Questions on the Borel Complexity of Characterized Subgroups

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Abstract

We propose various problems about Borel complexity of characterized subgroups of compact abelian groups, inspired by our forthcoming paper [DI2].

1 Introduction

In the sequel $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ will denote the circle group written additively, identifying $x \in \mathbb{T}$ with an element of the interval $[0,1)$. In these terms we consider the norm in $\mathbb{T}$ defined by $\|x\| = \min\{x, 1-x\}$ for $x \in \mathbb{T}$. It defines an invariant metric $d$ on $\mathbb{T}$ by letting $d(x, y) = \|x - y\|$, for $x, y \in \mathbb{T}$.

For a topological abelian group $X$, a continuous homomorphism $\chi : X \to \mathbb{T}$ is called a character of $X$. Denote by $\hat{X}$ the group of all characters of $X$, that is the Pontryagin dual of $X$.

Definition 1.1. For a compact abelian group $X$ and a sequence $v = (v_n)_{n \in \mathbb{N}}$ of characters of $X$ let

$$s_v(X) := \{x \in X : v_n(x) \to 0 \text{ in } \mathbb{T}\}.$$ 

One can easily check that $s_v(X)$ is a subgroup of $X$.

Definition 1.2. ([DMT]) A subgroup $H$ of a compact abelian group $X$ is called characterized if there exists a sequence of characters $v$ such that $H = s_v(X)$. We also say that $v$ characterizes $H$ and denote, where some special $\mathsf{F}_\sigma$-subgroups $H$ of $X$ were shown to be characterized, the family of all characterized subgroups of $X$.

The following general problem has been studied in [DG].

Problem 1.3. Describe the Borel complexity of $s_v(X)$ for a compact abelian group $X$.

Notation 1.4. Let us recall the first six classes of the Borel Hierarchy.

- $\{\text{open sets}\} \subseteq \mathcal{G}_\delta = \{G_\delta\text{-sets}\} \subseteq \mathcal{G}_{\delta\sigma} = \{\text{countable unions of } G_\delta\text{-sets}\}$
- $\{\text{closed sets}\} \subseteq \mathcal{F}_\sigma = \{F_\sigma\text{-sets}\} \subseteq \mathcal{F}_{\sigma\delta} = \{\text{countable inters. of } F_\sigma\text{-sets}\}$

In this paper we are interested in Borel subgroups, hence we shall use the above notation for the relative class of subgroups instead of sets. (e.g., accordingly, an $F_{\sigma\delta}$-subgroup means a subgroup that is an $F_{\sigma\delta}$-set as a subset, etc.).
Remark 1.5. Every characterized subgroup $H$ of a compact abelian group $X$ is an $F_{\sigma\delta}$-subgroup of $X$, and hence $H$ is a Borel subset of $X$. Indeed, if $H$ is characterized by a sequence $v$, this fact directly follows from the equality

$$s_v(X) = \bigcap_{0 < M < \omega} \bigcup_{m \geq M} \left\{ x \in X : \|v_n(x)\| \leq \frac{1}{M} \right\},$$

where $\|\|$ is the usual norm on the torus.

Since the characterized subgroups are Borel, if $H \in \text{Char}(X)$ then either $|H| = \aleph_0$ or $|H| = \mathfrak{c}$ by Alexandroff-Hausdorff’s theorem [KK, §37, Theorem 3].

The above remark establishes an upper bound for the Borel complexity of characterized subgroups. When characterized subgroups are $G_\delta$ was already established in [DG]. The main issue discussed in this paper is when characterized subgroups are $F_{\sigma}$, as well as the opposite question: when an $F_{\sigma}$-subgroup of a compact abelian group is a characterized subgroup.

2 Characterized Subgroups of Compact Abelian Groups

2.1 Some results on Borel Complexity

It was proved by Bićó, Deshouillers and Sós [BDS] that all countable subgroups of $\mathbb{T}$ are characterized. These authors conjectured that this fact could be extended to all compact metrizable abelian groups, although they gave no rigorous definition of a characterized subgroup of a compact abelian group. The relevant definition [L2] was provided later only in [DMH], where cyclic subgroups of some compact metrizable abelian groups were proved to be characterized. The fact conjectured by Bićó, Deshouillers and Sós was proved to be true:

Theorem 2.1 ([DK, Theorem 1.4], [BSW]). The countable subgroups of a compact metrizable abelian group are characterized.

Fact 2.2. It was pointed out in [D2] that metrizability is a necessary condition in the above theorem. Indeed, for a sequence $v$ of characters of a compact abelian groups $X$, the subgroup $K_v = \bigcap_{n \in \omega} \ker v_n$ is a closed $G_\delta$-subgroup of $X$ contained in $s_v(X)$. Hence, if the (countable) subgroup $\{0\}$ of $X$ is characterized, then $X$ is metrizable.

According to the next theorem, the $G_\delta$-subgroups of a compact abelian group $X$ are precisely the closed characterized subgroups of $X$.

Theorem 2.3 ([DG, Theorem A]). Let $X$ be a compact abelian group.

