Modeling and analysing of spring-loaded double parallelogram mechanism using moment balance

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Abstract. Double Parallelogram Lifting Mechanism (DPLM) is a multi-linkage mechanism adopted as the lifting mechanism of the designed robot in this paper. DPLM is designed based on Double Parallelogram Mechanism (DPM), making it a compact and stable multi-linkage mechanism with a low retracted height and a great working height. This paper investigates the condition in which DPLM obtains static equilibrium at any height using Hookean springs and the minimization of the motor’s required torque exertion against the gravitational moment of the weight of the mechanism. A mathematical model describing all the factors involved in moment balancing of DPLM is derived and established in MATLAB and is verified through experiments with a virtual prototype conducted in SOLIDWORKS. The obtained results show that the magnitude of the moment exerted by the tension of a spring can be determined not only by its exact installation positions but also by its D value, which is the difference in the distances between the two installation points of a spring and their respective rotation axes. An optimization method based on minimizing the RMSE of the unbalanced moment for DPLM is introduced.

1. Introduction

Double Parallelogram Mechanism (DPM) is a compact and stable multi-linkage mechanism. This mechanism is widely adopted in both heavy-load and light-load situations because of its light weight and its large range of extension. Heavy-load applications of DPM mainly include lifting mechanisms and aerial work platforms [1], in which each parallelogram is driven independently by a hydraulic cylinder in the lifting process. DPM is also used in forklifts [2] in which it is driven by only one hydraulic cylinder, moving both parallelograms simultaneously. This mechanism is also used in the design of stack parking systems, in which the whole mechanism can be driven by a single motor to realize three motion periods, including lifting, translation, and fillet transition of vehicles[3].

Applications of DPM in light-load situations include Remote Center of Motion (RCM) Mechanisms for surgical operations[4], exoskeletons[5][6], and force-reflective robots[7]. Some novel studies also investigated the kinetic energy transmitted through small deformations of DPM.[8][9]

In this paper, a robot is designed with DPM as its lifting mechanism (Figure 1). Therefore, this mechanism will be called a Double Parallelogram Lifting Mechanism (DPLM) in the rest of this paper. The DPLM enables the robot to elevate along a substantially vertical path at a high speed only with a single motor installed on the mechanism, while having a great working height without increasing the retracted height. The load platform of the DPLM is maintained horizontal throughout the whole lifting process, making it desirable to be used as a platform for inspection purposes. For example, when the
lifting apparatus is used in inspecting large vehicles, such as aircraft and water vessels, inspection devices can be installed on the load platform.[10]

![Figure 1. The designed robot with a DPLM as its lifting mechanism.](image1)

Double Parallelogram Lifting Mechanism (DPLM) is composed of two four-bar linkages connected with a pair of meshing gears, which enable the two linkages to elevate simultaneously at high speed with only one drive motor. However, if the motion of the whole DPLM depends on only one motor, a motor with very high power will be needed, which will highly increase the weight and the cost of the DPLM. In addition, the motor is also more prone to overloading and fatigue in such a situation. Therefore, in order to reduce the load of the drive motor, Hookean springs, and other elastic materials, are often installed on the parallelogram pair to resist the gravitational moment due to the weight of the DPLM.

However, the optimal installation positions and the parameters of the Hookean springs are currently obtained through testings. Therefore, in this paper, the method of using the elastic force of Hookean spring to maintain DPLM in static equilibrium at any height is studied. Numerical simulation is used in optimizing the parameters of Hookean spring as well as their installation positions.

![Figure 2. DPLM with a positive rotation angle(θ).](image2)

![Figure 3. DPLM with a negative rotation angle(θ).](image3)
2. Description of DPLM

DPLM, as shown in Figure 2 and Figure 3, is composed of the lower four-bar linkage $O_3O_3'O_4O_4$ and the upper four-bar linkage $O_1O_2'O_2'O_4$. The lower four-bar linkage is fixed to the base through the hinge at $O_4$, while the upper four-bar linkage is connected to the lower four-bar linkage through a pair of meshing gears. Since the upper and lower parallelograms can move simultaneously, the drive actuator can be installed at any hinge of $O_1$, $O_2$, $O_3$, $O_4$, $O_3'$, $O_4'$.

$P_1$, $P_2$, and $P_3$ are Hookean springs installed on the upper or lower four-bar linkage to resist the gravitational moment. The Hookean springs can be ordinary springs, rubber bands, or gas springs.

