COMPLETE FAMILIES OF COMMUTING FUNCTIONS
FOR COISOTROPIC HAMILTONIAN ACTIONS

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ABSTRACT. Let G be an algebraic group defined over a field F of characteristic zero with $g = \text{Lie} G$. The dual space $g^*$ equipped with the Kirillov-Kostant bracket is a Poisson variety and each irreducible $G$-invariant subvariety $X \subset g^*$ carries the induced Poisson structure. We prove that there is a set $\{f_1, \ldots, f_l\} \subset F[X]$ of algebraically independent polynomial functions, which pairwise commute with respect to the Poisson bracket, such that $l = (\dim X + \text{tr.deg} F(X)^G)/2$. We also discuss several applications of this result to complete integrability of Hamiltonian systems on symplectic Hamiltonian $G$-varieties of corank zero and 2.

INTRODUCTION

In this paper, we study Hamiltonian actions of algebraic groups on affine varieties focusing on the non-reductive case. The ground field $F$ is assumed to be of characteristic zero, but not necessarily algebraically closed. Let us start with main definitions in the general algebraic setting.

Definition 1. Let $\mathcal{A}$ be a commutative associative $F$-algebra equipped with an additional anticommutative bilinear operation $\{ , \} : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ called a Poisson bracket such that

$$\{a, bc\} = \{a, b\}c + b\{a, c\},$$
$$\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0$$

for all $a, b, c \in \mathcal{A}$. Then $\mathcal{A}$ is called a Poisson algebra. An ideal $I \subset \mathcal{A}$ is said to be Poisson if $\{I, \mathcal{A}\} \subset I$; a homomorphism $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ of Poisson algebras is said to be Poisson if $\varphi(\{x, y\}) = \varphi(x) \varphi(y)$ for all $x, y \in \mathcal{A}$. The Poisson centre of $\mathcal{A}$ is the Poisson subalgebra $Z \mathcal{A} := \{a \in \mathcal{A} | \{a, \mathcal{A}\} = 0\}$. A subalgebra $\mathcal{B} \subset \mathcal{A}$ is said to be Poisson-commutative if $\{\mathcal{B}, \mathcal{B}\} = 0$.

Let $\mathcal{P}$ be a Poisson algebra. Assume that $\mathcal{P}$ has no zero-divisors and $\text{tr.deg} \mathcal{P} < \infty$. Let Der($\mathcal{P}, \text{Quot} \mathcal{P}$) stand for the set of all Quot-$\mathcal{P}$-valued derivations of the algebra $\mathcal{P}$ regarded just as a commutative associative algebra. This is a linear space over Quot-$\mathcal{P}$ of dimension $\text{tr.deg} \mathcal{P}$. Each $\varphi \in \mathcal{P}$ gives rise to a derivation ad($\varphi$), where $\text{ad}(\varphi) \cdot \psi = \{\varphi, \psi\}$ for all $\psi \in \mathcal{P}$. Let $V(\mathcal{P}) := \langle \text{ad}(\varphi) | \varphi \in \mathcal{P} \rangle$ be the subspace of Der($\mathcal{P}, \text{Quot} \mathcal{P}$) spanned by the inner derivations. Then $\dim V(\mathcal{P})$ (the dimension over Quot-$\mathcal{P}$) is said to be the rank of $\mathcal{P}$, usually denoted by $\text{rk} \mathcal{P}$.

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1
If the ground field \( \mathbb{F} \) is algebraically closed and the algebra \( \mathcal{P} \) is finitely generated, then \( \text{Der}(\mathcal{P}, \text{Quot}\mathcal{P}) \) can be viewed as the space of rational vector fields on the affine algebraic variety \( \text{Spec}\mathcal{P} \). The inner derivations of \( \mathcal{P} \) are then interpreted as the Hamiltonian vector fields.

Next, set \( \omega(\text{ad}(\varphi), \text{ad}(\psi)) := \{ \varphi, \psi \} \). Since \( \text{ad}(\varphi) = 0 \) for each \( \varphi \in \mathbb{Z}\mathcal{P} \), \( \omega \) is a non-degenerate skew-symmetric bilinear form on \( V(\mathcal{P}) \) over \( \text{Quot}\mathcal{P} \). Hence, in particular, \( \text{rk}\mathcal{P} \) is even. It is not difficult to see that \( V(\mathcal{P}) \) and \( \omega \) do not change if we pass to the localisation of \( \mathcal{P} \) by a multiplicative subset of \( \mathbb{Z}\mathcal{P} \).

**Definition 2.** A Poisson algebra \( \mathcal{P} \) is said to be symplectic, if \( V(\mathcal{P}) = \text{Der}(\mathcal{P}, \text{Quot}\mathcal{P}) \), or, in other words, if \( \text{rk}\mathcal{P} = \text{tr.deg}\mathcal{P} \).

**Definition 3.** A Hamiltonian action of a (finite-dimensional) Lie algebra \( \mathfrak{q} \) on a symplectic algebra \( \mathcal{P} \) is a linear map \( \rho : \mathfrak{q} \to \mathcal{P} \) such that \( \rho([\xi, \eta]) = \{ \rho(\xi), \rho(\eta) \} \) for all \( \xi, \eta \in \mathfrak{q} \) and each \( p \in \mathcal{P} \) is contained in an \( \text{ad}(\rho(\mathfrak{q})) \)-invariant finite-dimensional subspace of \( \mathcal{P} \).

In what follows, we assume that \( \rho \) is injective and consider \( \mathfrak{q} \) as a Lie subalgebra of \( \mathcal{P} \). The Poisson subalgebra \( \mathcal{P}(\mathfrak{q}) \subset \mathcal{P} \), generated by \( \mathfrak{q} \), is called the Noether subalgebra.

Let \( \mathcal{P} \) be a symplectic algebra and \( \mathcal{A} \subset \mathcal{P} \) a Poisson subalgebra. Let \( U(\mathcal{A}) \subset V(\mathcal{P}) \) be the subspace spanned over \( \text{Quot}\mathcal{P} \) by the derivations \( \text{ad}(\varphi) \) with \( \varphi \in \mathcal{A} \).

**Definition 4.** A Hamiltonian action \( \mathfrak{q} \hookrightarrow \mathcal{P} \) is said to be coisotropic if the subspace \( U(\mathcal{P}(\mathfrak{q})) \) is coisotropic with respect to \( \omega \).

The main result of the paper is the following theorem.

**Theorem 1.** For any coisotropic Hamiltonian action of a Lie algebra \( \mathfrak{q} \) on a symplectic algebra \( \mathcal{P} \), the subalgebra \( \mathcal{P}(\mathfrak{q}) \) contains a Poisson-commutative subalgebra of transcendence degree \( \frac{1}{2}\text{rk}\mathcal{P} \).

With a few preparations it follows from a more geometric statement. Let \( \mathfrak{g} = \text{Lie}G \) be the Lie algebra of a connected algebraic (or a Lie) group \( G \), and \( \mathcal{S}(\mathfrak{g}) \) be the symmetric algebra of \( \mathfrak{g} \). Then \( \mathcal{S}(\mathfrak{g}) \) is a Poisson algebra and the algebra \( \mathfrak{g} \) acts on it in the sense of Definition 3. The same holds for any quotient of \( \mathcal{S}(\mathfrak{g}) \) by a \( G \)-invariant ideal \( I \triangleleft \mathcal{S}(\mathfrak{g}) \) (which is automatically a Poisson ideal).

**Theorem 2.** Let \( I \triangleleft \mathcal{S}(\mathfrak{g}) \) be a prime \( G \)-invariant ideal. Set

\[
I(I) := \text{tr.deg}(\mathcal{S}(\mathfrak{g})/I) - \frac{1}{2}\text{rk}(\mathcal{S}(\mathfrak{g})/I).
\]

Then there are Poisson-commuting algebraically independent functions \( f_1, \ldots, f_{I(I)} \in \mathcal{S}(\mathfrak{g})/I \).

In case \( I = 0 \), the existence of a Poisson-commutative subalgebra \( \mathcal{A} \subset \mathcal{S}(\mathfrak{g}) \) with \( \text{tr.deg}\mathcal{A} = I(\mathfrak{g}^*) \) was conjectured by Mishchenko and Fomenko [13], and proved by Sadetov [15]. A clearer treatment of this result is given by Bolsinov [2]. Note that of course the image of a Poisson-commutative subalgebra \( \mathcal{A} \subset \mathcal{S}(\mathfrak{g}) \) remains Poisson-commutative in \( \mathcal{S}(\mathfrak{g})/I \). However, transcendence degree may sink far below \( I(I) \). Our proof of Theorem 2 follows the same strategy as the proofs of Sadetov and Bolsinov for \( \mathcal{S}(\mathfrak{g}) \). Note that
in the general case our functions \( f_1, \ldots, f_l(I) \in S(g)/I \) do not extend to Poisson-commuting functions in \( S(g) \).

