A Note on the Characterization of Two-Dimensional Quasi-Einstein Manifolds

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Received: 26 September 2020; Accepted: 10 November 2020; Published: 12 November 2020

Abstract: In this article, we aim to introduce new classes of two-dimensional quasi-Einstein pseudo-Riemannian manifolds with constant curvature. We also give a classification of 2D quasi-Einstein manifolds of warped product type working in local coordinates. All the results are obtained by elementary methods.

Keywords: Ricci soliton; quasi-Einstein metric; curvature; potential function

1. Introduction

Nonlinear differential equations arise in wide areas of research in pure and applied sciences. In this note, we study some non-linear differential equations whose solutions are quasi-Einstein metrics that appear in mathematical physics [1,2]. In our approach, we will use the techniques of mathematical analysis, more suitable in this case, as in article [3]. The quasi-Einstein metric is a generalization of the Einstein metric, it contains gradient Ricci solitons and it is also closely related to the construction of the warped product Einstein metrics. The study of quasi-Einstein metrics was initiated by Chaki and Maity in [4]. In [5], Chaki and Ghoshal studied some global properties of quasi-Einstein manifolds, while in [6], De and Ghosh gave some examples, proven their existence, and underline some properties. As applications, gravitational instantons are defined to be solutions of the quasi-Einstein equations [7,8].

To set forth our research in this article, we organized our discussion in two parts.

In the first part of this research, we consider a pseudo-Riemannian manifold $(M, \bar{g})$, which has constant sectional curvature $K$, [9]. Given a smooth real valued mapping $F$, defined on $M$ and $\mu$ a real constant, suppose that the tensor having the components $h_{ij} = F_{ij} - \mu F_i F_j$ is non-degenerate and has constant signature. With these assumptions, we establish a link between the Christoffel symbols $\Gamma^k_{ij}$ of $h$ and the Christoffel symbols $\hat{\Gamma}^k_{ij}$ of the initial metric $\bar{g}$.

When $\bar{g}$ has constant sectional curvature $K$, it is known that $\bar{R}_{ij} = (n - 1) K \bar{g}_{ij}$. The relation which defines the quasi-Einstein pseudo-Riemannian metrics:

$$\bar{R}_{ij} + F_{ij} - \mu F_i F_j = \rho \bar{g}_{ij}$$

can be written as:

$$F_{ij} - \mu F_i F_j = [\rho - (n - 1) K] \bar{g}_{ij}.$$

This means that the Christoffel symbols $\Gamma^k_{ij}$ of $h$ and the Christoffel symbols $\hat{\Gamma}^k_{ij}$ of $\bar{g}$ have to be identical. The equalities:

$$\Gamma^k_{ij} = \hat{\Gamma}^k_{ij}, \forall i, j, k = 1, n$$

imply necessary conditions such that the initial metric $\bar{g}$, with constant curvature $K$, becomes a quasi-Einstein pseudo-Riemannian metric.
As application, we shall show that $\mathbb{R}^2$, endowed with the pseudo-Riemann metric of Kruskal-type, having null sectional curvature, is a quasi-Einstein manifold.

The second part of this research aims to continue the study of A. Pitea, recently published in [10,11]. Following this research, we introduce the explicit form of the sectional curvature under the necessary conditions that a generalized Poincaré metric $\mathfrak{g}(x,y) = \text{diag}(g(y), f(y))$ becomes quasi-Einstein. We show that if the sectional curvature $K$ is constant, then the metric $\mathfrak{g}$ becomes a quasi-Einstein one.

As application, we consider the warped product Riemannian metrics $\mathfrak{g}(x,y) = \text{diag}(g(y), 1)$, with constant sectional curvature, and we obtain a complete classification of those metrics which are quasi-Einstein too.

2. A New Class of Quasi-Einstein Metrics

Finding quasi-Einstein metrics classes has attracted the attention of much research, since there were studied in different forms in the recent past [2,7,10–13].

Consider a smooth $n$-dimensional pseudo-Riemannian manifold $(M^n, \mathfrak{g})$ and fix $F : M \to \mathbb{R}$ a smooth real valued function on $M$. One denotes with $\nabla^2 F$ its Hessian with respect to $\mathfrak{g}$.

