Estimates for the higher order buckling eigenvalues in the unit sphere

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Abstract. We consider the higher order buckling eigenvalues of the following Dirichlet poly-Laplacian in the unit sphere \((-\Delta)^p u = \Lambda(-\Delta)u\) with order \(p \geq 2\). We obtain universal bounds on the \((k+1)\)th eigenvalue in terms of the first \(k\)th eigenvalues independent of the domains. In particular, for \(p = 2\), our result is sharper than estimates on eigenvalues of the buckling problem obtained by Wang and Xia in [19].

Keywords: eigenvalue, poly-Laplacian, buckling problem, unit sphere.
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1 Introduction

Let \(\Omega\) be a connected bounded domain in an \(n\)-dimensional complete Riemannian manifold \(M\).

Assume that \(\lambda_i\) is the \(i\)th eigenvalue of the Dirichlet poly-Laplacian with order \(p\):

\[
\begin{align*}
\begin{cases}
(-\Delta)^p u = \lambda u & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = \cdots = \frac{\partial^{p-1} u}{\partial \nu^{p-1}} = 0 & \text{on } \partial \Omega,
\end{cases}
\end{align*}
\]

(1.1)

where \(\Delta\) is the Laplacian in \(M\) and \(\nu\) denotes the outward unit normal vector field of \(\partial \Omega\). Let \(0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to +\infty\) denote the successive eigenvalues for (1.1), where each eigenvalue is repeated according to its multiplicity. When \(p = 1\), it is well known that the eigenvalue problem (1.1) is called a fixed membrane problem and it is called a clamped plate problem when \(p = 2\). For any \(p\) and \(M = \mathbb{R}^n\), Cheng-Ichikawa-Mametsuka proved in [5] the following inequality of the type of Yang:

\[
\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4p(2p + n - 2)}{n^2} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)\lambda_i.
\]

(1.2)

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In particular, when \( p = 1 \), the inequality \((1.2)\) becomes the following inequality of Yang in [22]:

\[
\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \lambda_i.
\]

In an excellent paper of Cheng-Ichikawa-Mametsuka [4], by introducing functions \( a_i \) and \( b_i \), they considered the eigenvalue problem \((1.1)\) with any order \( p \) and \( M = S^n(1) \). They proved that

\[
\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n^2} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \left\{ \left( \frac{1}{\lambda_i^p} + n \right)^p - \lambda_i \right\} + 4 \left( 2^p - (p + 1) \right) \lambda_i^p \left( \frac{1}{\lambda_i^p} + n \right)^{p-2} \left( \frac{\lambda_i^p + n^2}{4} \right).
\]

We remark that the inequality (2.19) in [6] of Cheng-Yang and inequality (4.16) in [18] of Wang-Xia are included in the inequality (1.3). For the related research and important improvement in eigenvalue problem \((1.1)\), we refer to [1–3, 7, 8, 10, 11, 14–17, 20, 21] and the references therein.

Now assume that \( \Lambda_i \) is the \( i \)th eigenvalue of the following Dirichlet poly-Laplacian with order \( p \) (\( \geq 2 \)):

\[
\begin{aligned}
(-\Delta)^p u &= \Lambda (-\Delta) u & \text{in } \Omega, \\
u &= \frac{\partial u}{\partial \nu} = \cdots = \frac{\partial^{p-1} u}{\partial \nu^{p-1}} = 0 & \text{on } \partial \Omega.
\end{aligned}
\]

It is well known that this problem has a discrete spectrum \( 0 < \Lambda_1 \leq \Lambda_2 \leq \Lambda_3 \leq \cdots \rightarrow +\infty \), where each eigenvalue is repeated according to its multiplicity. When \( p = 2 \), the eigenvalue problem \((1.4)\) is called a buckling problem. By introducing a new method to construct nice trial functions, Cheng-Yang obtained in [9] that, for \( p = 2 \) and \( M = \mathbb{R}^n \),

\[
\sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \leq \frac{4(n + 2)}{n^2} \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i) \Lambda_i.
\]

As a generalization of inequality \((1.5)\), Huang-Li [12] considered the problem \((1.4)\) with any order \( p \). In fact, for \( M = \mathbb{R}^n \), they proved that

\[
\sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \leq \frac{4(p - 1)(n + 2p - 2)}{n^2} \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i) \frac{2p-2}{\Lambda_i^{p-1}}.
\]

