An open problem concerning operator representations of frames

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Abstract

Recent research has shown that the properties of overcomplete Gabor frames and frames arising from shift-invariant systems form a precise match with certain conditions that are necessary for a frame in $L^2(\mathbb{R})$ to have a representation $\{T^k\varphi\}_{k=0}^{\infty}$ for some bounded linear operator $T$ on $L^2(\mathbb{R})$ and some $\varphi \in L^2(\mathbb{R})$. However, for frames of this type the existence of such a representation has only been confirmed in the case of Riesz bases. This leads to several open questions connecting dynamical sampling, coherent states, frame theory, and operator theory. The key questions can either be considered in the general functional analytic context of operators on a Hilbert space, or in the specific situation of Gabor frames in $L^2(\mathbb{R})$.

1 Introduction and motivation

A coherent state is a (typically overcomplete) system of vectors in a Hilbert space $\mathcal{H}$. In general it is given by the action of a class of linear operators on a single element in the underlying Hilbert space. In particular, it could be given by iterated action of a fixed operator on a single element, i.e., as $\{T^k\varphi\}_{k=0}^{\infty}$ for some $\varphi \in \mathcal{H}$ and a linear operator $T : \mathcal{H} \to \mathcal{H}$. Coherent states play an important role in mathematical physics [15, 17], operator theory, and modern harmonic analysis [9, 6]. In particular, a Gabor system (see the definition below) is a coherent state.

Systems of vectors on the form $\{T^k\varphi\}_{k=0}^{\infty}$ also appear in the more recent context of dynamical sampling [1, 2, 3, 18]. Here the goal is to analyse the frame properties of such systems for certain classes of operators, e.g., normal operators or self-adjoint operators. A different approach was taken in the papers [7, 8]: here, the starting point is a frame and the question is when and how it can be represented on the form

$\{T^k\varphi\}_{k=0}^{\infty}$ for some $\varphi \in \mathcal{H}$ and a bounded linear operator $T$. (1.1)
In particular the existence of such a representation was proved for the case where the frame is indeed a basis, so the following discussion deals with the overcomplete case. It turns out that the conditions for the desired representation to exist form a perfect match with the known properties of Gabor frames in $L^2(\mathbb{R})$. This raises the natural question whether (some or all) overcomplete Gabor frames indeed have a representation of the form $(1.1)$. A positive answer to the question would shed new light on Gabor frames, and also provide new insight in the context of dynamical sampling. In fact, only few overcomplete frames are known to have a representation on the form $(1.1)$. To the best knowledge of the authors, Gabor frames have not been considered in this context before, except for the examples appearing in the papers [7, 8].

The above questions will also be analyzed with a different indexing, i.e., considering systems on the form $\{T^k \varphi\}_{k=-\infty}^{\infty}$ instead of $\{T^k \varphi\}_{k=0}^{\infty}$. The indexing in terms of $\mathbb{Z}$ is natural for several well-known classes of frames, and the theoretical conditions for a frame having such a representation with a bounded operator $T$ are similar to the ones for systems indexed by $\mathbb{N}_0$. However, the change in indexing gives an interesting twist on the problem: indeed, a shift-invariant system always has a representation $\{T^k \varphi\}_{k=-\infty}^{\infty}$ with a bounded operator $T$, but it is an open problem whether it also has a representation with the indexing in $(1.1)$. For Gabor systems it is not known whether the chosen indexing play a role.

The question of the existence of a representation on the form $(1.1)$ is interesting for general frames as well. We will also formulate an open problem for general frames in an arbitrary separable Hilbert space; a negative answer to that question would also lead to a negative answer to the above specific questions.

In Section 2, we will provide the necessary background on frames and operator representations. The presentation is kept as short as possible, and only contains the information that is necessary in order to understand the open problems; these are outlined in Section 3. Section 4 contains further motivation, analysis of special cases, and results about certain orderings of the frame elements that must be avoided if we want to obtain a representation of the form $(1.1)$.