(a) $\mathfrak{G}_\delta(X) \subseteq \{\text{closed subgroups of } X\}$.

(b) $\mathfrak{G}_\delta(X) \subseteq \text{Char}(X)$; more precisely, a closed subgroup $H$ of $X$ is characterized if and only if $H$ is a $G_\delta$-subgroup.

Remark 2.4. Note that if $X$ is metric then the inclusion of item (a) of the theorem becomes an equality. In case $X = \mathbb{T}$, item (b) of the above theorem is an easy consequence of item (a), as the closed subgroups of $\mathbb{T}$ are precisely the finite ones and they have the form $\mathbb{T}[m] = \{x \in \mathbb{T} : mx = 0\}$ for some $m \in \mathbb{N}$ ($\mathbb{T}[m]$ is trivially characterized by the constant sequence $u = (m)_{n \in \mathbb{N}}$).
For a sequence \( v \) of characters of a compact abelian groups \( X \) the subgroup \( s_v(X)/K_v \) of \( X/K_v \) is a characterized. More generally one has:

**Theorem 2.5 (DG Theorem B).** For a subgroup \( H \) of a compact abelian group \( X \) the following are equivalent:

(a) \( H \) is characterized;

(b) \( H \) contains a closed (necessarily characterized) \( G_\delta \)-subgroup \( K \) of \( X \) such that \( H/K \) is a characterized subgroup of the compact metrizable group \( X/K \).

The previous theorem, reduces the study of characterizable subgroups of compact abelian groups, to the study of these subgroups in compact metrizable abelian groups.

Following [DG], call a subgroup \( H \) of a compact abelian group \( X \) **countable modulo compact** (briefly, CMC) if \( H \) has a compact subgroup \( K \) such that \( H/K \) is countable and \( K \) is a \( G_\delta \)-set of \( X \). Clearly, CMC subgroups are \( F_\sigma \), but they are also characterized subgroups as we shall see in Corollary 2.6 below.

**Corollary 2.6 (DG Corollary B1).** Let \( X \) be a compact abelian group and let \( H \) be a CMC subgroup of \( X \). Then \( H \) is characterized.

This follows easily from theorems 2.1 and 2.5 (see Proposition 2.20 for a stronger result). The above corollary was inspired by the following theorem. The subgroups characterized in the following theorem are particular CMC subgroups.

In fact they are **countable modulo a torsion compact subgroup**.

**Theorem 2.7 (DK Theorem 1.5).** Let \( X \) be a compact abelian group and \( \{F_n\}_{n \in \mathbb{N}} \) a family of closed subgroups of \( X \) such that, \( F_n \preceq F_{n+1} \) for all \( n \in \mathbb{N} \). Then \( H = \bigcup F_n \in \mathsf{Char}(X) \) if and only if there exists \( m \in \mathbb{N} \) such that \( X/F_m \) is metrizable and \( |F_{n+1} : F_n| < \infty \) for all \( n \geq m \).

It is proved in [DG], that every characterized subgroup \( H \) of \( X \) is CMC if and only if \( X \) has finite exponent.

A natural question arising from Corollary 2.6 is the following:

**Question 2.8.** When an \( F_\sigma \)-subgroup \( H \) of a compact metrizable abelian group \( X \) is characterized?

The next theorem sharpening Remark 1.5 was proved in [DG]:

**Theorem 2.9.** [DG] For every infinite compact abelian group \( X \), the following inclusions hold:

\[
\mathcal{G}_\delta(X) \subseteq \mathsf{Char}(X) \subseteq \mathcal{F}_\sigma(X) \quad \text{and} \quad \mathcal{F}_\sigma(X) \nsubseteq \mathsf{Char}(X).
\]

(1) If in addition \( X \) has finite exponent, then

\[
\mathsf{Char}(X) \subseteq \mathcal{F}_\sigma(X).
\]

(2) Clearly, one has \( \mathsf{Char}(X) \nsubseteq \mathcal{F}_\sigma(X) \) in (2), due to the second part of (1).

Gabriyelyan [G4] proved that the implication in the final part of the above Theorem can be inverted:
Theorem 2.10 ([G4]). \( \text{Char}(X) \subseteq \mathfrak{F}_\sigma(X) \) for a compact abelian group \( X \) if and only if \( X \) has finite exponent.

In other words, for every compact abelian group of infinite exponent, he produced a characterized subgroup that is not \( F_\sigma \). The first example of a characterized subgroup of \( T \) that is not an \( F_\sigma \)-subgroup of \( T \) was given by Bukovský, Kholshevikova, Repický [BKR] (see Example 3.7 in §3.3). This motivates the general question of when uncountable characterized subgroups of the compact metrizable groups are \( F_\sigma \)? This issue will be discussed in 2.4, while the case of characterized subgroups of \( T \) is discussed in more detail in §3.

In [G2] and [G3], Gabriyelyan used a useful property of characterized subgroups, noticed first by Biró [B3]. Namely they are Polishable, as we see in the next section.

\section*{2.2 Polishability}

Let us recall the notion of Polishable subgroup that plays a key role in the study of characterized subgroups. It was introduced in [KL].

