A CHAIN RULE FOR A CLASS OF EVOLUTIVE NONLOCAL HYPOELLIPTIC EQUATIONS

FEDERICO BUSEGHIN AND NICOLA GAROFALO

Abstract. We prove a chain rule of local type for a class of fractional hypoelliptic equations of Kolmogorov-Fokker-Planck type. We introduce a semigroup based notion of nonlocal \textit{car\`e du champ} which works successfully in situations in which the infinitesimal generator of the semigroup itself does not necessarily possess a gradient. Our results extend and sharpen the original 2004 chain rule due to A. C\'ordoba and D. C\'ordoba.

1. Introduction and statements of the results

The chain rule for the standard Laplacian states that, given a function $\varphi \in C^2(\mathbb{R})$ and a function $u \in C^2(\Omega)$, $\Omega \subset \mathbb{R}^n$ an open set, then

$$\Delta \varphi(u) = \varphi'(u)\Delta u + \varphi''(u)|\nabla u|^2.$$  \hfill (1.1)

As it is well-known, such property, which extends to more general second-order differential operators with non-smooth coefficients, plays a central role in the study of the regularity properties of generalised solutions, see \cite{7}, \cite{18} and \cite{16}, \cite{17}. It is obvious that if $\varphi$ is convex, then (1.1) implies

$$-\Delta \varphi(u) \leq \varphi'(u)(-\Delta u).$$  \hfill (1.2)

Consider now the fractional Laplacian defined by

$$(-\Delta)^s u(x) = \frac{\gamma(n, s)}{2} \int_{\mathbb{R}^n} \frac{2u(x) - u(x + y) - u(x - y)}{|y|^{n+2s}} dy,$$  \hfill (1.3)

with normalisation constant given by

$$\gamma(n, s) = \frac{s2^{2s}\Gamma \left( \frac{n+2s}{2} \right)}{\pi^{\frac{n+s}{2}} \Gamma(1 - s)}.$$  \hfill (1.4)

In \cite[Theorem 1]{4} the authors proved for this pseudo-differential operator the following inequality:

$$(-\Delta)^s u^2 \leq 2u(-\Delta)^s u,$$  \hfill (1.5)

for any $u \in \mathcal{S}(\mathbb{R}^n)$. They also presented some important applications of (1.5) to time-decay estimates for viscosity solutions of quasi-geostrophic equations. In the paper \cite{6} the authors

\begin{footnote}
\textit{Key words and phrases.} Kolmogorov equations, hypoelliptic operators of H\rmander type, nonlocal equations, chain rule.

The second author was supported in part by a Progetto SID (Investimento Strategico di Dipartimento) “Non-local operators in geometry and in free boundary problems, and their connection with the applied sciences”, University of Padova, 2017.
\end{footnote}
generalised the inequality (1.5) to any convex function \( \varphi \in C^1(\mathbb{R}) \) and to the fractional powers of the Laplacian on a compact manifold \( M \). Precisely, they showed that for any \( u \in C^\infty(M) \) one has

\[ (-\Delta)^s \varphi(u) \leq \varphi'(u)(-\Delta)^s u, \tag{1.6} \]

where now \((-\Delta)^s\) is suitably defined. We quote from [6]: “Despite its apparent simplicity its validity is quite surprising given the non-local character of the involved operators”. The inequality (1.6) represents a nonlocal version of (1.2) and, similarly to its local counterpart, it plays an important role in many problems from the applied sciences involving \((-\Delta)^s\). See for instance the works [5], on the two-dimensional quasi-geostrophic equation, and [3], on nonlinear evolution equations with fractional diffusion.

In this note we generalise these results to the fractional powers of a large class of evolutive hypoelliptic equations and show that, in fact, we can improve on the inequality (1.6) and obtain an equality similar to (1.1) above. We achieve this by introducing a semigroup based notion of nonlocal \( \text{carré du champ} \) which works successfully in situations in which the infinitesimal generator of the semigroup itself does not necessarily possess a gradient. A prototypical example of what we have in mind is given by the situation when \( \varphi(t) = t^2 \) treated in [4]. Given a function \( u \in \mathcal{S}(\mathbb{R}^n) \) consider the Aronszajn-Gagliardo-Slobedetzky \( s\)–energy of \( u \)