2 Technical background

2.1 Frame theory

Let $\mathcal{H}$ denote a separable Hilbert space. A sequence $\{f_k\}_{k \in I}$ in $\mathcal{H}$ is a frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that $A \|f\|^2 \leq \sum_{k \in I} |\langle f, f_k \rangle|^2 \leq B \|f\|^2, \forall f \in \mathcal{H}$; it is a frame sequence if the stated inequalities hold for all $f \in \text{span}\{f_k\}_{k \in I}$. The sequence $\{f_k\}_{k \in I}$ is a Bessel sequence if at least the upper frame condition holds. Also, $\{f_k\}_{k \in I}$ is called a Riesz sequence if there exist constants $A, B > 0$ such that $A \sum |c_k|^2 \leq \|\sum c_k f_k\|^2 \leq B \sum |c_k|^2$ for all finite scalar sequences $\{c_k\}_{k \in I}$. A Riesz basis is a Riesz sequence $\{f_k\}_{k \in I}$ for
which $\text{span}\{f_k\}_{k \in I} = \mathcal{H}$.

If $\{f_k\}_{k \in I}$ is a Bessel sequence, the *synthesis operator* is defined by

$$U : \ell^2(I) \rightarrow \mathcal{H}, \quad U\{c_k\}_{k \in I} := \sum_{k \in I} c_k f_k; \quad (2.1)$$

it is well known that $U$ is well-defined and bounded. A central role will be played by the kernel of the operator $U$, i.e., the subset of $\ell^2(I)$ given by

$$N_U = \left\{ \{c_k\}_{k \in I} \in \ell^2(I) \mid \sum_{k \in I} c_k f_k = 0 \right\}. \quad (2.2)$$

The *excess* of a frame is the number of elements that can be removed yet leaving a frame. It is well-known that the excess equals $\dim(N_U)$; see [4].

Given a Bessel sequence $\{f_k\}_{k \in I}$, the *frame operator* $S : \mathcal{H} \rightarrow \mathcal{H}$ is defined by $S := UU^*$. For a frame $\{f_k\}_{k \in I}$, the frame operator is invertible and $f = \sum_{k \in I} (f, S^{-1} f_k) f_k, \forall f \in \mathcal{H}$. The sequence $\{S^{-1} f_k\}_{k \in I}$ is also a frame; it is called the *canonical dual frame*.

Note that every orthonormal basis (or more generally, every Riesz basis) $\{f_k\}_{k \in I}$ is a frame. A frame that is not a basis is said to be overcomplete. We refer to [6] and [13] for more information about frames and Riesz bases.

### 2.2 Operator representations of frames

Dynamical sampling typically concern frame properties of sequences in a Hilbert space $\mathcal{H}$ of the form $\{T^k \varphi\}_{k=0}^{\infty}$, where $\varphi \in \mathcal{H}$ and $T : \mathcal{H} \rightarrow \mathcal{H}$ is a linear operator. We will also consider sequences indexed by $\mathbb{Z}$; thus, in the sequel the index set $I$ denotes either $\mathbb{Z}$ or $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$.

As in the papers [7, 8] we will look at the topic from the opposite side. Indeed, we will consider a given frame $\{f_k\}_{k \in I}$ in a Hilbert space $\mathcal{H}$ and ask for the existence of a representation on the form

$$\{f_k\}_{k \in I} = \{T^k \varphi\}_{k \in I}, \quad (2.3)$$

where $T : \text{span}\{f_k\}_{k \in I} \rightarrow \mathcal{H}$ is a bounded linear operator. Note that we will consider $\{f_k\}_{k \in I}$ as a sequence, i.e., as an ordered set.

The following result collects findings from the papers [7, 8], describing when a representation of the form $(2.3)$ is possible, and when the operator $T$ can be chosen to be bounded. We will need the right-shift operator on $\ell^2(I)$, defined by

$$T : \ell^2(I) \rightarrow \ell^2(I), \quad T\{c_k\}_{k \in I} = \{c_{k-1}\}_{k \in I};$$

for the case $I = \mathbb{N}_0$ we define $c_{-1} := 0$.  

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Proposition 2.1 Consider a frame \( \{ f_k \}_{k \in I} \) for an infinite-dimensional Hilbert space \( \mathcal{H} \), with \( I \) denoting either \( \mathbb{N}_0 \) or \( \mathbb{Z} \). Then the following hold:

(i) There exists a linear operator \( T : \text{span}\{f_k\}_{k \in I} \to \mathcal{H} \) such that (2.3) holds if and only if \( \{ f_k \}_{k \in I} \) is linearly independent.

