Quantum inequalities for the free Rarita-Schwinger fields in flat spacetime

Hongwei Yu \(^{a,b}\) and Puxun Wu \(^b\)

\(^a\)CCAST(World Lab.), P. O. Box 8730, Beijing, 100080, P. R. China.
\(^b\)Department of Physics and Institute of Physics, Hunan Normal University, Changsha, Hunan 410081, China\(^*\).

Abstract

Using the methods developed by Fewster and colleagues, we derive a quantum inequality for the free massive spin-3/2 Rarita-Schwinger fields in the four-dimensional Minkowski spacetime. Our quantum inequality bound for the Rarita-Schwinger fields is weaker, by a factor of 2, than that for the spin-1/2 Dirac fields. This fact along with other quantum inequalities obtained by various other authors for the fields of integer spin (bosonic fields) using similar methods lead us to conjecture that, in the flat spacetime, separately for bosonic and fermionic fields, the quantum inequality bound gets weaker as the the number of degrees of freedom of the field increases. A plausible physical reason might be that the more the number of field degrees of freedom, the more freedom one has to create negative energy, therefore, the weaker the quantum inequality bound.

\(^*\)Mailing address
I. INTRODUCTION

It is well established that the energy density of a field, which is strictly positive in classical physics, can become negative and even unbounded from below in quantum field theory due to quantum coherence effects [1]. Specific experimentally studied examples of quantum states exhibiting negative energy density are squeezed states of light in quantum optics [2] and the Casimir vacuum state of quantized fields [3]. As a result, all the known pointwise energy conditions in classical general relativity, such as the weak energy condition and null energy condition, can be violated.

However, if the negative energy density in quantum field theory is unconstrained, i.e., if an arbitrary amount of negative energy is allowed to persist for an arbitrary long period of time, then serious ramifications result. These include exotic phenomena such as violation of the second law of thermodynamics [4,5], traversable wormholes [6,7], "warp drive" [8], and even time machines [7,9]. Therefore, a lot of effort has been made toward determining the extent to which these violations of local energy are permitted in quantum field theory. One powerful approach is that of the quantum inequalities constraining the magnitude and duration of negative energy regions [10–30].

The work on quantum inequalities was pioneered by Ford [10], who derived an inequality type of bound on negative energy fluxes for the quantized, massless, minimally-coupled scalar fields in flat spacetime. Similar results for the sampled energy density have been subsequently established for both massless and massive scalar fields and electromagnetic fields in Minkowski spacetime [11,14] as well as in static curved spacetimes [16,17]. However, in all these works, a Lorentzian sampling function

\[ f(\tau) = \frac{\tau_0}{\pi(\tau^2 + \tau_0^2)} \]  

was employed in the calculations. Note that here \( \tau_0 \) sets the characteristic averaging timescale.

Progress has been made toward removing the restriction of the Lorentzian weight to include arbitrary sampling functions. In this regard, Flanagan [15] obtained optimal quantum inequalities for the massless scalar field in two dimensions for arbitrary smooth positive sampling functions. Fewster and colleagues [19,22,23] derived the quantum inequalities for the minimally-coupled scalar field in static curved spacetimes of any dimension for an arbitrary sampling function. More recently, Pfenning [27] established a quantum inequality for electromagnetic field in static curved spacetimes for arbitrary positive sampling functions using the techniques developed by Fewster and colleagues for scalar fields in [19] and [22].

