Wasserstein-Type Distances of Two-Type Continuous-State Branching Processes in Lévy Random Environments

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Abstract
Under natural conditions, we prove exponential ergodicity in the $L_1$-Wasserstein distance of two-type continuous-state branching processes in Lévy random environments with immigration. Furthermore, we express precisely the parameters of the exponent. The coupling method and the conditioned branching property play an important role in the approach. Using the tool of superprocesses, ergodicity in total variation distance is also proved.

Keywords Exponential ergodicity · Wasserstein distance · Branching process · Random environment · Superprocess

Mathematics Subject Classification (2020) 60J25 · 60J68 · 60J80 · 60J76

1 Introduction

The model of two-type continuous-state branching processes with immigration in Lévy random environments (two-type CBIRE processes) was established in [22]. And the authors provided a necessary and sufficient condition for ergodicity in the
$L_1$-weak convergence. Continuing the work, we are concerned in this paper with the convergence rate of the ergodicity in the $L_1$-Wasserstein distance and the ergodicity in the total variation distance.

CBIRE processes are derived from the model of classical continuous-state branching processes with immigration (CBI processes). The ergodic property is an important topic in the study of CBI processes. It was announced in [20] that, for a (sub)-critical single-type CBI process with branching mechanism $\phi$ strictly positive on $(0, \infty)$, the ergodicity holds if and only if

$$\int_0^\infty \frac{\psi(\lambda)}{\phi(\lambda)} d\lambda < \infty,$$

(1)

for some $z > 0$. Here, $\psi$ is the immigration mechanism defined by

$$\psi(\lambda) = h\lambda + \int_0^\infty (1 - e^{-z\lambda}) n(dz),$$

(2)

where $h \geq 0$ and $(1 \wedge z)n(dz)$ is a finite measure on $(0, \infty)$. In the subcritical case, the condition given in (1) is equivalent to $\int_1^\infty \log(u)n(du) < \infty$; see [14] for a proof of the result. Moreover, [17] proved the exponential ergodicity when the process is subcritical and the immigration mechanism $\psi(\lambda)$ takes the particular form of $a\lambda$, where $a > 0$. In fact, for the more general $\psi(\lambda)$ defined by (2), the conclusion is still valid (see pp.66–67 of [15]). And the proof used the method of coupling, which has been proved effective in the study of exponential ergodicity of Markov processes (see [1, 5, 7, 13, 16, 18] and references therein). We also refer to [3, 9, 10] for similar results of single-type CBI processes.

For branching processes in random environments, limited work has been done in the topic of ergodicity. The model of single-type CBIRE processes was established independently in [11, 19]. For a single-type CBIRE process with branching mechanism $\phi$, immigration mechanism $\psi$ and random environment $\xi$, where $\{\xi(t) : t \geq 0\}$ is a Lévy process, [11] proved that, if $P[\xi(1)]$ is strictly less than the drift coefficient of the branching mechanism $\phi$, then the sufficient and necessary condition of ergodicity is

$$\int_1^\infty \log(u)n(du) < \infty.$$  

Recently, [23] gave an exact description for the extinction speed of continuous state branching processes in heavy-tailed Lévy random environment with stable branching mechanism. As a by-product, the polynomial ergodicity in the total variation distance of such process with immigration can be easily derived. Furthermore, [10] also provided a sufficient condition for exponential ergodicity of CBIRE processes in the Wasserstein distance and the ergodicity in the total variation distance. The above results are concerned with the one-dimensional case, whereas in this paper, we consider similar problems in the two-dimensional setting.

Our main results consist of two parts, the exponential ergodicity in the $L_1$-Wasserstein distance and the ergodicity in the total variation distance. When proving
the exponential ergodicity of two-type CBIRE processes, the main difficulties lie in dealing with the random environment: Because of the random environment, it is more challenging to construct the test function as in [13], and unlike the processes in [16], the two-type CBIRE process no longer satisfies the branching property. Fortunately, owing to the conditioned branching property of such processes, it is still possible to use a coupling similar to [16] after some adjustments. Moreover, lightened by the skew convolution semigroup of CBI processes on p.66 of [14], we introduce a random skew convolution semigroup when constructing the processes with immigration in Lévy random environments, which we have not seen in previous researches. When proving the ergodicity in the total variation distance, the absence of the corresponding Grey’s condition in multi-dimensional cases is the most critical difficulty. We conquer it by linking the multi-dimensional cases to the single-dimensional case with the help of different spatial motions in the setting of Dawson–Watanabe superprocesses. Coincidentally, in the very recent work, the Grey’s condition for multi-type CBI processes was established in [2] by using the Lamperti representation instead of our tool of super-processes. But the main idea of constructing the local projection is similar. Besides, the Grey’s condition established in [2] is more general than what appears in this paper, since it is both necessary and sufficient. So it might seem more direct to follow the idea of [2] to solve our problem in the model of multi-type CBIRE processes. But in that case, further consideration has to be taken when estimating the distribution of the coalescence time of the coupling processes. We refer to Chapter 10 in [15] for a similar natural coupling.

In the next section, we present our main results after some necessary reviews of the two-type CBIRE processes and some basic knowledge on the Wasserstein distance. Theorem 2.1 gives a sufficient condition for the exponential ergodicity in $L_1$-Wasserstein distance, and its exact expression of the parameters is provided in its proof. Theorem 2.2 provides a sufficient condition for the ergodicity in the total variation distance. Sections 3 and 4 are devoted to the proofs of Theorem 2.1 and Theorem 2.2. Our method is strongly influenced by [16]. In the proofs, we also make full use of the conditioned branching property.

Throughout this paper, we let $|\cdot|$ denote the Euclidean norm and $\langle \cdot , \cdot \rangle$ stands for the inner product.

## 2 Preliminaries and Main Results

The model of two-type CBIRE processes can be seen as a combination of the two-type CBI processes and the single-type CBIRE processes. The branching and immigration mechanisms are inherited from the classical two-type CBI processes (see pp.44–45 in [14]). For ease of notation, we still use $\phi$ and $\psi$ in the two-dimensional version without confusion. Specifically, $b = (b_{ij})$ is a $(2 \times 2)$-matrix with

$$
\begin{align*}
b_{12} + \int_{\mathbb{R}^2_+} z_2 m_1(dz) & \leq 0, \\
b_{21} + \int_{\mathbb{R}^2_+} z_1 m_2(dz) & \leq 0,
\end{align*}
$$

(3) where $m_1, m_2$ are $\sigma$-finite measures on $\mathbb{R}^2_+$ supported by $\mathbb{R}^2_+ \setminus \{0\}$ satisfying
\[
\int_{\mathbb{R}^2_+} (z_1 \wedge z_1^2 + z_2)m_1(dz) + \int_{\mathbb{R}^2_+} (z_2 \wedge z_2^2 + z_1)m_2(dz) < \infty.
\]

The branching mechanism \( \phi = (\phi_1, \phi_2) \) is a function from \( \mathbb{R}^2_+ \) to itself with the following representations,

\[
\phi_1(\lambda) = b_{11}\lambda_1 + b_{12}\lambda_2 + c_1\lambda_1^2 + \int_{\mathbb{R}^2_+} (e^{-\langle \lambda, z \rangle} - 1 + \langle \lambda, z \rangle)m_1(dz),
\]

\[
\phi_2(\lambda) = b_{21}\lambda_1 + b_{22}\lambda_2 + c_2\lambda_2^2 + \int_{\mathbb{R}^2_+} (e^{-\langle \lambda, z \rangle} - 1 + \langle \lambda, z \rangle)m_2(dz).
\]

And we write \( \psi \) for the immigration mechanism. It is a function from \( \mathbb{R}^2_+ \) to \( \mathbb{R}_+ \) with representation

\[
\psi(\lambda) = \langle h, \lambda \rangle + \int_{\mathbb{R}^2_+ \setminus \{0\}} (1 - e^{-\langle \lambda, z \rangle})n(dz), \quad \lambda \in \mathbb{R}^2_+.
\]

In the above, \( h = (h_1, h_2) \), \( c = (c_1, c_2) \) are constants in \( \mathbb{R}^2_+ \), and \( n \) is a \( \sigma \)-finite measure on \( \mathbb{R}^2_+ \) supported by \( \mathbb{R}^2_+ \setminus \{0\} \) satisfying \( \int_{\mathbb{R}^2_+} (1 \wedge |z|)n(dz) < \infty \).

