Antibrackets and localization of (path) integrals.

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Abstract

The transparent way for the invariant (Hamiltonian) description of equivariant localization of the integrals over phase space is proposed. It uses the odd symplectic structure, constructed over tangent bundle of the phase space and permits straightforward generalization for the path integrals. Simultaneously the method of supersymmetrization for a wide class of the Hamiltonian systems is derived.

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1. Recently a number of papers were published (for example, \(^1-3\)), where exact evaluation of the phase space path integrals was studied using corresponding generalization \(^1\) of the Duistermaat—Heckman localization formula \(^4\) (DH-formula). In accordance with it, if on the compact manifold \(M\) provided with the symplectic structure \(\omega = \frac{1}{2\pi} \omega_{ij} dx^i \wedge dx^j\) the Hamiltonian \(H(x)\) defines the action of the group \(U(1) \sim S^1\), then

\[
Z_0 = \int_M e^{H(\omega)} = \sum_{dH = 0} \frac{e^{H} \sqrt{\det \omega_{ij}}}{\sqrt{\det \frac{\partial^2 H}{\partial x^i \partial x^j}}},
\]

(1)

Using its path integral generalization, one can localize the phase space path integral into (finite-dimensional) integral over classical phase space.

This approach turns out convenient for a number of problems \(^2\), in topological field theories particularly. It formed the basis for a conceptually new method of description of supersymmetric theories \(^3\).

In the present letter we propose a simple method of the invariant description of DH-localization. For this, following \(^1-3\) we present the integral (1) in the form

\[
Z_0 = \frac{1}{(2\pi)^N} \int_M e^{H(x)} \sqrt{\det \omega_{ij}} d^{2N}x = \frac{1}{\pi^N} \int_M e^{H-F} d^{2N}x d^{2N}\theta,
\]

(2)

where \(\theta^i\) are auxiliary Grassmannian fields \((p(\theta^i) = p(x^i) + 1)\), which correspond to 1-forms \(dx^i\), \(\mathcal{M}\) is the supermanifold associated with the tangent bundle of \(M\) \((z^A = (x^i, \theta^i)\) are the local coordinates on \(M)\),

\[
F(z) = -\frac{1}{2} \theta^i \omega_{ij} \theta^j.
\]

(3)

After that we shall define on \(\mathcal{M}\) the odd symplectic structure. The corresponding odd Poisson brackets (antibrackets) give the Hamiltonian description (and natural interpretation) of the DH-localization without introduction of the additional structures, used in the cited papers.

Besides we show that the use of antibrackets gives the simple supersymmetrization method for the Hamiltonian systems, which define the isometries of the Riemannian metric on the their phase space.

Finally, the present constructions can be generalized straightforwardly to the case, if \(M\) is a symplectic supermanifold. Moreover, they are completely symmetrical according to the relation to initial and auxiliary coordinates.

All constructions presented in Letter relate to the finite-dimensional integrals over compact symplectic manifolds. One can accomplish their generalization for the path integrals by the lifting on the loop space analogously \(^1-3\). It does not principally change the presented description scheme.

Notice that it is naturally connected with the Batalin—Vilkovisky quantization formalism \(^5\).

2. Let us provide the supermanifold \(\mathcal{M}\), which we defined above, with odd symplectic structure

\[
\Omega_1 = \omega_{ij} dx^i \wedge d\theta^j + \omega_{ij,k} \theta^j dx^i \wedge dx^k,
\]

(4)
where \( \omega_{ij} \) corresponds to the symplectic structure on \( M \).