In 2007, Wang and Xia [19] considered this problem when \( p = 2 \) and \( M = S^n(1) \). They proved that, for any \( \delta > 0 \),

\[
2 \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \leq \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \left( \delta \Lambda_i + \frac{\delta^2(\Lambda_i - (n-2))}{4(\delta \Lambda_i + n - 2)} \right)
+ \frac{1}{\delta} \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i) \left( \Lambda_i + \frac{(n-2)^2}{4} \right).
\]
We remark that the right hand side of inequality (1.7) depends on $\delta$. In a recent paper, by introducing a new parameter and using Cauchy inequality, Huang-Li-Cao [13] obtain the following stronger inequality than (1.7) which is independent of $\delta$:

$$
\sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \left( 2 + \frac{n-2}{\Lambda_i - (n-2)} \right) 
\leq 2 \left\{ \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \left( \Lambda_i - \frac{n-2}{\Lambda_i - (n-2)} \right) \right\}^{\frac{1}{2}} 
\times \left\{ \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i) \left( \Lambda_i + \frac{(n-2)^2}{4} \right) \right\}^{\frac{1}{2}},
$$

(1.8)

Motivated by the idea used in [4], we consider in this paper the eigenvalue problem (1.4) for any integer $p \geq 2$ when $M$ is $S^n(1)$. We obtain the following results:

**Theorem 1.1.** Let $\Omega$ be a connected bounded domain in an $n$-dimensional unit sphere $S^n(1)$. Assume that $\Lambda_i$ is the $i$th eigenvalue of the eigenvalue problem (1.4) with $p \geq 2$. Then, we have

$$
\sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \left( 2 + \frac{n-2}{\Lambda_i - (n-2)} \right) 
\leq 2 \left\{ \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \left( f(\Lambda_i, n) - \frac{\Lambda_i}{\Lambda_i - (n-2)} \right) \right\}^{\frac{1}{2}} 
\times \left\{ \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i) \left( \Lambda_i + \frac{(n-2)^2}{4} \right) \right\}^{\frac{1}{2}},
$$

(1.9)

where

$$
f(\Lambda_i, n) = \frac{1}{2(n-1)} \left( \left( \Lambda_i^{\frac{p-1}{p}} + n \right)^{p-1} - \left( \Lambda_i^{\frac{p-1}{p}} - n + 2 \right)^{p-1} \right) 
+ \frac{n}{(n-1)\Lambda_i^{\frac{p-1}{p}}} \left( \Lambda_i^{\frac{1}{p}} + n \right)^{p-2} - \frac{1}{n-1} \Lambda_i^{\frac{p-3}{p}} \left( \Lambda_i^{\frac{1}{p}} - n + 2 \right)^{p-2} 
+ 2 (2^{p-1} - p) \Lambda_i^{\frac{1}{p}} \left( \Lambda_i^{\frac{1}{p}} + n \right)^{p-3} 
+ 4 (2^{p-2} - (p-1)) \Lambda_i^{\frac{2}{p-1}} \left( \Lambda_i^{\frac{1}{p-1}} + n \right)^{p-4}.
$$

**Corollary 1.2.** Under the assumptions of Theorem 1.1, we have

$$
\sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \leq \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i) \left( f(\Lambda_i, n) - \frac{\Lambda_i}{\Lambda_i - (n-2)} \right) 
\times \left( \Lambda_i + \frac{(n-2)^2}{4} \right),
$$

(1.10)
Estimates for the higher order buckling eigenvalues

\[ \Lambda_{k+1} \leq S_{k+1} + \sqrt{S_{k+1}^2 - T_{k+1}}, \]  
(1.11)

and

\[ \Lambda_{k+1} - \Lambda_k \leq 2\sqrt{S_{k+1}^2 - T_{k+1}}, \]  
(1.12)

where

\[ S_{k+1} = \frac{1}{k} \sum_{i=1}^{k} \Lambda_i + \frac{1}{2k} \sum_{i=1}^{k} \left( f(\Lambda_i, n) - \frac{\Lambda_i}{\Lambda_i - (n-2)} \right) \left( \Lambda_i + \frac{(n-2)^2}{4} \right), \]

\[ T_{k+1} = \frac{1}{k} \sum_{i=1}^{k} \Lambda_i^2 + \frac{1}{k} \sum_{i=1}^{k} \Lambda_i \left( f(\Lambda_i, n) - \frac{\Lambda_i}{\Lambda_i - (n-2)} \right) \left( \Lambda_i + \frac{(n-2)^2}{4} \right). \]

Remark 1.1. When \( p = 2 \), we have \( f(\Lambda_i, n) = \Lambda_i + 1 \) and

\[ f(\Lambda_i, n) - \frac{\Lambda_i}{\Lambda_i - (n-2)} = \Lambda_i - \frac{n-2}{\Lambda_i - (n-2)}. \]

Hence, for \( p = 2 \), our inequality (1.9) becomes the inequality (1.8) of Huang-Li-Cao. Moreover, the inequality (1.9) is sharp than the inequality (1.7) of Wang and Xia in [19].

