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On the topology of the spaces of curvature constrained plane curves

Abstract: Plane curves with the same endpoints are homotopic; an analogous claim for plane curves with the same endpoints and bounded curvature still remains open. We find necessary and sufficient conditions for two plane curves with bounded curvature to be deformed into each other by a continuous one-parameter family of curves also having bounded curvature. We conclude that the space of these curves has either one or two connected components, depending on the distance between the endpoints. The classification theorem presented here answers a question raised in 1961 by L. E. Dubins.

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1 Introduction

Any two plane curves with the same endpoints are homotopic. Surprisingly, an analogous claim for plane curves with the same endpoints and a bound on the curvature still remains open. Since the plane is simply connected, all immersed plane curves connecting two different points in the plane are regularly homotopic. On the other hand, H. Whitney in [15] classified the regular homotopy classes of closed curves in the plane. A natural step forward is to study homotopy classes in spaces of curvature constrained plane curves. A $\kappa$-constrained curve is required to be smooth with the curvature bounded by a positive constant $\kappa$. We obtain necessary and sufficient conditions for any two $\kappa$-constrained plane curves to be deformed into each other by a continuous one-parameter family of $\kappa$-constrained plane curves. We pay special attention to the interaction between the bound on the curvature and the distance between the endpoints.

Our main result, Theorem 5.1, gives the number of homotopy classes in spaces of $\kappa$-constrained plane curves for any choice of end points in $\mathbb{R}^2$. Let $\kappa = 1/r$ be the bound on the curvature (with $r > 0$ being the minimum radius of curvature), and let $d$ be the euclidean distance between the end points of a curve. If $d = 0$, then any two closed $\kappa$-constrained plane curves are $\kappa$-constrained homotopic, i.e. they are homotopic and satisfy the same curvature bound throughout the deformation; for $0 < d < 2r$ we prove the existence of two homotopy classes of $\kappa$-constrained plane curves: one homotopy class includes the straight line between the end points, and the other one includes the two outer arcs of circles in the left part of Figure 2; finally if $d > 2r$, then any two $\kappa$-constrained plane curves are $\kappa$-constrained homotopic to each other. In addition, for $0 < d < 2r$ we prove in Theorem 4.21 the existence of a planar region where only embedded $\kappa$-constrained plane curves can be defined (see $I$ in Figure 2). Also, there are regions where $\kappa$-constrained plane curves cannot be defined (see $E$ in Figures 2 and 3).

Curves with a bound on the curvature and fixed end points and tangent vectors have been extensively studied. Most of the papers in this area focus on issues related to reachability and optimality, cf. [1; 2; 4; 7; 8; 10; 11; 13; 14]. This paper presents the first results on the spaces of curves with a bound on the curvature where the initial and final vectors are allowed to vary. The classification theorem presented here answers a question raised in 1961 by L. E. Dubins [9].

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2 Preliminaries

Throughout we consider a parametrised plane curve to be the continuous image in $\mathbb{R}^2$ of a closed interval.

**Definition 2.1.** An arc-length parameterised plane curve $\sigma : [0, s] \rightarrow \mathbb{R}^2$ is called a $\kappa$-constrained plane curve if:

- $\sigma$ is $C^1$ and piecewise $C^2$;
- $\|\sigma''(t)\| \leq \kappa$, for all $t \in [0, s]$ when defined, where $\kappa > 0$ is a constant.

The first condition means that a $\kappa$-constrained curve has continuous first derivative and piecewise continuous second derivative. The second condition means that a $\kappa$-constrained plane curve has absolute curvature bounded above (when defined) by a positive constant $\kappa = 1/r$ where $r > 0$ is the minimum radius of curvature. Let $I = [0, s]$. The length of $\sigma$ restricted to $[a, b] < I$ is denoted by $\mathcal{L}(\sigma, a, b)$ and $\mathcal{L}(y) = s$. Since our curves lie in $\mathbb{R}^2$, we sometimes refer to a $\kappa$-constrained plane curve just as a $\kappa$-constrained curve.

The interior, boundary and closure of a subset $X$ in a topological space are denoted by $\text{int}(X)$, $\partial(X)$ and $\text{cl}(X)$ respectively. The ambient space of our curves is $\mathbb{R}^2$ with the topology induced by the euclidean metric.

