Isovector Meson Masses from QCD Sum Rules

Nasrallah F. Nasrallah$^a$, Karl Schilcher$^{b,c}$

$^a$ Faculty of Science, Lebanese University, Tripoli 1300, Lebanon
$^b$ Institut für Physik, Johannes Gutenberg-Universität
Staudingerweg 7, D-55099 Mainz, Germany
$^c$ Centre for Theoretical Physics and Astrophysics
University of Cape Town, Rondebosch 7700, South Africa

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Abstract

We present a calculation of the masses of the isovector mesons (vector, scalar and pseudoscalar including the established recurrences) using a new method of finite energy QCD sum rules. The method is based on the idea of choosing a suitable integration kernel which minimizes the occurring integral over the cut in the complex energy (squared) plane. We obtain remarkably stable results in a wide range $R$, where $R$ is the radius of the integration contour. The sum rule predictions agree with the experimental values within the expected accuracy showing that QCD describes single resonances.

KEYWORDS: Sum Rules, QCD, meson masses.

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1 QCD sum rules and meson masses

The mass of the $\rho$-meson was first calculated from QCD and the operator product expansion (OPE) in the pioneering paper of Shifman, Vainshtein and Zakharov [1]. The calculation is based on QCD sum-rules of the Borel (or Laplace) type. Although the results were at the time rather spectacular, it was soon recognized [2] that, apart from the arbitrariness of the integration kernel, they suffer from instabilities related to the specific choice of the Borel variable and the assumption of a constant continuum of the isovector spectral function. There are two arbitrary parameters, the Borel parameter, called $M_0^2$, and the onset of the continuum. Note, in the original paper the authors based their analysis on a small QCD coupling, corresponding to a scale $\Lambda_{\text{QCD}} \approx 100$ MeV compared to the modern value $\Lambda_{\text{QCD}} \approx 350$ MeV).

A more stringent approach is based on finite energy sum rules [3]. We developed this idea further and used it in numerous applications [4]. Here we will use our approach to calculate all relevant hadron masses, starting with the $\rho$-meson [5]

$$m_\rho = 0.775.25 \pm 0.00026 \text{ GeV } m_\rho^2 = 0.6010 \text{ GeV}^2$$

Consider the isovector current

$$j_\mu = \frac{1}{2}(\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d) \quad (1)$$

with the quantum numbers of the $\rho$. The relevant spectral function is given by the absorptive part of the correlator

$$\Pi_{\mu\nu}(q) = i \int d^4xe^{iqx} \langle 0 | T j_\mu(x) j_\nu(0) | 0 \rangle \quad (2)$$

$$= (q_{\mu}q_{\nu} - g_{\mu\nu}q^2)\Pi(q^2) \quad (3)$$

Phenomenologically the spectral functions built up by the $\rho$ and the higher resonances $\rho'(1450), \rho''(1700),...$. Neglecting its width, the experimental spectral function of the $\rho$ is given by

$$\rho^{\text{exp}}(t) = \frac{m_\rho^2}{g_\rho^2} \delta(t - m_\rho^2) = \frac{1}{\pi} \text{Im} \Pi^{\text{exp}}(t) \quad (4)$$

where $g_\rho$ is defined by

$$\langle 0 | j_\mu(0) | \rho(p, s) \rangle = \frac{m_\rho^2}{g_\rho} \varepsilon_\mu$$

with $g_\rho = 4.97 \pm 0.07$ as determined from the leptonic decay of the rho-meson. The spectral function of Eq.(4) corresponds to an amplitude

$$\Pi^{\text{exp}}(t) = -\frac{m_\rho^2}{g_\rho^2} \frac{1}{(t - m_\rho^2)} \quad (5)$$
To lowest non-trivial order, the corresponding QCD expression is

$$\Pi_{\text{QCD}}(t) = -\frac{1}{8\pi^2}(1 + a_s)L + \frac{\langle m_u\bar{u}u + m_d\bar{d}d \rangle}{2t^2} + \frac{\langle a_sGG \rangle}{24t^2} + \frac{112\pi}{81t^3}\langle \sqrt{\alpha_s} \langle \bar{q}q \rangle \rangle^2 + \ldots$$

(6)

where $a_s = \alpha_s/\pi$ is the strong coupling at the scale $\mu$ and $L \equiv \ln \frac{-t}{\mu^2}$. We take $a_s = 0.1$ for $\mu$ of order 3 GeV$^2$ to 4 GeV$^2$ as measured in $\tau$-decay. The variation of $a_s$ in this region is of higher order.