Let \( (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}) \) be a filtered probability space satisfying the usual hypothesis. The random environment is described by an \((\mathcal{F}_t)\)-Lévy process \( \{\xi(t) : 0 < t < \infty\} \) with \( \xi(0) = 0 \), whose Lévy-Itô decomposition is given as follows:

\[
\xi(t) = at + \sigma W(t) + \int_0^t \int_{[-1,1]} z\tilde{N}(ds, dz) + \int_0^t \int_{[-1,1]} zN(ds, dz), \quad t \geq 0,
\]

where \( a \in \mathbb{R} \) and \( \sigma \geq 0 \) are given constants, \( \{W_t : t \geq 0\} \) is an \((\mathcal{F}_t)\)-Brownian motion and \( N(ds, dz) \) is an \((\mathcal{F}_t)\)-Poisson random measure on \( (0, \infty) \times \mathbb{R} \) with intensity \( ds \nu(dz) \) satisfying \( \int_{(0,\infty)} (1 \wedge z^2)\nu(dz) < \infty \). We denote the compensated measure of \( N(ds, dz) \) by \( \tilde{N}(ds, dz) \). Similar to the treatment in the single-type model of CBIRE processes [11], the environment can be extended to \( \{\xi(t) : -\infty < t < \infty\} \).

Given an interval \( I \subset \mathbb{R} \), for \( t \in I, \lambda \in \mathbb{R}^2_+ \), there exists \( r \mapsto u_{r,t}(\xi, \lambda) \in \mathbb{R}^2_+ \) as the unique positive strong solution to

\[
u_{r,t}^{(i)}(\xi, \lambda_1, \lambda_2) := e^{-\xi(t)} u_{r,t}^{(i)}(\xi, \xi(t)\lambda_1, \xi(t)\lambda_2), \quad \lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2_+, \quad i = 1, 2.
\]
We call $u_{r,t}$ and $v_{r,t}$ random cumulant semigroups. Define a stochastic transition semigroup $Q^\xi_{r,t}(x, dy)$ by

$$
\int_{\mathbb{R}^2_+} e^{-\langle \lambda, y \rangle} Q^\xi_{r,t}(x, dy) = \exp\{-\langle x, v_{r,t}(\xi, \lambda) \rangle\}, \quad \lambda, x \in \mathbb{R}^2_+.
$$

(10)

It is easy to check that $Q^\xi_{r,t}(x, \cdot)$ has the branching property, namely,

$$
Q^\xi_{r,t}(x + y, \cdot) = Q^\xi_{r,t}(x, \cdot) * Q^\xi_{r,t}(y, \cdot), \quad r < t \in \mathbb{R}, \ x, y \in \mathbb{R}^2_+,
$$

(11)

where $*$ means the convolution between measures. And we write $Q^\xi_t(x, \cdot)$ for the special case of $r = 0$. Furthermore, for $t > 0$, define the transition semigroup $\tilde{Q}_t(x, \cdot)$ by

$$
\int_{\mathbb{R}^2_+} e^{-\langle \lambda, y \rangle} \tilde{Q}_t(x, dy) = P\left[ \exp\left\{ -\langle x, v_{0,t}(\xi, \lambda) \rangle - \int_0^t \psi(v_{s,t}(\xi, \lambda))ds \right\} \right].
$$

(12)

In fact, $\tilde{Q}_t(x, \cdot)$ is the transition semigroup of $\{X(t) : t \geq 0\}$, which is a two-type CBRE process with branching mechanism $\phi$, immigration semigroup $\psi$, random environment $\xi$ and initial value $x \in \mathbb{R}^2_+$.

When $\psi(\lambda) \equiv 0$, we get a two-type CBRE process $\{Y(t) : t \geq 0\}$, and its transition semigroup $Q_t(x, \cdot)$ is given by

$$
\int_{\mathbb{R}^2_+} e^{-\langle \lambda, y \rangle} Q_t(x, dy) = P\left[ \exp\left\{ -\langle x, v_{0,t}(\xi, \lambda) \rangle \right\} \right].
$$

(13)

According to Theorem 2.2 in [22], if $P[\xi(1)] < 0$, $\int_{|z| \geq 1} \log(|z|)n(dz) < \infty$ and the eigenvalues of $b$ have strictly positive real parts, then there is a unique limiting distribution $\mu$ for $X(t)$ as $t \to \infty$. And for $Y(t)$ the limiting distribution is $\mu = \delta_0$, the Dirac measure for $(0, 0)$.

The main purpose of this paper is to prove that the convergence still holds in total variation distance and is exponential in $L^1$-Wasserstein distance. By $\mathcal{P}(\mathbb{R}^2_+)$, we denote the space of all Borel probability measures over $\mathbb{R}^2_+$. Let $d$ be a metric on $\mathbb{R}^2_+$ such that $(\mathbb{R}^2_+, d)$ is a Polish space and define

$$
\mathcal{P}_d(\mathbb{R}^2_+) = \left\{ \rho \in \mathcal{P}(\mathbb{R}^2_+) : \int_{\mathbb{R}^2_+} d(x, 0) \rho(dx) < \infty \right\}.
$$

(14)

The Wasserstein distance on $\mathcal{P}_d(\mathbb{R}^2_+)$ is defined by

$$
W_d(P_1, P_2) = \inf_{\Pi \in \mathcal{C}(P_1, P_2)} \int_{\mathbb{R}^2_+ \times \mathbb{R}^2_+} d(x, y) \Pi(dx, dy),
$$

(15)
where \( C(P_1, P_2) \) stands for the set of all coupling measures of \( P_1 \) and \( P_2 \), i.e., \( C(P_1, P_2) \) is the collection of measures on \( \mathbb{R}_+^2 \times \mathbb{R}_+^2 \) having \( P_1 \) and \( P_2 \) as marginals. It can be shown that this infimum is attainable. According to Theorem 6.16 in [21], \( \mathcal{P}_d(\mathbb{R}_+^2), W_d \) is also a Polish space. In the reminder of the article, we will use the following particular examples.

If \( d_{TV}(x, y) = 1_{\{x \neq y\}} \), then \( \mathcal{P}_{d_{TV}}(\mathbb{R}_+^2) = \mathcal{P}(\mathbb{R}_+^2) \) and

\[
W_{d_{TV}}(P_1, P_2) = \frac{1}{2} \| P_1 - P_2 \|_{TV} := \frac{1}{2} \sup_{A \subset \mathcal{B}(\mathbb{R}_+^2)} |P_1(A) - P_2(A)|
\]

is the total variation distance.

If \( d_1(x, y) = |x - y| \), then

\[
\mathcal{P}_{d_1}(\mathbb{R}_+^2) = \left\{ \rho \in \mathcal{P}(\mathbb{R}_+^2) : \int_{\mathbb{R}_+^2} |x| \rho(dx) < \infty \right\},
\]

and the corresponding \( W_{d_1} \) is the \( L^1 \)-Wasserstein distance written as \( W_{d_1} := W_1 \).

For a matrix \( A \), we denote its determinant by \( det(A) \), and its trace by \( tr(A) \). Moreover, define \( \beta := a + \frac{\sigma^2}{2} + \int_{[-1,1]}(e^z - 1 - z) \nu(dz) + \int_{[-1,1]}(e^z - 1) \nu(dz), \Delta := [tr(b)]^2 - 4 det(b) > 0 \) and \( \epsilon := \sqrt{\Delta} + b_{22} - b_{11} + 2b_{21} \).

