BIFFURCATION OF POSITIVE EQUILIBRIA IN NONLINEAR STRUCTURED POPULATION MODELS WITH VARYING MORTALITY RATES

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ABSTRACT. A parameter-dependent model involving nonlinear diffusion for an age-structured population is studied. The parameter measures the intensity of the mortality. A bifurcation approach is used to establish existence of positive equilibrium solutions.

1. INTRODUCTION

Let $u = u(t, a, x) \geq 0$ be the distribution density at time $t \geq 0$ of individuals of a population structured by age $a \in J := [0, a_m]$ and spatial position $x \in \Omega$, where $a_m \in (0, \infty)$ denotes the maximal age and $\Omega$ is a bounded and smooth domain in $\mathbb{R}^N$. Suppose that the individual’s movement is governed by a nonlinear diffusion term $\text{div}_x(D(U(t,x),a)\nabla_x u)$ with dispersal speed $D(U,a) > 0$ depending on age and on the total local population $U(t,x) := \int_0^{a_m} u(t,a,x) da$.

Let $b = b(U,a) \geq 0$ and $\bar{\mu} = \bar{\mu}(U,a) \geq 0$ denote respectively the density dependent birth and death rate. Then a simple model describing the evolution of the population with initial distribution $u^0 = u^0(a,x) \geq 0$ is

$$
\begin{align*}
\partial_t u + \partial_a u & = \text{div}_x(D(U(t,x),a)\nabla_x u) - \bar{\mu}(U(t,x),a) u, & t > 0, a \in J, x \in \Omega, \\
u(t,0,x) & = \int_0^{a_m} b(U(t,x),a) u(a) da, & t > 0, x \in \Omega, \\
\delta u(t,a,x) + (1-\delta)\partial_{\nu} u(t,a,x) & = 0, & t > 0, a \in J, x \in \partial\Omega, \\
u(0,a,x) & = u^0(a,x), & a \in J, x \in \Omega,
\end{align*}
$$

where either $\delta = 1$ or $\delta = 0$ corresponding to Dirichlet or Neumann boundary conditions. Models of this type have a long history and we refer to [22] for a survey of structured population models. The well-posedness of these equations and related population models involving nonlinear diffusion is investigated e.g. in [19]. Questions regarding the large time behavior are linked to equilibrium solutions. In this paper we thus shall focus on nontrivial nonnegative equilibrium solutions for such equations, that is, on time-independent solutions $u = u(a,x) \geq 0$ with $u \not\equiv 0$.

Positive equilibrium solutions for age-structured population models without diffusion are studied e.g. in [9, 10, 11] using bifurcation techniques or also in [15] using fixed point theorems in conical shells. A bifurcation approach to age-structured population models with linear diffusion and linear birth but nonlinear death rates is used in [13, 14]. For an approach to age-structured models including both nonlinear diffusion and nonlinear death and birth rates we refer to [20, 21], where local and global bifurcation, respectively, is shown for a bifurcation parameter measuring the intensity of the fertility similarly as in [9, 10, 11]. The aim of this paper is to demonstrate that also the intensity of the mortality can be treated as bifurcation parameter in age-structured models with nonlinear diffusion. Moreover, as expected and opposed to the results of e.g. [20], where the fertility intensity varies, in the present situation subcritical bifurcation occurs under realistic
assumptions.

The approach we choose is based on introducing a parameter \( \lambda \) measuring the intensity of the mortality without changing its structure; that is, we shall consider parameter-dependent death rates of the form \( \bar{\mu} = \lambda \mu(U, a) \) with \( \mu = \mu(U, a) \) being a fixed reference function. Thus we are looking for solutions \( u = u(a(x)) \) to the parameter-dependent problem

\[
\partial_a u = \text{div}_x \left( D(U(x), a) \nabla u \right) - \lambda \mu(U(x), a) u, \quad a \in J, \ x \in \Omega, \quad (1.1)
\]

\[
u(0, x) = \int_0^{a_m} b(U(x), a) u(a) \, da, \quad x \in \Omega, \quad (1.2)
\]

\[
\delta u(a, x) + (1 - \delta) \partial_a u(a, x) = 0, \quad a \in J, \ x \in \partial \Omega. \quad (1.3)
\]

Clearly, \( u \equiv 0 \) is a solution to (1.1)-(1.3) for any value of \( \lambda \). The main goal is then to establish existence of nontrivial solutions which are also nonnegative. Under suitable assumptions we shall prove that the theorem of Crandall-Rabinowitz [3] applies so that there is a unique value \( \lambda_0 > 0 \) for which a nontrivial branch \( \{(\lambda, u_\lambda); |\lambda - \lambda_0| \text{ small}\} \) bifurcates from the trivial branch \( \{ (\lambda, 0); \lambda \in \mathbb{R} \} \) at the critical point \( (\lambda_0, 0) \) and that at least one part of the nontrivial branch near the critical point consists of nonnegative solutions.

To be more precise, let \( \sigma_1 \) be the first eigenvalue of \( -\Delta x \) on \( \Omega \) subject to Dirichlet (if \( \delta = 1 \)) or Neumann (if \( \delta = 0 \)) boundary conditions, hence \( \sigma_1 > 0 \) in the first and \( \sigma_1 = 0 \) in the second case. Suppose that

\[
\int_0^{a_m} b(0, a) e^{-\sigma_1 \int_0^a \mu(z, r) dr} \, da > 1 \quad \text{and} \quad \mu(0, a) > 0 \text{ for } a \text{ near } 0. \quad (1.4)
\]

Roughly speaking, the first assumption in (1.4) may be interpreted as that for a zero death rate, the population is (locally) increasing. Letting \( \lambda_0 > 0 \) be such that

\[
\int_0^{a_m} b(0, a) e^{-\lambda_0 \int_0^a \mu(z, r) dr} e^{-\sigma_1 \int_0^a \mu(z, r) dr} \, da = 1,
\]

the following result on local bifurcation holds for equations (1.1)-(1.3):

**Theorem 1.1.** Let \( D \in C^{\infty, 1}(\mathbb{R} \times J) \) with \( D(z, a) \geq d_0 > 0 \) for \( z \in \mathbb{R} \) and \( a \in J \). Further, let \( \mu, b \in C^{\infty, 1}(\mathbb{R} \times J) \) be nonnegative and suppose (1.4). Then \( (\lambda_0, 0) \) is a bifurcation point for (1.1)-(1.3), that is, there is a unique local branch of nontrivial nonnegative solutions

\[
(\lambda, u) \quad \text{in} \quad \mathbb{R}^+ \times (C(J, C(\Omega)) \cap C^1(J, C(\bar{\Omega})) \cap C^2(J, C^2(\Omega)))
\]

emanating from the critical point \( (\lambda_0, 0) \), where \( J := J \setminus \{0\} \). In addition, if \( \delta = 0 \) and

\[
b(z, a) \leq b(0, a), \quad \mu(z, a) \geq \mu(0, a), \quad z \geq 0, \quad a \in J, \quad (1.5)
\]

then bifurcation is subcritical, i.e. \( \lambda \leq \lambda_0 \) for any nonnegative solution \( (\lambda, u) \).

Assumption (1.5) is a common modeling assumption stating that effects of population densities do not increase fertility nor decrease mortality. The result thus shows that lowering the intensity of mortality below a critical value leads to other equilibrium solutions than the trivial one. We also refer to Section 3 for an example where subcritical bifurcation occurs when \( \delta = 1 \) and (1.5) holds.

We shall emphasize that Theorem 1.1 is merely a consequence of the considerably more general Theorem 2.5 that includes general nonlinear elliptic diffusion operators not necessarily in divergence form (and also less regular data). The proof of Theorem 1.1 will be given as an application of Theorem 2.5 in Section 8.

To cover a great variety of applications we thus shall consider (1.1)-(1.3) as an abstract equation of the form

\[
\partial_a u + A(u, a) u = -\lambda h(a) u + g(\lambda, u, a) u, \quad a \in J, \quad (1.6)
\]

\[
u(0) = \int_0^{a_m} b(u, a) u(a) \, da, \quad (1.7)
\]
in an ordered Banach space $E_0$ with positive cone $E_0^+$ for the unknown function $u : J \rightarrow E_0^+$. Here, $A(u, a)$ defines for fixed $(u, a) \in E_0 \times J$ a bounded linear operator from a subspace $E_1$ of $E_0$ into $E_0$. Problem (1.1)-(1.3) then fits into this abstract framework by choosing

$$E_0 := L_q(\Omega), \quad E_1 := \{ v \in W_0^2(\Omega) : \delta v + (1 - \delta)\partial_v v = 0 \text{ on } \partial\Omega\}$$

for some $q \in (1, \infty)$ (where boundary values are interpreted in the sense of traces) and letting

$$A(u, a)w := -\text{div}_x (D(U, a)\nabla_x w), \quad h(a) := \mu(0, a), \quad \text{and } g(\lambda, u, a) := \lambda(\mu(U, a) - \mu(0, a)).$$

In Section 2 we consider the abstract equations (1.6), (1.7) and prove under suitable assumptions in Theorem 2.5 a local bifurcation result. In Section 3 we give applications of Theorem 2.5 and prove in particular Theorem 1.1. Finally, the appended Section 4 contains a result on the differentiability of superposition in the literature in this form.

