Numerical Solution of Wave Propagation in Viscoelastic rods
(Standard Linear Solid Model)

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Abstract. The studies is about the propagation of waves through a short rod (or slug) of viscoelastic material. The viscoelastic material are modelled as standard linear solids which involve three (3) material parameters and the motion is treated as one-dimensional. In this study, a viscoelastic slug is placed between two semi-infinite elastic rods and a wave initiated in the first rod is transmitted through the slug into the second rod. The objective is to relate the transmitted signal to the material parameters of the slug. We solve the governing system of partial differential equations using Laplace transform. We invert the Laplace transformed solution numerically to obtain the transmitted signal for several viscosity time constants and ratios of acoustic impedances. In inverting the Laplace transformed equations, we used the complex inversion formula because there is a branch cut and infinitely many poles within the Bromwich contour. Finally, we discussed the relationship between the viscosity time constants, ratios of acoustic impedances and the results of the interface velocity discontinuities.

1 Introduction
There are materials for which a suddenly applied and maintained state of uniform stress induces an instantaneous deformation followed by a flow process which may or may not be limited in magnitude as time grows [1]. These materials are said to exhibit both an instantaneous elasticity effect and a creep characteristic. This behavior clearly can not be described by either elasticity or viscosity theories alone as it combines features of each and is called viscoelastic. Viscoelasticity is a generalization of elasticity and viscosity. The ideal linear elastic element is the spring whilst the ideal linear viscous element is the dashpot. Energy is stored in springs as elastic strain energy and energy is dissipated in a dashpot as heat [2]. This paper deals with the transmission of waves through a viscoelastic material which is modeled by a small rod (slug) of the material placed between two semi-infinite elastic rods. The purpose of this paper is to determine the properties of the viscoelastic slug after a wave initiated in the first semi-infinite elastic rod is transmitted through the viscoelastic slug into the semi-infinite elastic rod.

In this problem, we investigate the behaviours of the longitudinal waves after the waves transmitted through a finite length viscoelastic slug. We model the viscoelastic material as a standard linear solid (Figure 1). Many dynamical systems are so complex that analytical solutions cannot be found and numerical solution is the only answer. Here, we numerically compute the waves transmitted through the slug into the second semi-infinite elastic rod. Results are obtained for several viscosity time constants and several ratios of acoustic impedances.

2 Mathematical Model of Wave Propagation in Viscoelastic Material
We model the system where a finite length viscoelastic slug is placed between two semi-infinite elastic rods as an idealization of the experimental work described by H. Kolsky [4] as shown in Figure 2. A velocity discontinuity wave is initiated in the first rod and propagates through the viscoelastic slug moving into the second semi-infinite elastic rod. There are multiple reflections and transmission of waves in the viscoelastic slug. When the waves arrive at the interface between the first rod and the slug, some of the waves are...
transmitted and some of them are reflected. The same situation happens when the waves reach the second interface between the slug and the second rod.

\[ u = \frac{\rho_2}{K_1 + \rho_2} \cdot \Delta \]

\[ \sigma = \frac{\rho_2}{K_1 + \rho_2} \cdot \sigma_1 \]

\[ \rho_2/2 \]

\[ \Delta = K_1 \cdot \eta_2 \]

\[ E = \frac{\Delta}{\eta_2} \]

\[ \eta_2 \]

3 Governing Equations

Let \( \bar{u} \) be the additional displacement in the first rod, and \( \bar{v} \) and \( \bar{w} \) be the displacements in the slug and in the rod following the wave propagation respectively, where the notation \( \bar{\cdot} \) indicates dimensional variables. Let \( \sigma_1 \) and \( \sigma_2 \) be the stress in the first rod and in the second rod respectively, \( E_1 \) and \( E_2 \) be young’s modulus in the slug and in the rod respectively, \( \rho_1 \) and \( \rho_2 \) be the density in the slug and in the rods respectively. We choose the origin of coordinates at the center of the first interface and axis \( X \) following the wave propagation respectively, where the notation \( \partial \) indicates partial derivatives.

We choose the origin of coordinates at the center of the first interface and axis \( X \) along the axis of the slug and we assume the wave reaches the viscoelastic slug at time \( t = 0 \). When the wave, initiated in the first rod with constant velocity discontinuity \( \gamma \), reaches the slug at time \( t = 0 \) and \( \dot{X} = 0 \), we can write the position at time \( t \) of the cross-section of the first rod which was at location \( X \) at time \( t = 0 \) as

\[ \ddot{x}(X, t) = \ddot{x}(X, 0) + \ddot{u}(X, t) \]

For \( 0 \leq X \leq h \),

\[ \ddot{x}(X, t) = \ddot{x}(X, 0) + \ddot{u}(X, t) \]

Where \( h \) is the length of the slug. We write the position at time \( t \) of the cross-section of the second rod which at location \( X \) at time \( t = 0 \) as

\[ \ddot{x}(X, t) = \ddot{x}(X, 0) + \ddot{w}(X, t) \]

For \( X \geq h \)