Let \( d^\theta \) be a metric on \( \mathbb{R}_+^2 \) given by

\[
d^\theta(x, y) = (1 + \theta)|x - y|
\]

for some positive constant \( \theta \). Obviously, \( (\mathbb{R}_+^2, d^\theta) \) is a complete separable metric space and \( (\mathcal{P}_{d^\theta}(\mathbb{R}_+^2), W_{d^\theta}) \) is also a Polish space.

It can be derived from Theorem 2.2 in [22] that, if \( \mathbf{P}[\xi(1)] < 0, \int_{|z| \geq 1} \log(|z|) n(dz) < \infty \) and the eigenvalues of \( b \) have strictly positive real parts, then there exists a probability measure \( \mu \) on \( \mathbb{R}_+^2 \) such that for any \( x \in \mathbb{R}_+^2 \), \( \bar{Q}_t(x, \cdot) \) converges weakly to \( \mu \) as \( t \to \infty \). And the Laplace transform of \( \mu \) is

\[
\int_{\mathbb{R}_+^2} e^{-(\lambda, y)} \mu(dy) = \mathbf{P}\left[ \exp\{- \int_{-\infty}^0 \psi(y, 0, \lambda) ds\}\right]. \tag{16}
\]

In this paper, we show that the convergence stays true in the total variation distance, and it is exponential in the \( L^1 \)-Wasserstein distance.

**Theorem 2.1** Suppose that \( \int_{|z| \geq 1} |z| n(dz) < \infty, \int_1^\infty e^{\epsilon} v(dz) < \infty \), and \( \beta < \frac{1}{2} (tr(b) - \sqrt{\Delta}) \). Then there exist positive constants \( \theta > 0, \rho > 0 \) (which has an explicit expression) and a unique stationary distribution \( \mu \in \mathcal{P}_1(\mathbb{R}_+^2) \) such that for any \( x \in \mathbb{R}_+^2 \) and \( t \geq 0 \),

\[
W_1(\delta_x \bar{Q}_t, \mu) \leq W_{d^\theta}(\delta_x, \mu)e^{-\rho t},
\]

where \( \delta_x \bar{Q}_t = \bar{Q}_t(x, \cdot) \). Moreover, the Laplace transform of \( \mu \) is given by (16).
Theorem 2.2 Suppose that Condition 3.5 (see Sect. 4 for details) is satisfied. If 
\[ \int_{|z| \geq 1} |z| n(dz) < \infty, \int_1^\infty e^{\gamma v}(dz) < \infty, \beta < \frac{1}{2}(tr(b) - \sqrt{\Delta}) \text{ and } \lim_{t \to \infty} \xi(t) = -\infty, \]
then there exists a unique stationary distribution \( \mu \in \mathcal{P}(\mathbb{R}_+^2) \) such that for any \( x \in \mathbb{R}_+^2 \) and \( t \geq 0, \)
\[
\lim_{t \to \infty} \| \delta_x \tilde{Q}_t - \mu \|_{TV} = 0.
\]

Moreover, the Laplace transform of \( \mu \) is given by (16).

3 Proofs

3.1 Exponential Ergodicity in the \( L_1 \)-Wasserstein Distance

In this section, we give the proofs of the exponential ergodicity in \( L_1 \)-Wasserstein distance. Our method of coupling is strongly influenced by [6, 16], and the key to the proof is the use of conditional branching property. For \( 0 < r < t \), define

\[
\pi_1'(r, t) = \frac{\partial u_{r,t}^{(1)}(\xi, \lambda)}{\partial \lambda_1}|_{\lambda=0^+} + \frac{\partial u_{r,t}^{(1)}(\xi, \lambda)}{\partial \lambda_2}|_{\lambda=0^+},
\]
\[
\pi_2'(r, t) = \frac{\partial u_{r,t}^{(2)}(\xi, \lambda)}{\partial \lambda_1}|_{\lambda=0^+} + \frac{\partial u_{r,t}^{(2)}(\xi, \lambda)}{\partial \lambda_2}|_{\lambda=0^+}.
\]

Differentiating both sides of (8), we get,

\[
\frac{\partial u_{r,t}^{(1)}(\xi, \lambda)}{\partial \lambda_1}|_{\lambda=0^+} = 1 - \int_r^t \left( b_{11} \frac{\partial u_{s,t}^{(1)}(\xi, \lambda)}{\partial \lambda_1}|_{\lambda=0^+} + b_{12} \frac{\partial u_{s,t}^{(2)}(\xi, \lambda)}{\partial \lambda_1}|_{\lambda=0^+} \right) ds,
\]
\[
\frac{\partial u_{r,t}^{(2)}(\xi, \lambda)}{\partial \lambda_1}|_{\lambda=0^+} = - \int_r^t \left( b_{21} \frac{\partial u_{s,t}^{(1)}(\xi, \lambda)}{\partial \lambda_1}|_{\lambda=0^+} + b_{22} \frac{\partial u_{s,t}^{(2)}(\xi, \lambda)}{\partial \lambda_1}|_{\lambda=0^+} \right) ds,
\]
\[
\frac{\partial u_{r,t}^{(2)}(\xi, \lambda)}{\partial \lambda_2}|_{\lambda=0^+} = 1 - \int_r^t \left( b_{22} \frac{\partial u_{s,t}^{(2)}(\xi, \lambda)}{\partial \lambda_2}|_{\lambda=0^+} + b_{21} \frac{\partial u_{s,t}^{(1)}(\xi, \lambda)}{\partial \lambda_2}|_{\lambda=0^+} \right) ds,
\]
\[
\frac{\partial u_{r,t}^{(1)}(\xi, \lambda)}{\partial \lambda_2}|_{\lambda=0^+} = - \int_r^t \left( b_{12} \frac{\partial u_{s,t}^{(2)}(\xi, \lambda)}{\partial \lambda_2}|_{\lambda=0^+} + b_{11} \frac{\partial u_{s,t}^{(1)}(\xi, \lambda)}{\partial \lambda_2}|_{\lambda=0^+} \right) ds.
\]

Then, it is not difficult to see,

\[
\pi_1'(r, t) = 1 - \int_r^t \left( b_{11} \pi_1'(s, t) + b_{12} \pi_2'(s, t) \right) ds,
\]
\[
\pi_2'(r, t) = 1 - \int_r^t \left( b_{22} \pi_2'(s, t) + b_{21} \pi_1'(s, t) \right) ds.
\]
For $0 < r < t$, set
\begin{align*}
\pi_1(r, t) &= \mathbf{P} \frac{\partial v_{r, t}^{(1)}(\xi, \lambda)}{\partial \lambda_1}|_{\lambda=0^+} + \mathbf{P} \frac{\partial v_{r, t}^{(1)}(\xi, \lambda)}{\partial \lambda_2}|_{\lambda=0^+}, \\
\pi_2(r, t) &= \mathbf{P} \frac{\partial v_{r, t}^{(2)}(\xi, \lambda)}{\partial \lambda_1}|_{\lambda=0^+} + \mathbf{P} \frac{\partial v_{r, t}^{(2)}(\xi, \lambda)}{\partial \lambda_2}|_{\lambda=0^+}. \tag{17}
\end{align*}

**Lemma 3.1** Suppose that $\int_1^\infty e^{-z} \nu(dz) < \infty$. For $t \geq 0$,
\[ \pi(0, t) = e^{\beta t} \pi'(0, t). \tag{18} \]