A natural extension of the Ricci tensor $\text{Ric}$ of $\mathfrak{g}$ is the $\mu$-Bakry–Emery Ricci tensor:

$$\text{Ric}^\mu_F := \text{Ric} + \nabla^2 F - \mu dF \otimes dF$$

and hence, if $F$ is constant then $\text{Ric}^\mu_F = \text{Ric}$. For more details on the $\mu$-Bakry–Emery Ricci tensor see [14,15].

The pseudo-Riemannian manifold $(M, \mathfrak{g})$ is called quasi-Einstein if the $\mu$-Bakry–Emery Ricci tensor is a constant multiple of the metric tensor:

$$\text{Ric}^\mu_F = \rho \mathfrak{g}, \rho \in \mathbb{R}. \quad (1)$$

In local coordinates, the relation (1) becomes a system of differential equations:

$$\mathcal{R}_{ij} + F_{,ij} - \mu F_{,i}F_{,j} = \rho \mathfrak{g}_{ij}$$

for some real constants $\mu$ and $\rho$, where [13]:

$$F_i = \frac{\partial F}{\partial x^i}; \quad F_{ij} = \frac{\partial^2 F}{\partial x^i \partial x^j} - \Gamma^l_{ij} F_{,l}; \quad F_{,ij} = \frac{\partial F_{,ij}}{\partial x^k} - \Gamma^l_{kj}F_{,li} - \Gamma^l_{ki}F_{,lj}.$$  

For $\mu = 0$ the relation (1) defines gradient Ricci solitons, [16,17].

Suppose that the tensor of components $h_{ij} = F_{ij} - \mu F_{,i}F_{,j}$ is non-degenerate and has constant signature. Then $h$ is a new pseudo-Riemannian metric, which has the Levi–Civita connection $\nabla_h$ and the Christoffel symbols $\Gamma^h_{ij}$, see [18,19]. We will derive the symbols $\Gamma^h_{ij}$ in terms of the $\Gamma^\mu_{ij}$ of $\mathfrak{g}_{ij}$:

Theorem 1. Let $h^{\mu k}$ be the contravariant components of the pseudo-Riemannian metric $h$ and $\text{Ric}^\mu_{ikj}$ the curvature tensor field of $\mathfrak{g}$. Then the Christoffel symbols $\Gamma^h_{ij}$ are given by:

$$\Gamma^h_{ij} = \Gamma^\mu_{ij} + \frac{1}{2} h^{\mu k} \left[ F_{,ijk} - 2 \mu F_{,ij}F_{,k} + \left( \text{Ric}^\mu_{ikl} + \text{Ric}^\mu_{jkl} \right) F_{,j} \right].$$
Proof. From:
\[
\frac{\partial h_{ij}}{\partial x^k} = \frac{\partial}{\partial x^k} (F_{ij} - \mu F_j F_j)
\]
\[
= \frac{\partial}{\partial x^k} \left( \frac{\partial^2 F}{\partial x^i \partial x^j} - \Gamma_{ij}^r F_r \right) - \mu \left( \frac{\partial^2 F}{\partial x^k \partial x^j} F_j + F_j \frac{\partial^2 F}{\partial x^k \partial x^j} \right)
\]
\[
= \frac{\partial^3 F}{\partial x^i \partial x^j \partial x^k} - \frac{\partial \Gamma_{ij}^r}{\partial x^k} F_r - \Gamma_{ij}^r \Gamma_{kr} F_r - \mu \left( \frac{\partial^2 F}{\partial x^k \partial x^j} F_j + F_j \frac{\partial^2 F}{\partial x^k \partial x^j} \right)
\]
and the similar formulas for \( \frac{\partial h_{jk}}{\partial x^i} \) and \( \frac{\partial h_{ik}}{\partial x^j} \), we have:
\[
2h_{pk} \Gamma_{ij}^p = \frac{\partial h_{jk}}{\partial x^i} + \frac{\partial h_{ij}}{\partial x^k} - \frac{\partial h_{ij}}{\partial x^k} = \frac{\partial \Gamma_{ij}^r}{\partial x^k} F_r + \frac{\partial \Gamma_{ij}^r}{\partial x^j} F_r + \Gamma_{ij}^r \Gamma_{kr} F_r - \mu \left( \frac{\partial^2 F}{\partial x^k \partial x^j} F_j + F_j \frac{\partial^2 F}{\partial x^k \partial x^j} \right)
\]
Substituting \( \frac{\partial^2 F}{\partial x^i \partial x^j} \) by \( F_{ij} + \Gamma_{ij}^r F_r \) and using the formula for \( F_{jk} \) we obtain:
\[
2h_{pk} \Gamma_{ij}^p = F_{jk} + \left( \Gamma_{ki}^j F_{lj} + \Gamma_{kj}^i F_{lj} + \Gamma_{ij}^l F_{kl} \right) + \frac{\partial \Gamma_{ij}^r}{\partial x^k} F_r + \Gamma_{ij}^r \Gamma_{kr} F_r + \mu \left( \frac{\partial^2 F}{\partial x^k \partial x^j} F_j + F_j \frac{\partial^2 F}{\partial x^k \partial x^j} \right)
\]
We reduce the terms \( \Gamma_{ki}^j F_{lj} \) with \( \Gamma_{ik}^j F_{lj} \) and \( \Gamma_{kj}^i F_{lj} \) with \( -\Gamma_{jk}^i F_{lj} \) and we find:
\[
2h_{pk} \Gamma_{ij}^p = F_{jk} - 2 \mu F_{ij} F_k + 2 \Gamma_{ij}^l (F_{kl} - \mu F_k F_j) + \left( \frac{\partial \Gamma_{ij}^r}{\partial x^k} F_r + \Gamma_{ij}^r \Gamma_{kr} F_r \right) F_j
\]
Finally, we obtain the formula:
\[
\Gamma_{ij}^p = \Gamma_{ij}^p + \frac{1}{2} h_{pk} \left[ F_{jk} - 2 \mu F_{ij} F_k + \left( \frac{\partial \Gamma_{ij}^r}{\partial x^k} F_r + \Gamma_{ij}^r \Gamma_{kr} F_r \right) F_j \right],
\]
and the proof is complete. \( \square \)