On the other hand, work is also being done for fields other than scalar and electromagnetic ones. Investigations on spin-\( \frac{1}{2} \) Dirac fields have been carried out by various authors [21,24,25,28,30]. Specific quantum states with negative energy density have been examined and shown to satisfy the quantum inequalities for the scalar field obtained with a Lorentzian sampling function [21,28]. Using arguments similar to those of Flanagan’s [15], Vollick [24] derived an optimal quantum inequality for the Dirac field in two dimensions. Fewster and Verch [25] have established the existence of quantum inequalities for the Dirac (and Majorana) field in general 4-dimensional globally hyperbolic spacetimes, and more recently Fewster
and Mistry [30] have presented an explicit quantum inequality bound for the Dirac field in four-dimensional Minkowski spacetime using the modified methods for scalar fields. Recently quantum inequalities have also been established for massive spin-one Proca fields in globally hyperbolic spacetimes whose Cauchy surfaces are compact and have trivial first homology group by Fewster and Pfenning [29]. As a further step along this line, we will present a quantum inequality for massive spin $-\frac{3}{2}$ Rarita-Schwinger fields in four-dimensional Minkowski spacetime for arbitrary, smooth positive sampling functions using the methods developed by Fewster and colleagues in [25,30]. The quantum inequality we are going to prove is, for any real-valued, smooth, compactly supported function $g$,

$$\int dt \langle \rho(t,x) \rangle g(t)^2 \geq -\frac{1}{12\pi^3} \int_m^\infty du |\hat{g}(u)|^2 u^4 Q_{RS}^3(u/m),$$

(2)

where $\langle \rho(t,x) \rangle$ is the quantum expectation value of the energy density of the field and

$$Q_{RS}^3(x) = 8 \left(1 - \frac{1}{x^2}\right)^{3/2} - 6Q_{3}^B(x),$$

(3)

and

$$Q_{3}^B(x) = \left(1 - \frac{1}{x^2}\right)^{1/2} \left(1 - \frac{1}{2x^2}\right) - \frac{1}{2x^4} \ln(x + \sqrt{x^2 - 1}).$$

(4)

We will work in the units where $c = \hbar = 1$ and take the signature of the metric to be $(+,−,−,−)$.

**II. RARITA-SCHWINGER FIELDS AND THE QUANTUM INEQUALITY**

Let us start with a review of the basics of free Rarita-Schwinger fields [31,32]. The Rarita-Schwinger fields describe particles of spin $\frac{3}{2}$ and they satisfy the following equations

$$(-i\gamma \cdot \partial + m)\psi^\mu = 0, \quad \gamma_\mu \psi^\mu = 0,$$

(5)

where $\gamma \cdot \partial = \gamma^\nu \partial_\nu$. The $\gamma$-matrices are given in terms of the Pauli matrices $\sigma_i$ by

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \quad (i = 1, 2, 3)$$

(6)

and obey $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. The Lagrangian for the field can be written as

$$L = \frac{i}{2} \bar{\psi} \gamma^\mu \gamma^\nu \partial_\mu \psi^\nu - m\bar{\psi}\psi + \frac{i}{6} \bar{\psi} \gamma^\mu \gamma^\nu \gamma^\rho (\partial_\mu \partial_\nu + \gamma_\mu \partial_\nu) \psi^\rho + \frac{i}{6} \bar{\psi} \gamma^\mu \gamma^\nu \gamma^\rho (\gamma_\mu \partial_\nu \psi^\rho + \frac{1}{3} m\bar{\psi}\gamma_\mu \gamma^\nu \psi^\rho).$$

(7)

A complete set of solutions for the field equations is given by

$$\mathcal{U}^\mu_{k\sigma} e^{ik\cdot x} \quad \sigma = 1,..4, \quad \mu = 0, 1, 2, 3.$$  

(8)
Here \(\mathcal{U}_{k\sigma}^{\mu}\) and \(\mathcal{V}_{k\sigma}^{\mu}\) can be expressed in terms of the Dirac spinors and a triad of four-vectors \(\epsilon_1(k), \epsilon_2(k), \epsilon_3(k)\) as

\[
\mathcal{U}_{k1} = \epsilon_1(k) \otimes u_{k1},
\]

\[
\mathcal{U}_{k2} = \sqrt{\frac{1}{3}} \epsilon_1(k) \otimes u_{k2} - \sqrt{\frac{2}{3}} \epsilon_3(k) \otimes u_{k1},
\]

\[
\mathcal{U}_{k3} = \sqrt{\frac{1}{3}} \epsilon_2(k) \otimes u_{k1} + \sqrt{\frac{2}{3}} \epsilon_3(k) \otimes u_{k2},
\]

