Factorization in Break-up and Recombination Processes for Atoms with a Large Scattering Length

Eric Braaten and Dongqing Zhang
Physics Department, The Ohio State University, Columbus, Ohio 43210, USA
(Dated: March 23, 2022)

Break-up and recombination processes for loosely-bound molecules composed of atoms with a large scattering length $a$ necessarily involve interactions that are nonperturbative in the exact $N$-body interaction. If these processes involve atoms with relative momenta much larger than $\hbar/|a|$, the leading contributions to their rates can be separated into short-distance factors that are insensitive to $a$ and long-distance factors that are insensitive to the range of the interaction. These factorization contributions can be obtained from the leading term in a perturbation expansion in the exact atom-atom scattering amplitude. The short-distance factors are atom-atom cross sections at a lower collision energy. In the special case of inclusive break-up cross sections for atom-molecule scattering, the long-distance factors simply count the number of atoms in the molecule.

PACS numbers: 12.38.-t, 12.38.Bx, 13.20.Gd, 14.40.Gx

Nonrelativistic particles with short-range interactions that have been tuned, either by the experimenter or fortuitously by nature, so that their S-wave scattering length $a$ is much larger than the range, have universal low-energy properties that depend on $a$ but are otherwise insensitive to their interactions at short distances. (See Ref. [1] and references therein.) If the particles form loosely-bound 2-body or higher $N$-body clusters whose sizes are comparable to $|a|$, the clusters also have universal properties. A classic example in atomic physics is $^4$He atoms whose scattering length $a \approx 100 \text{ Å}$ is much larger than their effective range $r_\ast \approx 7 \text{ Å}$. The $^4$He dimer and the excited state of the $^4$He trimer both have sizes comparable to $a$. A classic example in nuclear physics is nucleons. The deuteron is a bound state of the neutron and proton associated with a large scattering length in the spin-triplet channel.

The large scattering length implies that interactions between atoms whose relative momenta are in the 2-body scattering amplitude. A simple consequence of these strong interactions with $a > 0$ is the existence of a loosely-bound dimer whose binding energy is given by

$$E_D = \hbar^2/ma^2.$$  

In some cases (for example, identical bosons), the strong interactions produce the Efimov effect: as $a \to \pm \infty$, there are increasingly many loosely-bound trimers with an accumulation point at the 3-atom threshold $2\hbar$. In the limit $a = \pm \infty$, the trimers have an asymptotically exponential spectrum:

$$E_T^{(n)} \to (e^{-2\pi/s_0})^{n-n_\ast} \hbar^2 \kappa^2/m \quad \text{as } n \to \infty,$$  

where $s_0$ is a numerical constant whose value for identical bosons is 1.00624, $n_\ast$ is an arbitrary integer, and $\kappa$ is a 3-body parameter [4]. The strong interactions can also lead to intricate dependence on the scattering length $a$. An example in the case of identical bosons is the event rate constant in the low-energy limit for 3-body recombination into the loosely-bound dimer:

$$K_3 = \frac{768 \pi^2 (4\pi - 3\sqrt{3}) \hbar a^4/m}{\sinh^2(\pi s_0) + \cosh^2(\pi s_0) \cot^2(s_0 \ln(\kappa\ast) + 1.16)}.$$  

The log-periodic dependence on $a$ in Eq. (3) was discovered in Refs. 3. The completely analytic expression was derived more recently by Petrov 4. The coefficient of $\hbar a^4/m$ on the right side of Eq. (3) can range from 0 to 402.7 depending on the value of the 3-body parameter $\kappa\ast$. The dimer break-up cross section in atom-dimer scattering at a collision energy $E$ just above the threshold $E_D$ has the same log-periodic factor:

$$\sigma_{AD}^{(\text{break-up})}(E) = \frac{\sqrt{3}}{96\pi} \left( \frac{mK_3}{\hbar a^4} \right)^{2} \left( E - E_D \right)^2 E_D^{1/2} E_D^{3/2}. $$