In the following, we consider the case when \((M, \bar{g})\) has constant sectional curvature \( K \) and then:
\[
\bar{R}_{ij}^l = K \left( \bar{g}_{ij} \bar{g}_{kl} - \bar{g}_{jl} \bar{g}_{ik} \right) , \quad \bar{K}_{ij} = \bar{R}_{ij}^l = (n - 1) \bar{g}_{ij}.
\]
The relation (2) takes the form:
\[
F_{ij} - \mu F_j F_j = [\rho - (n - 1) K] \bar{g}_{ij}.
\]
From Theorem 1 we get that $\Gamma_{ij}^p = \Gamma_{ij}^p$ for all $i, j, p = 1, n$ if and only if:

$$F_{ijk} - 2\mu F_{ij} F_{ik} + K \left( 2\delta_{ij} F_{kj} - \delta_{ik} F_{j} - \delta_{ik} F_{j} \right) = 0, \quad \forall i, j, p = 1, n. \quad (3)$$

**Example 1.** We consider an open subset $M$ of $\mathbb{R}^2$ endowed with the pseudo-Riemannian metric of Kruskal-type $g(x,y) = \begin{pmatrix} 0 & \alpha(x) \beta(y) \\ \alpha(x) \beta(y) & 0 \end{pmatrix}$, where $\alpha, \beta: \mathbb{R} \to \mathbb{R}^*$ are smooth functions. In addition we assume that: $\alpha(x) > 0, \beta(y) > 0$ and $\mu > 0$. From [20] we have $K = 0$ and the Christoffel symbols:

$$\Gamma_{11}^1 = \frac{\alpha'(x)}{\alpha(x)}, \quad \Gamma_{22}^2 = \frac{\beta'(y)}{\beta(y)}, \quad \Gamma_{12}^1 = \Gamma_{21}^2 = \Gamma_{12}^2 = \Gamma_{21}^2 = 0.$$  

The system of differential Equation (3) becomes:

$$\frac{\partial}{\partial x} \left[ \frac{\partial^2 F}{\partial x^2} - \frac{\alpha'(x)}{\alpha(x)} \frac{\partial F}{\partial x} \right] - 2 \frac{\alpha'(x)}{\alpha(x)} \left[ \frac{\partial^2 F}{\partial x^2} - \frac{\alpha'(x)}{\alpha(x)} \frac{\partial F}{\partial x} \right] = 2\mu \left[ \frac{\partial^2 F}{\partial x^2} - \frac{\alpha'(x)}{\alpha(x)} \frac{\partial F}{\partial x} \right] \frac{\partial F}{\partial x} \quad (4)$$

