CONVERGENCE OF INTERVAL AOR METHOD FOR LINEAR INTERVAL EQUATIONS

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Abstract. A real interval vector/matrix is an array whose entries are real intervals. In this paper, we consider the real linear interval equations $Ax = b$ with $A, b$ respectively, denote an interval matrix and an interval vector. The aim of the paper is to study the numerical solution of the linear interval equations for various classes of coefficient interval matrices. In particular, we study the convergence of interval AOR method when the coefficient interval matrix is either interval strictly diagonally dominant matrices, interval $L$-matrices, interval $M$-matrices, or interval $H$-matrices.

1. Introduction. The problem of solving linear interval equations occurs in many real world problems, like solving engineering problems with uncertain data, global optimization and mathematical programming dealing with uncertainty etc., which can be reviewed in [1, 5, 8, 10, 14, 15]. In this paper we consider linear interval systems and analyze the convergence of interval iterative method, namely interval AOR method. In general, the computational complexity and feasibility of solving linear interval equations is NP-hard problem, so different methods are developed to find an enclosure of the solution set of the linear interval system, and the interval Jacobi, interval Gauss-Seidel, Krawczyk iteration methods, Bauer-Skeel and Hansen-Blek-Rohn methods are among the well-known iterative methods for solving linear interval systems (see [9, 11, 17, 19]). The enclosure obtained by the methods which may not be optimal. Also in [9, 11], authors used preconditioning techniques to provide as tight as possible bounds for the solution set of linear interval system.

In the paper $\mathbb{I} \mathbb{R}$, $\mathbb{I} \mathbb{R}^n$, and $\mathbb{I} \mathbb{R}^{m,n}$, respectively, denote the set of all real intervals, the set of $n$-dimensional real interval vectors and the set of real interval matrices of size $m \times n$. An $m \times n$ real interval matrix is defined as $A = \{ A \in \mathbb{R}^{m,n} : A \leq A \leq A \}$, for some given real matrices $A, A$, with $A \leq A$, and inequality is valid componentwise, equivalently, $A$ can be written as $m \times n$ rectangular array $A = (A_{ij})$, with $A_{ij}$ is in $\mathbb{I} \mathbb{R}$. In [3], author introduced interval AOR method solving linear interval equations, and provided sufficient conditions for the convergence of the method for various classes of interval matrices. Sufficient conditions for the

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convergence of AOR method for solving linear system for various classes of matrices can be found in [4, 6, 26]. Motivated by their work, in this paper we analyze the convergence of interval AOR method for solving linear interval equations of strictly diagonally dominant interval matrices, interval L-matrices, interval M-matrices and interval H-matrices.

Consider system of linear interval equations of the form

$$Ax = b$$

with given $A \in \mathbb{IR}^{n \times n}$, $b \in \mathbb{IR}^n$ and $x \in \mathbb{IR}^n$ is unknown. The system (1) is the family of linear equations $\tilde{A}\tilde{x} = \tilde{b}$ with $\tilde{A} \in \mathbb{A}$ and $\tilde{b} \in \mathbb{b}$. The solution set of (1) is defined as

$$\Sigma(A, b) := \{\tilde{x} \in \mathbb{R}^n : \tilde{A}\tilde{x} = \tilde{b} \text{ for some } \tilde{A} \in \mathbb{A}, \tilde{b} \in \mathbb{b}\}$$

The interval hull of the solution set $\Sigma(A, b)$, that is, the smallest interval enclosure of $\Sigma(A, b)$ with respect to inclusion, is denoted by $\Sigma := \square \Sigma(A, b) = [\inf(\Sigma(A, b)), \sup(\Sigma(A, b))]$. We now illustrate the solution set $\Sigma$ with the following example [19].

Consider the interval system (1) with $A := \begin{pmatrix} [2, 4] & [2, 4] \\ [-1, 1] & [2, 4] \end{pmatrix}$, and $b := \begin{pmatrix} [-3, 3] \\ 0 \end{pmatrix}$ so that

$$\Sigma(A, b) = \{x \in \mathbb{R}^2 : 2|x_2| \leq |x_1|, 2|x_1| \leq 3 + |x_2|\}$$

which is given by the following region:

From Figure 1, it is clear that $\Sigma = \begin{pmatrix} [-2, 2] \\ [-1, 1] \end{pmatrix}$.

The basic properties of interval arithmetic and elementary results on solutions of (1) can be found in [18, 19, 20]. Iterative methods to solve (1) are started by an initial enclosure $x^{(0)} \supset \Sigma$, and in general can be written as an operator
\( \mathcal{I} : \mathbb{IR}^n \to \mathbb{IR}^n \) such that

\[ x \cap \Sigma \subset \mathcal{I}(x) \]  

(2)
The AOR method for solving linear interval equations was introduced in [3], is similar to that for linear system, that is, defined with respect to the standard splitting of the coefficient matrix into its diagonal, strictly lower and strictly upper triangular parts. Consider the splitting \( \mathbf{A} = \mathbf{D} - \mathbf{E} - \mathbf{F} \) of the interval matrix \( \mathbf{A} \) into its diagonal, strictly lower and strictly upper triangular parts, respectively. If \( 0 \notin A_{ii} \), then for \( \omega, \sigma \in \mathbb{R} \) with \( \omega \neq 0 \), the interval AOR method, introduced in [3] for the numerical solution (1) is written as

\[ x^{(k+1)} = \mathbf{M}_{\sigma,\omega} x^{(k)} + \mathbf{d}, \quad k = 0, 1, 2, \ldots \]  

(3)

where \( \mathbf{M}_{\sigma,\omega} = (\mathbf{D} - \sigma \mathbf{E})^{-1}[(1-\omega)\mathbf{D} + (\omega - \sigma)\mathbf{E} + \omega \mathbf{F}] \) and \( \mathbf{d} = \omega(\mathbf{D} - \sigma \mathbf{E})^{-1}\mathbf{b} \). In (3) \( \sigma \) is called acceleration parameter and \( \omega \) is called relaxation parameter. Note that equation (3) provides an enclosure of \( \mathbf{D}^{-1} \mathbf{A} \) for interval matrices and review few results related to these interval matrices.

The present paper is organised as follows: In Section 2, we provide the background of various interval matrices and review few results related to these interval matrices. Section 3 presents the convergence analysis of interval AOR method for solving (1) for various classes of interval matrices. In Section 4, numerical illustrations are considered to compare the interval AOR-method with some existing methods. Lastly, concluding remark is given in Section 5.

