).

We can see that in algorithm 3 in contrast to what happens in [40], the final complexity is $O(\epsilon^{-1})$ instead of $O(\epsilon^{-2})$ since the procedure of [34] allows us to avoid Phase Estimation. This does not take into account the error due to the sparsification which adds an additional $O(\epsilon^{-1})$.

Taking into account all the points previously explained, we describe the QLSA step-by-step in algorithm 3.

F. On quantizing the termination.

If there exist a feasible and optimal solution, we have seen that the loop should terminate with either procedures (24) or (25). However it is unclear how to carry out this minimization. What we know is that it can be efficiently calculated, or substituted by any more modern and efficient method if found.

The cost of carrying out this termination by classical procedures should be not too big. In fact, according to [46] the overall cost is around that of one iteration of the main loop.

However, we can also propose a quantum method to finish this. It would consist on using a small Grover subroutine 17 to find all solutions of (24b) or (25b) in a small neighbourhood of the latest calculated point. After that, without reading out the state, one could apply 47 to calculate the one with the smallest distance to the calculated point, as in (24a) or (25a). In any case this should be no problem, and should be calculated efficiently.

G. Complexity

In this section we will indicate the complexity of our algorithm against other algorithms that can be used to solve Linear Programming problems. In particular, we will compare against the same Predictor-Corrector algorithm but using the fastest Classical Linear System Algorithm (conjugate gradient descent 48), and against the recent algorithm proposed by Brandão and Svore 20 for solving Semi Definite Programming problems, a more general class of problems than those studied here (Linear Programming).

Firstly, we must take into account that, as we are using the Predictor-Corrector algorithm 27, that means by construction $O(\sqrt{n L})$ iterations of the main loop, and therefore the final time complexity will carry such factor. For sparse problems (as those we are considering), we should also take into account the complexity of solving two Linear Systems of Equations. The QLSA we are using is 34, with complexity $O(\skpoly \log(n + m, k, s, \epsilon^{-1}))$.

Note that we are not using Amplitude Amplification to lower the complexity from $O(k^2)$ to $O(k)$, but this is because we will not be needing it since we do not amplify the amplitude at the end, so we will only have $O(k)$ time complexity. In contrast, the fastest comparable Classical Linear System Algorithm is
### Algorithms for LP

| Algorithm                  | Worst-case time complexity                                                                 | Space complexity | Average time complexity |
|----------------------------|------------------------------------------------------------------------------------------------|------------------|-------------------------|
| Multiplicative weights     | $O((\sqrt{n} (\frac{n+m}{n}) + \sqrt{m}) (\frac{n+m}{m})^3)$                                  | $O(1)$           | –                       |
| Pred-Corr. + Grad des.     | $O((n + m)\sqrt{nkLs^3 \log(\epsilon^{-1})\epsilon_s^{-1}}$                                 | $O(n + m)$       | $O((n + m)\sqrt{nks^3 \log(\epsilon^{-1})\epsilon_s^{-1}}$ |
| Pred-Corr. + Cholesky      | $O(\sqrt{n(n + m)L})$                                                                             | $O((n + m)^2)$  | $O(\sqrt{n(n + m)})$    |
| Pred-Corr. + QLSA          | $O(\sqrt{nLs^3 k\epsilon^{-1} \epsilon_s^{-1}}$                                               | $O(n + m)$       | $O(n + m)$              |

**TABLE I.** Comparison of complexity of different algorithms that can be used for solving LP problems. It includes only leading-order terms. QLSA stands for Quantum Linear System Algorithm, although it is not comparable to usual linear system algorithms, for a series of reasons explained in [49]. ‘Grad des.’ indicates Preconditioned Conjugate Gradient Descent, and ‘Cholesky’ stands for Cholesky decomposition. Note that both the algorithm of Multiplicative weights approach can be applied to more general problems concerning Semidefinite Programming, and that the sparsity they report equals 1 in Linear Programming, as Linear Programming is a particular case of Semidefinite Programming when all matrices are diagonal. Additionally, take into account that the $(O(s^3))$ sparsity contribution to the complexity of QLSA and Preconditioned Conjugate Gradient Descent cases comes mainly (otherwise $O(s)$) from the preconditioning, which we have chosen to be Sparse Approximate Inverse [38], for comparison purposes. $\epsilon_s^{-1}$ is the error coming from the sparsification. Finally, the average time complexity is the same as the worst case except for the fact that it takes into account the independence of the time complexity with respect to $L$ [41].

the conjugate gradient method [48], which in our case has time complexity $O((n + m)sk \log(\epsilon^{-1})\epsilon_s^{-1})$ or $O((n+m)s\sqrt{k}\log(\epsilon^{-1})\epsilon_s^{-1})$ depending on whether the matrix is positive definite or not ($\epsilon_s^{-1}$ comes from the sparsification).