\[ \delta_s(u) = \frac{\gamma(n,s)}{4} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))^2}{|x - y|^{n+2s}} \, dy \, dx, \tag{1.7} \]

which defines membership in the fractional Sobolev space \( W^{s,2}(\mathbb{R}^n) \), see e.g. [8]. The relevance of (1.7) is underscored by the fact that the nonlocal equation \((-\Delta)^s u = 0\) is the Euler-Lagrange equation of the functional \( u \to \delta_s(u) \). Given \( u \in \mathcal{S}(\mathbb{R}^n) \), we have in fact for every \( \varphi \in \mathcal{S}(\mathbb{R}^n) \)

\[ \frac{d}{dt} \delta_s(u + t\varphi)|_{t=0} = \int_{\mathbb{R}^n} (-\Delta)^s u(x) \varphi(x) \, dx. \]

This shows that \( u \) is a critical point of \( \delta_s(u) \) if and only if \((-\Delta)^s u = 0\). Now, the definition (1.7) of the energy suggests to define for every \( x \in \mathbb{R}^n \) the quantity

\[ \Gamma_s(u)(x) = \frac{\gamma(n,s)}{2} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))^2}{|x - y|^{n+2s}} \, dy. \tag{1.8} \]

One notable property of (1.8) is that it represents a nonlocal version of the P.A. Meyer \( \text{carré du champ} \) which is at the basis of the Bakry-Émery gamma calculus. One has in fact the remarkable identity (see [9, Lemma 20.2])

\[ \Gamma_s(u) = -\frac{1}{2} [(-\Delta)^s (u^2) - 2u(-\Delta)^s u]. \tag{1.9} \]

Notice that, since from [2, Lemma 2.3] we know that \( \lim_{s \to 1^-} (-\Delta)^s u(x) = -\Delta u(x) \), a direct consequence of (1.9) is that

\[ \lim_{s \to 1^-} \Gamma_s(u) = -\frac{1}{2s} \lim_{s \to 1^-} [(-\Delta)^s (u^2) - 2u(-\Delta)^s u] = -\frac{1}{2} [\Delta (u^2) + 2u \Delta u] = |\nabla u|^2, \tag{1.10} \]
so that $\Gamma_{(s)}(u)$ provides indeed a good nonlocal length of the “gradient”. But the most striking consequence of the identity (1.9) is that it gives the following sharper version of the chain rule (1.5).

**Proposition 1.1.** Let $0 < s < 1$. For every $u \in \mathcal{S}(\mathbb{R}^n)$ one has

\[(1.11) \quad (-\Delta)^s(u^2) = 2u(-\Delta)^s u - 2\Gamma_{(s)}(u).\]

In view of (1.8), it is clear that (1.11) trivially implies (1.5). It is also clear from (1.10) that, if in (1.11) we pass to the limit as $s \searrow 1^-$, we recover the local identity

\[\Delta u^2 = 2\Delta u + 2|\nabla u|^2,\]

which corresponds to the case $\varphi(t) = t^2$ of (1.1).

The first question that we address in the present paper is to what extent Proposition 1.1 continues to be valid when $\varphi(t) = t^2$ is replaced by a generic function $\varphi$. Because of the nonlocal nature of $(-\Delta)^s$, we should not expect formula (1.11) to generalise exactly, there is a tail. However, such tail vanishes in the limit as $s \searrow 1^-$. One has in fact the following result.

**Proposition 1.2.** Let $U \subset \mathbb{R}$ be an interval and suppose that $\varphi \in C^{1,1}(U) \cap C^2_{\text{loc}}(U)$, for some $\alpha > 0$. For any function $u \in \mathcal{S}(\mathbb{R}^n)$ such that $u(\mathbb{R}^n) \subset U$, one has

\[(1.12) \quad (-\Delta)^s \varphi(u(x)) = \varphi'(u(x))(-\Delta)^s u(x) - \varphi''(u(x))\Gamma_{(s)}(u)(x) + \mathcal{R}_{(s)}(u;\varphi)(x),\]

where for any $x \in \mathbb{R}^n$ we have

\[(1.13) \quad \lim_{s \to 1^-} \mathcal{R}_{(s)}(u;\varphi)(x) = 0.\]

