BSDEs driven by $|z|^2/y$ and applications

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October 16, 2018

Abstract

Quadratic backward stochastic differential equations with singularity in the value process appear in several applications, including stochastic control and physics. In this paper, we prove existence and uniqueness of equations with generators (dominated by a function) of the form $|z|^2/y$. In the particular case where the BSDE is Markovian, we obtain existence of viscosity solutions of singular quadratic PDEs with and without Neumann lateral boundaries, and rather weak assumptions on the regularity of the coefficients. Furthermore, we show how our results can be applied to optimization problems in finance.

Keywords: Domination condition; singularity at zero; viscosity solutions; probabilistic representation, decision theory in finance.

MSC 2010: 60H10, 60H20, 35K58, 35K67, 91G10.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space carrying a $d$-dimensional Brownian motion denoted $W$ and equipped with the $\mathbb{P}$-completion of the filtration $\sigma(W_s, s \leq t \leq T)$ generated by $W$, with $T \in (0, \infty)$. The aim of this article is to give conditions for existence and uniqueness of quadratic backward stochastic differential equations (BSDEs) of the form

$$Y_t = \xi + \int_t^T H(u, Y_u, Z_u) \, du - \int_t^T Z_u \, dW_u,$$

(1.1)

when the function $H$ (the generator) has a possible singularity at zero, and to investigate a few applications notably in singular PDE theory. Here is a tidbit of our results.

Proposition 1.1. Let $\xi$ be a random variable such that $\mathbb{E}[|\xi|^3] < \infty$, with $\xi > 0 \mathbb{P}$-a.s. or $\xi < 0 \mathbb{P}$-a.s., put $H(t, y, z) = |z|^2/y$. Then, the BSDE (1.1) has a solution $(Y, Z)$ such that $\mathbb{E}[\sup_{0 \leq t \leq T} |Y_t|^2] + \mathbb{E}[\int_0^T |Z_t|^2 \, dt] < \infty$. Moreover, uniqueness holds among solutions such that $Y$ is in the class $(D)$.

The definition of the class $(D)$ will be recalled in Section 2. We emphasize that the terminal condition $\xi$ is not assumed bounded, and can be arbitrarily close to zero. The generator in the statement of Proposition 1.1 is the canonical form of generators we consider. In particular, the generators will not necessarily be locally Lipschitz continuous, so that the equation will not be amenable to techniques involving Picard iterations, a priori estimates or localizations and approximations which are prevalent in the literature.

The method we develop in the present article is rather based on a combination of a simple change of variable technique akin to Zvonkin’s transform in the theory of stochastic differential equations and a domination method developed in Bahlali [3]. This allows to show (see Theorem 2.2) that when $H$ is continuous and satisfies

$$0 \leq H(t, \omega, y, z) \leq \alpha_t(\omega) + \beta_t(\omega)y + \gamma_t(\omega)z + \frac{1}{y}|z|^2 \quad \text{for } y > 0$$

(1.2)

for some (not necessarily bounded) processes $\alpha$, $\beta$ and $\gamma$, the BSDE (1.1) admits a strictly positive solution. Here again, $\xi$ is almost surely strictly positive. Also due to the singularity of the generator, the proof we give of uniqueness for generators of the form (1.2) does not rely on comparison principles as customary. We instead base our arguments on
convex duality techniques for BSDEs initiated by Drapeau et al. [22]. This allows us to guarantee uniqueness of the value process \( Y \). It is not known whether the control process \( Z \) is unique except in the canonical case of Proposition 1.1. The uniqueness of the value process will already be enough for our main applications.

To the best of our knowledge, the only papers dealing with existence (without uniqueness) of quadratic BSDEs of the form \( f(y)|z|^2 \) (with \( f \) not constant) are the papers of Bahlali et al. [7], Duffie and Epstein [24] and Bahlali [3]. In the works of Bahlali et al. [7] and Duffie and Epstein [24], the function \( f \) is assumed to be integrable (and even continuous in the second paper) and in Bahlali [3], it is assumed to be locally integrable. The case \( f(y) = 1/y \) requires a special treatment and is of special practical interest.

In financial mathematics and economics, BSDEs with generators of the form \(|z|^2/y\) naturally appear in problems of optimal investment and decision theory, (see e.g. Nutz [49], Heyne et al. [33], Xing [56] and Epstein and Zin [25]) but also in interest rates problems (see Hyndman [35]). We also refer to Subsection 5.1 below where we discuss a portfolio optimization problem. When specializing to the Markovian framework, BSDEs considered in the present article allow to derive viscosity solutions of some singular semilinear PDEs (with and without Neumann lateral boundaries) with quadratic nonlinearity in the gradient whose simplest model is

\[
\begin{cases}
  v_t + \frac{1}{2} \Delta v = -\frac{\nabla v|z|^2}{v} \\
  v(0, x) = h(x)
\end{cases}
\]

(1.3)

with the gradient and laplacian operators acting on the spatial variable. This type of equations is being the subject of a sustained research interest, see e.g. [1, 2, 10, 13, 29, 46, 47] and references therein. This is due to at least two reasons. As explained by Molino [47] and Boccardo and Orsina [10], Equation (1.3) is a simplified form of Euler-Lagrange equation. In physics, Equation (1.3) appears in modeling of quenching problems (Dong and Levine [21], Merle [45], Merle and Zaag [46], Chapman et al. [14]) and in the study of model gas flow in porous media (Giachetti et al. [28], Giachetti and Murat [29]). We emphasize that the existing literature deals with existence (and properties) of Sobolev solutions in the \( H^1_0 \)-sense. Note in passing that the link between BSDEs and Sobolev solutions of parabolic PDEs was well-explained by Barles and Lesigne [8].

In this work we are interested in existence of viscosity solutions in the sense of Crandall et al. [17]. It is well-known that existence of \( H^1_0 \) solutions does not imply that of viscosity solutions and vice versa. Moreover, we prove a probabilistic representation of solutions. This is particularly relevant for numerical computations of such PDEs using Monte-Carlo approximation or neural networks (see Gobet et al. [30] and Han et al. [32]).

Let us now say a few words on the extensive literature on BSDEs with and their connections to parabolic PDEs. When the generator \( H \) is Lipschitz continuous (in \((y,z)\)) and \( \xi \) is square integrable, Pardoux and Peng [51] proved existence and uniqueness of a square integrable solution \((Y,Z)\). The case where the generator can have quadratic growth in \( z \) (i.e. grows slower than \(|z|^2\)) is particularly relevant in several applications. It has been initially investigated by Kobylniski [39] for bounded terminal conditions \( \xi \). This result has been extended in a number of papers, including Briand and Hu [12], Cheridito and Nam [15], Delbaen et al. [20], Tevzadze [55] and Barrieu and El Karoui [9]. The approach of Briand and Hu [12] is based on a localization technique. Cheridito and Nam [15] show that the problem can be reduced to the Lipschitz case when the terminal condition has bounded Malliavin derivative, see also Hamadène [31]. Barrieu and El Karoui [9] derive existence from a stability property for the so-called quadratic semimartingales. As commonly assumed in the literature, all the above mentioned works assume \( H \) continuous in \((y,z)\), or even locally Lipschitz, and that the terminal condition is bounded, or has exponential moments.

In the next section we state the main existence and uniqueness results for BSDEs of the form \(|z|^2/y\). The proofs are given in Section 3. The final section is devoted to applications. We start by proving existence of viscosity solutions of singular parabolic PDEs first when there are no boundary conditions and then considering singular PDEs with lateral Neumann boundary conditions. In Section 5, we discuss applications to finance and economics.
2. Main results

Consider the following spaces and norms: For $p > 0$, we denote by $L^p_{\text{loc}}(\mathbb{R})$ the space of (classes) of functions $u$ defined on $\mathbb{R}$ which are $p$-integrable on bounded subsets of $\mathbb{R}$. We also denote, $W^2_{p, \text{loc}}(\mathbb{R})$ the Sobolev space of (classes) of functions $u$ defined on $\mathbb{R}$ such that both $u$ and its generalized derivatives $u'$ and $u''$ belong to $L^p_{\text{loc}}(\mathbb{R})$. By $C$ we denote the space of continuous and $\mathcal{F}_t$-adapted processes. By $S^p(\mathbb{R})$ we denote the space of continuous, $\mathcal{F}_t$-adapted processes $Y$ such that $E_{\tau < T} |Y_t|^p < \infty$, and $S^\infty$ the space of processes $Y \in S^p$ such that $\sup_{0 \leq t \leq T} |Y_t| \in L^\infty$.

The set $S^p(\mathbb{R})$ denotes the positive elements of $S^p(\mathbb{R})$. Let $\mathcal{M}^p(\mathbb{R})$ be the space of $\mathbb{R}^d$-valued $\mathcal{F}_t$-progressive processes $Z$ satisfying $\mathbb{E} \left[ (\int_0^T |Z_t|^2 dt)^{\frac{p}{2}} \right] < +\infty$. By $L^2$ we denote the space of $\mathcal{F}_t$-progressive processes $Z$ satisfying $\int_0^T |Z_t|^2 ds < +\infty$ $P$-a.s. BMO is the space of uniformly integrable martingales $M$ satisfying

$$\sup_{\tau} ||\mathbb{E}[M_T - M_{\tau}]||_{\infty} < \infty$$

where the supremum is taken over all stopping times $\tau$. A process $Y$ is said to belong to the class (D) if $\{Y_{\tau} : \tau$ stopping times $\}$ is uniformly integrable.

**Definition 2.1.** Given $\xi \in L^0$ and a measurable function $H : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, we denote by BSDE$(\xi, H)$ the BSDE with terminal condition $\xi$ and generator $H$, i.e.

$$Y_t = \xi + \int_t^T H(s, Y_s, Z_s) \, ds - \int_t^T Z_s \, dW_s, \quad t \in [0, T]. \quad (2.1)$$

A solution to BSDE$(\xi, H)$ is a process $(Y, Z)$ which belongs to $C \times \mathcal{L}^2$ such that $(Y, Z)$ satisfies equation BSDE$(\xi, H)$ for each $t \in [0, T]$ and $\int_0^T |H(s, Y_s, Z_s)| ds < \infty$ $P$-a.s.

Our first main result gives existence of BSDEs having generators with growth of the form $|z|^2/y$. Let $\alpha, \beta : [0, T] \times \Omega \to \mathbb{R}_+$ and $\gamma : [0, T] \times \Omega \to \mathbb{R}^d$ be progressively measurable processes and consider the following conditions:

(A1) $H : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is progressively measurable and continuous for each fixed $(t, \omega)$, and it holds

$$0 \leq H(t, \omega, y, z) \leq \alpha(t, \omega)y + \beta(t, \omega)z + \frac{1}{y}|z|^2 =: g(t, y, z) \quad (2.2)$$

for all $(y, z) \in \mathbb{R}_+ \times \mathbb{R}^d$.

(A2) $\alpha, \beta \in S^\infty_+, \gamma \in \mathcal{H}^6$, the terminal condition $\xi > 0$, there is $p > 1$ such that $\xi \in L^{3p}$, $e^{\int_0^T \gamma_s dW_s} \in L^q$ with $1/p + 1/q = 1$ and

$$\mathbb{E} \left[ \xi^{3p} e^{\int_0^T \lambda_\omega du} \right] < \infty \quad \text{and} \quad \mathbb{E} \left[ (\int_0^T e^{\int_s^T \lambda_\omega du} \alpha_s ds)^p \right] < \infty,$$

with $\lambda_t := 3(\alpha_t + \beta_t) + \frac{|\gamma_t|^2}{2r}$ for some $r \in (0, \frac{1\wedge(p-1)}{2})$.

(A3) $\alpha, \beta \in S^\infty_+, \gamma \in \mathcal{H}^6$, the terminal condition $\xi > 0$, there is $p > 1$ such that $\xi \in L^{3p}$, $e^{\int_0^T \gamma_s dW_s} \in L^q$ and

$$e^{\int_0^T \alpha_s + \beta_s du} \xi \in L^{3p} \quad \text{and} \quad \int_0^T e^{\int_0^T \alpha_s + \beta_s du} \alpha_s ds \in L^p$$

with $1/p + 1/q = 1$. 
(A1') \( H : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \) is progressively measurable and continuous for each fixed \((t, \omega)\), and it holds
\[
0 \geq H(t, \omega, y, z) \geq -\alpha_t(\omega) - \beta_t(\omega)y - \gamma_t(\omega)z - \frac{1}{y}|z|^2 =: g(t, y, z) \tag{2.3}
\]
for all \((y, z) \in \mathbb{R} \times \mathbb{R}^d\).

(A2') \( \xi < 0 \) and satisfies the integrability conditions stated in (A2) along with \( \alpha, \beta \) and \( \gamma \).

(A3') \( \xi < 0 \) and satisfies the integrability conditions stated in (A3) along with \( \alpha, \beta \) and \( \gamma \).

**Theorem 2.2.** If (A1) and (A2) hold, (resp. (A1') and (A2') hold), then the BSDE\((\xi, H)\) admits a solution \((Y, Z) \in S^\| \times L^2 \) such that \( 0 < Y \leq Y^g \), (resp. \( Y^{-g} \leq Y < 0 \)), where \((Y^g, Z^g)\) and \((Y^{-g}, Z^{-g})\) solve BSDE\((\xi, g)\) and BSDE\((\xi, -g)\), respectively.

If instead of (A2), condition (A3) holds (resp. (A3') instead of (A2')), then the solution \((Y, Z)\) satisfies
\[
\sup_{t \in [0, T]} \mathbb{E}[|Y_t|^3] < \infty \text{ and } Z \in L^2.
\]

The proof is given in Subsection 3.2 below. Under slightly stronger conditions, the BSDE with generator driven by \(|z|^2/y\) further admits a unique value process.

**Theorem 2.3.** Assume that \( \xi \in L^{\infty}, \alpha, \beta \in S^{\infty}, \int \gamma \, dW \) is a BMO martingale and that for each \((t, \omega)\), the function \(H(t, \omega, \cdot, \cdot)\) is jointly convex. If (A1) and \( \xi > 0 \), (resp. (A1') and \( \xi < 0 \) then, for every solution \((Y, Z)\) of BSDE\((\xi, H)\) such that \( 0 < Y \leq Y^g \) (resp. \( Y^{-g} \leq Y < 0 \)), the process \(Y\) is bounded and \(\int Z \, dW\) is a BMO martingale. Moreover, for every solutions \((Y, Z), (Y', Z')\) satisfying \(0 < Y, Y' \leq Y^g\) (resp. \(Y^{-g} \leq Y, Y' < 0\)) the processes \(Y\) and \(Y'\) are indistinguishable.

Along with the existence result, the above uniqueness result is crucial for existence of viscosity solutions of a class of singular parabolic PDEs. For instance, our results will allow to solve the PDE
\[
\begin{align*}
\frac{\partial v}{\partial t}(s, x) + b(t, x)\nabla v(s, x) + \frac{1}{2}\Delta v(s, x) + \frac{1}{4} \frac{\nabla v}{v}(s, x) & = 0, \quad \text{on } [0, T) \times \mathbb{R}^d, \\
v(t, x) > 0 & \quad \text{on } [0, T) \times \mathbb{R}^d, \\
v(T, x) & = h(x)
\end{align*}
\]
when the functions \(b\) and \(h\) are merely continuous in \(x\), with additional growth conditions, see Theorem 4.6 and Remark 4.7. This will be developed in Subsection 4.1 for parabolic PDEs with no boundary conditions and Subsection 4.3 for the case of PDEs with lateral Neumann boundaries. In Section 5 we provide two applications of the this existence result to economics, namely to utility maximization with random endowment and to the existence of Kreps-Porteus stochastic differential utility in continuous time.

