Kaestner Brackets

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Abstract

We introduce Kaestner brackets, a generalization of biquandle brackets to the case of parity biquandles. This infinite set of quantum enhancements of the biquandle counting invariant for oriented virtual knots and links includes the classical quantum invariants, the quandle and biquandle 2-cocycle invariants and the classical biquandle brackets as special cases, coinciding with them for oriented classical knots and links but defining generally stronger invariants for oriented virtual knots and links. We provide examples to illustrate the computation of the new invariant and to show that it is stronger than the classical biquandle bracket invariant for virtual knots.

Keywords: Biquandle brackets, Quantum enhancements, skein invariants, parity biquandles, virtual knots and links

2010 MSC: 57M27, 57M25

1 Introduction

While introducing virtual knots in [6], Kauffman noted that crossings in Gauss code representations of knot diagrams have a well-defined parity of even or odd which does not change under Reidemeister moves. In [5], this property of crossing parity was exploited to define parity biquandles, algebraic structures which may be understood as biquandles with different biquandle operations at the and odd parity crossings.

In [8], the second listed author and collaborators introduced biquandle brackets, skein invariants for biquandle-colored oriented knots and links. The resulting infinite family of invariants includes the classical quantum invariants as well as the quandle and biquandle cocycle invariants as special cases, as well as new invariants. In [9], the second listed author and collaborator introduced a graphical calculus for computing biquandle bracket invariants using trace diagrams to allow for recursive computation as opposed to state-sum style computation of biquandle bracket invariants.

Biquandle bracket invariants are well-defined for oriented virtual knots and links by the usual convention of ignoring virtual crossings when computing biquandle colorings and states in the state-sum expansion of the bracket (or, equivalently, working over a supporting surface of minimal genus). In this paper we apply the biquandle bracket idea to the case of parity biquandles, obtaining an algebraic structure with associated virtual link invariants that we call Kaestner brackets. This infinite family of oriented virtual link invariants includes classical biquandle bracket invariants as a special case – indeed, coincides with them on classical knots – but includes new generally stronger invariants for virtual knots and links.

The paper is organized as follows. In Section 2 we review the basics of biquandles and parity biquandles, and we recall a few facts about signed Gauss codes. In Section 3 we review the basics of biquandle brackets. In Section 4 we generalize biquandle brackets to the case of parity biquandles, defining Kaestner brackets. We provide examples of the computation of a Kaestner bracket polynomial and collect some results from computation via python, in particular showing that Kaestner bracket invariants can distinguish virtual knots which are not distinguished by the corresponding classical biquandle bracket. We conclude in Section 5 with some questions for future research.

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2 Biquandles and Parity Biquandles

We begin with a definition. (See [2] for more.)

**Definition 1.** Let $X$ be a set. A *biquandle structure* on $X$ consists of binary operations $\bowtie, \triangledown : X \times X \to X$ such that

(i) For all $x \in X$, $x \bowtie x = x \triangledown x$,

(ii) For all $x, y \in X$, the maps $\alpha_y : X \to X$ and $\beta_y : X \to X$ defined by

$$\alpha_y(x) = x \triangledown y, \beta_y(x) = x \bowtie y$$

are all invertible, and

(iii) For all $x, y, z \in X$, we have the following *exchange laws*:

$$((x \bowtie y) \bowtie (z \triangledown y)) = (x \bowtie (y \triangledown z)) \bowtie (y \triangledown z)$$

$$((x \bowtie y) \triangledown (z \triangledown y)) = (x \bowtie (y \triangledown z)) \bowtie (y \triangledown z)$$

$$((x \triangledown y) \bowtie (z \triangledown y)) = (x \triangledown (y \triangledown z)) \bowtie (y \triangledown z)$$

**Definition 2.** Let $L$ be an oriented virtual link diagram and let $X$ be a biquandle. An *$X$-coloring* or *biquandle coloring* of $L$ by $X$ is a labeling of all of the semiarcs of $L$ with elements of $X$ as shown:

Note that at virtual crossings the biquandle colors do not change, i.e we have

$$x \bowtie y = tx + (s-t)y \quad x \triangledown y = sy.$$

The biquandle axioms are chosen so that given a biquandle coloring of an oriented virtual link diagram before a Reidemeister move, there is a unique biquandle coloring of the diagram after the move which agrees with the original coloring outside the neighborhood of the move. In particular, we have the following result:

**Theorem 1.** Let $X$ be a finite biquandle, and let $L$ be an oriented virtual link diagram. Then the number of $X$-colorings of an oriented virtual link is an integer-valued invariant.