\[
\mathcal{U}_{k4} = \epsilon_2(k) \otimes u_{k2},
\]

and

\[
\mathcal{V}_{k1} = \epsilon_1(k) \otimes v_{k1},
\]

\[
\mathcal{V}_{k2} = \sqrt{\frac{1}{3}} \epsilon_1(k) \otimes v_{k2} - \sqrt{\frac{2}{3}} \epsilon_3(k) \otimes v_{k1},
\]

\[
\mathcal{V}_{k3} = \sqrt{\frac{1}{3}} \epsilon_2(k) \otimes v_{k1} + \sqrt{\frac{2}{3}} \epsilon_3(k) \otimes v_{k2},
\]

\[
\mathcal{V}_{k4} = \epsilon_2(k) \otimes v_{k2}.
\]

The triad of four-vectors can be written as

\[
\epsilon_\mu^\nu(k) = L_\mu^\nu(k)\epsilon_\nu^\nu(0),
\]

where \(\epsilon_\nu^\nu(0)\) are given by

\[
\epsilon_1(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix}, \quad \epsilon_2(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix}, \quad \epsilon_3(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},
\]

and \(L_\mu^\nu(k)\) by [33]

\[
L_1^i(k) = \delta_{ij} + (\gamma - 1)\tilde{k}_i\tilde{k}_j,
\]

\[
L_0^i(k) = L_0^0(k) = \tilde{k}_i\sqrt{\gamma^2 - 1},
\]

\[
L_0^0(k) = \gamma
\]

with

\[
\tilde{k}_i \equiv \frac{k_i}{|k|}, \quad \gamma \equiv \sqrt{k^2 + m^2} = \frac{\omega_k}{m}.
\]

Let us note that if the momentum is taken to be along the \(z\)-axis one has
\[
\epsilon_1(k) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix}, \quad \epsilon_2(k) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix}, \quad \epsilon_3(k) = \begin{pmatrix} |k| \\ 0 \\ 0 \\ \omega_k/m \end{pmatrix}.
\]

(22)

The Dirac spinors, \(u_{\kappa \alpha}\) and \(v_{\kappa \alpha}\), are

\[
u_{\kappa \alpha} = \left( \frac{\sqrt{\omega_k + m} \, \phi^\alpha}{\sqrt{2\omega_k(\omega_k + m)^2} \, \phi^\alpha} \right),
\]

(23)

\[
v_{\kappa \alpha} = \left( \frac{\sqrt{\omega_k + m} \, \phi^\alpha}{\sqrt{2\omega_k(\omega_k + m)^2} \, \phi^\alpha} \right)
\]

(24)

with \(\phi^{1+} = (1, 0), \phi^{2+} = (0, 1)\). Making use of the above results, one can show that

\[
\sum_{\sigma} U_{k \sigma}^{\mu} U_{\mu k \sigma} = \sum_{\sigma} V_{k \sigma}^{\mu} V_{\mu k \sigma} = \frac{4}{V}.
\]

(25)

For spin \(\frac{3}{2}\) fields, the canonical quantization procedure becomes rather awkward, because of the difficulty of separating dynamical degrees of freedom. In particular, \(\psi^{\mu}\) would have to be decomposed into its irreducible spin \(\frac{1}{2}\) and spin \(\frac{3}{2}\) parts and only the latter part is subject to canonical quantization. To avoid the difficult calculations which this entails, one can bypass the canonical procedure altogether and work directly with the creation and annihilation operators for the normal modes. A consistent quantization can be obtained [34] by expanding the field in terms of the complete set of solutions of Eq. (8) and Eq. (9)

\[
\psi^{\mu}(x) = \sum_{k} \sum_{\sigma}^{4} \left[ c_{k \sigma} U_{k \sigma}^{\mu} e^{ikx} + d_{k \sigma}^{\dagger} V_{k \sigma}^{\mu} e^{-ikx} \right],
\]