$$\frac{\partial}{\partial y} \left[ \frac{\partial^2 F}{\partial x^2} - \frac{\alpha'(x)}{\alpha(x)} \frac{\partial F}{\partial x} \right] = 2\mu \left[ \frac{\partial^2 F}{\partial x^2} - \frac{\alpha'(x)}{\alpha(x)} \frac{\partial F}{\partial x} \right] \frac{\partial F}{\partial y} \quad (5)$$

$$\frac{\partial^3 F}{\partial x^2 \partial y} - \frac{\alpha'(x)}{\alpha(x)} \frac{\partial^2 F}{\partial x^2} = 2\mu \frac{\partial^2 F}{\partial x \partial y} \frac{\partial F}{\partial x} \quad (6)$$

$$\frac{\partial^3 F}{\partial x \partial y^2} - \frac{\beta'(y)}{\beta(y)} \frac{\partial^2 F}{\partial y^2} = 2\mu \frac{\partial^2 F}{\partial x \partial y} \frac{\partial F}{\partial y} \quad (7)$$

$$\frac{\partial}{\partial x} \left[ \frac{\partial^2 F}{\partial y^2} - \frac{\beta'(y)}{\beta(y)} \frac{\partial F}{\partial y} \right] - 2 \frac{\beta'(y)}{\beta(y)} \left[ \frac{\partial^2 F}{\partial y^2} - \frac{\beta'(y)}{\beta(y)} \frac{\partial F}{\partial y} \right] = 2\mu \left[ \frac{\partial^2 F}{\partial y^2} - \frac{\beta'(y)}{\beta(y)} \frac{\partial F}{\partial y} \right] \frac{\partial F}{\partial y} \quad (8)$$

By integrating with respect to $y$ the Equation (5), we obtain

$$\frac{\partial^2 F}{\partial x^2} - \frac{\alpha'(x)}{\alpha(x)} \frac{\partial F}{\partial x} = e^{2\mu F} \alpha(x),$$

where $a$ is a function which depends on $x$ only. Substituting in Equation (4) and then integrating with respect to $x$, we get $\ln \left[ e^{\alpha(x)} e^{2\mu F} \right] - 2 \ln a(x) = 2\mu F + b(y)$, where $b$ is a function, which depends on $y$ only. The last equation leads us to $e^{\alpha(x)} = a^2(x) e^{b(y)}$, hence $\frac{\alpha(x)}{a^2(x)} = e^{b(y)}$ must be a positive constant. Then $a(x) = c + 2 \ln a(x)$, and $b(y) = c$, so

$$\frac{\partial^2 F}{\partial x^2} - \frac{\alpha'(x)}{\alpha(x)} \frac{\partial F}{\partial x} = e^{2\mu F} \alpha^2(x).$$

In a similar manner, from the last two equations we obtain

$$\frac{\partial^2 F}{\partial y^2} - \frac{\beta'(y)}{\beta(y)} \frac{\partial F}{\partial y} = e^{2\mu F} \beta^2(y).$$

From the Equation (6) we have $\frac{\partial^2 F}{\partial x \partial y} = \alpha(x) e^{2\mu F} b^2(y)$ and from the Equation (7) we deduce $\frac{\partial^2 F}{\partial y^2} = \beta(y) e^{2\mu F} \alpha^2(x)$. The last equalities lead us to $\frac{\alpha(x)}{a^2(x)} = e^{b(y)}$, so these expressions must be a positive constant $r$. 

Therefore \( e^{(x)} = r\alpha (x), e^{(y)} = r\beta (y) \) and \( \frac{\partial^2 F}{\partial x \partial y} = r\alpha (x) \beta (y) e^{2\mu_F} \). Turning back to the initial system (2) which defines quasi-Einstein metrics, we have

\[
F_{11} - \mu F_{1} F_{1} = \rho g_{11}, \tag{10}
\]

\[
F_{12} - \mu F_{1} F_{2} = \rho g_{12}, \tag{11}
\]

\[
F_{22} - \mu F_{2} F_{2} = \rho g_{22}. \tag{12}
\]

The Equation (10) leads us to

\[
\frac{\partial F}{\partial x} = \pm \frac{1}{\sqrt{\mu}} e^{\frac{x}{2}} e^{\mu_F} \alpha (x) .
\]

The Equation (12) is equivalent to

\[
\frac{\partial F}{\partial y} = \pm \frac{1}{\sqrt{\mu}} e^{\frac{y}{2}} e^{\mu_F} \beta (y) .
\]