First we prove Theorem 2 in case of a reductive \( g \), see Section 3. In the general case, we argue by induction on \( \dim g \), see Section 5. We remark that the number \( l(I) \) does not change under field extensions.

Some applications of Theorems 1 and 2 to integrable Hamiltonian systems are discussed in Section 2.

1. SYMPLECTIC ALGEBRAS AND HAMILTONIAN ACTIONS

Consider a Poisson algebra \( \mathcal{P} \). Assume that \( \mathcal{P} \) has no zero-divisors and that \( \text{tr.deg} \mathcal{P} < \infty \).

For each subalgebra \( C \subset \mathcal{P} \), let \( C^{-1} \mathcal{P} \) denote the localisation of \( \mathcal{P} \) by the subset of all non-zero elements of \( C \). Clearly \( C^{-1} \mathcal{P} \) is a subset of the field \( \text{Quot} \mathcal{P} \). The Poisson structure uniquely extends from \( \mathcal{P} \) to \( \text{Quot} \mathcal{P} \) and for any multiplicative system \( S \subset \mathcal{P} \) the localisation \( \mathcal{P}_S \) is a Poisson subalgebra of \( \text{Quot} \mathcal{P} \). In particular, this is true for \( C^{-1} \mathcal{P} \). If \( C \subset \mathcal{Z} \mathcal{P} \), then \( C^{-1} \mathcal{P} \) can be regarded as a Poisson algebra over the field \( \text{Quot} C \).

**Definition 5.** A Poisson algebra \( \mathcal{P} \) is said to be **separable** if \( \text{tr.deg} \mathcal{Z} \mathcal{P} + \text{rk} \mathcal{P} = \text{tr.deg} \mathcal{P} \).

Roughly speaking, \( \mathcal{P} \) is separable if generic symplectic leaves of the underlying Poisson affine variety \( X = \text{Spec}(\mathcal{P} \otimes_{\mathcal{F}} \mathcal{F}) \) are separated by the “central” functions, elements of \( \mathcal{Z} \mathcal{P} \otimes_{\mathcal{F}} \mathcal{F} \).

If \( \mathcal{P} \) is a separable Poisson algebra, then \( (\mathcal{Z} \mathcal{P})^{-1} \mathcal{P} \) is a symplectic algebra over \( \text{Quot} \mathcal{Z} \mathcal{P} \), see Definition 2.

**Example 6.** Let \( W \) be a finite-dimensional vector space over \( \mathcal{F} \) equipped with a non-degenerate skew-symmetric bilinear form \( \omega \). Then \( \omega \) defines a Poisson bracket on the symmetric algebra \( S(W) \) by the formula

\[
\{x, y\} := \omega(x, y) \quad \text{for all } x, y \in W.
\]

This Poisson algebra \( (S(W), \omega) \) is symplectic and \( V(S(W)) = \text{Quot}S(W) \otimes_{\mathcal{F}} W \) with the same (extended) form \( \omega \).

The algebra \( S(W) \) has a natural grading, with grading components being \( S^k(W) \), and \( \{S^k(W), S^l(W)\} \subset S^{k+l-2}(W) \). Hence \( q := \mathcal{F} + W + S^2(W) \) is a Lie subalgebra and \( n := \mathcal{F} + W \) is an ideal of \( q \). Note that \( n \) is a Heisenberg algebra. The map

\[
q \rightarrow \text{Der} n
\]

\[
\delta \mapsto \text{ad}(\delta)|_n
\]

is an epimorphism of Lie algebras with the kernel \( \mathcal{F} \) and \( S^2(W) \) is mapped isomorphically onto the Lie algebra \( \mathfrak{sp}(W) \) of the symplectic group \( \text{Sp}(W, \omega) \). Note also that the centraliser of \( n \) in \( S(W) \) coincides with \( \mathcal{F} \).

**Example 7.** Let \( q \) be a (finite-dimensional) Lie algebra over \( \mathcal{F} \). Then \( S(q) \) is a Poisson algebra with the usual Kirillov-Kostant bracket. The corresponding symplectic vector space \( V(S(q)) \) can be constructed as follows. Set \( \mathbb{K} := \text{Quot}S(q) \). Recall that \( \mathbb{K} \) is also a
Poisson algebra. Set further \( \tilde{\omega}(\xi, \eta) := [\xi, \eta] \) for all \( \xi, \eta \in \mathfrak{q} \). Since \( \tilde{\omega}(\xi, \eta) \in \mathbb{K} \), this formula defines a skew-symmetric bilinear form on a \( \mathbb{K} \)-vector space \( \tilde{V} := \mathfrak{q} \otimes_\mathbb{F} \mathbb{K} \). (In a basis of \( \mathfrak{q} \), \( \tilde{\omega} \) is just the structural matrix.) Then \( V(S(\mathfrak{q})) = \tilde{V}/\text{Ker}\tilde{\omega} \). Let us say that \( \mathfrak{q} \) is **separable** if \( \text{tr.deg} \mathfrak{z} S(\mathfrak{q}) = \text{tr.deg} \mathfrak{z} \mathbb{K} \). A Lie algebra \( \mathfrak{q} \) is separable if and only if \( S(\mathfrak{q}) \) is separable. In that case \( (\mathfrak{z} S(\mathfrak{q}))^{-1} S(\mathfrak{q}) \) is a symplectic algebra over \( \text{Quot}\mathfrak{z} S(\mathfrak{q}) \).

The next two statements follow easily from the construction of \( (V(P), \omega) \) and Definition 2.

**Proposition 8.** Let \( P \) be a symplectic algebra and \( \mathcal{A} \subset P \) a Poisson subalgebra. Let \( U(\mathcal{A}) \subset V(P) \) be a subspace spanned over \( \text{Quot} P \) by the derivations \( \text{ad}(\phi) \) with \( \phi \in \mathcal{A} \). Then

1. \( \dim_{\text{Quot} P} U(\mathcal{A}) = \text{tr.deg} \mathcal{A} \);
2. \( \text{rk} \omega_{U(\mathcal{A})} = \text{rk} \mathcal{A} \);
3. \( U(\mathcal{A})/\text{Ker} w_{U(\mathcal{A})} = \text{Quot} P \otimes_{\text{Quot} \mathcal{A}} V(\mathcal{A}) \).

**Proposition 9.** Let \( \mathcal{A}, \mathcal{B} \subset P \) be two Poisson subalgebras of a symplectic algebra \( P \). Then \( \{\mathcal{A}, \mathcal{B}\} = 0 \) if and only if the subspaces \( U(\mathcal{A}) \) and \( U(\mathcal{B}) \) are orthogonal with respect to \( \omega \).

Combining Propositions 8 and 9, we get that if \( \{\mathcal{A}, \mathcal{B}\} = 0 \), then \( \text{tr.deg} \mathcal{A} + \text{tr.deg} \mathcal{B} \leq \text{rk} P \).

From now on assume that \( P \) is symplectic and that we have a Hamiltonian action of a Lie algebra \( \mathfrak{q} \) on \( P \), see Definition 3. Set

\[ P^q := \{ \phi \in P \mid \{q, \phi\} = 0 \} = \mathfrak{z}_P(P(q)). \]

As above, \( \text{tr.deg} P(q) + \text{tr.deg} P^q \leq \text{rk} P \). A Hamiltonian action is said to be **separable** if \( \text{tr.deg} P(q) + \text{tr.deg} P^q = \text{rk} P \). It is possible to characterise separable Hamiltonian coisotropic actions.

**Proposition 10.** A separable Hamiltonian action \( q \mapsto P \) is coisotropic if and only if \( \{P^q, P^q\} = 0 \).

**Proof.** Recall that for a separable action, the orthogonal complement of \( U(P(q)) \) coincides with \( U(P^q) \). Hence the action \( q \mapsto P \) is coisotropic if and only if \( U(P^q) \subset V(P) \) is an isotropic subspace. According to Proposition 9, this condition is equivalent to the Poisson-commutativity of \( P^q \).

