ON THE INTERACTION OF ELECTROMAGNETIC WAVES WITH CONDUCTORS.

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We study the interaction of electromagnetic waves with electrons. Our results can be applied to radio waves in the ionosphere or to lasers impinging on metals causing melting. We generalize the classical analysis of Zener to the case which includes the interactions of the electrons with lattice vibrations or the positive ions. We use the induced polarization to give a globally coherent and unifying analysis of the two cases, where collisions are important and where they are negligible.

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INTRODUCTION

Born and Wolf in their famous book “Optics” have devoted a chapter on the “Optics of metals”. They have discussed the subject exhaustively, explaining the Drude theory of conductivity, taking into account the free electrons, which are free to move inside the metal, except for occasional collisions with lattice vibrations. The electrons have a finite conductivity as a consequence of these collisions with the lattice vibrations. Born and Wolf have calculated reflection and transmission coefficients of electromagnetic waves.

A very simple model of the electrons in metals, which assumes no collisions, was given by Zener. Zener’s model is applicable where the electron density is very low and there are almost no collisions with the positive ions. Zener’s theory is actually well suited for applications to the much heavier positive ions. We use the induced polarization to give a globally coherent and unifying fashion

Although Zener’s theory is restrictive to certain conditions, it provides a good model for understanding the process, and is useful for students at the undergraduate level. In the present paper, we extend Zener’s theory to include collisions with the lattice vibrations. Our results are applicable to the case when the frequency of the electromagnetic wave is much lower than the frequency of the collisions of the electrons with the lattice vibrations. The merit of Zener’s approach is in its simplicity; and our approach is just as simple and gives a clear picture of the process. Thus teachers can use our theory to teach

at the undergraduate level.

In another experimental situation, an intense laser beam can produce local melting in metal plates. We can conclude from our discussions that the frequency of laser should be lower that the collision frequency, in that way leading to the transfer of energy to the lattice from the laser through collisions of electrons with the lattice vibrations, and induce the melting.

Early experiments on reflection and transmission of electromagnetic radiation from thin plates of alkali metals were done by Woods. His observations were explained by Zener, who assumed that the electrons in alkali metals are completely free, and they respond to the electric field of the incident radiation. It was observed by Kronig that although the electrons in these metals are free to move inside, they are in fact scattered by lattice vibrations. They are not completely free. If τ is the mean time between successive collisions, then the electrons are really free, only for a time t ≪ τ; and consequently, Zener’s theory is only good as long as the period of oscillation of the radiation 1/ω is smaller than τ. That means ωτ ≫ 1. If ωτ ≪ 1, then Zener’s theory cannot be applied. In the present paper we extend Zener’s analysis exactly to the case ωτ ≪ 1.

Our analysis presents this interaction through the use of the polarization that the electromagnetic wave induces in the conductor. We can then present the two cases where collisions are important and where they are negligible in a single, logically coherent and unifying fashion

POLARIZATION

When an electric field is applied to a conducting medium, the mobile charges immediately react to the field and rearrange in an effort to cancel it. In the cases that we shall consider, it is the electrons that move, while the much heavier positive ions are essentially fixed in position. They may actually be stuck in place, as in a solid conductor or, simply being much heavier than the electrons, their motion occurs on a much longer time scale
and hence is negligible.

On the other hand even the mobile charges are assumed to be at thermal equilibrium. Such an equilibrium is only established through the existence of collisions. However, there is a time scale associated with the collisions and a length scale which depend on the temperature and the density. Taking a Maxwell-Boltzmann distribution for the mobile charge carriers, which is a very good approximation for atmospheric plasmas and for metals that are not too cold (so that the electrons are not degenerate), the relations are

$$v \sim (k_B T/m_e)^{1/2}, \quad l \sim 1/(n\sigma), \quad \tau \sim l/v \quad (1)$$

