A SUMMARY: QUANTUM SINGULARITIES IN STATIC AND CONFORMALLY STATIC SPACETIMES

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This is a summary of how the definition of quantum singularity is extended from static space-times to conformally static space-times. Examples are given.

1. Introduction

The question addressed in this review is: What happens if instead of classical particle paths (time-like and null geodesics) one uses quantum mechanical particles to identify singularities in conformally static as well as static spacetimes? A summary of the answer is given together with a couple of example applications. This conference proceeding is based primarily on an article by the authors.

2. Types of Singularities

2.1. Classical Singularities

A classical singularity is indicated by incomplete geodesics or incomplete paths of bounded acceleration in a maximal spacetime. Since, by definition, a spacetime is smooth, all irregular points (singularities) have been excised; a singular point is a boundary point of the spacetime. There are three different types of singularity: quasi-regular, non-scalar curvature and scalar curvature. Whereas quasi-regular singularities are topological, curvature singularities are indicated by diverging components of the Riemann tensor when it is evaluated in a parallel-propagated orthonormal frame carried along a causal curve ending at the singularity.

2.2. Quantum Singularities

A spacetime is QM (quantum-mechanically) nonsingular if the evolution of a test scalar wave packet, representing the quantum particle, is uniquely determined by
the initial wave packet, manifold and metric, without having to put boundary conditions at the singularity. Technically, a static ST (spacetime) is QM-singular if the spatial portion of the Klein-Gordon operator is not essentially self-adjoint on $C^\infty_0(\Sigma)$ in $L^2(\Sigma)$ where $\Sigma$ is a spatial slice. This is tested (see, e.g., Konkowski and Helliwell) using Weyl’s limit point - limit circle criterion that involves looking at an effective potential asymptotically at the location of the singularity. Here a limit-circle potential is quantum mechanically singular, while a limit-point potential is quantum mechanically non-singular.

This definition of quantum singularity has been utilized in the analysis of several timelike spacetime singularities; three examples by the authors include asymptotically power-law spacetimes, spacetimes with diverging higher-order curvature invariants and a two-sphere singularity.

3. Conformally Static Space-Times

The Klein-Gordon with general coupling of a scalar field to the scalar curvature is given by

$$ (\Box - \xi R)\Phi = M^2\Phi $$

where $M$ is the mass if the scalar particle, $R$ is the scalar curvature, and $\xi$ is the coupling ($\xi = 0$ for minimal coupling and $\xi = 1/6$ for conformal coupling). Using the natural symmetry of conformally static space-times the radial equation easily separates allowing it to be put into so-called Schrödinger form to identify the potential, allowing easy analysis of the quantum singularity structure.

4. Friedmann-Robertson-Walker Space-Times with Cosmic String

A metric modeling a Friedmann-Robertson-Walker cosmology with a cosmic string can be written as

$$ ds^2 = a^2(t)(-dt^2 + dr^2 + \beta^2 r^2 d\phi^2 + dz^2) $$

where $\beta = 1 - 4\mu$ and $\mu$ is the mass per unit length of the cosmic string. This metric is conformally static (actually conformally flat). Classically it has a scalar curvature singularity times when $a(t)$ is zero and a quasiregular singularity when $\beta^2 \neq 1$. We focus on resolving the timelike quasiregular singularity.

For the quantum analysis, the Klein-Gordon equation with general coupling can be separated into mode solutions with the radial equation changed to Schrödinger form,

$$ u'' + (E - V(x))u = 0 $$

*Only the time equation contains the coupling constant $\xi$.\)
where $E$ is a constant, $x = r$, and the potential

$$V(x) = \frac{m^2 - \beta^2/4}{\beta^2 x^2}$$

Near zero one can show that the potential $V(x)$ is limit point if $m^2/\beta^2 \geq 1$. So any modes with sufficiently large $m$ are limit point, but $m = 0$ is limit circle and thus generically this conformally static space-time is quantum mechanically singular.

5. Roberts Solution

The Roberts metric

$$ds^2 = e^{2t}(-dt^2 + dr^2 + G^2(r)d\Omega^2)$$

where $G^2(r) = 1/4[1 + p - (1 - p)e^{-2r}](e^{2r} - 1)$ is conformally static, self-similar, and spherically symmetric. It has a classical scalar curvature singularity at $r = 0$ for $0 < p < 1$ that is timelike.

The massive minimally coupled Klein-Gordon equation can be separated into mode solutions and by changing both dependent and independent variables ($r = x$), we get an appropriate inner product and a one-dimensional Schrödinger equation similar to Eq.(3) where again $E$ is a constant but, here, near zero, $V(x)$ goes like $-1/4x^2 < 3/4x^2$ so the potential is limit circle and there is a quantum singularity.

Acknowledgments

One of us (DAK) thanks B. Yaptinchay for discussions.

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