But we also have to take into account other procedures. Those are: the preconditioning using [38] requires $O(n + m)$ space complexity and $O(s^3)$ time complexity. Since this is a classical procedure needed to control $k$, it will be the same for the classical algorithm if we substituted QLSA for conjugate gradient descent, for example. The preparation of quantum states has time complexity $O(1)$, but the readout procedure (Amplitude Estimation, algorithm 2) has $O(n + m)$ space complexity (in each solution we can only read out one amplitude) and $O(\epsilon^{-1}\epsilon_s^{-1})$ time complexity.

Let us therefore compare the complexity of different ways of solving Linear Programming. To leading order, we have table [1].

To summarize all the components in our quantum Predictor-Corrector algorithm and the interrelations among them, we show a diagram in Fig. 3 in the form of a flow chart of actions from the initialization to the termination of the quantum algorithm providing the solution to the given (LP) and (LD) problems in [2] and [3].
FIG. 3. Flow chart of the algorithm
IV. OVERALL STRUCTURE OF THE ALGORITHM

A. Initialization

The initialization procedure consists in preparing the matrix $M$, and the state $f$.

Algorithm 4 Quantum interior point algorithm initialization.

1: procedure Initialization
2: Problem: Solve the following dual problems
3:   minimize $c^Tx$, subject to $Ax \geq b$, $x \geq 0$.  \hspace{1cm} (57)
4:   and
5:   maximize $b^Ty$, subject to $A^Ty \geq c$. \hspace{1cm} (58)
6: Input: Sparse matrix $A \in \mathbb{R}^{m \times n}$, sparse vector $c \in \mathbb{R}^m$, vector $b \in \mathbb{R}^n$.
7: Output: Dual solutions $y \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$, or a signal that the problem is infeasible.
8: Initialization: Want to form the matrix (19).
9: Define $\tau = \kappa = \theta = 1$.
10: Set $x^0 = z^0 = \epsilon \times 1_{n \times 1}$, and $y^0 = 0$.
11: Calculate $\bar{g}$ classically, $O(\log n)$.
12: Set $\bar{c} = c$ and $\bar{b} = b$ (sparsification).
13: Set $t = 0$.

B. Termination

In the termination we propose one possible way of using Grover to run the termination explained in [27]. Any other classical termination is also possible.

Algorithm 5 Quantum interior point algorithm termination.

1: procedure Termination
2: In this section we propose a termination technique using Grover algorithm [17] and [47] to find the optimal solution. We suppose the search space is small enough to allow for this ‘brute force’ search without affecting the complexity class of the main loop. This technique can be nevertheless substituted by any other efficient classical termination.
3: if termination of algorithm 6 was due to 2nd criterion then
4:   \hspace{1cm} (2) or (3) do not have feasible solutions such that $\|(x,z)\| \leq 1/(2\epsilon_3) - 1$. The problem is infeasible or unbounded. Check feasibility with the latest available step.
5: if termination of algorithm 6 was due to 1st criterion then
6:   if $\tau^t \geq \kappa^t$ then
7:     Use Grover search algorithm [17] to find all possible solutions to (24b), without reading them out.
8:     If there is more than one solution, use Grover Search minimum finding algorithm [47] to find the minimum of the possible states.
9:   if $\tau^t < \kappa^t$ then
10:    Use Grover search algorithm [17] to find all possible solutions to (25b), without reading them out.
11:    Use Grover Search minimum finding algorithm [47] to find the minimum of the possible states.
C. Main loop

The main loop consists in two steps called predictor and corrector. The structure of them is very similar:

1. Calculate $f$ and $M$.
2. Use SPAI \[38\] to precondition $M$ and update $M$ and $f$.
3. Prepare $|f\rangle$ and solve $M|d\rangle = |f\rangle$ with QLSA.
4. Read $|d\rangle \rightarrow d$ and calculate the new vector $(y^{t+1},x^{t+1},\tau^{t+1},\theta^{t+1},\kappa^{t+1})$

Note that we have omitted the term $\epsilon_{\delta}$ (error due to sparsification of $M$) to avoid overloading the notation. To restore it, substitute each $\epsilon$ for $\epsilon_{\delta}$.