2. Notation and Preliminaries. Throughout the paper boldface is used to denote intervals. The underscores and overscores notations are adopted to denote lower bounds and upper bounds of intervals, respectively. For any interval \( \mathbf{x} = [\underline{x}, \overline{x}] \), the quantity \( \langle \mathbf{x} \rangle \) denotes the magnitude of \( \mathbf{x} \) and is defined as \( \langle \mathbf{x} \rangle := \min\{|x| : x \in \mathbf{x}\} = \min\{|\underline{x}|, |\overline{x}|\} \), whereas \( |\mathbf{x}| := \max\{|x| : x \in \mathbf{x}\} = \max\{|\underline{x}|, |\overline{x}|\} \) denotes the magnitude of \( \mathbf{x} \). Similarly, min- and max-definitions are extended to comparable sets of intervals. For a given \( \mathbf{A} = (A_{ij}) \in \mathbb{IR}^{n,n} \), we denote \( |\mathbf{A}| := (|A_{ij}|) \in \mathbb{IR}^{n,n} \), and we use \( \langle \mathbf{A} \rangle \) to represent the comparison matrix of \( \mathbf{A} \) with entries

\[ \langle \mathbf{A} \rangle_{ii} = \langle A_{ii} \rangle \quad \text{and} \quad \langle \mathbf{A} \rangle_{ij} = -|A_{ij}|, \quad \text{for } i \neq j \]

We now define various classes of interval matrices under consideration.

Definition 2.1. [20, 22, 23] An interval matrix \( \mathbf{A} \in \mathbb{IR}^{n,n} \) is said to be regular if every \( A \in \mathbf{A} \) is nonsingular. In case, \( \mathbf{A} \) is regular, then its inverse \( \mathbf{A}^{-1} \) is defined as

\[ \mathbf{A}^{-1} := \mathbb{I} \{ A^{-1} : A \in \mathbf{A} \} \]

where \( \mathbb{I} \Sigma := \{ \inf \Sigma, \sup \Sigma \} \) denotes the hull of a bounded set \( \Sigma \) of matrices. Note that \( \mathbf{A}^{-1} \) is the smallest interval matrix containing the set \( \{ A^{-1} : A \in \mathbf{A} \} \).
**Definition 2.2.** [20] For any two real intervals \( x = [x, \bar{x}] \) and \( y = [y, \bar{y}] \), the interval multiplication \( xy \) is defined by the following table:

| *          | \( y \geq 0 \) | \( y \geq 0 \) | \( y \leq 0 \) |
|------------|-----------------|-----------------|-----------------|
| \( x \geq 0 \) | \([xy, \bar{xy}]\) | \([\bar{x}y, \bar{xy}]\) | \([\bar{x}y, \bar{xy}]\) |
| \( x \geq 0 \) | \([\min\{\bar{xy}, \bar{y}x\}, \max\{xy, \bar{xy}\}]\) | \([\bar{x}y, \bar{xy}]\) | \([\bar{x}y, \bar{xy}]\) |
| \( x \leq 0 \) | \([xy, \bar{xy}]\) | \([\bar{x}y, \bar{xy}]\) | \([\bar{x}y, \bar{xy}]\) |

The interval addition and subtraction are defined, respectively, as

\[
x + y = [x + y, \bar{x} + \bar{y}], \quad \text{and} \quad x - y = [x - \bar{y}, \bar{x} - \bar{y}]
\]

**Definition 2.3.** [20] If \( A \in \mathbb{IR}^{m \times n} \) and \( B \in \mathbb{IR}^{n \times p} \), then \( AB \in \mathbb{IR}^{m \times p} \) is defined as

\[
AB = \square\{\hat{A} \hat{B} : \hat{A} \in A, \hat{B} \in B\}
\]

Note that if \( A = (A_{ij}) \) and \( B = (B_{ij}) \), then

\[
(AB)_{ik} = \sum_{j=1}^{n} A_{ij} B_{jk}
\]

If \( A, B \in \mathbb{IR}^{m \times n} \), addition and multiplication for interval matrices are defined, respectively, as

\[
A + B = \square\{A + B : A \in A, B \in B\}
\]

\[
A - B = \square\{A - B : A \in A, B \in B\}
\]

It is known that if \( A = [A, \overline{A}] \) and \( B = [B, \overline{B}] \), then

\[
A + B = [A + B, \overline{A + B}], \quad \text{and} \quad A - B = [A - B, \overline{A - B}]
\]

**Example 2.1.** [20] Consider the regular interval matrix

\[
A := \begin{pmatrix}
2 & [-1, 0] \\
[-1, 0] & 2
\end{pmatrix}.
\]

Then any \( \hat{A} \in A \) has the form \( \hat{A} := \begin{pmatrix} 2 & -a_1 \\ -a_2 & 2 \end{pmatrix} \) with \( a_1, a_2 \in [0, 1] \). Hence by the definition

\[
A^{-1} = \square\left\{ \frac{1}{4 - a_1 a_2} \begin{pmatrix} 2 & a_1 \\ a_2 & 2 \end{pmatrix} : a_1, a_2 \in [0, 1] \right\} = \begin{pmatrix} [\frac{1}{2}, \frac{3}{2}] & [0, \frac{1}{3}] \\ [0, \frac{1}{3}] & [\frac{1}{2}, \frac{3}{2}] \end{pmatrix}
\]

**Definition 2.4.** [3, 20] Let \( A \in \mathbb{IR}^{n \times n} \) and \( 0 \notin A_{ii} \) for all \( i \). Then \( A \) is called an interval strictly diagonally dominant (SDD) matrix if its comparison matrix \( \langle A \rangle \) is strictly diagonally dominant, that is, if \( \langle A_{ii} \rangle > \sum_{j \neq i} |A_{ij}| \), for all \( i \). For example,

\[
A = \begin{pmatrix}
[4, 5] & [-2, 2] & 1 \\
[1, 2] & 5 & [-2, 2] \\
[-2, 2] & [-1, 1] & 5
\end{pmatrix}
\]

is an interval SDD matrix.

**Definition 2.5.** [2, 20] A real matrix \( A \in \mathbb{IR}^{n \times n} \) is called an \( L \)-matrix if its diagonal entries are positive and off-diagonal entries are non-positive.