If $\varphi(t) = at^2 + bt + c$, then we have $\mathcal{R}_{(s)}(u;\varphi) \equiv 0$. As a consequence of (1.12), (1.13), we obtain

\[(1.14) \quad - \lim_{s \to 1^-} (-\Delta)^s \varphi(u(x)) = \varphi'(u(x))\Delta u(x) + \varphi''(u(x))|\nabla u(x)|^2.\]

Our main objective in this note is generalising Propositions 1.1 and 1.2 to the fractional powers of the following class of nonlocal Kolmogorov-Fokker-Planck operators in $\mathbb{R}^{N+1}$, recently studied in [10, 11, 12, 13]:

\[(1.15) \quad \mathcal{H} u = \mathcal{A} u - \partial_t u \overset{\text{def}}{=} \text{tr}(Q\nabla^2 u) + < BX, \nabla u > - \partial_t u,
\]

where the $N \times N$ matrices $Q$ and $B$ have real, constant coefficients, and $Q = Q^* \geq 0$. We assume throughout that $N \geq 2$, and we indicate with $X$ the generic point in $\mathbb{R}^N$, with $(X,t)$ the one in $\mathbb{R}^{N+1}$. The class (1.15) was introduced by Hörmander in his celebrated 1967 hypoellipticity paper [14], where he proved that $\mathcal{H}$ is hypoelliptic if and only if the covariance matrix

\[(1.16) \quad K(t) = \frac{1}{t} \int_0^t e^{sB} Q e^{sB^*} \, ds\]

is invertible (i.e., strictly positive) for every $t > 0$. The hypothesis $K(t) > 0$ will be henceforth tacitly assumed. We note that, in the special case when $Q = I_N$ and $B = O_N$, then (1.15) becomes the standard heat operator $H = \Delta - \partial_t$ in $\mathbb{R}^{N+1}$. One should note that, even in this seemingly simple example, one lacks an obvious notion of “gradient”. In fact, because of the evolutive nature of $H$ a tool like the P.A. Meyer carré du champ $\Gamma^H(u) = \frac{1}{2}[H(u^2) - 2uHu]$ is
not effective here since $\Gamma^H(u) = |\nabla u|^2$, which does not provide any control on the time variable $t$. The lack of a gradient is not only caused by the time variable. Even for the time-independent operator $\mathcal{A}$ in (1.15), we have $\Gamma^\mathcal{A}(u) = \frac{1}{2}[\mathcal{A}(u^2) - 2u \mathcal{A} u] = \langle Q \nabla u, \nabla u \rangle$, and this quantity fails to control the directions of the drift, or those of non-ellipticity of $Q$, in the degenerate case.

Since the operators $\mathcal{A}$ and $\mathcal{H}$ in (1.15) are not variational, with the goal of generalising Propositions 1.1 and 1.2 a crucial point is to understand what replaces the nonlocal energy $\Gamma(s)$ in (1.11), (1.12). We introduce a general notion of nonlocal energy which is exclusively semigroup based and therefore prescinds from the existence of a “gradient”. Specialised to time-independent functions, such energy perfectly recaptures that in (1.8) for the fractional Laplacian, see Proposition 1.5 below.

In order to introduce the relevant notion we recall the semigroup $P_t = e^{-t\mathcal{A}}$ defined on a function $f \in \mathcal{S}(\mathbb{R}^N)$ by

$$P_tf(X) = \int_{\mathbb{R}^N} p(X,Y,t) f(Y) dY,$$

where

$$p(X,Y,t) = (4\pi)^{-\frac{N}{2}} (\det(tK(t)))^{-1/2} \exp \left( -\frac{\langle K(t)^{-1}(Y - e^{tB} X), Y - e^{tB} X \rangle}{4t} \right)$$

is the fundamental solution of the operator $\mathcal{H}$ constructed by Hörmander in [14]. We recall that $\int_{\mathbb{R}^N} p(X,Y,t) dY = 1$ for every $X \in \mathbb{R}^N$ and $t > 0$ (however, one has $\int_{\mathbb{R}^N} p(X,Y,t) dX = e^{-t \text{tr} B}$).

