On lower semicontinuity of the entropic disturbance and its applications in quantum information theory

M. E. Shirokov and A. S. Holevo

Abstract. We consider an important characteristic of a quantum channel called the entropic disturbance. It is defined as the difference between the \( \chi \)-quantity of a generalized ensemble and that of the image of the ensemble under the channel. We prove the lower semicontinuity of the entropic disturbance for any infinite-dimensional quantum channel on its natural domain. A number of useful corollaries of this property are established, in particular, the existence of a \( \chi \)-optimal ensemble for any quantum channel and the continuity of the output \( \chi \)-quantity under the energy-type input constraint.

Keywords: von Neumann entropy, \( \chi \)-quantity, ensemble of quantum states, quantum channel, classical capacity.

§ 1. Introduction

The concept of a completely positive map is widely used in modern functional analysis [1], [2]. In mathematical models of quantum information theory as well as in the theory of open systems a central role is played by the notion of a quantum channel (dynamical map) defined as a linear completely positive trace-preserving transformation of the Banach space of trace-class operators on a Hilbert space. In physics such transformations describe (in general) irreversible evolution of quantum systems affected by external influence and noise [3]. The study of various entropic and information characteristics of quantum channels includes an analysis of continuity properties indicating the robustness of these characteristics with respect to small perturbations of states and of channels, unavoidable in real situations. Such an analysis is of special importance in the infinite-dimensional case both from the mathematical point of view and due to applications to ‘quantum systems with continuous variables’ [4].

Among the basic characteristics used in the study of information properties of quantum systems and channels, the \( \chi \)-quantity of quantum state ensembles plays a central role.\(^1\) It is this characteristic that is used, directly or implicitly, in expressions for the various capacities of a quantum channel ([3], Chs. 8, 9).

\(^1\)This quantity (called the Holevo quantity in the literature) gives an upper bound for the classical information in discrimination of quantum states by quantum measurements [5].

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In this paper we consider the entropic disturbance defined as the difference between the $\chi$-quantity of a generalized ensemble of quantum states and the $\chi$-quantity of the image of this ensemble under a quantum channel $\Phi$ (see [6] and the references therein). We prove the lower semicontinuity of the entropic disturbance in the topology of weak convergence on the set of generalized ensembles where it is well defined (Theorem 1) and establish a number of useful corollaries of this property, in particular, the existence of a $\chi$-optimal ensemble for any quantum channel under an energy-type input constraint (a problem stated in [7]).

§ 2. Preliminaries

Let $\mathcal{H}$ be a separable Hilbert space, $\mathfrak{B}(\mathcal{H})$ the algebra of all bounded operators on $\mathcal{H}$ with the operator norm, $\mathfrak{T}(\mathcal{H})$ the Banach space of all trace-class operators on $\mathcal{H}$ with the trace norm $\| \cdot \|_1 = \text{Tr} | \cdot |$. Let $\mathfrak{T}_+(\mathcal{H})$ be the cone of positive operators in $\mathfrak{T}(\mathcal{H})$ and $\mathfrak{S}(\mathcal{H})$ the convex set of density operators, that is, operators in $\mathfrak{T}_+(\mathcal{H})$ with unit trace, describing quantum states [3], [8].

The support $\text{supp} A$ of a positive operator $A$ is the orthogonal complement of its kernel. Its dimension is called the rank of the operator: $\text{rank} A = \dim \text{supp} A$. The range of an arbitrary operator $A$ will be denoted by $\text{Ran} A$.

We will denote by $I_{\mathcal{H}}$ the identity operator on a Hilbert space $\mathcal{H}$ and by $\text{Id}_{\mathcal{H}}$ the identity transformation of the Banach space $\mathfrak{T}(\mathcal{H})$.

A finite or countable collection $\{ \rho_i \}$ of states with a probability distribution $\{ \pi_i \}$ is called an ensemble and denoted by $\{ \pi_i, \rho_i \}$. The state $\rho = \sum_i \pi_i \rho_i$ is called the average state of the ensemble. A generalized ensemble is a Borel probability measure on the set of quantum states, so that ensembles as previously defined correspond to discrete probability measures. We denote by $\mathcal{P}(\mathcal{H})$ the set of all Borel probability measures on $\mathfrak{S}(\mathcal{H})$ equipped with the topology of weak convergence [9], [7]. The set $\mathcal{P}(\mathcal{H})$ is a complete separable metric space [10]. The average state of a generalized ensemble $\mu \in \mathcal{P}(\mathcal{H})$ is the barycentre of the measure $\mu$ defined by the Bochner integral

$$\bar{\rho}(\mu) = \int_{\mathfrak{S}(\mathcal{H})} \rho \mu(d\rho).$$

We will use the following compactness criterion for subsets of $\mathcal{P}(\mathcal{H})$ ([7], Proposition 2).

**Proposition 1.** A closed subset $\mathcal{P}_0$ of $\mathcal{P}(\mathcal{H})$ is compact if and only if the set $\{ \bar{\rho}(\mu) \mid \mu \in \mathcal{P}_0 \}$ is a compact subset of $\mathfrak{S}(\mathcal{H})$.

The von Neumann entropy of a quantum state $\rho \in \mathfrak{S}(\mathcal{H})$ is defined as $H(\rho) = \text{Tr} \eta(\rho)$, where $\eta(x) = -x \log x$ for $x > 0$ and $\eta(0) = 0$. It is a non-negative, concave and lower-semicontinuous function on the set $\mathfrak{S}(\mathcal{H})$ [11]–[13].

The quantum relative entropy of states $\rho$ and $\sigma$ in $\mathfrak{S}(\mathcal{H})$ is defined as follows (see[11]):

$$H(\rho \parallel \sigma) = \sum_{i=1}^{+\infty} \langle i | \rho \log \rho - \rho \log \sigma | i \rangle,$$
There exists a sequence of channels $\Lambda_n: A \rightarrow A$ strongly converging\footnote{Throughout the paper we use the Dirac notation (see, for example, \cite{3, 8}), in which an orthonormal set of vectors is conventionally denoted by $\{|i\rangle\}_{i \in I}$, where $I = \{1, 2, \ldots, n\}$ or $I = \mathbb{N}$.} to the identity channel $\text{Id}_A$ such that $\Lambda_n(\rho) \in \mathcal{S}(\mathcal{H}_A^n)$ for all $\rho \in \mathcal{S}(\mathcal{H}_A)$, where $\mathcal{H}_A^n$ is a finite-dimensional subspace of $\mathcal{H}_A$ for each $n$.

If quantum systems $A$ and $B$ are described by Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, then the composite system $AB$ is described by the tensor product of these spaces: $\mathcal{H}_{AB} \equiv \mathcal{H}_A \otimes \mathcal{H}_B$. For a state $\omega_{AB} \in \mathcal{S}(\mathcal{H}_{AB})$, the partial states are $\omega_A = \text{Tr}_B \omega_{AB}$ and $\omega_B = \text{Tr}_A \omega_{AB}$, where $\text{Tr}_B$ is the partial trace over $\mathcal{H}_B$ (and similarly for $A$).

