Semi-stable fibrations of generic $p$-rank 0

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1 Introduction

Let $k$ be an algebraically closed field and $\pi : X \to C$ be a semi-stable fibration of a connected proper smooth surface to a connected proper smooth curve over $k$. If the base field $k$ is a subfield of $\mathbb{C}$, the filed of complex numbers, the following semi-positivity theorem holds.

**Theorem.** (Semi-Positivity Theorem, Xiao) If $\pi : X \to C$ is a fibration of a proper smooth surface to a proper smooth curve over $\mathbb{C}$, then all the quotient bundles of $\pi_*\omega_{X/C}$ are of non-negative degree. \cite{10}, p.1

In general the semi-positivity theorem is not valid over a field of positive characteristic. In \cite{7}, Moret-Bailly constructed a semi-stable fibration $\pi_M : X_M \to \mathbb{P}^1$ of fiber genus 2 such that $R^1\pi_M_*\mathcal{O}_{X_M} = \mathcal{O}(1) \oplus \mathcal{O}(-p)$ where $p$ is the characteristic of the base filed. In the previous work, \cite{6} we have proved that for a semi-stable fibration $\pi : X \to C$, if the generic fiber is ordinary, then the semi-positivity theorem holds. Precisely, when the generic fiber of $\pi$ is ordinary, all the Harder-Narasimhan slopes of $R^1\pi_*\mathcal{O}_X$ are non-positive. For Moret-Bailly’s example, the $p$-rank of the generic fiber is 0. In particular, every special fiber of $\pi_M$ is a supersingular smooth curve of genus 2 or a union of two supersingular elliptic curves which intersect at a point transversally. In this paper we prove the following theorem which generalizes the failure of the semi-positivity theorem for Moret-Bailly’s example.

**Theorem 1.** Let $\pi : X \to C$ be a non-isotrivial semi-stable fibration of proper smooth surface to a proper smooth curve over a field of positive characteristic. If the generic $p$-rank of $\pi$ is 0, then $F_C^n R^1\pi_*\mathcal{O}_X$ has a positive Harder-Narasimhan slope for a sufficiently large $n \in \mathbb{N}$. In particular, if the genus of $C$ is 0 or 1, $R^1\pi_*\mathcal{O}_X$ has a positive Harder-Narasimhan slope.

As an application of the theorem, we obtain a result on a distribution of $p$-ranks of reductions of a certain non-closed point in the moduli space of curves over a number field.
Corollary 2.8. Suppose that $\pi : X \rightarrow C$ is a non-isotrivial semi-stable fibration of base genus 0 or 1 defined over a number field $F$, and that $U \subseteq C$ is the smooth locus of $\pi$. $\pi$ defines a non-constant morphism $f : U \rightarrow \mathcal{M}_{g,F}$. Let $P$ be the image of the generic point of $U$ under $f$. Then the reduction $P_\nu$ is not contained in the $p$-rank 0 strata for almost all $\nu$.

This result can be considered as a variation of Serre’s ordinary reduction conjecture. It is a weak statement since it is only about some non-closed points in the moduli space and 0 $p$-rank. But it is a somewhat interesting phenomenon that the semi-positivity theorem, which is concerned with a coherent module on a fiber of characteristic 0, encodes an information of $p$-ranks of the reductions.

2 Proof of Theorem 1.

We follow the terminology of [6]. Let us recall the definition of a semi-stable curve. Let $k$ be an algebraically closed field and $C$ be a projective curve over $k$.

Definition 2.1. $C$ is (semi-)stable if

1. It is connected and reduced.
2. All the singular points are normal crossing.
3. An irreducible component, which is isomorphic to $\mathbb{P}^1$, meets other components in at least 3 (resp. 2) points.

For an arbitrary base scheme, we define a (semi-)stable curve as follows.

Definition 2.2. A proper flat morphism of relative dimension 1 of schemes $\pi : X \rightarrow S$ is a (semi-)stable curve if every geometric fiber of $\pi$ is a (semi-)stable curve in the sense of definition 2.1.

In this paper, we assume $\pi : X \rightarrow C$ is a generically smooth semi-stable fibration of a proper smooth surface to a proper smooth curve over a field $k$ unless it is stated otherwise.