where $v$ is the root mean square velocity, $l$ is the mean free path, and $\tau$ is the collision time, which are functions of $k_B$ the Boltzmann constant, $T$ the temperature, $m_e$ the electron mass, $n$ the density of the electrons, and $\sigma$ the total scattering cross-section. But for the Coulomb interaction without screening, the total cross-section is straightforward and yields $\pi l_D$, where $l_D$ is the distance over which charge is screened within a conductor. The derivation of the Debye length is straightforward and yields

$$l_D \sim (k_B T/n e^2)^{1/2}. \quad (2)$$

Calculation of the scattering cross-section crucially requires taking into account the screening length, otherwise the results are divergent.

Now for an electromagnetic wave, the electric field is oscillating back and forth with frequency $\omega$, and depending on how high the frequency is, it is important to either neglect collisions or to take collisions into account. Thus there are two natural regimes, $\omega \tau \gg 1$ and $\omega \tau \ll 1$. In the former, collisions are unimportant and we treat the electrons in the section below called Free Electrons. In the latter, the collision are important, and our corresponding section following is called Scattered Electrons.

In both regimes, the electrons are able to move relative to the heavy or stationary positive ions. This relative movement induces a local, microscopic dipole moment throughout the conductor, albeit for a finite amount of time that is proportional the inverse of the frequency of the electromagnetic radiation. This local, microscopic dipole moment is entirely analogous to the way an electric field induces a polarization in a dielectric (non-conducting) medium, although the clear difference is that a dielectric can support static electric fields and static dipole moments, while a conductor cannot. The local, microscopic dipole moment fluctuates wildly over distances comparable with the inter-particle separation, however it is the averaged value of the field produced that is important. The analysis is again very straightforward, we refer the interested reader to the classic reference by Jackson. The result is that there is a polarization that is induced throughout the medium which tries to compensate the applied electric field. This polarization field, $\vec{P}$, is given by

$$\vec{P} = n \langle \vec{p}_{mol} \rangle \quad (3)$$

where $\vec{p}_{mol}$ is the dipole moment of each microscopic molecule and $\langle \cdots \rangle$ indicates averaging over a macroscopically small volume, which nevertheless contains many microscopic molecules. The induced electric field is simply $4\pi \vec{P}$ and the displacement is defined as

$$\vec{D} = \vec{E} + 4\pi \vec{P}. \quad (4)$$

Computing the average polarization is very easy. The dynamics of the individual electron motion is governed by Newton’s law; the forces acting on the electron are the electric field (the magnetic field also exercises a force on the electrons, but this is down by a power of $v/c$) and the frictional forces if there are collisions. If $x(t)$ is the electron position (considering for simplicity motion along the $x$ direction), the microscopic dipole moment is $p_{mol} = -e x(t)$ as the corresponding positive ion is assumed to stay immobile at $x = 0$, and the consequent polarization per unit volume is obtained by multiplying by the number density, $P = -n e x(t)$.

**FREE ELECTRONS**

As stated above, Zener’s model is well suited for treating electrons in the ionosphere. The density of electrons in the ionosphere is very low, and the electrons hardly ever collide with the ions. Then, Zener’s approach is most appropriate for dealing with the reflection of electro-magnetic waves. It is well known that radio signals can be sent across oceans or over long distances across continents because they can be reflected off the ionosphere. The ionosphere is a collection of electrons and an equal number of protons forming a shell around the earth. The theory of reflections by free-electrons is easy to understand: If an electromagnetic wave is incident on the ionosphere, the electrons which are much lighter than the ions, are accelerated by the electric field of the electromagnetic wave.

