Orthogonal polynomials on the disk in the absence of finite moments

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Abstract

We introduce a new family of orthogonal polynomials on the disk that has emerged in the context of wave propagation in layered media. Unlike known examples, the polynomials are orthogonal with respect to a measure all of whose even moments are infinite.

1 Introduction

For each $\alpha > -1$ there is a corresponding family of disk polynomials that are orthogonal with respect to the measure $(1-x^2-y^2)^\alpha dx\,dy$ on the unit disk $\mathbb{D}$; these are sometimes referred to as generalized Zernike polynomials, named for the case $\alpha = 0$ introduced in [8]. The well-established theory of disk polynomials is detailed in [4, 6, 1]. The constraint $\alpha > -1$ stems from the requirement that the measure $(1-x^2-y^2)^\alpha dx\,dy$ have finite moments, which is necessary for meaningful evaluation of the corresponding scalar product

$$\langle p, q \rangle_\alpha = \int_\mathbb{D} p(x, y)q(x, y)(1-x^2-y^2)^\alpha dx\,dy \quad (1.1)$$

on arbitrary polynomials $p$ and $q$. Recent work on the propagation of waves in layered media [2, 3] has brought to light a family of polynomials orthogonal with respect to $(1-x^2-y^2)^{-1}dx\,dy$. Since

$$\int_\mathbb{D} x^{2m}y^{2n}(1-x^2-y^2)^{-1}dx\,dy = \infty \quad (1.2)$$

for every pair of nonnegative integers $m$ and $n$, the scalar product (1.1) is not defined for arbitrary polynomials in the case $\alpha = -1$. Nevertheless, polynomials—which we term scattering polynomials—comprise an orthogonal basis for $L^2(\mathbb{D}, dx\,dy/(1-x^2-y^2))$. The purpose of the present paper is to present the details of this result.

2 Definition and properties of scattering polynomials

Referring to the notation $z = x + iy$ for points in the unit disk $\mathbb{D}$, one has the option of working with euclidean $x, y$-coordinates, or with complex coordinates $z$ and $\bar{z}$. As far as orthogonal polynomials are concerned these are essentially equivalent, as elaborated in [7]; the present paper uses whichever coordinates are most convenient for the task at hand.

We define scattering polynomials by a Rodrigues type formula, as follows. For every $(p, q) \in \mathbb{Z}^2$ with $\min\{p, q\} \geq 1$, set

$$\varphi^{(p,q)}(z) = \frac{(-1)^p}{q(p + q - 1)!}(1-z\bar{z})^{p+q} \frac{\partial^{p+q}}{\partial z^p \partial \bar{z}^q}(1-z\bar{z})^{p+q-1} \quad (2.1)$$

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The chosen normalization simplifies the formulation of the boundary Green’s function for scattering in layered media (see [3]) and so is physically natural, although not important for present considerations.

Note that disk polynomials satisfy a Rodrigues formula similar to that of scattering polynomials, but there is a qualitative difference: it follows directly from (2.1) that \( \varphi^{(p,q)}(z) = 0 \) for every \( z \) on the unit circle \( T \), whereas all disk polynomials have constant non-zero modulus on \( T \), cf. [4]. Our main result concerns completeness of scattering polynomials, as follows.

**Theorem 1** Scattering polynomials \( \varphi^{(p,q)} \) defined by (2.1), where \( p, q \in \mathbb{Z}^2 \) and \( \min\{p,q\} \geq 1 \), comprise an orthogonal basis for \( L^2(D, dxdy/(1 - x^2 - y^2)) \).

In §2.1 and §2.2 below we show that scattering polynomials are eigenfunctions of a second order differential operator and may be expressed in terms of Jacobi polynomials; these results contribute to a proof of Theorem 1 completed in §2.3.

**2.1 Eigenfunctions of** \(- (1 - x^2 - y^2)\Delta / 4 \)**

Let \( \tilde{\Delta} \) denote the modified laplacian

\[
\tilde{\Delta} = (1 - z\bar{z}) \frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1 - x^2 - y^2}{4} \Delta,
\]

where \( \Delta \) is the usual (euclidean) laplacian. Direct computation using (2.1) shows that for all integers \( p, q \geq 1 \),

\[
- \tilde{\Delta} \varphi^{(p,q)} = pq \varphi^{(p,q)}.
\]

Letting \( \sigma_0 : \mathbb{Z}_+ \to \mathbb{Z}_+ \) denote the divisor function, there is thus a family of \( \sigma_0(k) \) eigenfunctions of \( - \tilde{\Delta} \) of the form \( \varphi^{(p,q)} \) corresponding to each positive integer eigenvalue \( k \). We show in the next section that these eigenfunctions are linearly independent.

**2.2 Representation in terms of Jacobi polynomials**

Like disk polynomials, scattering polynomials have a representation in terms of Jacobi polynomials, but again, there is a qualitative difference. The disk polynomials corresponding to parameter \( \alpha > -1 \) can be expressed in terms of Jacobi polynomials \( P_n^{(\alpha,\beta)} \) for nonnegative integer values of \( \beta \). Since there is no Jacobi polynomial corresponding to \( \alpha = -1 \), the same cannot be true for scattering polynomials. Indeed it turns out that scattering polynomials can be formulated in terms of \( P_n^{(1,\beta)} \), where \( \beta \) is a nonnegative integer, as follows.