**Proof** By Lemma 3.2 in [12], if $\int_1^\infty e^{-z} \nu(dz) < \infty$, then $\mathbf{P} e^{\xi(t)} = e^{\beta t}$ for all $t \geq 0$. And for all $t \geq r \in I$,
\[ \frac{\partial v_{r, t}(\xi, \lambda)}{\partial \lambda}|_{\lambda=0^+} = e^{\xi(t) - \xi(r)} \frac{\partial u_{r, t}(\xi, \lambda)}{\partial \lambda}|_{\lambda=0^+}. \]
Since $\frac{\partial u_{r, t}(\xi, \lambda)}{\partial \lambda}|_{\lambda=0^+} = e^{b(r-t)}$ is not random, where
\[ e^{b(r-t)} := I_{2 \times 2} + b(r-t) + \frac{(r-t)^2}{2!} b^2 + \cdots + \frac{(r-t)^k}{k!} b^k + \cdots, \]
and $I_{2 \times 2}$ is the $2 \times 2$ identity matrix, we get (18). \qed

**Proposition 3.2** Suppose that $\int_{|z| \geq 1} |z| n(dz) < \infty$. Let $(\mathcal{Q}_t)_{t \geq 0}$ be the transition semigroup of a two-type CBIRE process. Then for all $x, y \in \mathbb{R}^2_+$ and $t \geq 0$ we have
\[ |\langle x - y, \pi(0, t) \rangle| \leq W_1(\delta_x \mathcal{Q}_t, \delta_y \mathcal{Q}_t) \leq \sum_{i=1}^2 |x_i - y_i| \pi_i(0, t). \tag{19} \]

**Proof** For $i = 1, 2$, denote $\gamma_i = h_i + \int_{\mathbb{R}^2_+} z_i n(dz)$. Taking derivatives with respect to $\lambda = 0^+$ on both sides of (12), we get,
\[ \int_{\mathbb{R}^2_+} (y_1 + y_2) \mathcal{Q}_t(x, dy) = \langle x, \pi(0, t) \rangle + \int_0^t \langle y, \pi(s, t) \rangle ds. \]
It follows from Theorem 5.10 in [4] that,
\[ W_1(\delta_x \mathcal{Q}_t, \delta_y \mathcal{Q}_t) \geq \int_{\mathbb{R}^2_+} (z_1 + z_2) \left( \mathcal{Q}_t(x, dz) - \mathcal{Q}_t(y, dz) \right) = \langle x - y, \pi(0, t) \rangle. \]
Symmetrically, $W_1(\delta_x \mathcal{Q}_t, \delta_y \mathcal{Q}_t) \geq \langle y - x, \pi(0, t) \rangle$, then the first inequality follows. On the other hand, for $x, y \in \mathbb{R}^2_+$, put $(x - y)_\pm := ((x_1 - y_1)_\pm, (x_2 - y_2)_\pm)$, and
\( x \wedge y := x - (x - y)_+ = y - (x - y)_- \). Let \( Q^t_i(x, y, d\eta_1, d\eta_2) \) be the image of the product measure
\[
Q^t_i(x \wedge y, d\gamma_0)Q^t_i((x - y)_+, d\gamma_1)Q^t_i((x - y)_-, d\gamma_2)
\]
under the mapping \((\gamma_0, \gamma_1, \gamma_2) \mapsto (\eta_1, \eta_2) := (\gamma_0 + \gamma_1, \gamma_0 + \gamma_2)\). Define \( Q_t(x, y, d\eta_1, d\eta_2) \) on \( \mathbb{R}_+^4 \) by
\[
Q_t(x, y, d\eta_1, d\eta_2) = P Q^t_i(x, y, d\eta_1, d\eta_2).
\]

It is not hard to see that \( Q_t(x, y, d\eta_1, d\eta_2) \) is a coupling of \( Q_t(x, d\eta_1) \) and \( Q_t(y, d\eta_2) \).

By (11) and Theorem 1.35 in [14], for \( x \in \mathbb{R}_+^2 \) and \( r \in [0, t] \) there exists \( a^x_{r,t} \in \mathbb{R}_+^2 \) and a finite measure \( (1 \wedge |z|)I^x_{r,t}(dz) \), such that,
\[
\langle x, v_{r,t}(\xi, \lambda) \rangle = \langle a^x_{r,t}, \lambda \rangle + \int_0^t (1 - e^{-\langle \lambda, z \rangle})I^x_{r,t}(dz), \quad \lambda \in \mathbb{R}_+^2,
\]
where \( v_{r,t}(\xi, \lambda) \) is defined by (10) with \( x \) replaced by a càdlàg function \( \zeta = \{ \xi(t) : t \in \mathbb{R} \} \). By (6), (21) and Theorem 1.37 in [14], for \( r \in [0, t] \), there exists \( A^x_{r,t} \in \mathbb{R}_+^2 \) and a finite measure \( (1 \wedge |z|)L_{r,t}(dz) \), such that,
\[
J_{r,t}(\xi, \lambda) := \int_r^t \psi(v_{s,t}(\xi, \lambda))ds = \langle A^x_{r,t}, \lambda \rangle
\]
\[
+ \int_{\mathbb{R}_+^2 \setminus \{0\}} (1 - e^{-\langle \lambda, z \rangle})L_{r,t}(dz), \quad \lambda \in \mathbb{R}_+^2.
\]

In fact, by (6), (10) and taking \( x = h \) in (21), we have,
\[
\psi(v_{s,t}(\xi, \lambda)) = \langle h, v_{s,t}(\xi, \lambda) \rangle + \int_{\mathbb{R}_+^2 \setminus \{0\}} (1 - e^{-\langle v_{s,t}(\xi, \lambda), y \rangle})n(dy)
\]
\[
= \langle a^h_{s,t}, \lambda \rangle + \int_{\mathbb{R}_+^2 \setminus \{0\}} (1 - e^{-\langle v_{s,t}(\xi, \lambda), y \rangle})I^h_{s,t}(dz)
\]
\[
+ \int_{\mathbb{R}_+^2 \setminus \{0\}} (1 - e^{-\langle v_{s,t}(\xi, \lambda), y \rangle})n(dy)
\]
\[
= \langle a^h_{s,t}, \lambda \rangle + \int_{\mathbb{R}_+^2 \setminus \{0\}} (1 - e^{-\langle v_{s,t}(\xi, \lambda), y \rangle})I^h_{s,t}(dz)
\]
\[
+ \int_{\mathbb{R}_+^2 \setminus \{0\}} n(dy) \int_{\mathbb{R}_+^2 \setminus \{0\}} (1 - e^{-\langle \lambda, z \rangle})Q^\xi_{s,t}(y, dz).
\]

For \( a_s = (a^{(1)}_s, a^{(2)}_s) \in \mathbb{R}^2 \), we denote \( \int_r^t a_s ds := (\int_r^t a^{(1)}_s ds, \int_r^t a^{(2)}_s ds) \) for simplicity. Integrating (23) over \((r, t]\), we get,
\[
A_{r,t} = \int_r^t a^h_{s,t} ds,
\]
\[ L_{r,t}(dz) = \int_r^t \left[ \mathbb{E}_x^t (dz) + \int_{\mathbb{R}_+^2 \setminus \{0\}} \mathcal{Q}^\xi_{s,t}(y, dz)n(dy) \right] ds. \]

Following Theorem 1.35 in [14], for a càdlàg function \( \xi \) and \( r \in [0, t] \), we can define an infinitely divisible measure \( \Upsilon_{r,t}^\xi \) on \( \mathbb{R}_+^2 \) by

\[ \int_{\mathbb{R}_+^2} e^{-(\lambda, y)} \Upsilon_{r,t}^\xi(dy) = J_{r,t}(\xi, \lambda). \] (24)

It is easy to verify that

\[ J_{r,t}(\xi, \lambda) = J_{r,s}(\xi, v_{s,t}(\xi, \lambda)) + J_{s,t}(\xi, \lambda). \]

When the function \( \xi \) is reduced to the case \( \xi(t) \equiv 0 \), \( (v_{r,t}(\xi, \cdot))_{r \leq t} \) goes back to the cumulant semigroup of a classical CB-process, and \( (\Upsilon_{r,t}^\xi)_{r \leq t} \) goes back to the skew convolution semigroup of a classical CBI process.