The quantum mutual information of an infinite-dimensional composite quantum system in a state $\omega_{AB}$ is defined as (see \cite{14})

$$I(A:B)_\omega = H(\omega_{AB} \parallel \omega_A \otimes \omega_B) = H(\omega_A) + H(\omega_B) - H(\omega_{AB}),$$

where the second formula is valid if $H(\omega_{AB}) < +\infty$. It is well known that

$$I(A:B)_\omega \leq 2 \min\{H(\omega_A), H(\omega_B)\}$$

(1)

for any state $\omega_{AB}$ \cite{15, 16}.

The $\chi$-quantity of a generalized ensemble $\mu \in \mathcal{P}(\mathcal{H})$ is defined as (see \cite{7})

$$\chi(\mu) = \int H(\rho \parallel \overline{\rho}(\mu)) \mu(d\rho) = H(\overline{\rho}(\mu)) - \int H(\rho) \mu(d\rho),$$

(2)

where the second formula is valid if $H(\overline{\rho}(\mu)) < +\infty$. For a discrete ensemble of states $\{\pi_i, \rho_i\}$ the $\chi$-quantity is equal to

$$\chi(\{\pi_i, \rho_i\}) = \sum_i \pi_i H(\rho_i \parallel \overline{\rho}) = H(\overline{\rho}) - \sum_i \pi_i H(\rho_i),$$

(3)

where the second formula is valid if $H(\overline{\rho}) < +\infty$.

A quantum channel $\Phi$ from a system $A$ to a system $B$ is a completely positive trace-preserving linear map $\mathcal{F}(\mathcal{H}_A) \rightarrow \mathcal{F}(\mathcal{H}_B)$, where $\mathcal{H}_A$ and $\mathcal{H}_B$ are Hilbert spaces associated with the systems $A$ and $B$. In this case we write $\Phi: A \rightarrow B$ \cite{3, 8}.

For any quantum channel $\Phi: A \rightarrow B$, Stinespring’s theorem (see \cite{1}) implies the existence of a Hilbert space $\mathcal{H}_E$ (the environment) and an isometry $V: \mathcal{H}_A \rightarrow \mathcal{H}_B \otimes \mathcal{H}_E$ such that

$$\Phi(\rho) = \text{Tr}_E V \rho V^*, \quad \rho \in \mathcal{F}(\mathcal{H}_A).$$

(4)

The minimal dimension of $\mathcal{H}_E$ is called the Choi rank of $\Phi$. The quantum channel

$$\mathcal{F}(\mathcal{H}_A) \ni \rho \mapsto \hat{\Phi}(\rho) = \text{Tr}_B V \rho V^* \in \mathcal{F}(\mathcal{H}_E)$$

(5)

is said to be complementary to the channel $\Phi$ (\cite{3}, Ch.6).

Throughout the paper we will use the following simple assertion.

**Remark 1.** There exists a sequence of channels $\Lambda_n: A \rightarrow A$ strongly converging to the identity channel $\text{Id}_A$ such that $\Lambda_n(\rho) \in \mathcal{S}(\mathcal{H}_A^n)$ for all $\rho \in \mathcal{S}(\mathcal{H}_A)$, where $\mathcal{H}_A^n$ is a finite-dimensional subspace of $\mathcal{H}_A$ for each $n$.\footnote{This means that $\lim_{n \rightarrow \infty} \Lambda_n(\rho) = \rho$ for any $\rho \in \mathcal{S}(\mathcal{H}_A)$ \cite{17}.}
Such a sequence can be constructed using any sequence \{P_n\} of finite-rank projectors strongly converging to the unit operator \(I_A\) as follows:

\[ \Lambda_n(\rho) = P_n \rho P_n + \sigma \text{Tr}(I_A - P_n) \rho, \]

where \(\sigma\) is a fixed state.

For an ensemble \(\mu \in \mathcal{P}(\mathcal{H}_A)\) its image \(\Phi(\mu)\) under a quantum channel \(\Phi: A \to B\) is defined as the ensemble in \(\mathcal{P}(\mathcal{H}_B)\) corresponding to the measure \(\mu \circ \Phi^{-1}\) on \(\mathcal{S}(\mathcal{H}_B)\), that is, \(\Phi(\mu)[\mathcal{S}_B] = \mu[\Phi^{-1}(\mathcal{S}_B)]\) for any Borel subset \(\mathcal{S}_B \subseteq \mathcal{S}(\mathcal{H}_B)\), where \(\Phi^{-1}(\mathcal{S}_B)\) is the pre-image of \(\mathcal{S}_B\) under the map \(\Phi\). If \(\mu = \{\pi_i, \rho_i\}\) then \(\Phi(\mu) = \{\pi_i, \Phi(\rho_i)\}\).

We will use the following continuity condition for the output \(\chi\)-quantity \(\chi(\Phi(\mu))\) ([17], Corollary 1).

**Proposition 2.** Let \(\Phi: A \to B\) be an arbitrary quantum channel. If the function \(\mu \mapsto H(\Phi(\rho(\mu)))\) is continuous on a subset \(\mathcal{P}_0 \subseteq \mathcal{P}(\mathcal{H}_A)\), then the function \(\mu \mapsto \chi(\Phi(\mu))\) is continuous on \(\mathcal{P}_0\).

**Remark 2.** We will say that the local continuity of a function \(f\) implies that of a function \(g\) if for any sequence \(\{x_k\}\) converging to \(x_0\) such that \(\lim_{k \to \infty} f(x_k) = f(x_0) \neq \pm \infty\) we have

\[ \lim_{k \to \infty} g(x_k) = g(x_0) \neq \pm \infty. \]

We will repeatedly use the following simple fact.

**Lemma 1.** Let \(f_1, \ldots, f_n\) be a collection of non-negative lower-semicontinuous functions on a metric space. Then the local continuity of \(\sum_{k=1}^n f_k\) implies that of all the functions \(f_1, \ldots, f_n\).

§ 3. Lower semicontinuity of the entropic disturbance

For a given channel \(\Phi: A \to B\) and a generalized ensemble \(\mu\), the monotonicity of the relative entropy implies that

\[ \chi(\Phi(\mu)) \leq \chi(\mu), \]

where \(\Phi(\mu)\) is the image of the ensemble \(\mu\) under action of the channel \(\Phi\). Thus the quantity

\[ \Delta^\Phi \chi(\mu) \equiv \chi(\mu) - \chi(\Phi(\mu)), \]

called the entropic disturbance in [6], is a non-negative function on the set of generalized ensembles for which the value of \(\chi(\Phi(\mu))\) is finite.

**Theorem 1.** For an arbitrary quantum channel \(\Phi: A \to B\), the function \(\Delta^\Phi \chi(\mu)\) is lower semicontinuous on the set \(\{\mu \in \mathcal{P}(\mathcal{H}_A) \mid \chi(\Phi(\mu)) < +\infty\}\).

If either the input dimension \(d_A\) or the Choi rank \(d_E\) of the channel \(\Phi\) is finite, then the function \(\Delta^\Phi \chi(\mu)\) is continuous on the above set and bounded above by \(\min\{\log d_A, 2 \log d_E\}\).
Proof. Let $E$ be an environment for $\Phi$ with the minimal dimension $d_E$ and $V: \mathcal{H}_A \to \mathcal{H}_{BE}$ the Stinespring isometry in the representation (4). We will use the identity

$$\chi(\mu) + I(B:E)V_\pi \rho V^* = \chi(\Phi(\mu)) + \chi(\hat{\Phi}(\mu)) + \int I(B:E)_{V_\rho V^*} \mu(d\rho),$$

which is valid for any $\mu \in \mathcal{P}(\mathcal{H}_A)$ (with values $+\infty$ possible on both sides).