### 3. Proofs of the main results

#### 3.1. Preliminaries

Since BSDEs with linear generators play an important role in our arguments, we summarize below some of their properties.

**Lemma 3.1.** Let \( \xi \) be an \( F_T \)-measurable random variable and \( \gamma : [0, T] \times \Omega \to \mathbb{R}^d \) a progressively measurable process. Consider the BSDE
\[
Y_t = \xi + \int_t^T \gamma_s Z_s \, ds - \int_t^T Z_s \, dW_s. \tag{3.1}
\]
Definition 3.2. \(H\)

\[\text{Proof.} \quad \text{Since the probability measure}\]
\[\text{representation theorem, see e.g. [53, Corollary 5.2.3]. Moreover,} \]
\[\text{ξ ∈ L}^1(\Omega) \text{ and holds}\]
\[Y_t = \mathbb{E}_{\xi}^{Q^\gamma}[\xi | \mathcal{F}_t]. \quad \text{(3.2)}\]

This shows in particular, due to Hölder’s inequality, that \(\sup_{0 \leq t \leq T} \mathbb{E}[|Y_t|] < \infty\). That \(\int Z \, dW\) is a local martingale

follows by Girsanov’s theorem.

If \(\xi > 0\), then by (3.2) and the fact that \(Q^\gamma\) is equivalent to \(P\) we have \(Y > 0\). The latter argument further shows that

if \(\xi = 0\), then \(Y = 0\) and \(Z = 0\), and if \(\xi \neq 0\) we must have \(Y \neq 0\).

Now, assume \(\gamma = 0\). If \((Y, Z)\) is a solution such that \(Y > 0\), then \(\xi > 0\). Let \(\tau_n\) be a localizing sequence such that \(\int_0^{\tau_n} Z \, dW\) is a martingale. Then, it holds \(Y_0 = \mathbb{E}[Y_{\tau_n}]\) and by Fatou’s lemma and continuity of \(Y\), this implies \(\mathbb{E}[\xi] \leq Y_0 < \infty\).

Reciprocally, if \(\xi \in L^1\), the proof goes as in [3]. \(\Box\)

Another important tool in our arguments is a so-called existence by domination result which we now present.

Definition 3.2. (Domination conditions) We say that the data \((\xi, H)\) satisfy a domination condition if there exist two progressively measurable processes \(H_1\) and \(H_2\) and two \(\mathcal{F}_T\)-measurable random variables \(\xi_1\) and \(\xi_2\) satisfying

(a) \(\xi_1 \leq \xi \leq \xi_2\)

(b) BSDE(\(\xi_1, H_1\)) and BSDE(\(\xi_2, H_2\)) have two solutions \((Y^1, Z^1)\) and \((Y^2, Z^2)\) respectively, such that:

(i) \(Y^1 \leq Y^2\),

(ii) for every \((t, \omega), y \in [Y_t^1(\omega), Y_t^2(\omega)]\) and \(z \in \mathbb{R}^d\), it holds \(H_1(t, y, z) \leq H(t, y, z) \leq H_2(t, y, z)\) and \(|H(t, \omega, y, z)| \leq \eta(\omega) + C(\omega)|z|^2\).
where $C$ and $\eta$ are $\mathcal{F}_t$-adapted processes such that $C$ is continuous and $\eta$ satisfies for each $\omega$, $\int_0^T |\eta_t(\omega)|ds < \infty$.

**Lemma 3.3.** *(Existence by domination)* Let $H$ be continuous in $(y, z)$ for a.e. $(t, \omega)$. Assume moreover that $(\xi, H)$ satisfy the domination conditions (D1)–(D2). Then, BSDE$(\xi, H)$ has at least one solution $(Y, Z) \in C \times L^2$ such that $Y^1 \leq Y \leq Y^2$. Moreover, among all solutions which lie between $Y^1$ and $Y^2$, there exist a maximal and a minimal solution.

This lemma, whose proof can be found in [3], is an intermediate value-type theorem. It directly gives the existence of solutions. Neither a priori estimates nor approximations are needed. The idea of the proof consists in deriving the existence of solutions for the BSDE without reflection from solutions of a suitable quadratic BSDE with two reflecting barriers obtained by [26, Theorem 3.2]. Note that the latter result is established without assuming any integrability conditions on the terminal value.

### 3.2. Existence

We start by giving the argument for the proof of Proposition 1.1.

**Proof (of Proposition 1.1).** The function

$$u(y) := \frac{1}{3} y^3$$

is a twice continuously differentiable function which is one to one from $\mathbb{R}$ onto $\mathbb{R}$. Moreover, its inverse $v := u^{-1}$ is also twice continuously differentiable from $\mathbb{R}^*$ to $\mathbb{R}$. Therefore, Itô’s formula shows that BSDE$(\xi, \frac{1}{y} |z|^2)$ has a solution if and only if BSDE$(\frac{1}{y} \xi^3, 0)$ has a strictly positive (or a strictly negative) solution. According to Dudley’s representation theorem, BSDE$(\frac{1}{y} \xi^3, 0)$ has a solution for any $\mathcal{F}_T$-measurable random variable $\xi$. No integrability is needed for $\xi$. But, in order to apply Itô’s formula to the function $u^{-1}(x) = (3x)^{\frac{1}{3}}$, we need that BSDE$(\frac{1}{y} \xi^3, 0)$ has a strictly positive (or strictly negative) solution. This holds when $\xi$ belongs to $L^3$, $\xi \neq 0$ and has a constant sign (> 0 or < 0). In this case, according to Lemma 3.1, BSDE$(\frac{1}{y} \xi^3, 0)$ has a unique solution $(\bar{Y}, \bar{Z})$ such that $\bar{Y}$ belongs to class $(D)$ and $\bar{Z}$ belongs to $\mathcal{M}^p$ for each $0 < p < 1$. Putting $\bar{Y} := (3\bar{Y})^{\frac{1}{3}}$, there is $Z$ such that $(\bar{Y}, Z)$ solves BSDE$(\frac{1}{y} |z|^2)$ and we have

$$\sup_{0 \leq \tau \leq T} \mathbb{E} |(\bar{Y}_\tau)^3| = \frac{1}{3} \sup_{0 \leq \tau \leq T} \mathbb{E} |\bar{Y}_\tau| < \infty.$$  \hfill (3.4)

where the supremum is taken over all stopping time $\tau \leq T$.

We shall show that $(Y, Z)$ belongs to $S^2(\mathbb{R}) \times \mathcal{M}^2(\mathbb{R}^d)$. Itô’s formula gives

$$|Y_t|^2 = |\xi|^2 + \int_t^T |Z_s|^2 ds - 2 \int_t^T Y_s Z_s dW_s.$$  For $R > 0$, let $\tau_R := \inf \{ t \geq 0; \int_t^T |Y_s Z_s|^2 ds \geq R \} \wedge T$. It holds $\mathbb{E}[Y_0] = \mathbb{E}[\xi^2] + \mathbb{E} \int_0^{T \wedge \tau_R} |Z_s|^2 ds$. Since $\tau_R \to \infty$ as $R \to \infty$, we deduce that there is a constant $K \geq 0$ such that

$$\mathbb{E} \int_0^T |Z_s|^2 ds \leq \mathbb{E}[\xi^2] + K.$$  \hfill (3.5)

We now prove that $Y$ belongs to $S^2$. Using Itô’s formula and Doob’s inequality, it follows that there exists a universal
constant \( \ell \) such that for any \( \varepsilon > 0 \)
\[
\mathbb{E} \left[ \sup_{0 \leq s \leq T} |Y_t|^2 \right] \leq \mathbb{E}[|\xi|^2] + \mathbb{E} \int_0^T |Z_s|^2 ds + 2\mathbb{E} \left[ \sup_{0 \leq s \leq T} |\int_t^T Y_s Z_s dW_s| \right]
\]
\[
\leq \mathbb{E}[|\xi|^2] + \mathbb{E} \int_0^T |Z_s|^2 ds + 2\varepsilon \mathbb{E} \left[ \sup_{0 \leq s \leq T} |Y_t|^2 \right] + \varepsilon \mathbb{E} \int_0^T |Z_s|^2 ds.
\]
Taking \( \varepsilon = 4\ell \), we deduce that
\[
\mathbb{E} \left[ \sup_{0 \leq s \leq T} |Y_t|^2 \right] \leq 2\mathbb{E}[|\xi|^2] + 2(4\ell + 1)\mathbb{E} \int_0^T |Z_s|^2 ds
\]
which shows that \( Y \) belongs to \( S^2(\mathbb{R}) \).

Assume \( \xi > 0 \) and let \( (\tilde{Y}, \tilde{Z}), (\bar{Y}, \bar{Z}) \in S^2(\mathbb{R}) \times \mathcal{M}^2(\mathbb{R}^d) \) be two strictly positive solutions of BSDE\((\xi, \frac{1}{y}|z|^2)\).

Applying Itô’s formula shows that there are progressive processes \( \tilde{Z}' \) and \( \bar{Z}' \) such that \( (\tilde{Y}', \tilde{Z}') \) and \( (\bar{Y}', \bar{Z}') \) solve BSDE\((\xi, 0)\), with \( \tilde{Y}' = \frac{1}{\bar{Y}'} \tilde{Y} \) and \( \bar{Y}' = \frac{1}{\tilde{Y}'} \bar{Y} \). Since \( \tilde{Y}' \) and \( \bar{Y}' \) are of class \( (D) \), it follows by Lemma 3.1 that \( \tilde{Y}' = \bar{Y}' \) and \( \tilde{Z}' = \bar{Z}' \). The case \( \xi < 0 \) is proved analogously. Proposition 1.1 is proved. \( \square \)

3.2.1. BSDE\((\xi, g)\)

In this subsection the quadratic BSDE under consideration is
\[
Y_t = \xi + \int_t^T \left( \alpha_s + \beta_s Y_s + \gamma_s Z_s + \frac{1}{Y_s} |Z_s|^2 \right) ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T \tag{3.6}
\]
where \( \alpha, \beta : [0, T] \times \Omega \rightarrow \mathbb{R}_+ \) and \( \gamma : [0, T] \times \Omega \rightarrow \mathbb{R}^d \) are progressively measurable processes and \( \delta \in \mathbb{R}_+ \). We put
\[
g(t, \omega, y, z) = \alpha_t(\omega) + \beta_t(\omega)y + \gamma_t(\omega)z + \frac{1}{y} |z|^2. \tag{3.7}
\]

**Proposition 3.4.** Assume that \( (A2) \) holds and \( g \geq 0 \) on \( \mathbb{R}_+ \times \mathbb{R}^d \) (resp. \( (A2') \) holds and \( g \leq 0 \) on \( \mathbb{R}_- \times \mathbb{R}^d \)). Then, the BSDE (3.6) has a solution \( (Y, Z) \in S^{2p} \times \mathcal{M}^2 \) such that \( Y > 0 \) (resp. \( Y < 0 \)).

If \( (A2) \) is replaced by \( (A3) \) (resp. \( (A2') \) replaced by \( (A3') \)), then the solution \( (Y, Z) \) satisfies the integrability \( \sup_{t \in [0, T]} \mathbb{E}[|Y_t|^p] < \infty \) and \( Z \in \mathcal{M}^2 \).

**Proof.** We consider only the case \( \xi > 0 \), i.e. \( (A2) \). The second case goes similarly.

The BSDE
\[
Y_t = \xi + \int_t^T \left( \alpha_s + \beta_s Y_s + \gamma_s Z_s + \frac{1}{Y_s} |Z_s|^2 \right) ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T \tag{3.8}
\]
admits a solution if and only if the BSDE
\[
\tilde{Y}_t = \tilde{\xi} + \int_t^T (9^{1/3} \alpha_s(Y_s)^{2/3} + 3\beta_s \tilde{Y}_s + \gamma_s \tilde{Z}_s) ds - \int_t^T \tilde{Z}_s dW_s \tag{3.9}
\]
also does, where $\xi := \frac{1}{3}\xi^3$. This follows from Itô’s formula applied to $\frac{1}{3}Y^3$ and $(3Y)^{1/3}$. Note that in order to apply Itô’s formula to $(3Y)^{1/3}$, we need that $Y$ be strictly positive (or strictly negative). We will show below that this holds when $\xi > 0$ (or $\xi < 0$).

In order to prove existence of a solution to (3.9), we will apply Lemma 3.3 with $\xi_1 = \xi_2 = \xi$, $H_1(t, z) = \gamma tz$, $H_2(t, y, z) = 9^{1/3}\alpha_1(y)^{2/3} + 3\beta ty + \gamma tz$ and $H_2(t, y, z) = 3(\alpha_1 + (\alpha_1 + \beta_1)y) + \gamma tz$. Since $\xi^3 \in L^p$, and $\exp \left(\int_0^T \gamma_0 dW_t\right) \in L^q$, we clearly have

- $Y' = \mathbb{E}_{Q_1}[\xi | \mathcal{F}_t]$ is a solution of BSDE($\xi, H_1$) and $Y' > 0$,
- $H_2(t, y, z) \leq H_2(t, y, z)$ for every $y > 0$ and every $z \in \mathbb{R}^d$.

By [6, Theorem 2.1] the BSDE($\xi_2, H_2$) admits a unique solution $(Y'', Z'') \in S^p \times \mathcal{H}^p$. Since $\alpha$ and $\beta$ are positive, we have $Y'' \geq Y'$ and since $\xi > 0$ (which is equivalent to $\xi > 0$), we further have $Y'' > 0$. It follows from Lemma 3.3 that Equation (3.9) admits a solution $(Y, Z)$ such that $Y' \leq Y \leq Y''$. In particular, $Y \in S^p$. Consequently, Equation (3.8) admits a solution $(Y, Z)$ and, we have $\mathbb{E} \sup_{0 \leq t \leq T} |Y_t|^p = 3\mathbb{E} \sup_{0 \leq t \leq T} |Y_t| < \infty$ and $Y > 0$.

Let us show that $Z \in \mathcal{M}^2$. Since $(Y, Z)$ satisfies Equation (3.8) and $Y > 0$, applying Itô’s formula to $Y^2$ yields

$$Y_t^2 = \xi^2 + \int_0^T 2\alpha_s Y_s + 2\beta_s Y_s^2 + 2Y_s \gamma_s Z_s + Z_s^2 \, ds - 2 \int_0^T Y_s Z_s \, dW_s.$$ 

Let $n \in \mathbb{N}$ and consider the stopping time

$$\tau_n := \inf \left\{ t > 0 : \int_0^t |Y_s|^2 |Z_s|^2 \, ds > n \right\} \wedge T.$$ 

Then, there is a constant $C > 0$ such that for every $n \in \mathbb{N}$ and every $\varepsilon > 0$ we have

$$\mathbb{E} \left[ \int_0^{\tau_n} |Z_s|^2 \, ds \right] \leq Y_0^2 + \mathbb{E} \left[ \int_0^{\tau_n} \left( 2\alpha_u Y_u + 2\beta_u Y_u^2 + 2Y_u \gamma_u Z_u \right) \, du \right]$$

$$\leq Y_0^2 + C \mathbb{E} \left[ \int_0^{\tau_n} \alpha_u^2 + Y_u^2 + 2\beta_u^3 + \frac{4}{3} Y_u^3 + \frac{2}{3\varepsilon^2} |Z_u|^2 \, du + \frac{1}{3\varepsilon^2} \left( \int_0^{\tau_n} |\gamma_u|^2 \, du \right)^3 \right]$$

where the inequality follows by Hölder’s inequality. This shows that there is a constant $C > 0$ such that

$$\mathbb{E} \left[ \int_0^{\tau_n} |Z_s|^2 \, ds \right] \leq C \left( 1 + \sup_{t \in [0, T]} \mathbb{E}[Y_0^3] + \mathbb{E} \left[ \int_0^{\tau_n} \alpha_u^2 + \beta_u^3 + \left( \int_0^T |\gamma_u|^2 \, du \right)^{3/2} \right] \right).$$

Taking the limit on both sides, it follows by continuity of $Y$ and Lebesgue dominated convergence theorems that

$$\mathbb{E} \left[ \int_0^T |Z_s|^2 \, ds \right] \leq C \left( 1 + \sup_{t \in [0, T]} \mathbb{E}[Y_0^3] + \mathbb{E} \left[ \xi^2 + \int_0^T \alpha_u^2 + \beta_u^3 + \left( \int_0^T |\gamma_u|^2 \, du \right)^{3/2} \right] \right) < \infty.$$

That is, $Z \in \mathcal{M}^2$. 