2. The abstract bifurcation result

Studying the nonlinear problem (1.6), (1.7) demands an investigation of its linearization around $u = 0$. We first state the precise assumptions required.

2.1. Assumptions. Given Banach spaces $E$ and $F$ we let $\mathcal{L}(E, F)$ denote the space of all bounded and linear operators from $E$ into $F$, and $\mathcal{L}(E) := \mathcal{L}(E, E)$, we write $\mathcal{L}is(E, F)$ for the subspace of $\mathcal{L}(E, F)$ consisting of all topological isomorphisms and $\mathcal{K}(E, F)$ for the subspace of compact operators.

For the remainder of this section let $J := (0, a_m)$ with $a_m \in (0, \infty]$ and note that $J$ may be bounded or unbounded. Moreover, we fix an ordered Banach space $E_0$ with positive cone $E_0^+$ and a dense subspace $E_1$ thereof which is also supposed to be compactly embedded in $E_0$. This latter property we express by writing $E_1 \hookrightarrow E_0$. Given $\theta \in [0, 1]$ and an admissible interpolation functor $(\cdot, \cdot)_{\theta}$ we equip the interpolation space $E_{\theta} := (E_0, E_1)_{\theta}$ with the order induced by the positive cone $E^+_{\theta} := E_0 \cap E^+_1$. Note that $E_{\theta} \hookrightarrow E_0$ for $0 < \theta < \theta' \leq 1$ according to [5] Thm.2.11.1. In particular, we fix $p \in (1, \infty)$ and set $E_{\varsigma} := (E_0, E_1)_{1-1/p, p}^\varsigma$; that is, $E_{\varsigma}$ is the real interpolation space between $E_0$ and $E_1$ of exponent $\varsigma := 1 - 1/p$. We then assume that

$$\text{int}(E_{\varsigma}^+) \neq \emptyset,$$

where $\text{int}(E_{\varsigma}^+)$ denotes the topological interior of the cone $E_{\varsigma}^+$. We set

$$E_0 := L_p(J, E_0) \quad \text{and} \quad E_1 := L_p(J, E_1) \cap W_0^{1, p}(J, E_0)$$

and recall that $E_1 \hookrightarrow BUC(J, E_1)$ (see [5]). Thus, the trace operator $\gamma_0 u := u(0)$ for $u \in E_1$ is a well-defined operator $\gamma_0 \in \mathcal{L}(E_1, E_\varsigma)$. We also set $E_{\varsigma}^+ := E_1 \cap L_p^+(J, E_0)$. Suppose that

$$F$$

is a Banach space ordered by a positive cone $F^+$ with $F \cdot E_1 \hookrightarrow E_{\varsigma}$ and $F^+ \cdot E_0^+ \hookrightarrow E_{\varsigma}^+$,

$$\text{(2.2)}$$

where e.g. $F \cdot E_1 \hookrightarrow E_{\varsigma}$ means a continuous bilinear mapping (i.e. a multiplication) $F \times E_1 \rightarrow E_{\varsigma}$, $(f, e) \mapsto f \cdot e$. Let $\Sigma$ be a fixed ball in $E_1$ centered at 0 of some positive radius and assume that

$$g \in C^1(\mathbb{R}^+ \times \Sigma, L_\infty(J, F)) \text{ with } g(\lambda, 0) \equiv 0 \text{ for } \lambda \in \mathbb{R}^+$$

$$\text{(2.3)}$$

and

$$h \in L_+^1(J, \mathbb{R}) \cap L_\infty(J, \mathbb{R}) \text{ with } h > 0 \text{ near } a = 0.$$ 

$$\text{(2.4)}$$

Observe that (2.3) guarantees that we may interpret $g(\lambda, u)$ as an element of $L_\infty(J, \mathcal{L}(E_1, E_0))$ for $(\lambda, u)$ in $\mathbb{R}^+ \times \Sigma$ fixed. Suppose then that

$$A \in C^1(\Sigma, L_\infty(J, \mathcal{L}(E_1, E_0)))$$

$$\text{(2.5)}$$
is such that
\[ A(u) + \lambda h - g(\lambda, u) \in L_\infty(J, \mathcal{L}(E_1, E_0)) \]
generates a positive parabolic evolution operator \( \Pi(\lambda, u)(a, \sigma), 0 \leq \sigma \leq a < a_m \), on \( E_0 \) with regularity subspace \( E_1 \) for each \( (\lambda, u) \in \mathbb{R}^+ \times \Sigma \).

We refer to [5] for a definition and properties of parabolic evolution operators. Note that, due to (2.3) and (2.4), the parabolic evolution operator \( \Pi_0 := \Pi(0,0) \) is simply generated by \( A(0) \) and
\[ \Pi(\lambda, 0)(a, \sigma) = e^{-\lambda \int_0^a b(r) \, dr} \Pi_0(a, \sigma), \quad 0 \leq \sigma \leq a < a_m. \] (2.7)

We further suppose that
\[ \text{there are } \gamma \in \mathbb{R}, \rho, \omega > 0, \kappa \geq 1 \text{ such that } \gamma + A(0) \in C^0(J, \mathcal{H}(E_1, E_0; \kappa, \omega)) \] (2.8)
and that
\[ A(0) + \lambda h \in L_\infty(J, \mathcal{L}(E_1, E_0)) \text{ possesses maximal } L_p\text{-regularity on } J, \]
that is, \( (\partial_a + A(0) + \lambda h, \gamma_0) \in \text{Lip}(\Omega(a, \sigma), \mathcal{L}(E_1, E_2)), \) for each \( \lambda > 0. \) (2.9)

We refer again to [5] for a definition of the space \( \mathcal{H}(E_1, E_0; \kappa, \omega) \) and details about operators having maximal \( L_p \)-regularity. We agree upon the notation \( A(u, a) := A(u)(a) \) and e.g. \( (hu)(a) := h(a)u(a) \) for \( a \in J \) and \( u \in E_1. \) We point out that, owing to (2.6), (2.9), and [5, III.Prop.1.3.1], the linear problem
\[ \partial_a u + (A(0, a) + \lambda h(a)) u = f(a), \quad a \in J, \quad u(0) = u^0 \]
adopts for each datum \((f, u^0) \in E_0 \times E_\gamma \) and \( \lambda > 0 \) a unique solution \( u \in E_1 \) given by
\[ u(a) = \Pi(\lambda, 0)(a, 0) u^0 + \int_0^a \Pi(\lambda, 0)(a, \sigma) f(\sigma) \, d\sigma, \quad a \in J, \] (2.10)
satisfying for some \( c_0 = c_0(\lambda) > 0 \)
\[ \|u\|_{E_1} \leq c_0(\|f\|_{E_0} + \|u^0\|_{E_\gamma}). \] (2.11)
Moreover, invoking [5, II.Lem.5.1.3] it follows from (2.8) that there are \( M_0 \geq 1 \) and \( \omega_0 \in \mathbb{R} \) such that for \( 0 \leq \gamma < \beta < \alpha \leq 1 \)
\[ \|\Pi_0(a, \sigma)\|_{\mathcal{L}(E_\gamma, E_\beta)} + (\alpha - \sigma)^{\alpha - \gamma} \|\Pi_0(a, \sigma)\|_{\mathcal{L}(E_\beta, E_\alpha)} \leq M_0 e^{\omega_0(a - \sigma)}, \quad 0 \leq \sigma < a < a_m. \] (2.12)

We also assume that
\[ \Pi_0(a, 0) \text{ is strongly positive for each } a \in (0, a_m), \] (2.13)
that is, \( \Pi_0(a, 0) \phi \in \text{int}(E_\gamma^+) \) for \( \phi \in E_\gamma^+ \setminus \{0\} \) and \( a \in (0, a_m), \) and that
\[ \Pi_0(a, 0) \Pi_0(\sigma, 0) = \Pi_0(\sigma, 0) \Pi_0(a, 0), \quad 0 \leq a, \sigma < a_m. \] (2.14)
The latter condition means that the operators \( \{A(0, a) ; a \in J\} \) commute with each other. Finally, we assume that
\[ b \in C^1(\Sigma, L_p^+(J, F)) \text{ with } 0 \neq b_0 := b(0, \cdot) \in L_p^+(J, \mathbb{R}) \]
and
\[ \int_0^{a_m} b_0(a) e^{\omega_0 a} \, da < \infty, \] (2.15)
where \( p' \) is the dual exponent of \( p, \) i.e. \( 1/p + 1/p' = 1. \) The last condition in (2.15) is obviously superfluous if \( a_m < \infty \) or \( \omega_0 < 0. \)

