DESINGULARIZATION OF SURFACE MAPS

ERICA CLAY, BORIS HASSELBLATT, AND ENRIQUE PUJALS

(Communicated by Svetlana Katok)

ABSTRACT. We prove a result for maps of surfaces that illustrates how singular-hyperbolic flows can be desingularized if a global section can be collapsed to a surface along stable leaves.

1. INTRODUCTION

The Lorenz attractor was not only a prominent motivation for the inception of “chaos theory,” but it also provided a proving ground for extending methods from uniformly hyperbolic dynamics to hyperbolic systems with singularities. Here, the singularity is a fixed point, whereas uniformly hyperbolic flows are fixed-point free.

As illustrated in Figure 1 a local section near the singularity is an important tool in the analysis of this system, and the presence of the singularity entails unbounded return times, manifested in the figure by the distortion of rectangles into pointy triangles under the return map. It is useful that the local section foliates into stable leaves, and collapsing these provides a 1-dimensional system for analysis.

Received by the editors November 15, 2016 and, in revised form, January 11, 2017.

2000 Mathematics Subject Classification. Primary: 37D20; Secondary: 57N10.

Key words and phrases. Singularization, hyperbolicity.

©2016 American Institute of Mathematical Sciences
An obvious counterpart in higher dimension is a *generalized Lorenz attractor* obtained by taking a Cartesian product with a contracting flow. However, the aforementioned resulting 1-dimensional expanding system is unchanged under this extension; there are merely more stable directions to collapse.

The definition of a singular-hyperbolic set includes the requirement that the fixed point be a hyperbolic fixed point, as is the case with the Lorenz attractor \([7, 8]\), and several constructions of such examples emerged some 2 decades ago, such as by Morales and Pumariño ([5], stated as [6, Theorem 4]). They produce an isotopy of vector fields in which a hyperbolic attractor is deformed to a singular-hyperbolic attractor, and for all isotopy parameters there is an attractor which is hyperbolic or singular hyperbolic. Bonatti, Pumariño, and Viana [2] constructed examples with more than one expanding direction by introducing a singularity to a uniform hyperbolic attractor, a process known as singularization: from an expanding map of a surface construct the (invertible) geometric inverse limit as in Smale's attractors that are derived from expanding maps [10, p. 788] and suspend the resulting diffeomorphism to obtain a flow. Then perform a homotopy that involves reversing the flow direction on part of a chosen orbit to create hyperbolic fixed points as illustrated in Figure 2; this reversal corresponds to a saddle-node bifurcation or Cherry-flow construction.

![Figure 2. Singularization (from [1])](image)

This paper is motivated by the question of whether all singular-hyperbolic attractors arise in this way, i.e., whether any singular-hyperbolic attractor is isotopic to a uniformly hyperbolic one. We note that this question does not arise with respect to the Lorenz attractor itself because that would produce a hyperbolic attractor in a rectangle (the section), and this is known to be impossible (3 punctures are needed, as for the Plykin attractor). So, more generally, the question arises only in the context of singular-hyperbolic attractors with more than one expanding direction.

Our main result indicates that this seems to be the case when the expanding direction is 2-dimensional and there is a global section of the right kind, namely, such that collapsing stable manifolds produces a compact surface. Consider a hyperbolic attractor with a transverse section \(S \times E^n\), where \(S\) is a surface such that the first-return map induces a “punctured diffeomorphism” and such that one can quotient by the stable foliation to get an expanding punctured endomorphism on \(S\). Then the problem is reduced to desingularizing surface endomorphisms. Our main theorem does the latter, and reversing the projection process produces a (nonsingular) uniformly hyperbolic flow.
We note that the proof of this theorem is purely topological and therefore applies more generally to continuous locally injective maps.