An interval matrix \( A = [A, \overline{A}] \) is an interval \( L \)-matrix if each \( A \in A \) is an \( L \)-matrix, equivalently, if \( A_{ii} > 0 \) for all \( i \) and \( \overline{A}_{ij} \leq 0 \), for \( i \neq j \).
Definition 2.6. [2] A matrix $A \in \mathbb{R}^{n,n}$ is said to be a Z-matrix if off-diagonal entries $A$ are non-positive. A Z-matrix is called an M-matrix if can be written as $A = sI - B$, where $s > \rho(B)$, the spectral radius of $B$.

From now onwards, we simply write $M$-matrix for non-singular $M$-matrix. It is known that a Z-matrix $A$ is an M-matrix if and only if there exists a $u > 0$ such that $Au > 0$. Motivated by this property of $M$-matrix, an interval $M$-matrix is defined.

We now state characterization of $M$-matrices, which motivates to define interval $M$-matrices.

Theorem 2.7. [2] Let $A \in \mathbb{R}^{n,n}$ be a Z-matrix. Then following statements are equivalent:

(i) $A$ is an $M$-matrix.
(ii) $A^{-1} \geq 0$.
(iii) There exists $u > 0$ such that $Au > 0$.

Definition 2.8. [20] An interval $M$-matrix is a square interval matrix $A \in \mathbb{I}^{n,n}$ such that $A_{ik} \leq 0$, that is, every element in $A_{ik}$ is non-positive, for all $i \neq k$ and $Au > 0$ for some positive vector $u \in \mathbb{R}^n$.

Definition 2.9. [20] An interval matrix $A \in \mathbb{I}^{n,n}$ is called an interval H-matrix if its comparison matrix $\langle A \rangle$ is an $M$-matrix. Equivalently, we say that $A$ is an interval H-matrix if and only if $\langle A \rangle u > 0$ for some $u > 0$.

Definition 2.10. [2] A splitting of a matrix $A \in \mathbb{R}^{n,n}$ is defined as $A = M - N$, with non-singular $M$. A splitting $A = M - N$ of the matrix $A$ is called

(i) regular if $M^{-1} \geq 0$ and $N \geq 0$.
(ii) weak regular if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$.
(iii) $M$-splitting if $M$ is a $M$-matrix and $N \geq 0$.

Definition 2.11. A splitting of a square interval matrix $A \in \mathbb{I}^{n,n}$ is defined as $A = M - N$, with regular $M$.

We now state two basic results on matrices, which are subsequently needed to prove our results in the next section.

Proposition 1. [20, 12] If $A, B \in \mathbb{R}^{n,n}$, and $|A| \leq B$, then $\rho(A) \leq \rho(B)$.

Proposition 2. [12] If $A \in \mathbb{R}^{n,n}$ is a nonnegative matrix and $x \geq 0$ and $x \neq 0$ such that $Ax \geq \alpha x$, for some $\alpha \in \mathbb{R}$, then $\rho(A) \geq \alpha$.

Theorem 2.12. [20] Let $A$ be an $M$-matrix and $B \geq 0$ then $A - B$ is an $M$-matrix if and only if $\rho(A^{-1}B) < 1$.

Theorem 2.13. [2] Let $A$ be an $M$-matrix and let $A = M - N$ be a regular or weak regular splitting of $A$, then $\rho(M^{-1}N) < 1$.

Theorem 2.14. [27] Let $A = M - N$ be an $M$-splitting of $A$. Then $\rho(M^{-1}N) < 1$ if and only if $A$ is a non-singular $M$-matrix.

Followings are some few well-known results on interval matrices.

Theorem 2.15. [20] Let $A, B \in \mathbb{I}^{n,n}$. Then following statements hold:

(i) If $A$ is an $M$-matrix and $B \subseteq A$, then $B$ is an $M$-matrix. In particular each $\tilde{A} \in A$ is an $M$-matrix.
Theorem 2.16. [3] For an interval matrix $A$ we have
(i) if $A$ is interval triangular (lower/upper) matrix, $A$ is an interval $H$-matrix.
(ii) if $A$ is an interval $H$-matrix, then $|A^{-1}| \leq \langle A \rangle^{-1}$. Equality holds if $A$ is an interval $M$-matrix.

Theorem 2.17. [20] For $A \in \mathbb{R}^{n,n}$ the following conditions are equivalent
(i) $A$ is an $H$-matrix.
(ii) $0 \leq u \in \mathbb{R}^n, ((A)u \leq 0) \Rightarrow (u = 0)$. 

Proposition 3. [20] For $A, B \in \mathbb{R}^{n,n}$ and $C \in \mathbb{R}^{n,p}$, following properties hold:
(i) $\langle A \rangle = \langle \tilde{A} \rangle$, for some $\tilde{A} \in A$.
(ii) $|AB| \leq |A||B|$
(iii) $\langle A \pm B \rangle \geq \langle A \rangle - |B|$
(iv) $|A| - |B| \leq |A \pm B| \leq |A| + |B|$.
(v) $|AC| \geq |A||C|$.

Theorem 2.18. [3] Let $A, B \in \mathbb{R}^{n,n}$ satisfy $\rho(|A||B|) < 1$. Then for any $f \in \mathbb{R}^n$, the following statements hold:
(i) The equation $x = A(Bx + f)$ has a unique algebraic solution $x \in \mathbb{R}^n$
(ii) For any starting vector $x^0 \in \mathbb{R}^n$, the iteration 
$$x^{(k+1)} = A(Bx^{(k)} + f), \quad k = 0, 1, \ldots$$
converges to the solution $x$ of the equation $x = A(Bx + f)$.

3. Convergence of interval AOR method. In [3] authors provided sufficient conditions for convergence of interval AOR method defined in (3) for interval SDD-matrices and for interval $H$-matrices. Motivated by their work and by convergence properties of AOR method for linear system, our aim is study the convergence of interval AOR method defined in (3) for various classes of interval matrices defined in Section 2 and Theorem 2.18 is used to achieve the goal. Now onwards, for $A \in \mathbb{R}^{n,n}$, we assume that $0 \notin A_{ii}$ so that equation (3) is well-defined.

If $\tilde{M} = D - \sigma E$, $\tilde{N} = (1 - \omega)D + (\omega - \sigma)E + \omega F$ and $H_{\sigma, \omega} = |\tilde{M}^{-1}| \cdot |\tilde{N}|$, by Theorem 2.18, interval AOR method (3) converges for the linear interval equations (1) if $\rho(H_{\sigma, \omega}) < 1$.