Definition 2.3. For a generically smooth semi-stable fibration $\pi : X \rightarrow C$ defined over a field of positive characteristic, the generic $p$-rank of $\pi$ is the $p$-rank of a geometric generic fiber of $\pi$.

2.1 Self duality of $B\omega^1$

Let $k$ be a perfect field of positive characteristic and $\pi : X \rightarrow C$ be a generically smooth semi-stable curve. Let $\omega^1_{X/C}$ be the relative dualizing line bundle for $\pi$. There is the canonical inclusion $i : \Omega^1_{X/C} \hookrightarrow \omega^1_{X/C}$ at $\nu$. At a relative smooth point for $\pi$, $i$ is an
isomorphic. On the other hand, at a relative singular point, where étale locally \( \pi \) is given by

\[
\text{Spec } k[x, y, t]/(xy - t) \rightarrow \text{Spec } k[t],
\]

\( \omega^1_{X/C} \) is a free module of rank 1 generated by \( dx/x = -dy/y \) and \( \Omega^1_{X/C} \) is a submodule of \( \omega^1_{X/C} \) generated by \( dx \) and \( dy \) via \( i \). Composing with the inclusion \( i \), we have the differential morphism \( d: \mathcal{O}_X \rightarrow \omega^1_{X/C} \). When \( F_{X/C} \) is the relative Frobenius morphism for \( \pi \), \( \pi \)

\[
\begin{align*}
X & \xrightarrow{F_{X/C}} X^p & X \\
\downarrow & \downarrow \pi^p & \downarrow \pi \\
C & \xrightarrow{F_C} C,
\end{align*}
\]

\( d: F_{X/C}^* \mathcal{O}_X \rightarrow F_{X/C}^* \omega^1_{X/C} \) is \( \mathcal{O}_{X^p} \)-linear. The kernel of \( d \) is the image of \( F_{X/C}^* : \mathcal{O}_{X^p} \rightarrow F_{X/C}^* \mathcal{O}_X \) and the image of \( d \) is denoted by \( B^1 \omega_{X/C} \) or \( B \omega^1 \). \( B \omega^1 \) is flat over \( \mathcal{O}_C \). Let \( U \hookrightarrow X \) be the smooth locus for \( \pi \). The usual Cartier isomorphism

\[
C: \Omega^1_{U/C}/B^1 \Omega_{U/C} \rightarrow \Omega_{U^p/C}
\]

is extended to an isomorphism

\[
C: \omega^1_{X/C}/B^1 \omega \rightarrow \omega^1_{X^p/C}. \quad [5], \text{p.381}
\]

Using this Cartier morphism, we have an \( \mathcal{O}_{X^p} \)-linear paring

\[
F_{X/C}^* \mathcal{O}_X \otimes F_{X/C}^* \omega^1_{X/C} \rightarrow \omega^1_{X^p/C}, \quad (\alpha, \omega) \mapsto C(\alpha \omega).
\]

This pairing induces a pairing

\[
(F_{X/C}^* \mathcal{O}_X/\mathcal{O}_{X^p}) \otimes B^1 \omega \rightarrow \omega^1_{X^p/C}.
\]

On \( U \), this pairing gives a perfect self duality. In particular, we have

\[
B^1 \Omega_{U/C} \simeq \text{Hom}(B^1 \Omega_{U/C}, \Omega^1_{U^p/C}).
\]

**Proposition 2.4.** If \( X \) is a smooth surface over a perfect field \( k \) which admits a semi-stable fibration, \( \pi: X \rightarrow C \), to a smooth curve \( C \) over \( k \), then

\[
B^1 \omega_{X/C} \simeq \text{Hom}(B^1 \omega_{X/C}, \omega^1_{X^p/C}).
\]

**Proof.** It’s enough to check that the paring is perfect at the relative singular points in \( X \). Let \( x \in X \) be a relative singular point. Étale locally, we may assume \( X = \text{Spec } A \), \( C = \text{Spec } B \) where

\[
A = k[x, y, t]/(xy - t) \simeq k[x, y], \quad B = k[t]
\]

3
and $\pi$ is the canonical morphism

$$k[t] \to k[x, y, t]/(xy - t).$$

Let $A^p = A \otimes_B (B, F_B)$. Then $A^p = k[X, Y, t]/(XY - t^p)$ and the relative Frobenius morphism is a $k$-algebra morphism $F_{A/B} : A^p \to A$ given by

$$X \mapsto x^p, \ Y \mapsto y^p \text{ and } t \mapsto xy.$$

We may regard $A^p$ is a $k$-subalgebra of $A$ generated by $x^p, y^p, xy$. As an $A^p$-module, $B^1\omega = A/A^p$ is generated by $x, x^2, \ldots, x^{p-1}, y, \ldots, y^{p-1}$.