For the remainder of this section we assume that conditions (2.1)-(2.6), (2.8), (2.9), (2.13)-(2.15) hold and refer to Section [3] for examples where these conditions are met. In particular, they hold in case of Theorem [1.1]
2.2. The linear problem. We begin by investigating the linearization of (1.6), (1.7) around \( u = 0 \), that is, by investigating the problem
\[
\partial_t u + \left( A(0, a) + \lambda h(a) \right) u = 0, \quad a \in J,
\]
\[
u(0) = \int_0^{a_m} b_0(a) u(a) \, da.
\]

It readily follows from the previous observations that any solution \((\lambda, u) \in \mathbb{R}^+ \times \mathbb{E}_1\) of (2.16), (2.17) is of the form
\[
u(a) = e^{-\lambda \int_0^a h(r) \, dr} \Pi_0(a, 0) u(0), \quad a \in J, \quad \nu(0) = Q_\lambda \nu(0),
\]
where the operator \(Q_\lambda\) is given by
\[
Q_\lambda := \int_0^{a_m} b_0(a) e^{-\lambda \int_0^a h(r) \, dr} \Pi_0(a, 0) \, da
\]
and enjoys the following properties:

**Lemma 2.2.** For any \( \lambda \geq 0, Q_\lambda \in \mathcal{K}(E_\lambda) \) is strongly positive. Hence, the spectral radius \( r(Q_\lambda) > 0 \) of \( Q_\lambda \) is a simple eigenvalue of \( Q_\lambda \) and of its dual operator \( Q_\lambda^* \) with eigenvector \( B_\lambda \in \text{int}(E_\lambda^+) \) and strictly positive eigenfunctional \( B_\lambda^* \in E_\lambda^* \), respectively. Moreover, \( r(Q_\lambda) \) is the only eigenvalue with a positive eigenvector.

**Proof.** Let \( \theta \in (0, 1/p) \) and set \( \vartheta := \theta + \varsigma > \varsigma \). Then we derive from (2.12) and (2.15) that \( Q_\lambda \in \mathcal{L}(E_\varsigma, E_\vartheta) \) for \( \lambda \geq 0 \) and thus, since \( E_\vartheta \hookrightarrow E_\varsigma \) by [5] 1. Thm. 1.11, and we have \( Q_\lambda \in \mathcal{K}(E_\varsigma), \lambda \geq 0 \). Hence the assertion follows from the Krein-Rutman theorem (e.g., see [12] Thm. 12.3) and assumption (2.1) provided we can show that \( Q_\lambda \in \mathcal{K}(E_\varsigma) \) is strongly positive. To fill this gap let \( f' \) be any nontrivial element of the dual space \( E_\varsigma^* \) of \( E_\varsigma \) with \( \langle f', \phi \rangle_{E_\varsigma} \geq 0 \) for \( \phi \in E_\varsigma^+ \). Let \( \varphi \in E_\varsigma^+ \setminus \{0\} \). Then it follows from [7] Prop. A.2.7, Prop. A.2.10 and (2.1) that \( \langle f', \Pi_0(a, 0) \varphi \rangle_{E_\varsigma} > 0 \) for \( a \in (0, a_m) \) since \( \Pi_0(a, 0) \varphi \in \text{int}(E_\varsigma^+) \) by (2.13), and thus
\[
\langle f', Q_\lambda \varphi \rangle_{E_\varsigma} = \int_0^{a_m} b_0(a) e^{-\lambda \int_0^a h(r) \, dr} \langle f', \Pi_0(a, 0) \varphi \rangle_{E_\varsigma} \, da > 0
\]
owing to (2.15). Hence \( Q_\lambda \varphi \) is an interior point of \( E_\varsigma^+ \) again due to [7] Prop. A.2.7, Prop. A.2.10 and assumption (2.1). This yields the strong positivity of \( Q_\lambda \).

We assume in the sequel that
\[
r(Q_0) > 1.
\]

Observe that (2.18) implies that \( u(0) \) is (if nonzero) an eigenvector of \( Q_\lambda \) to the eigenvalue 1. If \( u \) is nonnegative, i.e., \( u \in E_\varsigma^+ \), then necessarily \( u(0) \in E_\varsigma^+ \) and so \( r(Q_\lambda) = 1 \) by the previous lemma. The next lemma shows that \( r(Q_\lambda) \) is strictly decreasing in \( \lambda \). Hence, if (2.19) does not hold, there is no admissible (i.e., positive) value of \( \lambda \) for which the linearized problem (2.16), (2.17) admits a nonnegative nontrivial solution. The interpretation of the operator \( Q_\lambda \) is that it contains information about the spatial distribution of the expected number of newborns that a population produces when the birth and death processes are described by \( b(0, \cdot) \) and \( \lambda h = \lambda \mu(0, \cdot) \), respectively, and spatial movement is governed by \( A(0, \cdot) \). Hence, at equilibrium these processes yield exact replacement. Roughly speaking, assumption (2.19) may be interpreted as that the population subject to birth processes and spatial dispersal increases locally if \( \lambda = 0 \), that is, if no deaths occur.

Under assumption (2.19), the following lemma guarantees the existence of a unique value \( \lambda_0 > 0 \) with \( r(Q_{\lambda_0}) = 1 \).

The following auxiliary result uses the ideas of [14]:

**Lemma 2.2.** The mapping \( [\lambda \mapsto r(Q_\lambda)] : [0, \infty) \to (0, \infty) \) is continuous, strictly decreasing, and \( \lim_{\lambda \to \infty} r(Q_\lambda) = 0 \). In particular, there is a unique \( \lambda_0 \in (0, \infty) \) with \( r(Q_{\lambda_0}) = 1 \). 


Given $\lambda \geq 0$, let $B_\lambda \in \int(\mathcal{E}_c^+)$ and $B_\lambda' \in \mathcal{E}_c'$ be the eigenvectors and strictly positive eigenfunctionals introduced in Lemma 2.1. Then, for $\xi > \lambda \geq 0$, we deduce from (2.20) that

$$r(Q_\lambda) (B_\lambda', B_\xi)_{E_c} = \langle Q_\lambda B_\lambda', B_\xi \rangle_{E_c} = \langle B_\lambda', Q_\lambda B_\xi \rangle_{E_c} = r(Q_\xi) (B_\lambda', B_\xi)_{E_c},$$

whence $r(Q_\lambda) > r(Q_\xi) \Rightarrow (\lambda \rightarrow r(Q_\lambda))$ is strictly decreasing. Next, let $\lambda > 0$ (the case $\lambda = 0$ is analogous) and consider a sequence $(\lambda_j)$ such that $0 \leq \lambda_j \rightarrow \lambda$. Given $\varepsilon > 0$ sufficiently small we may assume that $0 \leq \lambda - \varepsilon < \lambda_j < \lambda + \varepsilon$ for all $j \in \mathbb{N}$. Note then that (2.4) implies

$$Q_{\lambda-\varepsilon} B_{\lambda} \leq e^{\varepsilon \|h\|_1} Q_{\lambda} B_{\lambda} = e^{\varepsilon \|h\|_1} r(Q_\lambda) B_{\lambda}$$

with $\|h\|_1$ denoting the $L_1$-norm of $h$. Since $B_\lambda \in \mathcal{E}_c^+$ we derive

$$r(Q_\lambda) e^{\varepsilon \|h\|_1} > r(Q_{\lambda-\varepsilon})$$

from [12 Cor.12.4] and (2.1). Conversely, we have

$$Q_{\lambda} B_{\lambda+\varepsilon} \leq e^{\varepsilon \|h\|_1} Q_{\lambda+\varepsilon} B_{\lambda+\varepsilon} = e^{\varepsilon \|h\|_1} r(Q_{\lambda+\varepsilon}) B_{\lambda+\varepsilon}$$

and thus, invoking again [12 Cor.12.4],

$$e^{\varepsilon \|h\|_1} r(Q_{\lambda+\varepsilon}) > r(Q_{\lambda}).$$

Therefore, combining (2.21), (2.22) and recalling that $r(Q_\lambda)$ is strictly decreasing in $\lambda$, we obtain

$$e^{-\varepsilon \|h\|_1} r(Q_{\lambda}) < r(Q_{\lambda+\varepsilon}) < r(Q_{\lambda-\varepsilon}) < e^{\varepsilon \|h\|_1} r(Q_{\lambda}).$$