**Example 1.** Let $R$ be a commutative ring with identity and two distinguished units $t, s$. Then $R$ is a biquandle (called an *Alexander biquandle*) under the operations

$$x \bowtie y = tx + (s-t)y \quad x \triangledown y = sy.$$
Example 2. In fact, we note that we can relax the “commutative ring with identity” condition somewhat; any module over a ring \( R \) with commuting units \( s, t \) forms an Alexander biquandle. For example, the ring \( M_n(\mathbb{F}) \) of \( n \times n \) matrices over a field \( \mathbb{F} \) becomes an Alexander biquandle with a choice of two commuting invertible matrices \( t, s \). When \( \mathbb{F} \) is finite, so is the resulting biquandle.

Example 3. Let \( X = \{1, 2, \ldots, n\} \). We can specify a biquandle structure on \( X \) by giving the operation tables of \( \uplus \) and \( \uplus' \) explicitly, e.g.

\[
\begin{array}{c|ccc}
\uplus & 1 & 2 & 3 \\
1 & 1 & 3 & 2 \\
2 & 3 & 2 & 1 \\
3 & 2 & 1 & 3 \\
\end{array}
\quad \begin{array}{c|ccc}
\uplus' & 1 & 2 & 3 \\
1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 \\
\end{array}
\]

Next, we recall **parity biquandles** (see [5]):

**Definition 3.** Let \( X \) be a set together with four binary operations \( \uplus_0, \uplus_1, \uplus'_0, \uplus'_1 \). Then we call \((X, \uplus_0, \uplus_1, \uplus'_0, \uplus'_1)\) a **parity biquandle** iff

- \((X, \uplus_0, \uplus'_0)\) is a biquandle (note, \((X, \uplus_1, \uplus'_1)\) need not be a biquandle),
- For all \( x, y \in X \), the maps \( \alpha^1_y, \beta^1_y : X \to X \) and \( S^1 : X \times X \to X \times X \) defined by
  \[
  \alpha^1_y(x) = x \uplus'_1 y, \beta^1_y(x) = x \uplus'_1 y \quad \text{and} \quad S^1(x, y) = (y \uplus^1 x, x \uplus'^1 y)
  \]
  are all invertible, and
- For all \((a, b, c) \in \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}\) and for all \( x, y, z \in X \), we have the **mixed exchange laws**:
  \[
  (x \uplus^a y) \uplus^b (x \uplus^c y) = (x \uplus^b x) \uplus^a (y \uplus^c x),
  (x \uplus^a y) \uplus^b (x \uplus^c y) = (x \uplus^b z) \uplus^a (y \uplus^c z),
  (y \uplus^a x) \uplus^b (z \uplus^c x) = (y \uplus^b z) \uplus^a (x \uplus^c z)
  \]

**Example 4.** Biquandles are parity biquandles where \( \uplus_0 = \uplus_1 \) and \( \uplus'_0 = \uplus'_1 \).

**Example 5.** The operation tables

\[
\begin{array}{c|ccc}
\uplus_0 & 1 & 2 & 3 \\
1 & 1 & 3 & 3 \\
2 & 2 & 2 & 2 \\
3 & 3 & 1 & 1 \\
\end{array}
\quad \begin{array}{c|ccc}
\uplus'_0 & 1 & 2 & 3 \\
1 & 1 & 3 & 3 \\
2 & 2 & 2 & 2 \\
3 & 3 & 1 & 1 \\
\end{array}
\quad \begin{array}{c|ccc}
\uplus_1 & 1 & 2 & 3 \\
1 & 1 & 3 & 3 \\
2 & 2 & 2 & 2 \\
3 & 3 & 1 & 1 \\
\end{array}
\quad \begin{array}{c|ccc}
\uplus'_1 & 1 & 2 & 3 \\
1 & 1 & 3 & 3 \\
2 & 2 & 2 & 2 \\
3 & 3 & 1 & 1 \\
\end{array}
\]

define a parity biquandle structure on the set \( \{1, 2, 3\} \).