Define $M = I - \sigma L = D^{-1}\tilde{M}$ and $N = (1 - \omega)I + (\omega - \sigma)L + \omega U = D^{-1}\tilde{N}$. Observe that $\langle M \rangle = \langle I - \sigma L \rangle = I - |\sigma| \cdot |L|$ is an $M$-matrix, as $|\sigma| \cdot |L|$ is a nonnegative matrix and $1 > 0 = \rho(|\sigma| \cdot |L|)$. If $C_{\sigma, \omega} = \langle M \rangle^{-1}|N|$, by Theorem 2.7, $\langle M \rangle^{-1} \geq 0$. Therefore $C_{\sigma, \omega} \geq 0$, and thus by Perron-Frobenious theorem [12], $\rho(C_{\sigma, \omega})$ is an eigenvalue and there exists an associated non-negative eigenvector.

We begin our result by obtaining a bound of $\rho(H_{\sigma, \omega})$.

Lemma 3.1. Let $D \in \mathbb{R}^{n,n}$ be an interval diagonal matrix with $0 \notin D_{ii}$. Then for any $A \in \mathbb{R}^{n,n}$ followings hold:
(i) $|DA| = |D| \cdot |A|$
(ii) $\langle D \rangle \langle A \rangle \leq \langle DA \rangle \leq |D| \langle A \rangle$. 
Theorem 3.2. For $\sigma \geq 0$, $\rho(\tilde{H}_{\sigma,\omega}) \leq \rho(C_{\sigma,\omega})$. 

Proof. (i) For any $i, j$ we have $|DA|_{ij} = \max\{|d_i, \tilde{d}_i|a_{ij}, \tilde{a}_{ij}\}$. For simplicity, we write $d = D_{ii} = [d_i, \tilde{d}_i]$ and $a = A_{ij} = [a_{ij}, \tilde{a}_{ij}]$. We now consider the following cases:

Case 1: Let $d > 0$, $a \geq 0$, so that $|d_i, \tilde{d}_i|a_{ij}, \tilde{a}_{ij}| = [d_i, \tilde{d}_i, \tilde{d}_i, \tilde{d}_i] \geq 0$. Hence

$$|DA|_{ij} = \max\{|d_i, \tilde{d}_i|a_{ij}, \tilde{a}_{ij}|\} = \tilde{d}_i, a_{ij}, \tilde{a}_{ij} = |D|_{ii}A_{ij}$$

Case 2: Assume that $d > 0$, $a \geq 0$. Then $|D|_{ii} = \tilde{d}_i$ and

$$|d_i, \tilde{d}_i|a_{ij}, \tilde{a}_{ij}| = \tilde{d}_i, a_{ij}, \tilde{a}_{ij}$$

Hence

$$|DA|_{ij} = \max\{|d_i, \tilde{d}_i|a_{ij}, \tilde{a}_{ij}|\} = \tilde{d}_i, a_{ij}, \tilde{a}_{ij} = |D|_{ii}A_{ij}$$

Case 3: Let $d > 0$, $a \geq 0$. Then $|D|_{ii} = \tilde{d}_i$, $|A|_{ij} = |a_{ij}|$ and

$$|d_i, \tilde{d}_i|a_{ij}, \tilde{a}_{ij}| = \tilde{d}_i, a_{ij}, \tilde{a}_{ij} \leq 0$$

Thus $|DA|_{ij} = \tilde{d}_i, a_{ij} = \tilde{d}_i, a_{ij} = |D|_{ii}A_{ij}$. 

Case 4: Suppose that $d < 0$, $a \geq 0$. Then $|d_i, \tilde{d}_i|a_{ij}, \tilde{a}_{ij}| = [d_i, \tilde{d}_i, \tilde{d}_i, \tilde{d}_i] \leq 0$ and hence

$$|DA|_{ij} = \max\{|d_i, \tilde{d}_i|a_{ij}, \tilde{a}_{ij}|\} = \tilde{d}_i, a_{ij}, \tilde{a}_{ij} = |D|_{ii}A_{ij}$$

Similarly it can be verified for $d < 0$, $a \geq 0$ and $d < 0$, $a \leq 0$. Thus $|DA| = |D| \cdot |A|$. 

(ii) For any two indices $i, j$ with $i \neq j$, and using Part (i) we obtain that

$$\langle DA \rangle_{ij} = -|DA|_{ij} = -|D|_{ii}|A|_{ij} = |D|_{ii}(A)_{ij} = ||D|_{ii}(A)|_{ij}$$

Again,

$$\langle DA \rangle_{ij} = |D|_{ii}(A)_{ij} \geq |D|_{ii}(A)_{ij} = \langle D \rangle_{ii}(A)_{ij}$$

Let $i = j$. We use the notations of Part (i) and the proof is also similar to that of Part(i). If $d > 0$, $a \geq 0$, then

$$\langle DA \rangle_{ii} = \min\{|d_i, \tilde{d}_i|a_{ii}, \tilde{a}_{ii}|\} = d_i, a_{ii} = d_i, a_{ii} = \langle D \rangle_{ii}(A)_{ii} \leq |D|_{ii}(A)_{ii}$$

Assume $d > 0$, $a \geq 0$. Then we have

$$\langle DA \rangle_{ii} = \min\{|d_i, \tilde{d}_i|a_{ii}, \tilde{a}_{ii}|\} = \tilde{d}_i, \min\{|a_{ii}, |\tilde{a}_{ii}|\} = \langle D \rangle_{ii}(A)_{ii} \leq |D|_{ii}(A)_{ii}$$

Similarly, considering other cases we can show that

$$\langle D \rangle_{ii}(A)_{ii} \leq \langle DA \rangle_{ii} \leq |D|_{ii}(A)_{ii}$$

