Directed polymer near a hard wall and KPZ equation in the half-space

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Abstract – We study the directed polymer with fixed endpoints near an absorbing wall, in the continuum and in the presence of disorder, equivalent to the KPZ equation on the half-space with droplet initial conditions. From a Bethe Ansatz solution of the equivalent attractive boson model we obtain the exact expression for the free-energy distribution at all times. It converges at large time to the Tracy-Widom distribution $F_4$ of the Gaussian symplectic ensemble (GSE). We compare our results with numerical simulations of the lattice directed polymer, both at zero and high temperature.

Progress was achieved recently in finding exact solutions in one dimension for noisy growth models in the Kardar-Parisi-Zhang (KPZ) universality class [1,2], and for the closely related equilibrium statistical-mechanics problem of the directed polymer (DP) with quenched disorder [3]. The KPZ class was explored in several recent experiments [4,5], and applications to the DP range from biophysics [6] to describe the glass phase of pinned vortex lines [7] and magnetic walls [8]. The height of the growing interface, $h(x,t)$, corresponds to the free energy of a DP of length $t$ starting at $x$, under a mapping which is exact in the continuum (Cole-Hopf), and for some discrete realizations. Not only the scaling exponents $h \sim t^{1/3}, x \sim t^{2/3}$ are known [9,10], but also the one-point (and in some cases the many-point) probability distribution (PDF) of the height $[11,12]$. Their dependence on the initial condition exhibits remarkable universality at large time, with only a few subclasses, most being related to Tracy-Widom (TW) distributions $[13]$ of the largest eigenvalues of random matrices. Most of these subclasses were initially discovered in a discrete growth model (the PNG model) $[14–16]$ which can be mapped onto the statistics of random permutations $[17]$, and a zero-temperature lattice DP model $[10]$. Recently, exact solutions were obtained directly in the continuum at arbitrary time $t$, for the droplet $[18–21]$, flat [22,23] and stationary [24] initial conditions. The PDF of the height $h(x,t)$ converges at large time to $F_2$, the Gaussian unitary ensemble (GUE), and to $F_1$, the Gaussian orthogonal ensemble (GOE) universal TW distributions, for droplet and flat initial conditions, respectively. One useful method which led to these solutions introduces $n$ replicas and maps the DP problem to the Lieb Liniger model, i.e., the quantum mechanics of $n$ bosons with delta-function attraction, a model solved using the Bethe Ansatz.

The KPZ equation on the half-line $x > 0$, equivalently a DP in the presence of a wall, is also of great interest. In the statistical-mechanics context, constrained fluctuations are important for the study of fluctuation-induced (Casimir) forces $[25,26]$ and for extreme value statistics. In the surface growth context one can study an interface pinned at a point, or an average growth rate which jumps across a boundary. The half-space problem was previously studied in a discrete version, for the (symmetrized) random permutations/PNG model $[27,28]$ and found to involve also TW distributions in the limit of large system size. In order to exhibit full KPZ universality, it is important to solve the problem directly in the continuum, i.e., for the KPZ equation itself. Previous approaches did not address the finite time behavior which is also universal$^1$.

The aim of this letter is to present a solution of the directed polymer problem in the continuum in the presence of a hard wall (absorbing wall) using the Bethe Ansatz (BA). Equivalently, we obtain the one-point height probability distribution for the KPZ equation on the

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$^1$In the large diffusivity, weak noise regime, equivalently high temperature regime for the DP, see below.
half-line $x > 0$ with fixed large negative value of $h$ or of $- \nabla h$ (i.e., a small contact angle) at $x = 0$. For simplicity we study a DP with both endpoints fixed—which corresponds to the droplet initial condition in KPZ—near the wall. We do not consider the case of the attractive wall (we briefly mention it at the end). We obtain an exact expression for the generating function of the moments of the DP partition sum as a Fredholm Pfaffian, from which we extract the PDF of the free energy of the DP (height of KPZ) at all times. We then show that this PDF converges to $F_\delta$, the Tracy-Widom distribution of the largest eigenvalue of the Gaussian symplectic ensemble (GSE). Calculations are performed for the DP, consequences for the KPZ equation are detailed at the end. Our results are checked against numerics on a discrete DP model, both at high and zero temperature, confirming universality. Consequences for extreme value statistics are discussed. This is the first occurrence of the $F_\delta$ distribution and of the GSE within a continuum BA calculation. It is consistent with the results of [27,28] for the discrete model and confirms that these belong to the same universality class as the continuum KPZ equation on the half-space, solved here for all times.