\textbf{Definition 2.11.} A Polishable subgroup \( H \) of a Polish group \( G \) is a subgroup that satisfies one of the following equivalent conditions:

(a) there exists a Polish group topology \( \tau \) on \( H \) having the same Borel sets as \( H \) when considered as a topological subgroup of \( G \);

(b) there exists a continuous isomorphism from a Polish group \( P \) to \( H \);

(c) there exists a continuous surjective homomorphism from a Polish group \( P \) onto \( H \).

The topology witnessing the polishability of a subgroup is unique [S].

Answering negatively the first named author’s question on whether \( \mathfrak{F}_\sigma(T) \) is contained in \( \text{Char}(T) \), Biró proved the more precise theorem below. Before stating the theorem, let us recall the definition of Kronecker set.

\textbf{Definition 2.12.} A non empty compact subset \( K \) of an infinite compact metrizable abelian group \( X \) is called a Kronecker set, if for every continuous function \( f : K \to T \) and \( \varepsilon > 0 \) there exists a character \( v \in \hat{X} \) such that

\[
\max \{ \| f(x) - v(x) \| : x \in K \} < \varepsilon.
\]

\textbf{Theorem 2.13 ([B3 Theorem 2]).} If \( K \) is an uncountable Kronecker set of \( T \), then \( \langle K \rangle \in \mathfrak{F}_{\sigma}(T) \setminus \text{Char}(T) \).

To this end he proved that every characterized subgroup of \( T \) is Polishable, while the subgroup \( \langle K \rangle \) generated by any uncountable Kronecker set \( K \), is not polishable and hence not characterized. Unaware of this result, Gabriyelyan [G2] generalized significantly this theorem in several directions.

\textbf{Theorem 2.14.} Let \( X \) be a compact metrizable abelian group. Then

(a) [G2 Theorem 1] \( s_\psi(X) \) is Polishable for every sequence \( \psi \) of characters of \( X \);
if $K$ is an uncountable Kronecker set in $X$, then $\langle K \rangle$ is not Polishable; in particular, $\langle K \rangle$ is not characterizable.

The topology witnessing the polishability of $s_\nu(X)$ is induced by the following metric.

**Definition 2.15.** Let $X$ be a compact metrizable abelian group, $\delta$ be a compatible invariant metric on $X$ and $\nu = (v_n)$ be a sequence of characters of $X$. Let $x, y \in X$ and

$$\varrho_\nu(x, y) = \sup_{n \in \mathbb{N}} \{ \delta(x, y), d(v_n(x), v_n(y)) \}.$$ 

Denote by $\tau_{\varrho_\nu}$ the topology of $X$ generated by the metric $\varrho_\nu$. By the uniqueness of the Polish topology, the restriction $\tau_{\varrho_\nu} \upharpoonright s_\nu(X)$ does not depend on $\nu$, in the sense that $\tau_{\varrho_\nu} \upharpoonright s_\mu(X) = \tau_{\varrho_\nu} \upharpoonright s_\nu(X)$ whenever $s_\mu(X) = s_\nu(X)$. This is why we denote by $\tau_{\varrho_u}$ on the whole group $X$ appeared for the first time only in [DG].

The assignments $u \mapsto s_u(X)$ and $u \mapsto \tau_{\varrho_u}$ give rise to two natural equivalence relations between sequences of characters of $X$.

**Definition 2.16.** For two sequences of characters $u$ and $v$ of a compact abelian group $X$ we write

- $u \sim v$, if $s_u(X) = s_v(X)$
- $u \approx v$, if $\tau_{\varrho_u} = \tau_{\varrho_v}$.

As mentioned above, $u \sim v$ always implies also $\tau_{\varrho_u} \upharpoonright s_u(X) = \tau_{\varrho_v} \upharpoonright s_v(X)$, but we do not know if $u \sim v$ implies in general $u \approx v$ (see Question 4.6). We shall see below that the answer is positive if the group $s_u(X) = s_v(X)$ is $F_\sigma$ (Corollary 2.24).

Let $\tau$ be the compact metrizable topology on $X$, it is clear that $\tau = \tau_{\varrho_u} \upharpoonright s_u(X)$ and hence $\tau(H) \supseteq \tau \upharpoonright H$ for $H \in \mathcal{Char}(X)$. Let us mention that $\tau(H) = \tau \upharpoonright H$, if and only if $H$ is closed. Indeed, $\tau \upharpoonright H$ is Polish if and only if $H \in \mathfrak{S}(X)$. By item (a) of Theorem 2.3 the class $\mathfrak{S}(X)$ coincides with class of closed subgroups of $X$. Hence, we can conclude, by the uniqueness of $\tau(H)$.

In the light of the above mentioned results of Biró and Gabriyelyan, it makes sense to introduce also the notation $\mathfrak{Pol}(X)$ for the collection of all Polishable subgroups of $X$. Clearly, Theorem 2.14 (a) can be written briefly as

$$\mathcal{Char}(X) \subseteq \mathfrak{Pol}(X).$$

In [G1, G2] Gabriyelyan produced a compact group $X$ witnessing $\mathcal{Char}(X) \neq \mathfrak{Pol}(X)$.