**Algorithm 6** Quantum interior point algorithm loop.

1: procedure **Main Loop**
2: **Main loop:** Loop $O(L\sqrt{n})$ times over $t$ until one of the following two criteria are fulfilled: Choose $\epsilon_1, \epsilon_2, \epsilon_3$ small numbers and
3: 1. $(x^t/\tau^t)^T(z^t/\tau^t) \leq \epsilon_1$ and $(\theta^t/\tau^t)||\langle\bar{b},\bar{c}|| \leq \epsilon_2$.
4: 2. $\tau^t \leq \epsilon_3$.
5: We will have to iterate $O(\bar{t})$ times: $\bar{t} = \max\{\log(((x^0)^Tz^0)/(\epsilon_1\epsilon_3^2))\},\log(||\langle\bar{b},\bar{c}||/\epsilon_2\epsilon_3)\}$.
6: **Preconditioning and recalculating the state:**
7: 4. Use the algorithm SPAI \[38\] as done in \[40\] to sparsify the matrix $M$. It will take $O(s^3)$, and we will control $k$ as expressed in the preconditioning section.
8: 5. Generate $f$ with $\gamma = 0$ calculating $Xz$ and $\mu$ classically. $O(\log(n + m))$, due to the summation of the inner product.
9: 6. Multiply $f$ by the preconditioning matrix, and make $O(((n + m)k/\epsilon)\sqrt{\log(k/\epsilon)})$ classical copies. $O((\log(((n + m)k/\epsilon)\sqrt{\log(k/\epsilon)})$).
10: **Predictor step:**
11: 8. Generate $O(((n + m)k/\epsilon)\sqrt{\log(k/\epsilon)})$ copies in parallel of $|f\rangle$ as in \[40\]. Use algorithm \[1\].
12: 9. Use the Fourier approach of \[34\] as a QLSA to solve \[19\] $O(n + m)$ times in parallel, without reading the ancilla qubit of its last step. Time complexity: $O(sk\text{polylog}(n + m),\epsilon^{-1})$.
13: 10. Read the results using Amplitude Estimation (algorithm \[2\]). To do that, estimate the amplitude of each of the elements of the result vector in one of the results of the previous steps. This and the previous step take time-complexity $O((k\log(n + m))/\epsilon)$. $A$ is the algorithm composed by the two previous steps: preparing the state $|f\rangle$ and applying QLSA to it. Calculate also the relative sign between entries using the procedure of \[40\].
14: 11: Estimate the modulus and global sign of the classical vector by calculating the first entry of $Ax$ and comparing it with the expected $b$. Use it to update every entry of $x$.
15: 12: Use binary search to find the $\delta$ that fulfills that \[21\] $\in \mathcal{N}(1/2)$.
16: 13: Calculate the values of $(y^{t+1},x^{t+1},\tau^{t+1},\theta^{t+1},\kappa^{t+1})$ using \[20\].
17: 14: $t \leftarrow t + 1$.
18: **Preconditioning and recalculating the state:**
19: 15: Use the algorithm SPAI \[38\] as done in \[40\] to sparsify the matrix $M$. It will take $O(s^3)$, and we will control $k$ as expressed in the preconditioning section.
20: 16: Generate $f$ with $\gamma = 1$ calculating $Xz$ and $\mu$ classically. $O(\log(n + m))$, due to the summation of the inner product.
21: 17: Multiply $f$ by the preconditioning matrix, and make $O(((n + m)k/\epsilon)\sqrt{\log(k/\epsilon)})$ classical copies. $O((\log(((n + m)k/\epsilon)\sqrt{\log(k/\epsilon)})$).
22: **Corrector step:**
23: 20: Generate $O(((n + m)k/\epsilon)\sqrt{\log(k/\epsilon)})$ copies in parallel of $|f\rangle$ as in \[40\]. Use algorithm \[1\].
24: 21: Use the Fourier approach of \[34\] as a QLSA to solve \[19\] $O(n + m)$ times in parallel, without reading the ancilla qubit of its last step. Time complexity: $O(sk\text{polylog}(n + m),\epsilon^{-1})$.
25: 22: Read the results using Amplitude Estimation (algorithm \[2\]). To do that, estimate the amplitude of each of the elements of the result vector in one of the results of the previous steps. This and the previous step take time-complexity $O((ks\log(n + m))/\epsilon)$. $A$ is the algorithm composed by the two previous steps: preparing the state $|f\rangle$ and applying QLSA to it. Calculate also the relative sign between entries using the procedure of \[40\].
26: 23: Estimate the modulus and global sign of the classical vector by calculating the first entry of $Ax$ and comparing it with the expected $b$. Use it to update every entry of $x$.
27: 24: Calculate the values of $(y^{t+1},x^{t+1},\tau^{t+1},\theta^{t+1},\kappa^{t+1})$ using \[21\].
28: 25: $t \leftarrow t + 1$. 