**Directed polymer: analytical solution.** We consider the partition function of a DP at temperature $T$ in the continuum, i.e., the sum over positive paths $x(\tau) \in R^+$ starting at $x(0) = y$ and ending at $x(t) = x$,

$$ Z(x,y,t) = \int_{x(0)=y}^{x(t)=x} Dx(\tau) e^{-\int_0^t \int_0^\infty [\eta(\xi) + V(x(\tau),\tau)]}, $$

with initial condition $Z(x,y,0) = \delta(x - y)$. The hard wall requires $Z(0,y,t) = Z(x,0,t) = 0$. The random potential $V(x,t)$ is a centered Gaussian with correlator $\langle V(x,t)V(x',t') \rangle = \delta(t-t')\delta(x-x')$. The natural units for the continuum model are $t^* = 2T^2/\Xi$ and $x^* = T^3/\Xi c$ which allow to remove $T$ and set $\Xi = 1$ (see footnote 2). The time (i.e., polymer length) dependence is embedded in a single dimensionless parameter:

$$ \lambda = (t/4t^*)^{1/3} $$

(2)

defined in our previous works [19,22,23] and in [20].

Repeating (1) and averaging over disorder, the $n$-th integer moment of the DP partition sum can be expressed [29] as a quantum-mechanical expectation for $n$ particles described by the attractive Lieb-Liniger Hamiltonian [30]

$$ H_n = -\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} - 2\epsilon \sum_{1 \leq i < j \leq n} \delta(x_i - x_j). $$

(3)

in natural units (for the moment not rescaling by $c$, as in [19]). The moments of the partition sum with both endpoints fixed at $x$ can be written as

$$ Z(x,x,t) = \sum_{n=0}^\infty \left| \Psi_n(x,\ldots,x) \right|^2 e^{-TE_n}/\|\mu\|^n, $$

(4)
i.e., a sum over the un-normalized eigenfunctions $\Psi_n$ (of norm denoted $\|\mu\|$) of $H_n$ with energies $E_n$. Here we used the fact that only symmetric (i.e., bosonic) eigenstates contribute. In the presence of a hard wall at $x = 0$, $\Psi_n(x_1,\ldots,x_n)$ vanishes when any of the $x_j$ vanishes. It is solved by a generalization of the standard BA [31,32]. The Bethe states $\Psi_n$ are superpositions of plane waves [30] over permutations $P$ of the rapidities $\lambda_j (j = 1,\ldots,n)$, with here an additional summation over $\pm \lambda_j$. The eigenfunctions read

$$ \Psi_n(x_1,\ldots,x_n) = \frac{1}{(2\pi)^{n-1}} \sum_{P \in S_n} \epsilon_2 \cdots \epsilon_n \epsilon_2 \cdots \epsilon_n \times A_{\lambda_{P_1}\epsilon_2\epsilon_2\cdots\epsilon_n\lambda_{P_n}} \sin(x_1 \lambda_1) \prod_{j=2}^n e^{i\epsilon_j x_j \lambda_{P_j}}, $$

(5)

$$ A_{\lambda_1\cdots\lambda_n} = \prod_{n \geq \ell > k \geq 1} \left( 1 + \frac{i\bar{c}}{\lambda_k - \lambda_\ell} \right) \left( 1 + \frac{i\bar{c}}{\lambda_k + \lambda_\ell} \right) $$

(6)

for $x_1 < \ldots < x_n$, recalling that $\Psi_n(x_1,\ldots,x_n)$ is symmetric in its arguments. Imposing a second boundary condition at $x = L$, e.g., also a hard wall, one gets Bethe equations [31] which determine the possible sets of $\lambda_j$. The large-$L$ limit was studied in [32] and we do not reproduce the analysis here. The structure of the states is very similar to the standard case, i.e., the general eigenstates are built by partitioning the $n$ particles into a set of $n_\ell$ bound states formed by $m_j \geq 1$ particles with $n = \sum_{j=1}^n m_j$. Each bound state is a perfect string [33], i.e., a set of rapidities $\lambda_j^\ell = k_j + \frac{\bar{c}}{2}(m_j - 1)$. The difference with the standard case is that the states are now invariant by flipping any of the momenta $\lambda_j \rightarrow -\lambda_j$, i.e., $k_j \rightarrow -k_j$.