If $\dim \mathcal{H}_A, \dim \mathcal{H}_B < +\infty$ then the validity of (6) can be verified directly, since in this case $I(B:E)_{V_\rho V^*} = H(\Phi(\rho)) + H(\hat{\Phi}(\rho)) - H(\rho)$ for any input state $\rho$. In the general case the identity (6) can be proved by approximation (see §8). It implies that

$$\chi(\mu) - \chi(\Phi(\mu)) = \chi(\hat{\Phi}(\mu)) + \int I(B:E)_{V_\rho V^*} \mu(d\rho) - I(B:E)_{V_\pi \rho V^*}$$

(7)

for any ensemble $\mu$ with finite $\chi(\Phi(\mu))$ and $I(B:E)_{V_\pi \rho V^*}$.

Assume first that the Choi rank $d_E \equiv \dim \mathcal{H}_E$ of the channel $\Phi$ is finite. In this case the output entropy of the channel $\hat{\Phi}: A \to E$ is continuous on $\mathcal{S}(\mathcal{H}_A)$, so that the function $\chi(\hat{\Phi}(\mu))$ is continuous on $\mathcal{P}(\mathcal{H}_A)$ by Proposition 2.

The assumption $\dim \mathcal{H}_E < +\infty$ also implies the continuity on $\mathcal{P}(\mathcal{H}_A)$ of the other terms on the right hand side of (7). Indeed, the upper bound (1) and part A of Theorem 1 in [18] show that $\rho \mapsto I(B:E)_{V_\rho V^*}$ is a continuous bounded function on $\mathcal{S}(\mathcal{H}_A)$. Hence the continuity of the second (integral) term in (7) follows from the definition of the topology of weak convergence on $\mathcal{P}(\mathcal{H}_A)$, while the continuity of the third follows from that of the barycentre map $\mu \mapsto \bar{\pi}(\mu)$.

To prove the upper bound

$$\Delta^\Phi \chi(\mu) \equiv \chi(\mu) - \chi(\Phi(\mu)) \leq 2 \log d_E,$$

(8)

note that the triangle inequality $|H(\rho) - H(\Phi(\rho))| \leq H(\hat{\Phi}(\rho)) \leq \log d_E$ (see [3], [8]) directly implies (8) for any finite ensemble $\mu = \{\pi_i, \rho_i\}$ such that $H(\rho_i) < +\infty$ for all $i$, since in this case

$$\chi(\mu) - \chi(\Phi(\mu)) = [H(\bar{\rho}) - H(\Phi(\bar{\rho}))] - \sum \pi_i[H(\rho_i) - H(\Phi(\rho_i))].$$

The validity of (8) for an arbitrary ensemble $\mu$ follows from the density of the finite ensembles in $\mathcal{P}(\mathcal{H}_A)$ and the continuity of $\Delta^\Phi \chi(\mu)$ proved before.

We can now prove the first assertion of the theorem. By the Stinespring representation we may assume that $\mathcal{H}_A = \mathcal{H}_{BE}, \Phi = \text{Tr}_E(\cdot)$ and $\hat{\Phi} = \text{Tr}_B(\cdot)$. Let $\mu$ be an arbitrary ensemble in $\mathcal{P}(\mathcal{H}_{BE})$. Consider a sequence of channels $\Lambda^n_E: E \to E$ strongly converging to the identity channel $\text{Id}_E$ such that $\Lambda^n_E(\mathcal{S}(\mathcal{H}_E)) \subseteq \mathcal{S}(\mathcal{H}^n_E)$ for some finite-dimensional subspace $\mathcal{H}^n_E$ of $\mathcal{H}_E$ (see Remark 1). Let $\mu_n$ be the image of a given ensemble $\mu$ under the channel $\text{Id}_B \otimes \Lambda^n_E$.

For each $n$ the ensemble $\mu_n$ is supported by the subspace $\mathcal{H}_B \otimes \mathcal{H}^n_E$. Thus, when speaking about the action of the channel $\Phi$ on this ensemble we may assume that this channel has finite Choi rank $\dim \mathcal{H}^n_E$. Since $\Phi(\mu) = \Phi(\mu_n)$ for all $n$ and the map $\mu \mapsto \mu_n$ is continuous, the above part of the proof shows that the function

$$\mu \mapsto \chi(\mu_n) - \chi(\Phi(\mu))$$
is continuous on the set of all ensembles $\mu$ with finite $\chi(\Phi(\mu))$. Thus, to prove the lower semicontinuity of the function $\mu \mapsto \chi(\mu) - \chi(\Phi(\mu))$ on this set it suffices to show that

$$\chi(\mu_n) \leq \chi(\mu)$$

for all $n$ and

$$\lim_{n \to \infty} \chi(\mu_n) = \chi(\mu)$$

for any $\mu \in \mathcal{P}(\mathcal{H}_{BE})$. These relations follow from the lower semicontinuity of the function $\mu \mapsto \chi(\mu)$ on the set $\mathcal{P}(\mathcal{H}_{BE})$ and its monotonicity under the action of quantum channels.

To complete the proof of the theorem it suffices to say, by Lemma 1 and the lower semicontinuity of the function $\Delta^\Phi \chi(\mu)$ proved above, that in the case $d_A \leq \dim \mathcal{H}_A < +\infty$ the function $\mu \mapsto \chi(\mu)$ is continuous on the set $\mathcal{P}(\mathcal{H}_A)$ and is bounded above by $\log d_A$. □

Theorem 1 implies the following condition for the local continuity of the output $\chi$-quantity.

**Corollary 1.** For an arbitrary quantum channel $\Phi: A \to B$ the local continuity of $\chi(\mu)$ implies that of $\chi(\Phi(\mu))$, that is,

$$\lim_{n \to \infty} \chi(\mu_n) = \chi(\mu_0) < +\infty \quad \Rightarrow \quad \lim_{n \to \infty} \chi(\Phi(\mu_n)) = \chi(\Phi(\mu_0)) < +\infty$$

for any sequence $\{\mu_n\} \subset \mathcal{P}(\mathcal{H}_A)$ converging to an ensemble $\mu_0 \in \mathcal{P}(\mathcal{H}_A)$.

**Proof.** By Proposition 1 in [17] and Theorem 1, all the terms in the equality

$$\chi(\Phi(\mu)) + \Delta^\Phi \chi(\mu) = \chi(\mu)$$

are lower-semicontinuous functions on the set of all ensembles $\mu$ with $\chi(\mu)$ finite. Thus, the assertion of the corollary follows from Lemma 1. □

Corollary 1 states that **local continuity of the $\chi$-quantity is preserved by quantum channels.**

Combining Corollary 1 and Proposition 2 we obtain the following continuity condition for the output $\chi$-quantity, which is more convenient for applications.