(26)

and imposing the following anticommutation relations on the creation and annihilation operators:

\[
\{ c_{k \sigma}, c_{k' \sigma'}^{\dagger} \} = \{ d_{k \sigma}, d_{k' \sigma'}^{\dagger} \} = \delta_{kk'} \delta_{\sigma \sigma'}, 
\]

(27)

\[
\{ c_{k \sigma}, c_{k' \sigma'} \} = \{ d_{k \sigma}, d_{k' \sigma'} \} = \{ c_{k \sigma}, d_{k' \sigma'} \} = \{ c_{k \sigma}, d_{k' \sigma'} \} = 0.
\]

(28)

In order to establish the quantum inequality for the Rarita-Schwinger fields, we will use the following symmetrized energy momentum tensor \(T^{\mu \nu}\) known as Belinfante tensor [35].

\[
T^{\mu \nu} = \partial^\nu \bar{\psi}^\alpha \frac{\partial L}{\partial (\partial_\mu \psi_\alpha)} + \frac{\partial L}{\partial (\partial_\mu \psi_\alpha)} \partial^\nu \bar{\psi}^\alpha - g^{\mu \nu} L + \frac{1}{2} \partial_\beta \left[ \frac{\partial L}{\partial (\partial_\beta \psi_\alpha)} J^{\mu \nu \psi_\alpha} + \bar{\psi}^\alpha j^{\mu \nu} \frac{\partial L}{\partial (\partial_\beta \psi_\alpha)} \right] - \frac{\partial L}{\partial (\partial_\mu \psi_\alpha)} J^{3 \nu \psi_\alpha} - \bar{\psi}^\alpha j^{3 \nu} \frac{\partial L}{\partial (\partial_\mu \psi_\alpha)} - \frac{\partial L}{\partial (\partial_\nu \psi_\alpha)} J^{3 \beta \psi_\alpha} - \bar{\psi}^\alpha j^{3 \beta} \frac{\partial L}{\partial (\partial_\nu \psi_\alpha)} \right],
\]

(29)
where $J^\mu\nu$ and $\bar{J}^\mu\nu$ are the generators of the Lorentz transformations for $\psi$ and $\bar{\psi}$ respectively. Consider an infinitesimal Lorentz transformation

$$x'\mu = x\mu + \omega^{\mu\nu}x_\nu.$$  

We have

$$\delta\psi = \frac{1}{2}\omega_{\alpha\beta}(S^{\alpha\beta} + L^{\alpha\beta})\psi,$$  

$$\delta\bar{\psi} = \frac{1}{2}\omega_{\alpha\beta}(S^{\alpha\beta} - L^{\alpha\beta})\psi,$$

where $S^{\alpha\beta}$ and $L^{\alpha\beta}$, which operate on the spacetime vector and internal spinor indices of $\psi$ respectively, are given by

$$(S^{\alpha\beta})^{\mu\nu} = \eta^{\alpha\mu}\eta^{\beta\nu} - \eta^{\alpha\nu}\eta^{\beta\mu},$$  

$$L^{\alpha\beta} = \frac{1}{4}\{\gamma^\alpha, \gamma^\beta\}.$$  

So, it follows that

$$J^{\alpha\beta} = S^{\alpha\beta} + L^{\alpha\beta},$$  

$$\bar{J}^{\alpha\beta} = S^{\alpha\beta} - L^{\alpha\beta}.$$  

The energy momentum tensor is obtained, after a concrete calculation using the above results and taking the equations of motion into account as

$$T_{\mu\nu} = \frac{i}{4}[\bar{\psi}^{\mu}\gamma_\nu\bar{\psi}_\rho + \bar{\psi}^{\nu}\gamma_\mu\bar{\psi}_\rho].$$  

Hence the energy density is

$$T_{00} = \frac{i}{2}[\psi^{\mu}\psi^{\nu} - \bar{\psi}^{\mu}\bar{\psi}^{\nu}].$$  