Let \( (\Upsilon_{r,t}^\xi)_{r \leq t} \) be the random skew convolution semigroup associated with \( (\mathcal{Q}^\xi_{r,t})_{r \leq t} \) defined by (24) with \( \xi = \xi \). Let \( \mathcal{Q}^\xi_{r,t}(x, \cdot) := \mathcal{Q}^\xi_{r,t}(x, \cdot) \ast \Upsilon_{r,t}^\xi, r \leq t \). It is clear that for \( x \in \mathbb{R}_+^2, t \geq 0 \),

\[ \bar{\mathcal{Q}}_t(x, \cdot) = \mathcal{P} \left[ \mathcal{Q}^\xi_{0,t}(x, \cdot) \right]. \] (25)

For more details on skew convolution semigroups, see Chapter 9 in [14]. Let \( \bar{\mathcal{Q}}_t^\xi(x, y, d\sigma_1, d\sigma_2) \) be the image of \( \Upsilon_t(d\eta_0) \mathcal{Q}^\xi_t(x, y, d\eta_1, d\eta_2) \) under the mapping \( (\eta_0, \eta_1, \eta_2) \mapsto (\sigma_1, \sigma_2) = (\eta_0 + \eta_1, \eta_0 + \eta_2) \). Define \( \mathcal{Q}_t(x, y, d\sigma_1, d\sigma_2) \) on \( \mathbb{R}_+^2 \) by

\[ \bar{\mathcal{Q}}_1(x, y, d\sigma_1, d\sigma_2) := \mathcal{P} \mathcal{Q}^\xi_1(x, y, d\sigma_1, d\sigma_2), \] (26)

Then by the relation (25), we can verify that \( \bar{\mathcal{Q}}_t(x, y, d\sigma_1, d\sigma_2) \) is the coupling measure of \( \bar{\mathcal{Q}}_t(x, \sigma_1) \) and \( \bar{\mathcal{Q}}_t(y, d\sigma_2) \). For similar construction in the setting of measure-valued processes, see [16]. Finally,

\[ W_1(\delta_x \bar{\mathcal{Q}}_t, \delta_y \bar{\mathcal{Q}}_t) \leq \int_{\mathbb{R}_+^2 \times \mathbb{R}_+^2} |\sigma_1 - \sigma_2| \bar{\mathcal{Q}}_t(x, y, d\sigma_1, d\sigma_2) \]
\[ \leq \mathcal{P} \left( \int_{\mathbb{R}_+^2} (\gamma_1 + \gamma_2) \mathcal{Q}^\xi_t((x - y)_+, d\gamma_1) + \int_{\mathbb{R}_+^2} (\gamma_2 + \gamma_2) \mathcal{Q}^\xi_t((y - x)_-, d\gamma_2) \right) \]
\[ = \mathcal{P} \left( \int_{\mathbb{R}_+^2} (\xi_1 + \xi_2) \mathcal{Q}^\xi_t(\{ |x_1 - y_1|, |x_2 - y_2| \}, d\xi) \right) \]
\[ = \int_{\mathbb{R}_+^2} (\xi_1 + \xi_2) \mathcal{Q}_t(\{ |x_1 - y_1|, |x_2 - y_2| \}, d\xi) \]
\[ = \sum_{i=1}^2 |x_i - y_i| \pi_i(0, t), \] (27)
where the first equality comes from the conditioned branching property. \(\square\)

We are now in a position to prove our first main result.

**Proof of Theorem 2.1** Because \(\mathbf{P}[\xi(1)] < \beta < \frac{1}{2} (tr(b) - \sqrt{\Delta})\), and \(\frac{1}{2} (tr(b) - \sqrt{\Delta})\) is the smallest eigenvalue of \(b\), we may adjust the parameters so that \(\mathbf{P}[\xi(1)] < 0\), and the eigenvalues of \(b\) are strictly positive. According to Eq. (10–11) in [22], this adjustment is without loss of generality. Thus, by the result of Theorem 2.2 in [22], for \(x \in \mathbb{R}_+^d\), \(Q_t(x, \cdot)\) converges weakly to \(\mu\) as \(t \to \infty\).

By assumption, the equation

\[
\lambda^2 + (b_{11} + b_{22})\lambda + b_{11}b_{22} - b_{12}b_{21} = 0
\]

has two different solutions on \(\mathbb{R}_-\), which we denote by \(\lambda_1\) and \(\lambda_2\). Recall that \(\Delta = (b_{22} - b_{11})^2 + 4b_{12}b_{21} > 0\), we have \(\lambda_1 = \frac{1}{2}(-b_{11} - b_{22} + \sqrt{\Delta}), \lambda_2 = \lambda_1 - \sqrt{\Delta}\). Note that (3) implies \(b_{12}, b_{21} \leq 0\).

**Case 1** \(b_{12}b_{21} = 0\). It is straightforward to show that

\[
\pi_1'(0, t) = e^{-b_{11}t}, \quad \pi_2'(0, t) = e^{-b_{22}t}.
\]

**Case 2** \(b_{12}b_{21} \neq 0\). Some simple calculations yield

\[
\pi_1'(0, t) = \frac{b_{12}(b_{22} - b_{11} + 2b_{21} - \sqrt{\Delta})}{\sqrt{\Delta}(-\sqrt{\Delta} + b_{11} - b_{22})} e^{\lambda_1t} - \frac{b_{12}(\sqrt{\Delta} + b_{22} - b_{11} + 2b_{21})}{\sqrt{\Delta}(\sqrt{\Delta} + b_{11} - b_{22})} e^{\lambda_2t},
\]

\[
\pi_2'(0, t) = \frac{b_{11} - b_{22} - 2b_{21} + \sqrt{\Delta}}{2\sqrt{\Delta}} e^{\lambda_1t} + \frac{\sqrt{\Delta} + b_{22} - b_{11} + 2b_{21}}{2\sqrt{\Delta}} e^{\lambda_2t}.
\]

If \(\epsilon := \sqrt{\Delta} + b_{22} - b_{11} + 2b_{21} < 0\), then there exist \(\theta_{11} = -\frac{b_{12}(2\sqrt{\Delta} - \epsilon)}{\sqrt{\Delta}(\sqrt{\Delta} + b_{11} - b_{22})} \in (0, 1)\) and \(\theta_{12} = \frac{2\sqrt{\Delta} - \epsilon}{2\sqrt{\Delta}} > 0\) such that

\[
\pi_1'(0, t) = \theta_{11} e^{\lambda_1t} + (1 - \theta_{11}) e^{\lambda_2t}, \quad \pi_2'(0, t) \leq \theta_{12} e^{\lambda_1t} + e^{\lambda_2t}.
\]

If \(\epsilon > 0\), there exist \(\theta_{21} = -\frac{b_{12}(2\sqrt{\Delta} - \epsilon)}{\sqrt{\Delta}(\sqrt{\Delta} + b_{11} - b_{22})} > 0\) and \(\theta_{22} = \frac{2\sqrt{\Delta} - \epsilon}{2\sqrt{\Delta}} \in (0, 1)\) such that

\[
\pi_1'(0, t) \leq \theta_{21} e^{\lambda_1t} + e^{\lambda_2t}, \quad \pi_2'(0, t) = \theta_{22} e^{\lambda_1t} + (1 - \theta_{22}) e^{\lambda_2t}.
\]

If \(\epsilon = 0\), then \(\pi_1'(0, t) = \pi_2'(0, t) = e^{\lambda_1t}\).