**Corollary 2.** Let $\mathcal{S}_0$ be a subset of $\mathcal{S}(\mathcal{H}_A)$ on which the entropy is continuous. Then the output $\chi$-quantity $\chi(\Phi(\mu))$ of any quantum channel $\Phi: A \to B$ is continuous on the set $\{\mu \in \mathcal{P}(\mathcal{H}_A) \mid \rho(\mu) \in \mathcal{S}_0\}$.

In other words, Corollary 2 states that

$$\lim_{n \to \infty} H(\rho(\mu_n)) = H(\rho(\mu_0)) < +\infty \quad \Rightarrow \quad \lim_{n \to \infty} \chi(\Phi(\mu_n)) = \chi(\Phi(\mu_0)) < +\infty$$

for any quantum channel $\Phi: A \to B$ and any sequence $\{\mu_n\} \subset \mathcal{P}(\mathcal{H}_A)$ converging to an ensemble $\mu_0 \in \mathcal{P}(\mathcal{H}_A)$.

It is well known (see [12], [13]) that the entropy is continuous on the set of states $\rho$ satisfying the inequality

$$\text{Tr} \, e^{-\lambda H} < +\infty$$

for all $\lambda > 0$. (9)

Hence Corollary 2 implies the following useful observation.

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4The value of $\text{Tr} \, H \rho$ (finite or infinite) is defined as $\sup_n \text{Tr} \, \rho P_n H P_n$, where $P_n$ is the spectral projector of $H$ corresponding to the interval $[0, n]$. 

---
Corollary 3. Let $\Phi : A \rightarrow B$ be a quantum channel and $\tilde{\Phi}$ its complementary channel. If the Hamiltonian $H_A$ of system $A$ satisfies condition (9), then the functionals

$$\mu \mapsto \chi(\Phi(\mu)) \quad \text{and} \quad \mu \mapsto \chi(\Phi(\mu)) - \chi(\tilde{\Phi}(\mu))$$

(10)

are continuous on the set of all generalized ensembles with bounded energy of average states (that is, on the set $\{\mu \in \mathcal{P}(\mathcal{H}_A) \mid \text{Tr} H_A \rho(\mu) \leq E\}$).

The condition of Corollary 3 is valid if $A$ is a system of quantum oscillators and $B$ is any system, in particular, if $B = A$; see [13], [19]. The first functional in (10) is connected to the unassisted classical capacity of a quantum channel, while the second is connected to the private classical capacity [3], [16].

§ 4. On the existence of a $\chi$-optimal ensemble for an arbitrary channel

When we are considering the transmission of classical information over an infinite-dimensional quantum channel $\Phi : A \rightarrow B$ we have to impose constraints on the states used for encoding information to be consistent with the physical implementation of the process. A typical constraint is the requirement of the bounded energy of states at the input of a channel expressed by the linear inequality

$$\text{Tr} H_A \rho \leq \mathcal{E},$$

(11)

where $H_A$ is a positive self-adjoint operator (the Hamiltonian of the input quantum system $A$) and $\mathcal{E} > 0$.

The $\chi$-capacity of the channel $\Phi$ with the constraint (11) can be defined as follows:

$$\overline{C}(\Phi, H_A, \mathcal{E}) = \sup_{\text{Tr} H_A \rho(\mu) \leq \mathcal{E}} \chi(\Phi(\mu)),$$

(12)

where $\chi(\Phi(\mu))$ is the output $\chi$-quantity of an ensemble $\mu$ and the supremum is over all ensembles in $\mathcal{P}(\mathcal{H}_A)$ whose average state satisfies (11); see [7].

An interesting question concerns the attainability of the supremum in (12). It was formulated in [7] in the following more general form: under what conditions is there an ensemble $\mu_*$ such that

$$\sup_{\rho(\mu) \in \mathcal{S}_c} \chi(\Phi(\mu)) = \chi(\Phi(\mu_*)) \quad \text{and} \quad \overline{\rho}(\mu_*) \in \mathcal{S}_c$$

(13)

for a given subset $\mathcal{S}_c$ of $\mathcal{S}(\mathcal{H}_A)$. The theorem in [7] guarantees the existence of such an ensemble (called a $\chi$-optimal ensemble) if the set $\mathcal{S}_c$ is compact and the output entropy $H(\Phi(\rho))$ is continuous on $\mathcal{S}_c$. The last condition is difficult to verify, since in general the local continuity of the entropy is not preserved by quantum channels, that is, the continuity of the entropy on some set of input states does not imply the continuity (or even the finiteness!) of the output entropy on this set.

The results in § 3 make it possible to obtain a simpler condition for the existence of a $\chi$-optimal ensemble which does not depend on a channel $\Phi$.

5The importance of this condition is shown in [7], where it is proved that a $\chi$-optimal ensemble does not exist for some compact set $\mathcal{S}_c$ and channel $\Phi$. 
Proposition 3. Let \( \Phi: A \to B \) be a quantum channel and \( \mathcal{S}_c \) a compact subset of \( \mathcal{S}(\mathcal{H}_A) \). If the entropy is continuous on \( \mathcal{S}_c \), then (13) holds for some ensemble \( \mu_\ast \in \mathcal{P}(\mathcal{H}_A) \) supported by pure states.

Proof. By Proposition 1 the set \( \mathcal{P}_c = \{ \mu \in \mathcal{P}(\mathcal{H}_A) \mid \overline{\rho}(\mu) \in \mathcal{S}_c \} \) is compact. By Corollary 2 the function \( \mu \to \chi(\Phi(\mu)) \) is continuous on the set \( \mathcal{P}_c \). Hence this function achieves its finite maximum on the set \( \mathcal{P}_c \), that is, (13) holds for some ensemble \( \mu_0 \). By Corollary 6 in [20] there is an ensemble \( \mu_\ast \) supported by pure states such that \( \mu_\ast \succ \mu_0 \), where \( \succ \) is the Choquet partial order (see [21], [20]) on the set \( \mathcal{P}(\mathcal{H}_A) \). Since \( \overline{\rho}(\mu_\ast) = \overline{\rho}(\mu_0) \), the convexity and lower semicontinuity of the function \( \rho \mapsto H(\Phi(\rho) \| \Phi(\sigma)) \) imply, by Lemma 1 in [20], that \( \chi(\Phi(\mu_\ast)) \geq \chi(\Phi(\mu_0)) \). Thus, (13) also holds for the ensemble \( \mu_\ast \). \( \square \)

Remark 3. If the set \( \mathcal{S}_c \) is convex, then Proposition 4 in [7] shows that the \( \chi \)-optimal ensemble \( \mu_\ast \) is characterized by the property

\[
\int H(\Phi(\rho) \| \Phi(\overline{\rho}(\mu_\ast))) \nu(d\rho) \leq \int H(\Phi(\rho) \| \Phi(\overline{\rho}(\mu_\ast))) \mu_\ast(d\rho) = \chi(\Phi(\mu_\ast))
\]

for any ensemble \( \nu \in \mathcal{P}(\mathcal{H}_A) \) such that \( \overline{\rho}(\nu) \in \mathcal{S}_c \). This property can be considered as a generalization of the maximal-distance property of a \( \chi \)-optimal ensemble for unconstrained finite-dimensional channels [22].