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In case the integrability condition (A2) is replaced by the condition (A3), we notice that the BSDE($\xi_2, H_2$) admits a solution ($Y''$, $Z''$) if and only if the equation
\[ \bar{Y}_t = e^{\frac{1}{3} \int_0^t \alpha_u + \beta_u \, du} \bar{\xi} + \int_t^T e^{\frac{1}{3} \int_u^T \alpha_s + \beta_s \, ds} \gamma_s \bar{Z}_s \, ds - \int_t^T \bar{Z}_s \, dW_s \tag{3.10} \]
also does. This follows again as application of Itô’s formula to $Y''_t = e^{\frac{1}{3} \int_0^t \alpha_u + \beta_u \, du} \bar{Y}_t$ and $\bar{Y}_t = e^{\frac{1}{3} \int_0^t \alpha_u + \beta_u \, du} Y''_t$. Moreover, since there is $p > 1$ such that $e^{\frac{1}{3} \int_0^t \alpha_u + \beta_u \, du} \in L^p$ and $\exp \left( \frac{1}{3} \int_0^T \gamma_t \, dW_t \right) \in L^p$, it follows that Equation (3.10) admits a solution ($\bar{Y}$, $\bar{Z}$) which is given by
\[ \bar{Y}_t = \mathbb{E}_{Q^\gamma} \left[ e^{\frac{1}{3} \int_0^T \alpha_u + \beta_u \, du} \bar{\xi} + \int_0^T e^{\frac{1}{3} \int_u^T \alpha_s + \beta_s \, ds} \gamma_s \bar{Z}_s \, ds \mid \mathcal{F}_t \right]. \]

It follows by Hölder’s inequality that $\sup_{0 \leq t \leq T} \mathbb{E}[\bar{Y}_t] < \infty$, and by Girsanov’s theorem that $\bar{Z} \in L^2$. Thus, since $\alpha$ and $\beta$ are positive, BSDE($\xi_2, H_2$) admits a solution ($Y''$, $Z''$) satisfying $Y''_t \geq Y'_t$. Since $\xi > 0$ (which is equivalent to $\xi > 0$), we further have that $Y''_t > 0$. Hence, $0 \leq Y'' \leq \bar{Y}$, showing that $\sup_{0 \leq t \leq T} \mathbb{E}[|Y''_t|^p] < \infty$ and arguing as above we get $\sup_{0 \leq t \leq T} \mathbb{E}[|Y'_t|^p] < \infty$ and $Z \in \mathcal{M}^2$. This concludes the proof.

3.2.2. Proof of Theorem 2.2

We consider only the case where $\xi > 0$ and $H \geq 0$. The negative case is symmetric. The proof follows by the domination argument given in Lemma 3.3. Indeed, put $g(t, y, z) = \alpha_t + \beta_t y + \gamma_t z + \frac{1}{9} |z|^2$. Since $\xi > 0$, by Proposition 3.4, the BSDE($\xi, g$) admits a solution ($Y', Z'$) such that $0 < Y'$. It follows by Lemma 3.3 that BSDE($\xi, H$) admits a solution ($Y, Z$) such that $0 \leq Y \leq Y'$. In particular, this shows that if (A2) is satisfied, then $Y \in S^{3p}$ and if (A3) is satisfied, then $\sup_{t \in [0, T]} \mathbb{E}[|Y_t|^p] < \infty$. Since $H \geq 0$ and $\xi > 0$, it holds $Y > 0$.

3.3. Uniqueness

The proof of uniqueness relies on convex duality arguments. Due to the nonstandard form of the generator, we prove only uniqueness of the value process.

**Proposition 3.5.** Assume that $\xi \in L^\infty$, $\alpha, \beta \in S^\infty$, and $\int \gamma \, dW \in \text{BMO}$ If $\xi > 0$ and $H$ satisfies (A1), (resp. $\xi < 0$ and $H$ satisfies (A1')), then every solution $(Y, Z)$ of BSDE($\xi, H$) such that $0 < Y \leq Y'$ (resp. $Y' \leq Y < 0$) satisfies $Y \in S^{3p}$ and $\int Z \, dW \in \text{BMO}$.

**Proof.** We treat only the case $\xi > 0$ and $H$ satisfies (A1). The second case is treated similarly. It follows by assumption that $\int \gamma \, dW$ satisfies the so-called reverse Hölder inequality, see e.g. [38, Theorem 3.4]. Thus, there is $q > 1$ such that $\exp \left( \frac{1}{3} \int_0^T \gamma_t \, dW_t \right) \in L^q$. Let $(Y, Z) \in S^2 \times L^2$ be a solution of BSDE($\xi, H$) such that $0 \leq Y \leq Y'$, where $(Y', Z')$ is solution to BSDE($\xi, g$) with $g(t, y, z) = \alpha_t + \beta_t y + \gamma_t z + \frac{1}{9} |z|^2$. As shown in the proof of Proposition 3.4, $0 < Y \leq (3Y)^{1/3}$ with $\bar{Y}$ such that
\[ \bar{Y}_t = \bar{\xi} + \int_t^T (9\alpha_u \bar{Y}_u)^{2/3} + 3\beta_u \bar{Y}_u + \gamma_u \bar{Z}_u \, du - \int_t^T \bar{Z}_u \, dW_u, \]
and $\xi := \frac{1}{3} \bar{\xi}^3$. Thus, it holds
\[ \bar{Y}_t \leq \mathbb{E}_{Q^\gamma} \left[ \bar{\xi} + \int_t^T 9\alpha_u + 9(\alpha_u + \beta_u) \bar{Y}_u \, du \mid \mathcal{F}_t \right]. \]
This shows e.g. by Gronwall inequality that $\bar{Y}$ is bounded, and therefore that $Y^g$ is bounded.

Let $\tau$ be a $[0, T]$-valued stopping time. Let $v : \mathbb{R} \to \mathbb{R}$ be an increasing function belonging to the Sobolev space $\mathcal{W}^2_1(\mathbb{R})$ to be specified. Itô-Krylov’s formula applied to $v(Y)$ yields (here $v''$ is understood as the density of the generalized second derivative of $v$)

$$v(Y_\tau) = v(\xi) + \int_\tau^T v'(Y_u)H(\xi, Y_u, Z_u) - \frac{1}{2} v''(Y_u)|Z_u|^2\,du - \int_\tau^T v'(Y_u)Z_u\,dW_u$$

$$\leq v(\xi) + \int_\tau^T v'(Y_u)\left(\alpha_u + \beta_uY_u + \frac{1}{2}\gamma_u^2\right)\,du$$

$$- \int_\tau^T |Z_u|^2\left(\frac{1}{2} v''(Y_u) - \frac{1}{2} v'(Y_u) - \frac{1}{Y_u} v'(Y_u)\right)\,du - \int_\tau^T v'(Y_u)Z_u\,dW_u. \tag{3.11}$$

Let $f(x) := \frac{1}{2}$, fix $\varepsilon > 0$ and set

$$K(y) := \int_0^y \exp\left(-2\frac{x}{\varepsilon}\int f(r)\,dr\right)\,dx.$$  

The function

$$v(x) := \int_0^x K(y)\exp\left(2\frac{y}{\varepsilon}\int f(r)\,dr\right)\,dy$$

is increasing, belongs to $\mathcal{W}^2_1(\mathbb{R})$ and satisfies the ordinary differential equation $\frac{1}{2}v''(x) - f(x)v'(x) = \frac{1}{2}$, see e.g. [7]. Let $n \in \mathbb{N}$ and consider the stopping time

$$\tau_n := \inf\left\{ t > 0 : \int_0^t |v'(Y_s)Z_s|^2\,ds > n \right\} \wedge T.$$

It is readily checked that $\tau_n \uparrow T$. From (3.11), for every $n \in \mathbb{N}$ we have

$$\frac{1}{2} \int_\tau^{T \land \tau_n} |Z_u|^2\,du \leq v(Y_{T \land \tau_n}) - v(Y_{\tau_n}) + \int_\tau^{T \land \tau_n} v'(Y_u)\left(\alpha_u + \beta_uY_u + \frac{1}{2}\gamma_u^2\right)\,du - \int_\tau^{T \land \tau_n} v'(Y_u)Z_u\,dW_u$$

$$\leq C_1 + C_2 \int_\tau^{T \land \tau_n} \frac{1}{2}|\gamma_u|^2\,du + \int_\tau^{T \land \tau_n} v'(Y_u)Z_u\,dW_u,$$

for some constants $C_1, C_2 > 0$ depending only on the uniform bounds of $\alpha$, $\beta$ and $\xi$. Thus, taking expectation on both sides yields

$$\mathbb{E}\left[\int_\tau^{\tau_n \land T} |Z_u|^2\,du \mid \mathcal{F}_\tau\right] \leq C_1 + C_2\mathbb{E}\left[\int_\tau^{\tau_n \land T} |\gamma_u|^2\,du \mid \mathcal{F}_\tau\right]$$

so that by Fatou’s lemma and Lebesgue dominated convergence theorem we have

$$\mathbb{E}\left[\int_\tau^T |Z_u|^2\,du \mid \mathcal{F}_\tau\right] \leq C_1 + C_2\mathbb{E}\left[\int_\tau^T |\gamma_u|^2\,du \mid \mathcal{F}_\tau\right]. \tag{3.12}$$
Therefore, $\int Z \, dW$ is a BMO martingale.

The next result gives a convex dual representation of the value process $Y$ of the solution $(Y, Z)$ of BSDE($\xi, H$). The main idea of the proof is taken from [22]. Here however, new BMO estimates are needed. We denote by $H^*$ the convex conjugate of the function $H$ given by

$$H^*(t, \omega, b, a) := \sup_{y, z} (by + az - H(t, \omega, y, z)).$$

The function $H^*$ is convex and lower semicontinuous, and since $H$ is continuous, it can be checked that the function $H^*$ is progressively measurable. Moreover, given a $\mathbb{R}^d$-valued progressively measurable process $a$ such that $\int a \, dW$ is in BMO, we denote by $Q^a$ the probability measure

$$\frac{dQ^a}{da} := \exp \left( \int_0^T a_u \, dW_u - \frac{1}{2} \int_0^T |a_u|^2 \, du \right).$$

**Proposition 3.6.** Assume that $\xi \in L^\infty$, $\alpha, \beta \in S^\infty$, that $\int \gamma \, dW$ is a BMO martingale and that for each $(t, \omega)$, the function $H(t, \omega, \cdot, \cdot)$ is jointly convex. Further assume that $\xi > 0$ and (A1) holds. Then, every solution $(Y, Z)$ of BSDE($\xi, H$) such that $0 < Y < Y^g$ admits the convex dual representation

$$Y_t = \text{ess sup}_{a, b} \mathbb{E}^Q_a \left[ e^{\int_t^T b_u \, du} \xi - \int_t^T e^{\int_u^T b_s \, ds} H^*(u, a_u, b_u) \, du | \mathcal{F}_t \right],$$

(3.13)

where the supremum is over progressively measurable processes $a : [0, T] \times \Omega \rightarrow \mathbb{R}^d$ and $b : [0, T] \times \Omega \rightarrow \mathbb{R}$ such that $\int a \, dW$ is in BMO and $|b| \leq \sup_{0 \leq t \leq T} |\beta_t|$.

We have the same representation when $\xi < 0$ and (A2) holds, for all solutions between $Y^g$ and 0.

**Proof.** Let $(Y, Z)$ be a solution of BSDE($\xi, H$) such that $0 < Y \leq Y^g$. Let $b$ be a real-valued progressively measurable process satisfying $|b| \leq \sup_{0 \leq t \leq T} |\beta_t|$, and $a$ an $\mathbb{R}^d$-valued progressive process such that $\int a \, dW$ is in BMO. Applying Itô’s formula to $e^{\int_t^T b_u \, du} Y_u$, it follows by Girsanov change of measure that

$$Y_t = \mathbb{E}^Q_a \left[ e^{\int_t^T b_u \, du} \xi - \int_t^T e^{\int_u^T b_s \, ds} (b_u Y_u + a_u Z_u - H(u, Y_u, Z_u)) \, du | \mathcal{F}_t \right].$$

(3.14)

It now follows by definition of $H^*$ and the fact that $a$ and $b$ were taken arbitrary that

$$Y_t \geq \text{ess sup}_{a, b} \mathbb{E}^Q_a \left[ e^{\int_t^T b_u \, du} \xi - \int_t^T e^{\int_u^T b_s \, ds} H^*(u, a_u, b_u) \, du | \mathcal{F}_t \right].$$

(3.15)

It remains to show that the above inequality is in fact an equality. Since $H(t, \omega, \cdot, \cdot)$ is convex, it has a nonempty subgradient at every point of the interior of its domain, see for instance [57, Theorem 47.A]. In particular, there is $\dot{a}(t, \omega), \dot{\beta}(t, \omega)$ such that

$$H(t, \omega, Y_t(\omega), Z_t(\omega)) = \dot{b}(t)(\omega) Y_t(\omega) + \dot{a}(t)(\omega) Z_t(\omega) - H^*(t, \omega, \dot{b}(t)(\omega), \dot{a}(t)(\omega)).$$

(3.16)

It follows by measurable selection arguments that $\dot{a}$ and $\dot{b}$ can be chosen progressively measurable, see e.g. [22] for details. Let us show in addition that $|b| \leq \|\sup_{0 \leq t \leq T} |\beta_t||_{L^\infty}$, and $\int \dot{a} \, dW$ is in BMO. If $|b| > \|\sup_{0 \leq t \leq T} |\beta_t||_{L^\infty}$, then for all $y, z$, it holds

$$H^*(t, \dot{b}_t, \dot{a}_t) \geq \dot{a}_t z + \dot{b}_t y - H(t, y, z) \geq \dot{a}_t z + \dot{b}_t y - \alpha_t y - \beta_t y - \frac{1}{y} |z|^2.$$

(3.17)
In particular, taking $z = 0$ and $y = nb_t$ yields

$$H^*(t, b_t, a_t) \geq n|b_t|(|b_t| - \beta_t) - \alpha_t$$

which goes to infinity as $n$ goes to infinity. This is a contradiction, since $H$ being finite, $H^*(t, b_t, a_t)$ is also finite. Thus, $|b_t| \leq \sup_{0 \leq t \leq T} |\beta_t| \in L^\infty$. Taking $y = 2$ and $z = a_t$ in (3.17) yields

$$H^*(t, b_t, a_t) \geq |\bar{a}_t|^2 + 2(\bar{b}_t - \beta_t) - \alpha_t - \frac{1}{2}|a_t|^2 \geq |\bar{a}_t|^2 + 2(\bar{b}_t - \beta_t) - \alpha_t. \quad (3.18)$$