### 2.3 Further properties of the polish topology of a characterized subgroup

A subset $A$ of a topological abelian group $G$ is called quasi-convex if for every $g \in G \setminus A$ there exists $\chi \in \hat{G}_d$ such that

$$\chi(A) \subseteq \mathcal{T}_+, \text{ but } \chi(g) \not\in \mathcal{T}_+,$$
where $\mathbb{T}_+$ is the image of the segment $[-\frac{1}{2}, 0]$ with respect to the natural quotient map $\mathbb{R} \to \mathbb{T}$ and $G_d$ is the the group $G$ equipped with the discrete topology. A topological group $(G, \tau)$ (as well as its topology $\tau$) is called \textit{locally quasi-convex} if $G$ has a basis of neighbourhoods of $0$ consisting of quasi-convex subsets of $(G, \tau)$.

In connection to the inclusion $\text{Char}(X) \subseteq \text{Pol}(X)$ Gabriyelyan in [G2] added in item (a) of Theorem 2.14 one more property of the finer Polish topology of a characterized subgroup $H$ of a compact metrizable abelian group $X$; namely, it is also locally quasi convex. On the other hand, he produced a compact metrizable abelian group $X$ with a locally quasi convex Polishable $F_\sigma$-subgroup that is not characterized.

A natural class of groups having both properties (namely, being Polish and locally quasi convex) is the class of second countable locally compact abelian groups. Therefore, it makes sense to consider the subfamily $\mathcal{L}(X)$ of $\text{Pol}(X)$, consisting of those subgroups $H \in \text{Pol}(X)$ whose Polish topology is \textit{locally compact}. This is justified by the following result from [N], answering a question from [G3]:

**Theorem 2.17** ([N]). If $G$ is a second countable locally compact abelian group and $p : G \to X$ is a continuous injective homomorphism into a compact metrizable abelian group $X$, then $p(X)$ is a characterized subgroup of $X$.

**Corollary 2.18.** $\mathcal{L}(X) \subseteq \text{Char}(X)$ for every compact metrizable abelian group $X$.

Indeed, if $H \in \mathcal{L}(X)$ and $G$ denotes the group $H$ equipped with the finer locally compact group topology, then the inclusion map $p : G \hookrightarrow X$ is a continuous injective homomorphism, so $H = p(G)$ is a characterized subgroup of $X$ by Theorem 2.17.

The next chain of inclusions summarizes [3] and the above corollary

$$\mathcal{L}(X) \subseteq \text{Char}(X) \subseteq \text{Pol}(X).$$

Recall that a topological space is $\sigma$-\textit{compact} if it is a countable union of compact subspace. Note that if $H \in \mathcal{L}(X)$, then $H$ is $\sigma$-compact (see [K]) and hence obviously an $F_\sigma$-subgroup. Therefore, the following more precise inclusion holds

$$\mathcal{L}(X) \subseteq \text{Char}(X) \cap \overline{\text{F}_\sigma}(X).$$

(4)

**Example 2.19** ([DI3]).

$$\mathcal{L}(\mathbb{Z}_p) = \{\text{countable subgroups}\} \cup \{\text{clopen subgroups}\}.$$

It is easy to check, that every CMC subgroup of $X$ is in $\mathcal{L}(X)$:

**Proposition 2.20.** The CMC subgroups of a compact metrizable abelian group $X$ belong to $\mathcal{L}(X)$, so they are characterized $F_\sigma$-subgroups of $X$.

The next theorem describes when the implication of the above proposition can be inverted.

**Theorem 2.21.** [DI3] For a compact metrizable abelian group $X$ the following are equivalent:
(a) there exist no continuous injective homomorphisms $\mathbb{R} \to X$;

(b) all subgroups in $\mathcal{E}(X)$ are CMC.

Obviously, $X$ satisfies (a) whenever $X$ is totally disconnected. More precisely, $X$ satisfies (a) if and only if its arc-component is either trivial or isomorphic to $\mathbb{T}$ (for more detail see [DI3]).

### 2.4 When characterized subgroups of the compact metrizable groups are $F_\sigma$?

The following topology, obtained by a modification of $\tau_\varphi$, was introduced in [DI2].

**Definition 2.22.** Let $\varphi$ be a sequence of characters on a compact metrizable abelian group $(X, \tau)$ and $	au^*_\varphi$ be the topology on $X$ having as a filter of neighborhoods of 0 in $X$ the family

$$\left\{ W_n = \overline{B_1^\tau(0)} : n \in \mathbb{N} \right\},$$

where $\overline{\cdot}$ denotes the closure in $(X, \tau)$.

We can refer to $\tau^*_\varphi$ as the $F_\sigma$-test topology with respect to the sequence $\varphi$. Note that this topology is metrizable since it has a countable local base. Moreover if $\tau$ is the starting compact topology on $X$, the following inclusions hold.

$$\tau \subseteq \tau^*_\varphi \subseteq \tau_\varphi.$$

Clearly, $\tau_\varphi$ is discrete if and only if $\tau^*_\varphi$ is discrete.