**Proposition 11.** Theorem 1 follows from Theorem 2.

**Proof.** The subalgebra \( P(q) \subset P \) is isomorphic to some Poisson quotient \( S(q)/I \), where \( I \lhd S(\mathfrak{g}) \) is a Poisson ideal. In particular, \( I \) is \( G \)-invariant. Since \( P \) is a domain, the algebra \( S(q)/I \) is also a domain. By Theorem 2, \( S(q)/I \) contains a Poisson-commutative subalgebra \( \mathcal{A} \) with \( \text{tr.deg} \mathcal{A} = l \), where

\[ l = \text{tr.deg} P(q) - \frac{1}{2} \text{rk} P(q). \]

Combining Definition 4 and Proposition 8, we see that

\[ \text{rk} P(q) = \text{rk} P - 2(\text{rk} P - \text{tr.deg} P(q)) = 2\text{tr.deg} P(q) - \text{rk} P \]

and therefore \( l = \frac{1}{2} \text{rk} P \). \( \square \)
2. Geometric realisation and Applications

Suppose that \( X \) is an irreducible affine variety defined over \( \mathbb{F} \). Let \( X(\mathbb{F}) \) denote the set of its points over the algebraic closure of \( \mathbb{F} \). As usual \( \mathbb{F}[X] \) and \( \mathbb{F}(X) := \text{Quot} \mathbb{F}[X] \) stand for the algebras of regular and rational functions on \( X \), respectively. Our convention is that \( \mathbb{F}[X(\mathbb{F})] = \mathbb{F}[X] \otimes_{\mathbb{F}} \mathbb{F} \). All subvarieties of \( X \), all differential forms on \( X \), and all morphisms of \( X \) are supposed to be defined over \( \mathbb{F} \).

Let \( G \) be a connected linear algebraic group over \( \mathbb{F} \) with \( g = \text{Lie} G \). An algebraic action of \( G \) on \( X \) gives rise to a representation of \( G \) (and of \( g \)) on \( \mathbb{F}[X] \).

**Definition 12** (Geometric version of Definition 3). Suppose that \( Y \) is an affine variety such that \( \mathbb{F}[Y] \) is a Poisson algebra. An algebraic action \( G \times Y \rightarrow Y \) is said to be **Hamiltonian** if there is a \( G \)-equivariant map, called the moment map, \( \mu: Y \rightarrow g^* \) such that \( \mu^*: S(g) \rightarrow \mathbb{F}[Y] \) is a Poisson homomorphism and \( \{\mu^*(\xi), h\} = \xi \cdot h \) for all \( \xi \in g \), \( h \in \mathbb{F}[Y] \).

Suppose that we have a Hamiltonian action \( G \times Y \rightarrow Y \). Then each function in \( \mu^*(S(g)) \) is called a **Noether integral** on \( Y \). Their most important property is given by the Noether theorem: \( \{\mathbb{F}(Y)^G, \mu^*(S(g))\} = 0 \). The kernel of \( \mu^* \) is a Poisson ideal of \( S(g) \), say \( I \), and therefore \( S(g)/I \) is a Poisson quotient of \( S(g) \).

Let \( I \subset S(g) \) be a \( G \)-invariant prime ideal. Being \( G \)-invariant implies that \( \{g, I\} \subset I \). In other words \( I \) is a Poisson ideal. Set \( X = \text{Spec}(S(g)/I) \). Then \( \mathbb{F}[X] \) is a Poisson algebra and \( X \) is a Poisson variety. Set

\[
I(X) := (\dim X + \text{tr.deg} \mathbb{F}(X)^G)/2.
\]

It follows from Rosenlicht’s theorem, that

\[
I(X) = \frac{1}{2} (\dim X + (\dim X - \max_{\gamma \in X(\mathbb{F})} \dim (g \gamma))) = \dim X - \frac{1}{2} \max_{\gamma \in X(\mathbb{F})} \dim (g \gamma),
\]

where \( X(\mathbb{F}) \subset (g \otimes_{\mathbb{F}} \mathbb{F})^* \) and \( g \gamma = T_\gamma(G \gamma) \). Let \( R \subset \mathbb{F}[X] \) be a Poisson commutative subalgebra. Take \( \gamma \in X(\mathbb{F}) \) such that the orbit \( G \gamma \) is of maximal possible dimension. The subspace

\[
\langle d_{\gamma} a \mid a \in R \rangle \subset T_\gamma^* X(\mathbb{F}) \subset g \otimes_{\mathbb{F}} \mathbb{F},
\]

spanned over \( \mathbb{F} \) by the differentials \( d_{\gamma} a \), is isotropic with respect to the symplectic form \( \gamma(x, y) = \gamma([x, y]) \) (here \( x, y \in g \otimes_{\mathbb{F}} \mathbb{F} \)). Hence the dimension of this subspace is less than or equal to \( I(X) \) and also \( \text{tr.deg} R \leq I(X) \). A family \( \{f_1, \ldots, f_i(X)\} \subset \mathbb{F}[X] \) is said to be **complete** if \( \{f_i, f_j\} = 0 \) for all \( i, j \) and \( f_1, \ldots, f_{i(X)} \) are algebraically independent.

From now until the end of this section, assume that the geometric points of an irreducible affine variety \( M \) form a dense subset of \( M(\mathbb{F}) \). Suppose further that there is a non-degenerate closed regular 2-form \( \varnothing \) on the smooth locus of \( M(\mathbb{F}) \). Then \( \varnothing \) induces a Poisson bracket on \( \mathbb{F}(M) \), and \( \mathbb{F}(M) \) is a symplectic Poisson algebra in the sense of Definition 2. The variety \( M \) is said to be **symplectic** if \( \mathbb{F}[M] \) is a Poisson subalgebra of \( \mathbb{F}(M) \), i.e., if \( \{\mathbb{F}[M], \mathbb{F}[M]\} \subset \mathbb{F}[M] \). In that case \( \mathbb{F}[M] \) is a symplectic algebra as well. This is always the case for normal affine varieties. Set \( 2n := \dim_{\mathbb{F}} M = \text{tr.deg} \mathbb{F}(M) \). A family of functions \( \{f_1, \ldots, f_n\} \subset \mathbb{F}(M) \) such that \( \{f_i, f_j\} = 0 \) for all \( i \) and \( j \) is said to be **complete** if the
$f_i$'s are algebraically independent. A complete family of functions generates a Poisson-commutative subalgebra $\mathcal{A} \subset \mathbb{F}(M)$ with $\text{tr.deg}\, \mathcal{A} = n$.

The simplest example of a symplectic variety is an even-dimensional vector space $V$ equipped with a non-degenerate skew-symmetric bilinear form $\omega$. Each Lagrangian decomposition $V = V_+ \oplus V_-$ gives us a complete family of linear functions on $V$, namely, one has to take a basis of $V_+$. Another familiar example is the cotangent bundle of a smooth irreducible affine variety $Y$, $M = T^*Y$, equipped with the canonical symplectic structure. Here $\mathbb{F}[Y]$ is a Poisson commutative subalgebra of $\mathbb{F}[M]$. Since $\dim M = 2\dim Y$, the subalgebra $\mathbb{F}[Y]$ contains a complete family of functions on $M$.

It is a challenging open problem to prove that for each affine symplectic variety $M$, the Poisson algebra $\mathbb{F}[M]$ contains a complete family.

Suppose $h \in \mathbb{F}[M]$. Let $\eta_h$ be the vector field on the smooth locus of $M$ uniquely defined by the formula $dh = \omega(\eta_h, \cdot)$. Then $\eta_h$ defines a Hamiltonian dynamical systems on $M$, and any function $f$ on $M$ such that $\{h, f\} = 0$ is called a first integral of this system. The intersection of the level hypersurfaces of first integrals is stable with respect to the flow generated by $\eta_h$. Thus, to understand dynamical properties of $\eta_h$, it is desirable to construct as many independent first integrals as possible. The triple $(M, \omega, h)$ is said to be completely integrable if there are algebraically independent first integrals $f_1, \ldots, f_n$ such that $\{f_i, f_j\} = 0$ and $2n = \dim M$.

Let $(M, \omega)$ be a symplectic affine variety and $G \times M \to M$ a Hamiltonian algebraic action. Write $M^{\text{reg}}$ for the smooth locus of $M$. For each $x \in M^{\text{reg}}$, let $(gx)^\perp \subset T_xM$ denote the orthogonal complement of $gx$ taken with respect to $\omega$. The function

$$x \mapsto \dim (gx \cap (gx)^\perp)$$

is constant on a non-empty open subset $U \subset M^{\text{reg}}$ and its value $d$ on $U$ is called the defect of the action $G \times M \to M$ (see [16, Chapter II, §3]).

**Definition 13.** The corank of $G \times M \to M$, denoted by $\text{cork}\, M$, is defined by the formula

$$\text{cork}\, M := \min_{x \in M^{\text{reg}}} \dim (gx)^\perp - d.$$

In other words, it equals the rank of the form $\omega|_{(gx)^\perp}$ for generic $x \in M^{\text{reg}}$. A Hamiltonian action of $G$ on a symplectic variety $M$ is said to be coisotropic if $\text{cork}\, M = 0$, i.e., if $(gx)^\perp \subset gx$ for generic $x \in M^{\text{reg}}$.

**Theorem 3** (A geometric version of Theorem 1). Let $G \times M \to M$ be a coisotropic Hamiltonian action on a symplectic variety $M$ and let $\mu : M \to g^*$ be the corresponding moment map. Then there are functions $f_1, \ldots, f_n \in S(g)$, where $n = \dim M/2$, such that $\{\mu^*(f_1), \ldots, \mu^*(f_n)\}$ is a complete family on $M$.

