Mathematical Modeling and Characteristic Analysis on the Dynamics of Slider Flapping Wing Mechanism

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Abstract: This paper addresses the modelling of a distinct flapping wing mechanism sans any gears deploying only the primitive slider and hinge type joints. The proposed contraption is meticulously designed so as to hamper vibrational flutter and enhance durability. Initially, equations of forces enacting at significant motion transformation junctions are articulated in terms of Differential Algebraic Equations (DAE) and then these DAEs culminate as the characteristic mathematical model for the mechanism. The derived mathematical model is then harnessed to optimize the performance characteristics of the system. Further, results attained from the mathematical formulation are assessed and validated with graphical analysis.

Keywords: Flapping Wing Mechanism, Differential Algebraic Equations, Mathematical Model

1. Introduction
A seamless flapping wing mechanism for Micro Wing Aerial Vehicle (MAV) is designed to accomplish an unhindered flapping action with enhanced durability. The mechanism is designed with the considerations of the implicational and geometrical conditions regards to the application of Micro-Aerial Vehicle (MAV) Flight. Neglecting the use of gears, the links and hinge joints are equipped. The mechanism transforms the rotational motion from a motor into a periodic symmetric flapping action. The Links implied as wing of the MAV mechanism are enforced with a slider motion besides the hinge joints. This is done in order to bring out the proper inline flapping motion and eliminating the phase lag between both the links. In order to optimize the performance of the mechanism, optimum flapping angle and suitable dimensions of varied linkages have to be chosen.

A mathematical model in the form a Differential Algebraic Equation is derived for arriving at a controlled stability analysis for the proposed mechanical system. Conventionally, the Euler Lagrange method has been followed by many authors to arrive at their mathematical formulation. But this method yields a very complex solution which is very disadvantageous in scenarios where we have to compute a unique solution for the derived model. In this paper, we adopt a finite element approach where we break our mechanism into sub mechanisms at the nodes of motion transformations and analyze them individually and then combine them as complete formulation. The relation between the different linkages are assessed and ultimately the coherence between optimum flapping angle, minimum auxiliary displacement and maximum torque is established in a graphical representation. A unique solution for the optimum flapping angle is finally attained which aids in extrapolating the other characteristic performances like flapping frequency.
In the aforementioned proposed mechanism two primary motion transformations take place –

a. Rotational Motion of the Crank Link (denoted by $L_C$ in figure) is transformed into Vertical Oscillatory Linear Motion of the Body Plate (the hexagonal shaped linkage)

b. Vertical Oscillatory Motion of the Body Plate is transformed into a combination of Sliding and Rotational Motion of the Wing Link (denoted by $L_W$)

Adopting a Finite elemental approach for the purpose of reducing complexity, the three Rigid bodies (Crank Link, Body Plate and Wing Link) are separately assessed.

1.1 Geometrical Transformation (2-Translation Variables_2-Rotational Angles)

Let $s =$ Vertical Displacement of the Body Plate from its initial position at time $t$

$x =$ Horizontal Displacement of the Hinge Point of the Wing Link at the slider from its initial position at time $t$

$\theta =$ Angle made by the Crank Link from the horizontal at time $t$

$\phi =$ Angle made by the Wing Link from the horizontal at time $t$

The crank link has mass of $m_C$ and length $L_C$. The mass of the body link is denoted by $m_B$. The mass and length of the wing link is represented by $m_W$ and $L_W$ respectively.

From the mechanism assembly, the relationship between the crank rotation angle($\theta$) and vertical displacement of the body plate(s) is given by

$$s = L_C \sin \theta$$  \quad \quad \quad (1)

Therefore, \quad $$\dot{s} = L_C \cos \theta \dot{\theta}$$  \quad \quad \quad (2)

$$\ddot{s} = -L_C \sin \theta \ddot{\theta}$$ \quad \quad \quad (3)

From the mechanism assembly, the relationship between the wing rotation angle($\phi$) and vertical displacement of the body plate(s) is given by

$$s = L_W \sin \phi$$ \quad \quad \quad (4)

Therefore, \quad $$\dot{s} = L_W \cos \phi \dot{\phi}$$ \quad \quad \quad (5)

$$\ddot{s} = -L_W \sin \phi \ddot{\phi}$$ \quad \quad \quad (6)

From the mechanism assembly, the relationship between the horizontal displacement of the hinge joint along the slider(x) and vertical displacement of the body plate(s) is given by

$$x = L_W - \sqrt{L_W^2 - \dot{s}^2}$$ \quad \quad \quad (7)
1.2 Crank Link
The Crank link exhibits a Pure Rotational Motion rotating about the z-axis as

**Force Equation**
Initially a Torque is applied to the crank link of length $L_c$ which initiates the rotary motion of the crank with a particular angular acceleration.

