Algebraic construction of Weyl invariant $E_8$ Jacobi forms

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Abstract

We study the ring of Weyl invariant $E_8$ weak Jacobi forms. Wang recently proved that the ring is not a polynomial algebra. We consider a polynomial algebra which contains the ring as a subset and clarify the necessary and sufficient condition for an element of the polynomial algebra to be a Weyl invariant $E_8$ weak Jacobi form. This serves as a new algorithm for constructing all the Jacobi forms of given weight and index. The algorithm is purely algebraic and does not require Fourier expansion. Using this algorithm we determine the generators of the free module of Weyl invariant $E_8$ weak Jacobi forms of given index $m$ for $m \leq 20$. We also determine the lowest weight generators of the free module of index $m$ for $m \leq 28$. Our results support the lower bound conjecture of Sun and Wang and prove explicitly that there exist generators of the ring of Weyl invariant $E_8$ weak Jacobi forms of weight $-4m$ and index $m$ with all $12 \leq m \leq 28$.

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1. Introduction and summary

Jacobi forms are functions which have the characteristics of both an elliptic function and a modular form. Eichler and Zagier initiated the systematic study of Jacobi forms [1]. Wirthmüller investigated a generalization of Jacobi forms associated with root systems [2]. Let $R$ be an irreducible root system of rank $r$ and $W(R)$ the Weyl group of $R$. A Weyl invariant $R$ Jacobi form, or a $W(R)$-invariant Jacobi form is a holomorphic function of variables $\tau \in \mathbb{H}$ and $z \in \mathbb{C}^r$ which is quasi-periodic in $z$, invariant under $W(R)$ acting on $z$ and satisfies the modular transformation law.

$W(R)$-invariant Jacobi forms of integral weight and integral index for $m$ a bigraded algebra $J_{s,*}^R$, are a polynomial algebra for any irreducible root system $R$ except $E_8$ [2]. Generators of the polynomial ring have been explicitly constructed for all irreducible root systems $R$ other than $E_8$ [2–8]. On the other hand, for $R = E_8$ the structure of the bigraded algebra $J_{s,*}^{E_8}$ has not yet been fully understood.

The investigation of $W(E_8)$-invariant Jacobi forms was first developed in physics. The systematic construction of $W(E_8)$-invariant Jacobi forms was initiated in [9] in the study of E-strings. In [10], nine $W(E_8)$-invariant meromorphic Jacobi forms $a_2, a_3, a_4, b_1, b_2, b_3, b_4, b_5, b_6$ were constructed as the coefficients of the Seiberg–Witten curve for the E-string theory. Among many applications, the curve can be used in particular to generate the topological string partition function of the Calabi–Yau threefolds known as the local $1/2$ K3 or the local rational elliptic surfaces [11]. $a_i, b_j$ are meromorphic because they have poles at the zero points of Eisenstein series $E_4(\tau)$. In [11], $a_i, b_j$ were expressed in terms of nine simple $W(E_8)$-invariant holomorphic Jacobi forms $A_1, A_2, A_3, A_4, A_5, B_2, B_3, B_4, B_6$ and $E_4, E_6$.

In physics, it is deduced by symmetry consideration [9] that the elliptic genus $Z_n$ of $n$ E-strings multiplied by $\Delta^{n/2}$ ($\Delta = (E_4^3 - E_6^2)/1728$) has to be a $W(E_8)$-invariant quasi Jacobi form (i.e. involving $E_2(\tau)$) of weight $6n - 2$ and index $n$. Moreover, a prescription to express $\Delta^{n/2}Z_n$ as a polynomial of $a_i, b_j, E_2, E_4, E_6$ for any $n$ was given in [10]. Therefore, a natural guess from physics is that $a_i, b_j$ — though they are meromorphic and are not exactly Jacobi forms themselves — play the role of “generators” of $W(E_8)$-invariant Jacobi forms, in the sense that any $W(E_8)$-invariant Jacobi form is written as a polynomial of $a_i, b_j, E_4, E_6$. This was explicitly conjectured in [6]. A proof of this conjecture (Theorem 3.1) is one of the main results of this paper.

In [12], Wang initiated the detailed investigation of the bigraded algebra $J_{s,*}^{E_8}$ of
$W(E_8)$-invariant Jacobi forms. In particular, it was explicitly proved that $J_{E_8}^*$ is not a polynomial algebra over $M_*$. In [12] and other earlier works such as [13][14], $W(E_8)$-invariant Jacobi forms are written in terms of the above mentioned $A_i, B_j$ introduced in [11]. It is natural to use $A_i, B_j$ rather than $a_i, b_j$ as the building blocks because $A_i, B_j$ are simpler and they themselves are Jacobi forms. On the other hand, it was pointed out in [13] that certain $W(E_8)$-invariant Jacobi forms cannot be expressed as polynomials of $A_i, B_j$ unless further multiplied by $E_4$. In [14] it was found that the $W(E_8)$-invariant holomorphic Jacobi form of weight 16 and index 5 given by

$$P_{16,5} = 864A_1^3A_2 + 3825A_1B_2^2 - 770E_6A_3B_2 - 840E_6A_2B_3 + 60E_6A_1B_4 + 21E_6^2A_5$$

(1.1)

vanishes at the zero points of $E_4$. The authors of [14] then conjectured that the quotient $P_{16,5}/E_4$ is also a $W(E_8)$-invariant holomorphic Jacobi form. They further conjectured that any polynomial of $E_6, A_i, B_j$ which vanishes at the zero points of $E_4$ is divisible by $P_{16,5}$. These conjectures were proved recently by Sun and Wang [15]. Moreover, Sun and Wang proved that every $W(E_8)$-invariant Jacobi form of index $t$ can be expressed uniquely as

$$\sum_{j=0}^{t_1} (P_{16,5}/E_4)^{t_1-j} P_j \Delta^{N_t}$$

(1.2)

where $t_1, N_t$ are certain non-negative integers determined by $t$, $\{P_j\}_{j=0}^{t_1}$ are polynomials of $E_6, A_i, B_j$, and $P_{t_1}$ is a polynomial of $E_4, E_6, A_i, B_j$.

This theorem is a powerful tool to investigate the structure of the bigraded algebra $J_{*}^{E_8}$ of $W(E_8)$-invariant Jacobi forms. In fact, it is of practical use in constructing $W(E_8)$-invariant Jacobi forms, in particular, of higher index: For given index, the polynomial ansatz has a finite number of coefficients, which are to be constrained so that the Fourier expansion of the ansatz satisfies the condition of being a Jacobi form. The condition boils down to solving linear equations. Sun and Wang determined the generators of the free module of $W(E_8)$-invariant Jacobi forms of given index $m$ for $m \leq 13$ [13]. A technical drawback of this construction is that the Fourier expansion involves manipulation of Weyl orbits of $E_8$, which gets more and more complicated when one constructs Jacobi forms of higher index.

In this paper we propose another efficient method of constructing $W(E_8)$-invariant Jacobi forms. Our method is purely algebraic and does not require Fourier expansion. We consider the polynomial algebra $R$ generated by $a_i, b_j$ over $M_* = \mathbb{C}[E_4, E_6]$. By our Theorem 3.1, $R$ contains the bigraded algebra $J_{*}^{E_8}$ as a subset. In fact, any
element of $R$ satisfies almost all of the properties of $W(E_8)$-invariant Jacobi forms. The only issue is the holomorphicity: some of the elements have poles at the zero points of $E_4$, which prevents them from being Jacobi forms. Therefore, we have only to check whether a given element of $R$ is holomorphic at these points or not. This can be done easily by using our Theorem 3.2. It turns out that the necessary and sufficient condition for an element of $R$ to be a $W(E_8)$-invariant Jacobi form is that it can be written as a polynomial of $\Delta^{-1}, E_4, E_6, A_i, B_j, (P_{16,5}/E_4)$. This gives an efficient algorithm for constructing all the Jacobi forms of given weight and index.

Using this algorithm we will determine the generators of the free module $J_{E_8}^* \otimes R$ of $W(E_8)$-invariant weak Jacobi forms of index $m$ for $m \leq 20$. We will also determine the lowest weight generators of $J_{E_8}^* \otimes R$ for $m \leq 28$. Our results confirm the lower bound conjecture of Sun and Wang [15] that the weight of non-zero $W(E_8)$-invariant weak Jacobi forms of index $m$ is not less than $-4m$. Moreover, our results prove explicitly that the generators of the ring $J_{E_8}^* \otimes R$ must include those of index $m$ and weight $-4m$ for all $12 \leq m \leq 28$. This means that the structure of $J_{E_8}^* \otimes R$ is highly complicated, as already recognized in [15], but perhaps more highly than expected.