To reproduce such intricate dependence on $a$ requires accurate numerical methods that are capable of solving the problem “exactly”, that is, to any desired precision. The Schrödinger equation for 2 or 3 atoms interacting through a short-range potential can be solved exactly. Systems of 4 atoms are at the frontiers of few-body physics: the binding energies of tetramers can be calculated accurately, but there are some scattering observables for which effective calculational methods have yet to be developed. In any process that involves loosely-bound molecules, strong interactions involving the momentum scale $\hbar/|a|$ are unavoidable. This suggests that accurate calculations of the rates for such processes may be possible using present methods only if the total number of atoms involved is at most 3 or 4.

In this Letter, we point out that this obstacle can be avoided for processes involving atoms whose momenta relative to the loosely-bound molecules are set by a scale $Q \gg \hbar/|a|$. Examples of such processes are the break-up of the molecule in a collision with an energetic atom...
and the formation of the molecule in a collision involving energetic atoms. The separation of scales $Q \gg h/a$ can be exploited by using factorization to separate the rate into short-distance factors that are insensitive to $a$ and long-distance factors that do not involve the scale $Q$. The strong interactions associated with large $a$ enter only in the long-distance factors. The rate can be calculated accurately using present methods provided the long-distance factors involve at most 3 or 4 atoms. In some cases, even this limitation is unnecessary because the long-distance factors can be determined analytically. Thus factorization significantly expands the list of processes whose rates can be calculated accurately using present methods.

Factorization has proved to be a powerful tool in quantum chromodynamics (QCD). The Nobel Prize in physics for 2004 was awarded to Gross, Politzer, and Wilczek for the discovery of the asymptotic freedom of QCD. Asymptotic freedom refers to the decrease of the strength of the interaction between quarks as their separation decreases, or equivalently, as their relative momentum increases. For selected observables, asymptotic freedom can be exploited by using factorization to separate the observables into short-distance factors that involve only weak QCD interactions between quarks and gluons and long-distance factors that involve strong QCD interactions. The short-distance factors involve a large momentum transfer $Q$ and can be calculated using perturbation theory in the running coupling constant $\alpha_s(Q)$ of QCD. The long-distance factors involve only momentum transfers small compared to $Q$. In most cases, systematic methods for calculating the long-distance factors have not been developed. The bulk of the quantitative evidence that QCD describes the strong force is the experimental verification that the short-distance factors are correctly predicted by perturbative QCD.

Atoms with a large scattering length have a property analogous to asymptotic freedom: the strength of their interaction, as measured by the magnitude of the elastic cross section, decreases as their relative momentum $h/k$ increases. This behavior of the atom-atom interaction motivates our use of the factorization strategy that has been so successful in QCD. The $S$-wave elastic scattering amplitude has the form

$$f_k = \left(-1/a + \frac{k^2}{2} + \ldots - ik\right)^{-1}, \quad (5)$$

where $r_s$ is the effective range. For collision energies $E$ at which higher partial waves can be neglected, the elastic cross section for identical bosons is

$$\sigma_{AA}(E) = \frac{8\pi}{(-1/a + \frac{k^2}{2} + \ldots + mE/h^2)^2 + mE/h^2}. \quad (6)$$

The cross section has its maximum value $8\pi a^2$ at $E = 0$. In the scaling region $h^2/ma^2 \ll E \ll h^2/mr_s^2$, the cross section scales like $1/E$:

$$\sigma_{AA}(E) \approx \frac{8\pi h^2}{mE}. \quad (7)$$

Consider the break-up of a loosely-bound dimer through a collision with an atom. We take all three atoms to be identical bosons. The two atoms inside the dimer have typical separation $a$. The atom-dimer collision energy is $E = \frac{3}{4}Q^2/m$, where $Q$ is the momentum of the atom or the dimer in the center-of-mass frame. If $Q \gg h/a$, the leading contribution to the break-up of the dimer comes from one of its constituent atoms being knocked out by the energetic incoming atom, while the other acts as a spectator. This process can be represented by the Feynman diagram in Fig. 1 which is the leading term in a perturbation expansion in the exact atom-atom scattering amplitude. The expression for the $T$-matrix element is