Therefore Input Force is given by,

$$ F_{in} = \frac{r}{L_c} \quad \text{(10)} $$

In order to sustain the rotational motion, a centripetal force is also generated at the crank link. This force complements the vertical linear movement of the Body Plate.

Hence the Force generated at the Crank Link, $F_c = m_c L_c \dot{\theta}^2 \quad \text{(11)}$

**Body Plate:**
The Body Plate exhibits a Pure Translational Motion moving linearly along the y-axis

**Force Equation**
The Body Plate exhibits a vertical oscillatory motion due to the influence of the crank and geometry of the set up.

Therefore Force experienced by the Body Plate, $F_B = m_B \ddot{s} \quad \text{(12)}$

1.3 Wing Link
The Wing link is imparted a Combined Translational and Rotational Motion along x-axis and about z-axis respectively to acquire the flapping action.

**Force Equation**
The wing link undergoes combined translational and rotational motion. Hence two types of forces act on the wing link. The first kind is due to Linear acceleration of the joint in the slider. The second kind is because of the angular acceleration of the wing link.

For the Force due to Rotational Motion Inertia term is to be included

Since the wing link is of rectangular cross section, $I = \frac{m_W L_w^2}{12}$

Therefore, Force generated by the Wing Link, $F_w = m_w \ddot{x} + \frac{1}{L_w} I \ddot{\phi}$

$$ F_w = m_w \ddot{x} + \frac{m_w L_w \ddot{\phi}}{12} \quad \text{(13)} $$

2. Mathematical Formulation:
*Transformation 1:* Rotary Motion of Crank to Linear Vertical Motion of Body Plate

Equations (8), (9) and (10) constitute to this Transformation.

The equilibrium force equation for Transformation 1 is given by,

$$ F_{in} = F_c + F_B $$

$$ F_{in} = \frac{r}{L_c} = m_c L_c \dot{\theta}^2 + m_B \ddot{s} \quad \text{(14)} $$
Transformation 2: Linear Vertical Motion of Body Plate to Combined Sliding and Rotational Oscillation of Wing Link

Equations (3) and (4) constitute to this Transformation.
The equilibrium force equation for Transformation 2 is given by,

\[ F_b = F_w \]

\[ m_b \ddot{s} = m_w \ddot{x} + \frac{m_w L_w \ddot{\phi}}{12} \]  \hspace{1cm} (15)

Substituting Equation (6) in Equation (5), we get

\[ F_{in} = m_c L_c \dot{\theta}^2 + m_w \ddot{x} + \frac{m_w L_w \ddot{\phi}}{12} \]  \hspace{1cm} (16)

Representing Equation (16) as a function of \( s \),

\[ F_{in} = m_c \frac{\ddot{s}^2}{L_c \cos^2 \theta} + m_w \left( \frac{s}{L_w} \right) \ddot{s} + \frac{L_w^2}{L_w^2 - s^2} \ddot{s}^2 - \frac{m_w L_w \ddot{s}}{12 \sin \phi} \]  

On Further Simplification,

\[ F_{in} = m_c \frac{\ddot{s}^2}{L_c \cos^2 \left( \frac{s}{L_c} \right)} + m_w \left( \frac{s}{L_w} \right) \ddot{s} + \frac{L_w^2}{L_w^2 - s^2} \ddot{s}^2 - \frac{m_w L_w \ddot{s}}{12 \sin \left( \frac{s}{L_w} \right)} \]  

\[ \text{---------------------- (17)} \]