Using $P_t$ we next consider the evolutive semigroup $P^\mathcal{H}_t$ introduced in [10]. For a function $u \in \mathcal{S}(\mathbb{R}^{N+1})$, we let

$$P^\mathcal{H}_t u(X,t) = \int_{\mathbb{R}^N} p(X,Y,t) u(Y,t - \tau) dY.$$

For the main properties of the semigroup $\{P^\mathcal{H}_t\}_{t>0}$ we refer the reader to [10]. Here, we note that if for $u \in \mathcal{S}(\mathbb{R}^{N+1})$ we define $U((X,t),\tau) = P^\mathcal{H}_\tau u(X,t)$, then $U \in C^\infty(\mathbb{R}^{N+1} \times (0,\infty))$ and it solves the Cauchy problem

$$\begin{cases}
\partial_\tau U = \mathcal{H} U \\
U((X,t),0) = u(X,t) & (X,t) \in \mathbb{R}^{N+1}.
\end{cases}$$

As in [10], given $s \in (0,1)$, we now define the nonlocal operator $(-\mathcal{H})^s$ for $u \in \mathcal{S}(\mathbb{R}^{N+1})$ and $(X,t) \in \mathbb{R}^{N+1}$, as follows

$$(-\mathcal{H})^s u(X,t) = -\frac{s}{\Gamma(1-s)} \int_0^\infty \tau^{-1-s} \left[ P^\mathcal{H}_\tau u(X,t) - u(X,t) \right] d\tau.$$

We note from (1.17), (1.19) that, if $u$ does not depend on $t$, i.e., $u(X,t) = v(X)$, then $P^\mathcal{H}_t u(X,t) = P_t v(X)$. The next definition is central to this note. It introduces a semigroup based notion of nonlocal carré du champ which works successfully in situations in which the infinitesimal generator of the semigroup itself does not necessarily possess a gradient.
Definition 1.3. Given $s \in (0, 1)$, we define the nonlocal evolutive carré du champ of a function $u : \mathbb{R}^{N+1} \to \mathbb{R}$ as

$$
\Gamma^\mathcal{X}_{(s)}(u)(X,t) = \frac{s}{2\Gamma(1-s)} \int_0^\infty \frac{1}{\tau^{1+s}} P^\mathcal{X}_\tau((u - u(X,t))^2)(X,t) d\tau.
$$

We notice the obvious consequence $\Gamma^\mathcal{X}_{(s)}(u)(X,t) \geq 0$ of the positivity of the semigroup $P^\mathcal{X}_\tau$. More importantly, the relevance of Definition 1.3 is connected with the following Besov spaces, see also [12] for the time-independent case.

Definition 1.4. For $p \geq 1$ and $\alpha \geq 0$, we define the evolutive Besov space $\mathcal{B}^{\alpha,p}(\mathbb{R}^{N+1})$ as the collection of those functions $u \in L^p(\mathbb{R}^{N+1})$, such that the seminorm

$$
\mathcal{M}^{\alpha,p}(u) = \left( \int_0^\infty \frac{1}{\tau^{2s+1}} \int_{\mathbb{R}^{N+1}} P^\mathcal{X}_\tau(|u - u(X,t)|^p)(X,t) dXdt d\tau \right)^{\frac{1}{p}} < \infty.
$$

We endow the space $\mathcal{B}^{\alpha,p}(\mathbb{R}^{N+1})$ with the following norm

$$
||u||_{\mathcal{B}^{\alpha,p}(\mathbb{R}^{N+1})} \overset{def}{=} ||u||_{L^p(\mathbb{R}^{N+1})} + \mathcal{M}^{\alpha,p}(u).
$$

We are interested in the situation in which $p = 2$, and $\alpha = s \in (0,1)$. We remark explicitly that, in such case, we have

$$
\mathcal{M}^{s,2}(u)^2 = \frac{2\Gamma(1-s)}{s} \int_{\mathbb{R}^{N+1}} \Gamma^\mathcal{X}_{(s)}(u)(X,t) dXdt.
$$

We have the following result that shows that, specialised to functions which do not depend on $t$, and to the standard heat operator, we recover from $\Gamma^\mathcal{X}_{(s)}(u)(X,t)$ the carré du champ (1.9), (1.8).

Proposition 1.5. Suppose that $Q = I_N$ and $B = O_N$, and thus $\mathcal{X} = \Delta - \partial_t$, and let $u(X,t) = v(X)$. Then,

$$
\Gamma^\mathcal{X}_{(s)}(u)(X,t) = \Gamma^\mathcal{X}_{(s)}(v)(X) = \frac{\gamma(N,s)}{2} \int_{\mathbb{R}^N} \frac{(v(X) - v(Y))^2}{|X - Y|^{N+2s}} dY,
$$

see (1.8).