2 Proof of the main theorem

Let \( u_i \) be the \( i \)th orthonormal eigenfunction of the problem (1.4) corresponding to the eigenvalue \( \Lambda_i \), that is, \( u_i \) satisfies

\[ \begin{cases} (-\Delta)^p u_i = \Lambda_i (-\Delta) u_i & \text{in } \Omega, \\ u_i = \frac{\partial u_i}{\partial \nu} = \cdots = \frac{\partial^{p-1} u_i}{\partial \nu^{p-1}} = 0 & \text{on } \partial \Omega, \\ \int_{\Omega} \langle \nabla u_i, \nabla u_j \rangle = \delta_{ij}. \end{cases} \]

(2.1)

Let \( x_1, x_2, \ldots, x_{n+1} \) be the standard Euclidean coordinate functions of \( \mathbb{R}^{n+1} \). Then the unit sphere is defined by

\[ S^n(1) = \left\{ (x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{a=1}^{n+1} x_a^2 = 1 \right\}. \]

Then by a rather long computation and a careful analysis, we are able to derive a sequence of inequalities which can be successfully used to prove the following key proposition of the present paper:
Proposition 2.1.

\[
\sum_{\alpha=1}^{n+1} \int_{\Omega} ((\nabla x_{\alpha}, \nabla u_i) + x_{\alpha} \Delta u_i) (-\Delta)^{p-2} ((\nabla x_{\alpha}, \nabla u_i) + x_{\alpha} \Delta u_i) \\
\leq \frac{1}{2(n-1)} \left( \left( \Lambda^\frac{1}{p-1} + n \right)^{p-1} - \left( \Lambda^\frac{1}{p-1} - n + 2 \right)^{p-1} \right) \\
+ \frac{n}{(n-1)} \Lambda^\frac{1}{p-1} \left( \Lambda^\frac{1}{p-1} + n \right)^{p-2} - \frac{1}{n-1} \Lambda^\frac{1}{p-1} \left( \Lambda^\frac{1}{p-1} - n + 2 \right)^{p-2} \\
+ 2 \left( 2^{p-1} - p \right) \Lambda^\frac{1}{p-1} \left( \Lambda^\frac{1}{p-1} + n \right)^{p-3} \\
+ 4(2^{p-2} - (p-1)) \Lambda^\frac{2}{p-1} \left( \Lambda^\frac{1}{p-1} + n \right)^{p-4}.
\] 

(2.2)

We should remark that the main idea in proving Proposition 2.1 is similar to that in reference [4]. However, here in our case, it seems a little more complicated than in the case they considered.

For functions \( f \) and \( g \) defined on \( \Omega \), we define the Dirichlet inner product \((f, g)_D\) by

\[
(f, g)_D = \int_{\Omega} (\nabla f, \nabla g)
\]

and the Dirichlet norm of \( f \) by

\[
\|f\|_D = ((f, g)_D)^{1/2} = \left( \int_{\Omega} |\nabla f|^2 \right)^{1/2}.
\]

Define \( H^2_p(\Omega) \) by

\[
H^2_p(\Omega) = \{ f : f, |\nabla f|, \ldots, |\nabla^p f| \in L^2(\Omega) \},
\]

where

\[
|\nabla^p f|^2 = \sum_{i_1, \ldots, i_p=1}^{n} |\nabla_{i_1} \nabla_{i_2} \cdots \nabla_{i_p} f|^2.
\]

Then \( H^2_p(\Omega) \) is a Hilbert space with respect to the norm \( \| \cdot \|_p \):

\[
\|f\|_p = \left( \int_{\Omega} \left( f^2 + |\nabla f|^2 + \cdots + |\nabla^p f|^2 \right) \right)^{1/2}.
\]

Consider the subspace \( H^2_{p,D}(\Omega) \) of \( H^2_p(\Omega) \) defined by

\[
H^2_{p,D}(\Omega) = \left\{ f \in H^2_p(\Omega) : f = \frac{\partial f}{\partial n} = \cdots = \frac{\partial^{p-1} f}{\partial n^{p-1}} = 0 \text{ on } \partial \Omega \right\}.
\]

Then the operator \((-\Delta)^p\) defines a self-adjoint operator acting on \( H^2_{p,D}(\Omega) \) for the eigenvalue problem \((1.4)\) and eigenfunctions \( \{u_i\}_{i=1}^{\infty} \) defined in \((2.1)\) form a complete orthonormal basis for the Hilbert space \( H^2_{p,D}(\Omega) \). For vector-valued functions

\[
F = (f_1, f_2, \ldots, f_{n+1}), \quad G = (g_1, g_2, \ldots, g_{n+1}) : \Omega \to \mathbb{R}^{n+1},
\]
we define the inner product \((F, G)\) by

\[
(F, G) = \int_{\Omega} \langle F, G \rangle = \int_{\Omega} \sum_{\alpha=1}^{n+1} f_\alpha g_\alpha.
\]

The norm of \(F\) is given by

\[
\|F\| = (F, F)^{\frac{1}{2}} = \left( \int_{\Omega} \sum_{\alpha=1}^{n+1} f_\alpha g_\alpha \right)^{\frac{1}{2}}.
\]