Hence (ii) holds. \hfill \square
Proof. Note that \( \tilde{M} \) is a lower triangular interval matrix and hence by Theorem 2.16, \( \tilde{M} \) is an interval \( H \)-matrix and
\[
|\tilde{M}^{-1}| \leq (\tilde{M})^{-1}, \tag{5}
\]
Since \( \tilde{H}_{\sigma,\omega} \geq 0 \), by Perron-Frobenius theorem choose a nonnegative vector \( x \) such that \( \tilde{H}_{\sigma,\omega}x = \rho x \) where \( \rho = \rho(\tilde{H}_{\sigma,\omega}) \). From equation (5) we obtain that
\[
\rho x = \tilde{H}_{\sigma,\omega}x = |\tilde{M}^{-1}| : |\tilde{N}|x \leq (\tilde{M})^{-1} : |\tilde{N}|x \tag{6}
\]
Again by Lemma 3.1 we have that \( \langle M \rangle = \langle D^{-1}\tilde{M} \rangle \leq |D^{-1}|\langle \tilde{M} \rangle \). Since \( (\tilde{M})^{-1} \) and \( \langle M \rangle^{-1} \) are nonnegative, so \( (\tilde{M})^{-1} \leq (M)^{-1}|D^{-1}| \). Therefore equation (6) reduces to
\[
\rho x \leq (M)^{-1}|D^{-1}| : |\tilde{N}|x = (M)^{-1}|D^{-1}|\tilde{N}|x = (M)^{-1}|N|x = C_{\sigma,\omega}x
\]
Hence by Proposition 2.18, \( \rho \leq \rho(C_{\sigma,\omega}) \).
\( \square \)

The above theorem implies that in order to prove the convergence of the iterative scheme (3), it suffices to show \( \rho(C_{\sigma,\omega}) < 1 \), which mostly used in the paper to show that convergence of interval AOR method for the interval system (1).

In [4], authors considered generalized AOR (GAOR) method and provided a bound on the spectral radius of the GAOR iteration matrix for interval SDD-matrices. Following theorem provides a bound of \( \rho(C_{\sigma,\omega}) \) and hence of spectral radius of the iteration matrix \( \tilde{H}_{\sigma,\omega} \) of (1) for interval SDD-matrices, which is similar to that of discussed in Theorem 1 of [4].

**Theorem 3.3.** If \( A \) is an interval SDD matrix, then
\[
\rho(C_{\sigma,\omega}) \leq \max_i \left\{ \frac{|1 - \omega| + |\omega - \sigma|L_i + |\omega|U_i}{1 - |\sigma|L_i} : L_i \neq 0, |\sigma| < \frac{1}{L_i} \right\}, \tag{7}
\]
where \( L_i \) and \( U_i \) denote the absolute value of \( i \)-th row sum of \( L \in L \) and \( U \in U \), respectively.

**Proof.** Let \( \lambda \) be the eigenvalue of \( C_{\sigma,\omega} \). Choose an \( x \neq 0 \in \mathbb{R}^n \) such that
\[
C_{\sigma,\omega}x = \lambda x \quad \Rightarrow \quad (I - \sigma L)L^{-1}|(1 - \omega)I + (\omega - \sigma)L + \omega U|x = \lambda x
\]
\[
\Rightarrow \quad |(1 - \omega)I + (\omega - \sigma)L + \omega U|x = \lambda(I - \sigma L)x
\]
\[
\Rightarrow \quad |1 - \omega|I + |\omega - \sigma|L + |\omega|U|x = \lambda(I - |\sigma|L)x
\]
\[
\Rightarrow \quad (\lambda - |1 - \omega|)x - (|\omega - \sigma| + |\lambda| |\sigma|) |L|x - |\omega| |U|x = 0
\]
\[
\Rightarrow \quad \left[ I - \frac{|\omega - \sigma| + |\lambda| |\sigma|}{\lambda - |1 - \omega|} |L| - \frac{|\omega|}{\lambda - |1 - \omega|} |U| \right] x = 0 \tag{8}
\]
This shows that \( Q = \left[ I - \frac{\omega - \sigma}{\lambda - |1 - \omega|} \mathbf{L} - \frac{|\omega|}{\lambda - |1 - \omega|} \mathbf{U} \right] \) is singular, which implies that \( Q \) is not SDD, and hence there exist an index \( k \) such that

\[
1 \leq \left| \frac{\omega - \sigma}{\lambda - |1 - \omega|} \right| L_k + \left| \frac{|\omega|}{\lambda - |1 - \omega|} \right| U_k
\]

\[
\Rightarrow |(\lambda - |1 - \omega|)| \leq |(|\omega - \sigma| + \lambda|\sigma|)| L_k + |\omega|U_k
\]

\[
\Rightarrow |\lambda| - |1 - \omega| \leq |\omega - \sigma|L_k + |\lambda| |\sigma| L_k + |\omega|U_k
\]

\[
\Rightarrow |\lambda| - |\lambda||\sigma| L_k \leq |1 - \omega| + |\omega - \sigma|L_k + |\omega|U_k
\]

\[
\Rightarrow |\lambda| \leq \frac{|1 - \omega| + |\omega - \sigma|L_k + |\omega|U_k}{1 - |\sigma|L_k} \quad \text{if } |\sigma| < \frac{1}{L_k}, L_k \neq 0
\]

\[
\leq \max_i \left\{ \frac{|1 - \omega| + |\omega - \sigma|L_i + |\omega|U_i}{1 - |\sigma|L_i} : L_i \neq 0, |\sigma| < \frac{1}{L_i} \right\}
\]  \quad \text{(9)}

Since \( \lambda \) is chosen an arbitrarily eigenvalue of \( C_{\sigma,\omega} \), so (7) holds.

In [6], author proved that interval AOR method converges for irreducible weak diagonally dominant matrices, whenever \( 0 \leq \sigma, \omega \leq 1 \), which motivates us to the convergence of interval AOR method for interval SDD matrices.

**Theorem 3.4.** If \( \mathbf{A} \) is interval SDD matrix, then interval AOR method (3) converges for all \( 0 \leq \sigma \leq \omega \leq 1 \) with \( \omega \neq 0 \).

**Proof.** As \( \mathbf{A} \) is an interval SDD, so \( \mathbf{D} \) is regular and \( \mathbf{D}^{-1}\mathbf{A} \) is also an interval SDD matrix. Hence it suffices to prove the convergence of the iterative scheme (4), or equivalently, to show that \( \rho(C_{\sigma,\omega}) < 1 \).