(ii) Assume that \( \{ f_k \}_{k \in I} \) is linearly independent. Then the operator \( T \) in (2.3) is bounded if and only under right-shifts; in particular \( T \) is bounded if \( \{ f_k \}_{k \in I} \) is a Riesz basis.

(iii) Assume that \( \{ f_k \}_{k \in I} \) is linearly independent and overcomplete. If the operator \( T \) in (2.3) is bounded, then \( \{ f_k \}_{k \in I} \) has infinite excess.

Let us mention an example of an overcomplete frame that indeed has a representation on the form (1.1); the example is due to Aldroubi et al. [2].

Example 2.2 Consider the matrix \( T = [a_{ij}]_{i,j \in \mathbb{N}} \) given by \( a_{jj} = 1 - 2^{-j}, a_{ij} = 0, i \neq j \), and the sequence \( g := \{ \sqrt{1 - (1 - 2^{-j})^2} \}_{j \in \mathbb{N}} \). Then the system \( \{ T^k g \}_{k=0}^\infty \) is a frame for \( \ell^2(\mathbb{N}) \) but not a basis. □

2.3 Shift-invariant systems and Gabor frames

Shift-invariant systems and Gabor systems are defined in terms of certain classes of operators on \( L^2(\mathbb{R}) \). For \( a \in \mathbb{R} \), define the translation operator \( T_a \) acting on \( L^2(\mathbb{R}) \) by \( T_a f(x) := f(x - a) \) and the modulation operator \( E_a \) by \( E_a f(x) := e^{2\pi iax} f(x) \). Both operators are unitary. Furthermore, defining the Fourier transform of \( f \in L^1(\mathbb{R}) \) by \( \hat{f}(\gamma) = \mathcal{F} f(\gamma) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i\gamma x} dx \) and extending it in the standard way to a unitary operator on \( L^2(\mathbb{R}) \), we have \( T_a \mathcal{F} = \mathcal{F} E_a \).

Given a function \( \varphi \in L^2(\mathbb{R}) \) and some \( b > 0 \), the associated shift-invariant system is given by \( \{ T_{kb} \varphi \}_{k \in \mathbb{Z}} \). The following result collects the necessary information about such systems. Consider the function \( \Phi(\gamma) := \sum_{k \in \mathbb{Z}} |\hat{\varphi}(\frac{\gamma + k}{b})|^2, \gamma \in \mathbb{R} \).

Proposition 2.3 Let \( \varphi \in L^2(\mathbb{R}) \setminus \{0\} \), and \( b > 0 \) be given. Then the following hold:

(i) \( \{ T_{kb} \varphi \}_{k \in \mathbb{Z}} \) is linearly independent.

(ii) \( \{ T_{kb} \varphi \}_{k \in \mathbb{Z}} \) is a Riesz basis if and only if there exist \( A, B > 0 \) such that \( A \leq \Phi(\gamma) \leq B, \text{ a.e. } \gamma \in [0, 1] \).

(iii) \( \{ T_{kb} \varphi \}_{k \in \mathbb{Z}} \) is a frame sequence if and only if there exist \( A, B > 0 \) such that \( A \leq \Phi(\gamma) \leq B, \text{ a.e. } \gamma \in [0, 1] \setminus \{ \gamma \in [0, 1] \mid \Phi(\gamma) = 0 \} \).
(iv) If \( \{T_{kb}\varphi\}_{k \in \mathbb{Z}} \) is an overcomplete frame sequence, it has infinite excess.

(v) \( \{T_{kb}\varphi\}_{k \in \mathbb{Z}} = \{(T_k)^k\varphi\}_{k \in \mathbb{Z}} \); i.e., the system \( \{T_{kb}\varphi\}_{k \in \mathbb{Z}} \) has the form of an iterated system indexed by \( \mathbb{Z} \).

The result in (i) is well-known, and (ii) & (iii) are proved in [5]; (iv) is proved in [4, 8], and (v) is clear.