If \( \mathcal{S}_c \) is the set defined by the inequality (11), then the entropy is continuous on \( \mathcal{S}_c \) for all \( \mathcal{E} > 0 \) if and only if the operator \( H_A \) satisfies the condition (9). This condition also implies the compactness of \( \mathcal{S}_c \) (by the lemma in [19]). Thus, we obtain the following result from Proposition 3 and Remark 3.

Corollary 4. Let \( \Phi: A \to B \) be an arbitrary quantum channel. If the Hamiltonian \( H_A \) of the system \( A \) satisfies the condition (9), then there is an ensemble \( \mu_\ast \in \mathcal{P}(\mathcal{H}_A) \) supported by pure states such that \( \text{Tr} H_A \overline{\rho}(\mu_\ast) \leq \mathcal{E} \),

\[
\chi(\Phi(\mu_\ast)) = \overline{\mathcal{C}}(\Phi, H_A, \mathcal{E}) \quad \text{and} \quad \int H(\Phi(\rho) \| \Phi(\overline{\rho}(\mu_\ast))) \nu(d\rho) \leq \overline{\mathcal{C}}(\Phi, H_A, \mathcal{E})
\]

for any ensemble \( \nu \in \mathcal{P}(\mathcal{H}_A) \) such that \( \text{Tr} H_A \overline{\rho}(\nu) \leq \mathcal{E} \).

If \( A \) is the system of quantum oscillators with quadratic Hamiltonian and \( B \) is any system, in particular, \( B = A \), then Corollary 4 shows the existence of a \( \chi \)-optimal ensemble supported by pure states for an arbitrary channel \( \Phi: A \to B \) with the energy constraint (11).

§ 5. On the properties of constrained \( \chi \)-capacity

In the analysis of the classical capacity of a quantum channel \( \Phi: A \to B \) and of its relations to other capacities, it is convenient to introduce the function

\[
\overline{\mathcal{C}}(\Phi, \rho) \defeq \sup_{\overline{\rho}(\mu) = \rho} \chi(\Phi(\mu))
\]

(14)
on the set $\mathcal{S}(\mathcal{H}_A)$ of input states. This function can be called the constrained $\chi$-capacity or simply the $\chi$-function\(^6\) of the channel $\Phi$ \cite{7, 17}. The $\chi$-capacity of the channel $\Phi$ with the constraint (11) can be defined via this function as follows:

$$
\overline{C}(\Phi, H_A, \mathcal{E}) = \sup_{\text{Tr} H_A \rho \leq \mathcal{E}} \overline{C}(\Phi, \rho).
$$

Proposition 3 implies the following useful observation.

**Proposition 4.** For any state $\rho \in \mathcal{S}(\mathcal{H}_A)$ with finite entropy, the supremum in (14) is attained at some ensemble supported by pure states.

For an arbitrary quantum channel $\Phi$, the non-negative function $\rho \mapsto \overline{C}(\Phi, \rho)$ is concave and lower semicontinuous on $\mathcal{S}(\mathcal{H}_A)$ \cite{17}. By Proposition 5 in \cite{17}, the continuity of this function on some subset (for example, a convergent sequence) of input states follows from that of the output entropy $H(\Phi(\rho))$ on this set.

The results in §3 make it possible to show that the continuity of the function $\rho \mapsto C(\Phi, \rho)$ on some subset of input states also follows from the continuity of the input entropy $H(\rho)$ on this set.

**Proposition 5.** If the entropy is continuous on a subset $\mathcal{S}_0 \subset \mathcal{S}(\mathcal{H}_A)$, then the function $\rho \mapsto \overline{C}(\Phi, \rho)$ is continuous on $\mathcal{S}_0$ for any channel $\Phi$.

**Proof.** By Proposition 4 in \cite{17}, the function $\rho \mapsto \overline{C}(\Phi, \rho)$ is lower semicontinuous on $\mathcal{S}(\mathcal{H}_A)$. Thus, it suffices to prove, by Lemma 1, that the function

$$
\rho \mapsto H(\rho) - \overline{C}(\Phi, \rho)
$$

is lower semicontinuous on the set of all states $\rho$ with $H(\rho)$ finite.

Assume there is a sequence $\{\rho_n\} \subset \mathcal{S}(\mathcal{H}_A)$ converging to a state $\rho_0$ such that $H(\rho_n) < +\infty$ for all $n \geq 0$ and the limit

$$
\lim_{n \to \infty} [H(\rho_n) - \overline{C}(\Phi, \rho_n)] < [H(\rho_0) - \overline{C}(\Phi, \rho_0)]
$$

exists. By Proposition 4, for each $n$ there exists an ensemble $\mu_n$ in $\mathcal{P}(\mathcal{H}_A)$ supported by pure states such that $\overline{C}(\Phi, \rho_n) = \chi(\Phi(\mu_n))$ and $\mathcal{P}(\mu_n) = \rho_n$. Since the set $\{\rho_n\}_{n \geq 0}$ is compact, Proposition 1 in §2 implies the relative compactness of the sequence $\{\mu_n\}$. Thus, we may assume (by passing to a subsequence) that the sequence $\{\mu_n\}$ converges to a particular ensemble $\mu_0 \in \mathcal{P}(\mathcal{H}_A)$ supported by pure states. The continuity of the map $\mu \mapsto \mathcal{P}(\mu)$ implies that $\mathcal{P}(\mu_0) = \rho_0$. Since $H(\rho_n) = \chi(\rho_n)$ and $H(\rho_0) = \chi(\mu_0)$, Theorem 1 shows that

$$
\lim_{n \to \infty} [H(\rho_n) - \overline{C}(\Phi, \rho_n)] = \lim_{n \to \infty} [\chi(\mu_n) - \chi(\Phi(\mu_n))]
$$

$$
\geq \chi(\mu_0) - \chi(\Phi(\mu_0)) \geq H(\rho_0) - \overline{C}(\Phi, \rho_0),
$$

where the last inequality follows from the definition (14). This contradicts (15). $\square$

---

\(^6\)In \cite{7}, \cite{17} this function is denoted by $\chi_{\Phi}(\rho)$. \n
Note that Proposition 5 implies that
\[
\overline{C}(\Phi, H_A, \mathcal{E}) = \max_{\text{Tr} H_A \rho \leq \mathcal{E}} C(\Phi, \rho),
\]
provided that the operator $H_A$ satisfies the condition (9), since in this case the set of states such that $\text{Tr} H_A \rho \leq \mathcal{E}$ is compact by the lemma in [19].

§ 6. On the gain of entanglement assistance

The rate of transmission of classical information over a quantum channel can be increased by using an entangled state between the input and output of a channel as an additional resource. A detailed description of the corresponding protocol can be found in [3], [8], [16]. The ultimate rate of information transmission (with $\varepsilon$-small error) by this protocol is called the entanglement-assisted classical capacity.

If $\Phi: A \rightarrow B$ is a finite-dimensional quantum channel, then the Bennett–Shor–Smolin–Thapliyal (BSST) theorem [23] gives the following expression for its entanglement-assisted classical capacity:

\[
C_{ea}(\Phi) = \sup_{\rho \in \mathcal{S}(H_A)} I(\Phi, \rho),
\]
where $I(\Phi, \rho) = H(\rho) + H(\Phi(\rho)) - H(\hat{\Phi}(\rho))$ is the quantum mutual information of the channel $\Phi$ in a state $\rho$.