The corresponding to (4) odd Poisson brackets (antibrackets) are defined by the conditions:

\[
\{ x^i, x^j \}_1 = 0, \quad \{ x^i, \theta^j \}_1 = \{ \theta^i, x^j \}_1 = \omega^{ij}, \quad \{ \theta^i, \theta^j \}_1 = -\{ \theta^j, \theta^i \}_1 = \omega_{ij},
\]

where \( \omega^i_j \omega_{jk} = \delta^i_k \). The antibrackets (5-6) satisfy the Jacobi identity:

\[
(-1)^{(p(f)+1)(p(h)+1)} \{ f, \{ g, h \}_1 \}_1 + \text{cycl.perm.}(f, g, h) = 0.
\]

Let us map the functions on \( M \) to the odd functions on \( M \):

\[
f(x) \rightarrow Q_f(z) = \{ f(x), F(z) \}_1,
\]

where \( F \) is defined by the expression (3). It puts the Hamiltonian dynamics \((H(x), \omega, M)\), into odd one \((Q, \Omega_1, \mathcal{M})\), where

\[
Q = \{ H, F \}_1,
\]

with the equation of motion

\[
\frac{d}{dt} x^i = \{ x^i, Q \}_1 = \{ x^i, H_0 \}_0 \equiv \xi^i_H, \quad \frac{d}{dt} \theta^i = \{ \theta^i, Q \}_1 = \frac{\partial \xi^i_H}{\partial x^j} \theta^j.
\]

This dynamics is supersymmetric: from the closeness of \( \omega \) follows \( \{ F, F \}_1 = 0 \), and taking into account (8) we obtain the simplest superalgebra

\[
\{ H + F, H - F \}_1 = \{ H \pm F, Q \}_1 = 0.
\]

The following correspondence is obvious:

\[
\{ H, \}_1 = \xi^i_H \frac{\partial}{\partial \theta^i} \rightarrow \iota_H - \text{the operator of interior product on } \xi_H;
\]

\[
\{ F, \}_1 = \theta^i \frac{\partial}{\partial x^i} \rightarrow d - \text{the operator of exterior differentiation};
\]

\[
\{ Q, \}_1 = \xi^i_H \frac{\partial}{\partial x^i} + \xi^i_H \theta^k \frac{\partial}{\partial \theta^i} \rightarrow \mathcal{L}_H - \text{the Lie derivative along } \xi_H.
\]

Taking into account the Jacobi identity (7) we have:

\[
\{ H, F \}_1 = Q \rightarrow d \iota_H + \iota_H d = \mathcal{L}_H - \text{homotopy formula}.
\]

As we see, the supersymmetry of \((Q, \Omega_1, \mathcal{M})\) corresponds to the equivariant differentiation \(d_H = d + \iota_H\).
Following the papers \(^1\)^2, let us assume that on \(M\) the Riemannian metrics \(g_{ij}\) which is Lie-derived with \(\xi_H\) is defined also. Then the odd function

\[ \hat{Q} = \xi_H g_{ij} \theta^j \equiv \xi_i \theta^i \quad (12) \]

is the integral of motion of (4):

\[ \mathcal{L}_g = 0 \to \{ Q, \hat{Q} \} = 0. \quad (13) \]

We have also:

\[ \{ F, \hat{Q} \} = -F_2, \quad \{ H, \hat{Q} \} = H_2 \]

where

\[ H_2 = \xi^i_H g_{ij} \xi^j_H, \quad F_2 = \frac{1}{2} \theta^i \omega_{(2)ij} \theta^j, \quad \omega_{(2)ij} = \frac{\partial \xi_i}{\partial x^j} - \frac{\partial \xi_j}{\partial x^i}. \]

3. Now we shall demonstrate the derivation of DH-formula (1) using the constructions presented above.

Let us consider the integral

\[ Z = \frac{1}{(\pi)^N} \int_M \exp(H - F - \lambda \{ H + F, \hat{Q} \}) d^{4N}z, \quad (14) \]

where \(\lambda\) is arbitrary numerical parameter.

As in the first item, we assume that \(M\) is associated with the tangent bundle of the compact symplectic manifold \(M\), and define on it the odd symplectic structure (3). We also assume that the Hamiltonian \(H(\vec{x})\) defines on \(M\) the action of \(U(1) \sim S^1\), that \(M\) provided with the Riemannian structure \(g_{ij}\), which is Lie-derived with \(\xi_H\), and that \(F\) and \(\hat{Q}\) define by the expressions (3), (12).