**Proof.** Let $I < S(g)$ be the kernel of $\mu^*$. Then $I$ is a prime Poisson ideal of $S(g)$. Set $X := \text{Spec}(S(g)/I)$ and take $x \in M$. Using the fact that $M$ is a symplectic variety and the property $\{\mu(\xi), h\} = \xi \cdot h$ of the moment map, see Definition 12, one deduces that the kernel of $d\mu_x$ coincides with $(gx)^\perp$, cf. [16, Chapter II, §2]. Therefore, $\dim X = \max_{x \in M} \dim (gx)$.
and $\max_{Y \in \mathcal{F}} \dim_{\mathcal{F}}(gY) = \max_{x \in M} \dim_{\mathcal{F}}(gx) - d$. Choose $x \in M$ such that $\dim(gx)$ is maximal. Then

$$l(X) = l(I) = \dim(gx) - \frac{1}{2}(\dim(gx) - d) = (\dim(gx) + d)/2 = (\dim M - \text{cork} M)/2.$$  

Clearly, $2l(X) = \dim M$ if and only if the action $G \times M \to M$ is coisotropic. By virtue of Theorem 2, there is a complete family $\{f_i\}$ in $S(g)/I = \mathbb{F}[X]$. Since $\text{cork} M = 0$ and $\mu^*$ is a Poisson homomorphism, $\{\mu^*(f_i)\}$ is a complete family on $M$. \quad \square

**Corollary.** A Hamiltonian action $G \times M \to M$ is coisotropic if and only if there is a complete family of Noether integrals on $M$; or, equivalently, each $G$-invariant Hamiltonian system on $M$ is completely integrable in the class of Noether integrals.

**Theorem 4.** Let $G \times M \to M$ be a Hamiltonian action with $\text{cork} M = 2$. Then there is a complete family in $\mathbb{F}(M)$. If in addition generic $G(\mathbb{F})$-orbits on $M(\mathbb{F})$ are separated by regular invariants, then there is a complete family in $\mathbb{F}[M]$.

**Proof.** Let $I \subset S(g)$ be the kernel of $\mu^*$. Set $X := \text{Spec}(S(g)/I)$. Then $l(X) = \dim M/2 - 1$. By Theorem 2, there are functions $f_1, \ldots, f_{l(X)} \in S(g)$ such that their restrictions to $X$ form a complete family. Set $R := \mu^*(S(g))$. Let $\langle d_i R \rangle$ be the subspace of $T^*_x M$ spanned over $\mathbb{F}$ by all differentials $d_i f$ with $f \in R$. Since $(gX)^\perp$ is the kernel of $d\mu_x$, we have $\langle d_i R \rangle = \text{Ann}((gX)^\perp)$. By Rosenlicht’s theorem, generic $G(\mathbb{F})$-orbits on $M(\mathbb{F})$ are separated by rational invariants. Therefore, $\langle d_i (\mathbb{F}(M)^G) \rangle = \text{Ann}(gX)$ for generic $x \in M$. Since the action $G \times M \to M$ is not coisotropic, $(gX)^\perp \not\subset gX$ and there is at least one $h \in \mathbb{F}(M)^G$ such that functions $\{h, \mu^*(f_1), \ldots, \mu^*(f_{l(X)})\}$ are algebraically independent. Recall that $\{\mathbb{F}(M)^G, R\} = 0$. Thus, $\{h, \mu^*(f_1), \ldots, \mu^*(f_{l(X)})\}$ is a complete family on $M$. If generic $G(\mathbb{F})$-orbits on $M(\mathbb{F})$ are separated by regular invariants, then $\mathbb{F}(M)^G = \text{Quot} \mathbb{F}[M]^G$ and we can choose $h$ in $\mathbb{F}[M]^G$. \quad \square

Let us say a few words about cotangent bundles. It was already mentioned that a complete family always exists here. But the construction of Theorems 2 and 3 provides other examples of complete families, which can be useful for other Hamiltonian systems.

Suppose that $M = T^*X$, where $X$ is a $G$-variety. Then $M$ possesses a canonical $G$-invariant symplectic structure such that the action of $G$ is Hamiltonian. If the action $G \times M \to M$ is coisotropic, then $X$ has an open $G$-orbit [5]. For reductive $G$ one can say more.

Suppose $\mathbb{F}$ is algebraically closed, $G$ is reductive, and $X$ is smooth. By a result of Knop [8, Sections 6&7], the action of $G$ on $T^*X$ is a coisotropic if and only if a Borel subgroup $B$ of $G$ has on open orbit on $X$. Normal varieties having an open $B$-orbit are said to be spherical. It was known before that if $X$ is spherical and $X = G/H$, where $H$ is a reductive subgroup of $G$, then each $G$-invariant Hamiltonian system on $T^*X$ is integrable within the class of Noether integrals, see [5, 11, 7]. Here we lift the assumption that $H$ is reductive. Smooth affine spherical varieties are classified (under mild technical constraints) in [9]. It would be interesting to study complete families on their cotangent bundles.
By the same result of Knop [8], the action of $G$ on $T^*X$ is of corank 2 if and only if $\text{tr}.\deg \mathcal{F}(X)^B = 1$, i.e., $X$ has complexity 1. Theorem 4 provides also (hopefully) interesting completely integrable systems for these cotangent bundles.

Other well-studied coisotropic actions on cotangent bundles are related to Gelfand pairs. Suppose that $\mathcal{F} = \mathbb{R}$ and $M = T^*X$, where $X = G/K$ is a Riemannian homogeneous space. Then $X$ is called commutative or the pair $(G,K)$ is called a Gelfand pair if the action $G \times M \to M$ is coisotropic. Gelfand pairs can be characterised by the following equivalent conditions.

(i) The algebra $\mathcal{D}(X)^G$ of $G$-invariant differential operators on $X$ is commutative.

(ii) The algebra of $K$-invariant measures on $X$ with compact support is commutative with respect to convolution.

(iii) The representation of $G$ on $L^2(X)$ has a simple spectrum.

Theorem 3 and its corollary provide two more equivalent conditions.

(iv) There is a complete family of Noether integrals on $T^*X$.

(v) Each $G$-invariant Hamiltonian system on $M$ is completely integrable in the class of Noether integrals.

According to [16], if $G/K$ is a Gelfand pair and $G = L \rtimes N$ is a Levi decomposition of $G$ such that $K \subset L$, then $\mathbb{R}[n]^L = \mathbb{R}[n]^K$ and $n$ is at most two-step nilpotent. These conditions guarantee that the construction of a complete family on $\mu(M)$ would have at most three induction steps. Thus, one can hope for explicit formulas for our commuting families and applications to physical problems. Gelfand pairs are partly classified in [17, 19] and completely in [20].

3. THE REDUCTIVE CASE

In this section, $G$ is a connected reductive algebraic group. Here one can apply a very powerful tool, the so called “argument shift method”. It was used by Manakov [10], Mishchenko and Fomenko [12], and Bolsinov [1] in constructions of complete families on $g^*$ and coadjoint $G$-orbits. The reader is referred to [6, Chapter 4] for a thorough exposition and historical remarks. Let us briefly outline this method.

Let $r$ be the rank of $g$. Choose any set $F_1, \ldots, F_r$ of free generators of $\mathbb{F}[g^*]^G$. For any $a \in g^*$, let $\mathcal{F}_a$ denote the finite set

$$\{F_i, \partial_a F_i, \partial_a^2 F_i, \ldots, \partial_a^{k(i)} F_i \mid i = 1, \ldots, r, k(i) = \deg F_i - 1\} \subset S(g).$$

Then $\{\mathcal{F}_a, \mathcal{F}_a\} = 0$, see e.g. [14, Sections 1.12, 1.13]. Here we should mention that this fact is stated in [14] for $\mathbb{F} = \mathbb{C}$, but the proofs are valid over all fields of characteristic zero.

Recall that the index of a Lie algebra $q$ is the minimum of dimensions of stabilisers $q_\xi$ over all covectors $\xi \in q^*$, i.e., $\text{ind}q = \min_{\xi \in q^*} \dim q_\xi$. Note that $\text{ind}q = \text{tr}.\deg \mathcal{F}(q^*)^g$ and that $\dim q - \text{ind}q$ is the rank of the Poisson algebra $S(q)$ as defined in the Introduction.
Proposition 14. [1, Theorem 2] Suppose that $\mathfrak{g}$ is a complex reductive Lie algebra and $\xi \in \mathfrak{g}^*$. Then there is $a \in \mathfrak{g}^*$ such that the restriction of $\mathcal{F}_a$ to the coadjoint orbit $G\xi$ contains $\frac{1}{2}\dim(G\xi)$ algebraically independent functions if and only if $\text{ind}\, \mathfrak{g}_\xi = \text{ind}\, \mathfrak{g}$.