V. CONCLUSIONS

Quantization of Linear Programming problems thus far have been achieved by using multiplicative weight methods as in the pioneering work of Brandão and Svore for Semidefinite Programming (SDP) problems [20], which are more general than LP problems. In this work, we have enlarged the range of applicability of quantum algorithms for LP problems by using Interior Point methods instead. Specifically, our quantum algorithm relies on a type of IP algorithm known as the Predictor-Corrector method that is very well behaved with respect to the feasibility, optimality conditions of the output solution and the iteration complexity.

The core of our quantum IP algorithm is the application of a quantum approximate linear algorithm [33, 40] to an auxiliary system of equations that comprises an homogeneous self-dual primal-dual problem associated to the original LP problem. This is the basis of the Predictor-Corrector method, from which many of its good properties derive. In particular, the iteration complexity of the classical part scales as the square root of the size \( n \) of the cost function. Then, the advantage of the quantum part of the Predictor-Corrector algorithm amounts to not incur in a further penalty in the efficiency of the process, unlike the classical counterpart that incurs in an extra linear term in \( n \). Altogether, we have that the quantum PC algorithm behaves as \( O(\sqrt{nLs^3k}\epsilon^{-1}) \) whereas the best classical IP algorithm has a worst-time complexity of \( O((n+m)\sqrt{nKs^3\log(\epsilon^{-1}))} \) or \( O(\sqrt{n}(n+m)L) \), depending on the method employed.

Hence, this quantum PC algorithm is an hybrid algorithm, partially classical, partially quantum. Applying the HHL quantum algorithm [33] or one of its QLSA variants [40] is not an easy task if we want to achieve a real exponential advantage. These algorithms come with several shortcomings, some of which have been recently overcome [40]. Namely, even though the solution to the system of linear equations can be obtained in a quantum state, then it is not easy to extract all the information provided by the solution. One has to be satisfied by obtaining partial information from the encoded solution such as an expectation value of interest or a single entry of the vector solution. We use the latter strategy to obtain the above quantum advantage in the time complexity at the expense of a trade-off in the space complexity, which turns out to be \( O(n+m) \), in contrast to the \( O(1) \) space complexity of the quantum Brandão-Svore SDP algorithm. This is clearly a matter for future investigations and improvement. Nonetheless, our quantum PC algorithm performs very well with respect to their classical counterpart that is also \( O(n+m) \) or even \( O(n+m)^2 \) if the Cholesky decomposition is used. [33]

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FIG. 4. Scheme of the algorithm

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[50] Upon completion of this work we became aware of a recent work on quantization methods for Interior Point algorithms for both SDP and LP that differs from our work in several crucial features: (1) Their IP algorithms do not belong to the homogeneous Predictor-Corrector class as the ones used here, but rather works by iteratively solving the Newton linear system of equations. (2) Their quantization method relies on QRAM methods of multi-dimensional phaseamacroscopic quantum computations instead of the QLSA that we use to quantize the Predictor-Corrector method, and do not use SPAI algorithms to precondition their linear system of equations before solving it. (3) They report a time complexity $O(n^{1.5} \epsilon^{-2} \mu k^3 \log(\epsilon^{-1}))$, (59) where they assume $m = O(n)$, $\mu$ is upper bounded by $\sqrt{n}$, and $\epsilon^{-1}$ is the precision in the objective function. On the other hand the classical parallelism allows us to report a time complexity on $n$ of $O(\sqrt{n})$, while having a space complexity linear on $n$. Our algorithm does however have the limitation of needing the system to be sparse, as we need to precondition
the system carrying a time complexity $O(s^3)$, which theirs does not.

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