Expanding the binomial \((1 - z\bar{z})^{p+q-1}\) in the formula (2.1), and then applying the derivative \( \partial^{p+q}/\partial z^p \partial \bar{z}^q \), yields

\[
\varphi^{(p,q)}(z) = \frac{(-1)^{q+\nu+1}}{q} (1 - z\bar{z})^m z^{m+\nu-p+1} \bar{z}^{m+\nu-q+1} \sum_{j=0}^{\nu} (-1)^j \frac{(j + \nu + m + 1)!}{j!(j + m)!(\nu - j)!} (z\bar{z})^j,
\]

where \( m = |p - q| \) and \( \nu = \min\{p, q\} - 1 \); the latter notation will be used in the remainder of this section. Switching to polar form \( z = re^{i\theta} \), it follows from (2.1) that

\[
\varphi^{(p,q)}(re^{i\theta}) = e^{i(q-p)\theta} f^{(p,q)}(r),
\]
where
\[ f^{(p,q)}(r) = (-1)^{q+\nu+1} \frac{\nu!}{q} (1 - r^2)^m \sum_{j=0}^{\nu} (-1)^j \frac{(j + \nu + m + 1)!}{j!(j + m)!((\nu - j))!} r^{2j}. \]

(2.6)

The radial functions \( f^{(p,q)} \) were first discovered in [2], as was the following connection to Jacobi polynomials, valid for \( \nu \geq 0 \):

\[ f^{(p,q)}(r) = \frac{(-1)^{q+m+\nu+1}(m + \nu + 1)}{q} (1 - r^2)^m P^{(1,m)}(2r^2 - 1). \]

(2.7)

Combined with (2.5) this yields the representation

\[ \varphi^{(p,q)}(re^{i\theta}) = \frac{(-1)^{q+\max\{p,q\}} \max\{p,q\}}{q} (1 - r^2)^{\nu} P^{(1,|p-q|)}_{\min\{p,q\}-1} (2r^2 - 1) e^{i(q-p)\theta}. \]

(2.8)

Note that the angular part of \( \varphi^{(p,q)}(re^{i\theta}) \), namely \( e^{i(q-p)\theta} \), is a pure frequency. Therefore if \( q - p \neq q' - p' \), then \( \varphi^{(p,q)} \) and \( \varphi^{(p',q')} \) are orthogonal in

\[ L^2\left( D, \frac{dx dy}{1 - x^2 - y^2} \right) = L^2\left( D, \frac{dr d\theta}{1 - r^2} \right). \]

In particular, if \( pq = p'q' \) and \( (p,q) \neq (p',q') \), then \( \varphi^{(p,q)} \) and \( \varphi^{(p',q')} \) are orthogonal, so the set of scattering polynomials corresponding to any fixed eigenvalue of \(-\Delta\) is linearly independent.

### 2.3 Completeness in \( L^2\left( D, \frac{dr d\theta}{1 - r^2} \right) \)

In general, given a measure \( \mu \) on a locally compact metric space \( X \) and a positive measurable weight function \( w : X \to \mathbb{R}_+ \),

\[ L^2(X, w d\mu) = \frac{1}{\sqrt{w}} L^2(X, d\mu), \]

(2.9)

and a sequence \( \{b_\nu\}_{\nu=0}^\infty \) is an orthogonal basis for \( L^2(X, w d\mu) \) if and only if the corresponding sequence \( \{\sqrt{w} b_\nu\}_{\nu=0}^\infty \) is an orthogonal basis for \( L^2(X, d\mu) \). In particular, setting \( d\mu = rdr d\theta/(1-r^2) \),

\[ L^2(D, d\mu) = \sqrt{1 - r^2} L^2(D, rdr d\theta). \]

(2.10)

Also, since for any nonnegative integer \( m \), \( \{P^{(1,m)}_\nu(u)\}_{\nu=0}^\infty \) is an orthogonal basis for

\[ L^2([-1,1], (1-u)(1+u)^m du), \]

it follows that the quasipolynomials

\[ Q^{(1,m)}_\nu(u) = \left( \frac{1 - u}{2} \right)^{\frac{1}{2}} \left( \frac{1 + u}{2} \right)^{\frac{m}{2}} P^{(1,m)}_\nu(u) \]

(2.11)

comprise an orthogonal basis for \( L^2([-1,1], du) \); see [5].