Let $0 < \varepsilon < \frac{1}{2}$. It holds

$$\left(\frac{1}{2} - \varepsilon\right)|\bar{a}_t|^2 = \frac{1}{2}|\bar{a}_t|^2 - \varepsilon|\bar{a}_t|^2 \leq H^*(t, \bar{a}_t, \bar{b}_t) + 2(\beta_t - \bar{b}_t) + \alpha_t - \varepsilon|\bar{a}_t|^2 = \bar{a}_tZ_t + \bar{b}_tY_t - H(t, Y_t, Z_t) + 2(\beta_t - \bar{b}_t) + \alpha_t - \varepsilon|\bar{a}_t|^2 \leq \frac{1}{2\varepsilon}|Z_t|^2 + \frac{\varepsilon}{2}|\bar{a}_t|^2 + \bar{b}_tY_t + 2(\beta_t - \bar{b}_t) + \alpha_t - \varepsilon|\bar{a}_t|^2 \leq \frac{1}{2\varepsilon}|Z_t|^2 + \bar{b}_tY_t + 2(\beta_t - \bar{b}_t) + \alpha_t.$$

Since $\int Z \, dW$ is a BMO martingale and the processes $b, Y, \alpha$ and $\beta$ are bounded, it follows that $\int \bar{a} \, dW$ is in BMO. Applying Itô’s formula to $e^{\int_{t}^{u} b_s \, ds}Y_u$, it follows by Girsanov’s change of measure that

$$Y_t = \mathbb{E}_{Q^a} \left[ e^{\int_{t}^{T} b_s \, ds} \xi - \int_{t}^{T} e^{\int_{s}^{u} b_s \, ds} \left( \bar{b}_uY_u + \bar{a}_uZ_u - H(u, Y_u, Z_u) \right) \, du \mid F_t \right]$$

$$= \mathbb{E}_{Q^a} \left[ e^{\int_{t}^{T} b_s \, ds} \xi - \int_{t}^{T} e^{\int_{s}^{u} b_s \, ds} H^*(u, \bar{b}_u, \bar{a}_u) \, du \mid F_t \right],$$

where the second equality follows by (3.16). This shows that the inequality in (3.15) is in fact an equality. \hfill \Box

**Proof of Theorem 2.3.** If $(Y, Z)$ and $(Y', Z')$ are two solutions of BSDE($\xi, H$), then by Proposition (3.6) we have

$$Y_t = \text{ess sup}_{u,b} \mathbb{E}_{Q^a} \left[ e^{\int_{t}^{T} b_s \, ds} \xi - \int_{t}^{T} e^{\int_{s}^{u} b_s \, ds} H^*(s, b_s, a_s) \, ds \mid F_t \right] = Y'_t. \quad (3.19)$$

Since both $Y$ and $Y'$ are continuous, this shows that the two processes are indistinguishable. \hfill \Box

## 4. Applications to parabolic PDEs

In this section we provide a few applications of our existence and uniqueness results. We first consider semi-linear partial differential equations without boundary conditions, and then equations with lateral boundaries of Neumann type.

### 4.1. Probabilistic formulas for singular parabolic PDEs

In this part, we assume that $\alpha, \beta$ and $\gamma$ are deterministic, i.e. depend only on $t$. We are concerned with the semilinear PDE associated to the Markovian version of our BSDE Let $\sigma, \mu$ be two measurable functions defined on $[0, T] \times \mathbb{R}^d$.
with values in $\mathbb{R}^{d \times d}$ and $\mathbb{R}^d$ respectively. Let $h$ be a measurable function defined on $\mathbb{R}^d$ with values in $\mathbb{R}$. Define the differential operator $\mathcal{L}$ by

$$\mathcal{L} := \sum_{i,j=1}^{d} (\sigma \sigma')_{ij}(s,x) \frac{1}{2} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} \mu_i(s,x) \frac{\partial}{\partial x_i}.$$ 

Consider the following semilinear PDE

$$\begin{cases}
\frac{\partial v}{\partial s}(s,x) + \mathcal{L}v(s,x) + H(s,v(s,x)), \sigma \sigma v(s,x) &= 0, \quad \text{on } [0,T] \times \mathbb{R}^d, \\
v(T,x) &= h(x)
\end{cases}$$

and the following conditions

(A4) $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$ and $\mu : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ are continuous functions and there exists $C > 0$ such that

$$|\sigma(t,x)| + |\mu(t,x)| \leq C(1 + |x|)$$

for every $(t,x) \in [0,T] \times \mathbb{R}^d$.

(A5) The SDE

$$X_t = x + \int_0^t \mu(u,X_u) \, du + \int_0^t \sigma(u,X_u) \, dW_u$$

admits a unique strong solution.

(A6) The terminal condition $h$ is continuous, bounded and satisfies $h > 0$.

**Theorem 4.1.** Assume that $H : [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is convex in the last two components $(y,z)$, and that (A1) and (A4)-(A6) are satisfied. Then, $v(t,x) := Y_{t,x}^t$ is a viscosity solution of the PDE (4.1).

**Remark 4.2.** Some comments on assumption (A4) are in order. First recall that existence of a unique strong solution means that on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with the Brownian motion $W$, there is a $W$-adapted process $X$ satisfying (4.2) and such that $X$ is indistinguishable to any other solution on this probability basis.

General conditions on the coefficients $\mu$ and $\sigma$ guaranteeing existence of a unique strong solution of (4.2) are well known. Strong existence and uniqueness holds when $\mu$ and $\sigma$ are Lipschitz continuous, but also for much rougher coefficients. For instance, (A4) already implies (A5) when $\sigma$ is a constant and non-zero, see [43]. We further refer to [41, 44], [16, Chapter 1] and the references therein for other conditions. The point here is that we obtain existence of (4.1) under much weaker regularity conditions than the standard Lipschitz continuity conditions usually assumed.

The following lemma will be needed to show that the function $v$ defined above is continuous.

**Lemma 4.3.** Assume that $H : [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is convex in the last two components $(y,z)$ and that (A1) is satisfied. Let $(\xi^{t,x})$ be a bounded family of $\sigma(W_r - W_t, t \leq r \leq T)$-measurable and strictly positive random variables such that $\xi^{t^n,x^n} \to \xi^{t,x}$ $\mathbb{P}$-a.s. for all $(t^n,x^n) \to (t,x)$. Then, $(t,x) \mapsto Y_{t,x}^t$ is continuous, where $(Y_{s,x}^t, Z_{s,x}^t)_{s \in [t,T]}$ solves BSDE$(\xi^{t,x}, H)$.

**Proof.** Let $(t^n, x^n) \to (t,x)$, and assume without loss of generality that $t^n \downarrow t$. It follows by Proposition 3.6 that for every $(t,x) \in [0,T] \times \mathbb{R}$, the solution $(Y_{s,x}^t, Z_{s,x}^t)$ of BSDE$(\xi^{t,x}, H)$ admits the representation

$$Y_{t,x}^t = \text{ess sup}_{a,b} \mathbb{E}^\mathbb{Q}_t \left[ e^{\int_t^T b_r \, ds} \xi^{t,x} - \int_t^T e^{\int_r^T b_s \, ds} H^*(u,b_0,a_u) \, du \mid \mathcal{F}_t \right],$$

(4.3)
where the supremum is over progressively measurable processes $a : [0, T] \times \Omega \to \mathbb{R}^d$ and $b : [0, T] \times \Omega \to \mathbb{R}$ such that $|b| \leq \sup_{0 \leq t \leq T} |\beta_t|$, and $\int a \, dW$ is in BMO. $Y^{t,x}_t$ is clearly deterministic, since $Y^{x}_s$ is measurable w.r.t. the sigma algebra $\sigma(W_r - W_t, t \leq r \leq s)$. Thus, (4.3.3) takes the form

$$Y^{t,x}_t = \sup_{a,b} \mathbb{E}_{Q^n} \left[ e^{\int_t^T b_s \, dW_s} - \int_t^T e^{\int_t^s b_r \, dW_r} \right].$$

(4.4.3)

By definition of $H^*$, for every $a, b$ it holds $H^*(u, b, x) \geq b_u - \beta_a - \alpha_u \geq -2||\beta||_{\infty} - ||\alpha||_{\infty} =: -C$ for some $C \geq 0$. Thus by (4.4.3), for every $n \in \mathbb{N}$, and every $a, b$, it holds

$$Y^{t,x}_t \geq \mathbb{E}_{Q^n} \left[ e^{\int_t^T b_s \, dW_s} - \int_t^T e^{\int_t^s b_r \, dW_r} (H^*(u, b, x) + C) - \int_t^T b_s \, dW_s C \right].$$

Applying Beppo-Levy theorem and the Lebesgue dominated convergence theorem, we get $\lim \inf_{n \to \infty} Y^{t,x}_t \geq Y^{t,x}_t$.

On the other hand, let $a^n, b^n$ be such that

$$Y^{t,x}_t \leq \mathbb{E}_{Q^n} \left[ e^{\int_t^T b^n_s \, dW_s} - \int_t^T e^{\int_t^s b^n_r \, dW_r} H^*(u, b^n_s, a^n_s \, dW_s) \right] + \frac{1}{n}.$$

(4.5.3)

Since $\xi^{t,x,n}$ and $Y^{t,x}_t$ are bounded (see Proposition 3.5), this implies

$$\mathbb{E}_{Q^n} \left[ \int_t^T e^{\int_t^s b^n_r \, dW_r} H^*(u, b^n_s, a^n_s \, dW_s) \right] \leq C \text{ for some } C \geq 0.$$

Arguing as in the computations leading to (3.18), and using that $\alpha, \beta$ and the sequence $b^n$ are bounded, we find two positive constants $C_1, C_2$ with $C_1 > 0$ such that

$$H^*(u, b^n_s, a^n_s) \geq C_1 |a^n_s|^2 + \delta b^n_s - C_2.$$

Hence, it follows $\mathbb{E}_{Q^n} \left[ \int_t^T \frac{1}{2} |a^n_s|^2 \, dW_s \right] \leq C$ for some (possibly different) constant $C \geq 0$. This shows in particular, due to Girsanov’s theorem, that $\mathbb{E} \left[ Z^{a^n}_T \log (Z^{a^n}_T) \right] \leq C$, with $Z^{a^n}_T := \exp \left( \int_t^T a^n_s \, dW_s - \frac{1}{2} \int_t^T |a^n_s|^2 \, dW_s \right)$. Thus, by the criterion of de la Vallée Poussin, $(Z^{a^n}_T)_n$ is uniformly integrable and therefore there exists $K \in L^1$ such that $(Z^{a^n}_T)_n$ converges weakly to $K$ (i.e. w.r.t. the topology $\sigma(L^1, L^\infty)$). Since the sequence $(\xi^{t,x,n} - \xi^{t,x})_n$ is uniformly bounded and converges to $0$ P.a.s., letting $C \in \mathbb{R}$ be such that $e^{\int_t^T b^n_s \, dW_s} \leq C$, one has

$$\lim_{n \to \infty} \sup \mathbb{E} \left[ Z^{a^n}_T e^{\int_t^T b^n_s \, dW_s} (\xi^{t,x,n} - \xi^{t,x}) \right] \leq C \lim_{n \to \infty} \mathbb{E} \left[ Z^{a^n}_T (\xi^{t,x,n} - \xi^{t,x}) \right] = 0,$$

where the equality follows from [11, Lemma 2.8] (after pushing forward the measures $Q^n$ to the canonical space $C([0, T], \mathbb{R}^d)$). Hence, for every $\varepsilon > 0$, there is $n$ large enough such that

$$\mathbb{E}_{Q^n} \left[ e^{\int_t^T b^n_s \, dW_s} \xi^{t,x} \right] = \mathbb{E} \left[ Z^{a^n}_T e^{\int_t^T b^n_s \, dW_s} \xi^{t,x} \right] \leq \mathbb{E} \left[ Z^{a^n}_T \xi^{t,x} \right] + \varepsilon = \mathbb{E}_{Q^n} \left[ e^{\int_t^T b^n_s \, dW_s} \xi^{t,x} \right] + \varepsilon.$$

That is, up to a subsequence, (4.5.3) yields

$$Y^{t,x}_t \leq \mathbb{E}_{Q^n} \left[ e^{\int_t^T b^n_s \, dW_s} \xi^{t,x} - \int_t^T e^{\int_t^s b^n_r \, dW_r} H^*(u, b^n_s, a^n_s \, dW_s) \right] + \frac{1}{n} + \varepsilon.$$

(4.5.4)
Taking the limit as $n$ goes to infinity and then dropping $\varepsilon$ shows that $\limsup_{n \to \infty} Y^{t^n, x^n}_t \leq Y^{t, x}_t$. This finishes the proof of the continuity.

Classically the existence of viscosity solution is established via the comparison property of the corresponding BSDE. The comparison theorem is far from obvious in our situation (which justified our proof of uniqueness by convex duality). On the other hand the touching property given by Kobylanski [39, Theorem 3.9] cannot be used in our situation since the variable $Z$ is not continuous in $t$. To overcome these difficulties, we give a new proof of the existence of viscosity solution (Theorem 4.1). This proof is partially based on a new variant of the touching property which is more adapted to our situation. It is moreover direct and simpler than the proofs given in the previous papers.

**Lemma 4.4. (Touching)** Let $(\xi_t)_{0 \leq t \leq T}$ be a continuous adapted process such that

$$d\xi_t = \beta_t dt + \alpha_t dW_t,$$

where $\beta$ and $\alpha$ are adapted processes such that $\beta(\omega) \in L^1(dt)$ and $\alpha(\omega) \in L^2(dt)$ for $P$-almost all $\omega$. If $\xi_t \geq 0$ for all $t$, then it holds

$$1_{\{\xi_t = 0\}}\alpha_t = 0 \quad \text{and} \quad 1_{\{\xi_t = 0\}}\beta_t \geq 0 \quad P \otimes dt\text{-a.e.}$$

**Proof.** For every $n \in \mathbb{N}$, let $\tau_n$ be defined by

$$\tau_n := \inf \left\{ t > 0 : \int_0^t |\alpha_u|^2 du \geq n \right\} \wedge T.$$

Since $(\xi_t)$ is continuous and adapted, $(\tau_n)$ is a sequence of stopping times such that $\tau_n \uparrow T$ $P$-a.s. Put $\eta^n_a := \xi_{t \wedge \tau_n}$ and denote by $L^n_a(\eta^n)$ the local time of $\eta^n$ at $a \in \mathbb{R}$. Since $\eta^n$ satisfies $d\eta^n_a = \beta_t 1_{[0, \tau_n]} dt + \alpha_t 1_{[0, \tau_n]} dW_t$, it follows by the occupation time formula that

$$\mathbb{E} \int_0^T 1_{\{\eta^n_a = 0\}} \alpha^2_t 1_{[0, \tau_n]} da = \mathbb{E} \int_0^T 1_{\{\eta^n_a = 0\} \cap \{a = 0\}} d(\eta^n)_a = \mathbb{E} \int_0^T 1_{\{a = 0\}} L^n_t(\eta^n)(da) = 0.$$

This shows that $1_{\{\eta^n_a = 0\}} \alpha^2_t 1_{[0, \tau_n]} = 0$ $P \otimes dt$-a.e. Thus, $1_{\{\xi_t = 0\}} \alpha^2_t 1_{[0, \tau_n]} = 0$ $P \otimes dt$-a.e. To conclude, take the limit as $n$ goes to infinity.