We note the following results; see [5].

**Lemma 2.** Let \( L \) be an oriented virtual knot or link diagram and let \( c_i \) be a crossing in \( L \). Then virtual Reidemeister moves preserve the parity of \( c_i \).

**Lemma 3.** Let \( L \) be as above. Then for any Reidemeister III move, either all three crossings are even, or two are odd and one is even.

Next, we briefly recall **signed Gauss codes**; see [6, 3].

**Definition 4.** Let \( L \) be an oriented link represented as a planar diagram with \( n \) crossings. Then we encode \( L \) in a string of symbols by the following scheme:
• Pick some starting point \( p_0 \) on \( K \) and begin transversing \( K \) in the positive direction as specified by the orientation. Label new crossings as with \( 1, \ldots, n \) in the order that they’re encountered. Each crossing should be visited exactly twice; we only label a crossing the first time.

• Whenever we encounter a crossing, we record three pieces of information:
  – The crossing label,
  – whether we’re on the under/overstrand, and
  – the sign of the crossing.

We can write this compactly by \( k^x_\epsilon \), where \( k \in \{1, \ldots, n\} \) is the label we’ve assigned our crossing, \( x \in \{u, o\} \) denotes whether we’re on the understrand or overstrand, and \( \epsilon \in \{+, -\} \) denotes the sign of \( k \).

Repeat for each component of \( L \). The resulting set of ordered strings is the **signed Gauss code** of \( L \).

**Example 6.** The signed Gauss code

\[
1^- u, 2^- o, 3^- u, 4^- o, 5^- u, 6^- o, 7^- u, 1^- o, 6^+ o, 5^+ o, 4^+ u, 3^+ o, 2^- u, 7^- o
\]

corresponds to a diagram for the knot 72:

Recall (see [3, 6]) that the Reidemeister moves on signed Gauss codes are as follows:

• Reidemeister I moves correspond to insertion/deletion of an adjacent pair:

\[
(\ldots, i_u^+, i_u^-, \ldots) \quad (\ldots, i_o^-, i_o^+, \ldots) \quad (\ldots, i_o^-, i_u^+, \ldots) \quad (\ldots, i_u^+, i_o^-, \ldots)
\]

• Reidemeister II moves correspond to insertion/deletion of a pair of pairs:

\[
(\ldots, i_u^+, j_o^-, \ldots) \quad (\ldots, i_u^-, j_o^+, \ldots) \quad (\ldots, i_o^+, j_u^-, \ldots) \quad (\ldots, i_o^-, j_u^+, \ldots)
\]

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\]

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\[
(\ldots, i_u^+, j_o^-, \ldots) \quad (\ldots, i_u^-, j_o^+, \ldots) \quad (\ldots, i_o^+, j_u^-, \ldots) \quad (\ldots, i_o^-, j_u^+, \ldots)
\]
• Reidemeister III moves correspond to flipping three pairs. We have two cases, depending on the connectivity:

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\begin{array{3 Biquandle Brackets
Next, we recall a definition (see [8, 9]).

**Definition 5.** Let \((X, \Sigma, \Delta)\) be a biquandle and \(R\) a commutative ring with identity. Let \(w \in R^x, \delta \in R,\) and \(A, B : X \times X \to R^x\) such that

- For all \(x \in X,\)
  \[-A_{x,x}^2 B_{x,x}^{-1} = w,\]

- For all \(x, y \in X,\)
  \[-A_{x,y}^{-1} B_{x,y} - A_{x,y} B_{x,y}^{-1} = \delta,\]