3. Calculation model

A coordinate system is established with $O_4'$ as the origin as shown in Figure 2 and Figure 3. The angle between the linkage $O_4'O_4$ and the negative X-axis is $\theta$.

3.1. Naming

- Centroids:
  - $m$ (e.g. $m_{O_1O_2}$ represents the mass of the linkage $O_1O_2$)
- Forces:
  - $H$ (e.g. $H_{O_1}$ represents the horizontal force at pivot $O_1$)
  - $V$ (e.g. $V_{O_1}$ represents the vertical force at pivot $O_1$)
  - $f$ (e.g. $f_1$ represents the tension exerted by a spring)
- Length:
  - $L$ (e.g. $L_{O_1O_2}$ represents the length of linkage $O_1O_2$)
  - $X$ (e.g. $X_{O_1O_1'}$ represents the horizontal distance between the centroid of linkage $O_1O_1'$ and its rotational axis)

3.2. Force analysis

Each part of DPLM is analyzed separately, and an equation describing all the factors involved in moment balancing is established at the end according to the principles of theoretical mechanics.[11]

The free body diagrams (FBD) of all linkages in the DPLM are shown below in Figure 4 to 9:

![Figure 4. FBD of linkage $O_1'O_2'$.](image)

![Figure 5. FBD of linkage $O_2O_4$.](image)
3.2.1. Linkage $O'_1O'_2$. By the conditions for equilibrium of rigid bodies, as shown in Figure 4, three balance equations (1)(2)(3) are established.

$$\sum H_y = H_{O_1} - H_{O_2} = 0$$  \hspace{1cm} (1)

$$\sum V_y = V_{O_1} - V_{O_2} = 0$$  \hspace{1cm} (2)

$$\sum M_{O_2} = H_{O_1} - L_{O_2} \cdot G_{O_1} = 0$$  \hspace{1cm} (3)

The expressions of $H_{O_1}$ and $H_{O_2}$ are derived as in formula (4) by substituting formula (3) into formula (1).

$$\frac{G_{O_1} \cdot X_{O_2}}{L_{O_2}} = H_{O_1} = H_{O_2}$$  \hspace{1cm} (4)

3.2.2. Linkage $O_2O'_2$. By the conditions for equilibrium of rigid bodies, as shown in Figure 7, three balance equations (5)(6)(7) are derived.

$$\sum V_y = V_{O_2} + f_1 \cdot \sin(\phi - \theta) - G_{O_2} = 0$$  \hspace{1cm} (5)

$$\sum H_{O_2} = f_1 \cdot \cos(\phi - \theta) = H_{O_2}$$  \hspace{1cm} (6)

$$\sum M_{O_2} = f_1 \cdot L_{O_2} \cdot G_{O_2} \cdot X_{O_1} \cdot H_{O_2} = 0$$  \hspace{1cm} (7)

3.2.3. Linkage $O_1O'_1$. By the conditions for equilibrium of rigid bodies, as shown in Figure 6, three balance equations (8)(9)(10) are derived.

$$\sum V_y = V_{O_2} + f_1 \cdot \sin(\phi - \theta) - G_{O_1} = 0$$  \hspace{1cm} (8)

$$\sum H_{O_1} = f_1 \cdot \cos(\phi - \theta) + H_{O_1}$$  \hspace{1cm} (9)

$$\sum M_{O_1} = f_1 \cdot L_{O_1} \cdot F_1 \cdot G_{O_1} \cdot X_{O_1} \cdot \cos \theta + H_{O_1} \cdot L_{O_1} \cdot \sin \theta - V_{O_2} \cdot L_{O_2} \cdot \cos \theta \cdot M = 0$$  \hspace{1cm} (10)

3.2.4. Linkage $O_2O_4$. By the conditions for equilibrium of rigid bodies, as shown in Figure 5, two balance equations (11)(12) are derived.

$$\sum V_y = V_{O_2} + f_1 \cdot \sin(\phi - \theta) - G_{O_1} = 0$$  \hspace{1cm} (11)

$$\sum H_{O_1} = f_1 \cdot \cos(\phi - \theta) + H_{O_1}$$  \hspace{1cm} (12)
\[
\sum H_i = -H_{O_1} - H_{O_2} + H_{O_3} = 0
\]  
\[
\sum V_y = -V_{O_1} - V_{O_2} + V_{O_3} = 0
\]

3.2.5. **Linkage \( O_1 \) \( O_1' \).** By the conditions for equilibrium of rigid bodies, as shown in Figure 8, a balance equation (13) is derived.