The renormalized expectation value of the energy density, i.e., $\langle \rho \rangle = \langle : T_{00} : \rangle$, in an arbitrary quantum state, is given by

$$\langle \rho(t, x) \rangle = \frac{1}{2}\sum_{k,k'}\sum_{\sigma\sigma'} \left\{ (\omega_k + \omega_{k'})\left[ \langle c_{k\sigma} c_{k'\sigma'}\rangle U_{k\sigma}^\dagger U_{k'\sigma'} e^{i(k-k')x} 
\right.ight.$$  

$$+ \langle d_{k\sigma} d_{k'\sigma'}\rangle V_{k\sigma}^\dagger V_{k'\sigma'} e^{-i(k-k')x} \n\right. \right.$$  

$$+ (\omega_k - \omega_{k'})\left[ \langle d_{k\sigma} c_{k'\sigma'}\rangle V_{k\sigma}^\dagger U_{k'\sigma'} e^{-i(k+k')x} \n\right. \right.$$  

$$- \left. \langle c_{k\sigma} d_{k'\sigma'}\rangle U_{k\sigma}^\dagger V_{k'\sigma'} e^{i(k+k')x} \right\}. $$

Consider the sampled energy density measured by a stationary observer at the spatial origin

$$\langle \rho \rangle_f = \int_{-\infty}^{\infty} \langle \rho(t, 0) \rangle f(t) \, dt,$$
where $f$ is a non-negative sampling function. Then

$$
\langle \rho \rangle_f = \frac{1}{2} \sum_{k,k'} \sum_{\sigma,\sigma'} \left\{ (\omega_k + \omega_{k'}) \langle c_{k\sigma}^\dagger c_{k'\sigma'} \rangle U_{k\sigma}^{\mu} U_{k'\sigma'}^{\mu} \hat{f}(\omega_k - \omega_{k'}) + (d_{k\sigma}^\dagger d_{k'\sigma'} \rangle V_{k\sigma}^{\mu} V_{k'\sigma'}^{\mu} \hat{f}(\omega_k - \omega_{k'}) \right\} ,
$$

(40)

where $\hat{f}$ is the Fourier transform of $f$ defined by

$$
\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt .
$$

(41)

Let $f = g^2$ and define a family of operators by

$$
\mathcal{O}_{\lambda i}^{\mu} = \sum_{k'\sigma'} \left\{ \overline{g(-\omega_{k')} + \lambda} \ c_{k'\sigma'}^{\dagger} U_{k'\sigma'}^{\mu} \right\} + \overline{\hat{g}(\omega_{k'} + \lambda) \ d_{k'\sigma'} \ V_{k'\sigma'}^{\mu} \hat{f}(\omega_k - \omega_{k'})} ,
$$

(42)

where $i = 1, \ldots, 4$ is the spinor index and $\overline{g}$ is the conjugate of the Fourier transform. Using the anticommutation relations and the fact that

$$
\sum_{\sigma} U_{k\sigma}^{\mu} U_{\mu k\sigma} = \sum_{\sigma} V_{k\sigma}^{\mu} V_{\mu k\sigma} = \frac{4}{V} ,
$$

(43)

we find

$$
\mathcal{O}_{\lambda i}^{\mu} \mathcal{O}_{\mu i} = S_\lambda + \sum_{kk'} \sum_{\sigma,\sigma'} \left\{ \overline{\hat{g}(\omega_k + \lambda)} \overline{\hat{g}(\omega_{k'} + \lambda)} \ c_{k\sigma}^{\dagger} c_{k'\sigma'} U_{k\sigma}^{\mu} U_{k'\sigma'}^{\mu} - \overline{\hat{g}(\omega_k + \lambda)} \overline{\hat{g}(\omega_{k'} + \lambda)} \ d_{k\sigma}^\dagger d_{k'\sigma'} V_{k\sigma}^{\mu} V_{k'\sigma'}^{\mu} \right\} ,
$$

(44)

where

$$
S_\lambda = \frac{4}{V} \sum_k |\hat{g}(\omega_k + \lambda)|^2 .
$$

(45)