$$T = \frac{(32\pi \hbar^2/m)(\pi/a)^{1/2}(1 + \frac{3}{2}r_s/a + \ldots)}{(q_3^2 + mE_D)(-1/a + \frac{k^2}{2} + \ldots - ik_{12})}. \quad (8)$$

where $E_D = \hbar^2/(ma^2)(1 + r_s/a + \ldots)$ is the dimer binding energy including range corrections, $q_3$ is the relative momentum of the two atoms in the dimer (one of which is the outgoing atom labelled 3), and $k_{12} = \frac{3}{2}Q$ is the relative momentum of the incoming atom and the scattered atom from the dimer (which is also the relative momentum of the outgoing atoms labelled 1 and 2). The sum is over cyclic permutations of the three outgoing atoms. The effects of subsequent interactions between the scattered atoms and the spectator atom in the dimer enter through higher-order diagrams in the perturbation expansion in the exact atom-atom scattering amplitude. These diagrams are all suppressed by a factor of $\hbar/(aQ)$ and can be neglected if the collision energy is sufficiently high. If we take the limit $Q \gg h/a$, the $T$-matrix element in Eq. (8) can be expressed as the product of a short-distance factor that involves only the large momentum scales $Q$ and $h/r_s$ and a long-distance factor that involves only the small momentum scales $q$ and $h/a$:

$$T \approx \sum_{(123)} \frac{4\hbar^2(\pi/a)^{1/2}}{q_3^2 + h^2/a^2} \times \frac{8\pi h/m}{(\frac{1}{2}r_s k_{12}^2 + \ldots - ik_{12})}. \quad (9)$$

The long-distance factor is proportional to the

![FIG. 1: Feynman diagram for the break-up of a dimer by the hard scattering of one of its constituent atoms and a colliding atom. The blob represents the wavefunction of the dimer and the dot represents the exact atom-atom scattering amplitude.](image)
momentum-space wavefunction of the dimer in the zero-range limit. The cross section is obtained by squaring the T-matrix element and integrating over the momenta of the three identical outgoing atoms. The cross section is dominated by the squares of the terms in Eq. (9), which we call diagonal terms. The terms in the cross section that correspond to the interference between terms in Eq. (9), which we call cross terms, are suppressed by a power of $\hbar/(aQ)$. We can take the limit $Q \gg \hbar/a$ in the momentum integrals. The result for the break-up cross section at collision energy $E \gg h^2/(ma)^2$ can be expressed in the factored form

$$\sigma_{\text{AD}}^{\text{(break-up)}}(E) \approx 2 \sigma_{AA}(\frac{2}{3}E). \quad (10)$$

The coefficient 2, which is the long-distance factor, is obtained by using the normalization integral for the dimer wavefunction. The factorization form in Eq. (10) applies at arbitrarily large $E$ provided $\sigma_{AA}$ is the total atom-atom cross section. In the scaling region $E_D \ll E \ll h^2/(ma)^2$, this breakup cross section scales as $1/E$:

$$\sigma_{\text{AD}}^{\text{(break-up)}}(E) \approx \frac{64\pi\hbar^2}{3mE}. \quad (11)$$

The result can be generalized to the inclusive break-up cross sections for Efimov trimers and other loosely-bound molecules whose constituents all have separations of order $|a|$. The leading contribution comes from one of the atoms in the molecule being knocked out by a collision with the energetic incoming atom. If the molecule contains $N$ loosely-bound atoms, the atom-molecule collision energy is $E = \frac{N+1}{2N}Q^2/m$, where $Q$ is the momentum in the center-of-mass frame. The short-distance factor in the T-matrix element is the same as in Eq. (10), except that the relative momentum is $\hbar k = \frac{N+1}{2N}Q$. For any specific final state, the long-distance factor may be a very complicated function of $a$ and the 3-body parameter $\kappa_3$ that appears in Eq. (10). It involves the wavefunction of the molecule and the wavefunction of the $(N - 1)$-atom system that remains after the hard scattering. Some of those $N - 1$ atoms may be bound into dimers or other clusters. However, if we sum over all these $(N - 1)$-atom states, we can use completeness relations to show that the long-distance factor is simply the number $N$ of atoms in the molecule. Our result for the inclusive break-up cross section as a function of the collision energy $E$ is