We now prove that $1_{\{\xi_t = 0\}} \beta_s \geq 0$ $P \otimes dt$-a.e. According to [52, Theorem 1.7 p. 225], we have for every $t \in [0, T]$

$$L^n_t(\xi) - L^n_t(0) \geq 2 \int_0^t 1_{\{\xi_s = 0\}} d\xi_s \geq 2 \int_0^t 1_{\{\xi_s = 0\}} \beta_s ds.$$

Since $L^n_t(\xi) = 0$ (because $\xi \geq 0$) and $L^n_t(0) \geq 0$ for each $t$, it follows that $\int_0^t 1_{\{\xi_s = 0\}} \beta_s ds \geq 0$ for each $t$. This shows that $1_{\{\xi_t = 0\}} \beta_s \geq 0$ $P \otimes dt$-a.e. \hfill $\square$

**Proof (of Theorem 4.1).** By Proposition A.2, the map $(t, x) \mapsto X^{t, x}$ is continuous. Thus, due to Lemma 4.3, the map $v(t, x) := Y^{t, x}_t$ is continuous in $(t, x)$. We shall show that $v$ is a viscosity subsolution of the PDE (4.1). Let $(t_0, x_0)$ be fixed. If $t_0 = T$, then $v(t_0, x_0) = h(x_0)$. Let us assume $t_0 < T$. For simplicity, we set $(X_t, Y_t, Z_t) := (X^{t_0, x_0}_t, Y^{t_0, x_0}_t, Z^{t_0, x_0}_t), t \geq t_0$. Since $v(t, x) = Y^{t, x}_t$, the Markov property of $X$ and the uniqueness of $Y$ show that for every $t \in [t_0, T]$,

$$v(t, X_t) = Y_t^t.$$  \hfill (4.6)
Let \( \phi \in C^{1,2}_0 \), (i.e. twice continuously differentiable with bounded derivatives) such that \( v - \phi \) admits a local maximum at \((t_0, x_0)\), which we assume global and equal to 0 (see e.g. [27]), that is:

\[
\phi(t_0, x_0) = v(t_0, x_0) \quad \text{and} \quad \phi(t, x) \geq v(t, x) \quad \text{for each} \ (t, x).
\]

Together with (4.6), this implies that for every \( t \in [t_0, T] \), it holds

\[
\phi(t, X_t) \geq Y_t. \tag{4.7}
\]

Let us assume by contradiction that there is \( \delta > 0 \) such that

\[
\frac{\partial \phi}{\partial t}(t_0, x_0) + \mathcal{L}\phi(t_0, x_0) + H(t_0, v(t_0, x_0), \sigma' \nabla \phi(t_0, x_0)) = -\delta < 0.
\]

By continuity, there exists a strictly positive constant \( \varepsilon \) such that for every \((t, x)\) in the set \((t_0 - \varepsilon, t_0 + \varepsilon) \times \{ y : |y - x_0| \leq \varepsilon \}\), it holds

\[
\frac{\partial \phi}{\partial t}(t, x) + \mathcal{L}\phi(t, x) + H(t, v(t, x), \sigma' \nabla \phi(t, x)) \leq -\delta/2. \tag{4.8}
\]

We define

\[
\tau_1 := \inf \{ t \geq t_0 : |X_t - x_0| \geq \varepsilon \} \wedge t_0 + \varepsilon, \quad \tau_2 := \inf \{ t \geq t_0 : Y_t \neq \phi(t, X_t) \} \wedge T,
\]

\[
\tau_3 := \inf \left\{ t \geq t_0 : \int_{t_0}^{t} |H(s, Y_s, Z_s)| \, ds \geq 1 \right\} \wedge T
\]

and

\[
\tau := \tau_1 \wedge \tau_2 \wedge \tau_3. \tag{4.9}
\]

Since \( t \mapsto \int_{t_0}^{t} |H(s, Y_s, Z_s)| \, ds \) is continuous and increasing, we have \( \tau_3 > t_0 \). Combined with \( X_{t_0} = x_0 \) and \( Y_{t_0} = \phi(t_0, X_{t_0}) \), with probability one, it holds \( \tau > t_0 \). Furthermore, by continuity of the paths of \( X \), it holds \( |X_{t \wedge \tau} - x_0| \geq \varepsilon \) for all \( t \in [t_0, T] \). Thus, by Itô’s formula we have

\[
\phi(t \wedge \tau, X_{t \wedge \tau}) = \phi(t, X_t) - \int_{t \wedge \tau}^{\tau} \left( \frac{\partial \phi}{\partial t} + \mathcal{L}\phi \right)(r, X_r) \, dr - \int_{t \wedge \tau}^{\tau} \sigma \nabla \phi(r, X_r) \, dW_r, \quad t_0 \leq t \leq T. \tag{4.10}
\]

Thus, taking expectation above yields

\[
\phi(t \wedge \tau, X_{t \wedge \tau}) = \mathbb{E} \left[ \phi(t, X_t) - \int_{t \wedge \tau}^{\tau} \left( \frac{\partial \phi}{\partial t} + \mathcal{L}\phi \right)(r, X_r) \, dr \mid F_{t \wedge \tau} \right].
\]

On the other hand, since \( Y \) satisfies

\[
Y_{t \wedge \tau} = Y_{\tau} + \int_{t \wedge \tau}^{\tau} H(r, v(r, X_r), Z_r) \, dr - \int_{t \wedge \tau}^{\tau} Z_r \, dW_r = Y_{\tau} + \int_{t \wedge \tau}^{\tau} H(r, v(r, X_r), Z_r) \, dr - \int_{t \wedge \tau}^{\tau} Z_r \, dW_r,
\]

taking expectation again and using Lemma 4.4 and Equation (4.8), (recall that by Proposition 3.5 \( \int \tau dW \) is a martingale) we get

\[
Y_{t \wedge \tau} = \mathbb{E} \left[ Y_{\tau} + \int_{t \wedge \tau}^{\tau} H(r, v(r, X_r), Z_r) \, dr \mid F_{t \wedge \tau} \right]
\]

\[
\leq \mathbb{E} \left[ \phi(t, X_t) - \int_{t \wedge \tau}^{\tau} \left( \frac{\partial \phi}{\partial t} + \mathcal{L}\phi \right)(r, X_r) \, dr \mid F_{t \wedge \tau} \right] - \frac{\delta}{2} \mathbb{E} \left[ (\tau - t \wedge \tau) \mid F_{t \wedge \tau} \right]
\]

\[
= \phi(t \wedge \tau, X_{t \wedge \tau}) - \frac{\delta}{2} \mathbb{E} \left[ (\tau - t \wedge \tau) \mid F_{t \wedge \tau} \right].
\]
Letting \( t = t_0 \), we obtain \( v(t_0, x_0) = Y_{t_0} = Y_{t_0}^{\tau - \tau} \leq \phi(t_0, x_0) - \frac{\delta}{2}E[(\tau - t_0) | F_{t_0}] \), which is a contradiction, since \( (\tau - t_0) \) is strictly positive. Thus,
\[
\frac{\partial \phi}{\partial t}(t_0, x_0) + \mathcal{L}\phi(t_0, x_0) + H(t_0, \phi(t_0, x_0), \sigma' \nabla \phi(t_0, x_0)) \geq 0,
\]
which shows that \( v \) is a viscosity subsolution.

The viscosity supersolution property is proved similarly. \( \square \)

### 4.2. The case of the canonical generator \( \frac{1}{y}|z|^2 \)

As pointed out in Proposition 1.1, in the special case where the generator is of the form \( H(y, z) := |z|^2/y \) existence and uniqueness can be obtained under weaker conditions. In the same vein, PDEs with nonlinearity of the form \(|\nabla v|^2/v\) can be treated with more general assumptions. Since such equations are of particular interest in applications (see e.g. \[18\]), we dedicate this subsection to their analysis. Thus, we consider the equation
\[
\begin{align*}
\frac{\partial v}{\partial s}(s, x) + \mathcal{L}v(s, x) + \frac{|\sigma' \nabla v|^2}{v}(s, x) &= 0, & \text{on } [0, T) \times \mathbb{R}^d, \\
v(t, x) &> 0 & \text{on } [0, T) \times \mathbb{R}^d, \\
v(T, x) &= h(x).
\end{align*}
\tag{4.11}
\]

The following notions are well-known, they are recalled here for the reader’s convenience.

**Definition 4.5.** The SDE (4.2) admits a weak solution \((\hat{X}, \hat{W})\) if there is a filtered probability space \((\bar{\Omega}, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \bar{\mathbb{P}})\) on which \( \hat{W} \) is an adapted Brownian motion and \( \hat{X} \) is adapted and such that \((\hat{X}, \hat{W})\) satisfies (4.2).

There is weak uniqueness (or uniqueness in law) if given two weak solutions \((\hat{X}, \hat{W})\) and \((\check{X}, \check{W})\), possibly on different probability spaces one has \( \text{Law}(\hat{X}) = \text{Law}(\check{X}) \).

Consider the following conditions:

- **(A5')** The SDE (4.2) satisfies existence and uniqueness in law.
- **(A6')** The terminal condition \( h \) is continuous, of polynomial growth and satisfies \( h > 0 \).

Let \((X_{t}^{t, x}, W_{s})_{s \leq t \leq T}\) be the unique (weak) solution to SDE (4.2) starting from \((t, x)\) on the probability space \((\bar{\Omega}, \mathcal{F}, \bar{\mathbb{P}})\). Let \((Y_{s}^{t, x}, Z_{s}^{t, x})_{s \leq t \leq T}\) be the unique solution of BSDE \((h(X_{T}^{t, x}), |z|^2/y)\) on \((\bar{\Omega}, \mathcal{F}, \bar{\mathbb{P}})\).

**Theorem 4.6.** Assume that (A4), (A5') and (A6') hold. Then, \( v(t, x) := Y_{t}^{t, x} \) is a viscosity solution of PDE (4.11).

**Remark 4.7.** Theorem 4.6 is specific for quadratic PDEs with the nonlinearity \(|\nabla v|^2/v\). This restriction in the nonlinearity covered by the previous result Theorem 4.1 allows an important gain in the integrability of terminal value, as well as in the regularity to be imposed on the coefficients, since we only assume existence and uniqueness in law. For instance by a well-known result of Stroock and Varadhan \[54\] or \[16, Proposition 1.14\], Assumption (A5') is already satisfied when \( \mu \) and \( \sigma \) are bounded continuous and such that \(|\sigma(t, x)\lambda| \geq \varepsilon(t, x)|\lambda|\) for every \( \lambda \in \mathbb{R}^d \) and some constant \( \varepsilon(t, x) \).

On the other hand, the literature dealing with singular PDEs of the form treated in this paper (which amount to initial value problems by a change of time) often assume the initial condition (corresponding to \( h \) in our case) to be bounded and bounded away from zero. See \[18\], where further restrictions are made on the coefficient \( \sigma \). \( \diamond \)

**Proof.** Let \((X_{t}^{t, x}, \hat{W})\) be a solution of (4.2) on a probability space \((\bar{\Omega}, \mathcal{F}, \bar{\mathbb{P}})\). Let \( p > 0 \) be such that \(|h(x)| \leq C(1 + |x|^p)\) for some \( C \geq 0 \). By Assumption (A4) it is easily checked that \( X_{t}^{t, x} \in L^{3p}(\bar{\mathbb{P}}) \), thus \( h(X_{T}^{t, x}) \in L^{3}(\bar{\mathbb{P}}) \). Hence, by Proposition 1.1, there is \((Y_{t}^{t, x}, Z_{t}^{t, x})\) solving BSDE \((h(X_{T}^{t, x}), |z|^2/y)\) driven by the Brownian motion \( \hat{W} \) on \((\bar{\Omega}, \mathcal{F}, \bar{\mathbb{P}})\).
Notice that the function \( v(t, x) := Y^{t,x}_{s} \) is well-defined and continuous (in particular, does not depend on the underlying probability basis). Indeed, put \( u(y) := \frac{1}{2} y^2 \), and let \( (Y^{s}_s, Z^{s}_s) \) be the unique solution of BSDE\( u(h(X^{t,x}_T)), 0 \) in \( S^2 \times \mathcal{M}^2 \). It satisfies \( Y^{s}_s = \mathbb{E}_s[u(h(X^{t,x}_T))] | \mathcal{F}_s = \mathbb{E}_s[u(h(X^{t,x}_T)) | X^{t,x}_s] \) where the second equality follows by the Markov property. Thus, \( Y^{t,x}_t \) is deterministic. That is,

\[
v(t, x) = Y^{t,x}_t = u^{-1}(\hat{Y}^{t,x}_t) = u^{-1}(\mathbb{E}_t[u(h(X^{t,x}_T))]) = u^{-1}\left( \int_{\mathbb{R}^d} u \circ h(r) \nu^{t,x}(dr) \right),
\]

where \( \nu^{t,x} \) denotes the law of the solution \( X^{t,x}_T \) which by \((A5')\) is unique. By \((A5')\), any other solution of \((4.2)\) has the same law, showing that \( v \) is well-defined. In particular, it does not depend on the probability basis on which \((Y^{s}_s, Z^{s}_s)\) is defined. Furthermore, since \( u \) and \( h \) are continuous, it follows by Proposition A.3 that if \((t^n, x^n) \to (t, x)\), then \( u \circ h(X^{t^n,x^n}_T) \to u \circ h(X^{t,x}_T) \) in law. Applying [50, Theorem 6.1] shows that \( \hat{Y}^{t^n,x^n}_t \) converges to \( \hat{Y}^{t,x}_t \) in law.

Therefore, \( v(t^n, x^n) = u^{-1}(\hat{Y}^{t^n,x^n}_t) \to u^{-1}(\hat{Y}^{t,x}_t) = v(t, x) \).

The rest of the proof follows exactly the proof of Theorem 4.1, except that it is applied on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and the stopping time \( \tau_3 \) should be replaced by

\[
\tau_3 := \inf \left\{ t \geq t_0 : \int_{t_0}^{t} |\tilde{Z}_s|^2 \, ds + \int_{t_0}^{t} |\tilde{Z}_s|^2 \, ds \geq 1 \right\} \wedge \hat{T}
\]

since in this case \( \int Z \, dW \) is not necessarily a martingale.

\( \square \)

### 4.3. Probabilistic formulas for singular parabolic PDEs with lateral Neumann boundary conditions

In this section, \( \mathcal{O} \subseteq \mathbb{R}^d \) is an open, convex, connected and bounded subset of \( \mathbb{R}^d \). We assume that there is a function \( \Phi \in C^2_b(\mathbb{R}^d) \) such that \( \mathcal{O} = \{ \Phi > 0 \}, \partial \mathcal{O} = \{ \Phi = 0 \}, \mathbb{R}^d \setminus \overline{\mathcal{O}} = \{ \Phi < 0 \} \) and \( |\nabla \Phi(x)| = 1 \) for all \( x \in \partial \mathcal{O} \). We consider the following parabolic PDE with lateral Neumann boundary conditions:

\[
\frac{\partial u}{\partial t} + \mathcal{L} u + H(t, u, \sigma \nabla u) = 0 \quad \text{for } (t, x) \in [0, T] \times \mathcal{O}
\]

\[
\frac{\partial u}{\partial n} = 0 \quad \text{for } (t, x) \in [0, T] \times \partial \mathcal{O}
\]

\[
u u(T, x) = h(x) \quad \text{for } x \in \partial \mathcal{O},
\]

with

\[
\frac{\partial}{\partial n} := \sum_{i=1}^{d} \frac{\partial \Phi}{\partial x_i} \frac{\partial}{\partial x_i}.
\]

Consider the reflected SDE

\[
\begin{align*}
X^{t,x}_s &= x + \int_{t}^{s} \mu(u, X^{t,x}_u) \, du + \int_{t}^{s} \sigma(u, X^{t,x}_u) \, dW_u + \int_{t}^{s} \nabla \Phi(X^{t,x}_u) \, dK^{t,x}_u \\
K^{t,x}_s &= \int_{t}^{s} 1_{\{X^{t,x}_u \in \partial \mathcal{O}\}} \, dK^{t,x}_u \quad \text{and } K^{t,x} \text{ is nondecreasing}.
\end{align*}
\]

(4.13)

with \((s, x) \in [t, T] \times \partial \mathcal{O} \).