The proof of Theorem 2 in [1] uses only linear algebra and can be repeated for any algebraically closed field of characteristic zero. We are going to use the result also for $\mathbb{F} \neq \mathbb{C}$.

Proposition 15. If $\mathfrak{g}$ is reductive and $\xi \in \mathfrak{g}^*$, then $\text{ind}\, \mathfrak{g}_\xi = \text{ind}\, \mathfrak{g}$.

The statement of Proposition 15 is known as Elashvili’s conjecture. For the classical Lie algebras, it is proved in [18] under the assumption that $\text{char}\, \mathbb{F}$ is good for $\mathfrak{g}$. W. de Graaf used a computer program to verify the conjecture for the exceptional Lie algebras, see [4]. An almost conceptual proof of Elashvili’s conjecture is given in [3]. (The authors still have to rely on computer calculations for a few orbits.)

Let $\hat{\mathcal{V}}_{a,\xi} \subseteq T^*_\xi(\mathfrak{g}^*)$ be the $\mathbb{F}$-linear span of the differentials $\{d_\xi F \mid F \in \mathcal{F}_a\}$ and let $\mathcal{V}_{a,\xi}$ be the restriction of $\hat{\mathcal{V}}_{a,\xi}$ to $T^*_\xi(G\xi) = \mathfrak{g}_\xi$. Since the orbit $G\xi$ is a symplectic variety and the subspace $\mathcal{V}_{a,\xi}$ is isotropic, we get $2\dim \mathcal{V}_{a,\xi} \leq \dim(G\xi)$. The restriction of $\mathcal{F}_a$ to $G\xi$ contains a complete family if and only if there is $a' \in \mathfrak{a}$ such that $2\dim \mathcal{V}_{a',\xi} = \dim(G\xi)$.

Combining Propositions 14 and 15, we obtain the following assertion.

Proposition 16. Suppose that $\mathbb{F} = \overline{\mathbb{F}}$. Then for each $\xi \in \mathfrak{g}^*$, there is $a \in \mathfrak{g}^*$ such that $2\dim \mathcal{V}_{a,\xi} = \dim(G\xi)$.

Proof of Theorem 2 in the reductive case. Let $I$ be a prime Poisson ideal of $\mathcal{S}(\mathfrak{g})$ and $X(\overline{\mathbb{F}})$ a closed subvariety of $(\mathfrak{g} \otimes \overline{\mathbb{F}})^*$ defined by $I$.

Choose a set of homogeneous generators $\{F_1, \ldots, F_r\} \subseteq \mathbb{F}[\mathfrak{g}^*]^G$. Let $\hat{F}_i$ denote the restriction of $F_i$ to $X$. Each fibre of the quotient morphism $X(\overline{\mathbb{F}}) \to X(\overline{\mathbb{F}})//G(\overline{\mathbb{F}})$ contains finitely many $G$-orbits. Hence for generic $\xi \in X(\overline{\mathbb{F}})$ the differentials $\{d_\xi \hat{F}_i \mid i = 1, \ldots, r\}$ generate a subspace of dimension $m := \dim X - \dim(G\xi)$. According to Proposition 16, there is an element $a \in (\mathfrak{g} \otimes \overline{\mathbb{F}})^*$ such that the restriction of $\mathcal{F}_a$ to $G(\overline{\mathbb{F}})\xi$ contains a complete family, i.e., $2\dim \mathcal{V}_{a,\xi} = \dim(G(\overline{\mathbb{F}})\xi)$. There is an open subset of such elements. In particular, we may (and will) assume that $a \in \mathfrak{g}^*$. Then $\mathcal{F}_a$ is a subset of $\mathcal{S}(\mathfrak{g})$. Each differential $d_\xi \hat{F}_i$ is zero on $\mathfrak{g}_\xi$. Therefore

$$\dim \langle d_\xi f \mid f \in \mathcal{F}_a/(\mathcal{F}_a \cap I) \rangle = m + \dim(G\xi)/2 = (m + \dim X)/2 = \ell(X)$$

and the restriction of $\mathcal{F}_a$ to $X$ contains a complete family. \hfill $\Box$

4. Auxiliary results

In this section, we collect several facts concerning structural properties of algebraic Lie algebras. They will be used in the proof of the main theorem.

Recall that a $(2n+1)$-dimensional Heisenberg Lie algebra over $\mathbb{F}$ is a Lie algebra $\mathfrak{h}$ with a basis $\{x_1, \ldots, x_n, y_1, \ldots, y_n, z\}$ such that $n \geq 1$, $[x_i, x_j] = [y_i, y_j] = 0$, $[h, z] = 0$, and $[x_i, y_j] = \delta_{ij}z$. Recall also that a Lie ideal $a \triangleleft \mathfrak{q}$ is said to be a characteristic ideal if it is stable under all automorphisms of the Lie algebra $\mathfrak{q}$. 
Lemma 17. Suppose that $n$ is a nilpotent Lie algebra such that each commutative characteristic ideal of $n$ is one-dimensional. Then $n$ is a Heisenberg algebra.

Proof. Let $z$ be the centre of $n$. Then $\dim z = 1$. Consider the upper central series of $n$

$$z = n_0 \subset n_1 \subset n_2 \subset \cdots \subset n_{k-1} \subset n_k = n,$$

i.e., $n_i/n_{i-1}$ is the centre of $n/n_{i-1}$. The centre of $n_1$ is a commutative characteristic ideal of $n$. Hence, it is one-dimensional and coincides with $z$. Therefore $n_1$ is a Heisenberg algebra. Let $z_n(n_1)$ be the centraliser of $n_1$ in $n$. Clearly, $z_n(n_1)$ is an ideal in $n$ and $n_1 \cap z_n(n_1) = z$.

We claim that $n = n_1 + z_n(n_1)$. Indeed, let $z \in n$. Then $[z, n_1] \subset n_0$ and there is an element $z_0 \in n_1$ such that $[z - z_0, n_1] = 0$.

Let $z_0$ be the centre of $z_n(n_1)/z$. Since $n/z = (n_1/z) \oplus (z_n(n_1)/z)$ is the direct sum of two ideals, $z_0$ lies in the centre of $n/z$. Thus, $z_0 \subset (n_1/z)$ and $z_0 = 0$. Since $z_n(n_1)/z$ is a nilpotent Lie algebra, we have $z_n(n_1)/z = 0$, and $n = n_1$ is a Heisenberg algebra. □

Let $N$ be the unipotent radical of an affine algebraic group $G$. Set $n := \text{Lie}N$. For any action $P \times Y \rightarrow Y$ let $Y/P$ stand for the set of $P$-orbits on $Y$.

Lemma 18. Suppose that $n$ is a Heisenberg Lie algebra and the centre $z$ of $n$ lies in the centre of $g$.

Given a non-zero $\alpha \in z^*$, set $Y_\alpha := \{\gamma \in g^* \mid \gamma(z) = \alpha\}$. Then $Y_\alpha/N = \text{Spec} \mathbb{F}[Y_\alpha]^N$; the natural action of $G/N$ on $Y_\alpha/N$ is Hamiltonian in the sense of Definition 12 and the moment map $\mu : Y_\alpha/N \rightarrow (g/n)^*$ is a $G$-isomorphism.

Proof. Choose a Levi decomposition $G = L \ltimes N$ and let $V$ be an $L$-invariant complement of $z$ in $n$. Set $S_\alpha := \{\gamma \in g^* \mid \gamma(V) = 0, \gamma(z) = \alpha\}$. In other words, $S_\alpha = (g/n)^* + \tilde{\alpha}$, where $\tilde{\alpha} \in g^*$, $\tilde{\alpha}(V) = 0$, $\tilde{\alpha}(l) = 0$, and $\tilde{\alpha}(z) = \alpha$. Clearly $S_\alpha \subset Y_\alpha$. Each point $\gamma \in Y_\alpha$ can be uniquely presented as a sum

$$\gamma = \beta + \text{ad}^* (\eta) \cdot \tilde{\alpha} + \tilde{\alpha}, \text{ where } \beta(n) = 0 \text{ and } \eta \in V.$$

Thus $N\gamma \cap S_\alpha = \gamma - \text{ad}^* (\eta) : \gamma + \frac{1}{2} (\text{ad}^* (\eta))^2 : \gamma = \{pr\}$. We obtain the isomorphism $\mu : Y_\alpha/N \rightarrow (g/n)^*$, where $\mu(N\gamma)$ is the unique point in $N\gamma \cap S_\alpha$. Therefore $Y_\alpha/N$ is an algebraic variety (an affine space) and $\mathbb{F}[Y_\alpha]^N = \mathbb{F}[Y_\alpha]^N$. For the rest of the proof, we fix the isomorphism $(g/n)^* \cong \Gamma^*$ given by the Levi decomposition $g = l \oplus n$ and the induced isomorphisms $S_\alpha \cong (\Gamma^* + \tilde{\alpha}) \cong \Gamma^*$, where the last one is given by choosing $\tilde{\alpha}$ as the origin.