**Theorem 1.** Let $S$ be a compact orientable surface, $p \in S$, and $f : S \setminus \{p\} \to S$ a continuous locally injective map which admits an open disk $B$ around $p$ whose boundary $\Gamma$ is a circle such that

- $[A1]$ there is a point with more than 3 preimages in $S^o := S \setminus B$,
- $[A2]$ $f|_\Gamma : \Gamma \to C := f(\Gamma)$ is a homeomorphism.

Then there is a covering map $F : S \to S$ which agrees with $f$ off $B$.

**Remark 2.**
- The conclusion implies that $S$ is a torus (if $k \neq 1$, which is necessary if we wish $F$ to be expanding, which in turn is needed for hyperbolicity): if there is a $k$-sheeted covering $\Sigma \to S$, then
  \[ \chi(\Sigma) = k\chi(S), \]  
  where $\chi(\cdot)$ denotes the Euler characteristic, so $\Sigma = S$ implies $\chi(S) = 0$.
- If $k = 1$, then $f|_{S^o}$ is a homeomorphism onto its image which is hence the complement of a disk in $S$, so the extension to a self-homeomorphism is obvious. (And $S$ can be any surface.) That is, $[A1]$ in Theorem 1 could be replaced by “$f$ is injective or there is a point with more than 3 preimages in $S^o := S \setminus B$."
- The proof of Theorem 1 involves showing that $S = \mathbb{T}^2$. We extend $f|_{S^o}$ to a covering $\Sigma \to S$ for some $\Sigma$ and then show that $\Sigma = S = \mathbb{T}^2$. This latter part is where the assumption on the number of preimages is needed. Indeed, we prove Theorem 3 below along the way (Remark 10), and the purpose of $[A1]$ is to establish assumption $[A1']$ in Theorem 3.
- If instead of proving it we assume in Theorem 1 that $S = \mathbb{T}^2$, then we can dispense with $[A1]$ and weaken $[A2]$ by requiring only that $f(\Gamma)$ is a circle. Consequently, we have Theorem 4 below.
- Maybe assumption $[A1]$ in Theorem 1 can be weakened to “at least 3” because we use “greater” solely to eliminate the possibility (in Proposition 11) that the set of points with the maximal number of preimages is a disk (which might be doable without this assumption). Our construction shows that “at least 2” is not enough for Theorem 1, but it might suffice for treating expanding $f$.

**Theorem 3.** Theorem 1 holds if instead of [A1] we assume

- $[A1']$ the set of points with the smallest number of preimages in $S^o$ is a disk.

**Theorem 4.** Let $p \in \mathbb{T}^2$ and $f : \mathbb{T}^2 \setminus \{p\} \to \mathbb{T}^2$ be a continuous locally injective map that admits a disk $B$ around $p$ whose boundary $\Gamma$ is a circle such that

- $[A2']$ $C := f(\Gamma)$ is a circle.

Then there is a covering map $F : \mathbb{T}^2 \to \mathbb{T}^2$ which agrees with $f$ off $B$.

**Remark 5.** It seems that $[A2']$ could be restated as requiring that $\Gamma$ can be chosen in such a way that there are only 2 possible values for the number of preimages under $f|_{S^o}$—the arguments in Proposition 8 should show that in this case the number of preimages in $\Gamma$ of points in $C$ is constant, which gives $[A2']$. 
2. Proof of the main theorem

We recall that the plan is to extend $f|_{S^\circ}$ to a covering $\Sigma \to S$ for some $\Sigma$ and to then show that $\Sigma = S = \mathbb{T}^2$.

To construct the extension of $f$ we establish that $C$ separates $S$ into $2$ subsurfaces $R$ and $P$, consisting respectively of points with $k$ or $k-1$ preimages, where a posteriori $k$ is the degree of the eventual cover. We then take $\Sigma = S \# P$ (connected sum), and $F := f \# \text{Id}$ is then continuous and locally injective and such that all points have $k$ preimages. Then (1) and the assumption [A1] (that $k > 3$) imply that $P$ is a disk and hence $\Sigma = S$. (We alert the reader to an abuse of notation: the connected sum of closed surfaces is defined by the deletion of disks and identification of the resulting boundaries. Here we often include summands that are not closed, but where a circle boundary is already available for identification. This should not cause confusion.)