Letting $\varepsilon \rightarrow 0$ implies $\lim_{j \rightarrow \infty} r(Q_{\lambda_j}) = r(Q_{\lambda})$, whence the continuity of the function $\lambda \rightarrow r(Q_\lambda)$. Finally, the assumption that $h > 0$ near $a = 0$ together with (2.12) and (2.15) easily entails that

$$0 < r(Q_\lambda) \leq \|Q_\lambda\|_{\mathcal{L}(\mathcal{E}_c)} \rightarrow 0, \quad \lambda \rightarrow \infty,$$

from which the assertion follows in view of (2.19). }

### 2.3. The nonlinear problem.

To investigate the nonlinear problem (1.6), (1.7) we apply the theorem of Crandall-Rabinowitz [3]. Clearly, the solutions $(\lambda, u) = (\lambda_0 + t, u)$ of (1.6), (1.7) are the zeros of the function

$$F(t, u) := \left( \frac{\partial_t u + A(u) + (\lambda_0 + t)hu - g(\lambda_0 + t, u)u}{u(0) - \int_0^{\infty} b(u, a)u(a)da} \right).$$

Assumptions (2.2), (2.3), (2.4), (2.5), and (2.15) imply that

$$F : (-\lambda_0, \infty) \times \Sigma \rightarrow \mathbb{R}_+ \times \mathcal{E}_c \quad \text{with} \quad F(t, 0) = 0, \quad t > -\lambda_0.$$

Moreover, it is easily seen that all partial derivatives $F_t$, $F_u$, and $F_{tu}$ exist and are continuous and that, since $g(\lambda, 0) \equiv 0$, the Fréchet derivatives at $(t, u) = (0, 0)$ applied to $\varphi \in \mathcal{E}_c$ are given by

$$F_u(0, 0) \varphi = \left( \frac{\partial_u \varphi + (A_0 + \lambda_0 h)\varphi}{\varphi(0) - \int_0^{\infty} b_0(a)\varphi(a)da} \right),$$

where $A_0 := A(0, \cdot)$, and

$$F_{tu}(0, 0) \varphi = \left( \frac{h \varphi}{0} \right).$$
Recall that \( B_{\lambda_0} \in \text{int}(E_+^\ast) \) with \( \ker(1 - Q_{\lambda_0}) = \text{span}\{B_{\lambda_0}\} \). Then (2.9) implies that \( \Pi_{(\lambda_0,0)}(\cdot, 0)B_{\lambda_0} \) belongs to \( E_+^\ast \). Moreover, for

\[
\ell_0(u) := \int_0^a b_0(a)u(a) \, da ,
\]

\[
(K_0 f)(a) := \int_0^a \Pi_{(\lambda_0,0)}(a, \sigma) f(\sigma) \, d\sigma , \quad f \in E_0 ,
\]

we deduce from (2.2), (2.10), (2.11), and (2.15) that

\[
\ell_0 \in L(E_1, E_1) , \quad K_0 \in L(E_0, E_1).
\] (2.25)

With these notations we can state the following result.

**Lemma 2.3.** \( L := F_0(0, 0) \in L(E_1, E_0 \times E_1) \) is a Fredholm operator of index 0. In fact,

\[
\ker(L) = \text{span}\{\Pi_{(\lambda_0,0)}(\cdot, 0)B_{\lambda_0}\} ,
\]

\[
\text{rg}(L) = \{(\varphi, \psi) \in E_0 \times E_1 : \psi + \ell_0(K_0 \varphi) \in \text{rg}(1 - Q_{\lambda_0})\}
\]

are both closed and \( \dim(\ker(L)) = \text{codim}(\text{rg}(L)) = 1 \).

**Proof.** This is a reformulation of [20, Lem.2.1] using (2.23), (2.25), (2.9), Lemma 2.1, and Lemma 2.2. □

This lemma also allows us to validate the transversality condition from [8].

**Lemma 2.4.** We have \( F_{ts}(0, 0)\Pi_{(\lambda_0,0)}(\cdot, 0)B_{\lambda_0} \notin \text{rg}(L) \).

**Proof.** According to (2.23) and Lemma 2.3 we have to check that

\[
z := \ell_0 \left( K_0 \left( h \Pi_{(\lambda_0,0)}(\cdot, 0)B_{\lambda_0} \right) \right) \notin \text{rg}(1 - Q_{\lambda_0}) .
\]

Due to assumptions (2.4), (2.15), and properties of evolution operators, we compute

\[
z = \int_0^a \ell_0(a) \int_0^a \Pi_{(\lambda_0,0)}(a, \sigma) h(\sigma) \Pi_{(\lambda_0,0)}(\sigma, 0) B_{\lambda_0} \, d\sigma \, da
\]

\[
= \int_0^a b_0(a) \int_0^a h(\sigma) \, d\sigma \Pi_{(\lambda_0,0)}(a, 0) B_{\lambda_0} \, da .
\]

Thus \( z \neq 0 \) due to (2.4), (2.13), and (2.15). Using the commuting condition (2.14) we derive on interchanging the order of integration that

\[
Q_{\lambda_0} z = \int_0^a \int_0^a b_0(s) \Pi_{(\lambda_0,0)}(s, 0) b_0(a) \left( \int_0^a h(\sigma) \, d\sigma \right) \Pi_{(\lambda_0,0)}(a, 0) B_{\lambda_0} \, ds \, da 
\]

\[
= \int_0^a b_0(a) \left( \int_0^a h(\sigma) \, d\sigma \right) \Pi_{(\lambda_0,0)}(a, 0) \int_0^a b_0(s) \Pi_{(\lambda_0,0)}(s, 0) B_{\lambda_0} \, ds \, da .
\]

Hence, simplifying the integral by recognizing \( Q_{\lambda_0} B_{\lambda_0} = B_{\lambda_0} \) in the integrand and reversing the computations we obtain \( Q_{\lambda_0} z = z \), that is, \( z \in \ker(1 - Q_{\lambda_0}) \). But then \( z \notin \text{rg}(1 - Q_{\lambda_0}) \) since \( r(Q_{\lambda_0}) = 1 \) is a simple eigenvalue of the compact operator \( Q_{\lambda_0} \). □

Occurrence of local bifurcation in (1.6), (1.7) is then a consequence of Lemma 2.3, Lemma 2.4, and [8, Thm.1.7].

**Theorem 2.5.** Suppose (2.1), (2.6), (2.8), (2.9), (2.13), (2.15), (2.19). Further let \( \lambda_0 > 0 \) with \( r(Q_{\lambda_0}) = 1 \). Then \( (\lambda_0, 0) \) is a bifurcation point for (1.6), (1.7). More precisely, there is \( \varepsilon_0 > 0 \) and a unique branch \( \{(\lambda(\varepsilon), u(\varepsilon)) : |\varepsilon| < \varepsilon_0\} \) in \( \mathbb{R}^+ \times E_1 \) emanating from \( (\lambda_0, 0) \) with \( u(\varepsilon) \neq 0 \) if \( \varepsilon \neq 0 \) of the form

\[
u(\varepsilon) = \varepsilon \left( \Pi_{(\lambda_0,0)}(\cdot, 0) B_{\lambda_0} + z(\varepsilon) \right) , \quad |\varepsilon| < \varepsilon_0 .
\] (2.26)

Both \( \lambda : (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^+ \) and \( z : (-\varepsilon_0, \varepsilon_0) \to Z \) are continuous, where \( E_1 = \ker(L) \oplus Z \) with an arbitrary complement \( Z \). Moreover, \( u(\varepsilon) \in E_1^+ \) and \( \gamma_0 u(\varepsilon) \in \text{int}(E_1^+) \) for \( \varepsilon \in (0, \varepsilon_0) \).
Proof. The existence of a nontrivial branch follows from Lemma 2.3 and Lemma 2.4 by applying [8] Thm.1.7. It remains to prove the positivity assertion for \( \varepsilon \in (0, \varepsilon_0) \). Note that from (2.6) we have, for \( \varepsilon \in (0, \varepsilon_0) \) with \( \varepsilon_0 \) sufficiently small,

\[
\frac{1}{\varepsilon} \gamma_0 u(\varepsilon) = B_{\lambda_0} + \gamma_0 \varphi(\varepsilon) \in \text{int}(E_\varepsilon^+)
\]

since \( \varphi(\varepsilon) \to 0 \) in \( E_1 \hookrightarrow \text{BUC}(J, E_\varepsilon) \) and \( B_{\lambda_0} \in \text{int}(E_\varepsilon^+) \). On the one hand, \( u(\varepsilon) = \Pi_{(\lambda(\varepsilon), u(\varepsilon))}(\cdot, 0) \gamma_0 u(\varepsilon) \in E_1^+ \), \( \varepsilon \in (0, \varepsilon_0) \).