- For all \(x, y, z \in X,\) we have
  \[A_{x,y} A_{x,y,z} A_{y,z} = A_{y,z} A_{x,z} A_{x,y,z} A_{x,z} A_{x,z},\]
  \[A_{x,y} B_{x,y,z} B_{y,z} = B_{y,z} A_{y,z} A_{x,z} A_{x,y} A_{x,y,z},\]
  \[B_{x,y} B_{y,z} A_{y,z} A_{x,z} = A_{x,y} B_{x,y,z} B_{y,z} A_{y,z},\]
  \[A_{x,y} B_{x,y,z} A_{y,z} A_{x,z} = A_{y,z} A_{x,z} A_{x,z} A_{x,z} A_{y,z} A_{x,z} A_{x,z},\]
  \[B_{y,z} B_{x,y,z} A_{y,z} A_{x,z} = A_{x,y} B_{x,y,z} B_{y,z} A_{y,z},\]
  \[B_{x,y} B_{y,z} A_{y,z} A_{x,z} = A_{x,y} B_{x,y,z} B_{y,z} A_{y,z},\]

where we write \(A(x, y)\) and \(B(x, y)\) as \(A_{x,y}\) and \(B_{x,y}\) respectively. Then we call \((A, B)\) an \(X\)-bracket over \(R.\)

The biquandle bracket axioms were chosen such that the state sum obtained by recursively simplifying a
knot or link diagram $L$ according to the biquandle-colored skein relations

then replacing each smoothed component with $\delta$ and multiplying by the writhe correction factor $w^{n-p}$ (where $n, p$ are the numbers of negative and positive crossings respectively) is invariant under biquandle-colored Reidemeister moves. It follows that the multiset of state-sum values over the set of biquandle colorings is an invariant of oriented classical and virtual knots and links.

Let us write $[D]$ for the state-sum value of a diagram $D$. Then it is straightforward to verify that we have

as well as

hence we can define the invariant as follows:

**Definition 6.** Let $L$ be an oriented knot or link diagram with a coloring by a finite biquandle $X$ and let $\beta$ be an $X$-bracket. Denote the number of positive crossings by $p$, the number of negative crossings by $n$, and the total number of crossings by $N$. 
• A state of \( L \) is a choice of a type \( A \) or type \( B \) smoothing at each crossing \( j = 1, \ldots, N \) in \( L \). We denote a state by \( S \in \{A, B\}^N \), and denote the coefficient of the \( j \)th smoothing in \( S \) by \( C^S_j \).

• For each state \( S \in \{A, B\}^N \), let \( m \) be the number of simple closed curves in \( S \), and define

\[
C_S = \delta^m \prod_{j=1}^N C^S_j.
\]

Then the state-sum \( \beta(L) \) given by

\[
\beta(L) = w^{n-p} \sum_{S \in \{A, B\}^N} C_S
\]

is invariant under \( X \)-colored Reidemeister moves. See \([8, 9]\) for more.

Before moving on, it’s worth noting that the smoothings defined above have two undesirable effects. First, both type \( A \) and type \( B \) smoothings can break colorings, as they concatenate incompatibly-colored semiarcs. Second, type \( B \) smoothings leave us with no well-defined orientation to our knots. This makes the idea of “recursively” constructing the biquandle bracket invariant appear nonsensical.

One possible solution is to simply “perform all the smoothings simultaneously” and obtain the state-sum form of the invariant. However, this loses one of the primary benefits of skein-relation-based invariants: namely, that partially-smoothed states can be simplified using the Reidemeister moves, thus reducing computations we need to perform.

To address these problems, in \([9]\) the second listed author and collaborator reformulated biquandle brackets in terms of oriented spatial graphs called trace diagrams. The idea is to encode oriented links as graphs where the vertices correspond to crossings, and the directed edges correspond to oriented semiarcs. By extending biquandle colorings to these graphs, we obtain a straightforward way to formalize valid manipulations on partially-smoothed states.