Without loss of generality, take the direction of the electric field to be along the $x$ direction. All quantities will have a periodic variation of angular frequency $\omega$. Thus, the electric field will be

$$E_x = E e^{i(kz-\omega t)}. \quad (5)$$

We also know this periodic variation exists in all quantities we deal with as the equations are linear. We will choose not to write the exponential which is there in all quantities of interest. Of course the electric field is actually a real quantity, hence what we mean is the real part
of Equation (5). This does not mean that the periodic behaviour is always necessarily \( \cos(kz - \omega t) \), we allow \( E \) to be in general a complex number, which then permits any real combination of cosines and sines. Indeed, there may appear explicit factors of \( i \) in the subsequent equations, this simply corresponds to a relative phase difference of \( \pi/2 \), i.e. cosines become sines and vice versa.

In this electric field, electrons will be accelerated according to Newton’s law, the following equation:

\[
\ddot{x} = -\frac{eE}{m} \quad (6)
\]

or

\[
-\omega^2 x = -\frac{eE}{m} \quad (7)
\]

The contribution to the dipole moment by one electron is then

\[
-e x = -\frac{e^2 E}{m \omega^2} . \quad (8)
\]

If \( n \) is the density of electrons per unit volume, the total dipole moment per unit volume is given simply by

\[
P = -\frac{n e^2 E}{m \omega^2} . \quad (9)
\]

The dielectric constant at frequency \( \omega \) is easily found through the relation

\[
\epsilon E = D = E + 4\pi P . \quad (10)
\]

This gives

\[
\epsilon(\omega) = 1 + \frac{4\pi n e^2}{m \omega^2} = 1 - \frac{\omega_p^2}{\omega^2} , \quad (11)
\]

where \( \omega_p^2 = \frac{4\pi n e^2}{m} \), is known as the (square of the) plasma frequency. The electromagnetic wave equation in a nonmagnetic, isotropic medium is

\[
\ddot{D} = c^2 \nabla^2 E . \quad (12)
\]

Which yields

\[
-\omega^2 \epsilon = -c^2 k^2 \quad (13)
\]

that is

\[
\omega^2(1 - \frac{\omega_p^2}{\omega^2}) = c^2 k^2 . \quad (14)
\]

This is called the dispersion relation. At frequencies less than \( \omega_p \), \( k^2 \) will be negative and hence \( k \) will be imaginary. If \( k \) is imaginary, the wave cannot propagate and it will be totally reflected.

The above analysis can be directly applied to the case of free electrons in metals. The plasma frequency, \( \omega_p \), of the conduction electrons in metals is very high and much higher than the frequency of visible light, which is reflected by metals. Thus, metals look shiny.

SCATTERED ELECTRONS

We are successful in explaining why the metals shine, but we note that the free electrons in conducting solids are free only for a short interval of time. They interact with lattice vibrations and are scattered after a time \( \tau \), the average time between successive collisions. The above analysis can only be applied to conductors if the frequency is large \( \omega \tau \gg 1 \). The electrons can be accelerated only for a time \( \tau \). At low frequencies, the period of oscillation of the electric field is much larger than the time between successive collisions \( \tau \). As a consequence of suffering many collisions, the electrons are not accelerated indefinitely, and acquire a drift velocity

\[
\dot{x} = -\frac{eE \tau}{m} . \quad (15)
\]

Equation (15) is the defining relation for the collision time \( \tau \). At low frequencies \( \omega \tau \ll 1 \). We must replace equation (6) by equation (15) giving us

\[
-i\omega x = -\frac{eE \tau}{m} , \quad (16)
\]

that is

\[
x = \frac{e \tau}{i \omega m} E . \quad (17)
\]

Since the polarization, \( P \) is given by

\[
P = -ne \frac{-e^2 \tau}{i \omega m} E , \quad (18)
\]

we get

\[
\epsilon E = \left( 1 - \frac{4\pi n e^2 \tau}{i \omega m} \right) E \quad (19)
\]

which gives

\[
\epsilon = 1 - \frac{4\pi n e^2 \tau}{i \omega m} \quad (20)
\]

that is

\[
\epsilon = 1 - \frac{\omega_p^2 \tau}{i \omega} . \quad (21)
\]