On the other hand, \( \frac{1}{\varepsilon} \gamma_0 u(\varepsilon) \) and thus also \( \gamma_0 u(\varepsilon) \) are quasi-interior points of \( E_\varepsilon^+ \), that is, \( (f, \gamma_0 u(\varepsilon))_{E_\varepsilon} > 0 \) for each \( f \in E_\varepsilon^+ \setminus \{0\} \) with \( f \geq 0 \). So (2.1) and [7, Prop.A.2.10] imply that \( \gamma_0 u(\varepsilon) \) is an interior point of \( E_\varepsilon^+ \).

Remark 2.6. We proved a local bifurcation result for (1.6), (1.7) under the assumption that \( h \geq 0 \). However, the statement of Theorem 2.5 still holds true if \( h \leq 0 \). The only modification consists of replacing \( h \) by \( -h \) in assumption (2.4) so that the spectral radius \( r(Q_\lambda) \) is strictly increasing in \( \lambda \) (see Lemma 2.2) and one thus has, in addition, to replace (2.19) by the assumption that \( r(Q_0) < 1 \).

Moreover, if \( h < 0 \) one can even prove a global bifurcation result using the Rabinowitz alternative [16] provided that the nonlinearities in the operator \( A(u, a) \) are of “lower order”, that is, if \( A(u, a) \) is a sum of operators \( A_0(a) + A_\varepsilon(u, a) \), where \( A_0(a) \in \mathcal{L}(E_1, E_0) \) and \( A_\varepsilon(u, a) \in \mathcal{L}(E_0, E_0) \) with \( \varepsilon \in [0, 1) \). The approach is similar to [21]. We also refer to [14] where the case \( h = -1 \) is considered with linear diffusion.

3. Examples

Let \( \Omega \subset \mathbb{R}^N, \) \( N \geq 1 \), be a bounded and smooth domain lying locally on one side of \( \partial \Omega \). Let the boundary \( \partial\Omega \) be the distinct union of two sets \( \Gamma_0 \) and \( \Gamma_1 \), both of which are open and closed in \( \partial\Omega \). Let the maximal age be finite, i.e. let \( a_m \in (0, \infty) \) and set \( J := [0, a_m] \).

3.1. A general example. Consider a second order differential operator of the form

\[
A(U(x), a)w := -\text{div}_x\left(D(U(x), a)\nabla x w\right) + d(U(x), a) \cdot \nabla x w, \tag{3.1}
\]

where, for some \( \rho > 0 \),

\[
D \in C^{5-\rho}(\mathbb{R} \times J) \quad \text{with} \quad D(z, a) \geq d_0 > 0, \quad z \in \mathbb{R}, \quad a \in J, \tag{3.2}
\]

and

\[
d \in C^{4-\rho}(\mathbb{R} \times J, \mathbb{R}^N) \quad \text{with} \quad d(0, \cdot) \equiv 0. \tag{3.3}
\]

For simplicity we refrain from an explicit dependence of \( A \) on \( x \in \Omega \). Let

\[
\nu_0 \in C^1(\Gamma_1), \quad \nu_0(x) \geq 0, \quad x \in \Gamma_1, \tag{3.4}
\]

and let \( \nu \) denote the outward unit normal to \( \Gamma_1 \). Let

\[
B(x)w := \begin{cases} w, & \text{on } \Gamma_0, \\ \frac{\partial}{\partial \nu} w + \nu_0(x) w, & \text{on } \Gamma_1. \end{cases}
\]

Fix \( p, q \in (1, \infty) \) with

\[
\frac{2}{p} + \frac{N}{q} < 1, \tag{3.5}
\]

and let \( E_0 := L_q := L_q(\Omega) \) be ordered by its positive cone of functions that are nonnegative almost everywhere. Observe that

\[
E_1 := W^{2,q}_{q,B} := W^{2,q}_{q,B}(\Omega) := \{ u \in W^{2,q}_{q}; Bu = 0 \} \hookrightarrow L_q = E_0,
\]
where $W^2_0(\Omega)$ is the usual Sobolev space of order 2 over $L_q(\Omega)$. Also note that, up to equivalent norms, the real interpolation spaces between $E_0$ and $E_1$ are subspaces of the Besov spaces $B^{2\xi}_{q,p} := B^{2\xi}_{q,p}(\Omega)$, that is,

$$E_\xi := (L_q, W^2_{q,B})_{\xi,p} \doteq B^{2\xi}_{q,p} := \left\{ \begin{array}{ll}
 B^{2\xi}_{q,p}, & 0 < 2\xi < 1/q , \\
 \{ w \in B^{2\xi}_{q,p} : u|_{\Gamma_0} = 0 \}, & 1/q < 2\xi < 1/1 + 1/q , \\
 \{ w \in B^{2\xi}_{q,p} : Bu = 0 \}, & 1 + 1/q < 2\xi < 2 , 
\end{array} \right.$$  

(see e.g. [17]). In particular, due to (3.3) we have $E_\xi \doteq B^{2-2/\xi}_{q,p} \hookrightarrow C^1(\Omega)$ for $\xi = 1 - 1/p$ and some $\epsilon > 0$. So int$(E_\xi^+)$ $\neq 0$ yielding (2.1). Fix any $\kappa \in (2 - 2/p, 2) \setminus \{1\}$ and set $F := B^{\kappa}_{q,p,B}$ with order induced by the cone of $L_q$. Then pointwise multiplication

$$B^{\kappa}_{q,p,B} \cdot W^1_{q,B} \hookrightarrow B^{2(1-1/p)}_{q,p,B} \doteq E_\xi$$

is continuous according to (3.5) and [4, Thm.4.1]. Thus (2.2) holds. Let

$E_1 := L_p(J, W^2_{q,B}) \cap W^1_p(J, L_q)$ and $E_0 := L_p(J, L_q)$

and note that

$$U := \int_0^m u(a) da \in E_1 = W^2_{q,B} , \quad u \in E_1 .$$

Suppose that

$$\mu \in C^{4-r}(\mathbb{R} \times J) , \quad \mu \geq 0 , \quad \mu(0, a) > 0 \text{ for } a \text{ near } 0 . \quad (3.6)$$

Set $h(a) := \mu(0, a)$ and $q(\lambda, u)(a) := \lambda(\mu(U(a) - h(a)))$ for $\lambda \in \mathbb{R}$, $u \in E_1$, and $a \in J$. Then (3.6) and Proposition 4.1 from the appendix ensure (2.3) and (2.4). Further suppose that

$$\bar{b} \in C^{4-0}(\mathbb{R} \times J) , \quad \bar{b} \geq 0 , \quad \bar{b}(0, \cdot) \neq 0 . \quad (3.7)$$

and set $b(u, a) := \bar{b}(U(a), a)$ for $u \in E_1, a \in J$. Then (3.7) and Proposition 4.1 ensure (2.15). Define

$$A(u, a) w := A(U(a), w) , \quad w \in E_1 , \quad u \in E_1 .$$

Proposition 4.1 (2.2), and (3.3) entail that the superposition operators induced by $D, \partial_1 D$, and $d$ (again labeled $D, \partial_1 D$, and $d$) satisfy $D, \partial_1 D, d \in C^1(W^2_{q,B}, L_\infty(J, C^{1+r}(\Omega)))$. This yields

$$A \in C^1(\{W^2_{q,B}, L_\infty(J, L(W^2_{q,B}, L_q))\}) ,$$

whence (2.5). If $u \in E_1$ and $\lambda \geq 0$ are fixed, then $A(u, \cdot) + \lambda \mu(U, \cdot) \in C^0(J, H(W^2_{q,B}, L_q))$ from which we conclude (2.6) and (2.8) due to [5, I.Corr.1.3.2, II.Corr.4.4.2] and the compactness of $J$. Noticing that $A(0, a) = -D(0, a) \Delta_x$ by (3.3) it follows from [11, Sect.7, Thm.11.1] that for $a \in J$ and $\lambda > 0$ fixed, $-A(0, a) - \lambda h(a)$ is resolvent positive, generates a contraction semigroup of negative type on each $L_r(\Omega)$, $r \in (1, \infty)$, and is self-adjoint on $L_2(\Omega)$. Hence $A(0, \cdot) + \lambda h$ possesses maximal $L_p$-regularity on $J$ according to [5, III.Ex.4.7.3, III.Thm.4.10.8], whence (2.9). Moreover, since