With Definition 1.3 in hands, we are now ready to state the evolutive counterpart of Proposition 1.1.

Proposition 1.6. Let $u \in \mathcal{S}(\mathbb{R}^{N+1})$. Then, for every $(X,t) \in \mathbb{R}^{N+1}$ we have

$$
(-\mathcal{X})^s(u^2)(X,t) = 2u(X,t)(-\mathcal{X})^s u(X,t) - 2\Gamma^\mathcal{X}_{(s)}(u)(X,t).
$$

In particular, the following extension of (1.5) holds:

$$
(-\mathcal{X})^s(u^2)(X,t) \leq 2u(X,t)(-\mathcal{X})^s u(X,t).
$$

More in general, for every convex function $\varphi \in C^1(\mathbb{R})$, one has

$$
(-\mathcal{X})^s \varphi(u)(X,t) \leq \varphi'(u)(X,t)(-\mathcal{X})^s u(X,t).
$$

We next consider the appropriate generalisation of Proposition 1.2.
Proposition 1.7. Let $U \subset \mathbb{R}$ be an interval and suppose that $\varphi \in C^{1,1}(U) \cap C^{2,\alpha}_{\text{loc}}(U)$, for some $\alpha > 0$. For any function $u \in \mathcal{S}(\mathbb{R}^{N+1})$ such that $u(\mathbb{R}^{N+1}) \subset U$, one has

$$(-\mathcal{X})^s \varphi(u(X,t)) = \varphi'(u(X,t))(-\mathcal{X})^s u(X,t) - \varphi''(u(X,t)) \Gamma_{(s)}(u)(X,t)$$

where for any $(X,t) \in \mathbb{R}^{N+1}$ we have

$$\lim_{s \to 1^-} \mathcal{R}_s(u; \varphi)(X,t) = 0.$$  

When $\varphi(t) = at^2 + bt + c$, we have $\mathcal{R}_s(u; \varphi) \equiv 0$.

In Section 2 we present the proofs of Propositions 1.5, 1.6 and 1.7. In view of Proposition 1.5, Proposition 1.2 is contained in the more general Proposition 1.7, and we thus omit its proof.

Acknowledgment: We thank Giulio Tralli for his interest in this note and for helpful discussions.

2. Proof of the results

In this section we present the proofs of the results. We begin with the

Proof of Proposition 1.5. Under the assumptions of the proposition we know that $p(X,Y,t) = (4\pi t)^{-\frac{N}{2}} \exp\left(-\frac{|X-Y|^2}{4t}\right)$. Since from (1.17), (1.19) we have $P_{\tau}(u - u(X,t))^2(X,t) = P_{\tau}((v - v(X))^2)(X)$, we find

$$\Gamma_{(s)}(u)(X,t) = \frac{s}{2\Gamma(1-s)} \int_0^\infty \tau^{-1-s} P_{\tau}((v - v(X))^2)(X)d\tau$$

$$= \frac{s^{2-N} \pi^{-\frac{N}{2}}}{2\Gamma(1-s)} \int_{\mathbb{R}^N} (v(Y) - v(X))^2 \int_0^{+\infty} \tau^{-1-s-\frac{N}{2}} \exp\left(-\frac{|X-Y|^2}{4\tau}\right)d\tau dY.$$  

Now, a simple computation gives

$$\int_0^{+\infty} \tau^{-1-s-\frac{N}{2}} \exp\left(-\frac{|X-Y|^2}{4\tau}\right)d\tau = \frac{2^{N+2s}}{|X-Y|^{N+2s}} \Gamma\left(\frac{N+2s}{2}\right).$$

Substituting this identity in the above equation, recalling (1.4) which gives

$$\gamma(N,s) = \frac{s2^{2s} \Gamma(s + \frac{N}{2})}{\pi^{\frac{N}{2}} \Gamma(1-s)},$$

and keeping (1.8) in mind, we reach the desired conclusion $\Gamma_{(s)}(u)(X,t) = \Gamma_{(s)}(v)(X)$.  