The above equation depicts the relation between Input Force and vertical displacement \( s \). Since we need a relation involving the flapping angle \( \phi \) and Torque \( \tau \) to optimize parameters, the above equation is further modified as

\[ F_{in} = m_c \frac{\ddot{s}^2}{L_c \cos^2 \theta} + m_w \left( \frac{s}{L_w \cos \phi} \right) \ddot{s} + \frac{L_w^2}{L_w^2 \cos^3 \phi} \ddot{s}^2 - \frac{m_w L_w \ddot{s}}{12 \sin \phi} \]  

\[ F_{in} = m_c \frac{L_c}{L_c^2 - L_w^2} \frac{\ddot{s}^2}{\sin^2 \phi} + m_w \left( \frac{s}{L_w \cos \phi} \right) \ddot{s} + \frac{1}{L_w \cos^3 \phi} \ddot{s}^2 - \frac{m_w L_w \ddot{s}}{12 \sin \phi} \]  

\[ F_{in} = m_c \frac{L_c L_w^2 \cos^2 \phi}{L_c^2 - L_w^2 \sin^2 \phi} \ddot{\phi}^2 + m_w L_c \left( \sec \phi \right) \ddot{\phi}^2 - \frac{m_w L_c L_w \left( \sin \phi \tan \phi \right) \ddot{\phi}}{12} + \frac{m_w \ddot{L}_w}{12 \sin \phi} \]  

\[ \tau_{in} = \frac{m_c L_c^2 L_w^2 \cos^2 \phi}{L_c^2 - L_w^2 \sin^2 \phi} \ddot{\phi}^2 + m_w L_c L_w \left( \sec \phi \right) \ddot{\phi}^2 - \frac{m_w L_c L_w \left( \sin \phi \tan \phi \right) \ddot{\phi}}{12} + \frac{m_w L_w L_c \ddot{\phi}}{12} \]  

\[ \tau_{in} = \frac{m_c L_c^2 L_w^2 \left( \cot^2 \phi \right) \ddot{\phi}^2 + m_w L_c L_w \left( \sec \phi \right) \ddot{\phi}^2 - \frac{m_w L_c L_w \left( \sin \phi \tan \phi \right) \ddot{\phi}}{12} + \frac{m_w L_w L_c \ddot{\phi}}{12} \]  

\[ \text{------------------------ (18)} \]
Equation (18) is the required mathematical model that establishes a relation between input torque and output flapping angle.

Since the same kind of material is employed for the whole mechanism and the density of the links are very low when compared to the length dimensions ($m_c \ll L_c$ and $m_w \ll L_w$), we can neglect $m_c$ and $m_w$.

Let $A = \frac{L_c^2 L_w^2}{L_c^2 - L_w^2}$, $B = L_c L_w$, $C = \frac{L_w L_c}{12}$

Therefore, $\tau_{in} = A(\cot^2 \phi \dot{\theta}^2 + B[(\sec \phi) \dot{\theta}^2 - (\sin \phi \tan \phi) \dot{\theta}] + C \dot{\theta}^2)$  \hspace{1cm} (19)

Equation (19) is the mathematical model of the proposed Flapping Wing Mechanism with an input of $F_{in}$.

3. Stability Analysis

For the proposed mechanism, three combinations for the Crank Link ($L_c$) and the Wing Link ($L_w$) are possible. In order to deduce the optimum flapping angle, the following cases are considered.

(i) $L_c = L_w$  \hspace{1cm} (ii) $L_c > L_w$  \hspace{1cm} (iii) $L_c < L_w$

**Case (i) $L_c = L_w$**

From Equation (18), $A = \frac{L_c^2 L_w^2}{L_c^2 - L_w^2}$

At $L_c = L_w$ the value of $A$ becomes infinity and hence the system will be unstable.

If the Length of the Crank Link is equal to the Length of the Wing Link, then during Transformation 1 (Rotary Motion of Crank to Linear Vertical Motion of Body Plate), the angular acceleration of the Crank Link will be discontinuous and the Crank Link will cease to rotate. Thus, the Flapping action will be restricted and unstable.

**Case (ii) $L_c > L_w$**

From Equation (7), $x = L_w - \sqrt{L_w^2 - s^2}$

At $\theta = 90^\circ$, $s = L_c$ $x = L_w - \sqrt{L_w^2 - L_c^2}$

Since $L_c > L_w$ $x = L_w - \sqrt{L_c^2 - L_w^2}$

Thus, the system has a discontinuity at $\theta = n\pi/2$ and hence the system will be unstable.