\[
\sum M_{O_1} = f_2 L_{O_1} + G_{O_1} X_{O_1} \cdot \cos \theta + H_{O_3} \cdot L_{O_1} \cdot \sin \theta \cdot V_{O_3} - L_{O_1} \cdot \cos \theta - M_c = 0
\]  

3.2.6. **Linkage \( O_4 \) \( O_4' \).** By the conditions for equilibrium of rigid bodies, as shown in Figure 9, a balance equation (14) is derived.

\[
\sum M_{O_4} = f_2 L_{O_4} + G_{O_4} X_{O_1} \cdot \cos \theta + H_{O_3} \cdot L_{O_4} \cdot \sin \theta \cdot V_{O_4} - L_{O_4} \cdot \cos \theta + M_d = 0
\]

The \( M_d \) in formula (14) is the moment exerted by the actuator, which is zero when the actuator is not in use.

3.2.7. **Finding tensions**. In order to obtain the relation between tension \( f_1 \) and \( f_2 \), by adding formula (7) and formula (13), then substituting formula (4) into the equation, we further derive formula (15):

\[
f_1 \left( L_{O_1} - L_{O_2} \right) - G_{O_1} X_{O_1} \cdot \cos \theta + H_{O_3} \cdot L_{O_1} \cdot \sin \theta + V_{O_3} \cdot L_{O_1} \cdot \cos \theta = 0
\]  

By adding formula (10) and formula (14) we obtain the temporary equation below:

\[
f_2 \left( L_{O_1} + L_{O_2} \right) + G_{O_1} X_{O_1} \cdot \cos \theta + \left( H_{O_3} + H_{O_4} \right) \cdot L_{O_1} \cdot \sin \theta \cdot M_c + M_d = 0
\]

Then by substituting formulae (5)(6)(8)(9)(11) into the temporary equation, after further simplifications we obtain formula (16):

\[
f_2 \left( L_{O_1} + L_{O_2} \right) + G_{O_1} X_{O_1} + G_{O_1} X_{O_1} + \left( G_{O_1} + G_{O_2} + G_{O_1} + G_{O_2} \right) \cdot L_{O_1} \cdot \cos \theta = 0
\]  

3.2.8. **Establishing the relationship between \( f_1 \) and \( f_2 \)**. Subtracting (15) from (16), \( M_C \) will be eliminated and we obtain (17):

\[
f_1 \left( L_{O_1} + L_{O_2} \right) + f_2 \left( L_{O_1} - L_{O_2} \right) - G_{O_1} X_{O_1} - G_{O_1} X_{O_1} - G_{O_2} X_{O_1} + L_{O_1} \cdot \cos \theta - G_{O_1} X_{O_1} - G_{O_2} X_{O_1} - G_{O_2} X_{O_1} = 0
\]  

Notice that when tension \( f_1 \) and \( f_2 \) in formula (17) are substituted with arrays of tensions of multiple Hookean springs, the derivation of (17) isn’t affected. By this we obtain (18):

\[
\sum f_{i_1} \cdot \sin \phi_{i_1} \left( L_{O_1} + L_{O_2} \right) + \sum f_{i_2} \cdot \sin \phi_{i_2} \left( L_{O_2} + L_{O_3} \right) - G_{O_1} X_{O_1} - G_{O_1} X_{O_1} - G_{O_2} X_{O_1} + \left( G_{O_1} + G_{O_2} + G_{O_1} + G_{O_2} \right) \cdot L_{O_1} \cdot \cos \theta = 0
\]

The \( L_{O_1} + L_{O_2} \) and \( L_{O_2} + L_{O_3} \) in formula (18), which are the differences in the distances between the two installation points of a spring and their respective rotation axes, are named respectively \( D_i \) and \( D_j \).

Formula (18) also suggests that the magnitude of the moment of the tension of a spring can be determined not only by its exact installation position but also by its \( D \) value. The absence of \( X_{O_1} \) and
$X_{O_2O_4}$ in formula (18) also suggests that the gravitational moment of the whole DPLM is independent of the position of the center of mass of linkages $O_1'O_2'$ and $O_2O_4$.

4. **Virtual Prototype Validation**

Here the validity of the proposed model in (18) is checked by comparing

1. the gravitational moment at $O_4'$ at any angle without installation of any Hookean spring on the mechanism, and

2. the tension required for equilibrium with different installation positions of spring at multiple angles

obtained in the MATLAB model (Figure 11) with those obtained in experiments conducted with a virtual prototype in SOLIDWORKS (SW).