Example 2.4 Letting

\[
\text{sinc}(x) := \begin{cases} \frac{\sin(\pi x)}{\pi x} & \text{if } x \neq 0; \\ 1 & \text{if } x = 0, \end{cases}
\] (2.5)

Shannon’s sampling theorem states that the shift-invariant system \( \{T_k\text{sinc}\}_{k \in \mathbb{Z}} \) form an orthonormal basis for the Paley-Wiener space

\[
P_W := \left\{ f \in L^2(\mathbb{R}) \mid \text{supp} \hat{f} \subseteq \left[ -\frac{1}{2}, \frac{1}{2} \right] \right\}. \quad (2.6)
\]

It follows from this that the system \( \{T_{k/2}\text{sinc}\}_{k \in \mathbb{Z}} \) is an overcomplete frame for the Paley-Wiener space; indeed, \( \{T_{k/2}\text{sinc}\}_{k \in \mathbb{Z}} = \{T_k\text{sinc}\}_{k \in \mathbb{Z}} \cup \{T_{1/2}T_k\text{sinc}\}_{k \in \mathbb{Z}} \), which shows that the system is a union of two orthonormal bases for the Paley-Wiener space. □

A collection of functions in \( L^2(\mathbb{R}) \) of the form \( \{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}} \) for some \( a, b > 0 \) and some \( g \in L^2(\mathbb{R}) \) is called a Gabor system. Gabor systems play an important role in time-frequency analysis; we will just state the properties that are necessary for the flow of the current paper, and refer to [12, 10, 11, 6] for much more information.

Proposition 2.5 Let \( g \in L^2(\mathbb{R}) \setminus \{0\} \). Then the following hold:

(i) \( \{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}} \) is linearly independent.

(ii) If \( \{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}} \) is a frame for \( L^2(\mathbb{R}) \), then \( ab \leq 1 \).

(iii) If \( \{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}} \) is a frame for \( L^2(\mathbb{R}) \), then \( \{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}} \) is a Riesz basis if and only if \( ab = 1 \).

(iv) If \( \{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}} \) is an overcomplete frame for \( L^2(\mathbb{R}) \), then \( \{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}} \) has infinite excess.

The result in (i) was proved in [16] (hereby confirming a conjecture stated in [14]); (ii) & (iii) are classical results [12, 6], and (iv) is proved in [4].

Note that it immediately follows from Proposition [2.1] and Proposition [2.5] that any Gabor frame \( \{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}} \) can be represented on the form \( \{T^k\varphi\}_{k \in I} \)
for some linear operator $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ and some $\varphi \in L^2(\mathbb{R})$. Whether there exists a \textit{bounded} operator $T$ with this property is a much harder question, and indeed the topic for the next section. Let us apply Proposition \ref{prop:bddness} and show an example of a concrete Gabor frame and a concrete ordering that leads to an unbounded operator.

\textbf{Example 2.6} Consider the Gabor frame $\{E_{m/3}T_{n}\chi_{[0,1]}\}_{m,n\in\mathbb{Z}}$, which is the union of the three orthonormal bases $\{E_{k/3}E_{m}T_{n}\chi_{[0,1]}\}_{m,n\in\mathbb{Z}}$, $k = 0, 1, 2$. Re-order the frame as $\{f_k\}_{k\in\mathbb{Z}}$ in such a way that the elements $\{f_{2k+1}\}_{k\in\mathbb{Z}}$ corresponds to the orthonormal basis $\{E_{m}T_{n}\chi_{[0,1]}\}_{m,n\in\mathbb{Z}}$. By construction, the elements $\{f_{2k}\}_{k\in\mathbb{Z}}$ now forms an overcomplete frame. By Proposition \ref{prop:bddness} there is an operator $T : \text{span}\{f_k\}_{k\in\mathbb{Z}} \to \text{span}\{f_k\}_{k\in\mathbb{Z}}$ such that $\{f_k\}_{k\in\mathbb{Z}} = \{T^k f_0\}_{k\in\mathbb{Z}}$. Since the subsequence $\{f_{2k}\}_{k\in\mathbb{Z}}$ is an overcomplete frame, there is a non-zero sequence $\{c_{2k}\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ such that $\sum_{k\in\mathbb{Z}} c_{2k} f_{2k} = 0$. Defining $c_k = 0$ for $k \in 2\mathbb{Z} + 1$, we have $\sum_{k\in\mathbb{Z}} c_k f_k = \sum_{k\in\mathbb{Z}} c_{2k} f_{2k} = 0$. On the other hand, since $\{f_{2k+1}\}_{k\in\mathbb{Z}}$ is a Riesz basis and $\{c_k\}_{k\in\mathbb{Z}}$ is non-zero, $\sum_{k\in\mathbb{Z}} c_k f_{k+1} = \sum_{k\in\mathbb{Z}} c_{2k} f_{2k+1} \neq 0$. This shows that $N_T$ is not invariant under right-shifts; thus, $T$ is unbounded \hfill \qed