To next order in QCD and the $\overline{MS}$ scheme, the perturbative part of the vector correlator is given by [6]

$$8\pi^2\Pi_{\text{QCD}} = -\left[1 + L + aL + a^2(F_3L + \frac{\beta_1}{4}L^2)\right] + \ldots$$

(7)

where

$$\beta_1 = -\frac{1}{2}(11 - \frac{2}{3}n_f), \quad F_3 = 1.9857 - 0.1153n_f$$

The basis of FESR is Cauchy’s theorem applied to the contour of Fig. 1

![Fig. 1: Integration contour of FESR](image)

which implies

$$-\frac{m_{\rho}^2}{g_{\rho}^2}P(m_{\rho}^2) = \frac{1}{\pi} \int_{\text{cut}}^R dt \ P(t)Im\Pi_{\text{QCD}}^\text{exp}(t) + \frac{1}{2\pi i} \oint_{|t|=R} dt \ P(t)\Pi_{\text{QCD}}(t)$$

(8)

where $P(t)$ is an entire function, e.g. a polynomial. Over the circle of large radius $R$ the correlator $\Pi(t)$ has been replaced by its QCD expression. The
principal unknown in Eq. (8) is the integral over the cut, i.e. over the higher vector-isovector resonances with mass \( m^2_\rho \leq R \). To minimize this integral (before neglecting it), a judicious choice of the weight-function \( P(t) \) has to be made. With the classic choice \([1]\) \( P(t) = \exp(-t/M^2_0) \) the Borel variable \( M^2_0 \) cannot be chosen too large because it would minimize the contribution of the \( \rho \)-meson. Also \( M^2_0 \) cannot be too small because the unknown condensates in Eq. (6) would explode. It was hoped in \([1]\) that a region of stability at an intermediate \( M^2_0 \) can be found. This can be shown to be not the case \([2]\). In our FESR approach we take \( P(t) \) to be a polynomial

\[
P(t) = \sum_{n=0}^{n_{\text{max}}} c_n t^n.
\]

It is clear that the order \( n_{\text{max}} \) cannot be chosen arbitrarily high because of the contribution of unknown condensates. We choose a polynomial \( P_1(t) \) which vanishes at the mass \( m_1 = (1465 \pm 25) \text{ MeV} \) \([5]\) \( (m_1^2 = 2.15 \text{ GeV}^2) \) of the first resonance and at the integration radius \( R \). Explicitly, we take

\[
P_1(t, R) = (1 - \frac{t}{m_1^2})(1 - \frac{t}{R})
\]

\[
= 1 - a_1 t - a_2 t^2 \quad \text{where} \quad a_1(R) = \frac{1}{m_1^2} + \frac{1}{R}, \quad a_2(R) = -\frac{1}{m_1^2 R}
\]

When necessary we take \( R \gtrsim 3 \text{ GeV}^2 \) (\( \sim m_1^2 \)) as we know from the \( \tau \)-decay analysis \([7]\) that global duality is valid there. Later we will need \( P_1(m_1^2, R) = 0.6 \text{ GeV}^2, \quad R = 3 \text{ GeV}^{-2} \) = 0.576 and \( a_1(R = 3 \text{ GeV}^2) = 0.799 \) \( \text{GeV}^{-2} \) and \( a_2(R = 3 \text{ GeV}^2) = -0.155 \text{ GeV}^{-4} \).

If we neglect the contribution of the physical continuum for \( 0 \leq t \leq R \) and use Cauchy’s theorem we arrive at the sum-rule

\[
\frac{m_\rho^2}{g_\rho^2} P_1(m_\rho^2, R) = \frac{1}{2\pi i} \int_{|t|=R} dt P_1(t, R) \Pi^{\text{QCD}}(t)
\]

\[
= \frac{1}{8\pi^2}(1 + a_s) \int_0^R dt P_1(t, R) + a_1 \frac{\langle uu + dd \rangle}{2} - a_1 \frac{\langle a_s GG \rangle}{24}.
\]

As a check of our method we plot the integral appearing in Eq. (12). As can be seen from Fig. 2.