The paper is organized as follows. In section 2 we collect known results about $W(E_8)$-invariant Jacobi forms. In section 3 we prove our main theorems. In section 4 we explain our algorithm in detail and present the results about the generators of $J_{E_8}^* \otimes R$ and $J_{E_8}^* \otimes R$ for $m \leq 28$. There are two appendices on conventions.

2. Preliminaries

2.1. Definitions

Let $\mathbb{H} = \{ \tau \in \mathbb{C} | \text{Im} \tau > 0 \}$ be the upper half plane and set $q = e^{2\pi i \tau}$. Let $E_4(\tau), E_6(\tau)$ denote the Eisenstein series of weight 4, 6 respectively (see Appendix B). It is well known that $E_4, E_6$ generate the graded algebra $M_*$ of all SL$_2(\mathbb{Z})$ modular forms:

$$M_* = \mathbb{C}[E_4, E_6].$$

We will also use the cusp form

$$\Delta = \eta^{24} = \frac{1}{1728} (E_4^3 - E_6^2)$$

frequently, where $\eta(\tau)$ is the Dedekind eta function.

**Definition 2.1** (Weyl invariant Jacobi form). Let $R$ be an irreducible root system of rank $r$ and $W(R)$ the Weyl group of $R$. Let $L_R$ be the root lattice of $R$ and
\( L^*_R \) the dual lattice of \( L_R \). A holomorphic function \( \varphi_{k,m} : \mathbb{H} \times \mathbb{C}^r \to \mathbb{C} \) is called a \( W(R) \)-invariant weak Jacobi form of weight \( k \) and index \( m \) \((k \in \mathbb{Z}, \ m \in \mathbb{Z}_{\geq 0})\) if it satisfies the following properties \[^1\][^2]:

(i) Weyl invariance:
\[
\varphi_{k,m}(\tau, w(z)) = \varphi_{k,m}(\tau, z), \quad w \in W(R).
\]

(ii) Quasi-periodicity:
\[
\varphi_{k,m}(\tau, z + \tau \alpha + \beta) = e^{-m\pi i(\tau \alpha^2 + 2z \alpha)} \varphi_{k,m}(\tau, z), \quad \alpha, \beta \in L_R.
\]

(iii) Modular transformation law:
\[
\varphi_{k,m} \left( \frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) = (c\tau + d)^k \exp \left( \frac{m\pi i}{c\tau + d} \frac{z^2}{c^2} \right) \varphi_{k,m}(\tau, z),
\]
\[
\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{Z}).
\]

(iv) \( \varphi_{k,m}(\tau, z) \) admits a Fourier expansion of the form
\[
\varphi_{k,m}(\tau, z) = \sum_{n=0}^{\infty} \sum_{w \in L^*_R} c(n, w) e^{2\pi i w \cdot z} q^n.
\]

If \( \varphi_{k,m}(\tau, z) \) further satisfies the condition that the coefficients \( c(n, w) \) of the Fourier expansion \[^2\][^6] vanish unless \( w^2 \leq 2mn \), it is called a \( W(R) \)-invariant holomorphic Jacobi form. In this paper a Jacobi form means a weak Jacobi form unless otherwise specified.

**Remark.** In this paper we also introduce the notion of meromorphic Jacobi forms: we call function \( \psi(\tau, z) \) a meromorphic Jacobi form if \( \psi \) itself is not a Jacobi form but there exists a modular form \( f \in M_* \) such that \( f \psi \) is a Jacobi form.

Let the vector space of \( W(R) \)-invariant weak Jacobi forms of weight \( k \) and index \( m \) be denoted by
\[
J^R_{k,m}.
\]

\( J^R_{k,m} \) constitute the free module
\[
J^R_{*,m} := \bigoplus_{k \in \mathbb{Z}} J^R_{k,m}
\]
and the bigraded algebra
\[
J^R_{*,*} := \bigoplus_{m=0}^{\infty} J^R_{*,m}
\]
over \( M_* \). In this paper we will study \( J^R_{*,m} \) and \( J^R_{*,*} \) for \( R = E_8 \).
2.2. \( W(E_8) \)-invariant Jacobi forms

Nine independent \( W(E_8) \)-invariant holomorphic Jacobi forms were constructed in [11]. The basic building block is the theta function of the root lattice \( L_{E_8} \):

\[
\Theta_{E_8}(\tau, z) := \sum_{w \in L_{E_8}} \exp(\pi i\tau w^2 + 2\pi i z \cdot w) = \frac{1}{2} \sum_{k=1}^{4} \prod_{j=1}^{8} \vartheta_k(z_j, \tau). \tag{2.10}
\]

\( \vartheta_k(z, \tau) \) are the Jacobi theta functions (see Appendix [B]). The nine \( W(E_8) \)-invariant holomorphic Jacobi forms are given by [11, Appendix A]

\[
\begin{align*}
A_1(\tau, z) &= \Theta_{E_8}(\tau, z), & A_4(\tau, z) &= A_1(\tau, 2z), \\
A_m(\tau, z) &= \frac{m^3}{m^3 + 1} \left( A_1(m\tau, mz) + \frac{1}{m^3} \sum_{k=0}^{m-1} A_1(\frac{r+k}{m}, z) \right), & m &= 2, 3, 5, \\
B_2(\tau, z) &= \frac{48}{\tau} \left( e_1(\tau)A_1(2\tau, 2z) + \frac{1}{24} e_3(\tau)A_1(\frac{7}{3}, z) + \frac{1}{24} e_2(\tau)A_1(\frac{\tau+1}{2}, z) \right), \\
B_3(\tau, z) &= \frac{8}{8\tau} \left( h_0(\tau)^2 A_1(3\tau, 3z) - \frac{1}{24} \sum_{k=0}^{2} h_0(\frac{\tau+k}{3})^2 A_1(\frac{\tau+k}{3}, z) \right), \\
B_4(\tau, z) &= \frac{16}{15} \left( \vartheta_4(2\tau)^4 A_1(4\tau, 4z) - \frac{1}{24} \vartheta_4(2\tau)^4 A_1(\tau + \frac{1}{2}, 2z) - \frac{1}{24} \sum_{k=0}^{3} \vartheta_2(\frac{\tau+k}{2})^4 A_1(\frac{\tau+k}{2}, z) \right), \\
B_6(\tau, z) &= \frac{9}{10} \left( h_0(\tau)^2 A_1(6\tau, 6z) + \frac{1}{24} \sum_{k=0}^{1} h_0(\tau + k)^2 A_1(\frac{3\tau+3k}{2}, 3z) - \frac{1}{36} \sum_{k=0}^{2} h_0(\frac{\tau+k}{3})^2 A_1(\frac{2\tau+2k}{3}, 2z) - \frac{1}{36} \sum_{k=0}^{3} h_0(\frac{\tau+k}{3})^2 A_1(\frac{\tau+k}{6}, z) \right). \tag{2.11}
\end{align*}
\]

Here, functions \( e_j(\tau) \) and \( h_0(\tau) \) are defined as

\[
\begin{align*}
e_1(\tau) &= \frac{1}{12} \left( \vartheta_3(\tau)^4 + \vartheta_4(\tau)^4 \right), \\
e_2(\tau) &= \frac{1}{12} \left( \vartheta_2(\tau)^4 - \vartheta_4(\tau)^4 \right), \\
e_3(\tau) &= \frac{1}{12} \left( -\vartheta_2(\tau)^4 - \vartheta_3(\tau)^4 \right), \\
h_0(\tau) &= \vartheta_3(2\tau) \vartheta_3(6\tau) + \vartheta_2(2\tau) \vartheta_2(6\tau). \tag{2.12}
\end{align*}
\]

\( A_m, B_m \) are of weight 4,6 respectively and of index \( m \). If we set \( z = 0 \), these Jacobi forms reduce to the Eisenstein series \( E_4, E_6 \):

\[
A_m(\tau, 0) = E_4(\tau), & \quad B_m(\tau, 0) = E_6(\tau). \tag{2.13}
\]

Based on the conjectures of [14], Sun and Wang proved the following theorem, which will be used in the proof of our main theorems.