For each $\gamma \in S_\alpha$ and $l \in L$, we have $\mu(l \cdot N\gamma) = l\gamma - \tilde{\alpha} = l(\gamma - \tilde{\alpha}) = l \cdot \mu(N\gamma)$. This shows that $\mu$ is $G$-equivariant.

It remains to prove that $\mu^*$ is a homomorphism of the Poisson algebras $S(g/n)$ and $\mathbb{F}[Y_\alpha]^N$, i.e., to show that $\{\mu^*(f_1), \mu^*(f_2)\} = \mu^*(\{f_1, f_2\})$ for all $f_1, f_2 \in S(g/n)$.

Let $\gamma \in S_\alpha$. The identification $S(g/n) \cong S(l) \subset S(g)$ gives us that

$$\{f_1, f_2\} (\gamma) = \{f_1, f_2\} (\gamma - \tilde{\alpha}) = \{f_1, f_2\} (\mu(N\gamma)) = \mu^*(\{f_1, f_2\})(N\gamma).$$

The last step is to prove that $\{\mu^*(f_1), \mu^*(f_2)\}(N\gamma) = \{f_1, f_2\}(\gamma)$. It is well-known that $G\gamma$ is a symplectic leaf of $Y_\alpha$ and $g^*$. Also $L\gamma$ is a symplectic leaf of $S_\alpha$. We have

$$T_\gamma(G\gamma) = T_\gamma(L\gamma) \oplus T_\gamma(N\gamma),$$

where $T_\gamma(L\gamma) = l\gamma$ and $T_\gamma(N\gamma) = n\gamma$ are orthogonal,
one can deduce that

\[ \{ \mu^*(f_1), \mu^*(f_2) \} (N\gamma) = \{ F_1, F_2 \} (\gamma). \]

Clearly, the functions \( F_1 \) and \( F_2 \) are \( N \)-invariant, hence \( d_Hf_i(\eta\gamma) = 0 \). Thus \( \{ F_1, F_2 \} (\gamma) = \{ F_1\vert_{LY}, F_2\vert_{LY} \} (\gamma) = \{ f_1, f_2 \} (\gamma) \) and we are done. \( \square \)

**Corollary.** In the setting of Lemma 18, we have

\[ (\mathbb{F}[g^*][1/z])^N \cong \mathcal{S}(g/n) \otimes_\mathbb{F} \mathbb{F}[z, 1/z] \subset \mathcal{S}(g/n) \otimes_\mathbb{F} \mathbb{F}(z^*), \]

where \( z \) is a non-zero element of \( z \). Moreover, if \( X(\mathbb{F}) \subset (g \otimes_\mathbb{F} \mathbb{F})^* \) is a closed \( G(\mathbb{F}) \)-invariant subset defined over \( \mathbb{F} \) and such that \( z|_X(\mathbb{F}) \neq 0 \), then \( (\mathbb{F}[X][1/z])^N \) is a Poisson quotient of \( \mathcal{S}(g/n) \otimes_\mathbb{F} \mathbb{F}[z, 1/z] \).

**Proof.** Suppose first that \( \mathbb{F} = \overline{\mathbb{F}} \). Then \( X(\mathbb{F}) = X \). Set \( X_\alpha := X \cap Y_\alpha \). Then \( S_\alpha \cap X_\alpha \) defines a section of \( X_\alpha/N \), i.e., \( X_\alpha/N \cong X_\alpha \cap S_\alpha \cong S_\alpha \), where \( S_\alpha \subset (g/n)^* \) is a \( G \)-invariant (Poisson) subvariety. Therefore \( (\mathbb{F}[X][1/z])^N \) is a Poisson quotient of \( \mathcal{S}(g/n) \otimes_\mathbb{F} \mathbb{F}[z, 1/z] \).

Consider now the general case. The Galois group \( \text{Gal}_\mathbb{F}(\overline{\mathbb{F}}) \) of the field extension \( \mathbb{F} \subset \overline{\mathbb{F}} \) acts on \( (\mathbb{F}[X][1/z])^N \) and on \( \mathcal{S}(g/n) \otimes_\mathbb{F} \mathbb{F}[z, 1/z] \). Taking its fixed points on both sides, we see that the statement holds. \( \square \)

**Remark 19.** From Lemma 18 one can deduce that \( \mathbb{F}(g^*)^G = \mathbb{F}((g/n)^*)^{G/N} \otimes_\mathbb{F} \mathbb{F}(z^*) \). In particular, in this case \( \mathbb{F}(g^*)^G \) is a rational field.

Let \( H \triangleleft N \) be a connected commutative normal subgroup of \( G \) with \( \text{Lie} H = h \).

**Lemma 20.** Fix \( \alpha \in h^* \) and let \( Y_\alpha \) be the preimage of \( \alpha \) under the natural restriction \( g^* \to h^* \). Then \( Y_\alpha/H = \text{Spec} \mathbb{F}[Y_\alpha]^H \) and the restriction map \( \pi_\alpha : Y_\alpha \to (g_\alpha)^* \) defines an isomorphism \( Y_\alpha/H \cong (g_\alpha/h)^* \times \{ \alpha \} \).

**Proof.** Let \( \gamma \in g_\alpha, \xi \in h, \eta \in g \). Then

\[ (\xi - \gamma)(\eta) = \gamma(\eta, \xi) - \alpha(\eta, \xi) = -\langle \eta, \alpha \rangle(\xi). \]

Note that \( \xi(\gamma) = 0 \) and therefore \( H\gamma = \gamma + h\gamma = \gamma + (g_\alpha)^* \). Each non-zero fibre of the natural \( G_\alpha \)-equivariant restriction \( \pi_\alpha : Y_\alpha \to (g_\alpha)^* \) is exactly one \( H \)-orbit. Let us fix a decomposition \( g = g_\alpha \oplus m \). Choose any \( \alpha \in g^* \) such that \( \alpha(m) = 0 \) and \( \alpha|_h = \alpha \). Then \( Y_\alpha = (g/h)^* + \alpha \) and \( \pi_\alpha(Y_\alpha) \cong (g_\alpha/h)^* \times \{ \alpha \} \cong (g_\alpha/h)^* \times \{ \alpha \} \). \( \square \)

Until Lemma 23, we assume that \( \mathbb{F} = \overline{\mathbb{F}} \). Suppose that \( X \subset g^* \) is a closed \( G \)-invariant subset. Let \( r_\alpha \subset h^* \) denote the image of \( X \) under the restriction \( g^* \to h^* \). Set \( K := \mathbb{F}(r_\alpha) \) and

1. \( \hat{g} := \{ \xi \in g \otimes_\mathbb{F} K \mid \bar{\xi}(h) = 0 \} \),
2. \( \hat{h} := \{ \xi \in h \otimes_\mathbb{F} K \mid \alpha(\bar{\xi}(\alpha)) = 0 \text{ for each } \alpha \in r_\alpha \text{ such that } \bar{\xi}(\alpha) \text{ is defined} \} \).

Then \( \hat{g} \) is the Lie algebra of all rational maps \( \xi : r_\alpha \to g \) such that \( \bar{\xi}(\alpha) \in g_\alpha \) whenever \( \bar{\xi}(\alpha) \) is defined.
Since $\mathfrak{h}$ is a commutative ideal of $\mathfrak{g}$, we have $\mathfrak{h} \otimes \mathbb{K} \triangleleft \mathfrak{g}$. Moreover, $\mathfrak{h}$ is also an ideal of $\mathfrak{g}$. The main object of our interest is the quotient Lie algebra $\mathfrak{g} := \hat{\mathfrak{g}}/\hat{\mathfrak{h}}$. Another way to define this Lie algebra is to say that $\hat{\mathfrak{g}} := \{\xi \in \mathfrak{g} \otimes \mathbb{K} \mid [\xi, \mathfrak{h}] (\mathfrak{h}) = 0\}$.

Set $\mathcal{A} := (\mathbb{F}[X] \otimes \mathbb{F}[\mathfrak{h}])^H = \mathbb{F}[X]^H \otimes \mathbb{F}[\mathfrak{h}] \mathbb{K}$. Then the algebra $\mathcal{A}$ carries a natural Poisson structure induced from $\mathbb{F}[X]$.

**Lemma 21.** Suppose that $\mathbb{F} = \overline{\mathbb{F}}$. Then $\mathcal{A}$ is a Poisson quotient of $S(\tilde{\mathfrak{g}})$.

*Proof.* The elements of $\mathcal{A}$ and $S(\tilde{\mathfrak{g}})$ are linear combinations of rational functions on $\mathfrak{h}$ with coefficients from $\mathbb{F}[X]^H$ or $S(\mathfrak{g})$, respectively. Thus, it suffices to verify the claim at generic $\alpha \in \mathfrak{h}$.