$B^1\omega$ is a torsion free $A^p$-module and the Frac$(A^p)$-dimension of $B^1 \otimes_{A^p} \text{Frac}(A^p)$ is $p - 1$. Therefore there are only $p - 1$ obvious relations

$$t^{p-1}x = Xy^{p-1}, \ t^{p-2}x^2 = Xy^{p-2}, \ldots, tx^{p-1} = Xy$$

among the generators and we have an $A^p$-module decomposition

$$B^1\omega = \oplus_{i=1}^{p-1} < x^i, y^{p-i}>.$$

On the other hand, $\omega^1_{A/B}$ is a rank 1 free $A$-module generated by $dx/x = -dy/y$ and $\omega^1_{A^p/B}$ is a rank 1 free $A^p$-module generated by $dX/X = -dY/Y$. The Cartier morphism is the $A^p$-linear morphism satisfying

$$
\begin{align*}
\frac{dx}{x} &\mapsto dX/X, \\
\frac{xdx}{x} &\mapsto 0, \\
\vdots &\\
\frac{x^{p-1}dx}{x} &\mapsto 0, \\
\frac{ydx}{x} &\mapsto 0, \\
\vdots &\\
\frac{y^{p-1}dx}{x} &\mapsto 0.
\end{align*}
$$

In the above decomposition of $B^1\omega$, it’s easy to see that the dual of $< x^i, y^{p-i} >$ is $< x^{p-i}, y^i >$ and that the pairing $B^1\omega \otimes B^1\omega \to \omega^1_{X^p/C}$ gives a perfect duality of the dual components of both sides. This proves the claim.  

**Corollary 2.5.** Let $k$ be a perfect field of positive characteristic. Let $\pi : X \to C$ be a semi-stable fibration of a proper smooth surface to a proper smooth curve over $k$. Let $M$ and $T$ be the free part and the torsion part of $R^1\pi_*B^1\omega_{X^p/C}$ respectively and $N = R^1\pi_*\mathcal{O}_X$. Then there exists an exact sequence of coherent modules on $C$

$$0 \to M^* \to F^p_CN \to N \to M \oplus T \to 0.$$
Proof. The exact sequence of coherent $\mathcal{O}_{X^p}$-modules
\[ 0 \rightarrow \mathcal{O}_{X^p} \rightarrow F_{X/C}^* \mathcal{O}_X \rightarrow B^1 \omega_{X/C} \rightarrow 0 \]
gives a long exact sequence for the $\pi^p_*$ functor
\[ 0 \rightarrow \mathcal{O}_C \cong \mathcal{O}_C \rightarrow \pi^p_* B^1 \omega_{X/C} \rightarrow R^1 \pi^p_* \mathcal{O}_{X^p} \rightarrow R^1 \pi^p_* F_{X/C}^* \mathcal{O}_X \rightarrow R^1 \pi^p_1 B^1 \omega_{X/C} \rightarrow 0. \]
Because the Frobenius morphism of $C$ is finite flat, $R^1 \pi^p_* \mathcal{O}_{X^p} = F^*_C R^1 \mathcal{O}_X$. And the relative Frobenius morphism $F : X \rightarrow X^p$ is finite affine, so $R^1 \pi^p_* F_{X/C}^* \mathcal{O}_X = R^1 \pi_* \mathcal{O}_X$. By proposition 2.4, $B^1 \omega_{X/C} \cong H^0(B^1 \omega_{X/C}, \omega_{X^p/C})$. Since $\pi$ is relative 1-dimensional, by the relative duality theorem
\[ \pi^p_1 B^1 \omega_{X/C} = \text{Hom}(R^1 \pi^p_* B^1 \omega_{X/C}, \mathcal{O}_C). \]
Therefore the claim follows. \( \square \)