We remark that as this work evolved, we initially intended $F$ to serve in a purely auxiliary role to constrain the topology of $S$, and we did not expect it to be the map that defines the dynamics of interest. Indeed, when we discuss extensions at the end, there is no corresponding coincidence.

We begin by showing that $C$ separates $S$ by showing that $S \setminus C$ is a finite disjoint union of open sets. This does not use either of the assumptions [A1] or [A2].

With $f|_{S^\circ}$ in mind we note that the number of preimages of a point $y$ under a continuous locally injective map $f$ between compact Hausdorff spaces is finite since $f^{-1}(\{y\})$ is compact by continuity and discrete by local injectivity. Thus $I: S \to \mathbb{N}$ given by $y \mapsto \text{card}(f^{-1}(\{y\}) \cap S^\circ)$ is well defined. We will show that it is continuous, hence locally constant, on $S \setminus C$, so its level sets form the desired partition. First we show that $I$ is upper semicontinuous:

**Lemma 6.** If $f: X \to Y$ is continuous and locally injective and $X$ is a compact Hausdorff space, then the set $Y_l$ of points with at least $l$ preimages is closed.

**Proof.** For $y \in \overline{Y}_l$ consider $y_i \in Y_l$ with $y_i \to y$ and $\{x_{i,1},\ldots,x_{i,l}\} \subset f^{-1}(\{y_i\})$. Passing to subsequences, compactness of $X$ gives $x_{i,j} \to x_j \in X$ as $i \to \infty$ for $1 \leq j \leq l$. By continuity $f(x_i) = \lim_{i \to \infty} f(x_{i,j}) = \lim_{i \to \infty} y_i = y$, so $\{x_{1,\ldots,x_{i,l}} \subset f^{-1}(\{y\})$, and the $x_j$ are pairwise distinct since $f$ is locally injective: if $m \neq n$ and $U$ is a local injectivity neighborhood of $x_m$ and $N$ such that $x_{i,m} \in U$ for $i \geq N$, then $x_{i,n} \notin U$ for any $i \geq N$, so $x_{i,n} \not\to x_m$ since $X$ is Hausdorff. □

We note that this implies boundedness of $I$, although we don’t need this. Next we show that $I$ is lower semicontinuous on $S \setminus C$:

**Lemma 7.** Let $M,N$ be manifolds, $f: M \to N$ a local homeomorphism, and $l \in \mathbb{N}$. Then the set $Y_l$ of points with at least $l$ preimages is open.

**Proof.** Let $y \in Y_l$ and $\{x_{1,\ldots,x_l} \subset f^{-1}(\{y\})$. There are pairwise disjoint neighborhoods $O_j$ of $x_j$ that are homeomorphically mapped by $f$ to neighborhoods $U_j$ of $y$. Then each point in the neighborhood $U := \bigcap_{j=1}^{l} U_j$ has at least $j$ preimages. □

We have shown that $I|_{S \setminus C}$ is continuous and hence locally constant, so $S \setminus C$ is partitioned by the open sets $L_i := I^{-1}(i) \cap C$. (So each connected component of $S \setminus C$ is contained in one of these sets.) While we remarked earlier that there are only finitely many of these, assumption [A2'] in Theorem 4 (and therefore
assumption [A2] in Theorem 1), namely, the fact that \( C \) is a connected manifold embedded in \( S \) as a closed subset, implies that there are at most 2 of them by [3, Proposition 6.1]. We now use assumption [A2] to show that there are at least two level sets.

**Proposition 8.** If assumption [A2] holds and \( y \in C \), then there are arbitrarily small disks \( D \) around \( y \) with \( I(D) = \{k, k-1\} \) for some \( k \in \mathbb{N} \).