If $\Phi$ is an infinite-dimensional quantum channel, then constraints of the type (11) must necessarily be imposed. The definition of the entanglement-assisted classical capacity $C_{ea}(\Phi, H_A, \mathcal{E})$ of an infinite-dimensional quantum channel $\Phi$ with such a constraint is given in [19], where the generalization of the BSST theorem is proved under special restrictions on the channel $\Phi$ and the constraint operator $H_A$. A general version of the BSST theorem for infinite-dimensional constrained channels without any simplifying restrictions, which is proved in [24], states that

\[
C_{ea}(\Phi, H_A, \mathcal{E}) = \sup_{\text{Tr} H_A \rho \leq \mathcal{E}} I(\Phi, \rho) \leq +\infty
\]
for an arbitrary channel $\Phi$ and an arbitrary constraint operator $H_A$, where $I(\Phi, \rho)$ is the quantum mutual information defined by the formula

\[
I(\Phi, \rho) = H(\Phi \otimes \text{Id}_R (|\varphi_\rho\rangle\langle\varphi_\rho|) \parallel \Phi(\rho) \otimes \varrho),
\]
in which $|\varphi_\rho\rangle$ is a purification of the state $\rho$ in $H_A \otimes H_R$ and $\varrho = \text{Tr}_A |\varphi_\rho\rangle\langle\varphi_\rho|$. The results in § 4 make it possible substantially to strengthen and simplify the conditions for equality between the entanglement-assisted classical capacity and the $\chi$-capacity of an infinite-dimensional quantum channel with a linear constraint presented in [24], Theorem 2. This equality implies zero gain in the classical capacity due to entanglement assistance.

In what follows we will assume that the operator $H_A$ satisfies the condition (9). Thus, Corollary 4 guarantees the existence of a $\chi$-optimal ensemble for an arbitrary channel $\Phi$ with the constraint (11), that is, an ensemble $\mu_\ast \in \mathcal{P}(H_A)$ such that

\[
\overline{C}(\Phi, H_A, \mathcal{E}) = \chi(\Phi(\mu_\ast)) \quad \text{and} \quad \text{Tr} H_A \overline{\rho}(\mu_\ast) \leq \mathcal{E}.
\]
We will give the new conditions for the equality
\[
\overline{C}(\Phi, H_A, \mathcal{E}) = C_{\text{ea}}(\Phi, H_A, \mathcal{E}),
\]  
which means zero gain due to entanglement assistance.

**Definition 1.** A channel \( \Phi: A \to B \) is called a discrete classical-quantum (briefly, discrete c-q) channel if it has the representation
\[
\Phi(\rho) = \sum_k \langle k|\rho|k \rangle \sigma_k,
\]  
where \( \{|k\} \) is an orthonormal basis in \( \mathcal{H}_A \) and \( \{\sigma_k\} \) is a collection of states in \( \mathcal{S}(\mathcal{H}_B) \).

**Definition 2.** Let \( \mathcal{H}_A^0 \) be a subspace of \( \mathcal{H}_A \). The restriction of a channel \( \Phi: A \to B \) to the subspace \( \mathcal{S}(\mathcal{H}_A^0) \) is called the subchannel of \( \Phi \) corresponding to the subspace \( \mathcal{H}_A^0 \) and is denoted by \( \Phi_{\mathcal{H}_A^0} \).

**Definition 3.** A subspace \( \mathcal{H}_A^0 \) of \( \mathcal{H}_A \) is called a sufficient subspace for a channel \( \Phi: A \to B \) if
\[
\overline{C}(\Phi, H_A, \mathcal{E}) = \overline{C}(\Phi_{\mathcal{H}_A^0}, H_A, \mathcal{E}) \quad \text{and} \quad C_{\text{ea}}(\Phi, H_A, \mathcal{E}) = C_{\text{ea}}(\Phi_{\mathcal{H}_A^0}, H_A, \mathcal{E}).
\]

The following theorem is an infinite-dimensional version of Theorem 2 in [25].

**Theorem 2.** Let \( H_A \) be a positive operator satisfying the condition (9) and \( \mathcal{E}_\mathcal{m} = \inf_{\|\varphi\|=1} \langle \varphi|H_A|\varphi \rangle \) the infimum of the spectrum of \( H_A \).

i) If \( \Phi: A \to B \) is an arbitrary channel and (19) holds for some \( \mathcal{E}_m > \mathcal{E}_m \), then there is a sufficient subspace \( \mathcal{H}_A^0 \) for \( \Phi \) such that \( \Phi_{\mathcal{H}_A^0} \) is a discrete c-q channel (20) for some basis \( \{|k\} \) of \( \mathcal{H}_A^0 \). The subspace \( \mathcal{H}_A^0 \) can be defined as the minimal subspace of \( \mathcal{H}_A \) containing the supports of all the ensembles \( \mu_* \) satisfying (18).

ii) If \( \Phi: A \to B \) is a degradable channel, then (19) holds for some \( \mathcal{E}_m > \mathcal{E}_m \) if and only if \( \Phi \) is a discrete c-q channel (20), where \( \{|k\} \) is a basis of eigenvectors of \( H_A \) and \( \{\sigma_k\} \) is a collection of states with mutually orthogonal supports.

**Remark 4.** The presence of ‘sufficient subspace’ in Theorem 2 is natural, since in general the equality \( \overline{C}(\Phi, H_A, \mathcal{E}) = C_{\text{ea}}(\Phi, H_A, \mathcal{E}) \) cannot give information about the action of the channel \( \Phi \) on states absent from the codes determining \( \overline{C}(\Phi, H_A, \mathcal{E}) \) and \( C_{\text{ea}}(\Phi, H_A, \mathcal{E}) \). This is confirmed by the example of non-entanglement-breaking finite-dimensional channel \( \Phi \) such that \( \overline{C}(\Phi) = C_{\text{ea}}(\Phi) \) proposed in [23] and described in [25], Example 2.

**Proof of Theorem 2.** i) This assertion follows from Theorem 2 in [24] and Corollary 4. It suffices to note that if (18) holds for ensembles \( \mu_*^1, \mu_*^2, \ldots \), then it holds for any convex combination \( \sum_k p_k \mu_*^k \) of them (as probability measures).

ii) By Corollary 4 the equality \( \overline{C}(\Phi, H_A, \mathcal{E}) = C_{\text{ea}}(\Phi, H_A, \mathcal{E}) \) implies the existence of a generalized ensemble with the average state \( \rho_* \) such that \( \overline{C}(\Phi, \rho_*) = I(\Phi, \rho_*) = C_{\text{ea}}(\Phi, H_A, \mathcal{E}) \) and \( \text{Tr} H_A \rho_* \leq \mathcal{E} \). Since for any degradable channel \( \Phi \) we have \( \overline{C}(\Phi, \rho) \leq H(\rho) \leq I(\Phi, \rho) \) for any state \( \rho \), it is easy to see that \( \rho_* \) is
the state with maximal entropy under the condition $\text{Tr} H_A \rho_\ast \leq \mathcal{E}$, that is, the Gibbs state $[\text{Tr} e^{-\lambda^* H_A}]^{-1} e^{-\lambda^* H_A}$, where $\lambda^*$ is a solution of the equality $\mathcal{E} \text{Tr} e^{-\lambda^* H_A} = \text{Tr} H_A e^{-\lambda^* H_A}$. Thus, $\rho_\ast$ is a full-rank state and Theorem 2 in [24] shows that $\Phi$ is a discrete c-q channel.