Let \(H^2_{p-1}(\Omega)\) be the Hilbert space of vector-valued functions given by

\[
H^2_{p-1}(\Omega) = \left\{ F = (f_1, f_2, \ldots, f_{n+1}) : f_\alpha, |\nabla f_\alpha|, \ldots, |\nabla^{p-1} f_\alpha| \in L^2(\Omega), \right. \\
\text{for } \alpha = 1, \ldots, n+1 \}
\]

with norm

\[
\|F\|_{p-1} = \left\{ \|F\|^2 + \int_{\Omega} \left( \sum_{\alpha=1}^{n+1} |\nabla f_\alpha|^2 + \cdots + \sum_{\alpha=1}^{n+1} |\nabla^{p-1} f_\alpha|^2 \right) \right\}^{\frac{1}{2}}.
\]

Observe that a vector field on \(\Omega\) can be regarded as a vector-valued function from \(\Omega\) to \(\mathbb{R}^{n+1}\). Let \(H^2_{p-1,D}(\Omega)\) be a subspace of \(H^2_{p-1}(\Omega)\) spanned by the vector-valued functions \(\{\nabla u_i\}_{i=1}^{\infty}\) which form a complete orthonormal basis of \(H^2_{p-1,D}(\Omega)\). For any \(f \in H^2_{p,D}(\Omega)\), we have \(\nabla f \in H^2_{p-1,D}(\Omega)\) and for any \(X \in H^2_{p-1,D}(\Omega)\), there exists a function \(f \in H^2_{p,D}(\Omega)\) such that \(X = \nabla f\).

Let \(x_1, x_2, \ldots, x_{n+1}\) be the standard Euclidean coordinate functions of \(\mathbb{R}^{n+1}\), and \(u_i\) be the \(i\)-th orthonormal eigenfunction of the problem \((1.4)\) corresponding to the eigenvalue \(\Lambda_i\) (see \((2.1)\)). For any \(\alpha = 1, 2, \ldots, n+1\) and each \(i = 1, \ldots, k\), we decompose the vector-valued functions \(x_\alpha \nabla u_i\) as

\[
x_\alpha \nabla u_i = \nabla h_{\alpha i} + W_{\alpha i}, \tag{2.3}
\]

where \(h_{\alpha i} \in H^2_{p,D}(\Omega)\), \(\nabla h_{\alpha i}\) is the projection of \(x_\alpha \nabla u_i\) in \(H^2_{p-1,D}(\Omega)\), \(W_{\alpha i} \perp H^2_{p-1,D}(\Omega)\). Thus we have

\[
(W_{\alpha i}, \nabla u) = \int_{\Omega} \langle W_{\alpha i}, \nabla u \rangle = 0, \text{ for any } u \in H^2_{p,D}(\Omega). \tag{2.4}
\]

By the denseness of \(H^2_{p,D}(\Omega)\) in \(L^2(\Omega)\) and \(C^1(\Omega)\) is dense in \(L^2(\Omega)\), we conclude that

\[
(W_{\alpha i}, \nabla h) = 0, \forall h \in C^1(\Omega) \cap L^2(\Omega), \tag{2.5}
\]

which implies from the divergence theorem that

\[
\int_{\Omega} h \text{ div}(W_{\alpha i}) = 0,
\]

where \(\text{div}(Z)\) denotes the divergence of \(Z\). Consequently, we get

\[
\text{div}(W_{\alpha i}) = 0. \tag{2.6}
\]
Define $\phi_{\alpha i}$ by

$$
\phi_{\alpha i} = h_{\alpha i} - \sum_{j=1}^{k} b_{\alpha ij} u_j, 
$$

(2.7)

where

$$
b_{\alpha ij} = \int_{\Omega} x_{\alpha} \langle \nabla u_i, \nabla u_j \rangle = b_{\alpha ji}.
$$

Then we have

$$
\phi_{\alpha i} = \partial \phi_{\alpha i} / \partial \nu = \cdots = \partial^{p-1} \phi_{\alpha i} / \partial \nu^{p-1} = 0
$$

and

$$
(\phi_{\alpha i}, u_j)_D = \int_{\Omega} \langle \nabla \phi_{\alpha i}, \nabla u_j \rangle = 0, \quad \text{for any } j = 1, \ldots, k. 
$$

(2.8)

It follows from the Rayleigh-Ritz inequality that

$$
\Lambda_{k+1} \leq \int_{\Omega} \phi_{\alpha i} (-\Delta)^p \phi_{\alpha i} \frac{\|\nabla \phi_{\alpha i}\|^2}{\|x_{\alpha} \nabla u_i\|^2},
$$