\[
I_0(R) = \int_0^R (1 - \frac{t}{m_1^2})(1 - \frac{t}{R}) dt = \frac{R}{2} \left( 1 - \frac{1}{3} \frac{R}{m_1^2} \right)
\]
the other hand we do not expect higher resonances to contribute significantly to the sum rule in this region. Our results do not depend on the precise choice of $R$ as an be seen from Fig. 2 as long as $2.5 \text{ GeV}^2 \leq R \leq 3.5 \text{ GeV}^2$. We will use Eq. (12) in this region in the chiral limit

$$
\frac{m^2}{g^2} P_1(m^2, R) = \frac{1}{8\pi^2} (1 + a_s) \int_0^R dt P_1(t, R), - \frac{a_1}{24} (a_s GG) \quad (14)
$$

$$
\frac{m^2}{g^2} 0.583 = \frac{1.1}{8\pi^2} 0.801 \text{ GeV}^2 - \frac{0.799 \text{ GeV}^2}{24} (a_s GG) \quad (15)
$$

for $R \sim 3 \text{ GeV}^2$

Experimentally, $g_\rho$ can be determined from its leptonic decay width

$$
\Gamma(\rho \rightarrow e^+ e^-) = \frac{1}{3} \alpha^2 m_\rho \frac{4\pi}{g^2} = 7.04 \pm 0.06 \text{ KeV}
$$

$$
\frac{g^2}{4\pi} = 1.96 \pm 0.02 \quad (16)
$$

Neglecting the gluon condensate and setting $m_\rho^2 = 0.6 \text{ GeV}^2$, Eq. (15) gives

$$
\frac{g^2}{4\pi} = \frac{0.6 \text{ GeV}^2}{4\pi} \left( \frac{1}{0.583} - \frac{1.1 \times 0.801 \text{ GeV}^2}{8\pi^2} \right)^{-1} = 2.494
$$

This result is consistent with the experimental one Eq. (16) considering that the former involves the difference of two large numbers. The error cannot be estimated reliably as it arises mainly from the Ansatz of a narrow $\rho$-resonance for the spectral function.
The error due to $\Pi_{\text{pert}}^{QCD}$ is small. To order $a^2 = (\alpha_s/\pi)^2$ the correlator is given in Eq. (7). The relevant integrals can be found in [8]. The error due to neglected higher order perturbative terms turns out to be of order 2% to 4%, significantly smaller than the further errors to be discussed below.

2 The Isovector Vector Mesons

We propose a sum rule method that is optimally suited to calculate all resonance masses from QCD. We define a polynomial $P_i(t)$

$$P_i(t) = \left(1 - \frac{t}{m_i^2}\right) \left(1 - \frac{t}{R}\right)$$

which vanishes at the mass $m_i$ and at the integration radius $R$. For example $m_1 = (1465 \pm 25)$ MeV [5] ($m_1^2 = 2.15$ GeV$^2$) is the first resonance recurrence. In order to get the mass of the $\rho$-meson we take the first moment integral

$$\frac{m_\rho^4}{g_\rho^2} P_1(m_\rho^2, R) = \frac{1}{2\pi i} \int_{|t|=R} dt \frac{t}{P_1(t, R)} \Pi^{QCD}(t)$$

$$= \frac{1}{8\pi^2} (1 + \alpha_s) \int_0^R dt \left. P_1(t, R) \right|_{t=R} - \frac{\langle m_u\bar{u}u + m_d\bar{d}d\rangle}{2} - \frac{\langle a_sGG\rangle}{24} \tag{17}$$

Consider the integral

$$I_1(R, m_1^2) = \int_0^R (1 - \frac{t}{m_1^2})(1 - \frac{t}{R}) t dt = \frac{R^3}{12} (2 - \frac{R}{m_1^2}) \tag{18}$$

For $R = 3.0$ GeV$^2$ and $m_1 = 1.465$ GeV the result is $I_1(3 \text{ GeV}^2) = 0.451$ GeV$^4$. The result is still stable for $2.5 \text{ GeV}^2 \leq R \leq 3.5 \text{ GeV}^2$, see Fig.2. With the standard values of the condensates

$$\langle a_sGG\rangle = 0.013 \text{ GeV}^4 \ , \ \langle m_u\bar{u}u + m_d\bar{d}d\rangle = -1.67 \times 10^{-4} \text{ GeV}^4$$

and choosing $R = 3.0$ GeV$^2$ Eqs. (12) and (17) give

$$m_\rho = 0.73 \text{ GeV}$$

Our choice (Eq.10) for $P_1(t)$ provides a good damping for the contribution of the continuum in the interval $2.5$ GeV$^2 \leq t \leq 3.5$ GeV$^2$: The contribution of the resonances $\rho(1450)$ and $\rho(1700)$ (almost) vanishes and that of the $\rho(1580)$ is shrunk by a factor of $P_1(1.582 \text{ GeV}^2) / P_1(0.6 \text{ GeV}^2) = -5.7 \times 10^{-2}$. This renders the contribution of the continuum negligible.