In order to show that

\[ B = \{ \varphi^{(p,q)} \mid (p,q) \in \mathbb{Z}^2 \& \ \min\{p,q\} \geq 1 \} \]

(2.12)
is an orthogonal basis of $L^2(\mathbb{D}, r dr d\theta/(1-r^2))$, we first argue that the functions $\varphi^{(p, q)}$ are orthogonal, and then that the span of $\mathcal{B}$ is dense. It was proven in \cite{22} that $\varphi^{(p, q)}$ and $\varphi^{(p', q')}$ are orthogonal if $pq = p'q'$ and $(p, q) \neq (p', q')$. On the other hand, if $pq \neq p'q'$, then orthogonality of $\varphi^{(p, q)}$ and $\varphi^{(p', q')}$ follows from the fact that they are eigenfunctions, corresponding to distinct eigenvalues, of the self-adjoint operator $-\Delta$; self-adjointness of $-\Delta$ follows from that of $-\Delta$ by \cite{22}

It remains to show that span $\mathcal{B}$ is dense in $L^2(\mathbb{D}, r dr d\theta/(1-r^2))$. Toward this end, suppose that $h \in L^2(\mathbb{D}, r dr d\theta/(1-r^2))$ is orthogonal to every member of $\mathcal{B}$. By \eqref{2.10} there exists $g \in L^2(\mathbb{D}, r dr d\theta)$ such that

$$h(r, \theta) = \sqrt{1-r^2} g(r, \theta).$$

\label{2.13}

Let $\alpha_{p, q} = (-1)^{q+\max\{p, q\}} \max\{p, q\}/q$ denote the coefficient occurring on the right-hand side of \eqref{2.8}. Then for each fixed $n = q - p \in \mathbb{Z}$, for every $\nu = \min\{p, q\} - 1 \geq 0$,

$$0 = \int_{\mathbb{D}} h(r, \theta) \varphi^{(p, q)}(r, \theta) \frac{r dr d\theta}{1-r^2}$$

$$= \alpha_{p, q} \int_{0}^{1} \left( \int_{0}^{2\pi} h(r, \theta) e^{-in\theta} d\theta \right) \left( 1 - r^2 \right) r^n |P_{\nu}^{(1, |n|)}(2r^2 - 1) \frac{rdr}{1-r^2} \quad (\text{by \eqref{2.7}})$$

$$= \alpha_{p, q} \int_{0}^{1} \left( \int_{0}^{2\pi} g(r, \theta) e^{-in\theta} d\theta \right) \sqrt{1-r^2} r^n |P_{\nu}^{(1, |n|)}(2r^2 - 1) rdr \quad (\text{by \eqref{2.13}})$$

$$= \frac{\alpha_{p, q}}{4} \int_{-1}^{1} \left( \int_{0}^{2\pi} g \left( \sqrt{\frac{1+u}{2}}, \theta \right) e^{-in\theta} d\theta \right) \sqrt{\frac{1-u}{2}} \sqrt{\frac{1+u}{2}} |P_{\nu}^{(1, |n|)}(u) du \quad (u = 2r^2 - 1)$$

$$= \frac{\alpha_{p, q}}{4} \int_{-1}^{1} \left( \int_{0}^{2\pi} g \left( \sqrt{\frac{1+u}{2}}, \theta \right) e^{-in\theta} d\theta \right) Q_{\nu}^{(1, |n|)}(u) du \quad (\text{as in \eqref{2.11}}).$$

Since the quasipolynomials $Q_{\nu}^{(1, |n|)}$ are an orthogonal basis for $L^2([-1, 1], du)$, it follows that for each $n \in \mathbb{Z}$,

$$\int_{0}^{2\pi} g \left( \sqrt{\frac{1+u}{2}}, \theta \right) e^{-in\theta} d\theta = 0 \quad (\text{\eqref{2.14}})$$

for every $u \in [-1, 1]$ outside a set $E_n$ of measure zero. Since $\{e^{in\theta}\}_{n \in \mathbb{Z}}$ is an orthogonal basis of $L^2([0, 2\pi], d\theta)$, it follows in turn that for $u \notin \cup E_n$

$$g \left( \sqrt{\frac{1+u}{2}}, \theta \right) = 0,$$

for almost every $\theta \in [0, 2\pi]$. Thus $g(r, \theta) = 0$ for almost every $(r, \theta) \in \mathbb{D}$ and $g = 0$ as a function in $L^2(\mathbb{D}, r dr d\theta)$, whence $h = 0$ also. This proves that the orthogonal complement of $\mathcal{B}$ in $L^2(\mathbb{D}, r dr d\theta/(1-r^2))$ is empty, and hence that $\mathcal{B}$ is an orthogonal basis.

\section{3 Conclusions}

Since the vector space $L^2(\mathbb{D}, r dr d\theta/(1-r^2)) = \sqrt{1-r^2}L^2(\mathbb{D}, r dr d\theta)$ is dense in

$$L^2(\mathbb{D}, r dr d\theta) = L^2(\mathbb{D}, dx dy),$$

and convergence in the former space implies convergence in the latter, scattering polynomials comprise a (non-orthogonal) basis for $L^2(\mathbb{D}, dx dy)$ consistent with Dirichlet boundary values. From the
perspective of analysis of functions on the disk, this provides an alternative to Zernike polynomials—and their generalizations the disk polynomials—which are non-zero on the boundary circle and so inconsistent with Dirichlet conditions.

More generally, scattering polynomials illustrate that orthogonal polynomials can comprise an orthogonal basis for a function spaces $L^2(X, d\mu)$ in which not all polynomials are integrable. A natural question for further investigation is the existence and extent of other such examples.

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