2.2 Proof of the theorem

**Theorem 1.** Let $\pi : X \rightarrow C$ be a non-isotrivial semi-stable fibration of proper smooth surface to a proper smooth curve over a field of positive characteristic. If the generic $p$-rank of $\pi$ is 0, then $F^n_X/C \mathcal{O}_X$ has a positive Harder-Narasimhan slope for a sufficiently large $n \in \mathbb{N}$. In particular, if the genus of $C$ is 0 or 1, $R^1 \pi_* \mathcal{O}_X$ has a positive Harder-Narasimhan slope.

Proof. The $n$-iterative relative Frobenius morphism in the diagram
\[
\begin{array}{ccc}
X & \xrightarrow{F^n_{X/C}} & X^n \\
\downarrow \pi^n & & \downarrow \pi^n \\
C & \xrightarrow{F^n_C} & C
\end{array}
\]
is the composition of relative Frobenius morphisms
\[ X \xrightarrow{F^n_{X/C}} X^p \xrightarrow{F^n_{X^p/C}} \cdots \xrightarrow{F^n_{X^{p^n-1}/C}} X^{p^n}. \]
$F^n_{X/C}$ gives an exact sequence of coherent $\mathcal{O}_{X^{p^n}}$-modules
\[ (*) \quad 0 \rightarrow \mathcal{O}_{X^{p^n}} \xrightarrow{F_{X/C}^{n*}} F_{X/C}^{n} \mathcal{O}_X \rightarrow E_n \rightarrow 0. \]
Here $E_n$ is flat over $\mathcal{O}_C$ and $E_1 = B^1 \omega_{X/C}$. If we denote $N = R^1 \pi_* \mathcal{O}_X$, $R^1 \pi^{p^n}_* \mathcal{O}_{X^{p^n}} = F^{n*}_C N$. Let $\lambda_n : F^{n*}_C N \rightarrow N$ be the morphism induced by $F^n_{X/C}$ in $(*)$. Because the relative Frobenius morphism commutes with a base change and the Frobenius morphism
of $C$ is flat, $\lambda_n$ is the composition of Frobenius pullbacks of $\lambda_1 : F^*_C N \to N$,

$$\lambda_n = \lambda_1 \circ \cdots \circ F^{(n-2)}_C \lambda_1 \circ F^{(n-1)}_C \lambda_1 : F^{(n-1)*}_C N \to F^{(n-1)*}_C N \to \cdots \to F^*_C N \to N.$$ 

On the other hand, the restriction of $(\ast)$ to a special fiber $X_s$ is

$$0 \to \mathcal{O}_{X_s^{p^n}} \to F^n_{X_s/k(s)} \mathcal{O}_{X_s} \to E_{n,s} \to 0.$$ 

Furthermore we have the long exact sequence

$$\cdots \to H^1(\mathcal{O}_{X_s^{p^n}}) \to H^1(\mathcal{O}_{X_s}) \to H^1(E_{n,s}) \to 0.$$ 

Since we have assumed that the generic $p$-rank of $\pi$ is 0, by the Grothendieck specialization theorem, for all $s \in C$, the $p$-rank of $X_s$ is 0. It follows that there exists $n \in \mathbb{N}$, such that $H^1(X_s, \mathcal{O}_{X_s^{p^n}}) \to H^1(X_s, \mathcal{O}_{X_s})$ is the zero morphism for all $s \in C$ such that $X_s$ is smooth. Hence $\dim H^0(X_s, E_{n,s}) = \dim H^1(X_s, E_{n,s}) = \text{gen } X_s = g$ for such $s$. But by the semi-continuity theorem, $\pi^n_* E_n$ and $R^1 \pi^n_* E_n$ are vector bundles of rank $g$. Because $N$ is also a vector bundle of rank $g$ on $C$, considering the exact sequence

$$0 \to \pi^n_* E_n \to F^{n*}_C N \xrightarrow{\lambda_n} N \to R^1 \pi^n_* E_n \to 0,$$