Suppose that \( \lambda \) is an eigenvalue of \( C_{\sigma,\omega} \) and \( |\lambda| \geq 1 \). Then as shown in (8), we have \( \det(Q) = 0 \), where \( Q = I - \frac{\omega - \sigma + \sigma\lambda}{\lambda - 1 + \omega} \mathbf{L} - \frac{\omega}{\lambda - 1 + \omega} \mathbf{U} \). We now prove that \( Q \) is SDD, which leads to a contradiction that \( Q \) is singular. It suffices to show that

\[
\begin{align*}
|\lambda - 1 + \omega|^2 - |\omega - \sigma + \sigma\lambda|^2 &= \lambda(\lambda - (1 - \omega)) - (\omega - \sigma + \sigma\lambda)(\omega - \sigma + \sigma\lambda) \\
&= \lambda^2\lambda - \lambda(1 - \omega) + \lambda(1 - \omega) + (1 - \omega)^2 - ((\omega - \sigma)^2 + r^2\sigma^2 + 2\sigma(\omega - \sigma)\cos \theta) \\
&\geq r^2 - 2r(1 - \omega)\cos \theta + (1 - \omega)^2 - (\omega - \sigma - r^2\sigma^2 - 2\sigma(\omega - \sigma)) \\
&\geq (r - 1 + \omega)^2 - (\omega - \sigma + \sigma r)^2 \\
&= [(r - 1)(1 + \sigma) + 2\omega](r - 1)(1 - \sigma) > 0
\end{align*}
\]
Again for the second inequality we have
\[ |\lambda - 1 + \omega|^2 - |\omega|^2 = [r^2 - 2r(1 - \omega) \cos \theta + (1 - \omega)^2] - \omega^2 \geq (r + \omega - 1)^2 - \omega^2 = (r + 2\omega - 1)(r - 1) > 0 \]

Hence the theorem holds. \( \square \)

**Example 3.1.** Consider 
\[ \mathbf{A} = \begin{pmatrix} 3 & [-2, 2] & 1 \\ [-1, 1] & 2 & [1, 2] \\ [-2, 2] & 1 & [1, 2] \end{pmatrix} \]
which is not an interval SDD matrix. Take \( \sigma = 0.2, \omega = 0.6 \). Then \( \rho(\widetilde{H}_{\sigma, \omega}) = \rho(C_{\sigma, \omega}) = 1.6085 > 1 \). So interval AOR method doesn’t converge.

**Theorem 3.5.** If \( \mathbf{A} \) is an interval SDD matrix and \( \omega \geq \sigma \geq 0 \) (\( \omega \neq 0 \)), then a sufficient condition for the convergence of interval AOR method is
\[ 0 < \omega < \frac{2}{1 + \max \{L_i + U_i\}} \quad (10) \]
where \( L_i \) and \( U_i \) are the sum of modulus of entries of \( i \)-th row of \( L \in \mathbf{L} \) and \( U \in \mathbf{U} \) respectively.

**Proof.** If \( \omega \leq 1 \), the conclusion follows from Theorem 3.4. Assume that \( \omega > 1 \). Then by the hypothesis we have
\[ \omega < \frac{2}{1 + \max \{L_i + U_i\}} \leq \frac{2}{1 + L_i + U_i} \text{ for all } i \]
that is, \( \omega + \omega L_i + \omega U_i < 2 \), which implies that
\[ (\omega - 1) + (\omega - \sigma)L_i + \omega U_i < 1 - \sigma L_i, \quad \text{for all } i \]
Or,
\[ \max_i \left\{ \frac{|\omega - \sigma L_i| + |\omega| U_i}{1 - \sigma L_i} : L_i \neq 0, \sigma < \frac{1}{L_i} \right\} < 1 \]
Hence \( \rho(C_{\sigma, \omega}) < 1 \) by Theorem 3.3, and so the interval AOR method converges. \( \square \)

**Example 3.2.** Let us consider the interval SDD matrix
\[ \mathbf{A} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \]
Then any \( L \in \mathbf{L} \) and \( U \in \mathbf{U} \) have the form
\[ L = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \end{pmatrix} \]
where \( a \in [-\frac{1}{2}, \frac{1}{2}] \) so that \( \max \{L_i + U_i\} = \frac{3}{4} \). Hence equation (10) asserts that AOR method converges if \( 0 \leq \sigma \leq \omega \leq \frac{8}{7} \), which of course provides a bigger range for \( \omega \) and \( \sigma \) than proposed in Theorem 3.4 for convergence of interval AOR method.

Following result is due to [6] about the convergence of AOR-method for solving linear system \( \mathbf{A} \mathbf{x} = \mathbf{b} \) with the coefficient matrix \( \mathbf{A} \) as an \( \mathbf{L} \)-matrix.

**Theorem 3.6.** If \( \mathbf{A} \) is an \( \mathbf{L} \)-matrix, then for all \( \sigma \) and \( \omega \) such that \( 0 \leq \sigma \leq \omega \leq 1(\omega \neq 0) \), then AOR-method converges for the linear system \( \mathbf{A} \mathbf{x} = \mathbf{b} \) if and only if Jacobi-method converges for the system.
Following theorem provides similar result stated in Theorem 3.6 for linear system of $L$-matrices to linear interval system of interval $L$-matrices.

**Theorem 3.7.** If $A$ is an interval $L$-matrix, then for $0 \leq \sigma \leq \omega \leq 1$ with $\omega \neq 0$, interval AOR method converges for (1) if and only if interval Jacobi method converges for (1).

**Proof.** Suppose that interval Jacobi converges, that is, $\rho(H_{0,1}) < 1$. Note that $A$ is an interval $L$-matrix implies so is $D^{-1}A$. Hence it’s enough to show that $\rho(C_{\sigma,\omega}) < 1$. As discussed in the beginning of the section, $\rho(C_{\sigma,\omega})$ be the eigenvalue of $C_{\sigma,\omega}$.