To simplify the problem, we restrict here to a DP with endpoints near the wall, i.e., we define the partition sum for $x = e^0$, $Z = \lim_{x \to 0} Z_V(x,t,t^*)/x^2$. Then, one factor in (4) drastically simplifies as $\lim_{x \to 0} \|\mu\|^n \langle \Psi_n(x,\ldots,x) \rangle /x^{2n} = e^{n/2} \lambda_1^2 \cdots \lambda_n^2$. The last needed factor in (4) is the norm, usually not trivial to obtain [34]. With some amount of heuristics we arrive at the following formula [35] (we now fully use the natural units, hence setting $\bar{c} = 1$):

$$ \|\mu\|^2 = n! \lambda_1^{2n} \prod_{i=1}^n S_{k_i} \prod_{1 \leq i < j \leq n} D_{k_i,k_j,m_i,m_j} L^{n_i}, $$

$$ S_{k,m} = \frac{m^2}{2(m-1)} \prod_{p=1}^{\lfloor m/2 \rfloor} \frac{k^2 + (m + 1 - 2p)^2/4}{k^2 + (m - 2p)^2/4}, $$

$$ D_{k_1,k_2,m_1,m_2} = 4(k_1 - k_2)^2 + (m_1 + m_2)^2 $$

$$ \times \frac{4(k_1 + k_2)^2 + (m_1 + m_2)^2}{4(k_1 + k_2)^2 + (m_1 - m_2)^2}. $$

(7)

(8)
We now have a starting formula for the integer moments
\[
\mathbb{Z}^n = \sum_{n=1}^{\infty} n! \left( \prod_{m_j \geq 1} b_{m_j} \right) \mathcal{P}(m_1, \ldots, m_n)
\]
with
\[
b_{m_j} = \prod_{j=0}^{m_j-1} \left( 4k^2 + j^2 \right).
\]
Here \((m_1, \ldots, m_n)_n\) stands for all the partitioning of \(n\) such that \(\sum_{j=1}^{m_j} = n\) with \(m_j \geq 1\) and we used \(\sum_{j < n} \to m_j L \int \frac{dt}{\pi} \) which holds also here in the large \(L\) limit.

This formula allows for predictions at small time. Defining\(^3\) \(z = \mathbb{Z}/\mathbb{Z}\) we obtain \(z^\infty = z^2 - 1\) as
\[
z^\infty = 42.99376\lambda^3,
\]
and the short time (i.e., small \(\lambda\)) expansion of \(z^\infty = 1 + \mathcal{O}(\lambda^2)\) terms. The skewness of the PDF of \(z\) behaves at short time as
\[
\gamma_1 = \frac{\text{var}(z)}{\text{var}(z)^2} \approx 0.079863175 \lambda^{3/4}.
\]
It is interesting to compare with the same results in ref. [19] in the absence of the hard wall (full space) and we find the universal ratio of the variances at small time:
\[
\rho = \frac{\text{var}(z)^{HS}}{\text{var}(z)^{PS}} \approx 3/2 - 0.076597089 \lambda^{3/2} + \mathcal{O}(\lambda^3)
\]
and of the skewness \(\gamma_1^{HS}/\gamma_1^{PS} = 0.63689604 + \mathcal{O}(\lambda^3/2)\).