The Equation (11)

\[
r\alpha (x) \beta (y) e^{2\mu F} - \mu \frac{\partial F}{\partial x} \frac{\partial F}{\partial y} = \rho \alpha (x) \beta (y)
\]

becomes \( e^{2\mu_F} \left( r \pm e^{\frac{x}{2}} e^{\frac{y}{2}} \right) = \rho \). If \( r \pm e^{\frac{x}{2}} e^{\frac{y}{2}} \neq 0 \), then \( F \) is a constant, a trivial case. So we have to impose \( r = e^{\frac{x}{2}} e^{\frac{y}{2}} = 0 \) and \( \rho = 0 \). Then we choose \( \frac{\partial F}{\partial x} = -\frac{1}{\sqrt{\mu}} e^{\mu_F} e^{\frac{x}{2}} \alpha (x) \) and \( \frac{\partial F}{\partial y} = -\frac{1}{\sqrt{\mu}} e^{\mu_F} e^{\frac{y}{2}} \beta (y) \).

The Equation (10) becomes

\[
\frac{\partial F}{\partial x} e^{-\mu F} = -\frac{1}{\sqrt{\mu}} e^{\frac{x}{2}} \alpha (x),
\]

hence

\[
e^{-\mu F} = e^{\frac{x}{2}} \sqrt{\mu} \int \alpha (x) \, dx + t (y).
\]

The relation (11) can be written

\[
\frac{\partial F}{\partial y} e^{-\mu F} = -\frac{1}{\sqrt{\mu}} e^{\frac{y}{2}} \beta (y),
\]

hence

\[
e^{-\mu F} = e^{\frac{y}{2}} \sqrt{\mu} \int \beta (x) \, dx + u (x).
\]

These last two equations lead us to

\[
e^{\frac{y}{2}} \sqrt{\mu} \int \alpha (x) \, dx - u (x) = e^{\frac{x}{2}} \sqrt{\mu} \int \beta (y) \, dy - t (y).
\]

Since the left-hand side is a function with one only variable \( x \), and the right-hand side is a function with one only variable \( y \), then

\[
\left\{ e^{\frac{x}{2}} \sqrt{\mu} \int \alpha (x) \, dx - u (x) = -l
\right\}
\]

\[
\left\{ e^{\frac{y}{2}} \sqrt{\mu} \int \beta (y) \, dy - t (y) = -l
\right\}
\]

\( l \) real constant. Thus

\[
u (x) = e^{\frac{x}{2}} \sqrt{\mu} \int \alpha (x) \, dx + l, t (y) = e^{\frac{y}{2}} \sqrt{\mu} \int \beta (y) \, dy + l
\]

and

\[
e^{-\mu F} = \sqrt{\mu} \left( e^{\frac{x}{2}} \int \alpha (x) \, dx + e^{\frac{y}{2}} \int \beta (y) \, dy \right) + l.
\]
Finally, the potential function is

$$ F(x, y) = -\frac{1}{\mu} \ln \left| \sqrt{\mu} \left( e^{\frac{1}{2}} \int \alpha(x) \, dx + e^{\frac{1}{2}} \int \beta(y) \, dy \right) + 1 \right|, $$

which corresponds to $\mu \neq 0$ and $\rho = 0$.

3. Generalizing a Question of Besse

An important question was stated by Besse [21], which was how to determine examples of Einstein manifolds, which are warped products [16]. A natural generalization asks to finding examples of quasi-Einstein manifolds of warped product type. In the following, we continue the study started by Pitea [10], by finding new classes of explicit quasi-Einstein Riemannian manifolds, endowed with generalized Poincaré metrics, which also have constant sectional curvature. We consider the manifold $M$ as an open subset of $\mathbb{R}^2$, endowed with metric of diagonal type $g(x, y) = \text{diag}(g(y), f(y))$, where $g$ and $f$ are strictly positive smooth functions [16,22,23]. In order to find $g$, $f$ and $F$ which satisfy (2), Pitea [10] shows that in the case when the potential function $F$ depends only on $y$, $F$ has the form:

$$ F(y) = 2\rho \int \frac{f(y) g'(y)}{g(y)} \, dy + \ln \left( \frac{|g'(y)|}{\sqrt{f(y) g(y)}} \right). $$