Then, for all \(t \geq 0\) and \(x, y \in \mathbb{R}_+^d\), there exists \(\theta = \max\{\theta_{12}, \theta_{21}\}\) such that

\[
\sum_{i=1}^{2} |x_i - y_i| \pi'_i(0, t) \leq d^\theta(x, y) e^{\lambda_1t}.
\]  

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Notice that under the assumption $\int_{|z|\geq 1} |z|^n \, n(\mathrm{d}z) < \infty$ and $\beta < \frac{1}{2} (tr(b) - \sqrt{\Delta})$, the limiting distribution $\mu$ has finite expectation. Therefore, $W_1(\delta_x \tilde{Q}_t, \mu), W_{d^\theta}(\delta_x, \mu)$ are well-defined. By Equs. (18), (19), (28) and the convexity of the Wasserstein distance, we obtain,

$$W_1(\delta_x \tilde{Q}_t, \mu) \leq \int_{\mathbb{R}_+^4} W_1(\delta_x \tilde{Q}_t, \delta_y \tilde{Q}_t) \, H(\mathrm{d}x, \mathrm{d}y)$$

$$\leq e^{(\beta+\lambda_1)t} \int_{\mathbb{R}_+^4} W_{d^\theta}(x, y) \, H(\mathrm{d}x, \mathrm{d}y)$$

$$= e^{(\beta+\lambda_1)t} W_{d^\theta}(\delta_x, \mu),$$

where $H$ is the optimal coupling measure of $(\delta_x, \mu)$; see for instance, Chapter 5 in [4]. Observing that $\lambda_1 = -\frac{1}{2} (tr(b) - \sqrt{\Delta})$, we arrive at the conclusion by setting $\rho = \frac{1}{2} (tr(b) - \sqrt{\Delta}) - \beta > 0$. □

**Corollary 3.3** Suppose that $\int_1^\infty e^z \nu(\mathrm{d}z) < \infty$, $\beta < \frac{1}{2} (tr(b) - \sqrt{\Delta})$. Then there exist $\rho > 0, \theta > 0$ such that for any $x \in \mathbb{R}_+^2$ and $t \geq 0$,

$$W_1(\delta_x Q_t, \delta_0) \leq W_{d^\theta}(\delta_x, \delta_0) e^{-\rho t}.$$

### 3.2 Ergodicity in the Total Variation Distance

In this section, we prove the ergodicity in the total variation distance. The key to the proof is the finiteness of the random cumulant semigroup. And we deal with it using the tool of Dawson–Watanabe superprocess. For the sake of completeness, we make a brief introduction to the inhomogeneous superprocesses, which is mainly summarized from [14]. Throughout this section, we use the notation that for any topological space $S$, $B(S)$ is the Banach space of bounded Borel functions on $S$ endowed with the supremum norm, and $B(S)^+$ is the subset of positive elements of $B(S)$. Moreover, $M(S)$ denotes the space of finite Borel measures on $S$ and $M(S)^0$ stands for $M(S)$ subtracting the null measure.

Suppose that $\tilde{E}$ is a Borel subset of $I \times E$, where $I \subset \mathbb{R}_+$ is an interval and $E$ is a Lusin topological space. $(P_{r,t} : t \geq r \in I)$ is an inhomogeneous Borel right transition semigroup with global state space $\tilde{E}$. The system $\Pi = (\Omega, \mathcal{F}, \mathcal{F}_{r,t}, \Pi_t, \mathcal{P}_{r,x})$ is a right continuous inhomogeneous Markov process realizing $(P_{r,t} : t \geq r \in I)$. For any $t \in I$, let $I_t = [0, t] \cap I$ and $E_t = \{x \in E : (t, x) \in \tilde{E}\}$. According to Theorem 6.10 in [14], for every $t \in I$ and $f \in B(E_t)^+$, there is a unique bounded positive solution $(r, x) \mapsto v_{r,t}(x) = V_{r,t} f(x)$ to the integral equation.

$$v_{r,t}(x) = P_{r,x} f(\Pi_t) - \int_r^t P_{r,x} [\Phi(s, \Pi_s, v_{s,t})] \, \mathrm{d}s, \quad r \in I_t, x \in E_r.$$  (29)
In the equation, $\Phi$ is an inhomogeneous branching mechanism defined as

$$
\Phi(s, x, f) = b(s, x)f(x) + c(s, x)f(x)^2 - \int_{E_s} f(y)g(s, x, dy) + \int_{M(E_s)^{\circ}} [e^{-v(f)} - 1 + v(f)]H(s, x, dv), (s, x) \in \tilde{E}, f \in B(E_s)^{+},
$$

(30)

where $b \in B(E)$ and $c \in B(E)^{+}$, $g(s, x, dy)$ is a bounded kernel on $\tilde{E}$. And $H(s, x, dv)$ is a $\sigma$-finite kernel from $\tilde{E}$ to $M(\tilde{E})^\circ$. For every $(s, x) \in \tilde{E}$, we assume that $g(s, x, dy)$ is supported by $\{s\} \times E_s$ and $H(s, x, dv)$ is supported by $M(\{s\} \times E_s)^{\circ}$. Then $g(s, x, dv)$ and $H(s, x, dv)$ can be seen as measures on $E_s$ and $M(E_s)^{\circ}$, respectively. In addition, we assume

$$
\sup_{(s, x) \in \tilde{E}} [b(s, x)| + c(s, x)| + g(s, x, E_s) + \int_{M(E_s)^{\circ}} [v(1) \wedge v(1)^2 + v(1)]H(s, x, dv)] < \infty,
$$

where $v_x(dy)$ denotes the restriction of $v(dy)$ to $E_s \setminus \{x\}$. A Dawson–Watanabe superprocess with spatial motion $\Pi$ and branching mechanism $\Phi$ is a Markov process in $M(E)$ with transition semigroup $(Q_{r,t})_{t \geq r \geq 0}$, defined by

$$
\int_{M(E_t)} e^{-v(f)}Q_{r,t}(\mu, dv) = \exp\{-\mu(V_{r,t}f)\}, \quad f \in B(E_t)^{+}.
$$

In this case, we can rewrite (29) into

$$
v_{r,t}(x) = P_{r,t}f(x) - \int_{0}^{t} ds \int_{E} \Phi(s, y, v_{s,t})P_{t-s}(x, dy), \quad x \in E, t \geq r \geq 0.
$$

(31)

Following by similar arguments to the proof of Corollary 5.18 in [14] in inhomogeneous setting, we get the comparison theorem.

**Proposition 3.4** Suppose that $\Phi_1$ and $\Phi_2$ are two branching mechanisms given by (30) satisfying $\Phi_1(x, f) \geq \Phi_2(x, f)$ for all $x \in E$ and $f \in B(E)^{+}$. Let $(t, x) \mapsto v_{i}(t, x)$ be the solution of (31) with $\Phi$ replaced by $\Phi_i$, for $i = 1, 2$. Then $v_{1}(t, x) \leq v_{2}(t, x)$ for all $t \geq 0$ and $x \in E$.

In our model of two-type continuous-state branching processes in Lévy random environments, $E = \{1, 2\}$, and the parameters in (30) take the particular form

$$
b(1) = b_{11}, g(s, 1, f) = \int_{[1,2]} f(y)g(s, 1, dy) = -b_{12}f(2),
$$

$$
b(2) = b_{22}, g(s, 2, f) = \int_{[1,2]} f(y)g(s, 2, dy) = -b_{21}f(1).
$$

(32)
Since $g$ is homogeneous here, we write $g(s, i, f) = g(i, f)$ for simplicity. Furthermore, take $\Phi(x, s, f) = e^{\xi(s)} \varphi(e^{-\xi(s)} f)$, and $\Pi(t) \equiv \Pi(0)$, $t \in I$, we get (8). Note that the domain of $f$ is exactly $\{1, 2\}$, so we sometimes use $\lambda \in \mathbb{R}_+$ instead of $f$ by letting $\lambda_i = f(i)$, $i = 1, 2$.