$$\sigma_{\text{AM}}^{\text{(break-up)}}(E) \approx N \sigma_{AA}(\frac{N+1}{2N}E). \quad (12)$$

This expression is valid when the collision energy satisfies $E \gg h^2/(ma)^2$ and $E \gg E_M$, where $E_M$ is the binding energy of the molecule with respect to the $N$-atom threshold.

The cross section for the break-up of $^4$He dimers in collisions with Xe atoms has been studied in Ref. [1] for collision energies $E$ ranging from 46 K to 348 K. Since the Xe atom is so much heavier than a He atom, each of the atoms in the He dimer carries approximately $\frac{1}{4}$ the collision energy $E$. The analog of our factorization formula in Eq. (10) predicts that if $E$ is much larger than the He dimer binding energy, which is about 1.6 mK, the $^4$He–Xe break-up cross section should be approximately 2 times the He–Xe cross section at collision energy $\frac{1}{4}E$. This prediction agrees reasonably well with the VCC-RIOS approximation studied in Ref. [1]. By comparing the cross section $\sigma_{\text{VCC-RIOS}}^{\text{tot}}$ in Table 2 of Ref. [1] with the appropriate interpolated values of $\sigma_{\text{He-Xe}}^{\text{el}}$, we can see that the difference is less than 9% in the energy range from 93 K to 348 K. In the Independent-Atoms model considered in Ref. [1], the total cross section is 2 times the He–Xe cross section at collision energy $E$ instead of $\frac{1}{4}E$. This model underestimated the VCC-RIOS cross section by an amount that ranges from 11% at 93 K to 30% at 348 K.

Factorization can also be applied to the rate for 3-body recombination into the loosely-bound dimer when the three incoming atoms all have relative momenta much greater than $\hbar/a$. The leading contribution comes from a hard scattering of two of the atoms that scatters one of them into a state with small momentum relative to the third atom, followed by the coalescence of those two atoms into a dimer. The coalescence probability is substantial only if the relative momentum of the scattered atom and the third atom is of order $\hbar/a$. At leading order in the perturbation expansion in the exact atom-atom scattering amplitude, the T-matrix element for this process can be expressed as the sum of three Feynman diagrams obtained by time-reversing the diagram in Fig. 1 and summing over cyclic permutations of the three incoming atoms. We work in the center-of-mass frame, taking the momenta of the three incoming atoms to be $\vec{p}_1$, $\vec{p}_2$, and $\vec{p}_3$ and the momenta of the outgoing dimer and atom to be $+\vec{Q}$ and $-\vec{Q}$. The collision energy is $E = (p_1^2 + p_2^2 + p_3^2)/(2m)$. For large collision energy $E$, the leading contribution to the T-matrix element is given by Eq. (11). Each of the three terms in the sum is the product of a short-distance factor and a long-distance factor. The hard scattering of atoms 1 and 2 followed by the coalescence of one of them with atom 3 gives the term shown explicitly in Eq. (11), where $\hbar k_{12} = \frac{1}{2}(\vec{p}_1 - \vec{p}_2)$ is the large relative momentum of atoms 1 and 2 and $q_3 = \vec{p}_3 - \frac{1}{2}\vec{Q}$ is the small relative momentum between atom 3 and the atom it coalesces with. The other two terms are obtained by cyclic permutations of 1, 2, and 3. The 3-body recombination rate is obtained by squaring the T-matrix element and integrating over the final momenta of the outgoing atom and dimer. In the limit $E \gg h^2/(ma)^2$, the rate $R$ for forming dimers per volume and per time is

$$R \approx \frac{256\pi^2\hbar^3}{m^2} \sigma_{AA}(\frac{1}{4}E) \sum_{\text{(123)}} \delta(E - 3p_3^2/m). \quad (13)$$