**Definition 7.** A trace diagram is an encoding of a partially smoothed oriented knot in a decorated directed graph as follows:

• Let \( L \) be an oriented knot or link diagram with crossings \( \{c_1, \ldots, c_n\} \). Since each crossing \( c_i \) involves an overstrand and an understrand, we need to add two corresponding vertices \( u_i \) and \( o_i \) to our digraph to determine which is which. Thus, let \( V = \{u_1, o_1, \ldots, u_n, o_n\} \).

• For each oriented semiarc connecting crossings \( (c_i, c_j) \) in \( L \), add a directed edge from \( u_i/o_i \) to \( u_j/o_j \) as appropriate. If \( L \) is colored by some biquandle \( X \), color all these edges accordingly.

• For each \( c_i \), add a directed edge \( (u_i, o_i) \) if \( c_i \) is positive, and \( (o_i, u_i) \) if \( c_i \) is negative. We call these edges traces, and will draw them in our diagrams with a dashed line. Traces corresponding to positive crossings are called positive traces, and analogously in the negative case.

**Remark 1.** In the case of classical knots and links, trace diagrams may be visualized in \( \mathbb{R}^3 \)

The purpose of including the directed trace edges is to allow us to maintain biquandle colorings after we perform smoothings. Note that \( \delta \) has been chosen such that when drawing the \( o_i \) on top of the \( u_i \), positive traces point out of the page.

\[1\] Virtual crossings are permitted.

\[2\] However in practice, we will simply draw \( u_i \) and \( o_i \) on top of each other and use breaks in our edges to denote this instead.
Example 7. The trefoil can be encoded in the following trace diagram:

By drawing the \( o_i, u_i \) on top of each other, the correspondence becomes easier to see:

Trace diagrams offer us a nice way to encode smoothings. We choose the following convention:

Definition 8. Let \( X \) be a biquandle, and let \( L \) be an \( X \)-colored knot/link with trace diagram \( G = (V, A) \). Let \( c_i \) be a crossing in \( L \), and let \( (\cdot, o_i), (o_i, \cdot), (\cdot, u_i), (u_i, \cdot) \), be the non-trace directed edges associated to \( o_i \) and \( u_i \). Then we define a smoothing at \( c_i \) by the rules

| Sign of \( c_i \) | Type A | Type B |
|------------------|--------|--------|
| + \( (o_i, \cdot) \mapsto (u_i, \cdot) \) & \( (\cdot, o_i) \mapsto (\cdot, u_i) \) |        |        |
| - \( (\cdot, o_i) \mapsto (\cdot, u_i) \) & \( (o_i, \cdot) \mapsto (o_i, \cdot) \) |        |        |

which we can represent diagrammatically by

for positive crossings, and
for negative crossings. Flattening the 3-D perspective, we see that these rules reflect the same smoothings employed in the biquandle bracket map:

\[
\begin{align*}
\text{Flattening perspective:} & \\
\text{Positive crossing:} & \quad \text{(same as in the biquandle bracket map)} \\
\text{Negative crossing:} & \quad \text{(same as in the biquandle bracket map)}
\end{align*}
\]

Note that the rules we have chosen result in trace arrows pointing in positive \(x, y\) for positive crossings, and negative \(x, y\) for negative crossings.

**Definition 9.** Given a virtual knot or link \(L\), we construct the trace diagram for \(L\) identically to in the classical case, now decorating each vertex with parity information and denoting virtual crossings with circles.

**Remark 2.** In the case of virtual knots and links, the three-dimensional visualization above should be understood locally within a neighborhood of each classical crossing. Since the virtual Reidemeister moves do not change either the biquandle colorings or connectivity in an oriented virtual link diagram, it follows that starting with a biquandle-colored oriented virtual link diagram we can apply the biquandle-bracket skein relations (either with or without traces) to obtain a state-sum value \(\beta\) which is invariant under virtual Reidemeister moves.

### 4 Kaestner Brackets

We will now generalize biquandle brackets to the case of parity biquandles. Note that the perhaps natural term for this construction, “parity biquandle bracket”, is in fact already in use in the literature with a different meaning – in [4], the previously defined parity bracket construction from [7] was enhanced with biquandle colorings, where in our case we are enhancing biquandle brackets by replacing biquandles with parity biquandles. After briefly considering introducing a distinction between “(parity biquandle) brackets” and “parity (biquandle brackets)”, we opted instead to name the new structure in honor of Aaron Kaestner, who introduced parity biquandles in [5].