Define another branching mechanism $\tilde{\varphi} = (\tilde{\varphi}_1, \tilde{\varphi}_2)$ as a function from $\mathbb{R}_+^2$ to itself with the following representations,

\[
\tilde{\varphi}_1(\lambda) = b_{11}\lambda_1 + b_{12}\lambda_2 + c_1\lambda_1^2 + \int_{\mathbb{R}_+^2} (e^{-z_1\lambda_1} - 1 + z_1\lambda_1)m_1(\text{d}z),
\]

\[
\tilde{\varphi}_2(\lambda) = b_{21}\lambda_1 + b_{22}\lambda_2 + c_2\lambda_2^2 + \int_{\mathbb{R}_+^2} (e^{-z_2\lambda_2} - 1 + z_2\lambda_2)m_2(\text{d}z).
\]

Similarly, there exists $r \mapsto \tilde{U}_{r,t}(\xi, \lambda) \in \mathbb{R}_+^2$ as the unique positive strong solution to

\[
\tilde{U}_{r,t}^{(i)}(\xi, \lambda) = \lambda_i - \int_r^t e^{\xi(s)} \tilde{\varphi}_i(e^{-\xi(s)} \tilde{U}_{s,t}(\xi, \lambda)) \text{d}s, \quad i = 1, 2, \quad r \in I \cap (-\infty, t].
\]

The local projection of $\varphi$ defined by (4)-(5) is the function $\varphi^*$ from $\mathbb{R}_+$ to $\mathbb{R}_+^2$ defined by

\[
\varphi^*_1(x) = (b_{11} + b_{12})x + c_1x^2 + \int_{\mathbb{R}_+^2} (e^{-xz_1} - 1 + xz_1)m_1(\text{d}z),
\]

\[
\varphi^*_2(x) = (b_{21} + b_{22})x + c_2x^2 + \int_{\mathbb{R}_+^2} (e^{-xz_2} - 1 + xz_2)m_2(\text{d}z).
\]

**Condition 3.5** For $x = 1, 2$, $\varphi^*_x \geq \varphi(z)$, where $\varphi$ is a branching mechanism of a single-type continuous-state continuous-time branching process in Lévy random environment satisfying $\int_0^\infty \varphi(z)^{-1} \text{d}z < \infty$. The corresponding random cumulant semigroup is $w_{r,t}$ given by

\[
w_{r,t}(\xi, \lambda) = \lambda - \int_r^t e^{\xi(s)} \varphi(e^{-\xi(s)} w_{s,t}(\xi, \lambda)) \text{d}s, \quad r \in I \cap (-\infty, t].
\]

**Theorem 3.6** Suppose that Condition 3.5 is satisfied. Then

\[
u_{r,t}^{(i)}(\xi, \lambda) \leq w_{r,t}(\xi, |\lambda|), \quad \text{for any } i = 1, 2, \lambda \in \mathbb{R}_+^2.
\]

Furthermore, if $\liminf_{t \to \infty} \tilde{\varphi}(t) = -\infty$, then $\lim_{t \to \infty} \tilde{v}_{0,t}^{\xi} = 0$, $P$-a.s., where $u_{r,t}$ and $v_{r,t}$ are defined by (8) and (9), $\tilde{v}_{0,t}^{\xi} := \lim_{\lambda \to \infty} v_{0,t}(\xi, \lambda)$.

**Proof** By Proposition 2.9 in [14], Equation (35) can be rewritten into

\[
\tilde{U}_{r,t}^{(1)}(\xi, \lambda) = e^{b_{12}(t-r)}\lambda_1 - \int_r^t e^{b_{12}(s-r)} \left[ e^{\xi(s)} \tilde{\varphi}_1(e^{-\xi(s)} \tilde{U}_{s,t}(\xi, \lambda)) + b_{12} \tilde{U}_{s,t}^{(1)}(\xi, \lambda) \right] \text{d}s,
\]
\[U^{(2)}_{r,t}(\xi, \lambda) = e^{b_{21}(t-r)} \lambda_2 - \int_r^t e^{b_{21}(s-r)} \left[ e^{\xi(s)} \tilde{\phi}_2(e^{-\xi(s)} \tilde{U}_{s,t}(\xi, \lambda)) + b_{21} \tilde{U}^{(2)}_{s,t}(\xi, \lambda) \right] ds.\]

It is clear that
\[\tilde{\phi}_1(\lambda) = \phi^*_1(\lambda_1) - b_{12} \lambda_1 + b_{12} \lambda_2,\]
\[\tilde{\phi}_2(\lambda) = \phi^*_2(\lambda_2) - b_{21} \lambda_2 + b_{21} \lambda_1.\]

Hence,
\[\tilde{U}^{(1)}_{r,t}(\xi, \lambda) = e^{b_{12}(t-r)} \lambda_1 - \int_r^t e^{b_{12}(s-r)} \left[ e^{\xi(s)} \phi^*_1(e^{-\xi(s)} \tilde{U}^{(1)}_{s,t}(\xi, \lambda)) + b_{12} \tilde{U}^{(2)}_{s,t}(\xi, \lambda) \right] ds,\]
\[\tilde{U}^{(2)}_{r,t}(\xi, \lambda) = e^{b_{21}(t-r)} \lambda_2 - \int_r^t e^{b_{21}(s-r)} \left[ e^{\xi(s)} \phi^*_2(e^{-\xi(s)} \tilde{U}^{(2)}_{s,t}(\xi, \lambda)) + b_{21} \tilde{U}^{(1)}_{s,t}(\xi, \lambda) \right] ds.\]

Define
\[P^g_{r,t} f (1) = e^{b_{12}(t-r)} f (1), \quad P^g_{r,t} f (2) = e^{b_{21}(t-r)} f (2),\]
and use the relation \(\lambda_i = f (i), \ i = 1, 2,\) the above equations can be rewritten into
\[\tilde{U}^{(i)}_{r,t}(\xi, \lambda) = P^g_{r,t} f (i) - \int_r^t ds \int_{[1,2]} \left[ e^{\xi(s)} \phi^*_1(e^{-\xi(s)} \tilde{U}^{(1)}_{s,t}(\xi, \lambda)) - g(i, \tilde{U}^{(1)}_{s,t}(\xi, \lambda)) \right] P^g_{s,t} (i, dy). \quad (38)\]

Let \((\tilde{P}_{r,t} : t \geq r \geq 0)\) be an inhomogeneous Borel right semigroup defined by
\[\tilde{P}_{r,t} f (x) = P^g_{r,t} f (x) + \int_r^t ds \int_E g(y, \tilde{P}^{g s}_{s,t}) P^g_{s,t} (x, dy), \quad (39)\]
where \(g(s, y, f)\) is defined by \((32)\). We claim that the unique solution of \((39)\) is given by
\[\tilde{P}_{r,t} f (x) = P^g_{r,t} f (x) + \sum_{k=1}^{\infty} \int_r^t ds_1 \int_{s_1}^t ds_2 \cdots \int_{s_{k-1}}^t P^g_{r,s_{k-1}} g P^g_{s_{k-1},s_k} \cdots g P^g_{s_{k-1},s_k} f (x) ds_k. \quad (40)\]

Indeed, the \(i\)-th term of the series in \((40)\) is bounded by \(K_{r,t}\), where \(K_{r,t} := -[b_{12}(t-r) e^{b_{12}(t-r)} \wedge b_{21}(t-r) e^{b_{21}(t-r)}]\). Then the series converges uniformly on \([0, \infty) \times [1, 2]\). For the uniqueness of the solution, suppose that \((r, x) \mapsto z_{r,t}(x)\) is a locally bounded solution of \((39)\) with \(z_{r,t}(x) \equiv 0\). Let \(\| \cdot \|\) denote the supremum norm of functions, then
\[\|z_{r,t}\| \leq K_{r,t} \sup_{r \leq s \leq t} \|z_{s,t}\|.\]
Notice that $K < 1$, we have $\|z_{r,t}\| = 0$ for $0 \leq r \leq t$. That gives the uniqueness and hence the unique solution of (39) is given by (40).