It is nevertheless worthwhile to assess the influence of the variation of $P(t)$ in the result for $m_\rho$ in order to estimate the error inherent in the method.
One choice would be

\[ P(t) = \left( 1 - \frac{t}{2.1 \text{ GeV}^2} \right) \left( 1 - \frac{t}{2.5 \text{ GeV}^2} \right) \left( 1 - \frac{t}{2.89 \text{ GeV}^2} \right) \]

yielding \( m_\rho = 0.71 \text{ GeV} \). Or

\[ P(t) = \left( 1 - \frac{t}{2.5 \text{ GeV}^2} \right)^2 \]

which gives \( m_\rho = 0.79 \text{ GeV} \). We have tried several other polynomials, so we give finally

\[ m_\rho = (0.74 \pm 0.04) \text{ GeV} \quad (19) \]

Taking an additional moment yields the mass \( m_1 \) of the \( \rho_1(1450) \). In addition to Eqs. (12) and (17) we have, neglecting the higher condensates

\[
\frac{m_\rho^6}{g_\rho^2} P_1(m_\rho^2, R) = \frac{(1 + a_s)}{8\pi^2} \int_0^R dt t^2 P_1(t, R) \quad (20)
\]

The mass \( m_1 \) can be determined by the two ratios

\[
m_\rho^2 = \frac{\text{rhs of Eq. (20)}}{\text{rhs of Eq. (17)}} = \frac{\int_0^R dt t^2 P_1(t, R)}{\int_0^R dt t P_1(t, R)}
\]

all at \( R = 3.0 \text{ GeV}^2 \). The mass \( m_1 \) is determined by the equating the two ratios using Eqs. (19), (13) and

\[
I_2(R, m_1^2) = \int_0^R \left( 1 - \frac{t}{m_1^2} \right) (1 - \frac{t}{R}) t^2 dt = \frac{R^3}{60} (5 - \frac{3R}{m_1^2}) \quad (21)
\]

From

\[
\frac{I_1}{I_0} = \frac{I_2}{I_1}
\]

we obtain

\[ m_1 = m_\rho(1450) = 1.42 \pm 0.10 \text{ GeV} \]

The error is again estimated by varying \( R \) by 10%.

We can proceed further and consider the integral over the contour indicated in Fig. 3 with the kernel

\[
P_3(t, R) = \left( 1 - \frac{t}{m_1^2} \right) \left( 1 - \frac{t}{R} \right) \quad (22)
\]

In a calculation of \( m_1 \) we choose \( m_{i-1}^2 \) as the lower limit of integration because the contour starts there. In addition we have a check of the consistency.
of the full set of isovector meson mass determination. \( R \) (and \( a_s \)) is again chosen in the stability region \( (R \sim 3.5 \text{ GeV}^2) \).

We have to assume here that global duality is for \( t \geq m_1^2 = 2.15 \text{ GeV}^2 \).

Neglecting all condensates the mass of the \( \rho_2 = \rho(1580) \) is obtained from the sum rules

\[
\frac{m^2_2}{g^2_{\rho_2}} P_3(m^2_2, R = 3.5) = \frac{1 + a_s}{8\pi^2} \int_{m_1^2}^{3.5 \text{ GeV}^2} dt P_3(t, R) \tag{23}
\]

\[
\frac{m^4_2}{g^2_{\rho_2}} P_5(m^2_2, R = 3.5) = \frac{1 + a_s}{8\pi^2} \int_{m_1^2}^{3.5 \text{ GeV}^2} dt tP_3(t, R) \tag{24}
\]

taking the mass \( m_3 = m_{\rho}(1700) \) (entering via \( P_3 \)) as given. The ratio of the above integrals gives

\[
m^2_2 = \int_{m_1^2}^{3.5 \text{ GeV}^2} dt \frac{tP_3(t, R)}{P_3(t, R)} \tag{25}
\]

in the stability region \( (R \sim 3.5 \text{ GeV}^2) \)

\[
m_2 = m_{\rho}(1580) = (1.50 \pm 0.05) \text{ GeV} \tag{26}
\]