Also, Pitea [10] proves that if we introduce new functions $h = \frac{g'}{g}$, $p = \frac{f'}{f}$, then from (2) $h$ and $p$ must satisfy the equation:

$$ 2\rho \left( \frac{f}{h} \right)' + \left( \frac{h'}{h} \right)' - \frac{1}{2} h' \rho' - \mu \left( \frac{2\rho f}{h} + h' + \frac{h^2}{2} - \frac{1}{2} \rho \right)^2 = \rho f + \frac{1}{4} h^2 + \rho p f \frac{1}{h} + \frac{1}{2} p h - \frac{1}{4} p^2. \quad (13) $$

Now, we introduce the expression of the sectional curvature:

$$ K = \frac{1}{f(y)} \left[ - \left( \frac{g'(y)}{2g(y)} \right)' + \frac{g'(y)}{2f(y)} \left( \frac{f'(y)}{2f(y)} - \frac{g'(y)}{2g(y)} \right) \right] $$

in the Equation (13). It follows $K = \frac{h}{2f} \left( -\frac{h'}{h} + \frac{p - h}{2} \right)$, hence

$$ \frac{h'}{h} = \frac{p - h}{2} - 2Kf. $$

Substituting in (13) we obtain:

$$ \left( \frac{2\rho f}{h} + \frac{p - h}{2} - \frac{2Kf}{h} \right)' - \frac{1}{2} h' - \mu \left( \frac{2\rho f}{h} + \frac{p - h}{2} - \frac{2Kf}{h} + h - p \right)^2 = \rho f + \frac{1}{4} h^2 + \rho p f \frac{1}{h} + p \left( \frac{p - h}{2} - 2Kf \right) - \frac{1}{4} p^2, $$

which is equivalent to:

$$ \left( \frac{2f(p - K)}{h} \right)' - \frac{1}{2} h' - \mu \frac{4f^2(p - K)^2}{h^2} = \rho f + \frac{1}{4} h^2 + \rho p f - Kpf - \frac{1}{4} p^2. $$
Substituting again $h'$ by $\frac{h(p-h)}{2} - 2Kf$ we find:

$$2(\rho-K) \left( \frac{f}{h} \right)' - 4\mu(\rho-K)^2 \frac{f^2}{h^2} = (\rho-K) f + \frac{pf(\rho - K)}{h}$$

or:

$$(\rho-K) \left[ 2f' - 4f\frac{h'}{h} - 4\mu(\rho-K) \left( \frac{f}{h} \right)^2 - f - pf \right] = 0.$$ 

But $\frac{h'}{h} = p - h - 2Kf$ and $pf = f'$. Finally, we get:

$$(\rho-K)(K - \mu \rho + \mu) = 0. \quad (14)$$

The sectional curvature $K$ must be either $K = \rho$ or $K = \frac{\mu \rho}{1+\mu}$, $\mu \neq -1$. Therefore the use of $K$ from (14) seems be the most suitable way to find quasi-Einstein manifolds with constant sectional curvature, endowed with generalized Poincaré metrics.

Hence we expressed the relation (13) according to sectional curvature $K$, obtaining the relation (14) and the following theorem.

**Theorem 2.** An open subset $M$ of $\mathbb{R}^2$ with the metric $g(x,y) = \text{diag}(g(y),1)$, having the constant sectional curvature $K = \rho$ or $K = \frac{\mu \rho}{1+\mu}$, $\mu \neq -1$ is a quasi-Einstein Riemannian manifold corresponding to the real constants $\rho$ and $\mu$.

**CASE STUDY.** In the following, we shall determine all metrics of the form $g(x,y) = \text{diag}(g(y),1)$ with constant sectional curvature. Taking $f(y) = 1$ in the above formula we obtain the sectional curvature:

$$K = -\frac{1}{2} \left( \frac{g'(y)}{g(y)} \right)' - \frac{1}{4} \left( \frac{g'(y)}{g(y)} \right)^2,$$

and the expression of potential function:

$$F(y) = 2\rho \int \frac{g(y)}{g'(y)} dy + \ln \left( \frac{g'(y)}{\sqrt{g(y)}} \right).$$

The relation $h = \frac{g'}{g}$ yields:

$$2h' = -h^2 - 4K. \quad (15)$$

The solutions of Equation (15), obtained in the general case, enable us to classify the two dimensional quasi-Einstein Riemannian metrics $g(x,y) = \text{diag}(g(y),1)$ with constant sectional curvature $K$.