**Theorem 2.2** (Sun and Wang [15] Theorem 1.1).

1. \( P_{16,5}/E_4 \) with \( P_{16,5} \) given by [11] is a \( W(E_8) \)-invariant holomorphic Jacobi form of weight 12 and index 5.
(2) For any $W(E_8)$-invariant Jacobi form $P \in \mathbb{C}[E_6, \{A_i\}, \{B_j\}]$, if $P/E_4$ is holomorphic on $\mathbb{H} \times \mathbb{C}^8$, then

$$\frac{P}{P_{16,5}} \in \mathbb{C}[E_6, \{A_i\}, \{B_j\}]$$

(2.14)

(3) Every $W(E_8)$-invariant Jacobi form of index $t$ can be expressed uniquely as

$$\sum_{j=0}^{t} \frac{(P_{16,5}/E_4)^{t_j - j} P_j}{\Delta_{N_t}},$$

(2.15)

where $t_1, N_t \in \mathbb{Z}_{\geq 0}$ are such that $t_1 = [t/5]$, $N_t - 5t_0 = 0,0,1,2,3,4$ for $t - 6t_0 = 0,1,2,3,4,5$ respectively with $t_0 = [t/6]$, $[x]$ is the integer part of $x$ and

$$\{P_j\}_{j=0}^{t_1-1} \in \mathbb{C}[E_6, \{A_i\}, \{B_j\}], \quad P_i \in \mathbb{C}[E_4, E_6, \{A_i\}, \{B_j\}],$$

(2.16)

### 2.3. $W(E_8)$-invariant meromorphic Jacobi forms

In [10] nine meromorphic Jacobi forms $\{a_i\}_{i=2}^4, \{b_j\}_{j=1}^6$ were explicitly constructed. Later they were expressed in terms of the above $A_i, B_j$ as [11] Appendix A]

\begin{align*}
a_2 &= \frac{6}{E_4 \Delta} \left(-E_4 A_2 + A_1^2\right), \\
a_3 &= \frac{1}{9E_4^2 \Delta^2} \left(-7E_4^2 E_6 A_3 - 20E_3^2 B_3 - 9E_4 E_6 A_1 A_2 + 30E_3^2 A_1 B_2 + 6E_6 A_1^3\right), \\
a_4 &= \frac{1}{864E_4^3 \Delta^3} \left((E_4^6 - E_4^3 E_6^3) A_4 + (56E_4^5 - 56E_4^2 E_6^2) A_1 A_3 - 27E_4^5 A_2^2 \\
&\quad - 90E_3 E_4 E_6 A_2 B_2 - 75E_4^2 B_2^2 + (180E_4^4 - 36E_4 E_6^2) A_1^2 A_2 \\
&\quad + 240E_4 E_6 A_1^2 B_2 + (-210E_4^3 + 18E_6^2) A_1^4\right), \\
b_1 &= -\frac{4}{E_4} A_1, \quad b_2 = \frac{5}{6E_4 \Delta} \left(E_4^2 B_2 - E_6 A_1^2\right), \\
b_3 &= \frac{1}{108E_4^2 \Delta^2} \left(-7E_4^5 A_3 - 20E_3^2 E_6 B_3 \\
&\quad - 9E_4^4 A_1 A_2 + 30E_4^2 E_6 A_1 B_2 + (16E_4^3 - 10E_6^2) A_1^2\right), \\
b_4 &= \frac{1}{1728E_4^3 \Delta^3} \left((-5E_4^7 + 5E_4^4 E_6^2) B_4 + (80E_4^6 - 80E_4^3 E_6^2) A_1 B_3 \\
&\quad + 9E_4^5 E_6 A_2^2 + 30E_4^6 A_2 B_3 + 25E_4 E_6 B_2^2 - 48E_4^4 E_6 A_1^2 A_2 \\
&\quad + (-140E_4^3 + 60E_4^2 E_6^2) A_1^2 B_2 + (74E_3^3 E_6 - 10E_6^3) A_1^4\right),
\end{align*}
\[ b_5 = \frac{1}{72E_4^4\Delta^5} \left( (-21E_4^4 + 21E_4^4E_6^0)A_5 - 294E_4^6A_2A_3 - 770E_4^6E_6B_2A_3 \\
- 840E_4^6E_6A_2B_3 - 2200E_4^5B_2B_3 + 168E_4^2A_2^3 + 480E_4^6E_6A_2^3 \\
- 621E_4^5A_2^2 + 3525E_4^4A_4B_2^2 + 1224E_4^4A_2^2A_2 - 24E_4^6E_6A_2^2B_2 \\
+ (-456E_4^3 + 24E_4^6)A_1^3 \right), \]

\[ b_6 = \frac{1}{13436928E_6^0\Delta^5} \left( (-20E_4^4 + 40E_4^6E_6^0 - 20E_4^6E_4^4)B_6 \\
+ (-189E_4^{10}E_6 + 378E_4^7E_6^3 - 189E_4^{10}E_6^5)A_1A_5 \\
+ (-9E_4^{10}E_6 + 9E_4^7E_6^3)A_2A_4 + (-15E_4^{11} + 15E_4^8E_6^2)B_2A_4 \\
+ (-180E_4^{11} + 180E_4^8E_6^2)A_2B_4 + (-300E_4^9E_6 + 300E_4^6E_6^3)B_2B_4 \\
+ (22E_4^9E_6 - 22E_4^7E_6^3)A_2^2A_4 + (150E_4^{10} + 120E_4^7E_6^2 - 270E_4^6E_6^3)A_2^2B_4 \\
+ (196E_4^{10}E_6 - 196E_4^7E_6^3)A_2^3 + (1120E_4^{11} - 1120E_4^8E_6^2)A_3B_3 \\
+ (1600E_4^9E_6 - 1600E_4^7E_6^3)B_2^2 + (-2982E_4^9E_6 + 2982E_4^6E_6^3)A_1A_2A_3 \\
+ (-2520E_4^{10} - 4410E_4^7E_6^2 + 6930E_4^4E_6^3)A_1B_2A_3 \\
+ (3360E_4^{10} - 10920E_4^7E_6^2 + 7560E_4^4E_6^3)A_1A_2B_3 \\
+ (-19800E_4^8E_6 + 19800E_4^5E_6^3)A_1B_2B_3 + (2016E_4^8E_6 - 2016E_4^5E_6^3)A_1^2A_3 \\
+ (-5920E_4^9 + 7360E_6^0E_4^2 - 1440E_4^3E_6^3)A_1^2B_3 + (405E_4^9E_6 + 162E_4^6E_6^3)A_2^3 \\
+ (1215E_4^{10} + 1620E_4^7E_6^3)A_2^3B_2 + 4725E_4^6E_6A_2B_2 \\
+ (1125E_4^9 + 1500E_4^6E_6^3)B_2^3 + (-9477E_4^8E_6 + 5103E_4^5E_6^3)A_1^2A_2 \\
+ (-19800E_4^9E_6 + 5400E_4^6E_6^3)A_1^2A_2B_2 + (20925E_4^7E_6 - 33075E_4^4E_6^3)A_1^2B_2^2 \\
+ (20304E_4^7E_6 - 9072E_4^4E_6^3)A_1^3A_2 \\
+ (12780E_4^8 + 5400E_4^5E_6^3 + 540E_4^2E_6^4)A_1^4B_2 \\
+ (-11076E_4^6E_6 + 1512E_4^3E_6^3 - 36E_6^0)A_1^5 \right). \]  

(2.17)

One can easily invert these relations and express \( A_1, B_2 \) in terms of \( a_k, b_l \) as

\[ A_1 = -\frac{E_4^6b_1}{4}, \quad A_2 = \frac{3E_4^6b_1^2 - 8\Delta a_2}{48}, \]

\[ A_3 = \frac{-21E_4^6b_1^3 - 12\Delta E_4b_3 + \Delta E_6a_3 - 72\Delta a_2b_1}{1344}, \]

\[ A_4 = \frac{1}{2304}(-\Delta E_4^2a_2^2 + 9E_4^6b_1 - 288\Delta E_4b_1b_3 + 1444E_4^6b_2^2 - 24\Delta E_6a_2b_2 + 24\Delta E_6a_3b_1 \\
+ 1296\Delta a_2b_1^2 + 1152\Delta^2a_4), \]