The following theorem, providing a simple criterion for the solution of Problem [13], justifies the term test-topology for $\tau^*_\varphi$.

**Theorem 2.23 ([DI2 Theorem A]).** Let $X$ be a metrizable compact abelian group and $\varphi \in \hat{X}^\mathbb{N}$. Then

$$s_\varphi(X) \in F_\sigma(X) \iff s_\varphi(X) \in \tau^*_\varphi.$$  

An important consequence of the above theorem is the independence of the topology $\tau_\varphi$ on the choice of the characterizing sequence whenever $s_\varphi(X)$ is an $F_\sigma$-subgroup.

**Corollary 2.24 ([DI2 Corollary A1]).** Let $X$ be a metrizable compact abelian group and $\varphi$ be a sequence of characters such that $s_\varphi(X) \in \mathfrak{F}(X)$. If $\psi$ is a sequence of characters such that $\varphi \sim \psi$, then $\varphi \approx \psi$.

**Proof.** Let $H := s_\varphi(X) = s_\psi(X)$. Recall that $\tau_\varphi$ and $\tau_\psi$ coincide when restricted to $H$, since the topology witnessing the polishability of $H$ is unique. Moreover, as $H \subseteq \mathfrak{F}(X)$, $H$ is $\tau^*_\varphi$-open and hence also $\tau^*_\psi$-open. Analogously, $H$ is $\tau^*_\psi$-open. Hence $\tau^*_\varphi$ and $\tau^*_\psi$ coincide on a subgroup that is open in both topologies. Therefore, they coincide on the whole group $X$.  

It is not known if the conclusion of the above corollary remains valid without the assumption that $H$ is an $F_\sigma$-subgroup (see Question [14]).
Corollary 2.25. Let $X$ be a compact metrizable abelian group. If $H \in \mathcal{LC}(X)$, then for every pair of characterizing sequences $u$ and $v$ for $H$ the topologies $\tau_{\theta u}$ and $\tau_{\theta v}$ coincide.

Proof. Follows from Corollary 2.24 and (4).

From Theorem 2.23 we deduce now that $\tau_{\theta v}$ is discrete on the whole $X$ whenever $s_{v}(X)$ is countable.

Theorem 2.26 (D12 Theorem B]). Let $X$ be a metrizable compact abelian group and $v \in \hat{X}$. Then the following are equivalent:

(a) $s_{v}(X)$ is countable;
(b) $|s_{v}(X)| < \aleph$;
(c) $\tau_{\theta v}$ is discrete;
(d) $\tau_{\theta v}^{*}$ is discrete;
(e) $\tau_{\theta v}^{*} |_{s_{v}(X)}$ is discrete;
(f) $\tau(s_{v}(X))$ is discrete.

Proof. (a)$\Rightarrow$(b). It is a consequence of Remark 1.5 (a)$\Rightarrow$(c). Indeed if $s_{v}(X)$ is countable, then it is obviously in $\mathcal{F}_v(X)$. Hence, $s_{v}(X)$ is a $\tau_{\theta v}$-open, by Theorem 2.23. Moreover $\tau_{\theta v}$ is discrete when restricted to $s_{v}(X)$ since the topology that witnesses the polishability of $s_{v}(X)$ is unique. Thus, $s_{v}(X)$ is both $\tau_{\theta v}$-open and $\tau_{\theta v}$-discrete. Consequently, $\tau_{\theta v}$ is discrete.

(c)$\Rightarrow$(d). It follows from the definition of $\tau_{\theta v}^{*}$.

(d)$\Rightarrow$(e). It is obvious.

(e)$\Rightarrow$(f). It follows from the fact that $\tau(s_{v}(X)) = \tau_{\theta v}^{*} |_{s_{v}(X)}$ is finer then $\tau_{\theta v}^{*} |_{s_{v}(X)}$.

(f)$\Rightarrow$(a). It follows from the fact that $(s_{v}(X), \tau(s_{v}(X)))$ is separable.

This corollary shows, that $u \approx v$ does not imply $u \sim v$. Indeed, of $H$ and $H'$ two distinct countable characterized subgroups, then $\tau_{\theta u}$ and $\tau_{\theta v}$ are discrete, so coincide, while $H' \neq H$.

3 Characterized subgroups of the Circle Group

3.1 Preliminaries

Definition 3.1. For a topological abelian group $X$ and a sequence $u = (u_n)_{n \in \mathbb{N}}$ of integers, let

$$t_{u}(X) = \{x \in X : u_n x \to 0 \text{ in } X\}.$$ 

This subset is actually a subgroup of $X$ and has been studied in [D1, BDMW2]. Here we consider only the case when $X = T$. Since $\hat{T} = \mathbb{Z}$, the notion defined above coincides in the case of $X = T$ with the notion of characterized subgroup of $T$. In fact $t_{u}(T) = s_{v}(T)$ where for all $n \in \mathbb{N}$, $v_n : x \mapsto u_n x$, i.e., $H \leq T$ is characterized if and only if there exists $u \in \mathbb{Z}^{\mathbb{N}}$ such that $H = t_{u}(T)$. If
\( u \) does not have any constant subsequences one can find a strictly increasing sequence \( u^* \) of non-negative integers such that \( u^* \sim u \) (see \[BDMW1\] Proposition 2.5).