□

Next, we present the
Proof of Proposition 1.6. We only prove (1.23), since (1.21) will follow from Proposition 1.7, and (1.22) is a trivial consequence of (1.23). By the assumption of convexity of \( \varphi \) we have for any \( \gamma, \sigma \in \mathbb{R} \),
\[
\varphi'(\sigma)(\gamma - \sigma) \leq \varphi(\gamma) - \varphi(\sigma).
\]
Applying this inequality with \( \gamma = u(Y, t - \tau) \) and \( \sigma = u(X, t) \), we obtain
\[
P_T^X \varphi \circ u(X, t) - \varphi \circ u(X, t) = \int_{\mathbb{R}^N} p(X, Y, \tau)(\varphi(u(Y, t - \tau)) - \varphi(u(X, t))) dY \geq \varphi'(u(X, t)) \int_{\mathbb{R}^N} p(X, Y, \tau)(u(Y, t - \tau) - u(X, t)) dY = \varphi'(u(X, t))(P_T^X u(X, t) - u(X, t)).
\]
Multiplying both sides of the latter inequality by \(-\frac{s}{1(1-s)}\) and integrating in \( \tau \) over \((0, \infty)\), if we keep (1.20) in mind we immediately obtain the desired conclusion (1.23).

Finally, we give the

Proof of Proposition 1.7. For every \((X, t) \in \mathbb{R}^{N+1}\) fixed, we have
\[
(2.1) \quad P_T^X \varphi(u)(X, t) - \varphi(u)(X, t) = \int_{\mathbb{R}^N} p(X, Y, \tau)(\varphi(u(Y, t - \tau)) - \varphi(u(X, t))) dY.
\]
By the Taylor expansion of \( \varphi(u) \) we can write
\[
\varphi(u(Y, t - \tau)) - \varphi(u(X, t)) = \varphi'(u(X, t))(u(Y, t - \tau) - u(X, t)) + \frac{1}{2} \varphi''(u(X, t))(u(Y, t - \tau) - u(X, t))^2 + \frac{1}{2} \varphi''(\tilde{u})(\varphi''(u(X, t)))(u(Y, t - \tau) - u(X, t))^2,
\]
where \( \tilde{u} \) is a point (depending on \( X, Y, t, \tau \)) between \( u(X, t) \) and \( u(Y, t - \tau) \). Substituting in the above identity, we find
\[
P_T^X \varphi(u)(X, t) - \varphi(u)(X, t) = \varphi'(u)(X, t)(P_T^X u(X, t) - u(X, t)) + \frac{1}{2} \varphi''(u(X, t))(u - u(X, t))^2(X, t) + \frac{1}{2} \int_{\mathbb{R}^N} P_T^X p(X, Y, \tau) \varphi''(\tilde{u})(\varphi''(u(X, t)))(u(Y, t - \tau) - u(X, t))^2 dY.
\]
Multiplying both sides of the latter equation by \(-\frac{s}{1(1-s)}\) and integrating in \( \tau \) over \((0, \infty)\), if we keep (1.20) and (1.3) in mind we obtain (1.24), where we have let
\[
\mathcal{R}_{(s)}^X(u; \varphi)(X, t) = -\frac{s}{2\Gamma(1-s)} \int_0^{+\infty} \frac{1}{\tau^{1+s}} \int_{\mathbb{R}^N} p(X, Y, \tau)(\varphi''(\tilde{u}) - \varphi''(u(X, t)))(u(Y, t - \tau) - u(X, t))^2 dY.
\]
From the latter equation it is obvious that, when \( \varphi(t) = at^2 + bt + c \), then \( \varphi'' \equiv 2a \), and therefore we have \( \mathcal{R}_{(s)}^X(u; \varphi)(X, t) \equiv 0 \). Combined with (1.24), this proves in particular the chain rule.
Finally, we need to show (1.25). With this objective in mind, we write

$$\mathcal{R}(u; \varphi)(X, t) = -\frac{s}{2\Gamma(1-s)} \int_0^1 \int_{\mathbb{R}^N} p(X, Y, \tau)(\varphi''(\tilde{u}) - \varphi''(u(X, t)))(u(Y, t - \tau) - u(X, t))^2dYd\tau$$

$$- \frac{s}{2\Gamma(1-s)} \int_1^\infty \int_{\mathbb{R}^N} p(X, Y, \tau)(\varphi''(\tilde{u}) - \varphi''(u(X, t)))(u(Y, t - \tau) - u(X, t))^2dYd\tau$$