\section{The open problems}

In this section we continue our convention of letting the index set $I$ denote either $\mathbb{N}_0$ or $\mathbb{Z}$. The results in Section \ref{sec:linearindep} show that there is a perfect match between the stated necessary conditions for a frame to be represented in terms of a bounded operator as in \ref{eq:bddness}, and the properties of Gabor frames and shift-invariant systems. Indeed, by Proposition \ref{prop:bddness} a frame for an infinite-dimensional Hilbert space has the form \ref{eq:bddness} for a (not necessarily bounded) operator $T$ if it is linearly independent; by Proposition \ref{prop:linind} and Proposition \ref{prop:overcomp} this condition is satisfied for the Gabor frames and the shift-invariant frames. Also, Proposition \ref{prop:bddness} shows that if a frame $\{f_k\}_{k \in I}$ for $\mathcal{H}$ is linearly independent and overcomplete, then the operator $T$ can only be bounded if $\{f_k\}_{k \in I}$ has infinite excess; again, by Proposition \ref{prop:linind} and Proposition \ref{prop:overcomp} these conditions match the Gabor frames and the shift-invariant frames. For the case of a shift-invariant frame we have already seen in Proposition \ref{prop:linind} that it has the form $\{(T_b)^k \varphi\}_{k \in \mathbb{Z}}$, i.e., as an iterated system indexed by $\mathbb{Z}$. We will therefore first state the open problems for Gabor frames, and return to shift-invariant systems afterwards.

On short form, the key problem is as follows:

\textbf{Problem 1:} Does there exist overcomplete Gabor frames $\{E_{mb}T_{na}\varphi\}_{m,n\in\mathbb{Z}}$ and an appropriate ordering on the form $\{f_k\}_{k \in I}$, such that the operator $T$ in \ref{eq:bddness} is bounded?

From the formulation of Problem 1 it is already clear that the question
contains several subproblems. Assuming that Problem 1 has a positive answer for a certain overcomplete Gabor frame \( \{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}} \), we can formulate them as follows:

(P1-a) Is a representation of the form (2.3) with \( T \) bounded available for both indexings \( I = \mathbb{N}_0 \) and \( I = \mathbb{Z} \)?

(P1-b) Can we characterize the orderings \( \{f_k\}_{k \in I} \) of \( \{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}} \) for which a representation as in (2.3) is available with a bounded operator \( T \)?

(P1-c) Can we characterize the Gabor frames \( \{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}} \) for which an appropriate ordering on the form \( \{f_k\}_{k \in I} \) leads to a bounded operator \( T \) in (2.3), i.e., what are the conditions on \( a, b > 0 \) and \( g \in L^2(\mathbb{R}) \) for this to be possible?

(P1-d) Is there a universal ordering on the form \( \{f_k\}_{k \in I} \) that applies for all the Gabor frames in (P1-c)?

As far as we know, no information is available in the literature concerning question (P1-a). Concerning (P1-b), recall that an ordering that lead to a negative conclusion was considered in Example 2.6.

Let us now turn our attention to shift-invariant systems \( \{T_k b \varphi\}_{k \in \mathbb{Z}} \). Using Proposition 2.3 (v) we immediately see that such a system always has the representation \( \{(T_k \varphi\}_{k \in \mathbb{Z}} \), with the translation operator \( T_b \) being bounded. Thus Problem 1 trivially has a positive answer for the index set \( I = \mathbb{Z} \). However, it is unclear whether the same is true for the index set \( I = \mathbb{N}_0 \). Let us formulate the key question:

**Problem 2:** Is it possible to order an overcomplete frame of translates \( \{T_k b \varphi\}_{k \in \mathbb{Z}} \) on the form \( \{f_k\}_{k=0}^{\infty} = \{T^k \varphi\}_{k=0}^{\infty} \) for a bounded operator \( T \)?