The three terms in the sum are diagonal terms that come from the squares of the terms in Eq. (9). The cross terms corresponding to the interference between terms in Eq. (9) have been neglected because they are suppressed by a factor of $(E_D/E)^{1/2}$.
The event rate \( R \) in Eq. (13) can be reduced to a function \( K_3 \) of the collision energy \( E \) by averaging over the momentum hyperangle \( \alpha_3 \) defined by \( p_k = (\frac{4}{5}mE)^{3/2} \cos \alpha_k \) and over the angle \( \beta_3 \) between the vectors \( \vec{p}_3 \) and \( \vec{p}_1 - \vec{p}_2 \). The hyperangular average of \( R \) can be expressed as

\[
\langle R \rangle = \frac{2}{\pi} \int_0^{\pi/2} d\alpha_3 \sin^2(2\alpha_3) \int_0^\pi d\beta_3 \sin \beta_3 \, R. \tag{14}
\]

The hyperangular average of the rate in Eq. (16) is

\[
K_3 \equiv \langle R \rangle \approx \frac{384\sqrt{3}\pi h^3}{m^2E^2} \sigma_{AA}(\frac{3}{4}E). \tag{15}
\]

This result is valid when \( E \gg E_D \). In the scaling region \( E_D \ll E \ll \hbar^2/mr_a^2 \), the rate in Eq. (15) scales as \( 1/E^2 \):

\[
K_3 \approx \frac{12288\sqrt{3}\pi^2h^5}{3m^3E^2}. \tag{16}
\]

Suno et al. have calculated the 3-body recombination rate of \(^4\)He atoms into the dimer at collision energies up to 10 mK using accurate solutions of the 3-body Schrödinger equation for the HFD-B3-FCH potential model. In this model, the scattering length is \( a = 91.0 \text{ Å} \), the effective range is \( r_s \approx 7 \text{ Å} \), and the dimer binding energy is \( E_D = 1.6 \text{ mK} \). The scattering length deduced from the universal formula for the dimer binding energy in Eq. (16) is \( a_D = 87.0 \text{ Å} \). In Fig. 2 we compare the result of Ref. [9] for \( K_3 \) with the scaling approximation in Eq. (16). At \( E = 10 \text{ mK} \), the highest collision energy considered in Ref. [9], our scaling approximation in Eq. (16) is larger than the total result in Ref. [9] by a factor of 28.8. Our factorization approximation in Eq. (15) is larger than the total result in Ref. [9] by a factor of 23.7.

We now consider why the scaling approximation in Eq. (16) overestimates the 3-body recombination rate at 10 mK by more than an order of magnitude. We will show that the scaling approximation can be accurate only for collision energies that are at least an order of magnitude greater than 20 mK. We calculate the sum of the contributions to \( R \) at leading order in the perturbation expansion in the exact atom-atom scattering amplitude without making the factorization approximations. The T-matrix element is given by Eq. (5). We set the effective range \( r_s \) to zero to make the scaling behavior of the individual contributions at large energy \( E \) more evident. The resulting expression for the 3-body recombination rate is

\[
R = \frac{32\sqrt{3}\pi h^3}{m^2E^2} [E_D(E + E_D)]^{1/2} \sum_{(123)} \sigma_{AA}(E \sin^2 \alpha_3) (\cos^2 \alpha_3 - \frac{1}{4} + \frac{3}{4}E_D/E)^2 + \frac{3}{4}E_D/E(1 + E_D/E)^{-1}
\]

\[
+ 2 \sigma_{AA}(E \sin^2 \alpha_1) \sigma_{AA}(E \sin^2 \alpha_2) \sigma_{AA}(E \sin \alpha_1 \sin \alpha_2)
\]

\[
\times \int_0^1 dt \left[(\cos^2 \alpha_{12}(t) - \frac{1}{4} + \frac{3}{4}E_D/E)^2 + \frac{3}{4}(E_D/E)(1 + E_D/E) + 3t(1-t)(1 + E_D/E)\sin^2 \alpha_3\right]^{-1}. \tag{17}
\]