The error is obtained by varying \( R \) by \( \pm 10\% \).
Similarly we can obtain \( m_3 = m_\rho(1700) \) from the kernel

\[
P_4(t, R) = \left( 1 - \frac{t}{m_4^2} \right) \left( 1 - \frac{t}{R} \right)
\]

and integrating over the contour of Fig. 3. Assuming QCD duality is valid from \( m_2^2 = 1.465^2 = 2.146 \text{ GeV}^2 \) the corresponding sum rule reads

\[
m_3^2 = \frac{\int_{m_3^2}^R dt \, t P_4(t, R)}{\int_{m_3^2}^R dt \, P_4(t, R)}
\]

(27)

For \( m_4 = 2.150 \text{ GeV} \) the stability region being \( R \sim 4.5 \text{ GeV}^2 \). The result is

\[ m_3 = 1.71 \text{ GeV} \]

The mass \( m_3 \) can also be obtained from \( P_3(t, R = 3.5 \text{ GeV}^2) = \left( 1 - \frac{t}{m_3^2} \right) \left( 1 - \frac{t}{R} \right) \) from

\[
m_3^2 = \frac{\int_{m_3^2}^{R=3.5 \text{ GeV}^2} dt \, t^2 P_3(t, R)}{\int_{m_3^2}^{R=3.5 \text{ GeV}^2} dt \, P_3(t, R)} = \frac{\int_{m_3^2}^{R=3.5 \text{ GeV}^2} dt \, t P_3(t, R)}{\int_{m_3^2}^{R=3.5 \text{ GeV}^2} dt \, P_3(t, R)}
\]

The equality of the ratios gives

\[ m_3 = 1.74 \text{ GeV} \] (28)

Combining the two results we obtain

\[ m_3 = m_\rho(1700) = 1.73 \pm 0.05 \text{ GeV} \]

We finally proceed to calculate \( m_4 = m_\rho(2150) \) using the kernel

\[
P_4(t, R) = \left( 1 - \frac{t}{m_4^2} \right) \left( 1 - \frac{t}{R} \right)
\]

in the stability region \( R \sim 6.0 \text{ GeV}^2 \). The mass \( m_4 \) follows from the sum rule

\[
m_4^2 = \frac{\int_{m_4^2}^{6.0 \text{ GeV}^2} dt \, t^2 P_4(t, R)}{\int_{m_4^2}^{6.0 \text{ GeV}^2} dt \, P_4(t, R)} = \frac{\int_{m_4^2}^{6.0 \text{ GeV}^2} dt \, t P_4(t, R)}{\int_{m_4^2}^{6.0 \text{ GeV}^2} dt \, P_4(t, R)} \text{ all at } R = 6.0 \text{ GeV}^2
\]

(29)

Equality of the ratios give

\[ m_4 = m_\rho(2150) = 2.18 \pm 0.09 \text{ GeV} \]

The error is obtained by varying \( R \) by 10%.

We collect our results together with the experimental numbers from [5] in a table.

**Table of Results:**

9
| Resonance | Result for the mass in GeV | Experimental value in GeV |
|-----------|---------------------------|--------------------------|
| $\rho(770)$ | 0.74 ± 0.04               | 0.77511 ± 0.00034        |
| $\rho(1450)$ | 1.42 ± 0.10               | 1.465 ± 0.025            |
| $\rho(1570)$ | 1.50 ± 0.05               | 1.570 ± 0.036 ± 0.062    |
| $\rho(1700)$ | 1.73 ± 0.05               | 1.720 ± 0.020            |
| $\rho(2150)$ | 2.18 ± 0.09               | 2.155 ± 0.021            |

We conclude that the QCD sum rules can predict the masses of all established higher $\rho$ recurrences. QCD describes in this case a single resonance.

3 The Isovector Pseudoscalars

The following isovector pseudoscalars have been observed $\pi$ with $m_\pi \approx 0$ GeV, $\pi_1(1300)$ with $m_1 = 1.3 \pm 0.1$ GeV, $\pi_2(1810)$ with $m_2 = 1.81 \pm 0.01$ GeV, $\pi_3(2370)$ with $m_3 = 2.360 \pm 25$ GeV [5].