\[ A_5 = \frac{1}{64512}(3\Delta E_4^2a_2^2b_1 - 63E_4^6b_1^2 + 216\Delta E_4^6b_1^2b_3 - 144\Delta E_4b_1b_2^2 - 24\Delta E_6a_2b_1b_2 \\
+ 110\Delta E_6a_3b_1^2 - 1200\Delta a_2b_1^3 - 128\Delta^2E_4b_5 - 1344\Delta^2a_2b_3 + 2112\Delta^2a_3b_2), \]
\[ B_2 = \frac{5E_6b_1^2 + 96\Delta b_2}{80}, \quad B_3 = \frac{-\Delta E_4^2a_3 - 60E_6b_1^3 + 12\Delta E_6b_3 - 1728\Delta b_1b_2}{3840}, \]
\[ B_4 = \frac{1}{34560}(-24\Delta E_4^2a_2b_2 + 36\Delta E_4^2a_3b_1 + \Delta E_4E_6a_2^2 + 135E_6b_1^4 - 432\Delta E_6b_1b_3 + 144\Delta E_6b_2^2 + 5184\Delta b_1b_2 - 6912\Delta^3b_4), \]
\[ B_6 = \frac{1}{552960}(-\Delta E_4^2E_6a_4b_1^2 + 72\Delta E_4^2a_2b_1^2b_2 - 216\Delta E_4^2a_3b_1^3 - 9\Delta E_4E_6a_2b_1^2
+ 135E_6b_1^3 - 96\Delta^2E_4^2a_2b_1 + 72\Delta^2E_4^2a_3b_1 - 144\Delta^2E_4^2a_4b_1 + 12\Delta^2E_4E_6a_2a_4
- 3\Delta^2E_4E_6a_3^2 - 144\Delta^2E_4a_3^2b_2 + 288\Delta^2E_4a_2a_3b_1 + 12\Delta^2E_6a_2 + 12\Delta^2E_6b_1^2b_4
- 216\Delta E_6b_1b_3 + 7776\Delta b_1^3b_4 - 2592\Delta^2E_6b_1b_5 + 1152\Delta^2E_6b_2b_4 - 432\Delta^2E_6b_3^2
+ 10368\Delta b_1^2b_4 - 124416\Delta^3b_6). \]  

(2.18)

**Proposition 2.3.** $A_i, B_j$ are polynomials of $E_4, E_6, a_k, b_l$. 

**Proof.** This is clear from \((2.2)\) and \((2.18)\). \(\square\)

$a_m, b_m$ are of weight $4 - 6m, 6 - 6m$ respectively and of index $m$. They are meromorphic and are not exactly Jacobi forms. Indeed, the lowest weight of non-zero $W(E_6)$-invariant weak Jacobi forms of index $m$ for $m = 1, 2, 3, 4, 5, 6$ is $4, -4, -8, -16, -16, -24$ respectively \([12],[15]\) (or see \((1.16)\)), but the weight of every $a_m, b_m$ is lower than this bound. More specifically, $a_m, b_m$ have poles at the zero points of $E_4$. On the other hand, they behave well at the cusp:

**Proposition 2.4.** $a_i, b_j$ admit a Fourier expansion of the form

\[ a_i(\tau, z) = \sum_{n=0}^{\infty} a_i^{(n)}(z)q^n, \quad b_j(\tau, z) = \sum_{n=0}^{\infty} b_j^{(n)}(z)q^n. \]  

(2.19)

**Proof.** By construction the $q$-expansions of $a_i, b_j$ are integral. The absence of negative powers can be shown by direct calculation. \(\square\)

$a_i^{(0)}, b_j^{(0)}$ were explicitly computed in \([10]\) Appendix B:\n
\[ a_2^{(0)} = -2 \frac{w_1 + 12w_8 - 1440}{3}, \quad a_3^{(0)} = -2w_2 + 96w_4 - 1152w_8 + 103680, \]
\[ a_4^{(0)} = \frac{4}{3}w_1^2 - 4w_3 - 16w_6 - 48w_1w_8 - 144w_8^2
+ 400w_2 + 1440w_7 + 1728w_1 + 41472w_8 - 2073600, \]
\[ b_1^{(0)} = -4, \quad b_2^{(0)} = -\frac{1}{18}w_1 - 3w_8 + 840, \]
\[ b_3^{(0)} = -\frac{1}{6}w_2 - 4w_7 - 8w_1 + 528w_8 - 79680, \]
\[
b_4^{(0)} = \frac{2}{9}w_1^2 - \frac{1}{3}w_3 - \frac{16}{3}w_6 - 24w_1w_8 - 120w_8^2 \\
\quad + \frac{424}{3}w_2 + 1272w_7 + 4608w_1 - 25920w_8 + 3939840,
\]
\[
b_5^{(0)} = \frac{2}{3}w_1w_2 - 4w_5 - 16w_1w_7 + 64w_2w_8 + 288w_7w_8 - 96w_1^2 - 60w_3 - 160w_6 \\
\quad + 3456w_8^2 + 800w_2 - 24480w_7 - 108480w_1 + 933120w_8 - 97873920,
\]
\[
b_6^{(0)} = -\frac{8}{27}w_1^3 + w_2^2 + \frac{4}{3}w_1w_3 - 4w_4 - \frac{32}{3}w_1w_6 - 48w_1^2w_8 + 48w_2w_7 + 288w_7^2 \\
\quad - 40w_3w_8 - 480w_6w_8 - 2592w_1w_8^2 - 9792w_8^3 + \frac{1124}{3}w_1w_2 + 548w_5 \\
\quad + 6688w_1w_7 + 1884w_2w_8 + 25632w_7w_8 + 24576w_1^2 + 12920w_3 + 88320w_6 \\
\quad + 578688w_1w_8 + 1714176w_8^2 - 1694400w_2 - 8460000w_7 - 30102720w_1 \\
\quad - 104198400w_8 + 726126800. \tag{2.20}
\]

Here, \(w_j\) (\(j = 1, \ldots, 8\)) denote the Weyl orbit characters associated with the fundamental weights \(\Lambda_j^{E_8}\) of \(E_8\) (see Appendix A for our convention)

\[
w_j(z) := \sum_{w \in \text{Weyl orbit of } \Lambda_j^{E_8}} e^{2\pi i w \cdot z}. \tag{2.21}
\]

**Proposition 2.5.** \(\{a_i^{(0)}\}_{i=2}^4, \{b_j^{(0)}\}_{j=2}^6\) are algebraically independent over \(\mathbb{C}\).

**Proof.** This follows from the algebraic independence of the Weyl orbit characters \(\{w_j\}_{j=1}^8\). Using the expression (2.20) one can compute the Jacobian determinant

\[
\begin{vmatrix}
\frac{\partial (a_2^{(0)}, a_3^{(0)}, a_4^{(0)}, b_2^{(0)}, b_3^{(0)}, b_4^{(0)}, b_5^{(0)}, b_6^{(0)})}{\partial (w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8)}
\end{vmatrix} = \frac{16384}{3} \neq 0, \tag{2.22}
\]

which proves the proposition. \(\square\)

### 3. Main theorems

In this section we will prove the main theorems of this paper.

**Theorem 3.1** (Conjecture of [6, Sec.3.2]). The bigraded algebra \(J_{c, x}^{E_8}\) of \(W(E_8)\)-invariant weak Jacobi forms is a proper subset of the polynomial algebra generated by \(\{a_i\}_{i=2}^4, \{b_j\}_{j=1}^6\) over \(M_x = \mathbb{C}[E_4, E_6]\). In other words,

\[
J_{c, x}^{E_8} \subsetneq \mathbb{C}[E_4, E_6, a_2, a_3, a_4, b_2, b_3, b_4, b_5, b_6]. \tag{3.1}
\]

**Proof.** Let \(R\) denote the polynomial algebra generated by \(a_i, b_j\) over \(M_x\):

\[
R := \mathbb{C}[E_4, E_6, a_2, a_3, a_4, b_2, b_3, b_4, b_5, b_6]. \tag{3.2}
\]
By Theorem 2.2 (3) of Sun and Wang, any $W(E_8)$-invariant weak Jacobi form $\phi_t$ of index $t$ is expressed as in (2.15):

$$\phi_t = \sum_{j=0}^{t_1} \frac{(P_{16,5}/E_4)^{t_1-j} P_j}{\Delta^{N_t}}. \quad (3.3)$$