The angle \( \alpha_{12}(t) \) is defined by

\[
\cos^2 \alpha_{12}(t) = t \cos^2 \alpha_1 + (1-t) \cos^2 \alpha_2. \tag{18}
\]

The two terms inside the sum in Eq. (17) are a diagonal term, which corresponds to the square of a term in Eq. (5), and a cross term, which corresponds to the
The hyperangular average of the rate $R$ in Eq. (17) can then be calculated using Eq. (14). In Fig. 2 we show $K_3 = \langle R \rangle$ as a function of the collision energy $E$, as well as the contributions to $K_3$ from the diagonal terms and from the cross terms. The cross terms are larger than the diagonal terms at low energies and smaller at higher energies, with the crossover occurring near $E = 20\, \text{mK}$. At very high energy, the cross terms scale like $E^{-5/2}$. They eventually become negligible compared to the diagonal terms, which scale like $E^{-2}$. The factorization approximation in Eq. (14) is a high energy approximation to the contribution from the diagonal terms. From the crossover point and the scaling behavior, we can infer that the factorization approximation can be a good approximation only if the collision energy exceeds the 20 mK by more than an order of magnitude.

We can get further insights into the factorization approximation by considering the angular-momentum decomposition of the 3-body recombination rate. The rate calculated by Suno et al. in Ref. [9] was obtained by adding the contributions with total angular momentum quantum number $J = 0, 1, 2$ and 3. These contributions are shown in Fig. 3. Note that at $E = 10\, \text{mK}$, the highest energy for which they were calculated, the $J = 0$ and 1 contributions are increasing and they are smaller than the $J = 2$ and 3 terms which are decreasing. The angular-momentum decomposition of the rate in Eq. (17) is

$$R = \frac{32\sqrt{3\pi} \hbar^3}{m^2 E} \left( \frac{E_D}{E + E_D} \right)^{1/2} \sum_{J=0}^{\infty} (2J + 1) \sum_{\{123\}} \left( \sigma_{AA} (E \sin^2 \alpha_3) \frac{Q_J(z_3)^2}{\cos^2 \alpha_3} \right.$$  

$$+ 2 \frac{\sigma_{AA} (E \sin^2 \alpha_1) \sigma_{AA} (E \sin^2 \alpha_2)}{\sigma_{AA} (E \sin \alpha_1 \sin \alpha_2)} \frac{Q_J(z_1)Q_J(z_2)}{\cos \alpha_1 \cos \alpha_2} P_J(z_{3,12}) \right), \quad (20)$$

where the $Q_J(z)$ are Legendre functions of the second kind with branch cuts on the interval $-1 < z < +1$. Their arguments are

$$z_i = \frac{\cos^2 \alpha_i + \frac{1}{2} + E_D/E}{\sqrt{1 + E_D/E \cos \alpha_i}}. \quad (21)$$

The argument of the Legendre polynomial $P_J(z)$ in Eq. (20) is

$$z_{3,12} = \frac{\cos^2 \alpha_3 - \cos^2 \alpha_1 - \cos^2 \alpha_2}{2 \cos \alpha_1 \cos \alpha_2}. \quad (22)$$