Now one has an additional tool to use in the study of characterized subgroups, namely the sequence of ratios \( (q_n) := (\frac{u_{n+1}}{u_n}) \). Furthermore, let

\[
q^u := \limsup_n \frac{u_{n+1}}{u_n} \quad \text{and} \quad q_u := \liminf_n \frac{u_{n+1}}{u_n}.
\]

Note that the sequence of ratios \( (q_n) \) is bounded precisely when \( q^u \) is finite, while \( (q_n) \) converges to infinity precisely when \( q_u = \infty \). In these terms, Eggleston proved the following theorem in \[Egg\] (see also \[BDMW1\]).

**Theorem 3.2** (Eggleston). Let \( u = (u_n) \) a sequence of integers.

- If \( q^u < \infty \) (i.e., the sequence of ratios \( (q_n) \) is bounded), then \( t_u(T) \) is countable.
- If \( q_u = \infty \) (i.e., the sequence of ratios \( (q_n) \) converges to infinity), then \( t_u(T) \) is uncountable.

The following remark shows that the implications of Eggleston Theorem 3.2 cannot be inverted

**Remark 3.3.** \[BDMW1\] If \( H \) is an infinite characterized subgroup of \( T \), with characterizing sequence \( u \in \mathbb{Z}^N \), then for every \( m \in \mathbb{N} \) one can find a strictly increasing sequence \( u^* \sim u \) such that \( q^{u^*} = \infty \) (i.e., \( \limsup_n \frac{u_{n+1}^{u^*}}{u_n^*} = \infty \)) and \( q_{u^*} = m \) (i.e., \( \liminf_n \frac{u_{n+1}^{u^*}}{u_n^*} = m \)).

This remark shows that the properties of having bounded ratios, or having ratios converging to \( \infty \) are not \( \sim \)-invariant. Actually, if one takes a sequence \( u \) with bounded ratios, and then a sequence \( u^* \sim u \) as in the remark, then one will have also \( u^* \approx u \), according to Corollary 2.24 while \( u^* \) will fail to have the property of having bounded ratios.

If \( (q_n) \) is bounded, then \( t_u(T) \) is countable and hence obviously \( F_\sigma \). One can study the relations between these three properties, that are not equivalent in general (see Theorem 3.11 and remarks 3.13 and 3.12). To this end we distinguish two cases; the first one is the case of a general sequence of integers. The second one is the case of a particular kind of sequences of integers, namely the arithmetic sequences (see Definition 3.8).

### 3.2 General Sequences

According to Corollary 2.26 when \( s_\varphi(X) \) is countable, then the topology \( \tau_\varphi \) is discrete on \( X \), where \( X \) is a compact metrizable abelian group. In case \( X = T \), by imposing a stronger condition on the sequence, one can say something more precise than Corollary 2.26.

**Theorem 3.4** \([DI2\] Theorem C\]). Let \( u \) be a strictly increasing sequence of positive integers such that its sequence of ratios \( (q_n) \) is bounded. In that case \( \tau_\varphi \) is discrete.

In particular if \( 0 < C \in \mathbb{R} \) and \( q_n \leq C \) for all \( n \in \mathbb{N} \), then \( B_\varphi^{\mathbb{R}}(0) = \{0\} \) in \( T \).
The proof of the next theorem uses a property of \( T \) first noticed by Hewitt \([H]\), and then extended to all subgroups of \( T \) (and elsewhere) in \([DI3]\): if \( H \) is a subgroup of \( T \) and \( \tau \) is a strictly finer locally compact group topology on \( H \), then \( \tau \) is discrete. Hence one can add to Corollary 2.26 one more equivalent condition in the case \( X = T \), given by the next theorem.

**Theorem 3.5** (\([DI3\), Theorem 2.1]). Let \( H \leq T \). Then \( H \in \mathcal{L}(T) \), if and only if \( H \) is countable.

Let us recall that \( \mathcal{L}(T) \subset \text{Char}(T) \) and that \( H = \text{tu}(T) \) coincides with \( T \) if and only if \( u \) is definitely 0. It turns out that this theorem cannot be proved in any more general situation:

**Theorem 3.6** (\([DI3\), Theorem 2.2]). Let \( X \) be an infinite compact abelian group. Then TFAE:

(a) \( X \cong T \);

(b) whenever \( H \in \mathcal{L}(X) \) and \( H \neq X \), then \( H \) is countable.

### 3.3 Arithmetic Sequences

In the case of \( T \) one has the following two examples of characterized non-\( F_\sigma \)-subgroups:

**Example 3.7.** (a) For the sequence \( u = (2^n) \) Bukovský, Kholshevikova, Repický \([BKR]\) proved that the characterized subgroup \( \text{tu}(T) \) of \( T \) is not an \( F_\sigma \)-set.

(b) For the sequence \( u = (n!) \) Gabriyelyan \([G4]\) proved that \( \text{tu}(T) \) of \( T \) is not an \( F_\sigma \)-subgroup of \( T \).