We now study arbitrary time, i.e., any \(\lambda\), and to this aim we define the generating function of the distribution \(P(f)\) of the scaled free energy \(\ln \mathbb{Z} = -\lambda \gamma\):
\[
g(s) = \exp(-e^{-s} \lambda \mathbb{Z}) = 1 - \sum_{n=1}^{\infty} \frac{(-e^{-s} \lambda) n!}{n!} \mathbb{Z}^n
\]
from which \(P(f)\) is immediately extracted at \(\lambda \to \infty\):
\[
\lim_{\lambda \to \infty} g(s) = \theta(f + s) = \text{Prob}(f > s)
\]
and below we recall how it is extracted at finite \(\lambda\). The constraint \(\sum_{i=1}^{n} m_i = n\) in (9) can then be relaxed by reorganizing the series according to the number of strings:
\[
g(s) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \mathbb{Z}^n
\]
Solvability arises from the Pfaffian identity:
\[
\prod_{1 \leq i < j \leq n} n! \mathcal{D}_{b_{m_i}, m_j}( \lambda) = \frac{\mathbb{Z}^n}{\lambda^3} \prod_{j=1}^{n} \frac{X_j}{\mathcal{Z}^2}
\]
where \(X_{2p-1} = m_p + 2ikp\), \(X_{2p} = m_p - 2ikp\), \(p = 1, \ldots, n\), a consequence of Schur’s identity as used in refs. [22, 23] to which we refer for details. We recall that the pfaffian of an antisymmetric matrix \(A\) is defined as \(\text{pf}A = \sqrt{\det A}\). Equation (18) allows to write the \(n_s\) string partition sum as
\[
Z(n_s, s) = \sum_{n_{1, \ldots, n_s = 1}}^{\infty} (-1)^n \prod_{p=1}^{n_s} \frac{\mathbb{Z}^{w_p}}{2\pi} e^{m_{w_p,k_p}\mathcal{D}_{b_{m_p}, s}(\lambda)}
\]
Now, as in refs. [22, 23] we use the representation
\[
\mathbb{Z}^{w_p} = \mathbb{Z}^{\mathcal{D}_{b_{m_p}, s}(\lambda)}
\]
and standard properties of the Pfaffian allow to take the integral over the \(2n_s\) variables outside the Pfaffian. After manipulations very similar to refs. [22, 23] the integration and summation over \(k_j, m_j\) can be performed, leading to
\[
Z(n_s, s) = \frac{1}{\mathcal{D}(2n_s - 1 !)} \prod_{j=1}^{2n_s - 1} \mathbb{Z}^{w_p}
\]
where \((2n_s - 1)! = (2n_s)!/((n_s)!^2)\) is the number of pairings of \(2n_s\) objects, with the kernel:
\[
f(v_1, v_2) = \sum_{m=1}^{\infty} \frac{\mathbb{Z}^{w_p}}{2\pi} e^{m_{w_p,k_p}\mathcal{D}_{b_{m_p}, s}(\lambda)}
\]
We used that in the natural units \(t(\equiv \frac{t}{\lambda}) = 4\lambda^2\). \(g(s)\) has now the form of a Fredholm Pfaffian. One shows [35]
\[
g(s) = \sqrt{\text{Det}(I + \mathcal{K})},
\]
It is interesting that \(g(s)^2\) is precisely the generating function for the two independent half-spaces (on each side of the hard wall) and that it is itself a Fredholm determinant (FD). Performing the rescaling \(v_i \to \alpha_i v_i\) and \(k_j \to k_j/\lambda\) leaves the result (22) unchanged with the scaled
\[\mathbb{Z} = 1/\sqrt{4\pi}\] for full space and \(\mathbb{Z} = 1/\sqrt{4\pi}\) with a hard wall.

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\(^4\)We have performed the usual shift \(Z = e^{-c^2/12\lambda^2}\) (we drop the hat below) which does not affect \(z = \mathbb{Z}/\mathbb{Z}\).
function $g(s) = \frac{1}{2} \left( F_1(s) + F_2(s) \right) = F_4(s = -2^{-2/3}s)$ (29)

in the conventions of ref. [38]. To summarize, for the continuum DP model with fixed endpoints near the hard wall we find

$$\ln z = 2^{2/3}\lambda \xi_t,$$  \hspace{1cm} (30)

where $z = Z/\mathbb{Z}$ and $\xi_t$ converges at large time in distribution to the GSE Tracy-Widom distribution $F_4$. The same formula holds for the full space but with $\xi_t$ converging at large time to the GUE distribution $F_2$.