Thus, Lemma 2 below makes it possible to reduce assertion ii) to the following:

\[ C(\Pi, H_A, \mathcal{E}) = C_{ea}(\Pi, H_A, \mathcal{E}) \iff \langle k | H_A | k' \rangle = 0 \text{ for all } k \neq k', \]

where $\Pi(\rho) = \sum_k \langle k | \rho | k \rangle \langle k |$ and $\mathcal{E} > \mathcal{E}_m$, which was proved in [25], Example 3. □

**Lemma 2** [25]. A discrete c-q channel (20) is degradable if and only if the collection $\{\sigma_k\}$ consists of states with mutually orthogonal supports. In this case $C(\Phi, H_A, \mathcal{E}) = C(\Pi, H_A, \mathcal{E})$ and $C_{ea}(\Phi, H_A, \mathcal{E}) = C_{ea}(\Pi, H_A, \mathcal{E})$ for any operator $H_A$ and $\mathcal{E} > 0$, where $\Pi(\rho) = \sum_k \langle k | \rho | k \rangle \langle k |$.

It is easy to show that a channel $\Phi$ has a discrete c-q subchannel of the form (20) if and only if $\Phi(\langle k | \langle k' \rangle) = 0$ for all $k \neq k'$. Hence Theorem 2 implies sufficient conditions for the strict inequality

\[ C_{ea}(\Phi, H_A, \mathcal{E}) > C(\Phi, H_A, \mathcal{E}), \quad (21) \]

which means that using an entangled state between the input and output increases the ultimate speed of information transmission over the channel $\Phi$ and gives a gain in the size of an optimal code.

**Corollary 5.** Let $H_A$ be a positive operator satisfying the condition (9) and let $\mathcal{E}_m = \inf_{\|\varphi\|=1} \langle \varphi | H_A | \varphi \rangle$. Then (21) holds for a channel $\Phi$ and $\mathcal{E} > \mathcal{E}_m$ if one of the following conditions holds:

1) $\Phi(\langle \varphi \rangle \langle \psi \rangle) \neq 0$ for any orthogonal unit vectors $\varphi$ and $\psi$;
2) $\Phi$ is a degradable channel which is not a discrete c-q channel;
3) $\Phi$ is a degradable channel and $\Phi(\langle \varphi \rangle \langle \psi \rangle) \neq 0$ for at least two orthogonal eigenvectors $\varphi$ and $\psi$ of the operator $H_A$ corresponding to different eigenvalues;
4) $\Phi$ is not a discrete c-q channel and the maximum in (16) is attained at some full-rank state.

Note that condition 1) means that $\Phi^*(\mathcal{B}(\mathcal{H}_B))$ is a transitive subspace of $\mathcal{B}(\mathcal{H}_A)$ [27].

Consider an application of Corollary 5 to the class of the Bosonic Gaussian channels playing a central role in continuous-variable quantum information theory [4].

Let $\mathcal{H}_X (X = A, B, \ldots)$ be the space of an irreducible representation of the Canonical Commutation Relations

\[ W_X(z)W_X(z') = \exp \left( -\frac{i}{2} z^\top \Delta_X z' \right) W_X(z' + z) \]

with a symplectic space $(Z_X, \Delta_X)$ and the Weyl operators $W_X(z)$ ([3], Ch. 12). Denote by $s_X$ the number of modes of the system $X$, that is, $2s_X = \dim Z_X$. 
A Bosonic Gaussian channel $\Phi : \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ is defined via the action of its dual $\Phi^* : \mathfrak{B}(\mathcal{H}_B) \to \mathfrak{B}(\mathcal{H}_A)$ on the Weyl operators:

$$\Phi^*(W_B(z)) = W_A(Kz) \exp \left( il^T z - \frac{1}{2} z^T \alpha z \right), \quad z \in \mathbb{Z}_B,$$

where $K$ is a linear operator $\mathbb{Z}_B \to \mathbb{Z}_A$, $l$ is a $2s_B$-dimensional real row and $\alpha$ is a real symmetric $(2s_B) \times (2s_B)$ matrix satisfying the inequality

$$\alpha \geq \pm i \left[ \Delta_B - K^\top \Delta_A K \right].$$

By applying unitary displacement transformations, an arbitrary Gaussian channel can be transformed into the Gaussian channel with $l = 0$ and the same matrices $K$ and $\alpha$ (such a channel will be called centred and denoted by $\Phi_{K,\alpha}$).

It follows from Proposition 5 in [24] that $\Phi_{K,\alpha}$ is a discrete c-q channel if and only if $K = 0$ (that is, if and only if $\Phi_{K,\alpha}$ is a completely depolarizing channel). Proposition 3 in [26] shows that the first condition in Corollary 5 holds if and only if $\text{Ran} \ K = Z_A$ (that is, $\text{rank} \ K = \text{dim} \ Z_A$). Thus, Corollary 5 implies the following sufficient condition for (21).

**Corollary 6.** Let $H_A$ be a positive operator satisfying the condition (9) and let $\mathcal{E}_m = \inf_{\|\varphi\|=1} \langle \varphi | H_A | \varphi \rangle$. Then (21) holds for the channel $\Phi_{K,\alpha}$ and $\mathcal{E} > \mathcal{E}_m$ if one of the following conditions holds:

1) $\text{Ran} \ K = Z_A$ (that is, $\text{rank} \ K = \text{dim} \ Z_A$);
2) $\Phi_{K,\alpha}$ is a degradable channel;
3) $K \neq 0$ and the maximum in (16) is attained at some full-rank state.

The last condition in Corollary 6 holds if $\Phi_{K,\alpha}$ is a non-trivial gauge covariant or contravariant channel and $H_A = \sum_{ij} \epsilon_{ij} a_i^\dagger a_j$ is a gauge invariant Hamiltonian (here $[\epsilon_{ij}]$ is a positive matrix), since in this case the maximum in (16) is attained at a non-degenerate Gaussian state, the average state of the $\chi$-optimal ensemble supported by coherent states [28], [29].

**§ 7. Lower semicontinuity of the coherent information for degradable channels**

The coherent information

$$I_c(\Phi, \rho) = H(\Phi(\rho)) - H(\hat{\Phi}(\rho)) \quad (22)$$

of a channel $\Phi$ in a state $\rho$ is an important characteristic related to the quantum capacity of the channel [3], [8], [16].

The function $\rho \mapsto I_c(\Phi, \rho)$ is continuous on any set on which the input entropy $H(\rho)$ is continuous ([18], Corollary 14), but in general it is not upper or lower semicontinuous on the set of all input states (where the difference in (22) is well defined).

---

7The gauge invariance condition for $H_A$ can be replaced by the requirement that condition (18) in [30] holds as a strict operator inequality.
It is known that $I_c(\Phi, \rho)$ is non-negative for any degradable channel $\Phi$ (that is, any channel satisfying $\Phi = \Theta \circ \Phi$ for some channel $\Theta : B \to E$). We will show that in this case the coherent information $I_c(\Phi, \rho)$ is lower semicontinuous as a function of $\rho$.