**Definition 4.8.** A strong solution to the reflected SDE (4.13) is a couple \((X^{t,x}_s, K^{t,x}_s)_{s \geq t}\) of adapted processes satisfying equation (4.13) and such that

\[
\int_{t}^{T} |\mu(s, X^{t,x}_u)| \, du + \int_{t}^{T} |\sigma(s, X^{t,x}_u)|^2 \, du + \int_{t}^{T} |\nabla \Phi(X^{t,x}_u)| \, dK^{t,x}_u < \infty \quad \mathbb{P}\text{-a.s.}
\]

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The solution \((X^{t,x}, K^{t,x})\) is said to be unique if it is indistinguishable to any other solution. (A5”) The SDE (4.13) admits a unique strong solution.

We will extend the solution process to \([0, T]\) by denoting

\[
X_s^{t,x} := x, \quad K_s^{t,x} := 0, \quad \forall s \in [0, t).
\]  

(4.14)

It follows by Lions and Sznitman [42] that if \(\mu : [0, T] \times \mathcal{O} \to \mathbb{R}^d\) and \(\sigma : [0, T] \times \mathcal{O}\) are uniformly Lipschitz continuous, that is, there is \(C \geq 0\) such that

\[
|\mu(t, x) - \mu(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq C|x - y| \quad \text{for all } x, y,
\]

then there is a unique pair \((X^{t,x}_s, K^{t,x}_s)\) satisfying (4.13). Just as in the case of Equation 4.2, assumption (A5”) also holds under much weaker conditions, see e.g. [48, 58].

Let \((Y^{t,x}_s, Z^{t,x}_s)\) satisfy

\[
Y_s^{t,x} = h(X_T^{t,x}) + \int_s^T H(u, Y_u^{t,x}, Z_u^{t,x}) \, du - \int_s^T Z_u^{t,x} \, dW_u, \quad s \in [t, T].
\]

Then, we have

**Theorem 4.9.** Assume that \(H : [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}\) is convex in \((y, z)\) and that (A1), (A4), (A5”) and (A6) are satisfied. Then, \(v(t, x) := Y^{t,x}_t\) is a viscosity solution of (4.12).

**Proof.** By Proposition A.1 \(X^{t,x}_T \to X^{t,x}_T\) in \(L^2\) whenever \((t^n, x^n) \to (t, x)\). Thus, due to Lemma 4.3, the mapping \((t, x) \mapsto v(t, x) := Y^{t,x}_t\) is continuous. Let us show the viscosity subsolution property. The viscosity supersolution property is similarly proved. Let \((t_0, x_0)\) be fixed. If \(t_0 = T\), then \(v(t_0, x_0) = h(x_0)\). Let us assume \(t_0 < T\). For simplicity, we denote \((X_t, K_t, Y_t, Z_t) := (X^{t_0,x_0}_t, K^{t_0,x_0}_t, Y^{t_0,x_0}_t, Z^{t_0,x_0}_t), t \geq t_0\). Since \(v(t, x) = Y^{t,x}_t\), then the Markov property of \(X\) and the uniqueness of \(Y\) show that for every \(t \in [t_0, T]\),

\[
v(t, X_t) = Y_t. \quad (4.15)
\]

Let \(\phi \in C_b^{1,2}(\mathcal{O}),\) (i.e. twice continuously differentiable with bounded derivatives) such that \(v - \phi\) admits a local maximum at \((t_0, x_0)\), which we assume global and equal to 0 (see e.g. [27]), that is:

\[
\phi(t_0, x_0) = v(t_0, x_0) \quad \text{and} \quad \phi(t, x) \geq v(t, x) \quad \text{for each } (t, x).
\]

If \(x \in \mathcal{O}\), then the argument is similar to the proof of Theorem 4.1, if not for the fact that Lemma 4.4 should be applied differently. Let us assume by contradiction that there is \(\delta > 0\) such that

\[
\frac{\partial \phi}{\partial t}(t_0, x_0) + \mathcal{L} \phi(t_0, x_0) + H(t_0, v(t_0, x_0), \sigma' \nabla \phi(t_0, x_0)) = -\delta < 0.
\]

By continuity, there exists a constant \(\varepsilon \in (0, T-t_0)\) such that for every \((t, x)\) in the set \((t_0 - \varepsilon, t_0 + \varepsilon) \times \{y : |y - x_0| \leq \varepsilon\}\), it holds

\[
\frac{\partial \phi}{\partial t}(t, x) + \mathcal{L} \phi(t, x) + H(t, v(t, x), \sigma' \nabla \phi(t, x)) \leq -\delta/2.
\]

The constant \(\varepsilon\) can, in addition, be chosen so that \(\{y : |y - x_0| \leq \varepsilon\} \subseteq \mathcal{O}\) and \((t_0, x_0)\) is a maximum of \(v - \phi\) in the set \((t_0 - \varepsilon, t_0 + \varepsilon) \times \{y : |y - x_0| \leq \varepsilon\}\). Consider the stopping time \(\tau\) defined in (4.9). Since \(X_{t_0} = x_0\), with probability one.
it holds $\tau > t_0$, and by continuity of the paths of $X$, it holds $|X_{t\wedge \tau} - x_0| \geq \varepsilon$ and $X_{t\wedge \tau} \in \mathcal{O}$ $\mathbb{P}$-a.s. for all $t \in [t_0, T]$. Thus, arguing as in the proof of Theorem 4.1, we arrive at

$$\phi(t \wedge \tau, X_{t\wedge \tau}) = \mathbb{E} \left[ \phi(t, X_t) - \int_{t \wedge \tau}^{\tau} \left( \frac{\partial \phi}{\partial r} + \mathcal{L} \phi \right) (r, X_r) dr - \int_{t \wedge \tau}^{\tau} \frac{\partial \phi}{\partial n}(r, X_r) dK_r \mid \mathcal{F}_{t\wedge \tau} \right]$$

and

$$Y_{t\wedge \tau} = \mathbb{E} \left[ Y_\tau + \int_{t \wedge \tau}^{\tau} H(r, v(r, X_r), Z_r) dr \mid \mathcal{F}_{t\wedge \tau} \right]. \quad (4.16)$$

Using that the quadratic variation of the process $\phi(t, X_t) - Y_t$ is given by $\sigma' \nabla_x \phi(t, X_t) - Z_t$, applying the argument of the first part of the proof of Lemma 4.4 yields

$$1_{\{\phi(t, X_t) - Y_t \geq 0\}} (\sigma' \nabla_x \phi(t, X_t) - Z_t) = 0 \quad \mathbb{P} \otimes dt\text{-a.e.}$$

Coming back to (4.16) and keeping in mind the definition of $\tau$, we have

$$Y_{t\wedge \tau} \leq \mathbb{E} \left[ \phi(t, X_t) - \int_{t \wedge \tau}^{\tau} \left( \frac{\partial \phi}{\partial r} + \mathcal{L} \phi \right) (r, X_r) dr \mid \mathcal{F}_{t\wedge \tau} \right] - \mathbb{E} \left[ \frac{\delta}{2} (\tau - t \wedge \tau) \mid \mathcal{F}_{t\wedge \tau} \right],$$

where the last equality comes from the fact that $K_{t\wedge \tau} = 0$, since $X_{t\wedge \tau} \in \mathcal{O}$. Letting $t = t_0$, we obtain $v(t_0, x_0) = Y_{t_0} = Y_{t_0\wedge \tau} \leq \phi(t_0, x_0) - \frac{\delta}{2} \mathbb{E}[\tau - t_0] \mid \mathcal{F}_{t_0}]$, which is a contradiction. Thus,

$$\frac{\partial \phi}{\partial t}(t_0, x_0) + \mathcal{L} \phi(t_0, x_0) + H(t_0, v(t_0, x_0), \sigma' \nabla \phi(t_0, x_0)) \geq 0.$$

If $x_0 \in \partial \mathcal{O}$, let us assume by contradiction that there is $\delta > 0$ such that

$$\max \left\{ \left( \frac{\partial \phi}{\partial t} + \mathcal{L} \phi \right)(t, x) + H(t, \phi(t, x), \sigma' \nabla_x \phi(t, x)), \frac{\partial \phi}{\partial n}(t, x) \right\} = -\delta < 0.$$

Let $\varepsilon \in (0, T - t)$ be such that

$$\sup_{1 \leq s \leq t + \varepsilon} \max \left\{ \left( \frac{\partial \phi}{\partial t} + \mathcal{L} \phi \right)(s, y) + H(s, \phi(s, y), \sigma' \nabla_x \phi(s, y)), \frac{\partial \phi}{\partial n}(s, y) \right\} \leq -\delta / 2.$$

Such an $\varepsilon$ can be found since $\phi \in C^{1,2}$ and $H$ is continuous. Consider the stopping time $\tau$ defined by (4.9). The problem in this case is that one cannot guarantee that $X_{t\wedge \tau} \in \mathcal{O}$, thus one can have $\mathbb{P}(K_{t\wedge \tau} > 0) > 0$. To overcome this issue, note that by the choice of $\phi$ and $\tau$, one has

$$\left( \frac{\partial \phi}{\partial t} + \mathcal{L} \phi \right)(t, X_t) + H(t, \phi(t, X_t), \sigma' \nabla_x \phi(t, X_t)) < -\delta / 2 \quad \text{and} \quad \frac{\partial \phi}{\partial n}(t, X_t) < -\delta / 2 \quad \text{on} \quad \{t \leq \tau\}.$$

Now, put $\xi_t := \phi(t, X_t) - Y_t$, $t \in [0, T]$. Arguing as above, $1_{\{\xi_t = 0\}} (\sigma' \nabla_x \phi(t, X_t) - Z_t) = 0 \mathbb{P} \otimes dt\text{-a.e.}$ Since $\xi \geq 0,$
it follows as in the argument in the second part of the proof of Lemma 4.4 that for every \( t \),

\[
0 \leq L^0_t(\xi) - L^0_t(\xi) = 2 \int_0^t \{ \frac{\partial \phi}{\partial t} + \mathcal{L} \phi \}(s, X_s) + H(s, \phi(s, X_s), Z_s) \} \, ds + 1_{\{\xi_s=0\}} \frac{\partial \phi}{\partial n}(s, X_s) \, dK_s
\]

\[
= 2 \int_0^t \{ \frac{\partial \phi}{\partial t} + \mathcal{L} \phi \}(s, X_s) + H(s, \phi(s, X_s), \sigma' \nabla_x \phi(s, X_s)) \} \, ds
\]

\[
+ 1_{\{\xi_s=0\}} \frac{\partial \phi}{\partial n}(r, X_r)(s, X_s) \, dK_s.
\]

Therefore, for every \( t \),

\[
0 \leq \int_0^{t \wedge \tau} \{ \frac{\partial \phi}{\partial t} + \mathcal{L} \phi \}(s, X_s) + H(s, \phi(s, X_s), \sigma' \nabla_x \phi(s, X_s)) \} \, ds + 1_{\{\xi_s=0\}} \frac{\partial \phi}{\partial n}(s, X_s) \, dK_s
\]

which contradicts \( \frac{\partial \phi}{\partial n}(t, X_t) < -\delta/2 \) on \( \{ t \leq \tau \} \) and concludes the proof of the viscosity subsolution property. □

**Remark 4.10.** In view of Theorem 4.6 and its proof, the PDE

\[
\begin{cases}
\frac{\partial u}{\partial t} + \mathcal{L} u + \frac{\sigma'(|\nabla u|^2)}{\sigma} = 0 & \text{for } (t, x) \in [0, T] \times \mathcal{O} \\
\frac{\partial u}{\partial n} = 0 & \text{for } (t, x) \in [0, T] \times \partial \mathcal{O} \\
u(T, x) = h(x) & \text{for } x \in \overline{\mathcal{O}},
\end{cases}
\tag{4.17}
\]

can also be solved when \( h \) is continuous, has polynomial growth and \( h > 0 \) and when \( b \) and \( \mu \) are two continuous functions of linear growth such that the SDE (4.13) admits a unique solution in law. ♦

## 5. Applications to decision theory

In this final part we provide applications to decision theory in finance, including to expected utility maximization and to existence of stochastic differential utility.

### 5.1. Utility maximization with multiplicative terminal endowment

In this subsection and the ensuing one, we discuss some applications of our existence result for BSDEs driven by \( |z|^2/y \) in financial economics.

We consider a market with \( m \) stocks available for trading \((m \leq d)\) and following the dynamics

\[
dS_t = S_t (b_t \, dt + \sigma_d \, dW_t), \quad S_0 = s_0 \in \mathbb{R}^m_+
\]

where \( b \) and \( \sigma \) are bounded predictable processes valued in \( \mathbb{R}^m \) and \( \mathbb{R}^{m \times d} \), respectively. We assume that the matrix \( \sigma \sigma' \) is of full rank and put \( \theta := \sigma'(\sigma \sigma')b \), the so-called market price of risk. Let us denote by \( \pi \) the trading strategy, i.e. \( \pi_t^i \)
is the part of total wealth at time $t$ invested in the stock $S^i$. We denote by $\mathcal{A}$ the set of admissible trading strategies. It is given by

$$\mathcal{A} := \left\{ \pi : [0, T] \times \Omega \to \mathbb{R}^m, \text{ progressive and } \int_0^T |\pi_t \sigma_t|^2 \, dt < \infty \right\}. $$

Let $x > 0$ be the initial wealth. For every $\pi \in \mathcal{A}$, the wealth process $X^\pi$ given by

$$X^\pi_t = x + \int_0^t \sum_{i=1}^d \frac{X^\pi_t \pi^i_s}{S^i_s} dS^i_s = x + \int_0^T X^\pi_t \pi_s \sigma_s \, ds + dW_s$$

is well-defined and positive. To ease the notation, we put $p := \pi \sigma$ and by abuse of notation we will write $p \in \mathcal{A}$. It is well-known that in the above setting, the market is free of arbitrage, see e.g. [19]. In particular, $X^p$ is a local martingale under the equivalent probability measure $Q^p$.

The aim of this section is to solve the utility maximization problem from the terminal wealth of an investor with power or logarithmic utility functions and non-trivial terminal endowment $\xi \in L^0$. More precisely, we consider the utility maximization problem

$$V(x) = \sup_{p \in \mathcal{A}} E[U(X_t^p(\xi))]$$

with $U(x) = x^\delta / \delta$ or $U(x) = \log(x)$. In (5.1), one can think of $\xi$ as some random charge or tax that the investor is required to pay. Such problems with multiplicative endowment also arise when the investor pays/receives a terminal portion of the terminal wealth, i.e. $F = X^p_T$. In this case, the terminal utility becomes $U(X^p_T + F) = U(X^p_T(1 + \eta))$, see e.g. [36].