For the question of an operator representation indexed by \( \mathbb{N}_0 \), there is an even more fundamental problem underlying Problem 1 & 2. The problem can be phrased in the setting of a general Hilbert space. Indeed, the known cases of overcomplete frames having a representation \( \{f_k\}_{k=0}^{\infty} = \{T^k f_0\}_{k=0}^{\infty} \) in terms of a bounded operator have the property that \( f_k \to 0 \) as \( k \to \infty \); see, e.g., Example 2.2

**Problem 3:** Let \( \mathcal{H} \) denote a separable Hilbert space. Do there exist overcomplete frames for \( \mathcal{H} \) that are norm-bounded below and have a representation \( \{f_k\}_{k=0}^{\infty} = \{T^k f_0\}_{k=0}^{\infty} \) for a bounded operator \( T : \mathcal{H} \to \mathcal{H} \)?

Note that since Gabor frames and shift-invariant systems consist of vectors with equal norm, a negative answer to Problem 3 would automatically lead to a negative answer to Problem 1 (for the index set \( I = \mathbb{N}_0 \) and Problem 2.)
4 Frames in Hilbert spaces

The purpose of this section is to shed light on the questions in Section 3 from a general Hilbert space angle. We first show that an overcomplete frame with a representation on the form (1.1) must have a very particular property:

Lemma 4.1 Assume that \( \{f_k\}_{k=0}^\infty \) is an overcomplete frame and that \( \{f_k\}_{k=0}^\infty = \{T^k f_0\}_{k=0}^\infty \) for some bounded linear operator \( T : \mathcal{H} \to \mathcal{H} \). Then there exists an \( N \in \mathbb{N}_0 \) such that \( \{f_k\}_{k=0}^N \cup \{f_k\}_{k=M}^\infty \) is a frame for \( \mathcal{H} \) for all \( M > N \).

Proof. Choose some coefficients \( \{c_k\}_{k=0}^\infty \in \ell^2(\mathbb{N}_0) \) such that \( \sum_{k=0}^\infty c_k f_k = 0 \). Letting \( N := \min\{k \in \mathbb{N}_0 \mid c_k \neq 0\} \), we have that

\[
-c_N f_N = \sum_{k=N+1}^\infty c_k f_k,
\]

so \( f_N \in \text{span} \{f_k\}_{k=N+1}^\infty \). Thus \( \{f_k\}_{k=N+1}^\infty \) is a frame for \( \text{span} \{f_k\}_{k=N}^\infty \). Applying the operator \( T \) on (1.1) shows that \( f_{N+1} \in \text{span} \{f_k\}_{k=N+2}^\infty \). By iterated application of the operator \( T \) this proves that for any \( M > N \), the family \( \{f_k\}_{k=M}^\infty \) is a frame for \( \text{span} \{f_k\}_{k=N}^\infty \), which leads to the desired result. \( \square \)

Intuitively, Lemma 4.1 says that a frame having the desired type of representation must have "infinite excess in all directions."

We will now consider frames with a special structure and examine the existence of a representation on the form (1.1). Remember from Proposition 2.1 that any orthonormal basis has a representation on the form (1.1); on the other hand, for a sequence consisting of a union of a basis and finitely many elements, a representation on this form is not possible with a bounded operator \( T \). In order to analyse the existence of a representation (1.1) it is therefore natural to consider the union of two orthonormal bases; the indexing of these orthonormal bases is not relevant, and we will write them as \( \{e_k\}_{k=1}^\infty \), respectively, \( \{\varepsilon_k\}_{k=1}^\infty \). Note that this setup actually covers certain special Gabor frames, and thereby directly relate to the questions in Section 3 for example, the Gabor frame \( \{E_{m/2} T_n \chi_{[0,1]}\}_{m,n \in \mathbb{Z}} \) is indeed the union of the two orthonormal bases \( \{E_m T_n \chi_{[0,1]}\}_{m,n \in \mathbb{Z}} \) and \( \{E_{1/2} E_m T_n \chi_{[0,1]}\}_{m,n \in \mathbb{Z}} \).