**Proof.** With the notations from the proof of Lemma 7 we can up to relabeling take \( x_1 \in \Gamma \), so \( O_1 \) intersects both \( S^o \) and \( B \). Take \( k \) such that points in \( f(S^o \cap O_1) \) (and hence those on \( C \)) have \( k \) preimages. Then points in \( f(B \cap O_1) \) have \( k-1 \) preimages. \( \square \)

Thus, \( S \setminus C \) is the disjoint union of \( R := L_k \) ("regular part") and \( P := L_{k-1} \) ("singular part"), and these are, moreover, the connected components of \( S \setminus C \).

We are now in a position to extend \( f|S^o \) to a covering. To obtain a map to \( S \) with constant number of preimages we "add in" the missing preimage of points in \( P \) by gluing a copy of \( P \) onto \( S^o \) along \( C \) and mapping that copy to \( P \). That way we obtain a \( k \)-sheeted covering \( F : \Sigma \to S \), where \( \Sigma := S \setminus C \).

To allow some flexibility later and to avert confusion now we do not literally use \( P \) itself:

**Proposition 9.** Let \( h : \hat{P} \to P \cup C \) be a homeomorphism and represent \( \Sigma = S \setminus P \) as the identification space \( S^o \cup \hat{P} \) with the relation \( x \sim y \iff f(x) = h(y) \in C \) for \( x \in \Gamma \) and \( y \in \partial \hat{P} \). Then

\[
F : \Sigma \to S, \quad x \mapsto \begin{cases} 
  f(x) & \text{if } x \in S^o \\
  h(x) & \text{if } x \in \hat{P}
\end{cases}
\]

is a covering map.

**Proof.** \( F \) is well defined by construction and continuous as well as locally injective (this only requires checking on the common boundary, where it is clear). Therefore it is a covering (invariance of domain), clearly of degree \( k \). \( \square \)

**Remark 10.** If \( P \) is a disk, then Theorem 1 follows because this implies that \( \Sigma = S \), so \( F \) is as claimed. To be specific, take \( \hat{P} = B \) and \( h \) such that \( h|\Gamma = f|\Gamma \); then the construction above gives \( F : S \to S \). Thus, we have proved Theorem 3.

To see that \( P \) is a disk if we assume [A1], we use that this construction puts us in a position to apply the Euler characteristic formula (1). Together with \( \chi(\cdot) = 2-2g(\cdot) \) it gives

\[
2 - 2g(S) - 2g(P) = 2 - 2g(\Sigma) = k\chi(S) = k(2 - 2g(S))
\]

or

\[
g(P) = (k-1)(g(S) - 1), \tag{2}
\]

which implies that \( P \) is a disk if \( S = \mathbb{T}^2 \) or \( k = 1 \). More generally, rewrite (2) as

\[
g(P) = (k-1)(g(R) + g(P) - 1)
\]

to get

\[
(k-1)g(R) + (k-2)g(P) = k - 1 \tag{3}
\]

and hence the following:
Proposition 11. If $k > 1$ is as in Proposition 8, then (1) becomes
\[ g(R) + \frac{k-2}{k-1} g(P) = 1, \] (4)
so $g(R) \leq 1$ and hence we have one of the following:

1) $g(R) = 1$ and $P$ is a disk (hence $S = T^2$).
2) $g(R) = 1$ and $k = 2$.
3) $g(R) = 0$ and $\frac{k-1}{k-2} = g(P) \in \mathbb{Z}$, so $k = 3$ (and $g(S) = g(P) = 2$).

Remark 12. 
1. The purpose of assumption [A1] is to rule out the second and third cases, so $P$ is a disk and Theorem 1 is now proved.
2. It seems that the third scenario is impossible—a triple-covering of a genus-2 surface by a genus-4 surface containing a circle separating the genus-4 surface into two “halves” that is being homeomorphically mapped to a circle bounding a disk. A proof of this would allow us to relax assumption [A1] from “more than” to “at least.”
3. Although we were unable to prove it, it seems that the double cover in the second case has to be of such a nature as to be incompatible with $f$ being expanding, so in that case it should be possible to dispense with assumption [A1] (since “expanding” implies $k > 1$).
4. Assuming that $S = T^2$ would rule out the third possibility in Proposition 11 and thus imply that $g(R) = 1$ and hence $P$ is a disk, making the assumption on $k$ unnneeded. This gives Theorem 4 but with assumption [A2] rather than merely [A2’].