Making use of the following relation which was proven by Fewster and colleagues [25,30] for real-valued, smooth, compactly supported $g = f^{1/2}$

$$
(\omega + \omega') \hat{f}(\omega - \omega') = \int_{-\infty}^{\infty} \frac{\lambda}{\pi} \overline{\hat{g}(\omega - \lambda)} \overline{\hat{g}(\omega' - \lambda)} d\lambda ,
$$

(46)

we can show that
\[
\langle \rho \rangle_f = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\langle O_{\lambda i}^{\mu} O_{\mu,i} \rangle - S_{\lambda}) \lambda \, d\lambda.
\]  

(47)

Now let us calculate the anticommutator \( \{ O_{\lambda i}^{\mu}, O_{\mu,i} \} \) to get

\[
\{ O_{\lambda i}^{\mu}, O_{\mu,i} \} = \sum_{kk'} \sum_{\sigma \sigma'} \left\{ \hat{g}(\omega_k + \lambda) \hat{g}(\omega_{k'} + \lambda) \right\} \left\{ c_{k\sigma}^\dagger c_{k'\sigma'}^\dagger \right\} U_{k\sigma}^{\mu} U_{k'\sigma'}
\]

\[
- \hat{g}(\omega_k + \lambda) \hat{g}(\omega_{k'} + \lambda) \left\{ d_{k\sigma}^\dagger d_{k'\sigma'} \right\} V_{k\sigma}^{\mu} V_{k'\sigma'}
\]

\[
= \sum_k |\hat{g}(\omega_k + \lambda)|^2 \sum_{\sigma} U_{k\sigma}^{\mu} U_{k\sigma} + \sum_k |\hat{g}(\omega_k + \lambda)|^2 \sum_{\sigma} V_{k\sigma}^{\mu} V_{k\sigma}
\]

\[
= \frac{4}{V} \sum_k |\hat{g}(\omega_k + \lambda)|^2 + \frac{4}{V} \sum_k |\hat{g}(\omega_k - \lambda)|^2
\]

\[
= S_{-\lambda} + S_{\lambda}.
\]  

(48)

Here we have used the anticommutation relations Eqs. (27, 28) and appealed to the fact that \( |\hat{g}(x)| \) is an even function since \( g \) is real. An application of the above result leads to

\[
\langle \rho \rangle_f = \frac{1}{2\pi} \int_{0}^{\infty} (\langle O_{\lambda i}^{\mu} O_{\mu,i} \rangle - S_{\lambda}) \lambda \, d\lambda + \frac{1}{2\pi} \int_{0}^{\infty} \left[ (S_{\lambda} + S_{-\lambda}) - (\langle O_{\lambda i}^{\mu} O_{\mu,i} \rangle - S_{\lambda}) \right] \lambda \, d\lambda
\]

\[
= \frac{1}{2\pi} \int_{0}^{\infty} (\langle O_{\lambda i}^{\mu} O_{\mu,i} \rangle - S_{\lambda}) \lambda \, d\lambda + \frac{1}{2\pi} \int_{-\infty}^{0} (S_{-\lambda} - (\langle O_{\lambda i}^{\mu} O_{\mu,i} \rangle)) \lambda \, d\lambda.
\]  

(49)

For all quantum states in which \( \langle O_{\lambda i}^{\mu} O_{\mu,i} \rangle \geq 0 \), the following inequality holds

\[
\langle \rho \rangle_f \geq -\frac{1}{2\pi} \int_{0}^{\infty} \lambda S_{\lambda} \lambda \, d\lambda + \frac{1}{2\pi} \int_{-\infty}^{0} \lambda S_{-\lambda} \lambda \, d\lambda
\]

\[
= -\frac{1}{\pi} \int_{0}^{\infty} \lambda S_{\lambda} \lambda \, d\lambda = -\frac{4}{\pi} \int_{0}^{\infty} d\lambda \frac{1}{V} \sum_k |\hat{g}(\omega_k + \lambda)|^2.
\]  

(50)