Proposition 2.3 states that $A_i, B_j \in \mathcal{R}$, so that $P_j \in \mathcal{R}$. Moreover, substituting (2.18) into (1.1) one obtains

$$\frac{P_{16,5}}{E_4} = \frac{1}{9216} (24\Delta E_4^2 E_6 a_2 b_1 b_2 - 18\Delta E_4^2 E_6 a_3 b_1^2 + 20736\Delta^2 E_4^2 a_2 b_1^3 + 5\Delta E_4 E_6^2 a_2^2 b_1$$

$$- 28440 E_6^2 b_1^5 - 336\Delta^2 E_4 E_6 a_2 a_3 + 4824\Delta^2 E_6^2 b_1 b_3 - 1008\Delta E_4^2 b_1 b_2^2$$

$$- 991872\Delta E_6 b_1^2 b_2 - 13436928\Delta b_1^3 - 384\Delta^2 E_6^2 b_5 + 27648\Delta^2 E_6 b_1 b_4$$

$$+ 76032\Delta^2 E_6 b_2 b_3 - 12690432\Delta^2 b_1 b_2^3), \quad (3.4)$$

which means that $P_{16,5}/E_4 \in \mathcal{R}$. Hence, (3.3) implies that $\Delta^{N_t} \phi_t \in \mathcal{R}$, i.e. it is written as some polynomial $Q$ of $E_4, E_6, a_i, b_j$:

$$\Delta^{N_t} \phi_t = Q(E_4, E_6, \{a_i\}, \{b_j\}). \quad (3.5)$$

Since $\Delta = q + O(q^2)$ and any $W(E_8)$-invariant Jacobi form has a regular power series expansion in $q$, the $q$-expansion of $\Delta^{N_t} \phi_t$ starts at the order of $q^n$, $n \geq N_t$. Therefore, the $O(q^n)$ part of $Q(E_4, E_6, \{a_i\}, \{b_j\})$ has to vanish, i.e. there is an algebraic relation among the $O(q^n)$ parts of $E_4, E_6, a_i, b_j$, denoted by $E_4^{(0)}, E_6^{(0)}, a_i^{(0)}, b_j^{(0)}$. We see that $E_4^{(0)} = 1$, $E_6^{(0)} = 1$, $b_1^{(0)} = -4$, while $\{a_i^{(0)}\}_{i=2}^4, \{b_j^{(0)}\}_{j=2}^6$ are algebraically independent by Proposition 2.3. Since $b_i$ is of weight 0 and index 1 while $E_4, E_6$ are of weight 4, 6 respectively and of index 0, the only possible algebraic relation compatible with the bigrading is of the form $(E_4^{(0)})^3 - (E_6^{(0)})^2 = 0$ multiplied by some polynomial of $E_4^{(0)}, E_6^{(0)}, a_i^{(0)}, b_j^{(0)}$. This means that the polynomial $Q$ is divisible by $\Delta = (E_4^2 - E_6^2)/1728$, i.e. it is written as $Q = \Delta Q_1$ with $Q_1 \in \mathcal{R}$.

The Fourier expansion of $Q_1$ starts at the order of $q^n$, $n \geq N_t - 1$. One can repeat the same discussion as above and show that $Q_1$ is written as $Q_1 = \Delta Q_2$ with $Q_2 \in \mathcal{R}$. In this way, one can show that $Q$ is written as $Q = \Delta Q_1 = \Delta^2 Q_2 = \cdots = \Delta^{N_t} Q_{N_t}$ with $Q_{N_t} \in \mathcal{R}$, which gives $\phi_t$. Thus we have proved that any $\phi_t \in J_{E_8}^{E_8}$ satisfies $\phi_t \in \mathcal{R}$. Since $a_i, b_j$ are meromorphic and are not the elements of $J_{E_8}^{E_8}$, $\mathcal{R}$ is bigger than $J_{E_8}^{E_8}$. Hence $J_{E_8}^{E_8} \subseteq \mathcal{R}$. \hfill \Box

Remark. A similar embedding of $J_{E_8}^{E_8}$ into a polynomial algebra was considered in [16, Theorem 5.1], where the polynomial algebra is generated by $A_i/g^i, B_j/g^j$ with $g$ being the product of $\Delta^8 E_4$ and a modular form of weight 72. Our Theorem 3.1 is stronger than this theorem.
The following theorem is rather a corollary of Theorem 2.2 of Sun and Wang, but is very useful when combined with Theorem 3.1.

**Theorem 3.2.** Suppose that $P \in \mathcal{R} = \mathbb{C}[E_4, E_6, a_2, a_3, a_4, b_1, b_2, b_3, b_4, b_5, b_6]$. Then the following two conditions are equivalent

$$ P \in J_{E_8} \iff P \in \mathbb{C}[\Delta^{-1}, E_4, E_6, A_1, A_2, A_3, A_4, A_5, B_2, B_3, B_4, B_5, B_6, (P_{16,5}/E_4)]. $$

(3.6)

Proof. ($\Rightarrow$) This is a direct consequence of Theorem 2.2 (3). ($\Leftarrow$) Any element of $\mathcal{R}$ satisfies properties (i)-(iii) of Definition 2.1 by construction and also (iv) by Proposition 2.4. Therefore, we have only to show that $P$ is holomorphic on $\mathbb{H} \times \mathbb{C}^8$. The generators other than $(P_{16,5}/E_4)$ are holomorphic by construction. $(P_{16,5}/E_4)$ is holomorphic by Theorem 2.2 (1).

4. Constructing $W(E_8)$-invariant Jacobi forms of higher index

4.1. Algorithm

Using the theorems proved in the last section we can formulate an efficient algorithm for constructing all $W(E_8)$-invariant weak Jacobi forms of given weight $k$ and index $m$, as described below:

**Algorithm 4.1.**

1. Let $P$ be the most general polynomial of $E_4, E_6, a_i, b_j$ of weight $k$ and index $m$. This is our ansatz. More specifically, $P$ can be easily constructed by taking the $O(x^k y^m)$ part of the generating series

$$ \frac{1}{(1 - x^4 E_4)(1 - x^6 E_6) \prod_{i=2}^{4}(1 - x^{4-6i} y^i a_i) \prod_{j=1}^{6}(1 - x^{6-6j} y^j b_j)} $$

(4.1)

and then inserting undetermined coefficient $c_i$ in front of every $i$th monomial.

2. Substitute (2.17) into $P$ to express it in terms of $\Delta, E_4, E_6, A_i, B_j$. In general, $P$ contains negative powers of $\Delta$ and $E_4$. Let $-n$ be the lowest degree of $\Delta$. Then multiply $P$ by $\Delta^n$, so that $\Delta^n P$ no longer contains negative powers of $\Delta$.

3. Express $\Delta^n P$ in the form

$$ \Delta^n P = \sum_{l=1}^{l_1} Q_l(E_6, \{A_i\}, \{B_j\}) \frac{E_4}{E_4} + R(E_4, E_6, \{A_i\}, \{B_j\}), \quad (4.2) $$

where $l_1$ is some positive integer and $Q_l, R$ are some polynomials.
(4) For every \( l = 1, \ldots, l_1 \), let \( S_l(E_6, \{ A_i \}, \{ B_j \}) \) be the most general polynomial of weight \( k + 12n - 12l \) and index \( m - 5l \) with undetermined coefficients \( d_{l,i} \). More specifically, \( S_l \) can be easily constructed by taking the \( O(x^{k+12n-12l}y^m-5l) \) part of the generating series

\[
\frac{1}{(1 - x^b E_6) \prod_{i=1,2,3,4,5}(1 - x^4 y^i A_i) \prod_{j=2,3,4,6}(1 - x^b y^j B_j)}
\]

and then inserting \( d_{l,i} \) in front of every \( i \)th monomial.

(5) Solve the linear equations among \( c_i \) and \( d_{l,i} \) in such a way that

\[
Q_l(E_6, \{ A_i \}, \{ B_j \}) = (P_{16,5})^l S_l(E_6, \{ A_i \}, \{ B_j \}) \quad (l = 1, \ldots, l_1)
\]

hold identically.

(6) Substitute the general solution back into \( P \). This gives the most general linear combination of \( W(E_8) \)-invariant weak Jacobi forms of weight \( k \) and index \( m \).

As an illustration, let us first construct all \( W(E_8) \)-invariant weak Jacobi forms of weight \(-16\) and index \( 5 \). The ansatz takes the form

\[
P = c_1 E_4^2 b_5 + c_2 E_6 a_2 a_3 + c_3 E_4 a_2 b_3 + c_4 E_4 a_3 b_2 + c_5 E_4 a_4 b_1 + c_6 a_5^2 b_1.
\]

By substituting (2.17) into \( P \), one finds that the lowest degree of \( \Delta \) is \(-3\). Therefore, expanding \( \Delta^3 P \) as in (1.2) one obtains

\[
\begin{align*}
Q_1 &= (-\frac{10}{3} c_1 + 20 c_2 + \frac{5}{3} c_3 - \frac{20}{9} c_4 - \frac{10}{9} c_5) E_6 A_3^2 B_2 \\
&\quad + (-\frac{14}{3} c_2 + \frac{35}{3} c_3 + \frac{7}{27} c_5) E_6 A_1^2 A_3 + (6 c_2 + \frac{1}{12} c_6) E_6 A_4 A_1^2, \\
Q_2 &= (-10 c_2 + \frac{8}{3} c_3 + \frac{2}{5} c_4 + \frac{1}{3} c_5 - \frac{1}{6} c_6) E_6 A_1^2 A_2, \\
Q_3 &= (\frac{1}{3} c_1 + 4 c_2 - \frac{5}{3} c_3 - \frac{5}{3} c_4 - \frac{1}{12} c_5 + \frac{1}{12} c_6) E_6 A_1^5.
\end{align*}
\]