A virtual prototype of DPLM, as shown in Figure 10, is set up in SW and the position of linkage $O_2'O_4'$ is fixed. The parameters of the prototype are shown in Table 1.[12]

**Figure 10.** A virtual prototype of DPLM in SW.  
**Figure 11.** A MATLAB model of DPLM.

| Linkage | Length(mm) | Mass(g) |
|---------|------------|---------|
| $O_1'O_1'O_2'$ | 762 | 847.34661 |
| $O_1'O_2'$ | 254 | 298.70661 |
| $O_2O_4'$ | 615 | 679.04403 |
| $O_3'O_3'O_4'$ | 648 | 724.22661 |

### 4.1. Validating the moment at $O_4'$

An actuator is added at $O_4'$ in the SW virtual prototype and it is made to rotate at a constant angular velocity with $\theta$ within the range [-20,60], causing the whole mechanism to rise. Then, the moment exerted by the actuator with respect to $\theta$, which by formula (18) is equal to the sum of gravitational moment of all linkages, is exported from SW. This exported moment is compared with the gravitational moment calculated in MATLAB.

The results of the SW simulation and MATLAB calculation are plotted in Figure 12, with the root-mean-square error(RMSE) between the two results being only 8.7051. This has shown that the MATLAB model (18) is completely in accordance with the SW simulation results.

Note that no Hookean spring is installed on the DPLM in this simulation.
4.2. Validating the tension required for the DPLM prototype to obtain equilibrium with different installation positions of spring.

To show that the tension part of the model (18) is in accordance with the SW simulation, multiple pairs of $(D, \theta)$ are selected randomly and the respective tensions required for equilibrium are calculated in MATLAB. These forces are then used as anchor points in finding the force required for equilibrium in the SW simulation. The tested equilibrium point (at which the mechanism stays in equilibrium), the tested falling point (at which the mechanism falls), and the respective ranges of error between the MATLAB model and the SW simulation results are presented in Table 2.

**Table 2. Results from SW simulation.**

| $D$  | $\theta$ | Required force calculated in MATLAB model (N) | Tested equilibrium point from SW (N) | Tested falling point from SW (N) | Range of error between MATLAB SW results (N) |
|------|----------|-----------------------------------------------|--------------------------------------|----------------------------------|---------------------------------------------|
| 260  | 60       | 60.7905                                       | 60.70                                | 60.68                            | [+0.0905, +0.1105]                           |
| 260  | 40       | 99.2001                                       | 99.05                                | 99.02                            | [+0.1501, +0.1801]                           |
| 260  | 20       | 134.6107                                      | 134.37                              | 134.35                           | [+0.2407, +0.2607]                           |
| 260  | 0        | 165.9367                                      | 165.70                              | 165.60                           | [+0.2367, +0.3367]                           |
| 260  | -20      | 192.2237                                      | 191.90                              | 191.85                           | [+0.3237, +0.2237]                           |
| 100  | 60       | 207.3691                                      | 207.37                              | 207.36                           | [-0.0009, +0.0091]                           |
| 100  | 40       | 242.8567                                      | 242.86                              | 242.85                           | [-0.0033, +0.0067]                           |
| 100  | 20       | 283.7354                                      | 283.74                              | 283.73                           | [-0.0046, +0.0054]                           |
| 100  | 0        | 324.0134                                      | 324.02                              | 324.01                           | [-0.0066, +0.0034]                           |
| 100  | -20      | 359.8104                                      | 359.82                              | 359.81                           | [-0.0096, +0.0004]                           |

The table shows that the error between our MATLAB model and the SOLIDWORKS simulation is less than +0.4N under all tested $(D, \theta)$ pairs.
5. Numeric analysis

With our model validated in the previous section, this section will propose an optimization method for the moment balance of the DPLM loaded with one spring. The equation of moment balance of such a DPLM (19) is derived from formula (18):

\[
k (L_{p1}P_{2} l) \cdot \sin \varphi \cdot D - G_{O_1O_2} \cdot X_{O_1O_2} - G_{O_1O_1} \cdot X_{O_1O_1} - G_{O_1O_1} \cdot L_{O_1O_2} \cdot \cos \theta \cdot G_{O_1O_1} \cdot X_{O_1O_1} - G_{O_1O_1} \cdot X_{O_1O_1} - G_{O_1O_1} \cdot L_{O_1O_1} \cdot \cos \theta + M_d = 0
\]

(19)