(2.9)

where $\|f\|^2 = \int_{\Omega} |f|^2$. It is easy to see from (2.7) and (2.8) that

$$
\int_{\Omega} \phi_{\alpha i} (-\Delta)^p \phi_{\alpha i} = \int_{\Omega} \phi_{\alpha i} \left( (-\Delta)^p h_{\alpha i} - \sum_{j=1}^{k} b_{\alpha ij} \Lambda_j (-\Delta) u_j \right)
$$

$$
= \int_{\Omega} \phi_{\alpha i} (-\Delta)^p h_{\alpha i}
$$

$$
= \int_{\Omega} \left( h_{\alpha i} - \sum_{j=1}^{k} b_{\alpha ij} u_j \right) (-\Delta)^p h_{\alpha i}
$$

$$
= \int_{\Omega} h_{\alpha i} (-\Delta)^p h_{\alpha i} - \sum_{j=1}^{k} b_{\alpha ij} \int_{\Omega} u_j (-\Delta)^p h_{\alpha i}
$$

$$
= \int_{\Omega} h_{\alpha i} (-\Delta)^p h_{\alpha i} - \sum_{j=1}^{k} b_{\alpha ij} \int_{\Omega} h_{\alpha i} (-\Delta)^p u_j
$$

$$
= \int_{\Omega} h_{\alpha i} (-\Delta)^p h_{\alpha i} - \sum_{j=1}^{k} \Lambda_j b_{\alpha ij}^2.
$$

(2.10)

Since

$$
\|x_{\alpha} \nabla u_i\|^2 = \int_{\Omega} x_{\alpha}^2 |\nabla u_i|^2 = \|\nabla h_{\alpha i}\|^2 + \|W_{\alpha i}\|^2,
$$

(2.11)

$$
\|\nabla h_{\alpha i}\|^2 = \|\nabla \Phi_{\alpha i}\|^2 + \sum_{j=1}^{k} b_{\alpha ij}^2.
$$

(2.12)

Therefore, (2.10) can be written as

$$
\int_{\Omega} \phi_{\alpha i} (-\Delta)^p \phi_{\alpha i} = \int_{\Omega} h_{\alpha i} (-\Delta)^p h_{\alpha i} - \Lambda_i \|x_{\alpha} \nabla u_i\|^2
$$

$$
+ \Lambda_i \left( \|\nabla \phi_{\alpha i}\|^2 + \|W_{\alpha i}\|^2 + \sum_{j=1}^{k} b_{\alpha ij}^2 \right) - \sum_{j=1}^{k} \Lambda_j b_{\alpha ij}^2.
$$

(2.13)
Inserting (2.13) into (2.9) yields

\[
(\Lambda_{k+1} - \Lambda_i) \| \nabla \Phi_{\alpha_i} \|^2 \leq \int_\Omega h_{\alpha i} (-\Delta)^p h_{\alpha i} - \Lambda_i \| x_\alpha \nabla u_i \|^2 + \Lambda_i \| W_{\alpha i} \|^2 + \sum_{j=1}^k (\Lambda_i - \Lambda_j) b_{\alpha ij}^2
\]

\[= p_{\alpha i} + \| \langle \nabla x_\alpha, \nabla u_i \rangle \|^2 + \Lambda_i \| W_{\alpha i} \|^2 + \sum_{j=1}^k (\Lambda_i - \Lambda_j) b_{\alpha ij}^2, \tag{2.14}\]

where

\[
p_{\alpha i} = \int_\Omega h_{\alpha i} (-\Delta)^p h_{\alpha i} - \Lambda_i \| x_\alpha \nabla u_i \|^2 - \| \langle \nabla x_\alpha, \nabla u_i \rangle \|^2.
\]

**Lemma 2.1.** [18] Let

\[
c_{\alpha ij} = \int_\Omega \langle Z_{\alpha i}, u_j \rangle,
\]

where \(Z_{\alpha i} = \nabla \langle x_\alpha, \nabla u_i \rangle - \frac{n-2}{2} x_\alpha \nabla u_i\). Then we have

\[
c_{\alpha ij} = -c_{\alpha ji}.
\]

Note that

\[
-2 \int_\Omega \langle x_\alpha \nabla u_i, Z_{\alpha i} \rangle = -2 \int_\Omega \langle x_\alpha \nabla u_i, \nabla \langle x_\alpha, \nabla u_i \rangle \rangle + (n - 2) \int_\Omega x_\alpha^2 |\nabla u_i|^2
\]

\[= 2 \int_\Omega \langle x_\alpha, \nabla u_i \rangle^2 + \int_\Omega \langle x_\alpha^2, \nabla u_i \rangle \Delta u_i + (n - 2) \int_\Omega x_\alpha^2 |\nabla u_i|^2. \tag{2.15}\]