$$\Pi_0(a, \sigma) = e^{\int_0^\sigma D(0,r)dr \Delta_x} , \quad 0 \leq \sigma \leq a \leq a_m ,$$

where $\{e^{a\Delta_x} : a \geq 0\}$ is the semigroup associated with $(-\Delta_x, \mathcal{B})$, condition (2.13) follows from the maximum principle and (2.14) is obvious. Finally, let $\sigma$ be the first eigenvalue of $(-\Delta_x, \mathcal{B})$ and let $\varphi_1 \in W^2_{q,B}$ be a corresponding positive eigenfunction (e.g. see [11]). Then

$$e^{\int_0^a D(0,r)dr \Delta_x} \varphi_1 = e^{-\sigma} e^{\int_0^a \mu(0,r)dr} e^{\int_0^a D(0,r)dr \Delta_x} \varphi_1 \quad 0 \leq a \leq a_m ,$$

and thus

$$Q_\lambda \varphi_1 = \int_0^a b(0,a) e^{-\lambda \int_0^a \mu(0,r)dr} e^{\int_0^a D(0,r)dr \Delta_x} \varphi_1 da = k(\lambda) \varphi_1 ,$$

where

$$k(\lambda) := \int_0^a b(0,a) e^{-\lambda \int_0^a \mu(0,r)dr} e^{-\sigma} e^{\int_0^a D(0,r)dr \Delta_x} da .$$
Since the spectral radius \(r(Q_\lambda)\) is the only eigenvalue with positive eigenfunction for the strongly positive compact operator \(Q_\lambda \in \mathcal{K}(B_{q,p}^{\alpha-2p'})\) by the Krein-Rutman theorem, we have \(r(Q_\lambda) = k(\lambda)\). To satisfy (2.19) we assume that
\[
k(0) = \int_0^{a_m} b(0, a) e^{-\sigma_1 J_a^m D(0,r) dr} da > 1
\]and then choose \(\lambda_0 > 0\) such that \(k(\lambda_0) = 1\).

Summarizing what we have just shown and referring to Theorem 2.5, we can state:

**Proposition 3.1.** Suppose (3.1)–(3.8). Then \((\lambda_0, 0)\) with \(k(\lambda_0) = 1\) is a bifurcation point for the problem
\[
\begin{align*}
\partial_t u + A(U(x), a)u + \lambda_0 U(x), a)u &= 0, & a \in J, x \in \Omega, \\
u(0, x) &= \int_0^{a_m} b(U(x), a)u(a) da, & x \in \Omega, \\
B(u(a, x) &= 0, & a > 0, x \in \partial\Omega, \\
U(x) &= \int_0^{a_m} u(a, x) da, & x \in \Omega.
\end{align*}
\]

There are \(\varepsilon_0 > 0\) and a unique branch \(\{(\lambda(\varepsilon), u(\varepsilon)); |\varepsilon| < \varepsilon_0\}\) of solutions emanating from \((\lambda_0, 0)\) with \(u(\varepsilon) \in L_p(J, W^{2,2}_{q,0}) \cap W^{1,1}_p(J, L_q), \quad u(\varepsilon) \neq 0\) if \(\varepsilon \neq 0\), of the form
\[
u(\varepsilon) = \varepsilon \left( e^{-\lambda_0} \int_0^{a_m} \mu(0, r) dr e^{-\sigma_1 J_a^m D(0,r) dr} \varphi_1 + z(\varepsilon) \right), \quad |\varepsilon| < \varepsilon_0.
\]
Both \(\lambda: (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^+\) and \(z: (-\varepsilon_0, \varepsilon_0) \to L_p(J, W^{2,2}_{q,0}) \cap W^{1,1}_p(J, L_q)\) are continuous. Moreover, \(u(\varepsilon)(a, x) \geq 0\) for \(\varepsilon \in (0, \varepsilon_0)\) and \((a, x) \in J \times \Omega\).

**Remark 3.2.** A local dependence of the data on \(u\) with respect to age is also possible. For example, one may apply Theorem 2.5 for diffusion terms of the form \(\text{div}_x(D(u(a, x)) \nabla u)\) as well (see [20, Ex.3.1]) for details. Moreover, the functions \(D\) and \(d\) in (3.1) may also depend on \(x \in \Omega\) provided the dependence is sufficiently smooth. In this case one needs Remark 4.2(d) to verify (2.5).

3.2. **Proof of Theorem 1.1** Applying Proposition 3.1 to the problem
\[
\begin{align*}
\partial_t u &= \text{div}_x(D(U(x), a) \nabla_x u) - \lambda \mu(U(x), a) u, & a \in J, x \in \Omega, \\
u(0, x) &= \int_0^{a_m} b(U(x), a)u(a) da, & x \in \Omega, \\
\delta u(a, x) + (1 - \delta) \partial_x u(a, x) &= 0, & a \in J, x \in \partial\Omega.
\end{align*}
\]
considered in the introduction, we obtain under the assumptions of Theorem 1.1 a branch of nontrivial solutions
\[
\{(\lambda(\varepsilon), u(\varepsilon)); |\varepsilon| < \varepsilon_0\} \quad \text{in} \quad \mathbb{R}^+ \times (C(J, C(\Omega)) \cap C^1(J, C(\Omega)) \cap C(J, C^2(\Omega))),
\]
where the regularity of \(u(\varepsilon)\) is due to standard parabolic regularity theory (e.g. see [3, Thm.9.2]), where \(J := J \setminus \{0\}\).

Let now \(\delta = 0\) in (3.12) and assume (1.5). Then \(\sigma_1 = 0\) and, for any nonnegative solution \((\lambda, u)\) to (3.10)–(3.12), we have
\[
\frac{d}{da} \int_{\Omega} u(a, x) dx = -\lambda \int_{\Omega} \mu(U(x), a)u(a, x) dx \leq -\lambda \mu(0, a) \int_{\Omega} u(a, x) dx,
\]
whence
\[
z(a) \leq z(0)e^{-\lambda J_a^m \mu(0,r) dr} \quad \text{for} \quad z(a) := \int_{\Omega} u(a, x) dx.
\]
Moreover, by (3.11) and (1.5),
\[
z(0) = \int_\Omega \int_0^{a_m} b(U(x), a)u(a, x) \, da \, dx \leq \int_0^{a_m} b(0, a)z(a) \, da \leq z(0) \int_0^{a_m} b(0, a)e^{-\lambda f_0^a \mu(0,r)dr} \, da ,
\]
and thus
\[
k(\lambda) = \int_0^{a_m} b(0, a)e^{-\lambda f_0^a \mu(0,r)dr} \, da \geq 1 .
\]
Since \(k(\lambda)\) is strictly decreasing in \(\lambda\) and \(k(\lambda_0) = 1\), we conclude \(\lambda \leq \lambda_0\), that is, subcritical bifurcation occurs in (3.10)-(3.12) in this case. This proves Theorem 1.1.

### 3.3. Subcritical bifurcation for Dirichlet boundary conditions.

We consider an example involving Dirichlet boundary conditions. More precisely, let us consider
\[
\partial_a u = D(U) \Delta_x u - \lambda \mu(U(x), a)u ,
\]
\[a \in J , \ x \in \Omega; \quad (3.13)\]
\[u(0, x) = \int_0^{a_m} b(U(x), a)u(a) \, da ,\]
\[x \in \Omega; \quad (3.14)\]
\[u(a, x) = 0 ,\]
\[a \in J , \ x \in \partial\Omega; \quad (3.15)\]

Note that the diffusion coefficients are independent of \(a \in J\). Suppose that \(D \in C^1(W_{q,B}^2(\Omega), (0, \infty))\). If (1.5) still holds, then we easily derive for any positive solution \((\lambda, u)\) of (3.13)-(3.15) analogously as above that
\[
z'(a) + \sigma_1 D(U)z(a) \leq -\lambda \mu(0, a)z(a) \quad \text{for} \quad z(a) := \int_\Omega \varphi_1(x)u(a, x) \, dx ,
\]
where \(\varphi_1\) is a positive eigenfunction to the principal eigenvalue \(\sigma_1 > 0\) of \(-\Delta_x\) subject to Dirichlet boundary conditions on \(\partial\Omega\). Thus, if in addition \(D(U) \geq D(0)\) for \(0 \leq U \in W_{q,B}^2(\Omega)\), then we deduce again that
\[
z(0) \leq \int_0^{a_m} b(0, a)e^{-\sigma_1 D(U)\mu(0,r)dr} \, da z(0) = k(\lambda) z(0) .
\]

Hence \(\lambda \leq \lambda_0\) and thus subcritical bifurcation occurs also in this case.