**Definition 2.2.** For $x, y \in \mathbb{R}^2$, the space of all $\kappa$-constrained plane curves from $x$ to $y$ is denoted by $\Sigma(x, y)$.

Throughout this note we consider the space $\Sigma(x, y)$ together with the $C^1$-metric. Suppose a $\kappa$-constrained curve is continuously deformed under a parameter $p$. For each $p$ we reparametrise the corresponding curve by its arc-length. Thus $\sigma : [0, s_p] \rightarrow \mathbb{R}^2$ describes a deformed curve at parameter $p$, and $s_p$ corresponds to its arc-length.

**Definition 2.3.** Let $\sigma, y \in \Sigma(x, y)$. A $\kappa$-constrained homotopy between $\sigma : [0, s_0] \rightarrow \mathbb{R}^2$ and $y : [0, s_1] \rightarrow \mathbb{R}^2$ corresponds to a continuous one-parameter family of immersed plane curves $H : [0, 1] \rightarrow \Sigma(x, y)$ such that

1. $H(0) = \sigma(t)$ for $t \in [0, s_0]$ and $H(1) = y(t)$ for $t \in [0, s_1]$
2. $H(p) : [0, s_p] \rightarrow \mathbb{R}^2$ is an element of $\Sigma(x, y)$ for all $p \in [0, 1]$.

We say that the curves $\sigma$ and $y$ are $\kappa$-constrained homtopic.

**Remark 2.4** (Homotopy classes in $\Sigma(x, y)$). Let $x, y \in \mathbb{R}^2$. Then

- two curves are $\kappa$-constrained homtopic if there exists a $\kappa$-constrained homotopy from one curve to the other. The previously described relation defined by $\sim$ is an equivalence relation;
- a homotopy class in $\Sigma(x, y)$ corresponds to an equivalence class in $\Sigma(x, y)/\sim$;
- a homotopy class is a maximal path connected set in $\Sigma(x, y)/\sim$;
- we denote by $|\Sigma(x, y)|$ the number of homotopy classes in $\Sigma(x, y)$.

**Definition 2.5.** A fragmentation for a curve $\sigma : I \rightarrow \mathbb{R}^2$ corresponds to a finite sequence $0 = t_0 < t_1 \cdots < t_m = s$ of elements in $I$ such that $\mathcal{L}(\sigma, t_{i-1}, t_i) \leq r$ with $\sum_{i=1}^{m} \mathcal{L}(\sigma, t_{i-1}, t_i) = s$. The restriction of $\sigma$ to the interval determined by two consecutive elements in the fragmentation is called a fragment.

The following results are presented for 1-constrained curves and can be found in [4]. These give lower bounds for the length of curves when compared with arcs in unit circles and line segments. These results can be easily adapted for $\kappa$-constrained curves. Consider $\sigma(t) = (r(t) \cos \theta(t), r(t) \sin \theta(t))$ in polar coordinates.

**Lemma 2.6** (cf. Lemma 2.8 in [4]). For every curve $\sigma : [0, s] \rightarrow \mathbb{R}^2$ with $\sigma(0) = (1, 0)$, $r(t) \geq 1$ and $\theta(s) = \eta$, one has $\mathcal{L}(\sigma) \geq \eta$.

**Lemma 2.7** (cf. Lemma 2.9 in [4]). For every $C^1$ curve $\sigma : [0, s] \rightarrow \mathbb{R}^2$ with $\sigma(0) = (0, 0)$, $\sigma(s) = (x, z)$ and $z \geq 0$, one has $\mathcal{L}(\sigma) \geq z$.

**Lemma 2.8** (cf. Lemma 7.5 in [6]). If a 1-constrained curve $\sigma : [0, s] \rightarrow \mathbb{R}^2$ lies in a disk $D$ with radius 1, then either $\sigma$ is entirely in $\partial(D)$, or the interior of $\sigma$ is disjoint from $\partial(D)$. 
3 A fundamental lemma

We emphasise that a $\kappa$-constrained plane curve has absolute curvature bounded above (when defined) by a positive constant $\kappa = 1/r$ where $r > 0$ is the minimum radius of curvature.