The obvious common features between (a) and (b) are \( u_n | u_{n+1} \) and \( q_n = \frac{u_{n+1}}{u_n} \to \infty \). We are going to use the first one to define the following new notion.

**Definition 3.8.** A strictly increasing sequence of positive integers \( u = (u_n) \) is called arithmetic (or briefly, an \( a \)-sequence) if \( u_n | u_{n+1} \) for all \( n \in \mathbb{N} \).

**Notation 3.9.** Let \( \mathbb{P} \) be the set of all prime numbers. If \( p \in \mathbb{P} \), then \( \mathbb{Z}(p^\infty) \) is the Prüfer \( p \)-group and \( \mathbb{Z}(p^n) \) the cyclic group of order \( p^n \) where \( n \in \mathbb{N} \).

Note that in this case \( (q_n) \) is a sequence of integers. The problem of the description of the structure of \( \text{tu}(T) \) in the case of arithmetic sequences has been raised in \([DPS\), Chap.4\], where some partial results can be found. Much earlier, Armacost \([Arm]\) considered two special kinds of \( a \)-sequences, \( u_n = n! \) and \( u_n = p^n \) for some \( p \in \mathbb{P} \). The subgroup characterized by the former type is \( \mathbb{Z}(p^\infty) \), as shown in \([Arm]\). The subgroup characterized by \( u_n = n! \) was denoted by \( T! \), its description was left as an open problem in \([Arm]\), resolved independently by Borel \([B62]\) and \([DPS]\). Further results on subgroups of \( T \) characterized by \( a \)-sequences can be found in \([DdS\), \([DI]\].

**Example 3.10.** \([BDMW3]\) If \( u \) is an \( a \)-sequence and \( (q_n) \) is bounded, then \( \text{tu}(T) \cong \bigoplus_{i=1}^{s} \mathbb{Z}(p_i^{k_i}) \bigotimes_{j=1}^{r} \mathbb{Z}(t_j^\infty) \); where \( p_i \in \mathbb{P} \), \( k_i = |\{ n : p_i | q_n \} | < \infty \) and \( t_j \) are primes that divide infinitely many \( q_n \). It is easy to see that the subgroups of these form of \( T \) are countably many.
In the case of arithmetic sequences one can add some further equivalent conditions to the conditions of Corollary 2.26 as follows.

**Theorem 3.11** ([DI2 Theorem E]). The following are equivalent for an a-sequence \( u \) in \( \mathbb{Z} \):

(a) \( t_u(T) \leq \mathbb{Q}/\mathbb{Z} \);
(b) \( (q_n) \) is bounded;
(c) \( t_u(T) \) is countable;
(d) \( t_u(T) \in \mathfrak{g}_u(T) \).

The implications \((b) \Rightarrow (c) \Rightarrow (d)\) are obvious, while the equivalence between (a) and (b) proved in [DdS, DI1]. The last implication \((d) \Rightarrow (b)\) is proved in [DI2].

From the equivalence of (a) and (b) in the above theorem and Example 3.10, we deduce that only countably many subgroups of \( \mathbb{Q}/\mathbb{Z} \) can be characterized by mean of an a-sequence \( u \) in \( \mathbb{Z} \).

Let us see that some of the implications of Theorem 3.11 do not hold for a general \( u \in \mathbb{Z}^N \).

**Remark 3.12** \((b) \not\Rightarrow (a) \not\Rightarrow (c)\). If \( u \) is the Fibonacci’s sequence, then \( (q_n) \) is bounded, so \( t_u(T) \) is countable (by virtue of Theorem 3.11), i.e. (b) and (c) of Theorem 3.11 holds. Indeed \( u_n = u_{n-1} + u_{n-2} \) for all \( n > 1 \) and \( u_0 = u_1 = 1 \). Hence, \( q_n = \frac{u_{n+1}}{u_n} = 1 + \frac{u_{n-2}}{u_{n-1}} \leq 2 \) for all \( n \in \mathbb{N} \).

In [L, BDMW3, BDS] it is proved that \( t_u(T) \) is the infinite cyclic group generated by the fractional part of the golden ratio. In particular, \( t_u(T) \) is not torsion.

The following remark shows that \((a) \not\Rightarrow (b) \not\Rightarrow (c)\) in Theorem 3.11.

**Remark 3.13** \((a) \not\Rightarrow (b) \not\Rightarrow (c)\). Take any infinite subgroup \( H \) of \( \mathbb{Q}/\mathbb{Z} \). By Remark 3.3 \( H \) has a characterizing sequence such that its sequence of ratios is unbounded. This proves the non-implication \((a) \not\Rightarrow (b)\) in Theorem 3.11. As \( (a) \) implies \((c)\), this witnesses also the non-implication \((c) \not\Rightarrow (b)\).

We do not know if the remaining implication \((d) \Rightarrow (c)\) is true in the general case (see Question 4.4).

The following diagram summarizes Theorem 3.11 and remarks 3.12 and 3.13, where the question mark denotes the unknown implication and the slashes denote the failing implications.
4 Questions and Remarks

We start with some questions and remarks concerning characterizing sequences.