**Proposition 6.** If $\Phi : A \to B$ is a degradable channel, then the function $\rho \mapsto I_c(\Phi, \rho)$ is lower semicontinuous on the set

$$\mathcal{S}_* = \{\rho \in \mathcal{S}(\mathcal{H}_A) \mid H(\hat{\Phi}(\rho)) < +\infty\}.$$  

**Proof.** Since $H(\Phi(\rho)) = H(\hat{\Phi}(\rho))$ for any pure state $\rho$ and $\hat{\Phi} = \Theta \circ \Phi$, we have

$$I_c(\Phi, \rho) = \chi(\Phi(\mu)) - \chi(\Phi(\mu)) = \Delta^E \chi(\Phi(\mu))$$  

(23)

for any $\rho \in \mathcal{S}(\mathcal{H}_A)$ and any ensemble $\mu \in \mathcal{P}(\mathcal{H}_A)$ of pure states such that $H(\Phi(\rho)) < +\infty$ and $\bar{\rho}(\mu) = \rho$. By using the approximation technique [17], it is easy to show that (23) holds for any $\rho$ in $\mathcal{S}_*$ and the corresponding ensemble $\mu$.

Let $\{\rho_n\} \subset \mathcal{S}_*$ be a sequence converging to a state $\rho_0 \in \mathcal{S}_*$. Take any sequence $\{\mu_n\}$ of ensembles of pure states converging to an ensemble $\mu_0$ such that $\bar{\rho}(\mu_n) = \rho_n$ for all $n$ (such a sequence can be constructed using the spectral decompositions of the states $\rho_n$). Then (23) and Theorem 1 imply that

$$\liminf_{n \to +\infty} I_c(\Phi, \rho_n) = \liminf_{n \to +\infty} \Delta^E \chi(\Phi(\mu_n)) \geq \Delta^E \chi(\Phi(\mu_0)) = I_c(\Phi, \rho_0).$$  

**Corollary 7.** The continuity of the output entropy $H(\Phi(\rho))$ of a degradable channel $\Phi : A \to B$ on some subset $\mathcal{S}_0 \subset \mathcal{S}(\mathcal{H}_A)$ implies that of the input entropy $H(\rho)$ and of the entropy exchange $H(\hat{\Phi}(\rho))$ on $\mathcal{S}_0$.

**Proof.** Since

$$H(\Phi(\rho)) = I_c(\Phi, \rho) + H(\hat{\Phi}(\rho)),$$

the continuity of the entropy exchange $H(\hat{\Phi}(\rho))$ on $\mathcal{S}_0$ follows from Proposition 6 and Lemma 1. The continuity of the input entropy $H(\rho)$ on $\mathcal{S}_0$ follows from Proposition 9 in [18]. □

Corollary 7 shows, in particular, that

$$\lim_{n \to -\infty} H(\rho_n) \neq H(\rho_0) \Rightarrow \lim_{n \to -\infty} H(\Phi(\rho_n)) \neq H(\Phi(\rho_0))$$

for a degradable channel $\Phi$ and any sequence $\{\rho_n\}$ of input states converging to a state $\rho_0$ with $H(\Phi(\rho_0))$ finite. This means that **degradable channels preserve the local discontinuity of the input entropy**.

§ 8. Appendix: proof of the equality (6)

To relax the condition $\dim \mathcal{H}_A, \dim \mathcal{H}_B < +\infty$, consider sequences of channels $\Lambda_n^B : B \to B$ and $\Lambda_n^E : E \to E$ with finite-dimensional outputs strongly converging to the identity channels $\text{Id}_B$ and $\text{Id}_E$ (see Remark 1). Since

$$I(B:E)_{\Pi_n(V\rho V^*)} = H(\Lambda_n^B \circ \Phi(\rho)) + H(\Lambda_n^E \circ \hat{\Phi}(\rho)) - H(\Pi_n(V\rho V^*)), $$
where \( \Pi_n \doteq \Lambda_n^B \otimes \Lambda_n^E \), we have
\[
\chi(\Pi_n(V\mu V^*)) + I(B:E)\Pi_n(V\overline{\mu}V^*) \\
= \chi(\Lambda_n^B \circ \Phi(\mu)) + \chi(\Lambda_n^E \circ \widehat{\Phi}(\mu)) + \int I(B:E)\Pi_n(V\rho V^*) \mu(d\rho). \tag{24}
\]

By using Proposition 4 in [17] and the chain rule for the \( \chi \)-quantity it is easy to show that
\[
\lim_{n \to \infty} \chi(\Pi_n(V\mu V^*)) = \chi(V\mu V^*) = \chi(\mu) \leq +\infty,
\]
\[
\lim_{n \to \infty} \chi(\Lambda_n^B \circ \Phi(\mu)) = \chi(\Phi(\mu)) \leq +\infty,
\]
\[
\lim_{n \to \infty} \chi(\Lambda_n^E \circ \widehat{\Phi}(\mu)) = \chi(\widehat{\Phi}(\mu)) \leq +\infty.
\]

Thus, to derive (6) from (24) it suffices to show that
\[
\lim_{n \to \infty} I(B:E)\Pi_n(V\overline{\mu}V^*) = I(B:E)V\overline{\mu}V^* \leq +\infty, \tag{25}
\]
\[
\lim_{n \to \infty} \int I(B:E)\Pi_n(V\rho V^*) \mu(d\rho) = \int I(B:E)V\rho V^* \mu(d\rho) \leq +\infty. \tag{26}
\]

The limit relation (25) follows directly from the lower semicontinuity of the quantum mutual information \( I(B:E) \) and the fact that it is non-increasing under the action of the channel \( \Pi_n \). The limit relation (26) can be rewritten as follows:
\[
\lim_{n \to \infty} \int I(B:E)_\omega \nu_n(d\omega) = \int I(B:E)_\omega \nu(d\omega) \leq +\infty, \tag{27}
\]
where \( \nu_n \) and \( \nu \) are the images of the ensemble \( \mu \) under the channels \( \Pi_n(V(\cdot)V^*) \) and \( V(\cdot)V^* \) respectively. By noting that the strong convergence of a sequence of channels implies its uniform convergence on compact subsets of states, it is easy to show the weak convergence of the sequence \( \{\nu_n\} \) to the ensemble \( \nu \) (see the proof of Lemma 1 in [17]). Since the lower semicontinuity and non-negativity of the function \( \omega \mapsto I(B:E)_\omega \) imply the lower semicontinuity of the functional \( \nu \mapsto \int I(B:E)_\omega \nu(d\omega) \) [9], to prove (27) (and hence (26)) it suffices to note that the fact that \( I(B:E) \) is non-increasing under the action of the channel \( \Pi_n \) implies that
\[
\int I(B:E)_\omega \nu_n(d\omega) \leq \int I(B:E)_\omega \nu(d\omega)
\]
for all \( n \).

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Maxim E. Shirokov
Steklov Mathematical Institute of the Russian Academy of Sciences, Moscow
E-mail: msh@mi.ras.ru

Alexander S. Holevo
Steklov Mathematical Institute of the Russian Academy of Sciences, Moscow
E-mail: holevo@mi.ras.ru