We now obtain the PDF of the free energy at finite time. We follow the method in [19]. It is written as a convolution, i.e., $Z = \ln Z_0 + \lambda u_0$ is the sum of two independent random variables, where in $Z_0$ has a unit Gumbel distribution (i.e., $P(Z_0) = e^{-e^{-y}}$). Then the PDF of $u$ is obtained by analytical continuation $p(u) = \frac{1}{2\pi} \text{Im} g(s)|_{e^{2\pi i s}ightarrow e^{2\pi i s} + i\lambda}$. Using (22), (23) and (24) and some complex analysis we find the free-energy distribution as the difference of two (complex) Fredholm Pfaffians (FP):

$$p(u) = \frac{\lambda}{2\pi} \left( \sqrt{\text{Det}[I + P_0K_uF_0]} - \sqrt{\text{Det}[I + P_0K_u^*F_0]} \right)$$ \hspace{1cm} (31)

with the kernel

$$K_u(v_i, v_j) = \frac{d}{dv_i} \int \frac{dk}{2\pi} \int_y \text{Ai}(y + u + v_i + v_j + 4k^2)$$ \hspace{1cm} (32)

$$\times \frac{\sin(2(v_i - v_j)k)}{k} \left[ f_{k/\lambda}(e^{\lambda y}) + if_{k/\lambda}(e^{\lambda y}) \right],$$

$$f_{k}(z) = \frac{\pi k}{\sinh(2\pi k)} \left( I_{-4ik} \left( 2\sqrt{\frac{1}{z}} \right) + I_{4ik} \left( 2\sqrt{\frac{1}{z}} \right) \right)$$ \hspace{1cm} (33)

$$-1 F_2 \left( 1; 1 - 2ik, 1 + 2ik; \frac{1}{z} \right).$$

Note that the same formula (31) with each FP replaced by its square, i.e., the FD, holds for the free energy associated to the union of the two independent half-spaces.

**Numerical simulations:** Here we call $l$ the (integer) polymer length. At high temperature, we follow [19,36,39] and define the partition sum (PS) $Z(l) = \sum_{\gamma} e^{-i\sum_{\langle x,y \rangle} V(y,x)}$ of paths $\gamma$ directed along the diagonal of a square lattice from $(0,0)$ to $(l/2, l/2)$ with only $(1,0)$ or $(0,1)$ moves. We denote space $x = (i - j)/2$ and time $\tau = i + j$. An i.i.d. random number $V(x, \tau)$ is defined at each site of the lattice (we use a unit centered Gaussian). The disorder averaged full space PS is $\overline{Z} = N(e^{\beta^2 l^2/2}$ where $N_{F} \approx 2^{l} \sqrt{2/(\pi \hat{l})}$ is the number of paths of length $l$. The half-space PS is obtained
by summing only on paths with $x > 0$, equivalent to an absorbing wall (hard wall), with $N_{t}^{HS} \sim 2(2t)^{3/2}/\sqrt{\pi}$.

We compute $\ln z$ with $z = Z/Z$ with the transfer matrix algorithm. As established in [19,36,39] in the high-$T$ limit at fixed $\lambda$, where $\lambda = (t/2T^{4})^{1/3}$ for the lattice model, $\ln z$ can be directly compared—with no free parameter—with the analytical predictions of the continuum model, the variable $s$ in all figures is called $s$ in the text.

In fig. 1 we show the convergence to the GSE TW distribution both for i) $T = 0$ and large polymer length $t$ and ii) at $T > 0$ and large $\lambda$. The agreement is very good. The variation for $T > 0$ as a function of $\lambda$ is shown in more detail in fig. 2 where the (small) differences in the cumulative distributions (CDF) are shown on a larger scale. As in fig. 1 the mean and variance of the numerical PDFs are adjusted to those of $F_{t}$. In fig. 3 we show the ratio of half-space (HS) to full-space (FS) variances as a function of $\lambda$. Since the two TW distributions have variances $\sigma_{F_{2}} = 0.8131947928$ and $\sigma_{F_{3}} = 1.03544744$, the ratio $\rho$ should converge to the value $1.273308$ at large time, which is apparent in fig. 3, up to finite $t$ effects discussed there. Similarly, the two TW distributions have skewness $\gamma_{F_{2}} = 0.2240842$ and $\gamma_{F_{3}} = 0.16550949$ hence the skewness ratio is predicted to increase from 0.636896 at small time (see above) to 0.738604 at large time.