3. Generalizations: An embedded circle versus embedding a circle

We now replace assumption [A2] in Theorem 1 by assumption [A2’] in Theorem 4, which implies that $f|_\Gamma$ is a covering of $C$. Our approach mirrors that in the proof of Theorem 1 in that we first extend $f$ to a covering map (a ramified one in this case, alas) and then obtain the formula (6) analogous to (4). Since a ramified cover has a singularity, this does not lead to broader results of interest, and (6) implies that $P$ is not a disk (Remark 17.4), so the ramified covering remains auxiliary. Accordingly, we give replacements for [A1] which together with [A2’] yield the main theorem. The most prominent result of this is Theorem 4, but some of those appearing on page 8 may be of interest.

Recalling now that Proposition 8 was the first occasion on which assumption [A2] was used, we note that its proof can be adapted to show

Proposition 13. If assumption [A2’] holds and $y \in C$, then there are arbitrarily small disks $D$ around $y$ with $I(D) = \{k, k-l\}$ for some $k, l \in \mathbb{N}$.

Here, $l$ is necessarily the degree of $f|_\Gamma : \Gamma \rightarrow C$. Then the connected components of $S \setminus C$ are $R := L_k$ and $P := L_{k-l}$, i.e., the points of $P$ have $l$ fewer preimages than those in $R$. (In particular, $f|_{S^o}$ is surjective if and only if $l < k$.) We now wish to glue $l$ copies of $P$ into $S^o$.

To that end, note that a disk admits a $l$-sheeted ramified self-covering (given by $(r, \theta) \mapsto (r, l\theta)$ in polar coordinates), so gluing $l$ copies of $P$ symmetrically onto a disk gives an $l$-fold ramified covering of $P$ with a single ramification point whose boundary map is an $l$-fold covering of a circle by a circle. As in Proposition 9 this
gives an extension of \( f \) to a \( k \)-sheeted ramified covering of \( S \) by \( \Sigma := S \# P \# \cdots \# P \). 

The \textit{Riemann–Hurwitz formula} then is 
\[
\chi(\Sigma) = k\chi(S) - (l - 1).
\] (5)

We immediately note that since \( \chi(\Sigma) \) and \( \chi(S) \) are even,

**Proposition 14.** \( l \) is odd.

We now derive a counterpart to (4).

**Proposition 15.** If \( k > 1 \) then 
\[
g(R) + \frac{k - 1 - l}{k - 1}g(P) = 1 - \frac{1}{2} \frac{l - 1}{k - 1} \in (0, 1].
\] (6)

**Remark 16.** Note that for \( l = 1 \) this is (4).

**Proof.** \( g(S) = g(R) + g(P) \) and \( g(\Sigma) = g(S) + l \cdot g(P) = g(R) + (l + 1)g(P) \), so 
\[
2 - 2g(R) - 2(l + 1)g(P) = k(2 - 2g(R) - 2g(P)) - (l - 1),
\]
and 
\[
2(k - 1)g(R) + 2(k - l - 1)g(P) = 2(k - 1) - (l - 1) \leq 2(k - 1).
\] (7) \( \square \)

This immediately gives Theorem 4:

**Proof of Theorem 4.** If \( g(S) = 1 \), then either \( g(R) = 1 \) and \( g(P) = 0 \), so \( l = 1 \) by (7) or \( g(R) = 0 \) and \( g(P) = 1 \), in which case (7) implies \( l = -1 \), which is impossible. Thus, our construction proceeds as in the proof of Theorem 1. \( \square \)