$\lambda_n = 0$. Now assume all the Harder-Narasimhan slopes of $F^*_C N$ are non-positive for all $i$. Let $M$ be the free part of $R^1 \pi^n_* B^1 \omega_{X/C}$. Then $\pi^n_* B^1 \omega_{X/C} = M^* = \text{Hom}_{\mathcal{O}_X}(M, \mathcal{O}_C)$.(Corollary 2.4) All the Harder-Narasimhan slopes of $F^i_* M^*$ are non-positive since $F^i_* M^*$ is a sub-bundle of $F^*_C N$, so all the Harder-Narasimhan slopes of $F^i_* M$ are non-negative. Let us consider an exact sequence

$$0 \to \text{Im } \lambda_{i-1}/ \text{Im } \lambda_i \to N/ \text{Im } \lambda_i \to N/ \text{Im } \lambda_{i-1} \to 0. \quad (i \geq 2)$$

We can also think of the free part of the above exact sequence

$$0 \to V'_i \to V_i \to V''_i \to 0.$$ 

Here $V_i$ is the free part of $N/ \text{Im } \lambda_i$ and $V''_i$ is the free part of $N/ \text{Im } \lambda_{i-1}$. $V'_i$ is the saturation of the free part of $\text{Im } \lambda_{i-1}/ \text{Im } \lambda_i$ in $V_i$. We can see $V''_i = M$ is of non-negative degree by the assumption. On the other hand, since $V'_i$ is a saturation of a quotient bundle of $F^{(i-1)*} M$, it is also of non-negative degree. Therefore, by induction $V_i$ is of non-negative degree. Since $\lambda_n = 0$ for a sufficiently large $n$, the degree of $V_n = N$ is non-negative. But since $\pi$ is non-isotrivial, $\deg N$ is strictly negative.\cite{9},p.173 It is contradiction, so $F^i_* N$ has a positive Harder-Narasimhan slope for some $i$. If the genus of the base $C$ is 0 or 1, the Frobenius pull back $F^*_C$ preserves the semi-stability of vector bundles, so $R^1 \pi_* \mathcal{O}_X$ has a positive Harder-Narasimhan slope. \[ \square \]

**Remark 2.6.** The failure of the semi-positivity theorem for Moret-Bailly’s example in the introduction is a special case of Theorem 2. Using Theorem 2, we may construct a lot of counterexamples of the semi-positivity theorem. Assume $k$ is algebraically
closed of positive characteristic. In \( M_{g,k} \), the moduli space of smooth proper curves of genus \( g \) over \( k \), the \( p \)-rank 0 strata is a closed subscheme which is purely \( 2g - 3 \) dimensional. \[3\], p.120 Let \( P \) be a 1 dimensional point in the \( p \)-rank 0 strata. By the semi-stable reduction theorem, \[1\], p.3 there is a semi-stable fibration \( \pi : X \to C \) such that the morphism \( C \to M_{g,k} \) induced by \( \pi \) sends the generic point of \( C \) to \( P \). Since the generic \( p \)-rank of \( \pi \) is 0, for a suitable Frobenius base change \( \pi^{p^n} : X^{p^n} \to C \), \( R^1 \pi^{p^n}_* \mathcal{O}_{X^{p^n}} \) has a positive slope by Theorem 1. \( X^{p^n} \) may contain isolated singularities. But the composition of \( \pi^{p^n} \) and the desingularization \( X^{(n)} \to X^{p^n}, \pi^{(n)} : X^{(n)} \to C \) is a semi-stable fibration and \( R^1 \pi^{(n)}_* \mathcal{O}_{X^{(n)}} = R^1 \pi^{p^n}_* \mathcal{O}_{X^{p^n}} \). Hence \( \pi^{(n)} \) is a counterexample to the semi-positivity theorem.

**Corollary 2.7.** Let \( F \) be a number field and suppose a semi-stable fibration \( \pi : X \to C \) is defined over \( F \). There is an integral model of \( \pi, \pi_A : X_A \to C_A \) defined over \( \text{Spec} A \), an affine open set of \( \text{Spec} \mathcal{O}_F \). Let \( \pi_v : X_v \to C_v \) be the reduction of \( \pi_A \) at a place \( v \in \text{Spec} A \). If the genus of \( C \) is 0 or 1, then the generic \( p \)-rank of \( \pi_v \) is not 0 for all but finitely many places \( v \).