**Lemma 3.1.** A $\kappa$-constrained plane curve $\sigma : I \to \mathcal{B}$, where $\mathcal{B} = \{(x, y) \in \mathbb{R}^2 \mid -r < x < r, y \geq 0\}$, cannot satisfy both

- $\sigma(0), \sigma(s)$ are points on the $x$-axis;
- if $C$ is a circle of radius $r$ with centre on the negative $y$-axis and $\sigma(0), \sigma(s) \in C$, then some point in $\text{Im}(\sigma)$ lies above $C$.

**Proof.** Suppose that such a curve exists. Let $h : \text{Im}(\sigma) \to \mathbb{R}_{\geq 0}$ be the projection onto the $y$-axis. Since $\text{Im}(\sigma)$ is compact and $h$ is continuous, there exists $p \in I$ such that $h(\sigma(p)) = \max_{t \in I} h(\sigma(t))$. Consider a continuous one-parameter family of circles $C_u$ obtained by translating $C$ along the $y$-axis by $u \geq 0$. Note that by continuity there exists a $v \geq 0$ such that $\sigma$ lies inside $C_v$ and is tangent to $C_v$ at some point. By viewing $C_v$ as $\partial(D)$ in Lemma 2.8 (near the point of tangency) we immediately obtain a contradiction. \hfill $\square$

**Definition 3.2.** A plane curve $\sigma$ has parallel tangents if there exist $t_1, t_2 \in I$, with $t_1 < t_2$, such that $\sigma'(t_1)$ and $\sigma'(t_2)$ are parallel and point in opposite directions.

**Definition 3.3.** Let $L_1$ and $L_2$ be the lines $x = -r$ and $x = r$ respectively. A line joining two points of $\sigma$ with distance at least $2r$, one to the left of $L_1$ and one to the right of $L_2$, is called a cross section (see Figure 1 right).

The next result gives conditions for the existence of parallel tangents.

**Corollary 3.4.** Suppose a $\kappa$-constrained plane curve $\sigma : I \to \mathbb{R}^2$ satisfies:

- $\sigma(0), \sigma(s)$ are points on the $x$-axis;
- if $C$ is a circle of radius $r$ with centre on the negative $y$-axis and $\sigma(0), \sigma(s) \in C$, then some point in $\text{Im}(\sigma)$ lies above $C$.

Then $\sigma$ admits parallel tangents and therefore a cross section.
Proof. Consider such a $\kappa$-constrained curve $\sigma$. By Lemma 3.1, $\sigma$ is not entirely contained in the band $B$ (see Figure 1 left). It is not hard to see that if $\sigma$ is tangent to $L_1$ and $L_2$ from the inside of $B$, then a pair of parallel tangents is obtained (see second illustration in Figure 1). Suppose that $\sigma$ is tangent to $L_2$ and crosses $L_1$ twice (see third illustration in Figure 1). By rotating counterclockwise the parallel lines $L_1$ and $L_2$ simultaneously (by a sufficiently small angle) each of these parallel lines cuts $\sigma$ in at least in two points. Suppose that $\sigma$ intersects $L_1$ at $P$ and $P'$, and $L_2$ at $Q$ and $Q'$ (see Figure 1 right). Since $\sigma$ is $C^1$, the intermediate value theorem for the derivatives of $\sigma$ between $P$ and $P'$ and between $Q$ and $Q'$ yields the existence of parallel tangents. We ensure that the directions of the parallel vectors are opposite by considering the subarcs of $\sigma$ between the first time it leaves $B$ and the first time it reenters $B$ and then considering $\sigma$ between the last time it leaves $B$ and the last time $\sigma$ reenters $B$. Since $\sigma$ has a point to the left of $L_1$ and a point to the right of $L_2$, there exists a cross section.

In general, it is not an easy task to construct $\kappa$-constrained homotopies between two given curves; see [5]. Proposition 3.8 shows that the existence of parallel tangents leads to a method for constructing $\kappa$-constrained homotopies.

**Definition 3.5.** Let $\sigma$ be a $C^1$ curve. The affine line generated by $\langle \sigma'(t) \rangle$ is called the tangent line at $\sigma(t)$, for $t \in I$. The ray containing $\langle \sigma'(t) \rangle$ is called the positive ray.

The next definition can be easily adapted for arc length-parametrised curves; we leave the details to the reader.