In the present case, all \( S_i \) are trivial, i.e. \( S_1 = S_2 = S_3 = 0 \) because there are no polynomials of \( E_6, A_i, B_j \) of (weight, index) = (8, 0), (-4, -5), (-16, -10). Thus, (1.4) become simply \( Q_l = 0 \) for \( l = 1, 2, 3 \). In order for these equations to hold identically, \( \{ c_i \}_{i=1}^6 \) have to satisfy five linear equations (which are not entirely independent with each other). The equations are solved as

\[
c_3 = \frac{18}{5} c_1 + \frac{36}{5} c_2, \quad c_4 = -\frac{24}{5} c_1 - \frac{108}{5} c_2, \quad c_5 = 12 c_1 + 72 c_2, \quad c_6 = -72 c_2.
\]
Substituting this back into the original ansatz (4.5), one obtains
\[
c_1 \left( E_1^2 b_5 + \frac{18}{5} E_4 a_2 b_3 - \frac{24}{5} E_4 a_3 b_2 + 12 E_4 a_4 b_1 \right) \\
+ c_2 \left( E_6 a_2 a_3 + \frac{36}{5} E_4 a_2 b_3 - \frac{108}{5} E_4 a_3 b_2 + 72 E_4 a_4 b_1 - 72 a_2^2 b_1 \right). \tag{4.8}
\]

This is the most general linear combination of \(W(E_8)\)-invariant weak Jacobi forms of weight \(-16\) and index 5. Clearly, \(\dim J^E_{-16,5} = 2\).

Next, let us consider the case of weight \(-26\) and index 7 as another example. The ansatz takes the form
\[
P = c_1 E_1^2 a_3 a_4 + c_2 E_6 a_2 b_5 + c_3 E_6 a_3 b_4 + c_4 E_6 a_4 b_3 \\
+ c_5 E_4 a_2^2 a_3 + c_6 E_4 a_1 b_6 + c_7 E_4 b_2 b_5 + c_8 E_4 b_3 b_4 \\
+ c_9 a_2 b_1 b_4 + c_{10} a_2 b_2 b_3 + c_{11} a_3 b_1 b_3 + c_{12} a_3 b_2^2 + c_{13} a_4 b_1 b_2. \tag{4.9}
\]

In this case, one finds that \(n = 5\), \(l_1 = 6\) and \(S_i\) are constructed as
\[
S_1 = d_{1,1} E_6^3 A_2, \quad S_2 = S_3 = S_4 = S_5 = S_6 = 0. \tag{4.10}
\]

The linear equations among \(\{c_i\}_{i=1}^{13}\) and \(d_{1,1}\) are solved as
\[
c_2 = 20736 d_{1,1}, \quad c_3 = -\frac{41472}{5} d_{1,1}, \quad c_6 = 746496 d_{1,1}, \quad c_7 = -\frac{746496}{5} d_{1,1}, \\
c_8 = \frac{746496}{25} d_{1,1}, \quad c_9 = -\frac{4478976}{5} d_{1,1}, \quad c_{10} = \frac{4478976}{25} d_{1,1}, \quad c_{11} = \frac{4478976}{5} d_{1,1}, \quad c_{12} = -\frac{8057052}{25} d_{1,1}, \quad c_1 = c_4 = c_5 = c_{13} = 0. \tag{4.11}
\]

Substituting this back into (4.9) and setting \(d_{1,1} = 25/20736\), one obtains
\[
25 E_6 a_2 b_5 - 10 E_6 a_3 b_4 + 900 E_4 b_1 b_6 - 180 E_4 b_2 b_5 + 36 E_4 b_3 b_4 \\
- 1080 a_2 b_1 b_4 + 216 a_2 b_2 b_3 + 1080 a_3 b_1 b_3 - 432 a_3 b_2^2. \tag{4.12}
\]

This is the unique generator of the vector space \(J^E_{-26,7}\).

### 4.2. Free modules of given index

It was proved in \cite{[12]} Theorem 4.1 that the space \(J^E_{s,m} = \bigoplus_{k \in \mathbb{Z}} J^E_{k,m}\) is a free module over \(M_s\) and the rank \(r(m)\) is given by the generating series
\[
\frac{1}{(1 - x)(1 - x^2)^2(1 - x^3)^2(1 - x^4)^2(1 - x^5)(1 - x^6)} = \sum_{m=0}^{\infty} r(m) x^m. \tag{4.13}
\]

To understand the structure of \(J^E_{s,m}\) for given index \(m\), it is sufficient to determine \(r(m)\) generators. It was proved in \cite{[12]} Proposition 6.1 that the weight of non-zero
$W(E_8)$-invariant weak Jacobi forms of index $m$ is not less than $-5m$. It was also proved in [15, Proposition 6.4] that for any $m \geq 2$, the free module $J_{s,m}^{E_8}$ is generated by Jacobi forms of non-positive weight. Therefore, it is sufficient to construct Jacobi forms of weight $k$ with $-5m \leq k \leq 0$ for $m \geq 2$.

In [15], Sun and Wang determined all generators of $J_{s,m}^{E_8}$ for $1 \leq m \leq 13$. Using our algorithm we determine all generators of $J_{s,m}^{E_8}$ for $1 \leq m \leq 20$.

**Proposition 4.2.** Let $d_{k,m}$ denote the number of generators of weight $k$ of $J_{s,m}^{E_8}$. For $1 \leq m \leq 20$, the Laurent polynomials

$$P_m^w := \sum_{k \in \mathbb{Z}} d_{k,m} x^k$$

are determined as

\[
\begin{align*}
P_1^w &= x^4, \\
P_2^w &= x^4 + x^{-2} + 1, \\
P_3^w &= x^{-8} + x^{-6} + x^{-4} + x^{-2} + 1, \\
P_4^w &= x^{-16} + x^{-14} + x^{-12} + x^{-10} + 2x^{-8} + x^{-6} + x^{-4} + x^{-2} + 1, \\
P_5^w &= 2x^{-16} + 2x^{-14} + 3x^{-12} + 2x^{-10} + 2x^{-8} + x^{-6} + x^{-4} + x^{-2} + 1, \\
P_6^w &= 2x^{-24} + 2x^{-22} + 3x^{-20} + 3x^{-18} + 3x^{-16} + 3x^{-14} + 3x^{-12} + 2x^{-10} + 2x^{-8} + x^{-6} + x^{-4} + x^{-2} + 1, \\
P_7^w &= x^{-26} + 3x^{-24} + 5x^{-22} + 7x^{-20} + 4x^{-18} + 4x^{-16} + 4x^{-14} + 3x^{-12} + 2x^{-10} + 2x^{-8} + x^{-6} + x^{-4} + x^{-2} + 1, \\
P_8^w &= 2x^{-32} + 4x^{-30} + 7x^{-28} + 6x^{-26} + 7x^{-24} + 6x^{-22} + 6x^{-20} + 5x^{-18} + 5x^{-16} + 4x^{-14} + 3x^{-12} + 2x^{-10} + 2x^{-8} + x^{-6} + x^{-4} + x^{-2} + 1, \\
P_9^w &= x^{-36} + 2x^{-34} + 8x^{-32} + 10x^{-30} + 11x^{-28} + 9x^{-26} + 9x^{-24} + 7x^{-22} + 7x^{-20} + 6x^{-18} + 5x^{-16} + 4x^{-14} + 3x^{-12} + 2x^{-10} + 2x^{-8} + x^{-6} + x^{-4} + x^{-2} + 1, \\
P_{10}^w &= 4x^{-40} + 7x^{-38} + 11x^{-36} + 12x^{-34} + 14x^{-32} + 12x^{-30} + 12x^{-28} + 11x^{-26} + 10x^{-24} + 8x^{-22} + 8x^{-20} + 6x^{-18} + 5x^{-16} + 4x^{-14} + 3x^{-12} + 2x^{-10} + 2x^{-8} + x^{-6} + x^{-4} + x^{-2} + 1, \\
P_{11}^w &= 5x^{-42} + 15x^{-40} + 19x^{-38} + 20x^{-36} + 16x^{-34} + 17x^{-32} + 15x^{-30} + 14x^{-28} + 12x^{-26} + 11x^{-24} + 9x^{-22} + 8x^{-20} + 6x^{-18} + 5x^{-16} + 4x^{-14} + 3x^{-12} + 2x^{-10} + 2x^{-8} + x^{-6} + x^{-4} + x^{-2} + 1, \\
P_{12}^w &= 8x^{-48} + 13x^{-46} + 21x^{-44} + 22x^{-42} + 22x^{-40} + 22x^{-38} + 22x^{-36} + 20x^{-34} + 20x^{-32} + 17x^{-30} + 15x^{-28} + 13x^{-26} + 12x^{-24} + 9x^{-22} + 8x^{-20} + 6x^{-18} + 5x^{-16} + 4x^{-14} + 3x^{-12} + 2x^{-10} + 2x^{-8} + x^{-6} + x^{-4} + x^{-2} + 1,
\end{align*}
\]
\[P_w^{13} = 2x^{-52} + 10x^{-50} + 24x^{-48} + 32x^{-46} + 37x^{-44} + 28x^{-42} + 29x^{-40} + 28x^{-38} + 26x^{-36} + 23x^{-34} + 22x^{-32} + 18x^{-30} + 16x^{-28} + 14x^{-26} + 12x^{-24} + 9x^{-22} + 8x^{-20} + 6x^{-18} + 5x^{-16} + 4x^{-14} + 3x^{-12} + 2x^{-10} + 2x^{-8} + x^{-6} + x^{-4} + x^{-2} + 1,\]