Remark 4.1. If $\mathbf{aChar}(\mathbb{T})$ denotes the set of all subgroups of $\mathbb{T}$ characterized by an a-sequence, then

$$\{\text{closed subgroups}\} = \mathbf{G}_\delta(\mathbb{T}) \subsetneq \mathbf{aChar}(\mathbb{T}) \cap \mathbf{G}_\sigma(\mathbb{T}).$$

An example witnessing the above proper inclusion is $t_u(\mathbb{T}) \in \mathbf{G}_\delta(\mathbb{T}) \setminus \mathbf{G}_\sigma(\mathbb{T})$ where $u = (p^n)$. More generally all countably infinite $t_u(\mathbb{T}) \in \mathbf{aChar}(\mathbb{T})$ witness that proper inclusion, where $u$ is an a-sequence.

To give an exhaustive description of $\mathbf{Char}(\mathbb{T})$ and $\mathbf{aChar}(\mathbb{T})$, in the terms of Problem 1.3, it is needed to establish if there exists a sequence $u$ such that $t_u(\mathbb{T}) \notin \mathbf{G}_\sigma(\mathbb{T})$. This kind of sets is called $F_{\sigma\delta}$-complete (or $\Pi^0_3$-complete using the Descriptive Set-Theoretic terminology, see [K]), that is in simple terms, a set among the most complex in $\mathbf{G}_{\sigma\delta}(\mathbb{T})$. Hence the following questions remain open.

Question 4.2. Does there exists a sequence $u$ such that $t_u(\mathbb{T}) \notin \mathbf{G}_\sigma(\mathbb{T})$? What about a-sequences $u$?

Question 4.3. Does there exists an explicit method to distinguish whether $t_u(\mathbb{T}) \notin \mathbf{G}_\sigma(\mathbb{T})$ for a sequence $u$? What about a-sequences $u$?

Since $t(\mathbb{T}) = \mathbb{Q}/\mathbb{Z}$ is countable, all ($\forall$ many) torsion subgroups of $\mathbb{T}$ can be characterized, by Theorem 2.11. On the other hand, not all the torsion subgroups of $\mathbb{T}$ can be characterized by an a-sequence. Indeed, by Example 3.10 and Theorem 3.11 all torsion subgroups of $\mathbb{T}$ of the form $\bigoplus_{i \in I} \mathbb{Z}(p_i)$ with $p_i \in \mathbb{N} \cup \{\infty\}$ for all $i \in I$, $p_i \in \mathbb{P}$, and $p_i \neq p_j$ for $i \neq j$ and $|I| = \aleph_0$ cannot be characterized by an a-sequence. Hence, as far as only torsion subgroups of $\mathbb{T}$ are concerned, one can say that $\mathbf{aChar}(\mathbb{T})$ is much smaller with respect to $\mathbf{Char}(\mathbb{T})$ (as $\mathbf{aChar}(\mathbb{T})$ contains only countably many torsion subgroups, while $\mathbf{Char}(\mathbb{T})$ contains $\forall$ many torsion subgroups of $\mathbb{T}$).

The next question is related to the implication $(d) \Rightarrow (c)$ in Theorem 3.11.

Question 4.4. If $u \in \mathbb{Z}^\mathbb{N}$ is not definitely 0 and $t_u(\mathbb{T}) \in \mathbf{G}_\sigma(\mathbb{T})$, must $t_u(\mathbb{T})$ be necessarily countable?

The following question presents a weaker form of the implication $(c) \Rightarrow (b)$, that fails in general (see Remark 3.13).

Question 4.5. If $H$ is a countable subgroup of $\mathbb{T}$, does there exist a characterizing sequence $u \in \mathbb{Z}^\mathbb{N}$ of $H$ with bounded sequence of ratios $(q_n)$?

The final questions concern Polishable subgroups.

According to Corollary 2.23 the topology $\tau_{\phi_0}$ does not depend on the choice of $v$ in case $s_v(X)$ is an $F_\sigma$-subgroup of $X$, however the questions remain open in the general case:

Question 4.6. If $s_v(X) = s_u(X) \notin \mathbf{G}_\sigma(X)$, is $\tau_{\phi_0} = \tau_{\phi_0}$ in the whole $X$?

As we recalled in 2.23 in [G1, G2] Gabriyelyan proved that $\phi_0(X) \notin \mathbf{Char}(X)$ for some $X$. But for $X = \mathbb{T}$ the situation remains unclear. Hence one can ask the following question.
Question 4.7. Does the inclusion $\mathcal{P}(T) \subseteq \mathcal{C}(T)$ hold true?

In the light of Biró’s result it is natural to ask:

Question 4.8. Which of the following two inclusions hold true:

(a) $\mathcal{P}(T) \cap \mathcal{F}_\sigma(T) \subseteq \mathcal{C}(T)$;

(b) $\mathcal{P}(T) \cap \mathcal{F}_{\sigma\delta}(T) \subseteq \mathcal{C}(T)$?

The last two questions have a negative answer for a general compact metrizable abelian group $X$, according to the example found by Gabriyelyan (see [29]).

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