Taking the continuum limit \( \frac{1}{V} \sum_k \rightarrow \int \frac{d^3k}{(2\pi)^3} \) and following the same steps as in Ref. [30], we can show that

\[
\langle \rho \rangle_f \geq -\frac{1}{\pi^3} \int_{m}^{\infty} du |\hat{g}(u)|^2 \left( \frac{2}{3} u(u^2 - m^2)^{3/2} - \frac{1}{2} u^4 Q_3^B \left( \frac{u}{m} \right) \right),
\]  

(51)

where

\[
Q_3^B(x) = \left( 1 - \frac{1}{x^2} \right)^{1/2} \left( 1 - \frac{1}{2x^2} \right) - \frac{1}{2x^4} \ln(x + \sqrt{x^2 - 1}).
\]  

(52)

Using the translational invariance of the theory, the quantum inequality can be expressed by the sampled energy density measured by a stationary observer at an arbitrary space time point as
\[ \int dt \langle \rho(t, x) \rangle g(t)^2 \geq -\frac{1}{12\pi^3} \int_{m}^{\infty} du |\hat{g}(u)|^2 u^4 Q^RS_3(u/m). \] (53)

Here

\[ Q^RS_3(x) = 8 \left( 1 - \frac{1}{x^2} \right)^{3/2} - 6Q^B_3(x). \] (54)

The bound is finite since \(|\hat{g}(u)|^2\) decays faster than any polynomial in \(u\) and \(u^4 Q^RS_3(u/m)\) grows like \(u^4\) as \(u \to \infty\). Comparing the above result with that obtained by Fewster and Mistry [30] for the Dirac field, i.e.,

\[ \int dt \langle \rho(t, x) \rangle g(t)^2 \geq -\frac{1}{12\pi^3} \int_{m}^{\infty} du |\hat{g}(u)|^2 u^4 Q^D_3(u/m), \] (55)

one can see that

\[ Q^RS_3(x) = 2Q^D_3(x). \] (56)

So the quantum inequality bound for the free massive Rarita-Schwinger field is weaker, by a factor of 2, than that of the Dirac field.

III. DISCUSSIONS

In conclusion, we have derived a quantum inequality for the free massive Rarita-Schwinger field in Minkowski spacetime for arbitrary smooth positive sampling functions following methods developed by Fewster and colleagues [25,30]. Our quantum inequality bound for Rarita-Schwinger fields is weaker, by a factor of 2, than that for the Dirac field. This seems to be a result of the fact that massive Rarita-Schwinger fields have twice as many number of field degrees of freedom as the Dirac fields. Recall the quantum inequalities that have been established for quantized fields of integer spin in four dimensional Minkowski spacetime using similar methods [19,22,27,29]

\[ \int dt \langle \rho(t, x) \rangle g(t)^2 \geq -\frac{S}{16\pi^3} \int_{m}^{\infty} du |\hat{g}(u)|^2 u^4 Q^B_3(u/m), \] (57)

where \(S\) is just the number of the field degrees of freedom. \(S = 1\) for scalar fields (spin zero), 2 for electromagnetic fields (spin 1) and 3 for massive Proca fields (spin 1). Note, however, that in general curved spacetimes the quantum inequalities of these theories may not simply related by an overall factor. In the same spirit, quantum inequalities obtained so far for the half-integral spin fields can also be cast into the following unified form

\[ \int dt \langle \rho(t, x) \rangle g(t)^2 \geq -\frac{S}{24\pi^3} \int_{m}^{\infty} du |\hat{g}(u)|^2 u^4 Q^F_3(u/m), \] (58)

where
$$Q^F_3(x) = 4 \left(1 - \frac{1}{x^2}\right)^{3/2} - 3Q^B_3(x).$$

(59)

Here $S = 2$ for Dirac fields and 4 for Rarita-Schwinger fields.