\[P_w^{14} = 9x^{-56} + 22x^{-54} + 37x^{-52} + 38x^{-50} + 39x^{-48} + 37x^{-46} + 38x^{-44} + 36x^{-42} + 35x^{-40} + 32x^{-38} + 29x^{-36} + 25x^{-34} + 23x^{-32} + 19x^{-30} + 17x^{-28} + 14x^{-26} + 12x^{-24} + 9x^{-22} + 8x^{-20} + 6x^{-18} + 5x^{-16} + 4x^{-14} + 3x^{-12} + 2x^{-10} + 2x^{-8} + x^{-6} + x^{-4} + x^{-2} + 1,\]

\[P_w^{15} = 5x^{-60} + 19x^{-58} + 44x^{-56} + 55x^{-54} + 55x^{-52} + 48x^{-50} + 49x^{-48} + 46x^{-46} + 46x^{-44} + 42x^{-42} + 39x^{-40} + 35x^{-38} + 31x^{-36} + 26x^{-34} + 24x^{-32} + 20x^{-30} + 17x^{-28} + 14x^{-26} + 12x^{-24} + 9x^{-22} + 8x^{-20} + 6x^{-18} + 5x^{-16} + 4x^{-14} + 3x^{-12} + 2x^{-10} + 2x^{-8} + x^{-6} + x^{-4} + x^{-2} + 1,\]

\[P_w^{16} = 16x^{-64} + 37x^{-62} + 58x^{-60} + 63x^{-58} + 65x^{-56} + 60x^{-54} + 62x^{-52} + 61x^{-50} + 59x^{-48} + 54x^{-46} + 52x^{-44} + 46x^{-42} + 42x^{-40} + 37x^{-38} + 32x^{-36} + 27x^{-34} + 25x^{-32} + 20x^{-30} + 17x^{-28} + 14x^{-26} + 12x^{-24} + 9x^{-22} + 8x^{-20} + 6x^{-18} + 5x^{-16} + 4x^{-14} + 3x^{-12} + 2x^{-10} + 2x^{-8} + x^{-6} + x^{-4} + x^{-2} + 1,\]

\[P_w^{17} = 6x^{-68} + 32x^{-66} + 73x^{-64} + 89x^{-62} + 90x^{-60} + 76x^{-58} + 79x^{-56} + 76x^{-54} + 75x^{-52} + 71x^{-50} + 67x^{-48} + 60x^{-46} + 56x^{-44} + 49x^{-42} + 44x^{-40} + 38x^{-38} + 33x^{-36} + 28x^{-34} + 25x^{-32} + 20x^{-30} + 17x^{-28} + 14x^{-26} + 12x^{-24} + 9x^{-22} + 8x^{-20} + 6x^{-18} + 5x^{-16} + 4x^{-14} + 3x^{-12} + 2x^{-10} + 2x^{-8} + x^{-6} + x^{-4} + x^{-2} + 1,\]

\[P_w^{18} = 26x^{-72} + 59x^{-70} + 95x^{-68} + 102x^{-66} + 98x^{-64} + 96x^{-62} + 99x^{-60} + 96x^{-58} + 96x^{-56} + 90x^{-54} + 85x^{-52} + 79x^{-50} + 73x^{-48} + 64x^{-46} + 59x^{-44} + 51x^{-42} + 45x^{-40} + 39x^{-38} + 34x^{-36} + 28x^{-34} + 25x^{-32} + 20x^{-30} + 17x^{-28} + 14x^{-26} + 12x^{-24} + 9x^{-22} + 8x^{-20} + 6x^{-18} + 5x^{-16} + 4x^{-14} + 3x^{-12} + 2x^{-10} + 2x^{-8} + x^{-6} + x^{-4} + x^{-2} + 1,\]
\[ P_{19}^w = 12x^{-76} + 56x^{-74} + 112x^{-72} + 139x^{-70} + 140x^{-68} + 117x^{-66} + 122x^{-64} \]
\[ + 122x^{-62} + 119x^{-60} + 113x^{-58} + 110x^{-56} + 100x^{-54} + 93x^{-52} + 85x^{-50} \]
\[ + 77x^{-48} + 67x^{-46} + 61x^{-44} + 52x^{-42} + 46x^{-40} + 40x^{-38} + 34x^{-36} + 28x^{-34} \]
\[ + 25x^{-32} + 20x^{-30} + 17x^{-28} + 14x^{-26} + 12x^{-24} + 9x^{-22} + 8x^{-20} + 6x^{-18} \]
\[ + 5x^{-16} + 4x^{-14} + 3x^{-12} + 2x^{-10} + 2x^{-8} + x^{-6} + x^{-4} + x^{-2} + 1, \]
\[ P_{20}^w = 34x^{-80} + 93x^{-78} + 151x^{-76} + 159x^{-74} + 152x^{-72} + 145x^{-70} + 151x^{-68} \]
\[ + 149x^{-66} + 149x^{-64} + 143x^{-62} + 137x^{-60} + 127x^{-58} + 120x^{-56} + 108x^{-54} \]
\[ + 99x^{-52} + 89x^{-50} + 80x^{-48} + 69x^{-46} + 62x^{-44} + 53x^{-42} + 47x^{-40} + 40x^{-38} \]
\[ + 34x^{-36} + 28x^{-34} + 25x^{-32} + 20x^{-30} + 17x^{-28} + 14x^{-26} + 12x^{-24} + 9x^{-22} \]
\[ + 8x^{-20} + 6x^{-18} + 5x^{-16} + 4x^{-14} + 3x^{-12} + 2x^{-10} + 2x^{-8} + x^{-6} + x^{-4} + x^{-2} + 1. \]
\[ (4.15) \]

For \( 1 \leq m \leq 13 \), our results of \( P_m^w \) are in perfect agreement with those of (the revised version of) \([15, \text{Theorem 4.2}]\). As explained in \([15]\), \( P_m^w \) gives the dimension of the space \( J_{k,m}^{E_8} \) of weak Jacobi forms of arbitrary weight \( k \) and given index \( m \) as
\[
\frac{P_m^w}{(1 - x^4)(1 - x^6)} = \sum_{k \in \mathbb{Z}} \dim J_{k,m}^{E_8} x^m. \tag{4.16}
\]

4.3. Lowest weight generators of free modules

We further construct all \( W(E_8) \)-invariant Jacobi forms of weight \( k \leq -4m \) and index \( m \) for \( m \leq 28 \). We find that there are no \( W(E_8) \)-invariant Jacobi forms of weight \( k < -4m \) and index \( m \) for \( 1 \leq m \leq 28 \). This serves as a further supporting evidence of the following conjecture:

**Conjecture 4.3** (Sun and Wang \([15, \text{Conjecture 6.1}]\)). *The weight of non-zero \( W(E_8) \)-invariant weak Jacobi forms of index \( m \) is not less than \(-4m \).*

On the other hand, for every \( 12 \leq m \leq 28 \), we find \( W(E_8) \)-invariant Jacobi forms of weight \(-4m \) and index \( m \). We summarize our results as follows:

**Proposition 4.4.** Let the dimension of the space \( J_{-4m,m}^{E_8} \) be described by the series
\[
J_{-4m,m}^{w, \text{lb}} := \sum_{m=0}^{\infty} \dim J_{-4m,m}^{E_8} x^m. \tag{4.17}
\]
Table 1: Number of generators of $J_{E,s}^b$

$J^{w,b}$ is determined up to the order of $x^{28}$ as

$$J^{w,b} = 1 + x^4 + 2x^6 + 2x^8 + x^9 + 4x^{10} + 8x^{12} + 2x^{13} + 9x^{14} + 5x^{15} + 16x^{16} + 6x^{17} + 26x^{18} + 12x^{19} + 34x^{20} + 23x^{21} + 52x^{22} + 31x^{23} + 80x^{24} + 53x^{25} + 105x^{26} + 83x^{27} + 154x^{28} + O(x^{29}).$$

(4.18)

Based on these results, we consider the subspace of $J_{E,s}^b$ given by

$$J_{E,s}^b := \bigoplus_{m=0}^\infty J_{E,s}^{b,-4m,m}.$$  

(4.19)

Clearly, $J_{E,s}^b$ is a graded subalgebra of $J_{s,s}^b$ over $M_*$. Since Conjecture 4.3 is verified for $1 \leq m \leq 28$, at least within this range of $m$, determining generators of weight $-4m$ and index $m$ of $J_{s,s}^E$ is equivalent to determining those of $J_{E,s}^b$. We determine generators of index $m$ of $J_{E,s}^b$ for $1 \leq m \leq 28$.