Set $\lambda = \rho(C_{\sigma,\omega})$. Suppose that $\lambda \geq 1$. Choose a nonegative vector $v \neq 0 \in \mathbb{R}^n$, such that

$$C_{\sigma,\omega}v = \lambda v \Rightarrow (I - \sigma L)^{-1} (1 - \omega)I + (\omega - \sigma)L + \omega U |v = \lambda v$$

$$\Rightarrow | (1 - \omega)I + (\omega - \sigma)L + \omega U |v = (I - \sigma L)\lambda v$$

$$\Rightarrow \left( \frac{\omega - \sigma + \sigma \lambda}{\omega} |L| + |U| \right) v = \frac{\lambda - 1 + \omega}{\omega} v$$

Thus $\frac{\lambda - 1 + \omega}{\omega}$ is an eigenvalue of the nonnegative matrix $\left( \frac{\omega - \sigma + \sigma \lambda}{\omega} |L| + |U| \right)$ and hence

$$\frac{\lambda - 1 + \omega}{\omega} = \left| \frac{\lambda - 1 + \omega}{\omega} \right| \leq \rho\left( \frac{\omega - \sigma + \sigma \lambda}{\omega} |L| + |U| \right) \quad (11)$$

Note that $\frac{\omega - \sigma + \sigma \lambda}{\omega} \geq 1$, and $H_{0,1} = C_{0,1} = |L + U| = |L| + |U|$. Thus we have that

$$0 \leq \frac{\omega - \sigma + \sigma \lambda}{\omega} |L| + |U| \leq \frac{\omega - \sigma + \sigma \lambda}{\omega} (|L| + |U|) = \frac{\omega - \sigma + \sigma \lambda}{\omega} H_{0,1} \quad (12)$$

So, Proposition 1 implies that

$$\rho\left( \frac{\omega - \sigma + \sigma \lambda}{\omega} |L| + |U| \right) \leq \frac{\omega - \sigma + \sigma \lambda}{\omega} \rho(H_{0,1}) < \frac{\omega - \sigma + \sigma \lambda}{\omega}.$$ 

Thus from equation (11) we obtain $\lambda - 1 + \omega < \omega - \sigma + \sigma \lambda$, that is, $\sigma > 1$, which is a contradiction. Hence if interval Jacobi converges then so does the interval AOR method.

The converse part is obvious. \(\square\)

Next we review the convergence analysis of interval AOR method for interval $M$-matrices.

**Theorem 3.8.** If $0 \leq \sigma \leq \omega \leq 1$ and $\omega \neq 0$, then interval AOR method for solving (1) converges for interval $M$-matrix $A$.

**Proof.** Let $A$ be an interval $M$-matrix. Then $A$ is an interval $H$-matrix and hence there exists $u > 0$ such that $\langle A \rangle u > 0$. Again $D \geq 0$ implies that $D^{-1} \geq 0$. Thus by Lemma 3.1 we have that

$$\langle D^{-1}A \rangle u \geq \langle D^{-1} \rangle \langle A \rangle u > 0 \quad (13)$$

Since off-diagonal entries of $\langle D^{-1}A \rangle$ are non-positive, equation (13) implies that $\langle D^{-1}A \rangle$ is an $M$-matrix and hence $\langle D^{-1}A \rangle^{-1} \geq 0$. 


Now,
\[ (\mathbf{D}^{-1}\mathbf{A}) = \begin{cases} \langle \mathbf{D}^{-1}\mathbf{A} \rangle_{ii} = \mathbf{D}^{-1}_{ii} \mathbf{A}_{ii} > 0 & \text{as } \mathbf{D}^{-1}_{ii} > 0, \mathbf{A}_{ii} > 0 \\ -|\mathbf{D}^{-1}\mathbf{A}|_{ij} & \leq 0 \end{cases} \]

From Theorem 2.7 we can say \( (\mathbf{D}^{-1}\mathbf{A}) \), and hence \( \omega(\mathbf{D}^{-1}\mathbf{A}) \) is an \( M \)-matrix. Set \( M = (I - \sigma\mathbf{L}) = I - \sigma|\mathbf{L}| \) and \( N = |(1 - \omega)I + (\omega - \sigma)\mathbf{L} + \omega \mathbf{U}| \) so that \( \omega(\mathbf{D}^{-1}\mathbf{A}) = M - N \) and \( C_{\sigma,\omega} = M^{-1}N \). Note that \( M \) is an \( M \)-matrix and \( N \geq 0 \), so Theorem 2.12 implies that \( \rho(C_{\sigma,\omega}) < 1 \). Thus interval AOR method converges for interval \( M \)-matrices.

We now end with the convergence analysis of AOR method for interval \( H \)-matrices.

**Theorem 3.9.** If \( 0 \leq \sigma \leq \omega \leq 1 \) \( (\omega \neq 0) \), then interval AOR method converges for interval \( H \)-matrices.

**Proof.** Let \( \mathbf{A} \) be an interval \( H \)-matrix. Then \( \langle \mathbf{A} \rangle = (\mathbf{D}) - |\mathbf{E}| - |\mathbf{F}| \) is an \( M \)-matrix, and hence by Theorem 2.7, there exists \( u > 0 \) such that \( \langle \mathbf{A} \rangle u > 0 \). We write \( M = (\mathbf{D}) - \sigma|\mathbf{E}| \) and \( N = (1 - \omega)|\mathbf{D}| + (\omega - \sigma)|\mathbf{E}| + \omega|\mathbf{F}| \). Note that \( M \) is a \( Z \)-matrix and

\[
M u = \langle \mathbf{A} \rangle u + (1 - \sigma)|\mathbf{E}|u + |\mathbf{F}|u > 0
\]

Therefore, \( M \) is an \( M \)-matrix and hence \( M^{-1} \geq 0 \). Furthermore \( N \geq 0 \). Thus \( \rho(C_{\sigma,\omega}) = \rho(M^{-1}N) < 1 \) by Theorem 2.13. This shows that interval AOR method converges for interval \( H \)-matrices. \( \square \)

4. **Numerical Examples.** In this section we consider numerical examples to compare the interval AOR method with the existing method. As interval Gauss-Seidel can be obtained by taking \( \sigma = \omega = 1 \), we compare the method with interval Gauss-Seidel method. The computations are carried out in MATLAB(2018b) with the latest released interval toolbox INTLAB v12 [24] and on a machine with Processor 2.3 GHz Intel Core i5 and CPU 4 GB 1600 MHz. The computations are rounding to four digits error bound is taken as \( 10^{-2} \).