**Definition 3.6.** Suppose that $\sigma, y : [0, 1] \to \mathbb{R}^2$ with $\sigma(1) = y(0)$. The concatenation of $\sigma$ and $y$ is denoted by $y \# \sigma$ and is defined by

$$(y \# \sigma)(t) = \begin{cases} \sigma(2t) & 0 \leq t \leq \frac{1}{2} \\ y(2t-1) & \frac{1}{2} < t \leq 1. \end{cases}$$

**Remark 3.7** (Train track displacement). The next result gives a direct method for obtaining $\kappa$-constrained homotopies. Suppose that a $\kappa$-constrained curve $\sigma$ has parallel tangents at $t_1, t_2 \in I$. The tangent lines at $\sigma(t_1)$ and $\sigma(t_2)$ may work as train tracks for the displacement of the portion of $\sigma$ in between $\sigma(t_1)$ and $\sigma(t_2)$; see Figure 4.

**Proposition 3.8.** The train track displacement obtained from the existence of parallel tangents in Remark 3.7 induces a $\kappa$-constrained homotopy.

**Proof.** Suppose that $\sigma$ is a $\kappa$-constrained curve having parallel tangents at parameters $t_1$ and $t_2$. Consider the restriction $\tilde{\sigma} : [t_1, t_2] \to \mathbb{R}^2$. Subdivide $\sigma$ in such a way that $\sigma = \sigma_2 \# \tilde{\sigma} \# \sigma_1$. Consider the parametrised lines $\psi_1, \psi_2 : [0, 1] \to \mathbb{R}^2$ defined by

$$\psi_1(r) = \tilde{\sigma}(t_1)(1-r) + Pr,$$  $$\psi_2(r) = \tilde{\sigma}(t_2)(1-r) + Qr,$$

where $P$ belongs to the positive ray of $\langle \sigma'(t_1) \rangle$ and $Q$ belongs to the negative ray of $\langle \sigma'(t_2) \rangle$ with $d(P, \sigma(t_1)) = d(Q, \sigma(t_2))$. Call $\tilde{\gamma}$ the translation of $\tilde{\sigma}$ obtained by adding the vector from $\tilde{\sigma}(t_1)$ to $\psi_1(r)$. Define $\varphi_r = \psi_2 |_{[0,r]} \# \tilde{\gamma} \# \psi_1 |_{[0,r]}$ for each $r \in [0, 1]$, where $\psi_2(r) = \psi_2(1-r)$. Define the homotopy $\tilde{\mathcal{H}}(r) = \sigma_2 \# \varphi_r \# \sigma_1$. In this fashion we have that $\tilde{\mathcal{H}}(0) = \sigma_2 \# \varphi_0 \# \sigma_1 = \sigma$, and $\tilde{\mathcal{H}}(r) = \sigma_2 \# \varphi_r \# \sigma_1 = y$ are both $\kappa$-constrained curves for $r \in [0, 1]$, after reparametrisation. □

The homotopy in Proposition 3.8 defines an operation on $\kappa$-constrained curves called operation of type III (see Remark 4.5). We say that the curves $\sigma$ and $y$ in the previous result are parallel homotopic.
4 Homotopy classes in spaces of $\kappa$-constrained plane curves

In order to determine the number of homotopy classes in $\Sigma(x, y)$ we first study the case where the end points of a $\kappa$-constrained curve are different. Then we study closed $\kappa$-constrained curves. As a consequence of the curvature bound, if the end points are different we have two scenarios, namely $0 < d < 2r$ or $d \geq 2r$, where the minimum radius of curvature is $r = 1/\kappa$. We will see that for $0 < d < 2r$ there exist two planar regions where no $\kappa$-constrained curve can be defined (see Figure 3). In addition, for $0 < d < 2r$ there exists a planar region that traps $\kappa$-constrained curves. That is, no $\kappa$-constrained curve defined in the trapping region can be made $\kappa$-constrained homotopic to a curve having a point in the complement of the trapping region. In particular, we conclude that these trapped curves correspond to a homotopy class of embedded $\kappa$-constrained curves.

**Figure 2:** Left: An illustration for $0 < d < 2r$. The region shaped like a lens corresponds to $I$. The union of the lighter shaded regions correspond to $E$. Right: An illustration for $d = 2r$.