$$= I(X, t; s) + II(X, t; s).$$

Since by assumption $\varphi'' \in L^\infty(U)$, using the fact that $\int_{\mathbb{R}^N} p(X, Y, \tau)dY = 1$, we easily find for every $(X, t) \in \mathbb{R}^{N+1}$,

$$|II(X, t; s)| \leq \frac{4s}{\Gamma(1-s)} ||\varphi''||_\infty ||\varphi||_\infty \int_1^\infty \int_{\mathbb{R}^N} p(X, Y, \tau)dYd\tau$$

$$= \frac{4s}{\Gamma(1-s)} ||\varphi''||_\infty ||\varphi||_\infty \int_1^\infty \int_{\mathbb{R}^N} d\tau = \frac{4}{\Gamma(1-s)} ||\varphi''||_\infty ||\varphi||_\infty \rightarrow 0$$

when $s \rightarrow 1^-$. Here, we have used the fact that $\Gamma(1-s) \approx 1/(1-s)$ as $s \rightarrow 1^-$. Estimating $I(X, t; s)$ requires some additional care. Because $u \in \mathcal{Y}(\mathbb{R}^{N+1})$, Taylor’s formula gives

$$|u(Y, t - \tau) - u(X, t)| \leq |\nabla_{(X,t)} u(\tilde{Y}, \tilde{\tau})|||Y - X| + \tau| \leq C_1(|Y - X| + \tau),$$

where $C_1 = ||\nabla_{(X,t)} u||_\infty$ and $\tilde{Y}, \tilde{\tau}$ respectively are a point on the segment joining $X$ to $Y$, and a number in the interval joining $t$ and $t - \tau$. The hypothesis $\varphi \in C^{2,\alpha}_{loc}(U)$ thus gives

$$|\varphi''(\tilde{u}) - \varphi''(u(X, t))| \leq C_2|u(Y, t - \tau) - u(X, t)|^\alpha \leq C_3(|Y - X| + \tau)^\alpha,$$

where $C_2, C_3$ are two positive constants. Then, there exist a constant $C_4 > 0$ such that

$$|u(Y, t - \tau) - u(X, t)|^2 |\varphi''(\tilde{u}) - \varphi''(u(X, t))| \leq C_4(|Y - X|^{2+\alpha} + |Y - X|^2\tau^\alpha + |Y - X|^\alpha\tau^2 + \tau^{2+\alpha}).$$

Using this, and (1.18), we find

$$|I(X, t; s)| \leq \frac{sC_4}{\Gamma(1-s)} \int_0^1 \int_{\mathbb{R}^N} 1 \tau^{1+s} (\det(\tau K(\tau)))^{1/2} \exp \left(-\frac{|K(\tau)|^{-1/2}(Y - e^{\tau B}X)^2}{4\tau}\right)$$

$$\left(|Y - X|^{2+\alpha} + |Y - X|^2\tau^\alpha + |Y - X|^\alpha\tau^2 + \tau^{2+\alpha}\right)dYd\tau.$$

We now make the change of variable $Z = K^{-1/2}(\tau)(X - e^{\tau B}Y)/\sqrt{\tau}$ in the integral over $\mathbb{R}^N$. This gives $Y = e^{\tau B}(X - (\tau K(\tau))^{1/2}Z)$, and therefore $dY = (\det(\tau K(\tau)))^{1/2}e^{\tau B}dZ$. Notice also that, in terms of the new variable $Z$, one has

$$|Y - X| = |(e^{\tau B} - I_N)X + e^{\tau B}(\tau K(\tau))^{1/2}Z| \leq |(e^{\tau B} - I_N)X| + |e^{\tau B}(\tau K(\tau))^{1/2}Z|.$$

Since for $0 < \tau < 1$ we easily find

$$|(e^{\tau B} - I_N)X| \leq C|X|\tau \leq C|X|\tau^{1/2}, \quad |e^{\tau B}(\tau K(\tau))^{1/2}Z| \leq C|Z|\tau^{1/2},$$

for some constant $C > 0$, we have

$$|(e^{\tau B} - I_N)X + e^{\tau B}(\tau K(\tau))^{1/2}Z| \leq C \max\{1, |X|\}(1 + |Z|)\tau^{1/2}. $$
It is then clear that there exists a constant $C_5 > 0$, depending on the point $X$, and on $N, \alpha$, such that, in terms of the new variable $Z$, one has for all $Z \in \mathbb{R}^N$ and $\tau \in (0, 1)$

$$|Y - X|^{2+\alpha} + |Y - X|^2 \tau^\alpha + |Y - X|^\alpha \tau^2 + \tau^{2+\alpha} \leq C_5 (1 + |Z|^{2+\alpha}) \tau^{1+\alpha/2}.$$ 