For convenience, we make the notation that $\tilde{U}_{r,t}(\xi, f)(i) = \tilde{U}_{r,t}^{(i)}(\xi, f) = \tilde{U}_{r,t} f(i)$, for $i = 1, 2$. From Eqs. (39) and (38) it follows that,

$$
\tilde{U}_{r,t}(\xi, f)(1) = \tilde{P}_{r,t} f(1) - \int_t^r \int_{s_1}^t \cdots \int_{s_{k-1}}^t \left[ e^{\xi(s)} \phi_1^*(e^{-\xi(s)} \tilde{U}_{s_i,t} f(1)) \right] ds_1 \cdots ds_{k-1} ds_t \geq 0, \quad \Delta_t = \sup_{i < j \leq n} |\Delta_i|.
$$

Using the above relation successively,

$$
\tilde{U}_{r,t}(\xi, f)(1) = \tilde{P}_{r,t} f(1) - \int_t^r \int_{s_1}^t \cdots \int_{s_{k-1}}^t \left[ e^{\xi(s)} \phi_1^*(e^{-\xi(s)} \tilde{U}_{s_i,t} f(1)) \right] ds_1 \cdots ds_{k-1} ds_t \geq 0, \quad \Delta_t = \sup_{i < j \leq n} |\Delta_i|.
$$

where

$$
\varepsilon_n(r, t, 1) = \int_t^r ds_1 \int_{s_1}^t \cdots \int_{s_{k-1}}^t \left[ e^{\xi(s)} \phi_1^*(e^{-\xi(s)} \tilde{U}_{s_i,t} f(1)) \right] ds_1 \cdots ds_{k-1} ds_t \geq 0, \quad \Delta_t = \sup_{i < j \leq n} |\Delta_i|.
$$

By Proposition 5.1 in [22], there exist constants $C_1 > 0$, $C_2 > 0$, such that $\|\tilde{U}_{r,t} f\| \leq C_1 \|f\| e^{-C_2(t-r)}$. Therefore,

$$
\|\varepsilon_n(r, t, \cdot)\| \leq (1 + C_1 e^{-C_2(t-r)}) \|f\| \|g(\cdot, 1)\| \int_t^r ds_1 \int_{s_1}^t \cdots \int_{s_{n-1}}^t ds_{n-1} \quad \Delta_t = \sup_{i < j \leq n} |\Delta_i|.
$$

Letting $n \to \infty$ and using (40) we get,

$$
\tilde{U}_{r,t} f(x) = \tilde{P}_{r,t} f(x) - \int_t^r \int_{s_1}^t \cdots \int_{s_{k-1}}^t \left[ e^{\xi(s)} \phi_1^*(e^{-\xi(s)} \tilde{U}_{s_i,t} f(1)) \right] ds_1 \cdots ds_{k-1} ds_t \geq 0, \quad \Delta_t = \sup_{i < j \leq n} |\Delta_i|.
$$

Hence, we may see $(\tilde{U}_{r,t})_{t \geq r \geq 0}$ as the cumulant semigroup of a Dawson–Watanabe superprocess with branching mechanism $\phi*$ and underlying transition semigroup.
$(\tilde{P}_{r,t})_{t \geq r \geq 0}$. By Proposition 3.4 we have,

$$\tilde{U}_{r,t} f(x) \leq \tilde{U}_{r,t} \|f\| \leq w_{r,t}(\xi, \|f\|).$$

Similarly, $u_{r,t}^{(i)}(\xi, \lambda) \leq \tilde{U}_{r,t}^{(i)}(\xi, \lambda)$ for $i = 1, 2, \lambda \in \mathbb{R}_+^2$, since $\tilde{\phi}_i(\lambda) \leq \phi_i(\lambda)$.

Furthermore, if $\lim \inf \xi(t) = -\infty$, according to Corollary 4.4 in [11], we have

$$\lim_{t \to \infty} \lim_{\lambda \to \infty} w_{r,t}(\xi, \lambda) = 0, \text{ P-a.s.} \quad \text{Thus, } \lim_{t \to \infty} |\bar{v}^\xi_{0,t}| = 0, \text{ P-a.s.} \hspace{1cm} \square$$

We are now in a position to prove our second main result.

**Proof of Theorem 2.2** Because $P[\xi(1)] < \beta < \frac{1}{2}(tr(b) - \sqrt{\Delta})$, and $\frac{1}{2}(tr(b) - \sqrt{\Delta})$ is the smallest eigenvalue of $b$, we may adjust the parameters so that $P[\xi(1)] < 0$, and the eigenvalues of $b$ are strictly positive. Thus, $\xi(t) \to \infty$, as $t \to \infty$. By Theorem 3.6, $w_{r,t}(\xi, \lambda) < \infty$, P-a.s. Let $\{X_t(x) : t \geq 0\}$ be the two-type CBIRE process with transition semigroup $\tilde{Q}_t$ and initial value $x$. As in the proof of Theorem 3.2 in [8], one can show that for $y \geq x \geq 0$ and $t \geq 0$, we have $X_t(y) \geq X_t(x)$, P-a.s. and the process $\{X_t(y) - X_t(x) : t \geq 0\}$ is identically distributed with $\{X_t(y-x) : t \geq 0\}$. Let $T(x, y) = \inf\{t \geq 0 : X_t(x) = X_t(y)\} = \inf\{t \geq 0 : X_t(y) - X_t(x) = 0\}$. Then $X_t(x) = X_t(y)$, P-a.s. for $t \geq T(x, y)$. Thus, for any bounded Borel function $f$ on $\mathbb{R}_+^2$ and any $t \geq 0$,

$$|\tilde{Q}_t f(x) - \tilde{Q}_t f(y)| \leq P[f(X_t(x)) - f(X_t(y))|1_{\{T(x,y) > t\}}$$

$$\leq 2\|f\|P[T(x, y) > t]$$

$$= 2\|f\|P[1 - \exp\{-((y - x), \bar{v}_{0,t}(\xi))\}]$$

$$\leq 2\|f\|P\min[1, |(y - x), \bar{v}_{0,t}(\xi)|]]. \hspace{1cm} (42)$$

Let $H$ be any coupling of $(\delta_x, \mu)$. By convexity of the Wasserstein distance, we obtain

$$\|\delta_x \tilde{Q}_t - \mu\|_{TV} \leq \int_{\mathbb{R}_+^2} \|\tilde{Q}_t(x, \cdot) - \tilde{Q}_t(y, \cdot)\|_{TV} H(dx, dy)$$

$$\leq 2 \int_{\mathbb{R}_+^2} P[1 - \exp\{-((y - x), \bar{v}_{0,t}(\xi))\}]H(dx, dy). \hspace{1cm} (43)$$

Since $\lim \inf \xi(t) = -\infty$, $\lim_{t \to \infty} |\bar{v}^\xi_{0,t}| = 0$ according to Theorem 3.6. By dominated convergence,

$$\lim_{t \to \infty} \|\delta_x \tilde{Q}_t - \mu\|_{TV} = 0.$$

By choosing $H$ as the optimal coupling of $(\delta_x, \mu)$ with respect to $W_1$ and using

$$\int_{\mathbb{R}_+^2} P[\min(1, |x - y|, |\bar{v}^\xi_{0,t}|)]H(dx, dy) \leq P[|\bar{v}^\xi_{0,t}|] \int_{\mathbb{R}_+^2} |x - y|H(dx, dy),$$

we finish the proof. \hspace{1cm} \square
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Declarations

Conflict of interest  The authors have no conflicts of interest to declare. All co-authors have seen and agree with the contents of the manuscript. The data that support the findings of this study are available on request from the corresponding author.

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