**Proof.** Since the harder-Narasimhan filtration of \( R^1 \pi_{A*} \mathcal{O}_{X_A} \) on the generic fiber of \( C_A \to \text{Spec} A \) extends to a non-empty open set of \( \text{Spec} A, R^1 \pi_{v*} \mathcal{O}_{X_v} \) has no positive Harder-Narasimhan slope for almost all \( v \in \text{Spec} A \). Therefore the generic \( p \)-rank of \( \pi_v \) is not 0 by Theorem 1.

Let \( M_{g,F} \) be the moduli space of proper smooth curves over \( \text{Spec} \mathcal{O}_F \). \( M_{g,F} \), the moduli space over \( F \), is the generic fiber of \( M_{g,\mathcal{O}_F} \to \text{Spec} \mathcal{O}_F \). When \( P \) is a geometrically irreducible point of \( M_{g,F} \), the closure of \( P \) in \( M_{g,\mathcal{O}_F} \) has a geometrically irreducible reduction at almost all places \( v \). Let us denote the generic point of the reduction at \( v \) by \( P_v \). \( P_v \) is contained in a \( p \)-rank strata in \( \mathcal{O}_{g,k_v} \), the moduli space over the residue field at \( v \). We may ask the distribution of the \( p \)-ranks of \( P_v \). Serre’s ordinary reduction conjecture is a problem for closed points in the moduli space. In the language of the moduli space, Corollary 2.7 can be stated as follow.

**Corollary 2.8.** Suppose that \( \pi : X \to C \) is a non-isotrivial semi-stable fibration of base genus 0 or 1 defined over a number field \( F \), and that \( U \subseteq C \) is the smooth locus of \( \pi, \pi \) defines a non-constant morphism \( f : U \to M_{g,F} \). Let \( P \) be the image of the generic point of \( U \) under \( f \). Then the reduction \( P_v \) is not contained in the \( p \)-rank 0 strata for almost all \( v \).

**Remark 2.9.** If the fiber genus of \( \pi : X \to C \) is 2, Corollary 2.7 holds for arbitrary base \( C \). Ekedahl showed that if \( \pi : X \to C \) is a generically super-singular semi-stable fibration of fiber genus 2, then there exist a finite étale cover \( f : D \to C \) and a morphism \( g : D \to \mathbb{P}^1 \) such that \( \pi_f : X \times_C D \to D \) is isomorphic to \( \pi_{M,g} : X_M \times_{\mathbb{P}^1} D \to D \) where \( \pi_M : X_M \to \mathbb{P}^1 \) is Moret-Bailly’s fibration. \[2\], p.173 Since a pullback by a finite separable morphism of curves preserves the semi-stability of vector bundles, \( R^1 \pi_* \mathcal{O}_X \) has a positive Harder-Narasimhan slope. Considering the construction of a semi-stable fibration from a 1-dimensional point in the moduli space( Remark 2.6) and the Grothendieck specialization theorem, Corollary 2.8 holds for an arbitrary non-closed point in \( M_{2,F} \).
It is natural to expect that Corollary 2.8 holds for an arbitrary non-closed point in $\mathcal{M}_{g,F}$ for any $g$, or equivalently that Corollary 2.7 holds for arbitrary base curve $C$. If the result of Proposition 2.12 in [6] is valid over a filed of characteristic 0, i.e. the slope 0 part of $R^1\pi_*O_X$ is potentially trivial, this expectation is valid. Indeed, in the situation of Corollary 2.7 without the assumption of the base genus, if the slope 0 part, $(R^1\pi_*O_X)_0$ is potentially trivial, $(R^1\pi_*O_{X_v})_0$ is strongly semi-stable for almost all $v \in \text{Spec } A$. On the other hand, by [8], p.660, the negative slope part, $F^n_v(R^1\pi_*O_{X_v})^-$ has only negative Harder-Narasimhan slopes for any $n$ if the residue characteristic of $v$ is sufficiently large. Therefore the claim follows.

**Question 2.10.** For an arbitrary geometrically irreducible non-closed point $P$ in $\mathcal{M}_{g,F}$, is the $p$-rank of the reduction $P_v$ nonzero for almost all $v \in \text{Spec } O_F$?

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