**Definition 10.** Let \(X = (X, \oplus^0, \ominus^0, \ominus^1, \oplus^1)\) be a parity biquandle, and let \(R\) be a commutative ring with identity. Let \(A_0, B_0, A_1, B_1 : X \times X \to R^\times\). Then the collection \((X, A_0, B_0, A_1, B_1)\) is a Kaestner bracket if it satisfies the following conditions:

- \((X, A_0, B_0)\) is a biquandle bracket,
- \(A_1, B_1 : X \times X \to R^\times\) are invertible,
- For all \(x, y \in X\),
  \[
  \delta = -A_{1,x,y} \cdot B_{1,x,y} - A_{1,x,y}^{-1} \cdot B_{1,x,y}
  \]
• For all \((a, b, c) \in \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}\), we have the following (brace labels correspond to the trace diagrams in the following section):

\[
\begin{align*}
(i) \quad & A_{a,x,y} \cdot A_{b,x,z} \cdot A_{c,y,z} = A_{c,y,z} \cdot A_{b,x,z} \cdot A_{a,x,y} \\
(V) \quad & B_{a,x,y} \cdot B_{b,x,z} \cdot A_{c,y,z} = A_{c,y,z} \cdot B_{b,x,z} \cdot B_{a,x,y} \\
(VI) \quad & B_{a,x,y} \cdot B_{b,x,z} \cdot A_{c,y,z} = A_{c,y,z} \cdot B_{b,x,z} \cdot B_{a,x,y} \\
(VII) \quad & A_{a,x,y} \cdot A_{b,x,z} \cdot A_{c,y,z} = A_{a,x,y} \cdot A_{b,x,z} \cdot A_{c,y,z} \\
(VIII) \quad & B_{a,x,y} \cdot B_{b,x,z} \cdot B_{c,y,z} = B_{a,x,y} \cdot B_{b,x,z} \cdot B_{c,y,z}
\end{align*}
\]

This definition is motivated by the following skein relations

\[
\begin{align*}
& x \xrightarrow{y^a} y \xleftarrow{x^a} \\
& y \xrightarrow{y^a} y \xleftarrow{x^a}
\end{align*}
\]

with the same values of \(\delta\) and \(w\) as before. Then we obtain the conditions above by considering trace diagram
reductions of both sides of the positive Reidemeister III move. Applying the smoothing rules, the LHS gives

while the RHS gives

**Example 8.** Let $X$ be the parity biquandle in Example 5 and let $R = \mathbb{Z}_5$. The reader can verify that the coefficients in the tables

| $A_0$ | 1 | 2 | 3 |
|-------|---|---|---|
| 1     | 1 | 4 | 4 |
| 2     | 4 | 1 | 4 |
| 3     | 4 | 4 | 1 |

| $B_0$ | 1 | 2 | 3 |
|-------|---|---|---|
| 1     | 1 | 4 | 1 |
| 2     | 1 | 4 | 1 |
| 3     | 1 | 1 | 4 |

| $A_1$ | 1 | 2 | 3 |
|-------|---|---|---|
| 1     | 1 | 2 | 1 |
| 2     | 2 | 2 | 3 |
| 3     | 4 | 2 | 4 |

| $B_1$ | 1 | 2 | 3 |
|-------|---|---|---|
| 1     | 1 | 2 | 1 |
| 2     | 2 | 2 | 3 |
| 3     | 4 | 2 | 4 |

define a Kaestner bracket over $R$ with $w = -(1)^2(4^{-1}) = 1$ and $\delta = -(1^{-1})(4) - 4^{-1}(1) = 1 + 1 = 2$. This data encodes 54 skein relations depending on the parity, crossing sign and colors at a crossing, including for
We can now state our main definition.