The individual contributions from $J = 0, 1, 2,$ and 3 are shown in Fig. 3. In the high energy limit, each individual contribution scales like $E^{-5/2}$. Since the total $\langle R \rangle$ scales like $E^{-2}$, the sum over $J$ must provide the additional contribution.
factor of $E^{1/2}$. At $E = 10$ mK, the $J = 0$, 1, 2, and 3 terms are larger than the results of Ref. 9 by factors of 711, 12.0, 1.6, and 1.7, respectively. The mismatch in the $J = 2$ and 3 curves in Fig. 3 is small. The large mismatch in the $J = 1$ curves is easy to understand. The cross terms in Eq. (20) are constructive for even $J$ and destructive for odd $J$, and they are much smaller than the diagonal terms for $J \geq 3$. Thus the $J = 1$ channel is the only one with substantial destructive interference. The sum of the diagonal and cross terms is 47% of the exact atom-atom scattering amplitude. The production of deuterons, $^3$He nuclei, and the corresponding antimolecules has been observed in ultrarelativistic heavy-ion collisions 11. These collisions are believed to produce a thermalized state with very high energy density which, as it expands and cools, makes transitions to a quark-gluon plasma and then to a hadron gas. Analogs of the factorization formulas in Eqs. (10) and (15) can be applied to the breakup and formation of deuterons and antideuterons in the hadron gas phase.

The strong interactions between atoms with large scattering lengths implies that high numerical accuracy is required to calculate rates for processes involving loosely-bound molecules. Using traditional methods, the list of processes for which accurate calculations are possible is restricted to those involving at most 3 or 4 atoms. By using factorization, the list can be expanded to include all those for which the long-distance factors involve $N \leq 3$ or 4 atoms. In some cases, such as the break-up cross sections in Eq. (12), the long-distance factors can be determined analytically and accurate calculations are possible even for $N > 4$.

In the examples of $^4$He molecules and the deuteron, the large scattering length $a$ arises from a fortuitous fine tuning by nature. Another exciting application of factorization is to alkali atoms near a Feshbach resonance, which allows $a$ to be tuned to arbitrarily large values by varying a magnetic field. The factorization formulas give the leading term in a systematic expansion in powers of $1/a$. As the scattering length is tuned to be increasingly large, the factorization approximation becomes increasingly accurate and it applies at increasingly lower colli-
sion energies.

We acknowledge useful discussions with R. Furnstahl. We thank B. Esry for providing us with the results of Ref. [9]. This research was supported in part by the Department of Energy under grants DE-FG02-91ER4069 and DE-FG02-05ER15715.

[1] E. Braaten and H. W. Hammer, arXiv:cond-mat/0410417
[2] V. Efimov, Phys. Lett. 33B, 563 (1970).
[3] E. Nielsen and J. H. Macek, Phys. Rev. Lett. 83, 1566 (1999); B. D. Esry, C. H. Greene, and J. P. Burke, Phys. Rev. Lett. 83, 1751 (1999); P. F. Bedaque, E. Braaten, and H.-W. Hammer, Phys. Rev. Lett. 85, 908 (2000).
[4] D. Petrov, “Three-boson problem near a narrow Feshbach resonance”, talk at OCTS Workshop on Strongly Interacting Quantum Gases, Ohio State University, April 2005 (octs.osu.edu/images/Gases/Talks/Petrov.pdf).
[5] D. J. Gross and F. Wilczek, Phys. Rev. Lett. 30, 1343 (1973); H.D. Politzer, Phys. Rev. Lett. 30, 1346 (1973).
[6] J. C. Collins and D. E. Soper, Ann. Rev. Nucl. Part. Sci. 37, 383 (1987).
[7] E. Buonomo, F.A. Gianturco, and F. Ragnetti, J. Phys. Chem. 100, 9206 (1996).
[8] F.A. Gianturco, S. Serna, G. Delgado-Barrio, and P. Villareal, J. Chem. Phys. 95, 5024 (1991).
[9] H. Suno, B.D. Esry, C.H. Greene, and J.P. Burke, Jr., Phys. Rev. A 65, 042725 (2002).
[10] L.W. Bruch, W. Schöllkopf, and J.P. Toennies, J. Chem. Phys. 117, 1544 (2002); R. Kariotis, L.W. Bruch, and O. Kornilov, J. Chem. Phys. 121, 3044 (2004).
[11] C. Adler et al. [STAR Collaboration], Phys. Rev. Lett. 87, 262301 (2004); J. Nystrand [PHENIX Collaboration], arXiv:nucl-ex/0409006.