**CLASS 1.** If $-h^2 - 4K = 0$ then $h' = 0$. Hence $h(y) = c$ a real constant, the metric component $g(y) = e^{\rho y}$ and the potential function is $F(y) = \frac{4\rho + c^2}{2c} y + \ln |c|$. If $K = -\rho$ then $\rho = -\frac{c^2}{4}$ and $F(y) = \ln |c|$ is trivial. If $K = \frac{\mu \rho}{1+\mu} = -\frac{c^2}{4}$ then $F(y) = -\frac{c^2}{4} y + \ln |c|$.

Therefore, we just proved the following proposition.
Proposition 1. The Riemannian manifold \((\mathbb{R}^2, g(x, y) = \text{diag}(e^y, 1))\) with sectional curvature \(K = -\frac{c^2}{4}\) is a quasi-Einstein manifold corresponding to the real constants \(\rho\) and \(\mu\), with \(\frac{\mu \rho}{1 + \mu} = -\frac{c^2}{4} < 0\), \(\mu \notin \{-1, 0\}\) and the potential function:

\[F(y) = -\frac{c}{2\mu} y + \ln |c| .\]

Class 2. If \(-h^2 - 4K \neq 0\) then we may write \(-\frac{2h'}{h^2 - 4K} = 1\) or \(\frac{h'}{h^2 + 4K} = -\frac{1}{2}\).

Class 2.1. If \(K > 0\), then we integrate \(\frac{h'(y)}{h^2(y) + (2\sqrt{K})^2} = -\frac{1}{2}\) and we have \(h(y) = -2\sqrt{K} \tan (y \sqrt{K}), g(y) = \cos^2 y \sqrt{K}\) and the potential function

\[F(y) = \left(\frac{-\rho}{K} + 1\right) \ln |\sin y \sqrt{K}| + \ln \left(2\sqrt{K}\right) .\]

If \(K = \rho\) then \(F(y) = \ln \left(2\sqrt{K}\right)\) is trivial. If \(K = \frac{\mu \rho}{1 + \mu}, \) then \(F(y) = -\frac{1}{\mu} \ln |\sin y \sqrt{K}| + \ln \left(2\sqrt{K}\right) .\)

Therefore, we just proved the following proposition.

Proposition 2. The Riemannian manifold \((\mathbb{R} \times [-\frac{\pi}{2K}, \frac{\pi}{2K}], g(x, y) = \text{diag}(\cos^2 y \sqrt{K}, 1))\) with the sectional curvature \(K > 0\) is a quasi-Einstein manifold corresponding to the real constants \(\rho\) and \(\mu\) with \(\frac{\mu \rho}{1 + \mu} = K > 0\), \(\mu \notin \{-1, 0\}\) and the potential function:

\[F(y) = -\frac{1}{\mu} \ln |\sin y \sqrt{K}| + \ln \left(2\sqrt{K}\right) .\]

Class 2.2. If \(K = 0\) then \(\frac{h'}{h^2} = -\frac{1}{2}\) hence \(h(y) = -\frac{2}{y}, g(y) = cy^2, c > 0\) and the potential function:

\[F(y) = \frac{\rho}{2} y^2 + \ln 2\sqrt{c} .\]

If \(K = \rho = 0\) then \(F(y) = \ln 2\sqrt{c}\) is trivial. If \(K = \frac{\mu \rho}{1 + \mu} = 0\) then we choose \(\mu = 0\) and in this case we obtain a gradient Ricci soliton.

Hence we just demonstrated the following proposition.

Proposition 3. The Riemannian manifold \((\mathbb{R}^2 \setminus \{0\}, g^c(x, y) = \text{diag}(cy^2, 1)), c > 0\) with the sectional curvature \(K = 0\), is quasi-Einstein corresponding to the real constants \(\rho\) and \(\mu = 0\) and with the potential function:

\[F(y) = \frac{\rho}{2} y^2 + \ln 2\sqrt{c} .\]

In fact, \((\mathbb{R}^2 \setminus \{0\}, g^c)\) is a gradient Ricci soliton; more precisely, \(g^c_1\) is the Euclidean metric in polar coordinates.