If the Length of the Crank Link is greater than the Length of the Wing Link, then during Transformation 2 (Linear Vertical Motion of Body Plate to Combined Sliding and Rotational Oscillation of Wing Link), the sliding motion of the Wing Link will be hindered due to geometry when the Crank Link is rotated to an angle of $n\pi/2$. Hence a Singularity occurs at that point and the mechanism ceases to exhibit the desired Flapping action.

**Case (iii) $L_c < L_w$**

If the Length of the Crank Link is lesser than the Length of the Wing Link, all the derived equations satisfy. Thus, the drawbacks of the above mentioned geometries can be overhauled and both the Transformations 1 & 2 will be executed seamlessly resulting in a stable mechanism with uniform Flapping action. Hence, we can infer that the length of the wing link should always be greater than that of the crank link in order for the mechanism to produce the desired flapping action. Further analysis is made taking into consideration the stability criterion and the characteristic parameters are assessed and deduced by means of graphical analysis utilizing the mathematical formulation. The iterative results are plotted to attain optimum parameters.
4. Graphical Analysis
Considering the stable case of $L_W > L_C$, a graph is plotted for Flapping Angle versus Vertical Body Plate with the derived mathematical model for different combinations of $L_C$ and $L_W$ and a similar periodic pattern is obtained.

![Flapping Angle Vs Auxiliary Displacement](image)

**Figure 2.** Flapping Angle Vs Auxiliary Displacement of Body Plate when $L_W = 2L_C$

![Flapping Angle Vs Auxiliary Displacement](image)

**Figure 3.** Flapping Angle Vs Auxiliary Displacement when $L_W = 3L_C$

From the results, it is found that the minimum body displacement ($s_{min}$) occurs at a flapping angle of 135°.

Hence 135° is the optimum flapping angle which can provide the required flapping frequency with reduced body displacement which in turn minimizes the overall vibrations that incur during the dynamic motion.

From the derived mathematical model, a graph is plotted between Flapping Angle and Torque and the observed results are as below.
The maximum torque can be generated at the optimum flapping angle. In our case the optimum flapping angle is at 135° which yields a maximum torque of 448.01 Nm and 414.83 Nm respectively in both the observed cases.

It is pertinent for the Flapping Wing Mechanism to generate maximum torque at a minimum auxiliary displacement in order to minimize the vibrational flutter, reduce the wear rate and enhance the flapping frequency. Thus, from the graphical analysis, we deduce that a flapping angle of 135° serves the purpose of optimization.

5. Results and Discussions
The relations between the lengths of the crank link $L_C$ and wing link $L_W$ are attained. It is found out that the stability of the mechanism completely depends upon the relation between the two links. The mechanical system exhibits a stable periodic and symmetric flapping action if the length of the Wing Link is designed in such a manner that its dimensions are greater than the length of the Crank Link (i.e., $L_W > L_C$). In order to obtain optimum performance characteristics like high flapping frequency and less vibrations with reduced wear, it is necessary that the mechanism generates high torque with least possible link displacement. By employing the formulated mathematical model, the different torque values for various flapping angles are computed for different combinations of Crank Link $L_C$ and Wing Link $L_W$ dimensions. Since the flapping angle has an explicit relation to the auxiliary displacement of the body plate, the different range of values for auxiliary displacement is also computed. When the results are plotted in a graph the optimum Flapping angle for the proposed contraption is found out to be 135° for which auxiliary displacement is minimum and torque generated is maximum.
6. Conclusion
Mathematical Modelling Approaches like Denavit-Hartenberg matrix transformations, Euler-Lagrange and Hamiltonian Approaches can only yield effective results in case of simple mechanisms. But for a more complex mechanism such as the proposed slider flapping wing mechanism, the conventional approaches cannot be applied to conclusively predict the performance parameters as all of these approaches yield only general equations without any unique solutions. Adopting the proposed methodology yields an effective mathematical formulation which aids in optimizing the mechanism and enhancing its efficiency by obtaining a unique solution unlike the other methodologies. Since all the linkages are explicitly connected to the flapping angle, the design dimensions of interconnected links $L_c$ and $L_w$ can be determined. The flapping angle deduced as the optimum one can be utilized to acquire the flapping frequency of the mechanism which is root parameter around which a Micro Aerial Vehicle design revolves upon.

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