Formula (19) consists of three parts: \(k (L_{p1}P_{2} l) \cdot \sin \varphi \cdot D\) is the moment of tension exerted by a spring (\(l\) is the initial length of the spring; \(k\) is the spring constant of the spring),

\[-G_{O_1O_2} \cdot X_{O_1O_2} - G_{O_1O_1} \cdot X_{O_1O_1} - G_{O_1O_1} \cdot L_{O_1O_2} \cdot \cos \theta \cdot G_{O_1O_1} \cdot X_{O_1O_1} - G_{O_1O_1} \cdot X_{O_1O_1} - G_{O_1O_1} \cdot L_{O_1O_1} \cdot \cos \theta \]

is the gravitational moment of the whole mechanism, which is known, and \(M_d\) is the moment exerted by the actuator to compensate for the unbalanced moment.

The maximum control accuracy of the motion of the mechanism can be achieved through DTC (Direct torque control) when the output torque of the actuator is required to exert changes linearly with \(\theta\). Therefore, the objective of the optimization here is to find a set of \([D \ l \ k]\) that makes the unbalanced moment best resemble a straight line. In other words, the objective is to minimize the RMSE between the unbalanced moment and its linear regression line.

This optimization considers the moment balance of the DPLM with \(\theta\) changes within [-20, 60].

Three variables are involved in this optimization: \(k\), \(l\), and \(D\). Considering that the deformation of the spring has to be limited to maintain its spring rate in a constant range and prevent it from irreversible deformation, the maximum length of it should not exceed three times its initial length. Thus, we have constraint: \(MAX(L_{p1}P_{2}) \leq 3l\), in which \(L_{O_1O_2}\) is related to the value of \(D\).

The independent effects of the three variables are respectively observed by plotting the moment of tension with respect to one variable while the other two remain constant:

Figure 13 shows that the increase in the magnitude of the moment is the main change due to the increase of \(D\). Figure 14 shows that the increase in the magnitude of the moment is the only change with the increase of \(k\). Figure 15 shows that the decrease of \(l\) not only causes the magnitude of the moment of tension but also shifts the maximum moment to a greater \(\theta\) (i.e. to the right), closing up the gap between it and the moment of weight, \(M_g\).

![Figure 13. Moment of tension with respect to \(D\).](image1)

![Figure 14. Moment of tension with respect to \(k\).](image2)
To find the optimal set of variables for the moment of tension within the spring extension range, \([k \ l \ D]\) are first iterated within the ranges \([0.1, 2]\), \([100, 300]\), and \([0, 400]\) with relatively big steps being 0.2, 10 and 20 respectively. Variable sets with relatively small RMSE (\([k \ l \ D]\) respectively being \([0.5 \ 140 \ 270]\), \([0.5 \ 140 \ 280]\), \([0.7 \ 130 \ 210]\), \([0.5 \ 150 \ 290]\) as shown in Figure 16) found in the first iteration are used as anchors for a second iteration.

**Figure 15.** Moment of tension with respect to \(l\).

**Figure 16.** Results from the first iteration.
The second iteration is performed within the ranges \([0.45 \text{ 135 260}]\) to \([0.55 \text{ 145 200}]\), \([0.65 \text{ 120 200}]\) to \([0.75 \text{ 140 220}]\) and \([0.45 \text{ 140 280}]\) to \([0.55 \text{ 160 300}]\) with relatively small steps being 0.01, 1, and 2. The optimal set of variables found in this iteration, which has a RMSE of 80.76, is \([k \ l \ D] = [0.68 \text{ 127 210}]\) with \(\text{MAX}(L_{p1}, L_{p2}) = 2.9994l\), as shown in Figure 17.

![Figure 17. The optimal result found in the second iteration.](image)

6. Conclusion
In this paper, a mathematical model of a spring-loaded Double Parallelogram Lifting Mechanism is derived and developed based on the principle of moment. This model shows that the magnitude of the moment exerted by the tension of a spring can be determined not only by its exact installation positions but also by its \(D\) value, which is the difference in the distances between the two installation positions of a spring and their respective rotational axes. Experiments of the virtual prototype of this mechanism are performed in SOLIDWORKS. The experiment results are essentially identical with the proposed model. An optimization method based on minimizing the RMSE of the unbalanced moment of a Double Parallelogram Lifting Mechanism loaded with one spring is proposed and performed in MATLAB.

7. Future
In the future, the effects of the changes in centers of mass of linkages on the moment balance of DPLM, the moment balance of DPLM when it is tilted at a certain angle, and the optimization method for DPLM with multiple Hookean springs installed will be further discussed and investigated based on the model and method proposed in this paper.

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