**Proposition 5.1.** Assume that $U(x) = \log(x)$. Further assume that $\xi \in L^2$ and $\xi > 0$. Then, the value function is given by $V(x) = \log(xY_0)$, where $Y_0$ is the initial value of a solution $(Y, Z)$ of the BSDE $(\xi, H)$, where

$$H(t, y, z) = \begin{cases} \frac{1}{2} \frac{|z|^2}{y} & \text{if } y > 0 \\ +\infty & \text{else} \end{cases},$$

with $\sup_{0 \leq t \leq T} \mathbb{E}[|Y_t|^2] < \infty$ and $Z \in L^2$. Moreover, there exists an optimal admissible trading strategy $p^*$ given by

$$p^*_t = \theta_t, \quad t \in [0, T].$$

**Proof.** The proof relies on application of the martingale optimality principle initiated by [34] and the existence theorems for BSDEs derived above. Indeed, let us construct a family of processes $R^p$ such that for all $p \in \mathcal{A}, R^p_t = U(X^p_t(\xi))$; $R^0_t = R_0$ does not depend on $p$; $R^p$ is a supermartingale for all $p \in \mathcal{A}$ and, there is $p^* \in \mathcal{A}$ such that $R^{p^*}$ is a martingale. It can be checked that if such a family is constructed, then $p^*$ is the optimal strategy and $R_0 = V(x)$ is the value function. See [34] for details.

Put $R^p_t := U(X^p_t Y_t)$ where $(Y, Z)$ is a solution to BSDE $(\xi, g)$ such that $Y > 0$, $\sup \mathbb{E}[|Y_T|^2] < \infty$ and $Z \in L^2$, for some function $g : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ to be determined. Applying Itô’s formula, we obtain

$$dR^p_t = \left(U'(X^p_t Y_t) \{X^p_t g(t, Y_t, Z_t) + Y_t p_t X^p_t \theta_t + X^p_t p_t Z_t\} + \frac{1}{2} U''(X^p_t Y_t)(X^p_t)^2 |Z_t + p_t Y_t|^2 \right) \, dt$$

$$+ U'(X^p_t Y_t) X^p_t (Z_t + p_t Y_t) \, dW_t.$$

For all $n \in \mathbb{N}$, define the stopping time

$$\tau_n := \inf \left\{ t > 0 : \int_0^t \left(U'(X^p_s Y_s) X^p_s (Z_s + Y_s p_s)\right)^2 \, ds \geq n \right\} \wedge T.$$
Since $\int_0^{\tau_n} U'(x_t, Y_t) X_t^2 dW_t$ is a martingale, it follows that if
\[
g(t, Y_t, Z_t) \leq \left\{ \frac{1}{2} U''(x_t, Y_t) X_t^2 + \frac{1}{2} U''(x_t, Y_t) Y_t^2 \right\}
\] on $\{ t \leq \tau_n \},$ (5.2)
then the process $(R^p_{t \wedge \tau_n})_{t \in [0, T]}$ is a supermartingale. Thus, it follows from Fatou’s lemma that $R^p$ is a supermartingale, since $\tau_n \uparrow T$ and $R^p$ has continuous paths. When $U(x) = \log(x),$ the condition (5.2) amongst to
\[
g(t, Y_t, Z_t) \leq \frac{1}{2} \frac{1}{Y_t^2} |Z_t + p_t Y_t|^2 - Y_t p_t \theta_t - p_t Z_t.
\]
Therefore, we put
\[
g(t, Y_t, Z_t) := \inf_{p \in A} \left( \frac{1}{2} \frac{1}{Y_t^2} |Z_t + p_t Y_t|^2 - Y_t p_t \theta_t - p_t Z_t \right).
\]
Since $Y > 0,$ the above infimum is attained by $p^* = \theta$ and $g(t, Y_t, Z_t)$ is given by
\[
g(t, Y_t, Z_t) = \frac{1}{2} \frac{|Z_t|^2}{Y_t^2} - \frac{1}{2} \theta^2 Y_t.
\]
In particular, $p^*$ is admissible. Let us show that $R^{p^*}$ is a martingale. By construction, the drift term of $R^{p^*}$ is zero, so that $(R^{p^*}_{t \wedge \tau_n})_{t \in [0, T]}$ is a Martingale. Thus, we can conclude by dominated convergence that $R^{p^*}$ is a martingale if we show that the set $\{ R^{p^*}_{\tau^*} : \tau \text{ stopping time} \}$ is uniformly integrable. For every $[0, T]$-valued stopping time $\tau,$ we have
\[
\mathbb{E}\left[ \exp(R^{p^*}_\tau) \right] \leq C \mathbb{E}\left[ X^\tau_T \right] \leq C \mathbb{E}\left[ X^\tau_T \right]^{1/2} \mathbb{E}\left[ Y^\tau_T \right]^{1/2}
\leq C \sup_{\tau} \mathbb{E}\left[ X^\tau_T \right]^{1/2} \sup_{\tau} \mathbb{E}\left[ Y^\tau_T \right]^{1/2} < \infty,
\]
where the second inequality follows by Hölder’s inequality and the last one the fact that $\theta$ is bounded and $\xi \in L^3.$ Therefore, $\{ R^{p^*}_{\tau^*} : \tau \text{ stopping time} \}$ is uniformly integrable.

It remains to show that BSDE($\xi, H$) admits a solution $(Y, Z)$ such that $Y > 0,$ $\sup_{t \in [0, T]} \mathbb{E}[Y^2_t] < \infty$ and $Z \in L^2.$ The generator $g$ does not satisfy the conditions of Theorem 2.2, but by the change of variable $Y = \exp(-\int_0^t \frac{1}{2} \theta_u^2 du) Y_t,$ solving BSDE($\xi, g$) amongst to solving the equation
\[
\dot{Y}_t = e^{-\int_0^t \frac{1}{2} \theta_u^2 du} \xi + \frac{1}{2} \frac{1}{Y_u} du - \int_0^T \frac{1}{Y_u} dW_u.
\]
Even though this equation does not satisfy the conditions of Proposition 1.1 or Theorem 2.2, it can be treated in a similar way. In fact, consider the function $u(x) := \frac{1}{2} x^2.$ Applying Itô’s formula to $\tilde{Y} := u(Y)$ shows that $Y$ exists if and only if $\tilde{Y}$ solves BSDE($\frac{1}{2} e^{-\int_0^t \frac{1}{2} \theta_u^2 du} \xi, 0$) which by Lemma 3.1 admits a unique solution $(\tilde{Y}, \tilde{Z})$ such that $\sup_{t \in [0, T]} \mathbb{E}[|\tilde{Y}_t|] < \infty$ and $\tilde{Y} > 0.$ Therefore, $(\tilde{Y}, \tilde{Z})$ solving BSDE($\frac{1}{2} e^{-\int_0^t \frac{1}{2} \theta_u^2 du} \xi, \frac{1}{2} |z|^2 / y$) exists and satisfies $\sup_{t \in [0, T]} \mathbb{E}[|\tilde{Y}_t|^2] < \infty$ and $\tilde{Y} > 0,$ showing that $(Y, Z)$ solves BSDE($\xi, H$), $Y > 0$ and since $\theta$ is bounded, it holds $\sup_{t \in [0, T]} \mathbb{E}[|Y|^2] < \infty.$

5.2 Stochastic differential utility

In economics and decision theory, Epstein-Zin preferences [25] refer to a class of (dynamic and) recursively defined utility functions (or preference specifications) given by
\[
U_t(c) := F \left( \epsilon_t, f^{-1}\left( \mathbb{E}[f(U_{t+1}(c)) | \mathcal{F}_t] \right) \right).
\]
Here, $U_t(c)$ is the time-$t$ utility of the consumption $c = (c_t, c_{t+1}, \ldots)$, which is assumed to be an adapted sequence of real-valued random variables, $F : \mathbb{R}^2 \to \mathbb{R}_+$ is a given function and $f : \mathbb{R} \to \mathbb{R}$ is a utility function, i.e. a strictly increasing and concave function. Epstein-Zin preferences are mostly important because they allow to disentangle risk aversion (modeled by $f$) and intertemporal substitution (modeled by $F$). The continuous-time analogue (know as stochastic differential utility) of Epstein-Zin preferences was developed by Duffie and Epstein [24] and defined as the unique adapted solution $(U_t)_{0 \leq t \leq T}$ (when it exits) of the integral equation

$$U_t(c) = \mathbb{E} \left[ - \int_t^T g(c_s, U_s(c)) + \frac{1}{2} A(U_s(c)) \sigma_U^2(s) \, ds \mid \mathcal{F}_t \right].$$

(5.3)

Here, $c : \Omega \times [0, T] \to \mathbb{R}$ is an adapted consumption process, $\sigma_U^2$ is the “volatility” of the unknown process $U$, and $A : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given functions modeling risk aversion and intertemporal substitution, respectively. When considering utility of a (terminal) position $\xi$ in addition to that of a consumption process $c$, the stochastic differential utility takes the form

$$U_t(\xi, c) = \mathbb{E} \left[ \xi - \int_t^T g(c_s, U_s(\xi, c)) + \frac{1}{2} A(U_s(\xi, c)) \sigma_U^2(s) \, ds \mid \mathcal{F}_t \right].$$

(5.4)

It was shown in [24] that when the function $A$ is continuous and integrable, the function $g(c, u)$ is Lipschitz continuous in $u$ and of linear growth in $c$, the integral equation (5.3), which of course coincides with the BSDE

$$dY_t = g(c_t, Y_t) + \frac{1}{2} A(Y_t)|Z_t|^2 \, dt - Z_t \, dW_t, \quad Y_T = 0$$

admits a unique square integrable solution. We also refer to [7] for extensions of this result, for $A$ integrable. Moreover, this class utility functions are important in the context of asset pricing, (see Duffie and Epstein [23]) the case $A(u) := 1/u$ being of particular interest, as the continuous time analogue of the Kreps-Porteus utility.

A direct consequence of our main result is the following extension of the existence of a class of dynamic differential utilities beyond the class of Lipschitz continuous intertemporal functions $g$.

**Proposition 5.2.** Let $\xi$ be strictly positive, and $c : [0, T] \times \Omega \to \mathbb{R}$ an adapted process. Assume that $g : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is progressively measurable and satisfies

$$|g(t, \omega, c, y)| \leq \alpha_t(\omega) + \beta_t(\omega) y$$

for two positive and progressively measurable processes $\alpha$ and $\beta$ such that there is $p > 1$ such that

$$E \left[ \xi^p e^{\int_0^T \alpha_s + \beta_s \, ds} \right] < \infty \quad \text{and} \quad E \left[ \left( \int_0^T e^{\frac{1}{2} \int_0^s \alpha_u + \beta_u \, du} \, \alpha_s \, ds \right)^p \right] < \infty,$$

Then, there exists a stochastic differential utility $U \in \mathcal{S}^{3p}$ satisfying

$$U_t(\xi, c) = \mathbb{E} \left[ \xi - \int_t^T g(c_s, U_s(\xi, c)) + \frac{1}{U_s(\xi, c)} \sigma_U^2(s) \, ds \mid \mathcal{F}_t \right].$$

Moreover, if $\xi$ and $\alpha$ are bounded, then $U$ is bounded and unique.
Example 5.3. The continuous-time analogue of the Kreps-Porteus utility (see e.g. Duffie and Epstein [23] and Kreps and Porteus [40] for the discrete-time case) is obtained by setting

$$A(u) := \frac{\alpha - 1}{u} \quad \text{and} \quad g(c, u) := \frac{\beta}{\rho} \cdot \frac{c^\rho - u^\rho}{u^{\rho-1}},$$

where $\alpha, \beta$ and $\rho$ are constants satisfying $0 < \alpha \leq 1$, $0 \leq \beta$ and $0 < \rho \leq 1$. Existence and uniqueness of such stochastic differential utilities in continuous time follows as a consequence of Proposition 5.2.

A. Continuity of SDE solutions w.r.t. initial parameters

In this section we present the main arguments of the proofs of continuity results of SDE solutions in their initial conditions. The first two results concern strong solutions of SDE with and without reflection, their proofs are modest extensions of the main result of [4]. The last result concerns a form of continuity for weak solutions. This result seems to be new.

Proposition A.1. Assume that (A4) and (A5') are satisfied, let $(t^n, x^n)$ be a sequence in $[0, T] \times \mathcal{O}$ converging to $(t, x)$. If the pathwise uniqueness holds for Equation 4.13, then the sequence of processes $(X^{t_n, x_n}, K^{t_n, x_n})$ converges in $\mathcal{S}^2(\mathbb{R})$ to $(X^{t, x}, K^{t, x})$ which is the unique solution of the SDE (4.13) starting at $x$ at time $t$.

Proof. We follow the idea of the proofs given in [4, 37], we also use some computations from [5]. Assume that the conclusion of Proposition A.1 is false. Then there exist a positive number $\delta$ and a sequence $(t_n, x_n)$ converging to $(t, x)$ such that

$$\inf_n \mathbb{E} \left[ \sup_{0 \leq s \leq T} \left| (X^{t_n, x_n}(s), K^{t_n, x_n}(s)) - (X^{t, x}(s), K^{t, x}(s)) \right|^2 \right] \geq \delta. \quad (A.1)$$

Without loss of generality, we assume that $t_n \geq t$ for each $n$. Arguing as in the proof of [5, Lemma 3.8, p. 12] and using assumption (A4) and boundedness of the domain $\mathcal{O}$, we show that the sequence $(X^{t_n, x_n}, K^{t_n, x_n}, X^{t, x}, K^{t, x}, W)$ is tight in $C([0, T], \mathbb{R}^d)$. Hence, according to Skorohod’s representation theorem, there exists a sequence of processes $(X^n, K^n, Y^n, K^{1,n}, W^n)_{n \geq 1}$ and a process $(\bar{X}, \bar{K}, \bar{Y}, \bar{K}^1, \bar{W})$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that for each $n$

$$\text{Law}(X^n, K^n, Y^n, K^{1,n}, W^n) = \text{Law}(X^{t_n, x_n}, K^{t_n, x_n}, X^{t, x}, K^{t, x}, W) \quad (A.2)$$

and there exists a subsequence still denoted $(X^n, K^n, Y^n, K^{1,n}, W^n)$, such that

$$\lim_{n \to \infty} (X^n, K^n, Y^n, K^{1,n}, W^n) = (X, K, Y, K^1, W) \text{ uniformly on every finite interval } \mathbb{P}\text{-a.s.} \quad (A.3)$$

Let $\mathcal{F}_t^n$ (resp. $\mathcal{F}_t$) be the $\sigma$-algebra $\sigma(X^n_s, Y^n_s, W^n_s; s \leq t)$ (resp. $\sigma(X_s, Y_s, W_s; s \leq t)$) completed with $\mathbb{P}$-null sets. Hence $(W^n_t, \mathcal{F}_t^n)$ and $(W_t, \mathcal{F}_t)$ are $\mathbb{P}$ Brownian motions and the processes $X^n, K^n, Y^n, K^{1,n}$ (resp. $\bar{X}, \bar{K}, \bar{Y}, \bar{K}^1$) are adapted to $\mathcal{F}_t^n$ and $\mathcal{F}_t$ respectively.