**Definition 4.1.** Suppose that $0 < d < 2$. Let $D_1$ and $D_2$ be disks with radius $r$, let $C_1 = \partial(D_1)$ and $C_2 = \partial(D_2)$ and $C_1 \cap C_2 = \{x, y\}$. Set $I = \text{int}(D_1 \cap D_2)$ and $U = D_1 \cup D_2$. Then define $E = \text{int}(U) \setminus \text{cl}(I)$. Also, denote by $\partial(I)$ the union of the shorter circular arcs of $C_1$ and $C_2$ joining $x$ and $y$ (see Figure 2).

**Remark 4.2.** When we study $I$, $E$ or $U$ we implicitly assume that $\Sigma(x, y)$ is such that $x \neq y$ and $d < 2r$.

**Figure 3:** Let $C_1$ and $C_2$ be circles with radius $r$. Does $\sigma$ represent a $\kappa$-constrained curve in $J$, $E$ or $U$? See Figure 2.

**Remark 4.3** (On piecewise constant curvature $\kappa$-constrained curves). As [5] shows, constructing explicit $\kappa$-constrained homotopies is not a simple matter. In Subsection 4.1 below we discuss a process applied to $\kappa$-constrained curves called normalisation; see [4; 2; 5]. The normalisation of a $\kappa$-constrained curve $\sigma$ is a piecewise constant curvature $\kappa$-constrained curve corresponding to a finite number of concatenated pieces called components. These components are arcs of circles with radius $r$ and line segments. The number of components is called the complexity of the curve. It is important to note that both $\sigma$ and its normalisation are curves in the same connected component in $\Sigma(x, y)$. Our efforts in [4; 2; 5] have been made in order to overcome the difficulty of constructing explicit $\kappa$-constrained homotopies. After normalising $\sigma$ we apply a reduction pro-
cess, also described in Subsection 4.1. This process consists of manipulations of piecewise constant curvature \( \kappa \)-constrained curves while reducing length and complexity and without violating the curvature bound.

**Definition 4.4.** A piecewise constant curvature \( \kappa \)-constrained curve whose circular components lie in circles with radius \( r \) is called a \( cs \) curve.

With the intention of simplifying our arguments in [5], we defined the so-called operations of type I and II. The **operations of type III** will be defined to be the ones performed under the existence of parallel tangents (see Proposition 3.8). Note that the first two operations are applied to \( cs \) curves and the third one may be applied only to \( \kappa \)-constrained curves with parallel tangents.

**Remark 4.5 (Operations on \( cs \) curves, see [5] and Figure 4).**

- **Operations of type I:** In order to perform operations of type I we consider a point \( z \) in the image of a \( cs \) curve as rotation point. We then consider two disks with radius \( r \) (pushing disks) both tangent to the \( cs \) curve at \( z \). Once the rotation point is chosen, we twist the initial \( cs \) curve along the boundary of the two pushing disks in a clockwise or counterclockwise fashion\(^1\).
- An example of an **operation of type II** is illustrated in Figure 4 upper-right; for an in-depth description, we refer to [5].
- **Operations of type III:** these operations are defined to be the ones performed under the existence of parallel tangents (see Proposition 3.8).

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**Figure 4:** Upper-left: An illustration of an operation of type I under a clockwise rotation. Upper-right: An illustration of an operation of type II. Lower: An illustration of a train track displacement (an operation of type III). Note that \( \sigma \) has parallel tangents. In addition, observe that \( \sigma'(t_1) = \gamma'(t'_1) \) and \( \sigma'(t_2) = \gamma'(t'_2) \) for some parameters \( t_1 \) and \( t_2 \).

Next, we establish that the larger circular arcs connecting \( x \) and \( y \) in \( C_1 \) and \( C_2 \) (see Figure 2 left) can be deformed into each other without violating the prescribed curvature bound\(^2\).

**Proposition 4.6.** Suppose that \( 0 < d < 2r \). The larger circular arcs in \( C_1 \) and \( C_2 \) joining \( x \) and \( y \) are \( \kappa \)-constrained homotopic in \( \mathbb{R}^2 \).