We start with the correlator

$$\Pi(q) = i \int d^4x e^{iqx} \langle 0| T j_5(x) j_5(0)|0 \rangle$$

of the pseudoscalar current

$$j_5 = i\gamma_5 q, \quad j_5 = \frac{1}{2m_q} \partial^\mu A_\mu$$

where $m_q$ is the quark mass, $q = u$ or $d$. The QCD expression for the correlator is

$$\Pi^{\text{QCD}}(t) = -\frac{3}{8\pi^2}(1 + \frac{11}{3} a_s) t \ln(-t) + \frac{\langle m_q^2 q q \rangle}{t} + \frac{\langle a_s G G \rangle}{8t} + ...$$

The scale $\mu^2$ only enters at order $\alpha_s^2$. The method used for the vector mesons is repeated here. We use the kernel $P_2(t, R) = (1 - \frac{t}{m_1^2})(1 - \frac{t}{R})$ which vanishes at $t = m_1^2$ and $t = R$ to get the sum rule

$$m_1^2 = \frac{1}{8\pi^2}(1 + \frac{11}{3} a_s) \int_0^R dt t^2 P_2(t, R)$$

$$= \frac{\int_0^R dt t^2 P_2(t, R)}{\int_0^R dt t P_2(t, R)} - \delta$$

where

$$\delta = \frac{\pi^2 \langle a_s G G \rangle}{(1 + \frac{11}{3} a_s)} = 0.094 \text{ GeV}^4$$

With the stability region $R \approx 4.0 \text{ GeV}^2$ this gives

$$m_1 = m_\pi(1300) = 1.22 \text{ GeV}.$$
Taking an additional moment with the kernel

\[ P_2(t, m^2) = \left(1 - \frac{t}{m^2} \right) \left(1 - \frac{t}{R} \right) \text{ at } R = 4.2 \text{ GeV}^2, \quad (34) \]

we obtain the consistency condition

\[ \frac{\int_0^R dt t^3 P_2(t, R)}{\int_0^R dt t^2 P_2(t, R)} = \frac{\int_0^R dt t^2 P_2(t, R)}{\int_0^R dt t P_2(t, R) - \delta}, \text{ all at } R = 4.2 \text{ GeV}^2 \quad (35) \]

This yields

\[ m_2 = m_\pi(1810) = 1.77 \text{ GeV} \]

Alternatively one can use the kernel

\[ P_3(t, R) = \left(1 - \frac{t}{m^2_3} \right) \left(1 - \frac{t}{R} \right) \quad (36) \]

in the sum rule

\[ m_2^2 = \frac{\int_{1.69}^R dt t^2 P_3(t, R)}{\int_{1.69}^R dt t P_3(t, R) - \delta}, \quad (37) \]

assuming optimistically that QCD duality is valid from \( R = 1.69 \text{ GeV}^2 \). At stability \((R \approx 7 \text{ GeV}^2)\) this gives

\[ m_2 = m_\pi(1810) = 1.74 \text{ GeV} \quad (38) \]

One can calculate the mass \( m_3 \) making use of the kernel \( P_3(t, m^2) \) at \( R = 4.4 \text{ GeV}^2 \), and imposing the condition

\[ \frac{\int_{m_2^2}^{7.1 \text{ GeV}^2} dt t^3 P_3(t, m^3)}{\int_{m_2^2}^{7.1 \text{ GeV}^2} dt t^2 P_3(t, m^3)} = \frac{\int_{m_2^2}^{7.1 \text{ GeV}^2} dt t^3 P_3(t, m_3)}{\int_{m_2^2}^{7.1 \text{ GeV}^2} dt t^2 P_3(t, m_3) - \delta} \quad (39) \]

yields

\[ m_3 = m_X(2370) = 2.66 \text{ GeV} \]

This is an argument for the isovector pseudoscalar nature of the \( X(2370) \).