**Remark 17.** We list a few additional consequences of (6) (or (7)).

1. \( g(R) \leq 1 \) unless \( l = k \), i.e., unless \( f \mid_{S^c} \) is not surjective.
2. Put differently, if \( f \mid_{S^c} \) is surjective, then \( g(R) \leq 1 \). (See also items 11 and 12 below.)
3. If \( k = 1 \) then \( l = 1 \) and hence \( g(P) = 0 \) from (7) (or (3)).
4. If \( g(P) = 0 \) then \( l = 1 \) because either \( k = 1 \) or else (6) gives \( l = 1 \).
5. If \( l = k - 1 \) then (7) becomes 
\[
2(k - 1)g(R) = 2(k - 2) \quad \text{or} \quad g(R) = k - 1,
\]
and 
\[
2(k - 1)g(R) + 2(k - l - 1)g(P) = 2(k - 1) - (l - 1) \leq 2(k - 1).
\] (7) \( \square \)

6. If \( l > 1 \) then (6) implies that \( g(R) = 0 \Leftrightarrow l < k - 1 \).
7. If \( g(R) > 0 \) and \( l > 1 \), then \( l = k \) by 5 and 6 (and hence \( f \mid_{S^c} \) is not surjective, and \( k \) is odd by Proposition 14).
8. The contrapositive of 7 is that if \( g(R) > 0 \) and \( l \neq k \) then \( l = 1 \).
9. If \( l = k \) (so \( f \mid_{S^c} \) is not surjective) then (7) becomes 
\[
2(k - 1)g(R) - 2g(P) = k - 1,
\]
and then \( k > 1 \) implies \( g(R) > 0 \) and \( g(P) = \frac{k - 1}{2} (2g(R) - 1) > 0 \), so 
\[
2 \leq g(S) = kg(R) = \frac{k - 1}{2}.
\]
10. If \( 1 < l < k \), then \( l < k - 1 \) by 5, so \( g(R) = 0 \) by 6.
11. If \( g(R) = 0 \), then \( l \neq k - 1 \) by 5, and \( g(S) = g(P) \). Thus (7) becomes 
\[
2(k - l - 1)g(S) = 2(k - 1) - (l - 1) = k + (k - l - 1),
\]
so 
\[
2g(S) - 1 = \frac{k}{k - l - 1} \neq 1.
\] (This gives a laborious alternate proof of Theorem 4: if \( g(S) = 1 \), then \( l \in \{1, k\} \) by 10 and 11, but also \( l = k \Rightarrow k = 1 \) by 9.)
12. If \( g(R) = 1 \), then (7) implies that either \( l = 1 \) (and \( g(P) = 0 \), or \( k = 2 \), so we are in the context of Theorem 1) or \( k = l > 1 \) (hence \( f|_{S^0} \) is not surjective) and \( 2 \leq g(S) = \frac{k+1}{2} \) by 9.

Items 10 and 11 of Remark 17 give a slightly odd extension of Theorem 1:

**Theorem 18.** Theorem 1 holds when we assume \([A2']\) from Theorem 4 instead of \([A2]\) in Theorem 1 and replace \([A1]\) by the assumption that every point has at least 3 preimages in \( S^0 \) and the maximum number of such preimages is a prime.

**Proof.** With the usual terminology, if \( l = 1 \), then the second assumption implies that there is a point with more than 3 preimages, and the Main Theorem applies. We cannot have \( l > 1 \) because the assumption on preimages implies \( k - l \geq 3 \), so \( \frac{k}{k-l-1} \notin \mathbb{Z} \), contrary to 10 and 11. \( \square \)

**Remark 19.** Instead of assuming that \( k \) (the maximum number of preimages) is prime one could here assume that \( 2g(S) - 1 \) and \( k - l - 1 \) (1 less than the minimum number of preimages) do not both divide \( k \) (or, to state the precise contrapositive of 11, that their product is not \( k \)).

In light of item 2 in Remark 12, we now assume that \( g(R) > 0 \).