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1 For a full rotation see Figure 3 in [5].
2 An analogous construction was pointed out by N. Kuiper to L. E. Dubins in [9] page 480 (proof not added in the manuscript).
Proof. We label the steps in Figure 5 from left to right, with labels 1 to 8. Step 1 starts with $C_2$. Since $0 < d < 2r$ the length of $C_2$ is greater than $\pi r$, thus $C_2$ has parallel tangents. In step 2 we apply Proposition 3.8 by performing an operation of type III. In step 3 we consider the rotation point $z = x$ and start a clockwise operation of type I. In steps 4 and 5 we rotate the pushing disk so that it coincides with $C_1$ in step 6. Since $C_1$ also has parallel tangents, we apply an operation of type III in step 6 to obtain the curve in step 7. We apply an operation of type II to the curve in step 7 to obtain the curve in step 8. Since the curve in step 8 is symmetrical we can obtain the curve in step 8 by applying the same operations as before starting from $C_1$ in place of $C_2$, which concludes the proof.

Figure 5: The larger circular arcs joining $x$ and $y$ in $C_1$ and $C_2$ are $\kappa$-constrained homotopic.

Definition 4.7. A curve $\sigma : I \to \mathbb{R}^2$ is said to be in $X$ if $\sigma(I) \subseteq X$; otherwise $\sigma$ is said to be not in $X$.

Theorem 4.8. A $\kappa$-constrained plane curve $\sigma : (0, s) \to \mathcal{E}$ with $\sigma \in \Sigma(x, y)$ cannot exist (see Figure 3).

Proof. Suppose there exists such a $\kappa$-constrained curve $\sigma$ (see Figure 3 centre). Consider an arc in $\partial(I)$ as the arc of the circle $C$ from $p$ to $q$ in Lemma 3.1. Corollary 3.4 implies that $\sigma$ admits a cross section. The result follows as $\mathcal{E}$ is contained in $\mathcal{B}$, but the existence of a cross section implies that $\sigma$ is not contained in $\mathcal{B}$.

Corollary 4.9. The only $\kappa$-constrained plane curves in $\mathcal{U}$ are

- the $\kappa$-constrained plane curves in $\text{cl}(\mathcal{U})$;
- the $\kappa$-constrained plane curves having their image in $C_1$ or $C_2$.

Proof. Suppose that there exists a $\kappa$-constrained curve $\sigma$ in $\mathcal{U}$ having a point in the portion of $\mathcal{E}$ enclosed by $C_1$ and a point in the portion of $\mathcal{E}$ enclosed by $C_2$. By continuity, $\sigma$ has points lying in each of the arcs in $\partial(\mathcal{U})$. By considering in turn each of the arcs in $\partial(\mathcal{U})$ as $C$ in Lemma 3.1 the result follows.

Proposition 4.10. A $\kappa$-constrained plane curve in $\text{cl}(\mathcal{U})$ not being an arc in $\partial(\mathcal{U})$ cannot be made $\kappa$-constrained homotopic to a plane curve in $\text{cl}(\mathcal{U})$ having a single common point with $\partial(\mathcal{U})$ other than $x$ and $y$.

Proof. Immediate from Lemma 2.8.

Proposition 4.11. Consider $x, y \in \mathbb{R}^2$ such that $0 < d < 2r$. The arcs in $\partial(\mathcal{U})$ cannot be made $\kappa$-constrained homotopic to a curve not in $\text{cl}(\mathcal{U})$.

Proof. Suppose that there exists a $\kappa$-constrained homotopy between the upper arc in $\partial(\mathcal{U})$ and a curve $\gamma$ not in $\text{cl}(\mathcal{U})$. Let $C$ be the circle with radius $r$ containing the upper arc in $\partial(\mathcal{U})$; see Figure 1. Since the curve $\gamma$ has a point not in $\text{cl}(\mathcal{U})$ lying above $C$ (or below $C$, see Theorem 4.8 and Figure 3 centre), by applying Lemma 3.1 we obtain immediately a contradiction.

Remark 4.12. If the distance between the endpoints satisfies $0 < d < 2r$, then Propositions 4.10 and 4.11 yield a lower bound $> 1$ for the number of homotopy classes in $\Sigma(x, y)$. 

It is not hard to see that the arcs in $\partial(I)$ can be made $\kappa$-constrained homotopic to the line segment joining $x$ and $y$. See [5] for a rigorous continuity argument for the preservation of the curvature bound under continuous deformations.

**Proposition 4.13.** Suppose that $x \neq y$. The shortest $\kappa$-constrained plane curve in $Σ(x, y)$ is the line segment joining $x$ and $y$.

**Proof.** Immediate from Lemma 2.7. □

The following result can be found in [2] for $1$-constrained curves. The proof for $\kappa$-constrained curves follows the same lines.