An interesting point to note from the above results is that, separately for fields of integer spin (bosonic fields) and those of half-integral spin (fermionic fields), the quantum inequality bound gets weaker as the number of degrees of freedom of the field increases. However, since none of these bounds are optimal, this observation is now more a conjecture than a conclusion. Optimal bounds for all these fields have to be found to see if this is true or even if it is true for both bosonic and fermionic fields combined. Nevertheless, we would like to point out that this is physically plausible, since the more the number of field degrees of freedom, the more freedom one has to create negative energy, therefore, the weaker the quantum inequality bound ought be.

ACKNOWLEDGMENTS

We would like to acknowledge the support by the National Natural Science Foundation of China under Grants No. 10075019 and No. 10375023.
REFERENCES

[1] H. Epstein, V. Glaser and A. Jake, Nuovo Cimento 36, 1016 (1965).
[2] L.-A Wu, H.J. Kimble, J.L. Hall, and H. Wu, Phys. Rev. Lett. 57, 2520(1986).
[3] S.K. Lamoreaux, Phys. Rev. Lett. 78, 5(1997); U. Mohideen and A. Roy, Phys. Rev. Lett. 81, 4549(1998).
[4] L. H. Ford, Proc. R. Soc. Lond. A. 364, 227-236 (1978).
[5] P. C. W. Davies, Phys. Lett. B 11, 215(1982).
[6] M. Morris and K. Thorne, Am. J. Phys. 56, 395(1988).
[7] M. Morris, K. Thorne, and U. Yurtsever, Phys. Rev. Lett. 61, 1446(1988).
[8] M. Alcubierre, Class. Quantum Grav. 11, L73(1994).
[9] A. Everett, Phys. Rev. D 53, 7365(1996).
[10] L. H. Ford, Phys. Rev. D43, 3972 (1991).
[11] L. H. Ford and T. A. Roman, Phys. Rev. D51, 4277(1995).
[12] L. H. Ford and T. A. Roman, Phys. Rev. D 53, 1988(1996).
[13] L. H. Ford and T. A. Roman, Phys. Rev. D 53, 5496(1996).
[14] L.H. Ford and T.A. Roman, Phys. Rev. D55, 2082(1997).
[15] E. E. Flanagan, Phys. Rev. D 56 , 4922(1997).
[16] M.J. Pfenning and L. H. Ford, Phys. Rev. D55, 4813(1997).
[17] M.J. Pfenning and L. H. Ford, Phys. Rev. D 57, 3489(1998).
[18] L.H. Ford, M.J. Pfenning and T.A. Roman, Phys. Rev. D57, 4839(1998).
[19] C.J. Fewster and S.P. Eveson, Phys. Rev. D58 084010 (1998) 1970
[20] H. Yu, Phys. Rev. D 58, 064017(1998).
[21] Dan N. Vollick, Phys. Rev. D 57, 3484 (1998).
[22] C.J. Fewster and E. Teo, Phys. Rev. D 59, 104016(1999).
[23] C.J. Fewster, Class. Quant. Gravit. 17, 1897(2000).
[24] Dan N. Vollick, Phys. Rev. D 61, 084022 (2000).
[25] C. Fewster and R. Verch, Commun. Math. Phys. 225 , 331(2002).
[26] E. E. Flanagan, Phys. Rev. D 66, 104007(2002).
[27] M.J. Pfenning, Phys. Rev. D 65, 024009(2002).
[28] H. Yu and W. Shu, Phys. Lett. B570, 123(2003).
[29] C.J. Fewster and M.J. Pfenning, J. Math. Phys. 44, 4480(2003).
[30] C.J. Fewster and B. Mistry, Phys. Rev. D 68, 105010(2003).
[31] W. Rarita and J. Schwinger, Phys. Rev. 60, 61(1941).
[32] S. Kusaka, Phys. Rev. 60, 61(1941).
[33] S. Weinberg, *The Quantum Theory of Fields, Foundations*, Vol. 1 (Cambridge University Press, 1995), P.68.
[34] D. Lurie, *Particles and Fields*, ( John and Wiley Sons, New York, 1968).
[35] F, Belinfante, Physica (Amsterdam) 6, 887 (1939).