Interestingly, the difference of the means $\mu$ of the GSE and GUE TW distributions gives information about extreme value properties of the DP. $p \equiv Z^{HS}/Z^{FS}$ is the probability that, in the full space and for endpoints fixed at position $x > 0$ the DP does not cross $x = 0$. $p$ is defined for each disorder realization, with $p \approx x^{2} \hat{p}/t$ for small $x$. Then at large time (i.e., large $\lambda$) one has $\ln \hat{p} = 2^{3/2}(\lambda^{1/3} - \mu_{F_{2}})$, where $\mu_{F_{2}} = -3.2624279$ while $\mu_{F_{3}} = -1.7710868$. At small time (i.e., small $\lambda$) one finds from above and [19] $\ln \hat{p} = -\frac{1}{2} \sqrt{2} \lambda^{3/2} - 0.0082964\lambda^{3} + \ldots$ hence $-\ln \hat{p}$ crosses over from $\sim t^{1/2}$ to $\sim t^{1/3}$. Note that $p$ is highly non-self-averaging at low temperature: at $T = 0$ it is either 0 or 1, and a numerical study [35] indicates that $p = \text{Prob}(p = 1)$ decays algebraically with time. Computing the PDF of $p$ seems a hard, although interesting, task.

**KPZ equation:** let us now detail how our results translate in terms of the KPZ equation,

$$\partial_{x} h = \nu \nabla h + \frac{\lambda_{0}}{2} (\nabla h)^{2} + \eta(x,t),$$  \hspace{1cm} (34)

where $\eta(x,t)\eta(x',t') = R_{\eta}(x-x')\delta(t-t')$, with Gaussian noise correlator $R_{\eta}(x) = D \delta(x)$. The Cole-Hopf mapping generally implies $\frac{\lambda_{0}}{2\nu} \hat{h} = \ln \hat{Z}$ and $\hat{c} = D \lambda_{0}^{2}$. To be more specific, the initial condition (1) corresponds to a wedge $h(x,0) = -w|y-x|^{\nu}$ in the limit $w \to \infty$, before $y \to 0$. Because of the hard wall one has $\frac{\lambda_{0}}{2\nu} \hat{h}(x,t) = \ln(xy) + \frac{\lambda_{0}}{2\nu} \hat{h}(x,t)$, where $\hat{h}$ is not singular when both $x$ and $y$
approach zero, and the correspondence is really $\frac{\lambda}{2\nu} \hat{h}(0, t) = \ln Z$. Schematically the boundary conditions (BC) can be stated as $h(0, t) = -\infty$ or $\nabla h(0, t) = +\infty$ (see more general ones below). Hence from (30):

$$\frac{\lambda_0}{2\nu} \hat{h}(0, t) = \ln Z + 2^{2/3} \lambda \xi_t$$  \hspace{1cm} (35)$$

with, at large $t$, $\ln Z \approx v_\infty t$. From footnotes 3 and 4, $v_\infty = \frac{\Delta E_{\text{ini}}}{4\pi v^2} - \frac{D^2 \lambda^2}{2}$ is the same non-universal constant (see discussion in [36]) in both HS and FS cases, the difference in $\ln Z$ being only sublinear in time, as $\sim \ln t$.

Finally, we discuss the universality of our results. Another standard BC is the reflecting wall (RW) $\nabla Z = 0$, i.e., $\nabla h = 0$ (contact angle $\pi/2$) which may be of experimental interest. For the DP it is achieved considering two symmetric half-spaces, i.e., $V(-x, t) = V(x, t)$ (see footnote 6). At $T = 0$ there is no difference in the optimal path energy between the hard and reflecting wall. At $T > 0$ they become different, since there is more entropy in the RW. However, the longer the polymer, the closer it becomes, effectively, to $T = 0$. Hence we expect different $g(s)$, which become equal at large time. In fact all BC such that $\nabla h \geq 0$ should converge to $F_4$. This is consistent with the results of [28] translated into the $T = 0$ lattice DP model (although the equivalent of the hard wall was not explicitly considered there). In the PNG model it corresponds to an absent or weak enough boundary source [27]. We will not discuss here the case of BC $\nabla h < 0$ which leads to an unbinding transition. A similar transition was studied in the random permutation model [28] and in the PNG model [14,27], but not using the BA (see, however, [40]). Work on that case is in progress.

Our results apply to the conductance $g$ of disordered 2D conductors deep in the localized regime. Extending ref. [41] we predict that $L^{-1/3} \ln g$ should be distributed as $F_4$ if the leads are small, separated by $L$, and placed near the frontier of the sample (which occupy, say, a half-space).

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$^6$Note that if one chooses $V(-x, t) = V(x, t)$ the usual imaging method works, i.e., $Z(x, y, t) = Z(x, -y, t)$ is the PS in the half-space with reflecting (respectively, absorbing) BC.