**Example 4.1.** Consider the linear interval system \( \mathbf{A}\mathbf{x} = \mathbf{b} \), with the coefficient interval matrix

\[
\mathbf{A} = \begin{pmatrix} 2 & [-1,0] \\ [-1,0] & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1.2 \\ -1.2 \end{pmatrix}.
\]

The function \texttt{verifylss} from the package INTLAB [24] provides the enclosure

\[
\mathbf{x} = ([0.2399, 0.7201], [-0.7201, -0.2399])^T
\]

Observe that \( \mathbf{A} \) is an interval \( M \)-matrix (hence interval \( L \)-matrix), which is further an interval \( H \)-matrix. Thus by Theorem 3.7, 3.8 or, 3.9 for \( 0 \leq \sigma \leq \omega \), the interval AOR method converges. Take \( \sigma = 0.2 \) and \( \omega = 0.6 \). Starting with the initial guess \( \mathbf{x}_0 = ([ -1,1], [-1,1])^T \), interval Gauss-Seidel and interval AOR methods, respectively, yield the enclosures \( \mathbf{x}_{GS} \) and \( \mathbf{x}_{AOR} \), after 10th iteration steps, which are given by

\[
\mathbf{x}_{GS} = ([0.1999, 0.7601], [-0.8001, -0.1599])^T
\]
\[
\mathbf{x}_{AOR} = ([0.0323, 0.6933], [-0.6793, -0.4602])^T
\]

This shows that interval AOR method provides a tighter enclosure than interval Gauss-Seidel method and the using the \texttt{verifylss} function.
Furthermore, the solution set is given by [19]

\[ \Sigma(A, b) = \left\{ \left( \begin{array}{c} 1.2(2 - a_1)/(4 - a_1a_2) \\ 1.2(a_2 - 2)/(4 - a_1a_2) \end{array} \right) : a_1, a_2 \in [0, 1] \right\} \]

and bounded region is shown in the following Figure 4.1:

**Example 4.2.** Consider the linear interval system \( Ax = b \), with the coefficient interval matrix

\[ A = \begin{pmatrix} 2 & -1 \\ [-6, -2] & 3.5 \end{pmatrix} \text{ and } b = \begin{pmatrix} 3 \\ 3 \end{pmatrix}. \]

The bounded region of the solution set \( \Sigma(A, b) \) is given by the following figure:
From the above figure we observe that

$$\square \Sigma(A, b) := ([−2.5, 7.5], [−8, 12])^T$$

The function `verifylss` from the package INTLAB [24] provides the enclosure

$$x = ([−8.1001, 13.5001], [−25.6800, 30.4800])^T$$

Observe that $A$ is an interval $M$-matrix. Taking $\sigma = 0.5$ and $\omega = 0.5$ and with the starting initial guess $x_0 = ([−1, 2], [−2, 2])^T$, interval Gauss-Seidel and interval AOR methods, respectively, yield the enclosures $x_{GS}$ and $x_{AOR}$, after 100th iteration steps, which are given by

$$x_{GS} = ([−8.1000, 13.5000], [−19.2000, 24.0000])^T$$

$$x_{AOR} = ([−4.0757, 9.6370], [−8.6734, 12.6575])^T$$

This shows that interval AOR method provides a tighter enclosure than interval Gauss-Seidel method and the using the `verifylss` function.

**Example 4.3.** [20] Consider the linear interval system (1) with

$$A := \begin{pmatrix} [2, 4] & [−1, 1] \\ [−1, 1] & [2, 4] \end{pmatrix}, \quad \text{and} \quad b := \begin{pmatrix} [−3, 3] \\ 0 \end{pmatrix}.$$  

Then the solution set $\Sigma(A, b)$ of (1) is given by the Figure 1 and

$$\square \Sigma(A, b) := ([−2, 2], [−1, 1])^T$$

Observe that $A$ is an interval SDD matrix and an interval $H$-matrix also. The function `verifylss` from the package INTLAB [24] yields the enclosure

$$x = ([−2.0010, 2.0010], [−1.0020, 1.0020])^T$$
Taking the initial guess \( x_0 = ([−1, 1], [−1, 1])^T \), interval Gauss-Seidel converges after 1st iteration and gives the enclosure

\[ x_{GS} = ([−2.0001, 2.0001], [−1.0001, 1.0001])^T \]

But with \( \sigma = 0.2 \) and \( \omega = 0.6 \) interval AOR method converges after 20th iteration and yields the enclosures

\[ x_{AOR} = ([−3.4265, 3.4265], [−1.7129, 1.7129])^T \]

This shows that interval GS method provides the most tightest enclosure of the solution set, whereas interval AOR gives the loosest one among these.

**Example 4.4.** [20] Consider the linear interval system (1) with

\[
A := \begin{pmatrix}
3 & [−2, 2] & 0 \\
0 & 3 & [−2, 2] \\
[−2, 2] & 0 & 3
\end{pmatrix}, \quad \text{and} \quad b := \begin{pmatrix}
6 \\
6 \\
6
\end{pmatrix}.
\]

Note that \( A \) is an interval SDD matrix, which is also an interval \( H \)-matrix, and the function `verifylss` from the package INTLAB [24] gives the enclosure

\[ x = ([−2.0001, 6.0001], [−2.0001, 6.0001], [−2.0001, 6.0001])^T \]

Taking the initial guess \( x_0 = ([−1, 3], [−1, 3], [−1, 3])^T \), interval Gauss-Seidel converges after 20th iteration and gives the enclosure

\[ x_{GS} = ([−2.0000, 6.0000], [−2.0000, 6.0000], [−2.0000, 6.0000])^T \]

But with \( \omega = 0.9 \) and \( \sigma = 0.8 \), interval AOR methods converges after 20th iteration and yields the enclosures

\[ x_{AOR} = ([−2.7848, 6.7848], [−2.4597, 6.4597], [−2.0206, 6.0206])^T \]

In this example also interval GS method provides the most tightest enclosure of the solution set.

### 5. Conclusion.

In this paper we considered interval AOR method introduced in [3] for solving system of linear interval equations. In particular we discussed the convergence analysis of interval AOR method for interval strictly diagonally dominant matrices, interval \( L \)-matrices, interval \( M \)-matrices and interval \( H \)-matrices, by restricting the acceleration parameter \( \sigma \) and the relaxation parameter \( \omega \). We also obtained a bound for the spectral radius of iteration matrix of interval AOR method. Lastly, numerical examples are considered for each class of matrices under reviewed, which illustrated the fact that the interval AOR method yields a tighter enclosure of the solution set.

Furthermore, we observed from the numerical illustrations that interval AOR may not give the best enclosure for interval SDD matrices, whereas interval GS method provides the tightest enclosure of the solution set. Since interval GS is a special case of interval AOR method and Theorem 3.4 assures that for \( 0 \leq \sigma \leq \omega \leq 1 \), interval AOR method converges, it is an open problem to investigate that interval AOR method provides the tightest enclosure for interval SDD matrices if \( \omega = \sigma = 1 \).
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