Observe that one may replace the diffusion term \(D(U) \Delta_x u\) in (3.13) by \(\text{div}_x(D(U(x)) \nabla x u)\) depending locally with respect to \(x\) on \(U\) and derive the same conclusion of subcritical bifurcation provided that \(\sigma_1(U) \geq \sigma_1(0)\) for \(0 \leq U \in W_{q,B}^2(\Omega)\), where \(\sigma_1(U)\) is the first eigenvalue of \(w \mapsto -\text{div}_x(D(U(x)) \nabla x w)\) and using a corresponding positive eigenfunction \(\varphi_1 = \varphi_1(U)\) in the definition of \(z\).

### 3.4. An example with Holling-Tanner type nonlinearities.

As noted in Remark 2.6 one can also allow for \(h < 0\) in (1.6). We conclude with an example, which has been investigated in [14] in the case of linear diffusion:
\[
\partial_a u + A(U(x), a)u + \mu(U(x), a)u = \lambda u - \frac{u}{1 + u} ,
\]
\[a \in J , \ x \in \Omega; \quad (3.16)\]
\[u(0, x) = \int_0^{a_m} b(U(x), a)u(a) \, da ,\]
\[x \in \Omega; \quad (3.17)\]
\[Bu(a, x) = 0 ,
\]
\[a > 0 , \ x \in \partial\Omega; \quad (3.18)\]
\[U(x) = \int_0^{a_m} u(a, x) \, da ,\]
\[x \in \Omega; \quad (3.19)\]

with \(A\) and \(B\) as in Subsection 3.1. We impose the same assumptions (3.1)-(3.7) as there, where we take \(q = p\) for simplicity. The strict positivity of \(\mu(0, a)\) in (3.6) is not needed here. We also use the same spaces as in Subsection 3.1.

\[
E_1 := W_{p,B}^2 := W_{p,B}^2(\Omega) : = \{ u \in W_{p}^2; Bu = 0 \} \hookrightarrow L_p = : E_0 , \quad E_\infty = W_{p,B}^{2-2/p} ,
\]
\[
E_1 := L_p(J,W_{p,B}^2) \cap W_{p}^1(J, L_p) \quad \text{and} \quad E_0 := L_p(J, L_p) .
\]
Noticing that the “Holling-Tanner-type” nonlinearity can be written in the form
\[ \frac{u}{1+u} = u - \frac{u^2}{1+u}, \]
problem \((3.16)-(3.19)\) fits in the abstract form \((1.6), (1.7)\) by setting
\[ A(u, a) := A(U, a) + \mu(U, a) \equiv 1 \in L(W^2_{p,B}, L_p), \quad h(a) := -1, \quad g(u) := \frac{u}{1+u}. \]
Let \(V := (-1, 1)\). Then \((3.5)\) (with \(p = q\)), Remark \((2.6)\) from the appendix, and \([6, VII.\text{Thm.6.4}]\) ensure that the superposition operator of \(g\) (still denoted by \(g\)) belongs to \(C^1(C(J, V_p), C(J, W^{2-2/p}_{p,B}))\), where we define \(V_p := W^{2-2/p}_{p,B} \cap C(\bar{\Omega}, V)\). Also note that \(g(0) = 0\). Recalling that \(E_1 \hookrightarrow C(J, E_r)\), this implies \(g \in C^1(\Sigma, C(J, F))\) for \(\Sigma := \mathbb{B}_{E_1}(0, R)\) with \(R > 0\) sufficiently small and \(F := E_r\). To satisfy \(r(Q_0) < 1\) (see Remark \((2.6)\)), we assume that
\[ k(0) < 1 \quad (3.20) \]
for
\[ k(\lambda) := \int_0^{a_m} b(0, a) e^{(\lambda+1)a} e^{-\int_0^a \mu(0,r)dr} e^{-\sigma_1 \int_0^a D(0,r)dr} da, \]
where \(\sigma_1\) is the first eigenvalue of \((-\Delta_x, B)\). Let \(\varphi_k \in W^2_{p,B}\) be a corresponding positive eigenfunction. As in Subsection \((3.1)\) we have \(Q_\lambda \varphi = k(\lambda) \varphi_1\) with
\[ Q_\lambda := \int_0^{a_m} b(0, a) e^{(\lambda+1)a} e^{-\int_0^a \mu(0,r)dr} e^{\int_0^a D(0,r)dr \Delta_x} da, \]
whence \(r(Q_\lambda) = k(\lambda)\), in particular, \(r(Q_0) < 1\) in view of \((3.20)\). Thus we may invoke Theorem \((2.5)\) and Remark \((2.6)\) to conclude the existence of a branch of nontrivial solutions to \((3.16)-(3.19)\) emanating from the critical point \((\lambda_0, 0)\), where \(\lambda_0 > 0\) with \(k(\lambda_0) = 1\).

This local bifurcation result generalizes the (global) one of \([14]\) in that nonlinear diffusion and nonlinear death and birth rates may be considered. However, we shall point out that in \([14]\) a death rate depending on local position is considered (what can be considered in the present situation as well but requires some additional effort).

4. Appendix

We prove a result on the differentiability of superposition operators in Sobolev-Slobodeckii spaces that is used in the previous examples but might be of interest in other applications as well. The proof is similar to \([18, \text{Lem.2.7}]\) or \([2, \text{Prop.15.4}]\), where continuity properties are derived.

To set the stage let \(\bar{\Omega}\) be an open and bounded subset of \(\mathbb{R}^k\), let \(V\) be an open neighborhood of 0 in \(\mathbb{R}^k\), and let \(I\) be a compact interval in \(\mathbb{R}\). Given a function \(f: I \times V \rightarrow \mathbb{R}\) define the superposition operator \(F\) of \(f\) by
\[ F[u](a)(x) := f(a, u(x)) \quad \text{for } u: \Omega \rightarrow V \text{ and } a \in I, \ x \in \Omega. \]
We write \(f \in C^{0,k-1}(I \times V)\) provided that \(\partial_x^{k-1} f \in C(I \times V)\) is Lipschitz continuous in \(x \in V\) uniformly with respect to \(a \in I\).

Recall the definition of the norm in \(W^q_\xi(\Omega, \mathbb{R}^k)\) (e.g. see \([17]\)): if \(q \in (1, \infty)\) and \(\xi \in (0, 1)\), then
\[ \|u\|_{W^q_\xi(\Omega, \mathbb{R}^k)} = \|u\|_{L^q(\Omega, \mathbb{R}^k)} + \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^q}{|x-y|^{n+\xi q}} \, d(x, y) \]
and if \(\xi \in (1, 2)\), then
\[ \|u\|_{W^q_\xi(\Omega, \mathbb{R}^k)} = \|u\|_{L^q(\Omega, \mathbb{R}^k)} + \sum_{j=1}^n \|\partial^j_x u\|_{W^{q-1}_\xi(\Omega, \mathbb{R}^k)}^q. \]
Proposition 4.1. Let $q \in (n, \infty)$, $\xi \in (n/q, 2)$, and $\eta \in (0, \xi)$. Further, let $V_\xi := W^q_\xi(\Omega, B_\infty) \cap C(\bar{\Omega}, V)$ be equipped with the $W^q_\xi$-topology. Then $F \in C^2(V_\xi, L_\infty(I, W^q_\eta(\Omega)))$ provided that $f \in C^{0,4-}(I \times V)$. The Fréchet derivative $DF[u]$ at $u \in V_\xi$ is given by

$$ (DF[u]h)(a)(x) = \partial_2 f(a, u(x))h(x), \quad a \in I, \quad x \in \Omega, \quad h \in W^q_\xi(\Omega, B_\infty) . $$

Proof. We may assume $k = 1$.