On the other hand, from (2.3), (2.5) and (2.8), we obtain

\[
-2 \int_\Omega \langle x_\alpha \nabla u_i, Z_{\alpha i} \rangle = -2 \int_\Omega \langle \nabla h_{\alpha i} + W_{\alpha i}, Z_{\alpha i} \rangle
\]

\[= -2 \int_\Omega \langle \nabla h_{\alpha i}, Z_{\alpha i} \rangle + (n - 2) \int_\Omega \langle W_{\alpha i}, x_\alpha \nabla u_i \rangle
\]

\[= -2 \int_\Omega \langle \nabla \phi_{\alpha i} + \sum_{j=1}^k b_{\alpha ij} \nabla u_j, Z_{\alpha i} \rangle + (n - 2) \int_\Omega \langle W_{\alpha i}, x_\alpha \nabla u_i \rangle
\]

\[= -2 \int_\Omega \langle \nabla \phi_{\alpha i}, Z_{\alpha i} \rangle - 2 \sum_{j=1}^k b_{\alpha ij} c_{\alpha ij} + (n - 2) \| W_{\alpha i} \|^2
\]

\[= -2 \int_\Omega \langle \nabla \phi_{\alpha i}, Z_{\alpha i} \rangle - \sum_{j=1}^k c_{\alpha ij} \nabla u_j - 2 \sum_{j=1}^k b_{\alpha ij} c_{\alpha ij}
\]

\[+ (n - 2) \| W_{\alpha i} \|^2. \tag{2.16}\]
From (2.15) and (2.16), we obtain

\[
 r_{\alpha i} + 2 \sum_{j=1}^{k} b_{\alpha ij} c_{\alpha ij} = -2 \int_{\Omega} \langle \nabla \phi_{\alpha i}, Z_{\alpha i} - \sum_{j=1}^{k} c_{\alpha ij} \nabla u_j \rangle + (n - 2) \| W_{\alpha i} \|^2, \quad (2.17)
\]

where

\[
 r_{\alpha i} = 2 \int_{\Omega} \langle x_{\alpha i}, \nabla u_i \rangle^2 + \int_{\Omega} \langle \nabla x_{\alpha i}, \nabla u_i \rangle \Delta u_i + (n - 2) \int_{\Omega} x_{\alpha i}^2 |\nabla u_i|^2.
\]

Multiplying (2.17) by \((\Lambda_{k+1} - \Lambda_i)^2\), one obtains from the Schwarz inequality and (2.14) that

\[
(\Lambda_{k+1} - \Lambda_i)^2 \left( r_{\alpha i} + 2 \sum_{j=1}^{k} b_{\alpha ij} c_{\alpha ij} \right) \\
= (\Lambda_{k+1} - \Lambda_i)^2 \left( -2 \int_{\Omega} \langle \nabla \phi_{\alpha i}, Z_{\alpha i} - \sum_{j=1}^{k} c_{\alpha ij} \nabla u_j \rangle + (n - 2) \| W_{\alpha i} \|^2 \right) \\
\leq \delta (\Lambda_{k+1} - \Lambda_i)^3 \| \nabla \phi_{\alpha i} \|^2 + \frac{1}{\delta} (\Lambda_{k+1} - \Lambda_i) \left\| Z_{\alpha i} - \sum_{j=1}^{k} c_{\alpha ij} \nabla u_j \right\|^2 \\
+ (n - 2) (\Lambda_{k+1} - \Lambda_i)^2 \| W_{\alpha i} \|^2 \\
\leq \delta (\Lambda_{k+1} - \Lambda_i)^2 \left( p_{\alpha i} + \| \langle \nabla x_{\alpha i}, \nabla u_i \rangle \|^2 + \Lambda_i \| W_{\alpha i} \|^2 + \sum_{j=1}^{k} (\Lambda_i - \Lambda_j) b_{\alpha ij}^2 \right) \\
+ \frac{1}{\delta} (\Lambda_{k+1} - \Lambda_i) \left( \| Z_{\alpha i} \|^2 - \sum_{j=1}^{k} c_{\alpha ij}^2 \right) + (n - 2) (\Lambda_{k+1} - \Lambda_i)^2 \| W_{\alpha i} \|^2. \quad (2.18)
\]

Since \(b_{\alpha ij} = b_{\alpha ji}\) and \(c_{\alpha ij} = -c_{\alpha ji}\), summing over \(i\) from 1 to \(k\) for (2.18) yields

\[
\sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 r_{\alpha i} \\
\leq \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \left( \delta p_{\alpha i} + \delta \| \langle \nabla x_{\alpha i}, \nabla u_i \rangle \|^2 + (\delta \Lambda_i + n - 2) \| W_{\alpha i} \|^2 \right) \\
+ \frac{1}{\delta} \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i) \| Z_{\alpha i} \|^2. \quad (2.19)
\]