**Theorem 4.14** (cf. Theorem 4.6 in [2]). The shortest closed $\kappa$-constrained plane curve corresponds to the boundary of a disk with radius $r$.

In general, if $σ : I \to \mathbb{R}^2$ is a length minimiser, the image of $σ$ restricted to $[a, b] \subset l$ is also a length minimiser. From Theorem 4.14 we immediately have the following result.

**Corollary 4.15.** Let $0 < d < 2r$. The minimal length $\kappa$-constrained plane curves not in $\text{cl}(I)$ are the longer arcs between $x$ and $y$ in $C_1$ and $C_2$.

### 4.1 Normalising and reducing $\kappa$-constrained curves

Here we discuss some crucial ideas presented in [4; 2; 5]. Recall from Definition 2.5 that a fragmentation for a $\kappa$-constrained curve $σ$ corresponds to a partition of the image of $σ$ in such a way that each piece, or fragment, has length less than $r = 1/\kappa$. The idea is to consider fragments of length less than $r$ in order to allow the construction of a specially convenient type of curves called replacements. A replacement is a cs curve, i.e. a $\kappa$-constrained curve with fixed end points and vectors corresponding to a concatenation of three consecutive pieces, these being an arc in a circle with radius $r$ followed by a line segment followed by an arc in a circle with radius $r$. The following two results are important.

**Proposition 4.16** (Proposition 3.6. in [5]). A fragment is bounded-homotopic to its replacement.

**Lemma 4.17** (Lemma 2.14 in [4]). The length of a replacement is at most the length of the associated fragment, with equality if and only if these are identical.

The normalisation process replaces any $\kappa$-constrained curve $σ$ with a prescribed fragmentation by a cs curve, called its normalisation, which is $\kappa$-constrained homotopic to $σ$. We $\kappa$-constrained homotope each fragment to a cs curve (the replacement). Note that the complexity of the normalisation will depend on the fragmentation. The reduction process corresponds to a sequence of $\kappa$-constrained homotopies so that at each step an initial cs curve is $\kappa$-constrained homotoped to a non-longer cs curve having no higher complexity than the initial one. We start with the normalisation, and after finitely many steps we end up with a length minimiser in the homotopy class of $σ$; see [5].

**Proposition 4.18.** A $\kappa$-constrained plane curve $σ$ is $\kappa$-constrained homotopic to a cs curve of length at most the length of $σ$.

**Proof.** Consider a fragmentation for $σ \in Σ(x, y)$ and consider for each fragment a replacement which by Lemma 4.17 is of length at most the length of the fragment. We apply Proposition 4.16 to conclude that each fragment is bounded-homotopic to its replacement. After a reparametrisation we obtain that $σ$ is $\kappa$-constrained homotopic to a cs curve of length at most the length of $σ$. □

**Theorem 4.19.** The space $Σ(x, x)$ corresponds to a single homotopy class of $\kappa$-constrained plane curves (see Figure 6).
Proof. Consider a fragmentation for \( \sigma \in \Sigma(x, x) \). Applying the reduction process in [5] to \( \sigma \) we find that \( \sigma \) is \( \kappa \)-constrained homotopic to a minimal length element in its homotopy class, i.e. to a circle with radius \( r \) containing the base point \( x \) (cf. Theorem 4.14). On the other hand, considering a different element \( y \in \Sigma(x, x) \) and applying the reduction process to \( y \), we conclude that \( y \) is also \( \kappa \)-constrained homotopic to a minimal length element in its homotopy class (it is easy to see that there are infinitely many such circles, all of them \( \kappa \)-constrained homotopic to each other). By transitivity, we conclude that \( \sigma \) and \( y \) are \( \kappa \)-constrained homotopic.

Remark 4.20. Recall from Definition 4.1 that for \( x, y \in \mathbb{R}^2 \) with \( 0 < d < 2 \) we have the set \( I = \text{int}(D_1 \cap D_2) \). Here \( D_1 \) and \( D_2 \) are the two circles with radius \( r \) containing both \( x \) and \( y \) in their boundaries.

Theorem 4.21. Choose \( x, y \in \mathbb{R}^2 \) such that \( 0 < d < 2r \). Then the space of \( \kappa \)-constrained plane curves in \( \text{cl}(I) \) corresponds to a homotopy class of embedded curves in \( \Sigma(x, y) \).