Define \( \alpha = \frac{\omega_p^2 \tau}{\omega} \), then

\[
\epsilon = 1 + i\alpha . \quad (22)
\]

The dispersion relation in equation (14) becomes

\[
\omega^2 (1 + i\alpha) = c^2 k^2 . \quad (23)
\]

If \( k = k_1 + ik_2 \), then

\[
\omega^2 (1 + i\alpha) = c^2 (k_1^2 - k_2^2 + 2ik_1k_2) . \quad (24)
\]
Equating the real and imaginary parts, we find
\[
\frac{\omega^2}{c^2} = k_1^2 - k_2^2, \tag{25}
\]
\[
\frac{\omega^2}{c^2} \alpha = 2k_1k_2, \quad (k_1^2 - k_2^2) \alpha = 2k_1k_2; \tag{26}
\]
\[
k_2^2 + \frac{2k_1}{\alpha} k_2 - k_1^2 = 0. \tag{27}
\]
Then we find for the imaginary part
\[
k_2 = -\frac{k_1}{\alpha} \pm \sqrt{\frac{k_1^2}{\alpha^2} + k_1^2} \tag{28}
\]
\[
= -\frac{k_1}{\alpha} \pm k_1 \left(1 + \frac{1}{\alpha^2}\right)^{\frac{1}{2}} \tag{29}
\]
\[
\approx \frac{k_1}{\alpha} \pm k_1 \left(1 + \frac{1}{2\alpha^2}\right) \tag{30}
\]
where the approximation is valid for \(\alpha \gg 1\) which is the same as assuming a low frequency \(\omega\).
\[
k_2 = k_1 \left(1 + \frac{1}{\alpha} + \frac{1}{2\alpha^2}\right). \tag{31}
\]

The imaginary part is slightly larger than \(k_1\). This indicates that the wave is absorbed in a distance of less than one wave length, and does not penetrate into the conductor.

We have chosen to take the + sign of the radical in equation (28) because the negative sign will give us a negative value of \(k_2\) which would imply the unphysical result that the wave is amplified. There is no reason for the amplification. In our analysis, we have introduced collisions of electron which is a dissipative mechanism. We expect the wave to lose energy and not gain it.

**DISCUSSION AND CONCLUSION**

In conclusion, we observe introduction of scattering and interactions of electrons brusquely impacts the behaviour of electromagnetic wave in conducting media. The analysis of Zener for the free electron model cannot be applied, specifically in the case \(\omega \tau \ll 1\). However a modification of that analysis can be applied to this region taking into account the collisions and the drift velocity that they produce. The analysis presented here is similar in spirit to Drude’s original analysis [2], however there one passes through the conductivity that is induced in the electron current, rather than the more direct route presented here of the polarization that is induced and its effect on the Maxwell equations.

Thus we see, in a very simple and pedagogical manner, that at radio frequencies an electromagnetic wave will travel a great distance along the surface of the earth due to multiple reflections off the ionosphere. Here the free electron model is sufficient to explain this behaviour. On the other hand, the same electromagnetic wave will be damped in a real conductor, in which electrons interact with the lattice. Here the scattering of the electrons with the lattice vibrations cannot be neglected. Thus, for example, a superconductor, where electron scattering from the lattice is completely absent, will reflect radio waves, but if it becomes normal it will suddenly absorb the waves. Although it is well understood that superconductivity owes its existence to electron lattice interactions, nevertheless, the electrons in the superconducting state absorb energy, due to interactions with the phonons, but at an exponentially suppressed rate compared to normal conduction electrons [12]. Hence in the superconductor, for temperatures significantly below the transition temperature, the attenuation of ultrasonic waves vanishes. In this sense, it is correct to interpret this fact simply as the complete transparency of the superconducting electrons to sound vibrations impressed upon the thermal phonon distribution.