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Let $\rho$ be a positive constant. Then we have

\[
\rho\|\langle \nabla x_\alpha, \nabla u_i \rangle\|^2 = \rho \int_\Omega \langle \nabla x_\alpha, \nabla u_i \rangle^2 \\
= -\rho \int_\Omega x_\alpha \text{div}(\langle \nabla x_\alpha, \nabla u_i \rangle \nabla u_i) \\
= -\rho \int_\Omega \langle x_\alpha \nabla u_i, \nabla \langle \nabla x_\alpha, \nabla u_i \rangle \rangle - \rho \int_\Omega \langle \nabla x_\alpha, \nabla u_i \rangle x_\alpha \Delta u_i \\
= -\rho \int_\Omega \langle \nabla h_{\alpha i}, \nabla \langle \nabla x_\alpha, \nabla u_i \rangle \rangle - \frac{\rho}{2} \int_\Omega \langle \nabla x_{\alpha}^2, \nabla u_i \rangle \Delta u_i \\
\leq (\delta \Lambda_i + n - 2)\|\nabla h_{\alpha i}\|^2 + \frac{\rho^2}{4(\delta \Lambda_i + n - 2)}\|\nabla \langle \nabla x_\alpha, \nabla u_i \rangle\|^2 \\
- \frac{\rho}{2} \int_\Omega \langle \nabla x_{\alpha}^2, \nabla u_i \rangle \Delta u_i. \tag{2.20}
\]

Applying (2.20) to (2.19) yields

\[
\sum_{i=1}^{k}(\Lambda_{k+1} - \Lambda_i)^2 r_{\alpha i} \leq \sum_{i=1}^{k}(\Lambda_{k+1} - \Lambda_i)^2 \left(\delta p_{\alpha i} + (\delta \Lambda_i + n - 2)\|W_{\alpha i}\|^2 + (\delta - \rho)\|\langle \nabla x_\alpha, \nabla u_i \rangle\|^2 + \rho\|\langle \nabla x_\alpha, \nabla u_i \rangle\|^2\right) \\
+ \frac{1}{\delta} \sum_{i=1}^{k}(\Lambda_{k+1} - \Lambda_i)\|Z_{\alpha i}\|^2 \\
\leq \sum_{i=1}^{k}(\Lambda_{k+1} - \Lambda_i)^2 \left(\delta p_{\alpha i} + (\delta \Lambda_i + n - 2)(\|W_{\alpha i}\|^2 + \|\nabla h_{\alpha i}\|^2) \\
+ (\delta - \rho)\|\langle \nabla x_\alpha, \nabla u_i \rangle\|^2 + \frac{\rho^2}{4(\delta \Lambda_i + n - 2)}\|\nabla \langle \nabla x_\alpha, \nabla u_i \rangle\|^2 \\
- \frac{\rho}{2} \int_\Omega \langle \nabla x_{\alpha}^2, \nabla u_i \rangle \Delta u_i \right) + \frac{1}{\delta} \sum_{i=1}^{k}(\Lambda_{k+1} - \Lambda_i)\|Z_{\alpha i}\|^2 \\
= \sum_{i=1}^{k}(\Lambda_{k+1} - \Lambda_i)^2 \left(\delta p_{\alpha i} + (\delta \Lambda_i + n - 2)\|x_\alpha \nabla u_i\|^2 \\
+ (\delta - \rho)\|\langle \nabla x_\alpha, \nabla u_i \rangle\|^2 + \frac{\rho^2}{4(\delta \Lambda_i + n - 2)}\|\nabla \langle \nabla x_\alpha, \nabla u_i \rangle\|^2 \\
- \frac{\rho}{2} \int_\Omega \langle \nabla x_{\alpha}^2, \nabla u_i \rangle \Delta u_i \right) + \frac{1}{\delta} \sum_{i=1}^{k}(\Lambda_{k+1} - \Lambda_i)\|Z_{\alpha i}\|^2. \tag{2.21}
\]

Since

\[
\Delta h_{\alpha i} = \text{div}(\nabla h_{\alpha i}) = \text{div}(x_\alpha \nabla u_i) = \langle \nabla x_\alpha, \nabla u_i \rangle + x_\alpha \Delta u_i,
\]
we get from Proposition 2.1 that
\[
\sum_{\alpha=1}^{n+1} p_{\alpha i} = \sum_{\alpha=1}^{n+1} \left( \int_{\Omega} h_{\alpha i}(-\Delta)^p h_{\alpha i} - \Lambda_i \|x_{\alpha} \nabla u_i\|^2 - \|\langle \nabla x_{\alpha}, \nabla u_i \rangle\|^2 \right)
\]
\[
= \sum_{\alpha=1}^{n+1} \int_{\Omega} h_{\alpha i}(-\Delta)^p h_{\alpha i} - (\Lambda_i + 1)
\]
\[
= \sum_{\alpha=1}^{n+1} \int_{\Omega} \left( (\nabla x_{\alpha}, \nabla u_i) + x_{\alpha} \Delta u_i \right) (-\Delta)^p \left( (\nabla x_{\alpha}, \nabla u_i) + x_{\alpha} \Delta u_i \right) - (\Lambda_i + 1)
\]
\[
\leq f(\Lambda_i, n) - (\Lambda_i + 1).
\]