**Theorem 20.** Theorem 1 holds if \( g(R) > 0 \) and

- we assume \([A2']\) from Theorem 4 instead of \([A2]\) in Theorem 1,
- we replace \([A1]\) by the stronger assumption that the maximum number of preimages in \( S^0 \) of any point is even and exceeds 2.

**Proof.** \( k \) even \( \Rightarrow l \neq k \), so 8 in Remark 17 \( \Rightarrow l = 1 \), and Theorem 1 applies. \( \square \)

Since our motivation yields expanding maps, we explore the assumption that \( f|_{S^0} \) is surjective. In this case \( l \neq k \), so the preceding argument also proves

**Theorem 21.** Theorem 1 holds if \( g(R) > 0 \), \( f|_{S^0} \) is surjective, and

- we assume \([A2']\) from Theorem 4 instead of \([A2]\) in Theorem 1.

**Theorem 22.** Theorem 1 holds if \( g(R) > 0 \), \( f|_{S^0} \) is surjective, and

- we replace \([A1]\) by the weaker assumption that there is a point with more than 2 preimages in \( S^0 \),
- we assume \([A2']\) from Theorem 4 instead of \([A2]\) in Theorem 1.

**Proof.** Items 2 and 12 of Remark 17 imply \( l = 1 \), and \( g(R) > 0 \) also excludes the last case in Proposition 11, so the proof of Theorem 1 goes through. \( \square \)

**References**

[1] C. Bonatti, L. J. Díaz and M. Viana, *Dynamics Beyond Uniform Hyperbolicity. A Global Geometric and Probabilistic Perspective*, Encyclopaedia of Mathematical Sciences, 102, Mathematical Physics, III, Springer-Verlag, Berlin, 2005. MR 2105774

[2] C. Bonatti, A. Pumariño and M. Viana, *Lorenz attractors with arbitrary expanding dimension*, C. R. Acad. Paris Sér. I Math., 325 (1997), 883–888. MR 1485910

[3] R. J. Daverman and G. A. Venema, *Embeddings in Manifolds*, Graduate Studies in Mathematics, 106, American Mathematical Society, Providence, RI, 2009. MR 2561389

[4] B. Hasselblatt and A. Katok, *A First Course in Dynamics. With a Panorama of Recent Developments*, Cambridge University Press, New York, 2003. MR 1995704

[5] C. A. Morales and M. J. Pacifico, *Strange attractors arising from hyperbolic flows*, preprint.

[6] C. A. Morales, M. J. Pacifico and E. R. Pujals, *Global attractors from the explosion of singular cycles*, C. R. Acad. Sci. Paris Sér. I Math., 325 (1997), 1317–1322. MR 1490422
[7] C. A. Morales, M. J. Pacifico and E. R. Pujals, Singular Hyperbolic Systems, Proc. Amer. Math. Soc., 127 (1999), 3393–3401. MR 1610761
[8] R. Metzger and C. Morales, Sectional-Hyperbolic Systems, Ergodic Theory Dynam. Systems, 28 (2008), 1587–1597. MR 2449545
[9] S. Newhouse, On simple arcs between structurally stable flows, in Dynamical Systems Warwick 1974 (Proc. Sympos. Appl. Topology and Dynamical Systems, Univ. Warwick, Coventry, 1973/1974; presented to E. C. Zeeman on his fiftieth birthday), Lecture Notes in Math., Vol. 468, Springer, Berlin, 1975, 209–233. MR 0650638
[10] S. Smale, Differentiable dynamical systems, Bull. Amer. Math. Soc., 73 (1967), 747–817. MR 0228014

E-mail address: Erica.Waite@tufts.edu

Department of Mathematics, Tufts University, Medford, MA 02155, USA

E-mail address: Boris.Hasselblatt@tufts.edu

Department of Mathematics, Tufts University, Medford, MA 02155, USA

E-mail address: enrique@impa.br

IMPA, Estrada Dona Castorina 110, Rio de Janeiro, Brasil 22460-320