Proof. Consider a fragmentation for \( \sigma \in \Sigma(x, y) \) in \( \text{cl}(I) \). By applying to \( \sigma \) the reduction process described in Remark 4.1 we obtain that \( \sigma \) is \( \kappa \)-constrained homotopic to the unique minimal length element in its homotopy class i.e., the line segment joining \( x \) and \( y \) (cf. Theorem 4.13). On the other hand, by considering a fragmentation for a different element \( y \in \Sigma(x, y) \) in \( \text{cl}(I) \) and by applying the reduction process to it, we conclude that \( y \) is also \( \kappa \)-constrained homotopic to the minimal length element in its homotopy class. Therefore, by transitivity, we conclude that \( y \) and \( \sigma \) are \( \kappa \)-constrained homotopic curves. To check that such curves are actually embedded, let \( \sigma \in \Sigma(x, y) \) be in \( \text{cl}(I) \) having self intersections. Consider \( \sigma \) in between the first self intersection. By the Pestov–Ionin Lemma, see [12], there exists a disk with radius \( r \) in the interior component of \( \sigma \) in between the considered self intersection. Corollary 3.4 implies that \( \sigma \) has a cross section. Since \( d > 0 \) the diameter of \( \text{cl}(I) \) is less than \( 2r \), hence \( \sigma \) is a curve not in \( \text{cl}(I) \); this leads to a contradiction.

Note that for \( 0 < d < 2r \) we have \( |\Sigma(x, y)| \geq 2 \). The next result shows that indeed \( |\Sigma(x, y)| = 2 \).

Theorem 4.22. Choose \( x, y \in \mathbb{R}^2 \) such that \( 0 < d < 2r \). Then the space of \( \kappa \)-constrained plane curves not in \( \text{cl}(I) \) is a homotopy class in \( \Sigma(x, y) \).

Proof. Consider a fragmentation for a curve \( \sigma \in \Sigma(x, y) \) not in \( \text{cl}(I) \). By applying the reduction process in Remark 4.1 to \( \sigma \) we obtain that \( \sigma \) is \( \kappa \)-constrained homotopic to the minimal element in its homotopy class. By Proposition 4.11 such a curve cannot be the line segment joining \( x \) and \( y \) since the latter is in \( \text{cl}(I) \). By Corollary 4.15 the minimal \( \kappa \)-constrained curve not in \( \text{cl}(I) \) is the longer arc joining \( x \) and \( y \) in \( C_1 \) or \( C_2 \). By considering different element \( y \in \Sigma(x, y) \) not in \( \text{cl}(I) \) and by applying the reduction process to it, we conclude that \( y \) is also \( \kappa \)-constrained homotopic to a minimal length element in its homotopy class i.e., one of the
longer arcs in $C_1$ or $C_2$ joining $x$ and $y$. By Proposition 4.6 the longer arcs joining $x$ and $y$ in $C_1$ and $C_2$ are $\kappa$-constrained homotopic. By transitivity, we conclude that $y$ and $\sigma$ are $\kappa$-constrained homotopic.

**Theorem 4.23.** If $d \geq 2r$, then $|\Sigma(x, y)| = 1$.

**Proof.** The proof is identical to the proof of Theorem 4.21.

## 5 Main result

**Theorem 5.1.** Let $x, y \in \mathbb{R}^2$. Then $|\Sigma(x, y)| = 1$ if $d = 0$ or $d \geq 2r$, and $|\Sigma(x, y)| = 2$ if $0 < d < 2r$.

**Proof.** If $d = 0$, then $|\Sigma(x, y)| = 1$ by Theorem 4.19. If $0 < d < 2$, by applying Theorem 4.21 and Theorem 4.22, we conclude that $|\Sigma(x, y)| = 2$. If $d \geq 2r$, then $|\Sigma(x, y)| = 1$ by Theorem 4.23.

In other words, for $d = 0$ any two closed $\kappa$-constrained curves are $\kappa$-constrained homotopic to each other. For $0 < d < 2r$ any two $\kappa$-constrained curves are $\kappa$-constrained homotopic to each other if and only if they are either both in $\text{cl}(\mathcal{C})$ or both not in $\text{cl}(\mathcal{C})$. Finally, for $d \geq 2r$ any two $\kappa$-constrained curves are $\kappa$-constrained homotopic to each other.

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