Fix a vector space decomposition $\mathfrak{g} = \mathfrak{g}_\alpha \oplus \mathfrak{m}$ and let $s : \{\alpha\} \times (\mathfrak{g}_\alpha/\mathfrak{h})^* \to \text{Ann}(\mathfrak{m}) \subset \mathbb{Y}_\alpha$ be the corresponding section of $\pi_\alpha$. Then $S_\alpha := \text{Im} s$ is a closed subset of $\mathbb{Y}_\alpha$ and by Lemma 20, $S_\alpha \cap X \cong \pi_\alpha(\mathbb{Y}_\alpha \cap X) \cong (X \cap \mathbb{Y}_\alpha)/H$. Let $\mathcal{A}_\alpha \subset \mathcal{A}$ be the subset of elements that are defined at $\alpha$. Then for generic $\alpha \in \mathfrak{h}$, we have a surjective map

$$
\varepsilon_\alpha : \mathcal{A}_\alpha \rightarrow \mathbb{F}[\mathbb{Y}_\alpha \cap X]^H \cong \mathbb{F}[S_\alpha \cap X].
$$

At the same time, $\tilde{\mathfrak{g}}(\alpha) := \{\xi(\alpha) \mid \xi \in \hat{\mathfrak{g}}, \xi(\alpha) \text{ is defined}\} = \mathfrak{g}_\alpha$ for generic $\alpha \in \mathfrak{h}$. The algebra $\tilde{\mathfrak{g}}(\alpha) := \{\xi(\alpha) \mid \xi \in \tilde{\mathfrak{g}}, \xi(\alpha) \text{ is defined}\}$ is a 1-dimensional central extension of $\mathfrak{g}_\alpha/\mathfrak{h}$. We have $(\mathfrak{g}_\alpha/\mathfrak{h}) \oplus \mathbb{F}w$ with the Lie bracket

$$
[\xi + \mathfrak{h}, \eta + \mathfrak{h}] := ([\xi, \eta] + \mathfrak{h}) + \tilde{\mathfrak{g}}(\alpha)[\xi, \eta]w \text{ for all } \xi, \eta \in \mathfrak{g}_\alpha,
$$

where $\tilde{\alpha} \in \mathfrak{g}_\alpha^*$ is a linear function such that $\tilde{\alpha}_w = \alpha$. Hence $(\tilde{\mathfrak{g}}(\alpha))^* = (\mathfrak{g}_\alpha/\mathfrak{h})^* \times \mathbb{F}\tilde{\alpha}$ and $S_\alpha \cap X$ is a closed subset of $(\tilde{\mathfrak{g}}(\alpha))^*$.

Therefore $\mathcal{A}$ is a quotient of $S(\tilde{\mathfrak{g}})$. Since the Poisson structure on $\mathcal{A}$ is induced from $\mathbb{F}[X]$ and $X$ is a Poisson subvariety of $\tilde{\mathfrak{g}}^*$, it is indeed a Poisson quotient. \hfill $\Box$

**Remark 22.** Informally speaking, $\mathcal{A}$ is the algebra of functions on the set $\tilde{X}$ of all rational morphisms $\psi : \mathfrak{h} \rightarrow X$ such that $\psi(\alpha) \in (X \cap S_\alpha)$. Here $\tilde{X}$ is also a set of the $H$-invariant rational morphisms $\psi : \mathfrak{h} \rightarrow X$ such that $\psi(\alpha) \in (X \cap \mathbb{Y}_\alpha)$.

If $\mathbb{F} \neq \overline{\mathbb{F}}$, then it is better to work with ideals. Let $I \subset S(\mathfrak{g})$ be a $G$-invariant prime ideal. Set $I_0 = I \cap S(\mathfrak{h})$ and let $\mathfrak{h} \subset \mathfrak{h}^*$ be the subvariety defined by $I_0$. Now $\mathbb{K} = \text{Quot}S(\mathfrak{h})/I_0$ and

$$
\tilde{\mathfrak{g}} := \{\xi \in \mathfrak{g} \otimes \mathfrak{h} \mathbb{K} \mid \{\xi, \mathfrak{h}\} \subset I_0 \otimes \mathbb{K}\}.
$$

Finally set $\tilde{\mathcal{P}} := (S(\tilde{\mathfrak{g}})/I)^{\mathfrak{h}} \otimes \mathbb{F}[\mathfrak{h}] \mathbb{K}$.

**Lemma 23.** Let $\mathbb{F}$ be any field of characteristic zero. Then $\tilde{\mathcal{P}}$ is a Poisson quotient of $S(\tilde{\mathfrak{g}})$.

*Proof.* In case $\mathbb{F} = \overline{\mathbb{F}}$, $\tilde{\mathfrak{g}}$ coincides with the quotient $\hat{\mathfrak{g}}/\hat{\mathfrak{h}}$, where $\hat{\mathfrak{g}}$ and $\hat{\mathfrak{h}}$ are defined by Formulas (1) and (2). In the general case, we have $\tilde{\mathfrak{g}} \otimes \overline{\mathbb{F}} = \mathfrak{g} \otimes \mathbb{F}/h \otimes \mathbb{F}$. By Lemma 21, $\tilde{\mathcal{P}} \otimes \overline{\mathbb{F}}$ is a Poisson quotient of $S(\tilde{\mathfrak{g}}) \otimes \mathbb{F}$. The Galois group $\text{Gal}_\mathbb{F}(\overline{\mathbb{F}})$ of the field extension $\mathbb{F} \subset \overline{\mathbb{F}}$ acts on both these Poisson algebras. By taking fixed points of $\text{Gal}_\mathbb{F}(\overline{\mathbb{F}})$, we conclude that $\tilde{\mathcal{P}}$ is a Poisson quotient of $S(\tilde{\mathfrak{g}})$. \hfill $\Box$
5. Inductive Argument

Let $I \triangleleft S(\mathfrak{g})$ be a $G$-invariant (i.e., Poisson) prime ideal, set $X = \text{Spec}(S(\mathfrak{g})/I)$. Then $\mathcal{P} := \mathbb{F}[X] = S(\mathfrak{g})/I$ is a Poisson algebra. In this section, we construct a complete family in $\mathbb{F}[X]$. 

Proof of Theorem 2. Set $n := \dim X = \text{tr.deg} \mathcal{P}$, $m := \dim X - \text{rk} \mathcal{P}$. Then $n - m$ is the dimension of a generic $G(\mathbb{F})$-orbit on $X(\mathbb{F})$, and $l = l(X) = (n + m)/2$. The task is to construct $l$ functions $f_i \in S(\mathfrak{g})$ such that $\{f_i, f_j\} \in I$ and their restrictions to $X(\mathbb{F})$ are algebraically independent. We argue by induction on $\dim \mathfrak{g}$. At first it is assumed that $\mathfrak{g}$ is algebraic. The case of a non-algebraic Lie algebra $\mathfrak{g}$ is treated at the very end.

- In case of a reductive $G$, Theorem 2 is proved in Section 3. Assume therefore that $G$ is not reductive. If $I$ contains a non-trivial ideal $\mathfrak{c} \triangleleft \mathfrak{g}$, then $\mathbb{F}[X]$ is Poisson quotient of $S(\mathfrak{g}/\mathfrak{c})$ and we can replace $\mathfrak{g}$ by $\mathfrak{g}/\mathfrak{c}$ without loss of generality. Below we assume that $I \cap \mathfrak{g} = 0$.

- Let $n$ be the nilpotent radical of $\mathfrak{g}$ and $N \subset G$ the connected subgroup with $\text{Lie} N = n$.

- Suppose that $n$ is a Heisenberg Lie algebra and $\mathfrak{z} = [n, n]$ is a central subalgebra of $\mathfrak{g}$. Then Lemma 18 applies. Let $z \in \mathfrak{z}$ be a non-zero element. Since $\mathfrak{z} \not\subset I$, we have $z_{|X(\mathbb{F})} \neq 0$.

Set $\mathcal{P} := (\mathcal{P}[1/z])^N$. By Lemma 18, $\mathcal{P}$ is a Poisson quotient of $S(\mathfrak{g}/n) \otimes_{\mathbb{F}} \mathbb{F}[z, 1/z]$. The Lie algebra $\mathfrak{g}/n$ is reductive, therefore there are pairwise commuting functions $f_1, \ldots, f_k \in S(\mathfrak{g}/n) \otimes \mathbb{F}(\mathfrak{z}^*)$ such that their images form a complete family in $\mathcal{P}$. After multiplying by a common denominator, we may assume that each $f_i$ lies in $S(\mathfrak{g}/n)$.