The stress in the body is denoted by \( \sigma_1(\dot{X}, X) \) and at the cross-section \( \dot{X} \), the cross-sectional area is \( A \).

\[ \left( \sigma + \frac{\partial \sigma}{\partial \dot{X}} \frac{\partial \dot{X}}{\partial X} \right) A - \sigma A = \rho A \frac{\partial^2 \ddot{x}}{\partial t^2} + \frac{\partial \sigma}{\partial \dot{X}} \frac{\partial \dot{X}}{\partial X} = \rho \frac{\partial^2 \ddot{w}}{\partial t^2} \]

Then the equation of motion [6] in the first rod is

\[ \frac{\partial \sigma}{\partial \dot{X}} = \rho \frac{\partial^2 \ddot{x}}{\partial t^2} \]

The equation of motion in the slug is

\[ \frac{\partial \sigma}{\partial \dot{X}} = \rho \frac{\partial^2 \ddot{x}}{\partial t^2} \]

and the equation of motion in the second rod is

\[ \frac{\partial \sigma}{\partial \dot{X}} = \rho \frac{\partial^2 \ddot{w}}{\partial t^2} \]

We model the slug as standard linear solid so that the constitutive equation in the slug is

\[ \sigma_1 + \frac{\rho A}{\alpha} \frac{\partial \sigma_1}{\partial \dot{X}} = E \left( \frac{\partial \ddot{x}}{\partial \dot{X}} + \frac{\partial^2 \ddot{w}}{\partial \dot{X}^2} \right) \]

The stress-strain relations in the rods are

\[ \sigma_1 = E \left( \frac{\dot{Y}}{c} - \frac{\dot{X}}{c} \right) H(t - X) + \frac{\rho}{c} \frac{\partial \ddot{w}}{\partial \dot{X}} \]

\[ \sigma_2 = E \left( \frac{\dot{Y}}{c} - \frac{\dot{X}}{c} \right) H(t - X) + \frac{\rho}{c} \frac{\partial \ddot{w}}{\partial \dot{X}} \]

Where \( c^2 = \frac{E}{\rho} \), \( \gamma^2 = \frac{E}{\rho} \), \( \alpha = \frac{c}{\gamma} \) and \( \alpha = \frac{c}{\gamma} \)

Now we non-dimensionalize equations (1), (2) and (3), we obtained

\[ x = X + \frac{\rho c}{\gamma} \left( t - X \right) H(t - X) + \frac{\rho}{c} \ddot{w}(X, t) \]

for \( X < 0 \)

\[ x = X + \frac{\rho c}{\gamma} \alpha \ddot{w}(X, t) \]

for \( 0 \leq X \leq 1 \)
\( x = X + \frac{t}{\ell} w(X,t) \quad \text{for} \quad X > 1 \) (12)

We then non-dimensionalize (5) – (9), take Laplace transform the equations to obtain and solve the differential equations for the displacement transforms \( \hat{u}, \hat{v}, \hat{w} \) in the \( s \) domain. We obtain

\[
\hat{u}(X,s) = a(s)e^{\alpha sX},
\]

\[
\hat{v}(X,s) = b(s) \exp \left( -\frac{\alpha sX}{\beta(s)} \right) + d(s) \exp \left( \frac{\alpha sX}{\beta(s)} \right)
\]

\[
\hat{w}(X,s) = f(s)e^{-\alpha sX}
\]

Where \( \beta^{-1}(s) = \frac{1}{1 + \frac{1}{2} \pi i} \).

In order to find \( a(s), b(s), d(s) \), and \( f(s) \), we apply the boundary conditions described

(i) The particle velocity in the first rod is equal to the velocity in the slug at the first interface \((X = 0)\)

(ii) The stresses in the first rod and in the slug at \( X = 0 \) are equal

(iii) The velocities in the slug and in the second rod are equal at the second interface \((X = 1)\)

(iv) The stresses in the slug and in the second rod at \( X = 1 \) are equal

We obtain

\[
\hat{X} - \hat{X} = -\frac{2}{\pi^2} \sinh(\pi X) + \frac{2e^{-\alpha sX}}{\alpha^2 s^2} \left[ e^{\frac{2\alpha sX}{\beta(s)}} (e^{\frac{2\alpha sX}{\beta(s)}} + \frac{\alpha sX}{\beta(s)}) - (\frac{\alpha sX}{\beta(s)})^2 \right]
\]

\[
\hat{v}(X,s) = \frac{2e^{-\alpha sX}}{\alpha s^2} \left[ e^{\frac{2\alpha sX}{\beta(s)}} (e^{\frac{2\alpha sX}{\beta(s)}} + \frac{\alpha sX}{\beta(s)}) - (\frac{\alpha sX}{\beta(s)})^2 \right]
\]

\[
\hat{w}(X,s) = \frac{4e^{-\alpha sX}}{\alpha s^2} \left[ e^{\frac{2\alpha sX}{\beta(s)}} (e^{\frac{2\alpha sX}{\beta(s)}} + \frac{\alpha sX}{\beta(s)}) - (\frac{\alpha sX}{\beta(s)})^2 \right]
\]