The vector fields (11) preserve the “volume form” \(d^{4N}z = d^{2N}x d^{2N}\theta\). From (10), (13) we deduce

\[ \{ H + F, e^{H - F - \lambda \{ H + F, \hat{Q} \}} \} = 0, \quad \{ Q, e^{H - F - \lambda \{ H + F, \hat{Q} \}} \} = 0. \]

Therefore the integral (14) is invariant under equivariant and Lie transformations along \(\xi_H\). We have also

\[ \{ Q, \hat{Q} e^{H - F - \lambda \{ H + F, \hat{Q} \}} \} = 0. \]

Using these expressions and the fact that the integral of an equivariantly exact form vanishes, we show that

\[ \frac{dZ_\lambda}{d\lambda} = -\lambda \frac{1}{(\pi)^N} \int_M \{ H + F, \hat{Q} \} e^{H - F - \lambda \{ H + F, \hat{Q} \}} d^{4N}z = \]

\[ = -\lambda \frac{1}{(\pi)^N} \int_M \{ H + F, \hat{Q} e^{H - F - \lambda \{ H + F, \hat{Q} \}} \} d^{4N}z + \]

\[ + \lambda \frac{1}{(\pi)^N} \int_M \hat{Q} \{ H + F, e^{H - F - \lambda \{ H + F, \hat{Q} \}} \} d^{4N}z = 0 \]

Thus, provided the limits \(\alpha \to 0, \alpha \to \infty\) and taking into account that

\[ \delta(\xi^i_H) = \frac{1}{\pi^{2N}} \lim_{\lambda \to \infty} \sqrt{\lambda^{2N} \det g_{ij} e^{-\lambda \xi_H g_{ij} \xi_H}}, \]
we obtain the DH-localization formula:

\[ Z_0 = \frac{1}{(2\pi)^N} \int_M e^H \sqrt{\det \omega_{ij}} d^{2N}x = \lim_{\alpha \to \infty} \frac{1}{\pi^N} \int_M e^{H - F - \lambda(H^2 - F^2)} d^{4N}Z = \]

\[ = \int_M e^H \delta(\xi_H) \sqrt{\det \omega_{ij}} \sqrt{\det \frac{\partial \xi_i^j}{\partial x^j}} d^{2N}x. \]

Generalization of the presented constructions for the path integrals can be accomplished by the lifting on the loop space similarly to \[1^{-3}.\]

Then \( H \to \int A_i dx^i - H dt \) (where \( dA = \omega \)), \( \xi_H^i \to \xi_S^i = (\dot{x}^i - \xi_H^i) \), and the path integral localizes in the ordinary integral over the classical phase space.

Note that the representation of the initial integral in the form (4) formally coincides with the form of the integral from differential forms in the case where \( M \) is the supermanifold \[6.\] Note also, that the present description is symmetric according to the relation to initial and auxiliary coordinates. Then it can be generalize for the super-Hamiltonian systems.

4. If on \( M \) both symplectic and Riemannian structures are definite, then on \( \mathcal{M} \) one can also construct even symplectic structures

\[ \Omega_\alpha = \frac{1}{2} (\omega_{(\alpha)ij} + R_{ijkl} \theta^k \theta^l) dx^i \wedge dx^j + g_{ij} D\theta^i \wedge D\theta^j, \quad \alpha = 0, 2 \]

where \( D\theta^i = d\theta^i + \Gamma^i_{kl} \theta^k dx^l \); \( R_{ijkl}, \Gamma^i_{kl} \) – correspondingly curvature and connection associated with the metric \( g_{ij} \) on \( M \), \( \omega_{(0)ij} \equiv \omega_{ij} \).

It is easy to see that \((H_0 + F_2, \Omega_0, \mathcal{M}), (H_2 + F_2, \Omega_2, \mathcal{M})\), and \((Q, \Omega_1, \mathcal{M})\), define the same supersymmetric dynamics \((9)\), if \( g_{ij} \) is Lie-derived with \( \xi_H^i \).

The example of supersymmetric dynamics provided with both even and odd Hamiltonian structure (one dimensional Witten dynamics) was first proposed by D. V. Volkov et al. \[7.\] Such dynamics were considered also in \[8.\]
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