From (A.2) and (4.13) we have

$$\begin{align*}
X^n_s &= x_n + \int_{t_n}^s \mu(r, X^n_r)dr + \int_{t_n}^s \sigma(r, X^n_r)dW^n_r + \int_{t_n}^s \nabla_x \Phi(X^n_r) dK^n_r \quad \mathbb{P}\text{-a.s.} \quad (A.4) \\
K^n_s &= \int_{t_n}^s 1_{\{X^n_r \in \partial \mathcal{O}\}} d\bar{K}_r^n
\end{align*}$$

and

$$\begin{align*}
Y^n_s &= x + \int_{t_n}^s \mu(r, Y^n_r)dr + \int_{t_n}^s \sigma(r, Y^n_r)dW^n_r + \int_{t_n}^s \nabla_y \Phi(Y^n_r) dK^{1,n}_r \quad \mathbb{P}\text{-a.s.} \\
K^{1,n}_s &= \int_{t_n}^s 1_{\{Y^n_r \in \partial \mathcal{O}\}} d\bar{K}^{1,n}_r
\end{align*} \quad (A.5)$$
see [4] and [5] for details. Using (A.3), (A.4) and (A.5), we show that

\[
\begin{align*}
\dot{X}_s &= x + \int_t^s \mu(r, \dot{X}_r) dr + \int_t^s \sigma(r, \dot{X}_r) dW_r + \int_t^s \nabla_x \Phi(X^n_r) d\bar{K}_r & \bar{P}\text{-a.s.} \\
\bar{K}_s &= \int_t^s 1_{\{X_s \in \partial \Omega\}} d\bar{K}_u
\end{align*}
\]

and

\[
\begin{align*}
\dot{Y}_s &= x + \int_t^s \mu(r, \dot{Y}_r) dr + \int_t^s \sigma(r, \dot{Y}_r) dW_r + \int_t^s \nabla_x \Phi(Y^n_r) d\bar{K}_r^1 & \bar{P}\text{-a.s.} \\
\bar{K}_s^1 &= \int_t^s 1_{\{\bar{Y}_s \in \partial \Omega\}} d\bar{K}_u^1
\end{align*}
\]

Thus, \((\bar{X}, \bar{K})\) and \((\bar{Y}, \bar{K}^1)\) satisfy then the same SDE with the same Brownian motion and the same initial value. Therefore the pathwise uniqueness property shows that \((\bar{X}, \bar{K})\) and \((\bar{Y}, \bar{K}^1)\) are indistinguishable. Returning back to (A.1), we use (A.2), (A.3) and the uniform integrability to get

\[
\delta \leq \lim \inf_n \mathbb{E}\left[ \sup_{t \leq s \leq T} \left| \left( X^{\tau_n,x_n}_s, K^{\tau_n,x_n}_s \right) - \left( X^{t,x}_s, K^{t,x}_s \right) \right|^2 \right]
\]

\[
= \lim \inf_n \mathbb{E}\left[ \sup_{t \leq s \leq T} \left| \left( \bar{X}^n_s, \bar{K}^n_s \right) - \left( \bar{Y}^n_s, \bar{K}^1_s \right) \right|^2 \right]
\]

\[
\leq \mathbb{E}\left[ \sup_{t \leq s \leq T} \left| \left( \bar{X}_s, \bar{K}_s \right) - \left( \bar{Y}_s, \bar{K}_s^1 \right) \right|^2 \right]
\]

\[
= 0
\]

which is a contradiction. The proof is finished. \( \square \)

**Proposition A.2.** Assume that (A3) and (A4) are satisfied, let \((t^n, x^n)\) be a sequence in \([0, T] \times \mathbb{R}^d\) converging to \((t, x)\). Then the sequence of processes \((X^{t_n,x_n})\) converges in \(S^2(\mathbb{R})\) to \((X^{t,x})\) which is the unique strong solution of the SDE (4.2) starting at \(x\) at time \(t\).

The proof is similar (and simpler) than that of Proposition A.1.

**Proposition A.3.** Assume that (A4) is satisfied, let \((t^n, x^n)\) be a sequence in \([0, T] \times \mathcal{O}\) converging to \((t, x)\). If the uniqueness in law holds for Equation (4.13), then the sequence of processes \((X^{t_n,x_n})\) converges in law to \(X^{t,x}\) which is the unique solution (in law) of the SDE (4.13) starting at \(x\) at time \(t\).

**Proof.** Without loss of generality, we assume that \(t_n \geq t\) for each \(n\). Using assumption (A4) , we show that the sequence \((X^{t_n,x_n}, W)\) is tight in \(C([0, T], \mathbb{R}^d)\). Hence, according to Skorohod’s representation theorem, there exists a sequence of processes \((\bar{X}^n, \bar{W}^n)\) and a process \((\bar{X}, \bar{W})\) defined on some probability space \((\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})\) such that for each \(n\)

\[
\text{Law}(\bar{X}^n, \bar{W}^n) = \text{Law}(X^{t_n,x_n}, W)
\]

and there exists a subsequence still denoted \((\bar{X}^n, \bar{W}^n)\), such that

\[
\lim_{n \to \infty} (\bar{X}^n, \bar{W}^n) = (\bar{X}, \bar{W}) \text{ uniformly on every finite interval } \bar{P}\text{-a.s.}
\]

Let \(\bar{\mathcal{F}}^n\) (resp. \(\bar{\mathcal{F}}\)) be the \(\sigma\)-algebra \(\sigma(\bar{X}^n_s, \bar{W}^n_s; s \leq t)\) (resp. \(\sigma(\bar{X}_s, \bar{W}_s; s \leq t)\)) completed with \(\bar{P}\)-null sets. Hence \((\bar{W}_t, \bar{\mathcal{F}}^n)\) and \((\bar{W}_t, \bar{\mathcal{F}})\) are \(\bar{P}\) Brownian motions and the processes \(\bar{X}^n\) (resp. \(\bar{X}\)) are adapted to \(\bar{\mathcal{F}}^n\) and \(\bar{\mathcal{F}}\) respectively.
From (A.8) and (4.2) we have

\[ \bar{X}_s^n = x_n + \int_{t_n}^{s} \mu(r, \bar{X}_r^n) dr + \int_{t_n}^{s} \sigma(r, \bar{X}_r^n) dW_r^n \quad \mathbb{P}\text{-a.s.} \quad (A.10) \]

Using (A.8), (A.9) and (A.10) we show that

\[ \bar{X}_s = x + \int_{t}^{s} \mu(r, \bar{X}_r) dr + \int_{t}^{s} \sigma(r, \bar{X}_r) dW_r \quad \mathbb{P}\text{-a.s.} \quad (A.11) \]

The uniqueness in law shows that \( \bar{X} \) and \( X^{t,x} \) have the same law and that the whole sequence \( X^{t_n,x_n} \) converges in law to \( X^{t,x} \).

References

[1] B. Abdellaoui and D. Giachetti and I. Peral and M. Walia. Elliptic problems with nonlinear term depending on the gradient and singular on the boundary. *Nonlinear Anal.*, 74,1355–1371, 2011.

[2] D. Arcoya, L. Boccardo, T. Leonori, and A. Porretta. Some elliptic problem with singular natural growth lower order terms. *J. Diff. Equ.*, 249:2771–2795, 2010.

[3] K. Bahlali. Unbounded quadratic BSDEs: Existence by domination. *Preprint*, 2018.

[4] K. Bahlali, B. Mezerdi, and Y. Ouknine. Pathwise uniqueness and approximation of solutions of stochastic differential equations. *Séminaire de Probabilité (Strasbourg)*, 32:166–187, 1998.

[5] K. Bahlali, L. Maticiuc, and A. Zalinescu. Penalization method for a nonlinear Neumann PDE via weak solutions of reflected SDEs. *Elect. J. Probab.*, 18(102), 2013.

[6] K. Bahlali, E. Essaky, and M. Hassani. Existence and uniqueness of multidimensional BSDEs and of systems of degenerate PDEs with superlinear growth generator. *SIAM J. Math. Anal.*, 47(6):4251–4288, 2015.

[7] K. Bahlali, M. Eddahbi, and Y. Ouknine. Quadratic BSDEs with \( L^2 \)-terminal data Krylov’s estimate and Itô–Krylov’s formula and existence results. *Ann. Probab.*, 45(4):2377–2397, 2017.

[8] G. Barles and E. Lesigne. SDEs, BSDEs and PDE. *Backward Stochastic Differential Equations*, El Karoui, N., Mazliak (Editors) Longman 47–82, 1997.

[9] P. Barrieu and N. El Karoui. Monotone stability of quadratic semimartingales with applications to unbounded general quadratic BSDEs. *Ann. Probab.*, 41(3B):1831–1863, 05 2013.

[10] L. Boccardo and L. Orsina. Semilinear elliptic equations with singular nonlinearities. *Cal. Var. PDE.*, 37(3-4):363–380, 2010.

[11] M. Boué and P. Dupuis. A variational representation for certain functionals of Brownian motion. *Ann. Probab.*, 26(4):1641–1659, 1998.

[12] P. Briand and Y. Hu. BSDE with quadratic growth and unbounded terminal value. *Probab. Theory Related Fields*, 136(4):604–618, 2006.

[13] J. Carmona and T. Leonori. A uniqueness result for a singular elliptic equation with gradient term. *Proceedings of the Royal Society of Edinburgh, to appear*, 2018.

[14] S. J. Chapman, B. J. Hunton, and R. Ockendon. Vortices and boundaries. *Quart. Appl. Math.*, 56(3):507–519, 1998.

[15] P. Cheridito and K. Nam. BSDEs with terminal conditions that have bounded Malliavin derivative. *J. Funct. Anal.*, 266(3):1257–1285, 2014.

[16] A. S. Cherny and H.-J. Engelbert. *Singular Stochastic Differential Equations*. Lecture Notes in Mathematics. Springer, 2005.

[17] M. Crandall, H. Ishii, and P.-L. Lions. User’s guide to viscosity solutions of second order partial differential equations. *Bulletin A.M.S.*, 27(1), 1992.

27
[18] A. Dall’Aglio, L. Orsina, and F. Petitta. Existence of solutions of degenerate parabolic equations with singular terms. *Nonlinear Anal.*, 131:273–288, 2016.

[19] F. Delbaen and W. Schachermayer. A general version of the fundamental theorem of asset pricing. *Math. Ann.*, 300(3):463–520, 1994.

[20] F. Delbaen, Y. Hu, and X. Bao. Backward SDEs with Superquadratic Growth. *Probab. Theory Related Fields*, 150(1-2): 145–192, 2011.

[21] K. Dong and H. A. Levine. On the blow-up of $u_t$ at quenching. *Proc. A.M.S.*, 106:1049–1056, 1989.

[22] F. Delbaen, Y. Hu, and X. Bao. Backward SDEs with Superquadratic Growth. *Probab. Theory Related Fields*, 150(1-2): 145–192, 2011.

[23] D. Duffie and L. G. Epstein. Asset pricing with stochastic differential utility. *Econometrica*, 5(3):411–436, 1992.

[24] D. Duffie and L. G. Epstein. Stochastic differential utility. *Econometrica*, 60(2):353–94, 1992.

[25] L. G. Epstein and S. E. Zin. Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework. *Econometrica*, 57(4):937–69, July 1989.

[26] E. Essaky and M. Hassani. Generalized BSDE with 2–reflecting barriers and stochastic quadratic growth. *J. Differential Equations*, 254(3):1500–1528, 2013.

[27] W. H. Fleming and H. M. Soner. *Controlled Markov processes and viscosity solutions*, volume 25 of *Stochastic Modelling and Applied Probability*. Springer, New York, second edition, 2006.

[28] D. Giachetti, F. Petitta, and S. S. de León. Elliptic equations having a singular quadratic gradient term and a changing sign datum. *Comm. Pure. Appl. Anal.*, 11:1875, 2012.

[29] D. Giachetti and F. Murat. An elliptic problem with a lower order term having singular behavior. *Bollettino U.M.I.*, 9(II):349–370, 2009.

[30] E. Gobet, J.-P. Lelong, and X. Warin. A regression-based Monte Carlo method to solve backward stochastic differential equations. *Ann. Appl. Probab.*, 15(3):2172–2202, 2005.

[31] S. Hamadène. Multi-dimensional BSDEs with uniformly continuous coefficients. *Bernoulli*, 9(3):517–534, 2003.

[32] J. Han, A. Jentzen, and W. E. Solving high-dimensional partial differential equations using deep learning. *To appear in Proc. Natl. Acad. Sci.*, 2017.

[33] G. Heyne, M. Kupper, and L. Tangpi. Portfolio optimization under nonlinear utility. *Int. J. Theor. Appl. Fin.*, 19(5):1650029, 2016.

[34] Y. Hu, P. Imkeller, and M. Müller. Utility maximization in incomplete markets. *Ann. Appl. Probab.*, 15(3):1691–1712, 2005.

[35] C. B. Hyndman. A forward-backward SDE approach to affine models. *Math. Financ. Econ.*, 2009.

[36] P. Imkeller, A. Réveillac, and J. Zhang. Solvability and numerical simulation of BSDEs related to BSPDEs with applications to utility maximization. *Int. J. Theor. Appl. Finance*, 14:635–667, 2011.

[37] H. Kaneko and S. Nakao. A note on approximation for stochastic differential equations. *Séminaire de Probabilité (Strasbourg)*, XXII:155–162, 1988.

[38] N. Kazamaki. *Continuous Exponential Martingales and BMO*, volume 1579 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1994.

[39] M. Kobylanski. Backward Stochastic Differential Equations and Partial Differential Equations with Quadratic Growth. *Ann. Probab.*, 28(2):558–602, 2000.

[40] D. Kreps and E. Porteus. Temporal resolution of uncertainty and dynamic choice theory. *Econometrica*, 46:185–200, 1978.

[41] N. Krylov and M. Röckner. Strong solutions of stochastic equations with singular time dependent drift. *Probab. Theory Related Fields*, 131:154–196, 2005.

[42] P. L. Lions and A. S. Sznitman. Stochastic differential equations with reflecting boundary conditions. *Comm. Pure. Appl. Math.*, 37:511–537, 1984.

[43] O. Menoukeu-Pamen and S. E. A. Mohammed. Flows for singular stochastic differential equations with unbounded drifts. *Preprint*, 2017.
[44] O. Menoukeu-Pamen, Y. Ouknine, and L. Tangpi. Pathwise uniqueness of non-uniformly elliptic SDEs with rough coefficients. Preprint, 2017.

[45] F. Merle. Solution of a nonlinear heat equation with arbitrary given blow-up points. Comm. Pure. Appl. Math., 45:263–300, 1992.

[46] F. Merle and H. Zaag. Reconnection of vortex with the boundary and finite time quenching. Nonlinearity, 10:1497–1550, 1997.

[47] A. Molino. Gelfand type problem for singular quadratic quasilinear equations. NoDEA, 23:56, 2016.

[48] T. Nilssen and T. Zhang. A note on the Malliavin differentiability of one-dimensional reflected stochastic differential equations with discontinuous drift. arXiv:1410.0520v1 Preprint, 2014.

[49] M. Nutz. The Bellman equation for power utility maximization with semimartingales. Ann. Appl. Probab., 22(1):363–406, 2012.

[50] E. Pardoux. BSDEs, weak convergence and homogenization of semilinear PDEs. In F. Clarke and R. Stern, editors, Nonlinear Analysis, Differential Equations and Control, pages 503–549. Kluwer Academic Publishers, 1999.

[51] E. Pardoux and S. G. Peng. Adapted solution of a backward stochastic differential equation. System Controll Lett., 14:55–61, 1990.

[52] D. Revuz and M. Yor. Continuous Martingales and Brownian Motion, volume 293 of Grundlehren der Mathematischen Wissenschaften. Springer-Verlag, Berlin, 3 edition, 1999.

[53] S. E. Shreve. Stochastic Calculus for Finance II Continuous time models. Springer, 2004.

[54] D. Stroock and S. Varadhan. Diffusion processes with continuous coefficients, I, II. Comm. Pure. Appl. Math., 22(345-400): 479–530, 1969.

[55] R. Tevzadze. Solvability of backward stochastic differential equations with quadratic growth. Stoch. Proc. Appl., 118(503-515), 2008.

[56] H. Xing. Consumption investment optimization with Epstein-Zin utility in incomplete markets. Finance Stoch., 21(1):227–262, 2017.

[57] E. Zeidler. Nonlinear Functional Analysis and its Applications III: Variational Methods and Optimization. Springer-Science +Buissiness Media, LLC, New York, 1985.

[58] T. Zhang. On the strong solutions of one-dimensional stochastic differential equations with reflecting boundary. Stoch. Proc. Appl., 50:135–147, 1994.

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