Our presentation makes use of the induced polarization in the conductor to analyze the interaction in the two regimes in a globally coherent, unifying and logical fashion. We find that this way of presenting the analysis bypasses the necessity of introducing the conductivity, which is only defined without subtlety in one of the regimes. In such analyses, the conductivity is computed, and then fed back in to Maxwell’s equations to obtain the dispersion relation [11].

The conductivity \(\sigma\) is defined by the relation
\[
\vec{J} = \sigma \vec{E}. \tag{32}
\]
This relation is the microscopic basis of electrical resistivity. For a linear, homogeneous conductor of length \(L\) and sectional area \(A\), we get (taking all vectors in one given direction for simplicity and so dropping the vectorial signs)
\[
I = J \times A = \frac{\sigma \times A}{L} E \times L = \frac{1}{R} \times V \tag{33}
\]
where \(I\) is the current, \(V\) is the voltage difference and \(R\) is the resistivity. In a collisionless plasma, the resistance is zero, thus the notion of conductivity is subtle. In fact it can be defined, but turns out to be purely imaginary. Thus it represents in fact a phase lag between the applied electric field and the corresponding current. Thus using the conductivity, for example as in [11], to approach the limit of a collisionless electron gas to arrive at the plasma frequency is not as direct an approach, as we have done in our section on Free Electrons.

The polarization however is a globally well defined concept in both regimes whether collisions are negligible or
important. Therefore we find that our presentation is simpler, globally coherent and a logical method for exposing and studying the interaction of electromagnetic waves on conductors.

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[1] M. Born and E. Wolf, Principles of Optics. Oxford: Pergamon, 6th ed., (1980)
[2] P. Drude, “Zur Elektronentheorie der Metalle, 1.Teil”, Annalen der Physik, 1, 566-613, (1900); “Zur Elektronentheorie der Metalle, 2.Teil”, ibid, 3, 369-402 (1900).
[3] C. Zener, “Remarkable optical properties of the alkali metals”, Nature, 968-968, (1933).
[4] R. de L. Kronig, “Remarkable optical properties of the alkali metals”, Nature, 601-610, (1933).
[5] P. N. Saeta, J. K. Wang, Y. Siegal, N. Bloembergen, and E. Mazur, “Ultrafast electronic disordering during femtosecond laser melting of GaAs”, Phys. Rev. Lett., 67, 1023-1026, (1991); D. v. d. Linde and K. Sokolski-Titten, “The physical mechanisms of short-pulse laser ablation”, Appl. Surf. Sci, vol. 154-155, 1-10, (2000).
[6] R. W. Woods, “Remarkable optical properties of the alkali metals”, Physical Review, 44, 353-360, (1933)
[7] see for example the textbook “A Course in Statistical Thermodynamics”, J. Kestin and J. R. Dorfman, Academic Press, (1971).
[8] P. Debye and E. Hückel “Zur Theorie der Elektrolyte. I. Gefrierpunktserneniedrigung und verwandte Erscheinungen [The theory of electrolytes. I. Lowering of freezing point and related phenomena]” Physikalische Zeitschrift 24: 185–206(1923).
[9] “Statistical Physics Part 1, Landau and Lifshitz Course of Theoretical Physics Volume 5”, enlarged and revised by E. M. Lifshitz and L. P. Pitaevski, Pergamon Press (1980).
[10] “Classical Electrodynamics”, J. D. Jackson, John Wiley (1962).
[11] “Solid State Physics”, N. W. Ashcroft and N. D. Mermin, Holt, Rinehart and Winston, (1976).
[12] J. Bardeen, L. N. Cooper and J. R. Schrieffer, “Theory of the superconductivity”, Physical Review, 108, 1175-1204, (1955); T. Tsumoto, “Ultrasonic attenuation in superconductors”, Physical Review, 121, 402-415, (1961); L. P. Kadanoff and A. B. Pippard, “Ultrasonic attenuation in superconductors”, Proc. Royal Soc. London A, 31, vol. 292, no. 1430, 299-313, (1966)