**Proposition 4.5.** Let $d_m^b$ denote the number of generators of index $m$ of $J_{E,s}^b$. For $1 \leq m \leq 28$, $d_m^b$ are determined as in Table 1.

One sees that $d_m^b > 0$ for all $12 \leq m \leq 28$. This implies the following corollary:

**Corollary 4.6.** The generators of $J_{E,s}^b$ must include those of weight $-4m$ and index $m$ with all $12 \leq m \leq 28$.

It is worth noting that $J_{E,s}^b$ is not freely generated, i.e. some elements of $J_{E,s}^b$ are not uniquely expressed in terms of the generators. This is clearly seen from

$$\prod_{m=1}^\infty \frac{1}{(1-x^m)^{d_m^b}} - J^{w,b} = 3x^{24} + 2x^{25} + 5x^{26} + 6x^{27} + 14x^{28} + O(x^{29}).$$

(4.20)

One sees that some algebraic relations among polynomials of generators start appearing at index 24. This also means that $J_{E,s}^b$ is not freely generated.

The above fact and the existence of a large number of generators give enough reason to conclude that the structure of $J_{E,s}^b$ is highly complicated. On the other hand, the modest behavior of $d_m^b$ as a function of $m$ at large $m$ might be an indication that $J_{E,s}^b$ is finitely generated, as conjectured in [15, Conjecture 6.7].
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A. Simple roots and fundamental weights of $E_8$

Let $\{e_j\}$ ($j = 1, 2, \ldots, 8$) be the orthonormal basis of $\mathbb{C}^8$. We take the simple roots of $E_8$ as

$$
\alpha_{E_8}^1 = \frac{1}{2} (e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8), \\
\alpha_{E_8}^2 = e_1 + e_2, \\
\alpha_{E_8}^j = -e_{j-2} + e_{j-1} \quad (j = 3, 4, \ldots, 8).
$$

(A.1)

The fundamental weights of $E_8$ are then given by

$$
\Lambda_{E_8}^1 = 2e_8, \\
\Lambda_{E_8}^2 = \frac{1}{2} e_1 + \frac{1}{2} e_2 + \frac{1}{2} e_3 + \frac{1}{2} e_4 + \frac{1}{2} e_5 + \frac{1}{2} e_6 + \frac{1}{2} e_7 + \frac{1}{2} e_8, \\
\Lambda_{E_8}^3 = -\frac{1}{2} e_1 + \frac{1}{2} e_2 + \frac{1}{2} e_3 + \frac{1}{2} e_4 + \frac{1}{2} e_5 + \frac{1}{2} e_6 + \frac{1}{2} e_7 + \frac{1}{2} e_8, \\
\Lambda_{E_8}^4 = e_3 + e_4 + e_5 + e_6 + e_7 + 5e_8, \\
\Lambda_{E_8}^5 = e_4 + e_5 + e_6 + e_7 + 4e_8, \\
\Lambda_{E_8}^6 = e_5 + e_6 + e_7 + 3e_8, \\
\Lambda_{E_8}^7 = e_6 + e_7 + 2e_8, \\
\Lambda_{E_8}^8 = e_7 + e_8.
$$

(A.2)

B. Special functions

The Jacobi theta functions are defined as

$$
\vartheta_1(z, \tau) := i \sum_{n \in \mathbb{Z}} (-1)^n y^n q^{(n+1)/2}, \\
\vartheta_2(z, \tau) := \sum_{n \in \mathbb{Z}} y^n q^{(n+1)/2}, \\
\vartheta_3(z, \tau) := \sum_{n \in \mathbb{Z}} y^n q^{n/2}, \\
\vartheta_4(z, \tau) := \sum_{n \in \mathbb{Z}} (-1)^n y^n q^{n/2}.
$$

(B.1)
where
\[ y = e^{2\pi i z}, \quad q = e^{2\pi i \tau} \]
and \( z \in \mathbb{C}, \tau \in \mathbb{H} \). We often use the following abbreviated notation
\[ \vartheta_k(\tau) := \vartheta_k(0, \tau). \]  
(B.3)

The Dedekind eta function is defined as
\[ \eta(\tau) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \]  
(B.4)

The Eisenstein series are given by
\[ E_{2n}(\tau) = 1 - \frac{4n}{B_{2n}} \sum_{k=1}^{\infty} \frac{k^{2n-1} q^k}{1 - q^k} \]  
for \( n \in \mathbb{Z}_{>0} \). The Bernoulli numbers \( B_k \) are defined by
\[ \frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k. \]  
(B.6)

We often abbreviate \( \eta(\tau), E_{2n}(\tau) \) as \( \eta, E_{2n} \) respectively.

References

[1] M. Eichler and D. Zagier, “The Theory of Jacobi forms,” Prog. in Math. 55, Birkhäuser-Verlag, 1985.
[2] K. Wirthmüller, “Root systems and Jacobi forms,” Comp. Math. 82 (1992) 293–354.
[3] I. Satake, “Flat structure for the simple elliptic singularity of type \( \tilde{E}_6 \) and Jacobi form,” hep-th/9307009.
[4] M. Bertola, “Jacobi groups, Jacobi forms and their applications,” PhD. Thesis, SISSA, Trieste, 1999.
[5] M. Bertola, “Frobenius manifold structure on orbit space of Jacobi groups; Part I,” Differ. Geom. Appl. 13 (2000) 19–41.
[6] K. Sakai, “\( E_n \) Jacobi forms and Seiberg–Witten curves,” Commun. Num. Theor. Phys. 13 (2019) 53–80 [arXiv:1706.04619 [hep-th]].
[7] D. Adler and V. Gritsenko, “The $D_8$-tower of weak Jacobi forms and applications,” J. Geom. Phys. 150 (2020) 103616 [arXiv:1910.05226 [math.AG]].

[8] D. Adler, “The structure of the algebra of weak Jacobi forms for the root system $F_4$,” [arXiv:2007.07116 [math.AG]].

[9] J. A. Minahan, D. Nemeschansky, C. Vafa and N. P. Warner, “$E$-Strings and $N = 4$ Topological Yang-Mills Theories,” Nucl. Phys. B 527 (1998) 581–623 [hep-th/9802168].

[10] T. Eguchi and K. Sakai, “Seiberg–Witten Curve for the $E$-String Theory,” JHEP 05 (2002) 058 [hep-th/0203025].

[11] K. Sakai, “Topological string amplitudes for the local $\frac{1}{2}K3$ surface,” PTEP 2017 (2017) no.3, 033B09 [arXiv:1111.3967 [hep-th]].

[12] H. Wang, “Weyl invariant $E_8$ Jacobi forms,” Commun. Num. Theor. Phys. 15 (2021) no.3, 517–573 [arXiv:1801.08462 [math.NT]].

[13] M. X. Huang, A. Klemm and M. Poretschkin, “Refined stable pair invariants for $E_r$, $M$- and $[p, q]$-strings,” JHEP 11 (2013) 112 [arXiv:1308.0619 [hep-th]].

[14] M. Del Zotto, J. Gu, M. X. Huang, A. K. Kashani-Poor, A. Klemm and G. Lockhart, “Topological Strings on Singular Elliptic Calabi-Yau 3-folds and Minimal 6d SCFTs,” JHEP 03 (2018) 156 [arXiv:1712.07017 [hep-th]].

[15] K. Sun and H. Wang, “Weyl invariant $E_8$ Jacobi forms and $E$-strings,” [arXiv:2109.10578 [math.NT]].

[16] H. Wang, “Weyl invariant Jacobi forms: A new approach,” Adv. Math. 384 (2021) 107752 [arXiv:2007.16033 [math.NT]].