A direct calculation yields (see (2.44), (2.45), (2.46) and (2.47) in [18])
\[
\sum_{\alpha=1}^{n+1} r_{\alpha i} = n,
\]
\[
\sum_{\alpha=1}^{n+1} \|x_{\alpha} \nabla u_i\|^2 = \sum_{\alpha=1}^{n+1} \|\langle \nabla x_{\alpha}, \nabla u_i \rangle\|^2 = 1,
\]
\[
\sum_{\alpha=1}^{n+1} \|\nabla (\nabla x_{\alpha}, \nabla u_i)\|^2 = \Lambda_i - (n - 2),
\]
\[
\sum_{\alpha=1}^{n+1} \|Z_{\alpha i}\|^2 = \Lambda_i + \frac{(n - 2)^2}{4}.
\]

Therefore, summing up (2.21) over \(\alpha\) from 1 to \(n + 1\), one gets
\[
\sum_{i=1}^{k} \left( \Lambda_{k+1} - \Lambda_i \right)^2
\]
\[
\leq \sum_{i=1}^{k} \left( \Lambda_{k+1} - \Lambda_i \right)^2 \left( \delta f(\Lambda_i, n) - (\Lambda_i + 1) \right) + (\delta \Lambda_i + n - 2) + (\delta - \rho)
\]
\[
+ \frac{\rho^2}{4(\delta \Lambda_i + n - 2)} (\Lambda_i - (n - 2)) + \frac{1}{\delta} \sum_{i=1}^{k} \left( \Lambda_{k+1} - \Lambda_i \right) \left( \Lambda_i + \frac{(n - 2)^2}{4} \right).
\]

That is,
\[
2 \sum_{i=1}^{k} \left( \Lambda_{k+1} - \Lambda_i \right)^2
\]
\[
\leq \sum_{i=1}^{k} \left( \Lambda_{k+1} - \Lambda_i \right)^2 \left( \delta f(\Lambda_i, n) - \rho + \frac{\rho^2}{4(\delta \Lambda_i + n - 2)} (\Lambda_i - (n - 2)) \right)
\]
\[
+ \frac{1}{\delta} \sum_{i=1}^{k} \left( \Lambda_{k+1} - \Lambda_i \right) \left( \Lambda_i + \frac{(n - 2)^2}{4} \right). \tag{2.22}
\]

Taking
\[
\rho = \frac{2(\delta \Lambda_i + n - 2)}{\Lambda_i - (n - 2)}
\]
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in (2.22) yields

\[ 2 \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \leq \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \left( \delta f(\Lambda_i, n) - \frac{\delta \Lambda_i + n - 2}{\Lambda_i - (n - 2)} \right) \]

\[ + \frac{1}{\delta} \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i) \left( \Lambda_i + \frac{(n - 2)^2}{4} \right). \]

Hence, we obtain

\[ \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \left( 2 + \frac{n - 2}{\Lambda_i - (n - 2)} \right) \]

\[ \leq \delta \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \left( f(\Lambda_i, n) - \frac{\Lambda_i}{\Lambda_i - (n - 2)} \right) \]

\[ + \frac{1}{\delta} \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i) \left( \Lambda_i + \frac{(n - 2)^2}{4} \right). \]  \hspace{1cm} (2.23)

Minimizing the right hand side of (2.23) as a function of \( \delta \) by choosing

\[ \delta = \left( \frac{\sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i) \left( \Lambda_i + \frac{(n - 2)^2}{4} \right)}{\sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \left( f(\Lambda_i, n) - \frac{\Lambda_i}{\Lambda_i - (n - 2)} \right)} \right)^{\frac{1}{2}} \]

concludes the proof of Theorem 1.1.

**Proof of Corollary 1.2**

It is easy to see from (1.9) that

\[ \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \leq \left\{ \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \left( f(\Lambda_i, n) - \frac{\Lambda_i}{\Lambda_i - (n - 2)} \right) \right\}^{\frac{1}{2}} \]

\[ \times \left\{ \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i) \left( \Lambda_i + \frac{(n - 2)^2}{4} \right) \right\}^{\frac{1}{2}}. \]  \hspace{1cm} (2.24)

One can check by induction that

\[ \left\{ \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \left( f(\Lambda_i, n) - \frac{\Lambda_i}{\Lambda_i - (n - 2)} \right) \right\} \left\{ \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i) \left( \Lambda_i + \frac{(n - 2)^2}{4} \right) \right\} \]

\[ \leq \left( \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \right) \left\{ \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i) \left( f(\Lambda_i, n) - \frac{\Lambda_i}{\Lambda_i - (n - 2)} \right) \left( \Lambda_i + \frac{(n - 2)^2}{4} \right) \right\}, \]

which together with (2.24) yields inequality (1.10).

Solving the quadratic polynomial of \( \Lambda_{k+1} \) in (1.10), we obtain inequality (1.11) and (1.12). It completes the proof of Corollary 1.2.
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