**Definition 11.** Let $X$ be a parity biquandle, $R$ a commutative ring with identity, and $\beta$ a Kaestner bracket on $X$ with coefficients in $R$. Then for any oriented virtual link $L$, we define the *Kaestner bracket polynomial* of $L$ to be the sum of contributions of a formal variable $u$ to the power of the state sum $\beta(L_f)$ over the set of $X$-colorings $L_f$ of $L$, i.e.,

$$\Phi^\beta_X(L) = \sum_{L_f \in \mathcal{C}(L,X)} u^{\beta(L_f)}.$$

By construction, we have the following:

**Proposition 4.** For any finite parity biquandle $X$ and Kaestner bracket $\beta$, $\Phi^\beta_X(L)$ is a invariant of oriented virtual links.

**Example 9.** The virtual trefoil knot 2.1 has three colorings by the parity biquandle in Example 5 as shown.
Let us compute the state-sum value $\beta$ for the first one.

$$\Phi_X^\beta(K) = \begin{array}{c|c}
\Phi_X^\beta(K) & K \\
\hline
3 & 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 4.9, 4.10, 4.11, 4.12, 4.13, 4.14, 4.15, \\
& 4.16, 4.17, 4.18, 4.19, 4.20, 4.21, 4.22, 4.23, 4.24, 4.29, 4.30, 4.31, \\
& 4.32, 4.33, 4.34, 4.35, 4.36, 4.37, 4.38, 4.39, 4.40, 4.41, 4.42, 4.48, \\
& 4.49, 4.50, 4.51, 4.52, 4.57, 4.58, 4.59, 4.60, 4.61, 4.62, 4.63, 4.64, \\
& 4.65, 4.66, 4.67, 4.68, 4.69, 4.70, 4.71, 4.72, 4.78, 4.79, 4.82, 4.83, \\
& 4.84, 4.85, 4.86, 4.87, 4.88, 4.89, 4.90, 4.91, 4.92, 4.93, 4.94, 4.95, \\
& 4.96, 4.97, 4.98, 4.99, 4.103, 4.104, 4.105, 4.106, 4.107, 4.108 \\
3u & 2.1, 4.1, 4.2, 4.3, 4.6, 4.7, 4.8, 4.25, 4.26, 4.43, 4.46, 4.47, 4.53, \\
& 4.55, 4.56, 4.73, 4.75, 4.76, 4.77, 4.80, 4.91, 4.100, 4.102 \\
3u^2 & 4.4, 4.5, 4.27, 4.28, 4.44, 4.45, 4.54, 4.74, 4.81, 4.101 \\
\end{array}$$

The four states contribute $1 + 2 + 2 + 1 = 1$ for a state-sum contribution of $u^1$ to the invariant value. Repeating with other two colorings, we obtain $\Phi_X^\beta(2.1) = 3u$. This result distinguishes $2.1$ from the unknot, which has invariant value $\Phi_X^\beta(\text{unknot}) = 3u^2$.

**Example 10.** We computed $\Phi_X^\beta$ using the parity biquandle and Kaestner bracket from Example 8 for the virtual knots in the knot atlas $[\text{I}]$. The results are in the table. Note that the counting invariant alone does not distinguish any of the virtual knots on the table, while this particular Kaestner bracket sorts them into three classes.

In particular, treating all crossings as even and evaluating the resulting classical biquandle bracket in this example yields the value $3$ for all of the virtual knots on the table; hence this example shows that Kaestner brackets define a stronger invariant in general than their classical biquandle bracket counterparts. Moreover, since the unknot has invariant value $3u^2$, this invariant detects the nontriviality of all of the virtual knots in the table!
5 Questions

We conclude in this section with some possibilities for future directions of research and interesting questions surrounding Kaestner brackets and other types of biquandle brackets. Namely,

- Is the current list of axioms minimal? It seems likely that the answer is “no.” Finding a simpler set of axioms could improve the performance of computer search programs by allowing fewer condition checks.
- Limiting our search to brackets with coefficients in finite rings allows for computer searches; we are extremely interested in finding brackets with coefficients in infinite rings ($\mathbb{Z}$, polynomials over infinite fields, etc.)
- What ways are there to modify the value $\delta$ of a state component to depend on the biquandle coloring as well?

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