Thus, let us now consider a frame \( \{f_k\}_{k=1}^\infty \) in a Hilbert space \( \mathcal{H} \), formed as the union of the elements in two orthonormal bases \( \{e_k\}_{k=1}^\infty \) and \( \{\varepsilon_k\}_{k=1}^\infty \). In order to describe the elements in \( \{f_k\}_{k=1}^\infty \) let us introduce the index sets

\[
I_1 := \{k \in \mathbb{N} \mid f_k \in \{e_j\}_{j=1}^\infty \text{ and } f_{k+1} \in \{\varepsilon_j\}_{j=1}^\infty \}; \\
I_2 := \{k \in \mathbb{N} \mid f_k \in \{e_j\}_{j=1}^\infty \text{ and } f_{k+1} \in \{e_j\}_{j=1}^\infty \}; \\
I_3 := \{k \in \mathbb{N} \mid f_k \in \{\varepsilon_j\}_{j=1}^\infty \text{ and } f_{k+1} \in \{e_j\}_{j=1}^\infty \}; \\
I_4 := \{k \in \mathbb{N} \mid f_k \in \{\varepsilon_j\}_{j=1}^\infty \text{ and } f_{k+1} \in \{\varepsilon_j\}_{j=1}^\infty \}.
\]
Note that the sets $I_k, k = 1, \ldots, 4$ form a disjoint covering of the index set $\mathbb{N}$ of the frame $\{f_k\}_{k=1}^{\infty}$, and that the sets $I_1$ and $I_3$ always are nonempty and infinite sets. By symmetry we can assume that $f_1 = e_1$. The following result shows that if it is possible for the frame $\{f_k\}_{k=1}^{\infty}$ to have a representation as in (1.1) with $I = \mathbb{N}_0$, then necessarily $I_2 \neq \emptyset$ and $I_4 \neq \emptyset$.

**Lemma 4.2** In the setup described above, assume that either $I_2 = \emptyset$ or $I_4 = \emptyset$. Then $T$ is unbounded.

**Proof.** Assume that $\{f_k\}_{k=1}^{\infty}$ has a representation on the form (1.1), with $T$ being bounded. Then $T$ is surjective. Indeed, since $f_1 = e_1$, then the set $\{T f_k\}_{k=1}^{\infty}$ contains all the vectors $\{e_k\}_{k=1}^{\infty}$. This implies that $T$ is surjective.

Now assume that $I_2 \neq \emptyset$ and $I_1 = \emptyset$. Then $\mathbb{N} = I_1 \cup I_2 \cup I_3$, and $\{T e_k\}_{k=1}^{\infty} \subseteq \{e_k\}_{k=2}^{\infty}$. Since $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$, this implies that $\mathcal{R}(T) \subseteq \text{span}\{e_k\}_{k=2}^{\infty} \neq \mathcal{H}$. This contradicts the surjectivity of $T$. Thus the operator $T$ can not be bounded in this case. The case where $I_2 = \emptyset$ and $I_4 \neq \emptyset$ is similar. In this case there exist some $\ell, \ell' \in \mathbb{N}$ such that $T e_\ell = e_\ell'$, and thus $\{T e_k\}_{k=1}^{\infty} \subseteq \{e_k\}_{k \neq \ell}^{\infty}$. Again this contradicts the surjectivity of $T$. Thus, again $T$ can not be bounded. □

In particular, Lemma 4.2 shows that the operator $T$ can not be bounded if the elements in the frame $\{e_k\}_{k=1}^{\infty} \cup \{e_k\}_{k=1}^{\infty}$ are ordered as $\{e_1, e_1, e_2, e_2, \ldots\}$. This shows a significant difference between the availability of a representation as in (1.1) for $I = \mathbb{N}_0$ and $I = \mathbb{Z}$: indeed, Example 2.4 yields a concrete case where a union of two orthonormal bases, ordered as $\{\ldots, e_0, e_0, e_1, e_1, e_2, e_2, \ldots\}$, actually has a representation as in (1.1) with $I = \mathbb{Z}$. This difference between the indexing in terms of $I = \mathbb{N}_0$ and $I = \mathbb{Z}$ is precisely our motivation for Problem (P1-a) and Problem 2 from the previous section.

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