**Table of Results:**

| Resonance | Result for the mass in GeV | Experimental value in GeV |
|-----------|---------------------------|--------------------------|
| \( \pi_1(1300) \) | 1.22 | 1.300 |
| \( \pi_2(1810) \) | 1.77 ± 0.04 | 1.810 |
| \( \pi_3 = \pi_3(2370) \) | 2.66 | 2.370 |

The predictions, although qualitatively in agreement with the data, are not as good as in the vector meson case mainly because QCD perturbation theory is less convergent.
4 The isovector scalar mesons

The spectrum of the scalar mesons is $a_0(980)$, $a_0(1450)$, $X(1835)$, $a_0(1950)$ where the status of the $X(1835)$ is uncertain. We start with the correlator Eq. (30) with the scalar current

$$j(x) = \bar{q}(x)q(x)$$

(40)

The QCD expression is the same as given in Eq. (32) except for the negligible $m_q^2 \langle \bar{q}q \rangle$ term. The method gives

$$f_1^2 P_2(m_1^2, R) = \frac{1}{8\pi^2} \left( 1 + \frac{11}{3} a_s \right) \int_0^R dt \frac{P_2(t, R)}{t} - \frac{\langle a_s GG \rangle}{8} + \ldots$$

(41)

and

$$f_1^2 m_1^2 P_2(m_1^2, R) = \frac{1}{8\pi^2} \left( 1 + \frac{11}{3} a_s \right) \int_0^R dt \frac{t^2 P_2(t, R)}{t} - \frac{\langle a_s GG \rangle}{8} + \ldots$$

(42)

with $P_2(m_1^2, R)$ given by Eq. (31) with the input $m_2^2 = 2.10$ GeV$^2$. At stability ($R \approx 2.7$ GeV$^2$) the ratio of the above equations gives

$$m_1 = m_{a_0}(980) = 0.93 \text{ GeV}$$

(43)

We emphasize that we assume that the correlator is given by QCD perturbation theory is for $|t| \geq R = 2.7 \text{ GeV}^2$ as phenomenologically supported by the analysis of $\tau$-decay [7].

Taking an additional moment with the kernel $P_2(t, R) = (1 - \frac{t}{m_2^2}) (1 - \frac{t}{2.7 \text{ GeV}^2})$ and the condition

$$m_1^2 = \frac{\int_0^R dt \frac{t^3 P_2(t, R)}{t^2 P_2(t, R)} - \frac{\langle a_s GG \rangle}{8}}{\int_0^R dt \frac{t^2 P_2(t, R)}{t^2 P_2(t, R) - \delta}}$$

all at $R = 2.7 \text{ GeV}^2$

(44)

Equating the two ratios yields

$$m_2 = m_{a_0}(1450) = 1.51 \text{ GeV}$$

We next use the kernel $P_4(t, R) = (1 - \frac{t}{m_4^2}) (1 - \frac{t}{4.7 \text{ GeV}^2})$ with $m_4 = m_{a_0}(1950)$ to determine $m_3$:

$$m_3^2 = \frac{\int_{m_2^2}^R dt \frac{t^3 P_4(t, R)}{t^2 P_4(t, R) - \delta}}{\int_{m_2^2}^R dt \frac{t^2 P_4(t, R)}{t^2 P_4(t, R) - \delta}}$$

all at $R = 4.7 \text{ GeV}^2$

(45)

The result is $m_3 = m_X(1835) = 1.80 \text{ GeV}$. 

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\( m_4 \) can be determined by

\[
\frac{\int_R^{m_2^2} dt \ t^3 P_4(t, R)}{\int_{m_2^2}^R dt \ t^2 P_4(t, R)} = \frac{\int_R^{m_2^2} dt \ t^2 P_4(t, R)}{\int_{m_2^2}^R dt \ t \ P_4(t, R) - \delta}, \text{ all at } R = 4.7 \text{ GeV}^2 \quad (46)
\]

with the result

\[
m_4 = m_{a_0}(1950) = 1.80 \text{ GeV}
\]
Table of results:

| Resonance   | Result for the mass in GeV | Experimental value in GeV |
|-------------|----------------------------|----------------------------|
| $a_0(980)$  | 0.93                       | 0.98                       |
| $a_0(1450)$ | 1.51                       | 1.45                       |
| $X(1835)$   | 1.80                       | 1.895                      |
| $a_0(1950)$ | 1.80                       | 1.930                      |

Conclusions: We have calculated the masses of the isovector (vector, pseudoscalar, scalar) mesons and their recurrences with a new variant of QCD finite energy sum rules. The method works well for all similar systems such as the nucleon resonances. The main source of error is the zero width approximation for the resonances. We have estimated this error by allowing the radius entering the sum rule to vary by $\pm 10\%$. Order $\alpha_s$ corrections are included, order $\alpha_s^2$ are calculated and found to be negligible. The sum rule predictions are compared with the experimental numbers and agreement within the expected accuracy is found. It can be concluded that QCD is applicable to single resonances and their recurrences.

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