Choose a decomposition $n = V_+ \oplus V_- \oplus \mathfrak{z}$, where $V_+$ and $V_-$ are commutative subalgebras. Recall that $n_\gamma = \mathfrak{z}$ for generic $\gamma \in X$. In case $\mathbb{F} = \mathbb{F}$, one can say immediately that a generic $(G/N)$-orbit on $\tilde{X} := \text{Spec} \mathcal{P}$ has dimension $(n - d) - (\dim n - 1)$. Hence $l(\tilde{X}) = l(X) - (\dim n - 1)/2$ in case $S(\mathfrak{z}) \cap I \neq 0$; and $l(\tilde{X}) = l(X) - (\dim + 1)/2$ otherwise. Since the numbers $l(X)$ and $l(\tilde{X})$ do not change under field extensions, the same equalities hold over any $\mathbb{F}$.

If $S(\mathfrak{z}) \cap I = 0$, then $f_1, \ldots, f_k$ together with a basis of $V_+ \oplus \mathfrak{z}$ give us a complete commutative family on $X$. If $S(\mathfrak{z}) \cap I \neq 0$, then we add a basis of $V_+$ to $\{f_i\}$ and again obtain a complete family on $X$.

- If the previous case does not hold, then either $n$ is a Heisenberg Lie algebra such that $[n, n]$ is not a central subalgebra of $\mathfrak{g}$, or $n$ contains a commutative characteristic ideal $\mathfrak{h}$ such that $\dim \mathfrak{h} > 1$, see Lemma 17. In both cases, there is a commutative ideal $\mathfrak{h} \subset n$ of $\mathfrak{g}$ such that either $[\mathfrak{g}, \mathfrak{h}] \neq 0$ or $\dim \mathfrak{h} > 1$. Set $I_0 := I \cap S(\mathfrak{h})$ and let $\mathfrak{r}_0$ be the subvariety of $\mathfrak{h}^*$ defined by $I_0$. By definition, $\mathfrak{r}_0(\mathbb{F})$ coincides with the image of the natural projection $X(\mathbb{F}) \to (\mathfrak{h} \otimes_{\mathbb{F}} \mathbb{F})^*$. The connected unipotent subgroup $H = \exp(\mathfrak{h}) < G$ will play a rôrle in the proof.

Set $\mathbb{K} := \mathbb{F}(\mathfrak{r}_0) = \text{Quot} S(\mathfrak{h})/I_0$. Consider the Lie algebra $\bar{\mathfrak{g}} := \{\xi \in \mathfrak{g} \otimes_{\mathfrak{h}} \mathbb{K} \mid \{\xi, \mathfrak{h}\} \subset I_0 \otimes \mathbb{K}\}$ and also set $\mathcal{P} := \mathcal{P}^h \otimes_{\mathbb{F}[\mathfrak{r}_0]} \mathbb{K}$. By Lemma 23, $\mathcal{P}$ is a Poisson quotient of $S(\bar{\mathfrak{g}})$.

We claim that $\mathcal{P}$ contains no zero-divisors. Indeed, suppose $x, y \in \mathcal{P}$ and $xy = 0$. After multiplying $x$ and $y$ by suitable elements of the field $\mathbb{F}(\mathfrak{r}_0)$, we may assume that $x, y \in \mathcal{P}^h$. Since $\mathcal{P}$ is a domain, either $x = 0$ or $y = 0$. 

Set $\tilde{X} := \text{Spec} \tilde{F}$. Then $\tilde{X} \subset \tilde{g}^*$ is Poisson subvariety defined over $\mathbb{K}$. Let us compute $l(\tilde{X})$. In order to simplify notation, we do it in case $F = \overline{F}$. (The numbers $l(X)$ and $l(\tilde{X})$ do not change under field extensions.)

Let $k$ be the dimension of a generic $H$-orbit on $X$. Note that $k$ is also the dimension of a generic $G$-orbit in $F_k$. Since $\mathfrak{h}$ is an algebraic Lie algebra consisting of nilpotent elements, we have $F(X)^H = \text{Quot} F[X]^H$. Therefore generic $H$-orbits on $X$ are separated by regular $H$-invariants and $\text{tr.deg} F^\mathfrak{h} = n - k$. Hence $\text{tr.deg} \tilde{F} = n - k - \dim F^\mathfrak{h}$.

Next, $\mathbb{K}(\tilde{X}) = F(X)^H \otimes_{\mathbb{K}} \mathbb{K}$. Recall that $\tilde{X}$ is a Poisson subvariety of $\tilde{g}^*$. In particular, the Poisson centre $\mathbb{Z}\mathbb{K}(\tilde{X})$ of $\mathbb{K}(\tilde{X})$ coincides with $\mathbb{K}(\tilde{X})^{\mathfrak{h}}$. Because $\mathfrak{h}$ is commutative, $\mathfrak{h} \subset F[X]^H$. Therefore the Poisson centre $\mathbb{Z}F(X)^H$ is equal to the Poisson centraliser

$$R := \{ f \in F(X) \mid \{ f, F[X]^H \} = 0 \}.$$ 

Clearly $R$ contains both $F[F^\mathfrak{h}]$ and $\mathbb{Z}F(X) = F(X)^{\mathfrak{h}}$. For generic $\gamma \in X$ we have $\dim(\mathfrak{h}|_{\mathfrak{g}^\gamma}) = \dim(\mathfrak{h}\gamma) = k$. Since all functions in $F(X)^{\mathfrak{h}}$ are constant on $G$-orbits, the subspace of $T_\gamma X$ generated by $d_F[F^\mathfrak{h}]$ and $d_F(F(X)^{\mathfrak{h}})$ has dimension $d + k$. Hence, $\text{tr.deg} R \geq d + k$. By a simple dimension reason $\text{tr.deg} R = d + k$. Since $\mathbb{Z}\mathbb{K}(\tilde{X}) = \mathbb{Z}F(X)^H \otimes_F \mathbb{K}$, we get $\text{tr.deg} \mathbb{K}(\tilde{X})^{\mathfrak{h}} = d + k - \dim F^\mathfrak{h}$. Thus

$$l(\tilde{X}) = (\dim \tilde{X} + \text{tr.deg.} \mathbb{K}(\tilde{X})^{\mathfrak{h}})/2 = (n - k - \dim F^\mathfrak{h} + d + k - \dim F^\mathfrak{h})/2 =$$

$$= (n + d)/2 - \dim F^\mathfrak{h} = l(X) - \dim F^\mathfrak{h}.$$ 

It remains to show that the dimension of $\tilde{g}$ over $F(F^\mathfrak{h})$ is less than $\dim g$. If this is not the case, then $\tilde{g} = g \otimes_F \mathbb{K}$ and $\hat{\mathfrak{h}} = 0$ (here $\hat{\mathfrak{h}}$ is the same as in (2)). From the first equality we get $[\mathfrak{g}, \mathfrak{h}] \subset I_0$, hence $[\mathfrak{g}, \mathfrak{h}] = 0$; and by the second one, $\dim \mathfrak{h} = 1$. Together these conclusions contradict the initial assumptions on $\mathfrak{h}$.

Applying the inductive hypothesis to $\tilde{X}$, we construct $l(X) - \dim F^\mathfrak{h}$ functions $\tilde{f}_i \in S(\tilde{g})$ such that their restrictions give us a complete commutative family on $\tilde{X}$. After multiplying them by a suitable element of $\mathbb{K}$, we may assume that $\tilde{f}_i \in S(g)$. The remaining $\dim F^\mathfrak{h}$ functions we get from $S(\mathfrak{h})$.

- Suppose now that $g$ is a non-algebraic Lie algebra. If the nilpotent radical $n < g$ contains a characteristic ideal $\mathfrak{h}$ such that either $\dim \mathfrak{h} > 1$ or $[\mathfrak{g}, \mathfrak{h}] \neq 0$, then the above “commutative” part of the proof (decreasing of $\dim g$) goes without any alteration. If $n = 0$, then $g$ is reductive and algebraic. It remains to consider the “Heisenberg” case.

Choose any decomposition $n = V \oplus J$ and any non-zero $\alpha \in n^*$ with $\alpha(V) = 0$. Then $\tilde{l} := \{ \xi \in g \mid [\xi, V] \subset V \}$ is a subalgebra such that $\tilde{l} \cap n = J$ and $\tilde{l} + n = g$. Since $J$ lies in the centre of $g$, the subgroup $\tilde{L} = \exp(\tilde{l}) \subset G$ can play the role of $L$ in the proof of Lemma 18. Here the conclusion is that $(\mathbb{F}[X][1/z])^{\tilde{Y}} \cong \mathbb{F}[\tilde{X}][1/z]$, where $\tilde{X} = X \cap \tilde{l}^\circ$ is the intersection in the scheme sense, and $\tilde{l}^* = \text{Ann}(V) \subset g^*$. The reduction from $g$ to $\tilde{l}$ still works, because $l(\tilde{X}) = l(X) - \frac{1}{2} \dim V$. 

\[\square\]

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