(i) First, let $\xi \in (n/q, 1)$ and note that $W^q_\xi(\Omega) \hookrightarrow C(\bar{\Omega})$. Fix $u \in V_\xi$ and choose an open neighborhood $R$ of $u(\bar{\Omega})$ in $V$ such that its closure $\bar{R}$ is compact and contained in $V$. Then, since $f \in C^{0,3-}(I \times V)$, there is $c_0(R) > 0$ with

$$ |\partial_2 f(a, r)| \leq c_0(R), \quad r \in \bar{R}, \quad a \in I,$$

$$ |\partial_2 f(a, r) - \partial_2 f(a, s)| + |\partial^2_2 f(a, r) - \partial^2_2 f(a, s)| \leq c_0(R)|r - s|, \quad r, s \in \bar{R}, \quad a \in I. $$

In the following we suppress the (fixed) variable $a \in I$ in $f$ and its derivatives for the sake of readability and we set $f' := \partial_2 f$. Let $h \in W^q_\xi(\Omega)$ with $\|h\|_{W^q_\xi(\Omega)}$ sufficiently small so that $u(\bar{\Omega}) + h(\bar{\Omega}) \subset \bar{R}$. Then

$$(F'[u]h)(a)(x) := \partial_2 f(a, u(x))h(x) = f'(u(x))h(x)$$

by convention. The mean value theorem implies for $x, y \in \Omega$:

$$ |F[u + h](a)(x) - F[u](a)(x) - (F'[u]h)(a)(x) - F[u + h](a)(y) - F[u](a)(y) - (F'[u]h)(a)(y)| $$

$$ \leq \left| \int_0^1 \left[ f'(u(x) + \tau h(x)) - f'(u(x)) \right] d\tau (h(x) - h(y)) \right| $$

$$ + \left| \int_0^1 \int_0^1 f''(u(x) + \tau s h(x)) ds \sigma h(x) ds h(y) - \int_0^1 \int_0^1 f''(u(y) + \tau s h(y)) ds \sigma h(y) ds h(y) \right| $$

$$ \leq c_0(R) \|h\|_\infty |h(x) - h(y)| + \left| \int_0^1 \int_0^1 [f''(u(x) + \tau s h(x)) - f''(u(y) + \tau s h(y)) ] ds \sigma h(x) ds h(y) \right| $$

$$ \leq c(R) \|h\|_\infty |h(x) - h(y)| + c_0(R) \|h\|_\infty^q \left( |u(x) - u(y)| + |h(x) - h(y)| \right) . $$

Therefore, we obtain by definition of the norm in the Sobolev-Slobodeckii space $W^q_\xi(\Omega)$ that

$$ \|F[u + h](a)(x) - F[u](a)(x) - (F'[u]h)(a)(x)\|_{W^q_\xi(\Omega)}^q $$

$$ \leq c(R) \|h\|_\infty^q + c(R) \|h\|_\infty^q \|h\|_{W^q_\xi(\Omega)}^q + c(R) \|h\|_{W^q_\xi(\Omega)}^q \left\{ \|u\|_{W^q_\xi(\Omega)}^q + \|h\|_{W^q_\xi(\Omega)}^q \right\} . $$

Recalling the embedding $W^q_\xi(\Omega) \hookrightarrow C(\bar{\Omega})$ we thus deduce that

$$ \|F[u + h] - F[u] - F'[u]h\|_{L_\infty(I, W^q_\xi(\Omega))} = o(\|h\|_{W^q_\xi(\Omega)}) , \quad (h \to 0) , $$

whence $F : V_\xi \to L_\infty(I, W^q_\xi(\Omega))$ is Fréchet differentiable at $u \in V_\xi$ with derivative $DF[u]h = F'[u]h$ for $h \in W^q_\xi(\Omega)$. Moreover, since pointwise multiplication $W^q_\xi(\Omega) \times W^q_\xi(\Omega) \to W^q_\xi(\Omega)$ is continuous due to [2] Thm.4.1 and $\xi > n/q$, we have for $v \in W^q_\xi(\Omega)$ with sufficiently small norm that (writing here and in the following e.g. $f''(u)$ for the superposition operator at $u$ induced by $f'$)

$$ \|DF[u + v] - DF[u]\|_{L_\infty(I, W^q_\xi(\Omega))} = \sup_{h \in W^q_\xi(\Omega)} \frac{\|f''(u + v)h - f''(u)h\|_{L_\infty(I, W^q_\xi(\Omega))}}{\|h\|_{W^q_\xi(\Omega)}} $$

$$ \leq c \|f''(u + v) - f''(u)\|_{L_\infty(I, W^q_\xi(\Omega))} \leq c(R) \|v\|_{W^q_\xi(\Omega)} . $$
where the last inequality can be shown similarly as above (or also follows from [18 Lem.2.7]). This implies $F \in C^{2,-}(V_\xi, L_\infty(I, W^\frac{\xi}{q}_q(\Omega)))$.

(ii) Now let $\xi \in (1, 2)$ and $\eta \in (1, \xi)$. Choose $\tau \in (n/q, 1)$ with $\tau > \eta - 1$. Then pointwise multiplication $W^\frac{\xi}{q}_q(\Omega) \times W^{\xi-1}_q(\Omega) \to W^{\eta-1}_q(\Omega)$ is continuous, see [4 Thm.4.1]. Therefore, taking $\xi = \tau$ in (i) and using the chain rule we obtain for $u \in V_\xi$ and $h \in W^\xi_q(\Omega)$ with $\|h\|_{W^\xi_q(\Omega)}$ sufficiently small that

$$
\|F[u + h] - F[u] - F'[u]h\|_{L_\infty(I, W^\eta_q(\Omega))} \\
\leq \|F[u + h] - F[u] - F'[u]h\|_{L_\infty(I, L_q(\Omega))} \\
+ \sum_{j=1}^n \|\left(f'(u + h) - f'(u) - f''(u)h\right) \partial_j u\|_{W^{\eta-1}_q(\Omega)} \\
+ \sum_{j=1}^n \|\left(f'(u + h) - f'(u)\right) \partial_j h\|_{W^{\eta-1}_q(\Omega)} \\
\leq o(\|h\|_{W^\xi_q(\Omega)}) + c \|f'(u + h) - f'(u) - f''(u)h\|_{W^{\eta}_q(\Omega)} \|u\|_{W^\xi_q(\Omega)} \\
+ c \|f'(u + h) - f'(u)\|_{W^{\eta}_q(\Omega)} \|h\|_{W^\xi_q(\Omega)} \\
\leq o(\|h\|_{W^\xi_q(\Omega)}) + o(\|h\|_{W^\eta_q(\Omega)}) \|u\|_{W^\xi_q(\Omega)} + c \|h\|_{W^\eta_q(\Omega)} \|h\|_{W^\xi_q(\Omega)} \\
= o(\|h\|_{W^\xi_q(\Omega)}), \quad (h \to 0)
$$

the last equality being due to the embedding $W^\xi_q(\Omega) \hookrightarrow W^\eta_q(\Omega)$. This shows that $F : V_\xi \to L_\infty(I, W^\eta_q(\Omega))$ is Fréchet differentiable at $u \in V_\xi$ with derivative $DF[u]h = F'[u]h$ for $h \in W^\xi_q(\Omega)$.

Finally, to prove the Lipschitz continuity of $DF$ choose $\xi \in (\eta, \xi)$ and note that pointwise multiplication $W^\xi_q(\Omega) \times W^\xi_q(\Omega) \to W^\eta_q(\Omega)$ is continuous due to [4 Thm.4.1]. Hence it follows for $v \in W^\xi_q(\Omega)$ with sufficiently small norm that

$$
\|DF[u + v] - DF[u]\|_{L(W^\xi_q(\Omega), L_\infty(I, W^\eta_q(\Omega)))} = \sup_{h \in W^\xi_q(\Omega)} \frac{\|f'(u + v)h - f'(u)h\|_{L_\infty(I, W^\eta_q(\Omega))}}{\|h\|_{W^\xi_q(\Omega)}} \\
\leq c \|f'(u + v) - f'(u)\|_{L_\infty(I, W^\xi_q(\Omega))} \leq c(R) \|v\|_{W^\xi_q(\Omega)},
$$

where the last inequality stems from [18 Lem.2.7]. So $F \in C^{2,-}(V_\xi, L_\infty(I, W^\xi_q(\Omega)))$. The case $\xi = 1$ is obvious. $\Box$

Remarks 4.2. (a) If $\xi \in (n/q, 1)$ and $f \in C^{0,3,-}(I \times V)$, then $F \in C^{2,-}(V_\xi, L_\infty(I, W^\xi_q(\Omega)))$.

Proof. See part (i) of the proof of Proposition [4.1] $\Box$

(b) If $\xi \in (1 + n/q, 2)$ and $f \in C^{0,4,-}(I \times V)$, then $F \in C^{2,-}(V_\xi, L_\infty(I, W^\xi_q(\Omega)))$.

Proof. This follows exactly as in part (ii) of the proof of Proposition [4.1] by observing that pointwise multiplication $W^\tau_q(\Omega) \times W^{\tau-1}_q(\Omega) \to W^{\xi-1}_q(\Omega)$ for $\tau \in (n/q, 1)$ with $\tau > \xi - 1$ is continuous according to [4 Thm.4.1]. $\Box$

(c) For simplicity we refrain form taking into account an explicit dependence of $f$ on $x \in \Omega$, that is, we do not consider functions $f : I \times \bar{\Omega} \times V \to \mathbb{R}$ in Proposition 4.1. Such a dependence can be included provided that $f$ (and its derivatives) are Hölder continuous with respect to $x$, see [2 Prop.15.4, Prop.15.6].

Acknowledgement

Part of this paper was written while visiting the University of Strathclyde in Glasgow. I would like to thank for the kind hospitality and support.
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