Class 2.3. If \(K < 0\) then we have \(\frac{h'}{h^2 - (2\sqrt{-K})^2} = -\frac{1}{2}\) hence \(h(y) - 2\sqrt{-K} = e^{-2\sqrt{-K} y}\) or \(h(y) + 2\sqrt{-K} = \pm e^{-2\sqrt{-K} y}.\)
If \( \frac{h(y) - 2\sqrt{K}}{h(y) + 2\sqrt{K}} = e^{-2\sqrt{-Ky}} \) then:

\[
h(y) = 2\sqrt{K} \frac{e^{2\sqrt{-Ky}}}{e^{2\sqrt{-Ky}}} = 2\sqrt{-K} \frac{\cosh(\sqrt{-Ky})}{\sinh(\sqrt{-Ky})},
\]

and the potential function:

\[
g(y) = c\sinh^2(\sqrt{-Ky}), \quad c > 0
\]

If \( K = \rho = 0 \) then \( F(y) = \ln(2\sqrt{-cK}) \) is trivial. If \( K = \frac{\mu \rho}{1 + \mu} < 0 \) then \( F(y) = -\frac{1}{\mu} \ln|\cosh\sqrt{-Ky}| + \ln(2\sqrt{-cK}) \).

We just showed the following proposition.

**Proposition 4.** The Riemannian manifold \( (\mathbb{R}^2, \bar{g}(x, y) = \text{diag}(c\sinh^2(\sqrt{-Ky}), 1)) \), \( c > 0 \) with sectional curvature \( K < 0 \) is a quasi-Einstein corresponding to the real constants \( \rho \) and \( \mu \) with \( \frac{\mu \rho}{1 + \mu} = K, \mu \not\in \{-1, 0\} \) and the potential function:

\[
F(y) = -\frac{1}{\mu} \ln|\cosh\sqrt{-Ky}| + \ln(2\sqrt{-cK})
\]

Subclass 2.3.2. If \( \frac{h(y) - 2\sqrt{K}}{h(y) + 2\sqrt{K}} = -e^{-2\sqrt{-Ky}} \), then:

\[
h(y) = 2\sqrt{-K} \frac{\sinh(\sqrt{-Ky})}{\cosh(\sqrt{-Ky})}, \quad g(y) = c\cosh^2(\sqrt{-Ky}), \quad c > 0
\]

and the potential function:

\[
F(y) = \left( -\frac{\rho}{K} + 1 \right) \ln|\sinh\sqrt{-Ky}| + \ln(2\sqrt{cK})
\]

If \( K = \rho < 0 \), then \( F(y) = \ln(2\sqrt{-cK}) \) is trivial. If \( K = \frac{\mu \rho}{1 + \mu} < 0 \) then \( F(y) = -\frac{1}{\mu} \ln|\sinh\sqrt{-Ky}| + \ln(2\sqrt{-cK}) \).

Thus, we obtain the following result.

**Proposition 5.** The Riemannian manifold \( (\mathbb{R}^2, \bar{g}(x, y) = \text{diag}(c\cosh^2(\sqrt{-Ky}), 1)) \), \( c > 0 \) with the sectional curvature \( K < 0 \) is quasi-Einstein corresponding to the real constants \( \rho \) and \( \mu \), with \( \frac{\mu \rho}{1 + \mu} = K, \mu \not\in \{-1, 0\} \) and the potential function:

\[
F(y) = -\frac{1}{\mu} \ln|\sinh\sqrt{-Ky}| + \ln(2\sqrt{-cK})
\]

**Remark 1.** The pseudo-Riemannian metric \( \bar{g}(x, y) = \text{diag}(c\cosh^2(\sqrt{-Ky}), 1) \) corresponding to \( c = -1 \) is exactly the two-dimensional version of anti de Sitter metric of General Relativity, a very important model in that theory.
4. Conclusions

In our work, we provided new classes of two-dimensional quasi-Einstein manifolds, endowed with generalized Poincaré metrics and having constant sectional curvature $K$.

These metrics generalize some remarkable metrics (for instance, Kruskal-type metric and anti de Sitter metric) and have applications in Theoretical and Experimental Physics [24–30], where geometric methods are used for physical modeling.

This use is due to the fact that the Ricci tensor for these manifolds looks like that of a perfect fluid model of general relativity.

**Funding:** The APC was funded by “Dunărea de Jos” University of Galați, Romania.

**Conflicts of Interest:** The author declares that he has no competing interests.

**Availability of Data and Material:** The datasets used or analyzed during the current study are available from the corresponding author on reasonable request.

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