We conclude from this estimate that

$$|I(X, t; s)| \leq \frac{sC_6}{\Gamma(1-s)} \int_0^1 \frac{\tau^{1+\alpha/2}}{\tau^{1+s}} d\tau \int_{\mathbb{R}^N} (1 + |Z|^{2+\alpha}) e^{-|Z|^2} dZ,$$

$$\leq \frac{1}{\Gamma(1-s) (1-s) + \frac{\alpha}{2}} \rightarrow 0,$$

as $s \rightarrow 1^-$. This proves (1.25).

\[ \Box \]

References

1. A.V. Balakrishnan, *An operational calculus for infinitesimal generators of semigroups*. Trans. Amer. Math. Soc. 91 (1959), 330-353.
2. A.V. Balakrishnan, *Fractional powers of closed operators and the semigroups generated by them*. Pacific J. Math. 10 (1960), 419–437.
3. L. A. Caffarelli & A. Vasseur, *Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation*, Ann. of Math. (2) 171 (2010), no. 3, 1903-1930.
4. A. Córdoba & D. Córdoba, *A pointwise estimate for fractionary derivatives with applications to partial differential equations*, Proc. Natl. Acad. Sci. USA 100 (2003), no. 26, 15316-15317.
5. A. Córdoba & D. Córdoba, *A maximum principle applied to quasi-geostrophic equations*, Comm. Math. Phys. 249 (2004), no. 3, 511-528.
6. A. Córdoba & A. Martínez, *A pointwise inequality for fractional Laplacians*, Adv. Math. 280 (2015), 79-85.
7. E. De Giorgi, *Sulla differenziabilità e l’analiticità delle estremali degli integrali multipli regolari*. (Italian) Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3) 3 (1957), 25–43.
8. E. Di Nezza, G. Palatucci & E. Valdinoci, *Hitchhiker’s guide to the fractional Sobolev spaces*, Bull. Sci. Math. 136 (2012), no. 5, 521-573.
9. N. Garofalo, *Fractional thoughts*. New developments in the analysis of nonlocal operators, 1-135, Contemp. Math., 723, Amer. Math. Soc., Providence, RI, 2019.
10. N. Garofalo & G. Tralli, *A class of nonlocal hypoelliptic operators and their extensions*, ArXiv: 1811.02968.
   To appear in Indiana Univ. Math. J.
11. N. Garofalo & G. Tralli, *Hardy-Littlewood-Sobolev inequalities for a class of non-symmetric and non-doubling hypoelliptic semigroups*, ArXiv: 1904.12982
12. N. Garofalo & G. Tralli, *Functional inequalities for class of nonlocal hypoelliptic equations of Hörmander type*, to appear in Nonlinear Anal., Special Issue ‘Fractional and Nonlocal equations’ (ArXiv: 1905.08887)
13. N. Garofalo & G. Tralli, *Nonlocal isoperimetric inequalities for Kolmogorov-Fokker-Planck operators*, ArXiv: 1907.02281
14. L. Hörmander, *Hypoelliptic second order differential equations*. Acta Math. 119 (1967), 147-171.
15. N. S. Landkof, *Foundations of modern potential theory*. Translated from the Russian by A. P. Doohovskoy. Die Grundlehren der mathematischen Wissenschaften, Band 180. Springer-Verlag, New York-Heidelberg, 1972.
16. J. Moser, *A new proof of De Giorgi’s theorem concerning the regularity problem for elliptic differential equations*. Comm. Pure Appl. Math. 13 (1960), 457-468.
17. J. Moser, *On Harnack’s theorem for elliptic differential equations*. Comm. Pure Appl. Math. 14 (1961), 577-591.
18. J. Nash, *Continuity of solutions of parabolic and elliptic equations*. Amer. J. Math. 80 (1958), 931-954.