4 The Signal Transmitted Through Viscoelastic Slug

Complex inversion formula is applied to invert the Laplace equation (18). Firstly, we apply the Bromwich contour to lay-out the calculation of the complex integrals along the contour and determine the poles and branch points. Secondly, we find all the roots in the real and complex plane until their contribution to the solution is insignificant. After applying complex inversion formula to (18), we obtain

\[
w(X,t) = \frac{1}{2\pi i} \int_{\Gamma} e^{sX} \hat{w}(X,s) ds = \frac{1}{2\pi i} \int_{\Gamma} e^{sX} \hat{w}(X,s) ds
\]

We consider a closed curve in the left half of the complex plane and we note that there are branch points of the integrand at \( s = -\frac{1}{\alpha} \pi i \) and \( s = -\frac{1}{\alpha} \pi i \) (see figure 3). Then we make a cut along \(-\frac{1}{\alpha} \pi i\) and \(-\frac{1}{\alpha} \pi i\), and modify the contour in order to avoid crossing the branch cut as shown in figure 3. We let \( \chi \) be the closed contour ABCDEFGHJKLM and \( \Gamma \) to be the contour BCD, DE, EF, FG, GH, HJ, JK, KL, LMA. In the closed contour \( \chi \), there are a second degree pole at \( s = 0 \) and poles when the denominator of the integrand (30) is equal to zero. Then it follows from (30), on putting \( T = \sqrt{R^2 - Y^2} \), that

\[
n(X,t) = \lim_{R \to \infty} \frac{1}{2\pi i} \int_{\Gamma} e^{sX} \hat{w}(X,s) ds = \frac{1}{2\pi i} \int_{\Gamma} e^{sX} \hat{w}(X,s) ds
\]

\[
= \sum \text{ residues inside } \chi \int_{\Gamma} e^{sX} \hat{w}(X,s) ds
\]

In finding the roots, we used Maple software routine. We found 150 poles in the upper half plane and consequently their complex conjugates. This gave 300 poles in total which gives reasonably accurate results. The residue of the second order pole at \( s = 0 \) is
\[
\lim_{s \to s_n} \frac{\alpha \beta (s)}{s - s_n} N(s) \quad \text{Residue} = \lim_{s \to s_n} \frac{(s - s_n)N(s)}{s^2 D(s)}
\]

In calculating the other residues we use

\[
\chi = \frac{\alpha (\gamma - \rho)}{2} + 2 - X + \frac{\alpha (1 - z)}{2Z} - \alpha Z (\gamma - \rho)
\]

\[
\sum \text{residue inside } \chi
\]

5 Results and Discussions

Having found the required numbers of poles to evaluate the displacement of the wave in the second rod, we obtain the displacement in the time domain

\[
w(X, t) = \frac{\alpha (\gamma - \rho)}{2} + 2 - X + \frac{\alpha (1 - z)}{2Z} - \alpha Z (\gamma - \rho) + \sum \text{residue inside } \chi
\]

The results also show that the first jump at \( X = 1 \) increases as the ratio of acoustic impedances, \( z \) increases until the \( z \) effective, \( z = \sqrt{\frac{\rho}{\rho}} \). As the \( z \) effective \( z = \sqrt{\frac{\rho}{\rho}} \) approaches 1, the graph shows no jumps after the first jump. Then the jumps appear again for values of \( z \) effective, \( z = \sqrt{\frac{\rho}{\rho}} > 1 \). The numerical results also show that the subsequent jumps for each ratio of viscosity parameters are getting smaller as time increases. However figures 5 shows that the discontinuities are so small that the velocity graphs appear to be increasing smoothly to a horizontal asymptote at \( v = 1 \). Actually there are jumps according to the viscoelastic discontinuity analysis. Since the numerical values are very small compared to the other jumps, they are not apparent in the results.

For bigger viscosity time constants which are displayed in figures 5 and 6, the velocity curves remain constant in between two discontinuity jumps. However when the viscosity time constants are \( \tau = 5 \), \( \rho = 2 \) and \( \rho = 1 \), the velocity curves show a linear increase between two discontinuity jumps. Furthermore figures 5 and 6 show that the velocity curves are increasing smoothly until the curves settle down as the velocities approaches one.

Another significant finding shown by figures 4, 5 and 6 is that the jumps are bigger for large viscosity time constants. The discussion shows that the effective ratios of acoustic impedance \( z \) and the viscosity time constants \( \tau \) and \( \rho \) play a very important role in determining the behaviours of the wave transmitted through the viscoelastic slug.

References

[1] Christensen R M 1971 Theory of Viscoelasticity: An Introduction Academic Press
[2] Bland D R 1960 The Theory Of Linear Viscoelasticity International Series of Monographs on Pure and Appl. Math. Pergamon Press
[3] Kolsky H 1963 Stress Waves in Solids Dover Publication
[4] Kolsky H 1949 Proc. Phys. Soc. London B62 676
[5] Spiegel M R 1983 Advanced Mathematics for Engineers and Scientist McGraw-Hill Book Co.
[6] Stronge W J 2000 Impact Mechanics Cambridge University Press